Stability of asymptotically de Sitter and anti-de Sitter black holes in 4D regularized Einstein-Gauss-Bonnet theory

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The regularized four-dimensional Einstein-Gauss-Bonnet model has been recently proposed in [D. Glavan and C. Lin, Phys. Rev. Lett. 124, 081301 (2020)] whose formulation is different of the Einstein theory, allowing us to bypass the Lovelock theorem. The action is formulated in higher dimensions ($D > 4$) by adding the Gauss-Bonnet correction to the conventional Einstein-Hilbert action with a cosmological constant. The four-dimensional spacetime is constructed through dimensional regularization by taking the limit $D \to 4$. We find explicitly the parametric regions of stability of black holes for the asymptotically flat and (anti-)de Sitter spacetimes by analyzing the time-domain profiles for gravitational perturbations in both vector and scalar channels. In addition to the known eikonal instability we find the instability due to the positive cosmological constant. On the contrary, asymptotically anti-de Sitter black holes have no other instability than the eikonal one.

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I. INTRODUCTION

One of the most promising alternative theory of gravity is the higher-dimensional ($D > 4$) Lovelock theory [1] which is constructed by adding higher-curvature corrections to the Einstein action. Due to the Lovelock theorem the model is reduced to Einstein theory in $D = 4$. In five or six dimensions, there is an additional term in the action, called the Gauss-Bonnet term, which is quadratic in curvature and can be interpreted as a quantum correction in the low energy limit of the heterotic string theory. For higher-dimensional spacetimes, higher order corrections appear.

Stability of black holes against small perturbations is a crucial condition for viability of a black-hole model [2]. Linear black hole perturbations can be formulated as a superposition of their modes labeled by the multiple number $\ell$. When higher growing rate modes appear for larger multiple number $\ell$, the threshold of instability is called the eikonal instability emphasizing the fact that it happens in the regime of geometrical optics. In the general Lovelock theory, there were found instabilities of gravitational perturbations for sufficiently small black holes [3–9]. For the Einstein-Gauss-Bonnet black holes in addition to the eikonal instability, which appears in the asymptotically flat case [10] [11], there is an instability, which develops at lower multipoles when the cosmological constant is sufficiently large [12], called, therefore, the $\Lambda$-instability.

Furthermore, the Lovelock theorem states that the four-dimensional Einstein tensor with the cosmological constant forms the unique combination which is divergence-free, symmetric and second-order in its equation of motion [13]. Recently it has been proposed a way to circumvent the Lovelock theorem and avoid the Ostrogradsky instability [14]. Namely, we define a four-dimensional spacetime in the limit $D \to 4$ of higher-dimensional Lovelock theory, after the re-scaling of the Gauss-Bonnet coupling constant $\alpha_2 \to 2\alpha_2/(D-4)$ in the Lagrangian. The further generalization, which includes the higher-curvature correction due to the Einstein-Lovelock theory, has been proposed in [15] [16]. Some properties of black holes in the asymptotically flat $4D$ regularized Einstein-Gauss-Bonnet theory, such as instability, quasinormal modes and shadows have been studied in [17, 18]. Thermodynamics of the $4D$ Einstein-Gauss-Bonnet-anti-de Sitter black holes with electric charge was discussed in [19] and quasinormal modes of the neutral asymptotically AdS black hole were considered in [20]. Parametric region of the eikonal instability for $4D$ Einstein-Gauss-Bonnet and Einstein-Lovelock black holes was obtained in [21]. Some further properties, such as the innermost stable orbit, axial symmetry, Hawking radiation and thermodynamics were studied in [22] [27].

In the present paper we find the parametric region of linear stability for black holes in the asymptotically de Sitter and anti-de Sitter spacetimes, by systematically studying perturbation profiles in the time domain. For the stable black holes we also calculate the dominant quasinormal modes of the scalar-type (polar) and vector-type (axial) gravitational perturbations.

