Bohr Almost Periodic Sets of Toral Type

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Abstract

A locally finite multiset (Λ, c), Λ ⊂ R^n, c : Λ → {1, ..., b} defines a Radon measure μ := ∑λ∈Λ c(λ)δ_λ that is Bohr almost periodic in the sense of Favorov if the convolution μ * f is Bohr almost periodic every f ∈ C_c(R^n). If it is of toral type: the Fourier transform $\hat{\mu}$ equals zero outside of a rank $m < \infty$ subgroup, then there exists a compactification $\psi : R^n \to T^m$ of $R^n$, a foliation of $T^m$, and a pair $(K, \kappa)$ where $K := \overline{\psi(\Lambda)}$ and $\kappa$ is a measure supported on $K$ such that $\hat{\kappa} = (\hat{\mu}) \circ \hat{\psi}$ where $\hat{\psi} : \hat{T}^m \to \hat{R}^n$ is the Pontryagin dual of $\psi$. If $(\Lambda, c)$ is uniformly discrete Bohr almost periodic and $c = 1$, we prove that every connected component of $K$ is homeomorphic to $T^{m-1}$ embedded transverse to the foliation and the homotopy of its embedding is a rank $m - n$ subgroup $S$ of $\hat{Z}^m$, and we compute the density of $\Lambda$ as a function of $\psi$ and the homotopy of components of $K$. For $n = 1$ and $K$ a nonsingular real algebraic variety, this construction gives all Fourier quasicrystals (FQ) recently characterized by Olevskii and Ulanovskii and suggest how to characterize FQ for $n > 1$.

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1 Preliminaries

This section defines notation and summarizes basic facts. Section 2 introduces toral compactification concepts illustrated by examples based on the work of Favorov and Kolbasina, Kurasov and Sarnak, Lagarias, Lev, Olevskii and Ulanovskii, and Meyer. Section 3 contains our main result: a detailed description of $m$-dimensional toral compactifications $K \subset T^m$ of Bohr almost periodic Delone subsets $\Lambda \subset R^n$. We prove that the density of $\Lambda$ is determined by the homotopy classes of connected components of $K$ and the spectrum of $\Lambda$ is determined by a measure on $K$. Section 4 formulates research questions.

: means ‘is defined to equal’ and iff means ‘if and only if’. $\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are the natural, integer, rational, real, and complex numbers, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. For $m, n \in \mathbb{N}$ and $A$ an integral domain, $A^{m \times n}$ is the set of $m$ by $n$ matrices over $A$, and $GL(m, A) \subset A^{m \times m}$ is its group of invertible matrices. $SO(n, \mathbb{R})$ is the special orthogonal group. For a subgroup $S \subset \mathbb{Z}^m$, $S_1 := \{k \in \mathbb{Z}^m : ak \in S \text{ for some } a \in \mathbb{N}\}$ is the smallest projective subgroup containing $S$. There exists a unique subgroup $S_2$ with $S_1 \oplus S_2 = \mathbb{Z}^m$. Smith’s normal form theorem [28, 33] implies: (i) card $S_1 / S < \infty$, (ii) if $r := \text{rank } S = m$, then $S_1 = \mathbb{Z}^m$, (iii) if $r < m$, there exists $K_1 \subset \mathbb{Z}^{m \times r}$, $K_2 \subset \mathbb{Z}^{m \times (m-r)}$ with $[K_1, K_2] \subset GL(m, \mathbb{Z})$, $S_1 = K_1 \mathbb{Z}^r$, $S_2 := K_2 \mathbb{Z}^{m-r}$.

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For $n \in \mathbb{N}$, $\mathbb{R}^n$ is an $n$-dimensional Euclidean space consisting of column vectors, $B(0, R) \subset \mathbb{R}^n, R > 0$ is the open ball of radius $R$ with center 0 and $V_n(R) := \frac{a^n}{n!} R^n$ is its volume. We identify $\mathbb{R}^{m \times n}$ with the set of continuous homomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^m$, which equals the set of $\mathbb{R}$–linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. The Stone–Weirestrass theorem ([35], Theorem 1.7) implies that the following limit exists

\[ \lim_{k \to \infty} \frac{\sum_{|\mu| \leq k} \mu \eta}{\sum_{|\mu| \leq k} |\mu|^p} \]

for every $\eta \in \mathbb{R}^n$ and equipped with the appropriate Fréchet space topology. The Fourier transform

\[ \hat{\eta}(y) := \lim_{R \to \infty} \frac{\int_{\mathbb{R}^n} f \chi_{-y}}{V_n(R)}, \quad y \in \mathbb{R}^n \]

and defines a function $\hat{\eta} : \mathbb{R}^n \to \mathbb{C}$ that is nonzero only on a countable set $\Omega(f)$ called the spectrum of $f$. This gives the formal expansion

\[ f \sim \sum_{y \in \Omega(f)} \hat{f}(y) \chi_y. \]

Every $f \in B^2(\mathbb{R}^n)$ satisfies Parseval’s equation $\|f\|_{B^2}^2 = \sum_{y \in \Omega(f)} |\hat{f}(y)|^2$, and every square summable $g : \mathbb{R}^n \to \mathbb{C}$ equals $\hat{f}$ for some $f \in B^2(\mathbb{R}^n)$. $\Omega(f)$ is the smallest set such that $f$ can be approximated by elements in $\mathbb{R}^n$. If the space $C_c(\mathbb{R}^n)$ is equipped with the inductive limit topology ([37], p. 515), then its dual is the space of Radon measures ([37], p. 217–222). The Fourier transform $\mathfrak{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ is defined by $(\mathfrak{F} f)(y) := \int_{\mathbb{R}^n} f \chi_{-y}$. $S(\mathbb{R}^n)$ is the Schwartz space ([33], consisting of smooth functions $f : \mathbb{R}^n \to \mathbb{C}$ all of whose derivatives vanish faster than $(1 + |x|)^{-N}$ for every $N \in \mathbb{N}$ and equipped with the appropriate Fréchet space topology. The Fourier transform $\mathfrak{F} : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ is a bijection. The dual space $S'(\mathbb{R}^n)$ is the Fréchet space of tempered distributions ([37], p. 272, Definition 25.2). The Fourier transform extends to give a bijection $\mathfrak{F} : S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)$

\[ (\mathfrak{F} \eta)(f) := \eta(\mathfrak{F} f), \quad \eta \in S'(\mathbb{R}^n), \ f \in S(\mathbb{R}^n). \]
Remark 2 Lagarius proved ([15], Theorem 1.1) that type. A multiset (Λ is a Meyer set.

\[ \Lambda \]\(\] is uniformly finite if \\[ |\Lambda| \in \mathbb{N} \] and \(c : \Lambda \to \mathbb{C} \) such that \( \mu = \sum_{\lambda \in \Lambda} c(\lambda) \delta_\lambda \), where \( \delta_\lambda \) is the point measure at \( \lambda \). A crystalline measure is an atomic measure \( \mu \) that is also a tempered distribution whose Fourier transform is also an atomic measure, so \( \mathcal{F}(\mu) = \sum_{\omega \in S} a(\omega) \delta_\omega \) is an atomic measure, and both \( \Lambda \) and \( \Omega(\mu) \) have no limit points (locally finite). \( \mu \) is then a Fourier quasicrystal if both \( |\mu| := \sum_{\lambda \in \Lambda} |c(\lambda)| \delta_\lambda \) and \( |\mathcal{F}(\mu)| := \sum_{\omega \in S} |a(\omega)| \delta_\omega \) are tempered distributions.

