INVARIANTS OF HYPERBOLIC 3-MANIFOLDS IN RELATIVE GROUP HOMOLOGY

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ABSTRACT. Let $M$ be a complete oriented hyperbolic 3–manifold of finite volume. Using classifying spaces for families of subgroups we construct a class $\beta_P(M)$ in the Hochschild relative homology group $H_3([PSL_2(\mathbb{C}) : \bar{P}]; \mathbb{Z})$, where $\bar{P}$ is the subgroup of parabolic transformations which fix $\infty$ in the Riemann sphere. We prove that the group $H_3([PSL_2(\mathbb{C}) : \bar{P}]; \mathbb{Z})$ and the Takasu relative homology group $H_3(PSL_2(\mathbb{C}), \bar{P}; \mathbb{Z})$ are isomorphic and under this isomorphism the class $\beta_P(M)$ corresponds to Zickert’s fundamental class. This proves that Zickert’s fundamental class is well-defined and independent of the choice of decorations by horospheres. We also construct a homomorphism from $H_3([PSL_2(\mathbb{C}) : \bar{P}]; \mathbb{Z})$ to the extended Bloch group $\hat{B}(\mathbb{C})$ which is isomorphic to $H_3(PSL_2(\mathbb{C}); \mathbb{Z})$. The image of $\beta_P(M)$ under this homomorphism is the $PSL$–fundamental class constructed by Neumann and Zickert.

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1. Introduction

Given a compact oriented hyperbolic 3–manifold $M$ there is a canonical representation. $\tilde{\rho}: \pi_1(M) = \Gamma \to PSL_2(\mathbb{C})$. To this representation corresponds a map $B\tilde{\rho}: M \to BPSL_2(\mathbb{C})$ where $BPSL_2(\mathbb{C})$ is the classifying space of $PSL_2(\mathbb{C})$ considered as a discrete group. There is a well-known invariant $[M]_{PSL}$ of $M$ in the...
group $H_3(PSL_2(\mathbb{C}); \mathbb{Z})$ given by the image of the fundamental class of $M$ under the homomorphism induced in homology by $B\tilde{\rho}$.

In the case when $M$ has cusps, a construction of $[M]_{PSL}$ was first described by Neumann in [13, Theorem 14.2] and a detailed proof is given by Zickert in [22]. Suppose that the complete oriented non-compact hyperbolic 3-manifold of finite volume $M$ (i.e., with cusps) is triangulated. From the triangulation a class in the (Takensu) relative homology group $H_3(PSL_2(\mathbb{C}), \bar{P}; \mathbb{Z})$ is constructed where $\bar{P}$ is the image in $PSL_2(\mathbb{C})$ of the subgroup $P$ of $SL_2(\mathbb{C})$ of matrices of the form $(\frac{\pm 1}{\pm 1} \ 0 \ \pm 1)$. Then a natural splitting of the map

$$H_3(PSL_2(\mathbb{C}); \mathbb{Z}) \to H_3(PSL_2(\mathbb{C}), \bar{P}; \mathbb{Z})$$

is used to obtain $[M]_{PSL}$. The class in $H_3(PSL_2(\mathbb{C}), \bar{P}; \mathbb{Z})$ a priori depends on choices of horoballs at the cusps, but the image $[M]_{PSL} \in H_3(PSL_2(\mathbb{C}); \mathbb{Z})$ is independent of this choice.

In the present article we generalize the construction given in Cisneros-Molina–Jones [3] using classifying spaces for families of isotropy subgroups to give an alternative construction of the class $[M]_{PSL}$ for an hyperbolic 3-manifold with cusps.

A family $\mathfrak{F}$ of subgroups of a discrete group $G$ is a set of subgroups of $G$ which is closed under conjugation and subgroups. A classifying space $E_{\mathfrak{F}}(G)$ for the family $\mathfrak{F}$ is a terminal object in the category of $G$-sets with isotropy subgroups in the family $\mathfrak{F}$. We consider the case $G = PSL_2(\mathbb{C})$ and the family $\mathfrak{F}(P)$ of subgroups generated by the subgroup $\bar{P}$. Using general properties of classifying spaces for families of isotropy subgroups and the canonical representation $\tilde{\rho} : \pi_1(M) \to PSL_2(\mathbb{C})$ we construct a canonical map $\tilde{\psi}_{\bar{P}} : \tilde{M} \to B_{\mathfrak{F}(\bar{P})}(G)$ where $\tilde{M}$ is the end compactification of $M$ and $B_{\mathfrak{F}(\bar{P})}(G)$ is the orbit space of $E_{\mathfrak{F}(\bar{P})}(G)$. We have that $H_3(\tilde{M}; \mathbb{Z}) \cong \mathbb{Z}$ and the image of the generator under the homomorphism $(\tilde{\psi}_{\bar{P}})_* : H_3(\tilde{M}; \mathbb{Z}) \to H_3(B_{\mathfrak{F}(\bar{P})}(G); \mathbb{Z})$ gives a well defined element $\beta_{\bar{P}}(M) \in H_3(B_{\mathfrak{F}(\bar{P})}(G); \mathbb{Z})$. Then we notice that the group $H_3(B_{\mathfrak{F}(\bar{P})}(G); \mathbb{Z})$ is the Hochschild relative homology group $H_3([PSL_2(\mathbb{C}) : \bar{P}]; \mathbb{Z})$ defined by Hochschild [10, §4] which in general is different from the Takasu relative homology group $H_3(PSL_2(\mathbb{C}), \bar{P}; \mathbb{Z})$ defined by Takas in [15].

Then we construct a homomorphism from $H_3([PSL_2(\mathbb{C}) : \bar{P}]; \mathbb{Z})$ to the extended Bloch group $\tilde{B}(\mathbb{C})$ which is isomorphic to $H_3(PSL_2(\mathbb{C}); \mathbb{Z})$ by Neumann [13, Theorem 2.6]. The image of $\beta_{\bar{P}}(M)$ under this homomorphism is the fundamental class $[M]_{PSL}$.

Using the relative homological algebra of [10] and results in [15] we prove that in the case of $PSL_2(\mathbb{C})$ and the subgroup $\bar{P}$ both relative homology groups coincide, that is

$$H_3([PSL_2(\mathbb{C}) : \bar{P}]; \mathbb{Z}) \cong H_3(PSL_2(\mathbb{C}), \bar{P}; \mathbb{Z}).$$

We also give an explicit expression for this isomorphism using the construction of $H_3(PSL_2(\mathbb{C}), \bar{P}; \mathbb{Z})$ given by Zickert in [22] using a complex of truncated simplices. Under such isomorphism we prove that the class in $H_3(PSL_2(\mathbb{C}), \bar{P}; \mathbb{Z})$ defined by Zickert coincides with the class $\beta_{\bar{P}}(M)$ proving that Zickert’s class is independent of the choice of horoballs.

This construction of $[M]_{PSL}$ has some advantages:

1. It does not require a triangulation of $M$. 

(2) It shows that the class $\beta_P(M)$ obtained in the (Hochschild) relative homology group $H_n([SL_2(C) : P] ; Z)$ is a well-defined invariant of $M$ which lifts the classical Bloch invariant $\beta(M)$.

(3) Choosing an ideal triangulation of $M$ one can give a explicit formula for the class $\beta_P(M)$. Computationally is seems to be more efficient to use the group $H_3([PSL_2(C) : P] ; Z)$ rather than the group $H_3(PSL_2(C), P; Z)$ since it necessary less information to describe elements in the former group than in the later one.

In [13] Proposition 2.5, Theorem 2.6] Neumann defined a homomorphism $\hat{L}: \hat{\mathbb{P}}(C) \to \mathbb{C}/\pi^2\mathbb{Z}$ and proved that it corresponds to the Cheeger–Chern–Simons class of the universal flat $PSL_2(C)$ over the classifying space $BPSL_2(C)$, where $PSL_2(C)$ is considered as a discrete group. Hence the image of the $PSL$–fundamental class under $\hat{L}$ gives the complex volume of $M$ (times $i$), ie

$$\hat{L}([M]_{PSL}) = i(\mathrm{Vol}(M) + i\mathrm{CS}(M)) \in \mathbb{C}/i\pi^2\mathbb{Z},$$

where $\mathrm{Vol}(M)$ is the volume of $M$ and $\mathrm{CS}(M)$ is (a multiple) of the Chern–Simons invariant.

Our construction also works in the general context of $(G,H)$–representations of tame $n$–manifolds considered in [22]. In Section 10 we prove that any $G$–representation mapping boundary curves to conjugates of a fixed subgroup $H$ gives a well-defined class in the Hochschild relative homology group $H_n([G : H]; Z)$. In this general context the Hochschild and Takasu relative homology groups do not necessarily coincide, the class in $H_n(G,H; Z)$ constructed by Zickert in [22] §5 depends of a choice of decoration and different classes given by different choices are all mapped to the class in $H_n([G : H]; Z)$ by a canonical homomorphism $H_n(G,H; Z) \to H_n([G : H]; Z)$. So in this general context it is more appropriate to use Hochschild relative group homology than Takasu relative group homology because we obtain classes independent of choice.

In the case of boundary-parabolic $PSL_2(C)$–representations $\rho$ of tame 3–manifolds the construction for the geometric representation generalizes and we obtain a well-defined $\beta_P(\rho)$ class in $H_3(PSL_2(C), P; Z) \cong H_3([PSL_2(C) : P] ; Z)$ and therefore a well-defined $PSL$–fundamental class $[\rho]_{PSL} \in H_3(PSL_2(C); Z)$. Then we can define as in Zickert [22] §6] the complex volume of a boundary-parabolic representation $\rho$ by

$$i(\mathrm{Vol}(\rho) + i\mathrm{CS}(\rho)) = \hat{L}([\rho]_{PSL}).$$

Using an ideal triangulation of $M$ we obtain an explicit formula for the complex volume of $\rho$.

2. Classifying spaces for families of isotropy subgroups

In this section we define the classifying spaces for families of subgroups and we give their main properties. We recommend the survey article by Lück [12] and the bibliography there for more details.

Let $G$ be a discrete group. Let $X$ be a $G$–space. For each subgroup $H$ of $G$, we define the set $X^H = \{ x \in X \mid h \cdot x = x \text{ for all } h \in H \}$ of fixed points of $H$. We denote by $G_x = \{ g \in G \mid g \cdot x = x \}$ the isotropy subgroup fixing $x \in X$. More generally, let $Y \subset X$ be a subspace, then $G_Y = \bigcap_{y \in Y} G_y$ is the isotropy subgroup (pointwise) fixing $Y$. We also denote by $G_{(Y)} = \{ g \in G \mid g \cdot Y = Y \}$ the subgroup leaving $Y$ invariant. Note that in general $G_Y \subset G_{(Y)}$. 
Since $G$ is discrete, a $G$–$CW$–complex is an ordinary $CW$–complex $X$ together with a continuous action of $G$ such that,

1. for each $g \in G$ and any open cell $\sigma$ of $X$, the translation $g \cdot \sigma$ is again an open cell of $X$.
2. if $g \cdot \sigma = \sigma$, then the induced map $\sigma \rightarrow \sigma$ given by the translation $x \mapsto g \cdot x$ is the identity, i.e., if a cell is fixed by an element of $G$, it is fixed pointwise, in other words $G(\sigma) = G\sigma$.

See for instance tom Dieck [19, Proposition II.1.15].

Remark 2.1. Notice that in a $G$–$CW$–complex $X$ for each open cell $\sigma$ of $X$ one has $G\sigma = Gx$ for every $x \in \sigma$. Hence

$$\{G_\sigma \mid \sigma \text{ is a cell of } X\} = \{G_x \mid x \in X\},$$

ie the set of isotropy subgroups of the points of $X$ is the same as the set of isotropy subgroups of the cells of $X$.

A family $\mathcal{F}$ of subgroups of $G$ is a set of subgroups of $G$ which is closed under conjugation and taking subgroups. Let $\{H_i\}_{i \in I}$ be a set of subgroups of $G$, we denote by $\mathcal{F}(H_i)$ the family consisting of all the subgroups of the $\{H_i\}_{i \in I}$ and all their conjugates by elements of $G$.

Let $\mathcal{F}$ be a family of subgroups of $G$. A model for the classifying space for the family $\mathcal{F}$ of subgroups is a $G$–$CW$–complex $E_{\mathcal{F}}(G)$ which has the following properties:

1. All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$.
2. For any $G$–$CW$–complex $Y$, whose isotropy groups belong to $\mathcal{F}$, there is up to $G$–homotopy a unique $G$–map $Y \rightarrow E_{\mathcal{F}}(G)$.

In other words, $E_{\mathcal{F}}(G)$ is a terminal object in the $G$–homotopy category of $G$–$CW$–complexes, whose isotropy groups belong to $\mathcal{F}$. In particular two models for $E_{\mathcal{F}}(G)$ are $G$–homotopy equivalent and for two families $\mathcal{F}_1 \subseteq \mathcal{F}_2$ there is up to $G$–homotopy precisely one $G$–map $E_{\mathcal{F}_1}(G) \rightarrow E_{\mathcal{F}_2}(G)$.

Remark 2.2. There is another version for the classifying space for the family $\mathcal{F}$ in the category of $\mathcal{F}$–numerable $G$–spaces (see Luck [12, Definition 2.1] or tom Dieck [20, page 47] for the definition), but both versions are $G$–homotopy equivalent when $G$ is a discrete group [12, Theorem 3.7]. Moreover, any $G$–$CW$–complex with all its isotropy groups in the family $\mathcal{F}$ is $\mathcal{F}$–numerable [12, Lemma 2.2], thus we can work with any of the two versions.

There is a homotopy characterization of $E_{\mathcal{F}}(G)$ which allows us to determine whether or not a given $G$–$CW$–complex is a model for $E_{\mathcal{F}}(G)$.

Theorem 2.3 ([12, Theorem 1.9]). A $G$–$CW$–complex $X$ is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy groups belong to $\mathcal{F}$ and the $H$–fixed point set $X^H$ is weakly contractible for each $H \in \mathcal{F}$ and is empty otherwise. In particular, $E_{\mathcal{F}}(G)$ is contractible.

There are different ways of constructing models for $E_{\mathcal{F}}(G)$. We consider the following.
2.1. Simplicial Construction. The following proposition gives a simplicial construction of a model for $E_\mathfrak{F}(G)$, compare with Farrell–Jones [9, Theorem A.3].

Let $\{H_i\}_{i \in I}$ be a set of subgroups of $G$ such that every group in $\mathfrak{F}$ is conjugate to a subgroup of an $H_i$, that is, $\mathfrak{F} = \mathfrak{F}(H_i)$. Consider the disjoint union $\Delta_{\mathfrak{F}} = \coprod_{i \in I} G/H_i$, or more generally, let $X_{\mathfrak{F}}$ be any $G$–set such that $\mathfrak{F}$ is precisely the set of subgroups of $G$ which fix at least one point of $X_{\mathfrak{F}}$. Notice that $\Delta_{\mathfrak{F}}$ is an example of such a $G$–set.

**Proposition 2.4.** Let $X_{\mathfrak{F}}$ be as above. A model for $E_{\mathfrak{F}}(G)$ is the geometric realization $Y$ of the simplicial set whose $n$–simplices are the ordered $(n + 1)$–tuples $(x_0, \ldots, x_n)$ of elements of $X_{\mathfrak{F}}$. The face operators are given by

$$d_i(x_0, \ldots, x_n) = (x_0, \ldots, \widehat{x_i}, \ldots, x_n),$$

where $\widehat{x_i}$ means omitting the element $x_i$. The degeneracy operators are defined by

$$s_i(x_0, \ldots, x_n) = (x_0, \ldots, x_i, \ldots, x_n).$$

The action of $g \in G$ on an $n$–simplex $(x_0, \ldots, x_n)$ of $Y$ gives the simplex $(gx_0, \ldots, gx_n)$.

**Proof.** Let $\sigma = (x_0, \ldots, x_n)$ be an $n$–simplex of $Y$. By the definition of the action of $G$ and since $\mathfrak{F}$ is closed with respect to subgroups we have that

$$G_\sigma = \bigcap_{i=0}^n G_{x_i} \subset \mathfrak{F}.$$

Hence by Remark 2.1 we have that all the isotropy subgroups of points of $Y$ belong to the family $\mathfrak{F}$. Let $H \in \mathfrak{F}$, then its set of fixed points is given by

$$Y^H = \left\{ (x_0, \ldots, x_n) \mid H \subset \bigcap_{i=1}^n G_{x_i}, \ n = 0, 1, \ldots \right\}$$

Notice that if $\sigma$ is a cell in $Y^H$ then $d_i \sigma \subset Y^H$ and $s_i \sigma \subset Y^H$, and therefore $Y^H$ is a simplicial subset of $Y$. Let $x \in X_{\mathfrak{F}}$ such that $h \cdot x = x$ for every $h \in H$, that is, $x$ is fixed by $H$, in other words, $H \subset G_x$. Such an $x$ exists by definition of $X_{\mathfrak{F}}$ and since $H \in \mathfrak{F}$. We shall see that $Y^H$ is contractible defining a contracting homotopy $c$ of $Y^H$ to the vertex $(x)$. Let $\sigma = (x_0, \ldots, x_n)$ be an arbitrary cell in $Y^H$, define

$$c(\sigma) = (x, x_0, \ldots, x_n).$$

It is straightforward to see that $d_0 \circ c = Id$ and $d_{i+1} \circ c = c \circ d_i$ for $i > 1$. Therefore $c$ indeed defines a contracting homotopy. This shows $Y^H$ is contractible and, therefore by Theorem 2.3 $Y$ is a model for $E_{\mathfrak{F}}(G)$, see also [9, Theorem A.2].

**Remark 2.5.** Let $H$ be a subgroup of $G$ and consider the family $\mathfrak{F}(H)$ which consists of all the subgroups of $H$ and their conjugates by elements of $G$. In this case we can take $X_{\mathfrak{F}(H)} = G/H$.

**Remark 2.6.** When $\mathfrak{F} = \{1\}$, the above construction corresponds to the universal bundle $EG$ of $G$, i.e., it corresponds to the standard $G$–resolution of $Z$.

The $G$–orbit space of $EG$ is the classical classifying space $BG$ of $G$. In analogy with $BG$, we denote by $B_{\mathfrak{F}}(G)$ the $G$–orbit space of $E_{\mathfrak{F}}(G)$. Thus when $\mathfrak{F} = \{1\}$, we have that $B_{\{1\}}(G) = BG$. 


Let $H$ be a subgroup of $G$. For a $G$–space $X$, let $\text{res}^G_H X$ be the $H$–space obtained by restricting the group action. If $\mathcal{H}$ is a family of subgroups of $G$, let $\mathcal{H}/H = \{ L \cap H \mid L \in \mathcal{H} \}$ be the induced family of subgroups of $H$.

**Proposition 2.7** ([19 Proposition 7.2.4], [9 Proposition A.5]).

$$\text{res}^G_H E_\mathcal{H}(G) = E_{\mathcal{H}/H}(H).$$

### 3. Relative group homologies

In this section we define Hochschild and Takasu group homologies and we give a condition where both homologies coincide.

Since a $G$–set $X$ is equivalent to a homomorphism from the group $G$ into the group of permutations of $X$, in the literature the pair $(G, X)$ is sometimes called a *permutation representation*. For any $G$–set $X$ we construct a complex $(C_\ast(X), \partial_\ast)$ of abelian groups by letting $C_n(X)$ be the free abelian group generated by the ordered $(n + 1)$–tuples of elements of $X$. Define the $i$–th face homomorphism $d_i : C_n(X) \to C_{n-1}(X)$ by

$$d_i(x_0, \ldots, x_n) = (x_0, \ldots, \hat{x}_i, \ldots, x_n),$$

where $\hat{x}_i$ denotes deletion, and the boundary homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$ by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i.$$

The standard direct computation shows that $\partial_n \circ \partial_{n+1} = 0$ proving that $C_n(X)$ is indeed a complex. Define $C_{-1}(X) = \mathbb{Z}$ as the infinite cyclic group generated by $(\ )$ and define $\partial_0(x) = (\ )$ for any $x \in X$. Notice that this extended complex is precisely the augmented complex

$$C_n(X) \to C_{n-1}(X) \to \cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$  

with $\varepsilon = \partial_0$ the augmentation homomorphism.

The action of $G$ on $X$ induces an action of $G$ on $C_n(X)$ with $n \geq 0$ given by

$$g \cdot (x_0, \ldots, x_n) = (g \cdot x_0, \ldots, g \cdot x_n)$$

which endows $C_n(X)$ with the structure of a $G$–module. We also let $G$ act on $C_{-1}(X) = \mathbb{Z}$ trivially.

For each $x \in X$ and $n \geq -1$ define the map $s^x_n : C_n(X) \to C_{n+1}(X)$ given by

$$(3.1) \quad s^x_n(x_0, \ldots, x_n) = (x, x_0, \ldots, x_n).$$

We shall use later the following lemma (see [16 Proposition 1.1] and [1 §6])

**Lemma 3.1.** Let $G_x$ be the isotropy subgroup of $x$. Then $s^x_n$ is a $G_x$–homomorphism.

**Proof.** Let $g \in G_x$. Then we have

$$s^x_n(g(x_0, \ldots, x_n)) = s^x_n(g \cdot g \cdot x_0, \ldots, g \cdot x_n) = (x, g \cdot x_0, \ldots, g \cdot x_n) = (g \cdot x, g \cdot x_0, \ldots, g \cdot x_n) = g s^x_n(x_0, \ldots, x_n).$$

$\square$

**Proposition 3.2.** The augmented complex $(C_\ast(X), \partial_\ast)$ is acyclic. Hence it is a $G$–resolution of the trivial $G$–module $\mathbb{Z}$. 

Proof. Let $x \in X$ and consider the homomorphisms $s^x_n : C_n(X) \to C_{n+1}(X)$ given by (\ref{homology}) for $n \geq -1$. It is straightforward to see that $d_0 \circ s^x_n = 1d$ and $d_{i+1} \circ s^x_n = s^x_{n-1} \circ d_i$ for $i > 1$. Therefore $s^x$ defines a contracting homotopy and the augmented complex is acyclic. \hfill \Box

Denote by $(B_\ast(X), \partial, \otimes id_Z)$ the complex given by

(3.2) \hspace{1cm} B_\ast(X) = C_\ast(X) \otimes_{Z[G]} Z.

As usual, here we see $C_\ast(X)$ as a right $G$–module by defining

$$(x_0, \ldots, x_n) \cdot g := g^{-1} \cdot (x_0, \ldots, x_n).$$

The groups

$$H_n(G, X; Z) = H_n(B_\ast(X)),$$

are the homology groups of the permutation representation $(G, X)$. The corresponding cohomology groups were defined by Snapper in \cite{16}.

Remark 3.3. One can define the homology groups of the permutation representation $(G, X)$ with coefficients in a $G$–module $A$ by defining $B_\ast(X; A) = C_\ast(X) \otimes_{Z[G]} A$ and

$$H_n(G, X; A) = H_n(B_\ast(X; A)),$$

but in the present article we only will use integer coefficients.

The following lemma is a small generalization of Dupont–Zickert \cite[Lemma 1.3]{8}.

Lemma 3.4. Let $\bar{C}_\ast(X)$ be a $G$–subcomplex of the augmented complex $C_\ast(X)$. Suppose that for each cycle $\sigma$ in $\bar{C}_n(X)$ there exists a point $x(\sigma) \in X$ such that $s^{x(\sigma)}_n(\sigma) \in \bar{C}_{n+1}(X)$, where $s^n_\ast$ is given by (3.1). Then $\bar{C}_\ast(X)$ is acyclic.

Same proof as \cite[Lemma 1.3]{8}.

3.1. Group homology. If $X$ is a free $G$–set, then $C_\ast(X)$ is a free $G$–module and $C_\ast(X)$ is a free $G$–resolution of the trivial $G$–module $Z$ \cite[Remark 1.3]{10}. In particular, if $X$ is the group $G$ acting on itself by left multiplication, then $C_\ast(G)$ is the standard free $G$–resolution of the trivial $G$–module $Z$ (see for instance Brown \cite[§1.5]{11}). Hence the complex $B_\ast(G)$ computes the homology of $G$.

