Hybrid Impulsive Control for Closed Quantum Systems

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The state transfer problem of a class of nonideal quantum systems is investigated. It is known that traditional Lyapunov methods may fail to guarantee convergence for the non-ideal case. Hence, a hybrid impulsive control is proposed to accomplish a more accurate convergence. In particular, the largest invariant sets are explicitly characterized, and the convergence of quantum impulsive control systems is analyzed accordingly. Numerical simulation is also presented to demonstrate the improvement of the control performance.

1. Introduction

One of major concerns in quantum control is how to steer quantum states to a desired target state precisely and efficiently. A solution to this quantum state transfer problem will help us to advance some promising applications such as quantum computation and quantum chemistry. The main difficulty in quantum control is due to the limitations on the application of observation and feedback in quantum systems. Open-loop control has therefore been a commonly adopted approach in quantum control, where recorded control signals obtained from numerical simulations are implemented to real quantum systems. Among existing open-loop control design methods, the Lyapunov method could be the most popular one and has been tested in real applications [1–6]. Despite great advances have been made in Lyapunov methods, they may fail to achieve the control goal if the internal Hamiltonian is not strong regular [7]. This nonideal case means that distances between the eigenvalues of the internal Hamiltonian are not distinct. It is worth pointing out that this nonideal case does exist in many practical quantum systems such as coupled spin systems and harmonic oscillator systems [3, 4].

In particular, this paper will study the state transfer for closed quantum systems modeled as the following Schrödinger equation:

$$i \frac{\partial \psi (t)}{\partial t} = H_0 \psi (t),$$

where $H_0$ is the internal Hamiltonian. For quantum systems, the control is implemented to the system through electromagnetic fields. Our basic problem is to transfer a quantum state from an initial state to a desired target state. The difficulty for the Lyapunov control of nonideal quantum system mainly comes from the fact that the system could be driven to undesired limit points in the invariant set. There exist a few results in the recent literature handling such a nonideal case. In [7], the complete controllability of the quantum systems with twofold degeneracy was investigated, and the basic idea is to apply a weak constant field to eliminate the degeneracy. In [4], the implicit Lyapunov method was used to deal with such a nonideal case. However, it is difficult to characterize invariant sets which are critical for the following convergence analysis. Therefore, we propose a new hybrid impulsive control strategy for closed quantum systems under the nonideal case. Nowadays, the impulsive control has proved to be an effective method to accomplish...
good performance for classical systems [8–12]. This motivates us to apply such a control idea to quantum systems.

The basic idea of the hybrid impulsive control is to divide the control into a piecewise continuous open-loop coherent control \( u_l(t) \) and an impulsive control \( u_i(t) \). This design is similar to the traditional Lyapunov control which drives states to invariant sets. Specifically, the system under the piecewise continuous control can be described by

\[
i |\psi(t)\rangle = \left[H_0 + \sum_{l=1}^{r} H_l u_{jl}(t)\right] |\psi(t)\rangle, \quad t \neq t_k, \quad (2)
\]

where \( H_j \) is the control Hamiltonian and \( u_{jl}(t) \) is real-valued control function (\( l \in J = \{1, 2, \ldots, r\} \)). The continuous-time coherent control \( u_l(t) \) is implemented through the control Hamiltonians \( H_l \) when \( t \neq t_k \). Due to the nonideal quantum system, this control cannot guarantee the convergence to the desired target state. After a certain instant \( t_k \), the controlled state would sufficiently approach an undesired limit point.

In practice, we do not have much freedom to choose the control Hamiltonian due to the structure limitations of the control fields [3, 4]. The impulsive control Hamiltonian cannot be chosen arbitrarily to achieve the state transfer instantaneously. Thus in this paper, \( H_0 \) is fixed and assumed to be known beforehand. Denote \( B_k = e^{-iB_k t} \), which is unitary. With the hybrid impulsive control fields \( u(t) \), system (1) becomes a closed quantum impulsive control system as follows:

\[
i |\psi(t)\rangle = \left[H_0 + \sum_{l=1}^{r} H_l u_{jl}(t)\right] |\psi(t)\rangle, \quad t \neq t_k.
\]

(4)

In the following, we denote \( H = H_0 + \sum_{l=1}^{r} H_l u_{jl}(t) \).

