ABSTRACT MATRIX-TREE THEOREM

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Abstract. The classical matrix-tree theorem discovered by G. Kirchhoff in 1847 relates the principal minor of the \( n \times n \) Laplace matrix to a particular sum of monomials of matrix elements indexed by directed trees with \( n \) vertices and a single sink. In this paper we consider a generalization of this statement: for any \( k \geq n \) we define a degree \( k \) polynomial \( \det_{n,k} \) of matrix elements and prove that this polynomial applied to the Laplace matrix gives a sum of monomials indexed by acyclic graphs with \( n \) vertices and \( k \) edges.

1. Introduction and the main results

1.1. Principal definitions. Denote by \( \Gamma_{n,k} \) the set of all directed graphs with \( n \) vertices numbered 1, \ldots, \( n \) and \( k \) edges numbered 1, \ldots, \( k \). We will write \( e = [ab] \) if \( e \) is an edge from vertex \( a \) to vertex \( b \); in particular, \( [aa] \) means a loop attached to the vertex \( a \). We will treat elements of \( \Gamma_{n,k} \) as sequences of edges: \( G = (e_1, \ldots, e_k) \in \Gamma_{n,k} \) means a graph where the edge \( e_\ell \) has number \( \ell \), for all \( \ell = 1, \ldots, k \). By a slight abuse of notation \( e \in G \) will mean that \( e \) is an edge of \( G \) (regardless of number).

Let \( G \in \Gamma_{n,k} \) and \( e \in G \). By \( G \setminus e \), \( G/e \) and \( G/_e \) we will denote the graph \( G \) with \( e \) deleted, \( e \) contracted and \( e \) reversed, respectively. Note for correctness that since \( G \setminus e \in \Gamma_{n,k-1} \), one has to change the edge numbering in \( G \) after deleting \( e \): namely, if \( e \) bears number \( s \) in \( G \) then the numbers of the edges are preserved if they are less than \( s \) and lowered by 1 otherwise. For \( G/e \in \Gamma_{n-1,k-1} \) the same renumbering is applied both to the edges and to the vertices. The contracted edge \( e \) should not be a loop.

A graph \( H \in \Gamma_{n,m} \) is called a subgraph of \( G \in \Gamma_{n,k} \) (notation \( H \subseteq G \)) if \( H \) is obtained from \( G \) by deletion of several (possibly zero) edges.

Denote by \( G_{n,k} \) a vector space over \( \mathbb{C} \) spanned by \( \Gamma_{n,k} \). The direct sum \( G_n \defeq \bigoplus_{k=0}^\infty G_{n,k} \) bears the structure of an associative algebra: one defines a product of the graphs \( G_1 = (e_1, \ldots, e_{k_1}) \in \Gamma_{n,k_1} \) and \( G_2 = (h_1, \ldots, h_{k_2}) \in \Gamma_{n,k_2} \) as \( G_1 * G_2 \defeq (e_1, \ldots, e_{k_1}, h_1, \ldots, h_{k_2}) \in \Gamma_{n,k_1+k_2} \); then \( * \) is extended to the whole \( G_n \) as a bilinear operation. Note that \( G_1 * G_2 \neq G_2 * G_1 \) (the edges are the same but the edge numbering is different), so the algebra \( G_n \) is not commutative.

We call a graph \( G \in \Gamma_{n,k} \) strongly connected if every two of its vertices can be joined by a directed path. A graph is strongly semiconnected if every its connected component (in the topological sense) is strongly connected; equivalently, if every its edge is a part of a directed cycle. A strongly semiconnected graph may contain isolated vertices (i.e. vertices not incident to any edge); by \( \mathcal{S}_{n,k}^{i_1, \ldots, i_s} \) we denote the set of strongly semiconnected graphs \( G \in \Gamma_{n,k} \) such that the vertices \( i_1, \ldots, i_s \), and only they, are isolated. By \( \mathcal{S}_{n,k} \defeq \bigcup_{I \subseteq \{1, \ldots, n\}} \mathcal{S}_{n,k}^I \) we will denote the set of all strongly semiconnected graphs.
We call a graph $G \in \Gamma_{n,k}$ acyclic if it contains no directed cycles. Recall that a vertex $a$ of the graph $G$ is called a sink if $G$ has no edges starting from $a$. Note that an isolated vertex is a sink but a vertex with a loop attached to it is not. We denote by $\mathfrak{A}_{n,k}^{\{i_1,\ldots,i_s\}}$ the set of acyclic graphs $G \in \Gamma_{n,k}$ such that the vertices $i_1,\ldots,i_s$, and only they, are sinks. By $\mathfrak{A}_{n,k} = \bigcup_{s=1}^{n} \mathfrak{A}_{n,k}^{\{i_1,\ldots,i_s\}}$ we will denote the set of all acyclic graphs.

**Example 1.1.** If a vertex of a strongly semiconnected graph $G \in \mathfrak{G}_{n,k}$ is not isolated then there is at least one edge starting from it; so if $I = \{i_1,\ldots,i_s\}$ and $\mathfrak{A}_{n,k}^{I} \neq \emptyset$ then $k \geq n - s$.

Let $k = n - s$. If $G \in \mathfrak{G}_{n,k}^{I}$ then for any vertex $i \notin I$ there is exactly one edge $[i,\sigma(i)]$ starting at it and exactly one edge $[j,\sigma(j)] = [i, j]$ finishing at it (that is, $\sigma(j) = i$). Hence $\sigma$ is a bijection $\{1,\ldots,n\} \setminus I \to \{1,\ldots,n\} \setminus I$ (a permutation of $k = n - s$ points).

