Quantum Speed Limit and Optimal Control of Many-Boson Dynamics

Ioannis Brouzos,1 Alexey I. Streltsov,2 Antonio Negretti,3 Ressa S. Said,1 Tommaso Caneva,4 Simone Montangero,1 and Tommaso Calarco1
1Center for Integrated Quantum Systems, Universität Ulm, D-89069 Ulm, Germany
2Theoretische Chemie, Physikalisches Institut, Universität Heidelberg, D-69120 Heidelberg, Germany
3Zentrum für Optische Quantentechnologien and The Hamburg Centre for Ultrafast Imaging, Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, Germany
4ICFO-Institut de Ciencies Fotoniques, Mediterranean Technology Park, S-08860 Castelldefels (Barcelona), Spain

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We extend the concept of quantum speed limit – the minimal time needed to perform a driven evolution – to complex interacting many-body systems. We investigate a prototypical many-body system, a bosonic Josephson junction, at increasing levels of complexity: (a) within the two-mode approximation corresponding to a nonlinear two-level system, (b) at the mean-field level by solving the nonlinear Gross-Pitaevskii equation in a double well potential, and (c) at an exact many-body level by solving the time-dependent many-body Schrödinger equation. We propose a control protocol to transfer atoms from the ground state of a well to the ground state of the neighbouring well. Furthermore, we show that the detrimental effects of the inter-particle repulsion can be eliminated by means of a compensating control pulse, yielding, quite surprisingly, an enhancement of the transfer speed because of the particle interaction – in contrast to the self-trapping scenario. Finally, we perform numerical optimisations of both the nonlinear and the (exact) many-body quantum dynamics in order to further enhance the transfer efficiency close to the quantum speed limit.

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The maximum speed of evolution of a quantum system has been a subject of theoretical investigation since decades. This research activity – mainly focused on state-to-state transfer for single-particle quantum dynamics – resulted in the introduction of the concept generally known as quantum speed limit (QSL), namely the minimum time needed to achieve a certain quantum state transformation [1, 2]. In the last few years, this fascinating theoretical concept, which arises from an energy-time uncertainty relation, has started to have practical relevance especially for the development of quantum technologies and because quantum gas laboratories are now capable of probing it [3, 4]. Indeed, recent experiments with cold atoms on a chip reported the achievement of time-optimal processes at the QSL [4]. However, the QSL for nonlinear and many-body quantum systems has not been clearly defined yet and, in particular, the influence of the inter-particle interaction on the QSL has to be still clarified. Here, we provide an exhaustive theoretical analysis on this subject and we apply it to the relevant scenario of ultracold bosons in a double well that has been realized [5, 6] and controlled [7] experimentally. We characterize the QSL of nonlinear and many-body systems of bosons, and show that optimal control might be effectively applied to achieve time-optimal transformations. In particular, we show that optimal control strategies can be developed to cancel the decelerating effects of the repulsive atom interactions. These strategies differ substantially in the vicinity of the QSL in such a way that the combination of an exact many-body quantum description and optimal control is instrumental in order to achieve the best performance.

In our study different degrees of complexity in the description of the system are gradually considered (see Fig. 1). (a) We analyse the optimal dynamics for a Bosonic Josephson Junction (BJJ) [8, 9] within the two-mode approximation, where one mode (i.e., orbital) per well, solution of the time-independent Gross-Pitaevskii Equation (GPE), is assumed [10]. (b) We investigate the mean-field scenario for which the time-dependent GPE dynamics in the double well potential in coordinate space is solved, and finally (c) we solve the full many-body Schrödinger equation by means of the ab initio Multi-Configurational Time-Dependent Hartree method for Bosons (MCTDHB) [11]. The latter approach allows us to include possible excitations to higher orbitals (i.e., beyond the GPE), which are responsible for quantum depletion and fragmentation.

