FULLY REPRESENTABLE AND *-SEMISimple
TOPOLOGICAL PARTIAL *-ALGEBRAS

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Abstract. We continue our study of topological partial *-algebras, focusing our attention to *-semisimple partial *-algebras, that is, those that possess a multiplication core and sufficiently many *-representations. We discuss the respective roles of invariant positive sesquilinear (ips) forms and representable continuous linear functionals and focus on the case where the two notions are completely interchangeable (fully representable partial *-algebras) with the scope of characterizing a *-semisimple partial *-algebra. Finally we describe various notions of bounded elements in such a partial *-algebra, in particular, those defined in terms of a positive cone (order bounded elements). The outcome is that, for an appropriate order relation, one recovers the $\mathcal{M}$-bounded elements introduced in previous works.

1. Introduction

Studies on partial *-algebras have provided so far a considerable amount of information about their representation theory and their structure. Many results have been obtained for concrete partial *-algebras, i.e., partial *-algebras of closable operators (the so-called partial $\mathcal{O}^*$-algebras), but a substantial body of knowledge has been gathered also for abstract partial *-algebras. A full analysis has been developed by Inoue and two of us some time ago and it can be found in the monograph [1], where earlier articles are quoted.

In a recent paper [4], we have started the analysis of certain types of bounded elements in a partial *-algebra $\mathfrak{A}$ and their incidence on the representation theory of $\mathfrak{A}$. It was shown, in particular, that the crucial condition is that $\mathfrak{A}$ possesses sufficiently many invariant positive sesquilinear forms (ips-forms). The latter, in turn, generate *-representations, that is, *-homomorphisms into a partial $\mathcal{O}^*$-algebra, via the well-known GNS construction. As in the particular case of a partial $\mathcal{O}^*$-algebra, a spectral theory can then be developed, provided the partial *-algebra has sufficiently many bounded elements. To that effect, we have introduced in [4] the notion of $\mathcal{M}$-bounded elements, associated to a sufficiently large family $\mathcal{M}$ of ips-forms.

We continue this study in the present work, focusing on topological partial *-algebras that possess what we call a multiplication core, that is, a
dense subset of universal right multipliers with all the regularity properties necessary for a decent representation theory. In particular, we will require that our partial *-algebra has sufficiently many *-representations, a property usually characterized, for topological *-algebras, in terms of the so-called *-radical. When the latter is reduced to \( \{0\} \), the partial *-algebra is called *-semisimple, the main subject of the paper. According to what we just said, *-semisimplicity is defined in terms of a family \( \mathcal{M} \) of ips-forms. Since it may be difficult to identify such a family in practice, we examine in what sense ips-forms may be replaced by a special class of continuous linear functionals, called \textit{representable}. This leads to identify a class of topological partial *-algebras for which representable linear functionals and ips-forms can be freely replaced by one another, since every representable linear functional comes (as for *-algebras with unit) from an ips-form. These partial *-algebras are called \textit{fully representable} (extending the analogous concept discussed in \[7\] for locally convex quasi *-algebras) and the interplay of this notion with *-semisimplicity is investigated.

This being done, we may come back to bounded elements of a *-semisimple partial *-algebra, more precisely to elements bounded with respect to some positive cone, thus defined in purely algebraic terms. Early work in that direction has been done by Vidav \[13\] and Schmüdgen \[9\], then generalized in our previous paper \[4\]. Here we consider several types of order on a partial *-algebra and analyze the corresponding notion of order bounded elements. The outcome is that, under appropriate conditions, the correct notion reduces to that of \( \mathcal{M} \)-bounded ones introduced in \[4\]. Therefore, when the partial *-algebra has sufficiently many such elements, the whole spectral theory developed in \[3\] and \[4\] can be recovered.

The paper is organized as follows. Section 2 is devoted to some preliminaries about partial *-algebras, taken mostly from \[1\] and \[3, 4\]. In addition, we introduce the notion of multiplication core and draw some consequences. We introduce in Section 3 the notion of *-semisimple partial *-algebra and discuss some of its properties. In Section 4 we compare the respective roles of ips-forms and representable linear functionals, with particular reference to fully representable partial *-algebras, and discuss the relationship of the latter notion with that of *-semisimple partial *-algebra. Finally, Section 5 is devoted to the various notions of bounded elements, from \( \mathcal{M} \)-bounded to order bounded ones.

2. Preliminaries

The following preliminary definitions will be needed in the sequel. For more details we refer to \[1, 8\].

A partial *-algebra \( \mathfrak{A} \) is a complex vector space with conjugate linear involution * and a distributive partial multiplication \( \cdot \), defined on a subset \( \Gamma \subset \mathfrak{A} \times \mathfrak{A} \), satisfying the property that \( (x, y) \in \Gamma \) if, and only if, \( (y^*, x^*) \in \Gamma \) and \( (x \cdot y)^* = y^* \cdot x^* \). From now on we will write simply \( xy \) instead of \( x \cdot y \) whenever \( (x, y) \in \Gamma \). For every \( y \in \mathfrak{A} \), the set of left (resp. right) multipliers of \( y \) is denoted by \( L(y) \) (resp. \( R(y) \)), i.e., \( L(y) = \{ x \in \mathfrak{A} : (x, y) \in \Gamma \} \) (resp.
that respect to the following operations: the usual sum\(X + A\) denote by multiplication\(\lambda X\) a weaker form of associativity holds. More precisely, we say that\( A\) is semi-associative if \(y \in R(x)\) implies \(yz \in R(x)\), for every \(z \in R\mathfrak{A}\), and
\[
(xy)z = x(yz).
\]

The partial *-algebra \(\mathfrak{A}\) has a unit if there exists an element \(e \in \mathfrak{A}\) such that \(e = e^*, e \in R\mathfrak{A} \cap L\mathfrak{A}\) and \(xe = ex = x\), for every \(x \in \mathfrak{A}\).

Let \(\mathcal{H}\) be a complex Hilbert space and \(\mathcal{D}\) a dense subspace of \(\mathcal{H}\). We denote by \(L^\dagger(\mathcal{D}, \mathcal{H})\) the set of all (closable) linear operators \(X\) such that \(D(X) = \mathcal{D}\), \(D(X^*) \supseteq \mathcal{D}\). The set \(L^\dagger(\mathcal{D}, \mathcal{H})\) is a partial *-algebra with respect to the following operations: the usual sum \(X_1 + X_2\), the scalar multiplication \(\lambda X\), the involution \(X \mapsto X^\dagger := X^* \upharpoonright \mathcal{D}\) and the (weak) partial multiplication \(X_1 \ast X_2 := X_1^\dagger X_2\), defined whenever \(X_2\) is a weak right multiplier of \(X_1\) (we shall write \(X_2 \in R^w(X_1)\) or \(X_1 \in L^w(X_2)\)), that is, whenever \(X_2 \mathcal{D} \subset D(X_1^\dagger)\) and \(X_1 \ast \mathcal{D} \subset D(X_2^\dagger)\).

It is easy to check that \(X_1 \in L^w(X_2)\) if and only if there exists \(Z \in L^\dagger(\mathcal{D}, \mathcal{H})\) such that
\[
(1) \quad \langle X_2 \xi | X_1^\dagger \eta \rangle = \langle Z \xi | \eta \rangle, \quad \forall \xi, \eta \in \mathcal{D}.
\]

In this case \(Z = X_1 \ast X_2\), \(L^\dagger(\mathcal{D}, \mathcal{H})\) is neither associative nor semi-associative. If \(I\) denotes the identity operator of \(\mathcal{H}\), \(I_\mathcal{D} := I \upharpoonright \mathcal{D}\) is the unit of the partial *-algebra \(L^\dagger(\mathcal{D}, \mathcal{H})\).

If \(\mathfrak{N} \subseteq L^\dagger(\mathcal{D}, \mathcal{H})\) we denote by \(R^w\mathfrak{N}\) the set of right multipliers of all elements of \(\mathfrak{N}\). We recall that
\[
(2) \quad RL^\dagger(\mathcal{D}, \mathcal{H}) \equiv R^w L^\dagger(\mathcal{D}, \mathcal{H}) = \{A \in L^\dagger(\mathcal{D}, \mathcal{H}) : A \text{ bounded and } A : \mathcal{D} \rightarrow \mathcal{D}^*\},
\]

where
\[
\mathcal{D}^* = \bigcap_{X \in L^\dagger(\mathcal{D}, \mathcal{H})} D(X^\dagger) = \bigcap_{X \in L^\dagger(\mathcal{D}, \mathcal{H})} X^* \upharpoonright \mathcal{D}.
\]

We denote by \(L^\dagger_b(\mathcal{D}, \mathcal{H})\) the bounded part of \(L^\dagger(\mathcal{D}, \mathcal{H})\), i.e., \(L^\dagger_b(\mathcal{D}, \mathcal{H}) = \{X \in L^\dagger(\mathcal{D}, \mathcal{H}) : X \text{ is a bounded operator}\} = \{X \in L^\dagger(\mathcal{D}, \mathcal{H}) : X \in B(\mathcal{H})\}\).

A \(\uparrow\)-invariant subspace \(\mathfrak{M}\) of \(L^\dagger(\mathcal{D}, \mathcal{H})\) is called a (weak) partial \(O^*\)-algebra if \(X \ast Y \in \mathfrak{M}\), for every \(X, Y \in \mathfrak{M}\) such that \(X \in L^w(Y)\). \(L^\dagger(\mathcal{D}, \mathcal{H})\) is the maximal partial \(O^*\)-algebra on \(\mathcal{D}\).

The set \(L^\dagger(\mathcal{D}) := \{X \in L^\dagger(\mathcal{D}, \mathcal{H}) : X, X^\dagger : \mathcal{D} \rightarrow \mathcal{D}\}\) is a *-algebra; more precisely, it is the maximal \(O^*\)-algebra on \(\mathcal{D}\) (for the theory of \(O^*\)-algebras and their representations we refer to [3]).

In the sequel, we will need the following topologies on \(L^\dagger(\mathcal{D}, \mathcal{H})\):

- The strong topology \(t_s\) on \(L^\dagger(\mathcal{D}, \mathcal{H})\), defined by the seminorms
  \[
p_\xi(X) = \|X\xi\|, \quad X \in L^\dagger(\mathcal{D}, \mathcal{H}), \xi \in \mathcal{D}.
\]
• The strong\(^*\) topology \(t_{\ast}\) on \(L^1(\mathcal{D}, \mathcal{H})\), defined by the seminorms
\[
p_\xi^*(X) = \max\{\|X\xi\|, \|X^\dagger\xi\|\}, \xi \in \mathcal{D}.
\]

A \(*\)-representation of a partial \(*\)-algebra \(\mathfrak{A}\) in the Hilbert space \(\mathcal{H}\) is a linear map \(\pi : \mathfrak{A} \rightarrow L^1(\mathcal{D}(\pi), \mathcal{H})\) such that: (i) \(\pi(x^*) = \pi(x)^\dagger\) for every \(x \in \mathfrak{A}\); (ii) \(x \in L(y)\) in \(\mathfrak{A}\) implies \(\pi(x) \in L^w(\pi(y))\) and \(\pi(x) \circ \pi(y) = \pi(xy)\).

The subspace \(\mathcal{D}(\pi)\) is called the domain of the \(*\)-representation \(\pi\). The \(*\)-representation \(\pi\) is said to be bounded if \(\pi(x) \in B(\mathcal{H})\) for every \(x \in \mathfrak{A}\).

Let \(\varphi\) be a positive sesquilinear form on \(D(\varphi) \times D(\varphi)\), where \(D(\varphi)\) is a subspace of \(\mathfrak{A}\). Then we have
\[
\varphi(x, y) = \overline{\varphi(y, x)}, \quad \forall x, y \in D(\varphi),
\]
\[
|\varphi(x, y)|^2 \leq \varphi(x, x)\varphi(y, y), \quad \forall x, y \in D(\varphi).
\]

We put
\[
N_\varphi = \{x \in D(\varphi) : \varphi(x, x) = 0\}.
\]

By (3), we have
\[
N_\varphi = \{x \in D(\varphi) : \varphi(x, y) = 0, \quad \forall y \in D(\varphi)\},
\]
and so \(N_\varphi\) is a subspace of \(D(\varphi)\) and the quotient space \(D(\varphi)/N_\varphi := \{\lambda_\varphi(x) \equiv x + N_\varphi; x \in D(\varphi)\}\) is a pre-Hilbert space with respect to the inner product
\[
\langle \lambda_\varphi(x) | \lambda_\varphi(y) \rangle = \varphi(x, y), \quad x, y \in D(\varphi).
\]

We denote by \(\mathcal{H}_\varphi\) the Hilbert space obtained by completion of \(D(\varphi)/N_\varphi\).

