Factoring the Characteristic Polynomial

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Motivating Example

Quotient Posets

The Standard Equivalence Relation

The Main Theorem

Partitions Induced by Chains
All posets $P$ will be finite and have a unique minimal element $\hat{0}$. All $P$ will also be ranked meaning that for every $x \in P$, all saturated $\hat{0}–x$ chains will have the same length, $\rho(x)$. We also define the rank of $P$ to be

$$\rho(P) = \max_{x \in P} \rho(x).$$

If $\mu$ is the Möbius function of $P$ then the characteristic polynomial of $P$ is

$$\chi(P) = \chi(P; t) = \sum_{x \in P} \mu(x) t^{\rho(P) - \rho(x)}.$$ 

Many ranked posets have characteristic polynomials whose roots are nonnegative integers. Why? Reasons have been given by Saito and Terao, Stanley, Zaslavsky, Blass and S, and others.

Proposition

Let $P, Q$ be ranked posets.

1. $P \cong Q \implies \chi(P; t) = \chi(Q; t)$.
2. $P \times Q$ is ranked and $\chi(P \times Q; t) = \chi(P; t)\chi(Q; t)$. 
Let $\Pi_n$ be the lattice of set partitions of $[n] = \{1, \ldots, n\}$ ordered by refinement.

**Theorem**

$$\chi(\Pi_n, t) = (t - 1)(t - 2) \cdots (t - n + 1).$$

**Ex.** Consider $\Pi_3$.

$$\chi(\Pi_3, t) = t^2 - t - t - t + 2 = t^2 - 3t + 2 = (t - 1)(t - 2).$$
The *claw*, $CL_n$, consists of a $\hat{0}$ together with $n$ atoms.

$$\begin{array}{cccccc}
-1 & a_1 & a_2 & -1 & \cdots & a_n & -1 \\
\end{array}$$

Thus

$$\chi(\overset{\circ}{\text{CL}_n}) = t - n.$$ 

So the characteristic polynomial of $CL_n$ can give us any positive integer root as $n$ varies.
Let us consider the product $CL_1 \times CL_2$.

We have

$$\chi(CL_1 \times CL_2) = \chi(CL_1)\chi(CL_2) = (t - 1)(t - 2) = \chi(\Pi_3).$$
Clearly $\Pi_3$ and $CL_1 \times CL_2$ are not isomorphic. What if we identify the top two elements of $CL_1 \times CL_2$?

\[ \begin{array}{c}
\begin{array}{cccc}
12/3 & 13/2 & 1/23 \\
\end{array}
\end{array} \]

Note that the Möbius values of $(a, b)$ and $(a, c)$ added to give the Möbius value of $(a, b) \sim (a, c)$. So $\chi(CL_1 \times CL_2)$ did not change after the identification since characteristic polynomials only record the sums of the Möbius values at each rank.
General Method.
Suppose $P$ is a ranked poset and we wish to prove

$$
\chi(P) = (t - r_1) \ldots (t - r_n)
$$

where $r_1, \ldots, r_n$ are positive integers.

1. Construct the poset

$$
Q = CL_{r_1} \times \cdots \times CL_{r_n}.
$$

2. Identify elements of $Q$ to form a poset $Q/\sim$ in such a way that
   
   (a) $\chi(Q/\sim) = \chi(Q) = (t - r_1) \ldots (t - r_n),$
   (b) $(Q/\sim) \cong P.$

3. It follows that

$$
\chi(P) = \chi(Q/\sim) = (t - r_1) \ldots (t - r_n).
$$
Let $P$ be a poset and let $\sim$ be an equivalence relation on $P$. We define the quotient, $P/\sim$, to be the set of equivalence classes with the binary relation $\leq$ defined by

$$X \leq Y \text{ in } P/\sim \iff x \leq y \text{ in } P \text{ for some } x \in X \text{ and some } y \in Y.$$ 

Quotients of posets need not be posets.

Ex. Consider

$$C_2 = \begin{array}{c}
2 \\
1 \\
0
\end{array}$$

Put an equivalence relation on $C_2$ with classes

$$X = \{0, 2\}, \quad Y = \{1\}.$$ 

Then $X < Y$ since $0 < 1$ and $Y < X$ since $1 < 2$. 
Let $P$ be a poset and let $\sim$ be an equivalence relation on $P$. We say the quotient $P/\sim$ is a \textit{homogeneous quotient} if

1. $\hat{0}$ is in an equivalence class by itself, and
2. $X \leq Y$ in $P/\sim$ implies that for all $x \in X$ there is a $y \in Y$ with $x \leq y$.

\textbf{Lemma (Hallam-S)}

\textit{If $P/\sim$ is a homogeneous quotient then $P/\sim$ a poset.}
How do we determine a suitable equivalence relation? If $P$ is a lattice, then there is a canonical choice.

