SYMPLECTIC REALIZATIONS OF HOLOMORPHIC POISSON MANIFOLDS

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ABSTRACT. Symplectic realization is a longstanding problem which can be traced back to Sophus Lie. In this paper, we present an explicit solution to this problem for an arbitrary holomorphic Poisson manifold. More precisely, for any holomorphic Poisson manifold \((X, \pi)\), we prove that there exists a holomorphic symplectic structure in a neighborhood \(Y\) of the zero section of \(T^*X\) such that the projection map is a symplectic realization of the given Poisson manifold, and moreover the zero section is a holomorphic Lagrangian submanifold. We describe an explicit construction for such a new holomorphic symplectic structure on \(Y \subseteq T^*X\).

1. Introduction

The notion of “symplectic realizations" can be traced back to Sophus Lie who used the name “function group". In \[18\], Lie defined a “function group" as a collection of functions of the canonical variables \((q_1, \ldots, q_n, p_1, \ldots, p_n)\) which is a subalgebra under the canonical Poisson bracket and generated by a finite number of independent functions \(\phi_1, \ldots, \phi_r\). In modern language, this means that \(C^r\) has a Poisson structure induced from the canonical symplectic structure \(C^{2n}\) in the sense that \(\Phi = (\phi_1, \ldots, \phi_r) : C^{2n} \to C^r\) is a Poisson map. In the \(C^\infty\)-context, a symplectic realization of a Poisson manifold \(M\), as defined by Weinstein \[29\] (called a full symplectic realization), is a Poisson map from a symplectic manifold \(V\) to \(M\) which is a surjective submersion. Since Sophus Lie’s treasure work on the theory of transformation group \[18\], the following has become a central question:

Problem A. Does a symplectic realization always exist for a given Poisson manifold?

In fact, this question is closely related to Lie’s theory on Lie groups. To get a flavor on this, consider the Lie Poisson manifold \(g^T\) corresponding to a Lie algebra \(g\). A natural choice of a symplectic realization is \(q : T^*G \to g^T\) with the canonical cotangent symplectic structure on \(T^*G\) and \(q\) being the left translation, where \(G\) is a Lie group with Lie algebra \(g\), and \(g^T \cong T^*_eG\). Lie himself proved that symplectic realization always exists locally for any smooth Poisson manifold of constant rank \[18\]. The local existence theorem of symplectic realizations for general \(C^\infty\) Poisson manifolds was proved by Weinstein in 1983 \[29\]. Subsequently, Karasev \[9\] and Weinstein \[30\] independently proved the global existence theorem by gluing methods. Indeed they proved a stronger result: for any general \(C^\infty\) Poisson manifold, there exists an essentially unique distinguished symplectic realization, which possesses a local groupoid structure (in general there is an obstruction for the existence of a global groupoid structure \[3\]) compatible with the symplectic structure \[30\], which is now called a symplectic local groupoid. Its Lie algebroid, the infinitesimal of a local Lie groupoid as introduced by Pradines \[25\], of this local Lie groupoid can be proved, as shown by Coste-Dazord-Weinstein \[2\], to be isomorphic to the cotangent bundle of the Lie groupoid \((T^*M)_\pi\), which is canonically associated to the given Poisson manifold \((M, \pi)\). The bracket of this Lie algebroid \((T^*M)_\pi\) essentially extends the natural Lie bracket relation on exact forms: \([df, dg] = d\{f, g\}\) in an obvious way. For a Poisson manifold \((M, \pi)\), the pair of Lie
algebroids \(((T^* M)_\pi, TM)\) constitutes an example of the so called Lie bialgebroids [20]. From the theory of integration of Lie bialgebroids of Mackenzie–Xu [21] (which extends the classical theory of Drinfeld [6, 7] for integrating Lie bialgebras), it follows that, under some topological assumption, a Lie groupoid with Lie algebroid \((T^* M)_\pi\) carries automatically a compatible symplectic structure, and therefore is a symplectic groupoid. As a consequence, a local Lie groupoid with Lie algebroid \((T^* M)_\pi\), which always exists according to [25], is automatically a symplectic realization of the underlying Poisson manifold. In this way, the Mackenzie–Xu integration method provided an alternative proof of the existence of global symplectic realizations [21]. However, all these results are existence results and are not constructive. In 2001, based on their study of Poisson sigma models, Cattaneo–Felder discovered an explicit construction of the local symplectic groupoid, and therefore a symplectic realization, of an arbitrary Poisson manifold \((M, \pi)\) as a certain quotient space of the so called A-paths on the Lie algebroid \((T^* M)_\pi\) [1].

This important result of Cattaneo–Felder and their local symplectic groupoids inspired many important works in Poisson geometry in the past 15 years, among which the solution to the problem of integrability of Lie algebroids by Crainic-Fern [3].

Although there has been a lot of works on symplectic realizations in the \(C^\infty\)-context, very little is known in the holomorphic context. A holomorphic Poisson manifold is a complex manifold \(X\) whose sheaf of holomorphic functions \(O_X\) is a sheaf of Poisson algebras. Symplectic realizations can be defined in a similar fashion as in the \(C^\infty\) case. A natural question then is

**Problem B.** Does a symplectic realization always exist for a given holomorphic Poisson manifold? And, if so, is it possible to describe an explicit construction of a certain class of distinguished ones?

To any holomorphic Poisson manifold \(X\), one associates two \(C^\infty\) Poisson bivector fields. To see this, one writes the holomorphic Poisson tensor \(\pi \in \Gamma(\wedge^2 \mathfrak{t}^{1,0} X)\) as \(\pi_R + i\pi_I\), where \(\pi_R, \pi_I \in \Gamma(\wedge^2 \mathfrak{t} X)\) are bivector fields on the underlying real manifold \(X\). Then both \(\pi_R\) and \(\pi_I\) are \(C^\infty\) Poisson bivector fields [16]. In 2009, Laurent-Gengoux, Stiénon and Xu proved that a holomorphic Poisson manifold is integrable if and only if either \((X, \pi_R)\), or \((X, \pi_I)\) are integrable as a real \(C^\infty\) Poisson manifold (Theorem 3.22 [17]). Since any \(C^\infty\) Poisson manifold admits a symplectic local groupoid, as a consequence, this result of Laurent-Gengoux, Stiénon and Xu implies that symplectic realizations do exist for any holomorphic Poisson manifolds. However, the conclusion is not constructive. The purpose of the present paper is to describe an explicit construction of such a holomorphic symplectic local groupoid and therefore gives an affirmative answer to Problem B.

Our approach is based on the observation that a holomorphic Poisson manifold \((X, \pi)\) gives rise to a Poisson–Nijenhuis [13, 22] structure \((X, \pi_I, J)\) on the underlying real manifold \(X\) such that \(\pi^2_R = \pi^2_I \circ J^T\) [16], where \(J : TX \rightarrow TX\) is the underlying almost complex structure. Indeed, holomorphic Poisson manifolds are equivalent to a special class of Poisson–Nijenhuis manifolds, i.e. those where the Nijenhuis tensor is almost complex. Therefore, holomorphic symplectic local groupoids are equivalent to a special class of symplectic Nijenhuis local groupoids in the sense of Stiénon–Xu [26]. Our goal is to describe an explicit construction of such a symplectic Nijenhuis local groupoid. For this purpose, it only suffices to construct explicitly two compatible symplectic structures on the local groupoid. At this point, we must mention the recent beautiful work of Crainic-Mărcut [4], where they present a very simple explicit construction of a symplectic realization of an arbitrary \(C^\infty\)-Poisson manifold \((M, \pi)\) on an open neighborhood of \(T^* M\). However, they [4] proved the main result by a direct computation and it remains mysterious how such an explicit formula of symplectic structure can be derived.
Another goal of our paper is to present a conceptual proof of Crainic–Mărcuț theorem. The idea is quite simple indeed. Given a local Lie groupoid $\Sigma$ with Lie algebroid $A$, it is well-known that, by choosing an $A$-connection on $A$, one can construct a local diffeomorphism, called the exponential map, from an open neighborhood of the zero section of $A$ onto an open neighborhood of the unit space in $\Sigma$. Now if $\Sigma$ is a Cattaneo–Felder local symplectic groupoid, its Lie algebroid $A$ is known to be isomorphic to $(T^*M)_\pi$. By pulling back the symplectic form on $\Sigma$ via such an exponential map, one obtains a symplectic form on an open neighborhood of the zero section of $(T^*M)_\pi$. One can verify directly that this coincides with the formula given by Crainic–Mărcuț. By applying a combination of techniques developed in the study of symplectic Nijenhuis local groupoids and the theory of Lie bialgebroids and Poisson groupoids, we are able to describe explicitly the two compatible symplectic structures on the local groupoid as desired, and thus obtain the following main result of the paper.

**Theorem 1.1.** Let $X$ be a holomorphic Poisson manifold with almost complex structure $J$ and holomorphic Poisson tensor $\pi \in \Gamma(\wedge^2 T^{1,0} X)$. By considering $X$ as a real $C^\infty$-manifold, we choose an affine connection $\nabla$ on $X$ and let $\xi \in \mathfrak{X}(T^*X)$ be the associated geodesic vector field on $T^*X$ defined in terms of the holomorphic Poisson tensor (see Definition 4.2 and Section 9 for details). Denote by $\varphi^\nabla_t$ the flow of $\xi$. The following then holds.

(i) There is an open neighborhood $Y \subset T^*X$ of the zero section of $T^*X$ such that the two-forms $\omega_R, \omega_I \in \Omega^2(Y)$ given by

\[
\omega_I(u,v) = \int_0^1 ((\varphi^\nabla_t)^* \omega_{can})(u,v) \, dt
\]

\[
\omega_R(u,v) = -\int_0^1 ((J^* \circ \varphi^\nabla_t)^* \omega_{can})(u,v) \, dt
\]

forall $u,v \in T_Y \lambda$, $\lambda \in Y$, are well defined and symplectic, and the $(1,1)$-tensor

\[
J = (\omega_R^\flat)^{-1} \omega_I^\flat : TY \to TY
\]

is an integrable almost complex structure on $Y$. In particular, $Y$ endowed with $J$ is a complex manifold. Here $\omega_{can} \in \Omega^2(T^*X)$ denotes the standard $C^\infty$-symplectic 2-form on $T^*X$.

(ii) The two-form $\omega \in \Omega^2(Y) \otimes \mathbb{C}$:

\[
\omega(u,v) := \frac{1}{4} (\omega_R(u,v) - i \omega_I(u,v))
\]

is holomorphic symplectic on $Y$ with respect to the new complex structure $J$, and the natural projection $\text{pr}|_Y : Y \to X$ is a holomorphic symplectic realization of the holomorphic Poisson manifold $(X, \pi)$.

