EHRHART THEORY OF POLYTOPES AND SEIBERG–WITTEN INVARIANTS OF PLUMBED 3–MANIFOLDS

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ABSTRACT. Let $M$ be a rational homology sphere plumbed 3–manifold associated with a connected negative definite plumbing graph. We show that its Seiberg–Witten invariants equal certain coefficients of an equivariant multivariable Ehrhart polynomial. For this, we construct the corresponding polytopes from the plumbing graphs together with an action of $H_1(M, \mathbb{Z})$ and we develop Ehrhart theory for them. At an intermediate level we define the ‘periodic constant’ of multivariable series and establish their properties. In this way, one identifies the Seiberg–Witten invariant of a plumbed 3–manifold, the periodic constant of its ‘combinatorial zeta–function’, and a coefficient of the associated Ehrhart polynomial. We make detailed presentations for graphs with at most two nodes. The two node case has surprising connections with the theory of affine monoids of rank two.

1. INTRODUCTION

1.1. The main motivation of the present article is the combinatorial computation of the Seiberg–Witten invariants of negative definite plumbed 3–manifolds. The final output is the identification of these invariants with certain coefficients of a multivariable equivariant Ehrhart polynomial.

Let $\Gamma$ denote the connected negative definite decorated plumbing graph with vertices $V$, which determines the oriented plumbed 3–manifold $M = M(\Gamma)$. We assume that $\Gamma$ is a tree, and all the plumbed surfaces have genus zero, that is, $M$ is a rational homology sphere. We denote by $\text{sw}_\sigma(M)$ the Seiberg–Witten invariants of $M$ indexed by the spin–$c$–structures $\sigma$ of $M$.

In the last years several combinatorial expressions were established regarding the Seiberg–Witten invariants. In [Ni04] Nicolaescu proved (based on the surgery formulas of [MW02]) that they are equivalent with Turaev’s torsion normalized by the Casson–Walker invariant. In terms of $\Gamma$, a combinatorial formula for the Casson–Walker invariant can be deduced from Lescop’s book [L96], while the Turaev’s torsion is determined in [NN02] in terms of a Dedekind–Fourier sum.

For some special graphs, when the Heegaard–Floer homology is determined, we obtain the Seiberg–Witten invariant as the normalized Euler characteristic of the Heegaard–Floer homology [OSz03b, OSz04a, OSz04b]. They can be determined inductively by surgery formulae as well, see e.g. [OSz03b, N05, R04]. [BN10] provides a different type of surgery formula (which is not induced by an exact triangle, but involves the periodic constant of a series — more in the spirit of the present work). In parallel, one can rely on the lattice cohomology too (introduced in [N05, N08a]): in [NT1] the second author proved that the Seiberg–Witten invariant is the normalized Euler characteristic of the lattice cohomology of $M$. Hence, the surgery formulae [N10], and closed formulae for specific families [NR10, N07] provide further examples.

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1.2. The starting point of the present article is the result of [N11], when the Seiberg–Witten invariant appears as the periodic constant of a multivariable series. Next we provide some details.

Let us consider the plumbed 4–manifold $X$ associated with $\Gamma$. Its second homology $L$ is freely generated by the 2–spheres $\{E_v\}_{v \in V}$, and its second cohomology $L'$ by the (anti) dual classes $\{E_v^*\}_{v \in V}$; the intersection form $I = ( , )$ embeds $L$ into $L'$. Set $x^2 := (x, x)$.

Let $K \in L'$ be the canonical class (given by the adjunction relations), $\tilde{\sigma}_{can}$ the canonical $spin^c$–structure on $\tilde{X}$ with $\sigma_{can} = -K$, and $\sigma_{can} \in Spin^c(M)$ its restriction on $M$. Set $H := H_1(M, \mathbb{Z}) = L'/L$. Then $Spin^c(M)$ is an $H$–torsor, with action denoted by $\ast$.

Next, consider the multivariable Taylor expansion $Z(t) = \sum p_v t^l$ at the origin of

$$\prod_{v \in V} (1 - t^{E_v^*})^{\delta_v - 2},$$

where for any $l' = \sum_v l_v E_v \in L'$ we write $t^{l'} = \prod_v t_v^{l_v}$, and $\delta_v$ is the valency of $v$. This lives in $\mathbb{Z}[L']$, the submodule of formal power series $\mathbb{Z}[t_1^{\pm 1/d}, \ldots, t_N^{\pm 1/d}]$ in variables $\{t_v^{\pm 1/d}\}_{v \in V}$, where $d = \det(-I)$.

It has a natural decomposition $Z(t) = \sum_{h \in H} Z_h(t)$, where $Z_h(t) = \sum_{[l'] = h} p_v t^{l'}$ (where $[l']$ is the class of $l'$). Then $\text{sw}_{h \ast \sigma_{can}}(M)$ can be deduced from $Z_h$ as follows [N11].

Assume that $l' = \sum_v a_v E_v^*$ satisfies $a_v \geq -(E_v^2 + 1)$. Then

$$\sum_{l \in L', l \geq 0} p_{l+l'} = -\frac{(K + 2l')^2 + |V|}{8} - \text{sw}_{[-l'] \ast \sigma_{can}}(M).$$

The left hand side appears as a counting function of the coefficients of $Z_h$ associated with a special truncation, while the right hand side is a multivariable quadratic polynomial whose free term is the normalized Seiberg–Witten invariant. In order to guarantee the validity of the formula, the vector $l'$ should sit in a special chamber described by the inequalities of the assumption. This, after we establish the necessary bridges, will read as follows: ‘the third degree’ coefficient of a multivariable Ehrhart polynomial associated with a certain polytope and specific chamber can be identified with the SW invariant.

In fact, the way how one recovers the needed information from the series can be done at several levels. The first one is entirely at the level of series (or Taylor expansions of rational functions). We develop a theory which associates with any series the counting function of its coefficients (given by a truncation of the monomials) — like the right hand side of (1.2.1). This, usually is a piecewise quasipolynomial. Once we fix a chamber, the free term of the counting function is the so called ‘periodic constant’ (denoted by $pc$). In this terminology, the Seiberg–Witten invariant can be interpreted as the multivariable periodic constant $pc(Z)$ of the series $Z(t)$, where the chosen chamber is described by the inequalities of the assumption (a part of the ‘Lipman cone’). The ‘periodicity’ is related with the quasipolynomial behavior of the counting function.) The ‘periodic constant’ of one variable series was introduced in [NO09] [O08], and it had several applications (see e.g. [NO08] [NO09] [N11] [BN10]).

Here we create the general theory, which carries necessarily several difficult technical ingredients (e.g. one has to choose the ‘right’ truncation and summation procedure of the coefficients, which in the context of general series is not automatically motivated, and also it depends on the chamber decomposition of the space of exponents). The theory has some similarities with the theory of vector partition functions.

On the other hand, there is a more sophisticated way to generalize the identity (1.2.1) too.

From any Taylor expansion of a multivariable rational function with denominator of type $\prod_i (1 - t^{a_i})$ we construct a polytope situated in a lattice which carries also a representation
of a finite abelian group $H$. Associated with these data we consider the equivariant multivariable Ehrhart piecewise quasipolynomials, whose existence, main properties (like the Ehrhart–MacDonald–Stanley type reciprocity law or chamber decompositions) will also be established. This applied to the series $Z(t)$ above, and to the quasipolynomial of those chambers which belong to the Lipman cone shows that the first three top–degree coefficients (at least) will carry geometrical/topological meaning, including the SW invariants of the link. (This coefficient identifications, and in fact (1.2.1) too, supplies an additional addendum to the intimate relationship between lattice point counting and the Riemann–Roch formula, exploited in global algebraic geometry by toric geometry.)

Here is a schematic picture of these connections and areas we target:

1.3. The number of terms in the denominator $\prod_i (1 - t^{a_i})$ of the series equals the number of variables of the corresponding partition function (associated with vectors $a_i$), and it is also the rank of the lattice where the corresponding polytope sit. In the case of the series $Z(t)$ associated with plumbing graph, this is the number of end vertices of $\Gamma$. On the other hand, the number of variables of $Z(t)$ is the number $|V|$ of vertices of $\Gamma$. Furthermore, in the Ehrhart theoretical part, the associated (non–convex) polytope will be a union of $|V|$ simplicial polytopes. Hence, with the number of vertices, the number of facets and the complexity of the polytope increases considerably as well.

Nevertheless, the Reduction Theorem [5.4.2] eliminates a part of this abundance of parameters: it says that from the periodic constant point of view, the number of variables of the series, and also the number of simplicial polytopes in the union, can be reduced to the number of nodes of the graph. Hence, in fact, the complexity level is measured by the number of nodes.

In the body of the article, besides the general theory, we make detailed computations for graphs with less than two nodes. Even in the special case of graphs without nodes (that is, the case of lens spaces) the description of the equivariant Ehrhart quasipolynomials is new. In the one node case (start shaped graphs) we provide a detailed presentation of all the involved (SW and Ehrhart) invariants, and we establish closed formulae in terms of the Seifert invariants. Here we make connection with already known topological results regarding the Seiberg–Witten invariants of Seifert 3–manifolds, and also with analytic invariants of weighted homogeneous singularities.

In the two node case again we make complete presentations in terms of the analogs of the Seifert invariants of the chains and star–shaped subgraphs, including closed formulae for $\text{sw}(M)$. But, this case has a very interesting additional surprise in store. It turns out that the corresponding combinatorial series $Z(t)$ associated with $\Gamma$, reduced to the two variables of the nodes, is
the Hilbert (characteristic) series of an affine monoid of rank two (and some of its modules). In particular, the Seiberg–Witten invariant appears as the periodic constant of Hilbert series associated with affine monoids (and certain modules indexed by $H$), and, in some sense, measures the non–normality of these monoids.

1.4. It is important to emphasize that the origin (and main motivation) of the identity (1.2.1) was an analytic identity. Recall that the manifolds $M$ appear as links of complex normal surface singularities, and several of the above objects have their analytic counterparts. For example, the analogue of the series $Z(t)$ is the Hilbert series associated with the multivariable equivariant divisorial filtration of the local ring of the singular germ, and its equivariant periodic constants are the equivariant geometric genera. In the body of the paper we emphasize this parallelism as well, whenever the corresponding analytic invariants coincide with the topological ones. This happens e.g. in the case of star–shaped graphs and the weighted homogeneous analytic structures carried by them. For further relations with analytic structures (e.g. for the Seiberg–Witten Invariant Conjecture targeting these type of connections), see [NN02, N03].

The relevant terminology and additional connections with theory of complex singularities can be found in [AGV84, CHR04, CDG04, EN85, N99, N08b, N08c], its connection with Seiberg Witten theory in [BN10, BN07, N05, N07, N08a, N11, NN02, NN04, Ni04]. For some results in Ehrhart theory, relevant to the present work, see [Bar94, BP99, B99, B02, B04, BS02, BR07, BDR02, CL98, DR97], while for partition functions, see [BV97, SZV03, St95].

1.5. The titles of the sections are the following; they show also the organization of the paper.

2. Normal surface singularities. The main motivation
3. Equivariant multivariable Ehrhart theory
4. Multivariable rational functions and their periodic constants
5. The case of rational functions associated with plumbing graphs
6. The one–node case, star–shaped plumbing graphs
7. The two–node case
8. Ehrhart theoretical interpretation of the SW invariant (the general case).

2. Normal surface singularities. The main motivation

2.1. Surface singularities and their links and graphs.

Let $(X, o)$ be a complex normal surface singularity whose link $M$ is a rational homology sphere. Let $\pi : \tilde{X} \to X$ be a good resolution with dual graph $\Gamma$ whose vertices are denoted by $V$. Hence $\Gamma$ is a tree and all the irreducible exceptional divisors have genus 0. We will write $s$, or $|V|$, for the number of vertices and $H := H_1(M, \mathbb{Z})$.

Set $L := H_2(\tilde{X}, \mathbb{Z})$. It is freely generated by the classes of the irreducible exceptional curves $\{E_v\}_{v \in V}$. $L$ will also be identified with the group of integral cycles supported on $E = \pi^{-1}(o)$. We set $I_{vw} = (E_v, E_w)$. The vertex $v$ of the graph is decorated by $I_{vv}$. The intersection matrix $I = \{I_{vw}\}$ is negative definite, and any connected plumbing graph with negative definite intersection form appears in this way for some singularity. The graph may also serve as the plumbing graph of the link $M = \partial \tilde{X}$. In this case $\tilde{X}$ is the plumbed 4–manifold associated with $\Gamma$, and one might consider this topological starting setup instead of the analytic one.

If $L'$ denotes $H^2(\tilde{X}, \mathbb{Z})$, then the intersection form provides an embedding $L \hookrightarrow L'$ with factor $H^2(\partial \tilde{X}, \mathbb{Z}) \simeq H; [\ell]$ denotes the class of $\ell$. The form $( , )$ extends to $L'$ (since $L' \subset L \otimes \mathbb{Q}$). The module $L'$ over $\mathbb{Z}$ is freely generated by the (anti-)duals $\{E^*_v\}_v$, where we prefer the convention
\( (E'_v, E_w) = -1 \) for \( v = w \), and 0 otherwise. We write \( \det(I) := \det(-I) \). The inverse of \( I \) has entries \((I^{-1})_{vw} = (E'_v, E'_w)\), all of them are negative. Furthermore, cf. [EN85, page 83 and §20],

\[
-|H| \cdot (E'_v, E'_w) \text{ equals the determinant of the subgraph obtained from } \Gamma \text{ by eliminating the shortest path connecting } v \text{ and } w.
\]

The canonical class \( K \in L' \) is defined by the adjunction formulae

\[
(K + E_v, E_v) + 2 = 0 \quad \text{for all } v \in V.
\]

We set \( \chi(l') := -(l', l' + K)/2 \) for any \( l' \in L' \). If \( l \in L \) is effective then by Riemann-Roch theorem \( \chi(l) = h^0(\mathscr{O}) - h^1(\mathscr{O}) \). The integer \( \chi(l') \) has similar analytic interpretation via line bundles of \( \tilde{X} \), cf. [N07, 2.2.8]. The expression \( K^2 + |\mathcal{V}| \) will appear in several formulae. One has the following combinatorial expression in terms of the graph, cf. [NN02]:

\[
K^2 + |\mathcal{V}| = \sum_{v \in V} (E'_v, E'_v) + 3|\mathcal{V}| + 2 + \sum_{v,w \in V} (2 - \delta_v)(2 - \delta_w)I^{-1}_{vw},
\]

where \( \delta_v \) is the valency of the vertex \( v \).

For \( l_1, l_2 \in L \otimes \mathbb{Q} \) one writes \( l_1 \geq l_2 \) if \( l_1 - l_2 = \sum r_v E_v \) with all \( r_v \in \mathbb{Q}_{\geq 0}. \) Denote by \( S' \) the Lipman cone \( \{l' \in L' : (l', E_v) \leq 0 \text{ for all } v\} \). It is generated over \( \mathbb{Z}_{\geq 0} \) by the elements \( E'_v \).

Since all the entries of \( E'_v \) are strict positive, for any fixed \( a \in L' \) one has:

\[
\{l' \in S' : l' \not\in a\} \text{ is finite.}
\]

For any class \( h \in H \) there exists a unique minimal element of \( \{l' \in L' : [l'] = h\} \cap S' \), cf. [N05, 5.4], it will be denoted by \( s_h \). Furthermore, we set \( \square = \{\sum v l'_v E_v \in L' : 0 \leq l'_v < 1\} \) for the ‘semi-open cube’, and for any \( h \in H = L'/L \) we consider the unique representative \( r_h \in \square \) with \( [r_h] = h \). One has \( s_h \geq r_h \), and usually \( s_h \neq r_h \) (see e.g. [N07, 4.5]). Moreover, using the generalized Laufer computation sequence of [N07, 4.3.3] connecting \(-r_h\) with \(-s_h\) one gets

\[
\chi(s_h) \leq \chi(r_h).
\]

Denote by \( \theta : H \to \tilde{H} \) the isomorphism \([l'] \mapsto e^{2\pi i (l', \gamma)} \) of \( H \) with its Pontrjagin dual \( \tilde{H} \). For more details on the resolution graphs see e.g. [N99, N05, N07].

2.1.6. Spin\(^c\)-structures and the Seiberg–Witten invariant of \( M \). Let \( \tilde{\sigma}_{can} \) be the canonical \( \text{spin}^c \)-structure on \( \tilde{X} \); its first Chern class \( c_1(\tilde{\sigma}_{can}) = -K \in L' \), cf. [GS99, p.415]. The set of \( \text{spin}^c \)-structures \( \text{Spin}^c(\tilde{X}) \) of \( \tilde{X} \) is an \( L' \)-torsor; if we denote the \( L' \)-action by \( l' \ast \tilde{\sigma} \), then \( c_1(l' \ast \tilde{\sigma}) = c_1(\tilde{\sigma}) + 2l' \). Furthermore, all the \( \text{spin}^c \)-structures of \( M \) are obtained by restrictions from \( \tilde{X} \). \( \text{Spin}^c(M) \) is an \( H \)-torsor, compatible with the restriction and the projection \( L' \to H \).

The canonical \( \text{spin}^c \)-structure \( \sigma_{can} \) of \( M \) is the restriction of \( \tilde{\sigma}_{can} \).

We denote the Seiberg–Witten invariant by \( \mathfrak{sw} : \text{Spin}^c(M) \to \mathbb{Q}, \sigma \mapsto \mathfrak{sw}_\sigma \).

2.2. Motivation: \( \mathfrak{sw}_\sigma(M) \) as the constant term of a ‘combinatorial Hilbert series’.

Consider the multivariable Taylor expansion \( Z(t) = \sum \sum \sum \sum \sum \) at the origin of

\[
\prod_{v \in V} (1 - t^{E'_v})^{\delta_v - 2},
\]

where for any \( l' = \sum l'_v E_v \in L' \) we write \( t^{l'} = \prod l'_{v_1}^{l'_{v_1}} \) and \( \delta_v \) is the valency of \( v \) as above. This lives in \( \mathbb{Z}[[L']] \), the submodule of formal power series \( \mathbb{Z}[[t^{\pm 1/H}]] \) in variables \( \{t_{v'_{-1/H}}\}_{v'} \).
Theorem 2.2.2. [N11] Fix some $l' \in L'$. Assume that for any $v \in V$ the $E^*_v$-coordinate of $l'$ is larger than or equal to $-(E^*_v + 1)$. Then

$$\sum_{l \in L, l \geq 0} p_{v \cdot l} = -\sigma_{[-v] \cdot \sigma_{\text{can}}}(M) - \frac{(K + 2l')^2 + |V|}{8},$$

where $\cdot$ denotes the torsor action of $H$ on $\text{Spin}^c(M)$. In particular,

$$-\sigma_{[-v] \cdot \sigma_{\text{can}}}(M) - \frac{K^2 + |V|}{8}$$

appears as the constant term of a ‘combinatorial multivariable Hilbert polynomial’ (the right hand side of (2.2.3)).

Since $Z(t)$ is supported on the Lipman cone, by (2.1.4) the sum (2.2.3) is finite.

Note also that the series $Z(t)$ decomposes in several series indexed by elements of $H$. Indeed, $Z(t) = \sum_{h} Z_h(t)$, where $Z_h(t) = \sum_{v : [v] = h} p_{v \cdot t}$. The identity (2.2.3) involves only $Z_{[v]}$.

In fact, the above topological theorem 2.2.2 was motivated by a similar theorem which targets the analytic invariants of the singularity. In order to have a complete picture and possibility to interpret the subsequent results via analytic invariants, we recall briefly this setup as well.

2.3. The analytic motivation: multivariable Hilbert series of divisorial filtrations.

One of the strongest analytic invariants of $(X, o)$ is its equivariant divisorial Hilbert series $\mathcal{H}(t)$. This is defined as follows (for more details, see [N08c, N08b]).

Fix a resolution $\pi$ of $(X, o)$ as in (2.1), let $c : (Y, o) \to (X, o)$ be the universal abelian cover of $(X, o)$ with Galois group $H = H_1(M, \mathbb{Z})$, $\pi_Y : \tilde{Y} \to Y$ the normalized pullback of $\pi$ by $c$, and $\tilde{c} : \tilde{Y} \to \tilde{X}$ the morphism which covers $c$. Then $O_{Y,o}$ inherits the divisorial multi-filtration:

$$\mathcal{F}(l') := \{ f \in O_{Y,o} | \text{div}(f \circ \pi_Y) \geq \tilde{c}^*(l') \}.$$ 

Let $h(l')$ be the dimension of the $\theta([l'])$–eigenspace of $O_{Y,o}/\mathcal{F}(l')$. Then the equivariant divisorial Hilbert series is

$$\mathcal{H}(t) = \sum_{l' \in L'} h(l') t_1^{l_1} \cdots t_8^{l_8} = \sum_{l' \in L'} h(l') t^{l'} \in \mathbb{Z}[[L']].$$

In $\mathcal{H}(t)$ the exponents $l'$ of the terms $t^{l'}$ reflect the $L'/L \simeq H$ eigenspace decomposition too. E.g., $\sum_{l \in L} h(l) t^l$ corresponds to the $H$–invariants, hence it is the Hilbert series of $O_{X,o}$ associated with the $\pi^{-1}(o)$-divisorial multi-filtration (see e.g. [CHR04, CDG04]).

If $l'$ is in the special ‘Kodaira vanishing zone’ $l' \in -K + S$, then by vanishing (of a certain first cohomology), and by Riemann-Roch, one obtains (see [N08c]) that the expression

$$h(l') + \frac{(K + 2l')^2 + |V|}{8}$$

depends only on the class $[l'] \in L'/L$ of $l'$. The key bridge connecting $\mathcal{H}(t)$ with the topology of the link and with $\Gamma$ is done by the series (cf. [CDG04, CDG08, N08b, N08c], [N08c])

$$P(t) = -\mathcal{H}(t) \cdot \prod_v (1 - t_v^{-1}) \in \mathbb{Z}[[L']].$$

Moreover, this identity can be ‘inverted’ (cf. [N08c, (3.2.6)])

$$h(l') = \sum_{l \in L, l \geq 0} \bar{p}_{v \cdot l}, \text{ where } P(t) = \sum_{l'} \bar{p}_{l'} t^{l'}.$$
Theorem 3.1.2. of Stanley [S74], McMullen [M78] and Beck [B99, B02].

expression \( \left( \text{We also suppose that} \right) \)

means that there is a bijection between their faces that preserves the inclusion relation. (This im-

plies that they are connected by homeomorphisms, which preserve the stratification of the faces.)

for any \( l' \in -K + S' \), where \( \text{const}_{[-l']} \) depends only on the class \([-l']\) of \(-l'\). The right hand side can be interpreted as a ‘multivariable Hilbert polynomial’ of degree 2 associated with the series \( \mathcal{H}(t) \), or with \( \mathcal{P}(t) \). Its constant term is the (normalized) equivariant geometric genera of the universal abelian cover \( Y \), that is (cf. [N08c])

\[
\dim(H^1(\widetilde{Y}, \mathcal{O}_Y)_{\theta(h)}) = -\text{const}_{[-r_h]} - \frac{(K + 2r_h)^2 + |\mathcal{V}|}{8}.
\]

The point is that the topological candidate of \( \mathcal{P}(t) \) is exactly \( Z(t) \) from the previous subsection; they agree for several singularities, see e.g. [CDG08, N08b, N08c]. The identification of their constant terms (for ‘nice’ analytic structures) is the subject of the ‘Seiberg–Witten Invariant Conjecture’, cf. [NN02, N03, N07]. Hence, when \( \mathcal{P}(t) = Z(t) \), then \( \text{const}_{[-l']} = \text{sw}_{[-l']s_{\text{can}}}(M) \) too, and (2.3.3) creates the bridge between the combinatorial/ topological Seiberg–Witten theory of the analytic counterpart. The identity \( \mathcal{P}(t) = Z(t) \) is valid e.g. for splice quotient singularities [N08c], which include all the rational singularities (when the links \( M \) are \( L \)-spaces), minimally elliptic singularities, or weighted homogeneous singularities.

