“LARGE” STRANGE ATTRACTORS
IN THE UNFOLDING OF A HETEROCLINIC ATTRACTION

ALEXANDRE RODRIGUES
Centro de Matemática da Univ. do Porto
Rua do Campo Alegre, 687
4169-007 Porto, Portugal

(Communicated by Jairo Bochi)

Abstract. We present a mechanism for the emergence of strange attractors in a one-parameter family of differential equations defined on a 3-dimensional sphere. When the parameter is zero, its flow exhibits an attracting heteroclinic network (Bykov network) made by two 1-dimensional connections and one 2-dimensional separatrix between two saddles-foci with different Morse indices. After slightly increasing the parameter, while keeping the 1-dimensional connections unaltered, we concentrate our study in the case where the 2-dimensional invariant manifolds of the equilibria do not intersect. We will show that, for a set of parameters close enough to zero with positive Lebesgue measure, the dynamics exhibits strange attractors winding around the “ghost” of a torus and supporting Sinai-Ruelle-Bowen (SRB) measures. We also prove the existence of a sequence of parameter values for which the family exhibits a superstable sink and describe the transition from a Bykov network to a strange attractor.

1. Introduction. Homoclinic and heteroclinic bifurcations constitute the core of our understanding of complicated intermittent behaviour in dynamical systems. It has started with Poincaré on the late XIX century, with subsequent contributions by the schools of Andronov, Shilnikov, Smale and Palis. These results rely on a combination of geometrical and analytical techniques used to understand the qualitative behaviour of the dynamics.

Heteroclinic cycles and networks are flow-invariant sets that can occur robustly in dynamical systems and are frequently associated with intermittent behaviour. The rigorous analysis of the dynamics associated to the structure of the nonwandering sets close to heteroclinic networks is still a challenge. We refer to [14] for an overview of homoclinic bifurcations and for details on the dynamics near different types of heteroclinic structures. In this article, we establish connections between the theory of rank-one attractors and a classical dynamical scenario related to heteroclinic attractors. We present a mechanism that produce “large” strange attractors in the

2020 Mathematics Subject Classification. Primary: 34C28; 34C37; 37D05; Secondary: 37D45; 37G35.

Key words and phrases. Heteroclinic bifurcations, Heteroclinic network, Strange attractors, Rank-one dynamics, Superstable sinks.

*AR was partially supported by CMUP (UID/MAT/00144/2020), which is funded by FCT with national (MCTES) and European structural funds through the programs FEDER, under the partnership agreement PT2020. AR also acknowledges financial support from Program INVESTIGADOR FCT (IF/00107/2015).
A compact attractor is said to be strange if it contains a dense orbit with at least one positive Lyapunov exponent. A dynamical phenomenon in a one-parameter family of maps is said to be persistent if it occurs for a set of parameters of positive Lebesgue measure. Persistence of chaotic dynamics is physically relevant because it means that a given phenomenon is numerically observable with positive probability.

Strange attractors are of fundamental importance in dynamical systems; they have been observed and recognized in many scientific disciplines [4, 11, 13, 17, 28]. Atmospheric physics provides one of the most striking examples of strange attractors observed in natural sciences. We address the reader to [29] where the authors established the emergence of strange attractors in a low-order atmospheric circulation model. Among the theoretical examples that have been studied are the Lorenz and Hénon attractors, both of which are closely related to suitable one-dimensional reductions.

The rigorous proof of the strange character of an invariant set is a great challenge and the proof of the persistence (in measure) of such attractors is an involved task.

For families of autonomous differential equations in $\mathbb{R}^3$, the persistence of strange attractors can be proved near homo or heteroclinic cycles whose first return map to a cross section exhibits a homoclinic tangency to a dissipative point [13, 17, 20]. In this paper we give a further step towards this analysis. We provide a criterion for the existence of abundant strange attractors (in the terminology of [20]) near a specific heteroclinic configuration, using the theory of rank-one attractors developed by Q. Wang and L.-S. Young [32, 33, 34, 36]. This technique is quite general and may be applied other heteroclinic bifurcations with a single direction of instability.

1.2. Rank-one attractors theory: an overview. We briefly summarise how the theory of rank-one attractors fits in the existing literature. In 1976, Hénon [12] proposed the following two-parameter family of maps on $\mathbb{R}^2$

$$f_{(a,b)}(x,y) = (1 - ax^2 + y, bx),$$

for which numerical experiments for $(a,b) = (1.4, 0.3)$ suggested the existence of a global attractor. Hénon conjectured that this dynamical system should have a strange attractor and that it might be more amenable for analysis than the Lorenz system.

Benedicks and Carleson [5] managed to prove that Hénon’s conjecture was true, not for the parameters $(a,b) = (1.4, 0.3)$ but for $b > 0$ small. In fact, for such small $b$-values, the map (1) is strongly dissipative, and may be seen as an “unfolded” version of the quadratic map on the interval. It was shown that, for these values of $b$, there is a forward invariant region which accumulates on a topological attractor that coincides with the topological closure of the unstable manifold of a fixed point of saddle-type $p$, $\overline{W^u(p)}$. Results in [5] state that, as long as $b > 0$ is kept sufficiently small, there is a positive Lebesgue measure set of parameters $a \in [1, 2]$ (close to $a = 2$) for which there is a dense orbit in $\overline{W^u(p)}$ along which the derivative grows exponentially fast. The techniques developed in [5] promoted the emergence of several results not specific to the context of Hénon maps, among which the work by Mora and Viana [20] assumes a crucial importance.
L. Mora and M. Viana [20] proposed a renormalization scheme that, when applied to a generic unfolding of a homoclinic tangency associated to a dissipative saddle, reveals the presence of Hénon-like families. This means that chaotic attractors arise abundantly in a specific dynamical scenario. Continuing the study of the dynamical properties of Hénon maps, Benedicks and Young [6] developed these techniques to obtain that every attractor occurring for suitable parameters \((a, b)\) of (1) supports a unique SRB measure. Young [38] extended these results to dynamical systems that admit a horseshoe with infinitely many branches. In doing so, the author provided a general scheme that unifies the proofs of these results in several dynamical situations.

The theory of rank-one maps, systematically developed by Wang and Young [32, 33, 35, 36], concerns the dynamics of maps with some instability in one direction of the phase space and strong contraction in all other directions of the phase space. This theory originated with the work of Jackobson [15] on the quadratic family and the analysis of strongly dissipative Hénon maps by Benedicks and Carleson [5].

It is a comprehensive theory for a nonuniformly hyperbolic setting that is flexible enough to be applicable to concrete systems of differential equations and has experienced unprecedented growth in the last 20 years in the context of non-autonomous systems. It provides checkable conditions that imply the existence of nonuniformly hyperbolic dynamics and SRB measures in parametrized families \(F_\lambda\) of dissipative embeddings in \(\mathbb{R}^n\) for \(n \geq 2\). Roughly speaking, the theory asserts that, under certain checkable conditions, there exists a set \(\Delta \subset \mathbb{R}\) of values with positive Lebesgue measure such that if \(\lambda \in \Delta\), then \(F_\lambda\) has a strange attractor supporting a SRB measure. This theory has already been applied to several non-autonomous dynamical scenarios, including systems with stable foci and limit cycles subject to pulsate drives [21, 22, 33, 34] and heteroclinic bifurcations [19, 26, 31]. Although our setting is formulated to give rigorous results, the theory can also provide justification for various mathematical statements about the strange attractors found in [9].

1.3. Structure of the article. Motivated by the bifurcation scenario involving a Bykov attractor [17, 18], in Section 3, we enumerate the main assumptions concerning the configuration of an attracting network, preceding the presentation of the main results in Section 4. In Section 5, we give a descriptive summary of the rank-one attractors theory in dimension 2, after the introduction of a Misiurewicz-type map. We also introduce some important dynamical and ergodic concepts.

The coordinates and other notation used in the rest of the article are presented in Section 6. In Sections 7, 8 and 9, we prove the main results of this paper. In Section 8, we add an extra subsection with some complementary remarks. Finally, in Section 10, we relate our results with others in the literature and we point out some bifurcations in the family of vector fields, emphasising the role of the twisting number in the sequel.

Throughout this paper, we have endeavoured to make a self contained exposition bringing together all topics related to the proofs. We revive some useful results from the literature; we hope this saves the reader the trouble of going through the entire length of some referred works to achieve a complete description of the theory. We have drawn illustrative figures to make the paper easily readable.

2. Preliminaries. To make this paper self-contained and readable, we recall some definitions and results on heteroclinic bifurcations, adapted to our purposes. For
$\varepsilon > 0$ small enough and $k \geq 4$, consider the one-parameter family of $C^k$–smooth autonomous differential equations

$$\dot{x} = f_\lambda(x) \quad x \in S^3 \quad \lambda \in [0, \varepsilon] \quad (2)$$

where $S^3$ denotes the unit 3-dimensional sphere, endowed with the $C^k$–topology.

### 2.1. Bykov network.

Suppose that $O_1$ and $O_2$ are two hyperbolic saddle-foci of (2) with different Morse indices (dimension of the unstable manifold). We say that there is a **heteroclinic cycle** associated to $O_1$ and $O_2$ if

$$W^u(O_1) \cap W^s(O_2) \neq \emptyset \quad \text{and} \quad W^u(O_2) \cap W^s(O_1) \neq \emptyset.$$ 

For $i, j \in \{1, 2\}$, the non-empty intersection of $W^u(O_i)$ with $W^s(O_j)$ is called a **heteroclinic connection** between $O_i$ and $O_j$, and will be denoted by $[O_i \to O_j]$.