Geometrically $G$ is a union of disjoint directed cycles passing through all vertices except $i_1,\ldots,i_s$.

**Example 1.2.** Let $n > k$; then any graph $G \in \Gamma_{n,k}$ contains at least $n - k$ connected components. If $G$ is acyclic then every its connected component contains a sink. So for $I = \{i_1,\ldots,i_s\}$ if $\mathfrak{A}_{n,k}^{I} \neq \emptyset$ then $k \geq n - s$.

Let $k = n - s$. Then the elements of $\mathfrak{A}_{n,k}^{I}$ are forests of $s$ components, each component containing exactly one vertex $i_\ell \in I$ (for some $\ell = 1,\ldots,s$), which is its only sink. This component is a tree and every its edge is directed towards the sink $i_\ell$.

### 1.2. Determinants and minors.

Let $W = (w_{ij})$ be a $n \times n$-matrix; denote by $\langle W | G \rangle \defeq \prod_{[ij] \in G} w_{ij}$.

Note that $\langle W | G \rangle$ is independent of the edge numbering in $G$; in particular, $\langle W | G_1 \ast G_2 - G_2 \ast G_1 \rangle = 0$ for all $G_1, G_2$.

For a function $f : \bigcup \Gamma_{n,s} \to \mathbb{C}$ and a graph $G \in \Gamma_{n,k}$ introduce the notation

$$s(f; G) \defeq \sum_{H \subseteq G} f(H).$$

For a set of graphs $\mathfrak{B} \subseteq \Gamma_{n,k}$ denote

$$U(\mathfrak{B}) \defeq \sum_{G \in \mathfrak{B}} G \in \mathfrak{G}_{n,k},$$

$$X(\mathfrak{B}) \defeq \sum_{G \in \mathfrak{B}} (-1)^{\beta_0(G)}G \in \mathfrak{G}_{n,k};$$

$\beta_0(G)$ here means the 0-th Betti number of $G$, i.e. the number of its connected components (in the topological sense).

**Definition 1.3.** The element

$$\det_{n,k} \defeq \frac{(-1)^k}{k!} X(\mathfrak{A}_{n,k}^{I}) \in \mathfrak{G}_{n,k}$$
is called a universal diagonal $I$-minor of degree $k$; in particular, $\det^\otimes_{n,k}$ is called a universal determinant of degree $k$.

The element
\[ \det^{i/j}_{n,k} = \frac{(-1)^k}{k!} X \{ (G \in \mathcal{G}_{n,k} \mid ([ij]) \in \mathcal{S}^\otimes_{n,k+1} \} \]
is called a universal (codimension 1) $(i,j)$-minor of degree $k$.

Example 1.4. Example [1.4] implies that if $I = \{i_1, \ldots, i_s\}$ and $k < n - s$ then $\det^I_{n,k} = 0$.

Let $k = n$ and $I = \emptyset$. By Example [1.4] the graphs $G \in \mathcal{S}_{n,n}$ are in one-to-one correspondence with permutations $\sigma$ of $\{1, \ldots, n\}$. It is easy to see that $(-1)^{\delta_0(G)}$ is equal to $(-1)^n$ if $\sigma$ is even and to $-(-1)^n$ if it is odd. Geometrically $G$ is a union of disjoint directed cycles. If the order of vertices in all the cycles is fixed, then there are $n!$ ways to assign numbers $\{1, \ldots, n\}$ to the edges; this implies the equality
\[ \langle W \mid \det^\otimes_{n,n} \rangle = \sum_{\sigma} (-1)^{\text{parity of } \sigma} w_{1\sigma(1)} \cdots w_{n\sigma(n)} = \det W \]
for any matrix $W = (w_{ij})$. Similarly, for any set $I = \{i_1, \ldots, i_s\}$ the value $\langle W \mid \det^I_{n,n-s} \rangle$ is equal to the diagonal minor of the matrix $W$ obtained by deletion of the rows and the columns $i_1, \ldots, i_s$. Also $\langle W \mid \det^{i/j}_{n,n-1} \rangle$ is equal to the codimension 1 minor of $W$ obtained by deletion of the row $i$ and the column $j$. This explains the terminology of Definition 1.3.

The elements $\det^I_{n,k}$ exhibit some properties one would expect from determinants and minors:

Proposition 1.5.

1. (generalized row and column expansion)
\[ \det^\otimes_{n,k} = \frac{1}{k} \sum_{i,j=1}^{n} ([ij]) \ast \det^{i/j}_{n,k-1} . \]

2. (partial derivative with respect to a diagonal matrix element) Let matrix elements $w_{ij}$, $i, j = 1, \ldots, n$, of the matrix $W$ be independent (commuting) variables. Then for any $i = 1, \ldots, n$ and any $m = 1, \ldots, k$ one has
\[ \frac{\partial^m}{\partial w_{ii}^m} \langle W \mid \det^\otimes_{n,k} \rangle = \langle W \mid \det^\otimes_{n,k-m} \rangle + \langle W \mid \det^\otimes_{n,k-m} \rangle. \]

See [2] Lemma 86 for a formula similar to (1.3) (with $m = 1$ and a finite difference instead of a derivative).

1.3. Main results. Let $G \in \Gamma_{n,k}$, $p \in \{1, \ldots, k\}$ and $i, j \in \{1, \ldots, n\}$. Denote by $R_{abp}G \in \Gamma_{n,k}$ the graph obtained from $G$ by replacement of its $p$-th edge by the edge $[ab]$ bearing the same number $p$.

