FREE ASSOSYMMETRIC ALGEBRAS AS MODULES OF GROUPS

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Abstract. An algebra with identities \((a, b, c) = (a, c, b) = (b, a, c)\) is called assosymmetric, where \((x, y, z) = (xy)z - x(yz)\) is associator. We study \(S_n\)-module, \(A_n\)-module and \(GL_n\)-module structures of free assosymmetric algebra.

1. Introduction

Let \(K\) be an algebraically closed field of characteristic 0. All algebras, vector spaces, modules and tensor products we consider will be over field \(K\).

Let \(X = \{x_1, x_2, \ldots\}\) be a set of generators and \(K\{X\}\) be the absolutely free nonassociative algebra. A polynomial \(f(x_1, x_2, \ldots, x_n) \in K\{X\}\) is called polynomial identity or identity for the \(K\)-algebra \(R\) if \(f(r_1, r_2, \ldots, r_n) = 0\) for all \(r_1, r_2, \ldots, r_n \in R\).

Let \(\{f_i \in K\{X\} | i \in I\}\) be a set of elements in \(K\{X\}\). The class \(V\) of all algebras satisfying the polynomial identities \(f_i = 0, i \in I\) is called the variety defined by the system of polynomial identities \(\{f_i | i \in I\}\). The set \(T(V)\) of all polynomial identities satisfied by the variety \(V\) is called the \(T\)-ideal or verbal ideal of \(V\).

A nonassociative algebra \((R, \cdot)\) is called assosymmetric, if for any \(a, b, c \in R\) the following identities are hold

\[(a, b, c) = (a, c, b) = (b, a, c),\]

where \((x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)\) is associator.

The factor algebra \(F(X) = K\{X\}/T(R)\) is called free assosymmetric algebra.

Assosymmetric algebras was studied in [19], [25], [2], [17], [7].

Basis of free assosymmetric algebras was constricted in [17]. Moreover, this paper contains multiplication rule of basis elements that allows to present an element of free assosymmetric algebra as a linear combination by basis elements. In [7] it was proved that assosymmetric algebras under Jordan product satisfy Lie triple and Glennie identities.

In polynomial identities theory there are two main questions: 1) describe algebras with identities; 2) describe identities in algebras. The language of varieties allows one to freely pass from identity to algebra and from algebra to identity. Therefore studying varieties of algebras is one of the important problem in modern algebras. In 1950, A.I. Malcev [23] and W.Specht [28] first time and independently used...
the representation theory of symmetric group to classify polynomial identities of algebraic structures. If \( \text{char} K = 0 \), then every polynomial is equivalent to a finite set of multilinear polynomials.

For several classes of algebras \( S_n \), \( GL_n \)-module structures on multilinear parts of free algebras are studied. Some cases these structures can be easy described. For example, multilinear parts of free associative, free Zinbiel and free Leibniz algebras of degree \( n \) as \( S_n \)-module are isomorphic to regular module \( KS_n \). In case of Lie algebras module structures are slightly complicated. In [20] it was found list of irreducible \( S_n \)-representations that are involved in decomposition of multilinear parts of free Lie algebras. Description of multiplicities of irreducible \( S_n \)-representations in decomposition of multilinear part of free Lie algebra by language of major indices of standard Young tableaux is given in [21].

Let us give some references for other papers where representations of groups on free algebras are studied.

In [30] full described varieties of associative algebras with identity of degree three by the methods of the theory of representations of symmetric group and general linear group. In [1] is given the criterion to the distributivity of the lattice of subvarieties of varieties of associative algebras by using the methods of the theory of representations of symmetric group and general linear group. In [24] is given the criterion to the distributivity of the lattice of subvarieties of varieties of alternative algebras by using the methods of the theory of representations of symmetric group and general linear group. In [6] proved Specht problem (or finite basis problem) for varieties of bicommutative algebras over field of characteristic 0 by using methods of the theory of the representations of symmetric group.

In [9] constructed basis of free bicommutative algebras, described multiplication of basis elements, found cocharacter sequence, codimension sequence, calculated Hilbert series. It is also proved that bicommutative operad is not Koszul and the growth of codimension sequence of bicommutative algebra is equal to 2. In [5] is given alternative proof of the formula for the cocharacter sequence of bicommutative algebra.

In [8] studied module structures of free Novikov algebra over symmetric group and general linear group. It is given criteria for Novikov admissibility of any partition of positive integer.