Remark 1 Meyer proved that crystalline measures are Bohr almost periodic tempered distributions ([23], Lemma 2), but not necessarily Bohr almost periodic measures ([23], Theorems 4 and 5).

If \( \mu \) is an atomic measure and \( \mu * C_c(\mathbb{R}^n) \subset B^1(\mathbb{R}^n) \) then the limit

\[
\hat{\mu}(y) := \lim_{R \to \infty} \frac{\sum_{\lambda \in \Lambda \cap B(0,R)} c(\lambda) \chi_{-y}(\lambda)}{V_n(R)}, \quad y \in \mathbb{R}^n
\]

exists and equals 0 except at a countable set called the spectrum \( \Omega(\mu) \) of \( \mu \). This gives the following formal expansions:

\[
\mu \sim \sum_{y \in \Omega(\mu)} \hat{\mu}(y) \chi_y, \quad \mathcal{F} \mu \sim \sum_{y \in \Omega(\mu)} \hat{\mu}(y) \delta_y.
\]

A multiset is a pair \((\Lambda, c)\), where \( \Lambda \subset \mathbb{R}^n \) is nonempty and \( c : \mathbb{R}^n \to \mathbb{N} \) satisfies \( c(\lambda) > 0 \) iff \( \lambda \in \Lambda \). They form a commutative semigroup under the binary operation \((\Lambda_1, c_1) + (\Lambda_2, c_2) := (\Lambda, c)\) where \( \Lambda := \Lambda_1 \cup \Lambda_2 \) and \( c := c_1 + c_2 \). \( \Lambda \) is a multiset \((\Lambda, 1)\) where \( \Lambda \) is a translate of a discrete rank \( n \) subgroup of \( \mathbb{R}^n \). A multiset is trivial if it is the sum of lattices.

\[
R_1(\Lambda) := \inf \{|\lambda_2 - \lambda_1| : \lambda_1, \lambda_2 \in \Lambda, \lambda_1 \neq \lambda_2\} \in [0, \infty),
\]

\[
R_2(\Lambda) := \inf \{R \in [0, \infty] : \bigcup_{\lambda \in \Lambda} B(\lambda, R) = \mathbb{R}^n\},
\]

\[
b(\Lambda, c) := \lim_{R \to 0} \sup_{x \in \mathbb{R}^n} \sum_{\lambda \in \Lambda \cap B(x, R)} c(\lambda) \in \mathbb{N} \cup \{\infty\},
\]

Definition 1 A Radon measure \( \mu \) on \( \mathbb{R}^n \) is Bohr; Besicovitch almost periodic if convolution with it satisfies \( \mu * C_c(\mathbb{R}^n) \subset U(\mathbb{R}^n); B^p(\mathbb{R}^n) \). A tempered distribution \( \mu \) on \( \mathbb{R}^n \) is Bohr; Besicovitch almost periodic if \( \mu * S(\mathbb{R}^n) \subset U(\mathbb{R}^n); B^p(\mathbb{R}^n) \).

Definition 2 \( \Lambda \subset \mathbb{R}^n \) is uniformly discrete; syndetic if \( R_1(\Lambda) > 0; R_2(\Lambda) < \infty \). Delone (aka Delaunay) ([7]) if it is uniformly discrete and syndetic, finite type if there exists a finite \( F \subset \mathbb{R}^n \) with \( \Lambda - \Lambda \subset \Lambda + F \), quasiregular ([12]) if both \( \Lambda \) and \( \Lambda - \Lambda \) are Delone, and a Meyer set ([27]) if it is Delone and finite type. A multiset \((\Lambda, c)\) is uniformly finite if \( b(\Lambda, c) < \infty \).
A uniformly finite multiset \((\Lambda, c)\) defines a Radon measure
\[
\mu(\Lambda, c) := \sum_{\lambda \in \Lambda} c(\lambda) \delta_\lambda.
\]
(10)

Then \(\mu := \mu(\Lambda, c)\) defines, by convolution, the map \(\mu* : C_c(\mathbb{R}^n) \to C_b(\mathbb{R}^n)\)
\[
(\mu * f)(x) = \sum_{\lambda \in \Lambda} c(\lambda) f(x - \lambda), \quad f \in C_c(\mathbb{R}^n), \ x \in \mathbb{R}^n
\]
(11)
\(\mu\) is also a tempered distribution and \(\mathfrak{F}(\mu * f) = (\mathfrak{F} \mu)(\mathfrak{F} f), \ f \in C_c(\mathbb{R}^n)\).

**Definition 3** A uniformly finite multiset \((\Lambda, c)\) is Bohr; Besicovitch almost periodic if \(\mu * C_c(\mathbb{R}^n) \subset U(\mathbb{R}^n); B^p(\mathbb{R}^n)\). We define it to be a Fourier quasicrystal if \(\mu := \mu(\Lambda, c)\) is a quasicrystal.

If \((\Lambda, c)\) is trivial, then for every \(f \in C_c(\mathbb{R}^n), \mu * f\) is a sum or periodic functions hence \((\Lambda, c)\) is Bohr almost periodic.

**Remark 3** Lagarias ([16], Problem 4.4) asked if every Bohr almost periodic multiset is trivial. Favorov answered no by proving that every Bohr almost periodic perturbation of a lattice is Bohr almost periodic [18, 19]. He also proved that every Bohr almost periodic Meyer set is trivial [10], [11].

The Poisson summation formula ([35], Theorem 2.4) implies that trivial multisets are Fourier quasicrystals. Olevskii and Ulanovskii extended work with Lev [19, 20, 21] and work of Meyer [24, 25] and Kurasov and Sarnak [14] by proving the following deep result ([30], Theorem 8):

**Theorem 1** For \(n = 1\), \((\Lambda, c)\) is a Fourier quasicrystal iff there exists a trigonometric polynomial whose extension to an entire function only has real roots \(\Lambda\) with multiplicities \(c\).