3. Equivariant multivariable Ehrhart theory

3.1. Preparatory results on Ehrhart theory.

In this section we generalize the classical Ehrhart theory to the equivariant multivariable version, involving non-convex polytopes, which will fit with our comparison with the equivariant multivariable series provided by plumbing graphs.

Let us start with a \( d \)-dimensional lattice \( \mathcal{X} \subset \mathbb{R}^d \) and a group homomorphism \( \rho : \mathcal{X} \to \mathfrak{h} \) to a finite abelian group \( \mathfrak{h} \). We consider a rational vector–dilated polytope with parameter \( l = (l_1, \ldots, l_r), l_v \in \mathbb{Z}^{m_v} \),

\[
P^{(l)} = \bigcup_{v=1}^r P_v^{(l_v)}, \quad \text{where } P_v^{(l_v)} = \{ x \in \mathbb{R}^d : A_v x \leq l_v \},
\]

with \( A_v \in M_{m_v,d}(\mathbb{Z}) \) (integral \( m_v \times d \) matrices). If \( \{ A_v,\lambda \} \) and \( \{ l_v,\lambda \} \) are the entries of \( A_v \) and \( l_v \), then the inequality \( A_v x \leq l_v \) in (3.1.1) reads as \( \sum_{i=1}^d x_i A_v,\lambda i \leq l_v,\lambda \) for any \( \lambda = 1, \ldots, m_v \).

We will vary the parameter \( l \) in some ‘chambers’ (described below for the needed cases) such that the polytopes \( P^{(l)} \) remain combinatorial equivalent when \( l \) runs in the same chamber. This means that there is a bijection between their faces that preserves the inclusion relation. (This implies that they are connected by homeomorphisms, which preserve the stratification of the faces.) We also suppose that \( P^{(l)} \) is homeomorphic to a \( d \)-dimensional manifold. Denote the set of all closed facets of \( P^{(l)} \) by \( \mathcal{F} \) and let \( \mathcal{T} \) be a subset of \( \mathcal{F} \), such that \( \bigcup_{F^{(l)} \in \mathcal{T}} F^{(l)} \) is homeomorphic to a \( (d - 1) \)-manifold. Then we have the following generalization to the equivariant version of results of Stanley [S74], McMullen [M78] and Beck [B99, B02].

Theorem 3.1.2. For any \( h \in \mathfrak{h} \) and \( \mathcal{T} \subset \mathcal{F} \) let

\[
\mathcal{L}_h(A, \mathcal{T}, l) := \text{cardinality of } \left( (P^{(l)} \setminus \bigcup_{F^{(l)} \in \mathcal{T}} F^{(l)}) \cap \rho^{-1}(h) \right).
\]

(a) If \( l \) moves in some region in such a way that \( P^{(l)} \) stays combinatorially stable then the expression \( \mathcal{L}_h(A, \mathcal{T}, l) \) is a quasipolynomial in \( l \in \mathbb{Z}^{\sum_{m_v}} \).
(b) For a fixed combinatorial type of $P^{(l)}$ and for a fixed $\mathcal{T}$, the quasipolynomials $L_h(A, \mathcal{T}, l)$ and $L_{-h}(A, \mathcal{F} \setminus \mathcal{T}, 1)$ satisfy the Ehrhart–MacDonald–Stanley reciprocity law

\begin{equation}
L_h(A, \mathcal{T}, 1) = (-1)^d \cdot L_{-h}(A, \mathcal{F} \setminus \mathcal{T}, 1)|_{1=1} = \cdot L_{-h}(A, \mathcal{F} \setminus \mathcal{T}, 1)|_{1=1}.
\end{equation}

Proof. The statements for $H = 0$ are identical with those of Beck from [B02]. Part (a) above can be proved identically as in [B02]. Or, we notice that via standard additivity formulae, cf. [B02, § 2], it is enough to prove the statement for each convex $P^{(l)}_{V^{(l)}}$. But, considering $P^{(l)}_{V^{(l)}}$ and $K := \ker(\rho)$, for any $r \in X$ one has the isomorphism

$$\{x \in K + r : A_v x \leq l_v\} \approx \{y \in K : A_v y \leq l_v - A_v r\}.$$  

Hence [CL98, Theorem 2] can be applied, which shows (a). Next, part (b) can also be reduced to [B02]. Indeed, we can reduce the discussion again to $P^{(l)}_{V^{(l)}}$. We drop the index $v$, we choose $r_h \in X$ with $\rho(r_h) = h$, and we fix some $l_0$. Then for $x \in K \pm r_h$ with $Ax \leq l_0$ we take $y := x \mp r_h$ and $k := l_0 \mp Ar_h$, which satisfy $y \in K$ and $Ay \leq k$. Therefore, using [B02] for this polytope and the lattice $K$, and the natural notations:

$$L_h(A, \mathcal{T}, 1)|_{1=l_0} = L(Ay \leq m, \mathcal{T}, y \in K)|_{m=k} = (-1)^d \cdot L_{-h}(A, \mathcal{F} \setminus \mathcal{T}, 1)|_{1=1}.$$  

\[\square\]

Definition 3.1.5. The quasipolynomial $L_h(A, \mathcal{T}, l)$ considered in Theorem 3.1.2 associated with a fixed combinatorial type of $P^{(l)}$, is called the \textit{equivariant multivariable quasipolynomial} associated with the corresponding data.

If we vary $l$ in $\mathbb{Z}^{\sum m_e}$ (hence we allow the variation of the combinatorial type) we obtain the \textit{equivariant multivariable piecewise quasipolynomial} $L_h(A, \mathcal{T}, 1)$ (see also Theorem [4.3.9] and Corollary [4.3.11] below).

Remark 3.1.6. Parallel to the collection \{$L_h$\}_h defined in (3.1.3) one can consider their Fourier transforms as well: for any character $\xi \in \mathfrak{S} = \text{Hom}(\mathfrak{S}, \mathbb{S}^1)$, one defines

\begin{equation}
L_\xi(A, \mathcal{T}, l) := \sum \xi^{-1}(\rho(x)) \text{, sum over } x \in (P^{(l)} \setminus \cup_{F^{(l)} \in \mathcal{T}} F^{(l)}) ,
\end{equation}

which satisfies $L_\xi = \sum_h L_h : \xi^{-1}(h)$, and $|\mathfrak{S}| \cdot L_h = \sum_\xi L_\xi : \xi(h)$. Hence, the above properties of $L_h$ can be obtained from similar properties of $L_\xi$ as well. Hence, Theorem 3.1.2 can be deduced from [BV97, § 4.3] too.

Remark 3.1.8. (a) In the sequel we will not consider polytopes with this high generality: our polytopes will be special ones associated with the denominators of type $\prod_i (1 - t^{n_i})$ of multivariable rational functions, or their Taylor series. In order to avoid unnecessary technical details, the stability of the combinatorial type of $P^{(l)}$, and the corresponding chamber decomposition of $\mathbb{R}^{\sum m_e}$ will also be treated for this special polytopes, see [4.3.7].

(b) To avoid any confusion regarding the expression of (3.1.4) we note: the two quasipolynomials in (3.1.4) are associated with that domain of definition (chamber) which corresponds to the fixed combinatorial type. Usually for $-1$ the combinatorial type of $P^{(l)}$ is different, hence the right hand side of (3.1.4) \textit{does not equal} $(-1)^d \cdot L_{-h}(A, \mathcal{F} \setminus \mathcal{T}, 1)$. This last expression is the value at $-1$ of the quasipolynomial associated with the chamber which contains $-1$. 

\[\square\]
4. Multivariable rational functions and their periodic constants

4.1. Historical remark: the one–variable case [NO09 3.9], [O08 4.8(1)].

Let \( S(t) = \sum_{l \geq 0} c_l t^l \in \mathbb{Z}[[t]] \) be a formal power series. Suppose that for some positive integer \( p \), the expression \( \sum_{i=0}^{p-1} c_i \) is a polynomial \( P_p(n) \) in the variable \( n \). Then the constant term \( P_p(0) \) of \( P_p(n) \) is independent of the ‘period’ \( p \). We call \( P_p(0) \) the periodic constant of \( S \) and denote it by \( \text{pc}(S) \). For example, if \( l \mapsto Q(l) \) is a quasipolynomial and \( S(t) := \sum_{l \geq 0} Q(l)t^l \), then one can take for \( p \) the period of \( Q \), and one shows that \( \text{pc}(\sum_{l \geq 0} Q(l)t^l) = 0 \).

Assume that \( S(t) \) is the Hilbert series associated with a graded algebra/vector space \( A = \bigoplus_{l \geq 0} A_l \) (i.e. \( c_l = \dim A_l \)), and the series \( S \) admits a Hilbert quasipolynomial \( Q(l) \) (that is, \( c_l = Q(l) \) for \( l \gg 0 \)). Since the periodic constant of \( \sum_{l} Q(l)t^l \) is zero, the periodic constant of \( S(t) \) measures exactly the difference between \( S(t) \) and its ‘regularized series’ \( S_{reg}(t) := \sum_{l \geq 0} Q(l)t^l \).

That is: \( \text{pc}(S) = (S(t) - S_{reg}(t))|_{t=1} \) collecting all the anomalies of the starting elements of \( S \).

Note that \( S_{reg}(t) \) can be represented by a rational function of negative degree with denominator of type \( A(t) = \prod l_i (1 - t^{a_i}) \), and \( S(t) - S_{reg}(t) \) is a polynomial. Conversely, one has the following reinterpretation of the periodic constant [BN10 7.0.2]. If \( \sum_{l} c_l t^l \) is a rational function \( B(t)/A(t) \) with \( A(t) = \prod_{l} (1 - t^{a_i}) \), and one rewrites it as \( C(t) + D(t)/A(t) \) with \( C \) and \( D \) polynomials and \( D(t)/A(t) \) of negative degree, then \( \text{pc}(S) = C(1) \). From this fact one also gets that \( \text{pc}(S(t)) = \text{pc}(S(t^N)) \) for any \( N \in \mathbb{Z}_{>0} \). We will refer to \( C(t) \) as the polynomial part of \( S \).

As an example, consider a subset \( S \subset \mathbb{Z}_{>0} \) with finite complement. Then \( S(t) = \sum_{s \in S} t^s \) rewritten is \( 1/(1-t) - \sum_{s \notin S} t^s \), hence \( \text{pc}(S) = -\#(\mathbb{Z}_{>0} \setminus S) \). In particular, if \( S \) is the semigroup of a local irreducible complex plane curve singularity, then \( -\text{pc}(S) \) is the delta–invariant of that germ. Our study below includes the generalization of this fact to surface singularities.

4.2. The setup for the multivariable generalization.

4.2.1. We wish to extend the definition of the periodic constant to the case of Taylor expansions at the origin of multivariable rational functions of type

\[
(4.2.2) \quad f(t) = \frac{\sum_{k=1}^{r} t_k^{b_k}}{\prod_{l=1}^{d} (1 - t^{a_l})} \quad (t_k \in \mathbb{Z}).
\]

Let us explain the notation. Let \( L \) be a lattice of rank \( s \) with fixed bases \( \{E_v\}_{v=1}^s \). Let \( L' \) be an overlattice of it with same rank, \( L \subset L' \subset L \otimes \mathbb{Q} \) with \( |L'/L| = \emptyset \). Then, in (4.2.2), \( \{b_k\}_{k=1}^d \), \( \{a_i\}_{i=1}^d \) \( \in L' \) and for any \( l' = \sum_{v} l'_v E_v \in L' \) we write \( t^{l'} = t_1^{l'_v} \ldots t_s^{l'_v} \). We also assume that all the coordinates \( a_{i,v} \) of \( a_i \) are strict positive. Hence, in general, the coefficients \( l'_v \) are not integral, and the Laurent expansion \( Tf(t) \) of \( f(t) \) at the origin is

\[
Tf(t) = \sum_{l'} p_{l'} t^{l'} \in \mathbb{Z}[[t_1^{1/0}, \ldots, t_s^{1/0}]] [t_1^{-1/0}, \ldots, t_s^{-1/0}] := \mathbb{Z}[[t^{1/0}]] [t^{-1/0}].
\]

We also consider the natural partial ordering of \( L \otimes \mathbb{Q} \) (defined as in 2.1). If all vectors \( b_k \geq 0 \) then \( Tf(t) \) is in \( \sum_{l'} p_{l'} t^{l'} \in \mathbb{Z}[[t^{1/0}]] \). Sometimes we will not make difference between \( f \) and \( Tf \).

4.2.3. This will be extended to the following equivalent case. We fix a finite abelian group \( G \), and for each \( g \in G \) a series (or rational function) \( Tf_g \in \mathbb{Z}[[t^{1/0}]] [t^{-1/0}] \) as in 4.2.1 and we set

\[
Tf^e(t) := \sum_{g \in G} Tf_g(t) \cdot [g] \in \mathbb{Z}[[t^{1/0}]] [t^{-1/0}] [G].
\]
Sometimes this equivariant extension is given automatically in the context of 4.2.1. Indeed, if in 4.2.1 we set $H := L'/L$, and for

\begin{equation}
T f = \sum_{\nu'} p_{\nu'} t'' \quad \text{we define} \quad T f_h := \sum_{[\nu'] = h} p_{\nu'} t'' ,
\end{equation}

we obtain a decomposition of $T f$ as a sum $\sum_h T f_h \in \mathbb{Z}[[t^{1/3}]]t^{-1/3}[H]$ (with $\delta = |H|$).

In our cases we always start with this group $L'/L = H$ (hence $f$ determines its decomposition $\sum_h f_h$). Nevertheless, some alterations will appear. First, we might consider the nonequivariant case, hence we can forget the decomposition over $H$. Another case appears as follows. In order to simplify the rational function we will eliminate some of its variables (e.g., we substitute $t_i = 1$ for certain indices $i$), or we restrict $f$ to a linear subspace $V$. Then, after this substitution, the restricted function $f|_{t_i=1}$ will not determine anymore the restrictions $(f_h)|_{t_i=1}$ of the ‘old’ components $f_h$. That is, the new pair of lattices $(L_V, L'_V) = (L \cap V, L' \cap V)$ and the ‘old group’ $H = L'/L$ become rather independent. In such cases we will keep the old group $H = L'/L$ (and the ‘old’ decomposition $f_h$) without asking any compatibility with $L'_V/L_V$.

**4.2.5.** Since all the coordinates $a_{i,v}$ of $a_i$ are strict positive, for any $T f(t) = \sum_{\nu'} p_{\nu'} t''$ we get a well defined **counting function** of the coefficients,

\[ l' \mapsto Q(l') := \sum_{\nu' \geq l'} p_{\nu'} . \]

If $T f = \sum_h T f_h$, then each $T f_h$ determines a counting function $Q_h$ defined in the same way.

If $H = L'/L$ and $T f$ decomposes into $\sum_h T f_h$ under the law from (4.2.4), then

\begin{equation}
\sum_{\nu' \geq l'} p_{\nu'} \cdot [l''] = \sum_{h \in H} Q_h(l')[h].
\end{equation}

The definitions are motivated by formulae (2.2.3) and (2.3.2). The functions $Q_h(l')$ will be studied in the next subsections via Ehrhart theory.

**4.3. Ehrhart quasipolynomials associated with denominators of rational functions.**

First we consider the case $d > 0$, the special case $d = 0$ will be treated in [4.3.19].

**4.3.1. The polytope associated with $\{a_i\}_{i=1}^d$.** In order to run the Ehrhart theory we have first to fix the lattice $X$ and the representation $\rho : X \rightarrow \mathcal{H}$, cf. section [3]. First, we set $X = \mathbb{Z}^d$ and $\alpha : X \rightarrow L'$ given by $\alpha(x) = \sum_{i=1}^d x_i a_i \in L'$. For $(\mathcal{H}, \rho)$ there are two possibilities:

(a) $\mathcal{H} = H = L'/L$ and $\rho$ is the composition $X \xrightarrow{\alpha} L' \rightarrow L'/L$.

(b) $\mathcal{H} = 0$ and $\rho = 0$.

This choice has an effect on the equivariant decomposition $f^e = \sum_g f_g[g]$ of $f$ too. In case (a) usually we have $G = H$ and the decomposition is given by (4.2.4). In case (b) we can take either $G = 0$ (this can happen e.g. when we forget the decomposition in case (a), and we sum up all the components), or we can take any $G$ (by specifying each $f_g$). In this latter case each fixed $f_g$ behaves like a function in the nonequivariant case $G = 0$, hence can be treated in the same way.

Since the case (b) follows from case (a) (by forgetting the extra information from $\mathcal{H}$), in the sequel we treat the case (a), hence $\mathcal{H} = G = L'/L$.

Consider the matrix $A$ with column vectors $[H]a_i$ and write $A_v$ for its rows. Then the construction of subsection [3] can be repeated (eventually completing each $A_v$ to assure the inequalities
\( x_i \geq 0 \) as well. For \( l \in \sum v l_v E_v \in L \) consider

\begin{equation}
(4.3.2) \quad P_v^a := \{ x \in (\mathbb{R} \geq 0)^d : |H| \cdot \sum_i x_i a_{i,v} < l_v \} \quad \text{and} \quad P^a := \bigcup_{v=1}^s P_v^a.
\end{equation}

The closure \( P_v \) of \( P_v^a \) is a dilated convex (simplicial) polytope depending on the one-dimensional parameter \( l_v \). Moreover, \( P^a \) is described via the partial ordering of \( L \otimes \mathbb{Q} \) as the set \( \sum_i x_i a_i \nless l/|H| \). Since \( L' \subseteq L/|H| \), we can restrict ourselves to the lattice \( L' \) (preserving all the general results of section 3). Hence for any \( l' \in L' \) we set

\begin{equation}
(4.3.3) \quad P(l'), a := \{ x \in (\mathbb{R} \geq 0)^d : \sum_i x_i a_i \nless l' \}, \quad P(l') = \text{closure of } (P(l'), a).
\end{equation}

The combinatorial type of \( P(l') \) might vary with \( l' \). Nevertheless, by definition, the facets will be grouped for all different combinatorial types by the same principle: we consider the coordinate

facets \( F \), \( l \in L' \). Moreover, \( P(l') \) has the collection of all other facets. Hence \( P(l'), a := P(l') \cup \{ F \} \in T F(l') \). The construction is motivated by the summation from (2.2.3) (although in the general statements the choice of \( T \) is irrelevant).

Then [2.2.2] and [4.1] lead to the next counting function defined in the group ring \( \mathbb{Z}[H] \) of \( H \):

\begin{equation}
(4.3.4) \quad L^e(A, T, l') := \sum_{h \in H} L_h(A, T, l') \cdot [h] := \sum \text{1} [l''] \in \mathbb{Z}[H],
\end{equation}

where the last sum runs over \( l'' \in \left( P(l') \cup \{ F \} \right) \cap L' = P(l'), a \cap L' \).

The corresponding nonequivariant counting function, corresponding to \( G = 0 \) is denoted by

\begin{equation}
L_{ne}(A, T, l') := \sum_{h \in H} L_h(A, T, l') \in \mathbb{Z}.
\end{equation}

Similarly, we set \( L^e(A, F \setminus T, l') \) too. For both of them Theorem 3.1.2 applies.

By the very construction, we have the following identity. Consider the equivariant Taylor expansion at the origin of the function determined by the denominator of \( f \), namely

\begin{equation}
(4.3.5) \quad f^{(t)}(t) = \frac{1}{\prod_{i=1}^d (1 - [a_i] t^{a_i})} = \sum_{l''} \tilde{p}_{l''} t^{l''} \cdot [l''] \in \mathbb{Z}[([t^{1/|H|}] [H] ]]
\end{equation}

Note that since all the \( \{ E_v \} \)-coefficients of each \( a_i \) are strict positive, for any \( l' \in L' \) the set \( \{ l'' : \tilde{p}_{l''} \neq 0, l'' \nless l' \} \) is finite. Then, by the above construction,

\begin{equation}
(4.3.6) \quad \sum_{l'' \nless l'} \tilde{p}_{l''} \cdot [l''] = L^e(A, T, l').
\end{equation}

### 4.3.7. Combinatorial types, chambers.

Next, we wish to make precise the combinatorial stability condition. The result of Sturmfels [St95], Brion–Vergne [BV97], Clauss–Loechner [CL98] and Szénes–Vergne [SZV03] implies that \( L^e \) from (4.3.6) (that is, each \( L_h \) is a piecewise quasipolynomial on \( L' \): the parameter space \( L \otimes \mathbb{R} \) decomposes into several chambers, the restriction of \( L^e \) on each chamber is a quasipolynomial, and \( L^e \) is continuous. The chambers are described as follows.

Notice that the combinatorial type of \( P(l') \) in (4.3.3) vary in the same way as the closure of its convex complement in \( \mathbb{R}^d_{\geq 0} \), namely

\begin{equation}
(4.3.8) \quad \{ x \in (\mathbb{R} \geq 0)^d : \sum_i x_i a_i \geq l' \},
\end{equation}
since both are determined by their common boundary \( \mathcal{T} \). The inequalities of (4.3.8) can be viewed as a vector partition \[ \sum_i x_i a_i + \sum_v y_v (-E_v) = l', \] with \( x_i \geq 0 \) and \( y_v \geq 0 \). Hence, according to the above references, we have the following chamber decomposition of \( L \otimes \mathbb{R} \).

