TOPICAL REVIEW

Superconformal mechanics

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Abstract
We survey the salient features and problems of conformal and superconformal mechanics and portray some of its developments over the past decade. Both classical and quantum issues of single- and multiparticle systems are covered.

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3 On leave of absence from V.N. Karazin Kharkov National University, Ukraine.
1. Introduction

The sustained interest in conformal mechanics and its various superconformal extensions is caused by several interconnected reasons.

Superconformal mechanics models are characterized by a simple non-compact non-Abelian group of dynamical symmetry, namely $\text{SL}(2, \mathbb{R})$ or one of its supersymmetric extensions. Although the unitary representations of unimodular real groups were constructed by Bargmann [8] in the 1940s, physical applications of these group-theoretical results in (super)conformal mechanics models only became apparent much later.

One of the first descriptions of a simple conformal mechanical system (with an additional oscillator potential) appeared in the textbook by Landau and Lifshitz [84]. Later on, at the end of the 1960s and in the 1970s, interest in models with $d = 1$ conformal symmetry was supported from two independent sources. First, conformal symmetry characterizes an important class of integrable many-particle systems discovered by Calogero in his pioneering papers [25, 26]. Second, models of conformal mechanics were studied as examples of one-dimensional ($d = 1$) field theories. One of the forerunners in this direction was [9]. However, the beginning of concrete studies and applications of conformal mechanics, at the classical and the quantum level, was marked by the seminal paper of de Alvaro, Fubini and Furlan [5]. Their model can be recovered from the two-particle Calogero model [25, 26] after eliminating the center-of-mass coordinate, establishing a link between these two early forays into applications of $d = 1$ conformal symmetry.

The 1980s brought supersymmetry to the models of conformal mechanics. The models of superconformal mechanics are important particular cases of supersymmetric quantum mechanics [107]. The papers [4, 45] found the $\mathcal{N} = 2$ supersymmetric generalization of the de Alvaro–Fubini–Furlan (AFF) model. Two different (but closely related) $\mathcal{N} = 4$ superconformal mechanics models based on $\text{SU}(1, 1|2)$ were constructed in [45, 70]. At approximately the same time, the authors of [43] constructed the $\mathcal{N} = 2$ supersymmetric extension of the multi-particle Calogero system.

The interest in conformal and superconformal mechanics increased in connection with the AdS/CFT correspondence [89, 60, 109], when it was discovered that such models describe (super)particles moving in near-horizon (AdS) geometries of black-hole solutions to supergravities in diverse dimensions. Namely, it was suggested in [28] that the radial motion of a massive charged particle near the horizon of an extremal Reissner–Nordström black hole is described by some ‘relativistic’ type of conformal mechanics, which reduces to that of [5] in a ‘non-relativistic’ limit. The target variable of this conformal mechanics is the radial AdS coordinate of an $\text{AdS}_2 \times S^2$ background. The latter is the bosonic body of the maximally supersymmetric near-horizon extremal Reissner–Nordström solution of $\mathcal{N} = 2$ $d = 4$ supergravity [28], with the full isometry supergroup being $\text{SU}(1, 1|2)$. This observation led [28] to a conjecture that the full dynamics of a superparticle in the near-horizon geometry of an extremal Reissner–Nordström black hole is governed by $\mathcal{N} = 4$ superconformal mechanics.

This tight relation to the AdS/CFT correspondence spurred further works on $\mathcal{N} = 4$ superconformal mechanics with $\text{SU}(1, 1|2)$ symmetry at the end of the 1990s. Such models were constructed in the framework of nonlinear realizations and transferred to the black-hole

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4 See also [77].
context in [6], partly recreating earlier results of [70]. In [57], it was then argued that the large-$n$ limit of an $n$-particle generalization of SU(1, 1|2) superconformal mechanics may provide a microscopic description of the near-horizon dynamics in a multi-center extremal Reissner–Nordström black-hole geometry. Further evidence in favor of the proposal of [28] was adduced in [71, 10] by a canonical transformation linking the radial motion of a (super)particle on $\text{AdS}_2 \times S^2$ to $\mathcal{N} = 0, \mathcal{N} = 2$ [71] and $\mathcal{N} = 4$ [10] superconformal mechanics.

First attempts [110, 11, 12] to construct multi-particle systems with $\mathcal{N} = 4$ superconformal invariance revealed a surprising rigidity, which renders it a difficult problem. In contrast to the $\mathcal{N} = 2$ superconformal case with a single prepotential $U$, a second prepotential, $F$, appears. Both are not only subject to homogeneity conditions, but must also solve a coupled system of quadratic partial differential equations which are prominent in mathematical physics: the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation (for $F$) [108, 34] and the associated twisted-period equation (for $U$) [35, 39]. As documented in a number of works [49–51, 19, 80, 86, 20], even taking the $F$ solution from the WDVV literature, it proved to be very tedious to explicitly solve the $U$ equation for more than three particles in the case of SU(1, 1|2) symmetry.

These technical difficulties are not the only reason to look beyond SU(1, 1|2) to the most general $\mathcal{N} = 4$ superconformal group in one dimension, which is the exceptional one-parameter supergroup $D(2, 1; \alpha)$ depending on a real parameter $\alpha$ [44]. It reduces to SU(1, 1|2) × $\supset$SU(2) at $\alpha = 0$ and $\alpha = -1$. In fact, the isometry supergroup of a near-horizon $M$-brane solution of $d = 11$ supergravity was determined as $D(2, 1; \alpha) \times D(2, 1; \alpha)$ [55], and $D(2, 1; \alpha)$ is physically realized for any value of the parameter $\alpha$ in near-horizon $M$-theory solutions [104]. The general $\mathcal{N} = 4$ superconformal group may be relevant to the multi-black-hole system, since the corresponding moduli spaces of $n$ black holes in four- and five-dimensional supergravities are described by sigma-model-type multi-black-hole mechanics respecting $D(2, 1; \alpha)$ invariance [93, 92, 90, 24].

It turns out that, in order to go beyond $\alpha = 0$ or $\alpha = -1$, it is necessary to enlarge the degrees of freedom by a set of ‘semi-dynamical’ harmonic isospin variables [36, 15, 37, 38]. This slight generalization allows one to find a $U$ solution for many WDVV solutions $F$, but only for a particular value of $\alpha$ depending on $F$ [79]. The relation to previously found solutions in the special cases of SU(1, 1|2) [80] or OSp(4|2) [36] remains unclear.

Another method of constructing many-particle systems with superconformal symmetry was proposed in [36–38]. There, such systems arise from a superconformally-invariant gauging of certain supersymmetric matrix mechanics models, which contain the semi-dynamical isospin degrees of freedom just mentioned, in addition to the dynamical and purely auxiliary ones. These semi-dynamical variables are described by a Chern–Simons (or Wess–Zumino) mechanical action [42, 65, 100] and, after quantization, represent isospin (or spin) degrees of freedom. The resulting isospin-extended superconformal many-particle models exist for any value of $\alpha$, but the particle coordinates parametrize a non-flat target space, except for $\alpha = -\frac{i}{2}$, i.e. the case of OSp(4|2).

Finally, any compact supersymmetric mechanics system can be enhanced to a superconformal one by coupling it to a super-dilaton [61, 62]. In this way, the system to start with provides the ‘angular part’ and the dilaton yields the ‘radial part’ of the combined superconformal mechanics. In particular, any $\mathcal{N} = 4$ extension of an angular system can be lifted to a $D(2, 1; \alpha)$ invariant mechanics [61].

This review provides an overview of recent results obtained by us and other authors in their studies of $d = 1$ superconformal models, mainly classical but also quantum-mechanical, mainly single-particle but also multi-particle. Given the large number of works devoted to the subject, we naturally focus mostly on those which, in our opinion, have been most influential.
or prospective for future investigations. Our review can be considered as being complementary
to already existing reviews (see, e.g., [24, 56]).

The plan of the review is as follows.

We start, in section 2, with the detailed description of the $d = 1$ conformal group
$\text{SO}(2, 1) \simeq \text{SU}(1; 1) \simeq \text{SL}(2, \mathbb{R})$, its representations and the renowned AFF model of
conformal mechanics together with its multi-particle generalization, i.e. the rational Calogero
model. Besides presenting some well-known geometrical, classical and quantum results in this
area, we also include developments based on our recent papers. This concerns a derivation
of conformal mechanics and Calogero models from gauging specific matrix models by non-
propagating $d = 1$ gauge fields as well as an extension of the standard conformal mechanics
by ‘semi-dynamical’ isospin variables, which provides a new mechanism for generating
the conformal potential. These considerations form a prerequisite for the methods used in
subsequent sections covering superconformal mechanics.

Section 3 is devoted to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superextensions of the conformal group
and to the characterization of the corresponding superconformal mechanics models, both in
the superfield and in the component language. Again, besides addressing previously known
models (like the Freedman–Mende $\mathcal{N} = 2$ super-Calogero models), we discuss more recently
proposed types of such models. We elaborate on new $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superconformal
extensions of the bosonic Calogero model obtained through supersymmetric versions of the
gauging procedure introduced in the section before.

$\mathcal{N} = 4$ superconformal models are the subject of section 4. This section is the most
extensive and largely based upon the results obtained with participation of the authors. We
start with presenting the necessary facts about $\mathcal{N} = 4$ superconformal algebras, which can all
be treated as particular cases of the general $D(2, 1; \alpha)$ superalgebra. Then, we discuss various
models of $\mathcal{N} = 4$ superconformal mechanics, both in the one- and multi-particle cases,
starting either from the ordinary $\mathcal{N} = 4$ superspace formulation or from the component one.
Subsequently, we develop a formulation in harmonic $\mathcal{N} = 4, d = 1$ superspace, which often
makes the geometric properties of such models more transparent and yields their new types.
In particular, we describe a new $D(2, 1; \alpha)$ invariant super-Calogero model extending the so-
called spinning Calogero model. This model involves the semi-dynamical isospin variables and
realizes the $\mathcal{N} = 4$ extension of the previously defined $d = 1$ gauging procedure. Finally we
list the known applications of $\mathcal{N} = 4$ superconformal mechanics models within the AdS/CFT
correspondence and in black-hole physics.

In section 5, we summarize what is known to date about superconformal mechanics with
$\mathcal{N}$ larger than 4.

Section 6 contains some final remarks and lists possible directions for further
investigations of the subjects discussed in the review.

2. Conformal mechanics

2.1. The one-dimensional conformal algebra and its representations

The conformal algebra in one dimension is $o(2, 1)$ spanned by the Hermitian generators $H, K$
and $D$ satisfying the commutation relations

\[ [D, H] = -iH, \quad [K, H] = -2iD, \quad [D, K] = iK. \tag{2.1} \]

Let us define the dimensionless $O(2, 1)$ vector $T_r, r, s, t = 0, 1, 2,$

\[ T_0 = \frac{1}{2} (mK + m^{-1}H), \quad T_1 = \frac{1}{2} (mK - m^{-1}H), \quad T_2 = D, \tag{2.2} \]
where \( m \) is a constant of mass dimension\(^5\). Using these generators, one obtains another representation of the same \( o(2, 1) \) algebra (2.1)

\[
[T_r, T_s] = i\epsilon_{rst} T_t,
\]

with \( \epsilon_{012} = +1, T' = g'^2 T, g_{\alpha\nu} = \text{diag}(-++) \).

For applications to supersymmetric theories, it is important to be aware of a spinorial representation of the \( d = 1 \) conformal algebra. Introducing the \( SU(1, 1) \cong O(2, 1) \) bispinor

\[
T_{\alpha\beta} = T_{\beta\alpha}, \quad \alpha, \beta, \gamma = 1, 2,
\]

one gets the spinorial representation of the \( o(2, 1) \) commutation relations (2.1)

\[
[T_{\alpha\beta}, T_{\gamma\delta}] = i(\epsilon_{\alpha\gamma} T_{\beta\delta} + \epsilon_{\beta\delta} T_{\alpha\gamma}),
\]

where \( \epsilon_{12} = -\epsilon_{21} = 1 \).

The second-order Casimir operator of the \( o(2, 1) \) algebra is given by the following expression:

\[
T^2 = \frac{1}{2} [H, K] - D^2 = -T' T_r = \frac{1}{2} T'^{\alpha\beta} T_{\alpha\beta}.
\]

The noncompact group \( SL(2, \mathbb{R}) \cong SU(1, 1) \cong O(2, 1) \) has only infinite-dimensional unitary representations. They are characterized by the eigenvalues of the Casimir operator and the compact generator \( T_0 \). The infinite-dimensional unitary representations of the discrete series of the universal covering of the group \( SU(1, 1) \) are labeled by positive numbers \( r_0 \) which can be an integer or half-integer \([8, 97, 5]\). On the same states, the Casimir operator (2.6) takes the value

\[
T^2 = r_0 (r_0 - 1).
\]

### 2.2. The AFF model and its interpretation

The AFF conformal mechanics is described by the action [5]

\[
S_0 = \int dt \left( \dot{x}^2 - \gamma^2 x^{-2} \right) \equiv \int dt \mathcal{L}_0,
\]

which implies the equation of motion

\[
\ddot{x} = \gamma^2 x^{-3}.
\]

The canonical dimension of \( x \) is \([x] = L^{1/2}\) whereas the constant \( \gamma \) is dimensionless, \([\gamma] = L^0\).

The action (2.8) is invariant under the \( d = 1 \) conformal transformations

\[
\delta t = f(t), \quad \delta x = \frac{1}{2} \dot{f} x \quad \text{with} \quad \delta^3 f(t) = 0,
\]

whence

\[
\delta S_0 = \int dt \dot{\Lambda} \quad \text{with} \quad \Lambda = \frac{1}{2} x^2 \ddot{f}.
\]

This ‘passive’ form of \( d = 1 \) conformal transformations treats the time variable \( t \) and the field \( x(t) \) on equal footing, which matches with the geometric interpretation of both as different parameters of the group \( SL(2, \mathbb{R}) \) (see below). It is very convenient for checking invariance of various actions and will be applied for this purpose in other cases, including the models of

\(^5\) When considering physical applications of conformal symmetry, the generators \( H \) and \( K \) have physical origin and possess opposite dimensions: \([H] = L^{-1}, [K] = L\).
superconformal mechanics. On the other hand, for deriving the conserved currents, the ‘active’ form of the same transformations is more convenient,
\[
\delta t = 0, \quad \delta x = \frac{1}{2} \dot{f} x - f \dot{x},
\] (2.12)
under which
\[
\delta S_0 = \int dt \tilde{\Lambda} \quad \text{with} \quad \tilde{\Lambda} = \frac{1}{2} x^2 \dddot{f} - f \dddot{L}_0.
\] (2.13)
Identifying the constant parameters \(a, b\) and \(c\) of the \(d = 1\) translations, dilatations and conformal boosts with the coefficients in the \(t\)-expansion of \(f(t)\) as
\[
f(t) = a + bt + ct^2,
\] (2.14)
it is easy to read off from (2.13) the corresponding conserved Noether charges
\[
H = \frac{1}{4} p^2 + \frac{\gamma^2}{x^2}, \quad D = -\frac{1}{2} xp + iH, \quad K = x^2 - txp + i^2H,
\] (2.15)
where \(p = 2i\). Note that in the Hamiltonian formalism the conservation is understood to be with respect to the full time derivative, i.e.
\[
\frac{d}{dt} K = \frac{\partial}{\partial t} K + \{K, H\}_r = 0, \quad \frac{d}{dt} D = \frac{\partial}{\partial t} D + \{D, H\}_r = 0.
\] (2.16)
With respect to canonical Poisson brackets the charges (2.15) form the algebra \(\text{sl}(2, \mathbb{R})\)
\[
\{H, D\}_r = H, \quad \{K, D\}_r = -K, \quad \{H, K\}_r = 2D,
\] (2.17)
which thus defines the symmetry of the model. Note that the explicit dependence on \(t\) in the generators \(D\) and \(K\) can be absorbed into the similarity transformation \((D, K) = e^{iH}(D_0, K_0)e^{-iH}\), where the commutators are understood as Poisson brackets, and \(D_0 = -\frac{1}{2} xp, \quad K_0 = x^2\) are the \(t\)-independent parts of \(D\) and \(K\). Together with \(H\) they satisfy the same Poisson-bracket algebra (2.17). In what follows, we will basically consider only these main terms in the generators of the conformal algebra and its superextensions.

The expressions for the Noether charges (2.15) imply the following value for the classical Casimir operator (2.6),
\[
T^2 = \gamma^2,
\] (2.18)
and the Hamiltonian can be rewritten as
\[
H = \frac{1}{4} p^2 + \frac{T^2}{x^2}.
\] (2.19)
As was noted in [61, 62], this expression provides a general classical Hamiltonian of conformal mechanics models, including multi-dimensional and supersymmetric extensions of the one-dimensional AFF mechanics. The canonical variables \(p\) and \(x\) represent the radial degree of freedom, while the additional angular and fermionic coordinates and their canonical momenta are hidden in the conformal Casimir \(T^2\) which, on its own, can be treated as a Hamiltonian of some ‘angular’ mechanical system.

The AFF conformal mechanics [5] admits a nice geometric interpretation. As was demonstrated in [69], it can be obtained by applying the Maurer–Cartan (or nonlinear realizations) method to the algebra \(o(2, 1)\). We choose the exponential parametrization for the element of the group \(\text{SO}(2, 1)\):
\[
G_0 = e^{iH} e^{iCK} e^{iD},
\] (2.20)
and construct the left-covariant Maurer–Cartan (MC) one-forms
\[
g_0^{-1} dG_0 = i(\omega_H H + \omega_K K + \omega_D D),
\] (2.21)
with
\[ \omega_H = e^{-\mu t} dt, \quad \omega_K = e^\mu (dz + z^2 dt), \quad \omega_D = du - 2z \, dt. \]  

(2.22)

In the conformal mechanics [5], like in the construction of unitary representations of the group SO(2,1), one is led to choose the basis (2.2) in the \( o(2, 1) \) algebra. The MC one-forms associated with the generators \( T_\tau \) are, respectively,
\[ \omega_0 = m^{-1} \omega_K + m \omega_H, \quad \omega_1 = m^{-1} \omega_K - m \omega_H, \quad \omega_2 = \omega_D. \]  

(2.23)

The dynamics of the AFF conformal mechanics is obtained by imposing the following constraints on the one-dimensional coset fields \( z(t) \) and \( u(t) \) [69]:
\[ (a) \quad \omega_1 = 0, \quad (b) \quad \omega_2 = 0. \]  

(2.24)

Equation (2.24(b)) is the inverse Higgs [75] constraint trading the field \( z(t) \) for the time derivative of the dilaton \( u(t) \),
\[ z = \frac{1}{2} \dot{u}, \]  

(2.25)

while (2.24(a)) is the dynamical constraint which leads to the equation of motion (2.9) for the newly introduced single independent variable
\[ x = \mu^{-1/2} e^{\mu/2}, \]  

(2.26)

where \( \mu \) is the dimensionful part of \( m \), \( [\mu] = cm^{-1} \) and \( m = \mu \gamma \), \( [\gamma] = cm^0 \). Being constraints on the left-invariant Cartan 1-forms, equations (2.24) enjoy manifest \( d = 1 \) conformal SO(2,1)symmetry.

In the formalism of the MC one-forms, the conformal mechanics action can be rewritten as [69]
\[ S_0 = -\gamma \int \omega_0 = -\int dt [\mu^{-1} e^{\mu}(\dot{z} + z^2) + \mu \gamma^2 e^{-\mu}]. \]  

(2.27)

We see that the action (2.27) is specified by the remaining non-vanishing MC one-form in \( o(2, 1) \). Both the kinematical constraint (2.25) \( (\omega_2 = 0) \) and the dynamical equation (2.9) \( (\omega_1 = 0) \) follow from the action (2.27) as the equations of motion. Substituting the kinematical solution (2.25) into (2.27) and passing to the variable \( x \) by (2.26), we recover the original AFF conformal mechanics action (2.8). Note that this equivalence is valid modulo a total \( t \)-derivative under the integral, which explains why the action (2.27) is exactly invariant with respect to the ‘passive’ conformal transformations (2.10), while (2.8) is invariant up to a total \( t \)-derivative, equation (2.11).

Note that (2.24) define a class of geodesics on the SO(1, 2) group manifold, such that the geodesics are generated by the right action of the one-parameter compact subgroup with the generator \( T_0 \) [69]. Only such a class leads to the standard conformal mechanics with good quantum properties [5], as opposed to any other non-trivial choice of the constraints (for example, the choice of \( \omega_0 = 0 \) instead of (2.24(a))). This is the reason why in our considerations we stick just to the constraints (2.24). The coordinate \( \tau \) associated with the generator \( T_0 \) in the exponential parametrization of this compact subgroup is the natural parameter along the geodesic curve. In the black-hole interpretation of (super)conformal mechanics (see below), \( \tau \) plays the role of the test-particle proper time near the horizon of the extremal black hole.

Let us make a few comments on the Hamiltonian formulation of the model (2.27).

The definition of the momenta yields the second-class constraints
\[ p_u \approx 0, \quad p_z + \mu e^{\mu} \approx 0. \]  

(2.28)
These constraints allow one to eliminate the phase space variables \((p_z, p_u)\). The Dirac brackets for the surviving pair of the phase space variables \((u, z)\) and the Hamiltonian take the form

\[
\{u, z\}_D = \mu e^{-u}, \quad H = \frac{1}{4\mu} (e^u z^2 + 4\gamma^2 \mu^2 e^{-u}).
\] (2.29)

After introducing the variables \(x = \mu^{-1/2} e^{u/2}, p = 2\mu^{-1/2} e^{u/2} z\) which possess standard Dirac brackets,

\[
\{x, p\}_D = 1,
\] (2.30)

we find that the system (2.27) is described by the Hamiltonian

\[
H = \frac{1}{4} p^2 + \gamma^2 x^{-2},
\] (2.31)

which also follows from the action (2.8).

The basic features of the quantization based on the standard Hamiltonian (2.31) are as follows [5, 45, 24].

- The spectrum of \(H\) is continuous: if \(H |E\rangle = E |E\rangle\), then \(H e^{i\alpha D} |E\rangle = e^{2\alpha} E |E\rangle\).
- The spectrum includes all \(E > 0\) eigenvalues of \(H\); for each of them, there exists a plane-wave normalizable state.
- The state with \(E = 0\) is not even plane-wave normalizable and so it cannot be chosen as the ground state.

The absence of the normalizable ground state implies that the description of quantum conformal mechanics in terms of the \(H\) eigenstates is ill-defined. The correct description of the model is achieved by choosing the compact operator \(T_0\) as the Hamiltonian with respect to which one should define the energy spectrum [5]:

\[
\tilde{H} := 2mT_0 = H + m^2K = \frac{1}{4} p^2 + \gamma^2 x^{-2} + m^2 x^2.
\] (2.32)

It contains an oscillator-like term and so has a well-defined ground state. The corresponding Hilbert space is the space of functions which span infinite-dimensional unitary representations of SU(1,1), as described in the previous subsection. The spectrum of the Hamiltonian (2.32) is discrete. Its ground state is not invariant under the whole SU(1,1), but breaks it spontaneously [5, 103].

The passing to the phase with the new evolution operator \(T_0\) and spontaneously broken SU(1,1)\(\sim\)SL(2,\(\mathbb{R}\)) symmetry can be interpreted as a redefinition of the time coordinate [5, 28, 24, 103]. Namely, if we introduce a new evolution parameter \(\tau\) and coordinate \(q\) through the relations [5]

\[
\tau := \frac{1}{m} \arctan(mt), \quad q(\tau) := x(t)/\sqrt{1 + m^2 t^2},
\] (2.33)

the action (2.8), up to boundary terms, takes the form

\[
S_0 = \int d\tau (\dot{q}^2 - \gamma^2 q^{-2} - m^2 q^2), \quad \dot{q} := dq/d\tau,
\] (2.34)

which is the action corresponding to the Hamiltonian (2.32). Such a coordinate change has a nice geometrical interpretation in the application to black-hole dynamics. The near-horizon geometry is that of AdS\(_2\) with SL(2,\(\mathbb{R}\)) as the isometry group. The good global time coordinate for the particle moving on this near-horizon AdS\(_2\) is precisely the evolution parameter \(\tau\) (for details, see [28, 24, 103]). Note that the action (2.34), despite the presence of the oscillator term, is still invariant under a particular nonlinear realization of the underlying SL(2,\(\mathbb{R}\)) group [1].
The quantum counterparts of the $o(2, 1)$ generators (2.15) at $t = 0$ read
\[ H = \frac{1}{4} \hat{p}^2 + \frac{\gamma^2}{\hat{x}^2}, \quad D = -\frac{1}{4} [\hat{x}, \hat{p}], \quad K = \hat{x}^2, \] (2.35)
where
\[ [\hat{x}, \hat{p}] = i. \] (2.36)
The Casimir operator (2.6) takes the value
\[ T^2 = \gamma^2 - \frac{3}{16}, \] (2.37)
Therefore, the strength $\gamma$ of the conformal potential is related to the constant $r_0$ labeling the $SL(2, \mathbb{R})$ representation as
\[ r_0 = \frac{1}{2} (1 + \sqrt{4\gamma^2 + \frac{1}{4}}). \] (2.38)
The choice of positive $r_0$ guarantees a good behavior of the ground state wavefunction at the origin [5]. Note that, using the quantum Casimir operator (2.37), we can cast the quantum Hamiltonian (2.35) in the form
\[ H = \frac{1}{4} \hat{p}^2 + \frac{T^2 + \frac{3}{16}}{\hat{x}^2}. \] (2.39)
Comparing it with the classical expression (2.19), we observe the appearance of a constant shift in the numerator of the potential term.

### 2.3. Conformal mechanics from $d = 1$ gauging

It is interesting that the AFF conformal mechanics action and its version (2.34) with the oscillator-type term can be reproduced by applying some $d = 1$ gauging procedure to the very simple complex free-particle action [66]. This example is a prototype of more complicated (super)conformal models which can also be constructed by applying $d = 1$ gauge procedure and its supersymmetric generalizations (see sections 2.5, 3.3.2, 4.3.2 and 4.3.3).