(iii) The zero section is a Lagrangian submanifold of $(Y, \omega)$.

Moreover, different choices of the affine connection $\nabla$ give rise to isomorphic holomorphic symplectic realizations.

Note that when the holomorphic Poisson structure is trivial, i.e. $\pi = 0$, $Y$ reduces to the standard holomorphic symplectic structure on $T^*X$. Therefore the holomorphic symplectic manifold $(Y, \omega)$ can be considered as a deformation of the standard holomorphic symplectic manifold $(Y \subset T^*X, \omega_{can}, J)$ (and therefore, in particular, a deformation of the standard complex structure on $T^*X$), that is parameterized by the holomorphic Poisson structure $\pi$. It would be interesting to investigate how our result is related to Kodaira theory of deformation of complex structures.

The present paper was influenced in large measure by Petalidou’s splendid work on symplectic realizations of non-degenerate Poisson–Nijenhuis manifolds. Making use of the computational...
approach of [4], Petalidou discovered an explicit expression for the 2-forms on the symplectic realization. However, her proof of the compatibility is, to the best of our understanding, not entirely sound. In our approach, which is more conceptual, tracing the hidden underlying groupoid structures reveals crucial for proving the compatibility.

Finally, we would like to point out that our approach draws from various integration results valid only in the differential context. It is not clear whether this method will be of any use in the algebraic context. So the analogue of Problem 3 for algebraic Poisson varieties remains open.

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2. Holomorphic Poisson manifolds and symplectic realizations

Definition 2.1. A holomorphic Poisson manifold is a complex manifold $X$ whose structure sheaf $\mathcal{O}_X$ is endowed with a bracket $\{\cdot, \cdot\}_U : \mathcal{O}_X(U) \times \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ such that $(\mathcal{O}_X, \{\cdot, \cdot\})$ is a sheaf of Poisson algebras.

The following two facts are well known [16, 17].

Proposition 2.2. Let $X$ be a complex manifold. Then $X$ is a holomorphic Poisson manifold if and only if there is a bivector field $\pi \in \Gamma(\wedge^2 T^1, 0 X)$ with $\bar{\partial}\pi = 0$ and $[\pi, \pi] = 0$.

Lemma 2.3. Let $X$ be a complex manifold with almost complex structure $J$. Let $\pi = \pi_R + i\pi_I \in \Gamma(\wedge^2 T^1, 0 X)$ where $\pi_R, \pi_I \in \Gamma(\wedge^2 TX)$ are (real) bivectors. The following are equivalent:

(i) the pair $(X, \pi)$ defines a holomorphic Poisson manifold,
(ii) the pair $(\pi_I, J)$ defines a Poisson–Nijenhuis structure on $X$ (see Appendix B) and $\pi^\#_R = \pi^\#_I \circ J^\tau$.

Definition 2.4. A holomorphic Poisson manifold $(X, \pi)$ is called a holomorphic symplectic manifold if $\pi$ is non-degenerate.

Remark 2.5. If $\pi$ is non-degenerate, $\pi^\# : T^* X \to TX$ is an isomorphism of holomorphic vector bundles, and therefore induces an isomorphism $\pi^\# : \wedge^* T^* X \to \wedge^* TX$ between the complex exterior powers. As usual, let $\omega$ be the inverse image of $\pi$ under $\pi^\#$. Then $\omega \in \Omega^{2,0}(X)$ with $d\omega = 0$ and $\bar{\partial}\omega = 0$. Hence a holomorphic symplectic manifold can equivalently be described as a complex manifold endowed with a closed non-degenerate holomorphic two-form.

Definition 2.6. Let $X$ be a holomorphic Poisson manifold. A holomorphic symplectic realization of $X$ is a holomorphic symplectic manifold $Y$ together with a Poisson holomorphic surjective submersion $q : Y \to X$.

Example 2.7. Let $X$ be a complex manifold equipped with the zero Poisson structure. Then $T^* X$ with the canonical symplectic and complex structures is a holomorphic symplectic realization of $X$, where $q : T^* X \to X$ is the projection map.

Example 2.8. Let $\mathfrak{g}$ be a complex Lie algebra. Consider the Lie Poisson $\mathfrak{g}^\tau$, which is a holomorphic Poisson manifold. Let $G$ be a complex Lie group with Lie algebra $\mathfrak{g}$. Then $T^* G$ equipped with the canonical symplectic structure is a symplectic realization of $\mathfrak{g}^\tau$, where $q : T^* G \to \mathfrak{g}^\tau$ is the left translation $T^* G \to T^*_e G \cong \mathfrak{g}^\tau$.
3. Cattaneo–Felder symplectic local groupoids

In this section, we briefly recall the construction of Cattaneo–Felder [11] for the symplectic local groupoid associated to a Poisson manifold \((M, \pi)\). The fundamental idea is to construct it as a certain quotient space of the Banach manifold of all \(A\)-paths in the cotangent algebroid of \(M\). We will first focus on the part of the construction that is valid for an arbitrary Lie algebroid.

Let \(A\) be a Lie algebroid with anchor \(\rho : A \to TM\) and bracket \([\cdot, \cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)\). We will denote by \(\tilde{P}(A)\) the space of \(C^1\)-paths \(I \to A\), where \(I\) is the unit interval \([0, 1]\). Then \(\tilde{P}(A)\) is an infinite dimensional Banach manifold [14].

A \(C^1\)-path \(a : I \to A\) is further said to be an \(A\)-path if

\[
\rho(a(t)) = \frac{d \gamma(t)}{dt},
\]

where \(\gamma(t) = (p \circ a)(t)\) is the base path \((p : A \to M\) is the bundle projection\). The set of \(A\)-paths, denoted by \(P(A)\), is an infinite dimensional Banach submanifold of \(\tilde{P}(A)\). We will write the inclusion \(\iota : P(A) \hookrightarrow \tilde{P}(A)\).

It will be important to have an alternate description for the tangent bundles \(T\tilde{P}(A)\) and \(TP(A)\) of \(\tilde{P}(A)\) and \(P(A)\), respectively. For that, let us define a natural isomorphism (of Banach manifolds)

\[
\tau : T\tilde{P}(A) \to \tilde{P}(TA) : v \mapsto \tau v
\]

in the following way. For a given \(v \in T\tilde{P}(A)\), take any curve \(\theta : I \to \tilde{P}(A) : s \mapsto \theta_s \in \tilde{P}(A)\) such that \(v = \frac{d}{ds}|_{s=0} \theta_s\). Then let

\[
(\tau v)(t) \equiv \frac{d}{ds}|_{s=0} (\theta_s(t)) \in T_{\theta_0(t)}A.
\]

Explicitly, \(\tau\) gives identifications

\[
\tau : T_a\tilde{P}(A) \to \{ X \in \tilde{P}(TA) \mid X(t) \in T_{a(t)}A \}
\]

for all \(a \in \tilde{P}(A)\).

Now, let \(\Sigma \rightrightarrows M\) be a local Lie groupoid with Lie algebroid \(A\), source and target maps \(\alpha, \beta : \Sigma \to M\), and unit map \(\varepsilon : M \to \Sigma\). We now want to recall the well known fact [14] that \(\Sigma \rightrightarrows M\) can be reconstructed as a certain quotient of the \(A\)-path space \(P(A)\). To that end, we first define the map \(\hat{ad} : \Gamma(A) \to \mathfrak{X}(A)\) that will be used to describe the equivalence classes of that quotient.

Let \(\exp : \Gamma(A) \to \text{Bis}(\Sigma \rightrightarrows M)\) be the usual exponential where \(\text{Bis}(\Sigma \rightrightarrows M)\) is the set of local bisections of \(\Sigma \rightrightarrows M\) [19]. Recall that \(\text{Bis}(\Sigma \rightrightarrows M)\) acts on \(A\) by (the derivative of) conjugation. Let us denote, loosely, that action by a map \(\text{Ad} : \text{Bis}(\Sigma \rightrightarrows M) \to \text{Aut}(A)\). Then we let

\[
\hat{ad}(X)|_{a_0} := \frac{d}{dt}|_{t=0} \text{Ad}_{\exp(tX)}(a_0)
\]

for all \(X \in \Gamma(A)\) and all \(a_0 \in A\).

We are now able to define the distribution \(\mathcal{D}(PA) \subset TP(A)\) by which we will be quotienting in \(P(A)\) (see [14] for a more detailed treatment). Let, for \(a \in A\):

\[
H_a := \left\{ t \mapsto \hat{ad}(\xi(t))|_{a(t)} + \frac{d\xi(t)}{dt}|_{\gamma(t)} \in \tilde{P}(TA) \mid \xi : I \to \Gamma(A), \xi(0) = \xi(1) = 0 \right\},
\]
where $\gamma(t)$ is the base path of $a \in P(A)$ and
\[
\frac{d\xi(t)}{dt}\bigg|_{\gamma(t)} \in A_{\gamma(t)}
\] is thought of as belonging to the vertical elements in $T_{a(t)}A$. Then define, for $a \in A$,
\[
\mathfrak{D}_a(PA) := \tau^{-1}(H_a).
\]
We summarize some well-known important facts in the following theorem \cite{9, 10, 1, 3, 8}.

**Theorem 3.1.** The following holds.

- (i) $\mathfrak{D}(PA)$ is a finite codimensional integrable distribution on $P(A)$.
- (ii) Let $\mathcal{F}(A)$ be the foliation integrating $\mathfrak{D}(PA)$. Then there is an open neighborhood $P_0(A) \subset P(A)$ of the natural embedding
  \[
c_0 : M \hookrightarrow P(A) : x \in M \mapsto [t \in I \mapsto 0_x],
\]
  where the space of leaves
  \[
P_\circ(A) := P_0(A)/\mathcal{F}(A) \cap P_0(A)
\]
is a finite dimensional smooth manifold.
- (iii) The maps
  \[
  \alpha : P(A) \to M : a \mapsto a(0),
  \beta : P(A) \to M : a \mapsto a(1),
  \varepsilon : M \to P(A) : m \mapsto a(t) \equiv 0_m,
\]
descend to smooth maps $\bar{\alpha} : \bar{P}_0(A) \to M, \bar{\beta} : \bar{P}_0(A) \to M, \bar{\varepsilon} : M \to \bar{P}_0(A)$. There is also an open neighborhood of the natural embedding
  \[
  M \hookrightarrow M \times M \hookrightarrow \bar{P}_0(A)_{\bar{\beta}} \times \bar{\alpha} \bar{P}_0(A)
\]
where the concatenation of paths $(a, b) \mapsto a \cdot b$ descends to a well defined local multiplication $\bar{\mu}$ on $\bar{P}_0(A)$ and yielding, on $\bar{P}_0(A) \rightrightarrows M$, the structure of local Lie groupoid with Lie algebroid $A$.

