Toward a structure theory for Lorenzen dialogue games

Jesse Alama

Theory and Logic Group
Technical University of Vienna
alama@logic.at

Abstract. Lorenzen dialogues provide a two-player game formalism that can characterize a variety of logics: each set $S$ of rules for such a game determines a set $D(S)$ of formulas for which one of the players (the so-called Proponent) has a winning strategy, and the set $D(S)$ can coincide with various logics, such as intuitionistic, classical, modal, connexive, and relevance logics. But the standard sets of rules employed for these games are often logically opaque and can involve subtle interactions among each other. Moreover, $D(S)$ can vary in unexpected ways with $S$; small changes in $S$, even logically well-motivated ones, can make $D(S)$ logically unusual. We pose the problem of providing a structure theory that could explain how $D(S)$ varies with $S$, and in particular, when $D(S)$ is closed under modus ponens (and thus constitutes at least a minimal kind of logic).

1 Introduction

Lorenzen dialogue games [12] were offered as an alternative game-theoretic formalism for intuitionistic logic (both propositional and first-order). The first player, Proponent ($P$), lays down a logical formula and strives to successfully respond to the assaults of Opponent. The motion of the game is determined by rules that depend on the structure of a formula appearing in the game (which is always a subformula or, in the case of a first-order game, an instance of, the initial formula played by $P$), as well as by rules that depend less on the form of the formula at issue but rather concern the global structure of the game and what kinds of roles can permissibly be played by $P$ and Opponent (who are not merely dual to one another, as the players often are in other logic games [5]). Although Felscher’s equivalence theorem cleanly relates winning strategies of Lorenzen games to intuitionistic validity, the rules for these games are not entirely straightforward and indeed some of them appear to be arbitrary.

Lorenz claimed that Lorenzen’s dialogue games offer a new type of semantics for intuitionistic logic and asserts the equivalence between dialogical validity
(defined in terms of winning strategies for the $P$) and intuitionistic derivability [10,11]. Lorenz’s proof contained some gaps, and later authors sought to fill these gaps; a complete proof can be found in [3].

Dialogue games are not restricted to intuitionistic logic. By modifying the rules of the game, the dialogue approach can also provide a semantics for classical logic. The dialogical approach can be adapted equally well to capture validity for other logics, such as paraconsistent, connexive, modal and linear logics [8,14]. All of these extensions of Lorenzen’s and Lorenz’s initial formulation of dialogue games are achieved by modifying the rules of the game while maintaining the overall dialogical flavor.

The fact that there is no principled restriction on how the dialogical rules can be modified naturally raises the question of when the set of $S$-valid formulas, for a particular set $S$ of dialogical rules, actually corresponds to a logic. That is, we are interested in identifying desirable properties of the set of $S$-valid formulas in order to give it some logical sensibility. One such desirable property is that the set be closed under modus ponens: If $\varphi$ and $\varphi \to \psi$ are $S$-dialogically valid, then so should $\psi$ be. We propose to call the problem of resolving whether a set $S$ of rules for dialogue games satisfies this property the composition problem for $S$.

The structure of this paper is as follows. The next section introduces dialogue games and provides a few examples to make the reader familiar with the basic definitions and notation. Section 3 poses the composition problem. We generalize the problem and motivate it from two perspectives of dialogues. Section 4 presents the results of three initial experiments that bear on the composition problem. Section 4.1 describes a curious dialogical logic called $N$. Section 4.2 describes a failed (but apparently well-motivated) attempted dialogical characterization of the intermediate logic $LQ$ of weak excluded middle. Section 4.3 takes on the problem of giving a dialogical characterization of stable logic (the intermediate logic in which the principle $\neg\neg p \to p$ holds for atoms $p$). Section 4.4 motivates the problem of giving independent rulesets and argues that it may be a useful first step toward solving (instances of) the composition problem. Section 5 concludes and offers a few open problems for consideration.

## 2 Dialogue games

We largely follow Felscher’s approach to dialogical logic [3]. For an overview of dialogical logic, see [8].

We work with a propositional language; formulas are built from atoms and $\neg$, $\lor$, $\land$, and $\to$. In addition to formulas, there are the three so-called symbolic attack expressions, $?$, $\land_L$, and $\land_R$, which are distinct from all the formulas and connectives. Together formulas and symbolic attacks are called statements; they are what is asserted in a dialogue game.

The rules governing dialogues are divided into two types. Particle rules say how statements can be attacked and defended depending on their main connective. Structural rules define what sequences of attacks and defenses count as dialogues. Different logics can be obtained by modifying either set of rules.
The standard particle rules are given in Table 1. According to the first row, there are two possible attacks against a conjunction: The attacker specifies whether the left or the right conjunct is to be defended, and the defender then continues the game by asserting the specified conjunct. The second row says that there is one attack against a disjunction; the defender then chooses which disjunct to assert. The interpretation of the third row is straightforward. The fourth row says that there is no way to defend against the attack against a negation; the only appropriate “defense” against an attack on a negation \( \neg \varphi \) is to continue the game with the new information \( \varphi \).

Further constraints on the development of a dialogue are given by the structural rules. In this paper we keep the particle rules fixed, but we shall consider a few variations of the structural rules.

**Definition 1.** Given a set \( S \) of structural rules, an \( S \)-dialogue for a formula \( \varphi \) is a dialogue commencing with \( \varphi \) that adheres to the rules of \( S \). \( P \) wins an \( S \)-dialogue if \( P \) made the last move in the dialogue and no moves are available for \( O \) by which the game could be extended.

**Remark 1.** According to this definition, if the dialogue can go on, then neither player is said to win; the game proceeds as long as moves are available. \( P \) wins by making a winning move; in other presentations of dialogue games such as Fermüller’s [4], \( P \) wins when Opponent makes a losing move.

Winning strategies for dialogue games can be used to capture notions of validity.

