Measuring the black hole spin direction in 3D Cartesian numerical relativity simulations

Vassilios Mewes,1 José A. Font,1,2 and Pedro J. Montero3

1Departamento de Astronomía y Astrofísica, Universitat de València,
Dr. Moliner 50, 46100, Burjassot (València), Spain
2Observatori Astronòmic, Universitat de València,
C/ Catedrático José Beltrán 2, 46980, Paterna (València), Spain
3Max-Planck-Institute für Astrophysik, Karl-Schwarzschild-Str. 1, 85748, Garching bei München, Germany

We show that the so-called flat-space rotational Killing vector method for measuring the Cartesian components of a black hole spin can be derived from the surface integral of Weinberg’s pseudotensor over the apparent horizon surface when using Gaussian normal coordinates in the integration. Moreover, the integration of the pseudotensor in this gauge yields the Komar angular momentum integral in a foliation adapted to the axisymmetry of the spacetime. As a result, the method does not explicitly depend on the evolved lapse $\alpha$ and shift $\beta^i$ on the respective timeslice, as they are fixed to Gaussian normal coordinates, while leaving the coordinate labels of the spatial metric $\gamma_{ij}$ and the extrinsic curvature $K_{ij}$ unchanged. Such gauge fixing endows the method with coordinate invariance, which is not present in integral expressions using Weinberg’s pseudotensor, as they normally rely on the explicit use of Cartesian coordinates.

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I. INTRODUCTION

Since the binary black hole (BBH) merger breakthrough simulations of about a decade ago [1–3], ever-growing computational resources and advances in the numerical methods used to simulate these systems have made the exploration of the vast initial parameter space possible (see e.g. [4] and references therein for a recent overview of the status of BBH simulations). The initial parameters of BBH simulations are the BH mass ratio and the six components of their initial spin vectors. The investigation of these initial parameters has led to significant discoveries, as the occurrence of the orbital hang-up [3] and the presence of the so-called super-kicks where the final BH is displaced from the orbital plane after its formation [4].

The BH spin and in particular its orientation may also play a non-negligible role in non-vacuum spacetimes involving BHs surrounded by matter as, e.g. in the form of accretion disks, a situation commonly encountered in binary neutron star merger simulations. Recently, we have performed numerical relativity simulations of tilted self-gravitating accretion disks around BHs, investigating the precession and nutation the BH undergoes as it accretes mass and angular momentum from the torus [5]. In order to carry out a quantitative analysis of such kind of simulations, it is obviously necessary to measure both the magnitude of the BH spin and also its direction in space. Such a study inspired the work we present in this paper.

One of the standard methods in numerical relativity to measure the magnitude of the angular momentum of the BH horizon is described in [5]. This method is based on the so-called isolated horizon formalism [6] and the generalization to dynamical horizons [10]. In this approach the BH spin is calculated by performing the following surface integral on the apparent horizon (AH) of the BH

$$J_{AH} = \frac{1}{8\pi} \int_S (\psi^a R^b K_{ab}) \, dS,$$  \hspace{1cm} (1)

where $\psi^a$ is an approximate rotational Killing vector on the horizon that has to be determined numerically (see [6] for a method of finding $\psi^a$), $R^b$ is the outward pointing unit vector normal to the horizon, $K_{ab}$ is the extrinsic curvature on the horizon surface and $dS$ is the surface element. This method does not, however, give the direction of the BH spin in the 3D Cartesian reference frame of the computational grid of a numerical relativity simulation.

The direction of the BH spin in numerical relativity is commonly measured by the approach suggested by [11]. In this approach the BH spin direction is simply defined as the Euclidean unit vector tangent to the coordinate line joining the two poles on the horizon (i.e. the two points where the axially symmetric vector $\psi^a$ vanishes). The approximate Killing vector field $\psi^a$ on the horizon is obtained numerically using spherical-polar coordinates and the accuracy in the spin direction is typically about a few angular grid zones. This definition of the spin vector reproduces the Bowen-York spin parameter on the initial slice and gives satisfactory results as long as the BH horizon does not become too distorted. Moreover, [11] present another method for finding the spin magnitude and direction, using flat-space coordinate rotational Killing vectors to calculate the Cartesian components of the BH spin and its magnitude from the Euclidean norm of the resulting vector. The flat-space Killing vector method has the practical advantage that the vector $\psi^a$ used in the surface integral [11] is given analytically and is constant, therefore it does not have to be found numerically on each timeslice.

