OBSERVABILITY OF N-DIMENSIONAL INTEGRO-DIFFERENTIAL SYSTEMS

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Abstract. The aim of the paper is to show a reachability result for the solution of a multidimensional coupled Petrovsky and wave system when a nonlocal term, expressed as a convolution integral, is active. Motivations to the study are in linear acoustic theory in three dimensions. To achieve that, we prove observability estimates by means of Ingham type inequalities applied to the Fourier series expansion of the solution.

1. Introduction. In this paper we solve a reachability problem for systems of the following type

\[
\begin{aligned}
&u_{1tt} - \Delta u_1 + \beta \int_0^t e^{-\eta(t-s)} \Delta u_1(s, x) ds + a \Delta u_2 = 0 \\
u_{2tt} + \Delta^2 u_2 + bu_1 = 0
\end{aligned}
\]

in \((0, T) \times \Omega\), \(1.1\)

where \(T > 0, \Omega\) is an open ball of radius \(R\) in \(\mathbb{R}^N\), \(N \geq 2, \eta > \beta > 0\) and \(a, b\) are real constants.

According to several authors, see for example [9], a reachability problem for system (1.1) consists in determining boundary controls \(g_1\) and \(g_2\) which steer the solution of (1.1), subject to the boundary conditions

\[
u_1 = g_1, \quad u_2 = 0, \quad \Delta u_2 = g_2, \quad \text{on } (0, T) \times \partial \Omega, \quad 1.2\]

from the null initial conditions

\[
u_1(0, \cdot) = u_{10}(0, \cdot) = u_{20}(0, \cdot) = u_{21}(0, \cdot) = 0, \quad 1.3\]

to a given final target at time \(T\)

\[
u_1(T, \cdot) = u_{10}, \quad u_{1t}(T, \cdot) = u_{11}, \quad u_2(T, \cdot) = u_{20}, \quad u_{2t}(T, \cdot) = u_{21}.
\]

Exponential kernels are used to model viscoelastic systems, as well as in the analysis of Maxwell fluids or Poynting-Thomson solids, see [14]. The proposed system models the vibration of viscoelastic membranes with exponential kernels coupled with plates. In the present paper we introduce a coupling term of a different nature. In the second equation of (1.1) we assume the linear dependence on the displacement of the membrane. This coupling term is already used in the literature (see [5] and reference therein for string–beam coupling and [12] for viscoelastic string–beam

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coupling). The main novelty consists in considering a coupling term of higher order in the $N$-dimensional case, because in the first equation of (1.1) the coupling term is $a\triangle u_2$ instead of the term $au_2$. To justify this new term, we recall that the $\triangle$ operator measures the deficit (or concentration if we consider $-\triangle$) of the function with respect to its integral average, the so-called local anomaly (see e.g. [13]).

The coupling proposed requires an accurate spectral analysis, the inequalities leading to the reachability result are obtained with new proofs, adapted from previous results (see [12]).

Moreover, we study the model described by (1.1) in the $N$-dimensional case when the domain is an open ball of radius $R$. For this coupled system we will show a reachability result for time $T > 2R$, that is the same condition on $T$ as in the uncoupled case.

More precisely, we prove the following

**Theorem 1.1.** Let $\eta > 3\beta/2$. For any $T > 2R$, $(u_{10}, u_{11}) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $(u_{20}, u_{21}) \in H_0^1(\Omega) \times H^{-1}(\Omega)$ there exist $g_i \in L^2((0, T) \times \partial\Omega)$, $i = 1, 2$, such that the weak solution $(u_1, u_2)$ of system (1.1) verifies the final conditions

$$
\begin{align*}
  u_1(T, \cdot) &= u_{10}, & u_{11}(T, \cdot) &= u_{11}, & u_2(T, \cdot) &= u_{20}, & u_{21}(T, \cdot) &= u_{21}.
\end{align*}
$$

A remark about the assumption $\eta > 3\beta/2$ is now in order. The proof of Theorem 1.1 (see Section 5) requires the Ingham type inequalities proved in [12] and recalled in this paper as Theorem 3.1. To apply Theorem 3.1 to our system we have to verify, in particular, an assumption involving the eigenvalues of the integro-differential operator, that is $r_n \leq -3\omega_n$ for large $n$, and that assumption is satisfied if the condition $\eta > 3\beta/2$ holds true.

We also note that the shape of domain $\Omega$ is crucial to solve the corresponding eigenvalue problem in the $N$-dimensional case by means of the Bessel functions.

Our approach to the controllability of systems modeled by linear partial differential equations goes back to the survey paper by D. Russell [16], to the books by J.-L. Lions [8, 9], and to the book related to applications by J. Lagnese and J.-L. Lions [6]. We refer to [5] for the approach to controllability based on Fourier analysis and Ingham type inequalities (for the seminal paper by Ingham see [3]).

As is well known, to solve a reachability problem like ours is equivalent to establish observability estimates for the solution of the adjoint problem (see [16]).

We refer to [10] and [11] to get observability of a single viscoelastic equation in the 1-dimensional case and multidimensional case respectively. In [12] we explore, in 1-d, the problem with a weak coupling in a viscoelastic medium and we study the ensuing system, answering in a positive way to the question of exact controllability in time greater than $2\pi/\gamma$, $\gamma$ being the gap between eigenvalues. We refer to [4] for the exact controllability of membrane–plate coupled systems in the $N$-dimensional case without viscosity. For an overview concerning viscoelastic models and integro-differential equations see [15, 14]. Other applications are related to acoustic theory [2].

The plan of our paper is the following. In Section 2 we introduce some notations, a preliminary result, recall the structure of the eigenfunctions of the Laplace operator in a ball and briefly describe the Hilbert Uniqueness Method. In Section 3 we recall Ingham type inequalities. In Section 4 we give a Fourier series expansion for the solution of the adjoint system. Finally, in Section 5 we prove the reachability result for coupled systems with memory terms like (1.1).
2. Preliminaries. Throughout the paper, we will adopt the convention to write $f \asymp g$ if there exist two positive constants $c_1$ and $c_2$ such that $c_1 f \leq g \leq c_2 f$.

