Error Analysis for a Statistical Finite Element Method

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Abstract
The recently proposed statistical finite element (statFEM) approach synthesises measurement data with finite element models and allows for making predictions about the true system response. We provide a probabilistic error analysis for a prototypical statFEM setup based on a Gaussian process prior under the assumption that the noisy measurement data are generated by a deterministic true system response function that satisfies a second-order elliptic partial differential equation for an unknown true source term. In certain cases, properties such as the smoothness of the source term may be misspecified by the Gaussian process model. The error estimates we derive are for the expectation with respect to the measurement noise of the $L^2$-norm of the difference between the true system response and the mean of the statFEM posterior. The estimates imply polynomial rates of convergence in the numbers of measurement points and finite element basis functions and depend on the Sobolev smoothness of the true source term and the Gaussian process model. A numerical example for Poisson’s equation is used to illustrate these theoretical results.

1 Introduction
The finite element method has become an indispensable tool for solving partial differential equations in engineering and applied sciences. Today, the design, manufacture and maintenance of most engineering products rely on mathematical models based on finite element discretised partial differential equations (PDEs). These models depend on a wide range of parameters, including material, geometry and loading, which are inevitably subject to both epistemic and aleatoric uncertainties. Consequently, the response of the actual engineering product and the inevitably misspecified mathematical model often bear little resemblance to each other, resulting in inefficient designs and overtly cautious operational decisions. Fortunately, more and more engineering products are equipped with sensor networks providing operational measurement data (e.g., Febrianto et al., 2022). The recently proposed statistical finite element method (statFEM) allows us to infer the true system response by synthesising limited measurement data with the misspecified the finite element model (Girolami et al., 2021). By adopting a Bayesian approach, the prior probability measure of the finite element solution is obtained from the misspecified finite element model by solving a probabilistic forward problem. Although any parameters of the finite element model can be random, in this article only the source term of the respective PDE is random and Gaussian, so that the finite element solution is Gaussian. The assumed data-generating process for determining the likelihood of the measured data is additively composed of the random finite element solution, the known random measurement noise, and, possibly, an unknown random discrepancy component. The chosen prior and the likelihood ensure that the posterior finite element probability density conditioned on the measurement data is Gaussian and easily computable.

More concretely, we consider the following mathematical problem. Suppose that the system response $u$ one believes generated the measurement data is given by the solution of

$$L u = f$$

on a bounded domain $\Omega$ with Dirichlet boundary conditions. The statistical component of the statFEM solution arises from the placement of a stochastic process prior on the forcing term $f$ and, possibly, the
differential operator \( L \) or some of its parameters. Doing this induces a stochastic process prior over the solution \( u \). After hyperparameter estimation and inclusion of additional levels of statistical modelling (Kennedy and O’Hagan, 2002), which may account for various modelling discrepancies, one uses Bayesian inference to obtain a posterior distribution over the PDE solution given the measurement data. The posterior can then predict the system behaviour at previously unseen data locations and provide associated uncertainty quantification. See Girolami et al. (2021) and Duffin et al. (2021) for applications of this methodology to different types of PDEs and Abdulle and Garegnani (2021) for a somewhat different approach focusing on random meshes. In any non-trivial setting, computation of the prior for \( u \) from that placed on \( f \) requires solving the PDE (1.1). In statFEM the PDE is solved using finite elements. Due to their tractability, Gaussian processes (GPs) are often the method of choice for modelling physical phenomena. In the PDE setting we consider Gaussian processes are particularly convenient because a GP prior, \( f_{GP} \), on \( f \) induces a GP prior, \( u_{GP} \), on \( u \) if the PDE is linear (see Figure 1). The induced prior \( u_{GP} \) has been studied in Owhadi (2015); Raissi et al. (2017) and Cockayne et al. (2017, Section 3.1.2). Although \( u_{GP} \) is generally not available in closed form, it is straightforward to approximate its mean and covariance functions from those of \( f_{GP} \) by using finite elements.

In this article, we provide estimates of the predictive error for the GP-based statFEM when the data are noisy evaluations of some deterministic true system response function \( u_t \) which is assumed to be the solution of (1.1) for an unknown—but deterministic—true source term \( f_t \). Due to the complexity and difficulty of analysing a full statFEM approach, we consider a prototypical version that consists of a GP prior on \( f \) and, possibly, a GP discrepancy term. Scaling and other parameters these processes may have are assumed fixed. Despite recent advances in understanding the behaviour of GP hyperparameters and their effect on the convergence of GP approximation (Karvonen et al., 2020; Teckentrup, 2020; Wynne et al., 2021), these results are either not directly applicable in our setting or too generic in that they assume that the parameter estimates remain in some compact sets, which has not been verified for commonly used parameter estimation methods, such as maximum likelihood.

As mentioned, finite elements are needed for computation of the induced prior \( u_{GP} \) and the associated posterior. But why not simply use a readily available and explicit GP prior for \( u \), such as one with a Matérn covariance kernel, instead of something that requires finite element approximations? The main reason (besides this being the first step towards analysing the full statFEM) is that a prior \( u_{GP} \), for which \( Lu_{GP} = f_{GP} \), satisfies the structural constraints imposed by the PDE model and can therefore be expected to yield more accurate point estimates and more reliable uncertainty quantification than a more arbitrary prior if the data are generated by a solution of (1.1) for some source term. We give a detailed description of the considered method in Section 2.

1.1 Contributions

Our contribution consists of a number of error estimates for the statFEM approach sketched above. Suppose that the measurements are \( y_i = u_t(x_i) + \varepsilon_i \) for \( n \) locations \( x_i \in \Omega \subset \mathbb{R}^d \) and independent Gaussian noises.
\( \varepsilon_i \sim N(0, \sigma_e^2) \). The regression error estimates we prove are of the form

\[
\mathbb{E}\left[ \| u_t - \hat{m} \|_{L^2(\Omega)} \right] \leq C_1 n^{-1/2+a} + C_2 n^{-q/2},
\]

where \( \hat{m} \) is a posterior mean function obtained from statFEM and the expectation is with respect to the measurement noise. The constant \( a \in (0, 1/2) \) depends on the smoothness of \( f \) and \( q > 0 \) is the dimension \( d \) dependent characteristic rate of convergence of the finite element approximation with \( n_{\text{FE}} \) elements. In (1.2) it is assumed that the points \( x_i \) cover \( \Omega \) sufficiently uniformly. In Section 6.2 we present error estimates for four different variants of statFEM, each of which corresponds to a different \( \hat{m} \):

- Theorem 3.2 assumes that no finite element discretisation is required for computation of \( \hat{m} \). In this case \( C_1 > 0 \) and \( C_2 = 0 \). It is required that \( f \) be at least as smooth as the prior \( f_{\text{GP}} \).
- In Theorem 3.4, the more realistic assumption that \( \hat{m} \) is constructed via a finite element approximation is used. In this case \( C_1, C_2 > 0 \). It is required that \( f \) be at least as smooth as the prior \( f_{\text{GP}} \).
- Theorems 3.6 and 3.7 concern versions which include a GP discrepancy term \( v_{\text{GP}} \) (i.e., the prior for \( u \) is \( u_{\text{GP}} + v_{\text{GP}} \)) and do not use or use, respectively, finite element discretisation to compute \( \hat{m} \). These theorems allow the priors to misspecify the source term and system response smoothness as it is not required that \( f \) be at least as smooth as \( f_{\text{GP}} \) or that \( u \) be at least as smooth as \( v_{\text{GP}} \) or \( u_{\text{GP}} \).

As discussed in Remark 3.3, these rates are likely slightly sub-optimal. Some numerical examples for the one-dimensional Poisson equation are given in Section 4.

The proofs of these results are based on reproducing kernel Hilbert space (RKHS) techniques which are commonly used to analyse approximation properties of GPs (van der Vaart and van Zanten, 2011; Cialenco et al., 2012; Cockayne et al., 2017; Karvonen et al., 2020; Teckentrup, 2020; Wang et al., 2020; Wynne et al., 2021). Our central tool is Theorem 6.5, which describes the RKHS associated to the prior \( f_{\text{GP}} \) in terms of the maximal difference of the kernels. The proofs of these results are based on reproducing kernel Hilbert space (RKHS) techniques which are commonly used to analyse approximation properties of GPs (van der Vaart and van Zanten, 2011; Cialenco et al., 2012; Cockayne et al., 2017; Karvonen et al., 2020; Teckentrup, 2020; Wang et al., 2020; Wynne et al., 2021). Our central tool is Theorem 6.5, which describes the RKHS associated to the prior \( f_{\text{GP}} \) in terms of the maximal difference of the kernels.

1.2 Related Work

Solving PDEs with kernel-based methods goes back at least to Kansa (1990); see Fasshauer (1996) and Franke and Schaback (1998) as well as Chapter 16 in Wendland (2005) for a more general treatment. In the language of GPs, this radial basis function collocation approach is essentially based on modelling \( u \) as a GP with a given covariance kernel and conditioning on the derivative observations \( Lu(x_i) = f(x_i) \). Typically no synthesis of actual measurement data is present (though this could be easily included). For convergence results in a well-specified setting, see for example Theorem 16.15 in Wendland (2005). In a GP setting similar methods have been proposed and analysed in Graepel (2003); Cialenco et al. (2012); Cockayne et al. (2017); and Raissi et al. (2017). For some error estimates, see Lemma 3.4 and Proposition 3.5 in Cialenco et al. (2012). Priors and covariance kernels derived from Green’s function have been considered by Fasshauer and Ye (2011, 2013) and Owhadi (2015). Furthermore, Papandreou et al. (2023) have recently derived bounds on the Wasserstein distance \( W_2 \) between the ideal prior and posterior (see Section 2.1 in the present article) and their finite element approximations.

