Fractal Space Time and Variation of Fine structure Constant

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The effect of fractal space time of the quantum particles on the variation of the fine structure constant $\alpha$ has been studied. The variation of fine structure constant has been investigated around De Broglie length $\lambda$ and compton length $\lambda_c$ and it has been suggested that the variation may be attributed to the dimensional transition of the particle trajectories between these two quantum domains. Considering the Fractal universe with a small inhomogeneity in the mass distribution in the early universe, the variation of the fine structure constant have been investigated between matter and radiation dominated era. The fine structure constant shows a critical behaviour with critical exponent which is fractional and shows a discontinuity. It has been suggested that the variation of the fine structure constant may be attributed to the intrinsic scale dependance of the fundamental constants of nature.

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The possibility of variation of fundamental constants of nature has been a long standing problem and widely addressed by a number of authors [1] including Dirac himself [2]. It has been suggested that the fine structure constant \( \alpha = \frac{e^2}{c \hbar} \) varies with the age of the universe and the idea get a new impetus after the discovery of CMB data and some cosmological observation of data from supernova. With the progress of the observational cosmology the experimental verification of the suggestions seems to be plausible. Recent observations of the distant quasars have suggested that the fine structure constant varies with cosmological time scale and the variation is \( \approx \frac{d\alpha}{\alpha} \approx 10^{-5} \) [4] over the time period since the emission of the quasar light. Recent experimental observation from Keck telescope has suggested that \( \alpha \) varies with the direction and it is not same all over the universe giving an indication that the law of physics are not same everywhere [5]. The problem like horizon, flatness can be resolved if it can be considered that speed of light was faster in the past. Huey et al [6] studied CMB anisotropy spectrum dependence on fine structure constant and equation of state (EOS) of the dark energy component of the total energy of the universe. They have pointed out that the varying \( \alpha \) can be partially compensated by adjusting the EOS of dark energy and obtained \( \frac{\alpha - \alpha_0}{\alpha_0} = \frac{\Delta \alpha}{\alpha_0} = 10^{-2} \) to \( 10^{-3} \) where \( \alpha_0 \) is the current value. Barrow et al [7] have studied the dynamics of varying \( \alpha \) theories by introducing an exponential or inverse power law for scalar field which allows the time variation of \( \alpha \). They have observed \( \frac{\dot{\alpha}}{\alpha_0} \approx 1.2.10^{-16}/\text{yr} \). In a subsequent work [8]they have studied the red shift dependence of \( \alpha \) analyzing supernova results. They have observed that during matter dominated epoch VSL mechanism works where in the matter dominated era the varying \( c \) effect switched off allowing \( \Lambda \) to eventually surface resulting a accelerating universe. They have found that the residual variation in \( c \) induces a variation in \( \alpha \) which agrees with the observation. Anchordoqui et al [9] have investigated a cosmological model which includes variation of \( \alpha \) in short time scale and observed a phase transition in neutrino mass at a red shift \( z=0.5 \) which induces a phase transition in \( \alpha \). Moffat [10] has studied the cosmological evolution of \( \alpha \) analyzing the absorption spectra of distant quasars considering a cosmological scenario where the speed of light varies. Varying \( \alpha \) usually attributed to the varying speed of
light whereas varying $e$ is a less radical approach. Bakenstein [11] studied a theory which gives a varying $e$ which preserves the local gauge and Lorentz invariance and generally covariant. Berman et al [12] investigated the time variation of $\alpha$ due to varying electrical and magnetic permittivities. They have argued that the present accelerating universe and exponential inflation may be correlated with that variation. Sanvik et al [13] have investigated varying $\alpha$ with varying $e$. They have observed that $\alpha$ remains constant in radiation era, undergoes a small change in matter dominated era and approaches a constant value when universe starts accelerating due to positive cosmological constant. Cingoz et al [14] have presented direct measurement of temporal variation of $\alpha$ with atomic Disposium (Dy). The result shows a fractional variation of $(2.7 \pm 2.6) \times 10^{-15}$ without assumption of constancy of other fundamental constants. Uzan [15] has made an review on varying $\alpha$ and indicated that the variation may induce a new cosmological problem as varying $\alpha$ cannot be naturally explained from field theoretical approach.

