ON HADAMARD TYPE INEQUALITIES INVOLVING SEVERAL KIND OF CONVEXITY

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ABSTRACT. In this paper, we not only give the extensions of the results given in [7] by Gill et al. for log-convex functions, but also obtain some new Hadamard type inequalities for log-convex, \( m \)-convex and \((\alpha, m)\)-convex functions.

1. Introduction

The following inequality is well known in the literature as Hadamard’s inequality:

\[
\frac{f \left( \frac{a+b}{2} \right)}{2} \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2},
\]

where \( f : I \to \mathbb{R} \) is a convex function on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). This inequality is one of the most useful inequalities in mathematical analysis. For new proofs, noteworthy extension, generalizations and numerous applications on this inequality, see ([1], [4], [5], [6], [9], [12]) where further references are given.

Let \( I \) be an interval in \( \mathbb{R} \). Then \( f : I \to \mathbb{R} \) is said to be convex if for all \( x, y \in I \) and \( t \in [0, 1] \),

\[ f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y) \]

(see [9, P.1]). Geometrically, this means that if \( K, L \) and \( M \) are three distinct points on the graph of \( f \) with \( L \) between \( K \) and \( M \), then \( L \) is on or below chord \( KM \).

Recall that a function \( f : I \to (0, \infty) \) is said to be log-convex function, if for all \( x, y \in I \) and \( t \in [0, 1] \), one has the inequality (see [9, P.3])

\[ f \left( tx + (1 - t) y \right) \leq \left[ f(x) \right]^t \left[ f(y) \right]^{(1-t)}. \]

It is said to be log-concave if the inequality in (1.2) is reversed.

In [8], G. Toader defined \( m \)-convexity as follows:

Definition 1. The function \( f : [0, b] \to \mathbb{R} \), \( b > 0 \) is said to be \( m \)-convex, where \( m \in [0, 1] \), if we have

\[ f(t x + m (1 - t) y) \leq t f(x) + m (1 - t) f(y) \]

for all \( x, y \in [0, b] \) and \( t \in [0, 1] \). We say that \( f \) is \( m \)-concave if \( -f \) is \( m \)-convex.

Denote by \( K_m (b) \) the class of all \( m \)-convex functions on \([0, b]\) for which \( f(0) \leq 0 \). Obviously, if we choose \( m = 1 \), definition (1) recaptures the concept of standard convex functions on \([0, b]\).

In [8], V. G. Miheșan defined \((\alpha, m)\)-convexity as in the following:

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Definition 2. The function \( f : [0, b] \rightarrow \mathbb{R} \), \( b > 0 \), is said to be \((\alpha, m)\)−convex, where \((\alpha, m) \in [0, 1]^2\), if we have

\[
f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)
\]

for all \( x, y \in [0, b] \) and \( t \in [0, 1]\).

Denote by \( K^m_\alpha(b) \) the class of all \((\alpha, m)\)−convex functions on \([0, b]\) for which \( f(0) \leq 0 \). It can be easily seen that for \((\alpha, m) = (1, m)\), \((\alpha, m)\)−convexity reduces to \(m\)−convexity and for \((\alpha, m) = (1, 1)\), \((\alpha, m)\)−convexity reduces to the concept of usual convexity defined on \([0, b]\), \( b > 0 \).

For recent results and generalizations concerning \(m\)−convex and \((\alpha, m)\)−convex functions, see [2], [3], [10].

In [7], P.M. Gill et al. established the following results:

Theorem 1. Let \( f \) be a positive, log-convex function on \([a, b]\). Then

\[
\frac{1}{b-a} \int_a^b f(t) \, dt \leq L\left(f(a), f(b)\right)
\]

where

\[
L(p, q) = \frac{p - q}{\ln p - \ln q} \quad (p \neq q)
\]

is the Logarithmic mean of the positive real numbers \( p, q \) (for \( p = q \), we put \( L(p, p) = p \)).

For \( f \) a positive log-concave function, the inequality is reversed.