The paper is organized as follows. In Sec. II we present a general framework of a spherically symmetric, static black hole solution within the $4D$ regularized Einstein-Gauss-Bonnet model and its gravitational perturbations in the scalar and vector sectors. In Sec II we present the numerical method, which was used to obtain the temporal profiles for the gravitational perturbations of the black hole. In section IV we are discussing the parametric region of stability for the $4D$ regularized Einstein-Gauss-Bonnet-(anti)de Sitter black holes. Finally, in the conclusion we summarize the results and give a brief outlook of open problems.
II. THEORETICAL FRAMEWORK

A. Static black hole solution

The Lovelock theory formulated in \[1\], is given by the Lagrangian density

\[ \mathcal{L} = -2\lambda + \sum_{m=1}^{\infty} \frac{\alpha_m}{m} \mathcal{L}_m, \]  

(1)

where \( \lambda \) is the cosmological constant, \( \mathcal{L}_m \) are the Lovelock terms and \( \alpha_m \) are arbitrary parameters of the theory, such that \( 2m < D \). For \( D = 4 \) we consider \( m = 1 \) in the summation, for \( D = 5 \) or \( 6 \) we have also the second order in curvature, \( m = 2 \). Higher than six dimensions can also contain higher-order Lovelock terms. Following \[16\], we contain higher-order Lovelock terms. Following \[16\], we define the parameters \( \bar{\alpha}_m \) such as

\[ \bar{\alpha}_m = \frac{\alpha_m}{m} \prod_{k=1}^{m-2} (D - 2 - k). \]  

(2)

In the limit \( D \to 4 \), the Gauss-Bonnet coupling constant is

\[ \alpha_2 \to \frac{2\bar{\alpha}_2}{(D - 4)}, \]  

(3)

so that in order to regularize the four dimensional Einstein-Lovelock theory, we shall consider finite value of \( \bar{\alpha}_2 \). Such approach has been proposed in \[14\] although initially suggested by Y. Tomozawa in \[28\]. We limit our consideration by the second order in curvature, i.e., consider only the Gauss-Bonnet correction, then the Lagrangian density can be written as

\[ \mathcal{L} = -2\lambda + R + \alpha_2 \mathcal{L}_2 \]  

(4)

\[ = -2\lambda + R + \alpha_2 \left( R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \right). \]

In general, the coupling constant \( \alpha_2 \) is a real parameter bounded by some values of the theory. Despite \( D \to 4 \) has no Lagrangian formalism, (the Lagrangian diverges), the second order equations for the metric and consequently the static black-hole solution are well defined, and the metric can be described by the following line element,

\[ ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \]

\[ f(r) = 1 - r^2 \psi(r). \]  

(5)

The radial function \( \psi(r) \) satisfies the following relation,

\[ P[\psi(r)] = \psi(r) + \frac{\bar{\alpha}_2}{2} \psi(r)^2 = \frac{2M}{r^3} + \frac{\lambda}{3}, \]  

(6)

which could be obtained after substituting the metric \[5\] \[29\] into the vacuum Einstein-Gauss-Bonnet equation and considering the limit \( D \to 4 \). The constant of integration \( M \) is the asymptotic mass. Furthermore, from \[6\] we can see that the solution has two branches, and one of them is perturbative in \( \alpha_2 \),

\[ f(r) = 1 - \frac{1}{r^3} \left( -1 + \sqrt{1 + 4\bar{\alpha}_2 \left( \frac{2M}{r^3} + \frac{\lambda}{3} \right)} \right), \]  

(7)

while the other solution diverges as \( \alpha_2 \to 0 \). Measuring all quantities in units of the event horizon radius \( r_H \), we find the mass

\[ 2M = r_H \left( 1 + \frac{\bar{\alpha}_2}{r_H^2} - \frac{\Lambda r_H^3}{3} \right). \]  

(8)

