On Selections of Set-Valued Inclusions in a Single Variable with Applications to Several Variables

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Abstract. We present some applications of the result corresponding to the existence of a unique selection of a set-valued function satisfying inclusions in a single variable to the inclusions in several variables, especially the general linear inclusions or quadratic inclusions.

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1. Introduction

The stability theory of functional equations has developed in connection with a problem set by S.M. Ulam during his talk at a conference at the Wisconsin University in 1940. The first answer was given in 1941 by Hyers [5] who proved the following theorem:

Let $X$ be a linear normed space, $Y$ a Banach space and $\epsilon > 0$. Then for every function $f : X \to Y$ satisfying the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon, \quad x, y \in X$$

there exists a unique additive function $g : X \to Y$ such that

$$\|f(x) - g(x)\| \leq \epsilon, \quad x \in X.$$  

From now on the subject has been intensively investigated by many authors (see for example: [1,3,6,7,10,11,16]).
Smajdor [18] and Gajda and Ger [4] observed that if \( f \) satisfies (1), then the set-valued function \( F \): \( X \to n(Y) \) (\( n(Y) \) denotes the family of all non-empty subsets of \( Y \)) given by

\[
F(x) = f(x) + \overline{B}(0, \epsilon), \quad x \in X,
\]

where \( \overline{B}(0, \epsilon) \) is the closed ball of radius \( \epsilon \) centered at 0, is subadditive (i.e., \( F(x + y) \subset F(x) + F(y), x, y \in X \)) and the function \( g \) from the relation (2) is an additive selection of \( F \) (i.e., \( g(x + y) = g(x) + g(y) \) and \( g(x) \in F(x) \) for \( x, y \in X \)).

Now one may ask under what conditions a subadditive set-valued function admits an additive selection. We recall the result of Gajda and Ger [4] (\( \delta(F(x)) \) denotes the diameter of the set \( F(x) \)).

**Theorem 1.** Let \((S, +)\) be a commutative semigroup with zero, \( X \) a real Banach space and \( F \): \( S \to 2^X \) a set-valued map with convex and closed values such that

\[
F(x + y) \subset F(x) + F(y), \quad x, y \in S
\]

and \( \sup \{\delta(F(x)) : x \in S\} < \infty \). Then \( F \) admits a unique additive selection.

Later the above result was extended by Nikodem and Popa to the set-valued functions satisfying the following general linear inclusions:

\[
\begin{align*}
F(ax + by + c) &\subset pF(x) + qF(y) + C, \\
pF(x) + qF(y) &\subset F(ax + by + c) + C,
\end{align*}
\]

where \( a, b, p, q \in \mathbb{R} \), \( X \) is a real vector space, \( Y \) is a real Banach space, \( F \): \( X \to n(Y) \), \( c \in X \), \( C \in 2^Y \) (see [9, 13–15]).

The aim of this paper is to give some modification of Theorem 1 in [12] and its applications. We also show that our theorem generalizes the above results.

## 2. Main Results

Let \((Y, d)\) be a metric space. We will denote by \( n(Y) \) the family of all non-empty subsets of \( Y \). We understand the convergence of sets with respect to the Hausdorff metric derived from the metric \( d \). The number \( \delta(A) = \sup \{d(x, y) : x, y \in A\} \) is said to be the diameter of \( A \subset Y \). For \( F \): \( K \to n(Y) \) we denote by \( \text{cl} F \) the multifunction defined as \( (\text{cl} F)(x) = \text{cl} F(x), x \in K \). A function \( f \): \( K \to Y \) such that \( f(x) \in F(x) \) for all \( x \in K \) is called a selection of the multifunction \( F \). We write \( a^0(x) = x \) for \( x \in K \) and \( a^{n+1} = a^n \circ a \) for all \( n \in \mathbb{N}_0 \).