3.2. Hochschild relative group homology. If $H$ is a subgroup of $G$ and $X = G/H$ then we have that the homology of the complex $B_\ast(G/H)$ gives the Hochschild relative group homology groups

(3.3) \hspace{1cm} H_i([G : H]; Z) = H_i(B_\ast(G/H)), \hspace{1cm} n = 0, 1, 2, \ldots,

defined by Hochschild in \cite[§4]{13}. We use the notation analogous to the one used by Adamson in \cite{1} for the corresponding cohomology groups. More generally, let $X_{(H)}$ be a $G$–set, such that the action of $G$ on $X$ is transitive and the set of isotropy subgroups of points in $X_{(H)}$ is the conjugacy class of $H$ in $G$. Then $X_{(H)}$ and $G/H$ are isomorphic as $G$–sets and we have that the homology of the transitive permutation representation $(G, X_{(H)})$ is the Hochschild relative group homology, that is,

$$H_n([G : H]; Z) = H_n(G, X_{(H)}; Z) = H_n(B_\ast(X_{(H)})), \hspace{1cm} n = 0, 1, 2, \ldots.$$ 

Notice that when $H = \{1\}$ we recover the homology groups $H_n(G; Z)$ of $G$.

Analogous to group homology, the relative homology groups $H_n([G : H]; Z)$ can be computed using relative projective resolutions. What follows summarizes the
results we need about relative homological algebra given in [10] §1, §2 and §4, for
the particular case of the ring $\mathbb{Z}[G]$ and its subring $\mathbb{Z}[H]$ for a discrete group $G$ and
a subgroup $H$. For completeness, we also give proofs of some results which are only
indicated in [10].

An exact sequence of $\mathbb{Z}[G]$–homomorphisms between $\mathbb{Z}[G]$–modules,

$$N_{i+1} \xrightarrow{t_{i+1}} N_i \xrightarrow{t_i} N_{i-1}$$

is called $(\mathbb{Z}[G], \mathbb{Z}[H])$–exact if, for each $i$, the kernel of $t_i$ is a direct $\mathbb{Z}[H]$–module
summand of $N_i$,

**Proposition 3.5.** A sequence $N_{i+1} \xrightarrow{t_{i+1}} N_i \xrightarrow{t_i} N_{i-1}$ of $\mathbb{Z}[G]$–homomorphisms is
$(\mathbb{Z}[G], \mathbb{Z}[H])$–exact, if and only if, for each $i$:

1. $t_i \circ t_{i+1} = 0$, and
2. there exists a contracting $\mathbb{Z}[H]$–homotopy, i.e., a sequence of $\mathbb{Z}[H]$–homomorphisms

$$h_i : M_i \rightarrow M_{i+1}$$

such that $t_{i+1} \circ h_i + h_{i-1} \circ t_i$ is the identity map of $M_i$

Proof. Let $N_{i+1} \xrightarrow{t_{i+1}} N_i \xrightarrow{t_i} N_{i-1}$ be a $(\mathbb{Z}[G], \mathbb{Z}[H])$–exact sequence of $\mathbb{Z}[G]$–homomorphisms. As usual, we denote $Z_i = \ker t_i$ and $B_i = \operatorname{im} t_{i+1}$. Since $t_i$

is $(\mathbb{Z}[G], \mathbb{Z}[H])$–exact, it is an exact sequence of $\mathbb{Z}[G]$–homomorphisms. Hence

$t_i \circ t_{i+1} = 0$ and $Z_i/B_i = 0$ for every $i$. Thus $B_i = Z_i$ and the short sequences

$$0 \rightarrow Z_{i+1} \hookrightarrow N_{i+1} \xrightarrow{t_{i+1}} Z_i \rightarrow 0$$

are exact. Since $t_i$ is $(\mathbb{Z}[G], \mathbb{Z}[H])$–exact, we have that $N_{i+1} = Z_{i+1} + C_{i+1}$ as

$\mathbb{Z}[H]$–modules for some $\mathbb{Z}[H]$–module $C_{i+1}$. Hence the sequences (3.4) split and

there exists a contracting $\mathbb{Z}[H]$–homotopy (see [3] Proposition (0.3)).

Conversely, suppose the sequence $N_{i+1} \xrightarrow{t_{i+1}} N_i \xrightarrow{t_i} N_{i-1}$ of $\mathbb{Z}[G]$–homomorphisms

satisfies (1) and (2). By (1) we have that $\operatorname{im} t_{i+1} \subset Z_i$ and by (2), if $x \in Z_i$ we have that

$$t_{i+1}(h(x)) = x.$$ 

Thus $x \in \operatorname{im} t_{i+1}$ and the sequence $t_i$ is exact, that is, $Z_i = B_i$. Hence the short sequences

$$0 \rightarrow Z_{i+1} \hookrightarrow N_{i+1} \xrightarrow{t_{i+1}} Z_i \rightarrow 0$$

are exact. Moreover, by (3.5) we also have that $h|Z_i$ is a section of $t_{i+1}$ and therefore the sequence splits. Since $h$ is an $\mathbb{Z}[H]$–homomorphism, we have that $Z_{i+1}$ is a direct $\mathbb{Z}[H]$–module summand of $N_{i+1}$. Thus $t_i$ is $(\mathbb{Z}[G], \mathbb{Z}[H])$–exact. \qed

A $G$–module $A$ is said to be $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective if, for every $(\mathbb{Z}[G], \mathbb{Z}[H])$–

exact sequence $0 \rightarrow U \xrightarrow{p} V \xrightarrow{q} W \rightarrow 0$, and every $G$–homomorphism $\psi : A \rightarrow W$,

there is a $G$–homomorphism $\psi' : A \rightarrow V$ such that $q \circ \psi' = \psi$. This is shown in the

following commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\psi} & W \\
\downarrow{\psi'} & & \\
0 & \xrightarrow{p} & V & \xrightarrow{q} & W & \rightarrow 0.
\end{array}
$$

**Lemma 3.6 ([10] Lemma 2).** For every $\mathbb{Z}[H]$–module $A$, the $\mathbb{Z}[G]$–module $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A$

is $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective.
If $N$ is any $G$–module, the natural map
\[
\theta: \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N \rightarrow N
\]
(3.6)
\[g \otimes n \mapsto gn\]
gives rise to an exact sequence (see [10, III (3.4)])
\[
0 \rightarrow K_N \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N \rightarrow N \rightarrow 0.
\]
(3.7)
The map $\phi: N \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$ given by $n \mapsto 1 \otimes n$ is an $H$–homomorphism since
\[
\phi(hn) = 1 \otimes hn = h \otimes n = h(1 \otimes n) = h\phi(n), \quad \text{for every} \ h \in H.
\]
Notice that if $h$ is in $\mathbb{Z}[G]$ but not in $\mathbb{Z}[H]$ we cannot perform the second step, so $\phi$ in general is not an $G$–homomorphism. Since $\theta \circ \phi(n) = \theta(1 \otimes n) = n$, $\phi$ is a section of $\theta$ and it is an $H$–isomorphism of $N$ onto a $\mathbb{Z}[H]$–module complement of $K_N$ in $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$, showing that the exact sequence is actually $(\mathbb{Z}[G], \mathbb{Z}[H])$–exact.

If $N$ is $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective, considering the identity map on $N$, there exists $\psi': N \rightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$ which makes the following diagram of $G$–homomorphisms commute

\[
\begin{array}{ccc}
0 & \longrightarrow & K_N \\
\downarrow & & \downarrow id \\
\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N & \longrightarrow & N & \longrightarrow & 0.
\end{array}
\]

It follows that the sequence is $(\mathbb{Z}[G], \mathbb{Z}[G])$–exact, so that $N$ is $G$–isomorphic with a direct $G$–module summand of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$.

**Proposition 3.7** ([10] page 249). A direct $G$–module summand of a $(\mathbb{Z}[G],\mathbb{Z}[H])$–projective module is also $(\mathbb{Z}[G],\mathbb{Z}[H])$–projective.

**Proof.** Let $A$ be a direct $G$–module summand of the $(\mathbb{Z}[G],\mathbb{Z}[H])$–projective module $N$. Let $p: N \rightarrow A$ be the projection and $i: A \rightarrow N$ be the inclusion. Let
\[
0 \rightarrow U \rightarrow V \xrightarrow{q} W \rightarrow 0
\]
be an $(\mathbb{Z}[G],\mathbb{Z}[H])$–exact sequence and let $\psi: A \rightarrow W$ be an $G$–homomorphism. Consider the composition $\psi \circ p: N \rightarrow W$. Since $N$ is $(\mathbb{Z}[G],\mathbb{Z}[H])$–projective there exists a $G$–homomorphism $\phi: N \rightarrow V$ such that the following diagram commutes

\[
\begin{array}{ccc}
N & \xrightarrow{i} & A \\
\downarrow \phi & & \downarrow \psi \\
0 & \longrightarrow & U \\
\end{array}
\]

Thus $\psi' = \phi \circ i: A \rightarrow V$ is such that
\[
q \circ \psi' = q \circ \phi \circ i = \psi \circ p \circ i = \psi.
\]
Therefore $A$ is $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective. \qed

**Corollary 3.8** ([10] page 249). A $G$–module $N$ is $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective if and only if it is $G$–isomorphic with a direct $G$–module summand of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$, or if and only if $K_N$ is a direct $G$–module summand of $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$. 

A $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective resolution of a $\mathbb{Z}[G]$–module $N$ is a $(\mathbb{Z}[G], \mathbb{Z}[H])$–exact sequence $\cdots \to C_1 \to C_0 \to N \to 0$ in which each $C_i$ is $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective.

**Proposition 3.9** ([10, page 250]). Every $\mathbb{Z}[G]$–module $N$ has a $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective resolution.

**Proof.** By Lemma 3.6 $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$ is $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective. Take $C_0 = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$, with the natural map $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N \to N$ which by the exact sequence (3.7) is an epimorphism. Then proceed in the same way from the kernel $K_N$ of this map in order to obtain $C_1 = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} K_N$, with the map $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} K_N \to K_N \hookrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N$, etc.

The $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective resolution obtained in the proof of Proposition 3.9 is called in [10, §2] the standard $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective resolution of $N$.

**Proposition 3.10** ([10, page 260]). The augmented complex $C_*(G/H)$ is a $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective resolution of $\mathbb{Z}$.

**Proof.** The $\mathbb{Z}[G]$–module $C_n(G/H)$ as abelian group is generated by $n + 1$–tuples $(g_0 H, \ldots, g_n H)$ of cosets of $H$. By Lemma 3.1 the homomorphism

$$s_n^H : C_n(G/H) \to C_{n+1}(G/H)$$

$$s_n^H(g_0 H, \ldots, g_n H) = (H, g_0 H, \ldots, g_n H)$$

is a $\mathbb{Z}[H]$–homomorphism. By the proof of Proposition 3.2 $C_*(G/H)$ is exact and $s_n^H$ is a contracting $\mathbb{Z}[H]$–homotopy. Hence, by Proposition 3.3 $C_*(G/H)$ is $(\mathbb{Z}[G], \mathbb{Z}[H])$–exact. There is a $\mathbb{Z}[G]$–isomorphism

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} C_{n-1}(G/H) \to C_n(G/H)$$

$$g \otimes (g_1 H, \ldots, g_n H) \mapsto (gH, g_1 g_1 H, \ldots, g_1 g_n H).$$

Its inverse is given by

$$C_n(G/H) \to \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} C_{n-1}(G/H)$$

$$(g_0 H, \ldots, g_n H) \mapsto g_0 \otimes (g_0^{-1} g_1 H, \ldots, g_0^{-1} g_n H).$$

Another representative of $g_0 H$ is of the form $g_0 h$ with $h \in H$ and

$$(g_0 h H, \ldots, g_n H) \mapsto g_0 h \otimes (h^{-1} g^{-1}_0 g_1 H, \ldots, h^{-1} g^{-1}_0 g_n H)$$

$$\mapsto g_0 \otimes (h^{-1} g^{-1}_0 g_1 H, \ldots, h^{-1} g^{-1}_0 g_n H)$$

$$\mapsto g_0 \otimes (g_0^{-1} g_1 H, \ldots, g_0^{-1} g_n H),$$

so it is well defined. Hence by Lemma 3.6 each $C_n(G/H)$ is $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective and therefore $C_*(G/H)$ is a $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective resolution of $\mathbb{Z}$.

**Proposition 3.11.** Let $G$ be a group and $H$ be a subgroup of $G$. Consider $\mathbb{Z}$ as a trivial $G$–module and let $\cdots \to C_1 \to C_0 \to \mathbb{Z} \to 0$ be a $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective resolution of $\mathbb{Z}$. Then

$$H_n((G : H); \mathbb{Z}) = H_n(C_* \otimes_{\mathbb{Z}[G]} \mathbb{Z}), \quad n = 0, 1, 2, \ldots.$$
and it is proved that this definition is independent of the \((\mathbb{Z}[G], \mathbb{Z}[H])\)-projective resolution of \(A\). In \([10, \S 4]\) the relative homology groups for \((G, H)\) are defined by

\[
H_n((G : H); \mathbb{Z}) = \text{Tor}_n(\mathbb{Z}[G], \mathbb{Z}[H])(\mathbb{Z}, \mathbb{Z}),
\]

and by Proposition \ref{prop:projective-resolution} the complex \(\mathcal{C}_*(G/H)\) is a \((\mathbb{Z}[G], \mathbb{Z}[H])\)-projective resolution of \(\mathbb{Z}\).

**Remark 3.12.** Let \(G\) be a discrete group and let \(\mathfrak{F}\) be a family of subgroups. By Proposition \ref{prop:simplicial-complex} we have that the simplicial chain complex of \(G\) is given by \(\mathcal{B}_*(X_{\mathfrak{F}})\). In particular, given a group \(G\) and a subgroup \(H\) of \(G\), consider the family \(\mathfrak{S}(H)\) generated by \(H\), then by Remark \ref{rem:simplicial-complex} we have that \(X_{\mathfrak{S}(H)} = X_{(H)}\) and the simplicial chain complex of \(B_{\mathfrak{S}(H)}(G)\) is precisely \(\mathcal{B}_*(X_{(H)}) \cong \mathcal{B}_*(G/H)\).

Then the following is immediate.

**Proposition 3.13.** Let \(G\) be a discrete group. Let \(H\) be a subgroup of \(G\). Consider the family of subgroups \(\mathfrak{S}(H)\) generated by \(H\). Then

\[
H_n(B_{\mathfrak{S}(H)}(G); \mathbb{Z}) \cong H_n([G : H]; \mathbb{Z}), \quad n = 0, 1, 2, \ldots,
\]

where \(B_{\mathfrak{S}(H)}(G)\) is the orbit space of the classifying space \(E_{\mathfrak{S}(H)}(G)\).

When \(H = \{1\}\) we recover the following well-known fact

**Corollary 3.14.** Let \(G\) be a group. Then

\[
H_n(BG; \mathbb{Z}) \cong H_n(G; \mathbb{Z}), \quad n = 0, 1, 2, \ldots.
\]

So we can say that the space \(B_{\mathfrak{S}(H)}(G)\) is a classifying space for the permutation representation \((G, X_{(H)})\), compare with Blowers \[3\] where a classifying space for an arbitrary permutation representation is constructed for Snapper’s cohomology \[10\].

**Proposition 3.15.** Let \(K\) be a normal subgroup of \(G\) contained in \(H\). Then we have an isomorphism of relative homology groups

\[
H_n([G : H]; \mathbb{Z}) \cong H_n([G/K : H/K]; \mathbb{Z}), \quad n = 0, 1, 2, \ldots.
\]

**Proof.** We have a bijection of sets

\[
G/H \cong (G/K) \sslash (H/K)
\]

\[
gH \leftrightarrow gK(H/K),
\]

which is equivariant with respect to the actions of \(G\) on \(G/H\) and of \(G/K\) on \((G/K) \sslash (H/K)\) via the natural projection \(G \to G/K\). This bijection induces an isomorphism between the chain complexes \(\mathcal{B}_*(G/H)\) and \(\mathcal{B}_*((G/K) \sslash (H/K))\) which commutes with the boundary operators. Hence we get the desired isomorphism of homology groups.

**Remark 3.16.** The statement of Proposition \ref{prop:relative-homology} with coefficients in an arbitrary \(\mathbb{Z}[G]\)-module \(A\) is

\[
H_i([G : H]; A) \cong H_i([G/K : H/K]; A_K),
\]

where \(A_K\) is the group of \(K\)-coinvariants of \(A\), compare with Adamson \[1, \text{Theorem 3.2}\].

**Corollary 3.17.** If \(H\) is a normal subgroup of \(G\), then we have an isomorphism

\[
H_n(G/H; \mathbb{Z}) \cong H_n([G : H]; \mathbb{Z}), \quad n = 0, 1, 2, \ldots.
\]
3.2.1. **Natural G–maps and induced homomorphism.** Let $H$ and $K$ be subgroups of $G$. There exists a $G$–map $G/H \to G/K$ if and only if there exists $a \in G$ such that $a^{-1}Ha \subset K$ and is given by

$$R_a: G/H \to G/K,$$

$$gH \mapsto gaK.$$ 

Any $G$–map $G/H \to G/K$ is of the form $R_a$ for some $a \in G$ such that $a^{-1}Ha \subset K$ and $R_a = R_b$ only if $ab^{-1} \in K$, see tom Dieck [20, Proposition 1(1.14)].

Let $H$ and $K$ be subgroups of $G$ such that $H$ is conjugate to a subgroup of $K$, then there is a $G$–map

$$h^K_H: X(H) \to X(K).$$

This induces a $G$–homomorphism

$$(h^K_H)_*: C_*(X(H)) \to C_*(X(K)),$$

which in turn induces a homomorphism of homology groups

$$H_n((G:H); \mathbb{Z}) \to H_n([G:K]; \mathbb{Z}).$$

**Remark 3.18.** Let $H$ and $K$ be subgroups of $G$. Consider the families $\mathcal{G}(H)$ and $\mathcal{G}(K)$ generated by $H$ and $K$ respectively and suppose that $\mathcal{G}(H) \subset \mathcal{G}(K)$. Then there exists a $G$–map unique up to $G$–homotopy $E_{\mathcal{G}(H)}(G) \to E_{\mathcal{G}(K)}(G)$. Notice that $\mathcal{G}(H) \subset \mathcal{G}(K)$ implies that $H$ is conjugate to a subgroup of $K$ and therefore there exists a $G$–map $h^K_H: X(H) = X_{\mathcal{G}(H)} \to X_{\mathcal{G}(K)} = X_{\mathcal{G}(K)}$. Using the Simplicial Construction of Proposition 2.4 we can see the $G$–map $E_{\mathcal{G}(H)}(G) \to E_{\mathcal{G}(K)}(G)$ as the simplicial $G$–map given in $n$–simplices by

$$(x_0, \ldots, x_n) \mapsto (h^K_H(x_0), \ldots, h^K_H(x_n)), \quad x_i \in X_{\mathcal{G}(H)}, \; i = 0, \ldots, n.$$ 

This map induces a canonical map $B_{\mathcal{G}(H)}(G) \to B_{\mathcal{G}(K)}(G)$ between the corresponding $G$–orbit spaces. This map in turn induces a canonical homomorphism in homology

$$H_n(B_{\mathcal{G}(H)}(G); \mathbb{Z}) \to H_n(B_{\mathcal{G}(K)}(G); \mathbb{Z}),$$

which by Proposition 3.13 corresponds to the homomorphism $(h^K_H)_*$ in (3.8).

Let $C_{h^K_H}^*(X(H))$ be the subcomplex of $C_*(X(H))$ generated by tuples mapping to different elements by the homomorphism $h^K_H$. We call this subcomplex the $h^K_H$–subcomplex of $C_*(X(H))$. As before, set

$$B_{h^K_H}^*(X(H)) = C_{h^K_H}^*(X(H)) \otimes_{\mathbb{Z}[G]} \mathbb{Z}.$$ 

**Lemma 3.19.** Let $H$ and $K$ be subgroups of $G$ such that $H$ is conjugate to a subgroup of $K$ and suppose $K$ has infinite index in $G$ (and therefore also $H$). Then the $h^K_H$–subcomplex $C_{h^K_H}^*(X(H))$ is acyclic.

**Proof.** Since $K$ has infinite index in $G$, given an $n$–cycle $\sigma = \sum n_i(x_0^{(i)}, \ldots, x_n^{(i)})$ in $C_{h^K_H}^n(X(H))$ there exists $y(\sigma) \in X(K)$ such that $y(\sigma)$ is different from all the $h^K_H(x_j^{(i)})$, $j = 0, \ldots, n$. Let $x(\sigma) \in h^K_H^{-1}(y) \subset X(H)$, then we have that $s_{x(\sigma)}^{x_j(\sigma)}(\sigma) \in C_{n+1}^h(X(H))$ and by Lemma 3.4 we get the result. \(\square\)
**Proposition 3.20.** Let $H$ and $K$ be subgroups of $G$ such that $H$ is conjugate to a subgroup of $K$ and suppose $K$ has infinite index in $G$ (and therefore also $H$). Then

$$H_n(B_n^{K\neq}(X(H))) \cong H_n([G : H]; \mathbb{Z}), \quad n = 0, 1, 2, \ldots .$$

**Proof.** Since $K$ has infinite index in $G$ by Lemma 3.19 we have that $C_n^{h_K}(X(H))$ is a resolution of $\mathbb{Z}$.

Since $C_n(X(H))$ is $(\mathbb{Z}[G], \mathbb{Z}[H])$–exact, $\ker \partial_n$ is a direct $H$–module summand of $C_n(X(H))$, so we can write

$$C_n(X(H)) = \ker \partial_n \oplus N$$

for some $H$–module $N$ (actually $N = \text{coker} \partial_{n+1}$). Clearly we can also write $C_n(X(H))$ as a direct sum of $G$–modules

$$(3.9) \quad C_n(X(H)) = C_n^{h_K}(X(H)) \oplus C_n^{h_K}(X(H))$$

where $C_n^{h_K}(X(H))$ is generated by tuples $(x_0, \ldots, x_n)$ such that $h_K(x_i) = h_K(x_j)$ for some $i \neq j$. Restricting the action to $H$ this can be seen as a direct sum of $H$–modules. Hence we can see $C_n^{h_K}(X(H))$ as a direct sum of $H$–modules

$$C_n^{h_K}(X(H)) = (C_n^{h_K}(X(H)) \cap \ker \partial_n) \oplus (C_n^{h_K}(X(H)) \cap \ker \partial_n),$$

and since $\ker \partial_n |_{C_n^{h_K}(X(H))}$ is also $(\mathbb{Z}[G], \mathbb{Z}[H])$–exact, $C_n^{h_K}(X(H))$ is a $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective module for $n \geq 0$ by (3.9) and Proposition 3.7 we have that $C_n^{h_K}(X(H))$ is also a $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective module for $n \geq 0$. Hence $C_n^{h_K}(X(H))$ is a $(\mathbb{Z}[G], \mathbb{Z}[H])$–projective resolution of $\mathbb{Z}$ and therefore $B_n^{h_K}(X(H))$ computes $H_n([G : H]; \mathbb{Z})$. \hfill $\square$

**Remark 3.21.** In the case $H = K$, we can take $h_K$ to be the identity. Hence $C_n^{h_K}(X(H))$ is generated by tuples of distinct elements, so we just denote it by $C_n(X(H))$. By Proposition 3.20 we have that

$$H_n(B_n^{K}(X(H))) \cong H_n([G : H]; \mathbb{Z}), \quad n = 0, 1, 2, \ldots .$$

### 3.3. Takasu relative group homology

Recall that the homology of a discrete group $G$ is equal to the homology of its classifying space $BG$ (Corollary 3.14). Let $H$ be a subgroup of $G$, the classifying space $BG$ can be regarded as a subspace of the classifying space $BG$. We define the **Takasu relative homology groups** denoted by $H_n(G,H)$, as the homology of the pair $H_n(BG,BH;\mathbb{Z})$, or equivalently, let $\text{Cof}(i)$ denote the cofibre (mapping cone) of the map $BH \to BG$ induced by the inclusion $i: H \to G$ and take the reduced singular homology groups $\tilde{H}_n(\text{Cof}(i);\mathbb{Z})$ for $n = 0, 1, 2, \ldots$. Algebraically this is done as follows (see the Topological Remark in Weibel [21 p. 19]). Let $B_n(H)$ and $B_n(G)$ be as in (3.3) and let $i_*: B_n(H) \to B_n(G)$ be the chain map induced by the inclusion. The **mapping cone** of $i_*$ is the chain complex

$$D_n(G,H) := B_{n-1}(H) \oplus B_n(G)$$
with boundary map given by the matrix \( \begin{pmatrix} -\partial_{n-1} & 0 \\ \varepsilon & -\partial_n \end{pmatrix} \). Hence the Takasu relative homology groups are given by

\[
H_n(G, H; \mathbb{Z}) = H_n(\mathcal{D}_*(G, H)).
\]

There is a description of these relative homology groups in terms of \( G \)-projective (in particular \( G \)-free) resolutions: Let \( H \) be a subgroup of \( G \). Given a \( G \)-module \( N \) we consider the epimorphism given in 3.6

\[
\theta: \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} N \to N \quad g \otimes n \mapsto gn
\]

and define the \( G \)-module

\[
I_{(G,H)}(N) = \ker \theta.
\]

We have that \( I_{(G,H)}(N) \) is a covariant exact functor from the category of left \( G \)-modules to itself with respect to the variable \( N \), see Takasu [18, Proposition 1.1 (i)].

