On exclusive Racah matrices $\bar{S}$ for rectangular representations

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ABSTRACT

We elaborate on the recent observation that evolution for twist knots simplifies when described in terms of triangular evolution matrix $B$, not just its eigenvalues $\Lambda$, and provide a universal formula for $B$, applicable to arbitrary rectangular representation $R = [r^s]$. This expression is in terms of skew characters and it remains literally the same for the 4-graded rectangularly-colored hyperpolynomials, if characters are substituted by Macdonald polynomials. Due to additional factorization property of the differential-expansion coefficients for the double-braid knots, explicit knowledge of twist-family evolution leads to a nearly explicit answer for Racah matrix $\bar{S}$ in arbitrary rectangular representation $R$. We also relate matrix evolution to existence of a peculiar rotation $U$ of Racah matrix, which diagonalizes the $Z$-factors in the differential expansion – what can be a key to further generalization to non-rectangular representations $R$.

Knot polynomials [1] belong to an advanced chapter of modern mathematical physics. Their understanding would provide a non-trivial extension of our knowledge from two to three dimensions and is foreseeably important for various branches of science. One of the intimately related is the theory of Racah matrices [2] which was supposed to help in the study of knot polynomials. Somewhat surprisingly, however, the inspiration went in the opposite direction – it turns simpler to find the knot polynomials and then convert them into the new results for Racah matrices. This paper describes a spectacular result of this kind.

1. Conceptually, knot invariants are the Wilson loop averages

$$\mathcal{H}_R^C(q, A) = \left< \text{Tr}_R P \exp \oint_C A \right>$$

i.e. the central objects in the Yang-Mills theory, but they are evaluated in one of its simplest versions – the topological 3d Chern-Simons theory [3]. This reduces the most interesting dependence on the integration contour $C$ to that on its topological class (knotting). Still, even this dependence is highly non-trivial – nothing to say about that on the coupling constant $g$, on the rank of gauge group $SU(N)$ and on its representation. One of the spectacular facts is that the average can be calculated exactly, and the it turns to be a polynomial of peculiar non-perturbative variables $q = \exp\left(\frac{2\pi i}{g+1}\right)$ and $A = q^N$. The reason which makes the theory solvable is that it describes the peculiar time-evolution of 2d conformal blocks [6], defined by their monodromies. This makes the study of knot polynomials the next step after a deep understanding of conformal blocks, achieved recently through the theory of Nekrasov functions [7] (instanton calculus in 4d and 5d supersymmetric Yang-Mills theories [8]) and the AGT relations [9]. On this way one can expect new insights and discoveries – and they are indeed being intensively produced and shed new light on a variety of subjects: from non-perturbative calculations and associated extension of integrability theory to the hard problems in conventional representation theory.
One of such problems is calculation of Racah matrices ($6j$-symbols) – intensively studied in theoretical physics and included into the standard textbooks like [4]. Conformal block is a contribution of particular intermediate state into the correlator, e.g. in the 4-point case

\[
\sum_X \rightarrow \sum_Y
\]

where two different ways of decomposition are shown, and correlator is the sum over $Y$ or, alternatively, over $X$. The blocks in the $t$-channel are related to those in the $S$-channel by a linear transformation – and it is defined by the Racah matrix. In fact, when external lines belong to irreducible representations of the symmetry group, Racah matrix depends only on representations, not on the choice of particular elements and can be considered as defining the associativity relation in representation products:

\[
\text{Racah : } (R_1 \otimes R_2) \otimes R_3 \rightarrow R_4 \rightarrow (R_1 \otimes (R_2 \otimes R_3) \rightarrow R_4)
\]

(2)

It is a matrix w.r.t. to the intermediate representations $X \in R_1 \otimes R_2$ and $Y \in R_2 \otimes R_3$. Clearly Racah matrices are crucial for all kinds of dualities in string theory and their study for various kinds of groups and representations remains one of the central problems of modern theory. Unfortunately, this is a very hard calculational problem and Racah matrices are not actually known even for the finite dimensional irreducible representations of $\text{Sl}_N$ group, labeled by Young diagrams – nothing to say about the more important example of DIM algebras, with representations, labeled by plane partitions.

Calculation of colored (i.e. with non-trivial diagrams $R$) Wilson averages in Chern-Simons theory is reduced to convolutions of Racah matrices by the modern version of Reshetikhin-Turaev formalism [5] – however, until recently these matrices were actually unknown for Young diagrams $R$ which have more than a single column or line. In [10] it was suggested to reverse the logic – and extract Racah matrices from non-trivial properties of knot polynomials for some relatively simple knot families, like their evolution and differential expansion properties. This appeared a very fruitful idea, and it led to explicit evaluation of Racah matrices for all 2-line representations $R$, some of which are presented in [11]. However, originally it was a hard task, and progress depended on the understanding of the structure of the formulas, which has been slowly revealed in a sequence of works. A new progress was achieved in a recent ref. [12] (KNTZ), which, as we are going to demonstrate in the present paper, leads to a great simplification and provides explicit answers for “exclusive” Racah matrices $\bar{S}$ (see below) in arbitrary rectangular representations $R$: as we explain, they have the form of finite sums

\[
\bar{S}_{\mu \nu}^R \sim \sum_{\lambda \in R} Z^\lambda_R \cdot E_{\lambda \mu} E_{\lambda \nu}
\]

(3)

over sub-diagrams $\lambda$ of $R$ with explicitly known factors $Z$ and matrices $E$, which are the eigenfunction (diagonalization) matrices of triangular $\mathcal{B}$, explicitly expressed through the well-known skew Schur functions. This is a tremendously simple expression, which can hardly be further simplified: in general case there are no more cancellations between the terms of the remaining sum.

