Concept formulation and university teaching methodology for dynamic chaos

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Abstract. Chaos is a typical property of dynamical systems that is widespread in all areas of modern physics. We can say that it is a very rare phenomenon when systems demonstrate only regular behavior. Researches on chaos have highlighted the study of a whole series of issues that have general physical and general scientific significance in a new way and provided new impulses. These changes have caused a number of methodological issues that need to be addressed in the process of development of methods of teaching modern concepts and phenomena. First of all, it is important to carefully select the material and adapt it in order to achieve a combination of scientific character and accessibility for students. This paper presents methodological techniques of teaching dynamic chaos on the basis of the authors’ experience. This technique enhances formulation of modern ideas on physical picture of the world, skills of working with nonlinear models of physics, and develop practical skills of working with computer models.

1. Introduction

One of the most important discoveries in modern science is a dynamic chaos. Dynamic or deterministic chaos is a non-periodic oscillation in non-linear deterministic systems and shows high sensitivity to the initial conditions [1-2]. From the very beginning of the formation of dynamic chaos as a scientific direction, a great interest was shown in exploring this phenomenon in a wide range of applied problems from physics, astronomy, radio electronics, telecommunications, biology, in particular, neurodynamics, to financial analysis. The reasons for the continued interest lie both in studying the fundamental properties of chaos, and in finding ways of application of this phenomenon. Research on chaos supplemented the existing picture of the world in terms of natural science. Previously it was believed that world structure in general terms can be explained on the basis of a small number of fundamental principles. All the rest was perceived as the getting consequences. Studies of chaos have shown that obtaining the consequences can lead to profound conceptual changes in science [2-4]. The work of Mushkin [5] provides a review of how a new language and system for describing dynamic chaos has developed, with emphasis on attributes that have a physical and general scientific significance. It is also important to note that chaos studies have interdisciplinary character.
This present article is motivated by the need to understand the student's conceptualization of dynamic chaos and highlight the challenges to the formulation of teaching chaos at undergraduate level. The aim of the paper is to provide justifications for the chosen methodology in the authors' experience.

2. Concept of dynamic chaos

Dynamic chaos is one of the basic concepts of nonlinear physics [6-7]. The Nobel Prize winner V.L. Ginzburg paid special attention to non-equilibrium processes, chaos, attractors and underlined their importance and fundamental character in science, noting that the study of dynamic chaos can be called nonlinear physics and is undoubtedly the current trend of physics development [1].

In the literature there are several different definitions of the concept of chaos. We proceed from the following definition that chaos is defined as some random process that is observed in dynamic systems that are not affected by noise or any random forces. We note the main properties of the chaotic system [1]:

- unpredictability (exponential instability),
- indecomposability (transitivity),
- element of irregularity (cycle density).

The theory of chaos is usually considered as part of the theory of dynamical systems.

A set of dynamic variables is specified for a dynamic system, that describes the state of the system during the time period before and after using the initial state, as well as subject to some rule which is set by evolution operator:

\[
\frac{dx_1}{dt} = F_1(x_1, \ldots, x_N, \mu),
\]

\[
\frac{dx_2}{dt} = F_2(x_1, \ldots, x_N, \mu),
\]

\[
\frac{dx_N}{dt} = F_N(x_1, \ldots, x_N, \mu),
\]

here the values \(x_1, x_2, \ldots, x_N\) are system variables, \(\mu\) is vector of control parameters, \(F_1, F_2, \ldots, F_N\) are some functions. If we consider the values \(x_1, x_2, \ldots, x_N\) as coordinates of the point \(x\) in \(N\)-dimensional space, then we get a visual geometric representation of the state of a system in the form of this point (The state of a system is defined by specifying values of a set of measurable properties sufficient to determine all other properties. The state of a system is defined by specifying values of a set of measurable properties sufficient to determine all other properties. For fluid systems, typical properties are pressure, volume and temperature. More complex systems may require the specification of more unusual properties. As an example, the state of an electric battery requires the specification of the amount of electric charge it contains.). This point is called the image or phase point, the values \(x_1, x_2, \ldots, x_N\) are called phase coordinates of the point or phase variables of the system, and the state space is called system phase space. The motion of the phase point along a certain line, which is called the phase trajectory responds to the changed state of the system in time. \(F_1, F_2, \ldots, F_N\) determine the speed of motion of image point in the \(N\)-dimensional phase space.

