Generalized Shannon-Khinchin Axioms and Uniqueness Theorem for Pseudo-additive Entropies

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Abstract
We consider the Shannon-Khinchin axiomatic systems for the characterization of generalized entropies such as Sharma-Mittal and Frank-Daffertshofer entropy. We provide the generalization of Shannon-Khinchin axioms and give the corresponding uniqueness theorem. The previous attempts at such axiomatizations are also discussed.

Keywords: Shannon-Khinchin axioms, Information measures, Shannon entropy, Rényi entropy, Nath entropy, Tsallis entropy, Sharma-Mittal entropy, Frank-Daffertshofer entropy, Generalized entropies.

1. Introduction
In the past, there was extensive work on defining the information measures which generalize the Shannon entropy [11] and on deriving the axiomatic systems for their characterization. Shannon entropy follows the composition rule by which the entropy of a joint system can be represented as the sum of the entropy of one system and the conditional entropy of another, with respect to the first one, with standardly defined expectation operator. The composition rule is shown to be of the essential properties of Shannon entropy and the axiomatic characterization based on this property has been established in literature as the Shannon-Khinchin axiomatic system [11]. Since then, there has been extensive work on generalizing the Shannon entropy and its axiomatic characterization.

Rényi introduced a one-parameter generalization of the Shannon entropy in the way that expectation operator is defined as quasi-linear mean [14], and the generalization of Shannon-Khinchin axioms for the Rényi entropy is presented by Jizba and Arimitsu [10]. Another approach for the generalization of the Shannon-Khinchin axioms is to keep the linear expectation form, but to replace the real addition in the composition rule by a more general one. Such approach was first proposed by Abe [1], who used the so-called ⊕γ addition [13] and obtained the Havrda-Charvát-Tsallis entropy [6], [17] as the unique entropy which satisfy his axiomatic system.

In this paper, we first relax the so-called normalization axiom from the Jizba-Arimitsu axiomatic system [10], and show that such a system characterizes a two-parameter generalization of Shannon and Rényi entropy previously introduced by Nath [12]. After that, we show that ⊕γ operation in Abe’s axiomatic system [1] can be replaced with a more general pseudo-addition operation without affecting the resulting entropy form. Accordingly, the Havrda-Charvát-Tsallis entropy is a unique entropy which can be characterized with generalized Shannon-Khinchin axioms if the linear expectation is used. Finally, we present the generalized Shannon-Khinchin axiomatic system based on the combination of the two mentioned approaches. A similar class of entropies has previously been characterized in [2] and [7] by the set of axioms which assume that pseudo-additive entropy is represented as quasi-linear mean-value of pseudo-additive information content.

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and the statistical properties of such entropies are examined in [3]. A similar attempt for the generalization has been made in [9], where the authors use pseudo-linear expectation and addition for the composition rule, but the class of entropies derived in [9] is incomplete, which is also discussed in the present paper.

The paper is organized as follows. In section 2 we review the Shannon-Khinchin axioms. The generalization of the Shannon-Khinchin axioms for the Nath and Rényi entropy is presented in section 3 and for the Havrda-Charvát-Tsallis entropy in section 4. In section 5 we consider the generalization of Shannon-Khinchin axioms for the class of entropies which can be represented as a nonlinear transformation of the Nath entropy. Two special cases, Sharma-Mittal and the Frank-Daffertshofer entropies, are considered in subsection 5.1.

2. Shannon-Khinchin axioms

Let the function

\[ S_n(P) = \tau \cdot \sum_{k=1}^{n} p_k \log_2 p_k, \quad \tau < 0, \]

The following theorem characterizes the Shannon entropy by the so-called Shannon-Khinchin axioms [11].

**Theorem 2.1.** Let the function \( S_n : \Delta_n \to \mathbb{R}^+ \) satisfy the following Shannon-Khinchin axioms, for all \( n \in \mathbb{N} \), \( n > 1 \):

[SA1] \( S_n \) is continuous in \( \Delta_n \);

[SA2] \( S_n \) takes its largest value for the uniform distribution, \( U_n = (1/n, \ldots, 1/n) \in \Delta_n \), i.e. \( S_n(U_n) = S_n(1/n) \), for any \( P \in \Delta_n \);

[SA3] \( S_n \) is expandable: \( S_{n+1}(p_1, p_2, \ldots, p_n, 0) = S_n(p_1, p_2, \ldots, p_n) \) for all \( (p_1, \ldots, p_n) \in \Delta_n \);