Although heteroclinic cycles involving equilibria are not a generic feature within differential equations, they may be structurally stable within families of vector fields which are equivariant under the action of a compact Lie group $G \subset O(n)$, due to the existence of flow-invariant subspaces [11]. A heteroclinic cycle between two hyperbolic saddle-foci of different Morse indices, where the invariant manifolds coincide, is called a **Bykov network** [8, 18].

### 2.2. Rotational horseshoe.

Let $\mathcal{H}$ stand for the infinite annulus $\mathcal{H} = S^1 \times [0, 1]$, endowed with the usual inner product from $\mathbb{R}^2$. We denote by $\text{Homeo}^+(\mathcal{H})$ the set of homeomorphisms of the annulus which preserve orientation. Given a homeomorphism $f : X \to X$ and a partition of $m \in \mathbb{N}\{1\}$ elements $R_0, ..., R_{m-1}$ of $X \subset \mathcal{H}$, the itinerary function

$$\Upsilon : X \to \{0, ..., m-1\}^\mathbb{Z} = \Sigma_m$$

is defined by

$$\Upsilon(x)(j) = k \iff f^j(x) \in R_k, \quad \text{for every} \quad j \in \mathbb{Z}.$$ 

We say that a compact invariant set $\Lambda \subset \mathcal{H}$ of $f \in \text{Homeo}^+(\mathcal{H})$ is a **rotational horseshoe** if it admits a finite partition $P = \{R_0, ..., R_{m-1}\}$ with $R_i$ open sets of $\Lambda$ so that:

- the itinerary $\Upsilon$ defines a semi-conjugacy between $f|_\Lambda$ and the full-shift $\sigma : \Sigma_m \to \Sigma_m$, that is $\Upsilon \circ f = \sigma \circ \Upsilon$ with $\Upsilon$ continuous and onto;
- for any lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of $f$, there exist $k > 0$ and $m$ vectors $v_0, ..., v_{m-1} \in \mathbb{Z} \times \{0\}$ so that\(^1:\)

$$\left\| (F^n(\hat{x}) - \hat{x}) - \sum_{i=0}^{n} v_{\Upsilon(x)(i)} \right\| < k \quad \text{for every} \quad \hat{x} \in \pi^{-1}(\Lambda), \quad n \in \mathbb{N},$$

where $\pi : \mathbb{R}^2 \to \mathcal{H}$ denotes the usual projection map and $\hat{x} \in \pi^{-1}(\Lambda)$ is the lift of $x$.

The existence of a rotational horseshoe for a map implies positive **topological entropy** at least $\log m$.

---

\(^1\) The usual norm of $\mathbb{R}^2$ is denoted by $\| \star \|$. 
2.3. **Strange attractors and SRB measures.** Based on [33], we formalize the notion of strange attractor for a two-parameter family $F(a,b)$ defined on $M = S^1 \times [0,1]$, endowed with the induced topology. Recall that, if $A \subset M$ then $\overline{A}$ is the topological closure of $A$.

Let $F(a,b)$ be an embedding such that $F(a,b)(U) \subset U$ for some non-empty open set $U \subset M$. In the present work we refer to

$$\Omega = \bigcap_{m=0}^{+\infty} F_m(a,b)(U).$$

as an attractor and $U$ as its basin. The attractor $\Omega$ is irreducible if it cannot be written as the union of two (or more) disjoint attractors.

We say that $\Omega$ is a *strange attractor* for $F(a,b)$ if, for Lebesgue-a.e $(x,y) \in U \subset M$, the orbit of $(x,y)$ has a positive Lyapunov exponent. In other words, denoting by $\| \star \|$ the usual norm of $\mathbb{R}^2$, we have:

$$\lim_{n \in \mathbb{N}} \frac{1}{n} \log \| D F^n(a,b)(x,y) \| > 0.$$

We say that $F(a,b)$ possesses a *strange attractor supporting an ergodic SRB measure* $\nu$ if:

1. $F(a,b)$ has a strange attractor,
2. the conditional measures of $\nu$ on unstable manifolds are equivalent to the Riemannian volume on these leaves and
3. for Lebesgue-a.e. $(x,y) \in U \subset M$ and for every continuous function $\varphi : U \to \mathbb{R}$, we have:

$$\lim_{n \in \mathbb{N}} \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ F_i(a,b)(x,y) = \int \varphi \, d\nu. \quad (3)$$

We say that $F(a,b)$ converges in distribution (with respect to $\nu$) to the normal distribution if, for every ergodic SRB measure $\nu$ and every Hölder continuous function $\varphi : \Omega \to \mathbb{R}$, the sequence $\{ \varphi \left( F_i(a,b) \right) : i \in \mathbb{N} \}$ obeys a central limit theorem; in other words, if $\int \varphi \, d\nu = 0$, then the sequence $\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ F_i(a,b)$ converges (in distribution with respect to $\nu$) to the normal distribution. The variance of the limiting normal distribution is strictly positive unless $\varphi \circ F(a,b) = \Psi \circ F(a,b) - \Psi$ for some $\Psi$.

Suppose that $F(a,b)$ possesses a strange attractor and support a unique ergodic SRB measure $\nu$. The dynamical system $(F,\nu)$ is *mixing* if it is isomorphic to a Bernoulli shift.

3. **Setting.** We will enumerate the main assumptions concerning the configuration of an attracting heteroclinic network.
3.1. The organising center. For \( \varepsilon > 0 \) small enough and \( r \geq 4 \), consider the one-parameter family of \( C^r \)-smooth differential equations

\[
\dot{x} = f_\lambda(x) \quad x \in \mathbb{S}^3 \quad \lambda \in [0, \varepsilon]
\]

where \( \mathbb{S}^3 \) denotes the unit 3-dimensional sphere, endowed with the usual \( C^r \)-topology. Denote by \( \varphi_\lambda(t, x), t \in \mathbb{R}, \) the associated flow\(^2\), satisfying the following hypotheses for \( \lambda = 0 \):

(P1) There are two different equilibria, say \( O_1 \) and \( O_2 \).

(P2) The eigenvalues of \( Df_0(X) \) are:

- \( P2a \) \( E_1 \) and \(-C_1 \pm \omega_1 i \) where \( C_1 > E_1 > 0, \quad \omega_1 > 0, \) for \( X = O_1; \)
- \( P2b \) \(-C_2 \) and \( E_2 \pm \omega_2 i \) where \( C_2 > E_2 > 0, \quad \omega_2 > 0, \) for \( X = O_2. \)

The equilibrium point \( O_1 \) possesses a 2-dimensional stable and 1-dimensional unstable manifold that will be denoted by \( W^s(O_1) \) and \( W^u(O_1) \), respectively. Dually, \( O_2 \) possesses a 1-dimensional stable and 2-dimensional unstable manifold and the terminology is \( W^s(O_2) \) and \( W^u(O_2) \). For \( M \subset \mathbb{S}^3 \), denoting by \( \overline{M} \) the topological closure of \( M \), we also assume that:

(P3) The manifolds \( W^s(O_2) \) and \( W^s(O_1) \) coincide and \( \overline{W^u(O_2)} \cap W^s(O_1) \) consists of a two-sphere (also called the 2D-connection).

(P4) There are two trajectories, say \( \gamma_1, \gamma_2 \), contained in \( W^u(O_1) \cap W^s(O_2) \), one in each connected component of \( \mathbb{S}^3 \backslash \overline{W^u(O_2)} \) (also called the 1D-connections).

For \( \lambda = 0 \), the two equilibria \( O_1 \) and \( O_2 \), the 2-dimensional heteroclinic connection from \( O_2 \) to \( O_1 \) referred in (P3) and the two trajectories listed in (P4) build a heteroclinic network we will denote hereafter by \( \Gamma \). This network consists of two cycles and has an attracting character\(^3\)\ (see references therein), this is why it will be called a Bykov\(^3\) attractor.

We say that \( \Gamma \subset \mathbb{S}^3 \) is asymptotically stable if we may find an open neighbourhood \( \mathcal{U} \) of the heteroclinic network \( \Gamma \) having its boundary transverse to the flow of \( \dot{x} = f_0(x) \) and such that every solution starting in \( \mathcal{U} \) remains in it and is forward asymptotic to \( \Gamma \).

**Lemma 3.1**\(^{[18]}\). The set \( \Gamma \) is asymptotically stable.

There are two different possibilities for the geometry of the flow around \( \Gamma \), depending on the direction trajectories turn around the 1-dimensional heteroclinic trajectories from \( O_1 \) to \( O_2 \). To make this rigorous, we need some new concepts adapted from\(^[17]\). Let \( V_1 \) and \( V_2 \) be small disjoint neighbourhoods of \( O_1 \) and \( O_2 \) with disjoint boundaries \( \partial V_1 \) and \( \partial V_2 \), respectively. These neighbourhoods have been constructed with detail in Section 6. Trajectories starting at \( \partial V_1 \backslash W^s(O_1) \) near \( W^s(O_1) \) go into the interior of \( V_1 \) in positive time, then follow one of the solutions in \([O_1 \rightarrow O_2]\), go inside \( V_2 \), come out at \( \partial V_2 \) and then return to \( \partial V_1 \). This trajectory is not closed since \( \Gamma \) is attracting. Let \( Q \) be a piece of trajectory like this from \( \partial V_1 \) to \( \partial V_1 \). Within \( \partial V_1 \backslash W^s(O_1) \), join its starting point to its end point by a

\(^2\)Since \( \mathbb{S}^3 \) is a compact set without boundary, the solutions of (4) may be extended to \( \mathbb{R} \).