In a linear two-level system the magnitude of the coupling $J$ between the two levels characterizes the QSL of a state-to-state transfer. Hence, the minimum time for the transfer between two orthogonal states is given by half the Rabi period, that is, $T_{QSL}^L = \pi \hbar / 2J [1, 2]$. Such time-optimal dynamics occurs along a geodesic on the Bloch sphere connecting the initial and the target state [see Fig. 1(d)]. Our main goal is to extend the geodesic interpretation of the QSL to interacting bosonic many-body quantum systems. At the beginning we show that at the level of the two-mode BJJ model, which in the presence of inter-particle interactions becomes nonlinear [12], the time-optimal dynamics does follow a geodesic too. This can be achieved by canceling the deviations from
the dynamics of the bosonic Josephson junction and analytically computable Compensating Control Pulses (CCP). We then study the range of validity of the CCP theory for a general Liouvillian evolution [18, 19].

The Hamiltonian of a linear two-level system reads: \( H(t) = \hbar J_x \sigma_x + hD(t)\sigma_z \), where the drift term expresses the coupling strength \( J \) between the two levels and \( D(t) \) is the (controllable) detuning. A prototypical process that can be driven in such a system is a state-to-state transfer [7, 16]. In Fig. 1 we represent the quantum state \( |\psi(t)\rangle = \cos(\theta(t)/2)|\psi_0\rangle + \sin(\theta(t)/2)e^{i\phi(t)}|\psi_T\rangle \) on a Bloch sphere. The speed of a quantum state moving in the Hilbert space is: \( \dot{s}(t) = 2\Delta H(t)/\hbar \), where \( \Delta H(t) = \sqrt{(H^2(t))} - (H(t))^2 \), and \( s \) is the geodesic distance between two states \( |\psi\rangle, |\chi\rangle \), defined as \( |\langle\chi|\psi\rangle|^2 = \cos^2(s/2) \) [20]. In a two-level system, the speed \( \dot{s}(t) = \sqrt{\dot{\theta}(t)^2 + \sin^2(\theta(t))\dot{\phi}(t)^2} \) has two components: \( \dot{\theta}(t)/J = \sin(\phi(t)) \), which drives the state along the meridians of the Bloch sphere, and \( \dot{\phi}(t)/J = D(t) + \cot(\theta(t))\cos(\phi(t)) \) along the parallels. In order to reach the time-optimal transfer between north and south pole of the Bloch sphere (see also Fig. 1), only the speed term \( \dot{\theta}(t) \) should be maximized: that is, \( \dot{\theta} = J \) at \( \phi(t) = \pm \pi/2 \), which implies \( D(t) = 0 \). Thus, the pulse given by the condition \( D(t) = 0 \) minimizes the infidelity (i.e., \( \varepsilon \to 0 \)), the transfer time \( T_{\text{QSL}} \), and the path length \( S = \int \dot{s} dt = \pi \) following the geodesic \( \phi(t) \) is constant. In case constraints are present on the control pulse, e.g., certain initial and final values of the detuning are assumed and can not be changed instantaneously, the optimal strategy is still the one that fulfills as much as possible the condition \( |D| = 0 \), as shown numerically in Ref. [16]. Here we have assumed that \( J \) is time-independent, since, if \( J \) can be varied, the time optimal solution is trivially the one that maximizes its value, and consequently minimizes \( T_{\text{QSL}} \).

Everything we have discussed so far applies to time-dependent or time-independent, linear or nonlinear.
Hamiltonians, provided that the coupling term $J$ is $\phi$-independent. For instance, the BJJ Hamiltonian has an additional term with respect to the linear case $H_L$, that is, $H_{BJJ} = H_L + [U - \Delta U \cos \theta(t)] \| + [\Delta U - U \cos \theta(t)] \sigma_z$, where $U$ is the nonlinear interaction strength and $\Delta U$ the interaction detuning between the two modes (see Supplemental Material for details) [10]. The nonlinear terms affect the speed on the parallels so that $D(t)$ has to be replaced by $\Delta(t) = D(t) + \Delta U + U \cos \theta(t)$. Now even for $D(t) = 0$ there is a non zero speed component that drives the state out of the geodesic path ($U \neq 0$ implies $\phi \neq 0$), as shown by the trajectory depicted in Fig. 1 (red line).