Consider \( \mathbb{Z} \) as a trivial \( G \)-module. For any free \( G \)-resolution \( F_* \) of \( I_{(G,H)}(\mathbb{Z}) \), we have a canonical isomorphism

\[
H_n(G, H) \cong H_{n-1}(F_* \otimes_{\mathbb{Z}[G]} \mathbb{Z}), \quad n = 1, 2, \ldots.
\]

That is, \( H_n(G, H) = \text{Tor}_{n-1}^{\mathbb{Z}[G]}(I_{(G,H)}(\mathbb{Z}), \mathbb{Z}) \), see Takasu [18, Def. 2 (i) & Proposition 3.2] or Zickert [22, Theorem 2.1].

**Remark 3.22.** Notice that \( \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \mathbb{Z} \cong \mathbb{Z}[G/H] = C_0(G/H) \) and with this isomorphism the epimorphism \( \theta \) corresponds to the augmentation \( \varepsilon: C_0(G/H) \to \mathbb{Z} \). Therefore

\[
I_{(G,H)}(\mathbb{Z}) = \ker \varepsilon.
\]

The relative homology groups fit in the long exact sequence [18, Proposition 2.1]

\[
\cdots \to H_i(H; \mathbb{Z}) \to H_i(G; \mathbb{Z}) \to H_i(G, H; \mathbb{Z}) \to H_{i-1}(H; \mathbb{Z}) \to \cdots
\]

which corresponds to the long exact sequence for the pair \((BG, BH)\).

Let \( H \) and \( K \) be a subgroup of \( G \) such that \( H \leq K \leq G \). There exists an induced homomorphism [18, Proposition 1.2]

\[
I_{(G,H)}(\mathbb{Z}) \to I_{(G,K)}(\mathbb{Z}),
\]

which in turn, by the functoriality of \( \text{Tor}_{n-1}^{\mathbb{Z}[G]} \) induces a homomorphism

\[
H_n(G, H; \mathbb{Z}) \to H_n(G, K; \mathbb{Z}).
\]

### 3.4. Comparison of Hochschild and Takasu relative group homologies.

Hochschild and Takasu relative group homologies do not coincide in general. The following examples were suggested to us by Francisco González Acuña. Consider \( G = \mathbb{Z} \times \mathbb{Z} \) and \( H = \mathbb{Z} \). Since \( \mathbb{Z} \) is normal in \( \mathbb{Z} \times \mathbb{Z} \) by Corollary 3.17 we have that

\[
H_n([\mathbb{Z} \times \mathbb{Z} : \mathbb{Z}] ; \mathbb{Z}) = H_n(\mathbb{Z} \times \mathbb{Z} / \mathbb{Z} ; \mathbb{Z}) = H_n(\mathbb{Z} ; \mathbb{Z}) = H_n(S^1, \mathbb{Z}).
\]

While on the other hand, since \( B(\mathbb{Z} \times \mathbb{Z}) = T^2 \), where \( T^2 \) is the 2-torus and \( B(\mathbb{Z}) = S^1 \),

\[
H_n(\mathbb{Z} \times \mathbb{Z}, \mathbb{Z} ; \mathbb{Z}) = H_n(B(\mathbb{Z} \times \mathbb{Z}), B(\mathbb{Z}) ; \mathbb{Z}) = \tilde{H}_n(T^2 / S^1 ; \mathbb{Z}) = \tilde{H}_n(S^2 \vee S^1 ; \mathbb{Z}),
\]

where \( \tilde{H}_n \) denotes reduced homology.
From the definitions of $H_n(G,H;\mathbb{Z})$ and $H_n([G:H];\mathbb{Z})$ in terms of resolutions we get a canonical homomorphism between them.

Let $F$ be any free $G$–resolution of $I_{(G,H)}(\mathbb{Z})$. By Remark 3.22 we have that the complex
\[
\cdots \rightarrow C_3(G/H) \xrightarrow{\partial_3} C_2(G/H) \xrightarrow{\partial_2} C_1(G/H) \xrightarrow{\partial_1} I_{(G,H)}(\mathbb{Z}),
\]
is a $G$–resolution of $I_{(G,H)}(\mathbb{Z})$. By the Comparison Theorem for Resolutions (see for instance Weibel [21, Theorem 2.2.6] or Brown [4, Lemma I.7.4]) there is a chain map $F_n \rightarrow C_{n+1}(G/H)$, $n \geq 0$, unique up to chain homotopy which by (3.10) and (3.3) induces a homomorphism
\[
H_n(G,H;\mathbb{Z}) \rightarrow H_n([G:H];\mathbb{Z}), \quad n = 2, 3, \ldots.
\]

The following proposition gives a case when both relative homologies agree.

**Proposition 3.23.** Let $H$ and $K$ be subgroups of $G$ such that $H$ is conjugate to a subgroup of $K$ and suppose $K$ has infinite index in $G$. Also assume that for any $g \notin K$ we have $H \cap gHg^{-1} = \{I\}$ where $I \in G$ is the identity element. Consider the $h^K$–subcomplex $C^{h^K}_n(X_{(H)})$ defined in Subsection 3.2.1. Then $C^{h^K}_n(X_{(H)})$ is a free resolution of $I_{(G,H)}(\mathbb{Z})$ and therefore we have an isomorphism
\[
H_n(G,H;\mathbb{Z}) \cong H_n(B^{h^K}_n(X_{(H)})), \quad n = 2, 3, \ldots.
\]

**Proof.** Firstly, we claim that $C^{h^K}_n(X_{(H)})$ is a free $G$–module for $n \geq 1$. By Remark 2.1 it is enough to compute the isotropy subgroup of an $n$–simplex. Without loss of generality we can consider an $n$–simplex $\sigma = (x_0, \ldots, x_n) \in C^{h^K}_n(X_{(H)})$ such that $G_{x_0} = H$ since the $G$–orbit of any $n$–simplex has an element of this form, and isotropy subgroups of elements in the same $G$–orbit are conjugate. We have that $G_{x_i} = g_iHg_i^{-1}$ for some $g_i \in G$ and $g_0 = I$. The isotropy subgroup of $\sigma$ is given by
\[
G_{\sigma} = \bigcap_{i=0}^{n} g_iHg_i^{-1}.
\]

By the definition of $C^{h^K}_n(X_{(H)})$ we have that $g_i \notin K$ for $i = 1, \ldots, n$ and by hypothesis the intersection of $H$ with any conjugate $gHg^{-1}$ with $g \notin K$ is the identity. Therefore $G_{\sigma} = \{I\}$ and $C^{h^K}_n(X_{(H)})$ is a free $G$–module for $n \geq 1$.

Now, we have that $C^{h^K}_0(X_{(H)}) = C_0(X_{(H)})$ and by Lemma 3.19 the augmented $h^K$–subcomplex
\[
\cdots \rightarrow C^{h^K}_2(X_{(H)}) \xrightarrow{\partial_2} C^{h^K}_1(X_{(H)}) \xrightarrow{\partial_1} C_0(X_{(H)}) \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0
\]
is exact, so $\ker \varepsilon = I_{(G,H)}(\mathbb{Z})$ and therefore
\[
\cdots \rightarrow C^{h^K}_2(X_{(H)}) \xrightarrow{\partial_2} C^{h^K}_1(X_{(H)}) \xrightarrow{\partial_1} I_{(G,H)}(\mathbb{Z}) \rightarrow 0
\]
is a free $G$–resolution of $I_{(G,H)}(\mathbb{Z})$.

**Corollary 3.24.** Let $H$ and $K$ be subgroups of $G$ such that $H$ is conjugate to a subgroup of $K$. Also assume that for any $g \notin K$ we have $H \cap gHg^{-1} = \{I\}$ where $I \in G$ is the identity element. Then
\[
H_n(G,H;\mathbb{Z}) \cong H_n([G:H];\mathbb{Z}), \quad n = 2, 3, \ldots
\]
4. The group $SL_2(\mathbb{C})$ and some of its subgroups

We denote by $\mathbb{C}^\times$ the multiplicative group of the field of complex numbers. Now we study the particular case when $G = SL_2(\mathbb{C})$ and $H$ is one of the following subgroups:

- $\pm I = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$,
- $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^\times \right\}$,
- $U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{C} \right\}$,
- $P = \left\{ \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} \mid b \in \mathbb{C} \right\}$,
- $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^\times, b \in \mathbb{C} \right\}$.

By abuse of notation, we denote by $I$ the identity matrix and also the subgroup of $G$ which consists only of the identity matrix. Denote by $\bar{G} = G/\pm I = PSL_2(\mathbb{C})$. Given a subgroup $H$ of $G$ denote by $\bar{H}$ the image of $H$ in $\bar{G}$. Notice that $\bar{U} = \bar{P}$.

We denote by $\bar{g}$ the element of $\bar{G}$ with representative $g \in G$. We shall use this notation through the rest of the article except in Section 10. As usual we consider all groups with the discrete topology.

We list some known facts about these groups. Their proofs are in Lang [11].

- $SL_2(\mathbb{C})$ is generated by elementary matrices [11, Lem. XIII.8.1].
- $B$ is a maximal proper subgroup [11, Proposition XIII.8.2].
- Bruhat decomposition: [11, XIII §8, p. 538]. Let $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. There is a decomposition of $SL_2(\mathbb{C})$ into disjoint subsets $SL_2(\mathbb{C}) = B \cup BwB$.
- The subgroups $U$ and $P$ are normal in $B$ and we have the exact sequences

$$I \to U \xrightarrow{i} B \to T \cong \mathbb{C}^\times \to I \quad (4.1)$$

$$I \to P \xrightarrow{j} B \to T \cong \mathbb{C}^\times \to I.$$

- $PSL_2(\mathbb{C})$ is a simple group [11, Theorem XIII.8.4]. Hence, $\pm I$ is the only normal subgroup of $SL_2(\mathbb{C})$.

**Lemma 4.1.** Let $g \notin B$. Then

$$U \cap gUg^{-1} = I.$$

$$P \cap gPg^{-1} = \pm I.$$

**Proof.** Since $g \notin B$ by Bruhat decomposition we have that $g$ can be written as $g = g_1wg_2$, $g_1, g_2 \in B$.

Then we have that

$$gUg^{-1} = g_1wg_2Ug_2^{-1}w^{-1}g_1^{-1} = g_1wg_1^{-1}Uw^{-1}g_1^{-1}.$$
Let us analyze the elements in $g_1 w U w^{-1} g_1^{-1}$. Consider $h = \left( \begin{smallmatrix} 1 & e \\ 0 & 1 \end{smallmatrix} \right) \in U$ and $g_1 = \left( \begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix} \right) \in B$, so $a \in \mathbb{C} \times$. We have that
\[ g_1 w h w^{-1} g_1^{-1} = \left( \begin{array}{cc} 1 + a^{-1} e & -b^2 e \\ a^{-2} e & 1 - a^{-1} e \end{array} \right). \]

The only way to have $g_1 w h w^{-1} g_1^{-1} \in U$ is to have $e = 0$ and in that case $g_1 w h w^{-1} g_1^{-1} = I$. Analogously if $h = \left( \begin{smallmatrix} \pm 1 & e \\ 0 & \pm 1 \end{smallmatrix} \right) \in P$ then $g_1 w h w^{-1} g_1^{-1} = \left( \begin{array}{cc} \pm 1 + a^{-1} e & -b^2 e \\ a^{-2} e & \pm 1 - a^{-1} e \end{array} \right)$, and the only way to have $g_1 w h w^{-1} g_1^{-1} \in P$ is to have $e = 0$ and in that case $g_1 w h w^{-1} g_1^{-1} = \pm I$. 

Now we give models for the $G$–sets $X_{(H)}$ with $H = U, P, B$.

**Remark 4.2.** Recall that for $H = P, B$ we have bijections of sets $G/H \cong \overline{G}/\overline{H}$ which are equivariant with respect to the actions of $G$ on $G/H$ and of $\overline{G}$ on $\overline{G}/\overline{H}$ via the natural projection $G \to \overline{G}$. Thus, we have that $X_{(H)} = X_{(\overline{H})}$ as sets, the subgroup will indicate whether we are considering the action of $G$ or $\overline{G}$ on it.

4.1. **The $G$–set $X_{(U)}$.** Consider the action of $G$ on $\mathbb{C}^2 \setminus \{(0, 0)\}$ given by left matrix multiplication.

**Proposition 4.3.** The group $G$ acts transitively on $\mathbb{C}^2 \setminus \{(0, 0)\}$.

**Proof.** Let $(x, y)$ and $(z, w)$ be elements of $\mathbb{C}^2 \setminus \{(0, 0)\}$. Then a matrix $g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in SL_2(\mathbb{C})$ which send $(x, y)$ to $(z, w)$ is given by:

*If $x \neq 0$: we have two cases:

1. $w \neq 0$:
   \[ a = \frac{zw + xy}{xw}, \quad b = -\frac{x}{w}, \quad c = \frac{w}{x}, \quad d = 0. \]
   That is $g = \left( \begin{array}{cc} \frac{zw + xy}{xw} & -\frac{x}{w} \\ \frac{w}{x} & 0 \end{array} \right)$

2. $w = 0$: which implies that $z \neq 0$
   \[ a = \frac{z - by}{x}, \quad c = -\frac{y}{z}, \quad d = \frac{x}{z}, \quad b = $ any complex number.
   That is (simplifying taking $b = 0$)
   \[ g = \left( \begin{array}{cc} \frac{z}{x} & 0 \\ -\frac{y}{z} & \frac{x}{z} \end{array} \right) \]

*If $x = 0$: which implies $y \neq 0$, then we have

\[ b = \frac{z}{y}, \quad d = \frac{w}{y}, \quad a = \begin{cases} 0 & \text{if } z \neq 0, \\
\frac{w}{y} & \text{if } w \neq 0, \end{cases} \quad c = \begin{cases} -\frac{w}{z} & \text{if } z \neq 0, \\
0 & \text{if } w \neq 0. \end{cases} \]
That is
\[ g = \begin{pmatrix} 0 & z \\ -\frac{w}{z} & \frac{y}{z} \end{pmatrix} \quad \text{if } z \neq 0, \]
\[ g = \begin{pmatrix} \frac{w}{w} & \frac{z}{w} \\ 0 & \frac{y}{y} \end{pmatrix} \quad \text{if } w \neq 0. \]
This proves the transitivity of the action. \[\square\]

Hence we have that

**Proposition 4.4.** The isotropy subgroup of \((1,0)\) is \(U\). Therefore, there is a \(G\)-isomorphism between \(SL_2(\mathbb{C})/U\) and \(X(U)\) given by
\[ SL_2(\mathbb{C})/U \rightarrow X(U) \]
\[ gU \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

**Proof.** Let \(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})\). Then
\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
implies that \(a = 1\) and \(c = 0\), but since the determinant of \(g\) is 1 we have that \(d = 1\) and therefore \(g = \begin{pmatrix} 1 & \frac{1}{d} \end{pmatrix} \in U\). \[\square\]

Therefore we can set \(X(U) = \mathbb{C}^2 \setminus \{(0,0)\}\).

4.2. **The \(G\)-set \(X(P)\).** Now consider the action of \(\mathbb{Z}_2\) on \(\mathbb{C}^2 \setminus \{(0,0)\}\) such that \(-1\) sends the pair \((x, y)\) to its antipodal point \((-x, -y)\). Consider the orbit space \(\mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z}_2\) and denote by \([x, y] = \{(x, y), (-x, -y)\}\) the orbit of the pair \((x, y)\).

Let \(g \in G\) and \([x, y] \in \mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z}_2\). We define an action of \(G\) on \(\mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z}_2\) given by
\[ g \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}. \]
This gives a well-defined action since
\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (ax + by) \\ (cx + dy) \end{pmatrix} \sim \begin{pmatrix} -ax - by \\ -cx - dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -x \\ -y \end{pmatrix}. \]
Since \(-I\) acts as the identity, this action descends to an action of \(\bar{G}\). By the definition of the action of \(G\) on \(\mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z}_2\) and Proposition 4.3 we have

**Corollary 4.5.** The group \(G\) acts transitively on \(\mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z}_2\).

**Proposition 4.6.** The isotropy subgroup of \([1,0]\) is \(P\). Therefore, there is a \(G\)-isomorphism between \(SL_2(\mathbb{C})/P\) and \(\mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z}_2\) given by
\[ SL_2(\mathbb{C})/P \rightarrow \mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z}_2 \]
\[ gP \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]
Proof. Let \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \) such that
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
This implies that \( a = \pm 1 \) and \( c = 0 \), but since \( \det g = 1 \) we have that \( d = \pm 1 \) and therefore \( g = ( \pm 1 \ b \ 0 \ \pm 1 ) \in \mathcal{P} \).
\( \square \)

Therefore we can set \( X(P) = \mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z}_2 \). By Remark 4.2 we also have that \( X(\bar{P}) = \mathbb{C}^2 \setminus \{(0,0)\}/\mathbb{Z}_2 \).

There is another model for \( X(P) \) which we learned from Ramadas Ramakrishnan. Consider the set \( \text{Sym} \) of \( 2 \times 2 \) non-zero symmetric complex matrices with determinant zero. The set \( \text{Sym} \) is given by matrices of the form
\[
\text{Sym} = \left\{ \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix} \bigg| (x, y) \in X(U) \right\}.
\]
Let \( g \in SL_2(\mathbb{C}) \) and \( S \in \text{Sym} \). We define an action of \( G \) on \( \text{Sym} \) by
\[
g \cdot S = gSg^T,
\]
where \( g^T \) is the transpose of \( g \). The action is well-defined because transpose conjugation preserves symmetry and the determinant function is a homomorphism. Since \( -I \) acts as the identity, this action descends to an action of \( \bar{G} \).

**Proposition 4.7.** The group \( G \) acts transitively on \( \text{Sym} \).

Proof. Let \( \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix} \) and \( \begin{pmatrix} z^2 & zw \\ wz & w^2 \end{pmatrix} \) be elements of \( \text{Sym} \). Then the matrix \( g' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \) which send \( \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix} \) to \( \begin{pmatrix} z^2 & zw \\ wz & w^2 \end{pmatrix} \) is given by:

If \( x \neq 0 \): we have two cases:

\( w \neq 0 \):
\[
a = -\frac{zw + xy}{xw}, \quad b = \frac{x}{w}, \quad c = -\frac{w}{x}, \quad d = 0.
\]
That is
\[
g' = \begin{pmatrix} -\frac{zw + xy}{xw} & \frac{z}{w} \\ -\frac{w}{x} & 0 \end{pmatrix}
\]

If \( w = 0 \): which implies that \( z \neq 0 \)
\[
a = -\frac{z}{x}, \quad c = \frac{y}{z}, \quad d = -\frac{x}{z}, \quad b = 0
\]
That is
\[
g' = \begin{pmatrix} -\frac{z}{x} & 0 \\ \frac{y}{z} & -\frac{x}{z} \end{pmatrix}
\]

If \( x = 0 \): which implies that \( y \neq 0 \), then we have
\[
b = -\frac{z}{y}, \quad d = -\frac{w}{y}, \quad a = \begin{cases} 0 & \text{if } z \neq 0, \\
-\frac{y}{w} & \text{if } w \neq 0. \end{cases} \quad c = \begin{cases} \frac{w}{y} & \text{if } z \neq 0, \\
0 & \text{if } w \neq 0. \end{cases}
\]
That is
\[
g' = \begin{pmatrix} 0 & -\frac{z}{y} \\ -\frac{w}{y} & -\frac{w}{y} \end{pmatrix}
\]
if \(z \neq 0\),
\[
g' = \begin{pmatrix} -\frac{w}{y} & y \\ 0 & -\frac{w}{y} \end{pmatrix}
\]
if \(w \neq 0\).
\[\square\]

Remark 4.8. Notice that the matrices \(g'\) found in the proof of Proposition 4.7 are the negatives of the corresponding matrices \(g\) found in the proof of Proposition 4.3, that is \(g' = -g\). In the proof of Proposition 4.7 one can also take the corresponding matrices \(g\).

Proposition 4.9. The isotropy subgroup of \((\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})\) \(\in\) Sym is \(P\). Therefore, there is a \(G\)-isomorphism between \(SL_2(\mathbb{C})/P\) and Sym given by
\[
SL_2(\mathbb{C})/P \to \text{Sym}
\]
\[
gP \mapsto g \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} g^T.
\]

Proof. Let \(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})\) such that
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 & ac \\ ac & c^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]
This implies that \(a = \pm 1\), \(c = 0\) and \(d = \pm 1\) since \(\det g = 1\). Thus \(g = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} \in P\) as claimed.
\[\square\]

We denote \(\bar{X}(P) = \text{Sym}\) to distinguish it from \(X(P)\). By Remark 4.2 we have that Sym is also a model for \(\bar{G}/\bar{P}\). We use the notation \(\bar{X}(\bar{P}) = \text{Sym}\) to distinguish it from \(X(\bar{P})\) and emphasize the action of \(\bar{G}\).

Corollary 4.10. The sets \(X(P)\) and \(\bar{X}(P)\) are isomorphic as \(G\)-sets.

Proof. This is immediate from Propositions 4.6 and 4.9. We can give an explicit isomorphism by using the transitivity of the actions of \(G\) on each of these \(G\)-sets. Let \([x,y] \in X(P)\), then
\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & z \\ y & w \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]
where \(z\) and \(w\) are complex numbers such that \(xw - yz = 1\). On the other hand, we have
\[
\begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix} = \begin{pmatrix} x & z \\ y & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]
We define the isomorphism by
\[
\rho: X(P) \to \bar{X}(P)
\]
\[(4.3) \begin{bmatrix} x \\ y \end{bmatrix} \leftrightarrow \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}.
\]
It is equivariant because
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.
\]
is sent by the isomorphism to
\[
\begin{pmatrix}
(ax + by)^2 & (ax + by)(cx + dy) \\
(ax + by)(cx + dy) & (cx + dy)^2
\end{pmatrix}
= \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}
\begin{pmatrix} a & c \\ b & d \end{pmatrix}.
\]
\[\square\]

4.3. The $G$–set $X_B$. Let $\hat{C} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Let $\text{LF}(\hat{C})$ be the group of fractional linear transformations on $\hat{C}$. Let $\phi: G = \text{SL}_2(\mathbb{C}) \to \text{LF}(\hat{C})$ be the canonical homomorphism. Let $g \in G$ act on $\hat{C}$ by the corresponding fractional linear transformation $\phi(g)$. It is well-known that this action is transitive. Abusing of the notation, given $g \in G$ and $z \in \hat{C}$ we denote the action by $g \cdot z = \phi(g)(z)$.

We can also identify $\mathbb{CP}^1$ with $\hat{C}$ via $[z_1 : z_2] \leftrightarrow \frac{z_1}{z_2}$, where $[z_1 : z_2] \in \mathbb{CP}^1$ is written in homogeneous coordinates. In this case an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ acts in an element $[z_1 : z_2]$ in $\mathbb{CP}^1$ by matrix multiplication
\[
g \cdot [z_1 : z_2] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = [az_1 + bz_2 : cz_1 + dz_2].
\]

Proposition 4.11. The isotropy subgroup of $\infty \in \hat{C}$ is $B$. Therefore, there is a $G$–isomorphism between $\text{SL}_2(\mathbb{C})/B$ and $\hat{C}$ given by
\[
\text{SL}_2(\mathbb{C})/B \to \hat{C}
\]
$gB \mapsto g \cdot \infty$.

