The Wigner function in the relativistic quantum mechanics

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Abstract

A detailed study is presented of the relativistic Wigner function for a quantum spinless particle evolving in time according to the Salpeter equation.

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1. Introduction

The Wigner function also referred to as the Wigner quasi-probability function is one of the most important concepts of nonrelativistic quantum mechanics. Its applications range from nonequilibrium quantum mechanics, quantum optics, quantum chaos and quantum computing to classical optics and signal processing. As far as we are aware, in spite of the fact that the paper by Wigner was dated 1932 [1], the relativistic generalization of the Wigner function in the simplest case of the spinless particle was introduced by Zavialov and Malokostov only in 1999 [2]. The dynamics of that relativistic Wigner function was studied in recent papers [3, 4] by Larkin and Filonov. Clearly, the difficulties in extending the concept of the Wigner function to the relativistic domain sometimes considered as unattainable [5] are closely related with problems in finding the relativistic counterpart of the Schrödinger equation, that is constructing the relativistic quantum mechanics. In this work we discuss the advantages and limitations of the Wigner function introduced by Zavialov and Malokostov and analyze an alternative relativistic generalization of the Wigner function based on the standard nonrelativistic formula that was applied earlier in the case of the Dirac particle. Both of these approaches utilize the relativistic quantum dynamics described by the
spinless Salpeter equation. The theory is illustrated by concrete examples of relativistic Wigner function for a spinless free particle.

2. The Zavialov-Malokostov Wigner function

2.1. Definition of the Wigner function

We now summarize the basic facts about the Wigner function introduced in ref. 2. The point of departure in [2] was the following form of the non-relativistic Wigner function

\[ W(x, p, t) = \frac{1}{(2\pi)^3 \hbar^3} \int d^3p_1 d^3p_2 \tilde{\phi}^*(p_1, t) \tilde{\phi}(p_2, t) \delta(p - \frac{1}{2}(p_1 + p_2)) e^{i \frac{(p_2 - p_1) \cdot x}{\hbar}}, \]

(2.1)

where \( \tilde{\phi}(p, t) \) is the Fourier transform of the wave function \( \phi(x, t) \), that is

\[ \tilde{\phi}(p, t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3x e^{-i \frac{p \cdot x}{\hbar}} \phi(x, t). \]

(2.2)

Furthermore, Zavialov and Malokostov restrict to the case of the free relativistic evolution described by the Salpeter equation (see [6] and references therein)

\[ i\hbar \frac{\partial \tilde{\phi}(p, t)}{\partial t} = \sqrt{p^2 c^2 + m^2 c^4} \tilde{\phi}(p, t), \]

(2.3)

and demand that the relativistic Wigner function has the basic properties of the nonrelativistic one referring to integration over the spatial and momentum variables such that

\[ \int d^3p W(x, p, t) = |\phi(x, t)|^2 = \rho(x, t), \]

(2.4)

\[ \int d^3x W(x, p, t) = \frac{1}{\hbar^3} |\tilde{\phi}(p, t)|^2 = \rho_p(p, t), \]

(2.5)

and satisfy the evolution law

\[ W(x, p, t + \tau) = W(x - \frac{cp}{p_0} \tau, p, t), \]

(2.6)

where \( p_0 = E/c = \sqrt{p^2 + m^2c^2} \). It is easy to verify that the evolution law (2.6) is the global form of the local relation

\[ \frac{\partial W(x, p, t)}{\partial t} + \frac{cp}{p_0} \nabla W(x, p, t) = 0 \]

(2.7)
generalizing the nonrelativistic equation that is valid in the case of the free evolution
\[
\frac{\partial W(x,p,t)}{\partial t} + \frac{p}{m} \cdot \nabla W(x,p,t) = 0.
\]
(2.8)

With these assumptions the following relativistic generalization of (2.1) was obtained in [2]:
\[
W(x,p,t) = \frac{1}{(2\pi)^3 \hbar^6} \int d^3 p_1 d^3 p_2 \tilde{\phi}^*(p_1,t)\tilde{\phi}(p_2,t) \delta(p - (p_1 \oplus p_2)) e^{i \frac{(p_2 - p_1) \cdot x}{\hbar}},
\]
(2.9)
where \(p_1 \oplus p_2\) is a counterpart of the sum on the mass hyperboloid. More precisely, \(p_1 \oplus p_2\) is the spacial part of the fourvector \(p_1 \oplus p_2\) on the mass hyperboloid \(p_i^2 = m_i^2 c^2\) of the form
\[
p_1 \oplus p_2 = mc \frac{p_1 + p_2}{\sqrt{(p_1 + p_2)^2}},
\]
(2.10)
so \((p_1 \oplus p_2)^2 = m^2 c^2\), and is given by
\[
p_1 \oplus p_2 = mc \frac{p_1 + p_2}{\sqrt{2(m^2 c^2 + p_0(p_1)p_0(p_2) - p_1 \cdot p_2)}},
\]
(2.11)
where \(p_0(p_i) = \sqrt{p_i^2 + m_i^2 c^2}\), \(i = 1, 2\). The formula (2.9) can be immediately generalized to involve the particle in the external potential \(V(x)\) [3] by demanding that \(\tilde{\phi}(x,t)\) in (2.9) fulfils the Salpeter equation in the momentum representation
\[
\frac{i\hbar}{\partial t} \tilde{\phi}(p,t) = \left[\sqrt{m^2 c^4 + p^2 c^2} + V(i\hbar \nabla p)\right]\tilde{\phi}(p,t).
\]
(2.12)

