Site–bond representation and self-duality for totalistic probabilistic cellular automata in one dimension

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Abstract. We study the one-dimensional two-state totalistic probabilistic cellular automata (TPCA) having an absorbing state with long range interactions, which can be considered as a natural extension of the Domany–Kinzel model. We establish the conditions for existence of a site–bond representation and self-dual property. Moreover we present an expression for a set-to-set connectedness between two sets, a matrix expression for a condition of the self-duality, and a convergence theorem for the TPCA.

Keywords: percolation problems (theory), phase transitions into absorbing states (theory)

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1. Introduction

Probabilistic versions of cellular automata can be considered as discrete-time Markov processes with parallel updating, which are useful in a large number of scientific areas [1]. As a special case, the Domany–Kinzel (DK) model introduced and studied by [2] is one of the simplest examples of a probabilistic cellular automaton. The model is defined on a lattice, and the two different states for a site can be said to be empty or occupied. The state of a given site depends on the number of the occupied states in its two nearest neighbours at the previous time step, where this property can be expressed with the totalistic rule [1, 3, 4]. In the DK model, the probability of transition from an empty neighbourhood to an occupied state is zero, so the empty configuration is an absorbing state. The totalistic probabilistic cellular automaton (TPCA) is a natural extension of the DK model, where we extend the number of neighbours to a finite integer $N (\geq 2)$. Remark that $N = 2$ case becomes the DK model. Concerning the DK model, various results are known (see our recent papers [5]–[8] and references therein). As for the $N = 3$ case with two absorbing states, a rich phase diagram is reported, by using mean-field approximations and numerical simulations (see [9, 10], for examples). However there are few rigorous results on the TPCA for a general $N \geq 3$. In this situation, the purpose of this paper is to give some rigorous results for a two-state $N$-neighbour TPCA with an absorbing state. More precisely, we reveal a relation between the site–bond representation and the self-duality for the TPCA. Furthermore we present an expression for a set-to-set connectedness between two sets, a matrix expression for a condition of self-duality, and a convergence theorem for the TPCA. It is known that the self-duality is a very useful technique in the study of stochastic interacting models. Because problems in uncountable state space (typically configurations of $\{0, 1\}^{\mathbb{Z}}$) can be reformulated as problems in countable state space (typically finite subsets of $\mathbb{Z}$). For some applications of the self-duality for the discrete-time case (e.g., the oriented bond percolation model) and the continuous-time case (e.g., the contact process), see [11]–[14]. So our results here might be useful in obtaining rigorous results on the phase diagram and critical properties of such models.
We have organized the present paper as follows. Section 2 is devoted to the definition of our TPCA. In section 3, we explain a site–bond representation and a self-duality for the TPCA. Moreover an expression for a set-to-set connectedness for the TPCA is given. Section 4 treats a matrix expression for the self-duality of the model. Section 5 contains a convergence theorem. Finally, concluding remarks and discussions are presented in section 6.

2. Definition of the TPCA

First we give the definition of the DK model (N = 2 case). Let $\xi^A_n \subset \mathbb{Z}$ be the state of the process with parameters $(p_1, p_2) \in [0, 1]^2$ at time $n$ starting from a set $A \subset 2\mathbb{Z}$. Its evolution is described by

$$P(x \in \xi^A_{n+1} | \xi^A_n) = f(\xi^A_n(x - 1) + \xi^A_n(x + 1)),$$

and given $\xi^A_n$, the events $\{x \in \xi^A_{n+1}\}$ are independent, where $P(B|C)$ is the conditional probability that $B$ occurs given that $C$ occurs, and

$$f(0) = 0, \quad f(1) = p_1, \quad f(2) = p_2.$$  

If we write $\xi(x, n) = 1$ for $x \in \xi_n^A$ and $\xi(x, n) = 0$ otherwise, each realization of the process is identified with a configuration $\xi \in \{0, 1\}^S$ with $S = \{s = (x, n) \in \mathbb{Z} \times \mathbb{Z}_+ : x + n \text{ even}\}$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. The model with $(p_1, p_2) = (1, 0)$ becomes Wolfram’s rule 90. For more detailed information, see section 5 in [12].

