Collective Excitations and Robust Stability in a Landau-Majorana Liquid

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We extend the theoretical machinery of Landau-Fermi liquid theory to a general system of interacting Majorana fermions. By adiabatically connecting the interacting eigenstates to those of the non-interacting Majorana-Schwinger gas, a Landau-Majorana-Silin kinetic equation is introduced, which reveals a Lifshitz transition in the Landau-Majorana liquid (LML) at large screening. The dispersion of the zero sound collective mode and an expression for the compressibility in the screened limit are derived. Our calculations predict the onset of Pomeranchuk instabilities in the LML for highly attractive interactions, supporting the hypothesis that Landau-like quasiparticles are a robust feature of a low-temperature quantum liquid of self-adjoint fermions.

Introduction. Majorana fermions were originally introduced as neutral solutions to a symmetrized Dirac equation. First proposed in the context of fundamental particle physics, Majorana particles have experienced a renaissance in the condensed matter community with the proposal that collective excitations in symmetry protected topological phases of matter support self-conjugate edge excitations known as Majorana zero modes (MZMs). More recently, quantum spin liquids (highly disordered spin systems with intrinsic topological order) have been shown to support Majorana excitations that are free to propagate through the bulk of the lattice. Whereas the long-range entanglement of MZM pairs forbids the coherent definition of a Majorana number operator, the Majorana quasiparticle found in gapless spin liquids behaves more akin to a conventional complex fermion. The non-Abelian phase of the Kitaev spin liquid on the hyperoctagon lattice (a proposed model of ) is suggested to host a “Majorana metal” with a well-defined neutral Fermi surface, while recent Raman spectroscopy measurements on Majorana excitations in the Abelian phase of (a proposed realization of a 2D Kitaev honeycomb lattice) hints at low-temperature fermionic behavior.

The Majorana representation of some general spin-1/2 disordered state is not limited to the field of Kitaev materials. One of the first condensed matter applications of the Majorana representation beyond the simple spin-1/2 antiferromagnet was to a simple description of the low-energy spin Hamiltonian of the two-channel Kondo problem. The Kondo insulator SmB6 has become of particular interest, as recent experiments appear to indicate the presence of a Fermi surface in the insulating phase of the material. Similar physics is seen in the fractionalized Fermi liquid theory, where a sharp Fermi surface is shown to exist while the system simultaneously violates the Luttinger count. As such, we cannot associate the sharp discontinuity in the Fermi momentum distribution with a traditional Landau quasiparticle weight, as it appears that such a sharply defined Fermi surface might be a robust feature inherent to the Majorana physics describing the bulk. Interestingly, ARPES measurements seem to suggest that this conducting state is not the result of some topological properties of this specific Kondo insulator. This has prompted the suggestion that SmB6 (as well as certain Kitaev materials) harbors a “Majorana-Fermi sea”, where the Majorana reality condition imposes a severe retardation of hole-like excitations. A “Landau-Majorana liquid” would explain why Fermi liquid-like properties remain in the bulk of the low-temperature Kondo insulator.

In this paper, we propose a natural extension of Landau-Fermi liquid theory to investigate the collective features of an ensemble of interacting Majorana fermions. Although the concept of a Majorana liquid has previously been suggested to unify the two competing views (fermionic vs. bosonic spinons) of electron fractionalization in spin liquids, there has been no attempt to describe the non-equilibrium excitations in a liquid of self-conjugate fermions via the Landau quasiparticle paradigm. In a previous work, we explicitly derived the low-temperature momentum distribution function of non-interacting, anti-symmetric particles that obey the Majorana reality condition. We find that mutual pairwise annihilation of these “Majorana-Schwinger fermions” (so-called because they exhibit an analogous spin-statistics relation) results in an increased stability of the Majorana Fermi surface against thermodynamic smearing. This tells us that the robust Fermi surface found in the Kondo insulator SmB6 and certain Kitaev materials might be a universal feature of many-body Majorana-Schwinger systems, and motivates us to build a Landau-Majorana liquid theory by suppressing the hole contribution in the Landau-Silin kinetic equation.

Majorana-Landau-Silin Kinetic Equation. In order to build the theory of a Landau-Majorana liquid (LML), we must first be able to coherently describe the eigenstates of the non-interacting system. This was done in a previous work, where the low-temperature momentum distribution of the free Majorana-Schwinger gas.

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in the thermodynamic limit is found to be

$$\bar{n}_{k\sigma}^0 \approx \Theta_<(k_F - k) + \Theta_>(k - k_F)n_{k\sigma}^0$$

(1)

where \(n_{k\sigma}^0\) is the momentum distribution of the free Fermi gas, and \(\Theta_<(k_F - k)\) and \(\Theta_>(k - k_F)\) are Heaviside step functions that are discontinuous and continuous at \(k = k_F\), respectively. Even though the eigenstates of an LML lack an isomorphism with the eigenstates of a non-interacting Fermi gas, we may still impose a one-to-one correspondence between the density fluctuations of the non-interacting and interacting Majorana systems. This is done by assuming, much as Landau did for the Landau-Fermi liquid, that the bare momentum distribution changes as an analytic function of the interaction strength. This is similar to the Landau phenomenology considered in the ferromagnetic Fermi liquid, where an isomorphism is constructed between the eigenstates of the interacting spin excitation spectrum and the equilibrium magnetic system, as opposed to that of the traditional Fermi-Dirac distribution. Although the self-conjugacy of the Majorana-Schwinger particles leads to the possibility of mutual annihilation if we perturb the system, we assume the fraction of particles that annihilate is not dependent on the interaction strength. This is apparent from the low-\(T\) description of the Kondo insulator SmB\(_6\), as a robust Fermi surface (the hallmark of the Majorana-Schwiner gas) and a LFL-like linear-\(T\) specific heat is found in the bulk even in the presence of strong interactions.

The above argument allows us to write the density fluctuation \(\delta \bar{n}_{k\sigma}\) in the LML as

$$\bar{n}_{k\sigma} - \bar{n}_{k\sigma}^0 \equiv \delta \bar{n}_{k\sigma} = \Theta_<(k - k_F)\delta n_{k\sigma}$$

(2)

We see that, as in previous descriptions of the Majorana-Fermi sea, the quasihole state is significantly suppressed. Because the Heaviside step function is an approximation only valid in the thermodynamic limit, we rewrite it as a Fermi-like function near the Majorana-Fermi surface:

$$\Theta_>(k - k_F) \approx \frac{1}{1 + e^{-\frac{k - k_F}{\alpha}}} \equiv \tilde{F}(k, \alpha)$$

(3)

where \(\alpha\) is a tunable parameter that we call the Majorana constant or parameter. Such a constant parametrizes the finite-system effects not taken into account in the derivation of the Majorana-Schwinger Boltzmann entropy, and may therefore be interpreted as the length scale at which Majorana-like effects become emergent. To ensure the exponential remains unitless, we take \(\alpha\) to have the same units as \(k_F\).