Consider now a linear operator $B_p : \mathcal{G}_{n,k} \rightarrow \mathcal{G}_{n,k}$ acting on every basic element $G \in \Gamma_{n,k}$ as follows:
\[ B_p(G) = \begin{cases} G, & \text{if the } p\text{-th edge of } G \text{ is not a loop,} \\ - \sum_{b \neq a} R_{abp}G, & \text{if the } p\text{-th edge of } G \text{ is the loop } [aa]. \end{cases} \]

In particular, $B_p = 0$ if $n = 1$ (and $k > 0$).
Definition 1.6. The product $\Delta \overset{\text{def}}{=} B_1 \ldots B_k : \mathcal{G}_{n,k} \to \mathcal{G}_{n,k}$ is called Laplace operator.

If $n = 1$ and $k > 0$ then $\Delta = 0$; also take $\Delta = \text{id}$ by definition if $k = 0$.

Remark. The operators $B_p$, $p = 1, \ldots, k$, are commuting idempotents: $B_p^2 = B_p$ and $B_p B_q = B_q B_p$ for all $p, q = 1, \ldots, k$. Therefore, $\Delta$ is an idempotent, too: $\Delta^2 = \Delta$.

Let $W = (w_{ij})_{i,j=1}^n$ be a $n \times n$-matrix, like in Example 1.4 and Proposition 1.5. Denote by $\hat{W}$ the corresponding Laplace matrix, i.e. a matrix with nondiagonal elements $w_{ij}$ ($1 \leq i \neq j \leq n$) and diagonal elements $-\sum_{i \neq j} w_{ij}$ ($1 \leq i \leq n$). It follows from Definition 1.6 that

$$\langle \hat{W} | X \rangle = \langle W | \Delta(X) \rangle$$

for any $X \in \mathcal{G}_{n,k}$. This equation explains the name “Laplace operator” for $\Delta$. Note that since $\Delta(X)$ is a sum of graphs containing no loops, one is free to change diagonal entries of $W$ in the right-hand side; in particular, one can use $\hat{W}$ instead.

The main results of this paper are the following two theorems:

Theorem 1.7 (abstract matrix-tree theorem for diagonal minors).

(1.4) $\Delta(\det_{i,k}^I) = \frac{(-1)^n}{k!} U(\mathcal{G}_{n,k}^I)$. and

Theorem 1.8 (abstract matrix-tree theorem for codimension 1 minors).

(1.5) $\Delta(\det_{i,j}^{I,j}) = \frac{(-1)^n}{k!} U(\mathcal{G}_{n,k}^{I,j})$.

Applying the functional $\langle \hat{W} | \rangle$ to equation (1.4) with $k = n - s$ and to equation (1.5) with $k = n - 1$ and using Examples 1.4 and 1.2 one obtains

Corollary 1.9. The diagonal minor of the Laplace matrix obtained by deletion of the rows and columns numbered $i_1, \ldots, i_s$ is equal to $\frac{(-1)^{n-s}}{(n-s)!} U(\mathcal{G}_{n,n-s}^I)$, that is, to $(-1)^{n-s}$ times the sum of monomials $w_{a_1b_1} \ldots w_{a_{n-s}b_{n-s}}$ such that the graph $([a_1b_1], \ldots, [a_{n-s}b_{n-s}])$ is a $s$-component forest where every component contains exactly one vertex $i_\ell$ for some $\ell = 1, \ldots, s$, and all the edges of the component are directed towards $i_\ell$.

and

Corollary 1.10. The minor of the Laplace matrix obtained by deletion of its $i$-th row and its $j$-th column is equal to $(-1)^{n-1}$ times the sum of monomials $w_{a_1b_1} \ldots w_{a_{n-1}b_{n-1}}$ such that the graph $([a_1b_1], \ldots, [a_{n-1}b_{n-1}])$ is a tree with all the edges directed towards the vertex $i$.

Corollaries 1.9 and 1.10 are particular cases of the celebrated matrix-tree theorem first discovered by G. Kirchhoff [3] in 1847 (for symmetric matrices and diagonal minors of codimension 1) and proved in its present form by W. Tutte [7].
Consider now the following functions on $\Gamma_{n,k}$:

$$
\sigma(G) = \begin{cases} (-1)^{\beta_0(G)}, & G \in \Gamma_{n,k} \text{ is strongly semiconnected,} \\ 0, & \text{otherwise}, \end{cases}
$$

and

$$
\alpha(G) = \begin{cases} (-1)^k, & G \in \Gamma_{n,k} \text{ is acyclic,} \\ 0, & \text{otherwise}. \end{cases}
$$

Theorem 1.17 follows from the two equivalent statements (see Section 2 for details):

**Theorem 1.11.** $S(\alpha; G) = (-1)^k \sigma(G)$ for $G \in \Gamma_{n,k}$.

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Here $S$ is summation over subgraphs, as defined by (1.1). These theorems are essentially [1, Proposition 6.16]. We will nevertheless give their proofs in Section 2 thus answering a request for a direct proof expressed in [1] (the original proof in [1] is a specialization of a much more general identity).

