A RANDOM COCYCLE WITH NON HÖLDER LYAPUNOV EXPONENT

PEDRO DUARTE
Departamento de Matemática and CMAFCIO
Faculdade de Ciências, Universidade de Lisboa
Campo Grande, Edifício C6, Piso 2, 1749-016 Lisboa, Portugal

SILVIUS KLEIN
Departamento de Matemática
Pontifícia Universidade Católica do Rio de Janeiro (PUC-Rio)
Rua Marquês de São Vicente 225, Rio de Janeiro, RJ, 22430-060, Brazil

MANUEL SANTOS
Departamento de Matemática
Instituto Superior Técnico, Universidade de Lisboa
Avenida Rovisco Pais, 1049-001 Lisboa, Portugal

(Communicated by Sylvain Crovisier)

Abstract. We provide an example of a Schrödinger cocycle over a mixing Markov shift for which the integrated density of states has a very weak modulus of continuity, close to the log-Hölder lower bound established by W. Craig and B. Simon in [6]. This model is based upon a classical example due to Y. Kifer [15] of a random Bernoulli cocycle with zero Lyapunov exponents which is not strongly irreducible. It follows that the Lyapunov exponent of a Bernoulli cocycle near this Kifer example cannot be Hölder or weak-Hölder continuous, thus providing a limitation on the modulus of continuity of the Lyapunov exponent of random cocycles.

1. Introduction. This paper is concerned with providing limitations on the modulus of continuity of the (maximal) Lyapunov exponent (LE) of random linear cocycles. By a random linear cocycle we understand the skew-product dynamical system defined by a Bernoulli or a Markov shift on the base and a locally constant linear fiber map. We fix the base dynamics and vary the fiber map relative to the uniform norm, thus continuity is with respect to the fiber map.

Continuity of the LE in a generic setting (i.e. assuming irreducibility and contraction, which in particular imply simplicity of the LE) was first established by H.
Furstenberg and Y. Kifer [12]. Recently, the genericity assumption was removed by C. Bocker-Neto and M. Viana [4] (in the two-dimensional Bernoulli setting) and by E. Malheiro and M. Viana [18] (in a certain two-dimensional Markov setting). A higher dimensional version of the result in [4] was announced by A. Avila, A. Eskin and M. Viana [24, Note 10.7]. All of these results are not quantitative, i.e. they do not provide a modulus of continuity for the LE.

The first quantitative result, namely Hölder continuity of the LE, was obtained by E. Le Page [17] in the generic, Bernoulli setting. This result refers to a one-parameter family of random linear cocycles; as such, it has been widely used in the theory of discrete, random, one-dimensional or strip Schrödinger operators (which give rise to such one-parameter families of cocycles). Still in the generic setting, extensions of this result were obtained by P. Duarte and S. Klein [8] and by A. Baraviera and P. Duarte [3]. See also P. Duarte and S. Klein [9] for a simpler approach in the two-dimensional setting.

More recently, the first two authors of this paper considered the problem of obtaining a modulus of continuity of the LE for two-dimensional random Bernoulli linear cocycles in the absence of any genericity assumption (see [10]), but assuming the simplicity of the maximal LE. Under this assumption, the results in [10] establish local weak-Hölder continuity of the maximal LE in the most “degenerate” situation (i.e. in the vicinity of a diagonalizable cocycle) and local Hölder continuity elsewhere. There is work in progress establishing similar results for Markov cocycles.

A natural question arising from these developments is determining how weak the modulus of continuity of the LE can be. An example of B. Halperin, made rigorous by B. Simon and M. Taylor in [20], shows that at this level of generality, the LE cannot be more regular than Hölder, and in fact the Hölder exponent may be arbitrarily close to zero. When the LE is simple (that is, positive, in the SL$_2(\mathbb{R})$ setting), by [10] it is at least weak-Hölder continuous. Can it be much weaker than this when the LE is not simple?

Y. Kifer [15] considered the random Bernoulli cocycle $(C, D; p, 1 - p)$ generated by the matrices

\[
C := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad D := \begin{bmatrix} e & 0 \\ 0 & e^{-1} \end{bmatrix}
\]

(1.1)

with probabilities $(p, 1 - p)$. A simple calculation shows that if $p > 0$ then the corresponding Lyapunov exponent is 0, while when $p = 0$, the Lyapunov exponent is 1, thus implying the discontinuity of the Lyapunov exponent as a function of the probability vector $(p, 1 - p)$ at the boundary of the simplex. In this work we provide an upper-bound for the regularity of the LE as a function of the matrices at $(C, D; \frac{1}{2}, \frac{1}{2})$. A similar upper-bound should hold for any probability $0 < p < 1$.

So far the only available method for proving limitations on the regularity of the LE for random cocycles is that of Halperin. This method in fact relies on the Thouless formula, which relates the LE to another quantity called the integrated density of states (IDS). The Thouless formula is only available for Schrödinger (and Jacobi) cocycles, which is not the case with Kifer’s example. Our idea was then to embed Kifer’s example into a family of Schrödinger cocycles (thus making the Thouless formula applicable) but over a finite type mixing Markov shift.

\[\text{1Independent and with different methods, the same problem has also been studied by E. Y. Tall and M. Viana.}\]
The example in this paper shows a huge breakdown on the regularity of the IDS (Theorem 1) which implies a similar breakdown for the LE in Kifer’s example (Theorem 2). In this example the two assumptions of the classical result of Le Page (and of its extensions) fail, namely the cocycle is not strongly irreducible and it has zero Lyapunov exponent.

Furthermore, Proposition 11 shows that given a cocycle with zero LE, if it is strongly irreducible, then the LE must be pointwise Lipschitz at that cocycle. Therefore, in some sense it is the simultaneous failing of the two assumptions that produces the break in regularity.

2. Basic concepts.

Linear cocycles. Consider a probability space \((X, \mu)\) and an ergodic measure preserving transformation \(T: X \to X\) on \((X, \mu)\). An \(\text{SL}(2, \mathbb{R})\)-linear cocycle over \(T\) is any map \(F_A : X \times \mathbb{R}^2 \to X \times \mathbb{R}^2\) defined by a measurable function \(A: X \to \text{SL}(2, \mathbb{R})\) through the expression

\[
F_A(x, v) := (Tx, A(x)v).
\]

When the base map \(T\) is fixed we identify \(F_A\) with \(A\).

The forward iterates \(F^n_A\) are given by

\[
F^n_A(x, v) = (T^n x, A^n(x)v),
\]

where

\[
A^n(x) := A(T^{n-1}x) \ldots A(Tx)A(x) \quad (n \in \mathbb{N}).
\]

The Lyapunov exponent (LE) of \(F_A\) is defined as the \(\mu\)-almost sure limit

\[
L(A) := \lim_{n \to +\infty} \frac{1}{n} \log\|A^n(x)\|,
\]

whose existence follows by Furstenberg-Kesten’s theorem [11].

Schrödinger operators and cocycles. Let \(T: X \to X\) be an ergodic transformation over a probability space \((X, \mu)\). Denote by \(l^2(\mathbb{Z})\) the usual Hilbert space of square summable sequences of real numbers \((\psi_n)_{n \in \mathbb{Z}}\). Note that \(\lim_{n \to \pm \infty} \psi_n = 0\) for all \(\psi \in l^2(\mathbb{Z})\).

Given some bounded measurable function \(\nu: X \to \mathbb{R}\), at every site \(n\) on the integer lattice \(\mathbb{Z}\) we define the potential

\[
v_n(x) := \nu(T^nx).
\]

The \textit{discrete ergodic Schrödinger operator} with potential \(n \mapsto v_n(x)\) is the operator \(H_x\) defined on \(l^2(\mathbb{Z})\) \(\ni \psi = \{\psi_n\}_{n \in \mathbb{Z}}\) as follows:

\[
[H_x \psi]_n := - (\psi_{n+1} + \psi_{n-1}) + v_n(x) \psi_n.
\] (2.1)

Due to the ergodicity of the system, the spectral properties of the family of operators \(\{H_x : x \in X\}\) are independent of \(x\) \(\mu\)-almost surely.

Consider the Schrödinger eigenvalue equation

\[
H_x \psi = E \psi,
\] (2.2)

for some eigenvalue \(E \in \mathbb{R}\) and eigenvector \(\psi = \{\psi_n\}_{n \in \mathbb{Z}}\).

The associated \textit{Schrödinger cocycle} is the cocycle \(A_E\) defined by

\[
A_E(x) := \begin{bmatrix}
u(x) - E & -1 \\ 1 & 0\end{bmatrix} \in \text{SL}_2(\mathbb{R}).
\]
Note that the Schrödinger equation (2.2) is a second order finite difference equation. An easy calculation shows that its formal solutions are given by

\[
\begin{bmatrix}
\psi_n \\
\psi_{n-1}
\end{bmatrix} = A_E^{(n)}(x) \begin{bmatrix}
\psi_0 \\
\psi_{-1}
\end{bmatrix}.
\] (2.3)

Denote by \( P_n : l^2(\mathbb{Z}) \to \mathbb{C}^{n+1} \) the coordinate projection to \( \{0, 1, 2, \ldots, n\} \subset \mathbb{Z} \), by \( P_n^* \) its adjoint and let

\[
H_x^{(n)} := P_n H(x) P_n^*.
\] (2.4)

This finite rank operator is called the \textit{n-truncation} of \( H_x \). By ergodicity, the following limit exists for \( \mu \)-a.e. \( x \in X \):

\[
N(E) := \lim_{n \to \infty} \frac{1}{n+1} \#((-\infty, E] \cap \text{Spectrum of } H_x^{(n)}).
\]

The function \( E \mapsto N(E) \) is called the \textit{integrated density of states} (IDS) of the family of ergodic operators \( \{ H_x : x \in X \} \) (see [7]).

