Radiation and evolution of small relativistic dipole in QED

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Abstract

We study in the quasiclassical approximation the radiation reaction and it’s influence on the space-time evolution for the small relativistic dipole moving in a constant external electromagnetic field in QED.

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I. INTRODUCTION

The problem of the radiation losses (radiation reaction) of the particle moving in the given external field is the classical problem both in quantum and in classical physics that had always attracted a lot of attention. The classical physics studies are thoroughly reviewed in refs. [1–3]. Since the creation of QED this problem was thoroughly studied on the quantum level by several groups of authors, the results reviewed in refs. [4–10]. The two different lines of approach were developed, one based on the use of the exact wave-functions in the external field [5,6,8], and the other based on the quasi-classical approach [7].

In particular, the classical results for the radiation reaction were extended to the quantum case, and it was shown that for ultra-relativistic particles such that \( FE/m^3 \gg 1 \) (\( F \) is the field strength, \( E \) – the energy and \( m \) the mass of the particle), the law of radiation reaction changes drastically compared to the classical case due to the strong recoil effects.

Recently a new version of the quasi-classical approach based on the use of the quasi-classical Schrodinger wave functions was developed in refs. [9,10] and references therein.

Although the theory of a particle in the external field seems to be thoroughly developed there is still a lot of interest in the subject. The reasons, apart from the internal beauty of the subject include a number of practical reasons. First, the external field is the simplest model of the media. Second, the QED results can be viewed as a starting point for the discussion of the propagation of the QCD particles in the media, the subject being extremely popular recently due to the recent interest in quark-gluon plasma [11]. Next, it was realized that the space-time evolution of the point charge in the external field is closely connected with the fundamental properties of QED, leading to the concept of the semi-bare electron [12].

The above research was devoted, however, to the radiation reaction of the charged particle in the external field. Much less is known about the dipole propagation in the external field. The experimental research of fast \( e^+ - e^- \) pairs propagating in the media continues since fifties, including the famous experiments by Perkins [13] in 1957. Theoretical investigation of the fast \( e^+ - e^- \) pairs leading to the concept of charge transparency, was started in refs. [14], [15]. Recently, there was a renewed theoretical interest in the study of the relativistic dipole in QED. The reasons are both practical (explanation of the experimental data on \( e^+ - e^- \) pairs), and theoretical. In particular, it was realized that the quantum effects play much bigger role in the propagation of the dipole in external field than that of the single particle, leading to the discovery of the quantum diffusion [16]. The essence of the latter phenomena is the diffusion-type law of the fast dipole expansion in the weak external field due to the noncoulombic quantum photon exchange between the components of the dipole. Thus it was realized, that the study of the propagating QED dipole, and in particular of it’s space-time evolution, is important for the understanding of the fundamental properties of the QED, in light of ref. [12]. Moreover, the study of the propagating dipole is extremely important due to it’s possible generalization to QCD, where the dipole, due to confinement, may be basic degree of freedom [17–19]. This approach led to the discovery of color transparency phenomena in QCD [17]. Moreover, the QED dipole is identical to the QCD dipole connected to the deep-inelastic scattering on the longitudinal virtual photons [20].

However, there is still very few knowledge about the properties of the propagating relativistic dipole (in particular relative to what we know about the propagation of the single charged particles).
The main goal of the present paper is to study the radiation reaction, and in particular the pattern of the charge transparency, and it’s influence on the evolution of the small ultra-relativistic dipole in the arbitrary strong external field in QED. In particular we shall be interested in the influence on the radiation reaction of the interference between the fields created by different components of the dipole. For simplicity we shall consider the case of the dipole containing two oppositely charged scalar particles of the same mass, moving in the constant external field whose direction is orthogonal to the direction of the motion of the center of mass of the dipole. We shall assume that two particles were created at the time \( T = 0 \) in the same point of the space-time \( \vec{r}(0) = 0 \).

The main goal of this paper is to take into account the influence of the quantum effects on the radiation reaction of the dipole. We shall be able to take into account the quantum effects connected with the recoil. We will not be able to take into account the quantum effects connected with the quantum character of the motion of the dipole, in particular we shall not be able to take into account the spread of the dipole wave packets and the quantum diffusion. We will not take into account the spin of the particle, limiting ourselves to the scalar particle case.

Throughout the paper we use the quasi-classical wave functions first derived in refs. [9,10].

We shall see that there are three distinct time scales: \( 1/E \ll T \ll m/F \) (this time regime exists for the dipole such that the initial transverse motion of its components is non-relativistic), \( m/F \ll T \ll E/F, T \gg E/F \). For the first regime (we shall call it very small dipole regime) the radiation reaction is strongly suppressed by interference. The interference also decreases a number of emitted photons. For the second regime the decrease in radiation reaction, relative to the sum of radiation reactions of two independent particles, depends on the Lorentz invariant parameter \( \chi = \frac{FE}{m^3} \). For \( \chi \ll 1 \) the interference quickly decreases starting from \( T \sim m/F \). For \( \chi \gg 1 \) the interference starts to decrease only starting from the larger time \( T^* \sim (E/F^2)^{1/3} \). In the latter case the dipole effects especially change the photon spectrum. The relevant photons first are concentrated near the end-point of the spectrum, and not in the middle, as for a single charged particle. The maximum of the radiation reaction spectral curve moves towards saturation at frequencies \( \omega \sim 0.4E \).

The interference radically changes the frequency distribution of the number of radiated photons. Instead of unbounded increase at small frequencies, it now goes to zero as \( \omega \to 0 \), and has a maximum at finite frequency, of the order of the maximum of the radiation reaction.

Finally in the third regime, the interference does not influence the radiation reaction, but still cuts off the soft photons with \( \omega \ll 1/T \), and the photon distributions will have a finite maximum at \( \omega \sim 2/T \).

Our results, derived in the approximation of the constant external field, can be translated to the model-independent language of the propagation of the dipole through the arbitrary external media. Indeed, the Lorentz-invariant parameter \( \chi = \frac{FE}{m^3} \) is really a ratio of two parameters: the parameter \( l_c = E/m^2 \), which is (up to non-important here numerical coefficient ) a coherence length, and \( l_F = m/F \), which is the field regeneration length (or time between successful interactions with the external field). Thus the parameter \( \chi \) actually measures a number of collisions once the dipole propagates through the coherence length. In
particular color transparency and quantum diffusion considered in refs. [16,17,19] correspond to the case $\chi \ll 1$. The regime of the very small dipole corresponds to the case $T \leq m/F$, i.e., in the model independent language, to the case when the propagation time is less than a time $T_F$ one needs to meet an external field photon. In other words, $l_F$ is analogous to the mean free path in the media language. Then it is clear that in this regime the radiation is always suppressed, independent of the parameter $\chi$. However, the later time evolution depends on the parameter $\chi$. If $\chi \ll 1$, and this corresponds to the case considered in refs. [16,17,19], the coherence length is much less than $l_F$, the radiation suppression ends, as we shall see, at $T \sim m/F$, and apart from the small time interval in the beginning $\sim E/m^2 \ll m/F$, one can use for the study of the radiation reaction and the spectra of the emitted photons a quasi-classical approximation. However, in the opposite case, $\chi \gg 1$, we have the situation of the multiple collisions during the coherence length. In this case, we were able to develop a quasi-classical theory of the radiation emission taking into account recoil. Our results show the suppression of the radiation reaction and the photon emission up to the time $T^* \gg T_F$. This looks quite similar to the Landau-Pomeranchuk effect for the propagation of the fast particle in the media. There the effect also appears when the coherence length is much bigger than the free mean path [9]. However, as we discuss below, the classical approximation may be not applicable to the situation when $l_F \ll l_c$ for the dipole. This is the case that occurs in the statistical mechanics, when the coherence length is bigger than free mean path. Then there is a number of the important effects that arise only beyond the quasi-classical approximation [21]. In this paper we shall only study what one obtains sticking to the quasi-classical approximation. The results may be considered as a starting point for the future study.

The paper is organized in the following way. In Chapter 2 we shall consider small classical dipole, but will derive it’s radiation reaction using relativistic quantum mechanics, and check that the classical approach corresponds to the recoiless limit of the quasi-classical theory. We shall review the results for a single particle, then consider the case of the radiation of the arbitrary dipole, and then derive the radiation reaction in the small and very small dipole limits. In Chapter 3 we shall briefly review the classical wave-functions method of refs. [9,10] and extend it to the case of the arbitrary dipole. Next we shall assume that the dipole is small (in the plane transverse to the direction of it’s center of mass motion) and derive the general formula for the radiation reaction of such small dipole. In the sections 4 and 5 we shall use the above formulae to study the radiation reaction in two important limiting cases. In Chapter 4 we shall study the frequency distribution of radiation and the time dependence of the total radiated energy for the limit of very small times, when the dipole own field was not generated yet. We shall call this regime a very small dipole regime. This regime can be also characterized as the regime when the particle deflection angle due to external field is less that the radiation angle. In Chapter 5 we shall consider the scale of times when the dipole is still small, but it’s field has already been generated. The particle deflection angle is much bigger than the radiation angle. We shall study the frequency distribution of the photons and the radiation reaction also in this case. We shall see that the radiation reaction depends on the parameter $\chi = E/m^3$. (Recall that for the single fast moving charged particle radiation reaction qualitatively depends on this parameter that is a Lorentz invariant: $\chi = \sqrt{(F_{\mu\nu}p^\nu)^2/m^6}$ [1].) In Chapter 6 we shall study the total back-force
acting on the dipole for very small times and it’s influence on both the transverse and the longitudinal evolution of the dipole. In Chapter 7 we shall make some qualitative comments on the influence of the quantum nature of the dipole motion on the radiation reaction, in particular on the possibility to go beyond the quasi-classical approximation. Our results, the directions for the future work and possible implications for QCD will be summarized in the conclusion.

II. RADIATION REACTION OF THE FAST RELATIVISTIC DIPOLE.

A. Radiation of the single scalar particle

Let us start by briefly recalling the basic quasi-classical formalism for radiation of photons by relativistic charged particle without taking into account recoil [4]. The results are the same as obtained by using classical electromagnetism theory [1,9], but we shall use from the beginning not the wave but the photon formalism, that will be easily extended in the next chapter to the case when we need to take into account recoil and the classical electromagnetism theory will be unapplicable.

The matrix element of the interaction between the electromagnetic field and the scalar particle is given by

\[ S^{(1)} = -iq \int d^4x A_\mu(x) J^\mu(x), \]  

where \( J_\mu(x) \) is the current density operator in the external field,

\[ J^\mu = \Phi^*(P^\mu \Phi) - (P^\mu \Phi^*) \Phi. \]

The operator \( P^\mu \) is the generalized momentum operator in the external field. Consequently, the matrix element for the emission of the photon with the frequency \( \omega \), wave vector \( \vec{k} \) and polarization vector \( \vec{e} \) is given by

\[ M_{fi} = -iq \int_0^T dt \int d^3\vec{r} \sqrt{\frac{2\pi}{\omega}} \frac{1}{E_i E_f} \phi_f^*(\vec{r}, t)(\vec{e} \cdot \vec{P}) \exp (i(\omega t - \vec{k} \cdot \vec{r})) \phi_i(\vec{r}, t) \]

Here \( \phi_i \) is the initial and \( \phi_f \) is the final state wave functions, normalized by the condition

\[ \int d^3\vec{r} \phi^*(\vec{r}) \phi(\vec{r}) = 1. \]

The operator \( \vec{P} \) is

\[ \vec{P} = \vec{p} - q \vec{A}, \]

\( \vec{p} = -\frac{\partial}{\partial x_i} \) is the momentum, \( q \) is the charge of the particle and \( A(\vec{r}, t) \) is the vector potential. \( E_i \) and \( E_f \) are the energies of the initial and the final states.

