Decreasing families of dynamically determined intervals in the power-law family

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Abstract

We study the rate of growth of ratios of intervals delimited by the post-critical orbit of a map in the quasi-quadratic family $x \mapsto - |x|^{\alpha} + a$. The critical order $\alpha$ is an arbitrary real number $\alpha > 1$. The range of the parameter $a$ is confined to an interval $(1, a_\alpha)$ of length depending on the critical order. We prove that in every power-law family there is a unique parameter $p_\alpha$ corresponding to the kneading sequence $RLRRRLRC$. Subsequently, we obtain monotonicity results concerning ratios of all intervals labeled by infinite post-critical orbit in the case of the kneading sequence $RLRL...$. This extends the results from [9], via refinement of the tools based on special properties of power-law mappings in non-euclidean metric.

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1 Introduction

In this paper we continue our work done in [9] on families of unimodal quasi-quadratic maps of the form $f_a(x) = - |x|^{\alpha} + a$, with a real parameter $a$ and an arbitrary – in general non-integer – fixed exponent $\alpha > 1$.

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The problem of monotone behaviour of the dynamics in such a family has
been first successfully solved for the strictly quadratic case $\alpha = 2$. The tools
initially developed for the quadratic case (see e.g [2], [6], [10], also [11], and
an independent attempt, partly relying on numerical evidence in [11]) were
broadly generalized in the work of Kozlovski-Shen-van Strien, see [3]. There
are also very interesting recent results by G.Levin, concerning uniqueness
of appearance of periodic orbits of given multiplier in the quadratic family
$z^2 + c$. Not only was he able to give a simple proof of Douady-Hubbard-
Sullivan theorem (cf.[4]), but he could continue somewhat beyond the hyper-
bolic domains in the Mandelbrot set also, see [5]. In this work, we focus on
questions closely related to these of Levin’s, though only orbits of periods 2
or 4 appear here. In return, working with real variable tools, we can do all
critical degrees, integer or non-integer, indiscriminately.

Despite of a great deal of progress achieved in the aforementioned papers,
and in other works as well, virtually all those developments are inherently
limited to the case of integer critical degrees. Non-integers clearly require
a fresh and different approach. For any real number $\alpha > 1$ the power-law
map $x \mapsto |x|^\alpha$ has negative Schwarzian derivative, and hence it expands the
non-euclidean lengths. This observation has long become one of the key tools
in one-dimensional dynamics. However, the power-law is not just a negative
Schwarzian map. It is a homogeneous map, and in the Poincaré metric with
the element $\frac{dt}{t}$ on the positive half-line $(0, \infty)$, it is nothing but a linear
map acting as multiplication by the coefficient $\alpha$, once we set the origin of
the Poincaré coordinates at 1. This simple fact is rather hard to make use
of in a direct way, but carries some strong consequences that can be applied
in a dynamical setting.

In our previous paper on this subject (see [9]), we introduced the tech-
nique of indirect use of linearity of the power-law map in the non-euclidean
metric and exemplified its usefulness in dynamics. There, we studied maps
in the one-parameter quasi-quadratic family $f_a$ with the kneading sequence
$RRR \ldots$, that is for the value of the parameter $a$ smaller than 1. For the
infinite decreasing family of intervals with endpoints labeled by the succe-
ssive points of the post-critical orbit we proved that the ratio of any two such
intervals is a function monotone in parameter $a$. This means, we studied the
situation which arises before the orbit of the critical point becomes a super-
stable orbit of period 2. It is clear that this period 2 super-sink situation
arises only once in our family.

In the current work, we further develop our tools in order to examine
the case of some parameters greater than 1, where the length of the interval \((1, a_\alpha)\) to which those parameters are confined depends on the critical order \(\alpha > 1\). In particular we shall be able to deal with kneading sequences of the form \(RLRL \ldots\), proving monotonicity of the ratios, in the respective decreasing families, of intervals delimited by the post-critical orbit. As a step in the build-up of the above techniques, we shall also establish uniqueness of the period 8 super-sink, corresponding to the kneading sequence \(RLRRRLRC\), in every power-law family (even when it does not admit a holomorphic extension!); uniqueness of the period 4 super-sink \(RLRC\) is elementary and follows along the way.

2 Notation and preliminaries

To begin with, we set some notation in conformance with that of [9]. The names non-euclidean and Poincaré we be used interchangeably.

The Poincaré coordinate of a point \(x\) in an oriented, open interval \((p, q)\) will be denoted by

\[ p_{p,q}(x) = \ln \frac{x - p}{q - x}, \]

and respectively \(p_{p,\infty}(x) = \ln(x - p); \) also \(p_{p,-\infty}(x) = \ln|x - p|\).

To single out the non-euclidean metric on the half-line, which turns the mapping \(h\) into a linear map, we will coin the term nonlinearity of an interval for the length of this interval measured in the Poincaré metric on \((0, \infty)\). Under this convention, the integral of nonlinearity of \(h\) over an interval \((p, q)\) equals, up to a multiplicative constant, to the nonlinearity of the domain of integration.

Given an orientation preserving homeomorphism \(\varphi : (p, q) \to (r, s)\) we shall observe the ‘bar’ notation for its counterpart in the non-euclidean coordinates, i.e. the mapping \(\overline{\varphi}_{p,q} : \mathbb{R} \to \mathbb{R}\) defined by the formula

\[ \overline{\varphi}_{p,q}(t) = p_{r,s} \left( \varphi \left( p_{p,q}^{-1}(t) \right) \right). \]

The non-euclidean push of \(\varphi\) at a point \(x \in (p, q)\) is, by definition, the quantity

\[ p_{r,s} \left( \varphi(x) \right) - p_{p,q}(x). \]

By the strength of a push we mean its absolute value.
By $\varphi_{p,q}^+$ and $\varphi_{p,q}^-$ we denote the finite or infinite limits
\[
\varphi_{p,q}^+ = \lim_{t \to +\infty} (\varphi(t) - t) \\
\varphi_{p,q}^- = \lim_{t \to -\infty} (\varphi(t) - t),
\]
provided they exist.

When $\varphi$ is the restriction of the homogeneous map $h(x) = x^\alpha$ to an interval $(p, q) \subseteq (0, \infty)$ we shall always put $h$ or $\overline{h}$ in place of $\varphi$ or $\overline{\varphi}$ respectively.

For a fixed exponent $\alpha > 1$, let $f_a = -|x|^\alpha + a$, and the successive points of the orbit of the critical point will simply be denoted by $n_a = f_a^n(0)$. Moreover, homogeneity of the power-law map allows for the linear change of coordinates, $n_a \mapsto n_a / 1_a$, so that we can set $0_a = 0, 1_a = 1$ and the dependence on the parameter $a$ turns into the dependence on the value of $2_a$ in these new coordinates. For $a > 1$ this rescaled value of $2_a$ is in the interval $(-1, 0)$ – so long as the post-critical orbit does not escape to infinity – and the quantity $p_0,1(2_a)$, which for obvious reason we will denote by $\overline{a}$, is increasing simultaneously with $a$. Throughout this work, this very quantity will be chosen as our new parameter, and it is always tacitly assumed that the rescaling $n_a \mapsto n_a / 1_a$ has been done.

We now record several observations concerning one-dimensional non-euclidean coordinates. Below, they are stated as propositions, verifiable by elementary calculations derived directly from the definition of the Poincaré metric.

**Proposition 2.1** For any $x \in (-\infty, 0)$ the following two Poincaré coordinates coincide
\[
p_{0,-\infty}(x) = p_{x,1}(0).
\]

*Proof.* We have $p_{0,-\infty}(x) = \ln(-x) = \ln \frac{0-x}{1-0} = p_{x,1}(0)$. \qed

**Proposition 2.2** For any $x \in (-1, 0)$ the following two Poincaré coordinates coincide
\[
p_{1,-1}(x) = p_{x,-x} \left( p_{x,1}^{-1}(p_{0,-1}(x)) \right).
\]

*Proof.* The identity in question is tantamount to $\ln \frac{x-1}{1-x} = \ln \frac{x-c}{x-c}$, or
\[
\frac{x-1}{1-x} = \frac{c-x}{x+c}, \quad (2.1)
\]
where $c = p_{x,1}^{-1}(p_{0,-1}(x))$, i.e. $p_{x,1}(c) = p_{0,-1}(x)$. But this last equality means $\frac{c-x}{1-c} = \frac{x}{1-x}$, and further $c = x^2$, so that (2.1) follows. \qed

\[4\]
Given a point \( x \in (-\infty, 0) \), we then pick a point \( y \in (x, 0) \). We shall let the point \( x \) vary, by which we mean a choice of another point \( \tilde{x} \in (-\infty, 0) \). The discrepancy in the non-euclidean coordinate will be denoted by

\[
\Delta \vartheta = p_{0,-\infty}(\tilde{x}) - p_{0,-\infty}(x).
\]

A broader version of Proposition 2.1 is the following

**Proposition 2.3** In the above notation we have

\[
p_{\tilde{x},0} \left( p_{\tilde{x},1}^{-1}(p_{x,1}(y) + \Delta \vartheta) \right) - p_{x,0}(y) = p_{1,-\infty}(\tilde{x}) - p_{1,-\infty}(x).
\]

