From Elastica to Floating Bodies of Equilibrium

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Abstract

A short historical account of the curves related to the two-dimensional floating bodies of equilibrium and the bicycle problem is given. Bor, Levi, Perline and Tabachnikov found, quite a number had already been described as Elastica by Bernoulli and Euler and as Elastica under Pressure or Buckled Rings by Levy and Halphen. Auerbach already realized that Zindler had described curves for the floating bodies problem. An even larger class of curves solves the bicycle problem.

The subsequent sections deal with some supplemental details: Several derivations of the equations for the elastica and elastica under pressure are given. Properties of Zindler curves and some work on the problem of floating bodies of equilibrium by other mathematicians are considered. Special cases of elastica under pressure reduce to algebraic curves, as shown by Greenhill. Since most of the curves considered here are bicycle curves, a few remarks concerning them are added.

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1 Introduction

Miklós Rédei introduces in his paper "On the tension between mathematics and physics" [37] the "supermarket picture" of the relation of mathematics and physics: that mathematics is like a supermarket and physics its customer.

When in 1938 Auerbach solved the problem of floating bodies of equilibrium [2] for the density $\rho = 1/2$ he could go to the supermarket 'Mathematics' and found the solution in form of the Zindler curves [56].

When I thought about this problem for densities $\rho \neq 1/2$ I found a differential equation, went to the supermarket and found elliptic functions as ingredients for the solution. But in the huge supermarket I did not find the finished product. Later Bor, Levi, Perline, and Tabachnikov [6] showed me the shelf, where the boundary curves as Elastica under Pressure had been put already in the 19th century. The linear limit, which I had also considered had been put there as Elastica already in the 17th century by James Bernoulli [4, 5] and in the 18th century by Leonhard Euler [13]. Fortunately good mathematics has no date of expiry.

Section 2 presents a short historical survey of the curves and their applications, called Elastica and Elastica under Pressure or Buckled Rings. These curves, known since the seventeenth and nineteenth century, first as solutions of elastic problems, have shown up as solutions of quite a number of other problems, in particular as boundaries of two-dimensional bodies which can float.
in equilibrium in all orientations; this later problem is also solved by Zindler curves.

These classes of curves yield solutions to the bicycle problem. In this problem one asks for front and rear traces of a bicycle, which do not allow to conclude, in which direction the bicycle went, that is the traces are identical in both directions. The curves that give the traces of the front wheel and the traces of the rear wheels are the boundary and the envelope of the water lines, resp., of the floating bodies of equilibrium.

The subsequent sections deal with some supplemental details: In section 3 several derivations of the equations for the elastica and elastica under pressure are given. Section 4.1 deals with the Zindler curves and work on the problem of floating bodies of equilibrium by other mathematicians, including criticism. Section 5 is devoted to work by Greenhill, who found that special cases of the elastica under pressure can be represented by algebraic curves. Since most of the curves considered here are special cases of bicycle curves, section 6 brings some remarks on these curves.

2 Historical survey

In this paper we consider a class of planar curves, which surprisingly show up in quite a number of different physical and mathematical problems. These curves are not generally known, since they are represented by elliptic functions, although special cases can be represented by more elementary functions.

2.1 The Curves

These curves appear in two flavors; they obey in the linear form in Cartesian coordinates \((x, y = y(x)), y' = dy/dx\) the equation

\[
\frac{1}{\sqrt{1 + y'^2}} = ay^2 + b, \tag{1}
\]

The circular form is described in polar coordinates \((r, \phi)\) by

\[
\frac{1}{\sqrt{r^2 + r'^2}} = ar^2 + b + \frac{c}{r^2}, \tag{2}
\]

with \(r = r(\phi)\) and \(r' = dr/d\phi\). Eq. (1) yields the curvature \(\kappa\),

\[
\frac{1}{y'} \frac{d}{dx} \frac{1}{\sqrt{1 + y'^2}} = -\frac{y''}{(1 + y'^2)^{3/2}} = \kappa = 2ay. \tag{3}
\]

From (2) we obtain the curvature

\[
\frac{1}{rr'} \frac{d}{d\phi} \frac{r^2}{\sqrt{r^2 + r'^2}} = \frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}} = \kappa = 4ar^2 + 2b. \tag{4}
\]
We relate the polar coordinates to Cartesian coordinates and shift \( y \) by \( r_0 \),

\[
  r \cos \phi = r_0 + y, \quad r \sin \phi = x. \tag{5}
\]

Then (4) reads

\[
  \kappa = 4a r_0^2 + 8a r_0 y + 4a(x^2 + y^2) + b. \tag{6}
\]

If we now choose

\[
  b = -2a r_0^2, \quad a = \tilde{a}/(4 r_0), \tag{7}
\]

and perform the limit \( r_0 \to \infty \), then eq. (6) reads

\[
  \kappa = 2\tilde{a} y, \tag{8}
\]

which is the linear form (3). Thus the linear form is a limit of the circular form, where the radius \( r_0 \) goes to infinity.

The equation of the curves can be formulated coordinate-independent,

\[
  2\kappa'' + \kappa^3 - \mu \kappa - \sigma = 0, \tag{9}
\]

where the prime now indicates the derivative with respect to the arc length. Multiplication by \( \kappa' \) allows integration,

\[
  \kappa'^2 + \frac{\kappa^4}{4} - \frac{\mu \kappa^2}{2} - \sigma \kappa = 2E. \tag{10}
\]

The coefficient \( \sigma \) vanishes in the linear case. The derivation will become apparent, when we formulate the physical and/or mathematical problems solved by the curves. But the relation between eqs. (1, 2) and eqs. (9, 10) can also be given directly.[6]

### 2.2 Linear Elastica

The linear curves show up in *Elastica*. The question is: How does an elastic beam (or wire or rod) of given length bend? It may be fixed at two ends and the direction of the wire at both ends are given, or it is fixed at one end and loaded at the other end. Bending under load was already considered in the 13th century by Jordanus de Nemore, around 1493 by Leonardo da Vinci, in 1638 by Galileo Galilei, and in 1673 by Ignace-Gaston Pardies. [33, 47], although they could not give correct results. [33, 46] James Bernoulli following Hooke’s ideas obtained a correct differential equation assuming that the curvature is proportional to the bending moment, and partially solved it in 1691-1692. [4],

\[
  \frac{dy}{dx} = \frac{x^2}{\sqrt{a^2-x^2}}. \tag{11}
\]

Huygens [27] argued in 1694 that the problem had a larger variety of solutions and sketched several of them. The more general differential equation was given by Bernoulli [5] in 1694-1695,

\[
  \frac{dy}{dx} = \frac{x^2 \pm ab}{\sqrt{a^4-(x^2 \pm ab)^2}} \tag{12}
\]
This equation yields
\[
\frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \frac{x^2 \pm ab}{a^2},
\] (13)

which is equation (1) with \(x\) and \(y\) exchanged and different notation for the constants. James Bernoulli also realized "... among all curves of a given length drawn over a straight line the elastic curve is the one such that the center of gravity of the included area is the furthest distant from the line, just as the catenary is the one such that the center of gravity of the curve is the furthest distant ..." [4]. Thus he found that the cross section of a volume of water contained in a cloth sheet is bounded (below the water line) by the elastica curve.

In 1742 Daniel Bernoulli (nephew of James) proposed in a letter to Leonhard Euler that the potential energy of a bent beam is proportional to the integral of the square of the curvature \(\kappa\) integrated over the arclength \(s\) of the beam \(\int ds\kappa^2\). In a 1744 paper Euler [13] used variation techniques to solve the problem. He found eq. (12), classified the solutions, discussed the stability, and realized that the curvature is proportional to \(x\),
\[
\kappa = \frac{y''}{(1 + y'^2)^{3/2}} = \frac{2x}{a^2}.
\] (14)

This property plays a role in at least two other physical phenomena:

In 1807 Pierre Simon Laplace [32] investigated the shape of the capillary. The surface of a fluid trapped between two parallel vertical plates obeys also (14), since the difference of pressure inside and outside the fluid is proportional to the curvature of the surface.

Charges in a linearly increasing magnetic field move due to the Lorentz force on trajectories with curvature proportional to the magnetic field and thus again on trajectories given by the curves of elastica. Without being aware of this equivalence Evers, Mirlin, Polyakov, and Wölfe considered them in their paper on the semiclassical theory of transport in a random magnetic field [14].

In 1859 Kirchhoff [29] introduced a kinetic analogue by showing that the problem of elastica is related to the movement of a pendulum. Set
\[
\dot{x} = \cos \theta, \quad \dot{y} = \sin \theta,
\] (15)

where the dot indicates the derivative with respect to the arc \(s\). Then one obtains
\[
\dot{\theta} = \kappa, \quad y' = \tan \theta.
\] (16)

Thus using eqs. (9) and (1) one obtains
\[
\dot{\theta}^2 = 4a^2y^2 = 4a(\cos \theta - b).
\] (17)

Multiplication by \(mr^2/2\) (with \(m\) for the mass and \(r\) for the length of the pendulum) and a corresponding choice of the constants \(a\) and \(b\), yields
\[
\frac{mr^2}{2} \dot{\theta}^2 - mgr \cos \theta = E.
\] (18)
Figure 1: Examples of linear Elastica (i)

Figure 2: Examples of linear Elastica (ii)

Figure 3: Examples of linear Elastica (iii)
This is the energy of a pendulum, if we substitute time $t$ for the arc $s$ in the derivative. Thus $\theta(t)$ is the time dependence of the angle of the pendulum against its lowest position. The periodic movement is obvious. Suppose $a$ is positive. Then for $-1 < b < 1$ the pendulum will swing in a finite interval $-\theta_0 \leq \theta \leq +\theta_0$. These are the inflectional solutions. If $b < -1$ then the pendulum will move across the highest point $\theta = \pi$. These are the non-inflectional solutions. The limit case $b = -1$ yields a non-periodic solution (infinite period).

Some examples of elastica are shown in figures 1 to 3. The first two rows of figure 1 show non-inflectional cases where the pendulum moves across the highest point. The third row shows the aperiodic limit case $b = -1$. It is called *syntractrix of Poloni* (1729). The last row of figure 1 and figures 2 and 3 show inflectional cases corresponding to periodic oscillations without reaching the highest point. This yields a large variety of shapes including the *Eight* in the middle of figure 2 called *lemnoid*. The second drawing in figure 3 corresponds to the case where the pendulum moves up to a horizontal position. It is the *rectangular elastica*. The pendulum swings below the horizontal position in the last two rows of figure 3. In all cases the curves show equilibrium positions of the elastic beam. However, only sufficiently short pieces of the curves correspond to a stable equilibrium or even the absolute minimum of the potential energy.

