DYNAMICS OF A ROTATING FLAT ELLIPSOID WITH A STOCHASTIC OBLATENESS

ETIENNE BEHAR¹, JACKY CRESSON¹,² AND FRÉDÉRIC PIERRET¹

Abstract. We derive a model for the motion of a rotating flat ellipsoid with a stochastic flattening based on an invariance theorem for stochastic differential equations. A numerical study of a toy-model is performed leading to an intriguing coincidence with observational data.

(1) Laboratoire de Mathématiques Appliquées de Pau, Université de Pau et des Pays de l’Adour, avenue de l’Université, BP 1155, 64013 Pau Cedex, France
(2) SYRTE UMR CNRS 8630, Observatoire de Paris and University Paris VI, France

1. Introduction

The irregularities in the Earth’s rotation (see Fig. 1 and [1], [11]) are described by the variations of the Earth rotation speed, polar motion, and variations in the direction of the rotation axis in space (precession and nutations). The major effect of these irregularities are caused by the irregular variation of Earth’s oblateness (see Fig. 2 and [2]). Many complex mechanisms induce this irregular flattening such as the motion of oceans, atmosphere and rotation of inner Earth’s liquid core.

In order to take into account these irregularities, classical models of Earth’s dynamics are constructed on geophysical considerations such as oceans and atmosphere dynamics and the Earth’s neighborhood like the Moon, the planets and the Sun (see [1], [11]).

In these models the short time irregularities are badly modelled due to the complexity of the phenomena (see [3]). With this problem we are led to the following question: Does an

2010 Mathematics Subject Classification. 60H10; 60H30; 65C30; 92B05.

Key words and phrases. Invariance criteria; stochastic differential equations, model validation, stochastic models in astronomy, celestial mechanics.
alternative approach of Earth’s rotation model is possible?

The irregular motion and the complex mechanisms which are involved led to think that the Earth’s rotation behaves very randomly on short period of time which should impact considerably the future of its motion contrary to the classical model. An example of such considerations is the two-body problem with a stochastic perturbation studied in [5].

In this work we model a flat ellipsoid of revolution which could represent the Earth and where its oblateness is varying with a stochastic component.

The plan of this paper is as follows:

In section 2, we remind the classical equations of motion for a rigid ellipsoid. Section 3 deals with the case of an ellipsoid with a time variable flattening: deterministic or stochastic. In particular we discuss the notion of admissible deformations based on the invariance criterion for (stochastic) differential equations. Section 4 is devoted to the numerical exploration of a toy-model obtained by a particular deformation equation of the flattening. The main constraint on simulations is to respect the invariance needed in the derivation of our model. We prove also that for sufficiently small time-step, this invariance is preserved. In section 5 we conclude and give some perspectives.

2. Free motion of a rigid ellipsoid

In this section we remind the equations of motion for a rigid ellipsoid. We refer to chapter 4 and 5 of [8] and chapter 6 of [14] for full details.

We consider an ellipsoid of revolution $E$ of major axis $a$ and $c$ of mass $M_E$ and volume $V_E$. Let $L$ be the angular momentum of $E$ with $L = I\Omega$ where $I$ is the inertia matrix of $E$ and $\Omega$ is the rotation vector. The equation of free motion for $E$ is

\[ \frac{dL}{dt} + \Omega \wedge L = 0. \]

We remind that free motion means that there is no external moments acting on the body $E$. In the principal axes which are the reference frame attached to the center of $E$ and where the inertia is diagonal whose coefficients are directly linked to the major axis $a$ and $c$. Indeed, in the case of an ellipsoid of revolution, the inertia matrix is expressed as

\[ I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \]

where $I_2 = I_1$, $I_1 = \frac{1}{5}M_E(a^2 + c^2)$ and $I_3 = \frac{2}{5}M_Ea^2$. The volume satisfy the classical formula $V_E = \frac{4}{3}\pi a^2 c$.

In the principal axes, the equation of free motion is expressed as

\begin{align*}
\frac{d\Omega_1}{dt} &= \frac{I_1 - I_3}{I_1} \Omega_3 \Omega_2, \\
\frac{d\Omega_2}{dt} &= -\frac{I_1 - I_3}{I_1} \Omega_3 \Omega_1, \\
\frac{d\Omega_3}{dt} &= 0.
\end{align*}
3. Motion of an ellipsoid with flattening

We are interested by variations of the flattening and we want to derive the perturbed Euler equation of motion under the following assumptions:

(H1) Conservation of the ellipsoid mass $M_E$.

