Elliptic polynomials orthogonal on the unit circle with a dense point spectrum

Alexei Zhedanov

Donetsk Institute for Physics and Technology, Donetsk 83114, Ukraine

Abstract

We introduce two explicit examples of polynomials orthogonal on the unit circle. Moments and the reflection coefficients are expressed in terms of Jacobi elliptic functions. We find explicit expression for these polynomials in terms of a new type of elliptic hypergeometric function. We show that obtained polynomials are orthogonal on the unit circle with respect to a dense point measure, i.e. the spectrum consists from infinite number points of increase which are dense on the unit circle. We construct also corresponding explicit systems of polynomials orthogonal on the interval of the real axis with respect to a dense point measure. They can be considered as an elliptic generalization of the Askey-Wilson polynomials of a special type.
1. Introduction

Let $\mathcal{L}$ be some linear functional defined on all possible monomials $z^n$ by the moments

$$c_n = \mathcal{L}(z^n), \quad n = 0, \pm 1, \pm 2 \ldots$$

(1.1)

(in general the moments $c_n$ are arbitrary complex numbers). Then the functional $\mathcal{L}$ is defined on the space of generic Laurent polynomials $P(z) = \sum_{n=-N_1}^{N_2} a_n z^n$ where $a_n$ are arbitrary complex numbers and $N_{1,2}$ are arbitrary integers. Namely we have (by linearity of the functional)

$$\mathcal{L}\{P(z)\} = \sum_{n=-N_1}^{N_2} a_n c_n.$$

The monic Laurent biorthogonal polynomials $P_n(z)$ are defined by the determinantal formula [13]

$$P_n(z) = (\Delta_n)^{-1} \begin{vmatrix} c_0 & c_1 & \ldots & c_n \\ c_{-1} & c_0 & \ldots & c_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1-n} & c_{2-n} & \ldots & c_1 \\ 1 & z & \ldots & z^n \end{vmatrix},$$

where $\Delta_n$ are defined as

$$\Delta_n = \begin{vmatrix} c_0 & c_1 & \ldots & c_{n-1} \\ c_{-1} & c_0 & \ldots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1-n} & c_{2-n} & \ldots & c_0 \end{vmatrix}.$$  

(1.2)

It is obvious from the definition (1.2) that the polynomials $P_n(z)$ satisfy the orthogonality property

$$\mathcal{L}\{P_n(z)z^{-k}\} = h_n \delta_{kn}, \quad 0 \leq k \leq n,$$  

(1.3)

where the normalization constants are

$$h_0 = c_0, \quad h_n = \Delta_{n+1}/\Delta_n.$$  

(1.4)

This orthogonality property can be rewritten in terms of biorthogonal relation [19], [13],

$$\mathcal{L}\{P_n(z)Q_m(1/z)\} = h_n \delta_{nm}$$  

(1.5)

where $h_n = \Delta_{n+1}/\Delta_n$ and the polynomials $Q_n(z)$ are defined by the formula

$$Q_n(z) = (\Delta_n)^{-1} \begin{vmatrix} c_0 & c_{-1} & \ldots & c_{-n} \\ c_1 & c_0 & \ldots & c_{1-n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & \ldots & c_{-1} \\ 1 & z & \ldots & z^n \end{vmatrix},$$  

(1.6)
i.e. the polynomials $Q_n(z)$ are again Laurent biorthogonal polynomials with moments $c_n^{(Q)} = c_{-n}$.

Laurent biorthogonal polynomials satisfy three-term recurrence relation [13]

$$P_{n+1}(z) + d_n P_n(z) = z(P_n(z) + b_n P_{n-1}(z))$$

(1.7)

with some recurrence coefficients $b_n, d_n$. In fact, recurrence relation (1.7) uniquely determine Laurent biorthogonal polynomials $P_n(z)$ under the standard initial conditions $P_{-1} = 0, P_0 = 1$.

An important special case of the Laurent biorthogonal polynomials is obtained if

$$c_{-n} = \bar{c}_n$$

(1.8)

where $\bar{c}_n$ means complex conjugated moments with respect to $c_n$.

In this case the biorthogonal partners are $Q_n(z) = \bar{P}_n(z)$. If additionally we demand that

$$\Delta_n > 0, \quad n = 0, 1, 2, \ldots$$

then there exists a nondecreasing measure $\mu$ on the unit circle $|z| = 1$ such that

$$\int_0^{2\pi} P_n(e^{i\theta})\bar{P}_m(e^{-i\theta})d\mu(\theta) = h_n \delta_{nm}$$

(1.9)

Corresponding polynomials are called the Szegö polynomials orthogonal on the unit circle. In what follows we will denote the Szegö polynomials by $\Phi_n(z)$ instead of $P_n(z)$.

The Szegö polynomials satisfy the recurrence relation

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{a}_n z^n \Phi_n(1/z),$$

(1.10)

where $a_n = -\bar{\Phi}_{n+1}(0)$ are so-called reflection parameters (sometimes called Schur, Geronimus, Verblunsky... parameters). The normalization constants $h_n$ in the orthogonality relation (1.9) are expressed in terms of the reflection parameters as

$$h_n = (1 - |a_0|^2)(1 - |a_1|^2)\ldots(1 - |a_{n-1}|^2)$$

(1.11)

If $|a_n| \neq 1$ then the orthogonal polynomials are nondegenerate, i.e. $h_n \neq 0, n = 0, 1, \ldots$ If however $|a_N| = 1$ for some $N > 1$ then we have a finite system of orthogonal polynomials on the unit circle: $\Phi_0(z), \Phi_1(z), \ldots \Phi_N(z)$.

From the Szegö recurrence relation (1.10) one can derive three term recurrence relation

$$\Phi_{n+1}(z) + d_n \Phi_n(z) = z(\Phi_n(z) + b_n \Phi_{n-1}(z)),$$

(1.12)

where

$$d_n = -\frac{\bar{a}_n}{\bar{a}_{n-1}}, \quad b_n = d_n(1 - |a_{n-1}|^2)$$

Note that relation (1.12) is a special case of three-term recurrence relation (1.7) for the Laurent biorthogonal polynomials.
In what follows we restrict ourselves with the so-called real case, i.e. we will assume that \( c_0 = 1 \) (the standard normalization condition) and all the moments \( c_n \) are real and symmetric \( c_{-n} = c_n \), whereas the reflection parameters \( a_n \) are real and satisfy the restriction

\[
-1 < a_n < 1, \quad n = 0, 1, \ldots
\]  

(1.13)

In this case the polynomials \( \Phi_n(z) \) have the orthogonality property

\[
\int_0^{2\pi} \Phi_n(e^{i\theta})\Phi_m(e^{-i\theta})d\mu(\theta) = h_n \delta_{nm},
\]  

(1.14)

where the measure possesses symmetric property

\[\mu(2\pi - \theta) = -\mu(\theta)\]

In case if the measure admits existence of the weight function \( \rho(\theta) \)

\[d\mu(\theta) = \rho(\theta)d\theta\]

we have that \( \rho(\theta) \) is a real nonnegative function on the unit circle (i.e. \( \rho(\theta) \geq 0, \quad 0 \leq \theta \leq 2\pi \)) which is symmetric with respect to the reflection

\[\rho(2\pi - \theta) = \rho(\theta)\]

Note that in the case (1.13) all expansion coefficients of the polynomials \( P_n(z) \) are real, i.e. polynomials \( P_n(z) \) take real values for real \( z \).

Orthogonality condition (1.14) in this case is equivalent to the conditions

\[
\int_0^{2\pi} \Phi_n(e^{i\theta})e^{-im\theta}d\mu(\theta) = 0, \quad m = 0, 1, \ldots n - 1
\]  

(1.15)

The recurrence relation for this case becomes

\[
\Phi_{n+1}(z) = z\Phi_n(z) - a_n z^n \Phi_n(1/z),
\]  

(1.16)

with the initial condition \( \Phi_0(z) = 1 \). Note that if the sequence of the real reflection parameters \( a_n, \quad n = 0, 1, 2, \ldots \) is given then the monic polynomials \( \Phi_n(z) \) are uniquely determined through the recurrence relation (1.16). Vice versa, if the real moments \( c_n \) are given \( (c_0 = 1) \) with the symmetry condition \( c_{-n} = c_n \) and with the positivity condition \( \Delta_n > 0, \quad n = 1, 2, \ldots \) then all the reflection coefficients \( a_n \) are determined uniquely and satisfy the restriction \( -1 < a_n < 1, \quad n = 0, 1, \ldots \)

In what follows we will use a simple

**Proposition 1** Assume that \( -1 < a_n < 1 \) for all \( n = 0, 2, \ldots \) and corresponding polynomials \( \Phi_n(z) \) have the expression

\[
\Phi_n(z) = \sum_{s=0}^{n} W_{ns} z^s
\]
with some expansion coefficients \( W_{ns} \) (\( W_{nn} = 1 \) due to monicity of polynomials \( \Phi_n(z) \)). Assume that there exists a real sequence \( \tilde{c}_n \), \( n = 0, 1, 2, \ldots \) such that \( \tilde{c}_0 = 1 \) and

\[
\sum_{s=0}^{n} W_{ns} \tilde{c}_s = 0, \quad n = 1, 2, 3, \ldots
\]  

(1.17)

Then the sequence \( \tilde{c}_n \) coincides with the (unique) sequence of moments \( c_n \) corresponding to the polynomials \( \Phi_n(z) \).