An excellent survey with many figures on the history of the elastica has been given by Raph Levin. Also Todhunter and Truesdell give reviews of the history of elasticity. Many details are found in the treatise by Love on the mathematical theory of elasticity. In his PhD thesis (1906) Max Born investigated elastic wires theoretically and experimentally in the plane and also in three dimensional space. The solution of eq. (1) or equivalently eqs. (11, 12) was given by Saalschütz in 1880 by means of elliptic functions. The elliptic functions were developed by Abel and Jacobi mainly in the years 1826 – 1829 in several articles in Crelles Journal. Abel died 1829, Jacobi published his fundamental work in 1829. The explicit solutions are not given here. They can be found, e.g., in sect. 263 of Love, in sect. 13 of and in ref. Engineers often call ‘Bernoulli-Euler beam theory’ approximations, in which the beam is only slightly bent.

### 2.3 Elastica under Pressure (buckled rings)

By now we considered elastica to which only forces acted at the ends. A more general problem considers elastic wires, on which forces act along the arc. Maurice Levy realized 1884 that the case, where a constant force $P$ per arc length acts perpendicularly on the wire, yields the differential equation (2). He showed that this problem could be solved by elliptic functions and found two types of solutions. They are called buckled rings, if the wire is closed. The constraints are equivalent if the perimeter and the area inside the ring are given. Halphen worked out the results in some detail in the same year and included it in his ‘Traité des fonctions elliptic et de leurs applications’. Some elastica under pressure are shown in figure 4. Their symmetry is given by the dieder groups $D_p$ with $p = 2, 3, 3$ in the first row, $p = 4, 4, 5, 5$ in the second row and $p = 5, 5, 6$ in...
the third row. All of them are solutions of equation (19). Whereas those in the first and second row can be considered as deformed circles, those in the third row show double points. Similarly as for the elastica the curves show equilibrium configurations. But often only small parts of them are in stable equilibrium.

Greenhill[21] considered the same problem in 1899 and looked particularly for curves that can be expressed by pseudo-elliptic functions. Thus some of the solutions are algebraic curves. The simplest example besides the circle is given by

\[ r^3 = a^3 \cos(3\phi) \]  

with the curvature

\[ \kappa = 4r^2/a^3 \]  

in polar coordinates \((r, \phi)\), which may be written

\[ (x^2 + y^2)^3 = a^3 x(x^2 - 3y^2) \]  

in Cartesian coordinates \((x, y)\). This ‘cloverleaf’ is shown in fig. 5.

**Area-constrained planar elastica in biophysics**

Cells in biology have usually nearly constant volume and constant surface area. Their shape is to a large extend determined by the minimum of the membrane bending energy, see e.g. Helfrich[25] and Svetina and Zeks[43]. The
two-dimensional analogue was considered by Arreaga, Capovilla et al.\cite{1, 10} and by Goldin et al.\cite{20}. Since pressure and area are conjugate quantities, the shapes are also given by those of elastica under pressure. Now of course, constant area and constant length of the bounding loop are given. The authors refer for the determination of the shape to the Lagrange equations \cite{31} as reported by Langer and Singer for elastica\cite{31} and for buckled rings \cite{30}, see also the references \cite{6, 42}. The equations for the elastica under pressure can also be obtained by considering the forces and momenta in the rods.\cite{45} The solution of the equation \cite{2} for buckled rings in terms of elliptic functions can be found in \cite{34, 23, 24, 21} and in \cite{53, 54}. Reference \cite{54} contains further figures.

2.4 Floating Bodies of Equilibrium

2.4.1 Ulam’s Problem in two dimensions

The curves mentioned in the previous subsections appear in two other problems, the problem of Floating Bodies of Equilibrium and the Bicycle Problem. The first of these problems is related to the problem 19 in the Scottish Book by Ulam\cite{49}: "Is a solid of uniform density which will float in water in every position a sphere?" The two-dimensional version of the problem concerns a cylinder of uniform density $\rho$ which floats in water in equilibrium in every position with its axis parallel to the water surface. Sought is the curve different from a circle confining the cross section of the cylinder perpendicular to its axis.

The density of the log be $\rho$ (more precisely $\rho$ is the ratio of the density of the log over that of the liquid). The area of the cross section be $A$, the part above and below the water-line are denoted by $A_1$ and $A_2$. Then Archimedes’ law requires

$$A_1 = (1 - \rho)A, \quad A_2 = \rho A. \quad (22)$$

The distance of the center of gravity of the cross section above the water-line be $h_1$, that below the water-line $h_2$, the length of the log $L$. Then the potential energy is

$$\rho(1 - \rho)ALg(h_1 + h_2). \quad (23)$$

Thus $h_1 + h_2$ has to be constant. The line connecting the two centers of gravity has to be perpendicular to the water-line. Rotation by an infinitesimal angle.
yields that the length $2l$ of the water-line obeys
\[
\frac{2}{3} \frac{1}{A_1} \left( \frac{1}{A_1} + \frac{1}{A_2} \right) = h_1 + h_2.
\] (24)

Thus the length of the water-line does not depend on the orientation of the log. The conditions that the area below the water-line and the length of the water-line are constant implies that the part of the perimeter of the cross-section below the water-line is constant. It also implies that the envelope of the water-lines is given by the midpoints of the water-lines.

This two-dimensional problem has attracted many mathematicians.

![Figure 6: Lower part of the boundary (black) of the floating body](image)

Two waterlines $P_0Q_0$ and $P_1Q_1$ are in blue and cyan. The midpoints $N_0$ and $N_1$ lie on the envelope (red) of the water-lines.

### 2.4.2 Density $\rho = 1/2$

There is a large class of solutions for $\rho = 1/2$. The solutions are not related to the elastica, but it seems worthwhile to mention them. Basically the solutions were found by Zindler [56], although he did not consider this physical problem, but found convex curves which have the property that chords between two points on the boundary which bisect the perimeter have constant length $2l$ and simultaneously cut the enclosed area in two halves. They can be parametrized by

\[
x(\alpha) = l \cos(\alpha) + \xi(\alpha), \quad y(\alpha) = l \sin(\alpha) + \eta(\alpha),
\] (25)

\[
\xi(\alpha) = \int_\alpha^\alpha \cos(\beta) \rho(\beta) \, d\beta, \quad \eta(\alpha) = \int_\alpha^\alpha \sin(\beta) \rho(\beta) \, d\beta,
\] (26)

where $\xi$ and $\eta$ obey

\[
\xi(\alpha + \pi) = \xi(\alpha), \quad \eta(\alpha + \pi) = \eta(\alpha).
\] (27)

The envelopes of the water lines are parametrized by $(\xi, \eta)$. Typically they have an odd number of cusps.

Condition (27) implies $\rho(\beta + \pi) = -\rho(\beta)$. The chords run from $(x(\alpha), y(\alpha))$ to $(x(\alpha + \pi), y(\alpha + \pi))$. Zindler did not consider the centers of gravity of both
halves of the area. Otherwise he would have realized that their distance does not
depend on the angle $\alpha$ and the line between them is always perpendicular to
the chord. This class of curves was also found by Auerbach\cite{2} in 1938 and by
Geppert\cite{17} in 1940. Special cases were given by Salkowski\cite{41} in 1934 and by
Salgaller and Kostelianetz\cite{40} in 1939. Examples of Zindler curves are shown
in figures 7 to 10. They are due to Auerbach\cite{2}, Zindler\cite{56}, and Salgaller and
Kostelianetz\cite{40}. The envelopes of the water lines are shown in red, the water
lines in blue and cyan.
2.4.3 Density $\rho \neq 1/2$

For a long time it was not clear, whether solutions for $\rho \neq 1/2$ exist. Gilbert\cite{gilbert} in his nice article 'How things float' claims in section 3 'Different heart-shaped cross sections work for other densities (he means densities different from 1/2) and there are other solutions that are not heart-shaped.' Indeed, there are cross-sections that are not heart-shaped for density 1/2 and densities different from 1/2. But I do not know a heart-shaped solution for density different from 1/2 and I doubt that at the time he wrote the paper a solution for densities different from 1/2 was known. At least he does not give reference to such a solution.

Attempts to find solutions for $\rho \neq 1/2$ by Salkowski\cite{salkowski} in 1934, Gericke\cite{gericke} in 1936, and Ruban\cite{ruban} in 1939 failed. As I will explain later, it seems that Ruban was close to a solution. It was proven by several authors that chords, which form a triangle or a quadrangle yield only circles.

Bracho, Montejano, and Oliveros\cite{bracho, montejano, oliveros} were probably the first to find solutions for densities different from 1/2. They consider a carousel, which is a dynamical equilateral polygon in which the midpoint of each edge travels parallel to it. The trace of the vertices describe the boundary and the midpoints outline the envelope of the waterline. In this way they found solutions where the chords form an equilateral pentagon. However, their solutions were not sufficiently convex, since the waterline cuts the cross section in some positions several times.

For special densities $\rho$ one can deform the circular cross-section into one with $p$-fold symmetry axis and mirror symmetry.

$$r(\phi) = r_0(1 + 2\epsilon \cos(p\phi) + 2 \sum_{n=2}^{\infty} c_n \cos(n\phi)), \quad (28)$$

where the coefficients $c_n$ are functions of $\epsilon$ and $p$ with $c_n = O(\epsilon^n)$. The corresponding $p - 2$ densities depend on $\epsilon$. Surprisingly, the perturbation expansion in $\epsilon$ yielded (up to order $\epsilon^7$) one and the same solution for all $p - 2$ densities, although it had to be expected only for pairs with density $\rho$ and $1 - \rho$. The present author reported this result in \cite{author}. This result was unexpected. It was probably true to all orders in $\epsilon$ and thus deserved further investigation. (In eq. (83) of \cite{author} $v_3$ should be replaced by $s_{80}$).

In a first step I considered the limit $p \to \infty$ with $r_0 \propto p$ and $\epsilon \propto 1/p$. This corresponds to the transition from the circular case to the linear case mentioned in subsection 2.1. In this limit only terms $\sum_{n,k} c_{n,k} p^{n+2k-1} \epsilon^{n+2k} \cos(n\phi)$ in the expansion (28) contribute (with odd $n$).

**Property of constant distance** These curves have a remarkable property, which I call the property of constant distance:

Consider two copies of the curves. Choose an arbitrary point on each curve. Then in the linear case there exists always a length $\delta u$ by which the curves can be shifted against each other, and in the circular case there exists an angle $\delta \phi$ by which the two curves can be rotated against each other, so that the distance $2l$ between the two points stays constant, if they move on both curves by the same arc distance $s$. 

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Considering this procedure the other way round, we may shift curves in the linear case continuously against each other and watch how the distance $2l$ increases with $\delta u$, or we may rotate the curves in the circular case continuously against each other and watch how $2l$ varies with $\delta \phi$. When $\delta \phi$ is increased by $2\pi/p$, then both curves fall unto themselves, and a solution for the floating bodies is found, provided the curve is sufficiently convex, so that the chord between the two points does not intersect the curve at another point.

Figure 11: Property of constant distance
The first figure of 4 and figure 5 are shown in two copies rotated against each other by $45^0$, $60^0$ and $30^0$.