A positive sesquilinear form \(\varphi\) on \(\mathfrak{A} \times \mathfrak{A}\) is said to be invariant, and called an ips-form, if there exists a subspace \(B(\varphi)\) of \(\mathfrak{A}\) (called a core for \(\varphi\)) with the properties

\begin{enumerate}
\item[(ips1)] \(B(\varphi) \subseteq R\mathfrak{A}\);
\item[(ips2)] \(\lambda_\varphi(B(\varphi))\) is dense in \(\mathcal{H}_\varphi\);
\item[(ips3)] \(\varphi(xa, b) = \varphi(a, xa^*b), \forall x \in \mathfrak{A}, \forall a, b \in B(\varphi)\);
\item[(ips4)] \(\varphi(xa^*b, y) = \varphi(a, (xy)b), \forall x \in L(y), \forall a, b \in B(\varphi)\).
\end{enumerate}

In other words, an ips-form is an everywhere defined biweight, in the sense of [1].

To every ips-form \(\varphi\) on \(\mathfrak{A}\), with core \(B(\varphi)\), there corresponds a triple \((\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)\), where \(\mathcal{H}_\varphi\) is a Hilbert space, \(\lambda_\varphi\) is a linear map from \(B(\varphi)\) into \(\mathcal{H}_\varphi\) and \(\pi_\varphi\) is a \(*\)-representation of \(\mathfrak{A}\) in the Hilbert space \(\mathcal{H}_\varphi\). We refer to [1] for more details on this celebrated GNS construction.

Let \(\mathfrak{A}\) be a partial \(*\)-algebra and \(\pi\) a \(*\)-representation of \(\mathfrak{A}\) in \(\mathcal{D}(\pi)\). For \(\xi \in \mathcal{D}(\pi)\) we put
\[
\varphi_\pi^* (x, y) := \langle \pi(x) \xi | \pi(y) \xi \rangle, \quad x, y \in \mathfrak{A}.
\]

Then, \(\varphi_\pi^*\) is a positive sesquilinear form on \(\mathfrak{A} \times \mathfrak{A}\).

Let \(\mathfrak{B} \subseteq R\mathfrak{A}\) and assume that \(\pi(\mathfrak{B}) \subseteq L^1(\mathcal{D}(\pi))\). Then it is easily seen that \(\varphi_\pi^*\) satisfies the conditions (ips3) and (ips4) above. However, \(\varphi_\pi^*\) is not necessarily an ips-form since \(\pi(\mathfrak{B})\xi\) may fail to be dense in \(\mathcal{H}\). For this
reason, the following notion of regular *-representation was introduced in\cite{11}.

**Definition 2.1.** A *-representation $\pi$ of $\mathcal{A}$ with domain $\mathcal{D}(\pi)$ is called $\mathcal{B}$-regular if $\varphi^*_B$ is an ips-form with core $\mathcal{B}$, for every $\xi \in \mathcal{D}(\pi)$.

**Remark 2.2.** The notion of regular *-representation was given in\cite{2} for a larger class of positive sesquilinear forms (biweights) referring to the natural core

$$B(\varphi^*_B) = \{ a \in R\mathcal{A} : \pi(a) \in \mathcal{D}^{**}(\pi) \}$$

(we refer to\cite{1} for precise definitions). If $\pi(\mathcal{B}) \subset \mathcal{L}^1(\mathcal{D}(\pi))$, the $\mathcal{B}$-regularity implies that $\varphi^*_B$ is an also ips-form with core $B(\varphi^*_B)$. We will come back to this point in Proposition 2.8.

Let $\mathcal{A}$ be a partial *-algebra. We assume that $\mathcal{A}$ is a locally convex Hausdorff vector space under the topology $\tau$ defined by a (directed) set $\{p_\alpha\}_{\alpha \in \mathcal{I}}$ of seminorms. Assume that\footnote{Condition (cl) was called (t1) in\cite{3}.}

(cl) for every $x \in \mathcal{A}$, the linear map $L_x : R(x) \to \mathcal{A}$ with $L_x(y) = xy$, $y \in R(x)$, is closed with respect to $\tau$, in the sense that, if $\{y_\alpha\} \subset R(x)$ is a net such that $y_\alpha \to y$ and $xy_\alpha \to z \in \mathcal{A}$, then $y \in R(x)$ and $z = xy$.

For short, we will say that, in this case, $\mathcal{A}$ is a topological partial *-algebra. If the involution $x \mapsto x^*$ is continuous, we say that $\mathcal{A}$ is a *-topological partial *-algebra.

Starting from the family of seminorms $\{p_\alpha\}_{\alpha \in \mathcal{I}}$, we can define a second topology $\tau^*$ on $\mathcal{A}$ by introducing the set of seminorms $\{p^*_\alpha(x)\}$, where

$$p^*_\alpha(x) = \max\{p_\alpha(x), p_\alpha(x^*)\}, \quad x \in \mathcal{A}.$$ 

The involution $x \mapsto x^*$ is automatically $\tau^*$-continuous. By (cl) it follows that, for every $x \in \mathcal{A}$, both maps $L_x$, $R_x$ are $\tau^*$-closed. Hence, $\mathcal{A}[\tau^*]$ is a *-topological partial *-algebra.

In this paper we will consider the following particular classes of topological partial *-algebras.

**Definition 2.3.** Let $\mathcal{A}[\tau]$ be a topological partial *-algebra with locally convex topology $\tau$. Then,

1. A subspace $\mathcal{B}$ of $R\mathcal{A}$ is called a multiplication core if
   (d1) $e \in \mathcal{B}$ if $\mathcal{A}$ has a unit $e$;
   (d2) $\mathcal{B} \cdot \mathcal{B} \subseteq \mathcal{B}$;
   (d3) $\mathcal{B}$ is $\tau^*$-dense in $\mathcal{A}$;
   (d4) for every $b \in \mathcal{B}$, the map $x \mapsto xb$, $x \in \mathcal{A}$, is $\tau$-continuous;
   (d5) one has $b^*(xc) = (b^*x)c$, $\forall x \in \mathcal{A}, b, c \in \mathcal{B}$.

2. $\mathcal{A}[\tau]$ is called $\mathcal{A}_0$-regular if it possesses\footnote{In\cite{4} it was only supposed that $\mathcal{A}_0$ is $\tau$-dense in $\mathcal{A}$.} a multiplication core $\mathcal{A}_0$ which is a *-algebra and, for every $b \in \mathcal{A}_0$, the map $x \mapsto bx$, $x \in \mathcal{A}$, is $\tau$-continuous (\cite{4} Def. 4.1)).
Remark 2.4. A simple limiting argument shows that, if $A$ is an algebra (i.e., it is also associative), then $A$ is a $B$-right module, i.e.,

$$(xa)b = x(ab), \forall x \in A, a, b \in B.$$ 

If $A$ is $A_0$-regular then, in a similar way,

$$(xa)b = x(ab), (ax)b = a(xb) \text{ for every } x \in A, a, b \in A_0.$$ 

Remark 2.5. We warn the reader that an $A_0$-regular topological partial *-algebra $A[\tau]$ is not necessarily a locally convex partial *-algebra in the sense of [1, Def. 2.1.8]. Neither need it be topologically regular in the sense of [4, Def. 2.1.8], which is a more restrictive notion.

Remark 2.6. Let $A[\tau]$ be an $A_0$-regular topological partial *-algebra. Then, for every $b \in A_0$, the maps $x \mapsto xb$ and $x \mapsto bx$, $x \in A$, are also $\tau$-continuous. However, the density of $A_0$ in $A[\tau^*]$ may fail. Thus $A[\tau^*]$ need not be an $A_0$-regular *-topological partial *-algebra.

Examples 2.7. The three notions given in Definition 2.3 are really different.

1. Take $A = L^1(D, H)$. Then, $R\mathbb{A}$ is given in [2], so that we have an example where $R\mathbb{A} \cdot R\mathbb{A} \not\subset R\mathbb{A}$.

2. Take again $A = L^1(D, H)[t_{s^*}]$. Then $L^1(D, H)$ is $A_0$-regular for $A_0 = L^1_0(D)$.

3. Assume $L^1(D, H)[t_{s^*}]$ is self-adjoint, i.e. $D = D^*$ (for instance, when $D = D^\infty(A)$ for a self-adjoint operator $A$). Then $R\mathbb{L}^1(D, H) = \{X \in L^1_0(D, H) : X : D \to D\}$ is an algebra, but it is not *-invariant. Hence it is a multiplication core, since it is $t_{s^*}$-dense in $L^1(D, H)$, but $L^1(D, H)$ is not $R\mathbb{L}^1(D, H)$-regular.

The case of a locally convex quasi *-algebra ($\mathbb{A}, A_0$) was studied in [7] and a number of interesting properties have been derived. Some of these extend to the general case of a partial *-algebra, as we shall see in the sequel.

Proposition 2.8. Let $A[\tau]$ be a topological partial *-algebra and $B$ a multiplication core. Then every $(\tau, t_{s^*})$-continuous *-representation of $A$ is $B$-regular.

Proof. First we may assume that $\pi(B) \subset \mathbb{L}^1(D(\pi))$. Indeed, put

$$D(\pi_1) := \left\{ \xi_0 + \sum_{i=1}^n \pi_1(b_i)\xi_i : b_i \in B, \xi_i \in D(\pi); i = 0, 1, \ldots, n \right\},$$

$$\pi_1(x) \left( \xi_0 + \sum_{i=1}^n \pi_1(b_i)\xi_i \right) := \pi(x)\xi_0 + \sum_{i=1}^n \langle\pi(x)\xi, \pi(b_i)\xi_i\rangle.$$

Then, exactly as in [2] we can prove that $\pi_1$ is a *-representation of $A$ with $\pi_1(B) \subset \mathbb{L}^1(D(\pi_1))$.

If $\pi$ is $(\tau, t_{s^*})$-continuous, then $\pi_1$ is $(\tau, t_{s^*})$-continuous too (recall that domains are different!). Indeed, if $x_\alpha \mapsto x$, then $\pi(x_\alpha)\xi \to \pi(x)\xi$, for every 

\( \xi \in \mathcal{D}(\pi) \). The continuity of the right multiplication then implies that \( x, b \mapsto xb \), for every \( b \in \mathcal{B} \). Thus, by the continuity of \( \pi \), we get, for every \( b \in \mathcal{B} \), \( \pi(x, b) \xi \rightarrow \pi(xb) \xi \), for every \( \xi \in \mathcal{D}(\pi) \) or, equivalently, \( (\pi(x) \triangleleft \pi(b)) \xi \rightarrow (\pi(x) \triangleleft \pi(b)) \xi \), for every \( \xi \in \mathcal{D}(\pi) \). Hence

\[
\pi_1(x_{\alpha}) \left( \sum_{i=1}^{n} \pi(b_i) \xi_i \right) = \sum_{i=1}^{n} (\pi(x) \triangleleft \pi(b_i)) \xi_i \rightarrow \\
\sum_{i=1}^{n} (\pi(x) \triangleleft \pi(b_i)) \xi_i = \pi_1(x) \left( \sum_{i=1}^{n} \pi(b_i) \xi_i \right).
\]

Thus, every \((\tau, t_s)\)-continuous \(*\)-representation \( \pi \) extends to a \((\tau, t_s)\)-continuous \(*\)-representation \( \pi_1 \) with \( \pi_1(\mathcal{B}) \subset L^1(\mathcal{D}(\pi_1)) \). Finally we prove the \( \mathcal{B} \)-regularity of \( \pi \). If \( x \in \mathcal{A} \) then there exists a net \( \{b_{\alpha}\} \subset \mathcal{B} \) such that \( b_{\alpha} \xrightarrow{\tau} x \). Then we have

\[
||\lambda_{\varphi^\xi}(x) - \lambda_{\varphi^\xi}(b_{\alpha})||^2 = \varphi^\xi(x - b_{\alpha}, x - b_{\alpha}) \leq p(x - b_{\alpha})^2 \rightarrow 0,
\]

where \( p \) is a convenient \( \tau \)-continuous seminorm. This implies that \( \lambda_{\varphi^\xi}(\mathcal{B}) \) is dense in \( \mathcal{H}_{\varphi^\xi} \). Hence \( \varphi^\xi \) is an ips-form with core \( \mathcal{B} \).

Let \( \mathcal{A}[\tau] \) be a topological partial \(*\)-algebra with multiplication core \( \mathcal{B} \) and \( \varphi \) a positive sesquilinear forms on \( \mathcal{A} \times \mathcal{A} \) for which the conditions \((\text{ips}_1)\), \((\text{ips}_3)\) and \((\text{ips}_4)\) are satisfied (with respect to \( \mathcal{B} \)). Suppose that \( \varphi \) is \( \tau \)-continuous, i.e., there exist \( p_{\alpha}, \gamma > 0 \) such that:

\[
|\varphi(x, y)| \leq \gamma p_{\alpha}(x) p_{\alpha}(y) \quad \forall x, y \in \mathcal{A}.
\]

Then \((\text{ips}_2)\) is also satisfied and, therefore, \( \mathcal{B} \) is a core for \( \varphi \), so that \( \varphi \) is an ips-form. We denote by \( \mathcal{P}_{\mathcal{B}}(\mathcal{A}) \) the set of all \( \tau \)-continuous ips-forms with core \( \mathcal{B} \).