From now on, we introduce a long range TPCA. In order to clarify the definition, we consider two cases of $N = \text{even}$ and $N = \text{odd}$ respectively, where $N$ is the number of neighbours.

(i) $N = 2L$ ($L = 1, 2, \ldots$) case. The space of sites is denoted by $S_0 = \{s = (x, n) \in \mathbb{Z} \times \mathbb{Z}_+ : x + n \text{ even}\}$ and the space of bonds is

$$B_0 = \{(x, n + 1), (x - 2L + 1, n), \ldots, ((x, n + 1), (x - 1, n)), ((x, n + 1), (x + 1, n)), \ldots, ((x, n + 1), (x + 2L - 1, n)) : (x, n + 1) \in S_0\}.$$  

For any initial set $A \subset 2\mathbb{Z}$, the process $\xi^A_n$ satisfies

$$P(x \in \xi^A_{n+1} | \xi^A_n) = f\left(\sum_{k=-(L-1)}^{L} \xi^A_n(x - 2k + 1)\right),$$

and given $\xi^A_n$, the events $\{x \in \xi^A_{n+1}\}$ are independent, where

$$f(0) = p_0 = 0, \quad f(1) = p_1, \ldots, f(2L) = p_{2L},$$  

with $p_1, p_2, \ldots, p_{2L} \in [0, 1]$. This process is considered on the space $S_0$. The case $L = 1$ is equivalent to the DK model.

(ii) $N = 2L + 1$ ($L = 1, 2, \ldots$) case. The space of sites is denoted by

$$S_1 = \{s = (x, n) \in \mathbb{Z} \times \mathbb{Z}_+\},$$

and the space of bonds is

$$B_1 = \{(x, n + 1), (x - L, n), \ldots, ((x, n + 1), (x - 1, n)), ((x, n + 1), (x, n)), ((x, n + 1), (x + 1, n)), \ldots, ((x, n + 1), (x + L, n)) : (x, n + 1) \in S_1\}.$$
For any initial set $A \subset \mathbf{Z}$, the process $\xi_n^A$ satisfies

$$P(x \in \xi_{n+1}^A|\xi_n^A) = f\left(\sum_{k=-L}^{L} \xi_n^A(x+k)\right),$$

and given $\xi_n^A$, the events $\{x \in \xi_{n+1}^A\}$ are independent, where

$$f(0) = p_0 = 0, \quad f(1) = p_1, \ldots, f(2L+1) = p_{2L+1},$$

with $p_1, p_2, \ldots, p_{2L+1} \in [0, 1]$. This process is considered on the space $S_1$.

If $p_0 = 0 \leq p_1 \leq p_2 \leq \cdots \leq p_{N-1} \leq p_N$, then the $N$-neighbour TPCA is said to be attractive. If not, it is said to be non-attractive. The attractiveness means that having more particles at one time implies there will be more particles at the next time(s).

In general, fewer rigorous results on the non-attractive model are known compared with the attractive model. Much more information on the results for the non-attractive $N = 2$ case is given for instance in [7]. So studying a general model including the non-attractive case, such as is the one in this paper, is very important.