**Proof.** Under condition \(-1 < a_n < 1\) we have that the polynomials \( \Phi_n(z) \) have a nondecreasing measure on the unit circle with some unique moments \( c_n \) \( (c_0 = 1) \). The orthogonality relation for these polynomials can be presented in the form

\[
\langle L \Phi_n(z), z^{-m} \rangle = 0, \quad m = 0, 1, 2, \ldots, n - 1
\]  

(1.18)

or in an equivalent form

\[
\sum_{s=0}^{n} W_{ns} c_{s-m} = 0, \quad n = 1, 2, \ldots, m = 0, 1, \ldots, n - 1,
\]  

(1.19)

where \( c_n = \langle L, z^n \rangle \) are the unique moments corresponding to the linear functional \( L \). In particular, for \( m = 0 \) we have the relation

\[
\sum_{s=0}^{n} W_{ns} c_s = 0, \quad n = 1, 2, \ldots,
\]  

(1.20)

Now consider relation (1.17) for \( n = 1 \). We have \( \tilde{c}_1 - a_0 \tilde{c}_0 = 0 \). Hence \( \tilde{c}_1 = a_0 = c_1 \) because \( \tilde{c}_0 = 1 \). We see that the moments \( \tilde{c}_1 \) and \( c_1 \) coincide. In order to prove that \( \tilde{c}_n = c_n \) for all \( n = 1, 2, 3, \ldots \) it is sufficient to apply induction with respect to condition (1.17). This finishes proving of the proposition. The importance of this proposition is that if we know the only orthogonality condition (1.17) then the general orthogonality relations (1.19) and (1.18) holds automatically and moreover the sequence \( \tilde{c}_n \) coincides with the moment sequence \( c_n \).

Together with given polynomials \( \Phi_n(z) \) one can introduce the “reflected sign” polynomials \( \tilde{\Phi}_n(z) = (-1)^n \Phi_n(-z) \).

**Proposition 2** If \( \Phi_n(z) \) are polynomials orthogonal on the unit circle with real reflection coefficients \( a_n \) then the polynomials \( \tilde{\Phi}_n(z) \) are again polynomials orthogonal on the unit circle with the reflection coefficients \( \tilde{a}_n = (-1)^{n+1} a_n \). The expansion coefficients for the polynomials \( \tilde{\Phi}_n(z) \) are \( \tilde{W}_{ns} = (-1)^{n+s} W_{ns} \). The moments \( \tilde{c}_n \) corresponding to the polynomials \( \tilde{\Phi}_n(z) \) are \( \tilde{c}_n = (-1)^n c_n \).

The proof of this proposition is trivial. We will use this proposition in order to reduce the expansion coefficients \( W_{ns} \) and reflection parameters \( a_n \) to the most convenient form.

Note also that for the case of real parameters \( a_n \) we have simple expressions for \( \Phi_n(\pm 1) \):

\[
\Phi_n(1) = \prod_{s=0}^{n-1} (1 - a_s)
\]  

(1.21)

and

\[
\Phi_n(-1) = (-1)^n \prod_{s=0}^{n-1} (1 + (-1)^s a_s)
\]  

(1.22)

These formulas follow directly from recurrence relations (1.10).
2. Elliptic binomial coefficients and their properties

Define the so-called ”elliptic binomial coefficients” (EBC) $E^n_j$. (In what follows we will often omit dependence of the Jacobi elliptic functions like $\text{sn}(z;k)$ on the modulus $k$). If $0 \leq j \leq n$ then

\begin{equation}
E^n_0 = 1, \quad E^n_j = \prod_{s=0}^{j-1} \frac{\text{sn}(w(n-s))}{\text{sn}(w(s+1))},
\end{equation}

and $E^n_j = 0$ otherwise. In (2.1) $w$ is an arbitrary real parameter satisfying the restriction

\begin{equation}
w \neq 4KM_1/M_2
\end{equation}

with some nonzero integers $M_1, M_2$. Recall that $K \equiv K(k)$ is the complete elliptic integral of the first kind.

The elliptic modulus $k$ is assumed to belong to the standard interval $0 < k < 1$. In this case all EBC are real for all $j, n = 0, 1, \ldots$

We have the obvious symmetry property

\begin{equation}
E^n_{n-j} = E^n_j, \quad j = 0, 1, \ldots, n
\end{equation}

Moreover, in the limit $w = 0$ we obtain the ordinary binomial coefficients

\[ \lim_{w \to 0} E^n_j = \binom{n}{j} = \frac{n!}{j!(n-j)!} \]

In the limit $k = 0$ we get trigonometric degeneration of the elliptic Jacobi functions:

\begin{equation}
\lim_{k \to 0} E^n_j = \prod_{s=0}^{j-1} \frac{\sin(w(n-s))}{\sin(w(s+1))} = q^{\frac{w^2}{2}} \left[ \begin{array}{c} n \\ j \end{array} \right]_q, \quad q = e^{-2iw},
\end{equation}

where $\left[ \begin{array}{c} n \\ j \end{array} \right]_q$ are so-called the Gauss polynomials or q-binomial coefficients [3], [16] defined as

\begin{equation}
\left[ \begin{array}{c} n \\ j \end{array} \right]_q = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-j+1})}{(1 - q)(1 - q^2) \cdots (1 - q^j)},
\end{equation}

where $(a)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ is standard Pochhammer q-symbol (q-shifted factorial).

Another limit $k = 1$ leads to hyperbolic functions degeneration. In this limit we have

\begin{equation}
\lim_{k \to 1} E^n_j = \frac{(q)_n}{(q)_j(q)_{n-j}} \frac{(-q)_{n-j}}{(-q)_n} \frac{(-q)_j(q^{-n})_j}{(-q)_n},
\end{equation}

where $q = e^{-2w}$.

A less trivial property of the EBC is the recurrence relation:

\begin{equation}
\text{cn}(wj)\text{dn}(w(n-j))E^n_j = \text{cn}(wn)E^{n-1}_{j-1} + \text{dn}(wn)E^n_{j-1}
\end{equation}

In order to prove the relation (2.7) we need addition formulas for the Jacobi elliptic functions [24] from which the identity follows

\[ \text{sn}(wn)\text{dn}(w(n-j))\text{cn}(wj) = \text{cn}(wn)\text{sn}(wj) + \text{dn}(wn)\text{sn}(w(n-j)) \]
which is valid for all (complex) values of \( w, n, j \). Using this identity we immediately arrive at recurrence relation (2.7). Note that in the limit \( w = 0 \) relation (2.7) becomes the well known recurrence relation for the ordinary binomial coefficients:

\[
\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}
\]  

(2.8)

There are other recurrence relations for EBC of the similar type (2.7). For example if one replaces \( j \to n-j \) in (2.7) and apply the symmetry property (2.3) then one obtains

\[
\text{cn}(w(n-j)) \text{dn}(wj) E_{j}^{n} = \text{cn}(wn) E_{j-1}^{n-1} + \text{dn}(wn) E_{j-1}^{n-1}
\]  

(2.9)

which differs from (2.7) by interchanging of the functions \( \text{cn} \) and \( \text{dn} \). We present also the relation

\[
\text{cn}(w(n-j)) \text{dn}(wj) E_{j}^{n} = \text{cn}(wn) E_{j-1}^{n-1} + \text{dn}(wn) \text{cn}(wj) E_{j-1}^{n-1}
\]  

(2.10)

and the similar one

\[
\text{cn}(wj) E_{j}^{n} = \text{dn}(wj) E_{j-1}^{n-1} + \text{cn}(wn) \text{dn}(w(n-j)) E_{j-1}^{n-1}
\]  

(2.11)

obtained from (2.10) by the change \( j \to n-j \).

Another pair of relations

\[
\text{dn}(w(n-j)) E_{j}^{n} = \text{cn}(w(n-j)) E_{j-1}^{n-1} + \text{dn}(wn) \text{cn}(wj) E_{j-1}^{n-1}
\]  

(2.12)

and

\[
\text{dn}(wj) E_{j}^{n} = \text{cn}(wj) E_{j-1}^{n-1} + \text{dn}(wn) \text{cn}(w(n-j)) E_{j-1}^{n-1}
\]  

(2.13)

is formally obtained from (2.10) and (2.11) by interchanging \( \text{cn} \) and \( \text{dn} \) functions.

All these relations are derived in a similar way using addition formulas for the elliptic Jacobi functions. Moreover, all these relations are reduced to the standard one (2.8) the limit \( w \to 0 \).

Note that our definition of the "elliptic binomial coefficients" differs from the one given, e.g. in [11].

3. \textbf{cn-polynomials on the unit circle}

Introduce the polynomials

\[
\Phi_{n}^{(C)}(z) = \sum_{s=0}^{n} W_{ns} z^{s},
\]  

(3.1)

where the expansion coefficients \( W_{ns} \) are defined as follows. If \( n = 0, 2, 4, \ldots \) is even then

\[
W_{ns} = (-1)^{s} \text{dn}(w(n-s)) E_{s}^{n}
\]  

(3.2)

If \( n = 1, 3, 5, \ldots \) is odd then

\[
W_{ns} = (-1)^{s+1} \text{cn}(w(n-s)) E_{s}^{n}
\]  

(3.3)

It is seen that \( W_{nn} = 1 \) for all \( n = 1, 2, 3, \ldots \) and hence the polynomials \( P_{n}(z) = z^{n} + O(z^{n-1}) \) are monic.
Define the sequence \(a_n = -\Phi_{n+1}^{(C)}(0), \ n = 0, 1, \ldots\) From definition (3.1) it follows that
\[
a_{2n} = \text{cn}(w(2n+1)), \quad a_{2n+1} = -\text{dn}(w(2n+2)), \quad n = 0, 1, 2, \ldots
\]
(3.4)

For the reciprocal polynomials \(\Phi_n^*(z) = z^n\Phi_n(1/z)\) we have (by symmetry property of the elliptic binomial coefficients) similar expression
\[
\Phi_n^{(C)*}(z) = \sum_{s=0}^{n} (-1)^s \text{dn}(ws) E_s^n z^s
\]
for even values of \(n\) and
\[
\Phi_n^{(C)*}(z) = \sum_{s=0}^{n} (-1)^s \text{cn}(ws) E_s^n z^s
\]
for odd values of \(n\).

**Proposition 3** The polynomials \(\Phi_n^{(C)}(z)\) satisfy the recurrence relation
\[
\Phi_{n+1}^{(C)}(z) = z\Phi_n^{(C)}(z) - a_n \Phi_n^{(C)*}(z), \quad n = 0, 1, 2, \ldots
\]
(3.5)

Indeed, comparison of terms \(z^s\) leads to the conclusion that for the even values of \(n\) the recurrence relation (3.5) is equivalent to
\[
-a_n \text{dn}(ws) E_s^n = -cn(w(n+1)).
\]
But this relation coincides with (2.10). Quite analogously, for the odd values of \(n\) we see that condition of vanishing of all terms \(z^s\) in expression (3.5) is equivalent to relation (2.12).