2.2. The Wigner function and probability current

An interesting property of the relativistic Wigner function (2.9) that was not recognized neither in [2] nor [3, 4] is the following easily proven relation
\[
j(x,t) = \int d^3 p \frac{cp}{p_0} W(x,p,t),
\]
(2.13)
where \(j(x,t)\) is the relativistic probability current introduced by us in ref. 6, describing the conservation of the probability in the Salpeter equation in the coordinate representation
\[
\frac{i\hbar}{\partial t} \phi(x,t) = [\sqrt{m^2 c^4 - \hbar^2 c^2 \Delta} + V(x)]\phi(x,t),
\]
(2.14)
where $\Delta \equiv \nabla^2$, via the continuity equation
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0,
\] (2.15)
where $\rho(x, t) = |\phi(x, t)|^2$ is the probability density, such that
\[
j(x, t) = \frac{c}{(2\pi)^3 h^6} \int d^3p d^3k \frac{p + k}{p_0(p) + p_0(k)} e^{i \frac{(k-p) \cdot x}{\hbar}} \tilde{\phi}^*(p, t) \tilde{\phi}(k, t).
\] (2.16)
Of course, (2.13) is the relativistic generalization of the relation
\[
j(x, t) = \int d^3p \frac{p}{m} W(x, p, t),
\] (2.17)
leading via the formula for quantum expectation values of an observable $\hat{A}$
\[
\langle \phi | \hat{A} \tilde{\phi} \rangle = \int d^3x d^3p A(x, p) W(x, p),
\] (2.18)
where $A(x, p)$ is the Weyl transform of the operator $\hat{A}$, to the well-known expression describing the connection of the integral of the probability current and average velocity in the given state
\[
\int j(x, t) d^3x = \langle \phi | \hat{v} \phi \rangle,
\] (2.19)
where $\hat{v} = p/m$ is the velocity operator. It must be borne in mind that (2.13) cannot be regarded as the definition of the relativistic probability current. The proper definition of the probability current that holds regardless of the accepted definition of the Wigner function and ensures the validity of the continuity equation (2.15) is (2.16). Nonetheless, the formula (2.13) is really remarkable.

2.3. Massless limit of the Wigner function

As we have seen the relativistic Wigner function (2.9) has some nice properties as (2.4), (2.5), (2.7) and (2.13). Nevertheless, the authors of [2] and [3, 4] seem to be unaware of its problematic behavior in the limit $m = 0$. Indeed consider for simplicity the case of a free relativistic particle on a line. The relativistic Wigner function takes then the form
\[
W(x, p, t) = \frac{1}{2\pi \hbar^2} \int dp_1 dp_2 \tilde{\phi}^*(p_1, t) \tilde{\phi}(p_2, t) \delta(p - (p_1 \oplus p_2)) e^{i \frac{(p_2 - p_1) \cdot x}{\hbar}},
\] (2.20)
where $\tilde{\phi}(p, t)$ satisfies the Salpeter equation

$$i\hbar \frac{\partial \tilde{\phi}(p, t)}{\partial t} = \sqrt{p^2c^2 + m^2c^4} \tilde{\phi}(p, t),$$

(2.21)

and

$$p_1 \oplus p_2 = mc \frac{p_1 + p_2}{\sqrt{2(m^2c^2 + p_0(p_1)p_0(p_2) - p_1p_2)}},$$

(2.22)

where $p_0(p_i) = \sqrt{p_i^2 + m^2c^2}$, $i = 1, 2$. Now we have the parametric form of the Wigner function introduced in ref. [2] that can be easily obtained from (2.20) by switching to coordinates $p_{1,2} = mc \sinh \gamma_{1,2}$ on the mass-shell hyperboloid:

$$W(x, p, t) = \frac{mc}{\pi \hbar^2 \cosh \kappa} \int_{-\infty}^{\infty} d\beta \cosh(\kappa + \beta) \cosh(\kappa - \beta) \tilde{\phi}^*(mc \sinh(\kappa + \beta), t) \times \tilde{\phi}(mc \sinh(\kappa - \beta), t) \exp \left\{ \frac{imc}{\hbar} [\sinh(\kappa - \beta) - \sinh(\kappa + \beta)] x \right\}, \quad (2.23)$$

where $p = mc \sinh \kappa$. From (2.23) we can derive the following formula for the Wigner function

$$W(x, p, t) = \frac{1}{\pi \hbar^2 \sqrt{p^2 + m^2c^2}}$$

$$\times \int_{-\infty}^{\infty} d\beta (p^2 + m^2c^2 \cosh^2 \beta) \tilde{\phi}^*(p \cosh \beta + \sqrt{p^2 + m^2c^2} \sinh \beta, t) \times \tilde{\phi}(p \cosh \beta - \sqrt{p^2 + m^2c^2} \sinh \beta, t) \exp \left( -\frac{2ix}{\hbar} \sqrt{p^2 + m^2c^2} \sinh \beta \right). \quad (2.24)$$