1.4. A digression: undirected graphs and the universal Potts partition function. Denote by $\Upsilon_{n,k}$ the set of all undirected graphs with $n$ vertices numbered $1, \ldots, n$ and $k$ edges numbered $1, \ldots, k$. Denote by $| \cdot | : \Gamma_{n,k} \to \Upsilon_{n,k}$ the “forgetful” map replacing every edge by its undirected version; the edge numbering is preserved. By $\Upsilon_{n,k}$ denote a vector space spanned by $\Upsilon_{n,k}$; then $| \cdot |$ is extended to the linear map $\mathcal{G}_{n,k} \to \Upsilon_{n,k}$. The notion of a subgraph and the notation $S$ (see (1.1)) for undirected graphs are similar to those for $\mathcal{G}_{n,k}$. One can also define the operators $B_p : \Upsilon_{n,k} \to \Upsilon_{n,k}$, $p = 1, \ldots, k$, and the Laplace operator $\Delta : \Upsilon_{n,k} \to \Upsilon_{n,k}$ for undirected graphs exactly as in Definition 1.6.

For any $G \in \Upsilon_{n,k}$ consider the two-variable polynomial:

$$(1.6) \quad Z_G(q, v) = S(q^{\beta_0(H)} v^{|\text{# of edges of } H|} G).$$

called Potts partition function. It is related [6, Eq. (2.26)] to the Tutte polynomial $T_G$ of the graph $G$ as

$$
T_G(x, y) = (x - 1)^{-\beta_0(G)} (y - 1)^{-n} Z((x - 1)(y - 1), y - 1; G).
$$

Values of $Z_G$ in some points have a special combinatorial interpretation, in particular

**Proposition 1.12.** [9, Eq. (8) and (10)].

$$
Z_G(-1, 1) = (-1)^{\beta_0(G)} 2^{\text{# of loops of } G} \# \{ \Phi \in \mathcal{S}_{n,k} \subset \Gamma_{n,k} \mid |\Phi| = G \},
$$

$$
Z_G(-1, -1) = (-1)^n \# \{ \Phi \in \mathcal{A}_{n,k} \subset \Gamma_{n,k} \mid |\Phi| = G \}.
$$

(Recall that by $\mathcal{S}_{n,k}$ and $\mathcal{A}_{n,k}$ we denote the sets of all strongly semiconnected and acyclic graphs in $\Gamma_{n,k}$, respectively.)

**Corollary 1.13.** For any graph $G \in \Upsilon_{n,k}$ one has

$$
\# \{ \Phi \in \mathcal{S}_{n,k} \subset \Gamma_{n,k} \mid |\Phi| = G \} = (-1)^{\beta_0(G)} Z_{\hat{G}}(-1, 1)
$$

where $\hat{G}$ is the graph $G$ with all the loops deleted.
The definition (1.6) of the Potts partition function implies immediately that $Z_G(q,v) = (v+1)^{\#\text{loops}} Z_G(q,v)$. □

Consider now the \textit{universal} Potts partition function $Z_{n,k}(q,v) \overset{\text{def}}{=} \sum_{G \in \Upsilon_{n,k}} Z_G(q,v)$ and its “shaved” version $\hat{Z}_{n,k}(q,v) \overset{\text{def}}{=} \sum_{G \in \Upsilon_{n,k}} \hat{Z}_G(q,v)$.

**Proposition 1.14.**

$\Delta \hat{Z}_{n,k}(-1,1) = (-1)^k Z_{n,k}(-1,-1)$.

Note that by Proposition 1.12 the right-hand side of the equality contains only graphs without loops, as does the left-hand side.

**Proof.** Corollary 1.13 implies that $\hat{Z}_{n,k}(-1,1) = (-1)^k k! \sum_{I \subseteq \{1,\ldots,n\}} |\det^I_{n,k}|$.

Apply now the Laplace operator $\Delta$ to both sides of the equality. Apparently, $\Delta$ commutes with the forgetful map: $|\Delta(x)| = |\Delta(x)|$ for any $x \in \Upsilon_{n,k}$. Therefore by Theorem 1.7 and Proposition 1.12

$\Delta \hat{Z}_{n,k}(-1,1) = (-1)^k k! \sum_{I \subseteq \{1,\ldots,n\}} |\Delta \det^I_{n,k}| = (-1)^{n+k} \sum_{I \subseteq \{1,\ldots,n\}} |U(\mathcal{A}^I_{n,k})|$  

$= (-1)^k Z_{n,k}(-1,-1)$.

□

Proposition 1.14 admits several generalizations. The author is planning to write a separate paper considering action of the Laplace operator on the universal Potts functions and their oriented-graph versions.

**1.5. An application: invariants of 3-manifolds.** Universal determinants have an application in 3-dimensional topology, due to M. Polyak. We describe it briefly here; see [4] and the MSc. thesis [2] for detailed definitions, formulations and proofs.

A \textit{chainmail graph} is defined as a planar graph, possibly with loops but without parallel edges; the edges (including loops) are supplied with integer weights. We denote by $w_{ij} = w_{ji}$ the weight of the edge joining vertices $i$ and $j$; $w_{ii}$ is the weight of the loop attached to the vertex $i$. If the edge $[ij]$ is missing then $w_{ij} = 0$ by definition.

There is a way (see [4]) to define for every chainmail graph $G$ a closed oriented 3-manifold $M(G)$; any closed oriented 3-manifold is $M(G)$ for some $G$ (which is not unique). To the chainmail graph $G$ with $n$ vertices one associates two $n \times n$-matrices: the adjacency matrix $W(G) = (w_{ij})$ and the Laplace (better to say, Schroedinger) matrix $L(G) = (l_{ij})$ where $l_{ij} \overset{\text{def}}{=} w_{ij}$ for $i \neq j$ and $l_{ii} \overset{\text{def}}{=} w_{ii} - \sum_{j \neq i} w_{ij}$. If all $w_{ii} = 0$ (such $G$ is called a balanced graph) then $L(G)$ is the usual (symmetric, degenerate) Laplace matrix $\hat{W}$ from Section 1.3.

