Statistical properties of worm algorithms for two dimensional frustrated Ising models

Geet Rakala,1 Kedar Damle,2 and Deepak Dhar3

1Okinawa Institute of Science and Technology Graduate University, Onna-son, Okinawa 904-0412, Japan
2Tata Institute of Fundamental Research, 1 Homi Bhabha Road, Mumbai 400005, India
3Indian Institute of Science Education and Research, Homi Bhabha Road, Pashan, Pune 411008, India

We study the distribution of lengths and other statistical properties of worms constructed by worm algorithms used in Monte Carlo simulations of frustrated triangular and kagome lattice Ising antiferromagnets, focusing on the behaviour of the associated persistence exponent $\theta$ in the critical phase associated with the two-step melting of three-sublattice order in these systems. We establish that these properties of the worms depend only on the universal long-distance equilibrium properties of the underlying critical phase, and not on the details of the worm algorithm or microscopic Hamiltonian. Viewing each step of the worm construction as a position increment (step) of a random walker, we argue that the power-law correlations of the underlying equilibrium system give rise to two related properties of this walk: First, the walk is conducted in a logarithmiccentral potential whose strength is set by the power-law exponent $\eta_m$ of the equilibrium defect-antidefect correlation function. Second, the steps of the walk are correlated. We demonstrate that our results imply a dynamical exponent $\eta > 2$ for this walk, which is related to the measured deviation of $\theta$ from the value $\eta_m/2$ via the scaling relation $\eta = (1 - \theta)/(2 - \eta_m)$.

PACS numbers: 75.10.Jm

I. INTRODUCTION

Worm algorithms are very useful as a means of generating non-local updates in Monte Carlo simulations of various lattice models (for a brief review, see Section 5.1 of Ref.1). The ‘worm’ construction typically starts by creating a defect and an antidefect next to each other in the initial configuration. The location of the defect defines the fixed tail of the worm, while the head of the worm corresponds to the antidefect, which is moved through the lattice in a way which satisfies detailed balance conditions in a larger configuration space that allows for one defect-antidefect pair. The construction ends when the head reaches the tail again and annihilates it. All variables encountered during the motion of the worm are updated as a result of this construction.

An early implementation of a worm algorithm in the context of classical Monte Carlo simulations used the high-temperature expansion representation, and updated closed path configurations through the motion of end points of a disconnected path.2 A similar idea was also used to develop a worm algorithm for the quantum rotor model in $d = 2$ spatial dimensions using the link-current representation (divergence-free configurations of current variables on links of the equivalent classical $d + 1 = 3$ dimensional space-time lattice).3,4 The construction creates a charged defect (with non-zero divergence of the link current) at the tail, and a corresponding antidefect at the head. In this case, the worm maintains detailed balance in the configuration space relevant to the sampling of the single-particle Green function of the system.5 In quantum Monte Carlo simulations of other bosonic systems, a similar worm algorithm has been used both in the framework of imaginary time worldline formulations6,7 and the stochastic series expansion (SSE) approach8 to perform non-local changes in the configuration. In this case too, the defects at the head and the tail of the worm correspond to creation and annihilation of a particle,9 allowing access to configurations relevant to the sampling of the single-particle Green function.

‘Dual’ worm algorithms have also been used to construct cluster updates for two-dimensional classical Ising models.10 These algorithms work by updating dimer configurations (which encode bond energies of the original model) along a closed loop on the corresponding dual lattice. The updated bond energies are used to obtain a new spin configuration in which all spins in the interior of this closed loop have been flipped in one step. Recently, this approach has been used11 to obtain efficient cluster updates for frustrated Ising models for which the usual cluster updates12,13 are known to perform poorly.14 For instance, for the antiferromagnetic Ising model on the triangular lattice, bond-energy configurations correspond to dimer configurations on the dual honeycomb lattice, with dimers intersecting frustrated bonds on the direct lattice. At $T = 0$, the dimer configuration is a perfect matching (each dual lattice site is touched by exactly one dimer). In this case, the defect at the head of the worm corresponds to a monomer, i.e. a site on the dual lattice with no dimers touching it. The antidefect corresponds to an antimonomer on the same sublattice, i.e. a site with two dimers touching it. The initial defect-antidefect pair is created by simply picking a site at random and pivoting the dimer touching it to another unoccupied link. The antidefect is then propagated by pivoting successive dimers along a closed path, with probabilities chosen to preserve detailed balance. The updated dimer configuration of the dual lattice is then mapped back to a new spin configuration after the worm construction is complete. This flips an entire cluster of spins. At nonzero temper-

arXiv:1812.11861v1 [cond-mat.stat-mech] 31 Dec 2018
ature, the dimer configuration is not a perfect matching since dual lattice sites with three dimers touching them are allowed, and the worm construction is suitably generalized to work with more general defect anti-defect pairs.11

The fact that all these worm constructions preserve detailed balance in a larger configuration space with one defect-antidefect pair allows for an interesting and simple method to calculate the corresponding correlation functions: The equilibrium defect-antidefect correlation function is simply proportional to the histogram of the head-to-tail separations measured during the worm construction.9,15,16 In the quantum rotor case, and in the context of worldline and SSE methods for bosonic systems, this corresponds to the imaginary time single-particle Green function.4,9,15,16 As we detail below, in the example studied in our work here, this corresponds to the correlation function between half-vortices (with vorticities ±1/2) in the argument \( \phi \) of the complex three-sublattice order parameter of the spin system.11

Apart from measuring the defect-antidefect correlator during worm construction, one can also measure various statistical properties of the worms themselves; the simplest of these is the distribution of worm lengths. This is of interest because the Monte-Carlo autocorrelation properties of such worm algorithms depend on the number of variables updated in a single worm construction, which in turn depends on the distribution of worm lengths. For instance, the fractal structure and scaling properties of worms defined within the high temperature expansion have been studied previously.17 Properties of spin clusters defined by other cluster algorithms12,13 have been numerically studied in the case of the critical two-dimensional Ising model18 and found to be in agreement with theoretical predictions.19–23 Following the generalization of cluster algorithms to the fully frustrated square lattice,24 the properties of such clusters have also been studied extensively in that setting.25 Since closed worms on the dual lattice define a cluster on the original lattice, properties of these clusters are also interesting from this point of view. Statistics of worms constructed by a direct worm algorithm for a three dimensional spin ice model have also been studied, but less information seems to be available on worms in the corresponding two dimensional model.26

Part of the motivation for the present study is our earlier observation that the autocorrelation properties of two rather different dual worm algorithms (the Deposition-Evaporation-Pivoting or DEP algorithm and the myopic algorithm)11 seem to be universally determined by the power-law exponent of the equilibrium spin-spin correlation function, when used to simulate frustrated Ising models on two different two-dimensional lattices (triangular and kagome) over a range of temperatures and further neighbor interactions for which the system displays power-law three-sublattice order. Since the worm length distribution is expected to control the manner in which successive configurations decorrelate with each other, we attempt to understand this universality by focusing here on the persistence exponent \( \theta \) that characterizes the power-law form of the worm length distribution in this interesting regime. In addition, we also study other statistical properties of the worms.