(H2) Conservation of the ellipsoid volume $V_E$.

(H3) Bounded variation of the flattening.

Those assumptions are physically consistent with observations and the physical considerations as we are only interested in a first approach by the effect of an homogeneous flattening.

The flattening that we consider is a geometric variation, a temporal evolution of his shape. The entire dynamic will be encode and describe with the major axis $c_t$ through the formula of the inertia matrix and the volume. The basic idea to approach variation of the flattening is that there exist a "mean" deformation of the flattening and a lower and a upper variation around it. The characterization of admissible deformations under the assumptions (H3) depends on its nature, i.e. deterministic or stochastic.

3.1. Deterministic flattening.

3.1.1. Deterministic variation of the flattening. Let $c_t$ satisfying the differential equation

$$\frac{dc_t}{dt} = f(t, c_t)$$

where $f \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$.

**Consequence of assumption (H1)**: Computing the derivative of the volume formula $V_E$ we obtain

$$a_t^2 = \frac{3V_E}{4\pi \frac{1}{c_t}}$$

we obtain

$$\frac{d(a_t^2)}{dt} = \frac{3V_E}{4\pi} \left( -\frac{1}{c_t^2} \frac{dc_t}{dt} \right) .$$

Thus using the expression of $\frac{dc_t}{dt}$ we obtain the following lemma:

**Lemma 3.1.** Under assumption (H1) the variation of $a$ is given by

$$\frac{d(a_t^2)}{dt} = \frac{3V_E}{4\pi} \left( -\frac{f(t, c_t)}{c_t^2} \right) .$$

We can now determine the variation of the inertia matrix coefficients $I_1$ and $I_3$.

**Consequence of assumption (H2)**: Computing the derivative of the expression of $I_1$ and $I_3$ gives the following lemma
Lemma 3.2. Under assumption (H2) the variation of $I_1$ and $I_3$ are given by

$$\frac{dI_3}{dt} = \frac{3MEV}{10\pi} \left( -\frac{f(t,c_t)}{c_t^2} \right)$$

and

$$\frac{dI_1}{dt} = \frac{ME}{5} \left( -\frac{3VE f(t,c_t)}{4\pi} + 2c_t f(t,c_t) \right).$$

3.1.2. Deterministic equations of motion. In order to formulate the equations of motion of $\mathcal{E}$ with a deterministic flattening, we first rewrite the equations of motion as

$$\frac{dL_i}{dt} = l_i(I,\Omega),$$

with $l_1(I,\Omega) = (I_1 - I_3)\Omega_2\Omega_3$, $l_2(I,\Omega) = -(I_1 - I_3)\Omega_1\Omega_3$ and $l_3(I,\Omega) = 0$.

Taking into account our deterministic variation of the flattening, we get the full set of the deterministic equations of motion for $\mathcal{E}$ as

$$\frac{dL_i}{dt} = l_i(I,\Omega),$$

$$\frac{dI_i}{dt} = k_i(c_t),$$

for $i = 1, 2, 3$ where

$$k_1(c_t) = \frac{ME}{5} \left( -\frac{3VE f(t,c_t)}{4\pi} + 2c_t f(t,c_t) \right),$$

$$k_2(c_t) = 0,$$

$$k_3(c_t) = \frac{3MEV}{10\pi} \left( -\frac{f(t,c_t)}{c_t^2} \right).$$

A deterministic version of the Euler equation induced by the deterministic flattening can then be obtained. As we consider only variation of the flattening, we still have a rotational symmetry. Hence, we have $L_i = I_i\Omega_i$ or equivalently $\Omega_i = \frac{L_i}{I_i}$ for $i = 1, 2, 3$. Computing the derivative for each component of $\Omega$ we obtain the following definition:

Definition 3.3. We call Deterministic Euler equations for an ellipsoid with a deterministic flattening the following equations

$$\frac{d\Omega_i}{dt} = \left( \frac{l_i(I,\Omega)}{I_i} - \frac{\Omega_i}{I_i} k_i(c_t) \right),$$

$$\frac{dI_i}{dt} = k_i(c_t),$$

$$\frac{dc_t}{dt} = f(t,c_t)$$

for $i = 1, 2, 3$.

