FLUCTUATIONS OF THE SPECTRUM IN ROTATIONALLY INVARIANT RANDOM MATRIX ENSEMBLES

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Abstract. We investigate traces of powers of random matrices whose distributions are invariant under rotations (with respect to the Hilbert–Schmidt inner product) within a real-linear subspace of the space of $n \times n$ matrices. The matrices we consider may be real or complex, and Hermitian, antihermitian, or general. We use Stein’s method to prove multivariate central limit theorems, with convergence rates, for these traces of powers, which imply central limit theorems for polynomial linear eigenvalue statistics. In contrast to the usual situation in random matrix theory, in our approach general, nonnormal matrices turn out to be easier to study than Hermitian matrices.

1. Introduction

The limiting behavior of the eigenvalues of random matrices is a central problem in modern probability, with applications and connections in statistics, physics, and beyond. The eigenvalues of the classical ensembles have been studied extensively, and much is known. However, there are many other ensembles which are natural in applied contexts that have been less thoroughly explored. In this paper, we study the eigenvalues of rotationally invariant random matrix ensembles; i.e., probability measures on real-linear spaces of $n \times n$ matrices which are invariant under rotations within those spaces. We emphasize that this is different from the more common assumption of invariance under conjugation by orthogonal or unitary $n \times n$ matrices; ensembles with the latter invariance property are most often referred to as “matrix models” or as “orthogonally invariant” or “unitarily invariant” ensembles, respectively, but are unfortunately also sometimes referred to as rotationally invariant. The spaces we consider include the spaces of all real or complex $n \times n$ matrices, the space of all $n \times n$ Hermitian matrices, or others. The classical Gaussian random matrix ensembles are of this type, and so are random matrices chosen uniformly from the sphere with respect to the Hilbert–Schmidt norm. Beyond the classical Gaussian cases, such ensembles have been studied in the physics literature (see, e.g., [11, 14, 21, 26, 32]), frequently under the names “fixed trace ensembles” (for matrices uniformly distributed on a sphere for the Hilbert–Schmidt norm) or “norm-dependent ensembles”; fixed trace ensembles have also been investigated in the numerical analysis literature [10, 14, 15] and in more mathematically oriented work on random matrix theory [16, 20, 19].

In this paper we investigate the fluctuations of traces of powers of such random matrices, showing that these fluctuations have a jointly Gaussian distribution, under certain hypotheses, in the high-dimensional limit. This implies, in particular, that linear eigenvalue statistics $\sum_{j=1}^{n} f(\lambda_j)$ are asymptotically Gaussian, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of our random matrix and $f$ is a polynomial function. Gaussian limits for fluctuations of linear eigenvalue statistics have been studied intensively for other random matrix ensembles; we mention in particular [2, 3, 23, 27, 38, 39] for Wigner-type matrices (random Hermitian matrices whose entries on and above the diagonal are independent),
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for Haar-distributed random matrices from the classical compact groups, and [8, 33, 35, 36, 37] for the typically most difficult case of random matrices with all independent entries.

Our proofs are based on the infinitesimal or continuous version Stein’s method of exchangeable pairs, which has found a number of applications in random matrix theory, and which is particularly well suited to the analysis of settings like ours that exhibit continuous geometric symmetries. This method has been used to prove central limit theorems for linear eigenvalue statistics for various random matrix ensembles in [12, 22, 40, 41, 42]; other applications in random matrix theory appear in [7, 17, 28, 30]. We also mention [6], which does not apply Stein’s method for distributional approximation but uses a continuous family of exchangeable pairs to prove identities for expectations, similar to our proof of Theorem 1 below; and [5, 33], which apply other versions of Stein’s method to investigate linear eigenvalue statistics of random matrices.

An unusual feature of our proofs is that they allow a unified approach to both the Hermitian and non-Hermitian cases. More surprisingly, it turns out that, in contrast to the usual situation in random matrix theory, the non-Hermitian case is easier to handle here, for reasons that will be discussed below.

One can reasonably object that in the non-Hermitian case it is natural to consider more general linear eigenvalue statistics; for example in the polynomial setting one should allow the test function $f$ to be a polynomial in both $z$ and $\overline{z}$. However, as in the present work, most known central limit theorems for linear eigenvalue statistics for non-normal random matrices require $f$ to be analytic or otherwise highly restricted (as in, e.g., [33, 34, 35, 36]), and even so, the proofs are more difficult than in the Hermitian case. An exception is the very recent work [8], which handles random matrices with i.i.d. complex entries and test functions with only $2 + \epsilon$ derivatives.

We now turn to a more precise description of the random matrix ensembles we consider and our results.

The random matrices we consider are drawn from a real-linear subspace $V$ of the space $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ matrices over $\mathbb{C}$. We take $V$ to be one of the following: $\mathcal{M}_n(\mathbb{C})$ itself; the space $\mathcal{M}_n(\mathbb{R})$ of $n \times n$ matrices over $\mathbb{R}$; the space $\mathcal{M}_n^s(\mathbb{R})$ of real symmetric $n \times n$ matrices; the space $\mathcal{M}_n^s(\mathbb{C})$ of complex Hermitian $n \times n$ matrices; the space $\mathcal{M}_n^a(\mathbb{R})$ of real antisymmetric $n \times n$ matrices; and the space $\mathcal{M}_n^a(\mathbb{C})$ of complex anti-Hermitian $n \times n$ matrices. All of these spaces are real inner product spaces with respect to the inner product $\langle A, B \rangle = \text{Re} \text{tr} (AB^*)$, and have the associated Hilbert–Schmidt norm $\|A\| = \sqrt{\text{tr}(AA^*)}$. (We will also make some use of the complex (Hilbert–Schmidt) inner product, and the operator norm $\|A\|_{op}$.)

The distributions we consider on $V$ are rotationally invariant in the sense that they are invariant under linear isometries of the entire space $V$ equipped with this inner product; this is stronger than the more commonly considered property of invariance under multiplication or conjugation by a unitary matrix in $\mathcal{M}_n(\mathbb{C})$. If $X \in V$ has a rotationally invariant distribution, then we can write $X = \|X\| \tilde{X}$, where $\tilde{X}$ is uniformly distributed on the unit sphere (with respect to the Hilbert–Schmidt norm) of $V$ and is independent from $\|X\|$. (In fact, in the proofs below it will be convenient to use a slightly different normalization for $\tilde{X}$.)

Rotationally invariant distributions can also be described concretely in terms of orthonormal bases on each space. Let $E_{jk}$ denote the $n \times n$ matrix with a one in the $(j, k)$ position and zeroes everywhere else. For $j < k$, let $F_{jk} = \frac{1}{\sqrt{2}}(E_{jk} + E_{kj})$ and $G_{jk} = \frac{1}{\sqrt{2}}(E_{jk} - E_{kj})$. 
Let $d$ be the (real) dimension of $V$. We denote by $\{B_\alpha\}_{\alpha=1}^d$ orthonormal bases (with respect to the real inner product $\langle A, B \rangle = \text{Re} \, \text{tr}(AB^*)$) for the spaces $V$ above, as follows.

| $V$          | $\{B_\alpha\}_{\alpha=1}^d$ |
|--------------|-------------------------------|
| $\mathcal{M}_n(\mathbb{C})$ | $\{E_{jk}\}_{1 \leq j,k \leq n} \cup \{iE_{jk}\}_{1 \leq j,k \leq n}$ |
| $\mathcal{M}_n(\mathbb{R})$ | $\{E_{jk}\}_{1 \leq j,k \leq n}$ |
| $\mathcal{M}_n^s(\mathbb{R})$ | $\{E_{jj}\}_{1 \leq j \leq n} \cup \{F_{jk}\}_{1 \leq j < k \leq n}$ |
| $\mathcal{M}_n(\mathbb{C})$ | $\{E_{jj}\}_{1 \leq j \leq n} \cup \{F_{jk}\}_{1 \leq j < k \leq n} \cup \{iG_{jk}\}_{1 \leq j < k \leq n}$ |
| $\mathcal{M}_n^s(\mathbb{C})$ | $\{G_{jk}\}_{1 \leq j < k \leq n}$ |
| $\mathcal{M}_n^a(\mathbb{C})$ | $\{iE_{jj}\}_{1 \leq j \leq n} \cup \{G_{jk}\}_{1 \leq j < k \leq n} \cup \{iF_{jk}\}_{1 \leq j < k \leq n}$ |

For each choice of $V$, consider a random vector $\{X_\alpha\}_{\alpha=1}^d$ with a rotationally invariant distribution in $\mathbb{R}^d$, normalized so that $\mathbb{E} \sum_{\alpha=1}^d X_\alpha^2 = n$, and define

$$X = \sum_{\alpha=1}^d X_\alpha B_\alpha.$$  

The random matrix $X \in V$ then has a rotationally invariant distribution in $V$ and satisfies $\mathbb{E} \|X\|^2 = n$. Note that choosing the random vector $\{X_\alpha\}_{\alpha=1}^d$ according to a Gaussian distribution results in various classical random matrix ensembles: in the case of unrestricted real or complex matrices, we have the real, respectively complex Ginibre ensembles, and in the case of real symmetric or complex Hermitian matrices, we have the Gaussian Orthogonal Ensemble (GOE) and Gaussian Unitary Ensemble (GUE), respectively.

Our first main result identifies the means of the random variables $W_p = \text{tr} X^p$ for $p \in \mathbb{N}$. This result is essentially known, and can easily be deduced from the Gaussian cases, where the classical proofs make essential use of the independence of the entries. Here we give an independent proof which is an easy by-product of the analysis of the exchangeable pair used to prove Theorems 2 and 4 below. (As noted above, a similar approach was used in [6] to prove identities for expectations of functions of random orthogonal matrices.)

**Theorem 1.** Let $X$ be a random matrix in $V \subseteq \mathcal{M}_n(\mathbb{C})$ as above, whose distribution is invariant under rotations of $V$.

Suppose that $\mathbb{E} \|X\|^2 = n$ and that for each $k$, there is a constant $\alpha_k$ depending only on $k$ such that

$$t_k(X) = \left| n^{-k/2} \mathbb{E} \|X\|^k - 1 \right| \leq \frac{\alpha_k}{n}.$$  

For $p \in \mathbb{N}$, let $W_p = \text{tr}(X^p)$. In all cases, if $p$ is odd, then $\mathbb{E} W_p = 0$. For $p = 2r$,

$$\mathbb{E} W_p = \begin{cases} 0 & \text{if } V = \mathcal{M}_n(\mathbb{C}), \\ 1 + O\left(\frac{1}{n}\right) & \text{if } V = \mathcal{M}_n(\mathbb{R}), \\ nC_r + O(1) & \text{if } V = \mathcal{M}_n^s(\mathbb{C}) \text{ or } \mathcal{M}_n^s(\mathbb{R}), \\ (-1)^r nC_r + O(1) & \text{if } V = \mathcal{M}_n^a(\mathbb{C}) \text{ or } \mathcal{M}_n^a(\mathbb{R}), \end{cases}$$

where $C_r = \frac{1}{r+1} \binom{2r}{r}$ is the $r$th Catalan number.
In Theorem 1 as well as all the following results, the $O$ terms refer to $n \to \infty$, with implied constants that may depend on $p$ (or $m$ below) and the constants $\alpha_k$, but do not otherwise depend on the precise distribution of $X$.