The subsequent theorem is a simple modification of Theorem 1 in [12]. However, we prove it for the convenience of the readers.
Theorem 2. Assume that \( K \) is a nonempty set, \((Y,d)\) is a metric space. Let \( F: K \to n(Y), \Psi: Y \to Y, a: K \to K, \lambda \in (0, +\infty), \)

\[
d(\Psi(x),\Psi(y)) \leq \lambda d(x,y) \quad \text{for } x, y \in Y
\]

and

\[
\lim_{n \to \infty} \lambda^n \delta(F(a^n(x))) = 0 \quad \text{for } x \in K.
\]

(1) If \( Y \) is complete and

\[
\Psi(F(a(x))) \subset F(x), \quad x \in K,
\]

then, for each \( x \in K \), the limit \( \lim_{n \to \infty} \cl \Psi^n \circ F \circ a^n(x) = f(x) \) exists and \( f \) is a unique selection of the multifunction \( \cl F \) such that \( \Psi \circ f \circ a = f \).

(2) If

\[
F(x) \subset \Psi(F(a(x))), \quad x \in K,
\]

then \( F \) is a single-valued function and \( \Psi \circ F \circ a = F \).

Proof. (1) Let us fix \( x \in K \). Replacing \( x \) by \( a^n(x) \) in (5) we get

\[
\Psi(F(a^{n+1}(x))) \subset F(a^n(x))
\]

for all \( n \in \mathbb{N}_0 \). Hence

\[
\Psi^{n+1}(F(a^{n+1}(x))) \subset \Psi^n(F(a^n(x))) \quad \text{for } n \in \mathbb{N}_0.
\]

Thus \( (\cl \Psi^n(F(a^n(x))))_{n \in \mathbb{N}_0} \) is a decreasing sequence of closed sets in a complete metric space. Moreover, in virtue of (4),

\[
\delta(\cl \Psi^n(F(a^n(x)))) \leq \lambda^n \delta(F(a^n(x))),
\]

so \( \lim_{n \to \infty} \delta(\cl \Psi^n(F(a^n(x)))) = 0 \). Therefore

\[
\lim_{n \to \infty} \cl \Psi^n(F(a^n(x))) = \bigcap_{n \in \mathbb{N}_0} \cl \Psi^n(F(a^n(x))) =: f(x)
\]

is a singleton. Of course, \( f(x) \in \cl F(x) \) and as \( \Psi \) is continuous we have

\[
\Psi(f(a(x))) = \Psi\left( \lim_{n \to \infty} \cl \Psi^n(F(a^n(a(x)))) \right) \subset \lim_{n \to \infty} \cl \Psi^{n+1}(F(a^{n+1}(x))) = f(x),
\]

thus \( \Psi \circ f \circ a = f \).

It remains to show the uniqueness of \( f \). Suppose that \( f, g \) are selections of \( \cl F \) and \( \Psi \circ f \circ a = f, \Psi \circ g \circ a = g \). By induction we obtain that \( \Psi^n \circ f \circ a^n = f \) and \( \Psi^n \circ g \circ a^n = g \) for \( n \in \mathbb{N}_0 \). Hence, for \( x \in K, n \in \mathbb{N}_0, \)

\[
d(f(x),g(x)) = d(\Psi^n \circ f \circ a^n(x), \Psi^n \circ g \circ a^n(x))
\]

\[
\leq \lambda^n d(f(a^n(x)),g(a^n(x))) \leq \lambda^n \delta(F(a^n(x))).
\]

As \( \lim_{n \to \infty} \lambda^n \delta(F(a^n(x))) = 0 \), we have \( f = g \).