**Definition 2.** For a set \( S \) of dialogue rules and a formula \( \varphi \), the relation \( \models_S \varphi \) means that \( P \) has an \( S \)-winning strategy for \( \varphi \). If \( \nvdash_S \varphi \), then we say that \( \varphi \) is \( S \)-invalid. \( D(S) \) is the set \( \{ \varphi : \models_S \varphi \} \).

Note that, like usual proof-theoretic characterizations of validity, dialogue validity is an existential notion, unlike the usual model-theoretic notions of validity, which are universal notions.

We now consider two standard rule sets from the dialogue literature.

**Definition 3.** The rule set \( D \) is comprised of the following structural rules [3, p. 220]:

\[(D10) \text{ P may assert an atomic formula only after it has been asserted by O before.}\]
Table 2. An E-dialogue for \( p \rightarrow (q \rightarrow p) \):

| Move | Player | Move |
|------|--------|------|
| 0    | P      | \( p \rightarrow (q \rightarrow p) \) (initial move) |
| 1    | O      | \( p \) [A,0] |
| 2    | P      | \( q \rightarrow p \) [D,1] |
| 3    | O      | \( q \) [A,2] |
| 4    | P      | \( p \) [D,3] |

Table 3. An E-dialogue for excluded middle

| Move | Player | Move |
|------|--------|------|
| 0    | P      | \( p \lor \neg p \) (initial move) |
| 1    | O      | ? [A,0] |
| 2    | P      | \( \neg p \) [D,1] |
| 3    | O      | \( p \) [A,2] |

Definition 4. The rule set \( D + E \) is \( D \) plus the following rule:

(E) O can react only upon the immediately preceding P-statement.

Definition 5. The rule set CL is E – \{D11, D12\}.

To give a sense of how these games proceed, let us look at a few concrete examples of them. In the following, note that we are working with concrete formulas; “\( p \)” and “\( q \)” in the following are concrete atomic formulas (atoms) and should not be read schematically (indeed, if one were to substitute more complex formulas for \( p \) and \( q \) in what follows, the examples would become incomplete in the sense that they no longer necessarily represent wins or losses for P).

Example 1. Let us consider a simple intuitionistic validity, the K-formula. Table 2 lays out a concrete game for this formula. This dialogue adheres to the E-rule because O is always responding to the immediately prior statement of P. Note that P is permitted to assert the atom \( p \) at move 3 because O already asserted it at move 1. P wins this game; O can make no further moves: the E-forces O to respond to move 4 (in fact, it must be attacked), but, in light of the particle rules, attacks on atoms are not permitted.

Example 2. Table 3 treats the classical law of the excluded middle, \( p \lor \neg p \). This short E-dialogue (which, incidentally, is also a D-dialogue) leads to a loss for P: as in the previous example, P is stuck.

Example 3. Returning to excluded middle, let’s see how the game goes when we change from intuitionistic to classical rules; see Table 4. The difference between
0 P p ∨ ¬p (initial move)
1 O ? [A,0]
2 P ¬p [D,1]
3 O p [A,2]
4 P p [D,1]

Table 4. A CL-dialogue for excluded middle

this dialogue, which P wins, and the previous dialogue, which P lost, is that P can now return to earlier attacks and defend against them in a new way. The absence of rule D11 from CL makes the difference.

These examples should serve to give the reader a sense for how dialogue games proceed, as one varies the rules. Despite their apparent lack of logical meaning, the rule sets D and E have the following property:

**Theorem 1 (Felscher).** For all formulas \( \varphi \), the following are equivalent:

- \( \varphi \) is intuitionistically valid.
- \( \models_D \varphi \).
- \( \models_E \varphi \).

The proof goes by converting deductions in an intuitionistic sequent calculus to D-winning strategies (via tableaux), and vice versa. Moreover, the ruleset CL has the following significance:

**Theorem 2 (Felscher).** For all formulas \( \varphi \), we have that \( \varphi \) is a classical tautology iff \( \models_{CL} \varphi \).

In other words, dropping D11 and D12 from the ruleset E moves us from intuitionistic to classical logic.

### 3 The composition problem

One can view dialogue games in two (compatible) ways. These games can be a kind of rational dialogue between two players, or they can be viewed as a kind of logical calculus. In this section we shall describe a problem about dialogues that bears on them no matter which view one takes about dialogues.

The statement of the problem does not depend on which viewpoint we adopt:

**Problem 1 (Composition)** Given a set \( S \) of structural rules, determine whether \( \mathcal{D}(S) \) is closed under modus ponens, that is, whether it is true that \( \varphi \in \mathcal{D}(S) \) and \( \varphi \to \psi \in \mathcal{D}(S) \) implies \( \psi \in \mathcal{D}(S) \).

One approach to the composition problem is to simply give positive solutions for each ruleset \( S \) that one is interested in. A more unifying problem is available, though:
Problem 2 (Uniform composition) Give criteria for a set $S$ of dialogue rules (perhaps coming from some delimited class of rulesets) such that modus ponens is admissible for $\mathcal{D}(S)$.

Instead of focusing on particular rulesets, the uniform composition problem asks for criteria for a ruleset which, if satisfied for any ruleset $S$, ensure that we have a positive solution to the composition problem for $S$.

The qualifier “(perhaps coming from some delimited class of rulesets)” in the statement of the uniform composition problem permits one to restrict the range of rulesets of interest (e.g., such as those coming from various dialogical characterizations of modal logic [8]). A totally general solution to the uniform composition problem seems to be out of the question, putting aside the question of what a “dialogue rule” in general is, which makes it unclear over what the problem quantifies.

We now consider this problem from the two points of view about dialogues.

3.1 Dialogues as rational interaction

If dialogues are to be for a (stylized) kind of rational interaction, then one ought to have a criterion according to which one can say that certain (sets of) dialogue rules support or undermine the rational behavior of the players.