In [4] M. Bremner studied varieties of anticommutative \( n \)-ary algebras and found the "correct" generalization of the Jacobi identity. It is also formulated several conjectures. One of them is module structures of \( P_n \) (\( S_{2n-1} \)-module spanned by the \( n \)-ary anticommutative monomials involving two pairs of brackets). In [26] M. Rotkiewicz proved above mentioned Bremner’s conjecture. In [3] classified varieties of anti-commutative algebras defined by identities of degree \( n \leq 7 \). Author classified using computing the decomposition of the \( S_n \)-module of multilinear polynomials of degree \( n \) into irreducible submodules. Until today in science is unknown the module structures of free anti-commutative algebra in general case.

In [22] calculated characters of representations of symmetric group on free right-symmetric and right-commutative algebras respectively.

In [15] showed explicitly decomposition of the group algebra of the alternating group into direct sum of minimal left ideals. In [16] studied Weyl modules for the Schur algebra of the Alternating Group. In [14] calculated the \( A \)-codimensions and the \( A \)-cocharacters of the infinite dimensional Grassmann (exterior) algebra.
Authors conjectured a finite generating set of the $A_n$-identities for the Grassmann algebra. In [13] proved Henke-Regev conjecture.

In this paper we consider multilinear part of free assosymmetric algebra $F(X)$ as $S_n$-module, $A_n$-module and $GL_n$-module. We find dimension of homogeneous component, sequence of dimensions of multilinear components or codimension sequence, colength sequence, cocharacter sequence in $S_n$-case, cocharacter sequence in $A_n$-case for assosymmetric algebras.

2. Statement of main result

Let $n$ be a positive integer. The sequence of positive integers $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ is called partition of $n$, if (1) $\lambda_1 + \lambda_2 + ... + \lambda_k = n$, (2) $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$, and denoted by $\lambda \vdash n$. Length of partition $\lambda \vdash n$ is the number of parts in $\lambda$ and denoted by $\ell(\lambda)$. It is known that between partitions of $n$ and Young diagrams with $n$ boxes exist one-to-one correspondence. We denote Young diagram with $\lambda$-shape by $Y_\lambda$.

Let $\lambda, \mu \vdash n$. Partition $\lambda$ is conjugate to partition $\mu$, if $Y_\mu$ is obtained from $Y_\lambda$ by turning the rows into columns and denoted by $\mu = \lambda'$. A partition that is conjugate to itself is said to be a self-conjugate partition, that is $\lambda = \lambda'$.

Let $S_n$ be symmetric group on set $\{1, 2, ..., n\}$ and $A_n$ be alternating subgroup of $S_n$. The symmetric group $S_n$ and alternating group $A_n$ acts on multilinear part of free assosymmetric algebra in natural way (left action or variable action).

Let $R$ be a PI-algebra. For $n \geq 1$, the $S_n$-character of $P_n(R) = P_n/(P_n \cap T(R))$ is called the $n$-th cocharacter of $R$ and denoted by $\chi_n(R)$, and

$$\chi_n(R) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where $\chi_\lambda$ is the irreducible $S_n$-character associated to the partition $\lambda \vdash n$ and $m_\lambda \geq 0$ is the corresponding multiplicity.

Let $T(R)$ be $T$-ideal of $R$. Then the non-negative integer

$$c_n(R) = \dim(P_n/P_n \cap T(R)),$$

is called the $n$-th codimension of the algebra $R$.

Let $R$ be a PI-algebra and

$$\chi_n(R) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda.$$

Then the non-negative integer

$$l_n(R) = \sum_{\lambda \vdash n} m_\lambda$$

is called the $n$-th colength of $R$.

For more information about codimension sequence, cocharacter sequence, colength sequence see [12].

We denote irreducible $S_n$-module or Specht module associated to partition $\lambda \vdash n$ by $S^\lambda$ and dimension of $S^\lambda$ by $d_\lambda$, irreducible $A_n$-module associated to non-self-conjugate partition $\lambda \vdash n$ by $S^\lambda_A$ and to self-conjugate partition $\lambda \vdash n$ by $S^{\lambda \pm}_A$, irreducible $GL_n$-module or Weyl module associated to partition $\lambda \vdash n$ by $W^\lambda$ in $S_n$-case and by $W^\lambda_A$ in $A_n$-case.

For more information about the theory of representations of $S_n$, $A_n$ and $GL_n$ see [10], [11], [18], [27], [14], [15], [16]
Free base of assosymmetric algebras was found in [17]. We use this result to find formulas for dimensions of free assosymmetric algebras. Let $F(r)$ be free assosymmetric algebra generated by $r$ elements $a_1, \ldots, a_r$. Let $F_{\ell_1,\ldots,\ell_r}(r)$ be a subspace of free assosymmetric algebra generated by $l_i$ elements $a_i$, where $i = 1, \ldots, r$, and $F_n(r)$ be a subspace of free assosymmetric algebra $F(r)$ of degree $n$ and $F_n^{\text{multi}} = F_{1,\ldots,1}(n)$ be multi-linear part of $F_n(n)$.