### 3. Site–bond representation and self-duality

From now on we consider only the $(p_1, p_2) \in D_s$ case, where the subset of parameter space $D_s$ is defined as follows:

$$D_s = \{(p_1, p_2) : 0 < p_1 \leq 1, 0 < p_2 \leq 1 \text{ and } p_2 < 2p_1\}.$$ 

A reason for introducing this set is that for any point outside of $D_s$, it will be shown that there exists no site–bond representation for the TPCA (see proposition 1 below). Suppose that $p_1, p_2 \in D_s$ are given. We put $\alpha = p_1^2/(2p_1 - p_2), \beta = 2 - p_2/p_1$. Let $(S, B) = (S_0, B_0)$ or $(S_1, B_1)$ and let $\mathbf{R}$ denote the set of real numbers. If there exist $\alpha, \beta \in \mathbf{R}$ such that $p_n \in [0, 1]$ is given by

$$p_n = \alpha[1 - (1 - \beta)^n],$$

for $n = 3, 4, \ldots, N$, then we say that an $N$-neighbour TPCA with $\{p_n : 0 \leq n \leq N\}$ has a site–bond representation with $\alpha$ and $\beta'$. In fact, when $\alpha, \beta \in (0, 1]$, the TPCA can be considered as an oriented mixed site–bond percolation with a long range interaction in the following way. On the space $(S, B)$ we define

$$X(S) = \{0, 1\}^S, \quad X(B) = \{0, 1\}^B, \quad X = X(S) \times X(B).$$

For given $\zeta = (\zeta_1, \zeta_2) \in X$, we say that $s = (y, n+k) \in S$ can be reached from $s' = (x, n) \in S$ and write $s' \to s$, if there exists a sequence $s_0, s_1, s_2, \ldots, s_k$ of members of $S$ such that $s' = s_0, s = s_k$, and

$$\zeta_i(s_i) = 1, \quad i = 0, 1, \ldots, k; \quad \zeta_2((s_i, s_{i+1})) = 1, \quad i = 0, 1, \ldots, k - 1.$$ 

We also say that $G \subset S$ can be reached from $G' \subset S$ and write $G' \to G$, if there exist $s \in G$ and $s' \in G'$ such that $s' \to s$.

We introduce the signed measure $m$ on $X$ defined by

$$m(\Lambda) = \alpha^k(1 - \alpha)^{\lambda_1} \beta^{k_2}(1 - \beta)^{\lambda_2},$$

$$\lambda_1 + \lambda_2 = k + 1.$$
for any cylinder set
\[ \Lambda = \{(\zeta_1, \zeta_2) \in X : \zeta_1(s_i) = 1, i = 1, 2, \ldots, k_1; \zeta_1(s'_i) = 0, i = 1, 2, \ldots, j_1; \]
\[ \zeta_2(b_i) = 1, i = 1, 2, \ldots, k_2; \zeta_2(b'_i) = 0, i = 1, 2, \ldots, j_2 \}; \]
where \( s_1, s_2, \ldots, s_k, s'_1, s'_2, \ldots, s'_j \) are distinct elements of \( S \), \( b_1, b_2, \ldots, b_k, b'_1, b'_2, \ldots, b'_j \) are distinct elements of \( B \), and \( \alpha = p_1^2/(2p_1 - p_2), \beta = 2 - p_2/p_1 \).

We should note the next three facts. If \( p_2 < 2p_1 \) and \( p_2 > 2p_1 - p_2^2 \), then \( \alpha > 1 \) and \( \beta \in (0, 1) \). If \( p_2 \leq 2p_1 - p_2^2 \) and \( p_2 \geq p_1 \), then \( \alpha, \beta \in (0, 1] \). Moreover, if \( p_2 < p_1 \), then \( \alpha \in (0, 1) \) and \( \beta \in (1, 2] \). From the above observation, we see that the measure is not a probability measure in the first and third cases, since \( \alpha > 1 \) and \( 1 - \beta < 0 \) respectively. See [5, 7] for more details on the site–bond representation.

