EXTENSION OF MITTAG-LEFFLER FUNCTION

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Abstract. In this paper, we present extension of Mittag-Leffler function by using extension of beta functions (Özergin et al. in J. Comput. Appl. Math. 235 (2011), 4601-4610) and obtain some integral representation of this newly defined function. Also we present the Mellin transform of this function in terms of Wright hypergeometric function. Furthermore, we show that the extended fractional derivative of the usual Mittag-Leffler function gives the extension of Mittag-Leffler function.

1. Introduction and Preliminaries

The Mittag-Leffler function occurs naturally in the solution of fractional order and integral equation. The importance of such functions in physics and engineering is steadily increasing. Some application of the Mittag-Leffler is carried out in the Study of Kinetic Equation, Study of Lorenz System, Random Walk, Levy Flights and Complex System and also in applied problems such as fluid flow, electric network, probability and statistical distribution theory.

We begin with the Gosta and Wiman Mittag-Leffler functions $E_\rho(z)$ and $E_{\rho,\sigma}(z)$ are defined by the following series as:

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + 1)}, z \in \mathbb{C}; \Re(\rho) > 0$$  \hspace{1cm} (1.1)

and

$$E_{\rho,\sigma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \sigma)}, z, \sigma \in \mathbb{C}; \Re(\rho) > 0,$$  \hspace{1cm} (1.2)

respectively. For further study of $E_\rho(z)$ and $E_{\rho,\sigma}(z)$ such as generalizations and applications, the readers may refer to the recent work of researchers [2, 10, 3, 4, 6, 11] and the work of Saigo and Kilbas [17]. In recent years, the function defined in (1.1) and some of generalizations have been numerically considered in the complex plane (see [9, 19]). Prabhakar [16] have introduced a generalization of the function $E_{\rho,\sigma}(z)$ defined in (1.2) as follows:

$$E^\gamma_{\rho,\sigma}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n z^n}{\Gamma(\rho n + \sigma) n!}, z, \sigma \in \mathbb{C}; \Re(\rho) > 0,$$  \hspace{1cm} (1.3)

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In this paper, we define further extension of Mittag-Leffler function as:

$$(\delta)_n = \begin{cases} 1, (n = 0, \delta \in \mathbb{C}) \\ \delta(\delta + 1) \cdots (\delta + n - 1), (n \in \mathbb{N}, \delta \in \mathbb{C}) \end{cases}$$

Obviously, the following special cases are satisfied:

$$E_{\rho,\sigma}^1(z) = E_{\rho,\sigma}^1(z) = E_{\rho,1}^1(z) = E_{\rho}^1(z). \quad (1.4)$$

In recent times many researchers have investigated the importance and great consideration of Mittag-Leffler function in the theory of special functions for exploring the generalization and some applications. Many extensions for these functions are found in [5, 21-24]. Shukla and Prajapati [20] (see also [25]) defined and investigated the function $E_{\rho,\sigma}^{\delta,q}(z)$, which is defined as:

$$E_{\rho,\sigma}^{\delta,q}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n}{\Gamma(pm + q)} \frac{z^n}{n!}, \quad (1.5)$$

where $z, \sigma, \delta \in \mathbb{C}$; $\Re(\rho) > 0$; $q > 0$. In the same paper, they have used the well-known Riemann-Liouville right-sided fractional integral, derivative and generalized Riemann-Liouville derivative operators (see [21, 22]). Very recently Özarslan and Yilmaz [13] have investigated an extended Mittag-Leffler function $E_{\rho,\sigma}^{\delta,c}(z;p)$, which is defined as:

$$E_{\rho,\sigma}^{\delta,c}(z;p) = \sum_{n=0}^{\infty} \frac{B_p(\delta+n,c-\delta)}{B(\delta,c-\delta)} \frac{(c)_n}{\Gamma(pm + \sigma)} \frac{z^n}{n!}, \quad (1.6)$$

where $p \geq 0$, $\Re(c) > \Re(\delta) > 0$ and $B_p(x,y)$ is extended beta function defined in (1.7) as follows:

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{pt}{1-t}}dt, \quad (1.7)$$

where $\Re(p) > 0$, $\Re(x) > 0$ and $\Re(y) > 0$.