The LE and the IDS are related via the Thouless formula:

\[
L(E) = \int_{\mathbb{R}} \log |E - E'| dN(E').
\]

**Random cocycles.** Let \( \Sigma = \{1, \ldots, s\} \) be a finite alphabet, let \( X = \Sigma^\mathbb{Z} \) be the compact product space of bi-infinite sequences of symbols in the set \( \Sigma \) and let \( T : X \to X \) be the \textit{full shift map}, \( T\{x_n\}_{n \in \mathbb{Z}} := \{x_{n+1}\}_{n \in \mathbb{Z}} \). Given a probability vector \( q = (q_1, \ldots, q_s) \) on \( \Sigma \), consider the product probability measure \( \mathbb{P}_q = q^\mathbb{Z} \) on \( X \). The map \( T \) determines an ergodic transformation on \( (X, \mathbb{P}_q) \) called the \textit{two-sided Bernoulli shift}.

Next we introduce the broader class of Markov shifts. Recall that a \textit{stochastic matrix} is any square matrix \( P = (p_{ij}) \in \text{Mat}_s(\mathbb{R}) \) such that:

1. \( p_{ij} \geq 0 \) for all \( i, j = 1, \ldots, s \),
2. \( \sum_{j=1}^s p_{ij} = 1 \) for all \( i = 1, \ldots, s \).

A \textit{P-stationary vector} is any probability vector \( q \in \mathbb{R}^s \) such that \( q = Pq \), that is, \( q_i = \sum_{j=1}^s p_{ij} q_j \) for all \( i = 1, \ldots, s \). Each power \( P^n \) is itself a stochastic matrix. Given a pair \( (P,q) \) where \( P \) is a stochastic matrix and \( q \) is a \( P \)-stationary probability vector there exists a unique probability measure \( \mathbb{P} = \mathbb{P}_{(P,q)} \) on \( X = \Sigma^\mathbb{Z} \) such that the stochastic process \( \{e_n : X \to \Sigma\}_{n \in \mathbb{Z}}, e_n(x) := x_n, \) has constant distribution \( q \) and transition probability matrix \( P \), i.e., for all \( i, j = 1, \ldots, s \),

1. \( \mathbb{P}[e_n = i] = q_i \),
2. \( \mathbb{P}[e_n = i | e_{n-1} = j] = p_{ij} \).

The support of \( \mathbb{P}_{(P,q)} \) is the space of admissible sequences

\[
B(P) := \{x \in X : p_{x_n x_{n-1}} > 0 \forall n \in \mathbb{Z}\}
\]

commonly referred to as the \textit{sub-shift of finite type} defined by \( P \). The stochastic matrix \( P \) is called \textit{primitive} if \( P^n > 0 \) for some \( n \geq 1 \) (that is all the entries of \( P^n \) are positive). If \( P \) is primitive then the two-sided shift \( T : X \to X \) is a mixing measure preserving transformation on \( (X, \mathbb{P}_{(P,q)}) \), called a \textit{mixing Markov shift}.

A (locally constant) \textit{random cocycle} is any cocycle \( A : X \to \text{SL}(2, \mathbb{R}) \) over a Bernoulli or Markov shift \( T \) such that \( A(\{x_n\}) \) depends only on the first coordinate \( x_0 \in \Sigma \). Once the base dynamics given by the full shift is fixed, a random cocycle is completely determined by a list of \( s \) matrices, \( A_1, \ldots, A_s \in \text{SL}(2, \mathbb{R}) \), such that \( A(\{x_n\}) = A_{x_0} \).
Modulus of continuity. Any continuous and strictly-increasing function $\omega: [0, +\infty) \to [0, +\infty)$ with $\omega(0) = 0$ will be referred to as a modulus of continuity. Given a metric space $(X, d)$, we say that a function $f: X \to \mathbb{R}$ has modulus of continuity $\omega$ if

$$|f(x) - f(y)| \leq \omega(d(x, y)), \quad \forall x, y \in X.$$

Let us recall some common moduli of continuity. A function $f: X \to \mathbb{R}$ is Hölder continuous if it has modulus of continuity $\omega(r) = Cr^\alpha = Ce^{-\alpha \log \frac{1}{r}}$ for some pair of constants $C < \infty$ and $0 < \alpha \leq 1$. When $\alpha = 1$ this corresponds to Lipschitz continuity.

A function $f$ is weak-Hölder continuous if it has modulus of continuity $\omega(r) = Ce^{-\alpha (\log \frac{1}{r})^\theta}$ for some constants $C < \infty$, $0 < \alpha \leq 1$ and $0 < \theta \leq 1$. When $\theta = 1$, this corresponds to Hölder continuity.

A function $f$ is log-Hölder continuous if it has modulus of continuity $\omega(r) = C (\log \frac{1}{r})^{-1}$ for some constant $C < \infty$.

Additionally, we define a stronger modulus of continuity than log-Hölder.

**Definition 1.** We say that a function $f$ is $(\gamma, \beta)$-log-Hölder continuous if it has modulus of continuity $\omega(r) = Ce^{-\beta (\log \log \frac{1}{r})^\gamma}$ for some constants $C < \infty$, $\gamma \geq 1$ and $\beta \geq 1$.

Note that when $\gamma = 1$ and $\beta = 1$, this corresponds to log-Hölder continuity, while as these parameters increase, the modulus of continuity improves.

Below we summarize the relations between these moduli of continuity.

Hölder $\Rightarrow$ weak-Hölder $\Rightarrow$ $(\gamma, \beta)$-log-Hölder $\Rightarrow$ log-Hölder

For discrete ergodic Schrödinger operators, W. Craig and B. Simon [6] proved that the IDS is always log-Hölder continuous.

M. Goldstein and W. Schlag [14, Lemma 10.3] showed that any singular integral operator on a space of functions preserves the modulus of continuity, as long as it is sharp enough. This applies to $(\gamma, \beta)$-log-Hölder continuity with $\gamma > 1, \beta \geq 1$ (and so to weak-Hölder and Hölder as well) but not to log-Hölder (or to a slightly stronger) modulus of continuity.

Since the Thouless formula

$$L(E) = \int_{\mathbb{R}} \log|E - E'| dN(E')$$

relates the IDS and the LE via such a singular integral operator (essentially the Hilbert transform), we conclude the following.

**Proposition 1.** Given $\gamma > 1, \beta \geq 1$, the LE is $(\gamma, \beta)$-log-Hölder continuous if and only if the IDS is $(\gamma, \beta)$-log-Hölder continuous.

However, the mere log-Hölder continuity of the IDS has no implications on the regularity of the LE (which in general may even be discontinuous).

3. Main results. Consider $\Sigma = \{0, a, b, c\}$ and the following Markov chain.

Let $X = \Sigma^\mathbb{Z} = \{0, a, b, c\}^\mathbb{Z}$, $T: X \to X$ be the shift map and let

$$q = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

(3.1)
be a probability vector on $\Sigma$. The transition probability matrix

$$P = \begin{bmatrix}
\frac{1}{2} & 0 & 0 & \frac{1}{4} \\
\frac{1}{2} & 0 & 0 & \frac{1}{4} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}$$

is primitive since $P^5 > 0$ and the vector $q$ is $P$-stationary. Therefore the pair $(P, q)$ determines a unique probability measure $\mathbb{P} = \mathbb{P}(P, q)$ on $X$, and with this measure the map $T: X \to X$ is a mixing Markov shift.

Consider now the function $v: \Sigma \to \mathbb{R}$ defined by

$$v(a) = v(c) = -e, \ v(b) = -e^{-1} \text{ and } v(0) = 0. \quad (3.3)$$

This function determines the locally constant random potential $v: X \to \mathbb{R}$ defined by $v(x) = v(x_0)$ for all $x \in X$, which in turn determines the family of Schrödinger cocycles

$$A_E(x) := \begin{bmatrix} v(x) - E & -1 \\ 1 & 0 \end{bmatrix} \in \text{SL}_2(\mathbb{R})$$

depending on the parameter $E \in \mathbb{R}$, over the Markov shift above.

Consider the corresponding discrete Schrödinger operator

$$[H_x \psi]_n := -(\psi_{n+1} + \psi_{n-1}) + v(T^n x) \psi_n.$$

The following is the first main result of this paper.

**Theorem 1.** For any $\beta > 2$, the integrated density of states $N(E)$ of the discrete Schrödinger operator corresponding to the random Markov shift defined above is not $(1, \beta)$-log-Hölder continuous at $E = 0$.

Recall that W. Craig B. Simon [6] established log-Hölder continuity of the IDS in the general setting of ergodic Schrödinger operators. W. Craig [5], J. Pöschel [19] and more recently H. Krüger and Z. Gan [16] constructed examples showing that this result is optimal in the setting of Schrödinger operators with limit periodic potentials. By a result of A. Avila [1], (non periodic) limit periodic potentials can be obtained by sampling a continuous function along the orbits of a minimal translation of a Cantor group. Finally, we were made aware of the work in progress [2] by A.
Avila, Y. Last, M. Shamis and Q. Zhou, where log-H"older is proven to be the optimal modulus of continuity for cocycles over a torus translation.