In just the first case of Theorem 1 (when $V = M_n(\mathbb{C})$), the hypothesis on $t_k(X)$ can be replaced by the weaker assumption that $t_k(X) < \infty$ for each $k$; that is, simply that all moments of $\|X\|$ are finite. In the other cases that hypothesis can be weakened to assuming each $t_k(X)$ is $o(1)$, at the expense of more complicated version of the error terms. We have chosen here to assume a simple and quite mild hypothesis that lets us state a clean result.

Theorems 2 and 4 describe the fluctuations of the $W_p$, formulated as comparisons of integrals of $C^2$ test functions. In what follows, for $g \in C^2(\mathbb{R}^m)$,
\[ M_1(g) = \sup_{x \in \mathbb{R}^m} |\nabla g(x)| \]
denotes the Lipschitz constant of $g$ and
\[ M_2(g) = \sup_{x \in \mathbb{R}^m} \|\text{Hess} \,(g)(x)\|_{op} \]
the maximum operator norm of the Hessian of $g$. For $g : \mathbb{C}^m \to \mathbb{R}$, these quantities are computed by identifying $g$ with a function on $\mathbb{R}^{2m}$.

We begin with the cases of unrestricted real or complex $n \times n$ matrices.

**Theorem 2.** Let $X$ be a random matrix in $V = M_n(\mathbb{C})$ or $M_n(\mathbb{R})$, whose distribution is invariant under rotations of $V$.

Suppose that $\mathbb{E}\|X\|^2 = n$ and that for each $k$, there is a constant $\alpha_k$ depending only on $k$ such that
\[ t_k(X) = \left| n^{-k/2}\mathbb{E}\|X\|^k - 1 \right| \leq \frac{\alpha_k}{n}. \]

Fix $m \in \mathbb{N}$ with $m \geq 3$ and
\[ W = (W_1,W_2, \ldots ,W_m) = \left( \text{tr}(X),\text{tr}(X^2), \ldots ,\text{tr}(X^m) \right) \in \mathbb{R}^m. \]

1. If $V = M_n(\mathbb{C})$ and $G$ is a standard complex Gaussian random vector in $\mathbb{C}^m$, then for any $f \in C^2(\mathbb{C}^m)$,
\[ |\mathbb{E}f(W) - \mathbb{E}f(\Sigma^{1/2}G)| \leq \frac{\kappa_m M_2(f)}{n}, \]
where $\Sigma$ is the diagonal matrix with $p$-$p$ entry given by $\sigma_{pp} = p$ and $\kappa_m$ is a positive constant depending only on $m$ and $\alpha_1, \ldots ,\alpha_m$.

2. If $V = M_n(\mathbb{R})$ and $G$ is a standard Gaussian random vector in $\mathbb{R}^m$, then for any $f \in C^2(\mathbb{R}^m)$,
\[ |\mathbb{E}f(W - EW) - \mathbb{E}f(\Sigma^{1/2}G)| \leq \frac{\kappa_m(M_1(f) + M_2(f))}{n}, \]
where $\Sigma$ is the diagonal matrix with $p$-$p$ entry given by $\sigma_{pp} = p$ and $\kappa_m$ is a positive constant depending only on $m$ and $\alpha_1, \ldots ,\alpha_m$.

As in Theorem 1 the hypothesis on $t_k$ can be weakened somewhat, at the expense of a more complicated version of the conclusion.

Theorem 2 immediately implies the following.

**Corollary 3.** For each $n$, let $X_n$ be an $n \times n$ satisfying the hypotheses of Theorem 2 (with constants $\alpha_k$ independent of $n$). Given any polynomial function, $g : \mathbb{C} \to \mathbb{C}$, define $X_{g,n} = \text{tr} \, g(X_n)$. Then the stochastic process \[ \{X_{g,n} - ng(0)\}_{g} \]
indexed by polynomials converges
as \( n \to \infty \), in the sense of finite dimensional distributions, to a centered complex-valued Gaussian process \( \{ Z_g \}_g \) with covariance given by

\[
\mathbb{E} Z_g Z_h = \frac{1}{\pi} \int_D g'(z) h'(z) \, d^2 z,
\]

where \( D = \{ z \in \mathbb{C} \mid |z| \leq 1 \} \) and \( d^2 z \) refers to integration with respect to Lebesgue measure.

Results of the same form as Corollary 3 are proved in [36, 8] and [33] for random matrices with independent complex or real entries (satisfying some technical conditions), respectively. In [36] the test functions need not be polynomials, but are required to be analytic on a neighborhood of the disc of radius 4; in [8] this is weakened substantially to \( C^{2+\epsilon} \) test functions. One might hope to extend Corollary 3 from polynomials to analytic or still more general test functions by approximation (as is done, for example, in [12] in the case of Haar-distributed random unitary matrices). However, the dependence on the constants \( \kappa_m \) in Theorem 2 on \( m \) provided by our proofs is insufficient to carry out such an approximation argument. (Moreover, as seen in [37, 8], extending beyond analytic functions requires a more complicated description of the limiting covariance structure.)

There are several key differences between the Hermitian case and the case of unrestricted complex matrices, the most crucial of which is that \( W_2 = \text{tr}(X^2) = \text{tr}(XX^*) = \|X\|^2 \) when \( X \) is Hermitian. In particular, a multivariate central limit theorem cannot hold in general for the vector

\[
W = (W_1, W_2, \ldots, W_m)
\]

because the second component need not have Gaussian fluctuations. In the case of \( X \) uniformly distributed on the sphere of radius \( \sqrt{n} \) in \( M_n(\mathbb{C}) \), \( W_2 \) is deterministic, and so one could hope for a central limit theorem involving a covariance matrix of rank \( m-1 \) (and indeed this is the case).

A related difference from the non-Hermitian case is that \( \mathbb{E} W_p \) is of order \( n \) for all even \( p \) in the Hermitian case; a consequence of this fact is that it is necessary to make a stronger (though still rather mild) concentration hypothesis for \( \|X\| \) than in Theorems 1 and 2.

**Theorem 4.** Let \( X \) be a random matrix in \( V = M_n(\mathbb{C}) \) or \( M_n(\mathbb{R}) \), whose distribution is invariant under rotations of \( V \).

Suppose that \( \mathbb{E} \|X\|^2 = n \) and that for each \( k \), there is a constant \( \alpha_k \) depending only on \( k \) such that

\[
t_k(X) = \left| n^{-k/2} \mathbb{E} \|X\|^k - 1 \right| \leq \frac{\alpha_k}{n^2}.
\]

Fix \( m \in \mathbb{N} \) with \( m \geq 3 \) and

\[
W = (W_1, W_2, \ldots, W_m) = (\text{tr}(X), \text{tr}(X^2), \ldots, \text{tr}(X^m)) \in \mathbb{R}^m.
\]

Let \( Y_2 = \|X\|^2 - n \) and define

\[
Z = (Z_1, Z_3, Z_4, \ldots, Z_m) \quad Z_p = W_p - \mathbb{E} W_p - \frac{p \mathbb{E} W_p}{2n} Y_2.
\]

Let \( \Sigma = A^{-1}B \), where \( A \) and \( B \) are indexed by \( \{1, \ldots, m\} \setminus \{2\} \) with entries

\[
a_{pq} = \begin{cases} 
-2pC_{(p-2-q)/2} & \text{if } 1 \leq q \leq p-2 \text{ and } p-q \text{ is even}, \\
p & \text{if } q = p, \\
0 & \text{otherwise}.
\end{cases}
\]
and

\[ b_{pq} = 2pq \begin{cases} 
C_{(p+q-2)/2} - C_{p/2}C_{q/2} & \text{if } p \text{ and } q \text{ are both even}, \\
C_{(p+q-2)/2} & \text{if } p \text{ and } q \text{ are both odd}, \\
0 & \text{if } p \text{ and } q \text{ have opposite parities}.
\end{cases} \]

and \( C_r \) again denotes the \( r \)th Catalan number. Then for any \( f \in C^2(\mathbb{R}^{m-1}) \),

\[
|\mathbb{E}f(Z) - \mathbb{E}f(\Sigma^{1/2}G)| \leq \frac{\kappa_m(M_1(f) + M_2(f))}{n},
\]

where \( \kappa_m \) is a positive constant depending only on \( m \) and \( \alpha_1, \ldots, \alpha_m \), and \( G \) is a standard Gaussian random vector in \( \mathbb{R}^{m-1} \).

It is not obvious from the form given in the statement of Theorem 4 that the covariance matrix \( \Sigma \) is symmetric, let alone positive semidefinite. It will, however, follow from the proof of Theorem 4 that this is indeed the case.

Theorem 4 immediately implies a multivariate central limit theorem for traces of odd powers of \( X \). It also implies a central limit theorem for traces of powers other than 2 if \( X \) is uniformly distributed on the sphere of radius \( \sqrt{n} \) in \( M_n^a(\mathbb{C}) \) or \( M_n^a(\mathbb{R}) \), or more generally if \( \mathbb{E}|Y_2| = o(1) \). In either of those two situations, one can deduce a result analogous to Corollary 3 although with more complicated expressions both for the means (as seen in Theorem 4) and for the covariance; for brevity we omit a precise statement here. Analogous results for Wigner matrices, in various levels of generality, were proved in [23, 27, 38, 39].

Rotationally invariant ensembles of antihermitian matrices reduce to the Hermitian case: if \( X \) is a rotationally invariant Hermitian random matrix, then \( iX \) is a rotationally invariant antihermitian matrix, and in particular \( \text{tr}(iX)^p = i^p \text{tr}(X^p) \). A version of Theorem 4 for antihermitian matrices is therefore a formal consequence of Theorem 4 itself. The explicit statement will be somewhat complicated, however, since the random vector \( Z \) will be distributed in a particular \((m-1)\)-dimensional real subspace of \( \mathbb{C}^{m-1} \).

In contrast, the case of real antisymmetric matrices requires an independent analysis. Note in particular that if \( X \in M_n^a(\mathbb{R}) \) then \( \text{tr}(X^p) = 0 \) for every odd \( p \). We have the following result for such random matrices.

**Theorem 5.** Let \( X \) be a random matrix in \( V = M_n^a(\mathbb{R}) \) whose distribution is invariant under rotations of \( V \).

Suppose that \( \mathbb{E}||X||^2 = n \) and that for each \( k \), there is a constant \( \alpha_k \) depending only on \( k \) such that

\[
t_k(X) = \left| n^{-k/2} \mathbb{E}||X||^k - 1 \right| \leq \frac{\alpha_k}{n^2}.
\]

Let \( m \geq 4 \) be even and

\[
W = (W_4, W_6, \ldots, W_m) = (\text{tr}(X^4), \text{tr}(X^6), \ldots, \text{tr}(X^m)).
\]

Then for each even \( p \),

\[
\mathbb{E}W_p = (-1)^{p/2}nC_{p/2} + O(1),
\]

where \( C_{p/2} \) is the \( p/2 \) Catalan number.