Now, in order to recover the time-optimal trajectory on a geodesic (green line) we impose the condition $\Delta(t) = 0$ which here translates to [21]:

$$D(t) = -\Delta U - U \cos \theta(t).$$

(1)

Finding the solution of this equation appears not trivial since it is a nonlinear problem: $\cos \theta(t)$ depends on the parameters of the problem ($D(t)$, $U$, $\Delta U$) and the time evolution of $\phi(t)$ itself. However, from the solution of the linear case, we know that if $\Delta(t) = 0$ the system will move on a geodesic with maximum speed in a Rabi oscillation, i.e. $\cos \theta(t) = \cos^2(Jt)$. In the following we will refer to this pulse as compensating control pulse and investigate numerically its effectiveness and robustness.

Numerical results. We have verified numerically, within the two-mode approximation, that the CCP drives the system along a geodesic, efficiently canceling the nonlinear interactions, and therefore achieving the full transfer at $T_{QSL}$, as it is also shown in Fig. 1 (green path). If there are constraints on the pulse, e.g. boundary conditions on the initial and final values of $D(t)$, then the condition of Eq. (1) cannot be fulfilled at every time, as required by the CCP scheme. Nevertheless, one could then devise an optimal control pulse by means of numerical optimisation. A typical result is reported in Fig. 1 (blue path) which has been obtained by using the CRAB algorithm. Note that the dynamics generated by the constrained optimal control pulse approximately follows a trajectory similar to the one induced by the CCP, namely moving at any time as close as possible to a geodesic.

We now go beyond the two-mode approximation by applying the previous findings to a Bose-Einstein condensate (BEC) in a one-dimensional double-well potential, the most common experimental realization of an atomic BJJ [5–7]. In those setups it is possible to vary in time the tilt of the double well, which corresponds to controlling $D(t)$ in the two-mode model. Our goal is to obtain a complete transfer of the BEC, initially prepared in the left well of the double well potential, to the right well. In particular, we consider a system composed by $N = 100$ interacting bosons trapped in a double well consisting of two separated harmonic traps of equal frequency $\omega$ and length $\ell = \sqrt{\hbar/\mu \omega}$ [22] (see Fig. 1 and the Supplemental Material technical details). We have then applied the different control schemes introduced before (CCP/CRAB), for various interaction strengths $N_g$, transfer times $T$, and approaches to solve the many-body quantum dynamics (GPE/MCTDHB).

To begin with, we consider the GPE dynamics and in Fig. 2 we compare the infidelity for a symmetric (unbiased) double well potential (red curves) within the scenario where the CCP on the tilt has been applied (green curves), as a function of the transfer time $T$ as well as for various interaction strengths $N_g$. The time-independent scenario confirms the standard BJJ theory, which predicts a Rabi oscillation (on the geodesic) for vanishing interaction strength (black line), a full but delayed transfer for weak interactions $N_g = 0.2(\hbar \omega \ell)^{-1}$, and a nonlinear self-trapping regime, where less than half of the atoms are transferred to the right well, for strong interaction strengths $N_g = 0.5(\hbar \omega \ell)^{-1}$ (see also Supplemental Material). The situation changes drastically when we apply the CCP: for all cases the infidelity follows a $\cos^2(JT)$ behavior as all the fitting green lines in Fig. 2 show. These results demonstrate that the CCP drives the system along the geodesic, resulting in a successful time-optimal transfer within the GPE dynamics, for all values of $N_g$. Moreover, the CCP not only compensates the nonlinearity recovering the linear single-particle dynamics (black line in Fig. 2), but it increases the speed of the transfer as demonstrated in the inset of Fig. 2:
a perfect geodesic trajectory (green curves). We many-body Schrödinger equation in order to conclude that the geodesic behavior obtained by state transfer. We have applied ab-initio MCTDHB[22] it has been demonstrated that the deviations from the GPE predictions for all time scales, and especially for strong interactions, might occur. Indeed, deviations induced by many-body fragmentation and depletion depend on several factors: number of atoms, interaction strength, dimensionality, total time evolution, strength of the control driving, potential profile. In Fig. 2 one can clearly see that the undriven dynamics calculated with MCTDHB (red cross and plus symbols) do not have significant deviations from the red curves obtained via GPE. However, when we apply the CCP, MCTDHB (green cross and plus symbols) and the GPE dynamics agree only for short time scales, while close to the QSL [see also Fig. 3(b)] and for strong interaction strengths there is a substantial disagreement (of up to one order of magnitude). This discrepancy occurs as for long times, strong driving, and strong couplings the fragmentation becomes not negligible (depletion of the condensate orbital of the order of 10%). Hence, for these particular situations we need to rely on optimal control in order to further improve the transfer efficiency.