From results like Felscher’s we can see that there must be a positive solution to the composition problem: since intuitionistic logic is actually a logic, if $P$ has winning strategies for $\varphi$ and $\varphi \rightarrow \psi$, then $P$ must have a winning strategy for $\psi$.

Thus, by singling out the composition problem, we are not necessarily raising a genuinely new problem about dialogue games, at least not in all cases, where correspondence results are known, such as for intuitionistic and classical first-order logic, modal logics, and so forth. Rather, we are proposing a problem with a change of emphasis: rather than solving the composition problem as a corollary of considerably stronger results, we raise the following challenge for dialogue games: if a set of dialogue rules $S$ is supposed to actually be a coherent logic, we would like to have a direct proof of this fact; it should, ideally, be possible to give a direct solution of the composition problem for $S$ before a technically complex correspondence is established between the set of formulas for which $P$ has a winning strategy and the set of known validities.

Another way to approach the composition problem: if dialogue games based on a set $S$ of dialogue rules are supposed to be an autonomous foundation for some kind of logic $L$, then it should be possible to solve the composition problem for $S$ without reference to whatever “machinery” for $L$ has been built up outside of the dialogical approach.

What do we mean by “rational”? Various senses are available, for a ruleset $S$:

– An $S$-strategy for a formula $\varphi$ should correspond to some conclusive reasoning for $\varphi$;
– if $P$ has an $S$-winning strategy for $\varphi$, then $P$ does not have an $S$-winning strategy for $\neg \varphi$;
– if $P$ has an $S$-winning strategy for $\varphi$, then Opponent does not also have an $S$-winning strategy for $\varphi$;
– if $P$ has $S$-winning strategies for $\varphi$ and $\varphi \rightarrow \psi$, then $P$ has an $S$-winning strategy for $\psi$.

The fourth explication of $S$-strategy rationality is simply the same as the composition problem for $S$.

We can further distinguish two loci of rationality: games and strategies.

**Definition 6 (Game rationality).** A ruleset $S$ is game-rational if the development of $S$-dialogue games should have the form of a rational conversation between two opposing players.

**Definition 7 (Strategy rationality).** A ruleset $S$ is strategy-rational if $S$-winning strategies constitute some kind of rational argument.

One way to deflate the composition problem is to acknowledge that dialogue games are not in fact supposed to be an autonomous foundation of capturing validity in a logic. (One might even wonder what it means to be an autonomous foundation for a logic.) Or we are to drop the requirement about game-rationality or strategy-rationality for dialogue rulesets. And it would seem that neither of these desiderata can really be abandoned, if one wishes to see dialogue games as more than a mere calculus and having something to do with “rational dialogue”. It seems we lack a compelling account of the rationality of dialogues, in the sense that we lack a defense of certain sets of dialogue rules over others.¹ If one views dialogues as simply alternative calculi for working with different logics, then one might still be persuade by our call for “direct” solutions to the composition problem. This point of view is taken up in the next subsection.

### 3.2 Dialogues as calculi

Apart from treating dialogue games as a stylized debate or rational interaction between two opposing players, one can view these games as a logical calculus on a par with other formalisms for proofs such as Hilbert-style, natural deduction, tableaux, or sequent calculus. (These two points of view are, of course, compatible.) From this point of view, the composition problem for a ruleset $S$ is the problem of showing that modus ponens is an admissible rule of inference for $\mathcal{D}(S)$, the set of all formulas $\varphi$ for which $P$ has an $S$-winning strategy for $\varphi$.

One way to view the problem is that we have a handful of positive results: for a certain very limited number of dialogue rulesets, we know about them that they correspond to certain logics (and hence positively solve their associated composition problems). We may view these positive results as local maxima in a space populated by logics and non-logics alike. We wish to understand what happens when we step away from these local maxima in this space. Certainly,

¹ Woods has highlighted another problem concerning the rationality of dialogue games, different from ours, which is related to the problem of logical omniscience [20]. Walton also sketches some problems of rationality in dialogues [19].
some curiosities will result (see section 4.1 for an example). The perspective behind the uniform composition problem is to embrace these non-maxima (or perhaps even discovering new maxima) in the hopes of understanding the whole space: let us shift from a (very) discrete point of view to a “continuous” point of view, to see what the dialogical space is like.

One can evidently point to theorems such as Felscher’s to dispense with the composition problem for the rulesets D and E. However, Felscher’s theorem does not, prima facie, solve the uniform composition problem. Some positive results bearing on the uniform composition problem are those of Fermüller [4], who, using so-called parallel dialogue games, gives dialogical characterizations of a variety of intermediate logics. We shall return to Fermüller’s results later, in section 4.2.

We are also interested in the question of to what extent dialogue games actually offer a fine-grained division of different kinds of logics. If it turns out that only a handful of sets of dialogue rules are adequate for the purpose of generating a logic (i.e., for capturing some minimally rational meaning of a dialogue game), then this needs to be explained. That is, if it turns out that there is something unique about the standard sets of dialogue rules that have heretofore been investigated, then this serves as a critical point for the dialogue approach, because it shows that its apparent opportunity for logical generality is in fact highly constrained and tightly delimited.

### 3.3 Direct solutions to the composition problem

A positive solution to the composition problem for a set of dialogue rules tells us that our rule set gives rise to a logic, at least in a weak sense of the term “logic”. Of course, we are likely not interested in the case where all formulas are valid, in which case the composition problem is trivially solved in the affirmative.

We have indicated that we prefer direct solutions to the composition problem. We can certainly bring to bear whatever means we have toward establishing significant properties of a dialogical logic. But if a “direct” solution to a problem is available, it seems reasonable to provide one alongside whatever other methods we have. The problem is simply that we wish to have multiple proofs. Whether a direct proof that operates on winning strategies is “the same” as a proof of the same result using some other methods is not always clear. Even if a positive solution to the composition problem is “really” a disguised version of cut elimination, there may still be value in working directly with strategies rather with, say, sequent calculi derivations, since we don’t need to first do the work of showing that the sequent calculi really captures the strategies.

