ZERO DISTRIBUTION OF RANDOM POLYNOMIALS

By

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Abstract. We study global distribution of zeros for a wide range of ensembles of random polynomials. Two main directions are related to almost sure limits of the zero counting measures and to quantitative results on the expected number of zeros in various sets. In the simplest case of Kac polynomials, given by the linear combinations of monomials with i.i.d. random coefficients, it is well known that under mild assumptions on the coefficients, their zeros are asymptotically uniformly distributed near the unit circumference. We give estimates of the expected discrepancy between the zero counting measure and the normalized arclength on the unit circle. Similar results are established for polynomials with random coefficients spanned by different bases, e.g., by orthogonal polynomials. We show almost sure convergence of the zero counting measures to the corresponding equilibrium measures for associated sets in the plane and quantify this convergence. In our results, random coefficients may be dependent and need not have identical distributions.

1 Introduction

Zeros of polynomials of the form \( P_n(z) = \sum_{k=0}^{n} A_k z^k \), where \( \{A_n\}_{k=0}^{n} \) are random coefficients, have been studied by Bloch and Pólya, Littlewood and Offord, Erdős and Offord, Kac, Rice, Hammersley, Shparo and Shur, Arnold, and many other authors. The early history of the subject with numerous references is summarized in the books by Bharucha-Reid and Sambandham [10] and by Farahmand [12]. It is well known that, under mild conditions on the probability distribution of the coefficients, the majority of zeros of these polynomials accumulate near the unit circumference, being equidistributed in the angular sense. Introducing modern terminology, we call a collection of random polynomials \( P_n(z) = \sum_{k=0}^{n} A_k z^k, \ n \in \mathbb{N} \), the ensemble of Kac polynomials. Let \( \{Z_k\}_{k=1}^{n} \) be the zeros of a polynomial \( P_n \) of degree \( n \), and define the zero counting measure

\[ \tau_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{Z_k}. \]

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The equidistribution of the zeros of random polynomials can now be expressed via the weak convergence of $\tau_n$ to the normalized arclength measure $\mu_\mathbb{T}$ on the unit circumference, where $d\mu_\mathbb{T}(e^{it}) := dt/(2\pi)$. Namely, $\tau_n \xrightarrow{w} \mu_\mathbb{T}$ with probability 1 (abbreviated as a.s. or almost surely). More recent work on the global distribution of zeros of Kac polynomials includes papers of Hughes and Nikeghbali [17], Ibragimov and Zeitouni [18], Ibragimov and Zaporozhets [19], and Kabluchko and Zaporozhets [20, 21]. In particular, Ibragimov and Zaporozhets [19] proved that if the coefficients are independent and identically distributed, then the condition $\mathbb{E}[\log^+ |A_0|] < \infty$ is necessary and sufficient for $\tau_n \xrightarrow{w} \mu_\mathbb{T}$ almost surely. Here, $\mathbb{E}[X]$ denotes the expectation of the random variable $X$.

The majority of available results require the coefficients $\{A_k\}_{k=0}^n$ to be independent and identically distributed (i.i.d.) random variables. This assumption is certainly natural from a probabilistic point of view. However, it is not necessary, as the following simple example shows. If $A_k = \tilde{\xi}$, $k = 0, 1, 2, \ldots$, are identical (hence dependent), where $\tilde{\xi}$ is a complex random variable, then we are dealing with the family of polynomials

$$P_n(z) = \sum_{k=0}^n \tilde{\xi}^k = \tilde{\xi} \frac{z^n+1}{z-1}, \quad n \in \mathbb{N}.$$  

The zeros of $P_n$ are uniformly distributed on $\mathbb{T}$, being the $n + 1$-st roots of unity except $z = 1$. Furthermore, $\tau_n \xrightarrow{w} \mu_\mathbb{T}$ almost surely, provided $\tilde{\xi}$ does not vanish with positive probability. The assumption of identical distribution for coefficients is not necessary for $\tau_n \xrightarrow{w} \mu_\mathbb{T}$ a.s. either. Thus one of our main goals is to remove unnecessary restrictions and prove results on zeros of polynomials whose coefficients need not have identical distributions and may be dependent.

Another interesting direction is related to the study of zeros of random polynomials spanned by various bases, e.g., by orthogonal polynomials. These questions were considered by Shiffman and Zelditch [29]-[31], Bloom [6] and [7], Bloom and Shiffman [9], Bloom and Levenberg [8], Bayraktar [4], and others. It is of importance for us that many of the papers mentioned used the potential theoretic approach to study the limiting zero distribution, including that of multivariate polynomials. We develop such ideas here and extend them to more general bases and classes of random coefficients, but only for the univariate case.

We do not discuss the local scaling limit results on the zeros of random polynomials, as this falls beyond the scope of the paper. Instead, we direct the reader to recent interesting papers on this topic by Tao and Vu [36] and by Sinclair and Yattselev [32].
The rest of our paper is organized as follows. Section 2 deals with almost sure convergence of the zero counting measures for polynomials with random coefficients that satisfy only weak log-integrability assumptions. Section 3 develops the discrepancy results of [25] and [26] and establishes expected rates of convergence of the zero counting measures to the equilibrium measures. Again, the random coefficients in Section 3 are neither independent nor identically distributed, and their distributions satisfy only certain uniform bounds for the fractional and logarithmic moments. We also consider random polynomials spanned by general bases in Sections 2 and 3, which include random orthogonal polynomials and random Faber polynomials on various sets in the plane. All proofs are given in Section 4.

2 Asymptotic equidistribution of zeros

We first study the limiting behavior of the normalized zero counting measures for sequences of polynomials of the form

\[ P_n(z) = \sum_{k=0}^{n} A_k z^k, \quad n \in \mathbb{N}. \]

Let \( A_k, k = 0, 1, 2, \ldots \), be complex-valued random variables that are neither necessarily independent nor required to be identically distributed. Recall that the distribution function of \(|A_k|\) is defined by \( F_k(x) = \mathbb{P}(|A_k| \leq x), \ x \in \mathbb{R} \): see Gut [16, Section 2.1]. We use the following assumptions on random coefficients in this section.

**Assumption 1.** The \( A_k \) are distributed such that there exist \( N \in \mathbb{N} \) and a decreasing function \( f : [a, \infty) \to [0, 1], \ a > 1 \), such that

\[
\int_{a}^{\infty} \frac{f(x)}{x} \, dx < \infty \quad \text{and} \quad 1 - F_k(x) \leq f(x) \quad \text{for all} \quad x \in [a, \infty)
\]

for all \( k = 0, 1, 2, \ldots \).

**Assumption 2.** The \( A_k \) are distributed such that there exist \( N \in \mathbb{N} \) and an increasing function \( g : [0, b] \to [0, 1], \ 0 < b < 1 \), such that

\[
\int_{0}^{b} \frac{g(x)}{x} \, dx < \infty \quad \text{and} \quad F_k(x) \leq g(x) \quad \text{for all} \quad x \in [0, b]
\]

for all \( k = 0, 1, 2, \ldots \).

If \( F(x) \) is the distribution function of \(|X|\), where \( X \) is a complex random variable, then

\[
\mathbb{E}[\log^+ |X|] < \infty \quad \iff \quad \int_{a}^{\infty} \frac{1 - F(x)}{x} \, dx < \infty, \ a \geq 0,
\]

and
\[ \mathbb{E}[\log^{-} |X|] < \infty \iff \int_0^b \frac{F(x)}{x} \, dx < \infty, \quad b > 0; \]
see, e.g., Gut [16, Theorem 12.3, p. 76]. Hence, when all random variables
\(|A_k|, \quad k = 0, 1, \ldots, \)
are identically distributed, one can state (2.1)-(2.2) in the more compact equivalent form
\[ \mathbb{E}[|\log |A_0||] < \infty. \]
Condition (2.2) readily implies that \(\mathbb{P}(|A_k) = 0\) = 0 for all \(k\), i.e., the probability measures of the coefficients cannot have point masses at 0. But they can have point masses elsewhere, and need not possess densities.

Schehr and Majumdar [28] considered random polynomials with Gaussian coefficients \(A_k\) that have mean zero and variance \(\sigma^2_k = e^{-\alpha k}\) and found that the expected number of real zeros for \(P_n(z) = \sum_{k=0}^n A_k z^k\) is asymptotic to \(n\) for \(\alpha > 2\). Thus almost sure equidistribution of zeros near the unit circumference can clearly fail in absence of uniform assumptions on coefficients.

We show in Lemma 4.2 that if both (2.1) and (2.2) hold, then
\[ \lim_{n \to \infty} |A_0|^{1/n} = \lim_{n \to \infty} |A_n|^{1/n} = \lim_{n \to \infty} \max_{0 \leq k \leq n} |A_k|^{1/n} = 1 \quad \text{a.s.} \]
These facts allow us to apply potential theoretic techniques, which were developed to study the asymptotic zero distribution of deterministic polynomials (see Andrievskii and Blatt [2] for an overview). We start with the following result for the Kac ensemble.

**Theorem 2.1.** If the coefficients of \(P_n(z) = \sum_{k=0}^n A_k z^k\), \(n \in \mathbb{N}\), are complex random variables that satisfy Assumptions 1 and 2, the zero counting measures \(\tau_n\) for this sequence of polynomials converge almost surely to \(\mu_T\) as \(n \to \infty\).

