Effects of Extended Uncertainty Principle on the Relativistic Coulomb Potential

B. Hamil *
Département de TC de SNV, Université Hassiba Benbouali, Chlef, Algeria.

M. Merad †
Faculté des Sciences Exactes, Université de Oum El Bouaghi, 04000 Oum El Bouaghi, Algeria.

T. Birkandan ‡
Department of Physics, Istanbul Technical University, 34469 Istanbul, Turkey.

Abstract

The relativistic bound-state energy spectrum and the wavefunctions for the Coulomb potential are studied for de Sitter and anti-de Sitter spaces in the context of the extended uncertainty principle. Klein-Gordon and Dirac equations are solved analytically to obtain the results. The electron energies of hydrogen-like atoms are studied numerically.

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1 Introduction

The standard Heisenberg uncertainty principle (HUP) [1] of quantum mechanics represents one of the fundamental properties of quantum systems. According to HUP, there should be a fundamental limit for the measurement accuracy where certain pairs of physical observables, such as the positions and momenta or energy and time, cannot be simultaneously measured with full accuracy. A large number of studies have converged on the idea that the HUP should be reformulated for systems with energies close to the Planck scale $\kappa$, which incorporates the concept of minimum measurable length [2–6]. The minimum measurable length idea is predicted by different tentative approaches to quantum gravity such as string theory [7,8], quantum geometry [9,10], loop quantum gravity [11] and black hole physics [12,13]. This fundamental scale leads to a modification of the Heisenberg uncertainty principle to the so-called generalized uncertainty principle (GUP) [14–16]. The GUP ideas are characterized by a deformation of the classical Heisenberg uncertainty relation and the most widely adopted generalization of the Heisenberg uncertainty principle reads [14],

$$\langle \Delta X \rangle \langle \Delta P \rangle \geq \frac{\hbar}{2} \left( 1 + l_p (\Delta P)^2 \right). \tag{1}$$

If $l_p$ is a positive constant, the formula implies the existence of a minimal momentum uncertainty. If $l_p$ is a negative constant, no minimal uncertainty occurs.

On the other hand, there exists another form of the deformed Heisenberg uncertainty principle which is called the extended uncertainty principle (EUP) [17,22] which takes into account the large distances where, due to gravity, the spacetime is curved. We have,

$$\langle \Delta X_i \rangle \langle \Delta P_i \rangle \geq \frac{\hbar}{2} \left( 1 + \frac{\langle \Delta X_i \rangle^2}{l_H^2} \right), \tag{2}$$

where $l_H$ is the (anti-)de Sitter radius which is related to the cosmological constant ($\Lambda$) as $\Lambda = -3/l_H^2$. The EUP modifies the standard commutation relations between position and, the momentum and the coordinate representation of the momentum operators for this model become position-dependent.

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*hamilbilel@gmail.com
†meradm@gmail.com
‡birkandant@itu.edu.tr
S. Mignemi showed that EUP can be derived from the geometric properties of the (anti-)de Sitter spacetime, with a suitable parametrization [17]. In addition to this, it is shown that the EUP arises naturally from the first terms in the expansion of any metric, which means that the corrections to the Hawking temperature of Schwarzschild black hole can be computed by incorporating the gravitational interaction as an external force on a flat background, and neglecting the curvature of spacetime [18]. To our knowledge, only a few works have studied the influence of extended uncertainty principle on quantum mechanical problems [23–35].

In this paper, we study the problem of relativistic Coulomb potential in the framework of the extended uncertainty principle in (3+1) dimensional spacetimes. We solve the problem of Coulomb potential for the Klein-Gordon and Dirac equations to get the exact form of the energy levels and eigenfunctions. This paper is organized as follows: In Sect. 2, we introduce the main relations of quantum mechanics with the extended uncertainty principle. In sections 3 and 4, we solve the Klein-Gordon and Dirac equations exactly in (3+1) dimensions with the Coulomb-like interaction in the context of EUP in the position space representation. Section 5 contains the conclusion.