In this paper we show how the flat-space Killing vector method can be derived by performing a surface integral of
Weinberg’s energy-momentum pseudotensor \([12]\). By using the 3+1 split of spacetime and Gaussian coordinates, it is possible to express the angular momentum of a given volume using Weinberg’s energy-momentum pseudotensor in a simple form that allows for a straightforward calculation of the spin vector of the BH horizon. Weinberg’s energy-momentum pseudotensor is a symmetric pseudotensor derived by writing Einstein’s equations using a coordinate system that is quasi-Minkowskian, i.e. with the four-dimensional metric \(g_{\mu\nu}\) approaching the Minkowski metric \(\eta_{\mu\nu}\) at infinity. Although it is not generally covariant, the pseudotensor is Lorentz covariant, and with the appropriate choice of coordinates it provides a measure of the total angular momentum of the system. In the following Greek indices run from 0 to 3 while Latin indices run from 1 to 3. We use geometrized units \((G = c = 1)\) throughout.

II. 3+1 SURFACE INTEGRAL OF WEINBERGS’S PSEUDOTENSOR IN GAUSSIAN COORDINATES

In this section, we first briefly review the calculation of the angular momentum contained in a volume using Weinberg’s pseudotensor. Next, we express the resulting Gaussian coordinates the integral reduces in complexity on a given timeslice. Finally, we show that by choosing surface integral in terms of the 3+1 spacetime variables Weinberg’s pseudotensor. Next, we express the resulting calculation of the spin vector of the BH horizon. Weinberg’s energy-momentum pseudotensor is a symmetric pseudotensor derived by writing Einstein’s equations using a coordinate system that is quasi-Minkowskian, which is the total energy-momentum “tensor” of the matter fields, \(\tau_{\mu\nu}\), which is the total energy-momentum “tensor” of the gravitational field, \(t_{\lambda\kappa}\), and of the gravitational field, \(t_{\lambda\kappa}\), approach the Cartesian Minkowski metric \(\eta_{\mu\nu}\) at infinity as follows

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \]  

where \(h_{\mu\nu}\) does not necessarily have to be small everywhere. Then, by writing the Einstein equations in parts linear in \(h_{\mu\nu}\), one arrives at an energy-momentum pseudotensor \(\tau_{\mu\nu}\), which is the total energy-momentum “tensor” of the matter fields, \(T_{\lambda\kappa}\), and of the gravitational field, \(t_{\lambda\kappa}\),

\[ \tau_{\mu\nu} = \eta^{\mu\lambda} \eta^{\nu\kappa} [T_{\lambda\kappa} + t_{\lambda\kappa}] = \frac{1}{\xi^{\lambda}} \frac{\partial}{\partial x^{\lambda}} Q^{\sigma\mu\nu}, \]  

where \(Q^{\sigma\mu\nu}\) is the superpotential given by

\[ Q^{\sigma\mu\nu} = \frac{1}{2} \left( \frac{\partial h^{\lambda\sigma}}{\partial x_{\mu}} \eta^{\nu\tau} - \frac{\partial h^{\lambda\tau}}{\partial x_{\mu}} \eta^{\nu\sigma} - \frac{\partial h^{\lambda\nu}}{\partial x_{\mu}} \eta^{\tau\sigma} + \frac{\partial h^{\mu\lambda}}{\partial x^{\sigma}} \eta^{\nu\tau} - \frac{\partial h^{\mu\sigma}}{\partial x^{\lambda}} \eta^{\nu\tau} \right), \]  

and indices of linearized quantities are raised and lowered with \(\eta_{\mu\nu}\).

Using the pseudotensor, the volume integrals giving the total four-momentum of the volume are given by

\[ P_{\mu} = \int_{V} T^{\mu\nu} d^{3}x = -\frac{1}{8\pi} \int_{V} \left( \frac{\partial Q^{\mu\nu}}{\partial x^{\tau}} \right) d^{3}x. \]  

Furthermore, the pseudotensor \(\tau_{\mu\nu}\) defined by Eq. 3 is symmetric, which allows one to use it to calculate the total angular momentum in a volume \(V\) using the following volume integral:

\[ J_{\mu\nu} = \int_{V} (x^{\mu} \tau_{\nu0} - x^{\nu} \tau_{0\mu}) d^{3}x = -\frac{1}{8\pi} \int_{V} \left( x^{\mu} \frac{\partial Q^{\mu\nu}}{\partial x^{\tau}} - x^{\nu} \frac{\partial Q^{\mu\tau}}{\partial x^{\mu}} \right) d^{3}x. \]  

