Device and semi-device independent random numbers based on non-inequality paradox

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In this work, we propose device independent true random numbers generation protocols based on non-inequality paradoxes such as Hardy’s and Cabello’s non-locality argument. The efficiency of generating randomness in our protocols are far better than any other proposed protocols certified by CHSH inequality or other non-locality test involving inequalities. Thus, highlighting non-inequality paradox as an important resource for device independent quantum information processing in particular generating true randomness. As a byproduct, we find that the non-local bound of the Cabello’s argument with arbitrary dimension is the same as the one achieved in the qubits system. More interestingly, we propose a new dimension witness paradox based on the Cabello’s argument, which can be used for constructing semi-device-independent true random numbers generation protocol.

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I. INTRODUCTION

Randomness is an important basic feature of nature, which has wide applications in information processing including other prominent fields e.g., biology, chemistry, social science, etc.. At present, the protocol behind most of the true random number generators is based primarily on classical laws of physics and fundamentally they are all deterministic in some underline theory. Therefore, people are looking for new protocols which can produce genuine private randomness. On the other hand, one of the intrinsic feature of quantum mechanics is that it is a probabilistic theory inherently. This probabilistic feature of quantum theory does not come from the subjective ignorance about the pre-assigned value of a dynamical variable in a quantum state, rather it represents the probabilistic nature of finding a particular value of a dynamical variable if that dynamical variable is measured. However, in any real experiment the randomness of measurement outcomes of quantum systems is unavoidably mixed-up with an apparent randomness that results from noise or lack of control on the quantum devices. Therefore, it is not so easy to generate ideal randomness even with the assist of quantum technology and several attempts have been made in this regards. To generate the certified quantum random number, Colbeck 1, 2 introduced a device independent true random number generation (DITRNG) protocol based on the GHZ test 3, while Pironio et.al. 4 proposed another DITRNG protocol based on the violation of CHSH inequality 5.

In CHSH inequality based DITRNG protocol, players distrust the concerning bipartite entangled state and all the measurement devices as they might have been fabricated by a spiteful party Eve, the randomness can be guaranteed if the CHSH expression violate the local hidden variable (LHV) bound. The maximal min-entropy of the generated random number is about 1.23 and it occurs when the corresponding CHSH expression attains its maximum quantum bound $2\sqrt{2}$. Unfortunately, all the existing DITRNG protocols require entanglement, which has the negative impact on the complexity of the devices and the rate of the random number generation. To get higher key rate recently, Li et al. 6–8 proposed a semi-device independent true random number generation (SDITRNG) protocol, where the true randomness certified if the corresponding dimension witness inequality 9–12 violates its LHV bound.

In general, the device independent scenarios have also been proposed in the context of several other information processing tasks, for example, quantum cryptography 13, state estimation 14 etc.. However, all of these protocols are mostly based either on CHSH inequality 5, Mermin’s inequality 15 or Chained inequality 16, 17. Here we propose a DITRNG protocol based on some other non-locality tests without involving any statistical inequalities such as Hardy’s paradox 18 and Cabello’s paradox 19. The two paradoxes respectively shows a direct contradiction between quantum theory and local-realistic (LR) theory with the help of two-qubit correlations. We find that our DITRNG protocols based on Hardy paradox or Cabello paradox are more efficient than other proposed inequality based DITRNG protocols in various context.

We start with a description of a non-locality test without using any statistical inequality known as Hardy paradox, which will allow us to formulate the DITRNG pro-
tocol based on non-inequality paradox. The maximal bound of generated randomness in this case is larger than the CHSH based DITRNG protocol. Next we present the protocol based on another non-inequality test called Cabello paradox and show that the efficiency of this is even better than Hardy paradox. Finally, we propose a new dimension witness paradox without involving any inequality, which also allow us to generate much more random number compare to other proposed SDITRNG protocols. All in all, our result shows that there is a practical advantage of using non-inequality paradox based protocol to generate true randomness over the CHSH inequality based protocols.