It is noted that if an \( N \)-neighbour TPCA has a site–bond representation with \( \alpha \) and \( \beta \), then \( \alpha = p_1^2/(2p_1 - p_2), \beta = 2 - p_2/p_1 \), since equation (1) gives
\[ p_1 = \alpha \beta, \quad p_2 = \alpha [1 - (1 - \beta)^2]. \]

As we will see in the proof of proposition 1, the following result can be obtained: an \( N \)-neighbour TPCA having a site–bond representation with \( \alpha \) and \( \beta \) is attractive if and only if \( p_1 \leq p_2 \). In other words, an \( N \)-neighbour TPCA having a site–bond representation with \( \alpha \) and \( \beta \) is non-attractive if and only if \( p_1 > p_2 \). Then we have

**Proposition 1** We consider \( N \)-neighbour TPCA with \( \{p_n : 0 \leq n \leq N\} \). Assume \( (p_1, p_2) \in D_1 \). Let \( \alpha = p_1^2/(2p_1 - p_2) \) and \( \beta = (2p_1 - p_2)/p_1 \). (i) \( p_1 > p_2 \) (non-attractive case) and \( p_n = \alpha [1 - (1 - \beta)^n] (0 \leq n \leq N) \); then \( p_n \in [0, 1] (0 \leq n \leq N) \). (ii) \( p_1 \leq p_2 \) (attractive case), \( p_n = \alpha [1 - (1 - \beta)^n] (0 \leq n \leq N) \), and \( p_N \leq 1 \); then \( p_n \in [0, 1] (0 \leq n \leq N) \).

**Proof.** First we consider a relation among \( p_0 = 0, p_1, \ldots, p_N \). Using \( p_n = \alpha [1 - (1 - \beta)^n] \) for \( n = 0, 1, \ldots, N \), we see that
\[ p_n - p_{n-1} = \alpha \beta (1 - \beta)^{n-1}, \]
\[ p_n - p_{n-2} = \alpha (1 - \beta)^{n-2} \{1 - (1 - \beta)^2\}. \]
Recall that
\[ 1 - \beta = (p_2 - p_1)/p_1, \quad \alpha \beta = p_1. \]
Now we consider the \( p_1 > p_2 \) (non-attractive) case. In this case, equation (4) gives
\[ -1 < 1 - \beta < 0, \quad 0 < \alpha \beta \leq 1, \quad 0 < \alpha < 1. \]
From equations (2), (3), and (5), we see that
\[ p_0 = 0 < p_1 < p_2 < p_3 < p_4 < \cdots < p_{2n} < p_{2n-1} < \cdots < p_5 < p_3 < p_1 \leq 1. \]
Thus we obtain \( p_n \in [0, 1] \) for any \( n = 0, 1, \ldots, N \). Next we consider the \( p_1 \leq p_2 \) (attractive) case. It is easily checked that \( p_n \leq p_{n+1} \) for any \( n = 0, 1, \ldots, N - 1 \) by using equation (2) and \( 0 \leq 1 - \beta < 1 \). So we have
\[ p_0 = 0 < p_1 \leq p_2 \leq p_3 \leq \cdots \leq p_{N-1} \leq p_N. \]

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Therefore a necessary and sufficient condition for the site–bond representation in this case is \( p_N = \alpha[1 - (1 - \beta)^N] \leq 1 \). The proof of proposition 1 is complete. \( \square \)

Remark that \( p_n = \alpha[1 - (1 - \beta)^n] \to \alpha \) as \( n \to \infty \), since \( |1 - \beta| < 1 \). Noting that \( \alpha \leq 1 \) if and only if \( p_2 \leq 2p_1 - p_1^2 \), we conclude that a sufficient (and not necessary) condition for a site–bond representation of an \( N \)-neighbour TPCA is \( p_2 \leq 2p_1 - p_1^2 \). Furthermore, equation (2) implies:

Corollary 2 If an \( N \)-neighbour TPCA with \( \{p_n : 0 \leq n \leq N\} \) and \( N \geq 3 \) has the site–bond representation, then \( p_n \) can be expanded with \( p_1 \) and \( p_2 \) as follows:

\[
p_n = \frac{(p_2 - p_1)^{n-2}}{p_1} + \frac{(p_2 - p_1)^{n-3}}{p_1^2} + \cdots + \frac{(p_2 - p_1)^2}{p_1} + p_2,
\]

for any \( n = 3, 4, \ldots, N \).