For a trigonometric polynomial \(p\) let \(Z_r(p)\) be its set of (real) roots, \(Z_c(p)\) the set of complex roots of its entire extension \(P\), and \(\rho_r(p); \rho_c(p)\) their densities (counted with multiplicities). The following result characterizes trigonometric polynomials whose entire extensions have only real roots.

**Theorem 2** Let \(p(x) := \sum_{j=1}^d c(d) x_{y_j}, \ d \geq 2, \ y_1 < \cdots < y_d, \ c(j) \in \mathbb{C}\setminus\{0\},\) be a trigonometric polynomial. Then \(Z_r(p) = Z_c(p)\) iff \(\rho_r(p) = \rho_c(p)\). Furthermore, \(\rho_r(p) = y_d - y_1\).

**Proof 1** Clearly \(Z_r(p) \subset Z_c(p)\) so \(\rho_r(p) \leq \rho_c(p)\). If \(P\) had a nonreal root \(z\) then there would exists a circle \(C\) centered at \(z\) such that \(C \cap \mathbb{R} = \phi\), \(P\) has only the root \(z\) inside \(C\), \(|P|\) has a minimum value \(\epsilon > 0\) on \(C\), and the winding number of \(P\) on \(C\) equals the multiplicity of the root \(z\). Choose \(B > 0\) such that \(|y| < B\) for all \(x + iy \in C\). Since \(P(x + iy)\) is almost periodic in \(x\) uniformly for \(|y| \leq B\), there exists a syndetic set \(S \subset \mathbb{R}\) such that \(|P(x + iy) - P(x + s + iy)| < \epsilon\) for all \(x + iy \in C\) and \(s \in S\). By Rouche’s theorem \(P\) has a root inside the circle \(C + s\) for every \(s \in S\). Since \(S\) is syndetic, the density of non real zeros is \(> 0\) hence \(Z_r(p) > \rho(Z(p))\). This implies the first assertion. The second assertion is a classic result by Titchmarsh ([36], Theorem 1).
2 Toral Compactifications of $\mathbb{R}^n$ and $(\Lambda, c)$

Let $\mathbb{R}^n_\mathbb{R}$ be the group $\mathbb{R}^n$ equipped with the discrete topology. Its Pontryagin dual is the Bohr compactification $B(\mathbb{R}^n)$ (31, 1.8). Bohr showed that $U(\mathbb{R}^n)$ corresponds to $C(B(\mathbb{R}^n))$ [5, 6], and Besicovitch showed that $B^p(B(\mathbb{R}^n))$ corresponds to $L^p(B(\mathbb{R}^n))$ [3, 4]. The Bohr compactification is very large, in particular, it is not first countable. Many almost periodic functions and related concepts can be studied using smaller compactifications. These include trigonometric polynomials and the model sets pioneered by Meyer [22, 23] in the context of harmonic analysis and number theory and later discovered empirically as a class of quasicrystals by Shechtman and colleagues [32].

**Definition 4** A toral compactification of dimension $m$ of $\mathbb{R}^n$ is a pair $(T^m, \psi)$ where $\psi : \mathbb{R}^n \to T^m$ is a continuous homomorphism with dense image.

A toral compactification defines an ergodic action of $\mathbb{R}^n$ on $T^m x(z) := \psi(x)z$, $x \in \mathbb{R}^n, z \in T^m$ and an associated foliation whose leaves are the orbits of this action.

**Lemma 1** The toral compactifications of dimension $m$ of $\mathbb{R}^n$ coincide with the set of pairs $(T^m, \psi)$ where $\psi = \rho_m \circ M$ (composition) and $M \in \mathbb{R}^{m \times n}$ has rows that are linearly independent over $\mathbb{Q}$.

**Proof 2** Since $\rho_m : \mathbb{R}^m \to T^m$ is a covering space, $\mathbb{R}^n$ is path, locally path and simply connected, the lifting property ([13], Theorem 1.55) implies that there exists a unique continuous map $h : \mathbb{R}^n \to \mathbb{R}^m$ such that $\psi = \rho_m \circ h$ and $h(0) = 0$. Define $\eta : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ by $\eta(x, y) = h(x + y) - h(x) - h(y)$. Since $\psi$ and $\rho_m$ are homomorphisms, $\rho_m \circ \eta = 1$ hence the image of $\eta$ is a subset of $\mathbb{Z}^m = \ker \rho_m$. Since $\eta$ is continuous and its domain is connected, its image is connected. Since $\eta(0, 0) = 0$, $\eta = 0$, $h$ is a homomorphism so there exists $M \in \mathbb{R}^{m \times n}$ such that $h = M$, hence $\psi = \rho_m \circ M$. Then $\overline{\psi(\mathbb{R}^n)} \neq T^m$ iff there exists $k \in \mathbb{Z}^m \setminus \{0\}$ with $1 = \zeta_k \circ \psi = \chi_{M^k} \Leftrightarrow M^k 0 = 0$ iff the rows of $M$ are not linearly indepenndent over $\mathbb{Q}$.

Define $\Psi : C(T^m) \to C_b(\mathbb{R}^n)$ by $\Psi g := g \circ \psi$, and $\Omega_{\psi} := M^T \mathbb{Z}^m$.

**Lemma 2** $\Psi T(T^m) = \text{span} \{ \chi_x : x \in \Omega_{\psi} \}$ and $\Psi$ is an isometry. $\Psi C(T^m) = \{ f \in U(\mathbb{R}^n) : \Omega(f) \subset \Omega_{\psi} \}$. For $p \in [1, \infty)$, $\Psi$ extends to a linear isometry $\Psi : L^p(T^m) \to B^{p}(\mathbb{R}^n)$ and $\Psi L^p(T^m) = \{ f \in B^p(\mathbb{R}^n) : \Omega(f) \subset \Omega_{\psi} \}$.

**Proof 3** The first assertion is obvious, the second follows since $\psi$ has a dense image. If $g \in C(T^m)$, the Stone–Weierstrass theorem implies there exists a sequence $P_n \in T(T^m)$ with $||P_n - g|| \to 0$. Define $Q_n = \Psi P_n$ and $f := \Psi g$. Then $Q_n \in \text{span} \{ \chi_x : x \in G_{\psi} \}$ and since $\Psi$ is an isometry, $||Q_n - f|| \to 0$ so $f \in U(\mathbb{R}^n)$ and $\Omega(f) \subset G_{\psi}$. Conversely, if $f \in U(\mathbb{R}^n)$ and $\Omega(f) \subset G_{\psi}$ then there exists $Q_n \in \text{span} \{ \chi_x : x \in G_{\psi} \}$ with $||Q_n - f|| \to 0$. Then there exists a sequence $P_n \in T(T^m)$ with $\Psi P_n = Q_n$. Since $Q_n$ is a Cauchy sequence and $\Psi$ is a isometry, $P_n$ is a Cauchy sequence in the complete metric space $C(T^m)$ so there exists $g \in C(T^m)$ such that $||P_n - g|| \to 0$. Therefore $f = \Psi g$. The second equality is proved in a similar way using the $|| ||_{B, p}$ norm.

