"COMPOSITE PARTICLES" and the EIGENSTATES of CALOGERO-SUTHERLAND and RUIJSENAARS-SCHNEIDER

by

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Abstract: We establish a one-to-one correspondance between the "composite particles" with \( N \) particles and the Young tableaux with at most \( N \) rows. We apply this correspondance to the models of Calogero-Sutherland and Ruijsenaars-Schneider and we obtain a momentum space representation of the "composite particles" in terms of creation operators attached to the Young tableaux. Using the technique of bosonisation, we obtain a position space representation of the "composite particles" in terms of products of vertex operators. In the special case where the "composite particles" are bosons and if we add one extra quasiparticle or quasihole, we construct the ground state wave functions corresponding to the Jain series \( \nu = p/(2np \pm 1) \) of the fractional quantum Hall effect.

I. INTRODUCTION

In a recent publication\[^{[1]}\], we introduced the concept of "composite particles" as a quasi-geometric construction which distributes \( N = pm + r \) (\( 0 < r \leq m \)) particles over a set of \( d \) states. The particles are decomposed into \( p \) groups of \( m \) particles each called "composite particles", and the remaining \( r \) particles are called uncomplete "composite particles".
The rule which distributes the $N$ particles over the $d$ states are such that the number of possible configurations is

$$C_{d+N-1-lE(N-1 \over m)}$$

where $E(x)$ means integer part of $x$ and where $l \geq 0$ and $m > 0$ are two integers. We note that if $l = 0$ we obtain the statistic for the bosons, and if $l = m = 1$ we have the statistic for the fermions. When the number $N$ of particles is large, the function integer part in $E \left( N - 1 \over m \right)$ becomes diluted and the number of configurations is given by Haldane’s formula

$$C_{d+(1-g)(N-1)}$$

where the fraction $g = \frac{l}{m}$. In that sense, the “composite particles” reproduce the fractional statistic introduced by Haldane and extend it naturally for any finite $N$ and finite $d$; in addition it gives a geometrical interpretation of this statistic.

A ”composite particle” (fig.1) is a set of $m$ particles and $d \geq l$ states with the constraint that the $(l-1)$ bottom states are empty and the $l^{th}$ state (from the bottom) has at least one particle. We define an uncomplete ”composite particle” as a set of $0 < r \leq m$ particles with no constraint on the empty states.

The main constraint in this construction is that a non empty state is entirely included inside a ”composite particle”. The configurations where a non empty state should be splitted into two consecutive ”composite particles” are forbidden (fig.1).

The partition function for the ”composite particles” when all the states have the same energy $E$ has been calculated somewhere else[8]; we reproduced the well-known result[4–8] for the average number of particles per state in the thermodynamic limit

$$n(y, g) = \frac{1}{W(y, g) + g}$$

where $y = \exp (-E/kT)$ and where the function $W(y, g)$ satisfies the equation

$$[W(y, g)]^g [1 + W(y, g)]^{1-g} = y^{-1}$$

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When the states have a linear distribution of energy so that the state $i$ contributes to the partition function by a factor

$$x^{i-1} y = \exp \left[ - (E + (i - 1) v) / kT \right]$$

we proved in ref.[3] that the partition function, in the thermodynamic limit ($d \to \infty$, $x \to 1$), is

$$Z \sim \exp \left( \frac{d}{\ln \xi} \int_{\xi_y}^{\nu} \frac{du}{u} \ln \left[ 1 + W^{-1}(u, g) \right] \right)$$

where $x = \xi^2$ and where the above integral is transformed in the literature into a Rogers dilogarithm function. In ref.[3], we transformed the partition function into a large $d$ expansion and consequently, we determined the finite $d$-size corrections to (6). These results show essentially that our construction of the ”composite particles” reproduces perfectly well previous results by Hikami (effective central charge for a $g$-on gas with fractional exclusion statistic), by van Elburg and Schoutens (quasi-particles in fractional quantum Hall effect edge theories), by Kedem, Klassen, McCoy and Melzer in the context of conformal field theory. This partition function is called the ”universal chiral partition function for exclusion statistic” by Berkovitch and McCoy.

Since we believe that our definition of ”composite particles” is the basic construction which generates fractional statistic, it seems necessary to establish the link between ”composite particles” and the eigenstates of the Hamiltonian for the models of Calogero-Sutherland and Ruijsenaars-Schneider which are known to satisfy fractional statistic. It is the purpose of this publication to establish this correspondance and to study some of its consequences. From the complete knowledge of the eigenfunctions of these two models, we have been able to construct a momentum space representation of the ”composite particles” in terms of creation operators attached to the corresponding Young tableaux; also, using the well-known bosonisation procedure in terms of products of vertex operators, we have defined a position space representation of the ”composite particles”.

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As an application, we have constructed the ground state wave functions for the electron, the quasiparticle and the quasihole in a fractional quantum Hall effect. To achieve this goal, we apply the vertex operator formalism to the case where the \( p \) "composite particles" are bosons and made of \( m \) (even) quasiparticles while the uncomplete "composite particle" is either a quasiparticle or a quasihole (\( r = 1 \)). When we have only one "composite particle" \( (p = 1) \) we obtain for the ground state wave function the Laughlin wave function\(^{[15]} \); more generally, for \( p \) "composite particles", we obtain the ground state wave function as the product of a boson wave function with satisfies the discrete \( Z_p \) symmetry times the usual fermionic wave function for the extra quasiparticle or quasihole. The corresponding filling factor defined as the number of "composite particles" divided by the total number of quasiparticles minus the total number of quasiholes \( (\nu = \frac{p}{N_\pm}) \) becomes, in this special case where \( m = 2n \) is even,

\[
\nu = \frac{p}{2np \pm 1}
\]

This filling factor defines the Jain series\(^{[16]} \) which characterise in the fractional quantum Hall effect most of the different plateaus in the Hall resistance.

In section 2, we establish a one-to-one correspondance between the configurations of the "composite particles" with \( N \) particles and the Young tableaux with at most \( N \) rows. Each particle corresponds to one row of the Young tableau; the length of the row is the difference between the "momentum" which labels the state where the particle is located in the given configuration and the "momentum" which labels the state where this same particle is located in the ground state.

Then, in order to construct the link with the models of Calogero-Sutherland (Ruijsenaars-Schneider), we introduce the notion of shifted momentum (which is a consequence of the thermodynamic Bethe Ansatz\(^{[17]} \)) where the "momentum" of the particles inside a given "composite particle" are successively shifted by \( 0, \frac{1}{m}, \frac{2}{m}, ..., \frac{(m-1)y}{m} \). As a result, we obtain the momentum (rapidities) for the eigenstates of the Hamiltonian which describes the model.
of Calogero-Sutherland (Ruijsenaars-Schneider) up to a global shift which comes from the labelling convention of the momenta in the Fermi sea.