Consider a complex $d = 1$ field $z(t), \bar{z}(t)$ with the following Lagrangian:
\[ L_z = \dot{z} \dot{\bar{z}} + i m (\dot{z} \bar{z} - \bar{z} \dot{z}). \] (2.40)
The first term is the kinetic energy, and the second one is the simplest $d = 1$ WZ term. One of the symmetries of this system is the invariance under $U(1)$ transformations:
\[ z' = e^{-i\lambda} z, \quad \bar{z}' = e^{i\lambda} \bar{z}. \] (2.41)
Now we gauge this symmetry by promoting $\lambda \rightarrow \lambda(t)$. The gauge invariant action involves the $d = 1$ gauge field $A(t)$
\[ L_{\text{gauge}} = (\dot{z} + iA) (\dot{\bar{z}} - i\bar{A}) + im (\ddot{z} \bar{z} - \ddot{\bar{z}} z + 2iA\dot{z} \bar{z}) + 2\gamma A, \quad A' = A + \dot{\lambda}, \] (2.42)
where a ‘Fayet–Iliopoulos term’ $\propto \gamma$ has also been added. This term is gauge invariant (up to a total derivative) by itself.

The next step is to choose the appropriate gauge in $L_{\text{gauge}}$:
\[ z = \tilde{z} \equiv q(t). \] (2.43)
We substitute it into $L_{\text{gauge}}$ and obtain
\[ L_{\text{gauge}} = (\dot{q} + iAq) (\dot{\bar{q}} - i\bar{A}q) + 2imAq^2 + 2\gamma A = (\dot{q})^2 + A^2 q^2 - 2mAq^2 + 2\gamma A. \] (2.44)
The field $A(t)$ is the typical example of the auxiliary field: it can be eliminated by its algebraic equation of motion:
\[ \delta A: \quad A = m - \gamma q^{-2}. \] (2.45)
The final form of the gauge-fixed Lagrangian is as follows:

\[ L_{\text{gauge}} \Rightarrow (\dot{q})^2 - (mq - \gamma q^{-1})^2. \]  

(2.46)

Up to an additive constant \( \sim m\gamma \), this Lagrangian coincides with (2.34). At \( m = 0 \), one recovers the standard conformal mechanics

\[ L_{\text{gauge}}^{(\text{mod})} = (\dot{q})^2 - \gamma^2 q^{-2}. \]  

(2.47)

The initial action \( S_0 = \int dt L_0 \) at \( m = 0 \) is manifestly invariant under the conformal transformations \( \delta t = f(t), \delta z = \frac{1}{2} f z, (\delta t)^3 f = 0 \). The conformal invariance is preserved by the gauging procedure, provided that the gauge field \( A(t) \) transform as \( \delta \tilde{A}, \) i.e. \( \delta A(t) = -\tilde{f} A(t) \). As a result, the gauge-fixed action \( \int dt L_{\text{gauge}}^{(\text{mod})} \) also respects the conformal invariance.

This \( d = 1 \) gauging procedure can be interpreted as an off-shell Lagrangian analog of the well-known Hamiltonian reduction. In the present case, in the parametrization \( z = qe^{\varphi} \), the Hamiltonian reduction consists in imposing the constraints \( p_\varphi - 2\gamma \approx 0, \varphi \approx 0 \), upon which the Hamiltonian of the system (2.40) is reduced to the AFF Hamiltonian.

2.4. Conformal mechanics with additional isospin degrees of freedom

We can consider an extension of the AFF conformal mechanics model by additional isospin degrees of freedom. These additional degrees of freedom are described by Chern–Simons (or, in other terminology, Wess–Zumino) mechanical action [42, 65, 100, 99]. Using them, together with gauging some isometries of some simple initial actions, one can recover the Lagrangian models constructed earlier by other methods and to construct new dynamical systems [29, 30]. Such an approach reveals an interesting deviation from the standard conformal quantum mechanics: besides the standard dilatonic variable \( x(t) \) with the conformal potential, it also contains a fuzzy sphere [87, 88] described by additional isospin variables. As a result, the relevant wavefunctions form non-trivial \( \text{SU}(2) \) multiplets, as opposed to the \( \text{SU}(2) \) singlet wavefunction of the standard conformal mechanics. The strength of the conformal potential proves to coincide with the eigenvalue of the \( \text{SU}(2) \) Casimir operator (i.e. ‘spin’) and so it is quantized.

Let us consider the following action [37, 38]:

\[ \tilde{S}_0 = \int dt \left[ \dot{x} + \frac{1}{2}(\bar{z}_a \dot{z}^a - \bar{z}_a z^a) - \frac{\alpha^2 (\bar{z}_a \dot{z}^a)^2}{16\lambda^2} - A (\bar{z}_a \dot{z}^a - c) \right], \]  

(2.48)

where \( \alpha \) is some dimensionless parameter (as we will see below, it coincides with the parameter characterizing the most general \( N = 4 \) superconformal group \( D(2, 1; \alpha) \)). This action is invariant under the local \( \text{U}(1) \) transformations

\[ A' = A - \dot{\lambda} \zeta, \quad z^a' = e^{i\lambda \zeta} z^a, \quad \bar{z}_a' = e^{-i\lambda \zeta} \bar{z}_a. \]  

(2.49)

The \( d = 1 \) gauge connection \( A(t) \) in (2.48) is the Lagrange multiplier for the constraint

\[ \bar{z}_a \dot{z}^a = c. \]  

(2.50)

After varying with respect to \( A \), the action (2.48) is gauge invariant only by taking into account this algebraic constraint which is gauge invariant by itself. It is convenient to fully fix the residual gauge freedom by choosing the phases of \( z^1 \) and \( z^2 \) opposite to each other. In this gauge, the constraint (2.50) is solved by

\[ z^1 = \kappa \cos \frac{\gamma}{2} e^{i\beta/2}, \quad z^2 = \kappa \sin \frac{\gamma}{2} e^{-i\beta/2}, \quad \kappa^2 = c. \]  

(2.51)

6 The \( d = 1 \) field \( \gamma(t) \) should not be confused with the constant \( \gamma \) of the previous subsections.
In terms of the newly introduced fields $x(t)$, $\gamma(t)$ and $\beta(t)$, the action (2.48) takes the form

$$S_b = \int dt \left[ \dot{x}^2 - \frac{\alpha^2 c^2}{16x^2} - \frac{c}{2} \cos \gamma \dot{\beta} \right].$$

The action (2.52) contains the ‘true’ kinetic term only for one bosonic component $x$ which also possesses the conformal potential, whereas two other fields $\beta$ and $\gamma$ parametrizing the coset $SU(2)/U(1)$ are described by a WZ term and so become a sort of isospin degrees of freedom (target $SU(2)$ harmonics) upon quantization. The conformal invariance of the WZ term is evident in the notation (2.52), keeping in mind that the SU(2) fields $\beta(t)$ and $\gamma(t)$ have zero conformal weights. Note that the fact of conformal invariance of $d = 1$ WZ terms was pointed out for the first time by Jackiw [76].

It should be pointed out that the considered model realizes a new mechanism of generating conformal potential $\sim 1/x^2$ for the field $x(t)$. Before eliminating auxiliary fields, the component action contains no explicit term of this kind. It arises as a result of varying with respect to the Lagrange multiplier $A(t)$ and making use of the arising constraint (2.50). As we will see below, in quantum theory this new mechanism entails a quantization of the constant $c$.

The corresponding canonical Hamiltonian of the model (2.48) reads

$$H_0 = \frac{1}{4} \left[ p^2 + \alpha^2 (\bar{z}_k z^k)^2 \right] + A(\bar{z}_k z^k - c).$$

Here $p = 2\dot{x}$ is the canonical momentum for the coordinate $x$. The canonical momentum for the field $A$ is vanishing, $p_A = 0$. This constraint and the fact that the field $A$ appears in the action (2.48) linearly suggest that we treat $A$ as the Lagrange multiplier for the constraint

$$D^0 - c := \bar{z}_k z^k - c \approx 0.$$  

The expressions for the canonical momenta $p_i$ and $\bar{p}^i$ of the $z$-variables, $[\bar{z}^i, p_j]_\rho = \delta^i_j$, $[\bar{z}_i, p^j}_\rho = \delta^i_j$, follow from the second-class constraints

$$G_k := p_k - \frac{i}{2} \bar{z}_k \approx 0, \quad \bar{G}^k := \bar{p}^k + \frac{i}{2} z^k \approx 0, \quad [G_k, \bar{G}^l]_\rho = -i \delta^l_k.$$  

Using their Dirac brackets

$$[A, B]_\rho = [A, B]_\rho + i[A, G_k] [\bar{G}^k, B]_\rho - i[A, \bar{G}^k] [G_k, B]_\rho,$$

we eliminate the spinor momenta $p_i$ and $\bar{p}^i$. Dirac brackets for the residual variables are

$$[x_i, p_j]_\rho = 1, \quad [\bar{z}^i, \bar{z}_j]_\rho = -i \delta^i_j.$$  

To finish with the classical description, we point out that the spinor variables describe a two-sphere. Namely, using the first Hopf map, we introduce three $U(1)$ gauge invariant variables

$$y_a = \frac{1}{2} \bar{z}_a (\sigma_a) z^j,$$

where $\sigma_a$, $a = 1, 2, 3$, are Pauli matrices. The constraint (2.54) suggests that these variables parameterize a two-sphere with the radius $c/2$:

$$y_a y_a = (c^2 \bar{z}_k z^k)/4 \approx c^2/4.$$  

The group of motion of this 2-sphere is of course the SU(2) group acting on the doublet indices $i, k$ and triplet indices $a$. In terms of the new variables (2.58), the Hamiltonian (2.53), up to terms vanishing on the constraints, takes the form

$$H = \frac{1}{4} \left[ p^2 + \frac{\alpha^2 y_a y_a}{x^2} \right] \approx \frac{1}{4} \left[ p^2 + \frac{\alpha^2 c^2}{4x^2} \right].$$
At the quantum level, the algebra of the canonical operators obtained from the algebra of Dirac brackets is

$$\{\hat{x}, \hat{p}\} = i, \quad \{\hat{\mathcal{Z}}, \hat{\mathcal{Z}}\} = \delta_{ij}. \quad (2.61)$$

Then, it is easy to check that the quantum counterparts of the variables (2.58)

$$\hat{\mathcal{Y}}_{\alpha} = \frac{1}{2} \hat{\mathcal{Z}}_{\alpha} (\sigma_{ab})^j_\alpha \hat{\mathcal{Z}}^j \quad (2.62)$$

form the $su(2)$ algebra

$$[\hat{\mathcal{Y}}_{\alpha}, \hat{\mathcal{Y}}_{\beta}] = i\epsilon_{abc} \hat{\mathcal{Y}}_{c}. \quad (2.63)$$

Note that no ordering ambiguity is present in the definition (2.62).

Moreover, the direct calculation yields

$$\hat{\mathcal{Y}}_{\alpha} \hat{\mathcal{Y}}_{\alpha} = \frac{1}{2} \hat{\mathcal{Z}}_{\alpha} \hat{\mathcal{Z}}_{\bar{\alpha}} \left( \frac{1}{2} \hat{\mathcal{Z}}_{\alpha} \hat{\mathcal{Z}}_{\bar{\alpha}} + 1 \right) \quad (2.64)$$

and, due to the constraints (2.54) (for definiteness, we adopt $\hat{\mathcal{Z}}_{\alpha} \hat{\mathcal{Z}}_{\bar{\alpha}}$-ordering in it), one obtains

$$\hat{\mathcal{Y}}_{\alpha} \hat{\mathcal{Y}}_{\alpha} = \frac{c}{2} \left( \frac{c}{2} + 1 \right). \quad (2.65)$$

However, relations (2.63) and (2.65) are the definition of the fuzzy sphere coordinates. Thus, the angular variables, described, at the classical level, by spinor variables $z^i$ or vector variables $y_{\alpha}$, after quantization acquire a nice interpretation of the fuzzy sphere coordinates. Comparing the expressions (2.59) and (2.65), we observe that upon quantization the radius of the sphere changes as $\frac{c}{2} \rightarrow \frac{c}{2} (\frac{c}{2} + 1)$.

As suggested by relation (2.63), the fuzzy sphere coordinates $\mathcal{Y}_{\alpha}$ are the generators of $su(2)$ algebra and the relation (2.65) fixes the value of its Casimir operator, with $c$ being the relevant SU(2) spin (‘fuzzyness’). Then, it follows that $c$ is quantized, $c \in \mathbb{Z}$.

The wavefunctions inherit this internal symmetry through a dependence on additional SU(2) spinor degrees of freedom. Let us consider the following realization for the operators $\hat{\mathcal{Z}}_i$ and $\hat{\mathcal{Z}}_i$.

$$\hat{\mathcal{Z}}_i = v^+_i, \quad \hat{\mathcal{Z}}^i = \hat{\partial} / \hat{\partial} v^+_i, \quad (2.66)$$

where $v^+_i$ is a commuting complex SU(2) spinor. Then, the constraint (2.54) imposed on the wavefunction $\Phi(x, v^+_i)$,

$$D^0 \Phi = \hat{\mathcal{Z}}_i \hat{\mathcal{Z}}^i \Phi = v^+_i \frac{\partial}{\partial v^+_i} \Phi = c \Phi, \quad (2.67)$$

leads to the polynomial dependence of it on $v^+_i$:

$$\Phi(x, v^+_i) = \phi_{k_1 \ldots k_c}(x) v^{+_{i_1}} \ldots v^{+_{i_c}}. \quad (2.68)$$

Thus, as opposed to the model of [5], in our case the $x$-dependent wavefunction carries an irreducible spin $c/2$ representation of the group SU(2), being an SU(2) spinor of the rank $c$.

Using (2.60) and (2.65), we see that on physical states the quantum Hamiltonian is

$$H = \frac{1}{4} \left( \rho^2 + \frac{g^2}{\rho^2} \right) \quad \text{with} \quad g = \alpha^2 \frac{c}{2} \left( \frac{c}{2} + 1 \right). \quad (2.69)$$

It is easy to show that the SU(1,1) Casimir operator takes the value

$$T^2 = \frac{1}{4} g - \frac{3}{16}, \quad (2.70)$$

and the Hamiltonian (2.69) can be rewritten in the form (2.39). Thus, like in [5], on the fields $\phi_{k_1 \ldots k_c}(x)$ the unitary irreducible representations of the group SU(1,1) are realized, despite the fact that the wavefunction is now multi-component, with $(c + 1)$ independent components. Requiring the wavefunction $\Phi(v^+_i)$ to be single-valued once again leads to the condition
that $c \in \mathbb{Z}$. This quantization of parameter $c$ could be important for the possible black hole interpretation of the considered variant of conformal mechanics.

Note that the action (2.52) can be extended by adding the conformally and SU(2) invariant sigma-model kinetic term for the SU(2)/U(1) variables $\gamma$ and $\beta$:

$$S_b \Rightarrow S'_b = S_b + g' \int dt \, x^2 [ (\dot{\gamma})^2 + (\dot{\beta})^2 \sin^2 \gamma ],$$

(2.71)

where $g'$ is a renormalization constant. In such an extended model, the SU(2) fields lose their status of ‘semi-dynamical’ isospin variables; they become the full-fledged physical degrees of freedom and make a contribution to the corresponding Hamiltonian. The coefficient $c$ before the WZ term is still quantized on the topological grounds. This sort of the conformally invariant mechanics model with three physical degrees of freedom (the radial variable $x$ and the angular variables $\beta$ and $\gamma$) and the WZ term related to the strength of the conformal potential as in (2.52) appears as a bosonic part of the $\mathcal{N} = 4$ superconformal mechanics based on the $\mathcal{N} = 4, d = 1$ off-shell supermultiplet [67].

2.5. Conformally invariant multi-particle systems

Above we presented a model that has only one dynamic degree of freedom. There are many models which possess conformal invariance and describe many dynamical degrees of freedom that can be interpreted as degrees of freedom of different particles. An example of such systems is provided by the well-known many-body Calogero model [25, 26, 98], which describes $n$ identical particles interacting pairwise through an inverse-square potential. Here we present a formulation of the $n$-particle Calogero model as a matrix model with gauge $\text{U}(n)$ symmetry [100, 58, 59, 101, 94, 95, 36].

In this formulation the $n$-particle Calogero model is described by the Hermitian $(n \times n)$-matrix field $X^b_a(t)$, $(\bar{X}^b_a) = X^a_b$, and complex $\text{U}(n)$-spinor field $Z_a(t)$, $(\bar{Z}^a) = (Z_a)$, $a, b = 1, \ldots, n$, and involves $n^2$ gauge fields $A^b_a(t)$, $(\bar{A}^b_a) = A_a^b$. The action has the following form:

$$S_C = \int dt \left[ \text{tr} \left( \nabla X \nabla X \right) + \frac{i}{2} (\bar{Z} \nabla Z - \nabla \bar{Z} Z) + c \text{tr} A \right],$$

(2.72)

where the covariant derivatives are

$$\nabla X = \dot{X} + i[A, X], \quad \nabla Z = \dot{Z} + iAZ, \quad \nabla \bar{Z} = \dot{\bar{Z}} - i\bar{A}Z. \quad (2.73)$$

The last term (Fayet–Iliopoulos term) includes only the $\text{U}(1)$ gauge field, and $c$ is a real constant.

The action (2.72) is invariant under the $d = 1$ conformal $\text{SO}(2, 1)$ transformations

$$\delta t = a(t), \quad \delta \partial_t = -\dot{a} \partial_t,$$

$$\delta X^b_a = \frac{i}{2} \dot{a} X^b_a, \quad \delta Z_a = 0, \quad \delta A^b_a = -\dot{a} A^b_a;$$

(2.74)

where $a(t)$ obeys the constraint

$$\partial_t^2 a = 0. \quad (2.75)$$

This is in agreement with the well-known fact that the Calogero model is conformal. Any gauge-invariant potential term of the field $X^b_a$ evidently breaks the conformal invariance. Non-conformal though still integrable deformations of the Calogero model correspond to adding some specific gauge invariant monomials of the matrix $X^b_a$ to the action (2.72). In particular, the combination

$$n\text{tr} X^2 - (\text{tr} X)^2$$

(2.76)
in the gauge (2.79) (see below) yields the additional term in the final action
\[ \sum_{a<b}(x_a - x_b)^2, \] (2.77)
which corresponds to passing to the Calogero–Moser model.

The action (2.72) is invariant with respect to the local U(n) transformations, \( g(\tau) \in U(n), X \rightarrow gXg^+, \ Z \rightarrow gZ, \ \tilde{Z} \rightarrow \tilde{Z}g^+, \ A \rightarrow gAg^+ + ig\dot{g}. \) (2.78)

Using the gauge transformations (2.78) we can impose a (partial) gauge fixing
\[ X_a \delta_{a} = 0, \ a \neq b. \] (2.79)

In this gauge, the matrix variable \( X \) takes the form
\[
X_a^b = x_a \delta_a^b = \begin{pmatrix}
x_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & x_2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & x_3 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & x_{n-1} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & x_n & 0 \\
\end{pmatrix}.
\] (2.80)

Then,
\[ [X, A]_{a}^b = (x_a - x_b)A_a^b, \] (2.81)
and, therefore,
\[
\text{tr}[X, A] = 0, \quad \text{tr}([X, A][X, A]) = - \sum_{a,b}(x_a - x_b)^2 A_a^b A_b^a. \] (2.82)

As a result of this, the action (2.72) takes the form
\[
S_C = \int dt \sum_{a,b} \left[ \dot{x}_a \dot{x}_a + \frac{i}{2} (\dot{Z}^a Z_a - \dot{\tilde{Z}}^a \tilde{Z}_a) + (x_a - x_b)^2 A_a^b A_b^a - \dot{Z}^a A_b^b Z_b + c A_a^a \right]. \] (2.83)

In the third term in the action (2.83), there remain only non-diagonal elements of the matrix \( A_a^b \) with \( a \neq b \). Therefore, the action (2.83) has the residual invariance under the gauge Abelian \([U(1)]^n\) symmetry with local parameters \( \psi_a(t) \):
\[ Z_a \rightarrow e^{i\psi_a} Z_a, \quad \tilde{Z}^a \rightarrow e^{-i\psi_a} \tilde{Z}^a, \quad A_a^b \rightarrow A_a^b - \dot{\psi}_a \] (no sum with respect to \( a \)). (2.84)

Making use of this invariance, we can impose the further gauge
\[ \tilde{Z}^a = Z_a. \] (2.85)

In this gauge, the second term in the action (2.83) vanishes and the action (2.83) takes the form
\[
S_C = \int dt \sum_{a,b} \left[ \dot{x}_a \dot{x}_a + (x_a - x_b)^2 A_a^b A_b^a - Z_a Z_b A_a^b + c A_a^a \right]. \] (2.86)

Let us verify that under passage from the action (2.83) to the action (2.86), the equations of motion are preserved and we do not obtain additional equations/constraints.

The equation of motion for \( Z \), following from the action (2.83),
\[ \nabla Z_a = \dot{Z}_a - i \sum_b A_a^b Z_b = 0, \quad \nabla \tilde{Z}^a = \dot{Z}^a - i \sum_b \dot{Z}^b A_a^b = 0, \] (2.87)
\[ A_{a}^{b} = \frac{Z_{a}Z_{b}}{2(x_{a} - x_{b})^{2}} \quad \text{for } a \neq b, \quad (2.88) \]

\[ \tilde{Z}^aZ_a = c \quad \forall a \quad (\text{no sum with respect to } a) \quad (2.89) \]

in the gauge (2.85) become the equations

(a) \[ 2\tilde{Z}_a - i \sum_b (A_{a}^{b} - A_{b}^{a})Z_b = 0, \quad (b) \sum_b (A_{a}^{b} + A_{b}^{a})Z_b = 0, \quad (2.90) \]

\[ A_{a}^{b} = \frac{Z_{a}Z_{b}}{2(x_{a} - x_{b})^{2}} \quad \text{for } a \neq b, \quad (2.91) \]

\[ (Z_a)^2 = c \quad \forall a \quad (\text{no sum with respect to } a). \quad (2.92) \]

Equations (2.90(b)), (2.91), (2.92) are exactly the equations of motion, following from the action (2.86). And it is important that equations (2.90(a)) are the corollary of equations (2.91) and (2.92) (equations (2.92) imply \( \dot{Z}_a = 0 \)).

Equations (2.92) imply \( c > 0 \). Inserting (2.90(b)) (which give in fact the expressions for diagonal \( A_{a}^{a} \)), (2.91) and (2.92) in the action (2.83), we finally obtain the standard Calogero action

\[ SC = \frac{1}{2} \int dt \left[ \sum_a \dot{x}_a \dot{x}_a - \sum_{a \neq b} \frac{c^2}{(x_a - x_b)^2} \right]. \quad (2.93) \]

Let us consider the Hamiltonian formulation of this matrix model. Expressions of the momenta, obtained from the action (2.72), are

\[ P_X = 2\nabla X, \quad P_Z = i\tilde{Z}, \quad \tilde{P}_Z = -iZ, \quad P_A = 0. \quad (2.94) \]

Thus, there are the constraints

\[ G \equiv P_Z - i\tilde{Z} \approx 0, \quad \tilde{G} \equiv \tilde{P}_Z + iZ \approx 0, \quad (2.95) \]

\[ P_X \approx 0. \quad (2.96) \]

The Hamiltonian of the system has the following form:

\[ H = \frac{1}{4} \text{tr} (P_X P_X) + \text{tr} (AT), \quad (2.97) \]

where

\[ T \equiv i[X, P_X] - 2Z \cdot \tilde{Z} - cI_n \quad (2.98) \]

and \( I_n \) is the \((n \times n)\) unity matrix.

The preservation of the constraints (2.96) leads to the secondary constraints

\[ T \approx 0. \quad (2.99) \]

The fields \( A \) are Lagrange multipliers for these constraints.

Using canonical Poisson brackets

\[ [X_a^{b}, P_c^{d}]_r = \delta_a^d \delta_c^b, \quad [Z_a, P_b^{c}]_r = \delta_c^b, \quad [\tilde{Z}_a, \tilde{P}_b]_r = \delta_a^b, \quad (2.100) \]

we compute Poisson brackets of the constraints (2.95)

\[ [G^i, \tilde{G}_a]_r = -2i\delta_a^i. \quad (2.101) \]

Next, we introduce Dirac brackets

\[ [A, B]_r = [A, B]_r + \frac{i}{2} [A, G^i]_r \tilde{G}_a, B]_r - \frac{i}{2} [A, \tilde{G}_a]_r [G^i, B]_r \quad (2.102) \]

and treat the constraints (2.95) in the strong sense, eliminating \( P_Z, \tilde{P}_Z \).
The remaining variables have the following Dirac brackets:
\[ [X^a_b, P_x^c_d] = \delta^d_a \delta^b_c, \quad [Z^a, \bar{Z}^b] = -\frac{i}{2} \delta^b_a. \quad (2.103) \]

The remaining constraints (2.98) form the \((n \times n)\) Hermitian matrix,
\[ T = T^+, \quad (2.104) \]
which generates \(u(n)\) algebra with respect to the Dirac brackets
\[ [T^h_b, T^d_c] = i(\delta^d_c T^h_b - \delta^h_c T^d_b). \quad (2.105) \]

They generate \(U(n)\) gauge transformations of \(X^a_b, P_x^a, Z^a, \bar{Z}^a\). Let us fix the gauges with respect to these transformations.

In the notations
\[ x^a_a \equiv X^a_a, \quad p^a_a \equiv P_x^a_a \quad \text{(no summation over } a), \quad (2.106) \]
\[ x^a_b \equiv X^a_b, \quad p^a_b \equiv P_x^a_b \quad \text{for } a \neq b, \quad (2.107) \]
the constraints (2.98) take the form
\[ T^a_b = i(x^a - x^b)p^a_b - i(p^a - p^b)x^a_b + i \sum_c (x^c x^a_b - p^c p^a_b) - 2Z^a \bar{Z}^a \approx 0 \quad \text{for } a \neq b, \quad (2.108) \]

\[ T^a_a = i \sum_c (x^c x^a - p^c p^a) - 2Z^a \bar{Z}^a - c \approx 0 \quad \text{(no summation over } a). \quad (2.109) \]

The constraints (2.108) generate the transformations
\[ \delta x^a_b = [x^a_b, \epsilon_b^a T^a_b] \sim i(x^a - x^b)\epsilon_b^a. \quad (2.110) \]

When \(x^a \neq x^b\), we can impose the gauge
\[ x^a_a \approx 0. \quad (2.111) \]

For the constraints (2.108), (2.111), we introduce Dirac brackets. As a result, we eliminate \(x^a_b, p^a_b\) by these constraints:
\[ x^a_b = 0, \quad p^a_b = -\frac{2i}{(x^a - x^b)} Z^a \bar{Z}^b. \quad (2.112) \]

Thus,
\[ P_{x^a} = \begin{cases} p^a_b \delta^a_b & \text{for } a = b; \\ -\frac{2i}{(x^a - x^b)} Z^a \bar{Z}^b & \text{for } a \neq b \end{cases}. \quad (2.113) \]

After this gauge fixing, the constraint (2.109) becomes
\[ -2Z^a \bar{Z}^a - c \approx 0 \quad \text{(no summation over } a) \quad (2.114) \]
and it generates local phase transformations of \(Z^a\). We can impose the gauge
\[ Z^a = \bar{Z}^a \approx 0. \quad (2.115) \]

As a result, we completely eliminated \(Z^a\).