Let \( Y_2 = ||X||^2 - n \) and define

\[
Z = (Z_4, Z_6, \ldots, Z_m) \quad \quad Z_p = W_p - \mathbb{E}W_p - (-1)^{p/2}p\mathbb{E}W_p \frac{Y_2}{2n}.
\]
Let \( \Sigma = A^{-1}B \), where \( A \) and \( B \) have entries (for \( p, q \geq 4 \) even)

\[
a_{pq} = \begin{cases} 
(-1)^{(p-q)/2}2C_{(p-2-q)/2} & \text{if } 4 \leq q \leq p-2, \\
1 & \text{if } q = p, \\
0 & \text{otherwise,}
\end{cases}
\]

and

\[
b_{pq} = (-1)^{(p+q)/2}q(C_{(p+q-2)/2} - C_{p/2}C_{q/2}).
\]

Then for any \( f \in C^2(\mathbb{R}^{(m-2)/2}) \),

\[
\left| E f(Z) - E f(\Sigma^{1/2}G) \right| \leq \frac{\kappa_m(M_1(f) + M_2(f))}{n},
\]

where \( \kappa_m \) is a positive constant depending only on \( m \) and \( \alpha_1, \ldots, \alpha_m \), and \( G \) is a standard Gaussian random vector in \( \mathbb{R}^{(m-2)/2} \).

To our knowledge, the only previous paper whose results explicitly include a central limit theorem for linear eigenvalue statistics of real antisymmetric random matrices is [34], although the methods of most previous works on Hermitian random matrices could presumably be adapted to cover the real antisymmetric case as well.

Our results are proved using a general Gaussian approximation theorem for exchangeable pairs [29] [12]. In section 2 below we state the general approximation theorem, define the exchangeable pair for an arbitrary matrix subspace \( V \), and carry out as much of the analysis as possible without specifying \( V \); this may be characterized as the essentially “algebraic” part of our proofs. The remaining sections carry out the “asymptotic” part of the argument, on a case-by-case basis, for each of the subspaces \( V \) considered here. In sections 3 and 4 we prove Theorems 1 and 2 for the cases of \( V = M_n(\mathbb{C}) \) and \( M_n(\mathbb{R}) \), respectively. In section 5 we prove Theorems 1 and 4 for \( V = M_n^s(\mathbb{C}) \). In section 6 we indicate how to modify the proofs of section 5 for \( V = M_n^s(\mathbb{R}) \). The proof of Theorem 5 is yet another variation on the same theme, and is omitted.

2. Common framework: The exchangeable pair

As discussed in the introduction, the proof of Theorem 1 is essentially a by-product of the proofs of Theorems 2 and 4 and so we postpone the proof of Theorem 1 for the moment. The other main theorems are proved via a version of Stein’s method. The complex form of the multivariate infinitesimal version of Stein’s method of exchangeable pairs stated below is due to Döbler and Stolz [12], following earlier work of E. Meckes [29] in the real case.

**Theorem 6.** Let \( W \) be a centered random vector in \( \mathbb{C}^m \) and, for each \( \epsilon \in (0, 1) \), suppose that \( (W, W_\epsilon) \) is an exchangeable pair. Let \( \mathcal{G} \) be a \( \sigma \)-algebra with respect to which \( W \) is measurable. Suppose that there is an invertible matrix \( \Lambda \), a symmetric, non-negative definite matrix \( \Sigma \), a \( \mathcal{G} \)-measurable random vector \( E \in \mathbb{C}^m \), \( \mathcal{G} \)-measurable random matrices \( E', E'' \in M_m(\mathbb{C}) \), and a deterministic function \( s(\epsilon) \) such that

\[
\frac{1}{s(\epsilon)} \mathbb{E} \left[ W_\epsilon - W \left| \mathcal{G} \right. \right] \xrightarrow[\epsilon \to 0]{L_1} -\Lambda W + E,
\]

\[
\frac{1}{s(\epsilon)} \mathbb{E} \left[ (W_\epsilon - W)(W_\epsilon - W)^* \left| \mathcal{G} \right. \right] \xrightarrow[\epsilon \to 0]{L_1(\|\cdot\|)} 2\Lambda \Sigma + E',
\]
transformation of $X$ is constructed as follows. As above, let $\{W_\epsilon\}$ be a parametrized family of exchangeable pairs of random matrices in $\mathbb{C}^d$ with a rotationally invariant distribution and $\{X_1, Y_1, \ldots, X_m, Y_m\}$ be i.i.d. $N(0, \frac{1}{\epsilon})$, and $|E|$ denotes the Euclidean norm of the random vector $E$.

Remarks:

(1) To recover the real case of Theorem 6, one omits condition (3) and the term $\mathbb{E} \|E''\|$ in (1). The real case will be used for all the proofs below except for the case of $V = M_n(\mathbb{C})$.

(2) In practice, we typically replace condition (4) with the formally stronger condition

$$\lim_{\epsilon \to 0} \frac{1}{s(\epsilon)} \mathbb{E} |W_\epsilon - W|^3 = 0.$$ 

This condition is trivially satisfied in our applications, since $W_\epsilon$ is constructed so that $W_\epsilon - W = \epsilon Y$ for some random vector $Y$ with $\mathbb{E}|Y|^3 < \infty$.

A parametrized family $(X, X_\epsilon)$ of exchangeable pairs of random matrices can be constructed as follows. As above, let $X = \sum_{\alpha=1}^d X_\alpha B_\alpha$, where $\{X_\alpha\}_{\alpha=1}^d$ is a random vector in $\mathbb{R}^d$ with a rotationally invariant distribution and $\{B_\alpha\}_{\alpha=1}^d$ is an orthonormal basis of a $d$-dimensional subspace $V$ of $M_n(\mathbb{C})$. We assume that $\mathbb{E}\|X\|^2 = n$ and that $\mathbb{E}\|X\|^{2m} < \infty$. For a $d \times d$ matrix $A = [a_{jk}]_{j,k=1}^d$ in the orthogonal group $O(d)$, denote by $A(X)$ the transformation of $X$ given by

$$A(X) = \sum_{\alpha=1}^d \left( \sum_{\beta=1}^d a_{\alpha\beta} X_\beta \right) B_\alpha.$$ 

Now fix $\epsilon$, and let

$$R_\epsilon = \left[ \sqrt{1 - \epsilon^2} \quad \epsilon \right] \left[ -\frac{\epsilon}{\sqrt{1 - \epsilon^2}} \right] \oplus I_{d-2} \in O(d).$$ 

That is, $R_\epsilon$ represents a rotation by $\arcsin(\epsilon)$ in the plane spanned by the first two standard basis vectors of $\mathbb{R}^d$. Choose $U \in O(d)$ according to Haar measure, independent of $X$, and let

$$X_\epsilon = (UR_\epsilon U^T)(X).$$ 

That is, $X_\epsilon$ is a small random rotation (in matrix space) of the random matrix $X$, and so $(X, X_\epsilon)$ is exchangeable for each $\epsilon$. For each $p \in \{1, \ldots, m\}$, define

$$W_{\epsilon,p} := \text{tr}(X_{\epsilon,p});$$

the $m$-dimensional random vectors $(W, W_\epsilon)$ are then exchangeable for each $\epsilon$. 

\begin{enumerate}
\item $\frac{1}{s(\epsilon)} \mathbb{E} \left[ (W_\epsilon - W)(W_\epsilon - W)^T \right] = \frac{L_1(\|\cdot\|)}{\epsilon \to 0} E''.$
\item For each $\rho > 0,$

$$\lim_{\epsilon \to 0} \frac{1}{s(\epsilon)} \mathbb{E} \|W_\epsilon - W\|^2 I(|W_\epsilon - W|^2 > \rho) = 0.$$
\end{enumerate}
To apply Theorem 6, the difference $W_\epsilon - W$ must be expanded in powers of $\epsilon$. First,

$$UR_d U^T = U \left[ I_d + \epsilon C \oplus 0_{d-2} + \left( -\frac{\epsilon^2}{2} + O(\epsilon^4) \right) I_2 \oplus 0_{d-2} \right] U^T,$$

where $0_n$ is the $n \times n$ matrix of all zeroes, $C$ is the $2 \times 2$ matrix

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and the $O(\epsilon^4)$ is the deterministic error in replacing $\sqrt{1 - \epsilon^2} - 1$ by $-\frac{\epsilon^2}{2}$. Letting $K$ denote the first two columns of $U$ and $Q := KK^T$, we have

$$UR_d U^T = I_d + \epsilon Q + \left( -\frac{\epsilon^2}{2} + O(\epsilon^4) \right) KK^T.$$  

It follows that

(2) $W_{\epsilon,p} - W_p$

$$= \text{tr}(X_p^p - X^p)$$

$$= \text{tr}\left( \left[ X + \epsilon Q(X) + \left( -\frac{\epsilon^2}{2} + O(\epsilon^4) \right) KK^T(X) \right]^p - X^p \right)$$

$$= \epsilon \sum_{j=0}^{p-1} \text{tr}(X^j(Q(X)]X^{p-1-j})$$

$$+ \epsilon^2 \sum_{j=0}^{p-2} \sum_{k=0}^{p-2-j} \text{tr}\left( X^j(Q(X)]X^k[Q(X)]X^{p-2-j-k} \right) - \frac{1}{2} \sum_{j=0}^{p-1} \text{tr}(X^j[KK^T(X)]X^{p-1-j})$$

$$+ O(\epsilon^3)$$

$$= \epsilon p \text{tr}(X^{p-1}[Q(X)])$$

$$+ \epsilon^2 \sum_{\ell=0}^{p-2} (\ell + 1) \text{tr}\left( X^\ell(Q(X)]X^{p-2-\ell}[Q(X)] \right) - \frac{p}{2} \text{tr}(X^{p-1}[KK^T(X)])$$

$$+ O(\epsilon^3),$$

where the implied constant in the error $O(\epsilon^3)$ is a random variable (with all moments finite) depending on $X$ and $U$. (The $O(\epsilon^3)$ terms here and below may depend on $E \|X\|^{2m}$, and hence are not necessarily uniform in either $n$ or $m$ without more assumptions than have been made up to this point. However, in Theorem 6 the limits as $\epsilon \to 0$ are taken with $n$ and $m$ both fixed, so this poses no difficulty.)

Analyzing this expression comes down to integrals over the orthogonal group $O(d)$ and over the sphere $S^{d-1}$. The following concentration result from 31 plays an important technical role. Given a polynomial $Q(x,y)$ in two variables, we refer to a function $P(X) = Q(X, X^*)$ on $M_n(\mathbb{C})$ as a $*$-polynomial.

**Proposition 7.** Let $P$ be a $*$-polynomial of degree at most $p$, and let $X$ be a random $n \times n$ matrix uniformly distributed in a sphere of radius $\sqrt{n}$ in a subspace of $M_n(\mathbb{C})$ of dimension $d \geq cn^2$. Then

$$P[ \text{tr} P(X) - E \text{tr} P(X) | \geq t] \leq \kappa_p \exp[-c_p \min\{t^2, n^{2/p}\}]$$

and

$$\| \text{tr} P(X) - E \text{tr} P(X) \|_q \leq C_p \max \left\{ \sqrt{q}, \left( \frac{n}{q} \right)^{p/2} \right\}$$
for each \( q \geq 1 \). Here \( \kappa_p, c_p, C_p \geq 0 \) are constants depending only on \( p \) and \( c \), and \( \|Y\|_q = (\mathbb{E}|Y|^q)^{1/q} \) denotes the \( L_q \) norm of a random variable.

