Complex and bi-Hermitian structures on four-dimensional real Lie algebras

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Abstract

We give a new method for calculation of complex and bi-Hermitian structures on low-dimensional real Lie algebras. In this method, using non-coordinate basis, we first transform the Nijenhuis tensor field and bi-Hermitian structure relations on Lie groups to the tensor relations on their Lie algebras. Then we use adjoint representation for writing these relations in the matrix form; in this manner by solving these matrix relations and using automorphism groups of four-dimensional real Lie algebras we obtain and classify all complex and bi-Hermitian structures on four-dimensional real Lie algebras.

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1. Introduction

Calculation of complex structures on homogeneous complex manifolds and especially on Lie groups is important from both mathematical and physical point of view. Mathematically, classification of these manifolds is based on determination of possible complex structures. From the physical point of view these structures have an important role in the $N = (2, 2)$ supersymmetric sigma models [1]. It is shown that the $N = (2, 2)$ extended supersymmetry in the sigma model implies the existence of the bi-Hermitian structure on the target manifold such that the complex structures are covariantly constant with respect to torsionful affine connections (see for example [2] and references therein). Furthermore, it is shown that the algebraic structures related to these bi-Hermitian structures for $N = (2, 2)$ supersymmetric WZW models are the Manin triples [3, 4]. For these reasons, the calculation of the complex and the bi-Hermitian structures on manifolds, especially on Lie groups, is important. Samelson [5] shows that compact Lie groups always admit an invariant complex structure. As for the non-compact case, Morimoto [6] proves that there always exist invariant complex structures on any even-dimensional reductive Lie groups. In [7] and [8], complex structures on real
four-dimensional Lie algebras are classified. The method used in this work is special and does not seem to be adequate for the calculation in higher dimensions. In the present paper we give a new method for this purpose, which can be applied for low-dimensional Lie groups. In this method, using non-coordinate basis, we first transform the Nijenhuis tensor on Lie groups to algebraic tensor relations on their Lie algebras. Then, using adjoint representation we rewrite these relations in the matrix form. Finally, we solve these matrix relations using Maple. In this research we perform this for real four-dimensional Lie algebras. The results for some algebras are different and complete with respect to [8]. Furthermore, calculation of the bi-Hermitian structure for four-dimensional Lie algebras is new.

The paper is organized as follows. In section 2, using non-coordinate basis, we transform the Nijenhuis tensor relation on a Lie group to the algebraic tensor relation on its Lie algebra. Then, using adjoint representation, we write these relations in the matrix form. The relations can also be obtained from the definition of complex structures on Lie algebras as endomorphism of them. Then, in section 3 using Maple we solve these matrix relations to obtain complex structures on real four-dimensional Lie algebras. In this process, we apply automorphism groups of real four-dimensional Lie algebras for obtaining non-equivalent complex structures (table 1). We then compare our results with [7] and [8]. Note that here we use the Patra et al [9] classification of real four-dimensional Lie algebras. The list of Lie algebras and their automorphism groups [10] is given in the appendix. In section 4, we first transform the tensorial form of the bi-Hermitian relations on Lie groups into the algebraic tensorial relations on their Lie algebras. In this respect, we define the bi-Hermitian structure on a Lie algebra independently and give an equivalent relation for obtaining non-equivalent bi-Hermitian structures. Then using adjoint representation we rewrite these relations in the matrix form and solve them by Maple. Therefore, the present paper will be a continuation to the discussion of a bi-Hermitian structure on real four-dimensional Lie algebras. Some discussions are given in the conclusion section.

2. A brief review of complex structures on Lie groups

**Definition 1.** Let $M$ be a differentiable manifold; then the pair $(M, J)$ is called almost complex manifold if there exists a tensor field $J$ of type $(1,1)$ such that at each point $p$ of $M$, $J_p^2 = -1$; the tensor field $J$ is also called the almost complex structure. Furthermore, if the Lie bracket of any vector fields of type $(1,0)$ $X, Y \in T_pM^+$ is again of the same type, then the complex structure $J_p$ is said to be integrable, where $T_pM^+ = \{Z \in T_pM^C \mid J_pZ = +iZ\}$.

**Theorem (Newlander and Nirenberg [11]).** An almost complex structure $J$ on a manifold $M$ is integrable if and only if

$$N(X, Y) = 0, \quad \forall X, Y \in \chi(M), \quad (1)$$

where $\chi(M)$ is the set of vector fields on $M$ and the Nijenhuis tensor $N : \chi(M) \otimes \chi(M) \rightarrow \chi(M)$ is given by

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]. \quad (2)$$

In the coordinate basis, i.e the basis $\{e_\mu = \frac{\partial}{\partial x^\mu}\}$ and $\{dx^\mu\}$ for vectors and dual vectors (forms) respectively on $M$, the almost complex structures and Nijenhuis tensor are expressed as $J = J_\mu^\nu e_\nu \otimes dx^\mu$ and $N = N_{\mu\nu}^\lambda dx^\mu \otimes dx^\nu \otimes e_\lambda$, respectively and the integrability condition (2) can be rewritten as follows:

$$N_{\mu\nu}^\lambda = J_\mu^\nu (\partial_\lambda J_\nu^\mu) + J_\nu^\lambda (\partial_\mu J_\nu^\lambda) - J_\nu^\lambda (\partial_\mu J_\mu^\nu) - J_\mu^\nu (\partial_\lambda J_\nu^\mu) = 0. \quad (3)$$
Meanwhile, the relation $J^2 = -1$ can be rewritten as

$$J^\lambda_{\mu} J^\mu_{\nu} = -\delta^\lambda_{\nu}. \quad (4)$$

Furthermore one can rewrite the above equations using the non-coordinate bases $\{\hat{e}_\alpha\}$ and $\{\hat{0}^\alpha\}$ for vectors and forms on $M$. For these bases we have

$$\hat{e}_\alpha = e^\alpha_{\mu} \hat{e}_\mu, \quad e^\alpha_{\mu} \in GL(m, R). \quad (5)$$