Corollary 1. Let \( f \) be positive log-convex functions on \([a, b]\). Then

\[
\frac{1}{b-a} \int_a^b f(t) \, dt \leq \min_{x \in [a, b]} \frac{(x - a) L(f(a), f(x)) + (b - x) L(f(x), f(b))}{b - a}.
\]

If \( f \) is a positive log-concave function, then

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \geq \max_{x \in [a, b]} \frac{(x - a) L(f(a), f(x)) + (b - x) L(f(x), f(b))}{b - a}.
\]

For some recent results related to the Hadamard’s inequalities involving two log-convex functions, see [11] and the references cited therein. The main purpose of this paper is to establish the general version of the inequalities (1.3) and new Hadamard type inequalities involving two log-convex functions or two \(m\)-convex functions or two \((\alpha, m)\)-convex functions using elementary analysis.

2. Main Results

We start with the following Theorem.

Theorem 2. Let \( f_i : I \subset \mathbb{R} \rightarrow (0, \infty) \) \((i = 1, 2, ..., n)\) be log-convex functions on \( I \) and \( a, b \in I \) with \( a < b \). Then the following inequality holds:

\[
\frac{1}{b-a} \int_a^b \prod_{i=1}^n f_i(x) \, dx \leq L\left(\prod_{i=1}^n f_i(a), \prod_{i=1}^n f_i(b)\right)
\]
where \( L \) is a logarithmic mean of positive real numbers.

For \( f \) a positive log-concave function, the inequality is reversed.

**Proof.** Since \( f_i (i = 1, 2, ..., n) \) are log-convex functions on \( I \), we have

\[
(2.2) \quad f_i (ta + (1 - t) b) \leq [f_i (a)]^t [f_i (b)]^{1-t}
\]

for all \( a, b \in I \) and \( t \in [0, 1] \). Writing \( (2.2) \) for \( i = 1, 2, ..., n \), multiplying the resulting inequalities it is easy to observe that

\[
(2.3) \quad \prod_{i=1}^{n} f_i (ta + (1 - t) b) \leq \left[ \prod_{i=1}^{n} f_i (a) \right]^t \left[ \prod_{i=1}^{n} f_i (b) \right]^{1-t} = \prod_{i=1}^{n} f_i (b) \left( \prod_{i=1}^{n} f_i (a) / f_i (b) \right)^t
\]

for all \( a, b \in I \) and \( t \in [0, 1] \).

Integrating inequality \( (2.3) \) on \([0, 1]\) over \( t \), we get

\[
\int_0^1 \prod_{i=1}^{n} f_i (ta + (1 - t) b) \, dt \leq \prod_{i=1}^{n} f_i (b) \int_0^1 \left[ \prod_{i=1}^{n} f_i (a) / f_i (b) \right] ^t \, dt.
\]

As

\[
\int_0^1 \prod_{i=1}^{n} f_i (ta + (1 - t) b) \, dt = \frac{1}{b-a} \int_a^b \prod_{i=1}^{n} f_i (x) \, dx
\]

and

\[
\int_0^1 \left[ \prod_{i=1}^{n} f_i (a) / f_i (b) \right] ^t \, dt = \frac{1}{b-a} L \left( \prod_{i=1}^{n} f_i (a), \prod_{i=1}^{n} f_i (b) \right),
\]

the theorem is proved. \( \square \)

**Remark 1.** By taking \( i = 1 \) and \( f_1 = f \) in Theorem 2 we obtain \( (1.3) \).

**Corollary 2.** Let \( f_i : I \subset R \to (0, \infty) \) \( (i = 1, 2, ..., n) \) be log-convex functions on \( I \) and \( a, b \in I \) with \( a < b \). Then

\[
(2.4) \quad \frac{1}{b-a} \int_a^b \prod_{i=1}^{n} f_i (x) \, dx
\]

\[
\leq \min_{x \in [a,b]} \frac{(x-a) L \left( \prod_{i=1}^{n} f_i (a), \prod_{i=1}^{n} f_i (x) \right) + (b-x) L \left( \prod_{i=1}^{n} f_i (x), \prod_{i=1}^{n} f_i (b) \right)}{b-a}.
\]