(2) By (6) we obtain

\[
F(x) \subset \Psi^n(F(a^n(x))) \subset \Psi^{n+1}(F(a^{n+1}(x))), \quad n \in \mathbb{N}, \ x \in K.
\]
It follows that \((\mathcal{P}^n(F(a^n(x))))\) is an increasing sequence of sets in a metric space satisfying

\[
\delta(\mathcal{P}^n(F(a^n(x)))) \leq \lambda^n \delta(F(a^n(x))).
\]

Hence \(\delta(\mathcal{P}^n(F(a^n(x))))\) converges to 0 as \(n \to \infty\). Consequently, \(\mathcal{P}^n \circ F \circ a^n(x)\) is single-valued for all \(n \in \mathbb{N}_0, x \in K\) and \(\mathcal{P} \circ F \circ a = F\).  

Obviously, if \(\mathcal{P}\) is a contraction and \(\sup \{\delta(F(x)) : x \in K\} < \infty\), then the limit \(\lim_{n \to \infty} \lambda^n \delta(F(a^n(x))) = 0\) and the assertions of Theorem 2 are satisfied.

From now on we assume that \(Y\) is a real normed space. By \(\text{ccl}(Y)\) we denote the family of all nonempty, convex and closed subsets of \(Y\). For \(A, B \in n(Y)\) and \(\lambda \in \mathbb{R}\) we define

\[
A + B = \{a + b : a \in A, b \in B\} \quad \text{and} \quad \lambda A = \{\lambda a : a \in A\}.
\]

It is known (see [8]) that \(\lambda(A + B) = \lambda A + \lambda B\) and \((\lambda + \mu)A \subset \lambda A + \mu A\) for \(A, B \in n(Y)\) and \(\lambda, \mu \in \mathbb{R}\). If additionally \(A\) is convex and \(\lambda \mu \geq 0\), then

\[
(\lambda + \mu)A = \lambda A + \mu A.
\]

Now we give some applications of Theorem 2 to the problem of the stability of set-valued functional equations in several variables.

Notice that Theorem 1 follows from Theorem 2. Indeed, setting \(y = x\) in (3) we get

\[
F(2x) \subset F(x) + F(x), \quad x \in K.
\]

As the set \(F(x)\) is convex we have

\[
F(2x) \subset 2F(x), \quad x \in K
\]

and

\[
\frac{1}{2} F(2x) \subset F(x), \quad x \in K.
\]

By Theorem 2, with \(\mathcal{P}(x) = \frac{1}{2}x\) and \(a(x) = 2x\), the limit \(\lim_{n \to \infty} \mathcal{P}^n(F(a^n(x))) = \lim_{n \to \infty} \frac{1}{2^n} F(2^n x) = f(x)\) exists and \(f\) is the selection of \(F\). Moreover,

\[
\frac{1}{2^n} F(2^n (x+y)) \subset \frac{1}{2^n} F(2^n x) + \frac{1}{2^n} F(2^n y)
\]

for \(n \in \mathbb{N}\), so letting \(n \to \infty\) we obtain \(f(x + y) = f(x) + f(y)\). Theorem 2 gives the uniqueness of \(f\) as well.

If the inverse inclusion is satisfied, i.e,

\[
F(x) + F(y) \subset F(x + y) \quad \text{for} \quad x, y \in K,
\]

then \(F\) must be single-valued. This comes out from Theorem 2, too. We have

\[
F(x) \subset \frac{1}{2} F(2x), \quad x \in K,
\]
thus, with $\Psi(x) = \frac{1}{2}x$ and $a(x) = 2x$, we obtain that $F$ is single-valued and $F(x + y) = F(x) + F(y)$ for $x, y \in K$.

Next corollaries concern the general linear inclusions and correspond to the results in \cite{9,13}.

**Corollary 1.** Let $X$ be a real vector space, $Y$ be a real Banach space, $K$ be a convex cone in $X$, $a, b, p, q > 0$, $F : K \to \text{ccl}(Y)$,

$$F(ax + by) \subset pF(x) + qF(y) \quad \text{for} \ x, y \in K \quad (7)$$

and $\sup\{\delta(F(x)) : x \in K\} < \infty$.

(1) If $p + q > 1$, then there exists a unique selection $f : K \to Y$ of the multifunction $F$ such that

$$f(ax + by) = pf(x) + qf(y) \quad \text{for} \ x, y \in K.$$

(2) If $p + q < 1$, then $F$ is single-valued.

