Late-Time Cosmology of Scalar-Coupled $f(R, G)$ Gravity

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In this work by using a numerical analysis, we investigate in a quantitative way the late-time dynamics of scalar coupled $f(R, G)$ gravity. Particularly, we consider a Gauss-Bonnet term coupled to the scalar field coupling function $\xi(\phi)$, and we study three types of models, one with $f(R)$ terms that are known to provide a viable late-time phenomenology, and two Einstein-Gauss-Bonnet types of models. Our aim is to write the Friedmann equation in terms of appropriate statefinder quantities frequently used in the literature, and we numerically solve it by using physically motivated initial conditions. In the case that $f(R)$ gravity terms are present, the contribution of the Gauss-Bonnet related terms is minor, as we actually expected. This result is robust against changes in the initial conditions of the scalar field, and the reason is the dominating parts of the $f(R)$ gravity sector at late times. In the Einstein-Gauss-Bonnet type of models, we examine two distinct scenarios, firstly by choosing freely the scalar potential and the scalar Gauss-Bonnet coupling $\xi(\phi)$, in which case the resulting phenomenology is compatible with the latest Planck data and mimics the $\Lambda$-Cold-Dark-Matter model. In the second case, since there is no fundamental particle physics reason for the graviton to change its mass, we assume that primordially the tensor perturbations propagate with the speed equal to that of light's, and thus this constraint restricts the functional form of the scalar coupling function $\xi(\phi)$, which must satisfy the differential equation $\dot{\xi} = H\xi$. The latter equation is greatly simplified when late times are considered and can be integrated analytically to yield a relation for $\xi$, which depends solely on the Hubble rate, in a model independent way. This leads eventually to an elegant simplification of the Friedmann equation, which when solved numerically, yields a viable late-time phenomenology. A common characteristic of the Einstein-Gauss-Bonnet models we considered is that the dark energy era they produce is free from dark energy oscillations.

PACS numbers: 04.50.Kd, 95.36.+x, 98.80.-k, 98.80.Cq,11.25.-w

I. INTRODUCTION

The quest for understanding the mysterious late-time acceleration era, is still ongoing in modern theoretical physics. Many possible theoretical descriptions have been proposed in order to model the dark energy era, among which modified gravity has an elevated role in the successful description of the dark energy era, since apart from being able to describe the late-time era, it is also possible to describe inflation with the same theoretical framework, see for example Refs. [10–16, 73]. Modern modified gravity models are put into stringent test of viability when the dark energy era is considered, since the models must be confronted with the latest Planck 2018 data [13], and also the models have to be compatible to some inferior extent with the $\Lambda$-Cold-Dark-Matter ($\Lambda$CDM) model, which is the most successful model for describing the dark energy era. The $\Lambda$CDM model is basically based on the assumptions of the presence of a cosmological constant, and the presence of particle dark matter [19, 24], however both the ingredients of the model are in question of their existence. Moreover, although the $\Lambda$CDM is quite compatible with the Cosmic Microwave Background data, it has several theoretical shortcomings which cannot be harbored by Einstein-Hilbert gravity. At this point, modified gravity offers many possibilities for successful theoretical descriptions. In this line of research, Einstein-Gauss-Bonnet theory [25–64] could be a potentially correct description of both the early and late-time era. In this work we shall investigate in a quantitative way the exact effect of the Gauss-Bonnet coupling on the late-time era, in the context of $f(R, \phi)$ theories of gravity in general. In particular, we shall investigate the effect of the non-trivial Gauss-Bonnet coupling on a simple canonical scalar field theory, and on $f(R)$ gravity in the presence of a canonical scalar field. We shall perform a thorough numerical analysis of the Friedmann equation and we shall derive the behavior of several statefinder quantities and of several physical quantities of interest, as functions of the redshift. Accordingly our findings shall be compared with the $\Lambda$CDM and shall be confronted with the latest Planck constraints on the cosmological parameters [13]. Our findings indicate that when an $f(R)$ gravity theory is present along with the Gauss-Bonnet coupling, the latter does not significantly affect the late-time phenomenology. Also in the case of a simple Einstein-Gauss-Bonnet theory, we show that it is possible to obtain the phenomenological viability of the model under study, but this result could be highly model dependent, and has a minor disadvantage, since it is hard to describe inflation and dark energy with the same Einstein-Gauss-Bonnet model, in general though.

Finally, we make a novel assumption that may constrain the functional form of the scalar coupling function of
the scalar field to the Gauss-Bonnet coupling, and we investigate the late-time phenomenology in this case too. Particularly, we assume that there is a constraint coming from the requirement that the primordial gravitational wave speed is equal to unity, which imposes a functional constraint on the functional form of the Gauss-Bonnet scalar coupling function. The reason for demanding that the primordial gravitational wave speed is equal to unity in natural units, is coming from the GW170817 event [54], which indicated that the gamma rays and the gravitational waves arrived almost simultaneously. Thus assuming that there is no fundamental particle physics reason for the graviton to change its primordial mass, the gravity speed constraint imposed by the GW170817 event, must stretch back to the primordial inflationary era. For the late-time era, the constraint imposed by requiring that the gravity wave speed of the primordial tensor modes is equal to unity in natural units, results to an elegant expression for the time derivative of the scalar Gauss-Bonnet coupling, which is expressed in terms of the Hubble rate and the redshift, and thus it depends on statefinder quantities and acquires a model independent description. The late-time viability of such an Einstein-Gauss-Bonnet gravity is examined in detail, and our findings indicate that these can also provide a successful description of the late-time era. Finally, we also conclude that the Einstein-Gauss-Bonnet theories in general, provide a dark energy oscillations free late-time era, in contrast to $f(R)$ gravity models. However, this result seems to be highly model dependent, at least in the context of $f(R)$ gravity and needs to be further discussed in another context.