\(^3\)The terminology Bykov is a tribute to V. Bykov who dedicated his latest research works to cycles with two saddle-foci.
segment, forming a closed curve, which we call the loop of \( Q \). By construction, the loop of \( Q \) and \( \Gamma \) are disjoint closed sets.

**Definition 3.2.** ([17]) We say that the two saddle-foci \( O_1 \) and \( O_2 \) in \( \Gamma \) have the same chirality if the loop of every trajectory is linked to \( \Gamma \) in the sense that the two closed sets cannot be disconnected by an isotopy. Otherwise, we say that \( O_1 \) and \( O_2 \) have different chirality.

From now on, we assume the following technical condition:

**(P5)** The saddle-foci \( O_1 \) and \( O_2 \) have the same chirality.

### 3.2. Perturbing term.

When the system (4) is slightly perturbed, the 2-dimensional invariant manifolds are generically transverse (either intersecting or not), as a consequence of the Kupka-Smale Theorem. With respect to the effect of the parameter \( \lambda \) on the dynamics, we assume that:

**(P6)** For \( \lambda > 0 \), the two trajectories within \( W^u(O_1) \cap W^s(O_2) \) persist.

**(P7)** For \( \lambda > 0 \), the 2-dimensional manifolds \( W^u(O_2) \) and \( W^s(O_1) \) do not intersect.

and

**(P8)** There exist \( \varepsilon, \lambda_1 > 0 \) for which the global maps associated to the connections \( [O_1 \to O_2] \) and \( [O_2 \to O_1] \) are given, in local coordinates, by the Identity map and by

\[
(x, y) \mapsto (x + \xi + \lambda \Phi_1(x, y), y + \lambda \Phi_2(x, y)) \quad \text{for} \quad \lambda \in [0, \lambda_1]
\]

respectively, where \( \xi \in \mathbb{R} \),

\[
\Phi_1 : S^1 \times [-\varepsilon, \varepsilon] \to \mathbb{R}, \quad \Phi_2 : S^1 \times [-\varepsilon, \varepsilon] \to \mathbb{R}^+
\]

are \( C^3 \)-maps and \( \ln(\Phi_2(x, 0)) \) is a Morse function with finitely many nondegenerate critical points\(^4\). This assumption will be detailed in Section 6.

For \( r \geq 4 \), denote by \( \mathcal{X}_{\text{Byk}}(S^3) \), the family of \( C^r \)-vector fields on \( S^3 \) endowed with the \( C^r \)-Whitney topology, satisfying Properties (P1)–(P8).

### 3.3. Digestive remarks about the Hypotheses.

In this subsection, we discuss the Hypotheses (P1)–(P8), pointing out that they appear in specific contexts.

**Remark 1.** The full description of the bifurcations associated to the Bykov attractor is a phenomenon of codimension three. Nevertheless, the setting described by (P1)–(P8) is natural in \( \text{SO}(2) \)-symmetry-breaking contexts [16, 18] and also in the setting of some unfoldings of the Hopf-zero singularity [4]. Hypothesis (P6) corresponds to the partial symmetry-breaking considered in Section 2.4 of [16]. The setting described by (P1)–(P8) generalizes Case (4) of [25].

**Remark 2.** An explicit example of a vector field on \( S^3 \) satisfying Properties (P1)–(P7) may be found in Section 4.1.3.2 of Aguiar [2]. Adding a generic term to system (4.24) of [2], a 2-dimensional torus will break generically and our results of Section 4 are valid. A numerical analysis of this system has been performed in [9].

\(^4\)In Section 9, we list the critical points of \( \ln(\phi_2(x, 0)) \) as \( c^{(1)}, \ldots, c^{(q)} \), for some \( q \in \mathbb{N} \setminus \{2\} \).
Remark 3. In the simplest scenario for the splitting of the sphere defined by the coincidence of the two-dimensional invariant manifolds, one observes either heteroclinic tangles or empty intersection. The expression for the global map associated to \([O_2 \to O_1]\) of (P8) is generic in the second case. The distance between \(W^u(O_2)\) and \(W^s(O_1)\) in a section to \(\Gamma\) may be computed using the Melnikov integral \([11]\).

Remark 4. The analytical expressions for the transitions maps along the connections \([O_1 \to O_2]\) and \([O_2 \to O_1]\) could be written as a general Linear map as the one considered in \([8]\):

\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
\]

and by

\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \lambda \begin{pmatrix} \Phi_1(x,y) \\ \Phi_2(x,y) \end{pmatrix},
\]

respectively, where \(a \geq 1, \xi_1, \xi_2, b_1, b_2, c_1, c_2 \in \mathbb{R}\). For the sake of simplicity and, without loss of generality, we restrict to the case \(a = 1 = b_1 = c_2 = 1\) and \(\xi_2 = b_2 = c_1 = 0\). This simplifies the computations and is not a restriction (Section 6 of \([24]\)).

3.4. Constants. For future use, we settle the following notation on the saddle-value of \(O_1, O_2\) and \(\Gamma\):

\[
\delta_1 = \frac{C_1}{E_1} > 1, \quad \delta_2 = \frac{C_2}{E_2} > 1, \quad \delta = \delta_1 \delta_2 > 1
\tag{5}
\]

and on the twisting number defined as:

\[
K_{\omega} = \frac{E_2 \omega_1 + C_1 \omega_2}{E_1 E_2} > 0.
\tag{6}
\]

4. Presentation of the results. Let \(T\) be a neighborhood of the Bykov attractor \(\Gamma\) that exists for \(\lambda = 0\). For \(\lambda_0 > 0\) small enough and \(r \geq 4\), let \((f_\lambda)_{\lambda \in [0, \lambda_0]}\) be a one-parameter family of vector fields in \(\mathcal{X}_{Byk}(S^3)\) satisfying conditions (P1)–(P8).

Theorem 4.1. Let \(f_\lambda \in \mathcal{X}_{Byk}(S^3)\) with \(r \geq 4\). Then, there is \(\tilde{\varepsilon} > 0\) (small) such that the first return map to a given cross section \(\Sigma\) to \(\Gamma\) may be written (in local coordinates of \(\Sigma\)) by:

\[
F_\lambda(x,y) = \left[ x + \xi + \lambda \Phi_1(x,y) - K_{\omega} \ln(y + \lambda \Phi_2(x,y)) \pmod{2\pi}, (y + \lambda \Phi_2(x,y))^4 \right] + \ldots
\]

where \(\xi \in \mathbb{R}\),

\[
(x,y) \in D = \{ x \in \mathbb{R} \pmod{2\pi}, \ y/\tilde{\varepsilon} \in [-1,1] \text{ and } y + \lambda \Phi_2(x,y) > 0 \}
\]

and the ellipses stand for asymptotically small terms depending on \(x\) and \(y\) which converge to zero along with their derivatives.
The proof of Theorem 4.1 is done in Section 7 by composing local and global maps. Since \( \delta > 1 \), for \( \lambda \) small enough, the second component of \( F_{\lambda} \) is contracting and the dynamics of \( F_{\lambda} \) is dominated by the family of circle maps

\[
h_a(x) = x + a + \xi + K_{\omega} \ln |\phi_2(x, 0)|
\]

where \( \xi \in \mathbb{R} \), \( x \in S^1 \) and \( a \sim -K_{\omega} \ln \lambda \mod 2\pi \). Next result shows that, for any small smooth unfolding of \( f_0 \), in the \( C^4 \)-Whitney topology, there is a sufficiently large twisting number \( K_{\omega} \) which forces the emergence of a strange attractor (for \( F_{\lambda} \) defined in Theorem 4.1).

**Theorem 4.2.** Let \( f_{\lambda} \in \mathcal{X}_{B_{4k}}(S^3) \) with \( r \geq 4 \). Then there exists \( K_{\omega}^0 > 0 \) such that if \( K_{\omega} > K_{\omega}^0 \), there exists a set \( \Delta \subset [0, \lambda_0] \) of positive Lebesgue measure with the following properties: for every \( \lambda \in \Delta \) the map \( F_{\lambda} \) exhibits an irreducible strange attractor that supports a unique ergodic SRB measure \( \nu \). The orbit of Lebesgue almost all points on the cross section \( \Sigma \) has a positive Lyapunov exponent and is asymptotically distributed according to \( \nu \).

The \( F_{\lambda} \)-iterates of almost all points on the strange attractor of \( \Sigma \) wind around an annulus (as discussed in Section 10), justifying the title of this manuscript on “large” strange attractors according to the terminology of [7]. The dynamics are genuinely non-uniformly hyperbolic, a central limit theorem holds and correlations decay at an exponential rate [33]. The proof of Theorem 4.2 is performed in Section 8 by reducing the dynamics of the 2-dimensional first return map to the dynamics of a one-dimensional map, via the theory of rank-one attractors described in Section 5.

Following the reasoning of [13], the next novelty is the existence of a sequence of parameters converging to zero for which the flow of (4) exhibits a superstable 2-periodic orbit (i.e. the map \( F_{\lambda} \) has a 2-periodic point which is critical).

**Theorem 4.3.** Let \( f_{\lambda} \in \mathcal{X}_{B_{4k}}(S^3) \) with \( r \geq 4 \). There exists \( K_{\omega}^0 > 0 \) and a sequence \( (\lambda_n)_{n \in \mathbb{N}} \) converging to 0 such that, if \( K_{\omega} > K_{\omega}^0 \) then the flow of \( \dot{x} = f_{\lambda_n}(x) \) exhibits a superstable 2-periodic orbit.

Each one one these sinks persist within intervals around \( \lambda = \lambda_n, n \in \mathbb{N} \). The proof of Theorem 4.3 is performed in Section 9.