**Definition 5** A multiset $(\Lambda, c)$ is of toral type if it is Besicovitch almost periodic and $\Omega(\mu)$ generates a rank $m < \infty$ subgroup of $\mathbb{R}^n$.
Lemma 2 implies that \((\Lambda, c)\) is of toral type iff there exists a compactification \((\mathbb{T}^m, \psi)\) with \(\mu \ast C_c(\mathbb{R}^n) \subset \Psi L^p(\mathbb{T}^m)\). Then define \(\Theta : C_c(\mathbb{R}^n) \to L^p(\mathbb{T}^m)\) so
\[
\Psi \circ \Theta = \mu \ast : C_c(\mathbb{R}^n) \to B^p(\mathbb{R}^n),
\]
and observe that \((\Lambda, c)\) is Bohr almost periodic iff \(\Theta C_c(\mathbb{R}^n) \subset C(\mathbb{T}^m)\).

**Lemma 3** If a multiset \((\Lambda, c)\) is Bohr almost periodic of toral type and \(\Lambda\) is Delone then for every \(\lambda_1 \in \Lambda\) and \(\gamma \in \text{kernel} \psi\), there exists \(\lambda_2 \in \Lambda\) with \(c(\lambda_2) = c(\lambda_1)\) and \(\lambda_1 + \gamma = \lambda_2\). Therefore \(r := \text{rank kernel} \psi \leq n\) and rank \(M = n\). If \(r = n\) then \(\Lambda\) is trivial.

**Proof 4** Let \(f \in C_c(\mathbb{R}^n)\) be a nonzero functions with support \(f \subset B(0, R_1(\Lambda)/2)\). Then since \(\mu \ast f = (\Theta f) \circ \psi\), the function \(\mu \ast f\) is invariant under translation by elements in kernel \(\psi\). Therefore for every \(\gamma \in \text{kernel} \psi\),
\[
\sum_{\lambda \in \Lambda} c(\lambda) f(x + \lambda + \gamma) = \sum_{\lambda \in \Lambda} c(\lambda) f(x + \lambda), \quad x \in \mathbb{R}^n.
\]
Since the supports of the summands on both sides of this equation are disjoint, for every \(\lambda_1 \in \Lambda\) there exists \(\lambda_2 \in \Lambda\) such that
\[
c(\lambda_1) f(x + \lambda_1 + \gamma) = c(\lambda_2) f(x + \lambda_2), \quad x \in \mathbb{R}^n.
\]
This proves the first assertion. Since \(\Lambda\) is discrete, kernel \(\psi\) is discrete hence \(r \leq n\) and rank \(M = n\). If \(r = n\) then \(\Lambda\) consists of orbits in \(\Lambda\) of kernel \(\psi\). Since \(\Lambda\) is discrete there are a finite number of orbits so \(\Lambda\) is trivial.

**Definition 6** Let \((\Lambda, c)\) be a Besicovitch almost periodic multiset of toral type with associated compactification \((\mathbb{T}^m, \psi)\) of \(\mathbb{R}^n\). The compactification of \((\Lambda, c)\) is the pair \((K, \kappa)\) where
\[
K := \overline{\psi(\Lambda)}
\]
is the closure of \(\psi(\Lambda)\), and \(\kappa : C(\mathbb{T}^m) \to \mathbb{C}\) is the measure defined by
\[
\kappa g := \lim_{R \to \infty} \frac{\sum_{\lambda \in \Lambda \cap B(0, R)} c(\lambda) g(\psi(\lambda))}{V_n(R)}, \quad g \in C(\mathbb{T}^m).
\]
The existence of the limit in Equation [14] follows from Equation [2] with \(y = 0\) as follows. Let \(h_n \in C_c(\mathbb{R}^n)\) be an approximate identity sequence and \(\mu = \sum_{\lambda \in \Lambda} c(\lambda) \delta_\lambda\) be the Radon measure associated with the Besicovitch almost periodic multiset \((\Lambda, c)\). Then \(\mu \ast h_n \in B^1(\mathbb{R}^n)\) and \(g \circ \psi \in U(\mathbb{R}^n)\) so the product \(g_n := (\mu \ast h_n)(g \circ \psi) \in B^1(\mathbb{R}^n)\) hence \(\hat{g}_n(0)\) exists. Since \(g \circ \psi\) is uniformly continuous \(\lim_{n \to \infty} \hat{g}_n(0) = \kappa g\). Furthermore since \(\|\mu \ast h_n\|_{B^1}\) are uniformly bounded by some constant \(C > 0\), it follows that \(|\hat{g}_n(0)| \leq C|g \circ \psi| = C\max_{z \in \mathbb{T}^m} |g(z)|\) hence \(|\kappa g| \leq C\max_{z \in \mathbb{T}^m} |g(z)|\). Therefore the Riesz-Markov-Kakutani theorem implies that \(\kappa\) is a Borel measure on \(\mathbb{T}^m\). Clearly \(\kappa\) is positive, supported on \(K\), and satisfies
\[
\kappa \zeta_k = \hat{\mu}(M^T k), \quad k \in \mathbb{Z}^m.
\]
Let \(\sigma_1\) be Lebesgue measure on \(\mathbb{R}^n\) and \(\sigma_2\) be Haar measure on \(\mathbb{T}^m\). Then \(f \in C_c(\mathbb{R}^n) \implies f \sigma_1\) is a measure on \(\mathbb{R}^n\) hence \(\psi\) induces the measure \(\psi^*(f \sigma_1)\) on \(\mathbb{T}^m\) and
\[
\psi^*(f \sigma_1) \ast \kappa = (\Theta f) \sigma_2.
\]
Lemma 3 implies that $MTM \in \mathbb{R}^{n \times n}$ is symmetric and positive definite. Therefore there exists $O \in SO(n, \mathbb{R})$ with $(MO)^TMO = D^2$ where $D$ is a diagonal matrix with positive diagonal entries, so $MOD^{-1}$ has orthonormal columns. Henceforth we assume that a change of variables has been made that transforms a multiset ($\Lambda, c$) into $(DO^T\Lambda, c \circ OD^{-1})$ having the same properties described in Definitions 2 and 3, and transforms $M$ to satisfy $MTM = I_n$ (the $n$ by $n$ identity matrix). Construct $N \in \mathbb{R}^{m \times (m-n)}$ such that $[MN] \in SO(m, \mathbb{R})$, the $(m-n)$-form

$$\xi_N := (2\pi i)^{-(m-n)} \sum_{1 \leq i_1, \ldots, i_{m-n} \leq m} N_{i_1,1} \cdots N_{i_{m-n},m-n} \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_{m-n}}}{z_{i_{m-n}}},$$

(17)

and the $n$-form $\xi_M; \xi_N$ is determined by $MM^T; NN^T$. Furthermore $\xi_M \wedge \xi_N =$ volume form on $\mathbb{T}^m$.