For any $T > 0$, we denote by $L^1(0, T)$ (resp. $L^2(0, T)$) the usual space of measurable functions $\varphi: (0, T) \to \mathbb{R}$ such that

$$\int_0^T |\varphi(t)| \, dt < \infty \quad \text{(resp. } \int_0^T |\varphi(t)|^2 \, dt < \infty \text{)}.\)$$

We recall well-known results concerning integral equations, see for example [1, Theorem 2.3.5].

Lemma 2.1. For $k \in L^1(0, T)$ the following properties hold true.

(i) For any $\psi \in L^1(0, T)$ there exists a unique solution $\varphi \in L^1(0, T)$ of the integral equation

$$\varphi(t) - \int_t^T k(s - t)\varphi(s) \, ds = \psi(t), \quad 0 \leq t \leq T,$$

given by

$$\varphi(t) = \psi(t) + \int_t^T \rho_k(s - t)\psi(s) \, ds, \quad 0 \leq t \leq T,$$

where $\rho_k \in L^1(0, T)$ is the resolvent kernel of $k$.

(ii) For any $\varphi \in L^2(0, T)$ the function $\varphi(t) - \int_t^T k(s - t)\varphi(s) \, ds \in L^2(0, T)$ and vice versa. Moreover,

$$\int_0^T \left| \varphi(t) - \int_t^T k(s - t)\varphi(s) \, ds \right|^2 \, dt \asymp \int_0^T |\varphi(t)|^2 \, dt. \quad (2.1)$$

Let $\Omega$ be an open ball $\Omega$ of radius $R$ in $\mathbb{R}^N$, $N \geq 2$. In the following we consider $L^2(\Omega)$ and $H^1_0(\Omega)$ endowed with the standard norms

$$\| u \|^2 = \langle u, u \rangle_{L^2(\Omega)} = \int_{\Omega} |u(x)|^2 \, dx, \quad \| u \|^2_{H^1_0(\Omega)} = \int_{\Omega} |\nabla u(x)|^2 \, dx,$$

and $H^{-1}(\Omega)$ is endowed with the dual norm of $\| \cdot \|_{H^1_0(\Omega)}$.

For the sake of completeness, we bring to mind some well-known arguments regarding the eigenfunctions of the Laplace operator in a ball, of which we will take advantage in the next sections.

First, we recall some basic facts regarding Bessel type functions (see e.g. [5]), which will be fundamental to introduce the eigenfunctions of the Laplace operator in a ball. Let us introduce the Bessel functions of the first kind of any real order $p$ by the formula

$$J_p(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(p+j+1)} \left(\frac{x}{2}\right)^{p+2j}, \quad x \geq 0,$$

where $\Gamma$ denotes the gamma function.

Lemma 2.2. Let $p$ be a nonnegative real number. The following equality holds for every positive real number $c$:

$$2c^2 \int_0^1 r |J_p(cr)|^2 \, dr = c^2 |J'_p(c)|^2 + (c^2 - p^2) |J_p(c)|^2. \quad (2.2)$$

As for the location of the zeros of the Bessel functions, we have the next result.
Theorem 2.3. The following statements hold true.

(a) For any given real number \( p \), the positive zeros of \( J_p(x) \) are simple and they form an infinite strictly increasing sequence \( \{ \lambda_n \} \) tending to infinity.
(b) The difference sequence \( \{ \lambda_{n+1} - \lambda_n \} \) converges to \( \pi \).
(c) The sequence \( \{ \lambda_{n+1} - \lambda_n \} \) is strictly decreasing if \( |p| > 1/2 \), strictly increasing if \( |p| < 1/2 \) and constant if \( p = \pm 1/2 \).

We may assume without loss of generality that \( \Omega \) is the unit ball of \( \mathbb{R}^N \): the general case then follows easily by a linear change of variables. We shall consider the case \( N \geq 2 \). Let us also recall that the spherical harmonics of order \( m \in \mathbb{N} \) are the restrictions to the unit sphere \( \partial \Omega \) of the homogeneous polynomials of order \( m \).

Lemma 2.4. The spherical harmonics of order \( m \in \mathbb{N} \) form a finite-dimensional subspace \( S_m \) in \( L^2(\partial \Omega) \). These subspaces are mutually orthogonal and their linear hull is dense in \( L^2(\partial \Omega) \).

By using hyperspherical coordinates \( (\rho, \theta) \), \( 0 \leq \rho \leq 1 \) and \( \theta \in \partial \Omega \), we can describe the eigenfunctions of the Laplace operator.

Theorem 2.5. The eigenfunctions of \( -\triangle \) with homogeneous Dirichlet boundary conditions are the functions

\[ E_{mk}(\rho, \theta) := \rho^{m+\frac{N}{2}} J_{m-1+\frac{N}{2}}(\lambda_{mk}\rho) H_m(\theta), \quad 0 \leq \rho \leq 1, \quad \theta \in \partial \Omega, \]

where \( m \in \mathbb{N} \cup \{0\} \), \( k \in \mathbb{N} \), \( H_m \in S_m \) and for each \( m \) we denote by \( \{ \lambda_{mk} \}_{k \in \mathbb{N}} \) the strictly increasing sequence of positive zeros of the Bessel function \( J_{m-1+\frac{N}{2}} \). The corresponding eigenvalue of the eigenfunction \( E_{mk}(\rho, \theta) \) is \( \lambda_{mk}^2 \).

For reader’s convenience, we will describe the Hilbert Uniqueness Method for coupled systems with memory, when the integral kernel is a general function \( k \in L^1(0,T) \). For another approach see e.g. [7, 17].

