On simultaneous rational approximations to a real number, its square, and its cube

by

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Au Professeur Wolfgang Schmidt, avec mes meilleurs vœux et toute mon estime

1. Introduction. In a remarkable paper [3], H. Davenport and W. M. Schmidt showed that, for any integer \( n \geq 2 \) and for any real number \( \xi \) which is not algebraic over \( \mathbb{Q} \) of degree at most \( n - 1 \), there exist infinitely many algebraic integers \( \alpha \) of degree at most \( n \) satisfying

\[
|\xi - \alpha| \leq cH(\alpha)^{-\tau(n)},
\]

where \( c = c(n, \xi) > 0 \) is an appropriate constant depending only on \( n \) and \( \xi \), and where \( \tau(2) = 2, \tau(3) = (3 + \sqrt{5})/2, \tau(4) = 3 \) and \( \tau(n) = \lceil (n + 1)/2 \rceil \) if \( n \geq 5 \). For \( n = 2, 3 \), this value of \( \tau(n) \) cannot be improved (see [3] for the case \( n = 2 \) and [7] for the case \( n = 3 \)). For \( n \geq 4 \), M. Laurent showed in [4] that \( \tau(n) \) can be taken to be \( \lceil (n + 1)/2 \rceil \). However, at present, no optimal value for \( \tau(n) \) is known for any single value of \( n \geq 4 \). Furthermore, we possess no non-trivial upper bound for \( \tau(n) \) for \( n \geq 4 \), besides the estimate \( \tau(n) \leq n \) coming from metrical considerations (by an application of the Borel–Cantelli lemma as in the proof of [1, Thm. 3.3]). Although we shall not go into this, let us simply mention that the situation is similar in the case of approximation by algebraic numbers of degree at most \( n \). In this case, it is only for \( n \leq 2 \) that the optimal exponents are known, the case \( n = 2 \) being due once again to Davenport and Schmidt [2].

Several years ago, I started working on finding an optimal value for \( \tau(4) \) (in the above notation) and, in spite of much effort, I was not successful. My hopes were that this would lead to a new class of extremal numbers,

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similar to that of [5] or [6, §6], and that such a construction could be generalized to larger values of $n$ to provide a non-trivial upper bound for the corresponding values of $\tau(n)$, and maybe settle the question as to whether $\limsup_{n \to \infty} \tau(n)/n$ is equal to 1 or strictly smaller than 1. These problems remain open.

The method initiated by Davenport and Schmidt in [3] for estimating $\tau(n)$ is based on geometry of numbers and requires an upper bound on the uniform exponent of simultaneous approximation to the first $n-1$ consecutive powers of a real number $\xi$ by rational numbers with the same denominator. By [3, §2, Lemma 1], our main result below implies that $\tau(4)$ can be taken to be $\lambda_3^{-1} + 1 \approx 3.3556$, where

$$\lambda_3 = \frac{1}{2} \left( 2 + \sqrt{5} - \sqrt{7 + 2\sqrt{5}} \right) \approx 0.4245.$$ 

**Theorem.** Let $\xi \in \mathbb{R}$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > 3$, and let $c$ and $\lambda$ be positive real numbers. Suppose that for any sufficiently large value of $X$, the inequalities $|x_0| \leq X$, $|x_0 \xi - x_1| \leq cX^{-\lambda}$, $|x_0 \xi^2 - x_2| \leq cX^{-\lambda}$, $|x_0 \xi^3 - x_3| \leq cX^{-\lambda}$ admit a non-zero solution $x = (x_0, x_1, x_2, x_3) \in \mathbb{Z}^4$. Then $\lambda \leq \lambda_3$. Moreover, if $\lambda = \lambda_3$, then $c$ is bounded below by a positive constant depending only on $\xi$.

The rest of the paper is devoted to the proof of this result, which, through its weaker hypothesis on $\xi$, complements [3, Theorem 4a]. The tools that we use for the proof are the same as those of [3] together with results on heights of subspaces of $\mathbb{R}^n$ defined over $\mathbb{Q}$ that were developed around the same period of time by W. M. Schmidt in [8]. Using other tools, similar to the bracket $[x, y, z]$ in [6, §2], I discovered recently that the exponent $\lambda_3$ in the above theorem is not optimal. Since the argument is quite involved and does not seem to lead to a significant improvement in $\lambda_3$, I decided not to include this here.

2. **First considerations.** Throughout this paper, we fix a real number $\xi$ with $[\mathbb{Q}(\xi) : \mathbb{Q}] > 3$ and positive constants $\lambda$, $c$ satisfying the hypotheses of the Theorem. In all statements below, the implied constants in the symbols $\gg$, $\ll$ and $\asymp$ (the conjunction of $\gg$ and $\ll$) depend only on $\xi$ and $\lambda$ (not on $c$). In particular, we may assume that $c \ll 1$. Our goal is to show that $\lambda \leq \lambda_3$ and that $c \gg 1$ in case of equality. By [3, Theorem 4a], we already have $\lambda \leq 1/2$.