Let \( M \) be the matrix with column vectors \( \{a_i\}_{i=1}^d \) and \( \{-E_v\}_{v=1}^s \). A subset \( \sigma \) of indices of columns is called basis if the corresponding columns form a basis of \( L \otimes \mathbb{R} \); in this case we write \( \text{Cone}(M_{\sigma}) \) for the positive closed cone generated by them. Then the chamber decomposition is the polyhedral subdivision of \( L \otimes \mathbb{R} \) provided by the common refinement of the cones \( \text{Cone}(M_{\sigma}) \), where \( \sigma \) runs all over the basis. A chamber is a closed cone of the subdivision whose interior is non-empty. Usually we denote them by \( \mathcal{C} \), let their index set (collection) be \( \mathcal{C} \).

We will need the associated disjoint decomposition of \( L \otimes \mathbb{R} \) with relative open cones as well. A typical element of this disjoint decomposition is the \( \{\mathcal{C}_{\mathcal{T}}\} \) where \( \mathcal{C}_{\mathcal{T}} \) is the closure of the (abstract) quasipolynomial associated with the fixed chamber \( \mathcal{C} \). This also shows that for any two chambers \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) one has the continuity property

\[
\mathcal{L}_h^c(\mathcal{A}, \mathcal{T})|_{\mathcal{C}_1 \cap \mathcal{C}_2} = \mathcal{L}_h^c(\mathcal{A}, \mathcal{T})|_{\mathcal{C}_1 \cap \mathcal{C}_2}.
\]

(c) \( \mathcal{L}_h^c(\mathcal{A}, \mathcal{T}) \) and \( \mathcal{L}_h^c(\mathcal{A}, \mathcal{F} \setminus \mathcal{T}) \) as abstract quasipolynomials associated with a fixed chamber \( \mathcal{C} \), satisfy the reciprocity

\[
\mathcal{L}_h^c(\mathcal{A}, \mathcal{T}, l') = (-1)^d \cdot \mathcal{L}_h^c(\mathcal{A}, \mathcal{F} \setminus \mathcal{T}, -l').
\]

We have the following consequences regarding the counting function \( l' \mapsto Q_h(l') \) of \( f_{c}(t) \) defined in (4.2.6):

**Corollary 4.3.11.** (a) \( Q_h \) is a piecewise quasipolynomial. Indeed, for any \( h \in H \) and \( l' \in L' \)

\[
Q_h(l') = \sum_k t_k \cdot \mathcal{L}_{-h_{-b_k}}^c(\mathcal{A}, \mathcal{T}, l' - b_k).
\]

In particular, the right hand side of (4.3.12) is independent of the representation of \( f \) as in (4.2.2) (that is, of the choice of \( \{b_k, a_i\}_{k,i} \)), it depends only on the rational function \( f \).

(b) Fix a chamber \( \mathcal{C} \) of \( L \otimes \mathbb{R} \), cf. 4.3.9 and for any \( h \in H \) define the quasipolynomial

\[
Q_h^{\mathcal{C}}(l') := \sum_k t_k \cdot \mathcal{L}_{-h_{-b_k}}^c(\mathcal{A}, \mathcal{T}, l' - b_k).
\]
Then the restriction of \( Q_h(l') \) to \( \cap_k (b_k + C) \) is a quasipolynomial, namely

\[
(4.3.14) \quad Q_h(l') = \tilde{Q}_h^C(l') \quad \text{on} \quad \cap_k (b_k + C).
\]

Moreover, there exists \( l'_* \in C \) such that \( l'_* + C \subset \cap_k (b_k + C) \).

[Warning: \( \mathcal{L}_{h-[b_k]}^C(A, T, l' - b_k) \neq \mathcal{L}_{h-[b_k]}^C(A, T, l' - b_k) \) unless \( l' - [b_k] \in C \).]

(c) For any fixed \( h \in H \), the quasipolynomial \( \tilde{Q}_h^C(l') \) satisfies the following property: for any \( l' \in L' \) with \([l'] = h\), and any \( q \in \square \) (the semi-open unit cube), one has

\[
(4.3.15) \quad \tilde{Q}_h^C(l' - q) = \tilde{Q}_h^C(l').
\]

In particular, by taking \( l' = q = r_h \):

\[
(4.3.16) \quad \tilde{Q}_h^C(r_h) = \tilde{Q}_h^C(0).
\]

**Proof.** For (a) use (4.3.3) and the fact that \( b_k + \sum x_i a_i \not\geq l' \) if and only if \( \sum x_i a_i \not\geq l' - b_k \). Since the coefficients of the Taylor expansion depend only on \( f \), the second sentence follows too.

For (b) use part (a) and the fact that \( C \cap (\cap_k (b_k + C)) \) contains a set of type \( l'_* + C \).

(c) Consider those values \( l' \) in some \( l'_* + C \) for which all elements of type \( l' - b_k \) and \( l' - q - b_k \) are in \( C \). For these values \( l' \), (4.3.15) follows from the identity \( P(l') \subseteq \rho^{-1}(h) = P(l' - q) \cap \rho^{-1}(h) \) whenever \([l'] = h\). This is true since for any \( l'' \) with \([l''] = [l']\), \( l'' \geq l' \) is equivalent with \( l'' \geq l' - q \). Indeed, taking \( y = l'' - l' \), this reads as follows: for any \( y \in L \), \( y \geq 0 \) if and only if \( y \geq -q \).

Now, if two quasipolynomials agree on \( l'_0 + C \) then they are equal. \( \square \)

**Remark 4.3.17.** Thanks to [SZV03, Theorem 0.2], the continuity property 4.3.10 has the following extension (coincidence of the quasipolynomials on neighboring strips). Set \( \square(A) := \sum_i [0, 1)a_i \). Then for any two chambers \( C_1 \) and \( C_2 \), and \( S := (-\square(A) + C_1) \cap (-\square(A) + C_2) \)

\[
(4.3.18) \quad \mathcal{L}_{h}^{C_1}(A, T)|_S = \mathcal{L}_{h}^{C_2}(A, T)|_S.
\]

**4.3.19. The \( d = 0 \) case.** All the above properties can be extended for \( d = 0 \) as well. Although the polytope constructed in 4.3.3 does not exist, we can look at the polynomial \( f(t) = \sum_k t_k b_k \) itself. Then using notation of (4.2.6) we set

\[
\sum_{h \in H} Q_h(l')[h] = \sum_{l'' \geq l'} p_{l''} \cdot [l''] = \sum_{\{k : b_k \geq v\}} \epsilon_k [b_k].
\]

Moreover, we have the chamber decomposition of \( L \times \mathbb{R} \) defined by \( \{-E_v\}_{v=1}^s \) via the same principle as above. This means two chambers: \( C_0 := \mathbb{R}_{\geq 0}\langle -E_v \rangle \) and \( C_1 \), the closure of the complement of \( C_0 \) in \( \mathbb{R}^s \). Then \( Q_h(l') = \sum\{k : b_k = h\} \epsilon_k \) on \( \cap_k (b_k + C_1) \) and 0 on \( \cap_k (b_k + C_0) \).

**4.4. The definition of the multivariable equivariant periodic constant of a rational function.**

We consider the situation of 4.2.1 and 4.3.1(a). For each \( h \in H \) define \( r_h \in L' \) as in 2.1

**Definition 4.4.1.** Let \( K \subset L' \times \mathbb{R} \) be a closed real cone whose affine closure \( \text{aff}(K) \) has positive dimension. For any \( h \in H \) we assume that there exist

- \( l'_* \in K \)
- a sublattice \( \tilde{L} \subset L \) of finite index, and
- a quasipolynomial \( l' \mapsto \tilde{Q}_h(l') \), defined on \( \tilde{L} \cap \text{aff}(K) \) such that

\[
(4.4.2) \quad Q_h(l') = \tilde{Q}_h(l') \quad \text{for any} \quad \tilde{L} \cap (l'_* + K).
\]
Then we define the **equivariant periodic constant** of \( f \) associated with \( \mathcal{K} \) by

\[
(pc^{e,\mathcal{K}}(f)) = \sum_{h \in H} pc^{\mathcal{K}}(f)[h] := \sum_{h \in H} \tilde{Q}_h(0) \cdot [h] \in \mathbb{Z}[H],
\]

and we say that \( f \) **admits a periodic constant in** \( \mathcal{K} \). (Sometimes we will use the same notation for the real cone \( \mathcal{K} \) and for its lattice points \( \mathcal{K} \cap L' \) in \( L' \).)

**Remark 4.4.4.** The above definition is independent of the choice of the sublattice \( \tilde{L} \); it can be replaced by any sublattice of finite index. The advantage of such sublattices is that convenient restrictions of \( Q_h \) might have nicer forms which are easier to compute. The choice of \( \tilde{L} \) corresponds to the choice of \( p \) in [4.1] and it is responsible for the name ‘periodic’ in the name of \( pc^{e,\mathcal{K}}(f) \).

**Proposition 4.4.5.** (a) Consider the chamber decomposition of \( L \otimes \mathbb{R} \) given by the denominator \( \prod_i (1 - t^{\alpha_i}) \) of \( f \) as in Theorem 4.3.9 Then \( f \) admits a periodic constant in each chamber \( \mathcal{C} \) and

\[
(pc^{\mathcal{C}}(f)) = \tilde{Q}_h(r_h) = \tilde{Q}_h(0).
\]

(b) If two functions \( f_1 \) and \( f_2 \) admit periodic constant in some cone \( \mathcal{K} \), then the same is true for \( \alpha_1 f_1 + \alpha_2 f_2 \) and

\[
(pc^{\mathcal{K}}(\alpha_1 f_1 + \alpha_2 f_2)) = \alpha_1 pc^{\mathcal{K}}(f_1) + \alpha_2 pc^{\mathcal{K}}(f_2) \quad (\alpha_1, \alpha_2 \in \mathbb{C}).
\]

(c) If \( f \) admits periodic constants in two (top dimensional) cones \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \), and the interior \( \text{int}(\mathcal{K}_1 \cap \mathcal{K}_2) \) of the intersection \( \mathcal{K}_1 \cap \mathcal{K}_2 \) is non-empty, then \( pc^{\mathcal{K}_1}(f) = pc^{\mathcal{K}_2}(f) \).

In particular, if \( \{C_i\}_{i=1,2} \) are two chambers as in (a), and \( f \) admits a periodic constant in \( \mathcal{K} \), and \( \text{int}(C_i \cap \mathcal{K}) \neq \emptyset \) \((i = 1, 2)\), then \( pc^{C}\!(f) = pc^{C'}\!(f) \).

**Proof.** For (a) use Corollary 4.3.11 (b) is clear. For (c) we can assume that \( \mathcal{K}_2 \subset \mathcal{K}_1 \) (by considering \( \mathcal{K}_1 \) and \( \mathcal{K}_1 \cap \mathcal{K}_2 \)). Then if \( Q_h \) is quasipolynomial on \( l'_1 + \mathcal{K}_1 \) (with \( l'_1 \in \mathcal{K}_1 \)), then \( (l'_1 + \mathcal{K}_2) \cap \mathcal{K}_2 \) contains a set of type \( l'_2 + \mathcal{K}_2 \) with \( l'_2 \in \mathcal{K}_2 \), on which one can take the restriction of the previous quasipolynomial.

**Remark 4.4.7.** Note that in the rational presentation of \( f \) we might assume that \( a_i \in L \) for all \( i \). Indeed, take \( a_i \in \mathbb{Z}_{>0} \) such that \( a_i a_j \in L \), and amplify the fraction by \( \prod_i (1 - t^{a_i})/(1 - t^{a_i}) \).

Therefore, for each \( h \) we can write \( f_h(t) \) in the form

\[
f_h(t) = t^{r_h} \sum_k t_k \cdot \frac{t^{b_k}}{\prod_i (1 - t^{a_i})},
\]

where \( a_i, b_k \in L \), hence \( f_h(t)/t^{r_h} \in \mathbb{Z}[t][[t^{-1}]] \). Then if we consider the nonequivariant periodic constant \( pc^{\mathcal{C}} \) of \( f_h(t)/t^{r_h} \), 4.2.6, 4.3.14 and 4.4.6 imply that \( pc^{\mathcal{C}}(f(t)) = pc^{\mathcal{C}}(f_h(t)/t^{r_h}) \) for all chambers \( \mathcal{C} \) associated with \( \{a_i\}_i \).

**Example 4.4.8.** Assume that \( L = L' = \mathbb{Z} \) and \( \mathcal{K} = \mathbb{R}_{\geq 0} \), and consider \( S(t) \) as in [4.1] If \( S(t) \) admits a periodic constant in \( \mathcal{K} \), then the definition of \( pc(S) \) from 4.4.1 is compatible with the statements from [4.1].

**Example 4.4.9.** (a) (The \( d = 0 \) case) Assume that \( f(t) = \sum_{k=1}^r \alpha_k t^{b_k} \). Then, using 4.3.19 (and its notation), \( pc^{e,\mathcal{C}_0}(f) = 0 \) and \( pc^{e,\mathcal{C}_1}(f) = \sum_{k=1}^r \alpha_k [b_k] \in \mathbb{Z}[H] \).

(b) Assume that the rank is \( s = 2 \) and \( f(t) = t^b/(1 - t^a) \), with both the entries \((a_1, a_2)\) of \( a \) positive. We assume that \( a \in L \) while \( b \in L' \). Again, for \( h \neq [b] \) the counting function, hence its periodic constant too, is zero. Assume \( h = [b] \), and write \( b = (b_1, b_2) \). Then the denominator provides three chambers: \( \mathcal{C}_0 := \mathbb{Z}_{\geq 0}\langle -E_1, -E_2 \rangle, \mathcal{C}_1 := \mathbb{Z}_{\geq 0}\langle a, -E_2 \rangle, \mathcal{C}_2 := \mathbb{Z}_{\geq 0}\langle a, -E_1 \rangle \). Then
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the three quasipolynomials for $1/(1-t^a)$ are $L_h^{c_0} = 0$, $L_h^{c_i}(n_1, n_2) = [n_i/a_i]$; hence $pc_h^{c_0}(f) = 0$, $pc_h^{c_i}(f) = [-b_i/a_i]$ ($i = 1, 2$). In particular, $pc_h^{c_1}(f)$, in general, depends on the choice of $C$.

(c) Assume that $L = L'$ and $f(t) = \frac{\prod_{i=1}^{d} t^{b_i}}{(1-t_{i_1}t_{i_2}) \cdots (1-t_{i_l_1}t_{i_l_2})}$. Then the chambers associated with the denominator are: $C_0 := \mathbb{R}_{\geq 0}(-E_1, -E_2)$, $C_2 := \mathbb{R}_{\geq 0}(1, 1), C := \mathbb{R}_{\geq 0}(1, 2)$ and $C_1 := \mathbb{R}_{\geq 0}(2, 1, -E_2)$. Then, by a computation,

$$L^{c_0} = 0; \quad L^{c_2}(l_1, l_2) = \frac{l_1^2}{2} + \frac{l_2^2}{2} - \frac{l_1}{2} l_2; \quad L^{c_1}(l_1, l_2) = \frac{l_1^2}{4} + \frac{l_2^2}{4} + \frac{1 + (-1)^{l_1+1}}{8}.$$ 

Hence, by Proposition 4.4.5 and (4.3.13), one has $pc^{c_1}(f) = L^{c_1}(-b_1, -b_2)$.

**Example 4.4.11. Normal affine monoids.** Consider the following objects (cf. 4.2.1): a lattice $L$ with fixed bases $\{E_i\}_{i=1}^d$ (hence $s = d$) and with induced partial ordering $\leq$, $L' \subset L \otimes \mathbb{Q}$ an overlattice with finite abelian quotient $H := L'/L$ and projection $\rho : L' \to H$. Furthermore, let $\{a_i\}_{i=1}^d$ be linearly independent vectors in $L'$ with all their $\{E_i\}$–coordinates positive. Let $K$ be the positive real cone generated by the vectors $\{a_i\}_i$, and consider the Hilbert series of $K$

$$f(t) := \sum_{t' \in K \cap L'} t'.$$

Since $K$ depends only on the rays generated by the vectors $a_i$, we can assume that $a_i \in L$ for all $i$.

Set $\square(A) = \sum_{i=1}^d [0, 1)a_i$ as above, and consider the monoid $M := \mathbb{Z}_{\geq 0}\langle a_i \rangle$ (cf. e.g. [BG08, 2.C]). Then the normal affine monoid $K \cap L'$ is a module over $M$ and if we set $B := \square(A) \cap L'$, [BG08, Prop. 2.43] implies that

$$K \cap L' = \bigcup_{b \in B} b + M.$$ 

In particular, $f(t)$ equals $\sum_{b \in B} b^t / \prod_{i=1}^{d} (1 - t^{a_i})$ and has the form considered in 4.2.

If the rank $d \geq 3$ then $K$ usually is cut in more chambers. Indeed, take e.g. $d = 3$, $a_i = (1,1,1) + E_i$ for $i = 1, 2, 3$. Then $K$ is cut in its barycentric subdivision. Nevertheless, if $d = 2$ then $K$ consists of a unique chamber and $f$ admits a periodic constant in $K$. Indeed, one has:

**Lemma 4.4.12.** If $d = 2$ then $pc^K(f) = 0$ for all $h \in H$.

**Proof.** It is elementary to see that $K$ is one of the chambers (use the construction from 4.3.7). Take $B = \{b_k\}_k$, and write $f = \sum_k f_k$, where $f_k = t^{b_k} / (1 - t^{a_1})(1 - t^{a_2})$. The only relevant classes $h \in H$ are given by $\{[b_i] : b_i \in B\}$, otherwise already the Ehrhart quasipolynomials are zero (since $a_i \in L$). Fix such a class $h = [b_i]$. Let $L^K_h(T)$ be the quasipolynomial associated with the chamber $K$ and the denominator of $f$. Then, by (4.4.6) and (4.3.13), $pc^K(f_k) = L^K_{[b_i-b_k]}(T, -b_k)$.

This, by the Reciprocity Law 4.3.9(c) equals $L^K_{[b_k-b_i]}(F \setminus T, b_k)$. Again, since the denominator is a series in $L$, for $[b_k - b_i] \neq 0$ the series is zero; so we may assume $[b_k - b_i] = 0$. But, since $b_k \in K$, the value $L^K_{[b_k]}(F \setminus T, b_k)$ of the quasipolynomial carries its geometric meaning, it is the cardinality of the set $\{m = n_1a_1 + n_2a_2 : n_1 > 0, n_2 > 0, m \neq b_k\}$. But since for any such $m$ one has $m \geq a_1 + a_2 > b_k$, contradicting $m \neq b_k$, this set is empty.

**Example 4.4.13. General affine monoids of rank $d = 2$.** Consider the situation of Example 4.4.11 with $d = 2$, and let $N$ be a submonoid of $\hat{N} = K \cap L'$ of rank 2, and we also assume that $\hat{N}$ is the normalization of $N$. Set

$$f(t) := \sum_{t' \in N} t'. $$
Then $f(t)$ is again of type (4.2.2). Indeed, by [BG08 Prop. 2.35], $\hat{N} \setminus N$ is a union of finite family of sets of type (I) $b \in \hat{N}$, or (II) $b + Z\mathbb{Z}_{a_i}$, where $b \in \hat{N}, \ell \in \mathbb{Z}_{\geq 0}, i = 1$ or $2$. Obviously, two sets of type (II) with different $i$-values might have an intersection point of type (I). In particular,

$$f(t) = \sum_{v \in \hat{N}} t^v - \sum_i \frac{t^{b_i,1}}{1 - t^{b_i,1a_i}} - \sum_j \frac{t^{b_j,2}}{1 - t^{b_j,2a_2}} + \sum_k (\pm t^{b_k}).$$

Note that the periodic constant of the first sum is zero by Lemma [4.4.12] and the others can easily be computed (even with closed formulae) via Example [4.4.9] parts (a) and (b).

The computation shows that the periodic constant carries information about the failure of normality of $N$ (compare with the delta-invariant computation from the end of 4.1).

The situation is similar when we consider a semigroup of $\hat{N}$, that is, when we eliminate the neutral element of the above $N$ (or, when we consider a module over the submonoid $N \subset \hat{N}$).

**Example 4.4.14. Reduction of variables.** The next statement is an example when the number of variables of the function $f$ can be reduced in the procedure of the periodic constant computation. (For another reduction result, see Theorem [5.4.2]). For simplicity we assume $L' = L$.

**Proposition 4.4.15.** Let $f(t) = \frac{t^v}{\prod_{i=1}^{n}(1-t^{a_i})}$ and assume that $b = \sum_{v=1}^{s} b_v E_v \in \mathcal{C}$, where $\mathcal{C}$ is a chamber associated with the denominator.

We consider the subset $\text{Pos} := \{ v : b_v > 0 \}$ with cardinality $p$, and the projection $\mathbb{R}^s \to \mathbb{R}^p$, defined by $(r_v)_{v=1}^{s} \mapsto (r_v)_{v \in \text{Pos}}$ and denoted by $v \mapsto v^\top$. Accordingly, we set a new function $f^\top(z) := \frac{z^{a_1}}{\prod_{i=1}^{n}(1-z^{a_i})}$ in $p$ variables, and a new chamber $\mathcal{C}^\top := \mathbb{R}_{\geq 0}(\{w_j^\top\}_j)$, where $w_j$ are the generators of $\mathcal{C} = \mathbb{R}_{\geq 0}(\{w_j\}_j)$. Then $\text{pc}^\top(f) = \text{pc}^\mathcal{C}(f^\top)$.

**Proof.** This is a direct application of Theorem [3.1.2] (b). Indeed, by the Ehrhart–MacDonald–Stanley reciprocity law, we get $\text{pc}^\mathcal{C}(f) = \mathcal{L}^\mathcal{C}(A, \mathcal{T}, -b) = (-1)^d \mathcal{L}^\mathcal{C}(A, \mathcal{F} \setminus \mathcal{T}, b)$. Since $b \in \mathcal{C}$, by the very definition of $\mathcal{L}^\mathcal{C}(A, \mathcal{F} \setminus \mathcal{T})$, this (modulo the sign) equals the number of integral points of $P^{(b)} \setminus \bigcup_{F^{(b)} \in \mathcal{F} \setminus \mathcal{T}} F^{(b)} \subset \mathbb{R}^d$. But, if $v \notin \text{Pos}$, or $b_v \leq 0$, then in (3.1.1) $P_v^{(b)}$ has only non-positive integral points. Therefore we can omit these polytopes without affecting the periodic constant. Then, this fact and $b_v \in \mathcal{C}^\top$ imply that $\text{pc}^\mathcal{C}$ can be computed as $(-1)^d \mathcal{L}^\mathcal{C}(A, \mathcal{F} \setminus \mathcal{T}, b^\top, b^\top)$. □

**Remark 4.4.16.** Under the conditions of Proposition [4.4.15] we have the following application of the statement from Remark [4.3.17] (based on [SZV03]): Assume that $b \in \square(A) - \mathcal{C}$ and $b \geq 0$. Then $\text{pc}^\mathcal{C}(f) = 0$. Indeed, $\text{pc}^\mathcal{C}(f) = \mathcal{L}^\mathcal{C}(A, \mathcal{T}, -b) = \mathcal{L}^\mathcal{C}(-b)(A, \mathcal{T}, -b)$, where $\mathcal{C}(-b)$ is a chamber containing $-b$. But since $-b \leq 0$ one gets $\mathcal{L}^\mathcal{C}(-b)(A, \mathcal{T}, -b) = 0$ by [4.4.15].