2 The Extended Uncertainty Relation

In three-dimensional space, the modified Heisenberg algebra leading to EUP is given by the following deformed commutation relations [17],

\[
[X_i, P_j] = i\hbar (\delta_{ij} - s\lambda X_i X_j); \quad i = j = 1, 2, 3, \\
[X_i, X_j] = [P_i, P_j] = -i\hbar s\lambda L_{ij},
\]

where \( \lambda = -\frac{1}{l^2} \) is a small parameter of dimension of inverse distance squared, \( L_{ij} = X_i P_j - X_j P_i \), \( s = 1 \) for de Sitter space and \( s = -1 \) for anti-de Sitter space. These commutation relations lead to the extended uncertainty principle,

\[
(\Delta X_i)(\Delta P_i) \geq \frac{\hbar}{2} \left( 1 - s\lambda (\Delta X_i)^2 \right).
\]

In anti-de Sitter space \( (s = -1) \), the uncertainty relation (4) is characterized by the appearance of a non-zero minimal uncertainty in momentum (MUM),

\[
(\Delta P_k)_{\text{min}} = \frac{\hbar \sqrt{X}}{2}, \quad \forall k,
\]

and for the case of de Sitter space \( (s = 1) \), no lower bound on the measurable length arises. An explicit representation of the momentum and position operators obeying Eq. (3) is given by,

\[
X_i = \frac{x_i}{\sqrt{1 + s\lambda r^2}} \quad \text{where} \quad r = \sum_{i=1}^{3} x_i^2,
\]

\[
P_i = \frac{\hbar}{i} \sqrt{1 + s\lambda r^2} \frac{\partial}{\partial x_i},
\]

in the position representation. In (anti-)de sitter space, the scalar product is not the usual one, but it is defined as,

\[
\langle \phi | \psi \rangle = \int \frac{d\tilde{r}}{\sqrt{1 + s\lambda r^2}} \phi^\dagger (\tilde{r}) \psi (\tilde{r}),
\]

which preserves the hermiticity of the position operator.

3 Klein-Gordon Equation for the Hydrogen Atom

In this section, we will study the eigenvalue problem of the Klein-Gordon equation for a Coulomb-type interaction in (3+1) dimensional spacetime. We have,

\[
\left( E + \frac{Ze^2}{R} \right)^2 - c^2 P^2 - m^2 c^4 \psi = 0.
\]

In de Sitter space, the momentum squared and distance operators act in coordinate space as

\[
P^2 \psi = -\hbar^2 \left( 1 + \lambda r^2 \right) \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{\tilde{L}^2}{\hbar^2 r^2} \right) + \lambda r \frac{\partial}{\partial r} \psi,
\]

where \( \tilde{L}^2 = \sum_{i=1}^{3} L_{ij} \) denotes the Lorentzian angular momentum.
\[ R\psi = \frac{r}{\sqrt{1 + \lambda r^2}} \psi, \]  

(11)

where \( \hat{L} \) is the orbital angular momentum operator whose eigenfunctions are given in terms of the spherical harmonics,

\[ \hat{L}^2 Y_{\ell,\nu} (\theta, \varphi) = \hbar^2 (\ell + 1) Y_{\ell,\nu} (\theta, \varphi), \]  

(12)

where \( \ell \) and \( \nu \) are quantum numbers. Using (10) and (11) in Eq. (9), we obtain,

\[ \left[ (1 + \lambda r^2) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{\delta^2}{r^2} \right) + \lambda r \frac{\partial}{\partial r} + \frac{2Z\mu E \sqrt{1 + \lambda r^2}}{\hbar c} + (Z\mu)^2 \left( 1 + \frac{\lambda r^2}{r^2} \right) + \frac{E^2 - m^2c^4}{\hbar^2 c^2} \right] \psi = 0, \]  

(13)

where \( \mu = \frac{e^2}{\hbar} \simeq 1/137.03602 \) is Sommerfeld’s fine-structure constant and \( \frac{2\pi \hbar}{m} = 2.4263 \times 10^{-12} \text{m} \) is the Compton wavelength. For the wave function \( \psi \), we make the ansatz,

\[ \psi = \frac{F_\ell (r)}{\sqrt{r}} Y_{\ell,\nu} (\theta, \varphi). \]  

(14)