For each integer $n \geq 1$ and each point $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$, we define points $x^-$ and $x^+$ of $\mathbb{R}^n$ by

$$x^- = (x_0, \ldots, x_{n-1}) \quad \text{and} \quad x^+ = (x_1, \ldots, x_n).$$
We also put
\[ \|\mathbf{x}\| = \max_{0 \leq i \leq n} |x_i| \quad \text{and} \quad L(\mathbf{x}) = \max_{1 \leq i \leq n} |x_0 \xi^i - x_i|. \]

Finally, we say that a point \( \mathbf{x} \in \mathbb{Z}^{n+1} \) is primitive if it is non-zero and if the gcd of its coordinates is 1. Then the hypothesis implies that, for any sufficiently large \( X \), there exists a primitive point \( \mathbf{x} \in \mathbb{Z}^4 \) with
\[ (1) \quad \|\mathbf{x}\| \leq X \quad \text{and} \quad L(\mathbf{x}) \leq c_1 X^{-\lambda}, \]
where \( c_1 = 2 \max\{1, |\xi|\}^{3\lambda} \). The following lemmas extend results of Davenport and Schmidt in [3, §4].

**Lemma 2.1.** Let \( C \in \mathbb{Z}^2 \) and \( \mathbf{x} \in \mathbb{Z}^{n+1} \) with \( n \in \{1, 2, 3\} \). Then the point \( \mathbf{y} = C^+ \mathbf{x}^- - C^- \mathbf{x}^+ \) satisfies
\[ (2) \quad \|\mathbf{y}\| \leq \|\mathbf{x}\| L(C) + c_2 \|C\| L(\mathbf{x}) \quad \text{and} \quad L(\mathbf{y}) \leq c_2 \|C\| L(\mathbf{x}) \]
for some constant \( c_2 = c_2(\xi) \). Moreover, if \( \mathbf{y} = 0 \) and if \( C \) and \( \mathbf{x} \) are non-zero and primitive, then
\[ \|\mathbf{x}\| = \|C\|^n \quad \text{and} \quad L(\mathbf{x}) \asymp \|C\|^{n-1} L(C). \]

**Proof.** Write \( C = (a, b) \). Then the estimates in (2) follow respectively from the formulas \( \mathbf{y} = (b - a \xi) \mathbf{x}^- + a(\xi \mathbf{x}^- - \mathbf{x}^+) \) and \( \mathbf{y} = bx^- - ax^+ \), upon choosing \( c_2 \) so that \( \|\xi \mathbf{x}^- - \mathbf{x}^+\| \leq c_2 L(\mathbf{x}) \) and \( L(\mathbf{x}) + L(\mathbf{x}^+) \leq c_2 L(\mathbf{x}) \). If \( \mathbf{y} = 0 \) and \( C \neq 0 \), then \( \mathbf{x} \) is a rational multiple of the geometric progression \( (a^n, a^{n-1}b, \ldots, b^n) \). If furthermore \( C \) and \( \mathbf{x} \) are primitive, this progression is a primitive point of \( \mathbb{Z}^{n+1} \) and so it coincides with \( \pm \mathbf{x} \). This gives \( \|\mathbf{x}\| = \|C\|^n \) and \( L(\mathbf{x}) \asymp \|\mathbf{x}^+ - \xi \mathbf{x}^-\| = \|C\|^{n-1} L(C) \).

**Lemma 2.2.** Suppose that \( \lambda > 1/3 \). Then for any non-zero point \( C \in \mathbb{Z}^2 \) we have \( L(C) \gg \|C\|^{-1/\lambda} \).

**Proof.** Since \( \xi \notin \mathbb{Q} \), we have \( L(C) \neq 0 \) for any non-zero point \( C \in \mathbb{Z}^2 \). So, it suffices to prove that \( L(C) \gg \|C\|^{-1/\lambda} \) for primitive points \( C \in \mathbb{Z}^2 \) of sufficiently large norm. Let \( C \) be a primitive point of \( \mathbb{Z}^2 \), and let \( \mathbf{x} \in \mathbb{Z}^4 \) be a primitive solution of (1) for the choice of \( X = (2cc_1c_2\|C\|)^{1/\lambda} \), where \( c_2 \) is the constant introduced in Lemma 2.1. Since \( \lambda > 1/3 \), we have \( X < \|C\|^3 \) if \( \|C\| \gg 1 \), and then the second part of Lemma 2.1 shows that \( \mathbf{y} = C^+ \mathbf{x}^- - C^- \mathbf{x}^+ \) is a non-zero point of \( \mathbb{Z}^3 \). Applying the first part of the same lemma, we deduce that
\[ 1 \leq \|\mathbf{y}\| \leq XL(C) + cc_2\|C\|X^{-\lambda} \leq XL(C) + 1/2, \]
and so \( L(C) \geq (2X)^{-1} \gg \|C\|^{-1/\lambda} \).

**Lemma 2.3.** Suppose that \( \lambda > 1/3 \). Then there exist at most finitely many points \( \mathbf{x} \in \mathbb{Z}^4 \) with \( L(\mathbf{x}) \leq cc_1\|\mathbf{x}\|^{-\lambda} \) such that \( \mathbf{x}^- \) and \( \mathbf{x}^+ \) are linearly dependent over \( \mathbb{Q} \).