II. ESSENTIAL FEATURES OF $f(R, \phi)$ EINSTEIN-GAUSS-BONNET GRAVITY

The starting point of our work is the gravitational action, which for the $f(R, \phi)$ Einstein-Gauss-Bonnet gravity has the following form,

$$S = \int d^4x \sqrt{-g} \left( \frac{f(R, \phi)}{2\kappa^2} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V - \xi(\phi)G + \mathcal{L}_{(m)} \right), \quad (1)$$

where $R$ denotes the Ricci scalar, $\kappa = \frac{1}{M_P}$ is the gravitational constant with $M_P$ being the reduced Planck mass, while $V$ is the scalar potential, while $\xi(\phi)$ is the Gauss-Bonnet scalar coupling function. Also $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ denotes the Gauss-Bonnet invariant with $R_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ being the Ricci and Riemann curvature tensor respectively and finally, $\mathcal{L}_{(m)}$ specifies the Lagrangian density of both relativistic and non-relativistic perfect fluids. For now, the exact form of the function $f(R, \phi)$ shall remain unspecified, at least for the moment. Concerning the cosmological geometric background, we shall assume that it corresponds to that of a flat Friedman-Robertson-Walker (FRW) metric with the line element being,

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \quad (2)$$

where $a(t)$ denotes the scale factor. Consequently, the Ricci scalar and Gauss-Bonnet invariant for the FRW background take the forms $R = 6(2H^2 + \dot{H})$ and $G = 24H^2(H^2 + \dot{H})$, where $H = \frac{\dot{a}}{a}$ signifies the Hubble rate and the “dot” denotes differentiation with respect to the cosmic time $t$. Furthermore, in order to simplify our work, we shall make the reasonable assumption that the scalar field is homogeneous and thus it depends solely on the cosmic time.

Implementing the variation principle with respect to the metric tensor $g^{\mu\nu}$ and the scalar field $\phi$, we obtain the field equations for gravitational sector and the scalar field equation, which are,

$$\frac{3f_R H^2}{\kappa^2} = \rho_{(m)} + \frac{1}{2} \dot{\phi}^2 + V + \frac{f_R R - f}{\kappa^2} - \frac{3H \dot{f}_R}{\kappa^2} + 24\dot{\xi} H^3, \quad (3)$$

$$\frac{-2f_R \dot{H}}{\kappa^2} = \rho_{(m)} + P_{(m)} + \dot{\phi}^2 + \frac{\ddot{f}_R - H \dot{f}_R}{\kappa^2} = 16\dot{\xi} H \dot{H}, \quad (4)$$

$$V_\phi + \ddot{\phi} + 3H \dot{\phi} - \frac{\dot{f}_\phi}{2\kappa^2} + \xi_\phi G = 0, \quad (5)$$

where $\rho_{(m)}$ and $P_{(m)}$ denote the matter density and pressure respectively of any prefect fluid of non-relativistic (baryons, leptons, Cold Dark matter (CDM)) matter and relativistic matter (photons and neutrinos). In particular, for the purposes of this work, for which the focus will be on the dark energy era, we shall assume that the perfect fluids compose of CDM and radiation, so we have,

$$\rho_{(m)} = \frac{\rho \phi}{a^3} \left( 1 + \frac{\chi}{a} \right), \quad (6)$$
\[ P = \sum_{i} \omega_i \rho_i, \]  
(7)

where \( \chi = \frac{\rho_{r0}}{\rho_{d0}} \) with \( \rho_{r0} \) being the current density of relativistic matter and \( \omega_i \) the equation of state parameter for each kind or matter, with \( i \) running from relativistic to non-relativistic. As we already mentioned, all matter species are described by perfect fluids and a barotropic EoS. Since we have prefect fluids, the continuity equation for each of them reads,

\[ \dot{\rho}_i + 3H \rho_i (1 + \omega_i) = 0. \]  
(8)

In the following, we shall implement two replacements in order to align better our study with the late-time dynamics, with the first being the use of the redshift instead of the cosmic time as a dynamical parameter. From its definition,

\[ 1 + z = \frac{1}{a(t)}, \]  
(9)

where we assumed that the scale factor at present time is set to unity, the time derivatives can be expressed in terms of derivatives with respect to the redshift, using the following rule,

\[ \frac{d}{dt} = -H(1 + z) \frac{d}{dz}, \]  
(10)

By replacing cosmic time \( t \) with the redshift, one can recast the cosmological equations of motion in the following way,

\[ \frac{3f_R H^2}{\kappa^2} = \rho_{(m)} + \frac{1}{2} \dot{\phi}^2 + V + \frac{f_R R - f}{2\kappa^2} - \frac{3H \dot{f}_R}{\kappa^2} - 24(1 + z) \xi' H^4, \]  
(11)

\[ V_\phi + \ddot{\phi} + 3H \dot{\phi} - \frac{f_\phi}{2\kappa^2} + \xi_\phi G = 0, \]  
(12)

where the “prime” denotes differentiation with respect to the redshift. Here, only two equations where rewritten since they will be used in our study. In addition, every time derivative participating in the equations of motion above shall be replaced as well. Specifically, we have,

\[ \dot{H} = -H(1 + z) H', \]  
(13)

\[ \dot{\phi} = -H(1 + z) \phi', \]  
(14)

\[ \ddot{\phi} = H^2(1 + z)^2 \phi'' + H^2(1 + z) \phi' + HH'(1 + z)^2 \phi', \]  
(15)

\[ \dot{f}_R = \dot{R} f_{RR} + \dot{\phi} f_{R\phi}, \]  
(16)

\[ \dot{R} = 6H(1 + z)^2 \left( HH'' + (H')^2 - \frac{3H H'}{1 + z} \right), \]  
(17)