**Proof.** We have \( \Delta \vartheta = \ln \frac{\tilde{x}}{x} \) and \( p_{1,-\infty}(\tilde{x}) - p_{1,-\infty}(x) = \ln \frac{\tilde{x} - 1}{x - 1} \). Denote \( c = p_{\tilde{x},1}^{-1}(p_{x,1}(y) + \Delta \vartheta) \), a point characterized by

\[
\frac{c - \tilde{x}}{1 - c} = \frac{y - x}{1 - y} \cdot \frac{\tilde{x}}{x}.
\]

(2.2)

We will be done once we show \( \frac{c - \tilde{x}}{1 - c} \cdot \frac{y - x}{1 - y} = \frac{1 - \tilde{x}}{1 - x} \), or

\[
\left( 1 + \frac{\tilde{x}}{c - \tilde{x}} \right) \left( \frac{y - x}{y} \right) = \frac{1 - x}{1 - \tilde{x}}.
\]

(2.3)

From (2.2) we get \( \frac{\tilde{x}}{c - \tilde{x}} = \frac{1 - y}{y - x} \cdot \frac{x - \tilde{x}}{x - \tilde{x}} + \frac{\tilde{x}}{1 - \tilde{x}} \) and (2.3) can now be checked immediately. \( \square \)

Proposition 2.1 is what we get of Proposition 2.3, in place of subtracting two infinite terms, when we set \( y = 0 \). We generalize Proposition 2.2 in a similar way. Suppose we are given a point \( x \in (-1, 0) \), and a point \( y \in (x, -x) \). Again, we let the point \( x \) vary by choosing a new point \( \tilde{x} \in (-1, 0) \). The discrepancies in the appropriate Poincaré coordinates of the two points will be denoted by

\[
\Delta t = p_{1,-1}(\tilde{x}) - p_{1,-1}(x),
\]

and by

\[
\Delta \theta = p_{0,-1}(\tilde{x}) - p_{0,-1}(x)
\]

respectively. A statement parallel to Proposition 2.3 is the following

**Proposition 2.4** In the above notation we have

\[
p_{\tilde{x},-1} \left( p_{\tilde{x},1}^{-1}(p_{x,1}(y) + \Delta \vartheta) \right) - p_{x,-1}(y) = \Delta t.
\]
Proof. The point \( c = p_{x,1}^{-1}(p_{x,1}(y) + \Delta \theta) \) satisfies 
\[
\frac{\tilde{x}}{c - \tilde{x}} = \frac{1 - y}{y - x} \cdot \frac{1 + \tilde{x}}{1 - \tilde{x}} \cdot \frac{x}{1 - x} + \frac{\tilde{x}}{1 - \tilde{x}}.
\]
(2.4)

We will be done if we show that 
\[
\frac{\tilde{x} + c}{c - \tilde{x}} = \frac{y + x}{y - x} \cdot \frac{1 + \tilde{x}}{1 - \tilde{x}} \cdot \frac{x - 1}{1 + x}.
\]
(2.5)

Since \( \frac{\tilde{x} + c}{c - \tilde{x}} = 1 + \frac{x}{c - x} \), equation (2.5) follows immediately from (2.4). \( \square \)

**Proposition 2.5** Suppose \( x, x' \in (0,1) \) and \( y \in (x,1) \). Let \( y' \) be such a point in \( (x',1) \) that \( p_{x',1}(y') = p_{x,1}(y) \). Then

\[
p_{1,0}(x') - p_{1,0}(x) = p_{y',0}(x') - p_{y,0}(x).
\]
(2.6)

Proof. The point \( y' \) is chosen in such a way that \( \frac{y' - x'}{1 - y'} = \frac{y - x}{1 - y} \), or \( \frac{1 - x'}{1 - y'} = \frac{1 - x}{1 - y} \). Identity (2.6) is now immediate.

### 3 The period 4 super-sink

In this short section we describe the behavior of the point \( 4_a \) when we let the parameter \( \bar{a} \) vary in such a range, that \( 3_a \in (0,1) \) and the point \( 4_a \) stays within the interval \( (2a, -2a) \).

Let a positive number \( t \) be the Poincaré coordinate of \( 2_a \) in the oriented interval \( (1, -1) \), and we set

\[
g(t) = p_{2a,-2a}(4_a).
\]

The following theorem holds true.

**Theorem 3.1** The inverse function \( g^{-1} : \mathbb{R} \rightarrow \mathbb{R}_+ \) is strictly increasing, and \( g'(t) > 1 \). In particular, the value \( g(t) = 0 \), corresponding to the super-stable orbit with the kneading sequence RLRC is assumed only once.

Proof. Consider a pair of admissible parameter values \( \bar{a} \) and \( \bar{a}' \), i.e. such that the orbits \( 2_a, 3_a, 4_a \) (and respectively \( 2_{a'}, 3_{a'}, 4_{a'} \)) satisfy the restraints on the dynamics we set above. Then

\[
\Delta t = p_{1,-1}(2a') - p_{1,-1}(2a), \quad \text{while} \quad \Delta g = p_{2a,-2a}(4_a) - p_{2a,-2a}(4_a).
\]
(3.1)
Applying Proposition 2.4 to this case we get

\[ p_{2a^*,2a^*}(p_{2a^*,1}^{-1}(p_{2a^*,1}(4a) + (p_{0,-1}(2a^*) - p_{0,-1}(2a))) - p_{2a,-2a}(4a) = \Delta t, \quad (3.2) \]

so, because of monotonicity of the coordinate functions, we only need to establish that

\[ p_{2a^*,1}(4a^*) - p_{2a,1}(4a) > p_{0,-1}(2a^*) - p_{0,-1}(2a). \quad (3.3) \]

This inequality becomes clear once we split the procedure leading from point \( 2a \) (respectively \( 2a^* \)) to \( 4a \) (respectively to \( 4a^* \)) into three steps. In the first step, we act on the interval \((0, -1)\) by the restriction of the power-law map. Thus, due to negative Schwarzian derivative, the initial discrepancy \((p_{0,-1}(2a^*) - p_{0,-1}(2a))\) in the Poincaré coordinates gets increased. So we see that

\[ p_{1,2a^*}(3a^*) - p_{1,2a}(3a) \geq p_{0,-1}(2a^*) - p_{0,-1}(2a). \]

In the second step, we turn the interval \((0, -1)\) over, onto the interval \((1, 2a)\), or onto \((1, 2a^*)\) respectively, and then we truncate the image at the critical point \(0\). This cut-off only increases the Poincaré coordinate of every point, which after the turnover landed in \((1, 0)\), because we now read the Poincaré coordinate in the interval \((1, 0)\) rather than in a larger domain \((1, 2a)\), or \((1, 2a^*)\) respectively. Moreover, the increase in the Poincaré coordinate inflicted by cutting the domain interval short, is in the case of point \(3a\) smaller then in the case of \(3a^*\). This is so, because the endpoint \(2a\) is closer to the critical point, while the endpoint \(2a^*\) is further away to the left, so of two corresponding points with identical Poincaré coordinate within the respective domain intervals (with the other endpoint at \(1\)), the gain in the latter situation is larger than in the former. But instead of equal coordinates, we have even better inequality \(p_{1,2a^*}(3a^*) > p_{1,2a}(3a)\), which further enlarges the gain. Thus, in this second step, made of the turnover followed by truncation, the initial discrepancy grows even further and so

\[ p_{1,0}(3a^*) - p_{1,0}(3a) > p_{1,2a^*}(3a^*) - p_{1,2a}(3a). \]

In the last step, we again act by a negative Schwarzian map stretching the discrepancy between the Poincaré coordinates yet further, and finally we make the turnover onto \((1, 2a)\), and respectively onto \((1, 2a^*)\), to arrive at (3.3). Therefore \(\Delta g > \Delta t\) and the proof is complete. \(\square\)
4 The period 8 super-sink

In the previous section we have established that, when we vary the parameter \( \bar{a} \), the position of the point \( 4_a \) within the interval \( (2_a, -2_a) \) changes monotonically, with the derivative greater than 1. It clearly follows from the proof, that this derivative actually stays bounded away from 1, in a way that depends on the critical order \( \alpha \). In section 5 we will study in detail the case of \( p_{2_a,-2_a}(4_a) < 0 \), and describe the behavior of the intervals delimited by the post-critical orbit with the kneading sequence \( RLRL \ldots \).

In here, we will focus on these admissible parameters \( \bar{a} \), for which \( p_{2_a,-2_a}(4_a) > 0 \) and \( p_{4_a,-4_a}(8_a) \leq 0 \), i.e. we are past the (unique) parameter corresponding to \( RLRC \), but we do not cover the critical point yet another time. From now on, we are making our choice of the parameter subject to this restriction. We shall see that, as long as the above condition on the dynamics is satisfied, the movement of the point \( 8_a \) is also monotone in parameter, and in the non-euclidean metric in \( (4_a, -4_a) \) this point moves with the derivative strictly positive. It will follow that the \( RLRRRLRC \) super-stable orbit appears uniquely in every power-law family. It is a subject of an ongoing work, that goes beyond the scope of this paper, to examine whether a claim analogous to that of Theorem 3.1 can be fully extended to larger set of parameters.