Note that $P_0(A)$ can be chosen as the subspace of the $A$-path space $P(A)$ consisting of $A$-paths lying in an open neighborhood $U$ of the zero section of $A$.

The second part of the construction, which deals with the symplectic structure, is summarized in the following theorem, due to Cattaneo–Felder.

**Theorem 3.2 (I).** Let $(M, \pi)$ be a Poisson manifold and let $A = (T^*M)_\pi$ be the cotangent Lie algebroid of $M$. The following then holds:

- (i) Let $a \in \bar{P}(A)$ and $u, v \in T_a\bar{P}(A)$. The integral
  \[
  \bar{\omega}_{can}(u, v) = \int_0^1 \omega_{can}((\tau u)(t), (\tau v)(t))dt
  \]
defines a natural symplectic form on $\bar{P}(A)$.
- (ii) Further, there exists a symplectic form $\bar{\omega}$ on $\bar{P}_0(A)$ turning $\bar{P}_0(A) \rightrightarrows M$ into a symplectic local groupoid such that
  \[
  q^*\bar{\omega} = \iota^*\bar{\omega}_{can},
  \]
where $q : P_0(A) \to \bar{P}_0(A)$ is the quotient map, and $\iota : P_0(A) \hookrightarrow \bar{P}_0(A)$ is the natural inclusion.
Theorem 3.3. Under the same hypothesis as in Theorem 3.2, the symplectic local groupoid $(P_0(A) \Rightarrow M, \omega)$ is the Lie bialgebroid $((T^*M)_\pi, TM)$.

Before we close this section, let us record the following proposition, which we will need later on. Its proof follows immediately from standard discussions on $\bar{P}(A)$ (see, for example, [3]).

Proposition 3.4. Let $A$ and $B$ be Lie algebroids over the same base manifold $M$, and let $\psi : A \to B$ be a Lie algebroid morphism.

(i) The induced map on path spaces $P(\psi) : \tilde{P}(A) \to \tilde{P}(B) : [t \mapsto a(t)] \mapsto [t \mapsto \psi(a(t))]$

preserves the $A$-paths, and descends to a morphism of local Lie groupoids $\bar{P}(\psi) : \bar{P}_0(A) \to \bar{P}_0(B)$ making the diagram

$$
\begin{array}{ccc}
P_0(A) & \xrightarrow{P(\psi)} & P_0(B) \\
\downarrow{q} & & \downarrow{q'} \\
\tilde{P}_0(A) & \xrightarrow{\bar{P}(\psi)} & \tilde{P}_0(B)
\end{array}
$$

(iii) commutes, where $q, q'$ are the quotient maps of Theorem 3.1.

(ii) The diagram

$$
\begin{array}{ccc}
TP(A) & \xrightarrow{P(\psi)_*} & TP(B) \\
\downarrow{\tau \circ \iota_*} & & \downarrow{\tau \circ \iota_*} \\
\tilde{P}(TA) & \xrightarrow{\bar{P}(\psi)_*} & \tilde{P}(TB)
\end{array}
$$

commutes.

4. Exponential maps

The classical exponential map establishes a local diffeomorphism from some neighborhood of zero in a Lie algebra to the corresponding local Lie group. This construction extends to Lie algebroids and local Lie groupoids. Unlike the Lie algebra case, however, one needs to choose some geometrical structure, namely an $A$-connection.

Let $A$ be, as before, a Lie algebroid over $M$. Recall that an $A$-connection on $A$ is a bilinear map $\nabla : \Gamma(A) \times \Gamma(A) \to \Gamma(A) : (a, b) \mapsto \nabla_a b$

satisfying the conditions:

$$
\begin{align*}
\nabla_{fa} b &= f \nabla_a b, \\
\nabla_a (fb) &= (\rho(a)f)b + f \nabla_a b,
\end{align*}
$$

for all $a, b \in \Gamma(A)$ and $f \in C^\infty(M)$.

In particular, to any linear connection $\nabla^{TM}$ on the vector bundle $A \to M$, there is an associated $A$-connection on $A$ given by $\nabla_a b = \nabla^{TM}_{\rho(a)}b$, for all $a, b \in \Gamma(A)$. However, not every $A$-connection on $A$ is of this form.

Definition 4.1. An $A$-geodesic is an $A$-path $a : I \to A$ such that $\nabla_{a(t)} a(t) = 0$, $\forall t \in I$. 

An $A$-connection $\nabla$ on $A$ also defines a map $h : A \times_M A \to TA$, called a horizontal lifting \cite{15}:
\[ h(a,b) = a_\ast(\rho(b)) - \nabla_b a \]
for any $a, b \in A_x$ ($x \in M$) and any $\bar{a} \in \Gamma(A)$ with $\bar{a}(x) = a$. It is not hard to check $h(a,b)$ does not depend on the choice of $\bar{a}$.

**Definition 4.2.** The geodesic vector field of $\nabla$ is the vector field $\xi \in \mathfrak{X}(A)$ defined by the formula
\[ \xi(a) = h(a,a), \forall a \in A. \]  \hspace{1cm} (7)

**Proposition 4.3.** Let $A$ be a Lie algebroid, and $\nabla$ an $A$-connection on $A$. Let $\xi \in \mathfrak{X}(A)$ be the geodesic vector field of $\nabla$, whose flow we denote by $\varphi^\nabla_t$.

(i) There is a neighborhood $U \subset A$ of the zero section such that $\varphi^\nabla_t$ is defined for $t \in I$;
(ii) for all $a_0 \in U$, the path $[t \mapsto a(t) = \varphi^\nabla_t(a_0)]$ is $A$-geodesic.

**Proof.**

(i) We briefly outline the argument (a full treatment can be found in \cite{15}). Denote by $m_s : A \to A$ the fibrewise multiplication by $s \in \mathbb{R}$. By definition of $\xi$, we have that
\[ s\xi_a = (m_s)_\ast a \xi_{sa} \]
for all $s > 0$ and $a \in A$. It then follows that
\[ s\varphi^\nabla_{ta}(a) = \varphi^\nabla_{ta}(sa), \]
where one side is defined exactly when the other is. By rescaling locally, this yields the claim.

(ii) Fix $a_0 \in A$ and let $a(t) = \varphi^\nabla_t(a_0)$. Denote by $\gamma(t) = p(a(t))$ the underlying base path. We have that
\[ p\ast(\dot{a}(t)) = p\ast(\xi(a(t))) = p\ast(h(a(t),a(t))) = \rho(a(t)), \]
so $a(t)$ is an $A$-path. Choose any time dependent section $\bar{a} : I \times M \to A$ such that $\bar{a}(t,p(a(t))) = a(t)$. Then
\[ \nabla_{a(t)}a(t) = \frac{\partial}{\partial t} \bar{a}(t,\gamma(t)) + \nabla_{a(t)}\bar{a}(t,\gamma(t)), \]
\[ = [\ddot{a}(t) - \dot{\bar{a}}(\dot{\gamma}(t))] + \nabla_{a(t)}\bar{a}(t,\gamma(t)) \]
\[ = \dot{a}(t) - \xi(a(t)) \]
\[ = 0 \]
as needed. \hfill \Box

Letting $U \subset A$ as in Proposition \cite{3} (i) above, we define
\[ \Phi : U \to P(A) : a_0 \in U \mapsto [t \mapsto \varphi^\nabla_t(a_0)]. \] \hspace{1cm} (8)
One should think of $\Phi$ as a kind of “exponential” defined by $\nabla$ at the level of $A$-paths \cite{3}. There is also a counterpart, at the level of integrating local Lie groupoids, which we now turn to.

Let $\nabla$ be an $A$-connection on $A$ and let $\Sigma \rightrightarrows M$ be a local Lie groupoid integrating $A$ with source and target maps $\alpha, \beta$, respectively. For any $x \in M$, there is a (usual) connection on the source fiber $\Sigma_x = \alpha^{-1}(x)$, which we will denote by $\nabla^x$. It is defined \cite{23} by the property that
\[ \nabla^x_{X^L}(f \cdot Y^L) = X^L(f) \cdot Y^L + f \cdot (\nabla^x X^L) Y^L, \] \hspace{1cm} (9)
for any two $X, Y \in \Gamma(A)$ and any $f \in C^\infty(\Sigma_x)$. Here $X^L$ denotes the left-invariant vector field $X^L \in \mathfrak{X}(\Sigma)$ associated to $X \in \Gamma(A)$.
Definition 4.4 ([23]). Let $\nabla$ be an $A$-connection on $A$ and $\Sigma \rightrightarrows M$ a local Lie groupoid with Lie algebroid $A$. The groupoid exponential is the map $\exp^\nabla : A \to \Sigma$, defined in a neighborhood of the zero section in $A$, which on each fiber $A_x$ is given by the exponential map of the connection $\nabla^x$ on $\Sigma_x$.

It can be proved that $\exp^\nabla$ is smooth [23]. Also, note that, by definition,
$$\alpha \circ \exp^\nabla = p,$$
where $p : A \to M$ is the projection, and that, for any $a_0 \in A$, $[t \in I \mapsto \exp^\nabla(ta_0)]$ is a source-path in $\Sigma$.

The relation between $\Phi$ and the groupoid exponential $\exp^\nabla$ is summarized in the following simple proposition.

Proposition 4.5. Let $U \subset A$ be as in Proposition 4.3 and fix $a_0 \in U$. Let $a = \Phi(a_0)$, i.e. $a(t) = \varphi^\nabla_t(a_0) \ (t \in I)$ be a path in $A$, and $r = [t \in I \mapsto r(t) = \exp^\nabla(ta_0)]$ a path in $\Sigma$. Then $r$ is a source-path that satisfies
$$\begin{align*}
\{ [L_{r^{-1}(t)}]_s r(t) = a(t) & \quad \forall t \in I, \\
r(0) = \varepsilon(p(a_0)), \quad r'(0) = a_0.
\end{align*}$$
Here, for any $g \in \Sigma$, we let $L_g : \Sigma_{\beta(g)} \to \Sigma_{\alpha(g)} : g' \mapsto g \cdot g'$ be left multiplication by $g$.

Proof. From Eq. (10), it follows that
$$0 = \nabla^x \circ \tilde{r}(t) = \left[ L_{r'(t)} \right]_s \left[ \nabla_{[L_{r^{-1}(t)}]_s \tilde{r}(t)} [L_{r^{-1}(t)}]_s \tilde{r}'(t) \right].$$
Hence the $A$-path $[L_{r^{-1}(t)}]_s \tilde{r}'(t)$ is $A$-geodesic. Since we have $[L_{r^{-1}(0)}]_s \tilde{r}'(0) = \varepsilon(p(a_0)) \cdot r(0) = a_0$
by definition, the result follows from the unicity of $A$-geodesics [15].