Theorem 1 shows that the log-H"older continuity of the IDS obtained by Craig and Simon is nearly optimal at the other end of ergodic behavior, namely for Schr"odinger operators with random potentials.

Using the above considerations regarding the transfer of a modulus of continuity via the Thouless formula, we derive the following about the regularity of the LE for random Bernoulli cocycles.

**Theorem 2.** Consider the random Bernoulli cocycle (1.1) with probabilities \( (\frac{1}{2}, \frac{1}{2}) \). For any \( \gamma > 1 \), the Lyapunov exponent is not \( (\gamma, \beta) \)-log-H"older continuous at this cocycle. In particular, it is not weak-H"older or H"older continuous.

4. A probabilistic lemma. The purpose of this section is to prove the following key lemma.

**Lemma 1.** Consider the matrices \( C \) and \( D \) defined in (1.1). There exist \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), the event that a random i.i.d. sequence \( \{A_j\} \) with \( \mathbb{P}[A_j = D] = \mathbb{P}[A_j = C] = \frac{1}{2} \) satisfies

\[
A_n \cdots A_1 A_0 = \pm \begin{bmatrix} e^\kappa & 0 \\ 0 & e^{-\kappa} \end{bmatrix} \text{ with } \kappa \geq \frac{1}{10} \sqrt{n}
\]

has probability \( > \frac{4}{10} \).

This will be proved at the end of this section.

**Proposition 2.** The matrices \( C \) and \( D \) satisfy

1. \( C^{-1} = -C \) and \( C^{2n} = \pm I, C^{2n-1} = \pm C \) for all \( n \in \mathbb{Z} \).
2. \( CDC = -D^{-1} \).
3. Any product \( A_n \cdots A_1 A_0 \) with factors \( A_j \in \{D, C\} \) has the form

\[
(a) \pm \begin{bmatrix} e^\kappa & 0 \\ 0 & e^{-\kappa} \end{bmatrix} \text{ when the number of factors } A_j = C \text{ is even,}
\]

\[
(b) \pm \begin{bmatrix} 0 & -e^\kappa \\ e^{-\kappa} & 0 \end{bmatrix} \text{ when the number of factors } A_j = C \text{ is odd.}
\]

**Proof.** \( C^2 = -I \) so, \( C^{2n} = \pm I \) depending on whether \( n \) is even or odd, which proves (1).

Item (2) follows from the fact that \( DCDC = -I \).

Let us prove (3). Given a product \( A_n \cdots A_1 A_0 \) substitute any even length list of consecutive \( C \)'s in it by a sign \( \pm 1 \), to get a product of the form:

\[
\pm \ldots CD^m C D^n C D^m \ldots C...
\]

(4.1)

Since there was an even number of \( C \)'s cancellations, the number of factors \( A_j = C \) has the same parity as the number of \( C \)'s in (4.1). Items (1) and (2) imply that \( C^l D = D^{-l} C \) for all \( l \in \mathbb{Z} \).

Making use of these commutation relations combined with the identity \( C^2 = -I \), we can transform (4.1) into either \( \pm D^\kappa \) or \( \pm C D^\kappa \), for some \( \kappa \in \mathbb{Z} \). Since the number of \( C \)'s cancellations is even, the product (4.1) is equal to \( \pm D^\kappa \) if and only if the number of \( C \)'s in it is even. This proves item (3).

Consider a random i.i.d. process \( \{A_n\}_{n \geq 0} \), such that for all \( n \geq 0 \),

\[
\mathbb{P}[A_n = D] = \mathbb{P}[A_n = C] = \frac{1}{2}.
\]
Define the product process
\[ M_n := A_{n-1} \ldots A_1 A_0. \]

By Proposition 2, the process \( M_n \) takes value in the union of the following two disjoint classes of matrices.
\[
M_+ := \left\{ \pm \begin{bmatrix} e^\kappa & 0 \\ 0 & e^{-\kappa} \end{bmatrix} : \kappa \in \mathbb{Z} \right\}
\]
\[
M_- := \left\{ \pm \begin{bmatrix} 0 & -e^\kappa \\ e^{-\kappa} & 0 \end{bmatrix} : \kappa \in \mathbb{Z} \right\}
\]

By the same proposition we have
\[
C M_+ \subset M_-, \quad C M_- \subset M_+, \quad D M_+ \subset M_+ \quad \text{and} \quad D M_- \subset M_-
\]

Consider the sign valued process \( \{ \eta_n \}_n \)
\[
\eta_n := \begin{cases} 
+ & \text{if } M_n \in M_+ \\
- & \text{if } M_n \in M_-
\end{cases}
\]

as well as the real valued process \( \{ S_n \}_n \) characterized by
\[
M_n = \begin{cases} 
\pm \begin{bmatrix} e^{S_n} & 0 \\ 0 & e^{-S_n} \end{bmatrix} & \text{if } M_n \in M_+ \\
\pm \begin{bmatrix} 0 & -e^{S_n} \\ e^{-S_n} & 0 \end{bmatrix} & \text{if } M_n \in M_-
\end{cases}
\]

**Proposition 3.** The processes \( \{ \eta_n \} \) and \( \{ S_n \} \) relate to \( \{ A_n \} \) in the following way:

1. \( \eta_{n-1} = +, \quad A_n = C \) \( \Rightarrow \eta_n = -, \quad S_n = S_{n-1} \)
2. \( \eta_{n-1} = -, \quad A_n = C \) \( \Rightarrow \eta_n = +, \quad S_n = S_{n-1} \)
3. \( \eta_{n-1} = +, \quad A_n = D \) \( \Rightarrow \eta_n = +, \quad S_n = S_{n-1} + 1 \)
4. \( \eta_{n-1} = -, \quad A_n = D \) \( \Rightarrow \eta_n = -, \quad S_n = S_{n-1} - 1 \)

**Proof.** Straightforward argument. \( \square \)

Given a set of words \( A \subset \{ C, D \}^n \) and a letter \( a \in \{ C, D \} \) we define
\[
A * a = \{ (w, a) \in \{ C, D \}^{n+1} : w \in A \}.
\]

Each word \( w = (w_0, w_1, \ldots, w_{n-1}) \in \{ C, D \}^n \) determines the cylinder
\[
C(w) := \{ x \in \{ C, D \}^\mathbb{Z} : x_j = w_j, \quad \forall j = 0, 1, \ldots, n-1 \}.
\]
Throughout the paper we identify each set \( A \subset \{ C, D \}^n \) with the event \( \cup_{w \in A} C(w) \) determined by its words. Because all words are equi-probable, the probability of \( A \) (regarded as an event) is \( \mathbb{P}(A) = \frac{\#A}{2^n} \).

Define for each pair of integers \((n, i)\),
\[
A_+(n, i) := \{ w \in \{ C, D \}^n : S_n(w) = i \quad \text{and} \quad \eta_n(w) = + \}
\]
\[
A_-(n, i) := \{ w \in \{ C, D \}^n : S_n(w) = i \quad \text{and} \quad \eta_n(w) = - \}
\]
\[
A(n, i) := \{ w \in \{ C, D \}^n : S_n(w) = i \}
\]

Note that \( A(n, i) = A_+(n, i) \cup A_-(n, i) \). Let us write \( a_+(n, i) := \#A_+(n, i) \), \( a_-(n, i) := \#A_-(n, i) \) and \( a(n, i) := \#A(n, i) \).
Proposition 4. For any pair of integers \((n, i)\),

\[
\begin{align*}
  a(n, i) &= a_+(n, i) & \text{if } n + i \text{ is even} \\
  a(n, i) &= a_-(n, i) & \text{if } n + i \text{ is odd}.
\end{align*}
\]

Moreover, the function \((n, i) \mapsto a(n, i)\) is determined by the recursive relation

\[
a(n, i) = a(n - 1, i - 1) + a(n - 1, i + 1)
\]

and the initial conditions

\[
a(1, 0) = a(1, 1) = 1 \text{ and } a(1, i) = 0 \text{ for all } i \neq 0, 1.
\]

For all integers \((n, i)\)

\[
a(n, 2i) = \left( \frac{n - 1}{\lfloor \frac{n-1}{2} \rfloor} + i \right)
\]

\[
a(n, 2i + 1) = \left( \frac{n - 1}{\lfloor \frac{n-1}{2} \rfloor} \right).
\]

Proof. Note that \(\mathcal{A}(1, 0) = \mathcal{A}_-(1,0) = \{(C)\}\) and \(\mathcal{A}(1, 1) = \mathcal{A}_+(1,1) = \{(D)\}\). Hence \(a(1, 0) = a_-(1, 0) = 1\) and \(a(1, 1) = a_+(1,1) = 1\), while \(a(1, i) = a_+(1, i) = a_-(1, i) = 0\) for all other \(i\). This proves (4.3).

With the notation introduced, from Proposition 3 we get

\[
\begin{align*}
  \mathcal{A}_+(n, i) &= \mathcal{A}_+(n - 1, i - 1) * D \sqcup \mathcal{A}_-(n - 1, i) * C & \text{if } n + i \text{ is even} \\
  \mathcal{A}_-(n, i) &= \mathcal{A}_+(n - 1, i) * C \sqcup \mathcal{A}_-(n - 1, i + 1) * D & \text{if } n + i \text{ is odd}
\end{align*}
\]

where the symbol \(\sqcup\) stands for disjoint union. These identities imply that

\[
\begin{align*}
  a_+(n, i) &= a_+(n - 1, i - 1) + a_-(n - 1, i) & \text{if } n + i \text{ is even} \\
  a_-(n, i) &= a_+(n - 1, i) + a_-(n - 1, i + 1) & \text{if } n + i \text{ is odd}
\end{align*}
\]

From the initial conditions we see by induction in \(n\) that \(a_-(n, i) = 0\) and \(a(n, i) = a_+(n, i)\) when \(n + i\) is even, while \(a_+(n, i) = 0\) and \(a(n, i) = a_-(n, i)\) when \(n + i\) is odd.