Finally, using expressions (2.113) and conditions (2.115), we obtain the following expression for the Hamiltonian (2.97):
\[ H = \frac{1}{4} \text{tr}(P_{x^a} P_{x^b}) = \frac{1}{4} \left( \sum_a (p^a)^2 + \sum_{a \neq b} \frac{c^2}{(x^a - x^b)^2} \right), \quad (2.116) \]
i.e. the standard Calogero Hamiltonian [25, 26].
The Calogero model is an example of the solvable multi-particle system. It can be obtained by applying the reduction method to a matrix system corresponding to the $A_n$ root system of Lie algebras [98].

The quantization of the Calogero model was analyzed in the initial papers by Calogero [25, 26] where the ground-state wavefunction and energy were found. The wavefunctions of some higher excitation states were found in [97]. The progress in obtaining physical wavefunctions for the Calogero model can be traced back to [22] where it was suggested to use Dunkl operators in the quantization procedure.

3. $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superconformal mechanics

3.1. $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superconformal algebras and their representations

$\mathcal{N} = 1$ case. The $\mathcal{N} = 1$ superconformal algebra is constituted by the following set of generators ($\alpha, \beta = 1, 2$):

$$G^{(1)} = (Q_\alpha, T_{\alpha\beta}), \quad (Q_\alpha)\dagger = Q_\alpha, \quad (T_{\alpha\beta})\dagger = T_{\alpha\beta} = T_{\beta\alpha}. \quad (3.1)$$

They form the real $osp(1|2)$ superalgebra which thus defines graded symmetries of $\mathcal{N} = 1$ superconformal mechanics [6, 24]. The nonvanishing (anti)commutators are (2.5) and

$$\{Q_\alpha, Q_\beta\} = 2T_{\alpha\beta}, \quad [T_{\alpha\beta}, Q_\gamma] = -i\epsilon_{\gamma(\alpha Q_\beta)}.$$  

(3.2)

The fermionic $O(2, 1)$ spinor supercharges $Q_\alpha$ encompass the standard supercharges $Q = Q_1$ and the generators of superconformal boosts $S = Q_2$. The second-order Casimir operator of the supergroup $OSp(1|2)$ is given by the following expression:

$$C^{(N=1)}_2 = T^2 + \frac{i}{4} Q_\alpha Q^\alpha.$$  

(3.3)

$\mathcal{N} = 2$ case. The $\mathcal{N} = 2$ superconformal algebra involves the following set of the generators:

$$G^{(2)} = (Q_\alpha, \tilde{Q}_\alpha; T_{\alpha\beta}, J, C), \quad (3.4)$$

which define the $su(1, 1|1) \cong osp(2|2)$ superalgebra with one central charge (see, e.g., [4, 45, 70, 6])

$$\{Q_\alpha, \tilde{Q}_\beta\} = 2T_{\alpha\beta} + i\epsilon_{\alpha\beta}J + i\epsilon_{\alpha\beta}C, \quad (3.5)$$

$$[T_{\alpha\beta}, Q_\gamma] = -i\epsilon_{\gamma(\alpha Q_\beta)}\tilde{Q}, \quad [T_{\alpha\beta}, \tilde{Q}_\gamma] = -i\epsilon_{\gamma(\alpha \tilde{Q}_\beta)}, \quad (3.6)$$

$$[J, Q_\alpha] = Q_\alpha, \quad [J, \tilde{Q}_\alpha] = -\tilde{Q}_\alpha, \quad (3.7)$$

where the $SO(1, 2)$ relations (2.5) should also be added. All other (anti)commutators vanish. The Hermiticity properties of the $SU(1, 1|1)$ generators are as follows:

$$(Q_\alpha)\dagger = Q_\alpha, \quad (T_{\alpha\beta})\dagger = T_{\alpha\beta} = T_{\beta\alpha}, \quad (J)\dagger = J, \quad (C)\dagger = C. \quad (3.8)$$

The generators $T_{\alpha\beta}$ form $so(1, 2)$ algebra, whereas $J$ is the $O(2)$ generator. The generator $C$ is the central charge one. Here we used the realization of the $osp(2|2)$ algebra as in [70], with fermionic supercharges being complex $O(2)$ spinors. In particular, the standard supercharges $Q = Q_1, \tilde{Q} = \tilde{Q}_1$ and the generators of conformal supertranslations $S = Q_2, \tilde{S} = \tilde{Q}_2$ are combined into a single complex $O(2, 1)$ doublet supercharge $Q_\alpha$. 

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The second-order Casimir operator of the supergroup OSp (2|2) is given by the following expression:

$$C_2^{(N=2)} = T^2 + \frac{i}{4} [Q_x, \tilde{Q}^x] - \frac{1}{4} J^2 - \frac{1}{4} [J, C].$$

(3.9)

Unitary irreducible representations act on the eigenstates of the second order Casimir $C_2$ with a fixed value. Such representations are decomposed into an infinite tower of the representations of the compact supergroup including the compact generator $T_0$ (2.2) of the conformal group, similarly to the bosonic case. In the $N=2$ case, this compact sub-superalgebra of the $osp(2|2)$ superalgebra is spanned by the generators

$$T_0, \quad J, \quad C, \quad \Gamma \equiv m^{1/2} S + im^{-1/2} Q, \quad \tilde{\Gamma} \equiv m^{1/2} \tilde{S} - im^{-1/2} \tilde{Q},$$

(3.10)

with the following nonvanishing (anti)commutators:

$$[\Gamma, \tilde{\Gamma}] = 4 T_0 - 2 C - 2 J, \quad [T_0, \Gamma] = \frac{1}{2} \Gamma, \quad [T_0, \tilde{\Gamma}] = -\frac{1}{2} \tilde{\Gamma}, \quad [J, \Gamma] = \Gamma, \quad [J, \tilde{\Gamma}] = -\tilde{\Gamma}. \quad (3.11)$$

3.2. Single-particle mechanics: nonlinear realizations and actions

We will consider $N=1$ superspace parameterized by $(t, \theta)$ where $t$ is an even coordinate and $\theta$ is a real Grassmann coordinate, $\theta^2 = 0$, $\langle \bar{\theta} \theta \rangle = 0$. The covariant spinor derivative is

$$D = \partial_t + i\theta \partial_\theta, \quad \{D, D\} = 2i \partial_t. \quad (3.12)$$

Coordinates of the $N=2$ superspace are $(t, \theta, \bar{\theta})$ where $t$ is an even coordinate again and $\theta, \bar{\theta}$ are two Grassmanian coordinates, $\theta^2 = \bar{\theta}^2 = 0$, $\langle \bar{\theta} \theta \rangle = 0$. $N=2$ covariant spinor derivatives are

$$D = \partial_t - i\bar{\theta} \partial_\theta, \quad \bar{D} = -\partial_t + i\theta \partial_\bar{\theta}, \quad \{D, \bar{D}\} = 2i \partial_t. \quad (3.13)$$

As given in [28, 6], the $N=1$ superconformal one-particle model is described by the free particle action. It follows from the superfield action

$$S_{n=1}^{(N=1)} = -i \int dt \, d\theta \, \Phi \bar{D} \Phi,$$

(3.15)

where $\Phi(t, \theta)$ is a real superfield with the following off-shell $(1, 1, 0)$ component contents

$$\Phi(t, \theta) = x(t) + i \bar{\theta} \psi(t).$$

(3.16)

The action (3.15) yields free actions for boson $x$ and fermion $\psi$, i.e. provides a supersymmetrization of the AFF mechanics with the vanishing conformal potential.

The $N=2$ superconformal mechanics [4, 45] can be described by a real $N=2$ superfield with the off-shell contents $(1, 2, 1)$

$$\Phi(t, \theta, \bar{\theta}) = x(t) + i \bar{\theta} \psi(t) - \bar{\theta} \bar{\psi}(t) + \theta \bar{\psi}(t),$$

(3.17)

and with the action

$$S_{n=1}^{(N=2)} = \int dt \, d^2\theta \, \bar{D} \Phi \, \bar{D} \Phi - c \ln \Phi.$$

(3.18)

The superpotential

$$W = c \ln \Phi$$

(3.19)

produces the conformal potential for the bosonic component field. In components, the action (3.18) takes the form

$$S_{n=1}^{(N=2)} = \int dt \left[ \bar{x}^2 - i (\bar{\psi} \bar{\psi} - \bar{\psi} \bar{\psi}) + \bar{y}^2 - \frac{c y^2}{x} + \frac{c \bar{\psi} \bar{\psi}}{x^2} \right].$$

(3.20)
After eliminating the auxiliary field \( y \) we obtain
\[
S_{(N=2)}^{(N=2)} = \int dt \left[ \dot{x}^2 - i(\dot{\psi}\psi - \dot{\bar{\psi}}\bar{\psi}) - \frac{c(c - 4\dot{\psi}\bar{\psi})}{4x^2} \right],
\]
(3.21)
which is just the \( N = 2 \) supersymmetric generalization of the AFF mechanics considered firstly in [45] and in [4].

Similarly to the bosonic case, the formulation (3.18) has a nice geometric interpretation within the nonlinear realizations method [70]. One starts from the exponential parametrization of the coset \( SU(1, 1)/U(1) \)
\[
G = e^{iH} e^{i(\xi - \xi\bar{\psi})} e^{iK} e^{i(\xi - \xi\bar{\psi})} e^{i\eta D}.
\]
(3.22)

The coset parameters associated with the generators \( K, S, \bar{S}, D \) are treated as Goldstone superfields, \( \tau = \tau(t, \theta, \bar{\theta}) \), \( \xi = \xi(t, \theta, \bar{\theta}) \), \( \bar{\xi} = \bar{\xi}(t, \theta, \bar{\theta}) \), \( \bar{\tau} = \bar{\tau}(t, \theta, \bar{\theta}) \). The generator \( J \) corresponds to the vacuum stability subgroup \( U(1) \). Imposing the inverse Higgs constraints on the left-covariant MC forms, one can eliminate a part of Goldstone superfields and obtain the equations of motion for the residual fields. The correct set of such constraints was derived in [70] (see also [2]). It corresponds to vanishing of all MC forms except for those associated with the generators of the sub-superalgebra (3.10). As a result, the appropriate geodesic submanifold is singled out. The coset parameters \( \tau, \eta, \bar{\eta} \), corresponding to the generators \( T_0, \Gamma, \bar{\Gamma} \), parametrize this geodesic supermanifold. As a result of imposing the constraints on the MC forms, including the kinematical inverse Higgs ones, the only independent superfield is the dilaton superfield \( u(t, \theta, \bar{\theta}) \). The superfield (3.17) is expressed in terms of the dilaton as \( \Phi = \exp(u/2) \).

The quantum generators of the Poincaré superalgebra are calculated to be
\[
H = \frac{1}{4} \left( \hat{\rho}^2 + \left( \frac{dW}{dx} \right)^2 \right) + \frac{1}{2} \frac{d^2 W}{dx^2} (\hat{\psi}\bar{\psi} - \bar{\psi}\hat{\psi}),
\]
(3.23)
whereas the superconformal boost generators are
\[
S = -2\hat{\psi}\hat{x}, \quad \bar{S} = -2\bar{\psi}\hat{x}.
\]
(3.25)
Here, \( W(\hat{x}) = c \ln \hat{x} \). Using the algebra of the basic quantum operators
\[
[\hat{x}, \hat{\rho}] = i, \quad \{\hat{\psi}, \bar{\psi}\} = \frac{i}{2},
\]
(3.26)
we find the non-vanishing anticommutators of the odd generators (3.24) and (3.25):
\[
\{Q, \bar{Q}\} = 2H, \quad \{S, \bar{S}\} = 2K.
\]
(3.27)
\[
\{Q, S\} = 2D + iJ + iC, \quad \{\bar{Q}, S\} = 2D - iJ - iC.
\]
(3.28)
Here, the generators
\[
K = \hat{x}^2, \quad D = -\frac{1}{2} (\hat{\rho}\hat{x} + \hat{x}\hat{\rho})
\]
(3.29)

\[
J = \hat{\psi}\bar{\psi} - \bar{\psi}\hat{\psi}
\]
(3.30)
is the \( U(1) \) generator, and
\[
C = \hat{x} \frac{dW}{dx} = c
\]
(3.31)
is the central charge. The remaining commutation relations in which the generators (3.24), (3.25) and (3.29)–(3.31) appear coincide with those present in the $su(1,1|1)$ superalgebra (3.4)–(3.7). A convenient realization of the quantum fermionic operators is through Pauli matrices,

$$\hat{\psi} = \frac{1}{\sqrt{2}} (\sigma_1 + i\sigma_2), \quad \hat{\bar{\psi}} = \frac{1}{\sqrt{2}} (\sigma_1 - i\sigma_2), \quad \hat{\psi} \hat{\bar{\psi}} - \hat{\bar{\psi}} \hat{\psi} = \frac{1}{2} \sigma_3. \quad (3.32)$$

We see that, as opposed to the non-supersymmetric case, the energy spectrum in the one-particle $N = 2$ superconformal model is doubly degenerate and the quantum Hamiltonian (3.23) takes the following form:

$$H = \frac{1}{4} \left[ \hat{p}^2 + \frac{c(e - \sigma_3)}{\hat{x}^2} \right]. \quad (3.33)$$

Since the second term in the brackets can be rewritten as $l(l + 1)/\hat{x}^2$, where $l = \pm c$ for the $\sigma_3$ entries $\mp 1$, the strength $|c|$ was identified in [6] with the orbital angular momentum of a particle near the horizon (i.e. in the large mass limit) of the extreme Reissner–Nordström black hole.

The quantum spectrum (with spontaneously broken superconformal symmetry) can be found by the same token as in the bosonic case, using the group-theoretical information sketched in the previous subsection. As was noticed in [45], the ‘naïve’ application of the ‘standard’ quantization scheme to this system does not lead to the desired result. Indeed, defining the wavefunction $\Psi_0$ of the ground state by the conditions

$$Q \Psi_0 = 0, \quad \bar{Q} \Psi_0 = 0, \quad (3.34)$$

which amount to the single matrix equation

$$\left( \partial_x - \frac{c\sigma_3}{\hat{x}} \right) \Psi_0 = 0, \quad (3.35)$$

we immediately find that the two-component wavefunction $\Psi_0$ has the following entries:

$$\Psi_0 = \begin{pmatrix} A_+ x^c \\ A_- x^{-c} \end{pmatrix}, \quad (3.36)$$

with $A_\pm$ being some constants. Obviously, neither component of the ground-state wavefunction (3.36) is normalizable, while, at the same time, the non-zero energy states are described by the plane-wave normalizable Bessel wavefunctions (see [5, 45] for details).

The basic step in the correct quantization of the $N = 2$ superconformal mechanics is to single out the compact sub-superalgebra (3.10) in the full $osp(2|2)$ superalgebra of conserved charges. Then, to define the ground state, we impose, instead of (3.34), the following conditions [4, 45]:

$$\Gamma \Psi_0 = 0, \quad \bar{\Gamma} \Psi_0 = 0, \quad (3.37)$$

where $\Gamma, \bar{\Gamma}$ are defined in (3.10). Explicitly, these supercharges are as follows (cf (3.24))

$$m^{1/2} \Gamma = i\hat{\psi} \left( \hat{p} - \frac{i}{m} \frac{d\tilde{W}}{dx} \right), \quad \bar{\Gamma} = -i\hat{\psi} \left( \hat{p} + i \frac{d\tilde{W}}{dx} \right), \quad (3.38)$$

where

$$\tilde{W} := W - mx^2 = c \ln x - mx^2.$$  

Equations (3.37) amount to the condition

$$\left( \partial_x - \sigma_3 \frac{d\tilde{W}}{dx} \right) \Psi_0 = 0, \quad (3.39)$$

This limiting case corresponds to what is called the Bertotti–Robinson black hole.
which is solved by

\[ \Psi_0 = \begin{pmatrix} \Psi_0^{(+)0} \\ \Psi_0^{(-)0} \end{pmatrix} = \begin{pmatrix} A_+ x^c e^{-mc^2} \\ A_- x^{-c} q^{mc^2} \end{pmatrix}. \]

Assuming, without loss of generality, that the ‘mass’ parameter \( m \) is positive we can choose the wavefunction \( \Psi_0^{(+)0} \) as the true normalizable ground state wavefunction. For definiteness, we can also choose this vacuum state to be bosonic [45]. Taking into account the relation (3.11), we conclude that

\[ (T_0 - \frac{1}{2} C - \frac{1}{2} J) \Psi_0^{(+0)} = 0. \]

Since \( C = c \) and \( J = \frac{1}{2} \) on the vacuum wavefunction, the vacuum eigenvalue of the conformal operator \( T_0 \) is

\[ r_0 = \frac{1}{2} (c + \frac{1}{2}). \tag{3.40} \]

Note that the relations (3.37) can be interpreted as expressing the property that the whole \( \text{OSP}(2|2) \) superconformal symmetry is spontaneously broken down to the compact supergroup \( SU(1|1) \times (\Gamma, \bar{\Gamma}, T_0 - \frac{1}{2} C - \frac{1}{2} J) \).

The full spectrum of the \( \mathcal{N} = 2 \) model considered in this subsection was described in details in [4, 45]. It should be emphasized that the choice of the ‘true’ spontaneously broken phase can be treated as passing to the effective theory with the redefined time and dynamical variables [4], quite similarly to the analogous phenomenon in the bosonic conformal mechanics explained in section 2.2. Redefining, in action (3.21), the evolution parameter and bosonic variable as in (2.33), and making an additional redefinition of the fermionic variable as \( \psi(t) \rightarrow \tilde{\psi}(t) = \psi(t) \), we obtain a system which on the quantum level is described by the new Hamiltonian

\[ \hat{H} = 2mT_0 = H + m^2 K = \frac{1}{4} \left[ \hat{p}^2 + \frac{c(c - 2J)}{\hat{x}^2} \right] + m^2 \hat{x}^2. \tag{3.41} \]

It basically coincides with the compact operator \( T_0 \) and so has the discrete spectrum.

Finally, we remark that on shell the \( \mathcal{N} = 2 \) superconformal mechanics associated with the multiplet \((1, 2, 1)\) has an equivalent description in terms of the chiral multiplet \((2, 2, 0)\) [70].

3.3. Multi-particle mechanics

3.3.1. The Freedman–Mende model. A direct construction of multiparticle system possessed \( \mathcal{N} = 2 \) superconformal symmetry is achieved by using \( n \) real superfields \( \Phi_a, a = 1, 2, \ldots, n \) (3.17) describing \( n \) \((1, 2, 1)\) multiplets. The superfield action is as follows:

\[ S_n^{(N=2)} = \int d^4 \theta \left[ \sum_{a=1}^{n} \bar{D} \Phi_a D \Phi_a - W(\Phi_a) \right], \tag{3.42} \]

where the superpotential \( W(\Phi_a) \) is included. The component action of this model is

\[ S_n^{(N=2)} = \int dt \left[ \sum_{a=1}^{n} \left( \tilde{\psi}_a - i \tilde{\psi}_a \phi_a + i \tilde{\psi}_a \tilde{\psi}_a - \frac{1}{4} \partial_a W \partial_a W \right) + \sum_{a,b} \tilde{\psi}_a \tilde{\psi}_b \partial_a \partial_b W \right], \tag{3.43} \]

where \( \partial_a := \partial/\partial x_a \). Of course, the action (3.43) possesses \( \mathcal{N} = 2 \) Poincaré supersymmetry, whose fermionic charges are

\[ Q = \sum_{a} \psi_a (p_a - i \partial_a W), \quad \bar{Q} = \sum_{a} \tilde{\psi}_a (p_a + i \partial_a W). \tag{3.44} \]
The requirement of $\mathcal{N} = 2$ superconformal symmetry imposes strong constraints on the superpotential. If we consider the superconformal boost transformation of the $(1, 2, 1)$ multiplets \[ \delta' \Phi_a = -i (\eta \bar{\theta} + \bar{\eta} \theta) \Phi_a \] (3.45) and take into account the invariance of the measure $\delta'(dt d^2 \theta) = 0$, we find that the action (3.42) is invariant under superconformal boosts only if the superpotential satisfies the condition
\[ x_a \partial_a W(x) = C, \] (3.46)
where $C$ is a constant. As shown in [48] by detailed calculations, this constant $C$ coincides with the central charge in general $\mathcal{N} = 2$ superconformal algebra (3.5)–(3.7). The full set of the generators of the $\mathcal{N} = 2$ superconformal algebra, including the quantum case, was presented in [48]. In particular, besides the Poincaré supersymmetry generators (3.44), there are additional fermionic generators which correspond to the superconformal boosts and are given by the expressions
\[ S = t \bar{Q} - 2 \sum_a \psi_a x_a, \quad \bar{S} = t \bar{Q} - 2 \sum_a \bar{\psi}_a x_a. \] (3.47)
The closure of the generators (3.44), (3.47) reproduces the whole $\mathcal{N} = 2$ superconformal algebra. As shown in [48], in the case where quantum Hamiltonian contains only boson–fermion couplings without boson–boson interaction, the superpotential $W$ is defined by a harmonic homogeneous function of $x_a$ and the central charge $C$ satisfies some ‘quantization’ conditions.

An important particular case of the superfield action (3.42) corresponds to the superpotential of the form \[ W = \sum_{a \neq b} c \ln (\Phi_a - \Phi_b), \] (3.48)
which is a generalization of the one-particle superpotential (3.19). Then, the component action of this model is
\[ S_{n=2}^{(\mathcal{N} = 2)} = \int dt \left\{ \sum_{a=1}^{a} \left( c^2 - i \bar{\psi}_a \psi_a + i \bar{\psi}_a \bar{\psi}_a \right) - \sum_{a \neq b} \frac{c^2 + 4c(\bar{\psi}_a - \bar{\psi}_b)(\psi_a - \psi_b)}{4(x_a - x_b)^2} \right\}. \] (3.49)
The action (3.49) provides just $\mathcal{N} = 2$ superconformal generalization of the Calogero model proposed by Freedman and Mende [43]. Thus, in the Freedman–Mende model, the strength $c$ of the conformal potential links with the central charge $C$ in the $\mathcal{N} = 2$ superconformal algebra by
\[ C = n(n-1)c, \] (3.50)
which directly follows from the condition (3.46).

A different interesting new case of the $n$-particle system with $\mathcal{N} = 2$ superconformal symmetry was obtained in [49] via nontrivial nonlinear transformation from the free $\mathcal{N} = 2$ superconformal $n$-particle system. This new interacting system is described by the superpotential
\[ W(x) = \nu \ln \left( \sum_a x_a x_a \right), \] (3.51)
where $\nu$ is a constant. It presents an $\mathcal{N} = 2$ superconformal generalization of the motion of the $n$-particle center of mass.

\[ \text{8} \quad \text{The geometric meaning of such transformations for simple (super)conformal systems was explained in [63].} \]
3.3.2. Gauged models. Using the $d = 1$ gauging method applied earlier for deriving the Calogero system, one can construct new many-body systems with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superconformal symmetry. In the $\mathcal{N} = 2$ case, these new systems and the Freedman–Mende system differ in their fermionic sectors. This type of superconformal extension of the Calogero superconformal symmetry. In the $\mathcal{N} = 1$ multiparticle mechanics. We start with the model which uses the Hermitian $\mathcal{N} = 1$ $(n \times n)$-matrix superfield $X^a(t, \theta)$, $(\mathcal{X})^\dagger = \mathcal{X}$, and $\mathcal{N} = 1$ U$(n)$-spinor superfield $Z_a(t, \theta)$, $Z^a(t, \theta) = (Z_a)^\dagger$, $a, b = 1, \ldots, n$. Gauge superfields in the present case are the anti-Hermitian odd $(n \times n)$-matrix superfield $A_b^a(t, \theta)$, $A_b^a = -(A^a_b)^\dagger$, which are spinor connections covariantizing the spinor derivatives. The even gauge potential superfield is $A_\tau = -iDA - A\bar{A}$. This composite superfield is the gauge connection covariantizing the derivative $\bar{\partial}$.

The gauge invariant action has the following form:

$$S^{(\mathcal{N}=1)} = \int \frac{d\theta}{\mathcal{N}} \left[ \text{tr} (\nabla_i \mathcal{X}^i)^2 + \frac{i}{2} (\bar{\mathcal{Z}} D\mathcal{Z} - D\bar{\mathcal{Z}}\mathcal{Z}) + c \text{tr} A \right].$$  

(3.52)

Here, the covariant derivatives are defined as

$$D\mathcal{X} = D\mathcal{X} + i[A, \mathcal{X}], \quad \nabla_i \mathcal{X} = \partial_i \mathcal{X} + i[A_i, \mathcal{X}].$$

(3.53)

$$D\mathcal{Z} = D\mathcal{Z} + iA\mathcal{Z}, \quad D\bar{\mathcal{Z}} = D\bar{\mathcal{Z}} - i\bar{A}\mathcal{Z}.$$

(3.54)

Being composed only of the gauge covariant objects, the action (3.52) is invariant with respect to the local U$(n)$ transformations:

$$\mathcal{X}' = e^{it} \mathcal{X} e^{-it},$$

(3.55)

$$\mathcal{Z}' = e^{it} \mathcal{Z}, \quad \bar{\mathcal{Z}}' = \bar{\mathcal{Z}} e^{-it},$$

(3.56)

$$A' = e^{it} A e^{-it} - i e^{it} (\text{De}^{-it}), \quad A_i' = e^{it} A_i e^{-it} - i e^{it} (\partial_i e^{-it}),$$

(3.57)

where $\tau^a(t, \theta) \in u(n)$ is the Hermitian $(n \times n)$-matrix parameter, $(\tau)^\dagger = \tau$.