5. **Theory of rank-one maps.** We gather in this section a collection of technical facts used repeatedly in later sections. In what follows, let us denote by \( C^2(S^1, \mathbb{R}) \) the set of \( C^2 \)-maps from \( S^1 \) (unit circle) to \( \mathbb{R} \). For \( h \in C^2(S^1, \mathbb{R}) \), let

\[
C(h) = \{ x \in S^1 : h'(x) = 0 \}
\]

be the critical set of \( h \). For \( \delta > 0 \), let \( C_{\delta} \) be the \( \delta \)-neighbourhood of \( C(h) \) in \( S^1 \) and let \( C_{\delta}(c) \) be the \( \delta \)-neighbourhood of \( c \in C(h) \). The terminology \( dist \) denotes the euclidian metric on \( \mathbb{R} \).
5.1. Misiurewicz-type map. We say that $h \in C^2(S^1, \mathbb{R})$ is a Misiurewicz-type map if the following assertions hold:

1. There exists $\delta_0 > 0$ such that:
   \begin{enumerate}
   \item $\forall x \in C_{\delta_0}, h''(x) \neq 0$ and
   \item $\forall c \in C(h)$ and $n \in \mathbb{Z}^+, \text{dist}(h^n(c), C(h)) \geq \delta_0$.
   \end{enumerate}

2. There exist constants $b_0, \lambda_0 \in \mathbb{R}^+$ such that for all $\delta < \delta_0$ and $n \in \mathbb{N}$, we have:
   \begin{enumerate}
   \item if $h^k(x) \notin C_{\delta}$ for $k \in \{0, \ldots, n-1\}$, then $|(h^n)'(x)| \geq b_0 \delta \exp(\lambda_0 n)$.
   \item if $h^k(x) \notin C_{\delta}$ for $k \in \{0, \ldots, n-1\}$ and $h^n(x) \in C_{\delta_0}$, then $|(h^n)'(x)| \geq b_0 \exp(\lambda_0 n)$.
   \end{enumerate}

For $\delta > 0$, the set $S^1$ may be divided into two regions: $C_{\delta}$ and $S^1 \setminus C_{\delta}$. In $S^1 \setminus C_{\delta}$, $h$ is essentially uniformly expanding; in $C_{\delta} \setminus C$, although $|h'(x)|$ is small, the orbit of $x$ does not return to $C_{\delta}$ until its derivative has regained an amount of exponential growth.

![Figure 1](image.png)

**Figure 1.** Example of a Misiurewicz-type map $h : S^1 \to \mathbb{R}$. For $\delta > 0$, the set $C_{\delta}$ is a neighbourhood of the set of critical points $C$. These maps are among the simplest examples with non-uniform expansion.

5.2. Rank-one maps. Let $M = S^1 \times [0,1]$, induced with the usual topology. We consider the two-parameter family of maps $F_{(a,b)} : M \to M$, where $a \in [0, 2\pi]$ where $0$ and $2\pi$ are identified, and $b \in \mathbb{R}$ is a scalar. Let $B_0 \subset \mathbb{R}\setminus\{0\}$ with $0$ as an accumulation point. Rank-one theory asks the following hypotheses:
(H1) Regularity conditions: 1. For each \( b \in B_0 \), the function \((x, y, a) \mapsto F_{(a, b)}\) is at least \(C^3\)-smooth.
2. Each map \( F_{(a, b)} \) is an embedding of \( M \) into itself.
3. There exists \( k \in \mathbb{R}^+ \) independent of \( a \) and \( b \) such that for all \( a \in [0, 2\pi], \ b \in B_0 \) and \((x_1, y_1), (x_2, y_2) \in M\), we have:
\[
\frac{|\det DF_{(a, b)}(x_1, y_1)|}{|\det DF_{(a, b)}(x_2, y_2)|} \leq k.
\]

(H2) Existence of a singular limit: For \( a \in [0, 2\pi] \), there exists a map \( F_{(a, 0)} : M \to \mathbb{S}^1 \times \{0\} \) such that the following property holds: for every \((x, y) \in M\) and \( a \in [0, 2\pi] \), we have
\[
\lim_{b \to 0} F_{(a, b)}(x, y) = F_{(a, 0)}(x, y).
\]

(H3) \(C^3\)-convergence to the singular limit: For every choice of \( a \in [0, 2\pi] \), the maps \((x, y, a) \mapsto F_{(a, b)}\) converge in the \(C^3\)-topology to \((x, y, a) \mapsto F_{(a, 0)}\) on \( M \times [0, 2\pi] \) as \( b \) goes to zero.

(H4) Existence of a sufficiently expanding map within the singular limit: There exists \( a^* \in [0, 2\pi] \) such that \( h_{a^*}(x) \equiv F_{(a^*, 0)}(x, 0) \) is a Misiurewicz-type map (see Subsection 5.1).

(H5) Parameter transversality: Let \( C_{a^*} \) denote the critical set of a Misiurewicz-type map \( h_{a^*} \). For each \( x \in C_{a^*} \), let \( p = h_{a^*}(x) \), and let \( x(\tilde{a}) \) and \( p(\tilde{a}) \) denote the continuations of \( x \) and \( p \), respectively, as the parameter \( a \) varies around \( a^* \). The point \( p(\tilde{a}) \) is the unique point such that \( p(\tilde{a}) \) and \( p \) have identical symbolic itineraries under \( h_{a^*} \) and \( h_{\tilde{a}} \), respectively. We have:
\[
\frac{d}{da} h_{\tilde{a}}(x(\tilde{a}))|_{a=a^*} \neq \frac{d}{da} p(\tilde{a})|_{a=a^*}.
\]

(H6) Nondegeneracy at turns: For each \( x \in C_{a^*} \), we have
\[
\frac{d}{dy} F_{(a^*, 0)}(x, y)|_{y=0} \neq 0.
\]

(H7) Conditions for mixing: If \( J_1, \ldots, J_r \) are the intervals of monotonicity of a Misiurewicz-type map \( h_{a^*} \), then:
1. \( \exp(\lambda_0/3) > 2 \) (see the meaning of \( \lambda_0 \) in Subsection 5.1) and
2. if \( Q = (q_{im}) \) is the matrix of all possible transitions defined by:
\[
\begin{cases}
1 & \text{if } J_m \subset h_{a^*}(J_i) \\
0 & \text{otherwise},
\end{cases}
\]
then there exists \( N \in \mathbb{N} \) such that \( Q^N > 0 \) (i.e. all entries of the matrix \( Q^N \), endowed with the usual product, are positive).

Definition 5.1. Identifying \( \mathbb{S}^1 \times \{0\} \) with \( \mathbb{S}^1 \), we refer to \( F_{(a, 0)} \) the restriction \( h_a : \mathbb{S}^1 \to \mathbb{S}^1 \) defined by \( h_a(x) = F_{(a, 0)}(x, 0) \) as the singular limit of \( F_{(a, b)} \) (see (H4)).
5.3. Q. Wang and L.-S. Young’s reduction. For attractors with strong dissipation and one direction of instability, Wang and Young conditions \((H1)–(H7)\) are relatively simple and checkable; when satisfied, they guarantee the existence of strange attractors with a package of statistical and geometric properties as follows:

**Theorem 5.2** ([33], adapted). Suppose the family \(F(a,b)\) satisfies \((H1)–(H7)\). Then, for all sufficiently small \(b \in B_0\), there exists a subset \(\Delta \subset [0, 2\pi]\) with positive Lebesgue measure such that for \(a \in \Delta\), the map \(F(a,b)\) admits an irreducible strange attractor \(\tilde{\Omega} \subset \Omega\) that supports a unique ergodic SRB measure \(\nu\). The orbit of Lebesgue almost all points in \(\tilde{\Omega}\) has positive Lyapunov exponent and is asymptotically distributed according to \(\nu\).

In contrast to earlier results, the theory in [32, 33, 34] is generic, in the sense that the conditions under which it holds rely only to certain general characteristics of the maps and not to specific formulas or contexts.

6. Local and transition maps. In this section we will analyze the dynamics near the network \(\Gamma\) through local maps, after selecting appropriate coordinates in neighborhoods of the saddle-foci \(O_1\) and \(O_2\).

6.1. Local coordinates. In order to describe the dynamics around the Bykov cycles of \(\Gamma\), we use the local coordinates near the equilibria \(O_1\) and \(O_2\) introduced in [18].

In these coordinates, we consider cylindrical neighbourhoods \(V_1\) and \(V_2\) in \(\mathbb{R}^3\) of \(O_1\) and \(O_2\), respectively, of radius \(\rho = \varepsilon > 0\) and height \(z = 2\varepsilon\). After a linear rescaling of the variables, we may also assume that \(\varepsilon = 1\). Their boundaries consist of three components: the cylinder wall parametrised by \(x \in \mathbb{R}\) (mod \(2\pi\)) and \(|y| \leq 1\) with the usual cover

\[(x, y) \mapsto (1, x, y) = (\rho, \theta, z)\]

and two discs, the top and bottom of the cylinder. We take polar coverings of these disks

\[(r, \varphi) \mapsto (r, \varphi, \pm 1) = (\rho, \theta, z)\]

where \(0 \leq r \leq 1\) and \(\varphi \in \mathbb{R}\) (mod \(2\pi\)). The local stable manifold of \(O_1\), \(W^s(O_1)\), corresponds to the circle parametrised by \(y = 0\). In \(V_1\) we use the following terminology:

- \(\text{In}(O_1)\), the cylinder wall of \(V_1\), consisting of points that go inside \(V_1\) in positive time;
- \(\text{Out}(O_1)\), the top and bottom of \(V_1\), consisting of points that go outside \(V_1\) in positive time.