Now more importantly, instead of using the Hubble rate and its derivatives in order to quantify the cosmological evolution, we shall use a statefinder quantity defined as follows \[66–68, 73,\]

\[ y_H = \frac{\rho_{DE}}{\rho_{d0}}, \]  
(18)

with \( \rho_{DE} \) denoting the dark matter energy density and \( \rho_{d0} \) the current value of density for non-relativistic matter. Here, we shall assume that the dark energy density is comprised of all the geometric terms in the Friedmann equation. In particular,

\[ \rho_{DE} = \frac{1}{2} \dot{\phi}^2 + V + \frac{f_R R - f}{2\kappa^2} - \frac{3H \dot{f}_R}{\kappa^2} + 24\xi H^3 + \frac{3H^2}{\kappa^2} (1 - f_R), \]  
(19)
Similarly, from the Raychaudhuri equation, the corresponding pressure for the dark energy fluid is defined as,

\[ P_{DE} = -V - 24\dot{\xi}H^3 - 8\dot{\xi}H\dot{\dot{H}} - \frac{f_R R - f}{2\kappa^2} - \frac{2\dot{H}}{\kappa^2} (1 - f_R), \]  

(20)

where,

\[ \dot{\rho}_{DE} + 3H(\rho_{DE} + P_{DE}) = 0, \]  

(21)

Hence, equations (3) and (4) obtain the usual Friedmann equation-like form of Einstein-Hilbert gravity,

\[ \frac{3H^2}{\kappa^2} = \rho(m) + \rho_{DE}, \]  

(22)

\[ -\frac{2\dot{H}}{\kappa^2} = \rho(m) + P(m) + \rho_{DE} + P_{DE}, \]  

(23)

Consequently, the newly defined statefinder parameter \( y_H \) can be written in terms of the Hubble rate, and vice-versa. Specifically, we have,

\[ H^2 = m_s^2 \left( y_H(z) + \frac{\rho(m)}{\rho_{d0}} \right), \]  

(24)

where \( m_s^2 = \kappa^2 \frac{\rho_{d0}}{\rho_{m0}} = 1.87101 \cdot 10^{-67} \). This extends to the derivatives of the Hubble rate as well since now,

\[ HH' = \frac{m_s^2}{2} \left( y_H' + \frac{\rho_m'}{\rho_{d0}} \right), \]  

(25)

\[ H'^2 + HH'' = \frac{m_s^2}{2} \left( y_H'' + \frac{\rho_m''}{\rho_{d0}} \right), \]  

(26)

In the following, we shall numerically solve the system of differential equations (11) and (12) with respect to the statefinder quantity \( y_H \) and the scalar field \( \phi \). Afterwards, we shall compare the theoretical results with the observations. This can be achieved by utilizing further statefinder parameters. Concerning dark energy, we define the equation of state parameter \( \omega_{DE} \) and the density parameter \( \Omega_{DE} \) with respect to \( z \) and \( y_H \) as follows \[67, 68, 73\],

\[ \omega_{DE} = -1 + \frac{1 + z}{3} \frac{d\ln y_H}{dz}, \quad \Omega_{DE} = \frac{y_H}{y_H + \frac{\rho_{m0}}{\rho_{d0}}}, \]  

(27)

Furthermore, for the overall evolution, we shall use the following statefinder parameters \[68, 73\],

\[ q = -1 - \frac{\dot{H}}{H^2}, \quad j = \frac{\dot{H}}{H^2} - 3q - 2, \quad s = \frac{j - 1}{3(q - \frac{1}{2})}, \quad Om(z) = \frac{\left( \frac{H}{H_0} \right)^2 - 1}{(1 + z)^3 - 1}, \]  

(28)

which in the order of appearance above, are the deceleration parameter, the jerk, the snap parameter and \( Om(z) \), which is indicative of the current CDM energy density parameter.

### III. \( f(R) \) EINSTEIN-GAUSS-BONNET GRAVITY: UNIFYING EARLY AND LATE TIME

Let us commence our study by introducing the arbitrary functions of the previous models. Hereafter, we shall limit our work to only simple cases for the scalar functions, namely \( V(\phi) \) and \( \xi(\phi) \), for which it is known that the early-time can be described successfully. Suppose that the scalar functions of the previous section obtain the following forms, which are arbitrary for the moment, meaning that there is no fundamental relation between the scalar potential and the scalar coupling function,

\[ \xi(\phi) = e^{\frac{\phi}{m}}, \]  

(29)
FIG. 1: Solutions $y_H$ (left) and $\phi$ over reduced Planck mass (right) for the $f(R)$ case. The main difference seems to be the scalar field which does not oscillate. In general, the addition of a canonical scalar field and a linear Gauss-Bonnet topological invariant coupled to a scalar function do not suffice to nullify dark energy oscillation at large redshifts.

