Higher-Order Floquet Topological Insulators with Anomalous Corner States

Biao Huang¹, ‡ and W. Vincent Liu¹, ‡

¹Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh PA 15260, USA
²Wilczek Quantum Center, School of Physics and Astronomy and T. D. Lee Institute, Shanghai Jiao Tong University, Shanghai 200240, China

(Dated: November 5, 2018)

Higher order topological insulators have emerged as a new class of phases, whose robust in-gap “corner” modes arise from the bulk higher-order multipoles beyond the dipoles in conventional topological insulators. Despite rapid theoretical and experimental breakthroughs, all discussions have been constrained to the static scenario due to the lack of specific schemes to compute higher-order dynamical topological invariant. Here we provide a concrete model and explicit constructions of topological invariants for a Floquet-driven system exhibiting anomalous corner states. The bulk quadrupolar moment for the eigenstates of static Floquet operators vanishes identically, while the anomalous topological invariant associated with full-time evolution correctly describes the quantized corner charges. The signature of such a phase in cold atom experiments is discussed through corner particle dynamics and a Floquet-Bloch band tomography.

Introduction — The advent of topological insulators (TI) [1–3] have revolutionized our understanding in the phases of matter by the principle of bulk-boundary correspondence. The non-trivial topology of the bulk wave-function is predicted by the principle to render robust edge states occupying a region one dimension lower than that of the bulk. Such a picture can be understood as a quantized dipole, or “first-order”, polarization of the bulk wave function which results in excessive charge at the system’s boundary. A wide class of topological systems have since been identified, including the $\mathbb{Z}_2$ TI [4–7], the Weyl semimetals [8–11], quantum anomalous Hall effects [12, 13], and the topological superconductors [14, 15].

Analogous to electrodynamics, one naturally wonders about the generalization of dipoles to higher-order multipoles for bulk wave functions, which would modify the bulk-boundary correspondence. Such a novel class of “higher-order” topological insulators (HOTI) were constructed successfully in recent theories [16–25], and have quickly led to experimental realizations in photonic [26, 27], electric circuits [28] and solid state systems [29, 30]. With vanishing dipole but quantized $n$-th order multipoles in HOTI, both the bulk and edge exhibit gapped spectrum, while in-gap “corner” states emerge in a region being $n$-dimensional lower than that of the bulk. Inspired by such success, rapid progress has been made towards higher order semimetals [31–33], superconductivity [34–39], spin liquids [40] and symmetry protected topological phases [41, 42] in the past year.

So far, all discussions on HOTI have been focusing on static scenarios. It is known, however, that periodically driven (Floquet) systems far from equilibrium are fertile grounds for intriguing phenomena without static counterparts [43–46]. In particular, there exist “anomalous” Floquet insulators (AFI) [47–52] whose static topological invariants vanish for all Floquet-Bloch bands, but edge states still emerge due to winding numbers of evolution operators $U(k, t)$ with genuine time dependence. It is therefore tantalizing to explore the higher order extensions of AFI, which would open the door to a whole new set of non-equilibrium topological matters with multipole features. Yet, the current scheme for studying HOTI lacks a natural way to generalize the topological numbers built from Hamiltonian eigenstates to that from $U(k, t)$, prohibiting practical investigations of Bloch wave multipoles in dynamical systems. Clearly, an urgent need is posted to bridge the theoretical gap between HOTI and AFI, and to extend the experimental realization of AFI into the higher-order scenarios.

In this work, we explicitly construct the models and topological invariants for such a higher order Floquet topological insulator (HOFTI) exhibiting anomalous quadrupoles. Within this phase, the static nested polarization, constructed by replacing the Hamiltonian as in previous work with Floquet operator $U_F(k) \equiv U(k, T)$, vanish identically. But anomalous “corner” states still arise and are described by the dynamical quadrupoles contained in the full evolution of $U(k, t)$. The key step is to involve a Hermitian mapping of $U(N) \rightarrow \mathbb{Z}_2 \times U(N)$, which does not change the topological property of evolution operators $U(k, t)$ while allowing for projections of $U(k, t)$ onto the system’s boundary. Nested Wilson loops constructed from the evolution operators projected to the boundary correctly capture the dynamical edge topology and predict the quantized charge accumulating at the system’s corners, coinciding with numerical results. Further, a cold atom realization of a HOFTI is discussed, together with its signatures of anomalous corner states in detections. Our work paves the way for a systematic study on non-equilibrium topological matters of higher multipole nature.

Models, symmetries, and phase diagrams — As a minimal model, we consider a binary drive with Hamiltonians in two driving sectors illustrated in Fig. 1. To set the time reversal invariant point at time $t = 0$, we write them into a 3-step driving with period $T$, where the Hamiltonian $H(k, t + T) = H(k, t)$ reads

$$H(k, t) = \begin{cases} \gamma h_1, & t \in [0, T/4]; \\ \lambda h_{2k}, & t \in [T/4, 3T/4]; \\ \gamma h_1, & t \in [3T/4, T]. \end{cases}$$ (1)

Here $\gamma, \lambda$ are hopping constants, and the (dimensionless) in-
staneous Hamiltonians written in momentum space read

\[ h_1 = \tau_1 \sigma_0 - \tau_2 \sigma_2, \]
\[ h_{2k} = \cos k_x \tau_1 \sigma_0 - \sin k_y \tau_2 \sigma_3 - \cos k_y \tau_2 \sigma_1 - \sin k_y \tau_2 \sigma_2, \]

where \( \tau, \sigma \) are Pauli matrices spanning the basis for 4 sublattices \( \mathcal{W}_k = (\psi_{k_1}, \psi_{k_2}, \psi_{k_3}, \psi_{k_4})^T \) [53] as shown in Fig. 1. The evolution operator is then

\[ U(k, t) = P_y e^{-i \int_0^t \mathcal{H}(k, \tau) d\tau}, \]

where \( P_y \) denotes the path-ordering of time \( \tau \). Such a model enjoys high solvability [54] and rich phase diagrams including both the normal and anomalous Floquet topological phases.

The dynamical model defined in Eqs. (1)–(3) satisfies all of the time reversal \( \Theta = TK \), particle-hole \( \Gamma = CK \), and chiral symmetries \( S = \Theta \Gamma \) [48, 49]

\[ T^{-1} U(k, t) T = U^*(-k, -t), \quad T = \tau_0 \sigma_0; \]
\[ C^{-1} U(k, t) C = U^*(-k, t), \quad C = \tau_3 \sigma_0; \]
\[ S^{-1} U(k, t) S = U(-k, t), \quad S = \tau_3 \sigma_0. \]

And \( \Theta^2 = \Gamma^2 = S^2 = +1 \). Here \( K \) is complex conjugation \( Ki = -iK \), and \( T, C, S \) are unitary matrices. Thus, the system belongs to the BDI class which holds no topological indices for the conventional first-order Floquet TIs [48, 49] in two dimensions. That means the bulk dipoles always vanish, and the in-gap modes in an open-boundary system would be attributed to higher order multipoles.

We are interested in the system’s characters at spectroscopic time \( t = NT \), with \( N \) being integers. The bulk spectrum of the Floquet operator \( U_F(k) \equiv U(k, T) : U_F(k)|E_k \rangle = e^{i E_k t}|E_k \rangle \) can be obtained as [54]

\[ \exp(i E_{k_s}) = \exp(\pm E_k) = f_k \pm i \sqrt{1 - f_k^2}, \]
\[ f_k = \cos E_k = \cos(\sqrt{2} \gamma) \cos(\sqrt{2} \lambda), \]
\[ = -\frac{\cos k_x + \cos k_y}{2} \sin(\sqrt{2} \gamma) \sin(\sqrt{2} \lambda), \]

with each quasi-energy band \( E_{k_s} = \pm E_k \) being two-fold degenerate. The gap closes when \( f_k = 1 \), giving the topological phase boundaries

\[ \sqrt{2} \lambda = \pm \sqrt{2} \gamma + m\pi, \quad m \in \mathbb{Z}. \]

Therefore, one can divide the irreducible phase diagram into four distinct regions as shown in Fig. 2(a) [55]. To characterize the topological properties, two numerical results are presented for each phase [56].

First, exact diagonalization of the real-space Floquet operator \( U_F(x, y) \) with open boundary conditions are shown in Fig. 2(c). Enforced by particle-hole symmetries, the boundary modes only exist at quasienergy \( E = 0 \) and/or \( E = \pi \), represented by the red dots. Each set of boundary modes within one bulk gap involves four eigenstates localizing at the four corners of the sample respectively. Fig. 2(b) illustrates the amplitudes of one such mode.