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
Lie algebra & Complex structures \\
\hline
$I X \oplus R$ & $J e_1 = -e_2, \ J e_3 = e_4$ \\
$A_2 \oplus A_2$ & $J e_1 = -e_2, \ J e_3 = -e_4$ \\
$V \oplus R$ & $J e_1 = -e_4, \ J e_2 = e_3$ \\
$V I I_0 \oplus R$ & $J e_1 = -e_2, \ J e_3 = -e_4$ \\
$V I I_2 \oplus R$ & $J e_1 = -e_4, \ J e_2 = e_3$ \\
$V I I_4 \oplus R$ & $J e_1 = -e_4, \ J e_2 = e_3$ \\
$a \neq 0$ & \\
$V I I_6 \oplus R$ & $J e_1 = -e_2, \ J e_3 = p e_3 - (1 + p^2)e_4 \quad (p \in \mathbb{R})$ \\
$I X \oplus R$ & $J e_1 = -e_2, \ J e_3 = p e_3 - (1 + p^2)e_4 \quad (p \in \mathbb{R})$ \\
$A_{1,2}$ & $J e_1 = e_2, \ J e_3 = e_4$ \\
$A_{a, b}^{(a \leq 1, a \neq 0)}$ & $J e_1 = -e_4, \ J e_2 = e_3$ \\
$A_{a, b}^{(a \leq 1, a \neq 0)}$ & $J e_1 = e_3, \ J e_2 = e_4$ \\
$A_{1, 1}^{(a \leq 1)}$ & $J e_1 = e_3, \ J e_2 = -e_4$ \\
$A_{a, b}^{(a \leq 1)}$ & $J e_1 = e_4, \ J e_2 = e_3$ \\
$(a \neq 0, b \geq 0)$ & \\
$A_{4,7}$ & $J e_1 = e_2, \ J e_3 = -e_4$ \\
$A_{4,8}$ & $J e_1 = e_2, \ J e_3 = e_4$ \\
$A_{4,9}^{(b > 0)}$ & \\
$A_{4,9}^{(0 < b < 1)}$ & $J e_1 = -e_2, \ J e_3 = e_4$ \\
$A_{a, b}^{(a > 0)}$ & $J e_1 = e_4, \ J e_2 = -e_3$ \\
$A_{4,10}^0$ & $J e_1 = e_4, \ J e_2 = e_3$ \\
$A_{4,10}^{(a > 0)}$ & \\
$A_{4,11}$ & $J e_1 = e_4, \ J e_2 = -e_3$ \\
$(a > 0)$ & \\
$A_{4,12}$ & $J e_1 = e_4, \ J e_2 = e_3$ \\
\hline
\end{tabular}
\caption{Complex structures on four-dimensional real Lie algebras.}
\end{table}
where for the vierbeins $e^a_\mu$ and its inverse $e^a_\nu$ we have

$$e^a_\mu e^\mu_\beta = \delta^a_\beta, \quad e^a_\mu e^\nu_\alpha = \delta^a_\nu.$$  \hspace{1cm} (6)

The dual bases $\{\hat{\theta}^a\}$ are defined by

$$\langle \hat{\theta}^a, \hat{e}^\beta \rangle = \delta^a_\beta$$

and we have $\hat{\theta}^a = e^a_\mu \, dx^\mu$. Furthermore, the vierbeins satisfy the following relation:

$$f_{a\beta\gamma} = e^\gamma_\nu \left( e^a_\mu \partial_\mu e^\beta_\nu - e^\beta_\mu \partial_\mu e^a_\nu \right);$$  \hspace{1cm} (7)

if $M$ is a Lie group manifold $G$, then $f_{a\beta\gamma}$ are the structure constants of the Lie algebra $\mathfrak{g}$ of $G$. Now on these bases the tensor $J = J_\mu^\nu e^\nu_\nu \otimes dx^\mu$ can be rewritten as

$$J_\mu^\nu = e^\alpha_\mu J^{\alpha}_\beta e^\beta_\nu,$$  \hspace{1cm} (8)

where $J^{\alpha}_\beta$ is an endomorphism of $\mathfrak{g}$, i.e. $J : \mathfrak{g} \rightarrow \mathfrak{g}$. Now by applying this relation to (4) we have the following matrix relation for matrices $J^{\alpha}_\beta$:

$$J^2 = -I.$$  \hspace{1cm} (9)

Furthermore, by applying relations (7) and (8) to the tensor equation (3) and using (6) and assuming that $J^{\alpha}_\beta$ and $g_{a\beta}$ are independent of coordinates of $G$, after some calculations we have the following algebraic relation for (3):

$$f_{\beta\alpha\gamma} + J_\sigma^\gamma J^{\alpha}_\delta f_{\beta\delta\sigma} - J_\beta^\sigma J^{\alpha}_\delta f_{\sigma\delta\gamma} + J_\beta^\gamma J^{\delta}_\sigma f_{\sigma\delta\alpha} = 0.$$  \hspace{1cm} (10)