In figure 11 three examples for the property of constant distance are shown. Two copies of the first of the figures 4 are shown in black and blue. The lines of constant distance $2l$ are drawn in cyan and green switching color in the middle, where they touch the red envelope. Similarly these lines are shown for copies of the cloverleaf, figure 5, rotated against each other by $\delta \phi = 60^0$ and $\delta \phi = 30^0$, resp. The length $2l$ for the cloverleaf is given by $2l = a |\sin(3\delta \phi/2)|^{1/3}$.

The distance $2l$ shrinks for the buckled ring (first figure of figure 11) after rotation by $2\pi/p = 180^0$ to zero. Therefore it cannot serve as floating body of equilibrium. The buckled rings in figure 12 of symmetry $D_3$ and $D_4$ and those in figure 13 of symmetry $D_5$ are boundaries of floating bodies of equilibrium. Rotation by $2\pi/p$ yields a non-zero distance $l$. The waterlines are shown in green and cyan, the envelope of the waterlines in red. The figures with odd $p$ are also solutions for $\rho = 1/2$, thus special Zindler curves. The figures with

Figure 12: Floating bodies of equilibrium, $p = 3$ and $p = 4$
$p = 5$ are besides solutions for $\rho = 1/2$ also solutions for a density $\rho > 1/2$ and for a density $\rho < 1/2$.

![Floating bodies of equilibrium, $p = 5$](image)

We turn to the linear case with examples in figure 14. In the first to third row examples of figures from the second to fourth row of figure 1 are shown. They are shifted by a distance $\delta u$ and in one case one curve is reflected. This reflected curve is solution of eq. 1 with the same constants $a$ and $b$. In the fourth and fifth row two examples are shown, where the figures were shifted so far that they fall on each other, together with lines of length $2l$ and the envelopes.

The derivation of the differential equations (21) for the curves are contained in [50] based on [51, 52]. First the linear case was dealt with, where first large distances, then infinitesimal distances, and finally arbitrary distances were considered (section 2 of [52]). It yields eq. (1) (Eq. (17) of [52] and Eq. (27) of [50]).

The circular case is considered in section 3 of [52] and in section 3.2 of [50]. In deriving this equation the author assumed that also for non-integer periodicity $p$ such chords (of infinitesimal length) exist between the curves rotated against each other by nearly $2\pi$. This assumption yields a differential equation of order 3, (43) of [52] and (33) of [50]. It can be integrated easily to eq. (2) (Eq. (47) of [52] and Eq. (37) of [50]). Explicit solution of these equations showed that the property of constant distance holds.

The problem is originally non-local, since it connects end-points of the chords generally without the necessity of closing them to a polygon of chords. The eqs. (1, 2), however, reduce it to a local problem: The equations connect only locus and direction of the curve at the same point.

It came to my big surprise, when Bor, Levine, Perline, and Tabachnikov [6] pointed out, that the problem of elastica under pressure and the problem of floating bodies of equilibrium in two dimensions are governed by the same differential equation (2).

Explicit solution of the equations (1, 2) showed that the hypothesis of the property of constant distance holds. [53, 54]

The problem is originally non-local, since it connects end-points of the chords generally without the necessity of closing them to a polygon of chords. The eqs. (1, 2), however, reduce it to a local problem: The equations connect only locus and direction of the curve at the same point.
2.5 The Bicycle Problem

The bicycle problem is closely related to the problem of finding floating bodies of equilibrium. It was addressed by Finn\(^\text{15, 16, 44}\). The problem goes back to a criticism of the discussion between Sherlock Holmes and Watson in The Adventure of the Priory School\(^\text{12}\) on which way a bicycle went whose tires' traces are observed. Let the distance between the front and the rear wheel of the bicycle be \(l\). The end points of the tangent lines of length \(l\) to the trace of the rear wheel in the direction the bicycle went yields the points of the traces of the front wheel. Thus if the tangent lines in both directions end at the trace of the front wheel, it is open which way the bicycle went. Thus curves \(\gamma\) for the rear wheel (in red in Fig. 5) and \(\Gamma\) for the front wheel (in black in fig. 6) are solutions for such an ambiguous direction of the bicycle. The tire track problem consists in finding such curves \(\Gamma\) and \(\gamma\) different from the trivial solutions of circles and straight lines.
Obviously the solutions of the two-dimensional floating body problem solve the bicycle problem, but also the linear elastica and the Zindler curves solve the problem. There are more solutions to the bicycle curves. Finn argues that the variety of bicycle curves is much larger: Draw from one point \((N_0)\) of the rear tire track the tangent to the front wheel in both directions and give an arbitrary smooth tire track between these two points \((P_0)\) and \((Q_0)\) in figure 6. Then one can continue tire track curves in both directions [15, 16].

We shortly explain why Zindler curves are bicycle curves. Let the bicyclist go in one direction so that the front wheel is at \((x(\alpha), y(\alpha))\) given in eq. (25) and with the rear wheels at \((\xi(\alpha), \eta(\alpha))\). Then the bicyclist going in the opposite direction is with its front wheels at

\[
x_-(\alpha) = -l \cos(\alpha) + \xi(\alpha), \quad y_-(\alpha) = -l \sin(\alpha) + \eta(\alpha).
\]

Since \((x_-(\alpha), y_-(\alpha)) = (x(\alpha + \pi), y(\alpha + \pi))\) and the traces of the rear wheels agree due to (27), both bicyclists use the same traces for their wheels and one cannot determine, which way the bicyclist went.

Zindler curves, but also a number of curves from elastica and buckled rings yield envelopes, that is traces of the rear wheels with cusps. Then the rear wheel has to go back and forth. For these curves the front wheel has to be turned by more than the right angle. Driving along these traces requires artistic skills. Apart from the Zindler curves (figs. 7 to 10) this is the case for the first, second, and fourth buckled ring of figure 12, the inner rear trace of figure 13, and the elastica of the fourth row of fig. 14. However, the third traces of fig. 12, the outer trace of fig. 13 and the traces of the fifth row of fig. 14 can be easily traversed and constitute good solutions of the bicycle problem.

3 Derivation of the differential equations

The equations governing the elastic beam (wire) has been given in different ways. They are developed in this section.

3.1 Bernoulli - Huygens solution

James Bernoulli considered initially a beam AB loaded by a weight C at the end assuming the beam to be horizontally at the point of the load, see figure 14. He assumed the curvature \(\kappa\) to be a function \(f\) of the moment. Hence,

\[
\kappa = \frac{d\phi}{ds} = \frac{d}{dx} \frac{y'}{\sqrt{1 + (y')^2}} = f(x),
\]

where \(\phi\) is the angle of the tangent at the curve against the x-axis. Integration yields

\[
y' = \frac{S(x)}{\sqrt{1 - S^2(x)}}, \quad S(x) = \int_0^x d\xi f(\xi).
\]
Assuming a linear relation between the curvature and the moment gives
\[ f(x) = \frac{2x}{a^2}, \quad S(x) = \frac{x^2 \pm ab}{a^2} \] (32)
and thus equation (12). The rectangular elastica primarily considered by James Bernoulli is obtained for \( b = 0 \).

### 3.2 Forces and Pressure

Here we consider the force and torque acting in the elastic wire similar to that given by Levy [34] and derive eqs. (33, 34). Let us cut out a piece from \( r \) to \( r' \) (figure 16). At the ends act forces \( \mathbf{F} \) and \( -\mathbf{F}' \). In addition a force per length \( P \) perpendicular to the wire exerts the force
\[ \mathbf{F}_P = Pe_3 \times (r' - r) \] (33)
on the piece of wire, where \( e_3 \) is the unit vector perpendicular to the plane.

The total force vanishes in the static case,
\[ \mathbf{F} - \mathbf{F}' + Pe_3 \times (r' - r) = 0. \] (34)
Integration yields the force acting on the wire,
\[ \mathbf{F}(r) = \mathbf{F}_0 + Pe_3 \times r. \] (35)
Next we consider the torque acting on the piece of wire. Due to the curvature of the wires there are torques $M$ and $-M'$ at the ends of the wires. Moreover $r \times F$ and $-r' \times F'$ are exerted by the forces at the ends and the force on the piece exerts

$$M_P = \int dM_P = \int (-r \times dF_P) = -P \int r \times (e_3 \times dr)$$

$$= -P \int e_3(rdr) = -\frac{1}{2} Pe_3(r^2 - r'^2). \quad (36)$$

The total torque vanishes,

$$M - M' + r \times F - r' \times F' + M_P = 0. \quad (37)$$

This yields

$$M(r) = -r \times F_0 - \frac{1}{2} Pe_3 r^2 + M_0. \quad (38)$$

Let us introduce

$$M = e_3M, \quad M_0 = e_3M_0, \quad e_3 \times F_0 = F_n. \quad (39)$$

Multiplication of (38) by $e_3$ yields

$$M(r) = r \cdot F_n - \frac{1}{2} Pr^2 + M_0 = -\frac{1}{2} P(r - \frac{F_n}{P})^2 + \frac{F_n^2}{2P^2} + M_0. \quad (40)$$

The torque is proportional to the curvature $\kappa$

$$M = -EI\kappa, \quad (41)$$

where $E$ is the elastic modulus and $I$ the second moment with respect to the axis in $e_3$-direction through the center of gravity of the cross section of the wire. Love calls this the ordinary approximate theory and discusses it in sections 255 – 258 of his book.[35]

If there is no external force, $P = 0$, then the first equation (40) yields

$$\kappa = -\frac{1}{EI}(r \cdot F_n + M_0) \quad (42)$$

in agreement with eq. (3). Hence $\kappa$ increases linearly with $r$ parallel to $F_n$.

