Quantum Stiefel manifolds

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Abstract

Quantum analogs of Stiefel manifolds $SU_q(n)/SU_q(n-m)$ were introduced by Podkolzin & Vainerman. The underlying $C^*$-algebra $C(SU_q(n)/SU_q(n-m))$ can be described as the $C^*$-subalgebra of $C(SU_q(n))$ generated by elements of last $m$ rows of the fundamental matrix of $SU_q(n)$. Using $R$-matrix of type $A_{n-1}$, one can find certain relations involving elements of last $m$ rows only. In this paper, by analyzing these relations and using a result of Neshveyev & Tuset, we establish $C(SU_q(n)/SU_q(n-m))$ as a universal $C^*$-algebra given by finite sets of generators and relations.

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1 Introduction

Compact quantum groups and their homogeneous spaces naturally occur in noncommutative geometry. A source of examples comes from $q$-deformation of compact simply connected semisimple Lie groups and their quotient spaces. Many of the $C^*$-algebras arising in this context are universal $C^*$-algebras given by finite sets of generators and relations. This fact along with a knowledge of their generators and relations proved to be very useful in studying topology as well as geometry of these noncommutative spaces. For example, the description of the quantum group $SU_q(2)$ and the quantum odd dimensional spheres $S^{2n+1}_q$ as universal $C^*$-algebras given by finite sets of generators and relations was used by Vaksman & Soibelman ([12],[13]) to compute their $K$-theory. Later using this description, Chakraborty & Pal ([1],[3]) and Pal & Sundar ([7]) constructed spectral triples for these spaces to study their geometrical aspects. Recently
Saurabh [9] established the quotient space $C(SP_q(2n)/SP_q(2n-2))$ as a universal $C^*$-algebra given by finite sets of generators and relations. Analyzing the relations among the generators, we get a chain of short exact sequences of $C^*$-algebras and then using homogeneous extension theory, we proved in [10] that $C(SP_q(2n)/SP_q(2n-2))$ is isomorphic to $C(S_q^{2n-1})$.

The most well-known examples of quantum homogeneous spaces are quantum Stiefel manifolds $SU_q(n)/SU_q(n - m)$. Podkolzin & Vainerman [6] described all irreducible representations of the algebra $S_q^{n,m}$ underlying the manifold $SU_q(n)/SU_q(n - m)$. They proved that the algebra $S_q^{n,m}$ is generated by matrix entries of last $m$ rows of the fundamental matrix of $SU_q(n)$. Using $R$-matrix of type $A_{n-1}$, they found certain relations satisfied by these generators of $S_q^{n,m}$ and proved that these relations are full system of relations (see Theorem 1, [6]). In other words, the algebra $S_q^{n,m}$ is the universal algebra generated by $2nm$ number of generators satisfying these relations. Using the techniques similar to that used by Podkolzin & Vainerman in [6], one can easily extend some of these results to the $C^*$-algebra level. For example, one can obtain all irreducible representations of the $C^*$-algebra underlying the Stiefel manifolds $SU_q(n)/SU_q(n - m)$ (in fact Podkolzin & Vainerman obtained it in [6]). Similarly it can be proved that $C(SU_q(n)/SU_q(n - m))$ is generated by matrix entries of last $m$ rows of the fundamental matrix $SU_q(n)$. But extension of the fact that $S_q^{n,m}$ is the universal algebra generated by elements of last $m$ rows satisfying the relations given in Theorem 1, [6] to the $C^*$-algebra level is not obvious and demands a proof. For $m = 1$, this was proved by Vaksman & Soibelman [12]. In this paper, we attempt this problem for all $m \leq n$.

Here we should mention that to the best of our understanding, there seems to be a gap in the proof of part (2) of the Theorem 1 in [6]. Podkolzin & Vainerman [6] stated that monomials in \( \{ u_k^l \} \) having the lexicographic order form bases of the algebras $U_q(n)$ underlying the manifold $SU_q(n)$ and $S_q^{n,m}$ respectively. But since \( \sum_{k=1}^n u_k^l u_k^l = \delta_{ij} \), this is not true. In fact it follows from diamond lemma (see page 103, [4]) that reduced monomials in the generators of $U_q(n)$ and $S_q^{n,m}$ will form bases of the algebras $U_q(n)$ and $S_q^{n,m}$ respectively. But then it is not obvious that a reduced monomial of the generators \( \{ u_k^l \} \) in the algebra $S_q^{n,m}$ will be a reduced monomial in the algebra $U_q(n)$ as there are more relations among the generators of the algebra $U_q(n)$. One thing we should point out here that Podkolzin & Vainerman [6] used left coset space and as a consequence, elements of last $m$ columns of the fundamental matrix of $SU_q(n)$ are the generators of the underlying algebra $S_q^{n,m}$. Here we take right
coset space $SU_q(n)/SU_q(n-m)$ and hence elements of last $m$ rows are the generators of the underlying algebra $S_{q,n,m}^m$. But similar arguments hold in both cases.

To tackle the problem, we first write down all irreducible representations of the $C^*$-algebra $C(SU_q(n)/SU_q(n-m))$ by applying a result of Neshveyev & Tuset (5). We associate some diagrams to each irreducible representation and using this, we describe certain properties of these representations. We then define the $C^*$-algebra $C_{n,m}$ as a universal $C^*$-algebra generated by $nm$ generators satisfying those relations satisfied by the generators of $C(SU_q(n)/SU_q(n-m))$. Due to universal property of $C_{n,m}$, we get a surjective homomorphism from $C_{n,m}$ to $C(SU_q(n)/SU_q(n-m))$. By analyzing the relations carefully and using some properties of irreducible representations of $C(SU_q(n)/SU_q(n-m))$, we show that all irreducible representations of the $C^*$-algebra $C_{n,m}$ factor through this homomorphism. This implies that the homomorphism is injective. Therefore, the two $C^*$-algebras $C_{n,m}$ and $C(SU_q(n)/SU_q(n-m))$ are isomorphic which establishes $C(SU_q(n)/SU_q(n-m))$ as a universal $C^*$-algebra given by finite sets of generators and relations.

We set up some notations which will be used throughout this paper. The standard basis of the Hilbert space $L_2(\mathbb{N})$ will be denoted by $\{e_n : n \in \mathbb{N}\}$. We denote the left shift operator and the number operator on $L_2(\mathbb{N})$ by $S$ and $N$ respectively. Let $p$ denote the rank one projection sending $e_0$ to $e_0$. We denote by $T^n$ the $n$-dimensional torus. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, the symbol $|\alpha|$ denotes the number $\sum_{i=1}^n \alpha_i$. The number $q$ will denote a real number in the interval $(0, 1)$.

2 ODQS(q) relations

In this section, we describe some properties of ordered tuples of bounded operators satisfying certain relations. Let $T_1, T_2, \cdots, T_n$ be bounded linear operators acting on a Hilbert space $H$. We say that the ordered tuple $(T_1, T_2, \cdots, T_n)$ satisfies odd dimension quantum sphere relations with parameter $q$ abbreviated as $ODQS(q)$ relations if it obeys the following relations:

$$T_i T_j = q T_j T_i, \quad 1 \leq j < i \leq n,$$

$$T_i^* T_j = q T_j^* T_i^*, \quad 1 \leq i \neq j \leq n,$$

$$T_i^* T_i - T_i T_i^* = (1 - q^2) \sum_{j>i} T_j T_j^*, \quad \sum_{i=1}^n T_i T_i^* = 1.$$
It follows from above relations that there exists a unique number $\ell$ in $\{1, 2, \cdots, n\}$ such that $T_\ell$ is a nonzero normal operator and for $i > \ell$, $T_i = 0$. We call this number $\ell$ the rank of $(T_1, T_2, \cdots, T_n)$. We say that the tuple $(T_1, \cdots, T_n)$ is irreducible if there is no proper subspace of $H$ invariant under the action of $\{T_i, T_i^* : i \in \{1, 2, \cdots, n\}\}$.

**Proposition 2.1.** Suppose that the tuple $(T_1, T_2, \cdots, T_n)$ obeys ODQS(q) relations and of rank $\ell$. Let $\omega$ be the operator $T_i^*T_\ell$ and $H_0$ be the eigenspace of $\omega$ corresponding to the eigenvalue 1. Then one has

1. for $1 \leq i < \ell$, $T_i \omega = q^{-2} \omega T_i$ and $T_i^* \omega = q^2 \omega T_i^*$,
2. $\omega = I$ on $\bigcap_{i=1}^{\ell-1} \ker T_i^*$,
3. $1(q^{2m+2}q^{2m})(\omega) = 0 \quad \forall m \in \mathbb{N}$,
4. $\ker T_i \subseteq \ker T_k^*$ for $k \geq i$ and $1 \leq i \leq \ell$,
5. if $u$ is a nonzero eigenvector of $\omega$ corresponding to the eigenvalue $q^{2m}$, then $u \notin \ker T_i$ for $1 \leq i \leq \ell - 1$.
6. $\sigma(\omega) = \{q^{2m} : m \in \mathbb{N}\} \bigcup \{0\}$.
7. For $i \in \{1, 2, \cdots, \ell - 1\}$ and $m \in \mathbb{N} - \{0\}$, one has

$$T_i^* T_i^m = T_i^m T_i^* + (1 - q^{2m}) \sum_{j>i} T_i^{m-1} T_j T_j^*.$$

8. Let $H_{(\alpha_1, \cdots, \alpha_{\ell-1})} := T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{\ell-1}^{\alpha_{\ell-1}} H_0$ where $\alpha_i \in \mathbb{N}$ for all $i \in \{1, 2, \cdots, \ell - 1\}$. Then for different values of $\alpha = (\alpha_1, \cdots, \alpha_{\ell-1})$, the subspaces $H_{(\alpha_1, \cdots, \alpha_{\ell-1})}$ are nonzero and mutually orthogonal.
9. If in addition, $\ker(\omega) = \{0\}$, then one has

$$H = \bigoplus_{\alpha_i \in \mathbb{N}} H_{(\alpha_1, \cdots, \alpha_{\ell-1})} = \bigoplus_{\alpha_i \in \mathbb{N}} T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{\ell-1}^{\alpha_{\ell-1}} H_0.$$