3.1.3. Admissible deterministic deformations. We give the form of the differential equations governing a deformation respecting assumption (H3) in the deterministic case.

Definition 3.4. Let $d_{\text{min}} < 0$ and $d_{\text{max}} > 0$ fixed values which correspond to the minimum and maximum variation with respect to the initial value $c_0 > 0$, with $d_{\text{min}} + c_0 > 0$. If $c_t$ satisfies the condition $c_0 + d_{\text{min}} \leq c_t \leq c_0 + d_{\text{max}}$ for $t \geq 0$ then we say that $c_t$ is an admissible deterministic deformation.

In order to characterize admissible deterministic deformations we use the classical invariance theorem (see [21], [17]):
Theorem 3.5. Let $a, b \in \mathbb{R}$ such that $b > a$ and $\frac{dX(t)}{dt} = f(t, X(t))$ where $f \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. Then, the set 

$$K := \{ x \in \mathbb{R} : a \leq x \leq b \}$$

is invariant for $X(t)$ if and only if 

$$f(t, a) \geq 0, \quad f(t, b) \leq 0,$$

for all $t \geq 0$.

Lemma 3.6 (Characterization of admissible deterministic deformations). Let $c_t$ satisfying $\frac{dc_t}{dt} = f(t, c_t)$ then $c_t$ is an admissible deterministic variation if and only if 

$$f(t, c_0 + d_{\text{min}}) \geq 0, \quad f(t, c_0 + d_{\text{max}}) \leq 0, \quad \forall t \geq 0.$$

3.1.4. A deterministic toy-model. In order to perform numerical simulations, we define an ad-hoc admissible deformations given by 

$$f(x) = \alpha \cos(t)(x - (c_0 + d_{\text{min}}))((c_0 + d_{\text{max}}) - x), \quad e, \alpha \in \mathbb{R}^+.$$

As a consequence, the major axis $c_t$ satisfies the differential equation 

$$\frac{dc_t}{dt} = \alpha \cos(t)(c_t - (c_0 + d_{\text{min}}))((c_0 + d_{\text{max}}) - c_t).$$

Remark 3.7. It is reasonable to take a periodic deformation for the deterministic part as we observe such kind of variations for the Earth’s oblateness (see [1, 3]).

3.2. Motion of an ellipsoid with stochastic flattening.

3.2.1. Reminder about stochastic differential equations. We remind basic properties and definition of stochastic differential equations in the sense of Itô. We refer to the book [16] for more details.

A stochastic differential equation is formally written (see [16],Chap.V) in differential form as 

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t,$$

which corresponds to the stochastic integral equation 

$$X_t = X_0 + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s,$$

where the second integral is an Itô integral (see [16],Chap.III) and $B_t$ is the classical Brownian motion (see [16],Chap.II,p.7-8).

An important tool to study solutions to stochastic differential equations is the multi-dimensional Itô formula (see [16],Chap.III,Theorem 4.6) which is stated as follows:

An important tool to study solutions to stochastic differential equations is the multi-dimensional Itô formula (see [16],Chap.III,Theorem 4.6) which is stated as follows:

We denote a vector of Itô processes by $X_t^T = (X_{t,1}, X_{t,2}, \ldots, X_{t,n})$ and we put $B_t^T = (B_{t,1}, B_{t,2}, \ldots, B_{t,n})$ to be a $n$-dimensional Brownian motion (see [12],Definition 5.1,p.72), $dB_t^T = \ldots$
We consider the multi-dimensional stochastic differential equation defined by (18). Let $f$ be a $C^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$-function and $X_t$ a solution of the stochastic differential equation (18). We have

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + (\nabla^T_X f) dX_t + \frac{1}{2} (dX^T_t)(\nabla^2_X f) dX_t,$$

where $\nabla_X f = \frac{\partial f}{\partial X}$ is the gradient of $f$ w.r.t. $X$, $\nabla^2_X f = \nabla_X \nabla^T_X f$ is the Hessian matrix of $f$ w.r.t. $X$, $\delta$ is the Kronecker symbol and the following rules of computation are used: $dtdt = 0$, $dtdB_t = 0$, $dB_t dB_t = \delta_{ij} dt$.

### 3.2.2. Stochastic variation of the flattening.

Let $c_t$ be a stochastic process expressed as

$$dc_t = f(t, c_t) dt + g(t, c_t) dB_t$$

where $f, g \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$.