[SA4] Let \( P = (p_1, \ldots, p_n) \in \Delta_n \), \( PQ = (r_{11}, r_{12}, \ldots, r_{nm}) \in \Delta_{nm} \), \( n, m \in \mathbb{N} \), \( n, m > 1 \) such that \( p_i = \sum_{j=1}^{m} r_{ij} \) and \( Q_{jk} = (q_{1jk}, \ldots, q_{mk}) \in \Delta_{m} \), where \( q_{ijk} = r_{ijk} / p_i \). Then,

\[ S_{nm}(PQ) = S_n(P) + S_m(Q), \quad \text{where} \quad S_m(Q) = \sum_k p_k \cdot S_m(Q_{jk}). \]

Then, the function \( S_n \) is the Shannon entropy given by the class (2).

3. Generalized Shannon-Khinchin axioms for Nath and Rényi entropy

The Nath entropy of \( n \)-dimensional distribution is a function \( N_n : \Delta_n \to \mathbb{R}^+ \) from the family parameterized by \( \tau, \lambda, \alpha \in \mathbb{R}^+ \):

\[ N_n(P) = \begin{cases} 
\tau \cdot \sum_{k=1}^{n} p_k \log_2 p_k, & \tau < 0, \quad \text{for} \quad \lambda = 0, \\
\frac{1}{\lambda} \log_2 \left( \sum_{k=1}^{n} p_k^{\lambda} \right), & \alpha > 0, \quad \lambda \cdot (1 - \alpha) > 0, \quad \text{for} \quad \lambda \neq 0.
\end{cases} \]
If \( \lambda = 1 - \alpha \) and \( \tau = -1 \), the Nath entropy reduces to the Rényi entropy, which is a function \( R_n : \Delta_n \to \mathbb{R}^+ \),

\[
R_n(P) = \begin{cases} 
- \sum_{k=1}^{n} p_k \log_2(p_k), & \text{for } \alpha = 1, \\
\frac{1}{1-\alpha} \log_2 \left( \sum_{k=1}^{n} p_k^\alpha \right), & \alpha > 0, \text{ for } \alpha \neq 1.
\end{cases}
\] (5)

Previously, Jizba and Arimitsu [10] had given the characterization of Rényi entropy using the generalized Shannon-Khinchin axioms. In this section, we extend the results from [10] to the more general case of the Nath entropy.

**Theorem 3.1.** Let the function \( N_n : \Delta_n \to \mathbb{R}^+ \) satisfy the following generalized Shannon-Khinchin axioms for all \( n \in \mathbb{N}, n > 1 \):

[NSK1] \( N_n \) is continuous in \( \Delta_n \);

[NSK2] \( N_n \) takes its largest value for the uniform distribution, \( U_n = (1/n, \ldots, 1/n) \in \Delta_n \), \( N_n(U_n) \leq N_n(P) \), for any \( P \in \Delta_n \);

[NSK3] \( N_n \) is expandable: \( N_{n+1}(p_1, p_2, \ldots, p_n, 0) = N_n(p_1, p_2, \ldots, p_n) \) for all \( (p_1, \ldots, p_n) \in \Delta_n \);

[NSK4] Let \( P = (p_1, \ldots, p_n) \in \Delta_n \), \( PQ = (r_{11}, r_{12}, \ldots, r_{nm}) \in \Delta_{mn} \), \( n, m \in \mathbb{N}, n, m, m > 1 \) such that \( p_i = \sum_{j=1}^{m} r_{ij} \), and \( Q_k = (q_{1k}, \ldots, q_{mk}) \in \Delta_m \), where \( q_{jk} = r_{jk}/p_k \) and \( \alpha \in \mathbb{R}^+ \) is some fixed parameter. Then,

\[
N_{nm}(P) = N_n(P) + N_m(Q|P), \quad \text{where } N_m(Q|P) = f^{-1}\left( \sum_{k=1}^{n} p^{(\alpha)}_k f(N_m(Q_k)) \right),
\] (6)

where \( f \) is an invertible continuous function and the \( \alpha \)-escort distribution \( P^{(\alpha)} = (p^{(\alpha)}_1, \ldots, p^{(\alpha)}_n) \in \Delta_n \) of distribution \( P \in \Delta_n \) is defined with

\[
p^{(\alpha)}_k = \frac{p^{(\alpha)}_k}{\sum_{i=1}^{n} p^{(\alpha)}_i}, \quad k = 1, \ldots, n, \quad \alpha > 0.
\] (7)

