Analysis and improvement of a mathematical turbine model

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Abstract. A mathematical turbine model is necessary for dynamic and transient analysis of hydro power plants, also in the early stages of a project. Such models can be based on a first principles approach or on empirical data, many models are a combination. A first principles approach is practical when specific laboratory data are unavailable, for example in the design phase of a new plant. These models can easily be simplified and linearized without losing physical correctness or generality, even though accuracy may vary when applied to a specific system.

The model studied in this work, was developed several years ago from the Euler turbine equation and the opening degree definition, and has later been modified and linearized for simple implementation into simulation software. The model captures losses for varying rotational speed quite well, but struggles to capture losses for varying flow rate. The hydraulic efficiency is overpredicted for flow far off the design point, because irreversible hydraulic loss phenomena such as 3D turbulence and dissipation are not inherently included in this 1D model. Empirical relations or Hill charts for exact fit to a measured turbine can be included, however generality is then lost.

This paper analyzes and discusses the mathematical model compared to laboratory measurements, highlighting its performance in predicting the efficiency off best efficiency point (BEP). A mathematical improvement based on turbine data is presented and implemented for three specific Francis turbines. The possibility to generalize the procedure is also discussed.

1. Introduction
To ensure optimal design and operation of new and existing hydro power plants, modelling and analysis of the system is crucial. The necessary detail and accuracy of the model will vary according to the necessary detail and accuracy of the results. Typical analysis are stability analysis of turbine and its governor, transient analysis of turbine and waterways, or larger grid analysis. The overall objective is the ability to predict behaviour and test scenarios without performing full-scale testing, which may not be a feasible option.

A significant part of turbine modelling includes modelling losses or turbine efficiency when hydraulic energy in the water is transformed to mechanical energy on the shaft by the runner. For a reaction turbine like Francis, the water is a continuum from upper reservoir through the turbine to the lower reservoir. Pressure transients in the system are due to changes in the flow, which is defined by the turbine and depends on available pressure head, angular speed of rotation and guide vane opening [2]. The turbine’s torque on the shaft, which defines the mechanical power output and thus the efficiency with respect to hydraulic power, depends on the same
variables through the flow. These intrinsic functionalities call for accurate mathematical models.

This paper seeks to present the most important parts of the Master Thesis work [10], which continues the Project Thesis work [9]. The turbine model studied in these works was first developed by professor Nielsen in [1], however a steady version of it, and excluding empirical loss coefficients making it independent from specific data as in [2][3][4]. As demonstrated in these works, along lines of constant turbine opening degree, the ”pure” model already performs quite well. The losses present are mainly determined by the speed of the turbine, spinning too slow or too fast compared to optimal. On their most basic form, the equations include such losses. Also demonstrated, losses caused by irreversible flow phenomena like turbulence, residual swirl in the draft tube, etc., typically related to varying flow rate, are not inherent in the equations to the same extent, and should be modelled separately.

2. Governing equations
The first derivation of the model is found in [1], and later in [2][3][4], as well as thoroughly step-by-step in [9]. The starting point is the Euler turbine equation and the definition of turbine opening degree $y$, together with the associated runner inlet (1) and outlet (2) velocity diagrams for the mean streamline:

\[
\eta_h = \frac{\omega (r_1 c_{u1} - r_2 c_{u2})}{g H} = \frac{u_1 c_{u1} - u_2 c_{u2}}{g H}
\]

\[
y = \left( \frac{Q}{\sqrt{2gH}} \right) \left( \frac{Q_R}{\sqrt{2gH_R}} \right) \Rightarrow gH = gH_R \left( \frac{Q}{y Q_R} \right)^2
\]

Where net pressure head $H$ is the difference in energy grade line (sometimes referred to as total head [3][4]) between the defined turbine inlet and outlet, $\eta_h$ or simply just $\eta$ is the Euler or hydraulic efficiency, $\omega$ is the runner rotational speed and $Q$ is the flow rate. Subscript ”R” denotes rated/ nominal values, which are assumed to correspond to BEP. The velocity vector components come from the velocity diagrams:

![Figure 1. Francis runner inlet (1) and outlet (2) velocity diagrams. This figure is slightly off the best efficiency point (BEP), where $c_{u2} = 0$, and $v_1$ is perfectly aligned with the runner blade at inlet.](image-url)