Lemma 4 If $K$ is an $(m-n)$-dimensional submanifold of $\mathbb{R}^m$, then $\kappa = |\kappa_c|$, (the total variation measure), where $\kappa_c$ is the complex Borel measure

$$\kappa_c g := \int_K g \xi_N, \quad g \in C(\mathbb{T}^m),$$

(18)

where the connected components of $K$ are given arbitrary orientations ($\kappa_c$ depends on the orientations but $\kappa$ does not).

Proof 5 Follows from Definition 7 and the ergodic action of $\mathbb{R}^n$ on $\mathbb{T}^m$.

For Laurent polynomials $P_j : (\mathbb{C} \setminus \{0\})^m \to \mathbb{C}, j = 1, \ldots, q$ define

$$Z_c(P_1, \ldots, P_q) := \{ z \in (\mathbb{C} \setminus \{0\})^m : P_j(z) = 0, \ j = 1, \ldots, q \},$$

and $Z_c(P_1, \ldots, P_q) = Z_c(P_1, \ldots, P_q) \cap \mathbb{T}^m$.

Remark 4 If $K = Z_c(P_1, \ldots, P_q)$ then $\hat{\kappa} : \mathbb{Z}^m \to \mathbb{C}$ satisfies a system of $q$-linear difference equations having constant coefficients.

Let $D := \{ z \in \mathbb{C} : |z| < 1 \}$. A Laurent polynomial is self-dual if $P(z^{-1})/P(z)$ is a monomial, and stable if $D^m \cap Z(P) = \emptyset$. Lee-Yang polynomials are self-dual and stable [18]. We describe three toral compactifications where $n = 1, m = 2, \theta \in [-\pi, \pi], \ M = [\cos \theta, \sin \theta]^T$, and $\psi := \rho_2 \circ M$.

Example 1 This example belongs to a class of Fourier quasicrystals constructed by Meyer ([20], Definition 6) using Ahern measures [19]. Let $P$ be the Lee-Yang polynomial $2z_1z_2 + z_1 + z_2 + 2$ and $\Lambda = \psi^{-1}(Z_c(P))$. Then

$$K := \mathcal{K} = Z_c(P) = \left\{ [z_1, z_2]^T \in \mathbb{T}^2 : z_1 \in \mathbb{T}, \ z_2 = \frac{-2 + z_1}{1 + 2z_1} \right\}$$

is homeomorphic to $\mathbb{T}$, and $\Lambda$ is the set of roots of the trigonometric polynomial $T = 2\chi_{\cos \theta + \sin \theta} + \chi_{\cos \theta} + \chi_{\sin \theta} + 2$, which is the set of real roots of its entire extension. The density of complex zeros of its entire extension equals $\max\{\cos \theta + \sin \theta, \cos \theta, \sin \theta, 0\} - \min\{\cos \theta + \sin \theta, \cos \theta, \sin \theta, 0\}$. This equals $|\cos \theta + \sin \theta|$ if $\cos \theta$ and $\sin \theta$ have the same sign, and equals $\max\{|\cos \theta|, |\sin \theta|\}$ otherwise. The density of $\Lambda$ equals $|\cos \theta + \sin \theta|$, therefore Theorems 1 and 2 imply that $\Lambda$ is a quasicrystal iff $\tan \theta \in [0, \infty]$. We now derive this conclusion by computing the Fourier transform of $\kappa$. Let
z_j = e^{i\theta_j}, j = 1, 2, and orient Z_r(P) by \theta_1 increasing from 0 to 2\pi. Then \frac{d\theta_1}{dt} = -z_2^{-1} for \tan \theta \in [-3, -1/3], Z_r(P) is transverse to the foliation, \xi_N = (-\sin \theta) d\theta_1 + (\cos \theta) d\theta_2, so Equation 16 implies

\nu([k_1, k_2] M) = \kappa \zeta_{-[k_1, k_2]'} = -\int_K z_1^{-k_1} z_2^{-k_2} \xi_N = 0

whenever k_1 k_2 < 0. This implies that \Omega(\Lambda) = projection of the support of \hat{\kappa} onto the line M \Re is discrete whenever tan \theta \in (0, \infty) \setminus \Q, and then \Lambda is a FQ. Note that \Lambda is trivial iff \tan \theta \in \Q \cup \{\infty\}. \hat{\kappa} satisfies the difference equation

2\hat{\kappa}([k_1, k_2]^T) + \hat{\kappa}([k_1 - 1, k_2]^T) + \hat{\kappa}([k_1, k_2 - 1]^T) + 2\hat{\kappa}([k_1 - 1, k_2 - 1]^T) = 0. \hspace{1cm} (19)

**Example 2** This example includes as a special case one constructed by Olevskii and Ulamsonovskii (22), Example 1) Let P(z_1, z_2) := z_1 - z_1^{-1} - \delta(z_2 - z_2^{-1}) Note that Z_r(P) has two connected components that approach the circles z_1 = 1 and z_1 = -1 as \delta \to 0. Let \Lambda := \psi^{-1}(Z_r(P)). Then K := \Lambda = Z_r(P), and \Lambda is the set of roots of the trigonometric polynomial T := \chi_{\cos \theta} - \chi_{-\cos \theta} - \delta(\chi_{\sin \theta} - \chi_{-\sin \theta}). For |tan \theta| < 1/\delta, the foliation is transverse to K and the density of \Lambda equals \frac{2}{|\cos \theta|}. A more detailed geometric analysis shows that if \tan \theta \geq 1/\delta then the density is less than \frac{2}{|\sin \theta|}. The density of complex zeros of the entire extension T equals 2 max{\{\cos \theta\}, |\sin \theta|}. Therefore Theorems 1 and 2 imply that \Lambda is a FQ iff \tan \theta \in (0, 1) \setminus \Q. Equation 17 implies