Using the continuity of the multiplication and Remark 27, it is easily seen that if \( \varphi \in \mathcal{P}_{\mathcal{B}}(\mathcal{A}) \) and \( a \in \mathcal{B} \), then \( \varphi_a \in \mathcal{P}_{\mathcal{B}}(\mathcal{A}) \), where

\[
\varphi_a(x, y) := \varphi(xa, ya), \quad x, y \in \mathcal{A}.
\]

3. **Topological partial \(*\)-algebras with sufficiently many \(*\)-representations**

Throughout this paper we will be mostly concerned with topological partial \(*\)-algebras possessing sufficiently many continuous \(*\)-representations. In the case of topological \(*\)-algebras this situation can be studied by introducing the so-called (topological) \(*\)-radical of the algebra. Thus we extend this notion to topological partial \(*\)-algebras.

Let \( \mathcal{A}[\tau] \) be a topological partial \(*\)-algebra. We define the \(*\)-radical of \( \mathcal{A} \) as

\[
\mathcal{R}^*(\mathcal{A}) := \{ x \in \mathcal{A} : \pi(x) = 0, \text{ for all } (\tau, t_s)\text{-continuous } *\text{-representations } \pi \}.
\]

We put \( \mathcal{R}^*(\mathcal{A}) = \mathcal{A} \), if \( \mathcal{A}[\tau] \) has no \((\tau, t_s)\)-continuous \(*\)-representations.
Remark 3.1. The *-radical was defined in [4, Sec.5] as
\[ \mathcal{R}_*(\mathfrak{A}) := \{ x \in \mathfrak{A} : \pi(x) = 0, \text{ for all } (\tau, t_*)\text{-continuous } *\text{-representations } \pi \}. \]

However, the two definitions are equivalent. Indeed, since every \((\tau, t_*)\)-continuous *-representation is \((\tau, t)\)-continuous, we have \(\mathcal{R}_*(\mathfrak{A}) \subset \mathcal{R}^*(\mathfrak{A})\).

In order to prove that \(\mathcal{R}^*(\mathfrak{A}) \subset \mathcal{R}_*(\mathfrak{A})\), assume that \(x \not\in \mathcal{R}_*(\mathfrak{A})\), i.e., there is a \((\tau, t_*)\)-continuous *-representation \(\pi\) such that \(\pi(x) \neq 0\). But \(\pi\) is also \((\tau, t)\)-continuous, hence \(x \not\in \mathcal{R}^*(\mathfrak{A})\) as well.

The *-radical enjoys the following immediate properties:

1. If \(x \in \mathcal{R}^*(\mathfrak{A})\), then \(x^* \in \mathcal{R}^*(\mathfrak{A})\).
2. If \(x \in \mathfrak{A}\), \(y \in \mathcal{R}^*(\mathfrak{A})\) and \(x \in L(y)\), then \(xy \in \mathcal{R}^*(\mathfrak{A})\).

From now on, we denote by \(\text{Rep}_c(\mathfrak{A})\) the set of all \((\tau, t)\)-continuous *-representations of \(\mathfrak{A}\). It \(\mathfrak{A}\) has a multiplication core \(\mathfrak{B}\), we may always suppose that \(\pi(x) \in L^1(D(\pi))\) for every \(x \in \mathfrak{B}\), as results from the proof of Proposition 2.8.

Proposition 3.2. Let \(\mathfrak{A}[\tau]\) be a topological partial *-algebra with unit \(e\). Let \(\mathfrak{B}\) be a multiplication core. For an element \(x \in \mathfrak{A}\) the following statements are equivalent.

(i) \(x \in \mathcal{R}^*(\mathfrak{A})\).
(ii) \(\varphi(x, x) = 0\) for every \(\varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})\).

Proof. (i) \(\Rightarrow\) (ii): Let \(\varphi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})\) and \(\pi_\varphi\) the corresponding GNS representation. Then, for every \(x \in \mathfrak{A}\),
\[ \|\pi_\varphi(x)\lambda_\varphi(a)\|^2 = \varphi(xa, xa) = \varphi_a(x, x) \leq p(x)^2, \quad a \in \mathfrak{B} \]
for some continuous \(\tau\)-seminorm \(p\) (depending on \(a\)). Hence \(\pi_\varphi\) is \((\tau, t)\)-continuous. If \(x \in \mathcal{R}^*(\mathfrak{A})\), then \(\pi_\varphi(x) = 0\). Thus \(\varphi(xa, xa) = 0\), for every \(a \in \mathfrak{B}\). From \(e \in \mathfrak{B}\), we get the statement.

(ii) \(\Rightarrow\) (i): Let \(\pi \in \text{Rep}_c(\mathfrak{A})\). We assume \(\pi(\mathfrak{B}) \subset L^1(D(\pi))\). For \(x, y \in \mathfrak{A}\) and \(\xi \in D\), put, as before,
\[ \varphi_\pi^\xi(x, y) := \langle \pi(x)\xi, \pi(y)\xi \rangle, \quad x, y \in \mathfrak{A}. \]

Then,
\[ |\varphi_\pi^\xi(x, y)| = |\langle \pi(x)\xi, \pi(y)\xi \rangle| \leq \|\pi(x)\xi\|\|\pi(y)\xi\| \leq p(x)p(y) \]
for some \(\tau\)-continuous seminorm \(p\). Hence, \(\varphi_\pi^\xi\) is continuous.

Thus, \(\varphi_\pi^\xi \in \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})\) and, by the assumption, \(\|\pi(x)\xi\|^2 = \varphi_\pi^\xi(x, x) = 0\). The arbitrariness of \(\xi\) implies that \(\pi(x) = 0\).

As for topological *-algebras, the *-radical contains all elements \(x\) whose square \(x^*x\) (if well defined) vanishes.

Proposition 3.3. Let \(\mathfrak{A}\) be a topological partial *-algebra. Let \(x \in \mathfrak{A}\), with \(x^* \in L(x)\). If \(x^*x = 0\), then \(x \in \mathcal{R}^*(\mathfrak{A})\).
Hence $\pi(x) = 0$. \hfill \square

**Remark 3.4.** A sort of converse of the previous statement was stated in [4, Proposition 5.3]. Unfortunately, the proof given there contains a gap.

**Definition 3.5.** A topological partial $*$-algebra $\mathfrak{A}[\tau]$ is called $*$-semisimple if, for every $x \in \mathfrak{A} \setminus \{0\}$ there exists a $(\tau, t_s)$-continuous $*$-representation $\pi$ of $\mathfrak{A}$ such that $\pi(x) \neq 0$ or, equivalently, if $R^*(\mathfrak{A}) = \{0\}$.

By Proposition 3.2, $\mathfrak{A}[\tau]$ is $*$-semisimple if, and only if, for some multiplication core $\mathfrak{B}$, the family of ips-forms $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ is sufficient in the following sense [4].

**Definition 3.6.** A family $\mathcal{M}$ of continuous ips-forms on $\mathfrak{A} \times \mathfrak{A}$ is sufficient if $x \in \mathfrak{A}$ and $\varphi(x,x) = 0$ for every $\varphi \in \mathcal{M}$ imply $x = 0$.

The sufficiency of the family $\mathcal{M}$ can be described in several different ways.

**Lemma 3.7.** Let $\mathfrak{A}$ be a topological partial $*$-algebra with multiplication core $\mathfrak{B}$. Then the following statements are equivalent:

(i) $\mathcal{M}$ is sufficient.
(ii) $\varphi(xa,b) = 0$, for every $\varphi \in \mathcal{M}$ and $a,b \in \mathfrak{B}$, implies $x = 0$.
(iii) $\varphi(xa,a) = 0$, for every $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$, implies $x = 0$.
(iv) $\varphi(xa,y) = 0$, for every $\varphi \in \mathcal{M}$ and $y \in \mathfrak{A}$, $a \in \mathfrak{B}$, implies $x = 0$.
(v) $\varphi(xa,xa) = 0$ for every $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$, implies $x = 0$.

We omit the easy proof.

Of course, if the family $\mathcal{M}$ is sufficient, any larger family $\mathcal{M}' \supset \mathcal{M}$ is also sufficient. In this case, the maximal sufficient family (having $\mathfrak{B}$ as core) is obviously the set $\mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ of all continuous ips-forms with core $\mathfrak{B}$. Hence if a sufficient family $\mathcal{M} \subseteq \mathcal{P}_{\mathfrak{B}}(\mathfrak{A})$ exists, $\mathfrak{A}[\tau]$ is $*$-semisimple.

**Example 3.8.** As mentioned before, the space $L^1(D, \mathcal{H})$ is a $L_b^1(D)$-regular partial $*$-algebra, when endowed with the strong $*$-topology $t_{s*}$. The set of positive sesquilinear forms $\mathcal{M} := \{\varphi_\xi : \xi \in D\}$, where $\varphi_\xi(X,Y) = \langle X\xi|Y\xi\rangle$, $X,Y \in L^1(D, \mathcal{H})$, is a sufficient family of ips-forms with core $L_b^1(D)$. Indeed, if $\varphi_\xi(X,X) = 0$, for every $\xi \in D$, then $\|X\xi\|^2 = 0$ and therefore $X = 0$.

**Example 3.9.** As shown in [6], the space $L^p(X)$, $X = [0,1]$, endowed with its usual norm topology, is $L^\infty(X)$-regular and it is $*$-semisimple if $p \geq 2$. Indeed, in this case the family of all continuous ips-forms is given by $\mathcal{M} = \{\varphi_w : w \in L^{p/(p-2)}, w \geq 0\}$, where $\varphi_w(f,g) = \int_X f(t)\overline{g(t)}w(t)dt$, $f,g \in L^p(X)$,
Proposition 3.10. Let \( \mathfrak{A} \) be a topological partial *-algebra with multiplication core \( \mathfrak{B} \). If \( \mathfrak{A} \) possesses a sufficient family \( \mathcal{M} \) of ips-forms, an extension of the multiplication of \( \mathfrak{A} \) can be introduced in a way similar to [1, Sec.4].

We say that the weak multiplication \( x \circ y \) is well-defined (with respect to \( \mathcal{M} \)) if there exists \( z \in \mathfrak{A} \) such that:
\[
\varphi(ya, x^*b) = \varphi(za, b), \quad \forall a, b \in \mathfrak{B}, \forall \varphi \in \mathcal{M}.
\]
In this case, we put \( x \circ y := z \) and the sufficiency of \( \mathcal{M} \) guarantees that \( z \) is unique. The weak multiplication \( \circ \) clearly depends on \( \mathcal{M} \): the larger is \( \mathcal{M} \), the stronger is the weak multiplication, in the sense that if \( \mathcal{M} \subseteq \mathcal{M}' \subseteq \mathcal{P}_\mathfrak{B}(\mathfrak{A}) \) and \( x \circ y \) exists w.r. to \( \mathcal{M}' \), then \( x \circ y \) exists with respect to \( \mathcal{M} \) too.

A handy criterion for the existence of the weak multiplication is provided by the following

**Proposition 3.10.** Let \( \mathfrak{B} \) be an algebra, then the weak product \( x \circ y \) is defined (with respect to \( \mathcal{M} \)) if, and only if, there exists a net \( \{b_\alpha\} \) in \( \mathfrak{B} \) such that \( b_\alpha \overset{\tau}{\longrightarrow} y \) and \( xb_\alpha \overset{\tau^*}{\longrightarrow} z \in \mathfrak{A} \).

Here \( \tau^*_w \mathcal{M} \) is the weak topology determined by \( \mathcal{M} \), with seminorms \( x \mapsto |\varphi(xa, b)|, \varphi \in \mathcal{M}, a, b \in \mathfrak{B} \). It is easy to prove that \( \mathfrak{A} \) is also a partial *-algebra with respect to the weak multiplication.

Since it holds in typical examples, e.g. \( \mathcal{L}(\mathcal{D}, \mathcal{H}) \) [1, Prop. 3.2], we will often suppose that the following condition is satisfied:

\( \text{(wp)} \) \( xy \) exists if, and only if, \( x \circ y \) exists. In this case \( xy = x \circ y \).

In this situation it is possible to define a stronger multiplication on \( \mathfrak{A} \): we say that the strong multiplication \( x \odot y \) is well-defined (and that \( x \in L^*(y) \) or \( y \in R^*(x) \)) if \( x \in L(y) \) and:

\( \text{(sm}_1) \) \( \varphi((xy)a, x^*z^*b) = \varphi(ya, (x^*z^*)b), \quad \forall z \in L(x), \forall \varphi \in \mathcal{M}, \forall a, b \in \mathfrak{B}; \)
\( \text{(sm}_2) \) \( \varphi((y^*x^*)a, vb) = \varphi(x^*a, (yv)b), \quad \forall v \in R(y), \forall \varphi \in \mathcal{M}, \forall a, b \in \mathfrak{B}. \)

The same considerations on the dependence on \( \mathcal{M} \) of the weak multiplication apply, of course, to the strong multiplication.

