APPLICATION OF HILBERT’S PROJECTIVE METRIC 
ON SYMMETRIC CONES

KHALID KOUFANY

Abstract. Let Ω be a symmetric cone. In this note, we introduce 
Hilbert’s projective metric on Ω in terms of Jordan algebras and 
we apply it to prove that given a linear transformation g such that 
g(Ω) ⊂ Ω and a real number p, |p| > 1, then there exists a unique 
element x ∈ Ω satisfying g(x) = x^p.

1. Introduction

If A is a positive linear transformation on \( \mathbb{R}^n \), then the Perron The- 
orem says that there exists \( x_0 \in \mathbb{R}_+^n \setminus \{0\} \) such that for all \( x \in \mathbb{R}_+^n \), 
\( A^nx \) converges in direction to \( x_0 \), i.e. \( \frac{A^nx}{\|A^nx\|} \rightarrow \frac{x_0}{\|x_0\|} \). It has been shown 
by Birkhoff \( \text{[1]} \) that the Perron Theorem can be considered as a spe-
cial case of the Banach contraction theorem. Birkhoff’s approach is to 
consider a projective metric in a cone of positive elements in a Banach 
space. This metric was first introduced by Hilbert \( \text{[6]} \) when studying 
hyperbolic geometry. We begin with the definition of this metric in a 
general setting.

Let \( V \) be is a real Banach space and \( C \) be a closed convex pointed 
cone, where pointed means that \( C \cap -C = \{0\} \). Denote the corre-
sponding interior by \( \Omega \). The relation \( \preccurlyeq \) is defined on \( V \) by saying that 
\( x \preccurlyeq y \) if and only if \( y - x \in C \). Hence \( (V, \preccurlyeq) \) is a partially ordered 
linear space and it is Archimedean, that is, if \( ny \preccurlyeq y \) for all \( n \in \mathbb{N}^* \), 
then \( y \preccurlyeq 0 \).

For \( x \in V \) and \( y \in \Omega \) we let 
\[ M(x, y) := \inf \{ \lambda | x \preccurlyeq \lambda y \}, \]
and
\[ m(x, y) := \sup \{ \mu | \mu y \preccurlyeq x \}. \]

Hilbert’s projective metric is defined on \( \Omega \) by
\[ d(x, y) = \log \frac{M(x, y)}{m(x, y)}. \]
In the case of $\mathbb{R}_+^n$, Hilbert’s projective metric is

$$d(x, y) = \log \frac{\max_{i=1}^n x_i}{\min_{i=1}^n y_i}$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two vectors of $\mathbb{R}_+^n$.

The Hilbert projective metric may be applied to a variety of problems involving positive matrices and positive integral operators. For example, one can use it to solve some Volterra equations. It is also particularly useful in proving the existence of the fixed point for positive operators defined in a Banach space. In this way, it has been shown by Bushell [4] that Hilbert projective metric may be applied to prove that, if $T$ a real nonsingular $r \times r$ matrix, then there exists a unique real positive definite symmetric $r \times r$ matrix $A$ such that

$$T'AT = A^2.$$  

Notice that if $T$ is neither symmetric nor orthogonal the existence and the uniqueness of $A$ is not an elementary problem, even if $r = 3$.

In this note we will formulate the Hilbert projective metric on symmetric cones in a way most convenient for our purpose using Jordan algebra theory and extend Bushell’s Theorem to this class of convex cones.

## 2. Preparatory definitions

We summarize here a number of basic definitions and results concerning symmetric cones. For more detailed discussion, see Faraut and Korányi [5].

Let $\Omega$ be an open convex cone in a Euclidean vector space $V$ of dimension $n$. Let $G(\Omega)$ be the group of linear automorphisms of $\Omega$

$$G(\Omega) = \{g \in GL(V) \mid g(\Omega) = \Omega\}.$$  

Then $\Omega$ is said to be homogeneous if $G(\Omega)$ acts on it transitively. If $\Omega$ is pointed, then $\Omega$ is said to be symmetric if it is homogeneous and self-dual.