Proof. Let $g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in B$. Then
\[
g \cdot [1 : 0] = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} = [a : 0] = [1 : 0].
\]

It is easy to see that all the elements in $G$ that fix $[1 : 0]$ are of this form, i.e., that they are elements in $B$. \[\square\]

Therefore we set $X_B = \hat{C}$. Again, by Remark 4.2 we also have that $X_B = \hat{C}$.

4.4. The explicit $G$–maps. The inclusions
\[(4.4) \quad I \hookrightarrow U \hookrightarrow P \hookrightarrow B\]
induce $G$–maps
\[
G \to G/U \to G/P \to G/B
\]
$g \mapsto gU \mapsto gP \mapsto gB$.

Using the models $X_H$ for the $G$–sets $G/H$ with $H = U, P, B$ given in the previous subsections we give the explicit $G$–maps between them.
Let \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \). We have the \( G \)-maps

\[
(4.5) \quad G \xrightarrow{h_U^U} X(U) \xrightarrow{h_U^P} X(P) \xrightarrow{h_P^B} X(B)
\]

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{g(1)} \begin{pmatrix} a \\ c \end{pmatrix} \xrightarrow{g \cdot [1 : 0]} \begin{pmatrix} a \\ c \end{pmatrix} \xrightarrow{g \cdot \infty} \begin{pmatrix} a \\ c \end{pmatrix}.
\]

Notice that \( h_U^P : X(U) \to X(P) \) is just the quotient map given by the action of \( \mathbb{Z}_2 \).

On the other hand, we have that

\[
(4.6) \quad h_B^U = h_B^U \circ h_P^U,
\]

where \( h_B^U \) is the Hopf map

\[
h_B^U : X(U) \to X(B)
\]

\[
h_B^U(a, c) = \frac{a}{c}.
\]

Using \( \bar{X}(P) \) instead of \( X(P) \) we have

\[
(4.8) \quad G \xrightarrow{h_U^U} X(U) \xrightarrow{\bar{h}_U^U} \bar{X}(P) \xrightarrow{\bar{h}_P^B} X(B)
\]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{(a, c)} \begin{pmatrix} a^2 & ac \\ ac & c^2 \end{pmatrix} \xrightarrow{\frac{a^2}{ac} = \frac{ac}{c^2} = \frac{a}{c}}
\]

We have that

\[
\bar{h}_P^B \circ \bar{h}_U^U = h_B^U.
\]

It is also useful to write the \( G \)-map \( \bar{h}_P^B : \bar{X}(P) \to X(B) \) in terms of the entries of the matrix in \( \bar{X}(P) \) without writing it in the form given in (4.2). Let \( \begin{pmatrix} r & t \\ t & s \end{pmatrix} \in \bar{X}(P) \), that is, \( \begin{pmatrix} r & t \\ t & s \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) and \( rs = t^2 \). Then we have

\[
\bar{h}_P^U : \bar{X}(P) \to X(B)
\]

\[
\begin{pmatrix} r & t \\ t & s \end{pmatrix} \mapsto \frac{r}{t} = \frac{t}{s}.
\]

Remark 4.12. For the case of \( \bar{G} \) we have practically the same \( \bar{G} \)-homomorphisms as in (4.5) and (4.8) except that \( X(U) = \bar{X}(\tilde{U}) \).

Remark 4.13. Consider \( \infty \in X(B) \) and its inverse image under the Hopf map

\[
(4.7) \quad (h_B^U)^{-1}(\infty) = \{ (x, 0) : x \in \mathbb{C}^\times \} \subset X(U),
\]

which corresponds to the first coordinate complex line minus the origin. Since by Proposition 4.11 the isotropy subgroup of \( \infty \in X(B) \) under the action of \( G \) is \( B \), we have that \( (h_B^U)^{-1}(\infty) \) is a \( B \)-invariant subset of \( X(U) \). Since the short exact sequence (4.1) splits, any element of \( B \) can be written in a unique way as the product of an element in \( U \) and an element in \( T \)

\[
(4.9) \quad \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & ab \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.
\]
It is easy to see that \( U \) fixes pointwise the points of \((h_B^B)^{-1}(\infty)\), while \( T \) acts freely and transitively on \((h_B^B)^{-1}(\infty)\) where the matrix \(
abla_{a b, b}^{-1} \) acts multiplying \((x, 0)\) by \(a \in \mathbb{C}^\times\).

Another way to interpret this is to write \( \infty \in X(B) \) in homogeneous coordinates \([\lambda : 0]\), then the elements in \( U \) fix \( \infty \) and the homogeneous coordinates \([\lambda : 0]\), while an element in \((a b, b) \in T \) fix \( \infty \) but multiply the homogeneous coordinates by \(a \in \mathbb{C}^\times\) obtaining the homogeneous coordinates \([a \lambda : 0]\).

More generally, for any point \( z \in X(B) \), its isotropy subgroup \( G_z \) is a conjugate of \( B \), which can be written as the direct product of the corresponding conjugates of \( U \) and \( T \), which we denote by \( U_z \) and \( T_z \). Writing \( z \in X(B) \) in homogeneous coordinates \([\lambda z : \lambda]\) the elements of \( U_z \) fix \( z \) and the homogeneous coordinates, while the elements of \( T_z \) fix \( z \) but multiply the homogeneous coordinates by a constant.

**Remark 4.14.** Analogously, consider \( \infty \in X(B) \) and its inverse image under the \( G \)-map \( h_B^B \)

\[
(h_B^B)^{-1}(\infty) = \{ [x, 0] \mid x \in \mathbb{C}^\times \} \subset X(P).
\]

By (4.9) any element of \( P \) can be written in a unique way as the product of an element in \( U \) and an element in \( T \)

\[
(\pm 1 \ b \ 0) = \begin{pmatrix}
1 & \pm b \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\pm 1 & 0 \\
0 & \pm 1
\end{pmatrix}.
\]

Given a representative \((x, 0)\) of \([x, 0] \in (h_B^B)^{-1}(\infty)\), the matrix \((\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}) \in T\) changes the sign of the representative to \((-x, 0)\) but fixes its class \([x, 0]\) and the matrix \((\begin{smallmatrix} 1 & \pm b \\ 0 & 1 \end{smallmatrix}) \in U\) fixes any representative of \([x, 0]\), thus it fixes the class itself. Therefore, the elements in \( P \) fix pointwise the points in \((h_B^B)^{-1}(\infty)\) while \( T \) acts transitively on \((h_B^B)^{-1}(\infty)\) with isotropy \( \mathbb{Z}_2 \), where the matrix \((a b, b) \) acts multiplying \([x, 0]\) by \(a \in \mathbb{C}^\times\) obtaining \([ax, 0]\).

If we use instead \( h_B^B \) the inverse image of \( \infty \in X(B) \) is given by

\[
(h_B^B)^{-1}(\infty) = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \mid x \in \mathbb{C}^\times \right\} \subset X(P).
\]

The elements in \( P \) fix pointwise the points in \((h_B^B)^{-1}(\infty)\) while \( T \) acts transitively on \((h_B^B)^{-1}(\infty)\) with isotropy \( \mathbb{Z}_2 \), where the matrix \((a b, b) \) acts multiplying \((\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}) \) by \(a^2\) with \( a \in \mathbb{C}^\times\) obtaining \((\begin{smallmatrix} a^2 & 0 \\ 0 & 0 \end{smallmatrix}) \).

More generally, for any point \( z \in X(B) \), its isotropy subgroup \( G_z \) is a conjugate of \( B \), and let \( P_z \) denote the corresponding conjugate of \( P \). The elements of \( P_z \) fix pointwise the points in \((h_B^B)^{-1}(z)\) while \( T_z \) acts transitively on \((h_B^B)^{-1}(z)\) multiplying by a constant.

### 4.5. Canonical homomorphisms

As in Section 2, we denote by \( \mathfrak{F}(H) \) the family of subgroups of \( G \) generated by \( H \). The inclusions (4.4) induce the inclusions of families of subgroups of \( G \)

\[
\mathfrak{F}(I) \hookrightarrow \mathfrak{F}(U) \hookrightarrow \mathfrak{F}(P) \hookrightarrow \mathfrak{F}(B)
\]

and in turn, these inclusion give canonical \( G \)-maps between classifying spaces

\[
E G \to E_{\mathfrak{F}(U)}(G) \to E_{\mathfrak{F}(P)}(G) \to E_{\mathfrak{F}(B)}(G)
\]

which are unique up to \( G \)-homotopy. Taking the quotient by the action of \( G \) we get canonical maps

\[
B G \to B_{\mathfrak{F}(U)}(G) \to B_{\mathfrak{F}(P)}(G) \to B_{\mathfrak{F}(B)}(G).
\]
The homomorphisms induced in homology give the sequence

\[ H_i(BG) \to H_i(B_{\mathcal{U}}(G)) \to H_i(B_{\mathcal{P}}(G)) \to H_i(B_{\mathcal{B}}(G)), \]

which by Proposition 3.13 is the same as the sequence of homomorphisms

\[ H_i(G) \xrightarrow{(h^{\mathcal{U}})} H_i([G : U]) \xrightarrow{(h^{\mathcal{P}})} H_i([G : P]) \xrightarrow{(h^{\mathcal{B}})} H_i([G : B]). \]

Recall that we denote by \( \tilde{G} = G/\pm I = PSL_2(\mathbb{C}) \) and given a subgroup \( H \) of \( G \) we denote by \( \tilde{H} \) the image of the \( H \) in \( \tilde{G} \). Notice that \( U = P \). Analogously, the inclusions \( I \to \tilde{P} \to \tilde{B} \) induce a sequence of homomorphisms

\[ H_i(\tilde{G}) \to H_i([\tilde{G} : \tilde{P}]) \to H_i([\tilde{G} : \tilde{B}]). \]

The relation between the coset sets of \( G \) and \( \tilde{G} \) can be shown in the following diagram

\[
\begin{array}{cccc}
G & \xrightarrow{\eta} & G/U & \xrightarrow{\eta} & G/P = \tilde{G}/\tilde{P} & \xrightarrow{\eta} & G/B = \tilde{G}/\tilde{B} \\
\tilde{G} = G/\pm I & \xrightarrow{\eta} & \tilde{G}/\tilde{U} = G/\tilde{T}.
\end{array}
\]

In turn, by Proposition 3.15 this induces the following commutative diagram of relative homology groups

\[ (4.12) \qquad H_i(G) \xrightarrow{\eta} H_i([G : U]) \xrightarrow{\eta} H_i([G : P]) \xrightarrow{\eta} H_i([G : B]). \]

4.6. **Relative homology of \( SL_2(\mathbb{C}) \).** The following proposition states that for the cases of \( G \) and \( U \) and \( \tilde{G} \) and \( \tilde{P} \) the Hochschild and Takasu relative homology groups coincide, compare with [3, Remark. 3.6] and [22, §7]).

Consider the \( h^{\mathcal{U}} \)-subcomplex \( C_{\mathcal{U}}^{h^{\mathcal{U}}} (X) \) and the \( h^{\mathcal{P}} \)-subcomplex \( C_{\mathcal{P}}^{h^{\mathcal{P}}} (X) \) defined in Subsection 3.2.1.

**Proposition 4.15.** We have isomorphisms

\[
H_n(G, U; \mathbb{Z}) \cong H_n(C_{\mathcal{U}}^{h^{\mathcal{U}}} (X_{\mathcal{U}})), \quad n = 2, 3, \ldots.
\]

\[
H_n(\tilde{G}, \tilde{P}; \mathbb{Z}) \cong H_n(C_{\mathcal{P}}^{h^{\mathcal{P}}} (X_{\mathcal{P}})), \quad n = 2, 3, \ldots.
\]

**Proof.** By Lemma 4.1 \( U \cap gUg^{-1} = I \) for any \( g \notin B \) and \( \tilde{P} \cap \tilde{g}P\tilde{g}^{-1} = \tilde{I} \) for any \( g \notin \tilde{B} \). Hence, the result follows by Proposition 3.23. \( \Box \)

Putting together Propositions 3.20 and 4.15 we get the following corollary.

**Corollary 4.16.**

\[
H_n(G, U; \mathbb{Z}) \cong H_n([G : U]; \mathbb{Z}), \quad n = 2, 3, \ldots.
\]

\[
H_n(\tilde{G}, \tilde{P}; \mathbb{Z}) \cong H_n([\tilde{G} : \tilde{P}]; \mathbb{Z}), \quad n = 2, 3, \ldots.
\]
5. Invariants for finite volume hyperbolic 3–manifolds

In this section we generalize the construction given in Cisneros-Molina–Jones [5] to define invariants \( \beta_t(M) \in H_3([SL_2(\mathbb{C}) : H]; \mathbb{Z}) \) of a complete oriented hyperbolic 3–manifold of finite volume \( M \), where \( H \) is one of the subgroups \( P \) or \( B \) of \( SL_2(\mathbb{C}) \) defined in Section [4]

5.1. Hyperbolic 3–manifolds and the fundamental class of \( \widehat{M} \). Consider the upper half space model for the hyperbolic 3–space \( \mathbb{H}^3 \) and identify it with the set of quaternions \( \{ z + tj \mid z \in \mathbb{C}, \ t > 0 \} \). Let \( \mathbb{H}^3 = \mathbb{H}^3 \cup \tilde{\mathbb{C}} \) be the standard compactification of \( \mathbb{H}^3 \). The group of orientation preserving isometries of \( \mathbb{H}^3 \) is isomorphic to \( PSL_2(\mathbb{C}) \) and the action of \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{C}) \) in \( \mathbb{H}^3 \) is given by the linear fractional transformation

\[
\phi(w) = \frac{(aw + b)(cw + d)}{cw + d}, \quad w = z + tj, \quad ad - bc = 1,
\]

which is the Poincaré extension to \( \mathbb{H}^3 \) of the complex linear fractional transformation on \( \mathbb{C} \) given by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Recall that isometries of hyperbolic 3–space \( \mathbb{H}^3 \) can be of three types: elliptic if fixes a point in \( \mathbb{H}^3 \); parabolic if fixes no point of \( \mathbb{H}^3 \) and fixes a unique point of \( \mathbb{C} \) and hyperbolic if fixes no point of \( \mathbb{H}^3 \) and fixes two points of \( \mathbb{C} \).

A subgroup of \( SL_2(\mathbb{C}) \) or \( PSL_2(\mathbb{C}) \) is called parabolic if all its elements correspond to parabolic isometries of \( \mathbb{H}^3 \) fixing a common point in \( \mathbb{C} \). Since the action of \( SL_2(\mathbb{C}) \) (or \( PSL_2(\mathbb{C}) \)) in \( \mathbb{C} \) is transitive and the conjugates of parabolic isometries are parabolic [15 (4.7.1)] we can assume that the fixed point is the point at infinity \( \infty \) which we denote by its homogeneous coordinates \( \infty = [1 : 0] \) and therefore parabolic subgroups are conjugate to a group of matrices of the form \( \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} \), with \( b \in \mathbb{C} \), or its image in \( PSL_2(\mathbb{C}) \). In other words, a parabolic subgroup of \( SL_2(\mathbb{C}) \) or \( PSL_2(\mathbb{C}) \) is conjugate to a subgroup of \( P \) or \( \bar{P} \) respectively.

A complete oriented hyperbolic 3–manifold \( M \) is the quotient of the hyperbolic 3–space \( \mathbb{H}^3 \) by a discrete, torsion-free subgroup \( \Gamma \) of orientation preserving isometries. Since \( \Gamma \) is torsion-free, it acts freely on \( \mathbb{H}^3 \) [15 Theorem 8.2.1] and therefore it consist only of parabolic and hyperbolic isometries. Notice that since \( \mathbb{H}^3 \) is contractible it is the universal cover of \( M \) and therefore \( \pi_1(M) = \Gamma \) and \( M \) is a \( K(\Gamma, 1) \), ie \( M = \tilde{\Gamma} \), the classifying space of \( \Gamma \). To such an hyperbolic 3–manifold we can associate a geometric representation \( \tilde{\rho} \): \( \pi_1(M) = \Gamma \rightarrow PSL_2(\mathbb{C}) \) given by the inclusion, which is canonical up to equivalence. This representation can be lifted to a representation \( \rho \): \( \Gamma \rightarrow SL_2(\mathbb{C}) \) [4 Proposition 3.1.1]. There is a one-to-one correspondence between such lifts and spin structures on \( M \). We identify \( \Gamma \) with a subgroup of \( SL_2(\mathbb{C}) \) using the representation \( \rho \): \( \Gamma \rightarrow SL_2(\mathbb{C}) \) which corresponds to the spin structure.

Let \( M \) be an orientable complete hyperbolic 3–manifold of finite volume. Such manifolds contain a compact 3–manifold–with–boundary \( M_0 \) such that \( M - M_0 \) is the disjoint union of a finite number of cusps. Each cusp of \( M \) is diffeomorphic to \( T^2 \times (0, \infty) \), where \( T^2 \) denotes the 2–torus, see for instance [15] page 647, Corollary 4 and Theorem 10.2.1. The number of cusps can be zero, and this case corresponds when the manifold \( M \) is a closed manifold.

Let \( M \) be an orientable complete hyperbolic 3–manifold of finite volume with \( d \) cusps, with \( d > 0 \). Each boundary component \( T^2 \) of \( M_0 \) defines a subgroup \( \Gamma_i \) of \( \pi_1(M) \) which is well defined up to conjugation. The subgroups \( \Gamma_i \) are called the
peripheral subgroups of \( \Gamma \). The image of \( \Gamma_i \) under the representation \( \rho: \Gamma \to SL_2(\mathbb{C}) \) given by the inclusion is a free abelian group of rank 2 of \( SL_2(\mathbb{C}) \). The subgroups \( \Gamma_i \) are parabolic subgroups of \( SL_2(\mathbb{C}) \). Hence we have that \( \Gamma_i \in \mathfrak{g}(P) \). Therefore the image of \( \Gamma_i \) under the representation \( \tilde{\rho}: \Gamma \to PSL_2(\mathbb{C}) \) is contained in \( \mathfrak{g}(\tilde{P}) \).

Let \( M = \Gamma \backslash \mathbb{H}^3 \) be a non-compact orientable complete hyperbolic 3–manifold of finite volume. Let \( \pi: \mathbb{H}^3 \to \Gamma \backslash \mathbb{H}^3 = M \) be the universal cover of \( M \). Consider the set \( \mathcal{C} \) of fixed points of parabolic elements of \( \Gamma \) in \( \mathbb{C} \) and divide by the action of \( \Gamma \). The elements of the resulting set \( \tilde{\mathcal{C}} \) are called the cusp points of \( M \).

**Remark 5.1.** No hyperbolic element in \( \Gamma \) has as fixed point any point in \( \mathcal{C} \), otherwise the group \( \Gamma \) would not be discrete [15, Theorem 5.5.4].

Let \( \tilde{Y} = \mathbb{H}^3 \cup \mathcal{C} \) and consider \( \tilde{M} = \Gamma \backslash \tilde{Y} \). If \( M \) is closed \( \mathcal{C} = \emptyset \) and \( \tilde{M} = M \), if \( M \) is non-compact we have that \( \tilde{M} \) is the end-compactification of \( M \) which is the result of adding the cusps points of \( M \). We get an extension of the covering map \( \pi \) to a map \( \tilde{\pi}: \tilde{Y} \to \tilde{M} \).

Consider as well the one-point-compactification \( M_+ \) of \( M \) which consists in identifying all the cusps points of \( \tilde{M} \) to a single point. Since \( M \) is homotopy equivalent to the compact 3–manifold-with-boundary \( M_0 \) we have that \( M_+ = \tilde{M} / \tilde{\mathcal{C}} = M_0 / \partial M_0 \). By the exact sequence of the pair \( (\tilde{M}, \tilde{\mathcal{C}}) \) we have that \( H_3(\tilde{M}; \mathbb{Z}) \cong H_3(\tilde{\mathcal{C}}; \mathbb{Z}) \) and therefore we have that

\[
H_3(\tilde{M}; \mathbb{Z}) \cong H_3(\tilde{M}, \tilde{\mathcal{C}}; \mathbb{Z}) \cong H_3(\tilde{\mathcal{C}}; \mathbb{Z}) \cong H_3(M_+; \mathbb{Z}) \cong H_3(M_0, \partial M_0; \mathbb{Z}) \cong \mathbb{Z}.
\]

We denote by \([\tilde{M}]\) the generator and call it the fundamental class of \( \tilde{M} \).

### 5.2. Invariants of hyperbolic 3–manifolds of finite volume

Let \( M \) be a compact oriented hyperbolic 3–manifold. To the canonical representation \( \tilde{\rho}: \pi_1(M) \to PSL_2(\mathbb{C}) \) corresponds a map \( B\rho: M \to BPSL_2(\mathbb{C}) \) where \( BPSL_2(\mathbb{C}) \) is the classifying space of \( PSL_2(\mathbb{C}) \). There is a well-known invariant \([M]_{PSL} \) of \( M \) in the group \( H_3(PSL_2(\mathbb{C}); \mathbb{Z}) \) given by the image of the fundamental class of \( M \) under the homomorphism induced in homology by \( B\rho \).

As we said before, we generalize the construction given in Cisneros-Molina–Jones [3] to extend this invariant when \( M \) is a complete oriented hyperbolic 3–manifold of finite volume (ie \( M \) is compact or with cusps) to invariants \( \beta_H(M) \), but in this case \( \beta_H(M) \) takes values in \( H_3([SL_2(\mathbb{C}) : H]; \mathbb{Z}) \), where \( H \) is one of the subgroups \( P \) or \( B \) of \( G = SL_2(\mathbb{C}) \) defined in Section 4.

Let \( \Gamma \) be a discrete torsion-free subgroup of \( SL_2(\mathbb{C}) \). The action of \( \Gamma \) on the hyperbolic 3–space \( \mathbb{H}^3 \) is free and since \( \mathbb{H}^3 \) is contractible, by Theorem 2.3 it is a model for \( ET \).

The action of \( \Gamma \) on \( \tilde{Y} \) is no longer free. The points in \( \mathcal{C} \) have as isotropy subgroups the peripheral subgroups \( \Gamma_1, \ldots, \Gamma_d \) of \( \Gamma \) or their conjugates and any subgroup in \( \mathfrak{g}(\Gamma_1, \ldots, \Gamma_d) \) fixes only one point in \( \mathcal{C} \). Therefore, by Theorem 2.3 we have that \( \tilde{Y} \) is a model for \( E_{\mathfrak{g}(\Gamma_1, \ldots, \Gamma_d)}(\Gamma) \).

We have the following facts:

- Since \( \{I\} \subset \mathfrak{g}(\Gamma_1, \ldots, \Gamma_d) \) there is a \( \Gamma \)-map \( \mathbb{H}^3 \to \tilde{Y} \) unique up to \( \Gamma \)-homotopy. We can use the inclusion.
- By Proposition 2.7 \( \text{res}^G_E \) is a model for \( ET \). Therefore, there is a \( \Gamma \)-homotopy equivalence \( \mathbb{H}^3 \to \text{res}^G_E \) which is unique up to \( \Gamma \)-homotopy.
• Since $\mathfrak{g}(\Gamma_1, \ldots, \Gamma_d) = \mathfrak{g}(P)/\Gamma \subset \mathfrak{g}(B)/\Gamma$ we have $\Gamma$–maps

$$\hat{Y} \to \text{res}_i^\mathfrak{g} E_{\mathfrak{g}(P)}(G) \to \text{res}_i^\mathfrak{g} E_{\mathfrak{g}(B)}(G)$$

which are unique up to $\Gamma$–homotopy.

**Remark 5.2.** Since $\mathfrak{g}(\Gamma_1, \ldots, \Gamma_d) = \mathfrak{g}(P)/\Gamma$ by Proposition 2.7 we have that the $\Gamma$–space $\text{res}_i^\mathfrak{g} E_{\mathfrak{g}(P)}(G)$ is a model for $E_{\mathfrak{g}(\Gamma_1, \ldots, \Gamma_d)}(\Gamma)$. Therefore, the $\Gamma$–map $\hat{Y} \to \text{res}_i^\mathfrak{g} E_{\mathfrak{g}(P)}(G)$ is in fact a $\Gamma$–homotopy equivalence.

