A NOTE ON BISHOP-PHELPS-BOLLOBÁS PROPERTY FOR OPERATORS

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Abstract. The main purpose of this paper is to study Bishop-Phelps-Bollobás property for operators on $c_0$-sums of euclidean spaces. We show that the pair $(c_0 (\bigoplus_{k=1}^{\infty} \ell^2_k)), Y)$ has the Bishop-Phelps-Bollobás property for operators (in short BPBp for operators) whenever $Y$ is a uniformly convex Banach space.

1. Introduction

In 1961, Bishop and Phelps [4], proved that, for any Banach space, the subset of norm attaining functionals is dense in the topological dual space. This result is known as Bishop-Phelps theorem. These authors posed the problem of possible extensions of such result to operators. A lot of attention has been devoted to extending Bishop-Phelps theorem to operators and interesting results have been obtained about this topic. In 1970, Bollobás [5], proved a refinement of the Bishop-Phelps theorem, which states that, every norm one functional and its almost norming points can be approximated by a norm attaining functional and its norm attaining point. In 2008, Acosta, Aron, Garcia and Maestre [1], defined a new notion for a pair of Banach spaces, which is called the Bishop-Phelps-Bollobás property for operators, and provided many notable results. Many important reference works in the field have appeared, among others, [1], [2], [6], [7], [8], [9], [11].

In [1] they showed, that $(\ell^p_{\infty}, Y)$ satisfies the Bishop-Phelps-Bollobás property for operators for every $n \in \mathbb{N}$, whenever $Y$ is a uniformly convex Banach space. They also left the open question if $(c_0, Y)$ satisfies the BPBp for operators, whenever $Y$ is a uniformly convex Banach space. In this sense, S. K. Kim [11], answered the question in a positive way. More generally, G. Choi and S. W. Kim proved, in [6], that $(c_0 (\bigoplus_{k=1}^{\infty} X), Y)$ has BPBp for operators, if $X$ is a uniformly convex Banach space.

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Banach space and $Y$ is $C$-uniformly convex Banach space. Every uniformly convex complex space is $C$-uniformly convex and the converse is not true.

The purpose of this work is to show that $(c_0(\bigoplus_{k=1}^{\infty} \ell^n_2), Y)$ satisfies the BPBp for operators whenever $Y$ is uniformly convex complex Banach space. Considering the real case, when $Y$ is strictly convex, we prove that if $(c_0(\bigoplus_{k=1}^{\infty} \ell^n_2), Y)$ satisfies BPBp for operators then $Y$ is uniformly convex. In this sense, this is an improvement of the result presented in \cite{6}, since the $c_0$-sum is performed with different spaces.

Notice that, the Banach space $c_0(\bigoplus_{n=1}^{\infty} \ell^n_2)$, known as $c_0$-sums of the Euclidean $n$-space, is not isometric to $c_0$. The importance of such space is due to the fact that the result presented by C. Stegall, in \cite{14}, where he showed that $\ell_\infty(\bigoplus_{n=1}^{\infty} \ell^n_2)$ does not have Dunford-Pettis property, but its predual, $\ell_1(\bigoplus_{n=1}^{\infty} \ell^n_2)$, has it.

Each $x \in c_0(\bigoplus_{n=1}^{\infty} \ell^n_2)$ can be represented by $x = \sum_{n=1}^{\infty} \sum_{k \in I(n)} x_k e_k$, where for every $n \in \mathbb{N}$, $I(n) = \{ l \in \mathbb{N} : s(n-1)+1 \leq l \leq s(n) \}$ with $s : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ the auxiliar function defined by $s(n) = 0$ if $n = 0$ and $s(n) = 1 + 2 + \ldots + n$ if $n \neq 0$, and $(e_k)$ is the standard basis of $c_0(\bigoplus_{n=1}^{\infty} \ell^n_2)$.

The norm of $x$ is given by the formula $\|x\| := \sup_{n \in \mathbb{N}} \left( \sum_{k \in I(n)} |x_k|^2 \right)^{1/2}$.