Though exhaustive and explicit, this expression is just a conjecture, depending on a number of mysterious observations made in [10] and afterwards. It passed a great number of highly non-trivial cross-checks – beginning from unitarity of the so constructed matrix $\bar{S}$ – and there is a little doubt that the answer is correct. However, its derivation remains a big challenge for quantum field theory and, most probably, will continue to be a source of new inspiration – as it happened at the previous stages.
As already stated, the recent breakthrough in [12] almost completed the quest for description of rectangularly-colored knot polynomials for the double twist (double braid) knots, originated in [13][14] and [10][15], and advanced in [16]-[19]. The crucial observation of [10] was that the coefficients of the differential expansion for the double twist colored HOMFLY-PT polynomial [1] factorize into a product of those for the twist knots with \( n = 1 \):

\[
\mathcal{H}^{(m,n)}_R = \sum_{\lambda \subseteq R} Z_R^\lambda \cdot F_\lambda^{(m,n)} = \sum_{\lambda \subseteq R} Z_R^\lambda \cdot F_\lambda^{(m)} \cdot F_\lambda^{(n)}
\]

\[
F_\lambda^{(m)} = \sum_{\mu \subseteq \lambda} f_{\mu} \cdot \Lambda_\mu^m
\]

The sum here goes over all the sub-diagrams \( \lambda \) of the rectangular Young diagram \( R = [r^\ast] \), and the evolution eigenvalues \( \Lambda_\mu \) are expressed through the hook parameters of the diagram:

\[
\mu = (a_1, b_1|a_2, b_2|\ldots) \implies \Lambda_\mu = \prod_h (A q^{a_h} - b_h)^{2(a_h + b_h + 1)} = \prod_{h=1}^{\text{# of hooks}} (A^2 q^{2(\text{length of the } h\text{-th hook})})^{\text{leg}_h + \text{arm}_h + 1}
\]

The weights in the sum come from the differential expansion [15][16] for the figure-eight knot with \((m, n) = (1, -1)\):

\[
Z_\lambda^{(r)} = \left( -\frac{(q - q^{-1})^2}{A^2} \right)^{\lambda} \cdot \chi_\lambda^{(r)}(r) \cdot \chi_\lambda^{(N + r)}(s) \cdot \chi_\lambda^{(N - s)}(t) \cdot (\chi_\lambda^2)^2
\]

Here \( \chi_\lambda^{(r)} := \chi_\lambda(p^r) \) here and \( \chi_\lambda^{(s)} := \chi_\lambda(p^s) \), which we will also need in (16) below, denote restriction of the Schur functions to the special loci: with time-variables \( p_k \) substituted respectively by

\[
p^r_k := \frac{q^{kr} - q^{-kr}}{q^k - q^{-k}} \quad \text{and} \quad p^s_k := \frac{(q - q^{-1})^k}{q^k - q^{-k}}
\]

Inverse of \( \chi_\lambda^s \) differs by a power of \( q \) from the \( N \)-independent denominator of \( \chi_\lambda^s(N) \), and (6) combines the usual product of differentials, combinatorial factor and \( F^{(1)}_\lambda \). The arguments of the HOMFLY-PT polynomial are \( q \) and \( A = q^N \). Dependence on parameters \( m \) and \( n \) of the knot appears in (4) only through the powers of \( \Lambda \), while dependence on parameters \( r \) and \( s \) of representation – only through the knot-independent weights \( Z_\lambda^{(r)} \) in
the sum, which restrict the summation domain to \( \lambda \subset R = [r^s] \). The analogues \( C_{\mu \nu} \) of the Adams coefficients in the double-twist analogue of the Rosso-Jones formula \([20][21]\)

\[
\mathcal{H}_{r^s}^{(m,n)} = \sum_{\mu, \nu \subset [r^s]} C_{\mu \nu} \lambda_{\mu}^{m} \lambda_{\nu}^{n}
\]  

(8)

are therefore equal to

\[
C_{\mu \nu} = \sum_{\mu, \nu \subset \lambda \subset [r^s]} Z_{[r^s]}^{\lambda} f_{\mu} f_{\lambda \nu}
\]  

(9)

For the figure eight knot, unknot and the trefoil the \( F \)-functions are nearly trivial:

\[
F_{\lambda}^{(-1)} = 1, \quad F_{\lambda}^{(0)} = 0, \quad F_{\lambda}^{(1)} = (-A)^{|\lambda|} \lambda_{\lambda}^{1/2} = \prod_{i=1}^{\# \text{ of hooks}} (-A^2 q^{a_i - b_i} (a_i + b_i + 1)) : = \bar{\lambda}_{\lambda}
\]  

(10)

(the difference between the trefoil \( F \)-factor \( \bar{\lambda} \) and the \( T^2 \) eigenvalue \( \lambda \) from \([3]\) is in factor 2 in the power of \( q \)).

3. Arborescent calculus \([22][23]\) implies that the same HOMFLY polynomial is expressed through Racah matrix \( \tilde{S} \): \( (R \otimes R) \otimes R \rightarrow R \)  

\[
\tilde{S} \rightarrow \left( R \otimes (R \otimes R) \rightarrow R \right)
\]

by almost the same formula:

\[
\mathcal{H}_{R}^{(m,n)} = \sum_{\mu, \nu \subset R} \sqrt{\mathcal{D}_{\mu} \mathcal{D}_{\nu} / \chi_{R}(N)} \bar{S}_{\mu \nu} \lambda_{\mu}^{m} \lambda_{\nu}^{n}
\]  