If $P \neq 0$, then we replace $r - F_n/P \rightarrow r$ and obtain

$$\kappa = 4ar^2 + 2b, \quad a = P/(8EI), \quad b = -(F_n^2/(2P) + M_0)/(2EI). \quad (43)$$

Hence the curvature $\kappa$ increases quadratically with $r$ in accordance with eq. (1).
3.3 Equation for the curvature

We start with the Frenet-Serret formula for the tangential vector $t$ and the normal vector $n$ of the wire

\[
\frac{dt}{ds} = \kappa n, \quad \frac{dn}{ds} = -\kappa t. \tag{44}
\]

We express the force as

\[
F = F_t t + F_n n. \tag{45}
\]

Then going along the wire (beam) we obtain

\[
\frac{d}{ds}(F_t t + F_n n) = Pn. \tag{46}
\]

Thus

\[
\frac{dF_t}{ds} - \kappa F_n = 0, \tag{47}
\]

\[
\frac{dF_n}{ds} + \kappa F_t = P. \tag{48}
\]

The torque obeys

\[
\frac{dM}{ds} = -F_n. \tag{49}
\]

Finally the torque is assumed to be proportional to the curvature

\[
M = -EI\kappa. \tag{50}
\]

This yields

\[
EI\frac{d\kappa}{ds} = F_n. \tag{51}
\]

We insert this relation in eq. (47) and obtain

\[
\frac{dF_t}{ds} - EI\kappa \frac{d\kappa}{ds} = 0, \tag{52}
\]

which can be integrated to

\[
F_t - \frac{EI}{2}\kappa^2 = c'. \tag{53}
\]

(51) and (48) yield

\[
EI\frac{d^2\kappa}{ds^2} = \frac{dF_n}{ds} = -\kappa F_t + P = \kappa(-c' - \frac{EI}{2}\kappa^2) + P. \tag{54}
\]

Hence

\[
\frac{d^2\kappa}{ds^2} + \frac{1}{2}\kappa^3 + \frac{c'}{EI}\kappa - \frac{P}{EI} = 0. \tag{55}
\]

If we multiply this equation by $d\kappa/\kappa$ and integrate, then we obtain

\[
\frac{1}{2} \left( \frac{d\kappa}{ds} \right)^2 + \frac{1}{8}\kappa^4 + \frac{c'}{2EI}\kappa^2 - \frac{P}{EI}\kappa = E. \tag{56}
\]

These are the equations (9) and (10).
3.4 Geometrical derivation

In this subsection we will derive this equation requiring the extreme of the integral over the square of the curvature with appropriate side conditions. We perform a purely geometrical derivation. As a function of the arc parameter \( s \) we introduce the angle \( \phi \) of the tangent against the x-axis and the Cartesian coordinates. The origin is at \( s = 0 \). The curve starts with the angle \( \phi_0 \).

\[
\phi(s) = \phi_0 + \int_0^s ds' \kappa(s'),
\]

(57)

\[
x(s) = \int_0^s ds' \cos(\phi(s')),
\]

(58)

\[
y(s) = \int_0^s ds' \sin(\phi(s')).
\]

(59)

The length of the arc be \( s_0 \). The area between the arc and the straight line from the origin to the endpoint of the arc \((x(s_0), y(s_0))\) is given by

\[
A = \frac{1}{2} \int_0^{s_0} ds (x(s) \sin(\phi(s)) - y(s) \cos(\phi(s))).
\]

(60)

We may have several side conditions on the curve: the angle at the end point \( \phi(s_0) \), the coordinates of the end point and the area fixed. Thus the corresponding quantities have to be subtracted from the integral over \( \kappa^2/2 \) by Lagrange multipliers \( \lambda_1...\lambda_5 \),

\[
I = \frac{1}{2} \int_0^{s_0} ds \kappa^2(s) - \lambda_1 \phi(s_0) - \lambda_2 x(s_0) - \lambda_3 y(s_0) - \lambda_4 A - \frac{\lambda_5}{2} (x^2(s_0) + y^2(s_0)).
\]

(61)

In total these may be too many conditions. But for those we do not take into account, we set \( \lambda_i = 0 \). The variation of \( I \) has to vanish,

\[
\delta I = \int_0^{s_0} ds \delta \kappa(s) F(s),
\]

(62)

\[
F(s) = \kappa(s) - \lambda_1 + (\lambda_2 + \lambda_5 x(s_0)) \int_s^{s_0} ds' \sin(\phi(s'))
\]

\[\quad - (\lambda_3 + \lambda_5 y(s_0)) \int_s^{s_0} ds' \cos(\phi(s'))
\]

\[\quad - \frac{\lambda_4}{2} \left( \int_s^{s_0} ds' (x(s') \cos(\phi(s')) + y(s') \sin(\phi(s'))) \right)
\]

\[\quad + \frac{\lambda_4}{4} \left( \int_s^{s_0} ds' \sin(\phi(s')) \right)^2 + \frac{\lambda_4}{4} \left( \int_s^{s_0} ds' \cos(\phi(s')) \right)^2
\]

\[\quad = \kappa(s) - \lambda_1 + \lambda_2 (y(s_0) - y(s)) - \lambda_3 (x(s_0) - x(s))
\]

\[\quad + \frac{\lambda_4}{2} [x(s)(x(s) - x(s_0)) + y(s)(y(s) - y(s_0))]
\]

\[\quad + \lambda_5 (x(s_0)y(s) - y(s_0)x(s)).
\]

(63)
The equation $F(s) = 0$ has to be solved. One immediately sees from eq. (64) that $\kappa$ depends (for $\lambda_4 \neq 0$) quadratically on the distance from some point. The derivatives of $F$ with respect to $s$, indicated by a dot vanish, 

\[ \ddot{F}(s) = \dot{\kappa}(s) + K_1 = 0, \]  
\[ K_1 = \cos(\phi(s))[\lambda_3 + \lambda_4 x(s) - \frac{\lambda_4}{2} x(s_0) - \lambda_5 y(s_0)] + \sin(\phi(s))[-\lambda_2 + \lambda_4 y(s) - \frac{\lambda_4}{2} y(s_0) + \lambda_5 x(s_0)], \]  
\[ \ddot{F}(s) = \ddot{\kappa}(s) + K_2 \kappa + \lambda_4 = 0, \]  
\[ K_2 = -\sin(\phi(s))[\lambda_3 + \lambda_4 x(s) - \frac{\lambda_4}{2} x(s_0) - \lambda_5 y(s_0)] + \cos(\phi(s))[-\lambda_2 + \lambda_4 y(s) - \frac{\lambda_4}{2} y(s_0) + \lambda_5 x(s_0)], \]

Elimination of $K_1$ and $K_2$ yields

\[ \kappa \ddot{F}(s) - \frac{1}{\kappa^2} \dot{\kappa} \ddot{F}(s) + \frac{1}{\kappa} \dddot{F}(s) = \kappa \frac{d\kappa}{ds} - \frac{1}{\kappa^2} \dot{\kappa} \ddot{\kappa} - \frac{\lambda_4}{\kappa^2} \ddot{\kappa} + \frac{1}{\kappa} \dddot{\kappa} = 0. \]  

This expression is a complete derivative. Integration and multiplication by $\kappa$ gives

\[ \frac{1}{2} \dot{\kappa}^2 + \ddot{\kappa} + \lambda_4 + C_1 \kappa = 0. \]  

This is the Lagrange equation for the relative extrema of the bending energy. Multiplication of this expression by $\kappa$ and another integration yields

\[ \frac{1}{2} \dot{\kappa}^2 + \frac{1}{8} \kappa^4 + \lambda_4 \kappa + \frac{C_1}{2} \kappa^2 = C_2, \]  

which agrees with eqs. (65) and (66) by renaming the constants. Apart from eq. (70) also boundary conditions for $F = 0$, $\dot{F} = 0$, and $\ddot{F}(0)$ at $s = 0$ or $s = s_0$ have to be fulfilled. Thus $\kappa$, $\dot{\kappa}$, and $\ddot{\kappa}$ at $s = 0$ or $s = s_0$ are expressed by $\lambda_i$ and $x(s_0)$, $y(s_0)$, $\phi$ at $s = 0$ or $s = s_0$. Insertion in (71) and (72) yields the corresponding constants $C_1$ and $C_2$.

### 3.5 Water contained in a cloth sheet

James Bernoulli showed that water contained in a long cloth sheet (in $y$-direction) is bound in the $xz$-direction by an elastic curve. (See Levien [33], footnotes 3 and 5, and Truesdell [47], p. 201). This problem goes under the name of lintearia.

The cloth ends should be fixed at $(\xi_1, \zeta_1)$ and $(\xi_2, \zeta_2)$. The height of the water-line is $h$. The water surface ranges from $x_1$ to $x_2$ as in figures 17 and 18.

We give two derivations, a longer one using variational techniques, and a shorter one, considering forces and pressure as in subsection 3.3.
The cyan curve indicates in both cases the continuation of the elastica curve.

3.5.1 Variation of the potential energy

We denote the area of the cross section by $A$, the breadth of the cloth by $s$, the potential energy by $V$, the length of the cloth in $y$-direction $L_y$, the gravitational constant $g$, and the density of the water $\rho$. Then the area of the cross section $A$, the potential energy $V$ and the breadth $s$ of the cloth are given by

\[
A = \int_{x_1}^{x_2} dx(h(x) - z(x)), \tag{73}
\]
\[
V = \frac{L_y g \rho}{2} \int_{x_1}^{x_2} dx(h^2(x) - z^2(x)), \tag{74}
\]
\[
s = s_1 + s_2 + s_0, \tag{75}
\]
\[
s_1 := \sqrt{(\xi_1 - x_1)^2 + (\zeta_1 - h_1)^2}, \tag{76}
\]
\[
s_2 := \sqrt{(\xi_2 - x_2)^2 + (\zeta_2 - h_2)^2}, \tag{77}
\]
\[ s_3 := \int_{x_1}^{x_2} dx \sqrt{1 + z'^2}, \]  
\[ z' := \frac{dz}{dx}, \quad h' := \frac{dh}{dx}. \]  

(78)  

(79)

Of course we know that \( h(x) \) does not depend on \( x \). However, it is of advantage to have \( h(x_1) \) and \( h(x_2) \) as two variables, which can be varied independently.

We look for the extreme of \( V - \alpha A - \sigma S \). The variations are

\[ \delta A = \int_{x_1}^{x_2} dx (\delta h(x) - \delta z(x)), \]  
\[ \delta V = L_y g \rho \int_{x_1}^{x_2} dx (\delta h - z \delta z), \]  
\[ \delta s_1 = \frac{\delta x_1 (x_1 - \xi_1) + \delta h_1 (h_1 - \zeta_1)}{s_1}, \]  
\[ \delta s_2 = \frac{\delta x_2 (x_2 - \xi_2) + \delta h_2 (h_2 - \zeta_2)}{s_2}. \]  
\[ \delta s_3 = \delta x_2 \sqrt{1 + z'^2} - \delta x_1 \sqrt{1 + z_1'^2} + \int_{x_1}^{x_2} dx \frac{z'}{\sqrt{1 + z'^2}} \frac{d\delta z}{dx}, \]  
\[ z'_i := z'(x = x_i), \quad h'_i := h'(x = x_i). \]  

(80)  

(81)  

(82)  

(83)  

(84)  

(85)

We have not included contributions proportional to \( \delta x_i \) in \( \delta A \) and \( \delta V \), since \( h_i = z_i \) at \( x_i \). Partial integration transforms the integral to

\[ \int_{x_1}^{x_2} dx \frac{z'}{\sqrt{1 + z'^2}} \frac{d\delta z}{dx} = \delta z_2 \frac{z_2'}{\sqrt{1 + z_2'^2}} - \delta z_1 \frac{z_1'}{\sqrt{1 + z_1'^2}} - \int_{x_1}^{x_2} dx \frac{1}{\sqrt{1 + z'^2}} \frac{d}{dx} \left( \frac{z'}{\sqrt{1 + z'^2}} \right). \]  

(86)