Proof. Suppose on the contrary that the conclusion is false. Then there exist infinitely many primitive points \( x \) of \( \mathbb{Z}^4 \) with \( L(x) \leq cc_1 \|x\|^{-\lambda} \) for which \( x^- \) and \( x^+ \) are linearly dependent. For each of them, there exists a primitive point \( C \in \mathbb{Z}^2 \) such that \( C^+x^- - C^-x^+ = 0 \). By Lemma 2.1, we have \( \|x\| = \|C\|^3 \) and \( L(x) \propto \|C\|^2 L(C) \). Thus \( \|C\| \) tends to infinity with \( \|x\| \), and the condition \( L(x) \leq cc_1 \|x\|^{-\lambda} \) translates into \( L(C) \ll \|C\|^{-2-3\lambda} \). Since \(-2-3\lambda < -3 < -1/\lambda\), this contradicts Lemma 2.2. \( \blacksquare \)

Lemma 2.4. Let \( n \in \{1, 2, 3\} \) and let \( U \) be a proper subspace of \( \mathbb{R}^{n+1} \) defined over \( \mathbb{Q} \). Then the function \( L(x) \) is bounded from below by a positive constant on the set of all non-zero points \( x \) of \( U \cap \mathbb{Z}^{n+1} \).

Proof. As in the proof of [3, §3, Lemma 5], suppose on the contrary that there exists a sequence of non-zero integral points \( (x_i)_{i \geq 1} \) in \( U \) such that \( \lim_{i \to \infty} L(x_i) = 0 \). Then, for any sufficiently large index \( i \), the first coordinate \( x_{i,0} \) of \( x \) is non-zero and the product \( x_i^{-1}x_i \) converges to \((1, \xi, \ldots, \xi^n)\) as \( i \) tends to infinity. Thus, the point \((1, \xi, \ldots, \xi^n)\) belongs to \( U \). This is impossible since \( U \) is a proper subspace of \( \mathbb{R}^{n+1} \) defined over \( \mathbb{Q} \) while the coordinates of the point \((1, \xi, \ldots, \xi^n)\) are linearly independent over \( \mathbb{Q} \). \( \blacksquare \)

Finally, we note that there exists a sequence of non-zero points \( (x_i)_{i \geq 1} \) in \( \mathbb{Z}^4 \) with the following properties:

(a) the positive integers \( X_i := \|x_i\| \) form a strictly increasing sequence,
(b) the positive real numbers \( L_i := L(x_i) \) form a strictly decreasing sequence,
(c) if some non-zero point \( x \in \mathbb{Z}^4 \) satisfies \( L(x) < L_i \) for some \( i \geq 1 \), then \( \|x\| \geq X_{i+1} \).

We fix such a choice of sequence \( (x_i)_{i \geq 1} \) and refer to it as the sequence of minimal points for \( \xi \) although it is not unique and differs from the notion introduced by Davenport and Schmidt in [3, §4]. We note that, for each \( i \geq 1 \), \( x_i \) is a primitive point of \( \mathbb{Z}^4 \) and, since (1) admits a non-zero solution \( x \in \mathbb{Z}^4 \) for each \( X \) with \( X_i \leq X < X_{i+1} \) when \( i \) is sufficiently large, we deduce from condition (c) that

\[
L_i \leq cc_1 X_{i+1}^{-\lambda}
\]

for each large enough index \( i \). We will use this property repeatedly in what follows, either in this form or in the weaker form \( L_i \ll cX_{i+1}^{-\lambda} \ll X_{i+1}^{-\lambda} \).}

3. A family of planes in \( \mathbb{R}^4 \). For each integer \( n \geq 1 \) and each subspace \( S \) of \( \mathbb{R}^n \) defined over \( \mathbb{Q} \) of dimension \( p > 0 \), we define the height \( H(S) \) of \( S \) by \( H(S) = \|y_1 \wedge \cdots \wedge y_p\| \), where \( (y_1, \ldots, y_p) \) is a basis of the group \( S \cap \mathbb{Z}^n \) of integral points of \( S \) (upon identifying \( \wedge^p \mathbb{R}^n \) with \( \mathbb{R}_c^p \) through an ordering of the Grassmann coordinates, as in [9, Chap. 1, §5]). We also
define $H(0) = 1$. It then follows from [9, Chap. 1, Lemma 8A] that, for any pair of subspaces $S$ and $T$ of $\mathbb{R}^n$ defined over $\mathbb{Q}$, we have

$$H(S \cap T)H(S + T) \leq c(n)H(S)H(T)$$

with a constant $c(n) > 0$ depending only on $n$. We also recall the duality formula $H(S) = H(S^\perp)$ where $S^\perp$ stands for the orthogonal complement of $S$ in $\mathbb{R}^n$ (see [9, Chap. 1, §8]).

For each $i \geq 2$, we denote by $W_i$ the subspace of $\mathbb{R}^4$ of dimension 2 generated by $x_{i-1}$ and $x_i$. We also introduce a new parameter

$$\theta = \frac{1 - \lambda}{\lambda},$$

and note that $\theta \geq 1$ since $\lambda \leq 1/2$.

**Lemma 3.1.** For each $i \geq 2$, the points $x_{i-1}$ and $x_i$ form a basis of $W_i \cap \mathbb{Z}^4$, and we have $H(W_i) \asymp X_i L_{i-1} \ll X^{1-\lambda}.$

This follows by a simple adaptation of the proofs of [2, Lemma 2] and [6, Lemma 4.1], the difference being that here $X_i$ stands for the norm of $x_i$ instead of the absolute value of its first coordinate. We now look at the sums $W_i + W_{i+1}$.

**Lemma 3.2.** There exist infinitely many indices $i \geq 2$ such that $W_i \neq W_{i+1}$. For each of them, we have

$$H(W_i + W_{i+1}) \ll X_i^{-1}H(W_i)H(W_{i+1}) \ll H(W_i)^{-1/\theta}H(W_{i+1}).$$

**Proof.** If there were only finitely many $i \geq 2$ for which $W_i \neq W_{i+1}$, then all points $x_i$ with $i$ sufficiently large would lie in a fixed subspace $W$ of $\mathbb{R}^4$ defined over $\mathbb{Q}$ of dimension 2, contrary to Lemma 2.4. This proves the first assertion of the present lemma.