In our current case, the scheme of the argument we used to prove Theorem 3.1 alone will not suffice, and a more delicate technique must be employed. Yet, some understanding of the way Poincaré coordinates vary remains an important component. Since, due to the more intricate dynamics, the required property of the non-euclidean coordinates becomes less self-evident, we state it as a separate lemma. The points \( x, y, z \) below will correspond to the points \( 2_a, 4_a, 8_a \) of the post-critical orbit. The origin of the summands, which do not have equivalent in the statement of Proposition 2.4 will be explained later, in the course of the proof of Theorem 4.1 below. Here, we only indicate that the last term has to do with the limit strength of a non-euclidean push.

Lemma 4.1 Suppose we are given two triples of points, \( (x, y, z) \) and \( (\tilde{x}, \tilde{y}, \tilde{z}) \), satisfying the following conditions:

(i) \( x, \tilde{x} \in (0, -1) \) and \( p_{0,-1}(\tilde{x}) > p_{0,-1}(x) \),

(ii) \( y \in (0, -x), \tilde{y} \in (0, -\tilde{x}) \) and \( p_{\tilde{x},1}(\tilde{y}) \geq p_{x,1}(y) + (p_{0,-1}(\tilde{x}) - p_{0,-1}(x)) \),
(iii) \( z \in (y, 0], \ z \in (\bar{y}, -\bar{y}) \) and \( p_{\bar{y}, \bar{x}}(\bar{z}) \geq p_{y, x}(\bar{z}) + (p_{0, -\bar{x}}(\bar{y}) - p_{0, -x}(y)) + \ln \frac{\bar{y} - \bar{x}}{y - x} - ((p_{\bar{x}, 1}(\bar{y}) - p_{x, 1}(y)) - (p_{0, -1}(\bar{x}) - p_{0, -1}(x))) \).

Then \( p_{\bar{y}, -\bar{y}}(\bar{z}) > p_{y, -y}(z) \).

Proof. It is immediate to check that for arbitrary \( y, \bar{y} \in (0, 1) \) one has

\[
p_{\bar{y}, -1}(0) = p_{y, -1}(0) + (p_{0, 1}(\bar{y}) - p_{0, 1}(y)) + \ln \frac{1 + \bar{y}}{1 + y} - (p_{-1, 1}(\bar{y}) - p_{-1, 1}(y)).
\]

(4.1)

We now assume \( \bar{y} > y \), and allowing \( z \neq 0 \) we verify, that for any \( z \in (0, y) \) the following generalization of (4.1) holds

\[
p_{\bar{y}, -\bar{y}}(p_{\bar{y}, -1}(p_{\bar{y}, -1}(z) + (p_{0, 1}(\bar{y}) - p_{0, 1}(y)) + \ln \frac{1 + \bar{y}}{1 + y} - (p_{-1, 1}(\bar{y}) - p_{-1, 1}(y)))) \geq p_{y, -y}(z).
\]

(4.2)

In order to see this, notice that

\[
(p_{0, 1}(\bar{y}) - p_{0, 1}(y)) + \ln \frac{1 + \bar{y}}{1 + y} - (p_{-1, 1}(\bar{y}) - p_{-1, 1}(y)) = \ln \frac{\bar{y}}{y},
\]

and denote \( c = p_{\bar{y}, -1}(p_{\bar{y}, -1}(z) + \ln \frac{\bar{y}}{y}) \), which means \( \frac{c - \bar{y}}{c - y} = \bar{y}^c \cdot \frac{1 - \bar{y}}{1 - \bar{y}} \), or

\[
\frac{\bar{y}}{c - y} = \frac{y}{1 + \bar{y}} \cdot \frac{1 + z}{z - y} - \frac{\bar{y}}{1 + \bar{y}}.
\]

(4.3)

We will be done if we show that \( \frac{c - \bar{y}}{c - y} \geq \frac{z - y}{z - y} \), being equivalent to \( \frac{2\bar{y}}{c - y} + 1 \geq \frac{y}{z - y} \) or \( \frac{\bar{y}}{c - y} \geq \frac{y}{z - y} \). The last inequality follows from (4.3), once we recall \( \bar{y} \geq y \).

In the next step we extend formula (4.2), allowing \( \bar{x} \neq -1 \). Assuming \( 1 > -\bar{x} > \bar{y} > y > z \geq 0 \), we will now show that

\[
p_{\bar{y}, -\bar{y}}(p_{\bar{y}, \bar{x}}(z) + (p_{0, -\bar{x}}(\bar{y}) - p_{0, -x}(y)) + \ln \frac{\bar{y} - \bar{x}}{y - \bar{x}} - (p_{\bar{x}, 1}(\bar{y}) - p_{\bar{x}, 1}(y)))) > p_{y, -y}(z).
\]

(4.4)

We emphasize that the inequality in formula (4.4) is always sharp, even for \( z = 0 \).

This time, we set

\[
c = p_{y, x}(z) + (p_{0, -x}(\bar{y}) - p_{0, -x}(y)) + \ln \frac{\bar{y} - \bar{x}}{y - \bar{x}} - (p_{\bar{x}, 1}(\bar{y}) - p_{\bar{x}, 1}(y)),
\]

(4.5)
which means
\[
\frac{c - \tilde{y}}{x - c} = \frac{z - y}{x - z} \cdot \frac{\tilde{y}}{-\tilde{x} - y} \cdot \frac{-\tilde{x} - y}{y} \cdot \frac{1 - \tilde{y}}{1 - y}.
\]

We transform this identity into
\[
\frac{\tilde{x} - \tilde{y}}{c - \tilde{y}} - 1 = \frac{\tilde{x} - z}{z - y} \cdot \frac{\tilde{x} + \tilde{y}}{\tilde{y}} \cdot \frac{y}{\tilde{x} + y} \cdot \frac{1 - y}{1 - y}
\]
and further into
\[
\frac{\tilde{y}}{c - \tilde{y}} = \frac{\tilde{y}}{\tilde{x} - \tilde{y}} \left[ \frac{\tilde{x} - z}{z - y} \cdot \frac{\tilde{x} + \tilde{y}}{\tilde{y}} \cdot \frac{y}{\tilde{x} + y} \cdot \frac{1 - y}{1 - y} + 1 \right].
\]
We will be done if we show \( \frac{\tilde{y}}{c - \tilde{y}} > \frac{y}{z - y} \), which is equivalent to \( \frac{\tilde{y}}{c - \tilde{y}} > \frac{y}{z - y} \), and so it is enough to verify that
\[
\frac{\tilde{y}}{\tilde{x} - \tilde{y}} \left[ \frac{\tilde{x} - z}{z - y} \cdot \frac{\tilde{x} + \tilde{y}}{\tilde{y}} \cdot \frac{y}{\tilde{x} + y} \cdot \frac{1 - y}{1 - y} + 1 \right] > \frac{y}{z - y}.
\]
This inequality can be rewritten as
\[
\frac{\tilde{x} - z}{z - y} \cdot \frac{\tilde{x} + \tilde{y}}{\tilde{y}} \cdot \frac{y}{\tilde{x} + y} \cdot \frac{1 - y}{1 - y} > \frac{y}{z - y} - \frac{\tilde{y}}{\tilde{x} - \tilde{y}},
\]
or (recall that \( y < z, \tilde{x} < 0, \tilde{y} > 0 \))
\[
(\tilde{x} - z)(\tilde{x} + \tilde{y})y(1 - y) < (\tilde{x} + y)(1 - \tilde{y})(y\tilde{x} - z\tilde{y}),
\]
and further
\[
\tilde{x}y(1 + \tilde{x})(\tilde{y} - y) < z(\tilde{y} - y)(\tilde{x}\tilde{y} + \tilde{x}y - \tilde{x} + y\tilde{y}).
\]
To conclude, we cancel out \((\tilde{y} - y)\), and observe that
\[
\tilde{x}y + \tilde{x}\tilde{y} - \tilde{x} + y\tilde{y} > \tilde{x}(1 + \tilde{x}). \tag{4.6}
\]
This is so because (4.6) boils down to the inequality \( \tilde{x}^2 + \tilde{x}(2 - y - \tilde{y}) - y\tilde{y} < 0 \), which is elementarily true for all \( \tilde{x} \in (-1, 0) \) and \( y, \tilde{y} \in (0, 1) \). For completion of the proof we now consider an arbitrary point \( x \in (0, \tilde{x}) \), such that \( y < -x \). We consider the movement of \( x \)-variable from position \( x \) to \( \tilde{x} \) and apply Proposition 2.4 twice, first to the induced movement of \( y \)-variable, then to
the consequent movement of $z$-variable. By virtue of that Proposition, we see that points $\hat{y}$ and $\hat{z}$, determined by the identities

$$p_{\hat{x},1}(\hat{y}) = p_{x,1}(y) + (p_{1,-1}(\bar{x}) - p_{1,-1}(x))$$

$$p_{\hat{y},\hat{z}}(\hat{z}) = p_{y,x}(z) + (p_{1,-1}(\bar{x}) - p_{1,-1}(x))$$

satisfy $\hat{y} < \bar{y}$ and $p_{\hat{y},\hat{z}}(\hat{z}) = p_{y,-y}(z) + (p_{1,-1}(\bar{x}) - p_{1,-1}(x)) > p_{y,-y}(z)$. Thus obviously $\ln \frac{\bar{y} - \hat{x}}{y - x} > \ln \frac{\bar{y} - \hat{x}}{y - x}$.