An immediate consequence of (2.24) is the massless limit of the Wigner function such that

$$W_0(x, p, t) = \lim_{m \to 0} W(x, p, t)$$

$$= \frac{|p|}{\pi \hbar^2} \int_{-\infty}^{\infty} d\beta \tilde{\phi}^*(p \cosh \beta + |p| \sinh \beta, t) \tilde{\phi}(p \cosh \beta - |p| \sinh \beta, t) e^{-\frac{2ix}{\hbar}|p| \sinh \beta},$$

(2.25)
where \( \tilde{\phi}(p, t) \) fulfills the Salpeter equation

\[
i\hbar \frac{\partial \tilde{\phi}(p, t)}{\partial t} = c|p|\tilde{\phi}(p, t).
\] (2.26)

We point out that for a spinless particle we have no problems connected with procedures of contractions of representations of little groups corresponding to massive and massless particles. Indeed, we then deal in both cases with trivial representations. The limit (2.25) gives the correct density in the momentum representation i.e.

\[
\int_{-\infty}^{\infty} W_0(x, p, t) dx = \frac{1}{\hbar}|\tilde{\phi}(p, t)|^2.
\] (2.27)

Nevertheless, it leads to erroneous formula for the density in the coordinate representation and the probability current. We now illustrate this observation by the example of the “Lorentzian” wave packet \([6]\).

Consider the following normalized solution \([6]\)

\[
\tilde{\phi}(p, t) = \sqrt{a} e^{-\left(a+it\right)|p|},
\] (2.28)

where \( a > 0 \), to the Salpeter equation in the momentum representation for a massless particle moving in a line (2.26), where we set \( \hbar = 1 \) and \( c = 1 \).

The normalized wave function corresponding to (2.28) satisfying the Salpeter equation

\[
i \frac{\partial \phi(x, t)}{\partial t} = \sqrt{-\frac{\partial^2}{\partial x^2}} \phi(x, t),
\] (2.29)

is given by

\[
\phi(x, t) = \sqrt{\frac{2a}{\pi}} \frac{a + it}{x^2 + (a + it)^2}.
\] (2.30)

From (2.30) and the one-dimensional counterpart of (2.16) for \( m = 0 \) such that

\[
j(x, t) = \frac{1}{2\pi} \int dp dk \frac{p + k}{|p| + |k|} e^{i(k-p)x} \tilde{\phi}^*(p, t) \tilde{\phi}(k, t),
\] (2.31)

we immediately get the following formulas for the probability density and current, respectively \([6]\)

\[
\rho(x, t) = |\phi(x, t)|^2 = \frac{2a}{\pi} \frac{a^2 + t^2}{(x^2 - t^2 + a^2)^2 + 4a^2t^2},
\] (2.32)

\[
j(x, t) = \frac{a}{4\pi t^2} \ln \left( \frac{(x + t)^2 + a^2}{(x - t)^2 + a^2} \right) - \frac{ax}{\pi t} \frac{x^2 - 3t^2 + a^2}{(x^2 - t^2 + a^2)^2 + 4a^2t^2}.
\] (2.33)
We now return to (2.25). Inserting (2.28) into (2.25) where we set \( \hbar = 1 \), and \( c = 1 \), we find that the probability density and probability current corresponding to the massless limit are expressed by

\[
\rho_0(x,t) = \int_{-\infty}^{0} W_0(x,p,t) dp = \frac{a}{2\pi} \left[ \frac{1}{(x-t)^2 + a^2} + \frac{1}{(x+t)^2 + a^2} \right],
\]

\[ (2.34) \]

\[
j_0(x,t) = \int_{-\infty}^{0} \frac{p}{|p|} W_0(x,p,t) dp = \frac{a}{2\pi} \left[ \frac{1}{(x-t)^2 + a^2} - \frac{1}{(x+t)^2 + a^2} \right].
\]

\[ (2.35) \]