From (4.6) and (4.7) we get that

\[
a_-(n, i) = a_+(n - 1, i) + a_-(n - 1, i + 1) = a_+(n, i + 1)
\]

when \(n + i\) is odd and \(a_-(n, i) = 0 = a_+(n, i + 1)\) otherwise. Therefore, because of these equalities, if \(n + i\) is even then

\[
a(n, i) = a_+(n, i) = a_+(n - 1, i - 1) + a_-(n - 1, i)
\]

\[
= a_+(n - 1, i - 1) + a_+(n - 1, i + 1)
\]

\[
= a(n - 1, i - 1) + a(n - 1, i + 1).
\]

Similarly, if \(n + i\) is odd then

\[
a(n, i) = a_-(n, i) = a_+(n - 1, i) + a_-(n - 1, i + 1)
\]

\[
= a_-(n - 1, i - 1) + a_-(n - 1, i + 1)
\]

\[
= a(n - 1, i - 1) + a(n - 1, i + 1).
\]

This establishes identity (4.2).

Table 1 presents the calculation of the first five rows of \(a(n, i)\). The recursive relations (4.3) and (4.2) show that both sequences \(\{a(n, 2i) : -n + 1 \leq 2i \leq n\}\) and \(\{a(n, 2i) : -n + 1 \leq 2i + 1 \leq n\}\) have exactly \(n\) entries matching the binomial numbers \(\binom{n-k}{k} : 0 \leq k \leq n-1\). Formulas (4.4) and (4.5) hold because as \(2i\)
Table 1. Pascal’s triangle for the numbers $a(n, i)$

| $i$ | $\cdots$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $+1$ | $+2$ | $+3$ | $+4$ | $+5$ | $\cdots$ |
|-----|----------|------|------|------|------|-----|------|------|------|------|------|------|
| $a(1, i)$ |         |      |      |      |      |      |      |      |      |      |      |      |
| $a(2, i)$ | 1       | 1    |      |      |      |      |      |      |      |      |      |      |
| $a(3, i)$ | 1       | 1    | 2    | 1    |      |      |      |      |      |      |      |      |
| $a(4, i)$ | 1       | 1    | 3    | 3    | 3    | 1    |      |      |      |      |      |      |
| $a(5, i)$ | 1       | 1    | 4    | 6    | 4    | 4    | 1    |      |      |      |      |      |

ranges from $-n + 1$ to $n$ the variable $k = \lfloor \frac{n-1}{2} \rfloor + i$ ranges from 0 to $n$, while as $2i + 1$ ranges from $-n + 1$ to $n$ the variable $k = \lfloor \frac{n}{2} \rfloor + i$ ranges from 0 to $n$. \hfill $\square$

Proof of Lemma 1. \textbf{Consider the event} $E_n = \{ \eta_n = +, S_n \geq 10 \sqrt{n} \}$ whose probability we want to estimate, which can be identified with the following set of words

$$E_n = \bigcup_{i \geq 1} 10 \sqrt{n} A(n, i) = \bigcup_{i \geq 1} 10 \sqrt{n} A(n, i).$$

By (4.4) and Proposition 4

$$\#E_n = \sum_{2i \geq \frac{1}{n} \sqrt{n}} a(n, 2i) = \sum_{i \geq \frac{1}{n} \sqrt{n} + \lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{i} \quad \text{when } n \text{ is even.}$$

Similarly, by (4.5) and Proposition 4

$$\#E_n = \sum_{2i+1 \geq \frac{1}{n} \sqrt{n}} a(n, 2i + 1) = \sum_{i \geq \frac{1}{n} \sqrt{n} + \lfloor \frac{n}{2} \rfloor - \frac{1}{2}} \binom{n-1}{i} \quad \text{when } n \text{ is odd.}$$

Therefore

$$P(E_n) = \begin{cases} \sum_{2i \geq \frac{1}{n} \sqrt{n} + \lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{i} & \text{if } n \text{ is even} \\ \sum_{2i+1 \geq \frac{1}{n} \sqrt{n} + \lfloor \frac{n}{2} \rfloor - \frac{1}{2}} \binom{n-1}{i} & \text{if } n \text{ is odd.} \end{cases} \quad (4.10)$$

We now use the Central Limit Theorem (CLT) to estimate these sums. Consider an i.i.d. process $\{Y_n\}$ where each $Y_n$ is a Bernoulli random variable with probabilities $\left(\frac{1}{2}, \frac{1}{2}\right)$, that is $P(Y_n = 0) = P(Y_n = 1) = \frac{1}{2}$. All moments of $Y_n$ are equal to $\frac{1}{2}$ and so is its standard deviation $\sigma(Y_n) = \frac{1}{2}$.

Next consider the normalized sum process

$$T_n = \frac{Y_1 + \ldots + Y_n - \frac{n}{2}}{\sqrt{n}} = 2 \left( \frac{Y_1 + \ldots + Y_n}{\sqrt{n}} \right) - n.$$ 

The CLT says that $T_n$ converges in distribution to the standard normal $N(0, 1)$, whose cumulative distribution is given by

$$F(u) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-\frac{x^2}{2}} \, dx.$$ 

More precisely this means that for all $u \in \mathbb{R}$,

$$\lim_{n \to \infty} P(T_n \leq u) = F(u).$$
On the other hand because the random variables $Y_n$ are Bernoulli,
\[
\mathbb{P}(T_n \leq u) = \mathbb{P}\left(\sum_{j=1}^{n} Y_j \leq \frac{n}{2} + u \sqrt{\frac{n}{2}}\right) = \frac{1}{2^n} \sum_{i \leq \frac{n}{2} + u \sqrt{\frac{n}{2}}} \left(\begin{array}{c}
\frac{n}{2} \\
i
\end{array}\right).
\]

Hence
\[
\frac{1}{2^{n-1}} \sum_{i > \frac{n}{2} + u \sqrt{\frac{n}{2}}} \left(\begin{array}{c}
n-1 \\
i
\end{array}\right) = \mathbb{P}\left(T_{n-1} > \frac{1}{10}\right) \quad (4.11)
\]
converges to $1 - F(1/10) \approx 0.460172 > 0.4$. Comparing the sums in (4.10) and (4.11) we conclude that $\lim_{n \to \infty} \mathbb{P}(E_n) = 1 - F(1/10) > 0.4$.

The Berry-Esseen’s Theorem (see [23]) implies that there exists $C < \infty$ such that
\[
\mathbb{P}(T_n \leq u) - F(u) \leq \frac{C}{\sqrt{n}}.
\]

Using this fact, the threshold after which $\mathbb{P}(E_n) > 0.4$ holds can be explicitly computed.

5. **Proof of Theorem 1.** B. Halperin gave an example of a random Schrödinger cocycle where the IDS (hence also the LE), as a function of the energy $E$, cannot be better than Hölder continuous, with some explicitly given Hölder exponent. Our argument follows closely the proof of this result given by B. Simon and M. Taylor in [20].

**Lemma 2 (Temple’s Inequality).** Let $A$ be a self-adjoint operator in some Hilbert space. Assume $\{f_j\}_{j=1}^k$ is an orthonormal family such that:

1. $\langle f_i, Af_j \rangle = \langle Af_i, f_j \rangle = 0$ for all $i \neq j$ and
2. $\| (A - E_0) f_j \| \leq \varepsilon$,

for some $\varepsilon > 0$ and $E_0 \in \mathbb{R}$. Then $A$ has at least $k$ eigenvalues (counted with multiplicity) in the range $[E_0 - \varepsilon, E_0 + \varepsilon]$.

**Proof.** See [20, Lemma A.3.2].

**Definition 2.** Any vector $f \in H$ such that $\| (A - E_0) f \| \leq \varepsilon$ will be called an $(\varepsilon, E_0)$-quasi-eigenfunction of the operator $A$.

Consider now, throughout the rest of this section, the family of Schrödinger cocycles defined in (3.3) and (3.4) over the Markov shift defined by (3.1) and (3.2).

**Lemma 3.** In our example the cocycle satisfies the following relations
\[
C = \begin{bmatrix}
v(0) & -1 \\
1 & 0
\end{bmatrix},
\]
\[
D = \begin{bmatrix}
v(c) & -1 & v(b) & -1 & v(a) & -1 \\
1 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
\]

where $C$ and $D$ are the matrices in (1.1).

**Proof.** Straightforward calculation.

A bi-infinite sequence $w = (\ldots, w_0, w_1, \ldots, w_n, \ldots) \in \Sigma^\mathbb{Z}$ is called allowable if it is a path in the graph of Figure 1. A typical allowable sequence looks like
\[
\ldots 0 abc abc 00 abc 000 abc 0 \ldots
\]

Clearly the set of allowable sequence has full probability.
A word \( w \in \Sigma^n \) is called \textit{allowable} if it is a path in the graph of Figure 1. We denote by \( \mathcal{B}(n) \) the set of all allowable words \( w \in \Sigma^n \). A word \( w \in \Sigma^n \) is called \textit{admissible} if it is allowable and moreover it only contains full ‘abc’ blocks. For instance the word \((00\,abc\,ab)\) is allowable but not admissible. We denote by \( \mathcal{A}(n) \) the set of all admissible words \( w \in \Sigma^n \). We write \( b(n) = \# \mathcal{B}(n) \) and \( a(n) = \# \mathcal{A}(n) \).