We next consider more general ensembles of random polynomials
\[ P_n(z) = \sum_{k=0}^n A_k B_k(z) \]
spanned by various bases \(\{B_k\}_{k=0}^{\infty}\). Let \(B_k(z) = \sum_{j=0}^k b_{j, k} z^j\), where \(b_{j, k} \in \mathbb{C}\) for all \(j\) and \(k\), and \(b_{k, k} \neq 0\) for all \(k\), be a **polynomial basis**, i.e., a linearly independent set of polynomials. Note that \(\deg B_k = k\) for all \(k \in \mathbb{N} \cup \{0\}\). Given a compact set \(E \subset \mathbb{C}\) of positive logarithmic capacity \(\text{cap}(E)\) (cf. Ransford [27]), we assume that
\[ \limsup_{k \to \infty} \|B_k\|_E^{1/k} \leq 1 \quad \text{and} \quad \lim_{k \to \infty} |b_{k, k}|^{1/k} = 1/\text{cap}(E), \]
where $\|B_k\|_E := \sup_E |B_k|$. Condition (2.3) holds for many standard bases used for representing analytic functions on $E$, e.g., for various sequences of orthogonal polynomials (cf. Stahl and Totik [34]) and for Faber polynomials (see Suetin [35]). In the former case, random polynomials spanned by such bases are called random orthogonal polynomials. Their asymptotic zero distribution was recently studied in a series of papers by Shiffman and Zelditch [30], Bloom [6] and [7], Bloom and Shiffman [9], Bloom and Levenberg [8] and Bayraktar [4]. In particular, it was shown that the counting measures of zeros converge weakly to the equilibrium measure of $E$, denoted by $\mu_E$, which is a positive unit Borel measure supported on the outer boundary of $E$ [27]. Most of the papers mentioned also considered multivariate polynomials. The authors assumed that the basis polynomials are orthonormal with respect to a measure satisfying the Bernstein-Markov property and that the coefficients are complex i.i.d. random variables with uniformly bounded distribution density function with respect to the area measure and with proper decay at $\infty$.

We develop this line of research, using the results of Blatt, Saff and Simkani [5] for deterministic polynomials of a single variable. In particular, we relax conditions on the random coefficients and consider more general choices of the bases.

**Theorem 2.2.** Suppose that a compact set $E \subset \mathbb{C}$, $\text{cap}(E) > 0$, has empty interior and connected complement. If the coefficients $A_k$ satisfy (2.1)-(2.2) and the basis polynomials $\{B_k\}_{k=0}^\infty$ satisfy (2.3), then the zero counting measures of $P_n(z) = \sum_{k=0}^n A_k B_k(z)$ converge almost surely to $\mu_E$ as $n \to \infty$.

Two very interesting applications of this result are related to random orthogonal and random Faber polynomials. Orthogonality below is considered with respect to the weighted arclength measure $w(s) \, ds$ on $E$; the definition of Faber polynomials may be found in [35, Chapter 2].

**Corollary 2.3.** Assume that conditions (2.1)-(2.2) hold for the coefficients.

(i) Suppose that $E$ is a finite union of rectifiable Jordan arcs with connected complement. If the basis polynomials $B_k$ are orthonormal with respect to a positive Borel measure $\mu$ supported on $E$ such that the Radon-Nikodym derivative $w(s) = d\mu/ds > 0$ for almost every $s$, then (2.3) is satisfied and $\tau_n$ converge almost surely to $\mu_E$ as $n \to \infty$.

(ii) Suppose that $E$ is a compact connected set with empty interior and connected complement, and that $E$ is not a single point. If the basis polynomials $B_k$ are the Faber polynomials of $E$, then (2.3) holds true and $\tau_n$ converge almost surely to $\mu_E$ as $n \to \infty$. 
If the interior of $E$ is not empty, we often need extra conditions to prevent excessive accumulation of zeros there. However, these additional assumptions may be replaced by more specific choices of the basis and geometric properties of $E$, as in the following result. If $E$ is a finite union of rectifiable curves and arcs, we call the polynomials orthonormal with respect to the arclength measure of Szegő polynomials. When $E$ is a compact set of positive area (2-dimensional Lebesgue measure on $\mathbb{C}$), we call the polynomials orthonormal with respect to the area measure $dA$ on $E$ Bergman polynomials.

**Theorem 2.4.** Suppose that $E$ is the closure of a Jordan domain with analytic boundary, and that the basis $\{B_k\}_{k=0}^{\infty}$ is given either by Szegő, or by Bergman, or by Faber polynomials. If (2.1)-(2.2) hold for the coefficients $A_k$, then the zero counting measures of $P_n(z) = \sum_{k=0}^{n} A_k B_k(z)$ converge almost surely to $\mu_E$ as $n \to \infty$.

In a more general setting, we introduce an extra assumption (2.4) on the constant term $A_0$.

**Theorem 2.5.** Let $E \subset \mathbb{C}$ be a compact set of positive capacity. If (2.1)-(2.3) hold, and there exists $t > 1$ such that

\[(2.4) \quad \sup_{z \in \mathbb{C}} \mathbb{E}\left[(\log^{-1} |A_0 - z|)^t\right] < \infty,\]

then the zero counting measures of $P_n(z) = \sum_{k=0}^{n} A_k B_k(z)$ converge almost surely to $\mu_E$ as $n \to \infty$.

Condition (2.4) means that the probability measure of $A_0$ cannot be too concentrated at any point $z \in \mathbb{C}$. In particular, it rules out the possibility that $A_0$ takes any specific value with positive probability, so that $A_0$ cannot be a discrete random variable. On the other hand, if $A_0$ is a continuous random variable satisfying (2.4), its density need not be bounded. For example, if the probability measure $\nu$ of $A_0$ is absolutely continuous with respect to the area measure $dA$ and has density $d\nu/dA(w)$ uniformly bounded by $C/|w - z|^s$, $s < 2$, near every $z \in \mathbb{C}$, then (2.4) holds.

Since we used a sequence of random coefficients $\{A_k\}_{k=0}^{\infty}$, the polynomials $\sum_{k=0}^{n} A_k B_k(z)$ were essentially partial sums of a random series. We now discuss even more general sequences of random polynomials of the form

$$P_n(z) = \sum_{k=0}^{n} A_{k,n} B_k(z).$$
Here, we deal with a triangular array of coefficients $A_{k,n}$, $k = 0, 1, \ldots, n, n \in \mathbb{N}$, that are complex-valued random variables. As before, they need not be identically distributed. We denote the distribution function of $|A_{k,n}|$ by $F_{k,n}$. Assumptions 1 and 2 uniformly imposed on all coefficients $A_{k,n}$ suffice to obtain

$$\lim_{n \to \infty} |A_{0,n}|^{1/n} = \lim_{n \to \infty} |A_{n,n}|^{1/n} = 1 \quad \text{a.s.}$$

by Lemma 4.1. But we need a slightly stronger condition to prove the limit

$$\lim_{n \to \infty} \max_{0 \leq k \leq n} |A_{k,n}|^{1/n} = 1 \quad \text{a.s.}$$

Thus we introduce the following assumptions on the triangular array of random coefficients.

**Assumption 1**. There exists $N \in \mathbb{N}$ such that $\{|A_{k,n}|\}_{k=0}^{n}$ are jointly independent for each $n \geq N$, and there exists a function $f : [a, \infty) \to [0, 1]$, $a > 1$, such that $f(x) \log x$ is decreasing,

$$\int_{a}^{\infty} f(x) \frac{\log x}{x} \, dx < \infty \quad \text{and} \quad 1 - F_{k,n}(x) \leq f(x) \quad \text{for all} \quad x \in [a, \infty)$$

for all $k = 0, 1, \ldots, n$ and all $n \geq N$.

**Assumption 2**. There exist $N \in \mathbb{N}$ and an increasing function $g : [0, b] \to [0, 1]$, $0 < b < 1$, such that

$$\int_{0}^{b} g(x) \frac{x}{x} \, dx < \infty \quad \text{and} \quad F_{k,n}(x) \leq g(x) \quad \text{for all} \quad x \in [0, b]$$

for all $k = 0, 1, \ldots, n$ and all $n \geq N$.

Lemma 4.3 gives all necessary limits (4.11)-(4.13) that allow to extend Theorems 2.1, 2.2 and Corollary 2.3 by following similar ideas, but certainly replacing (2.1) and (2.2) (Assumptions 1 and 2) with (2.5) and (2.6) (Assumptions 1* and 2*). The corresponding analog of Theorem 2.5 also holds if we replace (2.1) and (2.2) by (2.5) and (2.6), and replace (2.4) by the condition

$$\limsup_{n \to \infty} \sup_{z \in \mathbb{C}} \mathbb{E} \left[ (\log |A_{0,n} - z|^t) \right] < \infty$$

for a fixed $t > 1$. Detailed proofs of these statements will be published elsewhere; in this paper, we confine ourselves to an outline of the necessary arguments.
3 Expected number of zeros of random polynomials

Results in this section provide quantitative estimates for the weak convergence of the zero counting measures of random polynomials to the corresponding equilibrium measures. In particular, we study the expected deviation of the normalized counting measure of zeros \( \tau_n \) from the equilibrium measure \( \mu_E \) on certain sets, which is often referred to as the discrepancy between those measures. We again assume that the complex-valued random variables \( A_k, k = 0, 1, 2, \ldots \), are not necessarily independent nor identically distributed. It is convenient first to discuss the simplest case of the unit circle, which was first discussed in [25]. A standard way to study the deviation of \( \tau_n \) from \( \mu_T \) is to consider the discrepancy of these measures in annular sectors of the form

\[
A_r(\alpha, \beta) = \{ z \in \mathbb{C} : r < |z| < 1/r, \alpha \leq \arg z < \beta \}, \quad 0 < r < 1.
\]

The recent paper of Pritsker and Yeager contains the following estimate of the discrepancy.