10. The tuple $(T_1, \cdots, T_n)$ is irreducible if and only if $H_0$ is one dimensional. Moreover in this case, $\ker(\omega) = 0$ and for a nonzero vector $h \in H_0$,

$$\left\{ \frac{T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{\ell-1}^{\alpha_{\ell-1}} h}{\|T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{\ell-1}^{\alpha_{\ell-1}} h\|} : \alpha_i \in \mathbb{N} \text{ for all } i \in \{1, 2, \cdots, \ell - 1\} \right\}$$
form an orthonormal basis of $H$. Further for $h \in H_0$, $T_i h = \theta h$ for some $t \in \mathbb{T}$.

We call $t \in \mathbb{T}$ the angle of the irreducible tuple $(T_1, \cdots, T_n)$.

11. For $1 \leq i < \ell$ and $h \in H_0$, one has

$$T_i^* h = 0, \quad \text{and} \quad T_i^* T_i = (1 - q^2)h.$$

**Proof:**
1. Easy to see from ODQS(q) relations.
2. It follows from the relation $\sum_{i=1}^{\ell} T_i T_i^* = 1$.
3. From the ODQS(q) relations, it follows that $T_i^* f(\omega) = f(q^2 \omega) T_i^*$ for all $i \in \{1, 2, \cdots, \ell - 1\}$ for all continuous functions $f$ and hence for all $L_\infty$ functions. Thus
   \[
   T_i^* 1_{(q^{2n+2}, q^{2n})}(\omega) = 1_{(q^{2n+2}, q^{2n})}(q^2 \omega) T_i^* \\
   = 1_{(q^{2n}, q^{2n-2})}(\omega) T_i^*.
   \]
   By repeated application and using the relation $\sum_{i=1}^{\ell} T_i T_i^* = 1$ and the fact that $\sigma(\omega) \subseteq [0, 1]$, it follows that $1_{(q^{2n+2}, q^{2n})}(\omega) = 0$.
4. Let $h \in \ker(T_i)$. Then we have
   \[
   \langle T_i^* T_i h, h \rangle = \left( T_i T_i^* h + (1 - q^2) \sum_{k>j} T_k T_k^* h, h \right).
   \]
   Hence
   \[
   \|T_i^* h\|^2 + (1 - q^2) \sum_{k>j} \|T_k^* h\|^2 = 0.
   \]
   Hence $\|T_k^* h\| = 0$ for all $k \geq i$, which means $h \in \ker(T_i^*)$ for all $k \geq i$.
5. From part (4), we have $\ker(T_i) \subseteq \ker(T_i^*) = \ker(T_i) = \ker(\omega)$. Now if $u$ is a non-zero eigenvector of $\omega$ corresponding to eigenvalue $q^{2m}$ for some $m \in \mathbb{N}$, then $u \notin \ker(T_i^*)$. Hence $u \notin \ker(T_i)$ for $1 \leq i \leq \ell$.
6. From part (3) and the fact that $\|\omega\| \leq 1$, it follows that the spectrum $\sigma(\omega)$ of $\omega$ is contained in $\{q^{2m} : m \in \mathbb{N}\} \cup \{0\}$. Define
   \[
   A = \{m \in \mathbb{N} : q^{2m} \in \sigma(\pi(\omega))\}.
   \]
   Since $\omega \neq 0$, $A \neq \emptyset$. Let $m_0 = \inf \{m \in \mathbb{N} : q^{2m} \in \sigma(\omega)\}$. Let $u$ be a nonzero eigenvector corresponding to $q^{2m_0}$. Assume $u \notin \ker(T_i^*)$ for some $i \in \{1, 2, \cdots, \ell - 1\}$. Then from part (1), it follows that $T_i^* u$ is a nonzero eigenvector corresponding to the eigenvalue $q^{2m_0 - 2}$, which contradicts the fact that $m_0$ is inf $A$. Hence $u \in \bigcap_{i=1}^{\ell-1} \ker(T_i^*)$. As $\omega = I$ on $\bigcap_{i=1}^{\ell-1} \ker(T_i^*)$, we get $m_0 = 0$. From part (5), it follows that $u \notin \ker(T_i)$ for any $i \in \{1, 2, \cdots, \ell\}$. Hence we have $T_i^m u$ is a nonzero eigenvector corresponding to eigenvalue $q^{2m}$ for all $m \in \mathbb{N}$. This proves the claim.
7. It follows by applying the relation $T_i^* T_i = T_i T_i^* + (1 - q^2) \sum_{j>i} T_j T_j^*$ repeatedly.
8. For any nonzero $h \in \mathcal{H}_0$, the vector $T_1^{\alpha_1} \cdots T_{\ell-1}^{\alpha_{\ell-1}} h$ is an eigenvector corresponding to eigenvalue $q^{2|\alpha|}$. Hence it follows from part (5) that $\mathcal{H}(\alpha_2, \cdots, \alpha_{\ell-1})$ is nonzero. Let
Let \( \alpha = (\alpha_1, \ldots, \alpha_{\ell-1}) \) and \( \alpha' = (\alpha'_1, \ldots, \alpha'_{\ell-1}) \) be two different tuple of positive integers. Without loss of generality, we assume that \( \alpha_1 \neq 0 \). For \( h, h' \in H_0 \), we have

\[
\langle T_1^{\alpha_1} \cdots T_{\ell-1}^{\alpha'_{\ell-1}} h, T_1^{\alpha'_1} \cdots T_{\ell-1}^{\alpha'_{\ell-1}} h' \rangle \\
= \langle T_1^{\alpha_1-1} \cdots T_{\ell-1}^{\alpha'_{\ell-1}} h, T_1^{\alpha'_1} \cdots T_{\ell-1}^{\alpha'_{\ell-1}} h' \rangle \\
= q^{\sum_{i=1}^{\ell-1} \alpha'_i (1 - q^{2\alpha'})(T_1^{\alpha_1-1} \cdots T_{\ell-1}^{\alpha'_{\ell-1}} h, T_1^{\alpha'_1} \cdots T_{\ell-1}^{\alpha'_{\ell-1}} h')}
\]

Last equality follows from part (7) and the ODQS(q) relations. Now by induction on \(|\alpha|\), we get the claim.

9. Let \( H_m \) be the eigenspace of \( \omega \) corresponding to the eigenvalue \( q^{2m} \). Then by spectral theorem and the fact that \( \ker \omega = \{0\} \), one has \( H = \bigoplus_{m \in \mathbb{N}} H_m \). So, to prove the claim, we need to show that for all \( m \in \mathbb{N} \),

\[
H_m = \bigoplus_{\{\alpha_i \in \mathbb{N} : \sum_{i=1}^{\ell-1} \alpha_i = m\}} T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{\ell-1}^{\alpha_{\ell-1}} H_0.
\]

From the ODQS(q) relations, we have \( \bigoplus_{\{\alpha_i \in \mathbb{N} : \sum_{i=1}^{\ell-1} \alpha_i = m\}} T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{\ell-1}^{\alpha_{\ell-1}} H_0 \subset H_m \). Take a nonzero vector \( h \) in \( H_m \) for \( m \neq 0 \). Then from the relation \( \sum_{i=1}^{\ell-1} T_i T_i^* = 1 \), it follows that there exists \( i \in \{1, 2, \ldots, \ell - 1\} \) such that \( T_i^* h \neq 0 \). Hence we get a nonzero vector \( T_i^* h \) in \( H_{m-1} \). Proceeding in this way and arranging term using ODQS(q) relations, we get \( \beta_i \in \mathbb{N} \) such that \( \sum_{i=1}^{\ell-1} \beta_i = m \) and \( h_0 := (T_{\ell-1}^*)^{\beta_{\ell-1}} \cdots (T_1^*)^{\beta_1} h \) is nonzero vector in \( H_0 \). Therefore, we get

\[
\langle h, T_1^{\beta_1} T_2^{\beta_2} \cdots T_{\ell-1}^{\beta_{\ell-1}} h_0 \rangle = \langle (T_{\ell-1}^*)^{\beta_{\ell-1}} \cdots (T_1^*)^{\beta_1} h, h_0 \rangle = \langle h_0, h_0 \rangle \\
\neq 0.
\]

This implies that no nonzero vector in \( H_m \) is orthogonal to the vector subspace \( \bigoplus_{\{\alpha_i \in \mathbb{N} : \sum_{i=1}^{\ell-1} \alpha_i = m\}} T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{\ell-1}^{\alpha_{\ell-1}} H_0 \). Hence we get

\[
H_m \subset \bigoplus_{\{\alpha_i \in \mathbb{N} : \sum_{i=1}^{\ell-1} \alpha_i = m\}} T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{\ell-1}^{\alpha_{\ell-1}} H_0.
\]

This proves the claim.