Given \( w \in X = \Sigma^\mathbb{Z} \) such that for some integers \( n, m \geq 0 \) such that the finite word \((w_0, w_1, \ldots, w_m)\) is admissible, we can decompose the iterate \( \mathcal{A}^m_n(w) \) (at the energy level \( E = 0 \)) as a product of matrices with factors \( C \) and \( D \).

Given an admissible word \( w = (w_0, w_1, \ldots, w_n) \) we define

\[
M_w := \begin{bmatrix}
v(w_n) & 1 \\
0 & 0
\end{bmatrix} \cdots \begin{bmatrix}
v(w_1) & 1 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
v(0) & 1 \\
0 & 0
\end{bmatrix}.
\]

**Proposition 5.** Let \( w^* = w_1 0 w_2 = (a_0, a_1, \ldots, a_n) \) be a finite admissible word such that for some integers \( k_1, k_2 > K > 0 \), the sub-words \( w_1 \) and \( w_2 \) satisfy

\[
M_{w_1} = \begin{bmatrix} e^{k_1} & 0 \\ 0 & e^{-k_1}
\end{bmatrix} \quad \text{and} \quad M_{w_2} = \begin{bmatrix} e^{k_2} & 0 \\ 0 & e^{-k_2}
\end{bmatrix}.
\]

Consider any \( w \in X \) in the cylinder determined by \( w^* \). The associated \( n \)-truncation of the Schrödinger operator, defined in (2.4), is given by:

\[
H^{(n)}_w = \begin{bmatrix}
v(a_0) & -1 & 0 & 0 \\
-1 & v(a_1) & -1 & 0 \\
0 & -1 & v(a_2) & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Let \( \psi = (\psi_j)_{j \in \mathbb{Z}} \in l^2(\mathbb{Z}) \) be the sequence defined recursively by

\[
\begin{bmatrix}
\psi_0 \\
\psi_{-1}
\end{bmatrix} = \begin{bmatrix} e^{-k_1} \\ 0
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
\psi_j+1 \\
\psi_j
\end{bmatrix} = \begin{bmatrix} v(a_j) & -1 \\ 1 & 0
\end{bmatrix} \begin{bmatrix}
\psi_j \\
\psi_{j-1}
\end{bmatrix} \quad \text{for} \quad 0 \leq j \leq n,
\]

with \( \psi_i = 0 \) for all \( i < -1 \) and \( i > n + 1 \).

Then
1. \( \| \psi \| \geq 1 \),
2. \( \psi \) is an \((\sqrt{2} e^{-K}, 0)\)-quasi-eigenfunction of \( H_w \), i.e., \( \| H_w \psi \| \leq \sqrt{2} e^{-K} \),
3. the truncation \( P_n \psi \) of \( \psi \) to the range \([0, n]\) is also an \((\sqrt{2} e^{-K}, 0)\)-quasi-eigenfunction of the truncated operator \( H^{(n)}_w \).

**Proof.** We only prove items (1) and (2). The proof of the third item is similar.

Consider \( \psi = (\psi_j)_{j \in \mathbb{Z}} \) under the assumptions of the proposition. Then for all \( 0 \leq j \leq n \), one has \( \psi_{j+1} = v(a_j) \psi_j - \psi_{j-1} \), which implies that

\[
(H_w \psi)_j = - (\psi_{j+1} + \psi_{j-1}) + v(a_j) \psi_j = 0.
\]

In particular, since

\[
\begin{bmatrix}
\psi_0 \\
\psi_{-1}
\end{bmatrix} = \begin{bmatrix} e^{-k_1} \\ 0
\end{bmatrix}
\]

we have

\[
\begin{bmatrix}
\psi_{n+1} \\
\psi_n
\end{bmatrix} = \begin{bmatrix} e^{k_2} & 0 \\ 0 & e^{k_2}
\end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0
\end{bmatrix} \begin{bmatrix} e^{k_1} & 0 \\ 0 & e^{k_1}
\end{bmatrix} \begin{bmatrix} e^{-k_1} \\ 0
\end{bmatrix} = \begin{bmatrix} 0 \\ e^{-k_2}
\end{bmatrix}.
\]
Thus, since $\psi_{-1} = \psi_{-2} = 0$ we have
\[
(H_w \psi)_{-1} = -(\psi_{-2} + \psi_0) + v(w_{-1}) \psi_{-1} = \psi_0 = -e^{-k_1}.
\]
Similarly, since $\psi_{n+1} = \psi_{n+2} = 0$ we have
\[
(H_w \psi)_{n+1} = -(\psi_{n+2} + \psi_n) + v(w_{n+1}) \psi_{n+1} = \psi_n = -e^{-k_2}.
\]
Finally, because $\psi_j = 0$ for all $j \not\in [0, n]$, we get $(H_w \psi)_j = 0$ for all $j \not\in [-1, n+1]$. Together, these bounds show that
\[
\|H_w \psi\| = \sqrt{e^{-2k_1} + e^{-2k_2}} \leq \sqrt{2} e^{-\kappa}
\]
and prove (2).

Denoting by $q$ the length of the word $w_1$ we have
\[
\begin{bmatrix}
\psi_q \\
\psi_{q-1}
\end{bmatrix} = \begin{bmatrix}
e^{k_1} & 0 \\
0 & e^{k_1}
\end{bmatrix} \begin{bmatrix}
\psi_0 \\
\psi_{-1}
\end{bmatrix} = \begin{bmatrix}
e^{k_1} & 0 \\
0 & e^{k_1}
\end{bmatrix} \begin{bmatrix}
e^{-k_1} \\
0
\end{bmatrix} = \begin{bmatrix}1 \\
0
\end{bmatrix}
\]
which implies that $\|\psi\| \geq |\psi_q| = 1$, thus proving (1).

Recall that $a(n)$ and $b(n)$ count, respectively, the admissible and allowable words of length $n$.

**Proposition 6.** The sequence $b(n)$ satisfies
\[
b(1) = 4, \ b(2) = 6, \ b(3) = 9, \ b(n) = b(n-1) + b(n-3) \quad \forall \ n \geq 4.
\]
The sequence $a(n)$ satisfies
\[
a(0) = a(1) = a(2) = 1, \ a(n) = a(n-1) + a(n-3) \quad \forall \ n \geq 3.
\]
Moreover $b(n) = a(n+4)$ for all $n \geq 1$ and
\[
\lim_{n \to +\infty} \frac{a(n)}{b(n)} = \lambda^{-4} = 0.216757 \ldots,
\]
where $\lambda > 1$ is the Pisot number root the polynomial equation $x^3 = x^2 + 1$.

**Proof.** Note that $A(0) = \{\emptyset\}$, $A(1) = \{(0)\}$ and $A(2) = \{(00)\}$, which implies that $a(0) = a(1) = a(2) = 1$.

Now, for $n \geq 3$, the decomposition
\[
A(n) = A(n-1) * 0 \uplus A(n-3) * (abc)
\]
shows that $a(n) = a(n-1) + a(n-3)$. The first values of $a(n)$, known as the Narayana’s cows sequence (see [21]), are shown in Table 2.

| $n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | \ldots |
|-----|----|----|----|----|----|----|----|----|----|-------|
| $a(n)$ | 1  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 9  | 13  | \ldots |

Table 2. Narayana’s cows sequence $a(n)$

For the second sequence, note that
\[
\mathcal{B}(1) = \{(0), (a), (b), (c)\}
\]
\[
\mathcal{B}(2) = \{(00), (0a), (ab), (bc), (ca)\}
\]
\[
\mathcal{B}(3) = \{(000), (00a), (0ab), (abc), (bc0), (bca), (c00), (c0a), (cab)\}
\]
which gives $b(1) = 4$, $b(2) = 6$ and $b(3) = 9$. 
Given an allowable word $w \in \Sigma^n$ of length $n$ define the augmented word $w'$ of length $n + 3$ as

$$w' := \begin{cases} w00a & \text{if } w \text{ ends in } 0 \\ wbca & \text{if } w \text{ ends in } a \\ wc0a & \text{if } w \text{ ends in } b \\ w00a & \text{if } w \text{ ends in } c \\ \end{cases}$$

and let $\mathcal{B}'(n + 3) := \{w' : w \in \mathcal{B}(n)\}$. Similarly, given an allowable word $w \in \Sigma^n$ of length $n$ define the augmented word $w^*$ of length $n + 1$ as

$$w^* := \begin{cases} w0 & \text{if } w \text{ ends in } 0 \\ wb & \text{if } w \text{ ends in } a \\ wc & \text{if } w \text{ ends in } b \\ w0 & \text{if } w \text{ ends in } c \\ \end{cases}$$

and let $\mathcal{B}^*(n + 1) := \{w^* : w \in \mathcal{B}(n)\}$. Note that

$$\#\mathcal{B}(n) = \#\mathcal{B}'(n + 3) = \#\mathcal{B}^*(n + 1).$$

Finally, since

$$\mathcal{B}(n) = \mathcal{B}'(n - 3) \cup \mathcal{B}^*(n - 1),$$

the sequence $b(n)$ satisfies $b(n) = b(n - 1) + b(n - 3)$.