2 Statistical Finite Element Methods

This section describes the statFEM approach that is analysed in Section 6.2 and discusses some extensions that are not covered by our analysis. We begin by defining the class of second-order elliptic PDE problems that are considered in this article.

Let \( d \in \mathbb{N} \) and \( \Omega \subset \mathbb{R}^d \) be an open and bounded set which satisfies an interior cone condition (e.g., Wendland, 2005, Definition 3.6) and has a Lipschitz boundary \( \partial \Omega \) (i.e., the boundary is locally the graph of
a Lipschitz function). Occasionally we also require an assumption that \( \partial \Omega \) be \( C^k \) or \( C^{k,\alpha} \), which means that its boundary can be interpreted locally as the graph of a function in \( C^k(\mathbb{R}^{d-1}) \) or in the Hölder space \( C^{k,\alpha}(\mathbb{R}^d) \), for which see Section 3.1.

Let \( \mathcal{L} \) be a second-order partial differential operator of the form

\[
\mathcal{L}u = -\sum_{i=1}^{d} \sum_{j=1}^{d} \partial_i (a_{ij} \partial_j u) + \sum_{i=1}^{d} b_i \partial_i u + cu
\]

(2.1)

for coefficient functions \( a_{ij} \), \( b_i \) and \( c \) which are bounded on the closure \( \Omega \). We further assume that \( a_{ij} \in C^1(\Omega) \) and \( a_{ij} = a_{ji} \) for all \( i, j \). The differential operator is assumed to be uniformly elliptic, which is to say that there is a positive constant \( \lambda \) such that \( \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij}(x) z_i z_j \geq \lambda \|z\|^2 \) for any \( x \in \Omega \) and \( z \in \mathbb{R}^d \). Moreover, our results use the following regularity assumption.

**Assumption 2.1 (Regularity).** For a given \( k \in \mathbb{N}_0 \), the boundary \( \partial \Omega \) is \( C^{k+2} \) and \( a_{ij}, b_i, c \in C^{k+1}(\bar{\Omega}) \) for all \( i, j \). We emphasise that here \( \sigma \) and \( \nu \) are important because they induce Sobolev spaces; see Section 3.1. We model \( f \) the gamma function and \( K \) kernels one which appears repeatedly in this article—classes of positive-semidefinite kernels is that of the Matérn kernels:

\[
K(x, y) = \sigma^2 2^{1-\nu} \Gamma(\nu) \left( \frac{2^\nu \|x - y\|}{\ell} \right)^\nu K_\nu \left( \frac{2^\nu \|x - y\|}{\ell} \right),
\]

(2.5)

where \( \nu > 0 \) is a smoothness parameter, \( \ell > 0 \) a length-scale parameter, \( \sigma > 0 \) a scaling parameter, \( \Gamma(\nu) \) the gamma function and \( K_\nu \) the modified Bessel function of the second kind of order \( \nu \). These kernels are important because they induce Sobolev spaces; see Section 3.1. We model \( f_t \) as a Gaussian process \( f_{\text{GP}} \sim \text{GP}(m, K) \) and assume that

(i) \( m \in \mathcal{H}_L(\Omega) \) and (ii) \( K(\cdot, x) \in \mathcal{H}_L(\Omega) \) and \( \mathcal{L}_x^{-1} K(\cdot, x) \in \mathcal{H}_L(\Omega) \) for every \( x \in \Omega \),

(2.6)

where the subscript denotes the variable with respect to which the linear operator is applied. These assumptions ensure that various functions that we are about to introduce are unique and pointwise well-defined.
Because $L$ is a linear differential operator, the above GP prior over $f$ induces the prior $u_{GP} \sim \text{GP}(m_u, K_u)$ over $u_t$ with the mean function $m_u = L^{-1}m$ and the covariance kernel $K_u$ which satisfies

$$L_x L_y K_u(x, y) = K(x, y)$$

(2.7)

for all $x, y \in \Omega$ as well as $LK_u(\cdot, y) = 0$ on $\partial \Omega$ for every $y \in \bar{\Omega}$. The existence and uniqueness of the mean and covariance are guaranteed by the assumptions in (2.6). Using the Green’s function $G_L$ of the the PDE (2.2), these functions can be formally written as

$$m_u(x) = \int_{\Omega} G_L(x, x')m(x')\,dx' \quad \text{and} \quad K_u(x, y) = \int_{\Omega} \int_{\Omega} G_L(x, x')K(x', y')G_L(y, y')\,dx'\,dy'. \quad (2.8)$$

To arrive at an ideal version of the GP-based statFEM we condition the GP $u_{GP}$ on the measurement data in (2.4). This yields the conditional process

$$u_{GP} \mid Y \sim \text{GP}(m_{u|Y}, K_{u|Y})$$

whose mean and covariance are

$$m_{u|Y}(x) = m_u(x) + K_u(x, X)^T(K_u(X, X) + \sigma_n^2I_n)^{-1}(Y - m_u(X)), \quad (2.9a)$$

$$K_{u|Y}(x, y) = K_u(x, y) - K_u(x, X)^T(K_u(X, X) + \sigma_n^2I_n)^{-1}K_u(y, X), \quad (2.9b)$$

where $K_u(X, X)$ is the $n \times n$ kernel matrix with elements $K_u(x_i, x_j)$, $K_u(x, X)$ and $m_u(x)$ are $n$-vectors with elements $K_u(x_i, x_j)$ and $m_u(x_i)$, respectively, and $I_n$ is the $n \times n$ identity matrix. However, the mean function and covariance kernel in (2.8) cannot be solved in closed form in all but the simplest of cases. This necessitates replacing their occurrences in (2.9) with finite element approximations.

### 2.2 Finite Element Discretisation

Let

$$B(u, v) = \int_{\Omega} \left( \sum_{i=1}^d \sum_{j=1}^d a_{ij}(x)[\partial_i u(x)][\partial_j v(x)] + \sum_{i=1}^d b_i(x)[\partial_i u(x)]v(x) + c(x)u(x)v(x) \right) \,dx$$

be the bilinear form associated with the elliptic differential operator $L$ in (2.1) and let $\phi_1, \ldots, \phi_{\text{ref}}$ be $n_{\text{FE}} \in \mathbb{N}$ finite element basis functions. The finite element approximation $u^{\text{FE}}$ of the solution $u$ of (2.2) for any sufficiently regular $f$ is

$$u^{\text{FE}}(x) = \sum_{i=1}^{n_{\text{FE}}} u_i \phi_i(x),$$

where the coefficient vector $u = (u_1, \ldots, u_{\text{ref}})$ is the solution of the linear system

$$\begin{pmatrix}
B(\phi_1, \phi_1) & \cdots & B(\phi_{\text{ref}}, \phi_1) \\
\vdots & \ddots & \vdots \\
B(\phi_{\text{ref}}, \phi_1) & \cdots & B(\phi_{\text{ref}}, \phi_{\text{ref}})
\end{pmatrix}
\begin{pmatrix}
u_1 \\
\vdots \\
u_{\text{ref}}
\end{pmatrix}
= \begin{pmatrix}
\int_{\Omega} f(x)\phi_1(x)\,dx \\
\vdots \\
\int_{\Omega} f(x)\phi_{\text{ref}}(x)\,dx
\end{pmatrix}.$$  

(2.10)

Because $Lm_u = m$, by solving $u$ from (2.10) with $f = m$ we obtain the mean approximation

$$m^{\text{FE}}(x) = \mu^T A^{-1} \phi(x) \approx m_u(x),$$

where $\phi(x) = (\phi_1(x), \ldots, \phi_{\text{ref}}(x))$ and $\mu = (\int_{\Omega} m(x)\phi_1(x)\,dx, \ldots, \int_{\Omega} m(x)\phi_{\text{ref}}(x)\,dx)$. Because $L_x L_y K_u(x, y) = K(x, y)$, we may approximate $K_u$ by first forming and approximation with $f = K(\cdot, y)$ in (2.10) and subsequently forming a second approximation with the first approximation as $f$ in (2.10).

From this we obtain the covariance approximation

$$K^{\text{FE}}_u(x, y) = \phi(x)^T A^{-1} MA^{-1} \phi(y), \approx K_u(x, y),$$
An additional motivation for using such a generic assumption is the presence of integral approximations, which is often desirable to include a discrepancy term \( v_{GP} \sim GP(m_d, K_d) \) to account for modelling errors. We do this by replacing the induced GP model \( u_{GP} \sim GP(m_u, K_u) \) for \( u_t \) with \( u_{GP} + v_{GP} \), so that the full GP model for \( u_t \) is

\[
  u_{GP} + v_{GP} \sim GP(m_u + m_d, K_u + K_d).
\]
Unlike \( u_{GP} \), which is induced by the GP prior \( f_{GP} \) over \( f \), and is thus accessible only by solving the PDE (2.2), the discrepancy term is typically taken to be a GP with some standard covariance kernel, such as a Matérn in (2.5). Denote \( m_{ud} = m_u + m_d \) and \( K_{ud} = K_u + K_d \). When the discrepancy term is included, the exact conditional moments in (2.9) become

\[
\begin{align*}
    m_{d,ud}|y(x) &= m_{ud}(x) + K_{ud}(x, X)^T(K_{ud}(X, X) + \sigma^2 I_n)^{-1}(y - m_{ud}(X)), \\
    K_{d,ud}|y(x, y) &= K_{ud}(x, y) - K_{ud}(x, X)^T(K_{ud}(X, X) + \sigma^2 I_n)^{-1}K_{ud}(y, X).
\end{align*}
\]

When a finite element approximation is employed, we get

\[
\begin{align*}
    m_{d,ud}^{FE}|y(x) &= m_{ud}^{FE}(x) + K_{ud}^{FE}(x, X)^T(K_{ud}^{FE}(X, X) + \sigma^2 I_n)^{-1}(y - m_{ud}^{FE}(X)), \\
    K_{d,ud}^{FE}|y(x, y) &= K_{ud}^{FE}(x, y) - K_{ud}^{FE}(x, X)^T(K_{ud}^{FE}(X, X) + \sigma^2 I_n)^{-1}K_{ud}^{FE}(y, X).
\end{align*}
\]

where \( m_{ud}^{FE} = m_u^{FE} + m_d \) and \( K_{ud}^{FE} = K_u^{FE} + K_d \).