Second, we compute the transverse polarization \( p_y(x) \) in a semi-infinite system (described by \( U_F(x, k_y) \)), which represents the shift of wave-function centers towards \( y \)-direction.
away from lattice sites [16, 17, 54],

\[ W_s(x, k_y) = P_{k_y} e^{-i \frac{\hbar}{2} \int_{A_t(x, k_y)}^{A_s(x, k_y)} e^{-2\pi i v_{A_s}(x)} (v_j(x, k_y))}, \]

\[ p_j(x) = \frac{1}{L_y} \sum_{m, j, k_y} |v_j(x, k_y)|^2 u_m(x, k_y) \quad (7) \]

Here \( W_s \) is the Wilson loop with the Bloch-wave Berry connection \( A_t \), whose non-Abelian components \( [A(x, k'_y)]_{mn} = \langle u_m(x, k'_y) | \partial_{k_y} | u_n(x, k'_y) \rangle \). \( k_y \) in the first equation is the basepoint for the loop integration \( \frac{\hbar}{2} \). \( p_j \) are eigenstates of \( U_F \) in lower or upper bands, which render \( p_j \)'s of opposite signs. \( P_{k_y} \) means \( k'_y \) path-ordering. If \( U_F \) contains bulk quadrupoles, there would appear non-zero \( p_j \) near the edges \( x = 0, L_y \), as it represents "nested" polarization resulting in corner modes. For parameters near phase transitions, corner modes could be more extended into the bulk, but the half-system total \( p_j \) is quantized to be half-integer (modulo integer) in the topological quadrupolar phase.

From the numerical results, one can readily identify that \( \odot \) and \( \boxplus \) in Fig. 2 are the (normal) topological and trivial insulating phases respectively. The in-gap corner modes in \( \odot \) arise from non-trivial polarization \( p_j \) appearing at the system's edge. In fact, \( \odot \) and \( \boxplus \) are smoothly connected to the static phases in Refs. [16, 17], as when \( \gamma = 0 \) in \( \odot \) (or \( \lambda = 0 \) in \( \boxplus \)), the Floquet model Eq. (1) is described by a static Hamiltonian \( h_{2k} \) (or \( h_1 \)) in Eq. (2), which holds non-trivial (or trivial) Bloch-wave quadrupoles [16]. Further, the phase \( \boxplus \) is also a normal topological one, but with opposite polarization compared with \( \odot \) and the corner modes appear at \( E = \pi \) gap. The representative \( U_F = h_{1/2} \) at \( \sqrt{2}(y, \lambda) = (\pi, \frac{\pi}{2}) \) is equivalent to \( h_{2k} \) up to a global gauge transformation by \( h_{1/2} \).

The most interesting phenomenon occurs in phase \( \boxplus \). The Floquet operator reduces to \( U_F = h_1 \sqrt{2} \) with the representative \( \sqrt{2}(y, \lambda) = (\frac{\pi}{2}, \pi) \), which seems like a topological trivial one. Indeed, \( p_j \) vanishes identically in this case, indicating that the static Floquet operator involves no quantized quadrupoles. However, the corner modes do show up in both the \( E = 0, \pi \) gaps. The contradiction signals that the anomalous corner modes in phase \( \boxplus \) result from the quantized quadrupoles associated with the full dynamics of \( U(t) \) throughout a period, as we will discuss next.

**Topological invariant** — To define a topological number with time \( t \) being an independent parameter, we need to periodize the evolution operator in time such that \( t \in [0, T] \) functions as \( S^1 \) in the parameter space. This can be constructed by using the return map [47–49]:

\[ U_x(t) = U(t) e^{-iH_{eff}^{(x)} / T}, \quad (8) \]

where \( H_{eff}^{(x)} = \sum_n -i \log(\lambda_n) |\lambda_n\rangle \langle \lambda_n| \), and \( \epsilon \) denotes the branch cut when taking the logarithm. Here \( |\lambda_n\rangle \)'s are eigenstates of the Floquet operator \( U_F |\lambda_n\rangle = \lambda_n |\lambda_n\rangle \), and therefore we have by construction \( U_x(0) = U_x(T) = I \) because \( e^{-iH_{eff}^{(x)}} = U_F^{-1} \).

Choices of branch cuts determine the non-analytic point in quasiequilibrium spectra, which is the gap that we check the possible existence of corner modes.

Let us focus on class BDI related to our model above. In the presence of chiral symmetry, \( U_x \) at half evolution period takes block diagonal/off-diagonal forms depending on branch cuts \( \epsilon \) [3, 49]

\[ U_{x=0} \left( \frac{T}{2} \right) = \begin{pmatrix} 0 & U_x \\ U_x^{-1} & 0 \end{pmatrix}, \quad U_{x=\pi} \left( \frac{T}{2} \right) = \begin{pmatrix} 0 & U_x \\ U_x^{-1} & 0 \end{pmatrix}, \quad (9) \]

where \( U_x \) are unitary matrices. We emphasize that these operators still carry the information of the full evolution through \( H_{eff} \) and is not simply a static one at half period.

Up to now, the procedures are standard for an AFM [47–49]. Our major result is the scheme to analyze the winding number of the unitary \( U_x \) within the subspaces of certain Wannier bands. To do so, we introduce the tool Hermitian operator [57]

\[ H_{tool} = \begin{pmatrix} 0 & U_x \\ U_x^{-1} & 0 \end{pmatrix}. \quad (10) \]

Note \( H_{tool}^2 = I \), the eigenvalues of the Hermitian matrix are \( \pm 1 \). Thus, it introduces a map of \( \mathcal{T}^D \rightarrow U(N) \rightarrow \mathbb{Z}_2 \times U(N) \), where \( \mathbb{T} \) denotes torus and \( D \) is the spatial dimension. The solution of \( H_{tool} \) can be written as

\[ |m \pm \rangle = \frac{1}{\sqrt{2}} \left( \pm |\alpha_m \rangle \right). \quad (11) \]

where \( |\alpha_m \rangle = (|\alpha_m(1)\rangle, |\alpha_m(2)\rangle, \ldots, |\alpha_m(N)\rangle)^T \) is an arbitrary normalized \( U(N) \) spinor \( |\alpha_m(1)\rangle |\alpha_m(2)\rangle \ldots |\alpha_m(N)\rangle = 1 \), and \( \pm \) denotes eigenvalues \( \pm 1 \). Since the \( \mathbb{Z}_2 \) is a trivial reproduction of the properties of the \( U(N) \), we can focus on one of the two branches, say, the + branch, and simplify the notation \( |m \rangle \equiv |m + \rangle \). The eigenvector \( |m \rangle \) serves to introduce the Berry connection of \( U_x \) within the subspace spanned by selected \( |\alpha_m \rangle \):

\[ i[A_{\mu}]_{mn} = \langle m | \partial_{k_y} | n \rangle = \frac{1}{2} \langle \alpha_m | (U_x^\dagger \partial_{k_y} U_x)|n\rangle + \langle \alpha_m | \partial_{k_y} | n\rangle. \quad (12) \]

Without any subspace projection, one natural choice is \( |\alpha_m \rangle = (0, 1, 2, \ldots, 0_{m-1}, 1_m, 0_{m+1}, \ldots, 0_N) \), where the subscript denotes the \( i \)-th element of the \( U(N) \) spinor, and \( m = 0, 1, \ldots, N \) ranges over the whole \( U(N) \) space. Then we recover the usual Berry connection for the unitary matrix

\[ i[A_{\mu}]_{mn} = \langle U_x \partial_{k_y} U_x^{-1} \rangle_{mn}. \quad (13) \]

where the \( m, n \) on the right-hand-side denotes matrix indices.

With the aid of Eq. (12), we can compute the higher order winding number using the following procedure in the discrete Brillouin zone.

**I** Calculate the (first-order) Wilson loop with "bare" Berry connections defined in Eq. (13),

\[ F_{x,k} = \frac{1}{2} (I + U_x \partial_{k_y} U_x^{-1} \partial_{k_y}. \quad (14) \]

\[ W_{x,k} = F_{x,k} U_x \partial_{k_y} U_x^{-1} \partial_{k_y} \partial_{k_y} F_{x,k} \cdot \cdots \cdot F_{x,k+(L_x-1)\Delta k, \epsilon}. \]
For our model, $W_{x,k}$ is an SU(2) matrix. It represents the polarization of Bloch waves towards $x$-direction.