To this aim, we applied the CRAB optimization to the MCTDHB simulation for transfer times in the vicinity of the QSL. We succeeded to reduce the infidelities from ε ≈ 10% to ε ≈ 1% (see Fig. 3(b) blue points). In the inset of Fig. 3(b) we present representative optimal pulses for the tilt of the double well close to the QSL. As it is shown, there is a large difference between CRAB optimizations performed with the GPE or exact many-body MCTDHB simulations.

In conclusion we have shown that the repulsive interaction between the particles is responsible for slowing down the dynamics of a bosonic Josephson junction, and also precludes a highly efficient state transfer. We have provided the conditions to design a compensating control pulse that eliminates the detrimental effects of the interaction and drive the system on a geodesic path at the quantum speed limit. Our driving scheme allows for an increase of the transfer speed for stronger interaction strengths as it exploits the constructive effects of the interaction. Close to the QSL, and for large interaction strengths, the mean field description does not accurately account for the dynamics, and a simple compensating control pulse strategy is not sufficient to achieve high transfer efficiency. Given this, we have to resort to optimal control of the many-body Schrödinger equation, which can be efficiently solved by the ab-initio MCTDHB.
method. Our study introduced the concepts to characterize the QSL for many-body systems at all levels of complexity of the description of the problem (two-mode, nonlinear mean-field, full many-body), and provided optimal control schemes to achieve the state-to-state transfer at minimum time. Our findings enable the investigation and experimental realization of different protocols at the QSL for many-body systems in different settings and experimental setups with current technology.

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Full many-body Hamiltonian and MCTDHB method.

The Hamiltonian describing a one-dimensional bosonic many-body ensemble with contact interaction reads:

\[ H = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N} V(x_i,t) + g_{1D} \sum_{i<j} \delta(x_i-x_j). \]  

For the sake of numerical convenience, we consider an external potential formed by two harmonic oscillators \( V_\perp(x) = \frac{1}{2} m \omega_\perp^2 (x^2 + 2) \) like in Ref. [1]. Here \( V_\perp \) denotes the potential for \( x < 0 \) \( (x > 0) \). The contact interaction potential is characterised by the interaction strength \( g_{1D} = 2 \hbar \omega_\perp a_{s_D}^2 C \), where \( C \) is defined in the seminal paper of Olshanii [2]. Here \( \omega_\perp \) is the transverse trap frequency and \( a_{s_D} \) is the three-dimensional s-wave scattering length, controllable parameters in the experiment via Feshbach or confinement induced resonances. In the following we rescale the energies by \( \hbar \omega_\perp \) and the lengths by the oscillator length \( \ell \equiv \sqrt{\hbar/m \omega_\perp} \). Thus, the interaction strength is rescaled as \( g = g_{1D}/\hbar \omega_\perp \ell \).

Initially \( N = 100 \) bosons are prepared in the ground state of the left well \( V_\perp \) potential (which extends then in the whole \( x \) range) \( |\psi_0\rangle = |L\rangle \). At time \( t = 0 \) the time-evolution starts in the full double well potential. The target state is the ground state of the right well \( V_\perp \) potential \( |\psi_T\rangle = |R\rangle \). Those states are only approximately orthogonal (\( \langle L|R \rangle^2 \approx 10^{-4} \)) and their exact profile (and overlap) depends on the interaction strength as displayed in Fig.1(a) of the paper: stronger interactions lead to broadening the wavefunction profile and increase of the overlap. The uncontrolled BJJ dynamics take place in the symmetric full potential \( V(x) \) while the control is applied as \( V_\perp(x) = (x \pm 2)^2/2 + E_\perp(t) \). The detunings \( E_\perp(t) \) of the local energy of each potential allow for a simple translation of our CCP control scheme which we induce from the two-mode dynamics as we show below.