The variation of \( V - \alpha A - \sigma S \) contains contributions proportional to \( \delta x_i, \delta h_i, \) and \( \delta z_i \) and an integral over \( x \),

\[ \int_{x_1}^{x_2} dx \left( L_y g \rho h(x) - \alpha \right) \delta h(x) \]  
\[ - \int_{x_1}^{x_2} dx \left( L_y g \rho z(x) - \alpha - \sigma \frac{d}{dx} \left( \frac{z'}{\sqrt{1 + z'^2}} \right) \right) \delta z(x). \]  

(87)  

(88)

The variation of \( \delta h(x) \) yields constant \( h \) as expected,

\[ \alpha = L_y g \rho h. \]  

(89)

The variation of \( \delta z(x) \) yields

\[ L_y g \rho (z(x) - h) = \sigma \frac{d}{dx} \left( \frac{z'}{\sqrt{1 + z'^2}} \right) = \sigma \frac{z''}{(1 + z'^2)^{3/2}} = \sigma \kappa. \]  

(90)

Thus the curvature increases proportional to the depth measured from the water-surface.
Since the variation of $h(x_i) = z(x_i)$ yields
\[ \delta h_i + h'_i \delta x_i = \delta z_i + z'_i \delta x_i, \quad \delta z_i = \delta h_i - z'_i \delta x_i, \quad (91) \]
we obtain the contributions
\[
\begin{align*}
\delta x_1 \left( \frac{x_1 - \xi_1}{s_1} \pm \frac{1}{\sqrt{1 + z'^2_1}} \right) + \delta h_1 \left( \frac{h - \xi_1}{s_1} \pm \frac{z'_1}{\sqrt{1 + z'^2_1}} \right) \\
+ \delta x_2 \left( \frac{x_2 - \xi_2}{s_2} \pm \frac{z'_2}{\sqrt{1 + z'^2_2}} \right) + \delta h_2 \left( \frac{h - \xi_2}{s_2} \pm \frac{z'_2}{\sqrt{1 + z'^2_2}} \right). \quad (92)
\end{align*}
\]
The factors of $\delta x_i$ and $\delta h_i$ have to vanish. They yield the direction of the lines from the lines of the suspension to the lines where the cloth touches the waterline. If $x_1 > \xi_1$ and $\xi_2 > x_2$ as in fig. 17, then the upper signs in (92) apply, and the slope is continuous across the waterline as expected.

If instead $x_1 < \xi_1$ and $\xi_2 < x_2$ as in fig. 18, then $z(x)$ is double-valued with values $z_-(x)$ and $z_+(x)$. Then the $x$-integral of $s_3$ reads
\[
s_3 = \int_{x_{\text{min}}}^{x_{\text{max}}} dx \sqrt{1 + z'^2(x)} + \int_{x_{\text{min}}}^{x_1} dx \sqrt{1 + z'^2(x)} + \int_{x_2}^{x_{\text{max}}} dx \sqrt{1 + z'^2(x)}. \quad (93)
\]
Accordingly the contributions from $s_3$ change sign and the lower signs in (92) apply. Again the slope is continuous across the waterline. The expression for the area and similarly for the potential have different signs in front of the integrals,
\[
A = \int_{x_{\text{min}}}^{x_{\text{max}}} dx (h(x) - z_-(x)) - \int_{x_{\text{min}}}^{x_1} dx (h(x) - z_+(x)) - \int_{x_2}^{x_{\text{max}}} dx (h(x) - z_+(x)). \quad (94)
\]

### 3.5.2 Considering forces and pressure

A simpler derivation can be given by considering the forces and the pressure as in subsection 3.3. Since the cloth can be bent without exerting any forces or torques, one has $F_n = 0$, $M = 0$, $c = 0$. Thus eqs. (47) and (48) read
\[
\frac{dF_t}{ds} = 0, \quad \kappa \frac{F_t}{L_y} = P, \quad (95)
\]
where $F_t$ is the total force over the length $L_y$. The pressure acts from inside and depends on $z$,
\[
P = -\rho g (h - z). \quad (96)
\]
This yields
\[
F_t = \text{const}, \quad \kappa = \frac{L_y \rho g (z - h)}{F_t} \quad (97)
\]
in agreement with equation (43).
Figure 19: Two halves of the area

The area is cut in halves by the diameter \( D(\alpha) = A_1 A A_2 \). The coordinates of \( A \) are \( \xi(\alpha), \eta(\alpha) \), and those of \( A_1, A_2, C_1, C_2 \) are \( (x, y) \) at \( \alpha, \alpha + \pi, \gamma, \gamma + d\gamma \), resp.

The infinitesimal triangle of eq. (99) is \( AC_1 C_2 \).

4 The case \( \rho = 1/2 \)

The boundaries of the two-dimensional floating bodies of equilibrium with density 1/2 are Zindler curves. We describe these closed curves in the following subsection. Chords of the curves bisect both the boundary and the enclosed area. The centers of gravity of these halves have constant distance and their connecting line is perpendicular to the chord (subsect. 4.2). In the following subsection 4.3 I comment on some papers which investigate plain regions with the property that chords of constant length cut the region in two pieces of constant areas. Some of them deal with the problem of floating bodies of equilibrium, others are purely geometrical.

4.1 Zindler curves

Zindler considers mainly in sects. 6, 7 and 10 of [56] convex plain areas with the property that any chord between two points bisecting the boundary has constant length and bisects the area.

The envelope of the chords is defined by Equation (26) with the constraint (27) which implies \( \xi(\pi) = \xi(0), \eta(\pi) = \eta(0) \). The boundary can be parametrized by equation (25). Hence

\[
x(\alpha + \pi) = \xi(\alpha) - l \cos(\alpha), \quad y(\alpha + \pi) = \eta(\alpha) - l \sin(\alpha).
\] (98)

The diameters \( D(\alpha) \) ranges from \( (x(\alpha), y(\alpha)) \) to \( (x(\alpha + \pi), y(\alpha + \pi)) \). It bisects the perimeter and the area enclosed by the boundary, provided \( l \) is sufficiently large, so that the diameters cut the boundary only at the end points of the diameter.
We consider now the two regions cut by the diameter \( D(\alpha) \). They consist of infinitesimal triangles (fig. 19) 

\[
(\xi(\alpha), \eta(\alpha)), (x(\gamma), y(\gamma)), (x(\gamma + d\gamma), y(\gamma + d\gamma)),
\]

with \( \gamma = \alpha_1 = \alpha_{\ldots\alpha} + \pi \) for region 1 and \( \gamma = \alpha_2 = \alpha - \pi_{\ldots\alpha} \) for region 2.

Then the perimeter is given by

\[
u_{1,2} = \int_{\alpha_{\ldots\alpha}}^{\alpha_{\ldots\alpha} + \pi} d\gamma \sqrt{x'(\gamma)^2 + y'(\gamma)^2} = \int_{\alpha_{\ldots\alpha}}^{\alpha_{\ldots\alpha} + \pi} d\gamma \sqrt{\rho^2(\gamma) + l^2}.\tag{100}\]

Since \( \rho(\gamma - \pi) = -\rho(\gamma) \), one obtains the same integral. Hence \( u_2 = u_1 \). The diameter bisects the perimeter.

The area of the infinitesimal triangle \((99)\) is given by

\[
dA = \frac{1}{2} d\gamma [(x(\gamma) - \xi(\alpha))y'(\gamma) - (y(\gamma) - \eta(\alpha))x'(\gamma)].\tag{101}\]

With the abbreviations

\[
\xi_{\gamma,\alpha} := \xi(\gamma) - \xi(\alpha), \quad \eta_{\gamma,\alpha} := \eta(\gamma) - \eta(\alpha)\tag{102}\]

we obtain

\[
x(\gamma) - \xi(\alpha) = l \cos(\gamma) + \xi_{\gamma,\alpha},
\]
\[
y(\gamma) - \eta(\alpha) = l \sin(\gamma) + \eta_{\gamma,\alpha},
\]
\[
x'(\gamma) = -l \sin(\gamma) + \rho(\gamma) \cos(\gamma),
\]
\[
y'(\gamma) = l \cos(\gamma) + \rho(\gamma) \sin(\gamma)\tag{103}\]

and

\[
dA = \frac{1}{2} d\gamma [l^2 + l \xi_{\gamma,\alpha} \cos(\gamma) + l \eta_{\gamma,\alpha} \sin(\gamma) + \xi_{\gamma,\alpha} \rho(\gamma) \sin(\gamma) - \eta_{\gamma,\alpha} \rho(\gamma) \cos(\gamma)].\tag{104}\]

One finds that both areas \( A_1 \) and \( A_2 \) are equal

\[
A_1 = A_2 = \frac{\pi}{2} l^2 - \Delta,
\]

\[
\Delta = \int_{\alpha}^{\alpha + \pi} d\gamma \eta(\gamma) \rho(\gamma) \cos(\gamma)
\]

\[
= -\int_{\alpha}^{\alpha + \pi} d\gamma \xi(\gamma) \rho(\gamma) \sin(\gamma).\tag{105}\]

They are independent of \( \alpha \), since the integrand is a periodic function of \( \gamma \) with period \( \pi \). The term proportional to \( l \) vanishes, since it is a total derivative of a function, which vanishes at the limits,

\[
\int_{\alpha_{\ldots\alpha}}^{\alpha_{\ldots\alpha} + \pi} d\gamma \xi_{\gamma,\alpha} \cos(\gamma) + \eta_{\gamma,\alpha} \sin(\gamma)
\]

\[
= \int_{\alpha_{\ldots\alpha}}^{\alpha_{\ldots\alpha} + \pi} d\gamma \frac{d}{d\gamma} \left[ \xi_{\gamma,\alpha} \sin(\gamma) - \eta_{\gamma,\alpha} \cos(\gamma) \right].\tag{106}\]
4.2 Centers of Gravity

Zindler did not consider the centers of gravity of the two half areas. It is important for the floating body problem that their distance is constant and that the straight line between the two centers is normal to the chord.