This correspondence being established, we remind in section 3, the formalism of bosonisation which has been developed by many authors\cite{18} and which is based on the algebra of the vertex operators

\[
V(z) = e^{\sqrt{\beta Q}} \exp \left( \sum_{n>0} \sqrt{\frac{1-t^n}{1-q^n}} \frac{a_n^+ z^n}{n} \right) z^{\sqrt{\beta a_0}} \exp \left( -\sum_{n>0} \sqrt{\frac{1-t^n}{1-q^n}} \frac{a_n z^{-n}}{n} \right)
\]

where \( t = q^\beta \), where the operators \( a_n \) and \( a_n^+ \) satisfy the commutation relations

\[
[a_n, a_{n'}^+] = n \delta_{n,n'}
\]

and where the operators \( a_0 \) and \( Q \) satisfy

\[
[a_0, Q] = 1
\]

all other commutators being null. In (8), \( t \) and \( q \) are two parameters which characterise the homogeneous, symmetric Macdonald polynomials\cite{19} of several variables. The product of these vertex operators at different values of \( z \) generates the wave functions for the eigenstates of the model of Ruijsenaars-Schneider. In the limit \( t = q^\beta \) and \( q \to 1 \), the Macdonald polynomials become the Jack polynomials and we generate the wave functions for the eigenstates of the model of Calogero-Sutherland.

The product of several vertex operators \( V_+ (z) \) (where + means the exponential term containing the creation operators \( a_n^+ \) in (8)) taken in different positions (around a circle of length \( L \)) can be expanded in terms of Macdonald polynomials attached to all possible Young tableaux. The coefficients of these polynomials are, for each Young tableau \( \lambda \), a combination of the creation operators \( a_n^+ \) which we call \( a^+_\lambda (q,t) \). Then, the states \( a^+_\lambda (q,t) \; | \; \Omega_N > \), where \( | \; \Omega_N > \) is the vacuum state for \( N \) particles, define an orthonormal basis which is a momentum space representation for the "composite particles".

In section 4, we propose a position space representation for the "composite particles". The normal order product : \( V(z_1) \ldots V(z_N) \) : which consists in writing all creation operators
(including Q) at the left of all annihilation operators (including \(a_0\)) provide a position space representation for a set of different particles in different positions. Here, we want to define a "composite particles" in a complexified position \(z\) with all its constituent particles in the same position \(z\); moreover, we wish to introduce a duality relation between the quasihole and the "composite particles" (see Ref.[10]). To achieve this property, we define shifted positions \(q^i z\) for the \(N\) quasiparticles and quasiholes; then, if we choose \(t = q^N\), we obtain the property that the vertex operator for the "composite particles" is the dual \((q \leftrightarrow t \text{ and change of sign})\) of the vertex operator for the quasihole. Now, the characteristic feature which tells that the quasiparticles organize themselves into \(p\) "composite particles" is that these ones develop a discrete \(Z_p\) symmetry. This symmetry can be achieved using a formalism developed by Uglov[20] which consists in taking the limit \(q \to \exp(2i\pi/p)\).

In the special case where \(m = 2n\) is even (in that case the "composite particles" are bosons) and where there is a unique quasiparticle or quasihole \((r = 1)\) in the uncomplete "composite particle", we obtain for \(p = 1\), the Laughlin ground state wave function

\[
< V_{N\pm} (z_1) V_{N\pm} (z_2) >= (z_1 - z_2)^{2n\pm1} \tag{11}
\]

corresponding to a filling factor \(\nu = \frac{1}{2n\pm1}\). More generally, for any \(p\), we obtain for the ground state wave function the product of a \(Z_p\) invariant bosonic wave function times a fermionic wave function with power \(\pm1\) depending whether we have one extra quasiparticle or quasihole

\[
< V_{N\pm} (z_1) V_{N\pm} (z_2) >= (z_{1p}^p - z_{2p}^p)^{2n} (z_1 - z_2)^{\pm1} \tag{12}
\]

and corresponding to a filling factor \(\nu = \frac{p}{2np\pm1}\).

These results seem to be appropriate to describe a physics which is sensible to the number of "composite particles" rather than to the number of the individual particles which constitute them.
II. YOUNG TABLEAUX AND "COMPOSITE PARTICLES"

Let us remind the reader that the $N$ particles are distributed over $p$ "composite particles" containing $m$ particles each, and one extra uncomplete "composite particle" containing $r$ particles ($0 < r \leq m$) so that $N = pm + r$. The states of the $(p + 1)$ "composite particles" are labelled from top equal to 1 to bottom equal to $d$; for each "composite particle" we label the states from top equal to 1 to bottom equal to $\delta$, so that $\sum_{k=1}^{p+1} \delta_k = d$. To each state $i$ we attach a momentum which is simply $(i - 1)$ and which runs from 0 to $(d - 1)$. We label the particles from top to bottom and from left to right (on a same state) by an integer $j$ which runs from 1 to $N$.

In this section, we wish to show that for a given set of integers $l$ and $m$, and for a given number of states $d$, there exists a one-to-one correspondance between the possible configurations of $N$ particles into "composite particles" and the Young tableaux with at most $N$ rows and with the length of the first row $\lambda_1 \leq d - pl - 1$.

The ground state (fig.2) is the unique configuration with $d = pl + 1$; if $d \geq pl + 1$, the ground state is characterized by the fact that the momentum of the particle $j$ is equal to $k_j^0 = lE \left( \frac{j-1}{m} \right)$. To the ground state, we naturally associate the empty Young tableau where the length of the rows are all null

$$\lambda_j = 0 \quad j = 1, ..., N$$

(13)

Now, to a given configuration of particles in "composite particles" we associate a Young tableau in the following way: to any particle $j$ of the configuration, we associate a row of the Young tableau with length $\lambda_{N-j+1}$ which is the difference between the momentum of $j$ in the given configuration and the momentum of $j$ in the ground state configuration. If there are $r_i$ particles on the same state $i$, it corresponds to $r_i$ rows with the same length in the Young tableau. The Young tableau may be partitionned from the bottom into $p$ blocks of $m$ rows each corresponding to each "composite particle", and an extra block of $r$ rows at the
top of the tableau corresponding to the incomplete ”composite particle”. It may be noted that the length of the upper row of a given block is equal to the length of the lower row of the next block up (if the corresponding state is non empty); we also note that the length of the first row of the Young tableau is $\lambda_1 = k_N - pl$ where $k_N$ is the momentum of the $N^{th}$ particle. We give an illustration of the correspondance between ”composite particles” and Young tableaux in fig.3. Of course, this correspondance is crucially dependant of the values of the integers $l$ and $m$; in fig.4, we show how a given Young tableau corresponds to different configurations for different choices of $(l, m)$.