From their definition it follows that the parameters \(a_n\) satisfy an obvious restriction \(-1 < a_n < 1\) for all \(n = 0, 1, \ldots\) Thus the polynomials \(P_n^{(C)}(z)\) belong to the class of the polynomials orthogonal on the unit circle. From general properties [12] it follows that there exists a positive measure on the unit circle providing orthogonality property.

We define a linear functional \(\sigma\) by its moments
\[
c_n \equiv \langle \sigma, z^n \rangle = c_n(wn)
\]
(3.6)

**Proposition 4** If the functional \(\sigma\) is defined as (3.6) then the polynomials \(\Phi_n^{(C)}(z)\) satisfy the orthogonality property
\[
\langle \sigma, \Phi_n^{(C)}(z) \rangle \equiv \sum_{s=0}^{n} W_{ns} c_s = 0, \quad n = 1, 2, \ldots
\]
(3.7)

**Proof.** If \(n\) is odd then the proof is trivial. Indeed, we have
\[
S = \sum_{s=0}^{n} W_{ns} c_s = -\sum_{s=0}^{n} (-1)^s \text{cn}(w(n-s)) \text{cn}(ws) E_s^n.
\]
Changing summation variable \(s \rightarrow n - s\) and using symmetry property (2.3) we have
\[
S = -\sum_{s=0}^{n} (-1)^{n-s} \text{cn}(ws) \text{cn}(w(n-s)) E_s^n = -S,
\]

whence $S = 0$.

Now consider the case when $n$ is even. In this case we have

$$S = \sum_{s=0}^{n} W_{ns} c_s = \sum_{s=0}^{n} (-1)^s \text{dn}(w(n-s)) \text{cn}(ws) E_s^n$$

We can apply relation (2.7) to present the sum $S$ in an equivalent form

$$S = \text{cn}(wn) \sum_{s=0}^{n} (-1)^s E_s^{n-1} + \text{dn}(wn) \sum_{s=0}^{n-1} (-1)^s E_s^{n-1}$$

But $E_s^{n-1} = 0$, hence we can replace $s \rightarrow s + 1$ in the first sum, then we have

$$S = (\text{dn}(wn) - \text{cn}(wn)) \sum_{s=0}^{n-1} (-1)^s E_s^{n-1} = 0$$

due to symmetry property (2.3) (note that in this case $n - 1$ is the odd and all terms in the sum vanish pairwise). Thus we proved the proposition for all values $n = 1, 2, \ldots$.

Now apply Proposition (1) in order to conclude that condition (3.7) is equivalent to more general orthogonality condition

$$\langle \sigma, \Phi_n^{(C)}(z) z^{-j} \rangle = 0, \quad j = 0, 1, \ldots, n - 1$$

or, equivalently to condition

$$\langle \sigma, \Phi_n^{(C)}(z) \Phi_m^{(C)}(1/z) \rangle = h_n \delta_{nm},$$

Moreover, we proved that the moments $c_n$ providing orthogonality relation for the polynomials $P_n^{(C)}(z)$ are given by (3.6) and the functional $\sigma$ defined by (3.6) coincides with the orthogonality functional $\mathcal{L}$.

In order to get the orthogonality measure $\mu(\theta)$ on the unit circle corresponding to the moments $c_n = \text{cn}(wn)$ we use the Fourier expansion of the function $\text{cn}(z)$ [24]:

$$\text{cn}(z; k) = \frac{\pi}{kK} \sum_{s=-\infty}^{\infty} \frac{\exp(i\pi z(s-1/2)/K)}{q^{s-1/2} + q^{-s+1/2}},$$

where

$$q = \exp(-\pi K'/K)$$

and $K' \equiv K(k')$, $k'^2 = 1 - k^2$.

We have

**Theorem 1** The polynomials $\Phi_n(z)$ corresponding to the reflection parameters (3.4) satisfy the orthogonality property on the unit circle

$$\sum_{s=-\infty}^{\infty} \rho_s \Phi_n^{(C)}(z_s) \Phi_m^{(C)}(1/z_s) = h_n \delta_{nm},$$

where

$$z_s = \exp(i\pi w(s-1/2)/K)$$
and the discrete weights \( \rho_s \) are given by
\[
\rho_s = \frac{\pi}{kK} \frac{1}{q^{s-1/2} + q^{-s+1/2}}
\]

**Proof.** It is sufficient to show that the measure given by (3.12) reproduces "true" moments \( c_n \), i.e. we should verify that
\[
c_n \equiv cn(wn, k) = \sum_{s=-\infty}^{\infty} \rho_s z_s^n, \quad n = 0, \pm 1, \pm 2, \ldots
\]
But property (3.15) follows immediately from formula (3.10) if we put \( z = wn \). Thus we verified that our measure gives correct values of all moments \( c_n = cn(wn) \). This finishes the proof, because in our case this measure is the unique due to the condition \(-1 < a_n < 1\).

In order to get expression for the normalization constants \( h_n \) we use formulas (1.11) and (3.4) which yields
\[
h_n = \mu_n \prod_{s=1}^{n} sn^2(ws),
\]
where \( \mu_n = k^n \) for the even \( n \) and \( \mu_n = k^{n-1} \) for the odd \( n \). Clearly, the normalization coefficients are all positive \( h_n > 0 \), \( n = 0, 1, 2, \ldots \).

Clearly, for all \( s = 0, \pm 1, \pm 2, \ldots \) the spectral points (i.e. the only points of growth of the orthogonality measure) \( z_s \) lie on the unit circle. Note that under the condition (2.2) we have that \( z_s \) are dense on the unit circle. This means that for every point \( z_s \) and for any small parameter \( \epsilon \) there exists another spectral point \( z_s' \) such that \( |z_s - z_s'| < \epsilon \). In recent monograph by Barry Simon [20] one can find general results concerning existence of the singular continuous and dense point measures on the unit circle. Our \( cn \)-polynomials \( \Phi_n(z) \) provide a very simple explicit example leading to such measures.

Sometimes the "reflected sign" polynomials \( \tilde{\Phi}_n^{(C)}(z) = (-1)^n\Phi_n^{(C)}(-z) \) are more convenient. Using Proposition 2 we find that polynomials \( \tilde{\Phi}_n^{(C)}(z) \) have the moments \( \tilde{c}_n = (-1)^n cn(wn) \) and the reflection parameters \( \tilde{a}_n = -cn(w(n + 1)) \) for the even \( n \) and \( \tilde{a}_n = -dn(w(n + 1)) \) for the odd \( n \). The expansion coefficients are \( \tilde{W}_{ns} = dn(w(n-s))E_n^{(s)} \) for even \( n \) and \( W_{ns} = cn(w(n-s))E_n^{(s)} \) for the odd \( n \). Transformation \( \Phi_n^{(C)}(z) \to \tilde{\Phi}_n^{(C)}(z) \) in this case is equivalent to the substitution \( w \to w + 2K \). Indeed, we have \( \tilde{c}_n = cn(wn + 2Kn) = (-1)^n cn(wn) \) due to the property \( cn(z + 2K) = -cn(z) \). This observation allows one to construct corresponding orthogonality relation for the polynomials \( \tilde{\Phi}_n^{(C)}(z) \). We have almost the same relation as (3.12):
\[
\sum_{s=-\infty}^{\infty} \rho_s \tilde{\Phi}_n^{(C)}(z_s) \tilde{\Phi}_m^{(C)}(1/z_s) = h_n \delta_{nm},
\]
with the same weights \( \rho_s \) and normalization coefficients and where
\[
z_s = \exp(i\pi w(s - 1/2)/K - i\pi) = -z_s
\]

4. **dn-elliptic polynomials**

In the previous section we considered the polynomials \( \Phi_n^{(C)}(z) \) orthogonal on the unit circle with respect to the moments sequence \( c_n = cn(wn), n = 0, \pm 1, \pm 2, \ldots \). Consider now the polynomials \( \Phi_n^{(D)}(z) \) corresponding to the
moments given by the expression
\[ c_n = \text{dn}(wn), \quad n = 0, \pm 1, \pm 2, \ldots \quad (4.1) \]

Clearly, the moments (4.1) are real and satisfy the symmetry property \( c_{-n} = c_n \) (it is assumed that \( 0 < k < 1 \)).

As in the previous section it is verified that the reflection parameters \( a_n \) corresponding to the moments (4.1) are given by the expression
\[ a_{2n} = \text{dn}(w(2n + 1); k), \quad a_{2n+1} = -\text{cn}(w(2n + 2); k), \quad n = 0, 1, 2, \ldots \quad (4.2) \]

The explicit expression of the polynomials \( \Phi_n^{(D)}(z) = \sum_{s=0}^{\infty} W_{ns} z^s \) in this case is very similar to expression for polynomials \( \Phi_n^{(C)}(z) \). If \( n = 0, 2, 4, \ldots \) is even then
\[ W_{ns} = (-1)^s \text{cn}(w(n - s)) E_s^n \quad (4.3) \]
If \( n = 1, 3, 5, \ldots \) is odd then
\[ W_{ns} = (-1)^{s+1} \text{dn}(w(n - s)) E_s^n \quad (4.4) \]

Hence in this case the coefficients \( W_{ns} \) are obtained from the coefficients (3.2) and (3.3) by simple interchanging the functions \text{cn} and \text{dn}.

For real values of the parameter \( w \) the reflection parameters \( a_n \) are real and satisfy the property \(-1 \leq a_n \leq 1\). If additionally the parameter \( w \) is chosen such that \( w \neq 2KM/N \) then we have \(-1 < a_n < 1\) for all \( n = 0, 1, 2, \ldots \) and hence the reflection parameters satisfy the condition (1.13) meaning that there exists a symmetric measure \( \mu(\theta) \) providing orthogonality of the polynomials \( \Phi_n^{(D)}(z) \) on the unit circle.