If $\hat{z} \geq 0$, i.e. $p_{\hat{y},\hat{z}}(\hat{z}) \leq 0$, then keeping $\bar{x}$ fixed, we then apply formula (4.4) with $\hat{y}$, $\hat{z}$ in place of $y$, $z$, to the effect of yet further increase of the Poincaré coordinate of $\bar{z}$ compared to that of $\hat{z}$ (and so of $z$ itself), measured within respective symmetric $y$-domains. In case of $p_{\hat{y},\hat{z}}(\hat{z}) > 0$ the image of point $z$ has already past the midpoint of the (varying) symmetric $y$-domain interval while $y$-variable has been changing from $y$ to $\bar{y}$. Again, we then keep $\bar{x}$ fixed, to move the $y$-variable further, from $\hat{y}$ to $\bar{y}$. This time, application of formula (4.4) can induce some decrease in the Poincaré coordinate of the outcome – the resulting point $p_{\hat{y},\hat{z}}^{-1}(c)$, with $c$ as in (4.3), can divide the $y$-domain interval $(\bar{y}, -\bar{y})$ in smaller proportion than $\hat{z}$ did in $(\bar{y}, -\bar{y})$. Anyway, due to sharp inequality in (4.4) for all $z$ such that $p_{y,-y}(z) < 0$, the midpoint could only be attained from the other side. In other words, inequality (4.4) guarantees that the derivative of the induced $z$-movement, measured in the respective Poincaré coordinates, is positive (and actually bounded away from 0) as long as the values assumed by the $z$-variable are non-positive. Thus, in particular the value 0 can be attained only once, and so if we put a point $\hat{z}$ with $p_{y,-y}(\hat{z}) > 0$ into the formula at the left-hand side of inequality (4.4), we necessarily end up with a point on the same side of 0. Because the starting point $z$ was on the other side, the lemma holds in this case too. \qed

With lemma (4.4) in place, we are in the position to state and prove the main result of this section.

**Theorem 4.1** In the power-law family $f_a : x \mapsto -|x|^{\alpha} + a$, with $\alpha > 1$, there exists unique parameter $a = a(\alpha)$ corresponding to the kneading sequence \RLRRLRC.

**Proof.** In the course of the proof we make use of the tools developed in section 2 of [9], where we pointed to some consequences of homogeneity of the power-law mappings. In particular, we had Lemma 2.1 there, asserting that for any two points $q, \bar{q} \in (0, 1)$ one has

$$\overline{h}_{q,1} - \overline{h}_{\bar{q},1} = (p_{0,1}(h(q)) - p_{0,1}(h(\bar{q}))) - (p_{0,1}(q) - p_{0,1}(\bar{q})).$$
Speaking colloquially, identity (4.7) tells, that when we move the endpoint of an interval \((1, q)\) in \((1, 0)\) towards the critical point, then an extra gain in the Poincaré coordinate, coming from the successive action of the power-law map, is just enough to make up for the loss (measured in non-euclidean metric in \((1, q)\) and \((1, \tilde{q})\) respectively) suffered because of the simultaneously increased strength of the limit non-euclidean push towards that moving endpoint. Other propositions and lemmas of section 2 of [9] served to establish, that this limit situation, corresponding to Poincaré coordinate close to \(-\infty\), is essentially the worst possible, and when we consider an interior point of a definite Poincaré coordinate rather than the limit case, then the balance of gains vs. losses is in our favor ("we are never in the red"). We will be sending upon those properties when necessary, without reproducing them in this paper.

Proceeding similarly to what we did in the proof of Theorem 3.1, we split the procedure leading from \(8_{a}\) to \(8_a\), and respectively from \(8_{a'}\) to \(8_{a'}\), into several steps. First, we increase \(\bar{a}\) to \(\bar{a}'\). Theorem 3.1 yields, in particular, that \(p_{0, -2a'}(4a') - p_{0, -2a}(4a) > 0\). Next, we act upon \(4_a\), and \(4_{a'}\), by the map \(h\), and under the action of \(h\) the above discrepancy gets enlarged. This is so, because due to homogeneity, we may for the purpose of performing this step, tentatively set each of the endpoints, \(-2a\) and respectively \(-2a'\), at 1. Then each of the Poincaré coordinates \(p_{0, -2a}(4a)\), \(p_{0, -2a}(4_{a'})\), is transformed by same, fixed negative Schwarzian map \(\mathcal{T}_{0, 1}\). In the following step, we turn each of the intervals \((0, h(2a'))\), \((0, h(2a))\) over, and stretch them onto \((1, 3a')\) and respectively \((1, 3a)\). The image of \(4_{a'}\) is \(5_{a'}\), and by the so far described steps, it is clear that \(p_{1, 3a'}(5_{a'}) - p_{1, 3-a}(5a) > p_{0, -2a'}(4_{a'}) - p_{0, -2a}(4a)\). By the truncation argument from the proof of Theorem 3.1, we know that \(p_{1, 0}(3_{a'}) - p_{1, 0}(3a) > \bar{a} - a\). In particular, the nonlinearity of the interval \((1, 3_{a'})\) is larger than that of \((1, 3a)\). Now, we act by the homogeneous map \(h\) again. Notice, that unlike in the case of \(\mathcal{T}_{0, 2a'}\), this time the mapping \(\mathcal{T}_{1, 3a'}\) does not coincide with \(\mathcal{T}_{1, 3a}\). Anyway, we can still claim that in this step the discrepancy of the respective Poincaré coordinates grows again, i.e

\[
p_{1, h(3_{a'})}(h(5_{a'})) - p_{1, h(3_a)}(h(5a)) > p_{1, 3_{a'}}(5_{a'}) - p_{1, 3a}(5a) .
\] (4.8)

To this end, we invoke Propositions 2.5 and 2.4 of [9]. From the former, it follows that the strength of the non-euclidean push generated by \(h\) restricted to some domain, is a monotone function of the nonlinearity of that domain, when measured for a fixed Poincaré coordinate within the varying domain.
From the latter, we derive that when the domain stays fixed, the strength of the non-euclidean push of $h$ is monotone in the Poincaré coordinate of the argument. We have noticed already that the nonlinearity of $(1,3_a)$ is increasing in parameter $\bar{a}$, and also that $p_{1,3_a}(5_{a'}) > p_{1,3_a}(5_a)$, so the principle of monotone behaviour of the strength of non-euclidean push can be applied to the triples of points we consider. This immediately implies the desired increase in the discrepancy of appropriate Poincaré coordinates, as stated in (4.8).

Making the next step, we turn the obtained triples $(1, h(5_{a'}), h(3_a))$ and $(1, h(5_a), h(3_a))$ over, onto $(2_{a'}, 6_{a'}, 4_{a'})$, and respectively onto $(2_a, 6_a, 4_a)$, and then truncate them at the critical point 0. In the proof of Theorem 3.1 as well as in a step above, we were satisfied to ascertain that this truncation increases the Poincaré coordinates discrepancy, which in current step would yield $p_{2_{a'}, 0}(6_{a'}) - p_{2_{a}, 0}(6_a) > p_{2_{a'}, 4_{a'}}(6_{a'}) - p_{2_{a}, 4_{a}}(6_a)$, because by Theorem 3.1 we know that $p_{2_{a'}, 4_{a'}}(6_{a'}) > p_{2_{a}, 4_{a}}(6_a)$.

To proceed further, one more observation is needed. It is fairly clear that we have following lower bound on the increase of the Poincaré coordinates discrepancy, generated by the cut-off at 0:

$$p_{2_{a'}, 0}(6_{a'}) - p_{2_{a}, 0}(6_a) > p_{2_{a'}, 4_{a'}}(6_{a'}) - p_{2_{a}, 4_{a}}(6_a) + \ln \frac{4_{a'} - 2_{a'}}{4_a - 2_a}. \quad (4.9)$$

The equality in (4.9) is the limit case, attained for infinitesimally short intervals placed at the left-hand endpoints, i.e. when $p_{2_{a'}, 4_{a'}}(6_{a'}) \to -\infty$ and simultaneously $p_{2_{a'}, 4_{a'}}(6_{a'}) \to -\infty$. For non-infinitesimal intervals satisfying $p_{2_{a'}, 4_{a'}}(6_{a'}) > p_{2_{a}, 4_{a}}(6_a)$, the same argument as in the proof of Theorem 3.1 obviously yields sharp inequality in (4.9), and so the growth of the discrepancy gained in the cut-off step is strictly larger than the logarithmic term.