Let \( \Omega \) be an open ball of radius \( R \) in \( \mathbb{R}^N \), \( N \geq 2 \). We consider the following coupled system

\[
\begin{cases}
    u_{1tt} - \triangle u_1 + \int_0^t k(t-s) \triangle u_1(s,x) ds + a \triangle u_2 = 0 & \text{in } (0,T) \times \Omega, \\
    u_{2tt} + \triangle^2 u_2 + bu_1 = 0 & \text{on } (0,T) \times \partial \Omega, \\
    u_1 = g_1, \quad u_2 = 0, \quad \triangle u_2 = g_2, & \text{on } (0,T) \times \partial \Omega,
\end{cases}
\]  

(2.3)

with null initial conditions

\[ u_1(0,\cdot) = u_{1t}(0,\cdot) = u_2(0,\cdot) = u_{2t}(0,\cdot) = 0. \quad (2.4) \]

For a reachability problem we mean the following.

Definition 2.6. Given \( T > 0 \) and \( u_{10}, u_{11}, u_{20} \) and \( u_{21} \) belonging to suitable spaces, a reachability problem consists in finding \( g_i \in L^2(0,T), \) \( i = 1, 2 \) such that the weak solution \( u \) of problem (2.3) – (2.4) verifies the final conditions

\[ u_1(T,\cdot) = u_{10}, \quad u_{1t}(T,\cdot) = u_{11}, \quad u_2(T,\cdot) = u_{20}, \quad u_{2t}(T,\cdot) = u_{21}. \quad (2.5) \]

One can solve such reachability problems by the Hilbert Uniqueness Method. To see that, we proceed as follows.
Given \((z_{10}, z_{11}, z_{20}, z_{21}) \in (C_c^\infty(\Omega))^4\), we introduce the adjoint system of (2.3), that is
\[
\begin{aligned}
    z_{1tt} - \Delta z_1 + \int_0^T k(s-t)\triangle z_1(s,x)ds + b z_2 &= 0 \\
    z_{2tt} + \Delta^2 z_2 + a\Delta z_1 &= 0
\end{aligned}
\] 
with final data
\[
    z_1(T,\cdot) = z_{10}, \quad z_{1t}(T,\cdot) = z_{11}, \quad z_2(T,\cdot) = z_{20}, \quad z_{2t}(T,\cdot) = z_{21}.
\] 
The above problem is well-posed, see e.g. [14]. Thanks to the regularity of the final data, the solution \((z_1, z_2)\) of (2.6) – (2.7) is regular enough to consider the nonhomogeneous problem
\[
\begin{aligned}
    \phi_{1tt} - \Delta \phi_1 + \int_0^T k(t-s)\triangle \phi_1(s,x)ds + a\Delta \phi_2 &= 0 \\
    \phi_{2tt} + \Delta^2 \phi_2 + b \phi_1 &= 0 \\
    \phi_1 &= \partial_t z_1 - \int_0^T k(s-t)\partial_t z_1(s,\sigma)ds \\
    \phi_2 &= 0, \quad \Delta \phi_2 = -\partial_{\nu} z_2
\end{aligned}
\] 
(2.8) 
As in the non integral case, it can be proved that problem (2.8) admits a unique solution \(\phi\). Therefore, we can introduce the following linear operator: for any \((z_{10}, z_{11}, z_{20}, z_{21}) \in (C_c^\infty(\Omega))^4\) we define
\[
\Psi(z_{10}, z_{11}, z_{20}, z_{21}) = (-\phi_{1t}(T,\cdot), \phi_1(T,\cdot), -\phi_{2t}(T,\cdot), \phi_2(T,\cdot)).
\] (2.9) 

The following identity holds
\[
\langle \Psi(z_{10}, z_{11}, z_{20}, z_{21}), (z_{10}, z_{11}, z_{20}, z_{21}) \rangle_{L^2(\Omega)} = \int_0^T \int_{\partial\Omega} \left| \partial_{\nu} z_1 - \int_0^T k(s-t)\partial_t z_1(s,\sigma)ds \right|^2 + \left| \partial_{\nu} z_2 \right|^2 d\sigma dt.
\]
(2.10) 

As a consequence, we can introduce a semi-norm on the space \((C_c^\infty(\Omega))^4\). Precisely, for any \((z_{10}, z_{11}, z_{20}, z_{21}) \in (C_c^\infty(\Omega))^4\), if we consider the solution \((z_1, z_2)\) of the system (2.6) – (2.7), then we define
\[
\|(z_{10}, z_{11}, z_{20}, z_{21})\|_F := \left( \int_0^T \int_{\partial\Omega} \left| \partial_{\nu} z_1 - \int_0^T k(s-t)\partial_t z_1(s,\sigma)ds \right|^2 + \left| \partial_{\nu} z_2 \right|^2 d\sigma dt \right)^{1/2}.
\] (2.10) 

Thanks to Lemma 2.1–(i), \(\| \cdot \|_F\) is a norm if and only if the following uniqueness result holds.
Theorem 2.7. If \((z_1, z_2)\) is the solution of problem \((2.6) - (2.7)\) such that
\[
\partial_\nu z_1 = \partial_\nu z_2 = 0, \quad \text{on } [0, T] \times \partial \Omega,
\]
then
\[
z_1 = z_2 = 0 \quad \text{in } [0, T] \times \Omega.
\]

If Theorem 2.7 holds true, then we can define the Hilbert space \(F\) as the completion of \((C^\infty_c(\Omega))^4\) for the norm \((2.10)\). So, the operator \(\Psi\) can be extended uniquely to a continuous operator, denoted again by \(\Psi\), from \(F\) to the dual space \(F'\) in such a way that \(\Psi : F \to F'\) is an isomorphism.