2. Hybrid Impulsive Control Based on the State Distance

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In the following, we denote \( H = H_0 + \sum_{l=1}^{r} H_l u_{jl}(t) \).

2.1. Hybrid Impulsive Control Design and Dynamical Properties of Controlled Systems. In quantum control, the goal state \( |\psi_f\rangle \) is usually chosen to be an eigenstate of \( H_0 \), that is, \( H_0|\psi_f\rangle = \lambda_f|\psi_f\rangle \). We select a Lyapunov function based on the Hilbert-Schmidt distance between \( |\psi\rangle \) and \( |\psi_f\rangle \), that is, \( V_1(|\psi(t)\rangle) = V_1(t) = (1/2)(1 - |\langle\psi|\psi_f\rangle|^2) \). When \( t \neq t_k \), the time derivative of \( V_1 \) is given by

\[
\dot{V}_1(t) = -\sum_{l=1}^{r} u_{jl} \Re \left( \langle\psi|\psi_f\rangle \langle\psi_f|H_l|\psi\rangle \right),
\]

(5)

where \( \Re(\cdot) \) and \( \Im(\cdot) \) denote the imaginary part and real part of a complex number, respectively. When \( t = t_k \), the time derivative is

\[
\dot{V}_1(t) = \lim_{h \to 0^+} \left( (V_1(t_k) - V_1(t_k - h))/h \right).
\]

The difference of \( V_1 \) is described as \( \Delta V_1(t) = V_1(t_k^+) - V_1(t_k^-) \). In order that the designed control can work in the case of the initial state being orthogonal to the goal state, we rewrite (5) as

\[
\dot{V}_1(t) = -\sum_{l=1}^{r} u_{jl} \Re(\langle\psi|\psi_f\rangle \langle\psi_f|H_l|\psi\rangle).
\]

(5)

We need to design the control law such that \( \dot{V}_1(t) \leq 0, t \neq t_k \) and \( \Delta V_1(t_k) \leq 0 \). Choose the piecewise continuous control law as follows:

\[
u_{jl}(t) = K_l f_l \left[ \Re(\langle\psi|\psi_f\rangle \langle\psi_f|H_l|\psi\rangle) \right], \quad l \in J, \quad t \neq t_k.
\]

(6)
where $K_j > 0$ is the control gain and the function $f_i(\cdot)$ passes through the origin of plane $x_i - y_i$ monotonically satisfying $f_i(x_i)x_i \geq 0$ with $x_i = \Re\{\langle \psi_i | H_i | \psi_i \rangle\}$. To avoid the confusion, we define $\mathcal{L}(\langle \psi_i | H_i | \psi_i \rangle) = 0$ if $\langle \psi_i | H_i | \psi_i \rangle = 0$. In addition, the impulsive control matrix $B_k$ should be chosen to satisfy $\Delta V_1(t_k) \leq 0$; that is,

$$B_k^* \rho_j B_k - \rho_j \geq 0, \quad (7)$$

where $\rho_j = |\psi_j\rangle \langle \psi_j|$ is the density matrix of target state $|\psi_j\rangle$.

Inequality (7) holds at least for the unitary matrix $B_k$ which commutes with $\rho_j$. The control law satisfying (6) and (7) is the designed control law. In the following, the properties of system (4) will be studied to show that the system leave the initial state even if $|\psi(0)\rangle$ is an eigenstate of $H_0$.

**Lemma 1.** For control law (6) and (7), if the initial state is an eigenstate of $H_0$ with $H_0|\psi(0)\rangle = \lambda_0|\psi(0)\rangle$ and $\langle \psi(0) | \psi(0) \rangle = 0$, then the following conclusions hold:

(i) if there exists $l \in J$ such that $\mathcal{F}(\psi_l | H_l | \psi_l(0)) \neq 0$, then $\langle \psi(0) | \psi(0) \rangle = 0$ (t > 0);

(ii) if $\mathcal{F}(\psi_l | H_l | \psi_l(0)) = 0$, for all $l \in J$, and there exists $l \in J$ such that $\mathcal{F}(\rho_j B_k)$ for all $l \in J$, then $\langle \psi(0) | \psi(0) \rangle \neq 0$, and consequently the designed control fields cannot achieve the state steering of the closed-loop system.