A Euclidean Jordan algebra is a Euclidean vector space $V$ equipped with a bilinear product such that

$$xy = yx$$

$$x(x^2y) = x^2(xy)$$

$$(xy|z) = (y|xz).$$

It is shown that the interior $\Omega$ of the set of squares in $V$ is a symmetric cone, and every symmetric cone is given in this way.

We define the (left) multiplication $L$ by $L(x)y = xy$ and the so-called quadratic representation $P$ by $P(x) = 2L^2(x) - L(x^2)$. For any $x \in V$, the endomorphisms $L(x)$ and $P(x)$ are self-adjoint.
Example 2.1. The space $\text{Sym}(r, \mathbb{R})$ of real symmetric $r \times r$ matrices is a Jordan algebra for the product $x \circ y = \frac{1}{2}(xy + yx)$ and it is Euclidean for the scalar product $(x|y) = \text{Tr}(xy)$. The associated symmetric cone is the set $\Omega_{\text{Sym}}$ of positive definite symmetric $r \times r$ matrices and $G(\Omega_{\text{Sym}})$ is the linear group $\text{GL}(r, \mathbb{R})$. In this case, $P(x)y = yx$.

The Euclidean Jordan algebra has and identity element denoted by $e$. Let $K(\Omega) := \{ g \in G(\Omega) \mid g(e) = e \} =$, then $K(\Omega)$ is a maximal compact subgroup of $G(\Omega)$. Let $r$ be the rank of $V$. A Jordan frame $\{c_1, \ldots, c_r\}$ of $V$ is a complete system of non-zero orthogonal primitive idempotents:

$$c_i^2 = c_i, \quad c_i \text{ indecomposable},$$

$$c_ic_j = 0 \text{ if } i \neq j,$$

$$c_1 + \ldots + c_r = e.$$  

We denote by $\mathcal{J}(V)$ the set of all primitive idempotents of $V$. Suppose $V$ is simple. In other words, there is no non-trivial ideal in $V$. Then each element $x$ in $V$ can be written as

$$x = k(\sum_{j=1}^{r} \lambda_j c_j), \; k \in K(\Omega), \; \lambda_j \in \mathbb{R},$$

where $K(\Omega)$ is the identity component of $K(\Omega)$. The determinant is defined by $\text{det}(x) = \prod_{j=1}^{r} \lambda_j$ and the trace form by $\text{tr}(x) = \sum_{j=1}^{r} \lambda_j$. The real numbers $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of $x$ and the spectral norm of $x$ is then defined by

$$|x| = \sup_i |\lambda_i|.$$  

In the case of $\text{Sym}(r, \mathbb{R})$, (2.1) corresponds to the polar decomposition (diagonalization) of symmetric matrices. The functions $\text{det}$ and $\text{tr}$ are respectively the usual determinant $\text{Det}$ and trace $\text{Tr}$ of matrices.

3. Hilbert’s metric on symmetric cones

If we consider the cone $\Omega_{\text{Sym}}$ of real symmetric positive definite $r \times r$ matrices, then one can easily express the Hilbert projective metric (1.1) in terms of eigenvalues of elements of $\Omega_{\text{Sym}}$. Indeed, if $A$ and $B$ are in $\Omega_{\text{Sym}}$, then

$$M(A, B) := \inf \{ \lambda \mid \lambda B - A \preceq 0 \} = \max_{\|x\|=1} \frac{(Ax|x)}{(Bx|x)},$$

where $\preceq$ denotes the Loewner partial order.
and
\[ m(A, B) := \sup \{ \lambda \mid \lambda B - A \preceq 0 \} = \min_{\|x\|=1} \frac{(Ax|x)}{(Bx|x)}, \]
which are respectively the greatest and the least eigenvalue of \( B^{-1}A \). Observe that eigenvalues of the matrix \( B^{-1}A \) are the same of matrix \( B^{-\frac{1}{2}}AB^{-\frac{1}{2}} = P(B^{-\frac{1}{2}})A \) (see Example 2.1 for this notation).