**Theorem 3.1** ([26, p. 90]). Suppose that the coefficients of \( P_n(z) = \sum_{k=0}^n A_k z^k \) are complex random variables that satisfy

1. \( \mathbb{E}[|A_k|^t] < \infty, \ k = 0, \ldots, n, \) for a fixed \( t \in (0, 1] \),
2. \( \mathbb{E}[\log |A_0|] > -\infty \) and \( \mathbb{E}[\log |A_n|] > -\infty \).

Then, for all large \( n \in \mathbb{N} \),

\[
(3.1) \quad \mathbb{E}\left[ \left| \tau_n(A_r(\alpha, \beta)) - \frac{\beta - \alpha}{2\pi} \right| \right] \leq C_r \left[ \frac{1}{n} \left( \log \sum_{k=0}^n \mathbb{E}[|A_k|^t] - \frac{1}{2} \mathbb{E}[\log |A_0 A_n|] \right) \right]^{1/2},
\]

where

\[
C_r := \sqrt{\frac{2\pi}{k}} + \frac{2}{1-r} \quad \text{and} \quad k := \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^2}
\]

is Catalan’s constant.

Introducing uniform bounds, [26] also provides the rates of convergence for the expected discrepancy as \( n \to \infty \).

**Corollary 3.2.** Let \( P_n(z) = \sum_{k=0}^n A_{k,n} z^k, \ n \in \mathbb{N} \), be a sequence of random polynomials. If

\[
M := \sup \{ \mathbb{E}[|A_{k,n}|^t] \mid k = 0, \ldots, n, \ n \in \mathbb{N} \} < \infty
\]
and

\[ L := \inf \{ \mathbb{E}[\log |A_{k,n}|] \mid k = 0 \& n \in \mathbb{N} \} > -\infty, \]

then

\[
\mathbb{E} \left[ \left| \tau_n(A_r(\alpha, \beta)) - \frac{\beta - \alpha}{2\pi} \right| \right] \leq C_r \left[ \frac{1}{n} \left( \frac{\log(n + 1) + \log M}{t} - L \right) \right]^{1/2} = O \left( \sqrt{\frac{\log n}{n}} \right)
\]
as \( n \to \infty \).

It is well known from the original work of Erdős and Turán [11] that the order \( \sqrt{\log n / n} \) is optimal in the deterministic case. The proofs of Theorem 3.1 and Corollary 3.2 are sketched for convenience of the reader in Section 4. The papers [25] and [26] explain how to obtain quantitative results about the expected number of zeros of random polynomials in various sets; see [26, Propositions 2.3-2.5]. The basic observation here is that the number of zeros of \( P_n \) in a set \( S \subset \mathbb{C} \), denoted by \( N_n(S) \), is equal to \( n\tau_n(S) \), and the estimates for \( \mathbb{E}[N_n(S)] \) readily follow from Theorem 3.1 and Corollary 3.2.

We now turn to random polynomials spanned by the general bases \( B_k(z) = \sum_{j=0}^{k} b_{j,k} z^j \), \( k = 0, 1, \ldots \), where \( b_{j,k} \in \mathbb{C} \) for all \( j \) and \( k \), and \( b_{k,k} \neq 0 \) for all \( k \). These bases are considered in conjunction with an arbitrary compact set \( E \) of positive capacity in the plane whose equilibrium measure is denoted by \( \mu_E \). It is known that in order to obtain the discrepancy results for the pair \( \tau_n \) and \( \mu_E \) on compact sets \( E \subset \mathbb{C} \), one inevitably needs to restrict the geometric properties of \( E \); see Andrievskii and Blatt [2]. Although assumption (2.3) is typically sufficient for the discrepancy to converge to 0 as \( n \to \infty \), we need a different assumption to obtain the rates of convergence as in Corollary 3.2. In fact, many important bases satisfy

\[ \|B_k\|_E = O(k^p) \quad \text{and} \quad |b_{k,k}|(\text{cap}(E))^k \geq c k^{-q} \quad \text{as} \quad k \to \infty, \]

with fixed positive constants \( c, p, q \).

Instead of the annular sectors \( A_r(\alpha, \beta) \), we use the “generalized sectors” \( A_r \), defined with help of the Green function and conformal mappings. As in the previous section, we begin with the case \( E \) has empty interior. Specifically, let \( E \) be a compact set with one connected component being a Jordan arc \( L \) such that the distance from \( L \) to \( E \setminus L \) is positive. Denote the Green function of \( \mathbb{C} \setminus E \) with pole at \( \infty \) by \( g_E(z) \), and denote its harmonic conjugate by \( \tilde{g}_E(z) \). One can find \( b_L > 0 \)
such that \( \Phi(z) = \exp[b_L (g_E(z) + i \tilde{g}_E(z))] \) defines a conformal bijection between an annular region \( U_L \) with inner boundary \( L \) and an annulus \( 1 < |w| < R, R > 1 \). The mapping \( \Phi \) extends to \( L \) with values in \( \mathbb{T} \) by a standard argument. Given any subarc \( J \subset L \) and \( r \in (1, R) \), we set
\[
\mathcal{A}_r = \mathcal{A}_r(J) = \{ z \in U_L : 1 \leq |\Phi(z)| \leq r \text{ and } \Phi(z)/|\Phi(z)| \in \Phi(J) \}.
\]
In other words, \( \mathcal{A}_r \) is a curvilinear strip around \( J \) that is bounded by the level curve \( |\Phi(z)| = r \); more details of this construction may be found in [2, Chapter 2].

A smooth Jordan curve is said to be Dini-smooth if the angle between its tangent line and the positive real axis is Dini-continuous as a function of arclength parameter, i.e., the modulus of continuity of this function satisfies the Dini condition [2, p. 32]. A Dini-smooth arc is defined as a proper subarc of a Dini-smooth curve; a Dini-smooth domain is a domain bounded by a Dini-smooth curve.

We use general discrepancy results for deterministic polynomials obtained by Andrievskii and Blatt [2, Chapter 2] to study the expected deviation of zero counting measures for random polynomials from the limiting equilibrium measures.

**Theorem 3.3.** Suppose that \( E \) is a compact set with Dini-smooth arc \( L \subset E \) such that the distance from \( L \) to \( E \setminus L \) is positive. For \( P_n(z) = \sum_{k=0}^n A_k B_k(z) \), let \( \{A_k\}_{k=0}^n \) be random variables satisfying \( \mathbb{E}[|A_k|^t] < \infty, \ k = 0, \ldots, n, \) for a fixed \( t \in (0, 1] \), and \( \mathbb{E} \log |A_n| > -\infty \). Then, for all large \( n \in \mathbb{N} \),
\[
\mathbb{E} \left[ \left| (\tau_n - \mu_E)(\mathcal{A}_r) \right| \right] \leq C \left[ \frac{1}{n} \left( \frac{1}{t} \log \left( \sum_{k=0}^n \mathbb{E}[|A_k|^t] \right) + \log \max_{0 \leq k \leq n} \|B_k\|_{\infty} - \mathbb{E} \log |A_n| \right) \right]^{1/2},
\]
where \( C > 0 \) depends only on \( E \) and \( r \). Furthermore, if \( E \) is a finite union of closed intervals on the real line, then (3.3) holds with \( C = 8 \) and \( \mathcal{A}_r \) being the union of vertical strips \( \{ z \in \mathbb{C} : \Re(z) \in E \} \).

**Corollary 3.4.** Let \( P_n(z) = \sum_{k=0}^n A_{k,n} B_k(z), \ n \in \mathbb{N}, \) be a sequence of random polynomials, and let \( E \) satisfy the assumptions of Theorem 3.3. Suppose that for \( t \in (0, 1] \),
\[
\limsup_{n \to \infty} \max_{0 \leq k \leq n} \mathbb{E}[|A_{k,n}|^t] < \infty
\]
and
\[
\liminf_{n \to \infty} \mathbb{E}[\log |A_{n,n}|] > -\infty.
\]
If the basis polynomials $B_k$ satisfy (2.3), then

\[(3.6) \quad \lim_{n\to\infty} \mathbb{E} \left[ |(\tau_n - \mu_E)(A_r)| \right] = 0.\]

Furthermore, if (3.2) is satisfied, then

\[(3.7) \quad \mathbb{E} \left[ |(\tau_n - \mu_E)(A_r)| \right] = O\left(\sqrt{\frac{\log n}{n}}\right) \text{ as } n \to \infty.\]

The conclusion of Corollary 3.4 stated in (3.7) holds for the bases of orthogonal polynomials with respect to the weighted arclength measure on $E$ and of Faber polynomials when $E$ is a single arc. One need only verify that (3.2) is satisfied in those cases.

**Corollary 3.5.** Assume that conditions (3.4)-(3.5) hold for the coefficients.

(i) Suppose that $E$ is a finite union of disjoint Dini-smooth arcs. If the basis polynomials $B_k$ are orthonormal with respect to a positive Borel measure $\mu$ such that $d\mu(s) = w(s) ds$, where $w(s) \geq c > 0$ for almost every point on $E$, then (3.2) is satisfied and (3.7) holds.

(ii) Suppose that $E$ is an arbitrary Jordan arc. If the basis polynomials $B_k$ are the Faber polynomials of $E$, then (3.2) holds. Hence (3.7) is valid, provided $E$ is a Dini-smooth arc.