Let us check the superconformal invariance of the action (3.52). We will consider only superconformal boosts since all superconformal transformations are contained in the closure of the latter and Poincaré supersymmetry which is manifest in the superfield description. The superconformal boost transformations of the coordinates are

$$\delta t' = -in(t), \quad \delta \theta' = \eta \theta, \quad \delta (dt \, d\theta) = (dt \, d\theta)(-in\theta), \quad \delta D = \text{in} \theta \ D,$$

(3.58)

whereas the corresponding transformations of the superfields read

$$\delta' \mathcal{X} = -in\theta \mathcal{X}; \quad \delta' A = \text{in} \theta A; \quad \delta' \mathcal{Z} = 0; \quad \delta' \bar{\mathcal{Z}} = 0.$$  

(3.59)

Note that

$$\delta'(D\mathcal{X}) = \text{in} \mathcal{X}, \quad \delta' (\nabla_i \mathcal{X}) = \text{i} \eta \theta (\nabla_i \mathcal{X}) - \eta (D\mathcal{X}).$$

(3.60)

Using the expressions (3.58)–(3.60), we find that the variation of the action (3.52) is a total derivative. For example,

$$\delta' \int dt \, d\theta \, \text{tr} (\nabla_i \mathcal{X}^i D\mathcal{X}) = -in \int dt \, d\theta \, \text{tr} (\mathcal{X} \nabla_i \mathcal{X}^i) = -\frac{i}{2} n \int dt \, d\theta \, \partial_i \text{tr} (\mathcal{X}^2).$$

(3.61)

The component contents of the $\mathcal{N} = 1$ superfields are

$$\mathcal{X} = X + i\theta \Psi, \quad \mathcal{Z} = Z + \theta \Upsilon, \quad \bar{\mathcal{Z}} = \bar{Z} - \theta \bar{\Upsilon}, \quad A = i(\Phi + \theta A).$$

(3.62)

Due to the gauge invariance (3.57), we can choose the WZ gauge

$$A = i\partial A(t).$$

(3.63)
Substituting this expression in the action (3.52), integrating over \( \phi (\int d\theta \; d\bar{\theta} = 1) \) and eliminating the auxiliary fields \( \Upsilon \) and \( \bar{\Upsilon} \) by their equations of motion, \( \Upsilon = 0 \) and \( \bar{\Upsilon} = 0 \), we obtain the physical component action

\[
S_{\text{phys}}^{(N-1)} = \int dt \left[ \text{tr} \left( \nabla_X \nabla_Y - i \text{tr} (\Psi \nabla \Psi) + \frac{i}{2} (\bar{Z} \nabla Z - \nabla \bar{Z} Z) + c \text{tr} A \right) \right].
\]  

(3.64)

where \( \nabla \Psi = \Psi + i[A, \Psi] \) is the \( U(n) \)-covariant derivative of matrix Grassmannian odd field \( \Psi \). First, third and fourth terms in (3.64) include only bosonic fields and precisely yield the action (2.72) which was the starting point of deriving the Calogero model by the \( d = 1 \) gauging approach. The second term with fermionic fields ensures \( N = 1 \) supersymmetrization of the Calogero model. Note that the system involves \( n^2 \) (real) fermionic degrees of freedom comprising the matrix fermionic field \( \Psi(t) \). The bosonic and fermionic terms involve the same \( d = 1 \) gauge field \( A(t) \), which, being integrated out, produces non-trivial interaction of the bosonic fields of the Calogero model with the fermions.

Equally, we can analyze the model in a supersymmetric gauge. Using the gauge \( \tau \) transformations (3.55), we can impose a (partial) gauge fixing

\[
\chi_a^0 = 0, \quad a \neq b.
\]  

(3.65)

In this gauge, the action (3.52) takes the form

\[
S_{\text{phys}}^{(N-1)} = -i \int dt d\theta \left[ \sum_a \hat{X}_a D \chi_a + i \sum_a (\bar{Z}^a D \bar{Z}_a - D \bar{Z}_a \bar{Z}_a) - i \sum_a (\chi_a - \chi_b) D A_a^b A_b^a \right.
\]

\[
- \sum_{a,b} (\chi_a - \chi_b)^2 (\bar{A} A_a^b A_b^a) + \sum_{a,b} \tilde{Z}^a A_a^b Z_b + c \sum_a \bar{A}_a^a \left].
\]  

(3.66)

Let us consider the model (3.66) for small \( n \).

In the \( n = 1 \) case, the generic action (3.66) yields the action \( S_{\text{phys}}^{(N-1)} = -i \int dt d\theta \; \hat{X} \hat{D} \chi \) of the free real \( \hat{N} = 1 \) supermultiplet. No potential term is present in the component action.

The first non-trivial case is \( n = 2 \). After imposing the gauge conditions

\[
\hat{Z}_1 = \hat{Z}_1, \quad \hat{Z}_2 = \hat{Z}_2
\]  

(3.67)

for the residual gauge symmetries and eliminating the auxiliary fields \( \hat{Z}_1, \hat{Z}_2 \), we obtain

\[
S_{\text{phys}}^{(N-1)} = -i \int dt d\theta \left[ \frac{1}{2} \hat{\chi} D \chi - \frac{i}{2} C D \chi + \frac{1}{2} \hat{X} D \chi - \frac{i}{2} B D B - c \epsilon_1 \epsilon_2 \frac{B}{\chi} \right].
\]  

(3.68)

where

\[
\chi = \chi_1 - \chi_2, \quad \chi = \chi_1 + \chi_2, \quad B = \chi (A_1^2 + A_2^1), \quad C = i \chi (A_1^2 - A_2^1).
\]  

(3.69)

Here the constants \( \epsilon_1 = \pm 1, \epsilon_2 = \pm 1 \) arise from the constraint \( \hat{Z}_1 \hat{Z}_2 = -\frac{i}{\epsilon_1 \epsilon_2} \) which follows from the equations of motion for \( \hat{Z}_1, \hat{Z}_2 \). The superfields \( \chi \) and \( \chi \) are bosonic, while \( B \) and \( C \) are fermionic.

The action (3.68) is a sum of the actions of two free \( \hat{N} = 1 \) multiplets (superfields \( \chi \) and \( \chi \)) and two mutually interacting \( \hat{N} = 1 \) multiplets (superfields \( \chi \) and \( B \)). It is easy to see that it is the \( \hat{N} = 1 \) superfield form of the off-shell action of \( N = 2 \) superconformal mechanics based on the \( \hat{N} = 2 \) multiplet \( (1, 2, 1) \), supplemented by a free action of an extra multiplet \( (1, 2, 1) \). The latter, from the viewpoint of the two-particle Calogero model, is the \( \hat{N} = 2 \) superextension of the action corresponding to the center-mass motion. The hidden \( \hat{N} = 1 \) supersymmetry completing the manifest one to \( \hat{N} = 2 \) and leaving the above action invariant (up to a total derivative under the integral) is realized as

\[
\delta \chi = i \epsilon C, \quad \delta C = i \epsilon \chi, \quad \delta \chi = i \epsilon B, \quad \delta B = i \epsilon \chi,
\]  

(3.70)
where $\epsilon = \bar{\epsilon}$ is a Grassmann transformation parameter. The action (3.68) is also invariant (each of its two constituent terms separately) under the $\mathcal{N} = 1$ superconformal transformations:

\[
\begin{align*}
\delta \chi &= a(t) + i \theta \chi(t), \quad \delta \theta = \chi(t) + \frac{1}{2} \theta \dot{a}(t), \quad (\partial_\chi)^3 a = (\partial_\theta)^2 \chi = 0, \\
\delta D &= -AD, \quad \delta (d \theta d\bar{\theta}) = (d \theta d\bar{\theta}) A, \quad A = \frac{i}{2} \dot{a} + i \theta \dot{\chi}, \\
\delta X &= A X, \quad \delta Y = A Y, \quad \delta C = \delta B = 0.
\end{align*}
\] (3.71)

Together with the hidden $\mathcal{N} = 1$ supersymmetry, these transformations close on the $\mathcal{N} = 2, d = 1$ superconformal symmetry. The component form of the action (3.68) contains a conformal potential for the physical bosonic field $x = X|_{\theta = 0}$.

In the $n = 3$ case, passing through the same steps as before finally yields the following action:

\[
\begin{align*}
S_{n=3}^{(N=1)} &= -i \int d\theta d\bar{\theta} \left[ \sum_{a=1}^{3} \dot{X}_a D X_a - \frac{i}{2} \sum_{a=1}^{3} (B_a DB_a + C_a DC_a) \\
&\quad + \frac{1}{2} \left( \frac{1}{X_1 - X_2} + \frac{1}{X_2 - X_3} + \frac{1}{X_3 - X_1} \right) (C_1 B_2 B_3 + B_1 C_2 B_3 - B_1 B_2 C_3 + C_1 C_2 C_3) \\
&\quad - c \frac{B_1}{(X_1 - X_2)^2} \epsilon_{12} \left[ 1 - \frac{i}{2c} (B_2 C_2 + B_3 C_3) - \frac{1}{4c^2} (B_2 C_2)(B_3 C_3) \right] \\
&\quad - c \frac{B_2}{(X_2 - X_3)^2} \epsilon_{23} \left[ 1 + \frac{i}{2c} (B_1 C_1 + B_3 C_3) - \frac{1}{4c^2} (B_1 C_1)(B_3 C_3) \right] \\
&\quad - c \frac{B_3}{(X_3 - X_1)^2} \epsilon_{31} \left[ 1 - \frac{i}{2c} (B_1 C_1 - B_2 C_2) + \frac{1}{4c^2} (B_1 C_1)(B_2 C_2) \right]
\right].
\end{align*}
\] (3.73)

where

\[
\begin{align*}
B_1 &= (X_1 - X_2)(A_1^1 + A_2^1), \quad C_1 = i(X_1 - X_2)(A_2^1 - A_1^1), \\
B_2 &= (X_2 - X_3)(A_2^1 + A_3^1), \quad C_2 = i(X_2 - X_3)(A_3^1 - A_2^1), \\
B_3 &= (X_3 - X_1)(A_3^1 + A_1^1), \quad C_3 = i(X_1 - X_3)(A_1^1 - A_3^1).
\end{align*}
\] (3.74)

The superfield action (3.73) produces a new $\mathcal{N} = 1$ superconformal invariant system, which reveals no clear links with the known $\mathcal{N} = 2$ or $\mathcal{N} = 3$ superconformally invariant systems, contrary to the two-particle ($n = 2$) case. In components, in the limit where all fermionic fields are omitted, (3.73) yields of course the 3-particle Calogero model for the fields $x_a = X_a|_{\theta = 0}$.

$\mathcal{N} = 2$ supersymmetric extension. Here we present the $\mathcal{N} = 2$ superconformal gauged matrix model, which produces the $\mathcal{N} = 2$ supersymmetric extension of the Calogero model. Such a system is described by the Hermitian $(n \times n)$-matrix superfield $X_a^b(t, \theta, \bar{\theta})$, $(X_a)^+ = X_a$, $a, b = 1, \ldots, n$, and the chiral $U(n)$-spinor superfield $Z_a(t, \theta, \bar{\theta})$, $\tilde{Z}^a(t, \bar{\theta}) = (Z_a)^+$, $t_{R,L} = t \mp i \theta \bar{\theta}$, $a, b = 1, \ldots, n$, subjected to the constraints

\[
DZ_a = 0, \quad \bar{D} Z^a = 0.
\] (3.75)

An important ingredient of the gauging procedure is gauge superfields. Similar to the $\mathcal{N} = 2$, $d = 4$ supersymmetric theories, they can be described either by complex $(n \times n)$ matrix bridge superfields

\[
\begin{align*}
b_a^b(t, \theta, \bar{\theta}), \quad \tilde{b}_a^b &\equiv (b_a^b)^\dagger \quad (\tilde{b} \equiv b^\dagger),
\end{align*}
\] (3.76)

or by the prepotential $V$ defined by

\[
e^{2V} = e^{-i\tilde{b}} e^{ib}.
\] (3.77)
The covariant derivatives of the superfield $\mathcal{X}$ are
\begin{equation}
\mathcal{D}\mathcal{X} = D\mathcal{X} - i[A, \mathcal{X}], \quad \mathcal{D}\mathcal{X} = D\mathcal{X} - i[\mathcal{D}, \mathcal{X}],
\end{equation}
where the potentials are constructed from the bridges in the standard way:
\begin{equation}
A = -i \epsilon^{ij}(D e^{-\phi}), \quad \mathcal{A} = -i \epsilon^{ij}(\mathcal{D} e^{-\phi}).
\end{equation}

The action we propose is
\begin{equation}
S^{(N=2)} = \int dt \, d^2\theta \left[ \text{tr} (\mathcal{D}\mathcal{X} D\mathcal{X}) + \frac{1}{2} \mathcal{Z} e^{2\mathcal{V}} - c \text{tr} \mathcal{V} \right].
\end{equation}
It is invariant with respect to the local $U(n)$ transformations
\begin{equation}
e^{ib} \rightarrow e^{i\tau} e^{ib} e^{-i\lambda}, \quad e^{ib} \rightarrow e^{i\tau} e^{ib} e^{-i\lambda}, \quad e^{2\mathcal{V}} \rightarrow e^{i\mathcal{X}} e^{2\mathcal{V}} e^{-i\mathcal{X}},
\end{equation}
where $\tau$ is the Hermitian $(n \times n)$-matrix parameter, $t_u(t, \theta, \bar{\theta}) \in u(n)$, $(\tau)^+ = \tau$, and $\lambda$ are $(n \times n)$ complex gauge parameters, $\lambda = (\lambda_a^b)$, which are (anti)chiral superfields $\lambda(t, \theta) \in u(n)$, $\lambda(t, \bar{\theta}) = (\lambda)^+ \in u(n)$.

Global invariances of the action (3.80) form the $\mathcal{N} = 2$ superconformal group $SU(1,1|1)$. The transformations of the $\mathcal{N} = 2$ superspace coordinates under the superconformal boosts of $SU(1,1|1)$ were found in [70]:
\begin{equation}
\delta' t = -i(\eta \bar{\theta} + \bar{\eta} \theta)t, \quad \delta' \theta = -\eta(t - i\bar{\theta}\bar{\theta}), \quad \delta' \bar{\theta} = -\bar{\eta}(t + i\theta\bar{\theta}), \quad \delta' (dt \, d^2\theta) = 0,
\end{equation}
where $\delta' = -i(\eta \bar{\theta} + \bar{\eta} \theta) \mathcal{X}$, $\delta' \mathcal{Z} = 0$, $\delta' b = 0$, $\delta' \mathcal{V} = 0$.

Similarly to the $\mathcal{N} = 2$ $d = 4$ supersymmetric theories, there is a possibility to deal only with the $\lambda$-group covariant objects. After passing to the new Hermitian $(n \times n)$-matrix superfield
\begin{equation}
\mathcal{X} = e^{-ib} \mathcal{X} e^{ib}, \quad \mathcal{X}' = e^{ib} \mathcal{X} e^{-ib},
\end{equation}
the action (3.80) takes the more economical form
\begin{equation}
S^{(N=2)} = \int dt \, d^2\theta \left[ \text{tr} (\mathcal{D}\mathcal{X} e^{2\mathcal{V}} D\mathcal{X} e^{2\mathcal{V}}) + \frac{1}{2} \mathcal{Z} e^{2\mathcal{V}} - c \text{tr} \mathcal{V} \right],
\end{equation}
where the covariant derivatives of the superfield $\mathcal{X}$ are
\begin{equation}
\mathcal{D}\mathcal{X} = D\mathcal{X} + e^{-2\mathcal{V}} (D e^{2\mathcal{V}}) \mathcal{X}, \quad \mathcal{D}\mathcal{X} = D\mathcal{X} - \mathcal{X} e^{2\mathcal{V}} (D e^{-2\mathcal{V}}).
\end{equation}

In the Abelian ($n = 1$) case, similarly to the $\mathcal{N} = 0, 1$ cases, the action (3.86) describes the free $\mathcal{N} = 2$ real supermultiplet. The nontrivial superconformal models start from $n \geq 2$. Let us find dynamical component contents of this generic $n \geq 2$ model.

The component expansions of the involved superfields are
\begin{equation}
V = v + \theta \Phi - \bar{\theta} \bar{\Phi} + \theta \bar{\Phi} A, \quad \mathcal{X} = X + \theta \Psi - \bar{\theta} \bar{\Psi} + \theta \bar{\Psi} \bar{Y}, \quad Z = Z + 2i\theta \gamma + i\bar{\theta} \bar{\gamma} \bar{Z}, \quad \bar{Z} = \bar{Z} + 2i\bar{\theta} \bar{\gamma} + i\theta \gamma \bar{Z},
\end{equation}
where $\Psi_a^b, \bar{\Psi}_a^b = (\Psi_a^b) (\Psi = \Psi^+), \Phi_a^b, \bar{\Phi}_a^b = (\Phi_a^b) (\Phi = \Phi^+)$ and $T_a, \bar{T}_a = (T_a)$ are fermionic component fields. Let us consider the action (3.86) in the Wess–Zumino gauge
\begin{equation}
V(t, \theta, \bar{\theta}) = \theta \bar{\theta} A(t),
\end{equation}
where
Eliminating the auxiliary fields $\Upsilon, \tilde{\Upsilon}$ in this gauge by their equations of motion, we find the component form of the action (3.86) ($\int d^2\theta (\theta \dot{\theta}) = 1$):

$$S^{NW}_{ac} = \int dt \left[ \text{Tr}(\nabla X \nabla X) + \frac{i}{2} (\dot{Z} \nabla Z - \nabla \dot{Z}) - c tr A + i \text{tr}(\bar{\Psi} \nabla \Psi - \nabla \bar{\Psi} \Psi) \right]. \tag{3.91}$$

The covariant derivatives $\nabla X, \nabla Z, \nabla \bar{\Psi}, \nabla \Psi$ were defined in (2.73) and

$$\nabla \Psi = \dot{\Psi} + i[\Psi, A], \quad \nabla \bar{\Psi} = \dot{\bar{\Psi}} + i[\bar{\Psi}, A]. \tag{3.92}$$

We see that the bosonic part of the model (3.91) is exactly the Calogero system (2.72).

The action (3.91) is invariant with respect to the residual local $U(n)$ transformations, $g(\tau) \in U(n)$, defined by (2.78) and

$$\Psi \rightarrow g \Psi g^T, \quad \bar{\Psi} \rightarrow g \bar{\Psi} g^T. \tag{3.93}$$

Exactly as in the pure bosonic case, this local $U(n)$ invariance eliminates the nondiagonal fields $X^b_a, a \neq b$, and all spinor fields $Z_a$. Thus, the physical fields in our $\mathcal{N} = 2$ supersymmetric generalization of the Calogero system are $n$ bosons $X^a_a$ and $2n^2$ fermions $\Psi^b_a$. These fields present on-shell content of $n$ multiplets $(1, 2, 1)$ and $n^2 - n$ multiplets $(0, 2, 2)$. These multiplets are produced from the original $n^2$ multiplets $(1, 2, 1)$ by the gauging procedure [29]. We can depict this multiplet structure on the plot

$$\begin{array}{ccc}
\chi^a \rightarrow (X^a_a, \Psi^a_a) & \chi^b \rightarrow (X^b_a, \Psi^b_a), & a \neq b \\
\downarrow \quad \text{gauging} \quad \downarrow & & \\
\chi^a \rightarrow (X^a_a, \Psi^a_a) & \Omega^b_a = (\Psi^b_a, B^a_b), & a \neq b.
\end{array} \tag{3.94}$$

Here, the bosonic fields $B^b_a$ are auxiliary components of the multiplets $(0, 2, 2)$, which are not present in the action (3.91), being eliminated by their equations of motion. Thus, we obtained some new $\mathcal{N} = 2$ extensions of the $n$-particle Calogero models with $n$ bosons and $2n^2$ fermions, in contrast to the standard $\mathcal{N} = 2$ super-Calogero system of Freedman and Mende which involves only $2n$ fermions [43].

It is easy to explicitly express the considered system in terms of physical variables only. This can be done either by going to the Hamiltonian formalism, like we did in the purely bosonic case, or by eliminating the auxiliary variables after the gauge fixing. Using the latter method, i.e. by effecting the gauge-fixing conditions (2.79), (2.85) and eliminating the auxiliary fields $A^b_a$, we obtain the physical action of the considered $\mathcal{N} = 2$ multiparticle superconformal model of Calogero type

$$S^{(N=2)}_{ac} = \int dt \left[ \sum_a \dot{x}^a_{\alpha} x^a_{\alpha} + i (\bar{\Psi}^b_a \Psi^a_a - \bar{\Psi}^b_a \Psi^a_a) - H \right]. \tag{3.94}$$

Here, the Hamiltonian has the form

$$H = \frac{1}{4} \sum_a (p_a)^2 + \sum_{a \neq b} \frac{4}{(x_a - x_b)^2} (Z_a \bar{Z}^b Z^b_a Z_a + 2 \bar{Z}^a \{\Psi, \bar{\Psi}\}_a^b Z_b + \{\Psi, \bar{\Psi}\}_a^b (\Psi, \bar{\Psi})_b^a). \tag{3.95}$$

In the expression (3.95), the variables $Z_a$ (which are real after gauge fixing (2.85), $\bar{Z}^a = Z_a$) are found from the equations of motion for diagonal components of $A^b_a$

$$(Z_a)^2 = c - R_a, \tag{3.96}$$
where
\[ R_a = \langle \Psi, \tilde{\Psi} \rangle \equiv \sum_b (\Psi^a_b \tilde{\Psi}^a_b + \bar{\Psi}^a_b \Psi^a_b) \] (no summation over \( a \)).

(3.97)

The quantities \( R_a \) contains \( \Psi^a_b \) and \( \bar{\Psi}^a_b \) with \( a \neq b \) only and has \( 2(n-1) \) terms, so that \( (R_a)^{2n-1} = 0 \). Therefore, the solutions of equations (3.96) are

\[ Z_a = \epsilon_a \sqrt{c} \sum_{k=0}^{2(n-1)} \alpha_k \frac{\epsilon_k}{i^k} (R_a)^k, \]

(3.98)

where \( \epsilon_a = \pm 1 \), independently for each \( a \), and \( \alpha_k \) are some constants. The first constants from this set are \( \alpha_0 = \alpha_1 = 1, \alpha_2 = -\alpha_3 = -\frac{1}{2}, \alpha_3 = -\frac{i}{2} \). Let us consider the \( n = 2 \) case as an example. In this case, the action (3.94) has the form

\[ S_{sc}^{(n=2)} = \int \text{d}r \left\{ \xi_0 \partial_0 + i (\tilde{\psi}_0 \psi_0 - \tilde{\psi} \psi_0) + \tilde{\chi} \partial_0 - 2 \tilde{\psi} \tilde{\phi}_0 \right\}

+ \rho \tilde{\phi} + i (\tilde{\psi} \tilde{\phi} - \tilde{\phi} \tilde{\psi}) + i (\tilde{\phi} \tilde{\phi} - \tilde{\phi} \tilde{\phi})

- \frac{1}{\rho^2} \left[ \frac{c^2}{4} + \epsilon_1 \epsilon_2 (\Psi \nabla + \Psi \nabla + (\Psi \nabla + \Psi \nabla)^2) \right], \]

(3.99)

where we used the notations

\[ x_0 = \frac{1}{\sqrt{2}} (x_1 + x_2), \quad \rho = \frac{1}{\sqrt{2}} (x_1 - x_2), \]

(3.100)

\[ \psi_0 = \frac{1}{\sqrt{2}} (\psi_1^1 + \psi_2^2), \quad \psi = \frac{1}{\sqrt{2}} (\psi_1^1 - \psi_2^2), \quad \chi = \frac{1}{\sqrt{2}} (\psi_1^2 + \psi_2^1), \quad \varphi = \frac{1}{\sqrt{2}} (\psi_1^2 - \psi_2^1). \]

(3.101)

After introducing new fermionic fields via the nonlinear transformations

\[ \phi_0 = \chi \left( 1 + \frac{1}{2m} (\psi \tilde{\phi} - \psi \tilde{\phi}) + \frac{1}{4m} (\psi \tilde{\phi} \psi \tilde{\phi}) \right), \quad \phi_1 = \psi + \frac{1}{2m} (\chi \psi \psi), \]

\[ \phi_2 = \tilde{\phi} - \frac{1}{2m} (\chi \psi \tilde{\phi}), \]

(3.102)

where \( m = -\frac{\epsilon_1 \epsilon_2}{2} \), the action (3.99) becomes

\[ S_{sc}^{(n=2)} = \int \text{d}r \left\{ \xi_0 \partial_0 + i (\tilde{\psi}_0 \psi_0 - \tilde{\psi} \psi_0) + i (\tilde{\phi}_0 \phi_0 - \tilde{\phi}_0 \phi_0)

+ \rho \tilde{\phi} + i (\tilde{\phi}_1 \phi_1 - \tilde{\phi}_1 \phi_1) + i (\tilde{\phi}_2 \phi_2 - \tilde{\phi}_2 \phi_2)

- \frac{1}{\rho^2} [m^2 - 2m \partial_0 \phi_2 + m \partial_0 \tilde{\phi}_0 + (\partial_0 \phi_1 + \partial_0 \tilde{\phi}_0)^2] \right\}. \]

(3.103)

This action is none other than a sum of the action of free \( N = 4 \) \((1, 4, 3)\) supermultiplet \( \Omega \) with the physical components \( (x_0; \psi_0, \phi_0) \) and the \( N = 4 \) superconformal mechanics action [70] for \( N = 4 \) \((1, 4, 3)\) supermultiplet \( Y \) with physical components \( (\rho; \phi_1, \phi_2) \). It can be recovered from the \( N = 4 \) superfield action

\[ S_{sc}^{(n=2)} = - \int \text{d}t \text{d}^4 \theta \left( \frac{1}{2} \Omega^2 + Y \Omega Y \right), \]

(3.104)

where \( \Omega \) and \( Y \) are some constrained. \( N = 4 \) superfields representing two independent off-shell \((1, 4, 3)\) supermultiplets (see the following section). This appearance of hidden higher-rank
superconformal symmetry is quite similar to what happens in the $\mathcal{N} = 1, n = 2$ case. No such intriguing feature arises in $\mathcal{N} = 2$ models with $n > 2$.