(11)

where for rectangular \( R = [r^s] \) the \( \mathcal{D}_{\mu} = \chi_{\mu}^0(N)^2 \cdot C_{\varnothing \mu} = \chi_{\mu, \mu}^0(N) \) are dimensions of the composite representations \( (\mu, \mu) \in R \otimes \bar{R} \), see eq.(28) in \([24]\) for an explicit expression. From \([11]\) we obtain:

\[
\bar{S}_{[r^s]}^{[r^s]} = \frac{\chi_{[r^s]}^0}{\sqrt{\mathcal{D}_{\mu} \mathcal{D}_{\nu}}} \sum_{\mu, \nu \subset [r^s]} \bar{Z}_{\lambda}^{[r^s]} f_{\mu} f_{\lambda \nu}
\]  

(12)

while another Racah matrix \( S \): \( (\bar{R} \otimes R) \otimes R \rightarrow R \)  

\[
\tilde{S} \rightarrow \left( \bar{R} \otimes (R \otimes R) \rightarrow R \right)
\]

diagonalizes the product \( T^2 \), i.e. solves the linear equation

\[
\sum_{\mu} \bar{T}_{\lambda} \bar{S}_{\mu \lambda} \bar{T}_{\nu} \bar{S}_{\mu \lambda} \bar{T}_{\nu} = S_{\lambda \nu}
\]  

(13)

where \( \bar{T} \) and \( T \) are known diagonal matrices, e.g. \( \bar{T}^2 = \Lambda_{\mu} \). Orthogonality \( \sum_{\nu} \bar{S}_{\mu \nu} S_{\nu \nu} = \delta_{\mu \nu} \) of symmetric matrix \( \bar{S} \) is equivalent to the sum rule

\[
\sum_{\rho \subset [r^s]} \frac{C_{\mu \rho} C_{\nu \rho}}{C_{\varnothing, \rho}} = \chi_{[r^s]}^0(N)^2 \cdot C_{\varnothing, \mu} : \delta_{\mu \nu}
\]  

(14)

for the combinations \( C_{\mu \nu} \) in \([9]\), which does not contain square roots \( \sqrt{\mathcal{D}_{\mu}} \), what simplifies computer checks.

4. For \( F \) and \( f \) ref. \([10]\) suggested explicit expressions in a variety of examples, which in \([17]\) were expressed in terms of skew characters. However, expression for generic \([r^s]\) was not found at that stage. This was done in a recent paper \([12]\). After some polishing, the observation there is that \( F_{\lambda}^{(m)} \) actually have a very simple shape, which one could (but did not) naturally anticipate from the evolution interpretation:

\[
F_{\lambda}^{(m)} = \sum_{\mu} (B^{m+1})_{\lambda \mu}^{\lambda}
\]  

(15)

where \( B_{\lambda \mu} \) is a triangular matrix with the eigenvalues \( \Lambda_{\mu} \) at diagonal and non-vanishing entries only for embedded Young diagrams \( \mu \subset \lambda \), which are explicitly given by

\[
B_{\lambda \mu} = (-)^{|\lambda| - |\mu|} \cdot \Lambda_{\lambda} \cdot \frac{\chi_{\lambda}^{\varnothing} \cdot \chi_{\lambda \nu}^{\varnothing}}{\chi_{\lambda}^{\varnothing}}
\]  

(16)
5. As a simplest example, the original result of \(14\) for symmetric representations \(R = [r]\) is reproduced by the following piece of \(\mathcal{B}\), associated with the single-column diagrams \([r]\):

\[
\mathcal{B}_{ij} = \begin{cases} 
\frac{(-1)^{i-j}}{q((i-2)(i-3))} \cdot \Lambda_{i-1} & \text{for } i \geq j \\
0 & \text{for } i < j
\end{cases}
\]

\[
\begin{pmatrix}
\Lambda_{[0]} & 0 & 0 & 0 & 0 & \ldots \\
\Lambda_{[1]} & \Lambda_{[4]} & 0 & 0 & \ldots \\
\frac{1}{q} \Lambda_{[2]} & - \frac{2}{q} \Lambda_{[2]} & \Lambda_{[2]} & 0 & \ldots \\
\frac{1}{q^2} \Lambda_{[3]} & - \frac{3}{q^2} \Lambda_{[3]} & \Lambda_{[3]} & 0 & \ldots \\
\frac{1}{q^3} \Lambda_{[4]} & - \frac{4}{q^3} \Lambda_{[4]} & - \frac{4}{q^3} \Lambda_{[4]} & \Lambda_{[4]} & \ldots \\
\end{pmatrix}
\]

Square brackets in this formula are used to denote both the Young diagrams and quantum numbers \([n] := \frac{q^n - q^{-n}}{q - q^{-1}}\), hopefully this does not cause a confusion. Antisymmetric representations are controlled by a similar piece of \(\mathcal{B}\), associated with the sequence \([1]\). In the case of \(R = [3, 3, 3]\) the relevant fragment of \(\mathcal{B}\) is \(20 \times 20\), since there are 20 sub-diagrams in \([3, 3, 3]\), but the entries remain simple factorized expressions – still this nicely reproduces the complicated formulas for \(F^{(m)}_\lambda\) from \(19\).