Looking at the first terms of $a(n)$ in Table 2, since $a(n)$ and $b(n)$ satisfy the same recursive relation, we get that $b(n) = a(n + 4)$ for all $n \geq 1$.

Now the characteristic equation of the linear recursive equation for $a(n)$ is the polynomial equation $-x^3 + x^2 + 1 = 0$. This polynomial has 3 roots, the Pisot number $\lambda = 1.46557 \ldots$ and two more complex roots $\sigma, \tau$ inside the unit circle. Hence there are constants $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{C}$ such that

$$a(n) = c_1 \lambda^n + c_2 \sigma^n + c_3 \tau^n \quad \forall n \in \mathbb{N}.$$ 

Therefore

$$\lim_{n \to \infty} \frac{a(n)}{b(n)} = \lim_{n \to \infty} \frac{a(n)}{a(n + 4)} = \lim_{n \to \infty} \frac{c_1 \lambda^n + o(1)}{c_1 \lambda^{n+4} + o(1)} = \frac{1}{\lambda^4}. \quad \square$$

**Corollary 7.** $\lim_{n \to \infty} \mathbb{P}(A(n)) = \frac{1}{\lambda^4} = 0.216757 \ldots$.

**Proof.** Because all transition probabilities in the graph of $\Sigma$ (see Figure 1) have the same probability $\frac{1}{2}$, all allowable words $w \in \mathcal{B}(w)$ are equi-probable. Hence $\mathbb{P}(A(n)) = \frac{a(n)}{b(n)}$. \hfill $\square$

**Proposition 8.** Given a $\Sigma$-valued stationary Markov chain $\{w_n\}_{n \in \mathbb{Z}}$ with stochastic transition matrix (3.2), there exists $n_0 \in \mathbb{N}$ such that for all $l \geq n_0$, the event that $(w_0, w_1, \ldots, w_l)$ is an admissible word and

$$M_{(w_0, w_1, \ldots, w_l)} = \pm \begin{bmatrix} e^\kappa & 0 \\ 0 & e^{-\kappa} \end{bmatrix} \text{ with } \kappa \geq \frac{1}{10} \sqrt{\frac{T}{3}}$$

has probability $> \frac{8}{100}$.

**Proof.** Consider the set $\mathcal{A}_{l+1} = A(l + 1)$ of all admissible words in $\Sigma^{l+1}$. By Corollary 7, if $n_0$ is large enough and $l \geq n_0$, $\mathbb{P}(A_{l+1}) > 0.216$.

By Proposition 2 we can define $\kappa : \mathcal{A}_{l+1} \to \mathbb{Z}$ such that for all $w \in \mathcal{A}_{l+1}$,

$$M_w = \pm \begin{bmatrix} e^{\kappa(w)} & 0 \\ 0 & e^{-\kappa(w)} \end{bmatrix} \text{ or } M_w = \pm \begin{bmatrix} 0 & e^{-\kappa(w)} \\ e^{\kappa(w)} & 0 \end{bmatrix}.$$
Define also the functions $\rho, N : \mathcal{A}_{l+1} \to \mathbb{N}$, where $\rho(w) := \# \{0 \leq j \leq l : w_j = 0\}$ and $N(w) := \rho(w) + \frac{l - \rho(w)}{3}$ counts the number of 0’s plus the number of $abc$ blocks in an admissible word $w$.

To finish we now derive the lower bound for the probability of the word set

$$\mathcal{B}_l := \left\{ w \in \mathcal{A}_{l+1} : \kappa(w) \geq \frac{1}{10} \sqrt{l/3}, \rho(w) \text{ even} \right\}.$$  

Applying the Law of Total Probabilities we have

$$P(\mathcal{B}_l) = \sum_{n=1/3}^{l} P[\mathcal{A}_{l+1} \cap \{N = n\}] P \left[ \kappa \geq \frac{1}{10} \sqrt{l/3}, \rho \text{ even} \left| \mathcal{A}_{l+1} \cap \{N = n\} \right. \right]$$

$$\geq \sum_{n=1/3}^{l} P[\mathcal{A}_{l+1} \cap \{N = n\}] P \left[ \kappa \geq \frac{1}{10} \sqrt{n}, \rho \text{ even} \left| \mathcal{A}_{l+1} \cap \{N = n\} \right. \right]$$

$$> \sum_{n=1/3}^{l} P[\mathcal{A}_{l+1} \cap \{N = n\}] \frac{4}{10} = \frac{4}{10} P(\mathcal{A}_{l+1}) > 0.4 \times 0.216 > 0.08.$$  

In the first step we have used that $n \geq l/3$. Also, by Lemma 1 we get

$$P \left[ \kappa \geq \frac{1}{10} \sqrt{n}, \rho \text{ even} \left| \mathcal{A}_{l+1} \cap \{N = n\} \right. \right] > \frac{4}{10}.$$  

This concludes the proof. ∎

Let $\mathcal{B}_l$ be the set of admissible words in $\Sigma^{l+1}$ such that $M_w = \pm \begin{bmatrix} e^\kappa & 0 \\ 0 & e^{-\kappa} \end{bmatrix}$ with $\kappa \geq \frac{1}{10} \sqrt{l/3}$. By Proposition 8, $P(\mathcal{B}_l) > 8/100$. Next consider the event $\mathcal{C}_l = \mathcal{B}_l \cap \mathcal{B}_l^c$ of all admissible words $(w_1, 0, w_2) \in \Sigma^{2l+3}$ with $w_1, w_2 \in \mathcal{B}_l$.

**Proposition 9.** $P(\mathcal{C}_l) = P(\mathcal{B}_l)^2 > \frac{4}{625}$.

**Proof.** Let $\mathcal{B}_l^-$ be the cylinder associated with admissible words

$$w = (w_0, \ldots, w_{2l+2}) \in \Sigma^{2l+3} \text{ such that } (w_0, \ldots, w_l) \in \mathcal{B}_l.$$  

Analogously, let $\mathcal{B}_l^+$ be the cylinder associated with admissible words

$$w = (w_0, \ldots, w_{2l+2}) \in \Sigma^{2l+3} \text{ such that } (w_{l+2}, \ldots, w_{2l+2}) \in \mathcal{B}_l.$$  

Note that if an admissible word $w$ lies in $\mathcal{B}_l^- \cap \mathcal{B}_l^+$ then its middle letter can only be ‘0’, i.e. $w_{l+1} = 0$. Therefore $\mathcal{C}_l = \mathcal{B}_l^- \cap \mathcal{B}_l^+$.  

Given an admissible word $w = (w_0, \ldots, w_{n-1})$ of length $n$, define

$$P(w) = \prod_{i=1}^{n-1} P_{w_i, w_{i-1}}$$  

where $P_{i,j}$ stands for the transition probability from state $j$ to state $i$.

Because the transition probabilities in $\Sigma$ are all equal to $1/2$ (see Figure 1), $P(w) = \frac{1}{2^n}$ for any admissible word $w$ of length $n$. The cylinder $C(w)$ determined by this word has probability $P(C(w)) = \frac{1}{4} P(w) = \frac{1}{2^{2n+1}}$ because the stationary probability on $\Sigma$ is $q = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$. Hence

$$P(\mathcal{C}_l) = \sum_{w \in \mathcal{C}_l} \frac{1}{4} P(w).$$
Given an admissible word \( w = (w, 0, w') \in \mathcal{C}_1 \), since
\[
P(w, 0, w') = P(w) \frac{1}{2} P(w')
\]
we have
\[
\mathbb{P}(\mathcal{C}_1) = \sum_{w, w' \in B_1} \mathbb{P}(C(w, 0, w')) = \sum_{w, w' \in B_1} \frac{1}{4} P(w, 0, w')
\]
\[
= \sum_{w, w' \in B_1} \frac{1}{4} P(w) \frac{1}{4} P(w') = \left( \sum_{w \in B_1} \frac{1}{4} P(w) \right)^2 = \mathbb{P}(B_1)^2.
\]
The inequality in the proposition statement follows from Corollary 7.

Fix now positive (large) integers \( l, m \) and set \( L = m(2l + 3) \). Break \([0, L]\) into \( m \) equal blocks \( I_1 = [0, 2l + 2], I_2 = [2l + 3, 4l + 5], I_3 = [4l + 5, 6l + 7], \) etc., with the last block being \( I_m = [2(m - 1)l + 3(m - 1), 2ml + 3m - 1] \). We refer to
\[
I_j := [2(j - 1)l + 3(j - 1), 2jl + 3j - 1]
\]
as the \( j \)-th block of the word \( w \).

Moreover, the block of length \( 2l + 1 \),
\[
I_j^* := [2(j - 1)l + 3(j - 1) + 1, 2jl + 3j - 2],
\]
obtained by removing the first and last symbols from the \( j \)-th block of \( w \), is called the inner \( j \)-th block of \( w \).

**Lemma 4.** Take positive integers \( l, m \) and \( L \) as above. Consider an admissible word \( w \in X \) and define
\[
n_{l,m}(w) := \# \{ 1 \leq j \leq m : \text{the inner } j \text{-th block of } w \text{ lies in } \mathcal{C}_1 \}.
\]
Then the \( L \)-truncation operator \( H_w^{(L)} \) has at least \( n_{l,m}(w) \) eigenvalues (counted with multiplicities) in the range \([-\sqrt{2}e^{-K_2}, \sqrt{2}e^{-K_2}]\), with \( K_1 := \frac{1}{10} \sqrt{L/3} \).