10. It follows from the relations \( \sum_{i=1}^{\ell} T_i T_i^* = 1 \) that \( T_i \) acts as a unitary operator on \( H_0 \). Hence there exists an orthonormal basis \( \{h_i\}_{i \in \mathbb{N}} \) of \( H_0 \) such that \( T_i h_i = t_i h_i \) for some \( t_i \in \mathbb{T} \). It is easy to see that for any \( i \in \mathbb{N} \), the subspace spanned by the nonzero vectors \( \{T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{\ell-1}^{\alpha_{\ell-1}} h_i : \alpha_i \in \mathbb{N}\} \) is an invariant subspace. This proves that if \( (T_1, \ldots, T_n) \) is irreducible then \( H_0 \) is one dimensional. To show the
other way, take $h \in \mathcal{H}$. As done in part (9), one can show that by applying $(T_i)^{*}$'s repeatedly for appropriately chosen $i \in \{1, 2, \cdots, \ell - 1\}$, one can take $h$ to $\mathcal{H}_0$. This implies that any nonzero invariant subspace contains a vector in $\mathcal{H}_0$. Since $\mathcal{H}_0$ is one dimensional, any nonzero invariant subspace contains $\mathcal{H}_0$. This implies that the only nonzero invariant subspace is $\mathcal{H}$ and the tuple $(T_1, T_2, \cdots, T_n)$ is irreducible.

From the ODQS$(Q)$ relations, one can easily see that $\ker(\omega)$ is an invariant subspace. Since $\omega \neq 0$, it follows that $\ker(\omega) = \{0\}$. Other parts of the claim follow from part (8) and part (9).

11. It follows from ODQS$(q)$ relations that for $1 \leq i < \ell$ and $h \in \mathcal{H}_0$, $\omega(T_i^* h) = \frac{1}{q} T_i^* h$. Hence from part (6), we get $T_i^* h = 0$. Using this fact and the relation $T_i^* T_i = T_i T_i^* + (1 - q^2) \sum_{j>i} T_j^* T_j$, we get $T_i^* T_i h = (1 - q^2) h$.

$\Box$

**Remark 2.2.** Note that the operators $T_i$'s occurring in the tuple $(T_1, \cdots, T_n)$ that satisfies ODQS$(q)$ relations are homomorphic image of the generators of the $C^*$-algebra $C(S_q^{2n-1})$ of continuous functions on odd dimensional quantum sphere. Using this fact along with the knowledge of irreducible representations of the $C^*$-algebra $C(S_q^{2n-1})$, one can prove above proposition but to make the paper self-contained, we are giving a direct proof.

We shall now define some notions related to the ordered $n$-tuples satisfying ODQS$(q)$ relations. Let $\{(T_1^i, T_2^i, \cdots, T_n^i)\}_{i \in I}$ be ordered $n$-tuples obeying ODQS$(q)$ relations with rank $\ell_i$. Their direct sum $\bigoplus_{i \in I} (T_1^i, T_2^i, \cdots, T_n^i)$ is defined as $(\bigoplus_{i \in I} T_1^i, \cdots, \bigoplus_{i \in I} T_n^i)$. Clearly rank of the direct sum of ordered $n$-tuples is $\max\{\ell_i : i \in I\}$. Moreover, one can show that if $\ker(\bigoplus_{i \in I} T_i^i) = \{0\}$ where $\ell := \max\{\ell_i : i \in I\}$ then for all $i \in I$, $\ell_i = \ell$. We say that two ordered $n$-tuple $(T_1, T_2, \cdots, T_n)$ and $(T'_1, T'_2, \cdots, T'_n)$ of operators acting on the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ respectively that satisfy ODQS$(q)$ relations are isomorphic if there exists unitary $U : \mathcal{H} \to \mathcal{H}'$ such that $UT_i U^* = T'_i$ for all $1 \leq i \leq n$. The following proposition says that isomorphism class of an irreducible ordered $n$-tuple is characterized by its rank and angle. More precisely,

**Proposition 2.3.** Let $(T_1, T_2, \cdots, T_n)$ and $(T'_1, T'_2, \cdots, T'_n)$ be two irreducible ordered $n$-tuple obeying ODQS$(q)$ relations on the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ respectively. Let their ranks be $\ell$ and $\ell'$ respectively and angles be $t$ and $t'$ respectively. Then $(T_1, T_2, \cdots, T_n)$ and $(T'_1, T'_2, \cdots, T'_n)$ are isomorphic if and only if $\ell = \ell'$ and $t = t'$. 


Proof: It is easy to show that if \((T_1, T_2, \cdots, T_n)\) and \((T'_1, T'_2, \cdots, T'_n)\) are isomorphic then \(\ell = \ell'\) and \(t = t'\). To show the other way, let \(\mathcal{H}_0\) and \(\mathcal{H}'_0\) be the eigenspaces of \(T'_\ell T_\ell\) and \((T'_\ell)'T'_\ell\) respectively corresponding to the eigenvalue 1. From part (10) of the proposition \(\ref{prop:ODQS(q)}\) \(\mathcal{H}_0\) and \(\mathcal{H}'_0\) are one dimensional. Let \(h\) and \(h'\) be unit vectors in \(\mathcal{H}_0\) and \(\mathcal{H}'_0\) respectively. Let

\[
\begin{align*}
    h_{(\alpha_1, \cdots, \alpha_{\ell-1})} &:= \frac{T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{\ell-1}^{\alpha_{\ell-1}} h}{\| T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{\ell-1}^{\alpha_{\ell-1}} h \|}, \\
    h'_{(\alpha_1, \cdots, \alpha_{\ell-1})} &:= \frac{(T'_1)^{\alpha_1} (T'_2)^{\alpha_2} \cdots (T'_{\ell-1})^{\alpha_{\ell-1}} h'}{\| (T'_1)^{\alpha_1} (T'_2)^{\alpha_2} \cdots (T'_{\ell-1})^{\alpha_{\ell-1}} h' \|}.
\end{align*}
\]

It follows from part (10) of proposition \(\ref{prop:ODQS(q)}\) that \(\{h_{(\alpha_1, \cdots, \alpha_{\ell-1})} : \alpha_i \in \mathbb{N} \text{ for } 1 \leq i \leq \ell - 1\}\) and \(\{h'_{(\alpha_1, \cdots, \alpha_{\ell-1})} : \alpha_i \in \mathbb{N} \text{ for } 1 \leq i \leq \ell - 1\}\) are orthonormal bases of \(\mathcal{H}\) and \(\mathcal{H}'\) respectively. Let \(U : \mathcal{H} \to \mathcal{H}'\) such that \(Uh_{(\alpha_1, \cdots, \alpha_{\ell-1})} = h'_{(\alpha_1, \cdots, \alpha_{\ell-1})}\). Using ODQS(q) relations and the fact that \(t = t'\), it is easy to verify that \(UT_i U^* = T'_i\) for all \(1 \leq i \leq n\). This completes the proof. \(\square\)

Remark 2.4. Suppose that \((T_1, \cdots, T_n)\) obeys ODQS(q) relations on \(\mathcal{H}\) and of rank \(\ell\). Then it is easy to see that the tuple \((\omega_1 T_1, \omega_2 T_2, \cdots, \omega_n T_n)\) also satisfies ODQS(q) relations with rank \(\ell\) where \(\omega_i \in \mathbb{T}\) for all \(1 \leq i \leq n\). Further, it follows from proposition \(\ref{prop:isomorphic-tuple}\) that the two tuples \((\omega_1 T_1, \omega_2 T_2, \cdots, \omega_n T_n)\) and \((\omega'_1 T_1, \omega'_2 T_2, \cdots, \omega'_n T_n)\) are isomorphic if and only if \(\omega_\ell = \omega'_\ell\).

3 Representation theory of \(C(SU_q(n)/SU_q(n - m))\)

In the present section, we recall representation theory of \(C(SU_q(n)/SU_q(n - m))\) using a result of Neshveyev & Tuset and then associate a diagram to each irreducible representation. Let the fundamental matrices of the compact quantum groups \(SU_q(n)\) and \(SU_q(n - m)\) be \(U := ((u'_k^l))\) and \(V := ((v'^k_l))\) respectively. Consider the following map:

\[
    \phi(u'_k^l) = \begin{cases} 
        v'_k & \text{if } 1 \leq l, k \leq n - m, \\
        1 & \text{if } n - m + 1 \leq l = k \leq n, \\
        0 & \text{otherwise}.
    \end{cases}
\]

One can check that \(\phi\) is a quantum group homomorphism from \(SU_q(n)\) onto \(SU_q(n - m)\) and hence \(SU_q(n - m)\) is a subgroup of \(SU_q(n)\). The \(C^*\)-algebra of continuous functions on the right coset space \(SU_q(n)/SU_q(n - m)\) is given by

\[
C(SU_q(n)/SU_q(n - m)) = \{a \in C(SU_q(n)) : (\phi \otimes \text{id})\Delta(a) = I \otimes a\} \quad (3.1)
\]

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where $\Delta$ is the comultiplication map of $SU(n)$. The following theorem gives finite number of generators of the quotient space $C(SU_q(n)/SU_q(n - m))$. It can be considered as an extension of Theorem 1, [6] from algebra level to $C^*$-algebra level. One can easily derive it from the proof of Theorem 1, [6] but for the sake of completeness, we are giving its proof here.