More generally, for symmetric cones, Hilbert’s projective metric can be also formulated in terms of extremal eigenvalues: let \( x \) and \( y \) be in \( \Omega \) and let \( \lambda_M(x, y) > 0 \) and \( \lambda_m(x, y) > 0 \) denote the greatest and the least eigenvalue of the element \( P(y^{-\frac{1}{2}})x \in \Omega \). Then one can prove the following (see [7, Theorem 4.2])

**Proposition 3.1.** We have
\[ (3.1) \quad \lambda_M(x, y) = \max_{c \in J(V)} \frac{(x|c)}{(y|c)}, \]
and
\[ (3.2) \quad \lambda_m(x, y) = \min_{c \in J(V)} \frac{(x|c)}{(y|c)}. \]

From this characterization one can easily prove the following

**Lemma 3.2.** If \( x, y \in \Omega \) and \( \alpha, \beta \geq 0 \), then
\begin{enumerate}
  \item \( \lambda_M(\alpha x + \beta y, y) = \alpha \lambda_M(x, y) + \beta \),
  \item \( \lambda_m(\alpha x + \beta y, y) = \alpha \lambda_m(x, y) + \beta \),
  \item \( \lambda_M(x, y)\lambda_m(y, x) = 1 \).
\end{enumerate}

**Proposition 3.3.** If \( x, y \in \Omega \), then the Hilbert metric of \( x \) and \( y \) is given by
\[ (3.3) \quad d(x, y) = \log \frac{\lambda_M(x, y)}{\lambda_m(x, y)} = \log \lambda_M(x, y)\lambda_M(y, x). \]

**Proof.** According to Proposition 3.1 we have
\[
M(x, y) = \inf \{ \lambda \mid x \preceq \lambda y \} \\
= \inf \{ \lambda \mid \lambda y - x \in \Omega \} \\
= \max \{ \frac{(x|c)}{(y|c)} \mid c \in J(V) \} \\
= \lambda_M(x, y),
\]
and
\[
m(x, y) = \sup \{ \mu \mid \mu y \preceq \lambda x \} \\
= \sup \{ \mu \mid x - \mu y \in \Omega \} \\
= \min \{ \frac{(x|c)}{(y|c)} \mid c \in J(V) \} \]
\[ = \lambda_m(x, y). \]
Applying Lemma 3.2, we obtain easily the second equality of the proposition. \(\square\)

Remark 3.4. It follows from Lemma 3.2 and Proposition 3.3 that \(d\) is constant on rays,

\[
d(\lambda x, \mu y) = d(x, y) \quad \text{for} \quad \lambda, \mu > 0.
\]

In particular, if \(\mu > 0\), then the map \(\xi_\mu : x \mapsto \mu x\) is an isometry of \(d\).

4. Contraction of Hilbert’s projective metric

Let \(x = \sum_{j=1}^r \lambda_j c_j\) be an element of \(\Omega\). If \(p \in \mathbb{R}\), we write \(x^p\) for \(\sum_{j=0}^r \lambda_j^p c_j\) and we state the following contraction result:

**Proposition 4.1.** Let \(p \in \mathbb{R}\) such that \(|p| \leq 1\). Then the map

\[
\omega_p : \Omega \to \Omega, \ x \mapsto x^p
\]

is a contraction of \(d\), that is, for any \(x, y \in \Omega\),

\[
d(x^p, y^p) \leq |p| d(x, y).
\]