6. A possible way to explain the somewhat mysterious formula \(15\) is to rewrite the original arborescent formula for the HOMFLY-PT polynomial of the twist knot

\[
H_R^{\text{twist}} = D_R \cdot \left( \bar{S}_R \bar{T}^2 \bar{S}_R \bar{T}^{2m} \bar{S}_R \right)_{\emptyset \emptyset}
\]

with symmetric and orthogonal Racah matrix \(\bar{S}_R, \bar{S}_R^2 = I\), as

\[
H_R^{\text{twist}} = D_R \cdot \left( \bar{T}^2 \left( \bar{T}^2 \bar{S} \right)^m \right)_{\emptyset \emptyset} = \sum_{\lambda} D_R \cdot \left( \bar{T}^2 \bar{S} \bar{T}^{-2} \bar{S} \right)_{\emptyset \lambda} \left( \left( \bar{T}^2 \bar{S} \right)^{m+1} \right)_{\lambda \emptyset}
\]

and then further decompose it by inserting the unity decomposition \(I = U^{-1} U\) with an auxiliary matrix \(U\):

\[
H_R^{\text{twist}} = \sum_{\lambda} \sum_{\mu} D_R \cdot \left( \bar{T}^2 \bar{S} \bar{T}^{-2} \bar{S} \right)_{\emptyset \lambda} \cdot \sum_{\mu} \left( \bar{T}^2 \bar{S} \bar{T}^{-2} \bar{S} \right)_{\emptyset \mu} \cdot U_{\mu \emptyset} = \sum_{\lambda} \sum_{\mu} \left( F^{(m)}_\lambda \right)_{\lambda \mu} \cdot \left( B^{m+1} \right)_{\lambda \mu}
\]

Thus we obtain at once, from a single \(U\), the decomposition formula \(14\) and the matrix-evolution rule \(19\). However, we still need to choose \(U\) appropriately, so that it provides decomposition with the necessary (empirically justified) properties. The last transition in \(19\) requires that \(U\) has unities everywhere in the first column,

\[
U_{\mu \emptyset} = 1
\]

while its other elements are adjusted to make the KNTZ matrix

\[
\mathcal{B} := U \cdot \bar{T}^2 \bar{S} \cdot U^{-1}
\]

triangular and satisfying the constraints

\[
\sum_{\mu} B_{\lambda \mu} = \delta_{\lambda, \emptyset} \quad \Rightarrow \quad \sum_{\lambda} (\bar{B}^2)_{\lambda \mu} = B_{\lambda \emptyset} \quad \forall \lambda
\]
7. Remarkably, after $U$ is adjusted to convert symmetric $\bar{S}T\bar{S}$ into triangular $B$, the matrix elements

$$Z^\lambda_R := D_R \cdot (\bar{S}T^2 \bar{S}T^{-2} S \cdot U^{-1})_{\lambda\lambda}$$

appear to be factorized for all rectangular representations $R$ (for non-rectangular $R$ they are sums of several factorized expressions, see [18]) and reproduce the hook formulas for the $Z$-factors in the differential expansions, in particular

$$Z^\varnothing_R = D_R \cdot (\bar{S}T^2 \bar{S}T^{-2} S U^{-1})_{\varnothing\varnothing} = 1 \quad \forall R$$

One more impressive fact is the factorization property, which was the starting observation of [10]:

$$H^\text{double braid}_{m,n} = D_R \cdot (\bar{S}T^{2n} \bar{S}T^{2m} \bar{S})_{\varnothing\varnothing} = \sum \lambda Z^\lambda_R \cdot \frac{F^{(m)}_\lambda \cdot F^{(n)}_\lambda}{F_\varnothing^{(1)}}$$

It is now equivalent to a mysterious identity

$$\left(\bar{S}T^{2n} \bar{S}T^{2m} \bar{S}\right)_{\varnothing\varnothing} = \sum \lambda \left(\bar{S}T^2 \bar{S}T^{-2} S U^{-1}\right)_{\lambda\lambda} \cdot \frac{\left(U \bar{S}T^{2(m+1)} \bar{S}\right)_{\lambda\varnothing} \cdot \left(U \bar{S}T^{2(n+1)} \bar{S}\right)_{\varnothing\lambda}}{\left(U \bar{S}T^2 S U^{-1}\right)_{\lambda\lambda}}$$

and can serve as a prototype of the ("gauge invariant") arborescent vertex [23]

$$\mathcal{V} := \sum \lambda \frac{\bar{S}U^{-1}|\lambda\rangle \otimes \langle \lambda|U\bar{S} \otimes \langle \lambda|U\bar{S}}{\lambda\lambda}$$

or, perhaps,

$$\mathcal{V} := \sum \lambda \frac{\bar{S}T^{-2} SU^{-1}|\lambda\rangle \otimes \langle \lambda|U\bar{S}T^2 \bar{S} \otimes \langle \lambda|U\bar{S}T^2 \bar{S}}{\lambda\lambda}$$

8. Like $\bar{S}$, matrix $U$ depends on $R$, we omitted the label $R$ to make the formulas readable. Universal ($R$-independent) are $\bar{T}$ and $B$, and just the first column $U_{Y\varnothing} = 1$ of $U$.

**Example of $R = [1]$**: From

$$\bar{S}_{[1]} = \frac{1}{[N]} \begin{pmatrix} 1 & \sqrt{[N+1][N-1]} & \sqrt{[N+1][N-1]} \\ \sqrt{[N+1][N-1]} & -1 \end{pmatrix}, \quad \bar{T}^2 = \begin{pmatrix} 1 & 0 \\ 0 & A^2 \end{pmatrix}$$

we get

$$U_{[1]} = \left( \begin{array}{c} 1 \\
\frac{\sqrt{[N+1][N-1]}}{\sqrt{[N+1][N-1]}} \end{array} \right) = \left( \begin{array}{c} 1 \\
\frac{\sqrt{[N+1][N-1]}}{\sqrt{[N+1][N-1]}} + \frac{A}{\sqrt{[N+1][N-1]}} \end{array} \right), \quad B = \begin{pmatrix} 1 & 0 \\ -A^2 & A^2 \end{pmatrix}$$

and

$$Z^\varnothing_{[1]} = 1, \quad Z_{[1]} = \{ Aq \} \{ A/q \} = \{ q \} \{ q \} [N+1][N-1]$$