If \( f_i \) \( (i = 1, 2, ..., n) \) are a positive log-concave functions, then

\[
(2.5) \quad \frac{1}{b-a} \int_a^b \prod_{i=1}^{n} f_i (x) \, dx
\]

\[
\geq \max_{x \in [a,b]} \frac{(x-a) L \left( \prod_{i=1}^{n} f_i (a), \prod_{i=1}^{n} f_i (x) \right) + (b-x) L \left( \prod_{i=1}^{n} f_i (x), \prod_{i=1}^{n} f_i (b) \right)}{b-a}.
\]
Proof. Let \( f_i \) \((i = 1, 2, \ldots, n)\) be a positive log-convex functions. Then by Theorem 2 we have that
\[
\int_a^b \prod_{i=1}^n f_i(t) \, dt = \int_a^x \prod_{i=1}^n f_i(t) \, dt + \int_x^b \prod_{i=1}^n f_i(t) \, dt \leq (x - a) L \left( \prod_{i=1}^n f_i(a), \prod_{i=1}^n f_i(x) \right) + (b - x) L \left( \prod_{i=1}^n f_i(x), \prod_{i=1}^n f_i(b) \right)
\]
for all \( x \in [a, b] \), whence (2.4). Similarly we can prove (2.5). □

Remark 2. By taking \( i = 1 \) and \( f_1 = f \) in (2.4) and (2.5), we obtain the inequalities of Corollary 1.

We will now point out some new results of the Hadamard type for log-convex, \( m \)-convex and \((\alpha, m)\)-convex functions, respectively.

Theorem 3. Let \( f, g : I \to (0, \infty) \) be log-convex functions on \( I \) and \( a, b \in I \) with \( a < b \). Then the following inequalities hold:
\[
\left(\frac{a + b}{2}\right) f \left(\frac{a + b}{2}\right) g \left(\frac{a + b}{2}\right) \leq \frac{1}{2} \left[ f \left(\frac{a + b}{2}\right) f \left(\frac{a + b}{2}\right) + g \left(\frac{a + b}{2}\right) g \left(\frac{a + b}{2}\right) \right]
\]

Proof. We can write
\[
\frac{a + b}{2} = \frac{ta + (1 - t) b}{2} + \frac{(1 - t) a + tb}{2}.
\]
Using the elementary inequality \( cd \leq \frac{1}{4} \left[ c^2 + d^2 \right] \) \((c, d \geq 0 \text{ reals})\) and equality (2.7), we have
\[
f \left(\frac{a + b}{2}\right) g \left(\frac{a + b}{2}\right) \leq \frac{1}{2} \left[ f^2 \left(\frac{a + b}{2}\right) + g^2 \left(\frac{a + b}{2}\right) \right]
\]
\[
= \frac{1}{2} \left\{ f^2 \left(\frac{ta + (1 - t) b}{2} + \frac{(1 - t) a + tb}{2}\right) + g^2 \left(\frac{ta + (1 - t) b}{2} + \frac{(1 - t) a + tb}{2}\right) \right\}
\]
\[
= \frac{1}{2} \left\{ \left[ f \left(\frac{a + b}{2}\right) \right]^2 + \left[ g \left(\frac{a + b}{2}\right) \right]^2 \right\}
\]
\[
= \frac{1}{2} \left[ f \left(\frac{a + b}{2}\right) f \left(\frac{a + b}{2}\right) + g \left(\frac{a + b}{2}\right) g \left(\frac{a + b}{2}\right) \right].
\]
Since \( f, g \) are log-convex functions, we obtain

\[
\frac{1}{2} \left[ f \left( ta + (1 - t) b \right) f \left( (1 - t) a + tb \right) + g \left( ta + (1 - t) b \right) g \left( (1 - t) a + tb \right) \right] \\
\leq \left\{ \frac{1}{2} \left[ f \left( a \right) \right]^t \left[ f \left( b \right) \right]^{1-t} \left[ f \left( a \right) \right]^{1-t} \left[ f \left( b \right) \right]^t + \left[ g \left( a \right) \right]^t \left[ g \left( b \right) \right]^{1-t} \left[ g \left( a \right) \right]^{1-t} \left[ g \left( b \right) \right]^t \right\}
\]

\[
= \frac{f \left( a \right) f \left( b \right) + g \left( a \right) g \left( b \right)}{2}
\]

for all \( a, b \in I \) and \( t \in [0, 1] \).