The angular and radial dependent parts can now be separated to yield the radial equation,

\[ \left[ (1 + \lambda r^2) \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\delta^2}{r^2} \right) + \lambda r \frac{d}{dr} + \frac{2Z\mu E \sqrt{1 + \lambda r^2}}{\hbar c} + \frac{E^2 - m^2c^4}{\hbar^2 c^2} - \frac{\lambda}{2} \right] F_\ell (r) = 0, \]  

(15)

with \( \delta^2 = (\ell + \frac{1}{2})^2 - (Z\mu)^2 \). Let us now transform Eq. (15) into a hypergeometric differential equation by using two successive changes of variables as

\[ \zeta = \sqrt{1 + \frac{\lambda r^2}{\lambda r}} \text{ and } y = \frac{1}{2} \left( 1 - \zeta \right). \]  

(16)

Equation (16) takes the form,

\[ \left[ y(1-y) \frac{\partial^2}{\partial y^2} + \left( \frac{1}{2} - y \right) \frac{\partial}{\partial y} - \frac{\delta^2 - \frac{2Z\mu E}{\hbar c \sqrt{\lambda r}} - \frac{E^2 - m^2c^4}{4y}}{4} - \frac{\delta^2 + \frac{2Z\mu E}{\hbar c \sqrt{\lambda r}} - \frac{E^2 - m^2c^4}{4(1-y)}}{4} + \frac{1}{4} + \frac{\delta^2}{4} \right] F_\ell (y) = 0. \]  

(17)

The latter equation possesses three regular singular points located at \( y = \{0, 1, \infty\} \). Applying the definition,

\[ F_\ell (y) = y^a (1-y)^b \Xi_\ell (y), \]  

(18)

the Eq. (17) will reduce to the hypergeometric type, namely,

\[ \left\{ y(1-y) \frac{\partial^2}{\partial y^2} + \left( \frac{1}{2} + 2a - y(1+2a+2b) \right) \frac{\partial}{\partial y} - [(a+b)^2 - \delta^2] \right\} \Xi_\ell (y) = 0, \]  

(19)

where \( a \) and \( b \) are given by

\[ a = \frac{1}{4} + \frac{1}{2} \sqrt{\delta^2 - \frac{2Z\mu E}{\hbar c \sqrt{\lambda r}} - \frac{E^2 - m^2c^4}{\hbar^2 c^2 \lambda} + \frac{3}{4}}, \]  

(20)

\[ b = \frac{1}{4} + \frac{1}{2} \sqrt{\delta^2 + \frac{2Z\mu E}{\hbar c \sqrt{\lambda r}} - \frac{E^2 - m^2c^4}{\hbar^2 c^2 \lambda} + \frac{3}{4}}. \]  

(21)

The regular solution at the origin \( y = 0 \) of Eq.(19) is written in terms of the hypergeometric function as,

\[ \Xi_\ell (y) = F \left( A; B; \frac{1}{2} + 2a; y \right), \]  

(22)

whose parameters are given by

\[ A = a + b - \delta; \quad B = a + b + \delta. \]  

(23)

The hypergeometric function becomes a polynomial of degree \( n \) when

\[ A = -n \quad \text{or} \quad B = -n \quad \text{where} \quad n = 0, 1, 2, \ldots \]  

(24)
In both cases, we have
\[
\frac{1}{2} + \frac{1}{2} \sqrt{\delta^2 + \frac{3}{4} + \frac{m^2c^4}{\hbar^2c^2\lambda} - \frac{2Z\mu E}{\hbar c\sqrt{\alpha}} - \frac{E^2}{\hbar^2c^2\lambda}} + \frac{1}{2} \sqrt{\delta^2 + \frac{3}{4} + \frac{m^2c^4}{\hbar^2c^2\lambda} + \frac{2Z\mu E}{\hbar c\sqrt{\alpha}} - \frac{E^2}{\hbar^2c^2\lambda}} + \delta = -n. \tag{25}
\]