\nu([k_1, k_2] M) = \kappa \zeta_{-[k_1, k_2]'} = -\int_K z_1^{-k_1} z_2^{-k_2} \xi_N = 0

whenever k_2 > -k_1 \geq 1 or k_2 < -k_1 \leq 0. This implies that \Omega(\Lambda) = projection of the support of \hat{\kappa} onto the line M \Re is discrete whenever tan \theta \in (0, 1) \setminus \Q, and then \Lambda is a FQ. \hat{\kappa} satisfies the difference equation

\hat{\kappa}([k_1 +, k_2]^T) - \hat{\kappa}([k_1 - 1, k_2]^T) - \delta \hat{\kappa}([k_1, k_2 + 1]^T) + \delta \hat{\kappa}([k_1, k_2 - 1]^T) = 0. \hspace{1cm} (20)

**Example 3** Let tan \theta \notin \Q \cup \{\infty\} and \ell > 0. Then

\Lambda := \{m \cos \theta + n \sin \theta : m, n \in \Z, |m \sin \theta - n \cos \theta| < \ell/2\}

is a cut and project set. It is a Meyer subset of \Re and nontrivial so Remark 3 implies that it is not Bohr almost periodic. K = \rho_2 (\{t[-\sin \theta, \cos \theta]^T : t \in (-\ell/2, \ell/2)\}) and

\kappa g = \int_{-\ell/2}^{\ell/2} g \circ \rho_2 (t[-\sin \theta, \cos \theta]^T) dt, \hspace{1cm} g \in C (\T^2),

so Equation 16 implies that for every f \in C_c (\Re), \Theta f \in \L^p (\T^2) so \mu \ast f \in \B^p (\Re) and \Lambda is Besicovitch almost periodic. But \Theta f has discontinuities on leaves of the foliation that intersect the boundary of K so \mu \ast f \notin U (\Re). \kappa \zeta_k = \text{sinc} (\pi \ell [-\sin \theta \cos \theta] k), k \in \Z^2 hence \Omega(\mu) = \cos \theta \Z + \sin \theta \Z is a dense subset of \Re so \Lambda is not a Fourier quasicrystal. We observe that if \alpha := \frac{1 + \sqrt{5}}{2}, \tan \theta = \alpha, and \Lambda := [0, 1; 1, 1] \in GL(2, \Z), then

\psi(\alpha x) = A \psi(x), \hspace{1cm} x \in \Re,

therefore \alpha \Lambda \subset \Lambda, so \Lambda is quasicrystal which admits a dilation.
3 Homotopy Class of $K$ and Density of $\Lambda$.

Assume that $\Lambda \subset \mathbb{R}^n$ is a Delone Bohr almost periodic set of toral type with associated compactification $(\mathbb{T}^n, \psi)$ of $\mathbb{R}^n$ and compactification $(K, \kappa)$ of $\Lambda$ ($c = 1$). We assume that a change of variables has been made so that $M^T M = I_n$ where $\psi = \rho_m \circ M$. If $m = n$ then $\Lambda$ is trivial, $K$ is finite, and $\kappa$ is counting measure on $K$. We henceforth assume that $m > n$. We will derive a precise description of $K$.

Define $H := \rho_m^{-1}(K)$, and $V := M \mathbb{R}^n$, so $M \Lambda + \mathbb{Z}^m = H$ and $V + \mathbb{Z}^m = \mathbb{R}^m$. Then $J_1 := M(\text{kernel } \psi) = V \cap \mathbb{Z}^m$ is a rank $r$ projective subgroup of $\mathbb{Z}^m$ so there exists a rank $m - r$ subgroup $J_2$ with $J_1 \oplus J_2 = \mathbb{Z}^m$. Therefore

$$M \Lambda + \mathbb{Z}^m = \bigcup_{j \in J_2} (M \Lambda + j), \quad V + \mathbb{Z}^m = \bigcup_{j \in J_2} (V + j) \quad (22)$$

where the unions are disjoint. $M \Lambda \subset V$ is Delone since $R_j(M \Lambda) = R_1(\Lambda)$, $j = 1, 2$. Define $\mu$, $\Psi$ and $\Theta$ as in Section 2. Let $U$ be the orthogonal complement of $V$ so $U \oplus V = \mathbb{R}^m$ and let $P : \mathbb{R}^M \to U$ and $Q : \mathbb{R}^m \to V$ be orthogonal projections.

Construct $\varphi : V + \mathbb{Z}^m \to M \Lambda + \mathbb{Z}^m$ such that

$$\varphi(v) \in M \Lambda, \quad v \in V, \quad (23)$$

and

$$||v - \varphi(v)|| = \min\{||x - p|| : p \in M \Lambda\}, \quad v \in V. \quad (24)$$

Use the fact that $V + \mathbb{Z}^m$ is the disjoint union of $V + j, j \in J_2$ to extend the domain of $\varphi$ from $V$ to $V + \mathbb{Z}^m$ by defining

$$\varphi(v + j) = \varphi(v) + j, \quad v \in V, \quad j \in J_2. \quad (25)$$

For $j \in J_2$, $M \Lambda + j \subset V + j$ is Delone so gives a Voronoï tesselation \[39\] of $V + j$ and $\varphi$ maps the open cell containing $p \in M \Lambda + j$ to $p$.

**Lemma 5** $j \in J_2, x \in V + j, p \in M \Lambda + j, \quad ||x - p|| < \frac{R_1(M \Lambda)}{2} \implies \varphi(x) = p.$

**Proof 6** If $q \in M \Lambda + j$ and $||x - q|| < \frac{R_1(M \Lambda)}{2}$ then $||p - q|| < R_1(M \Lambda)$, hence $p = q$ and $p = \varphi(x)$.

Choose $R < R_1(M \Lambda)/2$, construct $g \in C_c(V)$ by

$$g(v) := \begin{cases} 0, & \text{if } ||v|| \geq R \\ 1 - R^{-1}||v||, & \text{if } ||v|| \leq R, \end{cases} \quad (26)$$

and define $f := g \circ M$, $G := (\Theta f) \circ \rho_m$, and $O := \{x \in \mathbb{R}^m : G(x) > 0\}$. Clearly $G : \mathbb{R}^m \to [0, 1]$ is uniformly continuous, $G^{-1}(\{1\}) = H$, and the restriction $G|_V = M \Lambda g$.