Rewriting (2.8) and (2.9), we have

\[
f \left( a + b \right) \frac{1}{2} g \left( a + b \right) \frac{1}{2} \leq \frac{1}{2} \left[ f \left( ta + (1 - t) b \right) f \left( (1 - t) a + tb \right) + g \left( ta + (1 - t) b \right) g \left( (1 - t) a + tb \right) \right]
\]

and

\[
\frac{1}{2} \left[ f \left( ta + (1 - t) b \right) f \left( (1 - t) a + tb \right) + g \left( ta + (1 - t) b \right) g \left( (1 - t) a + tb \right) \right] \\
\leq \frac{f \left( a \right) f \left( b \right) + g \left( a \right) g \left( b \right)}{2}.
\]

Integrating both sides of (2.10) and (2.11) on \([0, 1]\) over \( t \), respectively, we obtain

\[
f \left( a + b \right) \frac{1}{2} g \left( a + b \right) \frac{1}{2} \leq \frac{1}{2} \left[ f \left( x \right) f \left( a + b - x \right) + g \left( x \right) g \left( a + b - x \right) \right] dx
\]

and

\[
\frac{1}{2} \left[ f \left( x \right) f \left( a + b - x \right) + g \left( x \right) g \left( a + b - x \right) \right] dx \\
\leq \frac{f \left( a \right) f \left( b \right) + g \left( a \right) g \left( b \right)}{2}.
\]

Combining (2.12) and (2.13), we get the desired inequalities (2.6). The proof is complete.

\[\square\]

**Theorem 4.** Let \( f, g : I \to (0, \infty) \) be log-convex functions on \( I \) and \( a, b \in I \) with \( a < b \). Then the following inequalities hold:

\[
2 f \left( a + b \right) \frac{1}{2} g \left( a + b \right) \frac{1}{2} \leq \frac{1}{b - a} \int_a^b \left[ f^2 \left( x \right) + g^2 \left( x \right) \right] dx \\
\leq \frac{f \left( a \right) + f \left( b \right)}{2} L \left( f \left( a \right), f \left( b \right) \right) + \frac{g \left( a \right) + g \left( b \right)}{2} L \left( g \left( a \right), g \left( b \right) \right)
\]

where \( L \left( , , \right) \) is a logarithmic mean of positive real numbers.
Proof. From the inequality (2.10), we have
\[
\frac{f}{2} \left( a + \frac{b}{2} \right) g \left( a + \frac{b}{2} \right) \leq \frac{1}{2} \left[ f \left( ta + (1 - t)b \right) f \left( (1 - t)a + tb \right) \right.
\]
\[
\left. + g \left( ta + (1 - t)b \right) g \left( (1 - t)a + tb \right) \right]
\]
for all \( a, b \in I \text{ and } t \in [0,1] \).

Using the elementary inequality \( acd \leq \frac{1}{2} \left[ c^2 + d^2 \right] \) (\( c, d \geq 0 \) reals) on the right side of the above inequality, we have
\[
(2.15) \quad f \left( a + \frac{b}{2} \right) g \left( a + \frac{b}{2} \right) \leq \frac{1}{4} \left[ f^2 \left( ta + (1 - t)b \right) + f^2 \left( (1 - t)a + tb \right) \right.
\]
\[
\left. + g^2 \left( ta + (1 - t)b \right) + g^2 \left( (1 - t)a + tb \right) \right].
\]