The energy spectrum can be obtained from (25) which leads to
\[
E^dS_{KG} = \frac{mc^2}{\hbar/c^2} \left[ 1 - \frac{\hbar^2\lambda}{mc^2} \left( N - \ell - \frac{1}{2} + \sqrt{\left( \ell + \frac{1}{2} \right)^2 - \left( Z\mu \right)^2} \right)^2 + \left( Z\mu \right)^2 - \ell \left( \ell + 1 \right) - 1 \right]. \tag{26}
\]

where \( N = n + \ell + 1 \) is the principal quantum number. We observe that the energy levels depend on the deformation parameter \( \lambda \). This is a natural consequence of the modified Heisenberg algebra. We also note that the energy levels depend on \( N^2 \), which is the feature of hard confinement. According to the energy levels, we remark the following:

- The constraint \( \ell + \frac{1}{2} \geq Z\mu \) is necessary for the existence of physical energy eigenvalues.
- For \( \ell = 0 \), we must impose \( Z \leq 69 \) and bound states do not exist for larger \( Z \)-values.
- For larger \( N \)-values, the energy spectrum would have an unphysical behavior.

The expansion of Eq. (26) up to the first order in \( \lambda \) yields
\[
E^dS_{KG} = \varepsilon_{KG} - \frac{\hbar^2\lambda}{2mc^2} \left[ \left( N - \ell - \frac{1}{2} + \sqrt{\left( \ell + \frac{1}{2} \right)^2 - \left( Z\mu \right)^2} \right)^2 + \left( Z\mu \right)^2 - \ell \left( \ell + 1 \right) - 1 \right]. \tag{27}
\]

where,
\[
\varepsilon_{KG} = mc^2 \left[ 1 + \frac{Z^2\mu^2}{\left( N - \ell - \frac{1}{2} + \sqrt{\left( \ell + \frac{1}{2} \right)^2 - \left( Z\mu \right)^2} \right)^2} \right]^{-\frac{1}{2}}. \tag{28}
\]

The first term in (27) is the energy spectrum of the ordinary three-dimensional Coulomb potential of spin-0 particles with no deformation, while the second term represents the correction due to the presence of the EUP. Expanding (27) in powers of \( (Z\mu) \) yields
\[
W^dS_{KG} = E^dS_{KG} - mc^2 = -\frac{mZ^2e^4}{2\hbar^2N^2} - \frac{\hbar^2\lambda}{2m} \left[ N^2 - \ell \left( \ell + 1 \right) - 1 \right] - \frac{m^2c^4}{2mc^2} \left[ \frac{N^2}{\ell + \frac{1}{2}} - \frac{3}{4} \right] \left\{ 1 - \frac{\hbar^2\lambda}{2mc^2} \left[ N^2 - \ell \left( \ell + 1 \right) - 1 \right] \right\} + \frac{\hbar^2\lambda}{2m} \left( \frac{2N^3}{\ell + \frac{1}{2}} \right) \left[ N^2 - \ell \left( \ell + 1 \right) - 1 \right] + (Z\mu)^2 \left[ 1 + \frac{N^3}{2 \left( \ell + \frac{1}{2} \right)^3} - \frac{N^2}{2 \left( \ell + \frac{1}{2} \right)^2} - \frac{N}{\left( \ell + \frac{1}{2} \right)} \right]. \tag{29}
\]

The first and the second terms in (29) represent the non-relativistic energy levels of hydrogen in dS space [35], while the other terms are relativistic corrections in dS space.

The same calculation can be performed for the case of anti-de Sitter space by taking the change of the sign in the deformation into the account by using \( \lambda > 0 \). The energy spectrum of the system is found as
\[
E^{AdS}_{KG} = \frac{mc^2}{\hbar/c^2} \left[ 1 - \frac{\hbar^2\lambda}{mc^2} \left( N - \ell - \frac{1}{2} + \sqrt{\left( \ell + \frac{1}{2} \right)^2 - \left( Z\mu \right)^2} \right)^2 + \left( Z\mu \right)^2 - \ell \left( \ell + 1 \right) - 1 \right]. \tag{30}
\]

We can plot our results for different scenarios. The contribution of the deformation is very small numerically for physical values of the parameters. Thus we take \( \hbar = c = m = 1 \) in order to avoid numerical errors and make the effect of the
deformation factor visible in the graphics. In Fig. 1 we plot the ratio $\frac{E^{(A) dS}}{\varepsilon_{KG}}$ with respect to $N$ for some values of the deformation parameter $\lambda$. Here, $\ell = 0$ and $Z = 50$. We see that the curves for the AdS case grow with $N$, while they decrease almost rapidly with $N$ in the dS case. The undeformed calculation shown with a solid line with $\lambda = 0$ gives the ratio as unity as expected.