We shall use the quasi-classical wave functions of the scalar particle in the external field:

\[ \phi(\vec{r}, t) = \sqrt{\frac{D}{E_i - qA_0}} \exp \left( \frac{i}{\hbar} S(r, p, t) \right). \]
Here $S(r, p, t)$ is the action of the particle with the momentum $\vec{p}$ calculated along the classical trajectory of the particle passing through the point with the coordinate $\vec{r}$ at the time $t$ and having the momentum $\vec{p}$ at $t=0$. $D$ is the Van-Vleck determinant:
\[
D = \sqrt{\left\| \frac{\partial^2 S(\vec{r}, \vec{p})}{\partial \vec{r} \partial \vec{p}} \right\|} = \frac{1}{E} \sqrt{\delta(\vec{r} - \vec{r}(t))}.
\]
(2.6)

The wave functions (2.5) can't be substituted directly into the matrix element (2.3), due to the appearance of the quickly oscillating factors
\[
\exp i(S(r, p_f, t) - S(r, p_i, t))/\hbar
\]
for $\hbar \to 0$. In order to avoid this difficulty we have to use the representation:
\[
\phi_\pi(\vec{r}, t) = \int d^3\vec{p} \phi_p(\vec{r}, t) S_{pp_\pi},
\]
(2.7)

where $\phi_p$ is the quasi-classical wave function of the particle in the external field possessing at $t \to \infty$ the asymptotics
\[
\phi_p(\vec{r}, t) \to \frac{1}{\sqrt{2E_p}} \exp (i(\vec{p} \vec{r} - Et))).
\]

$S_{pp_\pi}$ is the scattering matrix of the particle in the external field considered. If we neglect the recoil, and substitute the representation (2.7) for the final state wave function into the matrix element (2.3), we shall recover the classical amplitude for the radiation of the electromagnetic waves, and the classical expression for the energy loss during a time interval $T$. (see e.g. refs. [7,9] for details):
\[
dW_{cl} = \frac{2q^2}{\pi^2} (d^3k) \int_0^T \int_0^T dt dt' (\vec{v}(t) \cdot \vec{v}(t'))(\vec{v}^* \cdot \vec{v}(t'))
\]
\[
\times \exp (i\omega(t - t') - i\vec{k} \cdot (\vec{r}(t) - \vec{r}(t'))).
\]
(2.8)

After averaging over the photon polarization vectors we obtain
\[
dW_{cl} = q^2 \frac{1}{2\pi^2} d^3k \int_0^T \int_0^T dt dt' (\vec{v}(t) \cdot \vec{v}(t')) - (\vec{n} \cdot \vec{v}(t))(\vec{n} \cdot \vec{v}(t'))
\]
\[
\times \exp(i\omega(t - t') - i\omega\vec{n} \cdot (\vec{r}(t) - \vec{r}(t')))
\]
(2.9)

where $\vec{k} = \omega\vec{n}$. Note that $\vec{n} \cdot \vec{v}(t) = \vec{v}\nabla \vec{r} = \partial/\partial t$. Hence the terms in the latter equation containing $\vec{n}$ in the preexponential can be integrated by parts:
\[
dW_{fi} = q^2 \frac{1}{2\pi^2} d^3k (\int_0^T \int_0^T dt dt' (\vec{v}(t) \cdot \vec{v}(t')) - 1) \times \exp(i\omega(t - t') - i\omega\vec{n} \cdot (\vec{r}(t) - \vec{r}(t')))
\]
\[
+ \frac{4}{\omega} \int_0^T \sin (\omega T + \pi\vec{n}(T))/2 \cos (\omega(T - 2s) + \pi\vec{n} \cdot (\vec{r}(T) - 2\vec{r}(s)))/2
\]
\[
- \frac{2}{\omega^2}(1 - \cos \omega T + \omega\vec{n} \cdot \vec{r}(T)).
\]
(2.10)
It is straightforward to see however, that the last two lines in eq. (2.10) correspond to terms decreasing or bounded with $T$, while the expression in the first line increases with $T$. Thus the two last lines can be omitted if we are interested in large time intervals $T \gg 1/\omega$. Indeed, we can integrate eq. (2.10) over the photon direction $\vec{n}$ and obtain

$$dW_{cl} = \frac{q^2 \omega}{2} \int_0^T dt \left( \vec{v}(t) \cdot \vec{v}(t') - 1 \right) \times \cos (\omega (t - t') \frac{\sin (\omega |\vec{r}(t) - \vec{r}(t')|)}{|\vec{r}(t) - \vec{r}(t')|}$$

$$+ \frac{2}{\omega} \int_0^T \frac{\cos (\omega s - \omega r(s)) - \cos (\omega s + \omega r(s))}{r(s)}$$

$$+ \frac{2}{\omega} \int_0^T \frac{\cos (\omega (T - s) - \omega |\vec{r}(T) - \vec{r}(s)|) - \cos (\omega s + \omega |\vec{r}(T) - \vec{r}(s)|)}{|\vec{r}(T) - \vec{r}(s)|}$$

$$- \frac{2}{\omega^2} (1 - \frac{\sin \omega(T + r(T)) - \sin (\omega T - r(T))}{r(T)}).$$

(2.11)

It is easy to see that the last 3 lines in the eq. (2.11) are suppressed like $1/(\omega T)$ relative to the double integral in the first line, and thus can safely discarded if we are interested in the frequencies and time intervals $\omega T \gg 1$. In order to know numerically how big are these terms, we shall however keep them.

Finally, since we are usually interested in the energy losses in the unit of time, we can differentiate eq. (2.11) over time $T$ and obtain

$$\frac{dW_{cl}}{dT} = q^2 4 \frac{\omega}{\pi} \omega \omega \left( \int_0^T dt \left( \vec{v}(T) \cdot \vec{v}(t) - 1 \right) \times \cos (\omega (T - t) \frac{\sin (\omega |\vec{r}(T) - \vec{r}(t)|)}{|\vec{r}(T) - \vec{r}(t)|}$$

$$+ \frac{2}{\omega} \frac{\cos (\omega T - \omega r(T)) - \cos (\omega T + \omega r(T))}{r(T)}$$

$$+ \frac{2}{\omega} \int_0^T \frac{d}{dt} \frac{\cos (\omega s) - \omega |\vec{r}(T) - \vec{r}(T - s)| - \cos (\omega s + \omega |\vec{r}(T) - \vec{r}(T - s)|)}{|\vec{r}(T) - \vec{r}(T - s)|}$$

$$- \frac{2}{\omega^2} \frac{d}{dT} (1 - \frac{\sin \omega(T + r(T)) - \sin (\omega T - r(T))}{r(T)}).$$

(2.12)

Below we shall use the first line in the latter equation and check that the last 3 lines can be neglected:

$$\frac{dW}{dT} = q^2 4 \frac{\omega}{\pi} \omega \omega \left( \int_0^T dt \left( \vec{v}(T) \cdot \vec{v}(t) - 1 \right) \times \cos (\omega (T - t) \frac{\sin (\omega |\vec{r}(T) - \vec{r}(t)|)}{|\vec{r}(T) - \vec{r}(t)|}. \right.$$}

(2.13)

The latter equation, if the limits of integration are infinite, can be easily brought into the standard form of the classical electromagnetic theory [4,9].

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Let us now consider the radiation reaction of the relativistic dipole in the case we can neglect recoil, i.e. \( \omega \ll E \).

For simplicity we consider the symmetric dipole, whose center of mass moves with the speed \( v \sim c \) in the direction orthogonal to the direction of the constant external field, and which was created at time \( T = 0 \) in the point \( \vec{r}(0) = 0 \). We shall denote the components of the dipole as P and A (particle and antiparticle). Let us assume that the particles of the dipole have, after its creation, the same initial energy \( E_i \), and the orthogonal component of the velocity \( \vec{v}_0 \). Note that if \( u_0 \) is the velocity in the transverse plane in the center of mass reference frame, moving with dipole, then \( v_0 = \frac{m}{E} u_0 \), meaning that in any case \( v_0 \leq m/E \).

In our kinematics the two components of the dipole will have the same velocity component in the direction of c.m. motion and the opposite sign components in the transverse plane.

The amplitude of the radiation of the photon with the polarization vector \( \vec{e} \) the wave vector \( \vec{k} \) and the frequency \( \omega \) will be the difference (due to the different charges of the dipole components) of the amplitudes of the photon emission of the particle and the anti-particle components of the dipole. Using the equations of the previous chapter it is straightforward to write:

\[
M_{fi} = -i \sqrt{\frac{2\pi}{\omega}} \frac{1}{\sqrt{E_i E_f}} \int_0^T dt \langle \vec{e} \cdot \vec{v}_P(t) \rangle \exp i(\omega t - \vec{k} \cdot \vec{r}_P(t)) - \langle \vec{e} \cdot \vec{v}_A(t) \rangle \exp i(\omega t - \vec{k} \cdot \vec{r}_A(t)) \rangle.
\]

(2.14)

Here \( \vec{r}_P(t) \) and \( \vec{r}_A(t) \) are the radius vectors of the particle and antiparticle components of the dipole. The energy radiation loss during the time from the creation of the dipole at the time \( t = 0 \) till time equal \( T \) with the photons radiated in the frequency range \( d\omega \) and the solid angle range \( d\omega \) is

\[
dW = \frac{q^2}{4\pi^2} \omega^2 d\omega \int_0^T dt \int_0^T dt' \exp i\omega(t - t')(\vec{e} \cdot \vec{v}_P(t)) \exp i(\vec{k} \cdot \vec{r}_P(t)) -
\]

\[
- \langle \vec{e} \cdot \vec{v}_A(t) \rangle \exp i(\vec{k} \cdot \vec{r}_A(t)) \rangle(\vec{e}^* \cdot \vec{v}_P(t')) \exp -i(\vec{k} \cdot \vec{r}_P(t') - \vec{e}^* \cdot \vec{v}_A(t') \exp -i(\vec{k} \cdot \vec{r}_A(t')).
\]

(2.15)

Summing over the polarizations of the photon we obtain

\[
\frac{dW}{d\omega} = \frac{q^2}{4\pi^2} \omega^2 \int_0^T dt \int_0^T dt' \exp i(\omega(t - t')))
\]

\[
(v_P(t') \cdot v_P(t') - (\vec{n} \cdot \vec{v}_P(t))(\vec{n} \cdot \vec{v}_P(t'))) \exp i(\vec{k} \cdot (\vec{r}_P(t) - \vec{r}_P(t')) + (P \leftrightarrow A) -
\]

\[
- ((\vec{v}_P(t) \cdot \vec{v}_A(t') - (\vec{n} \cdot \vec{v}_P(t))(\vec{n} \cdot \vec{v}_A(t'))) \exp i(\vec{k} \cdot (\vec{r}_P(t) - \vec{r}_A(t')) + (P \leftrightarrow A)).
\]

(2.16)
Here $\vec{k} = \omega \vec{n}$. Using, as for the single particle, $\vec{n} \cdot \vec{v}(t) = \vec{v} \cdot \nabla \vec{r} = \partial / \partial t$, we can carry the integration by parts and obtain:

$$
\frac{dW}{d\omega} = \frac{q^2}{4\pi^2} \omega^2 \left( \frac{2}{\omega} \int_0^T dt \int_0^T dt' \exp i(\omega(t - t'))(\vec{v}_P(t) \cdot \vec{v}_P(t') - 1) \exp i\vec{k} \cdot (\vec{r}_P(t) - \vec{r}_P(t')) + (P \leftrightarrow A)
\right.
$$

$$
- (\vec{v}_P(t) \cdot \vec{v}_A(t') - 1) \exp i\vec{k} \cdot (\vec{r}_P(t) - \vec{r}_A(t')) + (P \leftrightarrow A)
$$

$$
+ \Delta W(\omega, T).
$$

(2.17)

The latter equation gives us the formula for the radiation of the arbitrary relativistic dipole. Note that it is a sum of two terms that correspond to the radiation of the single particle and two terms that correspond to the interference between the particle and the antiparticle.