In the following step we once more act by homogeneous map $h$, and because $\bar{h}_{0,2_{a'}}$ coincides with $\bar{h}_{0,2_a}$, the same argument as before gives

$$p_{h(2_{a'}, 0)}(h(6_{a'})) - p_{h(2_{a}), 0}(h(6_a)) > p_{2_{a'}, 0}(6_{a'}) - p_{2_{a}, 0}(6_a). \quad (4.10)$$

This adds yet an extra amount to the discrepancy we consider. We again turn the intervals $(h(2_{a'}), 0)$ and $(h(2_a), 0)$ over and stretch them onto $(1, 3_{a'})$ and $(1, 3_a)$, with $6_{a'}$ going onto $7_{a'}$ and $6_a$ going onto $7_a$ respectively. It remains to examine what happens in the last step, when we act by the respective (non-coinciding!) restrictions of $h$ to the obtained intervals, before we eventually return onto $(2_{a'}, 4_{a'})$ and onto $(2_a, 4_a)$ by linear rescaling. This is what we
need Lemma 4.1 for. In what follows we verify its assumptions are fulfilled in our setting.

In this last step we perform, the strength of the non-euclidean push induced by $h|_{(1,3a')}$, measured at $7_{a'}$, can be greater than the respective strength of $h|_{(1,3a)}$ at $7_a$. This means that the discrepancy accumulated in all the so far steps can now diminish. However, the identity (4.7) provides a bound from the above on the amount of possible loss. To see this, we recall that

$$\bar{a}' - \bar{a} = p_{0,-1}(2_{a'}) - p_{0,-1}(2_a) < p_{1,0}(3_{a'}) - p_{1,0}(3_a), \quad (4.11)$$

and according to (4.7) we have

$$(p_{2_{a'},1}(4_{a'}) - p_{2_{a},1}(4_a)) - (p_{1,0}(3_{a'}) - p_{1,0}(3_a)) = (\overline{h}_{1,3a} - \overline{h}_{1,3a'}) \quad (4.12)$$

We know that $p_{3_{a'},1}(7_{a'}) > p_{3_{a},1}(7_a)$ and the interval $(1,3_{a'})$ has larger non-linearity than $(1,3_a)$, so we are in a position to invoke Propositions 2.5 and 2.4 of [9] once more. By them we have

$$(p_{3_{a'},1}(7_{a'}) - p_{3_{a},1}(7_a)) - (p_{4_{a'},2_{a'}}(8_{a'}) - p_{4_{a},2_{a}}(8_a)) < (\overline{h}_{1,3a} - \overline{h}_{1,3a'}). \quad (4.13)$$

The inequalities (4.9), (4.11) and (4.13) put together, provide for fulfillment of condition (iii) of Lemma 4.1 with the points $x$, $y$ and $z$ assuming values $2_a$, $4_a$ and $8_a$, as indicated before the statement of the lemma. Now the claim of Theorem 4.1 follows directly from Lemma 4.1 and so we are done. □

We complete this section explicitly recording one extra property, which we actually proved along the way. Denote the variable $\tau = p_{4_{a},-4_a}(8_a)$ and let $\gamma = p_{2_{a},-2_a}(4_a)$. From the proofs of Theorem 4.1 and Lemma 4.1 there immediately follows

**Corollary 4.1** The function $\gamma = \gamma(\tau) : \mathbb{R}_- \to \mathbb{R}_+$ is strictly increasing in $\tau$, with the derivative $\gamma'(\tau)$ bounded away from 0 and $+\infty$. □

## 5 \textit{RLRLRLRLRL...}

In this section we let the parameter $\bar{a}$ vary in a range such that the kneading sequence is \textit{RLRL...}. From Theorem 3.1 it follows immediately that the range of admissible $\bar{a}$’s is always a half-line ($-\infty, \bar{a}_1$), with the specific value
of $\bar{a}_1$ depending on the critical order $\alpha$. Upholding the normalization $0_a = 0$, $1_a = 1$ we have set before, this means the post-critical orbit begins with $2_a \in (0, -1)$, $3_a \in (1, 0)$ and $4_a \in (2_a, 0)$. Then, we get two sequences of nested intervals, the odds: $(1, 3_a), (3_a, 5_a), (5_a, 7_a) \ldots$, and the evens: $(0, 2_a), (2_a, 4_a), (4_a, 6_a) \ldots$. In terms of multipliers, we either have a period 2 orbit with negative multiplier, or this periodic orbit had turned into a repeller and, by bifurcation, there was born a period 4 attracting periodic orbit with positive multiplier. In what follows, we shall see that the ratios of consecutive intervals within each of the two decreasing families are functions strictly monotone in parameter $\bar{a}$. Moreover, the initial increase of the parameter, i.e. $\bar{a}' - \bar{a}$, does not eventually vanish, but a definite part of it is preserved through all the steps. This will further provide, with some extra work, for monotonicity of the multipliers, also in the case of repelling period 2 orbit. This is a work in preparation. The remaining part of this paper is devoted to the proof of the following claim.

**Theorem 5.1** For $\bar{a} \in (-\infty, \bar{a}_1)$ and for all non-negative integers $n$, the ratio functions

$$r_e^n = \frac{|(2n + 4)_a - (2n + 2)_a|}{|2n + 2)_a - (2n)_a|} \quad \text{and} \quad r_o^n = \frac{|(2n + 5)_a - (2n + 3)_a|}{|2n + 3)_a - (2n + 1)_a|}$$

are strictly increasing in $\bar{a}$.

Moreover, when the parameter increases from $\bar{a}$ to $\bar{a}'$, then for every $n \in \mathbb{Z}_+$ the induced discrepancy of the Poincaré coordinates satisfies

$$p_{(n+2)_a, n_a'}((n + 4)_a') - p_{(n+2)_a, n_a}((n + 4)_a) > (p_{1,-1}(2'_a) - p_{1,-1}(2_a)).$$

**Proof.** As before, we divide the procedure into steps. Once we cover the most delicate step, which turns out to be the passage from $(5_a', 7_a')$ to $(6_a', 8_a')$, we will be in a position to continue inductively. We begin by moving the initial point $2_a$ to a new position $2_a'$, with $\bar{a}' > \bar{a}$. Then, by the truncation argument from the proof of Theorem 3.1 we have

$$(p_{1,0}(3_a') - p_{1,0}(3_a)) > \bar{a}' - \bar{a} = \Delta \bar{a} \geq \Delta t,$$

where we denoted $\Delta t = (p_{1,-1}(2_a') - p_{1,-1}(2_a))$. Since we then act by the homogeneous map $h$, by (4.7) we get

$$p_{2_a,1}(4_a') - p_{2_a,1}(4_a) = (p_{1,0}(3_a') - p_{1,0}(3_a)) + (h_{3_a,1} - h_{3_a',1}).$$
Passing from \(3_{a'}\) to \(4_{a'}\), we cannot directly apply the truncation argument again, because in this step the Poincaré coordinate \(p_{1,2_{a'}}(0)\) of the cut-off point decreases (cf. Proposition 2.1). That can be fixed by decomposing the step in two, and simultaneous use of Proposition 2.4, identity (4.7) and truncation. According to (5.3) and Proposition 2.4 of [9] do apply when we act by

\[
p_{2_{a'},-2_{a'}}\left(p^{-1}_{2_{a'},1}(p_{1,0}(3_{a'}))\right) - p_{2_{a'},-2_{a'}}\left(p^{-1}_{2_{a'},1}(p_{1,0}(3_{a}))\right) > t. \tag{5.5}
\]

Truncation at 0 obviously gives

\[
p_{2_{a'},0}\left(p^{-1}_{2_{a'},1}(p_{1,0}(3_{a'}))\right) - p_{2_{a'},0}\left(p^{-1}_{2_{a'},1}(p_{1,0}(3_{a}))\right) > t. \tag{5.6}
\]

Then, to the Poincaré coordinate of the point \(p^{-1}_{2_{a'},1}(p_{1,0}(3_{a'}))\), read in the domain \((2_{a'},1)\), we add the extra gain of \((\overline{h}_{3_{a},1} - \overline{h}_{3_{a'},1})\). The non-euclidean length of this same extra interval, read in the domain \((2_{a'},0)\) rather than in \((2_{a'},1)\), is of course larger, because of truncation. Thus

\[
p_{2_{a'},0}(4_{a'}) - p_{2_{a'},0}(4_{a}) > t + (\overline{h}_{3_{a},1} - \overline{h}_{3_{a'},1}). \tag{5.7}
\]

Doing the homogeneous mapping again, by (4.7) and (5.7) we get

\[
p_{3_{a'},1}(5_{a'}) - p_{3_{a},1}(5_{a}) = \overline{h}_{2_{a'},0}(p_{2_{a'},0}(4_{a'})) - \overline{h}_{2_{a},0}(p_{2_{a},0}(4_{a})) > t + (\overline{h}_{3_{a},1} - \overline{h}_{3_{a'},1}) + (\overline{h}_{4_{a},2_{a}} - \overline{h}_{4_{a'},2_{a'}}). \tag{5.8}
\]