**Theorem 2.1.** Let $p = \text{char} K \neq 2, 3$. Then

$$
\dim F_{\ell_1,\ldots,\ell_r}(r) = \binom{l_1 + \cdots + l_r}{l_1 \cdots l_r} + (l_1 + 1) \cdots (l_r + 1) - \binom{r+1}{2} - r - 1 + w,
$$

where $w = w(l_1, \ldots, l_r)$ is a number of $1$'s in the sequence $l_1 \ldots l_r$,

$$
\dim F_n(r) = n^r + \binom{n+2r-1}{n} - \binom{r+1}{2} \binom{n+r-3}{n-2} - r \binom{n+r-2}{n-1} - \binom{n+r-1}{n},
$$

and

$$
\dim F_n^{\text{multi}} = n! + 2^n - \binom{n+1}{2} - 1.
$$

\[\square\]

By Stirling formula $n! \sim \sqrt{2\pi n}(n/e)^n$, and therefore,

$$
\dim F_n^{1/n} \sim n/e.
$$

We divide the set of multilinear basic elements into two types. First type

$$
T_n = \{((x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)})\ldots)x_{\sigma(n)}|\sigma \in S_n\},
$$

Second type

$$
T_{k,n-k} = \{x_{\sigma(1)}(x_{\sigma(2)}\ldots x_{\sigma(k)}\ldots)(x_{\sigma(k+1)}x_{\sigma(k+2)}x_{\sigma(k+3)}x_{\sigma(k+4)}\ldots x_{\sigma(n)}\ldots)|\sigma(1) < \sigma(2) < \ldots < \sigma(k), \quad \sigma(k+1) < \sigma(k+2) < \ldots < \sigma(n)\}.
$$

**Theorem 2.2.** The group $S_n$ acts transitively on the sets $T_n$ and $T_{k,n-k}$, $k = 0, 1, \ldots, n-3$.

Let $KT_n$ and $KT_{k,n-k}$ be subspaces of $F_n^{\text{multi}} := P_n$ spanned by the sets $T_n$ and $T_{k,n-k}$, $k = 0, 1, \ldots, n-3$, respectively.

**Corollary 2.3.** As $S_n$-module

$$
P_n \cong KT_n \oplus \bigoplus_{k=0,1,\ldots,n-3} KT_{k,n-k}.
$$

**Theorem 2.4.** As $S_n$-module

$$
P_n \cong \bigoplus_{\lambda \vdash n} d_{\lambda}S^\lambda \oplus \bigoplus_{(\lambda_1, \lambda_2) \vdash n} m(\lambda_1, \lambda_2)S^{(\lambda_1, \lambda_2)},
$$

where

$$
m(\lambda_1, \lambda_2) = \begin{cases} n - 2 - \lambda_2, & \lambda_2 \leq 3, \\ n + 1 - 2\lambda_2, & \lambda_2 \geq 4. \end{cases}
$$
Example 2.5.

\[ P_1 \cong S^{(1)}; \]
\[ P_2 \cong S^{(2)} \oplus S^{(1,1)}; \]
\[ P_3 \cong 2 \ast S^{(3)} \oplus 2 \ast S^{(2,1)} \oplus S^{(1,1,1)}; \]
\[ P_4 \cong 3 \ast S^{(4)} \oplus 4 \ast S^{(3,1)} \oplus 2 \ast S^{(2,2)} \oplus 3 \ast S^{(2,1,1)} \oplus S^{(1,1,1,1)}; \]
\[ P_5 \cong 4 \ast S^{(5)} \oplus 6 \ast S^{(4,1)} \oplus 6 \ast S^{(3,2)} \oplus 6 \ast S^{(2,1,1,1)} \oplus 5 \ast S^{(2,2,1,1,1)} \oplus 4 \ast S^{(2,1,1,1,1,1)} \oplus S^{(1,1,1,1,1,1,1)}. \]

Let \( V \) be a vector space with dimension \( m \). Let \( F(V) \) be free assosymmetric algebra generated by basis elements of \( V \) and \( H_n(V) \) be homogeneous part of \( F(V) \) of degree \( n \).