Since \( f, g \) are log-convex functions, then we get
\[
(2.16) \quad \left[ f^2 \left( ta + (1 - t)b \right) + f^2 \left( (1 - t)a + tb \right) \right.
\]
\[
\left. + g^2 \left( ta + (1 - t)b \right) + g^2 \left( (1 - t)a + tb \right) \right] \leq \left\{ \left[ f \left( a \right) \right]^{2t} \left[ f \left( b \right) \right]^{(2-2t)} + \left[ f \left( a \right) \right]^{(2-2t)} \left[ f \left( b \right) \right]^{2t} \right.
\]
\[
\left. + \left[ g \left( a \right) \right]^{2t} \left[ g \left( b \right) \right]^{(2-2t)} + \left[ g \left( a \right) \right]^{(2-2t)} \left[ g \left( b \right) \right]^{2t} \right\}
\]
\[
= \left[ f^2 \left( b \right) \left[ \frac{f \left( a \right)}{f \left( b \right)} \right]^{2t} + f^2 \left( a \right) \left[ \frac{f \left( b \right)}{f \left( a \right)} \right]^{2t} \right.
\]
\[
\left. + g^2 \left( b \right) \left[ \frac{g \left( a \right)}{g \left( b \right)} \right]^{2t} + g^2 \left( a \right) \left[ \frac{g \left( b \right)}{g \left( a \right)} \right]^{2t} \right].
\]

Integrating both sides of (2.15) and (2.16) on \([0,1]\) over \( t \), respectively, we obtain
\[
(2.17) \quad 2f \left( a + \frac{b}{2} \right) g \left( a + \frac{b}{2} \right) \leq \frac{1}{b - a} \int_a^b \left[ f^2 \left( x \right) + g^2 \left( x \right) \right] dx
\]

and
\[
(2.18) \quad \frac{1}{b - a} \int_a^b \left[ f^2 \left( x \right) + g^2 \left( x \right) \right] dx
\]
\[
\leq \frac{1}{2} \left( f^2 \left( b \right) \int_0^1 \left[ \frac{f \left( a \right)}{f \left( b \right)} \right]^{2t} dt + f^2 \left( a \right) \int_0^1 \left[ \frac{f \left( b \right)}{f \left( a \right)} \right]^{2t} dt \right.
\]
\[
\left. + g^2 \left( b \right) \int_0^1 \left[ \frac{g \left( a \right)}{g \left( b \right)} \right]^{2t} dt + g^2 \left( a \right) \int_0^1 \left[ \frac{g \left( b \right)}{g \left( a \right)} \right]^{2t} dt \right)
\]
\[
= \frac{1}{2} \left( f^2 \left( b \right) \left[ \frac{f \left( a \right)}{2 \log \frac{f \left( a \right)}{f \left( b \right)}} \right]^{2t} \bigg|_0^1 + f^2 \left( a \right) \left[ \frac{f \left( b \right)}{2 \log \frac{f \left( b \right)}{f \left( a \right)}} \right]^{2t} \bigg|_0^1 \right.
\]
\[
\left. + g^2 \left( b \right) \left[ \frac{g \left( a \right)}{2 \log \frac{g \left( a \right)}{g \left( b \right)}} \right]^{2t} \bigg|_0^1 + g^2 \left( a \right) \left[ \frac{g \left( b \right)}{2 \log \frac{g \left( b \right)}{g \left( a \right)}} \right]^{2t} \bigg|_0^1 \right)
Proof. Since \( f(2.20) \leq 0 \)

Theorem 5. complete. \( \square \)

Combining (2.17) and (2.18), we get the required inequalities (2.14). The proof is complete.

Theorem 5. Let \( f, g : [0, \infty) \to [0, \infty) \) be such that \( fg \) is in \( L^1([a,b]) \), where \( 0 \leq a < b < \infty \). If \( f \) is non-increasing \( m_1 \)-convex function and \( g \) is non-increasing \( m_2 \)-convex function on \([a, b]\) for some fixed \( m_1, m_2 \in (0, 1)\), then the following inequality holds:

(2.19) \[
\frac{1}{b-a} \int_a^b f(x) g(x) \, dx \leq \min \{ S_1, S_2 \}
\]

where

\[
S_1 = \frac{1}{6} \left[ (f^2(a) + g^2(a)) + m_1 f(a) f \left( \frac{b}{m_1} \right) 
+ m_2 g(a) g \left( \frac{b}{m_2} \right) + m_1^2 f^2 \left( \frac{b}{m_1} \right) + m_2^2 g^2 \left( \frac{b}{m_2} \right) \right]
\]

\[
S_2 = \frac{1}{6} \left[ (f^2(b) + g^2(b)) + m_1 f(b) f \left( \frac{a}{m_1} \right) 
+ m_2 g(b) g \left( \frac{a}{m_2} \right) + m_1^2 f^2 \left( \frac{a}{m_1} \right) + m_2^2 g^2 \left( \frac{a}{m_2} \right) \right].
\]