FIG. 2: Cosmological parameters $q$ (upper left), $j$ (upper right), $s$, (bottom left) and $s$ (bottom right) as functions of redshift. Once again, the same results as in the pure $f(R)$ case are obtained.

and also assuming that there is no scalar potential present, we assume that the there is also an $f(R)$ gravity part present too, and has the form [68, 73],

$$f(R) = R + \left( \frac{R}{M} \right)^2 - \gamma \Lambda \left( \frac{R}{3m_s^2} \right)^{\delta},$$

(30)

where $\gamma$ a dimensionless parameter, $\Lambda$ a constant with mass dimensions $[m]^2$, $M = 1.5 \cdot 10^{-55} M_P$ with $N$ being the e-folding number and $\delta$ an exponent which satisfies the relation $0 < \delta < 1$, see Refs. [68, 73] for details. In this particular case we assumed for simplicity that the scalar potential is absent, as we already mentioned. In this general
framework however, there is no physical constraint that connects the scalar potential and the scalar coupling function, nevertheless if one takes into account the primordial gravitational wave speed constraints, these two scalar functions are interconnected fundamentally. For the moment though we assume that these can be freely chosen given that no constraints on the speed of gravitational waves are imposed, i.e \( c_s^2 \) does not necessarily coincide with unity. The \( f(R) \) model we chose, was chosen simply because it is capable of uniting early with late time acceleration era of our Universe, due to the fact that for large \( R, R^2 \) becomes dominant whereas for \( R \rightarrow 0, R^3 \) becomes the dominant term, see Ref. [68, 73] for a detailed analysis on this issue. It is therefore interesting to examine whether the addition of a scalar field can alter the dynamics of such model. Essentially, such a model predicts non negligible dark energy oscillations for \( z \geq 5 \) for the statefinder parameter \( y_H(z) \), which become even more dominant in higher order derivatives. Recently, it was showcased that the addition of a function depending on the Gauss-Bonnet topological invariant \( \mathcal{G} \), which alone describes an oscillation-free late-time era, cannot nullify such oscillations on the \( f(R) \) model, implying that the later is more dominant [69]. It is therefore sensible to try and examine whether the addition of a canonical scalar field coupled to the Gauss-Bonnet topological invariant can achieve such phenomenological behavior. Essentially, by using the same parameters for the \( f(R) \) gravity as in Ref. [73], meaning that \( \gamma = 2, \Lambda = 1, 1895 \cdot 10^{-66} eV^2, \delta = \frac{1}{100}, N = 60 \) with the initial conditions chosen as \( y_H(z = 10) = \frac{\Lambda}{3m_0^2} \left(1 + \frac{m_0^2}{1000}\right), \frac{dy_H}{dz}|_{z=10} = -\frac{\Lambda}{3m_0^2} \frac{1}{1000}, \phi(z = 10) = 10^{-16} M_p, \)

\[
\left. \frac{d\phi}{dz} \right|_{z=10} = -10^{-17} M_p, \text{ then by solving numerically equations (3) and (5) in the interval } [z_i, z_f] = [-0.9, 10] \text{ with respect to } y_H, \phi, \text{ it becomes apparent that simply adding a canonical scalar field cannot negate the dark energy oscillations. This result seems to be in agreement with the one obtained in Ref. [66] for the } f(R) + g(\mathcal{G}) \text{ case given that } \mathcal{G} \text{ is quite small in terms of the rest of the parameters. The results of our numerical analysis for the particular model at hand can be found in Figs. 1, 2 and 3, while in Table I we compare the values of several statefinder quantities at present time with the corresponding values of the } \Lambda \text{CDM model and we confront the values of the dark energy density parameter } \Omega_{DE}(0) \text{ and the dark energy EoS parameter } \omega_{DE}(0) \text{ with the latest constraints of the Planck 2018 collaboration on cosmological parameters [18]. As it can be seen from Table I, the resulting cosmological quantities and the statefinder values at present time corresponding to the model at hand are quite close to the } \Lambda \text{CDM values, and both } \Omega_{DE}(0) \text{ and } \omega_{DE}(0) \text{ are compatible with the observational data. In Fig. 1 we present the behavior of the statefinder } y_H \text{ (left) and } \phi \text{ (right) as functions of the redshift, for the model at hand. The main difference with the pure } f(R) \text{ gravity seems to be the scalar field which does not oscillate. In general, the addition of a canonical scalar field and a linear Gauss-Bonnet topological invariant coupled to a scalar function do not suffice to nullify dark energy oscillation at large redshifts. Also in Fig. 2 we present the behavior of the cosmological statefinder quantities } q \text{ (upper left), } j \text{ (upper right), } s \text{ (bottom left) and } s \text{ (bottom right) as functions of the redshift. Once again, the same qualitative behavior as in the pure } f(R) \text{ case are obtained. Finally, in Fig. 3 we present the dark energy variables, namely the EoS (left) and the dark energy density parameter } \Omega_{DE} \text{ (right) as functions of the redshift. Out of these two parameters, only the latter is free of oscillations, however neither the canonical scalar field nor the Gauss-Bonnet topological invariant are responsible for such feature. The } f(R) \text{ contribution is the dominant term as it can also be inferred from the rest results.}

| Parameter | \( f(R) \) | \( \Lambda \text{CDM Value} \) |
|-----------|------------|-----------------|
| \( q(z=0) \) | -0.520954 | -0.535 |
| \( j(z=0) \) | 1.00319 | 1 |
| \( s(z=0) \) | -0.00104169 | 0 |
| \( \Omega_m(0) \) | 0.319364 | 0.3153±0.07 |
| \( \Omega_{DE}(0) \) | 0.683948 | 0.6847±0.0073 |
| \( \omega_{DE}(0) \) | -0.995205 | -1.018±0.031 |

Thus for this particular class of potential-less models, the \( f(R) \) gravity part seems to dominate the late-time evolution. In the following sections we shall also introduce a potential, and in parallel we shall assume an Einstein-Hilbert \( f(R) \) term. At a later section we shall constrain the functional forms of the scalar potential and the scalar coupling function, in order to see how the late-time dynamics are affected by these changes.