The full derivation will be omitted, but different versions can be found in [1][2][3][4][5][9][10]. The resulting governing equations describe the flow or head, and the torque, and are presented in per unit (p.u.) with respect to rated values:
\[ q = y \sqrt{h - \sigma (\bar{\omega}^2 - 1)} = f(h, y, \bar{\omega}) \]  
(3)

\[ t = q (m_S - \psi \bar{\omega}) = g(q, y, \bar{\omega}) \]  
(4)

Where \( q, h, y, \bar{\omega} \) and \( t \) are p.u. turbine flow rate, net head, opening degree (also denoted \( \kappa \) in several works), rotational speed and torque, respectively. \( \alpha_1 \) is defined as the angle between guide vanes and peripheral direction and is equal to the absolute flow angle at runner inlet assuming the absolute flow leaves the guide vane perfectly aligned. For a Francis, \( \alpha_1 \) relates to the p.u. opening degree \( y \) (or \( \kappa \)) according to:

\[ y = \frac{\sin \alpha_1}{\sin \alpha_{1R}} \]  
(5)

The geometrical constant \( \sigma \) is:

\[ \sigma = \frac{\omega_R^2}{8 g H_R} (D_1^2 - D_2^2) \]  
(6)

The non-dimensional specific starting (when \( \bar{\omega} = 0 \)) torque \( m_S \) is:

\[ m_S = \frac{\xi}{y} \left( \cos \alpha_1 + \tan \alpha_{1R} \sin \alpha_1 \right) \]  
(7)

The two machine constants \( \psi \) and \( \xi \) are:

\[ \psi = \frac{u_{2R}^2}{\eta_R g H_R} \]  
(8)

\[ \xi = (1 + \psi) \cos \alpha_{1R} \]  
(9)

The hydraulic efficiency can be defined as mechanical power on the shaft divided by available hydraulic power in the water. The p.u. hydraulic efficiency, \( \eta \), is highly relevant:

\[ \eta = \frac{T \omega}{\rho g Q H} = \frac{t T_R \bar{\omega} \omega_R}{\rho g q Q_R h H_R} \Rightarrow \eta = \frac{\eta_R}{q h} \]  
(10)

Substituting for p.u. head \( h \) and torque \( t \) gives:

\[ \eta = \frac{(m_S - \psi \bar{\omega}) \bar{\omega}}{\left( \frac{q}{y} \right) + \sigma (\bar{\omega}^2 - 1)} \]  
(11)

Since flow rate \( q \) is functional dependent upon pressure head \( h \), it is included indirectly in the efficiency. For a given opening degree, turbine efficiency is a function of flow (or head) and rotational speed. It can be visioned like a 3D surface in a 3D diagram, which is the Hill diagram where contour lines represent constant efficiency levels [4]. It is common to present the abscissa and ordinate axes using unit speed and unit flow, such that the head is also included [10].
3. Incipient efficiency $\eta_i$

The main purpose of the incipient efficiency is to improve model accuracy in capturing irreversible hydraulic loss phenomena for varying flow rate. This mainly includes turbulence, viscous effects and losses due to wrong (sub-optimal) inflow angles. In [1], the origin of these model equations, the same losses which $\eta_i$ intend to capture, were modelled as head losses with empirical loss coefficients. They were subtracted from the torque equation, but the incipient efficiency will instead be multiplied into the torque equation. In its first use [2], $\eta_i$ is inserted directly into the efficiency equation, and in later works [3][4] as well as in [9][10], this is ensured by inserting it into the torque equation. $\eta_i$ clearly separates the "pure" model equations from all the irreversible losses. The modified torque and efficiency equations become:

$$t = \eta_i q (m_S - \psi \tilde{\omega})$$  \hspace{1cm} (12)$$

$$\tilde{\eta} = \frac{\eta_i t \tilde{\omega}}{q h} = \frac{\eta_i (m_S - \psi \tilde{\omega}) \tilde{\omega}}{\left(\frac{q}{y}\right)^2 + \sigma (\tilde{\omega}^2 - 1)}$$  \hspace{1cm} (13)$$

Where $\eta_i$ is a mathematical loss function which can take any form. For any turbine, there should exist an $\eta_i$ that will make the model equations fit with measurements [4]. According to [4], inserting it into the torque equations is mathematically equivalent to inserting it into the denominator of the head equation, expression (13) will be the same. The flow equation (3) remains unchanged with this implementation.