The Figures 2 and 3 show the behavior of the energy eigenvalues $E^{(A) dS}$ with respect to $Z$. We take $N = 1$ and $\ell = 0$ in Fig. 2. As we have calculated above, no bound states exist for $Z > 69$. In Fig. 3, for $N = 3$ and $\ell = 1$, we verify the condition $\ell + \frac{1}{2} \geq Z \mu$ graphically by having $Z = 206$ as the accumulation point.
4 Dirac Equation for the Hydrogen Atom

The Dirac equation in the presence of the Coulomb potential $V$ is

$$
\left[ c \alpha \cdot \vec{P} + \beta mc^2 - E + V \right] \psi = 0,
$$

where the matrices $\alpha$ and $\beta$ satisfy the anticommutation relations,

$$
\{ \alpha_i, \beta \} = 0; \quad \{ \alpha_i, \alpha_j \} = 2\delta_{ij}; \quad i, j = 1, 2, 3.
$$

An explicit familiar representation of $\alpha$ and $\beta$ is provided by

$$
\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}; \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

where $\sigma_i$ are $2 \times 2$ Pauli spin matrices. In spherical coordinates, the operator $\alpha \cdot \vec{P}$ can be written as

$$
\alpha \cdot \vec{P} = \sqrt{1 + \lambda r^2} \alpha \cdot \vec{p} = -i\hbar \sqrt{1 + \lambda r^2} \alpha_r \left( \frac{\partial}{\partial r} + \frac{1}{r} - \frac{\beta}{\hbar r} K \right),
$$

in the deformed case. Here,

$$
\alpha_r = \alpha \cdot \vec{r}; \quad K = \beta \left( \sum \vec{L} + \hbar \right); \quad \sum = \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix},
$$

where the last operator is absent in the Klein-Gordon equation. The solution ansatz of (31) can be chosen such that

$$
\psi = \begin{pmatrix} f_1(r) \mathcal{Y}_{\kappa,\nu}(\theta, \varphi) \\ i f_2(r) \mathcal{Y}^{\ast}_{\kappa,\nu}(\theta, \varphi) \end{pmatrix},
$$

where $\mathcal{Y}_{\kappa,\nu}(\theta, \varphi)$ are spinor spherical harmonics. Considering the action of the operator $K$, we have

$$
K \mathcal{Y}_{\kappa,\nu}(\theta, \varphi) = -\hbar \kappa \mathcal{Y}_{\kappa,\nu}(\theta, \varphi), \quad K \mathcal{Y}^{\ast}_{\kappa,\nu}(\theta, \varphi) = \hbar \kappa \mathcal{Y}^{\ast}_{\kappa,\nu}(\theta, \varphi),
$$

where the quantum number $\kappa$ is defined as

$$
\kappa = \pm \left( j + \frac{1}{2} \right) = \begin{cases} -\ell + 1 & \text{for } j = l + 1/2 \\ \ell & \text{for } j = l - 1/2 \end{cases}.
$$
Using the expression for $\frac{\sqrt{7}}{\sqrt{2}}\psi$ we can rewrite Eq. (31) as

$$
\left[-i\hbar \sqrt{1 + \lambda r^2} \alpha_r \left( \frac{\partial}{\partial r} + \frac{1}{r} - \frac{\beta}{\hbar r} K \right) + \beta mc^2 - E + V \right] \psi = 0.
$$

(39)

From this equation, we get two coupled differential equations:

$$
\frac{E - mc^2}{\hbar c} + Z\mu \frac{\sqrt{1 + \lambda r^2}}{r} f_1 (r) = \sqrt{1 + \lambda r^2} \left( \frac{\kappa - 1}{r} - \frac{d}{dr} \right) f_2 (r),
$$