**Theorem 3.1.** [6] The quotient space $C(SU_q(n)/SU_q(n - m))$ is the $C^*$-algebra generated by $\{u_k^l : n - m + 1 \leq l \leq n, 1 \leq k \leq n\}$.

**Proof:** Using the definition of quotient space, one can easily check that $u_k^l$’s are in $C(SU_q(n)/SU_q(n - m))$ for $n - m + 1 \leq l \leq n$ and $1 \leq k \leq n$. So,

$$C(SU_q(n)/SU_q(n - m)) \supseteq C^* \{u_k^l : n - m + 1 \leq l \leq n, 1 \leq k \leq n\}.$$ 

To show the equality, consider the co-multiplication action on $C(SU_q(n)/SU_q(n - m))$ by the compact quantum group $C(SU_q(n))$ given by

$$C(SU_q(n)/SU_q(n - m)) \rightarrow C(SU_q(n)/SU_q(n - m)) \otimes C(SU_q(n))$$

$$a \mapsto \Delta a.$$ 

By theorem 1.5, [8], we get

$$C(SU_q(n)/SU_q(n - m)) = \overline{\bigoplus_{\lambda \in SU(n)} \bigoplus_{i \in I_\lambda} W_{\lambda,i}}$$

where $\lambda$ represents a finite-dimensional irreducible co-representation $u^\lambda$ of $C(SU_q(n))$, $W_{\lambda,i}$ corresponds to $u^\lambda$ for all $i \in I_\lambda$ and $I_\lambda$ is the multiplicity of $u^\lambda$. Since all matrix entries of any finite-dimensional irreducible co-representation of $C(SU_q(n))$ lie in the algebra generated by $\{u_k^l : n - m + 1 \leq l \leq n, 1 \leq k \leq n\}$, it follows from Theorem 1, [6] that $\bigoplus_{\lambda \in SU(n)} \bigoplus_{i \in I_\lambda} W_{\lambda,i} \subseteq C^* \{u_k^l : n - m + 1 \leq l \leq n, 1 \leq k \leq n\}$. This proves the claim. □

We will write down all irreducible representations of $C(SU_q(n)/SU_q(n - m))$ using Theorem 2.2, [5]. For $i = 1, 2, \cdots, n - 1$, define the map $\pi_{s_i} : C(SU_q(n)) \rightarrow L(L_2(\mathbb{N}))$ by

$$\pi_{s_i}(u_k^l) = \begin{cases} 
\sqrt{1 - q^{2n+2}}S & \text{if } (k, l) = (i, i), \\
S^* \sqrt{1 - q^{2n+2}} & \text{if } (k, l) = (i + 1, i + 1), \\
-q^{N+1} & \text{if } (k, l) = (i, i + 1), \\
q^N & \text{if } (k, l) = (i + 1, i), \\
\delta_{kl} & \text{otherwise}.
\end{cases}$$
Each $\pi_{s_i}$ is an irreducible representation $C(SU_q(n))$. For any two representations $\varphi$ and $\psi$ of $C(SU_q(n))$, define $\varphi \ast \psi := (\varphi \otimes \psi) \circ \Delta$. Let $W$ be the Weyl group of the Lie algebra $su_n$ and $\vartheta \in W$ such that $s_{i_1}s_{i_2}\ldots s_{i_k}$ is a reduced expression for $\vartheta$. Then $\pi_\vartheta = \pi_{s_{i_1}} \ast \pi_{s_{i_2}} \ast \cdots \ast \pi_{s_{i_k}}$ is an irreducible representation which is independent of the reduced expression. Now for $t = (t_1, t_2, \cdots, t_{n-1}) \in \mathbb{T}^{n-1}$, define the map $\tau_t : C(SU_q(n)) \to \mathbb{C}$ by

$$\tau_t(u^k) = \begin{cases} t_{n-k+1} \delta_{kl} & \text{if } k > 1, \\ t_1t_2\cdots t_{n-1} \delta_{kl} & \text{if } k = 1, \end{cases}$$

Then $\tau_t$ is a $C^*$-algebra homomorphism. For $t \in \mathbb{T}^{n-1}, \vartheta \in W$, let $\pi_{(t,\vartheta)} = \tau_t \ast \pi_\vartheta$. Define $\eta_{(t,\vartheta)}$ to be the map $\pi_{(t,\vartheta)}$ restricted to $C(SU_q(n)/SU_q(n-m))$. Let $a = (a_1, a_2, \cdots, a_m)$ be an $m$-tuple of positive integers such that $1 \leq a_j \leq n-j+1$ for all $j \in \{1, 2, \cdots, m\}$. Define the word $w(a)$ of the Weyl group $W$ as follows:

$$w(a) = s_{n-m}s_{n-m-1}\cdots s_{a_m}s_{n-m+1}s_{n-m}\cdots s_{a_m-1}\cdots s_{n-1}s_{n-2}\cdots s_{a_1}$$

with the convention that for any $1 \leq j \leq m$, the string $s_{n-j}s_{n-j-1}\cdots s_{a_j}$ is empty if $a_j = n-j+1$. Clearly $w(a)$ is in a reduced form. For $t \in \mathbb{T}^m$, define $[t]_n$ to be $(t_1, \cdots, 1) \in \mathbb{T}^{n-1}$. The following theorem describes all irreducible representations of the $C^*$-algebra underlying the quotient space $SU_q(n)/SU_q(n-m)$. For the proof, we refer the reader to [5].

**Theorem 3.2.** Let $A_m = \{(a_1, \cdots, a_m) : 1 \leq a_j \leq n-j+1 \text{ for all } j \in \{1, 2, \cdots, m\}\}$. Then the set $\{\eta([t]_n, w(a)) : t \in \mathbb{T}^m, a \in A_m\}$ is the set of all irreducible representations of the $C^*$-algebra $C(SU_q(n)/SU_q(n-m))$.

Now we will associate some diagrams with the above representations which will be useful for our purpose. We will use the scheme followed by Chakraborty & Pal [2] with a few additions. For convenience, we use labeled lines to represent operators as given in the following table.

| Arrow type | Operator | Arrow type | Operator |
|------------|----------|------------|----------|
| ___        | $I$      | ___        | $M_t$    |
| __+        | $S^*\sqrt{I-q^{2N+2}}$ | ___        | $\sqrt{I-q^{2N+2}}S$ |
| \_\_\_\_\_ | $q^N$    | \_\_\_\_\_ | $-q^{N+1}$ |

Note that for $t \in \mathbb{T}$, $M_t$ represents the multiplication operator on $\mathbb{C}$ sending 1 to $t$. For other operators, the Hilbert spaces on which they act are given at the top of the diagram.
Let us describe how to use a diagram to represent the irreducible representations \( \pi_{s_i} \) and \( \tau_t \) where \( 1 \leq i \leq n - 1 \) and \( t \in \mathbb{T}^{n-1} \).

\[
L_2(\mathbb{N}) 
\]

\[ \mathbb{C} \]

\[
\begin{array}{ccc}
  n & n & n \\
  n-1 & t_1 & n-1 \\
  n-2 & t_2 & n-2 \\
  n-m+1 & t_m & n-m+1 \\
  n-m & n-m \\
  1 & t_1 \cdots t_m & 1 \\
\end{array}
\]

Diagram 1: \( \pi_{s_i} \)

Diagram 2: \( \tau_{[t]_n} \)

In these two diagrams, each path from a node \( k \) on the left to a node \( l \) on the right stands for an operator given as in the table acting on the Hilbert space given at the top of the diagrams. Now \( \pi_{s_i}(u^k_k) \) and \( \tau_{[t]_n}(u^k_k) \) are the operators represented by the path from \( k \) to \( l \) in diagram 1 and diagram 2 respectively and are zero if there is no such path. Thus, for example, \( \pi_{s_i}(u^1_1) \) is \( I \); \( \pi_{s_i}(u^2_1) \) is zero whereas \( \pi_{s_i}(u^i_{i+1}) = -q^{N+1} \) if \( i > 1 \). Similarly \( \tau_{[t]_n}(u^n_n) = M_{t_1} \) and \( \tau_{[t]_n}(u^{n-1}_n) = 0 \).

Next, let us explain how to represent \( \pi \ast \rho \) by a diagram where \( \pi \) and \( \rho \) are two representations of \( C(SU_q(n)) \) acting on the Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) respectively. Simply keep the two diagrams representing \( \pi \) and \( \rho \) adjacent to each other. Identify, for each row, the node on the right side of the diagram for \( \pi \) with the corresponding node on the left in the diagram for \( \rho \). Now, \( (\pi \ast \rho)(u^k_k) \) would be an operator on the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) determined by all the paths from the node \( k \) on the left to the node \( l \) on the right. It would be zero if there is no such path and if there are more than one paths, then it would be the sum of the operators given by each such path. In this way, we can draw diagrams for each irreducible representation of \( C(SU_q(n)) \) and \( C(SU_q(n)/SU_q(n-m)) \).