Moreover, if we prove observability estimates of the following type
\[
\int_0^T \int_{\partial \Omega} |\partial_\nu z_1|^2 + |\partial_\nu z_2|^2 \, d\sigma dt \leq ||z_{10}||^2_{H^1_0(\Omega)} + ||z_{11}||^2_{H^1_0(\Omega)} + ||z_{20}||^2_{H^1_0(\Omega)} + ||z_{21}||^2_{H^{-1}(\Omega)},
\]
then, by virtue of \((2.1)\) and \((2.10)\), we get
\[
F = H^1_0(\Omega) \times L^2(\Omega) \times H^1_0(\Omega) \times H^{-1}(\Omega)
\]
with the equivalence of the respective norms. Finally, we can solve the reachability problem \((2.3) - (2.5)\) for \((u_{10}, u_{11}) \in L^2(\Omega) \times H^{-1}(\Omega)\) and \((u_{20}, u_{21}) \in H^1_0(\Omega) \times H^{-1}(\Omega)\).

3. Ingham type theorems. To apply the Hilbert Uniqueness Method we need suitable observability estimates for the solution of the adjoint system, see \((2.11)\). In the proof of our reachability result (see Theorem 5.1), to obtain such estimates we will take advantage of Ingham type inverse and direct inequalities that we proved in \([12, \text{Theorems } 5.19 \text{ and } 5.21]\). For reader’s convenience, in this section we recall those results, presented them in a slight modified version.

In the next theorem we consider functions of the following type
\[
\begin{aligned}
u_1(t) &= \sum_{n=1}^\infty \left( R_n e^{\gamma_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\omega_n t} + D_n e^{ip_n t} + \overline{D_n} e^{-ip_n t} \right), \\
u_2(t) &= \sum_{n=1}^\infty \left( d_n D_n e^{ip_n t} + \overline{d_n} D_n e^{-ip_n t} + e^{-\left(i\eta + \gamma_p\right)n} \|\frac{D_n}{\eta + ip_n} + \frac{\overline{D_n}}{\eta - ip_n} \| \right),
\end{aligned}
\]
with \(r_n, R_n \in \mathbb{R}\) and \(\omega_n, C_n, p_n, D_n \in \mathbb{C}\).

Theorem 3.1. Let \(\{\omega_n\}_{n \in \mathbb{N}}, \{r_n\}_{n \in \mathbb{N}}\) and \(\{p_n\}_{n \in \mathbb{N}}\) be sequences of pairwise distinct numbers such that \(\omega_n \neq p_m, \omega_n \neq \overline{p_m}, r_n \neq \mp, r_n \neq \mp, r_n \neq -\eta, p_n \neq 0\), for any \(n, m \in \mathbb{N}\). Assume
\[
\lim_{n \to \infty} (\Re p_{n+1} - \Re p_n) = +\infty, \quad \lim_{n \to \infty} \Im p_n = 0,
\]
and for some \(\gamma > 0, \alpha \in \mathbb{R}, n' \in \mathbb{N}, \mu > 0, \nu > 1/2, m_1, m_2 > 0\)
\[
\lim_{n \to \infty} (\Re \omega_{n+1} - \Re \omega_n) = \gamma, \quad \lim_{n \to \infty} \Im \omega_n = \alpha, \quad r_n \leq -\Im \omega_n \quad \forall \ n \geq n',
\]
\[
|R_n| \leq \frac{\mu}{n^\nu} |C_n| \quad \forall \ n \geq n', \quad |R_n| \leq (\mu |C_n| \quad \forall \ n \leq n',
\]
\[
m_1 |p_n|^2 \leq |d_n| \leq m_2 |p_n|^2 \quad \forall n \in \mathbb{N}.
\]
Then, for any \( T > 2\pi/\gamma \) we have
\[
\int_0^T |u_1(t)|^2 + |u_2(t)|^2 \, dt \geq \sum_{n=1}^{\infty} |C_n|^2 + \sum_{n=1}^{\infty} |D_n|^2 |p_n|^4,
\] (3.2)
where \( u_1 \) and \( u_2 \) are defined in (3.1).

4. Fourier series expansion of the solution. Let \( \Omega \) be the unit ball of \( \mathbb{R}^N \), \( N \geq 2 \), and \( T > 0 \). Fix two real numbers \( a, b \). For any \( (u_{10}, u_{11}) \in H_1^0(\Omega) \times L^2(\Omega) \) and \( (u_{20}, u_{21}) \in H_1^0(\Omega) \times H^{-1}(\Omega) \) there exists a unique weak solution \( (u_1, u_2) \) of the following coupled system
\[
\begin{align*}
&u_{1t} - \Delta u_1 + \beta \int_0^t e^{-\eta(t-s)} \Delta u_1(s, x) \, ds + bu_2 = 0, \quad \text{in} \quad (0, T) \times \Omega, \\
&u_{2t} + \Delta^2 u_2 + a\Delta u_1 = 0, \quad \text{in} \quad (0, T) \times \Omega, \\
&u_1(t, \cdot) = u_2(t, \cdot) = 0, \quad \text{on} \quad [0, T] \times \partial \Omega, \\
&u_1(0, \cdot) = u_{10}, \quad u_{11}(0, \cdot) = u_{11}, \\
&u_2(0, \cdot) = u_{20}, \quad u_{21}(0, \cdot) = u_{21}.
\end{align*}
\] (4.1)

If we use hyperspherical coordinates \( (\rho, \theta) \) \( (0 \leq \rho \leq 1, \theta \in \partial \Omega) \) and expand the initial data according to the eigenfunctions of \( -\Delta \) (see Theorem 2.5), then for \( j = 1, 2 \) we obtain
\[
\begin{align*}
u_{j0}(\rho, \theta) &= \rho^{1-\frac{N}{2}} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} J_{m-1+\frac{1}{2}} (\lambda_{mk} \rho) \alpha_{jmk}(\theta), \\
u_{j1}(\rho, \theta) &= \rho^{1-\frac{N}{2}} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} J_{m-1+\frac{1}{2}} (\lambda_{mk} \rho) \chi_{jmk}(\theta),
\end{align*}
\] (4.2)
where \( \alpha_{jmk}(\theta), \chi_{jmk}(\theta) \) are suitable spherical harmonics of order \( m \). By applying the spectral analysis developed in Hilbert spaces, see [12, Section 4], to the operator \( -\Delta \) with null Dirichlet boundary conditions, we are able to write the solution \( (u_1, u_2) \) of problem (4.1) as Fourier series.