**Consequence of assumption (H1)**: Applying the Itô formula on the volume formula

$$a_t^2 = \frac{3V_{E}}{4\pi c_t},$$

we obtain

$$d(a_t^2) = \frac{3V_{E}}{4\pi} \left( -\frac{1}{c_t^2} dc_t + \frac{1}{c_t^3} (dc_t)^2 \right).$$

Thus using the expression of $dc_t$ we obtain the following lemma:

**Lemma 3.8.** Under assumption (H1) the variation of $a$ is given by

$$d(a_t^2) = \frac{3V_{E}}{4\pi} \left[ -f(t, c_t) c_t^2 + g(t, c_t)^2 \right] dt - g(t, c_t) c_t^2 dB_t.$$

We can now determine the variation of the inertia matrix coefficients $I_1$ and $I_3$.

**Consequence of assumption (H2)**: Applying the Itô formula on the expression of $I_1$ and $I_3$ leads to

**Lemma 3.9.** Under assumption (H2) the variation of $I_1$ and $I_3$ are given by

$$dI_3 = \frac{3M_{E}V_{E}}{10\pi} \left[ -f(t, c_t) c_t^2 + g(t, c_t)^2 \right] dt - g(t, c_t) c_t^2 dB_t,$$

and

$$dI_1 = \frac{M_{E}}{5} \left[ -\frac{3V_{E} f(t, c_t)}{4\pi c_t^2} + g^2(t, c_t) \left( 1 + \frac{3V_{E}}{4\pi c_t^2} \right) + 2c_t f(t, c_t) \right] dt + g(t, c_t) \left( 2c_t - \frac{3V_{E}}{4\pi c_t^2} \right) dB_t.$$
3.2.3. Stochastic equations of motion. In order to formulate the equations of motion of $E$ with a stochastic flattening, we first rewrite the equations of motion \( \overline{1} \) in differential form:

\[
dL_i = l_i(I, \Omega) dt,
\]

where \( l_i(I, \Omega) \) are the same as previous. Taking into account our stochastic variation of the flattening we get the full set of the stochastic equations of motion for $E$ as

\[
dL_i = l_i(I, \Omega) dt,
dI_i = h_i(c_t) dt + m_i(c_t) dB_t,
\]

for $i = 1, 2, 3$ where

\[
h_1(c_t) = \frac{Me}{\delta} \left( -\frac{3Vc}{4\pi} \frac{f(t,c_t)}{c_t^2} + g^2(t,c_t) \left( 1 + \frac{3Vc}{4\pi c_t^2} \right) + 2c_t f(t,c_t) \right),
\]

\[
h_3(c_t) = \frac{3MeVc}{10\pi} \left( -\frac{f(t,c_t)}{c_t^2} + \frac{g(t,c_t)}{c_t} \right),
\]

\[
h_2(c_t) = h_3(c_t),
\]

\[
m_1(c_t) = \frac{Me}{\delta} g(t,c_t) \left( 2c_t - \frac{3Vc}{4\pi c_t^2} \right),
\]

\[
m_3(c_t) = -\frac{3MeVc}{10\pi} \frac{g(t,c_t)}{c_t^2},
\]

\[
m_2(c_t) = m_3(c_t).
\]

A stochastic version of the Euler equation induced by the stochastic flattening is then obtained as follows: As we consider only variation of the flattening, we have a rotational symmetry during the deformation. Hence, we have $L_i = I_i \Omega_i$ or equivalently $\Omega_i = \frac{I_i}{I}$ for $i = 1, 2, 3$. Thus, using the Itô formula for each component of $\Omega$, we obtain:

**Definition 3.10.** We call Stochastic Euler equations for an ellipsoid with a stochastic flattening the following equations

\[
d\Omega_i = \left( \frac{l_i(I, \Omega)}{I} - \frac{\Omega_i}{I} h_i(c_t) + \frac{\Omega_i}{I} m_i^2(c_t) \right) dt - \frac{\Omega_i}{I} m_i(c_t) dB_t,
\]

\[
dI_i = h_i(c_t) dt + m_i(c_t) dB_t,
\]

\[
dc_t = f(t,c_t) dt + g(t,c_t) dB_t
\]

for $i = 1, 2, 3$.