The following theorem is a standard consequence of the construction of the local groupoid $\tilde{P}_0(A) \rightrightarrows M$ of Theorem 3.1 and of Propositions 4.3 and 4.5 (see [3, 8]).

Theorem 4.6. Let $\nabla$ be an $A$-connection on $A$ and $U \subset A$ be as in Propositions 4.3. Then there exists a sufficiently small $\tilde{P}_0(A) \subset P(A)$, as in Theorem 3.1 (ii), such that $\exp^\nabla_{|U} : U \to \tilde{P}_0(A)$ is a diffeomorphism onto its image and the diagram

$$\begin{array}{ccc}
U \subset A & \xrightarrow{\Phi} & P_0(A) \\
\downarrow \exp & & \downarrow q \\
\tilde{P}_0(A) & & \\
\end{array}$$

commutes. Here $q : P_0(A) \to \tilde{P}_0(A)$ is the quotient map, as in Theorem 3.1.

The following computational lemma will be useful in our subsequent discussions.

Lemma 4.7. Let $\nabla$ be an $A$-connection on $A$, $U \subset A$ and $P_0(A) \subset P(A)$ as in Theorem 4.6 and $\Phi : U \to \tilde{P}(A)$ be as in Eq. (8).

(i) Then for any $\nu \in T_aA$, $\alpha \in U$, we have
$$[\tau(\alpha^* (\varphi^\nabla_t (\nu))))(t) = (\varphi^\nabla_t)_{\alpha} v$$
$\forall t \in I$. Recall that $\iota : P(A) \hookrightarrow \tilde{P}(A)$ is the inclusion.
(ii) For any \( v \in T_aA, \ a \in U, \) we have
\[
\tau[t_*(P(\psi)_*)\Phi_*v](t) = (\psi_* (\varphi^\nabla_t)_*) v \quad (13)
\]
\( \forall t \in I. \)

**Proof.**

(i) This is immediate from Eq. (2).

(ii) According to Eq. (6), we have
\[
(\tau \circ \iota_*) (P\psi) = P(\psi \circ (\tau \circ \iota_*)).
\]
Then
\[
\tau[t_*(P\psi)_*\Phi_*v](t) = \tau[t_*(P\psi)_*\Phi_*v](t) = \psi_* (\varphi^\nabla_t)_* v,
\]
as claimed.

\[ \square \]

5. Symplectic realizations of Poisson manifolds

Let \((M, \pi)\) be a Poisson manifold and let \( A = (T^*M)_{\pi} \) be its cotangent Lie algebroid. Define \( \hat{P}_0(A) \rightrightarrows M \) to be the symplectic local groupoid of Theorems 3.1 and 3.2. Denote its symplectic form by \( \omega \in \Omega^2(\hat{P}_0(A)). \)

Now, fix \( \nabla \) an \( A \)-connection on \( A \) and let \( U \subset A \) be a sufficiently small neighborhood of the zero section as in Theorem 4.6. Set
\[
\omega := (\exp \nabla)^*(\bar{\omega}).
\]
(14)

Then \( \omega \) is a symplectic form on \( U \subset T^*M. \)

**Proposition 5.1.** The symplectic form \( \omega \) can be explicitly expressed as follows:
\[
\omega(u, v) := \int_0^1 (\varphi^\nabla_t)^* \omega_{\text{can}}(u, v) \ dt
\]
\( \forall u, v \in T_aT^*M, \forall a \in U \subset T^*M, \) where \( \varphi^\nabla_t \) is the flow of the geodesic vector field \( \xi \in \mathfrak{X}(T^*M) \) corresponding to the \( A \)-connection \( \nabla \) on \( A. \)

**Proof.** By Eq. (11) we have \((\exp \nabla)^* \circ q^* = \Phi^* \circ q^* \) and by Eq. (4), \( q^* \bar{\omega} = \iota^* \bar{\omega}_{\text{can}}. \) Thus
\[
\omega = \Phi^* \iota^* \bar{\omega}_{\text{can}}.
\]

On the other hand, by Eq. (12):
\[
\Phi^* \iota^* \bar{\omega}_{\text{can}}(u, v) = \int_0^1 \omega_{\text{can}}[[\tau[t_*(\Phi_*u)](t), [\tau[t_*(\Phi_*v)](t)] dt
\]
\[
= \int_0^1 \omega_{\text{can}}((\varphi^\nabla_t)_*(u), (\varphi^\nabla_t)_*(v)) dt
\]
\[
= \int_0^1 \omega_{\text{can}}(u, v) dt
\]
which concludes. \( \square \)

As an immediate consequence of Eq. (15), we recover the following theorem which was proved by Crainic-Mărcuţ by a direct computation (except for part (ii)).

**Theorem 5.2.** Let \((M, \pi)\) be a Poisson manifold and \( A = (T^*M)_{\pi} \) its cotangent Lie algebroid. Fix \( \nabla \) an \( A \)-connection on \( A \) and let \( \xi \in \mathfrak{X}(T^*M) \) be the associated geodesic vector field. Also let \( U \subset A \) be, as in Theorem 4.6, a small enough open neighborhood of the zero section in \( A. \) In particular, the flow \( \varphi^\nabla_t(a_0) \) of \( \xi \) is defined for \( t \in I \) on any \( a_0 \in U. \) Then
(i) The projection \( \text{pr} |_U : U \subset T^* M \to M \) together with the symplectic form \( \omega \in \Omega^2(U) \), as defined by Eq. (15), gives a symplectic realization of \( (M, \pi) \);

(ii) The zero section of \( A \) is a Lagrangian submanifold of \( U \).

In this case, the geodesic vector field \( \xi \in \mathfrak{X}(A) \) is also called a Poisson spray in [4, 24].

6. Symplectic Nijenhuis local groupoids

It is well-known that there is a one-to-one correspondence between Poisson manifolds and local symplectic groupoids (in fact, this is a special case of Theorem A.9). This correspondence can be extended to a one-to-one correspondence between Poisson-Nijenhuis manifolds and local symplectic Nijenhuis groupoids. This is a result due to Stiénon–Xu [26], which we recall in Theorem 6.3 below.

In this section, we briefly go over the main ideas of the proof of Theorem 6.3, which we will need later on.

For completeness, let us recall that a \((1,1)\)-tensor \( \tilde{N} : T \Sigma \to T \Sigma \) on a local Lie groupoid \( \Sigma \rightrightarrows M \) is multiplicative if it is a local Lie groupoid morphism w.r.t. \( T \Sigma \rightrightarrows TM \) seen as the tangent groupoid.

**Definition 6.1.** A symplectic Nijenhuis local groupoid is a symplectic local groupoid \( (\Sigma \rightrightarrows M, \omega) \) equipped with a multiplicative \((1,1)\)-tensor \( \tilde{N} : T \Sigma \to T \Sigma \) such that \( (\Sigma, \omega, \tilde{N}) \) is a symplectic Nijenhuis manifold.

**Remark 6.2.** Let \( (\Sigma \rightrightarrows M, \omega, \tilde{N}) \) be a symplectic Nijenhuis local groupoid and denote by \( \tilde{\pi} \in \mathfrak{X}^2(\Sigma) \) the Poisson bivector field associated to \( \omega \). Then \( (\Sigma \rightrightarrows M, \tilde{\pi}) \) is a Poisson local groupoid. Further, from Proposition B.2 it follows that the bivector \( \tilde{\pi}_N \) defined by \( \tilde{\pi}_N^\sharp = \tilde{\pi}^\sharp \circ \tilde{N} \) gives another multiplicative Poisson structure on \( \Sigma \rightrightarrows M \). Thus we have another Poisson local groupoid \( (\Sigma \rightrightarrows M, \tilde{\pi}_N) \) with the same underlying local Lie groupoid. Note, however, that in general \( \tilde{\pi}_N \) is not symplectic.

The following theorem is due to Stiénon–Xu [24].

**Theorem 6.3.**

(i) The unit space of a symplectic Nijenhuis local groupoid is a Poisson–Nijenhuis manifold.

(ii) Given a Poisson–Nijenhuis manifold \( (M, \pi, N) \), there is a unique, up to isomorphism, symplectic Nijenhuis local groupoid whose induced Poisson–Nijenhuis structure on the unit space is \( (M, \pi, N) \).

In other words, there is one-one correspondence between Poisson–Nijenhuis manifolds and symplectic Nijenhuis local groupoids.

Let us briefly sketch a proof of Theorem 6.3, which will be needed in the future.

For (i), let \( (\Sigma \rightrightarrows M, \omega, \tilde{N}) \) be a symplectic Nijenhuis local groupoid. Let \( \tilde{\pi} \) be the bivector field on \( \Sigma \) which is the inverse of \( \omega \). Since \( (\Sigma \rightrightarrows M, \omega) \) is a symplectic local groupoid, it is well-known that \( \pi := \alpha_\ast \tilde{\pi} \) is a well-defined Poisson bivector field on the unit space \( M \) [20], and that the Lie algebroid of \( \Sigma \rightrightarrows M \) is isomorphic to \( (T^* M)\pi \). In fact, \( ((T^* M)\pi, TM) \) is the Lie bialgebroid corresponding to the symplectic groupoid \( (\Sigma \rightrightarrows M, \omega) \).

Now, let \( \tilde{\pi}_N \in \mathfrak{X}^2(\Sigma) \) be the bivector field defined by \( \tilde{\pi}_N^\sharp = \tilde{N} \circ \tilde{\pi}^\sharp \). Recall (Proposition B.2) that \( \tilde{\pi}_N \) is also Poisson tensor on \( \Sigma \). Moreover, \( \tilde{\pi}_N \) is also multiplicative since \( \tilde{N} \) is a multiplicative \((1,1)\)-tensor and \( \tilde{\pi} \) is a multiplicative bivector field. In other words, \( (\Sigma \rightrightarrows M, \tilde{\pi}_N) \) is a Poisson local groupoid.
Let \( \pi' := \alpha_* \tilde{\pi}_N \). It is known [31] that \( \pi' \) is a well defined Poisson bivector field on \( M \). Since 
\[
[\tilde{\pi}_N, \tilde{\pi}_N'] = [\tilde{\pi}, \tilde{\pi}_N] = 0,
\]
it follows that 
\[
[\pi', \pi'] = [\pi, \pi'] = 0.
\]
On the other hand, the Lie groupoid morphism \( \tilde{N} : T\Sigma \to T\Sigma \) induces a map on the unit manifolds \( \tilde{N} : TM \to TM \) which is, clearly, a \((1, 1)\)-tensor. The Nijenhuis torsion free condition for \( N \) also follows from that on \( \tilde{N} \). Further, it is clear that \( \pi'' = N \circ \pi' \), thus making \((M, \pi, N)\) a Poisson–Nijenhuis manifold as desired.