As Weinberg remarks, the physically interesting Cartesian components of the angular momentum contained in the volume are

\[ J_{x} \equiv J_{23}, \quad J_{y} \equiv J_{31}, \quad J_{z} \equiv J_{12}. \]  

Using Gauss’ law the volume integral can be transformed to the following surface integral over the bounding surface:

\[ J_{ij} = -\frac{1}{16\pi} \int_{S} \left( -x_{i} \frac{\partial h_{0j}}{\partial x^{k}} + x_{j} \frac{\partial h_{0i}}{\partial x^{k}} + x_{i} \frac{\partial h_{jk}}{\partial t} - x_{j} \frac{\partial h_{ik}}{\partial t} + h_{0j} \delta_{ki} - h_{0i} \delta_{kj} \right) n^{k} dS, \]  

where \(n^{i}\) is the unit normal to the surface of integration and \(dS\) the surface element.

The convergence of the four-momentum volume integrals \([5]\) involving the pseudotensor \(\tau_{\mu\nu}\) critically depends on the rate at which the metric \(g_{\mu\nu}\) approaches the Minkowski reference metric at large distances. Given the following behaviour of \(h_{\mu\nu}\) as \(r \rightarrow \infty\),

\[ h_{\mu\nu} = O(r^{-1}), \]
\[ \frac{\partial h_{\mu\nu}}{\partial x^{\sigma}} = O(r^{-2}), \]
\[ \frac{\partial^{2} h_{\mu\nu}}{\partial x^{\sigma} \partial x^{\rho}} = O(r^{-3}), \]

where \(r = (x^{2} + y^{2} + z^{2})^{1/2}\), it can be shown that the energy-momentum “tensor” of the gravitational field, \(t_{\mu\nu}\), behaves at large distances as

\[ t_{\mu\nu} = O(r^{-4}), \]

which in turn shows that the four-momentum volume integral \([8]\) converges. The convergence of the total angular momentum volume integral \([8]\) and of the corresponding surface integral \([5]\) is more problematic, due to the appearance of \(x^{3}\) in the volume integral. This is also observed in the convergence properties of the integrals.
of the ADM quantities \[13\], where the surface integrals for the ADM mass and linear momentum converge when imposing fall-off conditions like those of Eq. \[9\], while the calculation of the ADM angular momentum generally requires stronger asymptotic fall-off conditions \[14\]. We shall return to the issue of the convergence of Eq. \[8\] after we have expressed it in terms of the 3+1 variables and in Gaussian normal coordinates in the next section.

\section{The angular momentum pseudotensor integral in Gaussian coordinates}

We can express the total angular momentum given by Eq. \[8\] in Gaussian normal coordinates (also called synchronous coordinates), which represent free-falling observers. We start by doing a 3+1 decomposition of the four-dimensional metric \(g_{\mu\nu}\),

\[
g_{\mu\nu} = \left( \begin{array}{cc} -\alpha^2 + \beta_i \beta^i \gamma_{ij} \gamma_{ij} & -\alpha \beta_i \gamma_{ij} \\ -\alpha \gamma_{ij} & \gamma_{ij} \end{array} \right), \tag{11}
\]

where \(\alpha\) is the lapse function, \(\beta^i\) the shift vector, and \(\gamma_{ij}\) the spatial metric induced on the hypersurface. From the requirement that the metric \(g_{\mu\nu}\) approaches Cartesian Minkowski space at infinity \[2\], we see that

\[
h_{\mu\nu} = \left( \begin{array}{cc} -\alpha^2 + \beta_i \beta^i + 1 \gamma_{ij} \gamma_{ij} & -\alpha \beta_i \gamma_{ij} \\ -\alpha \gamma_{ij} & \gamma_{ij} \end{array} \right). \tag{12}
\]

