From the Mahler conjecture to Gauss linking integrals

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Dedicated to my father, on no particular occasion

We establish a version of the bottleneck conjecture, which in turn implies a partial solution to the Mahler conjecture on the product \( v(K) = \langle \text{Vol} \rangle K / \langle \text{Vol} \rangle K^0 \) of the volume of a symmetric convex body \( K \in \mathbb{R}^n \) and its polar body \( K^0 \). The Mahler conjecture asserts that the Mahler volume \( v(K) \) is minimized (non-uniquely) when \( K \) is an \( n \)-cube. The bottleneck conjecture (in its least general form) asserts that the volume of a certain domain \( K^0 \subseteq K \times K^0 \) is minimized when \( K \) is an ellipsoid. It implies the Mahler conjecture up to a factor of \( (\frac{2}{\sqrt{\pi}})^n \gamma_n \), where \( \gamma_n \) is a monotonic factor that begins at \( \frac{2}{\sqrt{\pi}} \) and converges to \( \sqrt{\pi} \). This strengthens a result of Bourgain and Milman, who showed that there is a constant \( c \) such that the Mahler conjecture is true up to a factor of \( c^n \).

The proof uses a version of the Gauss linking integral to obtain a constant lower bound on \( \text{Vol} \) \( K^0 \), with equality when \( K \) is an ellipsoid. It applies to a more general conjecture concerning the join of any two necks of the pseudospheres of an indefinite inner product space. Because the calculations are similar, we will also analyze traditional Gauss linking integrals in the sphere \( S^{n-1} \) and in hyperbolic space \( H^{n-1} \).

1. INTRODUCTION

If \( K \subset \mathbb{R}^n \) is a centrally symmetric convex body, let \( K^0 \) denote its dual or polar body. (It is also the unit ball of the norm dual to the one defined by \( K \).) The product of the volumes

\[ v(K) = \langle \text{Vol} \rangle K / \langle \text{Vol} \rangle K^0 \]

is known as the Mahler volume of \( K \). Since it is both continuous in \( K \) and affinely invariant, it achieves a finite maximum and a non-zero minimum in each dimension \( n \). Mahler \([15]\) conjectured an upper bound and a lower bound for the value of \( v(K) \) in each fixed dimension. The upper bound was proven by Santaló and is known as the Blaschke-Santalo inequality \([3, 19, 20]\):

**Theorem 1.1 (Blaschke, Santaló).** In any fixed dimension \( n \), \( v(K) \) is uniquely maximized by ellipsoids.

The lower bound is still a conjecture:

**Conjecture 1.2 (Mahler).** In any fixed dimension \( n \), \( v(K) \) is minimized among centrally symmetric convex bodies by the cube \( C_n = [-1, 1]^n \).

The minimization problem is considered harder because the conjectured minimum (or the real minimum) is a much more complicated shape than a round sphere. For instance, Saint-Raymond \([19]\) observed that the Mahler volume of an \( n \)-cube \( C_n \) is tied not only by the volume of the dual \( C_n^* \), a cross-polytope, but also by other polytopes when \( n \) is large. (Nonetheless, the Mahler conjecture has been established in some special cases, including 1-unconditional convex bodies \([19]\) and zonoids \([9, 17]\).)

In a noted paper, Bourgain and Milman showed that the Mahler conjecture is true up to an exponential factor \([5]\):

\[ v(K) \geq c^n v(C_n) \]

The proof of Theorem 1.3 has been simplified by Pisier \([16]\) and independently by Milman. Although all of the proofs technically construct the constant \( c \), no good value for it is currently known. Note that the Mahler volume of a sphere and a cube differ only by a factor of \( c^n \). So the Bourgain-Milman theorem says that all Mahler volumes in \( n \) dimensions only differ by some exponential factor. (Indeed, the Bourgain-Milman theorem also holds without central symmetry; see below.)

In this article, we will establish a better value of the constant \( c \) by comparing the Mahler volume with another volume. We will minimize the volume of a certain body \( K^\circ \subset K \times K^0 \). Define the subsets

\[ K^\pm = \{ (x, y) \in K \times K^0 | x \cdot y = \pm 1 \} \]

of the hyperboloids

\[ H^\pm = \{ (x, y) | x \cdot y = \pm 1 \} \]

in \( \mathbb{R}^n \times \mathbb{R}^n \). Suppose that \( K \) and \( K^0 \) are both positively curved. (I.e., they have no boundary points with zero extrinsic curvature in any direction.) We previously showed that \( K^\pm \) is a spacelike section of \( H^+ \), that \( K^- \) is a timelike section of \( H^- \), and that their inclusions are homotopy equivalences \([13]\). Submanifolds of \( H^+ \) and \( H^- \) of this type are called necks. The body \( K^\circ \) is the filled join, defined in Section 2 of the two necks \( K^+ \) and \( K^- \). (In Reference 13, \( K^\circ \) was the convex hull of \( K^+ \) and \( K^- \). Since the filled join is the real object of study, we have renamed it \( K^\circ \). We do not know if it is ever non-convex.)

**Theorem 1.4 (Main theorem).** Let \( N^+ \) and \( N^- \) by any two necks of the positive and negative unit pseudospheres \( H^+ \) and \( H^- \) in any indefinite inner product space \( \mathbb{R}^{n, b} \), and let \( N^0 \) be their filled join. Then \( \text{Vol} N^\circ \) is minimized when \( N^+ \) and \( N^- \) are flat, orthogonal, and centered at the origin.

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Because of the geometry of the necks $N^+$ and $N^-$, we previously called Theorem 1.4 “the bottleneck conjecture.” We will interpret Theorem 1.4 as a new isoperimetric-type inequality for convex bodies:

**Corollary 1.5.** If $K \subset \mathbb{R}^n$ is a centrally symmetric convex body, then $\text{Vol } K^\diamond$ is minimized when $K$ is an ellipsoid.

Corollary 1.5 follows immediately from Theorem 1.4 when $K$ and $K^\circ$ are positively curved, because necks $K^+$ and $K^-$ that come from a symmetric convex body $K$ are flat, orthogonal, and centered when $K$ is an ellipsoid. By continuity, the bound must also hold for arbitrary $K$.

On the one hand, $\text{Vol } K^\diamond$ is minimized when the Mahler volume $\nu(K)$ is maximized. On the other hand, $K^\diamond$ is only moderately exponentially smaller than its superset $K \times K^\circ$. Explicitly, let $B_n$ be the round unit ball in $\mathbb{R}^n$. Then

$$\text{Vol } B_n^\diamond = \frac{2^n (n!)^2 \pi^n}{(2n)! (n/2)!^2}$$

because $B_n$ is the join of two round, orthogonal $n$-balls of radius $\sqrt{2}$ and

$$\text{Vol } B_n = \frac{\pi^{n/2}}{(n/2)!}.$$

In comparison,

$$\nu(C_n) = \frac{4^n}{n!}.$$

In conclusion:

**Corollary 1.6.** If $K \subset \mathbb{R}^n$ is a centrally symmetric convex body, then

$$\nu(K) \geq \frac{2^n (n!)^2 \pi^n}{(2n)! (n/2)!^2} = \gamma_n \left( \frac{\pi}{4} \right)^n\nu(C_n),$$

where

$$\frac{4}{\pi} \leq \gamma_n = \frac{n^{3/2} 2^n}{(2n)! (n/2)!^2} \rightarrow \sqrt{2}.$$

Thus, our value of $c$ in Theorem 1.3 is $\pi/4$. Since the Bourgain–Milman theorem is also often phrased in terms of $\nu(B_n)$, the Mahler volume of a round ball, we can also write Corollary 1.6 as

$$\nu(K) \geq \frac{2^n (n!)^2}{(2n)!} \nu(B_n) > 2^{-n} \nu(B_n).$$

Thus, our value for the Bourgain–Milman constant in this form is $\frac{4}{\pi}$. Mahler’s conjecture would imply $\frac{2}{\pi}$.