The term $\Delta W$ arises from the integration by parts (cf. the single particle) and is equal to

$$
\Delta W(\omega, T) = do \frac{q^2}{4\pi^2} \omega^2 \left( \frac{2}{\omega} \int_0^T \sin (\omega(T - s) + \omega \vec{n}(\vec{r}_P(T) - \vec{r}_P(s))) + (P \leftrightarrow A)
\right.
$$

$$
- \frac{2}{\omega} \int_0^T ds \sin (\omega(T - s) + \omega \vec{n}(\vec{r}_P(T) - \vec{r}_A(s))) + \sin (\omega(T - s) + \omega \vec{n}(\vec{r}_A(T) - \vec{r}_P(s)) +
$$

$$
- (P \leftrightarrow A) - \frac{2}{\omega^2} (1 - \cos \omega \vec{n} \cdot (\vec{r}_P(T) - \vec{r}_A(T))).
$$

(2.18)

We can integrate over the angle variable $do$ and obtain:

$$
\frac{dW}{d\omega} = \frac{q^2}{\pi} \int_0^T dt \int_0^T dt' \cos (\omega(t - t'))(\vec{v}_P(t) \cdot \vec{v}_P(t') - 1) \frac{\sin |\vec{r}_P(t) - \vec{r}_P(t')|}{|\vec{r}_P(t) - \vec{r}_P(t')|} + (P \leftrightarrow A)
$$

$$
- (\vec{v}_P(t) \cdot \vec{v}_A(t') - 1) \frac{\sin |\vec{r}_P(t) - \vec{r}_A(t')|}{|\vec{r}_P(t) - \vec{r}_A(t')|} - (P \leftrightarrow A)
$$

$$
+ \Delta G(\omega, T).
$$

(2.19)

Here the term $\Delta G$ corresponds to the integral of the $\Delta W$:

$$
\delta G = \frac{2q^3 \omega}{\pi} \left( \int_0^T ds \left( \frac{\cos (\omega(T - s) - \omega|\vec{r}_P(T) - \vec{r}_P(s)|)}{\omega|\vec{r}_P(T) - \vec{r}_P(s)|} - \frac{\cos (\omega(T - s) + \omega|\vec{r}_P(T) - \vec{r}_P(s)|)}{\omega|\vec{r}_P(T) - \vec{r}_P(s)|} \right)
\right.
$$

$$
+ (A \leftrightarrow P)
$$

$$
- \left( \frac{\cos (\omega(T - s) - \omega|\vec{r}_P(T) - \vec{r}_A(s)|)}{\omega|\vec{r}_P(T) - \vec{r}_A(s)|} - \frac{\cos (\omega(T - s) + \omega|\vec{r}_P(T) - \vec{r}_A(s)|)}{\omega|\vec{r}_P(T) - \vec{r}_A(s)|} \right) - (A \leftrightarrow P)
$$

$$
- \frac{1}{\omega} (1 - \frac{\sin \omega|\vec{r}_P(T) - \vec{r}_A(s)|}{\omega|\vec{r}_P(T) - \vec{r}_A(s)|}).
$$

(2.20)
In order to get the radiation reaction it is enough to differentiate the above equations over $T$.

The latter equations describe the radiation of the arbitrary relativistic dipole, integrated over the angles, for the time interval $T$.

Now we can move to our goal – to consider the case of the small relativistic dipole.

**C. Radiation of the small relativistic dipole.**

Consider small quasiclassical relativistic dipole, i.e. $v \gg v_t$, where $v$ is its center of mass velocity and $v_t$ is the transverse component of the velocity ($v_t$ can be both relativistic and nonrelativistic). For sufficiently small times one can estimate

$$v_t(T) \sim v_{0t} + FT/E.$$  \hfill (2.21)

Here $F$ is the external field:

$$\vec{F} = \vec{E} + \vec{v} \times \vec{H},$$  \hfill (2.22)

$\vec{E}$ is an electric and $\vec{H}$ is a magnetic field. Consequently, one considers dipole as small if

$$FT \ll E.$$  \hfill (2.23)

For bigger time scales,

$$FT \geq E,$$

the components of the dipole behave as independent particles and there is no interference.

Let us study the interference pattern in the small dipole.

Let us assume that the condition (2.23) is satisfied. Then the photons are radiated into the small cone around z axis (we choose the z axis in the direction of the propagation of the dipole), of order $m/E$ at $T \sim 1/E_i$, where $E_i$ is the initial energy of each of the components of the dipole. Later the radiated photons are concentrated in two cones round the directions of the components of the dipole. It is clear that there exist, even if the condition (2.23)) is satisfied, two distinct possibilities: the two radiation cones, generated by the dipole components overlap, and that they stop to overlap. Since the cone angle for the ultra-relativistic particle is $\theta \sim m/E_i$, we see that the condition that the cones overlap is

$$v_{0t} + FT/E \leq m/E$$  \hfill (2.24)

If we can neglect the initial transverse velocity, the latter condition becomes

$$T \leq m/F$$  \hfill (2.25)

If

$$m/F \ll T \ll E/F$$

the dipole is still small, but the cones do not overlap, and the interference must decrease drastically. There is also the self-consistency condition: since $T \gg 1/E$, we must have
for the possibility of considering the very small dipole, with the cones overlapping, quasi-
classically. We need the weaker condition

\[ E^2 \gg F, \quad (2.27) \]

for the possibility to consider small quasi-classical dipole. If the latter conditions are not
fulfilled, we must take into account the interference of the dressing by external field and
the generation of the self-field by bare particle. This is beyond the scope of this research.
(Although it can be that our analysis is qualitatively true even in the latter case, since the
self dressing generates usually quickly oscillating terms that can be singled out).

We conclude that the classical dipole has two regimes: 1) very small dipole, when the
radiation cones of the particle and the anti-particle overlap strongly, \( T \ll m/F \), and 2) small
dipole in the sense it still moves along z axis, but the cones of the radiation do not
overlap, and the interference decreases. Note that these two cases correspond to two possible
relations between the depletion angle of the single charged particle in the external field and
the radiation angle. The very small dipole corresponds to the case when the latter angle
is much bigger than the former and the small dipole—when the former is bigger than the
latter. Note also that for a relativistic in the c.m. of dipole transverse motion means (then
\( v_t \sim m/E \)) we have only small dipole regime.

Suppose we have the very small dipole. Let us analyze the interference pattern. Consider
the exponents in eq. (2.17). The exponents in the terms that contain only the particle or
only the anti-particle radiation are

\[ \omega(\cos \theta(z(T) - z(t)) + \sin \theta(y(T) - y(t))). \]

Here \( \theta \) is the angle between the photon wave vector and direction of the z axis, and

\[ z(T) - z(t) \sim v_z(t - T), \quad y(T) - y(t) \sim v_t(T - t) \]

It is clear that the corresponding integrals will be saturated by \( t \sim T \), and the first term
will be dominant since \( \sin \theta \ll 1 \) Consider now the exponents in the interference terms in
eq. (2.17). These exponents have the form, for the chosen kinematics

\[ \omega(\cos \theta(z(T) - z(t)) + \sin \theta \sin \phi(y(t) + y(T))). \]

Here \( \phi \) is the asimutal angle. In the first approximation we can put \( y(t) \sim y(T) = (T)/2 \)
in the latter equation, and instead of integrating, substitute \( \sin \theta \) by it’s characteristic value
\( m/E \). (\( d(T) \) is the scale of the dipole, i.e. the separation between the charges, which in our
kinematics is purely transverse). Then integral over the angle \( \phi \) gives the Bessel function:

\[ \frac{1}{2\pi} \int_0^{2\pi} \exp(i \sin \phi \omega d(T)m/E) d\phi = J_0\left(\frac{\omega}{E} d(T)m\right) \quad (2.28) \]

With the same accuracy we can substitute \( \cos \theta(z(t) - z(T)) \) with \( \cos \theta(z(t) - z(T)) + \sin \theta \sin \phi(y(t) - y(T)), \) i.e. after taking into account the interference term, the exponent in
the interference term will be the same as in the direct terms. Then for very small dipole we
can rewrite eq. 2.17 as

\[
\frac{dW}{d\omega dT} = \frac{4q^2}{\pi} \int_0^T dt \cos (\omega(T-t))(\vec{v}_P(T) \cdot \vec{v}_P(t) - 1) \frac{\sin |\vec{r}_P(T) - \vec{r}_P(t)|}{|\vec{r}_P(T) - \vec{r}_P(t)|} \\
\times (1 - J_0(\frac{\omega}{E}\theta d(T)))
\]  

(2.29)

In addition, there is contribution from the terms, that correspond to integration by parts,
where it is enough to do the same approximation:

\[
\frac{dG}{dT d\omega} = \frac{q^2}{2\pi} \int_0^T \frac{d}{dT} \cos (\omega s - \omega |\vec{r}(T) - \vec{r}(T-s)|) - \cos (\omega s + \omega |\vec{r}(T) - \vec{r}(T-s)|) \\
\times (1 - J_0(\frac{\omega}{E}\theta d(T)))
\]

\[
+ \frac{d}{dT} \frac{2}{\omega^2} \frac{\sin \omega d(T)}{T}.
\]  

(2.30)

Here \(\vec{d}(s)\) is the time derivative of the dipole moment, i.e. the relative velocity of the particle
and antiparticle:

\[
\dot{\vec{d}}(s) = \frac{\partial (\vec{r}_P(s) - \vec{r}_A(s))}{\partial s}.
\]  

(2.31)

The latter equations gives the radiation energy losses rate for the very small relativistic
dipole between times 0 and T, emitted in the particular interval of photon frequencies.

Note that our interference analysis could be made in terms not of the characteristic
radiation angles, but in terms of the longitudinal and transverse momenta. Our characteristic
angles \(m/E\) correspond to the characteristic transverse momentum of the emitted photons
\(q_t \sim m\omega/E\), in particular, if we consider photons, whose energy is a finite part of E, the
characteristic transverse momentum will be \(q_t \sim m\).

Consider now the next regime, \(E/F \gg T \gg m/F\). This is the case of the small, but not
very small dipole. In this case we can still consider the trajectory of each of the particles as
the almost straight line. We can follow the above derivation of the interference terms, but in
this case, although still \(\theta \ll 1\), we need to take as \(\theta\) the angle \(v_t/v \sim v_t \sim v_{0t} + FT/E\). We
then get the equation similar to eq. (2.29), but with the different argument for the Bessel
Function:

\[
\frac{dW}{d\omega dT} = \frac{4q^2}{\pi} \int_0^T dt \cos (\omega(T-t))(\vec{v}_P(T) \cdot \vec{v}_P(t) - 1) \frac{\sin |\vec{r}_P(T) - \vec{r}_P(t)|}{|\vec{r}_P(T) - \vec{r}_P(t)|} \\
\times (1 - J_0(\omega \theta(T)d(T)))
\]  

(2.32)
where

\[ \theta(T) = v_{0t} + FT/E = v_y(T). \]  

(2.33)

Since for the classical dipole \( d(T) \sim v_y(T)T \sim v_{0t}T + FT^2/(2E) \), we see that interference is suppressed as \( 1 - J_0(\frac{\omega}{E}(v_0^2TE + 3v_{0t}FT^2/2 + F^2T^3/(2E^2))) \) Since \( v_t \leq m/E \) (due to the relativistic law of the velocity summation), and \( T \gg m/F \), the third term in the argument of the Bessel function will be dominant, i.e. the interference decreases as \( J_0((\omega/E)(F^2T^3/E)) \), and quickly becomes negligible. Finally note that for very small frequencies one always has interference.