**Proof.** Let \(x, y \in \Omega\). Then \(\lambda_M(x^{-1}, y^{-1}) = 1/\lambda_m(x, y)\) and \(\lambda_m(x^{-1}, y^{-1}) = 1/\lambda_M(x, y)\). Hence the map

\[
i : \Omega \to \Omega : x \mapsto x^{-1}
\]

is an isometry of \(d\). It suffices then to prove (4.2) for \(0 \leq p \leq 1\). We know that \(x \preceq \lambda_M(x, y)y\), then by the Loewner-Heinz inequality, see [8, Corollary 9], we have \(x^p \preceq \lambda_M(x, y)^p y^p\). Therefore \(\lambda_M(x^p, y^p) \leq \lambda_M(x, y)^p\). Using the same arguments we prove that \(\lambda_m(x^p, y^p) \geq \lambda_m(x, y)^p\). Thus \(d(x^p, y^p) \leq p d(x, y)\). \(\square\)

5. Completeness

**Proposition 5.1.** \((\Omega, d)\) is a pseudo-metric space. In other words, for any \(x, y, z \in \Omega\), the following holds:

(a) \(d(x, y) \geq 0\)

(b) \(d(x, y) = d(y, x)\)

(c) \(d(x, z) \leq d(x, y) + d(y, z)\)

(d) \(d(x, y) = 0 \iff \exists \lambda > 0 : x = \lambda y\)

**Proof.** Let \(x, y\) and \(z\) be in \(\Omega\). The first propriety of the proposition is obvious, since \(\lambda_M(x, y) \geq \lambda_m(x, y)\).

(b) From Lemma 3.2 we obtain

\[
d(x, y) = \log \frac{\lambda_M(x, y)}{\lambda_m(x, y)}
= \log \lambda_M(x, y) \lambda_M(y, x)
= d(y, x).
\]
(c) Let \( c \) be a primitive idempotent of \( V \), then according to Proposition 3.1, \( 0 < \frac{(x|c)}{(y|c)} \leq \lambda_M(x, y) \) and \( 0 < \frac{(y|c)}{(z|c)} \leq \lambda_M(y, z) \). Hence \( \frac{(x|c)}{(z|c)} \leq \lambda_M(x, y) \lambda_M(y, z) \) and

\[
(5.1) \quad \lambda_M(x, z) \leq \lambda_M(x, y) \lambda_M(y, z).
\]

Similarly we prove that

\[
(5.2) \quad \lambda_m(x, z) \geq \lambda_m(x, y) \lambda_m(y, z).
\]

It follows then from (5.1) and (5.2) that \( d(x, z) \leq d(x, y) + d(y, z) \).

(d) If \( d(x, y) = 0 \), then \( \lambda_M(x, y) = \lambda_m(x, y) = \lambda \) and all the eigenvalues of \( P(y^\frac{1}{2})x \) are equal to \( \lambda \). Therefore, \( P(y^\frac{1}{2})x = \lambda e \) and \( x = \lambda P(y^\frac{1}{2})e = \lambda y \). Conversely, if \( x = \lambda y \) where \( \lambda > 0 \), then \( P(y^\frac{1}{2})x = \lambda e \) and \( \lambda_M(x, y) = \lambda_m(x, y) = \lambda \). \(\square\)

Let \( S(V) \) be the unite sphere in \( V \), \( S(V) = \{x \in V \mid |x| = 1\} \) with respect to the spectral norm introduced in (22).

**Lemma 5.2.** Let \( x \) and \( y \) be in \( \Omega \cap S(V) \). Then

(a) \( |x - y| \leq e^{d(x, y)} - 1 \).

(b) If \( |x - y| < \lambda_m(y) \) then \( |x - y| \geq \lambda_m(y) \tanh\left\{ \frac{1}{2}d(x, y) \right\} \), where \( \lambda_m(y) \) is the least eigenvalue of \( y \).