The following diagram is for the representation \( \eta_{([t]_n, w(a))} \) of \( C(SU_q(6)/SU_q(4)) \) where \( a = (3, 2) \) and \( t = (t_1, t_2) \).
Let $a = (a_1, a_2, \cdots, a_m) \in \mathcal{A}_m$. For $1 \leq i \leq m$, we denote by $w(a)_i$ the reduced word $s_{n-i}s_{n-i-1}\cdots s_{a_i}$. Therefore, one can write the reduced Weyl word $w(a)$ and hence the corresponding irreducible representation $\pi_{w(a)}$ of $C(SU_q(n))$ as follows.

$$w(a) = w(a)_m w(a)_{m-1} \cdots w(a)_1, \quad \pi_{w(a)} = \pi_{w(a)_m} \ast \pi_{w(a)_{m-1}} \ast \cdots \ast \pi_{w(a)_1}.$$ 

The following diagram is for the representation $\pi_{w(a)_i} := \pi_{s_{n-i}} \ast \pi_{s_{n-i-1}} \ast \cdots \ast \pi_{s_{a_i}}$ of $C(SU_q(n))$. 

![Diagram](image-url)
4 Quantum Stiefel manifolds

In this section, our aim is to establish $C(SU_q(n)/SU_q(n-m))$ as a universal $C^*$-algebra given by finite sets of generators and relations. We start with a definition.

**Definition 4.1.** We define $C^*$-algebra $C^{n,m}$ as the universal $C^*$-algebra generated by the elements \{${w_i^j : n - m + 1 \leq i \leq n, 1 \leq j \leq n}$\} satisfying the following relations:

\[
\begin{align*}
    w_k^i w_k^j &= q w_k^i w_k^j & \text{for } i < j, \\
    w_k^i w_l^i &= q w_k^i w_l^i & \text{for } k < l, \\
    w_k^i w_l^i &= w_l^i w_k^i & \text{for } i < j, k > l.
\end{align*}
\]

Diagram 4: $\pi(w(a)_i)$
\[ w^i_j w^j_i = w^i_j w^j_i + (q^{-1} - q) w^i_j w^j_i \quad \text{for } i > j, k > l, (4.4) \]
\[ (w^i_k)^* w^j_l = w^i_k (w^j_l)^* \quad \text{for } i \neq j, k \neq l, (4.5) \]
\[ (w^i_k)^* w^j_l + (1 - q^2) \sum_{j > i} (w^j_k)^* w^j_l = q(w^i_k)^* w^j_l \quad \text{for } k \neq l, (4.6) \]
\[ w^i_k (w^j_k)^* + (1 - q^2) \sum_{l < k} w^i_k (w^j_k)^* = q(w^i_k)^* w^j_k \quad \text{for } i \neq j, (4.7) \]
\[ (w^i_k)^* w^j_k + (1 - q^2) \sum_{j > i} (w^j_k)^* w^j_k = w^i_k (w^j_k)^* + (1 - q^2) \sum_{l < k} w^i_j (w^j_l)^* \quad (4.8) \]
\[ \sum_{k=1}^n w^i_k (w^j_k)^* = \delta_{ij}. \quad (4.9) \]

**Remark 4.2.** These relations will be called commutation relations. It can be easily obtained from the relations given in [6] or using \( R \)-matrix for type \( A_{n-1} \) (see [4]). Hence the generators \( \{ u^i_j : n - m + 1 \leq i \leq n, 1 \leq j \leq n \} \) of \( C(SU_q(n)/SU_q(n-m)) \) satisfy these relations. Using this fact and relation (4.9), one can prove that the \( C^* \)-algebra \( C^m_{n,m} \) is well defined.

Since the generators \( \{ u^i_j : n - m + 1 \leq i \leq n, 1 \leq j \leq n \} \) of \( C(SU_q(n)/SU_q(n-m)) \) satisfy the same relations satisfied by the generators \( \{ w^i_j : n - m + 1 \leq i \leq n, 1 \leq j \leq n \} \) of \( C^m_{n,m} \), it follows from the universal property of \( C^m_{n,m} \) that there exists a unique surjective homomorphism

\[ \Psi_m : C^m_{n,m} \to C(SU_q(n)/SU_q(n-m)) \]

sending \( w^j_j \) to \( u^j_j \) for all \( 1 \leq j \leq n \) and \( n - m + 1 \leq i \leq n \). We will show that for all \( 1 \leq m \leq n \), the map \( \Psi_m \) is an isomorphism. For that, we need to analyse an irreducible representation of the \( C(SU_q(n)/SU_q(n-m)) \) more closely.

Let \( \pi \) be an irreducible representation of \( C(SU_q(n)/SU_q(n-m)) \) on the Hilbert space \( \mathcal{H} \). Then for some \( t = (t_1, \ldots, t_m) \in \mathbb{T}^m \) and \( a = (a_1, \ldots, a_m) \in \mathcal{A}_m \), the representation \( \pi \) is equivalent to the representation \( \eta([t]_n, w(a)) \). Set

\[ \mathcal{H}^1_0 = \mathcal{H}, \quad F_1 = \{1, 2, \ldots, n\}, \quad \ell_1 = n - a_1 + 1, \quad c_1 = a_1. \]

Let \( T^1_k = \pi(u^i_{n-k+1}) \) for \( 1 \leq k \leq n \) acting on the Hilbert space \( \mathcal{H}^1_0 \). Using commutation relations, one can check that the operators \( \{ T^1_1, \ldots, T^1_n \} \) acting on \( \mathcal{H}^1_0 \) satisfy ODQS\( (q) \) relations with rank \( \ell_1 \). Suppose that \( \mathcal{H}^i_0, F_j, \ell_j, c_j \) and the operators \( \{ T^j_k : \text{for } 1 \leq k \leq n - j + 1 \} \) on \( \mathcal{H}^i_0 \) is defined for all \( j = 1, 2, \ldots, i - 1 \). Further assume that for \( j = 1, 2, \ldots, i - 1 \), the ordered tuple \( \{ T^j_1, \ldots, T^j_{n-j+1} \} \) satisfy ODQS\( (q) \) relations with rank
\( \ell_j \). Define 

\[
\mathcal{H}_0^i = \text{eigenspace of } T_{\ell_i-1}^{i-1} \text{ corresponding to eigenvalue 1},
\]

\( \ell_i = n + 2 - i - a_i \), 

\( F_i = \{1, 2, \cdots, n\} - \{c_1, c_2, \cdots, c_{i-1}\} \), 

\( g_i(f) = \#\{j : c_j < f\} \) for \( f \in F_i \), 

\( c_i = f_i^1 \), 

\[
T_k^i = \frac{(-1)^{g_i(f_k^i)}}{q^{g_i(f_k^i)}} \pi(u_n^{i-2}a_i)_{\mathcal{H}_0^i} \quad \text{for } 1 \leq k \leq n - i + 1,
\]

where \( f_1^i > f_2^i > \cdots > f_{n-i+1}^i \) are all elements of \( F_i \). In this way, we define the operators \( T_1^i, T_2^i, \cdots, T_{n-i+1}^i \) for all \( i \in \{1, 2, \cdots, m\} \). One thing we need to show that at each stage, the tuple \( (T_1^i, T_2^i, \cdots, T_{n-i+1}^i) \) satisfies ODQS(q) relations with rank \( \ell_i \). We will show this by identifying these operators using the diagram associated to the irreducible representation \( \eta([l], \omega(a)) \). First note the following facts from diagram 4.

1. The eigenspace of the operator \( \pi(w(a)_i)(u_n^{i-2}a_i)^{*} \) corresponding to eigenvalue 1 is one dimensional subspace \( K_0^i \) spanned by the vector \( e_0 \otimes e_0 \otimes \cdots \otimes e_0 \).

2. Let \( l < n - i + 1 \). Then on \( K_0^i \), starting from \( l \), one can go to either \( l + 1 \) or \( l \) depending on whether \( l \geq a_i \) or \( l < a_i \). Moreover, in this case we have 

\[
\pi(w(a)_i)(u_1^{l+1})_{K_0^i} = qI \quad \text{if } l \geq a_i
\]

\[
\pi(w(a)_i)(u_1^l)_{K_0^i} = I \quad \text{if } l < a_i
\]

Using these observations and the diagram associated with the representation \( \eta([l], \omega(a)) \), one can show that on the Hilbert space \( \mathcal{H}_0^i \), \( T_{\ell_i}^i = t_{n-i+1}^i I \) in case \( \ell_i = 1 \) and in case \( \ell_i > 1 \), we have 

\[
T_i^j = \begin{cases} 
t_{n-i+1}^i \otimes 1 \otimes \cdots \otimes 1 \otimes \sqrt{1 - q^{2N}S^*} \otimes 1 \otimes 1 \otimes \cdots \otimes 1 & \text{if } l = 1; \\
\Sigma_{j=1}^{m} \otimes \text{copies of } T_{\ell_j-1}^{j-1} & \\
\sum_{j=1}^{m} \otimes \text{copies of } T_{\ell_j-1}^{j-1} & \if 1 \leq l \leq \ell_i; \\
\sum_{j=1}^{m} \otimes \text{copies of } T_{\ell_j-1}^{j-1} & \if l = \ell_i; \\
0 & \if l > \ell_i;
\end{cases}
\]

Hence for \( 1 \leq i \leq m \), the tuple \( (T_1^i, T_2^i, \cdots, T_{n-i+1}^i) \) satisfies ODQS(q) relations with rank \( \ell_i \). Moreover,
1. The eigenspace $H_0^{m+1}$ of the operator $T_m(T_m)^*$ corresponding to eigenvalue 1 is one dimensional. It follows from part (10) of the proposition \[2.1\] that the tuple $(T_1^m, T_2^m, \ldots, T_{n+1-m}^m)$ is irreducible.