II. NON-LOCAL PARADOX WITHOUT INEQUALITY

A. Hardy paradox

In 1992, Hardy [18] suggested an non-locality test without using any statistical inequality for two two-level system in comparison with Bell-CHSH test. Consequently, Hardy’s test known as a non-locality test without inequality. Consider a physical system shared between two remote parties say, Alice and Bob. Let Alice can perform measurement on her subsystem chosen randomly from two positive-operator valued measurements (POVM) \( \{A_{+10}, A_{-10}\} \) and \( \{A_{+11}, A_{-11}\} \). The possible outcomes of each such measurement are + or −. Therefore, they can be assumed to be dichotomic observables. Similarly, Bob also can measure his subsystem chosen randomly from two ± valued POVM \( \{B_{+01}, B_{-01}\} \) and \( \{B_{+11}, B_{-11}\} \). To explain the non-local property, Hardy puts following three constraint on the joint probabilities:

\[
\begin{align*}
p(+, +|A_0, B_0) &= 0, \\
p(+, -|A_1, B_0) &= 0, \\
p(-, +|A_0, B_1) &= 0,
\end{align*}
\]

where, \( A_i \equiv A_{+i|1} - A_{-i|1} \) and \( B_i \equiv B_{+i|1} - B_{-i|1} \) for \( i, j \in \{0, 1\} \) and \( p(a, b|A_x, B_y) \) denotes the joint probability of observing result, with \( a, b = \pm \) under local setting \( x, y \in \{0, 1\} \). Note that under LHV theory each of the joint probability \( p(a, b|A_x, B_y) \) can be expressed as \( p(a, b|A_x, B_y) = \sum_{z} p_z(a|A_x)p_z(b|B_y) \). Therefore, one can easily get the following constraint

\[
p_{\text{Hardy}} \equiv p(+, +|A_1, B_1) = 0.
\]

However, Hardy showed that there exists two-qubits non-maximal correlation which satisfies all the three conditions of (1) but can give a non-zero value of \( p_{\text{Hardy}} \) i.e., \( p_{\text{Hardy}} \equiv p(+, +|A_1, B_1) > 0. \) More precisely, it has been proved that for a given pair of dichotomic observables on each site there exists an unique two-qubits non-maximally entangled state which satisfies all the three conditions of (1) and violate the condition of (2) and the maximal value of \( p_{\text{Hardy}} \) can go up to \( \frac{5\sqrt{2} - 1}{2} \) in the case of two qubits preparation [21]. Recently, Rabelo et al. [22] proved that the maximal Hardy probability \( (p_{\text{Hardy}}) \) has no advantage in higher-dimension quantum systems, they also proposed an non-ideal version of the paradox by introducing external noisy as the following section.

1. Noisy Hardy paradox

It is quite obvious that the three joint probabilities given in (1) may not be identically zero in practice due to the noise introduced by external environment and/or by imperfection of the devices. Therefore, it is worthy to study the imperfect case along with the ideal scenario of the Hardy test. Let us denote our noisy parameter as \( \epsilon \) and consider each of the concerning three joint probabilities are bounded by \( \epsilon \). Hence, the original constraints (1) on the joint probabilities reduce to following inequalities

\[
\begin{align*}
p(+, +|A_0, B_0) &\leq \epsilon, \\
p(+, -|A_1, B_0) &\leq \epsilon, \\
p(-, +|A_0, B_1) &\leq \epsilon.
\end{align*}
\]

From the famous CH inequality [23], or rather its left hand side

\[
-1 \leq p(+, -|A_1, B_0) + p(+, +|A_0, B_1) + p(-, -|A_0, B_0)
\]

\[
-p(-|A_0) - p(-|B_0) - p(+, +|A_1, B_1),
\]

we get the following LHV bound on the maximal value of the Hardy probability \( p_{\text{Hardy}} \)

\[
p_{\text{Hardy}} \leq p(+, +|A_0, B_0) + p(+, -|A_1, B_0) + p(-, +|A_0, B_1)
\]

\[
\leq 3\epsilon,
\]

the first inequality of the above expression can be reached immediately from (1), once one notices that

\[
p(+, +|A_0, B_0) = p(-, -|A_0, B_0) - p(-|A_0) - p(-|B_0) + 1.
\]

Note that \( p_{\text{Hardy}} \) attends its algebraic maximum when the corresponding noisy parameter \( \epsilon \) reaches the value \( \frac{1}{3} \). Hence, it is worthy to discuss the noise Hardy test for \( 0 \leq \epsilon < \frac{1}{3} \). If the noisy parameter is larger than \( \frac{1}{3} \), then the maximal bound of Hardy’s probability \( p_{\text{Hardy}} \) has no advantage over the LHV bound.