In this paper, we define further extension of Mittag-Leffler function as:

$$E_{\alpha,\beta}^{\gamma,\lambda,\rho}(z;p) = \sum_{n=0}^{\infty} \frac{B^\rho_p(\gamma+n,c-\gamma)}{B(\gamma,c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \quad (1.8)$$

where $p \geq 0$, $\Re(c) > \Re(\gamma) > 0$ and $B^\lambda_p$ is extension of extended beta function defined by

$$B^\lambda_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \frac{F_1\left[\lambda; \rho; -\frac{p}{t(1-t)}\right]}{t(1-t)} dt, \quad (1.9)$$

where $\Re(p) > 0$, $\Re(x) > 0$, $\Re(y) > 0$, $\Re(\lambda) > 0$, $\Re(\rho) > 0$. It is obvious that $B^\lambda_p(x,y) = B_p(x,y)$ and $B^\lambda_0(x,y) = B(x,y)$. For further details the readers are refer to the work of Özergin et al. [13].

**Remark.** (1) If setting $\rho = \lambda$ in (1.8), then it reduces to the well-known extended Mittag-Leffler function as defined in (1.0).

(ii) If setting $\rho = \lambda$ and $p = 0$ in (1.8), then it reduces to the well-known Mittag-Leffler function as defined in (1.3).
2. Properties of further extended Mittag-Leffler function

We start with the following theorem, which gives the integral representation of the extended Mittag-Leffler function.

**Theorem 2.1.** Let $c, \alpha, \beta, \gamma, \lambda, \rho \in \mathbb{C}$, $\Re(c) > \Re(\gamma) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\lambda) > 0$, $\Re(\rho) > 0$. Then for the extended Mittag-Leffler function, we have the following integral representation

$$E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = \frac{1}{B(\gamma,c-\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1} \cdot \frac{p}{t(1-t)} \cdot E_{\alpha,\beta}^{c;\gamma,\rho}(tz) dt. \quad (2.1)$$

**Proof.** Using equation (1.9) in equation (1.8), we have

$$E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = \sum_{n=0}^{\infty} \left\{ \int_0^1 t^{\gamma+n-1}(1-t)^{c-\gamma-1} \right. \times 1F_1 \left[ \left( \frac{c}{\gamma} \right)_n \frac{z^n}{B(\gamma,c-\gamma) \Gamma(\alpha+\beta)n!} \right],$$

Interchanging the order of summation and integration in above equation, we get

$$E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1} \times 1F_1 \left[ \left( \frac{c}{\gamma} \right)_n \frac{z^n}{B(\gamma,c-\gamma) \Gamma(\alpha+\beta)n!} \right].$$

Using equation (1.9) in above equation, we get the desired integral representation. □

**Corollary 2.2.** Substituting $t = \frac{u}{1+u}$ in Theorem 2.1 we get

$$E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = \frac{1}{B(\gamma,c-\gamma)} \int_0^\infty \frac{u^{\gamma-1}}{(u+1)^{\gamma-1}} \cdot 1F_1 \left[ \left( \frac{c}{\gamma} \right)_n \frac{z^n}{B(\gamma,c-\gamma) \Gamma(\alpha+\beta)n!} \right] du. \quad (2.2)$$

**Corollary 2.3.** Taking $t = \sin^2 \theta$ in Theorem 2.1 we get the following integral representation

$$E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = \frac{1}{B(\gamma,c-\gamma)} \left[ \int_0^{\infty} \sin^{2\gamma-1} \theta \cos^{2c-1} \theta \cdot 1F_1 \left[ \left( \frac{c}{\gamma} \right)_n \frac{z^n}{\sin^2 \theta \cos^2 \theta} \right] \right] \times E_{\alpha,\beta}^c \left( z \sin^2 \theta \right) d\theta. \quad (2.3)$$

Kurulay and Bayram [15] introduced the following recurrence relation for Prabhakar Mittag-Leffler function as:

$$E_{\alpha,\beta}^c(tz) = \beta E_{\alpha,\beta+1}^c(tz) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^c(tz).$$

Inserting the above recurrence relation into (2.1), we get the following recurrence relation for the newly defined extended Mittag-Leffler function.