Note also that the argument of the Bessel function can be represented as

\[ xb(\tau), \ b(\tau) = m\frac{dd^2(\tau)}{d\tau} \]

i.e. as a Lorentz invariant (see also the discussion below). Here \( \tau \) is the proper time in the reference frame of the c.m. of the dipole.

Finally, since the integrands in eqs. (2.29) and (2.32) are concentrated near \( T = s \), we can expand them in Taylor series near \( s = T \). Consider first the difference \( \vec{r}(T) - \vec{r}(s) \) in the argument of the exponents.

For small dipole it is possible to use the approximations [9,10]:

\[ \vec{v}(T) = v(0)(1 - v^2(T)/2v(0)^2) + \vec{v}_t(T) \]  

(2.34)

and

\[ \dot{\vec{v}}(T)_t = q\vec{F}/E. \]  

(2.35)

In this approximation up to the terms of order \( m/E \)

\[ \dot{\vec{v}}(T) \sim \dot{v}_t \sim q\vec{F}/E, \]

i.e. the vectors \( \vec{v} \) and \( d\vec{v}/dT \) are orthogonal. Also

\[ \frac{d^2\vec{v}}{dT^2} = -\omega_0^2\vec{v}(T), \]

\[ \omega_0 = qF/E. \]

Then

\[ |\vec{r}(T) - \vec{r}(s)| = \sqrt{v^2(T)(T - s)^2(1 - \omega_0^2(T - s)^2)^2 + \omega_0^2(T - s)^2/4} \]

\[ \sim v(T)(T - s)(1 + \omega_0^2(T - s)^2/24). \]  

(2.36)

Then we have in the standard way ( [9]):

\[ \vec{v}_p(T)\vec{v}_p(t) - 1 = -(1 - v^2(T) + (T - t)^2\omega_0^2/2) \]  

(2.37)
In the same approximation
\[
\dot{\mathbf{d}}(T) \cdot \dot{\mathbf{d}}(s) = 4(v_t(T)^2 - (T - s)\omega_0 v_t(T))
\]  \hspace{1cm} (2.38)

Thus we have our final result for the radiation of the small dipole:
\[
\frac{dW}{d\omega dT} = -\frac{4q^2}{\pi} \int_0^T dt \frac{(1 - v^2(T) + \omega_0^2(T - t)^2/2)}{T-t} \sin(\omega(1 - v)(T - t)) + \omega_0^2(T - t)^3/24 \times (1 - J_0(\omega\theta(T)d(T))).
\]  \hspace{1cm} (2.39)

Here
\[
\theta(T) = m/E \ T \ll m/F
\]
\[
\theta(T) \sim v_t(T) \sim m/E + FT/E, \ E/F \gg T \gg m/F.
\]  \hspace{1cm} (2.40)

In order to obtain the full radiation reaction we must integrate the latter formulae over \(\omega\). We thus obtained classical radiation reaction of the dipole using simple wave mechanics. In parallel we understood the nature of the interference in the transverse plane, that we shall use in the quantum case.

**III. RADIATION REACTION FOR THE RELATIVISTIC DIPOLE: RECOIL EFFECTS**

**A. Single Particle.**

In the previous section we studied, using the relativistic quantum mechanics method, the classical radiation from the classical dipole. Let us now move to the quantum effects. There are two types of quantum effects [9]: first, the effects due to the quantum character of the particle motion in the external field. This effect is characterized by the parameter \(F/E^2\) [9]. Second there are quantum effects, specifically due to the motion of the quantum dipole [16,17,19]. These effects we will not take into account. Third, there are the recoil effects, that arise if we take into account \(E_i \neq E_f\). The general theory of such effects was first derived in ref. [7]. Recently a new approach was derived by Akhiezer and Shulga [9,10]. Let us briefly review the idea of ref. [9]. We return to the derivation of matrix element of the radiation of photon (2.3). We still use the representation (2.7) for the quasi-classical wave functions, but when we substitute them into the matrix element (2.3) we take into account that the corresponding integral over \(\mathbf{p}\) is saturated not at \(\mathbf{p} = \mathbf{p}_f\), as we assumed when we neglected recoil, but at \(\mathbf{p} \sim \mathbf{p}_f + \mathbf{k}\), where as usual \(\mathbf{k}\) is the wave vector of the emitted photon. Then it is possible to prove that the generalized action \(S = S_f - (\omega t - \mathbf{k} \cdot \mathbf{r})\) satisfies the generalized Hamilton-Jacobi equation
\[
\frac{\partial S}{\partial t} = (\nabla S - q\mathbf{A} - \mathbf{k})^2 + m^2. \hspace{1cm} (3.1)
\]
Solving this equation and substituting the solution into the matrix element (2.3), where we use for the wave function $\phi_i$ the representation (2.7), one obtains the quasi-classical matrix element of the photon radiation where the recoil is taken into account:

$$M_{fi} = -iq \int_0^T dt \int d^3\vec{r} \sqrt{\frac{2\pi}{\omega}} \sqrt{\frac{E_i}{E_f}} (\vec{e} \cdot \vec{v}(t)) \exp \left( i \frac{E_i}{E_f} (\omega t - \vec{k} \cdot \vec{r}(t)) \right).$$ (3.2)

Here we can put $E_f = E_i - \omega$.

The corresponding radiation reaction will be the same as for the single particle in the previous section, except the rescaling of the frequency $\omega \rightarrow \omega E_i/E_f$ in the exponent and the general multiplier $E_i/E_f$:

$$dW_{fi} = q^2 \frac{2}{\pi^2} \frac{E}{E_f} d^3k \int_0^T dt \int_0^T dt' (\vec{e} \cdot \vec{v}(t)(\vec{e}' \cdot \vec{v}(t')) \times \exp \left( \frac{E}{E_f}(i\omega(t - t') - i\vec{k} \cdot (\vec{r}(t) - \vec{r}(t'))) \right).$$ (3.3)

After averaging over the photon polarisations and integrating by parts we obtain, as in the previous section, the equation for the radiation reaction of the single particle including the recoil effects:

$$\frac{dW}{dT} = \omega q^2 \frac{4\pi}{d\omega} (\int_0^T dt \langle \vec{v}(T) \cdot \vec{v}(t) - 1 \rangle \times \cos \left( \frac{E}{E_f} \omega(T - t) \right) \sin \left( \frac{E}{E_f} \omega |\vec{r}(T) - \vec{r}(t)| \right) \frac{\sin \left( \frac{E}{E_f} \omega |\vec{r}(T) - \vec{r}(t)| \right)}{|\vec{r}(T) - \vec{r}(t)|})$$

$$+ \frac{2}{E_f \omega} \frac{\cos \frac{E}{E_f}(\omega T - \omega r(T)) - \cos \frac{E}{E_f}(\omega T + \omega r(T))}{r(T)}$$

$$+ \frac{2}{E_f \omega} \int_0^T dT' \cos \frac{E}{E_f}(\omega s - \omega |\vec{r}(T) - \vec{r}(T - s)|) \cos \frac{E}{E_f}(\omega s + \omega |\vec{r}(T) - \vec{r}(T - s)|) \frac{\sin \left( \frac{E}{E_f} \omega s \right)}{|\vec{r}(T) - \vec{r}(T - s)|}$$

$$- \frac{d}{dT} \left( \frac{2}{E_f \omega} \right)^2 (1 - \frac{\sin \left( \frac{E}{E_f} \omega (T + r(T)) \right) - \sin \left( \frac{E}{E_f} \omega (T - r(T)) \right)}{r(T)}) .$$ (3.4)

All other formulae from the subsection A in the previous chapter are transformed in the same way: $\omega$ is rescaled except in the measure, and the general multiplier is added $E/E_f$. Note that the terms that arise from integration by parts (boundary effects) are suppressed now even stronger as $\omega TE/(E - \omega)$.

We keep the terms due to integration by parts so that we shall be able to check explicitly that they are small in our analysis of the eq. (3.4). For convenience, let us write the
latter equation without the backreaction term, that is the result that will be used in the calculations:

\[
\frac{dW}{dT} = q^2 4 \pi \omega d\omega \left( \int_0^T dt (\vec{v}(T) \cdot \vec{v}(t) - 1) \cos \left( \frac{E_f}{E_f^2} \omega (T - t) \right) \frac{\sin \left( \frac{E_f}{E_f} \omega |\vec{r}(T) - \vec{r}(t)| \right)}{|\vec{r}(T) - \vec{r}(t)|} \right). \tag{3.5}
\]

The recoil effects lead to the qualitative change of the spectrum of the single particle. The maximum of the radiation reaction will be shifted to \( \omega_m \sim 0.4E \) for large \( \chi \), and will be virtually \( \chi \) independent. For the opposite limit of small \( \chi \) the maximum will remain at the classical value of \( \sim E \chi \).

**B. Recoil effects in the dipole radiation.**

It is clear from the preceding sections that taking recoil into account will mean just rescaling \( \omega \) in the previous chapter. Consequently, we obtain:

\[
\frac{dW}{d\omega dT} = 4q^2 \pi \int_0^T dt \cos \left( \frac{E_f}{E_f} \omega (T - t) \right) (\vec{v}_P(T) \cdot \vec{v}_P(t) - 1) \sin \frac{|\vec{r}_P(T) - \vec{r}_P(t)|}{|\vec{r}_P(T) - \vec{r}_P(t)|} \times (1 - J_0(\frac{E_f}{E_f} \omega \theta(T) d(T))) \tag{3.6}
\]

where the function \( \theta(T) \) is given by eq. (2.40). Since the main contribution still comes from \( s \sim T \), we obtain:

\[
\frac{dW}{d\omega dT} = -\omega \frac{4q^2}{\pi} \int_0^T dt \left( 1 - v^2(T) + \frac{\omega_0^2 s^2}{24} \right) \frac{s}{\omega(1 - v(s)) + \omega_0^2(s^3/24)} \sin \frac{E_f}{E_f} \omega \times (1 - J_0(\frac{E_f}{E_f} \omega \theta(T) d(T))). \tag{3.7}
\]

One can obtain the full radiation reaction by integrating the above equation over all frequencies.