2.4 Extensions

In practice, a variety of additional levels of statistical modelling, or altogether a more complex PDE model, are typically used in staffFEM (Girolami et al., 2021). These can include an additional factor on the left-hand side of (2.2) which is modelled as a GP, the standard example being Poisson’s equation

\[
-\nabla(e^\mu \nabla u) = f
\]

(2.15)
in which a GP prior is placed on \( \mu \) (and the exponential ensures positivity of the diffusion coefficient, \( e^\mu \)) in addition to \( f \), as done in this article, and estimation of various parameters present in model, such as parameters of the covariance kernel \( K \) (e.g., \( \sigma, \ell \) and \( \nu \) of a Matérn kernel). If GP priors are placed on \( \mu \) and \( f \) in the model (2.15) or its generalisation of some form, the prior induced on \( u \) is no longer a GP. This would render most of the theoretical tools that we use inoperative, and this generalisation is not accordingly pursued here. While there is some recent theoretical work on parameter estimation in Gaussian process regression for deterministic data-generating functions and its effect on posterior rates of convergence and reliability (Karvonen et al., 2020; Teckentrup, 2020; Wang, 2021; Karvonen, 2023), the results that have been obtained are not yet sufficiently general to be useful in our setting.

3 Main Results

This section contains the main results of this article. The results provide rates of contraction, as \( n \to \infty \), of the expectation (with respect to the observation noise distribution) of the \( L^2(\Omega) \)-norm between the true source term \( u_t \) and the GP conditional means in (2.9a), (2.11a) and (2.14a). All proofs are deferred to Section 6. The results are expressed in terms of the fill-distance

\[
h_{X,\Omega} = \sup_{x \in \Omega} \min_{i=1, \ldots, n} \|x - x_i\|_2
\]

(3.1)
of the set of points \( X \subset \Omega \). The fill-distance cannot tend to zero with a rate faster than \( n^{-1/d} \), a rate which is achieved by, for example, uniform Cartesian grids.

3.1 Function Spaces

Let \( D^\alpha f \) denote the weak derivative of order \( \alpha \in \mathbb{N}_0^d \) of any sufficiently regular function \( f : \Omega \to \mathbb{R} \). Let \( k \in \mathbb{N}_0 \). The Sobolev space \( H^k(\Omega) \) consists of functions for which \( D^\alpha f \) exists for all \( |\alpha| \leq k \) and the norm

\[
\|f\|_{H^k(\Omega)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^2(\Omega)}^2 \right)^{1/2}
\]

is finite. The Hölder space \( C^{k,\alpha}(\Omega) \) consists of functions which are \( k \in \mathbb{N}_0 \) times differentiable on \( \Omega \) and whose derivatives of order \( k \) are Hölder continuous with exponent \( \alpha \in (0, 1] \).
Some of our assumptions are expressed in terms of reproducing kernel Hilbert spaces (RKHSs). By the classical Moore–Aronszajn theorem (Berlinet and Thomas-Agnan, 2004, p. 19) every positive-semidefinite kernel $K : \Omega \times \Omega \to \mathbb{R}$ induces a unique RKHS, $\mathcal{H}(K, \Omega)$, which consists of functions $f : \Omega \to \mathbb{R}$ and is equipped with an inner product $\langle \cdot, \cdot \rangle_K$ and norm $\lVert \cdot \rVert_K$. Two fundamental properties of this space are that (i) $K(\cdot, x)$ is an element of $\mathcal{H}(K, \Omega)$ for every $x \in \Omega$ and (ii) the reproducing property

$$\langle f, K(\cdot, x) \rangle_K = f(x) \quad \text{for every} \quad f \in \mathcal{H}(K, \Omega) \text{ and } x \in \Omega. \quad (3.2)$$

Our results will use an assumption that $\mathcal{H}(K, \Omega)$ is norm-equivalent to a Sobolev space.

**Definition 3.1** (Norm-equivalence). The RKHS $\mathcal{H}(K, \Omega)$ is norm-equivalent to the Sobolev space $H^k(\Omega)$, denoted $\mathcal{H}(K, \Omega) \simeq H^k(\Omega)$, if $\mathcal{H}(K, \Omega) = H^k(\Omega)$ as sets and there exist positive constants $C_K$ and $C'_K$ such that

$$C_K \|f\|_{H^k(\Omega)} \leq \|f\|_K \leq C'_K \|f\|_{H^k(\Omega)} \quad (3.3)$$

for all $f \in \mathcal{H}(K, \Omega)$.

The RKHS of a Matérn kernel of smoothness $\nu$ in (2.5) is norm-equivalent to $H^{k+d/2}(\Omega)$, if $k > d/2$, the Sobolev embedding theorem ensures that any kernel which is norm-equivalent to $H^k(\Omega)$ is continuous and that all functions in its RKHS are continuous. From now on we assume that $\mathcal{H}(K, \Omega) \subset H_C(\Omega)$, which is to say that the PDE in (2.2) admits a unique classical solution for every $f \in \mathcal{H}(K, \Omega)$.

### 3.2 Exact Posterior

Our first result concerns an ideal statFEM that uses no finite element discretisation is used. The relevant posterior moments are given in (2.9).

**Theorem 3.2.** Let $k > d/2$ and suppose that Assumption 2.1 holds and $c \leq 0$. If $\mathcal{H}(K, \Omega) \simeq H^k(\Omega)$, $m \in H^k(\Omega)$ and $f_t \in H^k(\mathbb{R}^d) \cap C^k(\mathbb{R}^d)$, then there are positive constants $C_1$ and $h_0$, which do not depend on $X$, such that

$$\mathbb{E}\left[ \|u_t - m_u\|_{L^2(\Omega)} \right] \leq C_1 \left( h_{X,\Omega}^{k+2} \sqrt{n} + h_{X,\Omega}^{d/2} n^{d/(4k)} \right) \quad (3.4)$$

whenever $h_{X,\Omega} \leq h_0$. If $h_{X,\Omega} = O(n^{-1/d})$, then there are positive constant $C_2$ and $n_0$, which do not depend on $X$, such that

$$\mathbb{E}\left[ \|u_t - m_u\|_{L^2(\Omega)} \right] \leq C_2 n^{-1/2 + d/(4k)} \quad (3.5)$$

if $n \geq n_0$.

**Remark 3.3.** The mini-max optimal rate for regression in $H^k(\Omega)$ is $n^{-k/(2k+d)}$ (Tsybakov, 2009, Chapter 2). Since

$$\frac{1}{2} - \frac{d}{4k} - \frac{1}{2} - \frac{d}{4k + 2d} = \frac{k}{2k + d},$$

the rate (3.5) that we have proved is slightly sub-optimal.

### 3.3 Finite Element Posterior

Next we turn to the analysis the effect of finite element discretisation and consider the posterior moments in (2.11). A straightforward combination of Theorem 3.2 and Proposition 6.12 yields an error estimate that combines the errors from GP regression and finite element discretisation.

**Theorem 3.4.** Let $k > d/2$. Suppose that Assumptions 2.1 and 2.2 and hold and that $c \leq 0$. If $\mathcal{H}(K, \Omega) \simeq H^k(\Omega)$, $m \in H^k(\Omega)$, $f_t \in H^k(\mathbb{R}^d) \cap C^k(\mathbb{R}^d)$ and $h_{X,\Omega} = O(n^{-1/d})$, then there are constant $C$ and $n_0$, which do not depend on $X$, such that

$$\mathbb{E}\left[ \|u_t - m_{u,\text{FE}}\|_{L^2(\Omega)} \right] \leq C \left( n^{-1/2 + d/(4k)} + (n_{\text{FE}}^{-q} + \sigma_\varepsilon^2) \sigma_\varepsilon^{-2} (\|f_t\|_{L^\infty(\Omega)} + \sigma_\varepsilon) n_{\text{FE}}^{-q} n^{3/2} \right) \quad (3.6)$$

if $n \geq n_0$. 

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Remark 3.5. To obtain the best possible rate of convergence in terms of $n$ in (3.6), we could set

$$n_{FE} = n^{(2-d/(4k))}/q.$$  \hfill (3.7)

By incorporating all other terms in the constant $C$, we then obtain the error estimate

$$\mathbb{E}\left[ \|u_t - m_{FE}[Y]\|_{L^2(\Omega)} \right] \leq C\left(n^{1/2+d/(4k)} + n_{FE}^{-q}n^{3/2}\right) = 2Cn^{-1/2+d/(4k)},$$

which is equal to the bound in (3.5) up to a constant factor.

Practical application of Remark 3.5 is difficult because, while (3.7) yields the best possible polynomial rate in (3.6), what one would actually like to obtain is the smallest possible right-hand side in (3.6). But finding $n_{FE}$ that minimises the right-hand side is difficult because the constant factors involved are rarely, if ever, available.