Proof. (i) For a sufficiently small $dt$, as $t \neq t_k$, we have $i|\psi(0)\rangle = i \lim_{dt \to 0} \left[ |\psi(dt)\rangle - |\psi(0)\rangle \right]/dt = H|\psi(0)\rangle$; that is, as $dt \to 0$, $|\psi(dt)\rangle = |(I - iHdt)|\psi(0)\rangle$. Since $\langle \psi(0) | \psi(0) \rangle = 0$, the inequality $\langle \psi(0) | \psi(dt) \rangle \neq 0$ is equivalent to $\sum_{l=1}^L u_{il}(\psi_l | H_l | \psi_l(0)) = \sum_{l=1}^L u_{il}(\mathcal{F}(\psi_l | H_l | \psi_l(0))) \neq 0$. By (6), we have $u_{il}(\mathcal{F}(\psi_l | H_l | \psi_l(0))) \neq 0$, for all $l \in J$. It follows from the assumption in case (i) that there exists $l \in J$ such that $u_{il}(\mathcal{F}(\psi_l | H_l | \psi_l(0))) \neq 0$, and consequently, $\langle \psi(0) | \psi(0) \rangle \neq 0$. Since $V_1 \leq 0$, we obtain that $\langle \psi(0) | \psi(0) \rangle = 0$, $t \in (0, t_1)$. By (7) we have $|\langle \psi | \psi(t_1) \rangle| = |\langle \psi | \psi(t_1) \rangle| \leq 0$. Hence, we obtain that $\langle \psi(0) | \psi(0) \rangle \neq 0$ (t > 0).

(ii) Initially, the system evolves freely because $u_{il}(0) = 0$, $l \in J$. For a sufficiently small $t < t_1$, one can obtain that $|\psi(t)\rangle = e^{-iHdt}|\psi(0)\rangle = e^{-i\lambda_0 t^*}|\psi(0)\rangle$. Moreover, as $dt \to 0$, we have $\langle \psi(t) | \psi(t^* + dt) \rangle = \langle \psi(t) | (I - iHdt)|\psi(t)\rangle \equiv dt (1 - \cos(\lambda_0 t^*) - \sin(\lambda_0 t^*)) \sum_{l=1}^L u_{il}(\psi_l | H_l | \psi_l(0))$. Noticing that there exists $l \in J$ such that $\langle \psi_l | H_l | \psi_l(0) \rangle \neq 0$, we have $u_{il}(t^*) = K_{f lj} e^{i\lambda_0 t^*} \langle \psi_l | H_l | \psi_l(0) \rangle = K_{f lj} (1 - \sin(\lambda_0 t^*) \langle \psi_l | H_l | \psi_l(0) \rangle \neq 0$. Similar to the discussion in case (i), we obtain that $\langle \psi(t) | \psi(t^* + dt) \rangle \neq 0$, and then $\langle \psi(0) | \psi(0) \rangle = 0$, $t \neq t^*$. This completes the proof.

For the characterization of invariant sets, properties of the states such that $V_1 = 0$ are studied. Since the proof is similar to that of Proposition 4 in [1], we omit it here.

**Lemma 2.** If $\langle \psi_i | \psi_i \rangle = 0$ and the conditions (i) or (ii) in Lemma 1 hold, the following conclusions are equivalent:

(i) $V_1(t) = 0$, $t \neq t_k$,

(ii) $i\langle \psi(t) | H | \psi(t) \rangle = 0$, $t \neq t_k$,

(iii) there exists $\lambda_1 \in \Re$ such that $\langle \psi_j | (\lambda I - H_j) \psi(t) \rangle = 0$, $t \neq t_k$, $l \in J$.

**Lemma 2** only characterizes the states guaranteeing that $V_1 = 0$ at specific instants. We need to characterize the states from which the system trajectories stay in the set $V_1 = 0$ and $\Delta V_1 = 0$. We first present the extensive LaSalle invariance principle for impulsive systems in [21].

**Lemma 3.** Consider the following differential impulsive system on an open set $\mathcal{D}$:

$$\dot{x}(t) = f_c(x(t)), \quad x(0) = x_0, \quad t \neq t_k,$n

$$\Delta x(t) = f_d(x(t)), \quad t = t_k.$$

If there exists a continuous function $V$ such that $V(x) f_c(x) \leq 0$, $t \neq t_k$ and $\Delta V(t_k) \leq 0$, then $x(t) \to \mathcal{M}$ as $t \to \infty$, where $\mathcal{M}$ is the largest invariant set contained in $\mathcal{D} \equiv \{ x : V(x) f_c(x) = 0 \} \cap \{ x : \Delta V(x) = 0 \}$.