Thus it turns out that \( \rho_0 \) and \( j_0 \) obtained from the massless limit of the Wigner function are different from the correct probability density \( \rho \) and probability current \( j \) given by (2.32) and (2.33), respectively. We point out that \( \rho_0 \) and \( j_0 \) can be written as

\[
\rho_0(x,t) = \frac{1}{2}(|\phi_+(x,t)|^2 + |\phi_-(x,t)|^2),
\]

\[ (2.36) \]

\[
j_0(x,t) = \frac{1}{2}[j_+(x,t) + j_-(x,t)],
\]

\[ (2.37) \]

where

\[
\rho_\pm(x,t) = |\phi_\pm(x,t)|^2 = \pm j_\pm(x,t) = \frac{a}{\pi} \frac{1}{(x \mp t)^2 + a^2},
\]

\[ (2.38) \]

and \( \rho_\pm \) is the probability density and \( j_\pm \) is the probability current related to the wave packet \( \phi_\pm \) referring to the particle moving to the right and left, respectively such that [6]

\[
\phi_\pm(x,t) = \sqrt{\frac{a}{\pi}} \frac{\pm i}{x \mp t \pm ia},
\]

\[ (2.39) \]

\[
\rho_\pm(x,t) = |\phi_\pm(x,t)|^2 = \pm j_\pm(x,t) = \frac{a}{\pi} \frac{1}{(x \mp t)^2 + a^2},
\]

\[ (2.40) \]

where

\[
\tilde{\phi}_\pm(p,t) = \sqrt{2a} \theta(\pm p) e^{-(a+it)|p|}
\]

\[ (2.41) \]

and \( \theta(p) \) is the Heaviside step function. Since

\[
\phi(x,t) = \frac{1}{\sqrt{2}}[\phi_+(x,t) + \phi_-(x,t)],
\]

\[ (2.42) \]
where $\phi(x, t)$ is given by (2.30), therefore $\rho_0(x, t)$ differs from the correct probability density in lack of the interference terms. We remark that $\rho_0(x, t)$ and $j_0(x, t)$ satisfy the continuity equation
\[
\frac{\partial \rho_0}{\partial t} + \frac{\partial j_0}{\partial x} = 0. \tag{2.43}
\]

In spite of their suggestive form, the problems with the massless limit of the relativistic Wigner function (2.9) are related to the conditions (2.7) and (2.13). To see this let us assume that (2.7) holds in the ultra-relativistic limit $m \to 0$. Furthermore, we confine to the particle on a line, so we have
\[
\frac{\partial W(x, p, t)}{\partial t} + \frac{cp}{|p|} \frac{\partial W(x, p, t)}{\partial x} = 0. \tag{2.44}
\]

Consider now the relation
\[
j(x, t) = \int_{-\infty}^{\infty} dp \frac{cp}{|p|} W(x, p, t), \tag{2.45}
\]
that is the massless limit of (2.13). Using (2.44) we get
\[
\frac{\partial j(x, t)}{\partial t} = -c^2 \frac{\partial}{\partial x} \int_{-\infty}^{\infty} dp W(x, p, t) = -c^2 \frac{\partial \rho(x, t)}{\partial x}, \tag{2.46}
\]
where $\rho(x, t) = |\phi(x, t)|^2$ is the probability density. Combining this with the continuity equation
\[
\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial j(x, t)}{\partial x} = 0, \tag{2.47}
\]
we arrive at the wave equation
\[
\frac{1}{c^2} \frac{\partial^2 \rho(x, t)}{\partial t^2} - \frac{\partial^2 \rho(x, t)}{\partial x^2} = 0. \tag{2.48}
\]

On the contrary, in view of (2.32) and (2.33) both relations (2.46) and so (2.48) are easily shown to be erroneous in the case with the free evolution of the massless particle described by the wavefunction (2.30).

The formula for the Wigner function referring to the solution (2.28) can be derived with the help of (2.25) where we set $\hbar = 1$, and the identity (see [7] and [8])
\[
\int_0^\infty dx \frac{\exp(-\alpha \sqrt{x^2 + \beta^2})}{\sqrt{x^2 + \beta^2}} \cos \gamma x = K_0(\beta \sqrt{\alpha^2 + \gamma^2}), \quad \text{Re}\alpha > 0, \text{Re}\beta > 0. \tag{2.49}
\]
Figure 1: The density plot illustrating the time evolution of the Wigner function (2.50) corresponding to the solution of the Salpeter equation given by the wave packet for a free massless particle on a line (2.31). The parameter \( a = 1 \). The stable maxima refer to the particle moving to the left and to the right described by (2.39).

Namely, we have

\[
W_0(x, p, t) = \frac{2a|p|}{\pi} K_0[2\sqrt{a^2p^2 + (pt - x|p|)^2}],
\]

where \( K_\nu(z) \) is the modified Bessel function (Macdonald function). We remark that the Wigner function (2.50) is nonnegative. The time evolution of the Wigner function is shown in Fig. 1. As one might expect in view of the form of the wave packet (2.42) we have two stable maxima of the quasiprobability function (2.50) — one moving to the left and one moving to the right with the same constant absolute value of the momenta.