We establish that these properties of the worms depend only on the universal long-distance properties of the underlying equilibrium ensemble, and not on the details of the worm algorithm or microscopic Hamiltonian. Viewing each step of the worm construction as a position increment (step) of a random walker, we argue that the power-law correlations of the underlying equilibrium system give rise to two related properties of this walk: First, the walk is conducted in a logarithmic central potential whose strength is set by the power-law exponent \( \eta_m \) of the equilibrium defect-antidefect correlation function. Second, the steps of the walk are correlated. Without the second effect, this random walk analogy would suggest \( \theta = \eta_m/2 \), which is seen to be inconsistent with our numerical results. We establish a scaling relation \( z = (1-\theta)/(2-\eta_m) \) which relates the deviation of \( \theta \) from this Markovian random walk value of \( \eta_m/2 \) to a dynamical exponent \( z \) different from \( z = 2 \). Our numerical results therefore imply a dynamical exponent \( z > 2 \) for the random walk defined by the worm construction process.

Our starting point is the well-known observation, alluded to in the foregoing, that the histogram of head-to-tail distances of the worm is given by the equilibrium defect-antidefect correlator, which in our case corresponds to the correlation function for half-vortices in the three-sublattice order parameter. Using standard results from the Kosterlitz-Thouless description27 of the power-law three-sublattice ordered phase, we conclude that this defect-antidefect correlator also has a power-law form, with exponent \( \eta_m = 1/4\eta_s \), where \( \eta_s \) is the power-law exponent for the correlations of. For such a Markovian random walker in two dimensions \((d = 2)\), \( \theta \) depends on the strength of the logarithmic potential via:

\[
\theta = \frac{\eta_m}{2}.
\]

We develop a scaling argument that relates the dynamical exponent \( z \) for this walk to \( \eta_m \) and \( \theta \), obtaining:

\[
z = \frac{2 - \eta_m}{1 - \theta}
\]

Thus, deviations of \( \theta \) from the Markovian random walk value of \( \eta_m/2 \) are indicative of a dynamical exponent \( z \) different from \( z = 2 \). Indeed, our numerical results yield a dynamical exponent \( z > 2 \) that increases monotonically with increasing \( \eta_m \) in this power-law ordered phase. The worms therefore define a discrete-time realization of a fractional brownian motion which has a conventional steady-state given by the equilibrium Gibbs distribution of a particle in a logarithmic central attractive potential.

The rest of this paper is organized as follows: In Section II we briefly describe the models which are simulated by the worm algorithms studied here. In Section III we model the worm length distribution (see
Eq. (1) by drawing an analogy between our worm algorithm and a Markovian random walker in a central logarithmic potential and provide a scaling argument that relates deviations of $\theta$ from the predictions of this Markovian model to a dynamical exponent $z$ different from 2. In Section V we provide precise definitions of various properties of the worm which are measured during the worm construction. In Section VI, we summarize our results for these statistical properties of the worms including the persistence exponent $\theta$ and the dynamical exponent $z$. Finally in Section VII, we discuss some promising directions for future work.

II. MODELS

Ising models on triangular and kagome lattices with antiferromagnetic nearest neighbor interactions are among the simplest models of geometric frustration.\textsuperscript{30,31} For these models, the pattern of nearest-neighbour bond energies can be represented in terms of dimer models on the corresponding dual lattice (honeycomb and dice respectively).\textsuperscript{11} When further neighbour interactions are absent, there is a macroscopic degeneracy of minimum energy spin configurations, which corresponds to a $T = 0$ ensemble of dimer configurations on the dual lattice. For the triangular lattice antiferromagnet, this $T = 0$ ensemble is made up of all perfect matchings (fully-packed

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{The lattice size $L$ dependence of the defect correlator $C_{\text{defect}}(e_x L^{24})$ at separation $e_x L^{24}$ on periodic $L \times L$ triangular lattices using the DEP worm algorithm for three values of $J_2/T$ at which the system is in the power-law ordered critical phase in the zero temperature limit $T \to 0$. Lines denote fits to a power-law form $\propto 1/L^{7m}$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{The lattice size $L$ dependence of the defect correlator $C_{\text{defect}}(e_x L^{24})$ at separation $e_x L^{24}$ on periodic $L \times L$ triangular lattices using the myopic worm algorithm for three values of $J_2/T$ at which the system is in the power-law ordered critical phase in the zero temperature limit $T \to 0$. Lines denote fits to a power-law form $\propto 1/L^{7m}$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{The lattice size $L$ dependence of the defect correlator $C_{\text{defect}}(e_x L^{24})$ at separation $e_x L^{24}$ on periodic $L \times L$ triangular lattices using the DEP worm algorithm for three values of $T$ at which the system is in the power-law ordered critical phase. Lines denote fits to a power-law form $\propto 1/L^{7m}$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{The lattice size $L$ dependence of the defect correlator $C_{\text{defect}}(e_x L^{24})$ at separation $e_x L^{24}$ on periodic $L \times L$ triangular lattices using the myopic worm algorithm for three values of $T$ at which the system is in the power-law ordered critical phase. Lines denote fits to a power-law form $\propto 1/L^{7m}$.}
\end{figure}
\[ C_m(\hat{e}_x L^{24}) + C_m(\hat{e}_y L^{24}) \]

\[ T = 1.36; \eta_m = 1.27(4) \]
\[ T = 1.30; \eta_m = 1.63(5) \]
\[ T = 1.24; \eta_m = 1.96(9) \]

\[ \eta_m \text{ extracted from the } L \text{ dependence of the defect correlator plotted as a function of } \eta_s \text{ in simulations employing the DEP and myopic worm algorithms. The line denotes the theoretically expected dependence } \eta_m = 1/\eta_s. \]

\[ \eta_m \text{ for the ensemble of fully-packed dimer configurations on the honeycomb lattice. In this representation, the spin operator at the three-sublattice wavevector corresponds to } \exp(i\pi h) \text{ (where } h \text{ is the height field) while the dimer operator has a uniform part given in terms of the gradient } \nabla h \text{ and a second piece } \exp(2\pi i h) \text{ at the three-sublattice wavevector. Dimer correlations at the uniform wavevector fall of as } 1/r^2 \text{ independent of the stiffness of the height model, while correlations at the three-sublattice wavevector decay with power-law exponent } \eta_d \text{ controlled by the stiffness of the height model. Spin correlations at the three-sublattice wavevector fall of as a power law} \]
with exponent \( \eta_s \) (with \( \eta_d = 4 \eta_s \)). When all fully-packed dimer configurations have equal weight (as is the case for the nearest neighbour antiferromagnet), \( \eta_d = 2 \).