We denote by \(\text{In}^+(O_1)\) the upper part of the cylinder, parametrised by \((x, y), y \in [0, 1]\) and by \(\text{In}^-(O_1)\) its lower part.

The cross-sections obtained for the linearisation around \(O_2\) are dual to these. The set \(W^u(O_2)\) is the \(z\)-axis intersecting the top and bottom of the cylinder \(V_2\) at the origin of its coordinates. The set \(W^u(O_2)\) is parametrised by \(z = 0\), and we use:

- \(\text{In}(O_2)\), the top and bottom of \(V_2\), consisting of points that go inside \(V_2\) in positive time;
- \(\text{Out}(O_2)\), the cylinder wall of \(V_2\), consisting of points that go inside \(V_2\) in negative time, with \(\text{Out}^+(O_2)\) denoting its upper part, parametrised by \((x, y), y \in [0, 1]\) and \(\text{Out}^-(O_2)\) its lower part.
We will denote by $W^u_{\text{loc}}(O_2)$ the portion of $W^u(O_2)$ that goes from $O_2$ up to $\text{In}(O_1)$ not intersecting the interior of $V$ and by $W^s_{\text{loc}}(O_1)$ the portion of $W^s(O_1)$ outside $V_2$ that goes directly from $\text{Out}(O_2)$ into $O_1$. The flow is transverse to these cross-sections and the boundaries of $V_1$ and of $V_2$ may be written as the closure of $\text{In}(O_1) \cup \text{Out}(O_1)$ and $\text{In}(O_2) \cup \text{Out}(O_2)$, respectively. The orientation of the angular coordinate near $O_2$ is chosen to be compatible with that the direction induced by Hypothesis (P4) and (P5).

6.2. Local maps near the saddle-foci. Following [10], the trajectory of a point $(x, y)$ with $y > 0$ in $\text{In}^+(O_1)$, leaves $V_1$ at $\text{Out}(O_1)$ at

$$L_1(x, y) = \left( y^{\delta_1} + S_1(x, y; \lambda), -\frac{\omega_1 \ln y}{E_1} + x + S_2(x, y; \lambda) \right) = (r, \varphi), \quad (7)$$

where $S_1$ and $S_2$ are smooth functions which depend on $\lambda$ and satisfy:

$$\left| \frac{\partial^{k+l+m}}{\partial x^k \partial y^l \partial \lambda^m} S_i(x, y; \lambda) \right| \leq C \gamma^{\delta_1 + \sigma - 1}, \quad (8)$$

and $C$ and $\sigma$ are positive constants and $k, l, m$ are non-negative integers. Similarly, a point $(r, \varphi)$ in $\text{In}(O_2) \setminus W^s_{\text{loc}}(O_2)$, leaves $V_2$ at $\text{Out}(O_2)$ at

$$L_2(r, \varphi) = \left( \frac{\omega_2 \ln r}{E_2} + \varphi + R_1(r, \varphi; \lambda), r^{\delta_2} + R_2(r, \varphi; \lambda) \right) = (x, y) \quad (9)$$

where $R_1$ and $R_2$ satisfy a condition similar to (8). The terms $S_1$, $S_2$, $R_1$, $R_2$ correspond to asymptotically small terms that vanish when the radial components $y$ and $r$ go to zero.

6.3. The transitions. The coordinates on $V_1$ and $V_2$ are chosen so that $[O_1 \rightarrow O_2]$ connects points with $z > 0$ (resp. $z < 0$) in $V_1$ to points with $z > 0$ (resp. $z < 0$) in $V_2$. Points in $\text{Out}(O_1) \setminus W^u_{\text{loc}}(O_1)$ and $W^s(O_1)$ are mapped into $\text{In}(O_2)$ along a flow-box around each of the connections $[O_1 \rightarrow O_2]$. We will assume that the transition

$$\Psi_{1 \rightarrow 2} : \text{Out}(O_1) \rightarrow \text{In}(O_2)$$

does not depend on $\lambda$ and is the Identity map, a choice compatible with hypothesis (P4) and (P5). Denote by $\eta$ the map

$$\eta = L_2 \circ \Psi_{1 \rightarrow 2} \circ L_1 : \text{In}(O_1) \rightarrow \text{Out}(O_2).$$

From (7) and (9), omitting high order terms in $y$ and $r$, we infer that, in local coordinates, for $y > 0$ we have

$$\eta(x, y) = (x - K_\omega \log y \pmod{2\pi}, y^\delta) \quad (10)$$

with

$$\delta = \delta_1 \delta_2 > 1 \quad \text{and} \quad K_\omega = \frac{C_1 \omega_2 + E_2 \omega_1}{E_1 E_2} > 0. \quad (11)$$

If $y < 0$, solutions that are not trapped by the stable manifold of $O_1$, enter in $\text{In}(O_1)$ after some positive time. Using (P7) and (P8), for $\lambda \in [0, \lambda_1]$, we still have a well defined transition map

$$\Psi_{2 \rightarrow 1}^\lambda : \text{Out}(O_2) \rightarrow \text{In}(O_1)$$

that depends on the parameter $\lambda$, given by (see (P8)):

$$\Psi_{2 \rightarrow 1}^\lambda(x, y) = (x + \xi + \lambda \Phi_1(x, y), y + \lambda \Phi_2(x, y)). \quad (12)$$
To simplify the notation, in what follows we will sometimes drop the subscript \( \lambda \), unless there is some of misunderstanding. By \((P8)\), one knows that \( \ln \Phi_2(x,0) \) has a finite number of nondegenerate critical points – see Figure 2. Hereafter, let us collect them as \( \{c^{(1)},\ldots,c^{(q)}\} \) where \( q \in \mathbb{N} \), \( c^{(i)} < c^{(i+1)} \) for \( i \in \{1,\ldots,q\} \) and \( c^{(q+1)} \equiv c^{(1)} \).

7. **Proof of Theorem 4.1.** The proof of Theorem 4.1 is straightforward by composing the local and global maps constructed in Section 6. Let 

\[
\mathcal{F}_\lambda = \eta \circ \Psi^\lambda_{2 \to 1} =: \mathcal{D} \subset \text{Out}(O_2) \rightarrow \mathcal{D} \subset \text{Out}(O_2)
\]

be the first return map to \( \mathcal{D} \), where \( \mathcal{D} \subset \Sigma = \text{Out}(O_2) \) is the set of initial conditions \((x,y) \in \text{Out}(O_2)\) whose solution returns to \( \text{Out}(O_2) \). Up to high order terms, composing \( \eta \) (10) with \( \Psi^\lambda_{2 \to 1} \) (12), the analytic expression of \( \mathcal{F}_\lambda \) is given by:

\[
\mathcal{F}_\lambda(x,y) = \left[ x + \xi + \lambda \Phi_1(x,y) - K_\omega \ln(y + \lambda \Phi_2(x,y)) \right] \pmod{2\pi}, (y + \lambda \Phi_2(x,y))^\delta
\]

The approximation of \( \mathcal{F}_\lambda \) may be performed in a \( C^3 \)-norm since the local maps \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) may be taken to be \( C^{r-1} \) (\( r \geq 4 \) is the class of differentiability of the initial vector field) and the global maps are assumed to be \( C^3 \)-embeddings.

**Remark 5.** When \( \lambda = 0 \), we may write (for \( y > 0 \)):

\[
\mathcal{F}_0(x,y) = \left( x + \xi - K_\omega \ln y \right) \pmod{2\pi}, \ y^\delta.
\]

This means that the \( y \)-component is contracting and thus the dynamics is governed by the \( x \)-component (circle maps). This is consistent to the fact that the network \( \Gamma \) is asymptotically stable (Lemma 3.1). Notice that the remaining theory does not hold for \( \lambda = 0 \) due to the change of coordinates (14).

8. **Proof of Theorem 4.2.**

8.1. **Insight into the reasoning.** In the proof of Theorem 4.2, we transform the map \( \mathcal{F}_\lambda \) into a two-parameter family of embeddings \( \mathcal{F}(a,b) \) satisfying the Hypotheses \((H1)-(H7)\) of \([33]\) (revisited in Subsection 5.2). For a given parameter \( a \in [0,2\pi[ \), we construct a sequence \( (b_n)_n \) of \( \lambda \)-values such that the singular limit is well defined. If this one-dimensional map has certain “good” properties, then some of them can
be passed back to the 2-dimensional system \((b > 0)\). They in turn allow us to prove results on strange attractors for a positive Lebesgue measure set of \(\lambda\). Our reasoning is similar to the one used in Section 3 of [31].

8.2. Change of coordinates. For \(\lambda \in [0, \lambda_1]\) fixed and \((x, y) \in \text{Out}(O_2)\), let us make the following change of coordinates:

\[
\begin{align*}
\pi &\mapsto x \quad \text{and} \quad \bar{y} \mapsto \frac{y}{\lambda}.
\end{align*}
\]

Taking into account that:

\[
\begin{align*}
F_1^\lambda(x, y) &= x + \xi + \lambda \Phi_1(x, y) - K_\omega \ln(y + \lambda \Phi_2(x, y)) \pmod{2\pi} \\
F_2^\lambda(x, y) &= y + \lambda \Phi_2(x, y) - K_\omega \ln\left(\frac{y}{\lambda} + \Phi_2(x, y)\right) \pmod{2\pi}
\end{align*}
\]

and

\[
\begin{align*}
F_1^a(x, \bar{y}) &= (y + \lambda \Phi_2(x, y))^\delta = \lambda^\delta \left(\frac{y}{\lambda} + \Phi_2(x, y)\right)^\delta,
\end{align*}
\]

we may write:

\[
\begin{align*}
F_1^\lambda(x, \bar{y}) &= x + \xi + \lambda \Phi_1(x, \bar{y}) - K_\omega \ln\lambda - K_\omega \ln\left(\left(\frac{y}{\lambda} + \Phi_2(x, \bar{y})\right)\right) \pmod{2\pi} \\
F_2^\lambda(x, \bar{y}) &= \lambda^{\delta-1} \left(\bar{y} + \Phi_2(x, \bar{y})\right)^\delta.
\end{align*}
\]

8.3. Reduction to a singular limit. In this subsection, we compute the singular limit of \(F_\lambda\) written in the coordinates \((x, \bar{y})\) studied in Subsection 8.2, for \(\lambda \in [0, \lambda_1]\). Let \(k : \mathbb{R}^+ \to \mathbb{R}\) be the invertible map defined by

\[
k(x) = - K_\omega \ln(x),
\]

whose graph is depicted in Figure 3. Define now the decreasing sequence \((\lambda_n)_n\) such that, for all \(n \in \mathbb{N}\), we have:

1. \(\lambda_n \in [0, \lambda_1]\) and
2. \(k(\lambda_n) \equiv 0 \pmod{2\pi}\).