Finally, using adjoint representations

$$f_{\beta a}^\gamma = -(\gamma^a)_\beta^\gamma, \quad f_{\beta a}^\gamma = -(\chi^a)_\beta^\gamma,$$  \hspace{1cm} (11)

relation (10) will have the following matrix form:

$$\gamma^a + J \gamma^\alpha J^{\alpha}_\beta + J^{\alpha}_\beta \gamma^\beta J^\beta - J \gamma^\alpha J^\beta = 0,$$  \hspace{1cm} (12)

or

$$\chi^a + J \chi^\alpha J + J^{\alpha}_\beta \chi^\beta J - J^{\alpha}_\beta J \chi^\beta = 0.$$  \hspace{1cm} (13)

Note that the above equation can also be obtained from the definition of the complex structure on the Lie algebra $\mathfrak{g}$ as follows.

**Definition 2.** An integrable complex structure on a real Lie algebra $\mathfrak{g}$ is an endomorphism $J$ of $\mathfrak{g}$ such that

(a) $J^2 = -Id$,

(b) $[X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0, \quad \forall X, Y \in \mathfrak{g}$.  \hspace{1cm} (14)

Now if we use $\{X_\alpha\}$ as basis for the Lie algebra $\mathfrak{g}$ with the following structure constants:

$$[X_\alpha, X_\beta] = f_{a\beta}^\gamma X_\gamma,$$  \hspace{1cm} (16)

and use the following relation for $J$:

$$JX_\alpha = J^{\alpha}_\beta X_\beta,$$  \hspace{1cm} (17)

then relations (14) and (15) can be rewritten as (9), (12) or (13). Now, in order to obtain algebraic complex structures $J^{\alpha}_\beta$, it is enough to solve equations (9) and (12) or (13) simultaneously. We do this for real four-dimensional Lie algebras in the next section.
3. Calculation of complex structures on four-dimensional real Lie algebras

In this section we use the Patera et al. classification [9] of four-dimensional real Lie algebras. The commutation relations and the automorphism groups of these Lie algebras are given in the appendix. Now one can write the adjoint representation (3') for these Lie algebras and then solve the matrix relations (9) and (12) for obtaining complex structures. We do this by Maple. Note that in this process one can obtain equivalent complex structures; to avoid these and in order to obtain an inequivalent complex structure we use the following equivalent relation:

Definition 3 ([8]). Two complex structures $J_1$ and $J_2$ of the Lie algebra $g$ are equivalent if there exists an element $A$ of an automorphism group of the Lie algebra $\text{Aut}(g)$ such that

$$J_2 = AJ_1A^{-1}. \quad (18)$$

Note that this relation is an equivalent relation.

In this way, we do this and obtain all non-equivalent complex structures on four-dimensional real Lie algebras. The results are classified in table 1. As the table shows 21 out of 30 real four-dimensional Lie algebras have complex structures. To compare these results with results of [8] first we must obtain the isomorphism relations between the four-dimensional real Lie algebras presented in [9] and those presented in [8]. According to the calculations in [8] and [12] we have isomorphism relations as summarized in the following table:

| $\mathfrak{a}$ | $\mathfrak{a}_4$ | $\mathfrak{a}_4$,1 | $\mathfrak{a}_4$,2 | $\mathfrak{a}_4$,3 | $\mathfrak{a}_4$,4 | $\mathfrak{a}_4$,5 | $\mathfrak{a}_4$,6 | $\mathfrak{a}_4$,7 | $\mathfrak{a}_4$,8 | $\mathfrak{a}_4$,9 | $\mathfrak{a}_4,10$ | $\mathfrak{a}_4,11$ | $\mathfrak{a}_4,12$ |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| $\mathfrak{a}_4$ | $\mathfrak{n}_4$ | $\mathfrak{r}_4,0$ | $\mathfrak{r}_4$ | $\mathfrak{r}_{4,a,b}$ | $\mathfrak{r}_{4,a,b}$ | $\mathfrak{h}_4$ | $\mathfrak{d}_4$ | $\mathfrak{d}_{4,1/1+b}$ | $\mathfrak{d}_{4,0}$ | $\mathfrak{d}_{4,a}$ | $\text{aff}(\mathbb{C})$ |
| $\mathfrak{a}_2 \oplus \mathfrak{a}_2$ | $\mathfrak{i}_1 \oplus \mathfrak{r}$ | $\mathfrak{i}_3 \oplus \mathfrak{r}$ | $\mathfrak{iv} \oplus \mathfrak{r}$ | $\mathfrak{v} \oplus \mathfrak{r}$ | $\mathfrak{v}_1 \oplus \mathfrak{r}$ | $\mathfrak{v}_1 \oplus \mathfrak{r}$ | $\mathfrak{v}_1 \oplus \mathfrak{r}$ | $\mathfrak{v}_1 \oplus \mathfrak{r}$ | $\mathfrak{v}_1 \oplus \mathfrak{r}$ | $\mathfrak{v}_1 \oplus \mathfrak{r}$ | $\mathfrak{v}_1 \oplus \mathfrak{r}$ | $\mathfrak{v}_1 \oplus \mathfrak{r}$ | $\mathfrak{v}_1 \oplus \mathfrak{r}$ |
| $\mathfrak{r}_2 \mathfrak{r}_3$ | $\mathfrak{r}_{3,0}$ | $\mathfrak{r}_{3}$ | $\mathfrak{r}_{3,1}$ | $\mathfrak{r}_{3,-1}$ | $\mathfrak{r}_{3,a}$ | $\mathfrak{r}_{3,0}$ | $\mathfrak{r}_{3,a}$ |
| $\mathfrak{r}_3 \mathfrak{r}_3$ | $\mathfrak{r}_{3,0}$ | $\mathfrak{r}_{3,1}$ | $\mathfrak{r}_{3,-1}$ | $\mathfrak{r}_{3,a}$ | $\mathfrak{r}_{3,0}$ | $\mathfrak{r}_{3,a}$ |