Conversely, to see \((ii)\), let \((M, \pi, N)\) be a Poisson–Nijenhuis manifold. Let \( A = (T^*M)_\pi \) be the cotangent Lie algebroid of \( \pi \) and \( \tilde{P}_0(A) \rightrightarrows M \) be the corresponding local Lie groupoid. Fix \( \tilde{\omega} \) to be the multiplicative symplectic two-form on \( \tilde{P}_0(A) \) as in Theorem [32] and let \( \tilde{\pi} \) be the Poisson bivector field on \( \tilde{P}_0(A) \) associated to \( \tilde{\omega} \). Then the Lie bialgebroid of the Poisson local groupoid \((\tilde{P}_0(A) \rightrightarrows M, \tilde{\pi})\) is just \(((T^*M)_\pi, TM)\) (see Theorem [A.9] ((iii))).

Now, out of \((M, \pi, N)\) we get another Lie bialgebroid, namely \(((T^*M)_\pi, (TM)_N)\) (see Proposition [B.4]) and thus there exists a multiplicative Poisson bivector field \( \tilde{\pi}_N \) on \( \tilde{P}_0(A) \) (see Theorem [A.9]), which makes \((\tilde{P}_0(A) \rightrightarrows M, \tilde{\pi}_N)\) into a Poisson local groupoid with Lie bialgebroid \(((T^*M)_\pi, (TM)_N)\). Let \( \tilde{\delta} : \Gamma(\wedge^*T^*M) \to \Gamma(\wedge^*T^*M) \) be the Lie algebroid coboundary of the Lie algebroid \((TM)_N\) (see Eq. (36)). Then one can check that \([\tilde{\delta}, d_{\text{DR}}] = 0\) (see Lemma 5.3 in [26]), where \(d_{\text{DR}}\) is the De Rham operator on \(\Gamma(\wedge^*T^*M) = \Omega^*(M)\). On the other hand, when \(A = TM\) is the tangent Lie algebroid of a manifold \(M\), its Lie algebroid coboundary is known to be the De Rham operator \(d_{\text{DR}}\). According to the Universal Lifting Theorem [3], we have 
\[
[\tilde{\pi}_N, \tilde{\pi}] = 0.
\]
Finally, let
\[
\tilde{N} = \tilde{\pi}_N^\sharp \circ \tilde{\omega}^\sharp : T\tilde{P}_0(A) \to T\tilde{P}_0(A).
\]
Then \(\tilde{N}\) is a multiplicative Nijenhuis tensor, and \((\tilde{P}_0(A) \rightrightarrows M, \tilde{\omega}, \tilde{N})\) is a symplectic Nijenhuis local groupoid. It is simple to check that the induced Poisson Nijenhuis structure on the unit space \(M\), as in (i) above, is indeed \((M, \pi, N)\).

Let us single out the following important fact that we will need later on.

**Proposition 6.4.** Let \((\Sigma \rightrightarrows M, \tilde{\omega}, \tilde{N})\) be a symplectic Nijenhuis local groupoid which induces the Poisson–Nijenhuis structure \((M, \pi, N)\) on the unit space. Then the source map \(\alpha : \Sigma \to M\) is a Poisson–Nijenhuis map. In particular, we have
\[
\alpha_* \tilde{\pi} = \pi, \quad \alpha_* \tilde{\pi}_N = \pi_N, \quad \alpha_* \circ \tilde{N} = N \circ \alpha_*
\]
where \(\tilde{\pi}\) denotes the bivector field on \(\Sigma\) which is the inverse of \(\tilde{\omega}\).

7. Relation with the complete lift to the cotangent bundle

We start by recalling the definition of the complete lift, to the cotangent bundle, of a \((1, 1)\)-tensor on a manifold \(M\).

Let \(N\) be a \((1, 1)\) tensor on a manifold \(M\). Denote by \(\langle \cdot, \cdot \rangle : T^*M \times_M TM \to M \times \mathbb{R}\) the canonical pairing. We define a 1-form \(\theta_N \in \Omega^1(T^*M)\) by
\[
\theta_N(u) = \langle \lambda, N \circ p_*(u) \rangle
\]
\(\forall u \in T\lambda T^*M\), where \(p : T^*M \to M\) is the projection. In particular, for \(N = \text{Id}\), \(\theta_N\) is just the Liouville form.

**Definition 7.1.** The complete lift of \(N\) (to the cotangent bundle) is the \((1, 1)\)-tensor \(N^c : TT^*M \to TT^*M\) on \(T^*M\) defined by the property that
\[
\omega_{\text{can}}(N^c u, v) = d\theta_N(u, v)
\]
\(\forall u, v \in T\lambda T^*M, \lambda \in T^*M\).
It can be checked, by a direct computation, that
\[
\omega_{\text{can}}(N^c u, v) = \omega_{\text{can}}((N^T) u, (N^T) v),
\]
for all \( u, v \in T\Lambda T^* M \), \( \lambda \in T^* M \).

**Lemma 7.2.** Let \( N : TM \to TM \) be a Nijenhuis tensor on \( M \). Denote by \( \tilde{\pi}_N \in \mathfrak{X}^2(T^* M) \) the Poisson bivector field of the Lie–Poisson structure, on \( T^* M \), corresponding to the Lie algebroid \((TM)_N \). Then
\[
\tilde{\pi}_N^b \omega_{\text{can}}^b = N^c
\]
where \( N^c : TT^* M \to TT^* M \) is the complete lift of \( N \) to \( T^* M \).

**Proof.** By definition, for any two \( X, Y \in \mathfrak{X}(M) \):
\[
\{\ell_X, \ell_Y\} \tilde{\pi}_N = \ell_{[NX,Y] + [X, NY] - N[X,Y]},
\]
\[
\{\ell_X, p^* f\} \tilde{\pi}_N = p^* ((NX) \cdot f),
\]
\[
\{p^* f, p^* g\} \tilde{\pi}_N = 0.
\]
where \( \ell_X, \ell_Y \in C^\infty(T^* M) \) denote the fibrewise linear functions defined by \( X, Y \), and where \( f, g \in C^\infty(M) \) are arbitrary and \( p : T^* M \to M \) is the bundle projection. The following relation is standard (see for example \([5]\)):
\[
N^c \mathcal{H}(\ell_X) - \mathcal{H}(\ell_{NX}) = \pi_{\text{can}}^b(\theta_{LX} N),
\]
where \( (L_X N)(Y) = [X, NY] - N[X,Y] \) is the usual Lie derivative and where, for any \( \psi \in C^\infty(T^* M) \), \( \mathcal{H}(\psi) \) denotes the Hamiltonian vector field of \( \psi \) with respect to the canonical Poisson structure on \( T^* M \). In other words, \( \mathcal{H}(\psi) = \pi_{\text{can}}^b(d\psi) \).

Also, it is easily checked that
\[
[\pi_{\text{can}}^b(\theta_{LX} N)](\ell_Y) = \ell_{(L_X N)(Y)}.
\]
Applying Eq. (21) to \( \ell_Y \) and using Eq. (22) immediately yields
\[
[N^c \mathcal{H}(\ell_X)](\ell_Y) = \ell_{[NX,Y]} + \ell_{(L_X N)(Y)} = \ell_{[NX,Y] + [X, NY] - N[X,Y]}.
\]
Since \( \pi_{\text{can}}^b \) sends horizontal forms to vertical vector fields and since \( \theta_{LX} N \) is a horizontal form, we have \([\pi_{\text{can}}^b(\theta_{LX} N)](p^* f) = 0\). Then, again from Eq. (21) but this time applied to \( p^* f \), it follows that
\[
[N^c \mathcal{H}(\ell_X)](p^* f) = p^* ((NX) \cdot f),
\]
Then using Eq. (18) and Eq. (19) on Eq. (23) and Eq. (24) gives
\[
N^c \pi_{\text{can}}^b(d\ell_X) = \pi_{\text{can}}^b(d\ell_X).
\]
The equality between the vector fields \( N^c \circ \pi_{\text{can}}^b(d(p^* f)) = \pi_{\text{can}}^b(d(p^* f)) \) is an easy consequence of the fact that \( N^c \) respects vertical vector fields (when evaluated on \( p^* g \) for some \( g \in C^\infty(M) \)) and of Eq. (21) by skew-symmetry (when evaluated on \( \ell_X \) for some \( X \in \mathfrak{X}(M) \)). This concludes the proof.

**Remark 7.3.** According to a theorem of Vaisman \([27]\), in the lemma above, \( \tilde{\pi}_N \) is compatible (in the sense of compatible Poisson structures) with the canonical Poisson structure \( \pi_{\text{can}} \) on \( T^* M \). Therefore it follows, from Lemma 7.2 above, that \((T^* M, \pi_{\text{can}}, N^c)\) defines a Poisson–Nijenhuis structure on \( T^* M \) and that its induced second Poisson structure \( \pi_{N^c} \) defined, as usual, by
\[
\pi_{N^c} = N^c \circ \pi_{\text{can}}^b
\]
coincides with the Lie–Poisson structure of the Lie algebroid \((TM)_N \).
We now need to recall the following well known fact from the general theory of Poisson groupoids \cite{20,21}.

For any Poisson local groupoid \((\Sigma \Rightarrow M, \bar{\pi})\) with Lie bialgebroid \((A, A^*)\), the diagram (of vector bundles and vector bundle morphisms)

\[
\begin{array}{ccc}
\text{Lie}(T^*\Sigma) & \xrightarrow{\text{Lie}(\bar{\pi}^\sharp)} & \text{Lie}(T\Sigma) \\
\downarrow j'_\Sigma & & \downarrow j_\Sigma \\
T^*A & \xrightarrow{\pi^\sharp_A} & TA
\end{array}
\]  

commutes. Here \(\pi_A\) denotes the Lie–Poisson structure on \(A\) induced by the Lie algebroid structure on \(A^*\) and \(j'_\Sigma, j_\Sigma\) are certain natural vector bundle isomorphisms which we shall not make explicit for brevity. See \cite{20,21} for more details.