As a final comment, we should mention that even though the value of the scalar field seems to increase with respect to time, the rate of increase is smaller and subdominant when it is compared to the rate of the \( f(R) \) gravity terms, and in particular from the term \( \sim R^3 \). Subsequently, the \( f(R) \) part is dominant in comparison to the scalar terms and thus the increasing value of \( \dot{\phi} \) does not contradict the overall phenomenology. For this exact reason, one observes dark energy oscillations in the high redshift area, a feature which arises in the pure \( f(R) \) case.
FIG. 3: Dark energy variables, the EoS (left) and the Density parameter $\Omega_{DE}$ (right). Out of these two parameters, only the latter is free of oscillations however neither the canonical scalar field nor the Gauss-Bonnet topological invariant are responsible for such feature. The $f(R)$ contribution is the dominant term as it can also be inferred from the rest results.

IV. EINSTEIN-GAUSS-BONNET GRAVITY IN THE PRESENCE OF A SCALAR POTENTIAL

Let us now proceed with a different approach. We shall assume that the $f(R)$ case is reduced to a simple $f(R) = R$ case and that the scalar potential is now present in the formalism. Let us assume that the potential has the following arbitrary form,

$$V(\phi) = \left(\frac{\phi}{M_P}\right)^4,$$  \hspace{1cm} (31)

while the scalar coupling function has the following form,

$$\xi(\phi) = \left(\frac{\phi}{M_P}\right)^2,$$  \hspace{1cm} (32)

In this case we shall assume simple power-law models for the scalar functions which are normalized with respect to the reduced Planck mass. Such models are frequently used in the inflationary era where the slow-roll conditions for the scalar field are usually assumed to hold true. In the late time era, there exists no need to apply the slow-roll conditions since they do not hold true. Let us proceed with the numerical results. In this case, the Einstein-Hilbert form of $f(R)$ implies that certain terms in Eq.(3) are discarded which facilitates our study. Furthermore, since $\dot{R}$ is now absent, there exists no second derivative of statefinder $y_H$. In fact, if it was not for the scalar field which has a term $\ddot{\phi}$ proportional to $y_H'$ in the continuity equation, the aforementioned statefinder function would need no initial conditions to be specified. Here, we shall only assume that $y_H(z = 10)$ is once again equal to $y_H(z = 10) = \frac{\Lambda}{3m_s^2} \left(1 + \frac{1+z}{1000}\right)$.
FIG. 5: Deceleration $q$ (upper left), jerk $j$ (upper right), snap $s$ (bottom left) and $Om$ (bottom right) as functions of redshift. In this case, no dark energy oscillations are present since such feature is generated from additional $R$ terms. All variables are in agreement with the ΛCDM however what is fascinating is the snap parameter which decreases with time.

and in addition, $\phi(z = 10) = M_P, \left. \frac{d\phi}{dz}\right|_{z=10} = M_P$ then the results of our analysis can be found in Figs. 4, 5 and 6. Also in Table II, as in the model of the previous section, we compare the values of several statefinder quantities at present time with the corresponding values of the ΛCDM model and we confront the values of the dark energy density parameter $\Omega_{DE}(0)$ and the dark energy EoS parameter $\omega_{DE}(0)$ with the latest constraints of the Planck 2018 collaboration on cosmological parameters [18]. As it can be seen from Table II, the resulting cosmological quantities and the statefinder values at present time corresponding to the model at hand are quite close to the ΛCDM values, and both $\Omega_{DE}(0)$ and $\omega_{DE}(0)$ are compatible with the observational data. It can easily be inferred from the plots

| Parameter      | $R$   | \( ΛCDM \) Value |
|----------------|-------|-------------------|
| $q(z=0)$      | -0.522521 | -0.535         |
| $j(z=0)$      | 1.0002  | 1                |
| $s(z=0)$      | -0.00006431 | 0              |
| $Om(z = 0)$   | 0.318319 | 0.3153±0.07     |
| $\Omega_{DE}(0)$ | 0.681713 | 0.6847±0.0073   |
| $\omega_{DE}(0)$ | -1    | -1.018±0.031    |

that the qualitative behavior of the model under study is quite close to the ΛCDM model. One striking feature is that the dark energy density $\rho_{DE}$ is nearly constant throughout the interval \([-0.9,10]\), as indicated by $y_H$, and therefore the EoS parameter $\omega_{DE}$ is also nearly equal to $-1$. This result is robust towards changing the free parameters for this particular model. The rest of the cosmological parameters however seem to have an infinitesimal evolution, for instance the jerk parameter is quite close to $j = 1$ but not exactly equal to unity as $\omega_{DE}$ is $-1$. As a final comment,
it should be noted that both models studied so far do not have a fixed value for the velocity of the primordial gravitational waves these models produce. Since \[ c_T^2 = 1 - \frac{Q_f}{2Q_t}, \]
with \( Q_f = 16(\dddot{\xi} - H\dot{\xi}) \) and \( Q_t = M_P^2 - 8\dot{\xi}H \), then it stands to reason that the velocity obtains arbitrary values.

Despite being arbitrary, its value is essentially equal to unity due to the fact that \( Q_f \ll 1 \), and thus an infinitesimal value is subtracted from unity in Eq. (33). In particular, \( Q_f \sim \mathcal{O}(10^{-33}) \) and \( Q_t \sim \mathcal{O}(10^{36}) \) hence the reason why the velocity is equal to unity, and this can also be seen in Fig. 7, but the fact that \( Q_f \ll 1 \), even in Planck units where \( \kappa = 1 \), implies that a relation of the form \( Q_f = 0 \) or \( \dot{\xi} = H\xi \) for the Gauss-Bonnet scalar coupling function is not a random choice. In the next section we shall examine the phenomenological implications of constraining the aforementioned velocity by letting \( Q_f = 0 \).