\section{Results}

It will be convenient to recall the following notation. Let $X$ and $Y$ be Banach spaces. We denote by $S_X$, $B_X$, $X^*$, and $\mathcal{L}(X,Y)$, the unit sphere, the closed unit ball, the topological dual space of $X$ and the space of all bounded linear operators from $X$ into $Y$, respectively. An operator $T \in \mathcal{L}(X,Y)$ is said to attain its norm at $x_0 \in S_X$, if $\|T\| = \|T(x_0)\|$. We denote by $NA(X,Y)$ the subset of $\mathcal{L}(X,Y)$ of all norm attaining operators between $X$ and $Y$. Now, we recall a few definitions.

\textbf{Definition 2.1.} Let $X$ and $Y$ be Banach spaces. We say that the pair $(X,Y)$ has the Bishop-Phelps-Bollobás property for operators (in shortly BPBp for operators) if given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that whenever $T \in S_{\mathcal{L}(X,Y)}$ and $x_0 \in S_X$ satisfy that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_X$ and an operator $S \in S_{\mathcal{L}(X,Y)}$ satisfying the following conditions

$$
\|Su_0\| = 1, \quad \|u_0 - x_0\| < \varepsilon, \quad \text{and} \quad \|S - T\| < \varepsilon.
$$

A Banach space $X$ is uniformly convex if for every $\varepsilon > 0$ there is a $0 < \delta < 1$ such that, if $\left\| \frac{x+y}{2} \right\| > 1 - \delta$ for every $x, y \in B_X$, then $\|x - y\| < \varepsilon$. In this case, the modulus of convexity is given by
δ(ε) = \inf \{ 1 - \|x + y\| : x, y \in B_X, \|x - y\| \geq \epsilon \}. A Banach space X is strictly convex if \(\|x + y\| < 1\) whenever \(x, y \in S_X\) and \(x \neq y\). We remark that uniformly convexity implies strictly convexity, but the converse is not true.

We observe that in the following results, we will use similar techniques found in [11, 6] and [11].

**Lemma 2.2.** Let \(Y\) be a strictly convex Banach space and let \(T \in L(c_0(\bigoplus_{k=1}^{\infty} \ell^2_k), Y)\). If \(\|T(x)\| = \|T\|\) for some \(x = \sum_{n=1}^{\infty} \sum_{k \in I(n)} x_ke_k \in S_{c_0(\bigoplus_{k=1}^{\infty} \ell^2_k)}\), so

\[ T(e_k) = 0 \quad \text{for all} \quad k \in \left\{ j \in \mathbb{N} : \left( \sum_{i \in I(j)} |x_i|^2 \right)^{1/2} < 1 \right\}. \]

**Proof.** Let \(x = \sum_{n=1}^{\infty} \sum_{k \in I(n)} x_ke_k\) be an element of \(S_{c_0(\bigoplus_{k=1}^{\infty} \ell^2_k)}\) such that \(\|T(x)\| = \|T\|\). The set \(\left\{ j \in \mathbb{N} : \left( \sum_{i \in I(j)} |x_i|^2 \right)^{1/2} < 1 \right\}\) is nonempty, since \(\|x\| = \sup_{n \in \mathbb{N}} \left( \sum_{i \in I(n)} |x_i|^2 \right)^{1/2} = 1\).

Now, we will show by a contradiction, and we assume that there exists \(k_0 \in \left\{ j \in \mathbb{N} : \left( \sum_{i \in I(j)} |x_i|^2 \right)^{1/2} < 1 \right\}\) such that \(T(e_{k_0}) \neq 0\). If we consider \(I(k_0) = \{k, \ldots, l\}\), where \(k, \ldots, l \in \mathbb{N}\), then the element \(v := (x_k \pm \left( 1 - \left( \sum_{i \in I(k_0)} |x_i|^2 \right)^{1/2} \right) , x_{k+1}, \ldots, x_l) \in \ell^2_{k_0}\) and \(\|v\| \leq 1\). It implies that

\[
\left\| x \pm \left( 1 - \left( \sum_{i \in I(k_0)} |x_i|^2 \right)^{1/2} \right) e_k \right\| = \sup_{n \in \mathbb{N}} \left\{ \left( \sum_{i \in I(n) \setminus \{k_0\}} |x_i|^2 \right)^{1/2}, \|v\|_2 \right\} \leq 1.
\]