In this respect, one can see that in [8] one complex structure is obtained for the Lie algebra $\mathfrak{v}_1 \mathfrak{l}_0 + \mathfrak{r}$ but according to our calculation this Lie algebra has two non-equivalent complex structures. For non-solvable Lie algebras $\mathfrak{v}_1 \mathfrak{l}_0 + \mathfrak{r}$ and $\mathfrak{v}_1 \mathfrak{l}_0 + \mathfrak{r}$ we obtain complex structures. In [8], two complex structures are obtained for the Lie algebra $\mathfrak{a}_4,12$ but here we obtain four complex structures to this Lie algebra. Furthermore, two complex structures are obtained for the Lie algebra $\mathfrak{a}_4,10$ in [8], but according to table 1 we obtain four complex structures for this Lie algebra. Meanwhile, in [8], two complex structures are obtained for the Lie algebra $\mathfrak{a}_4,12$ but here we obtain three complex structures for this Lie algebra. The results for other Lie algebras are the same as [8] (table 1).

4. Bi-Hermitian structures on four-dimensional real Lie algebras

Definition 4 ([1, 2]). If the complex manifold $M$ has two complex structures $J_{\pm}$ such that it is Hermitian with respect to both complex structures, i.e.
\[ J_\pm^2 = -1, \]  
\[ N_{\mu\nu} (J_\pm) = 0, \]  
\[ J_{\pm\mu} \gamma g_{\pm\nu} J_\pm = g_{\mu\nu}, \]  
and furthermore if these complex structures be covariantly constant with respect to certain connections \( \Gamma^\pm \)
\[ \nabla_{\pm\mu} J_{\pm\nu} \equiv J_{\pm\nu} \delta - \Gamma_{\pm\mu\nu}^\lambda J_{\pm\lambda} = 0, \]  
with
\[ \Gamma_{\pm\mu\nu}^\lambda = \Gamma_{\mu\nu\lambda}^\pm \pm T_{\mu\nu}^\pm, \quad T_{\mu\nu}^\pm = H_{\mu\nu\eta} g_{\eta\lambda}, \]
then it is said that \( M \) has a bi-Hermitian structure, shown by \((M, g, J_\pm)\).

In the above definition \( g_{\mu\nu}, \Gamma^\pm_{\mu\nu\lambda} \) and \( H_{\mu\nu\eta} \) are metric, Christoffel connection and antisymmetric tensors on \( M \), respectively. Using (22), the integrability condition (20) may be rewritten in the following form [2]:
\[ H_{\delta\nu\lambda} = J_{\delta\nu} \sigma J_{\sigma\lambda} + J_{\delta\lambda} \sigma J_{\sigma\nu} + J_{\delta\nu\sigma} J_{\lambda\sigma}, \]  
Furthermore, by introducing the Kähler forms
\[ \omega_{\pm\mu\nu} \equiv g_{\mu\nu} J_{\pm\lambda}, \]
and by the use of (22) one can find
\[ (d\omega_\pm)_{\rho\mu\nu} = \pm (H_{\sigma\rho\mu} J_{\pm\nu}^\sigma + H_{\sigma\rho\nu} J_{\pm\mu}^\sigma + H_{\sigma\nu\rho} J_{\pm\mu}^\sigma), \]  
where
\[ (d\omega_\pm)_{\lambda\sigma\gamma} = \frac{1}{2} (\partial_{\lambda} \omega_{\pm\sigma\gamma} + \partial_{\sigma} \omega_{\pm\lambda\gamma} + \partial_{\gamma} \omega_{\pm\lambda\sigma}). \]  
Finally, using (25) and (26) one can find
\[ H_{\mu\nu\rho} = -J_{\mu\nu} J_{\rho\sigma} (d\omega_\pm)_{\lambda\sigma\gamma} = -J_{\mu\nu} J_{\rho\sigma} (d\omega_\pm)_{\lambda\sigma\gamma}. \]  
In this respect, the target manifold \((M, g, J_\pm)\) is said to have a bi-Hermitian structure if two Hermitian complex structures \( J_\pm \) satisfy relation (28) (i.e. relation between \( (J_+, \omega_+) \) and \( (J_-, \omega_-) \)) which defines the torsion \( H \). Now, for the case where \( M \) is a Lie group \( G \), similar to the process presented in section 2, one can transform relations (19)–(22) and (24) to the algebraic relation using relations (6), (7) and the following relations:
\[ g_{\alpha\beta} = L^\alpha_{\mu} L^\beta_{\nu} g_{\mu\nu} = R_{\alpha}^\mu R_{\beta}^\nu g_{\mu\nu}, \quad g_{\mu\nu} = L^\alpha_{\mu} L^\beta_{\nu} g_{\alpha\beta} = R^\alpha_{\mu} R^\beta_{\nu} g_{\alpha\beta}, \]  
\[ H_{\mu\nu\rho} = \frac{1}{2} L^\alpha_{\mu} L^\beta_{\nu} L^\gamma_{\rho} H_{\alpha\beta\gamma} = \frac{1}{2} R_{\alpha}^\mu R_{\beta}^\nu R_{\gamma}^\rho H_{\alpha\beta\gamma}, \]  
\[ J_{\mu\nu} = R_{\alpha}^\mu J_{\beta} J_{\nu} J_{\rho} J_{\alpha} J_{\mu} J_{\nu} J_{\rho}, \]
where \( L^\alpha_{\mu}(R^\beta_{\nu}) \) and \( R_{\alpha}^\mu(R^\beta_{\nu}) \) are the left (right) invariant vierbeins and their inverses, respectively. Now using these relations, (21) and (24) transform to the following matrix relations:
\[ J g J^t = g, \]  
\[ H_{\alpha} = J (H_{\beta} J_{\alpha}^\beta) + J H_{\alpha} J^t + (H_{\beta} J_{\alpha}^\beta) J^t, \]  
(33)
where $(H_\alpha)_{\beta\gamma} = H_{\alpha\beta\gamma}$. Furthermore, using the following relations [13]:

$$\nabla^\beta L^\alpha_{\mu} = -\frac{1}{2} \left( f^{(\rho\mu)}_{\alpha} + f^{\rho\mu}_{\alpha} + T_{\alpha}^{(\rho\mu)} + T_{\rho\mu}^{\alpha} + L_\beta^\rho L^\mu_{\rho\nu} g_{\gamma\nu} + L_{\rho\nu}^\mu \nabla_\alpha g_{\beta\nu} - L_{\rho\nu}^\mu \nabla_\beta g_{\alpha\nu} \right), \quad (34)$$

$$\nabla^\beta R^\alpha_{\mu} = -\frac{1}{2} \left( -f^{(\rho\mu)}_{\alpha} - f^{\rho\mu}_{\alpha} + T_{\alpha}^{(\rho\mu)} + T_{\rho\mu}^{\alpha} + R_\beta^\rho R^\mu_{\rho\nu} \nabla_\alpha g_{\beta\nu} + R_{\rho\nu}^\mu \nabla_\beta g_{\alpha\nu} - R_{\rho\nu}^\mu \nabla_\rho g_{\alpha\nu} \right), \quad (35)$$

and assuming that $g_{\alpha\beta}$ are coordinate independent, relation (22) transforms to the following algebraic relation:

$$J(H_\alpha - \chi_\alpha g) = (J(H_\alpha - \chi_\alpha g))^t. \quad (36)$$

Note that the metric $g_{\alpha\beta}$ is the ad invariant metric on Lie algebras $g$, i.e. we have

$$\{X_\alpha, X_\beta\} = g_{\alpha\beta}, \quad (37)$$

or in matrix notation we have

$$\chi_\alpha g = -(\chi_\alpha g)^t. \quad (39)$$

Now, one can obtain bi-Hermitian structures on Lie algebras by solving relations (9), (12), (32), (33), (36) and (39) simultaneously. These relations can be applied on the Lie algebra as a definition of the algebraic bi-Hermitian structure on $g$.

**Definition 5.** If there exists endomorphism $J : g \rightarrow g$ of the Lie algebra with an invariant metric $g$ and antisymmetric bilinear map $H : g \otimes g \rightarrow g$ such that relations (9), (12), (32), (33), (36) and (39) are satisfied, then we have a bi-Hermitian structure $(J, g, H)$ on $g$.

Note that relation (33) is equivalent to the matrix relation of integrability condition, i.e. relation (12). For this reason, first it is better to obtain algebraic complex structures $J$, then solve relations (32), (36) and (39) and finally check them in (33). We do this for real four-dimensional Lie algebras using Maple. Note that similar to complex structures, in order to obtain non-equivalent bi-Hermitian structures we suggest the following equivalent relations.

**Definition 6.** Two bi-Hermitian structures $(J, g, H)$ and $(J', g', H')$ of Lie algebras $g$ are equivalent if there exists an element $A$ of the automorphism group of the Lie algebra $g$ (Auto $g$) such that

$$J' = AJA^{-1}, \quad (40)$$

$$g' = AgA^t, \quad (41)$$

$$H'_\alpha = A(H_\beta A_\alpha^\beta)A^t. \quad (42)$$

Note that in these relations all algebraic (target) indices are lowered and raised by $g_{\alpha\beta}$ ($g_{\mu\nu}$); furthermore, these indices transform into each other by $L_{\alpha\beta}^{\mu\nu}$ ($R_{\mu\nu}^{\rho\sigma}$) or $L_{\rho\nu}^{\mu\sigma}$ ($R_{\rho\nu}^{\mu\sigma}$). The symmetrization notation have the following form: $f_{\alpha}^{(\rho\mu)} = f_{\alpha\rho\mu} + f_{\alpha\mu\rho}$.