To solve the full many-body time-dependent Schrödinger equation, we apply the MCTDHB method as in Ref. [1]. The main idea of the method is to express the many-body wave function as a linear combination of time-dependent permanents [3]:

\[ |\Psi(t)\rangle = \sum_{\vec{n}} C_{\vec{n}}(t) |\vec{n},t\rangle. \]  

The expansion coefficients \( C_{\vec{n}}(t) \) and the corresponding orbitals needed to construct the permanents \( |\vec{n},t\rangle \) are determined by the Dirac-Frenkel time-dependent variational principle. When the initial state is specified we propagate the MCTDHB equation of motions and compute the evolution of the full many-body wave function. Within this approach at every time-slice we can diagonalize the reduced one-body density matrix and express it in terms of the \textit{natural} orbitals (NO) and occupations, i.e., its eigenstates \( \phi_k^{NO}(x,t) \) and eigenvalues \( n_k \):

\[ \rho(x,x',t) = \int \Psi^*(x',x_2...x_N,t)\Psi(x,x_2...x_N,t)dx_2...dx_N \]
\[ = \sum_{q,k} \rho_{q,k} \phi_k^* (x',t) \phi_q (x,t) \]
\[ = \sum_k n_k \phi_k^{NO} (x',t) \phi_k^{NO} (x,t). \]  

The many-body state is called condensed if only one natural eigenfunction – \textit{condensate orbital} is \( \approx 100\% \) occupied. In all cases examined in this paper the initial and target states are condensed \( n_1 > 99.99\% \). During the many-body dynamics especially the driven ones the condensed fraction is reduced, indicating on the \textit{depletion} or \textit{fragmentation} of the initial state. The BEC is called depleted when the non-condensed fraction, i.e., the total occupation of other than the condensate orbital \( \sum_{i>1} n_i \) is of the order of \( 10\% \). When several eigenfunctions have macroscopic occupation the system is called fragmented. To quantify the success of the driving process we use the following general definition of the Uhlmann fidelity based on the reduced one-body density matrix:

\[ F(\rho_T, \rho) = \text{Tr} \left[ \sqrt{\sqrt{\rho_T} \rho \sqrt{\rho_T}} \right]. \]  

Within the MCTDHB method we can calculate the square root of the density matrix and find the trace needed. We would like to stress that here, in contrast to a pure state, the exa...
tations to higher orbitals, i.e., the processes of depletion and fragmentation are explicitly taken into account.

For $M = 1$ the MCTDHB ansatz and equations are reduced to the mean-field Gross-Pitaevskii description where only the condensate orbital describes the physics (see next paragraph). We have used $M = 2,3$ orbitals in our MCTDHB calculations and optimizations, to take into account many-body effects and to check the region of applicability and validity of the mean-field GPE approximation.

**GPE mean-field description.** The time-dependent mean-field Gross-Pitaevskii equation in 1D reads:

$$
i\hbar \frac{\partial}{\partial t} \Phi(x,t) = H_{\text{GPE}}[\Phi] \Phi(x,t)$$

$$H_{\text{GPE}}[\Phi] := -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x,t) + g_{1D} N|\Phi(x,t)|^2$$

(4)

It is effectively a single-particle description where the nonlinear term $|\Phi(x,t)|^2$ accounts for the (mean-field) interactions. In general when $N$ is large and $gN$ is small the mean field dynamics is very accurate. But for larger values of $g_{1D}$ and mesoscopic ensembles of bosons, many-body effects become relevant, and beyond GPE descriptions (like MCTDHB) are needed. In our case the GPE describes very well the system for short times and weak (or absent) driving, while for times close to the quantum speed limit (i.e., half Rabi oscillation period) and strong drivings the many-body effects (depletion-fragmentation) lead to significant deviations from the GPE predictions.