Corollary 1.6 also improves a previous non-asymptotic estimate which was inspired by the bottleneck conjecture [12]. There we established that if $K \subset \mathbb{R}^n$ and $n \geq 4$, then

$$\nu(K) \geq (\log_2 n)^{-n} \nu(B_n).$$

Corollary 1.6 is strictly better for all $n \geq 4$.

There is also a version of the Mahler conjecture for asymmetric convex bodies that contain the origin in their interiors.

**Conjecture 1.7** (Mahler). In any fixed dimension $n$, $\nu(K)$ is minimized among convex bodies $K$, whose interiors contain the origin, by a centered simplex $\Delta_n$.

Although we will only directly study centrally symmetric convex bodies, Corollary 1.6 yields a corollary for general convex bodies using a standard technique.

**Corollary 1.8.** If $K \subset \mathbb{R}^n$ is a convex body with the origin in its interior, then

$$\nu(K) \geq \frac{4^n (n!)^4 \pi^n}{(2n)! (n/2)!^4} = \delta_n \left( \frac{\pi}{2e} \right)^n \nu(\Delta_n),$$

where

$$\frac{2e}{\pi} \leq \delta_n = \frac{8^n n!^2 e^n}{(2n)! (n/2)!^4 (n + 1)^{n+1}} \rightarrow \frac{2 \pi}{e}.$$

Thus, our value for the Bourgain–Milman constant in the asymmetric case is $\frac{\pi}{2e}$.

**Remark.** Even though the Bourgain–Milman inequality holds in the asymmetric case, the bound in Corollary 1.6 does not. In particular,

$$\lim_{n \to \infty} \left( \frac{\nu(\Delta_n)}{\nu(C_n)} \right)^{1/n} = e^{-1/4} < \frac{\pi}{4}.$$

The proof fails because when $K$ is not centrally symmetric, only $K^+$ exists as a subset of $K \times K^\circ$. The region $K^\diamond$ can be formed but its volume may be larger than $\nu(K)$. Nonetheless Theorem 1.4 does say something about asymmetric convex bodies. We can let $N^+ = K_1^+$ for one convex body $K_1$, and let $N^-$ be a natural isometric image of $K_2^+$ for another convex body $K_2$. Then the statement is that $\nu(N^\circ)$ is minimized when $K_1$ and $K_2$ coincide, and are a centered ellipsoid.

**Proof.** Given a convex body $K$, then $K - K$ is its difference body, by definition the set of differences between points in $K$. It is a centrally symmetric convex body. Rogers and Shephard [18] showed that

$$\text{Vol } K - K \leq \left( \frac{2n}{n} \right) \text{Vol } K,$$

with equality if and only if $K$ is a simplex. The polar body of $K - K$ is best described by a relation between the corresponding norms on $\mathbb{R}^n$:

$$||x||_{K - K} = ||x||_{K^\circ} + ||x||_{-K^\circ}.$$

Since

$$\text{Vol } K = \int_{S^{n-1}} \frac{1}{||x||_K} d\bar{x},$$

and since the integrand is convex, Jensen’s inequality tells us that

$$\text{Vol } (K - K)^\circ \leq 2^{-n} (\text{Vol } K^\circ),$$
with equality if and only if $K$ is centrally symmetric. Combining these estimates with Corollary 1.6 establishes the claim.

The numerical comparison to the simple case uses the elementary volume formula

$$v(\Delta_n) = \frac{(n+1)^{n+1}}{(n!)^2}.$$ 

Section 4 is a digression in which we will compute invariant linking forms for the sphere $S^{n-1}$ and for hyperbolic space $H^{n-1}$. The computations are similar to Lemma 3.2 used in the proof of Theorem 1.4. We also give a second derivation in the sphere case using elementary geometry and probability instead of Stokes’ theorem.

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2. A REVIEW

In this section, we review some relevant definitions and results in Reference 13 and set up our new arguments. As in the introduction, let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body such that $K$ and $K^\circ$ are both positively curved. (Note that $K$ is positively curved if and only if $K^\circ$ is.)

Let $V$ be an (non-degenerate) indefinite inner product space; we will write all inner products as dot products. Recall that a non-zero vector $\vec{x} \in V$ is spacelike if $\vec{x} \cdot \vec{x} > 0$, timelike if $\vec{x} \cdot \vec{x} < 0$, and null if $\vec{x} \cdot \vec{x} = 0$. (The definitions of spacelike and timelike can be switched in some contexts.) The positive and negative unit pseudospheres of $V$ are the solutions sets to the equations $\vec{x} \cdot \vec{x} = \pm 1$. Every indefinite inner product space is isometric to the standard inner product space $\mathbb{R}^{(a,b)}$ with inner product

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ay_a - x_{a+1}y_{a+1} - x_{a+2}y_{a+2} - \cdots - x_{a+b}y_{a+b}$$

with signature $(a,b)$, for some $a$ and $b$.

Recall that a pseudo-Riemannian manifold $M$ is a smooth manifold with a smooth field of non-degenerate inner products, which is called a metric. The metric on $M$ restricts to a metric on every submanifold $N \subset M$; however, the restriction may or may not be non-degenerate. A submanifold $N$ of a pseudo-Riemannian manifold $M$ is spacelike, timelike, or null if all of its non-zero tangent vectors are, respectively, spacelike, timelike, or null in their respective tangent spaces.

For example, the positive and negative pseudospheres $H^\pm$ of $\mathbb{R}^{(a,b)}$ are pseudo-Riemannian with signature $(a-1,b)$ and $(a,b-1)$. They are diffeomorphic to $S^{a-1} \times \mathbb{R}^b$ and $\mathbb{R}^a \times S^{b-1}$. The diffeomorphisms can be chosen so that every fiber of the first factor is spacelike and every fiber of the second factor is timelike.

If $A$ and $B$ are two sets in $\mathbb{R}^n$, we define their geometric join $A \ast B$ to be the union of all line segments that connect a point in $A$ and a point in $B$. The geometric join is a kind of partial convex hull. If $M \subset \mathbb{R}^n$ is a closed manifold of codimension 1, we define the filling $\overline{M}$ of $M$ to be the compact region in $\mathbb{R}^n$ that it encloses. If $A \ast B$ is such a manifold, then $\overline{A \ast B}$ is the filled join of $A$ and $B$.