Theorem 4.1. For any \( t \geq 0, 0 \leq \rho \leq 1 \) and \( \theta \in \partial \Omega \) we have
\[
\begin{align*}
u_1(t, \rho, \theta) &= \rho^{1-\frac{N}{2}} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \left( R_{mk}(\theta) e^{i\rho_{mk} t} + C_{mk}(\theta) e^{i\omega_{mk} t} + \overline{C_{mk}(\theta)} e^{-i\omega_{mk} t} \right) \\
&\quad + D_{mk}(\theta) e^{i\rho_{mk} t} + \overline{D_{mk}(\theta)} e^{-i\rho_{mk} t} \right) J_{m-1+\frac{1}{2}} (\lambda_{mk} \rho), \\
&= \rho^{1-\frac{N}{2}} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \left( d_{mk} D_{mk}(\theta) e^{i\rho_{mk} t} + \overline{d_{mk} D_{mk}(\theta)} e^{-i\rho_{mk} t} \right) J_{m-1+\frac{1}{2}} (\lambda_{mk} \rho),
\end{align*}
\] (4.3)
where the numbers \( r_{mk} \in \mathbb{R}, \omega_{mk}, p_{mk}, d_{mk} \in \mathbb{C} \) are defined by
\[
r_{mk} = \beta - \eta + O\left( \frac{1}{\lambda_{mk}^2} \right),
\]
\( \mathcal{R} \omega_{mk} = \lambda_{mk} + \frac{\beta}{2} \left( \frac{3}{4} \beta - \eta \right) \frac{1}{\lambda_{mk}^2} + O \left( \frac{1}{\lambda_{mk}^3} \right), \quad \Im \omega_{mk} = \frac{\beta}{2} + O \left( \frac{1}{\lambda_{mk}^8} \right), \) \hspace{1cm} (4.4)

\[ \mathcal{R} p_{mk} = \lambda_{mk}^2 + O \left( \frac{1}{\lambda_{mk}^4} \right), \quad \Im p_{mk} = O \left( \frac{1}{\lambda_{mk}^8} \right), \] \hspace{1cm} (4.5)

\[ d_{mk} = \frac{1}{b} \left( p_{mk} - \mathcal{R} p_{mk} + \frac{\beta}{\eta} \mathcal{R} p_{mk} \right), \]  
and the spherical harmonics \( R_{mk}(\theta) \in \mathbb{R}, C_{mk}(\theta), D_{mk}(\theta) \in \mathbb{C} \) are written in terms of the spherical harmonics of the initial data, see (4.2), in the following way

\[ R_{mk}(\theta) = \frac{\beta}{\lambda_{mk}} \left( \alpha_{1mk}(\theta)(\beta - \eta) + \chi_{1mk}(\theta) \right) + (\alpha_{1mk}(\theta) + \chi_{1mk}(\theta)) O \left( \frac{1}{\lambda_{mk}^4} \right), \]

\[ C_{mk}(\theta) = \frac{\alpha_{1mk}(\theta)}{2} - \frac{i}{4\lambda_{mk}} (\beta \alpha_{1mk}(\theta) + 2 \chi_{1mk}(\theta)) + (\alpha_{1mk}(\theta) + \chi_{1mk}(\theta)) O \left( \frac{1}{\lambda_{mk}^4} \right), \]

\[ D_{mk}(\theta) = \frac{b}{2\lambda_{mk}^4} \left( \alpha_{2mk}(\theta) - i \chi_{2mk}(\theta) \right) + (\alpha_{2mk}(\theta) + \chi_{2mk}(\theta)) O \left( \frac{1}{\lambda_{mk}^4} \right). \]

Moreover, for any \( m \in \mathbb{N} \cup \{0\}, k \in \mathbb{N} \) and \( \theta \in \partial \Omega \) one has

\[ \lambda_{mk}^2 |C_{mk}(\theta)|^2 \leq \lambda_{mk}^2 \alpha_{1mk}(\theta)^2 + \chi_{1mk}(\theta)^2, \]

\[ \lambda_{mk}^4 |D_{mk}(\theta)|^2 \leq \lambda_{mk}^2 \alpha_{2mk}(\theta)^2 + \frac{\chi_{2mk}(\theta)^2}{\lambda_{mk}^2}. \] \hspace{1cm} (4.6)

5. The reachability result. In this section we will prove a reachability result for the coupled system by taking advantage of the Hilbert Uniqueness Method, recalled at the end of Section 2, and of the Ingham type inequalities given by Theorem 3.1. We may assume without loss of generality that \( \Omega \) is the unit ball of \( \mathbb{R}^N \).

**Theorem 5.1.** Let \( \eta > 3\beta/2 \). For any \( T > 2 \), \( (u_{10}, u_{11}) \in L^2(\Omega) \times H^{-1}(\Omega) \) and \( (u_{20}, u_{21}) \in H^1_0(\Omega) \times H^{-1}(\Omega) \) there exist \( g_i \in L^2((0, T) \times \partial \Omega), i = 1, 2 \), such that the weak solution \( (u_1, u_2) \) of system

\[
\begin{cases}
   u_{1t} - \Delta u_1 + \beta \int_0^t e^{-\eta(t-s)} \Delta u_1(s, x)ds + a \Delta u_2 = 0 \\
   u_{2t} + \Delta^2 u_2 + bu_1 = 0 \\
   u_1 = g_1, \quad u_2 = 0, \quad \Delta u_2 = g_2, \quad \text{on} \ (0, T) \times \partial \Omega, \\
   u_1(0, \cdot) = u_{11}(0, \cdot) = u_2(0, \cdot) = u_{21}(0, \cdot) = 0,
\end{cases}
\]

verifies the final conditions

\[ u_1(T, \cdot) = u_{10}, \quad u_{11}(T, \cdot) = u_{11}, \quad u_2(T, \cdot) = u_{20}, \quad u_{21}(T, \cdot) = u_{21}. \]