The $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superextensions of the conformal Calogero model considered in this section (and sketched in [36]) are new and so need a more detailed analysis, including the study of their possible integrability (e.g. along the line of [21–23]). Though they contain non-minimal sets of fermionic fields as compared to the Freedman and Mende models, these sets are necessarily implied by the supersymmetric gauge procedure which is a well-defined generalization of the bosonic $d = 1$ gauge procedure yielding the ordinary Calogero model. It is an open question whether it is possible to somehow reduce the number of fermions—either by imposing some extra covariant conditions on the original superfields or through the proper enlargement of the underlying gauge invariance. In section 4.3, we will show that in the $\mathcal{N} = 4$ case, the similar gauging procedure naturally leads to the $\mathcal{N} = 4$ superconformal extension of the so-called spin Calogero model [36].

4. $\mathcal{N} = 4$ superconformal mechanics

4.1. A bestiary of $\mathcal{N} = 4, d = 1$ superconformal algebras

The full family of $\mathcal{N} = 4, d = 1$ superconformal algebras is spanned by the following generators in the spinorial basis:

\[ G^{(4)} = (Q_{\alpha'\beta'\gamma'}, T_{\alpha'\beta'}, J_{j\gamma'}, I_{j'\gamma'}) \]  

In general they form the superalgebra $D(2, 1; \alpha)$ (for more details, see e.g. [44, 7, 67])

\[ \{Q_{\alpha'\beta'}, Q_{\beta'\gamma'}\} = 2 (\epsilon_{\alpha\beta} \epsilon_{\gamma'} T_{\alpha'\beta'} + \alpha \epsilon_{\alpha\beta} \epsilon_{\gamma'} J_{j\gamma'} - (1 + \alpha) \epsilon_{\alpha\beta} \epsilon_{j'\gamma'} I_{j'\gamma'}), \]  

\[ [T_{\alpha'\beta'}, T_{\gamma'\delta'}] = i (\epsilon_{\alpha\gamma} T_{\beta\delta} + \epsilon_{\beta\delta} T_{\alpha\gamma}), \]  

\[ [J_{j\gamma'}, J_{k\delta'}] = i (\epsilon_{j\kappa} J_{\gamma\delta} + \epsilon_{j\delta} J_{\kappa\gamma}), \]  

\[ [I_{j'\gamma'}, I_{k'\delta'}] = i (\epsilon_{j'k'} I_{\gamma\delta} + \epsilon_{j'\delta'} I_{k'\gamma'}), \]  

\[ [T_{\alpha'\beta'}, Q_{\gamma'\delta'}] = -i \epsilon_{\gamma'\alpha'} Q_{\beta'\delta'}, \]  

\[ [J_{j\gamma'}, Q_{\kappa'\delta'}] = -i \epsilon_{\kappa'j} Q_{\gamma'\delta'}, \]  

\[ [I_{j'\gamma'}, Q_{\kappa'\delta'}] = -i \epsilon_{\kappa'j'\gamma'} Q_{\delta'}, \]  

with other commutators vanishing. All the $D(2, 1; \alpha)$ generators are assumed to be Hermitian, i.e. they satisfy the relations

\[ (Q_{\alpha'\beta'})^\dagger = \epsilon^{\gamma'\delta'} \epsilon_{\gamma'\delta'} Q_{\alpha'\beta'}, \]  

\[ (T_{\alpha'\beta'})^\dagger = T_{\alpha'\beta'}, \]  

\[ (J_{j\gamma'})^\dagger = \epsilon^{\delta'k'\gamma'} J_{j\delta'k'}, \]  

\[ (I_{j'\gamma'})^\dagger = \epsilon^{j'k'\gamma'} I_{j'k'\gamma'}. \]  

The generators $T_{\alpha'\beta'}$ form the $o(2, 1)$ algebra ($T_{11} = H$, $T_{22} = K$, $T_{12} = D$), whereas $J_{j\gamma'}$ and $I_{j'\gamma'}$ constitute two mutually commuting $o(3)$ algebras forming the $o(4)$ algebra. The indices $\alpha, \beta, \gamma = 1, 2$ are spinor $o(2, 1)$ indices and $i, j, k = 1, 2$; $\gamma', j' = 1, 2$ are doublet indices of two $o(3)$ algebras. Everywhere in this review, we take $\epsilon_{12} = \epsilon^{12} = 1$. The fermionic $o(2, 1)$ spinor supercharges $Q_{\alpha'\beta'}$ unify the standard $d = 1$ supercharges $Q_{\bar{\alpha}'} = Q_{\bar{\alpha}}^\dagger$ and the generators of superconformal boosts $S_{\alpha'} = Q_{2\alpha'}$. In the complex notation, with only one $O(3)$ symmetry being manifest, the supercharges are rewritten as

\[ Q_{\gamma'}^1 = -Q_{\gamma'}, \quad Q_{\gamma'}^2 = -\tilde{Q}_{\gamma'}, \quad (Q_{j'\gamma'})^\dagger = \bar{Q}_{j'\gamma'}. \]
4.2. Superconformal models from standard \( \mathcal{N} = 4 \) superspace

A natural arena for \( \mathcal{N} = 4, d = 1 \) supersymmetric theories is the \( \mathcal{N} = 4, d = 1 \) superspace [70]

\[
(t, \theta_i, \bar{\theta}^i), \quad \bar{\theta}^i = (\bar{\theta}^i), \quad (i = 1, 2). \tag{4.20}
\]

The corresponding spinor covariant derivatives have the form

\[
D^\prime = \frac{\partial}{\partial \theta_i} - i \bar{\theta}^i \partial_i, \quad \bar{D}_i = \frac{\partial}{\partial \bar{\theta}^i} - i \theta_i \partial_i = -\overline{(D^\prime)}. \tag{4.21}
\]

One of the two SU(2) factors of the full R-symmetry (automorphism) group SO(4)\(_R\) acts on the doublet indices \( i \) and will be denoted SU(2)\(_R\). The second SU(2) mixes \( \theta_i \) with their complex conjugates and is not manifest in the considered approach.
4.2.1. Single-particle models. The one-particle model is built on the multiplet (1, 4, 3), which is described by the even real superfield $X$ subjected to the constraints

$$D^i \chi_i = 0, \quad \bar{D}_i \bar{\chi}_i = 0, \quad [D^i, \bar{D}_i] \chi_i = 0.$$  \hspace{1cm} (4.22)

The superconformal action of the (1, 4, 3) multiplet is given by ($\alpha \neq 0$)

$$S_X = -\frac{1}{4(1+\alpha)} \int dt \ d^4 \theta \ (X^{-1/\alpha}).$$  \hspace{1cm} (4.23)

Note that the action (4.23) is in fact non-singular at $\alpha = -1$. Indeed, making use of the fact that $\int \mu \chi$ is an integral of the total derivative (in virtue of constraints (4.22)), we cast the action (4.23) in the equivalent form

$$S_X = -\frac{1}{4(1+\alpha)} \int dt \ d^4 \theta \ (X^{-1/\alpha} - \chi).$$  \hspace{1cm} (4.24)

Thus, in the limit $\alpha = -1$, we obtain the meaningful action

$$S_X \big|_{\alpha=1} = -\frac{1}{4} \int dt \ d^4 \theta \ X \ln X.$$

The action (4.23) is not defined at $\alpha = 0$, and this special case needs a separate analysis (see [29, 30]). In what follows, we assume that $\alpha \neq 0$.

The action (4.23) is invariant with respects to the rigid $N = 4$ superconformal symmetry $D(2, 1; \alpha)$. All superconformal transformations are contained in the closure of the supertranslations and superconformal boosts.

Invariance of the action (4.23) under the supertranslations ($\epsilon^i = (\epsilon_i)$)

$$\delta t = i \epsilon_k \bar{\theta}^k, \quad \delta \theta_k = \epsilon_k, \quad \delta \bar{\theta}^k = \epsilon^k$$  \hspace{1cm} (4.26)

is automatic because we work in the $N = 4$ superfield approach.

The coordinate realization of the superconformal boosts of $D(2, 1; \alpha)$ [67, 68, 72] is as follows ($\bar{\eta} = (\bar{\eta}_i)$):

$$\delta' t = i (\theta_k \bar{\eta}^k + \bar{\theta}^k \eta_k) - (1+\alpha) \theta_i \bar{\theta}^i (\theta_k \bar{\eta}^k + \bar{\theta}^k \eta_k),$$  \hspace{1cm} (4.27)

$$\delta' \theta_i = -\eta_i t - 2i \alpha \theta_i (\theta_k \bar{\eta}^k) + 2i (1+\alpha) \theta_i (\bar{\theta}^k \eta_k) - i(1+2\alpha) \eta_i (\bar{\theta}^k),$$  \hspace{1cm} (4.28)

$$\delta' \bar{\theta}^i = -\bar{\eta}^i t - 2i \alpha \bar{\theta}^i (\bar{\theta}^k \eta_k) + 2i (1+\alpha) \bar{\theta}^i (\theta_k \bar{\eta}^k) + i(1+2\alpha) \bar{\eta}^i (\theta_k \bar{\eta}^k),$$  \hspace{1cm} (4.29)

$$\delta' (d^4 \theta) = -\alpha^{-1} (d^4 \bar{\theta}) \Lambda_0,$$  \hspace{1cm} (4.30)

where

$$\Lambda_0 = 2 \alpha (\theta_k \bar{\eta}^k + \bar{\theta}^k \eta_k).$$  \hspace{1cm} (4.31)

Taking the superfield transformations in the form (here we use the ‘passive’ interpretation of them)

$$\delta' \chi = -\Lambda_0 \chi,$$  \hspace{1cm} (4.32)

it is easy to check the invariance of the action (4.23).

The solution of the constraint (4.22) is as follows:

$$X(t, \theta_i, \bar{\theta}^i) = x + \theta_i \psi^i + \bar{\psi} \bar{\theta}^i + \psi \bar{\theta}^i \partial_t \Lambda_0 + \frac{i}{2} (\theta)^2 \psi \bar{\psi}^i + \frac{i}{2} (\bar{\theta})^2 \theta_i \bar{\psi}^i + \frac{1}{4} (\theta)^2 (\bar{\theta})^2 \chi,$$  \hspace{1cm} (4.33)
where $(\theta)^2 \equiv \theta \theta^t$, $(\bar{\theta})^2 \equiv \bar{\theta} \bar{\theta}$. Inserting (4.33) in (4.23) and integrating over the Grassmann variables, we obtain

$$S_X = \frac{1}{4\alpha^2} \int dt \ x^{-\frac{1}{2}-2} \left[ \dot{x} + i(\bar{\psi}_k \dot{\psi}^k - \dot{\bar{\psi}}_k \bar{\psi}^k) - \frac{1}{2} N^{ik} \bar{N}_k \right] - \frac{1}{4\alpha^2} \left( \frac{1}{\alpha} + 2 \right) \int dt \ x^{-\frac{1}{2}-3} N^{ik} \bar{\psi}_k \psi \bar{\psi}_k \psi, \quad (4.34)$$

Eliminating the auxiliary fields $N^{ik}$ by their algebraic equations of motion,

$$N_{ik} = -\left( \frac{1}{\alpha} + 2 \right) x^{-1} \bar{\psi}_k \psi, \quad (4.35)$$

we obtain the on-shell form of the action (4.34)

$$S = \int dt \left[ \dot{x} + i(\bar{\psi}_k \dot{\psi}^k - \dot{\bar{\psi}}_k \bar{\psi}^k) \right] + \frac{2}{3} (1+2\alpha) \int dt \ x^{-\frac{1}{2}-4} \bar{\psi}_k \psi \bar{\psi}_k \psi. \quad (4.36)$$

This is the action of one-particle $\mathcal{N} = 4$ superconformal mechanics. It contains no pure conformal potential at the classical level. It can appear from the last term with fermions upon quantization.

In section 4.3.2, we will consider the system in which one-particle model (4.23) appears as a subsystem. Therefore, using formulas of section 4.3.2 (for example, (4.138)–(4.145)) and making the appropriate truncations, we can recover the generators of the $\mathcal{N} = 4$ superconformal $D(2, 1; \alpha)$ symmetry in the considered one-particle superconformal system (4.23) (see also [110, 50, 51] for such a realization of the $su(1, 1|2)$ superconformal algebra as a particular case of $D(2, 1; \alpha)$).

When only $su(1, 1|2)$ symmetry is required, while the $su(2)_c$ symmetry appearing in the semi-direct sum (see (4.13)) is allowed to be broken, the constraints (4.22) for the even real superfield $X$ can be weakened [70] by adding nonzero constants on their right-hand sides. These constants can always be rotated between equations (4.22), for example, to

(a) $D^i D_i X = 0, \quad \bar{D}_i \bar{D}^i X = 0$; \quad (b) $[D', \bar{D}_i] X = m, \quad (4.37)$

where the constant $m$ provides a central charge to the $su(1, 1|2)$ algebra. The solution to (4.37) is a sum of (4.33) and an additional term $-\frac{1}{2} \theta \bar{\theta} m$. Then, the action (4.23) (with $\alpha = -1$) gives rise to additional contributions to the physical component Lagrangian (4.36) which now becomes [70]

$$\tilde{S} = \int dt \left[ \dot{x} + i(\bar{\psi}_k \dot{\psi}^k - \dot{\bar{\psi}}_k \bar{\psi}^k) \right] + \frac{(m + \bar{\psi} \dot{\psi})^2}{x^2}. \quad (4.38)$$

The additional terms, proportional to $m^2/x^2$ and $m \bar{\psi} \dot{\psi}/x^2$, also appear in the Hamiltonian, and they are induced by the appropriate new terms in the Noether supercharges, which correspond to the $su(1, 1|2)$ algebra with a central charge proportional to $m$. Such a central-charge deformation is possible only for $\alpha = -1$ (and $\alpha = 0$). Thus in this case the conformal potential comes out at the classical level, and its strength is the square of the central charge $m$. Note that the action (4.38) and the action (3.103) for the superfield $Y$ are equivalent to each other. Instead of the constraints (4.37), the superfield $Y$ is subject to the constraints

(a) $D^i D_i Y = m, \quad \bar{D}_i \bar{D}^i Y = m$; \quad (b) $[D', \bar{D}_i] Y = 0. \quad (4.39)$

The constraints (4.37) and (4.39), as well as the actions (4.38) and (3.103), are related to each other by the (broken) SU(2) rotations which mix $\theta$ with $\bar{\theta}$ (see the details in [70]).
4.2.2. Multi-particle models with $D(2, 1; \alpha)$ symmetry. Despite the physical importance, multiparticle systems with $\mathcal{N} = 4$ superconformal symmetry are poorly understood to date. Unlike multi-particle $\mathcal{N} = 2$ superconformal systems, direct generalizations of $\mathcal{N} = 4$ one-particle superconformal systems do not yield $\mathcal{N} = 4$ superconformal systems of Calogero type. For this reason, until now, studies of the $\mathcal{N} = 4$ multi-particle superconformal systems were performed not only in ordinary superspace, but also at the component level. Furthermore, it turns out that the realization of $D(2, 1; \alpha)$ superconformal symmetry on the multi-particle phase space for $\alpha \neq -1$ or 0 requires at least one pair of (bosonic) isospin variables \( \{u', \bar{u} \}_i \) parametrizing an internal two-sphere. This subsection mainly summarizes the results of [79].

We consider $n$ particles on a real line, with coordinates and momenta \( \{x^a, p_a\}_{a = 1, \ldots, n} \) as well as associated complex pairs of fermionic variables \( \{\psi^a, \bar{\psi}^a\}_{a = 1, \ldots, n, i = 1, 2} \).

The basic nonvanishing Poisson brackets read

\[
\{x^a, p_b\} = \delta^a_b, \quad \{\psi^a, \bar{\psi}^b\} = -\frac{i}{2} \delta^a_b \delta_{ab}, \quad \{u'_i, \bar{u}_k\} = -i \delta^i_k. \tag{4.40}
\]

We would like to realize the $\mathcal{N} = 4$ superconformal algebra $D(2, 1; \alpha)$ on the (classical) phase space of this mechanical system, thereby severely restricting the particle interactions. It is convenient to start with an ansatz for the supercharges $Q_a$ and $\bar{Q}^a$. An important novelty of the $\mathcal{N} = 4$ supercharges compared to the $\mathcal{N} = 2$ ones is the necessity of a term cubic in the fermions besides the standard linear one in (3.44) and (3.47). Therefore, the ansatz should take the form

\[
Q_i = p_a \psi^a_i + iU_a(x)^i \psi^a + iF_{abc}(x)_{ij}^k \psi^a_j \bar{\psi}^b k, \tag{4.41}
\]

where $U_a(x)^i$ and $F_{abc}(x)_{ij}^k$ are homogeneous functions of degree $-1$ in \( \{x^1, \ldots, x^n\} \), to be determined.

For the $F$ functions, we adopt the simplest possibility,

\[
F_{abc}(x)_{ij}^k = F_{abc}(x) \epsilon_{ij}^k \epsilon^{ab}, \tag{4.42}
\]

while the analogous option $U_a(x)^i = W_a(x) \delta^i_k$ for the $U$ functions turns out to work only for the case of $\alpha = -1$ or 0 (see below). To open the way for a realization of the generic $D(2, 1; \alpha)$ superalgebra, one must generalize to

\[
iU_a(x)^i = U_a(x)J_i^k \quad \text{with} \quad J_i^k = \frac{1}{2} (u'_i \bar{u}_k + u_k \bar{u}'_i), \tag{4.43}
\]

utilizing the isospin $su(2)$ current. Therefore, the ansatz in [79] reads

\[
Q_i = p_a \psi^a_i + U_a(x)J_i^k \bar{\psi}^k + iF_{abc}(x)\psi^a \bar{\psi}^b k \bar{\psi}^c. \tag{4.44}
\]

The spin variables just serve to produce these currents and do not appear by themselves. In the quantum case, the cubic terms should be Weyl ordered.

Following [79], let us try to build the $D(2, 1; \alpha)$ algebra based on (4.44). First, the $\mathcal{N} = 4$ super-Poincaré subalgebra

\[
\{Q_i, Q_k\} = 0 \quad \text{and} \quad \{Q_i, \bar{Q}^k\} = 2i\delta_i^k H \tag{4.45}
\]

defines a Hamiltonian $H$ and enforces the following conditions on our functions $U_a$ and $F_{abc}$,

\[
\partial_a U_b - \partial_b U_a = 0, \quad \partial_a F_{bcd} - \partial_b F_{acd} = 0, \tag{4.46}
\]

\[
F_{ace}F_{bd} - F_{abe}F_{cad} = 0, \tag{4.47}
\]

\[
- \partial_a U_b + U_a U_b + F_{abc} U_c = 0. \tag{4.48}
\]

9 Viewed as a one-particle system, the bosonic target is $\mathbb{R}^n$. Its metric $(\delta_{ab})$ allows us to pull down all particle indices. Spinor indices are raised and lowered with the invariant tensor $\epsilon^{ab}$ and its inverse $\epsilon_{ab}$, respectively.
The integrability conditions (4.46) are solved by
\[ U_a = \partial_a U \quad \text{and} \quad F_{abc} = \partial_a \partial_b \partial_c F \] (4.49)
with two scalar prepotentials \( F(x) \) and \( U(x) \), and hence we read subscripts on \( U \) and \( F \) as derivatives.\(^{10}\) Thus, the other two conditions become nonlinear differential equations for \( F(x) \) and \( U(x) \), whose solutions define the various possible models. In particular, (4.47) is the celebrated WDVV equation \([108, 34]\), and (4.48) describes the (logarithm of) so-called twisted periods related to \( F \) \([35, 39]\). With the above conditions fulfilled, the Hamiltonian acquires the form
\[ H = \frac{1}{2} p_a p_a + \frac{1}{8} J^{ik} J_{ik} U_a U_a (x) - i U_{ab} (x) J_{ik} \psi^{ai} \psi^{jk} - \frac{1}{2} F_{abcd} (x) \psi^{ai} \psi^{bj} \psi^{ck} \psi^{dk}. \] (4.50)
One may check that \([H, J^{ik} J_{ik}] = 0\), and thus the Casimir \( J^{ik} J_{ik} =: g^2 \) appears as a coupling constant in the bosonic potential
\[ V = \frac{g^2}{8} U_a U_a. \] (4.51)
Second, for the full \( D(1, 2; \alpha) \) superconformal invariance, one has to realize the additional generators. This can be done via
\[ D = -\frac{1}{2} x^a p_a, \quad K = x^a x^a, \quad S_i = -2 x^a \psi_i^a, \quad \tilde{S}^i = -2 x^a \tilde{\psi}^{ia}, \] (4.52)

\[ J_{ik} = J_{ik} + 2 i \psi_i^a \tilde{\psi}^{ak}, \quad \text{and} \quad I_{11} = i \psi_i^a \psi_i^{ia}, \quad I_{22} = -i \tilde{\psi}^{ia} \tilde{\psi}^{ai}, \quad I_{12} = i \psi_i^a \tilde{\psi}^{ia}. \] (4.53)

Now, dilatation invariance requires homogeneity,
\[ (x^a \partial_a + 1) U_b = \partial_b (x^a U_a) = 0 \quad \text{and} \quad (x^a \partial_a + 1) F_{bcd} = \partial_b (x^a F_{acd}) = 0. \] (4.54)
Third, the remaining superalgebra commutators only fix the integration constants to
\[ x^a U_a = 2 \alpha \quad \text{and} \quad x^a F_{abc} = -(1 + 2 \alpha) \delta_{bc} \Rightarrow (x^a \partial_a - 2) F = -\frac{1}{2} (1 + 2 \alpha) x^a x^a. \] (4.55)
Clearly, \( F \) cannot vanish unless \( \alpha = -\frac{1}{2} \), the \( osp(4|2) \) case.

It is instructive to introduce the exponential of the prepotential \( U \), since this linearizes (4.48),
\[ W = e^{-U} \Rightarrow W_{ab} - F_{abc} W_c = 0 \quad \text{and} \quad (x^a \partial_a + 2 \alpha) W = 0. \] (4.56)
For \( \alpha = -1 \) or 0, the presence of a central charge \( m \) perturbs the homogeneity of \( U \) and \( W \), and it is better to work with \( W \) in these cases (see below). Clearly, the WDVV prepotential takes the form
\[ F(x) = -\frac{1}{2} (1 + 2 \alpha) x^2 \ln x^2 + F_0(x), \] (4.57)
while (for \( m = 0 \)) the other prepotential reads
\[ U(x) = \alpha \ln x^2 + U_0(x) \iff W(x) = x^{-2 \alpha} W_0(x), \] (4.58)
where \( F_0, U_0 \) and \( W_0 \) are homogeneous of degrees 2, 0 and 0, respectively, and \( x^2 \) is any expression quadratic in the coordinates. Obviously, the \( F \) prepotentials for any two values of \( \alpha \) are related by a mere rescaling as long as \( \alpha \neq -\frac{1}{2} \). The mathematical literature usually does not introduce a Euclidean metric \( \delta_{ab} \) but defines an induced metric \( G_{bc} = -\delta_a F_{abc} \) which is constant and nondegenerate. Hence, for any \( \alpha \neq -\frac{1}{2} \) we can import all known WDVV solutions \([91, 106, 27, 40, 41, 85, 86]\) up to constant coordinate transformations. The special case of

\(^{10}\) Note that \( U(x) \) and \( F(x) \) are defined only up to polynomials of degrees 0 and 2, respectively.
been constructed. Here, we present only the special case of polynomials which completely nontrivial with four particles. 

The system (4.46)–(4.48) is form-invariant under SO\((n)\) rotations. It can happen that such a rotation reduces the system (4.46)–(4.48) to two decoupled subsystems. For example, in the presence of global translation invariance, \(x^a \rightarrow x^a + \xi\), the center of mass, \(X = \sum_\alpha x^\alpha\), can be decoupled by passing to \(n-1\) appropriate relative coordinates. Furthermore, in any coordinates the contractions of (4.47) and (4.48) with \(x^\alpha\) are already a consequence of (4.55), which fixes the ‘radial’ dependence. This effectively reduces the dimensionality further to \(n-2\) ‘angular’ coordinates. The number of independent equations in (4.47) and (4.48) is \(\frac{1}{2}(n-1)(n-2)^2(n-3)\) and \(\frac{1}{2}(n-1)(n-2)\), respectively, so that for up to three particles, the WDVV equation (4.47) is empty, and (4.48) reduces to at most one angular condition, which can always be solved. Hence, the construction of irreducible multi-particle models becomes nontrivial with four particles.

All known WDVV solutions are of the form\(^{11}\)

\[
F(x) = \sum_\beta f_\beta K(\beta \cdot x) \quad \text{with} \quad f_\beta \in \mathbb{R} \quad \text{and} \quad \beta \cdot x = \beta(x) = \beta^a x_a, \tag{4.59}
\]

where the sum runs over a collection \(\{\beta\}\) of \(p\) non-parallel covectors comprising a deformed (super) Lie-algebra root system, and the function \(K\) is universal up to a quadratic polynomial,

\[
K''(z) = -\frac{1}{z} \Rightarrow K(z) = -\frac{1}{4} z^2 \ln z^2. \tag{4.60}
\]

Similarly, all known explicit solutions for \(W\) are of the form\(^{39}\)

\[
W(x) = \prod_\ell P_\ell(x)^{-u_\ell} \quad \Leftrightarrow \quad U(x) = \sum_\ell u_\ell \ln P_\ell(x) \quad \text{with} \quad \sum_\ell u_\ell u_\ell = 2a, \tag{4.61}
\]

where \(P_\ell\) is a homogeneous polynomial of degree \(n_\ell\).

However, it is a deep and unsolved mathematical problem to find all solutions \(F\) to the WDVV equation, and only for a part of the known solutions (4.59) have the twisted periods \(W\) been constructed. Here, we present only the special case of polynomials which completely factorize into linear factors (with the notation of (4.59))\(^{12}\)

\[
P_\ell(x) = \prod_\beta \beta_\ell \cdot x \quad \Leftrightarrow \quad W(x) = \prod_\beta (\beta \cdot x)^{-u_\beta} \quad \Leftrightarrow \quad U(x) = \sum_\beta u_\beta \ln \beta \cdot x, \tag{4.62}
\]

where we choose a collection of covectors \(\beta\) common to \(U\) and \(F\). Note however that not all covectors from \(\{\beta\}\) need to appear in \(F\) or \(U\), because some \(f_\beta\) or \(u_\beta\) may vanish. The normalizations (4.55) imposed by conformal invariance translate into simple conditions for the coefficients \(u_\beta\) and \(f_\beta\),

\[
\sum_\beta u_\beta = 2a \quad \text{and} \quad \sum_\beta f_\beta \beta_\beta = (1+2a)\delta_{\beta c} \Rightarrow \sum_\beta \beta \cdot \beta f_\beta = (1+2a)n \tag{4.63}
\]

with \(\beta \cdot \gamma = \delta_{ab} \beta_a \gamma_b\), and the bosonic potential becomes

\[
V(x) = \frac{g_2}{8} \sum_{\beta, \gamma} u_\beta u_\gamma \frac{\beta \cdot \gamma}{\beta \cdot x \gamma \cdot x}. \tag{4.64}
\]

One can actually employ the twisted-period equation (4.48) to solve for \(u_\beta\) in terms of \(f_\beta\). When inserting the forms (4.59) and (4.62) into (4.48), the vanishing of each double pole \((\beta \cdot x)^{-2}\) yields

\[
u_\beta (u_\beta + 1) = \beta \cdot \beta f_\beta u_\beta \Rightarrow u_\beta = 0 \quad \text{or} \quad u_\beta = \beta \cdot \beta f_\beta - 1 \tag{4.65}
\]

\(^{11}\) We disregard here the possibility of ‘radial’ terms, where \(z = \sqrt{\sum_\alpha x_\alpha^2}\) or \(z = \sqrt{\sum_{\alpha < \beta} (x_\alpha - x_\beta)^2}\) [50, 85].