We now introduce the shifted momenta as a consequence of the so-called Bethe Ansatz which applies to the Calogero-Sutherland and Ruijsenaars-Schneider models. The Bethe Ansatz equations are of the type

$$e^{2i\pi \tilde{k}_r} = \prod_{s \neq r} S(\tilde{k}_s - \tilde{k}_r)$$  \hspace{1cm} (14)

If we take the logarithm on both sides of (14), we get

$$\tilde{k}_r = k_r + \sum_{s \neq r} \Theta(\tilde{k}_s - \tilde{k}_r)$$  \hspace{1cm} (15)

where the $k_r$’s are integers which may be used to label the eigenstates and where the phase shifts $\Theta(\tilde{k})$ are

$$\Theta(\tilde{k}) = \frac{1}{2i\pi} \ln S(\tilde{k})$$  \hspace{1cm} (16)

In Calogero-Sutherland and Ruijsenaars-Schneider models, we have

$$\Theta(\tilde{k}_s - \tilde{k}_r) = \begin{cases} \frac{l}{m} & \text{if } s < r \\ 0 & \text{if } s \geq r \end{cases}$$  \hspace{1cm} (17)

Consequently, to a particle $j$ inside a given ”composite particle” we attach a shifted momentum which is
\[ \widetilde{k}_j = k_j + q \frac{l}{m} \]  

(18)

where \( k_j \) is the momentum which labels the states and \( q = 0, 1, ..., m - 1 \) is the number of particles which are on the left (same state) and above \( j \), inside the corresponding "composite particle". If we introduce the corresponding momentum \( k_j^0 \) of the particle in the ground state, we obtain

\[ \widetilde{k}_j = \lambda_{N-j+1} + (mE \left( \frac{j-1}{m} \right) + q) \frac{l}{m} = \lambda_{N-j+1} + (j-1) \frac{l}{m} \]  

(19)

In order to make the correspondence with the usual momenta in the model of Calogero-Sutherland, we still have to perform a global shift over all momenta equal to \(-\frac{N-1}{2} \frac{l}{m}\) because the fermi sea (ground state for the fermions \( l = m = 1 \)) used to have momenta which runs from \(-\frac{N-1}{2}\) to \(+\frac{N-1}{2}\) and not from 0 to \( N - 1 \). We obtain the characteristic expression for the Calogero-Sutherland momenta

\[ \widetilde{k}_j = \lambda_{N-j+1} + \left( j - \frac{N+1}{2} \right) \beta \]  

(20)

where the Calogero-Sutherland coupling constant is \( \beta = \frac{l}{m} \).

The same organisation applies identically to the relativistic model of Ruijsenaars-Schneider with the single difference that the momenta are \( sh(\tilde{k}_j) \) where \( \tilde{k}_j \) are the shifted rapidities.

III. THE STATES FOR "COMPOSITE PARTICLES" AND VERTEX OPERATORS.

We wish to use our complete knowledge of the eigenstates of the Calogero-Sutherland and Ruijsenaars-Schneider Hamiltonian to construct the space of states for the "composite particles". The technique used here is the description of the eigenstates from the bosonization procedure and from the corresponding vertex operators.
We first remind the reader the description of the eigenstates in the model of Calogero-Sutherland. We introduce the complex variables

\[ z_i = \exp\left(\frac{2i\pi}{L}x_i\right) \]  

where \( x_i \) is the circular coordinates of the particle \( i \) along the circle of length \( L \). The ground state is represented by the function

\[ \Delta (z) = \prod_{i<j} (z_i - z_j)^\beta \]  

where \( \beta \) is the coupling constant in the Hamiltonian. Using the technique of bosonisation, the above function can be represented in the following way: we introduce the vertex operators

\[ V (z) = e^{\sqrt{\beta}Q} \exp\left(\sqrt{\beta} \sum_{n>0} \frac{a_n^+}{n} z^n\right) z^{\sqrt{\beta}a_0} \exp\left(-\sqrt{\beta} \sum_{n>0} \frac{a_n z^{-n}}{n}\right) \]  

where the creation operators \( a_n^+ \) and the annihilation operators \( a_n \) satisfy the commutation relations

\[ [a_n, a_m^+] = n \delta_{n,m} \]  

and where the operators \( a_0 \) and \( Q \) satisfy

\[ [a_0, Q] = 1 \]  

all other commutators being nul.

We define the normal product

\[ : V (z_1) ... V (z_N) : \]  

as the product of the exponential operators where all exponentials for creation operators (including \( Q \)) are written at the left of all exponentials for annihilation operators (including \( a_0 \)). Clearly, we have

\[ V (z_1) ... V (z_N) = \prod_{i<j} (z_i - z_j)^\beta : V (z_1) ... V (z_N) : \]
We define a vacuum state $| \Omega >$ such that the action of all the annihilation operators upon $| \Omega >$ gives zero

$$a_n | \Omega > = 0, \quad n \geq 0$$  \hspace{1cm} (28)

Now, the action of the operator $\exp\left(\sqrt{\beta}Q\right)$ insures the charge conservation in the vacuum expectation values and defines a vacuum state for $N$ particles as

$$e^{N\sqrt{\beta}Q} | \Omega > = | \Omega \left( N\sqrt{\beta} \right) >$$  \hspace{1cm} (29)

As a result of the above definitions, the ground state wave function is found to be

$$< \Omega \left( N\sqrt{\beta} \right) | V(z_1) ... V(z_N) | \Omega > = \prod_{i<j}(z_i - z_j)^\beta$$  \hspace{1cm} (30)

and we naturally associate the vacuum state $| \Omega \left( N\sqrt{\beta} \right) >$, with $\beta = \frac{L}{m}$, to the configuration of "composite particles" which describes the ground state (fig.2).

The wave functions for the excited states of the Calogero-Sutherland Hamiltonian are

$$J_\lambda (z_i, \beta) \prod_{i<j}(z_i - z_j)^\beta$$  \hspace{1cm} (31)

where $J_\lambda (z_i, \beta)$ is the symmetric, homogeneous Jack polynomial\cite{19} attached to the Young tableaux $\lambda$ which describes the momenta of the corresponding excited state. Now, in the bosonisation procedure

$$: V(z_1) ... V(z_N) : | \Omega > = \exp \left[ \sqrt{\beta} \sum_{n>0} \frac{a_+^n}{n} \left( \sum_{i=1}^{N} z_i^n \right) \right] | \Omega \left( N\sqrt{\beta} \right) >$$  \hspace{1cm} (32)

can be expanded in terms of Jack polynomials; this expansion defines a set of orthogonal states $a_+^\lambda (\beta) | \Omega \left( N\sqrt{\beta} \right) >$ such that

$$: V(z_1) ... V(z_N) : | \Omega > = \sum_\lambda \sqrt{N_\lambda (\beta)} \ J_\lambda (z_i, \beta) \ a_+^\lambda (\beta) \ | \Omega \left( N\sqrt{\beta} \right) >$$  \hspace{1cm} (33)

where $N_\lambda (\beta)$ is a normalisation. As a consequence, the wave functions for excited states (31) may be written as

$$\frac{1}{\sqrt{N_\lambda (\beta)}} < \Omega \left( N\sqrt{\beta} \right) | a_\lambda (\beta) \ V(z_1) ... V(z_N) | \Omega >$$  \hspace{1cm} (34)
From the facts that the configurations for "composite particles" are in a one-to-one correspondence with the Young tableaux which describe the eigenstates of the Calogero-Sutherland Hamiltonian, we may conclude that the orthogonal basis $a_\lambda^+ (\beta) | \Omega \left(N \sqrt{\beta} \right) >$ defines a representation of the configurations for the "composite particles". This representation is a momentum space representation; the adjoint representation in the complexified position space is obtained from the primary fields $V(z)$ and their products.