Since \(k\) is an invertible map, for \(a \in \mathbb{S}^1 \equiv [0, 2\pi]\) fixed and \(n \geq n_0 \in \mathbb{N}\), let

\[
\lambda_{(a, n)} = k^{-1}(k(\lambda_n) + a) \in [0, \lambda_1],
\]

as shown in Figure 3. It is immediate to check that:

\[
k(\lambda_{(a, n)}) = - K_\omega \ln(\lambda_n) + a = a \pmod{2\pi}.
\]

The following proposition establishes the convergence of the map \(F_{\lambda_{(a, n)}}\) to a singular limit as \(n \to +\infty\), \((\|\|_r\|_{C^r}\) represents the norm in the \(C^r\)-topology for \(r \geq 2\):
Lemma 8.1. The following equality holds:

\[ \lim_{n \to \infty} \|F_{\lambda_{(n,a)}}(x, y) - (h_a(x, y), 0)\|_{C^0} = 0 \]

where 0 is the null map and

\[ h_a(x, y) = x + a - K_\omega \ln(y + \Phi_2(x, y)) + \xi. \]  

(17)

Proof. Using (16), note that

\[ F_{\lambda_{(n,a)}}^1(x, y) = x + \xi + \lambda_{(n,a)} \Phi_1(x, y) - K_\omega \ln \lambda_{(n,a)} - K_\omega \ln |\Phi_2(x, y)| \quad \text{(mod 2}\pi) \]

\[ = x + \xi + \lambda_{(n,a)} \Phi_1(x, y) + a - K_\omega \ln |\Phi_2(x, y)| \quad \text{(mod 2}\pi) \]

and

\[ F_{\lambda_{(n,a)}}^2(x, y) = \lambda_{(n,a)}^{\delta-1} (y + \Phi_2(x, y))^\delta. \]

Therefore, since \( \lim_{n \to \infty} \lambda_{(n,a)} = 0 \) we may write:

\[ \lim_{n \to \infty} F_{\lambda_{(n,a)}}^1(x, y) = x + \xi + a - K_\omega \ln |\Phi_2(x, 0)| \quad \text{(mod 2}\pi) \]

and

\[ \lim_{n \to \infty} F_{\lambda_{(n,a)}}^2(x, y) = 0, \]

and we get the result.
Remark 6. The map \( h_a(x) = x + \xi + a - K_{\omega} \ln \Phi_2(x, 0) = F_{\lambda(n,a)}^1(x, 0) \) has finitely many nondegenerate critical points (by Hypothesis (P8)).

8.4. Verification of the hypotheses of the theory of rank-one maps. We show that the family of flow-induced maps \( F_{\lambda(n,a)}^1 \equiv F_{(a,\lambda(n,a))}, a \in S^1 \) and \( n \geq n_0 \), satisfies Hypotheses (H1)–(H7) stated in Subsection 5.2. Since our starting point is an attracting network for \( \lambda = 0 \) (cf. Lemma 3.1), the absorbing sets defined in Subsection 2.4 of [33] are automatic here.

(H1): The first two items are immediate. We establish the distortion bound (H1)(3) by studying local maps and global maps separately. Direct computation using (10) and (12), implies that for every \( \lambda \in (0, \lambda_1) \) and \((x, y) \in \text{Out}(O_2)\), one gets:

\[
D\Psi_{2\rightarrow 1}^{\lambda}(x, y) = \begin{pmatrix}
1 + \lambda \frac{\partial \Phi_1(x, y)}{\partial x} & \lambda \frac{\partial \Phi_1(x, y)}{\partial y} \\
\lambda \frac{\partial \Phi_2(x, y)}{\partial x} & 1 + \lambda \frac{\partial \Phi_2(x, y)}{\partial y}
\end{pmatrix}
\]

and

\[
D\eta(x, y) = \begin{pmatrix}
1 - \frac{K_{\omega}}{y} \\
0 & \delta y^{\delta - 1}
\end{pmatrix}
\]

Since \( F_\lambda = \eta \circ \Psi_{2\rightarrow 1}^{\lambda}, \) taking \((X, Y) = \Psi_{2\rightarrow 1}^{\lambda}(x, y)\), we may write:

\[
|\det D\mathcal{F}_\lambda(x, y)| = |\det D\eta(X, Y)| \times |\det D\Psi_{2\rightarrow 1}^{\lambda}(x, y)|
\]

\[
= |\delta y^{\delta - 1}| \times \left| \left(1 + \lambda \frac{\partial \Phi_1(x, y)}{\partial x}\right) \left(1 + \lambda \frac{\partial \Phi_2(x, y)}{\partial y}\right) - \lambda \frac{\partial \Phi_2(x, y)}{\partial x} \frac{\partial \Phi_1(x, y)}{\partial y} \right|.
\]

Since \( \delta y^{\delta - 1} > 0 \) (by (P8)), we conclude that there exists \( \lambda_2 > 0 \) small enough such that:

\[
\forall \lambda \in [0, \lambda_2], \quad |\det D\mathcal{F}_\lambda(x, y)| \in [k_1^{-1}, k_1],
\]

for some \( k_1 > 1 \). This implies that hypothesis (H1)(3) is satisfied.

(H2) and (H3): These items follow from Lemma 8.1 where \( b = \lambda_{(n,a)} \).

The next proposition generalizes the results of Subsections 5.2 and 5.3 of [33] and will be used in the sequel.

Proposition 1 ([33, 34], adapted). Let \( A : S^1 \rightarrow \mathbb{R} \) be a \( C^3 \) function with non-degenerate critical points. Then there exist \( L_1 \) and \( \delta \) depending on \( A \) such that if \( L \geq L_1 \) and \( B : S^1 \rightarrow \mathbb{R} \) is a \( C^3 \) function with \( \|A\|_{C^3} \leq \delta \) and \( \|B\|_{C^0} \leq 1 \), then the family of maps

\[
\theta \mapsto \theta + a + L(A(\theta) + B(\theta)), \quad a \in [0, 2\pi], \quad \theta \in S^1
\]
satisfies \((H4)\) and \((H5)\). If \(L\) is sufficiently large, then Hypothesis \((H7)\) is also verified.

\((H4)\) and \((H5)\): These hypotheses are connected with the family of circle maps
\[ h_a : S^1 \to S^1 \]
defined in Remark (6). Taking into account the Proposition 1, the family
\[ h_a(x) = x + \xi + a + K\omega \log(\Phi_2(x,0)) \quad a \in [0, 2\pi] \]
satisfies Properties \((H4)\) and \((H5)\).

\((H6)\): The computation follows from direct computation using the expression of \(F_1^\lambda(x,0)\). Indeed, for each \(x \in C_{a^*}\) (set of critical points of the Misiurewicz-type map \(h_{a^*}\) whose existence is ensured in \((H4)\)), we have
\[ \frac{d}{dy} F_1^{a^*,0}(x,y)|_{y=0} = 1 + K\omega \frac{1 + \frac{\partial \Phi_2(x,y)}{\partial y}|_{y=0}}{1 + \Phi_2(x,y)} \neq 0, \]
as a consequence of \((P8)\).

\((H7)\): It follows from Proposition 1 if \(K\omega\) is large enough \((\Rightarrow \) the “big lobe” of \([30]\) is large enough).

Since the family \(F_{(a,b)}\) for \(b = \lambda_{(n,a)}\) satisfies \((H1)\)--\((H7)\) of \([33]\) (revisited in Theorem 5.2) then, for \(\lambda_0 = \min\{\lambda_1, \lambda_2\} > 0\), for \(K\omega > K_0\omega\); there exists a subset \(\Delta \in [0, \lambda_0]\) with positive Lebesgue measure such that for \(\lambda \in \Delta\), the map \(F_\lambda\) admits a strange attractor
\[ \Omega \subset \bigcap_{m=0}^{+\infty} F_\lambda^m(D) \tag{18} \]
that supports a unique ergodic SRB measure \(\nu\). The orbit of Lebesgue almost all points in \(\Omega\) has positive Lyapunov exponent and is asymptotically distributed according to \(\nu\).

Remark 7. From the reasoning of Section 3 of \([31]\), we also have:
\[ \liminf_{r \to 0^+} \frac{\text{Leb}\{\lambda \in [0, r] : F_\lambda \text{ has a strange attractor with a SRB measure}\}}{r} > 0. \tag{19} \]
where \(\text{Leb}\) denotes the one-dimensional Lebesgue measure. This means that the existence of strange attractors for \(F_\lambda\) is an abundant phenomenon in the terminology of \([20]\).