**IV. RADIATION REACTION FOR THE VERY SMALL DIPOLE.**

Let us analyse the above equations for different regimes discussed in the chapter 3. We consider in this section the case of the very small dipole: \( 1/\omega \ll T \ll m/F \). First, let us check, what time scales contribute to eq. (2.39) in this case. For the linear term in the argument of \( \cos \) in eq. (2.39) to be dominant we need:

\[
(1 - v) \gg \omega_0^2 s^2 / 24,
\]
or
\[ s \ll 2\sqrt{6} \frac{m}{\sqrt{2E}}(E/F) = 2\sqrt{3} \frac{m}{F} \sim 3.5 \frac{m}{F}. \]

Here \( s = T - t \). Since the latter condition is satisfied for the very small dipole for the entire integration region in \( s \), we can neglect the cubic terms in the arguments of the \( \cos \) as well as the nonleading terms in the preexponentials. The integrals over \( s \) in eq. (2.39) can be taken explicitly. As it is explained in the appendix A, in this case we can discard the terms in eq. (2.39) proportional to \( 1 - v^2 \), as well as the terms originated from the integration by parts. The reason is that up to the terms suppressed as \( m^2/E^2 \) these terms correspond to radiation reaction of the free charged particle moving with a constant velocity. The latter is of course a nonphysical phenomena (see discussion in the Appendix A), and must be substracted. We start with an integral

\[
\frac{dW}{d\omega} = -\frac{4q^2}{\pi} \int_0^T dt \int_0^t ds \frac{\omega_0 s \sin E}{E_f} \omega((1 - v)(s))
\]

\[
\times (1 - J_0(\frac{E}{E_f} \omega m E d(T)))
\]

(4.1)

Note that the latter integral, as it is well known from the classical theory [1] is proportional to \( F^2 \), i.e. to the square of the acceleration.

The latter integral can be taken explicitly under the assumption that the interference multiplier weakly depends on time \( T \) in the limit of integration, and thus can be taken outside of the integrand. We obtain:

\[
\frac{dW}{d\omega} = \omega_0^2 \frac{q^2}{\pi} \left( \frac{2(1 - \cos(\omega'(1 + v)T) - \omega(1 + v)T \sin(\omega'(1 + v)T)}{\omega'(1 + v)^3} \right)
\]

\[
\times \frac{2(1 - \cos(\omega'(1 - v)T) - \omega(1 - v)T \sin(\omega'(1 - v)T)}{\omega'(1 - v)^3} \right)((1 - J_0(\frac{E}{E_f} \omega m E d(T)))
\]

(4.2)

The corresponding spectral curve for the single particle (without taking into account radiation) is depicted in Fig. 1. The spectral curve for the same energy and field, but for the dipole, whose transverse motion velocity is \( v_t \ll 1 \) and the interference is taken into account is depicted in Fig. 2. We see the drastic decrease of the radiation for all frequencies. We also see, that increasing the factor \( \gamma \) (i.e. decreasing mass for given energy) leads the maximum to be shifted further to the end-point of the spectrum (Fig. 3). Indeed, the classical maximum of radiation is at

\[ \omega_{cl} \sim \frac{1}{T} \frac{E^2}{m^2} \]

(4.3)

and for sufficiently big energies is beyond the end-point. This is the case when the recoil effects are most important.
Note that at $T \sim m/F$ the classical radiation maximum $\omega_{cl}$ of eq. (4.3) reaches $\omega_H$ - the classical radiation maximum for small (but not very small) time regime.

We see that the effects of interference are the biggest if the transverse motion is nonrelativistic. For $v_0 t \sim 1$ the interference effects are small (see Fig. [4]).

The latter equation can be differentiated in $T$ and then integrated over $\omega$ to obtain the total radiation reaction of the very small dipole:

$$
\frac{dE}{dT} = \frac{q^2}{\pi} T^2 E^2 \omega_0^2 \int_0^\infty dx \frac{1}{x(1+x)^3} \sin(xb) - b \times x \cos(b \times x))/b^2
$$

$$
- (\sin(xa) - a \times x \cos(a \times x))/a^2)((1 - J_0(xmd(T)))
$$

Here $a = ET(1 - v), b = ET(1 + v)$. The latter integral, contrary to the single particle case (see Appendix A), can’t be taken explicitly. In order to estimate this integral, it is worthwhile to get rid of oscillating terms using the the representation

$$
\frac{1}{(1+x)^3} = 0.5 \int_0^\infty p^2 \exp(-p),
$$

and the equations (B7) and (B8) from Appendix B. We obtain

$$
\frac{dE}{dT} = 0.5 \frac{q^2}{\pi} \int_0^\infty p^2 \exp(-p)(G_1(p, b, md(T)) - G_2(p, b, md(T)))
$$

(4.5)

The corresponding time dependence is given in Fig. 6 We put the Figure below the radiation reaction curve for the single particle Fig. 5. The radiation reaction decreases drastically due to interference.

Finally, note that for $T \sim m/F$ radiation reaction for $\chi \to \infty$ behaves like $1/\chi$, while for $\chi \to 0$ like $\sqrt{\chi}$.

Numerically, it is easy to see that the condition for the interference to decrease significantly the total radiation reaction, is that the radiation maximum must occur for the frequencies where the interference is still strong, i.e. this frequency $\omega_m$ is such that

$$
\omega_m E/(E - \omega_m)md(T) \leq 1
$$

(4.6)

It is easy to see that this condition is equivalent to

$$
E/m \leq T/d(T) \sim 1/v_t
$$

(4.7)

In other words, the interference decreases the dipole radiation by order of magnitude if it is nonrelativistic in it’s c.m. reference frame, while the interference influences the total radiation reaction only lightly if the dipole transverse motion is relativistic ($v_{0t} \sim 1$).

It is worthwhile to describe qualitatively the position of the radiation maximum and the structure of the spectral curve for different $\chi$. First, consider $\chi \ll 1$. In this case we see that at $T \sim E/m^2 \ll m/F$ the maximum of radiation will be near the end-point of the spectrum. The radiation itself will be negligible. For larger times the total radiation slowly increases,
while the radiation maximum moves to \( \sim E\chi \), where it reaches at times \( \sim m/F \). Then the curve smoothly transforms itself into the curve for small dipole that is studied in the next section. The radiation maximum does not move anymore. Afterwards radiation reaction quickly increases, while the interference diminishes. The reason why for the nonrelativistic transverse motion the radiation is still suppressed at times \( \sim m/F \) is that the suppression factor in the maximum is \( 1 - J_0(\nu_0 x_m) \).

For the opposite case \( \chi \gg 1 \) the situation quite different. There is a complete suppression of radiation and the radiation maximum remains near the end-point well into the small dipole regime (see next section).

The results for the radiation reaction are in correspondence with the situation with the total number of the photons. Without interference (see Fig.7) the total number of radiated photons remains finite, and has a maximum. Including the interference cuts off the soft photons and decreases the total number of photons drastically (see Fig. 8). The maximum of the number of radiated photons distribution, as is it is seen from the figure is parametrically located at the same frequencies as that of radiation reaction, i.e. most of radiated photons are hard.

In this discussion we did not take into account the important effects of Sudakov form-factors and wave function renormalization, that generally tend to cancel out. For the number of photons we expect these factors be more important than for the energy. Consequently, our discussion is just a conjecture that needs further calculation.

V. SMALL DIPOLE.

In the previous section we discussed the case of very small dipole, corresponding to \( T \ll m/F \). The goal of this section will be to consider the opposite limiting case \( T \gg m/F \). In the latter case we can substitute the integration limits by infinity and discard the terms due to integration by parts. We immediately obtain for spectral density:

\[
\frac{dE}{dT d\omega} = \frac{2q^2 m^2}{\sqrt{\pi} E^2} \omega^2 \left( \frac{1}{2} \int_a^\infty \Phi(u) du + \frac{1}{a} \frac{\partial \Phi(a)}{\partial a} \right) (1 - J_0(\omega' \theta(T)d(T)))
\]

where

\[
a = \left( \frac{\omega}{\omega_H} \right)^{2/3} \quad \omega_H = \omega_0 (E/m)^3
\]

\[
\omega' = \frac{\omega E}{E - \omega}
\]

Function \( \Phi \) is the standard Airy function (see Appendix B). This result is the single particle answer times the interference multiplier.

Let us first consider the case of \( \chi \ll 1 \). The corresponding graphs are depicted in Figs. 9,10,11. We put the graphs 10 and 11 under the graph 9, that corresponds to the spectral curve of the radiation reaction for a free particle. The graph 10 corresponds to time \( T \sim m/F \), while 11 for time \( T \sim E/F \). We see that for the first graph the interference is very strong, while for the second case it is only slight.
The opposite limiting case $\chi \gg 1$ is depicted in Figures 12,13,14,15. We see that for $T \sim m/F$ the interference decreases dramatically the radiation reaction, for $T \sim E/F$ the decrease is only slight (if fact we see very slight increase). Most interesting, we see that interference remains important numerically even at $T \sim (E/F^2)^{1/3}$, where we see that it decreases the maximum by the order of 1.5 and clearly significantly decreases the total radiation reaction (the spectral curve is still "dipole-like"). Moreover, the interference leads to the further shift of the radiation maximum to the end-point of the spectrum.

Let us now consider the total radiation reaction. Integrating eq. (5.1) we obtain:

$$\frac{dE}{dT} = \frac{2q^2}{\sqrt{\pi}} m^2 \int_0^\infty dx \frac{x}{(1+x)^3} (0.5 \int_u^\infty \Phi(u) + \Phi'(u)/u)(1 - J_0(x Ed(T)\theta(T)))$$  (5.4)

Where the function $\theta(T)$ is given by eq. (2.40) above. The expression in the case without interference differs slightly from the one for the single particle, since we did not do the usual integration by parts.

$$u = \frac{x^{2/3}}{\chi^{2/3}}$$  (5.5)

In order to understand qualitatively the influence of the dipole interference let us change the integration variable to $u$. Then

$$\frac{dE}{dT} = \frac{3q^2 \chi^2}{\sqrt{\pi}} m^2 \int_0^\infty du \frac{u^2}{(1+\chi u^{2/3})^3} ((0.5 \int_u^\infty \Phi(s)ds + \Phi'(u)/u)(1 - J_0(u^{3/2} \chi Ed(T)\theta(T)))).$$  (5.6)

The radiation reaction is the function of two parameters: the relativistic invariant $\chi$ and relativistic invariant $b(\tau)$ which is equal to

$$b(\tau) = E \theta(T)d(T) = m \frac{dd^2(\tau)}{d\tau}$$  (5.7)

for small dipole and

$$b(\tau) = md(\tau)$$  (5.8)

for the very small dipole. The radiation reaction can be written as

$$\frac{dE}{dT} = F(\chi, c(\tau))$$  (5.9)

where

$$c(\tau) = b(\tau) \chi.$$  (5.10)

Here $\tau$ is the proper time in the c.m. reference frame of the dipole.

In order to understand qualitatively the influence of the dipole interference consider two limits: $\chi \ll 1$ and $\chi \gg 1$. For $\chi \ll 1$ we can use the equation (5.6). For the time $T \gg m/F$ we can estimate:

$$dd^2(T)/dT \sim F^2 T^3/E,$$
and the argument of the Bessel function in eq. (5.6) is just $u^{3/2}(T/T_0)^3$, where $T_F = m/F$. We then put $\chi = 0$ in the denominator in the latter equation and obtain: Thus we obtain

$$dE/dT = -\frac{2 F^2 E^2}{3 \rho^4} G(T/T_F), \quad (5.11)$$

where the first term corresponds to the classical Pomeranchuk effect, and the function $G(T/T_F)$ is defined by

$$G(s) = \frac{9}{2 \sqrt{pi}} \int_0^\infty u^2 (0.5 \int_u^\infty (\Phi(u) + \Phi'(u)/u)(1 - J_0(u^{3/2}(T/T_F)^3))) \quad (5.12)$$

It is clear that for $T \gg T_0 \ G(T) \rightarrow 1$. The radiation reaction in this case is depicted in Fig.16, where we see, the sharp increase of radiation reaction at $T \sim T_F$. We see that it becomes weakly time dependent numerically at $T \sim E/F$.