By assumption \(\|T(x)\| = \|T\|\), we have naturally \(\|T(2x)\| = 2\|T\|\),

\[
2\|T\| \leq \left\| T \left( x + \left( 1 - \left( \sum_{i \in I(k_0)} |x_i|^2 \right)^{1/2} \right) e_k \right) \right\| + \left\| T \left( x - \left( 1 - \left( \sum_{i \in I(k_0)} |x_i|^2 \right)^{1/2} \right) e_k \right) \right\| \leq 2\|T\|.
\]
So, \( \left\| T \left( x \pm \left( 1 - \left( \sum_{i \in I(k_0)} |x_i|^2 \right)^{1/2} \right) e_k \right) \right\| = \| T \| \), and
\[
\frac{T \left( x \pm \left( 1 - \left( \sum_{i \in I(k_0)} |x_i|^2 \right)^{1/2} \right) e_k \right)}{\| T \|} \in S_Y. \quad \text{Finally,}
\]
\[
\frac{\left\| T \left( x \pm \left( 1 - \left( \sum_{i \in I(k_0)} |x_i|^2 \right)^{1/2} \right) e_k \right) \right\|}{\| T \|} + \frac{T \left( x - \left( \sum_{i \in I(k_0)} |x_i|^2 \right)^{1/2} e_k \right)}{\| T \|} = \frac{\left\| 2T(x) \right\|}{2\| T \|} = 1.
\]
Contradicting the fact that \( Y \) is a strictly convex Banach space, then \( T(e_{k_0}) = 0 \).

As a consequence of the Lemma 2.2 we get the following theorem.

**Theorem 2.3.** Let \( c_0 \left( \oplus_{k=1}^{\infty} \ell^k_2 \right) \) a real Banach space and \( Y \) a strictly convex real Banach space. If \( \left( c_0 \left( \oplus_{k=1}^{\infty} \ell^k_2 \right), Y \right) \) satisfies the BPBp, then \( Y \) is an uniformly convex Banach space.

**Proof.** Suppose that \( Y \) is not an uniformly convex Banach space. Then there exists \( \epsilon > 0 \) and sequences \( (y_k), (z_k) \subset S_Y \) such that
\[
(1) \quad \lim_{k \to \infty} \left\| \frac{y_k + z_k}{2} \right\| = 1 \quad \text{and} \quad \| y_k - z_k \| > \epsilon, \quad \forall k.
\]

For each positive integer \( i \in \mathbb{N} \), we define \( T_i : c_0 \left( \oplus_{k=1}^{\infty} \ell^k_2 \right) \to Y \) by
\[
T_i(x) = \left( \frac{x_1 + x_2}{2} \right) y_i + \left( \frac{x_1 - x_2}{2} \right) z_i, \quad x = (x_k) \in X.
\]

For each \( i \in \mathbb{N} \) and each \( x \in S_{c_0(\oplus_{k=1}^{\infty} \ell^k_2)} \) we have that
\[
\| T_i(x) \| \leq \frac{1}{2} \{ |x_1 + x_2| + |x_1 - x_2| \} \leq 1.
\]
As \( \| T_i(e_1 + e_2) \| = 1 \), it follows that \( \| T_i \| = 1 \), for each \( i \in \mathbb{N} \). We observe that, for each \( i \in \mathbb{N} \), \( \| T_i(e_1) \| = \left\| \frac{y_i + z_i}{2} \right\| \), by \( \| T_i(e_1) \| \) converges to 1 when \( i \to \infty \). Then there is \( i_0 \in \mathbb{N} \) such that \( \| T_{i_0}(e_1) \| > 1 - \eta \left( \frac{\epsilon}{2} \right) \).

As the pair \( \left( c_0 \left( \oplus_{k=1}^{\infty} \ell^k_2 \right), Y \right) \) has the BPBp, there are an operator \( R \in S'_{c_0(\oplus_{k=1}^{\infty} \ell^k_2), Y} \) and a point \( u \in S_{c_0(\oplus_{k=1}^{\infty} \ell^k_2)} \) such that
\[
(2) \quad \| R(u) \| = 1, \quad \| R - T_{i_0} \| < \frac{\epsilon}{2}, \quad \| u - e_1 \| < \beta \left( \frac{\epsilon}{2} \right) < 1.
\]
Then \( \left( \sum_{i \in I(k)} |u_i|^2 \right)^{1/2} < 1 \), for all \( k \in \mathbb{N} \setminus \{1\} \), and by Lemma 2.2
\[
R(e_k) = 0, \quad \text{for all} \quad k \in \mathbb{N} \setminus \{1\}.
\]
Therefore, it can be assumed that \( u = e_1 \), so \( R(e_1) = R(e_1 + e_2) = R(e_1 - e_2) \), it implies that
\[
\|y_0 - z_0\| = \|T_{i_0}(e_1 + e_2) - T_{i_0}(e_1 - e_2)\|
= \|T_{i_0}(e_1 + e_2) - R(e_1 + e_2) + R(e_1 - e_2) - T_{i_0}(e_1 - e_2)\|
\leq \|T_{i_0} - R\||e_1 + e_2| + \|R - T_{i_0}\||e_1 - e_2|
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
It is a contradiction, so \( Y \) is an uniformly convex Banach space.