Dynamic systems are classified as conservative and dissipative. A system will be conservative if its energy is constant. The dissipative system is displayed by dissipation of energy under the influence of external conditions. The main property of dissipative systems is the contraction of the phase volume. This means that within the time period, starting from the dynamic equations, the original volume decreases. It is expressed in the form of the inequality \(\text{Div}(\nu) < 0\).

An important difference between dissipative system and conservative dynamical systems is that in the latter systems only that attracting sets occur. As \(i \to \infty\), all phase trajectories will converge to some subset of zero volume, which is called the attractor of dynamical system. Scientists are studying a significant number of different attractors, but for an initial view of this phenomenon, there are three options:
1. "Point attractor". An example of such a system is a swinging pendulum, which, with time, stops the friction force at one point. The system is "attracted" to the initial point of equilibrium.

2. "Limiting cycle". Suppose there is no friction. Then the pendulum will always fluctuate and represent a regular periodic system. Figure 1 shows this type of attractor. Motion on limit cycle reflects a complex process of energy changes in time that takes place in the system. If the external disturbance shifts the trajectory on the phase plane inside the limit cycle, then the introduced energy will exceed the dissipated one on average. Here the average value of divergence will be positive. Outside the limit cycle, the divergence is negative, which leads to the striving of phase trajectories to the limiting cycle from the outside.

3. "Strange attractor". If we randomly change the energy communicated to the pendulum through equal time intervals, the resultant motion will be different and non-periodic. However, it is limited by the maximum amplitude of the pendulum and by the laws of physics (gravitational force, etc.). The result of such a movement will be a chaotic, or strange attractor. It is usually assumed that a dynamical system possesses a strange attractor if in its phase space there is a limit set consisting of chaotic trajectories. Strange attractors are complex fractal sets attracting to themselves all trajectories from some adjacent area [8].

Here we briefly consider an example of a strange attractor of a system with chaotic dynamics on the Chua scheme (Fig. 2). It is necessary to have at least three variables corresponding to the reactive elements of the circuit for describing the chaotic dynamics. In this case, these elements are two capacitors and one inductor.

\[ \begin{align*}
\frac{dx}{dt} &= \alpha(y - x - f(x)), \\
\frac{dy}{dt} &= -y + x + z, \\
\frac{dz}{dt} &= -\beta y.
\end{align*} \]

\[ f(x) = \begin{cases} 
  bx + a - b, & x \geq 1. \\
  ax, & |x| \leq 1. \\
  bx - a + b, & x \leq -1.
\end{cases} \]

Figure 1. The limiting cycle arising in classical non-linear Van der Pol oscillator

Figure 2. Strange attractor of chaotic system

The active element of the Chua scheme is highlighted with a rectangle in Fig. 2. The mathematical model of the circuit is described by the following system of equations:
Here we introduce the following notation for dimensionless variables and parameters:

\[ x = \frac{V_{C1}}{E}, \]
\[ y = \frac{V_{C2}}{E}, \]
\[ z = \frac{I_L}{E}, \]
\[ \alpha = \frac{C_2}{C_1}, \]
\[ \beta = \frac{R_2 C_2}{L}, \]
\[ f(x) = V_{C1} + \frac{R G(V_{C1})}{E}, \]

where \( V_{C1} \) and \( V_{C2} \) are the voltages on the capacitors \( C_1 \) and \( C_2 \), \( I_L \) is the current flowing through the coil \( L \). We can obtain types of attractors by changing the parameter \( a \) in the range from \( a = -0.1 \) to \( a = -1.11 \) and for fixed values of the remaining parameters \( b = 0.714, \alpha = 15.6, \beta = 28 \).

An attractor of the "limit cycle" type is obtained for \( a = -0.1 \) (Fig. 3). The period-doubling bifurcation is observed with a further decrease of the parameter up to \( a = -1.045 \). (Fig. 4). The oscillations in the system become chaotic when the value of the parameter \( a \) reaches \( a = -1.079 \) and a random attractor of the Rössler type appears, this result is shown in the Figure 5. Finally, the merging of two spiral attractors into a single attractor of the "double curl" type occurs as the parameter decreases to \( a = -1.11 \) (Fig. 6).

3. Methodology

The phenomenon of dynamic chaos can be studied in four stages. Below are the stages of studying the phenomenon within a 15-week course consisting of 3 credits including lectures and laboratory works. This discipline is intended for bachelor students of the 3rd year of study, specialty 5B071900 "Radio engineering, electronics and telecommunications" at Al-Farabi Kazakh National University [9].