The vector space $\mathbb{R}^n \times \mathbb{R}^n$ has a relevant inner product (or dot product), with signature $(n,n)$, given by the formula

$$(x_1,y_1) \cdot (x_2,y_2) = \frac{x_1 \cdot y_2 + x_2 \cdot y_1}{2}. \quad (1)$$

The hyperboloids $H^+$ and $H^-$ from the introduction are the positive and negative unit pseudospheres with respect to this inner product. Note that this inner product does not have determinant $\pm 1$. We will use the standard volume structure on $\mathbb{R}^n \times \mathbb{R}^n$ rather than the one induced by $\mathbb{R}^n$. (It may have been a bit clearer to put $K$ in a vector space $V$ and then define this inner product on $V \oplus V^*$. We will use the notation $K \subset \mathbb{R}^n$ because it is standard in finite-dimensional convex geometry.)

![Figure 1: The geometry of $K^+$, $K^-$, and $K^\circ$ in $\mathbb{R}^n \times \mathbb{R}^n$.](image)

In this section, we will establish the facts that $K^+$ is a neck of $H^+$ (to be defined precisely below), that $K^-$ is a neck of $H^-$, and that their filled join $K^\circ = K^+ \ast K^-$ is a starlike body that lies between $H^+$ and $H^-$. The overall geometry is shown in Figure 1. As the figure suggests, $K^\circ$ is sometimes all of $K \times K^\circ$; for example, when $K = C_n$. (A useful exercise is to work out the geometry of $K^\circ$ when $K = C_2$. In this case, $K^+$ and $K^-$ are non-planar octagons that visit all 16 vertices of $K \times K^\circ$, which is a 4-cube.) But in other cases, $K^\circ$ is significantly smaller than $K \times K^\circ$.

Every point in $K^+$ is a pair consisting of a point $\vec{x} \in K$ and a dual vector $\vec{y} \in K^\circ$ that represents a supporting hyperplane of $\vec{x}$. In this situation, $\vec{x} \in \partial K$ and $\vec{y} \in \partial K^\circ$. Moreover, since $K$ and $K^\circ$ are both smooth, $\vec{x}$ uniquely and smoothly determines $\vec{y}$ and vice versa. Therefore $K^\circ$ is diffeomorphic to $\partial K$, and in particular, is diffeomorphic to the sphere $S^{n-1}$. Likewise $K^-$ is as well.
Lemma 2.1. $K^+$ is a spacelike submanifold of $H^+$ and $K^-$ is a timelike submanifold of $H^-$. 

Proof. (Sketch) The proof is from Reference [13]. The convex body $K$ has a unique osculating ellipsoid $E$ at the point $\bar{x}$. Because $E$ is an ellipsoid, $E^+$ is a flat ellipsoid surface in $H^+ \subset \mathbb{R}^n \times \mathbb{R}^n$, and in particular is a spacelike manifold. Since $E$ osculates $K$ at $\bar{x}$, it follows that

$$T_{(\bar{x},\bar{y})}K^+ = T_{(\bar{x},\bar{y})}E^+.$$ 

Thus, $K^+$ is spacelike, and likewise $K^-$ is timelike. \hfill \Box

In addition, $K^+$ is a spacelike section of $H^+$, in two senses. Since $H^+$ is a pseudo-Riemannian manifold of signature $(n-1,n)$, $K^+$ is a spacelike section in the sense of having maximal dimension. But also, the normal exponential map from any flat section $K^+$ of $H^+$ (such as $E^+$ if $E$ is an ellipsoid) makes $H^+$ a bundle of hyperbolic spaces over a sphere. Then $K^+$ is a section of this bundle, because it is both spacelike and isotopic to $K^+$. ($K^+$ is isotopic to $E^+$ by deforming $K$ to $E$, and then all flat sections are isotopic.) Likewise $K^-$ has the same properties in $H^-$. 

If $V$ is any indefinite inner product space and $H^\pm$ are its positive and negative unit pseudospheres, define a neck of $H^\pm$ to be a spacelike or timelike submanifold $\mathbb{N}^\pm$ which is a section in both of the above senses. If $V$ has signature $(a,b)$, then $\mathbb{N}^+$ has dimension $a-1$ and $\mathbb{N}^-$ has dimension $b-1$. 

For the main result, we will let $V = \mathbb{R}^{(a,b)}$. Each spacelike $a$-plane in $\mathbb{R}^{(a,b)}$ has a canonical orientation, as does each timelike $b$-plane. The group of isometries of $\mathbb{R}^{(a,b)}$ that preserve its total orientation is $\text{SO}(a,b)$. The subgroup that separately preserves the orientations of maximal spacelike and timelike planes is $\text{ISO}(a,b)$; it is a connected Lie group.

Let $\mathbb{N}^+ \subset \Phi \mathbb{N}^-$ be necks of $H^+$ and $H^-$. 

Lemma 2.2. The geometric join $N^+ \ast N^-$ is the boundary of a starlike body in $\mathbb{R}^n \times \mathbb{R}^n$.

Proof. The argument is a refinement of the one in Reference [13], where it was argued that $K^+ \ast K^-$ is starlike except possibly on a set of measure 0. 

Let $\mathbb{N}^+ \subset \mathbb{N}^-$ denote the abstract join of $N^+$ and $N^-$ as defined in topology (and usually written $N^+ \ast N^-$), and let

$$\Phi : \mathbb{N}^+ \subset \mathbb{N}^- \rightarrow \mathbb{R}^{(a,b)}$$

be the obvious continuous map with image $N^+ \ast N^-$, given by the formula

$$\Phi(\bar{x},\bar{y},t) = (1-t)\bar{x} + t\bar{y}.$$ 

Let

$$\rho : \mathbb{R}^{(a,b)} \rightarrow S^{a+b-1}$$

be the tautological radial projection.

The composition $\rho \circ \Phi$ is a map between $(a+b-1)$-spheres. We claim that it has degree 1, that it is smooth with non-zero Jacobian except on $N^+$ and $N^-$, and that it is a local homeomorphism at $N^+$ and $N^-$. The last two claims imply that $\rho \circ \Phi$ is a covering map and therefore a bijection by the first claim. This is equivalent to the assertion of the lemma.

It is elementary that $\rho \circ \Phi$ is smooth on the smooth points of its domain.

The claim that $\rho \circ \Phi$ has degree 1 is not really necessary, because $S^{a+b-1}$ is simply connected unless $a = b = 1$, in which case it is easy to show that $N^+ \ast N^-$ is a starlike quadrilateral. One way to show it is that $\rho \circ \Phi$ varies continuously as $N^+$ and $N^-$ are varied. If we take $N^+$ and $N^-$ to be orthogonal, flat, and centered, then it can be seen directly that $N^+ \ast N^-$ is starlike and indeed convex, which implies that $\rho \circ \Phi$ has degree 1.