The property that the states $a_\lambda^+ | \Omega \left(N \sqrt{\beta} \right) >$ form an orthonormal basis is non trivial. Another set of symmetric, homogeneous polynomials which has this remarkable property is the set of Macdonald’s polynomials $M_\lambda (z_i, q, t)$. These polynomials depend of two parameters $q$ and $t = q^\beta$ and are generated from the vertex operators

$$V (z_1) ... V (z_N) : | \Omega > = \sum_\lambda \sqrt{N_\lambda (q,t)} \ M_\lambda (z_i, q, t) \ a_\lambda^+ (q,t) | \Omega \left(N \sqrt{\beta} \right) >$$

where $N_\lambda (q,t)$ is a normalisation. The vertex operators in (35) are defined as

$$V (z) = e^{\sqrt{\beta} Q} \ exp \left( \sum_{n>0} \sqrt{1 - t^n} a_n^+ \ a_n \right) \ z^{\sqrt{\beta} a_0} \ exp \left( - \sum_{n>0} \sqrt{1 - q^n} a_n z^{-n} \right)$$

In (36), the commutation relations between creation and annihilation operators are the same as in (24-25). In the case where $t$ and $q$ are real, the wave function for the ground state may be found as

$$< \Omega \left(N \sqrt{\beta} \right) | V (z_1) ... V (z_N) | \Omega > = \prod_{i<j} (z_i - q^r z_j)$$

where

$$(x; q)_n = \prod_{i=0}^{n-1} (1 - q^i x)$$

Let us make the following remark: if $t = q^k$, $k \in N_+$ and if $q$ is real, then

$$< \Omega \left(N \sqrt{\beta} \right) | V (z_1) ... V (z_N) | \Omega > = \prod_{i<j} \prod_{r=0}^{k-1} (z_i - q^r z_j)$$

and especially if $q \to 1$, we get $\prod_{i<j} (z_i - z_j)^k$.  

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When $t$ and $q$ are real, the wave functions for the excited states are obtained from

$$
\frac{1}{\sqrt{N_\lambda(q,t)}} \langle \Omega \left( N \sqrt{\beta} \right) | a_\lambda(q,t) V(z_1) \ldots V(z_N) | \Omega \rangle (40)

= M_\lambda(z_i, q,t) \prod_{i<j} \frac{(z_j, \beta \sqrt{q})}{(z_i, \sqrt{q})} \prod_{j=1}^{N} z_j^{(N-j)\beta}
$$

For this system again, we may conclude that the orthonormal basis $a_\lambda^+(q,t) | \Omega \left( N \sqrt{\beta} \right) >$ is a momentum space representation of the configurations for the "composite particles". The primary fields $V(z)$ and their products form an adjoint representation in the complexified position space.

**IV. REPRESENTATION OF "COMPOSITE PARTICLES" IN POSITION SPACE FOR $L = 1$.**

In this section, we wish to find a representation in position space for $N = pm + r \ (0 < r \leq m)$ particles inside $p$ "composite particles" (+ one uncomplete), the whole thing being defined at the complexified coordinate $z$. In order to construct this representation, we use the vertex operators introduced in section 3. Then, we apply and extend our results to the case where $N = pm \pm 1$ with $m$ even and interpret the theory as a possible theory for the edge states of the quantum Hall effect at filling factor $\nu = \frac{p}{pm \pm 1}$.

We follow in our construction a similar approach as the one proposed in Ref.[10] by van Elburg and Schoutens. The quasihole with charge $-e/N > 0$ and the electron with charge $e < 0$ are defined as mutually dual quasiparticles. More precisely, in the Calogero-Sutherland model, duality means the exchange $\sqrt{\beta} \to -\frac{1}{\sqrt{\beta}}$ and has for consequence on the wave functions of the excited states that the Young tableaux $\lambda$ attached to the Jack polynomials are transformed into $\tilde{\lambda}$ symmetric of $\lambda$ with respect to its diagonal.

The authors of Ref.[10] consider mainly the case where the filling factor is $\nu = \frac{1}{m}$; in appendix A, they "briefly describe a quasiparticle formulation of the composite edge theories..."
corresponding to the filling fractions $\nu = \frac{p}{pm+1}$ of the Jain series”. They consider that a convenient basis should be one “which separates a single charged mode from a set of $(p-1)$ neutral modes, the latter being governed by an $su(p)_1$ affine Kac-Moody symmetry”. This program is achieved below (and extended to the case where the filling factor is $\nu = \frac{p}{pm+1}$) by performing the limit introduced by Uglov$^{[20]}$: $q \to \exp(2i\pi/p)$ in the model of Ruijsenaars-Schneider.

The electron is a ”composite particles” built from $p$ bosonic ”composite particles” each containing $m$ (even) quasiparticles + one extra quasiparticle (for $\nu = \frac{p}{pm+1}$) or quasihole (for $\nu = \frac{p}{pm-1}$) as the incomplete ”composite particle” ($r = 1$); while the electrons have a fermionic behaviour in the exchange of their positions, the quasiparticles and the quasiholes exhibit in this exchange a behaviour typical of a fractional statistic.