If we now express the angular momentum surface integral, Eq. \[8\], in terms of the 3+1 variables we find that \(J^{ij}\) can be written as

\[
J^{ij} = -\frac{1}{16\pi} \int_S \left( -x_i \frac{\partial(\gamma_j m \beta^m)}{\partial x^k} + x_j \frac{\partial(\gamma_i m \beta^m)}{\partial x^k} \\
+ x_i \frac{\partial(\gamma_{jk} - \delta_{jk})}{\partial t} - x_j \frac{\partial(\gamma_{ik} - \delta_{ik})}{\partial t} \\
+ \gamma_{jm} \beta^m \delta_{ki} - \gamma_{im} \beta^m \delta_{kj} \right) n^k dS. \tag{13}
\]

Moreover, in terms of the 3+1 variables, Gaussian coordinates are defined by the following choice of the lapse and shift vector:

\[
\alpha = 1, \quad \beta^i = 0, \tag{14}
\]

so that \(h_{00} = h_{0i} = h_{i0} = 0\). In this gauge, Eq. \[13\] considerably simplifies to

\[
J^{ij} = -\frac{1}{16\pi} \int_S \left( x_i \frac{\partial \gamma_{jk}}{\partial t} - x_j \frac{\partial \gamma_{ik}}{\partial t} \right) n^k dS. \tag{15}
\]

We can now use the definition of the extrinsic curvature \(K_{ij}\),

\[
K_{ij} = -\frac{1}{2\alpha} \left( \frac{\partial \gamma_{ij}}{\partial t} - \mathcal{L}_\beta \gamma_{ij} \right), \tag{16}
\]

where \(\mathcal{L}_\beta\) is the Lie derivative with respect to the shift vector \(\beta^i\), to see that the time derivative of the spatial metric \(\partial \gamma_{ij}/\partial t\) in Gaussian coordinates is simply

\[
\frac{\partial \gamma_{ij}}{\partial t} = -2K_{ij}. \tag{17}
\]

Substituting Eq. \[17\] in Eq. \[15\], we find that

\[
J^{ij} = \frac{1}{8\pi} \int_S (x_i K_{jk} - x_j K_{ik}) n^k dS. \tag{18}
\]

Finally, using Eq. \[7\], the three components of the Cartesian angular momentum vector of a volume are given by

\[
J_x = J^{23} = \frac{1}{8\pi} \int_S \left( y K_{3k} - z K_{2k} \right) n^k dS ;
J_y = J^{31} = \frac{1}{8\pi} \int_S \left( z K_{1k} - x K_{3k} \right) n^k dS ;
J_z = J^{12} = \frac{1}{8\pi} \int_S \left( x K_{2k} - y K_{1k} \right) n^k dS .
\tag{19}
\]

Introducing the components of the three Cartesian Killing vectors of the rotational symmetry of Minkowski space

\[
\xi_x = (0, z, y),
\xi_y = (z, 0, -x),
\xi_z = (-y, x, 0),
\tag{20}
\]

we can rewrite the surface integrals of the three Cartesian components of the angular momentum in the following way:

\[
J_i = \frac{1}{8\pi} \int_S K_{jk} (\xi_i^j) n^k dS . \tag{21}
\]

Thus, Weinberg’s identification of the (2,3), (3,1) and (1,2) components as being the physically interesting ones is now clearly seen from Eq. \[21\], as it is the rotational Killing vectors of Minkowski space that enter in the calculation of the Cartesian components of the total angular momentum of the volume.

Note that this form of the angular momentum is remarkably similar to that of the ADM angular momentum \[14\]:

\[
J_i = \frac{1}{8\pi} \lim_{r \to \infty} \int_S (K_{jk} - K \gamma_{jk}) (\xi_i^j) n^k dS . \tag{22}
\]

If the integration is done over a sphere, the components of the surface normal \(n^k\) are given by

\[
n^i = \left( \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right), \tag{23}
\]

so that \((\xi_i)^j n^k = (\xi_i)^k n_k = 0\ \forall i. \tag{24}\]
Therefore, the part of the integral containing the trace of \( K_{ij} \) in Eq. (22) vanishes for spherical surfaces and therefore equations (21) and (22) are identical. We have thus shown that by using Weinberg’s pseudotensor in Gaussian coordinates we obtain the total ADM angular momentum evaluated at spatial infinity, when the integration surface is a sphere. We might still need to impose a stricter asymptotic behaviour than the asymptotic Euclidean flatness of (14) (for instance the quasi-isotropic or asymptotic maximal gauge), but as [15] noted, the \( K_{jk}(ξ^i)n^i \) part of Eq. (22) converges in practice. We are, however, interested in evaluating Eq. (21) quasi-locally, that is, associated with finite 2-surfaces (in our actual applications, these will be apparent horizons of black holes [7]).