In a similar way as in the previous section we obtain the orthogonality measure

**Theorem 2** The polynomials \( \Phi_n^{(D)}(z) \) corresponding to the reflection parameters (4.2) satisfy the orthogonality property on the unit circle
\[ \sum_{s=-\infty}^{\infty} \rho_s \Phi_n^{(D)}(z_s) \Phi_m^{(D)}(1/z_s) = h_n \delta_{nm}, \quad (4.5) \]

where
\[ z_s = \exp(i\pi ws/K) \quad (4.6) \]
and the discrete weights \( \rho_s \) are given by
\[ \rho_s = \frac{\pi}{K} \frac{1}{q^s + q^{-s}} \quad (4.7) \]
with \( q \) given by (3.11).

The proof of this theorem follows immediately from the Fourier expansion of the Jacobi dn-function [24]
\[ \text{dn}(z; k) = \frac{\pi}{K} \sum_{s=-\infty}^{\infty} \frac{\exp(i\pi ws/K)}{q^s + q^{-s}}, \quad (4.8) \]
5. Elliptic derivative operator

The polynomials $\Phi_n^{(C)}(z)$ and $\Phi_n^{(D)}(z)$ are very close to one another in their explicit expression. There is a remarkable relation between them based on a notion of so-called "elliptic derivative operator" $\mathcal{E}$. We define the operator $\mathcal{E}$ on the space of all formal series $\sum_{s=0}^{\infty} c_s z^s$ by its action on monomials:

$$\mathcal{E} z^n = e_n z^{n-1}$$  \hspace{1cm} (5.1)

where

$$e_n = E_n = \frac{\text{sn}(wn)}{\text{sn}(w)}$$

is the "elliptic number". Note that in the limit $w = 0$ we have $e_n = n$ and the operator $\mathcal{E}$ becomes the ordinary derivative operator: $\mathcal{E}_{w=0} = \partial_z$. In the limit $k = 0$ we get the q-derivative operator

$$\mathcal{E} z^n = \frac{q^n - q^{-n}}{q - q^{-1}} z^{n-1}$$

where $q = \exp(iw)$.

The operator $\mathcal{E}$ can be presented explicitly as an infinite sum of q-derivative operators. Indeed, introduce a set of q-derivative operators $W_j$ which act by the formula

$$W_j f(z) = \frac{f(ze^{ij\theta}) - f(ze^{-ij\theta})}{2iz}, \quad j = 1, 2, 3, \ldots$$  \hspace{1cm} (5.2)

with some fixed real parameter $\theta$ (it is assumed that $\theta \neq 2\pi M/N$ with integer $M, N$). On the set of monomials $z^n$ the operator $W_j$ acts as

$$W_j z^n = \sin(nj\theta) z^{n-1}, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (5.3)

Hence the operator $W_j$ sends any polynomial of degree $n$ to a polynomial of degree $n - 1$.

Now we have

$$\mathcal{E} = \sum_{j=1}^{\infty} \beta_j W_{2j-1},$$  \hspace{1cm} (5.4)

where $\theta = \pi w/(2K)$ and

$$\beta_j = \frac{2\pi}{K k(q^{1/2-j} - q^{-1/2}) \text{sn}(w)}, \quad q = \exp(-\pi K'/K)$$

Indeed, formula (5.4) follows from the Fourier expansion of the elliptic sn function [24]:

$$\text{sn}(z; k) = \frac{2\pi}{K k} \sum_{j=1}^{\infty} \frac{\sin((2j-1)t)}{q^{1/2-j} - q^{-1/2}}, \quad t = \pi z/(2K)$$

The operator $\mathcal{E}$ transforms any polynomial in $z$ of degree $n$ to a polynomial of degree $n - 1$.

We have

**Proposition 5** The polynomials $\Phi_n^{(C)}(z)$ and $\Phi_n^{(D)}(z)$ are related as

$$e_n \Phi_n^{(D)}(z) = \mathcal{E} \Phi_n^{(C)}(z), \quad e_n \Phi_n^{(C)}(z) = \mathcal{E} \Phi_n^{(D)}(z)$$  \hspace{1cm} (5.5)
The proof is elementary and is based on explicit expression of the polynomials \( \Phi_n^{(C)}(z) \), \( \Phi_n^{(D)}(z) \).

As an elementary consequence of this proposition we obtain that the operator \( \mathcal{E}^2 \) transforms both families of polynomials \( \Phi_n^{(C)}(z) \) and \( \Phi_n^{(D)}(z) \) to themselves:

\[
\mathcal{E}^2 \Phi_n^{(C)}(z) = e_n e_{n-1} \Phi_{n-2}^{(C)}(z), \quad \mathcal{E}^2 \Phi_n^{(D)}(z) = e_n e_{n-1} \Phi_{n-2}^{(D)}(z)
\]

6. Limits \( k = 0 \) and \( k = 1 \)

The “trigonometric” limit \( k = 0 \) is not interesting because it leads to a degeneration. Indeed, in this limit, e.g.

for the \( \text{cn} \) polynomials we have \( a_{2n+1} = -1 \) for all \( n \). Similarly, for \( \text{dn} \) polynomials in this limit we have \( a_{2n} = 1 \).

This means that we have the degenerated case of the polynomials orthogonal on the unit circle. It is instructive to consider what happens with orthogonality relation in these degeneration cases. In the limit \( k = 0 \) we have [24] \( K(0) = \pi/2 \) and \( K' = \log(4/k) + O(k^2 \log(k)) \). Hence \( q \to k^2/16 \to 0 \).

For the case of \( \text{cn} \)-polynomials we have that only terms \( \rho_0 \) and \( \rho_1 \) in formula (3.14) remain nonzero in the limit \( k = 0 \). Indeed, in this limit we have only two spectral points \( z_0 = e^{-iw} \) and \( z_1 = e^{iw} \) having two equal concentrated masses \( \rho_0 = \rho_1 = 1/2 \).

For the case of \( \text{dn} \)-polynomials we have the only spectral point \( z_0 = 1 \) with the concentrated mass \( \rho_0 = 1 \). Thus in the trigonometric limit the problem becomes trivial.

Consider the “hyperbolic” limit \( k = 1 \). In this case the reflection parameters \( a_n \) for both \( \text{cn} \) and \( \text{dn} \) polynomials coincide:

\[
a_n = \frac{(-1)^n}{\cosh(w(n + 1))}
\]

(6.1)

The moments for both polynomials are

\[
c_n = 1/ \cosh(wn)
\]

(6.2)

Hence both polynomials coincide in this limit and we have from (2.6) the following expression for the power coefficients

\[
W_{ns} = 2\frac{(-1)^{n+s}q^{-n/2+s/2}}{1 + q^{-n}} \frac{(q^n)_s(-q)_s}{(q^{-1})_s(\mathcal{E}(\mathcal{E}(q))}_s
\]

(6.3)

We can present polynomials \( \Phi_n(z) \) in the form

\[
\Phi_n(z) = \sum_{s=0}^{n} W_{ns} z^s = \frac{(-1)^n q^{-n/2}}{1 + q^{-n}} 2\phi_1 \left( q^{-n}, q^{1-n}; -z q^{-1/2} \right),
\]

(6.4)

where \( 2\phi_1 \) stands for the basic hypergeometric function defined as [16]

\[
2\phi_1 \left( \alpha, \beta, \gamma; z \right) \equiv \sum_{s=0}^{\infty} \frac{(\alpha)_{s}(\beta)_{s}}{(\gamma)_{s}} z^s
\]

(clearly if \( \alpha = q^{-n} \) with positive integer \( n \) then we have a finite sum of \( n + 1 \) terms).

The reflection parameters \( a_n \) satisfy the condition \(-1 < a_n < 1, n = 0, 1, \ldots\), hence there exists a unique weight function \( \rho(\theta) \) on the unit circle such that

\[
c_n = 1/ \cosh(wn) = \int_{0}^{2\pi} \rho(\theta) \cos(n\theta) d\theta, \quad n = 0, 1, 2, \ldots
\]

(6.5)
Proposition 6 The weight function \( \rho(\theta) \) for the moments (6.2) is expressed in terms of the Jacobi dn function:

\[
\rho(\theta) = \frac{K(k)}{\pi^2} \text{dn}(K(k)\theta/\pi),
\]

(6.6)

where \( K(k) \) is complete elliptic integral of the first kind and the elliptic modulus \( k \) is determined from the transcendent equation

\[
w = \pi K'(k)/K(k)
\]

(6.7)

Proof of this statement immediately follows from the Fourier expansion (4.8) of the dn function. The function \( \rho(\theta) \) satisfies the required symmetry property \( \rho(2\pi - \theta) = \rho(\theta) \) which follows from the periodicity property \( \text{dn}(u + 2K) = \text{dn}(u) \). Note that to every value of \( w > 0 \) there corresponds a unique value of the modulus \( k \) in the canonical interval \((0, 1)\) such that \( k \to 1 \) when \( w \to 0 \) and \( k \to 0 \) when \( w \to \infty \).

The obtained polynomials \( P_n(z) \) coincide with a special case of the family of biorthogonal polynomials introduced by Pastro [19]. Indeed, the Pastro polynomials depend on two parameters \( \alpha, \beta \) and are expressed as

\[
P_n(z) = A_n \, 2\Phi_1 \left( q^{-n}, q^\alpha; z q^{3/2-\beta} \right)
\]

where \( A_n \) is a constant not depending on \( z \). Comparing with our expression (6.4) we find that \( q^{-1} = q^{\beta-1} = -1 \) (we need also to rescale the argument \( z \to -z/q \)). Thus our polynomials correspond to a special case of the Pastro polynomials with fixed values of the parameters \( \alpha, \beta \). It is interesting to note that the weight function \( \rho(\theta) \) is expressed in terms of the elliptic function. This can be also derived from the results of Pastro [19] who obtained explicit expression of the weight function \( \rho(\theta) \) for arbitrary values of the parameters \( \alpha, \beta \). Pastro presented his result for \( \rho(\theta) \) in terms of an infinite product. In our case this infinite product coincides with corresponding formula for the function \( \text{dn}(x) \).