Let us revisit $\Pi_3$. Label the atoms of $CL_1 \times CL_2$ with atoms from $\Pi_3$ as follows:

\[
\begin{array}{ccc}
12/3 & 13/2 & 1/23 \\
\hat{0} & \times & \hat{0} \\
\end{array}
= \begin{array}{ccc}
(12/3, 13/2) & (12/3, 1/23) \\
(\hat{0}, 13/2) & (12/3, \hat{0}) & (\hat{0}, 1/23) \\
(\hat{0}, \hat{0})
\end{array}
\]
Now relabel each element of the product with the join of its two coordinates.

\[(\frac{12}{3}, \frac{13}{2}) \quad (\frac{12}{3}, \frac{1}{23})\]

\[
\begin{align*}
(\hat{0}, \frac{13}{2}) & \quad (\frac{12}{3}, \hat{0}) & \quad (\hat{0}, \frac{1}{23}) \\
(\hat{0}, \hat{0}) & &
\end{align*}
\]

Finally, identify elements with the same label to obtain the same quotient we did before. Not only is the quotient isomorphic to \(\Pi_3\), it even has the same labeling.
An ordered partition of a set \( \mathcal{A} \) is a sequence of subsets \((A_1, \ldots, A_n)\) with \( \bigcup_i A_i = \mathcal{A} \). We write \((A_1, \ldots, A_n) \vdash \mathcal{A}\).

Let \((A_1, \ldots, A_n) \vdash \mathcal{A}(L)\), where \(\mathcal{A}(L)\) is the atom set of a lattice \(L\). Let \(CL_{A_i}\) be the claw with atom set \(A_i\). The standard equivalence relation on \(\prod_i CL_{A_i}\) is

\[
\mathbf{t} \sim \mathbf{s} \text{ in } \prod_{i=1}^n CL_{A_i} \iff \bigvee \mathbf{t} = \bigvee \mathbf{s} \text{ in } L.
\]

The atomic transversals of \(x \in L\) are the elements of the equivalence class

\[
T^a_x = \left\{ \mathbf{t} \in \prod_{i=1}^n CL_{A_i} : \bigvee \mathbf{t} = x \right\}.
\]

**Ex.** \((A_1, A_2) \vdash \mathcal{A}(\Pi_3)\) with \(A_1 = \{12/3\}, A_2 = \{13/2, 1/23\}\). Note that \(CL_{A_1}\) and \(CL_{A_2}\) were the claws used for \(\Pi_3\).
We need a condition on the standard equivalence relation which will make sure that the quotient is homogeneous and ranked. The support of \( t = (t_1, \ldots, t_n) \in \prod_i CL_{A_i} \) is

\[
\text{supp } t = \{ i : t_i \neq \hat{0} \}.
\]

Note that \(|\text{supp } t| = \rho(t)\) where the rank is taken in \(\prod_i CL_{A_i}\).

**Lemma (Hallam-S)**

*Let \( L \) be a lattice, \((A_1, \ldots, A_n) \vdash A(L) \) and \( Q = \prod_i CL_{A_i} \). Suppose that for all \( x \in L \) and all \( t \in T_x^a \) we have

\[
|\text{supp } t| = \rho(x).
\]

Then the standard equivalence relation is homogeneous, \( Q/ \sim \) is ranked, and

\[
\rho(T_x^a) = \rho(x).
\]
We wish to make sure that when identifying the elements in an equivalence class, the Möbius function of the class is the sum of the Möbius functions of its elements so that $\chi$ does not change. Given $x \in L$, let

$$A_x = \{ a \in A(L) : a \leq x \}.$$  

Lemma (Hallam-S)

Let lattice $L$, $(A_1, \ldots, A_n) \vdash A(L)$ and $Q = \prod_i CL_{A_i}$ satisfy the conditions of the previous lemma. Suppose, for each $x \neq \hat{0}$ in $L$,

$$|A_x \cap A_i| = 1. \quad (1)$$

Then for any $T_x^a \in Q/\sim$ we have

$$\mu(T_x^a) = \sum_{t \in T_x^a} \mu(t).$$
Our main theorem is as follows.

**Theorem (Hallam-S)**

Let $L$ be a lattice, $(A_1, \ldots, A_n) \vdash A(L)$ and $Q = \prod_i CL_{A_i}$. Suppose that the following three conditions hold.

(1) For all $x \in L$ we have $\mathcal{T}_x^a \neq \emptyset$.

(2) If $t \in \mathcal{T}_x^a$ then $|\text{supp } t| = \rho(x)$.

(3) For each $x \neq \hat{0}$ in $L$, there is $i$ such that $|A_x \cap A_i| = 1$.

Then for the standard equivalence relation we can conclude the following.

(a) $(Q/ \sim) \cong L$.