Direct solutions to dialogical problems may be the only solutions, if one is exploring a logic whose relation to other, differently characterized logics is unknown. A positive, direct solution to the composition problem for such a logic is given in [1] (see also Section 4).

To illustrate further what we have in mind by “direct” solutions to dialogical problems, we now give a positive solution to the composition problem for a dialogical characterization of classical propositional logic. We will first present
a set CL of dialogue rules, and then we will show that $\models_{\text{CL}} \varphi \to \psi$ and $\models_{\text{CL}} \varphi$ implies $\models_{\text{CL}} \psi$ by working with CL-winning strategies. We will not show that CL captures CL; see [3].

**Lemma 1.** No atom is CL-valid.

*Proof.* The CL does not permit a game to even get started with the assertion by P of an atom.

Such a result obviously holds for any set of dialogue rules that contains D10.

The next lemma is a kind of consistency result of classical logic, construed dialogically.

**Lemma 2 (No explosion).** There is no CL-valid formula $\varphi$ with the property that $\models_{\text{CL}} \varphi \to \psi$ for all formulas $\psi$.

Such a formula gives rise to an “explosion” in the sense that it entails (in the object language) all formulas. If there were such a formula $\varphi$, we would have, for example, that $\models_{\text{CL}} \varphi \to p$, even for atoms $p$ that do not occur in $\varphi$. Such a case is clearly untenable. We have not yet been able to find a direct proof of this lemma, but it does seem to us to be an important step toward a direct proof of the composition problem for CL. (Note that, by closure of CL-valid formulas under modus ponens, such an “explosion” formula does not exist, since it would imply, as we said, that all atoms would be valid, which of course violates Lemma 1.) Such a problem can clearly be solved quite easily using the truth table notion of classical validity. Less easy is a proof-theoretic solution to the problem using a sequent calculi adequate for classical logic; the solution apparently requires cut elimination.

**Theorem 3 (Attack-first).** If $\models_{\text{CL}} \varphi$, then there is a CL-winning strategy for $\varphi$ in which P’s defenses are delayed as far as possible.

*Proof.* The idea is that we consider CL-winning strategies $\tau$ that have the property that, for each branch $b$ of $\tau$, and each P move $m$ of $b$, if $m$ is a defense, then at $m$ it is not possible for P to attack. That is, we consider CL-winning strategies where P must defend; if P can attack, then he does.

The existence of such CL-winning strategies is clear. If there is a CL-winning strategy for $\varphi$, but only one, then it satisfies the attack-first condition because P has no alternatives available to him. For a CL-valid formula $\varphi$, there could even be multiple such strategies.

We can refine the attack-first strategy further by requiring that, if no attacks but multiple defenses are available for P, then we require that P defend against the most recent attack.

**Theorem 4 (Attack-first-defend-most-recent).** IF $\models_{\text{CL}} \varphi$, then there exists a CL-winning strategy in which P’s defenses are delayed as far as possible, and in which, if multiple defenses are possible for P, then the defense against the most recent attack is chosen.
The existence of such strategies for CL-valid formulas is again clear.

Note that, unlike proofs of analogous results via cut elimination, these “normal forms” for dialogues do not require the definition of a reduction relation and a proof that it is normalizing; the existence of CL-winning strategies adhering to these conditions is clear.

We have so far not been able to find a “direct” proof of Lemma 2. Such a result must hold, since, thanks to Felscher’s and other dialogical characterization of classical logic (e.g., [16]), we have $\models_{\text{CL}} \varphi$ iff $\varphi$ is a classical tautology. From the perspective of truth tables, such a statement clearly holds: a $\varphi$ with this property would be a contradiction such as $\bot$ or $p \land \neg p$, but such statements are not valid. The following is an outline of a proof using “direct” methods, using the ideas developed so far.

**Theorem 5.** If $\models_{\text{CL}} \varphi$ and $\models_{\text{CL}} \varphi \rightarrow \psi$, then $\models_{\text{CL}} \psi$.

*Proof (Sketch).* Consider a CL-winning strategy $d$ for $\varphi \rightarrow \psi$. It begins with the assertion by $P$ of $\varphi \rightarrow \psi$, then an attack by $O$ on this implication, asserting $\varphi$. The beginning is shown in Table 5. We do not know what the next step is; $P$ could attack $\varphi$ or defend against the initial assertion of $\psi$. There are three possibilities:

- $P$ never defends against the initial attack on $\varphi \rightarrow \psi$. In this case, evidently it makes no difference what $\psi$ is, so by simply changing the first step of $d$ from $\varphi \rightarrow \psi$ to $\varphi \rightarrow \chi$, we have a CL-winning strategy for $\varphi \rightarrow \psi$, no matter what $\psi$ is. $\varphi$ would thus a counterexample to Lemma 2.
- $P$ never attacks $O$’s assertion of $\varphi$. Then $d$ is evidently already a CL-winning strategy for $\psi$, provided we simply delete the initial two moves.
- $P$ does defend against the initial attack on $\varphi \rightarrow \psi$. This is the general case, and likely the most difficult. The main idea is to look for a suitable rewriting, or normal form, of the strategy into one from which we can extract a CL-winning strategy for $\varphi \rightarrow \psi$. It seems plausible that the *defend last* normal form defined earlier would be helpful. By adhering to that normal form, we defer $P$’s defense against the initial attack as long as possible, forcing $O$ to make the greatest number of commitments (viz., assert the most atoms) before coming to the defense against the initial attack.

Theorem 5 says that $\models_{\text{CL}} \varphi \rightarrow \psi$ if we can prove $\models_{\text{CL}} \varphi$ and $\models_{\text{CL}} \varphi \rightarrow \psi$. The part about $\models_{\text{CL}} \varphi$ is a tautology, and the part about $\models_{\text{CL}} \varphi \rightarrow \psi$ is clear to us as a semantic consequence.