The following lemma is key in applying Theorem 6.

**Lemma 8.** For \( W \) as above and \( p \) fixed,

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} [W_{\epsilon, p} - W_p | X] = \sum_{\ell=0}^{p-2} \frac{2(\ell + 1)\|X\|^2}{d(d-1)} \text{tr} \left( X^\ell \sum_{\alpha} B_\alpha X^{p-2-\ell} B_\alpha \right) - \frac{p(p + d - 2)}{d(d-1)} W_p, \tag{1}
\]

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} \left[ (W_{\epsilon} - W)_p (W_{\epsilon} - W)_q | X \right] = \frac{2pq}{d(d-1)} \left[ \|X\|^2 \sum_{\alpha=1}^d \text{tr}(X^{p-1}B_\alpha)\text{tr}(X^{q-1}B_\alpha) - \sum_{\alpha, \beta=1}^d X_\alpha X_\beta \text{tr}(X^{p-1}B_\alpha)\text{tr}(X^{q-1}B_\beta) \right], \tag{2}
\]

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} [(W_{\epsilon} - W)_p (W_{\epsilon} - W)_q | X] = \frac{2pq}{d(d-1)} \left[ \|X\|^2 \sum_{\alpha=1}^d \text{tr}(X^{p-1}B_\alpha)\text{tr}(X^{q-1}B_\alpha) - \sum_{\alpha, \beta=1}^d X_\alpha X_\beta \text{tr}(X^{p-1}B_\alpha)\text{tr}(X^{q-1}B_\beta) \right], \tag{3}
\]

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} |W_{\epsilon} - W|^3 = 0. \tag{4}
\]

*In each case, the convergence is in the \( L_1 \) sense.*

**Proof.** By the expansion of \( W_{\epsilon, p} - W_p \) in powers of \( \epsilon \) given in (2), it follows from the independence of \( X \) and \( U \) that

\[
\mathbb{E} [W_{\epsilon, p} - W_p | X] = \epsilon p \text{tr}(X^{p-1} \mathbb{E} \left[ Q(X) | X \right]) + \epsilon^2 \sum_{\ell=0}^{p-2} (\ell + 1) \mathbb{E} \left[ \text{tr} \left( X^\ell \mathbb{E} \left[ Q(X) | X \right] X^{p-2-\ell} \mathbb{E} \left[ Q(X) | X \right] \right) | X \right] - \frac{p}{2} \text{tr} \left( X^{p-1} \mathbb{E} \left[ KK^T(X) | X \right] \right) + O(\epsilon^3),
\]

where here and in what follows, the implied constants in the error term are random but bounded in \( L_1 \).

The entries of \( KK^T \) and \( Q \) are given in terms of the entries of \( U = [u_{jk}]_{j,k=1}^d \) by

\[
[KK^T]_{jk} = u_{j1}u_{k1} + u_{j2}u_{k2}, \quad [Q]_{jk} = u_{j1}u_{k2} - u_{j2}u_{k1}.
\]

From this it is easy to see that

\[
\mathbb{E}[KK^T] = \frac{2}{d} I_d, \quad \mathbb{E}[Q] = 0,
\]

where \( I_d \) is the identity matrix of dimension \( d \).
and thus

\[ E [ W_{\epsilon, p} - W_p | X ] = \epsilon^2 \left[ \sum_{\ell=0}^{p-2} (\ell + 1) E \left[ \text{tr} \left( X^\ell [Q(X)] X^{p-2-\ell} [Q(X)] \right) \right] X \right] - \frac{p}{d} W_p + O(\epsilon^3). \]  

(3)

Now,

\[ E \left[ \text{tr} \left( X^\ell [Q(X)] X^{p-2-\ell} [Q(X)] \right) \right] X = \text{tr} \left( X^\ell E \left[ [Q(X)] X^{p-2-\ell} [Q(X)] \right] X \right). \]

For notational convenience, write \( A := X^{p-2-\ell} \). If \( q_{\alpha\beta} \) denotes the \((\alpha, \beta)\) entry of \( Q \), then by expanding in the basis \( \{B_j\} \),

\[ [Q(X)] A [Q(X)] = \sum_{\alpha,\beta,\gamma,\delta=1}^{d} q_{\alpha\beta} q_{\gamma\delta} X^{\beta} \sum_{\alpha} B_{\alpha} A B_{\gamma} \]

and so

\[ E \left[ [Q(X)] X^{p-2-\ell} [Q(X)] \right] X = \sum_{\alpha,\beta,\gamma,\delta=1}^{d} E [q_{\alpha\beta} q_{\gamma\delta}] X^{\beta} \sum_{\alpha} B_{\alpha} A B_{\gamma}. \]

The formulae above for \( q_{\alpha\beta} \) in terms of the entries of \( U \) can be used to derive the following (see Lemma 9 of [7])

\[ E [q_{\alpha\beta} q_{\gamma\delta}] = \frac{2}{d(d-1)} [\delta_{\alpha\gamma} \delta_{\beta\delta} - \delta_{\alpha\delta} \delta_{\beta\gamma}], \]

and so

\[ E \left[ [Q(X)] X^{p-2-\ell} [Q(X)] \right] X = \frac{2}{d(d-1)} \left[ \sum_{\alpha,\beta} X^2 B_{\alpha} A B_{\beta} - \sum_{\alpha,\beta} X^{\beta} \sum_{\alpha} X_{\alpha} X_{\beta} A B_{\beta} \right] \]

\[ = \frac{2}{d(d-1)} \left[ \|X\|^2 \sum_{\alpha} B_{\alpha} A B_{\alpha} - XAX \right]. \]

It thus follows from (3) that

\[ E [ W_{\epsilon, p} - W_p | X ] \]

\[ = \epsilon^2 \left[ \sum_{\ell=0}^{p-2} \frac{2(\ell + 1)}{d(d-1)} \left( \|X\|^2 \text{tr} \left( X^\ell \sum_{\alpha} B_{\alpha} X^{p-2-\ell} B_{\alpha} \right) - W_p \right) - \frac{p}{d} W_p \right] + O(\epsilon^3) \]

\[ = \epsilon^2 \left[ \sum_{\ell=0}^{p-2} \frac{2(\ell + 1)}{d(d-1)} \|X\|^2 \text{tr} \left( X^\ell \sum_{\alpha} B_{\alpha} X^{p-2-\ell} B_{\alpha} \right) - \frac{p(p+d-2)}{d(d-1)} W_p \right] + O(\epsilon^3), \]

whence the statement of part [4] of the lemma.
For part 2 again using the expansion of $W_e - W$ in (2) yields
\[
\mathbb{E} \left[ (W_e - W)_p (W_e - W)_q | X \right] = \epsilon^2 pq \mathbb{E} \left[ \text{tr}(X^{p-1} [Q(X)] \text{tr}(X^{q-1} [Q(X)]) | X \right] + O(\epsilon^3)
\]
\[
= \epsilon^2 pq \mathbb{E} \left[ \text{tr} \left( \sum_{\alpha,\beta=1}^{d} q_{\alpha\beta} X_{\beta} X^{p-1} B_{\alpha} \right) \text{tr} \left( \sum_{\gamma,\delta=1}^{d} q_{\gamma\delta} X_{\delta} X^{q-1} B_{\gamma} \right) | X \right] + O(\epsilon^3)
\]
\[
= \epsilon^2 pq \sum_{\alpha,\beta,\gamma,\delta=1}^{d} \mathbb{E} [q_{\alpha\beta} q_{\gamma\delta}] X_{\beta} X_{\delta} \text{tr}(X^{p-1} B_{\alpha}) \text{tr}(X^{q-1} B_{\gamma}) + O(\epsilon^3).
\]
Making use of the moment formula for $Q$ given in (4) then gives that
\[
\mathbb{E} \left[ (W_e - W)_p (W_e - W)_q | X \right] = \frac{2\epsilon^2 pq}{d(d-1)} \left[ \sum_{\alpha,\beta=1}^{d} X_{\beta}^2 \text{tr}(X^{p-1} B_{\alpha}) \text{tr}(X^{q-1} B_{\alpha}) - \sum_{\alpha,\beta=1}^{d} X_{\alpha} X_{\beta} \text{tr}(X^{p-1} B_{\alpha}) \text{tr}(X^{q-1} B_{\beta}) \right] + O(\epsilon^3).
\]

Exactly the same argument for part 3 gives that
\[
\mathbb{E} \left[ (W_e - W)_p (W_e - W)_q | X \right] = \frac{2\epsilon^2 pq}{d(d-1)} \left[ \sum_{\alpha,\beta=1}^{d} X_{\beta}^2 \text{tr}(X^{p-1} B_{\alpha}) \text{tr}(X^{q-1} B_{\alpha}) - \sum_{\alpha,\beta=1}^{d} X_{\alpha} X_{\beta} \text{tr}(X^{p-1} B_{\alpha}) \text{tr}(X^{q-1} B_{\beta}) \right] + O(\epsilon^3).
\]

Finally, it is clear from the expansion in $\epsilon$ that
\[
\mathbb{E} |W_e - W|^3 = O(\epsilon^3),
\]
which completes the proof. \qed

At this point in the analysis, it is necessary to consider the various subspaces separately; this is carried out in the following sections.

3. Rotationally invariant ensembles in $M_n(\mathbb{C})$

We begin with the following technical lemma.

**Lemma 9.** Let $X$ be a random matrix in $M_n(\mathbb{C})$ whose distribution is invariant under rotations within $M_n(\mathbb{C})$. Suppose that $\mathbb{E} \|X\|^2 = n$ and that for each $k$, there is a constant $\alpha_k$ depending only on $k$ such that
\[
t_k(X) = \left| n^{-k/2} \mathbb{E} \|X\|^k - 1 \right| \leq \frac{\alpha_k}{n}.
\]
Then for $p, q \in \mathbb{N}$,
\[
\mathbb{E} \left[ \|X\|^2 \text{tr}(X^p (X^*)^q) \right] = \begin{cases} n^2 + O(n), & p = q; \\ 0, & \text{otherwise.} \end{cases}
\]

Here, the implied constant in the $O(n)$ may depend on $p, q$, and the constants $\alpha_k$. 


Proof. For \( p \neq q \), \( \mathbb{E} \left[ \|X\|^2 \text{tr}(X^p(X^q)^*) \right] = 0 \) by symmetry. We suppose from now on that \( p = q \).

By the rotational invariance of \( X \), we can write \( X = \frac{\|X\|}{\sqrt{n}} \tilde{X} \), where \( \tilde{X} \) is uniformly distributed on the sphere of radius \( \sqrt{n} \) in \( M_n(\mathbb{C}) \) and \( \tilde{X} \) is independent from \( \|X\| \). We then have

\[
\mathbb{E} \left[ \|X\|^2 \text{tr}(X^p(X^*)^p) \right] = \left( \frac{\mathbb{E} \|X\|^{2p+2}}{n^{p+1}} \right) n\mathbb{E} \text{tr}(\tilde{X}^p(\tilde{X}^*)^p),
\]

and thus

\[
(5) \quad \left| \mathbb{E} \left[ \|X\|^2 \text{tr}(X^p(X^*)^p) \right] - n\mathbb{E} \text{tr}(\tilde{X}^p(\tilde{X}^*)^p) \right| \leq nt_{2p+2}(X)\mathbb{E} \text{tr}(\tilde{X}^p(\tilde{X}^*)^p).
\]

It therefore suffices to prove the lemma under the assumption that \( X \) is uniformly distributed on the sphere of radius \( \sqrt{n} \) in \( M_n(\mathbb{C}) \); the general case follows from (5) and the assumption on \( t_k(X) \).