2.2 Convergence Analysis. The following theorem presents the characterization of the invariant set for the nonideal systems under the hybrid impulsive control, by which the invariant set is smaller compared with that obtained by the conventional Lyapunov method. In the following, the unitary matrix $B_k$ is designed such that it commutes with $H_0$, $k \in Z^+$. 

**Theorem 4.** Consider system (4) with the hybrid impulsive control satisfying (6) and (7). The largest invariant set is given by $\mathcal{G} = S^{2m-1} \cap E_1 \cap E_2$ with $E_1 = \{ |\psi\rangle : |\psi\rangle \in M_k^1 \}$, $E_2 = \{ |\psi\rangle : |\psi\rangle \in N_k, k = 1, 2, \ldots \}$, and

$$M_k^1 := \{ |\psi\rangle : \mathcal{F}(\langle \psi | \psi_f \rangle \langle \psi_f | X^*_k | \psi \rangle) = 0 \}

$$

$$N_k := \left\{ |\psi\rangle : \langle \psi | \prod_{j=1}^{k-1} B_j^* | \psi_f \rangle \langle \psi_f | X^*_k \prod_{j=k-1}^{1} B_j | \psi \rangle \right\} = 0,s_1 = 1,2,\ldots,m_k \biggr\}, \quad k \geq 2,$$

$$\prod_{j=1}^{k-1} B_j^* | \psi_f \rangle \langle \psi_f | X^*_k \prod_{j=k-1}^{1} B_j | \psi \rangle \right\} = 0,$$

$$n$$

where $X^*_k, X^*_k, \ldots, X^*_k$ constitute the basis of the set $\{ i^l |H_0^j, H_k, s = 0, 1, 2, \ldots, l \in J \}$. Hence, system (4) converges to $G$ under the hybrid impulsive control.
Proof. When $t = t_0$, from (6), we obtain that

$$
\dot{V}_1(t_0) = 0 \iff | \langle \psi(t_0) , \psi_f \rangle | \times \mathfrak{S} \left[ e^{i \langle \psi(t_0) , \psi_f \rangle} \cdot \langle \psi_f | H_l | \psi(t_0) \rangle \right] = 0, \quad l \in J
$$

$$
\iff \mathfrak{S} \left( \langle \psi(t_0) , \psi_f \rangle \cdot \langle \psi_f | H_l | \psi(t_0) \rangle \right) = 0.
$$

The main idea of the proof is sketched as follows. The interval $[t_{k-1}, t_k]$ is divided into $n_k$ sufficiently small intervals with duration $dt$. By Lemma 2, the requirements $\dot{V}_1(t) = 0$ ($t \neq t_k$) and $\Delta V_1(t_k) = 0$ for the whole system trajectory will be transformed to the conditions on the initial state. By the Taylor expansion and commutativity between $H_0$ and $B_k$, it yields that

$$
\dot{V}_1(t_0 + dt) = 0
$$

$$
\iff \mathfrak{S} \left( \langle \psi(t_0 + dt) , \psi_f \rangle \cdot \langle \psi_f | H_l | \psi(t_0 + dt) \rangle \right) = 0
$$

$$
\iff \mathfrak{S} \left( \langle \psi(t_0) \cdot (1 + iH_0 dt) , \psi_f \rangle \cdot \langle \psi_f | H_l | \psi(t_0) \rangle \right) = 0
$$

$$
\iff \mathfrak{S} \left( i \langle \psi(t_0) , \psi_f \rangle \cdot \langle \psi_f | [H_0, H_l] | \psi(t_0) \rangle \right) = 0
$$

$$
D^{-1} V_1(t_k) = 0
$$

$$
\iff \mathfrak{S} \left( (i)^s \langle \psi(t_0) , \psi_f \rangle \cdot \langle \psi_f | [H_0^{(s)}, H_l] \cdot \sum_{j=1}^{k-1} B_j | \psi(t_0) \rangle \right) = 0,
$$

$$
\iff \mathfrak{S} \left( (i)^s \langle \psi(t_0) , \psi_f \rangle \cdot \langle \psi_f | [H_0^{(s)}] \cdot \sum_{j=1}^{k-1} B_j | \psi(t_0) \rangle \right) = 0,
$$

$$
\iff \mathfrak{S} \left( (i)^s \langle \psi(t_0) , \psi_f \rangle \cdot \langle \psi_f | [H_0^{(s)}] \cdot \sum_{j=1}^{k-1} B_j | \psi(t_0) \rangle \right) = 0,
$$