In the present work we have investigated the variation of $\alpha$ both in the context of the microphysics and cosmology. As we go below the classical radius of electron the nature of interaction changes and consequently variation in $\alpha$ is induced. We have studied the variation of $\alpha$ in the quantum domain near De Broglie length ($\lambda$) and the relativistic domain near the Compton length ($\lambda_c$) for electron. It has been suggested that the variation in $\alpha$ may be attributed to the fractal character of the space-time of trajectories of the quantum particles. The time variation of $\alpha$ between radiation dominated era and matter dominated era has also been investigated in the framework of the fractal universe studied by Banerjee et al [16] where small inhomogeneity has been incorporated through a mass fractal dimension ($d$) in the density distribution of FRW universe. It has been suggested that the scale dependence and power law behaviour of the fundamental constants may be the manifestation of fractal character of the space-time indicating the fact that geometry plays a fundamental role in describing an interaction.
Variation of Fine structure constant in Quantum Domain—a Fractal Transition:

The De Broglie wavelength for an electron is given by:

\[ \lambda = \frac{h}{mv} \]  

(1)

The Compton wavelength runs as:

\[ \lambda_c = \frac{h}{2\pi mc} \]  

(2)

From (1) and (2) we get,

\[ \lambda_c = \frac{\lambda}{\lambda c} \]  

(3)

The first Bohr orbit can be expressed as:

\[ a_0 = \frac{\lambda c}{2\pi \alpha} \]  

(4)

from (3) and (4) we obtain:

\[ \alpha = \frac{\lambda}{4\pi^2 a_0} \left( \frac{v}{c} \right) \]  

(5)

It is well known that the trajectory of a quantum particle is irregular in the fine scale and the path of the quantum mechanical particle is non differentiable. Feynman path integral approach [17] is a come back to the real situation of the quantum particle trajectories whereas in Bohr concept the trajectory has been completely abandoned. Feynman and Hibbs pointed out that with \( \epsilon = \delta t \) is the resolution:

\[ < \frac{x_{k+1} - x_k}{\epsilon} >^2 = \frac{h}{im\epsilon} < l > \]  

(6)

The above expression (6) can be interpreted as that the transition element of the square of the velocity varies as \( 1/\epsilon \) and thus tends to infinity as \( \epsilon \rightarrow 0 \) which is characteristic feature of a fractal measure. The expression (6) shows that the velocity of the particle is purely quantum relation not classical any more and \( < v^2 > \) tends to zero as \( \epsilon \) tends to infinity [18].

Now from the expression (6) we get:

\[ \frac{dx}{dt} = < v^2 >^{1/2} = (\delta t)^{-1/2} \]  

(7)
We can recast the above expression as:

\[ \frac{dx}{dt} = (\delta t)^{\frac{1}{D}} \]  \hspace{1cm} (8)

where \( D \) is the fractal dimension. According to the fractal definition the velocity is characterized by \((dx/dt) \propto (\delta x)^{-1}\) [18]. The derivative of the fractal function diverges like \(((\delta x)^{D_T-D})\) which in turn can be represented as \((\delta t)^{\frac{1}{D}}\) depending on the choice of the variable which defined the resolution and \( D_T \) is the topological dimension. From expression (7) the fractal dimension of a quantum trajectories is identified as \( D=2 \) whereas the topological dimension is \( D_T=1 \) and as the particle undergoes transition from the classical domain to quantum domain the the transition may be explained as a consequence of a transition from non-fractal domain \( D=1 \) to fractal domain \( D=2 \). Now Equation (5) can be recast as:

\[ \alpha = \left(\frac{\lambda}{4\pi^2a_0c}\right)(\delta t)^{\frac{1}{D}} \sim (\delta t)^{-\frac{1}{2}} \]  \hspace{1cm} (9)

so that the first derivative of \( \alpha \) varies as:

\[ \dot{\alpha} = A'(\delta t)^{-\frac{3}{2}} \]  \hspace{1cm} (10)

Around the de Broglie length the fine structure constant varies as \(\delta t^{-\frac{1}{2}}\) as the particle undergoes a transition from classical to quantum domain. The time variation of \( \alpha \) shows a fractal character with \( \dot{\alpha} \to 0 \) as the resolution \( \delta t \) tends to zero.