V. PHENOMENOLOGY WITH THE CONSTRAINT \( \dot{\xi} = H\xi \)

The above formulation seems to be in general in good agreement with not only the observational data, but also with the \( \Lambda \)CDM model itself. As stressed in the last model however, the Einstein-Gauss-Bonnet models have a flaw, having to do with a production of a primordial tensor power spectrum, with propagation speed different from unity. The primordial gravitational wave speed however, must be equal to that of light’s in order to comply with the
GW170817 event \cite{62}. The deviation from unity for the above models is perhaps small in magnitude, implying that the effective value is $c_T \simeq 1$ however it is interesting to examine the cosmological implications on the late-time era when constraints on the velocity of tensor perturbations are imposed. Subsequently, following Ref. \cite{54,58} we shall proceed by assuming $\dot{\xi} = H \ddot{\xi}$. During the inflationary era, where the slow-roll conditions are assumed to hold true, the previous differential equation can define the time evolution of the scalar field, $\dot{\phi}$. This study though was performed using the slow-roll assumptions, which of course do not hold true in the present late-time context. Although one can easily imply that the scalar functions of the model continue to have the same primordial relation they had during the inflationary era, and also that the gravitational wave speed remains unity after the horizon crossing of the primordial tensor modes, here we shall adopt a different approach, and since the redshift is used as a variable, we shall solve analytically the equation. Since $\ddot{\xi} = H \dot{\xi}$, the solution reads,

$$
\dot{\xi} = \lambda e^{\int H dt},
$$

where $\lambda$ is an integration constant. Since the definition of redshift is $\frac{dz}{dt} = -H (1 + z)$, the above integral can be solved analytically, and thus the final solution is written as,

$$
\dot{\xi} = a(t) \lambda = \frac{\lambda}{1 + z}.
$$

This is a quite useful result, to say the least, since by simply imposing constraints on the velocity of gravitational waves in the late-time era, the degrees of freedom of the model are decreased by one, similar to the inflationary era, and obtain a functional constraint on $\dot{\xi}$ which seems to be model independent. In contrast to the previous sections, no definition for $\xi(\phi)$ is needed since essentially a transformation was performed which replaced $\dot{\xi}(\phi)$ with $\dot{\xi}(z)$, and given that in the equations (3) and (5) which we aim to solve numerically, only $\dot{\xi}$ is present, the overall phenomenology is now significantly altered. Now, the equations of motion are altered as shown below,

$$
\frac{3f_R H^2}{\kappa^2} = \rho_m + \frac{1}{2} \dot{\phi}^2 + V + \frac{f_R R - f}{2\kappa^2} - \frac{3H \dot{f}}{\kappa^2} + 24 \frac{\lambda}{1 + z} H^3,
$$

$$
- \frac{2f_R H}{\kappa^2} = \rho_m + P_m + \dot{\phi}^2 + \frac{\ddot{f}_R - H \dot{f}_R}{\kappa^2} - 16 \frac{\lambda}{1 + z} H \dot{H},
$$

$$
V_\phi + \ddot{\phi} + 3H \dot{\phi} - \frac{f_\phi}{2\kappa^2} + \frac{\lambda}{1 + z} \frac{\mathcal{G}}{\dot{\phi}} = 0.
$$

It should be noted that all the previous equations acquired in section II are still valid even when the constraint is applied. Furthermore, given that $\mathcal{G}$ is small from its nature, for certain values of $\lambda$, the phenomenology for such choice is indistinguishable from the one obtained without the constraint. For instance, if we recall the results for the $f(R)$ model studied previously, it was mentioned that the scalar field cannot alter the results. The same can be said about the case of $f(R)$ with $\xi = \frac{1}{1 + z}$ for a plethora of values for $\lambda$. By altering $\lambda$ and giving it a quite large value, say $\lambda = 10^{100}$, then the solution diverges and compatibility cannot be achieved. Let us proceed with a specific model and examine the impact the constraint has on the late-time evolution.

Consider a non-minimally coupled model of the form,

$$
f(R, \phi) = h(\phi) R,
$$

$$
h(\phi) = \frac{\dot{\phi}^2}{\phi},
$$

$$
V(\phi) = \frac{V_\phi}{\phi},
$$

and obviously

$$
\dot{\xi} = \frac{\lambda}{1 + z}.
$$
where $\phi_0$ and $V_0$ are auxiliary parameters with mass dimensions $[m]$ and $[m]^5$ respectively. In this case as well, since there exists only a linear $R$ term, only a single initial condition for $y_H$ is needed. As was the case with the previous two models, we shall use the same value, meaning $y_H(z=10) = \frac{\Lambda}{3m^2_{pl}} \left( 1 + \frac{1+z}{1000} \right)$. In consequence, letting $\phi_0 = 1$, $V_0 = 1$, $\lambda = 1$, $\phi(z=10) = 10^{-35} M_{Pl}$, $\frac{d\phi}{dz}\bigg|_{z=10} = -10^{-20} M_{Pl}$ then the results obtained are compatible with the $\Lambda$CDM model as shown in Fig. 9, while in Fig. 10 we present the behavior of the dark energy EoS parameter and the dark energy density parameter as functions of the redshift. The results of our numerical analysis corresponding to the values of the statefinders and of the dark energy EoS parameter and the dark energy density parameter at present time, can be found in Table III. As it can be seen in Table III our model is in good qualitative agreement with the $\Lambda$CDM model and also is compatible with the 2018 Planck constraints on the cosmological parameters, when the dark energy EoS parameter and the dark energy density parameter are considered.