To compare, before the $U$-"rotation", which converted it into triangular $B$, the original symmetric matrix was

$$\bar{S}_{[1]} \bar{T}^2 \bar{S}_{[1]} = \frac{1}{[2][N]} \begin{pmatrix} A^2 \cdot (q^2[N+1] + q^2[N-1]) & -A\{q^2\}\sqrt{[N+1][N-1]} \\
-A\{q^2\}\sqrt{[N+1][N-1]} & q^2[N+1] + q^{-2}[N-1] \end{pmatrix}$$

The first line in $U$ is the same as in $\bar{S}$, i.e. consists of the square roots of quantum dimensions of the relevant representations from $R \otimes \bar{R}$. However, while $\bar{S}$ is finite, $U$ is singular in the double-scaling limit when $q, A \to 1$ and $N$ is fixed. This is because $\bar{T}^2$ and thus $\bar{S}T\bar{S}$ in this limit tend to a unit matrix, which is preserved by any $U$-rotation, but does not satisfy [23] – thus, when approaching the limit, $U$ develops a singularity.
Example of $R = [2]$: Likewise from

$$S_{[2]} = \frac{[2]}{[N][N + 1]} \left( \begin{array}{ccc} 1 & \sqrt{[N + 1][N - 1]} & \frac{[N]}{[2]} \sqrt{[N + 3][N - 1]} \\ \sqrt{[N + 1][N - 1]} & 1 + \frac{[N + 1]}{[2][N + 2]} ([N + 3][N - 1] - 1) & -\frac{[N]}{[N + 2]} \sqrt{[N + 3][N + 1]} \\ \frac{[N]}{[2]} \sqrt{[N + 3][N - 1]} & -\frac{[N]}{[N + 2]} \sqrt{[N + 3][N + 1]} & 1 \end{array} \right), \quad \bar{T}^2 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & A^2 & 0 \\ 0 & 0 & q^4 A^4 \end{array} \right)$$

it follows that

$$U_{[2]} = \left( \begin{array}{ccc} 1 & \frac{[q^6 + q^4]}{q^3 + q^2} A^3 - \{q^6 + q^4 \} A^2 + q^2 & \frac{[N]}{[2]} \sqrt{[N + 3][N - 1]} \\ \frac{A^2 q^6 - q^4}{q^3 + q^2} \nu \frac{[N + 1]}{[N - 1]} & \sqrt{[N + 1][N - 1]} & \frac{[N]}{[2]} \sqrt{[N + 3][N - 1]} \\ \frac{[N]}{[2]} \sqrt{[N + 3][N - 1]} & -\frac{[N]}{[N + 2]} \sqrt{[N + 3][N + 1]} & 1 \end{array} \right), \quad B = \left( \begin{array}{ccc} 1 & 0 & 0 \\ -A^2 & A^2 & 0 \\ q^2 A^4 & -(q^2 + q^4) A^4 & q^4 A^4 \end{array} \right)$$

and

$$Z_{[2]}^0 = 1, \quad Z_{[2]}^1 = \{Aq^2\} \{A/q\} = \{q\} \{q^2\} [N + 2][N - 1], \quad Z_{[2]}^2 = \{Aq^3\} \{Aq^2\} \{A\} \{A/q\} = \{q\} [N + 3][N + 2][N][N - 1]$$

We see that, unlike $\bar{S}_R$ and $U_R$, the matrices $\bar{T}$ and $B$ for $R = [2]$ contain those for $R = [1]$ as sub-matrices – this is a manifestation of their universality.

9. De facto, the KNTZ claim (15) is that the switch from a diagonal evolution matrix $\bar{T}^2$ to triangular $B$, though looks like a complication, actually reveals the hidden structure of the differential expansion for twist knots and somehow trades the sophisticated Racah matrix $\bar{S}$ for a much simpler and universal (representation-independent) $B$. As we explained, the reason for this can be that the actual evolution matrix was not the simple diagonal $\bar{T}^2$, but rather a sophisticated symmetric $\bar{S}T^2\bar{S}$, and then the switch to triangular $B$ is indeed a simplification. In this approach the crucial role is played by the switching matrix $U$, and the central phenomenon is a drastic simplicity of the first line in a peculiar matrix $UT^2U^{-1}B^{-1}$: for rectangular representations its entries are just products of the differentials, and better understanding of the phenomenon can help to explain the linear combinations of those, which emerge in the non-rectangular case.

10. The formula (15) is very nice, still in this form it is not immediately suitable for construction of Racah matrices. Fortunately, this is easy to cure. The simplest way to raise a matrix to a power is to diagonalize it. If $\mathcal{E}$ is a triangular matrix of right eigenvectors of $B$,

$$\sum_\mu B_{\lambda \mu} \mathcal{E}_{\mu \nu} = \mathcal{E}_{\lambda \nu} \Lambda_{\nu}$$

then $B^{m+1} = \mathcal{E} \Lambda^{m+1} \mathcal{E}^{-1}$ and it follows from (15) that

$$f_{\lambda \mu} = \mathcal{E}_{\lambda \mu} \Lambda_{\mu} \sum_\nu \mathcal{E}_{\mu \nu}^{-1}$$

Triangular $\mathcal{E}$ is defined modulo right multiplication by a diagonal matrix

$$\mathcal{E}_{\mu \nu} \rightarrow \mathcal{E}_{\mu \nu} \cdot \xi_{\nu}$$

what can be used to convert diagonal elements of $\mathcal{E}$ into unities. Expression (35) is invariant of this Abelian "gauge" transformation and for our purposes we are not obliged to make this additional conversion of $\mathcal{E}$. When two or more eigenvalues $\Lambda_{\mu}$ coincide, the ambiguity in $\mathcal{E}$ increases. One can choose the corresponding block to be a unit matrix.
11. Generalization to 4-graded hyperpolynomials \[25-27\], i.e. the $\beta deformation \[28\] of \[1\], is straight-forward and follows the recipe of \[14\], \[26\] and \[16\], \[17\].