We have targeted CL here and a dialogical characterization (CL) of it because CL is somewhat relaxed compared to the rulesets D and E, which are known to be adequate for intuitionistic logic. That IL is closed under modus ponens is, of course, obvious. It is not clear to us whether the strategy normal forms that we have proposed (namely, the attack-first and its refinement, attack-first-defend-latest forms) have the same significance in the presence of rules D11 and D12.
as they do when these two rules are missing (which is the case for the standard
dialogical characterization of CL).

4 Varying dialogue rules

To illustrate our approach, let us look at some examples where one varies the
rule sets. This section reports on three such experiments.

4.1 Nearly classical logic

We have stated earlier that Felscher’s theorem shows the correspondence between
the D and E rule sets and intuitionistic logic IL. Since IL is closed under modus
ponens, Felscher’s theorem implies that \( D(D) \) and \( D(E) \) are likewise both closed
under modus ponens. It is also known that, if one drops Felscher’s D11 and D12
from D, but adds rule E, one obtains a dialogical characterization of classical
logic CL.

Is rule E necessary for modus ponens?

**Definition 8.** Let \( N \) be \( D - \{D11, D12\} \), and let \( N \) be the set of \( N \)-valid formulas.

Since dropping the E makes no difference when passing from E to D, it is
ture that closure under modus ponens is preserved if one drops E from \( D - \{D11, D12\} \cup \{E\} \) (which dialogically captures CL)? More simply, is \( N \) closed
under modus ponens?

The answer, curiously, is that \( N \) is closed under modus ponens but not un-
der uniform substitution. The following necessary conditions govern \( N \)'s valid
implications:

**Theorem 6.** If \( \models_N \varphi \rightarrow \psi \), then either

1. \( \models_N \psi \),
2. \( \varphi \) is atomic, or
3. \( \varphi \) is a negation.

(For details, see [1].) Using Theorem 6, many failures of uniform substitution for
\( N \) can be produced. We have, for example, that \( \models_N p \rightarrow \neg \neg p \) (this can be shown
by calculation), but \( \not\models_N (p \land p) \rightarrow \neg \neg (p \land p) \), because the antecedent meets none
of the necessary conditions listed in Theorem 6. (That \( \not\models_N \neg \neg (p \land p) \) can be
shown by calculation.)

Adding rule E to \( N \) restores uniform substitution (and maintains closure
under modus ponens), so despite appearances, there must be something about
rule E intimately tied to uniform substitution.

The presence of the E could be regarded as a mere technical necessity for es-
ablishing a correspondence between existence of winning strategies for dialogue
games and some notion of logical validity, characterized without using dialogues.
The E rule has no obvious correspondence with everyday dialogue; even if one
were inclined to adopt some kind of regimentation, the E appears to be a rather
strong constraint. A better understanding of its eliminability is wanted. Results such as the curious N show that, at least in one well-known setting (classical dialogue games), E cannot be entirely dropped, while in at least one other setting (intuitionistic logic) it can be dropped. One problem would be to find some relaxation of E that suffices for CL. It seems fruitful us to investigate the precise conditions under which repetitions are permitted. Already some work has been done in this direction (see [9]) for intuitionistic logic.

4.2 An attempted dialogical characterization of the logic LQ

In a Hilbert-style calculus for propositional logic, one can start with intuitionistic logic and obtains classical logic by adding additional axioms, such as Peirce’s formula, excluded middle, or double negation elimination (the precise details depend on which propositional signature one is interested in).

With dialogues, one moves from intuitionistic to classical logic not by adding but by removing dialogue rules. In the dialogical setting, classical logic can be obtained by relaxing the dialogue rules for intuitionistic logic. One might then naturally wonder if one can give dialogical characterizations of intermediate logics (i.e., propositional logics between IL and CL) by adding dialogue rules to the ruleset CL.

One natural experiment would be to try to capture a “simple” intermediate logic, such as Jankov’s logic LQ [7,17], which is IL together with the principle of weak excluded middle (WEM), \( \neg p \lor \neg \neg p \). This principle is obviously classically valid but it is independent of IL (one can see this using Kripke models).

Fermüller has given a dialogical characterization of LQ (and other logics) with the help of parallel dialogue games [4]. Fermüller matches winning strategies for parallel dialogue games with derivability in a calculus based on hypersequents due to Ciabattoni et al. [2]. Fermüller’s parallel dialogue games diverge from the “sequential” games employed in this paper.

Despite Fermüller’s solution, one might still seek out a “sequential” characterization of LQ, perhaps employing a non-hypersequent formulation of LQ, such as Hosoi’s [6]. Ideally, one would seek an intuitive, self-contained addition or modification to some known ruleset, such as the E, that would characterize LQ. A first step would be to find such a modification according to which \( \neg p \lor \neg \neg p \) is valid.

To motivate the new dialogue rule that will be introduced soon, let us consider the E-dialogue game for WEM; see Table 6. The two E-dialogues for \( \neg p \lor \neg \neg p \) of Table 6 show that P loses quickly no matter whether the initial attack is defended by asserting \( \neg p \) or \( \neg \neg p \). Since WEM is not intuitionistically valid, by Felscher’s theorem P does not have an E-winning strategy for it. Indeed, the

---

2 The precise claim is that one can obtain a dialogical characterization of classical logic by removing D11 and D12 from the ruleset E. We say “can be obtained” rather than “is obtained” because, depending on which ruleset one chooses for intuitionistic logic, our claim is false: the ruleset N and the set N that it generates shows that we do not obtain classical logic by simply dropping D11 and D12 from the ruleset D.
above two games, diverging at move 2, together make up all possible ways the

game could go; \( P \) loses in both. The obstacle seems to be D10, which blocks \( P \)
from asserting the atom \( p \) before Opponent has conceded it. We can see this by
comparing the two E-dialogues with how the game goes when playing the ruleset
CL for classical logic. In the ruleset CL, \( P \) can return to earlier attack and defend
against them, unlike in the D and E rulesets, in which multiple defenses are ruled
out. \( P \)'s ability to repeat earlier defenses makes all the difference, because he
can defend in move 4, \textit{in a different way}, using \( O \)'s “concession” of the atom \( p \)
in move 3. (The game of Table 7 is in fact a winning strategy for WEM.)