B. Cabello paradox

In 2002, Cabello introduced another logical structure to prove Bell’s theorem without inequality for three-particle GHZ and W states. Later Liang and Li demonstrated that the argument is also applicable for two two-level systems [24]. The mathematical formulation of Cabello’s non-locality argument for two two-level systems is
as follows:

\[
\begin{align*}
    p(+, + | A_0, B_0) &= q_1, \\
    p(+, - | A_1, B_0) &= 0, \\
    p(-, + | A_0, B_1) &= 0, \\
    p(+, + | A_1, B_1) &= q_4.
\end{align*}
\]

(6)

One can also check that this set of conditions cannot be satisfied by any LHV theory as long as \(q_4 > q_1\). Therefore, LHV bound on the probability of success of Cabello’s argument \(p_{\text{Cabello}} \equiv q_4 - q_1\) is given by

\[p_{\text{Cabello}} \leq 0.\]

Note that the ideal Hardy’s argument is a special case of the Cabello’s argument with \(q_1 = 0\). On the other hand, Cabello’s argument can also be explained as a particular case of noisy Hardy paradox, where the first joint probability of \(|\psi\rangle\) introduce a noise \(q_1\) and other two concerning joint probabilities do not introduce any noise.

In 2006, Kunkri et al. \[22\] proved that the maximum probability of success in Cabello’s argument for two-qubits system is approximately 11\%, which is larger than the original Hardy’s success probability. Here, we show that the maximal bound of success probability of Cabello’s case has also no advantage in higher-dimension quantum systems and it can be used to generate much more randomness compare to the original Hardy test. More interestingly, Cabello’s argument can also be transformed to the Dimension Witness (DW) paradox by considering that Alice has the free will assumption. Farther, this can be used to construct new SDITRNG protocol based on non-locality test without inequality.

III. DEVICE-INDEPENDENT RANDOMNESS BASED ON HARDY’S PARADOX

Consider that state \(\rho\) is shared between Alice and Bob. If Alice and Bob perform the measurement \((A_x, B_y)\) on their respective part of \(\rho\), then the joint probability of getting the outcome \((a, b)\) is

\[p(a, b | A_x, B_y) = Tr(\rho A_{a|x} \otimes B_{b|y}).\]

Note that there is no constraint on the dimension of the system. Therefore, without loss of generality, we assume that \(\rho\) is pure and all the concerning measurements are projective. It is obvious that the measurement outcomes must obey the causality principle i.e., the corresponding joint probabilities satisfy the following non-signalling conditions

\[
\begin{align*}
    p(a | A_x) = & \sum_b p(a, b | A_x, B_0) = \sum_b p(a, b | A_x, B_1), \\
    p(b | B_y) = & \sum_a p(a, b | A_0, B_y) = \sum_a p(a, b | A_1, B_y),
\end{align*}
\]

(7)

where \(p(a | A_x)\) and \(p(b | B_y)\) denote the marginal probability. In the original DITRNG protocol, non-local correlations are used to certify the presence of genuine randomness in quantum theory. Note that the randomness of measurement outcome is sovereign and it independent of entanglement. In this context, it has been proved that non-locality with tiny amount of entanglement can generate full randomness asymptotically \[23\]. The parameter that used to estimate randomness of the measurement outcomes \(a\) and \(b\) conditioned on the input values \(x\) and \(y\) is the min-entropy function \[23\]. Since the state preparation and the measurement have no restriction in device independent protocol, Alice and Bob’s measurement equipment can be assumed to be black boxes. Let \(x \in \{0, 1\}\) and \(y \in \{0, 1\}\) be the inputs of Alice and Bob’s measurement black box respectively, then the min-entropy function for the given inputs \((x, y)\) can be expressed as

\[H_\infty((a, b) | x, y) = - \log_2[\max_{(a,b)} p(a, b | A_x, B_y)].\]

(8)

where \(\max_{(a,b)} p(a, b | A_x, B_y)\) denotes the maximal probability over all possible outcomes \((a, b)\). The min-entropy can reach the maximal value if \(p(a, b | A_x, B_y) = \frac{1}{4}\) for all \(a\) and \(b\).