(a) in differentials \(\{Aq^i/t\}\) were \(i\) and \(j\) are associated with the arms and legs in the Young diagram, the positive and negative powers of \(q\) become the powers of \(t = q^3\) instead of \(t\): \(\{Aq^i/t\}\).

(b) in $Z$-factors \[30\] quantum dimensions are substituted by Macdonald ones,

(c) the two constituents of $Z$-factors acquire opposite powers \(\sigma^\pm 1\) of the forth grading parameter, which appears nowhere else in the formulas,

(d) skew Schur functions in \(F\)'s are substituted by skew Macdonals and

(e) dimensions $\Lambda$ are $\beta$-deformed, $\Lambda \to \tilde{\Lambda}$.

Minor new comments are needed only at the points (d) and (e). In (d) one should choose the proper version of Schur formula to deform. Suggested in \[12\] was a complicated version, based on the possibility to re-express skew Schur functions at the zero-locus through shifted Schur functions \[29\]— so that the $\beta$-deformation involves shifted (interpolating) Macdonald polynomials \[30\]. Remarkably, this works, but there is a much simpler option: transposed skew Schurs in \[14\] can be substituted by those of negative times — and then they can be substituted by skew Macdonals (at Macdonald level transposition differs from time-inversion by somewhat complicated additional factors — and it is best to find the formulation where they do not show up).

The point (e) is more tricky. The hook formula \[1\] is actually equivalent to the $q$-power of regularized Casimir \[31\]: $\kappa(\lambda,\lambda) - \kappa(\lambda^\nu)$, where $\kappa_R = 2 \sum_{(i,j) \in R} (i-j) \lambda_1$ is the longest line in $\lambda$. The $N^2$ contribution is eliminated by taking the difference, which is equal to $N|\lambda| + c$, and $c$ is best expressed in terms of hooks. As known since \[20\], the $\beta$-deformation splits $\kappa_R = 2(\nu_R - \nu)$ with $\nu_R = \sum_{(i,j) \in R} (i-1) = \sum (i-1)R_i$ and $\nu = \sum_{(i,j) \in R} (j-1) = \sum (j-1)R_i$. The actual formula for the $\beta$-deformed eigenvalue is

$$\tilde{\Lambda}_\lambda = q^{2(\nu_R - \nu - 1)} \cdot t^{-2(\nu_R - \nu - 1)} \cdot \left( \frac{A^2}{q^{2N-1}} \right)^{|\lambda|} \ (37)$$

and it is actually independent of $N$. For symmetric and antisymmetric representations $\lambda = [r]$ and $\lambda = [r^*]$ expressions \[37\] reproduces the $A_2 \to A_2^q/t$ prescription of \[14\], but for generic rectangular representations $\lambda = [r^*]$ it provides explicit, but less trivial expressions.

Putting everything together, we obtain the following generalization of eq.(40) of \[16\] for the rectangularly-colored 4-graded hyperpolynomial from the figure-eight to arbitrary twist knots:

$$\tilde{F}^{(m,n)}_{[r]}(A, q, t, \sigma) = \sum_{\lambda \subseteq [r^\nu]} M_{\lambda, \sigma} \cdot \left\{ \frac{Aq^{i+j}}{\sigma t} \right\} \cdot \tilde{\tilde{F}}^{(m,n)}_\lambda$$

(38)

with the factorized

$$\tilde{\tilde{F}}^{(m,n)}_\lambda = \sum_{\mu \subseteq \lambda} \left( \frac{\beta_{m+1}}{\beta_{n+1}} \right) \cdot \sum_{\nu \subseteq \lambda} \left( \frac{\beta_{m+1}}{\beta_{n+1}} \right) \cdot \tilde{\tilde{F}}^{(m,n)}_\lambda$$

(39)

and the $\beta$-deformed KNTZ-like triangular evolution matrix

$$\tilde{\beta}_{\lambda \mu} = \tilde{\Lambda}_\lambda \cdot \frac{M_{\lambda, \mu} \{-\tilde{p}^\circ\} \cdot M_{\mu, \nu} \{\tilde{p}^\circ\}}{M_{\lambda, \nu} \{\tilde{p}^\circ\}} \ (40)$$

and the $\beta$-deformed zero-locus

$$\tilde{p}^0_k = \left\{ \frac{q}{t} \right\}^k \ (41)$$

Here $M$ denotes Macdonald polynomials \[32\], $M^\nu(q, t) := M(t^{-1}, q^{-1})$, and we use the standard condensed notation $\{x\} := x - x^{-1}$. 

8
For example, the matrix $\mathcal{B}$ for representation $R = [2]$ from the second example in sec.8 gets $\beta$-deformed in the following way:

$$
\mathcal{B}_{[2]} = \begin{pmatrix}
1 & 0 & 0 \\
-A^2 & A^2 & 0 \\
q^2A^4 & -(q^2 + q^4)A^4 & q^4A^4 \\
\end{pmatrix}
\xrightarrow{t \rightarrow q} \tilde{\mathcal{B}}_{[2]} = \begin{pmatrix}
1 & 0 & 0 \\
-q\frac{A^2}{t} & \frac{A^2}{t} & 0 \\
q^4\frac{A^4}{t^4} & \frac{A^4(q^2+1)A^4}{t^4} & \frac{q^8A^4}{t^4} \\
\end{pmatrix}
$$