**Theorem ([4]; see details of the proof in [2]).**
(1) The rank of the homology group $H_1(\mathcal{M}(G),\mathbb{Z})$ is equal to $\dim \ker L(G)$.

(2) If $L(G)$ is nondegenerate (so that $\mathcal{M}(G)$ is a rational homology sphere and $H_1(\mathcal{M}(G),\mathbb{Z})$ is finite) then

\[ |H_1(\mathcal{M}(G),\mathbb{Z})| = |\det L(G)| = |\langle L(G) \mid \det_{n,n}^G \rangle|. \]

(3) If $L(G)$ is nondegenerate then

\[ \langle W(G) \mid \Theta_n \rangle = 12 \det L(G)(\lambda_{\text{CW}}(\mathcal{M}(G)) - \frac{1}{4} \text{sign}(L(G))) \]

where $\lambda_{\text{CW}}$ is the Casson–Walker invariant [8] of the rational homology sphere $\mathcal{M}(G)$, sign is the signature of the symmetric matrix $L(G)$, and $\Theta_n$ is an element of $\mathcal{G}_{n,n+1} \oplus \mathcal{G}_{n,n-1}$ defined as

\[ \Theta_n \overset{\text{def}}{=} \det_{n,n+1}^G - \sum_{1 \leq i \neq j \leq n} (\langle ij \rangle) * \det_{n,n-2}^{\langle ij \rangle} - \sum_{i=1}^{n} \det_{n,n-1}^{\langle i \rangle}. \]

Conjecturally, (1.7) and (1.8) begin a series of formulas for invariants of 3-manifolds. See [4] for details.

Applying $\Delta$ to the element $\Theta_n$ and using Theorem 1.7 and Corollary 1.9 one obtains

**Corollary 1.15.** $\Delta \Theta_n = -2U(\mathcal{G}_{n,n-1})$. Therefore if $G$ is balanced then $\langle L(G) \mid \Theta_n \rangle$ is equal to $-2$ times the codimension 1 diagonal minor of $L(G)$.

The last assertion is [2, Theorem 84].

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2. Proofs

We start with proving Proposition 1.5 (Section 2.1), to continue with Theorems 1.11 and 1.11’ (Sections 2.2 and 2.3). Theorem 1.7 will then follow from Theorem 1.11 (Section 2.4), and Theorem 1.8 from Theorem 1.7 and assertion 1 of Proposition 1.5 (Section 2.5).

For two vertices $a, b \in \Gamma_{n,k}$ we will write $a \succeq b$ if $G$ contains a directed path starting at $a$ and finishing at $b$; also $a \succeq a$ for any $a$ by definition.

2.1. Proof of Proposition 1.5

Proof of assertion 1. Denote by $\mathcal{S}_{n,k}^{0,0}$ the set of all graphs $G \in \mathcal{S}_{n,k}$ having $q$ loops ($0 \leq q \leq k$) attached to vertex $i$. The graph $\tilde{G}$ obtained from $G$ by deletion of all these loops (with the relevant renumbering of the remaining edges) belongs either to $\mathcal{S}_{n,k-q}^{i} \subset \mathcal{S}_{n,k-q}^{0}$ or, if $q > 0$, to $\mathcal{S}_{n,k-q}^{i}$. Vice versa, if $q > 0$ and
\[ G \in \mathcal{S}_{n,k}^{[\mu]} \cup \mathcal{S}_{n,k}^{\{i\}} \] then \( G \in \mathcal{S}_{n,k}^{[\mu]} \). Deletion of a loop does not break a graph, so \( \beta_0(G) = \beta_0(\hat{G}) \).

If \( G \in \mathcal{S}_{n,k}^{[\mu]} \) then there are \( \binom{k}{i} \) ways to assign numbers to the loops of \( G \) attached to \( i \). Since \( \langle W \mid G \rangle \) does not depend on the edge numbering, one has for \( q > 0 \)

\[
\langle W \mid X(\mathcal{S}_{n,k}^{[\mu]}) \rangle = \binom{k}{q} w_{ii}^q \langle W \mid X(\mathcal{S}_{n,k}^{\{i\}}) + X(\mathcal{S}_{n,k,q}^{\{i\}}) \rangle,
\]

so that

\[
\langle W \mid \det(\mathcal{S}_{n,k}) \rangle = \frac{1}{k!} \sum_{q=0}^{k} \langle W \mid X(\mathcal{S}_{n,k}^{[\mu]}) \rangle
\]

\[
= \frac{1}{k!} \langle W \mid X(\mathcal{S}_{n,k}^{[\mu]}) \rangle + \sum_{q=1}^{k} \frac{w_{ii}^q}{q!(k-q)!} \langle W \mid X(\mathcal{S}_{n,k}^{\{i\}}) + X(\mathcal{S}_{n,k,q}^{\{i\}}) \rangle
\]

\[
= \sum_{q=0}^{k} \frac{w_{ii}^q}{q!(k-q)!} \langle W \mid X(\mathcal{S}_{n,k}^{[\mu]}) \rangle + X(\mathcal{S}_{n,k,q}^{\{i\}}) \rangle - \langle W \mid \det(\mathcal{S}_{n,k}) \rangle.
\]

The expressions \( \langle W \mid X(\mathcal{S}_{n,k}^{[\mu]}) \rangle + X(\mathcal{S}_{n,k,q}^{\{i\}}) \rangle \) and \( \langle W \mid \det(\mathcal{S}_{n,k}) \rangle \) do not contain \( w_{ii} \). So, applying the operator \( \partial w_{ii}^m \) to equation (2.1) and using the equation again with \( k - m \) in place of \( k \) one gets (13).