(40)

$$
\frac{E + mc^2}{\hbar c} + Z\mu \frac{\sqrt{1 + \lambda r^2}}{r} f_2 (r) = \sqrt{1 + \lambda r^2} \left( \frac{\kappa + 1}{r} + \frac{d}{dr} \right) f_1 (r).
$$

(41)

These two equations can be solved exactly by a diagonalization procedure [36],

$$
\begin{pmatrix}
g_1 \\
g_2
\end{pmatrix} = \begin{pmatrix}
X & 1 \\
1 & X
\end{pmatrix} \begin{pmatrix}
f_1 \\
f_2
\end{pmatrix},
$$

(42)

where $X = \frac{\gamma}{\sqrt{2}}$ and $\gamma = \sqrt{\lambda^2 - (Z\mu)^2}$. which transforms Eqs. (40) and (41) into the following equations,

$$
\sqrt{1 + \lambda r^2} \frac{d}{dr} + (\gamma + 1) \frac{\sqrt{1 + \lambda r^2}}{r} - Z\mu \frac{E}{\gamma \hbar c} g_1 = \left[ \frac{mc^2}{\hbar c} + \frac{\kappa E}{\gamma \hbar c} \right] g_2,
$$

(43)

$$
\sqrt{1 + \lambda r^2} \frac{d}{dr} + (1 - \gamma) \frac{\sqrt{1 + \lambda r^2}}{r} + Z\mu \frac{E}{\gamma \hbar c} g_2 = \left[ \frac{mc^2}{\hbar c} - \frac{E \kappa}{\hbar c \gamma} \right] g_1.
$$

(44)

This system gives the following differential equation for the component $g_2 (r) = \frac{1}{\gamma r} \Xi (r)$,

$$
\left[ (1 + \lambda r^2) \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) + \lambda r \frac{d}{dr} - \left( \gamma - \frac{1}{2} \right)^2 \right] + 2 Z\mu E \sqrt{1 + \lambda r^2} \frac{1}{r} + \frac{E^2 - mc^4}{\hbar c^2} - \lambda \left( \gamma^2 - \frac{1}{4} \right) \Xi (r) = 0.
$$

(45)

Following the same procedure in the Klein-Gordon case, the solution is obtained as by,

$$
\Xi = y^{a_1} (1 - y)^{b_1} F \left( A_1; B_1; \frac{1}{2} + 2a, y \right),
$$

(46)

with $F$ being a hypergeometric function with the parameters

$$
A_1 = a_1 + b_1 - \left( \gamma - \frac{1}{2} \right); \quad B_1 = a_1 + b_1 + \left( \gamma - \frac{1}{2} \right),
$$

(47)

where

$$
a_1 = \frac{1}{4} + \frac{1}{2} \sqrt{\gamma^2 - \frac{2 \mu Z E}{\hbar c \sqrt{\lambda}} - \frac{E^2 - mc^4}{\hbar c^2 \lambda}}; \quad b_1 = \frac{1}{4} + \frac{1}{2} \sqrt{\frac{2 \mu Z E}{\hbar c \sqrt{\lambda}} + \gamma^2 - \frac{E^2 - mc^4}{\hbar c^2 \lambda}}.
$$

(48)

If $A_1 = -n$ or $B_1 = -n$, $(n = 0, 1, 2, ...)$, the hypergeometric function reduces to a polynomial in $y$ whose degree is $n$, and then we have

$$
\gamma + \frac{1}{2} \sqrt{\gamma^2 + \frac{m^2 c^4}{\hbar^2 c^2 \lambda} + \frac{2 \mu Z E}{\hbar c \sqrt{\lambda}} \frac{E^2}{\hbar^2 c^2 \lambda} + \frac{1}{2} \sqrt{\gamma^2 + \frac{m^2 c^4}{\hbar^2 c^2 \lambda} + \frac{2 \mu Z E}{\hbar c \sqrt{\lambda}} \frac{E^2}{\hbar^2 c^2 \lambda}} = -n.
$$

(49)