Proof. Since \( f \) is \( m_1 \)-convex function and \( g \) is \( m_2 \)-convex function, we have

(2.20) \[
f(ta + (1-t)b) \leq tf(a) + m_1(1-t) f \left( \frac{b}{m_1} \right)
\]

and

(2.21) \[
g(ta + (1-t)b) \leq tg(a) + m_2(1-t) g \left( \frac{b}{m_2} \right)
\]

for all \( t \in [0, 1] \). It is easy to observe that

(2.22) \[
\int_a^b f(x) g(x) \, dx = (b-a) \int_0^1 f(ta + (1-t)b) g(ta + (1-t)b) \, dt.
\]
Using the elementary inequality $cd \leq \frac{1}{2} \left( c^2 + d^2 \right)$ ($c, d \geq 0$ reals), (2.20) and (2.21) on the right side of (2.22) and making the charge of variable and since $f, g$ is non-increasing, we have

\begin{equation}
\int_a^b f(x)g(x) \, dx
\end{equation}

\begin{align*}
&\leq \frac{1}{2} (b-a) \int_0^1 \left[ \{f (ta + (1-t)b)\}^2 + \{g (ta + (1-t)b)\}^2 \right] \, dt \\
&\leq \frac{1}{2} (b-a) \int_0^1 \left[ \left( tf (a) + m_1 (1-t) f \left( \frac{b}{m_1} \right) \right)^2 \\
&\quad + \left( tg (a) + m_2 (1-t) g \left( \frac{b}{m_2} \right) \right)^2 \right] \, dt \\
&= \frac{1}{2} (b-a) \left[ \frac{1}{3} f^2 (a) + \frac{1}{3} m_1^2 f^2 \left( \frac{b}{m_1} \right) + \frac{1}{3} m_1 f (a) f \left( \frac{b}{m_1} \right) \\
&\quad + \frac{1}{3} g^2 (a) + \frac{1}{3} m_2^2 g^2 \left( \frac{b}{m_2} \right) + \frac{1}{3} m_2 g (a) g \left( \frac{b}{m_2} \right) \right] \\
&= \frac{(b-a)}{6} \left[ (f^2 (a) + g^2 (a)) + m_1 f (a) f \left( \frac{b}{m_1} \right) \\
&\quad + m_2 g (b) g \left( \frac{b}{m_2} \right) + m_1^2 f^2 \left( \frac{a}{m_1} \right) + m_2^2 g^2 \left( \frac{a}{m_2} \right) \right].
\end{align*}

Analogously we obtain

\begin{equation}
\int_a^b f(x)g(x) \, dx
\end{equation}

\begin{align*}
&\leq \frac{(b-a)}{6} \left[ (f^2 (b) + g^2 (b)) + m_1 f (b) f \left( \frac{a}{m_1} \right) \\
&\quad + m_2 g (b) g \left( \frac{b}{m_2} \right) + m_1^2 f^2 \left( \frac{a}{m_1} \right) + m_2^2 g^2 \left( \frac{a}{m_2} \right) \right].
\end{align*}

Rewriting (2.23) and (2.24), we get the required inequality in (2.19). The proof is complete.

**Theorem 6.** Let $f, g : [0, \infty) \to [0, \infty)$ be such that $f g$ is in $L^1 ([a,b])$, where $0 \leq a < b < \infty$. If $f$ is non-increasing $(\alpha_1, m_1) -$convex function and $g$ is non-increasing $(\alpha_2, m_2) -$convex function on $[a, b]$ for some fixed $\alpha_1, m_1, \alpha_2, m_2 \in (0,1]$. Then the following inequality holds:

\begin{equation}
\frac{1}{b-a} \int_a^b f(x)g(x) \, dx \leq \min \{ E_1, E_2 \}
\end{equation}
where