Now, in the above framework, let \(\bar{\bar{N}} : T\Sigma \rightarrow T\Sigma\) be a multiplicative \((1,1)\)-tensor. Then there is an associated \((1,1)\)-tensor \(TA \rightarrow TA\), which we will denote by \(\text{Lie}(\bar{\bar{N}})\) and call the infinitesimal of \(\bar{\bar{N}}\) following \cite{17}, on its corresponding Lie algebroid \(A\). In terms of the (usual) Lie functor it is given simply as

\[
\text{Lie}(\bar{\bar{N}}) = j_\Sigma \circ \text{Lie}(\bar{\bar{N}}) \circ j^{-1}_\Sigma.
\]

Let \((\Sigma \Rightarrow M, \varpi, \bar{\bar{N}})\) be a symplectic Nijenhuis local groupoid with induced Poisson–Nijenhuis structure \((M, \pi, N)\) on the unit manifold \(M\). We now show that the infinitesimal of the multiplicative \((1,1)\)-tensor \(\bar{\bar{N}}\) is the complete lift \(N^c : TT^*M \rightarrow TT^*M\) of \(N\).

**Proposition 7.4.** Let \((\Sigma \Rightarrow M, \varpi, \bar{\bar{N}})\) be a symplectic Nijenhuis local groupoid, and \((M, \pi, N)\) its corresponding Poisson–Nijenhuis structure on the unit space, as in Theorem 6.3. Then the equality

\[
\text{Lie}(\bar{\bar{N}}) = N^c
\]

holds.

**Proof.** By definition, \(\bar{\bar{N}} = \bar{\pi}^\sharp_N \circ \varpi^b\) and hence \(\text{Lie}(\bar{\bar{N}}) = j_\Sigma \circ \text{Lie}(\bar{\pi}^\sharp_N) \circ \text{Lie}(\varpi^b) \circ j^{-1}_\Sigma\).

Let \((M, \pi, N)\) be the Poisson–Nijenhuis structure on the unit space of \(\Sigma\) induced by its symplectic Nijenhuis structure, as in Theorem 6.3. It is clear that

\[
\text{Lie}(\varpi^b) = (j'_\Sigma)^{-1} \circ \omega^b_{\text{can}} \circ j_\Sigma.
\]  

Indeed, let \(\bar{\pi}\) be the Poisson bivector defined by \(\varpi\). Then \((\Sigma \Rightarrow M, \bar{\pi})\) is a Poisson groupoid and, by Theorem A.9 its Lie bialgebroid is \(((T^*M)_{\pi}, TM)\). Since the Lie–Poisson of the tangent bundle Lie algebroid \(TM \rightarrow M\) is the canonical Poisson structure on \(T^*M\), the commutativity of Diagram 26 implies \(\text{Lie}(\bar{\pi}^\sharp) = j^{-1}_\Sigma \circ \pi^\sharp_{\text{can}} \circ j'_\Sigma\), which is equivalent to Eq. (27).

Finally, recall that the Lie bialgebroid of \((\Sigma \Rightarrow M, \bar{\pi}_N)\) is isomorphic to \(((T^*M)_{\pi_N}, (TM)_N)\), by the discussion in Section 6. We thus have \(\text{Lie}(\bar{\pi}^\sharp_N) = j^{-1}_\Sigma \circ \pi^\sharp_{\text{can}} \circ j'_\Sigma\), from the commutativity of Diagram 26 and then

\[
\text{Lie}(\bar{\bar{N}}) = j_\Sigma \circ \text{Lie}(\bar{\pi}^\sharp_N) \circ \text{Lie}(\varpi^b) \circ j^{-1}_\Sigma = \bar{\pi}^\sharp_N \circ \omega^b_{\text{can}} = N^c
\]

by Lemma 7.2. \qed
8. Symplectic realizations of non-degenerate Poisson–Nijenhuis manifolds

In this section we conclude the proof of a more general version of Theorem 1.1, that is in the case of non-degenerate Poisson–Nijenhuis structures. The main result here is Theorem 8.3. From Lemma 2.3 it is clear that this includes the case of holomorphic Poisson manifolds.

Let \((M, \pi, N)\) be a Poisson–Nijenhuis manifold, and \(\pi_N\) its second Poisson structure as in Proposition 3.2. Let us assume, throughout this Section, that the Nijenhuis tensor \(N : TM \to TM\) is invertible. The following standard lemma is crucial in our future discussions.

**Lemma 8.1 ([12]).** Let \((M, \pi, N)\) be a Poisson Nijenhuis manifold. The pair of maps

\[
N^T : (T^*M)_{\pi_N} \to (T^*M)_\pi, \quad N : (TM)_N \to TM
\]

defines a morphism of Lie bialgebroids

\[
(N^T, N) : ((T^*M)_{\pi_N}, TM) \to ((T^*M)_\pi, (TM)_N).
\]

In particular, if \(N\) is invertible, \((N^T, N)\) defines an isomorphism of Lie bialgebroids.

In an attempt to lighten the notation in the remainder of this section, and unless otherwise justified for emphasis, we let \(A\) (resp. \(A_N\)) stand for the cotangent Lie algebroid \((T^*M)_\pi\) (resp. \((T^*M)_{\pi_N}\)) of \(\pi\) (resp. \(\pi_N\)).

The local groupoid \(\tilde{P}_0(A_N) \rightrightarrows M\) is a symplectic local groupoid, as before. Denote by \(\tilde{\omega}\) the associated symplectic form, as in Theorem 3.2, on \(\tilde{P}_0(A_N)\). Also let \(\tilde{\pi}'\) be the associated Poisson structure. We then have a Poisson local groupoid \((\tilde{P}_0(A_N) \rightrightarrows M, \tilde{\pi}')\) whose associated Lie bialgebroid is \((A_N, TM)\) according to Theorem 3.2 (iii).

On the other hand, we can construct another Poisson local groupoid \((\tilde{P}_0(A) \rightrightarrows M, \tilde{\pi}_N)\). In fact, by Theorem 3.2 there is a symplectic Nijenhuis groupoid structure \((\tilde{P}_0(A), \tilde{\omega}, N)\) on \(\tilde{P}_0(A) \rightrightarrows M\), which relates to \((M, \pi, N)\) as in Proposition 6.4. As in Section 6 we let \(\tilde{\pi}\) be the Poisson bivector field associated to \(\tilde{\omega}\), and \(\tilde{\pi}_N\) be defined by \(\tilde{\pi}_N^* = \tilde{\pi}^* N^T\). Also, as recalled there, the Lie bialgebroid of the Poisson groupoid \((\tilde{P}_0(A) \rightrightarrows M, \tilde{\pi}_N)\) is \(((T^*M)_\pi, (TM)_N)\).

Now, from Lemma 8.1 there is a Lie bialgebroid morphism

\[
(N^T, N) : ((T^*M)_{\pi_N}, TM) \to ((T^*M)_\pi, (TM)_N),
\]

thus, from Theorem 3.2 it follows that the corresponding morphism of local Lie groupoids

\[
\tilde{P}(N^T) : \tilde{P}(A_N) \to \tilde{P}(A),
\]

as defined in Proposition 3.4 is a Poisson map. Concisely:

\[
\tilde{P}(N^T)^* \tilde{\pi}' = \tilde{\pi}_N.
\]

Up to now, we did not actually need \(N\) to be invertible. Since \(N\) is assumed to be non-degenerate though, Eq. 28 is actually an isomorphism of Lie bialgebroids. Therefore Eq. 29 is an isomorphism of Poisson local groupoids. In particular, \(\tilde{\pi}_N\) is non-degenerate since \((\tilde{P}_0(A_N) \rightrightarrows M, \tilde{\pi}')\) was in fact symplectic.

Let \(\tilde{\omega}_N\) be the (necessarily multiplicative) symplectic form on \(\tilde{P}_0(A)\) whose Poisson bivector is \(\tilde{\pi}_N\). Then \((\tilde{P}_0(A) \rightrightarrows M, \tilde{\omega}_N)\) is a symplectic local groupoid and since \(\tilde{P}(N^T)\) is a Poisson isomorphism, we must have

\[
\tilde{\omega}_N = (\tilde{P}(N^T)^{-1})^* \tilde{\omega}'.
\]

Summarizing, we have proved

**Proposition 8.2.** \((\tilde{P}_0(A) \rightrightarrows M, \tilde{\omega}_N)\) is a symplectic local groupoid which, seen as a Poisson groupoid only, has Lie bialgebroid \(((T^*M)_\pi, (TM)_N)\).
Now fix $\nabla$ an $A$-connection on $A$ and let $U \subset T^*M$ be sufficiently small around the zero section, as in Theorem 4.4. Then define

$$\omega = (\exp^\nabla)^* \varphi,$$

$$\omega_N = (\exp^\nabla)^* \varphi_N,$$

with $\exp^\nabla : U \to \tilde{P}_0(A)$ being the groupoid exponential map associated to $\nabla$. The formula of Eq. (15) still holds for $\omega$ defined above since $(\tilde{P}_0(A) \equiv M, \varphi)$ is exactly as in Theorem 3.2. On the other hand, we also have

**Proposition 8.3.** The symplectic form $\omega_N$ can be explicitly expressed as follows.

$$\omega_N(u, v) = \int_0^1 [((N^T)^{-1} o \varphi_t^\nabla)^* \omega_{can}](u, v) \, dt$$

(33)

$\forall u, v \in T_{a_0}(T^*M), \forall a_0 \in U \subset T^*M$, where $\varphi_t^\nabla$ is the flow of the geodesic vector field $\xi \in \mathfrak{X}(T^*M)$ corresponding to $\nabla$.