The functional expansion of the Landau-Majorana free energy is analogous to the Landau-Fermi case, and may be written in two equivalent forms:

$$F - F_0 = \sum_{k}(\epsilon_k - \mu)\delta n_k + \frac{1}{2}\sum_{kk'} f_{kk'}\delta n_k \delta n_{k'}$$

(4)

$$= \sum_{k}(\tilde{\epsilon}_k - \mu \tilde{F}(k, \alpha))\delta n_k + \frac{1}{2}\sum_{kk'} \tilde{f}_{kk'}\delta n_k \delta n_{k'}$$

(5)

where \(\tilde{\epsilon}_k\) and \(\tilde{f}_{kk'}\) are the regular Landau-Fermi liquid quasiparticle energies and interaction parameters multiplied by \(\tilde{F}(k, \alpha)\) and \(\tilde{F}(k, \alpha)\tilde{F}(k', \alpha)\), respectively. Eq. (4) is interpreted as a suppression of quasihole excitations while maintaining LFL-like interactions. In contrast, Eq. (5) may be interpreted as a system with LFL-like excitations with a suppressed interaction for \(k < k_F\). We can therefore regard the LML as a regular Landau-Fermi liquid with either retarded quasihole excitations or with a suppressed interaction term below the Majorana-Fermi surface. This allows us to define the Landau-Majorana effective mass \(\tilde{m}^*\) in terms of the Landau-Fermi effective mass \(m^*\) (see Appendix A), which gives us \(\tilde{m}^*/m^* \approx k_F/8\alpha\). As such, the LML may likewise be considered to be a LFL with an effective mass rescaled by \(\alpha\), and subsequently a severely suppressed particle-hole continuum, given by \((\tilde{\epsilon}_k + q - \tilde{\epsilon}_k)/\epsilon_{k_F} = \frac{q^2}{\tilde{m}^*}(q^2 + 2k \cdot q)\).

A large effective mass is similarly seen from quantum oscillation measurements in the Kondo insulators YbB\(_2\) and the (011)-plane of SmB\(_6\), which have already been suggested to harbor a neutral Majorana metallic state.

Taking the above interpretation, we are now in a position to write down the collision integral for the LML, which is derived in Appendix B:

$$i\mathcal{I}(\tilde{n}_k) = \tilde{F}(k, \alpha) \left\{ \delta n_k \left( \omega - q \cdot v_k \right) + \left( q \cdot v_k \frac{\partial n_{k\sigma}^0}{\partial \epsilon_k} + q \left\{ \frac{1 - \tilde{F}(k, \alpha)}{\alpha} \right\} n_{k\sigma}^0 \right) \delta \epsilon_k \right\}$$

(6)



\[\text{Distortions of the Majorana-Fermi sea.}\] We now want to derive the behavior of collective excitations in the Landau-Majorana system. We may do this by expressing Eqn. (6) in terms of the dimensionless
parameters $s = \omega / q v_F$ and $\cos \theta = \frac{q \cdot \hat{q}}{q v_F}$. We take the unscreened and screened limits of the resultant equation, defined by $\lim_{q \to 0} \frac{\delta n_{k'}}{\delta n_k} = 0$ and $\lim_{q \to 0} \frac{\delta n_{k'}}{\delta n_k} = \frac{\partial n_{k'}}{\partial \epsilon_{k'}} a^*_k k'$ respectively, where $a^*_k k'$ is the quasiparticle amplitude. The LMS equation is then recast in terms of the Majorana-Fermi surface distortion $\nu_k$:

$$\nu_k + \frac{\cos \theta}{\cos \theta - s} \int_{k'} f_{kk'} \frac{\partial n^0_k}{\partial \epsilon_{k'}} \nu_{k'} + \frac{F(\omega, q)}{4 \alpha v_F} \left( \frac{1}{\cos \theta - s} \right) = 0$$

(7)

where we have taken $n^0_k (1 - \xi(k, \alpha)) \approx \frac{3}{4}$ for $k \gtrsim k_F$ in units $\eta$ of the particle density, and

$$F(\omega, q) = \begin{cases} \int_{C} f_{kk'} \frac{\partial n^0_k}{\partial \epsilon_{k'}} a^*_k k', & \text{screened} \\ \frac{C}{C}, & \text{unscreened} \end{cases}$$

(8)

where $C$ is some constant. Note that the first two terms in (7) are present in the Landau-Fermi liquid, while the third term is unique to the interacting Majorana system. Also note that in all calculations concerning zero sound in this paper, we assume the main interesting features in the collective excitations are from the $\ell = 0$ channel. We subsequently truncate all spherical harmonics to this order.

For the unscreened Landau-Majorana liquid, the non-interacting limit $f_{kk'} \to 0$ yields the following simplification:

$$\nu_k - \nu_k^{(0)} = \frac{\cos \theta}{s - \cos \theta} \int_{k'} f_{kk'} \frac{\partial n^0_k}{\partial \epsilon_{k'}} \nu_{k'}$$

(9)

where $\nu_k^{(0)}$ is the Majorana-Fermi surface distortion in the non-interacting limit. In the unscreened limit, the net difference of the Majorana-Fermi surface distortion from the non-interacting case (which might be non-zero from emerging mutual annihilation) is equivalent to the well-known Landau-Fermi liquid result.

The screened Landau-Majorana liquid is somewhat more interesting. We can rewrite the value of $\nu_k$ in such a limit as

$$\nu_k = \frac{\cos \theta}{s - \cos \theta} \left( \frac{1}{1 + \frac{\gamma}{\gamma(\cos \theta - s)}} \right) \int_{k'} f_{kk'} \frac{\partial n^0_k}{\partial \epsilon_{k'}} \nu_{k'}$$

(10)

where

$$\gamma = \frac{4 \alpha v_F}{\eta} \left( \int_{k'} f_{kk'} \frac{\partial n^0_k}{\partial \epsilon_{k'}} a^*_k k' \right)^{-1}$$

(11)

Note that the above term is unitless, as we have taken $\eta$ to be a unitfull constant in terms of the particle’s density.

The limit of $\gamma \to 0$ is interpreted as a maximized “Majorana-like” contribution to the momentum distribution function, and therefore a total suppression of quasi-hole excitations. For no interaction, the system will be “purely” Majorana-like. For a very strong repulsive interaction, the system will be equivalent to that of a Fermi-Dirac system or nearly so due to a suppressed annihilation cross-section.