We now discuss the solutions \( \phi_{\pm}(x, t) \) to the Salpeter equation (2.29) given by (2.39) referring to the particle moving to the right and left. Using (2.41) and proceeding as with (2.28) we get the following Wigner function (2.25) with \( \hbar = 1 \), corresponding to \( \phi_{\pm}(x, t) \), respectively

\[
W_{0\pm}(x, p, t) = \pm\theta(\pm p)\frac{4ap}{\pi} K_0[\pm2p\sqrt{(x \mp t)^2 + a^2}].
\]

As with (2.50) both Wigner functions (2.51) are nonnegative. Of course, the plot of the function \( W_{0\pm}(x, p, t) \) refers to the upper (lower) part of Fig. 1. As
mentioned earlier the functions $\phi_{\pm}(x, t)$ satisfy the condition (2.27), where we set $\hbar = 1$. Moreover, using the identity [8]
\[
\int_{0}^{\infty} x^{\mu} K_{\nu}(ax) dx = 2^{\mu-1} a^{-\nu-1} \Gamma\left(\frac{1 + \mu + \nu}{2}\right) \Gamma\left(\frac{1 + \mu - \nu}{2}\right),
\]
\[
\text{Re}(\mu + 1 \pm \nu) > 0, \quad \text{Re} a > 0,
\]
we find that, in opposition to (2.50), the Wigner functions (2.51) give the correct formulas for the density in the coordinate representation, that is we have
\[
\int_{-\infty}^{\infty} dp W_{0\pm}(x, p, t) = |\phi_{\pm}(x, t)|^2.
\]
We conclude that the Wigner functions corresponding to the wave packets $\phi_{\pm}(x, t)$ referring to the massless particle moving to the right and left respectively, satisfy all requirements imposed on the Wigner function valid in the massive case. It seems that such good behavior of the Wigner functions (2.51) and bad one of the Wigner function (2.50) are related to the fact that in the massless case one can define the sum of vectors on a cone $p \rightarrow |p|$ that remains on a cone, analogous to (2.10), only for vectors with the same direction (proportional ones). We finally remark that in opposition to the nonrelativistic case when Hudson theorem [9] holds, which states that the only wave packet with non-negative Wigner function is the exponential of a quadratic polynomial, the wave functions (2.39) corresponding to the non-negative Wigner functions (2.51) are rational. This is to the best of our knowledge, the first example in the literature of such wave packets for a relativistic spinless particle. For the relativistic spin one-half particle the wave functions violating the Hudson theorem were introduced recently in ref.[10].

2.4. The Wigner function for the massive case

We finally discuss the normalized solution [6]
\[
\tilde{\phi}(p, t) = \frac{1}{\sqrt{2mK_1(2ma)}} e^{-(a+it)\sqrt{p^2+m^2}},
\]
where $a > 0$, to the Salpeter equation in the momentum representation for a free massive particle moving in a line
\[
\frac{1}{\sqrt{m^2 + p^2}} \frac{\partial \tilde{\phi}(p, t)}{\partial t} = \sqrt{m^2 + p^2} \tilde{\phi}(p, t),
\]
where we set $\hbar = 1$ and $c = 1$. The solution (2.54) is the Fourier transform of the following normalized wave function [6]

$$\phi(x, t) = \sqrt{\frac{m}{\pi K_1(2ma)}} \frac{a + it}{\sqrt{x^2 + (a + it)^2}} K_1[m \sqrt{x^2 + (a + it)^2}]$$

(2.56)

satisfying the Salpeter equation of the form

$$i \frac{\partial \phi(x, t)}{\partial t} = \sqrt{m^2 - \frac{\partial^2}{\partial x^2}} \phi(x, t).$$

(2.57)

It can be easily demonstrated with the help of the asymptotic formula

$$K_1(z) = \frac{1}{z}, \quad z \to 0,$$

(2.58)

that (2.28) and (2.30) are the massless limits of (2.54) and (2.56), respectively. In this sense (2.54) and (2.56) are the massive generalizations of (2.28) and (2.30). Now, taking into account the definition in the parametric form (2.23), the identity (2.49) and elementary properties of the modified Bessel function we get after some calculation the following formula for the Wigner function corresponding to (2.54)

$$W(x, p, t) = \frac{m}{2\pi K_1(2ma)} \frac{1}{\sqrt{p^2 + m^2}} \times \left\{ \left( 1 + \frac{2p^2}{m^2} \right) K_0 \left[ 2\sqrt{a^2(p^2 + m^2) + (tp - x\sqrt{p^2 + m^2})^2} \right] + \frac{a^2(p^2 + m^2) - (tp - x\sqrt{p^2 + m^2})^2}{a^2(p^2 + m^2) + (tp - x\sqrt{p^2 + m^2})^2} \times K_2 \left[ 2\sqrt{a^2(p^2 + m^2) + (tp - x\sqrt{p^2 + m^2})^2} \right] \right\}.$$  

(2.59)

Using (2.58) we can verify that the Wigner function obtained for the massless particle (2.50) is indeed the limit $m \to 0$ of (2.59), i.e. we have

$$\lim_{m \to 0} W(x, p, t) = W_0(x, p, t) = \frac{2a|p|}{\pi} K_0[2\sqrt{a^2p^2 + (pt - x|p|)^2}].$$

(2.60)

The time development of the Wigner function (2.59) is depicted in Fig. 2. We remark that for a wide range of parameters it is qualitatively similar to
Figure 2: The time development of the Wigner function (2.59) referring to the case of a free massive particle on a line described by the solution to the Salpeter equation in the momentum representation (2.54). The mass $m = 1$ and $a = 1$. 