One of the key messages of the above examples (starting from 4.4.9) is the following: ‘if $b$ is small compared with the $a_i$’s, then the periodic constant is zero’ (compare with 4.1 too).

### 4.5. The polynomial part of rational functions with $d = s = 2$.

In this case $\text{rank}(L) = 2$, and we have two vectors in the denominator of $f$, namely $a_i = (a_{i,1}, a_{i,2}), i = 1, 2$. We will order them in such a way that $a_2$ sits in the cone of $a_1$ and $E_1$, that is, $\det(a_{2,1} a_{2,2}) < 0$. The chamber decomposition will be the following: $C_0 := \mathbb{R}_{\geq 0}(-E_1, -E_2)$, $C_2 := \mathbb{R}_{\geq 0}(-E_1, a_1)$, $\mathcal{C} := \mathbb{R}_{\geq 0}(a_1, a_2)$ and $\mathcal{C}_1 := \mathbb{R}_{\geq 0}(a_2, -E_2)$ (the index choice is motivated by the formulae from [4.4.9] (b)).

Our goal is to write any rational function (with denominator $(1 - t^{a_1})(1 - t^{a_2})$) as a sum of $f^+(t)$ and $f^-(t)$, such that $f^+ \in \mathbb{Z}[L]$ (the ‘polynomial part of $f$’), and $\text{pc}^\mathcal{C}(f^-) = 0$. This is a generalization of the decomposition in the one–variable case discussed in 4.1 and will be a
major tool in the computation of the periodic constant is section 7 for graphs with two nodes. The specific form of the decomposition is motivated by Examples 4.4.9(b) and 4.4.11.

As above, we set □(A) = [0, 1)a1 + [0, 1)a2 and for i = 1, 2 we also consider the strips

\[ Ξ_i := \{b = (b_1, b_2) ∈ L ⊗ \mathbb{R} \mid 0 ≤ b_i < a_i \}. \]

**Theorem 4.5.1.** (1) Any function \( f(t) = (\sum_{k=1}^r t_k b_k) / \prod_{i=1}^2 (1 - t^{a_i}) \) can be written as a sum \( f(t) = f^+(t) + f^-(t) \), where

(a) \( f^+(t) \) is a finite sum \( \sum_\ell \kappa_\ell t^\beta_\ell \), with \( \kappa_\ell ∈ \mathbb{Z} \) and \( \beta_\ell ∈ L' \);

(b) \( f^-(t) \) has the form

\[ f^-(t) = \frac{\sum_{k=1}^r t_k b_k^{'} t^{b_k^{'}}} {\prod_{i=1}^2 (1 - t^{a_i})} + \frac{\sum_{i=1}^{n_1} t_{1,i} b_{1,i}} {1 - t^{a_1}} + \frac{\sum_{i=1}^{n_2} t_{2,i} b_{2,i}} {1 - t^{a_2}}, \]

with \( b_k^{'} ∈ L' ∩ □(A) \) for all \( k \), and \( b_{i,j} ∈ L' ∩ Ξ_j \) for any \( i \) and \( j = 1, 2 \).

(2) Consider a sum

\[ \Sigma(t) := \frac{Q(t)} {\prod_{i=1}^2 (1 - t^{a_i})} + \frac{Q_1(t)} {1 - t^{a_1}} + \frac{Q_2(t)} {1 - t^{a_2}} + f^+(t), \]

where \( Q(t) := \sum_{k=1}^r t_k b_k^{'} \) with \( b_k^{'} ∈ L' ∩ □(A) \) for all \( k \); \( Q_1(t) = \sum_{i=1}^{n_1} t_{1,i} b_{1,i} \) with \( b_{i,j} ∈ L' ∩ Ξ_j \) for any \( i \) and \( j = 1, 2 \); and finally \( f^+ ∈ \mathbb{Z}[L'] \) is a polynomial as in part (a) above.

Then, if \( \Sigma(t) = 0 \), then \( Q(t) = Q_1(t) = Q_2(t) = f^+(t) = 0 \).

In particular, the decomposition in part (1) is unique.

(3) The periodic constant of \( f^-(t) \) associated with the chamber \( C \) is zero. Hence, in the decomposition (1) one also has \( pc^{C}(f) = pc^{C}(f^+) = \sum_\ell \kappa_\ell [\beta_\ell] ∈ \mathbb{Z}[H] \).

**Remark 4.5.4.** (a) In the expression of \( f \) and \( f^- \) above we wished to emphasize that even the summation \( \sum_{k=1}^r \) and coefficients \( t_k \) are preserved when we decompose \( f \) into \( f^- \) and \( f^+ \). In fact, for every \( b_k ∈ L' \) we have a unique \( b_k^{'} ∈ L' ∩ □(A) \) such that \( b_k - b_k^{'} ∈ \mathbb{Z}(a_i) \). This is how we get from the expression of \( f \) the first fraction in (4.5.2).

(b) This decomposition is associated with a choice of a chamber: here the chamber is \( C \), and the decomposition satisfies property (3) of the chamber \( C \), where this choice is hidden.

Although in the present work we will not use any other decomposition, we note that in general any chamber \( C \) provides a decomposition with \( f^+ ∈ \mathbb{Z}[H] \) and \( pc^{C}(f^+) = 0 \) (and usually these decompositions are different). For example, for \( C_0 \), \( pc^{C_0}(f) = 0 \), hence we can take \( f^+ = 0 \).

**Proof.** First one determines the first fraction of \( f^- \) as it is explained in Remark 4.5.4(a). Then one has to decompose fractions of type \( (1 - t^{k_{1a_1} + k_{2a_2}}) / \prod_{i=1}^2 (1 - t^{a_i}) \) which is again elementary.

Part (2) is again elementary algebra. Or, proceed as follows. The vanishing of \( Q \) follows again by the unicity of the choice of \( b_k^{'}, b_k \) in (4.5.4)(a). For the others, take a convenient filtration of \( \mathbb{Z}[L'] \) (e.g. by integral multiples of \( Ξ_1, \) resp. of \( Ξ_2 \)).

The vanishing of the periodic constant of the first fraction of \( f^- \) follows from the proof of Lemma 4.4.12. The vanishing of \( pc^{C}(f) \) of the other two fractions follows from Example 4.4.9(b). For the last expression see Example 4.4.9(a). □

5. The case of rational functions associated with plumbing graphs

5.1. A ‘classical’ connection between polytopes and gauge invariants (and its limits).

In the literature of normal surface singularities there is a sequence of results which connect the topology of the link with the number of lattice points in a certain polytope. Here are some details.
The first step is based on the theory of hypersurface singularities with Newton nondegenerate principal part, see e.g. [AGV84]. According to this, for such a germ one defines the Newton polytope $\Gamma_N^-$ using the nontrivial monomials of the defining equation of the germ, and one proves that several invariants of the germ can be recovered from $\Gamma_N^-$, see e.g. [BN07]. E.g., by a result of Merle and Teissier [MT80], the geometric genus $p_g$ equals with the number of lattice points in $(\mathbb{Z}_{\geq 0})^2 \cap \Gamma_N^-)$. The second step is provided by Laufer–Durfee formula, which determines the signature of the Milnor fiber $\sigma$ as $-8p_g-K^2-|\mathcal{V}|$ [D78]. Finally, there is a conjecture of Neumann and Wahl, formulated for hypersurfaces with integral homology sphere links [NW91], and proved e.g. for Brieskorn, suspension [NW91] and splice quotient [NO09] singularities, according to which $\sigma/8 = \lambda(M)$, the Casson invariant of the link. Therefore, if all these steps run, e.g. in the Brieskorn case, then the Casson invariant of the link, normalized by $K^2 + |\mathcal{V}|$, can be expressed as the number of lattice points of a polytope associated with the equation of the germ.

[For the computations of the lattice points in the case simplicial polytopes in terms of Dedekind sums see e.g. [BP99, B99, BS02, DR97] and the citations therein, for its relation with the Riemann Roch formulae see e.g. [CS94, KK92, P93] or literature of classical toric geometry, while for the sums see e.g. [BP99, B99, BS02, DR97] and the citations therein, for its relation with the Riemann signature of the Milnor fiber $\sigma$ as $-8p_g-K^2-|\mathcal{V}|$ [D78]. Final

The present article defines another polytope, which carries an action of the group $H$, and its Ehrhart invariants determine the Seiberg–Witten invariant in any case. It is not described from the equations of the germ, but from its multivariable ‘zeta-function’ $Z(t)$. Furthermore, the polytope is a union of several simplices, and those coefficients of the Ehrhart polynomial which carry the information about the Seiberg–Witten invariant will be determined.

5.2. The new construction. Applications of Section 4.

Consider the topological setup of a surface singularity, as in subsection 2.1. The lattice $L$ has a canonical basis $\{E_v\}_{v \in \mathcal{V}}$ corresponding to the vertices of the graph $\Gamma$. We investigate the periodic constant of the rational function $Z(t)$, defined in 2.2, from $\Gamma$. Since $Z(t)$ has the form (4.2.2), all the results of section 4 can be applied. In particular, if $\mathcal{E} = \{v \in \mathcal{V} : \delta_v = 1\}$ denotes the set of ends of the graph, then $A$ has column vectors $a_v = E_v^*$ for $v \in \mathcal{E}$. Hence, the rank of the lattice/space where the polytopes $P^{(t')} = \cup_v P_v$ sit is $d = |\mathcal{E}|$, and the convex polytopes $\{P_v\}$ are indexed by $\mathcal{V}$. Furthermore, the dilation parameter $t'$ of the polytopes runs in a $|\mathcal{V}|$–dimensional space. In the sequel we will drop the symbol $A$ from $L_n^h(A, T, t')$.

[The construction has some analogies with the construction of the splice quotient singularities [NW05]: in that case the equations of the universal abelian cover of the singularity are written in $\mathbb{C}^d$, together with an action of $H$. Nevertheless, in the present situation, we are not obstructed with the semigroup and congruence relations present in that theory.]}

In this new construction, a crucial additional ingredient comes from singularity theory, it is Theorem 2.2.2 (in fact, this is the main starting point and motivation of the whole approach). This combined with facts from Section 4 give:

**Corollary 5.2.1.** Let $S = S_\mathbb{R}$ be the (real) Lipman cone $\{x \in \mathbb{R}^{|\mathcal{V}|} : (x, E_v) \leq 0 \text{ for all } v\}$. 


(a) The rational function \( Z(t) \) admits a periodic constant in the cone \( S \), which equals the normalized Seiberg–Witten invariant

\[
\text{pc}^S_h(Z) = -\frac{(K + 2r_h)^2 + |\mathcal{V}|}{8} - \text{sw}_{-h+\sigma_{\text{con}}}(M).
\]

(b) Consider the chamber decomposition associated with the denominator of \( Z(t) \) as in Theorem 4.3.9 and let \( C \) be a chamber such that \( \text{int}(C \cap S) \neq \emptyset \). Then \( Z(t) \) admits a periodic constant in \( C \), which equals both \( \text{pc}^S_h(Z) \) (satisfying \( 5.2.2 \)) and also

\[
\text{pc}^C_h(Z) = \sum_k t_k \cdot \mathcal{L}_{h-\lambda}^C(T, -b_k) = \sum_k t_k \cdot \mathcal{L}_{h}^C(F \setminus T, b_k).
\]

In particular, \( \text{pc}^C_h(Z) \) does not depend on the choice of \( C \) (under the above assumption).

**Proof.** Write \( l = \tilde{l} + r_h \) with \( \tilde{l} \in L \) in \( 2.2.3 \). Since \( \sum_{l \in \mathbb{Z}, l \geq 0} p_{l} = \sum_{l \in \mathbb{Z}, l \geq 0} p_{l} \), (a) follows from Theorem 2.2.2. For (b) use Corollary 4.3.11 and Proposition 4.4.5. \( \square \)

We note that the Lipman cone \( S \) can indeed be cut in several chambers (of the denominator of \( Z \)). This can happen even in the simple case of Brieskorn germs. Below we provide such an example together with several exemplifying details of the construction.

**Example 5.2.4. Lipman cone cut in several chambers.** Consider the 3–manifold \( S^3_{1}(T_{2,3}) \) (where \( T_{2,3} \) is the right-handed trefoil knot), or, equivalently, the link of the hypersurface singularity \( z_1^2 + z_2^3 + z_3^7 = 0 \). Its plumbing graph \( \Gamma \) and matrix \(-I^{-1}\) are:

\[
\begin{array}{ccc}
E_1 & E_0 & E_3 \\
-2 & & -7 \\
E_2 & -3 & \\
\end{array}
\quad
-I^{-1} = \begin{pmatrix}
42 & 21 & 14 & 6 \\
21 & 11 & 7 & 3 \\
14 & 7 & 5 & 2 \\
6 & 3 & 2 & 1 \\
\end{pmatrix}
\]

where the row/column vectors of \(-I^{-1}\) are \( E_0^*, E_1^*, E_2^* \) and \( E_3^* \) in the \( \{E_0\} \) basis. The polytopes defined in \((3.1.1)\), or in \((4.3.2)\), with parameter \( l = (l_0, l_1, l_2, l_3) \subset \mathbb{Z}^4 \), sit in \( \mathbb{R}^3 \). Let \( u_1, u_2, u_3 \) be the basis of \( \mathbb{R}^3 \). Then the polytopes are the following convex closures:

\[
\begin{align*}
P_0^{(l)} &= \text{conv} \left( 0, (l_0/21) u_1, (l_0/14) u_2, (l_0/6) u_3 \right) \\
P_1^{(l)} &= \text{conv} \left( 0, (l_1/11) u_1, (l_1/7) u_2, (l_1/3) u_3 \right) \\
P_2^{(l)} &= \text{conv} \left( 0, (l_2/7) u_1, (l_2/5) u_2, (l_2/2) u_3 \right) \\
P_3^{(l)} &= \text{conv} \left( 0, (l_3/3) u_1, (l_3/2) u_2, (l_3) u_3 \right).
\end{align*}
\]

Since \( E_0^* + \varepsilon(-E_0^*) \) is in the interior of the (real) Lipman cone for \( 0 < \varepsilon \ll 1 \), we get that the Lipman cone is cut in several chambers. The periodic constant can be computed with any of them. In fact, by the continuity of the quasipolynomials associated with the chambers, any quasipolynomial associated with a chamber which contains any ray in the Lipman cone, even if it is situated at its boundary, provides the periodic constant. One such degenerated polytope provided by a ray on the boundary of \( S \) is of special interest. Namely, if we take \( l = \lambda E_0^* \in \mathbf{S} \) for \( \lambda > 0 \), then \( P^{(l)} = \bigcup_{v=0}^{a} P_v^{(l)} \) is the same as \( P_0^{(l)} = \text{conv}(0, 2\lambda u_1, 3\lambda u_2, 7\lambda u_3) \). Moreover, if \( C \) is any chamber which contains the ray \( \mathbb{R}_{\geq 0} E_0^* \) at its boundary, then for any \( l = \lambda E_0^* \) one has \( \mathcal{L}^C(A, T, l) = \mathcal{L}(\tilde{P}_0, T, \lambda) \), where the last is the classical Ehrhart polynomial of the tetrahedron \( \tilde{P}_0 := \text{conv}(0, 2u_1, 3u_2, 7u_3) \). Here we witness an additional coincidence of \( \tilde{P}_0 \) with the Newton polytope \( \Gamma^\perp_N \) of the equation \( z_1^2 + z_2^3 + z_3^7 \).
We compute $L(\tilde{P}_0, T, \lambda)$ as follows. From (2.3.2)–(2.3.3) and Corollary 4.3.11, we get that
\[(5.2.5) \quad \chi(\lambda E_0^*) + \text{geometric genus of } \{ z_1^2 + z_2^3 + z_3^7 = 0 \} = L(\tilde{P}_0, T, \lambda) - L(\tilde{P}_0, T, \lambda - 1).\]

Since this geometric genus is 1, and the free term of $L(\tilde{P}_0, T, \lambda)$ is zero (since for $\lambda = 0$ the zero polytope with boundary conditions contains no lattice point), and $-K = 2E_0 + E_1 + E_2 + E_3$, we get that $L(\tilde{P}_0, T, \lambda) = 7\lambda^3 + 10\lambda^2 + 4\lambda$. We emphasize that a formula as in (5.2.5), realizing a bridge between the Riemann–Roch expression $\chi$ (supplemented with the geometric genus) and the Ehrhart polynomial of the Newton diagram, was not known for Newton nondegenerate germs.

In the sequel we will provide several examples, when the Newton polytope is not even defined.

5.3. Example. The case of lens spaces.

5.3.1. As we will see in Theorem 5.4.2, the complexity of the problem depends basically on the number of nodes $N = \{ v \in V : \delta_v \geq 3 \}$ of $\Gamma$. In this subsection we treat the case when there are no nodes at all, that is $M$ is a lens space. In this case the numerator of the rational function $f(t)$ is 1, hence everything is described by the 2–dimensional polytopes determined by the denominator. In the literature there are several results about lens spaces fitting in the present program, here we collect the relevant ones completing with the new interpretations. This subsection also serves as a preparatory part, or model, for the study of chains of arbitrary graphs.

5.3.2. The setup. Assume that the plumbing graph is 
\[ -k_1 \quad -k_2 \quad \cdots \quad -k_{s-1} \quad -k_s \]
with all $k_v \geq 2$, and $p/q$ is expressed via the (Hirzebruch, or negative) continued fraction
\[(5.3.3) \quad [k_1, \ldots, k_s] = k_1 - 1/(k_2 - 1/(\cdots - 1/k_s)\cdots).\]

Then $M$ is the lens space $L(p, q)$. We also define $q'$ by $q'q \equiv 1 \mod p$, and $0 \leq q' < p$. Furthermore, we set $g_v = [E_v^*] \in H$. Then $g_s$ generates $H = \mathbb{Z}_p$, and any element of $H$ can be written as $ag_s$ for some $0 \leq a < p$. Recall the definitions of $r_h$ and $s_h$ from 2.1 as well.

From analytic point of view $(X, o)$ is a cyclic quotient singularity $(\mathbb{C}^2, o)/\mathbb{Z}_p$, where the action is $\xi * (x, y) = (\xi x, \xi^a y)$ (here $\xi$ runs over $p$–roots of unity).

5.3.4. The Seiberg–Witten invariant. Since $(X, o)$ is rational, in this case $Z(t) = P(t)$ (cf. subsection 2.3). Moreover, in (2.3.3) $H^1(O_{\tilde{V}}) = 0$, hence
\[(5.3.5) \quad \text{sw}_{-h*\sigma_{can}}(M) = -\frac{(K + 2r_h)^2 + |V|}{8} - \frac{K^2 + |V|}{8} + \chi(r_h).\]

On the other hand, in [N05, N08a] a similar formula is proved for the SW-invariant: one only has to replace in (5.3.5) $\chi(r_h)$ by $\chi(s_h)$. In particular, for lens spaces, and for any $h \in H$ one has
\[(5.3.6) \quad \chi(r_h) = \chi(s_h).\]

[Note that, in general, for other links, $\chi(r_h) > \chi(s_h)$ might happen, see Example 6.4.8. Here, (5.3.6) follows from the vanishing of the geometric genus of the universal abelian cover of $(X, o)$.]

In general, the coefficients of the representatives $s_{ag}$ and $r_{ag}$, $(0 \leq a < p)$ are rather complicated arithmetical expressions; for $s_{ag}$, see [N05] 10.3 (where $g_a$ is defined with opposite sign). The value $\chi(s_{ag})$ is computed in [N05] 10.5.1 as
\[(5.3.7) \quad \chi(s_{ag}) = \frac{a(1-p)}{2p} + \sum_{j=1}^{a} \left\{ \frac{jq'}{p} \right\}.\]
For completeness of the discussion we also recall that $K = E_1^* + E_s^* - \sum_v E_v$ and (5.3.11)

\[(K^2 + |\mathcal{V}|)/4 = (p - 1)/(2p) - 3 \cdot s(q, p),\]
cf. \cite[10.5]{N05}, where $s(q, p)$ denotes the Dedekind sum.

\[s(q, p) = \sum_{l=0}^{p-1} \left(\left(\frac{l}{p}\right)\right) \left(\frac{ql}{p}\right), \quad \text{where} \quad \left(\frac{x}{\cdot}\right) = \begin{cases} \{x\} - 1/2 & \text{if} \ x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if} \ x \in \mathbb{Z}. \end{cases}\]

In particular, $\text{sm}_{-h^*\sigma_{can}}(M)$ is determined via the formulæ (5.3.5) – (5.3.8).

The non-equivariant picture looks as follows: $\sum_h \text{sm}_{-h^*\sigma_{can}} = \lambda$, the Casson–Walker invariant of $M$, hence (5.3.5) gives

\[\lambda = -p(K^2 + |\mathcal{V}|)/8 + \sum_h \chi(r_h).\]

This is compatible with (5.3.8) and formulæ $\lambda(L(p, q)) = p \cdot s(q, p)/2$ and $\sum_h \chi(r_h) = (p - 1)/4 - p \cdot s(q, p)$, cf. \cite[10.8]{N05}.

### 5.3.9. The polytope and its quasipolynomial.

We compare the above data with Ehrhart theory. In this case $Z(t) = (1 - t^{E_1^*})^{-1}(1 - t^{E_s^*})^{-1}$. The vectors $a_1 = E_1^*$ and $a_s = E_s^*$ determine the polytopes $P(l')$ and a chamber decomposition.

For $1 \leq v \leq w \leq s$ let $n_{vw}$ denote the numerator of the continued fraction $[k_v, \ldots, k_w]$ (or, the determinant of the corresponding bamboo subgraph). For example, $n_{1s} = p$, $n_{2s} = q$ and $n_{1s-1} = q'$. We also set $n_{p+1,v} := 1$. Then $pE_1^* = \sum_{v=1}^s n_{v+1,s} E_v$ and $pE_s^* = \sum_{v=1}^s n_{1,v-1} E_v$.

In particular, for any $l' = \sum_v l'_v E_v \in S'$, the (non-convex) polytopes are

\[(5.3.10) \quad P(l') = \bigcup_{v=1}^s \left\{ (x_1, x_s) \in \mathbb{R}^2_{\geq 0} : x_1 n_{v+1,s} + x_s n_{1,v-1} - l'_v \leq 0 \right\} \subset \mathbb{R}^2_{\geq 0}.\]

The representation $\mathbb{Z}^2 \xrightarrow{\rho} \mathbb{Z}_p$ is $(x_1, x_s) \mapsto (q x_1 + x_s) g_s$.