**Proof.** Given a word \( w \in X \), for each \( 1 \leq j \leq m \) such that the inner \( j \)-th block of \( w \) lies in \( \mathcal{C}_1 \) we take the quasi-eigenfunction of Proposition 5 and shift it to become supported on \( I_j^* \). Let \( f_1, f_2, \ldots, f_{n_{l,m}(w)} \in \ell^2(\mathbb{Z}) \) be the list of functions thus obtained. Since each \( f_i \) vanishes outside some \( I_j^* \), by Proposition 5 the truncated function \( P_L f_i \) satisfies \( \|H_w^{(L)}(P_L f_i)\| \leq \sqrt{2}e^{-K_1} \). Hence each \( P_L f_i \) is a \((\sqrt{2}e^{-K_1}, 0)\)-quasi-eigenfunction of the truncated operator \( H_w^{(L)} \).

By construction \( P_L f_i \) vanishes at the endpoints of the block \( I_j^* \). It follows that \( H_w^{(L)}(P_L f_i) \) is also supported on the block \( I_j^* \). Because these blocks are pairwise disjoint, assumptions (2) of Lemma 2 are automatically satisfied. The conclusion follows then by Temple’s inequality.

**Proof of Theorem 1.** Consider a typical admissible word \( w = (w_n)_{n \in \mathbb{Z}} \in X \). By ergodicity,
\[
\lim_{m \to \infty} \frac{n_{l,m}(w)}{m} = \mathbb{P}(\mathcal{C}_1) \geq \frac{4}{625}.
\]
Hence by Lemma 4,
\[
N\left(\sqrt{2}e^{-K_1}\right) - N\left(-\sqrt{2}e^{-K_1}\right) \geq \lim_{L \to \infty} \frac{n_{l,m}(w)}{L} = \lim_{m \to \infty} \frac{n_{l,m}(w)}{m(2l + 3)} \geq \frac{4}{625} \frac{1}{2l + 3}.
\]
Given $\epsilon > 0$, consider the $(2 + \epsilon)$-log Hölder modulus of continuity
\[ \omega(r) := C \left( - \log r \right)^{-(2+\epsilon)}. \]
For sufficiently large $l$ by Lemma 4 one has $K_l \approx l^{1/2}$, so
\[ \omega(e^{-K_l}) = C K_l^{-(2+\epsilon)} \lesssim l^{-2+\epsilon} \ll \frac{4}{625 (2l + 3)}. \]
Thus
\[ N \left( \sqrt{2} e^{-K_l} \right) - N \left( -\sqrt{2} e^{-K_l} \right) \gg \omega(2 \sqrt{2} e^{-K_l}), \]
which means that the IDS is not $(2 + \epsilon)$-log Hölder continuous. $\square$

6. Proof of Theorem 2. Recall that $\Sigma = \{0, a, b, c\}$, $X = \Sigma^\mathbb{Z}$ and $T : X \to X$ denotes the two-sided shift. Let $\mathcal{A} = C(0) \cup C(a)$ be the union of cylinders determined by the one letter words '0' and 'a'. Let $N : \mathcal{A} \to \mathbb{N}$ be the first return time to $\mathcal{A}$ and $T_\mathcal{A} : \mathcal{A} \to \mathcal{A}$ be the induced (first return) map on $\mathcal{A}$. The function $N : \mathcal{A} \to \mathbb{N}$ takes two values
\[ N(x) = \begin{cases} 1 & \text{if } x_0 = 0 \\ 3 & \text{if } x_0 = a \end{cases} \]
and hence the induced map on $\mathcal{A}$ is given by
\[ T_\mathcal{A}(x) = T^{N(x)} x = \begin{cases} T x & \text{if } x_0 = 0 \\ T^3 x & \text{if } x_0 = a. \end{cases} \]
The family of Schrödinger cocycles $A_E : X \to \text{SL}_2(\mathbb{R})$ also induces a family of cocycles $\tilde{A}_E : \mathcal{A} \to \text{SL}_2(\mathbb{R})$ over $T_\mathcal{A} : \mathcal{A} \to \mathcal{A}$ defined by
\[ \tilde{A}_E(x) = A_E^{(N(x))}(x) = \begin{cases} A_E(x) & \text{if } x_0 = 0 \\ A_E^{(3)}(x) & \text{if } x_0 = a. \end{cases} \]

**Proposition 10.** For all $E \in \mathbb{R}$, $L(\tilde{A}_E) = \frac{3}{2} L(A_E)$.

**Proof.** The event $\mathcal{A}$ has probability $\mathbb{P}(\mathcal{A}) = \frac{2}{3}$. The induced measure $\mathbb{P}_\mathcal{A}$ on $\mathcal{A}$ is the conditional probability, $\mathbb{P}_\mathcal{A}(E) := \mathbb{P}(E|\mathcal{A}) = \frac{2}{3} \mathbb{P}(E \cap \mathcal{A})$. Hence the return time $N : \mathcal{A} \to \mathbb{N}$ has expected value
\[ \int_\mathcal{A} N \, d\mathbb{P}_\mathcal{A} = \frac{1}{\mathbb{P}(\mathcal{A})} = \frac{3}{2}. \]
Consider now the sum process $S_n : \mathcal{A} \to \mathbb{N}$, defined by $S_n(x) := \sum_{j=0}^{n-1} N(T_\mathcal{A}^j x)$. By the ergodicity of $(T, \mathbb{P})$ and $(T_\mathcal{A}, \mathbb{P}_\mathcal{A})$, for $\mathbb{P}$-almost every $x \in \mathcal{A}$,
\[ L(\tilde{A}_E) = \lim_{m \to +\infty} \frac{1}{m} \log \| \tilde{A}_E^{(m)}(x) \| = \lim_{m \to +\infty} \frac{1}{m} \log \| A_E^{(S_m(x))}(x) \| \]
\[ = \lim_{m \to +\infty} \frac{S_m(x)}{m} \lim_{m \to +\infty} \frac{1}{S_m(x)} \log \| A_E^{(S_m(x))}(x) \| \]
\[ = \left( \int_\mathcal{A} N \, d\mathbb{P}_\mathcal{A} \right) L(A_E) = \frac{3}{2} L(A_E). \]
This proves the proposition. $\square$
Therefore, by Theorem 1 and Proposition 1 the function $E$ by the single letter ‘1’. This map conjugates the return map $T_A : A \to A$ with the full Bernoulli shift $T : \{0, 1\}^\mathbb{Z} \to \{0, 1\}^\mathbb{Z}$. It also determines a conjugation between the family of cocycles $A_E$ over $T_A : A \to A$ and the family of random Bernoulli cocycles $A_E = (C(E), D(E))$ over the full shift $T : \{0, 1\}^\mathbb{Z} \to \{0, 1\}^\mathbb{Z}$ defined by the following matrices

$$C(E) = \begin{bmatrix} v(0) - E & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -E & -1 \\ 1 & 0 \end{bmatrix}$$

$$D(E) = \begin{bmatrix} v(c) - E & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v(b) - E & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v(a) - E & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -e - E & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{e} - E & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -e - E & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e - E p(E) & -E q(E) \\ E q(E) & \frac{1}{e} + E \end{bmatrix}$$

with $p(E) := E^2 + (2e + e^{-1}) E + e^2$ and $q(E) := E + e + e^{-1}$. For $E = 0$ we have $C(0) = C$ and $D(0) = D$ and hence $A_0$ coincides with the Kifer example (1.1). The family $A_E$ of random Bernoulli cocycles is analytic and has LE

$$L(\hat{A}_E) = L(\hat{A}) = \frac{3}{2} L(A_E).$$

Therefore, by Theorem 1 and Proposition 1 the function $E \mapsto L(\hat{A}_E)$ is not $(\gamma, \beta)$-log-Hölder continuous at $E = 0$ for any $\gamma > 1$. This proves Theorem 2.

7. Irreducible cocycles. In this section we consider random $SL_2(\mathbb{R})$ cocycles over a finite Bernoulli shift. Let $\Sigma = \{1, \ldots, s\}$ be a finite alphabet and fix some Bernoulli measure $\mathbb{P}_q = q^\Sigma$, where $q$ is a probability vector on $\Sigma$. Let $T : X \to X$ denote the two sided shift on the space of sequences $X = \Sigma^\mathbb{Z}$ endowed with the probability measure $\mathbb{P}_q$.

Recall that a random cocycle over the Bernoulli shift $T$ is defined by a locally constant measurable function $A : X \to SL_2(\mathbb{R})$, i.e., a function which depends only on the 0-th coordinate. This implies that $A$ is determined by a function $A : \Sigma \to SL_2(\mathbb{R})$, or, in other words, by a list of $s$ matrices $A_1, \ldots, A_s \in SL_2(\mathbb{R})$.

**Definition 3.** A random cocycle $A : \Sigma \to SL_2(\mathbb{R})$ is said to be irreducible if there is no point $L \in \mathbb{P}(\mathbb{R}^2)$ such that $A(x)L = L$ for all $x \in \Sigma$.

**Definition 4.** A random cocycle $A : \Sigma \to SL_2(\mathbb{R})$ is said to be strongly irreducible if there is no finite subset $L \subseteq \mathbb{P}(\mathbb{R}^2)$, $L \neq \emptyset$, such that for all $x \in \Sigma$, $A(x)L = L$.

Clearly, strongly irreducible cocycles are also irreducible. Irreducible cocycles which are not strongly irreducible will be referred to as simply irreducible cocycles.

**Remark 1.** Simply irreducible $SL_2(\mathbb{R})$-cocycles have zero Lyapunov exponents.

The following statement is a classical theorem of H. Furstenberg [13].