The "reflected sign" polynomials \( \tilde{\Phi}_n(z) = (-1)^n \Phi_n(-z) \) have the moments \( \tilde{c}_n = (-1)^n / \cosh(wn) \). It is easily seen that corresponding weight function on the unit circle for these polynomials is

\[
\tilde{\rho}(\theta) = \rho(\theta + \pi) = \frac{k'K(k)}{\pi^2} \frac{1}{\text{dn}(K(k)(\theta/\pi))}
\]

7. Special case of polynomials orthogonal on regular polygons

So far, we assumed that the parameter \( w \) satisfies the restriction \( w \neq 4KM_1/M_2 \) with some integer \( M_1, M_2 \). This restriction is necessary in order for polynomials \( \Phi_n(z) \) to be nondegenerate.

In this section we consider the case when this restriction is omitted. This means that a degeneration occurs, i.e. \( a_{N-1} = \pm 1 \) for some positive integer \( N \). Assume that under this condition the polynomial \( \Phi_N(z) \) has only simple zeros:

\[
\Phi_N(z) = (z - z_0)(z - z_1) \ldots (z - z_{N-1})
\]

(7.1)

i.e. \( z_i \neq z_k \) when \( i \neq k \). It can be easily showed [12] that under conditions \( |a_n| < 1, n = 0, 1, \ldots, N - 2 \) all these zeros \( z_n \) lie on the unit circle: \( |z_n| = 1 \).
In this case we can consider only a finite set of polynomials $\Phi_0(z), \Phi_1(z), \ldots, \Phi_{N-1}(z)$ which are orthogonal on the finite set of these zeros on the unit circle \[12\]

\[
\sum_{s=0}^{N-1} \rho_s \Phi_n(z_s)\Phi_m(1/z_s) = h_n \delta_{nm}, \quad n, m = 0, 1, \ldots, N - 1
\]  

(7.2)

where the discrete weights $\rho_s$ can be expressed as \[26\]

\[
\rho_s = \frac{h_{N-1}}{\Phi_{N-1}(1/z_s)\Phi_N'(z_s)}
\]  

(7.3)

As an example, consider the case of cn polynomials and condition

\[
w = KM/N, \quad M = 1, 3, \ldots, N - 2
\]  

(7.4)

(it is assumed that $M$ is co-prime with $N$). From (3.4) we see that under this condition $a_{2N-1} = -1$. Thus we need to calculate zeros of the polynomial $\Phi_{2N}(z)$. We cannot substitute directly the value (7.4) into the explicit expression for $\Phi_{2N}(z)$ because in this case there is indeterminacy in expansion coefficients. Nevertheless we can put $w = KM/N + \epsilon$ and then take the limit $\epsilon = 0$ (because from recurrence relation (3.5) it is seen that expansion coefficients of the polynomials $\Phi_n(z)$ depend on $w$ continuously). Then we easily obtain that under condition (7.4) only the last and the first terms survive in the power expansion of $\Phi_{2N}(z)$:

\[
\Phi_{2N}(z) = z^{2N} + 1
\]

and hence all zeros are simple:

\[
z_s = \exp(\pi i(s - 1/2)/N), \quad s = -N + 1, -N + 1, \ldots
\]  

(7.5)

Note that these zeros coincide with corresponding spectral points (3.13) when $w = KM/N$ and $M$ is co-prime with $N$.

We thus obtained explicitly spectral points $z_s$ in the finite-dimensional case. In order to get expression for the weights $\rho_s$ we should calculate the expression

\[
\Phi_{2N-1}(1/z_s) = \sum_{m=0}^{2N-1} (-1)^{m+1} c_n(K(2N - m - 1)M/N)E_{m}^{2N-1} \exp(-i\pi m(s - 1/2)/N)
\]  

(7.6)

However we can avoid direct calculation of the sum (7.6) if we can construct a finite discrete measure on the unit circle which gives us appropriate values for the moments. In what follows we will consider only the simplest case when $N$ is prime number and $M = 1$. We have the set of $2N$ moments

\[
c_n = cn(Kn/N), \quad n = 0, 1, \ldots, 2N - 1
\]

We should find such values $\rho_j$ that

\[
c_n = cn(Kn/N) = \sum_{j=-N+1}^{N} \rho_j z_j^n = \sum_{j=-N+1}^{N} \rho_j \exp(i\pi n(j - 1/2)/N)
\]  

(7.7)
To do this we return to the Fourier series (3.10) for the cn function and put \( z = K_n/N \). It is convenient to present the summation variable \( s \) modulo \( 2N \), i.e. \( s = j + 2Nm \), where \( j = -N + 1, -N + 2, \ldots N \) and \( m \) can take all integer values \(-\infty < m < \infty \). Then formula (3.10) can be presented in the form

\[
\text{cn}(K_n/N) = \frac{\pi}{k_K} \sum_{j=-N+1}^{N} e^{i\pi n(j-1/2)/N} S(j; N) = \frac{\pi}{k_K} \sum_{j=-N+1}^{N} \sum_{m=-\infty}^{\infty} 1 \frac{1}{q^{j+2Nm-1/2} + q^{-j-2Nm+1/2}} \tag{7.8}
\]

where \( S(j; N) = \sum_{m=-\infty}^{\infty} \frac{1}{q^{j+2Nm-1/2} + q^{-j-2Nm+1/2}} \tag{7.9} \)

We thus have

\[
\rho_j = \frac{\pi}{k_K} S(j; N) \tag{7.10}
\]

In order to calculate the sum \( S(j; N) \) explicitly we need the

**Lemma 1** For any \( q \) such that \( 0 < q < 1 \) and any real \( \alpha \) introduce the function

\[
F(\alpha; q) = \sum_{s=0}^{\infty} \frac{1}{q^{n+\alpha} + q^{-n-\alpha}}. \tag{7.11}
\]

Then the function \( F(\alpha; q) \) has two equivalent representations: either

\[
F(\alpha; q) = \frac{1}{q^\alpha + q^{-\alpha}} \frac{(-q^{1+2\alpha}, -q^{1-2\alpha}, q^2, q^2; q^2)_{\infty}}{(-q^{2+2\alpha}, -q^{2-2\alpha}, q, q; q^2)_{\infty}} \tag{7.12}
\]

or

\[
F(\alpha; q) = \frac{K}{\pi} \text{dn}(2\alpha K' ; k'), \tag{7.13}
\]

where as usual \( q = \exp(-\pi K'/K) \). We adopt standard notation for the \( q \)-shifted factorial \([11] \) \((z; q)_n = (1 - z)(1 - zq) \ldots (1 - zq^{n-1})\) and \((z_1, z_2, \ldots, z_N; q)_n\) stands for product of \( q \)-shifted factorials \((z_1; q)_n \ldots (z_N; q)_n\).

The proof of this Lemma follows directly form the Ramanujan summation formula for the bilateral \(_1\psi_1\) basic hypergeometric function \([11]\):

\[
\sum_{s=0}^{\infty} \frac{(c/b; q)_n b^n}{(aq; q)_n} = \frac{(c/q/c, abq/c, q; q)_{\infty}}{(b/aq, aq/c, bq/c; q)_{\infty}} \tag{7.14}
\]

where \((x; q)_n\) means \( q \)-shifted factorial. In our case it is sufficient to put \( c = ab \), \( a = -q^\alpha \), \( b = q^{1/2} \). Then the Ramanujan formula (7.14) gives

\[
F(\alpha; q^{1/2}) = \sum_{s=0}^{\infty} \frac{1}{q^{(n+\alpha)/2} + q^{-(n+\alpha)/2}} = \frac{1}{q^{\alpha/2} + q^{-\alpha/2}} \frac{(-q^{1/2+\alpha}, -q^{1/2-\alpha}, q, q; q)_{\infty}}{(-q^{1+\alpha}, -q^{1-\alpha}, q^{1/2}, q^{1/2}; q)_{\infty}} \tag{7.15}
\]

Then we replace \( q \rightarrow q^2 \) and obtain formula (7.12). Formula (7.13) is then obtained from the well-known representation of the Jacobi elliptic functions in terms of infinite products \([24]\). Another method to prove (7.13) is using the well known Poisson summation formula \([2]\)

\[
\sum_{s=-\infty}^{\infty} f(t + sT) = \frac{1}{T} \sum_{s=-\infty}^{\infty} \hat{f} \left( \frac{2\pi s}{T} \right) \exp(2\pi i sT) \tag{7.15}
\]
where the functions \( f(x), \hat{f}(x) \) are connected by the Fourier transform:

\[
\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{-ixt} dt
\]

Applying formula (7.15) to the function \( f(x) = 1/\cosh(\beta x) \) with \( q = e^{-\beta} \) and using the Fourier expansion (4.8) of the dn-function we arrive at formula (7.13).

Now we see from (7.9) and (7.12) that

\[
S(j; N) = F \left( \frac{j - 1/2}{2N}; q^{2N} \right)
\]

and we can express the sum \( S(j; N) \) in the one of two equivalent forms (7.12) or (7.13). In the "product form" (7.12) we have

\[
S(j; N) = \frac{1}{q^{j-1/2} + q^{1/2-j}} \frac{(-q^{2N+2j-1}, -q^{2N-2j+1}, q^{4N}, q^{4N}; q^{4N})_{\infty}}{(-q^{4N+2j-1}, -q^{4N-2j+1}, q^{2N}, q^{2N}; q^{4N})_{\infty}} \quad (7.16)
\]

In the "dn-form" we have

\[
S(j; N) = \frac{\tilde{K}}{\pi} \ dn \left( \frac{j + 1/2}{N}; \tilde{K}; \tilde{k'} \right) \quad (7.17)
\]

where \( \tilde{K}, \tilde{K}' \) and \( \tilde{k}, \tilde{k}' \) correspond to \( \tilde{q} = q^{2N} = \exp(-2\pi N K'/K) \) instead of \( q \). Such transformation (from \( q \) to \( q^{2N} \)) correspond to so-called main \( 2N \)-order transformation of Jacobi elliptic functions [1]. In more details, we have [1]

\[
\tilde{K} = \frac{K}{2N \mu}, \quad \tilde{K}' = \frac{K'}{\mu}, \quad \tilde{k} = k^{2N} \prod_{r=1}^{N} \sn^{4} \left( \frac{(2r-1)K}{2N}; k \right)
\]

(7.18)

where

\[
\mu = \prod_{r=1}^{N} \frac{\sn^{2} \left( \frac{(2r-1)K}{2N}; k \right)}{\sn^{2} \left( \frac{K}{2N}; k \right)}
\]

It is interesting to note that the dn-function in (7.17) can be explicitly expressed as a rational function

\[
S(j; N) = R_{N}(\phi(j))
\]

of the function

\[
\phi(j) = \dn^{2} \left( \frac{(j + 1/2)K'}{N}; k \right)
\]

We will not consider this form here (see, e.g. [1] for details, where relation with so-called Zolotarev rational functions of the best approximation is discussed).