(2) Diagonalize the first order Wilson loop $W_{x,k}$,

$$W_{x,k}|v_y(k)\rangle = e^{i2\pi\nu(k)|v_y(k)\rangle}. \quad (15)$$

The phases $v_y(k)$ are the Wannier band spectrum, and $|v_y\rangle$'s carry the information of edge topology.

(3) Obtain the nested Wilson loop $\tilde{W}_{x,k}$ as

$$\tilde{F}^{(j)}_{y,k} = \frac{1}{2} \langle v_y(k)|U_{-k}^\dagger e^{-i\Delta k_x} U_{-k} |v_y(k)\rangle + (v_y(k)|v_y(k + \Delta k_x, e_i)\rangle - \frac{1}{2}, \quad \Delta k_x = \frac{2\pi}{L_y},$$

$$\tilde{W}^{(j)}_{y,k} = \tilde{F}^{(j)}_{y,k} \tilde{F}^{(j)}_{y,k+\Delta k_x, e_i} \cdots \tilde{F}^{(j)}_{y,k+(L_y-1)\Delta k_x, e_i}. \quad (16)$$

Here the Wilson loop has projected Berry connection defined in Eq. (12), where $|\alpha_m\rangle$ is replaced by the Wannier wave functions $|v_y\rangle$. The nested Wilson loop $\tilde{W}^{(j)}_{x,k}$ involves the simultaneous polarization of Bloch waves towards the $x$ and $y$ direction, which leads to the corner states. In our case, $\tilde{W}^{(x)}_{y,k}$ and $\tilde{W}^{(y)}_{x,k}$ are two U(1) numbers and therefore no further diagonalization is needed [58]. The nested polarization can be obtained as

$$\tilde{p}^{(j)}_{x,y} = \frac{1}{L_x} \sum_{k_x} \tilde{p}^{(j)}_{x}(k_x); \quad \tilde{p}^{(j)}_{x}(k_x) = \frac{1}{2\pi} \text{arg}(\tilde{W}^{(j)}_{y,k}). \quad (17)$$

Note that an arbitrary $k_x$ in $\tilde{W}_{x,k}$ can be taken because the phase factor does not depend on the base point $k_x$ after the (path-ordered) integration over $k_x$ in Eq. (16).

![FIG. 3. The Wannier bands $v_x$ and their nested polarization $\tilde{p}_x$ for the $\varepsilon = 0$ (left) and $\varepsilon = \pi$ (right) gaps in phase $\varphi$. Parameters are the same as in Fig. 2.](image)

We apply the above procedure and compute the quadrupole strength of our model, as shown in Fig. 3. We see that for phase $\varphi$, both the $E = 0$ and $E = \pi$ gap involves two gapped Wannier bands (denoted as $j = \pm$). Therefore, each Wannier band carries its own topological number. Indeed, $\tilde{p}_x(k_x) = 1/2$ identically for all $k_x$ up to numerical accuracy, and therefore the quadrupole strength $\tilde{p}_{x,y} = 1/2$ for both gaps. Thus, $\tilde{p}_{x,y}$ correctly captures the corner charges as shown in Fig. 2(c) for phase $\varphi$.

Similar to the first order Floquet TI's [47-49], one can write the relation between dynamical and static quadrupoles as

$$\tilde{p}^{(x)}_{x,y} - \tilde{p}^{(x)}_{x,y} = \tilde{p}^{(x)_{\text{band}} - \varepsilon_1} \mod 1, \quad (18)$$

where $\tilde{p}^{(x)}_{x,y}$, $\tilde{p}^{(x)_{\text{band}} - \varepsilon_1}$ are dynamical quadrupoles computed above for gaps $\varepsilon_{1,2}$ respectively, and $\tilde{p}^{(x)_{\text{band}} - \varepsilon_1}$ is the static quadrupole computed by $U_F$ for the band sandwiched by $\varepsilon_1$ and $\varepsilon_2$. This explains that the static quadrupoles in phase $\varphi$ vanish for all Floquet-Bloch bands (see Fig. 2(c)) because $\tilde{p}^{(x)}_{x,y} = \tilde{p}^{(x)_{\text{band}}}$ is 1/2, as shown in Fig. 3. One can also verify this relation for other phases, where gaps with (or without) corner modes correspond to $\tilde{p}^{(x)}_{x,y} = 1/2$ (or $\tilde{p}^{(x)_{\text{band}} - \varepsilon_1} = 0$).

**Experimental proposals** — First, we point out that the photonic experiments on the static HOTIs [26, 27] can be generalized to our HOFTI case with little modifications. The $h_1$ and $h_2k$ in Eqs. (1) have already been realized in these experiments, and one may extend them by performing repeated quenches between the two Hamiltonians resembling the situation for first order AFIs [50, 52, 59]. The corner modes can be observed directly through the photon intensity at sample corners. In contrast to the static HOTI which has only one in-gap frequency peak for corner modes, our theory predicts that HOFTI would exhibit two such peaks both between and aside of the bulk modes.

![FIG. 4. Experimental signatures. (a)-(d) The density evolution from the initial configuration at $t = 0$ in (a) to the final state at $t = 100\tau$ in (b)-(d) for different phases in a lattice of $20 \times 20$ unit cells. (Only the corner $3 \times 3$ cells are shown for visibility. Phase $\varphi$ has the same results as $\varphi$ in (b)). Each pixel denotes one lattice site, and axes ticks denote unit cells. (e) The TOF measurement for $\chi_{k,l}$ (left), $\theta_k$ (middle) and $\varphi_k$ (right) in phase $\varphi$ (upper row) and phase $\varphi$ (lower row). In all figures, $\sqrt{2}(\gamma, \lambda) = \pi(0.5, 0.1), \pi(0.5, 0.05), \pi(0.9, 0.5), \pi(0.5, 0.95)$ for phases $\varphi$ ~ $\varphi$ respectively.](image)

Second, we discuss the signatures of HOFTI in cold atom experiments [60]. Here, the particle conservation could lead to novel dynamics showing the localization of corner modes, and a time-of-flight (TOF) density mapping (band tomography) could reveal the trivial static polarization in the HOFTI phase. See the results of simulations in Fig. 4.

A "corner" can be engineered via the microscope methods associated with digital mirror device [61, 62], which gives rise to a step-like potential barrier $V(x, y) = V_0\Theta(-x)\Theta(-y)$ up to single site accuracy. With high enough $V_0$, it can be represented by the open boundary conditions in our simula-
tions. We consider the initial states with particles concentrating around the sample corner [63], which overlap with both the corner and bulk eigenstates. Since the bulk spectrum is dispersive, systems with/without corner states would have finite/zero density remaining at the corner after long-time evolution in an in situ imaging, as shown in Fig. 4(a)–(e).

To distinguish corner states in HOFTI from the static ones in HOTI, one can apply band tomography and map out all the eigenstates for $U_F$, from which the bulk nested polarization can be backed up. We briefly describe the procedures below and leave more details in SM [54]. $U_F(k)$ and therefore its eigenstates can be parameterized by three angles $\chi_k, \theta_k \in [0, \pi], \varphi_k \in (-\pi, \pi)$ in $S^3$. Working in the regimes $\gamma \approx 0$ for phase \(1\) or $\sqrt{2} \lambda \approx \pi$ for \(2\), where eigenstates connect smoothly to those in static systems, one can populate the “lowest two” Floquet bands even in the slow-driving situation. After equilibration, the lattice depth is ramped up and the “lowest two” Floquet bands even in the slow-driving situations.

**Acknowledgment** — This work is supported by AFOSR Grant No. FA9550-16-1-0006, ARO Grant No. W911NF-11-1-0230, and NSF of China Overseas Scholar Collaborative Program Grant No. 11429402 sponsored by Peking University.