Then, \( f \) is a function from the class parameterized by \( b, c, d, \lambda, \gamma \in \mathbb{R} \):

\[
f(x) = \begin{cases} 
c \cdot x + b, & c \neq 0, \quad \text{for } \lambda = 0, \\
d \cdot 2^{\lambda x} - 1, & d, \gamma \neq 0, \quad \text{for } \lambda \neq 0,
\end{cases}
\] (8)

and the function \( N_n \) is the Nath entropy given by the class [11]. In addition, if the following normalization axiom holds:

[NSK5] \( N_2 \left( \frac{1}{2}, \frac{1}{2} \right) = 1 \),

\( N_n \) reduces to the Rényi entropy given by the class [5].

In the proof of the theorem we will use the following theorem from [10] and [7].

**Theorem 3.2.** Let \( g : \mathbb{R} \to \mathbb{R} \) be continuous invertible function and \( N_n : \Delta_n \to \mathbb{R}^+ \) is continuous function,

\[
N_n(P) = g^{-1}\left( \sum_{k=1}^{n} p^{(\alpha)}_k g(\lambda \log_2 p_k) \right), \quad \tau < 0, \quad \alpha > 0.
\] (9)
for all \( n \in \mathbb{N} \), \( n > 1 \) and \( P = (p_1, \ldots, p_n) \in \Delta_n \), and let \( N_n \) be additive, i.e. \( N_{nm}(P \ast Q) = N_n(P) + N_m(Q) \) for all \( P = (p_1, \ldots, p_n) \in \Delta_n \), \( Q = (q_1, \ldots, q_m) \in \Delta_m \), \( n, m \in \mathbb{N}, n, m > 1 \). Then, \( g \) is the function from the class parameterized by \( c, \lambda, \gamma \in \mathbb{R} \setminus \{0\} \):

\[
g(x) = \begin{cases} -c \cdot x, & \text{for } \lambda = 0 \\ \frac{2^{-\alpha x} - 1}{\gamma}, & \text{for } \lambda \neq 0 \end{cases}
\]

\[
g^{-1}(x) = \begin{cases} -\frac{1}{c} \cdot x, & \text{for } \lambda = 0 \\ \frac{1}{\lambda} \log_2 (\gamma x + 1), & \text{for } \lambda \neq 0 \end{cases}
\]  

(10)

The entropy is uniquely determined with

\[
N_n(P) = \begin{cases} \tau \cdot \sum_{k=1}^n p_k^{(n)} \log_2 p_k, & \text{for } \lambda = 0 \\ -\frac{1}{\lambda} \log_2 \left( \frac{\sum_{k=1}^n p_k^{(n)} \tau^{-1}}{\sum_{j=1}^m p_j^{(m)}} \right), & \text{for } \lambda \neq 0, \end{cases}
\]  

(11)

where \( \tau < 0, \alpha > 0 \) and \( \alpha - \tau \lambda > 0 \).

**Proof of the Theorem 3.1** Let \( \mathcal{L}(r) = N_n(U_r) \) denote the entropy of the uniform distribution \( U_r = (1/r, \ldots, 1/r) \in \Delta_r \). By successive usage of axioms [NSK2] and [NSK3] we conclude that \( \mathcal{L}(r) \leq \mathcal{L}(r + 1) \), i.e. \( \mathcal{L} \) is a non-decreasing function. Let for \( P = (p_1, \ldots, p_n) \in \Delta_n \) and \( Q = (q_1, \ldots, q_m) \in \Delta_m \) the direct product, \( P \ast Q \in \Delta_{nm} \), be defined as

\[
P \ast Q = (p_1 q_1, p_1 q_2, \ldots, p_n q_m).
\]  

(12)

Repeated application of axiom [NSK4] then leads to

\[
N_{nm}(U_r \ast U_r \ast \ldots \ast U_r) = \sum_{k=1}^m N_r(U_r) = m \cdot N_r(U_r) \text{ i.e. } \mathcal{L}(r^m) = m \cdot \mathcal{L}(r).
\]  

(13)

Since \( \mathcal{L} \) is non-decreasing, the equation (13) has unique solution (11)

\[
\mathcal{L}(r) = -\tau \cdot \log_2 r, \quad \tau < 0.
\]  

(14)