2. For any nonzero vector $h \in H_0^{m+1}$, $T_{\ell_j}^i h = t_i h$ for $1 \leq i \leq m$.

3. Let $1 \leq j \leq m$. Then on the Hilbert space $H_0^i$,

$$\ker(T_{\ell_j}^i(T_{\ell_j}^i)^*) = \ker(T_{\ell_j}^i) = \{0\} \quad (4.12)$$

4. Let $k < j \leq m$. Then on the Hilbert space $H_0^j$, one has

$$\pi(u_{c_k}^{n-j+1}) = 0 \quad (4.13)$$

5. Let $j \leq m$ and $k < c_j$. Then on the Hilbert space $H_0^j$, one has

$$\pi(u_{c_k}^{n-j+1}) = 0 \quad (4.14)$$

**Remark 4.3.** For any $1 \leq i \leq m$, the function $g_i$ and the set $F_i$ depends on the set \{a_1, a_2, \ldots, a_{i-1}\} only. Therefore, the expression of the operators $T_1^i, \ldots, T_{n-i+1}^i$ depends on \{a_1, a_2, \ldots, a_{i-1}\} only and hence we can define the operators $T_1^i, \ldots, T_{n-i+1}^i$ for the representation $\pi$ of $C(SU_q(n)/SU_q(n-m))$ if $\pi$ can be decomposed into those irreducible representations $\eta_{i(q_{a})}$'s which have same value of \{a_1, a_2, \ldots, a_{i-1}\}. In addition to this, one can define $H_0^i$'s and $c_j$'s for $j < i$ in the same manner.

To show that $C(SU_q(n)/SU_q(n-m))$ is isomorphic to the $C^*$-algebra $C^{m,m}$, we use induction on $m$. Fix $n$. We first take up the case of $m = 1$. Although the following proposition has been already proved (see \[12\]), we prove it for the sake of completeness.

**Proposition 4.4.** The map $\Psi_1 : C^{m,1} \rightarrow C(SU_q(n)/SU_q(n-1))$ sending $w_j^n$ to $u_j^n$ for all $1 \leq j \leq n$ is an isomorphism.

**Proof:** It is enough to show that the map $\Psi_1$ is injective. Take an irreducible representation $\pi$ of $C^{m,1}$. Using commutation relations, one can easily see that the tuple $(\pi(w_1^n), \pi(w_2^n), \ldots, \pi(w_n^n))$ satisfy ODQS(q) relations. Let $l_0$ and $t_0$ be the rank and the angle respectively of this ordered tuple. Define $a = n - l_0 + 1$. Consider the ordered tuple $(\eta_{i_0,w(a)}(u_1^n), \eta_{i_0,w(a)}(u_2^n), \ldots, \eta_{i_0,w(a)}(u_n^n))$. It is easy to check that rank and angle of this ordered tuple is $l_0$ and $t_0$ respectively. Hence by proposition \[2.3\] these ordered tuples are isomorphic which further implies that the representations $\pi$ and $\eta_{i_0,w(a)} \circ \Psi_1$ of $C^{m,1}$ are equivalent. This shows that any irreducible representation of $C^{m,1}$ factors through the map $\Psi_1$. Hence image of an element in $\ker(\Psi_1)$ under any irreducible representation
of $C^{m,1}$ is zero which implies that that element is zero. This proves the claim.

Suppose that for $m = 1, 2, \ldots k - 1$, the map $\Psi_m$ is an isomorphism between $C^{m,m}$ and $C(SU_q(n)/SU_q(n-m))$. Let $\pi$ be an irreducible representation of $C^{m,k}$ acting on a Hilbert space $\mathcal{H}$. Let $\eta$ be the representation $\pi$ restricted to the $C^*$-algebra generated by $\{w^i_j : n - k + 2 \leq i \leq n, 1 \leq j \leq n\}$ that is isomorphic to $C^{m,k-1}$. By induction hypothesis and by Theorem 3.2 we have

$$\eta \equiv \bigoplus_{(t,a)\in \mathbb{T}^{k-1} \times \mathcal{A}_{k-1}} \bigoplus_{j\in J_{(t,a)}} \eta_{[t]_n,w(a)}, \quad (4.15)$$

Here $J_{(t,a)}$ represents an index set of the multiplicity of the irreducible representation $\eta_{[t]_n,w(a)}$ in the representation $\eta$ and for each $j \in J_{(t,a)}$, $\mathcal{H}^j_{(t,a)}$ denotes the Hilbert space for the irreducible representation $\eta_{[t]_n,w(a)}$. We will show that $J_{(t,a)}$ is empty set for all but one value of $a$. More precisely,

**Lemma 4.5.** Suppose that the map $\Psi_m : C^{m,m} \to C(SU_q(n)/SU_q(n-m))$ sending $w^i_j$ to $u^i_j$ is an isomorphism for all $m = 1, 2, \ldots k - 1$. Then for any irreducible representation $\pi$ of $C^{m,k}$, the index set $J_{(t,a)}$ defined as above is empty set for all but one value of $a \in \mathcal{A}_{k-1}$.

**Proof:** Define the operators $T^1_i := \pi(w^i_1)$ for $1 \leq i \leq n$. From commutation relations, it follows that the ordered tuple $(T^1_1, T^1_2, \ldots, T^1_n)$ satisfies ODQS(q) relations. Let its rank be $\ell_1$ and let $a^0_1 := n - \ell_1 + 1$. This implies that for $a \in \mathcal{A}_{k-1}$ such that $a_1 < a^0_1$, $J_{(t,a)}$ is empty set. Also from the equation (4.15), we have

$$\ker(T^1_{\ell_1}) = \bigoplus_{\{(t,a)\in \mathbb{T}^{k-1} \times \mathcal{A}_{k-1} : a_1 > a^0_1\}} \bigoplus_{j\in J_{(t,a)}} \mathcal{H}^j_{(t,a)}.$$ 

Clearly $\ker(T^1_{\ell_1})$ is invariant under the action of $\{\pi(w^i_1), \pi((w^i_1)^*) : 1 \leq j \leq n, n + 2 - k \leq i \leq n\}$. Using commutations relations, one can show that $\ker(T^1_{\ell_1})$ is an invariant subspace of $\pi$. Since $T^1_{\ell_1}$ is nonzero operator, we have $\ker(T^1_{\ell_1}) = \{0\}$. This implies that if $a_1 > a^0_1$ then $J_{(t,a)}$ is empty set.

Assume that for $a \in \mathcal{A}_{k-1}$ such that $(a_1, a_2, \ldots, a_{i-1}) \neq (a^0_1, a^0_2, \ldots, a^0_{i-1})$, $J_{(t,a)}$ is empty set. Define the operators $T^1_1, T^2_1, \ldots T^1_{n-i+1}$ by equation (4.15) acting on the the Hilbert space $\mathcal{H}^0_{(t,a)}$ (see remark 4.3). Clearly the tuple $(T^1_1, T^2_1, \ldots T^1_{n-i+1})$ obeys ODQS(q) relations as it is a direct sum of some tuples of operators satisfying ODQS(q) relations. Let its rank be $\ell_i$ and $a^0_i := n + 2 - i - \ell_i$. This implies that for $\alpha \in \mathcal{A}_{k-1}$ such that $a_i < a^0_i$, $J_{(t,a)}$ is empty set. Let $H_i := \ker(T^1_{\ell_i})$. Then using equations (4.15), (4.12) and part (9) of the proposition 2.1 we get

$$\bigoplus_{\{(t,a)\in \mathbb{T}^{k-1} \times \mathcal{A}_{k-1} : a_1 > a^0_1\}} \bigoplus_{j\in J_{(t,a)}} \mathcal{H}^j_{(t,a)} = \bigoplus_{\alpha \in \mathcal{A}_{k-1} \times \mathcal{A}_{k-1}} (T^1_1)^{a_1_{\ell_1-1}}(T^2_1)^{a_2_{\ell_1-1}}(T^1_{\ell_1-1})^{a_{\ell_1-1}} \cdots (T^1_{i-2})^{a_{i-1}}(T^1_{i-1})^{a_i} \cdots (T^1_{n-i})^{a_{n-i}} H_i.$$
Let \( H_{>a} := \oplus \{ (t,a) \in \mathbb{T}^{k-1} \times A_{k-1} : a > a \} \oplus \mathcal{H}^j_{(t,a)} \). We will show that \( H_{>a} \) is an invariant subspace of \( \pi \). From the equation (4.17), it follows that \( H_{>a} \) is invariant under the action of \( \{ \pi(w_j^n), \pi(w_j^n)^* : 1 \leq j \leq n, n+2-k \leq l \leq n \} \). Using commutation relations and equation (4.14), one can show that \( H_i \) is invariant under \( \{ \pi(w_j^n-k+1), \pi(w_j^n-k+1)^* : 1 \leq j \leq n \} \). By applying relations (4.1), (4.3), (4.4), (4.5), and (4.7), it follows that \( H_{>a} \) is an invariant \( \pi(w_n^{n-k+1}) \) and \( \pi(w_n^{n-k+1})^* \). Assume that \( H_{>a} \) is invariant under the action of \( \{ \pi(w_n^{n-k+1}), \pi(w_n^{n-k+1})^* : r+1 \leq j \leq n \} \). Now using relations (4.1), (4.3), (4.4), (4.5), and (4.7), it follows that \( H_{>a} \) is invariant under the action of \( \pi(w_n^{n-k+1}) \) and \( \pi(w_n^{n-k+1})^* \). Hence by backward induction, one can show that \( H_{>a} \) is an invariant subspace of \( \pi \). Since \( T_i \) is nonzero operator, \( H_{>a} \neq \mathcal{H} \) and hence \( H_{>a} = \{ 0 \} \). This proves the claim.

\[ \square \]

**Remark 4.6.** It follows from the above proof that \( \ker(T_{i}^j(T_{i}^j)^*) = \ker(T_{i}^j) = \{ 0 \} \) for all \( 1 \leq j \leq k-1 \).