The collective $\ell = 0$ breathing mode (zero sound) for the Landau-Fermi ($\gamma \to \infty$) result is apparent in the above equations, and is the signature of dominant forward-scattering between quasiparticles. For the LML, the “purely” Majorana-Fermi surface retains its isotropic shape while simultaneously being shifted by some small amount. Such behavior is reminiscent of the $\ell = 1$ breathing mode (first sound) observed in the LFL, except with a net “backscattering” (i.e., $\theta = \pi$) contribution. Emergent backscattering is evident from the unusually large effective mass $\tilde{m}$ in the LML, which subsequently leads to an increased number of interactions with large momentum transfer.

![Figure 1: Fermi surface distortion for the Landau-Fermi liquid (dotted line) and Landau-Majorana liquid (solid lines) vs. $\theta$ for a single fixed value of $s = \omega / q v_F = 3$, $\eta = 1$, and several values of $\gamma$.](image)

In Fig. 1, we see the Fermi and Majorana-Fermi surface distortions for fixed values of the collective mode velocity $s = \omega / q v_F$ plotted vs. various values of $\gamma$. As the value of $\gamma$ passes through a certain threshold, the Majorana-Fermi surface experiences exponential divergence until it eventually settles into the regular zero sound behavior of an LFL. From Eq. (11), singularities occur when $k' \cdot \hat{q} = \cos^{-1} (s - 1 / \gamma)$. As $\gamma \to \infty$, we recover the standard Landau damping of a LFL.

We interpret this singularity as the onset of a change of the Fermi surface topology brought about by increasing $\gamma$ and, hence, the gradual screening of the effects of self-conjugacy in the quantum fluid. In a LFL, such divergences appear when $s$ is in the particle-hole channel, however we avoid such a change in topology (known formally as a Lifshitz transition) by assuming an angle-dependent interaction and subsequently performing a simple contour integration. This leads to the
prediction of Landau damping in the LFL. The robust stability of the Majorana-Fermi surface brought about by the self-adjoint behavior of the Majorana quasiparticles "lifts" the Lifshitz paradigm above the particle-hole channel. This results in an instability brought about by a Fermi sea distortion governed by a linear Volterra equation (see Appendix B), and hence the divergent behavior cannot be described by a Landau-like damping mechanism.

Zero Sound in a Landau-Majorana liquid. From the above analysis, it appears that, although the LML is highly stable, an increase in effective screening brought about by stronger repulsive interactions leads to a highly unstable Lifshitz transition, after which the reality condition is completely suppressed and LFL behavior is restored. To explore the system further, we derive explicitly the interaction-dependence of the LML zero sound.

The zero-sound in the screened Landau-Majorana system is described by an equation similar to that describing the LFL zero sound\[^{37,38}\] except now we have an additional term dependent on the scattering amplitude \(A_0^s = a_0^s N(0)\):

\[
1 + \frac{1}{2} \log \left( \frac{s - 1}{s + 1} \right) \left\{ s + \frac{A_0^s}{v} \right\} = -\frac{1}{F_0^s} \tag{12}
\]

where \(\bar{v} \approx 4\alpha N(0)v_F/\eta\) and \(N(0)\) is the bare-particle density of states. See Appendix C for derivation. We first solve the above equation for \( |s| < 1\). In such a limit, the value of \(A_0^s/\bar{v}\) dwarfs the \(s\) in the parentheses, which leads to the result that \(s \approx \coth (\bar{v}/A_0^s)\). Interestingly, the above result is achieved for all values of \( |F_0^s| < 1\). This is different from the LFL, where the regime \(-1 \leq F_0^s < 0\) is heavily Landau damped. Like the LFL, the LML appears to break the Landau paradigm as it approaches the Pomeranchuk instability at \(F_0^s + 1 = 0\)\[^{15}\]. However, it is important to note that, as we approach this instability, the value of \(s\) becomes on the scale of \(A_0^s/\bar{v}\), and thus the above approximation breaks down. We must therefore consider the large \(s\) limit to investigate the zero sound behavior further.

Assuming \(|F_0^s| \gg 1\) leads to \(|s| \gg 1\) (as in the LFL), Eqn. (12) simplifies to

\[
\frac{1}{F_0^s} = \frac{1}{3s^2} + \left( 3(1 + F_0^s/\bar{v}) \right) \frac{1}{s^3} + \left( \frac{F_0^s}{(1 + F_0^s/\bar{v})} \right) \frac{1}{s} \tag{13}
\]

The full solutions are rather lengthy, and are reproduced in full in Appendix C. There are three equations for the value of velocity \(s\), each with differing dependence on the interaction strength \(F_0^s\). This dependence is shown in Fig. 2.

For large positive values of \(F_0^s\), there are two possible dispersions. The upper branch behaves as that of a regular LFL, while we interpret the lower branch as originating from the collective shift of the Majorana-Fermi surface under backscattering between exceptionally massive Majorana-Schwinger quasiparticles.

Below \(F_0^s = 0\), we observe the coexistence of a real and imaginary portion to the dispersion. This regime corresponds to the Landau damped regime, and unlike the LFL system (where the imaginary term continues to grow as \(F_0^s\) becomes increasingly more negative, signaling the onset of a Pomeranchuk instability), the imaginary part of the renormalized LML zero sound velocity \(s\) dies at a particular value of attractive \(F_0^s\), after which the real part of \(s\) bifurcates into \(s_1\) and \(s_2\). Such bifurcating behavior is reminiscent of the unstable avoided level crossing of collective modes in a toy model of coupled spinor condensates\[^{43}\]. The “pitchfork” dispersion pattern occurs in these systems when the coupling strength becomes comparable to the interaction energy. In the LML, it would then be reasonable to assume that bifurcation occurs when the effects of mutual-annihilation becomes comparable to the interaction strength. Also note that, much as in the LFL, the region with a finite imaginary contribution are highly unstable due to presence of the suppressed particle-hole continuum. In the LML, however, this Landau-damped region encompasses a larger region of negative interactions.
LML compressibility, given by bifurcations. This can easily be done by looking at the vertical line which denotes the analytic solution to the bifurcation point $n_c^{(1)}$.