Presently, the Semi-Definite Programming (SDP) is a prominent method to solve the convex optimization problem, which concerned with linear objective function over the semi-definite matrices. Follow the method suggested by Navascues et al. \[29\], we solve the following maximal guessing probability optimization problem with SDP method

Minimize : \(\max_{(a,b)} p(a, b | A_0, B_0)\)

Subject to : \(\Delta_{\text{Hardy}} \geq 0,\)

\[
\begin{align*}
    p(+, + | A_0, B_0) &= 0, \\
    p(+, - | A_1, B_0) &= 0, \\
    p(-, + | A_0, B_1) &= 0, \\
    p(+, + | A_1, B_1) &= p_{\text{Hardy}},
\end{align*}
\]

(9)

where \(\Delta_{\text{Hardy}} = [\Delta_{ij}]\) is the positive semi-definite matrix with \(\Delta_{ij} = Tr(E_i^\dagger E_j \rho)\) for \(E_i, E_j \in \{I, A_{0|x}, B_{0|y}, A_{1|x}, B_{1|y}\}\). Note that these measurement operators should also satisfy the hermiticity \((A_{0|x})^\dagger = A_{0|x}^\dagger, B_{0|y} = B_{0|y}^\dagger\), orthogonality \((A_{a|x}^\dagger A_{a'|x}) = \delta_{aa'}, A_{b|x}^\dagger B_{b'|y} = \delta_{bb'}, B_{b'|y} = 1\), completeness \((\sum_a A_{a|x} = 1\), \(B_{b|y} = 1\)), commutativity \((A_{a|x}, B_{b|y}) = 0\) and the no-signalling conditions \[23\]. By applying the SDP method and the Sedumi toolbox \[34\], we calculate the maximal min-entropy bound \(H_\infty((a, b) | A_0, B_0)\) with different Hardy parameter \(p_{\text{Hardy}}\). Calculation result shows that the maximal randomness can reach up to 1.35 if the corresponding Hardy’s probability attends its maximal value \(\frac{5\sqrt{2} - 11}{2}\). Comparing with the CHSH inequality based DITRNG protocol, Hardy paradox generates much more true random number in optimal case.

By applying similar SDP optimization method, we can estimate the generated randomness based on Noisy
Hardy paradox

Minimize : \[ \max_{\{a,b\}} p(a,b|A_0, B_0) \]
Subject to : \[
\begin{align*}
\Delta_{\text{Noisy Hardy}} &\geq 0, \\
p(+, +|A_0, B_0) &\leq \epsilon, \\
p(+, -|A_1, B_0) &\leq \epsilon, \\
p(-, +|A_0, B_1) &\leq \epsilon, \\
p(+, +|A_1, B_1) &\geq \delta.
\end{align*}
\] (10)

where \(\Delta_{\text{Noisy Hardy}}\) is the positive semi-definite matrix, \(\epsilon\) is the noisy Hardy parameter, \(\delta\) is the maximal joint probability \(p(+, +|A_1, B_1)\) by considering the constraint on three joint probabilities, which will reduce to \(\frac{2\sqrt{2}}{2}-1\) when the noisy Hardy parameter reduces to 0. The noisy Hardy paradox can show contradictions with LR theory only when \(\delta\) is larger than \(3\epsilon\) i.e., when it’s violate the associate LHV bound. We apply this non-local property to estimate randomness of the measurement outcomes, the maximal min-entropy bound \(H_\infty(a,b|A_0, B_0)\) with different noisy Hardy parameter \(\epsilon\) is given in Fig.2. From the calculation result, we find that the maximal randomness can reach 1.58 when the associated noisy Hardy parameter reaches its maximal value 0.3333. Correspondingly, the maximal quantum value can reach 0.99995, which is larger than the LHV bound 0.99990. Comparing with the CHSH inequality and original Hardy paradox, the noisy Hardy paradox can generate much more random number when \(\delta\) reach the maximal value correspondingly.