Now, similarly to the final step in the proof of Theorem 4.1, we can argue that the so far acquired gain in the Poincaré coordinate is enough to make up for possible losses in the next two steps. This is fairly clear. The interval \((3_{a'},1)\) has larger nonlinearity than \((3_{a},1)\), and \(p_{3_{a},1}(5_{a'}) > p_{3_{a},1}(5_{a})\), so Propositions 2.5 and 2.4 of [9] do apply when we act by \(h|_{(1,3_{a'})}\) and \(h|_{(1,3_{a})}\). Therefore, in this step the discrepancy \((p_{3_{a'},1}(5_{a'}) - p_{3_{a},1}(5_{a}))\) can only be diminished by an amount smaller than \((\overline{h}_{3_{a},1} - \overline{h}_{3_{a'},1})\), yielding

\[
p_{4_{a'},2_{a'}}(6_{a'}) - p_{4_{a},2_{a}}(6_{a}) > t + (\overline{h}_{4_{a},2_{a}} - \overline{h}_{4_{a'},2_{a'}}). \tag{5.9}
\]

By (5.9), the nonlinearity of \((4_{a'},2_{a'})\) is larger than that of \((4_{a},2_{a})\), and also \(p_{4_{a'},2_{a'}}(6_{a'}) > p_{4_{a},2_{a}}(6_{a})\). Thus, when we act by \(h|_{(4_{a'},2_{a'})}\), and respectively by \(h|_{(4_{a},2_{a})}\), we certainly do not lose more than \((\overline{h}_{4_{a},2_{a}} - \overline{h}_{4_{a'},2_{a'}})\) in the outgoing discrepancy. Hence, by (5.9)

\[
p_{5_{a'},3_{a'}}(7_{a'}) - p_{5_{a},3_{a}}(7_{a}) > t. \tag{5.10}
\]
We can now make a shortcut towards completion of the current cycle. The nonlinearity of \((3_{a'}, 1)\) is larger than that of \((3_a, 1)\) and \(p_{3_{a'}, 1}(5_{a'}) > p_{3_a, 1}(5_a)\), which in turn gives that the nonlinearity of \((5_{a'}, 3_{a'})\) is larger than that of \((5_a, 3_a)\). Also \(p_{5_{a'}, 3_{a'}}(7_{a'}) > p_{5_a, 3_a}(7_a)\), so we can apply the argument about monotonicity of the strength of the non-euclidean push, which we recalled in the proof of Theorem 4.1, immediately arriving at

\[ p_{6_{a'}, 4_{a'}}(8_{a'}) - p_{6_a, 4_a}(8_a) > p_{5_{a'}, 3_{a'}}(7_{a'}) - p_{5_a, 3_a}(7_a) > \Delta t. \] (5.11)

However, the above argument alone turns out to be insufficient, when we want to do further iterates. To obtain an inequality which we could use inductively at all steps, we need more subtle understanding at this particular stage of our procedure. Here we go.

From (5.3) and Proposition 2.5 it follows that

\[ p_{p_{a'}, 1}(p_{3_a, 1}(5_a)), 0(3_{a'}) - p_{5_a, 0}(3_a) > \Delta \bar{a}, \] (5.12)

so by \(p_{3_{a'}, 1}(5_{a'}) > p_{3_a, 1}(5_a)\) we have

\[ p_{5_{a'}, 0}(3_{a'}) - p_{5_a, 0}(3_a) > \Delta \bar{a}. \] (5.13)

By the same argument applied to \((5_{a'}, 3_{a'})\) rather than \((1, 3_{a'})\), we get

\[ p_{p_{5_{a'}, 3_{a'}}}(p_{5_a, 3_a}(7_a)), 0(3_{a'}) - p_{7_a, 0}(3_a) > \Delta \bar{a}. \] (5.14)

Now we do the homogeneous mapping \(h\), and rescale the image onto \((1, 2_{a'})\). The image of \(3_{a'}\) is \(4_{a'}\), and by a version of the truncation argument alike that used before in the step leading from \((1, 3_{a'})\) to \((2_{a'}, 4_{a'})\), we use

\[ p_{2_{a'}, 0}(p_{2_{a'}, 1}(p_{1, 0}(3_{a'})))) - p_{2_a, 0}(p_{2_a, 1}(p_{1, 0}(3_a))) > \Delta t \] (5.15)

and (5.14) to get

\[ p_{p_{2_{a'}, 1}(p_{1, 0}(p_{5_{a'}, 3_{a'}}(p_{5_a, 3_a}(7_a))))}, 0(4_{a'}) - p_{8_a, 0}(4_a) > \Delta \bar{a}. \] (5.16)

This is so, because (5.14) implies

\[ p_h(p_{5_{a'}, 3_{a'}}(p_{5_a, 3_a}(7_a))), 0(h(3_{a'})) - p_h(7_a), 0(h(3_a)) > \Delta \bar{a}, \] (5.17)
and when we consider the interval \((c, d)\), where \(d = p_{2',1}^{-1}(p_{2,1}(4a) + \Delta \bar{a})\), and the point \(c\) is defined so that

\[
p_{c,1}(d) - p_{8a,1}(4a) = \Delta \bar{a} \tag{5.18}
\]

then, according to Proposition \([2.5]\) applied to the domain \((1, 2a')\) in place of \((0, 1)\), and with the points \(d\) and \(c\) singled out, we see that point \(c\) divides the interval \((d, 2a')\) at the same proportion as \(8a\) divided \((4a, 2a)\). Re-applying Proposition \([2.5]\) to the domain \((0, 2a')\) with the same singled out pair of points, we further see that

\[
p_{c,0}(d) > p_{8a,0}(4a) + \Delta t, \tag{5.19}
\]

because by Proposition \([2.4]\) \(p_{2',0}(d) - p_{2a,0}(4a) > \Delta t\). Recalling \((5.17)\) and taking into account that \(p_{2a',1}(4a') > p_{2',1}(d)\), which in turn gives \(p_{1,a'}(0) > p_{1,d}(0)\), we can now do the standard truncation argument, cutting-off at 0 to arrive at \((5.16)\).

This formula could do for the iterative procedure if we cared only for some, indefinite growth. To obtain definite growth, claimed in the statement of Theorem \([5.1]\) we need to work harder.

In the next step of the proof, we will see that the extra amount of \(\Delta t\) in formula \((5.16)\) allows us to move \(7a\) towards the endpoint by at least that much. To this end, we again consider the interval \((5a', 3a')\), but this time the point within we single out, is point \(e\) determined by

\[
p_{5a',3a'}(e) = p_{5a,3a}(7a) + \Delta t. \tag{5.20}
\]

From \((5.14)\), using Proposition \([2.1]\) with points \(0, 3a\) and \(e\) in place of \(1, 0\) and \(x\) respectively, or by a direct check, one gets

\[
p_{e,0}(3a') - p_{7a,0}(3a) > \Delta \bar{a} - \Delta t. \tag{5.21}
\]

Doing the homogeneous mapping, we have

\[
p_{h(e),0}(h(3a')) - p_{h(7a),0}(h(3a)) > \Delta \bar{a} - \Delta t. \tag{5.22}
\]

Again, we consider an interval \((f, d)\), where \(d\) has same meaning as above, and point \(f\) is defined by

\[
p_{f,1}(d) - p_{8a,1}(4a) = \Delta \bar{a} - \Delta t. \tag{5.23}
\]
From (5.18) and (5.23), it follows by Proposition 2.1 that 
$p_{d,-\infty}(c) - p_{d,-\infty}(f) = \Delta t$, and again by this same proposition $p_{c,0}(d) - p_{f,0}(d) = \Delta t$. Hence, by (5.19), we have $p_{f,0}(d) > p_{8a,0}(4)$. This, and (5.22) lead to

$$p_{p^{-1},4}(\bar{a}^{'},4) > p_{8a,0}(4).$$  (5.24)

We can describe what we have found so far in the following way. We move the parameter up, from $\bar{a}$ to $a'$. In the odd family, we see $3a$ moving to $3'a$ by more than $\Delta\bar{a}$. Consequently, the non-euclidean coordinate of $5a$ vary, within its dynamically determined base interval, by at least $\Delta t$, plus an additional increment which is sufficient to make up for the increased – due to larger nonlinearity of the new new domain intervals – strength of the non-euclidean push backwards. In the next odd return we do not let $7a$ move all the way to its new position $7a'$ at once. Instead, we first only add $\Delta t$ to its Poincaré coordinate. This corresponds to starting from the point $e$ in the already fully enlarged domain $(5a', 3a')$, rather than from $7a'$. We have just seen that not only is the nonlinearity of $(e, 3a')$ larger than that of $(7a, 3a)$, but the nonlinearity of $(\hat{e}, 4a')$ is larger than that of $(8a, 4a)$ also. Here $\hat{e}$ is the dynamical successor of $e$ on the even side. This latter estimate from the below on the the nonlinearity, turns out to be fundamental for the prospective iterates.