Making this assumption, we now consider the expansion

\[
(6) \quad \mathbb{E} \text{tr}(X^p(X^*)^p) = \sum_{i_1, \ldots, i_{2p}} \mathbb{E} \left[ x_{i_1i_2} x_{i_2i_3} \cdots x_{i_{2p-1}i_{2p}} x_{i_{2p}i_{2p+1}} x_{i_{2p+1}i_{2p+2}} \cdots x_{i_1i_2} \right].
\]

By rotational symmetry, a term on the right side of (6) is non-zero only if each \( x_{ij} \) appears the same number of times as \( x_{ij} \). Consider the contribution to the sum such that \( i_1, \ldots, i_{p+1} \) are distinct, and \( i_2 = i_2p, i_3 = i_{2p-1} \), and so on. The contribution of such terms is

\[
n(n-1) \cdots (n-p)\mathbb{E} \left[ |x_{i1}|^2 |x_{i2}|^2 \cdots |x_{i1p}|^2 \right] = n^p \frac{n(n-1) \cdots (n-p)}{(n^2 + p - 1) \cdots n^2} = n + O(1)
\]

making use of the standard formula for integrating polynomials over the sphere (see, e.g., Lemma 14 of [30]).

The sum of remaining terms of (6) is \( O(1) \), since they necessarily involve the choice of fewer indices from \( \{1, \ldots, n\} \), while the expectations on the right hand side which appear all have the same order in \( n \) (this is immediate from the formula in [30]). By (5) this completes the proof of the lemma.

Proof of Theorems 2 and 2 for \( V = M_n(\mathbb{C}) \). Recall that in this context, the orthonormal basis \( \{B_{\alpha}\}_{\alpha=1}^d \) is \( \{E_{jk}\}_{1 \leq j, k \leq n} \cup \{iE_{jk}\}_{1 \leq j, k \leq n} \). It follows that for \( A \in M_n(\mathbb{C}) \),

\[
\sum_{\alpha=1}^d B_{\alpha} A B_{\alpha} = \sum_{j, k=1}^n E_{jk} A E_{jk} + \sum_{j, k=1}^n (iE_{jk}) A (iE_{jk}) = 0.
\]

Part 1 of Lemma 8 then implies that

\[
(7) \quad \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} \left[ W_{\epsilon;p} - W_p \mid X \right] = -\frac{p(p+d-2)}{d(d-1)} W_p.
\]

Note that by taking expectations of both sides of Equation (7), the exchangeability of \( (W_p, W_{\epsilon;p}) \) implies that \( \mathbb{E} W_p = 0 \) for all \( p \); this is also apparent from symmetry considerations (and hence the \( M_n(\mathbb{C}) \) case of Theorem 1).

Equation (7) shows that the matrix \( \Lambda \) in the statement of Theorem 6 may be taken to be diagonal, with \( (p,p) \) entry given by \( \frac{p(p+d-2)}{d(d-1)} \), and that the random vector \( E = 0 \). In particular,

\[
\| \Lambda^{-1} \|_{op} = d.
\]
Next, consider part 2 of Lemma 8. Let \((A, B)_{HS}\) denote the complex Hilbert–Schmidt inner product \((A, B)_{HS} = \text{tr}(AB^*)\). Since \(\{B_\alpha\}_{\alpha=1}^d = \{E_{jk}\}_{j,k=1}^n \cup \{iE_{jk}\}_{j,k=1}^n\),
\[
\sum_\alpha \text{tr}(X^{p-1}B_\alpha)\text{tr}(X^{q-1}B_\alpha) = 2 \sum_{j,k=1}^n \text{tr}(X^{p-1}E_{jk})\text{tr}(X^{q-1}E_{jk}) \\
= 2 \sum_{j,k=1}^n \langle X^{p-1}, E_{kj}\rangle_{HS} \overline{\langle X^{q-1}, E_{kj}\rangle_{HS}} \\
= 2 \langle X^{p-1}, X^{q-1}\rangle_{HS} \\
= 2 \text{tr}(X^{p-1}(X^{q-1})^*)
\]
where the third equality follows from the fact that \(\{E_{jk}\}_{j,k=1}^n\) is an orthonormal basis for the complex inner product \(\langle \cdot, \cdot \rangle_{HS}\). Similarly,
\[
\sum_{\alpha=1}^d X_\alpha \text{tr}(X^{p-1}B_\alpha) = \sum_{j,k=1}^n \langle X, E_{jk}\rangle \text{tr}(X^{p-1}E_{jk}) + \sum_{j,k=1}^n \langle X, iE_{jk}\rangle \text{tr}(X^{p-1}iE_{jk}) \\
= \sum_{j,k=1}^n [(\langle X, E_{jk}\rangle + i \langle X, iE_{jk}\rangle) \text{tr}(X^{p-1}E_{jk}) \\
= \sum_{j,k=1}^n \langle X, E_{jk}\rangle_{HS} (X^{p-1})^*, E_{jk}\rangle_{HS} \\
= \langle X, (X^{p-1})^*\rangle_{HS} = \text{tr}(X^p).
\]
It therefore follows from Lemma 8 that
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E}[(W_\epsilon - W)_p(W_\epsilon - W)_q|X] = \frac{2pq}{d(d-1)} \left(2 \|X\|^2 \text{tr}(X^{p-1}(X^{q-1})^*) - W_pW_q\right).
\]
Note that if \(p \neq q\), the expectation of both terms on the right is zero by symmetry. If \(p = q\), then taking expectations of both sides of (8) gives that
\[
2\mathbb{E}[(\|X\|^2 \text{tr}(X^{p-1}(X^*)^{p-1})) - \mathbb{E}|W_p|^2] = \lim_{\epsilon \to 0} \frac{d(d-1)}{2p^2\epsilon^2} \mathbb{E}[(W_\epsilon - W)_p]^2 \\
= \lim_{\epsilon \to 0} \frac{-d(d-1)}{p^2\epsilon^2} \mathbb{E}[(W_\epsilon - W)_p W_p] \\
= \lim_{\epsilon \to 0} \frac{-d(d-1)}{p^2\epsilon^2} \mathbb{E}[(W_\epsilon - W)_p|W W_p] \\
= \frac{p + d - 2}{p} \mathbb{E}|W_p|^2,
\]
where the second line follows by exchangeability and the last line follows from formula (7) for \(\mathbb{E}[(W_\epsilon - W)_p|W]\). Since \(d = 2n^2\), combining this computation with Lemma 9 means that
\[
\mathbb{E}|W_p|^2 = \frac{2p}{2p + 2n^2 - 2} \mathbb{E}[(\|X\|^2 \text{tr}(X^{p-1}(X^*)^{p-1})) = p + O\left(\frac{1}{n}\right).
\]
and then by Equation (8),
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathbb{E}[(W_{\varepsilon} - W)_{\varepsilon} (W_{\varepsilon} - W)]_q = \left( \frac{2p^2(p + d - 2)}{d(d - 1)} \right) + O \left( \frac{1}{n^3} \right) \delta_{pq}.
\]
We define \( \Sigma \) to be the diagonal matrix with \( \sigma_{pp} = p \). Taking \( \mathbb{S} = \sigma(X) \) in Theorem 6, the random matrix \( E' \) then has \( (p,q) \) entry
\[
[E']_{pq} = \left( \frac{2pq}{d(d - 1)} \right) \left[ 2 \|X\|^2 \text{tr}(X^{p-1}(X^{q-1})^*) - W_q \right] - \left( \frac{2p^2(p + d - 2)}{d(d - 1)} \right) \delta_{pq}
\]
\[
= \left( \frac{2pq}{d(d - 1)} \right) \left[ 2 \|X\|^2 \text{tr}(X^{p-1}(X^{q-1})^*) \right] - \mathbb{E} \left[ 2 \|X\|^2 \text{tr}(X^{p-1}(X^{q-1})^*) \right] - W_q \mathbb{E} + \mathbb{E} (W_p W_q) + O \left( \frac{1}{n^2} \right) \delta_{pq}.
\]
We will estimate the expected Hilbert–Schmidt norm by
\[
\mathbb{E} \|E'\| \leq \mathbb{E} \sum_{p,q=1}^m |E'_{pq}|.
\]
We first have
\[
\mathbb{E} |W_p W_q - \mathbb{E} (W_p W_q)| \leq 2 \mathbb{E} |W_p W_q| \leq 2 \sqrt{\mathbb{E} |W_p|^2} \sqrt{\mathbb{E} |W_q|^2} = pq + O \left( \frac{1}{n} \right)
\]
by the Cauchy–Schwarz inequality and (12).
As in the proof of Lemma 9, we write \( X = \frac{X}{\sqrt{n}} \tilde{X} \), where \( \tilde{X} \) is uniformly distributed on the sphere of radius \( \sqrt{n} \) in \( \mathcal{M}_n(\mathbb{C}) \) and \( \tilde{X} \) is independent from \( \|X\| \). We then have
\[
\mathbb{E} \left| \|X\|^2 \text{tr}(X^{p-1}(X^{p-1})^*) - \mathbb{E} \left[ \|X\|^2 \text{tr}(X^{p-1}(X^{p-1})^*) \right] \right|
\]
\[
= n \mathbb{E} \left| \frac{\|X\|^2}{n^p} \text{tr}(X^{p-1}(X^{p-1})^*) - \left( \frac{\|X\|^2}{n^p} \right) \mathbb{E} \text{tr}(X^{p-1}(X^{p-1})^*) \right|
\]
\[
\leq n \mathbb{E} \text{tr}(X^{p-1}(X^{p-1})^*) \mathbb{E} \left| \frac{\|X\|^2}{n^p} - \frac{\|X\|^2}{n^p} \right|
\]
\[
+ n \left( \frac{\|X\|^2}{n^p} \right) \mathbb{E} \left| \text{tr}(X^{p-1}(X^{p-1})^*) - \mathbb{E} \text{tr}(X^{p-1}(X^{p-1})^*) \right|
\]
\[
\leq n \mathbb{E} \text{tr}(X^{p-1}(X^{p-1})^*) \left( \sqrt{t_{2p}(X) - 2t_{2p}(X) + t_{2p}(X)} \right)
\]
\[
+ n(1 + t_{2p}(X)) \mathbb{E} \left| \text{tr}(X^{p-1}(X^{p-1})^*) - \mathbb{E} \text{tr}(X^{p-1}(X^{p-1})^*) \right|.
\]
Lemma 9 implies that
\[
\mathbb{E} \text{tr}(X^{p-1}(X^{p-1})^*) = n + O(1),
\]
and Proposition 7 implies that
\[
\mathbb{E} \left| \text{tr}(X^{p-1}(X^{p-1})^*) - \mathbb{E} \text{tr}(X^{p-1}(X^{p-1})^*) \right| \leq \kappa_p.
\]
We therefore have
\[
\mathbb{E} \left| \frac{1}{n} \sum_{\alpha} \text{tr}(X^{p-1}B_{\alpha}) \text{tr}(X^{q-1}B_{\alpha}) \right| 
\leq \kappa n^2 \left( \sqrt{t_{4p}(X) - 2t_{2p}(X) + t_{2p}(X)} + \kappa_p n(1 + t_{2p}(X)) \right) = O(n).
\]

Similarly, recalling that when \( p \neq q \) the means are 0,
\[
\mathbb{E} \left| \frac{1}{n} \sum_{\alpha} \text{tr}(X^{p-1}E_{\alpha}) \text{tr}(X^{q-1}E_{\alpha}) \right| = n \mathbb{E} \left| \frac{1}{n} \sum_{\alpha} \text{tr}(X^{p-1}E_{\alpha}) \text{tr}(X^{q-1}E_{\alpha}) \right| 
\leq \kappa n(1 + t_{p+q}(X)) = O(n).
\]

Making use of the fact that \( \| \Lambda^{-1} \|_{op} = d = 2n^2 \), it now follows that
\[
\| \Lambda^{-1} \|_{op} \mathbb{E} \| E' \| \leq \frac{\kappa_m}{n}
\]
for some constant \( \kappa_m \) depending only on \( m \).