*phys.huang.biao@gmail.com

†vwl@pitt.edu

[1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
[2] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
[3] A. P. S. Ching-Kai Chiu, Jeffrey C.Y. Teo and S. Ryu, Rev. Mod. Phys. 88, 035005 (2016).
[4] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005).
[5] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005).
[6] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science 314, 1757 (2006), cond-mat/0611399v1.
[7] D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, and M. Z. Hasan, Nature 452, 970 (2009), 0902.1356v1.
[8] A. V. Xiangang Wan, Ari M. Turner and S. Y. Savrasov, Phys. Rev. B 83, 205101 (2011).
[9] B. Q. Lv, H. M. Weng, B. B. Fu, X. P. Wang, H. Miao, J. Ma, P. Richard, X. C. Huang, L. X. Zhao, G. F. Chen, Z. Fang, X. Dai, T. Qian, and H. Ding, Phys. Rev. X 5, 031013 (2015), 1502.04684v3.
[10] S.-Y. Xu, I. Belopolski, D. S. Sanchez, C. Guo, G. Chang, C. Zhang, G. Bian, Z. Yuan, H. Lu, Y. Feng, T.-R. Chang, P. P. Shibayev, M. L. Prokopovych, N. Alidoust, H. Zheng, C.-C. Lee, S.-M. Huang, R. Sankar, F. Chou, C.-H. Hsu, H.-T. Jeng, A. Bansil, T. Neupert, V. N. Strocov, H. Lin, S. Jia, and M. Z. Hasan, Science 1, 10 (2015), 1508.03102v2.
[11] B. Yan and C. Felser, Annual Review of Condensed Matter Physics 8, 337 (2017).
[12] C.-X. Liu, S.-C. Zhang, and X.-L. Qi, Annu. Rev. Condens. Matter Phys. 7, 301 (2016), 1508.07160v1.
[13] C.-Z. Chang et al., Science 340, 167 (2013).
[14] M. Sato and Y. Ando, Rep. Prog. Phys. 80, 076501 (2017), 1608.03395v3.
[15] P. Zhang, K. Yaji, T. Hashimoto, Y. Ota, T. Kondo, K. Okazaki, Z. Wang, J. Wen, G. D. Gu, H. Ding, and S. Shin, Science 360, 182 (2018), 1706.05163v2.
[16] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Science 357, 61 (2017).
[17] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Phys. Rev. B 96, 245115 (2017), 1708.04230v2.
[18] F. K. Kunst, G. van Miert, and E. J. Bergholtz, Phys. Rev. B 97, 241405 (2018), 1712.07911v1.
[19] F. Schindler, A. M. Cook, M. G. Vergniory, Z. Wang, S. S. Parkin, B. A. Bernevig, and T. Neupert, Sci. Adv. 4, eaa0346 (2018).
[20] Z. Song, Z. Fang, and C. Fang, Phys. Rev. Lett. 119, 246402 (2017), 1708.02952v5.
[21] J. Langbehn, Y. Peng, L. Trifunovic, F. von Oppen, and P. W. Brouwer, Phys. Rev. Lett. 119, 246401 (2017), 1708.03640v2.
[22] L. Trifunovic and P. Brouwer, 1805.02598v1.
[23] D. Calaguari, V. Juricic, and B. Roy, 1808.08965v1.
[24] R. Queiroz and A. Stern, 1807.04141v2.
[25] M. Ezawa, 1806.03007v1.
[26] M. Serra-Garcia, V. Peri, R. Süssstrunk, O. R. Bilal, T. Larsen, L. G. Villanueva, and S. D. Huber, Nature 555, 342 (2018), 1708.05015v1.
[27] C. W. Peterson, W. A. Benalcazar, T. L. Hughes, and G. Bahl, Nature 555, 346 (2018), 1710.03231v1.
[28] S. Imhof, C. Berger, F. Bayer, J. Brehm, L. Molenkamp, T. Kiessling, F. Schindler, C. H. Lee, M. Greiter, T. Neupert, and R. Thomale, 1708.03647v1.
[29] Z. Wang, B. J. Wieder, J. Li, B. Yan, and B. A. Bernevig, 1806.11116v1.
[30] F. Schindler, Z. Wang, M. G. Vergniory, A. M. Cook, A. Murani, S. Sengupta, A. Y. Kasumov, R. Deblock, S. Jeon, I. Drozdov, H. Bouchiat, S. Guron, A. Yazdani, B. A. Bernevig, and T. Neupert, Nat. Phys. 14, 918 (2018), 1802.02585v1.
[31] M. Lin and T. L. Hughes, 1708.08457v1.
[32] M. Ezawa, Phys. Rev. Lett. 120, 026801 (2018), 1709.08425v2.
[33] M. Ezawa, Phys. Rev. B 97, 155305 (2018), 1802.03571v1.
[34] H. Shapourian, Y. Wang, and S. Ryu, Phys. Rev. B 97, 094508 (2018), 1711.02122v2.
[35] Y. Wang, M. Lin, and T. L. Hughes, (2018), 1804.01531v2.
The similar “Hermitian map” is introduced in [48] to discuss the full degeneracy of corner states, we put in a static, infinitesimal onsite chemical potential biases (0, 2, 3, 1) × 10⁻⁴ for sublattices 1, 2, 3 and 4. The choice is to fix the signs of polarizations at x = 0 and x = L_x.

The similar “Hermitian map” is introduced in [48] to discuss the first order Floquet topological insulators, and the one-to-one correspondence between Hermitian maps and U(t), and the equivalence of topology for U(t) and for the Hermitian map are proved there.

If W̃_j is a matrix, one needs to diagonalize it similar to Eq. (15) and obtain the phase argument ˜ϕ_j as for ϕ_j in step (2).

Note Ref [16] has already elaborated the realization of γ_1 + χ_L with π-flux Hofstadter models in the presence of optical superlattices, and the Floquet driving in our case corresponds to repeated quenching the relative lattice depth for superlattices. Thus, we only focus on the detection features missing in the previous literature.

[36] E. Khalaf, Phys. Rev. B 97, 205136 (2018), 1801.10050v4.
[37] X. Zhu, Phys. Rev. B 97, 205134 (2018), 1802.00270v2.
[38] Z. Yan, F. Song, and Z. Wang, Phys. Rev. Lett. 121, 096803 (2018), 1803.08545v2.
[39] Q. Wang, C.-C. Liu, Y.-M. Lu, and F. Zhang, (2018), 1804.04711v1.
[40] V. Dwivedi, C. Hickey, T. Eschmann, and S. Trebst, 1803.08922v1.
[41] Y. You, T. Devakul, F. J. Burnell, and T. Neupert, 1807.09788v2.
[42] A. Rasmussen and Y.-M. Lu, 1809.07325v1.
[43] R. Nandkishore and D. A. Huse, Annu. Rev. Condens. Matter Phys. 6, 15 (2015).
[44] A. Eckardt, Rev. Mod. Phys. 89, 011004 (2017).
[45] K. Sacha and J. Zakrzewski, Reports on Progress in Physics 81, 016401 (2017).
[46] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, arXiv preprint arXiv:1804.11065 (2018).
[47] M. S. Rudner, N. H. Lindner, E. Berg, and M. Levin, Phys. Rev. X 3, 031005 (2013).
[48] R. Roy and F. Harper, Phys. Rev. B 96, 155118 (2017), 1603.06944v2.
[49] S. Yao, Z. Yan, and Z. Wang, Phys. Rev. B 96, 195303 (2017), 1708.05993v2.
[50] S. Mukherjee, A. Spracklen, M. Valiente, E. Andersson, P. Hberg, N. Goldman, and R. R. Thomson, Nat. Commun. 8, 13918 (2017), 1605.03877v1.
[51] Y.-G. Peng, C.-Z. Qin, D.-G. Zhao, Y.-X. Shen, X.-Y. Xu, M. Bao, H. Jia, and X.-F. Zhu, Nat. Commun. 7, 13368 (2016).
[52] L. J. Maczewsky, J. M. Zeuner, S. Nolte, and A. Szameit, Nat. Commun. 8, 13756 (2017), 1605.03877v1.
[53] The direct product of Pauli matrices τ, σ can be understood as, i.e. τ,σ = \begin{pmatrix} τ_1 & 0 \\ 0 & -τ_1 \end{pmatrix}.
[54] See supplemental materials for details.
[55] Note that the axis lines V/2, V/2, l = 0, π are not phase boundaries except for the special points satisfying Eq. (6).
[56] The lattice sizes in Fig. 2(c) are (L_x, L_y) = (20, 20) for the spectrum, and (20, 100) for transverse polarization p_x. To break the full degeneracy of corner states, we put in a static, infinitesimal onsite chemical potential biases (0, 2, 3, 1) × 10⁻⁴ for sublattices 1, 2, 3 and 4. The choice is to fix the signs of polarizations at x = 0 and x = L_x.
[57] If W̃_j is a matrix, one needs to diagonalize it similar to Eq. (15) and obtain the phase argument ˜ϕ_j as for ϕ_j in step (2).
[58] Q. Cheng, Y. Pan, H. Wang, C. Zhang, D. Yu, A. Gover, H. Zhang, T. Li, L. Zhou, and S. Zhu, Nat. Commun. (2018), 1804.05134v2.
[59] Note Ref [16] has already elaborated the realization of γ_1 + χ_L with π-flux Hofstadter models in the presence of optical superlattices, and the Floquet driving in our case corresponds to repeated quenching the relative lattice depth for superlattices. Thus, we only focus on the detection features missing in the previous literature.
[60] M. E. Tai, A. Lukin, M. Rispoli, R. Schittko, T. Menke, D. Borgnia, P. M. Preiss, F. Grusdt, A. M. Kaufman, and M. Greiner, Nature 546, 519 (2017).
[61] A. Quelle, C. Weitenberg, K. Sengstock, and C. M. Smith, New J. Phys. 19, 113010 (2017).
[62] Here, as an illustration, we consider uniform initial particle distribution at the corner 3 × 3 unit cells (totally 36 sites). None-uniform distributions do not change the qualitative features. Experimentally, the initial state can be prepared by an off-center harmonic trap rendering the corner sites lowest chemical potentials [65], or direction moving fermion particles to the edge via microscope-based optical tweezers [66].
[63] In fact, there are infinitely many choices to back up the angles. We demonstrate one such choice in [54].
[64] M. Aidelsburger, M. Lohse, C. Schweizer, M. Atala, J. T. Barreiro, S. Nascimbene, N. Cooper, I. Bloch, and N. Goldman, Nature Physics 11, 162 (2015).
[65] M. Endres, H. Bernien, A. Keesling, H. Levine, E. R. Anschuetz, A. Krajenbrink, C. Senko, V. Vuletic, M. Greiner, and M. D. Lukin, Science 354, 1024 (2016).
[66] P. Hauke, M. Lewenstein, and A. Eckardt, Phys. Rev. Lett. 113, 045303 (2014).
[67] N. Fläschner, B. Rem, M. Tarnowski, D. Vogel, D.-S. Lühmann, K. Sengstock, and C. Weitenberg, Science 352, 1091 (2016).
[68] M. Tarnowski, M. Nuske, N. Fläschner, B. Rem, D. Vogel, L. Freytag, K. Sengstock, L. Mathey, and C. Weitenberg, Phys. Rev. Lett. 118, 240203 (2017), 1703.02813v2.
Supplemental Material