**Proof.** First, we consider the adjoint system of (5.1):

\[
\begin{cases}
   z_{1t} - \Delta z_1 + \beta \int_0^t e^{-\eta(s-t)} \Delta z_1(s, x)ds + b z_2 = 0 \\
   z_{2t} + \Delta^2 z_2 + a \Delta z_1 = 0 \\
   z_1 = z_2 = \Delta z_2 = 0 \quad \text{on} \ [0, T] \times \partial \Omega \\
   z_1(T, \cdot) = z_{11}, \quad z_1(t, \cdot) = z_{11}, \quad z_2(T, \cdot) = z_{20}, \quad z_2(T, \cdot) = z_{21},
\end{cases}
\]

with \( (z_{10}, z_{11}) \in H^1_0(\Omega) \times L^2(\Omega) \) and \( (z_{20}, z_{21}) \in H^1_0(\Omega) \times H^{-1}(\Omega) \). It is easy to verify that the backward problem (5.2) is equivalent to a Cauchy problem of the
type (4.1) with \( u_i(t, x) = z_i(T-t, x) \), \( i = 1, 2 \). Therefore, we can apply Theorem 4.1 to write the solution \((z_1, z_2)\) of the adjoint system as Fourier series. If we expand the final data according to the eigenfunctions of \(-\Delta\), then we obtain expressions similar to formulas (4.2). Indeed, by using hyperspherical coordinates \((\rho, \theta)\), with \(0 \leq \rho \leq 1\) and \(\theta \in \partial \Omega\), for \(j = 1, 2\) we have

\[
\begin{align*}
    z_{j0}(\rho, \theta) &= \rho^{1-\frac{d}{2}} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} J_{m-1+\frac{d}{2}}(\lambda_{mk}\rho)\alpha_{mk}(\theta), \\
    z_{j1}(\rho, \theta) &= \rho^{1-\frac{d}{2}} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} J_{m-1+\frac{d}{2}}(\lambda_{mk}\rho)\chi_{mk}(\theta),
\end{align*}
\]

where \(\alpha_{mk}(\theta), \chi_{mk}(\theta)\) are suitable spherical harmonics of order \(m\). Moreover, the relative norms are given by

\[
\begin{align*}
    ||z_{10}||^2_{H^1(\Omega)} &= \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \lambda_{mk}^2 \int_0^1 \rho |J_{m-1+\frac{d}{2}}(\lambda_{mk}\rho)|^2 \, d\rho \int_{\partial \Omega} |\alpha_{mk}(\theta)|^2 \, d\theta, \\
    ||z_{11}||^2 &= \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \int_0^1 \rho |J_{m-1+\frac{d}{2}}(\lambda_{mk}\rho)|^2 \, d\rho \int_{\partial \Omega} |\chi_{mk}(\theta)|^2 \, d\theta, \\
    ||z_{20}||^2_{H^1(\Omega)} &= \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda_{mk}^2} \int_0^1 \rho |J_{m-1+\frac{d}{2}}(\lambda_{mk}\rho)|^2 \, d\rho \int_{\partial \Omega} |\alpha_{mk}(\theta)|^2 \, d\theta, \\
    ||z_{21}||^2_{H^1(\Omega)} &= \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda_{mk}^2} \int_0^1 \rho |J_{m-1+\frac{d}{2}}(\lambda_{mk}\rho)|^2 \, d\rho \int_{\partial \Omega} |\chi_{mk}(\theta)|^2 \, d\theta.
\end{align*}
\]

Therefore \((z_1, z_2)\) can be written as in formulas (4.3): for any \(t \in [0, T]\), \(\rho \in [0, 1]\) and \(\theta \in \partial \Omega\) we have

\[
\begin{align*}
    z_1(t, \rho, \theta) &= \rho^{1-\frac{d}{2}} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} (R_{mk}(\theta)e^{r_{mk}(T-t)} + C_{mk}(\theta)e^{i\omega_{mk}(T-t)} + \overline{C_{mk}(\theta)}e^{-i\omega_{mk}(T-t)}) \\
    &+ D_{mk}(\theta)e^{ip_{mk}(T-t)} + \overline{D_{mk}(\theta)}e^{-ip_{mk}(T-t)}) J_{m-1+\frac{d}{2}}(\lambda_{mk}\rho), \\
    z_2(t, \rho, \theta) &= \rho^{1-\frac{d}{2}} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} (d_{mk} D_{mk}(\theta)e^{ip_{mk}(T-t)} + \overline{d_{mk} D_{mk}(\theta)}e^{-ip_{mk}(T-t)}) J_{m-1+\frac{d}{2}}(\lambda_{mk}\rho) \\
    &- \frac{\beta}{\lambda} e^{-\eta(T-t)} \rho^{1-\frac{d}{2}} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \Re p_{mk} \left( \frac{D_{mk}(\theta)}{\eta + ip_{mk}} + \frac{\overline{D_{mk}(\theta)}}{\eta - ip_{mk}} \right) J_{m-1+\frac{d}{2}}(\lambda_{mk}\rho),
\end{align*}
\]

where the numbers \(r_{mk}, \omega_{mk}, p_{mk}, d_{mk}\) and the spherical harmonics \(R_{mk}(\theta), C_{mk}(\theta), D_{mk}(\theta)\) are given as in Theorem 4.1. Taking into account that for any \(m \in \mathbb{N} \cup \{0\}\) \(\lambda_{mk}\) are zeros of the Bessel function \(J_{m-1+\frac{d}{2}}\), we get