2.2. Theorems 1.11 and 1.11 are equivalent. Denote by \( E(G) \) the set of edges of the graph \( G \in \Gamma_{n,k}. \) The functions \( \alpha \) and \( \sigma \) do not depend on the edge numbering; so the summation in the left-hand side of both theorems is performed over the set \( 2^{E(G)} \) of subsets of \( E(G) \). The equivalence of the theorems is now a particular case of the Moebius inversion formula [5]. Namely, for any finite set \( X \) the Moebius function of the set \( 2^X \) partially ordered by inclusion is \( \mu(S,T) = (-1)^{\#(S,T)} \), where \( S,T \subseteq X \). Therefore one has

\[
S(\sigma;G) = (-1)^k \alpha(G)
\]

\[
\iff S(\mu(G,H)(-1)^{\#\text{edges of } H\alpha(H);G) = \sigma(G)}
\]

\[
\iff S((-1)^{-k} \#\text{edges of } H(-1)^{\#\text{edges of } H\alpha(H);G) = \sigma(G)}
\]

\[
\iff S(\alpha(H);G) = (-1)^k \sigma(G).
\]

2.3. Proof of Theorem 1.11. To prove the theorem we use simultaneous induction by the number of vertices and the number of edges of the graph \( G \). If \( \mathcal{R} \) is some set of subgraphs of \( G \) (different in different cases) and \( \chi_{\mathcal{R}} \) is the characteristic function of this set then for convenience we will write \( S(f,\mathcal{R}) = S(f\chi_{\mathcal{R}},G) = \sum_{H \in \mathcal{R}} H f(H) \) for any function \( f \) on the set of subgraphs.

Consider now the following cases:

2.3.1. \( G \) is disconnected. Let \( G = G_1 \sqcup \cdots \sqcup G_m \) where \( G_i \) are connected components. A subgraph \( H \subset G \) is acyclic if and only if the intersection \( H_i = H \cap G_i \) is acyclic for all \( i \). Hence \( \alpha(H) = \alpha(H_1) \cdots \alpha(H_m) \), and therefore \( S(\alpha,G) = S(\alpha,G_1) \cdots S(\alpha,G_m) \). By the induction hypothesis \( S(\alpha,G_i) = (-1)^{k_i} \sigma(G_i) \) where \( k_i \) is the number of edges of \( G_i \). So

\[
S(\alpha,G) = S(\alpha,G_1) \cdots S(\alpha,G_m) = (-1)^{k_1 + \cdots + k_m} \sigma(G_1) \cdots \sigma(G_m) = (-1)^k \sigma(G).
\]
Now it will suffice to prove Theorem 1 for connected graphs $G$.

2.3.2. $G$ is connected and not strongly connected. In this case $G$ contains an edge $e$ which is not contained in any directed cycle. For such $e$ if $H \subset G$ is acyclic and $e \notin H$ then $H \cup \{e\}$ is acyclic, too. The converse is true for any $e$: if an acyclic $H \subset G$ contains $e$ then $H \setminus \{e\}$ is acyclic. Therefore

\[ S(\alpha, G) = \sum_{H \subset G \setminus \{e\}, \text{H acyclic}} (-1)^{\text{edges of } H} + (-1)^{\text{edges of } H \cup \{e\}} = 0 = \sigma(G). \]

So it will suffice to prove Theorem 1 for strongly connected graphs $G$.

2.3.3. $G$ is strongly connected and contains a crucial edge. We call an edge $e$ of a strongly connected graph $G$ crucial if $G \setminus \{e\}$ is not strongly connected. Suppose $e = [ab] \in G$ is a crucial edge.

Denote by $R_e^-$ (resp., $R_e^+$) the set of all subgraphs $H \subset G$ such that $e \notin H$ (resp., $e \in H$). Let $H \in R_e^-$ be acyclic. Since $G \setminus \{e\}$ is not strongly connected and contains one edge less than $G$, one has by Clause [2.3.2] above

\[ S(\alpha, R_e^-) = S(\alpha, G \setminus \{e\}) = 0. \]

Let now $H \in R_e^+$ be acyclic; such $H$ contains no directed paths joining $b$ with $a$. Since $G \setminus \{e\}$ is not strongly connected, $G \setminus \{e\}$ does not contain a directed path joining $a$ with $b$ either. It means that such path in $H$ will necessarily contain $e$, and therefore the graph $H/e \subset G/e$ (obtained by contraction of the edge $e$) is acyclic. The converse is true for any $e$: if $e \in H$ and $H/e \subset G/e$ is acyclic then $H \subset G$ is acyclic, too. The graph $G/e$ is strongly connected, contains one edge less (and one vertex less) than $G$, and $\beta_1(G/e) = \beta_1(G)$, so $\sigma(G/e) = \sigma(G)$. The graph $H/e$ contains one edge less than $H$, so $\alpha(H/e) = -\alpha(H)$. Now by the induction hypothesis

\[ S(\alpha, R_e^+) = -S(\alpha, G/e) = -(-1)^{k-1}\sigma(G/e) = (-1)^k\sigma(G), \]

and then (2.2) implies

\[ S(\alpha, G) = S(\alpha, R_e^-) + S(\alpha, R_e^+) = 0 + (-1)^k\sigma(G) = (-1)^k\sigma(G). \]

2.3.4. $G$ is strongly connected and contains no crucial edges. Let $e = [ab] \in G$ be an edge and not a loop: $b \neq a$. Recall that $G_e^\vee$ will denote a graph obtained from $G$ by reversal of the edge $e$. Since $e$ is not a crucial edge, $G \setminus \{e\} = G_e^\vee \setminus \{e\}$ is strongly connected. So $G_e^\vee$ is strongly connected, too, implying $\sigma(G_e^\vee) = \sigma(G)$.