We require a set \( S \) of dialogue rules “between” the ruleset E and CL. \( P \)'s
ability to return to earlier defenses seems to be rather too strong. Let us consider
the following modified form of D10:

\[(D10^*) \quad P \text{ may assert an atom } p \text{ only if } O \text{ has asserted either } p \text{ or } \neg p \text{ before.}\]

Let E* be E except with D10* instead of D10. The idea is that WEM is a kind
of excluded middle, but only for \textit{negative} statements. We modify D10 according
to this intuition: once \( O \) reveals some negative information (i.e., concedes a
negated atom), \( P \) is permitted to proceed with this information as though it
were positive. Table 8 is a calculation showing that at least one instance of
\( \models_{E^*} \text{WEM is valid, and proceeds in an intuitive way (from the perspective of}
LQ): But this rule goes overboard: we have not captured LQ but something else,
because the formula \( \neg p \rightarrow p \) is E*-valid. This can easily be seen: \( O \)'s unique
opening move is to assert \( \neg p \), and now \( P \) has a unique response: to assert \( p \),
winning the game.

The lesson of this failure to capture the logic LQ using dialogues is that we
had a well-motivated modification to a basic dialogue rule, but the consequences
of adopting this rule were that unacceptable formulas became valid. Ideally, we
would be able to appeal to a structure theory that would explain the precise
4.3 Characterizing stable logic dialogically

Stable logic $S$ is the intermediate logic axiomatized by the stability principle

$$\neg\neg p \rightarrow p$$

for atoms $p$. The stability principle is not provable in intuitionistic logic. This is intuitively clear by considering the Brouwer-Heyting-Kolmogorov interpretation, but it may be definitively shown by, e.g., a suitable Kripke model. Although it has the flavor of being “inherently classical”, stable logic is in fact strictly weaker than classical logic. To obtain classical logic, it suffices to add the stability principle to Jankov’s $LQ$ discussed in Section 4.2. From the standpoint of the program considered in this paper, it is natural to ask how one can give a dialogical characterization of stable logic. We found in Section 4.2 that, when we translated our semantic intuition of the principle of weak excluded middle into the dialogical context, the naive attempt failed. In the case of stable logic, though, one’s semantic intuition can be easily expressed dialogically.

To get started, let us consider, from the dialogical perspective, why stability is not provable. Let us take the D rules; see Table 9. The game can develop in only one way. $P$ would like to assert $p$, and since $O$ has also asserted it (move 3), so rule D10 wouldn’t be violated. $P$ cannot because he cannot defend against the

Already in IL, one can prove $p \rightarrow \neg\neg p$. Adding the stability principle, we have $p \leftrightarrow \neg\neg p$, so $p$ and $\neg\neg p$ are, as it were, on a par with one another. Consider now the following new structural rule:

(D10$'$) $P$ may assert an atom $p$ only if $O$ has asserted either $p$ or $\neg\neg p$ before.
This is reminiscent of our failed dialogical characterization of Jankov’s LQ using D10∗. Here, though, instead of “semantically identifying” p and ¬p (which, in hindsight, is the root of the failure), we semantically identify p and its double negation ¬¬p. Referring to the non-winning play in Table 9, it is clear that, were D10′ in effect rather than D10, the game would be “short-circuited” because, once O asserts ¬¬p, P can pounce and assert p in defense of the initial attack; O then cannot reply.

Of course, it is quite possible that, these positive signs notwithstanding, the adoption of our modified D10 has unacceptable consequences, as we saw in the preceding section when ¬p → p became dialogically valid.

Interestingly, the motivation behind the formalization of stable logic as presented in Negri and von Plato [13] is to give a sequent characterization of the familiar principle of indirect proof (“if from ¬p one can derive a contradiction, then p is provable”), here the principle is, roughly, the “semantic identification” of p and ¬¬p.

**Definition 9.** Let \( E' \) be the set of \( E \setminus \{D10\} \cup \{D10'\} \)-valid formulas.

The aim is to show that \( E' \) equals \( S \). We are so far able to show part of this (see Theorem 7). The following lemmas show that our proposal stands a chance of dialogically capturing \( S \).

**Lemma 3.** If \( \varphi \) is \( E' \), then there exist atoms \( p_1, \ldots, p_n \) in \( \varphi \) such that

\[
\left( \bigwedge_{1 \leq i \leq n} \neg
\neg p_i \rightarrow p_i \right) \rightarrow \varphi
\]

belongs to IL.

**Proof.** A move in dialogue games is not labeled by what rule justifies the move (unlike, say, sequent calculi or natural deduction). The idea is that all structural rules govern all moves. Nonetheless, let us nonetheless call an “application” of D10′ a move by P that would be impossible if the standard D10 were in effect rather than D10′. Applications of D10′ thus look, schematically, like this: The idea is, for each application of D10′, to adjust the strategy—in fact, even the initial statement by P—so that these applications are “eliminated” in the sense that the same sequence of moves goes through if D10′ were replaced the more strict D10. Such rewrites are effected as follows:

– Before the initial statement of \( \varphi \) by P, insert the following moves: (Of course, the addition of new moves to the beginning of the game needs to be accompanied by relabeling any references. Thus, a reference to move 3 should now refer to move 7, etc.)
\[
0 \quad P \quad (\neg p \rightarrow p) \rightarrow \varphi \quad \text{(initial move)}
1 \quad O \quad \neg p \rightarrow p \quad \text{[A,0]}
2 \quad P \quad \neg p \quad \text{[A,1]}
3 \quad O \quad \neg p \quad \text{[A,2]}
4 \quad O \quad \varphi \quad \text{[D,1]}
\]

Table 10. Eliminating an application of D10' by explicitly postulating an instance of stability

\[
\begin{align*}
m & \quad O \quad \neg p \quad \text{[-,-]} \\
\vdots \\
n & \quad P \quad \neg p \quad \text{[A,m]} \\
n + 1 & \quad O \quad p \quad \text{[A,n]} \\
n + 2 & \quad P \quad p \quad \text{[D,1]}
\end{align*}
\]

Table 11. Eliminating an application of D10'

– Replace the application of D10' as in Table 11:
– Repeat until there are no more applications of D10'.