3.2.4. Admissible stochastic deformations. The main constraint on the deformation in the stochastic case comes from the boundedness assumption.

**Definition 3.11.** If $c_t$ satisfies the condition $\mathbb{P} \left( c_0 + d_{\text{min}} \leq c_t \leq c_0 + d_{\text{max}} \right) = 1$ for $t \geq 0$ then we say that $c_t$ is an admissible stochastic deformation where $\mathbb{P}$ is the underlying probability measure.

In order to characterize admissible stochastic deformations, we use the stochastic invariance theorem (see \( \overline{15} \)):

**Theorem 3.12.** Let $a, b \in \mathbb{R}$ such that $b > a$ and $dX(t) = f(t,X(t)) dt + g(t,X(t)) dB_t$ a stochastic process. Then, the set

\[
K := \{ x \in \mathbb{R} : a \leq x \leq b \} 
\]
is invariant for the stochastic process $X(t)$ if and only if
\[ f(t, a) \geq 0, \]
\[ f(t, b) \leq 0, \]
\[ g(t, x) = 0 \quad \text{for } x \in \{a, b\}, \]
for all $t \geq 0$.

As a consequence, we have:

**Lemma 3.13** (Characterization of admissible stochastic deformations). Let $c_t$ satisfying $dc_t = f(t, c_t)dt + g(t, c_t)dB_t$ then $c_t$ is an admissible deterministic variation if and only if
\[ f(t, c_0 + d_{\min}) \geq 0, \]
\[ f(t, c_0 + d_{\max}) \leq 0, \quad \forall t \geq 0, \]
\[ g(t, c_0 + d_{\min}) = g(t, c_0 + d_{\max}) = 0, \quad \forall t \geq 0. \]

### 3.2.5. A stochastic Toy-model

In order to perform numerical simulations, we introduce an ad-hoc deformation defined by
\[ f(x) = \alpha \cos(t)(x - (c_0 + d_{\min}))(c_0 + d_{\max} - x), \quad e, \alpha \in \mathbb{R}^+, \]
\[ g(x) = \beta(x - (c_0 + d_{\min}))(c_0 + d_{\max} - x), \beta \in \mathbb{R}^+, \]
where $g$ is designed to reproduce the observed stochastic behaviour of the oblateness of the earth. However, as pointed out in the introduction, we do not intend to produce an accurate model but mainly to study if such a modelisation using stochastic processes leads to a good agreement on the shape of the polar motion.

The major axis $c_t$ satisfies the stochastic differential equation
\[ dc_t = \alpha \cos(t)(c_t - (c_0 + d_{\min}))(c_0 + d_{\max} - c_t)dt + \beta(c_t - (c_0 + d_{\min}))(c_0 + d_{\max} - c_t)dB_t. \]

### 4. Simulations of the Toy-model

#### 4.1. Initial conditions

All the simulations are done under the following set of initial conditions:
- The major axis for $E$: $a_0 = 1, c_0 = 0.8$.
- Upper variation $d_{\max} = 0.1$.
- Lower variation $d_{\min} = -0.1$.
- Mass: $M_E = 1$.
- Volume: $V_E = 10$.
- Rotation vector $\Omega$ is chosen in the principal axis as $\Omega = \Omega \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)^T$ with $\Omega = 1$ the norm which correspond to the rotational velocity amplitude.
- Perturbation coefficients: $\alpha = \beta = 1$.

The reader can test different values of the initial conditions using the Scilab program designed by F. Pierret (see

http://syrte.obspm.fr/~pierret/flattening_problem_sto.html
4.2. Numerical scheme and the invariance property. As we do not perform simulations over a long time, we can use in the deterministic case the Euler scheme and in the stochastic case the Euler-Murayama scheme. However, in each case a difficulty appears which is in fact present in many other domains of modeling (see [6],[7]), namely the respect of the invariance condition under discretisation. Indeed, even if the continuous model satisfies the invariance condition leading to an admissible deformation, the discrete quantity can sometime produce unrealistic values leading to, for example, negative values of the major axis. We prove in Appendix A and B that thanks to an appropriate choice of the time step, it is possible (under some conditions) to obtain a numerical scheme satisfying the invariance property (with a probability which can be as close as we want to one in the stochastic case).

In the following, we denote by $h \in \mathbb{R}^+$ the time increment of the numerical scheme. For $n \in \mathbb{N}$, we denote by $t_n$ the discrete time defined by $t_n = nh$ and by $X_n$ the numerical solution compute at time $t_n$.