A second-neighbour ferromagnetic interaction \( J_2 \) on the triangular lattice, with \( |J_2| \propto T \) in the \( T \to 0 \) limit, favours three-sublattice ordered fully-packed dimer configurations over others, and gives rise to a \( \eta_d < 2 \) and \( \eta_s < 1/2 \). Indeed, \( \eta_s \) decreases monotonically with increasing \( |J_2|/T \) (in this zero temperature limit), until the system develops long-range three-sublattice order when \( \eta_s = 1/9 \) is reached.\(^{34} \) In the coarse-grained height representation, this is understood by noting that \( |J_2|/T \) tunes the stiffness of the height model, thereby influencing the value of \( \eta_s \) (and of \( \eta_d = 4 \eta_s \)). Monomers in this fully-packed dimer model correspond, in the Coulomb gas (CG) description of the coarse-grained height model,\(^{15,35} \) to a magnetic charge +1 (antimonomers have magnetic charge −1). As a result, the monomer-antimonomer correlator decays as a power law with an exponent \( \eta_m = 1/\eta_s = 1/4 \eta_s \).

In terms of the argument \( \phi \) of the complex three-sublattice order parameter of the spin model, these monomers are half-vortices.

A fixed nonzero value of second-neighbor ferromagnetic interaction induces long-range three-sublattice order on both lattices at low enough temperature. This melts

---

**FIG. 9.** Probability distribution \( P(\tau_r) \) of the number of sites \( \tau_r \) of the dual lattice visited in one completed worm of the DEP algorithm for a \( 600 \times 600 \) triangular lattice for three values of \( T \) at which the system is in the power-law ordered critical phase. Lines denote fits to a power-law form \( \propto 1/\tau_r^{1+\eta_{\text{measured}}} \).

**FIG. 10.** Probability distribution \( P(\tau_r) \) of the number of sites \( \tau_r \) of the dual lattice visited in one completed worm of the myopic algorithm for a \( 600 \times 600 \) triangular lattice for three values of \( T \) at which the system is in the power-law ordered critical phase. Lines denote fits to a power-law form \( \propto 1/\tau_r^{1+\eta_{\text{measured}}} \).

**FIG. 11.** Probability distribution \( P(\tau_r) \) of the number of sites \( \tau_r \) of the dual lattice visited in one completed worm of the myopic algorithm for a \( 288 \times 288 \) kagome lattice for three values of \( T \) at which the system is in the power-law ordered critical phase. Lines denote fits to a power-law form \( \propto 1/\tau_r^{1+\eta_{\text{measured}}} \).

**FIG. 12.** The persistence exponent \( \theta \) extracted from fits to \( P(\tau_r) \) displayed as a function of \( (\eta_{\text{predicted}}/2) = 1/(8 \eta_s) \) in simulations employing the DEP and myopic worm algorithms in the zero temperature limit \( T \to 0 \). The line corresponds to the Markovian random walk value of \( \eta_m/2 = 1/8 \eta_s \).
via a two-step process, wherein the intermediate state has power-law ordered spin correlations at the three-sublattice wavevector, with power-law exponent $\eta_d$ that ranges from 1/9 (at the low-temperature end) to 1/4 (at the high-temperature end).\cite{27,38,39} When spin correlations display power-law three-sublattice order, the dimer correlations also have a power-law form, with exponent $\eta_d = 4d_s$. At non-zero $T$, the dimer representation of bond-energies now allows three-coordinated sites touched by three dimers or one dimer, greatly increasing the entropy of allowed configurations. The worm algorithm now makes other defects (Section IV) apart from monomers and antimonomers. However, we can still think of these defects as half-vortices in the argument $\phi^{27,38,39}$ of the Fourier component of the spin density at the three-sublattice wavevector. Since the power-law ordered phase is described by a Gaussian theory for $\phi$, the defect-antidefect correlator is again expected to decay with exponent $\eta_d = 1/4d_s = 1/4d$ (where $d_d$, the dimer correlation exponent, is again related to the power-law exponent $\eta_d$ via $\eta_d = 4d_s$). [See Fig. 3, 4 and 5.]

With this background, we use the previously developed DEP and myopic algorithms\cite{11} in the triangular case and the myopic algorithm in the kagome case to simulate the

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]

\[ f(x) = x \]

\[ \frac{\theta_{measured}}{2} \]
$P(\tau_r) \propto 10^{-1/\theta}$

$P(\tau_r)$ of the number of sites of the direct lattice visited ($\tau_r$) in one completed nonwinding worm of the loop algorithm for the fully packed dimer model on a $72 \times 72$ cubic lattice. Line denotes fit to a power-law form $\propto 10^{-1/\theta}$. 

$\theta_{\text{measured}} = 0.48(3)$

$\theta_{\text{measured}}$ from Fig. (13) and Fig. (12) and $\eta_m$ of Fig. (6) as a function of $\eta_r$ for all the cases studied here. Also shown are the values of $z$ extracted from the finite-size dependence of the cutoff scale $\tau_{\text{cutoff}}$ for the DEP algorithm in the $T \to 0$. 

$H = J_1 \sum_{\langle RR' \rangle} \sigma_R \sigma_{R'} + J_2 \sum_{\langle\langle RR' \rangle\rangle} \sigma_R \sigma_{R'}$, 

where $\langle RR' \rangle$ and $\langle\langle RR' \rangle\rangle$ denote nearest-neighbor and next-nearest-neighbor links of the lattice in question, and $\sigma_R = \pm 1$ are the Ising spins on sites $R$ of the triangular or kagome lattice. In our convention, $J_{1/2} > 0$ corresponds to an antiferromagnetic coupling, while $J_{1/2} < 0$ corresponds to a ferromagnetic coupling. We focus here on the case with $J_1 > 0$ and $J_2 < 0$, and study the statistics of worms generated by these algorithms in the power-law three-sublattice ordered phase on both lattices.

III. RANDOM WALK CONSIDERATIONS

As mentioned earlier, our starting point is the well-known statement, alluded to in the foregoing, that the histogram of head-to-tail distances of the worm is given by the equilibrium defect-antidefect correlator. Since the defect remains static and the antidefect moves in a manner that preserves detailed balance, we may interpret this power-law correlation between the defect and antidefect to be the result of a static attractive logarithmic interaction $V(\vec{r}) \equiv -\ln(C_{\text{defect}}(\vec{r})) = \eta_m \ln(r)$ between the head and the tail. The equilibrium power-law correlations of the spin and bond energy variables also give rise to correlations between the successive steps taken by the head of the worm.

Thus, the worm construction can be thought of in terms of a random walk that starts at a site adjacent to the origin and ends when it returns to the origin for the first time while moving with correlated steps in an attractive central logarithmic potential. A crucial property of this walk is that it has a long-time steady state distribution of position given by $C_{\text{defect}}(\vec{r})$. The worm length in this picture is mapped to the time $\tau_r$ of first return to origin of this random walk. As a result, the worm length distribution is given by the probability distribution $P(\tau_r)$ of return times, and is expected to have a power-law form $P(\tau_r) \sim 1/\tau_r^{\theta+1}$, where $\theta$ is the persistence exponent.