We also give corresponding results for smooth domains (or closed curves). Suppose that $E$ is a compact set whose connected component $S$ is a closed Jordan domain such that $\operatorname{dist}(S, E \setminus S) > 0$. We define the “generalized sector” $A_r$ using the conformal mapping $\Phi$ from the annular region $U_S$ with inner boundary $\partial S$ to an annulus $1 < |w| < R$, $R > 1$, constructed in the same way as before Theorem 3.3. In addition, we introduce a conformal mapping $\phi$ from the interior Jordan domain $G$ of $S$ onto the unit disk $\mathbb{D}$ such that $\phi(z_0) = 0$ for a point $z_0 \in G$. It is known that both mappings $\Phi$ and $\phi$ extend continuously to $\partial S$, being bijections between $\partial S$ and $\mathbb{T}$. For any subarc $J \subset \partial S$ and $r \in (1, r_0)$, we define

\[
A_r = A_r(J) = \{ z \in U_S : 1 \leq |\Phi(z)| \leq r \text{ and } \Phi(z)/|\Phi(z)| \in \Phi(J) \}
\cup \{ z \in G : 1/r \leq |\phi(z)| \leq 1 \text{ and } \phi(z)/|\phi(z)| \in \phi(J) \}.
\]

Again, $A_r$ may be described as a curvilinear strip around $J$ that is bounded by the level curves $|\Phi(z)| = r$ and $|\phi(z)| = 1/r$, $r > 1$.

**Theorem 3.6.** Suppose that $E$ is a compact set whose connected component $S$ is a closed Dini-smooth domain such that $\operatorname{dist}(S, E \setminus S) > 0$, with an interior point $w \in S^\circ$. For $P_n(z) = \sum_{k=0}^n A_k B_k(z)$, let $\{A_k\}_{k=0}^n$ satisfy $\mathbb{E}[|A_k'|] < \infty$,
\( k = 0, \ldots, n \), for a fixed \( t \in (0, 1) \). If \( \mathbb{E}[\log |A_n P_n(w)|] > -\infty \), then for all large \( n \in \mathbb{N} \),

\[ (3.8) \quad \mathbb{E} \left[ \left| (\tau_n - \mu E)(A_r) \right| \right] \leq C \left[ \frac{1}{n} \left( \frac{2}{t} \log \left( \sum_{k=0}^{n} \mathbb{E}[|A_k|^t] \right) + \log \max_{0 \leq k \leq n} \|B_k\|_2^2 - \mathbb{E}[\log |A_n P_n(w)|] \right) \right]^{1/2}, \]

where \( C > 0 \) depends only on \( E \) and \( r \).

In particular, if \( \mathbb{E}[\log |A_n|] > -\infty \) and \( \mathbb{E}[\log |A_0 + z|] \geq L > -\infty \) for all \( z \in \mathbb{C} \), then

\[ (3.9) \quad \mathbb{E}[\log |A_n P_n(w)|] \geq \log |b_{0,0}| + \mathbb{E}[\log |A_n|] + L > -\infty, \]

and (3.8) holds.

If \( \nu \) is the probability measure of \( A_0 \), then the assumption

\[ \mathbb{E}[\log |A_0 + z|] \geq L > -\infty \text{ for all } z \in \mathbb{C} \]

may be interpreted in terms of the logarithmic potential of \( \nu \) as

\[ U^\nu(z) = -\int \log |t - z| \, dv(t) \leq -L < \infty \text{ for all } z \in \mathbb{C}. \]

Measures with uniformly bounded above potentials are well understood in potential theory; they represent a wide class that do not have large local concentration of mass, e.g., they cannot have point masses.

We next state the analog of Corollary 3.4.

**Corollary 3.7.** Let \( P_n(z) = \sum_{k=0}^{n} A_k(z) B_k(z), \ n \in \mathbb{N} \), be a sequence of random polynomials, and let \( E \) satisfy the conditions of Theorem 3.6. Suppose that assumptions (3.4), (3.5) and

\[ (3.10) \quad \lim_{n \to \infty} \inf_{z \in \mathbb{C}} \mathbb{E}[\log |A_{0,n} + z|] > -\infty \]

are satisfied for the coefficients. If the basis polynomials \( B_k \) satisfy (3.2), then (3.7) holds.

We give examples of typical bases satisfying (3.2) below.

**Corollary 3.8.** Assume that conditions (3.4), (3.5) and (3.10) hold for the coefficients.

(i) Suppose that \( E \) is a finite union of mutually exterior closed Dini-smooth domains. If the basis polynomials \( B_k \) are orthonormal with respect to a positive
Borel measure \( \mu \) supported on \( \partial E \) such that \( d\mu(s) = w(s)\,ds \), where \( w(s) \geq c > 0 \) for almost every point of \( E \) in \( ds \)-sense, then (3.2) is satisfied and (3.7) holds.

(ii) Suppose that \( E \) is the closure of an arbitrary Jordan domain. If the basis polynomials \( B_k \) are the Faber polynomials of \( E \), then (3.2) holds. Hence (3.7) is valid provided \( \partial E \) is a Dini-smooth curve.

(iii) Suppose that \( E \) is a finite union of mutually exterior closed Dini-smooth domains. If the basis polynomials \( B_k \) are orthonormal with respect to \( d\mu(z) = w(z)\,dA(z) \), where \( dA \) is the area measure on \( E \) and \( w(z) \geq c > 0 \) a.e. in \( dA \)-sense, then (3.2) is satisfied and (3.7) holds.

It is clear that if the coefficients have identical distributions, then conditions (3.4) and (3.5) reduce to those on the single coefficient \( A_0 \). One can relax conditions on the orthogonality measure \( \mu \) while preserving the results of Corollaries 3.5 and 3.8, e.g., one can show that (3.7) also holds for polynomials orthogonal with respect to the generalized Jacobi weights of the form \( w(s) = v(s) \prod_{j=1}^J |s - s_j|^{\alpha_j} \), where \( v(s) \geq c > 0 \) a.e., in terms of the inner product defined either by \( ds \) or by \( dA \). It is also possible to relax the geometric conditions on \( E \) significantly, using the discrepancy results from [2] for quasiconformal arcs and curves. Thus smoothness is not critical for the results of this section, but the square root in all discrepancy estimates should then be replaced with a different (smaller) power depending on the angles at the boundary of \( E \).

4 Proofs of results

4.1 Proofs of results in Section 2. One of the key ingredients in the study of asymptotic zero distribution of polynomials is known to be the \( n \)-th root limiting behavior of their coefficients; see [2] for details. We prove the following probabilistic version of such results. Let \( \{X_n\}_{n=1}^\infty \) be a sequence of complex-valued random variables, and let \( F_n \) be the distribution function of \( |X_n| \), \( n \in \mathbb{N} \). We use the assumptions on random variables \( X_n \) that match those of (2.1) and (2.2) in Section 2.

Lemma 4.1. If there exist \( N \in \mathbb{N} \) and a decreasing function \( f : [a, \infty) \to [0, 1], \ a > 1 \) such that

\[
\int_a^\infty \frac{f(x)}{x} \,dx < \infty \text{ and } 1 - F_n(x) \leq f(x) \text{ for all } x \in [a, \infty),
\]

for all \( n \geq N \), then

\[
\limsup_{n \to \infty} |X_n|^{1/n} \leq 1 \quad a.s.
\]
Furthermore, if there exist \( N \in \mathbb{N} \) and an increasing function \( g : [0, b] \to [0, 1] \), \( 0 < b < 1 \) such that
\[
\int_0^b \frac{g(x)}{x} \, dx < \infty \quad \text{and} \quad F_n(x) \leq g(x) \quad \text{for all} \quad x \in [0, b],
\]
for all \( n \geq N \), then
\[
\liminf_{n \to \infty} |X_n|^{1/n} \geq 1 \quad \text{a.s.}
\]
(4.2)

Hence, if both assumptions above are satisfied, then
\[
\lim_{n \to \infty} |X_n|^{1/n} = 1 \quad \text{a.s.}
\]
(4.3)

We use a standard method for finding the almost sure limits of (4.1)-(4.3) via the first Borel-Cantelli Lemma.

**First Borel-Cantelli Lemma** (see, e.g., [16, p. 96]). Let \( \{\mathcal{E}_n\}_{n=1}^\infty \) be a sequence of arbitrary events. If \( \sum_{n=1}^\infty \mathbb{P}(\mathcal{E}_n) < \infty \), then
\[
\mathbb{P}(\mathcal{E}_n \text{ occurs in finitely often}) = 0.
\]

**Proof of Lemma 4.1.** We first prove (4.1). For \( \varepsilon > 0 \), define the events \( \mathcal{E}_n = \{|X_n| > e^{\varepsilon n}\}, \quad n \in \mathbb{N} \). Using the first assumption and letting \( m := \max(N, \lfloor \frac{1}{\varepsilon} \log a \rfloor) + 2 \), we obtain
\[
\sum_{n=m}^\infty \mathbb{P}(\mathcal{E}_n) = \sum_{n=m}^\infty (1 - \mathbb{P}(\{|X_n| \leq e^{\varepsilon n}\})) = \sum_{n=m}^\infty (1 - F_n(e^{\varepsilon n})) \leq \sum_{n=m}^\infty f(e^{\varepsilon n})
\]
\[
\leq \int_{m-1}^\infty f(e^{\varepsilon t}) \, dt \leq \frac{1}{\varepsilon} \int_{m-1}^\infty \frac{f(x)}{x} \, dx < \infty.
\]

Hence \( \mathbb{P}(\mathcal{E}_n \text{ occurs infinitely often}) = 0 \), by the first Borel-Cantelli Lemma; so, for all large \( n \), the complementary event \( \mathcal{E}_n^c \) must occur with probability 1. This means that \( |X_n|^{1/n} \leq e^\varepsilon \) for all sufficiently large \( n \in \mathbb{N} \) almost surely. We obtain
\[
\limsup_{n \to \infty} |X_n|^{1/n} \leq e^\varepsilon \quad \text{a.s.,}
\]
and (4.1), follows because \( \varepsilon > 0 \) may be arbitrarily small.