3.4 Inclusion of a Discrepancy Term

Finally, we consider inclusion of a discrepancy term as described in Section 2.3. The following two theorems concern the posterior means in (2.13a) and (2.14a). In these theorems it is assumed that the points are quasi-uniform, which means that there is $C_{q_0} > 0$ such that

$$q_X \leq h_{X,\Omega} \leq C_{q_0}q_X,$$

where $h_{X,\Omega}$ is the fill-distance in (3.1) and $q_X$ is the separation radius

$$q_X = \frac{1}{2}\min_{i \neq j}\|x_i - x_j\|_2.$$

Quasi-uniformity implies that the mesh ratio $\rho_{X,\Omega} = h_{X,\Omega}/q_X$ is uniformly bounded from above and that $h_{X,\Omega} = O(n^{-1/d})$; see Chapter 14 in Wendland (2005).

Theorem 3.6. Let $k_1 \geq r - 2 \geq k_2 > d/2$ and suppose that Assumption 2.1 holds with $k = k_1$ and $c \leq 0$. If $\mathcal{H}(K, \Omega) \simeq H^{k_1}(\Omega), \mathcal{H}(K_d, \Omega) \simeq H^r(\Omega), m_d \in H^{k_2+2}(\Omega), m \in H^{k_2}(\Omega)$ and $f_1 \in H^{k_2}(\mathbb{R}^d) \cap C^{k_2}(\mathbb{R}^d)$, then there are positive constants $C_1$ and $h_0$, which do not depend on $X$, such that

$$\mathbb{E}\left[ \|u_t - m_{d,u}[Y]\|_{L^2(\Omega)} \right] \leq C_1 \left( h_{X,\Omega}^{k_2+2}r^{-k_2-2} + \sqrt{n}h_{X,\Omega} + n^{(k_2,r)}h_{X,\Omega}^{d/2} \right)$$ \hfill (3.8)

whenever $h_{X,\Omega} \leq h_0$. The constant $\kappa(k_2,r) \leq 1/2$ is given in (6.20). If the points are quasi-uniform, there are positive constant $C_2$ and $n_0$, which do not depend on $X$, such that

$$\mathbb{E}\left[ \|u_t - m_{d,u}[Y]\|_{L^2(\Omega)} \right] \leq C_2 \left( n^{-(k_2+2)/d} + n^{-r/d+1/2} + n^{-1/2+\kappa(k_2,r)} \right)$$ \hfill (3.9)

if $n \geq n_0$.

Theorem 3.7. Let $k_1 \geq r - 2 \geq k_2 > d/2$. Suppose that Assumptions 2.1 (with $k = k_1$) and 2.2 hold and that $c \leq 0$. If $\mathcal{H}(K, \Omega) \simeq H^{k_1}(\Omega), \mathcal{H}(K_d, \Omega) \simeq H^r(\Omega), m_d \in H^{k_2+2}(\Omega), m \in H^{k_2}(\Omega), f_1 \in H^{k_2}(\mathbb{R}^d) \cap C^{k_2}(\mathbb{R}^d)$ and the points are quasi-uniform, then there are positive constants $C$ and $n_0$, which do not depend on $X$, such that

$$\mathbb{E}\left[ \|u_t - m_{FE}[Y]\|_{L^2(\Omega)} \right] \leq C\left(n^{-(k_2+2)/d} + n^{-r/d+1/2} + n^{-1/2+\kappa(k_2,r)} + n_{FE}^{-q}n^{3/2}\right)$$ \hfill (3.10)

if $n \geq n_0$. The constant $\kappa(k_2,r) \leq 1/2$ is given in (6.20).

Unlike the results in Sections 3.2 and 3.3, these theorems are valid also when the smoothness of the source term is misspecified. That is, in Theorems 3.6 and 3.7 it is possible that $k_1$, the smoothness of the kernel $K$ which specifies the prior for source term, is larger than the smoothness, $k_2$, of the true source term $f_1$. Such misspecification results for GP regression in different settings can be found in, for example, Karvonen et al. (2020); Kanagawa et al. (2020); Teckentrup (2020); and Wynne et al. (2021).
4 Numerical Example

In this section we investigate the convergence of the posterior mean $m_{\text{FE}u|Y}$ to the true system response $u_t$ for different values of the kernel smoothness parameter $k$. We consider the one-dimensional Poisson’s equation

$$-u'' = f \quad \text{in} \quad \Omega = (0, 1) \quad \text{and} \quad u(0) = u(1) = 0. \quad (4.1)$$

The true source term is set as the constant function

$$f_t(x) = \frac{\pi^2}{5} \sin(\pi x) + \frac{49\pi^2}{50} \sin(7\pi x). \quad (4.2)$$

The respective true system response is given in closed form by

$$u_t(x) = \frac{1}{5} \sin(\pi x) + \frac{1}{50} \sin(7\pi x).$$

A similar example was used in Girolami et al. (2021).

For the source term, we use a zero-mean Gaussian prior $f_{GP} \sim \text{GP}(0, K)$ with the Matérn covariance kernel (2.5). We use the kernel hyperparameters

$$\ell \in \left\{ \frac{1}{2}, 1 \right\} \quad \text{and} \quad \nu \in \left\{ \frac{1}{2}, \frac{5}{2} \right\}.$$  

As explained in Section 3.1, the values of $\nu$ correspond to the values $k \in \{1, 3\}$ of the RKHS smoothness parameter. In order to facilitate comparison with standard GP regression based on a Matérn kernel, we set scaling parameter $\sigma$ such that the maximum of $K_u$ equals one (see Figure 1). For each $k$, the true source term in (4.2) is an element of the RKHS and of $C^k(\Omega)$. The selection of the hyperparameters can be automated by considering the marginal likelihood or cross-validation (Rasmussen and Williams, 2006, Chapter 5). For finite element analysis we use the standard piecewise linear basis functions centered at $n_{\text{FE}} \in \{32, 64, 128, 256\}$ uniformly placed points on $\Omega = (0, 1)$. To compute the conditional mean $m_{\text{FE}u|Y}$, we use the approximation in (2.12); see Girolami et al. (2021, Section 2.2) for more details. Observations of $u_t$ at $n \in \{2^1 - 1, 2^2 - 1, \ldots, n_{\text{FE}} - 1\}$ uniformly placed points are corrupted by Gaussian noise with variances $\sigma^2_{\varepsilon} \in \{10^{-2}, 10^{-4}\}$. For illustration, the true system response and the finite element approximation to the conditional mean $m_{\text{FE}u|Y}$ and the corresponding 95\% credible interval are shown in Figure 3 for $\nu = 1/2, \ell = 1/2, n_{\text{FE}} = 2048$, and $n = 7$.

Convergence results are depicted in Figures 4 to 7. In these results the $L^2$-norm is approximated by numerical quadrature and the expectation by an average over 100 independent observation noise realisations. For each $\nu$ and $n_{\text{FE}}$ we also plot the $L^2$-error of standard GP regression when $u_t$ is directly modelled as a purely data-driven GP whose kernel is a Matérn with smoothness $\nu + 2$ and parameters $\sigma_{\text{Matérn}} = 1$ and $\ell_{\text{Matérn}} = \ell$. The selection of the smoothness parameter of the Matérn prior for $u_t$ corresponds to the smoothness of the induced prior $u_{GP}$ in statFEM. Being purely data-driven, this Matérn model for $u_t$ does not incorporate the boundary conditions or other structural characteristics.
$$\ell = 1/2 \quad \nu = 1/2 \quad \sigma_\varepsilon^2 = 10^{-2}$$

Figure 4: Empirical approximations to $L^2$-errors over 100 noise realisations when $\ell \in \{1/2, 1\}$, $\nu = 1/2$ and $\sigma_\varepsilon^2 = 10^{-2}$.

$$\ell = 1 \quad \nu = 1/2 \quad \sigma_\varepsilon^2 = 10^{-2}$$

$$\ell = 1/2 \quad \nu = 5/2 \quad \sigma_\varepsilon^2 = 10^{-2}$$

$$\ell = 1 \quad \nu = 5/2 \quad \sigma_\varepsilon^2 = 10^{-2}$$

Figure 5: Empirical approximations to $L^2$-errors over 100 noise realisations when $\ell \in \{1/2, 1\}$, $\nu = 5/2$ and $\sigma_\varepsilon^2 = 10^{-2}$.
\[ \ell = 1/2 \quad \nu = 1/2 \quad \sigma^2 = 10^{-4} \]

\[ \ell = 1 \quad \nu = 1/2 \quad \sigma^2 = 10^{-4} \]

Figure 6: Empirical approximations to \( L^2 \)-errors over 100 noise realisations when \( \ell \in \{1/2, 1\} \), \( \nu = 1/2 \) and \( \sigma^2 = 10^{-4} \).

We see that statFEM outperforms the Matérn model in Figures 4 to 6, particularly for small \( n \). This is to be expected as the prior dominates when there is little data. In Figure 7 (\( \ell \in \{1/2, 1\} \), \( \nu = 5/2 \) and \( \sigma^2 = 10^{-4} \)) statFEM exhibits clear saturation. However, as the Matérn model behaves similarly \( \ell = 1 \), \( \nu = 5/2 \) end \( \sigma^2 = 10^{-4} \), it seems that the saturation effect is not specific to statFEM in this example. The plots also show that statFEM works well even when the number of finite element nodes, \( n_{FE} \), is small. This is important because, especially in higher dimensions, the number of data points will be significantly smaller than the number of finite element nodes so that it will become even more important to encode the PDE and its boundary conditions.

5 Concluding Remarks

We have analysed a particular formulation of the statFEM approach in a deterministic setting with generic points and finite elements. A different set of assumptions could be equally well used—the practical relevance of these assumptions would likely depend much on the application and whether or not the user views the data-generating process as an actual Gaussian process or some unknown deterministic function. Settings that we believe could or could not be analysed using similar or related techniques as those in this article include the following:

- We have considered a “mixed” case in which a GP is used to model a deterministic function. But one could alternatively assume that \( f_t \), and consequently \( u_t \), is a GP (or some other stochastic process) and proceed from there. This is how statFEM is formulated in Girolami et al. (2021).