Combining the previous $\Gamma$–maps with the $G$–maps given in (4.11) we have the following commutative diagram (up to equivariant homotopy)

$$
\begin{array}{ccc}
E\mathfrak{g} & \rightarrow & E_{\mathfrak{g}(P)}(G) \\
\uparrow & & \uparrow \\
\hat{Y} & \rightarrow & E_{\mathfrak{g}(B)}(G) \\
\end{array}
$$

Taking the quotients by $SL_2(\mathbb{C})$ and $\Gamma$ we get the following commutative diagram

(5.1)

$$
\begin{array}{ccc}
E\mathfrak{g} & \rightarrow & E_{\mathfrak{g}(P)}(G) \\
\uparrow & & \uparrow \\
\hat{Y} & \rightarrow & E_{\mathfrak{g}(B)}(G) \\
\uparrow & & \uparrow \\
BG & \rightarrow & B_{\mathfrak{g}(P)}(G) \\
\uparrow & & \uparrow \\
M & \rightarrow & \hat{M} \\
\end{array}
$$

where $f = B\rho: BG \to BG$ is the map between classifying spaces which on fundamental groups induces the representation $\rho: \Gamma \to SL_2(\mathbb{C})$ of $M$, and $\hat{\psi}_P$ and $\hat{\psi}_B$ are given by the compositions

$$
\begin{align*}
\hat{\psi}_P: \hat{M} & \rightarrow E_{\mathfrak{g}(P)}(G)/\Gamma \rightarrow E_{\mathfrak{g}(P)}(G), \\
\hat{\psi}_B: \hat{M} & \rightarrow E_{\mathfrak{g}(B)}(G)/\Gamma \rightarrow E_{\mathfrak{g}(B)}(G),
\end{align*}
$$

and they are well-defined up to homotopy.

The maps $\hat{\psi}_P$ and $\hat{\psi}_B$ induce homomorphisms

$$
\begin{align*}
(\hat{\psi}_P)_*: H_3(\hat{M}; \mathbb{Z}) & \rightarrow H_3(B_{\mathfrak{g}(P)}(G); \mathbb{Z}), \\
(\hat{\psi}_B)_*: H_3(\hat{M}; \mathbb{Z}) & \rightarrow H_3(B_{\mathfrak{g}(B)}(G); \mathbb{Z}).
\end{align*}
$$

We denote by $\beta_P(M)$ and $\beta_B(M)$ the canonical classes in $H_3(B_{\mathfrak{g}(P)}(G); \mathbb{Z})$ and $H_3(B_{\mathfrak{g}(B)}(G); \mathbb{Z})$ respectively, given by the images of the fundamental class $[\hat{M}]$ of $\hat{M}$

$$
\begin{align*}
\beta_P(M) &= (\hat{\psi}_P)_*([\hat{M}]), \\
\beta_B(M) &= (\hat{\psi}_B)_*([\hat{M}]).
\end{align*}
$$

By the commutativity of the lower triangle in (5.1) we have that $\beta_P(M)$ is sent to $\beta_B(M)$ by the canonical homomorphism from $H_3(B_{\mathfrak{g}(P)}(G); \mathbb{Z})$ to $H_3(B_{\mathfrak{g}(B)}(G); \mathbb{Z})$. 




Thus, by Proposition 3.13 we have the following

**Theorem 5.3.** Given a complete oriented hyperbolic 3–manifold of finite volume $M$ we have well-defined invariants

$$
\beta_p(M) \in H_3([G : P]; \mathbb{Z}), \\
\beta_B(M) \in H_3([G : B]; \mathbb{Z}).
$$

Moreover, we have that

$$
\beta_B(M) = (h^B_\beta)_*(\beta_p(M)),
$$

where $(h^B_\beta)_*: H_n([G : P]; \mathbb{Z}) \to H_n([G : B]; \mathbb{Z})$ is the homomorphism described in (3.8).

**Proposition 5.4.** The invariants $\beta_p(M)$ and $\beta_B(M)$ of $M$ only depend on the canonical representation $\bar{\rho}: \Gamma \to PSL_2(\mathbb{C})$ and not on the lifting $p: \Gamma \to SL_2(\mathbb{C})$. In other words, they are independent of the choice a spin structure of $M$.

**Proof.** Notice that in diagram (5.1) we can replace $G$ by $\bar{G}$. Since by Proposition 3.15 $H_3([G : P]; \mathbb{Z}) \cong H_3([\bar{G} : P]; \mathbb{Z})$ and $H_3([G : B]; \mathbb{Z}) \cong H_3([\bar{G} : B]; \mathbb{Z})$, by (4.12) we get the same invariants $\beta_p(M)$ and $\beta_B(M)$. □

**Remark 5.5.** The invariants $\beta_p(M)$ and $\beta_B(M)$ extend the invariant $[M]_{PSL}$ for $M$ closed in the following sense: when $M$ is compact $\hat{M} = M$, by the commutativity of the lower diagram in (5.1) (using $\bar{G}$ instead of $G$) and by Remark 3.18 we have that

$$
(\bar{\psi})_* = (h^B_\psi)_* \circ f_*,
$$

where $(h^B_\psi)_*$ and $(h^B_\beta)_*$ are the homomorphisms described in (3.8). Thus

$$
\beta_p(M) = (h^\bar{\psi})_*([M]_{PSL}),
$$

$$
\beta_B(M) = (h^B_\beta)_*([M]_{PSL}).
$$

6. Relation with the extended Bloch group

In the present section we recall the definitions of the Bloch and extended Bloch groups and the Bloch invariant. We see that the Bloch group is isomorphic to $H_3([G : B]; \mathbb{Z})$ and under this isomorphism the Bloch invariant is the invariant $\beta_B(M)$.

6.1. The Bloch group. The pre-Bloch group $\mathcal{P}(\mathbb{C})$ is the abelian group generated by the formal symbols $[z]$, $z \in \mathbb{C} \setminus \{0, 1\}$ subject to the relation

$$
[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right] = 0, \quad x \neq y.
$$

This relation is called the *five term relation*. By Dupont–Sah [7, Lemma 5.11] we also have the following relations in $\mathcal{P}(\mathbb{C})$

$$
[x] = \left[\frac{1}{1-x}\right] = \left[1 - \frac{1}{x}\right] = -\left[\frac{1}{x}\right] = -\left[\frac{x}{x-1}\right] = -[1-x].
$$

Using this relations it is possible to extend the definition of $[x] \in \mathcal{P}(\mathbb{C})$ allowing $x \in \hat{\mathbb{C}}$ and removing the restriction $x \neq y$ in (6.1). This is equivalent [7] after
Lemma 5.11] to define $\mathcal{P}(\mathbb{C})$ as the abelian group generated by the symbols $[z]$, $z \in \hat{\mathbb{C}}$ subject to the relations
\[
[0] = [1] = [\infty] = 0,
\]
\[
[x] - [y] + \left[ \frac{y}{x} \right] - \left[ \frac{1 - x}{1 - y} \right] + \left[ \frac{1 - x}{1 - y} \right] = 0.
\]

The pre-Bloch group can be interpreted as a Hochschild relative homology group. The action of $G$ on $\hat{\mathbb{C}}$ by fractional linear transformations (see §6.3) is not only transitive but triply transitive, that is, given four distinct points $z_0, z_1, z_2, z_3$ in $\hat{\mathbb{C}}$, there exists an element $g \in PSL_2(\mathbb{C})$ such that
\[
g \cdot z_0 = 0, \quad g \cdot z_1 = \infty, \quad g \cdot z_2 = 1, \quad g \cdot z_3 = z
\]
where $z = [z_0 : z_1 : z_2 : z_3]$ is the cross-ratio of $z_0, z_1, z_2, z_3$ given by
\[
(6.3)\quad [z_0 : z_1 : z_2 : z_3] = \frac{(z_0 - z_3)(z_1 - z_2)}{(z_0 - z_2)(z_1 - z_3)}.
\]

In other words, the orbit of a 4-tuple $(z_0, z_1, z_2, z_3)$ of distinct points in $\hat{\mathbb{C}}$ under the diagonal action of $G$ is determined by its cross-ratio.

If we extend the definition of the cross-ratio to $[z_0 : z_1 : z_2 : z_3] = 0$ whenever $z_i = z_j$ for some $i \neq j$, we get a well defined homomorphism
\[
\sigma : B_3(X(B)) = B_3(\hat{\mathbb{C}}) \to \mathcal{P}(\mathbb{C})
\]
\[
(6.4)\quad (z_0, z_1, z_2, z_3)_G \mapsto [z_0 : z_1 : z_2 : z_3]
\]
where $(z_0, z_1, z_2, z_3)_G$ denotes the $G$-orbit of the 3-simplex $(z_0, z_1, z_2, z_3) \in C_3(X(B))$. It is easy to see that the five term relation (6.1) is equivalent to the relation
\[
\sum_{i=0}^{4} (-1)^i [z_0 : \cdots : \hat{z}_i : \cdots : z_4] = 0.
\]

By the triply transitivity of the action of $G$ on $\hat{\mathbb{C}}$ we have that $B_2(X(B)) \cong \mathbb{Z}$ and $B_3(X(B))$ consists only of cycles. Thus $\sigma$ induces an isomorphism, compare with [17] Lemma 2.2
\[
(6.5)\quad H_3([G : B]; \mathbb{Z}) = H_3(B_*(X(B))) \cong \mathcal{P}(\mathbb{C}).
\]

**Remark 6.1.** If we consider the first definition of the pre-Bloch group where for the generators $[z]$ of $\mathcal{P}(\mathbb{C})$ we only allow $z$ to be in $\mathbb{C} \setminus \{0,1\}$, each generator $[z]$ corresponds to the $G$-orbit of a 4-tuple $(z_0, z_1, z_2, z_3)$ of distinct points in $\hat{\mathbb{C}}$. In this case we have that
\[
(6.6)\quad H_3(B_3^f(X(B))) \cong \mathcal{P}(\mathbb{C}),
\]
and the isomorphism between (6.5) and (6.6) is given by Remark 3.21. Also using this definition of the pre-Bloch group it is possible to prove that it is isomorphic to the corresponding Takasu relative homology group:

**Proposition 6.2.**
\[
H_3(G,B; \mathbb{Z}) = H_3([G : B]; \mathbb{Z}) \cong \mathcal{P}(\mathbb{C}).
\]

**Proof.** By [18] Theorem 2.2 we have that $H_3(G,B; \mathbb{Z}) \cong H_2(G, I_{(G,B)}(\mathbb{Z}))$ and by [17] (A27), (A28) we also have $H_2(G, I_{(G,B)}(\mathbb{Z})) \cong \mathcal{P}(\mathbb{C})$. □
The Bloch group $\mathcal{B}(\mathbb{C})$ is the kernel of the map

\[
\nu: P(\mathbb{C}) \rightarrow \bigwedge^2 \mathbb{C}^\times
\]

\[
[z] \mapsto z \wedge (1 - z).
\]

6.2. The Bloch invariant. An ideal simplex is a geodesic 3–simplex in $\mathbb{H}^3$ whose vertices $z_0, z_1, z_2, z_3$ are all in $\partial \mathbb{H}^3 = \hat{\mathbb{C}}$. We consider the vertex ordering as part of the data defining an ideal simplex. By the triply transitivity of the action of $G$ on $\mathbb{H}^3$ the orientation-preserving congruence class of an ideal simplex with vertices $z_0, z_1, z_2, z_3$ is given by the cross-ratio $z = [z_0 : z_1 : z_2 : z_3]$. An ideal simplex is flat if and only if the cross-ratio is real, and if it is not flat, the orientation given by the vertex ordering agrees with the orientation inherited from $\mathbb{H}^3$ if and only if the cross-ratio has positive imaginary part.

From (6.3) we have that an even (ie orientation preserving) permutation of the $z_i$ replaces $z$ by one of three so-called cross-ratio parameters,

\[
z, \quad z' = \frac{1}{1 - z}, \quad z'' = 1 - \frac{1}{z},
\]

while an odd (ie orientation reversing) permutation replaces $z$ by

\[
\frac{1}{z}, \quad \frac{z}{z - 1}, \quad 1 - z.
\]

Thus, by the relations (6.2) in $P(\mathbb{C})$ we can consider the pre-Bloch group as being generated by (congruence classes) of oriented ideal simplices.

Let $M$ be a non-compact orientable complete hyperbolic 3–manifold of finite volume. An ideal triangulation for $M$ is a triangulation where all the tetrahedra are ideal simplices.

Let $M$ be an hyperbolic 3–manifold and let $\triangle_1, \ldots, \triangle_n$ be the ideal simplices of an ideal triangulation of $M$. Let $z_i \in \mathbb{C}$ be the parameter of $\triangle_i$ for each $i$. These parameters define an element $\beta(M) = \sum_{i=1}^n [z_i]$ in the pre-Bloch group. The element $\beta(M) \in P(\mathbb{C})$ is called the Bloch invariant of $M$.

Remark 6.3. Neumann and Yang defined the Bloch invariant using degree one ideal triangulations, in that way it is defined for all hyperbolic 3–manifolds of finite volume, even the compact ones, see Neumann–Yang [14, §2] for details.

In [14 Theorem 1.1] it is proved that the Bloch invariant lies in the Bloch group $\mathcal{B}(\mathbb{C})$. An alternative proof of this fact is given in Cisneros-Molina–Jones [5 Cor. 8.7].

Remark 6.4. By (6.5) we have that $H_3([G : B]; \mathbb{Z}) \cong P(\mathbb{C})$ and in [5 Theorem 6.1] it is proved that $\beta_B(M)$ is precisely the Bloch invariant $\beta(M)$ of $M$, see Subsection 8.1.

6.3. The extended Bloch group. Given a complex number $z$ we use the convention that its argument $\text{Arg } z$ always denotes its main argument $-\pi < \text{Arg } z \leq \pi$ and $\text{Log } z$ always denotes a fixed branch of logarithm, for instance, the principal branch having $\text{Arg } z$ as imaginary part.

Let $\triangle$ be an ideal simplex with cross-ratio $z$. A flattening of $\triangle$ is a triple of complex numbers of the form

\[
(w_0, w_1, w_2) = (\text{Log } z + p\pi i, -\text{Log}(1 - z) + q\pi i, \text{Log}(1 - z) - \text{Log } z - p\pi i - q\pi i)
\]
with \( p, q \in \mathbb{Z} \). The numbers \( w_0, w_1 \) and \( w_2 \) are called \textit{log parameters} of \( \triangle \). Up to multiples of \( \pi i \), the log parameters are logarithms of the cross-ratio parameters.

**Remark 6.5.** The log parameters uniquely determine \( z \). Hence we can write a flattening as \([z;p,q]\). Note that this notation depends on the choice of logarithm branch.

Following [13] we assign cross-ratio parameters and log parameters to the edges of a flattened ideal simplex as indicated in Figure 1.

![Cross-ratio and log parameters of a flattened ideal simplex](image)

**Figure 1.** Cross-ratio and log parameters of a flattened ideal simplex

Let \( z_0, z_1, z_2, z_3 \) and \( z_4 \) be five distinct points in \( \hat{\mathbb{C}} \) and let \( \triangle_i \) denote the ideal simplices \((z_0, \ldots, \hat{z}_i, \ldots, z_4)\). Let \((w_0^i, w_1^i, w_2^i)\) be flattenings of the simplices \( \triangle_i \). Every edge \([z_iz_j]\) belongs to exactly three of the \( \triangle_i \) and therefore has three associated log parameters. The flattenings are said to satisfy the \textit{flattening condition} if for each edge the signed sum of the three associated log parameters is zero. The sign is positive if and only if \( i \) is even.

From the definition we have that the flattening condition is equivalent to the following ten equations:

\[
\begin{align*}
[z_0 z_1] & : w_0^2 - w_0^3 + w_0^4 = 0 & [z_0 z_2] & : -w_0^1 - w_2^3 + w_2^4 = 0 \\
[z_1 z_2] & : w_1^0 - w_1^3 + w_1^4 = 0 & [z_1 z_3] & : w_0^0 + w_1^2 + w_2^3 = 0 \\
[z_2 z_3] & : w_1^0 - w_1^1 + w_0^4 = 0 & [z_2 z_4] & : -w_2^0 - w_1^1 - w_0^3 = 0 \\
[z_3 z_4] & : w_0^0 - w_0^1 + w_0^3 = 0 & [z_3 z_0] & : -w_2^0 + w_2^3 + w_4^1 = 0 \\
[z_4 z_0] & : -w_1^1 + w_1^2 - w_3^3 = 0 & [z_4 z_1] & : w_1^0 + w_2^2 - w_3^3 = 0
\end{align*}
\]

(6.8)

The \textit{extended pre-Bloch group} \( \hat{\text{P}}(\mathbb{C}) \) is the free abelian group generated by flattened ideal simplices subject to the relations:

\textbf{Lifted five term relation:}

\[
\sum_{i=0}^{4} (-1)^i (w_0^i, w_1^i, w_2^i) = 0,
\]

if the flattenings satisfy the flattening condition.

\textbf{Transfer relation:}

\[
[z;p,q] + [z;p',q'] = [z;p,q'] + [z;p',q].
\]
The extended Bloch group $\widehat{B}(\mathbb{C})$ is the kernel of the homomorphism

$$\widehat{\nu}: \widehat{P}(\mathbb{C}) \to \Lambda^2 \mathbb{C}$$

$$(w_0, w_1, w_2) \mapsto w_0 \wedge w_1.$$ 

7. Mappings via configurations in $X_{(P)}$

In this section following ideas in Dupont and Zickert [8, §3] we define a homomorphism

$$H_3([G: P]; \mathbb{Z}) \to \widehat{B}(\mathbb{C}).$$

We use the models for $X_{(U)}$, $X_{(P)}$, $X_{(B)}$ and the explicit $G$-maps between them described in Section 4. We simplify notation by setting

$$h_U = h^B_U: X_{(U)} \to X_{(B)}$$

$$h_P = h^B_P: X_{(P)} \to X_{(B)}.$$ 

Consider the $h_U$–subcomplexes with $H = U, P, B$: $C^h_{s^U}(X_{(U)})$, $C^h_{s^P}(X_{(P)})$ and $C^h_{s^B}(X_{(B)})$ defined in Subsection 3.2.1.

Since $h^U_U$, $h_U$ and $h_P$ are $G$–equivariant they induce maps

$$(h^U_U)_*: C^h_{s^U}(X_{(U)}) \to C^h_{s^P}(X_{(P)}),$$

$$(h_U)_*: C^h_{s^U}(X_{(U)}) \to C^h_{s^B}(X_{(B)}),$$

$$(h_P)_*: C^h_{s^P}(X_{(P)}) \to C^h_{s^B}(X_{(B)}).$$

We start defining a homomorphism $C^h_{s^U}(X_{(U)}) \to \widehat{P}(\mathbb{C})$ which descends to a homomorphism

$$C^h_{s^P}(X_{(P)}) \to \widehat{P}(\mathbb{C}).$$

We assign to each 4–tuple $(v_0, v_1, v_2, v_3) \in C^h_{s^U}(X_{(U)})$ a combinatorial flattening of the ideal simplex $(h_U(v_0), h_U(v_1), h_U(v_2), h_U(v_3))$ in such a way that the combinatorial flattenings assigned to tuples $(v_0, \ldots, v_i, \ldots, v_4)$ satisfy the flattening condition.

Given $v_i = (v^i_1, v^i_2) \in X_{(U)}$ we denote by

$$\det(v_i, v_j) = \det \begin{pmatrix} v^i_1 & v^i_2 \\ v^j_1 & v^j_2 \end{pmatrix} = v^i_1 v^j_2 - v^i_2 v^j_1.$$ 

Let $(v_0, v_1, v_2, v_3) \in C^h_{s^U}(X_{(U)})$. As was noticed in [8, Section 3.1], the cross-ratio parameters $z, \frac{1}{1-z}$ and $\frac{z-1}{z}$ of the simplex $(h_U(v_0), h_U(v_1), h_U(v_2), h_U(v_3))$ can be
expressed in terms of determinants

\[ (7.1) \]
\[ z = \left[ h_U(v_0) : h_U(v_1) : h_U(v_2) : h_U(v_3) \right] = \left( \frac{v_1^3}{v_i^3} - \frac{v_2^3}{v_j^3} \right) \left( \frac{v_i^3}{v_1^3} - \frac{v_j^3}{v_2^3} \right) = \frac{\det(v_0, v_3) \det(v_1, v_2)}{\det(v_0, v_2) \det(v_1, v_3)}, \]

\[ (7.2) \]
\[ \frac{1}{1 - z} = \left[ h_U(v_1) : h_U(v_2) : h_U(v_0) : h_U(v_3) \right] = \left( \frac{v_1^3}{v_i^3} - \frac{v_2^3}{v_j^3} \right) \left( \frac{v_i^3}{v_1^3} - \frac{v_j^3}{v_2^3} \right) = \frac{\det(v_1, v_3) \det(v_0, v_2)}{\det(v_0, v_1) \det(v_2, v_3)}, \]

\[ (7.3) \]
\[ \frac{z - 1}{z} = \left[ h_U(v_0) : h_U(v_2) : h_U(v_3) : h_U(v_1) \right] = \left( \frac{v_0^3}{v_i^3} - \frac{v_2^3}{v_j^3} \right) \left( \frac{v_i^3}{v_0^3} - \frac{v_j^3}{v_2^3} \right) = \frac{\det(v_0, v_1) \det(v_2, v_3)}{\det(v_0, v_3) \det(v_2, v_1)}. \]

Hence we also have

\[ (7.4) \]
\[ \frac{1 - z}{z} = - \left[ h_U(v_0) : h_U(v_2) : h_U(v_3) : h_U(v_1) \right] = \left( \frac{v_0^3}{v_i^3} - \frac{v_2^3}{v_j^3} \right) \left( \frac{v_i^3}{v_0^3} - \frac{v_j^3}{v_2^3} \right) = \frac{\det(v_0, v_1) \det(v_2, v_3)}{\det(v_0, v_3) \det(v_1, v_2)}. \]

We have that

\[ h_U(v_i) \neq h_U(v_j) \iff \frac{v_i^3}{v_i^3} - \frac{v_j^3}{v_j^3} \neq 0 \iff \frac{v_i^3 v_j^3 - v_j^3 v_i^3}{v_i^3 v_j^3} \neq 0 \iff \det(v_i, v_j) \neq 0. \]

Then all the previous determinants are non-zero.

We define

\[ (7.5) \]
\[ \langle v_i, v_j \rangle = \frac{1}{2} \log \det(v_i, v_j)^2. \]

Using formulas \[ (7.1), (7.2), \] and \[ (7.3), \] we assign a flattening to \( (v_0, v_1, v_2, v_3) \in \mathbb{C}^3 \) by setting

\[ w_0(v_0, v_1, v_2, v_3) = \langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle - \langle v_0, v_2 \rangle - \langle v_1, v_3 \rangle, \]

\[ w_1(v_0, v_1, v_2, v_3) = \langle v_0, v_2 \rangle + \langle v_1, v_3 \rangle - \langle v_0, v_1 \rangle - \langle v_2, v_3 \rangle, \]

\[ w_2(v_0, v_1, v_2, v_3) = \langle v_0, v_1 \rangle + \langle v_2, v_3 \rangle - \langle v_0, v_3 \rangle - \langle v_1, v_2 \rangle. \]

Recall that for \( w \in \mathbb{C}^x \) we have that

\[ (7.6) \]
\[ \frac{1}{2} \log w \Rightarrow \begin{cases} \log w + \pi i & \text{if } \operatorname{Arg} w \in (-\pi, -\pi/2], \\ \log w & \text{if } \operatorname{Arg} w \in (-\pi, \pi/2), \\ \log w - \pi i & \text{if } \operatorname{Arg} w \in (\pi/2, \pi]. \end{cases} \]

By the addition theorem of the logarithm \[ [2 \text{ §3.4}] \] and \[ (7.6) \] we have that:

\[ w_0 = \log z + i k \pi, \quad w_1 = \log \left( \frac{1}{1 - z} \right) + i l \pi, \quad w_2 = \log \left( \frac{1 - z}{z} \right) + i m \pi, \]

for some integers \( k, l \) and \( m \). Hence, \( w_0, w_1, w_2 \) are respectively logarithms of the cross-ratio parameters \( z, \frac{1}{1 - z} \) and \( \frac{1 - z}{z} \) up to multiples of \( \pi i \) and clearly we have that \( \frac{1}{1 - z} + \frac{1}{z} + \frac{1}{z} = 0 \). Therefore \( \langle w_0, w_1, w_2 \rangle \) is a flattening in \( \mathbb{P}(\mathbb{C}) \) of the ideal simplex \( (h_U(v_0), h_U(v_1), h_U(v_2), h_U(v_3)) \).
This defines a homomorphism
\[\tilde{\sigma}: C^b_{h_U}(X(U)) \to \tilde{\mathcal{P}}(\mathbb{C})\]
with \((v_0, v_1, v_2, v_3) \mapsto (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2)\).