Randomness of the DITRNG protocol based on Cabello paradox can also be analyzed with the similar method,

Minimize : \[ \max_{\{a,b\}} p(a,b|A_0, B_0) \]
Subject to : \[
\begin{align*}
\Delta_{\text{Cabello}} &\geq 0, \\
p(+, -|A_1, B_0) & = 0, \\
p(-, +|A_0, B_1) & = 0, \\
p(+, +|A_1, B_1) - p(+, +|A_0, B_0) & = p_{\text{Cabello}},
\end{align*}
\] (11)

where \(\Delta_{\text{Cabello}}\) is positive semi-definite matrix, \(p_{\text{Cabello}}\) is the success probability of Cabello paradox, which is the joint probability deviation value \(p(+, +|A_1, B_1) - p(+, +|A_0, B_0)\). The SDP calculation result shows that maximal value of \(p_{\text{Cabello}}\) is 0.10784 in arbitrary high dimension quantum system, which prove that high dimension system has no advantage to improve \(p_{\text{Cabello}}\). The Cabello paradox has the non-local property when \(p_{\text{Cabello}}\) is larger than zero, in which case the protocol can generate the true random number, the maximal min-entropy bound of \(H_\infty(a,b|A_0, B_0)\) with different Cabello paradox parameter \(p_{\text{Cabello}}\) can be given in Fig.3. Calculation shows that the maximal randomness can reach up to value 1.56 and the maximum attends when the associate Cabello’s parameter reaches its maximal value 0.10784. Thus, the efficiency of generating randomness is higher.
compare to the original Hardy paradox in optimal case. More importantly, the Cabello paradox can be transformed to a dimension witness paradox (from entanglement based protocol to state preparation and measurement based protocol), which can be used for constructing one way SDITRNG protocol.

IV. SEMI-DEVICE-INDEPENDENT RANDOMNESS BASED ON HARDY INEQUALITY

In a recent work by Li et al.\(^8\) proved that CHSH inequality can be transformed to a dimension witness inequality by considering Alice has random measurement outcomes with arbitrary input random number (which is equal to free will assumption). By applying the similar technique, we present the first dimension witness paradox without inequality.

\[
p(+, +|A_0, B_0) = q_1, \\
p(+, -|A_1, B_0) = 0, \\
p(-, +|A_0, B_1) = 0, \\
p(a|A_x) = p(a|A_x, B_y) = \frac{1}{2}, \\
p(+, +|A_1, B_1) = q_4.
\]  

(12)

Similar to the LHV theory, the classical mechanics theory has the Cabello paradox parameter restriction \(p_{\text{Cabello}} \leq 0\), while \(p_{\text{Cabello}} > 0\) guarantee that the system can only be explained by the quantum mechanics.

Since the dimension witness can be used to generate \textit{semi-device independent true random number}, where the randomness has no more restriction about the state preparation and measurement except the fact that the dimension of Hilbert space associated to each subsystem is two. Then we apply the SDP method to calculate the maximal guessing probability \(\max_{a,b} p(b|A_x, a, B_y)\) by considering the quantum dimension witness based on Cabello paradox.

\[
\begin{align*}
\text{Minimize} & : \max_{a,b} p(b|A_0, a, B_0) \\
\text{Subject to} & : \Delta_{\text{Cabello}} \geq 0, \\
& : p(+, -|A_1, B_0) = 0, \\
& : p(-, +|A_0, B_1) = 0, \\
& : p(a|A_x) = p(a|A_x, B_y) = \frac{1}{2}, \\
& : p(+, +|A_1, B_1) - p(+, +|A_0, B_0) = p_{\text{Cabello}},
\end{align*}
\]

(13)

where the Cabello paradox parameter \(p_{\text{Cabello}}\) can reach the maximal value 0.08279, which is smaller than the original Cabello paradox parameter, the reason for which is that the original Cabello paradox can not get full random measurement outcome in one side black box. Since have proved that SDP method can also be used in SDITRNG protocol\(^3\), the maximal min-entropy bound of \(H_\infty(b|A_0, a, B_0)\) with different Cabello paradox parameter \(p_{\text{Cabello}}\) is given in Fig.4. From the calculation result, we find that the maximal randomness can reach 0.68 when the Cabello’s parameter reaches the maximal value. Thus efficiency of this new protocol is evidently larger than the original SDITRNG protocol, which can generate maximum 0.23 random number and the maximum value attends when the dimension witness inequality value is 2.828.

V. CONCLUSION

In conclusion, We have proposed \textit{device independent true random number generation} protocols based on Hardy paradox, Noisy Hardy paradox and Cabello paradox respectively. All of these protocols can generate much more true random numbers compare to all other DI protocols. In this regard, a new kind of proof for quantum key distribution based on Hardy’s paradox has been proposed very recently\(^{31}\). It is an open problem to discuss other quantum information protocols based on non-inequality paradoxes.

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