The proof of (4.2) proceeds in a similar way. For \( \varepsilon > 0 \), we define the events \( \mathcal{E}_n = \{|X_n| \leq e^{-\varepsilon n}\}, \quad n \in \mathbb{N} \). Using the second assumption and letting \( m := \max(N, \lfloor -\frac{1}{\varepsilon} \log b \rfloor) + 2 \), we have
\[
\sum_{n=m}^\infty \mathbb{P}(\mathcal{E}_n) = \sum_{n=m}^\infty F_n(e^{-\varepsilon n}) \leq \sum_{n=m}^\infty g(e^{-\varepsilon n}) \leq \int_{m-1}^\infty g(e^{-\varepsilon t}) \, dt \leq \frac{1}{\varepsilon} \int_0^b \frac{g(x)}{x} \, dx < \infty.
\]
Hence \( \mathbb{P}(\mathcal{E}_n \text{ i.o.}) = 0 \), and \( |X_n|^{1/n} > e^{-\varepsilon} \) holds for all sufficiently large \( n \in \mathbb{N} \) almost surely. We obtain

\[
\liminf_{n \to \infty} |X_n|^{1/n} \geq e^{-\varepsilon} \quad \text{a.s.,}
\]

and (4.2) follows by letting \( \varepsilon \to 0 \). \( \square \)

Lemma 4.1 implies that any infinite sequence of coefficients satisfying Assumptions 1 and 2 also satisfies (4.3). We state this as follows.

**Lemma 4.2.** Suppose that (2.1) and (2.2) hold for the coefficients \( A_n \) of random polynomials. Then

(4.4) \[
\lim_{n \to \infty} |A_n|^{1/n} = 1 \quad \text{a.s.,}
\]

(4.5) \[
\lim_{n \to \infty} |A_k|^{1/n} = 1 \quad \text{a.s., } k = 0, 1, 2, \ldots,
\]

and

(4.6) \[
\lim_{n \to \infty} \max_{0 \leq k \leq n} |A_k|^{1/n} = 1 \quad \text{a.s.}
\]

**Proof.** The limit (4.4) follows from Lemma 4.1 by letting \( X_n = A_n, \ n \in \mathbb{N} \). Similarly, for \( k \in \mathbb{N} \cup \{0\} \), set \( X_n = A_k, \ n \in \mathbb{N} \). Then (4.5) is immediate.

We now deduce (4.6) from (4.4). Let \( \omega \) be any elementary event such that

\[
\lim_{n \to \infty} |A_n(\omega)|^{1/n} = 1,
\]

which holds with probability 1. We immediately obtain

\[
\liminf_{n \to \infty} \max_{0 \leq k \leq n} |A_k(\omega)|^{1/n} \geq \liminf_{n \to \infty} |A_n(\omega)|^{1/n} = 1.
\]

On the other hand, elementary properties of limits imply that

\[
\limsup_{n \to \infty} \max_{0 \leq k \leq n} |A_k(\omega)|^{1/n} \leq 1.
\]

Indeed, by (4.4), for any \( \varepsilon > 0 \) there exists \( n_\varepsilon \in \mathbb{N} \) such that \( |A_n(\omega)|^{1/n} \leq 1 + \varepsilon \) for all \( n \geq n_\varepsilon \). Hence

\[
\max_{0 \leq k \leq n} |A_k(\omega)|^{1/n} \leq \max_{0 \leq k \leq n_\varepsilon} |A_k(\omega)|^{1/n}, \quad 1 + \varepsilon \quad \text{as } n \to \infty,
\]

and the result follows by letting \( \varepsilon \to 0 \). \( \square \)

We state a somewhat modified version of the result due to Blatt, Saff, and Simkani [5, pp. 309–310], which is used to prove all equidistribution theorems of Section 2.
Theorem BSS. Let $E \subset \mathbb{C}$ be a compact set, $\text{cap}(E) > 0$. If the sequence of polynomials $P_n(z) = \sum_{k=0}^{n} c_{k,n} z^k$ satisfies

$$\limsup_{n \to \infty} \|P_n\|_E^{1/n} \leq 1 \quad \text{and} \quad \lim_{n \to \infty} |c_{n,n}|^{1/n} = 1/\text{cap}(E),$$

and for any closed set $A$ in the bounded components of $\mathbb{C} \setminus \text{supp} \mu_E$,

$$\lim_{n \to \infty} \tau_n(A) = 0,$$

then the zero counting measures $\tau_n$ converge weakly to $\mu_E$ as $n \to \infty$.

It is known that (4.8) holds if every bounded component of $\mathbb{C} \setminus \text{supp} \mu_E$ contains a compact set $K$ such that

$$\liminf_{n \to \infty} \|P_n\|_K^{1/n} \geq 1;$$

see Bloom [6, p. 1706] and [7, p. 134] and, for the case of unbounded component of $\mathbb{C} \setminus \text{supp} \mu_E$, Grothmann [15, p. 352] (also [2]). In applications, this compact set $K$ is often selected as a single point.

Proof of Theorem 2.1. We apply Theorem BSS with $E = \mathbb{T}$. Recall that $\text{cap}(\mathbb{T}) = 1$ and $d\mu_T(e^{it}) = dt/(2\pi)$; see [27]. It is immediate that

$$\|P_n\|_{\mathbb{T}} \leq \sum_{k=0}^{n} |A_k| \leq (n+1) \max_{0 \leq k \leq n} |A_k|.$$  

Using (4.4) and (2.3), we conclude that (4.7) holds almost surely. On the other hand, (4.5) with $k = 0$ also gives

$$\lim_{n \to \infty} |P_n(0)|^{1/n} = \lim_{n \to \infty} |A_0|^{1/n} = 1 \quad \text{a.s.,}$$

meaning that (4.9) is satisfied for $K = \{0\}$ almost surely. Hence (4.8) holds a.s. for any compact subset $A$ of the unit disk, and the result follows. \qed

Proof of Theorem 2.2. Since $\text{supp} \mu_E \subset E$, we have that $\mathbb{C} \setminus \text{supp} \mu_E$ has no bounded components in this case, and (4.8) holds trivially. Thus we need only prove (4.7) for polynomials

$$P_n(z) = \sum_{k=0}^{n} A_k B_k(z) = A_n b_{n,n} z^n + \cdots, \quad n \in \mathbb{N}.$$  

Applying (4.4) and (2.3), we obtain for the leading coefficients of $P - n$ that

$$\lim_{n \to \infty} |A_n b_{n,n}|^{1/n} = 1/\text{cap}(E) \quad \text{a.s.}$$
Furthermore,
\[ \|P_n\|_E \leq \sum_{k=0}^{n} |A_k| \|B_k\|_E \leq (n + 1) \max_{0 \leq k \leq n} |A_k| \max_{0 \leq k \leq n} \|B_k\|_E. \]

Note that by a simple argument (already used in the proof of Lemma 4.2), (2.3) implies
\[ \limsup_{n \to \infty} \max_{0 \leq k \leq n} \|B_k\|_E^{1/n} \leq 1. \]
Combining this fact with (4.6), we obtain
\[ \limsup_{n \to \infty} \|P_n\|_E^{1/n} \leq 1 \quad \text{a.s.} \]

**Proof of Corollary 2.3.** Since the coefficient conditions (2.1)-(2.2) hold by our assumptions, we need only verify that the bases satisfy (2.3) in both cases (i) and (ii). Almost sure convergence of \(\tau_n\) to \(\mu_E\) then follows from Theorem 2.2.

(i) Our assumptions on the orthogonality measure \(\mu\) and set \(E\) imply that the orthogonal polynomials have regular asymptotic behavior expressed by (2.3), according to [34, Theorem 4.1.1 and Corollary 4.1.2]. (Corollary 4.1.2 of [34] is stated for a set \(E\) consisting of smooth arcs and curves, but its proof holds for the arbitrary rectifiable case, since \(\mu\) and \(\mu_E\) are both absolutely continuous with respect to the arclength \(ds\).) In fact, it is known that the density of the equilibrium measure is expressed via normal derivatives of the Green function \(g_E\) for the complement of \(E\) from both sides of the arcs:
\[ d\mu_E = \frac{1}{2\pi} \left( \frac{\partial g_E}{\partial n_+} + \frac{\partial g_E}{\partial n_-} \right) ds; \]
see [24, Theorem 1.1 and Example 1.2]. Furthermore, \(d\mu_E/ds > 0\) almost everywhere in the sense of arclength on \(E\); see [14, Chapter II].

(ii) The assumptions imposed on \(E\) imply that \(\text{cap}(E) > 0\), and that Faber polynomials are well defined. In particular, the Faber polynomials of \(E\) satisfy \(B_n(z) = z^n/(\text{cap}(E))^n + \cdots , n = 0, 1, \ldots\), by definition; see [35, Section 2.1]. Furthermore, Kövári and Pommerenke [22] showed that the Faber polynomials of any compact connected set do not grow too rapidly:
\[ \|B_n\|_E = O(n^\alpha) \quad \text{as } n \to \infty, \]
where \(\alpha < 1/2\). Hence (2.3) also holds in this case. \(\square\)
Proof of Theorem 2.4. It is known that in all three considered cases of Szegő, Bergman and Faber bases, (2.3) is satisfied. For the cases of Bergman and Szegő polynomials, see [33, pp. 288-290 and pp. 336-338, respectively]. The case of Faber polynomials was considered above in the proof of Corollary 2.3(ii). Arguing as in the proof of Theorem 2.2, we see that (4.7) holds for \( P_n(z) = \sum_{k=0}^n A_kB_k(z) \). Furthermore, for any compact set \( K \) in the interior \( E^\circ \) of \( E \), we have (cf. [33, pp. 290, 338] and [35, Section 2.3]) \( \limsup_{n \to \infty} \|B_n\|_K^{1/n} < 1 \). Since (4.4) holds with probability 1, we conclude that the series \( f(z) = \sum_{k=0}^\infty A_kB_k(z) \) converges uniformly on compact subsets of the analytic Jordan domain \( E^\circ \) with probability 1. Its limit is (almost surely) an analytic function \( f \) that cannot vanish identically because of (4.4) and the uniqueness of series expansions in Szegő, Bergman and Faber polynomials (see [33, pp. 293, 340] and [35, Section 6.3] for these facts). Hence, for each limit \( f \), there is a point \( z_f \in E^\circ \) such that \( f(z_f) \neq 0 \). This means \( \lim_{n \to \infty} P_n(z_f) = f(z_f) \neq 0 \), so that (4.9) is satisfied with \( K = \{z_f\} \). Thus (4.8) holds almost surely for any compact subset of \( E^\circ \) (the only bounded component of \( \mathbb{C} \setminus \text{supp} \mu_E = \mathbb{C} \setminus \partial E \)), and the result follows from Theorem BSS.