**Definition 2.6.** Let \( \text{inv}(S_n) \) be number of involutions,

\[ \text{inv}(S_n) = \# \{ \sigma \in S_n \mid \sigma^2 = e \} \]

**Corollary 2.7.**

(a.) \( \chi_{S_n}(P_n) \cong \sum_{\lambda^w} d_{\lambda} \chi_{S_n}(\lambda) + \sum_{(\lambda_1, \lambda_2)^w} m(\lambda_1, \lambda_2) \chi_{S_n}(\lambda_1, \lambda_2) \),

where

\[ m(\lambda_1, \lambda_2) = \begin{cases} n - 2 - \lambda_2, & \lambda_2 \leq 3, \\ n + 1 - 2\lambda_2, & \lambda_2 \geq 4. \end{cases} \]

(b.) \( \chi_{A_n}(P_n) = 2 \chi_{A_n}(KA_n) + \sum_{(\lambda_1, \lambda_2)^w} m(\lambda_1, \lambda_2) \chi_{A_n}(\lambda_1, \lambda_2) \),

where

\[ m(\lambda_1, \lambda_2) = \begin{cases} n - 2 - \lambda_2, & \lambda_2 \leq 3, \\ n + 1 - 2\lambda_2, & \lambda_2 \geq 4. \end{cases} \]

c. \( (S_n\text{-case}) \)

\[ H_n(V) \cong \bigoplus_{\lambda^w} d_{\lambda} W^\lambda \oplus \bigoplus_{(\lambda_1, \lambda_2)^w} m(\lambda_1, \lambda_2) W^\lambda, \]

where

\[ d_{\lambda} > 0, \quad m(\lambda_1, \lambda_2) > 0 \quad \text{and} \]

\[ m(\lambda_1, \lambda_2) = \begin{cases} n - 2 - \lambda_2, & \lambda_2 \leq 3, \\ n + 1 - 2\lambda_2, & \lambda_2 \geq 4. \end{cases} \]

if \( \dim V \geq \ell(\lambda), \ell((\lambda_1, \lambda_2)) \), and

\[ d_{\lambda} = 0, \quad m(\lambda_1, \lambda_2) = 0, \]

if \( \dim V < \ell(\lambda), \ell((\lambda_1, \lambda_2)) \).

d. \( (A_n\text{-case}) \)

\[ H_n(V) \cong \bigoplus_{\lambda \neq \lambda'} 2d_A W^\lambda_A \bigoplus \bigoplus_{\lambda_1=\lambda_2} 2 \left( d_{\lambda_1} W^{\lambda_1}_A + \frac{d_{\lambda_2}}{2} W^{\lambda_2}_A \right) \bigoplus \bigoplus_{(\lambda_1, \lambda_2)^w} m(\lambda_1, \lambda_2) W^\lambda_A, \]

where

\[ m(\lambda_1, \lambda_2) = \begin{cases} n - 2 - \lambda_2, & \lambda_2 \leq 3, \\ n + 1 - 2\lambda_2, & \lambda_2 \geq 4. \end{cases} \]

e. For \( 1 \leq n \leq 3 \)

\[ l_n(P_n) = \delta_{n,3} + \text{inv}(S_n), \]
where $\delta_{i,j}$ is Kronecker delta.

For $n \geq 4$

$$l_n(P_n) = \begin{cases} k^2 + 2k - 5 + \text{inv}(S_n), & n = 2k; \\ k^2 + 3k - 4 + \text{inv}(S_n), & n = 2k + 1. \end{cases}$$

**Example 2.8.**

$$\chi_{S_1}(P_1) = \chi_{S_1}(1);$$

$$\chi_{S_2}(P_2) = \chi_{S_2}(2) + \chi_{S_2}(1,1);$$

$$\chi_{S_3}(P_3) = 2*\chi_{S_4}(2) + \chi_{S_4}(1,1,1);$$

$$\chi_{S_4}(P_4) = 3*\chi_{S_4}(4) + 2*\chi_{S_4}(2,2) + \chi_{S_4}(2,1,1) + \chi_{S_4}(1,1,1,1);$$

$$\chi_{S_5}(P_5) = 4*\chi_{S_5}(5) + 6*\chi_{S_5}(4,1) + 6*\chi_{S_5}(3,2) + 5*\chi_{S_5}(3,1,1) + 4*\chi_{S_5}(2,1,1,1) + \chi_{S_5}(1,1,1,1,1).$$

**Example 2.9.**

$$\chi_{A_1}(P_1) = \chi_{A_1}(1);$$

$$\chi_{A_2}(P_2) = \chi_{A_2}(2) + \chi_{A_2}(1,1);$$

$$\chi_{A_3}(P_3) = 3*\chi_{A_3}(3) + \chi_{A_3}^+(2,1) + \chi_{A_3}^-(2,1);$$

$$\chi_{A_4}(P_4) = 4*\chi_{A_4}(4) + 7*\chi_{A_4}(3,1) + \chi_{A_4}^+(2,2) + \chi_{A_4}^-(2,2);$$

$$\chi_{A_5}(P_5) = 5*\chi_{A_5}(5) + 10*\chi_{A_5}(4,1) + 11*\chi_{A_5}(3,2) + 3*\chi_{A_5}^+(3,1,1) + 3*\chi_{A_5}^-(3,1,1).$$

**Example 2.10.**

$$l_1(P_1) = 1; \; l_2(P_2) = 2; \; l_3(P_3) = 5; \; l_4(P_4) = 13; \; l_5(P_5) = 32.$$

### 3. Proof of Theorem 2.11

In calculation of dimensions we need the following easy proved combinatorial results.