In the de Sitter spacetime (\( \Lambda > 0 \)) with the cosmological horizon \( r_C \), the cosmological constant reads as

\[ \frac{\Lambda r_C^3}{3} = \frac{r_H^3}{r_H^3 + r_C^3 H^2} \left( 1 - \frac{\bar{\alpha}_2}{r_H^2 r_C} \right). \]  

(9)

For asymptotically Anti-de Sitter spacetime (\( \Lambda > 0 \)), it is convenient to introduce the AdS-radius \( R \), which is defined via the asymptotic of the metric function \( f(r) \to 1 + r^2/R^2 \) at infinity. The cosmological constant measure in units of \( R \) is

\[ \frac{\Lambda R^2}{3} = -1 + \frac{\bar{\alpha}_2}{R^2}. \]  

(10)

B. Gravitational perturbation

After perturbing the background metric \( \delta g_{\mu\nu} \),

\[ g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \quad |\delta g_{\mu\nu}| \ll |g_{\mu\nu}|, \]  

(11)

the equations which govern the propagation of such perturbation can be reduced to differential wave-like equations,

\[ \left( -\frac{\partial^2}{\partial r_*^2} + \frac{\partial^2}{\partial T^2} - V(t, r_*) \right) \Psi(t, r_*) = 0 \]  

(12)

where \( r_* \) is the tortoise coordinate \( dr_* = dr/f(r) \) and \( V \) is the effective potential, where \( i \) stands for \( t \) (tensor), \( v \) (vector) and \( s \) (scalar). In the linear approximation, these kinds of perturbations can be treated independently owing to the transformation law of the rotation group on a \((D-2)\)-sphere. In the limit \( D \to 4 \), the tensor-type perturbation becomes purely gauge transformations, possessing no additional degrees of freedom. The effective potentials for the vector- and scalar-type perturbations were determined in \[7\] and can be expressed as

\[ V_v(r) = \frac{(\ell - 1)(\ell + 2)}{r T(r)} \frac{d T(r)}{dr_*} + R(r) \frac{d^2}{dr_*^2} \left( \frac{1}{R(r)} \right), \]  

(13)

\[ V_s(r) = \frac{\ell (\ell + 1)}{r^2 B(r)} \frac{d}{dr_*} \left( r B(r) \right) + B(r) \frac{d^2}{dr_*^2} \left( \frac{1}{B(r)} \right). \]  

(14)
where $\ell = 2, 3, 4, \ldots$ is the multiple number and

\[
T(r) = r (1 + 2\tilde{\alpha} g(r)),
\]
\[
R(r) = r \sqrt{T(r)},
\]
\[
B(r) = \frac{2(\ell - 1)(\ell + 2) - 2r^3\psi'}{R(r)}.
\]

III. CHARACTERISTIC INTEGRATION

In order to catch the threshold of instability we used the characteristic method proposed by Gundlach, Price, and Pullin in [10], which allows us to determine the temporal profiles of black holes for any given set of parameters. This method consists in a discretization of a certain region of the spacetime near the black hole, defined by two null surfaces $(u_0, v_0)$ (as we can see in figure 1) and calculation of the magnitude of the wave-function $|\Psi|$ on each point of the grid. Rewriting (12) in terms of the light-cone coordinates $du = dt - dr^*$ and $dv = dt + dr^*$, one obtains

\[
4 \frac{\partial^2 \Psi}{\partial u \partial v} = -V_i (v - u) \Psi,
\]

then, the discretization scheme has the following form

\[
\Phi(N) = \Phi(W) + \Phi(E) - \Phi(S)
\]
\[
-\frac{\Delta^2}{8} V(S) [\Phi(W) + \Phi(E)] + \mathcal{O}(\Delta^4),
\]

IV. PARAMETRIC REGION OF STABILITY

We obtain the threshold of instability for $\ell = 2$ by analysing the temporal profiles and looking for an unboundedly growing perturbations. We have observed such behavior in the scalar sector only, while the perturbations of vector type remained stable.