We can obtain the energy eigenvalues by solving this equation for $E$, namely,

$$
E^{\text{Dirac}}_{n} = \sqrt{1 + \frac{\hbar Z^2}{mc^2} \left[ N - j - \frac{1}{2} + \sqrt{(j + \frac{1}{2})^2 - (Z\mu)^2} \right]^2 + (Z\mu)^2 - (j + \frac{1}{2})^2}.
$$

(50)
where $N = n + j + \frac{1}{2}$ is the principal quantum number. The equation (50) gives the energy levels of hydrogen-like atoms in de Sitter space. The energy levels depend on the principal quantum number $N$, $j$, $Z$ and the deformation parameter $\lambda$. It should be emphasized that the energy depends on the quantum number $j$, which is associated with both orbital angular momentum and spin. The other thing to notice about Eq. (50) is that for states with $\lambda = 1$, the energy becomes zero, i.e. $E_N^{dS} = 0$.

In addition, if $Z\mu > j + \frac{1}{2}$ the energy levels in (50) become complex which means that there exists no regular polynomial solutions for $ns_{1/2}$ or $np_{1/2}$ states when the charge is greater than $Z > 137$.

Expanding Eq. (50) to the first order in $\lambda$, we obtain

$$E^{dS}_{\text{Dirac}} = \epsilon_{\text{Dirac}} - \frac{\hbar^2 \alpha_{\text{Dirac}}}{2m^2 c^2} \left[ N - j - \frac{1}{2} + \sqrt{(j + \frac{1}{2})^2 - (Z\mu)^2} \right]^2,$$

where

$$\epsilon_{\text{Dirac}} = mc^2 \left[ 1 + \frac{\mu^2 Z^2}{\left( N - j - \frac{1}{2} + \sqrt{(j + \frac{1}{2})^2 - (Z\mu)^2} \right)^2} \right].$$

Clearly, the first term is identical with the Dirac energy levels in the ordinary case. The second term represents the correction due to the presence of the extended uncertainty principle. Moreover, the series expansion of (52) in powers of $(Z\mu)$ reads

$$W^{dS}_{\text{Dirac}} = E_N^{dS} - mc^2 = \epsilon_{N,j} + \Delta \epsilon_{N,j},$$

where

$$\epsilon_{N,j} = -\frac{Z^2 e^4 m}{2 \hbar^2 N^2} - mc^2 \frac{\mu^2 Z^4}{2 N^4} \left( \frac{N}{j + \frac{1}{2}} - \frac{3}{4} \right).$$

Here, the first term is the energy spectrum of the non-relativistic hydrogen atom, the second term contains all of the details of the fine structure, and

$$\Delta \epsilon_{N,j} = -\frac{\hbar^2 \lambda}{2m} \left[ 1 - \frac{Z^2 \mu^2}{2 N^2} - \frac{\mu^2 Z^4}{2 N^4} \left( \frac{N}{j + \frac{1}{2}} - \frac{3}{4} \right) \right] \left[ N^2 - \left( j + \frac{1}{2} \right)^2 - (Z\mu)^2 \left( 1 + \frac{(Z\mu)^2}{4(j + \frac{1}{2})^2} \right) \left( \frac{N}{j + \frac{1}{2}} - 1 \right) \right].$$

The formula (54) shows the effect of the EUP on the non-relativistic energy levels for the one-electron atom. We observe that the EUP correction carries new terms associated with the relativistic correction, which do not exist in the undeformed case.

Consequently, we can calculate the electron energies of hydrogen-like atoms in de Sitter space using our results with the numerical values of $mc^2 = 511004.1$ eV and $\sqrt{\lambda} = 0.252 \times 10^6$ m$^{-1}$ [28,37]. The results are given in Table 1.