Similarly, for representation $R = [2, 2]$

$$
\tilde{\mathcal{B}}_{[2, 2]} = \begin{pmatrix}
0 & [1] & [1, 1] & [2] & [2, 1] & [2, 2] \\
1 & 0 & 0 & 0 & 0 & 0 \\
-q\frac{2^2}{t} & \frac{2^2}{t} & 0 & 0 & 0 & 0 \\
\frac{2^2A^4}{t^2} & \frac{2^2(q^2+1)A^4}{t^2} & \frac{q^2A^4}{t^2} & 0 & 0 & 0 \\
\frac{q^4A^4}{t^2} & \frac{q^4(q^2+1)A^4}{t^2} & \frac{q^8A^4}{t^2} & \frac{q^8A^4}{t^2} & 0 & 0 \\
\frac{q^8A^8}{t^8} & \frac{(q^2+1)(q^2+1)q^8A^8}{t^8} & \frac{(q^2+1)(q^2+1)q^8A^8}{t^8} & \frac{(q^2+1)(q^2+1)q^10A^8}{t^{10}} & \frac{(q^2+1)(q^2+1)q^10A^8}{t^{10}} & \frac{q^12A^8}{t^{12}} \\
\end{pmatrix}
$$

As clear from this example, the KNTZ matrix has non-polynomial entries. Still for all rectangular $R = [r^s]$ we get hyperpolynomials, moreover they become positive in the DGR variables [33] (boldfaced): $A^2 = -a^2t$, $q = -qt$, $t = q$, i.e. can actually pretend to be the superpolynomials – for all twist, and, actually, double-braid knots [12].

The entries of the first column in $\mathcal{B}$ control the trefoil, which is a member of the twist family with $m = 1$, since (22) survives after the $\beta$-deformation:

$$
\sum_\mu \tilde{\mathcal{B}}_{\lambda\mu} = \delta_{\lambda}, \implies \sum_\mu (\tilde{\mathcal{B}}^2)_{\lambda\mu} = \tilde{\mathcal{B}}_{\lambda\theta}
$$

They are always polynomial and given by a simple hook-type formula [16]:

$$
\tilde{\mathcal{B}}_{\lambda\theta} = \tilde{\lambda} \cdot \frac{\mathcal{M}_\lambda\{\vec{p}^\circ\}}{\mathcal{M}_\lambda\{\vec{p}^\circ\}} = \left(-\frac{A^2q}{t}\right)^{|\lambda|} \prod_{(i,j) \in \lambda} q^{2(R_i'-i) t-2(R_j-j)}
$$

12. Conclusion and open problems. The two conjectures [41] and [13] together with explicit formulas [30] and [16] provide a closed explicit expression for arbitrary rectangulary-colored HOMFLY-PT polynomials for the double-twist knots, while [38]–[40] do the same for their 4-graded hyperpolynomial deformation.

a) Still both conjectures need a proof, at least in the Reshetikhin-Turaev formalism [5], and finally – at the level of Chern-Simons theory [34]. Of most interest would be the (still lacking) explanation, why the coefficients of the differential expansion factorize in [4].

b) Generalization to other knots is desirable, especially to the next-in-the-line family of pretzel knots [34].

c) A better understanding of the Rosso-Jones-like [20,21] formula [8] would be useful, in particular the mysterious conspiracy between the somewhat strange projector [20] onto the vector (1, . . . , 1), and the factorization of Z-factors [24], provided by the $U$-rotation of the vector (0)$|ST^2\tilde{S}T^{-2}\tilde{S}|$, which by itself has quite complicated components.
As explained in sec.11, the generalization to hyperpolynomials is straightforward, once the three-level structure

\[ \text{factorized differential expansion} \rightarrow \text{matrix evolution} \rightarrow \text{skew characters formula} \]

is revealed. Since rectangularly-colored hyperpolynomials for all twist knots are positive, as expected, this highly extends the set of rectangularly-colored superpolynomials, which so far were rarely available beyond torus knots. Still it is interesting to understand the reasons and the proof of polynomiality and positivity – both are far from obvious in the present approach.

In the case of Racah matrices the formulas \([12], [35]\) and \([13]\) are currently less explicit: the procedure involves solving two linear systems – \([34]\) for \(E\) and then \([13]\) for \(S\).

e) Still, given the simple form \([16]\) of \(\mathcal{B}\), one can hope to get more explicit expressions for the universal triangular matrices \(E\) and \(E^{-1}\) of the right and the left eigenvectors and then – for \(\bar{S}\) and \(S\).

f) Orthogonality of \(\bar{S}\) is far from obvious – and serves as an indirect and non-trivial check of the three-step conjecture in d). Orthogonality of \(S\) is straightforward, once it is defined as a diagonalizing matrix from \([13]\) and properly normalized, i.e. Abelian gauge freedom like \([36]\) is used to make it orthogonal.

g) Developing the considerations of \([35]\), one can look for the hypergeometric interpretations of Racah matrices – at least for rectangular representations, where they are unambiguously defined.

h) One can wonder what happens to relation between \([8]\) and \([11]\) in the case of hyperpolynomials – and what is the hyper-analogue of \(\bar{S}\) which one could define in this way from explicit expressions \([69]–[70]\). It is already interesting to see what happens to orthogonality of this matrix.

The most puzzling is generalization from rectangular to arbitrary representations. For rectangular \(R = [r^s]\) all the representations in \(R \otimes \bar{R}\) are in one-to-one correspondence with the sub-diagrams \(\lambda\) of \(R\), what is heavily used in all the above formulas. For generic \(R\), this relation is no longer one-to-one – one can say that some \(\lambda \subset R\) enter with non-trivial multiplicities \(c_{\lambda}\). As explained and illustrated in details in [18] this is not a very big problem at the level of colored knot polynomials, still the general answer is not yet available.

i) One can now look for a generalization of the matrix evolution formulas \([15]\) for generic \(R\) and for explicit expression for the corresponding HOMFLY-PT, see [36] for a recent progress in this direction. An important issue here is construction of the \(U\)-matrix and further decomposition of the \(Z_R^X\) with the help of \([24]\).

j) One can expect that associated 4-graded hyperpolynomials will fail to be positive – and this would provide a whole family of non-trivial answers beyond the torus-knot set, what can help to better understand the problem of colored superpolynomials \([27], [33]\), as well as the relation to the evolution properties of Khovanov-Rozansky polynomials, studied in \([37]\).