Finally, we note that, since the GPE describes the dynamics of a single particle pure state, the Uhlmann fidelity simply corresponds to the overlap of the wavefunction with the target one, that is, $F(\rho_T, \rho) = \text{Tr} \sqrt{\sqrt{\rho_T} \rho \sqrt{\rho_T}} = |\langle \Phi(x,t) | \phi_T(x) \rangle|$.

**Two-mode approximation.** The simplest description of the BJJ dynamics is obtained within the two-mode approximation. In this model the atoms can occupy only two possible time-independent states $\phi_{L,R}(x)$ localized in each site of the double-well potential. Those states can be found as a combination of the symmetric and antisymmetric GPE condensate orbitals representing the many-body ground and first excited state of the double well which are thus completely orthogonal by construction. In our investigation we consider a more realistic scenario and take as the initial and target states (modes) the ground state of the left $\langle x | \phi_0 \rangle = \phi_L(x)$ and right wells $\langle x | \phi_T \rangle = \phi_R(x)$ correspondingly. Those two modes are only approximately orthogonal. The two-mode approximation is based on the assumption that the evolving Gross-Pitaevskii mean-field state can be described as a linear time-dependent combination of the above described modes:

$$\Phi(x,t) = c_L(t) \phi_L(x) + c_R(t) \phi_R(x),$$

(5)

\[ J, U, \Lambda \]

\[ J \text{ CCP} \]
\[ J \text{ NOCP} \]
\[ U \text{ CCP} \]
\[ U \text{ NOCP} \]
\[ \Lambda \text{ CCP} \]
\[ \Lambda \text{ NOCP} \]
\[ \Lambda_c \]

**FIG. 1:** The parameters $J, U, \Lambda$ of the two-mode approximation estimated according to the procedures described in the text from the undriven (NOCP) and the optimally driven dynamics (CCP). We note that in order to better illustrate the difference among the tunneling matrix element $J$ between the linear and nonlinear case on the same scale, we plot the percent deviation from the (linear) value $J^L$ as: $J = (J^{NL}/J^L - 1)\times 100\%$.

where $c_{L,R}(t) = \sqrt{n_{L,R}(t)} e^{i \theta_{L,R}(t)}$. Here $n_{L,R}(t)$ represent the time-dependent populations of each well and $\theta_{L,R}(t)$ the time evolution of their phases. The modes $\phi_{L,R}(x)$ are time-independent, i.e., they stay unchanged during the evolution. Inserting this ansatz into Eq. (4) we obtain the two-mode Hamiltonian which determines the dynamics of $c_{L,R}(t)$:

$$H_{2-t}(t) = \left( E_L(t) + U_L n_L(t) - J ight)$$

(6)

Here $U_{L,R} = g n^L |\phi_{L,R}(x)|^4$ is the onsite interaction strength, $J = \int dx \phi^*_L(x) \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \phi_R(x) + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \phi_L(x) + V(x) \phi_{L,R}(x)$ are the on-site energies. In our study we have assumed a symmetric interaction strength between the wells $U_L = U_R = U$, but our findings are applicable also in the more general scenario where there is interaction detuning $\Delta U = U_R - U_L \neq 0$. In our more realistic scenario we have derived all these quantities from the observed uncontrolled dynamics and use them afterward in the GPE and many-body studies. Within the two-mode description the uncontrolled dynamics can be solved analytically [4], so one can fit the computed time-evolution with the corresponding analytical functions in order to obtain all the parameters.