**Corollary 2.4.** Let $p \geq 0$, $\Re(c) > \Re(\gamma) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\lambda) > 0$, $\Re(\rho) > 0$, then the following relation holds:

$$E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = \beta E_{\alpha,\beta+1}^{\gamma,c;\lambda,\rho}(z;p) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma,c;\lambda,\rho}(z;p) \quad (2.4)$$
In next theorem, we define the Mellin transforms of the extended Mittag-Leffler function in terms of the Wright hypergeometric function which is defined by (see [27]-[29]) as:

\[
p_q(z) = p_q \left[ \begin{array}{c} \alpha, A_1 \\ \beta, B_1 \end{array} ; z \right]
\]

\[
= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_q + B_q n)} \frac{z^n}{n!}
\]

(2.5)

where \( \beta_r \) and \( \mu_s \) are real positive numbers such that

\[
1 + \sum_{s=1}^{p} B_s - \sum_{r=1}^{q} A_r \geq 0.
\]

Theorem 2.5. The Mellin transform of extended Mittag-Leffler function is given by

\[
\mathfrak{M}\left\{ E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p); s \right\} = \frac{\Gamma^{\lambda,\rho}(s) \Gamma(c + s - \gamma)}{\Gamma(c - \gamma)} \cdot \frac{1}{\Gamma^{\lambda,\rho}(s)} \left[ \begin{array}{c} (c, 1), (\gamma + s, 1) \\ (\beta, \gamma), (c + 2s, 1) \end{array} ; z \right]
\]

(2.6)

where \( p \geq 0, \Re(c) > \Re(\gamma) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\lambda) > 0, \Re(\rho) > 0. \)

Proof. Taking the Mellin transform of extended Mittag-Leffler function defined in (1.8), we have

\[
\mathfrak{M}\left\{ E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p); s \right\} = \int_{0}^{\infty} p^{s-1} E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) dp.
\]

(2.7)

Using equation (2.1) in equation (2.7), we have

\[
\mathfrak{M}\left\{ E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p); s \right\} = \frac{1}{B(\gamma, c - \gamma)} \int_{0}^{\infty} p^{s-1} \left\{ \int_{0}^{1} t^{\gamma-1} (1 - t)^{c-\gamma-1} \cdot F_1 \left[ \lambda; \rho; \frac{p}{t(1-t)} \right] \right\} E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(t); dt.
\]

(2.8)

Interchanging the order of integrations in equation (2.8), we have

\[
\mathfrak{M}\left\{ E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p); s \right\} = \frac{1}{B(\gamma, c - \gamma)} \int_{0}^{1} \left[ t^{\gamma-1} (1 - t)^{c-\gamma-1} E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(t) \right]
\]

\[
\times \int_{0}^{\infty} p^{s-1} \cdot F_1 \left[ \lambda; \rho; \frac{p}{t(1-t)} \right] dp dt.
\]

(2.9)

Now, taking \( u = \frac{p}{t(1-t)} \) in second integral of equation (2.9), we get

\[
\int_{0}^{\infty} p^{s-1} \cdot F_1 \left[ \lambda; \rho; \frac{p}{t(1-t)} \right] dp = \int_{0}^{\infty} u^{s-1} t^{s-1} \cdot F_1 \left[ \lambda; \rho; -u \right] du
\]

\[
= t^{s}(1 - t)^{s} \int_{0}^{\infty} u^{s-1} \cdot F_1 \left[ \lambda; \rho; -u \right] du
\]

(2.10)

where \( \Gamma^{\lambda,\rho}(s) \) is the extended gamma function defined by [14].