The overlap between the region of eikonal instability [21] and the $\Lambda$-instability (for $\ell = 2$) is shown in the figure 3. When the cosmological constant is small, black holes are unstable for sufficiently large value of $\tilde{\alpha}$-coupling parameter, and the threshold value is dominated by the eikonal instability, i.e., there exists a finite critical value $\tilde{\alpha}_{\text{crit}}$ for a given cosmological constant $\Lambda$, such that for $\tilde{\alpha} > \tilde{\alpha}_{\text{crit}}$ black-hole perturbations are unstable at sufficiently large $\ell$. The reason for this phenomenon is that, as $\ell$ grows, the negative gap in the effective potential becomes deeper as shown on Fig. 3. The increasing of the cosmological constant $\Lambda$ leads to increasing of $\tilde{\alpha}_{\text{crit}}$ until
FIG. 3. Complete parametric region of instability for 4-dimensional Gauss-Bonnet black hole. The dotted blue points correspond to the $\Lambda$-instability and the soft curve is the eikonal instability.

$$\tilde{\alpha}/r_H^2$$

FIG. 4. Effective potential to scalar-type gravitational perturbations for $r_H/r_C = 75/100$, $\tilde{\alpha}_2 = 0.4r_H^2$ and $\ell = 2(\ell = 10)$ for green(magenta) curve.

FIG. 5. Temporal profiles for scalar perturbation close to the merge of eikonal and $\Lambda$-instability at $r_H/r_C = 0.75$, $\tilde{\alpha}_2 = 0.125r_H^2$. The blue, orange, magenta and green profiles correspond to $\ell = 2, 3, 4$ and 20 respectively.

TABLE I. Critical values of $\tilde{\alpha}_2$ corresponding to the $\Lambda$-instability for scalar-type gravitational perturbation

| $r_H/r_c$ | $\tilde{\alpha}/r_H^2$ | $r_H/r_c$ | $\tilde{\alpha}/r_H^2$ |
|-----------|-----------------|-----------|-----------------|
| 0         | 0.131           | 6/10      | 0.127           |
| 1/20      | 0.131           | 7/10      | 0.124           |
| 1/10      | 0.131           | 8/10      | 0.120           |
| 1/5       | 0.131           | 9/10      | 0.115           |
| 3/10      | 0.130           | 95/100    | 0.110           |
| 4/10      | 0.129           | 99/100    | 0.106           |
| 5/10      | 0.128           | 999/1000  | 0.105           |

the perturbations of $\ell = 2$ become unstable. One can see that, when $r_H/r_C \lesssim 0.75$, the instability region is defined by $\ell = 2$ ($\Lambda$-instability), so that at the threshold the perturbations, corresponding to higher values of the multipole number are more stable. A similar behavior was observed for the five-dimensional Einstein-Gauss-Bonnet black hole [12].

As we can see in Fig. 3 close to $r_H/r_C \approx 0.75$, both threshold (of eikonal- and $\Lambda$-instability) curves cross. In this region, for $\tilde{\alpha} > \tilde{\alpha}_{crit}$ we observe both types of instability: the Fig. 5 shows that, when $\tilde{\alpha}_2 = 0.125r_H^2$, the time-domain profile for $\ell = 2$ shows unbounded growth while the perturbations for $2 < \ell < 16$ are stable, and for larger values of $\ell$ they are unstable.

In Table I we present the dominant quasinormal modes of gravitational perturbations for both scalar and vector channels in the region of stability. The quasinormal modes were obtained directly by fitting the time-domain data by a superposition of damping exponents, using the Prony method.

We have also tested stability of the asymptotically anti-de Sitter black holes. We did not find an indication of instability in any channel in the parametric region, where the black holes do not show the eikonal instability [21]. We conclude, therefore, that the anti-de Sitter black hole are stable for

$$-1 < \frac{2\tilde{\alpha}_2}{r_H^2} < \sqrt{6\sqrt{3} - 9 + \sigma^2 + 2\sigma - \sigma - 1},$$

where

$$\sigma = -(2\sqrt{3} - 3)\Lambda r_H^2.$$

V. CONCLUSION

While quasinormal modes and stability of the higher dimensional Einstein-Gauss-Bonnet [31–34], or four dimensional Einstein-dilaton-Gauss-Bonnet black holes [35–38] are relatively well, though not completely, studied, the analysis of stability and spectra of the novel $4D$
Einstein-Gauss-Bonnet black holes is limited by only a few studies, where only the eikonal type of instability was discussed \cite{17, 21}.