Let us now determine the entropy form for the distribution \( P = (p_1, \ldots, p_n) \in \Delta_n \) when \( p_i \) are rational numbers and the case for irrational numbers follows from the continuity of entropy. Let \( P = (p_1, \ldots, p_n) \in \Delta_n \), \( Q_k = (q_1, \ldots, q_m, \ldots) \in \Delta_{mk}, k = 1, \ldots, n \) and \( PQ = (r_{11}, r_{12}, \ldots, r_{nn}) \) be an \( n \times m \) matrix, for \( n, m \in \mathbb{N}, n, m > 1 \) and \( p_i = m_j/m_i \) for any \( i = 1, \ldots, n \) and \( j = 1, \ldots, m_i \). Then we have \( N_n(PQ) = N_n(U_m) = \mathcal{L}(m) = -\tau \cdot \log_2 m \) and \( N_{nm}(Q_k) = N_m(U_m) = \mathcal{L}(m) = -\tau \cdot \log_2 m_k \). Since \( p_i = \sum_{j=1}^m r_{ij} \) and \( q_{jk} = r_{jk}/p_k \), we can apply the axiom [NSK4] to obtain

\[
N_n(P) = -\tau \cdot \log_2 m - f^{-1} \left( \sum_{k=1}^n p_k^{(n)} f \left( -\tau \cdot \log_2 m_k \right) \right) = -\tau \cdot \log_2 m - f^{-1} \left( \sum_{k=1}^n p_k^{(n)} f \left( -\tau \cdot \log_2 p_k - \tau \cdot \log_2 m \right) \right).
\]  

(15)

Let us define \( f_y(x) = f(-x - y) \), and \( f_y^{-1}(z) = -y - f^{-1}(z) \). If we set \( y = \tau \cdot \log_2 m \) the equality (15) becomes

\[
N_n(P) = f_y^{-1} \left( \sum_{k=1}^n p_k^{(n)} f_y \left( \tau \cdot \log_2 p_k \right) \right), \quad \tau < 0.
\]  

(16)

Since \( f \) is continuous, both \( f_y \) and \( f_y^{-1} \) are continuous, as well as the entropy, and we may extend the result (16) from rational \( p_i \)'s to any real valued \( p_i \)'s defined in \([0,1]\). Now, if we the axiom [NSK4] is used with
$PQ = P \ast Q$, the conditions from the Theorem 3.2 are satisfied so that the function $f_y$ is uniquely determined by the class (10):

$$f_y(x) = \begin{cases} -c \cdot x, & \text{for } \lambda = 0, \\ 2^{-\lambda x} - 1, & \text{for } \lambda \neq 0, \end{cases}$$

(17)

where $c, \lambda, \gamma \in \mathbb{R} \setminus \{0\}$. The entropy is uniquely determined by the class (11) parameterized by $\tau, \alpha$ and $\lambda$.

The relationship between the parameters is determined by usage of the axiom [NSK4], in the same manner as in [8] and [10], which gives $\alpha = 1$ for $\lambda = 0$ and $\alpha - \tau \lambda = 1$ for $\lambda \neq 0$. Positivity of entropy implies $\tau < 0$ and $\lambda \cdot (1 - \alpha) > 0$. Thus, if a function satisfies [NSK1]-[NSK4], then it has the form (4). Conversely, if a function has the form (4), [NSK1], [NSK3] and [NSK4] are obviously satisfied. In addition the Rényi entropy attains maximum for uniform distribution, as shown in [10], so that [NSK4] is satisfied, since $N_m(P) = \frac{Q^\alpha}{\lambda} \cdot R_m(P)$ and $\lambda \cdot (1 - \alpha) > 0$.

The form of function $f$ can be determined by substitution of (4) in [NSK4]. Let $P = (p_1, \ldots, p_n) \in \Delta_n$, $PQ = (r_{11}, r_{12}, \ldots, r_{nm}) \in \Delta_{nm}$, $n, m \in \mathbb{N}$, $n, m > 1$ such that $p_1 = \sum_{j=1}^{m} r_{ij}$, and $Q_k = (q_{1k}, \ldots, q_{mk}) \in \Delta_m$, where $q_{jk} = r_{jk}/p_k$. If we introduce $f(x) = 2^{\alpha x}$ for $\lambda \neq 0$ and $\bar{f}(x) = x$ for $\lambda = 0$ (recall that in this case $\alpha = 1$), it is easy to obtain

$$N_{nm}(P \ast Q) - N_m(P) = \bar{f}^{-1}\left(\sum_{k=1}^{n} p_k^{(\alpha)} f(N_m(Q_k))\right) = f^{-1}\left(\sum_{k=1}^{n} p_k^{(\alpha)} f(N_m(Q_k))\right).$$

(18)

Accordingly, $f$ and $\bar{f}$ generate the same mean so that, as shown in [5], $f$ is a linear function of $\bar{f}$ and the form (8) follows (invertibility of implies $c, d \neq 0$).