Next we define a set-to-set connectedness for the TPCA from a set \( A \) to a set \( B \) by

\[
\sigma(A, B) = \lim_{n \to \infty} P(\xi_n^A \cap B = \emptyset),
\]

if the right-hand side exists. As in a similar argument for the \( N = 2 \) case (the DK model) [5], we can easily extend the case to a general \( N \)-neighbour TPCA:

Proposition 3 Let \( (p_1, p_2) \in D_* \). We assume that an \( N \)-neighbour TPCA with \( \{p_n : 0 \leq n \leq N\} \) has a site–bond representation with \( \alpha \) and \( \beta \), where \( \alpha = p_1^2/(2p_1 - p_2) \) and \( \beta = (2p_1 - p_2)/p_1 \). Then for any \( A \) with \( |A| < \infty \), we have

\[
\sigma(2\mathbb{Z}, A) = \sum_{D \subseteq A, D \neq \emptyset} \alpha^{|D|}(1 - \alpha)^{|A\setminus D|}\sigma(D, 2\mathbb{Z}),
\]

when \( N = 2\mathbb{Z} \), and

\[
\sigma(\mathbb{Z}, A) = \sum_{D \subseteq A, D \neq \emptyset} \alpha^{|D|}(1 - \alpha)^{|A\setminus D|}\sigma(D, \mathbb{Z}),
\]

when \( N = 2\mathbb{Z} + 1 \).

From now on, we consider a self-duality of the TPCA. An \( N \)-neighbour TPCA \( \xi_n \) is said to be self-dual with a self-duality parameter \( x \) if

\[
E(x^{\xi_n^A \cap B}) = E(x^{\xi_n^B \cap A}) \quad (n = 0, 1, \ldots)
\]

holds for any \( A, B \subset \mathbb{Z} \) with \( |A| < \infty \) or \( |B| < \infty \). The above equation is called a self-duality equation. Then we have:

Theorem 4 Let \( (p_1, p_2) \in D_* \) with \( \alpha[1 - (1 - \beta)^N] \leq 1 \). An \( N \)-neighbour TPCA with \( \{p_n : 0 \leq n \leq N\} \) is self-dual with a self-duality parameter \( (\alpha - 1) / \alpha \) if and only if this model has a site–bond representation with \( \alpha \) and \( \beta \), where \( \alpha = p_1^2/(2p_1 - p_2) \) and \( \beta = (2p_1 - p_2)/p_1 \).

Proof. Theorem 1 in [15] implies that an \( N \)-neighbour TPCA with transition probabilities \( \{p_n : 0 \leq n \leq N\} \) that is self-dual with a self-duality parameter \( x \) is equivalent to

\[
(p_i x + q_i)^j = (p_j x + q_j)^i, \quad (6)
\]
for $0 \leq i, j \leq N$, where $q_i = 1 - p_i$. It can be confirmed that $x = (\alpha - 1)/\alpha$ with 
\[ \alpha = p_1^2/(2p_1 - p_2) \]
satisfies equation (6). Moreover we see that equation (6) if and only if
\[(p_1 x + q_1)^n = p_n x + q_n, \quad (7)\]
for $0 \leq n \leq N$. In fact if we take $i = 1$ and $j = n$, equation (6) becomes equation (7). Conversely we have
\[(p_1 x + q_1)^j = (p_1 x + q_1)^{ij} = (p_j x + q_j)^i.\]
Therefore we put $y = x - 1$. Then equation (7) becomes $p_n y + 1 = (p_1 y + 1)^n$. So we see that
\[
p_n = \sum_{r=1}^{n} \binom{n}{r} y^{r-1} p_1^r
= \sum_{r=1}^{n} \binom{n}{r} \left( \frac{p_2 - 2p_1}{p_1^2} \right)^{r-1} p_1^r
= \left( \frac{p_2 - 2p_1}{p_1^2} \right)^{-1} \sum_{r=1}^{n} \binom{n}{r} \left( \frac{p_2 - 2p_1}{p_1} \right)^{r}
= \alpha \sum_{r=0}^{n} \binom{n}{r} \left( 1 - \frac{p_2 - 2p_1}{p_1} \right)^{r}
= \alpha [1 - (1 + (p_2 - 2p_1)/p_1)^n]
= \alpha [1 - (1 - \beta)^n].\]