The centers of gravity \((\hat{x}(\gamma), \hat{y}(\gamma))\) of the infinitesimal triangles \((99)\) are given by

\[
\begin{align*}
\hat{x}(\gamma) &= \frac{1}{3}\xi(\alpha) + \frac{2}{3}(l\cos(\gamma) + \xi(\gamma)) = \xi(\alpha) + \frac{2}{3}(l\cos(\gamma) + \xi_{\gamma,a}), \\
\hat{y}(\gamma) &= \frac{1}{3}\eta(\alpha) + \frac{2}{3}(l\sin(\gamma) + \eta(\gamma)) = \eta(\alpha) + \frac{2}{3}(l\sin(\gamma) + \eta_{\gamma,a}).
\end{align*}
\]

Thus the centers of gravity \((\bar{X}, \bar{Y})\) of the halves of the area are given by the integrals

\[
\begin{align*}
A_{1,2}\bar{X}_{1,2}(\alpha) &= \int dA\hat{x} = \int dA\xi(\alpha) + \frac{1}{3} \int d\gamma [l^3\cos(\gamma) \\
&\quad + l^2\xi_{\gamma,a}(1 + \cos^2(\gamma)) + l^2\eta_{\gamma,a}\cos(\gamma)\sin(\gamma) \\
&\quad + l\xi_{\gamma,a}\cos(\gamma) + l\xi_{\gamma,a}\eta_{\gamma,a}\sin(\gamma) \\
&\quad + l\xi_{\gamma,a}\rho(\gamma)\sin(\gamma) - l\eta_{\gamma,a}\rho(\gamma)\cos^2(\gamma) \\
&\quad + \xi_{\gamma,a}\rho(\gamma)\sin(\gamma) - \xi_{\gamma,a}\eta_{\gamma,a}\rho(\gamma)\cos(\gamma)], \\
A_{1,2}\bar{Y}_{1,2}(\alpha) &= \int dA\hat{y} = \int dA\eta(\alpha) + \frac{1}{3} \int d\gamma [l^3\sin(\gamma) \\
&\quad + l^2\xi_{\gamma,a}\cos(\gamma)\sin(\gamma) + l^2\eta_{\gamma,a}(1 + \sin^2(\gamma)) \\
&\quad + l\xi_{\gamma,a}\eta_{\gamma,a}\cos(\gamma) + l\eta_{\gamma,a}\sin(\gamma) \\
&\quad + l\xi_{\gamma,a}\rho(\gamma)\sin^2(\gamma) - l\eta_{\gamma,a}\rho(\gamma)\sin(\gamma)\cos(\gamma) \\
&\quad + \xi_{\gamma,a}\eta_{\gamma,a}\rho(\gamma)\sin(\gamma) - \eta_{\gamma,a}^2\rho(\gamma)\cos(\gamma)].
\end{align*}
\]

The result can be written

\[
\begin{align*}
A_{1,2}\bar{X}_{1,2}(\alpha) &= \pm(l^3\hat{x}_3 + l\hat{x}_1) + l^2\hat{x}_2 + \hat{x}_0, \\
A_{1,2}\bar{Y}_{1,2}(\alpha) &= \pm(l^3\hat{y}_3 + l\hat{y}_1) + l^2\hat{y}_2 + \hat{y}_0,
\end{align*}
\]

which yields

\[
\begin{align*}
\hat{x}_3 &= -\frac{2}{3}\sin(\alpha), & \hat{y}_3 &= \frac{2}{3}\cos(\alpha).
\end{align*}
\]

The integral for \(\hat{x}_1\) and \(\hat{y}_1\) can be written

\[
\begin{align*}
\hat{x}_1 &= \frac{2}{3} \int d\gamma \frac{d}{d\gamma} [\xi_{\gamma,a}(\xi_{\gamma,a}\sin(\gamma) - \eta_{\gamma,a}\cos(\gamma))], \\
\hat{y}_1 &= \frac{2}{3} \int d\gamma \frac{d}{d\gamma} [\eta_{\gamma,a}(\xi_{\gamma,a}\sin(\gamma) - \eta_{\gamma,a}\cos(\gamma))].
\end{align*}
\]

They vanish at the limits. Thus \(\hat{x}_1 = \hat{y}_1 = 0\). Thus the distance \(h\) between both centers of gravity obeys \(A_{1,2}h = \frac{4}{3}l^3\) and the line between both centers is perpendicular to the chord between the two areas.
4.3 Remarks on other papers

At least seven papers appeared from 1933 to 1940, which discuss (i) characteristic properties of the circle, and (ii) which (convex) plain regions have the property that chords between two points of the boundary of constant length cut the bounded region in two pieces of constant area. I shortly report on them. The first one by Hirakawa stated two theorems:

Theorem I. A closed convex plane curve with the property that all chords of fixed length span arcs of equal length, is a circle.

Theorem II. A plane oval, in which the areas cut off by chords of equal length have the same content, is a circle.

Apparently it is meant that this should hold for all lengths of chords. Salkowski emphasizes that considerable weakening of the conditions yields similar results.

4.3.1 Salkowski 1934

Salkowski started in 1934 with these two theorems and sharpened them. First he introduces what is now known as Darboux butterfly:

Consider a polygonal line $P_0, P_1, P_2, \ldots P_n$ with constant side length $P_iP_{i+1} = s$. Then one connects $P_0$ with $Q_0 = P_n$ by the line $P_0Q_0 = 2d$. Then a point $P_{n+1} = Q_1$ is determined so that $Q_0Q_1 = s$, $P_1Q_1 = P_0Q_0 = 2d$. $Q_1$ is the point which lies on the parallel to $P_1P_n$ through $P_0$. The arcs $P_0P_1$ and $Q_0Q_1$ in fig. 6 are equal. They are replaced by straight lines of equal length in the Darboux butterfly fig. 20. In the limit considered here, where $P_0P_1 = s$ tends to zero, the ratio of arc and distance tends to one. He argues that then Theorem I is equivalent to Theorem II and that this remains true when $s$ tends to 0 (and correspondingly $n$ to $\infty$). He restricts the corresponding curve $c$ to a curve without turning point. He considers the isosceles trapezoid $P_iP_{i+1}Q_iQ_{i+1}$ with circumcircles with centers $M_i$. This center is intersection point of the middle normals on $P_iP_{i+1}, Q_iQ_{i+1}$, but also on $P_iQ_i, P_{i+1}Q_{i+1}$. Denote the midpoint of $P_iQ_i$ by $N_i$ and that of $P_{i+1}Q_{i+1}$ by $N_{i+1}$. In the limit $s \to 0$ the points $N_i$ yield the curve ($N$). The point $M_i$ yield the evolute ($M$) of ($N$). (I think, here
Salkowski continues: It may happen that the trapezoid degenerates to a rectangle. In this case the curve (N) has a cusp. The tangents to the oval at the end points P and Q are parallel and perpendicular to PQ. If the arc PQ is less than half of the circumference of the total circumference, then there exists an arc P′Q′ of the same length with P′Q′ parallel to PQ, but shorter chord P′Q′ < PQ. Thus the cusp of (N) is only possible, if PQ bisects the circumference. Such an example for (N) is Steiner’s hypocycloid with three cusps.

He shows now

Theorem III. If a plane regular piece of curve has the property that three sets of chords of constant length 2d₁, 2d₂, 2d₃ cut off constant lengths of curve and form a triangle, then the curve is a circle.

(Gericke [29] gave in 1936 another proof of this theorem).

Theorem IV. If all chords over constant arcs of length 2s of a curve have the same length 2d and the chords over arcs of length s have the same length 2d′, then the curve is a circle.

Theorem IV’. If a set of quadrangles PQRS with constant side lengths 2d₁, 2d₂, 2d₃, 2d₄ can be inscribed in an oval with corresponding constant arcs of curve, then the oval is a circle. In particular one finds

Theorem V. If all chords of an oval, which cut off one fourth of the circumference, have the same length, then the curve is a circle.

Finally Salkowski asks the general question: Are there ovals c, in which an n-gon with equal edges 2d can be shifted so that the corner points divide the perimeter in equal parts? One realizes that the area of the n-gon has to have constant size, further at least one angle is larger than a right angle. Consider three consecutive corners P, Q, R of the figure with an obtuse angle at Q. Shift the chord PQ continuously to QR, then the midpoint N describes a piece of an oval (N) and its evolute describes the curve (M) of the midpoints of the circles, which touch the oval in the end points of the chords. Denote the midpoint of PQ by N₀, the midpoint of QR by N₁. M₀, M₁ are the intersections of the mid-normals on PQ and QR, resp. M is the intersection of the two mid-normals. Then the points M₀, M, M₁ constitute a triangle with obtuse angle at M. The curve (M) touches the mid-normals at M₀ and M₁.

So far I agree with the construction. But now Salkowski continues: Thus the piece of curve MM₀ (to my opinion it should read curve M₁M₀, since M is generally not in the curve) is longer than the chord M₀M₁, thus larger than the distance MM₀. Since M is the midpoint of the circumcircle of the isosceles triangle PQR, thus MN₀ = MN₁, such a configuration is not possible unless M, M₀, M₁ coincide. Then the triangle PQR transforms into its neighboring position by an infinitesimal rotation, thus it remains unchanged during the shift along the curve c, hence remaining a circle.

I do not see a reason, why M, M₀, M₁ coincide in general. They will coincide at points where the curvature of (P) at Q has an extreme. Then the neighboring
The curves \((P)\) are in black, \((N)\) in magenta, and \((M)\) in blue for \(s_Q = \pi/7, \pi/14, \text{ and } 0.\)

Salkowski has based his argument on an obtuse angle at \(\angle PQR.\) Such obtuse angles appear in several cases considered in ref. [51]. The curves \((28)\) are convex for values \(\epsilon\) up to approximately \(1/(2p^2)\). I choose the curve with \(p = 7\) fold symmetry and \(\delta_0 = 52.959^\circ,\) that is \(\theta_0 = 37.041^\circ.\) With \(\epsilon = 1/98\) the angle \(\angle PQR\) varies between \(102.99^\circ\) and \(108.85^\circ\) and thus is always obtuse. I use an approximation in linear order in \(\epsilon\) for the curve \((P),\)

\[
r(\phi) = 1 + 2\epsilon \cos(p\phi), \quad \phi = s - \frac{\epsilon}{p} \sin(ps)
\]

and choose

\[
s_P = s_Q - 2\theta_0, \quad s_R = s_Q + 2\theta_0.
\]

The corresponding curves are shown in figure 21 for \(s_Q = \pi/7, \pi/14, \text{ and } 0.\) The curves \((P)\) are in black, \((N)\) in magenta, and \((M)\) in blue.

For \(s_Q = 0\) and \(s_Q = \pi/7\) the three points \(M, M_0,\) and \(M_1\) coincide. But in between, in particular for \(s_Q = 1/14\) the three points \(M_0, M_1,\) and \(M\) differ. Note that curve \((M)\) has cusps. Thus the proof of his last theorem fails. This does not mean that his theorem is disproved. Since in terms of eqs. \((113)\) the range for \(\epsilon\) for convex boundaries becomes smaller with increasing \(p,\) it may be that there are not such ovals.

### 4.3.2 Auerbach 1938

Zindler had derived curves whose chords, which bisect the perimeter, have constant length and bisect the area. Auerbach[2] 1938 rederived the solution by Zindler, but he showed that these curves had also the property, that any chord acting as water-line yields the same potential energy, and thus all orientations
are in equilibrium. The first five sections are for general $\rho$, He obtains for the curvatures at $P$ and $Q$

$$\kappa_Q - \kappa_P = 2 \frac{d\theta}{ds}, \quad \kappa_Q + \kappa_P = \frac{4}{d} \sin \theta,$$

(115)

where $d/2$ is our $l$ and $\theta$ the angle between chord and tangent. From section 6 on he considers the case $\rho = 1/2$. Auerbach derives the expressions for the coordinates $x, y$ of the boundary, eqs. (25, 26). One has to replace $(d/2)\tan(\theta) \rightarrow \rho$ and $d/2 \rightarrow l$.