\[
E_1 = \frac{1}{2} \left[ \frac{1}{2\alpha_1 + 1} f^2(a) + \frac{2\alpha_1^2}{(\alpha_1 + 1)(2\alpha_1 + 1)} m_1^2 f^2 \left( \frac{b}{m_1} \right) 
+ \frac{2\alpha_1}{(\alpha_1 + 1)(2\alpha_1 + 1)} m_1 f(a) f \left( \frac{b}{m_1} \right) + \frac{1}{2\alpha_1 + 1} g^2(a)
+ \frac{2\alpha_2^2}{(\alpha_2 + 1)(2\alpha_2 + 1)} m_2^2 g^2 \left( \frac{b}{m_2} \right)
+ \frac{2\alpha_2}{(\alpha_2 + 1)(2\alpha_2 + 1)} m_2 g(a) \left( \frac{b}{m_2} \right) \right],
\]

\[
E_2 = \frac{1}{2} \left[ \frac{1}{2\alpha_1 + 1} f^2(b) + \frac{2\alpha_1^2}{(\alpha_1 + 1)(2\alpha_1 + 1)} m_1^2 f^2 \left( \frac{a}{m_1} \right)
+ \frac{2\alpha_1}{(\alpha_1 + 1)(2\alpha_1 + 1)} m_1 f(b) f \left( \frac{a}{m_1} \right) + \frac{1}{2\alpha_1 + 1} g^2(b)
+ \frac{2\alpha_2^2}{(\alpha_2 + 1)(2\alpha_2 + 1)} m_2^2 g^2 \left( \frac{a}{m_2} \right)
+ \frac{2\alpha_2}{(\alpha_2 + 1)(2\alpha_2 + 1)} m_2 g(b) \left( \frac{a}{m_2} \right) \right].
\]

Proof. Since \( f \) is \( (\alpha_1, m_1) \)-convex function and \( g \) is \( (\alpha_2, m_2) \)-convex function, then we have

\[
f(ta + (1 - t)b) \leq t^{\alpha_1} f(a) + m_1 (1 - t^{\alpha_1}) f \left( \frac{b}{m_1} \right)
\]

and

\[
g(ta + (1 - t)b) \leq t^{\alpha_2} g(a) + m_2 (1 - t^{\alpha_2}) g \left( \frac{b}{m_2} \right)
\]

for all \( t \in [0, 1] \). It is easy to observe that

\[
\int_a^b f(x) g(x) \, dx = (b - a) \int_0^1 f(ta + (1 - t)b) g(ta + (1 - t)b) \, dt.
\]

Using the elementary inequality \( cd \leq \frac{1}{2} (c^2 + d^2) \) \( (c, d \geq 0 \text{ reals}) \), (2.26) and (2.27) on the right side of (2.28) and making the charge of variable and since \( f, g \) is non-increasing, we have

\[
\int_a^b f(x) g(x) \, dx
\leq \frac{1}{2} (b - a) \int_0^1 \left[ \{ f(ta + (1 - t)b) \}^2 + \{ g(ta + (1 - t)b) \}^2 \right] \, dt
\]

\leq \frac{1}{2} (b - a) \int_0^1 \left[ \left( t^{\alpha_1} f(a) + m_1 (1 - t^{\alpha_1}) f \left( \frac{b}{m_1} \right) \right)^2
+ \left( t^{\alpha_2} g(a) + m_2 (1 - t^{\alpha_2}) g \left( \frac{b}{m_2} \right) \right)^2 \right] \, dt
\[
\begin{align*}
&= \frac{1}{2} (b - a) \left[ \frac{1}{2\alpha_1 + 1} f^2(a) \\
&+ \frac{2\alpha_1^2}{(\alpha_1 + 1)(2\alpha_1 + 1)} m_1^2 f^2 \left( \frac{b}{m_1} \right) \\
&+ \frac{2\alpha_1}{(\alpha_1 + 1)(2\alpha_1 + 1)} m_1 f(a) f \left( \frac{b}{m_1} \right) + \frac{1}{2\alpha_2 + 1} g^2(a) \\
&+ \frac{2\alpha_2^2}{(\alpha_2 + 1)(2\alpha_2 + 1)} m_2^2 g^2 \left( \frac{b}{m_2} \right) \\
&+ \frac{2\alpha_2}{(\alpha_2 + 1)(2\alpha_2 + 1)} m_2 g(a) g \left( \frac{b}{m_2} \right) \right].
\end{align*}
\]