The effect of these repeated rewrites is that any “exploit” by \( P \) of the extra freedom granted by the relaxed D10' gets turned into an extra

The atoms claimed to exist in Lemma 3 come from the winning strategy that witnesses that \( \varphi \in E' \). It may be possible that, depending on the strategy, we get a different set of atoms. In any case, adding stability for more atoms can’t hurt (weakening is clearly acceptable).

**Lemma 4.** If \( \varphi \in E' \), then

\[
\left( \bigwedge_{1 \leq i \leq n} \neg p_i \rightarrow p_i \right) \rightarrow \varphi
\]

belongs to \( \text{IL} \), where \( p_1, \ldots, p_n \) lists all the atoms occurring in \( \varphi \).

Lemma 4 gives a dialogical characterization of a familiar fact about stable logic (see, e.g., [18, Ch. 3]), namely that one can “reduce” some intermediate logics to \( \text{IL} \) provided one explicitly postulates salient features of the intermediate logic (for the case of LQ, see Hosoi [6], where the fact that explicitly postulate instances of the principle of weak excluded middle, playing the same role there as the stability principle does for us here). Lemma 4 combined with the fact that stable logic includes all instances of the stability principle yields:

**Lemma 5.** If

\[
\left( \bigwedge_{1 \leq i \leq n} \neg p_i \rightarrow p_i \right) \rightarrow \varphi
\]
is in \( \mathcal{L} \), then \( \varphi \in S \).

Lemmas 3 and 5 yield:

**Theorem 7.** If \( \varphi \in \mathcal{E}' \), then \( \varphi \in S \).

Evidently, if \( \varphi \in \mathcal{L} \), then \( \varphi \in \mathcal{E}' \) (the extra freedom of D10’ granted to \( \mathcal{P} \) compared to D10 need not be exercised). Moreover, all instances of the stability scheme are, by construction, in \( \mathcal{E}' \), so \( \mathcal{E}' \) properly extends \( \mathcal{L} \).

We have not yet been able to show the converse of Theorem 7, but we conjecture that our proposed characterization of \( S \) is indeed correct. To complete the proof that \( \mathcal{E}' \) is \( S \), it would suffice to show that (i) \( \mathcal{E}' \) is closed under modus ponens, and (ii) \( \mathcal{E}' \) is closed under uniform substitution. (Such an approach is grounded on a Hilbert-style approach to stable logic.) Another route would be to try to go in the opposite direction taken in the proof of Lemma 3. There, we defined the notion of “application” of D10’ and showed how to eliminate them by explicitly postulating instances of the stability principle and manipulating the way the winning strategy starts. Ideally, one would want to go the other way: show how, from an \( \mathcal{E} \)-winning strategy for

\[
\left( \bigwedge_{1 \leq i \leq n} \neg\neg p_i \rightarrow p_i \right) \rightarrow \varphi,
\]

to construct a D10’-winning strategy for \( \varphi \) alone by introducing (rather than eliminating) what we have called applications of D10’.

The chief difficulty is to characterize the possible ways that a stability hypothesis \( \neg\neg p \rightarrow p \) is used.

### 4.4 Independent rulesets

Felscher indicates that rule E implies D13 and, for odd positions, D11 and D12, too. This means that every dialogue that adheres to rule E also adheres to D13, and if we understand D11 and D12 as quantifying over move positions 0, 1, ..., then every dialogue that adheres to the E also adheres to D11 and D12 if the quantifiers in these rules are restricted to odd numbers.

The fact that standard dialogue rules can imply each other, wholly or partially, is an obstacle for solving the composition problem for subsets of standard rulesets. What we would seek are **independent** sets of dialogue rules, that is, sets of rules each member of which is not implied by the others.

The examples above of \( \mathcal{N} \) and the failed dialogical characterization of \( \mathcal{LQ} \) demonstrate the sensitive dependence of \( \mathcal{D}(S) \) on a set \( S \) of dialogue rules. Slight changes to a set \( S \) of dialogue rules can cause \( \mathcal{D}(S) \) to shift from being a familiar logic to a curiosity like \( \mathcal{N} \) or the result of the failed characterization of \( \mathcal{LQ} \) (which may not even be a logic at all, in the sense of not being closed under modus ponens). On the other hand, sometimes simple semantic intuitions do apparently lead to success, as in the case of stable logic in Section 4.3.
The demonstrated sensitivity may turn out to be an intrinsic feature of the dialogical approach. Moreover, sensitivity can be found outside dialogues, too: one can jump from intuitionistic to classical logic in a Hilbert-style calculus—an enormous leap, from the point of view of the lattice of intermediate logics—in a single step by adding a single new axiom (e.g., excluded middle or Peirce’s formula). And one can move from intuitionistic to classical logic by simply dropping a constraint on the number of formulas that can appear on the right-hand side of a sequent.\(^3\)

Nonetheless, the non-independence of standard sets of dialogue rules is an obstacle to solving both the uniform and non-uniform composition problems. From a foundation of an independent set of dialogue rules, the problem of exposing some structure becomes easier because we can gradually add or subtract rules with the confidence that we are not making impermissible “jumps” in the space of possibilities.