Consequently, it can be obtained that

$$
D^{-1} V_1(t_k) = 0
$$

$$
\iff \mathfrak{S} \left( (i)^s \langle \psi(t_0) , \psi_f \rangle \cdot \langle \psi_f | [H_0^{(s)}] \cdot \sum_{j=1}^{k-1} B_j | \psi(t_0) \rangle \right) = 0,
$$

$$
\iff \mathfrak{S} \left( (i)^s \langle \psi(t_0) , \psi_f \rangle \cdot \langle \psi_f | [H_0^{(s)}] \cdot \sum_{j=1}^{k-1} B_j | \psi(t_0) \rangle \right) = 0,
$$

where $s = 0, 1, \ldots, \sum_{i=1}^{k} n_i$. Noticing that the set $\{(i)^s | [H_0^{(s)}],

H_l, s = 0, 1, \ldots, \sum_{i=1}^{k} n_i, l \in J \}$ has finite dimension, we denote its basis to be $X_1^1, X_1^2, \ldots, X_1^{m_1}, l \in J$. Since the division of the interval $[t_{k-1}, t_k]$ is random, (14) can be rewritten as

$$
D^{-1} V_1(t_k) = 0
$$

$$
\iff \mathfrak{S} \left( \langle \psi(t_0) , \psi_f \rangle \cdot \langle \psi_f | [H_0^{(s)}] \cdot \sum_{j=1}^{k-1} B_j | \psi(t_0) \rangle \right) = 0,
$$

For convenience, the set of the states satisfying (15) is denoted as $M_k^s$ in (9), $l \in J, k \geq 2$.

In the following, we will discuss the conditions on the initial states from which the trajectories stay in the set $| \psi \rangle : \Delta V_1(t_k) = 0, k = 1, 2, \ldots$.:
achieve more accurate convergence under the nonideal case. This implies that the proposed hybrid impulsive control can still guarantee the stability of the system. From the viewpoint of physics, the introduction of the impulsive control can improve the convergence rate.

Let \( l \in J \)

\[ \sum_{j=1}^{k} B_j \psi_j \]

be the first \( k \) elements of the set \( J \), where \( k \) is finite. If the set \( J \) is finite, then the relationship between the elements of \( J \) can be obtained in finite steps, \( l \in J \). If the first \( n_j \) elements in the set are linearly independent, then they constitute the basic set. If \( [H_0^{(s+1)}, H_j] \) can be represented by the first \( s_j \) elements, it is easy to obtain that \( [H_0^{(k)}, H_j] \) can be represented by the linear combination of \( H_j, [H_0, H_j], \ldots, [H_0^{(s)}, H_j] \), for all \( k > s \).

We denote the set of the states guaranteeing (17) to be \( N_k \) in (9).

In conclusion, all the states which stay in the intersection \( E_1 \cap E_2 \) constitute the largest invariant set of system (4). By Lemma 3, we complete the proof.

It should be noticed that the basis of the set \( \{(i)^{s}\} [H_0^{(s)}, H_1] \), \( s = 0, 1, 2, \ldots, \sum_{j=1}^{k} n_j \) can be obtained in finite steps, \( l \in J \). If the first \( n_j \) elements in the set are linearly independent, then they constitute the basic set. If \( [H_0^{(s+1)}, H_j] \) can be represented by the first \( s_j \) elements, it is easy to obtain that \( [H_0^{(k)}, H_j] \) can be represented by the linear combination of \( H_j, [H_0, H_j], \ldots, [H_0^{(s)}, H_j] \), for all \( k > s \).

**Corollary 5.** Consider system (4) with control field (6) without the impulsive control, that is, \( B_k = I, k = 1, 2, \ldots \). The largest invariant set is \( E \cap S^{2m-1}, E = \{ ||\psi|| : |\psi| \in \mathcal{M}_k, l \in J, k = 0, 1, \ldots \} \), where \( \mathcal{M}_k = \{ |\psi| : \mathfrak{A}^k \langle \psi | \psi_j \rangle \langle \psi_j | [H_0^{(k)}, H_j] |\psi\rangle = 0, k \geq 0, x \in J \} \).