We now provide a simple scaling argument that relates the dynamical exponent $z = (2 - \eta_m)/(1 - \theta)$ extracted from $\theta_{\text{measured}}$ (of Fig. (13) and Fig. (12)) and $\eta_m$ (of Fig. (6)) as a function of $\eta_r$ for all the cases studied here. Also shown are the values of $z$ extracted from the finite-size dependence of the cutoff scale $\tau_{\text{cutoff}}$ for the DEP algorithm in the $T \to 0$. 

We note that the requirement that the walk has a long-time steady state distribution of position given by $C_{\text{defect}}(\vec{r})$ implies that the histogram of positions $\vec{r}$, accumulated during the walk must be proportional to $C_{\text{defect}}(\vec{r})$. If we choose a normalization convention whereby this histogram measures the ratio of the number of times the head to tail separation is $\vec{r}$ to the number of returns to the origin, i.e. the number of times the head to tail separation is $\vec{r}$ per worm, then the mean return time is given as $\langle \tau_r \rangle = \langle v \rangle$, with $\langle v \rangle \equiv \sum_{\vec{r}} C_{\text{defect}}(\vec{r})$, where the sum extends over $L^2$ sites of the finite lattice. If this sum is dominated by contributions near the upper cutoff in distance, we expect $\langle v \rangle \sim L^{2-z \eta_m}$. This is true for all $\eta_m < 2$.

Next we note that the average return time can also be expressed in terms of the probability distribution $P(\tau_r)$ by writing $\langle \tau_r \rangle = \sum_{\tau_r} \tau_r P(\tau_r)$. Assuming that the power-law form $P(\tau_r) \sim 1/\tau_r^{\theta+1}$ persists up to a system-size dependent cutoff scale $\tau_{\text{cutoff}}(L) \sim L^z$, where $z$ is the dynamical exponent for the random walk, we obtain $\langle \tau_r \rangle \sim L^{2(1-\theta)}$ whenever the sum is dominated by the contributions near the upper cutoff. This is true for all $\theta < 1$.

Comparing these two predictions for the $L$ dependence of the mean return time, we arrive at the following scaling relation

$$z = \frac{2 - \eta_m}{1 - \theta}.$$
valid whenever $\eta_m < 2$ and $\theta < 1$. Thus, the dynamical exponent $z$ is in general not fixed to the usual Markovian random walk value of $z = 2$. This appearance of a non-Markovian value of $z$ in our description of the worms is in general not fixed to the usual Markovian random walk value of $z = 2$. This appearance of a non-Markovian value of $z$ in our description of the worms should be interpreted in the following way: The underlying worm algorithm is Markovian. The probability table that guides the choice of the next step in the worm construction depends only on the current configuration of the system. However, when one only focuses on the position of the head relative to the fixed tail of the worm, one is tracing out all information about the rest of the system. This projected process is non-Markovian, in the sense that it depends in principle on the entire history of previous positions of the head. It is this dependence that is being described in terms of a $z$ different from $z = 2$.

When $z = 2$, as is the case for the usual Markovian random walk, this relation implies $\theta = \eta_m/2$. This can also be understood in the following way: \cite{28,29} The Fokker Planck equation for a Brownian walker in an attractive logarithmic central potential in dimension $d$ can be transformed using radial coordinates to the equation for a free Brownian walker in an effective dimension $d'$, with

$$d' = d - \eta_m \quad (5)$$

The probability distribution of the first return time $\tau_r$ of a Markovian random walker in dimension $d'$ can be obtained using the corresponding Green function with absorbing boundary conditions at the origin. This predicts the large-$\tau_r$ form:

$$P(\tau_r) \sim \begin{cases} \frac{1}{\tau_r^{2-(d'/2)}}, & \text{for } d' < 2 \\
\frac{1}{(\tau_r \ln^2(\tau_r))}, & \text{for } d' = 2 \\
\frac{1}{\tau_r^{d'/2}}, & \text{for } d' > 2. \end{cases} \quad (6)$$

Thus, for a Markovian random walker with no memory

$$\theta = \begin{cases} 1 - (d'/2), & \text{for } d' < 2 \\
(d'/2) - 1, & \text{for } d' > 2. \end{cases} \quad (7)$$

Since $d = 2$ in our case and $\eta_m \in (1/2, 9/4)$, $d' = 2 - \eta_m$ is always less than 2. For $\eta_m \geq 2$, i.e. close to the ordering transition at which the power-law spin order gives way to long range order, $d'$ turns negative and the analysis leading to Eqn 7 breaks down. Thus, when $z = 2$, these standard results for a Brownian walker provide an alternate interpretation of the scaling relation $\theta = \eta_m/2$, valid when $\eta_m \in (1/2, 2)$. For general values of $z$, our scaling argument exposes the relationship between $z$, $\theta$ and $\eta_m$, but stops short of predicting the value of $\theta$.

**IV. ALGORITHMS**

In this section we briefly outline the algorithms developed in Ref. 11, whose statistics we wish to study here.

As mentioned in the introduction, these worm algorithms are defined on the dual lattice, and work with the dimer representation of the frustrated Ising antiferromagnet on the triangular and kagome lattices. At non-zero temperatures on the triangular lattice, either one or three frustrated bonds can exist on a triangle.  This translates to either one or three dimers touching each lattice site of the dual honeycomb lattice. Similarly on the kagome lattice at non-zero temperatures, every spin configuration corresponds to either one or three dimers touching the three coordinated site of the dual dice lattice and any even number of dimers touching the six-coordinated sites. Thus at non-zero temperatures the configuration space of the dual dimer model is larger than that of the usual fully-packed dimer model.

As noted earlier, the power-law three-sublattice ordered phase is obtained in the triangular lattice both...
at $T = 0$ (with a finite negative value for $J_2/T$) and at $T > 0$ (associated with the melting of long-range three-sublattice order). However, on the kagome lattice, power-law three-sublattice order is obtained only at $T > 0$ (associated with the melting of long-range three-sublattice order). In Ref. 11, two worm algorithms were introduced for the triangular lattice model (the DEP and myopic algorithms), only one of which (the myopic algorithm) generalizes to the kagome lattice. Thus, we have five different settings in which we can test the idea that the statistics of worms in the power-law three-sublattice ordered phase depends only on the exponent $\eta_m$, independent of the details of the worm construction and form of Hamiltonian: One can study the worm statistics of both algorithms in the $T = 0$ power-law ordered phase as well as the $T > 0$ power-law ordered phase on the triangular lattice, and one can also study the worm statistics of the myopic algorithm in the $T > 0$ power-law ordered phase on the kagome lattice. Both the DEP and myopic worm algorithms, though developed for the larger dual configuration space at $T > 0$, reduce in an obvious way at $T = 0$ to previously known implementations of worm algorithms for dimer models on the honeycomb and dice lattices\cite{11}.