Using equation (2.10) and the definition of Prabhakar’s Mittag-Leffler function in equation (2.4), we get

\[
\mathfrak{M}\left\{ E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p); s \right\}
\]
For the case we get the following elegant complex integral representation

Taking the inverse Mellin transform on both sides of equation (2.6),

Corollary 2.8.

where \( \nu > 0 \) is defined by

The well-known Riemann-Liouville fractional derivative of order \( \mu \) is defined by

\[
\mathcal{D}^\mu_t = \frac{1}{\Gamma(-\mu)} \int_0^t (t-s)^{-\mu-1} f(s) \, ds, \quad \Re(\mu) > 0.
\]

For the case \( m - 1 < \Re(\mu) < m \) where \( m = 1, 2, \ldots \), it follows

\[
\mathcal{D}^\mu_z = \frac{d^m}{dz^m} \mathcal{D}_z^{\mu-m} \{ f(z) \}.
\]
For the case 

\[\Gamma(-\mu + m)\int_0^x f(t)(x-t)^{-\mu+m-1} dt, \Re(\mu) > 0. \quad (3.2)\]

**Definition 2.** (see [13]) The extended Riemann-Liouville fractional derivative of order \(\mu\) is defined by

\[\mathcal{D}_x^{\mu,p} = \frac{1}{\Gamma(-\mu)} \int_0^x f(t)(x-t)^{-\mu-1} \exp\left(-\frac{px^2}{t(x-t)}\right) dt, \Re(\mu) > 0. \quad (3.3)\]

For the case \(m - 1 < \Re(\mu) < m\) where \(m = 1, 2, \ldots\), it follows

\[\mathcal{D}_x^{\mu,p} = \frac{d^m}{dx^m} \mathcal{D}_x^{\mu-m,p} \left\{ f(z) \right\} \]

\[= \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(-\mu + m)} \int_0^x f(t)(x-t)^{-\mu+m-1} \exp\left(-\frac{px^2}{t(x-t)}\right) dt \right\}, \Re(\mu) > 0. \quad (3.4)\]

**Definition 3.** Here, we define the extension of extended Riemann-Liouville fractional derivative of order \(\mu\) as

\[\mathcal{D}_x^{\mu,p} = \frac{1}{\Gamma(-\mu)} \int_0^x f(t)(x-t)^{-\mu-1} \cdot F_1 \left[ \lambda; \rho; \left(-\frac{px^2}{t(x-t)}\right) \right] dt, \Re(\mu) > 0. \quad (3.5)\]

For the case \(m - 1 < \Re(\mu) < m\) where \(m = 1, 2, \ldots\), it follows

\[\mathcal{D}_x^{\mu,p} = \frac{d^m}{dx^m} \mathcal{D}_x^{\mu-m,p} \left\{ f(z) \right\} \]

\[= \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(-\mu + m)} \int_0^x f(t)(x-t)^{-\mu+m-1} \cdot F_1 \left[ \lambda; \rho; \left(-\frac{px^2}{t(x-t)}\right) \right] dt \right\}, \Re(\mu) > 0. \quad (3.6)\]

Obviously, if \(\lambda = \rho\), then definition [3] reduces to extended fractional derivative [4]. Similarly, if \(\lambda = \rho\) and \(p = 0\), then definition [4] reduces to the well-known Riemann-Liouville fractional derivative [11].