- Distribution of the points \( x_1, \ldots, x_n \) where the measurement data are obtained is quite generic in this article in that no reference is made to how these points might be selected or sampled and all results are formulated in terms of the fill-distance \( h_{X,\Omega} \) or, in the quasi-uniform case, \( n \). In applications the points may be sampled randomly from some distribution on \( \Omega \). We refer to Briol et al. (2019, Theorem 1) for a related result concerning random points.

- To remove the assumption that the measurement data are noisy is likely to be challenging if one is interested in including the effect of the finite element discretisation. It is straightforward to derive versions of Theorems 3.2 and 3.6 in the noiseless case, but no other result in Section 6.2 generalises readily. The reason is the presence of the factor \( \sigma^{-2} \) in (6.16) which is used to prove all theorems concerning finite element discretisations: if \( \sigma = 0 \), this bound is rendered meaningless.
\[ \ell = 1/2 \quad \nu = 5/2 \quad \sigma_n^2 = 10^{-4} \]

\[ \ell = 1 \quad \nu = 5/2 \quad \sigma_n^2 = 10^{-4} \]

Figure 7: Empirical approximations to \( L^2 \)-errors over 100 noise realisations when \( \ell \in \{1/2, 1\} \), \( \nu = 5/2 \) and \( \sigma_n^2 = 10^{-2} \).

- As already mentioned in Section 3.3, the bounds include a variety of non-explicit constants. We do not believe that the constants are computable in all but perhaps the simplest of special cases.

6 Proofs

This section contains the proofs of the theorems in Section 3.

6.1 Auxiliary Results

We first collect and derive a number of auxiliary results that we use to prove the error estimates. These results are of four types: (i) standard regularity results for solutions of elliptic PDEs; (ii) results on the RKHS of the kernel \( K_u \); (iii) sampling inequalities and (iv) results related to the concentration function and small ball probabilities of Gaussian measures. We use the notation \( C = C(\theta_1, \ldots, \theta_p) \) to indicate that a constant \( C \) depends only on the parameters \( \theta_1, \ldots, \theta_p \).

6.1.1 Regularity Results for Elliptic PDEs

Recall the function spaces from Section 3.1. Certain standard regularity results and estimates play a crucial role in the derivation of our results. The following regularity theorem can be found in, for example, Evans (1998, Theorem 5 in Section 6.3).

**Theorem 6.1.** Consider the elliptic PDE in (2.2). Let \( k \in \mathbb{N}_0 \) and suppose that Assumption 2.1 holds. If \( f \in H^k(\Omega) \cap H_L(\Omega) \), then \( u \in H^{k+2}(\Omega) \) and there is a constant \( C = C(k, \Omega, L) \) such that

\[ \|u\|_{H^{k+2}(\Omega)} \leq C \|f\|_{H^k(\Omega)}. \]

The following boundedness result is a combination of the a priori bound in Theorem 3.7 and the Schauder regularity result in Theorem 6.14 of Gilbarg and Trudinger (1983).

**Theorem 6.2.** Consider the elliptic PDE in (2.2). Suppose that \( \partial \Omega \) is \( C^{2,\alpha} \), \( a_{ij}, b_i, c \in C^{0,\alpha}(\bar{\Omega}) \) for all \( i, j = 1, \ldots, d \) and some \( \alpha \in (0, 1) \) and \( c \leq 0 \). If \( f \in C^{0,\alpha}(\bar{\Omega}) \), then \( u \in C^{2,\alpha}(\bar{\Omega}) \) and there is a constant \( C = C(\Omega, L) \) such that

\[ \|u\|_{L^{\infty}(\Omega)} \leq C \|f\|_{L^{\infty}(\Omega)}. \]

Note that Assumption 2.1 implies the regularity assumptions in Theorem 6.2.
6.1.2 Transformed Reproducing Kernel Hilbert Spaces

The following lemma justifies the assumption in (2.6) that $L^{-1}_x K(\cdot, x)$ is an element of $H(K, \Omega) \subset H^d(\Omega)$ for every $x \in \Omega$. Let $\delta_x$ be the point evaluation functional at $x \in \Omega$, which is to say that $\delta_x(f) = f(x)$. As earlier, whenever there is a risk of ambiguity we use subscripts to denote the variable to which a functional or an operator applies to. That is,

$$ Af(x) = A_x f(x) = (\delta_x \circ A)_x f(x') $$

for any operator $A$.

**Lemma 6.3.** Let $k > d/2$. Suppose that $H(K, \Omega) \simeq H^k(\Omega)$ and that Assumption 2.1 holds. Then the functional $\delta_x \circ L^{-1}$ is bounded on $H(K, \Omega)$ and $L^{-1}_x K(\cdot, x) \in H(K, \Omega)$ for every $x \in \Omega$.

**Proof.** Let $x \in \Omega$ and set $\ell_x = \delta_x \circ L^{-1}$. Since $H(K, \Omega)$ is norm-equivalent to $H^k(\Omega)$ and $H^{k+2}(\Omega)$ is continuously embedded in $C(\Omega)$, it follows from Theorem 6.1 that

$$ |\ell_x(f)| = |u(x)| \leq \|u\|_{L\infty(\Omega)} \leq C_1 \|u\|_{H^{k+2}(\Omega)} \leq C_1 C_2 \|f\|_{H^k(\Omega)} \leq C_1 C_2 C^{-1} \|f\|_K $$

for any $f \in H(K, \Omega)$ and constants $C_1 = C(k, \Omega)$ and $C_2 = C(k, \Omega, L)$. This proves that $\ell_x$ is bounded on $H(K, \Omega)$. Because $\ell_x$ is bounded, it follows from the Riesz representation theorem that there exists a unique function $l_x \in H(K, \Omega)$ such that $\ell_x(f) = \langle f, l_x \rangle_K$ for every $f \in H(K, \Omega)$. Setting $f = K(\cdot, y)$ and using the reproducing property (3.2) we get, for any $y \in \Omega$,

$$ L^{-1}_x K(y, x) = \ell_x(K(y, x)) = \langle K(y, \cdot), l_x \rangle_K = l_x(y). $$

That is, $L^{-1}_x K(\cdot, x) = l_x \in H(K, \Omega)$. \hfill \square

Next we want to understand how the RKHS of the kernel $K_x$ defined in (2.7) relates to that of $K$. We use the following general proposition about transformations of RKHSs. See Theorems 16.8 and 16.9 in Wendland (2005) or Section 5.4 in Paulsen and Raghupathi (2016) for similar results. A proof is included here for completeness and because our formulation differs slightly from those that we have found in the literature.

**Proposition 6.4.** Let $K$ be a positive-semidefinite kernel on $\Omega$ and $A$ an invertible linear operator on $H(K, \Omega)$ such that the functional $\delta_x \circ A$ is bounded on $H(K, \Omega)$ for every $x \in \Omega$. Then

$$ R(x, y) = A_x A_y K(x, y) = (\delta_x \circ A)_x (\delta_y \circ A)_y K(x', y') $$

defines a positive-semidefinite kernel on $\Omega$. Furthermore,

$$ H(R, \Omega) = A(H(K, \Omega)) = \{ A f : f \in H(K, \Omega) \} \quad \text{and} \quad \|A f\|_R = \|f\|_K \quad \text{for every} \ f \in H(K, \Omega). $$

**Proof.** Because the functional $\ell_y = \delta_y \circ A$ is bounded on $H(K, \Omega)$, the Riesz representation theorem implies that there exists a unique representor $l_y \in H(K, \Omega)$ such that $\ell_y(f) = \langle f, l_y \rangle_K$ for every $f \in H(K, \Omega)$. Therefore, by the reproducing property,

$$ l_y(x) = \langle K(x, \cdot), l_y \rangle_K = \ell_y(K(x, \cdot)) = (\delta_y \circ A)_u K(x, u) $$

for any $x, y \in \Omega$. Since $l_y$ is an element of $H(K, \Omega)$, $\ell_x(l_y) = \langle \ell_x, l_y \rangle_K$ and

$$ \ell_x(l_y) = (\delta_x \circ A)_x (l_y) = (\delta_x \circ A)_v (\delta_y \circ A)_u K(v, u) = R(x, y), $$

from which it follows that $R$ is a well-defined kernel. To verify that $R$ is positive-semidefinite, compute

$$ \sum_{i=1}^N \sum_{j=1}^N a_i a_j R(x_i, x_j) = \sum_{i=1}^N \sum_{j=1}^N a_i a_j \langle l_{x_i}, l_{x_j} \rangle_K = \left( \sum_{i=1}^N a_i l_{x_i} \right) \left( \sum_{i=1}^N a_i l_{x_i} \right)_K $$

$$ = \left\| \sum_{i=1}^N a_i l_{x_i} \right\|_K^2 \geq 0 $$

(6.1)
for any $N \in \mathbb{N}$, $a_i \in \mathbb{R}$, and $x_i \in \Omega$. To prove the claims related to $\mathcal{H}(R, \Omega)$ we use a classical characterisation (e.g., Paulsen and Raghupathi, 2016, Section 3.4) which states that $f \in \mathcal{H}(K, \Omega)$ if and only if there is $c > 0$ such that

$$K_c(x, y) = c^2 K(x, y) - f(x)f(y)$$  \hspace{1cm} (6.2)

defines a positive-semidefinite kernel. The smallest constant for which $K_c$ is positive-semidefinite equals the RKHS norm of $f$. Now, assuming that $f \in \mathcal{H}(K, \Omega)$ and applying $A$ twice on (6.2) yields the kernel

$$R_c(x, y) = c^2 R(x, y) - Af(x)Af(y),$$

which, by the argument used in (6.1), is positive-semidefinite if $K_c$ is. This establishes that $A(\mathcal{H}(K, \Omega)) \subset \mathcal{H}(R, \Omega)$ and $\|Af\|_R \leq \|f\|_K$. That these are indeed equalities follows directly from the invertibility of $A$: $\mathcal{H}(K, \Omega) \rightarrow A(\mathcal{H}(K, \Omega))$.