**Proof.** By Proposition 3.4, we have

$$P((N^T)^{-1})^* q^* = q^* P((N^T)^{-1})^*$$

where $q : P(A) \to \tilde{P}_0(A)$ and $q' : P(A_N) \to \tilde{P}_0(A_N)$ are the respective projections. On the other hand, $P((N^T)^{-1}) = (P(N^T))^{-1}$ and $\tilde{P}(N^T)^{-1} = (\tilde{P}(N^T))^{-1}$. Since $q^* \varphi = t^* \tilde{\omega}_{can}$, we get

$$q^*(P(N^T)^{-1})^* \varphi = P((N^T)^{-1})^* q^* \varphi = P((N^T)^{-1})^* t^* \tilde{\omega}_{can}.$$ 

By Eq. (11), $(\exp^\nabla)^* = \Phi^* q^*$, and thus it follows that

$$\omega_N = (\exp^\nabla)^* (P(N^T)^{-1})^* \varphi = \Phi^* q^* (P(N^T)^{-1})^* \varphi = \Phi^* P((N^T)^{-1})^* t^* \tilde{\omega}_{can}.$$ 

Now, take $u, v \in T_{a_0} A (a_0 \in U)$ and consider

$$\omega_N(u, v) = \omega_{can}(t^* P((N^T)^{-1})^* \Phi^* u, t^* P((N^T)^{-1})^* \Phi^* v)$$

$$= \int_0^1 \omega_{can}(\tau[t^* P((N^T)^{-1})^* \Phi^* u](t), \tau[t^* P((N^T)^{-1})^* \Phi^* v](t)) \, dt$$

$$= \int_0^1 \omega_{can}((N^T)^{-1} \varphi_t^\nabla)_* u, ((N^T)^{-1} \varphi_t^\nabla)_* v) \, dt$$

$$= \int_0^1 (N^T)^{-1} \varphi_t^\nabla)^* \omega_{can}(u, v) \, dt,$$

where the second to last equality follows from Eq. (13). \hfill \Box

Applying Proposition 6.4 and Proposition 5.2, we immediately get

**Theorem 8.4.** Let $(M, \pi, N)$ be a non-degenerate Poisson–Nijenhuis manifold, and $A = (T^*M)_\pi$ the cotangent Lie algebroid of the Poisson manifold $(M, \pi)$. Fix $\nabla$ an $A$-connection on $A$ and let $\xi \in \mathfrak{X}(T^*M)$ be the associated geodesic vector field. Also let $U \subset A$ be (as in Theorem 4.4) a small enough open neighborhood of the zero section in $A$, so that for any $a_0 \in U$, the flow $\varphi_t^\nabla$ is defined for all $t \in I$. Then

(i) The projection $\Pi_U : U \to M$, together with the symplectic form $\omega$ (resp. $\omega_N$), as defined in Eq. (15) (resp. Eq. (33)), gives a symplectic realization of $\pi$ (resp. $\pi_N$).

(ii) The $(1,1)$-tensor

$$N := (\omega_N^b)^{-1} \omega^b : TU \to TU$$

(34)

is a Nijenhuis tensor on $U$. Further, the triple $(U, \omega, N)$ is a symplectic Nijenhuis manifold such that the underlying Poisson–Nijenhuis structure $(U, \pi, N)$ together with the bundle projection $\Pi_U : U \to M$ is a Poisson–Nijenhuis map.
(iii) The zero section is a Lagrangian submanifold of \( U \) with respect to both \( \omega \) and \( \omega_N \).

Following [24], the symplectic Nijenhuis manifold \((U, \omega, N)\), with the bundle projection \( \text{pr}|_U: U \to M \), is called a symplectic realization of the non-degenerate Poisson–Nijenhuis manifold \((M, \pi, N)\). Also note that the symplectic realization of a Poisson–Nijenhuis manifold only exists if the Nijenhuis tensor is non-degenerate.

**Remark 8.5.** As an immediate consequence of Eq. (17), Eq. (33) can be rewritten as

\[
\omega_N(u, v) = \int_0^1 \left[ (\varphi^\tau)^* \omega_{can} \right] (\rho c)^{-1} u, v \, dt
\]

\(\forall u, v \in T_a(T^*M), \forall a \in U \subset T^*M\), where \( \varphi^\tau \) is the flow of the geodesic vector field \( \xi \in \mathfrak{X}(T^*M) \) corresponding to \( \nabla \).

Eq. (35) is due to Petalidou [24]. In fact, Theorem 8.4 (i)-(ii) essentially recovers a theorem claimed by Petalidou [24] (whose proof contains a gap) which was obtained by following closely the computational approach in [4]. According to the discussion in this Section, we see that the construction of \( \omega \) and \( \omega_N \) are in fact part of data of constructing a symplectic Nijenhuis local groupoid in disguise.

### 9. Holomorphic symplectic realizations of holomorphic Poisson manifolds

There only remains to explain how Theorem 1.1 follows from Theorem 8.4. First, recall the following standard fact.

**Proposition 9.1 ([16]).** Let \((X, \omega = \omega_R + i\omega_I)\) be a holomorphic symplectic manifold. Denote by \( \pi = \pi_R + i\pi_I \) the associated holomorphic Poisson bivector. Then the two (real) differential forms \( \omega_R, \omega_I \in \Omega^2(X) \) are symplectic, and the Poisson bivector fields corresponding to \( \omega_R \) and \( \omega_I \) are \( 4\pi_R \) and \( -4\pi_I \), respectively.

We are now ready to conclude the proof of the main theorem of this paper.

**Proof (of Theorem 1.1).** Let \( X \) be a holomorphic Poisson manifold with almost complex structure \( J \) and holomorphic Poisson bivector \( \pi = \pi_R + i\pi_I \). By Lemma 2.3, we have that \((X, \pi_I, J)\) is a Poisson–Nijenhuis manifold. Denote by \( A = (T^*X)_{\pi_I} \), the cotangent Lie algebroid of \((X, \pi_I)\) (as a real Poisson manifold). Also fix an affine connection \( \nabla^{TX} \) on \( X \) (seen as a real manifold) and denote by \( \nabla^{TX} \) the induced linear connection on \( T^*X \). Finally, for any two \( a, b \in \Gamma(A) \), let \( \nabla_a b = (\nabla_{\rho(a)}X) b \), where \( \rho : A \to TX \) is the anchor of the cotangent Lie algebroid. Then \( \nabla \) is a \( A \)-connection on \( A \).

From Theorem 8.4 (i), with \( \pi := \pi_I \) and \( N := J \), it follows that there is an open neighborhood \( Y \subset T^*X \) of the zero section where the two symplectic forms \( \omega_N \) and \( \omega \), as in Eq. (33) and Eq. (15), respectively (which, in this case, we will denote by \( \omega_R \) and \( \omega_I \) respectively), together with the projection \( \text{pr}|_Y: Y \to X \), give symplectic realizations of \( \pi_R, \pi_I \) respectively. Further, by Theorem 8.4 (ii), the \( (1,1) \)-tensor

\[
J : = (\omega_R)^{-1} \omega_I : TY \to TY
\]

defines a symplectic Nijenhuis structure \((Y, \omega_I, J)\) such that \( \text{pr}|_Y: Y \to X \) is a Poisson Nijenhuis map as indicated by Proposition 5.4.

The following lemma can be easily checked.

**Lemma 9.2.** The tensor \( J \) is an almost complex structure on \( Y \), i.e. \( J^2 = -1 \).
Proof. Recall that the \((1,1)\)-tensor \(J : TP_0(A) \to TP_0(A)\), defined as in Eq. (16), is a local groupoid morphism w.r.t. the groupoid structure \(P_0(A) \rightrightarrows M\) of Theorem 3.1. Also note that, by definition,

\[
\bar{J} = (\exp\nabla)^{-1}_{\ast} \circ J \circ (\exp\nabla)_{\ast}.
\]

Now, we have \(\text{Lie}(\bar{J}) = J^c\) by Lemma 7.4. On the other hand, it is standard (see [5] for example) that \(J\) being an integrable almost complex structure implies \((J^c)^2 = (J^2)^c = -1\). Thus \(\text{Lie}(J^2) = (\text{Lie}(J))^2 = -1\). Since \(\bar{J}\) is multiplicative on a local Lie groupoid, this implies that \(\bar{J}^2 = -1\). Then \(\bar{J}^2 = -1\) as well, which concludes the proof. \(\square\)

Since \(\bar{J}\) is a Nijenhuis tensor, from Lemma 9.2, it follows that \((Y,\bar{J})\) is indeed a complex manifold. Moreover, since \((Y,\omega_I,\bar{J})\) is a symplectic Nijenhuis manifold and its induced second Poisson structure is the Poisson structure corresponding to \(\omega_R\), it follows that \(\omega := \frac{1}{4} (\omega_R - i\omega_I) \in \Omega^2(Y) \otimes \mathbb{C}\) yields a holomorphic symplectic form on \(Y\) with respect to the new complex structure \(\bar{J}\). In particular, \((Y,\bar{J})\) is a holomorphic symplectic manifold. \((Y,\omega,\bar{J})\) is indeed the underlying holomorphic symplectic manifold of the holomorphic symplectic local groupoid integrating the given holomorphic Poisson structure \(\pi \in \Gamma(\wedge^2T^{1,0}X)\); see, in particular, Theorem 3.22 of [17] for the explanation of the factor \(\frac{1}{4}\)). Denote by \(\pi\) the associated holomorphic Poisson bivector. Then, from the second part of Theorem 8.3(ii), Proposition B.1 and Proposition B.7 the projection \(\text{pr}|_Y : Y \to X\) is a holomorphic Poisson map with respect to the holomorphic Poisson structures \((Y,\bar{J},\pi,\bar{J})\) and \((X,\pi,\bar{J})\). This concludes the proof. \(\square\)

Appendix A. Lie bialgebroids and Poisson groupoids

**Definition A.1.** A Poisson local groupoid \((\Sigma \rightrightarrows M, \bar{\pi})\) is a local Lie groupoid \(\Sigma \rightrightarrows M\) endowed with a bivector field \(\bar{\pi} \in \mathfrak{X}^2(\Sigma)\) such that the graph \(\Lambda\) of multiplication in \(\Sigma\)

\[
\Lambda \equiv \{(x,y,x \ast y) \mid (x,y) \in \Sigma \times \Sigma \text{ composable}\} \subset \Sigma \times \Sigma \times \Sigma
\]

is coisotropic w.r.t. \(\bar{\pi}\). Here \(\bar{\Sigma}\) denotes \(\Sigma\) endowed with the Poisson bivector \(-\bar{\pi}\).

A bivector field \(\bar{\pi} \in \mathfrak{X}^2(\Sigma)\) as in Definition A.1 is also called multiplicative. In that context, the following is standard [20].

**Proposition A.2.** Let \(A\) be the Lie algebroid of \(\Sigma \rightrightarrows M\). The bivector \(\bar{\pi}\) is multiplicative if and only if the map \(\bar{\pi}^\sharp : T^\ast\Sigma \to T\Sigma\) is a local Lie groupoid morphism w.r.t. \(T^\ast\Sigma \rightrightarrows A^\ast\) seen as the cotangent local Lie groupoid and \(T\Sigma \rightrightarrows TM\) as the tangent local Lie groupoid of \(\Sigma \rightrightarrows M\), respectively.

**Definition A.3.** A symplectic local groupoid is a Poisson local groupoid \((\Sigma \rightrightarrows M, \bar{\pi})\) such that \(\bar{\pi}\) is non-degenerate. In other words, \(\Lambda \subset \Sigma \times \Sigma \times \Sigma\) is a Lagrangian submanifold.

We now recall some fundamental facts about Lie bialgebroids. In the rest of this section, let \(A \to M\) be a Lie algebroid with anchor \(\rho\) and bracket \([\cdot,\cdot]\).