So the proof of theorem 4 is complete. \hfill \Box

From this theorem it can be seen that two conditions for the self-duality and for
the site–bond representation are equivalent under an assumption that $(p_1, p_2) \in D_*$ with
$p_N = \alpha [1 - (1 - \beta)^N] \leq 1$.

4. Matrix expression

In this section we consider a criterion for the self-duality based on a matrix expression. Let
\[ X(x) = \begin{bmatrix} 1 & 1 \\ 1 & x \end{bmatrix}, \]
and
\[ X_N(x) = X(x)^\otimes N,\]
where $\otimes$ indicates the tensor product. For example, in the case of $N = 2$, we have
\[ X_2(x) = X(x) \otimes X(x) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & x \\ 1 & 1 & x & x \\ 1 & x & x & x^2 \end{bmatrix}.\]
For fixed \( i, j \in 1, 2, \ldots, 2^N \), the values of \( i_1, i_2, \ldots, i_N \) and \( j_1, j_2, \ldots, j_N \in \{0, 1\} \) are defined by the binary expansion of \( i \) and \( j \) as follows:

\[
\begin{align*}
    i - 1 &= i_12^{N-1} + i_22^{N-2} + \cdots + i_{N-1}2^1 + i_N2^0, \\
    j - 1 &= j_12^{N-1} + j_22^{N-2} + \cdots + j_{N-1}2^1 + j_N2^0,
\end{align*}
\]

and \( I = i_1 + i_2 + \cdots + i_N \). Moreover we introduce a \( 2^N \times 2^N \) matrix \( P_N \) whose \((i, j)\) element is defined by

\[
p_I^{(j_1) \cdot j_2) \cdot \cdots \cdot p_I^{(j_N)}),
\]

where \( p_I^{(j_1)} = p_I, p_I^{(0)} = 1 - p_I = q_I \). For the \( N = 2 \) case, the above definition gives

\[
P_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
q_1^2 & p_1q_1 & p_1q_1 & p_1^2 \\
q_1^2 & p_1q_1 & p_1q_1 & p_1^2 \\
q_2^2 & p_2q_2 & p_2q_2 & p_2^2
\end{bmatrix},
\]

since \( p_0 = 0, q_0 = 1 \). Then we have the next result:

**Theorem 5** A necessary and sufficient condition for self-duality for an \( N \)-neighbour TPCA with \( \{p_n : 0 \leq n \leq N\} \) with a self-duality parameter \( x \) is that \( x \) satisfies

\[
P_NX_N(x) = t(P_NX_N(x)), \tag{8}
\]

where \( t \) is the transpose operator.

**Proof.** Let \( c_{ik}^{(N)}(x) \) denote the \((i, k)\) element of \( P_NX_N(x) \). First, \( i_1, i_2, \ldots, i_N \) and \( k_1, k_2, \ldots, k_N \) are defined by \( i - 1 = i_12^{N-1} + i_22^{N-2} + \cdots + i_{N-1}2^1 + i_N2^0 \) and \( k - 1 = k_12^{N-1} + k_22^{N-2} + \cdots + k_{N-1}2^1 + k_N2^0 \), respectively. We put \( I = i_1 + i_2 + \cdots + i_N \) and \( K = k_1 + k_2 + \cdots + k_N \). Moreover \( \{k_1, k_2, \ldots, k_N\} \) is divided into two sets such as \( \{k_{m_1}, k_{m_2}, \ldots, k_{m_K} : k_v = 1 \text{ for any } u \in \{m_1, m_2, \ldots, m_K\} \} \) and \( \{k_1, k_2, \ldots, k_{N-K} : k_v = 0 \text{ for any } v \in \{l_1, l_2, \ldots, l_{N-K}\} \} \). Then we see that