For the opposite case $\chi \gg 1$ the main contribution in the integral of eq. (5.6) comes from the $u \rightarrow 0$. In this case we can put the argument in the Airy functions to zero, and use $\Phi'(0) \sim -0.5$. We then obtain

$$dE/dT = -\frac{q^2 m^2 \rho^{2/3} \Phi'(0)}{\pi} F(T/T^*), \quad (5.13)$$

where

$$F(T/T^*) = \int_0^\infty x^{1/3}/(1 + x)^3 (1 - J_0(x(T/T)^3)). \quad (5.14)$$

i.e. Here $T^* = (E/F^2)^{1/3}$.

Note that for $\chi \gg 1 \ E/F \gg T^* \gg T_F = m/F$. Thus the argument of the Bessel function $x$ is multiplied by a number less than 1. Consequently, as it is clear from the Figure 17, the radiation reaction remains negligible up to $T \sim T^*$. Afterwards, it increases rapidly, and at $T \sim E/F$ becomes approximately time-independent and a sum of radiation reactions of the components of the dipole.

In both cases the qualitative dependence on the parameter $\chi$ for the dipole is the same as for the single particle.

It is worthwhile, as in the previous section, to follow the position of the maximum of radiation. For small $\chi$ as we saw it does not change, and the interference in this regime is small. The interesting case is the case of the large $\chi$. We see that up to $T \sim T^*$ the spectrum is shifted to the end-point. This is in contrast with the single particle, where, as it seen from Fig. 12, (see also refs. [9,10]) the spectrum is concentrated near $\omega_m \sim 0.4E$. The strong interference in the maximum is the reason of the suppression of the total back-reaction. When $T \geq T^*$ the maximum, as it is clear from the Figures above, starts to move to the position of the maximum of the single particle, i.e. $0.4E$, where it comes by $T \sim E/F$, and the radiation reaction quickly increases.

Consider now the number of radiated photons in $q^2$ approximation of the perturbation theory. It is clear that this number is drastically decreased by interference. We see in Fig. 18 the number of emitted photons without the interference, while in Fig. 19 the number of emitted photons is shown with the interference taken into account. We see that the
interference qualitatively changes the spectrum: instead of being infinite in the limit of
soft photons ($\omega \to 0$), now the spectrum has no infrared singularity for soft frequencies.
Instead a number of radiated photons $\to 0$ at $\omega \to 0$ and has a finite maximum at finite
frequency. Moreover, it is clear that up to numerical coefficient of the order one, the position
of this maximum will be the same, as the frequency that corresponds to the maximum of the
radiation reaction. In particular, for ultra-relativistic dipole $\chi \gg 1$ the relevant frequency
will shift to the endpoint of the spectrum. The photons will take almost the entire energy
of the radiating electron in a single radiation event for the time interval $T^* \ll T \ll E/F$.
For $\chi \sim 1$ one radiating event will take $\sim 1/2$ of the initial energy of the electron.

Even more interesting effect will take place for higher times $T \sim E/F$ (and $\chi \gg 1$).
Since the soft photons will be cut by the dipole effects, the number of photons will have the
maximum for the finite frequencies. Numerical analysis shows that in this case

$$\omega_m \sim 2/T$$

(5.15)

This means that there are two distinct groups of radiating events for large $T$: radiation of
large numbers of soft photons with frequencies given by eq. (5.15) and the radiating events
where the dipole loses approximately half of it’s energy each time, this half being carried by
a photon.

VI. BACK-REACTION AND EVOLUTION OF THE VERY SMALL DIPOLE

We can answer now how the back-reaction influences the evolution of the very small
dipole, and where does the energy loss due to radiation goes: to the relative motion of the
particles in the center of mass or to the loss of the total energy of the center of mass motion.

Our results show, that for very single particle and for the very small times the back-
reaction force behaves according to eq. (A7) in the Appendix A. For the dipole the back-
reaction force behaves much less singular. Note that for very small times one can write

$$dE/dT = \frac{q^2a^3}{\pi(1-v)^2} \int_0^\infty ds (\sin(s) - s \cos(s)) \ast (1 - J_0(2v_0s))/(s(s+a)^3).$$

(6.1)

Here as in the previous sections $a = ET(1-v)$. The equation (6.1) can be used to obtain
the first several terms in the expansion of the back-reaction force for small $T$:

$$dE/dT = \frac{q^2T^3m^2F^2v_0^2}{2\pi E}(1 - \frac{3m^2T\pi}{4E}) + 0(T^5 \log (m^2T/E))$$

(6.2)

Recall that $v_0$ is the transverse velocity in the c.m. frame.

The back-reaction force for the dipole is smaller than for the single particle, logarithmic
terms are present only starting from $T^5$, and it’s leading term is proportional to $T^3$. The
condition for the applicability of the expansion (6.2) is

$$T \ll \min E/m^2, m/F.$$ 

(6.3)

Since the photons are radiated almost parallel to the direction of the dipole, the cor-
responding back-reaction force is directed in the direction opposite to the direction of the
dipole and leads to the decrease of it’s center of mass velocity.
There is an additional back-reaction force, that slows the expansion of the dipole in the orthogonal direction. For very small times regime this force is evidently \( \sim \sin^2 \theta dE/dT/p_t \), where \( \theta \sim m/E \) is the radiation cone angle, and \( p_t \sim v_0 m \) is the transverse momentum. Consequently the orthogonal component of the back-reaction force is

\[
F_y(T) \sim \frac{q^2 F^2 m^3 T^3 v_0 t}{E^2} + 0(T^4, T^5 \log (m^2 T/E)).
\]  (6.4)

The influence of this force on the wave packet radius begins only from the terms of \( \sim T^5 \), i.e. for very small \( T \) the expansion due to quantum diffusion and external field is dominant.

**VII. THE QUANTUM DIPOLE.**

In the latter analysis we did not take into account the quantum character of the dipole motion. In fact, there is an additional effect that influences the motion of dipole, and this is the noncoulombic photon exchange between the components of dipole. This effect is significant if the distance between the components of the dipole is less then \( 1/m \), and leads to so called quantum diffusion: the distance between the components of the dipole increases not linearly or quadratically as in the usual relativistic quantum mechanics, but in the diffusion way, i.e. as \( \sim \sqrt{T} \) [16] (a simple qualitative explanation of this phenomena is contained in ref. [27]). Moreover, the dipole motion along the coherence length may stop to be quasiclassical, as it was assumed throughout this paper [16].

It is easy to see that the effect is important for ultra-relativistic dipole with \( \chi \gg 1 \). Indeed, the diffusion is important till the distance between the components of dipole is \( \sim 1/m \), where \( 1/m \) is the scale of bound state in QED. In order to take into account the external field we need to write the wave functions in the external field taking into account quantum noncoulombic exchange of Fig. 20. This is beyond the scope of the current paper. Here we shall try to build a qualitative model to indicate the influence of the quantum effects. In order to estimate the field influence on the diffusion let us note that the diffusion law,

\[
d^2(T) = \frac{2T}{E},
\]  (7.1)

can be obtained from the equation

\[
\dot{y}/E = \eta(t),
\]  (7.2)

where \( \eta \) is the random external force such that:

\[
< \eta(t) \eta(t') > = E\delta(t - t').
\]  (7.3)

In order to include the external field, we generalize this equation in an obvious way:

\[
\ddot{y}(t) + \dot{y}/E = \eta(T) + F/E
\]  (7.4)

This equation can be easily solved with the result:
\[ d^2(T) = 4 \leq y^2(T) \leq 2T/E + 2F^2T^2/E^4 + O(T^3), \] (7.5)

and

\[ < y(T) > = FT/E^2 \quad < v_y(T) > = F/E^2 \] (7.6)

We see that quantum diffusion changes the velocity and the distance between charges. Average velocity is small \( F/E^2 \) and constant. Diffusion is important till the distance between dipole components is \( 1/m \), i.e. \( 2T/E \sim 1/m^2 \), or

\[ T \sim (E/m)(2/m) \] (7.7)

Note that for \( \chi \gg 1 \) this time is \( \gg m/F \). Also note that for all reasonable times \( \leq E/F \) the first term in eq. (7.5) is dominant.

Consider now the interference in the case of diffusion. We need to average the product \( \sin (\theta)d \). There are two possibilities. If the angle between the direction of a component of the dipole and z axis is much bigger than \( m/E \), we use \( \sin \theta \sim v_y \). Then we need to average

\[ < v_y d(T) > = \frac{dd(T)^2}{dT} = 2/E, \] and we get as the argument of the Bessel function \( 2x = 2\omega/E \) (\( \omega'/E \) if we also take into account recoil effects) . The interference multiplier will be

\[ (1 - J_0(2x)). \] (7.8)

If the angle is \( m/E \) we shall get the argument \( \omega m < d(T) > /E \sim 2\omega mFT/E^3 \). We must choose the biggest of two arguments. It is easy to see that the first argument will be bigger up to times \( T \sim E^2/(mF) = (E/F)(E/m) \), i.e. for all times where the dipole notion has sense. Thus, in our simple model, in the diffusion regime, which lasts parametrically longer, as \( \chi \) increases, the interference depends on time only weakly.

For small \( \chi \) the diffusion law holds only for very small times \( \ll m/F \). Thus the biggest influence seems to occur for \( \chi \geq 1 \), when the interference multiplier may significantly change the radiation.

We have developed above a simple phenomenological model indicating are the effects connected with the quantum character of the dipole motion. Unfortunately, at the moment we can, on the basis of this model, only indicate, that they may be very important for large \( \chi \) and that they lead to the suppression of the dipole radiation , as in the quasi-classical dipole.

The reason of the difficulties we encounter is the inadequacy of the classical approximation. It is possible to estimate the area of reliability of the quasi-classical approximation: we must demand that the transverse velocity acquired in the classical approximation due to the action of the external field is bigger than the velocity due to the quantum diffusion. Quite, remarkably, this gives us the condition \( T \geq T^* \), i.e. the quantum diffusion effects are important for large \( \chi \) up to the scale, when quasi-classically radiation suppression stops, and the radiated energy quickly increases as we saw in the Chapter 4. This result is consistent with the conclusion above, that in the time interval when the quantum effects in the dipole motion are important, there is still a suppression of the radiation reaction (the charge transparency). The origin of the difficulty is clear. The parameter \( \chi = \omega c/l_F \), where \( \omega \sim E/m^2 \) is the coherence length, while \( l_F \sim m/F \) is the field regeneration length, in other words the
average distance the dipole must travel before colliding with the external field photon. It is clear that the external field does not break coherence. Thus we are in the situation when we have multiple coherence conserving collisions along the coherence length. In this situation it is well known (see e.g. ref. [21]) that the classical approximation is, generally speaking, not applicable. The quasiclassical approximation corresponds to neglecting the coherence conservation and thus can lead to the wrong results. The further analysis along the lines of ref. [21] is needed.

**VIII. CONCLUSION**

We have studied the back-reaction and its influence on the evolution of the relativistic dipole in the arbitrary strong external field using the quasi-classical approximation. We have taken into account the quantum recoil effects in radiation, but not quantum effects in the motion of the dipole, i.e. the quantum diffusion. We found that the dipole motion is governed by two invariant parameters, one of them describes the longitudinal motion and is equal to $\chi = EF/m^3$, another describes the motion in the transverse plane and is equal to

\[ b(\tau) = md(\tau) \quad \text{if} \quad T \ll m/F \]

\[ b(\tau) = m \frac{dd^2(\tau)}{dT} E/F \gg T \gg m/F. \]

It is quite possible that there exists a single formula for $b$, although we were not able to obtain it.