For an axisymmetric spacetime, the angular momentum can be calculated via the so-called Komar angular momentum [16], which is defined as (following again the notation of [14, 15]):

\[
J_K = \frac{1}{16\pi} \int_S \nabla^\mu \phi^\nu dS_{\mu\nu}, \tag{25}
\]

where \( \phi^\nu \) is the axial Killing vector. Note the extra factor of 2 in the denominator, known as Komar’s anomalous factor [17]. The Komar angular momentum integral does not have to be evaluated at spatial infinity, but is valid for every surface. In [14, 13] it is shown that using a slicing adapted to the axisymmetry of the spacetime, and expressing Eq. (25) in terms of the 3+1 variables, the Komar angular momentum becomes

\[
J_K = \frac{1}{8\pi} \int_S K_{ij} \phi^i n^j dS. \tag{26}
\]

In [15] the above integral is evaluated for a Kerr BH in spherical Boyer-Lindquist coordinates, and the angular momentum is found to be \( J_K = \mathcal{M}a \), as expected, where \( \mathcal{M} \) and \( a \) are the black hole mass and spin parameter, respectively. As the two integrals (21) and (26) have exactly the same structure, and the latter is coordinate (but not foliation) invariant, we arrive at the conclusion that the introduction of Gaussian coordinates has led to a coordinate invariant expression for the angular momentum derived from Weinberg’s pseudotensor, namely the Komar angular momentum. Note the absence of the anomalous factor of 2 in our final expression (21). It therefore seems that it is possible to relax the restriction of using Cartesian coordinates in calculations involving Weinberg’s pseudotensor.

C. Measuring the angular momentum in numerical relativity simulations

It is easy to check that not only the choice of Gaussian coordinates simplifies the calculation of the total angular momentum via Weinberg’s pseudotensor, but also that it makes straightforward the implementation of the above expressions in a numerical relativity 3D Cartesian code based on the 3+1 decomposition. For instance, if using the widely adopted BSSN formulation [18, 20], the extrinsic curvature \( K_{ij} \) of the spatial slices is closely related to one of the evolved variables, namely the traceless part of the conformally related extrinsic curvature. We note that in present-day numerical relativity simulations one does not typically use Gaussian coordinates for the actual numerical evolutions. This has to do with the fact that Gaussian coordinates can only be used in the close vicinity of a spatial hypersurface, as the geodesics emanating from the hypersurfaces will eventually cross and form caustics in a finite time [15]. Furthermore the foliation is not singularity-avoiding, which means Gaussian coordinates are unsuitable for the numerical evolution of spacetimes containing curvature singularities. Instead, the gauge conditions most commonly employed today in numerical relativity belong to the family of the so-called moving puncture gauges, which consist of the “1+log” condition for the lapse function [21] and the Gamma-driver condition for the shift vector [22]. However, one can use the numerical solution for the extrinsic curvature \( K_{ij} \) in Eq. (21) due to the freedom to choose any gauge for calculations done on each timeslice.

In addition, Eq. (21) is actually equivalent to the method proposed by [11] for the calculation of the angular momentum of a volume using flat-space coordinate rotational Killing vectors (cf. Eq. (20)). To see this, consider the definition of the Killing vectors in Cartesian coordinates given by [11]:

\[
\psi^a_x = [0, -(z - z_c), (y - y_c)], \\
\psi^a_y = [(z - z_c), 0, -(x - x_c)], \\
\psi^a_z = [-(y - y_c), (x - x_c), 0], \tag{27}
\]

where \((x_c, y_c, z_c)\) is the coordinate centroid of the apparent horizon, which has to be subtracted to avoid including contributions from a possible orbital angular momentum of the BH about the center of the computational grid in the calculation of its spin. Upon substituting their flat-space coordinate rotational Killing vectors into Eq. (11), we find that

\[
J_x = \frac{1}{8\pi} \int_S (yK_{3b} - zK_{2b}) n^b dS, \\
J_y = \frac{1}{8\pi} \int_S (zK_{1b} - xK_{3b}) n^b dS, \\
J_z = \frac{1}{8\pi} \int_S (xK_{2b} - yK_{1b}) n^b dS, \tag{28}
\]

where we have set \( x_c = y_c = z_c = 0 \) for simplicity. We see that the two sets of expressions for the Cartesian components of the angular momentum vector of the AH, those from Weinberg’s pseudotensor evaluated in Gaussian coordinates and those from the flat space rotational Killing vector method, are equivalent and equal to the Komar angular momentum in an axisymmetric spacetime.
III. DISCUSSION