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the time evolution of the nonrelativistic Wigner function for the (normalized) state that can be regarded as a counterpart of (2.54)

$$\tilde{\varphi}_{nrel}(p, t) = \left(\frac{a}{\pi m}\right)^{1/4} e^{-(a+it)p^2/2m},$$  (2.61)

where $a > 0$, such that

$$W_{nrel}(x, p, t) = \frac{1}{\pi} \exp \left[-a\frac{p^2}{m} - \frac{m}{a} \left(x - \frac{p}{m} t\right)^2\right].$$  (2.62)

In particular, for large $t$ the Wigner function is concentrated around $p = 0$ and goes to the uniform distribution along $x$-axis corresponding to its maximum. Nevertheless, as easily seen from (2.59) by considering the case of small $a$ and $p$, in opposition to the nonrelativistic case (2.62), the Wigner function (2.59) can take negative values. Such behavior is depicted in Fig. 3. We also point out that, in contrast to the nonrelativistic Wigner function, the function (2.59) is not constrained to be bounded via the inequality

$$|W(x, p, t)| \leq \frac{2}{\hbar},$$  (2.63)

where $\hbar$ is the Planck constant, that is a reflection of the uncertainty principle. On the other hand, it is unclear what is the form of the upper bound in the general case of the Wigner function (2.24).

We finally write down the following equivalent form of the Wigner function (2.24) that can be regarded as a relativistic generalization of the standard formula for the Wigner function in the momentum space (see (3.14) in the next section)

$$W(x, p, t) = \frac{1}{4\pi \hbar^2} \int_{-\infty}^{\infty} dk \frac{m^2 c^2 k^2 + 4(p^2 + m^2 c^2)^2}{(p^2 + m^2 c^2)^2} \tilde{\varphi}^* \left(\frac{f(p, k) - k}{2}, t\right)$$
$$\times \tilde{\varphi} \left(\frac{f(p, k) + k}{2}, t\right) e^{i\frac{p}{\hbar} x},$$  (2.64)

where

$$f(p, k) = \frac{p\sqrt{k^2 + 4(p^2 + m^2 c^2)}}{\sqrt{p^2 + m^2 c^2}}.$$  (2.65)
Figure 3: The plot of the Wigner function $W(x, p, t)$ given by (2.59) with fixed $p = 0.01$ and $t = 5$ illustrating its nonpositivity. The parameter $a = 0.01$ and $m = 1$.

Hence, taking the limit $m \to 0$, we get the counterpart of (2.25)

$$W_0(x, p, t) = \frac{|p|}{\pi \hbar^2} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{k^2 + 4p^2}} \tilde{\tilde{\phi}}^* \left( \frac{p}{|p|} \sqrt{k^2 + 4p^2 - k^2} \right) \times \tilde{\tilde{\phi}} \left( \frac{p}{|p|} \sqrt{k^2 + 4p^2 + k^2}, t \right) e^{i k x / \hbar}. \tag{2.66}$$

the relation (2.64) can be obtained from (2.24) by formal substitution

$$\sqrt{p^2 + m^2 c^2} \sinh \beta = -k/2. \tag{2.67}$$

Nevertheless, it can be also obtained from (2.20) by effective integration of the delta function, without usage of coordinates on the mass-shell hyperboloid and the formula (2.24).

3. The relativistic Wigner function based on the standard definition

In our opinion the plausible relativistic generalization of the Wigner function for a free spinless particle is given by

$$W(x, p, t) = \frac{1}{(2\pi \hbar)^3} \int d^3q \phi^*(x - q/2, t)\phi(x + q/2, t)e^{-i p \cdot q / \hbar}, \tag{3.1}$$
where $\phi(x, t)$ satisfies the Salpeter equation
\[ i\hbar \frac{\partial \phi(x, t)}{\partial t} = \sqrt{m^2c^4 - \hbar^2c^2} \Delta \phi(x, t), \tag{3.2} \]
so we apply the nonrelativistic formula for the Wigner function but we use the relativistic dynamics. We point out that the similar form of the relativistic Wigner function has been already utilized in [10] and [11] for the Dirac spin one-half particle. Evidently, we can also express the Wigner function in terms of the momentum representation
\[ W(x, p, t) = \frac{1}{(2\pi)^3\hbar^6} \int d^3k \tilde{\phi}^*(p - k/2, t)\tilde{\phi}(p + k/2, t)e^{i\frac{kr}{\hbar}}, \tag{3.3} \]
where $\tilde{\phi}(p, t)$ fulfills (2.3). It is also clear the Wigner function given by (3.1) or equivalently (3.3) can be immediately generalized to the case of a particle in a potential field by postulating that evolution of states is described by (2.14) and (2.12), respectively.