In both cases, we perform numerical integration with time step $h = 10^{-4}$ over two initial periods which correspond to the period of the unperturbed motion.

4.3. Deterministic case. In order to do numerical simulations we use the Euler scheme. Let $X_t$ a smooth function such that

$$X_t = X_0 + \int_0^t f(s, X_s) \, ds$$

where $f \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. The associated Euler scheme associate is given by

$$X_{n+1} = X_n + f(t_n, X_n) h.$$  

Fig. 3 displays the variation of the precession in color over two period of the unperturbed motion which is compared to the unperturbed motion in black. The rotational velocity is given in Fig. 5. In Fig. 4 we see the variation of the oblateness for our toy-model in the deterministic case.

4.4. Stochastic case. In order to do numerical simulations we use the Euler-Murayama scheme which is the stochastic counterpart to the Euler scheme for deterministic differential equations (see [10], [13]). Let $X_t$ be a stochastic process written as

$$X_t = X_0 + \int_0^t f(s, X_s) \, ds + \int_0^t g(s, X_s) \, dB_s$$

where $f, g \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. The Euler-Murayama scheme is given by

$$X_{n+1} = X_n + f(t_n, X_n) h + g(t_n, X_n) \Delta B_n,$$

where $\Delta B_n$ is a Brownian increment which is normally distributed with mean zero and variance $h$ for all $n \geq 0$.

For one brownian motion realization, we can see on Fig. 6 the variation of the precession in color over two period of the unperturbed motion which is compared to the unperturbed motion.
Figure 3. The first two components of $\Omega$ in black. The rotational velocity is given in Fig. 8 and the variation of the oblateness in Fig. 7.

Considering stochastic variations of the flattening induce the erratic motion of Earth’s rotation over very short period of time. It could explain why it is so difficult to predict his motion for more than one day period of time in the actual model.

5. Conclusion

There exist a remarkable coincidence between the general shape for the rotation velocity obtained using simulations of the stochastic toy-model and the observational curves. It shows that the model capture a part of the random effects inside the real observations. Obviously the hypothesis on the stochastic nature of the deformation which should be a diffusion process seems to be too restrictive. Indeed the real data suggest there is some times noise coloration (see [1]). Of course it is possible to adapt all the theoretical and numerical results of this work with colored noise (see [9] and [20] for example for a short introduction to colored noise using
the Ornstein-Uhlenbeck process and its white noise limit). Testing such a model with the real data and colored noise and also the actual model of the main perturbation of Earth’s rotation such as the oceanic and atmospheric excitation will be the subject of a future paper. Also the impact of such a model of flattening which induce a stochastic perturbation on an orbiting body is actually a work in progress using the Stochastic Gauss equations developed in [18].

**Appendix A. Domain invariance conditions for the Euler scheme**

In order to study the invariance describe in Theorem 3.3 for the Euler scheme, we restrict our intention to the domain $K^+ := \{ x \in \mathbb{R} : a \leq x \}$ and the domain $K^- := \{ x \in \mathbb{R} : x \leq b \}$. As $K = K^+ \cap K^-$ the condition of invariance for the numerical scheme of $K$ would be both conditions of invariance for $K^+$ and $K^-$. The methods being the same to prove the invariance of $K^+$ and $K^-$, we write only the proof for $K^+$. 
Proposition A.1. Let $K^+$ be an invariant domain for $X_t$ and $D = \sup_{x \in K^+} \left( \int_0^1 \left\| \frac{\partial f}{\partial x}(t_n, a + s(x - a)) \right\| ds \right)$. If $\frac{1}{h} < \frac{1}{D}$ then the Euler scheme preserve the invariance of $K^+$ for $X_n$, $n \geq 0$.

Proof. By hypothesis with have $f(a) \geq 0$ and we suppose for $n \geq 0$, $X_n \geq a$. We will show that $X_{n+1}$ remains in $K^+$ if $h < \frac{1}{D}$.