**Definition 3.11.** Let \( \mathfrak{A} \) be a partial *-algebra. A *-representation \( \pi \) of \( \mathfrak{A} \) is called quasi-symmetric if, for every \( x \in \mathfrak{A} \),
\[
\bigcup_{z \in L(x)} D((\pi(x)^* \upharpoonright \pi(z^*)\mathcal{D}(\pi))^*) = D(\overline{\pi(x)});
\]
\[
\bigcup_{v \in R(x)} D((\pi(x)^* \upharpoonright \pi(v)\mathcal{D}(\pi))^*) = D(\overline{\pi(x)\dagger}).
\]

Of course, the same definition can be given for any partial O*-algebra \( \mathfrak{M} \) (by considering the identical *-representation).

**Remark 3.12.** The conditions given in the previous Definition are certainly satisfied if, for every \( x \in \mathfrak{A} \), there exist \( s \in L(x), t \in R(x) \) such that
(\pi(x)^* \upharpoonright \pi(z)\pi(b)) = (\pi(z)^* \pi(b))\xi, \\
\forall z \in L(x), \forall a, b \in \mathcal{B};

\langle (\pi(y)^* \pi(x)\pi(a)\xi | \pi(v) \pi(b)\xi\rangle = (\pi(x)^* \pi(a)\xi | (\pi(y) \pi(v) \pi(b)\xi), \\
\forall v \in R(y), \forall a, b \in \mathcal{B}.

By taking a = b = e and using the polarization identity, one gets, for every \xi, \eta \in \mathcal{D}(\pi),

\langle (\pi(y)^* \pi(x)\pi(a)\xi | \pi(z)^* \pi(v)\eta\rangle = (\pi(y)^* \pi(x)\pi(a)\xi | (\pi(z)^* \pi(v)\eta), \\
\forall z \in L(x); \\
\langle (\pi(y)^* \pi(x)\pi(a)\xi | \pi(v)\eta\rangle = (\pi(x)^* \pi(a)\xi | (\pi(y) \pi(v)\eta), \\
\forall v \in R(y).

From these relations, it follows that

\pi(y) : \mathcal{D}(\pi) \rightarrow D((\pi(x)^* \upharpoonright \pi(z)\pi(b))\pi(\pi(y))\pi(v)\pi(b)\pi(\pi(y))\pi(v)\pi(b)\pi(\pi(y)), \forall \xi \in L(x), \forall a, b \in \mathcal{B};

\pi(x)^* : \mathcal{D}(\pi) \rightarrow D((\pi(x)^* \pi(y)^* \pi(v)\pi(b)\pi(\pi(y))\pi(v)\pi(b)\pi(\pi(y)), \forall v \in R(y).

By the assumption, it follows that \pi(y) : \mathcal{D}(\pi) \rightarrow D(\pi(x)^*) and \pi(x)^* : \mathcal{D}(\pi) \rightarrow D(\pi(y)^*). Thus, \pi(x) \circ \pi(y) is well-defined.

\hfill \Box

4. Representable functionals versus ips-forms

So far, we have used ips-forms in order to characterize *-semisimplicity of a topological partial *-algebra. The reason lies in the fact that ips-forms allow a GNS-like construction. However, from a general point of view it is not easy to find conditions for the existence of sufficient families of ips-forms, whereas there exist well-known criteria for the existence of continuous linear functionals that separate points of \mathfrak{A}. However, continuous linear functionals, which are positive in a certain sense, do not give rise, in general to a GNS
construction. This can be done, if they are \textit{representable} in the sense specified below. It is then natural to consider, in more details, conditions for the representability of continuous positive linear functionals.

\textbf{Definition 4.1.} \ Let $\mathfrak{A}[\tau]$ be a topological partial $^*$-algebra with multiplication core $\mathfrak{B}$. A continuous linear functional $\omega$ on $\mathfrak{A}$ is $\mathfrak{B}$-\textit{positive} if $\omega(a^*a) \geq 0$ for every $a \in \mathfrak{B}$.

The continuity of $\omega$ implies that $\omega(x) \geq 0$ for every $x$ which belongs to the $\tau$-closure $\mathfrak{A}^+(\mathfrak{B})$ of the set

$$\mathfrak{B}^{(2)} = \left\{ \sum_{k=1}^{n} x_k^* x_k, \ x_k \in \mathfrak{B}, \ n \in \mathbb{N} \right\}.$$ 

In the very same way as in [7, Theorem 3.2] one can prove the following

\textbf{Theorem 4.2.} \ Assume that $\mathfrak{A}^+(\mathfrak{B}) \cap (-\mathfrak{A}^+(\mathfrak{B})) = \{0\}$. Let $a \in \mathfrak{A}^+(\mathfrak{B})$, $a \neq 0$. Then there exists a continuous linear functional $\omega$ on $\mathfrak{A}$ with the properties:

1. $\omega(x) \geq 0, \ \forall \ x \in \mathfrak{A}^+(\mathfrak{B})$;
2. $\omega(a) > 0$.

The set $\mathfrak{A}^+(\mathfrak{B})$ will play an important role in Theorem 4.12 and in the analysis of order bounded elements in Section 5.2.

\textbf{Definition 4.3.} \ Let $\omega$ be a linear functional on $\mathfrak{A}$ and $\mathfrak{B}$ a subspace of $R\mathfrak{A}$. We say that $\omega$ is \textit{representable} (with respect to $\mathfrak{B}$) if the following requirements are satisfied:

1. $\omega(a^*a) \geq 0$ for all $a \in \mathfrak{B}$;
2. $\omega(b^*(x^*a)) = \omega(a^*(xb))$, $\forall \ a, b \in \mathfrak{B}, \ x \in \mathfrak{A}$;
3. $\forall x \in \mathfrak{A}$ there exists $\gamma_x > 0$ such that $|\omega(x^*a)| \leq \gamma_x \omega(a^*a)^{1/2}$, for all $a \in \mathfrak{B}$.

In this case, one can prove that there exists a triple $(\pi^{(\omega)}_{\mathfrak{A}}, \lambda^{(\omega)}_{\mathfrak{B}}, \mathcal{H}^{(\omega)}_{\mathfrak{B}})$ such that

1. $\pi^{(\omega)}_{\mathfrak{A}}$ is a *-representation of $\mathfrak{A}$ in $\mathcal{H}^{(\omega)}_{\mathfrak{B}}$;
2. $\lambda^{(\omega)}_{\mathfrak{B}}$ is a linear map of $\mathfrak{A}$ into $\mathcal{H}^{(\omega)}_{\mathfrak{B}}$ with $\lambda^{(\omega)}_{\mathfrak{B}}(\mathfrak{B}) = \mathcal{D}(\pi^{(\omega)}_{\mathfrak{A}})$ and $\pi^{(\omega)}_{\mathfrak{A}}(x)\lambda^{(\omega)}_{\mathfrak{B}}(a) = \lambda^{(\omega)}_{\mathfrak{B}}(xa)$, for every $x \in \mathfrak{A}, a \in \mathfrak{B}$.
3. $\omega(b^*(xa)) = \langle \pi^{(\omega)}_{\mathfrak{A}}(x)\lambda^{(\omega)}_{\mathfrak{B}}(a) | \lambda^{(\omega)}_{\mathfrak{B}}(b) \rangle$, for every $x \in \mathfrak{A}, a, b \in \mathfrak{B}$.

In particular, if $\mathfrak{A}$ has a unit $e$ and $e \in \mathfrak{B}$, we have:

1. $\pi^{(\omega)}_{\mathfrak{A}}$ is a cyclic *-representation of $\mathfrak{A}$ with cyclic vector $\xi_{\omega}$;
2. $\lambda^{(\omega)}_{\mathfrak{B}}$ is a linear map of $\mathfrak{A}$ into $\mathcal{H}^{(\omega)}_{\mathfrak{B}}$ with $\mathcal{D}(\pi^{(\omega)}_{\mathfrak{A}})$, $\xi_{\omega} = \lambda^{(\omega)}_{\mathfrak{B}}(e)$ and $\pi^{(\omega)}_{\mathfrak{A}}(x)\lambda^{(\omega)}_{\mathfrak{B}}(a) = \lambda^{(\omega)}_{\mathfrak{B}}(xa)$, for every $x \in \mathfrak{A}, a \in \mathfrak{B}$.
3. $\omega(a) = \langle \pi^{(\omega)}_{\mathfrak{A}}(x)\xi_{\omega} | \xi_{\omega} \rangle$, for every $x \in \mathfrak{A}$.

The GNS construction then depends on the subspace $\mathfrak{B}$. We adopt the notation $\mathcal{R}(\mathfrak{A}, \mathfrak{B})$ for denoting the set of linear functionals on $\mathfrak{A}$ which are representable with respect to the same $\mathfrak{B}$.

\textbf{Remark 4.4.} \ It is worth recalling (also for fixing notations) that the Hilbert space $\mathcal{H}_{\omega}$ is defined by considering the subspace of $\mathfrak{B}$

$$N_{\omega} = \left\{ x \in \mathfrak{B}; \ \omega(y^*x) = 0, \ \forall \ y \in \mathfrak{B} \right\}.$$
The quotient $\mathcal{B}/N_\omega \equiv \{\lambda_\omega^0(x) := x + N_\omega; x \in \mathcal{B}\}$ is a pre-Hilbert space with inner product
$$\langle \lambda_\omega^0(x) | \lambda_\omega^0(y) \rangle = \omega(y^* x), \quad x, y \in \mathcal{B}.$$ Then $\mathcal{H}_\omega$ is the completion of $\lambda_\omega^0(\mathcal{B})$. The representability of $\omega$ implies that $\lambda_\omega^0 : \mathcal{B} \to \mathcal{H}_\omega$ extends to a linear map $\lambda_\omega : \mathcal{A} \to \mathcal{H}_\omega$.

**Remark 4.5.** We notice that if $\pi$ is a *-representation of $\mathcal{A}$ on the domain $\mathcal{D}(\pi)$, and $\mathcal{B}$ is a subspace of $\mathcal{R}\mathcal{A}$ such that $\pi(\mathcal{B}) \subset \mathcal{L}^1(\mathcal{D}(\pi))$, then, for every $\xi \in \mathcal{D}(\pi)$, the linear functional $\omega^\xi_\pi$ defined by $\omega^\xi_\pi(x) = \langle \pi(x)\xi|\xi \rangle$ is representable, whereas the corresponding sesquilinear form $\varphi^\xi_\pi(x, y) = \langle \pi(x)\xi|\pi(y)\xi \rangle$ is not necessarily an ips-form; the latter fact leads to the notion of regular representation discussed above.

**Example 4.6.** A continuous linear functional $\omega$ whose restriction to $\mathcal{B}$ is positive need not be representable. As an example, consider $\mathcal{A} = L^1(I)$, $I$ a bounded interval on the real line, and $\mathcal{B} = L^\infty(I)$. The linear functional
$$\omega(f) = \int_I f(t) dt, \quad f \in L^1(I)$$
is continuous, but it is not representable, since $(r_3)$ fails if $f \in L^1(I) \setminus L^2(I)$.

Since multiplication cores play an important role for topological partial *-algebras, we restrict our attention to the case where $\mathcal{B}$ is a multiplication core and we omit explicit reference to $\mathcal{B}$ whenever it appears. We will denote by $\mathcal{R}_\omega(\mathcal{A}, \mathcal{B})$ the set of $\tau$-continuous linear functionals that are representable (with respect to $\mathcal{B}$).

Since both representable functionals and ips-forms define GNS-like representations it is natural to consider the interplay of these two notions, with particular reference to the topological case. We refer also to [5, 7] for more details.

**Proposition 4.7.** Let $\mathcal{A}$ be a topological partial *-algebra with multiplication core $\mathcal{B}$ which is an algebra. If $\varphi$ is a $\tau$-continuous ips-form on $\mathcal{A}$ then, for every $b \in \mathcal{B}$, the linear functional $\omega^b_\varphi$ defined by
$$\omega^b_\varphi(x) = \varphi(xb, b), \quad x \in \mathcal{A}$$is representable and the corresponding map $a \in \mathcal{B} \mapsto \lambda^0_{\omega^b_\varphi}(a) \in \mathcal{H}_{\omega^b_\varphi}$ is continuous.

Conversely, assume that $\mathcal{A}$ has a unit $e \in \mathcal{B}$. Then, if $\omega$ is a representable linear functional on $\mathcal{A}$ and the map $a \in \mathcal{B} \mapsto \lambda^0_\omega(a) \in \mathcal{H}_\omega$ is continuous, the positive sesquilinear form $\varphi_\omega$ defined on $\mathcal{B} \times \mathcal{B}$ by
$$\varphi_\omega(a, b) := \omega(b^* a),$$is $\tau$-continuous on $\mathcal{B} \times \mathcal{B}$ and it extends to a continuous ips-form $\overline{\varphi}_\omega$ on $\mathcal{A}$.

**Proof.** We prove only the second part of the statement.

For every $a, b \in \mathcal{B}$ we have
$$|\varphi_\omega(a, b)| = |\omega(b^* a)| = |\langle \pi_\omega(a)\lambda^0_\omega(e)|\pi_\omega(b)\lambda^0_\omega(e) \rangle| \leq \|\pi_\omega(a)\lambda^0_\omega(e)\|\|\pi_\omega(b)\lambda^0_\omega(e)\| = \|\lambda^0_\omega(a)\|\|\lambda^0_\omega(b)\| \leq p(a)p(b)$$
for some continuous seminorm $p$.