We thus have the following

**Proposition 7** Assume that \( w = K/N \), where \( N = 1, 2, 3, \ldots \). Define the reflection parameters

\[
a_{2n} = cn(w(2n + 1)), \quad a_{2n+1} = -dn(w(2n + 2)), \quad n = 0, 1, \ldots, N - 1
\]

such that \( a_{2N-1} = -1 \). Define the monic polynomials \( \Phi_{n}(z), n = 0, 1, \ldots, 2N \) by recurrence relation (1.16).
Then the polynomial $\Phi_{2N}(z) = z^{2N} + 1$ has simple zeros $z_s = \exp(\pi i (s-1)/N)$, $s = -N+1, -N+1, \ldots, N$. The polynomials $\Phi_n(z)$, $n = 0, 1, \ldots, 2N - 1$ are orthogonal on the finite set $z_s$ of spectral points on the regular $2N$-gon:

$$\sum_{j=-N+1}^{N} \rho_j \Phi_n(z_j) \Phi_m(1/z_j) = h_n \delta_{nm} \quad (7.19)$$

where the discrete weights $\rho_j$ are given by formula (7.10) with $S(j; N)$ given by (7.16) or (7.17).

Remark. One can consider the generic case $w = KM/N$ with arbitrary co-prime integers $M, N$. This will lead to slightly modified formulas for the weights $\rho_s$. The case of $dn$-circle polynomials can be considered in a similar way.

8. Polynomials orthogonal on the interval

Now we can relate with our polynomials orthogonal on the unit circle $P_n(z)$ some polynomials $S_n(x)$ orthogonal on an interval of the real axis. This relation was first proposed by Delsarte and Genin [7] (see also [25] for further results concerning the Delsarte-Genin transformation (DGT)).

Recall the main properties of the DGT. Given polynomials $P_n(z)$ with real reflection parameters $a_n$ satisfying the restriction $-1 < a_n < 1$ we construct the polynomials $S_n(x)$ by the formula

$$S_n(x) = \frac{z^{-n/2} \Phi_n(z) + z^{n/2} \Phi_n(1/z)}{2^n (1 - a_{n-1})}, \quad (8.1)$$

where $x = (z^{1/2} + z^{-1/2})/2$ and it is assumed that $z^{\pm 1/2} = r^{1/2} e^{\pm i \theta/2}$ when $z = re^{i \theta}$. It is easily verified that the polynomials $S_n(x)$ are monic $n$-th degree polynomials in $x$: $S_n(x) = x^n + O(x^{n-1})$. Moreover, the polynomials $S_n(x)$ are symmetric, i.e. $S_n(-x) = (-1)^n S_n(x)$ and satisfy the recurrence relation

$$S_{n+1}(x) + v_n S_{n-1}(x) = x S_n(x) \quad (8.2)$$

where

$$v_n = \frac{1}{4} (1 + a_{n-1})(1 - a_{n-2}) \quad (8.3)$$

Note that for $n = 0$ we have formally $v_0 = 0$ due to assumed condition $a_{-1} = -1$. Due to the same condition we have also $v_1 = (1 + a_0)/2$.

Under restriction $-1 < a_n < 1$ the recurrence coefficients $v_n$ are strictly positive $v_n > 0, n = 1, 2, \ldots$. Thus the polynomials $S_n(x)$ are orthogonal on the interval $[-1, 1]$ with respect to a positive weight function

$$\int_{-1}^{1} S_n(x) S_m(x) w(x) dx = \kappa_n \delta_{nm} \quad (8.4)$$

where

$$w(x) = \frac{\rho(\theta)}{\sin(\theta/2)}, \quad x = \cos(\theta/2) \quad (8.5)$$
\[ \kappa_n = v_1 v_2 \ldots v_n = 2^{1-2n} h_n (1 - a_{n-1}^2)^{-1} \]

The orthogonality relation can be presented in an equivalent form

\[ \int_{-1}^{1} w(x) S_n(x) S_m(x) + S_n(-x) S_m(-x) \, dx = \kappa_n \delta_{nm} \quad (8.6) \]

which follows from the property \( w(-x) = w(x) \) for all symmetric polynomials \( S_n(x) \). Relation (8.6) is more convenient when the measure is discrete one.

Introduce the moments

\[ M_n = \int_{-1}^{1} w(x) x^n \, dx, \quad n = 0, 1, 2, \ldots \]

corresponding to the weight function \( w(x) \) of the polynomials \( S_n(x) \). It is easily seen that the moments \( M_n \) are expressed in terms of the moments \( c_n \) as

\[ M_n = 2^{-n} \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} c_{j-n/2} \quad (8.7) \]

when \( n \) is even and \( M_n = 0 \) when \( n \) is odd.

In the case of \( cn \)-polynomials we obtain that corresponding symmetric polynomials \( S_n(x) \) are orthogonal

\[ \sum_{s=\pm \infty}^{\infty} S_n(x_s) S_m(x_s) + S_n(-x_s) S_m(-x_s) \rho_s = \kappa_n \delta_{nm} \quad (8.8) \]

on the dense set of the points

\[ x_s = \cos \left( \frac{\pi w(s - 1/2)}{2K} \right) \]

with the weights \( \rho_s \) is given by (3.14).

In the case of \( dn \)-polynomials we obtain similar orthogonality relation (8.8) but now the orthogonality grid is

\[ x_s = \cos \left( \frac{\pi w_s}{2K} \right) \]

and the discrete weights \( \rho_s \) are given by (4.7).

As expected, both \( cn \) and \( dn \)-type polynomials \( S_n(x) \) are orthogonal with respect to measures which are positive and dense on the interval \([-1, 1]\).

It is well known [6] that to any family of symmetric orthogonal polynomials \( S_n(x) \) one can associate two families of orthogonal polynomials \( P_n(x) \) and \( Q_n(x) \) by the formulas \( S_{2n}(x) = 2^{-n} P_n(2x^2 - 1) \) and \( S_{2n+1}(x) = 2^{-n} x Q_n(2x^2 - 1) \). Polynomials \( P_n(y) \) and \( Q_n(y) \) are monic \( P_n(y) = y^n + O(y^{n-1}) \), \( Q_n(y) = y^n + O(y^{n-1}) \) and orthogonal on the same interval \([-1, 1]\):

\[ \int_{-1}^{1} P_n(y) P_m(y) w(\sqrt{(y+1)/2})(y+1)^{-1/2} \, dy = 2^{2n-1} \kappa_{2n} \delta_{nm} \quad (8.9) \]

and

\[ \int_{-1}^{1} Q_n(y) Q_m(y) w(\sqrt{(y+1)/2})(y+1)^{1/2} \, dy = 2^{2n-1} \kappa_{2n+1} \delta_{nm} \quad (8.10) \]
The polynomials $P_n(x)$ satisfy the recurrence relation [6]

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = x P_n(x),$$

where

$$u_n = 4v_{2n}v_{2n-1}, \quad b_n = 2(v_{2n} + v_{2n+1}) - 1 \quad (8.12)$$

Analogously, polynomials $Q_n(x)$ satisfy the recurrence relation

$$Q_{n+1}(x) + b_n Q_n(x) + u_n Q_{n-1}(x) = x Q_n(x)$$

where

$$u_n = 4v_{2n}v_{2n+1}, \quad b_n = 2(v_{2n+2} + v_{2n+1}) - 1 \quad (8.14)$$

Using the representation (8.3) we obtain explicit formulas for the recurrence coefficients of the polynomials $P_n(x)$:

$$u_n = \frac{1}{4}(1 + a_{2n-1})(1 - a_{2n-2}^2)(1 - a_{2n-3}), \quad b_n = \frac{1}{2}(a_{2n}(1 - a_{2n-1}) - a_{2n-2}(1 + a_{2n-1}))$$  

Similarly, for the recurrence coefficients of the polynomials $Q_n(x)$ we have

$$u_n = \frac{1}{4}(1 + a_{2n})(1 - a_{2n-1}^2)(1 - a_{2n-2}), \quad b_n = \frac{1}{2}(a_{2n+1}(1 - a_{2n}) - a_{2n-1}(1 + a_{2n}))$$  

Formulas (8.15), (8.16) were first obtained by Geronimus [12] (see also [20]).

Explicitly polynomials $P_n(x)$ are expressed in terms of $\Phi_n(z)$ as

$$P_n(x) = \frac{z^{-n}\Phi_{2n}(z) + z^n\Phi_{2n}(1/z)}{2^n(1 - a_{2n-1})}, \quad x = (z + z^{-1})/2$$  

(formula (8.17) is due to Szegö). Similarly for polynomials $Q_n(x)$ we have

$$Q_n(x) = \frac{z^{-n-1/2}\Phi_{2n+1}(z) + z^{n+1/2}\Phi_{2n+1}(1/z)}{2^n(z^{1/2} + z^{-1/2})(1 - a_{2n})}, \quad x = (z + z^{-1})/2$$  

In what follows we will consider only the polynomials $P_n(x)$. All formulas for the companion polynomials $Q_n(x)$ are obtained in a similar way.