To confirm that $\rho \circ \Phi$ has non-zero Jacobian away from $N^+$ and $N^-$, let $\bar{x} \in N^+$ and $\bar{y} \in N^-$, and let

$$\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{a-1} \in T_{\bar{x}}N^+ \quad \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_{b-1} \in T_{\bar{y}}N^-$$

be bases. Suppose that they are chosen so that

$$\bar{x}, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{a-1} \quad \bar{y}, \bar{w}_1, \bar{w}_2, \ldots, \bar{w}_{b-1}$$

are both positively oriented bases of maximal timelike and spacelike planes. Given coordinates on $N^+$ and $N^-$ that induce these tangent bases, the Jacobian of $\rho \circ \Phi$ at the point $(\bar{x},\bar{y},t)$ is

$$J(\bar{x},\bar{y},t) = \det[(1-t)\bar{v}_1, (1-t)\bar{v}_2, \ldots, (1-t)\bar{v}_{a-1}, t\bar{w}_1, t\bar{w}_2, \ldots, t\bar{w}_{b-1}, \bar{y} - \bar{x}, (1-t)\bar{y} + t\bar{x}]$$

and so does not vanish.

Figure 2: $\pi(N^+)$ is starlike and encloses the hole of $\pi(H^+)$. 

Finally at a point $\bar{y} \in N^-$, the map $\Phi$ has a local conelike structure. It is (tautologically) smooth and non-singular along the tangent space $T_{\bar{y}}N^-$, while the direction of $\bar{y}$ is annihilated by the radial projection $\rho$. In the remaining directions, the map $\Phi$ is locally a conical extension of the restriction of the projection

$$\pi : \mathbb{R}^{(a,b)} \rightarrow (T_{\bar{y}}N^- + \langle \bar{y} \rangle)^\perp$$

to the other neck $N^+$. Because $N^+$ is a spacelike section, $\pi(N^+)$ must be starlike. This implies that its cone extension is a homeomorphism, and therefore that $\rho \circ \Phi$ is a local homeomorphism at $\bar{y}$. The same argument applies to $\bar{x} \in N^+$. \hfill \Box
Lemma 2.2 implies that $N^+$ and $N^-$ admit a filled join

$$N^\flat = N^+ \ast N^-.$$  

The lemma also means that the volume of $N^\flat$ can be expressed as an integral, by the method of infinitesimals:

$$\text{Vol } N^\flat = \frac{(a-1)!(b-1)!}{(a+b)!} \int_{\bar{\mathbb{R}}^{a+b}} \bar{x} \wedge \bar{y} \wedge dx^{a-1} \wedge dy^{b-1}. \quad (2)$$

**Remark.** Equation (2) is different in several ways from the formula in Reference [13]. The most important change is that the constant factor was not correct previously. In addition, the previous formula was for the volume of $K^\flat$, which is here called $K^\flat$. Finally the old formula had $d\bar{x}$ and $d\bar{y}$ instead of their wedge powers. While this was plausible notation, the wedge power notation is more literally correct and will be important to the proof of Theorem [1,4].

Equation (2) is technically only true up to sign, but we can make it exactly true with suitable orientations for the necks.

The differential forms in equation (2) are vector-valued and the wedge products are really “double wedges” in the algebra

$$A = \Lambda^*(\mathbb{R}^{a,b}) \otimes \Omega^*(\mathbb{R}^{a,b}).$$

Even though both factors are graded-commutative algebras, we define multiplication in $A$ by tensoring them as ungraded algebras. If $\omega_1, \omega_2 \in A$ have degrees $(j_1,k_1)$ and $(j_2,k_2)$, then

$$\omega_1 \wedge \omega_2 = (-1)^{j_1 j_2 + k_1 k_2} \omega_2 \wedge \omega_1.$$  

For example,

$$\bar{x} \wedge \bar{y} = -\bar{y} \wedge \bar{x} \quad \bar{x} \wedge d\bar{y} = -d\bar{y} \wedge \bar{x} \quad d\bar{x} \wedge d\bar{y} = d\bar{y} \wedge d\bar{x}.$$  

Equation (2) is established by assuming simplex shapes for the infinitesimal elements $d\bar{x}$ and $d\bar{y}$ of $N^+$ and $N^-$, so that the corresponding infinitesimal element of $N^\flat$ is also a simplex (subtended by $\bar{x}$, $d\bar{x}$, $\bar{y}$, and $d\bar{y}$). Strictly speaking, the integral on the right side of (2) is not scalar, but rather has a value in $\Lambda^{a+b}(\mathbb{R}^{a,b})$. However, it can be read as a scalar because $\Lambda^{a+b}(\mathbb{R}^{a,b})$ is 1-dimensional; we endow it with the standard basis vector

$$\tilde{e}_1 \wedge \tilde{e}_2 \wedge \cdots \wedge \tilde{e}_{a+b}.$$  

3. PROOF OF THEOREM [1,4]

The idea of the proof is to find a topological lower bound for the integral in equation (2). We can compactify the union $H^+ \cup H^-$ by radially projecting it onto the Euclidean-unit sphere $S^{a+b-1}$ in $\mathbb{R}^{a,b}$, similar to the proof of Lemma 2.2. The image is dense; the points in $S^{a+b-1}$ that are missed are null vectors. Thus we have a compactification

$$H^+ \cup H^- \cong S^{a+b-1}.$$

Our lower bound will be proportional to a Gauss-type integral for the linking number of $N^+$ and $N^-$ in $H^+ \cup H^-$, which is necessarily 1. Our Gauss linking form will be invariant under the non-compact group $SO(a, b)$ rather than the usual rotation group $SO(a+b)$.

If $X$ and $Y$ are two manifolds, then differentials on $X \times Y$ are doubly graded according to their degrees in the $X$ and $Y$ directions. The full differential $d$ also splits as $d = d_1 + d_y$ according to differentiation along $X$ and $Y$ separately. This refines the de Rham complex $\Omega^* (X \times Y)$ into a double complex. In our case, $X = H^+$ and $Y = H^-$. As before, we will use the vector coordinates $\bar{x} \in H^+$ and $\bar{y} \in H^-$ using the inclusions of $H^*$ in $\mathbb{R}^{a,b}$.

**Lemma 3.1.** Let $\omega$ be an $ISO(a,b)$-invariant differential form on $H^+ \times H^-$ of degree $(a-1,b-1)$, and let $N^+$ and $N^-$ be closed, smooth submanifolds or cycles of dimension $a$ and $b$ in $H^+$ and $H^-$. Also suppose that $a,b > 1$. Then

$$\int_{N^+ \times N^-} \omega$$

is invariant under chain homotopy of $N^+$ and $N^-$ if and only if $d_1 d_y \omega = 0$.

**Remarks.** Since $d_1$ and $d_y$ anticommutate, the condition on $\omega$ is symmetric in $\bar{x}$ and $\bar{y}$. In the context of double complexes, $\omega$ can be called “weakly closed”.

Our form $\omega$ will actually be $SO(a,b)$-invariant. But $SO(a,b)$ is not connected when $a,b > 0$. Its connected subgroup $ISO(a,b)$ is more natural for this lemma (and not really different).

DeTurck and Gluck [6] show that the double integral of a similar form $\omega$ (see Section 4) is chain-homotopy invariant if and only if it is weakly closed and

$$d_1 \omega = d_y \sigma \quad d_y \omega = d_1 \kappa$$

for some forms $\sigma$ and $\kappa$. This criterion does not require $\omega$ to be $ISO(a,b)$-invariant.

**Proof.** We will first consider the “if” direction, which is the one that we will need. Let

$$\omega_k = \int_{N^-} \omega.$$  

Then

$$d_1 \omega_k = \int_{N^-} d_1 \omega$$

is a differential form on $H^+$ of degree $a$. By Stokes’ theorem, it is invariant under chain homotopy of $N^-$, because $d_1$ annihilates the integrand. In particular, $d_1 \omega_k$ is ISO$(a,b)$-invariant.