Finally, consider part 3 of Lemma 8. Observe that
\[
\sum_{\alpha} \text{tr}(X^{p-1}B_{\alpha}) = \text{tr}(X^p).
\]

It thus follows from Lemma 8 that
\[
[E'_{\alpha}]_{p,q} = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E}[(W_\epsilon - W)^p(W_\epsilon - W)^q | X] = \frac{2pq}{d(d-1)}W_p W_q.
\]

By the Cauchy–Schwarz inequality and (3), \( \mathbb{E}|W_p W_q| \) is bounded independent of \( n \), and so
\[
\mathbb{E} \| E' \| \leq \frac{\kappa_m}{d(d-1)},
\]
where \( \kappa_m \) is a constant depending only on \( m \). This completes the proof of Theorem 2 in the case of \( M_n(\mathbb{C}) \). \( \square \)

4. Rotationally invariant ensembles in \( M_n(\mathbb{R}) \)

As in the previous section, we begin with a technical lemma.

**Lemma 10.** Let \( X \) be a random matrix in \( M_n(\mathbb{R}) \) whose distribution is invariant under rotations in \( M_n(\mathbb{R}) \). Suppose that \( \mathbb{E} \| X \|^2 = n \) and that for each \( k \), there is a constant \( \alpha_k \) such that
\[
t_k(X) = \mathbb{E} \left| n^{-k/2} \| X \|^k - 1 \right| \leq \frac{\alpha_k}{n}.
\]

Then for all \( p, q \in \mathbb{N} \),
\[
\mathbb{E} \left[ \| X \|^2 \text{tr}(X^p(X^T)^q) \right] = \begin{cases} n^2 + O(n) & \text{if } p = q, \\ O(n) & \text{if } p \neq q \text{ and } p - q \text{ is even}, \\ 0 & \text{if } p - q \text{ is odd}; \end{cases}
\]

where the implied constants depend on \( p, q, \) and the \( \alpha_k \).
Proof. First note that if $p - q$ is odd, then $E \left[ \|X\|^2 \text{tr}(X^p(X^T)^q) \right] = 0$ by symmetry.

If $p - q$ is even, then as in the proof of Lemma 9 we may first assume that $X$ is uniformly distributed on the sphere of radius $\sqrt{n}$ in $\mathcal{M}_n(\mathbb{R})$. We have the expansion

\begin{equation}
(10) \quad E \left[ \text{tr}(X^p(X^T)^q) \right] = \sum_{i_1, \ldots, i_{p-1} \atop j_1, \ldots, j_{q-1}} E \left[ x_{i_1}x_{i_1i_2} \cdots x_{i_{p-1}i}x_{ijij_1}x_{j_1j_2} \cdots x_{j_{q-1}j} \right]
\end{equation}

Consider first the case that $p = q$. A term on the right side of (10) is nonzero only if each matrix entry $x_{ij}$ appears an even number of times. The total contribution from terms in which the indices (including $i$ and $j$) are chosen such that $i_1 = j_1, \ldots, i_{p-1} = j_{p-1}$, but are otherwise distinct, is

\[ n(n-1) \cdots (n-p) \frac{n(n-1) \cdots (n-p)}{(n^2 + 2p - 2)(n^2 + 2p - 4) \cdots n^2} = n + O(1). \]

Each of the non-zero expectations has the same order in $n$, so this is the main contribution to the sum, since it involves the maximum number of distinct indices.

Now suppose that $p \neq q$ and $p - q$ is even; assume without loss of generality that $p < q$ and write $q = p + 2k$. Consider the contribution of those terms on the right side of (10) in which $i_\ell = j_\ell$ for $1 \leq \ell \leq p - 1$, $j_\ell = j_{p+k}$ for $p \leq \ell \leq p + k - 1$, and $j = j_{p} = j_{q-1}$, and the indices are distinct except for these restrictions. The contribution of these terms is

\[
\sum_{n(n-1) \cdots (n-p) = 1 + O(n^{-1})}
\]

The leading contribution to (10) is made by these terms, and others obtained by permuting the equality structure among the indices $j, j_p, \ldots, j_{q-1}$; these equality structures maximize the number of indices which can be chosen to be distinct in this group, and all nonzero terms are of the same order in $n$. Since we are not interested in the leading coefficient in (10) in this case, it suffices for our purposes to note that the number of permutations is bounded in terms of $p$ and $q$. \hfill \square

We now proceed with the proofs of the $\mathcal{M}_n(\mathbb{R})$ cases of Theorems 1 and 2.

Proof of Theorem 1 for $V = \mathcal{M}_n(\mathbb{R})$. Trivially, if $p$ is odd then $EW_p = 0$.

To treat the case that $p$ is even, we make use of Lemma 8. Recall that in $\mathcal{M}_n(\mathbb{R})$, the orthonormal basis $\{B_\alpha\}_{\alpha=1}^d = \{E_{jk}\}_{j,k=1}^n$, so that given $A \in \mathcal{M}_n(\mathbb{R})$,

\[
\sum_{\alpha=1}^d B_\alpha A B_\alpha = \sum_{j,k=1}^n E_{jk} A E_{jk} = A^T.
\]

It follows from this computation and part 1 of Lemma 8 that

\begin{equation}
(11) \quad \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} \left[ W_{\epsilon,p} - W_p \right] X = \sum_{\ell=0}^{p-2} \frac{2\|X\|^2(\ell + 1)}{d(d-1)} \text{tr} \left( X^\ell (X^T)^{p-\ell} \right) - \frac{p(p + d - 2)}{d(d-1)} W_p
\end{equation}

\[
\sum_{\ell=0}^{p-2} \frac{2\|X\|^2p}{d(d-1)} \text{tr} \left( X^\ell (X^T)^{p-\ell} \right) - \frac{p(p + d - 2)}{d(d-1)} W_p.
\]
where the second line follows by replacing \( \ell \) with \( p - 2 - \ell \), and averaging the resulting expression with the first.

Since the expectation of the left-hand side of (11) is zero by exchangeability, the expectation of the right-hand side is zero as well, and so taking the expectation of both sides of the formula above gives that

\[
\mathbb{E}W_p = \sum_{\ell=0}^{p-2} \frac{1}{(p + d - 2)} \mathbb{E} \left[ \|X\|^2 \operatorname{tr} \left( X^{\ell}(X^T)^{p-2-\ell} \right) \right].
\]

By Lemma 10

\[
\mathbb{E} \left[ \|X\|^2 \operatorname{tr} \left( X^{\ell}(X^T)^{p-2-\ell} \right) \right] = n^2 + O(n),
\]

and all the other terms in the above sum are \( O(n) \), and thus \( \mathbb{E}W_p = 1 + O(n^{-1}) \).

**Proof of Theorem 2 for** \( V = M_n(\mathbb{R}) \). We begin with condition (7) of Theorem 6. Starting from Equation (11) above, since both sides of the equation have mean zero, it follows that if \( Y_p := W_p - \mathbb{E}W_p \) and \( Y_{e,p} := W_{e,p} - \mathbb{E}W_{e,p} \), then

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} \left[ Y_{e,p} - Y_p | X \right] = \frac{p}{d(d-1)} \sum_{\ell=0}^{p-2} \left[ \|X\|^2 \operatorname{tr} \left( X^{\ell}(X^T)^{p-2-\ell} \right) - \mathbb{E} \left[ \|X\|^2 \operatorname{tr} \left( X^{\ell}(X^T)^{p-2-\ell} \right) \right] \right] - \frac{p(p + d - 2)}{d(d-1)} Y_p.
\]

It follows essentially as in the previous section that for any \( \ell \),

\[
\mathbb{E} \left[ \|X\|^2 \operatorname{tr}(X^\ell(X^T)^{p-2-\ell}) - \mathbb{E} \left[ \|X\|^2 \operatorname{tr}(X^\ell(X^T)^{p-2-\ell}) \right] \right] = O(n),
\]

and we therefore choose the matrix \( \Lambda \) in the statement of Theorem 6 to be diagonal with \( p \)th entry given by \( \sum_{\ell=0}^{p-2} \mathbb{E} \left[ \|X\|^2 \operatorname{tr} \left( X^{\ell}(X^T)^{p-2-\ell} \right) \right] \), the function \( s(\epsilon) = \epsilon^2 \), and the error \( E \) to have \( p \)th entry

\[
E_p = \frac{p}{d(d-1)} \left[ \|X\|^2 \operatorname{tr} \left( \sum_{\ell=0}^{p-2} X^{\ell}(X^T)^{p-2-\ell} \right) - \mathbb{E} \left[ \|X\|^2 \operatorname{tr} \left( \sum_{\ell=0}^{p-2} X^{\ell}(X^T)^{p-2-\ell} \right) \right] \right],
\]

so that

\[
\|\Lambda^{-1}\|_{op} \mathbb{E} |E| \leq \frac{\kappa_m}{n},
\]

with the constant \( \kappa_m \) depending only on \( m \).

Moving on to condition (8) of Theorem 6 it follows from Lemma 8 exactly as in the previous case that

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} \left[ (W_\epsilon - W)_p(W_\epsilon - W)_q | X \right] = \frac{2pq}{n^2(n^2 - 1)} \left[ \|X\|^2 \operatorname{tr}((X^T)^{p-1}X^{q-1}) - W_pW_q \right].
\]

It follows from Proposition 7 and the fact that \( \|E_p\| = O(1) \) that \( \mathbb{E}W_pW_q \leq \kappa_{p,q} \) for some constant depending only on \( p \) and \( q \), and so if we choose \( \Sigma \) to be diagonal with \( \sigma_{pp} = p \), the random matrix \( E' \) in the statement of Theorem 6 has \( p-q \) entry

\[
[E']_{pq} = \frac{2pq}{n^2(n^2 - 1)} \left[ 2\|X\|^2 \operatorname{tr}((X^T)^{p-1}X^{q-1}) - \mathbb{E} \left[ 2\|X\|^2 \operatorname{tr}((X^T)^{p-1}X^{q-1}) \right] - W_pW_q + \mathbb{E}W_pW_q + O(n) \right].
\]
By Proposition \(7\), \(\mathbb{E}[|W_p|^2 - \mathbb{E}[|W_p|^2]|^2\) is bounded independently of \(n\), and we have observed already that
\[
\mathbb{E}\left[\|X\|^2 \text{tr}((X^T)^{p-1}X^{q-1}) - \mathbb{E}[\|X\|^2 \text{tr}((X^T)^{p-1}X^{q-1})]\right] = O(n)
\]
and so (making use of the fact that \(\|\Lambda^{-1}\|_{op} = d = n^2\)),
\[
\|\Lambda^{-1}\|_{op}\mathbb{E}[\|E\|^2] \leq \frac{\kappa_m'}{n}
\]
for some constant \(\kappa'_m\) depending only on \(m\).