Figure 7 shows the work curriculum with the names of the topics of the lectures in the form of a flowchart. Each stage of lecture classes is accompanied by a laboratory computer practice [10]. Students conduct research within the framework of the mentioned topic and present the results. During the computer experiment the methods and models of dynamic programming, data visualization method, and analysis are used.

We also offer students the following topics for independent work. All themes of independent work are related to our scientific works: synchronization, fractal antennas, stochastic resonance effect in nonlinear dynamical systems [11-13].

All laboratory work is carried out in a computer lab. The MatLab system is used for modeling.
We consider one of these laboratory works as an example, and describe the results that students themselves should receive.

**Figure 7.** Block diagram of the distribution of topics for a 15-week course

4. **Practical work**

4.1 **Aim**

The study of pre-chaotic or post-chaotic changes in the dynamic system in variation of its parameters by means of bifurcation diagrams.
4.2 Theory

The reaction of a dynamical system to a small perturbation is determined by its state, and in some cases the perturbing factors affect the mode of system functioning insignificantly, in others they lead to a sharp difference in the nature of the disturbed motion compared with the initial one. In the first case, the state of the system is stable, in the second case it is not.

Most physical problems in their mathematical description lead to differential equations that depend on parameters beside the initial boundary conditions. Changing of the parameter can cause loss of stability by one mode of motion and transition of the system to a different state. This phenomenon is called bifurcation, and the value of the parameter at which it occurs is a bifurcation point. New stable modes of motion can arise in the system as it passes through bifurcation points. We are interested in such bifurcations.

The mathematical basis of the elementary theory of bifurcations is the theory of stability, through which it is possible to breakdown the phase space into trajectories, and identifying the parameter sets. In practice, this makes it possible to construct bifurcation diagrams that explain the mechanisms of rearrangements of the motion regimes in the phase space of a dynamical system in variation of its parameters.

The question of stability and bifurcations of periodic trajectories can be considered directly with respect to differential equations, when the limit solution corresponds to a particular solution, as well as by analyzing the stability of fixed points of the corresponding Poincaré map.

4.3 Logistic mapping bifurcation

The most obvious and convenient for numerical study is the method of analysis of the mapping. Consider the logistic equation that describes the population growth model:

\[ x_{n+1} = rx_n(1 - x_n), \ n = 1, 2, ... N, \]

where \( x_n \) is the realization of physical quantity, \( r \) is the control parameter. The realization of this equation is shown in Figure 8 (the order of numbering of the figures has been changed in accordance with this paper). The bifurcation diagram of the logistic mapping is shown in Fig. 9.

**Figure 8.** Realization of logistic equation

**Figure 9.** Bifurcation diagram of Logistic map

*Bifurcation in system with inertia*

The emergence of chaos results from the consecutive increase in the period of oscillations according to the law of natural row.

The system with inertia is described by the following system of equations [14]:

\[
\begin{align*}
\dot{X} &= Y + (m_1 + m_2)X - XZ, \quad X \leq q, \\
\dot{X} &= Y - m_2X - qZ, \quad X > q,
\end{align*}
\]

(2)
\[ \dot{Y} = -X, \]
\[ \dot{Z} = -gZ + gF(X)X^2, \]

where \( F(X) \) is the Heaviside function; \( m_1, m_2, q, g \) are the parameters of excitation, dissipation, limitation and inertia, respectively. This system differs from the Anishchenko-Astakhov’s generator by its dynamic characteristics of a nonlinear amplifier, which has a linear section at \( X \leq q \) and a saturation section at \( X > q \). The fourth equation describes an inertial one half-wave inertial transducer. Thus, the system dynamics is determined by two mechanisms for limiting oscillations. The first mechanism is inertial-free and is related to the nonlinear character of amplifying element. The second - inertial, which is caused by the influence of voltage from the output of the inertial transformation on the steepness of the nonlinear amplifier.

In Figure 10, the fragment of bifurcation diagram of changes of maximal values \([X]\) of the oscillatory process \(X(t)\) as the parameter \(g\) changes.