CONTENTS

References .......................................................... 5
Algebraic details for evolution operators ....................... 7
Floquet operator ...................................................... 7
Periodized evolution operator .................................... 8
Numerical algorithms .............................................. 9
Static nested polarization ......................................... 9
Simulation of dynamics for non-interacting particles ........ 10
Additional details for experimental proposals ............... 10
Sublattice density dynamics at the corner unit cell ......... 10
Tomography for Floquet-Bloch band through time-of-flight for four-band model ............................................. 10
Populating “lower” Floquet-Bloch bands under slow driving .................. 11
Four-band tomography with lowest two bands being degenerate .................. 11

ALGEBRAIC DETAILS FOR EVOLUTION OPERATORS

Floquet operator

First, we consider the bare evolution operator. Set \( t_1 = t_3 = 1/2, t_2 = 1 \), which means \( T = 2 \). When \( t \in [0, 1/2] \):

\[
U(t) = \cos(\sqrt{2}yt) - i \sin(\sqrt{2}yt) \frac{h_1}{\sqrt{2}}; \quad (S1)
\]

when \( t \in [1/2, 3/2] \),

\[
U(t) = \left( \cos(\sqrt{2} \lambda (t - 1/2)) - i \sin(\sqrt{2} \lambda (t - 1/2)) \frac{h_{2k}}{\sqrt{2}} \right) \left( \cos(\gamma/\sqrt{2}) - i \sin(\gamma/\sqrt{2}) \frac{h_1}{\sqrt{2}} \right); \quad (S2)
\]

and finally, when \( t \in [3/2, 2] \),

\[
U(t) = \left( \cos(\sqrt{2} \gamma (t - 3/2)) - i \sin(\sqrt{2} \gamma (t - 3/2)) \frac{h_1}{\sqrt{2}} \right) \left( \cos(\sqrt{2} \lambda) - i \sin(\sqrt{2} \lambda) \frac{h_{2k}}{\sqrt{2}} \right) \left( \cos(\gamma/\sqrt{2}) - i \sin(\gamma/\sqrt{2}) \frac{h_1}{\sqrt{2}} \right). \quad (S3)
\]

The Floquet operator is the evolution operator at the end of a full period:

\[
U_F = U(T) = \left( \cos(\sqrt{2} \gamma) - i \sin(\sqrt{2} \gamma) \frac{h_1}{\sqrt{2}} \right) \left( \cos(\sqrt{2} \lambda) - i \sin(\sqrt{2} \lambda) \frac{h_{2k}}{\sqrt{2}} \right) \left( \cos(\gamma/\sqrt{2}) - i \sin(\gamma/\sqrt{2}) \frac{h_1}{\sqrt{2}} \right). \quad (S4)
\]

Using the relations

\[
\frac{h_1 h_{2k} + h_{2k} h_1}{2} = (\cos k_x + \cos k_y) \tau_0 \sigma_0,
\]

\[
\frac{h_1 h_{2k} h_1}{2} = \cos k_z \tau_1 \sigma_0 + \sin k_z \tau_2 \sigma_3 - \cos k_z \tau_2 \sigma_2 + \sin k_z \tau_2 \sigma_1, \quad (S5)
\]

we have the analytical form for the Floquet operator as

\[
U_F = f_k \Gamma_1 + i(g_{1k} \Gamma_1 + g_{2k} \Gamma_2 + g_{3k} \Gamma_3 + g_{4k} \Gamma_4), \quad (S6)
\]
where

\[
f_k = \cos \sqrt{2} \gamma \cos \sqrt{2} \lambda - \sin \sqrt{2} \gamma \sin \sqrt{2} \lambda \frac{\cos k_x + \cos k_y}{2},
\]

\[
g_{1k} = \frac{1}{\sqrt{2}} \sin \sqrt{2} \lambda \sin k_y,
\]

\[
g_{2k} = \frac{1}{\sqrt{2}} \left( \sin \sqrt{2} \gamma \cos \sqrt{2} \lambda + \cos \sqrt{2} \gamma \sin \sqrt{2} \lambda \frac{\cos k_x + \cos k_y}{2} - \sin \sqrt{2} \lambda \frac{\cos k_x - \cos k_y}{2} \right),
\]

\[
g_{3k} = \frac{1}{\sqrt{2}} \sin \sqrt{2} \lambda \sin k_x,
\]

\[
g_{4k} = -\frac{1}{\sqrt{2}} \left( \sin \sqrt{2} \gamma \cos \sqrt{2} \lambda + \cos \sqrt{2} \gamma \sin \sqrt{2} \lambda \frac{\cos k_x + \cos k_y}{2} + \sin \sqrt{2} \lambda \frac{\cos k_x - \cos k_y}{2} \right)
\]

(S7)

and \(I = \tau_0 \sigma_0, \Gamma_{1,2,3} = \tau_2 \sigma_{1,2,3}, \Gamma_4 = \tau_1 \sigma_0\) are Dirac matrices being anticommuting with each other \([\Gamma_i, \Gamma_j] = 2 \delta_{ij}\), and \(f, g\)'s are real numbers satisfying \(f^2 + \sum_{j=1}^{4} g_j^2 = 1\). That means \(U_F\), and therefore its eigenstates, can be parameterized by three \(S^3\) angles \((\chi, \theta, \varphi)\) in addition to the quasienergy \(E\): \(f = \cos E, (g_1, g_2, g_3, g_4) = \sin E(\sin \chi \sin \theta \cos \varphi, \sin \chi \sin \theta \sin \varphi, \sin \chi \cos \theta, \cos \chi)\). The eigenvalues of \(U_F\) can be obtained via \((U_F - f)^2 = f^2 \sum_{j=1}^{4} g_j^2 = f^2(1 - f^2)\), which gives in our case \(U_F |E_k\rangle = e^{iE_k} |E_k\rangle\),

\[
\exp(iE_k) = \exp(\pm iE_k) = f_k \pm i \sqrt{1 - f_k^2},
\]

\[
f_k = \cos E_k = \cos(\sqrt{2} \gamma) \cos(\sqrt{2} \lambda) - \frac{\cos k_x + \cos k_y}{2} \sin(\sqrt{2} \gamma) \sin(\sqrt{2} \lambda),
\]

(S8)

with each band being two-fold degenerate. The gap closes at \(k_x, k_y = 0, \pi\) when

\[
f_k = \pm 1, \quad \Rightarrow \quad \sqrt{2}|\lambda| = \sqrt{2}|\gamma| + m\pi.
\]

(S9)

This gives the topological phase boundaries in the main text.