### 4.3.3 Ruban, Zalgaller, Kostelianetz 1939

Eugene Gutkin gives a few remarks of personal and socio-historical character at the end of his paper[22]. He reports the cruel death of Herman Auerbach under the Nazi regime. He also reports that the archimedean floating problem was popular among older mathematics students around 1939 in Leningrad. The results of three of them, Ruban, Zalgaller and Kostelianetz were published in the Proceedings (Doklady) of the Soviet Academy of Sciences in a Russian and a shorter French version. Ruban and Kostelianetz obtained solutions for $\rho = 1/2$ in agreement with Zindler, but Ruban claimed that there are none besides the circle. Ruban obtained the second equation (115) and correctly found that $\sin \theta$ of the angle $\theta$ between chord and tangent are equal at both ends of the tangent. Erroneously he concluded that both angles are equal, but they add up to $\pi$.

In sect. 5 of his paper[38] Ruban introduces the angle $\theta_0$ between chord and tangent and curvature $\kappa_0$ (Ruban uses $k$ instead of $\kappa$) of a circle of length $S$ and claims without proof or explanation: If

$$\theta_0 \cot \theta_0 \neq \frac{\pi nl}{S} \cot \frac{\pi nl}{S}$$

(116)

for $n = 1, 2, ..., \text{then there exists a number } \epsilon > 0 \text{ so that the inequality } |\kappa(s) - \kappa_0| < \epsilon \text{ yields the equality } \kappa(s) = \kappa_0, \text{ where } l \text{ is the length of the arc. Since } \theta_0 = \pi l/S, \text{ the inequality may be rewritten } C_n \neq 0 \text{ with}$

$$C_n = \cos(\theta_0) \sin(n\theta_0) - n \cos(n\theta_0) \sin \theta_0.$$  

(117)

It is likely that Ruban, who has derived the relation

$$b[\kappa(s + l) + \kappa(s)] = 4 \sin \theta(s)$$

(118)

in eq. (5) ($s$ is the arc parameter) and used a Fourier expansion for $\kappa(s)$ considered

$$\kappa(s) = \kappa_0 + a_n \cos \frac{2\pi ns}{S} + b_n \sin \frac{2\pi ns}{S}.$$  

(119)
Together with $2\theta(s) = \int_s^{s+l} dt \kappa(t)$ and restricting to $a_n$ and $b_n$ in linear order yields condition $C_n = 0$ for nontrivial solutions $a_n, b_n$. This is the starting point for non-circular perturbative solutions, as given in [51], where $\theta_0 = \pi/2 - \delta_0$. Thus Ruban was close to a solution, if he would have performed a perturbation expansion. Obviously $C_n = 0$ is always fulfilled for $n = 1$. It corresponds to a translation of the curve, compare sect. 4.2 of [51]. Thus Ruban’s statement should not include the case $n = 1$.

In 1940 Geppert [17] gave the solutions for $\rho = 1/2$, but erroneously argued that there are no solutions for $\rho \neq 1/2$. He simply overlooked that in this later case the points on the boundary are end-points of two different chords, not one.

A general obstacle to find solutions for $\rho \neq 1/2$ was that one expected that the circumference should be divided in an integer or at least rational number $n$ of equal parts. This was very good for $n = 2, \rho = 1/2$, but it is not at all necessary for $\rho \neq 1/2$.

5 Algebraic Curves by Greenhill

In his 1899 paper [21] Greenhill gives special solutions expressed by pseudo-elliptic functions, in which the cosine and sine of the angle $n\theta$ and $n\theta/2$, resp. are algebraic functions of the radius $r$. Thus the curves are algebraic.

I do not attempt to go through the theory of the pseudo-elliptic functions, but refer only to the main results.

Starting point is the expression for the polar angle $\theta$ as function of the radius $r$,

$$\theta = \frac{1}{2} \int \frac{dr}{\sqrt{R}}, \quad R = r^2 - (Ar^4 + Br^2 + C)^2. \quad (120)$$

The integral is divided into two contributions,

$$\theta = \theta' + (B - B')u, \quad (121)$$

with

$$\theta' = \frac{1}{2} \int \frac{dr}{\sqrt{R}}, \quad u = \frac{1}{2} \int \frac{dr}{\sqrt{R}}. \quad (122)$$

where $u$ is the arc length.

The shape of the curves depends on two independent parameters. These may be the dimensionless $AC^3$ and $BC$, or $\epsilon$ and $\mu$ in [54], or $x$ and $y$ or $\beta$ and $\gamma$ by Greenhill [21].

For a given periodicity in $\theta'$, that is by an increase of $\theta'$ by $2\pi/n$ (class I) or by $4\pi/n, n$ odd (class II), a certain relation between these parameters is fixed. Since at Greenhill’s time the integral for the arc length $u$, expressed as elliptic function of the first kind, was tabulated, one could easily calculate $\theta$ for these cases.

If moreover one requires $B = B'$, in order to obtain an algebraic curve, one has a second condition, which allows only for single solutions.
5.1 Class I

For class I one finds solutions of the form

\[ \sin(n\theta') = \frac{H(q)\sqrt{P_1}}{Qq^{n/2}}, \]  
\[ \cos(n\theta') = \frac{L(q)\sqrt{P_2}}{Qq^{n/2}}, \]

\[ H(q) = q^{n-1} + h_1q^{n-2} + ... + h_{n-1}, \]  
\[ L(q) = q^{n-1} + l_1q^{n-2} + ... + l_{n-1}, \]

\[ P_1 = -q^2 + 2(2\gamma + 1)q - 1, \]  
\[ P_2 = q^2 + 2(2\beta - 2\gamma)q + (2\beta - 1)^2, \]

\[ P = P_1P_2 = -(q^2 + 2(\beta - 2\gamma - 1)q - 2\beta + 1)^2 + 16\beta^2\gamma q. \]

For \( \gamma > 0 \) one uses \( q = r^2 \) and for \( \gamma < -1 \) one takes \( q = -r^2 \). \( P \) is related to \( R \) by

\[ P = 16\beta^2|\gamma|R = 16\beta^2\gamma r^2 - \left[ 4\beta\sqrt{|\gamma|}(Aq^2 \pm Bq + C) \right]^2. \]

Hence \( A, B, C \) are related to \( \beta \) and \( \gamma \) by

\[ 4\beta\sqrt{|\gamma|} = A, \quad \pm 8\beta\sqrt{|\gamma|}(\beta - 2\gamma - 1) = B, \quad 4\beta\sqrt{|\gamma|}(1 - 2\beta) = C. \]

The zeroes \( q_{1,2} \) of \( P_1 \) are obtained as

\[ q_{1,2} = 2\gamma + 1 \mp 2\sqrt{\gamma(\gamma + 1)}. \]

This yields the extreme values \( r_{1,2} \) of \( r \),

\[ r_{1,2} = \begin{cases} \sqrt{(\gamma + 1) \pm \sqrt{4\gamma}} & \gamma > 0, \\ \sqrt{\gamma} \pm \sqrt{\gamma + 1} & \gamma < -1. \end{cases} \]

The scale of \( q \) is chosen so that \( q_1q_2 = 1 \). The other extreme values \( q_{3,4} \) are the zeroes of \( P_2 \),

\[ q_{3,4} = -2\beta + 2\gamma + 1 \pm 2\sqrt{\gamma(2\beta - \gamma - 1)}. \]

One may exchange \( P_1 \) and \( P_2 \) by simultaneously rescaling \( q \). This yields

\[ P_1(\beta, \gamma, q) = -(1 - 2\beta)^2P_2(\beta', \gamma', q'), \]  
\[ P_2(\beta, \gamma, q) = -(1 - 2\beta)^2P_1(\beta', \gamma', q'), \]  
\[ P(\beta, \gamma, q) = (1 - 2\beta)^4P(\beta', \gamma', q'), \]  
\[ H(\beta, \gamma, q) = -(1 - 2\beta)^{n-1}L(\beta', \gamma', q'), \]  
\[ L(\beta, \gamma, q) = -(1 - 2\beta)^{n-1}H(\beta', \gamma', q'). \]

with

\[ \beta' = \frac{-\beta}{1 - 2\beta}, \quad \gamma' = \frac{\gamma}{1 - 2\beta}, \quad q' = \frac{q}{1 - 2\beta}. \]
The derivative of \( \theta' \) calculated from eqs. (123, 124) yields

\[
\frac{n}{dq} \frac{d\theta'}{dq} = \frac{2q \frac{dH}{dq} P_1 + q H \frac{dP_1}{dq} - nH P_1}{2q L \sqrt{P_1 P_2}} \tag{141}
\]

\[
= \frac{-2q \frac{dL}{dq} P_2 + q L \frac{dP_2}{dq} - nLP_2}{2q H \sqrt{P_1 P_2}}. \tag{142}
\]

Thus one requires

\[
2q \frac{dH}{dq} P_1 + q H \frac{dP_1}{dq} - nH P_1 = nLP', \tag{143}
\]

\[
2q \frac{dL}{dq} P_2 + q L \frac{dP_2}{dq} - nLP_2 = -nHP', \tag{144}
\]

\[
P' := q^2 + 2(\beta - 2\gamma' - 1)q - 2\beta + 1, \tag{145}
\]

so that

\[
\frac{d\theta'}{dq} = \frac{P'}{2q \sqrt{P}} = \frac{R'}{2q \sqrt{R}}, \tag{146}
\]

\[
P' = 4\beta \sqrt{|\gamma|R'} = Ar^2 + B'r^2 + C, \tag{147}
\]

\[
B - B' = \pm 16\beta \sqrt{\gamma} |(\gamma' - \gamma). \tag{148}
\]

If one multiplies eq. (123) by \( H \) and eq. (124) by \( L \), and add both equations, then one obtains

\[
q^{n+1} \frac{d}{dq} \frac{H^2 P_1 + L^2 P_2}{q^n} = 0, \tag{149}
\]

which yields after integration

\[
\frac{H^2 P_1 + L^2 P_2}{q^n} = \text{const.} \tag{150}
\]

Denoting this constant by \( Q^2 \), one obtains

\[
\cos^2(n\theta') + \sin^2(n\theta') = 1, \tag{151}
\]

as required.

From eqs. (123, 124) one obtains

\[
\cos(2n\theta') = \frac{L^2(q) P_2 - H^2(q) P_1}{Q^2 q^n}. \tag{152}
\]

Since

\[
q^n \cos(2n\theta') = \Re((x + iy)^{2n}) \tag{153}
\]

and \( q = x^2 + y^2 \), these curves are algebraic curves. Basically one has to solve eqs. (143, 145). The coefficients \( \beta \) and \( \gamma \) are solutions of algebraic equations with integer coefficients.