**Proof.** Let \( x, y \in \Omega \cap S(V) \) and let \( c \) be a primitive idempotent of \( V \), then by Proposition 3.1 we have

\[
(5.3) \quad \lambda_m(x, y)(y|c) \leq (x|c) \leq \lambda_M(x, y)(y|c).
\]

But \( |x| = 1 \) and \( |y| = 1 \), then there exists \( c_1, c_2 \in \mathcal{J}(V) \) such that \( (x|c_1) = 1 \) and \( (y|c_2) = 1 \), thus

\[
(5.4) \quad \lambda_m(x, y) \leq 1 \leq \lambda_M(x, y).
\]

Let \( c_0 \) be a primitive idempotent of \( V \) such that \( |x - y| = (x - y|c_0) \), then using (5.4) we have

\[
|x - y| = (x|c_0) - (y|c_0) \\
= \left\{ \frac{(x|c_0)}{(y|c_0)} - 1 \right\}(y|c_0) \\
\leq \frac{(x|c_0)}{(y|c_0)} - 1 \quad \text{since} \quad |y| = 1 \\
\leq \lambda_M(x, y) - 1 \\
\leq \lambda_M(x, y) - \lambda_m(x, y) \\
\leq \left\{ \lambda_M(x, y) - \lambda_m(x, y) \right\} \frac{1}{\lambda_m(x, y)} \\
= e^{d(x, y)} - 1.
\]
Moreover
\[
\lambda_M(x, y) = \max_{c \in \mathcal{J}(V)} \frac{(x|c)}{(y|c)}
= \max_{c \in \mathcal{J}(V)} \left\{ \frac{(x - y|c)}{(y|c)} + 1 \right\}
\leq \frac{|x - y|}{\lambda_m(y)} + 1.
\]
(5.5)

Now, if \(|x - y| < \lambda_m(y)|\), then \(1 - \frac{|x - y|}{\lambda_m(y)} > 0\). Therefore, from (5.5) and Lemma 3.2, we have
\[
(1 - \frac{|x - y|}{\lambda_m(y)})y \not\ll x.
\]
Hence
\[
(5.6)
\lambda_m(x, y) \geq 1 - \frac{|x - y|}{\lambda_m(y)}.
\]
Finally, it follows from (5.5) and (5.6) that
\[
|x - y| \geq \lambda_m(y) \tanh \left\{ \frac{1}{2}d(x, y) \right\}.
\]
\[
\square
\]

**Proposition 5.3.** \((\Omega \cap S(V), d)\) is complete metric space.

**Proof.** It is clear that if \(x, y \in (\Omega \cap S(V), d)\) such that \(d(x, y) = 0\), then \(x = y\). Thus \((\Omega \cap S(V), d)\) is a metric space.

Let \((x_k)_k\) be a Cauchy sequence in \((\Omega \cap S(V), d)\). Then from Lemma 5.2, \((x_k)_k\) is a Cauchy sequence in \((V, | \cdot |)\) and then converges to an element \(x \in \Omega \cap S(V)\). In order to prove that \(x\) is an element of \(\Omega \cap S(V)\), we observe, using (5.2) that \(\lambda_m(x_i) = \lambda_m(x_i, e) \geq \lambda_m(x_i, x_j)\lambda_m(x_j, e) = \lambda_m(x_j, e)^{d(x_i, x_j)} \lambda_M(x_i, x_j)\lambda_m(x_i, e)\). Therefore, from (5.4), we obtain \(\lambda_m(x_i) \geq \lambda_m(x_j, e)^{d(x_i, x_j)}\). Since \((x_k)_k\) converges in the spectral norm to \(x\), \((\lambda_m(x_k))_k\) converges to \(\lambda_m(x)\). It follows that for a fixed large \(j\), \(\lambda_m(x) = \lim_{i \to \infty} \lambda_m(x_i) \geq \alpha \lambda_m(x_j) > 0\), where \(\alpha > 0\). Hence \(x \in \Omega \cap S(V)\). Finally, it is easy to prove using the second proposition of Lemma 5.2 that \((x_k)_k\) converges to \(x\) in \((\Omega \cap S(V), d)\). \(\square\)

6. Application

If \(V\) is the Jordan algebra \(\text{Sym}(r, \mathbb{R})\), then \(\Omega\) is the symmetric cone of positive definite matrices, and \(G(\Omega)\) coincides with the linear group \(\text{GL}(r, \mathbb{R})\). Recall that in this case \(G(\Omega)\) acts on \(\Omega\) by : \(t \cdot a = t'at\).