5 Conclusion

Dialogue games can be viewed either as a stylized form of rational interaction or as alternative logical calculi. We have raised two problems—the modus ponens problem and the uniform substitution problem—that, on either view, pose challenges for the dialogician that are, so far, largely unaddressed. It seems that today much remains to be done for dialogues to give them their proper proof-theoretic foundation. We have argued, more precisely, that one important gap is that we lack a structure theory for dialogues that could help shed light on the problem of precisely what is the force of various dialogue rules.

We are not able to give a precise characterization of “direct” solutions to problems in dialogical logic. We gave a few results and conjectures (see, e.g., Lemma 2) along the intended lines. We in fact do not yet possess a direct solution. We left open the problem of showing, directly, that classical propositional logic is consistent, in the sense that there is no CL-valid formula \(\varphi\) such that \(\models_{\text{CL}} \varphi\) but satisfies \(\models_{\text{CL}} \varphi \to \psi\) for all formulas \(\psi\). It seems plausible to us that a direct solution to the problem is available; the solution may go via a normal form theorem for dialogue games.

One reason behind our preference for direct solutions to problems in dialogical logic is to stimulate the development of the dialogue formalism so that it becomes more systematic. In our view, resorting to external devices to establish basic results in dialogical logic is not a vote of confidence for dialogues, but is implicitly a concession that the formalism is awkward and difficult to work with.

\(^3\) The claim here is not that all Hilbert-style and sequent calculi for intuitionistic logic are such that adding one new principle or dropping exactly one structural condition are sufficient to capture classical logic; there are precise calculi for which these claims hold. A concrete example of a suitable Hilbert-style calculus is provided by the axioms \(B, C, K,\) and \(I\), with Peirce’s formula [15]; the calculus \(G2\) [18] is a suitable example of a sequent calculus.
Returning to the non-winning play that motivated our investigation (see Table 9, another option for proceeding would be to drop rule D11 but keep the other rules. This opens the door to $\mathbf{P}$ having a winning strategy, but it is not clear what the logical characterization is. More precisely: A known characterization of $\mathbf{CL}$ is the $\mathbf{E}$ minus rules D11 and D12; it is possible that dropping D11, but keeping D12, gives us stable logic or at least a logic in which stability is provable. We leave this as an open problem. We also leave open the problem of showing that $\mathbf{E}'$ is closed under uniform substitution and modus ponens. Solving both of these two problems would, combined with our soundness result, entail that the simple modification we made to the structural rules for intuitionistic logic do indeed yield stable logic.

References

1. Alama, J., Uckelman, S.L.: A curious dialogical logic and its composition problem (2010), preprint, http://arxiv.org/abs/1008.0080
2. Ciabattoni, A., Gabbay, D.M., Olivetti, N.: Cut-free proof systems for logics of weak excluded middle. Soft Computing 2, 147–156 (1999)
3. Felscher, W.: Dialogues, strategies, and intuitionistic provability. Annals of Pure and Applied Logic 28(3), 217–254 (May 1985)
4. Fermüller, C.G.: Parallel dialogue games and hypersequents for intermediate logic. In: Cialdea Mayer, M., Pirri, F. (eds.) Automated Reasoning with Analytic Tableaux and Related Methods, International Conference, TABLEAUX 2003, Rome, Italy, September 9-12, 2003. Proceedings. Lecture Notes in Computer Science, vol. 2796, pp. 48–64. Springer (2003)
5. Hodges, W.: Logic and games. In: Zalta, E.N. (ed.) Stanford Encyclopedia of Philosophy. CSLI Publications, Spring 2009 edn. (2009), http://plato.stanford.edu/archives/spr2009/entries/logic-games/
6. Hosoi, T.: Gentzen-type formulation of the propositional logic $LQ$. Studia Logica 47, 41–48 (1988)
7. Jankov, V.: The calculus of the weak excluded middle. Mathematics of the USSR 8(648–658) (1968)
8. Keiff, L.: Dialogical logic. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy. Summer 2009 edn. (2009), http://plato.stanford.edu/archives/sum2009/entries/logic-dialogical/
9. Krabbe, E.C.W.: Formal systems of dialogue rules. Synthese 63, 295–328 (1985)
10. Lorenz, K.: Arithmetik und Logik als Spiele. Ph.D. thesis, Universität Kiel (1961)
11. Lorenz, K.: Dialogspiele als semantische Grundlage von Logikkalkülen. Archiv Math. Logik Grundlagenforsch. 11, 32–55, 73–100 (1968)
12. Lorenzen, P.: Logik und agon. In: Arti del XII Congresso Internazionale de Filosofia. pp. 187–194 (1958)
13. Negri, S., von Plato, J.: Structural Proof Theory. Cambridge University Press (2001)
14. Rückert, H.: Dialogues as a Dynamic Framework for Logic. Ph.D. thesis, Universiteit Leiden (2007)
15. Seldin, J.P.: Basic Simple Type Theory. Cambridge Tracts in Theoretical Computer Science, vol. 42. Cambridge University Press (1997)
16. Sørensen, M.H., Urzyczyn, P.: Sequent calculus, dialogues, and cut elimination. In: Barendsen, E., Geuvers, H., Capretta, V., Niqui, M. (eds.) Reflections on Type Theory, λ-Calculus, and the Mind, pp. 253–261. Universiteit Nijmegen (2007)

17. Troelstra, A., van Dalen, D.: Constructivism in Mathematics: An Introduction. No. 121 in Studies in Logic and the Foundations of Mathematics, Elsevier (1988)

18. Troelstra, A., Schwichtenberg, H.: Basic Proof Theory, Cambridge Tracts in Theoretical Computer Science, vol. 43. Cambridge University Press (2000)

19. Walton, D.N.: New directions in the logic of dialogue. Synthese 63, 259–274 (1985)

20. Woods, J.: Ideals of rationality in dialogic. Argumentation 2(4), 395–408 (1988)