We define the vertex operators for the quasihole $h$ and the quasiparticle $\pi$ as

$$V_h(z) = e^{-\frac{i}{\sqrt{\beta}}Q} \exp \left(- \sum_{n>0} \varepsilon_n \frac{i}{n} \left( q, t \right) \frac{a_n^+}{n} z^n \right).$$

$$V_\pi(z) = e^{\frac{i}{\sqrt{\beta}}Q} \exp \left( \sum_{n>0} \varepsilon_n \frac{i}{n} \left( q, t \right) \frac{a_n^+}{n} z^n \right).$$

(41)

$$V_h(z) = e^{-\frac{i}{\sqrt{\beta}}Q} \exp \left( \sum_{n>0} \varepsilon_n \frac{i}{n} \left( q, t \right) \frac{a_n^+}{n} z^n \right).$$

$$V_\pi(z) = e^{\frac{i}{\sqrt{\beta}}Q} \exp \left( - \sum_{n>0} \varepsilon_n \frac{i}{n} \left( q, t \right) \frac{a_n^+}{n} z^n \right).$$

(42)

where $t = q^\beta$, and

$$\chi_n \left( q, t \right) = \left| \frac{q^{n/2} - q^{-n/2}}{t^{n/2} - t^{-n/2}} \right|.$$  

(43)

The symbol $\varepsilon_n$ defines the determination of the square roots when $|q| = |t| = 1$. We choose $\varepsilon_n = e^{i\frac{\pi}{2}\delta_n}$ where $\delta_n = 0$ or 1 depending whether $\left( q^{n/2} - q^{-n/2} \right) / \left( t^{n/2} - t^{-n/2} \right)$ is $> 0$ or $< 0$. The consequence of this choice is that

$$\left[ \varepsilon_n \chi_n \left( q, t \right) \right]^2 = \frac{q^{n/2} - q^{-n/2}}{t^{n/2} - t^{-n/2}}.$$  

(44)
This choice is done in order to avoid absolute values $|\ |$ and to preserve the analticity in $q$ and $t$ when summing over $n$ in the calculation of the product of vertex operators.

The duality property can be implemented in the theory of Ruijsenaars-Schneider by shifting the variable $z \rightarrow q^i z$ and by taking the product over $i$ as we show below. We define the vertex operators for a set of $N_+ = pm + r$ quasiparticles and for a set of $pm$ quasiparticles and $r$ quasiholes; we define $N_- = pm - r > 0$. By definition,

$$V_{N_+} (z) =: V_\pi (q^{-\frac{N_+ - 1}{2}} z) V_\pi \left( q^{-\frac{N_+ - 3}{2}} z \right) \ldots V_\pi \left( q^{-\frac{N_+ - 1}{2}} z \right) : \quad (45)$$

$$V_{N_-} (z) =: V_\pi (q^{-\frac{N_- - 1}{2}} z) V_\pi \left( q^{-\frac{N_- - 1}{2}} z \right) V_h \left( q^{-\frac{N_- + 1}{2}} z \right) \ldots V_h \left( q^{-\frac{N_- + 1}{2}} z \right) : \quad (46)$$

If we choose respectively $t = q^{N_\pm}$ that is $\beta = N_\pm$, we get

$$V_{N_\pm} (z) = e^{\sqrt{\beta Q}} \exp \left( \sum_{n>0} \frac{a_n^+}{\sqrt{\chi_n}} (t, q) z^n \right) z^{\sqrt{\beta a_0}} \exp \left( - \sum_{n>0} \frac{\epsilon_n^{-1}}{\sqrt{\chi_n}} (t, q) a_n z^{-n} \right) \quad (47)$$

Clearly, the $N_\pm$ quasiparticles are described by a vertex operator which is dual to the vertex operator for the quasihole. From (47), we obtain the ground state wave functions

$$< \Omega (2\sqrt{N_\pm}) \mid V_{N_\pm} (z_1) V_{N_\pm} (z_2) \mid \Omega > = \prod_{k=-(N_\pm - 1)/2}^{(N_\pm - 1)/2} (z_1 - q^k z_2) \quad (48)$$

where the sum over $k$ is step one, all $k'$s are integers if $N$ is odd and are half integers if $N$ is even. The duality property gives

$$< \Omega \left( \sqrt{N_\pm} - \frac{1}{\sqrt{N_\pm}} \right) \mid V_{N_\pm} (z_1) V_{N_\pm} (z_2) \mid \Omega > = (z_1 - z_2)^{-1} \quad (49)$$

while

$$< \Omega \left( \sqrt{N_\pm} + \frac{1}{\sqrt{N_\pm}} \right) \mid V_{N_\pm} (z_1) V_{N_\pm} (z_2) \mid \Omega > = (z_1 - z_2) \quad (50)$$

If the quasiparticles and the quasiholes are not organized in "composite particles" we may take the limit $q \rightarrow 1$ and we obtain

$$V_{h_\pm} (z) = e^{-\frac{1}{\sqrt{N_\pm}}} \exp \left( - \frac{1}{\sqrt{N_\pm}} \sum_{n>0} \frac{a_n^+}{\sqrt{n}} z^n \right) z^{-\frac{1}{\sqrt{N_\pm}}} \exp \left( \frac{1}{\sqrt{N_\pm}} \sum_{n>0} \frac{a_n}{\sqrt{n}} z^{-n} \right) \quad (51)$$
By choosing \( t = q^{N\pm} \), the vertex operators for the quasiholes and for the quasiparticles become \( N\pm \) dependent so that they know that they belong to a set of \( N_+ \) quasiparticles or a set of \( pm \) quasiparticles and \( r \) quasiholes; in the limit \( q \to 1 \), the \( r \) quasiholes simply destroy \( r \) quasiparticles. If \( 0 < r \leq m \), the vertex operators \( V_{N\pm} (z) \) provide a description of the ”uncomplete composite particles” alone, that is for \( N_+ \leq m \) or \( p = 0 \). If we assign to \( V_{h\pm} (z) \) a charge +1, the operator \( V_{\pi\pm} (z) \) has a charge \(-1\) and the operator \( V_{N\pm} (z) \) has a charge \(-N_\pm\).

We obtain the following operator products

\[
V_{\pi\pm} (z_1) V_{\pi\pm} (z_2) = (z_1 - z_2)^{N_\pm} : V_{\pi\pm} (z_1) V_{\pi\pm} (z_2) : \quad (54)
\]

\[
V_{N\pm} (z_1) V_{N\pm} (z_2) = (z_1 - z_2)^{N_\pm} : V_{N\pm} (z_1) V_{N\pm} (z_2) : \quad (55)
\]

\[
V_{N\pm} (z_1) V_{\pi\pm} (z_2) = (z_1 - z_2) : V_{N\pm} (z_1) V_{\pi\pm} (z_2) : \quad (56)
\]

and the ground state wave functions for the \( N_+ \) quasiparticles or the \( pm \) quasiparticles and the \( r \) quasiholes are

\[
< \Omega \left( 2 \sqrt{N_\pm} \right) | V_{N\pm} (z_1) V_{N\pm} (z_2) | \Omega >= (z_1 - z_2)^{N_\pm} \quad (57)
\]

This wave function has a zero of order \( N_\pm \) at \( z_1 = z_2 \) \((x_1 = x_2 \) on the circle) that is \( N_\pm \) times the vanishing property for one particle \( (N_\pm = 1) \).