Finally, if in addition [NSK5] holds, from (4) we get $\lambda = \alpha - 1$ and $c = -1$ and the theorem is proven.

4. Generalized Shannon-Khinchin axioms for Havrda-Charvát-Tsallis entropy

Let $h : \mathbb{R} \to \mathbb{R}$ be an increasing continuous (hence invertible) function such that $h(0) = 0$ and let the pseudo-addition operation $\oplus$ be defined as:

$$h(x + y) = h(x) \oplus h(y); \quad x, y \in \mathbb{R}.$$

(19)

If the mapping $h$ is parameterized by $a, \lambda, \gamma \in \mathbb{R}$ and defined with

$$h(x) = \begin{cases} a \cdot x, & a > 0, \quad \text{for } \lambda = 0, \\ 2^{\lambda x} - 1, & \lambda \cdot \gamma > 0, \quad \text{for } \lambda \neq 0, \end{cases}$$

(20)

the formula (19), defines $\oplus_\gamma$-addition [13],

$$u \oplus_\gamma v = u + v + \gamma uv; \quad u, v \in \mathbb{R}.$$

(21)

For the case $\gamma = 0$, $h$ reduces to a linear function and the $\oplus_\gamma$-addition reduces to ordinary addition.

In [16], Abe considers generalized Shannon-Khinchin axiomatic system in which the first three axioms are the same as [SA1]-[SA3], but in the fourth axiom, $\oplus_\gamma$-addition is used instead of the ordinary one and the expectation is taken with respect to the $\alpha$-escort distribution. More specifically, Abe’s fourth axiom is:

[ASK4] Let $P = (p_1, \ldots, p_n) \in \Delta_n$, $PQ = (r_{11}, r_{12}, \ldots, r_{nm}) \in \Delta_{nm}$, $n, m \in \mathbb{N}$, $n, m > 1$, such that $p_1 = \sum_{j=1}^{m} r_{ij}$, and $Q_k = (q_{1k}, \ldots, q_{mk}) \in \Delta_m$, where $q_{jk} = r_{jk}/p_k$. Then,

$$T_{nm}(PQ) = T_m(P) \oplus_\gamma T_m(Q|P), \quad \text{where } T_m(Q|P) = \sum_{k=1}^{n} p_k^{(\alpha)} T_m(Q_k), \quad \alpha > 0.$$

(22)
In [16], it is shown that the axiomatic system [SA1]-[SA3] and [ASK4], uniquely characterizes the Havrda-Charvát [6] and Tsallis [17] entropies.

\[ T_n(P) = \begin{cases} S_n(P) = \tau \cdot \sum_{k=1}^{n} p_k \log_2 p_k, & \tau < 0, \quad \text{for } \gamma = 0 \\ \frac{1}{\gamma} \left( \sum_{k} p_k^\tau - 1 \right), & \alpha > 0, \gamma(1-\alpha) > 0, \quad \text{for } \gamma \neq 0 \end{cases} \]  

(23)

In the following theorem we show that the entropy form (23) can be characterized with weaker assumption about the pseudo-addition defined with (19). Alternatively, \( \oplus_r \)-addition is the unique pseudoaddition operation which can be used in axiom [ASK4].

**Theorem 4.1.** Let the entropy \( T_n : \Delta_n \to \mathbb{R}^+ \) be defined as a function which for all \( n \in \mathbb{N}, n > 1 \) satisfies the following axioms:

- [ASK1] \( T_n \) is continuous in \( \Delta_n \);
- [ASK2] \( T_n \) takes its largest value for the uniform distribution, i.e. for any \( P \in \Delta_n, T_n(P) \leq T_n(U_n) \);
- [ASK3] \( T_n \) is expandable: \( T_{n+1}(p_1, p_2, \ldots, p_n, 0) = T_n(p_1, p_2, \ldots, p_n) \) for all \( (p_1, \ldots, p_n) \in \Delta_n \);
- [ASK4] Let \( P = (p_1, \ldots, p_n) \in \Delta_n, P Q = (r_{11}, r_{12}, \ldots, r_{nn}) \in \Delta_{nm}, n, m \in \mathbb{N}, n, m > 1 \) such that \( p_i = \sum_{j=1}^{m} r_{ij} \), and \( Q_k = (q_{1k}, \ldots, q_{nk}) \in \Delta_m \), where \( q_{jk} = r_{jk}/p_k \). Then,

\[ T_{nm}(P Q) = T_n(P) \oplus T_m(Q|P), \quad \text{where} \quad H_m(Q|P) = \sum_k p_k^{(\alpha)} H_m(Q_k), \quad \alpha > 0, \]  

(24)

where \( \oplus \)-addition is defined with (19).