Proof of Theorem 2.5. We use Theorem BSS again. Condition (4.7) is verified exactly as in the proof of Theorem 2.2, so we omit that argument. It remains to show that (4.8) holds almost surely as a consequence of (2.4), which is again done via (4.9). In particular, we prove that

\[
\liminf_{n \to \infty} |P_n(w)|^{1/n} \geq 1
\]

holds almost surely for every given \( w \in \mathbb{C} \). Define the events

\[
E_n = \{|P_n(w)| \leq e^{-\epsilon n}\} = \left\{ \frac{1}{\epsilon} \log^{-} |P_n(w)| \geq n \right\}, \quad n \in \mathbb{N}.
\]

For any fixed \( t > 1 \), Chebyshev’s inequality gives

\[
P(E_n) \leq \frac{1}{n^t} \mathbb{E} \left[ \left( \frac{1}{\epsilon} \log^{-} |P_n(w)| \right)^t \right], \quad n \in \mathbb{N}.
\]

Note that

\[
(\log^{-} |P_n(w)|)^t \leq \left( \log^{-} |b_{0,0}| + \log^{-} \left| A_0 + \sum_{k=1}^n \frac{A_k}{b_{0,0}} B_k(w) \right| \right)^t
\]

\[
\leq 2^{t} \left( (\log^{-} |b_{0,0}|)^t + \left( \log^{-} \left| A_0 + \sum_{k=1}^n \frac{A_k}{b_{0,0}} B_k(w) \right| \right)^t \right).
\]

Denoting the value of supremum in (2.4) by \( C \), we obtain from the above inequality that

\[
\mathbb{E} \left[ (\log^{-} |P_n(w)|)^t \right] \leq 2^t \left( (\log^{-} |b_{0,0}|)^t + C \right).
\]
It follows that
\[ \sum_{n=1}^{\infty} P(E_n) \leq \frac{2^t}{\epsilon^t} \left( (\log^{-1} |b_{0,0}|)^t + C \right) \sum_{n=1}^{\infty} \frac{1}{n^t} < \infty. \]
Hence \( P(E_n \text{ i.o.}) = 0 \) by the First Borel-Cantelli Lemma, and \( |P_n(w)|^{1/n} > e^{-\epsilon} \) holds with probability 1 for all sufficiently large \( n \in \mathbb{N} \). We obtain
\[ \lim_{n \to \infty} |P_n(w)|^{1/n} \geq e^{-\epsilon} \quad \text{a.s.,} \]
and (4.10) follows by letting \( \epsilon \to 0 \).

The following lemma serves as a substitute for Lemma 4.2 and is necessary for the proofs of analogs of results from Section 2 generalized under Assumptions 1* and 2*.

**Lemma 4.3.** Suppose that (2.5) and (2.6) hold for the coefficients \( A_{k,n} \) of random polynomials. Then

\[ \lim_{n \to \infty} |A_{n,n}|^{1/n} = 1 \quad \text{a.s.,} \]
\[ \lim_{n \to \infty} |A_{k,n}|^{1/n} = 1 \quad \text{a.s.,} \quad k \in \mathbb{N} \cup \{0\}, \]

and

\[ \lim_{n \to \infty} \max_{0 \leq k \leq n} |A_{k,n}|^{1/n} = 1 \quad \text{a.s.} \]

**Proof.** The limits (4.11) and (4.12) follow from Lemma 4.1; we correspondingly let \( X_n = A_{n,n}, \ n \in \mathbb{N}, \) and \( X_n = A_{k,n}, \ n \in \mathbb{N} \) for a fixed \( k \in \mathbb{N} \cup \{0\} \). In fact, this argument holds under weaker assumptions such as (2.1) and (2.2), and does not require independence of coefficients.

In order to prove (4.13), we introduce the random variable \( Y_n = \max_{0 \leq k \leq n} |A_{k,n}| \) and denote its distribution function by \( F_n(x), \ n \in \mathbb{N} \). Note that
\[ \lim_{n \to \infty} |Y_n|^{1/n} \geq \lim_{n \to \infty} |A_{n,n}|^{1/n} = 1 \quad \text{a.s.} \]
Using independence of \( |A_{k,n}|, \ k = 0, \ldots, n, \) for each \( n \geq N, \) and applying (2.5), we estimate
\[ F_n(x) = \prod_{k=0}^{n} F_{k,n}(x) \geq (1 - f(x))^{n+1} \geq 1 - (n+1)f(x), \quad x \geq a. \]
For $\varepsilon > 0$, define events $E_n = \{|Y_n| > e^{\varepsilon n}\}$, $n \in \mathbb{N}$. Let $m := \max(N, \lfloor \frac{1}{\varepsilon} \log a \rfloor) + 2$. Then we obtain from the above estimate and (2.5) that

$$\sum_{n=m}^{\infty} \mathbb{P}(E_n) = \sum_{n=m}^{\infty} (1 - \mathbb{P}(|Y_n| \leq e^{\varepsilon n}))) = \sum_{n=m}^{\infty} (1 - F_n(e^{\varepsilon n})) \leq \sum_{n=m}^{\infty} (n + 1) f(e^{\varepsilon n})$$

$$\leq \frac{2}{\varepsilon} \int_{m-1}^{\infty} t f(e^{\varepsilon t}) \, dt \leq \frac{2}{\varepsilon} \int_{a}^{\infty} \frac{f(x) \log x}{x} \, dx < \infty.$$ 

Hence $\mathbb{P}(E_n \text{ i.o.}) = 0$ by the First Borel-Cantelli Lemma, and $|Y_n|^{1/n} \leq e^\varepsilon$ for all sufficiently large $n \in \mathbb{N}$ almost surely. We obtain that

$$\limsup_{n \to \infty} |Y_n|^{1/n} \leq e^\varepsilon \quad \text{a.s.,}$$

and (4.13) follows by letting $\varepsilon \to 0$. \hfill $\Box$

### 4.2 Proofs of results in Section 3

The following lemma is used several times below.

**Lemma 4.4.** If $A_k$, $k = 0, \ldots, n$, are complex random variables satisfying $\mathbb{E}[|A_k|^t] < \infty$, $k = 0, \ldots, n$, for a fixed $t \in (0, 1)$, then

$$\mathbb{E}\left[ \log \sum_{k=0}^{n} |A_k| \right] \leq \frac{1}{t} \log \left( \sum_{k=0}^{n} \mathbb{E}[|A_k|^t] \right).$$

**Proof.** We first state an elementary inequality. If $x_i \geq 0$, $i = 0, \ldots, n$, $\sum_{i=0}^{n} x_i = 1$, and $t \in (0, 1]$, then

$$\sum_{i=0}^{n} x_i^t \geq \sum_{i=0}^{n} x_i = 1.$$ 

Applying this inequality with $x_i = |A_i|/\sum_{k=0}^{n} |A_k|$, we obtain

$$\left( \sum_{k=0}^{n} |A_k| \right)^t \leq \sum_{k=0}^{n} |A_k|^t \quad \text{and} \quad \mathbb{E}\left[ \log \sum_{k=0}^{n} |A_k| \right] \leq \frac{1}{t} \mathbb{E}\left[ \log \left( \sum_{k=0}^{n} |A_k|^t \right) \right].$$

Jensen’s inequality and the linearity of expectation now give

$$\mathbb{E}\left[ \log \sum_{k=0}^{n} |A_k| \right] \leq \frac{1}{t} \log \mathbb{E}\left[ \sum_{k=0}^{n} |A_k|^t \right] = \frac{1}{t} \log \left( \sum_{k=0}^{n} \mathbb{E}[|A_k|^t] \right). \hfill \Box$$
Proof of Theorem 3.1. We use the following version of the discrepancy theorem of Erdős and Turán stated in [25, Proposition 2.1] (see also [11], [13] and [2]):

\[ \left| \tau_n(A_r(\alpha, \beta)) - \frac{\beta - \alpha}{2\pi} \right| \leq \sqrt{\frac{2\pi}{k}} \sqrt{\frac{1}{n} \log \frac{\|P_n\|_T}{\sqrt{|A_0A_n|}}} + \frac{2}{n(1 - r)} \log \frac{\|P_n\|_T}{\sqrt{|A_0A_n|}}. \]