| Parameter       | $h(\phi)R$ | $\Lambda$CDM Value |
|-----------------|------------|---------------------|
| $q(z=0)$        | -0.520794  | -0.535              |
| $j(z=0)$        | 0.99987    | 1                   |
| $s(z=0)$        | -0.00004423| 0                   |
| $\Omega_{DE}(0)$| 0.319364   | 0.3153±0.07         |
| $\Omega_{DE}(0)$| 0.680671   | 0.6847±0.0073       |
| $\omega_{DE}(0)$| -0.99984   | -1.018±0.031        |

Genuinely speaking, the linear $R$ case we studied in the previous section, without constraints and the $h(\phi)R$ case with constraints studied in this section, do not differ so much. The main difference lies in the evolution of statefinder $y_H$ and subsequently the EoS on the latter case, however even when the EoS is dynamically evolving, it does so with
a small rate that its value essentially cannot be distinguished from unity.

As a comment, it should be noted that in this case no difference between the constrained and the unconstrained Gauss-Bonnet phenomenology is found, however the latter seems quite arbitrary from one perspective, meaning that the velocity of gravitational waves is dynamically evolving in various cosmological eras and \( c_f \) just happens to be unity since \( Q_f \ll Q_t \). However, even uncontrolled, the fact that \( Q_f \ll 1 \) even in Planck units implies that the relation \( \xi = H\dot{\xi} \) is satisfied one way or another, hence instead of coming to such numerical conclusion at the end, it is beneficial to begin with such statement as the degrees of freedom seem to decrease in the first place. The main idea was to examine a model with the constraint \( c_f^2 = 1 \) and prove that compatibility can be achieved by taking into consideration that the constraint imposed from the velocity of gravitational waves decreases the degrees of freedom such that the Gauss-Bonnet scalar coupling function can be replaced by a single parameter. In the literature \([70]\) such a question has been addressed, and the results were quite different quantitatively in comparison to the present model. In principle however, the compatibility is a model dependent feature.

Before closing we need to discuss an interesting scenario, in view of the unified description of inflation with the dark energy era that Einstein-Gauss-Bonnet theory, combined with the fact that it is a string theory originating theory. In the present paper we assumed that the dark matter perfect fluid consists of an unknown particle, but string theory has also offered the possibility of having axion like particles present even in the pre-inflationary era. In fact, in the axion like particle phenomenology with a primordial pre-inflationary era broken \( U(1) \) symmetry. The axion due to the breaking of this symmetry is frozen in its vacuum expectation value, but as the Universe expands, the axion behaves as a condensate and evolves as a dark matter perfect fluid, which makes it a perfect candidate for a low-mass weakly interactive massive dark matter particle. Such scenarios in the context of modified gravity have been studied in the literature \([71–74]\), so one interesting scenario is to have the combined presence of the axion coupled to the Gauss-Bonnet scalar. This would utterly change the symmetry breaking pattern of the primordial \( U(1) \) symmetry, due to the presence of the coupling \( \xi(\phi) \) in the axion equation of motion, even pre-inflationary. The calculation might get easier if it is assumed that the Gauss-Bonnet corrections and the flat four dimensional spacetime, FRW-like, are the resulting outcomes of the quantum era. Also, the presence of the Gauss-Bonnet non-minimal coupling would alter the post-inflationary evolution, and in addition, in this scenario, the axion \( U(1) \) symmetry might be unbroken during the inflationary era. These issues are interesting material for a focused future work.

VI. CONCLUSIONS

In this work we investigated the late-time phenomenology aspects of scalar-coupled \( f(R, \mathcal{G}) \) gravity. We focused on theories of Einstein-Gauss-Bonnet form, and we examined three types of models, \( f(R) \) gravity Einstein-Gauss-Bonnet models, and pure Einstein-Gauss-Bonnet models, with arbitrary choice of the scalar functions of the models and with constrained functions of the models. Our numerical analysis indicated that for the models containing the \( f(R) \) gravity, the late-time dynamics is very much affected by the \( f(R) \) gravity part, and thus in those cases, the Einstein-Gauss-Bonnet coupling does not affect the dynamics. For the pure Einstein-Gauss-Bonnet, we made a novel assumption related to the requirement that the primordial gravitational wave speed is equal to unity, which in turn imposed a functional constraint on the functional form of the Gauss-Bonnet scalar coupling function. The exiting feature in the late-time study by taking into account the gravitational wave speed constraint, is the fact that the functional
form of $\xi$ is model independent, and has a specific form given in terms of the Hubble rate and the redshift. This simplification is rather interesting to think that there is a strong motivation to assume that the primordial gravity wave speed should be set equal to unity for all the cosmic times after the first horizon crossing of the primordial tensor modes. This could in fact constrain the functional forms of the scalar potential and of the scalar coupling function, directly from the inflationary era and thereafter, but we did not go deep to this study, since it seems that the Gauss-Bonnet coupling as we studied it in section II does not play a significant role during the late-time era. It should actually, it is of the order $\sim H^4$, but we aimed to formally investigate the phenomenology of these models. The positive outcome we keep is that the primordial gravitational wave speed constraint has some effect on the late-time cosmological dynamics, and this is a motivation for us to go investigate astrophysical scenarios related to Einstein-Gauss-Bonnet models, with the potential and the scalar coupling functions being related in the way we demonstrated in [54] [58]. Such task we aim to address in a future work.