**Lemma 2.1.** If the graph $G$ is strongly connected and $e = [ab] \in G$ is not a crucial edge then $S(\alpha, G) = \sigma(G)$ if and only if $S(\alpha, G_e^\vee) = \sigma(G_e^\vee) = \sigma(G)$.

**Proof.** Acyclic subgraphs $H \subset G$ are split into five classes:

I. $e \notin H$, but $a \geq b$ in $H$ (that is, $H$ contains a directed path joining $a$ with $b$).
II. $e \notin H$, but $b \geq a$ in $H$.
III. $e \notin H$, and both $a \not\geq b$ and $b \not\geq a$ in $H$.
IV. $e \in H$, and $a \geq b$ in $H \setminus \{e\}$.
V. $e \in H$, and $a \not\geq b$ in $H \setminus \{e\}$.
Obviously, \( H \in \mathcal{I} \) if and only if \( H \cup \{e\} \in \mathcal{V} \). The number of edges of \( H \cup \{e\} \) is the number of edges of \( H \) plus 1, so
\[
S(\alpha, \mathcal{I} \cup \mathcal{V}) = \sum_{H \in \mathcal{I}} (-1)^{\# \text{ of edges of } H} (1 - 1) = 0.
\]

Also, \( H \in \mathcal{III} \) if and only if \( H \cup \{e\} \in \mathcal{V} \) and similar to (2.3) one has \( S(\alpha, \mathcal{III} \cup \mathcal{V}) = 0 \), and therefore
\[
S(\alpha, G) = S(\alpha, \mathcal{III} \cup \mathcal{III} \cup \mathcal{IV} \cup \mathcal{V}) = S(\alpha, \mathcal{II}).
\]

Like in Clause 2.3.3 if \( H \in \mathcal{V} \) then \( H/e \subset G/e \) is acyclic, and vice versa, if \( e \in H \) and \( H/e \subset G/e \) is acyclic then \( H \in \mathcal{V} \). The graph \( G/e \) is strongly connected, so by the induction hypothesis \( S(\alpha, \mathcal{V}) = -S(\alpha, G/e) = (-1)^{k-1} \sigma(G/e) = (-1)^k \sigma(G) \), hence \( S(\alpha, \mathcal{III}) = (-1)^k \sigma(G) \).

If \( e \notin H \) and \( H \) is acyclic, then \( H \) is an acyclic subgraph of the strongly connected graph \( G \setminus \{e\} \). The graph \( G \) is strongly connected, too, so \( e \) enters a cycle, and \( \beta_1(G \setminus \{e\}) = \beta_1(G) - 1 \), which implies \( \sigma(G \setminus \{e\}) = -\sigma(G) \). The graph \( G \setminus \{e\} \) contains \( k - 1 < k \) edges, so by the induction hypothesis
\[
S(\alpha, \mathcal{II} \cup \mathcal{II} \cup \mathcal{III}) = S(\alpha, G \setminus \{e\}) = (-1)^{k-1} \sigma(G \setminus \{e\}) = (-1)^k \sigma(G),
\]
and therefore
\[
S(\alpha, G) = S(\alpha, G') = 2(-1)^k \sigma(G) = (-1)^k (\sigma(G) + \sigma(G')), \tag{2.5}
\]

A subgraph \( H \subset G \) of class \( \mathcal{I} \) is at the same time a subgraph \( H \subset G_e' \) of class \( \mathcal{IV} \). So, (2.4) applied to \( G_e' \) gives \( S(\alpha, \mathcal{I}) = S(\alpha, G_e') \). It follows now from (2.4) and (2.5) that
\[
S(\alpha, G) + S(\alpha, G_e') = 2(-1)^k \sigma(G) = (-1)^k (\sigma(G) + \sigma(G_e')), \tag{2.6}
\]

which proves the lemma.

To complete the proof of Theorem 1.11 let \( a \) be a vertex of \( G \), and let \( e_1, \ldots, e_m \) be the complete list of edges finishing at \( a \). Consider the sequence of graphs \( G_0 = G, G_1 = G_{e_1}', G_2 = (G_1)'_{e_2}, \ldots, G_m = (G_{m-1})'_{e_m} \). The graphs \( G_0 \) and \( G_1 \) are strongly connected; the graph \( G_m \) is not, because \( a \not\prec b \) for any \( b \neq a \) in it. Take the maximal \( \ell \) such that \( G_\ell \) is strongly connected. Since \( \ell < m \), the graph \( G_{\ell+1} \) exists and is not strongly connected, and therefore \( G_\ell \setminus \{e_{\ell+1}\} = G_{\ell+1} \setminus \{e_{\ell+1}\} \) is not strongly connected either. So, the edge \( e_{\ell+1} \) is crucial for the graph \( G_\ell \), and by Clause 2.3.3 one has \( S(\alpha, G_\ell) = (-1)^k S(G_\ell) = (-1)^k \sigma(G) \). The graphs \( G_0 = G, \ldots, G_\ell \) are strongly connected, so for any \( i = 0, \ldots, \ell - 1 \) the edge \( e_{\ell+1} \) is not crucial for the graph \( G_i \). Lemma 2.4 implies now
\[
S(\alpha, G_{\ell-1}) = (-1)^k \sigma(G_{\ell-1}) \Rightarrow S(\alpha, G_{\ell-2}) = (-1)^k \sigma(G_{\ell-2}) \Rightarrow \cdots \Rightarrow S(\alpha, G) = (-1)^k \sigma(G).
\]