**Lemma 7.1.** Let \(v_i, v_j \in X(U)\). Then \(\det(v_i, v_j)\) is invariant under the action of \(G\) on \(X(U)\).

**Proof.** Let \(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})\) and set
\[\bar{v}_i = (\bar{v}^1_i, \bar{v}^2_i) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (v^1_i, v^2_i) = (av^1_i + bv^2_i, cv^1_i + dv^2_i)\]
We have that
\[
\det(\bar{v}_i, \bar{v}_j) = \det\begin{pmatrix} \bar{v}^1_i & \bar{v}^2_i \\ \bar{v}^1_j & \bar{v}^2_j \end{pmatrix} = \bar{v}^1_i \bar{v}^2_j - \bar{v}^2_i \bar{v}^1_j
\]
\begin{equation}
= (av^1_i + bv^2_i)(cv^1_j + dv^2_j) - (av^1_j + bv^2_j)(cv^1_i + dv^2_i)
\end{equation}
\[= (ad - bc)(v^1_i v^2_j - v^2_i v^1_j)\] since \(ad - bc = 1\).
\[
= \det(v_i, v_j).
\]
\(\square\)

Hence the homomorphism \(\tilde{\sigma}\) descends to the quotient by the action of \(SL_2(\mathbb{C})\)
\[\tilde{\sigma}: B^{b_U}_{3}(X(U)) \to \tilde{\mathcal{P}}(\mathbb{C}).\]

Now suppose that \((\tilde{w}^0_0, \tilde{w}^1_0, \tilde{w}^2_0), \ldots, (\tilde{w}^0_3, \tilde{w}^1_3, \tilde{w}^2_3)\) are flattenings defined as above for the simplices \((h_U(v_0), \ldots, h_U(v_3))\). We must check that these flattenings satisfy the flattening condition, that is, we have to check that the ten equations (6.8) are satisfied. We check the first equation, the others are similar:
\[\bar{z}_0\bar{z}_1;\]
\[
\tilde{w}^2_0 = \langle v_0, v_4 \rangle + \langle v_1, v_3 \rangle - \langle v_0, v_3 \rangle - \langle v_1, v_4 \rangle,
\]
\[-\tilde{w}^3_0 = -\langle v_0, v_4 \rangle - \langle v_1, v_2 \rangle + \langle v_0, v_2 \rangle + \langle v_1, v_4 \rangle,
\]
\[
\tilde{w}^2_0 = \langle v_0, v_3 \rangle + \langle v_1, v_2 \rangle - \langle v_0, v_2 \rangle - \langle v_1, v_4 \rangle.
\]
Having verified all the ten equations, it now follows from [8 Theorem 2.8] or [13 Lemma 3.4] that \(\tilde{\sigma}\) sends boundaries to zero and we obtain a homomorphism
\[\tilde{\sigma}: H_3(B^{b_U}_{3}(X(U))) \to \tilde{\mathcal{P}}(\mathbb{C}).\]

**7.1. The homomorphism \(\tilde{\sigma}\) descends to \(C^{b_U}_{3}(X(U))\).** Given \(v_i = (v^1_i, v^2_i) \in X(U)\) we denote by \(v_i = h_U^+(v_i) = [v^1_i, v^2_i]\) its class in \(X(U)\).

**Remark 7.2.** Notice that if \(v_i = (v^1_i, v^2_i) \in X(U)\), then
\begin{equation}
\det(v_i, v_j) = \det\begin{pmatrix} v^1_i & v^2_i \\ v^1_j & v^2_j \end{pmatrix} = v^1_i v^2_j - v^2_i v^1_j = \det(-v_i, -v_j),
\end{equation}
but in the other hand
\begin{equation}
\det(-v_i, v_j) = \det\begin{pmatrix} -v^1_i & -v^2_i \\ v^1_j & v^2_j \end{pmatrix} = -v^1_i v^2_j + v^2_i v^1_j = -\det(v_i, v_j).
\end{equation}
Thus, the quantity $\det(v_i, v_j)$ is just well-defined up to sign. However, its square $\det(v_i, v_j)^2$ is well-defined.

By Lemma 7.1, 7.8 and 7.9 we have:

**Lemma 7.3.** Let $v_i, v_j \in X(P)$. Then $\det(v_i, v_j)^2$ is invariant under the action of $G$ on $X(P)$.

So, we define

\[(v_i, v_j) = \frac{1}{2} \log \det(v_i, v_j)^2. \tag{7.10} \]

Let $(v_0, v_1, v_2, v_3) \in \mathbb{C}_3^{h\neq}(X(P))$. The homomorphism $\bar{\sigma}$ descends to a well-defined homomorphism

\[
\bar{\sigma} : \mathbb{C}_3^{h\neq}(X(P)) \to \hat{\mathbb{P}}(\mathbb{C})
\]

by assign to $(v_0, v_1, v_2, v_3)$ the flattening of the ideal simplex

\[
(h_P(v_0), h_P(v_1), h_P(v_2), h_P(v_3)) = (h_U(v_0), h_U(v_1), h_U(v_2), h_U(v_3))
\]

given by (7.5) and we obtain a homomorphism

\[
\bar{\sigma} : H_3(B_3^{h\neq}(X(P))) \to \hat{\mathbb{P}}(\mathbb{C}).
\]

**Proposition 7.4.** The image of $\bar{\sigma} : H_3(B_3^{h\neq}(X(P))) \to \hat{\mathbb{P}}(\mathbb{C})$ is in $\hat{\mathbb{B}}(\mathbb{C})$.

*Proof.* Define a map $\mu : B_2^{h\neq}(X(P)) \to \wedge_2^2(\mathbb{C})$ by

\[
(v_0, v_1, v_2)_G \mapsto \langle v_0, v_1 \rangle \wedge \langle v_0, v_2 \rangle - \langle v_0, v_1 \rangle \wedge \langle v_1, v_2 \rangle + \langle v_0, v_2 \rangle \wedge \langle v_1, v_2 \rangle.
\]

Recall that the extended Bloch group $\hat{\mathbb{B}}(\mathbb{C})$ is the kernel of the homomorphism

\[
\hat{\nu} : \hat{\mathbb{P}}(\mathbb{C}) \to \wedge_2^2(\mathbb{C})
\]

\[
(w_0, w_1, w_1) \mapsto w_0 \wedge w_1.
\]

A straightforward computation shows that the following diagram

\[
\begin{array}{ccc}
B_3^{h\neq}(X(P)) & \xrightarrow{\hat{\sigma}} & \hat{\mathbb{P}}(\mathbb{C}) \\
\downarrow{\hat{\sigma}} & & \downarrow{\hat{\nu}} \\
B_2^{h\neq}(X(P)) & \xrightarrow{\mu} & \wedge_2^2(\mathbb{C})
\end{array}
\]

is commutative. This means that cycles are mapped to $\hat{\mathbb{B}}(\mathbb{C})$ as desired. $\square$

Therefore $\bar{\sigma} : H_3(B_3^{h\neq}(X(P))) \to \hat{\mathbb{B}}(\mathbb{C})$. Then by Proposition 3.20 we have a homomorphism

\[
\bar{\sigma} : H_3([G : P]; \mathbb{Z}) \to \hat{\mathbb{B}}(\mathbb{C}). \tag{7.11}
\]
Let \( A_i, A_j \in \tilde{X}(p) \), then we have that
\[
A_i = \begin{pmatrix} r_i & t_i \\ t_i & s_i \end{pmatrix}, \quad r_i s_i = t_i^2.
\]
Define
\[
DS(A_i, A_j) = r_i s_j - 2 t_i t_j + r_j s_i.
\]
Recall from Corollary 4.10 that the \( G \)-isomorphism between \( X(p) \) and \( \tilde{X}(p) \) is given by
\[
\varrho: \ X(p) \to \tilde{X}(p),
\]
\[
\begin{bmatrix} u \\ v \end{bmatrix} \leftrightarrow \begin{bmatrix} u^2 & uv \\ uv & v^2 \end{bmatrix}.
\]

**Lemma 7.5.** Let \( v_i \) and \( v_j \) in \( X(p) \). Then
\[
DS(\varrho(v_i), \varrho(v_j)) = \det(v_i, v_j)^2.
\]

**Proof.** Let \( v_i = [u_i, v_i] \) and \( v_j = [u_j, v_j] \) in \( X(p) \). We have that
\[
\varrho(v_i) = \begin{pmatrix} u_i^2 & u_i v_i \\ u_i v_i & v_i^2 \end{pmatrix}, \quad \varrho(v_j) = \begin{pmatrix} u_j^2 & u_j v_j \\ u_j v_j & v_j^2 \end{pmatrix}
\]
Then
\[
DS(\varrho(v_i), \varrho(v_j)) = u_i^2 v_j^2 - 2 u_i v_i u_j v_j + u_j^2 v_i^2 = (u_i v_j - u_j v_i)^2 = \det(v_i, v_j)^2.
\]

Combining Corollary 4.10 and Lemma 7.5 we get the following corollary which can also be proved with a straightforward but tedious computation.

**Corollary 7.6.** Let \( A_i, A_j \in \tilde{X}(p) \) and \( g \in G \). Then \( DS(A_i, A_j) \) is \( G \)-invariant, that is,
\[
DS(gA_i g^T, gA_j g^T) = DS(A_i, A_j).
\]

We define
\[
\langle A_i, A_j \rangle = \frac{1}{2} \Log DS(A_i, A_j).
\]
So by Lemma 7.5 given \( v_i, v_j \in X(p) \) we have that \( \langle \varrho(v_i), \varrho(v_j) \rangle = \langle v_i, v_j \rangle \).

Let \( (A_0, A_1, A_2, A_3) \in C^h_{3,T}(\tilde{X}(p)) \). Then the homomorphism
\[
\tilde{\sigma}: C^h_{3,T}(\tilde{X}(p)) \to \tilde{\mathcal{P}}(\mathbb{C})
\]
\[
(A_0, A_1, A_2, A_3) \mapsto (\tilde{w}_0, \tilde{w}_1, \tilde{w}_2),
\]
is given by assigning to \( (A_0, A_1, A_2, A_3) \) the flattening defined by
\[
\tilde{w}_0 = \langle A_0, A_3 \rangle + \langle A_1, A_2 \rangle - \langle A_0, A_2 \rangle - \langle A_1, A_3 \rangle,
\]
\[
\tilde{w}_1 = \langle A_0, A_2 \rangle + \langle A_1, A_3 \rangle - \langle A_0, A_1 \rangle - \langle A_2, A_3 \rangle,
\]
\[
\tilde{w}_2 = \langle A_0, A_1 \rangle + \langle A_2, A_3 \rangle - \langle A_0, A_3 \rangle - \langle A_1, A_2 \rangle,
\]
which by Lemma 7.5 is the same as the one given in (7.5) and by Proposition 7.4 we obtain the homomorphism (7.11).

7.3. The fundamental class of $M$ in $H_3(PSL_2(\mathbb{C}); \mathbb{Z})$. Recall that by Neumann [13 Theorem 2.6] we have that $\hat{B}(\mathbb{C})$ is isomorphic to $H_3(PSL_2(\mathbb{C}); \mathbb{Z})$. Given a complete oriented hyperbolic 3-manifold of finite volume $M$ in Subsection 5.2, we construct a canonical class $\beta_P(M) \in H_3([G : P]; \mathbb{Z})$. Taking the image of $\beta_P(M)$ under the homomorphism $\bar{\sigma} : H_3([G : P]; \mathbb{Z}) \to \hat{B}(\mathbb{C})$ we obtain a canonical class

$$[M]_{PSL} = \bar{\sigma}(\beta_P(M)) \in \hat{B}(\mathbb{C}) \cong H_3(PSL_2(\mathbb{C}); \mathbb{Z}).$$

We call this class the $PSL$-fundamental class of $M$.

Hence we have the following commutative diagram

$$
\begin{array}{cccc}
H_3([G : U]; \mathbb{Z}) & \xrightarrow{\bar{\sigma}} & \hat{P}(\mathbb{C}) \\
\downarrow{\langle h^0_P \rangle} & & \uparrow{\bar{\sigma}} \\
H_3([G : P]; \mathbb{Z}) & \xrightarrow{(\hat{\psi}_P)^*} & \hat{P}(\mathbb{C}) \\
\downarrow{\langle h_P \rangle} & & \uparrow{\hat{\psi}_P} \\
H_3(M; \mathbb{Z}) & \xrightarrow{(\hat{\psi}_P)^*} & H_3([G : B]; \mathbb{Z}) & \xrightarrow{\sigma} & \hat{P}(\mathbb{C})
\end{array}
$$

Remark 7.7. By Corollary 4.16 and Proposition 3.15 we have the isomorphism $H_3([G : P]; \mathbb{Z}) \cong H_3(\hat{G}, \hat{P}; \mathbb{Z})$. In Proposition 14.3 Neumann proves that the long exact sequence (3.11) gives rise to a split exact sequence

$$0 \to H_3(\hat{G}; \mathbb{Z}) \xrightarrow{i_*} H_3(\hat{G}, \hat{P}; \mathbb{Z}) \xrightarrow{\partial} H_2(\hat{P}; \mathbb{Z}) \to 0. \quad (7.12)$$

We will see in Remark 9.11 that the homomorphism $\bar{\sigma}$ defines a splitting of sequence (7.12), that is, $\bar{\sigma}i_* = \text{id}$. Hence, in the case $M$ is closed the lower square of diagram (5.1) induces in homology the following commutative diagram

$$
\begin{array}{ccc}
H_3(\hat{G}; \mathbb{Z}) & \xrightarrow{i_*} & H_3([G : P]; \mathbb{Z}) \\
\uparrow{f_* = B\bar{\rho}} & & \uparrow{\bar{\psi}_P} \\
H_3(M; \mathbb{Z}) & \xrightarrow{\cong} & H_3(\hat{M}; \mathbb{Z})
\end{array}
$$

which shows that for $M$ closed both definitions of $[M]_{PSL}$ coincide.

8. Computing $\beta_P(M)$ and $\beta_B(M)$ using an ideal triangulation of $M$

Let $M$ be a non-compact orientable complete hyperbolic 3-manifold of finite volume. Let $\pi : \mathbb{H}^3 \to \Gamma\backslash\mathbb{H}^3 = M$ be the universal cover of $M$. Then $M$ lifts to an exact, convex, fundamental, ideal polyhedron $P$ for $\Gamma$ [15 Theorem 11.2.1]. An ideal triangulation of $M$ gives a decomposition of $P$ into a finite number of ideal tetrahedra $(z^i_0, z^i_1, z^i_2, z^i_3)$, $i = 1, \ldots, n$. Since $P = \{gP \mid g \in \Gamma\}$ is an exact tessellation of $\mathbb{H}^3$ [15 Theorem 6.7.1], this decomposition of $P$ gives an ideal triangulation of $\mathbb{H}^3$.

As in Subsection 5.1 let $\mathcal{C}$ be the set of fixed points of parabolic elements of $\Gamma$ in $\partial\mathbb{H}^3 = \mathcal{C}$ and consider $\hat{Y} = \mathbb{H}^3 \cup \mathcal{C}$, which is the result of adding the vertices of the ideal tetrahedra of the ideal triangulation of $\mathbb{H}^3$. Hence we can consider $\hat{Y}$ as
a simplicial complex with 0–simplices given by the elements of \( C \subset \hat{C} \). The action of \( G \) on \( \hat{Y} \) induces an action of \( \hat{G} \) on the tetrahedra of the ideal triangulation of \( \hat{Y} \). Taking the quotient of \( \hat{Y} \) by \( \hat{G} \) we obtain \( \hat{M} \) and we get an extension of the covering map \( \pi \) to a map \( \pi \hat{:} \hat{Y} \to \hat{M} \). The \( \hat{G} \)–orbits \( (z_0, z_1, z_2, z_3)_\Gamma \) of the tetrahedra \( (z_0, z_1, z_2, z_3) \) of the ideal triangulation of \( \hat{Y} \) correspond to the tetrahedra of the ideal triangulation of \( \hat{M} \). The \( \hat{G} \)–orbit set \( \hat{C} \) of \( C \) corresponds to the cusps points of \( M \), we suppose that \( M \) has \( d \) cups, so the cardinality of \( \hat{C} \) is \( d \).

8.1. Computation of \( \beta_B(M) \). Using the simplicial construction of \( E_{\hat{G}}(G) \) given by Proposition 2.4 we have that a model for \( E_{\hat{G}}(G) \) is the geometric realization of the simplicial set whose \( n \)–simplices are the ordered \((n + 1)\)–tuples \((z_0, \ldots, z_n)\) of elements of \( X_{\hat{G}}(B) = \hat{C} \) and the \( i \)–th face (respectively, degeneracy) of such a simplex is obtained by omitting (respectively, repeating) \( z_i \). The action of \( g \in G \) on an \( n \)–simplex \((z_0, \ldots, z_n)\) gives the simplex \((gz_0, \ldots, gz_n)\).

Considering \( \hat{Y} \) as the geometric realization of its ideal triangulation and since its vertices are elements in \( C \subset \hat{C} \) we have that the \( \Gamma \)–map \( \psi_B \): \( \hat{Y} \to E_{\hat{G}}(B)(G) \) in diagram \([5.1]\) is given by the (geometric realization of the) \( \Gamma \)–equivariant simplicial map

\[
\hat{Y} \xrightarrow{\psi_B} E_{\hat{G}}(B)(G)
\]

\[
(z_0, z_1, z_2, z_3) \mapsto (z_0, z_1, z_2, z_3).
\]

This induces the map \( \hat{\psi}_B : \hat{M} \to B_{\hat{G}}(B)(G) \) in diagram \([5.1]\). Furthermore, this induces on simplicial 3–chains the homomorphism \( (\hat{\psi}_B)_* : C_3(\hat{M}) \to B_3(B_{\hat{G}}(B)(G)) = B_3(X(B)) \) (see Remark 3.12) which we can compose with homomorphism \([6.4]\) to get

\[
C_3(\hat{M}) = C_3(\hat{Y})_\Gamma \xrightarrow{(\hat{\psi}_B)_*} B_3(X(B)) \xrightarrow{\partial_3} \mathcal{P}(C)
\]

\[
(8.1) \quad (z_0, z_1, z_2, z_3)_\Gamma \mapsto [z_0 : z_1 : z_2 : z_3],
\]

where \((z_0, z_1, z_2, z_3)_\Gamma\) (resp. \((z_0, z_1, z_2, z_3)_G\)) denotes the \( \Gamma \)–orbit (respectively \( G \)–orbit) of the 3–simplex \((z_0, z_1, z_2, z_3) \) in \( C_3(\hat{Y}) \) (respectively in \( B_3(X(B)) \)) and \([z_0 : z_1 : z_2 : z_3] = \frac{z_0}{z_3} : \frac{z_1}{z_3} : \frac{z_2}{z_3} \) is the cross-ratio parameter of the ideal tetrahedron \((z_0, z_1, z_2, z_3) \).

Let \((z_0^i, z_1^i, z_2^i, z_3^i)_\Gamma, i = 1, \ldots, n, \) be the ideal tetrahedra of the ideal triangulation of \( \hat{M} \) and let \( z_i = [z_0^i : z_1^i : z_2^i : z_3^i] \in C \) be the cross-ratio parameter of the ideal tetrahedron \((z_0^i, z_1^i, z_2^i, z_3^i) \) for each \( i \). Then the image of the fundamental class \([\hat{M}] \) under the homomorphism in homology given by \((8.1)\) is given by

\[
H_3(\hat{M}) \xrightarrow{(\hat{\psi}_B)_*} H_3([G : B]; \mathbb{Z}) \xrightarrow{\partial_3} \mathcal{P}(C)
\]

\[
[\hat{M}] = \sum_{i=1}^n [z_0^i : z_1^i : z_2^i : z_3^i]_\Gamma \mapsto \sum_{i=1}^n (z_0^i, z_1^i, z_2^i, z_3^i)_\Gamma \mapsto \sum_{i=1}^n [z_i],
\]

and we have that the invariant \( \beta_B(M) \) under the isomorphism \( \sigma \) corresponds to the Bloch invariant \( \beta(M) \), see Cisneros-Molina–Jones \([5, \text{Theorem 6.1}]\).

8.2. Computation of \( \beta_P(M) \). We want to give a simplicial description of the \( \Gamma \)–map \( \psi_P : \hat{Y} \to E_{\hat{G}}(P)(G) \) in diagram \([5.1]\). For this, we also use the Simplicial Construction of Proposition 2.4 to give a model for \( E_{\hat{G}}(P)(G) \) as the geometric realization of the simplicial set whose \( n \)–simplices are the ordered \((n + 1)\)–tuples of
elements of $X_{\mathbb{H}(P)} = X_P$ (or $X_{\mathbb{R}(P)} = \tilde{X}_P$) (see Remark 3.12 and Subsection 4.2). The $i$–th face (respectively, degeneracy) of such a simplex is obtained by omitting (respectively, repeating) the $i$–th element. The action of $g \in G$ on an $n$–simplex is the diagonal action.

Since the vertices of $\hat{Y}$ are elements in $C \subset \hat{C}$, to give a simplicial description of the $\Gamma$–map $\psi_P : \hat{Y} \to E_{\mathbb{R}(P)}(G)$ is enough to give a $\Gamma$–map

$$\Phi : C \to X_P, \quad \text{or} \quad \Phi : C \to \tilde{X}_P$$

and define

$$\hat{Y} \xrightarrow{\psi_P} E_{\mathbb{R}(P)}(G)$$

$$(z_0, z_1, z_2, z_3) \mapsto (\Phi(z_0), \ldots, \Phi(z_3)).$$

For $i = 1, \ldots, d$, every cusp point $\hat{c}_i \in \hat{C}$ of $M$ corresponds to a $\Gamma$–orbit of $C$. Choose $c_i \in C$ in the $\Gamma$–orbit corresponding to $\hat{c}_i \in \hat{C}$. Now choose an element $v_i \in (h_P)^{-1}(c_i) \subset X_P$ (or $A_i \in (h_P)^{-1}(c_i) \subset \tilde{X}_P$) and define

$$\Phi : C \to X_P, \quad \text{or} \quad \Phi : C \to \tilde{X}_P$$

$$c_i \mapsto v_i \quad \text{or} \quad c_i \mapsto A_i$$

and extend $\Gamma$–equivariantly by

$$\Phi(g \cdot c_i) = g \cdot v_i, \quad \text{or} \quad \Phi(g \cdot c_i) = gA_i g^T.$$ 

**Remark 8.1.** Suppose that for every cusp point $\hat{c}_i \in \hat{C}$ we have chosen $c_i \in C \subset \hat{C}$ in the $\Gamma$–orbit corresponding to $\hat{c}_i$. Using homogeneous coordinates we can write

$$c_i = [z_i : w_i].$$

So, one way to choose $v_i$ (or $A_i$) is given by

$$v_i = [z_i, w_i], \quad \text{or} \quad A_i = \left( \begin{array}{cc} z_i^2 & z_i w_i \\ z_i w_i & w_i^2 \end{array} \right).$$

The $\Gamma$–isotropy subgroups of the $c_i$ are conjugates of the peripheral subgroups $\Gamma_i$ and they consist of parabolic elements, that is, elements in a conjugate of $P$, and by Remark 4.14 they fix pointwise the elements of $(h_P)^{-1}(c_i)$. On the other hand, by Remark 5.1 no hyperbolic element in $\Gamma$ has as fixed point any point in $C$. Therefore $\Phi$ is a well-defined $\Gamma$–equivariant map and the map $\psi_P$ in (8.2) is a well-defined $\Gamma$–equivariant map. Since any two such $\Gamma$–maps are $\Gamma$–homotopic, $\psi_P$ is independent of the choice of the $c_i$ and the $A_i$ up to $\Gamma$–homotopy. This induces the map $\hat{\psi}_P : \hat{M} \to B_{\mathbb{R}(P)}(G)$ in diagram (5.1) and choosing different $c_i$ and $A_i$ we obtain homotopic maps. Thus, this induces a canonical homomorphism in homology $(\hat{\psi}_P)_* : H_3(\hat{M}) \to H_3([G : P]; \mathbb{Z})$ independent of choices.