Both algorithms proceed by first translating the spin configuration into a dual dimer configuration, with each dimer configuration thus assigned a Boltzmann weight of the parent spin configuration. Next, we update the dimer configuration using these worm algorithms. The DEP and myopic worm algorithms differ in the way they per-

---

FIG. 21. Average number of dual links $\langle p \rangle$ flipped by worms of length $\tau_r$ for the DEP algorithm in a $600 \times 600$ triangular lattice for three values of $T$ at which the system is in the power-law ordered critical phase. Lines denote fits to a power-law form $\propto \tau_r^{-\xi}$.

FIG. 22. Average number of dual links $\langle p \rangle$ flipped by worms of length $\tau_r$ for the myopic algorithm in a $600 \times 600$ triangular lattice for three values of $T$ at which the system is in the power-law ordered critical phase. Lines denote fits to a power-law form $\propto \tau_r^{-\xi}$.

FIG. 23. Average number of dual links $\langle p \rangle$ flipped by worms of length $\tau_r$ for the myopic algorithm in a $288 \times 288$ kagome lattice for three values of $T$ at which the system is in the power-law ordered critical phase. Lines denote fits to a power-law form $\propto \tau_r^{-\xi}$.

FIG. 24. $\zeta$ extracted from the $\tau_r$ dependence of the average number of flipped dual links $\langle p \rangle$, plotted as a function $\eta_m$ for all cases studied here. The dotted-line is a guide to the eye, suggesting that $\zeta$ depends in a universal way on $\eta_m$, and the dependence may be a power law.
form this update. While the DEP worm algorithm keeps track of the local dimer environment near the head of the worm at every step of the worm construction, the myopic worm algorithm does not keep track of the local dimer environment near the head of the worm at alternate steps (it is thus “myopic” or short-sighted at alternate steps). The details of the worm construction protocol for both algorithms, and the proofs that these protocols obey detailed balance, have already been discussed extensively in Ref. 11.

Since detailed balance is explicitly satisfied, the dimer configuration obtained after the worm is constructed can be accepted modulo one subtlety: We work with periodic boundary conditions along \( \hat{x} \) and \( \hat{y} \) of the triangular and kagome lattices. This translates to constraints on the parity of the global winding number of the corresponding dimer model (For details on preserving detailed balance and winding number constraints see Ref. 11). Thus, after the worm construction, only updated dimer configurations which satisfy this constraint can be translated back to the spin configuration. Therefore, one has to occasionally reject a worm which winds around the lattice, if the result leads to a dimer configuration in an illegal winding number sector.

The worm algorithm updates dimers along a one dimensional closed loop on the dual lattice. When translated back to the spin configuration, this defines a two dimensional cluster of spins contained by the closed loop. These spins are all flipped. Also note that each valid dimer configuration corresponds to two spin configurations related by a global spin flip. One of the configurations is randomly chosen when translating back to the spin configuration.

![Figure 25](image1.png)

**FIG. 25.** Average number of spins \( \langle a \rangle \) flipped by worms that flip \( p \) dual links for the DEP algorithm in a \( 600 \times 600 \) triangular lattice for three values of \( J_2/T \) at which the system is in the power-law ordered critical phase in the zero temperature limit \( T \to 0 \). Lines denote fits to a power-law form \( \propto p^D \).

![Figure 26](image2.png)

**FIG. 26.** Average number of spins \( \langle a \rangle \) flipped by worms that flip \( p \) dual links for the myopic algorithm in a \( 600 \times 600 \) triangular lattice for three values of \( J_2/T \) at which the system is in the power-law ordered critical phase in the zero temperature limit \( T \to 0 \). Lines denote fits to a power-law form \( \propto p^D \).

![Figure 27](image3.png)

**FIG. 27.** Average number of spins \( \langle a \rangle \) flipped by worms that flip \( p \) dual links for the DEP algorithm in a \( 600 \times 600 \) triangular lattice for three values of \( T \) at which the system is in the power-law ordered critical phase. Lines denote fits to a power-law form \( \propto p^D \).

### V. OBSERVABLES

**Defect-antidefect correlator:** During the worm construction, a defect-antidefect pair is created on the dual lattice, and the antidefect is then moved (keeping the defect fixed) through the dual lattice (in a manner satisfying detailed balance in the enlarged configuration space) until it returns to the location of the defect and annihilates it, producing a legal dimer configuration that can be mapped back to a spin configuration. As noted earlier, the defect-antidefect correlator \( C_{\text{defect}}(\vec{r}) \) is proportional to the histogram of the position \( \vec{r} \) of the head relative to the tail of the worm, which can be accumulated during the worm construction. We choose...
a normalization convention where this histogram, when summed over \( \vec{r} \), gives the mean length of worms constructed by the algorithm (in other words, we measure the number of times the head to tail separation is \( \vec{r} \) per worm). In the power-law three-sublattice ordered phase we expect \( C_{\text{defect}}(\vec{r}) \sim 1/\vec{r}^{\eta_m} \), with \( \eta_m = 1/4\eta_s \). During the worm construction, the worm can wind around the torus defined by the periodic boundary conditions used in our study. Even if the worm winds before annihilating, we always record the shortest geometric separation between the head and tail of the worm (modulo the lattice size \( L \) in each direction).

**Worm length or return time distribution:** The number of dual lattice sites (with multiplicities, if a site is visited more than once) visited by the head of the worm during the worm construction defines the length of the worm, which corresponds in our random walk analogy to the first-return time of the walk. A histogram of this gives us \( P(\tau_r) \), the probability distribution of first-return times.

**Average worm length:** As noted earlier, once our defect-antidefect correlator is normalized to give the number of times the head to tail separation is \( \vec{r} \) per worm, then \( \langle \tau_r \rangle = \langle \nu \rangle \equiv \sum_{\vec{r}} C_{\text{defect}}(\vec{r}) \). In our numerics we measure \( \langle \nu \rangle \), which is expected to scale as \( \sim L^{2-\eta_m} \) in the power-law ordered phase.

**Average number of flipped links per worm:** When a worm retraces its path, it flips the dimers along the retraced path again, in effect not flipping them in the first place. Thus, counting the number of flipped links is equivalent to measuring the perimeter of the closed path defined by the worm. This closed path is made up of a number of disconnected components in general. This is because every intersection of the worm with its own trace splits off a closed loop of flipped links. We measure the average number of flipped links per worm \( \langle p \rangle \) (summed over all closed components of that path) as a function of the return time \( \tau_r \) of the worm.

**Average number of flipped spins per worm:** After mapping back to the original spin configuration, we can measure the average number of spins on the direct lattice flipped by one worm update. This is equivalent to measuring the area enclosed by the closed path defined by the trace of the worm. Since the worm is on a torus, this area can be the either be the inner or outer area with respect to the closed path. We choose to always work with the smaller of these two areas, and count the corresponding number of flipped spins. In our measurements, we keep track of the average number of flipped spins \( \langle a \rangle \) defined in this way, and study its dependence on the number of flipped links \( p \) introduced earlier. In this measurement too, both \( \langle a \rangle \) and \( p \) are summed over all closed-path components of a worm.