Applying (3) with $S = W_i$ and $T = W_{i+1}$, we find

$$H(W_i \cap W_{i+1})H(W_i + W_{i+1}) \ll H(W_i)H(W_{i+1}).$$

For each index $i \geq 2$ such that $W_i \neq W_{i+1}$, we have $W_i \cap W_{i+1} = \langle x_i \rangle_{\mathbb{R}}$ and so $H(W_i \cap W_{i+1}) = X_i$. This leads to the first estimate in (4). For the second one, we simply use the lower bound $X_i \gg H(W_i)^{1/(1-\lambda)}$ coming from Lemma 3.1. 

**Notation.** We denote by $I$ the set of indices $i \geq 2$ for which $W_i \neq W_{i+1}$, ordered by increasing magnitude.

Thus, for each $i \in I$, the sum $W_i + W_{i+1} = \langle x_{i-1}, x_i, x_{i+1} \rangle_{\mathbb{R}}$ is a three-dimensional subspace of $\mathbb{R}^4$ defined over $\mathbb{Q}$. By Lemma 2.4 such a subspace of $\mathbb{R}^4$ contains at most finitely many minimal points. This leads to the first assertion of the next lemma.
Lemma 3.3. There exist infinitely many pairs of consecutive elements $i, j$ of $I$ with $i < j$ and $W_i + W_{i+1} \neq W_j + W_{j+1}$. For any such pair of integers $i$ and $j$, we have

\begin{align}
(5) & \quad X_i X_j \ll H(W_i) H(W_j) H(W_{j+1}), \\
(6) & \quad H(W_i) H(W_j) \ll H(W_{j+1})^\theta \text{ and } X_i X_j \ll X_{j+1}^\theta.
\end{align}

Proof. For consecutive elements $i < j$ of $I$, we have $W_i \neq W_{i+1} = W_j \neq W_{j+1}$. If $W_i + W_{i+1}$ and $W_j + W_{j+1}$ are distinct subspaces of $\mathbb{R}^4$, their sum is the whole of $\mathbb{R}^4$ and their intersection is $W_{i+1} = W_j$. Since $H(\mathbb{R}^4) = 1$, we deduce from (3) that

$$H(W_{i+1}) \ll H(W_i + W_{i+1}) H(W_j + W_{j+1}).$$

Combining this estimate with the upper bounds

\begin{align}
H(W_i + W_{i+1}) & \ll X_i^{-1} H(W_i) H(W_{i+1}), \\
H(W_j + W_{j+1}) & \ll X_j^{-1} H(W_j) H(W_{j+1})
\end{align}

provided by Lemma 3.2, we obtain (5). Then combining (5) with the standard upper bounds $H(W_i) \ll X_i^{1-\lambda}$ and $H(W_j) \ll X_j^{1-\lambda}$ coming from Lemma 3.1, we find

$$X_i^\lambda X_j^\lambda \ll H(W_{j+1}),$$

so $H(W_i) H(W_j) \ll (X_i X_j)^{1-\lambda} \ll H(W_{j+1})^\theta \ll X_{j+1}^{\theta(1-\lambda)}$, which proves (6). \hfill \blacksquare

4. A family of points in $\mathbb{Z}^2$. For each pair of points $x$ and $y$ in $\mathbb{Z}^4$, we define

$$C(x, y) = (\det(x^-, x^+, y^-), \det(x^-, x^+, y^+)) \in \mathbb{Z}^2.$$

To alleviate the notation, we also write

$$C_{i,j} = C(x_i, x_j)$$

for each pair of integers $i, j \geq 1$. These points $C_{i,j}$ play a crucial role in the proof of the inequality $\lambda \leq 1/2$ by Davenport and Schmidt in [3, §4]. They also play an important role in the present work. We first prove general estimates.

Lemma 4.1. For any pair of integers $i, j \geq 1$, we have

$$\|C_{i,j}\| \ll X_j L_i^2 + X_i L_i L_j \quad \text{and} \quad L(C_{i,j}) \ll X_i L_i L_j.$$

Proof. The estimate for $\|C_{i,j}\|$ is standard (see for example the proof of [3, §4, Lemma 7]). For the other quantity, we find

$$L(C_{i,j}) = |\det(x^{-}_i, x^{+}_i, x^{-}_j - \xi x^{+}_j)|$$

$$= |\det(x^{-}_i, x^{+}_i - \xi x^{-}_i, x^{+}_j - \xi x^{-}_j)| \ll X_i L_i L_j. \hfill \blacksquare$$
The next lemma provides a sharper upper bound for \( L(C_{i,i+1}) \) when \( i \in I \).