Analogously we obtain

\[
(2.30) \quad \int_a^b f(x) g(x) \, dx \leq \frac{1}{2} (b - a) \left[ \frac{1}{2\alpha_1 + 1} f^2(b) \\
+ \frac{2\alpha_1^2}{(\alpha_1 + 1)(2\alpha_1 + 1)} m_1^2 f^2 \left( \frac{a}{m_1} \right) \\
+ \frac{2\alpha_1}{(\alpha_1 + 1)(2\alpha_1 + 1)} m_1 f(b) f \left( \frac{a}{m_1} \right) + \frac{1}{2\alpha_2 + 1} g^2(b) \\
+ \frac{2\alpha_2^2}{(\alpha_2 + 1)(2\alpha_2 + 1)} m_2^2 g^2 \left( \frac{a}{m_2} \right) \\
+ \frac{2\alpha_2}{(\alpha_2 + 1)(2\alpha_2 + 1)} m_2 g(b) g \left( \frac{a}{m_2} \right) \right].
\]

Rewriting (2.29) and (2.30), we get the required inequality in (2.25). The proof is complete. □

**Remark 3.** In Theorem 6, if we choose \( \alpha_1 = \alpha_2 = 1 \), we obtain the inequality of Theorem 5.

**References**

[1] M. Alomari and M. Darus, On the Hadamard’s inequality for log-convex functions on the coordinates, *Journal of Inequalities and Applications*, vol. 2009, Article ID 283147, 13 pages, 2009.

[2] M. K. Bakula, M. E. Özdemir and J. Pečarić, Hadamard type inequalities for \( m \)-convex and \((\alpha, m)\) -convex functions, *J. Inequal. Pure & Appl. Math.*, 9(2008), Article 96.

[3] M. K. Bakula, J. Pečarić, and M. Ribičić, Companion inequalities to Jensen’s inequality for \( m \)-convex and \((\alpha, m)\) -convex functions, *J. Inequal. Pure & Appl. Math.*, 7(2006), Article 194.

[4] X.-M. Zhang, Y.-M. Chu and X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its application, *Journal of Inequalities and Applications*, vol. 2010, Article ID 507560, 11 pages, 2010.

[5] C. Dinu, Hermite-Hadamard inequality on time scales, *Journal of Inequalities and Applications*, vol. 2008, Article ID 287947, 24 pages, 2008.

[6] S.S. Dragomir and C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, *RGMIA Monographs*, Victoria University, 2000. [ONLINE: http://www.staff.vu.edu.au/RGMIA/monographs/hermite-hadamard.html].

[7] P.M. Gill, C.E.M Pearce and J. Pečarić, Hadamard’s inequality for \( r \)-convex functions, *J. Math. Anal. Appl.*, 215(1997), 461-470.
[8] V. G. Miheşan, A generalization of the convexity, *Seminar on Functional Equations, Approx. and Convex.*, Cluj-Napoca (Romania) (1993)

[9] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, Classical and New Inequalities in Analysis, *Kluwer Academic Publishers, Dordrecht*, 1993.

[10] M.E. Özdemir, M. Avcı and E. Set, On some inequalities of Hermite-Hadamard type via $m$-convexity, *Appl. Math. Lett.* (2010), doi: 10.1016/j.aml.2010.04.037 (in press).

[11] B.G. Pachpatte, A note on integral inequalities involving two log-convex functions, *Math. Inequal. Appl.*, 7(4) (2004), 511-515.

[12] E. Set, M.E. Özdemir and S.S. Dragomir, On the Hermite-Hadamard inequality and other integral inequalities involving two functions, *Journal of Inequalities and Applications, Articles in Press* (2010).

[13] G. Toader, Some generalizations of the convexity, *Proc. Colloq. Approx. Opt. Cluj-Napoca*, (1984), 329-338.

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