(b) $\chi(L; t) = \prod_{i=1}^{n} (t - |A_i|)$.

Condition (1) is used to prove that the map $(Q/ \sim) \to L$ by $\mathcal{T}_x^a \mapsto x$ is surjective.
Corollary
\[ \chi(\Pi_n; t) = (t - 1)(t - 2) \ldots (t - n + 1). \]

**Proof.** If \( i < j \) let \( \{ i, j \} \) be the atom of \( \Pi_n \) having this set as its unique non-singleton block. Let \((A_1, \ldots, A_{n-1}) \vdash A(\Pi_n)\) where
\[
A_i = \{\{1, i + 1\}, \{2, i + 1\}, \ldots, \{i, i + 1\}\}.
\]

We will verify the three conditions for \( x = \hat{1} \).

(1) \( (\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}) \in T_{\hat{1}}^a \).

(2) With any \( t \in Q \), associate a graph \( G_t \) with \( V = [n] \) and
\[
ij \in E \iff \{i, j\} \in t.
\]
I claim \( G_t \) is a forest. If \( C : \ldots i, m, j, \ldots \) is a cycle with \( m = \max C \), then \( \{i, m\}, \{j, m\} \in t \). But \( \{i, m\}, \{j, m\} \in A_{m-1} \).

Also, the vertices of the components of \( G_t \) are the blocks of \( \vee t \).
\[
\therefore t \in T_{\hat{1}}^a \implies G_t \text{ a tree} \implies |\text{supp } t| = n - 1 = \rho(\hat{1}).
\]

(3) \( A_1 = \{\{1, 2\}\} \) so \( |A_{\hat{1}} \cap A_1| = 1 \).
\[
\therefore \chi(\Pi_n; t) = (t - |A_1|) \ldots (t - |A_{n-1}|) = (t - 1) \ldots (t - n + 1). \quad \square
\]
How do we find an appropriate atom partition? We say 
\((A_1, \ldots, A_n) \vdash A(L)\) is \textit{induced by a chain} if there is a chain 
\(C : \hat{0} = x_0 < x_1 < x_2 < \cdots < x_n = \hat{1}\) such that

\[ A_i = \{ a \in A(L) : a \leq x_i \text{ and } a \not\leq x_{i-1} \}. \]

\textbf{Ex.} In \(\Pi_n\), our partition is induced by \(\hat{0} < [2] < [3] < \cdots < \hat{1}\) 
where \([i]\) is the partition having this set as its only non-trivial block.

When will the partition induced by such a chain give the roots of 
a factorization? For \(x \in L\) with \(x \neq \hat{0}\), let \(i\) be the index with 
\(x \leq x_i\) and \(x \not\leq x_{i-1}\). Say that \(C\) satisfies the \textit{meet condition} if, 
for every \(x \in L\) of rank at least 2,

\[ x \land x_{i-1} \neq \hat{0}. \]
Our second main theorem is as follows.

**Theorem (Hallam-S)**

Let $L$ be a lattice and $(A_1, \ldots, A_n)$ induced by a chain $C$. Suppose that for all $x \in L$ and $t \in T_x^a$ we have

$$|\text{supp } t| = \rho(x).$$

Under these conditions, the following are equivalent.

1. For each $x \neq \hat{0}$ in $L$, there is $i$ such that $|A_x \cap A_i| = 1$.
2. Chain $C$ satisfies the meet condition.
3. The characteristic polynomial of $L$ factors as

$$\chi(L, t) = t^{\rho(L)} - n \prod_{i=1}^{n} (t - |A_i|).$$
Any lattice \( L \) satisfies: for all \( x, y, z \in L \) with \( y \leq z \)
\[
y \lor (x \land z) \leq (y \lor x) \land z \quad \text{(modular inequality).} \quad (2)
\]
Call \( x \in L \) \textit{left-modular} if, together with any \( y \leq z \), we have equality in (2). A lattice is \textit{supersolvable} if it has a saturated \( \hat{0} \-- \hat{1} \) chain of left-modular elements.

**Lemma (Hallam-S)**

Let \( L \) be a lattice and \( C \) a \( \hat{0} \-- \hat{1} \) chain in \( L \) inducing \((A_1, \ldots, A_n)\).

1. \textit{If} \( C \) is saturated and consists of left-modular elements, then \( C \) satisfies the meet condition.

2. \textit{If} \( L \) is semimodular then for any \( x \in L \) and \( t \in T^a_x \) we have
\[
|\text{supp} \ t| = \rho(x).
\]

**Corollary (Stanley, 1972)**

Let \( L \) be a semimodular, supersolvable lattice and \((A_1, \ldots, A_n)\) be induced by a saturated chain of left-modular elements. Then
\[
\chi(L; t) = \prod_{i=1}^{n} (t - |A_i|).
\]