Let us discuss at this point the issue of choosing the scalar functions of the models we studied in this paper. In order to address this comment we need to elaborate further on the concept of gravitational waves. Theories containing a Gauss-Bonnet term coupled to an arbitrary scalar function are notorious for producing primordial gravitational waves which propagate with a velocity which is different from the speed of light. Our initial work on Gauss-Bonnet started by confronting this nasty feature during the inflationary era and coming up with ways in order to remedy the theory, see Ref. [58]. The reason that the theory needs to be rectified, from our point of view, is that there exists no known mechanism which could produce massive primordial gravitons which were later turned into massless according to the recent GW170817 event. Therefore, a reasonable assumption is to impose that the theory is described by massless gravitons throughout the evolution of the Universe. The main result extracted from this simple statement is that the Gauss-Bonnet scalar coupling function must satisfy the differential equation $\ddot{\xi} = H \dot{\xi}$ and since Hubble’s parameter is given from the Friedmann equation, it becomes apparent that in this scenario both scalar coupling functions are connected, i.e. $V(\phi)$ and $\xi(\phi)$ are chosen in such a way so that the aforementioned differential equation is satisfied properly. This can be seen easily in the slow-roll regime where many terms are assumed to be subleading but during the late-time era, we are interested in all of them. The main idea was to start without imposing the constraint derived from the velocity of gravitational waves and afterwards compare the results between a constrained and an unconstrained model. As it turns out, both assumptions lead to relative similar results given that in the set of equations, other terms are dominant during the late-time era. Moreover, in order to perform a fully self-consistent study, further auxiliary parameters such as the jerk, snap and $Om(z)$ where used in order to see the pros and cons of the models studied in this case and to see explicitly where and if some cosmological parameters deviate from observations, thus rendering the model unsuitable. Concerning the initial conditions at redshift $z = 10$, the first variable is designated in order to coincide with observations while the initial value of the scalar field, since it cannot be observed, is chosen arbitrarily. The previous value seems to produce a slow evolution for the scalar field with respect to cosmic time and moreover all the physical observables values of the model are compatible with the observational data.

Moreover, let us comment that in the present text, the early-time phenomenology was not considered, however in order have an inferior Gauss-Bonnet term in the late-time era, one would expect that the scalar field related terms are subleading from the first horizon crossing and thereafter. No comment can be made about the Planck era unfortunately, but at least in the inflationary era, the $R$ part of the equations of motion should at best receive mild corrections so that to obtain acceptable values for the scalar spectral index and the tensor-to-scalar ratio. Assuming that the scalar-field evolves with either a slow-roll or a constant rate of roll, then by implementing either similar slow-roll conditions for the Gauss-Bonnet coupling in the unconstrained case, or working in the constrained case and in the presence of an arbitrary scalar potential, then compatibility with the observational data can be achieved relatively easily, while simultaneously the order of magnitude of several scalar functions and their derivatives can be negligible. To summarize, a smooth early-time description which ensures that the scalar components of the model act as an effective dark energy density in the Friedmann equation can be achieved by working with the classical prescription of inflation and assume a potential driven inflation with either slow or constant-roll evolution. In either case, the $R$ part of the equations of motion becomes dominant in comparison to the scalar one as time flows and especially during the late-time era.

Before closing we need to discuss an importance issue, related with the choice of the arbitrary functions of our models. As we demonstrated, the Gauss-Bonnet coupling contributes slightly to the late-time era. The fact that the Gauss-Bonnet coupling does not affect so significantly the late-time phenomenology of the models studied in the unconstrained case, is a direct consequence of their relevance. It turns out that the Gauss-Bonnet term $G$ is not dominant in the dark energy era (it scales as $G \sim H^3$ for a flat FRW spacetime). In consequence, all the scalar terms manage to produce what is perceived as a nearly constant term in the Friedmann equation, which in turn is interpreted as an effective late-time cosmological constant. In principle, in order to see different results, one would need a really strong contribution from the scalar components of the model, however no such case was observed in the respective computer program which was developed for this purpose. In fact as we evinced, in the first case, the $f(R)$ contribution, and mainly from the third term $R^3$, is the dominant driving force. Similarly, in the second case, a simple
$R$ term even in the presence of a scalar potential is the driving force. Finally, in the constrained model where the non-minimally coupled case was examined, the realization that the scalar field itself is subleading is the factor which ensures that the $R$ contribution is once again dominant. The reason behind such designation for the scalar coupling was mainly the simplicity of the resulting expressions and also the production of a viable inflationary era, which can be ascertained from recent observations, hence the reason why power-laws and also dilatonic couplings were assumed. Finally, it is interesting to compare the results obtained in this paper, with the ones obtained in Ref. [72] using again a dilatonic Einstein-Gauss-Bonnet gravity, but in the context of an emergent gravity scenario. The results produced in our paper are different from the ones obtained in Ref. [75]. This can be attributed to the specific form of the scale factor that the authors have chosen and/or to the non-canonical kinetic term of the scalar field. In the present article, we did not assume a specific form for the scale factor and in fact the scale factor is directly derivable from the equations of motion once $\Phi(\phi),$ $\xi(\phi)$ and $\phi(t)$ are chosen. Moreover, the kinetic term is assumed to be canonical, meaning that $\phi$ is not coupled to $\phi$. These two different assumptions may be the reason why the results do not agree, since Ref. [72] deals with the emergent Universal scenario, which has its own attributes however, so this issue was worthy of mentioning.

Acknowledgments

This work was supported by MINECO (Spain), project PID2019-104397GB-I00 and PHAROS COST Action (CA16214) (SDO).

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