**Lemma 4.2.** Let \( i < j \) be consecutive elements of the set \( I \). Then \( C_{i,j} = bC_{i,i+1} \) for some non-zero integer \( b \) with \( |b| \asymp X_j/X_{i+1} \), and we have

\[
L(C_{i,i+1}) \ll X_j X_j^{-\lambda} X_{j+1}^{-\lambda}.
\]

**Proof.** Since \( i \) and \( j \) are consecutive in \( I \), we have \( W_{i+1} = W_j \). Moreover, since \( x_i \) and \( x_{i+1} \) form a basis of the group of integral points of \( W_{i+1} \), there exist integers \( a \) and \( b \) with \( b \neq 0 \) such that \( x_j = ax_i + bx_{i+1} \). If \( X_j > 3|b|X_{i+1} \), we deduce that

\[
|a|X_i = ||x_j - bx_{i+1}|| \geq X_j - |b|X_{i+1} > 2|b|X_{i+1},
\]

and so \( |a| > 2|b| \). Then, we find \( L_j \geq |a|L_i - |b|L_{i+1} > |b|L_{i+1} \geq L_{i+1} \), which is impossible. This contradiction shows that \( |b| \geq X_j/(3X_{i+1}) \). Since the point \( C(x, y) \) is a linear function of \( y \) and since \( C(x, x) = 0 \) for any \( x \in \mathbb{R}^3 \), we also have

\[
C_{i,j} = C(x_i, ax_i + bx_{i+1}) = bC_{i,i+1}
\]

and so, by Lemma 4.1, we obtain (since \( \lambda \leq 1/2 \leq 1 \))

\[
L(C_{i,i+1}) = |b|^{-1}L(C_{i,j}) \leq |b|^{-\lambda}L(C_{i,j}) \ll \frac{X_{i+1}^{\lambda}}{X_j^{\lambda}} X_i L_i L_j \ll X_i X_j^{-\lambda} X_{j+1}^{-\lambda}.
\]

**Remark.** Although we will not use this here, it is interesting to note that the identity

\[
\det(w, x, y)z - \det(w, x, z)y + \det(w, y, z)x - \det(x, y, z)w = 0,
\]

which holds for any quadruple of points \( (w, x, y, z) \) in \( \mathbb{R}^3 \), specializes to

\[
C_{i,j}^+ x_j^+ - C_{i,j}^- x_j^- = C_{j,i}^- x_i^+ - C_{j,i}^+ x_i^-
\]

when we apply it to the quadruple \( (x_i^-, x_i^+, x_j^-, x_j^+) \) for a choice of integers \( i, j \geq 1 \).

**5. A family of planes in \( \mathbb{R}^3 \).** From now on, we assume that \( \lambda > 1/3 \). Then, by Lemma 2.3, there exists an index \( i_0 \) such that \( x_i^- \) and \( x_i^+ \) are linearly independent for each \( i \geq i_0 \). For those values of \( i \), we denote by \( V_i \) the two-dimensional subspace of \( \mathbb{R}^3 \) spanned by these points:

\[
V_i = \langle x_i^-, x_i^+ \rangle_{\mathbb{R}}.
\]

Since \( \max\{L(x_i^-), L(x_j^+)\} \ll L_j \) tends to 0 as \( j \to \infty \), it follows from Lemma 2.4 that each \( V_i \) contains at most finitely many points of the form \( x_j^- \) or \( x_j^+ \), and so there are infinitely many indices \( i \geq i_0 \) such that \( V_i \neq V_{i+1} \). We also note that, for \( i, j \geq i_0 \), we have

\[
V_i = V_j \iff C_{i,j} = 0 \iff C_{j,i} = 0.
\]
by definition of the points $C_{i,j}$ (see §4). In [3, §4], Davenport and Schmidt argue that, for each $i \geq i_0$ such that $V_i \neq V_{i+1}$, we have $1 \leq \|C_{i,i+1}\| = X_{i+1}L_i^2 \ll X_{i+1}^{1-2\lambda}$ (see Lemma 4.1). Since $i$ can be taken to be arbitrarily large, this gives $1 - 2\lambda \geq 0$ and so $\lambda \leq 1/2$.

**Lemma 5.1.** There exist infinitely many integers $i > i_0$ for which $V_{i-1} \neq V_i$. For each of them, we have

$$H(W_{i+1}) \ll X_{i+1}^{1-\lambda} \ll H(W_i)^\theta \ll X_i^{\theta(1-\lambda)}.$$  

In particular, this leads to the symmetric estimates $X_{i+1} \ll X_i^\theta$ and $H(W_{i+1}) \ll H(W_i)^\theta$.

**Proof.** The first assertion being already settled, fix an index $i > i_0$ such that $V_{i-1} \neq V_i$. Then the integral point $C_{i,i-1}$ is non-zero and so its norm is bounded below by 1. The absolute values of its coordinates are:

$$|\det(x_i^-,x_i^+,x_{i-1}^-)| = |\det(x_{i-1}^-,x_i^-,x_i^+ - \xi x_i^-)| \ll \|x_{i-1}^- \wedge x_i^-\|L_i,$$

$$|\det(x_i^-,x_i^+,x_{i-1}^+)| = |\det(x_{i-1}^+,x_i^+,x_i^- - \xi^{-1} x_i^+)| \ll \|x_{i-1}^+ \wedge x_i^+\|L_i.$$  

Since $\|x_{i-1}^- \wedge x_i^+\|$ and $\|x_{i-1}^+ \wedge x_i^+\|$ are bounded above by $\|x_{i-1}^- \wedge x_i^+\| = H(W_i)$, this means that $\|C_{i,i-1}\| \ll H(W_i)L_i$. Thus we obtain

$$1 \leq \|C_{i,i-1}\| \ll H(W_i)L_i \ll H(W_i)X_{i+1}^{1-\lambda},$$

and so $X_{i+1} \ll H(W_i)^{1/\lambda}$. The conclusion follows by combining this result with the estimates $H(W_i) \ll X_i^{1-\lambda}$ and $H(W_{i+1}) \ll X_{i+1}^{1-\lambda}$ coming from Lemma 3.1. □

**Proposition 5.2.** Suppose that there exist infinitely many indices $i \geq i_0$ such that $V_i = V_{i+1}$. Then $\lambda \leq \sqrt{2} - 1 \approx 0.4142$. Moreover, if $\lambda = \sqrt{2} - 1$, then we also have $c \gg 1$.