Define \( a_0 := (a_0, a_2, \ldots, a_{k-1}) \). For \( 1 \leq j \leq k-1 \), let \( c_j \) be as defined in the equation (4.10) for the representation \( \eta([\eta], w(a)) \) of \( C(SU_q(n)/SU_q(n-k+1)) \) for any \( t \in \mathbb{T}^{k-1} \).

Let \( \mathcal{H}_0^k \) be the eigenspace of \( T_{k-1}^k(T_{k-1}^k)^* \) corresponding to the eigenvalue 1. Define

\[
F_k = \{ 1, 2, \ldots, n \} - \{ c_1, c_2, \ldots, c_{k-1} \}, \quad g_k(f) = \# \{ j : c_j < f \} \quad \text{for } f \in F_k,
\]

\[
T_i = (\frac{(-1)^{g_k(f)}}{q^{g_k(f)}})^{\pi(f_i^k)}|_{h_0^k} \quad \text{for } 1 \leq i \leq n - k + 1.
\]

**Proposition 4.7.** Let the map \( \Psi_m : C^{n,m} \to C(SU_q(n)/SU_q(n-m)) \) sending \( w_i \) to \( w_i \) be an isomorphism for all \( m = 1, 2, \ldots, k-1 \). Then the operators \( T_1^k, T_2^k, \ldots, T_{n-k+1}^k \) defined above are well defined. Moreover, the tuple \( (T_1^k, T_2^k, \ldots, T_{n-k+1}^k) \) satisfies ODQS(q) relations and is irreducible.

**Proof:** It follows from Lemma 4.5 that for each \( 1 \leq i < k \), the tuple \( (T_{i}^j, T_{i}^j, \ldots, T_{n-i+1}^j) \) satisfies ODQS(q) relations and of rank \( \ell_i := n+2-i-a_i^0 \). Therefore, to show \( T_1^k, T_2^k, \ldots, T_{n-k+1}^k \) are well defined operators on \( \mathcal{H}_0^k \), we need to show that for \( 1 \leq i \leq n-k+1 \) and \( h \in \mathcal{H}_0^k \), \( T_i^k h \in \mathcal{H}_0^k \) or equivalently \( T_{i}^j T_{i}^k h = T_{i}^k h \) for all \( j < k \). For all \( i \) such that \( f_i^k > f_{i}^{j} \), it follows directly from the relation (4.3). For \( i \) such that \( f_i^k < f_{i}^{j} \), we have

\[
\pi(w_i^{n-j+1})\pi(w_i^{n-k+1}) h = \pi(w_i^{n-k+1})\pi(w_i^{n-j+1}) h + (q^{-1} - q)\pi(w_i^{n-k+1})\pi(w_i^{n-j+1}) h \quad \text{ (from relation (4.4))}
\]
= \pi (w_{n-k+1}^r) \pi (u_{n-k}^t) h \quad \text{(from equation } 4.14) \\
= \pi (u_{n-k+1}^t) h \quad \text{(since } \mathcal{H}_k \subset \mathcal{H}_{j+1})

This shows that \( T_{i,j}^k T_j^k h = T_j^k h \) for \( h \in \mathcal{H}_0^k \) and \( j < k \). Now we shall show that the tuple \( (T_1^k, T_2^k, \ldots, T_{n-k+1}^k) \) satisfies ODQS(\( q \)) relations. From relation \( 4.11 \), it follows that \( T_i^k T_j^k = qT_j^k T_i^k \) for \( i < j \). It follows from part (11) of the proposition \( 2.1 \) that for \( h \in \mathcal{H}_0^k \subset \mathcal{H}_{j+1}^k \) and \( i \neq l \), one has \( (T_{i,j}^k)^*T_j^k h = qT_l^k (T_{i,j}^k)^* h = 0 \). Using this fact and the relation \( 4.6 \), we get \( (T_i^k)^*T_i^k = qT_i^k (T_i^k)^* \) for \( i \neq l \). Observe that for \( h \in \mathcal{H}_0^k \), we have 

\[
\frac{1}{q} \pi (u_{c_i}^{n-k+1}) h = \frac{1}{q} \pi (u_{c_i}^{n-k+1}) T_i^k h \quad \text{(since } \mathcal{H}_0^k \subset \mathcal{H}_i^k) \\
= T_i^k \pi (u_{c_i}^{n-k+1}) h \quad \text{(from equation } 4.11 \text{ and } c_i = f_{i,j}^k) \\
= 0 \quad \text{(from part (6) of the proposition } 2.1)\]

Hence on \( \mathcal{H}_0^k \), we have \( \pi (u_{c_i}^{n-k+1}) = 0 \). Let us assume that \( c_{i_1} < c_{i_2} < \cdots < c_{i_{k-1}} \) where \( \{i_1, i_2, \ldots, i_{k-1}\} = \{1, 2, \ldots, k-1\} \). Then on the Hilbert space \( \mathcal{H}_0^k \), we have 

\[
\pi (w_{c_i}^{n-k+1}) \pi (w_{c_i}^{n-k+1})^* \\
= \pi (w_{c_1}^{n-k+1} \pi (w_{c_1}^{n-k+1}) + (1 - q^2) \sum_{j > n-k+1} \pi (w_{c_i}^j) \pi (w_{c_i}^j) \\
- (1 - q^2) \sum_{l < c_{i_1}} \pi (w_{c_1}^{n-k+1}) \pi (w_{c_1}^{n-k+1})^* \\
= (1 - q^2) \sum_{j \in \{n-i_1+1, \ldots, n-i_{k-1}+1\}} \pi (w_{c_i}^j) \pi (w_{c_i}^j) + (1 - q^2) \pi (w_{c_i}^{n-i_1+1}) \pi (w_{c_i}^{n-i_1+1}) \\
+ (1 - q^2) \sum_{j \in \{n-i_{k-1}+1, \ldots, n-i_{k-1}+1\}} \pi (w_{c_i}^j) \pi (w_{c_i}^j) - (1 - q^2) \sum_{l < c_{i_1}} \pi (w_{c_1}^{n-k+1}) \pi (w_{c_1}^{n-k+1})^* \\
= (1 - q^2)^2 (1 + q^2 + \cdots + q^{2k-4}) + (1 - q^2) q^{2k-2} - (1 - q^2) \sum_{l < c_{i_1}} \pi (w_{c_1}^{n-k+1}) \pi (w_{c_1}^{n-k+1})^* \\
\]

(by equations \( 4.11, 4.14 \) and part (11) of proposition \( 2.1 \))

\[
= (1 - q^2) - (1 - q^2) \sum_{l < c_{i_1}} \pi (w_{c_1}^{n-k+1}) \pi (w_{c_1}^{n-k+1})^* \quad (4.16)
\]

A little algebra along with the relations \( 4.8, 4.9 \) and equation \( 4.16 \) will show that 

\[
(T_i^k)^* T_i^k = T_i^k (T_i^k)^* + (1 - q^2) \sum_{j > i} T_j^k (T_j^k)^*, \quad \sum_{i=1}^{n-k+1} T_i^k (T_i^k)^* = 1
\]

Let \( \ell_k \) be the rank of \( (T_1^k, T_2^k, \ldots, T_{n-k+1}^k) \). Let \( \mathcal{H}_{k+1}^0 \) be the eigenspace of the operator \( T_{\ell_k}^k (T_{\ell_k}^k)^* \) corresponding to eigenvalue 1. To show that the tuple \( (T_1^k, T_2^k, \ldots, T_{n-k+1}^k) \) is
irreducible, take a nonzero vector $h_0 \in \mathcal{H}_k^{0+1}$. Define

$$K_{h_0} := \mathop{\oplus}_{i,j \in \mathbb{N}(T_i^1)\alpha_1^1(T_j^2)\alpha_2^2 \cdots (T_{t_i-1}^1)\alpha_{i-1}^1 \cdots (T_{t_k}^k)\alpha_{k}^k \cdots (T_{k_1}^k)\alpha_{k-1}^k}h_0$$