**Proof.** (1) Setting $y = x$ in (7) we get

$$F((a + b)x) \subset (p + q)F(x), \quad x \in K.$$ 

Dividing both sides of the last inclusion by $p + q$ we have

$$\frac{1}{p + q}F((a + b)x) \subset F(x), \quad x \in K.$$

By Theorem 2, with $\Psi(x) = \frac{1}{p+q}x$, $a(x) = (a + b)x$, there exists the limit

$$\lim_{n \to \infty} \Psi^n(F(a^n(x))) = \lim_{n \to \infty} \frac{1}{(p+q)^n}F((a + b)^n x) = f(x),$$

$f$ is single-valued and $f(x) \in F(x)$ for $x \in K$. Moreover, the inclusion

$$\frac{F((a + b)^n(ax + by))}{(p+q)^n} \subset p \frac{F((a + b)^n x)}{(p+q)^n} + q \frac{F((a + b)^n y)}{(p+q)^n}, \quad x, y \in K,$$

with $n \to \infty$, yields

$$f(ax + by) = pf(x) + qf(y), \quad x, y \in K.$$

The uniqueness also follows from Theorem 2.

(2) Putting $y = x$ in (7) we have

$$F((a + b)x) \subset (p + q)F(x), \quad x \in K.$$ 

Now, replacing $x$ by $\frac{1}{a+b}x$ in the last inclusion we obtain

$$F(x) \subset (p + q)F\left(\frac{1}{a+b}x\right), \quad x \in K.$$ 

Using Theorem 2, with $\Psi(x) = (p+q)x$, $a(x) = \frac{1}{a+b}x$, we get that $F$ is single-valued and satisfies the equality $F(ax + by) = pF(x) + qF(y)$ for $x, y \in K$.  \(\square\)
By the same method as in the proof of Theorem 2.1 in [13] we can also obtain the same result for the inclusion
\[ F(ax + by + k) \subset pF(x) + qF(y), \quad x, y \in K, \]
where \( k \in K, a + b \neq 1 \). Taking \( x_0 = \frac{k}{1 - a - b} \) and defining a multifunction \( G: K - x_0 \to \text{ccl}(Y) \) by \( G(x) = F(x + x_0) \) we obtain
\[ G(ax + by) \subset pG(x) + qG(y) \quad \text{for } x, y \in K. \]

If \( F: K \to \text{ccl}(Y) \) satisfies, instead of (7), the inclusion
\[ F(ax + by + k) \subset pF(x) + qF(y) + C, \quad x, y \in K, \]
where \( C \) is a compact and convex subset of \( Y, a + b \neq 1, p + q > 1 \), then there exists a unique single-valued function \( f: K \to Y \) satisfying the equation
\[ f(ax + by + k) = pf(x) + qf(y), \quad x, y \in K \]
and
\[ f(x) \in F(x) + \frac{1}{p + q - 1}C, \quad x \in K. \]

It is sufficient, as in [13], to consider the multifunction \( G(x) = F(x) + \frac{1}{p+q-1}C \) and use Corollary 1.

**Corollary 2.** Let \( X \) be a real vector space, \( Y \) be a real Banach space, \( K \) be a convex cone in \( X, a, b, p, q > 0, F: K \to \text{ccl}(Y) \),
\[ pF(x) + qF(y) \subset F(ax + by) \quad \text{for } x, y \in K \tag{8} \]
and \( \sup\{\delta(F(x)) : x \in K\} < \infty \).