\[
\begin{align*}
    \partial_\nu z_1(t, 1, \theta) &= \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \lambda_{mk} J'_{m-1+\frac{d}{2}}(\lambda_{mk}) R_{mk}(\theta)e^{r_{mk}(T-t)} \\
    &+ 2 \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \lambda_{mk} J'_{m-1+\frac{d}{2}}(\lambda_{mk}) \Re \left( C_{mk}(\theta)e^{i\omega_{mk}(T-t)} + D_{mk}(\theta)e^{ip_{mk}(T-t)} \right).
\end{align*}
\]
\[\partial_{\nu}z_{2}(t, 1, \theta) = 2 \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \lambda_{mk} J'_{m-1 + \frac{\nu}{2}}(\lambda_{mk}) \Re \left( d_{mk} D_{mk}(\theta) e^{i\rho_{mk}(T-t)} \right) \]
\[\hspace{1cm} - \frac{2\beta}{b} e^{-\eta(T-t)} \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \lambda_{mk} J'_{m-1 + \frac{\nu}{2}}(\lambda_{mk}) \Re p_{mk} \Re \left( \frac{D_{mk}(\theta)}{\eta + i\rho_{mk}} \right).\]

Since the spherical harmonics of different order are mutually orthogonal in \(L^2(\partial\Omega)\), see Lemma 2.4, we have

\[\int_{\partial\Omega} |\partial_{\nu}z_{1}(t, 1, \theta)|^2 d\theta = 4 \sum_{m=0}^{\infty} \int_{\partial\Omega} \left| \sum_{k=1}^{\infty} \lambda_{mk} J'_{m-1 + \frac{\nu}{2}}(\lambda_{mk}) \Re \left( R_{mk}(\theta) e^{\rho_{mk}(T-t)} \right) \right|^2 d\theta,\]
\[\int_{\partial\Omega} |\partial_{\nu}z_{2}(t, 1, \theta)|^2 d\theta = 4 \sum_{m=0}^{\infty} \int_{\partial\Omega} \left| \sum_{k=1}^{\infty} \lambda_{mk} J'_{m-1 + \frac{\nu}{2}}(\lambda_{mk}) \Re \left( d_{mk} D_{mk}(\theta) e^{i\rho_{mk}(T-t)} \right) \right|^2 d\theta.\]

Integrating from 0 to \(T\) and applying the Fubini – Tonelli theorem (we make also a change of variable which replace \(T - t\) with \(t\), we get

\[\int_{0}^{T} \int_{\partial\Omega} |\partial_{\nu}z_{1}(t, 1, \theta)|^2 d\theta \hspace{0.1cm} dt = \sum_{m=0}^{\infty} \int_{\partial\Omega} \int_{0}^{T} \left| \sum_{k=1}^{\infty} \lambda_{mk} J'_{m-1 + \frac{\nu}{2}}(\lambda_{mk}) \Re \left( R_{mk}(\theta) e^{\rho_{mk}t} \right) \right|^2 dt \hspace{0.1cm} d\theta + 2\Re \left( C_{mk}(\theta)e^{i\omega_{mk}t} + D_{mk}(\theta)e^{i\rho_{mk}t} \right) \right|^2 d\theta,\]
\[\int_{0}^{T} \int_{\partial\Omega} |\partial_{\nu}z_{2}(t, 1, \theta)|^2 d\theta \hspace{0.1cm} dt = 4 \sum_{m=0}^{\infty} \int_{\partial\Omega} \int_{0}^{T} \left| \sum_{k=1}^{\infty} \lambda_{mk} J'_{m-1 + \frac{\nu}{2}}(\lambda_{mk}) \Re \left( d_{mk} D_{mk}(\theta) e^{i\rho_{mk}t} \right) \right|^2 dt \hspace{0.1cm} d\theta.\]

We also observe that in view of (4.4) one gets

\[\Re \omega_{m,k+1} - \Re \omega_{mk} = \lambda_{m,k+1} - \lambda_{mk} + \frac{\beta}{2} \left( \eta - \frac{3}{4} \beta \right) \left( \frac{1}{\lambda_{mk}} - \frac{1}{\lambda_{m,k+1}} \right) + O \left( \frac{1}{\lambda_{mk}^2} \right).\]

Therefore, thanks to the behavior of zeros of Bessel functions (see Theorem 2.3) and the assumption \(\eta > 3\beta/2\), the numbers \(r_{mk}, \omega_{mk}, p_{mk}, \lambda_{mk} J'_{m-1 + \frac{\nu}{2}}(\lambda_{mk}) R_{mk}(\theta), \lambda_{mk} J'_{m-1 + \frac{\nu}{2}}(\lambda_{mk}) C_{mk}(\theta), \lambda_{mk} J'_{m-1 + \frac{\nu}{2}}(\lambda_{mk}) D_{mk}(\theta) \) and \(\lambda_{mk} J'_{m-1 + \frac{\nu}{2}}(\lambda_{mk}) d_{mk}\) satisfy the conditions of Theorem 3.1. So, for any \(m \in \mathbb{N} \cup \{0\}\) and \(\theta \in \partial\Omega\) we can apply Theorem 3.1 to the functions

\[\sum_{k=1}^{\infty} \lambda_{mk} J'_{m-1 + \frac{\nu}{2}}(\lambda_{mk}) \Re \left( R_{mk}(\theta) e^{\rho_{mk}t} + 2\Re \left( C_{mk}(\theta)e^{i\omega_{mk}t} + D_{mk}(\theta)e^{i\rho_{mk}t} \right) \right).\]
and
\[ 2 \sum_{k=1}^{\infty} \lambda_{mk} J_{m-1+\frac{\lambda_{mk}}{2}} \left( \frac{d_{mk} D_{mk}(\theta)}{\eta} e^{-\eta t} \frac{D_{mk}(\theta)}{\eta + i \rho_{mk}} \right). \]