\[
c_{ik}^{(N)}(x) = \sum_{j_m, \ldots, j_{m_K} \in \{0, 1\}} \sum_{j_{n_J}, \ldots, j_{n_{N-J}} \in \{0, 1\}} p_I^{(j_{m_1}) \cdot j_2 \cdot \cdots \cdot p_I^{(j_{m_K})} \times p_I^{(j_{n_1}) \cdot j_2 \cdot \cdots \cdot p_I^{(j_{n_{N-J}})}}
\]

\[
\times x_{(j_{m_1}) \cdot j_2 \cdot \cdots \cdot x_{(j_{m_K})} \times x_{(j_{n_1}) \cdot j_2 \cdot \cdots \cdot x_{(j_{n_{N-J})}}} = \sum_{j_m, \ldots, j_{m_K} \in \{0, 1\}} p_I^{(j_{m_1}) \cdot j_2 \cdot \cdots \cdot p_I^{(j_{m_K})}} \times x_{(j_{m_1}) \cdot j_2 \cdot \cdots \cdot x_{(j_{m_K})} = (p_I^{(0)} x_{(0)})^K
\]

where \( x_{(1)} = x \) and \( x_{(0)} = 1 \). In a similar way, we have

\[
c_{ki}^{(N)}(x) = (p_Kx + q_K)^J.
\]

So we have

\[
P_NX_N(x) = t(P_NX_N(x)),
\]
if and only if
\[ c_{ik}^{(N)}(x) = c_{ki}^{(N)}(x), \]
for \( 1 \leq i, k \leq 2^N \). Furthermore it is shown that the last equation is equivalent to
\[ (p_1x + q_1)^K = (p_Kx + q_K)^I, \]
for \( 1 \leq I, K \leq N \). Combining the above result with theorem 1 in [15], we obtain the desired conclusion.

For the \( N = 2 \) case (the DK model), theorem 5 gives
\[ (p_1x + q_1)^2 = p_2x + q_2. \]
This result was shown in [8, 15, 16] by using different methods.

5. Convergence theorem

In order to obtain a convergence theorem for the TPCA with \( \{p_n : 0 \leq n \leq N\} \) having a site–bond representation with \( \alpha = p_1^2/(2p_1 - p_2) \) and \( \beta = (2p_1 - p_2)/p_1 \), which corresponds to the result given by [8], a new process \( \eta_n \) is introduced as follows. For simplicity, we first consider the DK model (the \( N = 2 \) case) with \( (p_1, p_2) \in D_* \). Put \( p_* = \max\{p_1, p_2\} \). A new process defined below is called \( p_*-DK-dual \). We can see the thinning relationship by coupling the DK model and \( p_*-DK-dual \). We split both the DK model and the \( p_*-DK-dual \) into two phases, and we will allow the first phase to occur at times \( n + (1/2) \) for \( n \in \mathbb{Z}_+ \).