**Lemma 6** If $x \in O \cap (V + \mathbb{Z}^m)$, then

$$||x - \varphi(x)|| = R(1 - G(x)). \quad (27)$$
Proof 7 Since \( x = v + j \) for unique \( v \in V \) and \( j \in J_2 \), \( G(x) = G(v) = \sum_{p \in MA} g(v - p) \), so there exists a unique \( p \in MA \) such that \( G(v) = g(v - p) \). Since \( g(v - p) > 0, ||v - p|| < R \) hence \( p = \varphi(v) \) and \( G(x) = g(v - p) = 1 - R^{-1} ||v - p|| \). Therefore \( ||x - \varphi(v)|| = ||v - \varphi(v)|| = ||v - p|| = R(1 - G(x)) \).

The modulus of continuity \( \omega : (0, \infty) \to [0, \infty] \) of \( G \), defined by

\[
\omega(t) := \sum \{|G(x) - G(y)| : ||x - y|| < t\},
\]

satisfies \( \lim_{\delta \to 0} \omega(\delta) = 0 \) since \( G \) is uniformly continuous.

Lemma 7 \( \varphi : O \cap (V + \mathbb{Z}^m) \to MA + \mathbb{Z}^m \) is uniformly continuous.

Proof 8 For \( i = 1, 2 \) assume that \( x_i = v_i + j_i \in O \) and where \( v_i \in V \) and \( j_i \in J_2 \) and let \( \Delta \subset U \) satisfy \( V + j_i + \Delta = V + j_2 \). Since \( ||\Delta|| \leq ||x_2 - x_1|| \) and \( ||x_2 - x_1 - \Delta|| \leq ||x_2 - x_1|| \), Lemma 5 implies that

\[
||x_2 - (\varphi(x_1) + \Delta)|| \leq ||x_1 - \varphi(x_1)|| + ||x_2 - x_1 - \Delta|| \leq R(1 - G(x_1)) + ||x_2 - x_1||.
\]

If \( \omega(||x_2 - x_1||) < 1 \), then \( ||\varphi(x_1) + \Delta - \varphi(x_1) + \Delta|| \leq R \omega(||x_2 - x_1||) \), therefore

\[
||x_2 - \varphi(x_1 + \Delta)|| \leq R(1 - G(x_1)) + ||x_2 - x_1|| + R \omega(||\Delta||).
\]

Lemma 4 implies that if \( ||x_2 - x_1|| + \omega(||x_2 - x_1||) \leq R_1(\text{MA})/2 - R \) then

\[
\varphi(\varphi(x_1) + \Delta) = \varphi(x_2),
\]

hence

\[
||\varphi(x_2) - \varphi(x_1)|| < R \omega(||x_2 - x_1||) + ||x_2 - x_1||,
\]

which concludes the proof.

Lemma 8 \( \varphi : O \to MA + \mathbb{Z}^m \) extends to a continuous function \( \Phi : \overline{O} \to H \). For every \( U_v \subset U \) with \( v + U_v \subset O \) there exists a continuous \( h_v : U_v \to V \) such that

\[
\Phi(v + u) = h_v(u) + u, \quad u \in U_v.
\]

Proof 9 The first assertion follows since \( \varphi \) is uniformly continuous on \( O \cap (V + \mathbb{Z}^m) \) which is dense in \( \overline{O} \). Assume that \( U_v \subset U \) satisfies \( v + U_v \subset O \) and define \( h_v(u) := \Phi(v + u) - u \) for \( u \in U_v \) so Equation 29 holds. To prove \( h_v(U_v) \subset V \), let \( u \in U_v \) and use the fact that \( P_{J_2} \) is dense in \( U \) choose a sequence \( j_n \in J_2 \) with \( v + j_n \in O \) and \( \lim_{n \to \infty} P_{J_2} j_n = u \). Then

\[
\Phi(v + u) = \lim_{n \to \infty} \varphi(v + P_{J_2} j_n) = \lim_{n \to \infty} (\varphi(v - Q_{J_2} j_n) + Q_{J_2} j_n) + u.
\]

Since \( \varphi(v - Q_{J_2} j_n) + Q_{J_2} j_n \in V \) and \( V \) is closed, \( h_v(u) = \lim_{n \to \infty} (\varphi(v - Q_{J_2} j_n) + Q_{J_2} j_n) \in V \).

Let \( K_c \) be a connected component of \( K \), \( H_c \) be a connected component of \( \rho_m^{-1}(K_c) \), and \( \Lambda_c = \psi^{-1}(K_c) \).

Theorem 3 There exists a continuous \( h_1 : U \to V \) such that the map \( u \to h_1(u) + u \) is a homeomorphism of \( U \) onto \( H_c \).
Proof 10 Clearly $H_c \subset O$. Choose $u_1 \in U$ and $v_1 \in V$ such that $u_1 + v_1 \in H_c$. Choose an open bounded $U_1 \subset U$ containing $u_1$ such that $v_1 + U_1 \subset O$ and $\Phi(v_1 + U_1) \subset H_c$. This is possible since $H_c$ is a connected component of $H$. Then define $h_1 : \overline{U_1} \to V \cap H_c$ by

$$h_1(u) := \Phi(v_1 + u) - u, \quad u \in \overline{U_1}.$$  \hspace{1cm} (31)

Lemma 7 implies that $h_1(\overline{U_1}) \subset V$ so the map $u \to h(u) + u$ is one-to-one and continuous from the compact set $\overline{U_1}$ onto its image hence is a homeomorphism. Since the point $u_1 + v_1$ was an arbitrary, $H_c$ is a manifold. The function $h_1$ can be extended to give a homeomorphism $h_1 : U \to H_c$ to conclude the proof.

The inclusion map $i : K_c \to \mathbb{T}^m$ induces a map on the fundamental groups $\pi_1(i) : \pi_1(K_c) \to \pi_1(\mathbb{T}^m) = \mathbb{Z}^m$.

Theorem 4 $K_c$ is homeomorphic to $\mathbb{T}^{m-n}$ and $\pi_1(i)$ is injective.

Proof 11 Define $S := \{ k \in \mathbb{Z}^m : H_c + k = H_c \}$. Then $S$ is a subgroup of $\mathbb{Z}^m$ and it acts freely on $H_c$. If $x, y \in H_c$ then $\rho_m(x) = \rho_m(y)$ iff there exists $k \in \mathbb{Z}^m$ such that $y = x + k$. Since $H_c$ is connected, $k \in S$. Therefore $\pi_1(K_c) = S$, $\pi_1(i)$ is the inclusion map hence is injective. $K_c$ is homeomorphic to the space of orbits of the action of $S$ on $H_c$. Since $H_c$ is homeomorphic to $U$, it has dimension $m - n$. Since $K_c$ is compact and rank $S = m - n$, $K_c$ is homeomorphic to $\mathbb{T}^{m-n}$.