**Periodized evolution operator**

To obtain the periodized evolution operators, one needs to compute the eigenstates of the Floquet operator. Note that all the \(\Gamma\) matrices are Hermitian ones, and the eigenstates of \(U_F = \cos E_k + i \sin E_k M\) are the same as those of the Hermitian matrix \(M: U_F |E_k\rangle = \exp(\pm iE_k) |E_k\rangle, M |E_k\rangle = \pm |E_k\rangle\), where

\[
M = \frac{1}{\sin E_k} \sum_{j=1}^{4} g_j \Gamma_j = \begin{pmatrix} 0 & V' \\ V & 0 \end{pmatrix}, \quad V = \frac{1}{\sin E_k} (g_1 \sigma_0 + i(g_1 \sigma_1 + g_2 \sigma_2 + g_3 \sigma_3)).
\]

(S10)

We have put in the factor \(\frac{1}{\sin E_k}\) such that the matrix \(V\) is an unitary one \(VV^\dagger = \sigma_0\). The eigenstates then can be easily constructed as

\[
|E_{k+1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2} \sin E_k} \begin{pmatrix} \sin E_k \\ 0 \end{pmatrix}, \quad \sin E_k = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2} \sin E_k} \begin{pmatrix} 0 \\ \sin E_k \end{pmatrix},
\]

\[
|E_{k-1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2} \sin E_k} \begin{pmatrix} \sin E_k \\ 0 \end{pmatrix}, \quad \sin E_k = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2} \sin E_k} \begin{pmatrix} 0 \\ \sin E_k \end{pmatrix},
\]

(S11)

with corresponding eigenvalues in Eq. (S8). The \(\uparrow, \downarrow\) denotes two degenerate bands with the same eigenvalue.

For our purposes, we only need the return map at half period \(t/T = 1/2\). So the periodized evolution operator is only computed at this point. In Eq. (S8), we choose \(E_k \in [0, \pi]\). Thus, when the branch cut \(\varepsilon = 0\), the two eigenvalues of the return map are \(\exp(-iE_k/2)\) and \(\exp(-i(2\pi - E_k)/2) = -\exp(iE_k/2)\). If the branch cut is \(\varepsilon = \pi\), we have eigenvalues \(\exp(-iE_k/2)\) and \(\exp(iE_k/2)\). Thus,

\[
\varepsilon = 0: \quad e^{-iE_k/2} = e^{-iE_k/2} (|E_{k+1}\rangle + |E_{k+1}\rangle)(|E_{k-1}\rangle + |E_{k-1}\rangle) - e^{iE_k/2}(|E_{k+1}\rangle)(|E_{k-1}\rangle),
\]

\[
\varepsilon = \pi: \quad e^{-iE_k/2} = e^{-iE_k/2} (|E_{k+1}\rangle + |E_{k+1}\rangle)(|E_{k-1}\rangle + |E_{k-1}\rangle) + e^{iE_k/2}(|E_{k+1}\rangle)(|E_{k-1}\rangle + |E_{k-1}\rangle).
\]
Here
\[
\begin{align*}
|E_{k+1}\rangle (E_{k+1})^\dagger &= \frac{1}{2} \left( \begin{array}{c}
0 & Q^\dagger \\
Q & 0
\end{array} \right), \\
|E_{k-1}\rangle (E_{k-1})^\dagger &= \frac{1}{2} \left( \begin{array}{c}
0 & Q^\dagger \\
Q & 0
\end{array} \right)
\end{align*}
\]
Thus, it can be simplified as
\[
\begin{align*}
\epsilon &= 0: \quad e^{-iH_{\text{eff}}^{(0)} / 2} = \left( -i \sin \frac{E_k}{2} \sigma_0 \quad Q \cos \frac{E_k}{2} \right), \\
\epsilon &= \pi: \quad e^{-iH_{\text{eff}}^{(0)} / 2} = \left( \cos \frac{E_k}{2} \sigma_0 \quad -iQ \sin \frac{E_k}{2} \right).
\end{align*}
\]
In summary, we can apply Eqs. (S7), (S8), (S12) and (S13) to compute the periodized evolution operator at half period \( t/T = 1/2 \):
\[
\begin{align*}
h_1 &= \tau_1 \sigma_0 - \tau_2 \sigma_2, \\
h_{2k} &= \cos k_x \tau_1 \sigma_0 - \sin k_x \tau_2 \sigma_3 - \cos k_y \tau_2 \sigma_2 - \sin k_y \tau_1, \\
U(\frac{T}{2}) &= \left( \begin{array}{c}
\cos \frac{\sqrt{2} \lambda}{2} - i \sin \frac{\sqrt{2} \lambda}{2} \\
\sin \frac{\sqrt{2} \lambda}{2} \cos \frac{\sqrt{2} \lambda}{2}
\end{array} \right) \\
U_{\epsilon=0}(\frac{T}{2}) &= U(\frac{T}{2}) e^{-iH_{\text{eff}}^{(0)} / 2}, \\
U_{\epsilon=\pi}(\frac{T}{2}) &= U(\frac{T}{2}) e^{-iH_{\text{eff}}^{(0)} / 2}.
\end{align*}
\]
NUMERICAL ALGORITHMS

Static nested polarization

To be self-contained, we briefly review the procedure to compute static nested polarization in previous literature. Consider a static Hamiltonian \( H(k_x, k_y) \) or a time-independent Floquet operator \( U(k_x, k_y, T) \), where \( T \) is the period of a drive (a fixed number). The construction of topological invariants is based on their eigenstates
\[
H |E_k\rangle = E_k |E_k\rangle, \quad \text{or} \quad U_{\text{F}} |E_k\rangle = e^{iH_{\text{F}}(k)} |E_k\rangle.
\]
Consider a 4-band model, where two gapped bands (with energy/quasi-energy) \( \pm E_k \) are doubly degenerate respectively, and the eigenstates can be denoted as \( |+E_k\rangle, |+E_k\rangle \) for upper bands, and similarly \( |-E_k\rangle, |-E_k\rangle \) for lower bands. For a system with filled lower two bands, the first order Wilson loop \( W_{x,k} \) can be obtained as
\[
[F_{x,k}]_{mn} = (-1)^m E_{k+mL_x} \Delta k_x, \quad W_{x,k} = F_{x,k} F_{x,k+\Delta k_x} \cdots F_{x,k+(L_x-1)\Delta k_x}, \quad \Delta k_x = \frac{2\pi}{L_x},
\]
where \( L_x \) is the lattice sites along \( x \) and \( m, n = \uparrow, \downarrow \) denotes the two filled bands. Then, one can obtain the eigenvalues and eigenstates of the Wilson loop
\[
W_{x,k} |\nu_j(k)\rangle = e^{2\pi i j} |\nu_j(k)\rangle,
\]
where \( j = \pm \) denotes two Wannier bands for edge wave functions. The edge Wannier wave function is constructed by
\[
|\nu_j(k)\rangle = \sum_m |\nu_j(k)|_m |E_{km}\rangle,
\]
where \( \nu_j(k)_m \) denotes the \( m \)-th element of the 2-component spinor \( |\nu_j\rangle \). Finally, the nested polarization is obtained through
\[
\begin{align*}
\tilde{\Phi}_{\gamma,k}^{(j)} &= \langle \nu_j(k) |\nu_j(k+\Delta k,e_x)\rangle, \\
\tilde{W}_{\gamma,k}^{(j)} &= \tilde{\Phi}_{\gamma,k}^{(j)} \tilde{\Phi}_{\gamma,k+\Delta k,e_x}^{(j)} \cdots \tilde{\Phi}_{\gamma,k+(L_x-1)\Delta k_x}^{(j)}, \\
\Delta k_y &= \frac{2\pi}{L_y}, \quad \Delta k_x = \frac{2\pi}{L_x}.
\end{align*}
\]
The above procedure can similarly be applied to the semi-infinite situation with \( U_{\text{F}}(x, k_y) \). In this case, we already have the real-space resolution for \( x \), and one only needs to compute the first order Wilson loop along \( y \) (exchange the roles of \( (k_x, k_y) \leftrightarrow (k_y, x) \) in Eqs. (S16) and (S17)) as a function of \( x \). The non-trivial \( \nu_{j,k,y} \) at \( x = 0, L_x \) represents the nested polarization.
Simulation of dynamics for non-interacting particles

Here we consider the many-body dynamics of free fermions or bosons. The Floquet operator can be written as

\[ U_F = e^{iH_F t}, \quad H_F = \Psi \dagger H \Psi, \quad \Psi = (c_1, \ldots, c_N)^T, \]  

with \( c_j \) the fermion operator at site \( j \) and \( H_F \) an \( N \times N \) matrix that can be decomposed into its eigenbasis

\[ H = [E_n]E_n\langle E_n|. \]

Note \( |E_n\rangle \) and \( e^{iE_n t} \) are eigenstates and eigenvalues of the Floquet operator in the first quantized form. Here the effective Floquet Hamiltonian \( H_F \) gives the same dynamics as \( U_F \) only at stroboscopic time \( t = Nt, N \in \mathbb{Z} \). Since we do not look at evolution during one period, the branch cut does not matter unlike in the main text.