**Remark 6.** If \( H_0 \) is strong regular, then the result in Corollary 5 reduces to Theorem 2 in [1]. For the nonideal case, it is clear that \( G \subset E \). From the viewpoint of physics, this implies that the proposed hybrid impulsive control can achieve more accurate convergence under the nonideal case. In general, the matrix \( B_k \) can be chosen to guarantee that \( E_1 \) and \( E_2 \) contain finite sets \( M_1^k \) and \( N_k \). It can be found that the invariant set \( G \) depends on the choice of impulsive control matrix \( B_k \). The optimal determination of \( B_k \) and impulsive instants \( t_k \) to minimize the invariant set are under study.

### 3. Hybrid Impulsive Control Based on the State Error

It is known that different Lyapunov functions may have different control effects. The relations among them were studied in our previous work [4]. In this section, we consider the hybrid impulsive control of quantum systems based on the state error between the controlled state and the goal state. Let \( V_2(\psi(t)) = V_2(t) = (1/2) \langle \psi(t) - \psi_j | \psi(t) - \psi_j \rangle = 1 - \mathfrak{R} \langle \psi_j | \psi(t) \rangle \). Similar to the hybrid control design in Section 1, we consider the following quantum impulsive control system which is different from (4):

\[
\begin{align*}
   i |\psi(t)\rangle &= (H_0 + u_i H_1 + \omega I) |\psi(t)\rangle, \quad t \neq t_k, \\
   |\psi(t_k)\rangle &= B_k |\psi(t_k)\rangle,
\end{align*}
\]

where \( \omega \) is a new real scalar control field. For the convenience of the computation, the introduced \( \omega \) may be used to adjust the global phase without changing the physical quantities regarding \( |\psi\rangle \). While in practical implementation, it is not necessary to be implemented to the system. Similar conclusion can be drawn if there exists more than one control Hamiltonian \( H_l, l \geq 2 \). The time derivative of \( V_2 \) is

\[
\begin{align*}
   \frac{dV_2}{dt} &= - (\lambda + \omega) \mathfrak{A} \langle \psi_j | \psi \rangle \\
   &\quad - \mathfrak{A} \langle \psi_j | H_1 |\psi\rangle u_1, \quad t \neq t_k.
\end{align*}
\]

Let \( u_0 = \lambda + \omega \). We design the following control to ensure \( V_2(t) \leq 0, t \neq t_k \):

\[
\begin{align*}
   \lambda + \omega &= u_0 = K_0 f_0 \mathfrak{A} \langle \psi_j | \psi \rangle, \\
   u_1 &= K_1 f_1 \mathfrak{A} \langle \psi_j | H_1 |\psi\rangle,
\end{align*}
\]

Theorem 7. Consider system (18) with control fields (7) and (20). The largest invariant set is given by \( K = F_1 \cap F_2 \cap S^{2m-1} \), where \( F_1 = \{ |\psi\rangle : |\psi\rangle \in U_k, k = 1, 2, \ldots \} \), \( F_2 = \{ |\psi\rangle : |\psi\rangle \in W_k, k = 1, 2, \ldots \} \), and \( U_k := \{ |\psi\rangle : \mathfrak{A} \langle \psi_j | X_j^s |\psi\rangle = 0, s = 1, 2, \ldots, m_1 \} \), \( W_k := \{ |\psi\rangle : \mathfrak{R} \langle \psi_j | \psi \rangle - \mathfrak{A} \langle \psi_j | B_j |\psi\rangle = 0, k \geq 1 \} \), where \( X_1, X_2, \ldots, X_m \) are the basis of the set \( \{ I, (i)^{s}[H_0^{(s)}, H_1] \}, s = 0, 1, 2, \ldots \). Therefore, system (18) converges to \( K \) with the hybrid impulsive control satisfying (7) and (20).