Now, in the general case where the \( pm \) quasiparticles organize themselves in \( p \) ”composite particles” of \( m \) quasiparticles each, we must implement the fact that the \( p \) ”composite particles” have an extra symmetry \( Z_p \). This can be achieved using a formalism due to Uglov\(^{[20]} \) and which consists in taking the limit \( q \to \exp (2i\pi/p) \) in \((41,42,47)\) together with
$t = q^{N\pm}$. In this limit we have in (48) a reorganization of the product over $k$ by grouping $m$ times a product of $p$ quantities, times (or divided by) a product of $r$ quantities:

$$
< \Omega \left( 2\sqrt{N_\pm} \right) \mid V_{N_\pm} (z_1) V_{N_\pm} (z_2) \mid \Omega > = \left( z_1^p + (-)^{N\pm} z_2^p \right)^m \left[ \prod_{k=-r}^{r-1} \left( z_1 - (-)^m q^k z_2 \right) \right]^{\pm 1}
$$

This is the ground state wave function for a "composite particles" of $N_+$ quasiparticles or of $pm$ quasiparticles and $r$ quasiholes ($N_- = pm - r$).

We now decompose the vertex operators in $p$ different modes, one charged mode and $(p - 1)$ neutral modes. We write

$$
\varepsilon^{-1}_n \chi^\frac{1}{2}_n \left( q^{N\pm}, q \right) \rightarrow \varepsilon^{-1}_k \sqrt{N_\pm} \quad \text{if } n = kp, \quad k \in \mathbb{Z}
$$

where $\varepsilon_k = e^{i \frac{\pi}{2} \delta k}$ with $\delta k = 0$ or 1 whether $(N - 1) k$ is even or odd, and

$$
\varepsilon^{-1}_n \chi^\frac{1}{2}_n \left( q^{N\pm}, q \right) = \varepsilon^{-1}_{k,s} \chi^\frac{1}{2}_s \left( q^{\pm}, q \right) \quad \text{if } n = kp + s, \quad s = 1, ..., p - 1 \quad k \in \mathbb{Z}
$$

where $\varepsilon_{k,s} = e^{i \frac{\pi}{2} s [(N\pm - 1)k + ms + \delta_{s,s}]}$ with $\delta_{s,s} = 0$ or 1 whether

$$
\left( q^{\pm rs/2} - q^{\pm rs/2} \right) / \left( q^{s/2} - q^{-s/2} \right) \text{ is } > 0 \text{ or } < 0.
$$

We obtain the vertex operators $V_{\pi \pm} (z)$ and $V_{N \pm} (z)$ as products of $p$ independant vertex operators

$$
V_{N \pm} (z) =: \mathcal{V}_{\pm,0} \left( z^p \right) \prod_{s=1}^{p-1} \mathcal{V}_{\pm,s} \left( z^p \right) \quad : (61)
$$

$$
V_{\pi \pm} (z) =: \mathcal{V}_{+,0} \left( z^p \right) \prod_{s=1}^{p-1} \mathcal{V}_{\pm,s} \left( z^p \right) \quad : (62)
$$

with

$$
\mathcal{V}_{\pm,0} (z) = e^{\sqrt{\frac{2 \pm \pi}{p} Q}} \exp \left( \sqrt{\frac{N_-}{p}} \sum_{k>0} \varepsilon^{-1}_k \frac{A^\pm_k}{k} z^k \right).
$$

$$
. \, \varepsilon^{\sqrt{\frac{N_+}{p} A_0}} \exp \left( -\sqrt{\frac{N_+}{p}} \sum_{k>0} \varepsilon^{-1}_k \frac{A^\pm_k}{k} z^{-k} \right) \quad (63)
$$
\[ \nabla_{\pm, s} (z) = \exp \left( \frac{1}{2} \left( q^{\pm r}, q \right) \sum_{k \geq 0} \varepsilon_{k,s} A_{k+s/p}^{+} z^{k+s/p} \right) \exp \left( -\frac{1}{2} \left( q^{\pm r}, q \right) \sum_{k \geq 0} \varepsilon_{k,s} A_{k+s/p} z^{-(k+s/p)} \right) \]  

(64)

\[ \nabla_{\pm, 0} (z) = e^{\frac{1}{\sqrt{p} N_{\pm}}} \exp \left( -\frac{1}{\sqrt{p} N_{\pm}} \sum_{k \geq 0} \varepsilon_{k} A_{k} z^{k} \right) \]  

(65)

\[ \nabla_{\pm, s} (z) = \exp \left( \frac{1}{2} \left( q, q^{\pm r} \right) \sum_{k \geq 0} \varepsilon_{k,s} A_{k+s/p}^{+} z^{k+s/p} \right) \exp \left( -\frac{1}{2} \left( q, q^{\pm r} \right) \sum_{k \geq 0} \varepsilon_{k,s} A_{k+s/p} z^{-(k+s/p)} \right) \]  

(66)

In (63-66) we defined the creation and annihilation operators \( A_{k+s/p}^{+} \), \( A_{k+s/p}^{+} \) and \( \overline{Q} \) as

\[ a_{kp+s} = \sqrt{p} A_{k+s/p}^{+} \]  

(67)

\[ a_{kp+s}^{+} = \sqrt{p} A_{k+s/p}^{+} \]  

(68)

\[ Q = \frac{1}{\sqrt{p} \overline{Q}} \]  

(69)

so that

\[ [A_{k+s/p}, A_{k'+s'/p}^{+}] = (k + s/p) \delta_{k,k'} \delta_{s,s'} \]  

(70)

\[ [A_{0}, \overline{Q}] = 1 \]  

(71)

It is clear that the operators \( \nabla_{\pm, s} (z) \) and \( \nabla_{\pm, s'} (z') \) (as well as \( \nabla_{\pm, s}^{+} (z) \) and \( \nabla_{\pm, s'}^{+} (z') \)) commute for \( s \neq s' \). The operator \( \nabla_{\pm, 0} (z) \) has a charge \( +N_{\pm} \) if the operator \( \nabla_{\pm, 0}^{+} (z) \) has a charge \( +1 \). The remaining \( (p - 1) \) operators \( \nabla_{\pm, s} (z) \) (or \( \nabla_{\pm, s}^{+} (z) \)) are neutral. It is interesting to note that \( V_{N_{\pm}} (z) \) and \( V_{\pi_{\pm}} (z) \) in (61,62) are really functions of \( z^{p} \); this will be commented later on.
From (63,65), we successively obtain for the charged mode

\[ V_{\pm,0}(z_1) V_{\pm,0}(z_2) = (z_1 + (-)^{N_{\pm}} z_2) \frac{N_{\pm}}{p} : V_{\pm,0}(z_1) V_{\pm,0}(z_2) : \]  

\[ (72) \]