In [4], Bushell prove that for any \(t \in \text{GL}(r, \mathbb{R})\) and any integer \(k \geq 1\), there exists a unique real positive definite matrix \(a\) such that \(t'at = a^{2^k}\).

For a general symmetric cone we have the following

**Theorem 6.1.** Let \(g \in G(\Omega)\) and \(p \in \mathbb{R}\) such that \(|p| > 1\). Then there exists a unique element \(a\) in \(\Omega\) such that \(g(a) = a^p\).
Proof. Since \( \iota : x \mapsto x^{-1} \) is an isometry of \( d \), see (4.3), we can assume that \( p > 1 \). For \( x \in \Omega \cap S(V) \) we put \( F(x) = \frac{1}{|g(x)|^p} g(x)^{\frac{1}{p}} \). Then \( F \) maps \( \Omega \cap S(V) \) into \( \Omega \cap S(V) \). Recall that Hilbert's projective metric is constant on rays, see (3.4), then we have for \( x, y \in \Omega \cap S(V) \),

\[
d(F(x), F(y)) = d((\omega_{1/p} \circ g)(x), (\omega_{1/p} \circ g)(y)),
\]

where \( \omega_{1/p} \) is the power map \( x \mapsto x^{\frac{1}{p}} \). We have already noticed in (4.1) that \( \omega_{1/p} \) is a contraction of the metric \( d \), hence

\[
d(F(x), F(y)) \leq \frac{1}{p} d(g(x), g(y)).
\]

Moreover, the eigenvalues of \( P(y^{-1/2})x \) are the unique solutions of the characteristic equation \( \det(\lambda y - x) = 0 \). But, using [5, Proposition III.4.3],

\[
\det(\lambda g(y) - g(x)) = \det(\lambda y - x) = \det(g(y))^{n/r} \det(\lambda y - x).
\]

Thus the eigenvalues \( P(y)^{-1/2}x \) and \( P(g(y))^{-1/2}g(x) \) are the same and \( g \) is also an isometry of \( d \). Finally, all this implies that \( F \) is a contraction of the Hilbert metric \( d \),

\[
d(F(x), F(y)) \leq \frac{1}{p} d(x, y).
\]

Therefore, by the Banach contraction mapping theorem and Proposition 5.3, the map \( F \) has a unique fixed point, let say \( u \). Then \( u^p = \frac{1}{|g(u)|^{1/p}} g(u) \) and the element \( a := |g(u)|^{1/p} u \in \Omega \) is the unique solution of the equation \( g(x) = x^p \).

\[\square\]

Corollary 6.2. Let \( h \in G(\Omega) \) and \( p \in \mathbb{R} \) such that \( |p| > 1 \). Then there exists a unique element \( a \) of \( \Omega \) such that \( h(a^p) = a \).

Final remark. The Hilbert original definition of the projective metric involved the logarithm of the cross-ratio of for points in the interior of a convex cone in \( \mathbb{R}^n \). It would be interesting to define the Hilbert projective metric on symmetric cones using the generalized cross-ratio introduced in [2], see also [7].

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Institut Elie Cartan, UMR 7502 (UHP-CNRS-INRIA), Université H. Poincaré, B.P. 239, F-54506 Vandœuvre-lès-Nancy Cedex, France
E-mail address: khalid.koufany@iecn.u-nancy.fr