From the equation (4.15), it follows that the subspace $K_{h_0}$ is invariant under the action of $\{\pi(w_j^i), \pi(w_j^i)^* : 1 \leq j \leq n, n + 2 - k \leq l \leq n\}$. By applying commutation relations and the fact that $\pi(w_i^{n-k+1}) = 0$ on $\mathcal{H}_0^k$, one can prove that $K_{h_0}$ is invariant under the action of $\pi(w_i^{n-k+1})$. Using backward induction and commutation relations, it follows that $K_{h_0}$ is invariant under the action of $\{\pi(w_i^{n-k+1}) : i = 1, 2, \cdots, n\}$. In the same way, one can show that for any $h \in \mathcal{H}_0^{k+1}, K_h$, defined in the same way as $K_{h_0}$ is invariant under the action of $\{\pi(w_i^{n-k+1}) : i = 1, 2, \cdots, n\}$. This implies that $\{\pi(w_i^{n-k+1}) : i = 1, 2, \cdots, n\}$ keep $K_{h_0}$ as well as its complement invariant. As a consequence, the subspace $K_{h_0}$ is invariant under the action of $\{\pi(w_i^{n-k+1}), \pi(w_i^{n-k+1})^* : i = 1, 2, \cdots, n\}$ and hence invariant under $\pi$. Since $\pi$ is irreducible, it follows that $\mathcal{H}_0^{k+1}$ is one dimensional. By part (10) of the proposition 2.1, we get the claim. \[\square\]

In Lemma 4.5, we proved that any irreducible representation $\pi$ of $C_m^k$ when restricted to $C_{m,k-1}$ decomposes into those irreducible representations of $C(SU_q(n)/SU_q(n-k+1))$ that have same value of $a \in A_{k-1}$. But it does not rule out the possibility that these irreducible representations may have different values of $t \in \mathbb{T}^{k-1}$. In the following lemma, we will show that in such decomposition, only one irreducible representation of $C(SU_q(n)/SU_q(n-k+1))$ occurs with certain (infinite) multiplicity.

**Lemma 4.8.** Suppose that the map $\Psi_m : C_m^m \rightarrow C(SU_q(n)/SU_q(n-m))$ sending $w_j^i$ to $u_j^i$ is an isomorphism for all $m = 1, 2, \cdots k - 1$. Then for any irreducible representation $\pi$ of $C_m^k$, the index set $J_{(t,a)}$ defined as above is empty set for all but one value of $(t, a) \in \mathbb{T}^{k-1} \times A_{k-1}$.

**Proof:** For $1 \leq i \leq k$, let $T_1^i, T_2^i, \cdots, T_{n-k+1}^i$ be as above. Take a nonzero vector $h_0 \in \mathcal{H}_0^{k+1}$. By part (10) of proposition 2.1, one has

$$\mathcal{H}_0^k = \mathop{\oplus}_{\alpha_i \in \mathbb{N}}(T_1^k)_{\alpha_1}^1(T_2^k)_{\alpha_2}^2 \cdots (T_{t_i}^k)_{\alpha_{t_i}}^{t_i-1}h_0$$

Since for $i \leq k$, $\mathcal{H}_0^{k+1} \subset \mathcal{H}_0^{i+1}$, there exists $t_i \in \mathbb{T}$ such that $T_i^j h_0 = t_i h_0$ for all $1 \leq i \leq k$. By applying commutation relations (4.3), (4.4) and equation (4.14), we observe that $T_i^j T_j^k = T_j^k T_i^j$ on $\mathcal{H}_0^k$. Using this fact and equation (4.17), one gets $T_i^j h = t_i h$ for $h \in \mathcal{H}_0^k$. Hence it follows from equation (4.15) that if $t' \neq (t_1, t_2, \cdots, t_{k-1})$, $J(t', a)$ is empty. This along with Lemma 4.5 proves that for only one value of $(t, a) \in \mathbb{T}^{k-1} \times A_{k-1}$, $J(t, a)$ is nonempty. \[\square\]
We call $t := (t_1, t_2, \ldots, t_k)$ and $\ell := (\ell_1, \ell_2, \ldots, \ell_k)$ defined as in above lemma the angle and the rank respectively of the irreducible representation $\pi$ of $C^{m,k}$. The following lemma says that angle and rank completely determine an irreducible representation $\pi$ of $C^{m,k}$.

**Lemma 4.9.** Assume that the map $\Psi_m : C^{n,m} \to C(SU_q(n)/SU_q(n-m))$ sending $w_j^i$ to $u_j^i$ is an isomorphism for all $m = 1, 2, \ldots, k-1$. Let $\pi$ and $\pi'$ be irreducible representations of $C^{m,k}$ on a Hilbert space $\mathcal{H}$ and $\mathcal{H}'$ respectively with same rank and same angle. Then $\pi$ and $\pi'$ are equivalent.

**Proof:** Let $h$ and $h'$ be nonzero vectors in $\mathcal{H}_0^{k+1}$ and $(\mathcal{H}_0^{k+1})'$ respectively. Using remark 4.6 and part (9) and part (10) of proposition 2.1 we get orthonormal bases for $\mathcal{H}$ and $\mathcal{H}'$ given by

\[
\{ w_\alpha := \frac{(T_1^1)^{\alpha_1}(T_2^1)^{\alpha_2} \cdots (T_{\ell_1-1}^1)^{\alpha_{\ell_1-1}} (T_1^k)^{\alpha_1^k} \cdots (T_{\ell_k-1}^k)^{\alpha_{\ell_k-1}^k} h}{\|(T_1^1)^{\alpha_1}(T_2^1)^{\alpha_2} \cdots (T_{\ell_1-1}^1)^{\alpha_{\ell_1-1}} (T_1^k)^{\alpha_1^k} \cdots (T_{\ell_k-1}^k)^{\alpha_{\ell_k-1}^k} h\|} \alpha_1, \ldots, \alpha_{\ell_k} \in \mathbb{N} \}
\]

and

\[
\{ w'_\alpha := \frac{(T_1^1')^{\alpha_1'}(T_2^1')^{\alpha_2'} \cdots (T_{\ell_1-1}^1')^{\alpha_{\ell_1-1}'} (T_1^k')^{\alpha_1'^k} \cdots (T_{\ell_k-1}^k')^{\alpha_{\ell_k-1}'^k} h'}{\|(T_1^1')^{\alpha_1'}(T_2^1')^{\alpha_2'} \cdots (T_{\ell_1-1}^1')^{\alpha_{\ell_1-1}'} (T_1^k')^{\alpha_1'^k} \cdots (T_{\ell_k-1}^k')^{\alpha_{\ell_k-1}'^k} h'\|} \alpha_1', \ldots, \alpha_{\ell_k}' \in \mathbb{N} \}
\]

respectively. Define

\[ U : \mathcal{H} \longrightarrow \mathcal{H}' \]
\[ w_\alpha \mapsto w'_\alpha \]

In the light of equation (4.15), we only need to show that for $1 \leq j \leq n$,

\[ U\pi(w_j^{n-k+1})U^* = \pi'(w_k^{n-k+1}). \]

Using equation (4.13) and the fact that angle of $\pi$ and $\pi'$ are same, one can show that

\[ U\pi(w_j^{n-k+1})U^* h = \pi'(w_j^{n-k+1})h' \quad \text{for all} \ 1 \leq j \leq n. \]  

(4.18)

By applying commutation relations (4.1), (4.2), (4.3) and equation (4.18), we have $U\pi(w_n^{n-k+1})w_\alpha = \pi'(w_n^{n-k+1})w'_\alpha$ for all $\alpha$ and hence $U\pi(w_n^{n-k+1})U^* = \pi'(w_n^{n-k+1})$. Assume the claim for $j = n, n-1, \ldots, i+1$. Again by applying commutation relation (4.1), (4.2), (4.3) and equation (4.18), we get $U\pi(w_i^{n-k+1})w_\alpha = \pi'(w_i^{n-k+1})w'_\alpha$ for all $\alpha$ and hence $U\pi(w_i^{n-k+1})U^* = \pi'(w_i^{n-k+1})$. This completes the proof. \(\square\)

Now we prove the main result of this paper.
**Theorem 4.10.** For $1 \leq m \leq n$, the map $\Psi_m : C^{n,m} \rightarrow C(SU_q(n)/SU_q(n-m))$ sending $w_j^i$ to $u_j^i$ for all $1 \leq j \leq n$ and $n - m + 1 \leq i \leq n$ is an isomorphism.

**Proof:** We apply induction on $m$. For $m = 1$, the claim is true thanks to proposition 4.4. We assume that the result is true for $m = 1, 2, \ldots, k-1$. We will show that for $m = k$, the map $\Psi_k$ is an isomorphism. Take an irreducible representation $\pi$ of $C^{n,k}$. Let $\ell = (\ell_1, \ell_2, \ldots, \ell_k)$ and $t = (t_1, t_2, \ldots, t_k)$ be the rank and the angle respectively of $\pi$. Define $a := (n+1-\ell_1, n-\ell_2, \ldots, n+2-k-\ell_k)$. Consider the irreducible representation $\eta_{([\ell], [a], w(a))} \circ \Psi_k$ of the $C^*$-algebra $C^{n,k}$. It is easy to check that rank and angle of this irreducible representation is $\ell$ and $t$ respectively. Hence by Lemma 4.9 the representations $\pi$ and $\eta_{([\ell], [a], w(a))} \circ \Psi_k$ of $C^{n,k}$ are equivalent. This shows that any irreducible representation of $C^{n,k}$ factors through the map $\Psi_k$. Hence image of an element in $\ker(\Psi_k)$ under any irreducible representation of $C^{n,k}$ is zero which implies that the element is zero. Therefore, the map $\Psi_k$ is injective. □

This establishes $C(SU_q(n)/SU_q(n-m))$ as the universal $C^*$-algebra given by finite sets of generators and relations.

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