1. If \( p + q < 1 \), then there exists a unique selection \( f: K \to Y \) of the multifunction \( F \) such that
\[ f(ax + by) = pf(x) + qf(y), \quad x, y \in K. \]

2. If \( p + q > 1 \), then \( F \) is single-valued.

**Proof.** (1) Putting \( y = x \) in (8) and taking into account that \( F \) has convex values we get
\[ (p + q)F(x) \subset F((a + b)x), \quad x \in K. \]
Replacing \( x \) by \( \frac{1}{a+b}x \) in the last inclusion we have
\[ (p + q)F\left(\frac{1}{a+b}x\right) \subset F(x), \quad x \in K. \]

Again by Theorem 2, with \( \Psi(x) = (p + q)x \) and \( a(x) = \frac{1}{a+b}x \), we get that the limit
\[ \lim_{n \to \infty} (p + q)^n F\left(\frac{1}{(a+b)^n}x\right) = f(x) \]
exists and \( f \) is the selection of \( F \).
Moreover, by
\[
p(p + q)^n F \left( \frac{1}{(a + b)^n} x \right) + q(p + q)^n F \left( \frac{1}{(a + b)^n} y \right) \\
\subset (p + q)^n F \left( \frac{1}{(a + b)^n} (ax + by) \right)
\]
we obtain
\[
 pf(x) + qf(y) = f(ax + by) \quad \text{for } x, y \in K.
\]
(2) Setting \( y = x \) in (8) and dividing both sides of (8) by \( p + q \) we get
\[
 F(x) \subset \frac{1}{p + q} F((a + b)x), \quad x \in K.
\]
By Theorem 2, \( F \) must be single-valued. \( \square \)

We can also obtain a similar result if \( F \) satisfies
\[
pF(x) + qF(y) \subset F(ax + by + k) + C, \quad x, y \in K + x_0,
\]
where \( x_0 = \frac{k}{1-a-b}, a+b \neq 1, p+q < 1 \). Then there exists a unique single-valued map \( f: K + x_0 \to Y \) such that
\[
 pf(x) + qf(y) = f(ax + by + k), \quad x, y \in K + x_0
\]
and
\[
 f(x) \in F(x) + \frac{1}{1-a-b} C, \quad x \in K + x_0
\]
(see [9]). To obtain this, we define a multifunction \( G: K \to \text{ccl}(Y) \) by
\[
 G(x) = F(x + x_0) + \frac{1}{1-a-b} C, \quad x \in K.
\]
Since the multifunction \( G \) satisfies (8) we can use Corollary 2.

Notice that if \( p + q = 1 \) the above method breaks down. Moreover, if \( a = b = \frac{1}{2} \) and \( p = q = \frac{1}{2} \), then we get the Jensen inclusions
\[
 F \left( \frac{x + y}{2} \right) \subset \frac{F(x) + F(y)}{2} \quad \text{or} \quad \frac{F(x) + F(y)}{2} \subset F \left( \frac{x + y}{2} \right).
\]
It easy to see that a multifunction \( F: \mathbb{R} \to \text{ccl}(\mathbb{R}) \) given by \( F(x) = [x-1, x+1] \) satisfies
\[
 F \left( \frac{x + y}{2} \right) = \frac{F(x) + F(y)}{2}, \quad x, y \in \mathbb{R}
\]
and each function \( f(x) = x + b \), where \( b \in [-1, 1] \) is a Jensen selection of \( F \).

Observe also that a constant set-valued function \( F(x) = M \), where \( M \in \text{ccl}(X) \) satisfies inclusions (7), (8) (in fact, \( F \) satisfies even the equality) if \( p + q = 1 \) and each constant function \( f(x) = m \), where \( m \in M \) satisfies \( f(ax + by) = pf(x) + qf(y) \).
Let \((T, \star)\) be a groupoid, where \(\star\) is square symmetric, i.e., \((x \star y) \star (x \star y) = (x \star x) \star (y \star y)\) for \(x, y \in T\). Then the map \(\rho : T \to T\) given by \(\rho(x) = x \star x\) for \(x \in T\) is an endomorphism of the groupoid \((T, \star)\). It is easy to check that

\[x \star y := ax + by + k, \quad a, b > 0, \quad x, y, k \in K,\]

where \(K\) is a convex cone, is square symmetric. The operation

\[x \star y := \alpha(x) + \beta(y) + \gamma_0, \quad x, y, \gamma_0 \in T\]

is square symmetric as well, where \(\alpha, \beta : T \to K\) are homomorphisms with \(\alpha \circ \beta = \beta \circ \alpha\). Next corollaries complement the above results and correspond to the Corollary 2.8 in [2].