4 This relation can also be obtained from the algebraic form of (28).

For semisimple Lie algebras, the Killing form is one of the nondegenerate solutions for this equation and other nondegenerate solution may exist. For nonsemisimple Lie algebras the Killing tensor degenerates and there may exist a nondegenerate solution for (39).
| Lie algebra | Complex structures | $g$ | Antisymmetric tensor |
|-------------|------------------|-----|---------------------|
| $A_{4,8}$   | $J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ | $g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ | $H_1 = \begin{pmatrix} 0 & b_1 & 0 & -b_2 \\ -b_1 & 0 & b_2 & 0 \\ b_2 & -b_1 & 0 & 0 \end{pmatrix}$ |
| $J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $g = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$ | $H_2 = \begin{pmatrix} 0 & c_5 - 1 & c_6 & c_5 \\ -c_5 + 1 & 0 & c_6 & c_5 \\ -c_6 & -c_7 & 0 & c_5 \\ -c_7 & -c_8 & -c_9 & 0 \end{pmatrix}$ |
| $VIII \oplus R$ | $J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ | $g = \begin{pmatrix} -a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$ | $H_3 = \begin{pmatrix} 0 & d_5 & d_6 & -d_7 \\ -d_5 & 0 & d_6 + a & 0 \\ -d_6 & -d_7 & 0 & d_5 \\ -d_7 & -d_6 & -a & -d_5 \end{pmatrix}$ |
| $\alpha \in R - \{0\}$ | $J = \begin{pmatrix} 0 & d_5 & d_6 & -d_7 \\ -d_5 & 0 & d_6 + a & 0 \\ -d_6 & -d_7 & 0 & d_5 \\ -d_7 & -d_6 & -a & -d_5 \end{pmatrix}$ | $H_4 = \begin{pmatrix} 0 & d_{15} & d_{14} & -d_{13} \\ -d_{15} & 0 & d_{14} & 0 \\ -d_{14} & -d_{13} & 0 & d_{15} \\ -d_{13} & -d_{14} & -d_{16} & 0 \end{pmatrix}$ |
such that and (32), (33), (36), (39) so the results of table 2 can also be solutions of these equations which must be consistent

Note that results of table 1 are solutions of equations (9) and (12). But the results of table 2 are solutions of (9), (12) and (32), (33), (36), (39) so the results of table 2 can also be solutions of these equations which must be consistent with \( H \) and \( g \).

| Lie algebra | Complex structures | \( g \) | Antisymmetric tensor |
|-------------|-------------------|--------|---------------------|
| \( IX \oplus R \) | \( J, g, H \) | \( \beta \in \mathbb{R} - \{0\} \) |
| \( J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \) | \( g = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -f - \beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix} \) | \( H_1 = \begin{pmatrix} 0 & 0 & f_1 & -f_2 - \beta \\ 0 & 0 & f_2 & f_3 \\ -f_1 & -f_2 & 0 & f_1 \\ f_2 + \beta & -f_1 & -f_3 & 0 \end{pmatrix} \) |
| \( H_2 = \begin{pmatrix} 0 & f_4 & f_5 & -f_6 \\ -f_4 & 0 & f_6 & f_5 - \beta \\ -f_5 & -f_6 & 0 & f_1 \\ f_6 & -f_5 + \beta & -f_1 & 0 \end{pmatrix} \) | \( H_3 = \begin{pmatrix} 0 & f_8 & f_9 & -f_{10} \\ -f_8 & 0 & f_{10} & f_{11} \\ -f_9 & -f_{10} & 0 & f_{11} \\ f_{10} & -f_9 & -f_{11} & 0 \end{pmatrix} \) |
| \( H_4 = \begin{pmatrix} 0 & f_{12} & f_{13} & -f_{14} \\ -f_{12} & 0 & f_{14} & f_{15} \\ -f_{13} & -f_{14} & 0 & f_{15} \\ f_{14} & -f_{13} & -f_{15} & 0 \end{pmatrix} \) |

Note that \( b_i, c_i, d_i, f_i \) are all real parameters.

These relations are equivalent relations and are satisfied in the equivalent conditions. Note that if \( f_{\beta'\gamma} = H_{\beta'\gamma} g^{\alpha\delta} \) or \( H \) is isomorphic with \( f \), i.e. if there exists an isomorphism matrix \( C \) such that

\[
CY^\alpha C^\beta = \tilde{Y}^\beta C_\beta^\alpha, \tag{43}
\]

where \((Y^\alpha)_{\beta'\gamma} = -f_{\beta'\gamma} \) and \((\tilde{Y})_{\beta'\gamma} = -H_{\beta'\gamma} g^{\alpha\delta} \), then \( (J, g, H) \) shows the Manin triple structure on \( g \)[4]. In this way bi-Hermitian structures on real four-dimensional Lie algebras can be classified as table 2. Note that according to the table for the Lie algebra \( A_{4,8} \), we have two non-equivalent bi-Hermitian structures \( (J, g, H) \) where the second bi-Hermitian structure shows the Manin triple structure of \( A_{4,8} \) (i.e. \( A_{4,8} \) is a Manin triple of two-dimensional Lie bialgebras (type B and semi-Abelian) [14]) for the following values of parameters:

\[
c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = c_9 = c_{10} = c_{11} = c_{13} = c_{14} = 0, \quad c_{12} = c_{15} = -1.
\]

For Lie algebras \( VIII \oplus R \), there is one bi-Hermitian structure where this structure, for the values

\[
d_1 = d_2 = d_4 = d_5 = d_7 = d_8 = d_{10} = d_{11} = d_{12} = d_{14} = d_{15} = d_{16} = 0, \quad d_9 \neq 0,
\]

is isomorphic with a two-dimensional Lie bialgebra type A [14]. There exists one bi-Hermitian structure for the Lie algebra \( IX \oplus R \). The results are given in table 2. Note that the isomorphism relation (43) (i.e. the bi-Hermitian structures which show Manin triple) is independent of the choice of special bi-Hermitian structures from an equivalent class of bi-Hermitian structures. In this way if relation (43) holds, then by \( Y^\alpha = -H_{\beta'\gamma} g^{\alpha\delta} \) and using relations (42) and (43) one can show that

\[
(AC)Y^\alpha (AC)'^\gamma = \tilde{Y}^\alpha (AC)^\alpha Y^\gamma. \tag{44}
\]
5. Conclusion

We offered a new method for calculation of complex and bi-Hermitian structures on low-dimensional Lie algebras. By this method, we obtain complex and bi-Hermitian structures on real four-dimensional Lie algebras. In this manner, one can obtain these structures on Lie groups using vierbeins. Some bi-Hermitian structures on real four-dimensional Lie algebras are equivalent to the Manin triple structure obtained in [14]. One can use these methods for obtaining complex and bi-Hermitian structures on real six-dimensional Lie algebras [15]. We also apply this method for calculation of generalized complex structures on four-dimensional real Lie algebras [16].

