MOEBIUS RIGIDITY FOR COMPACT DEFORMATIONS OF NEGATIVELY CURVED MANIFOLDS

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ABSTRACT. Let $(X, g_0)$ be a complete, simply connected Riemannian manifold with sectional curvatures $K_{g_0}$ satisfying $-b^2 \leq K_{g_0} \leq -1$ for some $b \geq 1$. Let $g_1$ be a Riemannian metric on $X$ such that $g_1 = g_0$ outside a compact in $X$, and with sectional curvatures $K_{g_1}$ satisfying $K_{g_1} \leq -1$. The identity map $id : (X, g_0) \rightarrow (X, g_1)$ is bi-Lipschitz, and hence induces a homeomorphism between the boundaries at infinity of $(X, g_0)$ and $(X, g_1)$, which we denote by $\hat{id}_{g_0, g_1} : \partial g_0 X \rightarrow \partial g_1 X$. We show that if the boundary map $\hat{id}_{g_0, g_1}$ is Moebius (i.e. preserves cross-ratios), then it extends to an isometry $F : (X, g_0) \rightarrow (X, g_1)$.

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1. INTRODUCTION

In various rigidity problems for negatively curved spaces, the interplay between the geometry of the space and the geometry of its boundary at infinity plays a prominent role. For a CAT(-1) space $X$, there is a positive function called the cross-ratio defined on the space of quadruples of distinct points in the boundary $\partial X$, and a well-known problem asks whether the cross-ratio in fact determines the space up to isometry. More precisely, if $f : \partial X \rightarrow \partial Y$ is a Moebius homeomorphism between boundaries of CAT(-1) spaces $X, Y$ (i.e. a homeomorphism which preserves cross-ratios), then the question is whether $f$ extends to an isometry $F : X \rightarrow Y$. It is a classical result that this holds when $X = Y = \mathbb{H}^n$, the real hyperbolic space, a fact which is often used in rigidity theorems for hyperbolic manifolds, for example in the Mostow Rigidity theorem [Most68]. More generally, Bourdon [Bou96] showed that if $X$ is a rank one symmetric space of noncompact type with the metric normalized such that the maximum of the sectional curvatures equals $-1$, and $Y$ is any CAT(-1) space, then any Moebius embedding $f : \partial X \rightarrow \partial Y$ extends to an isometric embedding $F : X \rightarrow Y$. For general CAT(-1) spaces $X, Y$, the problem remains open.
We should remark that one of the main motivations for studying this problem is its relation to the marked length spectrum rigidity conjecture of Burns and Katok, which asks whether two closed negatively curved manifolds $X, Y$ with the same marked length are necessarily isometric. Otal \cite{Ota90} and independently Croke \cite{Cro90} proved that marked length spectrum rigidity holds in two dimensions. It is well known that in fact $X, Y$ have the same marked length spectrum if and only if there is an equivariant Moebius map between the boundaries of the universal covers $f : \partial X \to \partial Y$, so a positive answer to the problem of extending Moebius maps to isometries would also give a solution to the marked length spectrum rigidity problem (see \cite{Ham92}). Equality of marked length spectra is also known to be equivalent to existence of a homeomorphism between the unit tangent bundles $\phi : T^1 X \to T^1 Y$ conjugating the geodesic flows of $X, Y$ (\cite{Ham92}). Proofs of these equivalences may be found in \cite{Bis15}, section 5. We remark that in related work Beyrer, Fioravanti and Incerti-Medici have constructed a cross-ratio on the Roller boundary of any CAT(0) cube complex, and have shown that any cross-ratio preserving bijection between geodesically complete cube complexes admits a unique extension to an isomorphism of cube-complexes, and have also proved a version of marked length spectrum rigidity for group actions on CAT(0) cube complexes \cite{BFIM18}.

In \cite{Bis15}, it was shown that a Moebius homeomorphism between the boundaries of proper, geodesically complete CAT(-1) spaces extends to a $(1, \log 2)$-quasi-isometry between the spaces. For $X, Y$ complete, simply connected manifolds of pinched negative curvature $-b^2 \leq K \leq -1$, this result was refined in \cite{Bis17} to show that the extension may be taken in this case to be a $(1, (1-1/b) \log 2)$-quasi-isometry. In fact the quasi-isometric extension of \cite{Bis15} and \cite{Bis17} was shown to be given by a certain natural extension of Moebius maps called the circumcenter extension, which is natural with respect to composition with isometries. In \cite{Bis17b}, it was shown that if $f : \partial X \to \partial Y$ and $g : \partial Y \to \partial X$ are mutually inverse Moebius homeomorphisms between boundaries of complete, simply connected manifolds $X, Y$ of pinched negative curvature $-b^2 \leq K \leq -1$, then the circumcenter extensions $F : X \to Y$ and $G : Y \to X$ of $f, g$ are $\sqrt{b}$-bi-Lipschitz homeomorphisms which are inverses of each other.