To better understand the LML zero sound bifurcation, we plot the value of $\frac{\omega}{q}$ vs. the particle number density $n$. This is shown in Fig. 3 and Fig. 4. The vertical lines are the analytic results for the critical densities $n_c^{(1)}$ and $n_c^{(2)}$ where bifurcation occurs (see Appendix C for derivation):

$$n_c^{(1)} \approx \frac{\pi}{8 \cdot 3^{1/4}} \left( \frac{\eta \sqrt{F_0}}{\alpha} \right)^{3/2}$$

$$n_c^{(2)} \approx \frac{\pi}{3^{1/4}} \left( \frac{\eta}{\alpha} \right)^{3/2}$$

Rearranging the first of the above, we find that the value of the Landau parameter $F_0^*$ at the point of bifurcation is given by

$$F_0^* = -\frac{16\alpha^2 k_F^2}{3\pi^4 \eta^2}$$

Therefore, for an interaction given by the above, the imaginary portion is suppressed and bifurcation occurs. In Fig. 4, a separate bifurcation is shown which occurs at $n_c^{(2)}$ and is independent of the interaction.

Compressibility of the Landau-Majorana liquid. From the above analysis, we have shown that the zero sound dispersion exhibits interesting bifurcation behaviors for certain critical terms of either the interaction or the density (depending on which we are holding fixed). We now wish to explain the physical meaning of these bifurcations. This can easily be done by looking at the LML compressibility, given by

$$\tilde{\kappa} = \frac{1}{n^2} \frac{\partial n}{\partial \mu} \bigg|_T \approx \frac{4}{n^2} \left( \frac{N^*(0)}{1 + \frac{4\pi}{4\alpha} + F_0^*} \right)$$

For a derivation, see Appendix A. The compressibility is positive as long as $F_0^* > -\left(1 + \frac{4\pi}{4\alpha} \right)$. This is rather different from the case of the LFL, where the minimum condition for a stable Fermi surface is given by $F_0^*+1 = 0$. We therefore find that the LML is more stable to Pomeranchuk instabilities at attractive interactions than the LFL. Moreover, up to insignificant constants that we may ignore for $\alpha << 1$, we note that this is the exact behavior of the attractive interaction we found at the point of bifurcation if we assume the numerical value of $\alpha$ goes as the inverse of the Fermi wave vector. This makes sense, as this would mean that the length scale that determines particle-particle annihilation (i.e., $\alpha$) is comparable to the inter-particle distance. Moreover, the above analysis leads us to conclude that the first point of bifurcation for $F_0^* < 0$ signals the onset of a breakdown of the Landau paradigm in an interacting Majorana-Schwinger liquid.

The $\omega/q$ vs. $n$ plots shown in Figs. 3 and 4 illustrate three regions of interest for $F_0^* << -1$. Region I in Fig. 3 (right of the vertical line) is the region where the density is at least comparable to $(|F_0^*|)^{3/4} \alpha^{-3/2}$, and is described by a heavily Landau-damped zero mode. In such a regime, the collective Pauli exclusion of the dense system “protects” the bulk of the liquid from experiencing massive mutual annihilation and a collapse of the Majorana-Fermi sea. Region II in Fig. 3 (left of the vertical line) occurs when $n \lesssim (|F_0^*|)^{3/4} \alpha^{-3/2}$, which results in a negative compressibility and hence a breakdown of the LML. Region III of Fig. 4 (left of the vertical line)
line) is the unique region where \( n \lesssim (\pi/3^{1/4}) \cdot (1/\alpha^{3/2}) \). Here, the density of Majorana particles is low enough such that the annihilation cross section is sufficiently repressed so as to support a Landau-damped LML for attractive interactions.

**Conclusions.** In this paper, we have extended the formalism of Landau-Fermi liquid theory to a general system of interacting fermions obeying a self-conjugacy relation (i.e., the interacting limit of the Majorana-Schwinger gas\(^2\)). At zero temperature, the equilibrium system mirrors the non-interacting system of complex fermions. However, as we perturb the system, the suppression of LFL quasihole excitations in the Majorana-Fermi sea manifests itself as an amplification of the LML quasiparticle effective mass. By introducing the length-scale \( \alpha \), we can tune this quasihole suppression to find a Lifshitz transition separating the crossover between the screened LML and the LFL. The interaction dependence and density independence of the dimensionless zero sound parameter \( s = \omega/qv_F \) shows evidence of an avoided level crossing between collective modes originating from small and large momentum transfers. A compressibility calculation confirms that the point of bifurcation for attractive interactions signals a Pomeranchuk instability and a breakdown of the Majorana quasiparticle picture. The explicit Pomeranchuk instability condition for the Majorana quasiparticles shows a robust stability of the LML even for exceptionally large, negative values of \( F_0 \).

We believe such a study provides a simple proposed model of highly complex materials with unconventional quasiparticle excitations, such as Kitaev spin liquids\(^{11,12,22}\) and the Kondo insulator SmB\(_6\)^{18,19}, where recent theoretical proposals and experimental evidence points to Landau-Fermi liquid-like behavior without adiabatic continuity to the non-interacting Fermi gas. The Landau-Majorana liquid proposed here is a rigorous attempt to understand these materials with the well-established formalism of Landau-Fermi liquid theory, and provides the first known attempt to derive universal experimental signatures of many-body Majorana physics via a description of the interaction dependence and density dependence of the \( \ell = 0 \), zero-temperature acoustic modes.