Theorem 1.11 is proved.

2.4. Theorem 1.7 follows from Theorem 1.11. Note first that the operation \( B_i \) and hence \( \Delta \), preserves the sinks of the graph: if \( \Delta H = \sum_G x_G G \) and \( x_G \neq 0 \) then \( G \) has the same sinks as \( H \). Therefore if \( I = \{i_1, \ldots, i_s\} \) then \( \Delta(\det_{a}^{x}) = \sum_G x_G G \) where all the graphs \( G \) in the right-hand side have the sinks \( i_1, \ldots, i_s \) and have no loops.
Let $G$ be a graph with sinks $i_1, \ldots, i_s$ and without loops, and let $\Phi \in S_{n,k}$ (a strongly semiconnected graph with the isolated vertices $i_1, \ldots, i_s$). Denote by $\hat{\Phi}$ the graph $\Phi$ with the loops deleted. A contribution of $\Phi$ into $x_G$ is equal to $\frac{1}{k!}(-1)^{\beta_0(\Phi)} \cdot \# \text{ of loops in } \Phi + n$ if $\hat{\Phi} \subset G$ and is 0 otherwise.

The number of edges of $\hat{\Phi}$ is $k - \# \text{ of loops of } \Phi$. The graph $\hat{\Phi}$ is strongly semiconnected if and only if $\Phi$ is. The Euler characteristics of $\hat{\Phi}$ is $\beta_0(\hat{\Phi}) - \beta_1(\hat{\Phi}) = n - \# \text{ of edges of } \hat{\Phi} = n - k + \# \text{ of loops of } \Phi$

and $\beta_0(\hat{\Phi}) = \beta_0(\Phi)$. Therefore, the contribution of $\Phi$ into $x_G$ is $\frac{(-1)^n}{k!} \cdot S(\sigma; G) = \frac{1}{k!} \alpha(G)$

by Theorem 1.11. This proves Theorem 1.7.

2.5. Proof of Theorem 1.8. Note that $\det_{n,k}^{i/i} = \det_{n,k}^\emptyset + \det_{n,k}^\{i\}$. Applying the operator $\Delta$ to equation (1.2) and using Theorem 1.7 with $I = \emptyset$ and $I = \{i\}$ one obtains

$0 = \sum_{i,j=1}^{n} \Delta([ij]) \cdot \det_{n,k}^{ij} = \sum_{i=1}^{n} \Delta([ii]) \cdot \Delta(\det_{n,k}^\emptyset + \det_{n,k}^\{i\}) + \sum_{i,j=1}^{n} (\delta_{ij}) \cdot \Delta(\det_{n,k}^{ij})$

$= \sum_{i,j=1}^{n} \Delta(\det_{n,k}^{ij} - \det_{n,k}^{ij}) = \sum_{i,j=1}^{n} (\delta_{ij}) \cdot \Delta(\det_{n,k}^{ij}) - \frac{(-1)^k}{k!} U(\Phi_{n,k})^\{i\}.$

The $(i,j)$-th term of the identity above consists of graphs where the edge $[ij]$ carries the number 1. Therefore different terms of the identity cannot cancel, so every single term is equal to 0.

References

[1] J. Awan and O. Bernardi, Tutte polynomials for directed graphs, arXiv:1610.01839v2.
[2] B. Epstein, A combinatorial invariant of 3-manifolds via cycle-rooted trees, MSc. thesis (under supervision of prof. M. Polyak), Technion, Haifa, Israel, 2015.
[3] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung det linearen Verteilung galvanischer Ströme geführt wird, Ann. Phys. Chem., 72 (1847), S. 497–508.
[4] M. Polyak, From 3-manifolds to planar graphs and cycle-rooted trees, talk at Arnold’s legacy conference, Fields Institute, Toronto, 2014.
[5] G.-C. Rota, On the foundations of combinatorial theory I: Theory of Mobius functions, Z. Wahrsch. Verw. Gebiete, 2 (1964) pp. 340–368.
[6] A. Sokal, The multivariate Tutte polynomial (alias Potts model) for graphs and matroids, in: Surveys in Combinatorics 2005, Cambridge University Press, Jul 21, 2005 — Mathematics — 258 pages.
[7] W.T. Tutte, The dissection of equilateral triangles into equilateral triangles, Math. Proc. Cambridge Phil. Soc., 44 (1948) no. 4, pp. 463–482.
[8] K. Walker, An extension of Casson’s invariant, Annals of Mathematics Studies, 126, Princeton University Press, 1992.
[9] D.J.A. Welsh and C. Merino, The Potts model and the Tutte polynomial, J. of Math. Physics, 41 (2000), no. 3, pp. 1127–1152.
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