5. Rotationally invariant ensembles in \(\mathcal{M}_n^\times(\mathbb{C})\)

We initially proceed via Lemma \(8\) as above. Since in \(\mathcal{M}_n^\times(\mathbb{C})\), the orthonormal basis is
\[
\{B_{\alpha}\}_{\alpha=1}^d = \{E_{jj}\}_{j=1}^n \cup \{E_{jk}\}_{1 \leq j < k \leq n} \cup \{iG_{jk}\}_{1 \leq j < k \leq n},
\]
for a given \(A \in \mathcal{M}_n^\times(\mathbb{C})\),
\[
\sum_{\alpha=1}^d B_{\alpha}AB_{\alpha} = \sum_{j=1}^n E_{jj}AE_{jj} + \frac{1}{2} \sum_{1 \leq j < k \leq n} (E_{jk} + E_{kj})A(E_{jk} + E_{kj}) - \frac{1}{2} \sum_{1 \leq j < k \leq n} (E_{jk} - E_{kj})A(E_{jk} - E_{kj})
\]
\[
= \sum_{j,k=1}^n E_{jk}AE_{kj}
\]
\[
= \text{tr}(A)I.
\]
It thus follows from Lemma \(8\) that
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E}\left[|W_{e,p} - W_p|X\right] = \sum_{\ell=0}^{p-2} \frac{2(\ell + 1)}{d(d - 1)} W_2 W_\ell W_{p-2-\ell} - \frac{p(p + d - 2)}{d(d - 1)} W_p
\]
\[
= \frac{p}{d(d - 1)} W_2 \sum_{\ell=0}^{p-2} W_\ell W_{p-2-\ell} - \frac{p(p + d - 2)}{d(d - 1)} W_p,
\]
where the second line follows by replacing \(\ell\) with \(p - 2 - \ell\), and averaging the resulting expression with the first.

We first use this expression to prove Theorem \(\|\)

Proof of Theorem \(\|\) for \(V = \mathcal{M}_n^\times(\mathbb{C})\). If \(p\) is odd, then \(\mathbb{E}W_p = \mathbb{E}[\text{tr}(X^p)] = 0\) by symmetry.

Suppose now that \(p\) is even. As in the proofs of Lemmas \(9\) and \(10\) we may assume that \(X\) is uniformly distributed in the sphere of radius \(\sqrt{n}\) in \(\mathcal{M}_n^\times(\mathbb{C})\), so that \(W_2 = n\) is constant. Equation \((12)\) and the fact that \((W_{e,p}, W_p)\) is exchangeable imply that
\[
\mathbb{E}W_p = \frac{n}{p + d - 2} \sum_{\ell=0}^{p-2} \mathbb{E}[W_\ell W_{p-\ell-2}].
\]

Proposition \(7\) implies that
\[
\mathbb{E}[W_\ell W_{p-\ell-2}] - (\mathbb{E}W_\ell)(\mathbb{E}W_{p-\ell-2}) = \text{Cov}(W_\ell, W_{p-\ell-2}) \leq \sqrt{\text{Var}(W_\ell) \text{Var}(W_{p-\ell-2})} = O(1),
\]
where
and so by (11),
\[
\frac{\mathbb{E} W_p}{n} = \frac{n^2}{p + d - 2} \left( \sum_{\ell=0}^{p-2} \frac{\mathbb{E} W_p \mathbb{E} W_{p-\ell-2}}{n} + O(n^{-2}) \right).
\]

Writing \( p = 2r \) and \( \beta_r = \frac{\mathbb{E} W_{2r}}{n} \), we therefore have that \( \beta_0 = \beta_1 = 1 \) and
\[
\beta_r = \left( \sum_{k=0}^{r-1} \beta_k \beta_{r-k-1} + O(n^{-2}) \right) (1 + O(n^{-2}))
\]
for \( r \geq 2 \). Recalling that the Catalan numbers \( C_r = \frac{1}{r+1} \binom{2r}{r} \) satisfy the recurrence \( C_0 = 1 \) and \( C_r = \sum_{k=0}^{r-1} C_k C_{r-k-1} \), it now follows by induction on \( r \) that \( \beta_r = C_r + O(n^{-2}) \), where the \( O \) term may also depend on \( r \). \( \square \)

Note that if \( X \) is uniformly distributed in the sphere of \( M_n^a(\mathbb{C}) \), then \( iX \) is uniformly distributed in the sphere of \( M_n^a(\mathbb{C}) \). The anti-Hermitian case of Theorem 4 thus follows immediately from the Hermitian case.

Recall that if \( X \) is uniformly distributed on the sphere of radius \( \sqrt{n} \) in \( M_n^a(\mathbb{C}) \), it follows from Proposition 7 that the \( W_p \) have bounded variance; the following proposition shows that this also holds under the concentration condition we have put on \( \|X\|^2 \).

**Proposition 11.** Let \( X \) be a random matrix in \( M_n^a(\mathbb{C}) \) as above, whose distribution is invariant under rotations in \( M_n^a(\mathbb{C}) \), and let \( W_p = \text{tr}(X^p) \). Suppose that \( \mathbb{E} \|X\|^2 = n \) and that for each \( k \), there is a constant \( \alpha_k \) depending only on \( k \) such that
\[
t_k(X) = \left| n^{-k/2} \mathbb{E} \|X\|^k - 1 \right| \leq \frac{\alpha_k}{n^2}.
\]
Then for each fixed \( p \in \mathbb{N} \), there are constants \( \kappa_{p,2} \) and \( \kappa_{p,4} \), depending on \( p \) and the \( \alpha_k \) but not \( n \), such that
\[
\mathbb{V} \mathbb{A} \mathbb{R} (W_p) \leq \kappa_{p,2} \quad \text{and} \quad \mathbb{E} (W_p - \mathbb{E} W_p)^4 \leq \kappa_{p,4} n^2.
\]

**Proof.** As above, we write \( X = \frac{\|X\|}{\sqrt{n}} \tilde{X} \), where \( \tilde{X} \) is uniformly distributed on the sphere of radius \( \sqrt{n} \) in \( M_n^a(\mathbb{C}) \) and \( \tilde{X} \) is independent from \( \|X\| \). Let \( R := \frac{\|X\|}{\sqrt{n}} \) and \( \tilde{W}_p = \text{tr}(\tilde{X}^p) \). We have
\[
W_p - \mathbb{E} W_p = R^p \tilde{W}_p - (\mathbb{E} R^p) (\mathbb{E} \tilde{W}_p)
\]
\[
= R^p (\tilde{W}_p - \mathbb{E} \tilde{W}_p) + (\mathbb{E} \tilde{W}_p) [((R^p - 1) - (\mathbb{E} R^p - 1)]
\]
and therefore
\[
(\mathbb{E} |W_p - \mathbb{E} W_p|^q)^{1/q} \leq (1 + t_{pq}(X))^{1/q} \left( \mathbb{E} |\tilde{W}_p - \mathbb{E} \tilde{W}_p|^q \right)^{1/q}
\]
\[
+ \left| \mathbb{E} \tilde{W}_p \right| \left( (\mathbb{E} |R^p - 1|^q)^{1/q} + t_p(X) \right)
\]
for any \( q \geq 1 \) by the \( L^q \) triangle inequality. By Proposition 7, Theorem 4 and the fact that \( t_k(X) = O(n^{-2}) \) for each \( k \), we have \( \left( \mathbb{E} |\tilde{W}_p - \mathbb{E} \tilde{W}_p|^q \right)^{1/q} = O(1) \), \( \left| \mathbb{E} \tilde{W}_p \right| = O(n) \), \( (\mathbb{E} R^p)^{1/q} = O(1) \), and \( |\mathbb{E} R^p - 1| = O(n^{-2}) \).

To complete the proof, observe that
\[
\mathbb{E} (R^p - 1)^2 = (\mathbb{E} R^{2p} - 1) - 2(\mathbb{E} R^p - 1) \leq t_{2p}(X) + 2t_p(X) = O(n^{-2})
\]
and similarly
\[ \mathbb{E}(R^p - 1)^4 \leq t_{4p}(X) + 4t_{3p}(X) + 6t_{2p}(X) + 4t_p(X) = O(n^{-2}). \]

**Proof of Theorem 4** for \( V = \mathcal{M}_n^\ast(C) \). We begin with condition (1) of Theorem 6. We write \( R := \frac{\|X\|}{\sqrt{n}} \), \( Y_p := W_p - \mathbb{E}W_p \), and \( Y_{\epsilon,p} := W_{\epsilon,p} - \mathbb{E}W_{\epsilon,p} \). By (12),

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} [Y_{\epsilon,p} - Y_p | X] = \sum_{\ell=0}^{p-2} \frac{p}{d(d-1)} \left( W_2 W_{\ell} W_{p-2-\ell} - \mathbb{E}[W_2 W_{\ell} W_{p-2-\ell}] \right) - \frac{p(p + d - 2)}{d(d-1)} Y_p
\]

\[
= \sum_{\ell=0}^{p-2} \frac{pn}{d(d-1)} \left( R^2 W_\ell W_{p-2-\ell} - \mathbb{E}[R^2 W_{p-2-\ell}] \right) - \frac{p(p + d - 2)}{d(d-1)} Y_p.
\]

Observe that
\[
R^2 W_\ell W_{p-2-\ell} - \mathbb{E}[R^2 W_{p-2-\ell}] = Y_\ell \mathbb{E}W_{p-2-\ell} + Y_{p-2-\ell} \mathbb{E}W_\ell + Y_2 \left( \frac{\mathbb{E}W_\ell \mathbb{E}W_{p-2-\ell}}{n} \right) + F_{p,\ell},
\]

where
\[
F_{p,\ell} := R^2 Y_\ell Y_{p-2-\ell} - \mathbb{E}[R^2 Y_\ell Y_{p-2-\ell}] + \left( (R^2 - 1)Y_\ell - \mathbb{E}[(R^2 - 1)Y_\ell] \right) \mathbb{E}W_{p-2-\ell}
\]

\[
+ \left( (R^2 - 1)Y_{p-2-\ell} - \mathbb{E}[(R^2 - 1)Y_{p-2-\ell}] \right) \mathbb{E}W_\ell.
\]