A more severe problem in non-rectangular case is that arborescent formula \([11]\) no longer distinguishes between the elements in entire \(c_{\mu} \times c_{\nu}\) blocks in the matrix \(\bar{S}\), thus one can not extract it directly from explicit expression for the double-twist family.

k) One can wonder if \(\bar{S}\) can still be found on this path, as it was done with the help of non-linear and therefore difficult unitarity requirement in the simplest cases of \(R = [2, 1]\) and \(R = [3, 1]\) in [18]. The most interesting, of course, is the case of \(R = [4, 2]\) – the first one where multiplicities are not separated by symmetries. Additional help here could be provided by a solution of the problem b).

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References

[1] J.W. Alexander, Trans. Amer. Math. Soc. 30 (2) (1928) 275-306
V.F.R. Jones, Invent. Math. 72 (1983) 1 Bull. AMS 12 (1985) 103 Ann. Math. 126 (1987) 335
L. Kauffman, Topology 26 (1987) 395
P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millet, A. Ocneanu, Bull. AMS. 12 (1985) 239
J.H. Przytycki and K.P. Traczyk, Kobe J. Math. 14 (1987) 115-139
A. Morozov, Theor. Math. Phys. 187 (2016) 447-454, arXiv:1509.04928

[2] G. Racah, Phys. Rev. 62 (1942) 438-462
E.P. Wigner, Manuscript, 1940, in: Quantum Theory of Angular Momentum, pp. 871-33, Acad. Press, 1965;
Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra, Acad. Press, 1959
L.D. Landau and E.M. Lifshitz, Quantum Mechanics: Non-Relativistic Theory, Pergamon Press, 1977
J. Scott Carter, D.E. Flath, M. Saito, The Classical and Quantum 6j-symbols, Princeton Univ. Press, 1995
S. Nawata, P. Ramadevi and Zodinmawia, Lett. Math. Phys. 103 (2013) 1389-1398, arXiv:1302.5143
A. Mironov, A. Morozov, A. Sleptsov, JHEP 07 (2015) 069, arXiv:1412.8432

[3] S. Chern and J. Simons, Ann. Math. 99 (1974) 48-69
E. Witten, Comm. Math. Phys. 121 (1989) 351-399

[4] L.D. Landau and E.M. Lifshitz, Quantum Mechanics: Non-Relativistic Theory (1977) Pergamon Press

[5] N. Reshetikhin and V. Turaev, Comm. Math. Phys. 127 (1990) 1-26
E. Guadagnini, M. Martellini and M. Mintchev, Clausthal 1989, Proc. 307-317; Phys. Lett. B235 (1990) 275
A. Mironov, A. Morozov and A. Morozov, in: Strings, Gauge Fields, and the Geometry Behind: The Legacy
of Maximilian Kreuzer, WS pub. (2013) 101-118, arXiv:1112.5754; JHEP 1203 (2012) 034, arXiv:1112.2654
JHEP 1203 (2012) 034, arXiv:1112.2654

[6] A. Belavin, A. Polyakov and A. Zamolodchikov, Nucl. Phys. B241 (1984) 333-380
A. Zamolodchikov, A. Zamolodchikov, Conformal field theory and critical phenomena in 2d systems, 2009
L. Alvarez-Gaume, Helvetica Physica Acta 64 (1991) 361
P. di Francesco, P. Mathieu and D. Senechal, Conformal Field Theory, Springer, 1997
A. Mironov, S. Mironov, A. Morozov, An. Morozov, Theor. Math. Phys. 165 (2010) 1662-1698,

[7] N. Nekrasov, Adv. Theor. Math. Phys. 7 (2004) 831-864, hep-th/0206161
R. Flume and R. Pogossian, Int. J. Mod. Phys. A18 (2003) 2541
N. Nekrasov and A. Okounkov, hep-th/0306238

[8] G. Moore, N. Nekrasov and S. Shatashvili, Nucl. Phys. B534 (1998) 549-611, hep-th/9711108
A. Losev, N. Nekrasov and S. Shatashvili, Comm. Math. Phys. 209 (2000) 97-121; ibid. 77-95,

[9] L. Alday, D. Gaiotto and Y. Tachikawa, Lett. Math. Phys. 91 (2010) 167197, arXiv:0906.3219
N. Wyllard, JHEP 0911 (2009) 002, arXiv:0907.2189
A. Mironov and A. Morozov, Nucl. Phys. B825 (2009) 137, arXiv:0908.2569

[10] A. Morozov, JHEP 1609 (2016) 135, arXiv:1606.06015 v8

[11] http://knotebook.org

[12] M. Kameyama, S. Nawata, R. Tao, H.D. Zhang, arXiv:1902.02275

[13] H. Itoyama, A. Mironov, A. Morozov and An. Morozov, JHEP 2012 (2012) 131, arXiv:1203.5978

[14] A. Mironov, A. Morozov and An. Morozov, AIP Conf. Proc. 1562 (2013) 123, arXiv:1306.3197
Mod. Phys. Lett. A 29 (2014) 1450183, arXiv:1408.3076

[15] A. Morozov, Nucl. Phys. B911 (2016) 582-605, arXiv:1605.09728

[16] Ya. Kononov and A. Morozov, Theor. Math. Phys. 193 (2017) 1630-1646, arXiv:1609.00143

[17] Ya. Kononov and A. Morozov, Mod. Phys. Lett. A Vol. 31, No. 38 (2016) 1650223, arXiv:1610.04778