Let $\{(z_0^i, z_1^i, z_2^i, z_3^i)^r, i = 1, \ldots, n\}$ be the ideal tetrahedra of the ideal triangulation of $\hat{M}$. The image of the fundamental class $[\hat{M}]$ under $(\hat{\psi}_P)_*$ is given by

$$H_3(\hat{M}) \xrightarrow{(\hat{\psi}_P)_*} H_3([G : P]; \mathbb{Z})$$

$$[\hat{M}] = \sum_{i=1}^n (z_0^i, z_1^i, z_2^i, z_3^i)^r \mapsto \beta_P(M) = \sum_{i=1}^n (\Phi(z_0^i), \Phi(z_1^i), \Phi(z_2^i), \Phi(z_3^i))_G.$$
obtaining an explicit formula for the invariant \( \beta_p(M) \). Furthermore, applying the homomorphism \( \Psi \) we get an explicit formula for the \( PSL \)-fundamental class of \( M \)

\[
[M]_{PSL} = \tilde{\sigma}(\beta_p(M))
\]

\[
= \sum_{i=1}^{n} \left( \tilde{w}_0 \left( \Phi(z_{i0}), \Phi(z_{i1}), \Phi(z_{i2}), \Phi(z_{i3}) \right), \tilde{w}_1 \left( \Phi(z_{i0}), \Phi(z_{i1}), \Phi(z_{i2}), \Phi(z_{i3}) \right), \tilde{w}_2 \left( \Phi(z_{i0}), \Phi(z_{i1}), \Phi(z_{i2}), \Phi(z_{i3}) \right) \right).
\]

### 9. The invariant \( \beta_p(M) \) is Zickert’s fundamental class

In \[22\] Zickert defines a complex \( \tilde{C}_*(G, \tilde{P}) \) of truncated simplices and proves that the complex \( \tilde{B}_*(G, \tilde{P}) = \tilde{C}_*(G, \tilde{P}) \otimes_{Z[G]} Z \) computes the Takasu relative homology groups \( H_*(\tilde{G}, \tilde{P}; Z) \) \[22\] Corollary 3.8. This complex is used to define a homomorphism \( \Psi : H_3(\tilde{G}, \tilde{P}; Z) \to \tilde{B}(\mathbb{C}) \) \[22\] Theorem 3.17. Given an ideal triangulation of an hyperbolic 3–manifold and using a developing map of the geometric representation to give to each ideal simplex a decoration by horospheres, it is defined a fundamental class in the group \( H_3(\tilde{G}, \tilde{P}; Z) \) \[22\] Corollary 5.6. \textit{A priori}, this fundamental class depends on the choice of decoration, but it is proved that its image under the homomorphism \( \Psi \) is independent of the choice of decoration \[22\] Theorem 6.10. In fact, in \[22\] it is considered the more general situation of a tame 3–manifold with a boundary parabolic \( PSL_2(\mathbb{C}) \)-representation, this will be considered in the following section.

In this section we compare the results in \[22\] with our construction of the invariant \( \beta_p(M) \). We give an explicit isomorphism between the complexes \( \tilde{C}_*(G, \tilde{P}) \) and \( C_*(\tilde{G}, \tilde{P}); \) this gives another proof of Corollary 4.16. Under this isomorphism, the homomorphisms \( \Psi \) and \( \tilde{\sigma} \) in \(7.11\) coincide. Moreover, we prove that Zickert’s fundamental class in \( H_3(\tilde{G}, \tilde{P}; Z) \) coincides with the invariant \( \beta_p(M) \) in \( H_3([\tilde{G} : \tilde{P}]; Z) \) under the aforementioned isomorphism. This proves that Zickert’s fundamental class is independent of the choice of decoration. Choosing two different decorations, Zickert’s construction indeed gives two different cycles in \( H_3(\tilde{G}, \tilde{P}) \) but they are homologous. From this, it is obvious that the image of such cycles under \( \Psi \) coincide.

**Remark 9.1.** Notice the difference of notation, in \[22\] \( G = PSL_2(\mathbb{C}) \) and \( P \) corresponds to the subgroup of \( PSL(\mathbb{C}) \) given by the image of the group of upper triangular matrices with 1 in the diagonal, that is, in our notation to the subgroup \( \tilde{U} = \tilde{P} \).

### 9.1. The complex of truncated simplices

Let \( \triangle \) be an \( n \)-simplex with a vertex ordering given by associating an integer \( i \in \{0, \ldots, n\} \) to each vertex. Let \( \overline{\Delta} \) denote the corresponding truncated simplex obtained by chopping off disjoint regular neighborhoods of the vertices. Each vertex of \( \overline{\Delta} \) is naturally associated with an ordered pair \( ij \) of distinct integers. Namely, the \( ij \)th vertex of \( \overline{\Delta} \) is the vertex near the \( i \)th vertex of \( \triangle \) and on the edge going to the \( j \)th vertex of \( \triangle \).

Let \( \overline{\Delta} \) be a truncated \( n \)-simplex. A \( G \)-vertex labeling \( \{\overline{g}^{ij}\} \) of \( \overline{\Delta} \) assigns to the vertex \( ij \) of \( \overline{\Delta} \) an element \( \overline{g}^{ij} \in G \) satisfying the following properties:

(i) For fixed \( i \), the labels \( \overline{g}^{ij} \) are distinct elements in \( G \) mapping to the same left \( \tilde{P} \)-coset.

(ii) The elements \( \overline{g}_{ij} = (\overline{g}^{ij})^{-1} \overline{g}^{ii} \) are counterdiagonal, that is, of the form \( \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix} \).
Let $\bar{C}_n(\bar{G}, \bar{P})$, $n \geq 1$, be the free abelian group generated by $\bar{G}$–vertex labelings of truncated $n$–simplices.

**Remark 9.2.** Since $X(\bar{P})$ is $\bar{G}$–isomorphic to the set of left $\bar{P}$–cosets, using the homomorphism $h_i^P$ given in (4.5) (see Remark 4.12) property (i) means that for fixed $i$ we have

$$h_i^P(\tilde{g}^{ij}) = [a_i, c_i]$$

for some fixed element $[a_i, c_i] \in X(\bar{P})$ and for all $j \neq i$. By the definition of $h_i^P$ we have that for fixed $i$ all the $\tilde{g}^{ij}$ have the form

$$\tilde{g}^{ij} = \left( \begin{array}{cc} a_i & b_{ij} \\ c_i & d_{ij} \end{array} \right), \text{ with } j \neq i \text{ and } b_{ij} \neq b_{ik} \text{ and } d_{ij} \neq d_{ik} \text{ for } j \neq k.$$

Left multiplication endows $\bar{C}_n(\bar{G}, \bar{P})$ with a free $\bar{G}$–module structure and the usual boundary map on untruncated simplices induces a boundary map on $\bar{C}_n(\bar{G}, \bar{P})$, making it into a chain complex. Define

$$\bar{B}_*(\bar{G}, \bar{P}) = \bar{C}_*(\bar{G}, \bar{P}) \otimes_{\mathbb{Z}[\bar{G}]} \mathbb{Z}.$$ 

Let $\{\tilde{g}^{ij}\}$ be a $\bar{G}$–vertex labeling of a truncated $n$–simplex $\bar{\Delta}$. We define a $\bar{G}$–edge labeling of $\bar{\Delta}$ assigning to the oriented edge going from vertex $ij$ to vertex $kl$ the labeling $(\tilde{g}^{ij})^{-1} \tilde{g}^{kl}$. It is easy to see that for any $\tilde{g} \in \bar{G}$, the $\bar{G}$–vertex labelings of $\bar{\Delta}$ given by $\{\tilde{g}^{ij}\}$ and $\{\tilde{g}^{ij}\}$ have the same $\bar{G}$–edge labelings. Hence, a $\bar{G}$–edge labeling represents a generator of $\bar{B}_*(\bar{G}, \bar{P})$. The labeling of an edge going from vertex $i$ to vertex $j$ in the untruncated simplex is denoted by $\tilde{g}_{ij}$, and the labeling of the edges near the $i$th vertex are denoted by $\tilde{g}_{i}$, $\tilde{g}_{j}$, and $\tilde{g}_{ij}$. These edges are called the long edges and the short edges respectively. By properties (i) and (ii) in the definition of $\bar{G}$–vertex labelings of a truncated simplex, the $\tilde{g}_{ij}$'s are nontrivial elements in $\bar{P}$ and the $\tilde{g}_{ij}$'s are counterdiagonal. Moreover, from the definition of $\bar{G}$–edge labelings we have that the product of edge labeling along any two-face (including the triangles) is $I$.

In [22 Corollary 3.8] it is proved that the complex $\bar{B}_*(\bar{G}, \bar{P})$ computes the groups $\bar{H}_*(\bar{G}, \bar{P})$. For this result, it is not necessary to have property (ii), nor to ask to have distinct elements in property (i) in the definition of $\bar{G}$–vertex labelings [22 Remark 3.2]. The reason for asking this extra properties on the $\bar{G}$–vertex labelings is to be able to assign to each generator a flattening of an ideal simplex. We shall see that this corresponds to our use in Section 7 of the complex $C_*^{\bar{H}^P}(X(\bar{P}))$ rather than the complex $C_*^\bar{P}(X(\bar{P}))$ to assign to each generator a flattening of an ideal simplex.

In what follows we need a more explicit version of [22 Lemma 3.5].

**Lemma 9.3 (22 Lemma 3.5).** Let $\tilde{g}_i\bar{P} = \frac{a_i}{c_i} \frac{b_i}{d_i} \bar{P}$ and $\tilde{g}_j\bar{P} = \frac{a_j}{c_j} \frac{b_j}{d_j} \bar{P}$ be $\bar{P}$–cosets satisfying the condition $\tilde{g}_i\bar{B} \neq \tilde{g}_j\bar{B}$. There exists unique coset representatives $\bar{g}_i\bar{x}_{ij}$ and $\bar{g}_j\bar{x}_{ji}$ satisfying the condition that $(\tilde{g}_i\bar{x}_{ij})^{-1} \tilde{g}_j\bar{x}_{ji}$ be counterdiagonal given by

\[
\bar{g}_i\bar{x}_{ij} = \left( \begin{array}{cc} a_i & a_j \\ c_i & c_j \end{array} \right), \quad \bar{g}_j\bar{x}_{ji} = \left( \begin{array}{cc} a_i & a_j \\ c_i & c_j \end{array} \right).
\]
Proof. We start by reproducing the proof of [22, Lemma 3.5] since it saves computations. Let \( \overline{g}_i^{-1} \overline{g}_j = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \), and let \( \overline{x}_{ij} = \left( \begin{array}{c} p_{ij} \\ 1 \end{array} \right) \) and \( \overline{x}_{ji} = \left( \begin{array}{c} p_{ji} \\ 1 \end{array} \right) \). We have
\[
\overline{x}_{ij}^{-1} \overline{g}_i^{-1} \overline{g}_j \overline{x}_{ji} = \left( \begin{array}{c} a - cp_{ij} \\ c \\ ap_{ij} + b - p_{ij}(cp_{ij} + d) \\ cp_{ij} + d \end{array} \right).
\]
Since \( \overline{g}_i \overline{B} \neq \overline{g}_j \overline{B} \), it follows that \( c \) is nonzero. This implies that there exists unique complex numbers \( p_{ij} \) and \( p_{ji} \) such that the above matrix is conterdiagonal. They are given by
\[
(9.2) \quad p_{ij} = \frac{a}{c}, \quad p_{ji} = -\frac{d}{c}.
\]
Now, using the explicit expressions for \( \overline{g}_i \) and \( \overline{g}_j \) we have
\[
(9.3) \quad \overline{g}_i^{-1} \overline{g}_j = \left( \begin{array}{cc} a_{i}d_{i} - b_{i}c_{i} \\ a_{i}c_{j} - a_{j}c_{i} \\ d_{i}b_{j} - b_{i}c_{j} \\ a_{i}d_{j} - b_{i}c_{j} \\ a_{i}c_{j} - a_{j}c_{i} \end{array} \right) = \left( \begin{array}{cc} a \\ b \\ c \\ d \end{array} \right).
\]
Substituting in (9.2) we get
\[
p_{ij} = \frac{a_{i}d_{i} - b_{i}c_{j}}{a_{i}c_{j} - a_{j}c_{i}}, \quad p_{ji} = \frac{a_{i}d_{j} - b_{i}c_{j}}{a_{i}c_{j} - a_{j}c_{i}}.
\]
Hence we have
\[
\begin{align*}
\overline{g}_i \overline{x}_{ij} &= \left( \begin{array}{cc} a_i & b_i \\ c_i & d_i \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \left( \begin{array}{c} a_i \\ c_i \end{array} \right), \\
\overline{g}_j \overline{x}_{ji} &= \left( \begin{array}{cc} a_j & b_j \\ c_j & d_j \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \left( \begin{array}{c} a_j \\ c_j \end{array} \right).
\end{align*}
\]
Notice that \( \overline{g}_i \overline{x}_{ij} \) is a well defined element in \( \overline{G} \), if we change the signs of \( a_i \) and \( c_i \), the whole matrix changes sign, while if we change the signs of \( a_j \) and \( c_j \) the matrix does not change. Analogously for \( \overline{g}_j \overline{x}_{ji} \).

\[ \Box \]

Remark 9.4. Notice that the expressions for \( \overline{g}_i \overline{x}_{ij} \) and \( \overline{g}_j \overline{x}_{ji} \) given in (9.1) only depend on the classes \([a_i, c_i]\) and \([a_j, c_j]\) in \( X(P) \), so indeed they only depend on the left \( \overline{P} \)-cosets \( \overline{g}_i \overline{P} \) and \( \overline{g}_j \overline{P} \). Also notice that by (9.3) the condition \( \overline{g}_i \overline{B} \neq \overline{g}_j \overline{B} \) is equivalent to \( a_i c_j - a_j c_i \neq 0 \) which is equivalent to \( h^B_P(\overline{g}_i \overline{P}) = \frac{a_i}{c_i} \neq \frac{a_j}{c_j} = h^B_P(\overline{g}_j \overline{P}) \).

Corollary 9.5. Let \( \overline{G} \) be a truncated \( n \)-simplex. A generator of \( C_n(G, P) \), ie a \( \overline{G} \)-vertex labeling \( \{\overline{g}^{ij}\} \) of \( \overline{G} \) has the form
\[
\overline{g}^{ij} = \left( \begin{array}{c} a_i \\ c_i \end{array} \right), \quad i, j \in \{1, \ldots, n\}, \quad j \neq i, \quad a_i c_j - a_j c_i \neq 0,
\]
and the class \([a_i, c_i] \in X(P)\) corresponds to the left \( \overline{P} \)-coset associated to the \( i \)-th vertex of \( \overline{G} \). Hence, a generator of \( B_n(\overline{G}, \overline{P}) \), ie a \( \overline{G} \)-edge labeling of \( \overline{G} \) has the form
\[
\begin{align*}
\overline{a}_{jk} &= \left( \begin{array}{c} \frac{a_i c_j - a_j c_k}{(a_i c_j - a_i c_k)(a_i c_k - a_j c_i)} \\ 1 \end{array} \right), \quad i, j, k \in \{1, \ldots, n\}, \quad i \neq j, k, \quad j \neq k, \\
\overline{g}_{ij} &= \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad i, j \in \{1, \ldots, n\}, \quad i \neq j.
\end{align*}
\]
Proof. Follows immediately from Remark 9.2, Lemma 9.3 and Remark 9.4. \qed

**Corollary 9.6.** There is a $\bar{G}$–isomorphism of of chain complexes

$$C^h_{n} \cong (X_{(\bar{P})}) \leftrightarrow C_{n}(\bar{G}, \bar{P})$$

$$((a_0, c_0), \ldots, (a_n, c_n)) \leftrightarrow \bar{g}^{ij} = \begin{pmatrix} a_i & a_j & a_{ij} \\ c_i & c_j & c_{ij} \end{pmatrix}.$$ 

Hence, there is an isomorphism of chain complexes $B^h_{n} \cong \bar{B}_{n}(\bar{G}, \bar{P})$ where the $\bar{G}$–orbit $([a_0, c_0], \ldots, [a_n, c_n])_G$ corresponds to the $\bar{G}$–edge labeling given by (9.4) and (9.5).

**Proof.** This is a refined version of [22, Corollary 3.6] and follows from Corollary 9.5.

By direct computation it is easy to see that the isomorphism is $\bar{G}$–equivariant. The only thing that remains to prove is that the isomorphism commutes with the boundary maps of the complexes, which is an easy exercise. \qed

As we mention above, the isomorphism given in Corollary 9.6 induces an isomorphism $H_\ast((\bar{G} : \bar{P}); \mathbb{Z}) \cong H_\ast(\bar{G}, \bar{P}; \mathbb{Z})$ obtaining another proof of Corollary 4.16.

**Remark 9.7.** From Corollary 9.6 to represent a generator of $C^h_{n} \cong (X_{(\bar{P})})$ we just need $2(n+1)$ complex numbers, while to represent a generator of $C_{n}(\bar{G}, \bar{P})$ we need $2(n+1)^2$ because there is a lot of redundant information in $\bar{g}^{ij}$, the entries $b_{ij}$ and $d_{ij}$ in $\bar{g}^{ij}$, see Remarks 9.2 and 9.4. So it is more efficient to use the complex $C^h_{n} \cong (X_{(\bar{P})})$ than the complex $C_{n}(\bar{G}, \bar{P})$ to compute $H_\ast((\bar{G} : \bar{P}); \mathbb{Z}) \cong H_\ast(\bar{G}, \bar{P}; \mathbb{Z})$.

Another advantage is that by Proposition 3.15 we can work with the action of $\bar{G}$ rather than with the action of $G$, as we did in Section 7.

**Remark 9.8.** If we denote by $v_i = [v_{i}^1, v_{i}^2]$ an element in $X_{(\bar{P})}$, as in Subsection 7.1, we have that the isomorphism of Corollary 9.6 is written as

$$(v_0, \ldots, v_n) \leftrightarrow \bar{g}^{ij} = \begin{pmatrix} v_{i}^1 & v_{j}^1 \\ v_{i}^2 & v_{j}^2 \end{pmatrix},$$

where $(v_0, \ldots, v_n)$ is an $(n+1)$–tuple of elements of $X_{(\bar{P})}$ such that $\det(v_i, v_j)^2 \neq 0$ for $i \neq j$, see Remark 9.4. We also have that in this notation the $\bar{G}$–edge labeling given by (9.4) and (9.5) is written as

$$\bar{\alpha}_{ik} = \begin{pmatrix} 1 & \det(v_k, v_i) \\ 0 & \det(v_i, v_k) \end{pmatrix}, \quad i, j, k \in \{1, \ldots, n\}, \quad i \neq j, k, \quad j \neq k,$$

and

$$\bar{g}_{ij} = \begin{pmatrix} 0 & 1 \\ \det(v_i, v_j) & 0 \end{pmatrix}, \quad i, j \in \{1, \ldots, n\}, \quad i \neq j.$$

(9.6)

Notice that although $\det(v_i, v_j)$ is only well-defined up to sign, see Remark 7.2, we get well-defined elements in $\bar{G}$. The fact that the matrices (9.4) and (9.5) of the $\bar{G}$–edge labeling are constant under the action of $\bar{G}$ is because $\det(v_i, v_j)$ is invariant (up to sign) under the action of $\bar{G}$, see Lemma 7.1.
9.2. Decorated ideal simplices and flattenings. Also in [22] it is proved that there is a one-to-one correspondence between generators of $B_3(\bar{G}, \bar{P})$ and congruence classes of decorated ideal simplices.

Remember that the subgroup $\bar{P}$ fixes $\infty \in \mathbb{H}^3$ and acts by translations on any horosphere at $\infty$. A horosphere at $\infty$ is endowed with the counterclockwise orientation as viewed from $\infty$. Since $\bar{G}$ acts transitively on horospheres, we get an orientation on all horospheres.

A horosphere together with a choice of orientation-preserving isometry to $\mathbb{C}$ is called an Euclidean horosphere [22, Definition 3.9]. Two horospheres based at the same point are considered equal if the isometries differ by a translation. Denote by $\bar{P}$ the Euclidean horosphere induced by projection. We let $\bar{G}$ act on Euclidean horospheres in the obvious way, this action is transitive and the isotropy subgroup of $H(\infty)$ is $\bar{P}$. Hence the set of Euclidean horospheres can be identified with the set $\bar{G}/\bar{P}$ of left $\bar{P}$–cosets, which is $\bar{G}$–isomorphic to $X_3(\bar{P})$, where an explicit $\bar{G}$–isomorphism is given by

$$
\{\text{Euclidean horospheres}\} \leftrightarrow X_3(\bar{P})
$$

$$
H(\infty) \leftrightarrow [1, 0],
$$

and extending equivariantly using the action of $\bar{G}$.

A choice of Euclidean horosphere at each vertex of an ideal simplex is called a decoration of the simplex. Having fixed a decoration, we say that the ideal simplex is decorated. Two decorated ideal simplices are called congruent if they differ by an element of $\bar{G}$.

Using the identification of Euclidean horospheres with left $\bar{P}$–cosets, we can see a decorated ideal simplex as an ideal simplex with a choice of a left $\bar{P}$–coset for each vertex of the ideal simplex.

**Proposition 9.9.** Generators in $C_{3}^{h_{3}}(X(\bar{P}))$ are in one-to-one correspondence with decorated simplices. Thus, generators of $B_{3}^{h_{3}}(X(\bar{P}))$ are in one-to-one correspondence with congruence classes of decorated simplices.

**Proof.** Consider the homomorphism $(h_{\bar{P}})_*: C_{3}^{h_{3}}(X(\bar{P})) \to C_{3}^{\bar{P}}(X(\bar{B}))$ and consider a generator $(v_0, \ldots, v_3)$ of $C_{3}^{h_{3}}(X(\bar{P}))$. Its image $((h_{\bar{P}})_*)((v_0), \ldots, (h_{\bar{P}})_*(v_3))$ is a 4–tuple of distinct points in $X(\bar{B})$, so it determines a unique ideal simplex in $\mathbb{H}^3$. Moreover, $v_i$ represents a left $\bar{P}$–coset which corresponds to the vertex $(h_{\bar{P}})_*(v_i)$ of such ideal simplex. Hence $(v_0, \ldots, v_3)$ represents a decorated simplex. $\square$

This together with the isomorphism given in Corollary 9.6 proves [22, Theorem 3.13] that there is a one-to-one correspondence between generators of $B_3(\bar{G}, \bar{P})$ and congruence classes of decorated ideal simplices, see [22, Remark 3.14].

For a matrix $g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$, let $c(g)$ denote the entry $c$. Let $\alpha$ be a generator of $\bar{B}(\bar{G}, \bar{P})$. By (9.6) we have that $c(\bar{g}_{ij}) = \pm \det(v_i, v_j)$, that is, it is only well-defined up to sign. But we have that

$$
c(\bar{g}_{ij})^2 = \det(v_i, v_j)^2
$$
is a well-defined non-zero complex number. Squaring formulas (7.1), (7.2) and (7.3) and using (9.8) we get
\[
\frac{c(\bar{g}_{01})^2 c(\bar{g}_{12})^2}{c(\bar{g}_{02})^2 c(\bar{g}_{13})^2} = z^2, \quad \frac{c(\bar{g}_{01})^2 c(\bar{g}_{02})^2}{c(\bar{g}_{01})^2 c(\bar{g}_{23})^2} = \left(\frac{1}{1-z}\right)^2, \quad \frac{c(\bar{g}_{01})^2 c(\bar{g}_{23})^2}{c(\bar{g}_{03})^2 c(\bar{g}_{12})^2} = \left(\frac{1-z}{z}\right)^2,
\]
which are the formulas of [22, Lemma 3.15]. Now, our choice of logarithm branch defines a square root of \(c(\bar{g}_{ij})^2\), see [22, Remark 3.4], given by
\[
\log c(\bar{g}_{ij}) = \frac{1}{2} \log c(\bar{g}_{ij})^2 = \frac{1}{2} \log \det(v_i, v_j)^2,
\]
which is the definition of \(\langle v_i, v_j \rangle\) given in (7.10).