But there are no non-zero $ISO(a,b)$-invariant $a$-forms on $H^+$. They are sections of $N^*(T^*H^+)$. Each fiber of $T^*H^+$ is a representation of the point stabilizer ISO$(a-1,b)$ and is isomorphic to the defining representation $V$. The exterior power $\Lambda^a (V)$ has no invariant vectors unless $\dim V = a$. This can be checked by passing to the action of the complexified Lie algebra so$(a+b-1,\mathbb{C})$ on $\Lambda^a (V) \otimes \mathbb{C}$. The claim is then a basic calculation in the representation theory of complex simple Lie algebras. Namely, $\Lambda^a (V) \otimes \mathbb{C}$ is an irreducible representation.
of so(a + b − 1, C) unless a = b = 1, in which case it is a direct sum of two irreducible representations. The only case with a, b > 0 in which it is the trivial representation is when b = 1, which we have excluded by hypothesis.

Thus, $\omega_x$ is a closed form on $H^+$, so

$$ \int_{N^+} \omega_x = \int_{N^+ \times N^+} \omega $$

is invariant under chain homotopy of $N^+$. By the same argument, it is also invariant under chain homotopy of $N^-$. This concludes the “if” part of the lemma.

To sketch the “only if” part, suppose that $N^+_0$ and $N^-_1$ are two necks of $H^+$ that agree except for a locus which is the boundary of an $a$-dimensional blister $L^+$. (By a blister, we mean any manifold with cusp boundary that connects two smooth manifolds that are only partly disjoint.) Likewise, suppose that $N^-_0$ and $N^+_1$ are two necks of $H^-$ that differ only at a $b$-dimensional blister $L^-$. By two applications of Stokes’ theorem,

$$ 0 = \sum_{p,q \in \{0,1\}} (-1)^{p+q} \int_{N^+_p \times N^-_q} \omega = \int_{L^+ \times L^-} d_x d_y \omega. $$

But then, the blisters $L^+$ and $L^-$ can be arbitrarily small and can lie in any direction, so that in the limit,

$$ d_x d_y \omega = 0. $$

□

**Lemma 3.2.** If

$$ \omega = \phi(\tilde{x} \cdot \tilde{y}) \tilde{x} \wedge \tilde{y} \wedge d\tilde{x}^{a-1} \wedge d\tilde{y}^{b-1} $$

is an SO(a, b)-invariant differential form on $H^+ \times H^-$ in $\mathbb{R}^{(a, b)}$, then $d_x d_y \omega = 0$ if and only if $f'(\alpha) = \phi(\sinh(\alpha))$ satisfies the ordinary differential equation

$$ f'' + (a + b)(\tan \alpha) f' + abf = 0. $$

**Proof.** We compute:

$$ d_x \omega = (\tilde{y} \cdot d\tilde{x}) \phi(\tilde{x} \cdot \tilde{y}) \tilde{x} \wedge \tilde{y} \wedge d\tilde{x}^{a-1} \wedge d\tilde{y}^{b-1} $$

$$ - \phi(\tilde{x} \cdot \tilde{y}) \tilde{x} \wedge d\tilde{x}^{a-1} \wedge d\tilde{y}^{b-1}. $$

Then

$$ d_x d_y \omega = (\tilde{x} \cdot d\tilde{y})(\tilde{y} \cdot d\tilde{x}) \phi(\tilde{x} \cdot \tilde{y}) \tilde{x} \wedge \tilde{y} \wedge d\tilde{x}^{a-1} \wedge d\tilde{y}^{b-1} $$

$$ + (\tilde{y} \cdot d\tilde{x}) \phi(\tilde{x} \cdot \tilde{y}) \tilde{x} \wedge \tilde{y} \wedge d\tilde{x}^{a-1} \wedge d\tilde{y}^{b-1} $$

$$ - (\tilde{x} \cdot d\tilde{y}) \phi(\tilde{x} \cdot \tilde{y}) \tilde{x} \wedge d\tilde{x}^{a-1} \wedge d\tilde{y}^{b-1} $$

$$ - (\tilde{y} \cdot d\tilde{x}) \phi(\tilde{x} \cdot \tilde{y}) \tilde{x} \wedge d\tilde{x}^{a-1} \wedge d\tilde{y}^{b-1} $$

$$ - \phi(\tilde{x} \cdot \tilde{y}) \tilde{x} \wedge d\tilde{x}^{a-1} \wedge d\tilde{y}^{b-1} = 0. $$

In addition, the constraints

$$ \tilde{x} \cdot \tilde{x} = 1 \quad \tilde{y} \cdot \tilde{y} = -1 $$

yield the differential conditions

$$ \tilde{x} \cdot d\tilde{x} = 0 \quad \tilde{y} \cdot d\tilde{y} = 0. $$

The form $\omega$ is explicitly invariant under the diagonal action of SO(a, b), which is transitive on pairs of vectors $(\tilde{x}, \tilde{y}) \in H^+ \times H^-$ with a prescribed dot product. (Indeed, the given form of $\omega$ is the general expression for an invariant form of degree $(a-1, b-1)$.) Therefore we can check the condition $d_x d_y \omega = 0$ for the standard vectors

$$ \tilde{x} = (1, 0, 0, \ldots, 0) \quad \tilde{y} = (\sinh \alpha, 0, 0, \ldots, \cosh \alpha, \ldots, 0). $$

Here the two non-zero coordinates of $\tilde{y}$ are $y_1$ and $y_{a+1}$ and $\alpha$ is a hyperbolic angle (or rapidly). Then the conditions (4) simplify to

$$ dx_1 = 0 \quad (\sinh \alpha) dy_1 - (\cosh \alpha) dy_{a+1} = 0. $$

Let $d\tilde{x}$ and $d\tilde{y}$ be $d\tilde{x}$ and $d\tilde{y}$ with the first and $(a + 1)$ st coordinates deleted. Then equation (3) becomes (term by term)

$$ (-1)^a d_x d_y \omega = $$

$$ (\cosh \alpha)^2 \phi'' dy_1 \wedge d\tilde{x}_{a+1} \wedge d\tilde{y}^{a-1} \wedge d\tilde{y}^{b-1} $$

$$ + (\cosh \alpha) \phi' dy_{a+1} \wedge d\tilde{x}_{a+1} \wedge d\tilde{y}^{a-1} \wedge d\tilde{y}^{b-1} $$

$$ - b(\cosh \alpha) \phi' dy_{a+1} \wedge d\tilde{x}_{a+1} \wedge d\tilde{y}^{a-1} \wedge d\tilde{y}^{b-1} $$

$$ + a(\sinh \alpha) \phi' dy_1 \wedge d\tilde{x}_{a+1} \wedge d\tilde{y}^{a-1} \wedge d\tilde{y}^{b-1} $$