**VI. RESULTS**

All our measurements are performed on lattice sizes of upto \( 600 \times 600 \) lattice sites for the triangular lattice antiferromagnet and upto \( 288 \times 288 \) unit cells (with three sites per unit cell) for the kagome lattice antiferromagnet. For studying the statistics of worms, we perform one worm update per Monte Carlo step (MCS) and measure all histograms and averages during the worm construction. If after the worm construction the dimer configuration is not physical (as explained in Section IV), we discard the measurements made during the construction of that particular worm. All our data is averaged over \( 1 \times 10^8 \) MCS.

We have performed such measurements in all five cases mentioned in Section IV: In the \( T \rightarrow 0 \) limit on the triangular lattice, we study both the DEP and myopic algorithms at three values of \( J_2/T \) (0.00, 0.05 and 0.10),
all of which are in the power-law ordered phase. To access the $T > 0$ power-law ordered phase, we set $J_1 = 1$ and $J_2 = -1$. On the triangular lattice, we study both the algorithms in the power-law ordered phase at $T = 4.3, 4.5$ and 4.6, and on the kagome lattice we study the myopic algorithm in the power-law ordered phase at $T = 1.24, 1.30$ and 1.36 (all temperatures are measured in units of $J_1 = 1$).

The defect-antidefect correlator $C_{\text{defect}}(c_x, y_x)$ is measured at separation $r = c_x, y_x$ (with $s = 2$ for the zero temperature measurements and $s = 24$ for the nonzero temperature measurements) on periodic $L \times L$ lattices as a function of lattice size $L$ for $L = 288, 360, 420$ and 600 on the triangular lattice and $L = 96, 144, 216$ and 288 on the kagome lattice ($c_x$ is one of the Bravais lattice vectors). Figs. 1, 2 show this correlator in the $T \rightarrow 0$ limit for the DEP and myopic worm algorithms respectively on the triangular lattice. Figs. 3, 4 show this correlator in the $T > 0$ regime for the DEP and myopic worm algorithms respectively on the triangular lattice. Fig. 5 shows the correlator in the $T > 0$ regime for the myopic worm algorithm on the kagome lattice. In all the above cases we extract $\eta_m$ by fitting a power law to the $L$ dependence of the correlator. Fig. 6 plots the best-fit $\eta_m$ obtained in this way versus the spin correlation exponent $\eta_s$ (this exponent is measured by fitting the equilibrium spin correlator at the three sublattice wavevector to a power-law form) for each of these five cases. As can be seen, the data agrees very well with the theoretical prediction of $\eta_m = 1/4\eta_s$ for the $T \rightarrow 0$ case. We note that for $T > 0$ cases, the agreement is less impressive but still reasonable.

We measured the probability distribution of return times $P(\tau_r)$ as a function of $\tau_r$ for $L = 600$ on the triangular lattice and $L = 288$ on the kagome lattice. Figs. 7, 8 show the return time distribution in the $T \rightarrow 0$ limit for the DEP and myopic worm algorithms respectively on the triangular lattice. Figs. 9, 10 show the distribution in the $T > 0$ power-law ordered phase for the DEP and myopic worm algorithms respectively on the triangular lattice. Fig. 11 shows the distribution in the $T > 0$ power-law ordered phase for the myopic worm algorithm on the kagome lattice. In all these cases, we extract $\theta$ by fitting to a power-law form with exponent $1 + \theta$.

If the dynamical exponent $z$ were to take on the value $z = 2$, then our scaling argument would predict that $\theta = \eta_m/2 \equiv 1/8\eta_s$. We highlight the deviations of $\theta$ from this value by plotting the measured value of $\theta$ as a function of $\eta_m^{\text{predicted}}/2 \equiv 1/(8\eta_s)$ for the $T \rightarrow 0$ and the $T > 0$ cases in Fig. 12 and Fig. 13 respectively. These deviations are evidence that $\eta_s \neq 2$. Using the scaling relation between $\theta$ and $z$, our results for $\theta$ can be used to obtain the corresponding values of $z$. Additionally, the value of $z$ can also be determined independently by a direct measurement of the scale $\tau_{\text{cutoff}}(L)$ at which the power-law form of $P(\tau_r)$ is cut off by finite-size effects.

This is shown in Fig. 18. From the figure, we see that the value of $z$ appears, within errors, to be determined solely (i.e. independent of microscopic details like the precise form of the Hamiltonian, and the worm construction rules) by the power-law exponent $\eta_s$ that characterises the long-distance behavior of the equilibrium spin correlations. Additionally, $z$ obtained directly from the finite-size cutoff in $P(\tau_r)$ matches within errors with the value of $z$ extracted from the measured value of $\theta$. As is clear from this figure, $z$ decreases monotonically with increasing $\eta_s$ and appears to approach the value of $z = 2$ in the limit of large $\eta_s$. However, since the largest value of $\eta_s$ accessed in our work is the free-dimer value of $\eta_s = 2$, $z > 2$ in the entire regime studied here.

Thus, the worms constructed by these algorithms can be said to constitute a particular realization of fractional Brownian motion, with a nontrivial subdiffusive dynamical exponent $z > 2$ that is universally determined by the power-law spin correlations of the equilibrium problem. A particular feature of this realization of fractional brownian motion is the fact that there is a long-time steady state characterized by the Gibbs distribution for a particle in a central attractive logarithmic potential of strength $\eta_m$.

By way of comparison with a more well-known example of worm constructions, we also studied the return time distribution of the worm algorithm for the fully-packed dimer model,\textsuperscript{16} on the three-dimensional cubic lattice. In this case, the worm creates a monomer-antimonomer pair, and propagates the antimonomer through the lattice until it recombines with the monomer at the starting site. The monomer-antimonomer correlator on the cubic lattice is controlled by the emergent Coulomb interaction between the monomer and antimonomer. Since this is a power-law potential rather than a logarithmic potential, the effective dimension $d'$ in this case is equal to the spatial dimension: $d' = d = 3$. If the dynamical exponent were to take on the usual Markovian random walk value of $z = 2$, the return time statistics would be expected to be identical to that of the usual random walk in three dimensions.\textsuperscript{28}

Fig. 17 displays a power-law fit for the large $\tau_r$ behavior of $P(\tau_r)$ in this case. The best-fit value $\theta = 0.48 \pm 0.03$ agrees within errors with the exact value of 1/2 predicted by Eq. 7 for $d' = 3$ and $z = 2$. This value of $\theta$ is also consistent with the results for the worm length distributions in Ref.\textsuperscript{26} for a worm algorithm on the pyrochlore lattice. Thus, in this case, the worm length distributions suggest that correlations between the spatial increments of the random walk renormalize to zero in the long-time limit, yielding a conventional value of $z = 2$ for the dynamical exponent.