**Corollary 3.** Let \((T, \star)\) be a groupoid, \(S \subset T, \rho(S) \subset S, a, b > 0, Y\) be a real Banach space, \(F : S \to \text{ccl}(Y),\)

\[F(x \star y) \subset pF(x) + qF(y)\quad \text{for } x, y \in S, \ x \star y \in S\quad (9)\]

and \(\sup\{\delta(F(x)) : x \in S\} < \infty.\)

1. If \(p + q > 1\), then there exists a unique selection \(f : S \to Y\) of the multifunction \(F\) such that

\[f(x \star y) = pf(x) + qf(y)\quad \text{for } x, y \in S, \ x \star y \in S.\]

2. If \(p + q < 1\) and \(\rho\) is an invertible function, then \(F\) is single-valued.

**Proof.** (1) Setting \(y = x\) in (9) and dividing both sides of (9) by \(p + q\) we get

\[\frac{1}{p + q}F(\rho(x)) \subset F(x), \quad x \in S.\]

Then, by Theorem 2 with \(\Psi(x) = \frac{1}{p+q}x, a(x) = \rho(x),\) there exists a limit

\[\lim_{n \to \infty} F(\rho^n(x)) = f(x)\]

and \(f\) is a unique selection of the multifunction \(F\) such that

\[f(x \star y) = pf(x) + qf(y), \quad x, y \in S, \ x \star y \in S.\]

(2) Putting \(y = x\) in (9) we get

\[F(\rho(x)) \subset (p + q)F(x), \quad x \in S.\]

As \(\rho\) is invertible we have

\[F(x) \subset (p + q)F(\rho^{-1}(x)), \quad x \in S.\]

By Theorem 2, \(F\) must be single-valued, which establishes the proof. \(\square\)

We observe that if

\[F(x \star y) \subset pF(x) + qF(y) + C\quad \text{for } x, y \in S, \ x \star y \in S,\]

where \(p + q > 1, C\) is a compact and convex subset of \(Y\), then \(G(x) = F(x) + \frac{1}{p+q-1}C, x \in S,\) satisfies the inclusion (9) (see [2]). Thus, by Corollary 3, there exists a unique selection \(f\) of \(G\) (that is \(f(x) \in F(x) + \frac{1}{p+q-1}C, x \in S\)) such that
\[ f(x \star y) = pf(x) + qf(y), \quad x, y \in S, \quad x \star y \in S. \]

**Corollary 4.** Let \((T, \star)\) be a grupoid, \(S \subset T, \rho(S) \subset S, a, b > 0, Y\) be a real Banach space, \(F: S \to \text{ccl}(Y)\)

\[ pF(x) + qF(y) \subset F(x \star y), \quad x, y \in S, \quad x \star y \in S \quad (10) \]

and \(\sup\{\delta(F(x)) : x \in S\} < \infty.\)

(1) If \(p + q < 1\) and \(\rho\) is an invertible function, then there exists a unique selection \(f: S \to Y\) of the multifunction \(F\) such that

\[ f(x \star y) = pf(x) + qf(y), \quad x, y \in S, \quad x \star y \in S. \]

(2) If \(p + q > 1\), then \(F\) is single-valued.

**Proof.** (1) Putting \(y = x\) in (10) we get

\[ (p + q)F(x) \subset F(\rho(x)), \quad x \in S. \]

As \(\rho\) is an invertible function we have

\[ (p + q)F(\rho^{-1}(x)) \subset F(x), \quad x \in S. \]

In the same manner by Theorem 2, with \(\Psi(x) = (p + q)x, a(x) = \rho^{-1}(x)\), we get the assertion.