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1. E. Majorana, *Il Nuovo Cimento* **14**, 171 (1937).
2. A. Kitaev, *Physics-Uspekhi* **44**, 131 (2001).
3. L. Fu and C. L. Kane, *Phys. Rev. Lett.* **100**, 096407 (2008).
4. S. Nadj-Perge et al., *Science* **346**, 602 (2014).
5. Q. L. He et al., *Science* **357**, 294 (2017).
6. S. B. Bravyi and A. Y. Kitaev, *Annals of Physics* **298**, 210 (2002).
7. Y.-J. Wu et al., *Phys. Rev. A* **90**, 022324 (2014).
8. S. Watanabe et al., *npj Quantum Information* **1**, 15001 (2015).
9. C. Nayak et al., *Rev. Mod. Phys.* **80**, 1083 (2008).
10. A. Kitaev, *Annals of Physics* **321**, 2 (2006).
11. M. Hermanns and S. Trebst, *Phys. Rev. B* **89**, 235102 (2014).
12. Y. Wang et al., (Under review), arXiv:1809.07782v2.
13. J. L. Martin and N. Kemmer, *Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences* **251** (1959).
14. A. M. Tsvelik, *Phys. Rev. Lett.* **69**, 2142 (1992).
15. B. S. Shastry and D. Sen, *Phys. Rev. B* **55**, 2988 (1997).
16. P. Coleman, E. Miranda, and A. Tsvelik, *Physica B: Condensed Matter* **186-188**, 362 (1993).
17. P. Coleman, L. B. Ioffe, and A. M. Tsvelik, *Phys. Rev. B* **52**, 6611 (1995).
18. B. S. Tan et al., *Science* **349**, 287 (2015).
19. M. Hartstein et al., *Nature Physics* **14**, 166 (2018).
20. T. Senthil, S. Sachdev, and M. Vojta, *Phys. Rev. Lett.* **90**, 216403 (2003).
21. P. Hlawenka et al., *Nature Communications* **9**, 517 (2018).
22. G. Baskaran, G. Senthosh, and R. Shankar, (Unpublished), arXiv:0908.1614v3.
23. D. Takikawa and S. Fujimoto, arXiv:1902.06433.
24. G. Baskaran, “Majorana Fermi Sea in Insulating SmB6: A proposal and a Theory of Quantum Oscillations in Kondo Insulators,” Invited Talk at the Workshop ‘Concepts and Discovery in Quantum Matter’, 12-15th July 2015, Cavendish Laboratory, Cambridge University, UK.
25. W. T. Fuhrman and P. Nikolić, arXiv:1807.00005v1.
26. N. Wakeham et al., *Phys. Rev. B* **94**, 035127 (2016).
27. P. Maraner, J. K. Pachos, and G. Palumbo, arXiv:1811.02258v2.
28. C. Xu and S. Sachdev, *Phys. Rev. Lett.* **105**, 057201 (2010).
29. J. Heath and K. Bedell, *Journal of Physics A: Mathematical and Theoretical* (2019).
30. L. Landau, *JETP* **3**, 1058 (1956).
31. I. E. Dzyaloshinskii and P. S. Kondratenko, *JETP* **43** (1976).
32. K. S. Bedell and K. B. Blagoev, *Philosophical Magazine Letters* **81** (2001).
33. Y. Zhang, P. Farinas, and K. Bedell, *Acta Physica Polonica A* **127**, 153 (2015).
34. H. Liu et al., *Journal of Physics: Condensed Matter* **30** (2018).
35. Y. Lu et al., *Phys. Rev. B* **91**, 075130 (2015).
36. V. Silin, *JETP* **6**, 1227 (1958).
37. L. Landau, *JETP* **5**, 101 (1957).
38. N. D. Mermin, *Phys. Rev.* **159**, 161 (1967).
39. L. Landau, *J. Phys. X* **25** (1945).
40. D. Pines and D. Bohm, *Phys. Rev.* **85**, 338 (1952).
41. I. Lifshitz, *JETP* **38**, 1569 (1960).
42. I. Pomeranchuk, *JETP* **35**, 524 (1958).
Supplemental Material

A Derivation of the Landau-Majorana compressibility and effective mass

We may now derive the Landau-Majorana compressibility. In general, the compressibility is given by

$$
\kappa = \left. \frac{1}{n^2} \frac{\partial n}{\partial \tilde{\mu}} \right|_T
$$

We now write down the expression for the partial functional derivative of the LML chemical potential:

$$
\frac{\partial \tilde{\mu}}{\partial n} = \frac{\partial \tilde{\epsilon}_k}{\partial k} \frac{\partial k}{\partial n} \bigg|_{k_F} + \frac{1}{V} \sum_{k', \sigma'} \tilde{f}_{\sigma \sigma'}(k, k') \left| \frac{\partial n_{k'}}{\partial k} \frac{\partial k}{\partial n} \right|_{k_F} \bigg|_{k_F}
$$

(18)

Where

$$
\frac{\partial \tilde{\epsilon}_k}{\partial k} \bigg|_{k_F} = \frac{\partial}{\partial k} (\epsilon_k \Theta(k - k_F)) \bigg|_{k=k_F}
$$

$$
= \epsilon_k \delta(k - k_F) \bigg|_{k_F} + \frac{\partial \epsilon_k}{\partial k} \Theta(k - k_F) \bigg|_{k_F}
$$

$$
\approx \epsilon_k \tilde{\Psi}(k_F, \alpha) \left( 1 - \frac{\tilde{\Psi}(k_F, \alpha)}{\alpha} \right) + \frac{\partial \epsilon_k}{\partial k} \tilde{\Psi}(k_F, \alpha)
$$

(19)

Note that $\tilde{\Psi}(k, \alpha) = 1$ if we take $k - k_F >> \alpha$ and $\tilde{\Psi}(k, \alpha) = 1/2$ for $k - k_F << \alpha$. Hence,

$$
\frac{\partial \tilde{\epsilon}_k}{\partial k} \approx \begin{cases} 
\frac{\partial \epsilon_k}{\partial k} \bigg|_{k=k_F}, & k - k_F >> \alpha \\
\epsilon_{k_F} \frac{1}{4\alpha} + \frac{1}{2} \frac{\partial \epsilon_k}{\partial k} \bigg|_{k=k_F}, & k - k_F << \alpha
\end{cases}
$$

If we take the latter case, then

$$
\frac{\partial \tilde{\epsilon}_k}{\partial k} \bigg|_{k_F} \approx \epsilon_{k_F} \frac{1}{4\alpha} + \frac{v_F}{2}
$$

$$
= \frac{k_F^2}{8m^*\alpha} + \frac{k_F}{2m^*}
$$

$$
= \frac{k_F}{2m^*} \left( 1 + \frac{k_F}{4\alpha} \right)
$$

$$
= \frac{v^*_F}{2} \left( 1 + \frac{k_F}{4\alpha} \right)
$$

$$
\equiv v^*_M
$$

(20)

Where we define for convenience $v^*_M$ as the Landau-Majorana velocity.

It is interesting to note that, from the above form of the Landau-Majorana velocity, we could effectively define a Majorana density of states, and from this obtain the Landau-Majorana specific heat. Much as in the non-interacting Majorana-Schwinger gas\textsuperscript{29}, the specific heat $C_v$ for the Majorana system has the same temperature-dependence as the Fermi-Dirac case. Because the temperature-dependence of $C_v$ doesn’t change as we go from a LFL to the LML,
Luttinger’s theorem is still satisfied in the latter, and thus the Landau-Fermi liquid picture is applicable to our system\textsuperscript{44}.

The other quantities in the expression for \( \frac{\partial \tilde{\mu}}{\partial n} \) are identical to the LFL. We then find that

\[
\frac{\partial \tilde{\mu}}{\partial n} = \frac{\partial \tilde{\epsilon}_k}{\partial k} \left. \frac{\partial k}{\partial n} \right|_{k_F} + \frac{1}{V} \sum_{k', \sigma'} \tilde{f}_{\sigma \sigma'}(k, k') \left. \frac{\partial n_{k'}}{\partial k} \right|_{k_F} \left. \frac{\partial k}{\partial n} \right|_{k_F} \\
\approx \frac{\pi^2}{k_F^2} \left( v_M^* + \frac{k_F^2}{2\pi^2} f_0^s \right) \\
= \frac{\pi^2}{k_F^2} \left( v_M^* + N^*(0) v_F^* f_0^s \right) \\
\approx \frac{1 + k_F^4 + F_0^s}{4N^*(0)} \tag{21}
\]

where \( N^*(0) = \frac{m^* k_F^2}{2\pi^2} \). This leads directly to Eq. \( \text{(16)} \).