**Theorem 3.1.** Let \(p \geq 0, \Re(\mu) > \Re(\delta) > 0, \Re(\alpha) > 0, \Re(\beta) > 0.\) Then

\[\mathcal{D}_z^{\delta-\mu,p} \left\{ z^{\delta-1} E_{\alpha,\beta}(z) \right\} = \frac{z^{\mu-1} B(\delta, \gamma-\delta)}{\Gamma(\mu-\delta)} E_{\alpha,\beta}(z; p) \quad (3.7)\]

**Proof.** Replacing \(\mu\) by \(\delta - \mu\) in the definition of extension of extended fractional derivative formula [13], we have

\[\mathcal{D}_z^{\delta-\mu,p} \left\{ z^{\delta-1} E_{\alpha,\beta}(z) \right\} \]

\[= \frac{1}{\Gamma(\mu-\delta)} \int_0^1 t^{\delta-1} E_{\alpha,\beta}(t)(z-t)^{-\delta+\mu-1} \cdot F_1 \left[ \lambda; \rho; \left(-\frac{p2^2}{t(z-t)}\right) \right] dt \]

\[= \frac{1}{\Gamma(\mu-\delta)} \int_0^1 t^{\delta-1} E_{\alpha,\beta}(t)(1-t)^{-\delta+\mu-1} \cdot F_1 \left[ \lambda; \rho; \left(-\frac{p2^2}{t(z-t)}\right) \right] dt \]

Taking \(u = \frac{t}{z}\) in above equation, we have

\[\mathcal{D}_z^{\delta-\mu,p} \left\{ z^{\delta-1} E_{\alpha,\beta}(z) \right\} \]

\[= \frac{z^{\mu-1}}{\Gamma(\mu-\delta)} \int_0^z u^{\delta-1} (1-u)^{-\delta+\mu-1} \cdot F_1 \left[ \lambda; \rho; \left(-\frac{p}{u(1-u)}\right) \right] \cdot E_{\alpha,\beta}(uz) du \quad (3.8)\]

Comparing equation [3.8] with equation [2.1], we get the desired result. \qed
In the following theorem we define the derivative properties of extended Mittag-Leffler function.

**Theorem 3.2.** For the extended Mittag-Leffler function, we have the following derivative formula:

\[
\frac{d^n}{dz^n} \left\{ E_{\gamma,c}^{\alpha,\beta}(z;p) \right\} = \frac{(c)_n (\lambda)_n}{(\rho)_n} E_{\gamma+n,c+n;\lambda+n,\rho+n}^{\alpha+n,\beta+n}(z;p).
\]  

**(3.9)**

**Proof.** Taking derivative of equation (1.8) with respect to \( z \), we have

\[
\frac{d}{dz} \left\{ E_{\gamma,c}^{\alpha,\beta}(z;p) \right\} = \frac{c \lambda}{\rho} E_{\gamma+1,c+1;\lambda+1,\rho+1}^{\alpha,\beta+1}(z;p).
\]  

**(3.10)**

Again taking derivative of equation (3.10), with respect to \( z \), we have

\[
\frac{d^2}{dz^2} \left\{ E_{\gamma,c}^{\alpha,\beta}(z;p) \right\} = \frac{c(c+1) \lambda(\lambda+1)}{\rho(\rho+1)} E_{\gamma+2,c+2;\lambda+2,\rho+2}^{\alpha+2,\beta+2}(z;p).
\]  

**(3.11)**

Continuing in this way up to \( n \), we get the required result. \( \Box \)

**Theorem 3.3.** The following differentiation formula holds for the extended Mittag-Leffler function

\[
\frac{d^n}{dz^n} \left\{ z^{\beta-1} E_{\alpha,\beta}^{\gamma,\lambda}(\mu z;\mu) \right\} = z^{\beta-n-1} E_{\alpha,\beta-n}^{\gamma+n,c+n;\lambda+n,\rho+n}(\mu z^n;\mu).
\]  

**(3.12)**

**Proof.** In equation (3.9), replace \( z \) by \( \mu z^\alpha \) and multiply by \( z^{\beta-1} \) and then taking \( nth \) derivative with respect to \( z \), we get the required result. \( \Box \)

### 4. Conclusion

In this paper, we established further extension of extended Mittag-Leffler recently introduced by [13]. We conclude that if \( \lambda = \rho \), then we get the results of extended Mittag-Leffler function.

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