Though $P(l')$ is a plane polytope, the direct computation of its equivariant Ehrhart multivariable polynomial (associated with a chamber, or just with the Lipman cone) is still highly non-trivial. Here we will rely again on Theorem \[2.2.2\]. On a subset of type $t_0^* + S'$ the identity \[2.2.3\] provides the counting function. The right hand side of \[2.2.3\] depends on all the coordinates of $l'$, hence all the triangles $P_v$ contribute in $P(l')$. Since this can happen only in a unique combinatorial way, we get that there is a chamber $C$ which contains the Lipman cone. Let $L^{c,C}$ be its quasipolynomial, and $L^{c,S}$ its restriction on $S$. Since the numerator of $Z(t)$ is 1, $Q_h C = L_h^c$. Since this agrees with the right hand side of \[2.2.3\] on a cone of type $t_0^* + S'$, and the Lipman cone is in $C$, we get that

\[(5.3.11) \quad Q_h(l') = \overline{Q}_{h}(l') = L_h^{S}(l') = -\text{sm}_{-h^*\sigma_{can}}(M) - \frac{(K + 2l')^2 + |\mathcal{V}|}{8}\]

for any $l' \in (r_h + L) \cap S'$ and $h \in H$. Using the identity \[5.3.5\], this reads as

\[(5.3.12) \quad L_h^{S}(C,l') = \chi(l') - \chi(r_h), \quad l' \in (r_h + L) \cap S'.\]

Note that for any fixed $h$ and any $l'$ there exists a unique $q = q_{v,h} \in \square$ such that $l' + q := l'' \in r_h + L$. Indeed, take for $q$ the representative of $r_h - l'$ in $\square$. Then \[4.3.15\] and \[5.3.12\] imply

\[(5.3.13) \quad L_h^{S}(C,l') = L_h^{S}(C,l'') = \chi(l' + q_{v,h} - \chi(r_h)).\]

This formula emphasizes the periodic behavior of $L_h^{S}(C,l')$ as well.

If $l'$ is an element of $L$ then $q_{v,h} = r_h$, hence \[5.3.13\] gives in this case

\[(5.3.14) \quad L_h^{S}(C,l) = \chi(l + r_h) - \chi(r_h) = \chi(l) - (l, r_h) \quad \text{for} \ l \in L \cap S'.\]
In particular, \( \text{pc} (L_h^C (T)) = \chi (r_h) - \chi (r_h) = 0 \) (a fact compatible with \( H^1 (O_Y) = 0 \)).

Even the non-equivariant case looks rather interesting. Let \( L_{ne}^S (T) = \sum_{h \in H} L_h^C (T) \) be the Ehrhart polynomial of \( P^{(l')} \) (with boundary condition \( T \)), where we count all the lattice points independently of their class in \( H \). Then, (5.3.14) gives for \( l \in L \cap S \)

\[
L_{ne}^S (T, l) = p \cdot \chi (l) - (l, \sum_h r_h) = -p \cdot (l, l)/2 - p \cdot (l, K)/2 - (l, \sum_h r_h).
\]

In fact, \( \sum_h r_h \) can explicitly be computed. Indeed, set \( d_v = \gcd (p, n_{1, v-1}) \) and \( p_v = p/d_v \). Then one checks that \( aE_v^* = \sum_v n_{1, v-1}^{-1} a E_v, r_h = \sum_v \{ n_{1, v-1}^{-1} a \} E_v \) and \( \sum_h r_h = \sum_v d_v p_v^{-1} E_v \).

The coefficients of the polynomial \( L_{ne}^S (T, l) \) can be compared with the coefficients given by general theory of Ehrhart polynomials applied for \( P^{(l)} \). E.g., the leading coefficient gives

\[-p \cdot (l, l)/2 = \text{Euclidian area of } P^{(l)} .\]

Knowing that in \( P^{(l)} \) all the \( P_v \)'s contribute, and it depends on \( s \) parameters, and the intersection of their boundary is messy, the simplicity and conceptual form of (5.3.15) is rather remarkable.

5.4. **Reduction of the variables of \( Z(t) \).**

Let \( \mathcal{N} \) denote the set of nodes \( \{ v \in V : \delta_v \geq 3 \} \). Let \( \mathcal{S}_\mathcal{N} \) be the positive cone \( \mathbb{R}_{\geq 0} \langle E_n^* \rangle_{n \in \mathcal{N}} \) generated by the dual base elements indexed by \( \mathcal{N} \), and \( V_\mathcal{N} := \mathbb{R} \langle E_n^* \rangle_{n \in \mathcal{N}} \) be its supporting linear subspace in \( L \otimes \mathbb{R} \). Clearly \( \mathcal{S}_N \subset \mathcal{S} \). Furthermore, consider \( L_\mathcal{N} := \mathbb{Z} \langle E_n \rangle_{n \in \mathcal{N}} \) generated by the node base elements, and the projection \( pr_\mathcal{N} : L \otimes \mathbb{R} \rightarrow L_\mathcal{N} \otimes \mathbb{R} \) on the node coordinates.

**Lemma 5.4.1.** The restriction of \( pr_\mathcal{N} \) to \( V_\mathcal{N} \), namely \( pr_\mathcal{N} : V_\mathcal{N} \rightarrow L_\mathcal{N} \otimes \mathbb{R} \), is an isomorphism.

**Proof.** Follows from the negative definiteness of the intersection form of the plumbing, which guarantees that any minor situated centrally on the diagonal is nondegenerate. \( \square \)

Our goal is to prove that restricting the counting function to the subspace \( V_\mathcal{N} \), the non-node variables of \( Z(t) \) and \( Q(t') \) became non-visible, hence they can be eliminated. This will provide a remarkable simplification in the periodic constant computation. But, before any elimination-substitution, we have first to decompose our series \( Z(t) \) into \( \sum_{h \in H} Z_h(t)[h] \) if we wish to preserve the information about all the \( H \) invariants, cf. the comment at the end of 4.2.3.

**Theorem 5.4.2.** (a) The restriction of \( L_h (A, T, l') \) to \( V_\mathcal{N} \) depends only on those coordinates which are indexed by the nodes (that is, it depends only on \( pr_\mathcal{N} (l') \) whenever \( l' \in V_\mathcal{N} \)).

(b) The same is true for the counting function \( Q_h \) associated with \( Z_h(t) \) as well. In other words, if we consider the restriction

\[ Z_h (t_\mathcal{N}) := Z_h (t) |_{t_v = 1 \text{ for all } v \notin \mathcal{N}} \]

then for any \( l' \in L_\mathcal{N} \), the counting functions \( \sum_{u' \not\geq v} p_{w'} [l'] \) of \( Z_h(t) \) and \( Z_h(t_\mathcal{N}) \) are the same.

(c) Consider the chamber decomposition of \( \mathcal{S}_\mathcal{N} \) by intersections of type \( \mathcal{C}_\mathcal{N} := \mathcal{C} \cap \mathcal{S}_\mathcal{N} \), where \( \mathcal{C} \) denotes a chamber (of \( Z \)) such that \( \text{int} (\mathcal{C} \cap \mathcal{S}) \neq \emptyset \), and the intersection of \( \mathcal{C} \) with the relative interior of \( \mathcal{S}_\mathcal{N} \) is also non-empty. Then

\[
(5.4.3) \quad \text{pc}^C (Z_h(t)) = \text{pc}^{C_\mathcal{N}} (Z_h(t_\mathcal{N})).
\]

The theorem applies as follows. Assume that we are interested in the computation of \( \text{pc}^C (Z(t)) \) for some chamber \( C \) (e.g. when \( C \subset \mathcal{S} \), cf. Corollary 5.2.1). Assume that \( C \) intersects the relative interior of \( \mathcal{S}_\mathcal{N} \). Then, the restriction to \( C \cap \mathcal{S}_\mathcal{N} \) of the quasipolynomial associated with \( C \) has two
properties: it still preserves sufficiently information to determine $p_{C_h}(Z(t))$ (via the periodic constant of the restriction, see (5.4.3)), but it also has the advantage that for these dilatation parameters $l'$ the union $\bigcup_{v \in V} P_v^{(l'),\alpha}$ equals the union of essentially much less polytopes, namely $\bigcup_{n \in N} P_n^{(l'),\alpha}$.

For example, when we have only one node, one has to handle only one convex simplicial polytope instead of a union of $|V|$ simplices.

**Proof.** (a) We show that for any $l' \in V_N$ one has the inclusions

$$P_v^{(l'),\alpha} \subset \bigcup_{n \in N} P_n^{(l'),\alpha} \text{ for any } v \notin N.$$  

We consider two cases. First we assume that $v$ is on a leg (chain) connecting an end $e(v) \in E$ with a node $n(v)$ (where $e(v) = v$ is also possible). Then, clearly, (5.4.4) follows from

$$P_v^{(l'),\alpha} \subset P_n^{(l'),\alpha} \text{ for any } l' \in S_N.$$ 

Let $E_{vw}^* = (E_e^*)_v = -(E_{e^*}^*)_v$ be the $v$–coordinate of $E_{vw}^*$. Note that $E_{vw}^* = E_{vu}^*$. Using the definition of the polytopes, (5.4.5) is equivalent with the implication (cf. 4.3.1)

$$\sum_{e \in E} x_e E_{ve}^* < l' \implies \sum_{e \in E} x_e E_{n(v)e}^* < l'(n(v)) \text{ for any } l' \in S_N \text{ and } x_e \geq 0.$$ 

Let $\mathcal{W}$ be the set of vertices on this leg (including $e(v)$ but not $n(v)$). Then, one verifies that there exist positive rational numbers $\alpha$ and $\{\alpha_w\}_{w \in \mathcal{W}}$, such that

$$E_v^* = \alpha E_{\alpha(v)}^* + \sum_{w \in \mathcal{W}} \alpha_w E_w^*.$$ 

The numbers $\alpha$ and $\{\alpha_w\}_{w \in \mathcal{W}}$ can be determined from the linear system obtained by intersecting the identity (5.4.7) by $\{E_w\}_w$ and $E_{n(v)}$. Intersecting (5.4.7) by $E_e^* (e \in E)$, we get that $E_{ve}^* = \alpha E_{n(v)e}^*$ for any $e \neq e(v)$, and $E_{v,e,v}^* = \alpha E_{n(v),e(v)}^* + \alpha_{e(v)}$. Hence

$$\sum_{e \in E} x_e E_{ve}^* = \alpha \sum_{e \in E} x_e E_{n(v)e}^* + x_{e(v)} \alpha_{e(v)}.$$ 

On the other hand, intersecting (5.4.7) with $E_{n,v}^*$, for $n \in N$, we get $E_{vn}^* = \alpha E_{n(v)n}^*$. Since $l'$ is a linear combination of $E_{n,v}^*$'s, we get that

$$-l'_v = (l', E_v^*) = \alpha (l', E_{n(v)}^*) = -\alpha l'_{n(v)}.$$ 

Since $x_{e(v)} \alpha_{e(v)} \geq 0$, (5.4.8) and (5.4.9) imply (5.4.6). This ends the proof of this case.

Next, we assume that $v$ is on a chain connecting two nodes $n(v)$ and $m(v)$. Let $\mathcal{W}$ be the set of vertices on this bamboo (not including $n(v)$ and $m(v)$). Then we will show that

$$P_v^{(l'),\alpha} \subset P_n^{(l'),\alpha} \cup P_m^{(l'),\alpha} \text{ for any } l' \in S_N.$$ 

This follows as above from the existence of positive rational numbers $\alpha$, $\beta$ and $\{\alpha_w\}_{w \in \mathcal{W}}$ with

$$E_v^* = \alpha E_{n(v)}^* + \beta E_{m(v)}^* + \sum_{w \in \mathcal{W}} \alpha_w E_w^*.$$ 

(b) follows from (a) and from the fact that all $b_k$ entries in the numerator of $Z(t)$ belong to $S_N$.

(c) If $p_{C_l}(Z_h(t))$ is computed as $\tilde{Q}_h(0)$ for some quasipolynomial $\widetilde{\mathcal{Q}}_h$ defined on $\tilde{L} \subset L$, then part (b) shows that $p_{C_N}(Z_h(t_N))$ can be computed as $(\tilde{Q}_h|_{\tilde{L} \cap S_N})(0)$, which equals $\tilde{Q}_h(0)$. □

**Example 5.4.12.** Consider the following graph $\Gamma$: 

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Ehrhart theory and Seiberg–Witten invariants
6. THE ONE–NODE CASE, STAR–SHAPEO PLUMBING GRAPHS

6.1. Seifert invariants and other notations. Assume that the graph is star–shaped with \( d \) legs. Each leg is a chain with normalized Seifert invariant \((\alpha_i, \omega_i)\), where \( 0 < \omega_i < \alpha_i \), \( \gcd(\alpha_i, \omega_i) = 1 \). We also use \( \omega'_i \) satisfying \( \omega_i \omega'_i \equiv 1 \pmod{\alpha_i} \), \( 0 < \omega'_i < \alpha_i \).

If we consider the Hirzebruch/negative continued fraction expansion, cf. \((5.3.3)\)

\[
\alpha_i/\omega_i = [b_{i1}, \ldots, b_{i\nu_i}] = b_{i1} - 1/(b_{i2} - 1/(\cdots - 1/b_{i\nu_i}) \cdots) \quad (b_{ij} \geq 2),
\]

then the \( i \)-th leg has \( \nu_i \) vertices, say \( v_{i1}, \ldots, v_{i\nu_i} \), with decorations \( -b_{i1}, \ldots, -b_{i\nu_i} \), where \( v_{i1} \) is connected by the central vertex. The corresponding base elements in \( L \) are \( \{E_{ij}\}_{j=1}^{\nu_i} \). Let \( b \) be the decoration of the central vertex; this vertex also defines \( E_0 \in L \). The plumbed 3–manifold \( M \) associated with such a star–shaped graph has a Seifert structure. It is a rational homology sphere if and only if the central vertex has genus zero; this fact will be assumed in the sequel.

The classes in \( H = L'/L \) of the dual base elements are denoted by \( g_{ij} = [E_{ij}^*] \) and \( g_0 = [E_0^*] \). For simplicity we also write \( E_i := E_{i\nu_i} \) and \( g_i := g_{i\nu_i} \). A possible presentation of \( H \) is

\[
H = ab\langle g_0, g_1, \ldots, g_d \mid -b \cdot g_0 = \sum_{i=1}^{d} \omega_i \cdot g_i; g_0 = \alpha_i \cdot g_i \ (1 \leq i \leq d) \rangle,
\]

cf. \[\text{[Ne81]}\] (or use \( \sum_k I_k g_k \) repeatedly, see also \((6.1.3)\)). The orbifold Euler number of \( M \) is defined as \( e = b + \sum_i \omega_i/\alpha_i \). The negative definiteness of the intersection form implies \( e < 0 \). We write \( \alpha := \text{lcm}(\alpha_1, \ldots, \alpha_d) \), \( \delta := |H| \) and \( \sigma \) for the order of \( g_0 \) in \( H \). One has (see e.g. \[\text{[Ne81]}\])

\[
\delta = \alpha_1 \cdots \alpha_d |e|, \quad \sigma = \alpha |e|.
\]

Each leg has similar invariants as the graph of a lens space, cf. Example \[\text{[5.3]}\] and we can introduce similar notation. For example, the determinant of the \( i \)-th leg is \( \alpha_i \). We write \( n_{ij}^{i+j} \) for the determinant of the sub-chain of the \( i \)-th leg connecting the vertices \( v_{ij} \), and \( v_{ij} \) (including these vertices too). Then, using the correspondence between intersection pairing of the dual base elements and the determinants of the subgraphs, cf. \((2.1.1)\) or \[\text{[N05]}\] 11.1, one has

\[
(a) \quad (E_{ij}^*, E_{ij}^* - n_{ij}^{i+j+1} E_{i\nu_i}^*) = 0 \quad (b) \quad g_{ij} = n_{ij}^{i+j+1} g_{i\nu_i} \quad (1 \leq i \leq d, \ 1 \leq j \leq \nu_i)
\]

\[
(c) \quad (E_i^*, E_0^*) = \frac{1}{\alpha_i} \quad (d) \quad (E_0^*, E_0^*) = \frac{1}{\sigma}.
\]

Part \( b \) explains why we do not need to insert the generators \( g_{ij} \) \( (j < \nu_i) \) in \((6.1.1)\).
For any \( l' \in L' \) we set \( \tilde{c}(l') := -(E_0^*, l') \), the \( E_0 \)-coefficient of \( l' \). Furthermore, if \( l' = c_0 E_0^* + \sum_{i,j} c_{ij} E_{ij}^* \in L' \), then we define its reduced transform by
\[
l'_{\text{red}} := c_0 E_0^* + \sum_{i,j} c_{ij} \cdot n_{j+1}^i E_i^*.
\]

By (6.1.3) we get \([l'] = [l'_{\text{red}}]\) in \( H \), \( \tilde{c}(l') = \tilde{c}(l'_{\text{red}}) \), and if \( l'_{\text{red}} = \sum_{i=0}^d c_i E_i^* \), then \( \tilde{c}(l'_{\text{red}}) \) is
\[
\tilde{c} := \frac{1}{|e|} \cdot (c_0 + \sum_{i=1}^d c_i \alpha_i).
\]

If \( h \in H \), and \( l'_h \in L' \) is any of its liftings (that is, \([l'_h]\) = \( h \)), then \( l'_{h,\text{red}} \) is also a lifting of the same \( h \) with \( \tilde{c}(l'_{h,\text{red}}) = \tilde{c}(l'_h) \). In general, \( \tilde{c} = \tilde{c}(l'_h) \) depends on the lifting, nevertheless replacing \( l'_h \) by \( l'_h \pm E_0 \) we modify \( \tilde{c} \) by \( \pm 1 \), hence we can always achieve \( \tilde{c} \in [0, 1) \), where it is determined uniquely by \( h \). For example, since \( r_h \in \Box \), its \( E_0 \)-coefficient \( \tilde{c}(r_h) \) is in \([0, 1)\).

Finally, we consider
\[
\gamma := \frac{1}{|e|} \cdot (d - 2 - \sum_{i=1}^d \frac{1}{\alpha_i}).
\]

It has several `names'. Since the canonical class is given by \( K = -\sum_v E_v + \sum_v (\delta_v - 2) E_v^* \), by (6.1.3) we get that the \( E_0 \)-coefficient of \(-K\) is \((K, E_0^*) = \gamma + 1\). The number \(-\gamma\) is sometimes called the `log discrepancy' of \( E_0 \), \( \gamma \) the `exponent' of the weighted homogeneous germ \((X, o)\), and \( \sigma \gamma \) is the Goto–Watanabe \( \sigma \)-invariant of the universal abelian cover of \((X, o)\), see [GW78 (3.14)] and [BH98 (3.6.13)]; while in [Ne81] \( e\gamma \) appears as an orbifold Euler characteristic.

### 6.2. The function \( Z \)

By the reduction Theorem [5.4.3] for the periodic constant computation, we can reduce ourself to the variable of the single node, it will be denoted by \( t \).

First we analyze the equivariant rational function associated with the denominator of \( Z^e \)
\[
Z^{/[H]}(t) = \prod_{i=1}^d (1 - t^{-(E_i^*, E_0^*)}[g_i])^{-1} = \sum_{x_1, \ldots, x_d \geq 0} t^{\sum_i x_i / (\alpha_i |e|)} \left[ \sum_i x_i g_i \right] \in \mathbb{Z}[[t^{1/\alpha}]]/[H].
\]

The right hand side of the above expression can be transformed as follows (cf. [NN04, §3]). If we fix a lift \( \sum_{i=0}^d c_i E_i^* \) of \( h \) as above, then using the presentation (6.1.1) one gets that \( \sum_{i=0}^d x_i g_i \) equals \( h \) if and only if there exist \( \ell, \ell_1, \ldots, \ell_d \in \mathbb{Z} \) such that
\[
\begin{align*}
(a) & \quad -c_0 = \ell_1 + \cdots + \ell_d - \ell b \\
(b) & \quad x_i - c_i = -\omega_i \ell - \alpha_i \ell_i \quad (i = 1, \ldots, d).
\end{align*}
\]

Since \( x_i \geq 0 \), from (b) we get \( \tilde{\ell}_i := \frac{\alpha_i \omega_i \ell}{\alpha_i} - \ell_i \geq 0 \). Moreover, if we set for \( c = (c_0, c_1, \ldots, c_d) \)
\[
N_c(\ell) := 1 + c_0 - \ell b + \sum_{i=1}^d \left[ \frac{c_i - \omega_i \ell}{\alpha_i} \right],
\]
then the number of realizations of \( h = \sum_i c_i g_i \) in the form \( \sum_i x_i g_i \) is given by the number of integers \( (\tilde{\ell}_1, \ldots, \tilde{\ell}_d) \) satisfying \( \tilde{\ell}_i \geq 0 \) and \( \sum_i \tilde{\ell}_i = N_c(\ell) - 1 \). This is \( \binom{N_c(\ell) + d - 2}{d - 1} \). Moreover, the non-negative integer \( \sum_i x_i / (\alpha_i |e|) \) equals \( \ell + \tilde{c} \). Therefore,
\[
Z^H_h(t) = \sum_{\ell \geq -\tilde{c}} \binom{N_c(\ell) + d - 2}{d - 1} t^{\ell + \tilde{c}}.
\]
This expression is independent of the choice of \(c = \{c_i\}_{i=0}^d\). Similarly, for any function \(\phi\), the expression \(\sum_{c \geq -\tilde{c}} \phi(N_c(\ell)) t^{\ell+\tilde{c}}\) is independent of the choice of \(c\), it depends only on \(h = \sum c_i g_i\).

Furthermore, one checks that \(N_c(\ell) \leq 1 + (\ell + \tilde{c})|e|\), hence if \(\ell + \tilde{c} < 0\) then \(N_c(\ell) \leq 0\), therefore \(\left(\frac{N_c(\ell) + d - 2}{d - 1}\right) = 0\) as well. Hence, in (6.2.2) the inequality \(\ell + \tilde{c} \geq 0\) below the sum, in fact, is not restrictive.