For an observable \( A \) that can also be expressed in the bilinear form (such as the density \( n_j = c_j^\dagger c_j \))

\[ A = \Psi \dagger A \Psi, \quad A(NT) = (U_F^{-1})^N A(U_F)^N = e^{iH_N N}e^{-iH_N N}, \]  

note \([AB, C] = A[B, C] - [A, C]B\) for fermions and \([AB, C] = (A[B, C] + [A, C]B\) for bosons, in both cases we have

\[ (U_F^{-1})^N \Psi \alpha (U_F)^N = \sum_{n=0}^{N} \binom{iN}{n} \langle \Psi \dagger a \rangle H_{n\alpha} \Psi \beta \rangle^n \Psi \alpha \rangle \]  

and therefore

\[ A(NT) = \Psi \dagger (e^{iH_N N}e^{-iH_N N}) \Psi. \]  

Now, if the Floquet Hamiltonian matrix can be diagonalized by the unitary matrix \( G \),

\[ G \dagger H G = \text{diag}(E_1, E_2, \ldots, E_D), \]

where \( D \) is the dimension of \( \mathcal{H} \), we have

\[ A(NT) = \sum_{\alpha, \beta = 1}^D \Gamma_\alpha \dagger G_{\alpha\beta} \Gamma_\beta e^{i(E_\alpha - E_\beta)NT}. \]

For our purposes, consider the initial state being a Fock one, \( |\psi_{\text{init}}\rangle = \prod_{i \text{corner \ 2x2 \ c}_i^\dagger |0\rangle \). Then

\[ \langle \psi_{\text{init}} | A(NT) | \psi_{\text{init}} \rangle = \sum_{i=1}^{N} n_i^{(0)} a_{\alpha i} e^{iE_\alpha N} \left( G_{\alpha\beta} \right)_{\alpha i} e^{-iE_\beta N} G_{\beta i}^\dagger, \]  

where \( \alpha, \beta \) are summed over all eigenstates, and \( n_i^{(0)} \) (being a real number) is the initial particle number at site \( i \). The above formula applies to both free bosons and fermions when the initial state is a Fock one, with the restriction that for spinless fermions, \( n_i^{(0)} \leq 1 \) due to Pauli’s principle of exclusion.

ADDITIONAL DETAILS FOR EXPERIMENTAL PROPOSALS

Sublattice density dynamics at the corner unit cell

In the main text, we show the dynamics of averaged density in the corner unit cell. Here we show the dynamics of density within the corner cell for each sublattices respectively. The corner states in phase ① have more weight in sublattices 1 and 2, compared with those in phase ① (different phases are defined in Fig. 2 of the main text). It is understandable that to accommodate two corner modes in phase ① , it requires more than one site (i.e. sublattice 4) for spinless particles.

Tomography for Floquet-Bloch band through time-of-flight for four-band model

Compared with the original band tomography method [67, 68], we encounter two major differences. First, our system involves a slow Floquet driving such that the evolution cannot be represented by an effective static Hamiltonian as in the original fast-driving scheme. Thus, without specific preparations, particles will not equilibrate into the “lowest” bands. To perform tomography, it is necessary to find a way to concentrate particles into one set of degenerate bands. Second, there are four bands in the model and the two sets of bands are doubly degenerate respectively, unlike in the original case involving only 2 bands without any degeneracy. We tackle these two differences in the following.
Populating “lower” Floquet-Bloch bands under slow driving

Since the purpose here is to distinguish the normal HOTI from the anomalous HOFTI, it is adequate to choose certain representative parameter regions in phase 1 and 3.

For phase 1, it is straightforward to notice that when $\gamma = 0$, the Floquet model reduces to a static one described by $h_{2k}$ alone, with artificial “identity evolution” during $t_1$ and $t_3$. Also, we note that $h_{2k}$ alone produces two completely flat bands. Thus, one can first equilibrate the system under static $h_{2k}$ at a temperature larger than the band width (which is zero), but smaller than the band gap $\lambda$, such that all momentum states at the well-defined lowest two bands are equally populated. Then, one deviates from the static regime by slowly introducing small $\lambda h_1$, which does not close the quasi-energy gap and therefore particles would still concentrate in the lower Floquet bands. For optimal effects, we choose $\sqrt{2} \lambda = \pi/2$ such that the quasi-energy band gap is maximal.

For phase 3, there is no nearby static fixed points with large quasi-energy gaps. But we notice that at $\sqrt{2} \lambda = \pi$, the Floquet operator reads $U_F = e^{\gamma h_1/\sqrt{2}}$, which is just the inverse of $U_F = e^{-\gamma h_1/\sqrt{2}}$ for $\lambda = 0$ and the two have identical structures for eigenstates. The latter case is a static one evolving under only $h_1$. Thus, we can take advantage of the one-to-one correspondence between parameters $(\sqrt{2} \lambda = 0, \gamma)$ and $(\sqrt{2} \lambda = \pi, \gamma)$. First, we equilibrate the system into lower bands of $h_1$. Then, we suddenly start the driving with $\sqrt{2} \lambda = \pi$ (and $\sqrt{2} \gamma = \pi/2$ such that the quasi-energy band gap is maximal). Since in this cases the Floquet eigenstates are exactly the same as static ones at $\lambda = 0$, and due to the large quasi-energy gap, the populations in two Floquet bands should largely remain unaffected. Then, similar to the situation in phase 1, we can slowly deviate slightly from the fixed point $(\sqrt{2} \gamma, \lambda) = \pi(0.5, 1)$, and the particle populations in two Floquet bands should be kept by the band gap.

Four-band tomography with lowest two bands being degenerate

Here we generalize the framework in Ref. [69] to 4-bands. First, we introduce the three angles in $S^3$, $(\theta_k, \varphi_k, \lambda_k)$, characterizing the eigenstates of $U_F$ as mentioned in the main text. Note the Floquet operator can be written in the form of Eq. (S10), which we quote below

$$U_F = \cos E_k - i \sin E_k, \quad M = \frac{1}{\sin E_k} \sum_{j=1}^4 g_j \Gamma_j = \begin{pmatrix} 0 & V^\dagger \\ V & 0 \end{pmatrix}, \quad V = \frac{1}{\sin E_k} (g_4 \sigma_0 + i (g_1 \sigma_1 + g_2 \sigma_2 + g_3 \sigma_3)), \quad (S28)$$
where \( \cos E_k = f_k, \sin E_k = \sqrt{1 - f_k^2} = \Sigma_{j=1}^{4} g_j^2 \) and \( f_k, g_k \) are defined in Eq. (S7). Write \( V \) and its eigenstates as

\[
V = \cos \chi_k + i \sin \chi_k \begin{pmatrix}
\cos \theta & \sin \theta e^{-i\phi} \\
\sin \theta e^{i\phi} & -\cos \theta
\end{pmatrix},
\]

\[
\cos \chi_k = \frac{g_4}{\sin E_k}, \quad \sin \chi_k = \frac{g}{\sin E_k}, \quad \cos \theta_k = \frac{g_3}{g}, \quad \sin \theta_k \cos \varphi_k = \frac{g_1}{g}, \quad \sin \theta_k \sin \varphi_k = \frac{g_2}{g}, \quad g = \sqrt{g_1^2 + g_2^2 + g_3^2}
\]

\[
V \uparrow = e^{i\varphi_k} \uparrow, \quad | \uparrow \rangle = \begin{pmatrix}
\cos \frac{g}{e^{i\varphi} \sin \frac{g}{E_k}} \\
\sin \frac{g}{e^{i\varphi} \sin \frac{g}{E_k}}
\end{pmatrix}, \quad V \downarrow = e^{-i\varphi_k} \downarrow, \quad | \downarrow \rangle = \begin{pmatrix}
- e^{-i\varphi} \sin \frac{g}{2} \\
\cos \frac{g}{2}
\end{pmatrix}.
\]

(S29)