**Proof.** Let \( \omega = -\lambda \). When the system satisfies \( V_2 = 0 \), that is, \( u_1 = 0 \), the evolution of system (18) becomes

\[
\begin{align*}
   i |\psi(t)\rangle &= (\mathcal{H}_0 - \lambda I) |\psi(t)\rangle, \quad t \neq t_k, \\
   |\psi(t_k)\rangle &= B_k |\psi(t_k)\rangle,
\end{align*}
\]
It follows from (22) that
\[
|\psi(t_{k-1} + dt)\rangle = |\psi(t_{k-1})\rangle + |\psi(t_{k-1})\rangle dt
\]
\[
= \left| I - i \left( H_0 - \lambda_j I \right) dt \right| |\psi(t_{k-1})\rangle.
\]
From (20), we obtain the following relation:
\[
V_2(t_0) = 0 \iff \mathfrak{S} \left( \langle \psi_j | \psi(t_0) \rangle \right) = 0,
\]
\[
\mathfrak{S} \left( \langle \psi_j | H_1 | \psi(t_0) \rangle \right) = 0.
\]
(24)

Similarly, we divide the interval \([t_{k-1}, t_k]\) into \(n_k\) sufficiently small intervals. From (22)–(24), we have \(D^\kappa V_2(t_k) = 0 \iff \mathfrak{S}(i^{n_k} \langle \psi_j | [H^{(n_k)}_0, H_1] \psi(t_o) \rangle) = 0\). According to the similar method in the proof of Theorem 4, when \(t = t_k\), it yields that
\[
D^\kappa V_2(t_k) = 0
\]
\[
\iff \mathfrak{S} \left( \langle \psi_j | \prod_{j=1}^{s} B_j | \psi(t_0) \rangle \right) = 0,
\]
\[
\mathfrak{S} \left( \langle \psi_j | H_1 | \psi(t_0) \rangle \right) = 0.
\]
(25)
where \(s = 0, \ldots, \sum_{j=1}^{k} n_j\). Denote the basis of the set \(\{i, i, \ldots\} [H^{(s)}_0, H_1]\), \(s = 0, 1, \ldots, \sum_{j=1}^{k} n_j\) to be \(X^1, X^2, \ldots, X^{m_k}\). Equation (25) can be rewritten as \(D^\kappa V_2(t_k) = 0 \iff \mathfrak{S}(\langle \psi_j | X^s | \prod_{j=1}^{s} B_j | \psi(t_0) \rangle) = 0, s = 1, 2, \ldots, m_1\). This equality is denoted as \(V_k\) in (21).

Next, we characterize the initial states from which the system trajectories stay in \(|\psi\rangle : \Delta V_k(t_1) = 0, k = 1, 2, \ldots\). From the definition of \(V_2\), we have \(\Delta V_2(t_1) = \mathfrak{R}(\langle \psi_j | \psi(t_1) \rangle) - \mathfrak{R}(\langle \psi_j | \psi(t_1) \rangle) = \mathfrak{R}(\langle \psi_j | B_1 | \psi(t_1) \rangle - \mathfrak{R}(\langle \psi_j | \psi(t_1) \rangle).\) The following relations can be obtained:
\[
\Delta V_2 (t_1) = 0
\]
\[
\iff \mathfrak{R}(\langle \psi_j | \psi(t_1) \rangle) - \mathfrak{R}(\langle \psi_j | B_1 | \psi(t_1) \rangle) = 0
\]
\[
\iff \mathfrak{R}(\langle \psi_j | (I - B_1) \left| I - i \left( H_0 - \lambda_j I \right) dt \right| \times |\psi(t_1 - dt)\rangle = 0
\]
\[
\iff \mathfrak{R}(\langle \psi_j | \psi(t_0) \rangle - \mathfrak{R}(\langle \psi_j | B_1 | \psi(t_0) \rangle) = 0.
\]
By similar deduction, it yields that \(\Delta V_2(t_k) = 0 \iff \mathfrak{R}(\langle \psi_j | \psi(t_0) \rangle) - \mathfrak{R}(\langle \psi_j | \prod_{j=1}^{k} B_j | \psi(t_0) \rangle) = 0\), which can be denoted as \(W_k\) in (21).

In conclusion, all the states which remain in the intersection \(F_1 \cap F_2\) constitute the largest invariant set of controlled system (18). By Lemma 3, the proof is completed.

Similar to the discussion in Corollary 5, Theorem 7 can be reduced to Theorem 8 in [1] if there is no impulsive control, and \(H_0\) is strong regular. We can see that our result reduces the invariant set for the nonideal case. This implies that the proposed hybrid impulsive control scheme can accomplish more accurate state transfer.