Polynomials $P_n(x)$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$w(x) = \frac{\rho(\theta)}{|\sin \theta|} = \frac{\rho(\theta)}{\sqrt{1 - x^2}} \quad (8.19)$$

Assume that polynomials $\Phi_n(z)$ have the expression

$$\Phi_n(z) = \sum_{s=0}^{n} W_{ns} z^s$$

with some coefficients $W_{ns}$. Then from (8.17) we obtain corresponding expansion formula for polynomials $P_n(x)$ [5]:

$$P_n(x) = 2^{1-n}(1 - a_{2n-1})^{-1} \left( W_{2n,n} + \sum_{s=1}^{n} (W_{2n,n+s} + W_{2n,n-s}) T_s(x) \right), \quad (8.20)$$
where \( T_n(x) = \cos(\theta n) \) is the Chebyshev polynomial of the first kind (it is assumed that \( x = \cos \theta \)).

Thus if polynomials \( \Phi_n(z) \) on the unit circle have explicit expression as expansion in terms of monomials \( z^n \) then the corresponding polynomials \( P_n(x) \) on the interval have explicit expression as expansion in terms of the Chebyshev polynomials \( T_n(x) \).

Return to our elliptic cn-polynomials. Corresponding polynomials \( P_n(x) \) on the interval have the explicit recurrence coefficients

\[
\begin{align*}
\alpha_n &= \frac{\sin^2(w(2n-1))(1 - \text{dn}(2wn))(1 + \text{dn}(2w(n-1)))}{4} \\
\beta_n &= 1 + \frac{\text{cn}(w(2n + 1))(1 + \text{dn}(2wn)) - \text{cn}(w(2n - 1))(1 - \text{dn}(2wn))}{2}
\end{align*}
\]

Using explicit expression (3.2) for the expansion coefficients \( W_{ns} \) we obtain in case of the cn polynomials

\[
P_n(x) = (-1)^n 2^{1-n} (1 + \text{dn}(2wn))^{-1} \left( \text{dn}(wn) E_n^2 + \sum_{s=1}^{\infty} (-1)^s E_n^{2s} \text{dn}(w(n-s)) + \text{dn}(w(n+s)) T_s(x) \right)
\]

(8.21)

These polynomials satisfy the orthogonality relation

\[
\sum_{s=-\infty}^{\infty} \rho_s P_n(x_s) P_m(x_s) = H_n \delta_{nm}, \quad H_n = u_1 u_2 \ldots u_n
\]

(8.22)

where the spectral points are

\[
x_s = \cos \left( \frac{\pi w(s-1/2)}{K} \right)
\]

and the weights \( \rho_s \) are given by (3.14).

The polynomials \( Q_n(x) \) defined by (8.18) are orthogonal on the same set of spectral points but the measure acquires an additional linear factor \( 1 + x \):

\[
\sum_{s=-\infty}^{\infty} \rho_s (1 + x_s) Q_n(x_s) Q_m(x_s) = 0, \quad \text{if} \quad n \neq m
\]

Clearly, we have the case of a dense point spectrum on the interval \([-1, 1]\).

In the limit \( k = 1 \) we obtain the recurrence coefficients for the polynomials \( P_n(x) \)

\[
\begin{align*}
\alpha_n &= \frac{(1 - q^n)^2(1 - q^{2n-1})^2(1 + q^{n-1})^2}{4 (1 + q^{2n-1})^2(1 + q^{2n})(1 + q^{2n-2})} \\
\beta_n &= q^{n-1/2} \frac{2(q + 1)q^n(1 - q)(1 - q^{2n})}{(1 + q^{2n-1})(1 + q^{2n+1})}
\end{align*}
\]

where \( q = e^{-2w} \) (of course, this \( q \) should not be confused with already introduced “elliptic” \( q \) by (3.11)).

It is easily verified that these recurrence coefficients correspond to the Askey-Wilson orthogonal polynomials \([4], [16]\) \( p_n(z; a, b, c, d|q) \) with the parameters \( a = c = q^{1/2}, b = -d = 1 \). When \( \max(|a|, |b|, |c|, |d|) \leq 1 \) then the Askey-Wilson polynomials are orthogonal on the interval \([-1, 1]\) with respect to a positive continuous measure \([4], [16]\)

\[
\int_{-1}^{1} \frac{w(x; a, b, c, d|q)}{\sqrt{1 - x^2}} p_n(z; a, b, c, d|q)p_m(z; a, b, c, d|q)dx = h_n \delta_{nm}
\]
where
\[
    w(x; a, b, c, d|q) = \frac{h(x; 1)h(x; -1)h(x; q^{1/2})h(x; -q^{1/2})}{h(x; a)h(x; b)h(x; c)h(x; d)}
\]
and
\[
    h(x; \alpha) = \prod_{s=0}^{\infty} [1 - 2\alpha x q^s + \alpha^2 q^{2s}]
\]
Thus in our case the weight function becomes
\[
    w(x) = \frac{h(x; -q^{1/2})}{h(x; q^{1/2})} = \prod_{s=0}^{\infty} \frac{1 + 2x q^{s+1/2} + q^{2s+1}}{1 - 2x q^{s+1/2} + q^{2s+1}} 
    \tag{8.23}
\]
Expression (8.23) can be compared with the expression of the Jacobi \(dn\)-elliptic function in terms of the infinite product [24]
\[
    dn(u; k) = \sqrt{k'} \prod_{s=0}^{\infty} \frac{1 + 2 \cos(2\theta) q^{2s+1} + q^{4s+2}}{1 - 2 \cos(2\theta) q^{2s+1} + q^{4s+2}}, \quad u = 2K \frac{\theta}{\pi} \tag{8.24}
\]
We see that the weight function \(w(x)\) for the corresponding Askey-Wilson polynomials is proportional to the elliptic \(dn(u)\) function under the identification \(q \rightarrow q^2, \ x \rightarrow \cos(2\theta)\).

For the case of \(Q\)-polynomials defined by recurrence coefficients (8.16) we have in the limit \(k = 1\) again the Askey-Wilson polynomials \(p_n(x; a, b, c, d|q)\) with the parameters \(a = c = q^{1/2}, b = 1, d = -q\). For the weight function we obtain
\[
    w(x) = \frac{h(x; -1)h(x; -q^{1/2})}{h(x; -q)h(x; q^{1/2})} = 2(1 + x) \frac{h(x; -q^{1/2})}{h(x; q^{1/2})}
\]
which differs from the weight function (8.23) only by a linear factor \(2(1 + x)\).

It is interesting to note that despite the hyperbolic limit \(k \rightarrow 1\) some "smell" of the elliptic functions remains in the expression for the weight function of the Askey-Wilson polynomials. Askey and Wilson themselves discussed intriguing cases when the weight function \(w(x)\) is expressible in terms of the Jacobi elliptic functions \(sn(u), cn(u), dn(u)\) [4]. We see that the case of \(dn\)-weight function of the Askey-Wilson polynomials has a natural "elliptic" generalization leading to orthogonal polynomials with dense measure on the same interval \([-1, 1]\). It would be interesting to search other possible elliptic generalizations of the Askey-Wilson polynomials.

9. A general scheme leading to a dense point spectrum on the unit circle

The examples of polynomials on the unit circle constructed in the previous sections can be generalized. In this section we propose a wide class of POC having a positive dense point spectrum on the unit circle.

Let \(f(x)\) be a continuous function of a real variable \(x\) with three main properties:

(i) \(f(x)\) is an even function \(f(-x) = f(x)\) normalized by the condition \(f(0) = 1\);

(ii) function \(f(x)\) is periodic with some real period \(T\): \(f(x + T) = f(x)\);
(iii) the function \( f(x) \) has the Fourier expansion

\[
f(x) = \sum_{n=-\infty}^{\infty} A_n e^{2\pi i nx/T}, \quad A_{-n} = A_n,
\]

such that all the Fourier coefficients are nonnegative: \( A_n \geq 0, \ n = 0, 1, 2, \ldots \). Moreover, we assume that there are infinity many positive coefficients \( A_n > 0 \) (otherwise the problem is trivial). Of course, we assume that the Fourier series converges everywhere on the real axis.

We define the moments by the formula

\[
c_n = f(w_n), \quad n = 0, 1, 2, \ldots
\]

(9.2)

where \( w \) is an arbitrary positive parameter such that

\[
w \neq TM/N
\]

for some integers \( M, N \). Then it is obvious that \( c_{-n} = c_n \) and the moments \( c_n, \ n = 0, 1, 2, \ldots \) are all distinct: \( c_n \neq c_m \) if \( n \neq m \). Using these moments \( c_n \) we can construct the Toeplitz determinants \( \Delta_n \) and corresponding OPC by (1.2).

**Theorem 3** Under conditions (i)-(iii) the Toeplitz determinants \( \Delta_n \) are all positive \( \Delta_n > 0 \), or equivalently all reflection parameters satisfy the restriction \(-1 < a_n < 1\). The polynomials \( P_n(z) \) are orthogonal on the unit circle with respect to a positive dense point spectrum

\[
\sum_{s=-\infty}^{\infty} \rho_s \Phi_n(z_s) \Phi_m(1/z_s) = h_n \delta_{nm}
\]

(9.4)

where the orthogonality grid on the unit circle is

\[
z_s = e^{2\pi i ws/T}
\]

(9.5)

and the discrete weights coincide with the Fourier coefficients

\[
\rho_s = A_s, \quad s = \pm 1, \pm 2, \ldots
\]

(9.6)

**Proof.** From the Fourier expansion (9.1) we have the representation for the moments in the form

\[
c_n = f(wn) = \sum_{s=-\infty}^{\infty} A_s z_s^n = \sum_{s=-\infty}^{\infty} \rho_s z_s^n = \int_C z^n d\mu(z), \quad n = 0, \pm 1, \pm 2, \ldots
\]

where \( z_s \) is given by (9.5) and \( d\mu(z) \) is the measure on the unit circle corresponding to the positive discrete weights \( \rho_s > 0 \). Whence the polynomials \( P_n(z) \) possess orthogonality property (9.4). The measure \( \mu(z) \) is positive and dense on the unit circle. Indeed, positivity follows from the property \( A_n > 0 \) for infinitely many values of \( n \). On the other hand we see that in any small vicinity of the point \( z_s \) on the unit circle there exists at least one point \( z_{s'} \) from the set of spectral points \( z_s \) due to condition (9.3). Thus the points \( z_s \) form a Cantor dense set on the unit circle. This means that the corresponding point spectrum is dense.
Now from general properties of the polynomials orthogonal on the unit circle we conclude that all \( \Delta_n \) are positive \( \Delta_n > 0 \) and equivalently, that the reflection parameters satisfy the restriction \(-1 < a_n < 1\).