Applying Jensen’s inequality, we obtain

\[ \mathbb{E} \left[ \left| \tau_n(A_r(\alpha, \beta)) - \frac{\beta - \alpha}{2\pi} \right| \right] \leq \sqrt{\frac{2\pi}{k}} \sqrt{\frac{1}{n} \mathbb{E} \left[ \log \frac{\|P_n\|_T}{\sqrt{|A_0A_n|}} \right]} + \frac{2}{n(1 - r)} \mathbb{E} \left[ \log \frac{\|P_n\|_T}{\sqrt{|A_0A_n|}} \right] \]

\[ \leq C_r \sqrt{\frac{1}{n} \mathbb{E} \left[ \log \frac{\|P_n\|_T}{\sqrt{|A_0A_n|}} \right]}, \]

where the last inequality holds for all sufficiently large \( n \in \mathbb{N} \). We observe that \( \|P_n\|_\infty \leq \sum_{k=0}^{n} |A_k| \) and use the linearity of expectation and (4.14) to estimate

\[ \mathbb{E} \left[ \log \frac{\|P_n\|_T}{\sqrt{|A_0A_n|}} \right] \leq \mathbb{E} \left[ \log \sum_{k=0}^{n} |A_k| \right] - \frac{1}{2} \mathbb{E}[\log |A_0A_n|] \]

\[ \leq \frac{1}{n} \log \left( \sum_{k=0}^{n} \mathbb{E}[|A_k|^t] \right) - \frac{1}{2} \mathbb{E}[\log |A_0A_n|]. \]

The latter bound is finite by our assumptions. \( \square \)

Proof of Corollary 3.2. The result follows immediately upon using the uniform bounds \( M \) and \( L \) in estimate (3.1). \( \square \)

Proof of Theorem 3.3. Note that the leading coefficient of \( P_n \) is \( A_n b_{n,n} \). Then [2, Chapter 2, Theorem 4.2] gives a discrepancy estimate of the form

\[ \left| (\tau_n - \mu_E)(A_r) \right| \leq C \sqrt{\frac{1}{n} \log \frac{\|P_n\|_E}{|A_n b_{n,n}|(\text{cap}(E))^n}}, \]

where the constant \( C \) depends only on \( E \) and \( r \). Using this estimate and Jensen’s inequality, we obtain

\[ \mathbb{E} \left[ \left| (\tau_n - \mu_E)(A_r) \right| \right] \leq C \sqrt{\frac{1}{n} \mathbb{E} \left[ \log \frac{\|P_n\|_E}{|A_n b_{n,n}|(\text{cap}(E))^n} \right]} \]

\[ \leq C \sqrt{\frac{1}{n} \left( \mathbb{E}[\log \|P_n\|_E] - \log(|b_{n,n}|(\text{cap}(E))^n) \right) - \mathbb{E}[\log |A_n|]}. \]

It is clear that

\[ \|P_n\|_E \leq \sum_{k=0}^{n} |A_k| \|B_k\|_E \leq \max_{0 \leq k \leq n} \|B_k\|_E \sum_{k=0}^{n} |A_k|. \]
Hence (4.14) yields
\[
\mathbb{E} \left[ \log \| P_n \|_E \right] \leq \mathbb{E} \left[ \log \sum_{k=0}^{n} |A_k| \right] + \log \max_{0 \leq k \leq n} \| B_k \|_E
\]
\[
\leq \frac{1}{t} \log \left( \sum_{k=0}^{n} \mathbb{E} [|A_k|^t] \right) + \log \max_{0 \leq k \leq n} \| B_k \|_E,
\]
and combining the above estimates yields (3.3).

When \( E \) is a finite union of closed non-intersecting intervals, one needs to use the discrepancy estimate of [2, Chapter 2, Theorem 5.1], which has the same form as (4.15) but with \( C = 8 \) and \( A_r \) being the union of vertical strips \( \{ z \in \mathbb{C} : \Re(z) \in E \} \). The rest of the proof remains identical. □

**Proof of Corollary 3.4.** To estimate the right hand side of (3.3), we make two immediate observations: that (3.4) implies
\[
\frac{1}{n} \log \left( \sum_{k=0}^{n} \mathbb{E} [|A_{k,n}|^t] \right) \leq O \left( \frac{\log n}{n} \right) \quad \text{as } n \to \infty,
\]
and that (3.5) implies
\[
-\frac{1}{n} \mathbb{E} [\log |A_{n,n}|] \leq O \left( \frac{1}{n} \right) \quad \text{as } n \to \infty.
\]
If (2.3) is satisfied, then
\[
\limsup_{n \to \infty} \left( \max_{0 \leq k \leq n} \| B_k \|_E \right)^{1/n} = \limsup_{n \to \infty} (\| B_k \|_E)^{1/n} \leq 1,
\]
and therefore
\[
\limsup_{n \to \infty} \frac{1}{n} \log \frac{\max_{0 \leq k \leq n} \| B_k \|_\infty}{|b_{n,n}|(\text{cap}(E))^n} \leq 0.
\]
Hence (3.6) follows from (2.3), (3.3) and the above inequalities. On the other hand, if (3.2) is satisfied, then
\[
\frac{1}{n} \log \frac{\max_{0 \leq k \leq n} \| B_k \|_\infty}{|b_{n,n}|(\text{cap}(E))^n} \leq O \left( \frac{\log n}{n} \right) \quad \text{as } n \to \infty,
\]
and (3.7) follows in the same manner. □

**Proof of Corollary 3.5.** In both cases, we need to verify (3.2) and then apply Corollary 3.4 to verify (3.7).

(i) The leading coefficient \( b_{n,n} \) of the orthonormal polynomial \( B_n \) provides the solution [34, p. 14] of the extremal problem
\[
|b_{n,n}|^{-2} = \inf \left\{ \int |Q_n|^2 \, d\mu : Q_n \text{ is a monic polynomial of degree } n \right\}.
\]
We use a monic polynomial $Q_n(z)$ that satisfies $\|Q_n\|_E \leq C_1 (\text{cap}(E))^n$, where $C_1 > 0$ depends only on $E$. Existence of such a polynomial for a set $E$ composed of finitely many smooth arcs and curves was first proved by Widom [38] (see also Totik [37]). Andrievskii [1] recently obtained much more general results for unions of arcs and curves that are not necessarily smooth. We estimate that

$$|b_{n,n}| \geq \left( \int |Q_n|^2 \, d\mu \right)^{-1/2} \geq (\mu(E))^{-1/2} \|Q_n\|^{-1}_E \geq C_1^{-1} (\mu(E))^{-1/2} (\text{cap}(E))^{-n}.$$  

Thus the second part of (3.2) is proved. For the proof of the first part, we apply the Nikolskii type inequality (see [23, Theorem 1.1 and comments on p. 689])

$$\|B_n\|_E \leq C_2 n \left( \int_E |B_n|^2 \, d\mu \right)^{1/2} \leq \frac{C_2}{\sqrt{c}} n \left( \int_E |B_n|^2 w(s) \, ds \right)^{1/2} = \frac{C_2}{\sqrt{c}} n.$$  

In the last step, we have also used the fact that $B_n$ is orthonormal with respect to $d\mu(s) = w(s) \, ds$.

(ii) In fact, (3.2) has already been verified for the Faber polynomials of any compact connected set $E$ in the proof of Corollary 2.3. Recall that the Faber polynomials of $E$ have the form $F_n(z) = z^n/(\text{cap}(E))^n + \cdots, \, n = 0, 1, \ldots$, by definition; see [35]. Furthermore, $\|F_n\|_E = O(n^\alpha)$ as $n \to \infty$, where $\alpha < 1/2$, by [22].

**Proof of Theorem 3.6.** This proof is similar to that of Theorem 3.3. Observe that the leading coefficient of $P_n$ is $A_n b_{n,n}$. Let $A_r$ be a “generalized curvilinear sector” (neighborhood) associated with a subarc $J$ of $\partial S$. We use [2, Chapter 2, Theorem 4.5] for the needed discrepancy estimate

$$(4.16) \quad |(\tau_n - \mu_E)(A_r)| \leq C \sqrt{\frac{1}{n} \log \frac{\|P_n\|_E}{|A_n b_{n,n}| (\text{cap}(E))^n}} + \frac{1}{n} \log \frac{\|P_n\|_E}{|P_n(w)|},$$

where the constant $C$ depends only on $E$ and $r$. We again apply Jensen’s inequality to obtain

$$\mathbb{E} \left[ |(\tau_n - \mu_E)(A_r)| \right] \leq C \sqrt{\frac{1}{n} \mathbb{E} \left[ \log \frac{\|P_n\|_E}{|A_n b_{n,n}| (\text{cap}(E))^n} \right]} + \frac{1}{n} \mathbb{E} \left[ \log \frac{\|P_n\|_E}{|P_n(w)|} \right].$$

It follows exactly as in the proof of Theorem 3.3 that

$$\mathbb{E} \left[ \log \|P_n\|_E \right] \leq \frac{1}{r} \log \left( \sum_{k=0}^{n} \mathbb{E}[|A_k|^r] \right) + \log \max_{0 \leq k \leq n} \|B_k\|_E$$

and

$$\mathbb{E} \left[ \log \frac{\|P_n\|_E}{|A_n b_{n,n}| (\text{cap}(E))^n} \right] \leq \frac{1}{r} \log \left( \sum_{k=0}^{n} \mathbb{E}[|A_k|^r] \right) + \frac{\max_{0 \leq k \leq n} \|B_k\|_E}{|b_{n,n}| (\text{cap}(E))^n} - \mathbb{E}[\log |A_n|].$$
Hence (3.8) follows as combination of the above estimates.