Here we have obtained the full parametric region of stability for the four-dimensional Einstein-Gauss-Bonnet (anti)de Sitter black holes. For the sufficiently large $\Lambda$, in addition to the eikonal instability, we have observed $A$-instability, expanding the instability region perviously reported in \cite{17, 21}. We did not find indications of $A$-instability for the asymptotically anti-de Sitter black holes, when they do not suffer from the eikonal instability.

| $r_H/r_C$ | $\tilde{\alpha}_2/r_H^2 = -0.2$ | $\tilde{\alpha}_2/r_H^2 = -0.1$ | $\tilde{\alpha}_2/r_H^2 = 0.05$ | $\tilde{\alpha}_2/r_H^2 = -0.2$ | $\tilde{\alpha}_2/r_H^2 = -0.1$ | $\tilde{\alpha}_2/r_H^2 = 0.05$ |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0        | 1.0282-0.2615i  | 0.8588-0.1987i  | 0.7076-0.1724i  | 1.3717-0.3608i  | 1.0055-0.2486i  | 0.6509-0.1675i  |
| 0.1      | 1.0159-0.2591i  | 0.8446-0.1967i  | 0.6904-0.1689i  | 1.3519-0.3532i  | 0.9888-0.2346i  | 0.6340-0.1646i  |
| 0.2      | 0.9817-0.2519i  | 0.8065-0.1907i  | 0.6455-0.1597i  | 1.2980-0.3329i  | 0.9439-0.2305i  | 0.5903-0.1568i  |
| 0.3      | 0.9282-0.2398i  | 0.7495-0.1807i  | 0.5816-0.4411i  | 1.2166-0.3039i  | 0.8768-0.2113i  | 0.5281-0.1441i  |
| 0.4      | 0.8563-0.2224i  | 0.6769-0.1663i  | 0.5056-0.1282i  | 1.1122-0.2690i  | 0.7918-0.1879i  | 0.4545-0.1280i  |
| 0.5      | 0.7664-0.1995i  | 0.5912-0.1478i  | 0.4227-0.1082i  | 0.9871-0.2303i  | 0.6920-0.1614i  | 0.3750-0.1086i  |
| 0.6      | 0.6581-0.1741i  | 0.4939-0.1251i  | 0.3366-0.0867i  | 0.8422-0.1893i  | 0.5792-0.1326i  | 0.2936-0.0873i  |
| 0.7      | 0.5304-0.1379i  | 0.3860-0.0987i  | 0.2499-0.0645i  | 0.6761-0.1465i  | 0.4541-0.1021i  | 0.2135-0.0650i  |
| 0.75     | 0.4587-0.1192i  | 0.3282-0.0842i  | 0.2069-0.0534i  | 0.5842-0.1244i  | 0.3870-0.0862i  | 0.1746-0.0538i  |
| 0.8      | 0.3812-0.0988i  | 0.2679-0.0689i  | 0.1643-0.0424i  | 0.4855-0.1017i  | 0.3167-0.0700i  | 0.1367-0.0427i  |
| 0.9      | 0.2066-0.0534i  | 0.1394-0.0359i  | 0.0807-0.0208i  | 0.2639-0.0538i  | 0.1660-0.0361i  | 0.0652-0.0209i  |

TABLE II. Fundamental quasinormal modes for vector- and scalar-type perturbations for $\ell = 2$.

In order to obtain the complete picture of the stability region, it would be interesting to extend out our analysis to the 4D Einstein-Gauss-Bonnet black holes with electric charge \cite{39}.

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