Applying Proposition 6.4 to $A = L^{-1}$ yields the following theorem.

**Theorem 6.5.** Let $k > d/2$ and consider the kernel $K_u$ in (2.7). If $\mathcal{H}(K, \Omega) \simeq H^k(\Omega)$ and Assumption 2.1 holds, then

(i) The kernel $K_u$ is positive-semidefinite on $\Omega$ and its RKHS is

$$\mathcal{H}(K_u, \Omega) = \{u : u \text{ is a solution of (2.2)} \text{ for some } f \in \mathcal{H}(K, \Omega)\}. \hspace{1cm} (6.3)$$

Furthermore, $\|u\|_{K_u} = \|f\|_K$.

(ii) It holds that $\mathcal{H}(K_u, \Omega) \subset H^{k+2}(\Omega)$ and there are constants $C_u = C(K, k, \Omega, L)$ and $C_u' = C(K, k, \Omega, L)$ such that

$$C_u \|u\|_{H^{k+2}(\Omega)} \leq \|u\|_{K_u} \leq C_u' \|u\|_{H^{k+2}(\Omega)}$$  \hspace{1cm} (6.4)

for all $u \in \mathcal{H}(K_u, \Omega)$.

**Proof.** Because $\mathcal{H}(K, \Omega) \subset L^2(\Omega)$, the linear operator $L$ is invertible on $\mathcal{H}(K, \Omega)$. Furthermore, by Lemma 6.3 the functionals $\delta_x \circ L^{-1}$ are bounded on $\mathcal{H}(K, \Omega)$. Therefore the first claim follows by applying Proposition 6.4 to $A = L^{-1}$. To verify the second claim, observe that it now follows from the norm-equivalence $\mathcal{H}(K, \Omega) \simeq H^k(\Omega)$ and Theorem 6.1 that, for a constant $C = C(k, \Omega, L)$,

$$\|u\|_{K_u} = \|f\|_K \geq C_k \|f\|_{H^k(\Omega)} \geq C_k C^{-1} \|u\|_{H^{k+2}(\Omega)}$$

and

$$\|u\|_{K_u} = \|f\|_K \leq C_k' \|f\|_{H^k(\Omega)} = C_k' \|Lu\|_{H^k(\Omega)} \leq C_k' C_L \|u\|_{H^{k+2}(\Omega)},$$

where $C_L = C_k(k, \Omega, L)$ and the last inequality follows from the fact that the differential operator $L$ is second-order and its coefficient functions are in $C^{k+1}(\Omega)$ by Assumption 2.1.

Finally, we will need the following result (e.g., Berliet and Thomas-Agnan, 2004, p. 24) on the RKHS of a sum kernel to analyse statFEM when a discrepancy term is included (recall Section 2.3).

**Theorem 6.6.** Let $K_1$ and $K_2$ be two positive-semidefinite kernels on $\Omega$. Then $R = K_1 + K_2$ is a positive-semidefinite kernel on $\Omega$ and its RKHS consists of functions which can be written as $f = f_1 + f_2$ for $f_1 \in \mathcal{H}(K_1, \Omega)$ and $f_2 \in \mathcal{H}(K_2, \Omega)$. The RKHS norm is

$$\|f\|^2_R = \min \{\|f_1|^2_{K_1} + \|f_2|^2_{K_2} : f = f_1 + f_2 \text{ s.t. } f_1 \in \mathcal{H}(K_1, \Omega), f_2 \in \mathcal{H}(K_2, \Omega)\}.$$  

6.1.3 Sampling Inequalities

Denote $(x)_+ = \max\{x, 0\}$ for $x \in \mathbb{R}$. The following sampling inequality is the main building block of our error estimates.

**Theorem 6.7** (Arcangéli et al. 2007, Theorem 4.1). Let $p \in [1, \infty]$, $k > d/2$ and $\gamma = \max\{p, 2\}$. If $g \in H^k(\Omega)$, then there are constants $C_1 = C(k, p, \Omega)$ and $h_0 = C(k, \Omega)$ such that

$$\|g\|_{L^p(\Omega)} \leq C \left( h^{-d/(1/2 - 1/p)} X, \Omega \|g\|_{H^k(\Omega)} + h^{d/\gamma} X, \Omega \|h(X)\|_2 \right)$$

whenever $h X, \Omega \leq h_0$. Here $g(X) = (g(x_1), \ldots, g(x_n)) \in \mathbb{R}^n$.

See Wang et al. (2020); Karvonen et al. (2020); Teckentrup (2020); Wynne et al. (2021); and Wang (2021) for a variety of applications of this and related sampling inequalities to error analysis of GP regression.
6.1.4 The Concentration Function and Small Ball Probabilities

Final ingredients that we need are certain results on the concentration function and small ball probabilities of Gaussian measures. Given \( u_0 = \mathcal{L}^{-1} f_0 \) for some \( f_0 \in \mathcal{H}_L(\Omega) \), define the concentration function

\[
\phi_{u_0}(\varepsilon) = \gamma_{u_0}(\varepsilon) + \beta(\varepsilon),
\]

where

\[
\gamma_{u_0}(\varepsilon) = \inf_{u \in \mathcal{H}(K_{u_0}, \Omega)} \{ \| u \|^2_{K_u} : \| u - u_0 \|_{L^\infty(\Omega)} < \varepsilon \}
\]

and

\[
\beta(\varepsilon) = - \log \Pi_u(\{ u : \| u \|_{L^\infty(\Omega)} < \varepsilon \}).
\]

Here \( \Pi_u \) denotes the Gaussian probability measure associated to the zero-mean Gaussian process with covariance kernel \( K_u \).

**Proposition 6.8.** Let \( k \geq r > d/2 \). Suppose that \( \mathcal{H}(K, \Omega) \simeq H^k(\Omega) \), that Assumption 2.1 holds and that \( c \leq 0 \). If there exists \( f_0 \in H^r(\mathbb{R}^d) \cap C^r(\mathbb{R}^d) \) such that \( u_0 = \mathcal{L}^{-1} f_0 |_{\Omega} \), then there is a constant \( C = C(f_0, K, k, r, \Omega, \mathcal{L}) \) such that

\[
\gamma_{u_0}(\varepsilon) \leq C \varepsilon^{-2(k-r)/r}
\]

when \( \varepsilon \) is sufficiently small.

**Proof.** Theorem 6.5 implies that

\[
\gamma_{u_0}(\varepsilon) = \inf_{f \in \mathcal{H}(K, \Omega)} \{ \| f \|_K : \| f - f_0 \|_{L^\infty(\Omega)} < \varepsilon \text{ for } u = \mathcal{L}^{-1} f \}.
\]

Since \( k > d/2 \), so that \( \mathcal{H}(K, \Omega) \) is embedded in a Hölder space, and \( r \geq 1 \), for any \( f \in \mathcal{H}(K, \Omega) \) the function \( f - f_0 |_{\Omega} \) has a unique continuous extension to \( \Omega \) that satisfies the assumptions of Theorem 6.2. Thus there is \( C_1 \equiv C(\Omega, \mathcal{L}) \) such that \( \| f - f_0 |_{\Omega} \|_{L^\infty(\Omega)} \leq C_1 \| f - f_0 |_{\Omega} \|_{L^\infty(\Omega)} \). Therefore \( \| u - u_0 \|_{L^\infty(\Omega)} < \varepsilon \) if \( \| f - f_0 |_{\Omega} \|_{L^\infty(\Omega)} < \varepsilon/C_1 \), which implies that

\[
\gamma_{u_0}(\varepsilon) \leq \inf_{f \in \mathcal{H}(K, \Omega)} \{ \| f \|_K : \| f - f_0 |_{\Omega} \|_{L^\infty(\Omega)} < \varepsilon/C_1 \}.
\]

Lemma 4 in van der Vaart and van Zanten (2011) and Lemma 23 in Wynne et al. (2021) bound the right-hand side as

\[
\inf_{f \in \mathcal{H}(K, \Omega)} \{ \| f \|_K : \| f - f_0 |_{\Omega} \|_{L^\infty(\Omega)} < \varepsilon/C_1 \} \leq C_2 C_1^{2(k-r)/r} \varepsilon^{-2(k-r)/r}
\]

for \( C_2 = C(f_0, K, k, r, \Omega) \) when \( \varepsilon \) is sufficiently small. This completes the proof. \(
\Box\)

Let \( \varepsilon > 0 \). The metric entropy of a compact subset \( A \) of a metric space \( (\mathcal{H}, d_{\mathcal{H}}) \) is defined as \( H_{\text{ent}}(A, d_{\mathcal{H}}, \varepsilon) = \log N(A, d_{\mathcal{H}}, \varepsilon) \) for the minimum covering number

\[
N(A, d_{\mathcal{H}}, \varepsilon) = \min \left\{ n \geq 1 : \text{there exist } x_1, \ldots, x_n \in A \text{ s.t. } A \subset \bigcup_{i=1}^{n} B_{\varepsilon}(x_i; \mathcal{H}, d_{\mathcal{H}}) \right\},
\]

where \( B_{\varepsilon}(x; \mathcal{H}, d_{\mathcal{H}}) \) is the \( x \)-centered \( \varepsilon \)-ball in \( (\mathcal{H}, d_{\mathcal{H}}) \). If \( (\mathcal{H}, d_{\mathcal{H}}) \) is a normed space, we have the scaling identity

\[
H_{\text{ent}}(\lambda A, d_{\mathcal{H}}, \varepsilon) = H_{\text{ent}}(A, d_{\mathcal{H}}, \varepsilon | \lambda|^{-1}) \quad (6.5)
\]

for any \( \lambda \neq 0 \); see, for example, Equation (4.171) in Giné and Nickl (2015).