| $N$ | $\ell$ | $j$ | Label | $\epsilon_{N,j}$ (eV) | $|\Delta \epsilon_{N,j}|$ (eV) |
|-----|-------|-----|-------|----------------------|-------------------------------|
| 1   | 0     | 1/2 | $1s_{1/2}$ | -13.605 | 0 |
| 2   | 0     | 1/2 | $2s_{1/2}$ | -3.40132 | 3.4150 x 10$^{-8}$ |
| 2   | 1     | 1/2 | $2p_{1/2}$ | -3.40132 | 3.4150 x 10$^{-8}$ |
| 2   | 1     | 3/2 | $2p_{3/2}$ | -3.40127 | 0 |
| 3   | 0     | 1/2 | $3s_{1/2}$ | -1.51169 | 1.9405 x 10$^{-8}$ |
| 3   | 1     | 1/2 | $3p_{1/2}$ | -1.51169 | 1.9405 x 10$^{-8}$ |
| 3   | 1     | 3/2 | $3p_{3/2}$ | -1.51168 | 1.2128 x 10$^{-8}$ |
| 3   | 2     | 3/2 | $3d_{3/2}$ | -1.51168 | 1.2128 x 10$^{-8}$ |
| 3   | 2     | 5/2 | $3d_{5/2}$ | -1.51167 | 0 |

Table 1: Energy levels of hydrogen-like atoms.
The same calculation can be performed for the case of anti-de Sitter space. According to the correspondence $\lambda \to -\lambda$, the energy levels of a spin-1/2 particle in a Coulomb potential in anti-de Sitter space can be written as

$$E_{\text{Dirac}}^{\text{AdS}} = mc^2 \sqrt{1 + \frac{\hbar^2 \lambda}{m^2 c^2} \left[ \left( N - j - \frac{1}{2} + \sqrt{\left( j + \frac{1}{2} \right)^2 - (Z\mu)^2} \right)^2 + (Z\mu)^2 - \left( j + \frac{1}{2} \right)^2 \right]} \sqrt{1 + \frac{\mu^2 Z^2}{\left( N - j - \frac{1}{2} + \sqrt{\left( j + \frac{1}{2} \right)^2 - (Z\mu)^2} \right)^2}}.$$  (57)

We can plot our results as in the Klein-Gordon case, following the same numerical procedure. We observe the same behavior in the energy curves.

In Fig. 4 we plot the ratio $E_{\text{Dirac}}^{(A)\text{dS}} / \varepsilon_{\text{Dirac}}$ with respect to $N$ for some values of $\lambda$. We see that the curves for the AdS case grow with $N$, while they decrease with $N$ in the dS case. The undeformed energy value corresponding to $\lambda = 0$ is shown with a solid line where $E_{\text{Dirac}}^{\text{AdS}} = \varepsilon_{\text{Dirac}}$.

The Figures 5 and 6 show the behavior of the energy eigenvalues $E_{\text{Dirac}}^{(A)\text{dS}}$ with respect to $Z$. In Fig. 5 we take $N = 2$ and $j = 1/2$ and using $Z\mu > j + \frac{1}{2}$, we see that the bound states are limited by $Z < 137$. The results are similar in Fig. 6 where $N = 4$, $j = 3/2$ and $Z < 274$.

Figure 4: $E_{\text{Dirac}}^{(A)\text{dS}} / \varepsilon_{\text{Dirac}}$ vs. $N$

Figure 5: $E_{\text{Dirac}}^{(A)\text{dS}}$ vs. $Z$ for $N = 2$, $j = 1/2$
5 Conclusion

We studied the Klein-Gordon and Dirac equations for the relativistic Coulomb potential in the framework of the extended uncertainty principle for de Sitter and anti-de Sitter spaces. We treated the problem analytically and obtained the energy spectra and the eigenfunctions exactly in both cases. Eventually, we obtained the relativistic bound-state energy spectra for hydrogenic atoms and the corresponding wave functions for the spin 0 and 1/2 cases.

We solved the Klein-Gordon and Dirac equations in terms of the hypergeometric functions and utilizing the conditions that truncate the series to yield a polynomial solution, we obtained the energy eigenvalues. Analyzing the energy spectra, we found the limits \( \ell + \frac{1}{2} \geq Z \mu \) and \( Z \mu < j + \frac{1}{2} \) which are necessary for the existence of physical energy eigenvalues in the Klein-Gordon and Dirac cases, respectively.

Furthermore, we used our analytical results to create a numerical table for the energy levels of hydrogen-like atoms and plot some graphics to present the effects of the deformation on the energy spectra and the accumulation of the curves to a certain \( Z \)-value in a visual way for de Sitter and anti-de Sitter spaces.

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