3.1. The simplest $\eta_i$ function

The first formulation was approximated as a function of flow rate solely, it is the parabola [2]:

$$\eta_i = q(2 - q) = 1 - (q - 1)^2$$ \hspace{1cm} (14)$$

Which is 0 for $q = 0$ and $q = 2$, and 1 for $q = 1$, corresponding to nominal efficiency at nominal flow rate. Despite its simplicity, equation (14) captures all the main physical aspects of losses along the q-axis [2][4]. It was demonstrated in [10] that equation (14) is a little too generic for high head Francis, but can be a quite good approximation for low head Francis.

![Figure 2. Incipient efficiency function $\eta_i(q) = q(2 - q)$, proposed in [2].](image)

3.2. Best fit $\eta_i$ functions

When measurements are available, surely a fitted $\eta_i(q)$ function can improve model accuracy. This was performed in [10] based on experimental data from Norwegian turbine manufacturer
Rainpower, for three different model Francis runners designed to operate under very different pressure heads. Curve fitting was performed at best efficiency speed and head, resulting in the following incipient efficiency curves for high, medium and low head Francis, respectively:

**Figure 3.** High head Francis best fit $\eta_i(q)$ to measured model turbine data at rated speed.

**Figure 4.** Medium head Francis best fit $\eta_i(q)$ to measured model turbine data at rated speed.

**Figure 5.** Low head Francis best fit $\eta_i(q)$ to measured model turbine data at rated speed.

These best fit functions are mathematically independent from runner speed $\tilde{\omega}$ or pressure head $h$, they depend only on flow rate $q: \eta_i(q)$. The accuracy of this approximation was discussed a lot in [10]. The mathematical expressions are a 9th degree polynomial, a 5 harmonics Fourier series and a 4th degree polynomial for the high, medium and low head turbine, respectively. Other curve fitting models can also be adequate for this purpose.

**4. Model simulations**

Inputs to the mathematical model consist of optimal guide vane angle $\alpha_{1R}$, geometrical constant $\sigma$ and machine constant $\psi$ (and $\xi$ calculated from $\psi$ and $\alpha_{1R}$). These can be found from the main dimensions and nominal operating point of a turbine, obtained by measuring the geometry or by performing a simplified design “from scratch”. In [10], two different approaches to set up the model based on measured Hill chart without knowing runner dimensions, was demonstrated. The second approach, based on only one measured operating point, the BEP, proved to be more accurate. The resulting input values are presented with two decimal place accuracy in table 1.

|                      | High head Francis | Medium head Francis | Low head Francis |
|----------------------|-------------------|---------------------|------------------|
| $\alpha_{1R}$ [$^\circ$] | 10.52             | 15.99               | 27.15            |
| $\sigma$ [-]         | 0.69              | 0.46                | 0.01             |
| $\psi$ [-]           | 0.20              | 0.45                | 1.12             |
| $\xi$ [-]            | 1.18              | 1.39                | 1.89             |

Presented in the next subsections are the complete Hill diagrams calculated by the mathematical model, compared to the diagram from measurements of the associated model turbine [10]. The calculated diagrams are presented both excluding and including the $\eta_i(q)$ curves presented above in Fig. 3, 4 and 5, to emphasize differences in model behaviour. Dashed rectangle defines the operating area where data are given (“validation area”).
4.1. High head Francis

Figure 6. High head Francis Hill chart calculated by the model without $\eta_i$.

Figure 7. High head Francis Hill chart calculated by the model with $\eta_i$ from Fig. 3.

Figure 8. High head Francis Hill chart from measurements.
4.2. Medium head Francis

**Figure 9.** Medium head Francis Hill chart calculated by the model without $\eta_i$.

**Figure 10.** Medium head Francis Hill chart calculated by the model with $\eta_i$ from Fig. 4.

**Figure 11.** Medium head Francis Hill chart from measurements.
4.3. Low head Francis

Figure 12. Low head Francis Hill chart calculated by the model without $\eta_i$.

Figure 13. Low head Francis Hill chart calculated by the model with $\eta_i$ from Fig. 5.

Figure 14. Low head Francis Hill chart from measurements.

5. Discussion

The model including a smooth $\eta_i(q)$ curve going from 0 through 1 and back to 0, will produce a Hill diagram shaped similar to a cone with a rectangular base. Towards zero efficiency, shape is more rectangular, while towards BEP, more oval. In a certain area around the peak, the model appears to predict general shape well, but towards zero efficiency, it is more off target.