Consequently, thanks to (3.2), for any \( T > 2 \) we have
\[ \int_0^T \left| \sum_{k=1}^{\infty} \lambda_{mk} J_{m-1+\frac{\lambda_{mk}}{2}} \left( R_{mk}(\theta) e^{i \rho_{mk} t} + 2 \Re \left( C_{mk}(\theta) e^{i \omega_{mk} t} + D_{mk}(\theta) e^{i \rho_{mk} t} \right) \right) \right|^2 dt \]
\[ + 4 \int_0^T \left| \sum_{k=1}^{\infty} \lambda_{mk} J_{m-1+\frac{\lambda_{mk}}{2}} \left( d_{mk} D_{mk}(\theta) e^{i \rho_{mk} t} - \frac{\beta}{\rho} e^{-\eta t} \Re \rho_{mk} \frac{D_{mk}(\theta)}{\eta + i \rho_{mk}} \right) \right|^2 dt \]
\[ \approx \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \lambda_{mk}^2 \left| J_{m-1+\frac{\lambda_{mk}}{2}} (\lambda_{mk}) \right|^2 \int_{\partial \Omega} |C_{mk}(\theta)|^2 + |D_{mk}(\theta)|^2 |\rho_{mk}|^4 d \theta. \]

Combining (5.4) and (5.5) with the previous estimates we get
\[ \int_0^T \int_{\partial \Omega} \left| \partial_{\nu} z_1(t, 1, \theta) \right|^2 + \left| \partial_{\nu} z_2(t, 1, \theta) \right|^2 d \theta dt \]
\[ \approx \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \lambda_{mk}^2 \left| J_{m-1+\frac{\lambda_{mk}}{2}} (\lambda_{mk}) \right|^2 \int_{\partial \Omega} |C_{mk}(\theta)|^2 + |D_{mk}(\theta)|^2 |\rho_{mk}|^4 d \theta. \]

In addition, thanks to (2.2) we have
\[ \left| J_{m-1+\frac{\lambda_{mk}}{2}} (\lambda_{mk}) \right|^2 = 2 \int_0^1 \rho \left| J_{m-1+\frac{\lambda_{mk}}{2}} (\lambda_{mk} \rho) \right|^2 d \rho, \]
so, in view also of (5.6) and (4.5), we get
\[ \int_0^T \int_{\partial \Omega} \left| \partial_{\nu} z_1(t, 1, \theta) \right|^2 + \left| \partial_{\nu} z_2(t, 1, \theta) \right|^2 d \theta dt \]
\[ \approx \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \int_0^1 \rho \left| J_{m-1+\frac{\lambda_{mk}}{2}} (\lambda_{mk} \rho) \right|^2 d \rho \int_{\partial \Omega} \lambda_{mk}^2 |C_{mk}(\theta)|^2 + \lambda_{mk}^4 |D_{mk}(\theta)|^2 d \theta. \]

Taking into account the estimates (4.6) and the norms (5.3) of the final data, we obtain
\[ \int_0^T \int_{\partial \Omega} \left| \partial_{\nu} z_1(t, 1, \theta) \right|^2 + \left| \partial_{\nu} z_2(t, 1, \theta) \right|^2 d \theta dt \]
\[ \approx \| z_{10} \|_{H^1_0(\Omega)}^2 + \| z_{20} \|_{H^2_0(\Omega)}^2 + \| z_{21} \|_{H^{-1}(\Omega)}^2, \]
that is, estimates (2.11) hold true. Therefore, the space \( F \) introduced at the end of Section 2 is
\[ H^1_0(\Omega) \times L^2(\Omega) \times H^1_0(\Omega) \times H^{-1}(\Omega). \]

We recall that the operator \( \Psi \) defined in (2.9) is an isomorphism from the space \( H^1_0(\Omega) \times L^2(\Omega) \times H^1_0(\Omega) \times H^{-1}(\Omega) \) into its dual space given by \( H^{-1}(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega) \times H^1_0(\Omega) \). Therefore, if we take \( (u_{10}, u_{11}, u_{20}, u_{21}) \in L^2(\Omega) \times H^{-1}(\Omega) \times H^2_0(\Omega) \times H^{-1}(\Omega) \), then there exists a unique \( (z_{10}, z_{11}, z_{20}, z_{21}) \in H^1_0(\Omega) \times L^2(\Omega) \times H^1_0(\Omega) \times H^{-1}(\Omega) \) such that
\[ \Psi(z_{10}, z_{11}, z_{20}, z_{21}) = (u_{11}, u_{10}, -u_{21}, u_{20}). \]
Denoting by \((z_1, z_2)\) the weak solution of problem (5.2) with final data \(z_{10}, z_{11}, z_{20}\) and \(z_{21}\) and taking into account (2.1), we have that the functions defined by

\[
g_1 = \partial_\nu z_1 - \beta \int_t^T e^{-\eta(t-s)} \partial_\nu z_1(s, \sigma) ds, \quad g_2 = -\partial_\nu z_2,
\]

are the boundary controls required by our statement. Indeed, if \((u_1, u_2)\) is the weak solution of the following problem

\[
\begin{cases}
  u_{1tt} - \triangle u_1 + \beta \int_0^t e^{-\eta(t-s)} \triangle u_1(s, x) ds + a\triangle u_2 = 0 & \text{in } (0, T) \times \Omega, \\
  u_{2tt} + \triangle^2 u_2 + b u_1 = 0 & \\
  u_1(0, \cdot) = u_{1t}(0, \cdot) = u_2(0, \cdot) = u_{2t}(0, \cdot) = 0 & \\
  u_1 = \partial_\nu z_1 - \beta \int_t^T e^{-\eta(t-s)} \partial_\nu z_1(s, \sigma) ds & \text{on } (0, T) \times \partial\Omega, \\
  u_2 = 0, \quad \triangle u_2 = -\partial_\nu z_2 &
\end{cases}
\]

thanks to the definition of operator \(\Psi\), see (2.9), and (5.7) we have that \((u_1, u_2)\) verifies the final conditions

\[
u_1(T, \cdot) = u_{10}, \quad u_{1t}(T, \cdot) = u_{11}, \quad u_2(T, \cdot) = u_{20}, \quad u_{2t}(T, \cdot) = u_{21}.
\]

So, our proof is complete.

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