Recall we defined $e$ by (5.20) so as to have $p_{5a',3a'}(e) = p_{5a,3a}(7a) + \Delta t$. The same way we derived (5.11) from (5.10) we also get

$$p_{6a',4}(\hat{e}) - p_{6a,4}(8a) > \Delta t.$$  (5.25)

This will be needed, when it comes to definite growth in both odd and even family. But now, for points $e$ and $\hat{e}$ we have stronger input: in both cases, we know that the nonlinearity of the remaining part of the base interval increased. Therefore, we will now be able to proceed pretty much like in the initial step, that led from $(1, 3a')$ to $(2a', 4a')$, rather than use the earlier described shortcut. Similarly to that initial step, we again want to know that the surplus exceeding $\Delta t$ in (5.25) will make up for possible loss, inflicted by increased nonlinearity of $(5a', 7a')$, upon next return to $(5a', 7a')$. However, we have to overcome a serious obstacle. Formula (4.7) we previously used to that goal, holds true only so long as the critical point is the endpoint corresponding to non-euclidean $+\infty$. This is of course not the case for $(5a, 3a)$, nor for all other intervals in our odd and even families, except for the initial
ones. For intervals not bounded by the critical point, we only know monotonicity of the strength of non-euclidean push and this, in general, does not give control over an amount of the gain in Poincaré coordinates discrepancy. Composing \( h \) mappings over two arbitrary, successive domains, yet worsens the the situation. Fortunately, all this can be fixed with (5.21) and (5.24) in place. Increased nonlinearity of that part of a domain interval which bounds us away from the endpoint, provides an effective replacement for the critical endpoint. In particular, we will see that the gain in the non-euclidean coordinates discrepancy is even better than that in formula (4.7). This is why we have striven for those nonlinearity inequalities. As soon as we are over with the part which takes \( 7_a \) to \( e \), the remaining part, in which we move \( e \) further to \( 7_a' \), will require only an easy estimate. All the above holds true for even successors, \( \hat{e} \) and eventually \( 8_a' \), as well. With one extra observation to make, we will be able to do arbitrarily long iterates, preserving the \( \Delta t \) discrepancy all along the way.

To carry out the above described strategy, we recall that in Proposition 2.2 of [9] we gave an explicite formula for the strength of non-euclidean push, which turns out to be

\[
|p_{r,s}(\varphi(x)) - p_{p,q}(x)| = |\varphi_{x,q} + \varphi_{p,x}^-| \quad (5.26)
\]

We also noticed there, that for the homogeneous mapping \( h \) restricted to some interval, the quantities \( \overline{h}^- \) and \( \overline{h}^+ \) depend solely on the nonlinearity of that domain interval. By monotonicity of the strength of the non-euclidean push as a function of the nonlineatity of the domain, also the limit values, \( \overline{h}^- \) and \( \overline{h}^+ \), behave monotonically. By all the above, taking (5.24) into account, we have

\[
p_{6_a',4_a'}(\hat{e}) - p_{6_a,4_a}(8_a) = \overline{h}_{5_a',3_a'}(p_{5_a,3_a}(7_a) + \Delta t) - p_{6_a,4_a}(8_a) =
\]

\[
p_{h(5_a'),h(3_a')}(h(e)) - p_{h(5_a),h(3_a)}(h(7_a)) > \Delta t + (\overline{h}_{5_a',e}^- - \overline{h}_{5_a,7_a}^-) \quad (5.27)
\]

The sign at the superscript of \( \overline{h} \) in (5.27) depends only on an orientation of the domain, so (5.27) provides a better estimate than we could derive from (4.7), if the endpoint \( 3_a' \) coincided with the critical point. Doing the successive \( h \)-map step on the even side, because of (5.24), we get in the same way

\[
p_{h(6_a'),h(4_a')}(h(\hat{e})) - p_{h(6_a),h(4_a)}(h(8_a)) > \Delta t + (\overline{h}_{5_a',e}^- - \overline{h}_{5_a,7_a}^-) + (\overline{h}_{6_a',e}^- - \overline{h}_{6_a,8_a}^-). \quad (5.28)
\]

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These are formulas analogous to (5.7) and (5.8), and what we want now, is a similar estimate where the input is \(7_{a'}\) and \(8_{a'}\), rather than \(e\) and \(\hat{e}\). To move from \(e\) to \(7_{a'}\) we could simply invoke Lemma 2.4 of [9]. However, there is no generalization of that lemma which could be used over two unrelated domains. We need to be a bit more careful, and use the dynamical relation between an interval and its image. Doing the mapping \(h\), by homogeneity and (4.7) we have

\[
P_{h(5_{a'}),0}(h(7_{a'})) - p_{h(5_{a'}),0}(h(e)) = (p_{5_{a'}},0(7_{a'})) - p_{5_{a'}},0(e)) + (\overline{h}_{5_{a'},7_{a'}} - \overline{h}_{5_{a'},e}).
\]  

(5.29)

Before we do another mapping \(h\), we take the image over onto \((1, -\infty)\), so that \(h(3_{a'})\) goes onto \((4_{a'})\), and cut off at 0. Because of this truncation

\[
P_{6_{a'}},0(8_{a'}) - p_{6_{a'}},0(\hat{e}) > (p_{5_{a'}},0(7_{a'})) - p_{5_{a'}},0(e)) + (\overline{h}_{5_{a'},7_{a'}} - \overline{h}_{5_{a'},e}).
\]  

(5.30)

Now, acting by homogeneous map, we get

\[
P_{h(6_{a'}),0}(h(8_{a'})) - p_{h(6_{a'}),0}(h(\hat{e})) > (p_{5_{a'}},0(7_{a'})) - p_{5_{a'}},0(e)) + (\overline{h}_{5_{a'},7_{a'}} - \overline{h}_{5_{a'},e}) + (\overline{h}_{6_{a'},8_{a'}} - \overline{h}_{6_{a'},\hat{e}}).
\]  

(5.31)

We neglect a positive summand \((p_{5_{a'}},0(7_{a'})) - p_{5_{a'}},0(e))\) and truncate at \(h(4_{a'})\) to arrive at

\[
P_{h(6_{a'}),h(4_{a'})}(h(8_{a'})) - p_{h(6_{a'}),h(4_{a'})}(h(\hat{e})) > (\overline{h}_{5_{a'},7_{a'}} - \overline{h}_{5_{a'},e}) + (\overline{h}_{6_{a'},8_{a'}} - \overline{h}_{6_{a'},\hat{e}}).
\]  

(5.32)

Similarly, neglecting a positive summand at (5.29), followed by cutting off at \(h(3_{a'})\) leads to

\[
P_{h(5_{a'}),h(3_{a'})}(h(7_{a'})) - p_{h(5_{a'}),h(3_{a'})}(h(e)) > (\overline{h}_{5_{a'},7_{a'}} - \overline{h}_{5_{a'},e}).
\]  

(5.33)

The inequalities (5.27) and (5.33), in conjunction with (5.28) and (5.32), give

\[
P_{h(5_{a'}),h(3_{a'})}(h(7_{a'})) - p_{h(5_{a'}),h(3_{a'})}(h(e)) > \Delta t + (\overline{h}_{5_{a'},7_{a'}} - \overline{h}_{5_{a'},e})\]

(5.34)

and also

\[
P_{h(6_{a'}),h(4_{a'})}(h(8_{a'})) - p_{h(6_{a'}),h(4_{a'})}(h(8_{a})) > \Delta t + (\overline{h}_{5_{a'},7_{a'}} - \overline{h}_{5_{a'},e}) + (\overline{h}_{6_{a'},8_{a'}} - \overline{h}_{6_{a'},8_{a}}).
\]  