Using this expression in (15) yields
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} [Y_{\epsilon,p} - Y_p | X] \]

\[
= \sum_{\ell=0}^{p-2} \frac{pn}{d(d-1)} \left[ Y_\ell \mathbb{E}W_{p-2-\ell} + Y_{p-2-\ell} \mathbb{E}W_\ell + Y_2 \left( \frac{\mathbb{E}W_\ell \mathbb{E}W_{p-2-\ell}}{n} \right) + F_{p,\ell} \right] - \frac{p(p + d - 2)}{d(d-1)} Y_p
\]

\[
= \frac{p}{d(d-1)} \left[ 2n \sum_{\ell=0}^{p-2} Y_\ell \mathbb{E}W_{p-2-\ell} + Y_2 \sum_{\ell=0}^{p-2} \mathbb{E}W_\ell \mathbb{E}W_{p-2-\ell} + n \sum_{\ell=0}^{p-2} F_{p,\ell} - (p + d - 2) Y_p \right].
\]

As in the statement of Theorem 4 take
\[ Z_p = Y_p - \frac{p \mathbb{E}W_p}{2n} Y_2, \]

for \( p \geq 0 \) (in particular, \( Z_0 = Z_2 = 0 \)), and
\[ Z_{\epsilon,p} = Y_{\epsilon,p} - \frac{p \mathbb{E}W_p}{2n} Y_{\epsilon,2} = Y_{\epsilon,p} - \frac{p \mathbb{E}W_p}{2n} Y_2, \]
where the last equality follows since \( W_{\epsilon, 2} = \| X_{\epsilon} \|^2 = \| X \|^2 = W_2 \), so that \( Z_{\epsilon, p} - Z_p = Y_{\epsilon, p} - Y_p \). We then have

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} [Z_{\epsilon, p} - Z_p | X] = \frac{p}{d(d-1)} \left[ 2n \sum_{\ell=0}^{p-2} Z_{\ell} \mathbb{E} W_{p-2-\ell} + (p+1) Y_2 \sum_{\ell=0}^{p-2} \mathbb{E} W_{\ell} \mathbb{E} W_{p-2-\ell} \right. \\
\left. + n \sum_{\ell=0}^{p-2} F_{p, \ell} - (p+d-2) \left( Z_p + \frac{p \mathbb{E} W_p Y_2}{2n} \right) \right].
\]

By Theorem 1

\[
\sum_{\ell=0}^{p-2} \mathbb{E} W_{\ell} \mathbb{E} W_{p-2-\ell} = n^2 \sum_{\ell=0}^{p-2} C_{\ell} C_{p-2-\ell} = O(n^2)
\]

if \( p \) is even, and is 0 otherwise. Equation (17) therefore implies that

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} [Z_{\epsilon, p} - Z_p | X] = \frac{p}{d(d-1)} \left[ 2n \sum_{\ell=0}^{p-2} Z_{\ell} \mathbb{E} W_{p-2-\ell} - (p+d-2)Z_p + n \sum_{\ell=0}^{p-2} F_{p, \ell} + O(n^2)Y_2 \right].
\]

We define a matrix \( \Lambda \) with entries indexed by \( p, q \in \{1, \ldots, m\} \setminus \{2\} \) as follows:

\[
[\Lambda]_{pq} = \begin{cases} 
-\frac{2pC_{(p-2-q)/2}}{d-1} & \text{if } 1 \le q \le p-2, q \neq 2, \text{ and } p-2-q \text{ is even;} \\
\frac{2p}{d-1} & \text{if } q = p; \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( \Lambda = \frac{1}{d-1} T \) where \( T \) is an invertible, lower triangular matrix which is independent of \( n \), and so \( \| \Lambda^{-1} \|_{op} = (d-1) || T^{-1} ||_{op} \le \kappa_m n^{-2} \).

We define the random error \( E \) to be

\[
E = \lim_{\epsilon \to 0} \mathbb{E} [Z_{\epsilon} - Z | X] + \Lambda Z,
\]

so that

\[
E_p = \frac{2p}{d-1} \sum_{\ell=0}^{p-2} Z_{\ell} \left( \frac{\mathbb{E} W_{p-2-\ell}}{n} - C_{(p-2-\ell)/2} \right) - \frac{p(p-2)}{d(d-1)} Z_p + \frac{2p}{n(d-1)} \sum_{\ell=0}^{p-2} F_{p, \ell} + O(n^{-2})Y_2.
\]

Since \( Y_{\ell} \) and \( Y_2 \) are centered with bounded variance and \( \mathbb{E} W_{\ell} = O(n) \), \( Z_{\ell} \) is centered with bounded variance as well. We also claim that \( \mathbb{E} |F_{p, \ell}| = O(1) \).

To see this, observe first that by Hölder’s inequality and Proposition 11

\[
\mathbb{E} \left| (R^2 - 1) Y_{p-2-\ell} \right| \le \left( \mathbb{E} (R^2 - 1)^2 \right)^{1/2} \left( \mathbb{E} Y_{\ell}^4 \right)^{1/4} \left( \mathbb{E} Y_{p-2-\ell}^4 \right)^{1/4} = O(1)
\]

where \( \mathbb{E} (R^2 - 1)^2 \) is bounded as in the proof of Proposition 11. The other terms in (18) are bounded in \( L^1 \) similarly. It then follows that \( \mathbb{E} |E| \le \kappa_m n^{-3} \).

Finally, we consider part 2 of Lemma 8. In the present context, the Hilbert–Schmidt inner product is real and therefore coincides with the real inner product \( \langle \cdot, \cdot \rangle \). Therefore

\[
\sum_{\alpha} \text{tr}(X^{p-1} B_\alpha) \text{tr}(X^{q-1} B_\alpha) = \sum_{\alpha} \langle X^{p-1}, B_\alpha \rangle \langle X^{q-1}, B_\alpha \rangle = \langle X^{p-1}, X^{q-1} \rangle = W_{p+q-2}
\]

and

\[
\sum_{\alpha} X_{\alpha} \text{tr}(X^{p-1} B_\alpha) = \sum_{\alpha} \langle X, B_\alpha \rangle \langle X^{p-1}, B_\alpha \rangle = \langle X, X^{p-1} \rangle = W_p.
\]
It thus follows from part (3) of Lemma 8 that
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} \left[ \left( W_\epsilon - W \right)_p \left( W_\epsilon - W \right)_q | X \right] = \frac{2pq}{d(d-1)} \left[ W_2 W_{p+q-2} - W_p W_q \right].
\]
As before, \( \mathbb{E} W_\epsilon = \mathbb{E} W \), so that \( W_\epsilon - W = Y_\epsilon - Y \), and since \( Y_{\epsilon,2} = Y_2 \), it is furthermore the case that \( Y_\epsilon - Y = Z_\epsilon - Z \). That is, for \( p, q \in \{1, \ldots, m\} \setminus \{2\} \),
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} \left[ (Z_\epsilon - Z)_p (Z_\epsilon - Z)_q | X \right] = \frac{2pq}{d(d-1)} \left[ W_2 W_{p+q-2} - W_p W_q \right].
\]

Theorem 11 Proposition 11 and symmetry imply that
\[
\mathbb{E} W_p W_q = \begin{cases} 
n^2 C_{p/2} C_{q/2} + O(n) & \text{if } p \text{ and } q \text{ are both even}, 
O(1) & \text{if } p \text{ and } q \text{ are both odd}, 
0 & \text{if } p \text{ and } q \text{ have opposite parities}.
\end{cases}
\]

We define the matrix \( \Gamma \) indexed by \( p, q \in \{1, \ldots, m\} \setminus \{2\} \) by
\[
\Gamma_{p,q} = \frac{pq}{d-1} \begin{cases} 
C_{(p+q-2)/2} - C_{(p/2)C_{q/2}} & \text{if } p \text{ and } q \text{ are both even}, 
C_{(p+q-2)/2} & \text{if } p \text{ and } q \text{ are both odd}, 
0 & \text{if } p \text{ and } q \text{ have opposite parities}.
\end{cases}
\]

and let \( \Sigma = \Lambda^{-1} \Gamma \). With these choices of \( \Lambda \) and \( \Sigma \), the random error matrix \( E' \) of Theorem 8 can be bounded as in the previous sections to complete the proof. As noted in the introduction, it is not obvious from this form that \( \Sigma \) is positive semidefinite. However, the argument above shows that \( \Sigma \) arises as the limit of a sequence of covariance matrices, and therefore must be positive semidefinite. \( \square \)

6. Rotationally invariant ensembles in \( \mathcal{M}_n^s(\mathbb{R}) \)

Proceeding by Lemma 8 as before, let \( A \in \mathcal{M}_n^s(\mathbb{R}) \) and recall that in the case of \( \mathcal{M}_n^s(\mathbb{R}) \), \( \{B_\alpha\}_{\alpha=1}^d = \{E_{jj}\}_{j=1}^n \cup \{F_{jk}\}_{1 \leq j < k \leq n} \). We have
\[
\sum_{\alpha=1}^d B_\alpha A B_\alpha = \sum_{j=1}^n E_{jj} A E_{jj} + \frac{1}{2} \sum_{1 \leq j < k \leq n} (E_{jk} + E_{kj}) A (E_{jk} + E_{kj})
= \frac{1}{2} \sum_{j,k=1}^n E_{jk} A E_{jk} + \frac{1}{2} \sum_{j,k=1}^n E_{jk} A E_{kj}
= \frac{1}{2} A + \frac{1}{2} \text{tr}(A) I,
\]
using the fact that \( A \) is symmetric. It thus follows from Lemma 8 that
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} \left[ W_{\epsilon,p} - W_p | X \right]
= \sum_{\ell=0}^{p-2} \left( \ell + 1 \right) ||X||^2 \left( W_{p-2} + W_\ell W_{p-2-\ell} \right) - \frac{p(p+d-2)}{d(d-1)} W_p
= \frac{p}{2d(d-1)} \left[ (p-1)W_2 W_{p-2} + W_2 \sum_{\ell=0}^{p-2} W_\ell W_{p-2-\ell} - 2(p+d-2)W_p \right],
\]
Compare with the corresponding expression (12) in the Hermitian case: the first term here is new, but the remaining two terms are, to top order, $\frac{1}{2}$ times the corresponding term in the Hermitian case (recall that in $M_n^d(\mathbb{R})$, $d = \frac{n(n-1)}{2}$). The first term in (19) is of smaller order than the remaining terms, and so the proofs of Theorems 1 and 4 are essentially the same as in the complex Hermitian case.

The proof of Proposition 11 carries over verbatim to this case.

Returning to condition (1) of Theorem 6, since the expectation of the left-hand side of (19) is zero, if $Y_p := W_p - EW_p$ and $Y_{\epsilon,p} := W_{\epsilon,p} - EW_{\epsilon,p}$, then

$$
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} [Y_{\epsilon,p} - Y_p | X] = \frac{p(p-1)}{2d(d-1)} (W_2 W_{p-2} - EW_2 W_{p-2}) \\
+ \frac{p}{2d(d-1)} \sum_{\ell=0}^{p-2} [W_2 W_{\ell} W_{p-2-\ell} - EW_2 W_{\ell} W_{p-2-\ell}] - \frac{p(p+d-2)}{d(d-1)} Y_p.
$$

Again, the first term is new and the second two are very similar to the Hermitian case, differing only in factors of 2 that correspond to the change in dimension. By centering and applying Proposition 7 it is straightforward to check that the new term is of smaller order than the others and can be incorporated into the constant in the final bound. The proof of Theorem 4 then proceeds identically to the Hermitian case.

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