Hence $\psi_0$ extends uniquely to $\mathfrak{A} \times \mathfrak{A}$. Let $\tilde{\psi}_0$ denote this extension. It is easily seen that $\tilde{\psi}_0$ is a positive sesquilinear form on $\mathfrak{A} \times \mathfrak{A}$ and

$$|\tilde{\psi}_0(x, y)| \leq p(x)p(y), \quad \forall x, y \in \mathfrak{A}.$$  

Hence the map $x \mapsto \lambda_{\tilde{\psi}_0}(x) \in \mathcal{H}_{\tilde{\psi}_0}$ is also continuous, since

$$\|\lambda_{\tilde{\psi}_0}(x)\|^2 = \tilde{\psi}_0(x, x) \leq p(x)^2, \forall x \in \mathfrak{A}.$$  

Thus, if $x = \tau - \lim_{\alpha} b_\alpha$, $b_\alpha \in \mathcal{B}$, we get

$$\|\lambda_{\tilde{\psi}_0}(x) - \lambda_{\tilde{\psi}_0}(b_\alpha)\|^2 = \tilde{\psi}_0(x - b_\alpha, x - b_\alpha) \leq p(x - b_\alpha)^2 \to 0.$$  

The conditions $(\text{ips}_3)$ and $(\text{ips}_4)$ are readily checked. Concerning $(\text{ips}_4)$, for instance, let $x \in L(y)$ and $a, b \in \mathcal{B}$. Then,

$$\tilde{\psi}_0(a, (xy)b) = \omega(((xy)b)^*a) = \omega(b^*(xy)^a)$$

$$= \langle \pi_\omega(xy)^\dagger \lambda_0^0(a) | \lambda_0^0(b) \rangle$$

$$= \langle (\pi_\omega(y)^\dagger \pi_\omega(x)^\dagger) \lambda_\omega^0(a) | \lambda_\omega^0(b) \rangle$$

$$= \langle \pi_\omega(x)^\dagger \lambda_\omega^0(a) | \pi_\omega(y) \lambda_\omega^0(b) \rangle$$

$$= \tilde{\psi}_0(x^a, yb).$$

\[\square\]

**Remark 4.8.** If $\omega$ is a representable linear functional on $\mathfrak{A}$ and the map $a \in \mathcal{B} \mapsto \lambda_\omega^0(a) \in \mathcal{H}_\omega$ is continuous, then $\omega$ is continuous. The converse is false in general.

However, the continuity of $\omega$ implies the $\tau^*$-closability of the map $\lambda_\omega^0 : a \in \mathcal{B} \mapsto \lambda_\omega^0(a) \in \mathcal{H}_\omega$ as the next proposition shows.

**Proposition 4.9.** Let $\omega$ be continuous and $\mathcal{B}$-positive. Then the map $\lambda_\omega^0 : a \in \mathcal{B} \mapsto \lambda_\omega^0(a) \in \mathcal{H}_\omega$ is $\tau^*$-closable.

**Proof.** Let $a_\delta \xrightarrow{\tau^*} 0$, $a_\delta \in \mathcal{B}$, and suppose that the net $\{\lambda_\omega^0(a_\delta)\}$ is Cauchy in $\mathcal{H}_\omega$. Hence it converges to some $\xi \in \mathcal{H}_\omega$ and

$$\langle \lambda_\omega^0(b) | \lambda_\omega^0(a_\delta) \rangle \to \langle \lambda_\omega^0(b) | \xi \rangle, \quad \forall b \in \mathcal{B}.$$  

Moreover,

$$\langle \lambda_\omega^0(b) | \lambda_\omega^0(a_\delta) \rangle = \omega(a_\delta^*b) \to 0, \quad \forall b \in \mathcal{B},$$

since $a_\delta^* \xrightarrow{\tau} 0$ and the right multiplication by $b \in \mathcal{B}$ and $\omega$ are both $\tau$-continuous. Thus $\langle \lambda_\omega^0(b) | \xi \rangle = 0$, for every $b \in \mathcal{B}$. This implies that $\xi = 0$ and, therefore, $\lambda_\omega^0(a_\delta) \to 0$. \[\square\]

Actually, it is easy to see that the closability of the map $\lambda_\omega^0$ is equivalent to the closability of $\tilde{\psi}_0$. Indeed, closability of the map $a \in \mathcal{B} \mapsto \lambda_\omega^0(a) \in \mathcal{H}_\omega$ means that if $a_\delta \xrightarrow{\tau^*} 0$ and $\{\lambda_\omega^0(a_\delta)\}$ is a Cauchy net, then $\lambda_\omega^0(a_\delta) \to 0$. But $\{\lambda_\omega^0(a_\delta)\}$ is a Cauchy net if and only if $\psi_\omega(a_\delta - a_\gamma, a_\delta - a_\gamma) \to 0$. This leads to the conclusion $\|\lambda_\omega^0(a_\delta)\|^2 = \psi_\omega(a_\delta - a_\gamma, a_\delta - a_\gamma) \to 0$.

Therefore, Proposition 4.9 generalizes [7, Prop. 2.7], which says that, for a locally convex quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$, the sesquilinear form $\varphi_\omega$ is closable if $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{A}_0)$.  

Thus, if $\omega$ is continuous and $\mathfrak{B}$-positive, the map $\lambda^0_\omega$ has a closure $\overline{\lambda^0_\omega}$ defined on
$$D(\overline{\lambda^0_\omega}) = \{x \in \mathfrak{A} : \exists\{a_\delta\} \subset \mathfrak{B}, a_\delta \xrightarrow{\tau^*_\omega} x, \{\lambda^0_\omega(a_\delta)\}\}$$ is a Cauchy net.

From the discussion above, it follows that $D(\overline{\varphi_\omega})$ coincides with the domain $D(\overline{\varphi_\omega})$ of the closure of $\varphi_\omega$.

For the case of a locally convex quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$, the following assumption was made in [7]:

\begin{equation}
\text{(fr) } \bigcap_{\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})} D(\overline{\varphi_\omega}) = \mathfrak{A}.
\end{equation}

Quasi *-algebras verifying the condition (fr) are called fully representable (hence the acronym). Some concrete examples have been described in [7] and several interesting structure properties have been derived. We maintain the same definition and the same name in the case of topological partial *-algebras and, in complete analogy, we say that a topological partial *-algebra $\mathfrak{A}[\tau]$, with multiplication core $\mathfrak{B}$ is fully representable if

\begin{equation}
\text{(fr) } D(\overline{\varphi_\omega}) = \mathfrak{A}, \text{ for every } \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}).
\end{equation}

Then we have the following generalization of Proposition 3.6 of [7].

**Proposition 4.10.** Let $\mathfrak{A}$ be a semi-associative *-topological partial *-algebra with multiplication core $\mathfrak{B}$. Assume that $\mathfrak{A}$ is fully representable and let $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$. Then, $\overline{\varphi_\omega}$ is an ips-form on $\mathfrak{A}$ with core $\mathfrak{B}$, with the property

$$\overline{\varphi_\omega}(xa, b) = \omega(b^*xa), \quad \forall x \in \mathfrak{A}, a, b \in \mathfrak{B}.$$

**Proof.** The continuity of the involution implies that, for every $a \in \mathfrak{B}$, the map $x \mapsto a^*x$ is continuous on $\mathfrak{A}$. Hence the linear functional $\omega_a$ defined by $\omega((a^*xa))$ is continuous. We now prove that $\omega_a$ is representable; for this we need to check properties (r1), (r2) and (r3). We have

$$\omega_a(b^*b) = \omega(a^*(b^*b)a) = \omega((a^*b^*)(ba)) \geq 0, \quad \forall b \in \mathfrak{B},$$

i.e., (r1) holds. Furthermore, for every $b, c \in \mathfrak{B}$, we have

$$\omega_a(c^*(xb)) = \omega(a^*(c^*(xb))a) = \omega(a^*((c^*xb))a) = \omega(a^*(b^*(x^*c))a) = \omega_a(b^*(x^*c)).$$

As for (r3), for every $x \in \mathfrak{A}$ and $b \in \mathfrak{B}$, we have

$$|\omega_a(x^*b)| = |\omega(a^*(x^*b)a)| = |\omega((a^*x^*)(ba))| \leq \gamma_{x,a}\omega(a^*(b^*b)a)^{1/2} = \gamma_{x,a}\omega(a^*(b^*b))^{1/2}.$$

Thus $\omega_a \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$, for every $a \in \mathfrak{B}$. By Proposition 4.9, $\overline{\varphi_{\omega_a}}$ is well-defined and, by the assumption, $D(\overline{\varphi_{\omega_a}}) = \mathfrak{A}$. Hence, if $x \in \mathfrak{A}$, there exists a net $\{x_\alpha\} \subset \mathfrak{B}$ such that $x_\alpha \xrightarrow{\tau^*_\omega} x$ and $\varphi_{\omega_a}(x_\alpha - x_\beta, x_\alpha - x_\beta) \to 0$ or, equivalently, $\varphi_{\omega_a}(x_\alpha - x_\beta)a, (x_\alpha - x_\beta)a \to 0$. Hence, by the definition of closure, for every $b \in \mathfrak{A}$,

$$\overline{\varphi_\omega}(xa, b) = \lim_{\alpha} \varphi_\omega(x_\alpha a, b) = \lim_{\alpha} \omega(b^*(x_\alpha a)) = \omega(b^*(xa)),$$
by the continuity of $\omega$. This easily implies that $\varphi_\omega(xa, b) = \varphi_\omega(a, x^*b)$, for every $x \in \mathcal{A}$ and $a, b \in \mathcal{B}$, so that $\varphi_\omega$ satisfies (ips$_3$).

Let now $x \in L(y)$ and $a, b \in \mathcal{B}$. Now let $\{x_n\}$ and $\{y_n\}$ nets in $\mathcal{B}$, $\tau^*$-converging, respectively, to $x^*$ and $y$ and such that $\varphi_\omega((x'_n - x'_\beta)b, (x'_\beta - x''_\beta)b) \to 0$ and $\varphi_\omega((y_n - y'_\beta)a, (y_n - y'_\beta)a) \to 0$. Then we get

$$
\varphi_\omega((xy)a, b) = \omega(b^*(xy)a) \\
= \langle \pi_\omega(xy)\lambda_\omega^0(a)|\lambda_\omega^0(b) \rangle \\
= \langle \pi_\omega(x)\pi_\omega(y)\lambda_\omega^0(a)|\lambda_\omega^0(b) \rangle \\
= \langle \pi_\omega(y)\lambda_\omega^0(a)|\pi_\omega(x^*)\lambda_\omega^0(b) \rangle \\
= \lim_{\alpha, \beta}(\pi_\omega(y_\alpha)\lambda_\omega^0(a)|\pi_\omega(x'_\beta)\lambda_\omega^0(b)) \\
= \lim_{\alpha, \beta} \varphi_\omega(y_\alpha a, x'_\beta b) = \varphi_\omega(ya, x^*b).
$$

Thus, (ips$_4$) holds. To complete the proof, we need to show that $\lambda_\varphi(B)$ is dense in the Hilbert space $\mathcal{H}_f$. This part of the proof is completely analogous to that given in [7, Proposition 3.6] and we omit it. \qed

If $\mathcal{A}$ is semi-associative and fully representable, every continuous representable linear functional $\omega$ comes from a closed ips-form $\varphi_\omega$, but $\varphi_\omega$ need not be continuous, in general, unless more assumptions are made on the topology $\tau$.

**Corollary 4.11.** Let $\mathcal{A}[\tau]$ be a fully representable semi-associative *-topological partial *-algebra with multiplication core $\mathcal{B}$. Assume that $\mathcal{A}[\tau]$ is a Fréchet space. Then, for every $\omega \in \mathcal{R}_c(\mathcal{A}, \mathcal{B})$, $\varphi_\omega$ is a continuous ips-form.

**Proof.** The map $\lambda_\varphi$ is closed and everywhere defined. The closed graph theorem then implies that $\lambda_\varphi^0$ is continuous. The statement follows from Proposition 4.7. \qed

Summarizing, we have

**Theorem 4.12.** Let $\mathcal{A}[\tau]$ be a fully representable *-topological partial *-algebra with multiplication core $\mathcal{B}$ and unit $e \in \mathcal{B}$. Assume that $\mathcal{A}[\tau]$ is a Fréchet space and the following conditions hold

- (rc) Every linear functional $\omega$ which is continuous and $\mathcal{B}$-positive is representable;
- (sq) for every $x \in \mathcal{A}$, there exists a sequence $\{b_n\} \subset \mathcal{B}$ such that $b_n \xrightarrow{\tau} x$ and the sequence $\{b_n^*b_n\}$ is increasing, in the sense of the order of $\mathcal{A}^+(\mathcal{B})$.

Then $\mathcal{A}$ is *-semisimple.