Demonstrated in [2][3][4][10] and evident from the presented Hill charts, the equations already perform well along constant $y$ curves, especially for $y = 1$. The main weakness is performance along the $q$-axis, as well as predicting accurate runaway conditions. Implementing $\eta_i(q)$ functions appear to improve the former weakness. In the above diagrams, hill shape along the $q$-axis is highly affected by its presence, and can, when chosen wisely, be greatly improved.

It is evident from the calculated Hill charts that the model struggles to capture runaway curves accurately. Comparing Fig. 10 to Fig. 11, runaway conditions agrees relatively well around $\tilde{\omega} = 1$ and $y$ low, and around $y = 1$ and $\tilde{\omega}$ high, but for the region in between, the
modelled curve deviates from the measured one. For general transient analysis including load rejection, some more work is required.

Perhaps the most essential losses intended for $\eta_i$ to capture, are losses due to wrong inflow angles, mainly incidence or impact loss at the runner inlet and residual swirl in the draft tube. The loss model for these in [1] depends on flow rate, guide vane angle and rotational speed; $F(Q, \alpha_1, \omega)$. It is reasonable to believe that not including $\tilde{\omega}$ as an independent variable for $\eta_i$, will impact its accuracy away from $\tilde{\omega} = 1$, for example with respect to runaway conditions.

The concept of generalizing $\eta_i$ based on turbine speed number $\Omega$ was briefly demonstrated and discussed in [10]. The speed number relates nominal flow, head and rotational speed in a single dimensionless parameter, and is often used as a reference value for classification, or may assist in deciding appropriate type, size and shape [7]:

$$\Omega = \frac{\omega Q^{1/2}}{(2gH)^{3/4}}$$  \hspace{1cm} (15)

From the experimental data in Fig. 3, 4 and 5, increasing speed number involves a sharper efficiency curve, whereas decreasing speed number involves a wider curve having a flow region of relatively flat efficiency data. This tendency is consistent with references like [6][8]. Consequently, for $\eta_i(q)$ curves based on speed number, increasing $\Omega$ should "sharpen" the curve, decreasing $\Omega$ should "widen" it. Demonstrated in [10], linear interpolation between two curves corresponding to a lower and a higher speed number is a first order approximation to obtain $\eta_i$ when specific data are unavailable:

$$\eta_i(q) = (1 - x) f_1(q) + x f_2(q) = f_1(q) + x (f_2(q) - f_1(q))$$  \hspace{1cm} (16)

Where

$$x(\Omega) = \frac{\Omega - \Omega_1}{\Omega_2 - \Omega_1}, \quad \Omega_1 \leq \Omega \leq \Omega_2$$  \hspace{1cm} (17)

$\Omega_1$ and $f_1(q)$ correspond to the lower speed number, i.e. the high head turbine, and $\Omega_2$ and $f_2(q)$ correspond to the higher speed number, i.e. the low head turbine. For any $\Omega$ in this range the resulting $\eta_i(q)$ will be a linear combination of the two curves.

The overall objective is to improve model accuracy without loss of generality, and for it to remain independent from empirical relations.

6. Conclusion

The incipient efficiency intends to collectively model all irreversible hydraulic losses for flow rates away from optimal conditions, without the use of empirical loss coefficients which require tuning to specific data. The mathematical turbine model including a general $\eta_i$ function, seems to be very promising.

The presented modified formulation of the equations can predict turbine behaviour around the best efficiency point and along lines of constant nominal speed or nominal opening degree in the Hill chart quite well. This implies that the model is suitable for stability- or grid analysis around $\tilde{\omega} = 1$ for any $q$ and $y$, or around $y = 1$ for any $\tilde{\omega}$. The turbine runaway curve is not accurately modelled by this formulation. For the purpose of predicting all parts of the Hill chart accurately, for example for general transient analysis including load rejection, start-up or shut-down, further model improvement is necessary. This can involve developing $\eta_i$ functions where $\tilde{\omega}$ is also an independent variable.
In addition to further development of more general incipient efficiency curves, further work includes software implementation of the improvement. For the purpose of stability analysis, the linearized equations, covered in [3][4][9][10], are convenient. More work on the effects of $\eta_i$ on the linear model for operation off the design point, is necessary. Last but not least, $\eta_i$ functions appropriate for other turbine types like Pelton or Kaplan, can be developed in further work.

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