Remark 8. Theorem B remains valid if we take into account the high order terms of \((7)\) and \((9)\). The proof’s arguments run along the same lines to those of \([31]\).

8.5. Ergodic remark. For the sake of completeness, we decided to add this statistical remark on the article based on the results of \([37]\). The dynamical system \((F_\lambda, \nu)\) has exponential decay of correlations for Hölder continuous observables: given an Hölder exponent \(\eta\), there exists \(\tau_\eta < 1\) such that for all Holder maps \(\varphi, \psi : \Omega \to \mathbb{R}\) with Hölder exponent \(\eta\), there exists \(K(\varphi, \psi)\) such that for all \(m \in \mathbb{N}\), we have:
\[ \left| \int (\varphi \circ F_\lambda^m)\psi d\nu - \int \varphi d\nu \int \psi d\nu \right| \leq K(\varphi, \psi)\tau^m. \]
A further analysis is outside the scope of the present work.
9. **Proof of Theorem 4.3.** In this section, we prove the existence of a superstable sink for a sequence of $\lambda$-values for the one-parameter family (4). For $q \in \mathbb{N}$ and $a \in [0, 2\pi]$, let $C(h_a) = \{c^{(1)}, ..., c^{(q)}\}$ be the set of critical points of $h_a$ defined in Remark 6. As the result of the rank-one theory, for $\lambda$ small, all points lying in $\Omega$ (see (18)) have at least one contracting direction.

Let $a^* \in [0, 2\pi]$ be such that the limit family $h_{a^*}$ is a Misiurewicz-type map (see Subsection 8.4). Therefore, from Subsection 5.1, we may define the numbers:

- $\delta_0$: size of the critical set of $h_{a^*}$;
- $\lambda_0$: logarithmic expansion of the orbits outside the critical set;
- $b_0$: pre-factor associated to the exponential growth outside the critical set.

For fixed $\lambda < \lambda_0/5$ and $\alpha > 0$ small, let $\Delta(\lambda, \alpha)$ be the set of $a \in [0, 2\pi]$ for which the following conditions hold for $c \in C(h_a)$ and $n \in \mathbb{N}$:

- **(CE1):** $\text{dist}(h_a^n(c), C(h_a)) \geq \min\{\delta_0/2, 2 \exp(-\alpha n)\}$;
- **(CE2):** $|\left(h_a^n\right)'(h_a(c))| \geq 2b_0\delta_0 \exp(\lambda n)$.

These assertions are usually called by $(\lambda, \alpha)$ Collet-Eckmann conditions. The next technical result says that $a^*$ is a Lebesgue density point of $\Delta(\lambda, \alpha)$. This means that, under precise conditions on $\alpha$ and $\lambda$, we may easily find elements $\hat{a} \in [0, 2\pi]$ for which the $(\lambda, \alpha)$ Collet-Eckmann conditions are verified.

**Lemma 9.1** ([35], adapted). For $a^* \in [0, 2\pi]$ as above, the following equality holds for fixed $\lambda < \lambda_0/5$ and $\alpha > 0$ small:

$$\liminf_{r \to 0^+} \frac{\text{Leb}\{\Delta(\lambda, \alpha) \cap [a^*, a^* + r]\}}{r} = 1.$$
To prove Theorem 4.3, we make use of the following result:

**Proposition 2** ([23], adapted). Let \( \{h_a : a \in [0, 2\pi]\} \) be a family of maps as above such that:

1. there exists \( a^* \) such that \( h_{a^*} \) is a Misiurewicz-type map;
2. \( [0, 2\pi] \subset h_{a^*}([c(i), c(i+1)]) \), for all \( i \in \{1, \ldots, q\} \);
3. \( \exp(\lambda_0) > \ln 10 \).

Hence, for every \( \alpha > 0 \) sufficiently small and every \( \hat{a} \in \Delta(\lambda, \alpha) \) sufficiently close to \( a^* \), there exists \( (a_n)_{n \in \mathbb{N}} \) converging to \( \hat{a} \) for which \( h_{a_n} \) admits a superstable periodic orbit.

Taking into account the proof of Theorem 4.2, for \( K_\omega > 0 \) large enough, we know that conditions (i) – (iii) of Proposition 2 hold. Therefore, we conclude that for every \( \alpha > 0 \) small and for every \( \hat{a} \in \Delta(\lambda, \alpha) \) close to \( a^* \), there exists a sequence \( (a_n)_{n \in \mathbb{N}} \) converging to \( \hat{a} \) for which \( h_{a_n} \) admits a superstable sink. By (16), for \( \lambda_n = \exp \left( \frac{a_n - 2n\pi}{K_\omega} \right) \).

It is easy to see that \( \lim_{n \to +\infty} \lambda_n = 0 \). For this sequence of values, the flow of \( f_{\lambda_n} \) has a superstable 2-periodic orbit. The way this periodic orbit, which is a critical point for \( h_{a_n} \), is obtained is illustrated in Figure 4.

10. **Mechanism to create rank-one attractors: a geometrical interpretation.** This manuscript serves to make a bridge between the theory of rank-one attractors developed in [32, 33, 34] and the theory of heteroclinic bifurcations involving saddle-foci or periodic solutions [16, 18, 19, 21, 31]. In addition, we exhibit a mechanism to construct rotational horseshoes (see §2.2) and strange attractors. This work may be seen as the natural extension of [25], where a two-parameter family of vector fields has been considered to model bifurcations associated to certain unfoldings of Hopf-zero singularities [4].

In this section, we compare and discuss our results with previous works in the literature. Borrowing the arguments used in [25], we may draw, in the first quadrant, two smooth curves, the graphs of \( t_1 \) and \( t_2 \) drawn in Figure 5, such that:

1. for all \( \omega \in \mathbb{R}^+_0 \), we have \( t_2(\omega) \leq t_1(\omega) \);
2. the region below the graph of \( t_2 \) corresponds to flows having an invariant and attracting torus with zero topological entropy (attracting invariant torus);
3. the region between the graphs of \( t_1 \) and \( t_2 \) corresponds to flows having rotational horseshoes, which might not be observable in numerical simulations;
4. for \( K_\omega \) sufficiently large, the region above the graph of \( t_1 \) correspond to flows exhibiting strange attractors.
10.1. From an invariant curve to a strange attractor. Forgetting temporarily its connection to equation (17), we might think of
\[ h_a(x) = x + \xi + a + K_\omega \ln(\Phi_2(x,0)), \]
\( x \in S^1 \) and \( a \sim -K_\omega \ln \mu \) (mod \( 2\pi \)), as an abstract circle map. The emergence of this map in the first return map to a Bykov network \( \Gamma \) is expected. The rotation forced by the existence of saddle-foci causes a trajectory to be thrown out in all directions in the associated eigenspace on successive approaches to the equilibria [3]. Several types of behaviours are known to be typical for \( h_a \):

(a): If \( h_a \) is a diffeomorphism, the classical theory of Poincaré and Denjoy may be applied. We point out a resemblance between \( h_a \) and the family of circle maps. Because of strong normal contraction, invariant curves are shown to exist independent of rotation number.

(b): If \( h_a \) is non-invertible, in general the rotation number of a given initial condition is not unique and hence an interval of rotation numbers may exist. According to Section 3 of [33], two types of dynamical behaviours are known to be prevalent:

(b1): maps with sinks or

(b2): maps with absolutely continuous invariant measures.

The passage of Cases (a) and (b1) in one dimension to uniform hyperbolicity in two dimensions is relatively simple. The passage of the case (b2) to two dimensions is the core of [32] and has been revisited in Section 5.2 of the present paper.
10.2. Phenomenological description. For \( \lambda, K_\omega \) small enough, the flow of (4) has an attracting hyperbolic torus, which persists under small smooth perturbations. This is consistent with the results stated in [26] about the existence of an attracting curve for the map \( F_\lambda \). The initial deformation of \( \eta \) (see (10)) on the curve \( \text{In}(O_1) \cap \Phi_2(x,0) \) is suppressed by the contracting force, and the attractor is a non-contractible closed curve obtained by applying the Afraimovich’s Annulus Principle.

For a fixed \( \lambda > 0 \), if the twisting number \( K_\omega \) increases, points at \( \text{In}(O_1) \cap \Phi_2(x,0) \) at different distances from \( W^s(O_1) \) rotate at different speeds. The attracting curve starts to disintegrate into a finite collection of periodic saddles and sinks; this phenomenon is part of the torus-breakdown theory and occurs within an Arnold tongue [27]. Once the rotational horseshoes develop, they persist and correspond to what the authors of [33] call transient chaos. The rotational horseshoes found in [25] and the associated invariant manifolds are (in general) not observable in numerical simulations because they form a set of Lesbegue measure zero; almost all solutions will be trapped by a sink.

As the twisting number \( K_\omega \) gets larger, the deformation on the curve \( \text{In}(O_1) \cap \Phi_2(x,0) \) is exaggerated further, giving rise to rank-one attractors created by stretch-and-fold type actions – sustained chaos [33]. Iterates do not escape and wander around the full torus (“large” strange attractors). These two phenomena (stretch-and-fold) are due to the network attracting features combined with the presence of complex eigenvalues, which force a strong distortion.