**Proof.** Since there are infinitely many indices $i > i_0$ for which $V_{i-1} \neq V_i$, the hypothesis of the proposition forces the existence of arbitrarily large indices $i$ with

$$V_{i-1} \neq V_i = V_{i+1}.$$  

Fix such an $i$. Let $px_0 + qx_1 + rx_2 = 0$ be an equation of $V_i$ with relatively prime coefficients $p, q, r \in \mathbb{Z}$, so that by duality $H(V_i) = \|(p,q,r)\|$. For any point $\mathbf{x} = (x_0, x_1, x_2, x_3)$ of $W_{i+1}$, we have

$$\mathbf{x}^- = (x_0, x_1, x_2) \in (x_i^-, x_{i+1}^-)_{\mathbb{R}} \quad \text{and} \quad \mathbf{x}^+ = (x_1, x_2, x_3) \in (x_i^+, x_{i+1}^+)_{\mathbb{R}},$$

therefore $\mathbf{x}^-$ and $\mathbf{x}^+$ both belong to $V_i + V_{i+1} = V_i$, and so the point $\mathbf{x}$ satisfies

$$px_0 + qx_1 + rx_2 = 0 \quad \text{and} \quad px_1 + qx_2 + rx_3 = 0.$$  

This means that the orthogonal complement of $W_i$ in $\mathbb{R}^4$ is the space $\langle (p,q,r,0), (0,p,q,r) \rangle_{\mathbb{R}}$ and so, applying the duality property of the height
again, we find
\[ H(W_{i+1}) = H((p, q, r, 0), (0, p, q, r)) \neq 2H(V_i)^2 \]
the relation \( H(V_i) \ll H(W_{i+1})^{1/2} \) also follows from [3, Thm. 3] since the equality \( V_i = V_{i+1} \) means that \((p, q, r)\) provides a three-term recurrence relation satisfied by both \(x_i\) and \(x_{i+1}\). We now argue as M. Laurent in the proof of [4, Lemma 5]. Define
\[ P(T) = p + qT + rT^2 \in \mathbb{Z}[T]. \]
For any point \( y = (y_0, y_1, y_2) \in \mathbb{Z}^3 \), we have
\[ |(py_0 + qy_1 + ry_2) - y_0P(\xi)| \leq 2H(V_i)L(y). \]
Applying this estimate to the point \( x_{i+1}^i = V_i \), we get
\[ X_{i+1}^i|P(\xi)| \ll H(V_i)\mathcal{L}_{i+1}. \]
Since \( V_{i-1} \neq V_i \), at least one of the points \( x_{i-1}^- \) or \( x_{i-1}^+ \) does not belong to \( V_i \). If \( y = (y_0, y_1, y_2) \) is such a point, then \( py_0 + qy_1 + ry_2 \) is a non-zero integer, and using successively (9), (10) and (8) we obtain
\[ 1 \leq |py_0 + qy_1 + ry_2| \ll X_{i-1}^-|P(\xi)| + H(V_i)\mathcal{L}_{i-1} \ll H(V_i)\mathcal{L}_{i-1} \ll cH(W_{i+1})^{1/2}X_i^{1-\lambda}. \]
Moreover, Lemma 5.1 gives \( H(W_{i+1}) \ll X_i^{\theta(1-\lambda)} \) and so the last estimate leads to
\[ 1 \ll cX_i^{(1-\lambda)^2/(2\lambda) - \lambda} = cX_i^{(2-(1+\lambda)^2)/(2\lambda)}. \]
As \( i \) can be taken to be arbitrarily large, this implies that \( 2 - (1 + \lambda)^2 \geq 0 \), and so \( \lambda \leq \sqrt{2} - 1 \). Moreover, we obtain \( c \gg 1 \) if \( \lambda = \sqrt{2} - 1 \).

**Corollary 5.3.** Suppose that \( \lambda > \sqrt{2} - 1 \). Then we have \( V_{i-1} \neq V_i \) for any sufficiently large integer \( i \), and the estimates (7) of Lemma 5.1 apply to all integers \( i \geq 1 \). Moreover, for any pair of consecutive integers \( i < j \) of I with \( W_i + W_{i+1} \neq W_j + W_{j+1} \), we also have
\[ H(W_i) \ll X_i^{1-\lambda} \ll H(W_j)^{\theta^2-1} \ll X_j^{(\theta^2-1)(1-\lambda)}, \]
\[ H(W_j) \ll X_j^{1-\lambda} \ll H(W_{j+1})^{\theta(1-\lambda)} \ll X_{j+1}^{\theta(1-\lambda)^2}. \]