\(^{12}\) For more general solutions involving symmetric polynomials of higher degree, see [39].
for each covector $\beta$. Inserting this into the ‘sum rule’ (4.63) for $u_\beta$, one obtains a second necessary condition for $\{f_\beta\}$, namely
\[
\sum_{\beta} \delta_\beta (\beta^2 f_\beta - 1) = 2\alpha \quad \text{with} \quad \delta_\beta \in [0, 1].
\] (4.66)

It restricts the $F$ solutions to those which may admit a $U$ solution as well. However, by no means does it guarantee that the single-pole terms in (4.48) work out as well. More concretely, in a moduli space of WDVV solutions $F_i$, where $i = (t_1, t_2, \ldots)$ represents continuous moduli parameters, the necessary condition (4.66) for a twisted period $W$ yields a hypersurface of co-dimension 1,
\[
h_\beta(i) = 0 \quad \text{for every yes/no collection} \{\delta_\beta\}.
\] (4.67)

This means that $D(2, 1; \alpha)$ symmetric models can exist only on such hypersurfaces in moduli space, or it can be turned around to fix the value of $\alpha$ for any given WDVV solution $F_i$; in such a way, families of solutions $(F, U)$ were found for deformed root systems of types $A_n, BCD_n$ and $EF_n$, as well as for (a reduction of) the super root system $AB(1, 3)$ [79].

4.2.3. Multi-particle models with $osp(4|2)$ symmetry. For the value $\alpha = -\frac{1}{2}$, the superconformal algebra $D(2, 1; \alpha)$ reduces to $osp(4|2)$. This is a special case, since some formulae of the preceding subsection become singular for $2\alpha+1 = 0$. In particular, $F$ becomes homogeneous of degree 2, implying that $\sum_n f_\beta \beta \otimes \beta = 0$. Thus, the induced metric degenerates, but the scale of $F$ is determined via (4.46). It also means that the covector collection $\{\beta\}$ degenerates to rank $n-1$. Among the known deformed root systems which solve the WDVV equation, there exists a degenerate limit in the moduli space of deformed $A_n$ roots [79]: fix the positive roots of an $A_{n-1}$ subalgebra spanning an $\mathbb{R}^{n-1}$ subspace and project the remaining $n$ positive roots onto this subspace. There, they form the fundamental weights of the $A_{n-1}$ subalgebra. Embedding this system again into $\mathbb{R}^n$, one arrives at the translation and permutation invariant collection
\[
\{\beta \cdot x\} = \left\{x^a - x^b, x^a - \frac{1}{n}X \mid 1 \leq a < b \leq n\right\}.
\] (4.68)

The corresponding prepotentials read [79]
\[
F(x) = \frac{1}{4n} \sum_{a < b} (x^a - x^b)^2 \ln(x^a - x^b)^2 - \frac{1}{4n^2} \sum_a (nx^a - X)^2 \ln(nx^a - X)^2,
\] (4.69)
\[
U(x) = -\frac{1}{2n} \sum_a \ln(nx^a - X)^2.
\] (4.70)

The bosonic potential then becomes
\[
V(x) = \frac{g^2}{8} \left\{ \sum_a \frac{1}{(nx^a - X)^2} - \frac{1}{n} \left( \sum_a \frac{1}{(nx^a - X)} \right)^2 \right\} = \frac{g^2}{8n} \sum_{a < b} \left( \frac{1}{nx^a - X} - \frac{1}{nx^b - X} \right)^2,
\] (4.71)

where only the fundamental weights of $A_{n-1}$ appear. Of course, the center-of-mass motion can still be decoupled.

The simplest nontrivial example occurs for $n = 4$, featuring the six positive roots and four fundamental weights of $A_3$. Due to the isometry $A_3 \simeq D_4$, which maps the fundamental

\[\text{It is now possible to put } F = 0, \text{ but it yields the trivial rank-1 solution } U = -\ln \beta \cdot x \text{ for a single covector } \beta.\]
follow and discuss only of 2 one-particle case above generalizes to the multi-particle case. With 14 The case

\[ A_1 \text{ weights to the fundamental spinor weights of } D_4, \text{ there exists a nice three-dimensional coordinate system after elimination of the center of mass (note that } a, b = 1, 2, 3 \text{ only)} \]:

\[ F(x) = \frac{1}{16} \sum_{a, b, \pm} (x^a \pm x^b)^2 \ln(x^a \pm x^b)^2 - \frac{1}{16} \sum_{a, \pm} (x^1 \pm x^2 \pm x^3)^2 \ln(x^1 \pm x^2 \pm x^3)^2 \]  

(4.72)

\[ U(x) = - \frac{1}{8} \sum_{a, \pm} \ln(x^1 \pm x^2 \pm x^3)^2, \]  

(4.73)

where the sums run over all sign choices indicated. Abbreviating \((x^1, x^2, x^3) = (x, y, z)\), we can write the bosonic potential as

\[ V = \frac{g^2}{512} \frac{x^3(x^2 - y^2 - z^2) + x^2(y^2 - z^2 - x^2) + z^2(z^2 - x^2 - y^2) + 6x^2y^2z^2}{(x + y - z)^2 (x + y + z)^2 (x + y - z)^2 (x - y - z)^2}. \]  

(4.74)

The singular loci of this rational function are best visualized by projecting onto the unit two-sphere \(x^2 + y^2 + z^2 = 1\). On this sphere, the singular lines form the edges of (the celestial projection of) a regular cuboctahedron [64].

### 4.2.4. Multi-particle models with \(su(1, 1|2)\) symmetry.

The other special case occurs at \(\alpha = -1\) or 0, where \(D(2, 1|\alpha)\) becomes a semidirect sum of \(su(1, 1|2)\) with \(su(2)\). Note that these two special values are related by the discrete flip \(\alpha \leftrightarrow -(1+\alpha)\), which flips the sign of \(2\alpha + 1\) and therefore the sign of the WDVV solution \(F\) (see (4.55)). In this subsection, we follow [80] and discuss only \(\alpha = 0\). 14 The possibility of a central charge \(m\) gives us a new option: it perturbs the homogeneity of \(W\) to

\[ x^a \partial_a W = -m \Rightarrow W(x) = -m \ln R + W_0(x) \]  

(4.75)

and a degree-0 homogeneous function \(W_0\). Suddenly, the naïve choice \(U_a(x)^k = W_a(x) \delta_i^k\) works in (4.41), and we can dispense it with the isospin degrees of freedom. As a result one obtains

\[ Q_i = p_i \psi_i^a + iW_a(x)\psi_i^a + iF_{abc}(x)\psi_i^a \psi_i^b \psi_i^c, \]  

(4.76)

\[ H = \frac{1}{2} p_a p_a + \frac{1}{8} W_a W_a(x) - W_{ab}(x)\psi_i^a \psi_i^b - \frac{1}{4} F_{abcde}(x)\psi_i^a \psi_i^b \psi_i^c \psi_i^d \]  

(4.77)

and the bosonic potential \(V = \frac{1}{4} W_a W_a\) depends on \(m\) and possibly other coupling constants.

In the \(su(1, 1|2)\) situation, the standard \(N = 4\) superspace description applied to the one-particle case above generalizes to the multi-particle case. With \(n\) copies \(A^n, a = 1, \ldots, n\), of the \((1, 4, 3)\) multiplet subject to the constraints

(a) \(D' D_a A^a = 0, \) \(\tilde{D}_i D_i A^a = 0\);

(b) \([D', D_i] A^a = 2m^a,\)

(4.78)

one can formulate the action

\[ S_A = -\int \bar{d}x^a \partial^a \partial G(A^a), \]  

(4.79)

where the superpotential \(G(x)\) is subject to the conformality condition

\[ x^a G_a = G = c_a x^a \]  

(4.80)

for some constants \(c_a\). The bosonic part of this action takes the sigma-model form

\[ S_B = \frac{1}{2} \int d^4 \bar{x} \partial^a \partial^b G_{ab}(x) \]  

(4.81)

The case \(\alpha = -1\) is obtained via a ‘duality’, interchanging \(w_i\) and \(w_{\alpha}\) in [80] and giving \(W = R^2 (W_0 - m \ln R)\).
To serve as an action for \(n\) particles on a line, the target-space metric \(G_{ab}\) must be flat. This yields an integrability condition, namely the vanishing of the Riemann tensor,

\[
G_{abc} G^d_{fcd} - G_{ace} G^d_{fbd} = 0 \quad \text{with} \quad G^{ij} G_{ij} = \delta^g_{g}.
\]  

(4.82)

If this is met, there exists an ‘inertial’ coordinate system in which the kinetic term takes the standard form \(\frac{1}{2} \delta_{ab} \dot{x}^a \dot{x}^b\), and the superpotential produces both prepotentials \(F\) and \(W\) [80]. Converting into the ‘inertial’ coordinates, the condition (4.82) is in fact equivalent to

\[
F_{abc} F_{cd} - F_{ace} F_{bd} = 0 \quad \text{and} \quad W_{ab} - F_{abc} W_c = 0.
\]

(4.83)

In these coordinates, the conformality condition (4.80) becomes

\[
x^a G_{a} - 2G = -\frac{1}{2} x^a x^a,
\]

(4.84)

which yields the homogeneity relations

\[
x^a W_{a} = -m \quad \text{and} \quad x^a F_{abc} = -\delta_{bc} \Rightarrow (x^a \theta_a - 2)F = -\frac{1}{2} x^a x^a,
\]

(4.85)

and the central charge comes out as \(m = -2c_n m^I\).

Unfortunately, it is very difficult to solve (4.83) together with (4.85) for an arbitrary number \(n\) of particles\(^{15}\). At \(n = 3\), the general solution depends on one free function and can be given. For \(n = 4\), only sporadic solutions, mostly involving hypergeometric functions, have been found [80].

4.3. Superconformal models from harmonic \(\mathcal{N} = 4\) superspace

In the previous subsections, we dealt with the formulations of the one- and many-body \(\mathcal{N} = 4\) superconformal Calogero models in the ordinary \(\mathcal{N} = 4\) superspace and in the component approach. Here we discuss which new possibilities for construction of such models are provided by harmonic \(\mathcal{N} = 4, d = 1\) superspace [72].\(^{16}\) Sometimes we will use the harmonic superfields in parallel with the ordinary \(\mathcal{N} = 4\) ones, hoping that this will not give rise to misunderstanding.

4.3.1. Harmonic description of the \(\mathcal{N} = 4\) representations.

Off-shell \(\mathcal{N} = 4, d = 1\) supermultiplets admit a concise formulation in the harmonic superspace (HSS) [72], an extension of (4.20) by the harmonic coordinates \(u_\alpha^i\):

\[
(t, \theta^{\pm}, \bar{\theta}^{\pm}, u^{\pm}_i), \quad \theta^{\pm} = \theta^i u^\pm_i, \quad \bar{\theta}^{\pm} = \bar{\theta}^i u^\pm_i, \quad u^i {u^i} = 1.
\]

(4.86)

The commuting SU(2) spinors \(u^\pm_i\) parametrize the 2-sphere \(S^2 \sim SU(2)_R/U(1)_R\). The salient property of HSS is the presence of an important subspace in it, the harmonic analytic superspace (ASS) with half of Grassmann co-ordinates as compared to (4.20) or (4.86):

\[
(\zeta, \mu) = (t_\alpha, \theta^{+}, \bar{\theta}^{-}, u^{+}_i), \quad t_\alpha = t + i(\theta^+ \bar{\theta}^- - \theta^- \bar{\theta}^+).
\]

(4.87)

It is closed under the \(\mathcal{N} = 4\) supersymmetry transformations. Most of the off-shell \(\mathcal{N} = 4, d = 1\) multiplets are represented by the analytic superfields, i.e. those ‘living’ on the subspace (4.87).

Spinor covariant derivatives in the analytic basis of HSS, namely \((\zeta, \mu, \theta^-, \bar{\theta}^-)\), take the form

\[
D^+ = \frac{\partial}{\partial \theta^-}, \quad \bar{D}^+ = -\frac{\partial}{\partial \bar{\theta}^-}, \quad D^- = -\frac{\partial}{\partial \theta^+} - 2i \bar{\theta}^- \partial_{\alpha}, \quad \bar{D}^- = \frac{\partial}{\partial \bar{\theta}^+} - 2i \theta^- \partial_{\alpha}.
\]

(4.88)

\(^{15}\) One cannot simply map the \((F, U)\) solutions of subsection 4.2.2 to \((F, e^{-U})\) since in \(Q\) and \(H\) there appears \(W\) instead of \(U\). Hence, the \(su(1,1/2)\) system without isospin variables differs essentially from the \(\alpha \to 0\) or \(\alpha \to -1\) limit of the isospin system of subsection 4.2.2.

\(^{16}\) For a description of \(\mathcal{N} = 4, d = 1\) superconformal models in bi-harmonic superspace, see [74].
In the central basis (4.86), the same derivatives are defined as the projections \( D^\pm = \mu_\pm \), and harmonic covariant derivatives in the analytic basis read

\[
D^{\pm \pm} = \partial^{\pm \pm} + 2i\bar{\theta}^\pm \frac{\partial}{\partial \theta^\pm} + \theta^\pm \frac{\partial}{\partial \bar{\theta}^\pm}.
\]

(4.89)

The integration measures are defined by

\[
\mu_H = du d\theta d^2\bar{\theta} = \mu_A^{(-2)}(D^+ \bar{D}^+) , \quad \mu_A^{(-2)} = du d\theta d\bar{\theta}^+ d\bar{\theta}^- = du d\bar{\theta}^+(D^- \bar{D}^-).
\]

(4.90)

Let us briefly sketch the harmonic superspace description of some \( \mathcal{N} = 4, d = 1 \) supermultiplets (for details, see [72, 29, 30]).

The \( \mathcal{N} = 4, d = 1 \) supermultiplets are described in harmonic superspace by the harmonic superfields \( \Phi^{(q)}(t, \theta^\pm, \bar{\theta}^\pm, u) \) with \( U(1) \) charge \( q \) reflecting local \( U(1) \) symmetry of the harmonic formulation

\[
D^0 \Phi^{(q)} = q \Phi^{(q)},
\]

(4.91)

where

\[
D^0 = \partial^0 + \theta^+ \frac{\partial}{\partial \theta^+} + \bar{\theta}^- \frac{\partial}{\partial \bar{\theta}^-} - \theta^- \frac{\partial}{\partial \theta^-} - \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^+}.
\]

(4.92)

Analytic superfields \( \Phi^{(q)}(\xi, u) \) depending on the supercoordinates (4.87) are defined by the Grassmann Cauchy–Riemann constraints

\[
D^+ \Phi^{(q)} = \bar{D}^+ \Phi^{(q)} = 0.
\]

(4.93)

Analytic superfields can satisfy generalized reality conditions which yields usual reality conditions for component fields. Generalized conjugation of the harmonic superfields, which is the combination of complex conjugation and antipodal reflection on harmonic two-sphere, is denoted by a tilde: \( \Phi^{(q)} \rightarrow \tilde{\Phi}^{(q)} \) (for details, see [54, 72]). Below we briefly present the superconformal description of the basic \( \mathcal{N} = 4, d = 1 \) supermultiplets.

The \((1,4,3)\) supermultiplet. The formulation of this supermultiplet in the standard \( \mathcal{N} = 4 \) superspace was given in section 4.2.1. In HSS, it is described by the even real harmonic superfield \( X(t, \theta^\pm, \bar{\theta}^\pm, u) \) which has a zero harmonic charge and is subjected to the constraints

\[
D^{+ +} X = 0,
\]

(4.94)

\[
D^+ D^- X = 0, \quad \bar{D}^+ \bar{D}^- X = 0, \quad (D^+ \bar{D}^- + \bar{D}^+ D^-) X = 0.
\]

(4.95)

The set of conditions (4.94) and (4.95) is equivalent to the standard constraints (4.22) in the central basis (4.86): equation (4.94) implies independence of the superfield \( X \) on harmonic variables, i.e. \( X = X(t, \theta^i, \bar{\theta}^i) \), and then equations (4.95) are reduced to the constraints (4.22).

There is another equivalent description of the same \((1, 4, 3)\) supermultiplet. It makes use of the real analytic gauge superfield \( \mathcal{V}(\xi, u) \), \( D^+ \mathcal{V} = \bar{D}^+ \mathcal{V} = 0 \), which is defined up to the Abelian gauge freedom

\[
\delta \mathcal{V} = D^{++} \lambda^{--}, \quad \lambda^{--} = \lambda^{--}(\xi, u).
\]

(4.96)

The analytic superfield \( \mathcal{V} = \mathcal{V}(\xi, u) \) plays the role of the prepotential for the \((1, 4, 3)\) multiplet and is related to the superfield \( X(t, \theta^i, \bar{\theta}^i) \) by the harmonic integral transform [29]

\[
X(t, \theta^i, \bar{\theta}^i) = \int du \mathcal{V}(t, \theta^+, \bar{\theta}^+, u) \big|_{\theta^i = \theta^i u^i, \bar{\theta}^i = \bar{\theta}^i u^i}.
\]

(4.97)

The prepotential representation (4.97) automatically solves the basic constraints (4.22).
The superconformal action for the (1, 4, 3) multiplet is given by the formula (4.23). As shown below, the manifestly analytic prepotential formulation of this multiplet allows one to construct new $\mathcal{N} = 4$ superconformally invariant models in which the (1, 4, 3) multiplet is coupled to other $\mathcal{N} = 4$ supermultiplets.

_The (3,4,1) tensor supermultiplet._ This supermultiplet is described by the even analytic gauge superfield $L^{++}(\xi, u)$, which satisfies the generalized reality condition, $L^{++} = L^{++}$, and is subjected to the additional harmonic constraint [72]

$$D^{++}L^{++} = 0.$$  \hfill (4.98)

The constraints (4.98) can be directly solved. The off-shell component content of the tensor multiplet is formed by the fields $v_{ij} = v_{ij}, B, \psi_i$ and $\tilde{\psi}_i$. They enter the $\theta$-expansion of the superfield $L^{++}$ subjected to (4.98) as [72]

$$L^{++} = v^{++} + \theta^+ \psi^+ + \bar{\theta}^+ \bar{\psi}^+ + 2i \theta^+ \bar{\theta}^-(\bar{\psi}^+ + B),$$  \hfill (4.99)

where $v^{++} = v^{ij}u^+_iu^+_j$, $v^{++} = v^{ij}u^+_iu^+_j$, $\psi^+ = \psi^iu^+_i$ and $\bar{\psi}^+ = \bar{\psi}^+u^+_i$ and the SU(2)-triplet $v^i$ describes three bosonic physical degrees of freedom.

In central basis, the superfield $L^{++}$ is represented in the form

$$L^{++} = u^+_i u^+_k L^{ik},$$  \hfill (4.100)

where $L^{ik} (t, \theta, \bar{\theta})$ is the usual superfield subjected to the constraints

$$D^{i}(L^{kl}) = 0, \quad \bar{D}^{i}(L^{kl}) = 0.$$  \hfill (4.101)

Sigma-model-type actions for $L^{++}$ are written as an integral over the whole harmonic superspace of the Lagrangian which is a function of $L^{++}$, $L^{+-} = \frac{1}{2} D^{-+} L^{++}$, $L^{-+} = \frac{1}{2} D^{-+} L^{++}$ and harmonics (see the details in [72]). The $\mathcal{N} = 4$ superconformal subclass of these actions have the form (4.23) in which we must make the substitution (see the details in [72, 67, 68, 29, 30])

$$\chi \to L^{-1}, \quad \text{where} \quad L := \sqrt{L^{ik}L^{ik}} = \sqrt{2[L^{++} L^{-+} - (L^{+-})^2]}.$$  \hfill (4.102)

The WZ action, generating the superpotential term for $L^{++}$, is given by the following integral over the analytic superspace:

$$S_{WZ} = \frac{i}{2} \gamma \int \mu^{(2)}_A L^{++} (L^{++}, u) = \gamma \int \text{d}t \left( \frac{1}{2} A_{ik} \partial^{ik} + \frac{i}{2} R_{ik} \bar{\psi}^{(i}( \bar{\psi}^{k)} + uB \right),$$  \hfill (4.103)

where

$$A_{ik} = 2 \int \text{d}u_+ u^+_i u^+_k \frac{\partial L^{++}}{\partial v^{++}}, \quad R_{ik} = \int \text{d}u_+ u^+_i u^+_k \frac{\partial^2 L^{++}}{\partial (v^{++})^2}, \quad U = \int \text{d}u_+ \frac{\partial L^{++}}{\partial v^{++}}.$$  \hfill (4.104)

From the definition of these potentials follow the relations between them:

$$\Delta_{\mathbb{R}} U = 0, \quad \Delta_{\mathbb{R}} A_{ik} = 0, \quad \frac{\partial}{\partial v^{ik}} A_{ik} = 0,$$  \hfill (4.105)

$$\bar{\partial}_{ik} A_{ik} - \bar{\partial}_{ki} A_{ik} = (\epsilon_{ik} \bar{\partial}_{i} + \epsilon_{ki} \bar{\partial}_{k}) U,$$  \hfill (4.106)

$$\bar{R}_{ik} = \bar{\partial}_{ik} U.$$  \hfill (4.107)

Here, $\bar{\partial}_{ik} = \frac{\partial}{\partial v^{ik}}$ and $\Delta_{\mathbb{R}} = \partial^2 / \partial v^{ik}$ is the Laplace operator on $\mathbb{R}^3$. Equations (4.105) and (4.106) are recognized as the equations defining the monopole (static) solution for a self-dual Maxwell or gravitation fields in $\mathbb{R}^3$. The new striking feature of the $\mathcal{N} = 4$ mechanics models associated with the multiplet (3, 4, 1) as compared with those based on the multiplet (1, 4, 3)
is just the appearance of coupling of the SU(2) vector physical bosonic component to the external magnetic 3-potential $A_3$ in (4.103).

The superconformal action of $L^{++}$ was constructed in [72] as the harmonic superspace reformulation of the model previously constructed in [67] (which used the ordinary $N = 4$ superspace). It consists of the sigma model part and the WZ part, each being superconformal separately (with respect to the most general $N = 4$ superconformal symmetry $D(2, 1; \alpha)$). The unique superconformal WZ term corresponds to the one-monopole potential $U$ in (4.103). The explicit expression for the superconformal superfield WZ term is given in the next subsection.

In components, after elimination of the auxiliary field $B$ from both the sigma model and WZ parts, there emerges the conformal potential for $v^{ik}$ and the SU(2)/U(1) WZ term, with the strengths specified by the coefficient before the superfield WZ term and so related to each other. Thus, in this case, a sort of three-dimensional superconformal mechanics arises. The full bosonic action, in the parametrization in which $v^{ik}$ is split into the radial and angular parts, is given by the expression (2.71) with $g = (2\alpha)^{-2}$.

The $(4,4,0)$ ‘root’ supermultiplet. From this multiplet all others may be obtained via a reduction process on either the component-action [17] or the superfield-action [29–31] levels. It is described by the harmonic charge-one complex analytic superfields $\tilde{Z}^+, \bar{\tilde{Z}}^+$, subjected to the constraint

$$D^{++}\tilde{Z}^+ = 0.$$ 

(4.108)

The solution of the constraint (4.108) is as follows:

$$\tilde{Z}^+ = \frac{1}{2}u^a_t + \theta^+ \phi - 2i \theta^+ \bar{\theta}^+ \partial_u u^+,$$

(4.109)

where the SU(2)-dublet $u^a_t(t)$ describes four bosonic physical degrees of freedom. We can combine the superfields $\tilde{Z}^+$ and $\bar{\tilde{Z}}^+$ into a doublet of some extra (‘Pauli–Gürsey’) SU(2)$_{PG}$ group according to

$$q^{+a} := (\tilde{Z}^+, \bar{Z}^+), \quad a = 1, 2.$$ 

(4.110)

In the central basis, the ‘root’ supermultiplet is represented in the form

$$q^{+a} = u^a_t q^u,$$

(4.111)

where $q^u(t, \theta, \bar{\theta})$ is the ordinary $N = 4$ superfield subjected to the constraints

$$D^i q^{+a} = 0, \quad \bar{D}^i q^{+a} = 0.$$ 

(4.112)

Sigma-model action of the ‘root’ supermultiplet can be written as an integral over the whole harmonic superspace of a function of $q^{+a}$, $q^{-a} = D^a q^{+a}$, and harmonics. Taking into account that properties of the product $a_{(cb)} q^{+c} q^{+b}$, where $a_{(cb)}$ are some constants, are similar to the tensor superfield $L^{++}$, it is natural to construct the $N = 4$ superconformal sigma-action for the multiplet $(4, 4, 0)$ in the same way as for the $(3, 4, 1)$ supermultiplet (4.102). More precisely, $N = 4$ superconformal actions for the $(4, 4, 0)$ multiplet have the form (4.23) in which one should make the substitution

$$X \rightarrow q^2, \quad \text{where} \quad q^2 := q^u q_{ua} = 2q^{-a} q^+_a.$$ 

(4.113)

More details on the structure of superconformal sigma-model actions for the $(4, 4, 0)$ multiplet can be found in [72, 68].
one cannot construct the $\mathcal{N} = 4$ superconformal mechanics model with the conformal potential for the physical bosonic fields, as distinct from the case of the multiplet $(3, 4, 1)$. Some ways of gaining conformal potential for the multiplet $(4, 4, 0)$, through its superconformal couplings to some other $\mathcal{N} = 4$ multiplets, are discussed in [29, 30]. In the following section, we will consider a modification of the $(4, 4, 0)$ WZ term, such that it involves coupling to the $(1, 4, 3)$ prepotential $V(\zeta, u)$. This modification will allow us to construct a new $\mathcal{N} = 4$ superconformal mechanics with a new mechanism of generating a conformal potential for the field $x(t)$.