10.3. Interpretation of Figure 6. The evolution of

\[
h_a(x) = x + \xi + a + K_\omega \ln(\Phi_2(x,0)),
\]

when \( \xi, \lambda > 0 \) are fixed and \( K_\omega \) varies, is suggested in Figure 6. In (A) and (B), we deduce the existence of an invariant curve (a torus in the flow of (4)). The existence of periodic orbits within the torus depends on the rotation number. In case (B) we observe the existence of the two fixed points, suggesting that the chosen parameters are within a resonant tongue. In (C), the map \( h_a \) is not a diffeomorphism meaning that the invariant curve broke up; it corresponds to the point when the unstable manifold of the saddle (in the “Arnold tongue” [27]) turns around. In case (D), Property (H7) holds, meaning that the unstable manifold of a saddle within the “Arnold tongue” crosses each leaf of the stable foliation of the saddles of the torus’s ghost. This corresponds to what the authors of [30] call a big lobe. This non-negative quantity determines the magnitude between the given minimal value \( h_a \) and the preceding maximal one. For \( a > 0 \) fixed, as the twisting number increases, the critical points of \( h_a(c^{(1)}) \) and \( h_a(c^{(2)}) \) move in opposite directions at rate \( \frac{1}{K_\omega} \).

10.4. Concluding remarks. This article studies global bifurcations associated to the one-parameter family \( \dot{x} = f_\lambda(x) \), defined on a 3-dimensional sphere and satisfying Properties (P1)–(P8). For \( \lambda = 0 \), there is an attracting network containing a 2-dimensional connection (continua of solutions [3]) associated to two saddle-foci with different Morse indices, whose invariant manifolds do not intersect for \( \lambda > 0 \). These 2-dimensional manifolds are forced to fold uniformly within the network’s basin of attraction.

The novelty of this article is that, assuming that the twisting number \( K_\omega \) is large enough, rank-one attractors arise and wind around the “ghost” of a 2-dimensional
torus whose existence is stated in Theorem B of [25]. The 2-dimensional invariant manifold $W^u(O_2)$ plays the essential role in the construction of the global attractor where the SRB measure is supported. The strange attractors in the present
paper are nonuniformly hyperbolic and structurally unstable (they are not \textit{robustly transitive} as the geometric Lorenz model).

This work is the natural continuation of \cite{25}. Using the powerful theory of Wang and Young, we formulate checkable hypotheses (in terms of the vector field \eqref{4}) under which the first return map to global cross section admits a strange attractor and superstable periodic orbits. The chaos is observable (it has a strange attractor), “large” (is not confined to a small portion of the phase space) and abundant (condition \eqref{19} holds in the space of parameters). As far as we know, it is the first time that this theory is applied to an autonomous family.

Results in this manuscript go further in the analysis of unfoldings of the heteroclinic attractors. These families may behave periodically, quasi-periodically or chaotically, depending on specific character of the perturbation. Techniques used in the present article are valid for more general heteroclinic networks containing 2-dimensional connections that are pulled apart an also in the context of periodically perturbed networks. The study of the ergodic consequences of this dynamical scenario is the natural continuation of the present work.

Acknowledgments. The author is grateful to Isabel Labouriau for the fruitful discussions. The author is indebted to the reviewer for the corrections and suggestions which helped to improve the readability of this manuscript.

REFERENCES

[1] V. S. Afraimovich, S.-B. Hsu and H. E. Lin, Chaotic behavior of three competing species of May–Leonard model under small periodic perturbations, \textit{Internat. J. Bifur. Chaos Appl. Sci. Engrg.}, 11 (2001), 435–447.
[2] M. Aguiar, Vector fields with heteroclinic networks, \textit{Ph.D. thesis, Departamento de Matemática Aplicada}, Faculdade de Ciências da Universidade do Porto, 2003.
[3] P. Ashwin and P. Chossat, Attractors for robust heteroclinic cycles with continua of connections, \textit{J. Nonlinear Sci.}, 8 (1998), 103–129.
[4] I. Baldomá, S. Ibáñez and T. Seara, Hopf-Zero singularities truly unfold chaos, Commun. Nonlinear Sci. Numer. Simul., 84 (2020), 105162.
[5] M. Benedicks and L. Carleson, The dynamics of the Hénon map, \textit{Ann. of Math. (2)}, 133 (1991), 73–169.
[6] M. Benedicks and L.-S. Young, Sinai-Bowen-Ruelle measures for certain Hénon maps, \textit{Invent. Math.}, 112 (1993), 541–576.
[7] H. Broer, C. Simó and J. C. Tatjer, Towards global models near homoclinic tangencies of dissipative diffeomorphisms, \textit{Nonlinearity}, 11 (1998), 667–770.
[8] V. V. Bykov, Orbit Structure in a neighborhood of a separatrix cycle containing two saddle-foci, \textit{Translations of the American Mathematical Society - Series 2}, 200 (2000), 87–97.
[9] M. L. Castro and A. A. P. Rodrigues, Torus-breakdown near a heteroclinic attractor: A case study, \textit{Internat. J. Bifur. Chaos Appl. Sci. Engrg.}, 31 (2021), 2130029.
[10] B. Deng, The shilnikov problem, exponential expansion, strong $\lambda$-lemma, $C^1$ linearisation and homoclinic bifurcation, \textit{J. Diff. Eqs.}, 79 (1989), 189–231.
[11] J. Guckenheimer and P. Holmes, \textit{Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields}, Applied Mathematical Sciences, 42, Springer-Verlag, New York, 1990.
[12] M. Hénon, A two dimensional mapping with a strange attractor, \textit{Comm. Math. Phys.}, 50 (1976), 69–77.
[13] A. J. Homburg, Periodic attractors, strange attractors and hyperbolic dynamics near homoclinic orbits to saddle-focus equilibria, \textit{Nonlinearity}, 15 (2002), 1029–1050.
[14] A. J. Homburg and B. Sandstede, Homoclinic and heteroclinic bifurcations in vector fields, \textit{Handbook of Dynamical Systems}, 3 (2010), 379–524.
[15] M. Jakobson, Absolutely continuous invariant measures for one parameter families of one-dimensional maps, \textit{Comm. Math. Phys.}, 81 (1981), 39–88.
[16] I. S. Labouriau and A. A. P. Rodrigues, Global generic dynamics close to symmetry, \textit{J. Diff. Eqs.}, 253 (2012), 2527–2557.
[17] I. S. Labouriau and A. A. P. Rodrigues, Dense heteroclinic tangencies near a Bykov cycle, *J. Diff. Eqs.*, **259** (2015), 5875–5902.

[18] I. S. Labouriau and A. A. P. Rodrigues, Global bifurcations close to symmetry, *J. Math. Anal. Appl.*, **444** (2016), 648–671.

[19] A. Mohapatra and W. Ott, Homoclinic loops, heteroclinic cycles, and rank one dynamics, *SIAM J. Appl. Dyn. Syst.*, **14** (2015), 107–131.

[20] L. Mora and M. Viana, Abundance of strange attractors, *Acta Math.*, **171** (1993), 1–71.

[21] W. Ott, Strange attractors in periodically-kicked degenerate Hopf bifurcations, *Comm. Math. Phys.*, **281** (2008), 775–791.

[22] W. Ott and M. Stenlund, From limit cycles to strange attractors, *Comm. Math. Phys.*, **296** (2010), 215–249.

[23] W. Ott and Q. Wang, Periodic attractors versus nonuniform expansion in singular limits of families of rank one maps, *Discrete Contin. Dyn. Syst.*, **26** (2010), 1035–1054.

[24] A. A. P. Rodrigues, Repelling dynamics near a Bykov cycle, *J. Dynam. Differential Equations*, **25** (2013), 605–625.

[25] A. A. P. Rodrigues, Unfolding a Bykov attractor: From an attracting torus to strange attractors, *J. Dynam. Differential Equations*, 2020.

[26] A. A. P. Rodrigues, Abundance of strange attractors near an attracting periodically perturbed network, *SIAM J. Appl. Dyn. Syst.*, **20** (2021), 541–570.

[27] A. A. P. Rodrigues, Dissecting a resonance wedge on heteroclinic bifurcations, *J. Stat. Phys.*, **184** (2021), Paper No. 25, 32 pp.

[28] D. Ruelle and F. Takens, On the nature of turbulence, *Commun. Math. Phys.*, **20** (1971), 167–192.

[29] A. Shilnikov, G. Nicolis and C. Nicolis, Bifurcation and predictability analysis of a low-order atmospheric circulation model, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **5** (1995), 1701–1711.

[30] A. Shilnikov, L. Shilnikov and D. Turaev, On some mathematical topics in classical synchronization: A tutorial, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **14** (2004), 2143–2160.

[31] Q. Wang and W. Ott, Dissipative homoclinic loops of two-dimensional maps and strange attractors with one direction of instability, *Comm. Pure Appl. Math.*, **64** (2011), 1439–1496.

[32] Q. Wang and L.-S. Young, Strange attractors with one direction of instability, *Commun. Math. Phys.*, **218** (2001), 1–97.

[33] Q. Wang and L.-S. Young, From invariant curves to strange attractors, *Commun. Math. Phys.*, **225** (2002), 275–304.

[34] Q. Wang and L.-S. Young, Strange attractors in periodically-kicked limit cycles and Hopf bifurcations, *Commun. Math. Phys.*, **240** (2003), 509–529.

[35] Q. Wang and L.-S. Young, Nonuniformly expanding 1D maps, *Commun. Math. Phys.*, **264** (2006), 255–282.

[36] Q. Wang and L.-S. Young, Toward a theory of rank one attractors, *Ann. of Math.(2)*, **167** (2008), 349–480.

[37] Q. Wang and L.-S. Young, Dynamical profile of a class of rank-one attractors, *Ergodic Theory Dynam. Systems*, **33** (2013), 1221–1264.

[38] L.-S. Young, Statistical properties of dynamical systems with some hyperbolicity, *Ann. Math. (2)*, **147** (1998), 585–650.

Received March 2021; revised October 2021; early access December 2021.

*E-mail address: alexandre.rodrigues@fc.up.pt*