In particular, $U$ can be determined by using the single particle ground state of a single well (i.e., the displayed
harmonic oscillator ground state). Furthermore, the coupling parameter $J$ is also equal to the energy splitting between the single-particle ground and first excited state of the double well. While this way of estimating the parameters provides a good description for the non-interacting case (we remind that $J$ determines the $T_{QSL}^U$ for the interacting many-body system $J$ effectively increases (leading to $T_{QSL}^N > T_{QSL}^L$) as the overlap between initial and target state increases (see also Fig.1(a) in the paper). A simple way to determine the increasing coupling strength $J$ is the following: we fit the curves of the infidelity $\varepsilon(t)$ obtained by solving the GPE in the double well potential when the dynamics is uncontrolled (red data points of Fig. 2 in the paper) with the analytical two-mode solution [4]. Since the analytical solution of the two-mode approximation depends solely on one critical parameter $\Lambda = UN/2J$, we use this as only fitting parameter. Now from the estimation of $U$ and $\Lambda$ as outlined above, we can then determine the corresponding $J$. The result of this fitting procedure is illustrated in Fig. 1 (red lines and symbols) as a function of the interaction strength $N\sigma$. We proceeded in a similar way for the estimation of the parameters $U$, $J$, $\Lambda$ in the scenario for which the dynamics is driven, that is, by fitting the infidelity curves of Fig. 1 of the paper with the analytical solution of the two-mode approximation. The result is illustrated in Fig. 1 (green lines and symbols). As it is shown in Fig. 1, we have a very good agreement between those estimations, especially concerning the critical parameter $\Lambda$ (see the small difference between the green and red lines), whose critical value $\Lambda_c = 2$ is shown by the black line. Beyond this critical value we have the non-linear self-trapping scenario where less than half of the atom population can be transferred to the other side without control.

**Control pulses and optimisation.** Our CCP scheme assumes as control parameter the zero point energy of the left (right) well $E_+(t)$ ($E_-(t)$). This can be easily controlled in our double well potential as $V_d(x) = (x \pm 2)^2/2 + E_\pm(t)$. In order to cancel the effect of the non-linear interaction one applies the control pulse $D_{CCP} = E_+(t) - E_-(t) = UN \cos^2(Jt)$ according to the condition (1) of the paper with the $U$ and $J$ parameters calculated as explained above. We note that in the experiments this task can be accomplished by controlling external magnetic or electric fields. Hence, in a certain experimental situation and given the exact configuration of the potential one has to adapt this control scheme correspondingly.

In the circumstances where the CCP approach does not work successfully, we employ optimal control techniques. In particular, we apply the chopped random basis algorithm (CRAB) [5], which consists in the following recipe: We choose as initial guess the CCP pulse $G(t) = D_{CCP}(t)$. We define the new control pulse as:

$$D_{CRAB}(t) = G(t) + G(t) \frac{\sum A_n \sin(\omega_n t) + B_n \sin(\omega_n t)}{\Pi(t)},$$

where $\omega_n$ are random frequencies, and $\Pi(t)$ assures that $D(0, T) = G(0, T)$. In our numerical simulations we used up to 4 random frequencies. Thus, the task of the optimization is to find the time-independent coefficients $A_n$, $B_n$ that minimize the infidelity. A general observation of our optimizations is that the CCP is a very good guess which facilitates the optimization scheme (compared to linear ramping or constant guess which we have also tried). This scheme can be easily adapted to any experimental setup and to any calculation method for the dynamics (two-mode, GPE, MCTDHB), and the calculations on this particular setup have been done as a proof of principle.

**Experimental verification/feasibility.** Although our results have been obtained with a simple double well model potential, we would like to point out that, the effective reduction of the transfer time induced by the interaction between bosons and its achievement through the application of an appropriate compensating control pulse, is not a consequence of the particular shape of the trapping potential. Moreover, our approach to reach the QSL can be easily verified in current experimental settings as the ones reported in Refs. [6–8]. Our method would also aid to reduce the detrimental effects of decoherence typically occurring at longer time scales. On the other hand, in order to experimentally observe the genuine many-body effects (i.e., beyond mean field) induced by the driving of the system dynamics, the experimental design might be more sophisticated, even though the difference (~9%) in terms of transfer efficiency between the optimised GPE and the exact many-body quantum dynamics is rather large. For instance, in a one-dimensional setting such that the ratio between the transverse and longitudinal trap frequencies is 5, we would have a trap frequency for the double well of $2\pi 14.5$ Hz for 100 Sodium atoms. With a barrier height of about $h 1.1$ Hz and a separation between the wells of about 8.6 $\mu$m we would get $UN/(2J) = 4$ ($J \simeq 0.025 h\omega$), namely the system would be in the self-trapping regime discussed in this paper. Larger particle numbers and interaction strengths might also yield the same phenomenon, but a detailed study concerning the reliance on $g$ and $N$ deserve a separated investigation and it is out of the scope of the present work.

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