(i) Let \( \mu \) be the distribution of the \( p_*-DK-dual \) at time 0.
(ii) At time \( n = 1/2 \), it undergoes a \( p_* \) thinning. In general, for \( p \in [0, 1] \), the \( p \) thinning of a set \( A \subset \mathbb{Z} \) is the random subset of \( A \) obtained by independently removing each element of \( A \) with probability \( 1 - p \).
(iii) Start the DK model at time \( n = 0 \) with the same configuration as the \( p_*-DK-model \) (which is defined by \( f(0) = 0, f(1) = p_1/p_2, f(2) = p_2/p_2 \)) at time \( n = 1/2 \).
(iv) Couple the processes together until time \( n_0 \) is reached for the DK model, \( n_0 \) for the \( p_*-DK-model \). This can be done because the transitions for the DK model are the same as those for the \( p_*-DK-model \) lagging by the time unit 1/2.
(v) Perform a \( p_* \) thinning for the DK model at time \( n_0 \).

The distribution of the DK model started and ended as a \( p_* \) thinning of the \( p_*-DK-dual \). As in the DK model, we can define a new process \( \eta_n \) as a \( p_*-TPCA-dual \) for the \( N \)-neighbour TPCA, where \( p_* = \max\{p_1, p_2, \ldots, p_N\} \). Recall that the following duality holds for the TPCA \( \xi_n^A \) and the \( p_*-TPCA-dual \( \eta_n^A \) starting from \( A \) (see theorem 2 (2) in [15]):

**Theorem 6 ([15])** We assume an \( N \)-neighbour TPCA with \( \{p_n : 0 \leq n \leq N\} \) having a site–bond representation with \( \alpha = p_1^2/(2p_1 - p_2) \) and \( \beta = (2p_1 - p_2)/p_1 \). Suppose \( (p_1, p_2) \in D_* \). For any \( A, B \) with \( |A| < \infty \) or \( |B| < \infty \), we have
\[ E(\tilde{x}^{\xi_n^A \cap B}) = E(\tilde{x}^{\eta_n^B \cap A}), \]
for any \( n \geq 0 \), if \( \tilde{x} = 1 - (2p_1 - p_2)p_*/p_1^2 \).
Note that if $p_1 > p_2$ (the non-attractive case for the TPCA with a site–bond representation), then $p_* = p_1$. By using the same argument for the DK model as in section 7 of [8], we can get the following convergence theorem for the TPCA:

**Theorem 7** Assume $p_1 > p_2$ with $(p_1, p_2) \in D_*$. If the initial measure $\nu$ of the TPCA is a.s. (almost surely) infinite, then we have

$$\xi^n_{\nu} \rightarrow \mu_\eta,$$

as $n \rightarrow \infty$, where the limit measure is uniquely determined by $E((-(p_1 - p_2)/p_1)^{|\mu_\eta \cap A|}) = P(|\eta^A_\infty| = 0)$ for any $A$ with $|A| < \infty$.

### 6. Conclusions and discussion

In this work we have presented rigorous results on the site–bond representation, the set-to-set connectedness, the self-duality, the matrix expression, and the convergence theorem for an $N$-neighbour TPCA with $\{p_n : 0 \leq n \leq N\}$, where $p_0 = 0$ and $N \geq 2$. An interesting feature of our model is that the only dominant parameters for some properties are $p_1$ and $p_2$, among $\{p_n : 0 \leq n \leq N\}$; see proposition 1 and corollary 2, for example.

Arrowsmith and Essam [17] gave an expansion formula for a point-to-point connectedness for the oriented mixed site–bond percolation, in which each term is characterized by a graph. Konno and Katori [6] extended this formula to the $N = 2$ case (the DK model). So it is shown that the site–bond representation, self-duality, and the above graphical expansion formula hold in the DK model. Thus one interesting future problem is extending the relation to a general $N$-neighbour TPCA, as considered here.

Finally we mention a relation between our discrete-time model and a continuous-time one corresponding to it. In the continuous-time case, an infinitesimal generator for the model corresponds to $P_N$ in theorem 5. In fact, equation (8) in theorem 5 corresponds to equation (17) in [18] for the $N = 2$ case (an extension of their result to an $N$-neighbour case can be easily obtained). Recent works on the duality for continuous-time interacting particle systems include [19]–[22].

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