**Proposition 9.10.** The following diagram commutes
\[
\begin{array}{ccc}
H_3([G : \overline{P}]; \mathbb{Z}) & \overset{\Psi}{\longrightarrow} & \overline{B}(\mathbb{C}) \\
\downarrow & & \downarrow \Psi \\
H_3(G, \overline{P}) & & \\
\end{array}
\]
where \(\Psi\) is the homomorphism given in [22, Theorem 3.17], \(\overline{\sigma}\) is the homomorphism given in (7.11) and the vertical arrow is given by the isomorphism of Corollary 9.6.

**Proof.** The definition of \(\Psi\) given by formula [22, (3.6)] coincides with the definition of \(\overline{\sigma}\) given by (7.5) via the isomorphism of Corollary 9.6. \(\square\)

**Remark 9.11.** In [22, Proposition 6.12] Zickert proves that \(\Psi\) defines a splitting of the sequence (7.12). This together with Proposition 9.10 proves the claim made in Remark 7.7 that \(\overline{\sigma}\) defines a splitting of the sequence (7.12).

9.3. The fundamental class. Now we compare the construction of Zicker’s fundamental class in \(H_3(G, \overline{P}; \mathbb{Z})\) with our computation of the invariant \(\beta_P(M)\) given in Subsection 8.2.

As in Section 8 consider an hyperbolic 3–manifold of finite volume \(M\) and let \(\bar{\rho}: \pi_1(M) \to PSL_2(\mathbb{C})\) be the geometric representation. Let \(\hat{\pi}: \hat{Y} \to \hat{M}\) be the extension of the universal cover of \(M\) to its end-compactification. A developing map of \(\bar{\rho}\) is a \(\bar{\rho}\)–equivariant map
\[
D: \hat{Y} \to \mathbb{H}^3
\]
sending the points in \(\mathcal{C}\) to \(\mathcal{C} \mathbb{H}^3\). Let \(\hat{c} \in \hat{C}\) and for each lift \(c \in \mathcal{C}\) of \(\hat{c}\), let \(H(D(c))\) be an Euclidean horosphere based at \(D(c)\). The collection \(\{H(D(c))\}_{c \in \hat{\pi}^{-1}(\hat{c})}\) of Euclidean horospheres is called a decoration of \(\hat{c}\) if the following equivariance condition is satisfied:
\[
H(D(\gamma \cdot c)) = \bar{\rho}(\gamma)H(D(c)), \quad \text{for } \gamma \in \pi_1(M), \; c \in \hat{\pi}^{-1}(\hat{c}).
\]
A developing map of \(\bar{\rho}\) together with a choice of decoration of each \(\hat{c} \in \hat{C}\) is called a decoration of \(\bar{\rho}\).

By [22, Corollary 5.16] a decoration of \(\bar{\rho}\) defines a fundamental class \(F(M)\) in \(H_3(G, \overline{P}; \mathbb{Z})\). This can be seen as follows. The decoration of \(\bar{\rho}\) endows each 3–simplex of \(M\) with the shape of a decorated simplex. By [22, Theorem 3.13] each congruence class of these decorated simplices corresponds to a generator of \(B_3(G, \overline{P})\) which is a truncated simplex with a \(\overline{G}\)–edge labeling. The decoration and
the \(G\)-edge labelings respect the face pairings so this gives a well-defined cycle \(\alpha\) in \(B_3(\tilde{G}, \tilde{P})\), see \[22\] p. 518 for details.

**Theorem 9.12.** The fundamental class \(F(M)\) corresponds to the invariant \(\beta_P(M)\) under the isomorphisms given in Corollary 9.6.

**Proof.** In Subsection 8.2 the inclusion \(\tilde{Y} \to \tilde{H}^3\) is a developing map of the geometric representation \(\tilde{\rho}: \pi_1(M) \to PSL_2(\mathbb{C})\). Using the bijection between the set of horospheres and \(X_{(P)}\) given in 9.7 the map \(\Phi\) given in \[8.3\] corresponds to a decoration of \(\tilde{\rho}\). So \(\Phi\) not only endows each 3–simplex of \(\tilde{M}\) with the shape of a decorated simplex, but also gives the \(\Gamma\)-equivariant map \(\psi_P\) in \[8.2\]. This induces the map \(\tilde{\psi}_P:\tilde{M} \to B_{\tilde{g}^1(P)}(G)\). Composing the homomorphism on complexes induced by \(\tilde{\psi}_P\) with the isomorphism of complexes of Corollary 9.6 we get

\[
C_3(\tilde{M}) \xrightarrow{(\tilde{\psi}_P)_*} B_3(\tilde{M}) \xrightarrow{\beta_P} (X_{(P)}) \equiv B_3(\tilde{G}, \tilde{P}).
\]

The image of the fundamental cycle \([\tilde{M}]\) of \(\tilde{M}\) gives the cycle \[8.4\] in \(B^{h_{\tilde{g}^1(P)}}(X_{(P)})\) which represents \(\beta_P(M)\) and this corresponds to a well-defined cycle \(\alpha\) in \(B_3(\tilde{G}, \tilde{P})\) which represents the fundamental class \(F(M)\) in \(H_3(\tilde{G}, \tilde{P}; \mathbb{Z})\).

**Corollary 9.13.** The fundamental class \(F(M) \in H_3(\tilde{G}, \tilde{P}; \mathbb{Z})\) does not depend on the choice of decoration of \(\tilde{\rho}\).

**Remark 9.14.** Choosing a different decoration of \(\tilde{\rho}\), that is, a different \(\Gamma\)-equivariant map \(\Phi'\) in \[8.3\], we get a different \(\Gamma\)-equivariant map \(\psi'_P\), a different homomorphism of complexes \(\tilde{\psi}'_P\), and a different cycle \(\alpha'\) in \(B_3(\tilde{G}, \tilde{P})\); but by the universal property of \(E^{3^1}_{(P)}(G)\) we have that \(\tilde{\psi}_P\) and \(\psi'_P\) are \(\Gamma\)-homotopic and therefore the cycles \(\alpha\) and \(\alpha'\) in \(B_3(\tilde{G}, \tilde{P})\) are homologous. Consider the natural homomorphism \(\partial: H_3(\tilde{G}, \tilde{P}; \mathbb{Z}) \to H_2(\tilde{P}; \mathbb{Z})\) of sequence \[7.12\], the images of \(\alpha\) and \(\alpha'\) under this homomorphism give different but homologous cycles in \(H_2(\tilde{P}; \mathbb{Z})\), compare with \[22\] Remark 5.19. Thus we have that the image \(\partial(F(M)) \in H_2(\tilde{P}; \mathbb{Z})\) is also an invariant of \(M\).

From Corollary 9.13 now it is obvious that the image of the fundamental class \(F(M) \in H_3(\tilde{G}, \tilde{P}; \mathbb{Z})\) under the homomorphism \(\Psi: H_3(\tilde{G}, \tilde{P}; \mathbb{Z}) \to \mathcal{B}(\mathbb{C})\) is independent of the choice of decoration \[22\] Theorem 6.10 and by Theorem 9.12 and Proposition 9.10 it coincides with the \(PSL\)-fundamental class defined in Subsection 7.3

**Remark 9.15.** Notice that the image of the \(PSL\)-fundamental class \([M]_{PSL}\) under the homomorphism \(h_{\tilde{T}}: H_3(PSL_2(\mathbb{C}); \mathbb{Z}) \to H_3([PSL_2(\mathbb{C}) : \tilde{T}]; \mathbb{Z})\) in diagram \[4.12\] is also an invariant of \(M\) which we can denote by \(\beta_T(M)\). This invariant is sent to the classical Bloch invariant \(\beta_B(M)\) by the homomorphism \(h_{\tilde{B}}: H_3([PSL_2(\mathbb{C}) : \tilde{B}]; \mathbb{Z}) \to H_3([PSL_2(\mathbb{C}) : \tilde{B}]; \mathbb{Z})\). It would be interesting to see which information carries this invariant and if it is possible to obtain it directly from the geometric representation of \(M\).

10. \((G, H)\)-representations

Our construction also works in the more general context of \((G, H)\)-representations of tame manifolds considered in \[22\]. Here we give the basic definitions and facts, for more detail see \[22\] §4.
A tame manifold is a manifold $M$ diffeomorphic to the interior of a compact manifold $\overline{M}$. The boundary components $E_i$ of $\overline{M}$ are called the ends of $M$. The number of ends can be zero to include closed manifolds as tame manifolds with no ends.

Let $M$ be a tame manifold. We have that $\pi_1(M) \cong \pi_1(\overline{M})$ and each end $E_i$ of $M$ defines a subgroup $\pi_1(E_i)$ of $\pi_1(M)$ which is well defined up to conjugation. These subgroups are called peripheral subgroups of $M$.

Let $\overline{M}$ be the compactification of $M$ obtained by identifying each end of $M$ to a point. We call the points in $\hat{\overline{M}}$ corresponding to the ends as ideal points of $M$. Let $\hat{\overline{M}}$ be the compactification of the universal cover $\hat{M}$ of $M$ obtained by adding ideal points corresponding to the lifts of the ideal points of $\overline{M}$. The covering map extends to a map from $\hat{\overline{M}}$ to $\hat{M}$. We choose a point in $M$ as a base point of $\hat{\overline{M}}$ and one of its lifts as base point of $\hat{M}$. With the base points fixed, the action of $\pi_1(M)$ on $\hat{\overline{M}}$ by covering transformations extends to an action on $\hat{M}$ which is not longer free. The stabilizer of a lift $\hat{e}$ of an ideal point $e$ corresponding to an end $E_i$ is isomorphic to a peripheral subgroup $\pi_1(E_i)$. Changing the lift $\hat{e}$ corresponds to changing the peripheral subgroup by conjugation.

Let $G$ be a discrete group, let $H$ be any subgroup and consider the family of subgroups $\mathfrak{H}(H)$ generated by $H$. Let $M$ be a tame manifold, a representation $\rho: \pi_1(M) \to G$ is called a $(G,H)$–representation if the images of the peripheral subgroups under $\rho$ are in $\mathfrak{H}(H)$.

In the particular case when $G = PSL_2(\mathbb{C})$ and $H = \hat{P}$ a $(G,H)$–representation $\rho: \pi_1(M) \to PSL_2(\mathbb{C})$ is called boundary-parabolic.

The geometric representation of a hyperbolic 3–manifold is boundary parabolic. For further examples see Zickert [22, §4].

Let $M$ be a tame $n$–manifold with $d$ ends and let $\rho: \pi_1(M) \to G$ be a $(G,H)$–representation. Let $\Gamma$ be the image of $\pi_1(M)$ in $G$ under $\rho$, also denote by $\Gamma_i$ the image of the peripheral subgroup $\pi_1(E_i)$ under $\rho$ and consider the family $\mathfrak{F} = \mathfrak{F}(\Gamma_1, \ldots, \Gamma_d)$ of subgroups of $G$. On the other hand, define $\Gamma'_i = \rho^{-1}(\Gamma_i)$ and consider the family $\mathfrak{F}' = \mathfrak{F}'(\Gamma'_1, \ldots, \Gamma'_d)$ of subgroups of $\pi_1(M)$.

**Proposition 10.1.** Consider the classifying space $E_{\mathfrak{F}}(\Gamma)$ as a $\pi_1(M)$–space defining the action by

$$\gamma \cdot x = \rho(\gamma) \cdot x, \quad \gamma \in \pi_1(M), \quad x \in E_{\mathfrak{F}}(\Gamma).$$

Then, with this action $E_{\mathfrak{F}}(\Gamma)$ is a model for the classifying space $E_{\mathfrak{F}'}(\pi_1(M))$.

**Proof.** Consider the $\Gamma$–set $\Delta_{\mathfrak{F}}$ defined in Subsection 2.1. It is enough to see that $\Delta_{\mathfrak{F}}$ seen as a $\pi_1(M)$–set using $\rho$ is $\pi_1(M)$–isomorphic to $\Delta_{\mathfrak{F}'}$. By the definition of $\Gamma'$ and $\Gamma'_i$ we have that $\Gamma'/\ker \rho \cong \Gamma$ and $\Gamma'_i/\ker \rho \cong \Gamma_i$. Then

$$\Gamma'/\Gamma'_i \cong (\Gamma'/\ker \rho) / (\Gamma'_i/\ker \rho) \cong \Gamma/\Gamma_i.$$

Therefore

$$\Delta_{\mathfrak{F}'} = \prod_{i=1}^{d} \Gamma'/\Gamma'_i \cong \prod_{i=1}^{d} \Gamma/\Gamma_i = \Delta_{\mathfrak{F}}.$$

So now we can use $\Delta_{\mathfrak{F}'}$ in the simplicial construction of $E_{\mathfrak{F}'}(\pi_1(M))$ and we obtain precisely $E_{\mathfrak{F}}(\Gamma)$. \qed
Since the action of $\pi_1(M)$ on $\widehat{M}$ has as isotropy subgroups the peripheral subgroups $\pi_1(E_i)$ and $\pi_1(E_i) \in \mathcal{F}'$ there is a $\pi_1(M)$–map unique up to $\pi_1(M)$–homotopy

\begin{equation}
\psi_\mathcal{F} : \widehat{M} \to \mathcal{F}(\pi_1(M)) \cong \mathcal{F}(\Gamma).
\end{equation}

Now consider the classifying space $E_{\mathcal{F}(H)}(G)$, by Proposition 2.7 restricting the action of $G$ to $\Gamma$ we have that $\text{res}_\Gamma^{E_{\mathcal{F}(H)}(G)} \cong \mathcal{F}(H)/\Gamma \cong \mathcal{F}(\Gamma)$. On the other hand, we have that $\mathcal{F} \subset \mathcal{F}(H)/\Gamma$, so we have a $\Gamma$–map unique up to $\Gamma$–homotopy

\begin{equation}
\psi_\mathcal{F} : E_{\mathcal{F}}(\Gamma) \to E_{\mathcal{F}(H)}(G).
\end{equation}

Composing (10.1) with (10.2) we obtain a $\rho$–equivariant map unique up to $\rho$–homotopy

\begin{equation}
\psi_\rho : \widehat{M} \to E_{\mathcal{F}(H)}(G).
\end{equation}

Taking the quotients by the actions of $\pi_1(M)$ and $G$ we get a map unique up to homotopy given by the composition

$$\hat{\psi}_\rho : \widehat{M} \to E_{\mathcal{F}(H)}(G)/\Gamma \to B_{\mathcal{F}(H)}(G).$$

Denote by $\beta_H(\rho)$ the image of the fundamental class $[\widehat{M}]$ of $\widehat{M}$ under the map induced in homology by $\hat{\psi}_\rho$

\begin{equation}
\begin{aligned}
H_n(\widehat{M}; \mathbb{Z}) & \xrightarrow{(\hat{\psi}_\rho)_*} H_n(B_{\mathcal{F}(H)}(G); \mathbb{Z}) \\
[\widehat{M}] & \mapsto \beta_H(\rho).
\end{aligned}
\end{equation}

Thus, by Proposition 3.13 we have:

**Theorem 10.2.** Given an oriented tame $n$–manifold with a $(G, H)$–representation $\rho : \pi_1(M) \to G$ we have a well-defined invariant

$$\beta_H(\rho) \in H_n([G : H]; \mathbb{Z}).$$

As before, one can compute the class $\beta_H(\rho)$ using a triangulation of $M$. A triangulation of a tame manifold $M$ is an identification of $\widehat{M}$ with a complex obtained by gluing together simplices with simplicial attaching maps. A triangulation of $M$ always exists and it lifts uniquely to a triangulation of $\widehat{M}$.

Let $M$ be a tame $n$–manifold with $d$ ends and let $\rho : \pi_1(M) \to G$ be a $(G, H)$–representation. In [22] §5.2, given a triangulation of $M$ it is constructed a $(G, H)$–cocycle, see [22] §5.2 for the definition, which defines a fundamental class $F(\rho)$ in $H_n(G, H; \mathbb{Z})$. The construction of the $(G, H)$–cocycle depends on a decoration of $\rho$ by conjugation elements. Such decorations are given as follows: for each ideal point $e_i \in \widehat{M}$ choose a lifting $\tilde{e}_i \in \widehat{M}$ and assign to this lifting an element $g_i(\tilde{e}_i) \in G$, or rather an $H$–coset $g_i(\tilde{e}_i)H$, then extend $\rho$–equivariantly by

$$g_i(\gamma \cdot \tilde{e}_i) = \rho(\gamma)g_i(\tilde{e}_i), \quad \gamma \in \pi_1(M).$$

Let $I$ denote the set of ideal point in $\widehat{M}$. Notice that a decoration by conjugation elements is equivalent to give a $\rho$–equivariant map

$$\Phi_\rho : I \to G/H.$$

The map $\Phi_\rho$ defines explicitly the $\rho$–map (10.3) and using the triangulation of $M$ gives also explicitly the homomorphism (10.4) as in Subsection 8.2.
Remark 10.3. For general $G$ and $H$ we do not necessarily have that $H_n(G, H; \mathbb{Z})$ coincides with $H_n([G : H]; \mathbb{Z})$, see Subsection 3.4. The construction of the $(G, H)$–cocycle a priori depends on the choice of decoration of $\rho$ by conjugation elements, so in principle, choosing different decorations one can obtain different fundamental classes in $H_n(G, H; \mathbb{Z})$, in that case, all this fundamental classes are mapped to $\beta_H(\rho) \in H_n([G : H]; \mathbb{Z})$ under the canonical homomorphism \([5.12]\) since $\beta_H(\rho)$ does not depend on the choice of decoration because the $\rho$–map \([10.3]\) given by the decoration is unique up to $\rho$–homotopy. So in this general context it is more appropriate to use Hochschild relative group homology than Takasu relative group homology because we obtain invariants independent of choice.

10.1. Boundary-parabolic representations. In the case of boundary-parabolic representations of tame 3–manifolds we can use a developing map with a decoration to compute $\beta_P(\rho)$. Let $M$ be a tame 3–manifold and let $\tilde{\rho}$ be a boundary-parabolic representation. A developing map of $\rho$ is a $\rho$–equivariant map $D_{\rho}: \widetilde{M} \to \mathbb{H}^3$ sending the ideal points of $\widetilde{M}$ to $\partial \mathbb{H}^3 = \mathbb{C}$. Taking a sufficiently fine triangulation of $M$ it is always possible to construct a developing map of $\rho$ \([22]\) Theorem 4.11]. Let $C$ be the image under $D_{\rho}$ of the set of ideal points $\mathcal{I}$ of $\widetilde{M}$. A decoration of $\rho$ is a $\rho$–equivariant map $\Phi_{\rho}:C \to X(\tilde{\rho})$ which can be obtained assigning a Euclidean horosphere to each element of $C$ as in Subsection 9.3 or as in Remark 8.1. Again, the decoration defines explicitly the $\rho$–map \([10.4]\) which gives explicitly the homomorphism \([10.3]\) with $G = \text{PSL}_2(\mathbb{C})$ and $H = P$. Since by Corollary \([4.16]\) $H_3(\text{PSL}_2(\mathbb{C}), \tilde{P}; \mathbb{Z}) \cong H_3([\text{PSL}_2(\mathbb{C}) : \tilde{P}]; \mathbb{Z})$ in this case $F(\rho) = \beta_P(\rho)$ is independent of the choice of decoration.

As in the case of the geometric representation we can define the $\text{PSL}$–fundamental class of $\rho$ by $[\rho]_{\text{PSL}} = \tilde{\sigma}(\beta_P(\rho)) \in \hat{\mathcal{P}}(\mathbb{C}) \cong H_3(\text{PSL}_2(\mathbb{C}); \mathbb{Z})$.

Remark 10.4. As in Remarks \([9.14]\) the image $\partial(F(\rho)) \in H_2(\tilde{P}; \mathbb{Z})$ under the homomorphism $\partial$ of sequence \([7.12]\) is also an invariant of $\rho$.

The image of $\beta_P(\rho)$ under $\left(h^P\right)_*: H_3([\text{PSL}_2(\mathbb{C}) : \tilde{P}]; \mathbb{Z}) \to H_3([\text{PSL}_2(\mathbb{C}) : \tilde{B}]; \mathbb{Z}) \cong \mathcal{P}(\mathbb{C})$ gives an invariant $\beta_B(\rho)$ in the Bloch group $\mathcal{B}(\mathbb{C})$. The invariant $\beta_B(\rho)$ can be computed using a developing map as in Subsection 8.1.

Also as in Remarks \([9.15]\) the image of the $\text{PSL}$–fundamental class $[\rho]_{\text{PSL}}$ under the homomorphism $\left(h^T\right)_*: H_3(\text{PSL}_2(\mathbb{C}); \mathbb{Z}) \to H_3([\text{PSL}_2(\mathbb{C}) : \tilde{T}]; \mathbb{Z})$ in diagram \([4.12]\) is also an invariant $\beta_T(\rho)$ of $\rho$.

11. Complex volume

Recall that Rogers dilogarithm is given by $L(z) = -\int_0^z \frac{\log(1-t)}{t} dt + \frac{1}{2} \log(z) \log(1-z)$.

In \([13]\) Proposition 2.5] Neumann defines the homomorphism $\hat{L}: \hat{\mathcal{P}}(\mathbb{C}) \to \mathbb{C}/\pi^2 \mathbb{Z}$,

\[
[\rho;p,q] \mapsto L(z) + \frac{\pi i}{2} (q \log z + p \log(1-z)) - \frac{\pi^2}{6}
\]
where \([z; p, q]\) denotes elements in the extended pre-Bloch group using our choice of logarithm branch, see Remark 6.5 but \((11.1)\) is actually independent of this choice, see \cite[Remark 1.9]{neumann}. In \cite[Theorem 2.6 or Theorem 12.1]{neumann} Neumann proves that under the isomorphism \(H_3(PSL_2(\mathbb{C}); \mathbb{Z}) \cong \widehat{B}(\mathbb{C})\) the homomorphism \(\hat{L}\) corresponds to the Cheeger–Chern–Simons class.

Since by Proposition 9.10 the homomorphism \(\tilde{\sigma}: H_3([PSL_2(\mathbb{C}): \hat{P}]; \mathbb{Z}) \to \widehat{B}(\mathbb{C})\) corresponds to the map \(\Psi: H_3(PSL_2(\mathbb{C}), \hat{P}; \mathbb{Z}) \to \widehat{B}(\mathbb{C})\) we have that for the geometric representation of a complete oriented hyperbolic 3–manifold of finite volume \(M\) we have

\[
\hat{L} \circ \tilde{\sigma}(\beta_\rho(M)) = i(\text{Vol}(M) + i \text{CS}(M)) \in \mathbb{C}/i\pi^2\mathbb{Z},
\]

where \(\text{Vol}(M)\) is the volume of \(M\) and \(\text{CS}(M) = 2\pi^2 cs(M) \in \mathbb{R}/\pi^2\mathbb{Z}\) with \(cs(M)\) the Chern–Simons invariant of \(M\). Usually \(\text{Vol}(M) + i \text{CS}(M)\) is called the complex volume of \(M\), see Neumann \cite{neumann} and Zickert \cite{zickert} for details.

For the case of a boundary-parabolic representation \(\rho: \pi_1(M) \to PSL_2(\mathbb{C})\) of a tame 3–manifold \(M\), following Zickert \cite[§6]{zickert} we define the complex volume of the representation \(\rho\) by

\[
i(\text{Vol}(\rho) + i \text{CS}(\rho)) = \hat{L} \circ \tilde{\sigma}(\beta_\rho(M)).
\]

Applying \(\hat{L}\) to a formula for \([\rho]_{\text{PSL}}\) analogous to \((8.5)\) in Subsection 8.2 but using the decoration \(\Phi_\rho: \mathcal{C} \to X_\rho\) of \(\rho\) of Subsection 10.1 we obtain an explicit formula for the complex volume of \(\rho\).

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