Next, we consider the numerator \((1 - [g_1|e|]d - 2)\) of \(Z^e(\ell)\). A similar computation as above done for \(Z^e(\ell)\) (see [Ne81] and [NN04, §3]), or by multiplying (6.2.2) by the numerator and using \(\sum_{k=0}^{d-2} (-1)^k (d-2)(N-k+d-2) = (N)!\), gives

\[
Z_h(t) = \sum_{\ell \geq -\tilde{c}} \max\{0, N_c(\ell)\} t^{\ell+\tilde{c}}.
\]

In order to compute the periodic constant of \(Z_h(t)\) we decompose \(Z_h(t)\) into its ‘polynomial and negative degree parts’, cf. 4.1. Namely, we write \(Z_h(t) = Z_h^+(t) + Z_h^-(t)\), where

\[
Z_h^+(t) = \sum_{\ell \geq -\tilde{c}} \max\{0, -N_c(\ell)\} t^{\ell+\tilde{c}} \quad \text{(finite sum with positive exponents)}
\]

(6.2.4)

\[
Z_h^-(t) = \sum_{\ell \geq -\tilde{c}} N_c(\ell) t^{\ell+\tilde{c}}.
\]

In \(Z_h^-\) it is convenient to fix a choice with \(\tilde{c} \in [0, 1)\), hence the summation is over \(\ell \geq 0\). Then a computation (left to the reader) shows that it is a rational function of negative degree

\[
Z_h^-(t) = \left(1 - \frac{e \cdot t}{1 - t} - \frac{e \cdot t}{(1 - t)^2} - \sum_{i=1}^{d} \sum_{r_i=0}^{\alpha_i-1} \left\{ \frac{c_i - \omega_i r_i}{\alpha_i} \right\} t^{\alpha_i} \cdot \frac{1}{1 - t^{\alpha_i}} \right) \cdot t^{\tilde{c}}.
\]

This expression can be compared with the Laurent expansion of \(Z_h\) at \(t = 1\) which was already considered in the literature. Dolgachev, Pinkham, Neumann and Wagreich [Do83, Pi77, Ne81, Wa83] determine the first two terms (the pole part), while [NN04, NO05] the third terms as well. Nevertheless the above \(Z_h^+ + Z_h^-\) decomposition does not coincide with the ‘pole+regular part’ decomposition of the Laurent expansion terms, and focuses on different aspects.

Since the degree of \(Z_h^-\) is negative (or by a direct computation) \(\text{pc}(Z_h^-) = 0\), cf. 4.1.

On the other hand, since \(e < 0\), in \(Z_h^+(t)\) the sum is finite. (The degree of \(Z_0^+\) is \(\leq \gamma\), see e.g. [NOT10]. Since \(N_{c(r_h, red)}(\ell) \geq N_0(\ell)\), the degree of \(Z_h^+\) is \(\leq \gamma + \tilde{c}(r_h)\) too). By 4.1

\[
\text{pc}(Z_h) = Z_h^+(1) = \sum_{\ell \geq -\tilde{c}} \max\{0, -N_c(\ell)\}
\]

(6.2.6)

for any lifting \(c\) of \(h = \sum c_i g_i\). In this sum the bound \(\ell \geq -\tilde{c}\) is really restrictive.

We consider the non-equivariant version, the projection of \(Z^e \in \mathbb{Z}[t^{1/|e|}][H]\) into \(\mathbb{Z}[t^{1/\gamma}]\) too

\[
Z_{ne}(t) = \sum_h Z_h(t) = \frac{(1 - t^{1/|e|})^{d - 2}}{\prod_{i=1}^{d} (1 - t^{1/|e|^{\alpha_i}})} \in \mathbb{Z}[t^{1/\gamma}].
\]

We can get its \(Z_{ne}^+ + Z_{ne}^-\) decomposition either by summation of \(Z_{h}^+\) and \(Z_{h}^-\), or as follows. Write

\[
Z_{ne}(t) = \frac{1}{(1 - t^{1/|e|})^2} \prod_{i=1}^{d} \frac{1 - t^{1/|e|^{\alpha_i}}}{1 - t^{1/|e|^{\alpha_i}}} = \frac{1}{(1 - t^{1/|e|})^2} \sum_{0 \leq i < \alpha_i} \sum_{0 \leq l \leq d} t^{1/|e|^{\alpha_i}} S(x),
\]

(6.2.7)
where $S(x) := \sum_i \frac{a_i}{\alpha_i}$. Then its decomposition into $Z_{ne}^+(t) + Z_{ne}^-(t)$ is

\begin{equation}
Z_{ne}^-(t) = \sum_{0 \leq x_i < \alpha_i, 0 \leq i \leq d} t^{\frac{1}{2}[S(x)]} \cdot \left( \frac{1}{(1 - t^{1/|v|})^2} - \frac{|S(x)|}{(1 - t^{1/|v|})} \right)
\end{equation}

\begin{equation}
Z_{ne}^+(t) = \sum_{0 \leq x_i < \alpha_i, 0 \leq i \leq d} t^{\frac{1}{2}[S(x)]} \cdot \left( \frac{1}{(1 - t^{1/|v|})^2} - \frac{|S(x)|}{(1 - t^{1/|v|})} + \frac{|S(x)| - 1}{(1 - t^{1/|v|})^2} \right).
\end{equation}

After dividing in $Z_{ne}^+(t)$ (or by L’Hospital rule), we get

\begin{equation}
\text{pc}(Z_{ne}) = Z_{ne}^+(1) = \frac{1}{2} \sum_{0 \leq x_i < \alpha_i, 0 \leq i \leq d} [S(x)] \cdot [S(x) - 1].
\end{equation}

6.3. Analytic interpretations.

Rational homology sphere negative definite Seifert 3–manifolds can be realized analytically as links of weighted homogeneous singularities, or by equisingular deformations of weighted homogeneous singularities provided by splice quotient equations [Ne81, NW05].

Consider the smooth germ at the origin of $\mathbb{C}^d$ with coordinate ring $\mathbb{C}\{z\} = \mathbb{C}\{z_1, \ldots, z_d\}$, where $z_i$ corresponds to the $i$th end. Then $H$ acts on it by the diagonal action $h \ast z_i = \theta(g_i)(h)z_i$. Similarly, we can introduce a multidegree $\deg(z_i) = E_i^* \in L'$, hence the Poincaré series of $\mathbb{C}\{z\}$ associated with this multidegree is $\prod_i (1 - t^{E_i^*})^{-1}$. Moreover, considering the action of $H$ on it, $\tilde{Z}(t) = \prod_i (1 - [g_i]t^{E_i^*})^{-1}$ is the equivariant Poincaré series of $\mathbb{C}^d$, the invariant part $\tilde{Z}_0(t)$ being the Poincaré series of the corresponding quotient space $\mathbb{C}^d/H$.

In $\mathbb{C}^d$ one can consider the ‘splice equations’ as follows. Consider a matrix $\{\lambda_{ij}\}_{ij}$ of full rank and of size $d \times (d - 2)$. Then the equations $\sum_{j=1}^d \lambda_{ij}z_j^{a_i} = 0$, for $j = 1, \ldots, d - 2$, determine in $\mathbb{C}^d$ an isolated complete intersection singularity $(Y, o)$ on which the group $H$ acts as well. Then $(X, o) = (Y, o)/H$ is a normal surface singularity with the topological type of the Seifert manifold we started with. The equivariant Poincaré series of $(Y, o)$ is $\tilde{Z}(t)$ [Ne81]. For $(X, o)$ [BN10] proves the identity $P(t) = Z(t)$ mentioned in subsection 2.3 hence $Z(t)$ is also the Poincaré series of the equivariant divisorial filtration associated with all the vertices.

Theorem 5.4.2 reduces the filtration to the $\mathbb{Z}$-filtration: the divisorial filtration associated with the central vertex. In the weighted homogeneous case this filtration is also induces by the weighted homogeneous equations. Then, $Z/H(t)$ is the Poincaré series of $\mathbb{C}^d/H$, $Z(t)$ is the equivariant Poincaré series of $Y$, hence $Z_0(t)$ is the Poincaré series of $X$, cf. [Do83, Ne81, Pi77].

By $\{\text{pc}(Z_h)\}_{h \in H}$ are the equivariant geometric genera of the universal abelian cover $Y$ of $X$, hence $\text{pc}(Z_0)$ and $\text{pc}(Z_{ne})$ are the geometric genera of $X$ and $Y$ respectively, cf. [N08c].

6.4. Seiberg–Witten theoretical interpretations.

Fix $h \in H$. Then, for any lifting $\sum_i c_i g_i$ of $h$, Corollary 5.2.1 and equation 6.2.6 give

\begin{equation}
\text{pc}(Z_h) = \sum_{\ell \geq 0} \max \{0, -N_\ell(\ell)\} = -8\text{sw}_{-\hat{h}*\sigma_{can}}(M) - \frac{(K + 2r_h)^2 + |V|}{8}.
\end{equation}
Recall that $\sum_h \text{sw}_{-h*\sigma_{can}}(M)$ is the Casson–Walker invariant $\lambda(M)$. Hence, for the non-equivariant version we get

\[(6.4.2) \quad \text{pc}(Z_{ne}) = \frac{1}{2} \cdot \sum_{\substack{0 \leq i < c_i \\ 0 \leq l < d}} [S(x)] \cdot [S(x) - 1] = -\lambda(M) - d \cdot \frac{K^2 + |V|}{8} + \sum_h \chi(r_h).\]

For explicit formulae of $\lambda(M)$ and $K^2 + |V|$ in terms of Seifert invariants see e.g. [NN04, 2.6]).

**Remark 6.4.3.** \((6.4.1)\) can be compared with a known formulae of the Seiberg–Witten invariants involving the representative $s_h$. This will also lead us to an expression for $\chi(r_h) - \chi(s)$ in terms of $N_c(\ell)$. Let $c(s_h) = (c_0, \ldots, c_d)$ be the coefficients of $s_h$, $\text{red}$, cf. \([6.1]\). The set of all reduced coefficients $c(s_h)$, when $h$ runs in $H$, is characterized in [N05, 11.5] by the inequalities

\[(6.4.4) \quad \left\{ c_0 \geq 0, \; \alpha_i > c_i \geq 0 \; (1 \leq i \leq d) \right\} \quad N_c(\ell) \leq 0 \; \text{for any} \; \ell < 0.

Moreover, for this special lifting $c(s_h)$ of $h$, in [N05, §11] is proved

\[(6.4.5) \quad \sum_{\ell \geq 0} \max \left\{ 0, -N_c(s_h)(\ell) \right\} = -\text{sw}_{-h*\sigma_{can}}(M) - \frac{(K + 2s_h)^2 + |V|}{8}.

Using the discussion from the end of \([6.1]\) this can be rewritten for any lifting $c$ of $h$ as

\[(6.4.6) \quad \sum_{\ell \geq -\bar{c} + [\bar{c}(s_h)]} \max \left\{ 0, -N_c(\ell) \right\} = -\text{sw}_{-h*\sigma_{can}}(M) - \frac{(K + 2s_h)^2 + |V|}{8}.

This compared with \((6.4.1)\) gives for any lifting $c$ of $h$

\[(6.4.7) \quad \sum_{l \geq -\bar{c} + [\bar{c}(s_h)] > l \geq -\bar{c}} \max \left\{ 0, -N_c(\ell) \right\} = \chi(r_h) - \chi(s_h).

**Example 6.4.8.** The sum in \((6.4.7)\), in general, can be non-zero. This happens, for example, in the case of the link of a rational singularity whose universal abelian cover is not rational. Here is a concrete example, cf. [N07, 4.5.4]: take the Seifert manifold with $b = -2$ and three legs, all of them with Seifert invariants $(\alpha_i, \omega_i) = (3, 1)$. For $h = \sum_{i=1}^{3} g_i$ one has $s_h = \sum_{i=1}^{3} E_i^\pm$, the $E_0$-coefficient of $s_h$ is $1$, $r_h = s_h - E_0$, and $\chi(s_h) = 0, \chi(r_h) = 1$.

### 6.5. Ehrhart theoretical interpretations.

We fix $h \in H$ as above and we assume that $\bar{c} \in [0, 1)$. Note that $Z_h(t)$ has the form $t^p \sum_{\ell \geq 0} p_\ell t^\ell$; here the exponents $\{p + \ell\}_{\ell \geq 0}$ are the possible $E_0$-coordinates of the elements $(r_h + L) \cap S'$.

Let us compute the counting function for $Z_h$. If $S(t) = \sum_{r'} p_r t^{r'}$ is a series, we write $Q(S)(r') = \sum_{r < r'} p_r$, for $r' \in \mathbb{Q}_{\geq 0}$.

**Lemma 6.5.1.** For any $n \in \mathbb{N}_{\geq 0}$ one has the following facts.

\[(a) \quad Q(Z_h(n)) = Q(Z_h(n + \bar{c})).\]

\[(b) \quad Q(Z_h^+)(n) \text{ is a step function (hence piecewise polynomial) with} \]

\[Q(Z_h^+)(n) = \text{pc}(Z_h) \quad \text{for} \; n > \deg(Z_h^+).

\[(c) \quad Q(Z_h^-)(n) \text{ is a quasipolynomial:} \]

\[(6.5.2) \quad Q(Z_h^-)(n) = (1 - \bar{c})n - e \cdot \frac{n(n - 1)}{2} - \sum_{i=1}^{N} \sum_{r_i=0}^{\alpha_i-1} \frac{\{c_i - \omega_i r_i\}}{\alpha_i} \left[ n - r_i \right].\]
\begin{equation}
\frac{-en^2}{2} + \frac{en}{2}(\gamma + 1 - 2\widehat{c}) - \sum_{i=1}^{d} \sum_{r_i=0}^{\alpha_i-1} \left\{ \frac{c_i - \omega_i r_i}{\alpha_i} \right\} \left( \frac{r_i - n}{\alpha_i} \right) - \frac{r_i}{\alpha_i}.
\end{equation}

In particular, if \( n = m\alpha \) for \( m \in \mathbb{Z} \), and \( n > \deg(Z_h^+) \), then the double sum is zero, hence
\begin{equation}
Q(Z_h)(n) = -\frac{en^2}{2} + \frac{en}{2}(\gamma + 1 - 2\widehat{c}) + pc(Z_h).
\end{equation}

This is compatible with the expression provided by Theorem 5.4.2 and the Reduction theorem 2.2.2. Indeed, let us fix any chamber \( \mathcal{C} \) such that \( \text{int}(\mathcal{C} \cap S') \neq \emptyset \), and \( \mathcal{C} \) contains the ray \( \mathcal{R} = \mathbb{R}_{\geq 0} \cdot E_0^* \). Since the numerator of \( f(t) = (1 - t E_0^*)^{d-2} \), all the \( b_k \)-vectors belong to \( \mathcal{R} \). In particular, \( \cap_k(b_k + \mathcal{C}) \) intersects \( \mathcal{R} \) along a semi-line \( \mathcal{R}^{\geq c} = \mathbb{R}_{\geq \text{const}} \cdot E_0^* \) of \( \mathcal{R} \). Since \( Q_h(l') \) is quasipolynomial on \( \cap_k(b_k + \mathcal{C}) \), cf. (4.3.14), and certain restriction of it is determined by (2.2.3) whose right hand side is a quasipolynomial too, we obtain that the identity (2.2.3) is valid on \( \mathcal{R}^{\geq c} \) as well.

Recall that for any \( h \in H \) and \( l' \in L' \) we have a unique \( q_{l',h}^* \in \mathcal{L} \) with \( l' + q_{l',h}^* \in r_h + L \). Hence we get
\begin{equation}
Q_h(l') = -\text{sw}_{-h*\text{can}}(M) - \frac{(K + 2l' + 2q_{l',h}^*)^2 + |V|}{8} \quad (l' \in \mathcal{R}^{\geq c}).
\end{equation}

The term \( q_{l',h}^* \) is responsible for the non-polynomial behavior. Nevertheless, if we assume that \( l' = m\alpha E_0^* \in \mathcal{R}^{\geq c} \cap L, m \in \mathbb{Z} \), then \( q_{l',h}^* = r_h \), hence by (6.4.1)
\begin{equation}
Q_h(l') = -\frac{(l', l' + K + 2r_h)}{2} + pc(Z_h).
\end{equation}

By the Reduction theorem 5.4.2 \( Q_h(l') \) from (6.5.5) depends only on the \( E_0^* \)-coefficient of \( l' = m\alpha E_0^* \), which is exactly \( m\alpha \). One sees that in fact (6.5.5) agrees with (6.5.3) if we set \( n = m\alpha \).

The non-equivariant version can be obtained by summation of (6.5.3). For this we need \( \sum_h \widehat{c}(r_h) \). Consider the group homomorphism \( \varphi : H \rightarrow \mathbb{Q}/\mathbb{Z} \) given by \( h \mapsto [\widehat{c}(r_h)] \), or \([E_0^*] \mapsto [-(E_0^*, E_0^*)] \). Its image is generated by the classes of \( 1/\alpha_i|e| \), hence its order is \( \alpha \). Hence, \( \widehat{c}(r_h) \) vanishes exactly \( d/\alpha \) times (whenever \( h \in \ker(\varphi) \)). In all other cases \( \widehat{c}(r_h) \neq 0 \), and \( \widehat{c}(r_h) + \widehat{c}(r_{-h}) = 1 \). In particular, \( 2 \sum_h \widehat{c}(r_h) = d - d/\alpha \). Therefore, the summation of (6.5.5) provides
\begin{equation}
Q(Z_{ne})(n) = -\frac{d\alpha n^2}{2} + \frac{d\alpha n}{2}(\gamma + 1 - \frac{1}{\alpha}) + pc(Z_{ne}) \quad (n \in \alpha \mathbb{Z}).
\end{equation}

Next, we will identify the coefficients of (6.5.3) and (6.5.6) with the first three coefficient of the Ehrhart quasipolynomial \( L_h^c(T) \) via the identity (4.3.14).

For simplicity we will assume that \( \sigma = 1 \), in particular all the \( b_k \)-vectors belong to \( L \).

If \( l' \in \mathcal{R} \), then by Reduction theorem the counting function \( L_h^c(T, l') \) of the polytope \( P(l') \) depends only on the \( E_0^* \)-coefficient of \( l' \); let us denote this coefficient by \( l_0' \).

Hence, this \( L_h^c(T, l_0') \) is the Ehrhart quasipolynomial of the \( d \)-dimensional simplicial polytope, being its \( h \)-class counting function. Via (6.1.3) the definition (4.3.2) of this polytope becomes
\begin{equation}
P_0 = \{(x_1, \ldots, x_d) \in (\mathbb{R}_{\geq 0})^d : \sum_{i} \frac{x_i}{|e|\alpha_i} < l_0' \}.
\end{equation}

Let
\begin{equation}
L_h^c(T, l_0') = \sum_{j=0}^{d} a_{h,j}(l_0') \frac{(l_0')^j}{j!}.
\end{equation}
be the coefficients of the Ehrhart quasipolynomial: each \(a_{h,j}(l'_0)\) is a periodic function in \(l'_0\) and is normalized by \(1/j!\). Since the numerator of \(f\) is \((1 - t^{1/|e|})^{d - 2}\), by (4.3.14) we obtain for \(l' \in \mathcal{R}\)

\[
Q_h(l') = \sum_{j=0}^{d} a_{h,j}(l'_0) \cdot \frac{1}{j!} \sum_{k=0}^{d-2} (-1)^k \binom{d-2}{k} \left( l'_0 - \frac{k}{|e|} \right)^j .
\]

This equals the expression (6.5.4) above. The non-polynomial behavior of these two expressions indicate that \(a_j(l'_0)\) is indeed nonconstant periodic, and can be determined explicitly.

Since we are interested primarily in the Seiberg–Witten invariant, namely in \(pc(Z_h)\), we perform this explicit identification only via the expressions (6.5.3) and (6.5.5). Hence, similarly as in these cases, we take \(l' = m \alpha E_0^* \in \mathcal{R}^{2c} \cap L\), and we identify (6.5.3) with (6.5.9) evaluated for \(l'\), whose \(E_0\)-coefficient is \(l'_0 = m \alpha = n\). In this case \(a_{h,j}(n)\) is a constant, denoted by \(a_{h,j}\), and

\[
\frac{1}{|e|} \frac{(d-2) \cdot \sum_{k=0}^{d-2} (-1)^k \binom{d-2}{k} \left( n - \frac{k}{|e|} \right)^j}{j!} = \begin{cases} 0 & \text{if } j < d - 2, \\ 1 & \text{if } j = d - 2, \\ \frac{(d-2)(d-1)/2}{(d-2)(d-1)d(3d-5)/24} & \text{if } j = d. \end{cases}
\]

We obtain

\[
a_{h,d} = \frac{1}{|e|}, \\
a_{h,d-1} = \frac{d-2}{2|e|} - \frac{1}{2} (\gamma + 1 - 2\tilde{c}), \\
a_{h,d-2} = pc(Z_h) + \frac{(d-2)(3d-7)/24}{|e|} - \frac{d-2}{4} (\gamma + 1 - 2\tilde{c}).
\]

In particular, the \(a_{h,d-2}\) can be identified (up to ‘easy’ extra terms) with \(pc(Z_h)\) (with analytical interpretation \(\dim(H^1(\tilde{Y} \cup \mathcal{O}_{\tilde{Y}})_{\theta(h)})\)) and Seiberg–Witten theoretical interpretation (6.4.1). The first coefficients can also be identified with the equivariant volume of \(P_0\), (a fact already known in the non-equivariant cases). Usually (in the non-equivariant case, and when we count the points of all the facets) the second coefficient can be related with the volumes of the facets. Here we eliminate from this count some of the facets, and we are in the equivariant situation as well. The expression of the second coefficient is also a novelty of the present article (to the best of author’s knowledge).

In the non-equivariant case, if \(\sum_{j=0}^{d} a_j n_j^{l'_0}\) is the classical Ehrhart polynomial of \(P_0\), then

\[
a_{d} = \prod \alpha_i, \\
a_{d-1} = \prod \alpha_i \cdot \left( \frac{1}{\alpha_i} + \sum \frac{1}{\alpha_i} \right)/2, \\
a_{d-2} = \prod \alpha_i \left( pc(Z_{\alpha}) \frac{|e|}{\prod \alpha_i} - \frac{(d-2)(3d-5)/24}{|e|} + \frac{d-2}{4} \left( \frac{1}{\alpha_i} + \sum \frac{1}{\alpha_i} \right) \right).
\]

In this non-equivariant case the identities (6.5.13) are valid even without the assumption \(\sigma = 1\) by the same proof.

The formulae in (6.5.12) and (6.5.13) can be further simplified if we replace \(P_0\) by \(|e|P_0\), or if we substitute in the Ehrhart polynomial the new variable \(\lambda := |e|l'_0\) instead of \(l'_0\); cf. Section
7. THE TWO–NODE CASE

7.1. Notations and the group $H$. We consider the following graph $\Gamma$:

![Graph Diagram]

The nodes $E_0$ and $\tilde{E}_0$ have decorations $b_0$ and $\tilde{b}_0$ respectively. Similarly as in the one–node case, we codify the decorations of maximal chains by continued fraction expansions. In fact, it is convenient to consider the two maximal star–shaped graphs $\Gamma_0$ and $\tilde{\Gamma}_0$, and the corresponding normalized Seifert invariants of their legs. Hence, let the normalized Seifert invariants of the legs with ends $E_i$ ($1 \leq i \leq d$) be $(\alpha_i, \omega_i)$, while of the legs with ends $\tilde{E}_j$ ($1 \leq j \leq \tilde{d}$) be $(\tilde{\alpha}_j, \tilde{\omega}_j)$.