$$ + abf \phi d\tilde{x}^{a-1} \wedge d\tilde{y}^{b-1} = 0 $$

This expression so far uses the relations

$$ dx_1 = 0 \quad dy_1 \wedge d\tilde{y}_{a+1} = 0 $$

to eliminate terms in the expansion. If we apply the remaining relation

$$ (\sinh \alpha) dy_1 - (\cosh \alpha) dy_{a+1} = 0 $$

and collect differential factors, we conclude that all of the terms are proportional to

$$ dy_1 \wedge d\tilde{x}_{a+1} \wedge d\tilde{y}^{a-1} \wedge d\tilde{y}^{b-1}, $$

and the scalar factor yields the equation

$$ (\cosh \alpha)^2 \phi'' + (a + b + 1) (\sinh \alpha) \phi' + abf = 0. $$

However, the notation of this differential equation is misleading, because $\phi$ is not directly a function of $\alpha$. Rather, $\phi = \phi(t)$, where

$$ t = \tilde{x} \cdot \tilde{y} = \sinh \alpha. $$

If we let $f(\alpha) = \phi(\sinh \alpha)$, then

$$ f'' + (a + b)(\tan \alpha) f' + abf = 0 $$

is the corresponding ODE for $f$. □
Lemma 3.3. If $f(\alpha)$ satisfies the differential equation
\[ f'' + (a+b)(\tanh \alpha)f' + abf = 0 \]
and $f'(0) = 0$, then $|f(\alpha)| < |f(0)|$ when $\alpha \neq 0$.

Figure 3 shows an example.

Proof. The differential equation for $f$ is that of a damped harmonic oscillator, where the damping is forwards in time for positive time $\alpha > 0$, and backwards in time for negative time $\alpha < 0$. Therefore if $f'(0)$, the oscillator will lose energy in both directions and never again reach $\pm f(0)$.

In detail, let
\[ E = \frac{(f')^2 + abf^2}{2} \]
be the energy of the oscillator. Then
\[ E' = f''f' + abf'f = -(a+b)((\tanh \alpha)(f')^2). \]
Thus $E' \leq 0$ when $\alpha > 0$ and vice-versa, with equality only when $f' = 0$. Thus, $E(\alpha) < E(0)$ for $\alpha \neq 0$. Moreover,
\[ |f| \leq \sqrt{\frac{2E}{ab}}, \]
with equality when $f' = 0$, as is the case when $\alpha = 0$. These relations together establish the lemma.

Proof of Theorem 3.2 When $a, b > 1$, we combine the lemmas to establish the theorem. We want to find the necks $N^+$ and $N^-$ that minimize the integral
\[ w(N^+, N^-) = \int_{N^+ \times N^-} x \wedge y \wedge dx^a y^{a-1} \wedge dy^b y^{b-1}. \]
We orient the necks so that the integrand — not just the integral — is positive. This is possible by the geometry established in the proof of Lemma 3.2; namely, the vector spaces $T_xN^+ + \langle \vec{x} \rangle$ and $T_yN^- + \langle \vec{y} \rangle$ are maximal timelike and spacelike subspaces of $\mathbb{R}^{(a,b)}$. Let $f$ be the solution to the differential equation
\[ f'' + (a+b)(\tanh \alpha)f' + abf = 0 \]
with $f(0) = 1$ and $f'(0) = 0$. Lemmas 3.1 and 3.2 say that
\[ \ell(N^+, N^-) = \int_{N^+ \times N^-} f(\sinh^{-1} x \cdot \vec{y})x \wedge y \wedge dx^a y^{a-1} \wedge dy^b y^{b-1} \]
is invariant under homotopy of the necks, and is therefore the same for all necks. By Lemma 3.2, $f(\alpha) < f(0) = 1$ when $\alpha \neq 0$. Therefore
\[ \ell(N^+, N^-) \leq w(N^+, N^-). \]
Equality is achieved if and only if $\vec{x} \cdot \vec{y} = 0$ always; in other words, when $N^+$ and $N^-$ are orthogonal, flat, centered necks, as desired.

Lemma 3.1 is false in the stated generality when $a = 1$ or $b = 1$. Suppose that $b = 1$ (say). In this case, $H^-$ is a two-sheeted hyperboloid and a valid neck $N^-$ consists of one point on each sheet. Lemma 3.1 does hold, by a modified argument, if we either require $N^-$ to be a neck, or if we assume an even solution $f$ to the ODE in Lemma 3.2. However, we will give the old geometric proof from Reference 13 and explain how it is actually the same proof. Let $\vec{y}$ be the difference between the two points of $N^-$; it is necessarily a timelike vector. After a Lorentz transformation,
\[ \vec{y} = (0, 0, 0, \ldots, 0, y) \]
with $y \geq 2$, with equality when $N^-$ is centered. Then $v(N^+, N^-)$ is proportional to the area enclosed by the projection $\pi$ of $N^+$ onto the first $a$ coordinates. Meanwhile $H^+$ projects onto the outside of the unit sphere $S^{n-1}$ in $\mathbb{R}^a$, and $N^+$ must also wind around the hole, as in Figure 2. The enclosed area is minimized when $N^+$ is the boundary of the hole, i.e., when $N^+$ is a flat, centered neck orthogonal to $\vec{y}$. The integral $v(N^+, N^-)$ also has a factor of $y$, and is minimized when $y = 2$, which occurs when $N^-$ is centered.

This completes the proof in all cases. We remark that if $\rho$ is the radial projection of $\mathbb{R}^a$ onto $S^{n-1}$ and $\mu$ is Haar measure on $S^{n-1}$, then the pullback $\pi^*(\rho^*(\mu))$ is the same form $\omega$ that is defined in Lemmas 3.1 and 3.2. The scalar kernel is
\[ f(\alpha) = \frac{1}{(\cosh \alpha)^n} \]
which explicitly satisfies the equation and estimate of Lemma 3.3.

4. OTHER LINKING FORMS

The calculation of Lemma 3.2 can be adapted to compute Gauss linking forms in the round $(n-1)$-sphere $S^{n-1}$, or in hyperbolic space $H^{n-1}$. If $M$ is one of these two geometries, the form $\omega$ is now an element of $\Omega^{(a-1,b-1)}(M \times^2 \Delta)$, where $\Delta$ is the diagonal of $M \times^2$ and $a + b = n$. The goal is to choose $\omega$ so that it is invariant under Isom($M$) and so that its double
integral is the linking number between \((a-1)\)-dimensional and \((b-1)\)-dimensional submanifolds \(N_1\) and \(N_2\) in \(M\),

\[
\text{lk}(N_1, N_2) = \int_{N_1 \times N_2} \omega,
\]

or a proportionality.

This goal is closely related to the problem of finding the propagator of generalized electromagnetism, or the generalized Biot-Savart law, in curved spaces. This question was recently considered by DeTurck and Gluck in \(S^3\) and \(\mathbb{H}^3\) [7]. It is not exactly the same question as ours, because if \(\omega\) is such a propagator, then it satisfies a Laplace-type equation instead of an exterior derivative equation and a symmetry condition. However, the propagator has to be an invariant linking form by Ampere’s law.

The configuration space \(M^{\times 2} \setminus \Delta\) is no longer a Cartesian product, but it is foliated both horizontally and vertically, so the de Rham complex \(\Omega^*(M^{\times 2} \setminus \Delta)\) still refines to a double complex. Lemma 3.1 still holds, except that the chain homotopy on \(N_1\) and \(N_2\) must be disjoint. This allows the integral of \(\omega\) to be proportional to \(\text{lk}(N_1, N_2)\), provided that \(d\alpha, d\omega = 0\). No other invariants are available for the allowed chain homotopies.