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Appendix. Real four-dimensional Lie algebras and their automorphism groups [9, 10]

\[ \Lambda_1 = \text{Rotation}_{xy} \text{Boost}_{xz} \text{Boost}_{yz} \mathcal{C} \]  \hspace{1cm} (A.1)

where

\[ \text{Rotation}_{xy} = \begin{pmatrix} \cos(a_1) & \sin(a_1) & 0 & 0 \\ -\sin(a_1) & \cos(a_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  \hspace{1cm} (A.2)

\[ \text{Boost}_{xz} = \begin{pmatrix} \cosh(a_2) & 0 & \sinh(a_2) & 0 \\ 0 & 1 & 0 & 0 \\ \sinh(a_2) & 0 & \cosh(a_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  \hspace{1cm} (A.3)

\[ \text{Boost}_{yz} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh(a_3) & 0 & \sinh(a_3) \\ 0 & \sinh(a_3) & \cosh(a_3) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  \hspace{1cm} (A.4)

\[ \mathcal{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix} \]  \hspace{1cm} (A.5)

\[ \Lambda_2 = \text{Rotation}_{xy} \text{Rotation}_{xz} \text{Rotation}_{yz} \mathcal{C} \]  \hspace{1cm} (A.6)

where

\[ \text{Rotation}_{yz} = \begin{pmatrix} \cos(a_2) & 0 & -\sin(a_2) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(a_2) & 0 & \cos(a_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  \hspace{1cm} (A.7)
Table A1. Classifications of four-dimensional real Lie algebras.

| Lie algebra | Non-vanishing structure constants | Automorphisms group |
|-------------|-----------------------------------|---------------------|
| $4A_1$      |                                   |                     |
| $III \oplus R \cong (A_2 \oplus 2A_1)$ | $f_{12}^2 = -1$, $f_{12}^3 = -1$, $f_{11}^3 = 1$, $f_{11}^3 = 1$ | $egin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & a_5 & a_4 & -a_6 \\ 0 & a_4 & a_5 & a_6 \\ 0 & a_7 & -a_7 & a_8 \end{pmatrix}$ |
| $2A_2$      |                                   |                     |
| $II \oplus R \cong (A_3,1 \oplus A_1)$ | $f_{13}^1 = 1$ | $egin{pmatrix} 1 & a_1 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$ |
| $IV \oplus R \cong (A_3,2 \oplus A_1)$ | $f_{12}^2 = -1$, $f_{12}^3 = 1$, $f_{13}^1 = 1$ | $egin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 & 0 \\ 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & a_6 \end{pmatrix}$ |
| $V \oplus R \cong (A_3,3 \oplus A_1)$ | $f_{12}^2 = -1$, $f_{13}^1 = -1$ | $egin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 & 0 \\ 0 & 0 & a_6 & a_7 \\ 0 & 0 & 0 & a_8 \end{pmatrix}$ |
| $VI_6 \oplus R \cong (A_3,4 \oplus A_1)$ | $f_{13}^2 = 1$, $f_{13}^3 = 1$ | $egin{pmatrix} a_2 & a_1 & 0 & 0 \\ a_1 & a_2 & 0 & 0 \\ a_3 & a_4 & 1 & a_5 \\ 0 & 0 & 0 & a_6 \end{pmatrix}$ |
| $VII_6 \oplus R \cong (A_3,5 \oplus A_1)$ | $f_{12}^2 = -a$, $f_{12}^3 = -1$, $f_{11}^3 = 1$, $f_{11}^3 = a$ | $egin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & a_5 & a_4 & 0 \\ 0 & a_4 & a_5 & 0 \\ 0 & 0 & 0 & a_6 \end{pmatrix}$ |
| $VIII_6 \oplus R \cong (A_3,6 \oplus A_1)$ | $f_{23}^1 = 1$, $f_{13}^3 = -1$ | $egin{pmatrix} a_2 & -a_1 & 0 & 0 \\ a_1 & a_2 & 0 & 0 \\ a_3 & a_4 & 1 & a_5 \\ 0 & 0 & 0 & a_6 \end{pmatrix}$ |
| $VIII_6 \oplus R \cong (A_3,7 \oplus A_1)$ | $f_{11}^2 = 1$, $f_{13}^3 = a$, $f_{13}^3 = -a$, $f_{12}^1 = 1$ | $egin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & a_5 & -a_4 & 0 \\ 0 & a_4 & a_5 & 0 \\ 0 & 0 & 0 & a_6 \end{pmatrix}$ |
| $IX \oplus R \cong (A_3,8 \oplus A_1)$ | $f_{11}^2 = 1$, $f_{12}^1 = 1$, $f_{13}^1 = 1$ | $A_1$ |
| $X \oplus R \cong (A_3,9 \oplus A_1)$ | $f_{11}^2 = 1$, $f_{12}^1 = 1$, $f_{13}^1 = 1$ | $A_2$ |
Table A1. (Continued.)