Then, the entropy has the form (23).

**Proof.** Let \( h^{-1} \) denote the inverse mapping of \( h \), let us denote \( N_n(P) = h^{-1}(T_n(P)) \Leftrightarrow T_n(P) = h(N_n(P)) \) and let us apply \( h^{-1} \) to [ASK1]-[ASK4]. Since \( h^{-1} \) is increasing and continuous [ASK1]-[ASK4] transforms into [NSK1]-[NSK4] (with \( f \equiv h \)) and we get

\[ N_n(P) = \begin{cases} \tau \cdot \sum_{k=1}^{n} p_k \log_2 p_k; & \tau < 0 \quad \text{for } \lambda = 1, \\
\frac{1}{\lambda} \log_2 \left( \sum_{k=1}^{n} p_k^\lambda \right); & \alpha > 0, \lambda(1-\alpha) > 0 \quad \text{for } \lambda \neq 0. \end{cases} \]  

(25)

The function \( h \) is given with (8), and using \( h(0) = 0 \) we get (20):

\[ h(x) = \begin{cases} a \cdot x, & a > 0, \quad \text{for } \lambda = 0, \\
\frac{2^{\lambda x} - 1}{\gamma}, & \lambda \cdot \gamma > 0, \quad \text{for } \lambda \neq 0, \end{cases} \]  

(26)

and since \( T_n(P) = h(N_n(P)) \), the theorem is proven.

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1. Actually, Abe consider the case of Tsallis entropy only, which is obtained for \( \gamma = 1 - \alpha \), and does not cover the case of Havrda-Charvát entropy obtained for \( \gamma = 2^{1-\alpha} - 1 \). However, the discussion holds in our general case, which can be straightforwardly shown by repeating the steps from [18].
5. Generalized Shannon-Khinchin Axioms for a nonlinear transformation of the Nath entropy

In this section we combine the axiomatic systems from sections 3 and 4. Thus, we consider the pseudo-addition generated by an increasing continuous function \( h : \mathbb{R} \rightarrow \mathbb{R} \) such that \( h(0) = 0 \) by the following equation:

\[
h(x + y) = h(x) \oplus h(y); \quad x, y \in \mathbb{R},
\]

and we characterizes the entropy \( \mathcal{H}_n : \Delta_n \rightarrow \mathbb{R}^+ \), as a function which for all \( n \in \mathbb{N}, n > 1 \) satisfies the following axioms:

[GSK1] \( \mathcal{H}_n \) is continuous in \( \Delta_n \);

[GSK2] \( \mathcal{H}_n \) takes its largest value for the uniform distribution, i.e. for any \( P \in \Delta_n \), \( \mathcal{H}(P) \leq \mathcal{H}(U_n) \);

[GSK3] \( \mathcal{H}_n \) is expandable: \( \mathcal{H}_{n+1}(p_1, p_2, \ldots, p_n, 0) = \mathcal{H}_n(p_1, p_2, \ldots, p_n) \) for all \( (p_1, \ldots, p_n) \in \Delta_n \);

[GSK4] Let \( P = (p_1, \ldots, p_n) \in \Delta_n, \mathcal{P}Q = (r_{11}, r_{12}, \ldots, r_{nm}) \in \Delta_{nm}, n, m \in \mathbb{N}, n, m > 1 \) such that \( p_i = \sum_{j=1}^m r_{ij} \) and \( \mathcal{Q}k = (q_{1k}, \ldots, q_{mk}) \in \Delta_m \), where \( q_{jk} = r_{jk}/p_k \) and \( \alpha \in \mathbb{R}^+ \) is some fixed parameter. Then,

\[
\mathcal{H}_{nm}(PQ) = \mathcal{H}_n(P) \oplus \mathcal{H}_m(Q), \quad \text{where} \quad \mathcal{H}_m(Q) = g^{-1}\left(\sum_k p_k^{(\alpha)} g(\mathcal{H}_m(Q_k))\right),
\]

where \( g \) is the invertible continuous function.