**Lemma 3.1.** For non-negative integers $\alpha, \beta$ and $n$ takes place the following formula

$$\sum_{i=0}^{n} \binom{i + \alpha}{i} \binom{n - i + \beta}{n - i} = \binom{n + \alpha + \beta + 1}{n}$$

In particular,

$$\sum_{i=0}^{n} \binom{i + \alpha}{i} \binom{n - i + \alpha}{n - i} = \binom{n + 2\alpha + 1}{n}$$

**Lemma 3.2.** Number of non-decreasing sequences of length $m$ with components in the set $I = \{1, 2, \ldots, r\}$ is $\binom{m+r-1}{m}$.

**Lemma 3.3.** Number of non-decreasing sequences of length $m$ with components in the set $I = \{1, 2, \ldots, r\}$ such that each $i \in I$ appears no more than $l_i$ times is $(l_1 + 1) \cdots (l_r + 1)$. 
In [17] is proved that a base of free assosymmetric algebras can be constructed by elements of two kinds. If \( S = \{a_1, \ldots, a_r\} \) is a set of generators, then in degree \( n \) the base consists elements of a form

\[
\cdots ((a_{i_1}a_{i_2})a_{i_3}) \cdots a_{i_n}, \quad a_i \in S,
\]

where \( i_1 \leq i_2 \leq \cdots \leq i_m, \quad j_1 \leq j_2 \leq \cdots \leq j_k, \quad m \geq 0, \quad k \geq 3. \]

Number of elements of first kind is \( r^n \). By Lemma 3.2, number of elements of second kind \( L \) is equal to

\[
L = \sum_{m+k=n, m \geq 0, k \geq 3} \binom{m+r-1}{m} \binom{k+r-1}{k} - \\
\sum_{m+k=n, m \geq 0, k \geq 0} \binom{m+r-1}{m} \binom{k+r-1}{k} - \\
\binom{n+r-3}{2} \binom{r+1}{2} - \binom{n+r-2}{2} \binom{r}{1} - \binom{n+r-1}{1} \binom{r-1}{0}.
\]

By Lemma 3.3,

\[
L = \left(\frac{n+2r-1}{n}\right) - \left(\frac{r+1}{2}\right) \left(\frac{n+r-3}{n-2}\right) - r \left(\frac{n+r-2}{n-1}\right) - \left(\frac{n+r-1}{n}\right).
\]

Therefore,

\[
dim F_n(r) = \\
r^n + \left(\frac{n+2r-1}{n}\right) - \left(\frac{r+1}{2}\right) \left(\frac{n+r-3}{n-2}\right) - r \left(\frac{n+r-2}{n-1}\right) - \left(\frac{n+r-1}{n}\right).
\]

Now suppose that any generator \( a_s, s = 1, 2, \ldots, r \), in each base element should enter \( l_s \) times. Then the number of base elements of first kind is

\[
\binom{l_1 + \cdots + l_n}{l_1 \cdots l_n} \frac{(l_1 + \cdots + l_r)!}{l_1! \cdots l_r!}.
\]

Let \( M \) be set of sequences \( \alpha = i_1 \ldots i_m j_1 j_2 \ldots j_k \) with components in \( I = \{1, 2, \ldots, r\} \) such that each \( i \in I \) appears exactly \( l_i \) times and \( i_1 \leq \cdots \leq i_m \), \( j_1 \leq \cdots \leq j_k \). For \( \alpha \in M \) call its subsequence of first \( m \) components \( i_1 \ldots i_m \) as \textit{head} and denote \( \hat{\alpha} \). Note that each \( \alpha \in M \) is uniquely defined by head \( \hat{\alpha} \). Denote set of heads by \( \hat{M} \). Note also that in the sequence \( \hat{\alpha} = i_1 \ldots i_m \) each \( i \in I \) enters no more than \( l_i \) times. Therefore by Lemma 3.3 the number of heads is

\[
|\hat{M}| = (l_1 + 1) \cdots (l_r + 1).
\]