We have studied the pattern of charge transparency in the external field. We have found that the interference effects can be taken into account by the use of the general interference multiplier $1 - J_0(xb(\tau))$, where $x = \omega/(E - \omega)$. For the arbitrary times the radiation reaction is given by eq. (2.39).

We have seen that there are three different time scales. First, the very small dipole regime, $T \ll m/F$. This time scale exists if the dipole transverse velocity $v_0 \ll 1$. In this regime the radiation reaction is strongly suppressed by interference, leading to the strong decrease of the back-reaction, i.e. the fast moving dipole does not lose its energy. In this case we were able to calculate analytically the back-reaction force analytically for both the entire regime, and for very small times (eq. (6.2)). For larger time scales $E/F \gg T \gg m/F$ the influence of interference on back-reaction depends on the value of the parameter $\chi = EF/m^3$. If $\chi \ll 1$, the radiation reaction quickly increases starting from $T \sim m/F$, and by the time $T \sim E/F$ it is a sum of radiation reaction of the components of the dipole (see Fig. 16). However for the opposite case $\chi \gg 1$, the radiation reaction starts to increase only from the times $T \sim T^* = (E/F^2)^{1/3}$, and then once again goes quickly to the sum of the radiation reactions of the components.

For the third regime $T \gg E/F$ the components of the dipole can be considered as independent particles. For each of the regimes we obtained the analytical expressions for both the spectral distribution of the radiation and the total back-force. The results are qualitatively shown in Figures 1-20.

The results for the radiation reaction are in the correspondence with the influence of the interference on the number of radiated photons and on the scattering cross-sections.
These physical quantities are qualitatively influenced by interference up to times $\sim E/F$. Without the interference the number of photons is maximum (infinite) at $\omega \to 0$. As a result of the interference the number of photons goes to zero when $\omega \to 0$. The number of photons distribution has maximum at $\omega^* \sim \gamma/d(T)$ for the very small dipole regime. I.e. for ultra-relativistic dipole the maximum of the number of radiated photons shifts to the end-point of the spectrum.

For larger times (small dipole regime) the maximum in the number of the radiated photons is finite and parametrically lies at the same frequencies as the maximum of the radiation reaction $-E\chi$ for $\chi \ll 1$, 0.4E for $\chi \gg 1$. Moreover, we have seen that for $\chi \gg 1$ the radiated particles carry $\sim 1/2$ of the dipole energy for arbitrary times $T \gg E/F$, when the particles move as the independent ones. There are two distinct maximums and two groups of photons. One group is responsible for the energy loss, and it’s spectral curve maximum is at $\omega \sim 0.4E$ for large $\chi$. Another group is the soft photons, responsible for a total number of photons emitted (and they may give the main contribution into the cross-sections). These photons in the regime under discussion are the soft ones, with the maximum of the spectral curve located at $\omega \sim 2/T$ for large $\chi$.

It is also interesting to summarize the behaviour of the radiation spectrum for different $\chi$. For $\chi \ll 1$, the relevant maximum lies near end-point if $T \leq E/m^2$, but the radiation is strongly suppressed. However if it occurs, the dipole will be immediately destroyed, since photon takes all of it’s energy. Then it moves to $E\chi$ by the time $m/F$, and only afterwards the radiation begins to increase.

In the opposite case $\chi \gg 1$ the maximum is near end-point till $T \sim T^*$, and only then begins to move to saturation 0.4E, that corresponds to the single particle maximum. For $T \leq T^*$ the radiation is suppressed, but if occurs it destroys the dipole (photon carries it’s entire energy). For the times $T \gg E/F$ the radiation reaction is a sum of the component radiation events, and at each radiation event on average half of the electron energy is taken by the photon.

We have seen, that our results, although they were obtained for the simple model of the constant transverse field, can be reformulated in the model independent way. The parameter $\chi = l_c/l_F$ is the ratio of the coherence and the field generation length. The very small dipole regime corresponds to the situation when the dipole travels the distance less than $l_F$. There exists the charge transparency in this region independently of $\chi$. However for large times the parameter $\chi$ starts to play an important role. If $\chi \ll 1$, (this is the situation considered in refs. [16,17,19]) one can see the external field as a small perturbation. The region of the quantum diffusion is small (since $l_c \ll l_F$) and charge transparency (i.e. radiation suppression) continue up to $T \sim m/F$, well beyond the quantum diffusion range. However, once $\chi \geq 1$, we are in a completely different situation. In the quasi-classical approach we have here the situation quite similar to the Landau-Pomeranchuk effect in the single particle dynamics of the fast particle moving through the amorphous media. The radiation reaction continues to be suppressed even after the field regeneration time, thus extending parametrically the charge transparency interval to times $T \sim T^*$. However, as it was noted in the previous section, the area of $T \leq T^*$ must be studied beyond the quasi-classical approximation, since we must take into account the multiple coherent collisions. We expect that the radiation will still be strongly suppressed in this time interval, but further analysis is needed to make the qualitative statements, and to compare the results with those from
the quasi-classical approach.

The main possible drawback of our paper is the validity of the quasiclassical approximation. For the $\chi \leq 1$ the quasiclassical approximation works for all times larger than $T_c \sim E/m^2 \leq T_F$. For $\chi \geq 1$ for dipole one may expect the significant corrections to quasiclassical approximation for all times in light of the results of ref. [16] (see also the previous section). Nevertheless the quasiclassical analysis is still important as a first step to understand the radiation patterns in this regime of parameters.

Our work certainly makes a number of questions open. First, this is the influence of the quantum effects in the dipole motion on the radiation reaction. This is important for the study of the quantum dipole. We have seen that such effects for the dipole may be much more important than for a single particle, and may require the analysis beyond the quasi-classical approximation due to the coherent multiple scattering.

Second, it will be interesting to study further the dependence of the number of the radiated photons on $\chi$, in particular taking into account the multiple photon radiation. Our results suggest that, since the electrons are born in pairs, i.e. as dipole, the infrared photons are always cut off, and the evolution goes on by a series of radiative events, such that in each of these events the electron loses approximately half of its energy. This is true at least if $\chi \gg 1$, i.e. the dipole is ultra-relativistic or the field is very strong. This is opposite to the scenario when the fast electron loses its energy by radiating the soft photons, with small energies loss in each of the radiating events. This result can be important for carrying the next-to-leading order logarithmic calculations. The results of this paper imply, roughly, that such dipole moves till times $T^*$ without radiation, then after the transition period (up to $T \sim E/F$), starts to radiate losing at each event $\sim 1/2$ of its energy.

Moreover, the numerical analysis shows, that for large times there are two parallel process for ultra-relativistic dipole. First, it emits soft photons. The maximum of the photon number distribution for large $\chi$, as numerical analysis of eq. (4.1) shows, lies at

$$\omega \sim 2/T, \ T \gg E/F. \quad (8.1)$$

These photon numbers make significant contribution to cross-sections. But the energy loss of the dipole goes via the series of different events, when $\sim 0.4E$ is lost in each event, and the relevant photons are hard for the ultra-relativistic dipole.

It will be interesting to understand if the different regimes of radiation discussed in this paper are connected with the theory of the production of the $e^+ - e^-$ pairs by fast particle in the external field discussed in ref. [28].

Finally, it will be interesting to study the implications of our results for QCD. In particular, our results are clearly relevant to the studies of the colour transparency phenomena, first discussed in refs. [16,17,19]. As it was noted in the introduction for the case of the deep-inelastic scattering on the longitudinal photons the charge transparency is directly translated into the color transparency [20]. Our results give qualitative bounds on the color transparency for the arbitrary external fields, and indicate the direction of the research one needs to extend the color transparency to the case of the arbitrary external field.

It will be especially interesting to extend our results to the gluon color dipole radiation, since then the shift of the spectrum to the end-point will mean that the dipole loses all its energy by a single radiative event for small time. It will also imply that the color
dipole loses it’s energy by a series of events in each of them the gluon looses half of it’s energy. Note however that the extension to color dipole is nontrivial since the mass of the gluon is zero, and we need the additional regularization. Moreover, the definition of QCD dipole is slightly different then the one in this paper. In this paper dipole is a system of two oppositely charged particles with interference, QCD dipole is only a quantum dipole, i.e. the times considered are always less than the time interval that corresponds to the coherence length. For such case, as we saw above there may be significant corrections to the quasiclassical approximation that need further study. Nevertheless our results imply that the recoil effects may be very important also for the color dipole, i.e. for the small x deep-inelastic scattering.
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APPENDIX A: RADIATION REACTION FOR SINGLE QUANTUM PARTICLE FOR SMALL TIMES (SMALL DEFLECTION ANGLES).

Although the article is devoted to the radiation reaction of a dipole, in this section we shall discuss the radiation reaction of a single particle for the small times, taking into account recoil effects. Although such problem may look un-physical, since charged particles are created by pairs, exactly the same problem appears if the particle goes through the line of the constant external field of the length \( L \leq m/F \). The purely classical case was discussed by Landau and Lifshits [1]. However we were not able to find the quantum case in the literature.

We start from eq. (2.10) for total energy radiated by a single particle during the time interval \( T \). Using the approximation of eqs. (2.38) and (2.39) we see that the radiation reaction is the sum of three terms: the term proportional to \( 1 - v^2 \), the term proportional \( \omega_0^2 \), and the term due to the integration by parts. Note that all cubic terms in the arguments are negligible and thus can be omitted. The terms that arise due to the integration by parts do not depend on the external field, up to the terms additionally suppressed as \( m^2/E^2 \), and are the same as for the free particle. So only the term proportional to \( \omega_0^2 \) remains. It is straightforward to find:

\[
\frac{dW}{d\omega} = \frac{\omega_0^2 q^2}{2\pi} \int_0^T \int_0^T dsds' \int_{-1}^1 d\cos\theta (s-s')^2 \cos(\omega(s-s') \cos\theta) \exp(i(\omega v(s-s') \cos\theta)).
\]  

(A1)

The latter triple integral can be easily taken. We obtain

\[
\frac{dW}{d\omega} = \omega_0^2 \frac{q^2}{v\pi} \frac{2(1 - \cos(\omega(1+v)T) - \omega(1+v)T \sin(\omega(1+v)T)}{\omega^2(1+v)^3)}
\]

\[
+ \omega_0 \frac{q^2}{\pi} \frac{2(1 - \cos(\omega(1-v)T) - \omega(1-v)T \sin(\omega(1-v)T)}{\omega^2(1-v)^3)}.
\]

(A2)

This is the classical formulae. the recoil is taken by first rewriting:

\[
\frac{d\omega}{\omega^2} = \frac{\omega d\omega}{\omega^3}.
\]

Then we need to rescale \( \omega \) in the r.h.s. of eq. (A2), except in the product \( \omega d\omega \), as discussed in the text:

\[
\omega \rightarrow \omega' = \omega E/(E - \omega)
\]

(A3)

We obtain:
\[
\frac{dW}{d\omega} = -\omega_0^2 q^2 \frac{2(1 - \cos (\omega'(1 + v)T) - \omega'(1 + v)T \sin (\omega'(1 + v)T)}{\omega'(1 + v)^3} \\
+ \omega_0^2 q^2 \frac{2(1 - \cos (\omega'(1 - v)T) - \omega'(1 - v)T \sin (\omega'(1 - v)T)}{\omega'(1 - v)^3}.
\] (A4)

The typical spectral curve is depicted in Fig 1. (In the figure we added also the contribution of the term proportional to \((1 - v^2)\)).