We need the next Lemma to show the main result.

**Lemma 2.4.** Let \( F \subset \mathbb{N} \) and \( A = \bigcup_{i \in F} I(i) \). Suppose that \( 0 < \epsilon < 1 \) and \( Y \) an uniformly convex Banach space with modulus of convexity \( \delta(\epsilon) \). If \( T \in S_{\mathcal{L}(c_0(\oplus_{k=1}^\infty \ell_2^k)), Y} \) satisfies that \( \|TP_A\| > 1 - \delta(\epsilon) \), then \( \|T(I - P_A)\| \leq \epsilon \).

**Proof.** Let \( 0 < \epsilon < 1 \) and \( T \in S_{\mathcal{L}(c_0(\oplus_{k=1}^\infty \ell_2^k)), Y} \) an operator such that \( \|TP_A\| > 1 - \delta(\epsilon) \). Then there exists \( x \in S_{c_0(\oplus_{k=1}^\infty \ell_2^k)} \cap P_A(c_0(\oplus_{k=1}^\infty \ell_2^k)) \) such that \( \|TP_A(x)\| > 1 - \delta(\epsilon) \). Fix an element \( y = \sum_{n=1}^{\infty} \sum_{k \in I(n)} y_n e_k \in B_{c_0(\oplus_{k=1}^\infty \ell_2^k)} \) with \( \text{supp} \ y \subset \mathbb{N} \setminus A \), then
\[
\|x \pm y\| = \sup_{j \in \mathbb{N}} \left\{ \left( \sum_{i \in I(j), j \in F} |x_i|^2 \right)^{1/2}, \left( \sum_{i \in I(j), j \in \mathbb{N} \setminus F} |y_i|^2 \right)^{1/2} \right\}
\leq 1,
\]
it implies that, \( \|T(x \pm y)\| \leq 1 \), for every \( y \in B_{c_0(\oplus_{k=1}^\infty \ell_2^k)} \) with \( \text{supp} \ y \subset \mathbb{N} \setminus A \). Notice that, for every \( z \in B_{c_0(\oplus_{k=1}^\infty \ell_2^k)} \), the support of the vector \( (I - P_A)(z) \) is a subset of \( \mathbb{N} \setminus A \) and then, \( \|T(x) \pm T(I - P_A)(z)\| \leq 1 \).
So
\[
\frac{\|T(x + (I - P_A)(z)) + T(x - (I - P_A)(z))\|}{2} = \|TP_A(x)\|
> 1 - \delta(\epsilon).
\]
As \( Y \) is an uniformly convex Banach space, it implies that
\[
\|T(x + (I - P_A)(z)) - T(x - (I - P_A)(z))\| < \epsilon.
\]
Proposition 2.5. If $Y$ is an uniformly convex Banach space, then the pair $(\ell_\infty (\bigoplus_{k=1}^n \ell^2_k), Y)$ satisfies the Bishop-Phelps-Bollobás for operators.

Proof. The proof can be performed analogously to the Theorem 2.4 in [4] with some changes, for example we need to use the convexity modulus equal the maximal in the convexity modulus of each space involved in the finite sum.

Theorem 2.6. If $Y$ is an uniformly convex Banach space, then the pair $(c_0 (\bigoplus_{k=1}^n \ell^2_k), Y)$ satisfies the Bishop-Phelps-Bollobás for operators.