We also measured the average number of flipped dual links per worm, $\langle p \rangle$, as a function of $\tau_r$ for $L = 600$ on the triangular lattice and $L = 288$ on the kagome lattice. Figs. 19, 20 show this functional dependence in the $T \rightarrow 0$ power-law ordered phase for the DEP and myopic worm algorithms respectively on the triangular lattice. Figs. 21, 22 show this function in the $T > 0$ power-law ordered phase for the DEP and myopic worm.
algorithms respectively on the triangular lattice. Figs. 23 shows this functional dependence in the $T > 0$ power-law ordered phase for the myopic worm algorithm on the kagome lattice. In all the above cases we find that $\langle p \rangle$ is a power of $\tau_T$. The best-fit value of $\zeta$ is shown in Fig. 24 as a function of $\eta_m^{\text{predicted}} = 1/8q_a$ for each of these five cases. Though we do not have a theoretical prediction for this dependence, we note that all the measured data points seem to fall on a single curve, as would be expected if the geometric properties of the worms were universally determined by the long-distance behaviour of equilibrium correlations.

FIG. 30. The lattice size $L$ dependence of the average number of dual lattice sites visited per worm $\langle v \rangle$ using the DEP worm algorithm on the triangular lattice for three values of $J_2/T$ at which the system is in the power-law ordered critical phase in the zero temperature limit $T \to 0$. Since $\langle v \rangle/L^2 \sim 1/L^{\eta_m}$, the power-law fits give us an alternate measurement of $\eta_m$.

FIG. 31. The lattice size $L$ dependence of the average number of dual lattice sites visited per worm $\langle v \rangle$ using the myopic worm algorithm on the triangular lattice for three values of $J_2/T$ at which the system is in the power-law ordered critical phase in the zero temperature limit $T \to 0$. Since $\langle v \rangle/L^2 \sim 1/L^{\eta_m}$, the power-law fits give us an alternate measurement of $\eta_m$.

FIG. 32. The lattice size $L$ dependence of the average number of dual lattice sites visited per worm $\langle v \rangle$ using the DEP worm algorithm on the triangular lattice for three values of $T$ at which the system is in the power-law ordered critical phase. Since $\langle v \rangle/L^2 \sim 1/L^{\eta_m}$, the power-law fits give us an alternate measurement of $\eta_m$.

FIG. 33. The lattice size $L$ dependence of the average number of dual lattice sites visited per worm $\langle v \rangle$ using the myopic worm algorithm on the triangular lattice for three values of $T$ at which the system is in the power-law ordered critical phase. Since $\langle v \rangle/L^2 \sim 1/L^{\eta_m}$, the power-law fits give us an alternate measurement of $\eta_m$.

We also measured the average number of flipped spins per worm on the direct lattice $\langle a \rangle$ as a function of $p$, the number of flipped dual links, for $L = 600$ on the triangular lattice and $L = 288$ on the kagome lattice. Figs. 25, 26 show the distribution in the $T \to 0$ limit for the DEP and myopic worm algorithms respectively on the triangular lattice. Figs. 27, 28 show the distribution in the $T > 0$ case for the DEP and myopic worm algorithms respectively on the triangular lattice. Figs. 29 shows the distribution in the $T > 0$ case for the myopic worm algorithm on the kagome lattice. In all the above cases we extract the exponent $D$ by fitting this functional dependence to a power law form. For worms that do not
intersect themselves before closing, this would amount to plotting the enclosed area as a function of perimeter of the worm, and the exponent \( D \) could then be interpreted as the fractal dimension of the cluster constructed by the worm. However, when we perform the fits, we find that the measured fractal dimension \( D \approx 1 \) in all five cases studied. To understand this better, we have looked at the actual traces of the worms in all cases, and found that the worms defined by these algorithms intersect themselves very often. The spin cluster obtained from such a worm consists of many small components and the area of the individual components does not scale with the measured total perimeter. For such worms, it is quite natural that the total perimeter and the area scale in the same way, \( i.e. D \approx 1 \).

We can also extract \( \eta_m \) from the lattice size \( L \) dependence of the average number \( \langle v \rangle \) of dual sites visited per worm (as discussed in Section V) using the relation \( \langle v \rangle / L^2 \sim 1/L^{\eta_m} \). Figs. 30, 31 show the power-law fits in the \( T > 0 \) limit for the DEP and myopic worm algorithms respectively on the triangular lattice. Figs. 32, 33 show the power-law fits in the \( T > 0 \) regime for the DEP and myopic worm algorithms respectively on the triangular lattice. The extracted value of \( \eta_m \) matches within error-bars with the value of \( \eta_m \) extracted from the defect-antidefect correlator as seen in Figs. 1, 2 and Fig. 3, 4 for \( T \to 0 \) and \( T > 0 \) respectively. In the noninteracting dimer model limit of the dual dimer model \( (T \to 0 \text{ and } J_2/T = 0) \), it is well known that \( \eta_m = 1/2 \) and we find that our measurement of \( \langle v \rangle / L^2 \sim 1/L^{0.53(2)} \) is in very good agreement. However we note that a previous study of a worm algorithm for the square ice model in the free dimer limit\(^{30}\) concluded that \( \langle v \rangle \sim L^{1.665(2)} \), which is at odds with what one would expect when \( \eta_m = 1/2 \) (the values of \( \eta_m \) and \( \eta_d \) are the same for the noninteracting dimer model on the honeycomb and the square lattice).

VII. OUTLOOK

Our results imply that the worms studied here define a discrete-time realization of a fractional brownian motion which has a conventional steady-state given by the equilibrium Gibbs distribution of a particle in a logarithmic central attractive potential. Stochastic equations related to such non-Markovian processes with correlated steps have been studied for some time now. Ageing and steady-state behaviour of solutions to such equations, particularly in the presence of a confining potential, have also been of interest.\(^{41,42}\) It would therefore be interesting to ask if a continuous-time stochastic equation of this type emerges as the correct description of some scaling limit of the worm construction process studied here. Also, a similar picture for the worm-length distribution is possible in other applications of worm algorithms to two-dimensional critical points/phases, and it would be interesting to study the values of dynamical exponent \( z \) and persistence exponent \( \theta \) associated with these worm constructions. It would also be interesting to develop a theory for predicting the values of \( z \) and \( \theta \), even for the simplest case of the standard worm algorithm for the noninteracting fully packed dimer model on the honeycomb lattice, which admits an extremely well-studied continuum description in terms of a height field theory.

VIII. ACKNOWLEDGMENTS

We acknowledge useful discussions with Fabien Alet and Pranay Patil on their own unpublished observations about worm statistics of similar worm algorithms. Our computational work relied on the computational facilities of the Dept. of Theoretical Physics of the Tata Institute of Fundamental Research. The work of Geet Rakala was supported by a graduate fellowship at the Tata Institute of Fundamental Research, and formed a part of his Ph.D thesis at the Tata Institute of Fundamental Research.