Several curves of class I are shown in figures 22–29. They are listed in table 1.
| $n$ | $\beta$    | $\gamma$    | $\frac{\beta}{n}$ and fig. in [21] | fig. this paper |
|-----|------------|-------------|-----------------------------------|-----------------|
| 2   | -1.36602540 | -1.57735027 | 23 8                              | 22              |
| 3   | -0.37948166 | -1.14356483 | 24 9                              | 23              |
| 4   | -0.19053133 | -1.06944356 | 26 10                             | 24              |
| 4   | 4.19179270  | 1.34344652  | -                                  | 24              |
| 5   | -0.11633101 | -1.04168414 | 27 11                             | 25              |
| 5   | -4.26375725 | -2.97763686 | -                                  | 26              |
| 6   | -0.07884831 | -1.02799337 | -                                  | 28              |
| 6   | 2.21204454  | 0.41789155  | -                                  | 29              |

Table 1: Constants of curves class I

Figure 22: Curve with $n = 2$

Figure 23: Curve with $n = 3$

Figure 24: Curve with $n = 4$

Figure 25: Curve with $n = 4$

Figure 26: Curve with $n = 5$

Figure 27: Curve with $n = 5$
5.2 Class II

For class II the angle $\theta'$ can be written for special values $A, B, C$ and odd $n$

$$\sin\left(\frac{n}{2}\theta'\right) = \frac{H_+ \sqrt{R_+}}{Q r^{n/2}}, \quad (154)$$

$$\cos\left(\frac{n}{2}\theta'\right) = \frac{H_- \sqrt{R_-}}{Q r^{n/2}}, \quad (155)$$

$$H_\pm = r^{n-2} \pm h_1 r^{n-3} + h_2 r^{n-4} \pm ... \pm h_{n-2}, \quad (156)$$

$$R_\pm = r \pm (Ar^4 + Br^2 + C), \quad (157)$$

$$R = R_+ R_-.$$  \( (158) \)

Differentiating (154, 155) one obtains

$$\frac{n \theta'}{2 dr} = \pm 2 \frac{dH_\pm}{dr} R_\pm + H_\pm \frac{dR_\pm}{dr} - nH_\pm R_\pm$$  \( (159) \)

This expression should yield

$$\frac{d\theta'}{dr} = A' r^4 + B' r^2 + C' \quad (160)$$

This requires

$$2r \frac{dH_\pm}{dr} R_\pm + H_\pm r \frac{dR_\pm}{dr} - nH_\pm R_\pm = \pm nH_\mp (A' r^4 + B' r^2 + C'). \quad (161)$$

If we multiply the equation with the upper signs by $H_+$ and that with the lower signs by $H_-$, and add both then we obtain

$$r^n \frac{d}{dr} \left( H^2_\pm R_+ + H^2_\pm R_- \right) = 0, \quad (162)$$

which after integration yields

$$H^2_\pm R_+ + H^2_\pm R_- = \text{const.} \quad (163)$$
which we set to $Q^2$. If we set

$$H_1 = r^{n-2} + h_2 r^{n-4} + ..., \quad H_2 = h_1 r^{n-3} + h_3 r^{n-5} + ..., \quad P = Ar^4 + Br^2 + C,$$

(164)

so that

$$H_{\pm} = H_1 \pm H_2, \quad R_{\pm} = r \pm P,$$

(165)

then

$$H_2^2 R_+ + H_2^2 R_- = 2(H_1^2 + H_2^2)r + 4H_1 H_2 P,$$

(166)

which is a polynomial of order $2n-1$ containing only terms with odd powers of $r$. Thus eq. (163) can only be fulfilled for odd $n$.

From eqs. (154, 155) one obtains

$$\cos(n\theta') = \frac{H_2 R_- - H_2 R_+}{Q^2 r^n}.$$  

(167)

Since

$$r^n \cos(n\theta') = \Re((x + iy)^n)$$  

(168)

and

$$H_2^2 R_- - H_2^2 R_+ = -2(H_1^2 + H_2^2)P - 4H_1 H_2 r$$

(169)

is a polynomial even in $r$, these curves are algebraic, too.

Eq. (161) is an equation for a polynomial of order $n+2$ in $r$. Equating the coefficients of the powers $n+2$ and $n+1$ yields $nA = nA'$ and $(n-2)Ah_1 = nA'h_1$. $A = 0$ would yield a constant curvature of the curve, thus only a circle or straight line, we require $A \neq 0$. Hence $A' = A$, $h_1 = 0$. The coefficients of the zeroth power in $r$ yield $-nC h_{n-2} = -nC'h_{n-2}$. Thus if $h_{n-2} \neq 0$, then $C' = C$.

The simplest case is given by $n = 3$. For this case one obtains $A' = A$, $B' = B/3$, $C' = C/3$. Here one does not obtain necessarily $C' = C$, since $h_1 = 0$. Requiring $B' = B$, $C' = C$ yields

$$\sin(3\theta/2) = \frac{r \sqrt{r + Ar^4}}{\sqrt{2} r^3} = \sqrt{\frac{1 + Ar^3}{2}},$$

(170)

$$\cos(3\theta/2) = \frac{r \sqrt{r - Ar^4}}{\sqrt{2} r^3} = \sqrt{\frac{1 - Ar^3}{2}}.$$  

(171)

Hence we obtain

$$\cos(3\theta) = -Ar^3$$

(172)

and with $A = -1/a^3$

$$r^3 = a^3 \cos(3\theta).$$

(173)

This curve is shown in figure 5.

For $n = 5$ we find

$$B = \frac{1}{4C} - 4AC^2, \quad q = \sqrt{8C}$$

(174)

37
and

\[ B' = -\frac{1}{20C} - \frac{12AC^2}{5}. \]

Thus \( B = B' \) is obtained for

\[ AC^3 = \frac{3}{16}, \quad BC = -\frac{1}{2}. \]

A remark on \( n = 5 \) of Greenhill\(^\text{21}\). §14: The ratio of minimal radius and maximal radius is \( (\sqrt[3]{10} - 1)/3 = 0.38481156 \) as given in the paper. §17 contains numerical errors: \( c = (\sqrt[3]{10} - 1)^2/3 = 0.44423982 \), and the ratio of maximal radius and minimal radius is \((1 + c)/(1 - c) = 2.59867451\), the inverse of the ratio of minimal and maximal radius in §14. The curves of §14 and §17 are not only of the same character, but they are the same.

If we follow the curve with a continuous change of the direction of its tangent, then we have to circle twice around the origin in the figures 30, 32, 33, until we return to the start of the curve, whereas in figure 31 we have to do this only once. The curves are listed in table 2.

6 Bicycle Curves

Most of the elastica with and without pressure and the Zindler curves are bicycle curves. Finn\(^\text{15}\) pointed out, that the class of bicycle curves is probably much
Table 2: Constants of curves class II

larger. Of course the curves we found for elastica under pressure have to be closed after one revolution. This is not required for bicycle curves. Also the Zindler curves can be generalized to a larger class of bicycle curves, which I call Zindler multicurves.

6.1 Zindler multicurves

We generalize the definition of the Zindler curves by replacing eqs. (25, 26) by

\[
x(\alpha) = l \cos(m\alpha) + \xi(\alpha), \quad y(\alpha) = l \sin(m\alpha) + \eta(\alpha),
\]

(177)
\[
\xi(\alpha) = \int_{\alpha} d\beta \cos(m\beta)\rho(\beta), \quad \eta(\alpha) = \int_{\alpha} d\beta \sin(m\beta)\rho(\beta),
\]  \hspace{1cm} (178)

where \(m\) is an odd integer. As in (27) we require
\[
\xi(\alpha + \pi) = \xi(\alpha), \quad \eta(\alpha + \pi) = \eta(\alpha),
\]  \hspace{1cm} (179)

which again implies \(\rho(\beta + \pi) = -\rho(\beta)\). The curves for \(m = 1\) are Zindler curves. For larger \(m\) the curves are no longer double point free. Generally they repeat only after \(m\) revolutions. These curves are also bicycle curves, since the argument around eq. (29) applies again.

Examples are
\[
\rho(\beta) = (m^2 - n^2) \sin(n\beta)
\]  \hspace{1cm} (180)

with odd \(n\), \(n \neq m\), and \(n, m\) coprime. One obtains for the envelopes (traces of the rear wheels)
\[
\xi(\alpha) = \frac{n-m}{2} \cos((n+m)\alpha) + \frac{n+m}{2} \cos((n-m)\alpha), \hspace{1cm} (181)
\]
\[
\eta(\alpha) = \frac{n-m}{2} \sin((n+m)\alpha) - \frac{n+m}{2} \sin((n-m)\alpha). \hspace{1cm} (182)
\]

With \(\alpha = u/2\) one may write
\[
z = x + iy = \frac{n-m}{2} e^{i(n+m)u/2} + \frac{n+m}{2} e^{i(m-n)u/2} + i e^{i\mu u/2},
\]  \hspace{1cm} (183)
a representation often given for the Zindler curve with \(m = 1, n = 3\),
\[
z = e^{2iu} + 2e^{-iu} + i e^{iu/2}. \hspace{1cm} (184)
\]

Examples of such Zindler curves, \(m = 1\), were shown in figures 8 and 9. We show four examples of Zindler multicurves with \(m = 3\) and \(m = 5\) in figures 34 to 37.

These curves turn \(m\) times round the origin. The envelope has \(n\) cusps. For \(n > m\) the cusps point outward, for \(n < m\) they point inward.

### 6.2 Other bicycle curves

We show some buckled rings, which turn around the center several times. The ratio of the maximal radius and the minimal radius is given. The buckled ring figure 48 is very similar to the Zindler multicurve figure 47.

Figures 49 to 55 show buckled rings which turn around the center nine times. These buckled rings have two different envelopes. The smaller one has five cusps pointing inward, figure 49. The outer envelope, figure 52 has no cusps. If the ratio of the largest distance to the smallest distance from the center is not too large, then the outer trace for the rear tire is without cusps. This is the case for the figures 50 to 52. If the ratio becomes larger, then cusps appear as seen in figures 53 to 55.
7 Conclusion

In this paper a short historical account of the curves related to the two-dimensional floating bodies of equilibrium and the bicycle problem is given. Bor, Levi, Perline and Tabachnikov found that quite a number of the boundary curves had already been described as *Elastica* and *Elastica under Pressure* or *Buckled*. 
Rings. Auerbach already realized that curves described by Zindler are solutions for the floating bodies problem of density $1/2$. An even larger class of curves solves the bicycle problem.

The subsequent sections deal with some supplemental details: Several derivations of the equations for the elastica and elastica under pressure are given. The properties of Zindler curves and some work on the problem of floating bodies of equilibrium by other mathematicians is discussed. Special cases of elastica under pressure lead to algebraic curves as shown by Greenhill. Since most of the curves considered here are bicycle curves, we added some remarks on them.

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