Taylor’s expansion up to the first order with integral remainder on $f$ gives

$$f(t_n, X_n) = f(t_n, a) + (X_n - a) \int_0^1 \frac{\partial f}{\partial x}(t_n, a + s(X_n - a)) ds.$$  

Inserting the Taylor’s expansion of $f$ in the Euler scheme gives

$$X_{n+1} - a = (X_n - a) \left( 1 + h \int_0^1 \frac{\partial f}{\partial x}(t_n, a + s(X_n - a)) ds \right) + hf(a)$$

By hypothesis $f(a) \geq 0$ so

$$X_{n+1} - a \geq (X_n - a)(1 - hD).$$

As $X_n \geq a$ then $X_{n+1}$ remains in $K^+$ if $1 - hD > 0$ that is to say $h < \frac{1}{D}$. \[\square\]

Appendix B. Domain invariance conditions of the Euler-Murayama scheme

We now study the invariance of $K^+$ by the Euler-Murayama scheme. There is a huge difference between the stochastic scheme and the classical scheme due to the fact that $\Delta B_n$ is almost surely unbounded (see [19]) for all $n \geq 0$. However, we can estimate the probability such that the Euler-Murayama scheme will respect the invariance of $K^+$.

As the classical case we can write the condition of invariance for the Euler-Murayama scheme :

Proposition B.1. Let $K^+$ be an invariant domain for $X_t$ and $D = \sup_{x \in K^+} \left( \int_0^1 \left\| \frac{\partial f}{\partial x}(t_n, a + s(x - a)) \right\| ds \right)$, $S = \sup_{x \in K^+} \left( \int_0^1 \left\| \frac{\partial g}{\partial x}(t_n, a + s(x - a)) \right\| ds \right)$. If $h < \frac{1}{D}$ and $\Delta B_n > -\frac{1}{S}$ then the Euler-Murayama scheme preserve the invariance of $K^+$ for $X_n$, for all $n \geq 0$.

Proof. The steps for the proof are the same that the classical case. By considering a Taylor expansion up to the first order of $f$ and $g$ with integral remainder, we get using the Euler-Murayama scheme

$$\Delta B_n \geq \frac{hD - 1}{S}.$$ 

As we assume $h < \frac{1}{D}$ and $h > 0$ then the Euler-Murayama scheme preserve the invariance of $K^+$ if

$$\Delta B_n > -\frac{1}{S}.$$
We quantify the probability that $\Delta B_n > -\frac{1}{S}$ occurs:

**Proposition B.2.** For all $\epsilon > 0$, there exist $h_0$ such that for all $h < h_0(\epsilon)$ then

$$P\left(\Delta B_n > -\frac{1}{S}\right) > 1 - \epsilon$$

for all $n \geq 0$.

**Proof.** Let $\epsilon > 0$. The probability function is determined by a normal law of zero mean and variance $h$. We have

$$P\left(\Delta B_n > -\frac{1}{S}\right) = 1 - P\left(\Delta B_n < -\frac{1}{S}\right).$$

and by definition

$$P\left(\Delta B_n < -\frac{1}{S}\right) = \frac{1}{\sqrt{2\pi h}} \int_{-\frac{1}{S}}^{-\infty} e^{-\frac{x^2}{2h}} dx = \frac{1}{2} \left(1 + \text{erf}\left(-\frac{1}{S} \sqrt{\frac{1}{2h}}\right)\right)$$

where erf is the classical error function defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

As $-\frac{1}{S} < 0$ we use an approximation of erf on $\mathbb{R}^-$ (see [22]) which induces a bounded value as

$$\text{erf}(x) < -\sqrt{1 - \exp\left(-x^2 \frac{4}{\pi} + ax^2\right)}$$

where $a = \frac{8(\pi-3)}{3\pi(4-\pi)}$. Let

$$\gamma(h) = \frac{1}{2} \left(1 - \sqrt{1 - \exp\left(-\frac{4}{\pi} + \frac{4a}{2hS^2} + \frac{a}{2hS^2}\right)}\right).$$

Inverting in term of $h$ with find out two solutions,

$$h_-(\gamma) = -\left(\frac{4}{\pi} + a \ln(4\gamma(1 - \gamma))\right) - 2\sqrt{\left(\frac{4}{\pi} + a \ln(4\gamma(1 - \gamma))\right)^2 - 4a \ln(4\gamma(1 - \gamma))} + 4S^2 \ln(4\gamma(1 - \gamma)),$$

$$h_+(\gamma) = -\left(\frac{4}{\pi} + a \ln(4\gamma(1 - \gamma))\right) + 2\sqrt{\left(\frac{4}{\pi} + a \ln(4\gamma(1 - \gamma))\right)^2 - 4a \ln(4\gamma(1 - \gamma))} + 4S^2 \log(4\gamma(1 - \gamma))$$