In the following theorem we establish the class of entropies characterized by the generalized Shannon-Khinchin axioms.

**Theorem 5.1.** A function \( \mathcal{H}_n : \Delta_n \rightarrow \mathbb{R}^+ \), satisfies generalized Shannon-Khinchin axioms [GSK1]-[GSK4] iff it belongs to the following family parameterized by \( \tau, \lambda, \alpha \in \mathbb{R} \):

\[
\mathcal{H}_n(P) = h \left( \mathcal{N}_n(P) \right)
\]

\[
\begin{align*}
&= h \left( \tau \cdot \sum_{k=1}^n p_k \log_2 p_k \right), \quad \tau < 0, \quad \text{for } \lambda = 0, \\
&= h \left( \frac{1}{\lambda} \log_2 \left( \sum_{k=1}^n p_k^\alpha \right) \right), \quad \alpha > 0, \lambda \cdot (1 - \alpha) > 0, \quad \text{for } \lambda \neq 0.
\end{align*}
\]

In addition, if the following normalization axiom holds:

[GSK5] Normalization axiom: \( \mathcal{H}_2(\frac{1}{2}, \frac{1}{2}) = h(1) \),

then \( \tau = -1, \lambda = 1 - \alpha \) and \( \mathcal{H}_n \) belongs to the family

\[
\mathcal{H}_n(P) = h \left( \mathcal{R}_n(P) \right)
\]

\[
\begin{align*}
&= h \left( -\sum_{k=1}^n p_k \log_2(p_k) \right), \quad \text{for } \alpha = 1, \\
&= h \left( \frac{1}{1 - \alpha} \log_2 \left( \sum_{k=1}^n p_k^\alpha \right) \right), \quad \alpha > 0, \alpha \neq 1.
\end{align*}
\]

**Proof:** Let \( h^{-1} \) denotes the inverse mapping of \( h \), let us denote \( \mathcal{N}_n(P) = h^{-1}(\mathcal{H}_n(P)) \Rightarrow \mathcal{H}_n(P) = h(\mathcal{N}_n(P)) \) and \( f = g \circ h \), where \( \circ \) denote the composition of functions. Then, [GSK1]-[GSK5] reduces to [NSK1]-[NSK5] and result follows.
5.1. Example - Sharma-Mittal and Frank-Daffertshofer entropies

The idea for combining axiomatic systems [NSK1]-[NSK5] and [ASK1]-[ASK5] has firstly been proposed in [9] for the case of $\oplus_\gamma$-addition and the function $h$ given by (20), which is the special case of the system [GSK1]-[GSK5] proposed in this section. If the function (20) is used and $a = 1, \lambda = 1 - q$, where $q \in \mathbb{R}$,

$$h(x) = \begin{cases} 
\frac{x}{2^{1-q} - 1}, & \text{for } q = 1, \\
\frac{x}{1 - \gamma}, & \text{for } q \neq 1,
\end{cases}$$

(31)

generalized entropy form (30) reduces to the two-parameter entropy:

$$\begin{align*}
\mathcal{S}_n^{(P)} &= -\sum_{k=1}^{n} p_k \log_2(p_k) & \text{for } q = 1, \alpha = 1, \\
\mathcal{G}_n^{(P)} &= \frac{1}{\gamma} \left( \prod_{k} p_k^{q-\frac{1}{q} p_k} - 1 \right), \quad \gamma \cdot (1 - q) > 0, & \text{for } q \neq 1, \alpha = 1, \\
\mathcal{R}_n^{(P)} &= \frac{1}{1 - \alpha} \cdot \log_2 \left( \sum_{k=1}^{n} p_k^\alpha \right) & \text{for } q = 1, \alpha \neq 1, \alpha > 0, \\
\mathcal{D}_n^{(P)} &= \frac{1}{\gamma} \cdot \left( \sum_{k} p_k^\alpha \right)^{1-\alpha} - 1), & \gamma \cdot (1 - q) > 0, \text{ for } q \neq 1, \alpha \neq 1, \alpha > 0,
\end{align*}$$

(32)

considered by Sharma and Mittal [15], for $\gamma = 2^{1-q} - 1$, and Frank and Daffertshofer [4], for $\gamma = 1 - q$. The special cases contained in (32) are the Shannon entropy, $\mathcal{S}_n$, Gaussian entropy, $\mathcal{G}_n$, and Renyi entropy, $\mathcal{R}_n$, [13]. The uniqueness theorem for $\oplus_\gamma$-addition generalized Shannon-Khinchin axioms straightforwardly follows from the theorem [5].