Then, similar to the construction in Eq. (S11), the Floquet eigenstates can be written as \( U_f|E_{\pm} \rangle = e^{iE_{\pm}}|E_{\pm} \rangle \) where \( |\alpha \rangle \) can be an arbitrary SU(2) spinor. But here we choose the two eigenstates of \( V \) rather than \((1, 0)\) and \((0, 1)\) for \( |\alpha \rangle \), so the Floquet eigenstates are

\[
|E_{k\uparrow} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix}
\pm \cos \frac{g}{2} & \pm e^{i\varphi} \sin \frac{g}{2} \\
\e^{i\varphi} \sin \frac{g}{2} & \e^{i\varphi} \cos \frac{g}{2}
\end{pmatrix}, \quad |E_{k\downarrow} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix}
\mp \sin \frac{g}{2} & \mp e^{-i\varphi} \cos \frac{g}{2} \\
-e^{-i\varphi} \cos \frac{g}{2} & -e^{-i\varphi} \sin \frac{g}{2}
\end{pmatrix}, \quad \left\{ \begin{array}{c}
U_f|E_{k\uparrow} \rangle = e^{i\varphi_k} E_{k\uparrow} \rangle, \\
U_f|E_{k\downarrow} \rangle = e^{i\varphi_k} |E_{k\downarrow} \rangle.
\end{array} \right.
\]

(S30)

The angles \((\theta_k, \varphi_k, \lambda_k)\) above are the ones used in the main text. With these solutions, we can write the transformation between fermion operators at sublattice \( i: c_{ki} \), and those at certain band \( c_{k\mu, \mu} = \uparrow, \downarrow, \uparrow \uparrow \), as

\[
\begin{pmatrix}
c_{k1} \\
c_{k2} \\
c_{k3} \\
c_{k4}
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\cos \frac{g}{2} & -\sin \frac{g}{2} e^{i\varphi} & -\cos \frac{g}{2} e^{i\varphi} & \sin \frac{g}{2} \\
\sin \frac{g}{2} e^{i\varphi} & \cos \frac{g}{2} & \cos \frac{g}{2} e^{-i\varphi} & -\sin \frac{g}{2} e^{-i\varphi} \\
\cos \frac{g}{2} & -\sin \frac{g}{2} e^{-i\varphi} & -\cos \frac{g}{2} e^{-i\varphi} & \sin \frac{g}{2} \\
\sin \frac{g}{2} e^{-i\varphi} & \cos \frac{g}{2} & \cos \frac{g}{2} e^{i\varphi} & -\sin \frac{g}{2} e^{i\varphi}
\end{pmatrix}\begin{pmatrix}
c_{k\uparrow} \\
c_{k\downarrow} \\
c_{k\uparrow \downarrow} \\
c_{k\downarrow \uparrow}
\end{pmatrix}
\]

(S31)

During time-of-flight, four plain waves formed in the four sublattices respectively will interfere with each other. The interference pattern can be described as (up to normalization constants for the total particle number)

\[
n_k = \langle \psi_{mi}|d_k^\dagger d_k|\psi_{mi} \rangle \quad d_k = c_{k1} + c_{k2} + c_{k3} + c_{k4}, \quad \langle d_k, d_k^\dagger \rangle = 1, \quad \int \frac{dk}{(2\pi)^2} n_k = \frac{1}{2} \quad \text{(half-filling)}.
\]

(S32)

Here, we consider the initial state where the two "−" bands are completely filled, while the two "+" bands are empty, i.e. \( |\psi_{mi} \rangle = \langle \prod_k c_{k-1}^\dagger | \prod_k c_{k-1} \rangle |0 \rangle \). Then, \( \langle c_{k-1}^\dagger c_{k-1} \rangle = \langle c_{k-1}^\dagger c_{k-1} \rangle = 1 \), and other terms all vanish. As such, \( n_k \) is the sum of two independent interference patterns of \( |E_{k\uparrow} \rangle \) and \( |E_{k\downarrow} \rangle \),

\[
n_k^{(4)} = \frac{1}{8} \left| -\cos \frac{\theta}{2} - \sin \frac{\theta}{2} e^{i\varphi} - \sin \frac{\theta}{2} e^{-i\varphi} + \cos \frac{\theta}{2} e^{i(\varphi+\varphi')} \right|^2 + \frac{1}{8} \left| -\sin \frac{\theta}{2} - \cos \frac{\theta}{2} e^{i\varphi} - \sin \frac{\theta}{2} e^{-i\varphi} + \cos \frac{\theta}{2} e^{i(\varphi-\varphi')} \right|^2
\]

\[
= \frac{1}{2} (1 - \cos \chi_k) \left( 1 - \frac{g_k}{\sin E_k} \right).
\]

(S33)

A direct time-of-flight, as we see above, does not involve enough information to back up all three angles \((\theta_k, \varphi_k, \lambda_k)\). So, similar to [67, 68], we introduce a quench to deep lattice regime where the states evolve under static onsite chemical potentials \( \mu_{1,2,3,4} \) only, before doing time-of-flight. As mentioned in the main text, we need three chemical potential profiles. For instance, take \( \mu_{\text{quench}} = \frac{1}{3} \tau_3 \sigma_3 \) and evolve for time \( \tau_1 \), then

\[
|E_{k\uparrow} \rangle \to e^{-i\Delta_1 \tau_1/2} \begin{pmatrix}
\frac{1}{\sqrt{2}} & -e^{-i\varphi} \sin \frac{g}{2} \\
e^{i\varphi} \sin \frac{g}{2} & e^{i\varphi} \cos \frac{g}{2}
\end{pmatrix}, \quad |E_{k\downarrow} \rangle \to e^{-i\Delta_1 \tau_1/2} \begin{pmatrix}
\frac{1}{\sqrt{2}} & -e^{i\varphi} \sin \frac{g}{2} \\
e^{-i\varphi} \sin \frac{g}{2} & e^{-i\varphi} \cos \frac{g}{2}
\end{pmatrix}
\]

(S34)

A time-of-flight after such a procedure produces

\[
n_k^{(1)} = \frac{1}{2} (1 - \cos \chi_k \cos \Delta_1 \tau_1 + \sin \chi_k \sin \theta_k \cos \varphi_k \sin \Delta_1 \tau_1)
\]

\[
= \frac{1}{2} (1 + \sin \chi_k \sin \theta_k \cos \varphi_k) = \frac{1}{2} \left( 1 - \frac{g_1}{\sin E_k} \right)
\]

(S35)
Similarly, take $H_{\text{quench}}_2 = \frac{\Delta}{4} (\tau_3 \sigma_0 + \tau_0 \sigma_3 + \tau_3 \sigma_3) = \frac{\Delta}{4} \text{diag}(3, -1, -1, -1)$ and evolve for time $\tau_2$, we have

$$n_k^{(2)} = \frac{1}{4} (2 - \cos \lambda_k (1 + \cos \Delta_2 \tau_2) - \sin \lambda_k (\sin \theta_k \sin \varphi_k (1 - \cos \Delta_2 \tau_2) + (\cos \theta_k + \sin \theta_k \cos \varphi_k) \sin \Delta_2 \tau_2))$$

$$\frac{\Delta \tau_2 = \pi}{\frac{1}{2} (1 - \sin \lambda_k \sin \theta_k \sin \varphi_k)} = \frac{1}{2} \left(1 - \frac{g_k}{\sin E_k}\right)$$

(S36)

Finally, quenching to $H_{\text{quench}}_3 = \frac{\Delta}{4} \tau_3 \sigma_3$ for time $\tau_3$, the corresponding time-of-flight signatures will be

$$n_k^{(3)} = \frac{1}{2} (1 - \cos \chi_k \cos \Delta_3 \tau_3 + \sin \chi_k \cos \theta_k \sin \Delta_3 \tau_3)$$

$$\frac{\Delta \tau_3 = \pi/2}{\frac{1}{2} (1 + \sin \chi_k \cos \theta_k)} = \frac{1}{2} \left(1 - \frac{g_k}{\sin E_k}\right)$$

(S37)

Up to this point, we see that $n_k^{(1,2,3,4)}$ contain adequate information to back up the matrix $V$ in Eq. (S28), and therefore the eigenstates for $U_F$. The three angles ($\chi_k$, $\theta_k$, $\varphi_k$) can be written as

$$\chi_k = \arccos \left(1 - 2n_k^{(4)}\right), \quad \theta_k = \arccos \frac{2n_k^{(3)} - 1}{\sqrt{1 - (2n_k^{(3)} - 1)^2}}, \quad \varphi_k = \arg \left((2n_k^{(1)} - 1) + i(1 - 2n_k^{(2)})\right)$$

(S38)