**Lemma 6.9.** Let \( k > d/2 \). Suppose that \( \mathcal{H}(K, \Omega) \simeq H^k(\Omega) \), that Assumption 2.1 holds and that \( c \leq 0 \). Let \( B_{\varepsilon}^a \) and \( B_{\varepsilon}^b \) denote the unit balls of \( (\mathcal{H}(K_{u_0}, \Omega), \| \cdot \|_{K_{u_0}}) \) and \( (H^k(\Omega), \| \cdot \|_{H^k(\Omega)}) \), respectively. Then there is a constant \( C = C(K, \Omega, \mathcal{L}) \) such that

\[
H_{\text{ent}}(B_{\varepsilon}^a, \| \cdot \|_{L^\infty(\Omega)}, \varepsilon) \leq H_{\text{ent}}(B_{\varepsilon}^b, \| \cdot \|_{L^\infty(\Omega)}, C \varepsilon).
\]
We have whenever where \( u \) is a \( \varepsilon \)-net in \( \mathcal{H}(K, \Omega) \) and \( \{ f_i \} \) is a \( \varepsilon \)-covering of \( \mathcal{H}(K, \Omega) \). Then let \( \psi \) be any \( \varepsilon \)-approximation to \( \mathcal{H}(K, \Omega) \) and any \( \varepsilon \)-covering of \( \mathcal{H}(K, \Omega) \). We have 

\[
L^{-1} B_1 (0; H^k(\Omega), \| \cdot \|_{H^k(\Omega)}) \subset L^{-1} B_1 (0; H^k(\Omega), \| \cdot \|_{L^\infty(\Omega)})
\]

(6.7)

We have \( \| u \|_{K_n} = \| f \|_{K_n} \geq C_K \| f \|_{H^k(\Omega)} \) by Theorem 6.5 and the norm-equivalence \( \mathcal{H}(K, \Omega) \simeq H^k(\Omega) \) and \( \| u \|_{L^\infty(\Omega)} \leq C_\infty \| f \|_{L^\infty(\Omega)} \) for \( C_\infty = C(\Omega, \mathcal{L}) \) by the Sobolev embedding theorem and Theorem 6.2. Therefore

\[
B_n^u = B_1 (0; \mathcal{H}(K_u, \Omega), \| \cdot \|_{K_u}) = L^{-1} B_1 (0; \mathcal{H}(K, \Omega), \| \cdot \|_K) \subset L^{-1} B_1 (0; H^k(\Omega), \| \cdot \|_{H^k(\Omega)})
\]

and

\[
L^{-1} B_1 (f; H^k(\Omega), \| \cdot \|_{L^\infty(\Omega)}) \subset B_{C_\infty u_1} (u; H(K, \Omega), \| \cdot \|_{L^\infty(\Omega)}),
\]

where \( u_1 = L^{-1} f_1 \). Applying these two inclusion relations to (6.7) and using the definition of metric entropy, together with (6.5), yields the claim. \( \square \)

**Proposition 6.10.** Let \( k > d/2 \). Suppose that \( \mathcal{H}(K, \Omega) \simeq H^k(\Omega) \), that Assumption 2.1 holds and that \( \varepsilon \leq 0 \). Let \( B_n^u \) denote the unit ball of (\( \mathcal{H}(K_u, \Omega), \| \cdot \|_{K_u} \)). Then there is a positive constant \( C \), which does not depend on \( \varepsilon \), such that

\[
\beta(\varepsilon) \leq C \varepsilon^{-2d/(2k-d)}
\]

for sufficiently small \( \varepsilon \).

**Proof.** It is a standard result that the metric entropy of the unit ball of \( H^k(\Omega) \) in Lemma 6.9 satisfies

\[
H_{\text{ent}} (B_1^k, \| \cdot \|_{L^\infty(\Omega)}, \varepsilon) \leq C_{\text{ent}} \varepsilon^{-d/k}
\]

for a positive constant \( C_{\text{ent}} = C(k) \) and any \( \varepsilon < 1 \). See, for instance, Theorem 4.3.36 in Giné and Nickl (2015), Theorem 3.3.2 in Edmunds and Triebel (1996), the proof of Lemma 3 in van der Vaart and van Zanten (2011) and Appendix F in Wynn et al. (2021). It follows from Equation (6.6) that

\[
H_{\text{ent}} (B_1^k, \| \cdot \|_{L^\infty(\Omega)}, \varepsilon) \leq C_{\text{ent}} C^{-d/k} \varepsilon^{-d/k}
\]

for sufficiently small \( \varepsilon \). According to Theorem 1.2 in Li and Linde (1999), the estimate (6.8) implies that

\[
\beta(\varepsilon) \leq C' \varepsilon^{-2d/(2k-d)}
\]

for a positive constant \( C' \) which does not depend on \( \varepsilon \). \( \square \)

A combination of Propositions 6.8 and 6.10 yields an estimate for \( \psi_{u_0}(\varepsilon) \). Define the function

\[
\psi_{u_0}(\varepsilon) = \frac{\psi_{u_0}(\varepsilon)}{\varepsilon^2}
\]

and let \( \psi_{u_0}^{-1}(n) = \sup \{ \varepsilon > 0 : \psi_{u_0}(\varepsilon) \geq n \} \) denote its (generalised) inverse.

**Theorem 6.11.** Let \( k \geq r > d/2 \). Suppose that \( \mathcal{H}(K, \Omega) \simeq H^k(\Omega) \), that Assumption 2.1 holds and that \( \varepsilon \leq 0 \). If there exists \( f_0 \in H^r(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d) \) such that \( u_0 = L^{-1} f_0 \), then there is a positive constant \( C \), which does not depend on \( \varepsilon \), such that

\[
\psi_{u_0}^{-1}(n) \leq C n^{-\frac{r-k-d}{2k}}
\]

for any sufficiently large \( n > 0 \).

**Proof.** By Propositions 6.8 and 6.10,

\[
\psi_{u_0}(\varepsilon) \leq C_0 \left( \varepsilon^{-2(k-r)/r-2} + \varepsilon^{-2d/(2k-d)-2} \right) \leq 2C_0 \varepsilon^{-2d/(2k-d)-2}
\]

whenever \( \varepsilon \) is sufficiently small, where the positive constant \( C_0 \) does not depend on \( \varepsilon \). It follows from the definition of \( \psi_{u_0}^{-1} \) that

\[
\psi_{u_0}^{-1}(n) \leq C n^{-\frac{r-k-d}{2k}}
\]

for \( C = (2C_0)^{-\frac{r-k-d}{2k}} \). \( \square \)
6.2 Proofs of Main Results

We are now ready to prove the theorems in Section 3. Given an \( n \)-vector \( Z \), we employ the interpolation operator notation

\[
I_X(Z)(x) = K_n(x, X)^T(K_n(X, X) + \sigma_n^2 I_n)^{-1}Z.
\]  

(6.9)

That is, the ideal conditional mean in (2.9a) can be written as

\[
m_{w|Y} = m_u - I_X(m_u(X)) + I_X(Y).
\]  

(6.10)

**Proof of Theorem 3.2.** By Theorems 6.1 and 6.5, \( u_k, m_u \in \mathcal{H}(K_n, \Omega) \subset H^{k+2}(\Omega) \). Therefore it follows from Theorem 6.7 with \( p = 2 \) and \( g = u_t - m_{w|Y} \) that there are constants \( C_1 \) and \( h_0 = C(k, \Omega) \), which do not depend on \( X \), such that

\[
\| u_t - m_{w|Y} \|_{L^2(\Omega)} \leq C_1 \left( h^{k+2}_X \| u_t - m_{w|Y} \|_{H^{k+2}(\Omega)} + h^{d/2}_X \| m_u(X) - m_{w|Y}(X) \|_2 \right)
\]  

(6.11)

whenever \( h_X \leq h_0 \). The decomposition in (6.10) gives

\[
\| u_t - m_{w|Y} \|_{H^{k+2}(\Omega)} \leq \| u_t - I_X(Y) \|_{H^{k+2}(\Omega)} + \| m_u - I_X(m_u(X)) \|_{H^{k+2}(\Omega)}.
\]  

(6.12)

The triangle inequality and Lemma 17 in Wynne et al. (2021), in combination with (6.4), yield

\[
\| u_t - I_X(Y) \|_{H^{k+2}(\Omega)} \leq C_u^{-1} C_u' \| m_u \|_{H^{k+2}(\Omega)} + \| I_X(Y) \|_{H^{k+2}(\Omega)}
\]

\[
\leq C_u^{-1} C_u' \| m_u \|_{H^{k+2}(\Omega)} + \sigma^{-1}_\varnothing \| \varepsilon \|_2,
\]  

(6.13)

where \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{R}^n \) is the noise vector. The second term in (6.12) has the bound

\[
\| m_u - I_X(m_u(X)) \|_{H^{k+2}(\Omega)} \leq 2C_u^{-1} C_u \| m_u \|_{H^{k+2}(\Omega)},
\]  

(6.14)

which is obtained in the same way as (6.13) but with \( \varepsilon \) set as the zero vector. From Theorem 22 in Wynne et al. (2021) and Theorem 6.11 with \( r = k \) and \( f_0 = f_t \) (i.e., \( u_0 = u_t \) and \( \min\{r, k-d/2\} = k-d/2 \)) we get

\[
\mathbb{E}[\| u_t(X) - m_{w|Y}(X) \|_2^2] \leq C_2 \sqrt{n} \psi_1^{-1}(n) \leq C_3 n^{d/4(4k)}
\]  

(6.15)

for positive constants \( C_2 \) and \( C_3 \) which do not depend on \( X \). Inserting the estimates (6.13)–(6.15) into (6.11) and using the bound \( \sigma^{-1}_\varnothing \mathbb{E}[\| \varepsilon \|_2^2] \leq \sqrt{n} \), which follows from the Gaussianity of the noise terms, yields

\[
\mathbb{E}[\| u_t - m_{w|Y} \|_{L^2(\Omega)}^2] \leq 2C_1 C_u^{-1} C_u' h^{k+2}_X \| m_u \|_{H^{k+2}(\Omega)}^2 + C_1 C_u^{-1} C_u' h^{d/2}_X \| \varepsilon \|_2^2 + C_1 C_2 C_3 n^{d/4(4k)}
\]

\[
\leq 2C_1 C_u^{-1} C_u' h^{k+2}_X \| m_u \|_{H^{k+2}(\Omega)}^2 + C_1 C_2 C_3 h^{d/2}_X n^{d/4(4k)}
\]

\[
+ C_1 C_u^{-1} C_u' h^{k+2}_X \sqrt{n} + C_1 C_2 C_3 h^{d/2}_X n^{d/4(4k)}.
\]  

This concludes the proof of (3.4).