**Proof.** Assume, on the contrary, that there exists $x \in \mathcal{A} \setminus \{0\}$ such that $\varphi(x, x) = 0$, for every $\varphi \in \mathcal{P}_d(\mathcal{A})$. If $\omega$ is continuous and $\mathcal{B}$-positive, then by assumption it is representable; thus $\varphi_\omega$, which is everywhere defined on $\mathcal{A} \times \mathcal{A}$, is continuous, by Corollary 4.11. Let $x = \lim_{n \to \infty} b_n$, with $b_n \in \mathcal{B}$ and $\{b_n^*b_n\}$ increasing. Then we have

$$0 \leq \lim_{n \to \infty} \omega(b_n^*b_n) = \lim_{n \to \infty} \varphi_\omega(b_n, b_n) = \varphi_\omega(x, x) = 0.$$
Then $\omega(b_n^*b_n) = 0$, for every $n \in \mathbb{N}$. But this contradicts Theorem 4.2. \hfill \Box

As we have seen in Example 4.6, condition (rc) is not fulfilled in general. To get an example of a situation where this condition is satisfied, it is enough to replace in Example 4.6 the normed partial *-algebra $L^1(I)$ with $L^2(I)$ (which is fully representable, as shown in [7]). It is easily seen that both condition (rc) and (sq) are satisfied in this case. It has been known since a long time that this partial *-algebra is *-semisimple [6].

5. BOUNDED ELEMENTS IN *-SEMISIMPLE PARTIAL *-ALGEBRAS

*-Semisimple topological partial *-algebras are characterized by the existence of a sufficient family of ips-forms. This fact was used in [4] and in [7] to derive a number of properties that we want to revisit in this larger framework.

5.1. $\mathcal{M}$-bounded elements. First we adapt to the present case the definition of $\mathcal{M}$-bounded elements given in [4] Def. 4.9] for an $\mathfrak{A}_0$-regular topological partial *-algebra.

**Definition 5.1.** Let $\mathfrak{A}$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$ and a sufficient family $\mathcal{M}$ of continuous ips-forms with core $\mathfrak{B}$. An element $x \in \mathfrak{A}$ is called $\mathcal{M}$-bounded if there exists $\gamma_x > 0$ such that

$$|\varphi(xa, b)| \leq \gamma_x \varphi(a, a)^{1/2} \varphi(b, b)^{1/2}, \forall \varphi \in \mathcal{M}, a, b \in \mathfrak{B}.$$  

An useful characterization of $\mathcal{M}$-bounded elements is given by the following proposition, whose proof is similar to that of [4 Proposition 4.10].

**Proposition 5.2.** Let $\mathfrak{A}[\tau]$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$. Then, an element $x \in \mathfrak{A}$ is $\mathcal{M}$-bounded if, and only if, there exists $\gamma_x \in \mathbb{R}$ such that $\varphi(xa, xa) \leq \gamma_x^2 \varphi(a, a)$ for all $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$.

If $x, y$ are $\mathcal{M}$-bounded elements and their weak product $x \odot y$ exists, then $x \odot y$ is also $\mathcal{M}$-bounded.

**Lemma 5.3.** Let the $\mathcal{M}$-bounded element $x \in \mathfrak{A}$ have a strong inverse $x^{-1}$. Then $\pi(x)$ has a strong inverse for every quasi-symmetric *-representation $\pi$.

**Proof.** Let $x \in \mathfrak{A}$ with strong inverse $x^{-1}$, i.e., $x \bullet x^{-1} = x^{-1} \bullet x = e$. Let $\pi$ be a *-representation with $\pi(e) = I$, then

$$I = \pi(e) = \pi(x \bullet x^{-1}) = \pi(x) \circ \pi(x^{-1}) = \pi(x) \circ \pi(x^{-1}).$$

It follows that the strong inverse $\pi(x)^{-1} := \pi(x^{-1})$ of $\pi(x)$ exists. \hfill \Box

Given $X \in \mathcal{L}^1(D, \mathcal{H})$, we denote by $\rho^D_\circ(X)$ the set of all complex numbers $\lambda$ such that $X - \lambda I_D$ has a strong bounded inverse [6 Section 3] and by $\sigma^D_\circ(X) := \mathbb{C} \setminus \rho^D_\circ(X)$ the corresponding spectrum of $X$.

If $\pi$ is a *-representation of $\mathfrak{A}$, from [6 Proposition 3.9] it follows that $\sigma^D_\circ(\pi(x)) = \sigma(\pi(x))$. If, in particular, $\pi$ is a quasi-symmetric *-representation, we can conclude, by Lemma 5.3, that $\rho^{\mathcal{M}}(x) \subseteq \rho(\pi(x)) = \rho^D_\circ(\pi(x))$, where
ρ^M(x) denotes the set of complex numbers λ such that the strong inverse 
(x - λe)^{-1} exists as an M-bounded element of A [1] Definition 4.28. Hence,

\[ \sigma(\pi(x)) \subseteq \sigma^M(x). \]

Exactly as for partial *-algebras of operators, there is here a natural distinction between hermitian elements x of A (i.e. x = x*) and self-adjoint elements (hermitian and with real spectrum).

**Definition 5.4.** The element x ∈ A is said M-self-adjoint if it is hermitian and \( \sigma^M(x) \subseteq \mathbb{R} \).

**Proposition 5.5.** If x ∈ A is M-self-adjoint, then for every quasi-symmetric \( \pi \in \text{Rep}_c(\mathfrak{A}, \mathfrak{B}) \), the operator \( \pi(x) \) is essentially self-adjoint.

**Proof.** If x ∈ A is M-self-adjoint, then, for every \( \pi \in \text{Rep}_c(\mathfrak{A}, \mathfrak{B}) \), the operator \( \pi(x) \) is symmetric and \( \sigma^M(x) \subseteq \mathbb{R} \). By (6) it follows that \( \pi(x) \) is self-adjoint, hence \( \pi(x) \) is essentially self-adjoint. \( \square \)

5.2. Order bounded elements.

5.2.1. Order structure. Let \( \mathfrak{A}[\tau] \) be a topological partial *-algebra with multiplication core \( \mathfrak{B} \). If \( \mathfrak{A}[\tau] \) is *-semisimple, there is a natural order on \( \mathfrak{A} \) defined by the family \( P_\mathfrak{B}(\mathfrak{A}) \) or by any sufficient subfamily \( \mathcal{M} \) of \( P_\mathfrak{B}(\mathfrak{A}) \), and this order can be used to define a different notion of boundedness of an element x ∈ \( \mathfrak{A} \) [7, 9, 13].

**Definition 5.6.** Let \( \mathfrak{A}[\tau] \) be a topological partial *-algebra and \( \mathfrak{B} \) a subspace of \( R\mathfrak{A} \). A subset \( \mathfrak{K} \) of \( \mathfrak{A}_h := \{ x \in \mathfrak{A} : x = x^* \} \) is called a \( \mathfrak{B} \)-admissible wedge if

1. \( e \in \mathfrak{K} \), if \( \mathfrak{A} \) has a unit e;
2. \( x + y \in \mathfrak{K}, \forall x, y \in \mathfrak{K} \);
3. \( \lambda x \in \mathfrak{K}, \forall x \in \mathfrak{K}, \lambda \geq 0 \);
4. \( (a^*x)a = a^*(xa) =: a^*xa \in \mathfrak{K}, \forall x \in \mathfrak{K}, a \in \mathfrak{B} \).

As usual, \( \mathfrak{K} \) defines an order on the real vector space \( \mathfrak{A}_h \) by \( x \leq y \iff y - x \in \mathfrak{K} \).

In the rest of this section, we will suppose that the partial *-algebras under consideration are semi-associative. Under this assumption, the first equality in (4) of Definition 5.6 is automatically satisfied.

Let \( \mathfrak{A} \) be a topological partial *-algebra with multiplication core \( \mathfrak{B} \). We put

\[ \mathfrak{B}^{(2)} = \left\{ \sum_{k=1}^{n} x_k^* x_k, x_k \in \mathfrak{B}, n \in \mathbb{N} \right\}. \]

If \( \mathfrak{B} \) is a *-algebra, this is nothing but the set (wedge) of positive elements of \( \mathfrak{B} \). The \( \mathfrak{B} \)-strongly positive elements of \( \mathfrak{A} \) are then defined as the elements of \( \mathfrak{A}^+(\mathfrak{B}) := \mathfrak{B}^{(2)} \), already defined in Section 4. Since \( \mathfrak{A} \) is semi-associative, the set \( \mathfrak{A}^+(\mathfrak{B}) \) of \( \mathfrak{B} \)-strongly positive elements is a \( \mathfrak{B} \)-admissible wedge.
We also define
\[ \mathfrak{A}^+_\text{alg} = \left\{ \sum_{k=1}^{n} x_k^* x_k, \; x_k \in R\mathfrak{A}, \; n \in \mathbb{N} \right\}, \]
the set (wedge) of positive elements of \( \mathfrak{A} \) and we put \( \mathfrak{A}^+_\text{top} := \overline{\mathfrak{A}^+_\text{alg}}^{\text{top}} \). The semi-associativity implies that \( R\mathfrak{A}, R\mathfrak{A} \subseteq R\mathfrak{A} \) and then \( \mathfrak{A}^+_\text{top} \) is \( R\mathfrak{A} \)-admissible.

Let \( \mathcal{M} \subseteq \mathcal{P}_B(\mathfrak{A}) \). An element \( x \in \mathfrak{A} \) is called \( \mathcal{M} \)-positive if
\[ \varphi(xa, a) \geq 0, \quad \forall \varphi \in \mathcal{M}, a \in \mathcal{B}. \]
An \( \mathcal{M} \)-positive element is automatically hermitian. Indeed, if \( \varphi(xa, a) \geq 0, \quad \forall \varphi \in \mathcal{M}, a \in \mathcal{B} \), then \( \varphi(a, x^*a) = \varphi(xa, a) \geq 0 \) and \( \varphi(x^*a, a) \geq 0 \); hence \( \varphi((x - x^*)a, a) = 0, \quad \forall \varphi \in \mathcal{M}, \forall a \in \mathcal{B} \). By (iii) of Lemma 3.7, it follows that \( x = x^* \).

We denote by \( \mathfrak{A}^\dagger_{\mathcal{M}} \) the set of all \( \mathcal{M} \)-positive elements. Clearly \( \mathfrak{A}^\dagger_{\mathcal{M}} \) is a \( \mathcal{B} \)-admissible wedge.

**Proposition 5.7.** The following inclusions hold
\[ \mathfrak{A}^\dagger_{\mathfrak{B}}(\mathfrak{B}) \subseteq \mathfrak{A}^\dagger_{\text{top}} \subseteq \mathfrak{A}^\dagger_{\mathcal{M}}, \quad \forall \mathcal{M} \subseteq \mathcal{P}_B(\mathfrak{A}). \]

**Proof.** We only prove the second inclusion. Let \( x \in \mathfrak{A}^\dagger_{\text{top}} \). Then \( x = \lim_{\alpha} b_\alpha \) with \( b_\alpha = \sum_{i=1}^{n} c_{\alpha,i}^* c_{\alpha,i}, c_{\alpha,i} \in R\mathfrak{A} \). Thus,
\[ \varphi(xa, a) = \lim_{\alpha} \varphi(b_\alpha a, a) = \lim_{\alpha} \varphi(\sum_{i} (c_{\alpha,i}^* c_{\alpha,i}) a, a) \]
\[ = \lim_{\alpha} \sum_{i} \varphi((c_{\alpha,i}^* c_{\alpha,i}) a, a) = \lim_{\alpha} \sum_{i} \varphi(c_{\alpha,i} a, c_{\alpha,i} a) \geq 0. \]
by (ips).

Of course, one expects that under certain conditions the converse inclusions hold, or that the three sets in (7) actually coincide. A partial answer is given in Corollary 5.16.