As we have shown, the flat-space rotational Killing vector method of \[11\] can be derived from Weinberg's pseudotensor when using Gaussian coordinates. These coordinates have two interesting properties that make them particularly useful for the evaluation of the angular momentum pseudotensor integral, Eq. (8). First, as we have shown, the complicated integral $\int$ reduces to the much simpler expressions given by Eq. (21) and this final expression is equal to the Komar angular momentum integral in a foliation adapted to the axisymmetry of the system. As a result, one does not need the knowledge of the shift vector and of its spatial derivatives on the surface of integration, which in practice would involve more quantities that one would need to interpolate onto the horizon surface for the calculation of the spin, thus also avoiding the numerical error associated with the computation of the finite difference approximation to those spatial derivatives. Second, Gaussian coordinates trivially satisfy the necessary falloff conditions for the lapse and shift. Moreover, by using Gaussian coordinates we recover the ADM angular momentum evaluated at spatial infinity, provided we use a spherical surface of integration.

It is generally known that the various energy-momentum pseudotensors proposed in the literature are not covariant and care has to be taken when evaluating them in different coordinate systems and gauges. (See \[23\] for a review on quasi-local mass and angular momentum in General Relativity, where the problems arising when using pseudotensors for the calculation of mass and angular momentum are also discussed.) The derivation of Weinberg’s pseudotensor relies crucially on the reference space being Cartesian Minkowski. In his textbook \[12\] Weinberg states that a spherical polar coordinate system would lead to a gravitational energy density concentrated at infinity. While being non-covariant is generally not desirable, Weinberg’s method is employed in a Cartesian grid, and Gaussian coordinates guarantee the correct asymptotic behaviour of the lapse and shift, irrespective of the asymptotic behaviour the evolved lapse and shift may possess, which are, as previously stated, not explicitly used in the calculation of the AH spin on the respective timeslice. Furthermore, we have shown that Gaussian coordinates transform the pseudotensor angular momentum surface integral \[13\] to the Komar angular momentum integral \[20\] which is coordinate independent. The use of Gaussian coordinates (as an explicit gauge-fixing) seems to therefore remove the coordinate restrictions of the pseudotensor.

When \[11\] introduced the flat-space rotational Killing vectors for the calculation of the BH spin direction, the authors stated that they could not guarantee the correct results for all times because the method is not gauge invariant. However, as we have seen, such method can be derived from the integration of Weinberg’s total angular momentum pseudotensor over the apparent horizon surface when using Gaussian normal coordinates in the integration. As a result, the method does not depend on the evolving lapse and shift, as the gauge is fixed to the Gaussian normal coordinates on the respective timeslice. We stress that the evolution of the lapse and shift during the free evolution of the spacetime does not enter the calculation, given the coordinates evolve in such a way that an AH is found at all times during the evolution, which is usually the case in puncture evolutions with the BSSN system. There is a dependence on the gauge evolution via the extrinsic curvature $K_{ij}$ that is interpolated onto the AH for the calculation of the spin direction, but the same is true for the expression of the spin magnitude in Eq. (1).

In \[24\] the authors have shown that Eq. (1) will give the spin magnitude provided an approximate Killing vector can be found on the horizon and is gauge independent on the respective time-slice if the approximate Killing vector field $\psi^a$ is divergence-free. Here we have shown that both methods (i.e. either via Weinberg’s pseudotensor in Gaussian coordinates or via flat-space rotational Killing vectors) yield the Komar angular momentum when the latter is expressed in a foliation adapted to the axisymmetry. We note that the restriction to axisymmetry turns out in practice not to be a major weakness, as numerical relativity simulations repeatedly show that the remnants of binary black hole mergers and perturbed Kerr black holes typically settle down to the axisymmetric Kerr solution quickly \[24, 25\]. Moreover, both methods provide a measure of the BH spin magnitude and direction that is not explicitly dependent on the lapse and the shift on the respective time-slice. As both methods use the fixed rotational Killing vectors of Minkowski space, they measure the spin contribution from the axisymmetry of the AH.

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