We now move onto the modified Landau parameter for \( \ell > 0 \). Interestingly, we can write the Majorana compressibility derived above in the form

\[
\kappa = 2 \frac{n^2}{n^2} \left( \frac{\tilde{N}^*(0)}{1 + F_0^s} \right) \tag{22}
\]

where we have defined

\[
\tilde{N}^*(0) = 2N^*(0) \tag{23}
\]

and

\[
\tilde{F}_0^s = \frac{k_F}{4\alpha} + F_0^s \tag{24}
\]

Looking at the interaction term, we see that

\[
\tilde{f}_0^s = \frac{k_F}{4\alpha N^*(0)} + f_0^s \\
= \frac{k_F}{4\alpha N^*(0)} + \frac{1}{2} \int_{-1}^{1} d(\cos \theta) f_{kk'}^s \\\n= \frac{1}{2} \int_{-1}^{1} d(\cos \theta) \tilde{f}_{kk'}^s \tag{25}
\]

where we have now defined

\[
\tilde{f}_{kk'}^s = \frac{k_F}{4\alpha N^*(0)} + f_{kk'}^s \tag{26}
\]

With this form, we can then see that

\[
\tilde{f}_l^s = \frac{3}{2} \int_{-1}^{1} d(\cos \theta) \cos \theta \tilde{f}_{kk'}^s \\
= \frac{3}{2} \int_{-1}^{1} d(\cos \theta) \cos \theta \frac{k_F}{4\alpha N^*(0)} + \frac{3}{2} \int_{-1}^{1} d(\cos \theta) \cos \theta f_{kk'}^s \\
= f_l^s \tag{27}
\]

Without loss of generality, we see that \( \tilde{f}_l^s = f_l^s \) for all \( \ell \neq 0 \).

We now move onto the derivation of the effective mass of quasiparticles in the LML. The energy functional of the LML is given by
\[ \delta E[\delta n] = \sum_{k\sigma} \overline{\epsilon}_k \delta n_{k\sigma} + \frac{1}{2V} \sum_{kk'\sigma\sigma'} \overline{f}_{\sigma\sigma'}(k, k') \delta n_{k\sigma} \delta n_{k'\sigma'} \]  

(28)

where we have used the picture of \( \overline{\epsilon}_k \) rather than the \( \overline{\delta n}_{k\sigma} \) picture. With the above, we then see that

\[ \frac{\delta E}{\delta n_{k\sigma}} = \overline{\epsilon}_k + \frac{1}{V} \sum_{k\sigma'} \overline{f}_{\sigma\sigma'} \delta n_{k'\sigma} \]  

(29)

where we have foregone the spin index in the quasiparticle energy. We now expand the relevant quantities, noting that \( \mathbf{q} \cdot \hat{k} = q \) and \( \mathbf{q} \cdot \hat{k}' = q \cos \theta \):

\[ \overline{\epsilon}_{k-q} \approx \overline{\epsilon}_k - \mathbf{q} \cdot \nabla_k \overline{\epsilon}_k \approx \epsilon_k f - q v_M^* \]  

(30)

\[ n^0_{k'-q} \approx n^0_{k'} - \mathbf{q} \cdot \nabla_{k'} \frac{\partial n^0_{k'}}{\partial \overline{\epsilon}_{k'}} \approx n^0_{k'} - q v_M^* \cos \theta \frac{\partial n^0_{k'}}{\partial \overline{\epsilon}_{k'}} \]  

(31)

We therefore see that

\[ \frac{\delta E}{\delta n_{k\sigma}} \approx \epsilon_k f - q v_M^* + \frac{1}{V} \sum_{k\sigma'} \overline{f}_{\sigma\sigma'}(k, k') \left\{ -q v_M^* \cos \theta \frac{\partial n^0_{k'}}{\partial \overline{\epsilon}_{k'}} \right\} \]  

(32)

Now, note that the current \( j_k = k_F/m \) is given by

\[ j_k = -\frac{\partial}{\partial q} \left( \frac{\delta E}{\delta n_{k-q}} \right) \bigg|_{q=0} \]  

(33)

Hence,

\[ j_k = v_M^* + \frac{1}{V} \sum_{k\sigma} \overline{f}_{\sigma\sigma'}(k, k') v_M^* \frac{\partial n^0_{k'}}{\partial \overline{\epsilon}_k} \cos \theta \]

\[ = v_M^* \left( 1 + \frac{F_1^*}{3} \right) \]

\[ = \frac{v_M^*}{2} \left( 1 + \frac{k_F}{4\alpha} \right) \left( 1 + \frac{F_1^*}{3} \right) \]  

(34)

where we have used the fact that \( \overline{F}_1^* = F_1^* \). Therefore, the LML effective mass \( \overline{m}^* \) is given by

\[ \frac{\overline{m}^*}{m} = \frac{1}{2} \left( 1 + \frac{k_F}{4\alpha} \right) \left( 1 + \frac{F_1^*}{3} \right) \]

\[ \approx \frac{k_F}{8\alpha} \left( 1 + \frac{F_1^*}{3} \right) \]

\[ = \frac{k_F m^*}{8\alpha} \]  

(35)

The ratio of the LML effective mass and the LFL effective mass is then given by

\[ \frac{\overline{m}^*}{m^*} \approx \frac{k_F}{8\alpha} \]  

(36)

This subsequently leads to the expression in the text.
B Derivation of the Landau-Majorana-Silin kinetic equation and the Majorana-Fermi surface distortion

The general form for the Landau-Majorana-Silin (LMS) kinetic equation is given by

\[ I(\tilde{n}_k) = \frac{d}{dt} \tilde{n}_k = \partial_{\tau} \tilde{n}_k - \{ \epsilon_k, \tilde{n}_k \}_PB \]  

(37)

Because we restrict ourselves to quasiparticle states just above the Majorana-Fermi surface, the Landau-Majorana dispersion relation becomes

\[ \tilde{n}_k \approx \Theta(\xi(k - k_F))n_k, \]

where \( n_k \) is the Landau-Fermi liquid dispersion, and the Poisson bracket is given by

\[ \{ \epsilon_k, \tilde{n}_k \}_PB = \nabla_r \epsilon_k \cdot \nabla_k \tilde{n}_k - \nabla_k \epsilon_k \cdot \nabla_r \tilde{n}_k \]

(38)