The chain connecting the nodes, viewed in $\Gamma_0$ has normalized Seifert invariants $(\alpha_0, \omega_0)$, while viewed as a leg in $\tilde{\Gamma}_0$, it has Seifert invariants $(\alpha_0, \tilde{\omega}_0)$. One has $\omega_0 \tilde{\omega}_0 = \alpha_0 \tau + 1$. Clearly, $\alpha_0$ is the determinant of the chain, and

$$\omega_0 := \det \left( \begin{array}{c} E_2 \\ \vdots \\ E_s \end{array} \right), \quad \tilde{\omega}_0 := \det \left( \begin{array}{c} \tilde{E}_1 \\ \vdots \\ \tilde{E}_{s-1} \end{array} \right), \quad \tau := \det \left( \begin{array}{c} E_2 \\ \vdots \\ E_{s-1} \end{array} \right).$$

We denote the orbifold Euler numbers of the star–shaped subgraphs $\Gamma_0$ and $\tilde{\Gamma}_0$ by

$$e = b_0 + \frac{\omega_0}{\alpha_0} + \sum_i \frac{\omega_i}{\alpha_i} \quad \text{and} \quad \tilde{e} = \tilde{b}_0 + \frac{\tilde{\omega}_0}{\tilde{\alpha}_0} + \sum_j \frac{\tilde{\omega}_j}{\tilde{\alpha}_j}.$$

Consider the orbifold intersection matrix $I^{\text{orb}} = \begin{pmatrix} e & 1/\alpha_0 \\ 1/\tilde{\alpha}_0 & \tilde{e} \end{pmatrix}$, cf. [BN07, 4.1.4].

Then, the negative definiteness of $I$ (or $\Gamma$) implies that $I^{\text{orb}}$ is negative definite too, hence

$$\varepsilon := \det I^{\text{orb}} = e\tilde{e} - \frac{1}{\alpha_0^2} > 0.$$}

Then the determinant of the graph is $\det(\Gamma) = \det(-I) = \varepsilon \cdot \alpha_0 \prod_i \alpha_i \prod_j \tilde{\alpha}_j$, cf. [BN07].

Using (2.1.1) we have the following intersection number of the dual base elements:

$$\begin{align*}
(E_0^*)^2 &= \frac{\tilde{\varepsilon}}{\varepsilon};
(E_0^*)^2 &= \frac{\varepsilon}{\tilde{\varepsilon}},
(E_0^* , E_0^*) &= -\frac{1}{\alpha_0 \varepsilon};
(E_0^* , E_i^*) &= \frac{\varepsilon}{\alpha_i \varepsilon};
(E_i^* , \tilde{E}_0^*) &= -\frac{1}{\alpha_0 \alpha_i \varepsilon};
(E_i^* , \tilde{E}_j^*) &= -\frac{1}{\alpha_0 \alpha_i \varepsilon};
(E_0^* , E_i^*) &= \frac{\varepsilon}{\alpha_j \varepsilon}.
\end{align*}$$

Similarly as in [5.3] or [6.1] we can write $n_{k_1,k_2}^i \tilde{n}_{k_1,k_2}^j$ resp. $n^i_{k_1,k_2}$ for the the determinant of the sub–chains of the ‘left’ $i^{th}$ leg, ‘right’ $j^{th}$ leg and connecting chain connecting the vertices $v_{k_1}$ and $v_{k_2}$. Let $\nu_i$ and $\tilde{\nu}_j$ be the number of vertices in the legs, cf. [6.1] Then (with the standard notations, where $E_{i\ell}$ and $\tilde{E}_{j\ell}$ are the vertices of the legs) one has the following slightly technical Lemma, but whose proof is standard based on the arithmetical properties of continued fractions:

Lemma 7.1.2. (a) $E_{i\ell}^* = n_{\ell+1,\nu_i}^i E_i^* + \sum_{\ell < r \leq \nu_i} \frac{n_{\ell+1,\nu_i}^i n_{r+1,\nu_i}^j - n_{r-1,\nu_i}^i n_{r+1,\nu_i}^j}{\alpha_i} E_{r\ell}$ for any $1 \leq \ell < \nu_i$. 

(There is a similar formula for $\tilde{E}_{j,k}^s$.)

(b) $\tilde{E}_{j,k}^* = \tilde{n}_{i,k-1}E_{i}^s - \tilde{n}_{2,k-1}E_{0}^s + \sum_{1 \leq r < k} \frac{1}{\alpha_0} \tilde{n}_{i,r-1}E_{r}^s - \tilde{n}_{1,k-1}E_{r}^s$, for $1 < k \leq s$.

(This is true even for $k = s + 1$ with the identification $\tilde{E}_{k+1}^* = \tilde{E}_{k}^*$.)

Next, we give a presentation of $H = L'/L$. Set $g_i := [E_{i}^s] (1 \leq i \leq d)$, $\tilde{g}_j := [\tilde{E}_j^s] (1 \leq j \leq d)$, $g_0 := [E_0^s]$ and $\tilde{g}_0 := [\tilde{E}_0^s]$. Moreover we need to choose an additional generator corresponding to a vertex sitting on the connecting chain: we choose $\tilde{g} := [\tilde{E}_1^s]$ (this motivates the choice in Lemma 7.1.2(b) too). The above lemma implies

\[(7.1.3) \quad [E_{i,k}^s] = n_{i+1,\nu}^i g_i, \quad [\tilde{E}_{j,k}^s] = \tilde{n}_{j+1,\nu} \tilde{g}_j \quad \text{and} \quad [E_{k}^s] = \tilde{n}_{1,k-1} \tilde{g} - \tilde{n}_{2,k-1} g_0;\]

and similar arguments as in the star–shaped case provides the following presentation for $H$

\[(7.1.4) \quad H = ab(g_0, \tilde{g}_0, g_i, \tilde{g}_j, \tilde{g}) \quad | \quad g_0 = \alpha_i \cdot g_i; \quad \tilde{g}_0 = \tilde{\alpha}_j \cdot \tilde{g}_j; \quad \alpha_0 \cdot \tilde{g} = \omega_0 \cdot g_0 + \tilde{g}_0; \quad -\tilde{g} - b_0 \cdot g_0 = \sum_i \omega_i \cdot g_i; \quad -\tilde{\omega}_0 \cdot \tilde{g} + \tau \cdot g_0 = \tilde{b}_0 \cdot \tilde{g}_0 = \sum_j \tilde{\omega}_j \cdot \tilde{g}_j).\]

Moreover, for any $l' \in L'$,

\[l' = c_0 E_0^s + \tilde{c}_0 \tilde{E}_0^s + \sum_k \tilde{c}_k \tilde{E}_k^s + \sum_i \tilde{c}_{i,\ell} E_{i,\ell}^s + \sum_{j,\ell} \tilde{c}_{j,\ell} \tilde{E}_{j,\ell}^s,\]

if we define its reduced transform $l_{\text{red}}'$ by

\[(c_0 - \sum_{k > 1} \tilde{c}_k) E_0^s + \tilde{c}_0 \tilde{E}_0^s + (\tilde{c}_1 + \sum_{k > 1} \tilde{c}_k) \tilde{E}_1^s + \sum_{i,\ell} c_{i,\ell} E_{i,\ell}^s + \sum_{j,\ell} \tilde{c}_{j,\ell} \tilde{E}_{j,\ell}^s,\]

then, by Lemma 7.1.2 $[l'] = [l_{\text{red}}']$ in $H$. Moreover, if for any $l' \in L'$ we distinguish the $E_0$ and $\tilde{E}_0$ coefficients, that is, we set $c(l') := -(E_0^s, l')$ and $\tilde{c}(l') := -(\tilde{E}_0^s, l')$, then $c(l') = c(l_{\text{red}}')$ and $\tilde{c}(l') = \tilde{c}(l_{\text{red}}')$ as well. Lemma 7.1.2(b) (applied for $k = s + 1$) provide these coefficients for $\tilde{E}_1$:

\[(7.1.5) \quad (E_1^s, E_0^s) = \frac{1}{\varepsilon \alpha_0} (\omega_0 \tilde{c} - \frac{1}{\alpha_0}), \quad (\tilde{E}_1^s, \tilde{E}_0^s) = \frac{1}{\varepsilon \alpha_0} (e - \frac{\omega_0}{\alpha_0}).\]

We will use the coefficients $c = (c_0, \tilde{c}, \overline{c}, c_i, \tilde{c}_i)$ to write an element $l_{\text{red}}' = c_0 E_0^s + \tilde{c}_0 \tilde{E}_0^s + \overline{c} \tilde{E}_1^s + \sum_i c_i E_i^s + \sum_j \tilde{c}_j \tilde{E}_j^s$. Then (7.1.1) and (7.1.5) imply that

\[(7.1.6) \quad \left( \frac{c}{\overline{c}} \right) = \left( \frac{c(l_{\text{red}}')}{{\tilde{c}(l_{\text{red}}')}} \right) = (-I_{\text{orb}})^{-1} \cdot \left( \frac{A}{\overline{A}} \right) = \frac{1}{\varepsilon} \left( \frac{-\tilde{c}}{1/\alpha_0} \right) \cdot \left( \frac{1}{e} \right) \cdot \left( \frac{A}{\overline{A}} \right),\]

where

\[A := c_0 + \sum_i \frac{c_i}{\alpha_i} + \frac{\omega_0}{\alpha_0} \overline{c}, \quad \overline{A} := \tilde{c}_0 + \sum_j \frac{\tilde{c}_j}{\alpha_j} + \frac{1}{\alpha_0} \overline{c}.\]

Therefore, any $h \in H$ has a lift of type $l_{h,\text{red}}'$. Although the corresponding coefficients $c$ and $\overline{c}$ depend on the lift, by adding $\pm E_0$ and $\pm \tilde{E}_0$ to $l_{h,\text{red}}'$ we can achieve $c, \overline{c} \in [0, 1)$, and these values are uniquely determined by $h$. For example, the reduced transform $(r_h)_{\text{red}}$ of $r_h$ satisfies $c((r_h)_{\text{red}}) = c(r_h) \in [0, 1)$ and $\tilde{c}((r_h)_{\text{red}}) = \tilde{c}(r_h) \in [0, 1)$ since $r_h \in \square$.

As we will see, for different elements of $h \in H$, we have to shift the rank two lattices by vectors of type $(c, \overline{c})$, hence the vectors $(c, \overline{c})$ will play a crucial role later.
7.2. The function $Z$. If we wish to compute the periodic constant of $Z^\ell(t)$, by Theorem 5.4.2, we can eliminate all the variables of $Z^\ell(t)$ except the variables of the nodes; these remaining two variables are denoted by $(t, \tilde{t})$. Therefore the equivariant form of reciprocal of the denominator is

$$Z^\ell_H(t, \tilde{t}) = \prod_i \left(1 - t^{-E_i^c, E_0}\tilde{t}^{-E_i^c, E_\alpha^0}\right)^{-1} \cdot \prod_j \left(1 - t^{-F_j^c, F_0}\tilde{t}^{-F_j^c, F_\alpha^0}\right)^{-1} \cdot \sum_{x_i, x_{\tilde{t}} \geq 0} \left[ \sum_j x_i \tilde{g}_i + \sum_j \tilde{x}_j \tilde{g}_j \right].$$

We fix a lift $c_0 E_0^c + \tilde{c}_0 E_\alpha^0 + \tilde{c} E_\alpha^c + \sum_i c_i E_i^c + \sum_j \tilde{c}_j \tilde{E}_j^c$ of $h$. Then the class of $\sum_i x_i E_i^c + \sum_j \tilde{x}_j \tilde{E}_j^c$ equals $h$ if and only if its difference with the lift is a linear combination of the relation in 7.1.4. In other words, if there exist $\ell_0, \tilde{\ell}_0, \ell, \tilde{\ell}, \ell_0, \tilde{\ell}_0, \ell_j, \tilde{\ell}_j \in \mathbb{Z}$ such that

(a) $-c_0 = \sum_i \ell_i - b_0 \ell_0 + \tilde{\omega} \tilde{\ell}_0 + \omega_0 \tilde{\ell}$

(b) $-\tilde{c}_0 = \sum_j \tilde{\ell}_j - \tilde{b}_0 \tilde{\ell}_0 + \ell$

(c) $x_i - c_i = -\omega_i \ell_0 - \alpha_i \ell_j$ \quad (i = 1, \ldots, d)

(d) $\tilde{x}_j - \tilde{c}_j = -\tilde{\omega}_j \tilde{\ell}_0 - \tilde{\alpha}_j \tilde{\ell}_j$ \quad (j = 1, \ldots, d)

From (e) we deduce that

$$(\ell_0 + \tilde{\omega}_0 \tilde{\ell}_0) \equiv \tilde{c} (\text{mod } \alpha_0).$$

Since $x_i, \tilde{x}_j \geq 0$, (c) and (d) implies $\tilde{c}_i - \omega_0 \alpha_i \ell_0 \geq \ell_i$, $\tilde{c}_j - \tilde{\omega}_j \tilde{\alpha}_j \tilde{\ell}_0 \geq \tilde{\ell}_j$. Recall also that $\omega_0 \tilde{\omega}_0 = \alpha_0 \tau + 1$.

Therefore if we set $m_i := \left[ \frac{\alpha_i - \omega_0 \alpha_i}{\omega_0} - \ell_i \right]$ and $m_j := \left[ \frac{\tilde{\alpha}_j - \tilde{\omega}_j \tilde{\alpha}_j}{\tilde{\omega}_0} - \tilde{\ell}_j \right]$ non-negative integers then the number of the realization of $h$ in the form $\sum_i x_i \tilde{g}_i + \sum_j \tilde{x}_j \tilde{g}_j$ is determined by the number of non-negative integral $(d + \tilde{d})$-tuples $(m_i, \tilde{m}_j)$ satisfying

$$N_c(\ell_0, \tilde{\ell}_0) := c_0 + \frac{\omega_0}{\alpha_0} \ell_0 - (b_0 + \frac{\omega_0}{\alpha_0}) \ell_0 - \frac{1}{\alpha_0} \tilde{\ell}_0 + \sum_i \left[ \frac{\alpha_i - \omega_0 \alpha_i}{\omega_0} \right] = \sum_i m_i,$n

$$\tilde{N}_c(\ell_0, \tilde{\ell}_0) := \tilde{c}_0 + \frac{\tilde{\omega}_0}{\tilde{\alpha}_0} \tilde{\ell}_0 - (\tilde{b}_0 + \frac{\tilde{\omega}_0}{\tilde{\alpha}_0}) \tilde{\ell}_0 - \frac{1}{\tilde{\alpha}_0} \ell_0 + \sum_j \left[ \frac{\tilde{\alpha}_j - \tilde{\omega}_j \tilde{\alpha}_j}{\tilde{\omega}_0} \right] = \sum_j \tilde{m}_j.$$

This number is $\left( N_{d-c}(\ell_0, \tilde{\ell}_0) + d+1 \right)$ $\left( \tilde{N}_{d-\tilde{d}}(\ell_0, \tilde{\ell}_0) + d+1 \right)$ if $N_c$ and $\tilde{N}_c \geq 0$, otherwise is 0. Note that (7.2.1) guarantees that both $N_c$ and $\tilde{N}_c$ are integers. Furthermore, (c) and (d) and (7.1.6) show that the exponent of $t$ and $\tilde{t}$ in the formula of $Z^\ell_H(t, \tilde{t})$ are equal to $\ell_0 + c$ and $\tilde{\ell}_0 + \tilde{c}$ respectively. Hence

$$Z^\ell_H(t, \tilde{t}) = \sum_{\ell, \tilde{\ell} \in \mathbb{Z}^2} \left( N_c(\ell, \tilde{\ell}) + d+1 \right) \left( \tilde{N}_c(\ell, \tilde{\ell}) + d+1 \right) t^{\ell+c} \tilde{t}^{\tilde{\ell}+\tilde{c}},$$

where the sum runs over $(\ell, \tilde{\ell}) \in \mathbb{Z}^2$ with $\ell + \omega_0 \ell_0 \equiv \tilde{c} (\text{mod } \alpha_0)$.

The numerator of $Z(t, \tilde{t})$ is $\left(1 - t^{-F_0^c, E_0^c}\tilde{t}^{-F_0^c, E_\alpha^0}\right)^{d+1} \cdot \left(1 - t^{-E_0^c, E_0^c}\tilde{t}^{-E_0^c, E_\alpha^0}\right)^{\tilde{d}+1}$. Hence we get $Z^\ell$ by multiplying this expression by $\sum_h Z^\ell_H[h]$. Recall that $h = h_0 g_0 + \tilde{c}_0 g_0 + \tilde{c} \tilde{g} + \sum_i c_i g_i + \sum_j \tilde{c}_j \tilde{g}_j$ is paired with $c$. Set $h' := h + k g_0 + \tilde{k} \tilde{g}_0$ which corresponds to $c' = c + (k, \tilde{k}, 0, 0, 0)$. Hence $Z_{h'}[h']$ is the next sum according to the decompositions $h' = h + k g_0 + \tilde{k} \tilde{g}_0$:

$$\sum_{k=0}^{d-1} \left( \frac{k}{k} \right) \sum_{\ell=0}^{d-1} \left( \frac{\ell}{k} \right) \frac{1}{k} \left( \frac{\ell}{k} \right) \sum_{h} \left( \sum_{\ell} \left( N_{c'}(\ell, \tilde{\ell} + k + d-1) \right) \left( \tilde{N}_{c'}(\ell, \tilde{\ell} - k + d-1) \right) t^{\ell+c} \tilde{t}^{\tilde{c}+k} \right) \left[ h' \right] = \sum_{k=0}^{d-1} \left( \frac{k}{k} \right) \sum_{\ell=0}^{d-1} \left( \frac{\ell}{k} \right) \frac{1}{k} \left( \frac{\ell}{k} \right) \sum_{h} \left( \sum_{\ell} \left( N_{c'}(\ell, \tilde{\ell} - k + d-1) \right) \left( \tilde{N}_{c'}(\ell, \tilde{\ell} + k + d-1) \right) t^{\ell+c} \tilde{t}^{\tilde{c}+\tilde{k}} \right) \left[ h' \right].$$
There exists Lemma 7.3.3.

Clearly (7.3.1) is elementary. By Corollary [BG08, 2.12] we get the entries of
\[
\begin{pmatrix}
A - c_0 \ell/\alpha_0 \\
\tilde{A} - \ell_0/\alpha_0 - \tilde{e}_0 \ell
\end{pmatrix} = -I_{\text{orb}} \begin{pmatrix} \ell + c \\
\tilde{\ell} + \tilde{c}
\end{pmatrix}.
\]

This motivates to define
\[
S_c := \left\{ \ell \in \mathbb{Z}^2 : I_{\text{orb}} \begin{pmatrix} \ell + c \\
\tilde{\ell} + \tilde{c}
\end{pmatrix} \geq 0 \text{ and } \ell \text{ satisfies } \equiv_e \right\}.
\]

It is straightforward to verify that the right hand side of (7.2.3) does not depend on the choice of c, it depends only on h. The identity (7.2.3) is remarkable: it realizes the bridge between the series $Z^e$ and the equivariant Hilbert series of affine monoids and their modules.

7.3. The structure of $S_c$. Recall that for any $h \in H$ and c we consider a lift of $h$ identified by certain $c$ which determins the pair $(c, \tilde{c})$ (cf. (7.1.6)), and the integers $N_c(l)$ and $\tilde{N}_c(l)$, where $l = (\ell, \tilde{\ell}) \in \mathbb{Z}^2$. We abridge the congruence condition $\ell + \omega_0 \tilde{\ell} \equiv \tau \pmod{\alpha_0}$ by $\equiv_e$.

If $h = 0$ then we always choose the zero lift with $c = 0$.

If in the definition of $N_c(l)$ and $\tilde{N}_c(l)$ we replace each $[y]$ by $y$, we get the entries of
\[
\begin{pmatrix}
A - c_0 \ell/\alpha_0 \\
\tilde{A} - \ell_0/\alpha_0 - \tilde{e}_0 \ell
\end{pmatrix} = -I_{\text{orb}} \begin{pmatrix} \ell + c \\
\tilde{\ell} + \tilde{c}
\end{pmatrix}.
\]

This motivates to define
\[
S_c := \left\{ I \in \mathbb{Z}^2 : I_{\text{orb}} \begin{pmatrix} \ell + c \\
\tilde{\ell} + \tilde{c}
\end{pmatrix} \geq 0 \text{ and } I \text{ satisfies } \equiv_e \right\}.
\]

Clearly $S_c \subseteq S_c$. We also consider $C_{\text{orb}}$, the real cone $\{ I \in \mathbb{R}^2 : I_{\text{orb}} \cdot I \geq 0 \}$. Then $S_c = (C_{\text{orb}} - (c, \tilde{c})) \cap \mathbb{Z}^2 \cap (\equiv_e)$, where $(\equiv_e)$ means that the elements satisfy the congruence $\equiv_e$ also.

Lemma 7.3.2. (1) $S_0$ and $\bar{S}_0$ are affine monoids. $\bar{S}_0$ is the normalization of $S_0$.

(2) $S_c$ and $\bar{S}_c$ are finitely generated $S_0$-modules, $S_c$ is a submodule of $\bar{S}_c$.

Proof. (1) is elementary. By Corollary [BG08, 2.12] $\bar{S}_c$ is finitely generated over $\bar{S}_0$, but $\bar{S}_0$ itself is finitely generated as an $S_0$ module. \hfill \Box

Lemma 7.3.3. There exists $v_1$ and $v_2$ elements of $\mathbb{Z}^2$ with the following properties:

(a) $v_1$ and $v_2$ belong to $S_0$ and $\mathbb{R}_{\geq 0} v_1 + \mathbb{R}_{\geq 0} v_2 = C_{\text{orb}}$.

(b) For any $l \in \bar{S}_c$ one has: (i) $N_c(l + v_1) = N_c(l)$; (ii) $N_c(l + v_2) \geq 0$; (iii) $\bar{N}_c(l + v_2) = \bar{N}_c(l)$; and (ii) $\bar{N}_c(l + v_1) \geq 0$.