Let \( N \) be a subset of \( M \) consisting of sequences with the following heads

\[
\begin{array}{c}
\underbrace{1 \ldots 1}_{l_1} \ldots \underbrace{i \ldots i}_{l_i} \ldots \underbrace{r \ldots r}_{l_r} \\
\end{array}
\]

(number of such sequences is 1)

\[
\begin{array}{c}
\underbrace{1 \ldots 1}_{l_1} \ldots \underbrace{i \ldots j}_{l_i} \ldots \underbrace{r \ldots r}_{l_r} \quad i \in I,
\end{array}
\]

(number of such sequences is \( r \))
\[1 \ldots 1 \ldots \hat{i} \ldots \hat{i} \ldots r \ldots r, \quad l_i > 1, \quad i \in I,\]
(number of such sequences is \(r - w\), where \(w\) is a number of 1’s in the sequence \(l_1 \ldots l_r\))

\[1 \ldots 1 \ldots \hat{i} \ldots \hat{j} \ldots j \ldots r \ldots r, \quad i < j, \quad i, j \in I\]
(number of such sequences is \(r(r - 1)/2\).

Let \(M_1 = M \setminus N\) be a supplement of \(N\) in the set \(M\). Then any \(\alpha = i_1 \ldots i_m j_1 \ldots j_k \in M_1\) has the property \(k \geq 3\) and any such sequence generates base element of free associosymmetric algebra of second kind. Hence the number of base elements of second kind is

\[
\dim F^{l_1, \ldots, l_r}(r) = \left| M_1 \right| = \left( \frac{l_1 + \cdots + l_r}{l_1 \ldots l_r} \right) + (l_1 + 1) \ldots (l_r + 1) - \left( \begin{array}{c} r + 1 \\ l_r \end{array} \right) - r - 1 + w.
\]

Dimension for multilinear part is an easy consequence of this formula.

4. PROOF OF THEOREM

Let \(A\) be the \(T\)-ideal in \(K\{X\}\) determined by identities

\[(x, y, z) = (x, z, y) = (y, x, z).\]

Let \(R\) be associosymmetric algebra and \(I = (R, R, R) + (R, R, R)R\) is the ideal generated by associators. Proof of Theorem 2.2 is based on the following two results of [17].

**Lemma 4.1 (17, Lemma 1).** The expression \([[[\ldots[[a_1, a_2, a_3], a_4], a_5] \ldots]a_n]\) is invariant, modulo \(A\), under all permutations of the arguments.

**Lemma 4.2 (17, Lemma 2).** If \(x \in I\), the expression \(a_1(a_2(a_3(\ldots a_n x)\ldots))\) is invariant, modulo \(A\), under all permutations of the \(a_i\)’s.

Now we give proof of Theorem 2.2.

First type. Let

\[(\ldots((a_{i_1} \cdot a_{i_2}) \cdot a_{i_3}) \cdot \ldots) \cdot a_{i_n} \in T_n, \quad i_j \in \{1, 2, \ldots, n\}.\]

Let \(\sigma = (i_k, i_{k+1})\) be a transposition in \(S_n\). Then

\[(i_k, i_{k+1}) : (\ldots((a_{i_1} \cdot a_{i_2}) \cdot a_{i_3}) \cdot \ldots) a_{i_n} \mapsto (\ldots((a_{i_1} \cdot a_{i_2}) \cdot a_{i_3}) \cdot \ldots) a_{i_{k+1}} \cdot a_{i_{k+1}} \cdot \ldots) \cdot a_{i_n}.\]

By definition of \(\ldots((a_{i_1} \cdot a_{i_2}) \cdot a_{i_3}) \cdot \ldots) \cdot a_{i_{k+1}} \cdot a_{i_{k+1}} \cdot \ldots) \cdot a_{i_n}\) is multilinear basis element in \(T_n\).

Second type. Let

\[w = a_{i_1} \cdot (a_{i_2} \cdot (\ldots a_{i_k} \cdot \ldots) \cdot [a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}, \ldots, a_{i_n}]) \in T_{k, n-k}, \quad i_j \in \{1, 2, \ldots, n\}.\]

We present it in a form

\[
\underbrace{a_{i_1} \cdot (a_{i_2} \cdot (\ldots a_{i_k} \cdot \ldots) \cdot \ldots)}_{A-part} \cdot \underbrace{[a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}, \ldots, a_{i_n}]}_{B-part}.
\]
It suffices to consider the action of transposition \( \sigma = (i_j, i_{j+1}) \in S_n \) in three cases:

**Case-1:** \( \sigma \) acts on A-part;

**Case-2:** \( \sigma \) acts on B-part;

**Case-3:** \( \sigma \) acts on A-part and B-part simultaneously.

**Case-1:**

\[
\sigma : a_i \left( \ldots a_{i_j} (a_{i_1} \ldots (a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), \ldots) \right) \mapsto \nonumber
\sigma a_i \left( \ldots a_{i_j+1} (a_{i_1} \ldots (a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), \ldots) \right).
\]

By Lemma 1.2, \( \sigma w \in T_{k,n-k} \) and \( \sigma w = w \).