Proof. Let $0 < \epsilon < 1$. As $(\ell_\infty (\bigoplus_{k=1}^n \ell^2_k), Y)$ has the BPBp for operators, then that there exist $0 < \eta(\epsilon) < \epsilon$ and $\beta(\epsilon) > 0$, with $\lim_{\epsilon \to 0} \beta(\epsilon) = 0$, such that, for every $Q \in S_{\mathcal{L}(\ell_\infty (\bigoplus_{k=1}^n \ell^2_k)), Y}$ and $z \in S_{\ell_\infty (\bigoplus_{k=1}^n \ell^2_k)}$ which satisfies that $\|Q(z)\| > 1 - \eta(\epsilon)$ there exist $\hat{Q} \in S_{\mathcal{L}(\ell_\infty (\bigoplus_{k=1}^n \ell^2_k), Y)}$ and $z_0 \in S_{\ell_\infty (\bigoplus_{k=1}^n \ell^2_k)}$ such that

\begin{equation}
\|\hat{Q}(z_0)\| = 1, \quad \|z - z_0\| < \beta(\epsilon), \quad \|\hat{Q} - Q\| < \epsilon.
\end{equation}

Now, let $T \in S_{\mathcal{L}(c_0 (\bigoplus_{k=1}^n \ell^2_k), Y)}$ and $x = \sum_{n=1}^{\infty} \sum_{k \in I(n)} x_k e_k \in S_{c_0 (\bigoplus_{k=1}^n \ell^2_k)}$ such that

\begin{equation}
\|T(x)\| > 1 - \eta(\epsilon) + \gamma(\epsilon),
\end{equation}

and

\begin{equation}
\|T(x)\| > 1 - \delta(\epsilon) + \gamma(\epsilon),
\end{equation}

where $\delta(\epsilon) > 0$ is the modulus of convexity of $Y$, $\gamma(\epsilon) > 0$ and $\lim_{\epsilon \to 0} \gamma(\epsilon) = 0$. Since $c_{00}$ is a dense subspace of $c_0 (\bigoplus_{k=1}^n \ell^2_k)$, we can be choose a vector $u \in S_{c_0 (\bigoplus_{k=1}^n \ell^2_k)}$, with finite support, such that $\|x - u\| < \gamma(\epsilon)$. Then

\begin{equation}
\|T(u)\| = \|T(x - x + u)\| \geq \|T(x)\| - \|T(x - u)\| \geq 1 - \eta(\epsilon).
\end{equation}
Similarly, \( \|T(u)\| > 1 - \delta(\epsilon) \). Let \( n = \min\{k \in \mathbb{N} : \text{supp } u \subseteq \bigcup_{j=1}^{k} I(j) \} \), and \( A = \bigcup_{k=1}^{n} I(k) \). Then

\[
\|TP_{A}\| \geq \|TP_{A}(u)\| = \|T(u)\| > 1 - \delta(\epsilon),
\]

and

(5) \( \|TP_{A}\| > 1 - \eta(\epsilon) \).

By Lemma 2.4,

(6) \( \|T(I - P_{A})\| \leq \epsilon \).

Let \( J : \ell_{\infty} \left( \bigoplus_{k=1}^{\infty} \ell_{2}^{k} \right) \to c_{0} \left( \bigoplus_{k=1}^{\infty} \ell_{2}^{k} \right) \) be the map defined by

\[
J(w) = \begin{cases} 
    w_{i}, & \text{se } i \in A \\
    0, & \text{se } i \in \mathbb{N} \setminus A.
\end{cases}
\]

We observe that \( \|J(w)\| = \max_{1 \leq j \leq n} \left( \sum_{i \in I(j)} |w_{i}|^{2} \right)^{1/2} = \|w\| \), for every \( w \in \ell_{\infty} \left( \bigoplus_{k=1}^{\infty} \ell_{2}^{k} \right) \). Let \( Q : \ell_{\infty} \left( \bigoplus_{k=1}^{\infty} \ell_{2}^{k} \right) \to Y \) be the bounded linear operator defined by \( Q(w) = \frac{TP_{A}J}{\|TP_{A}J\|}(w) \) and the element \( z = (z_{i}) \in \ell_{\infty} \left( \bigoplus_{k=1}^{\infty} \ell_{2}^{k} \right) \) given by \( z_{i} = u_{i} \), if \( i \in A \) and \( z_{i} = 0 \) if \( i \in \mathbb{N} \setminus A \). It is easy to check that \( \|Q\| = \|z\| = 1 \), and as \( \|TP_{A}J\| \leq 1 \), we get