**Proof.** The first assertion follows directly from Lemma 5.1 and the above proposition. To prove the second one, we fix consecutive integers \( i < j \) in I with \( W_i + W_{i+1} \neq W_j + W_{j+1} \), and go back to the general estimate (5) from Lemma 3.3:
\[ X_iX_j \ll H(W_i)H(W_j)H(W_{j+1}). \]
On the right hand side of this inequality, we apply the standard estimate \( H(W_i) \ll X_i^{1-\lambda} \) from Lemma 3.1 as an upper bound for \( H(W_i) \), and the estimate \( H(W_{j+1}) \ll H(W_j)^{\theta} \) coming from (7) as an upper bound for \( H(W_{j+1}) \).
On the left hand side, we use instead the estimate $H(W_j) \ll X_j^{1-\lambda}$ from Lemma 3.1 as a lower bound for $X_j$. This gives

$$X_i^\lambda \ll H(W_j)^{\theta+1-1/(1-\lambda)} = H(W_j)^{\theta-1/\theta},$$

and (11) follows. To prove (12), we note instead that, $i$ and $j$ being consecutive elements of $I$, we have $W_j = W_{i+1}$ and so (13) combined with Lemma 3.1 gives

$$X_i X_j \ll H(W_i) H(W_{i+1}) H(W_{j+1}) \ll (X_i X_{i+1})^{1-\lambda} H(W_{j+1}).$$

Moving all powers of $X_i$ to the left and using the estimate $X_{i+1} \ll X_i^\theta$ from (7) as a lower bound for $X_i$, we obtain

$$X_{i+1}^{\lambda/\theta} X_j \ll X_{i+1}^{1-\lambda} H(W_{j+1}).$$

Moving all powers of $X_{i+1}$ to the right and observing that the exponent $1 - \lambda - \lambda/\theta = 1 - 1/\theta$ is $\geq 0$ (since $\theta \geq 1$), we finally obtain

$$X_j \ll X_{i+1}^{1-1/\theta} H(W_{j+1}) \leq X_j^{1-1/\theta} H(W_{j+1}),$$

which implies (12). 

6. The set $J$. We assume from now on that $\lambda > \sqrt{2} - 1$. Then, for each sufficiently large index $i$, the subspace $V_i = \langle x_i^-, x_i^+ \rangle_\mathbb{R}$ of $\mathbb{R}^3$ has dimension 2 and, by Corollary 5.3, we have $V_i \neq V_{i+1}$. Consequently, $C_{i,i+1}$ is a non-zero point of $\mathbb{Z}^2$ for each $i \gg 1$.

Notation. Let $J$ be the set of all elements $i$ of $I$ whose successor $j$ in $I$ satisfies $W_j + W_{j+1} \neq W_i + W_{i+1}$.

By Lemma 3.3, the set $J$ is infinite. The next result studies a possible configuration of points.

Lemma 6.1. Suppose that $\lambda > \sqrt{2} - 1$, and that $h < i < j$ are three consecutive elements of $I$ with $h \in J$ and $i \in J$. Then we have

$$L(C_{i,i+1}) \ll X_{j+1}^{\alpha} \quad \text{where} \quad \alpha = \frac{-\lambda^4 + \lambda^3 + \lambda^2 - 3\lambda + 1}{\lambda(\lambda^2 - \lambda + 1)}.$$

Proof. By Lemma 4.2,

$$L(C_{i,i+1}) \ll X_{i}^{1-\lambda} X_{j}^{1-\lambda}. \quad (14)$$

Since $i \in J$, we have $W_i + W_{i+1} \neq W_j + W_{j+1}$, and the second part of (6) in Lemma 3.3 gives

$$X_i \ll X_j^{\theta}. \quad \text{Since} \quad h \in J, \text{we also have} \quad W_h + W_{h+1} \neq W_i + W_{i+1}, \text{and the estimates (12) of Corollary 5.3 applied to the pair} \ (h,i) \text{instead of} \ (i,j) \text{lead to} \quad X_i \ll X_{i+1}^{(1-\lambda)\theta} \leq X_j^{(1-\lambda)\theta}. $$
Put $\beta = (1 - \lambda)/(\lambda^2 - \lambda + 1)$. Since $\lambda \leq 1/2$, we have $\beta \geq 1 - \lambda \geq 1/2$. We consider two cases.

(a) If $X_j \geq X_{j+1}^\beta$, we substitute into (14) the first of the above two upper bounds for $X_i$. This gives

$$L(C_{i,i+1}) \ll X_j^{-1-\lambda}X_{j+1}^\theta - \lambda \leq X_{j+1}^{-(1+\alpha)\beta + \theta - \lambda} = X_{j+1}^\alpha.$$ 

(b) If on the contrary, we have $X_j < X_{j+1}^\beta$, we substitute instead into (14) the second upper bound for $X_i$. Again we find

$$L(C_{i,i+1}) \ll X_j^{\lambda - \lambda}X_{j+1}^{-\lambda} \leq X_{j+1}^{(1-\lambda)\theta - \lambda} = X_{j+1}^\alpha,$$

upon noting that the exponent $(1 - \lambda)\theta - \lambda = (1 - 2\lambda)/\lambda$ is $\geq 0$.

**Proposition 6.2.** Suppose that $\lambda > \lambda_2$ where $\lambda_2 \cong 0.4241$ denotes the positive root of the polynomial $P_2(T) = 3T^4 - 4T^3 + 2T^2 + 2T - 1$, and let $\alpha$ be as in Lemma 6.1. Then we have $1 - 2\lambda + \alpha < 0$ and, for any triple of consecutive elements $h < i < j$ of $I$ contained in $J$, with $i$ large enough, the points $C_{i,i+1}$ and $C_{j,j+1}$ are linearly dependent over $\mathbb{Q}$.