**Case-2:**

\[
\sigma : a_i \left( \ldots a_{i_j} (a_{i_1} \ldots (a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), \ldots) \right) \mapsto \nonumber
\sigma a_i \left( \ldots a_{i_j+1} (a_{i_1} \ldots (a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), \ldots) \right).
\]

By Lemma 1.1, \( \sigma w \in T_{k,n-k} \) and \( \sigma w = w \).

**Case-3:**

Let

\[
w = a_i \left( \ldots a_{i_k} (a_{i_1} \ldots (a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), \ldots) \right) \in T_{k,n-k}, \quad i_j \in \{1, 2, \ldots, n\}.
\]

Assume that \( a_i \) belong to A-part and \( a_{i_j+1} \) belong to B-part, i.e.

\[
w = a_i \left( \ldots a_{i_k} (a_{i_1} \ldots (a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), \ldots a_{i_j+1}, \ldots, a_{i_n}) \right).
\]

Then

\[
\sigma : a_i \left( \ldots a_{i_j} (a_{i_1} \ldots (a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), \ldots) \right) \mapsto \nonumber
\sigma a_i \left( \ldots a_{i_{j+1}} (a_{i_1} \ldots (a_{i_{k+1}}, a_{i_{k+2}}, a_{i_{k+3}}), \ldots) \right)
\]

\[
= (a_{p_1} (a_{p_2} (\ldots (a_{p_k} \ldots \ldots a_{p_{n-1}}) \ldots a_{p_m}) \ldots a_{p_{n-1}})),
\]

where \( \{i_1, i_2, \ldots, i_{j+1}, \ldots, i_k\} = \{p_1, p_2, \ldots, p_{k+1}, p_{k+2}, \ldots, p_m \mid p_1 < p_2 < \ldots < p_{k+1} < p_{k+2} < \ldots < p_m \} \).

As we have noticed \( \sigma w \neq w \).

**5. Proof of Theorem 2.4**

Let \( V(n) \) be a vector space with dimension \( n \). By Theorem 2.2, \( KT_n \) is isomorphic to \( V(1) \otimes V(1) \otimes \ldots \otimes V(1) \) as \( S_n \)-module. Therefore

\[
KT_n \cong \text{Ind}_{S_1 \times S_1 \times \ldots \times S_1}^{S_n} (1_{S_1} \otimes 1_{S_1} \otimes \ldots \otimes 1_{S_1}) \cong \bigoplus_{\lambda \vdash n} d_\lambda S^\lambda,
\]

where \( 1_{S_1} \) is one-dimensional trivial representation of \( S_1 \).

By Theorem 2.2, group of automorphisms of A-part of \( T_{k,n-k} \) is \( S_k \) and group of automorphisms of B-part is \( S_{n-k} \). Therefore \( S_k \times S_{n-k} \) is group of automorphisms of \( T_{k,n-k} \).
Let
\[ g_A = \sum_{\sigma \in S_k} \sigma \in KS_k, \quad g_B = \sum_{\tau \in S_{n-k}} \tau \in KS_{n-k}. \]
be elements of group algebras \( KS_k \) and \( KS_{n-k} \), respectively. Then by Theorem 2.2 \( g_{T,k,n-k} = g_A \otimes g_B \) is generator of all basic elements of \( KT_{k,n-k} \) and \( g_A, g_B \) are one-dimensional trivial representations of \( S_k \) and \( S_{n-k} \), respectively, and \( KT_{k,n-k} \) is \( S_k \times S_{n-k} \)-module. Therefore \( KT_{k,n-k} \) as \( S_n \)-module is isomorphic to
\[
\text{Ind}_{S_k \times S_{n-k}}^S (1_{S_k} \otimes 1_{S_{n-k}}) \cong \bigoplus_{(\lambda_1,\lambda_2)\vdash n} S^{(\lambda_1,\lambda_2)}, \quad \lambda_2 \leq \min\{k, n-k\},
\]
where \( 1_{S_k} = g_A, 1_{S_{n-k}} = g_B \).