It is straightforward to integrate the latter equation over \(\omega\) from 0 to \(E\) (for this we change the integration variable to \(y = \omega/(E - \omega)\)). After trivial integration we obtain:

\[
\frac{dE}{dT} = \frac{q^2}{\pi (1 - v)^2} \omega_0^2 \left((1 - v)^2/(1 + v)^2\right) (-b + Ci(b)(b \cos b - \sin(b) + b^2 \sin(b)/2 - b^3 \cos(b)/2) \\
+ \pi b(-1/2 + b^2/4) \sin b + \pi(-1/2 + b^2/4) \cos b \\
- (-a + Ci(a)(a \cos a - \sin(a) + a^2 \sin(a)/2 - a^3 \cos(a)/2) \\
+ Si(a)(\cos(a) - a \sin a - a^2 \cos(a)/2 - a^3 \sin(a)/2) + \\
+ \pi a(-1/2 + a^2/4) \sin a + \pi(-1/2 + a^2/4) \cos(a)).
\] (A5)

Here

\[
a = (1 - v)ET \sim m^2T/(2E) \quad b = (1 + v)ET \sim 2ET.
\] (A6)

The corresponding typical radiation reaction curve is depicted in Fig. 5.

Expanding the latter equation in powers in \(T\) we see that the back-reaction force for small times is very small:

\[
\frac{dE}{dT} = -\frac{q^2 m^2 F^2 T^3}{6\pi E} \log(m^2T/2E) + \gamma + 1 + O(T^4),
\] (A7)

where \(\gamma \sim 0.55\) is the Euler constant. This force is directed against the direction of the particle.

The latter equation works for the whole range of \(T \leq m/F\) if \(\chi \gg 1\). For \(T \sim m/F\), \(\chi \ll 1\), the parameter \(a \sim 1/\chi \gg 1\) and we have to use the whole equation (A5) in the limit \(a \gg 1\). For \(\chi \gg 1\) the expansion parameter \(a \sim m^2T/E \sim 1/\chi\) and is still small for \(T \sim m/F\). Then the back-reaction force is for this time scale:

\[
F_{b.r.} \sim \frac{q^2 m^5}{6\pi EF} \log(E^2/m^2) = \frac{q^2 m^2}{6\pi \chi} \log 1/\chi.
\]
We see that the back-reaction is strongly suppressed for single particle in the ultra-relativistic case.

Finally, let us make a comment on the discarded terms proportional to $(1 - v^2)$, and those originated from integration by parts. It may be strange from the first sight that these terms really exist, since they are nonzero for a particle moving with constant velocity and finite mass, i.e. a particle that does not emit any radiation field. In fact this is situation usual in quantum mechanics and quantum field theory. Indeed, when we calculate the transition rate due to photon radiation in standard perturbation theory between stationary states we encounter multiplier

$$\sin^2(E_f - E_i - \omega)T/T(E_f - E_i - \omega)$$

(A8)

For infinite T this term gives a delta function $\delta(E_f - E_i - \omega)$, insuring the law of the energy conservation. However for finite T and non-finite $\Delta E = E_f - E_i - \omega$ this will be a function of T decreasing as a function of T for fixed $\Delta E$. This decrease, as it is well known, just expresses the energy uncertainty principle. If we consider the system for finite time, the energy can’t be measured unambiguously: $\Delta E T \geq \hbar$. The discarded terms in the radiation reaction have exactly the same origin and the same character. They decrease for large T as $1/T$ or faster and thus disappear at infinite T altogether. They exhibit the ambiguity in the measurement of the electromagnetic field of the free particle due to the finite time of our process. In practice this leads to the finite width of spectral lines for finite time. It is interesting to study these terms in more detail in connection with the Landau-Pierels inequalities [29]. However, it is clear from above, that these terms must be discarded if we are interested in the radiation in external field. In other words, all quantum calculations must contain renormalization, meaning that a free particle does not radiate.

APPENDIX B: SOME USEFUL INTEGRALS AND THEIR PROPERTIES

Here we shall collect together some useful integrals and asymptotic expansions, as given in refs. [22–25]. We shall also collect the definitions of several special functions that differ in the literature by normalization constants. We use the following integrals, directly expressible through Airy functions:

$$\frac{1}{\sqrt{\pi}} \int_0^\infty ds \sin (a s + s^3) = -\frac{d}{da} \text{Ai}(a).$$

(B1)

$$\int_0^\infty ds \frac{1}{\sqrt{\pi}} \sin (a s + s^3) / s = -\int_a^\infty dz \text{Ai}(z).$$

(B2)

Here $\text{Ai}(z)$ is an Airy function [22,23]:

$$\int_0^\infty ds \frac{1}{\sqrt{\pi}} \cos (a s + s^3/3) = \text{Ai}(a).$$

(B3)

Note that Airy function decreases as $\sim \exp (-z^{3/2})/z^{1/4}$ for the positive $z \to \infty$. 

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We use integral of the Airy function:

\[
\int_0^\infty z^{b-1} \text{Ai}(z) dz = 3^{(4b-1)/6-1} \Gamma(a/3) \Gamma((a+1)/3). \tag{B4}
\]

We define the Integral sinus and cosinus as:

\[
\text{Si}(x) = \int_0^x \frac{\sin(x)}{x}, \tag{B5}
\]

\[
\text{Ci}(x) = -\int_x^\infty \frac{\cos(x)}{x}, \tag{B6}
\]

While studying the dipole radiation we used some formulae for the integrals of Bessel functions \[26\]. We use

\[
G_2(p, a, b) = \int_0^\infty \exp(-px) \sin(bx) J_0(\alpha x) / x = \arcsin(2b/r), \tag{B7}
\]

\[
G_1(p, a, b) = \int_0^\infty \exp(-px) \cos(bx) J_0(\alpha x)
= \frac{1}{\sqrt{p^2 + (b+a)^2} \sqrt{p^2 + (b-a)^2} \sqrt{(r^2/4 - b^2)}}. \tag{B8}
\]

Here

\[
r = \sqrt{(b + a)^2 + p^2 + \sqrt{p^2 + (b - a)^2}}. \tag{B9}
\]
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FIG. 1. The spectral curve for the spectral distribution of the radiated energy of the single particle $\frac{dW}{d\omega}$ (normalised by $q^2/\pi$) versus the photon energy $\omega$ for a single particle. The velocity $v=0.99$, $E=100\ \text{GeV}$, $m=14\ \text{GeV}$, The field $F=100\ \text{GeV}^2$, $T=0.1\ \text{GeV}^{-1}$.

FIG. 2. Spectral curve as a function of $\omega$ for the very small time regime $T \leq m/F$. The parameters are the same as in the previous picture: $E=100\ \text{GeV}$, $F=100\ \text{GeV}^2$, $v=0.99$. The transverse velocity in the c.m. frame of the dipole is $v_{t0}=0.2$.

FIG. 3. Spectral curve as a function of $\omega$ for the very small time regime $T \leq m/F$. The parameters are: $E=100\ \text{GeV}$, $F=100\ \text{GeV}^2$, $v=0.999$ ($\gamma=0.04$). The transverse velocity in the c.m. frame of the dipole is $v_{t0}=0.2$. 
FIG. 4. Spectral curve as a function of $\omega$ for the very small time regime $T \leq m/F$. The parameteres are the same as in the Fig. [2]: $E=100$ GeV, $F=100$ GeV$^2$, $v=0.99$. The transverse velocity in the c.m. frame of the dipole is $v_0t = 0.9$.

FIG. 5. Radiation Reaction of the single particle as a function of $a = m^2 T/2E$ for very small time regime $T \leq m/F$. For the picture $m^2/2E = 1$ GeV, $E=100$ GeV, $F=100$ GeV$^2$. The radiation reaction $dE/dT$ is normalized by $q^2/\pi$.

FIG. 6. Radiation Reaction of the dipole as a function of $a = m^2 T/2E$ for very small time regime $T \leq m/F$. For the picture $m^2/2E = 1$ GeV, $E=100$ GeV, $F=100$ GeV$^2$. The radiation reaction $dE/dT$ is normalized by $q^2/\pi$, and $v_0t = 0.2$. 
FIG. 7. The spectral curve for the spectral distribution of the number of the radiated photons for the single particle \( \frac{dN}{d\omega} \) (normalized by \( q^2/\pi \)) versus the photon energy \( \omega \) for a single particle. The velocity \( v=0.99, \ E=100 \ \text{GeV}, \ m=14 \ \text{GeV}, \) The field \( F=100 \ \text{GeV}^2, \ T=0.1 \ \text{GeV}^{-1}. \)

FIG. 8. Spectral curve of a number of the radiated photons as a function of \( \omega \) for the very small time regime \( T \leq m/F. \) The parameters are the same as in the previous picture: \( E=100 \ \text{GeV}, \ F=100 \ \text{GeV}^2, v=0.99. \) The transverse velocity in the c.m. frame of the dipole is \( v_0 = 0.2. \)

FIG. 9. Spectral curve as a function of \( x = \omega/E \) for a single charged particle. We choose here \( F=100 \ \text{GeV}^2, \ E=100 \ \text{GeV}, \ v=0.8, \ \chi = 0.04. \) (All graphs for this and 6 figures below depict \( dE/(dTd\omega), \) normalized by \( q^2/\sqrt{\pi}. \))
FIG. 10. Spectral curve as a function of \( x = \omega / E \) for the dipole in the small time regime. \( F, E, v \) are the same as for the previous Figure, \( \chi = 0.04, T = m / F = 25 \text{ GeV}^{-1} \).

FIG. 11. Spectral curve as a function of \( x = \omega / E \) for the dipole in the small time regime. \( F, E, v \) are the same as for the previous Figure, \( \chi = 0.04, T = E / F = 100 \text{ GeV}^{-1} \).

FIG. 12. Spectral curve as a function of \( x = \omega / E \) for a single charged particle. We choose here \( F = 100 \text{ GeV}^2, \ E = 100 \text{ GeV}, \ v = 0.999, \ \chi = 111.6 \).
FIG. 13. Spectral curve as a function of $x = \omega/E$ for the dipole in the small time regime. $F,E,v$ are the same as for the previous Figure, $\chi = 111$, $T = m/F = 0.045$ GeV$^{-1}$.

FIG. 14. Spectral curve as a function of $x = \omega/E$ for the dipole in the small time regime. $F,E,v$ are the same as for the previous Figure, $\chi = 111$, $T = (E/F^2)^{1/3} = 0.21$ GeV$^{-1}$.

FIG. 15. Spectral curve as a function of $x = \omega/E$ for the dipole in the small time regime. $F,E,v$ are the same as for the previous Figure, $\chi = 111$, $T = E/F = 100$ GeV$^{-1}$.
FIG. 16. Radiation Reaction of the dipole as a function of $T$ GeV$^{-1}$ for small time regime $E/F \geq T \geq m/F$. For the picture $\chi = 0.04$, $E=100$ GeV, $F=100$ GeV$^2$. The radiation reaction $dE/dT$ is normalized by $q^2/\sqrt{\pi}$.

FIG. 17. Radiation Reaction of the dipole as a function of $T$ GeV$^{-1}$ for small time regime $E/F \geq T \geq m/F$. For the picture $\chi = 111.4$, $E=100$ GeV, $F=100$ GeV$^2$, $T^* \sim 0.22$ GeV$^{-1}$. The radiation reaction $dE/dT$ is normalized by $q^2/\sqrt{\pi}$.

FIG. 18. Number of radiated photons in the small dipole regime as a function of frequency: no interference. (Normalized by $q^2/\sqrt{\pi}$.)
FIG. 19. Number of radiated photons in the small dipole regime. Interference is taken into account. (Normalized by $q^2/\sqrt{\pi}$).

FIG. 20. The graphs with the noncoulombic photon exchange that lead to quantum diffusion.