We note the \( k \)-dependent term in the Poisson bracket is given by

\[ \partial_{\tau} \tilde{n}_k = \Theta(\xi(k - k_F))\frac{\partial}{\partial \tau} n_k \]

(39)

While

\[ \nabla_k (\tilde{n}_k) = \left\{ \Theta(\xi(k - k_F))\nabla_k + \hat{q}\delta(k - k_F) \right\} n_k \]

(40)

In the above equation, the first term is the usual LFL result, except now it is only valid above the Majorana-Fermi surface. The second term is not seen in the Landau-Fermi system, and is the direct result of some external driving wave vector \( q \). From the form of the Majorana distribution given in the text, we can see that the external plane wave perturbation induces a particle-hole term in the \( \hat{q} \) direction that must lead to a divergence in the \( k \)-dependent rate of change of the quasiparticle distribution function. As this term only effects the quasiparticle distribution in the direction of \( \hat{q} \), we write it as \( \delta(k - k_F)n_k \hat{q} \). This is opposed to the \( k \)-dependent term for the LFL and in the \( k > k_F \) regime of the LML, where we lack the constraint of a “sharpened” Fermi surface from the self-conjugacy condition of the underlying particles, and we hence lack an explicit \( q \)-dependence in the gradient of the momentum distribution. We might also say that, in the LFL, the external perturbation “smears” the Fermi surface in the direction of \( \hat{q} \) as the result of the propagation of the particle-hole pair, while in the LML we require an additional term solely in the \( \hat{q} \)-direction to mitigate such “smearing”.

We find that the Poisson bracket becomes

\[ \{ \epsilon_k, \tilde{n}_k \}_PB = \nabla_r \epsilon_k \cdot \nabla_k \tilde{n}_k - \nabla_k \epsilon_k \cdot \nabla_r \tilde{n}_k \]

(41)

This permits us to write the LMS as

\[ I(\tilde{n}_k) = \Theta(k - k_F)I(n_k) - \delta(k - k_F) \left( n_k \frac{\partial \epsilon_k}{\partial \tau} \right) \hat{q} \cdot \hat{r} \]

(42)

The above tells us that the LML collision integral reduces to the LFL result for \( k > k_F \). On the Fermi surface, a divergence occurs relative to the direction of \( \hat{q} \).

We now make the approximation of the Heaviside theta function given in Eq. (3). Hence,

\[ \delta(k - k_F) = \frac{\partial}{\partial k} \Theta(k - k_F) \]

\[ \approx \frac{\partial}{\partial k} \tilde{\delta}(k, \alpha) \]

\[ = \frac{\tilde{\delta}(k, \alpha)(1 - \tilde{\delta}(k, \alpha))}{\alpha} \]

(43)

The LMS then becomes

\[ I(\tilde{n}_k) = \tilde{\delta}(k, \alpha) \left\{ I(n_k) - \frac{1 - \tilde{\delta}(k, \alpha)}{\alpha} \left( n_k \frac{\partial \epsilon_k}{\partial \tau} \right) \hat{q} \cdot \hat{r} \right\} \]

(44)
The Poisson bracket in the LFL term $I(n_k)$ is given by the well-known result

$$\{-\epsilon_k, n_k\}_{PB} = v_k \cdot \nabla \left( \delta n_k - \frac{\partial n_k^0}{\partial \epsilon_k} \delta \epsilon_k \right)$$  \hspace{1cm} (45)$$

where $v_k = \partial \epsilon_k / \partial k$. The LML contribution may be evaluated in a similar fashion:

$$- \frac{1 - \mathcal{F}(k, \alpha)}{\alpha} \left( n_k \frac{\partial \epsilon_k}{\partial r} \right) \approx - \frac{1 - \mathcal{F}(k, \alpha)}{\alpha} \left( \frac{\partial (n_k^0 \delta \epsilon_k)}{\partial \epsilon_k} \right)$$  \hspace{1cm} (46)$$

The LMS then takes the form

$$I(\tilde{n}_k) \approx \mathcal{F}(k, \alpha) \left\{ \frac{\partial}{\partial t} n_k + v_k \frac{\partial}{\partial r} \left( \delta n_k - \frac{\partial n_k^0}{\partial \epsilon_k} \delta \epsilon_k \right) \right. \hspace{1.5cm} \left. \frac{\partial}{\partial r} \left( \frac{1 - \mathcal{F}(k, \alpha)}{\alpha} n_k^0 \delta \epsilon_k \right) \hat{q} \cdot \hat{r} \right\}$$  \hspace{1cm} (47)$$

Taking a Fourier transform, we find the above reducing to Eq. (6). The unusual lack of angular independence in the Majorana-specific term in the LMS equation is the direct result of demanding an amplified $k$-dependent rate of change at $k = k_F$ solely in the direction of an external plane wave perturbation.

We are now in a position to derive the distortions of the Majorana-Fermi surface. First, we write the differential element $\delta \epsilon_k$ as

$$\delta \epsilon_k = U + \int d\Omega_{k'} f_{kk'} \delta n_{k'}$$

$$\equiv U + \int_{k'} f_{kk'} \delta n_{k'}$$  \hspace{1cm} (48)$$

Taking the limit of $U \to 0$ and using the fact that the Fermi surface distortion is given by

$$\delta n_{k'} = - \frac{\partial n_k^0}{\partial \epsilon_k} \nu_{k'}$$  \hspace{1cm} (49)$$

We obtain

$$\nu_k + \frac{\cos \theta}{\cos \theta - s} \int_k f_{kk'} \frac{\partial n_k^0}{\partial \epsilon_k} \nu_{k'}$$

$$+ \frac{1}{\cos \theta - s} \left\{ 1 - f(k, \alpha) \right\} \frac{n_k^0}{\alpha v_F} \int_{k'} f_{kk'} \frac{\partial n_k^0}{\partial \epsilon_k} \nu_{k'} = 0$$  \hspace{1cm} (50)$$

The last term is specific to the Majorana system. We can simplify it by noting that

$$\frac{\delta n_{k'}}{\delta n_k} = \frac{q \cdot v_{k'}}{\omega - q \cdot v_{k'}} \frac{\partial n_k^0}{\partial \epsilon_k} \delta \epsilon_{k'}$$  \hspace{1cm} (51)$$