By Corollary 2.3
\[
P_n \cong KS_n \oplus \bigoplus_{k=0,1,\ldots,n-3} KT_{k,n-k} \cong \bigoplus_{\lambda^\prime \vdash n} d_\lambda S^\lambda \oplus \bigoplus_{(\lambda_1,\lambda_2)\vdash n} m(\lambda_1, \lambda_2) S^{(\lambda_1,\lambda_2)},
\]
where
\[
m(\lambda_1, \lambda_2) = \begin{cases} n - 2 - \lambda_2, & \lambda_2 \leq 3, \\ n + 1 - 2\lambda_2, & \lambda_2 \geq 4. \end{cases}
\]

6. Proof of Corollary 2.7

a. Follows from Theorem 2.4.

b. \( KT_n \) as \( S_n \)-module is isomorphic to
\[ KT_n \cong \bigoplus_{\lambda^\prime \vdash n} d_\lambda S^\lambda. \]

\( KA_n \) as \( A_n \)-module is isomorphic to
\[
KA_n \cong \left[ \bigoplus_{\lambda \neq \lambda'} d_\lambda S^\lambda_A \right] \oplus \left[ \bigoplus_{\lambda = \lambda'} \left( \frac{d_\lambda}{2} S^\lambda_A^+ \oplus \frac{d_\lambda}{2} S^\lambda_A^- \right) \right],
\]
where \( S^\lambda_A \) is irreducible \( A_n \)-module.

If \( \lambda \vdash n \) is non-self-conjugate partition, then \( S^\lambda \) and \( S^{\lambda'} \) as \( A_n \)-modules are isomorphic to
\[
\text{Res}_{A_n}^{S_n} (S^\lambda) \cong S^\lambda_A, \quad \text{Res}_{A_n}^{S_n} (S^{\lambda'}) \cong S^{\lambda'}_A
\]
and
\[
S^\lambda_A \cong S^{\lambda'}_A,
\]
where \( \dim(S^\lambda_A) = \dim(S^{\lambda'}_A) = d_\lambda \).

If \( \lambda \vdash n \) is self-conjugate partition, then \( S^\lambda \) as \( A_n \)-module is isomorphic to
\[
\text{Res}_{A_n}^{S_n} (S^\lambda) \cong (S^\lambda_A^+ \oplus S^\lambda_A^-),
\]
where \( \dim(S^\lambda_A^+) = \dim(S^\lambda_A^-) = \frac{d_\lambda}{2} \). Details see [10].

Therefore
\[ KT_n \cong 2 \cdot KA_n. \]
Note that $KT_{k,n-k}$, $k = 0, 1, ..., n - 3$, as $S_n$-module is isomorphic to

$$KT_{k,n-k} \cong \bigoplus_{(\lambda_1,\lambda_2)\vdash n} S^{(\lambda_1,\lambda_2)}, \quad \lambda_2 \leq \min\{k, n-k\}.$$ 

Therefore $KT_{k,n-k}$ as $A_n$-module is isomorphic to

$$Res_{A_n}^S(KT_{k,n-k}) \cong Res_{A_n}^S\left(\bigoplus_{(\lambda_1,\lambda_2)\vdash n} S^{(\lambda_1,\lambda_2)}\right) \cong \bigoplus_{(\lambda_1,\lambda_2)\vdash n} Res_{A_n}^S S^{(\lambda_1,\lambda_2)}, \quad \lambda_2 \leq \min\{k, n-k\}.$$ 

c. $(S_n$-case) It is well known, that

$$W^\lambda \cong V\otimes K S_n S^\lambda.$$ 

Then

$$H_n(V) \cong V\otimes K S_n P_n \cong \bigoplus_{\lambda\vdash n} d_\lambda S^\lambda \oplus \bigoplus_{(\lambda_1,\lambda_2)\vdash n} m(\lambda_1,\lambda_2) S^{(\lambda_1,\lambda_2)},$$ 

$$\cong \bigoplus_{\lambda\vdash n} (V\otimes K S_n S^\lambda) \oplus \bigoplus_{(\lambda_1,\lambda_2)\vdash n} m(\lambda_1,\lambda_2) (V\otimes K S_n S^{(\lambda_1,\lambda_2)}),$$ 

$$\cong \bigoplus_{\lambda\vdash n} d_\lambda (V\otimes K S_n S^\lambda) \oplus \bigoplus_{(\lambda_1,\lambda_2)\vdash n} m(\lambda_1,\lambda_2) (V\otimes K S_n S^{(\lambda_1,\lambda_2)}).$$ 

d. $(A_n$-case) As in case c ( $S_n$-case )
e.

e. Follows from a and Corollary 7.13.9 in [29] □

**Acknowledgements**

- This project partially was carried out when the second named author visited the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences. He is very grateful for the creative atmosphere and the warm hospitality during his visit.
- Authors are grateful to V. Drensky for his kind interest in our results and for essential comments.
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