By differentiating both sides of (3.3) with respect to time and using (2.3) as well as the identity
\[ \int d^3q W(q, p, t)e^{-i\frac{qk}{\hbar}} = \frac{1}{\hbar^6} \tilde{\phi}^*(p - k/2, t)\tilde{\phi}(p + k/2, t), \tag{3.4} \]
we find
\[ \frac{\partial W(x, p, t)}{\partial t} = -\frac{2c^2}{(2\pi\hbar)^3} \times p \cdot \nabla \int d^3kd^3q \frac{W(q, p, t)e^{i\frac{kr}{\hbar}}}{\sqrt{(p - k/2)^2c^2 + m^2c^4 + (p + k/2)^2c^2 + m^2c^4}}. \tag{3.5} \]
It thus appears that in opposition to (2.9) the evolution of the Wigner function (3.1) in the case of a free particle is nonlocal. One finds easily that in the limit $c \to \infty$ (3.5) reduces to (2.8) that is we get the correct nonrelativistic evolution. Finally, putting in (3.5) $m = 0$ we obtain the massless limit such that
\[ \frac{\partial W(x, p, t)}{\partial t} = -\frac{2c}{(2\pi\hbar)^3} p \cdot \nabla \int d^3kd^3q \frac{W(q, p, t)e^{i\frac{kr}{\hbar}}}{|p - k/2| + |p + k/2|} \quad (m = 0). \tag{3.6} \]
where $|a|$ designates the norm of the vector $a$. The formula (3.5) enables to easily demonstrate the correctness of the nonrelativistic limit, nevertheless we have the simpler relation describing the evolution of the Wigner function that can be derived with the use of the identity [12]

$$
\int_{-\infty}^{\infty} \sqrt{x^2 + a^2} \, e^{ipx} \, dx = -\frac{2a}{|p|} K_1(a|p|)
$$  \hspace{1cm} (3.7)

and the differentiation rule satisfied by the Bessel functions of the form

$$
K_1'(z) = \frac{1}{z} K_1(z) - K_2(z).
$$  \hspace{1cm} (3.8)

Namely, we have

$$
\frac{\partial W(x, p, t)}{\partial t} = \frac{2m^2c^3}{(2\pi\hbar)^2} \int d^3q \, K_2 \left( \frac{2mc}{\hbar} |x-q| \right) \sin \frac{2p \cdot (x-q)}{\hbar} W(q, p, t).
$$

(3.9)

We remark that (3.9) has the form analogous to the Salpeter equation for a free particle written in the form of the integro-differential equation [6]. On taking the limit $m \to 0$ of Eq. (3.9) and using the asymptotic formula

$$
K_2(z) = \frac{2}{z^2}, \quad z \to 0,
$$  \hspace{1cm} (3.10)

we arrive at the following massless limit of Eq. (3.9)

$$
\frac{\partial W(x, p, t)}{\partial t} = \frac{c}{(2\pi)^2} \int d^3q \, \sin \frac{2p \cdot (x-q)}{\hbar} W(q, p, t).
$$

(3.11)

The one-dimensional version of the formula (3.9) corresponding to the case of a relativistic particle moving in a line is

$$
\frac{\partial W(x, p, t)}{\partial t} = \frac{2mc^2}{\pi\hbar} \int_{-\infty}^{\infty} dq \, K_1 \left( \frac{2mc}{\hbar} |x-q| \right) \sin \frac{2p(x-q)}{\hbar} W(q, p, t). \hspace{1cm} (3.12)

An immediate consequence of (3.12) and the asymptotic formula (2.55) is the following limit $m = 0$ of (3.12)

$$
\frac{\partial W(x, p, t)}{\partial t} = \frac{c}{\pi} \int_{-\infty}^{\infty} dq \, \sin \frac{2p(x-q)}{\hbar} W(q, p, t) \quad (m = 0).
$$

(3.13)
In order to illustrate the approach introduced in this section based on the definition of the Wigner function (3.1) and compare it with the Zavialov-Malokostov formalism discussed in the previous section we now consider a free particle on a line and the states (2.28) and (2.54) referring to the massless and massive particle, respectively. The relativistic Wigner function for the motion in a line takes the form

$$W(x,p,t) = \frac{1}{2\pi \hbar^2} \int_{-\infty}^{\infty} dk \tilde{\phi}^*(p-k/2,t)\tilde{\phi}(p+k/2,t)e^{\frac{i k x}{\hbar}},$$

(3.14)

where $\tilde{\phi}(p,t)$ satisfies (2.21) and (2.26) for the massive and massless case, respectively. Consider first the massless particle and the state (2.28). On inserting (2.28) into (3.14) with $\hbar = 1$ and making use of the identity [8]

$$\int_0^\infty e^{-px} \cos(qx + \lambda) dx = \frac{1}{p^2 + q^2}(p \cos \lambda - q \sin \lambda), \quad p > 0, \quad (3.15)$$

we arrive at the following formula for the Wigner function

$$W(x,p,t) = \frac{a}{\pi} e^{-2a|p|} \left\{ |p| \frac{\sin 2(pt - x|p|)}{pt - x|p|} + \frac{a}{a^2 + x^2} \left[ \cos 2(pt - x|p|) + \frac{x}{a} \cos 2(pt - x|p|) \right] \right\}. \quad (3.16)$$

The time evolution of the Wigner function (3.16) is demonstrated in Fig. 4 and Fig. 5. The characteristic feature of the Wigner function is the existence of the stable global maximum centered around $x = 0$, $p = 0$, and appearance of local extrema whose number increases as the time develops.