Finally, we point out that the WZ potential terms for all $\mathcal{N} = 4$ supermultiplets can be represented in manifestly $\mathcal{N} = 4$ superfield form only in harmonic superspace but not in the usual superspace\(^{17}\). Such terms are important for the description of the isospin supermultiplets recently introduced in the construction of new integrable supersymmetric systems, including superconformal ones. Also note that for each $\mathcal{N} = 4$ multiplet listed above, there exists a ‘mirror’ (or ‘twisted’) counterpart [68] for which the roles of the manifest $R$-symmetry group $SU(2)_R$ (acting on the doublet indices $i, k$ of the component fields and Grassmann coordinates) and a hidden $R$-symmetry group $SU(2)$ (joining, e.g., $\theta_i$ and $\bar{\theta}_i$ into doublets) are interchanged. When these two types of $\mathcal{N} = 4$ multiplets are considered together, new models of supersymmetric (and superconformal) mechanics can be constructed (see, e.g., [74]). Not too much is known to date about such models, so we will not discuss them here.

4.3.2. Single-particle models with isospin degrees of freedom. Here we present a non-trivial model of $\mathcal{N} = 4$ superconformal mechanics in which the conformal potential is generated by the gauging method by making use of the ‘semi-dynamical’ $\mathcal{N} = 4$ isospin supermultiplet. This model is the one-particle case of the Calogero-like multi-particle model [36] which will be considered in the next section.

The model is built on superfields corresponding to three off-shell $\mathcal{N} = 4$ supermultiplets: (i) the ‘radial’ multiplet $(1, 4, 3)$; (ii) the Wess–Zumino (‘isospin’) multiplet $(4, 4, 0)$; and (iii) the gauge (‘topological’) multiplet $(0, 0, 0)$. The action is a sum of three terms

$$ S = S_X + S_{FI} + S_{WZ}. \tag{4.114} $$

The first term in (4.114) is the standard free action (4.23) of the $(1, 4, 3)$ multiplet. The second term in (4.114) is the Fayet–Iliopoulos (FI) term

$$ S_{FI} = -\frac{i}{2} c \int \mu_A^{(-2)} V^{++} \tag{4.115} $$

for the gauge supermultiplet. The even analytic gauge superfield $V^{++}(\zeta, u), D^+ V^{++} = 0, \bar{D}^+ V^{++} = 0$, is subjected to the gauge transformations

$$ V^{++'} = V^{++} - D^+ \lambda, \quad \lambda = \lambda(\zeta, u), \tag{4.116} $$

which are capable to gauge away, locally, all the components from $V^{++}$. However, the latter contains a component which cannot be gauged away globally. This is the reason why this $d = 1$ supermultiplet was called ‘topological’ in [29].

The last term in (4.114) is the Wess–Zumino (WZ) term

$$ S_{WZ} = -\frac{1}{2} \int \mu_A^{(-2)} \mathcal{V} \bar{Z}^+ Z^+. \tag{4.117} $$

\(^{17}\)In the formulations in ordinary $\mathcal{N} = 4$ superspace, such terms either include manifest $\theta$s [67] or are expressed through appropriate unconstrained prepotentials with a complicated pregauge freedom.
Here, the complex analytic superfield $Z^+$, $\tilde{Z}^+$ ($D^+ Z^+ = \tilde{D}^+ Z^+ = 0$), is subjected to the harmonic constraints

$$D^{++} Z^+ \equiv (D^+ + i V^+) Z^+ = 0, \quad \tilde{D}^{++} \tilde{Z}^+ \equiv (\tilde{D}^- + i V^-) \tilde{Z}^+ = 0 \quad (4.118)$$

and represents a gauge-covariantized version of the $\mathcal{N} = 4$ multiplet $(4, 4, 0)$. The relevant gauge transformations are

$$Z^{+} = e^{i\lambda} Z^+, \quad \tilde{Z}^{+} = e^{-i\lambda} \tilde{Z}^+. \quad (4.119)$$

The superfield $V(\zeta, u)$ in (4.117) is a prepotential of the $(1, 4, 3)$ superfield $\mathcal{X}$. The coupling to the multiplet $(1, 4, 3)$ in (4.117) is introduced to ensure $\mathcal{N} = 4$ superconformal $D(2, 1; \alpha)$ invariance. The coordinate realization of the superconformal boosts of $D(2, 1; \alpha)$ in analytic subspace is given by [72, 29]

$$\delta t_A = \alpha^{-1} \Lambda t_A, \quad \delta \theta^+ = -\eta^+ t_A + 2i(1+\alpha)\eta^- \bar{\theta}^+, \quad \delta u^+_i = \Lambda^{++} u^-_i, \quad (4.120)$$

where

$$\Lambda = 2i\alpha (\bar{\eta}^- \bar{\theta}^+ - \eta^- \bar{\theta}^+), \quad \Lambda^{++} = D^{++} \Lambda = 2i\alpha (\bar{\eta}^- \bar{\theta}^+ - \eta^- \bar{\theta}^+). \quad (4.122)$$

Based on the results of [72], it is easy to find the appropriate transformation laws of all involved superfields

$$\delta \mathcal{V} = -2\Lambda \mathcal{V}, \quad \delta' \mathcal{V}^+ = \Lambda \mathcal{V}^+, \quad \delta' \mathcal{V}^{++} = 0. \quad (4.123)$$

The invariance of the action (4.114) and harmonic constraints under these transformations can be easily checked. Note that the constraints (4.94), (4.95) and (4.118), as well as the actions (4.115) and (4.117), are invariant with respect to the $D(2, 1; \alpha)$ transformations with an arbitrary $\alpha$. It is worth to point out that the action (4.117) is superconformally invariant just due to the presence of the analytic prepotential $\mathcal{V}$.

In order to clarify the off-shell superfield content of our model, it is instructive to fix the underlying $U(1)$ gauge freedom by choosing a gauge which preserves manifest $\mathcal{N} = 4$ supersymmetry. A gauge suitable for our purpose was used in [30]. To make contact with the consideration in [30], we introduce the $SU(2)_{PG}$ spinor field by (4.110) and rewrite the transformation law (4.119) and the constraints (4.118) as

$$\delta q^{+a} = \lambda c^a_{\dot{b}} q^{+\dot{b}}, \quad D^{++} q^{+a} + V^{++} c^a_{\dot{b}} q^{+\dot{b}} = 0. \quad (4.124)$$

Here, the traceless constant tensor $c^a_{\dot{b}}$ breaks $SU(2)_{PG}$ down to $U(1)$ which is just the symmetry to be gauged. In the frame where the only non-zero entries of $c^a_{\dot{b}}$ are $c^1_{\dot{2}} = -c^2_{\dot{2}} = -i$, we recover the transformation law (4.119) and the constraints (4.118). It is easy to show that

$$\hat{Z}^+ Z^+ = -\frac{i}{2} q^{+a} c_{ab} q^{+b}. \quad (4.125)$$

In [29] (following [54]) an invertible equivalence redefinition of $q^{+a} \Rightarrow (\omega, L^{++})$ has been used, such that the $U(1)$ gauge transformation in (4.124) is realized as

$$\delta \omega = -2\lambda, \quad \delta L^{++} = 0. \quad (4.126)$$

One can fully fix the $U(1)$ gauge freedom by imposing the manifestly $\mathcal{N} = 4$ supersymmetric gauge

$$\omega = 0. \quad (4.127)$$
In this gauge, the harmonic constraint in (4.124) amounts to the following relations:

(a) \[ q^{a'}c_{ab}q^{b'} = 4(c^{++} + L^{++}), \]
(b) \[ V^{++} = \frac{L^{++}}{(1 + \sqrt{1 + c^{-}L^{++}})\sqrt{1 + c^{-}L^{++}}} \]
(c) \[ D^{++}(c^{++} + L^{++}) = D^{++}L^{++} = 0, \] (4.128)

where \( c^{±±} = c^{(ab)}u^+_a u^+_b \). After substituting the expressions (4.128(a)) and (4.128(b)) into (117) and (115), the total superfield action (114) takes the form

\[ S = -\frac{1}{4(1+\alpha)}\int \mu H X^{-1/\alpha} + i \int \mu^{(-2)} \]
\[ \times \left[ \mathcal{V}(c^{++} + L^{++}) - \frac{c}{2} \frac{L^{++}}{(1 + \sqrt{1 + c^{-}L^{++}})\sqrt{1 + c^{-}L^{++}}} \right]. \] (4.129)

As we already mentioned, the superfield \( L^{++} \) with the constraint (4.128(c)) accommodates an off-shell \( \mathcal{N} = 4 \) multiplet \((3, 4, 1)\) [72]. So the action (4.129) describes a system of two interacting off-shell \( \mathcal{N} = 4, d = 1 \) multiplets: \((1, 4, 3)\) described by the superfield \( \mathcal{X} \) and \((3, 4, 1)\) described by the analytic superfield \( L^{++} \). This is the off-shell content of our \( D(2, 1; \alpha) \) model.

As distinct from the superconformal mechanics based on a single \((3, 4, 1)\) multiplet the action of which is a sum of the sigma-model-type term and superconformal WZ term of \( L^{++} \) [67, 72], the action (4.129) involves only the superconformal superfield WZ term of this multiplet (the last term in the square brackets). The interaction with the multiplet \((1, 4, 3)\) is introduced through a superconformal bilinear coupling of both multiplets (the first term in the square brackets). Note that due to the absence of the kinetic term for \( L^{++} \) in (4.129), the on-shell content of the model appears to be drastically different from the off-shell one: the eventual component action contains only three bosonic fields and four fermionic fields, which are combined into some new on-shell \((3, 4, 1)\) multiplet (see the following section).

Now we consider the model in the WZ gauge. Using the U(1) gauge freedom (4.116), (4.119), we can choose the WZ gauge

\[ V^{++} = 2i \theta^+ \bar{\theta}^+ A(t_a). \] (4.130)

Then, using the component expansion for the prepotential superfield \( \mathcal{V} \),

\[ \mathcal{V}(t_a, \theta^+, \bar{\theta}^+, u^k) = x(t_a) - 2 \theta^+ \bar{\psi}(t_a)u_i^- - 2 \bar{\theta}^+ \bar{\psi}(t_a)u_i^- + 3 \theta^+ \bar{\theta}^+ A^{kl}(t_a)u_i^- u_k^- \] (4.131)

and the solution of the constraint (4.118) in the WZ gauge (4.130)

\[ Z^+ = z^i u_i^+ + \theta^+ \bar{\psi} + \bar{\theta}^+ \psi - 2i \theta^+ \bar{\theta}^+ \nabla_i z^i u_i^- \]
\[ \nabla z^k := \tilde{z}^k + iA z^k, \] (4.132)

as well as eliminating auxiliary fields and making the redefinition

\[ x' = x^{1/\alpha}, \quad \psi' = -\frac{1}{2\alpha} x^{1/\alpha - 1} \psi, \quad \tilde{z} = x^{1/2} \tilde{z}, \] (4.133)

we arrive at the on-shell form of the action (4.114) in the WZ gauge (we omitted the primes on \( x, \psi \) and \( \tilde{z} \))\(^{18}\)

\[ S = \int dt \left[ p \dot{x} + i(\bar{\psi} \dot{\psi} - \psi \dot{\bar{\psi}}) + \frac{1}{2} (\tilde{z} \dot{\tilde{z}} - \tilde{\tilde{z}} \tilde{z}) - H \right]. \] (4.134)

The Hamiltonian \( H \) is

\[ H = \frac{1}{4} \rho^2 + \alpha^2 \left( \frac{\tilde{z} \dot{\tilde{z}}}{4x^2} - 2\alpha \frac{\psi \dot{\bar{\psi}} z^i \tilde{z}_i}{x^2} - (1 + 2\alpha) \frac{\psi \dot{\psi} + \bar{\psi} \dot{\bar{\psi}}}{2x^2} \right). \] (4.135)

\(^{18}\) For obtaining such systems via Hamiltonian reduction, see [78].
The field $A(t)$, playing the role of $d = 1$ $U(1)$-connection, is the Lagrange multiplier for the first-class constraint

$$D^0 - c \equiv \bar{z}_k z^k - c \approx 0,$$  \hspace{1cm} (4.136)

which should be imposed on the wavefunctions in the quantum case.

Quantum operators of physical coordinates and momenta satisfy the quantum brackets, obtained in the standard way from Dirac brackets. The variables of the model (4.134) satisfy the following quantum algebra:

$$[\hat{x}, \hat{p}] = i, \quad [\hat{z}^i, \hat{z}^j] = \delta^i_j, \quad \{\hat{\psi}^i, \hat{\psi}^j\} = \frac{1}{2} \delta^i_j.$$  \hspace{1cm} (4.137)

Quantum supertranslation and superconformal boost generators defined by the corresponding classical expressions are

$$Q^i = \hat{p} \hat{\psi}^i + 2i\alpha z^i(\hat{z}_k \hat{\psi}_k + i(1+2\alpha)\hat{\psi}_k \hat{\bar{\psi}}_k),$$  \hspace{1cm} (4.138)

$$\hat{Q} = \hat{p} \hat{\psi}^i - 2i\alpha z^i(\hat{z}_k \hat{\psi}_k + i(1+2\alpha)\hat{\psi}_k \hat{\bar{\psi}}_k),$$  \hspace{1cm} (4.139)

$$S^i = -2 \hat{x} \hat{\psi}^i + tQ^i, \quad \hat{S}_i = -2 \hat{x} \hat{\psi}_i + \hat{Q}_i,$$  \hspace{1cm} (4.140)

where the symbol $\{\ldots\}$ means Weyl ordering. Evaluating the anticommutators of the odd generators (4.138), (4.140), one determines uniquely the full set of quantum generators of the superconformal algebra $D(2, 1; \alpha)$. We obtain

$$H = \frac{1}{4} \hat{p}^2 + \alpha^2 \frac{(\hat{z}_k \hat{z}^k)^2 + 2\hat{z}_k \hat{z}^k}{4\hat{x}^2} - 2\alpha^2 \frac{z^i(\hat{z}_k \hat{\psi}_k + i(1+2\alpha)\hat{\psi}_k \hat{\bar{\psi}}_k)}{\hat{x}^2} - (1+2\alpha)^2 \frac{\hat{\psi}_k \hat{\bar{\psi}}_k \hat{\bar{\psi}}_k}{2\hat{x}^2} + \frac{(1+2\alpha)^2}{16\hat{x}^2},$$  \hspace{1cm} (4.141)

$$K = \hat{x}^2 - \frac{1}{2} [\hat{x}, \hat{p}] + t^2 H,$$  \hspace{1cm} (4.142)

$$D = -\frac{1}{4} [\hat{x}, \hat{p}] + t H,$$  \hspace{1cm} (4.143)

$$J^k = i[\hat{x}, \hat{z}^k],$$  \hspace{1cm} (4.144)

$$I^{1V} = i \hat{\psi}_k \hat{\bar{\psi}}^k, \quad I^{2V}_1 = -i \hat{\psi}_k \hat{\bar{\psi}}^k, \quad I^{1V}_2 = \frac{i}{2} [\hat{\psi}_k, \hat{\bar{\psi}}^k].$$  \hspace{1cm} (4.145)

It can be directly checked that the generators (4.138)–(4.145) indeed obey the (anti)commutation relations of the $D(2, 1; \alpha)$ superalgebra (4.2)–(4.6).

For the realization (4.138)–(4.145), the second-order Casimir operator (4.17) of $D(2, 1; \alpha)$ is given by the following expression:

$$C_2 = \frac{1}{4} \alpha (1+\alpha)[(\hat{z}_k \hat{z}^k)^2 + 2\hat{z}_k \hat{z}^k + 1].$$  \hspace{1cm} (4.146)

Thus, on the physical wavefunction which is subjected to the constraints (4.136)

$$D^0 \Phi = \hat{z}_k \hat{\bar{\Phi}}^k = c \Phi$$  \hspace{1cm} (4.147)

we use the normal ordering for the even SU(2)-spinor operators, with all operators $Z'$ standing on the right), the Casimir (4.146) takes a fixed value.

The Hamiltonian (4.141) and the SL(2, $R$) Casimir operator (4.18) can be represented in the quantum case as

$$H = \frac{1}{4} \left( \hat{p}^2 + \frac{\hat{g}}{\hat{x}^2} \right),$$  \hspace{1cm} (4.148)

$$T^2 = \frac{1}{4} \hat{g} - \frac{3}{16},$$  \hspace{1cm} (4.149)
The operators (4.148) and (4.149) formally appear as those given in the model of [5]. However, there is an essential difference. Whereas the quantity $\hat{g}$ is a constant in the model of [5], in our case $\hat{g}$ is an operator which takes fixed, but different, constant values on different components of the full wavefunction.

To find the quantum spectrum of (4.148) and (4.149), we make use of the realization

$$\hat{z}_i = v_i^+, \quad \hat{z}' = \partial/\partial v_i^+$$

(4.151) for the bosonic operators $Z$ and $\bar{Z}$, as well as the following realization of the odd operators $\Psi^i, \bar{\Psi}_j$:

$$\hat{\psi}^i = \psi^i, \quad \hat{\bar{\psi}}_j = \bar{\psi}_j = \frac{1}{i} \partial/\partial \psi^i,$$

(4.152) where $\psi^i$ are complex Grassmann variables. Then, the wave function is defined as

$$\Phi = A_1 + \psi^i B_i + \psi^j \bar{\psi}_j A_2.$$

(4.153)

Like in the bosonic limit considered in section 2, requiring the wavefunction $\Phi(v^+)$ to be single-valued results in the condition that the constant $c$ is an integer, $c \in \mathbb{Z}$. We take $c$ to be positive in order to ensure a correspondence with the bosonic limit where $c$ becomes SU(2) spin. Then, (4.147) tells us that the wavefunction $\Phi(v^+)$ is a homogeneous polynomial in $v_i^+$ of degree $c$:

$$\Phi = A_1^{(c)} + \psi^i B_i^{(c)} + \psi^j \bar{\psi}_j A_2^{(c)}.$$

(4.154)

$$A_1^{(c)} = A_{i_1 \ldots i_c} v^{i_1} \ldots v^{i_c},$$

(4.155)

$$B_i^{(c)} = B_i^{(c)} + B_i^{(c)} = v_i^+ B_i^{(c)} v^{i_k} \ldots v^{i_{k-1}} + B_i^{(c)} v^{i_k} \ldots v^{i_{k-1}}.$$  (4.156)

In (4.156) we singled out the SU(2) irreducible parts $B_i^{(c)}$ and $B_i^{(c)}$ of the component wavefunctions, with the SU(2) spins $(c - 1)/2$ and $(c + 1)/2$, respectively.

On the same states, the Casimir operators (4.18) of the bosonic subgroups SU(1, 1), SU(2)$_R$ and SU(2)$_L$ take the values given in table 1.

For different component wavefunctions, the quantum numbers $r_0$, $j$ and $i$, defined by

$$T^2 = r_0(r_0 - 1), \quad J^2 = j(j + 1), \quad \hat{F} = i(i + 1),$$

(4.157) take the values listed in table 2.

The fields $B_i^R$ and $B_i^L$ form doublets of SU(2)$_R$ generated by $J^R$, whereas the component fields $A_i = (A_1, A_2)$ form a doublet of SU(2)$_L$ generated by $F$. If the super-wavefunction (4.153) is bosonic (fermionic), the fields $A_i$ describe bosons (fermions), whereas the fields $B_i^R$, $B_i^L$ present fermions (bosons).
Each of the component wavefunctions $A^\alpha(x, v^+)$, $B^\alpha(x, v^+)$, $B''^\alpha(x, v^+)$ carries an infinite-dimensional unitary representation of the discrete series of the universal covering group of the one-dimensional conformal group SU(1,1). Such representations are characterized by positive numbers $r_0$ [8, 97] (for the unitary representations of SU(1,1) the constant $r_0 > 0$ must be a (half) integer). Recall that the basis functions of these representations are eigenvectors of the compact SU(1,1) generator
\begin{equation}
T_0 = \frac{1}{2} (mK + m^{-1}H),
\end{equation}
where $m$ is a constant of the mass dimension (see the definition in (2.2)). The corresponding eigenvalues are $r = r_0 + n, n \in \mathbb{N}$ [8, 97, 5].

Let us dwell on some peculiar features of the $D(2, 1; \alpha)$ quantum mechanics constructed.

- As opposed to the standard SU(1,1)[2] superconformal mechanics [70, 6, 110], the construction presented here essentially uses the variables $z_i$ (or $v^+_i$) parametrizing the two-sphere $S^2$, in addition to the standard (dilatonic) coordinate $x$.
- The presence of additional ‘(iso)spin’ $S^2$ variables in our construction leads to a richer quantum spectrum. Besides, the relevant wavefunctions involve representations of the two independent SU(2) groups, in contrast to the SU(1,1)[2] models of [70, 6, 110, 50, 51] where only the SU(2) realized on fermionic variables really matters.
- In a contradistinction to the previously considered models, there naturally emerges a quantization of the conformal coupling constant which is expressed as an SU(2) Casimir operator, with both integer and half-integer eigenvalues. This happens already in the bosonic sector of the model, and is ensured by the $S^2$ variables.
- The variables $v_i^+$ in the expansions (4.155) and (4.156) can be identified with a half of the target space harmonic-like variables $v_i^+$ (though without the standard constraint $v^+_i v_i^- \sim \text{const}$).

Finally, it is worthwhile to mention that the $D(2, 1; \alpha)$ invariant $\mathcal{N} = 4$ superconformal mechanics model considered in this section is the one-particle case of the $D(2, 1; \alpha)$ invariant Calogero-type multi-particle models constructed in [36] (see the following subsection). An almost identical model of $D(2, 1; \alpha)$ invariant mechanics was constructed in [15]. These two models differ somewhat in their treatment of the semi-dynamical isospin variables however. In [36], the gauged (4, 4, 0) multiplet yields the additional algebraic constraint (4.136), which is absent in [15]. In the quantum theory, this constraint (4.147) fixes the value of the second-order Casimir operator $C_2$ in (4.146) and so singles out only one irreducible superconformal representation in the spectrum. Without this constraint, the space of quantum states of the model (4.134) will contain an infinite tower of irreducible representations.

4.3.3. Multi-particle models with $D(2, 1; \alpha)$ symmetry. Matrix extensions of the gauged model considered in the previous section lead to multi-particle systems with $\mathcal{N} = 4$
superconformal symmetry. Such models are described by the following harmonic superspace action:

\[
S = -\frac{1}{4(1+\alpha)} \int \mu_u \text{tr} (\mathcal{X}_-^{-1} / \mathcal{X}_-^1) - \frac{1}{2} \int \mu_A^{-1} \lambda_0 \bar{Z}^a_+ Z_a^+ - \frac{1}{2} c \int \mu_A^{-1} \text{Tr} V^+ .
\]  

(4.159)

The first term in (4.159) is the gauged action of the (1, 4, 3) multiplets which are described by Hermitian \( n \times n \)-matrix superfields \( \mathcal{X} = (\mathcal{X}^{ab}_a) \), \( a, b = 1, \ldots, n \). They are in the adjoint of \( U(n) \) and are subject to appropriate gauge-covariant constraints

\[
\mathcal{D}^+ \mathcal{X} = 0, \quad \mathcal{D}^+ \mathcal{D}^- \mathcal{X} = 0, \quad (\mathcal{D}^+ \mathcal{D}^- + \mathcal{D}^+ \mathcal{D}^-) \mathcal{X} = 0.
\]

(4.160)

(4.161)

The second term in (4.159) is a Wess–Zumino (WZ) action describing \( n \) commuting analytic superfields \( Z^+_a \) which are in the fundamental of \( U(n) \). They represent off-shell \( N = 4 \) multiplets (4, 0) and are defined by the constraints

\[
\mathcal{D}^+ Z^+ = 0, \quad \mathcal{D}^+ Z^+ = 0, \quad \mathcal{D}^+ Z^+ = 0.
\]

(4.162)

The constraints (4.160) and (4.162) involve the covariant harmonic derivative \( \mathcal{D}^{++} = \mathcal{D}^+ + i V^+ \), where the \( U(n) \) gauge matrix connection \( V^+ (\xi, u) \) is an analytic superfield. The gauge connections entering the spinor covariant derivatives in (4.161) are properly expressed through \( V^+ (\xi, u) \) [29]. The parameters of the \( U(n) \) gauge group are analytic, which implies \( \mathcal{D}^+ = \mathcal{D}^+ \), \( \mathcal{D}^+ = \mathcal{D}^+ \). Note that \( \mathcal{X} \) is in the adjoint of \( U(n) \), so \( \mathcal{D}^{++} \mathcal{X} = \mathcal{D}^{++} \mathcal{X} + 2i [V^+, \mathcal{X}] \), etc.

The third term in (4.159) is a Fayet–Iliopoulos (FI) term for \( V^+ \) and the real constant \( c \) is its strength. Clearly, only the trace part of \( V^+ \) (i.e. \( U(1) \) gauge connection) makes a contribution to this FI term. The superfield \( \mathcal{V}_0 (\xi, u) \) is a real analytic gauge prepotential for the \( U(n) \) singlet (1, 4, 3) superfield \( \Theta_0 \equiv \text{tr} (\mathcal{X}) \). It is defined by the integral transform

\[
\Theta_0 (t, \bar{t}, \bar{u}) = \int \text{d}u \mathcal{V}_0 (tA, \theta^+, \bar{\theta}^+, \bar{u}^+) |_{\theta = \theta^+, \bar{\theta} = \bar{\theta}^+, \bar{u} = \bar{u}^+}.
\]

(4.163)

The action (4.159) is invariant under the \( N = 4 \) superconformal group \( D(2, 1; \alpha) \). To show this we should use the \( D(2, 1; \alpha) \) transformation laws given in section 4. Once again, this invariance is ensured by the presence of the superfield multiplier \( \mathcal{V}_0 \) in the second term of the action (4.159).