In the non-relativistic domain around the compton wave length the momentum of the quantum relativistic path varies like \( p \propto (\delta x)^{-2} \) where \( \delta x \propto \lambda_c [18] \) indicating a fast transition from quantum mechanical to quantum relativistic domain which leads to a fractal dimension 3 for the particle trajectory. With the similar argument as discussed above we get:

\[ \alpha \sim (\delta x)^{D_T-D} \sim (\delta t)^{\frac{1}{D}} \sim (\delta t)^{\frac{1}{2}} \sim (\delta t)^{-\frac{1}{2}} \]  \hspace{1cm} (11)

so that,

\[ \dot{\alpha} \sim (\delta t)^{-\frac{5}{2}} \]  \hspace{1cm} (12)
Showing fractal behavior as in the quantum domain. So it has been observed that the transition from non relativistic to relativistic domain represent a transition from fractal space of dimension of 2 to a fractal dimension of 3. The critical exponent for the time variation of $\alpha$ have been obtained as $3/2$ and $5/3$ respectively for two domains and shows a discontinuity. It is interesting to note that the loss of notion in the position should be described by new transition to different dimensional space. It would be interesting to point out here that according to Feynman prescription the trajectory back in time is interpreted as the negative particles production. The dependence on the resolution or the non differentiable path may be interpreted as virtual particle creation and its contribution.

**Variation of Fine structure constant in Fractal universe:-**

The FRW metric for homogeneous universe and Einstein Equation is given by [19]:

$$\left[\frac{dR(t)/dt}{R}\right]^2 = \frac{4\pi G \rho}{3} - \frac{K}{R(t)^2} + \frac{\Lambda}{3}$$  \hspace{1cm} (13)

where the symbols have their usual meanings. Assuming $\Lambda$ to be zero the above equation can be recast as:

$$\frac{\dot{R}^2}{2} = \frac{G \rho 4\pi R^2}{3R} = \frac{-K}{2}$$  \hspace{1cm} (14)

Assuming $K=0$ for a Euclidean universe, the energy conservation demands that:

$$\frac{d(\rho R^3)}{dt} + p \frac{d(R^3)}{dt} = 0$$  \hspace{1cm} (15)

or

$$\frac{(\rho R^3)}{dR} = -3pR^3$$  \hspace{1cm} (16)

where p is isotropic pressure. We have introduced a small inhomogeneity in the mass distribution of the early universe for large value of R where the energy density varies through $\rho(R) \sim R^{-d-3}$ where d is the mass fractal dimension and lies between $0 < d < 1$ for matter (d=0) and radiation dominated era (d=1) respectively in our previous work [16] so that we have obtained:

$$R(t) \sim t^3$$  \hspace{1cm} (17)
where $\beta = 2/(d+3)$ and

$$H = \beta t^{-1}$$  \hspace{1cm} (18)

$H$ is the Hubble factor with the conventional nomenclature of present time $t$ as in Weinberg [20] where $t_0 < H_0$, $H$ is the Hubble factor. In the context of the effective field theory and M theory, the change of the fine structure constant is obtained by coupling the dynamical scalar field $\phi$ to the photon kinetic term in the low energy effective action. Moreover to study the cosmology of the field $\phi$, it has been assumed that $\phi$ is governed by a lagrangian $L = (\delta \phi)^2 - V(\phi)$ where the potential energy is give by generic form $\mu^4 f(\phi/M)$ [21]. $\phi$ and $M$ are microphysical parameters. Considering slow time variation of $\alpha$ the expression for $d\alpha^{-1}$ can be recast as [21]:

$$d\alpha^{-1} = \frac{4\pi\epsilon\mu^4}{3M^2} \int dt \frac{f'(\phi/M)}{H}$$  \hspace{1cm} (19)

With $\phi/M$ approximately constant during matter dominated era we may recast the above equation as:

$$d\alpha^{-1} = K/\beta \int dt/H$$  \hspace{1cm} (20)

Now using the time dependance of Hubble parameter from the expression (18) obtained above we get,

$$d\alpha^{-1} = \frac{K}{2\beta} t^2$$  \hspace{1cm} (21)

Integrating above expression we get,

$$\alpha \sim \frac{2}{K(d+3)} t^{-3}$$  \hspace{1cm} (22)

where $\beta = 2/d+3$. The above expression shows the variation of $\alpha$ in the mixed phase of matter and radiation dominated era in the limit $0 < d < 1$. We have observed that $\alpha$ varies very sharp with time in the matter dominated era. It is interesting to observe that $\alpha$ shows power law behaviour with critical exponent $3$.