| Lie algebra | Non-vanishing structure constants | Automorphisms group |
|-------------|-----------------------------------|---------------------|
| $A_{4,1}$   | $f_{24}^1 = 1, f_{34}^1 = 1$      |                     |
|             |                                   |                     |
| $A_{4,2}^a$ | $f_{14}^1 = a, f_{24}^1 = 1,$    |                     |
|             | $f_{34}^1 = 1, f_{34}^1 = 1$      |                     |
| $A_{4,3}^1$ | $f_{14}^1 = 1, f_{34}^1 = 1$      |                     |
| $A_{4,4}$   | $f_{14}^1 = 1, f_{34}^1 = 1,$    |                     |
|             | $f_{24}^1 = 1, f_{34}^1 = 1, f_{34}^1 = 1$ |                     |
| $A_{4,5}^{a,b}$ | $f_{14}^1 = 1, f_{24}^1 = a,$    |                     |
|             | $f_{34}^1 = b$                     |                     |
| $A_{4,5}^{a,a}$ | $f_{14}^1 = 1, f_{24}^1 = a,$ |                     |
|             | $f_{34}^1 = a$                     |                     |
| $A_{4,5}^{a,1}$ | $f_{14}^1 = 1, f_{24}^1 = a,$ |                     |
|             | $f_{34}^1 = 1$                     |                     |
| $A_{4,5}^{1,1}$ | $f_{14}^1 = 1, f_{24}^1 = 1,$ |                     |
|             | $f_{34}^1 = 1, f_{34}^1 = 1$      |                     |
| $A_{4,6}^{a,b}$ | $f_{14}^1 = a, f_{24}^1 = b,$    |                     |
|             | $f_{34}^1 = -1, f_{34}^1 = 1, f_{34}^1 = b$ |                     |
| Lie algebra | Non-vanishing structure constants | Automorphisms group |
|-------------|----------------------------------|---------------------|
| $A_{4,7}$   | $f_{14}^1 = 2$, $f_{24}^2 = 1$, $f_{34}^2 = 1$, $f_{34}^3 = 1$, $f_{23}^1 = 1$ | $\begin{pmatrix} a_1^2 & 0 & 0 & 0 \\ -a_2 a_5 & a_2 & 0 & 0 \\ -a_2 a_5 + a_2 a_4 - a_1 a_5 & a_1 & a_2 & 0 \\ a_3 & a_4 & a_5 & 1 \end{pmatrix}$ |
| $A_{4,8}$   | $f_{24}^2 = 1$, $f_{34}^3 = -1$, $f_{23}^1 = 1$ | $\begin{pmatrix} a_1 a_2 & 0 & 0 & 0 \\ a_1 a_5 & a_1 & 0 & 0 \\ a_2 a_4 & 0 & a_2 & 0 \\ a_3 & a_4 & a_5 & 1 \end{pmatrix}$ |
| $A_{4,9}$   | $f_{14}^1 = 1 + b$, $f_{24}^2 = 1$, $f_{34}^3 = b$, $f_{23}^1 = 1$ | $\begin{pmatrix} a_1 a_4 - a_2 a_3 & 0 & 0 & 0 \\ a_1 a_6 - a_1 a_7 & 1 & 0 & 0 \\ a_3 & a_4 & a_5 & 1 \\ a_6 & a_4 & a_7 & 1 \end{pmatrix}$ |
| $A_{4,9}^0$ | $f_{14}^1 = 2$, $f_{24}^2 = 1$, $f_{34}^3 = 1$, $f_{23}^1 = 1$ | $\begin{pmatrix} a_2 a_3 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 \\ a_1 a_5 & 0 & a_3 & 0 \\ a_4 & a_5 & 0 & 1 \end{pmatrix}$ |
| $A_{4,10}$  | $f_{24}^3 = -1$, $f_{34}^1 = 1$, $f_{23}^1 = 1$ | $\begin{pmatrix} a_2^2 + a_3^2 & 0 & 0 & 0 \\ -a_1 a_2 - a_2 a_5 & a_1 & a_2 & 0 \\ a_2 a_3 - a_1 a_5 & a_2 & 0 & 0 \\ a_3 & a_4 & a_5 & 1 \end{pmatrix}$ |
| $A_{4,11}$  | $f_{14}^1 = 2a$, $f_{24}^2 = a$, $f_{34}^3 = -1$, $f_{34}^1 = 1$ | $\begin{pmatrix} a_2^2 + a_3^2 & 0 & 0 & 0 \\ a (a_1 a_4 - a_2 a_5) & a_1 & a_2 & 0 \\ a (a_2 a_4 - a_1 a_5) & a_1 & a_2 & 0 \\ a_3 & a_4 & a_5 & 1 \end{pmatrix}$ |
| $A_{4,12}$  | $f_{14}^1 = -1$, $f_{23}^1 = 1$, $f_{24}^2 = 1$, $f_{23}^1 = 1$ | $\begin{pmatrix} a_2 & -a_1 & 0 & 0 \\ a_1 & a_2 & 0 & 0 \\ -a_4 & a_3 & 1 & 0 \\ a_3 & a_4 & 0 & 1 \end{pmatrix}$ |

Rotation$_{a,c} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(a_3) & \sin(a_3) & 0 \\ 0 & -\sin(a_3) & \cos(a_3) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (A.8)
\[ C = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & a_4
\end{pmatrix}. \quad (A.9)\]

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