Proof. We choose
\[
(A) v_1 = (\ell_1, \tilde{\ell}_1) \in \mathbb{Z}^2 \cap (\equiv_0) \text{ such that } -\omega_i \ell_1/\alpha_i = 0 \text{ for all } i, \text{ and } N_0(v_1) = 0;
\]
\[
(B) v_2 = (\ell_2, \tilde{\ell}_2) \in \mathbb{Z}^2 \cap (\equiv_0) \text{ such that } -\omega_j \ell_2/\alpha_j = 0 \text{ for all } j, \text{ and } \bar{N}_0(v_2) = 0.
\]

Then $v_1$ and $v_2$ satisfy (a), and (b)(ii), and (b)(ii). Furthermore, note that $N_c(l + v_2) \geq N_c(l) + N_0(v_2)$ and for any $l \in \bar{S}_c$ one has $N_c(l) \geq -(d - 1)$. Hence, if we also assume $\bar{N}_0(v_1) \geq d - 1$ and $\bar{N}_0(v_2) \geq d - 1$, then all the conditions will be satisfied. \hfill \Box
Usually, the ‘universal restrictions’ \( \tilde{N}_0(v_1) \geq \tilde{d} - 1 \) and \( N_0(v_2) \geq d - 1 \) in the proof of Lemma 7.3.3 provide rather ‘large’ vectors \( v_1 \) and \( v_2 \). Nevertheless, usually much smaller vectors also satisfy (a) and (b). Here is another choice. Besides (A) and (B) we impose the following:

(C) Let \( \square = \square(v_1, v_2) = \{ l = (q_1 v_1 + q_2 v_2 : 0 \leq q_1, q_2 < 1 \} \) be the semi-open cube in \( C^{orb} \).

Then we require \( N_0(v_2) \geq 0 \) and \( \mathcal{N}_c(l_0 + v_2) \geq 0 \) for any \( l_0 \in (\square - (c, \tilde{c})) \cap \mathbb{Z}^2 \cap (\Xi_c) \); and symmetrically: \( \tilde{N}_0(v_1) \geq 0 \) and \( \tilde{N}_c(l_0 + v_1) \geq 0 \) for any \( l_0 \in (\square - (c, \tilde{c})) \cap \mathbb{Z}^2 \cap (\Xi_c) \).

The wished inequality for any \( l \in \mathcal{S}_c \) then follows from \( N_c(l_0 + k_1 v_1 + k_2 v_2 + v_2) = N_c(l_0 + k_2 v_2 + v_2) \geq N_c(l_0 + v_2) + k_2 \tilde{N}_0(v_2) \) (and its symmetric version).

In the sequel the next two subsets of \( \mathcal{S}_c \) will be crucial.

\[
\mathcal{S}_{c,1} := \{ l \in (\square - (c, \tilde{c})) \cap \mathbb{Z}^2 \cap (\Xi_c) : N_c(l) < 0 \},
\]

\[
\mathcal{S}_{c,2} := \{ l \in (\square - (c, \tilde{c})) \cap \mathbb{Z}^2 \cap (\Xi_c) : \tilde{N}_c(l) < 0 \}.
\]

Again, both sets \( \mathcal{S}_{c,1} \) and \( \mathcal{S}_{c,2} \) are independent of the choice of \( c \), they depend only on \( h \).

**Proposition 7.3.4.** With the above notations one has

\[
\begin{align*}
(1) \quad & \mathcal{S}_c = \bigcup_{l \in (\square - (c, \tilde{c})) \cap \mathbb{Z}^2 \cap (\Xi_c)} l + \mathbb{Z}_{\geq 0} v_1 + \mathbb{Z}_{\geq 0} v_2, \\
(2) \quad & \mathcal{S}_c \setminus \mathcal{S}_c = \left( \bigcup_{l \in \mathcal{S}_{c,1}} l + \mathbb{Z}_{\geq 0} v_1 \right) \bigcup \left( \bigcup_{l \in \mathcal{S}_{c,2}} l + \mathbb{Z}_{\geq 0} v_2 \right),
\end{align*}
\]

where \( \bigcup_{l \in \mathcal{S}_{c,1}} l + \mathbb{Z}_{\geq 0} v_1 \bigcap \bigcup_{l \in \mathcal{S}_{c,2}} l + \mathbb{Z}_{\geq 0} v_2 = \bigcup_{l \in \mathcal{S}_{c,1} \cap \mathcal{S}_{c,2}} l. \]

**Proof.** The statements follow from the choice of \( v_1 \) and \( v_2 \) and the above properties (a) and (b). Compare also with the structure theorem [BG08, 4.36] of \( S_0 \) modules. \( \square \)

### 7.4. The periodic constant and the SW invariant in the equivariant case.

Set \( t = (t, \tilde{t}) \). Using (7.2.3) and Proposition 7.3.4 one can write \( Z_h(t)/t^{(c, \tilde{c})} \) in the next form:

\[
\sum_{l \in (\square - (c, \tilde{c})) \cap \mathbb{Z}^2 \cap (\Xi_c)} \frac{t^l}{(1 - t v_1)(1 - t v_2)} - \sum_{l \in \mathcal{S}_{c,1}} \frac{t^l}{1 - t v_1} - \sum_{l \in \mathcal{S}_{c,2}} \frac{t^l}{1 - t v_2} + \sum_{l \in \mathcal{S}_{c,1} \cap \mathcal{S}_{c,2}} t^l.
\]

Next, we apply the decomposition established in subsection 4.5. Here it is important to choose \( c \) in such a way that \( c \in [0, 1) \) and \( \tilde{c} \in [0, 1) \).

Note that \( v_1 \in \mathbb{R}_{>0}(1/\alpha_0, -c) \) and \( v_2 \in \mathbb{R}_{>0}(-\tilde{c}, 1/\alpha_0) \), hence \( v_2 \) sits in the cone determined by \( v_1 \) and \( (1, 0) \). Then, as in 4.5., we set \( \Xi_1 := \{ (\ell, \tilde{\ell}) : 0 \leq \ell < \) first coordinate of \( v_1 \} \) and \( \Xi_2 := \{ (\ell, \tilde{\ell}) : 0 \leq \ell < \) second coordinate of \( v_2 \} \), and for any \( l \in \mathcal{S}_c \), the unique \( n_{i,j} \) such that \( l - n_{i,j} v_1 \in \Xi_i, i = 1, 2 \). Then subsection 4.5. provides the following decomposition

\[
Z^+_h(t) = t^{(c, \tilde{c})} \left( \sum_{l \in \mathcal{S}_{c,1}} \sum_{j=1}^{n_{1,1}} t^{1-j v_1} + \sum_{l \in \mathcal{S}_{c,2}} \sum_{j=1}^{n_{1,2}} t^{1-j v_2} + \sum_{l \in \mathcal{S}_{c,1} \cap \mathcal{S}_{c,2}} t^l \right),
\]

\[
Z^-_h(t) = t^{(c, \tilde{c})} \left( \sum_{l \in (\square - (c, \tilde{c})) \cap \mathbb{Z}^2 \cap (\Xi_c)} \frac{t^l}{(1 - t v_1)(1 - t v_2)} - \sum_{l \in \mathcal{S}_{c,1}} \frac{t^l}{1 - t v_1} - \sum_{l \in \mathcal{S}_{c,2}} \frac{t^l}{1 - t v_2} + \sum_{l \in \mathcal{S}_{c,1} \cap \mathcal{S}_{c,2}} t^l \right).
\]

Therefore, by 4.4.7 and Theorem 4.5.1 we get

\[
pch^{orb}_h(Z) = pc^{orb}_h(Z_h/t^{(c, \tilde{c})}) = Z^+_h(1, 1) = \sum_{l \in \mathcal{S}_{c,1}^+} n_{1,1} + \sum_{l \in \mathcal{S}_{c,2}^+} n_{1,2} + |\mathcal{S}_{c,1}^- \cap \mathcal{S}_{c,2}^-|.
\]
**Corollary 7.4.1.** Choose $c$ in such a way that $c \in [0, 1)$ and $\bar{c} \in [0, 1)$. Then one has the following combinatorial formula for the normalized Seiberg–Witten invariant of $M$

$$-(K + 2r_h)^2 + |\mathcal{V}| - \mathcal{w}_{h \ast \sigma_{can}}(M) = \sum_{i \in S_{c,1}} n_{i,1} + \sum_{i \in S_{c,2}} n_{i,2} + |S_{c,1}^{-} \cap S_{c,2}^{-}|.$$

**Proof.** Use Corollary 5.2.1 the reduction Theorem 5.4.2 and the above computation.

**Example 7.4.2.** Consider the following plumbing graph

$$E_1 \quad \begin{array}{c}
-2 \\
-1 \\
-1 \\
-5 \\
-5 \\
\end{array} \quad \begin{array}{c}
E_2 \\
-3 \\
-5 \\
\end{array} \quad \begin{array}{c}
\bar{E}_1 \\
\bar{E}_2 \\
\bar{E}_3 \\
\end{array}$$

The corresponding Seifert invariants are $\alpha_1 = 2$, $\alpha_2 = 3$, $\bar{\alpha}_j = 5$, $\alpha_0 = 9$ and $\omega_i = \bar{\omega}_j = \omega_0 = \bar{\omega}_0 = 1$ for all $i$ and $j$. Hence $e = -1/18$, $\bar{e} = -13/45$ and $\varepsilon = 1/(3^3 \cdot 10)$. For $h = 0$ we choose $c = 0$. Then

$$S_0 = \left\{ \begin{array}{l}
(\ell, \bar{\ell}) \in \mathbb{Z}^2 \\
8\ell - \ell + 9 \cdot \left(\frac{\ell}{2}\right) \geq 0 \\
8\ell - \ell + 27 \cdot \left(\frac{\ell}{3}\right) \geq 0 \\
\ell + \ell \equiv 0 \pmod{9}
\end{array} \right\} \quad \text{and} \quad \overline{S}_0 = \left\{ \begin{array}{l}
(\ell, \bar{\ell}) \in \mathbb{Z}^2 \\
\ell - 2\ell \geq 0 \\
-5\ell + 13\ell \geq 0 \\
\ell + \ell \equiv 0 \pmod{9}
\end{array} \right\}.$$

If we take the generators $v_1 = (60, 30)$ and $v_2 = (26, 10)$ (via conditions (A)-(B)-(C) following Lemma 7.3.3), one can calculate explicitly the sets

$$S_{0,1}^{-} = \left\{ \begin{array}{l}
(13, 5), (19, 8), (25, 11), (31, 14), \\
(37, 17), (43, 20), (49, 23), \\
(55, 26), (61, 29), (67, 32)
\end{array} \right\} \quad \text{and} \quad S_{0,2}^{-} = \left\{ \begin{array}{l}
(6, 3), (19, 8), (12, 6), \\
(25, 11), (24, 12), (37, 17), \\
(42, 21), (55, 26)
\end{array} \right\}.$$

This generates the next counting function of $\overline{S}_0 \setminus S_0$, namely $\sum_{(\ell, \bar{\ell}) \in \overline{S}_0 \setminus S_0} t^{\ell} \bar{t}^{\bar{\ell}} = \ldots$

which by 7.4 provides $Z_0^+(t, \bar{t}) = \ldots$ Hence $pc_{0}^{Corb}(Z) = Z_0^+(1, 1) = 13$.

[It can be verified that there exists a splice quotient type normal surface singularity whose link is given by the above graph. It is a complete intersection in $(\mathbb{C}^4, 0)$ with equations $z^3 + (y_2 + 2y_3)^2 - y_1y_2(2y_2 + 3y_3) = y_1^5 + (2y_2 + 3y_3)y_2y_3 = 0$.]

7.5. **The periodic constant in the non-equivariant case and $\lambda(M)$.**

Though the non-equivariant $Z_{ne}$ can be obtained by the sum $\sum_h Z_h$ treated in the previous subsection, here we provide a more direct procedure, which leads to a new formula. Write $J :=$
identification will be established for any negative definite plumbing graph with arbitrary number
of components. We will not provide here the formulae, since this

\[ Z_{ne}(t) = \frac{(1 - t^{I_{(0)}})^{d-1}(1 - t^{I_{(0)}})^{d-1}}{\prod_i(1 - t^{I_{(i)}}) \prod_j(1 - t^{I_{(j)}})} \]

Set \( S(x) := \sum_i x_i/\alpha_i \) and \( \tilde{S}(\tilde{x}) := \sum_j \tilde{x}_j/\tilde{\alpha}_j \). Similarly as in [6.2.7], \( Z_{ne}(t) \) can be written as

\[ \sum_{0 \leq x_i < \alpha_i, 0 \leq \tilde{x}_j < \tilde{\alpha}_j} f(x, \tilde{x}), \quad \text{where} \quad f(x, \tilde{x}) = \frac{t^{J_{(S(x))}}}{(1 - t^{I_{(0)}})(1 - t^{I_{(0)}})}. \]

By the substitution \( u_1 = t^{I_{(0)}} \) and \( u_2 = t^{I_{(0)}} \), \( f(x, \tilde{x}) \) transforms into \( u_1^{S(x)} u_2^{\tilde{S}(\tilde{x})}/(1 - u_1)(1 - u_2) \). The division of this fraction (with remainder) is elementary, hence \( f(x, \tilde{x}) \) equals

\[ t^{J_{(S_{rat})}} \left( \sum_{n=0}^{S_{int}-1} \sum_{k=0}^{S_{int}-1} t^{J_{(k)}} - \sum_{k=0}^{S_{int}-1} \frac{t^{J_{(k)}}}{1 - t^{I_{(0)}}} - \sum_{k=0}^{S_{int}-1} \frac{t^{J_{(k)}}}{1 - t^{I_{(0)}}} + \frac{1}{(1 - t^{I_{(0)}})(1 - t^{I_{(0)}})} \right), \]

where \( S_{int} := [S(x)], \tilde{S}_{int} := [\tilde{S}(\tilde{x})], S_{rat} := \{S(x)\} \) and \( \tilde{S}_{rat} := \{\tilde{S}(\tilde{x})\} \).

Then, by [4.4.12], \( \text{pc}^\text{orb}(t^{J_{(S_{rat})}}/(1 - t^{I_{(0)}})(1 - t^{I_{(0)}})) = 0 \). Moreover, [4.5] gives a unique integer \( s(k) \geq 0 \) for \( k \in \{0, \ldots, S_{int} - 1\} \) such that \( t^{J_{(k + S_{rat})}}/1 - t^{I_{(0)}} \) has vanishing periodic constant with respect to \( C_{\text{orb}} \). It turns out that \( s(k) = [-\tilde{\alpha}_0(\tilde{k} + \tilde{S}_{rat}) + S_{rat}] \) in the case of \( t^{J_{(k + S_{rat})}}/1 - t^{I_{(0)}} \). Therefore, by [4.5.1] for

\[ \text{pc}(Z_{ne}) = -\lambda(M) - d \frac{K^2 + |V|}{8} + \sum_h \chi(r_h) \]

we get

\[ \sum_{0 \leq x_i < \alpha_i, 0 \leq \tilde{x}_j < \tilde{\alpha}_j} \left( S_{int} \tilde{S}_{int} + \sum_{k=0}^{S_{int}-1} [-\tilde{\alpha}_0(k + S_{rat}) + \tilde{S}_{rat}] + \sum_{k=0}^{S_{int}-1} [-\alpha_0(\tilde{k} + \tilde{S}_{rat}) + S_{rat}] \right). \]

7.6. **Ehrhart theoretical interpretation.** In general, in contrast with the one–node case [5.5] the
direct determination of the counting function of \( Z_h(t) \), or equivalently, of the complete equivariant
Ehrhart quasipolynomial associated with the corresponding polytope, is rather hard. Nevertheless,
those coefficients which are relevant to us (e.g. those ones which contain the information about
the Seiberg–Witten invariants of the 3–manifold) can be identified using the right hand side of
[2.2.3] The computation is more transparent when \( L' = L \). In that case, the two-variable Ehrhart
polynomial has degree \( d + \tilde{d} \), and a specific \( d + \tilde{d} - 2 \) degree coefficient is exactly the normalized
Seiberg–Witten invariant of the 3–manifold. We will not provide here the formulae, since this
identification will be established for any negative definite plumbing graph with arbitrary number
of nodes, see section [8] where several other coefficients will be computed as well.
8. Ehrhart Theoretical Interpretation of the SW Invariant (the General Case)

8.1. Let $\Gamma$ be a negative definite plumbing graph, a connected tree as in [2.1]. Let $\mathcal{N}$ and $\mathcal{E}$ be the set of nodes and end–vertices as above. We assume that $\mathcal{N} \neq \emptyset$. If $\delta_n$ denotes the valency of a node $n$, then $|\mathcal{E}| = 2 + \sum_{n \in \mathcal{N}} (\delta_n - 2)$.

We consider the matrix $J$ with entries $J_{nm} := - (E_n^*, E_m^*)$ for $n, m \in \mathcal{N}$. By (2.1) it is a principal minor of $-I^{-1}$ (with rows and columns corresponding to the nodes).

Another incarnation of the matrix $J$ already appeared in subsection 7.5, as the negative of the inverse of the orbifold intersection matrix. Indeed, let for any $n \in \mathcal{N}$ take that component of $\Gamma \setminus \cup_{m \in \mathcal{N} \setminus \{m\}} \{m\}$ which contains $n$. It is a star shaped graph, let $e_n$ be its orbifold Euler number. Furthermore, for any two nodes $n$ and $m$ which are connected by a chain, let $\alpha_{nm}$ be the determinant of that chain (not including the nodes). Then define the orbifold intersection matrix (of size $\mathcal{N} \times \mathcal{N}$) by the transformation law

$$ L \times \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N} ,$$

otherwise; cf. [BN07, 4.1.4]. One can show (see [BN07, 4.1.4]) that $I_{nm}$ is invertible, negative definite, and $\det (-I_{nm})$ is the product of $\det (-I)$ with the determinants of all (maximal) chains and legs of $\Gamma$. This fact and (2.1) imply that $J = (-I_{nm})^{-1}$.

8.2. The Ehrhart Polynomial. In the sequel we assume that $L = L'$, that is $H = 0$.

By (5.2) $P^{(l)}$ sits in $\mathbb{R}[\mathcal{E}]$. Moreover, by the reduction theorem (5.4) we can take $l$ of the form $l = \sum_{n \in \mathcal{N}} \lambda_n E_n^*$, from the subcone of the Lipman cone generated by $\{E_n^*\}_{n \in \mathcal{N}}$.

Then (5.4) guarantees that the associated polytope is $P^{(l)} = \bigcup_{n \in \mathcal{N}} P_n^{(l_n)} \cup P^l_n$ depending only of the component $l_n = -(l, E_n^*)$. Note that the coefficients $\{\lambda_n\}_n$ and the entries $\{l_n\}_n$ are connected exactly by the transformation law $(l_n)_n = J (\lambda_n)_n$.

Take any chamber $\mathcal{C}$ such that $\text{int}(\mathcal{C} \cap S) \neq \emptyset$, as in (5.2). Let $\hat{\mathcal{L}}^C(P, T, (\lambda_n)_n)$ be the Ehrhart quasipolynomial $\mathcal{L}^C(P, T, (l_n)_n)$, associated with the denominator of $Z$, after changing the variables to $((\lambda_n)_n)$ via $(l_n)_n = J (\lambda_n)_n$. It is convenient to normalize the coefficient of $\prod_n \lambda_n^{m_n}$ by a factor $\prod_n m_n!$, hence we write

$$ \hat{\mathcal{L}}^C(P, T, (\lambda_n)_n) = \sum_{\sum_{n \in \mathcal{N}} \lambda_n E_n^*} \hat{a}^C_{(m_n)_n} \prod_n \frac{\lambda_n}{m_n!}, $$

for certain periodic functions $\hat{a}^C_{(m_n)_n}$ in variables $(\lambda_n)_n$. By (2.2.3), (4.3.11) and (5.4.2) (8.2.1)

$$ \chi(\lambda_n) = \sum_{\lambda_n \in \mathcal{N}} \lambda_n E_n^* + pc^S(Z) = \Delta((\lambda_n)_n), $$

where

$$ \Delta((\lambda_n)_n) = \sum_{0 \leq k_n \leq \delta_n - 2} (-1)^{k_n} \prod_n \frac{\delta_n - 2}{k_n} \hat{\mathcal{L}}^C(P, T, (\lambda_n - k_n)_n) = $$

$$ \sum_{\sum_{n \in \mathcal{N}} m_n \leq |\mathcal{E}|} \left( \sum_{0 \leq p_n \leq m_n} \right) (-1)^{\sum_n p_n} \prod_n \frac{m_n}{p_n} \left( \sum_{k_n = 0}^{\delta_n - 2} \frac{(\delta_n - 2)^{k_n}}{k_n} \right) \hat{a}^C_{(m_n)_n} \prod_n \frac{\lambda_n^{m_n - p_n}}{m_n!}. $$

On the other hand, since $\chi(l) = -(K + l, l)/2$, the left hand side of (8.2.1) is the quadratic function

$$ \sum_{n, m \in \mathcal{N}} \frac{J_{nm}}{2} \lambda_n \lambda_m + \sum_{n \in \mathcal{N}} \sum_{n \in \mathcal{N}} (-(K, E_n^*)/2) \lambda_n + pc^S(Z). $$
Now we identify these coefficients with those of $\Delta((\lambda_n)_n)$ above. The additional ingredient is the combinatorial formula (6.5.11), which also shows that for the non-zero summands one necessarily has $p_n \geq \delta_n - 2$ for any $n$. One gets the following result.

**Theorem 8.2.2.**

\[
\hat{a}_{(\delta_n, (\delta_m - 2)_{m \neq n})}^C = J_{nn},
\]

\[
\hat{a}_{(\delta_n - 1, \delta_m - 1, (\delta_q - 2)_{q \neq n, m})}^C = J_{nm} \text{ for } n \neq m,
\]

\[
\hat{a}_{(\delta_n - 1, (\delta_m - 2)_{m \neq n})}^C = -\frac{1}{2} (K, E_n^*) + \frac{1}{2} \sum_{m \in \mathcal{N}} (\delta_m - 2) J_{nm}
\]

\[
\hat{a}_{(\delta_n - 2)_n}^C = p_{C}(Z) - \sum_{n \in \mathcal{N}} \frac{(\delta_n - 2)(K, E_n^*)}{4} + \sum_{n \in \mathcal{N}} \frac{(\delta_n - 2)(3\delta_n - 7)J_{nn}}{24} + \sum_{n, m \in \mathcal{N}, m \neq n} \frac{(\delta_n - 2)(\delta_m - 2)J_{nm}}{8}.
\]

Recall that $p_{C}(Z) = -(K^2 + \lvert \mathcal{V} \rvert)/8 - \lambda(M)$, where $\lambda(M)$ is the Casson invariant of $M$. Hence $\hat{a}_{(\delta_n - 2)_n}^C$ equals the normalized Casson invariant modulo some ‘easy terms’.

We emphasize that these formulae also show that the above coefficients are constants (as periodic functions in $(\lambda_n)_n$) and independent of the chosen chamber $C$.

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