**Discussions and conclusions:** In the present work we have investigated the variation of fine structure constant both in microphysics and in the early universe. We have tried
to understand how the nature of the space time geometry particularly fractal space affects the variation $\alpha$ in the context of electrodynamics and different phases of early universe. Recently Culetto et al [22] studied the role of fractal geometry in scaling the fundamentals of electrodynamics. They have come across an expression for $\alpha$ which relates Feigenbaum’s universal number and Thue-Morse constant. They have argued that the involvement of these constant points towards a digital regime at the infinitesimal level and one could not escape geometrization. They have also pointed out that the charge quantization may be tied to the fractal geometry. Nottale [18] has shown that the fine structure in electrodynamics become logarithmically divergent below compton length and this new behaviour has been attributed to the electron positron pair creation and annihilation which mainly occurs as expected from Heisenberg relation. Naschie et al [23] have derived an expression for fine structure constant from $\xi(\alpha)$ theory and pointed out that $\alpha$ can be interpreted as quasi geometrical probability and $\frac{1}{\alpha}$ is described as Hausdorff dimension of corresponding subspace. Goldfain et al [24] have derived fine structure constant considering the fractional time evolution of stochastic electrodynamics in the asymptotic limit of QED which is defined as the large fluctuation limit of all relevant variables. They have pointed out that the loss of characteristics scale, typical anomalous diffusion and complex behaviour is linked with the theory of critical phenomena. In their work fine structure constant has been found to emerge from the fractional evolution of density matrix whereas Chang et al [25] have proposed that the Finsler space-time can account the fine structure constant variation obtained from quasar spectra and suggested that the variation can be attributed to the space time inhomogeneity and anisotropy. In the current work from the dimensional analysis it has been found that the fine structure constant becomes function of the resolution and obey a power law behaviour diverging with the resolution in the quantum domain which may be attributed to the creation of virtual pairs as the non differentiability of the trajectory of the quantum particle means the new particle creation with the Feynman prescription. In fractal space the trajectories are completely non differentiable and there is no lower cut off below which fractralization would stop. Hence there is always structure whatever the scale may be and one never reaches the non structured
limit that is assumed in standard theory. This reopens the hope that the internal quantum numbers are nothing but very internal structure of the particle trajectory. However Cannatan et al [26] have pointed out that the transition from quantum to relativistic quantum domain occurs from D=2 to D=1 again not in D=3 which means again coming back to non-fractal space time in the relativistic domain.

The variation of $\alpha$ in the early universe has been studied and it has been observed that the mass fractal dimension do not have much significant role. However it has been observed that the time variation of $\alpha$ obey an critical behaviour with critical exponent 3 in matter dominated era. BSBM [11] model predicts $\alpha \sim t^{1/2}$ in radiation era and $\log t$ in the matter dominated era predicting a constant value in accelerating era. Barrow et al [7-8] have made a detailed discussions on time variation of $\alpha$. Moffat [10] has obtained similar type of relation for $\Delta \alpha /\alpha(0)$ and considered the exponent as 1.5.

In the present work we have studied consequences of fractal space in the variation of fine structure constant and found that it may be termed as running coupling constant embedding the infinite distance limit characteristics. This may be the manifestation of the fact that the set has structure at every minute level and we can never end up without any structure however small the scale may be. The fractional charge may be also be related to the fractal path of the quantum particles [22]. It has been further suggested that the production of particle -anti particles may be the manifestation of intrinsic fractal behaviour of space in quantum domain. Starting from the cosmological scale to microphysics the variation of $\alpha$ represents the scale relativity which is supposed to be the fundamental law of nature and the knowledge of detailed space time structure is unavoidable in our understanding of fundamental laws of Physics. However it may be pointed out here that the study of the variation of the fine structure constant needs the study of a domain which includes cosmology, astrophysics, high energy physics. Much more theoretical and observational efforts are needed to understand the origin of the variations of the fundamental constants of nature.

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