The local \( U(n) \) transformations leaving the action (4.159) invariant are

\[
\mathcal{X}' = e^{\lambda_0} \mathcal{X} e^{-i \lambda_0}, \quad Z'^+ = e^{i \lambda_0} Z^+, \quad V'^+ = e^{i \lambda_0} V^+ e^{-i \lambda_0} = e^{i \lambda_0} (D^+ + e^{-i \lambda_0}),
\]

(4.164)

where \( \lambda_0 (\xi, \bar{u}) \) is the ‘Hermitian’ analytic matrix parameter, \( \lambda = \bar{\lambda} \). Using this gauge freedom we can choose the WZ gauge

\[
V^+ = 2i \theta^+ \bar{\theta}^+ A(tA).
\]

(4.165)

In what follows, we specialize to the case \( \alpha = -1/2 \), which corresponds to the free superconformally invariant action for \( \mathcal{X} \) in (4.159).

Inserting the component expressions of the superfields in the action (4.159) and eliminating auxiliary fields by their equations of motion, we obtain, in the WZ gauge, the component action

\[
S_4 = S_b + S_f, 
\]

(4.166)

\[
S_b = \int \text{d}t \left[ \text{tr} (\nabla X \nabla X + c A) + \frac{n}{8} (\bar{Z}^i \nabla Z^i)(\bar{Z}^i \nabla Z^i) + \frac{i}{2} \bar{X}_0 (\bar{Z} \nabla Z - \nabla \bar{Z} Z) \right]. 
\]

(4.167)

\[
S_f = i \text{tr} \int \text{d}t \left[ (\bar{\Psi}_k \nabla \Psi^k) - (\nabla \bar{\Psi}_k \Psi^k) - \int \text{d}t \frac{\Psi_0 \bar{\Psi}_0 (\bar{Z} \nabla Z)}{X_0} \right]. 
\]

(4.168)

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where

\[ X_0 := \text{tr}(X), \quad \Psi_0^i := \text{tr}(\Psi^i), \quad \bar{\Psi}_0^i := \text{tr}(\bar{\Psi}^i). \]  

(4.169)

Let us consider the bosonic limit of \( S_4 \), i.e. the action (4.167). We can impose the gauge \( X_0^a = 0, a \neq b \), using the residual invariance of the WZ gauge (4.165): \( X' = e^{i \alpha} X e^{-i \alpha}, \) \( Z_k^b = e^{i \lambda} Z_k^b, A' = e^{i \lambda} A e^{-i \lambda} - i e^{i \lambda}(\partial_i e^{-i \lambda}) \) where \( \lambda^b(t) \in u(n) \) are ordinary \( d = 1 \) gauge parameters. As a result of this, and after eliminating \( A_0^a, a \neq b \), by the equations of motion, the action (4.167) takes the following form (instead of \( Z_a^s \) we introduce the new fields \( Z_a^s = (X_0)^{1/2} Z_a^s \) and omit the primes on these fields):

\[ S_b = \int dt \left\{ \sum_a \dot{x}_a x_a + \frac{i}{2} \sum_a (\dot{Z}_a^s Z_a^s - \dot{\bar{Z}}_a^s \bar{Z}_a^s) + \sum_{a \neq b} \frac{\text{tr}(S_a S_b)}{4(x_a - x_b)^2} - \frac{n \text{Tr} (\dot{S} \bar{S})}{2(X_0)^2} \right\}. \]  

(4.170)

Here, the fields \( Z_a^s \) are subject to the constraints\(^{19}\)

\[ Z_a^s = c \quad \forall a, \]  

(4.171)

and carry the residual Abelian gauge [U(1)]\( ^n \) symmetry, \( Z_a^s \to e^{i \varphi_a} Z_a^s \), with local parameters \( \varphi_a(t) \). In (4.170), we use the following notation:

\[ (S_a)^i := \bar{Z}_a^i Z_a^s, \quad (\bar{S})_i := \sum_a (S_a)_i - \frac{1}{2} \delta_i^j (S_a)_k. \]  

(4.172)

Note that at \( c = 0 \) the constraint (4.171) implies \( Z_a^s = 0 \), i.e. a non-trivial interaction exists only for \( c \neq 0 \) as in the previous cases. The new feature of the \( N = 4 \) case is that not all out of the bosonic variables \( Z_a^s \) are eliminated by fixing gauges and solving the constraint; there survives a non-vanishing WZ term for them in equation (4.170). After quantization, these variables become purely internal (U(2)-spin) degrees of freedom.

In the Hamiltonian approach, the kinetic WZ term for \( Z \) in (4.170) gives rise to the following Dirac brackets:

\[ [\bar{Z}_a^i, Z_b^j]_D = i \delta_a^b \delta^i_j. \]  

(4.173)

With respect to these brackets, the quantities (4.172) for each index \( a \) form \( u(2) \) algebras

\[ [(S_a)^i, (S_b)^j]_D = i \delta_{ab} \left\{ \delta^i_j (S_a)_k - \delta^j_i (S_a)_k \right\}. \]  

(4.174)

The quantities \( (\bar{S})_i \) defined in (4.172) are time-independent Noether charges for the SU(2) invariance of the system (4.170), so the numerator of the term \( \sim (X_0)^{-2} \) in (4.170) is a constant on the equations of motion for \( Z_a^s, \bar{Z}_a^s \). So, as opposed to the \( N = 1, 2 \) cases, the \( N = 4 \) multpartile action contains a conformal potential even in the center-of-mass sector (like in [50, 51, 80]). Modulo this extra conformal potential (last term in (4.170)), the bosonic limit of the \( N = 4 \) system constructed is none other than the integrable U(2)-spin Calogero model in the formulation of [102].

It is worthwhile to note that in the considered model, as distinct from the \( D(2, 1; \alpha) \) invariant \( n \)-particle models discussed in section 4.2.2, there are \( n \) independent sets of the harmonic-like target variables \( Z_a^s, \bar{Z}_a^s \), while only one set of similar variables is present in the models of section 4.2.2. It is an open question whether it is possible, from the very beginning, to covariantly reduce the number of the ‘semi-dynamical’ superfields \( Z_a^s \) to one such superfield and to arrive, in this way, at the superfield formulation of the models of section 4.2.2 for which so far only the component formulation is known.

\(^{19}\) Here and in (4.172) we do not sum over the repeated index \( a \).
4.4. AdS/CFT correspondence in $\mathcal{N}=4$ mechanics and black holes

In the late 1990s, an unexpected application of (super)conformal mechanics was discovered. As shown in [28], the near-horizon bosonic geometry of the extreme Reissner–Nordström black hole coincides with that of the manifold $\text{AdS}_2 \times S^2$. It was demonstrated in [28] that the motion of the relativistic particle with mass $m$ and charge $q$ near the horizon of the extreme Reissner–Nordström black hole with large mass $M$, in the limit when the difference $(m-|q|)$ tends to zero, with $M^2 (m-|q|)$ being kept fixed, is described by AFF conformal mechanics (2.8). The radial coordinate of $\text{AdS}_2 \times S^2$ is identified with the conformal mechanics degree of freedom.

This result gave an explanation of some quantum properties of a particle moving near the horizon of black hole. This concerns, in particle, the necessity to redefine the Hamiltonian $H \rightarrow T_0$ and the existence of an infinite number of Hermitian quantum states [24]. The dynamics of superparticle moving near the extremal black holes in the same limit is in general described by extended superconformal mechanics [70, 6].

The fact that conformal mechanics describes the specific limit of the $\text{AdS}_2 \times S^2$ geometry of an extremal black hole is in agreement with the general concept of AdS/CFT correspondence. In [71], the meaning of this particular AdS/CFT correspondence was further clarified for the case when the angular $S^2$ degrees of freedom are ‘frozen’. It was shown there that the standard AFF conformal mechanics and the mechanics describing the radial motion of relativistic particle near the horizon of extremal black hole are described by two different but in fact equivalent nonlinear realizations of the $d=1$ conformal group $\text{SO}(2,1)$. Actually, they differ only in the choice of the parametrization of $\text{SO}(2,1)$ group elements. This equivalence holds for any finite value of the black hole mass, i.e. without assuming any specific limit.

The standard conformal mechanics is based on the algebra (2.1), the group parametrization (2.20) and the constraints (2.24) incorporated in the action (2.27). In [71], for the description of the radial particle motion near the horizon of extremal black hole, it was suggested to use the ‘AdS basis’ in the same $\text{so}(2,1)$ algebra

$$
\hat{\mathcal{P}} = H, \quad \hat{\mathcal{K}} = \frac{1}{\mathcal{R}} K - \mathcal{R} H, \quad \hat{\mathcal{D}} = \frac{1}{\mathcal{R}} D,
$$

(4.175)

where $\mathcal{R}$ is the AdS radius. The commutation relations of the conformal algebra (2.1) in this AdS basis are rewritten as

$$
[\hat{D}, \hat{\mathcal{P}}] = -\frac{i}{\mathcal{R}} \hat{\mathcal{P}}, \quad [\hat{\mathcal{K}}, \hat{\mathcal{P}}] = -2i \hat{\mathcal{D}}, \quad [\hat{D}, \hat{\mathcal{K}}] = \frac{i}{\mathcal{R}} \hat{\mathcal{K}} + 2i \hat{\mathcal{P}}.
$$

(4.176)

An element of $\text{SO}(2,1)$ in the AdS basis is defined to be

$$
G_0 = e^{i \mathcal{P} \Omega(t)} \hat{\mathcal{K}} e^{i \phi(t)} \hat{\mathcal{D}}.
$$

(4.177)

Like in the standard conformal mechanics, the field $\Omega(t)$ can be eliminated by imposing the inverse Higgs constraint

$$
\hat{\omega}_D = 0,
$$

(4.178)

and the radial motion near the horizon is described by the action [71]

$$
S_{\text{AdS}} = -m \int \hat{\omega}_D + q \int d\tau e^{-\phi/\mathcal{R}}.
$$

(4.179)

The second term in (4.179) plays the role of a cosmological term. Thus, two conformal mechanics models arise as nonlinear realizations of the same spontaneously broken $\text{SO}(2,1)$ symmetry, with the dilaton as the only essential Goldstone field.
In [71], a direct link between these two seemingly different models was established through the following invertible change of variables:

\[ t = \tau - R e^{\phi/R} \Lambda, \quad u = \frac{\phi}{R} + \ln(1 - \Lambda^2), \quad z = \frac{\Lambda}{R}, \quad \Lambda := \tanh \Omega. \quad (4.180) \]

Already in [71], this ‘radial’ \( \text{AdS}_2/\text{CFT}_1 \) correspondence was extended to the case of \( \mathcal{N} = 2 \) SU(1,1|1) superconformal mechanics. In [10], by the explicit canonical transformations in the Hamiltonian formalism, the equivalence of the \( \mathcal{N} = 4 \) superconformal mechanics model and a charge massive particle propagating near the extremal black hole was shown for any finite black hole mass and with both the radial and the angular degrees of freedom of the particle taken into account. A generalization of this equivalence to the case of the extremal black hole possessing both electric and magnetic charges was performed in [46]. Note that the interaction with the magnetic field of black hole is described by the WZ term, similarly to (2.48), (2.52) and (4.134).

Recently, (super)conformal mechanics was also used to describe other black hole solutions. This is based on the fact that the isometry group of (diverse) four-dimensional extremal black hole in the near-horizon limit contains the \( d = 1 \) conformal group SO(2,1) [81]. The extremal Reissner–Nordström was invoked in an analysis of \( \mathcal{N} = 4 \) superconformal mechanics coupled to several \( \mathcal{N} = 8, d = 1 \) vector multiplets, giving the component action, supercharges and Hamiltonian with all fermionic terms included [18]. The extremal Kerr throat solution and the Kerr, Kerr–Newman and Kerr–Newman–AdS–dS black holes were studied using superconformal mechanics in [47, 53, 16, 52]. In particular, in [52], a general method was proposed of how to explicitly define, at the Hamiltonian level, the canonical transformations related the conformal and AdS bases for various cases of interest.

There are other links of the black hole physics with the models of extended superconformal mechanics. This concerns describing the set of black holes and clarifies some other aspects of the \( \text{AdS}_2/\text{CFT}_1 \) correspondence. In particular, the authors of the paper [57] argued that the large-\( n \) limit of the \( n \)-particle SU(1,1|2) superconformal Calogero model may provide a microscopic description of the extreme Reissner–Nordström black hole in the near-horizon limit. This hypothesis is based on the assertion that for a large number of particles and in the limit when all coordinates of the Calogero model, except for one, are treated as ‘small’, the Calogero model reduces to the conformal mechanics for this ‘allocated’ coordinate. Other investigations were made in [92, 93, 90, 24, 96] where it was shown that the moduli spaces of \( n \) black holes in four- and five-dimensional supergravities are described by the sigma-model-type conformal quantum mechanics. Note that the construction of a self-consistent \( n \)-body generalization of black-hole quantum mechanics is a rather complicated problem beyond the one- and two-body cases. In order to have a normalizable ground state in the latter cases, one should apply a proper time redefinition, just as in conformal quantum mechanics [5]. If the general multi-black-hole quantum mechanics indeed amounts to supersymmetric Calogero models, one can employ the powerful machinery developed for integrable super-Calogero systems (see e.g. [43, 110, 22, 21, 23, 49–51]).

5. Sketch of \( \mathcal{N} > 4 \) superconformal systems

So far, most of studies related to the superconformal mechanics were in fact limited to the lower \( \mathcal{N} \) extensions, namely to the \( \mathcal{N} = 1, \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) ones (also, the \( \mathcal{N} = 3 \) case was not investigated, though the corresponding off-shell multiplets for this case likely coincide, by their component contents, with the \( \mathcal{N} = 4 \) ones). The only \( \mathcal{N} > 4 \) superconformal mechanics models for which off-shell actions are known are a few systems of \( \mathcal{N} = 8 \) superconformal mechanics. Below we give a brief outline of this class of \( \mathcal{N} > 4 \) superconformal mechanics.
Any supergroup underlying an extended superconformal mechanics should, first of all, contain $d = 1$ conformal group $\text{SL}(2, \mathbb{R})$ as its subgroup. It is the only non-compact bosonic subgroup, while other bosonic subgroups are compact subgroups of the relevant $R$-symmetries. The full list of simple supergroups of $\mathcal{N}$-extended superconformal quantum mechanics can be found in [24] (more complete information can be found in [44]). In contrast to the cases of $\mathcal{N} \leq 4$ where the corresponding superconformal groups are in fact unique\(^{20}\), there are four different $\mathcal{N} = 8$ superconformal groups: $\text{SU}(1, 1|4)$, $\text{OSp}(8|2)$, $\text{OSp}(4^*|4)$ and real exceptional supergroup $F(4)$ with the $R$-symmetry subgroup $\text{SO}(7)$. Until now, models of $\mathcal{N} = 8$ superconformal mechanics have been found for the supergroup $\text{OSp}(4^*|4)$ in [13, 14] and for the supergroup $F(4)$ in [32].\(^{21}\)

Also in [70], on-shell component actions for the superconformal mechanics models associated with the conformal supergroups $\text{SU}(1, 1|\mathcal{N}/2)$, $\mathcal{N} \geq 4$, were constructed (see also [3, 110]). Imposing the appropriate covariant constraints on MC forms which accomplish the covariant reduction of the conformal supergroups $\text{SU}(1, 1|\mathcal{N}/2)$ to the compact supergroup $\text{SU}(1, \mathcal{N}/2)$, in [70] the physical component contents of these models and their equations of motion were found. The on-shell action of such a $(1, \mathcal{N}, \gamma)$ supermultiplet is similar to the action (4.38),

$$S_{\mathcal{N} \geq 4} = \int dt \left[ \ddot{x} + i(\bar{\psi}_k \psi^k - \bar{\psi}_k \psi^k) - \frac{(m + \bar{\psi}_k \psi^k)^2}{x^2} \right],$$

(5.1)

where $\psi^k$ is the spinor in the fundamental representation of $\text{SU}(\mathcal{N}/2)$. As shown in [110], the generators of the conformal supergroups $\text{SU}(1, 1|\mathcal{N}/2)$, $\mathcal{N} > 4$, can be obtained from the $\mathcal{N} = 4$ generators by directly replacing the $\text{SU}(2)$ spinor $\psi^k$ by the $\text{SU}(\mathcal{N}/2)$ spinor. In particular, the generators of the Poincaré superalgebra are given by the expressions

$$Q^i = \psi^i \left( p - 2i(m + \bar{\psi}_k \psi^k) \frac{1}{x} \right), \quad \bar{Q}_i = \bar{\psi}_i \left( p + 2i(m + \bar{\psi}_k \psi^k) \frac{1}{x} \right),$$

(5.2)

$$H = \frac{p^2}{4} + \frac{(m + \bar{\psi}_k \psi^k)^2}{x^2}.$$  

(5.3)

Up to now, it is unclear how to recover the system (5.1) from some off-shell superfield formalism.

Although the $\mathcal{N} = 8$ superfield formalism was developed in [13, 14] and $\mathcal{N} = 8$ superconformal equations of motion (for some superfields) were obtained through nonlinear realization in [13], the simplest way of constructing the $\mathcal{N} = 8$ superconformal mechanics models is to deal with the $\mathcal{N} = 4$ superspace formalism and to represent irreducible $\mathcal{N} = 8$ multiplets as direct sums of the appropriate $\mathcal{N} = 4$ multiplets. This expansion of the $\mathcal{N} = 8$ superfields in terms of the $\mathcal{N} = 4$ superfields can be schematically written as the following splitting [13, 73]:

$$(\mathbf{n}, \mathbf{8}, \mathbf{8} - \mathbf{n}) = (\mathbf{n_1}, \mathbf{4}, \mathbf{4} - \mathbf{n_1}) \oplus (\mathbf{n_2}, \mathbf{4}, \mathbf{4} - \mathbf{n_2}), \quad \mathbf{n} = \mathbf{n_1} + \mathbf{n_2}. \quad (5.4)$$

Using the splitting

$$(\mathbf{3}, \mathbf{8}, \mathbf{5}) = (\mathbf{3}, \mathbf{4}, \mathbf{1}) \oplus (\mathbf{0}, \mathbf{4}, \mathbf{4}), \quad (5.5)$$

in the paper [13] the $\mathcal{N} = 8$ superconformal Lagrangian for the supermultiplet $(\mathbf{3}, \mathbf{8}, \mathbf{5})$ was constructed. The $\mathcal{N} = 4$ supermultiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ is described by the harmonic superfield $L^+^+$ (4.98) whereas the supermultiplet $(\mathbf{0}, \mathbf{4}, \mathbf{4})$ is described by the analytic superfield

\(^{20}\)The degeneracy in the $\mathcal{N} = 4$ case is somewhat trivial, since all possible $\mathcal{N} = 4$ superconformal groups correspond to different choices of the parameter $\alpha$ in the supergroup $D(2, 1; \alpha)$; an exception is the supergroup $\text{SU}(1, 1|2)$, but it enters as a multiplier in the semidirect product structure of the supergroup $D(2, 1; \alpha)$ at $\alpha = 0$ and $-1$.

\(^{21}\)Some further possibilities are discussed, from the group-theoretical point of view, in a recent paper [83].
The action is most simple in the second case, with the constituent $N$ described, respectively, by the real SU(2) vector superfield added for ensuring (coefficient before the WZ term of the multiplet the first one being a conformally covariantized action of the $d$ that of a charged particle moving in the Dirac monopole background (the last two terms, of two conformally invariant actions, that of conformal mechanics (the first two terms) and $\alpha$ superconformal mechanics coincides with the bosonic sector of the $N' = 4, D(2, 1; \alpha)$ superconformal mechanics of [67, 72] for the choice $\alpha = 1$.

Another $N = 4$ superfield splitting which yields the same $N = 8$ superconformal model is

$$(3, 8, 5) = (1, 4, 3) \oplus (2, 4, 2),$$

(5.7)

where the first supermultiplet is described by the $N = 4$ superfield $v$ subjected to the SU(1, 1|2) covariant constraints (4.37) (with $m \to -2m$) and the second one is the $N = 4$ chiral multiplet described by a complex superfield $\varphi$, $D\varphi = \bar{D}\bar{\varphi} = 0$. The $N = 4$ superfield action in this case has the form [13]

$$S_{B}^{(3, 8, 5)} = -\frac{1}{4} \int dt d^{4}\theta[v \log(v + \sqrt{v^{2} + 4\varphi\bar{\varphi}}) - \sqrt{v^{2} + 4\varphi\bar{\varphi}}].$$

(5.8)

It has the manifest SU(1, 1|2) superconformal symmetry and hidden $N = 8$ Poincaré supersymmetry which close on the same $N = 8$ superconformal group OSp(4$^*|4$) which includes SU(1, 1|2) as one of its subgroups, along with OSp(4$^*|2$). The bosonic component action of course coincides with (5.6) (after field redefinitions and elimination of auxiliary fields). The constant $m$ which specifies the strength of both the conformal potential and WZ term comes now from the constraint on $v$, while in the previous splitting option it explicitly enters he superfield action.

In the same paper [13], a model of $N = 8$ superconformal mechanics based on the off-shell supermultiplet (5, 8, 3) was also considered. This model respects the same superconformal group OSp(4$^*|4$) and have two equivalent $N = 4$ superfield descriptions based upon the splittings

(a) $$(5, 8, 3) = (1, 4, 3) \oplus (4, 4, 0),$$

(b) $$(5, 8, 3) = (3, 4, 1) \oplus (2, 4, 2).$$

(5.9)

The action is most simple in the second case, with the constituent $N = 4$ multiplets being described, respectively, by the real SU(2) vector superfield $W^{(b)}$, $D^{(b)}W^{(b)} = \bar{D}^{(b)}\bar{W}^{(b)} = 0$, and by chiral superfields $\Phi$, $\bar{\Phi}$, $D'\Phi = \bar{D}'\bar{\Phi} = 0$. The superconformal action reads

$$S_{B}^{(5, 8, 3)} = 2 \int dt d^{4}\theta \frac{\log(\sqrt{W^{2}} + \sqrt{W^{2} + \frac{1}{2} \Phi \bar{\Phi}})}{\sqrt{W^{2}}}. $$

(5.10)
In the bosonic sector, it yields a particular (conformally invariant) type of the SO(5) invariant $d = 1$ sigma models of [33]:

$$S_B^{(5, 8, 3)} = \int dt \frac{W^{ik} \dot{W}_{ik} + \frac{1}{2} \dot{\Phi} \dot{\Phi}}{(W^2 + \frac{1}{2} \Phi^2)},$$

(5.11)

where the involved quantities are first components of the original superfields. The $\mathcal{N} = 4$ superconformally invariant WZ action for the multiplet $(3, 4, 1)$ can be extended to $\mathcal{N} = 8$ superconformally invariant action, thus producing conformal potentials for the bosonic fields.

An example of $\mathcal{N} = 8$ superconformal mechanics based on a different $\mathcal{N} = 8$ superconformal group was constructed in [32]. It is based on the $\mathcal{N} = 8$ off-shell supermultiplet $(1, 8, 7)$, with the following $\mathcal{N} = 4$ superfield splitting:

$$S_N^{(1, 8, 7)} = (1, 4, 3) \oplus (0, 4, 4).$$

(5.12)

The supermultiplet $(1, 4, 3)$ is described by the real harmonic superfield $\mathcal{X}(t, \theta^\pm, \bar{\theta}^\pm, u)$ (4.94), (4.95) or (4.22). The supermultiplet $(0, 4, 4)$ is described by the odd analytic superfield $\Psi^+(\xi, u)$ as above. It was shown that the $D(2, 1; \alpha)$-invariant action is

$$S_N^{(1, 8, 7)} = -\frac{1}{4(1+\alpha)} \int \mu H \mathcal{X}^{-1/\alpha} + \frac{1}{2} \int \mu_1^{(-2)} \mathcal{Y} \bar{\Psi}^+ \Psi^+,$$

(5.13)

and it consists of the superconformal action (4.23) of the $(1, 4, 3)$ multiplet and Wess–Zumino-type action of the form (4.117). This action is invariant under full $\mathcal{N} = 8$ superconformal transformations only for $\alpha = -1/3$. The only $\mathcal{N} = 8$ superconformal group into which one can embed $D(2, 1; -1/3)$ is the exceptional $\mathcal{N} = 8$ superconformal group $F(4)$. It is still unclear whether it is possible to generate a conformal potential for the physical scalar field without breaking this underlying $F(4)$ superconformal symmetry.

6. Conclusion

In this review, we have presented a particular view of the current state of superconformal mechanics. We acknowledge that our view is biased and incomplete due to the vast amount of literature on modern research in this area. We have tried to combine two approaches employed in the study of these models. The first one starts with a classical (super)field description, based on the Lagrangian formulation of the system. In this approach, the superconformal symmetry is transparent but the quantization often faces significant challenges due to the generic nonlinear character of the systems. The second approach starts from a quantum description, studying superconformal quantum systems algebraically. In this case, the geometrical interpretation becomes obscure, even after a final reconstruction of the relevant quantum superconformal algebras. It must be admitted that most articles on superconformal mechanics settle only on one of the two approaches. However, to overcome the problems manifest in either treatment, it is desirable to analyze systems with superconformal symmetry by combining both approaches, i.e. starting from the analysis of the symmetries of the classical action and ending with the quantum picture, with a construction of the quantum superconformal algebra and uncovering the physical spectrum. Such a strategy will allow for a better understanding of conformal and superconformal mechanics and perhaps facilitate a more widespread application of these nice theories.

22 After field redefinitions, this action can be represented as a sum of the standard conformal mechanics action for the dilaton (the radial part of the 5-vector $(W^\Phi, \Phi, \bar{\Phi})$) without the potential term and some conformally invariant sigma model action for the remaining four angular variables.

23 The general $\mathcal{N} = 8$ supersymmetric component action of this multiplet was constructed in [82].
Let us try to predict the research development on superconformal mechanics in the near future. In our opinion, the following (approximate) problems will be addressed predominantly:

- construction of extended superconformal (quantum) mechanics with isospin degrees of freedom and with diverse dynamical supermultiplets;
- study of various many-particle systems with superconformal symmetry, including those with isospin degrees of freedom; obtaining versions of spin-Calogero systems and entirely new ones; analysis of the integrability properties of these systems;
- deeper understanding of the role of superconformal systems, beyond the simplest cases, in the AdS/CFT correspondence and in the physics of black holes;
- understanding of the significance of the (iso)spin variables in superconformal mechanics (are they just angular degrees of freedom in physical problems, or truly spin ones?);
- constructing new superconformal systems with $\mathcal{N} > 4$ supersymmetry and seeking out their applications.

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