**Example 5.8.** We give here two examples where the wedges considered above coincide.

1. The first example, very elementary, is obtained by considering the space \( L^p(X), \; p \geq 2 \). Indeed, it is easily seen that the \( \mathcal{M} \)-positivity of a function \( f \) simply means that \( f(t) \geq 0 \) a.e. in \( X \) (\( \mathcal{M} \) is here the family of ips-forms defined in Example 3.9). On the other hand it is well-known that such a function can be approximated in norm by a sequence of nonnegative functions of \( L^\infty(X) \).
2. Let \( T \) be a self-adjoint operator with dense domain \( D(T) \) and denote by \( E(\cdot) \) the spectral measure of \( T \). We consider the space \( \mathcal{L}^1(D, \mathcal{H}) \) where \( D := \mathcal{D}^\infty(T) = \bigcap_{n \in \mathbb{N}} D(T^n) \). We prove that if \( \mathcal{L}^1(D, \mathcal{H}) \) is endowed with the topology \( t_\ast \) and \( \mathcal{M} \) is the family of ips-forms defined in Example 3.8 then every \( X \in \mathcal{L}^1(D, \mathcal{H}) \) which is \( \mathcal{M} \)-positive, i.e., \( \langle X \xi | \xi \rangle \geq 0 \), for every \( \xi \in D \), is the \( t_\ast \)-limit of elements of \( L^1(D)^{(2)} \). Indeed, if \( D, D' \) are bounded Borel subsets of the real line, then \( E(\Delta(\xi)) \xi \in D \), for every \( \xi \in \mathcal{H} \). This implies
that $E(\Delta')YE(\Delta)$ is a bounded operator in $\mathcal{H}$, for every $Y \in \mathcal{L}^1(\mathcal{D}, \mathcal{H})$ and its restriction to $\mathcal{D}$ belongs to $L^1(\mathcal{D})$. Put $\Delta_N = (-N, N], N \in \mathbb{N}$ and $\delta_m = (m, m + 1], m \in \mathbb{Z}$. The $\mathcal{M}$-positivity of $X$ implies that $E(\Delta_N)XE(\Delta_N) = B_N^*B_N$ for some bounded operator $B_N$. Then, observing that $\sum_{m \in \mathbb{Z}} E(\delta_m) = I$, in strong or strong*-sense, we obtain

$$E(\Delta_N)XE(\Delta_N) = B_N^*B_N = E(\Delta_N)B_N^*B_NE(\Delta_N)$$

$$= E(\Delta_N)B_N^* \left( \sum_{m \in \mathbb{Z}} E(\delta_m) \right) B_N E(\Delta_N)$$

$$= \sum_{m \in \mathbb{Z}} (E(\Delta_N)B_N^*E(\delta_m))(E(\delta_m)B_N E(\Delta_N)).$$

This proves that $E(\Delta_N)XE(\Delta_N)$ belongs to the $t_*$-closure of $L^1_b(\mathcal{D})^{(2)}$. Now, if we let $N \to \infty$, we easily get $\|X\xi - E(\Delta_N)XE(\Delta_N)\xi\| \to 0$ and so the statement is proved.

An improvement of Theorem 4.2 is provided by the following

**Corollary 5.9.** Let $\mathcal{M}$ be sufficient. Then, for every $x \in \mathfrak{A}^+_\mathcal{M}$, $x \neq 0$, there exists $\omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B})$ with the properties

(a) $\omega(y) \geq 0$, $\forall y \in \mathfrak{A}^+_\mathcal{M}$;

(b) $\omega(x) > 0$.

**Proof.** By the previous proposition, if $x \in \mathfrak{A}^+_\mathcal{M}$, $x \neq 0$, there exist $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$ such that $\varphi(xa, a) > 0$. Hence the linear functional $\omega(y) := \varphi(ya, a)$ has the desired properties. \hfill \Box

**Proposition 5.10.** Let the family $\mathcal{M}$ be sufficient. Then, $\mathfrak{A}^+_\mathcal{M}$ is a cone, i.e., $\mathfrak{A}^+_\mathcal{M} \cap (-\mathfrak{A}^+_\mathcal{M}) = \{0\}$.

**Proof.** If $x \in \mathfrak{A}^+_\mathcal{M} \cap (-\mathfrak{A}^+_\mathcal{M})$, then $\varphi(xa, a) \geq 0$ and $\varphi((-x)a, a) \geq 0$, for every $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$. Hence $\varphi(xa, a) = 0$, for every $\varphi \in \mathcal{M}$ and $a \in \mathfrak{B}$. The sufficiency of $\mathcal{M}$ then implies $x = 0$. \hfill \Box

**Remark 5.11.** The fact that $\mathfrak{A}^+_\mathcal{M}$ is a cone automatically implies that $\mathfrak{A}^+(\mathfrak{B})$ is a cone too.

The following statement shows that $\mathcal{P}_\mathfrak{B}(\mathfrak{A})$-positivity is exactly what is needed if we want the order to be preserved under any continuous *-representation. A partially equivalent statement is given in $[7]$, Proposition 3.1]. For making the notations lighter, we put $\mathfrak{A}_p^+ := \mathfrak{A}^+_{\mathcal{P}_\mathfrak{B}(\mathfrak{A})}$.

**Proposition 5.12.** Let $\mathfrak{A}$ be a topological partial *-algebra with multiplication core $\mathfrak{B}$ and unit $e \in \mathfrak{B}$. Then, the element $x \in \mathfrak{A}$ belongs to $\mathfrak{A}_p^+$ if and only if the operator $\pi(x)$ is positive for every $(\tau, t_\alpha)$-continuous *-representation $\pi$ with $\pi(e) = I_{\mathcal{D}(\pi)}$. 


Proof. Let \( x \in \mathfrak{A}_\mathcal{P}^+ \) and let \( \pi \) be a \((\tau, t_o)\)-continuous \(*\)-representation of \( \mathfrak{A} \) with \( \pi(e) = I_{D(\pi)} \). The sesquilinear form \( \varphi_\pi^\xi \), defined by
\[
\varphi_\pi^\xi(x, y) := \langle \pi(x)\xi|\pi(y)\xi \rangle \quad x, y \in \mathfrak{A},
\]
is a continuous ips-form as shown in the proof of Proposition 3.2. Then,
\[
\varphi_\pi^\xi(xa, a) = \langle \pi(xa)\xi|\pi(a)\xi \rangle = \langle (\pi(x)\circ \pi(a))\xi|\pi(a)\xi \rangle;
\]
in particular, for \( a = e \), \( \langle \pi(x)\xi|\xi \rangle \geq 0 \).

Conversely, let \( \varphi \in \mathcal{P} \) and \( \pi_\varphi \) the corresponding GNS representation. Then, as remarked in the proof of Proposition 3.2, \( \pi_\varphi \) is \((\tau, t_o)\)-continuous. We have, for every \( a \in \mathfrak{B} \),
\[
\varphi(xa, a) = \langle \pi_\varphi(x)\lambda_\varphi(a)|\lambda_\varphi(a) \rangle \geq 0,
\]
i.e., \( x \in \mathfrak{A}_\mathcal{P}^+ \).
\( \square \)

Proposition 5.13. Let \( \mathfrak{A} \) be a fully-representable \(*\)-topological partial \(*\)-algebra with multiplication core \( \mathfrak{B} \) and unit \( e \in \mathfrak{B} \). Assume that \( \mathfrak{A}[\tau] \) is a Fréchet space. Then the following statements are equivalent:

(i) \( x \in \mathfrak{A}_\mathcal{P}^+ \);
(ii) \( \omega(x) \geq 0, \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}) \).

Proof. (i) \( \Rightarrow \) (ii): If \( x \in \mathfrak{A}_\mathcal{P}^+ \), then \( \varphi(xa, a) \geq 0, \forall \varphi \in \mathcal{P}_\mathfrak{B}(\mathfrak{A}) \), \( \forall a \in \mathfrak{A}_0 \).
If \( \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}) \), by the assumptions and by Proposition 1.10 it follows that \( \overline{\omega} \varphi \) is an everywhere defined ips-form and thus, by Corollary 4.11 it is continuous. Hence,
\[
\omega(a^*xa) = \overline{\omega}(xa, a) \geq 0, \quad \forall a \in \mathfrak{B}.
\]
For \( a = e \), we get that \( \omega(x) \geq 0 \).

(ii) \( \Rightarrow \) (i): If \( \omega(x) \geq 0, \forall \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}) \), then this also holds for every linear functional \( \omega_\varphi^a, a \in \mathfrak{B}, \) defined by \( \varphi \in \mathcal{P}_\mathfrak{B}(\mathfrak{A}) \) as in Proposition 4.7. Then
\[
\varphi(xa, a) = \omega_\varphi^a(x) \geq 0, \quad \forall \varphi \in \mathcal{P}_\mathfrak{B}(\mathfrak{A}).
\]
By definition, this means that \( x \in \mathfrak{A}_\mathcal{P}^+ \).
\( \square \)

In complete analogy with Proposition 3.9 of [7], one can prove the following

Proposition 5.14. Let \( \mathfrak{A}[\tau] \) be a \(*\)-semisimple \(*\)-topological partial \(*\)-algebra with multiplication core \( \mathfrak{B} \).
Assume that the following condition \((P)\) holds:

\((P)\) \( y \in \mathfrak{A} \) and \( \omega(a^*ya) \geq 0, \) for every \( \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}) \) and \( a \in \mathfrak{A}_0 \),

imply \( y \in \mathfrak{A}_+^+(\mathfrak{B}) \).

Then, for an element \( x \in \mathfrak{A} \), the following statements are equivalent:

(i) \( x \in \mathfrak{A}_+^+(\mathfrak{B}) \);
(ii) \( \omega(x) \geq 0, \) for every \( \omega \in \mathcal{R}_c(\mathfrak{A}, \mathfrak{B}) \);
(iii) \( \pi(x) \geq 0, \) for every \((\tau, t_w)\)-continuous \(*\)-representation \( \pi \) of \( \mathfrak{A} \).
Remark 5.15. In [7, Proposition 3.9] it was required that the family \( R_c(\mathcal{A}, \mathcal{A}_0) \) of continuous linear functionals does not annihilate positive elements. This is always true for \(*\)-semisimple partial \(*\)-algebras, because of Proposition 5.10.

The previous propositions allow to compare the different cones defined so far.

Corollary 5.16. Under the assumptions of Propositions 5.13 and 5.14, one has \( A^+ (\mathcal{B}) = \mathcal{A}_c^+ \).

5.2.2. Order bounded elements. Let \( \mathcal{A}[\tau] \) be a topological partial \(*\)-algebra with multiplication core \( \mathcal{B} \) and unit \( e \in \mathcal{B} \). As we have seen in Section 5.2.1, \( \mathcal{A}[\tau] \) has several natural orders, all related to the topology \( \tau \). Each one of them can be used to define \textit{bounded} elements. We begin in a purely algebraic way starting from an arbitrary \( \mathcal{B} \)-admissible cone \( K \).

Let \( x \in \mathcal{A} \); put \( \Re(x) = \frac{1}{2}(x + x^*), \Im(x) = \frac{1}{2i}(x - x^*) \). Then \( \Re(x), \Im(x) \in \mathcal{A}_h \) (the set of self-adjoint elements of \( \mathcal{A} \)) and \( x = \Re(x) + i\Im(x) \).

Definition 5.17. An element \( x \in \mathcal{A} \) is called \( \mathcal{K} \)-\textit{bounded} if there exists \( \gamma \geq 0 \) such that
\[
\pm \Re(x) \leq \gamma e; \quad \pm \Im(x) \leq \gamma e.
\]
We denote by \( \mathcal{A}_b(\mathcal{K}) \) the family of \( \mathcal{K} \)-bounded elements.

The following statements are easily checked.

(1) \( \alpha x + \beta y \in \mathcal{A}_b(\mathcal{K}), \quad \forall x, y \in \mathcal{A}_b(\mathcal{K}), \alpha, \beta \in \mathbb{C} \).

(2) \( x \in \mathcal{A}_b(\mathcal{K}) \Leftrightarrow x^* \in \mathcal{A}_b(\mathcal{K}) \).

Remark 5.18. If \( \mathcal{A} \) is a \(*\)-algebra then, as shown in [9, Lemma 2.1], one also has

(3) \( x, y \in \mathcal{A}_b(\mathcal{K}) \Rightarrow xy \in \mathcal{A}_b(\mathcal{K}) \).

(4) \( a \in \mathcal{A}_b(\mathcal{K}) \Leftrightarrow aa^* \in \mathcal{A}_b(\mathcal{K}) \).

These statements do not hold in general when \( \mathcal{A} \) is a partial \(*\)-algebra. They are true, of course, for elements of \( \mathcal{B} \).

For \( x \in \mathcal{A}_h \), put
\[
\|x\|_b := \inf \{ \gamma > 0 : -\gamma e \leq x \leq \gamma e \}.
\]
\( \| \cdot \|_b \) is a seminorm on the real vector space \( (\mathcal{A}_b(\mathcal{K}))_h \).

Lemma 5.19. Let \( \mathcal{M} \) be sufficient. If \( \mathcal{K} = \mathcal{A}_M^+ \), then \( \| \cdot \|_b \) is a norm on \( (\mathcal{A}_b(\mathcal{M}))_h \).

Proof. By Proposition 5.10, \( \mathcal{A}_M^+ \) is a cone. Put \( E = \{ \gamma > 0 : -\gamma e \leq x \leq \gamma e \} \). If \( \inf E = 0 \), then, for every \( \epsilon > 0 \), there exists \( \gamma_\epsilon \in E \) such that \( \gamma_\epsilon < \epsilon \). This implies that \( -\epsilon e \leq x \leq \epsilon e \). If \( \varphi \in \mathcal{M} \), we get \( -\epsilon \varphi(a, a) \leq \varphi(xa, a) \leq \epsilon \varphi(a, a) \), for every \( a \in \mathcal{B} \). Hence, \( \varphi(xa, a) = 0 \). By the sufficiency of \( \mathcal{M} \), it follows that \( x = 0 \). \( \square \)

Let \( \mathcal{A}[\tau] \) be a \(*\)-semisimple topological partial \(*\)-algebra with multiplica-
tion core \( \mathcal{B} \). We can then specify the wedge \( \mathcal{K} \) as one of those defined above.
Take first $\mathcal{A} = \mathcal{A}^+_M$, where $\mathcal{M} = \mathcal{P}_B(\mathcal{A})$ is the sufficient family of all continuous $\ast$-ips-forms with core $B$. For simplicity, we write again $\mathcal{P} := \mathcal{P}_B(\mathcal{A})$, hence $\mathcal{A}^+_\mathcal{P} := \mathcal{A}^+_\mathcal{P}_B(\mathcal{A})$ and $\mathcal{A}_0(\mathcal{P}) := \mathcal{A}_0(\mathcal{P}_B(\mathcal{A}))$.