Similarly, using the relation

\[ \sum_{k \geq 0} x^{k+\frac{1}{p}} = -\sum_{s=0}^{p-1} q^{-us} \ln \left[ 1 - q^u x^{1/p} \right], \quad s = 0, ..., p-1 \]  

\[ (75) \]

where \( q = \exp(2i\pi/p) \), we obtain for all the neutral modes

\[ W_{\pm}(z) = \prod_{s=1}^{p-1} V_{\pm,s}(z) \]  

\[ W'_{\pm}(z) = \prod_{s=1}^{p-1} V'_{\pm,s}(z) \]  

\[ (76) \]

the relations

\[ W_{\pm}(z_1) W_{\pm}(z_2) = (z_1 + (-)^{N_{\pm}} z_2)^{\mp \frac{1}{p} \left[ \prod_{k=-\frac{r+1}{2}}^{\frac{r-1}{2}} (z_1^{1/p} - (-)^m q^k z_2^{1/p}) \right]^{\pm 1}} : W_{\pm}(z_1) W_{\pm}(z_2) : \]  

\[ (77) \]

\[ W'_{\pm}(z_1) W'_{\pm}(z_2) = (z_1 + (-)^{N_{\pm}} z_2)^{\pm \frac{1}{p} \left[ \prod_{k=-\frac{r+1}{2}}^{\frac{r-1}{2}} \prod_{u=0}^{p-1} \prod_{s=0}^{p-1} \left[ z_1^{1/p} - (-)^m q^k + u z_2^{1/p} \right]^{\pm \rho_r(u)} \right]} : W'_{\pm}(z_1) W'_{\pm}(z_2) : \]  

\[ (78) \]

where

\[ \rho_r(u) = \frac{1}{p} \sum_{s=0}^{p-1} q^{-us} \sin^2 \left( \frac{2\pi s}{p} \right) \]  

\[ (79) \]

and

\[ W_{\pm}(z_1) W'_{\pm}(z_2) = \left[ \frac{z_1^{1/p} - z_2^{1/p}}{(z_1 - z_2)^{1/p}} \right] : W_{\pm}(z_1) W'_{\pm}(z_2) : \]  

\[ (80) \]
We now specify the special case where $N_{\pm} = pm \pm 1$ ($m$ even) and justify the application to the fractional quantum Hall effect with filling factor $\nu = p/(pm \pm 1)$. Hopefully, the above results simplify greatly at $r = 1$ since $\rho_1(u) = \delta_{u,0}$. The vertex operator for the electron is defined as

$$V_{N_{\pm}}(z) = e^{\frac{\phi_0'}{p} + \frac{\epsilon_{\pm}}{\sqrt{p}} \sum_{s=1}^{p-1} \phi_s(z^p)} : (81)$$

The vertex operator for the quasiparticle is defined as

$$V_{\pi_{\pm}}(z) = e^{\frac{\phi_0'}{p(pm \pm 1)} + \frac{\epsilon_{\pm}}{\sqrt{p}} \sum_{s=1}^{p-1} \phi_s(z^p)} : (82)$$

The vertex operator for the quasihole is defined as

$$V_{h_{\pm}}(z) = e^{-\frac{\phi_0'}{p(pm \pm 1)} - \frac{\epsilon_{\pm}}{\sqrt{p}} \sum_{s=1}^{p-1} \phi_s(z^p)} : (83)$$

where $\epsilon_{\pm} = e^{\frac{i\pi}{2} \delta_{\pm}}$ with $\delta_{+} = 0$ and $\delta_{-} = 1$. In (81-83), $\phi_0(z) = \phi_0^{+}(z) + \phi_0^{-}(z)$ with

$$\phi_0^{+}(z) = \bar{Q} + \sum_{k>0} \frac{A_k^+}{z^k}$$
$$\phi_0^{-}(z) = A_0 \ln z - \sum_{k>0} \frac{A_k^-}{z^k}$$

so that

$$[\phi_0^{-}(z_1), \phi_0^{+}(z_2)] = \ln (z_1 - z_2)$$

Similarly, $\phi_s(z) = \phi_s^{+}(z) + \phi_s^{-}(z)$ with

$$\phi_s^{+}(z) = \sum_{k \geq 0} \frac{A_{k+s/p}}{k+s/p} z^{(k+s/p)}, \quad s = 1, \ldots, p-1$$
$$\phi_s^{-}(z) = -\sum_{k \geq 0} \frac{A_{k+s/p}}{k+s/p} z^{-(k+s/p)}, \quad s = 1, \ldots, p-1$$

so that

$$[\phi_s^{-}(z_1), \phi_s^{+}(z_2)] = \delta_{s,s'} \sum_{u=0}^{p-1} q^{-us} \ln \left[ 1 - q^u \left( \frac{z_2}{z_1} \right)^{1/p} \right]$$

(87)
and consequently
\[
\sum_{s=1}^{p-1} \phi_{s}^{-} (z_1) \sum_{s'=1}^{p-1} \phi_{s'}^{+} (z_2) = \ln \left[ \frac{\frac{z_1^{1/p} - z_2^{1/p}}{p}}{(z_1 - z_2)} \right]
\]  
(88)

We obtain the ground state wave functions for the electrons, the quasiparticles and the quasiholes as
\[
<\Omega \left( 2\sqrt{N_{\pm}} \right) | V_{N_{\pm}} (z_1) V_{N_{\pm}} (z_2) | \Omega > = (z_1^p - z_2^p)^m (z_1 - z_2)^{\pm 1}
\]  
(89)

We note that at \( p = 1 \), we obtain the so-called Laughlin wave function.
\[
<\Omega \left( \frac{2}{\sqrt{N_{\pm}}} \right) | V_{N_{\pm}} (z_1) V_{N_{\pm}} (z_2) | \Omega > = (z_1^p - z_2^p)^{\pm m} (z_1 - z_2)^{\pm 1}
\]  
(90)

\[
<\Omega \left( \frac{1}{\sqrt{N_{\pm}}} \right) | V_{N_{\pm}} (z_1) V_{N_{\pm}} (z_2) | \Omega > = (z_1 - z_2)
\]  
(91)

\[
<\Omega \left( \sqrt{N_{\pm}} - \frac{1}{\sqrt{N_{\pm}}} \right) | V_{N_{\pm}} (z_1) V_{N_{\pm}} (z_2) | \Omega > = (z_1 - z_2)^{-1}
\]  
(92)

The electron wave function is antisymmetric in the exchange \( z_1 \leftrightarrow z_2 \) while the quasiparticle and the quasihole behave in this exchange according to the fractional statistic described by the exchange angle
\[
\frac{\theta}{\pi} = \pm 1 - \frac{\pm m}{pm \pm 1}
\]  
(93)

If we define
\[
m' = \frac{\mp m}{pm \pm 1}
\]  
(94)

we have the relation
\[
pm' \pm 1 = \frac{1}{pm \pm 1}
\]  
(95)
which is another manifestation of the duality between the electron and the quasihole. If the charge of the electron is $e$, then the charge of the quasiparticle is $e^* = \frac{e}{N_\pm} = \frac{e}{pm\pm 1} = \pm e (1 - m\nu)$ where we introduce the filling ratio $\nu = \frac{p}{N_\pm} = \frac{p}{pm\pm 1}$. The $U (1)$ charge comes from the field $\phi_0 (z)$ while the $(p - 1)$ remaining modes $\phi_s (z)$ are neutral. The charged sector is described by a Calogero-Sutherland model with $\beta = \frac{1}{p}$; the neutral sectors are described by Calogero-Sutherland models with $\beta = \frac{1}{2p}$ and without the zero modes (the terms $Q$ and $A_0$).