4. Numerical Simulation

Example 1. Consider the five-level system with the internal Hamiltonian and impulsive control Hamiltonian given by \(H_0 = \text{diag}[1.0, 1.2, 1.2, 2.0, 2.15]\) and \(B_\kappa = \text{diag}[0, 0, -\pi, 0, -\pi]\), respectively. The unitary operation \(B_k = e^{-i\pi R} = \text{diag}[1, 1, -1, -1, 1]\) can be realized by performing the planar rotation on system states. It can be found that the system is a nonideal system. The control Hamiltonians are given by
\[
H_1 = \left( \begin{array}{cc} 0 & 0.010 \ 0 & 0 \ -0.010 & 0 \ 0 & 0 \ -0 & 0 \ -0 & 0 \ \end{array} \right),
\]
\[
H_2 = \left( \begin{array}{cc} 0 & 0.010 \ -0.010 & 0 \ 0 & 0 \ -0 & 0 \ -0 & 0 \ -0 & 0 \ \end{array} \right).
\]
Let the target state be \(|\psi_f\rangle = [0 0 1 0 0]^T\), the initial state be \(|\psi_0\rangle = [0 1 0 0 0]^T\), and let the control state be \(|\psi\rangle = [c_1, c_2, c_3]^T\). Take the control function to be \(f_j(x) = x, i = 1, 2\). Choose the impulsive instant to be \(t_k = 3k - 1, k \in \mathbb{Z}^+\) and \(K_1 = K_2 = 0.2\). Using the hybrid impulsive control based on the state distance, simple computation yields that the invariant set \(G = \{1, 3\}\) (without regard to the global phase), which implies that under the hybrid impulsive control the system converges to \(|\psi_f\rangle\). The populations of the controlled system are illustrated in Figure 1.

Now we compare performance of the hybrid impulsive control with that of classical Lyapunov control. If the impulsive control is not applied to the system, then the hybrid impulsive control is reduced to the classical Lyapunov control, by which the performance of the controlled system is shown in Figure 2. Hence, the proposed hybrid impulsive control improves the control performance.

Example 2. Consider the five-level system with the same internal Hamiltonian as the previous example. Let the target state and the initial state be \(|\psi_f\rangle = [0 0 0 0 1]^T\) and \(|\psi_0\rangle = [1 0 0 0 0]^T\), respectively, and the impulsive control Hamiltonian \(B_\kappa = \text{diag}[-\pi, 0, 0, -\pi]\). The unitary operation is chosen as \(B_k = e^{-i\pi R} = \text{diag}[-1, 1, 1, -1, 1]\). The system is a nonideal system. The control Hamiltonian is given...
by $H_1 = \begin{pmatrix} 0 & i & 0 & 0 & i \\ -i & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & i \\ 0 & -i & 0 & 0 & i \\ -i & -i & -i & -i & 0 \end{pmatrix}$. Let the control function be $f_1(x) = x$, and $K_0 = 0.1$, $K_1 = 0.2$. Choose the impulsive instant as $t_k = 3k - 1$, $k \in \mathbb{Z}^+$. Using the hybrid impulsive control based on the state error, simple computation yields that the invariant set $K = \{|5\rangle\}$ (without regard to the global phase), which implies that under the hybrid impulsive control the system converges to $|\psi_f\rangle$. Simulation results are illustrated in Figure 3.

When the hybrid impulsive control is reduced to the classical Lyapunov control, the trajectory of the controlled system is plotted in Figure 4. Moreover, if the implicit Lyapunov control strategy in [4] is employed with the same parameters, it fails to drive the system, as illustrated in Figure 5. Therefore, the proposed hybrid impulsive control improves the control performance.

5. Conclusion

In this paper, the coherent hybrid impulsive control for closed quantum systems has been investigated for the nonideal case that $H_0$ is not strong regular. The dynamical properties of the resulted quantum impulsive control system have been discussed to facilitate the convergence analysis. Based on two kinds of Lyapunov functions, the largest invariant sets have been characterized explicitly. Consequently, more accurate convergence of the controlled system has been achieved by the extensive LaSalle invariance principle. Compared with some existing results, the improved control performance has been shown for the nonideal case. Since the practical implementation of impulsive control has been studied in known literature, we believe that it is feasible. The optimal determination of the impulsive control Hamiltonian and impulsive instants is worth to be explored in the future work.

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