We can suppose that \(M = S^{n-1}\) is the unit sphere in \(\mathbb{R}^{n,0}\), or that \(M = H^{n-1}\) is the positive unit pseudosphere in \(\mathbb{R}^{n,-1,1}\). In either case, we can write

\[
\omega = \phi(x \cdot y) x \wedge y \wedge dx^{n-1} \wedge dy^{n-1}
\]

and perform a calculation similar to the one in Lemma 3.2.

The calculations are similar enough that we will just give the conclusions.

If \(M = S^{n-1}\), then we let

\[
\begin{align*}
\bar{x} &= (1, 0, 0, \ldots, 0) \\
\bar{y} &= (\cos \alpha, \sin \alpha, 0, 0, \ldots, 0) \\
f(\alpha) &= \phi(\cos \alpha).
\end{align*}
\]

Then the resulting ODE in \(f\) is:

\[
f'' + (a + b) (\cot \alpha) f' - abf = 0.
\]

In this case, the equation has the boundary condition that \(f\) is non-singular at \(\pi\). With this boundary condition, we obtain

\[
\int_{N_1 \times N_2} \omega = f\left(\frac{\pi}{2}\right) (\text{Vol } S^{n-1}) (\text{Vol } S^{n-1}) \text{lk}(N_1, N_2),
\]

where \(\text{Vol } S^n\) denotes the volume of the round unit \(n\)-sphere. The constant factor is obtained by taking \(N_1\) and \(N_2\) to be orthogonal great spheres in \(S^{n-1}\) as a test case. Explicitly, if \(a = b = 2\),

\[
f(\alpha) = \frac{(\pi - \alpha) (\cos \alpha) + (\sin \alpha)}{4 \alpha^2 (\sin \alpha^3)}.
\]

This solution is equivalent to the one given by DeTurck and Gluck [7].

If \(M = H^{n-1}\), then we let

\[
\begin{align*}
\bar{x} &= (1, 0, 0, \ldots, 0) \\
\bar{y} &= (\cos \alpha, \sin \alpha, 0, 0, \ldots, 0) \\
f(\alpha) &= \phi(\cos \alpha).
\end{align*}
\]

Then the resulting ODE in \(f\) is:

\[
f'' + (a + b) (\coth \alpha) f' + abf = 0.
\]

There are two solutions, one even \(\alpha\) and the other odd in \(\alpha\). When \(a = b = 2\), the even and odd solutions are:

\[
f(\alpha) = \frac{\cosh \alpha}{(\sinh \alpha)^3} f(\alpha) = \frac{(\sinh \alpha) - \alpha (\cosh \alpha)}{(\sinh \alpha)^3}.
\]

If we rescale the odd solution to match the Gauss formula in flat \(\mathbb{R}^3\) as \(\alpha \to 0\), we obtain

\[
f(\alpha) = \frac{\cosh \alpha}{4 \pi (\sinh \alpha)^3}.
\]

This solution is also equivalent to the one given by Gluck and DeTurck. (It is the analytic continuation of the linking form on the spherical manifold \(\mathbb{R}^3\). We could also analytically continue equation (6) to obtain a different solution.)

There is also a geometric argument for equation (6). Let \(N_1\) and \(N_2\) be two knots in \(S^3\), and let \(\bar{p} \in S^3\) be a point that does not lie on \(N_1\), \(N_2\), or \(-N_1\). The cone \(C = C(N_1, \bar{p})\) over \(N_1\) with vertex \(\bar{p}\) is a well-defined geometric chain in \(S^3\), using geodesic segments to connect \(\bar{p}\) to points in \(N_1\). Thus the linking number \(\text{lk}(N_1, N_2)\) equals the homological intersection \(C(N_1, \bar{p}) \cap N_2\) of \(C\) with \(N_2\). This is still so if we choose \(\bar{p}\) at random on \(S^3\) with respect to Haar measure. Therefore if we divide \(N_1\) and \(N_2\) into infinitesimal segments \(d\bar{x}\) and \(d\bar{y}\), the expected value of the homological intersection,

\[
\omega = E_{\bar{p}} [C(d\bar{x}, \bar{p}) \cap d\bar{y}],
\]

is an invariant linking form.

The expectation (7) can be computed explicitly. Define angles \(\alpha\) and \(\beta\) by

\[
\cos \alpha = \bar{x} \cdot \bar{y} \quad \cos \beta = \bar{x} \cdot \bar{p}.
\]

The cone \(C(d\bar{x}, \bar{p})\) is an infinitesimally thin triangle which intersects \(d\bar{y}\) with some probability. If there is an intersection, its sign does not depend \(\bar{p}\), but only on the sign of \(\bar{x} \wedge \bar{y} \wedge d\bar{x} \wedge d\bar{y}\). There is no intersection when \(\beta < \alpha\). For fixed \(\beta > \alpha\), the probability of an intersection, times its sign, is given by the formula

\[
\frac{(\sin \beta - \alpha)(\sin \beta)}{2 \pi^2 (\sin \alpha)^3} \bar{x} \wedge \bar{y} \wedge d\bar{x} \wedge d\bar{y} \wedge d\beta.
\]
This can be seen by supposing that \(d\bar{x}\) and \(d\bar{y}\) are orthogonal to each other and to both \(\bar{x}\) and \(\bar{y}\). Then the intersection window for \(\bar{p}\) is a rectangle which is

\[
\frac{\sin \beta - \alpha}{\sin \alpha} |d\bar{x}| \quad \text{by} \quad \frac{\sin \beta}{\sin \alpha} |d\bar{y}|.
\]

(See Figure 4.) The factor \(\bar{x} \wedge \bar{y}\) also produces a numerical factor of \(\sin \alpha\) which must be cancelled. If we include variation of \(\beta\), the region for \(\bar{p}\) is a brick of thickness \(d\beta\). To get an overall expectation, we must integrate the volume of this brick with respect to \(\beta\) and divide by \(\text{Vol } S^3 = 2\pi^2\). Using

\[
\int_{\alpha}^{\pi} (\sin \beta - \alpha)(\sin \beta)d\beta = \frac{(\pi - \alpha)(\cos \alpha) + (\sin \alpha)}{2},
\]

the final answer is

\[
\omega = \frac{(\pi - \alpha)(\cos \alpha) + (\sin \alpha)}{4\pi^2(\sin \alpha)^3} \bar{x} \wedge \bar{y} \wedge d\bar{x} \wedge d\bar{y}.
\]

This formula agrees with equation 6.

5. OPEN PROBLEMS

5.1. Odds and ends

If the convex body \(K\) or its dual \(K^0\) is not positively curved, then \(K^+\) is a weakly spacelike section of \(H^+\), and \(K^-\) is a weakly timelike section of \(H^-\). In other words, both will have some null tangents. In general if \(N^\pm\) are weak necks of \(H^\pm\) in \(\mathbb{R}^{(a,b)}\), their join \(N^+ \ast N^-\) might no longer be embedded except on a measure 0. For example, \(K^+ \ast K^-\) is not embedded if \(K\) is a polytope.