We thus see that starting from arbitrary even periodic continuous function \( f(x) \) having infinitely many positive Fourier coefficients \( A_n \) we can construct corresponding OPC with a positive dense point spectrum on the unit circle.

The Toeplitz determinants have the expression

\[
\Delta_n = \begin{vmatrix}
1 & f(w) & \ldots & f(w(n-1)) \\
f(w) & 1 & \ldots & f(w(n-2)) \\
\vdots & \ldots & \ddots & \vdots \\
f(w(n-1)) & f(w(n-2)) & \ldots & 1
\end{vmatrix}
\] (9.7)

All the reflection coefficients of these polynomials will satisfy the restriction \(-1 < a_n < 1\). Of course, if the function \( f(x) \) and its Fourier coefficients \( A_n \) are known explicitly, we have explicit orthogonality relation (9.4). However, explicit expression for the reflection coefficients \( a_n \) can be obtained only for some exceptional cases.

As far as we know the first (and so far the only) explicit example of such type was proposed by Magnus [17], [18] who considered the function \( f(x) \) defined as

\[
f(x) = (-1)^{[x]+1}(2(x - [x]) + 1),
\]

where \([x]\) means integer part, i.e. the greatest integer less than or equal to \( x \). Equivalently, the function \( f(x) \) can be defined as \( f(x) = 1 - 2|x| \) for \(-1 \leq x \leq 1\) and then continued for the whole real axis by 2-periodicity.

Clearly the function \( f(x) \) is even \( f(-x) = f(x) \) and periodic with period 2: \( f(x + 2) = f(x) \). Moreover, there is an obvious property \( f(x + 1) = -f(x) \). If two numbers \( x \) and \( y \) have the same integer part \([x] = [y]\) then obviously

\[
\frac{f(x) - f(y)}{x - y} = -(-1)^{[x]} \tag{9.8}
\]

The function \( f(x) \) has the Fourier expansion

\[
f(x) = \frac{4}{\pi^2} \sum_{s=-\infty}^{\infty} e^{\pi i x(2s+1)} \frac{1}{(2s + 1)^2}
\]

Thus \( A_n = 4/(\pi^2 n^2) \) if \( n \) is odd and \( A_n = 0 \) if \( n \) is even. We see that all \( A_n \) are nonnegative, hence conditions for existence of a positive dense point spectrum are fulfilled and corresponding polynomials are orthogonal on the unit circle

\[
\sum_{s=-\infty}^{\infty} (2s + 1)^{-2} \Phi_n(z_s) \Phi_m(1/z_s) = h_n \delta_{nm} \tag{9.9}
\]

with some positive normalization coefficients \( h_n \). The spectral points on the unit circle are

\[
z_s = e^{i\pi(2s+1)w}
\]
Magnus showed [18] that the polynomials $\Phi_n(z)$ (and corresponding reflected parameters $a_n$) can be found explicitly. This explicit expression is closely related with the expansion of the irrational number $w$ to the continued fraction

$$w = q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \ldots}}}$$

(9.10)

Introduce corresponding convergents $P_i/Q_i$, $i = 0, 1, 2, \ldots$, where $P_0 = \lfloor w \rfloor, Q_0 = 1$ is ”zero-step” rational (integer in this case) approximation of the number $w$ and the rational number $P_n/Q_n$ is obtained when truncating the continued fraction (9.10) at the $n$-th step:

$$\frac{P_n}{Q_n} = q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \ldots + \frac{1}{q_n}}}$$

The $n$-th convergent $P_n/Q_n$ provides the best rational approximation of the irrational number in the following sense:

$$|Q_nw - P_n| < |yw - x|$$

for all possible integers $x, y$ with the restriction $y < Q_n$.

For any given positive integer $n$ we define a sequence of pairs of nonnegative integers $(n_i, m_i)$ in the following way. Let the pair $(n_i, m_i)$ provides the $i$-th order best rational approximation of the number $w$ under condition $1 \leq n_i \leq n$. In more details this means the following. Define positive (irrational) numbers $y_i = \lfloor wn_i - m_i \rfloor$. The pairs $(n_i, m_i)$, $i = 1, 2, \ldots, n$ correspond to the ordering $0 < y_1 < y_2 < y_3 < \ldots < y_n$, where $1 \leq n_i \leq n$, i.e. the pair $(n_1, m_1)$ yields the best rational approximation to the number $w$ (with denominators not exceeding $n$), the pair $(n_2, m_2)$ yields then second (next) best approximation etc. It is well known that the best approximation $(n_1, m_1)$ is given by an appropriate continued fraction of the number $w$: $m_1 = P_k$, $n_1 = Q_k$, where $Q_k \leq n$. The second approximation $(n_2, m_2)$ as well all other approximations $(n_i, m_i)$ can be easily expressed in terms of so-called intermediate fractions (see e.g. [15]). Now we can always present the polynomial $\Phi_n(z)$ in the following form

$$\Phi_n(z) = z^n + G_n^{(1)}z^{n-n_1} + G_n^{(2)}z^{n-n_2} + \ldots G_n^{(s)}z^{n-n_s} + \ldots + G_n^{(n)}$$

(9.11)

with some coefficients $G_n^{(s)}$, where $n_1, n_2, \ldots$ correspond to the succeeding best approximations of the number $w$. (Clearly, the set $n_1, n_2, \ldots, n_s$ is a permutation of the set $1, 2, \ldots, n$, i.e. $n_i \neq n_k$ if $i \neq k$ due to irrationality of $w$). It can be easily showed (see [18] for details) that only 3 first terms survive in (9.11), i.e. we have

$$\Phi_n(z) = z^n + G_n^{(1)}z^{n-n_1} + G_n^{(2)}z^{n-n_2}$$

(9.12)

where the coefficients $G_n^{(1)}, G_n^{(2)}$ can be found explicitly if $(n_1, m_1)$ and $(n_2, m_2)$ are known [18].
Note that in general we have \( a_n = 0 \) for "almost all" \( n \). The nonzero coefficients \( a_n \) appear only if for some \( n \) we have \( n_1 = n \) or \( n_2 = n \). We thus need information about the best approximations of the number \( w \). Unfortunately for almost all irrational numbers \( w \) it is impossible to present some "explicit" formula for \( n_1, n_2 \) (expressing \( n_1, n_2 \) as an analytic function of \( n \)).

In contrast, in our examples of \( cn \)- and \( dn \)-elliptic polynomials we have purely explicit presentation for polynomials \( \Phi_n(z) \), moments \( c_n \) and reflection parameters \( a_n \).

Is it possible to construct other explicit examples of the polynomials \( \Phi_n(z) \) corresponding to functions \( f(x) \) satisfying properties (i)-(iii)?

### 10. Remarks concerning "elliptic hypergeometric functions"

Consider \( cn \) or \( dn \) polynomials \( \Phi_n(z) \) on the unit circle as some "elliptic hypergeometric functions" depending on the argument \( z \) and discrete parameter \( n \). The parameter \( w \) plays the role of the "deformation" parameter: when \( w = 0 \) we have from (3.1)

\[
\Phi_n(z) = (z - 1)^n
\]

for both \( cn \) and \( dn \) polynomials. Thus in the limit \( w = 0 \) we have the simplest hypergeometric function

\[
\lim_{w \to 0} \Phi_n(z) = (-1)^n \, {}_1F_0(-n; z) = (z - 1)^n.
\]

This allows us to consider the functions \( \Phi_n^{(C)}(z) \) and \( \Phi_n^{(D)}(z) \) as "elliptic deformations" of the hypergeometric function \( {}_1F_0(-n; z) \).

Note that from the explicit expression (3.1) it is seen that functions \( \Phi_n^{(C)}(z) \) and \( \Phi_n^{(D)}(z) \) do not belong to a class of elliptic hypergeometric functions \( _1V_{11} \) introduced by Frenkel and Turaev [10] and studied intensively during last years (see e.g. [21], [11] and many others). In particular, these functions perhaps do not possess simple transformation formulas like Bailey transform [21]. Nevertheless, functions \( \Phi_n^{(C)}(z) \) and \( \Phi_n^{(D)}(z) \) have several nice properties which are very close to the classical ones. In particular the recurrence relation (1.12) can be considered as a contiguous relation for 3 elliptic hypergeometric functions \( \Phi_n(z) \) with parameters \( n, n \pm 1 \). The transformation properties (5.5) with respect to the operator \( E \) can be considered as an elliptic analogue of the derivatives of the hypergeometric functions. Finally, the functions \( \Phi_n^{(C)}(z) \) and \( \Phi_n^{(D)}(z) \) possess nice summation formulas for two values of the argument \( z = \pm 1 \). Indeed, from (1.21) we have, e.g. for the \( cn \)-polynomials

\[
\Phi_{2n}^{(C)}(1) = (1 - \cn(w))(1 + \dn(2w)) \cdots (1 - \cn(w(2n - 1)))(1 + \dn(2wn))
\]

and

\[
\Phi_{2n+1}^{(C)}(1) = (1 - \cn(w))(1 + \dn(2w)) \cdots (1 + \dn(2wn))(1 - \cn(w(2n + 1)))
\]

(similar expressions hold for \( \Phi_{2n}^{(D)}(\pm 1) \)).

Thus there exists a much simpler class of "elliptic hypergeometric functions" with nice properties similar to the classical ones.
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