**Proposition 5.20.** If $x \in \mathcal{A}_0(\mathcal{P})$, then $\pi(x)$ is a bounded operator, for every $(\tau, t_0)$-continuous $\ast$-representation of $\mathcal{A}$. Moreover, if $x = x^\ast$, $\|\pi(x)\| \leq \|x\|_b$.

**Proof.** This follows easily from Proposition 5.12 and from the definitions.

The following theorem generalizes [7, Theorem 5.5].

**Theorem 5.21.** Let $\mathcal{A}[\tau]$ be a fully representable, semi-associative $\ast$-topological partial $\ast$-algebra, with multiplication core $B$ and unit $e \in B$. Assume that $\mathcal{A}[\tau]$ is a Fréchet space. Then the following statements are equivalent:

1. $x \in \mathcal{A}_0(\mathcal{P})$.
2. There exists $\gamma_x > 0$ such that $|\omega(a^*xa)| \leq \gamma_x \omega(a^*a), \forall \omega \in \mathcal{R}_c(\mathcal{A}, B), \forall a \in B$.
3. There exists $\gamma_x > 0$ such that $|\omega(b^*xa)| \leq \gamma_x \omega(a^*a)^{1/2}\omega(b^*b)^{1/2}, \forall \omega \in \mathcal{R}_c(\mathcal{A}, B), \forall a, b \in B$.

**Proof.** It is sufficient to consider the case $x = x^\ast$.

(i) $\Rightarrow$ (iii) If $x = x^\ast \in \mathcal{A}_0(\mathcal{P})$, there exists $\gamma > 0$ such that $-\gamma e \leq x \leq \gamma e$; or, equivalently,

$$-\gamma \varphi(a, a) \leq \varphi(xa, a) \leq \gamma \varphi(a, a), \forall \varphi \in \mathcal{P}, a \in B.$$ 

Since $\mathcal{A}$ is fully representable, $D(\varphi_\omega) = \mathcal{A}$ and, by Corollary 4.11 it is a continuous $\ast$-ips-form with core $B$. Thus, as seen in the proof of Proposition 3.2 $\pi_\varphi(x)$ is $(\tau, t_0)$-continuous. Hence, by Proposition 5.20 $\pi_\varphi(x)$ is bounded and $\|\pi_\varphi(x)\| \leq \|x\|_b$. Therefore,

$$|\omega(b^*xa)| = |\varphi_\omega(xa, b)| \leq \varphi_\omega(xa, xa)^{1/2}\varphi_\omega(b, b)^{1/2} = \|\pi_\varphi(x)\lambda_\varphi(a)||\varphi_\omega(b, b)^{1/2} \leq \|x\|_b \gamma_x \omega(a^*a)^{1/2}\omega(b^*b)^{1/2}.$$ 

(iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) Assume now that there exists $\gamma_x > 0$ such that

$$|\omega(a^*xa)| \leq \gamma_x \omega(a^*a), \forall \omega \in \mathcal{R}_c(\mathcal{A}, B), a \in B.$$ 

Define

$$\tilde{\gamma} := \sup\{\|\omega(a^*xa)\| : \omega \in \mathcal{R}_c(\mathcal{A}, \mathcal{A}_0), a \in \mathcal{A}_0, \omega(a^*a) = 1\}.$$ 

Let $\varphi \in \mathcal{P}$ and $a \in B$. By Proposition 4.14 the linear functional $\omega_\varphi^a$ defined by $\omega_\varphi^a(x) = \varphi(xa, a), x \in \mathcal{A}$, is continuous and representable. If $\varphi(a, a) = 0$, then, by 4.8, $\varphi(xa, a) = 0$. If $\varphi(a, a) > 0$, we get

$$\varphi((\tilde{\gamma} e \pm x)a, a) = \tilde{\gamma} \varphi(a, a) \pm \varphi(xa, a) = \varphi(a, a)(\tilde{\gamma} \pm \varphi(xu, u)) \geq 0,$$

where $u = a\varphi(a, a)^{-1/2}$. Hence, by the arbitrariness of $\varphi$ and $a$, we have $x \in \mathcal{A}_0(\mathcal{P})$. 

$\square$
We can now compare the notion of order bounded element with that of $\mathcal{P}_B(\mathfrak{A})$-bounded element given in Definition 5.1.

**Theorem 5.22.** Let $\mathfrak{A}[\tau]$ be a $\ast$-semisimple topological partial $\ast$-algebra with multiplication core $\mathfrak{B}$ and unit $e \in \mathfrak{B}$. For $x \in \mathfrak{A}$, the following statements are equivalent.

(i) $x$ is $\mathcal{P}_B(\mathfrak{A})$-bounded.

(ii) $x \in \mathfrak{A}_b(\mathcal{P})$.

(iii) $\pi(x)$ is bounded, for every $\pi \in \text{Rep}_c(\mathfrak{A})$, and

$$\sup\{\|\pi(x)\|, \pi \in \text{Rep}_c(\mathfrak{A})\} < \infty.$$ 

**Proof.** It is sufficient to consider the case $x = x^*$.  

(i) $\implies$ (ii): If $x = x^*$ is $\mathcal{P}_B(\mathfrak{A})$-bounded, we have, for some $\gamma > 0$,

$$-\gamma \varphi(a, a) \leq \varphi(xa, a) \leq \gamma \varphi(a, a), \quad \forall \varphi \in \mathcal{P}, a \in \mathfrak{B}.$$ 

This means that $-\gamma e \leq x \leq \gamma e$ in the sense of the order induced by $\mathfrak{A}_p^+$. Hence $x \in \mathfrak{A}_b(\mathcal{P})$.

(ii) $\implies$ (iii): Let $\pi \in \text{Rep}_c(\mathfrak{A})$ and $\xi \in D(\pi)$. Define $\varphi^\xi_\pi$ as in the proof of Proposition 5.12. Then $\varphi^\xi_\pi \in \mathcal{P}$. Hence by (ii), $|\varphi^\xi_\pi(xa, a)| \leq \gamma_x \varphi^\xi_\pi(a, a)$, for some $\gamma_x > 0$ which depends on $x$ only. In other words, $|\langle \pi(x)\xi|\xi\rangle| \leq \gamma_x \|\xi\|^2$. This in turn easily implies that $|\langle \pi(x)\xi|\eta\rangle| \leq \gamma_x \|\xi\|\|\eta\|$, for every $\xi, \eta \in D(\pi)$. Hence $\pi(x)$ is bounded and $\|\pi(x)\| \leq \gamma_x$.

(iii) $\implies$ (i): Put $\gamma_x := \sup\{\|\pi(x)\|, \pi \in \text{Rep}_c(\mathfrak{A})\}$. Then

$$|\langle \pi(x)\xi|\xi\rangle| \leq \|\pi(x)\|\|\xi\| \leq \gamma_x \|\xi\|^2, \quad \forall \xi \in D_\pi.$$ 

This in particular holds for the GNS representation $\pi_\varphi$ associated to any $\varphi \in \mathcal{P}_B(\mathfrak{A})$, since $\pi_\varphi$ is $(\tau, t_x)$-continuous. Hence, for every $a \in \mathfrak{B}$, we get

$$|\varphi(xa, a)| = |\langle \pi_\varphi(x)\lambda_\varphi(a)\lambda_\varphi(a)\rangle| \leq \gamma_x \|\lambda_\varphi(a)\|^2 = \gamma_x \varphi(a, a).$$

Using the polarization identity, one finally gets

$$|\varphi(xa, b)| \leq \gamma_x \varphi(a, a)^{1/2} \varphi(b, b)^{1/2}, \quad \forall \varphi \in \mathcal{P}, a, b \in \mathfrak{B}.$$ 

This proves that $x$ is $\mathcal{P}_B(\mathfrak{A})$-bounded. 

**Example 5.23.** In particular, Theorem 5.22 shows that, under the assumptions we have made, order boundedness is nothing but the $\mathcal{M}$-boundedness studied in [4]. So all results proved there apply to the present situation (in particular those concerning the structure of the topological partial $\ast$-algebra under consideration and its spectral properties). Clearly, the crucial assumption is the existence of sufficiently many continuous ips-forms, that is, the $\ast$-semisimplicity.
So far we have considered the order boundedness defined by the cone \( \mathcal{P}^+ \), but other choices are possible. For instance we may consider the order induced by \( \mathcal{B} \). It is clear that if \( x \in \mathcal{A}_b(\mathcal{P}^+) \) then \( x \in \mathcal{A}_b(\mathcal{P}) \). On the other hand, if \( x \in \mathcal{A}_b(\mathcal{P}) \) and the assumptions of Theorem 5.21 hold, there exists \( \gamma_x > 0 \) such that
\[
|\omega(a^*xa)| \leq \gamma_x \omega(a^*a), \quad \forall \omega \in \mathcal{R}_{ec}(\mathcal{A}, \mathcal{B}), \forall a \in \mathcal{B}.
\]
Hence, if condition (P) holds too, we can conclude, by adapting the argument used in the proof of Theorem 5.21, that \( x \in \mathcal{A}_b(\mathcal{P}^+) \). We leave a deeper analysis of the general question to future papers.

6. Concluding remarks

As we have discussed in the Introduction, the notion of bounded element for a topological partial *-algebra plays an important role for the whole discussion. We have at hand two different notions, one (\( \mathcal{M} \)-boundedness) based on a sufficient family of ips-forms, and another one (order boundedness) based on some \( \mathcal{B} \)-admissible wedge, where \( \mathcal{B} \) is a multiplication core. Both seem very reasonable definitions and, as we have seen, they can be compared in many occasions. In the framework of (topological) *-algebras, it is even possible that every element is order bounded (see examples in [10, Section 5]). The analogous situation for partial *-algebras is unsolved (in other words we do not know if there exist topological partial *-algebras where every element is order bounded) and we conjecture that a complete topological partial *-algebra \( \mathcal{A} \) whose elements are all bounded is necessarily an algebra. This is certainly true in the case where \( \mathcal{M} \)-boundedness is considered, where \( \mathcal{M} \) is a well-behaved family of ips-forms in the sense of Definition 4.26 of [4]. Indeed, as shown there (Proposition 4.27), under these assumptions the set of \( \mathcal{M} \)-bounded elements is a C*-algebra. The same, of course, holds true in the situation considered in Theorem 5.22 if the family \( \mathcal{P}_{\mathcal{B}}(\mathcal{A}) \) is well-behaved. However, the general question is open.

References

[1] J-P. Antoine, A. Inoue and C. Trapani, Partial *-algebras and their operator realizations, Kluwer, Dordrecht, 2002.
[2] J-P. Antoine, C. Trapani and F. Tschinke, Continuous *-homomorphisms of Banach Partial *-algebras, Mediterr. j. math. 4 (2007), 357–373.
[3] J-P. Antoine, C. Trapani and F. Tschinke, Spectral properties of partial *-algebras Mediterr. j. math. 7 (2010), 123–142.
[4] J-P. Antoine, C. Trapani and F. Tschinke, Bounded elements in certain topological partial *-algebras, Studia Math. 203 (2011), 223–251.
[5] F. Bagarello, A. Inoue and C. Trapani, Representable linear functionals on partial *-algebras, Mediterr. j. math. 9 (2012), 153-163.
[6] F. Bagarello and C. Trapani, L^p-spaces as quasi *-algebras, J. Math. Anal. Appl. 197 (1996), 810-824.
[7] M. Fragoulopoulou, C. Trapani and S. Triolo, Locally convex quasi *-algebras with sufficiently many *-representations, J. Math. Anal. Appl. 388 (2012), 1180-1193.

\(^3\)The terminology adopted in that paper comes from algebraic geometry, so that an admissible cone is called there a quadratic module.
[8] K. Schmüdgen, *Unbounded operator algebras and representation theory*, Birkhäuser Verlag, Basel, 1990.

[9] K. Schmüdgen, *A strict Positivstellensatz for the Weyl algebra*, Math. Ann. 331 (2005), 779–794.

[10] K. Schmüdgen, *Noncommutative real algebraic geometry - Some basic concepts and first ideas*, in *Emerging Applications in Algebraic Geometry*, ed. by M. Putinar and S. Sullivant, Springer, 2009.

[11] C. Trapani, *-Representations, seminorms and structure properties of normed *-algebras*, Studia Mathematica, Vol. 186, 47-75 (2008).

[12] C. Trapani and F. Tschinke, *Unbounded C*-seminorms and biweights on partial *-algebras* Mediterr. j. math. 2 (2005) 301–313.

[13] I. Vidav, *On some *-regular rings*, Acad. Serbe Sci. Publ. Inst. Math. 13 (1959) 73–80.

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