**Theorem 3.** Given a random cocycle $A : \Sigma \to SL_2(\mathbb{R})$ assume:

(a) $A$ is strongly irreducible,

(b) the sub-semigroup generated by the matrices $A_1, \ldots, A_s$ of the cocycle $A$ is not compact.
Then $L(A) > 0$.

The next proposition refers to the canonical $L^\infty$-norm: $\|A\|_\infty := \max_{x \in \Sigma} \|A(x)\|$.

**Proposition 11.** Let $A: \Sigma \to \text{SL}_2(\mathbb{R})$ be a strongly irreducible cocycle such that $L(A) = 0$. Then there exists $C = C(A) < \infty$ such that for any other cocycle $B: \Sigma \to \text{SL}_2(\mathbb{R})$,

$$|L(A) - L(B)| \leq C \|A - B\|_\infty.$$

**Proof.** By Theorem 3, the sub-semigroup $S \subset \text{SL}_2(\mathbb{R})$ generated by the matrices $A_1, \ldots, A_s$ of the cocycle $A$ must be compact. Every matrix $A \in S$ is elliptic, i.e., it has eigenvalues in the unit circle, because otherwise the matrix $A \in \text{SL}_2(\mathbb{R})$ would have an eigenvalue $\lambda$ with $|\lambda| > 1$ and the sub-semigroup $\{A^n : n \geq 0\}$ would not be compact. Thus

$$A = C^{-1} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} C$$

for some $\theta \in [0, 2\pi]$ and $C \in \text{SL}_2(\mathbb{R})$, and $A$ belongs to the compact group

$$H := \left\{ C^{-1} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} C : t \in \mathbb{R} \right\}.$$

Applying the Poincaré Recurrence Theorem to the translation map $T: H \to H$, $T(X) = AX$, we see that there exists a sequence of integers $n_k \to +\infty$ such that $A^{n_k}$ converges to the identity matrix as $k \to +\infty$. This proves that

$$A^{-1} = \lim_{k \to +\infty} A^{n_k-1}$$

belongs to $S$ and, therefore, that $S$ is a group.

All matrices in $S$ are orthogonal with respect to the following inner product

$$\langle u, v \rangle' := \int_S \langle Au, Av \rangle dA$$

where the integral refers to the Haar measure on $S$ and $\langle \cdot, \cdot \rangle$ stands for the canonical inner product on $\mathbb{R}^2$. Hence, denoting by $\|\cdot\|$ the corresponding operator norm (on the space of matrices) one has $\|A_j\| = 1$ for all $j = 1, \ldots, s$.

For any cocycle $B: \Sigma \to \text{SL}_2(\mathbb{R})$ we have

$$0 \leq L(B) = \lim_{n \to +\infty} \frac{1}{n} \mathbb{E}_q[\log \|B^{(n)}\|] \leq \mathbb{E}_q[\log \|B\|'].$$

Note that $\mathbb{E}_q[\log \|A\|'] = 0$ for the cocycle $A$. Hence

$$|L(B) - L(A)| = L(B) \leq \mathbb{E}_q[\log \|B\|']$$

$$\leq \mathbb{E}_q[\log (\|A\|' + \|B - A\|')]$$

$$= \mathbb{E}_q[\log (1 + \|B - A\|')]$$

$$\leq \mathbb{E}_q[\|B - A\|'] \leq C \|A - B\|_\infty$$

for any constant $C < \infty$ such that $\|M\|' \leq C \|M\|$, for all $M \in \text{Mat}_2(\mathbb{R})$, where $\|\cdot\|$ stands for the canonical operator norm on $\text{Mat}_2(\mathbb{R})$. $\Box$

**Remark 2.** Proposition 11 only proves the pointwise Lipschitz continuity of the LE at $A$, but it does not imply that the LE is always Lipschitz in a neighborhood of $A$. 
For the reader’s convenience, we collect in the two tables below a summary of the available results on the continuity of the LE for $\text{GL}_2(\mathbb{R})$ random cocycles. The positive results refer to a minimum modulus of continuity of the LE, while negative results give counterexamples where the modulus of continuity of the LE lies below a certain threshold.

| Cocycle class          | Positive results | Negative results |
|------------------------|------------------|------------------|
| Non diagonalizable     | [17, Théorèmes 1, 2], [3, Theorem 1] | [20, Appendix 3] |
| Diagonalizable         | [10, Theorem 1.2], [22, Theorem B] |                  |

**Table 3.** Quantitative results on the continuity of the LE for $\text{GL}_2(\mathbb{R})$ random cocycles with strictly positive LE

| Cocycle class          | Positive results | Negative results |
|------------------------|------------------|------------------|
| Strongly irreducible    | This paper, Proposition 11 |                  |
| Non strongly irreducible| [22, Theorem C]  | This paper, Theorem 2 |

**Table 4.** Quantitative results on the continuity of the LE for $\text{GL}_2(\mathbb{R})$ random cocycles with zero LE

We conclude this paper with the following question. Given a random $\text{SL}_2(\mathbb{R})$-cocycle under the assumptions of Proposition 11, is the LE always uniformly (as opposed to just pointwise) Hölder continuous in a neighborhood of that cocycle?

**REFERENCES**

[1] A. Avila, On the spectrum and Lyapunov exponent of limit periodic Schrödinger operators, *Comm. Math. Phys.*, **288** (2009), 907–918.

[2] A. Avila, Y. Last, M. Shamis and Q. Zhou, On the Abominable Properties of the Almost Mathieu Operator, in preparation.

[3] A. Baraviera and P. Duarte, Approximating Lyapunov exponents and stationary measures, *J Dyn Diff Equat*, **31** (2019), 25–48.

[4] C. Bocker-Neto and M. Viana, Continuity of Lyapunov exponents for random two-dimensional matrices, *Ergodic Theory Dynam. Systems*, **37** (2017), 1413–1442.

[5] W. Craig, Pure point spectrum for discrete almost periodic Schrödinger operators, *Comm. Math. Phys.*, **88** (1983), 113–131.

[6] W. Craig and B. Simon, Log Hölder continuity of the integrated density of states for stochastic Jacobi matrices, *Comm. Math. Phys.*, **90** (1983), 207–218.

[7] D. Damanik, Schrödinger operators with dynamically defined potentials, *Ergodic Theory Dynam. Systems*, **37** (2017), 1681–1764.

[8] P. Duarte and S. Klein, Lyapunov Exponents of Linear Cocycles, Continuity via large deviations. Atlantis Studies in Dynamical Systems, 3. Atlantis Press, Paris, 2016.

[9] ——, Continuity of the Lyapunov Exponents of Linear Cocycles, *Publicações Matemáticas, 31º Colóquio Brasileiro de Matemática*, IMPA, 2017, [https://impa.br/wp-content/uploads/2017/08/31CBM_02.pdf](https://impa.br/wp-content/uploads/2017/08/31CBM_02.pdf)

[10] ——, Large deviations for products of random two dimensional matrices, preprint, 2018.

[11] H. Furstenberg and H. Kesten, Products of random matrices, *Ann. Math. Statist.*, **31** (1960), 457–469.

[12] H. Furstenberg and Y. Kifer, Random matrix products and measures on projective spaces, *Israel J. Math.*, **46** (1983), 12–32.

[13] H. Furstenberg, Noncommuting random products, *Trans. Amer. Math. Soc.*, **108** (1963), 377–428.

[14] M. Goldstein and W. Schlag, Hölder continuity of the integrated density of states for quasiperiodic Schrödinger equations and averages of shifts of subharmonic functions, *Ann. of Math.*, (2) **154** (2001), 155–203.
A RANDOM COCYCLE

[15] Y. Kifer, Perturbations of random matrix products, Z. Wahrsch. Verw. Gebiete, 61 (1982), 83–95.
[16] H. Krüger and Z. Gan, Optimality of log Hölder continuity of the integrated density of states, Math. Nachr., 284 (2011), 1919–1923.
[17] É. Le Page, Régularité du plus grand exposant caractéristique des produits de matrices aléatoires indépendantes et applications, Ann. Inst. H. Poincaré Probab. Statist., 25 (1989), 109–142.
[18] E. C. Malheiro and M. Viana, Lyapunov exponents of linear cocycles over Markov shifts, Stoch. Dyn., 15 (2015), 1550020, 27pp.
[19] J. Pöschel, Examples of discrete Schrödinger operators with pure point spectrum, Comm. Math. Phys., 88 (1983), 447–463.
[20] B. Simon and M. Taylor, Harmonic analysis on $\text{SL}(2,\mathbb{R})$ and smoothness of the density of states in the one-dimensional Anderson model, Comm. Math. Phys., 101 (1985), 1–19.
[21] N. J. A. Sloane and S. Plouffe, The Encyclopedia of Integer Sequences, Academic Press, Inc., San Diego, CA, 1995.
[22] E. H. Y. Tall and M. Viana, Moduli of Continuity for Lyapunov Exponents of Random GL(2) Cocycles, preprint, 2018, http://w3.impa.br/~viana/out/holder.pdf.
[23] T. Tao, Topics in Random Matrix Theory, Graduate Studies in Mathematics, vol. 132, American Mathematical Society, Providence, RI, 2012.
[24] M. Viana, Lectures on Lyapunov Exponents, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2014.

Received November 2018; revised February 2019.

E-mail address: pmduarte@fc.ul.pt
E-mail address: silviusk@impa.br
E-mail address: manuel.batalha.dos.santos@ist.utl.pt