The unscreened and screened limits are discussed in the text. These two limits yield the above to be recast in the form

$$\frac{\partial n_k^0}{\partial \epsilon_k} \frac{\partial n_k^0}{\partial \epsilon_k} = \delta n_{k'} \nu_k$$

$$\frac{\partial n_k^0}{\partial \epsilon_k} \frac{\partial n_k^0}{\partial \epsilon_k} \rightarrow \left\{ \begin{array}{ll} \frac{\partial n_k^0}{\partial \epsilon_k} \nu_k \delta \epsilon_{k'}, & q \to 0 \hspace{0.3cm}, \hspace{0.3cm} \omega \to 0 \vspace{0.3cm} \end{array} \right. \right.$$

$$q \to 0, \hspace{0.3cm} \omega \to 0$$

Where the quasiparticle scattering amplitude is given by

$$a_{kk'}^s \equiv \lim_{q \to 0} \frac{\delta \epsilon_{k'}}{\partial \delta n_{k'}}$$  \hspace{1cm} (52)$$

This leads to Eq. (7). Of particular interest to this paper is the onset of a Lifshitz transition for certain values of the parameter $\gamma$. At this transition, we note that

$$\frac{\eta \nu_k}{4 \alpha v_F} \left( \frac{1}{\cos \theta - s} \right) \int_{k'} f_{kk'} \frac{\partial n_k^0}{\partial \epsilon_k} \nu_{k'} = - \nu_k$$  \hspace{1cm} (53)$$
By seeing that

$$\cos \theta = s - \frac{\eta}{4\alpha\nu_F} \int_{k'} f_{kk'} \frac{\partial n_0}{\partial k} a_{kk'}^s$$  \hspace{1cm} (54)$$

We are able to reduce the LMS equation to

$$-\frac{\eta}{4\alpha\nu_F} \int_{k'} f_{kk'} \frac{\partial n_0}{\partial k} a_{kk'}^s - s \int_{k'} f_{kk'} \frac{\partial n_0}{\partial k} a_{kk'}^s \nu_{k'} = 0$$  \hspace{1cm} (55)$$

There are two possibilities for this singularity to happen. Either we can have

$$s = \frac{\eta}{4\alpha\nu_F} \int_{k'} f_{kk'} \frac{\partial n_0}{\partial k} a_{kk'}^s$$  \hspace{1cm} (56)$$

Or we can have

$$\int_{k'} f_{kk'} \frac{\partial n_0}{\partial k} \nu_{k'} = \int_{k'} f_{kk'} \delta n_{k'} = 0$$  \hspace{1cm} (57)$$

The latter occurs when $\nu_k = 0$, and thus the non-physical solution. For the former solution, we find that

$$\nu_k = \cos \theta \left( \frac{\int_{k'} f_{kk'} \frac{\partial n_0}{\partial k} \nu_{k'}}{1 + \frac{\eta}{4\alpha\nu_F} \left( \frac{1}{\cos \theta - s} \right) \int_{k'} f_{kk'} \frac{\partial n_0}{\partial k} a_{kk'}^s} \right)$$

$$= \cos \theta \left( \frac{\int_{k'} f_{kk'} \frac{\partial n_0}{\partial k} \nu_{k'}}{1 + \frac{\eta}{4\alpha\nu_F} \left( \frac{1}{\cos \theta - s} \right) \int_{k'} f_{kk'} \frac{\partial n_0}{\partial k} a_{kk'}^s} \right)$$

$$= -\int_{k'} f_{kk'} \frac{\partial n_0}{\partial k} \nu_{k'}$$  \hspace{1cm} (58)$$

This is essentially a linear, homogenous Volterra equation for the function $\nu_k$. We can map this to a linear homogenous second order differential equation, where one of the solutions is an exponential as a function of $k$. The other solution is for $\nu_k = 0$, which is what we see in a conventional Landau-Fermi liquid when $s = 0$. However, because $\nu_k$ experiences a singularity at these point, $\nu_k$ cannot be zero, and thus the Majorana-Fermi surface distortion must increase exponentially at the points identified in the text.

### C Expressions for the Landau-Majorana zero sound and the Pomeranchuck instability condition in the density

The equation describing the zero sound renormalized velocity $s$ when $s >> 1$ is given by Eq. (13). The solutions $s_1$, $s_2$, and $s_3$ are given below:

$$s_1 = \frac{a_0}{3v} \left\{ \frac{F_0^s}{3^{2/3} G^{2/3}} + \frac{G^{(1)} \sqrt{2}}{\sqrt{G^{(2)} + \sqrt{F_0^3 \{(G^{(2)})^2 - 4(G^{(1)})^3\}}} + \sqrt{G^{(2)} + \sqrt{F_0^3 \{(G^{(2)})^2 - 4(G^{(1)})^3\}}} \frac{F_0^s \sqrt{2}}{2^{4/3} F_0^s} \right\}$$  \hspace{1cm} (59a)$$

$$s_2 = \frac{a_0}{3v} \left\{ \frac{F_0^s}{2^{2/3} \sqrt{G^{(2)} + \sqrt{F_0^3 \{(G^{(2)})^2 - 4(G^{(1)})^3\}}} - \frac{(1 + i\sqrt{3})G^{(1)}}{2^{2/3} \sqrt{G^{(2)} + \sqrt{F_0^3 \{(G^{(2)})^2 - 4(G^{(1)})^3\}}} - \frac{(1 - i\sqrt{3})G^{(2)}}{2^{2/3} F_0^s} \frac{F_0^s \sqrt{2}}{2^{4/3} F_0^s} \right\}$$  \hspace{1cm} (59b)$$
where we define

\[ G^{(1)}(\tilde{v}, F_s^0) \equiv G^{(1)} = F_s^0 + \tilde{v}^2 \left( F_s^0 + 1 \right) \]  

(60a)

\[ G^{(2)}(\tilde{v}, F_s^0) \equiv G^{(2)} = 2F_s^0 + \tilde{v}^2 \left( 3F_s^0 + 15F_s^0 + 21F_s^0 + 9F_s^0 \right) \]  

(60b)

By assuming a large magnitude \(|F_s^0|\) of the interaction, we can solve for the Pomeranchuck instability condition by solving for \(\tilde{v}\) when

\[ 4 \left( 9(F_s^0 - \tilde{v} - F_s^0 \tilde{v}) - 9F_s^4 \right)^3 + \left( 54F_s^6 + 81F_s^5 \tilde{v}^2 + 405F_s^4 \tilde{v}^2 + 567F_s^3 \tilde{v}^2 + 243F_s^2 \tilde{v}^2 \right)^2 = 0 \]  

(61)

which will lead to a vanishing imaginary portion of \(s_2\) and \(s_3\). Excluding the trivial case of \(\tilde{v} = 0\), we find that

\[ \tilde{v} = \frac{1}{2} \sqrt{\frac{3}{2}} \sqrt{\frac{3}{2} \sqrt{-2F_s^0} = \frac{\sqrt{3}}{2} \sqrt{-F_s^0}} \]  

(62)

Invoking the form of \(\tilde{v}\) defined in the text, we can then readily solve for the expressions Eqn. (14).