The following proposition allows us to make use of Assumption 2.2 on the error of the finite element discretisation. Although this basic proposition must have appeared several times and in various forms in the literature on scalable approximations for GP regression, we have not been able to locate a convenient reference for it.

**Proposition 6.12.** Let \( R_1 \) and \( R_2 \) be any positive-semidefinite kernels, \( \sigma > 0 \), and \( Z \in \mathbb{R}^n \). If

\[
\sup_{x, y \in \Omega} |R_1(x, y) - R_2(x, y)| = \delta
\]

for some \( \delta > 0 \), then the functions

\[
m_1'(Z)(x) = R_1(x, X)(R_1(X, X) + \sigma^2 I_n)^{-1}Z,
\]

\[
m_2'(Z)(x) = R_2(x, X)(R_2(X, X) + \sigma^2 I_n)^{-1}Z
\]

satisfy

\[
\sup_{x \in \Omega} |m_1'(Z)(x) - m_2'(Z)(x)| \leq \| Z \|_2 \left( \delta + C \sigma^{-2} \right) \sigma^{-2} n,
\]  

(6.16)

where \( C = \sup_{x, y \in \Omega} |R_2(x, y)| \).
Proof. Write
\[
|m_1^Z(x) - m_2^Z(x)|
= |Z^T [(R_1(X, X) + \sigma^2 I_n)^{-1} R_1(x, X) - (R_2(X, X) + \sigma^2 I_n)^{-1} R_2(x, X)]|
\leq \|Z\|_2 \|[(R_1(X, X) + \sigma^2 I_n)^{-1} R_1(x, X) - (R_2(X, X) + \sigma^2 I_n)^{-1} R_2(x, X)]\|.
\]
Let \( R(x, y) = R_1(x, y) - R_2(x, y) \) so that \( \sup_{x,y \in \Omega} |R(x, y)| = \delta \) and
\[
\| (R_1(X, X) + \sigma^2 I_n)^{-1} R_1(x, X) - (R_2(X, X) + \sigma^2 I_n)^{-1} R_2(x, X) \|
= \| (R_2(X, X) + R(X, X) + \sigma^2 I_n)^{-1} (R(x, X) + R_2(x, X))
- (R_2(X, X) + \sigma^2 I_n)^{-1} R_2(x, X) \|
\leq \| (R_2(X, X) + R(X, X) + \sigma^2 I_n)^{-1} - (R_2(X, X) + \sigma^2 I_n)^{-1} R_2(x, X) \|
+ \| (R_2(X, X) + R(X, X) + \sigma^2 I_n)^{-1} R_2(x, X) \|
\leq \sqrt{n} \| (R_2(X, X) + R(X, X) + \sigma^2 I_n)^{-1} - (R_2(X, X) + \sigma^2 I_n)^{-1} \|
+ \sqrt{n} \| (R_2(X, X) + R(X, X) + \sigma^2 I_n)^{-1} \|.
\]
Since \( R_1(X, X) = R_2(X, X) + R(X, X) \) is positive-semidefinite, the largest singular value of the matrix \( (R_2(X, X) + R(X, X) + \sigma^2 I_n)^{-1} \) is \( (\sigma^2 + \lambda_{\min}(R_1(X, X)))^{-1} \). Therefore
\[
\sqrt{n} \| (R_2(X, X) + R(X, X) + \sigma^2 I_n)^{-1} \| = \sqrt{n} \delta (\sigma^2 + \lambda_{\min}(R_1(X, X)))^{-1} \leq \sqrt{n} \sigma^{-2} \delta.
\]
Finally,
\[
\| (R_2(X, X) + R(X, X) + \sigma^2 I_n)^{-1} - (R_2(X, X) + \sigma^2 I_n)^{-1} \|
= \| (R_2(X, X) + R(X, X) + \sigma^2 I_n)^{-1} R_2(X, X) (R_2(X, X) + \sigma^2 I_n)^{-1} \|
\leq \| R(X, X) \| \| (R_2(X, X) + R(X, X) + \sigma^2 I_n)^{-1} (R_2(X, X) + \sigma^2 I_n)^{-1} \|
\leq \sqrt{n} \sigma^{-2} \delta.
\]
The claim follows by putting these estimates together. \( \square \)

The proof of Theorem 3.4 is a straightforward combination of Theorem 3.2 and Proposition 6.12.

Proof of Theorem 3.4. The triangle inequality yields
\[
\|u_t - m_{u|Y}^{\text{FE}}\|_{L^2(\Omega)} \leq \|u_t - m_{u|Y}\|_{L^2(\Omega)} + \|m_{u|Y} - m_{u|Y}^{\text{FE}}\|_{L^2(\Omega)}, \tag{6.17}
\]
Theorem 3.2 bounds the expectation of the first term as
\[
\mathbb{E} \left[ \|u_t - m_{u|Y}\|_{L^2(\Omega)} \right] \leq C_1 n^{-1/2 + \epsilon/4k} \tag{6.18}
\]
for a constant \( C_1 > 0 \), while Proposition 6.12 and Assumption 2.2 give
\[
\|m_{u|Y} - m_{u|Y}^{\text{FE}}\|_{L^2(\Omega)} \leq C_2 \|Y\|_2 (n_{\text{FE}}^{-q} + \sigma^{-2}) \sigma^{-q} n_{\text{FE}} \tag{6.19}
\]
for a constant \( C_2 > 0 \). Since
\[
\|Y\|_2 \leq \|u_t(X)\|_2 + \mathbb{E}[\|\epsilon\|_2] \leq \|u_t\|_{L^\infty(\Omega)} + \sigma \sqrt{n},
\]
from Theorem 6.2 we obtain
\[
\mathbb{E} \left[ \|m_{u|Y} - m_{u|Y}^{\text{FE}}\|_{L^2(\Omega)} \right] \leq C_3 (n_{\text{FE}}^{-q} + \sigma^{-2}) \sigma^{-q} (\|f_i\|_{L^\infty(\Omega)} + \sigma) n_{\text{FE}} n^{3/2}, \tag{6.19}
\]
for a constant \( C_3 > 0 \). Taking expectation of (6.17) and using the bounds (6.18) and (6.19) concludes the proof. \( \square \)
Proof of Theorem 3.6. By Theorem 6.5, the norm-equivalence assumption and \( k_1 + 2 \geq r \), it holds that \( H(K_u, \Omega) \subset H(K_d, \Omega) \). From this inclusion, Theorem 6.6 and (6.4) it follows that \( H(K_{ud}, \Omega) \simeq H^r(\Omega) \).

By Theorem 6.5 and our assumptions, the functions \( m_d, m_u \) and \( u_t \) are in \( H^{k_2 + 2}(\Omega) \) and \( r \geq k_2 + 2 \). We can therefore apply Theorem 2 in Wynne et al. (2021) with

\[
k = K_{ud}, \quad f = u_t, \quad \tau_k^- = \tau_k^+, \quad \tau_f = k_2 + 2, \quad s = 0, \text{ and } \quad q = 2.
\]

This yields the estimate

\[
E\left[ \|u_t - m_d u|_X\|_{L^2(\Omega)} \right] \leq C h_{X, \Omega}^{d/2} (h_{X, \Omega}^{k_2 + 2 - d/2} \rho_{X, \Omega}^{-k_2 - 2} + \sqrt{h_{X, \Omega}} h_{X, \Omega}^{-d/2} + n^{\kappa})
\]

\[
= C (h_{X, \Omega}^{k_2 + 2} \rho_{X, \Omega}^{-k_2 - 2} + \sqrt{h_{X, \Omega}} h_{X, \Omega}^{k_2} + n^{\kappa(k_2, r)} h_{X, \Omega}^{d/2}),
\]

where

\[
\kappa(k_2, r) = \max \left\{ \frac{1}{2}, \frac{k_2 + 2}{2r}, \frac{d}{4(k_2 + 2)} \right\} \leq \frac{1}{2}, \quad \text{(6.20)}
\]

for a positive constant \( C \), which does not depend on \( X \), and any sufficiently small \( h_{X, \Omega} \). This proves (3.8) while (3.9) follows from \( h_{X, \Omega} = O(n^{-r/d}) \) and the fact that the mesh ratio \( \rho_{X, \Omega} \) is bounded for quasi-uniform points.

Proof of Theorem 3.7. The proof is identical to that of Theorem 3.4 expect that the bound (3.9) is used in place of (3.5).

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