(2) Setting \(y = x\) in (10) and dividing both sides of the (10) by \(p + q\) we get

\[ F(x) \subset \frac{1}{p + q} F(\rho(x)), \quad x \in S. \]

Therefore, by Theorem 2, the proof is complete.

We can also obtain a result similar to the above for \(F\) satisfying

\[ pF(x) + qF(y) \subset F(x \star y) + C, \quad x, y \in S, \quad x \star y \in S, \]

where \(p + q < 1, \rho\) is invertible and \(C\) is a compact and convex subset of \(Y\). Then, for \(G(x) = F(x) + \frac{1}{p+q-1}C, x \in S\), we have

\[ pG(x) + qG(y) \subset G(x \star y), \quad x, y \in S, \quad x \star y \in S \]

and by Corollary 4 there exists a unique selection \(f\) of the multifunction \(G\) (that is \(f(x) \in F(x) + \frac{1}{p+q-1}C, x \in S\)) such that

\[ f(x \star y) = pf(x) + qf(y), \quad x, y \in S, \quad x \star y \in S, \]

We end presenting an application of Theorem 2 to the quadratic inclusions.

**Corollary 5.** Let \(X\) be a real vector space, \(Y\) be a real Banach space, \(K\) be a set in \(X\) such that for \(x, y \in K, x + y \in K\) and \(x - y \in K, F: K \to \text{ccl}(Y)\) and \(\sup\{\delta(F(x)) : x \in K\} < \infty.\)

(1) If

\[ F(x + y) + F(x - y) \subset 2F(x) + 2F(y), \quad x, y \in K, \quad (11) \]
then there exists a unique selection \( f: K \to Y \) of the multifunction \( F \) such that
\[
f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad x, y \in K.
\]

(2) If
\[
2F(x) + 2F(y) \subset F(x + y) + F(x - y), \quad x, y \in K, \tag{12}
\]
then \( F \) is single-valued.

**Proof.** (1) Setting \( y = 0 \) in (11) we have
\[
F(x) + F(x) \subset 2F(x) + 2F(0) \quad \text{for} \quad x \in K.
\]
By the Rådström cancelation lemma [17] we get
\[
\{0\} \subset F(0).
\]
Next setting \( y = x \) in (11) and using the last inclusion we obtain
\[
F(2x) \subset F(2x) + F(0) \subset 4F(x), \quad x \in K
\]
and
\[
\frac{F(2x)}{4} \subset F(x) \quad \text{for} \quad x \in K.
\]
By Theorem 2, with \( \Psi(x) = \frac{1}{4}x, a(x) = 2x \), there exists the limit
\[
\lim_{n \to \infty} \Psi^n(F(a^n(x))) = \frac{F(2^n x)}{4^n} = f(x), f(x) \in F(x) \quad \text{for} \quad x \in K
\]
and as
\[
\frac{F(2^n(x + y))}{4^n} + \frac{F(2^n(x - y))}{4^n} \subset 2\frac{F(2^n x)}{4^n} + 2\frac{F(2^n y)}{4^n}
\]
we get \( f(x + y) + f(x - y) = 2f(x) + 2f(y) \) for \( x, y \in K \). Moreover, \( f \) is unique.

(2) Setting \( y = 0 \) in (12) and using the Rådström cancelation lemma we get
\[
F(x) + F(0) \subset F(x), \quad x \in K.
\]
Thus and by (12) with \( y = x \) we have
\[
4F(x) \subset F(2x) + F(0) \subset F(2x) \quad x \in K
\]
and
\[
F(x) \subset \frac{F(2x)}{4} \quad \text{for} \quad x \in K.
\]
By Theorem 2, with \( \Psi(x) = \frac{1}{4}x, a(x) = 2x \), \( F \) must be single-valued. \(\square\)

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