(7) \( \|Q(z)\| = \left\| \frac{TP_{A}J}{\|TP_{A}J\|}(z) \right\| \geq \|TP_{A}(u)\| = \|T(u)\| \),

and by (4) we have that \( \|Q(z)\| > 1 - \eta(\epsilon) \). By Proposition 2.5 the pair \( (\ell_{\infty} \left( \bigoplus_{k=1}^{\infty} \ell_{2}^{k} \right), Y) \) has the BPBp for operators, so there exist \( \tilde{R} \in S_{\ell_{\infty}(\bigoplus_{k=1}^{\infty} \ell_{2}^{k}), Y} \) and \( \tilde{u} \in S_{\ell_{\infty}(\bigoplus_{k=1}^{\infty} \ell_{2}^{k})} \), such that

(8) \( \|\tilde{R}(\tilde{u})\| = 1, \quad \left\| \tilde{R} - Q \right\| < \epsilon, \quad \|z - \tilde{u}\| < \beta(\epsilon) \).

Now, we consider \((e_{k}), (f_{k})\) be the canonical basis of \( c_{0} \left( \bigoplus_{k=1}^{\infty} \ell_{2}^{k} \right) \) and \( \ell_{\infty} \left( \bigoplus_{k=1}^{\infty} \ell_{2}^{k} \right) \), respectively, and \( R : c_{0} \left( \bigoplus_{k=1}^{\infty} \ell_{2}^{k} \right) \to Y \) be the bounded linear operator given by

\[
R(y) = \sum_{j=1}^{\infty} \sum_{i \in I(j)} y_{i} R(e_{i}),
\]

where

\[
R(e_{i}) = \begin{cases} 
    \tilde{R}(f_{i}), & \text{if } i \in A \\
    0, & \text{if } i \in \mathbb{N} \setminus A,
\end{cases}
\]

and the vector \( v = (v_{i})_{i} \in c_{0} \left( \bigoplus_{k=1}^{\infty} \ell_{2}^{k} \right) \) defined by

\[
v_{i} = \begin{cases} 
    \tilde{u}_{i}, & \text{if } i \in A \\
    x_{i}, & \text{if } i \in \mathbb{N} \setminus A.
\end{cases}
\]
So \( R \in S_{c_0(\oplus_{k=1}^{\infty} l_2^k), Y} \), \( v \in S_{c_0(\oplus_{k=1}^{\infty} l_2^k)} \) and \( \| R(v) \| = \| \tilde{R}(\tilde{u}) \| = 1 \).

It follows that \( R \) attain its norm in \( v \). Next we will show that \( \| R - T \| < \epsilon \).

\[
\| R - T \| \leq \left\| R - \frac{TP_A}{\| TP_A \|} \right\| + \left\| \frac{TP_A}{\| TP_A \|} - TP_A \right\| + \| TP_A - T \|
= \left\| \tilde{R} - \frac{TP_A J}{\| TP_A J \|} \right\| + \left\| \frac{TP_A}{\| TP_A \|} - TP_A \right\| + \| TP_A - T \|
= \left\| \tilde{R} - \frac{TP_A J}{\| TP_A J \|} \right\| + \left\| TP_A \right\| \left| \frac{1}{\| TP_A \|} - 1 \right| + \| TP_A - T \|
< \epsilon + 1 - 1 + \eta(\epsilon) + \epsilon < 3\epsilon.
\]

Finally, we show that the vectors \( v \) and \( x \) are close,

\[
\| v - x \| = \| P_A(v - x) \| = \max_{1 \leq j \leq n} \left( \sum_{i \in I(j)} |v_i - x_i|^2 \right)^{1/2}
= \max_{1 \leq j \leq n} \left( \sum_{i \in I(j)} |\tilde{u}_i - x_i|^2 \right)^{1/2}
\leq \max_{1 \leq j \leq n} \left( \sum_{i \in I(j)} |\tilde{u}_i - u_i|^2 \right)^{1/2}
+ \max_{1 \leq j \leq n} \left( \sum_{i \in I(j)} |u_i - x_i|^2 \right)^{1/2}
\leq \| \tilde{u} - z \| + \| u - x \| < \beta(\epsilon) + \gamma(\epsilon),
\]

where \( \lim_{\epsilon \to 0} \beta(\epsilon) + \gamma(\epsilon) = 0 \). Therefore \( (c_0(\oplus_{k=1}^{\infty} l_2^k), Y) \) satisfies the \( BPBP \) for operators. □

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