The set $H_c$ is the graph of the function $h_1 : U \to V$ and for $s \in S$ the action of $s$ takes the point $h_1(u) + u \in H_c$ to

$$(h_1(u) + Qs) + (u + Ps)$$

hence for this point to be in $H_c$ it follows that

$$h_1(u) + Qs = h_1(u + Ps).$$

Since the space of orbits in $H_c$ under the action of $S$ is homeomorphic to the compact set $K_c$, the set $PS$ is a rank $(m-n)$ subgroup of $U$. Let $W_S$ be the $(m-n)$-dimensional subspace of $\mathbb{R}^m$ spanned by elements in $S$.

Lemma 9 There exists a linear map $h_0 : U \to V$ such that the map $u \to u + h_0(u)$ is a bijection of $U$ onto $W_S$.

Proof 12 Choose a basis $s_1, \ldots, s_{m-n} \in S$ for $W_S$. Then $Ps_1, \ldots, Ps_{m-n}$ is a basis for $U$. Every $u \in U$ has a unique representation as $c_1Ps_1 + \cdots + c_{m-n}Ps_{m-n}$. Then define $h_0(u) := c_1Qs_1 + \cdots + c_{m-n}Qs_{m-n}$. Then $u + h_0(u) = c_1s_1 + \cdots + c_{m-n}s_{m-n} \in W_S$ and the map $u \to u + h_0(u)$ is bijective.

We observe that the smallest projective subgroup of $\mathbb{Z}^m$ containing $S$ is $S_1 = W_S \cap \mathbb{R}^m$. Then $G := \rho_m(W_S)$ is a connected compact subgroup of $\mathbb{T}^m$ of dimension $m-n$ hence isomorphic to $\mathbb{T}^{m-n}$ and $G^\perp = S_2$ where $S_1 \oplus S_2 = \mathbb{Z}^m$. The quotient group $S_1/S$ is finite and we denote its cardinality by $|S_1/S|$. Define $\Lambda_c := \psi^{-1}(K_c)$ and the multiset $(\Lambda_0, c)$ where $\Lambda_0 := \psi^{-1}(G)$ and $c = |S_1/S|$. Define the family of maps $F_t : U \to \mathbb{R}^m$, $t \in [0, 1]$ by $F_t(u) := u + (1-t)h_0(u) + th_1(u)$ and the family of subsets $\Lambda_t := \psi^{-1}(\rho_m(F_t(U)))$, $t \in [0, 1]$. Each $F_t$ is a continuous injection and the family of these functions is a
homotopy from $W_S$ to $H_e$. Therefore the family of $\rho_m \circ F_t$ gives a homotopy of the map $\rho_m : W_S \to \mathbb{T}^m$ to the map $\rho_m : H_e \to \mathbb{T}^m$ and the map $t \to \Lambda_t$ is continuous with respect to the Hausdorff metric on closed subsets of $\mathbb{R}^n$. However, each fiber of the map $\rho_m \circ F_0 : W_S \to G$ is the orbit of the action of $S_1$ by addition, but each fiber of the map $\rho_n \circ F_1 : H_e \to K_e$ is the orbit of the action of the possibly smaller subgroup $S \subset S_1$. Therefore the family of sets images $\rho_n \circ F_t(U)$ gives a $|S_1/S|$-to-one deformation of $K_e$ to $G$. This argument proves that the density of $\Lambda_e$ equals the density of the multiset $(\Lambda_0, c)$ which equals $|S_1/S|$ times the density of $\Lambda_0$.

**Theorem 5** The density of $\Lambda_e$ equals $|S_1/S| \det E^T M$ where $E \in \mathbb{Z}^{m \times n}$ satisfies $S_2 = E \mathbb{Z}^n$.

**Proof 13** It suffices to prove that the density of $\Lambda_0$ equals $| \det E^T M |$. Since $G = \{ z \in \mathbb{T}^m : \zeta_k(z) = 1 \text{ for all } k \in G^1 \}$ and $G^1 = S_2 = E \mathbb{Z}^n$ it follows that

\[ \Lambda_0 := \psi^{-1}(G) = \{ \lambda \in \mathbb{R}^n : \zeta_k \circ \rho_m(M \lambda) \neq 1 \text{ for all } k \in E \mathbb{Z}^n \} \]

\[ = \{ \lambda \in \mathbb{R}^n : E^T M \lambda \in \mathbb{Z}^n \}. \]

Therefore density $\Lambda_0 = | \det E^T M |$.

**Remark 5** If $X$ and $Y$ are topological spaces $[X, Y]$ is the set of homotopy classes of continuous maps from $X$ to $Y$. Let $A$ be an abelian group and $n \geq 1$. Then $\alpha \in [X, Y]$ gives a homomorphism between cohomology groups $H^n(\alpha) : H^n(Y; A) \to H^n(X; A)$. If $Y$ is the Eilenberg-Maclane space $K(A, n)$, then there exists special $c \in H^n(K(A, n), A)$ such that for every CW complex $X$, the map $\alpha \to H^n(\alpha)(c) \in H^n(X; A)$ is a bijection, see ([13], Theorem 4.57). This result implies that the map $\alpha \to \pi_1(\alpha)(\pi_1(T^{m-n})) \subset \pi_1(T^m)$ is a bijection between homotopy classes and subgroups of $\mathbb{Z}^m$.

4 Research Questions

1. Can the results in Section 3 be extended to locally finite multisets?

2. Example 3 includes a Besicovitch almost periodic set with a dilation and higher dimensional ones include Penrose and Amman tilings described by Baake in [2]. A conjecture of Lagarias and Wang that we proved in [17] shows this is impossible if $K$ is a real analytic set. Do any Bohr almost periodic sets admit dilations?

3. Does this conjecture hold: $\Lambda \subset \mathbb{R}^n$ is a FQ iff there exists a toral compactification $(\mathbb{T}^m, \psi)$, $p_1, \ldots, p_n \in T(\mathbb{T}^m)$, and $\Lambda$ is the set of common zeros in $\mathbb{C}^n$ of the entire extensions $P_j$ of $p_j \circ \psi$ for $j = 1, \ldots, n$? Theorem 4 validates this conjecture for $n = 1$. We suggest deriving a Poisson summation formula, as Lev and Olevskii did in [19] for $n = 1$, using multidimension residue methods described by Tsikh [33] to compute an integral $\int_{\mathbb{R}^n} \frac{F}{P_1 \cdots P_n}$ for a class of entire functions $F$. We record that the hypothesis of the conjecture implies the map : $\mathbb{C}^n \to \mathbb{C}^n$ defined by $(z_1, \ldots, z_n) \to (P_1(z_1, \ldots, z_n), \ldots, P_n(z_1, \ldots, z_n))$ satisfies conditions ensuring the existence of local Grothendieck residues ([33], p. 14).

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