We now proceed to the lower bound for the expectation of $\log |A_n P_n(w)|$ in (3.9) by estimating

\[
\mathbb{E}[\log |A_n P_n(w)|] = \mathbb{E} \left[ \log \left( A_n \sum_{k=0}^{n} A_k B_k(w) \right) \right] \\
= \mathbb{E}[\log |A_n|] + \mathbb{E} \left[ \log \left( \sum_{k=0}^{n} A_k B_k(w) \right) \right] \\
= \mathbb{E}[\log |A_n|] + \log |b_{0,0}| + \mathbb{E} \left[ \log \left( A_0 + \sum_{k=1}^{n} A_k B_k(w) \frac{b_0}{b_{0,0}} \right) \right] \\
\geq \log |b_{0,0}| + \mathbb{E}[\log |A_n|] + L > -\infty,
\]

where we have used the fact that $b_{0,0} \neq 0$ and $\mathbb{E}[\log |A_0 + z|] \geq L$ for all $z \in \mathbb{C}$. □

**Proof of Corollary 3.7.** We use (3.8) and proceed in the same way as in the proof of Corollary 3.4. We observe that (3.4) implies

\[
\frac{2}{m} \log \left( \sum_{k=0}^{n} \mathbb{E}[|A_{k,n}|^2] \right) \leq O \left( \frac{\log n}{n} \right) \quad \text{as } n \to \infty,
\]

and (3.5) implies

\[
-\frac{1}{n} \mathbb{E}[\log |A_{n,n}|] \leq O \left( \frac{1}{n} \right) \quad \text{as } n \to \infty.
\]

Moreover, our assumption (3.2) about the basis again gives

\[
-\frac{1}{n} \log \frac{\max_{0 \leq k \leq n} \|B_k\|^2_E}{|b_{n,n}|(\text{cap}(E))^n} \leq O \left( \frac{\log n}{n} \right) \quad \text{as } n \to \infty.
\]

The new component in this proof is added by (3.10):

\[
-\frac{1}{n} \mathbb{E}[\log |P_n(w)|] = -\frac{1}{n} \mathbb{E} \left[ \log \left( \sum_{k=0}^{n} A_{k,n} B_k(w) \right) \right] \\
= -\frac{1}{n} \left( \log |b_{0,0}| + \mathbb{E} \left[ \log \left( A_{0,n} + \sum_{k=1}^{n} A_k B_k(w) \frac{b_0}{b_{0,0}} \right) \right] \right) \\
\leq O \left( \frac{1}{n} \right) \quad \text{as } n \to \infty.
\]

Hence (3.7) holds in the settings of Corollary 3.7. □
**Proof of Corollary 3.8.** All parts of Corollary 3.8 follow from Corollary 3.7, provided we show that the corresponding bases satisfy (3.2). But for parts (i) and (ii), this is done by arguments essentially identical to those of proofs for parts (i) and (ii) of Corollary 3.5. Hence we do not repeat them.

(iii) The proof of this part is also similar to that of part (i) of Corollary 3.5. The leading coefficient $b_{n,n}$ of the orthonormal polynomial $B_n$ satisfies [34, p. 14]

$$|b_{n,n}|^2 = \inf \left\{ \int |Q_n|^2 \, d\mu : Q_n \text{ is a monic polynomial of degree } n \right\}.$$  

To prove the second part of (3.2), we again use a monic polynomial $Q_n(z)$ that satisfies $\|Q_n\|_E \leq C_1(\text{cap}(E))^n$; see [38], [37], and [1]. It follows that

$$|b_{n,n}| \geq \left( \int |Q_n|^2 \, d\mu \right)^{-1/2} \geq (\mu(E))^{-1/2} \|Q_n\|_E^{-1} \geq C_1^{-1} (\mu(E))^{-1/2} (\text{cap}(E))^{-n}.$$

The first part of (3.2) follows from the area Nikolskii type inequality (see [23, Theorem 1.3 and remark (i) on p. 689])

$$\|B_n\|_E \leq C_2 n \left( \int_E |B_n|^2 \, dA \right)^{1/2} \leq \frac{C_2}{\sqrt{c}} n \left( \int_E |B_n|^2 \, w \, dA \right)^{1/2} = \frac{C_2}{\sqrt{c}} n,$$

where we have used the fact that the weighted area $L_2$ norm of $B_n$ equals 1 by definition. 

\[\square\]

**REFERENCES**

[1] V. V. Andrievskii, *Chebyshev polynomials on a system of continua*, Constr. Approx. **43** (2016), 217–229.

[2] V. V. Andrievskii and H.-P. Blatt, *Discrepancy of Signed Measures and Polynomial Approximation*, Springer-Verlag, New York, 2002.

[3] L. Arnold, *Über die Nullstellenverteilung zufälliger Polynome*, Math. Z. **92** (1966), 12–18.

[4] T. Bayraktar, *Equidistribution of zeros of random holomorphic sections*, Indiana Univ. Math. J. **65** (2016), 1759–1793.

[5] H.-P. Blatt, E. B. Saff, and M. Simkani, *Jentzsch-Szegő type theorems for the zeros of best approximants*, J. London Math. Soc. (2) **38** (1988), 307–316.

[6] T. Bloom, *Random polynomials and Green functions*, Int. Math. Res. Not. **2005**, no. 28, 1689–1708.

[7] T. Bloom, *Random polynomials and (pluri)potential theory*, Ann. Polon. Math. **91** (2007), 131–141.

[8] T. Bloom and N. Levenberg, *Random polynomials and pluripotential-theoretic extremal functions*, Potential Anal. **42** (2015), 311–334.

[9] T. Bloom and B. Shiffman, *Zeros of random polynomials on $\mathbb{C}^m$*, Math. Res. Lett. **14** (2007), 469–479.

[10] A. T. Bharucha-Reid and M. Sambandham, *Random Polynomials*, Academic Press, Orlando, FL, 1986.
[11] P. Erdős and P. Turán, *On the distribution of roots of polynomials*, Ann. of Math. (2) **51** (1950), 105–119.

[12] K. Farahmand, *Topics in Random Polynomials*, Longman, Harlow, 1998.

[13] T. Ganelius, *Sequences of analytic functions and their zeros*, Ark. Mat. **3** (1954), 1–50.

[14] J. B. Garnett and D. E. Marshall, *Harmonic Measure*, Cambridge Univ. Press, New York, 2005.

[15] R. Grothmann, *On the zeros of sequences of polynomials*, J. Approx. Theory **61** (1990), 351–359.

[16] A. Gut, *Probability: A Graduate Course*, Springer, New York, 2005.

[17] C. P. Hughes and A. Nikeghbali, *The zeros of random polynomials cluster uniformly near the unit circle*, Compos. Math. **144** (2008), 734–746.

[18] I. Ibragimov and O. Zeitouni, *On roots of random polynomials*, Trans. Amer. Math. Soc. **349** (1997), 2427–2441.

[19] I. Ibragimov and D. Zaporozhets, *On distribution of zeros of random polynomials in complex plane*, Prokhorov and Contemporary Probability Theory, Springer, Heidelberg, 2013, pp. 303–323.

[20] Z. Kabluchko and D. Zaporozhets, *Roots of random polynomials whose coefficients have logarithmic tails*, Ann. Probab. **41** (2013), 3542–3581.

[21] Z. Kabluchko and D. Zaporozhets, *Asymptotic distribution of complex zeros of random analytic functions*, Ann. Probab. **42** (2014), 1374–1395.

[22] T. Kövari and Ch. Pommerenke, *On Faber polynomials and Faber expansions*, Math. Z. **99** (1967), 193–206.

[23] I. E. Pritsker, *Comparing norms of polynomials in one and several variables*, J. Math. Anal. Appl. **216** (1997), 685–695.

[24] I. E. Pritsker, *How to find a measure from its potential*, Comput. Methods Funct. Theory **8** (2008), 597–614.

[25] I. E. Pritsker and A. A. Sola, *Expected discrepancy for zeros of random algebraic polynomials*, Proc. Amer. Math. Soc. **142** (2014), 4251–4263.

[26] I. E. Pritsker and A. M. Yeager, *Zeros of polynomials with random coefficients*, J. Approx. Theory **189** (2015), 88–100.

[27] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge Univ. Press, Cambridge, 1995.

[28] G. Schehr and S. N. Majumdar, *Condensation of the roots of real random polynomials on the real axis*, J. Stat. Physics **135** (2009), 587–598.

[29] B. Shiffman and S. Zelditch, *Distribution of zeros of random and quantum chaotic sections of positive line bundles*, Comm. Math. Phys. **200** (1999), 661–683.

[30] B. Shiffman and S. Zelditch, *Equilibrium distribution of zeros of random polynomials*, Int. Math. Res. Not. **2003** no. 1, 25–49.

[31] B. Shiffman and S. Zelditch, *Random complex fewnomials, I.*, Notions of Positivity and the Geometry of Polynomials, Birkhäuser/Springer, Basel AG, 2011, pp. 375–400.

[32] C. D. Sinclair and M. L. Yattselev, *Root statistics of random polynomials with bounded Mahler measure*, Adv. Math. **272** (2015), 124–199.

[33] V. I. Smirnov and N. A. Lebedev, *Functions of a Complex Variable: Constructive Theory*, MIT Press, Cambridge, MA, 1968.

[34] H. Stahl and V. Totik, *General Orthogonal Polynomials*, Cambridge Univ. Press, Cambridge, 1992.

[35] P. K. Suetin, *Series of Faber Polynomials*, Gordon and Breach Science Publishers, Amsterdam, 1998.

[36] T. Tao and V. Vu, *Local universality of zeroes of random polynomials*, Int. Math. Res. Not. **2015**, no. 13, 5053–5139.
[37] V. Totik, *Asymptotics of Christoffel functions on arcs and curves*, Adv. Math. 252 (2014), 114–149.

[38] H. Widom, *Extremal polynomials associated with a system of curves in the complex plane*, Adv. Math. 3 (1969) 127–232.

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