Free Energy of ABJM Theory

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The free energy of ABJM theory has previously been computed in the strong and weak coupling limits. In this note, we report on results for the computation of the first non-vanishing quantum correction to the free energy, from the field theory side. The correction can be expressed in terms of a thermal mass for the scalar fields. This mass vanishes to 1-loop order, but there is a non-vanishing result to 2-loop order. Hence, the leading correction to the free energy is non-analytic in the 't Hooft coupling constant $\lambda$. The reason is that the infrared divergences necessitate a resummation of ring diagrams and a related reorganization of perturbation theory, in which already the leading correction receives contributions from all orders in $\lambda$. These results suggest that the free energy interpolates smoothly between weak and strong coupling.

1 Introduction

The gauge/gravity correspondence [1] has been studied for a long time from both sides of the correspondence and in various versions. However, until recently, studies of the field theory side of the $AdS_4/CFT_3$ version were hampered by the fact that it seemed difficult to write down a Lagrangian with all the right symmetries, notably superconformal $OSp(8|4)$ symmetry and parity invariance. In fact, many people believed such a Lagrangian was impossible to write down, based on various no-go results.

This changed with the work of Bagger, Lambert and Gustavsson (BLG) [2]. Bagger and Lambert wrote down an action in terms of an algebraic construct known as a “3-algebra”. However, even though it had all the right symmetries, closer scrutiny of the moduli space seemed to reveal subtle difficulties in matching it to any known gravity dual, even for $N = 2$ [3].

The obstruction towards describing more than two M2-branes manifested itself in various ways in the different formulations of BLG theory. In the original 3-algebra formulation, the so-called fundamental identity only had a single unique solution, corresponding to gauge group $SO(4)$. In van Raamsdonk’s $SU(2) \times SU(2)$ formulation [4], there was a reality condition on the scalars which does not make sense beyond $N = 2$. And finally, in a superspace formulation, the superpotential is only valid for $N = 2$.

This problem was finally solved in 2008, when Aharony, Bergman, Jafferis and Maldacena introduced what is now known as the ABJM model [5]. In this model, the superpotential was crucially rewritten in a way to allow generalization to arbitrary ranks of the gauge group, while still coinciding with the BLG choice for $N = 2$. There are differences even for $N = 2$, since the ABJM model allows certain $U(1)$ factors, which are not present in BLG. This is related to the crucial role played by monopole operators in ABJM theory, which appear to be essential to obtaining the correct moduli space.

In this note, we will be interested in the unique behaviour of the degrees of freedom in ABJM theory, in particular as captured by the free energy. In super Yang-Mills theory (SYM), the free energy has been computed at strong coupling, as a system of D-branes [6,7]. Corrections using supergravity were computed in [8]. At weak coupling, the free energy including loop corrections was computed in [9,10], revealing a screening phenomenon for both scalars and the gluons, related to the theory being strongly divergent in the infrared.

ABJM theory at finite temperature and at strong coupling was also studied recently. In [11], a dimensional reduction of type IIA supergravity was carried out. Various static length scales including the mass gap and Debye screening mass were computed in [12]. An interesting phase transition which breaks the R-symmetry was found in [13], and a domain wall solution was found in [14].

In [15], we computed the quantum corrections to the free field theory result of ABJM [5], allowing us to see whether similar phenomena as in the SYM case also appear in ABJM. Our primary motivation, though, was to see if we can make progress from the field theory side on understanding the strong suppression of the entropy at strong coupling [7] (corrections were computed in [16]).

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We now want to investigate the coupling dependence of the free energy. As discussed in [15], one of the technical
we will use Coulomb gauge. Coulomb gauge makes the combinatorics simpler. On the other hand, Coulomb gauge is often very convenient in thermal

To one-loop order, only scalars and fermions contribute. Summing up those contributions, we find

\[ \tilde{F}_1 = N^2 \left( \frac{1}{2} A_0 - \frac{1}{2} A_{1/2} \right) = -N^2 T^3 \frac{7\zeta(3)}{\pi}, \]

3 Infrared divergences

We now want to investigate the coupling dependence of the free energy. As discussed in [15], one of the technical
difficulties is the infrared divergences. Assuming that the free energy is analytic in \( \lambda = N/k \) around \( \lambda = 0 \), we write

\[ \tilde{F}(\lambda) = \tilde{F}_1 + \lambda \tilde{F}_2 + \lambda^2 \tilde{F}_3 + \cdots. \]
where

\[ A_\nu = \int \frac{d^2 p}{(2\pi)^2} T \sum_{n \in \mathbb{Z} + \nu} \log(p^2 + \omega_n^2), \]

\[ A_0 = -T^3 \frac{\zeta(3)}{\pi}, \quad A_{1/2} = T^3 \frac{3\zeta(3)}{4\pi}. \]

(3.3)

The reason that gauge fields and ghosts do not contribute is that the gauge field does not have proper kinetic terms, only Chern-Simons terms. At zero temperature, supersymmetry ensures that the free energy cancels out, and this is also true of the expression (3.2).

The result (3.2) was already obtained in [5]. To attempt to find further terms in the naive expansion (3.1), we use perturbation theory. Propagators and vertices follow from the action (2.1). At each order, connected and one-particle-irreducible diagrams contribute. At two loops, all such diagrams vanish, so we find that \( \tilde{F}_2 = 0 \). Hence, it would appear we need to proceed to three-loop order.

However, the perturbative expansion (3.1) breaks down somewhere between two and three loops, due to infrared divergences. For example, the 6-vertex will contribute a correction proportional to

\[ \left( \int \frac{d^2 p}{(2\pi)^2} T \sum_{n \in \mathbb{Z}} \frac{1}{p^2 + \omega_n^2} \right)^3. \]

(3.4)

These infrared divergences do not cancel.