![Bifurcation Diagram](image)

**Figure 10.** The bifurcation diagram of system (2)

### 4.4 Assignments

1. **Write programs for realization the logistic mapping and its bifurcation diagram**
   - Based on diagram, determine in which values of control parameter the period doubling mode is observed
   - Determine \(r\) values in which the trajectories with period 1 and 3 are observed

2. **Write the programs and build the dependencies of the dynamic variable on time**
   - Pick up such system parameters at which oscillatory modes are possible
   - Construct a bifurcation diagram of variation of maximum values of oscillatory process depending on inertia parameter \(g\), like it is shown in Fig. 5
   - Increasing the bifurcation diagram section, find the bifurcation point value

### 4.5 Analysis of the results of laboratory work

Here we briefly give the main results that students should obtain in the course of performing this work.
Below is a program in MatLab, the result of which is Figure 9.

```matlab
% argument:
% rb - initial value of r (r is control parameter)
% rh - step change of r
% N - number of steps
% default values
rb=0.01;
rh=0.01;
N=500;
rk=rh*(N-1)+rb; % calculate the final value of r
r=[rb:rh:rk]; % set r
x(1:N)=0.1; % initial value of x
% we skip 1000 steps for the mapping to reach the attractor
for i=1:1000
    x=(1-x).*x.*r;
end
%- now we draw points
for i=1:1000
    % draws points on the graph
    plot(r,x,'k.'); % until the graph is displayed
    hold on % we block the creation of a new window
    x=(1-x).*x.*r; % we calculate the following points
end
%- we finally draw points (the graph is displayed in the window)
plot(r,x,'k.'); % we set the limits for the x axis
xlabel('r');
ylabel('x');
```

Analytical analysis of equation (1) leads to the following conclusion. When $r > 1$, there are two equilibrium points (that is, $x = rx(1-x)$). To determine the stability of mapping $x_{n+1} = f(x_n)$, the incline value $|f'(x)|$ at the quiescent point should be calculated. If $|f'(x)| > 1$, the rest point is unstable. When $1 < r < 3$, the logistic equation has two quiescent points: $x = 0, (r-1)/r$. The origin is an unstable point, and the second quiescent point is stable. However, in $r = 3$, the incline at $x = (r-1)/r$ exceeds the unit ($f' = 2-r$) and both equilibrium points become unstable. In $r$ parameter values, contained between 3 and 4, this simple difference equation describes the set of multi-periodic and chaotic motions. In $r = 3$, the stationary solution becomes unstable, but bi-cycle or bivariate orbit remains stable. In further increase in $r$, the bivariate orbit becomes unstable and a emerges period 4 cycle, which due to bifurcation, is rapidly replaced by period 8 cycle with even higher $r$ values. This doubling process period continues until $r$ reaches the value $r = 3.56994...$

Graphical analysis of bifurcation diagram leads the students to the following. Students find bifurcation points of doubling period. It is observed at values below $r = 3.57$. Section where each branch of a tree splits into two. Starting with $r < 3$, they can see a trajectory with a period 1. To see longer trajectories, students mark the first 30-50 iterations with dots, and subsequent iterations with another symbol. Chaotic trajectories can be detected at $3.57 < r < 4.0$. In the area $r = 3.83$, one can find a trajectory with period 3.

In analyzing system (2), the students come to the following results. The development of the oscillatory process occurs at the parameters $m_1 = 1.6, m_2 = 0.1, q = 0.05$. A small change in the values $[X]$ on bifurcation diagram corresponds to the regular system dynamics, the random scatter of points –
to chaotic dynamics. In the interval $g \in [0.57; 0.65]$, the dynamics of the system is characterized by a stable limit cycle. The interval $g \in [0.56; 0.57]$ corresponds to chaotic oscillations. This can be seen in Figure 11.

![Figure 11. Increased section of bifurcation diagram in Figure 5](image)

The point $g = 0.56$ is a bifurcation point. In each bifurcation, the distances between critical values of the changing parameter are reduced, zones of chaotic oscillations are denser.

5. Conclusion
Modernized teaching using computers and other innovations requires careful formulation and implementation of various methods. In this regard, it is necessary to pay attention to the developing resources of the content of physical education, in particular, the content of individual concepts of modern science. Also it is necessary to note the effectiveness of the use of significant scientific and theoretical material, which form the basis for the formation of professional competence.

Every year lecture material and computer lab works are continuously improved and updated. In our opinion, the proposed methodology makes it possible to achieve a better formation of students’ knowledge of dynamic chaos, formation of the required skills in scientific knowledge of modern physics.

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