**Theorem 5.2.** Let the entropy $\mathcal{D}_n : \Delta_n \to \mathbb{R}^+$ be defined as a function which for all $n \in \mathbb{N}$, $n > 1$ satisfies the following axioms:

**[SM1]** $\mathcal{D}_n$ is continuous in $\Delta_n$;

**[SM2]** $\mathcal{D}_n$ takes its largest value for the uniform distribution, i.e. for any $P \in \Delta_n$, $\mathcal{D}_n(P) \leq \mathcal{D}_n(U_n)$;

**[SM3]** $\mathcal{D}_n$ is expandable: $\mathcal{D}_{n+1}(p_1, p_2, \ldots, p_n, 0) = \mathcal{D}_n(p_1, p_2, \ldots, p_n)$ for all $(p_1, \ldots, p_n) \in \Delta_n$;

**[SM4]** Let $P = (p_1, \ldots, p_n) \in \Delta_n$, $PQ = (r_1, r_1, \ldots, r_m) \in \Delta_m$, $n, m \in \mathbb{N}$, $n > 1$ such that $p_i = \sum_{j=1}^{m} r_{ij}$, and $Q_k = (q_{1k}, \ldots, q_{mk}) \in \Delta_m$, where $q_{jk} = r_{jk}/p_k$ and $\alpha \in \mathbb{R}^+$ is some fixed parameter. Then,

$$\mathcal{D}_m(PQ) = \mathcal{D}_n(P) \oplus_\gamma \mathcal{D}_m(Q|P),$$

where $\mathcal{D}_m(Q|P) = f^{-1} \left( \sum_k p_k^{\alpha} f(\mathcal{D}_m(Q|k)) \right)$,

(33)

where $f$ is invertible continuous function.

**[SM5]** Normalization axiom:

$$\mathcal{D}_n\left( \frac{1}{2} \right) = h(1) = \begin{cases} 
\frac{x}{2^{1-q} - 1}, & \text{for } q = 1, \\
\frac{x}{1 - \gamma}, & \text{for } q \neq 1,
\end{cases}$$

(34)

Then, the entropy has the form (32).
Remark 5.3. The axiomatic system [SM1]-[SM5] has firstly been proposed in [9]. However, the class they obtained is for \( \alpha = 1 \) solution only. The proof from [9] may be rederived by applying the mapping \((20)\) on [NSK1]-[NSK5] and by setting \( D_n = h \circ \mathcal{N}_n \), where \( h \) is given with \((31)\). The equality \((16)\) can equivalently be written as

\[
D_n = g_y^{-1} \left( \sum_{k=1}^{n} p_k^{(a)} g_y(\text{Log}_y p_k') \right),
\]

where \( \text{Log}_y = h^{-1} \circ \log_2 \), \( g_y = f_y \circ h^{-1} \) and \( f_y \) is given with \((17)\). At this point Jizba and Arimitsu assume that

\[
g_y^{-1} \left( \sum_{k=1}^{n} p_k^{(a)} g_y(\text{Log}_y p_k') \right) = g^{-1} \left( \sum_{k=1}^{n} p_k^{(a)} g(\text{Log}_y p_k') \right),
\]

where \( g = f \circ h^{-1} \) and \( f \) is given with \((5)\). However, this implies that \( g = f \circ h^{-1} \) and \( g_y = f_y \circ h^{-1} \) generate the same mean and, according to \((5)\), \( g \) must be a linear function of \( g_y \), which is the case only for \( \lambda = 1 - \alpha = 0 \).

6. Conclusion

In this paper we considered a generalization of the Shannon-Khinchin axiomatic system [11]. Previously, Jizba and Arimitsu provided the generalization of the Shannon-Khinchin axioms for the characterization of the Rényi entropy [10]. We modified Jizba-Arimitsu’s system by relaxing the normalization axiom and obtained the Nath entropy as the unique solution. On the other hand, Abe provided the axiomatic system for the characterization of the Tsallis entropy [16]. Abe’s axiomatic system is based on \( \oplus \), addition [13]. In this paper we generalized Abe’s axiomatic system by considering the more general pseudo-addition operation.

In addition, two approaches were combined and the corresponding uniqueness theorem was given. We obtained a generalized entropy which can be represented as the nonlinear transformation of the Nath entropy, whose special case is the Sharma-Mittal entropy [15]. Previously, a similar axiomatic system was discussed in [9], and the we commented on the uniqueness theorem from [9].

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