Let us mention here that we concentrated on the ground state wave function for two set of ”composite particles”, but the expansion of the products of vertex operators may provide the wave functions for the various excited states and for several set of ”composite particles” in different complexified coordinates $z_i$.

Let us close this section by trying to understand the relation between the fields $\phi_s (z), \ s = 0, \ldots, p - 1$ and the $p$ fields usually introduced in the Luttinger liquid in order to describe the edge states of the quantum Hall effect. According to Wen$^{[24]}$

$$S_{edge} = \frac{1}{4\pi} \int dtdx \sum_{I,J=1}^{p} [K_{IJ} \partial_t \bar{\varphi}_I \partial_x \varphi_J - V_{IJ} \partial_x \bar{\varphi}_I \partial_x \varphi_J]$$

where $V$ is a positive definite matrix which describes the velocity of the edge excitations, and where the $p \times p$ matrix $K_{IJ} = m + \delta_{I,J}$. The cinetic part of the Lagrangian disappear in the Hamiltonian

$$H_{edge} = \frac{1}{4\pi} \int dtdx \sum_{I,J=1}^{p} V_{IJ} \partial_x \bar{\varphi}_I \partial_x \varphi_J$$

However, the cinetic part of the Lagrangian is responsible for the canonical quantization of the system. The matrix $K_{IJ}$ is cyclic with eigenvalues $\lambda_0 = pm+1$ and $\lambda_s = 1, \ s = 1, \ldots, p-1$. We may diagonalize the cinetic part by Fourier transforming it

$$\bar{\varphi}_I = \frac{1}{p} \sum_{s=0}^{p-1} q^{-sI} \varphi_s$$

with $q = \exp(2i\pi/p)$. The cinetic part of the Lagrangian becomes
\[ L = \frac{pm + 1}{p} \partial_t \varphi_0 \partial_x \varphi_0 + \frac{1}{p} \sum_{s=1}^{p-1} \partial_t \varphi_s \partial_x \varphi_{p-s} \]  \hspace{1cm} (99)

so that the conjugate momenta are

\[ \pi_0 = \frac{pm + 1}{p} \partial_x \varphi_0 \]
\[ \pi_s = \frac{1}{p} \partial_x \varphi_{p-s} \]  \hspace{1cm} (100)

The equal time commutators

\[ [\varphi_s(x,0), \pi_{s'}(y,0)] = i \delta_{s,s'} \delta(x - y) \]  \hspace{1cm} (101)

give

\[ [\varphi_0(x,0), \varphi_0(y,0)] = -\frac{i}{2} \frac{p}{pm + 1} \epsilon(x - y) \]
\[ [\varphi_s(x,0), \varphi_{s'}(y,0)] = -\frac{i}{2} \delta_{s+s',0} p \epsilon(x - y) \]  \hspace{1cm} (102)

or equivalently

\[ [\psi_s(x,0), \psi_{s'}(y,0)] = i \pi \delta_{s+s',0} \epsilon(x - y) \]  \hspace{1cm} (103)

with

\[ \psi_0(x,t) = i \sqrt{2\pi} \sqrt{\frac{pm + 1}{p}} \varphi_0(x,t) \]
\[ \psi_s(x,t) = i \sqrt{2\pi} \sqrt{\frac{1}{p}} \varphi_s(x,t), \quad s = 1, ..., p - 1 \]  \hspace{1cm} (104)

If we split \( \psi_s(x,0) \) into annihilation and creation parts and the distribution \( \epsilon(x - y) \) accordingly, we get

\[ [\psi_{s}^{-}(x,0), \psi_{s'}^{+}(y,0)] = \delta_{s+s',0} \ln \left( e^{2i\pi \frac{x}{L}} - e^{2i\pi \frac{y}{L}} \right) \]  \hspace{1cm} (105)

where \( L \) is a regulator and \( -L/2 \leq x - y \leq L/2 \).

For \( s = s' = 0 \), the commutator (105) is the same than the commutator (85). Consequently, the fields \( \phi_0(z) \) and \( \psi_s(x,0) \) are unitarily equivalent. The main difference between
Wen’s theory and our results is that the fields $\psi_s(x,0)$ are equally charged (as can be seen from (105)) for $s = 0, ..., p-1$, while our fields $\phi_s(z)$ are neutral for $s = 1, ..., p-1$. One way of recovering one charged field and $p-1$ neutral fields from Wen’s fields $\psi_s(x,0)$ should be to distinguish in their Fourier decomposition and in the decomposition of

$$\epsilon(x-y) = \frac{1}{i\pi} \left[ \sum_{n \neq 0} \frac{1}{n} e^{2i\pi n \frac{x-y}{L}} + 2i\pi \frac{x-y}{L} \right]$$

(106)

the various modes $n = kp$ and $n = kp + s$, $s = 1, ..., p-1$ but this is exactly what we have done.

In Ref.[21 – 22] the authors introduce an extra transformation $z \to z^{1/p}$ which is suggested by the fact that the vertex operators $V_{N \pm}(z)$, $V_{\pi \pm}(z)$ and $V_{h \pm}(z)$ in (81-83) are functions of $z^p$. As a consequence, the wave functions (and the Green functions) become multivalued; however, this transformation allows a reinterpretation of the results in terms of twisted conformal field theory[23]. In this method the fields $\phi_s(z)$ are introduced by hands using the properties of conformal field theory; in our method the fields come naturally from Calogero-Sutherland and Ruijsenaars-Schneider theory via the specific limit $q \to \exp(2i\pi/p)$ which fits perfectly well the idea of ”composite particles”. In Ref[22], the authors conclude that ”any non-null wave function can be written as cluster of $p$ one-electrons fields”; clearly, our approach runs the other way round, from ”composite particles” to conformal field theory.

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FIGURES

Forbidden configuration for m=3

p "composite particles" for m=3, l=2

Fig.1: "composite particles"
Fig. 2: Ground state for p "composite particles" (+ one uncomplete) and N=pm+r particles
Fig.3: Correspondance between "composite particles" and Young tableaux for (fig.1), $m=3 \ l=2$
Fig. 4: Two different configurations $(m=3, l=2)$ and $(m=2, l=3)$ for the same young tableau