Considering $h_0(\epsilon) = \min\{\frac{1}{\epsilon}, h_-(\epsilon), h_+(\epsilon)\}$ then for all $h < h_0(\epsilon)$ we have

$$P\left(\Delta B_n < -\frac{1}{S}\right) < \epsilon,$$

and finally

$$P\left(\Delta B_n > -\frac{1}{S}\right) > 1 - \epsilon.$$
As a consequence, if the time increment is sufficiently small, then the Euler-Murayama scheme will preserve the positivity with a probability very close to one.

References

[1] C. Bizouard. Le mouvement du pôle de l’heure au siècle. Presses Académiques Francophones, 284 p.
[2] M. Cheng, J C. Ries, and B D. Tapley. Variations of the earth’s figure axis from satellite laser ranging and grace. Journal of Geophysical Research: Solid Earth, 116(B1), 2011.
[3] M. Cheng and B D. Tapley. Variations in the earth’s oblateness during the past 28 years. Journal of Geophysical Research: Solid Earth, 109(B9):n/a-n/a, 2004.
[4] J. Cresson, M. Efendiev, and S. Sonner. On the positivity of solutions of systems of stochastic PDEs. ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik, 93(6-7):414–422, 2013.
[5] J. Cresson, F. Pierret, and B. Puig. Stochastic perturbation of the two-body problem. ArXiv e-prints, February 2014.
[6] J. Cresson, B. Puig, S. Sonner, Validating stochastic models : invariance criteria for systems of stochastic differential equations and the selection of a stochastic Hodgkin-Huxley type model, Int. J. Biomath. Biostat. 2 (2013), 111-122.
[7] J. Cresson, B. Puig, S. Sonner, Stochastic models in biology and the invariance problem, preprint, 27.p, 2014.
[8] H. Goldstein. Classical Mechanics. Addison-Wesley, Reading, second edition, 1980.
[9] P. Hanggi and P. Jung. Colored noise in dynamical systems. Advances in chemical physics, 89:239-326, 1995.
[10] D. J. Higham. An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM review, 43(3):525–546, 2001.
[11] S. Jin, T. van Dam, and S. Wdowinski. Observing and understanding the earth system variations from space geodesy. Journal of Geodynamics, 72(0):1 – 10, 2013. SI: Geodetic Earth System.
[12] I. Karatzas and S. E. Shreve. Brownian Motion and Stochastic Calculus. Springer, Berlin, 2nd edition, 2000.
[13] P. E. Kloeden and E. Platen. Numerical solution of stochastic differential equations, volume 23. Springer, 1992.
[14] L. D. Landau and E. M. Lifshitz. Mechanics, Third Edition: Volume 1 (Course of Theoretical Physics). Butterworth-Heinemann, 3 edition, January 1976.
[15] A. Milian. Stochastic viability and a comparison theorem. In Colloq. Math, volume 68, pages 297–316, 1995.
[16] B. Øksendal. Stochastic differential equations. Springer, 2003.
[17] N.H. Pavel and D. Motreanu. Tangency, Flow Invariance for Differential Equations, and Optimization Problems. Chapman & Hall/CRC Pure and Applied Mathematics. Taylor & Francis, 1999.
[18] F. Pierret, Stochastic Gauss equations, Preprint, (2014).
[19] F. Pierret, Simulations of the stochastic flattening problem - Program package : [http://syrte.obspm.fr/~pierret/flattening_problem_sto.tar.gz](http://syrte.obspm.fr/~pierret/flattening_problem_sto.tar.gz), 2014.
[20] H. Riecke Introduction to Stochastic Processes and Stochastic Differential Equations Lecture Notes in Engineering Sciences and Applied Mathematics, 2010, [http://people.esam.northwestern.edu/~riecke/Vorlesungen/442/Notes/notes_442.pdf](http://people.esam.northwestern.edu/~riecke/Vorlesungen/442/Notes/notes_442.pdf).
[21] W. Walter. Ordinary Differential Equations. Graduate Texts in Mathematics. Springer New York, 1998.
[22] S. Winitzki. A handy approximation for the error function and its inverse. A lecture note obtained through private communication, 2008.