The bracket, \([\cdot,\cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)\), can be extended to a bilinear bracket of multisections \(\Gamma(\wedge^k A) \times \Gamma(\wedge^l A) \to \Gamma(\wedge^{k+l-2} A)\). We will write both the initial bracket and its extension in the same way as \([\cdot,\cdot]\). The triple \((\Gamma(\wedge^\bullet A), \wedge, [\cdot,\cdot])\) then forms a Gerstenhaber algebra [32].

Lie bialgebroids are a certain class of Lie algebroids \(A\) for which the dual vector bundle \(A^\ast\) also admits a compatible Lie algebroid structure. In order to define the compatibility condition,
recall the Lie algebroid coboundary of $A$ is the operator $d : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$ defined by

$$
(d\lambda)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^i \lambda(a_i, a_{i+1}, \ldots, a_{k+1}) + \sum_{i<j} (-1)^{i+j} \lambda([a_i, a_j], a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{k+1}).
$$

for any $\lambda \in \Gamma(\wedge^k A^*)$, $a_1, \ldots, a_{k+1} \in \Gamma(A)$.

**Example A.4.** When $A = TM$ is the tangent Lie algebroid of a manifold $M$, $d = d_{DR}$ is the De Rham operator on $\Gamma(\wedge^* T^* M) = \Omega^*(M)$.

When $A^*$ happens to be a Lie algebroid as well, we denote by $d_* : \Gamma(\wedge^k A) \to \Gamma(\wedge^{k+1} A)$ the associated coboundary operator (acting on sections of $\Gamma(\wedge^\bullet A)$).

**Definition A.5.** Let $A \to M$ be a Lie algebroid such that $A^* \to M$ also carries a Lie algebroid structure. Then $(A, A^*)$ is a Lie bialgebroid if and only if the bracket $[\cdot, \cdot]$ on $A$ and the coboundary operator $d_*$ of $A^*$ are compatible in the following sense: for any $a, a' \in \Gamma(A)$, one has

$$
d_*[a, a'] = [d_*a, a'] + [a, d_*a'].
$$

The compatibility condition (38) is equivalent to asking that $d_*$ is a derivation of the Gerstenhaber algebra structure on $(\Gamma(\wedge^\bullet A), \wedge, [\cdot, \cdot])$ [32].

**Example A.6.** Let $A = TM$ be the tangent Lie algebroid and $A^* = (T^* M)^\pi$ be the cotangent Lie algebroid of a Poisson manifold $(M, \pi)$. It is easy to see $(A, A^*)$ is a Lie bialgebroid. In fact we have that $[\cdot, \cdot] : \Gamma(\wedge^* A) \times \Gamma(\wedge^* A) \to \Gamma(\wedge^* A)$ is the Schouten bracket $[\cdot, \cdot] \equiv [\cdot, \cdot]_S$ on $M$ and that $d_* = [\pi, \cdot]_S$. Then (38) follows from the graded Jacobi identity.

**Definition A.7.** Let $(A, A^*)$ and $(B, B^*)$ be two Lie bialgebroids over $M$. A Lie bialgebroid morphism $(\psi, \psi^T) : (A, A^*) \to (B, B^*)$ is a vector bundle map $\psi : A \to B$ such that

i. $\psi : A \to B$ is a Lie algebroid morphism, and

ii. $\psi^T : B^* \to A^*$ is a Lie algebroid morphism.

One can prove (Theorem 3.10 of [20]) that the definition of a Lie bialgebroid $(A, A^*)$ is symmetric in $A$ and $A^*$. In particular, from the previous example, we get the

**Proposition A.8.** Let $M$ be a Poisson manifold with Poisson bivector field $\pi \in \mathfrak{X}^2(M)$. Then $((T^* M)^\pi, TM)$ is a Lie bialgebroid.

The “$d_*$” operator of the Lie bialgebroid in Proposition A.8 is simply the De Rham differential. The following theorem is standard [8, 20, 21] which extends a well known classical result of Drinfeld for Poisson Lie groups [6, 7].

**Theorem A.9.**

(i). Lie bialgebroids $(A, A^*)$ are in one-to-one correspondence with Poisson local groupoids $(\Sigma \rightrightarrows M, \bar{\pi})$.

(ii). The correspondence in (i) is functorial. More precisely, let $(A, A^*)$ and $(B, B^*)$ be two Lie bialgebroids over $M$. Then morphisms $(\psi, \psi^T) : (A, A^*) \to (B, B^*)$ of Lie bialgebroids are in one-to-one correspondence with morphisms of the associated Poisson local groupoids

$$
\bar{\psi} : (\Sigma_A \rightrightarrows M, \bar{\pi}_A) \to (\Sigma_B \rightrightarrows M, \bar{\pi}_B).
$$

\[\text{In the literature, the theorem is normally stated for global Lie groupoids, for which one needs to assume source connectedness and source simply connectedness. Without such topological assumptions, the conclusion (as well as the proof) holds for local Lie groupoids.}\]
(iii). Let \((\Sigma \rightrightarrows M, \varpi)\) be a symplectic local groupoid and let \(\bar{\pi}\) be the multiplicative Poisson structure induced by \(\varpi\). Then, as a Poisson groupoid, \((\Sigma \rightrightarrows M, \bar{\pi})\) has Lie bialgebroid \(((T^*M)_{\bar{\pi}}, TM)\). Here, \(TM \to M\) is the tangent bundle Lie algebroid of \(M\).

For completeness, let us recall that by a morphism \(\bar{\psi} : (\Sigma \rightrightarrows M) \to (\Sigma' \rightrightarrows M)\) of local Lie groupoids we mean a pair \((\psi_1, \psi_0)\) of smooth maps \(\psi_1 : \Sigma \to \Sigma'\) and \(\psi_0 : M \to M\) satisfying the usual axioms of a Lie groupoid morphism around a neighborhood of the unit submanifolds of \(\Sigma, \Sigma'\).

**Appendix B. Poisson–Nijenhuis manifolds**

Recall that a \((1,1)\)-tensor \(N : TM \to TM\) on a smooth manifold \(M\) is called *Nijenhuis* if its Nijenhuis torsion \(T_N : \wedge^2 TM \to TM\) vanishes, where

\[
T_N(X,Y) = [NX,NY] - N([NX,Y] + [X,NY]) + N^2[X,Y].
\]

**Definition B.1.** Let \(\pi\) be a Poisson bivector field on \(M\) and \(N\) be a Nijenhuis \((1,1)\)-tensor. We say that the triple \((M, \pi, N)\) is a Poisson–Nijenhuis manifold [22, 13] if \(\pi\) and \(N\) satisfy the following compatibility relations for all \(\alpha, \beta \in \Omega^1(M)\):

\[
N_{\pi}^{\pi} = \pi_{\pi}^N, \quad [\alpha, \beta]_{\pi_N} = [N^\pi \alpha, \beta]_{\pi} + [\alpha, N^\pi \beta]_{\pi} - N^\pi [\alpha, \beta]_{\pi}.
\]

(39)

Here \(\pi_N\) is the bivector defined by \(\pi_N^2 = \pi^2 \circ N^*\) and \([\cdot, \cdot]_{\pi_N}\) is the associated bracket.

The following is standard in the theory of Poisson–Nijenhuis manifolds [22, 13, 28].

**Proposition B.2.** Let \((M, \pi, N)\) be a Poisson–Nijenhuis manifold. Then the bivector \(\pi_N \in \mathcal{X}^2(M)\) defined by the property that \(\pi_N^2 = \pi^2 \circ N^*\) is a Poisson bivector.

An alternative description of the various compatibility relations between \(\pi\) and \(N\) is summarized in the following well-known result.

**Theorem B.3** ([27, 13]). Let \(\pi \in \mathcal{X}^2(M)\) be a Poisson bivector on a manifold \(M\) and let \(N : TM \to TM\) be any \((1,1)\)-tensor. Then the tensor \(\pi_N\) defined by

\[
\pi_N(\alpha, \beta) = \beta(N^\pi \alpha) \quad \forall \alpha, \beta \in \Omega^1(M)
\]

is skew-symmetric if and only if Eq. (39) holds. In this case, we have

(i) \([\pi, \pi_N] = 0\) if Eq. (40) holds, and the converse holds if \(\pi\) is non-degenerate;

(ii) \([\pi_N, \pi_N] = 0\) if and only if \(N\) is Nijenhuis.

It can be proved [12] that any Poisson–Nijenhuis manifold \((M, \pi, N)\) gives rise to a Lie bialgebroid

\(((T^*M)_\pi, (TM)_N)\).

Here \((T^*M)_\pi := (T^*M, \pi^2, [\cdot, \cdot]_\pi)\) is the cotangent Lie algebroid of \((M, \pi)\) and \((TM)_N := (TM, \rho_N, [\cdot, \cdot]_N)\) is the tangent Lie algebroid of \(M\) twisted by \(N\). More precisely, its anchor and bracket are:

\[
\rho_N(X) = NX,
\]

\[
[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y].
\]

In fact, the following is true.

**Proposition B.4** ([12]). Let \(M\) be a smooth manifold, \(\pi \in \mathcal{X}^2(M)\) a bivector field on \(M\) and \(N : TM \to TM\) a \((1,1)\)-tensor. Then \((M, \pi, N)\) is a Poisson–Nijenhuis manifold exactly when \(((T^*M)_\pi, (TM)_N)\) defined as above is a Lie bialgebroid.
Definition B.5. A symplectic-Nijenhuis manifold \((M, \omega, N)\) is a Poisson–Nijenhuis manifold \((M, \pi, N)\) whose Poisson bivector is non-degenerate.

Definition B.6. Let \((X, \pi_X, N_X)\) and \((Y, \pi_Y, N_Y)\) be two Poisson–Nijenhuis manifolds. A Poisson–Nijenhuis map, from \(X\) to \(Y\), is a smooth map \(f : X \to Y\) such that
\[
f_*\pi_X = \pi_Y.
\]

If \(f : X \to Y\) is a Poisson–Nijenhuis map, then \(f_*\pi_{N_X} = \pi_{N_Y}\) as well. The following is then easily seen.

Proposition B.7. Let \((X, \pi = \pi_R + i\pi_I)\) and \((Y, \pi' = \pi'_R + i\pi'_I)\) be two holomorphic Poisson manifolds with respective almost complex structures \(J_X, J_Y\) and let \(f : X \to Y\) a smooth map. Then \(f\) is a holomorphic Poisson map iff \(f\) is a Poisson–Nijenhuis map from \((X, \pi_I, J_X)\) to \((Y, \pi'_I, J_Y)\).

In particular, if \(f\) is holomorphic, \(f\) is a holomorphic Poisson map iff \(f_*\pi_I = \pi'_I\).

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