Nonetheless, formula 2 is still the volume of a region \(N^0\). The necks are Lipschitz by virtue of being weakly spacelike and timelike, so the integrand is \(L^\infty\). Moreover, the weak necks can be approximated by strictly spacelike and timelike necks, so the integrand is never negative; no volume is ever subtracted. We conjecture that in this generality, flat necks uniquely minimize the volume of \(N^0\). This is immediate from the proof of Theorem 4 for strict necks, but weak necks introduce new complications.

We do not know whether \(K^0\) is always convex when \(K\) is a centrally symmetric convex body. In light of the Mahler conjecture, we conjecture that \(K^0\) has the most volume when \(K = C_n\). (We do not even know this when \(n = 2\).) A related phenomenon is the fact that \(K^0 = K \times K^0\) when \(K\) is a Hanner polytope 11. We conjecture that they are the only symmetric convex bodies with this property, in light of the conjecture that they are the only convex bodies with minimum Mahler volume.

For two general necks \(N^\pm, N^0\) need not be convex and its volume is unbounded even in fixed dimensions. An upper bound may be possible if we suppose that \(N^0\) lies between \(H^+\) and \(H^-\). When \(a = b\) and \(H^+\) are contact manifolds, and the necks that come from convex bodies are Legendrian. We conjecture the converse (and it may not be hard to prove), that every Legendrian neck of \(H^+\) is \(K^+\) for a convex body \(K\). This extra geometric property of \(K^+\) could be important for other problems in convex geometry.

Reference 13 discusses a different generalized bottleneck conjecture which is still open when \(a, b > 2\). Instead of minimizing

\[
\int_{N^+ \times N^-} \bar{x} \wedge \bar{y} \wedge d\bar{x}^{a-1} \wedge d\bar{y}^{b-1},
\]

we can minimize

\[
\int_{N^+ \times N^-} (\bar{x} \wedge d\bar{x}^{a-1} \cdot (\bar{y} \wedge d\bar{y}^{a-1})
\]

for two necks of \(H^+\). The dot product on \(\Lambda^a(\mathbb{R}^{(a,b)})\) in this integral is induced from the one on \(\mathbb{R}^{(a,b)}\). As before, we conjecture that the minimum occurs when the necks are flat and centered, and in this case coincident rather than orthogonal. When \(N^+ = K^+\) for a symmetric convex body \(K\), the two integrals are equal.

We can consider variations of the Mahler conjecture for complex and quaternionic convex bodies, i.e., the unit balls of complex and quaternionic norms. The natural conjecture is that the unit balls of the complex and quaternionic \(\ell^1\) and \(\ell^\infty\) norms have the least Mahler volume. (The \(\ell^\infty\) ball is a polydisk, a Cartesian power of the Euclidean unit ball or disk in the complex numbers or quaternions.) Hanner polytopes also generalize analogously to Hanner bodies. If \(K\) is a complex or quaternionic convex body, then we can define not only \(N^\pm\), but more generally the neck \(K\) for any scalar \(|z| = 1\). This extra geometry could lead to a variant of Theorem 4 and an improvement to Corollary 6.

5.2. Philosophy

We can attempt to apply the philosophy behind Theorem 1.4 and Corollary 1.6 to other problems in asymptotic convex geometry. Suppose that we want to bound some affinely invariant function \(f(K)\) on convex bodies. Typically \(f(K)\) is maximized (respectively, minimized) by ellipsoids; the question is to find a lower bound (respectively, an upper bound) to say that all convex bodies \(K\) are “not too far from round”. Our philosophy is to find another roundness statistic \(g(K) \leq f(K)\) (respectively \(g(K) \geq f(K)\)) which is minimized (respectively maximized) when \(K\) is an ellipsoid. This yields a bound for the original statistic \(f(K)\).

For example, the isotropic constant conjecture asserts that if \(K \subset \mathbb{R}^n\) is a convex body, its isotropic constant \(L(K)\) is bounded above by some constant \(C\), independent of dimension 4. (See also Reference 11.) The centrally symmetric case of the conjecture implies the general case 11. The constant can be defined by minimizing an expectation:

\[
L(K) \equiv \min_{K' = T(K)} \frac{E g(K'[\bar{x}, \bar{y}])}{n(Vol K')^{\frac{1}{n}}}.
\]

where the minimum is taken over all affine positions \(K' = T(K)\), and the expectation is taken with respect to a random
point \( \vec{x} \in K' \). It is also not hard to show that \( L(K) \) is minimized when \( K \) is an ellipsoid. (This result is due to Blaschke [2], but the argument is very simple. We can minimize \( E_K[\vec{x} \cdot \vec{x}] \) over all \( K \) with fixed volume just by moving the measure of \( K \) as close to the origin as possible.)

**Conjecture 5.1.** If \( K \subset \mathbb{R}^n \) is centrally symmetric, then

\[
E_{K \times K'}[\vec{x} \cdot \vec{y}]^2
\]

is maximized when \( K \) is an ellipsoid.

A number of convex geometers, including Ball and Giannopoulos, have long known that Conjecture 5.1 implies the isotropic constant conjecture. It is not hard to show [8, Prop. 1.2.3] that

\[
E_{K \times K'}[\vec{x} \cdot \vec{y}]^2 \geq n^2 L(K)^2 L(K')^2 (\text{Vol } K)^{2/n} (\text{Vol } K')^{2/n}.
\]

The centrally symmetric isotropic constant conjecture then follows by combining Conjecture 5.1, Theorem 1.3 and Blaschke’s inequality applied to \( L(K') \). On the negative side, Conjecture 5.1 also sharpens the Blaschke-Santaló theorem, so it is more difficult and perhaps less likely.

We can propose even stronger and less likely conjectures than Conjecture 5.1. We conjecture that for every \( 0 < c < 1 \), the probability

\[
p_K(c) = P_{K \times K'}[\vec{x} \cdot \vec{y} \geq c]
\]

is maximized when \( K \) is an ellipsoid. We can even conjecture that

\[
q_K(1 - \epsilon) = \frac{(\text{Vol } K^+)(\text{Vol } B_{n-1}) e^{\epsilon^2}}{\text{vol}(K)} + o(e^{\epsilon^2}),
\]

and that \( p_K(1 - \epsilon) \) has similar asymptotics. Here \( V ol^{n} \) is the volume of \( K^+ \) as a manifold of \( \mathbb{R}^n \times \mathbb{R}^n \) with the inner product \( \langle \cdot, \cdot \rangle \) as a spacelike, and therefore Riemannian, submanifold of \( \mathbb{R}^n \times \mathbb{R}^n \), though they maximize the denominator. A related conjecture is the following:

**Conjecture 5.2.** If \( N^+ \subset H^+ \subset \mathbb{R}^{(a,b)} \) is a neck, then its Riemannian volume \( V ol N^+ \) is maximized when \( N^+ \) is flat.

We do not know if Conjecture 5.2 is hard to prove, or even genuinely an open problem, in the context of current methods in differential geometry [14].

Note that if \( N^+ \) is only a weak neck, then \( V ol N^+ \) can vanish. For example, if \( K \) is a polytope, then \( V ol K^+ = 0 \).

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