(5.35)
which are the desired estimates. Now, by the same argument which led from (5.7) and (5.8), through (5.9) to (5.10), we can see that (5.34) and (5.35) imply
\[ p_{9,a'}(11a') - p_{0,a}(11a) > \Delta t. \] (5.36)
In the same way we obtained (5.11) from (5.10), we can also derive
\[ p_{10,a'}(12a') - p_{10,a}(12a) > \Delta t. \] (5.37)
We have completed the second cycle. Those were necessary to initialize the inductive procedure. We are now in a position to do the final argument, which can be used repeatedly. We believe that, because all the elaborate notation of the first two cycles is already in place, it will be more instructive to present this argument in detail as the next cycle, rather than in general terms. It will be obvious that what we do, is tantamount to the inductive step.
We pick a point \( \epsilon \in (9a', 7a') \), such that
\[ p_{9,a'}(\epsilon) = p_{0,a}(11a) + \Delta t. \] (5.38)
We will prove that the nonlinearity of \((\epsilon, 7a')\) is larger than that of \((11a, 7a)\), and simultaneously the nonlinearity of \((\hat{\epsilon}, 8a')\) is larger than that of \((12a, 8a)\). As before, \( \hat{\epsilon} \) stands for the dynamical successor of \( \epsilon \) on the even side. This will permit to bypass the non-critical endpoint obstacle in the next cycle, the way we did earlier, with \( \epsilon \) and \( \hat{\epsilon} \). To show this nonlinearity increase, we proceed in several steps. First, in \((5a', e)\) we find a point \( \beta = p_{5,a'}^{-1}(p_{9,a}(9a)) \). Then, in \((\beta, e)\) we find \( \epsilon \), such that
\[ p_{\beta,3}(\epsilon) = p_{9,a}(11a) + \Delta t. \] We use the fact that Poincaré coordinate of the point \( e \), compared to that of \( 7a \), is already moved by \( \Delta t \) towards the endpoint, to ascertain that the nonlinearity of \((\epsilon, e)\) is larger than nonlinearity of \((11a, 3a)\); we have
\[ \frac{|(\beta, e)|}{|(e, 3a')|} = (\exp \Delta t) \frac{|(9a, 7a)|}{|(7a, 3a)|} > (\exp \Delta t) \frac{|(\beta, \delta)|}{|(\delta, \epsilon)|}. \] (5.39)
where \( \delta \in (\beta, e) \) is a point such that
\[ \frac{|(\delta, e)|}{|(e, 3a')|} = \frac{|(11a, 7a)|}{|(7a, 3a)|}. \] Thus, \( p_{\epsilon,3}(\epsilon) > p_{\delta,3}(\delta) \), and consequently \( p_{\epsilon,3}(\epsilon) > p_{\delta,3}(\delta) \) or, in other words, \( \frac{|(\epsilon, e)|}{|(e, 3a')|} > \frac{|(\delta, e)|}{|(\delta, 3a')|} \). Since \((e, 3a')\) has larger nonlinearity than \((7a, 3a)\), the nonlinearity
of \((\varepsilon, e)\) must be larger than nonlinearity of \((11_a, 7_a)\). Next, we do the mapping \(h\) and consider the situation on the even side. The interval \((\varepsilon, 3_a)\) has larger nonlinearity than \((11_a, 3_a)\) and \(p_{\varepsilon, 3_a}(e) > p_{11, 3_a}(7_a)\), so by principles of monotonicity of the strength of non-euclidean push in nonlinearity of the domain, as well as in the coordinate of the point, the action of \(T_{\varepsilon, 3_a}\) makes

\[
p_{\hat{\varepsilon}, 4_a}(\hat{\varepsilon}) > p_{12, 4_a}(8_a),
\]

where \(\hat{\varepsilon}\) is the dynamic successor of \(\varepsilon\). Because \((\hat{\varepsilon}, 4_a)\) has larger nonlinearity than \((8_a, 4_a)\), it follows that the nonlinearity of \((\hat{\varepsilon}, \hat{\varepsilon})\) is also larger than that of \((12_a, 8_a)\). By the same two principles applied to \(p_{\beta, \epsilon}(\epsilon)\), we get \(p_{\beta, \epsilon}(\hat{\epsilon}) > p_{10, 8_a}(12_a) + \Delta t\), but because of the proved nonlinearity increases, we can also claim that

\[
p_{h(\beta), h(e)}(h(\varepsilon)) - p_{h(9_a), h(7_a)}(h(11_a)) > \Delta t + (\overrightarrow{T}_{\beta, \varepsilon} - \overrightarrow{T}_{9_a, 11_a})
\]

and

\[
p_{h(\beta), h(\hat{\varepsilon})}(h(\hat{\varepsilon})) - p_{h(10_a), h(8_a)}(h(12_a)) > \Delta t + (\overrightarrow{T}_{\beta, \varepsilon} - \overrightarrow{T}_{9_a, 11_a}) + (\overrightarrow{T}_{\beta, \hat{\varepsilon}} - \overrightarrow{T}_{10_a, 12_a}).
\]

We are through with the first part of the inductive step. Now, our immediate plan is to move \(e\) to \(7_a\), then \(\beta\) up to \(9_a\), and to replace \(\hat{\varepsilon}\) by \(\varepsilon\), keeping all the above gains untouched, both on the odd and on the even side. Having done all that, we will easily be able to move \(e\) to \(11_a\), to complete the procedure.

Denote by \(\lambda\) the point determined by \(p_{5_a, 7_a}(\lambda) = p_{9_a, 7_a}(9_a)\), and let \(\tau \in (\lambda, 7_a)\) be such, that \(p_{\lambda, 7_a}(\tau) = p_{9_a, 7_a}(11_a) + \Delta t\). The interval \((5_a, 7_a)\) has larger nonlinearity than \((5_a, e)\), so consequently \((\lambda, 7_a)\) has larger nonlinearity than \((\beta, e)\), and \((\tau, 7_a)\) has larger nonlinearity than \((\varepsilon, e)\). Thus \((h(\tau), h(7_a))\) has larger nonlinearity than \((h(\varepsilon), h(e))\). The distance of \(8_a\) to the critical point 0 is smaller than similar distance for the point \(\hat{\varepsilon}\), so the truncation argument after cutting of at 0, implies that \((\hat{\tau}, 8_a)\) has larger nonlinearity than \((\hat{\varepsilon}, \hat{\varepsilon})\) and, in turn, larger than \((12_a, 8_a)\). Again, we increase the intervals in question, choosing \(9_a\) in place of \(\lambda\), and replacing \(\tau\) by \(e\). Then, of course, \((\epsilon, 7_a)\) has yet larger nonlinearity, so \((h(\epsilon), h(7_a))\) has larger nonlinearity than \((h(\tau), h(7_a))\) and, after truncation, \((\hat{\epsilon}, 8_a)\) has larger nonlinearity than \((\hat{\tau}, 8_a)\). It immediately implies

\[
p_{h(9_a), h(7_a)}(h(\varepsilon)) - p_{h(9_a), h(7_a)}(h(11_a)) > \Delta t + (\overrightarrow{T}_{9_a, \varepsilon} - \overrightarrow{T}_{9_a, 11_a}), \quad (5.40)
\]

and

\[
p_{h(10_a), h(8_a)}(h(\hat{\varepsilon})) - p_{h(10_a), h(8_a)}(h(12_a)) > \Delta t + (\overrightarrow{T}_{9_a, \varepsilon} - \overrightarrow{T}_{9_a, 11_a}) + (\overrightarrow{T}_{\beta, \hat{\varepsilon}} - \overrightarrow{T}_{10_a, 12_a}). \quad (5.41)
\]
This is what we aimed at. By the same argument that earlier let us replace $e$ by $7$ and $\hat{e}$ by $8$, to derive formulas (5.34) and (5.35), we can now replace $\epsilon$ by $11$ and $\hat{\epsilon}$ by $12$, arriving at

\[
ph_{h(9_a'),h(7_a')}(h(11_a)) - ph_{h(9_a),h(7_a)}(h(11_a)) > \Delta t + \left(\mathbb{T}_{9_a',11_a'}^{+} - \mathbb{T}_{9_a,11_a}^{+}\right),
\]

and

\[
ph_{h(10_a'),h(8_a')}(h(12_a')) - ph_{h(10_a),h(8_a)}(h(12_a)) > \Delta t + \left(\mathbb{T}_{10_a',11_a'}^{+} - \mathbb{T}_{10_a,11_a}^{+}\right) + \left(\mathbb{T}_{10_a',12_a'}^{+} - \mathbb{T}_{10_a,12_a}^{+}\right).
\]

Similarly to (5.36) and (5.37), we also get

\[
p_{13_a',11_a'}(15_a) - p_{13_a,11_a}(15_a) > \Delta t, \quad \text{and} \quad p_{14_a',12_a'}(16_a) - p_{14_a,12_a}(16_a) > \Delta t.
\]

This completes the inductive step. The claim of the theorem follows immediately. ∎

References

[1] Dragan, V., Jones, A., Stacey, P., Repeated radicals and the real Fatou theorem, Austral. Math. Soc. Gaz. 29 (2002), 259–268.

[2] Graczyk, J., Świątek, G., Induced expansion for quadratic polynomials, Ann. Sci. Ecole Norm. Sup. 29 (1996), 399–482.

[3] Kozlovski, O., Shen, W., van Strien, S., Rigidity for real polynomials, to appear in Ann. Math. (2007).

[4] Levin, G., On explicit connections between dynamical and parameter spaces, J. Anal. Math. 91 (2003), 297–327.

[5] Levin, G., Multipliers of periodic orbits of quadratic polynomials and the parameter plane, preprint (2007).

[6] Lyubich, M., Dynamics of quadratic polynomials. I, II. Acta Math. 178 (1997), no. 2, 185–247, 247–297.

[7] de Melo, W., van Strien, S., One-Dimensional Dynamics, Springer, Berlin 1993.
[8] Milnor, J., Thurston, W., *Iterated Maps of the Interval*, In: *Dynamical Systems*, Lect. Notes Math. 1342, Springer 1988, 465–563.

[9] Pžlin, W., *A Case of Monotone Ratio Growth for Quadratic-Like Mappings*, Bull. Pol. Acad. Sci. Math. 52 (2004), pp. 381–393.

[10] Shishikura, M., *Yoccoz puzzles, \( \tau \)-functions and their applications*, unpublished.

[11] Tsujii, M., *A simple proof for monotonicity of entropy in the quadratic family*, Ergodic Theory Dynam. Systems 20 (2000), pp. 925–933.