We finally discuss the case of a free massive particle on a line and the state (2.54). Substituting (2.54) into (3.14) and setting $\hbar = 1$ we get

$$W(x,p,t) = \frac{1}{2m\pi K_1(2ma)} \int_0^\infty dk \exp\{-a[\sqrt{(p-k/2)^2 + m^2} + \sqrt{(p+k/2)^2 + m^2}]\} \times \cos\{t[\sqrt{(p-k/2)^2 + m^2} - \sqrt{(p+k/2)^2 + m^2}] + k x\}. \quad (3.17)$$

The authors do not know any analytic expression for the integral from (3.17). The only exception besides $m = 0$ is the case $p = 0$ when it reduces to the
Figure 4: The behavior of the Wigner function (3.16) corresponding to the case of the free massless particle described by the solution (2.31). The parameters have the same values as in Fig. 1. For the details of the figure bottom right see Fig. 5.
Figure 5: Figure 4 bottom right shown in 3D presentation. The multiple local extrema arising as time develops and the non-positivity of the Wigner function are easily seen. The central global maximum is stable.

The integral of the form \[8\] (see also (2.49))

\[
\int_0^\infty dx \exp(-\alpha \sqrt{x^2 + \beta^2}) \cos \gamma x = \frac{\alpha \beta}{\sqrt{\alpha^2 + \gamma^2}} K_1(\beta \sqrt{\alpha^2 + \gamma^2}),
\]

\[\text{Re} \alpha > 0, \text{Re} \beta > 0. \quad (3.18)\]

The Wigner function is then given by

\[
W(x, 0, t) = \frac{1}{\pi K_1(2ma)} \frac{a}{\sqrt{a^2 + x^2}} K_1(2m \sqrt{a^2 + x^2}). \quad (3.19)
\]

The time development of the Wigner function (3.17) obtained by numerical calculation of the integral is shown in Fig. 6 and Fig. 7. In opposition to the massless case the global maximum is not stable in the limit of large \(t\) and behaves similarly as in the case of (2.59) that is preserves maximal value but flatten. Analogously as in the massless case we have also multiple local extrema whose number increases with time. As with (2.59) the Wigner function can take the negative values as well (see Fig. 7).
Figure 6: The time evolution of the Wigner function (3.17) referring to free massive particle in the state (2.54) the same as in Fig. 2. The values of the parameters $m = 1$ and $a = 1$ are the same as well. The details of the figure bottom right are shown in Fig. 7.
Figure 7: The 3D version of Fig. 6 bottom right. Analogously as for \( m = 0 \) (see Fig. 3) the Wigner function has multiple local extrema. The regions where the Wigner function takes the negative values are also easily observed.

4. Summary

In this work we discuss two approaches to relativistic generalization of the Wigner function for a free spinless particle based on the Salpeter equation. The first one introduced by Zavialov and Malokostov postulates the local law of evolution for the Wigner function in the case of a free particle. Its flaw is the problematic behavior of the Wigner function in the limit \( m = 0 \). The problems with the limit \( m = 0 \) are related to the definition of the Wigner function (2.9) utilizing without clear physical motivation, the “sum” of two fourvectors on the mass hyperboloid that remains on the hyperboloid. Such “sum” cannot be in general defined on a cone in the case of massless particles. On the other hand, we have demonstrated that the difficulties with the discussed formalism are connected with the simultaneous validity of (2.7) and (2.13). Therefore, the question naturally arises as to whether the Wigner function satisfying assumptions (2.4), (2.5) and (2.7) is unique. The second approach is based on the standard definition applied in nonrelativistic quantum mechanics but with states evolving according to the relativistic Salpeter equation. In opposition to (2.7) assumed by Zavialov and Malokostov, the dynamics of such Wigner function is nonlocal in the case of a free particle. On the one hand this is quite plausible in view of the
nonlocality of the Salpeter equation related only to the kinetic energy term [6]. On the other hand, we know for example that in spite of the nonlocality of the current (2.16), it satisfies the local continuity equation (2.15). Bearing in mind all pros and cons we find that there is no conclusive evidence which candidate for the relativistic Wigner function is better. Nonetheless, it seems to us that the observations obtained herein, especially the analytic expressions for the Wigner functions, would be of importance for the further studies of the subject. Finally, we point out possible application of the method for construction of the Wigner functions for general Lie groups developed in [13, 14, 15]. Indeed, the role of addition of vectors on the mass hyperboloid in the Zavialov-Malokostov approach to the relativistic Wigner function and results obtained with the help of the method in the nonrelativistic case suggest some group-theoretic context of the problem. Nevertheless, in spite of the fact that Wigner functions obtained by means of the method have many desirable properties such as for example the covariance, their physical interpretation can be unclear (see for instance [16]). For this reason, we defer this issue to our future work.

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