### 4 Thermal Mass

Since the perturbative expansion (3.1) breaks down, we need to reorganize perturbation theory to obtain a finite answer beyond two loops. Equivalently, we regularize the theory by introducing a thermal mass. How this works was already studied both in QCD [18][19] and in super Yang-Mills [10].

Specifically, the idea is that we write the action

\[ S = (S + \delta S_2) - \delta S_2, \]

where (in momentum space)

\[ \delta S_2 = \frac{k}{2\pi} \int \frac{d^2 p}{(2\pi)^2} T \sum_{n \in \mathbb{Z}} \text{tr} \left[ \frac{1}{2} Y_A(p)m^2 Y_A(-p) \right]. \]

(4.1)

Treating \( -\delta S_2 \) as a perturbation to \( S + \delta S_2 \) creates a thermal mass in the scalar propagator. A thermal mass can be interpreted as the appearance of a screening length \( r = \frac{1}{m} \) [19].

We compute the scalar self-energy by summing up all one-particle-irreducible diagrams. It will be sufficient for our purposes to do this in the static limit, corresponding to no external momentum, since this is the troublesome mode that we want to regularize.
To two-loop order, we find the result \[15\]
\[
m^2_Y(\lambda) = (2\pi T)^2 \mu^2(\lambda),
\]
\[
\mu^2(\lambda) = \frac{118}{3(2\pi)^2} \lambda^2 \log(\mu)^2 + \mathcal{O}(\lambda^2 \log(\lambda)),
\]
(4.2)
corresponding to a regularized scalar propagator
\[
\frac{1}{\mathcal{Z}(\bar{p})} = \frac{2\pi}{kT} \frac{1}{\bar{p}^2 + \omega_n^2 + m_Y^2}.
\]
(4.3)

Note that the computation needs to be done self-consistently by requiring that successive higher order perturbative corrections do not shift the pole in the propagator (cf. D’Hoker’s original calculation for QCD$_3$ \[20\]). Hence, \((4.2)\) is an equation which in general cannot be explicitly solved for \(m_Y^2\).

5 Reorganized Perturbation Expansion

We have seen that due to IR divergences, the naive expansion \((5.1)\) is not successful in finding quantum corrections to the free energy. Instead, we will have to use the reorganized perturbation expansion
\[
F(\lambda) = F_1(\lambda) + \lambda F_2(\lambda) + \lambda^2 F_3(\lambda) \cdot \cdot \cdot ,
\]
(5.1)
where the coefficients \(F_i(\lambda)\) now depend on \(\lambda\). The reason is that we are now using the massive scalar propagator \((4.3)\). At one-loop order, we find
\[
F_1(\lambda) = N^2 \left( \frac{1}{2} A_0(m_Y^2) - \frac{1}{2} A_{1/2}(m_Y^2) \right),
\]
(5.2)
which generalizes the zero-mass answer \((4.2)\). However, from the derivation in section \(4\) it follows that already the one-loop answer is sensitive to corrections to all orders in \(\lambda\). To obtain the first non-vanishing correction, it will be sufficient to know the contribution from the scalars (see figure \[2\]),
\[
A_0(m^2) = \sum_{n \in \mathbb{Z}} \int \frac{d^2 p}{(2\pi)^2} \log \left( \bar{p}^2 + (2\pi Tn)^2 + m^2 \right) = \sum_{n \in \mathbb{Z}} \int \frac{d^2 p}{(2\pi)^2} \log \left( \bar{p}^2 + (2\pi Tn)^2 + m^2 \right) = \frac{T^3}{4\pi} \left[ -4\zeta(3) - m^2 T^{-2} \log (m^2 T^{-2}) + m^2 T^{-2} + \mathcal{O}(m^4 T^{-4}) \right],
\]
(5.3)
which generalizes (5.3). No thermal mass is generated for the fermions, since they have no zero modes. Hence, they only contribute to lower orders. In this way, we obtain the free energy including the first non-vanishing quantum correction, as already reported on in [15]. Specifically, by combining equations (5.1), (5.2) and (5.3), we find the free energy density

\[ F = -N^2 T^3 f(\lambda), \] (5.4)

where

\[ f(\lambda) = \left[ \frac{7 \zeta(3)}{\pi} + \frac{m_2^2(\lambda)}{\pi T^2} \log \left( \frac{m_2^2(\lambda)}{T^2} \right) + O(\lambda^2 \log(\lambda)^2) \right]. \] (5.5)

The scalar thermal mass (4.2) can be solved for numerically. It and the free energy (5.5) are plotted in figure 3. The result is suggestive of a free energy which smoothly interpolates to the strong coupling answer [7]

\[ f_s(\lambda) = \left[ \frac{27/2}{9} \frac{\pi^2}{\sqrt{\lambda}} + \cdots \right], \] (5.6)

as expected.

### 6 Conclusions

In this note, we described how to calculate the free energy of ABJM theory on \( \mathbb{R}^2 \times S^1 \). The first non-vanishing quantum correction is non-analytic in \( \lambda \) [15]. The reason is that perturbation theory had to be reorganized to deal with the IR divergences (resummation of ring diagrams). Thus, the theory was regularized by a scalar thermal mass, generated by screening effects. This calculation suggests that the free energy interpolates smoothly between weak and strong coupling.

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1 Using the approximation \( \log(\mu) \approx \log(\lambda) \) (valid for very small \( \lambda \)), it can be confirmed that to lowest non-vanishing order, the \( \lambda \) dependence of the correction is \( \lambda^2 \log(\lambda)^3 \), i.e. the same as that found by Gaiotto and Yin for a related type of three-dimensional Chern-Simons-matter theory [21].
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