The fact that $P_2(T)$ admits exactly one positive root $\lambda_2$ follows by observing that its second derivative $P_2''(T) = (6T - 2)^2$ is non-negative on $\mathbb{R}$ and that $P_2(0)$ is negative. Consequently, if $\lambda > \lambda_2$, we have $P_2(\lambda) > 0$.

**Proof.** For any triple of consecutive elements $h < i < j$ of $I$ contained in $J$, Lemma 6.1 gives $L(C_{i,i+1}) \ll X_{j+1}^\alpha$ and $L(C_{j,j+1}) \ll X_{k+1}^\alpha$, where $k$ denotes the successor of $j$ in $I$. As the general estimates of Lemma 4.1 provide $\|C_{l,l+1}\| \ll X_{l+1}^{1-2\lambda}$ for each $l \geq 1$, we deduce that

$$|\det(C_{i,i+1}, C_{j,j+1})| \ll \|C_{i,i+1}\|L(C_{j,j+1}) + \|C_{j,j+1}\|L(C_{i,i+1}) \ll X_{i+1}^{1-2\lambda}X_{k+1}^\alpha + X_{j+1}^{1-2\lambda + \alpha} \ll X_{k+1}^{1-2\lambda + \alpha} + X_{j+1}^{1-2\lambda + \alpha}.$$ 

As a short computation gives $1 - 2\lambda + \alpha = -P_2(\lambda)/(\lambda(\lambda^2 - \lambda + 1)) < 0$, we conclude that the integer $\det(C_{i,i+1}, C_{j,j+1})$ vanishes if $i$ is sufficiently large.

**Corollary 6.3.** Suppose that $\lambda > \lambda_2$. Then the complement of $J$ in $I$ is infinite.

**Proof.** If $I \setminus J$ were a finite set, then, by the above proposition, all points $C_{i,i+1}$ with $i \in I$ sufficiently large would belong to the same one-dimensional subspace of $\mathbb{R}^2$. By Lemma 2.4, this would imply that $L(C_{i,i+1}) \gg 1$, against the estimates of Lemma 6.1 since $\alpha < 2\lambda - 1 \leq 0$.

**7. Proof of the Theorem.** We may assume that $\lambda > \lambda_2 \cong 0.4241 > \sqrt{2} - 1$. Then, by Corollary 6.3, there exist infinitely many triples of elements $g < i < j$ of $I$ with $i$ and $j$ consecutive satisfying
\[(15) \quad W_g + W_{g+1} = W_i + W_{i+1} \neq W_j + W_{j+1}.\]

Fix such a triple. Since \(i\) and \(j\) are consecutive elements of \(I\), we have \(W_{i+1} = W_j\) and so
\[W_j = (W_i + W_{i+1}) \cap (W_j + W_{j+1}) = (W_g + W_{g+1}) \cap (W_j + W_{j+1}).\]

Since the sum of \(W_g + W_{g+1}\) and \(W_j + W_{j+1}\) is the whole of \(\mathbb{R}^4\) and since \(H(\mathbb{R}^4) = 1\), an application of (3) gives
\[(16) \quad H(W_g + W_{g+1}) H(W_j + W_{j+1}).\]

By Lemma 3.2, we have
\[H(W_g + W_{g+1}) \ll H(W_g)^{-1/\theta} H(W_{g+1}),\]
\[H(W_j + W_{j+1}) \ll H(W_j)^{-1/\theta} H(W_{j+1}),\]
while the estimates (7) of Lemma 5.1 provide
\[H(W_{g+1}) \ll H(W_g)^\theta \quad \text{and} \quad H(W_{j+1}) \ll H(W_j)^\theta.\]

Using the latter relations respectively as a lower bound for \(H(W_g)\) and as an upper bound for \(H(W_{j+1})\) and substituting them into the former, we obtain
\[(17) \quad H(W_g + W_{g+1}) \ll H(W_g)^{1-1/\theta^2}. \quad H(W_j + W_{j+1}) \ll H(W_j)^{\theta-1/\theta}.\]

Since \(g < i\), we have \(X_{g+1} \leq X_i\) and so Lemma 3.1 gives
\[(18) \quad H(W_{g+1}) \ll c X_{g+1}^{1-\lambda} \leq c X_i^{1-\lambda}.\]

We also have
\[(19) \quad X_i^{1-\lambda} \ll H(W_j)^{\theta^2 - 1}\]
by the estimates (11) of Corollary 5.3. Combining (16)--(19), we find
\[(20) \quad H(W_j) \ll c^{1-1/\theta^2} H(W_j)^{(1-1/\theta^2)(\theta^2 - 1) + (\theta - 1/\theta)}.\]

Since (19) shows that \(H(W_j)\) tends to infinity with \(i\), we conclude that
\[(\theta - 1/\theta)^2 + (\theta - 1/\theta) \geq 1,\]
and so \(\theta - 1/\theta \geq 1/\gamma\) where \(\gamma = (1 + \sqrt{5})/2\) (because \(\theta - 1/\theta\) is \(\geq 0\) and we have \(1/\gamma^2 + 1/\gamma = 1\)). After simplifications, the latter relation implies
\[\lambda^2 - (1 + 2\gamma)\lambda + \gamma \geq 0.\]

Since the polynomial \(T^2 - (1 + 2\gamma)T + \gamma\) admits two positive real roots, \(\lambda_3 \cong 0.4245\) and \(\gamma/\lambda_3 \cong 3.811\), it follows that \(\lambda \leq \lambda_3\). Moreover, if \(\lambda = \lambda_3\), then (20) gives \(c \gg 1\), as announced. \hfill \blacksquare

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