---

1. H. G. Evertz, “The loop algorithm”, Adv. Phys. 52, 1 (2003).
2. N. Prokofev and B. Svistunov, “Worm algorithms for classical statistical models”, Phys. Rev. Lett. 87, 160601 (2001).
3. F. Alet and E. Sorenson, “Cluster Monte Carlo algorithm for the quantum rotor model”, Phys. Rev. E 67, 015701(R) (2003).
4. F. Alet and E. Sorenson, “Directed geometrical worm algorithm applied to the quantum rotor model”, Phys. Rev. E 68, 026702 (2003).
5. M. Wallin et al., “Superconductor-insulator transition in two-dimensional dirty boson systems”, Phys. Rev. B 49 12 115 (1994).
6. M. Suzuki, “Relationship between d-dimensional quantal spin systems and (d+ 1)-dimensional Ising systems: Equivalence, critical exponents and systematic approximants of the partition function and spin correlations. Prog. Theor. Phys. 56, 1454 (1976).
7. J. E. Hirsch, R. L. Sugar, D. J. Scalapino, and R. Blankenbecler “Monte Carlo simulations of one-dimensional fermion systems Phys. Rev. B 26, 5033 (1982).
8. A. W. Sandvik, “Stochastic series expansion method with operator-loop update,Phys. Rev. B 59, R14157 (1999).
9. A. Dorneich and M. Troyer, “Accessing the dynamics of large many-particle systems using the stochastic series expansion”, Phys. Rev. E 64, 066701 (2001).
10. P. Hitchcock, E. Sorenson, and F. Alet, “Dual geometric worm algorithm for two-dimensional discrete classical lattice models”, Phys. Rev. E 70, 016702 (2004).
11. G. Rakala, K. Danile, “Cluster algorithms for frustrated two dimensional Ising antiferromagnets via dual worm con...
strucions”, arXiv:1612.00851 (2016).
12 R. H. Swendsen and J-S. Wang, “Nonuniversal critical dynamics in Monte Carlo simulation”, Phys. Rev. Lett. 58, 86 (1987).
13 U. Wolff, “Lattice field theory as a percolation process”, Phys. Rev. Lett. 60, 1461 (1988).
14 P. W. Leung and C. L. Henley, “Percolation properties of the Wolff clusters in planar triangular spin models”, Phys. Rev. B 43, 752 (1991).
15 F. Alet, Y. Ikhlef, J.L. Jacobsen, M. Gregoire and V. Pasquier, “Classical dimers with aligning interactions on the square lattice”, Phys. Rev. E 74, 041124 (2006).
16 A. Sandvik and R. Moessner, “Correlations and confinement in nonplanar two-dimensional dimer models”, Phys. Rev. B 73, 144504 (2006).
17 “Critical loop gases and the worm algorithm”, Nuc. Phys. B. 829, 573 (2010).
18 “Fractal structure of spin clusters and domain walls in the two-dimensional Ising model”, Phys. Rev. E. 71, 036703 (2005).
19 A. L. Stella and C. Vanderzande, “Scaling and fractal dimension of Ising clusters at the $d = 2$ critical point”, Phys. Rev. Lett. 62, 1067 (1989).
20 H. Saleur and B. Duplantier, “Exact determination of the percolation hull exponent in two dimensions”, Phys. Rev. Lett. 58, 2325 (1987).
21 H. E. Stanley, “Cluster shapes at the percolation threshold: and effective cluster dimensionality and its connection with critical-point exponents”, J. Phys. A 10, L211 (1977).
22 C. Vanderzande and A. L. Stella, “Bulk, surface and hull fractal dimension of critical Ising clusters in $d = 2$”, J. Phys. A 22, L445 (1989).
23 B. Duplantier, “Conformally invariant fractals and potential theory”, Phys. Rev. Lett. 84, 1363 (2000).
24 D. Kandel, R. Ben-Av, and E. Domany, “Cluster dynamics for fully frustrated systems”, Phys. Rev. Lett. 65, 941 (1990).
25 G. Franzese, “Cluster analysis for percolation on a two-dimensional fully frustrated system”, J. Phys. A 29, 7367 (1996).
26 L. D. C. Jaubert, M. Haque, and R. Moessner, “Analysis of a fully packed loop model arising in a magnetic coulomb phase”, Phys. Rev. Lett. 107, 177202 (2011).
27 J. V. Jose, L. P. Kadanoff, S. Kirkpatrick and D. R. Nelson, “Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model”, Phys. Rev. B 16, 1217 (1977).
28 A. J. Bray, “Random walks in logarithmic and power-law potentials, nonuniversal persistence, and vortex dynamics in the two-dimensional XY model”, Phys. Rev. E 62, 103 (2000).
29 S. Redner, “A guide to first-passage processes” Cambridge Univ. press (2001).
30 G. H. Wannier, “Antiferromagnetism. The triangular Ising net”, Phys. Rev. 79, 357 (1950).
31 K. Kano and S. Naya, “Antiferromagnetism. The kagome Ising net”, Prog. Theor. Phys. 10, 158 (1953).
32 J. Stephenson, “IsingModel spin correlations on the triangular lattice”, J. Math. Phys. 5, 1009 (1964).
33 H. W. J. Blote and H. J. Hilhorst, “Roughening transitions and the zero-temperature triangular Ising antiferromagnet”, J. Phys. A: Math. Gen. 15 L631 (1982).
34 B. Nienhuis, H. J. Hilhorst, and H. W. J. Blote, “Triangular SOS models and cubic-crystal shapes”, J. Phys. A: Math. Gen. 17, 3559 (1984).
35 S. Papanikolaou, E. Luijten and E. Fradkin, “Quantum criticality, lines of fixed points, and phase separation in doped two-dimensional quantum dimer models”, Phys. Rev. B. 76, 134514 (2007).
36 D. P. Landau, “Critical and multiscalar behavior in a triangular-lattice-gas Ising model: Repulsive nearest-neighbor and attractive next-nearest-neighbor coupling”, Phys. Rev. B 27, 5604 (1983).
37 M. Wolf and K. D. Schotte, “Ising model with competing next-nearest-neighbor interactions on the kagome lattice”, J. Phys. A: Math. Gen. 21, 2195 (1988).
38 K. Damle, “Melting of Three-Sublattice Order in Easy-Axis Antiferromagnets on Triangular and kagome Lattices”, Phys. Rev. Lett. 115, 127204 (2015).
39 G. Chern and O. Tchernyshyov, “Magnetic charge and ordering in kagome spin ice”, Phil. Trans. R. Soc. A 370, 5718 (2012).
40 G. T. Barkema and M. E. J. Newman, “Monte Carlo simulation of ice models”, Phys. Rev. E. 57, 1155 (1998).
41 J. Kursawe, J. Schulz, and R. Metzler, “Transient aging in fractional Brownian and Langevin-equation motion”, Phys. Rev. E 88, 062124 (2013).
42 L. Li, J-G. Liu, and J. Lu, “Fractional Stochastic Differential Equations Satisfying Fluctuation-Dissipation Theorem”, J. Stat. Phys. 169, 316 (2017).