In the present article we consider compactly supported deformations of the metric on a complete, simply connected manifold $(X, g_0)$ of pinched negative curvature $-b^2 \leq K_{g_0} \leq -1$, i.e. we consider metrics $g_1$ on $X$ such that $g_1 = g_0$ outside a compact in $X$, and such that the sectional curvature of $g_1$ is bounded above by $-1$. The identity map $id : (X, g_0) \to (X, g_1)$ is clearly bi-Lipschitz, hence it induces a homeomorphism between boundaries which we denote by $id_{g_0,g_1} : \partial_{g_0} X \to \partial_{g_1} X$, and the problem in this context becomes the following: if $id_{g_0,g_1}$ is Moebius, then does it extend to an isometry $F : (X, g_0) \to (X, g_1)$? Partial results for this problem were obtained in \cite{Bis16}, where local and infinitesimal versions of the problem were considered, namely metrics $g_1$ such that the $C^2$ norm $||g_0 - g_1||_{C^2}$ is small, and one-parameter families of metrics $(g_t)_{0 \leq t \leq 1}$, and in both cases it was shown that if the boundary maps are Moebius then they extend to isometries. Our main theorem below gives a complete solution to this problem:

**Theorem 1.1.** Let $(X, g_0)$ be a complete, simply connected manifold of pinched negative curvature $-b^2 \leq K_{g_0} \leq -1$. Let $g_1$ be a metric on $X$ such that $g_1 = g_0$ outside a compact in $X$, and such that the sectional curvature of $g_1$ satisfies $K_{g_1} \leq$
Let \( \hat{\text{id}}_{g_0,g_1} : \partial g_0 X \to \partial g_1 X \) denote the homeomorphism between boundaries induced by the identity map \( \text{id} : (X,g_0) \to (X,g_1) \). Suppose \( \hat{\text{id}}_{g_0,g_1} \) is Moebius. Then the circumcenter extension of \( \hat{\text{id}}_{g_0,g_1} \) is an isometry \( F : (X,g_0) \to (X,g_1) \).

The key to the proof of the above theorem is a further study of properties of the circumcenter extension. In section 2 we briefly recall some facts about Moebius maps, geodesic conjugacies and circumcenter extensions. In section 3 we prove the results about the circumcenter extension which are used in the proof of the main theorem, while section 4 is devoted to the proof of the main theorem.

2. Preliminaries

We give below a brief outline of the background on Moebius maps which we will be needing, for details and proofs of the assertions below the reader is referred to [Bis15], [Bis17a], [Bis17b].

Let \((Z, \rho_0)\) be a compact metric space of diameter one. The cross-ratio with respect to a metric \( \rho \) on \( Z \) is the function on quadruples of distinct points in \( Z \) defined by
\[
[\xi, \xi', \eta, \eta'] \mapsto \frac{\rho(\xi, \eta) \rho(\xi', \eta')}{\rho(\xi, \eta') \rho(\xi', \eta)} , \quad \xi, \xi', \eta, \eta' \in Z
\]
Two metrics \( \rho_1, \rho_2 \) on \( Z \) are said to be Moebius equivalent if their cross-ratios are equal, \([..., ...,]_{\rho_1} = [...,...,]_{\rho_2} \). A metric \( \rho \) on \( Z \) is said to be antipodal if it has diameter one and for all \( \xi \in Z \) there exists \( \eta \in Z \) such that \( \rho(\xi, \eta) = 1 \). Assume that the metric \( \rho_0 \) is antipodal. We then define \( \mathcal{M}(Z, \rho_0) \) to be the set of all antipodal metrics \( \rho \) on \( Z \) which are Moebius equivalent to \( \rho_0 \). Then for any two metrics \( \rho_1, \rho_2 \in \mathcal{M}(Z, \rho_0) \), there is a positive continuous function on \( Z \) called the derivative of the metric \( \rho_2 \) with respect to the metric \( \rho_1 \), denoted by \( \frac{d\rho_2}{d\rho_1} \), such that
\[
\rho_2(\xi, \eta)^2 = \frac{d\rho_2}{d\rho_1}(\xi) \frac{d\rho_2}{d\rho_1}(\eta) \rho_1(\xi, \eta)^2
\]
for all \( \xi, \eta \in Z \). If \( \xi \) is not an isolated point of \( Z \), then
\[
\frac{d\rho_2}{d\rho_1}(\xi) = \lim_{\eta \to \xi} \frac{\rho_2(\xi, \eta)}{\rho_1(\xi, \eta)}
\]
Moreover
\[
\left( \max_{\xi \in Z} \frac{d\rho_2}{d\rho_1}(\xi) \right) \left( \min_{\xi \in Z} \frac{d\rho_2}{d\rho_1}(\xi) \right) = 1
\]
This allows us to define a metric on the set \( \mathcal{M}(Z, \rho_0) \) by
\[
d_M(\rho_1, \rho_2) := \max_{\xi \in Z} \log \frac{d\rho_2}{d\rho_1}(\xi)
\]
The metric space \( (\mathcal{M}(Z, \rho_0), d_M) \) is proper and complete. The following lemma follows from the proof of Lemma 2.6 of [Bis15], we include a proof for convenience:

**Lemma 2.1.** For \( \rho_1, \rho_2 \in \mathcal{M}(Z, \rho_0) \), let \( \xi, \eta \in Z \) be points where \( \frac{d\rho_2}{d\rho_1} \) attains its maximum and minimum values respectively. If \( \xi' \in Z \) is such that \( \rho_1(\xi, \xi') = 1 \), then \( \frac{d\rho_2}{d\rho_1} \) attains its minimum at \( \xi' \), and \( \rho_2(\xi, \xi') = 1 \). If \( \eta' \in Z \) is such that \( \rho_2(\eta, \eta') = 1 \), then \( \frac{d\rho_2}{d\rho_1} \) attains its maximum at \( \eta' \), and \( \rho_1(\eta, \eta') = 1 \).
Proof: Let λ, μ > 0 be the maximum and minimum values of \( \frac{dρ}{dρ_{1}} \) respectively, then we know that λ · μ = 1. For \( ξ' \in Z \) such that \( ρ_{1}(ξ, ξ') = 1 \), we have

\[
1 ≥ \rho_{2}(ξ, ξ')^{2} = \frac{dρ_{2}(ξ)}{dρ_{1}}(ξ') \frac{dρ_{2}(ξ)}{dρ_{1}}(ξ')ρ_{1}(ξ, ξ')^{2} ≥ λ · μ · 1 = 1
\]

so equality holds in the inequalities above, hence \( \frac{dρ_{2}(ξ)}{dρ_{1}(ξ')} = μ \) and \( ρ_{2}(ξ, ξ') = 1 \).

For \( η' \in Z \) such that \( ρ_{2}(η, η') = 1 \), we have

\[
1 = \rho_{2}(η, η')^{2} = \frac{dρ_{2}(η)}{dρ_{1}(η)}(η') \frac{dρ_{2}(η)}{dρ_{1}(η)}(η')ρ_{1}(η, η')^{2} ≤ μ · λ · 1 = 1
\]

so equality holds in the inequalities above, hence \( \frac{dρ_{2}(η')}{dρ_{1}(η')} = λ \) and \( ρ_{1}(η, η') = 1 \).

Let \( f : (Z_{1}, ρ_{1}) \rightarrow (Z_{2}, ρ_{2}) \) be a homeomorphism between metric spaces. We say \( f \) is Möbius if \( f \) preserves cross-ratios with respect to the metrics \( ρ_{1} \) and \( ρ_{2} \), i.e., \( [f(ξ), f(ξ'), f(η), f(η')]_{ρ_{2}} = [ξ, ξ', η, η']_{ρ_{1}} \) for all quadruples of distinct points \( ξ, ξ', η, η' \in Z_{1} \). Then the metrics \( ρ_{1} \) and \( f^{*}ρ_{2} \) (the pull-back of \( ρ_{2} \) by \( f \)) are Möbius equivalent, and we define the derivative of the Möbius map \( f \) with respect to the metrics \( ρ_{1}, ρ_{2} \) to be the function \( \frac{df^{*}ρ_{2}}{dρ_{1}} \).

Let \( X \) be a proper, geodesically complete CAT(-1) space (this means that every finite geodesic segment in \( X \) can be extended to a bi-infinite geodesic), with boundary at infinity \( \partial X \). The Busemann function of \( X \) is the function \( B : X \times X × \partial X \rightarrow \mathbb{R} \) defined by

\[
B(x, y, ξ) := \lim_{z \rightarrow ξ}(d(x, z) − d(y, z)), \ x, y \in X, ξ \in \partial X
\]

Note that \( |B(x, y, ξ)| ≤ d(x, y) \) for all \( x, y \in X, ξ \in \partial X \). For \( x \in X \) and \( ξ, η \in \partial X \), we denote by \([x, ξ] \subset X \) the unique geodesic ray joining \( x \) to \( ξ \), and we denote by \( (ξ, η) \subset X \) the unique bi-infinite geodesic joining \( ξ \) and \( η \). For every \( x \in X \), there is a metric \( ρ_{x} \) on \( \partial X \) called the visual metric on \( \partial X \) based at \( X \), defined by \( ρ_{x}(ξ, η) = e^{-\frac{1}{2}(ξ[η]x)}, \) where \( (ξ[η]x) \) is the Gromov inner product between \( ξ, η \in \partial X \) with respect to the basepoint \( x \in X \), defined by

\[
(ξ[η]x) := \lim_{y \rightarrow ξ, z \rightarrow η} \frac{1}{2}(d(x, y) + d(x, z) − d(y, z))
\]

The metric space \( (\partial X, ρ_{x}) \) is compact of diameter one, and the metric \( ρ_{x} \) is antipodal. We have \( ρ_{x}(ξ, η) = 1 \) if and only if the point \( x \) lies on the bi-infinite geodesic \( (ξ, η) \). Moreover, any two visual metrics \( ρ_{x}, ρ_{y} \) on \( \partial X \) are Möbius equivalent, so there is a canonical cross-ratio function on quadruples of distinct points in \( \partial X \), which we will denote by simply \( [\cdot, \cdot, \cdot, \cdot] \). The derivative \( \frac{dρ_{2}}{dρ_{1}} \) is given by

\[
\frac{dρ_{2}}{dρ_{1}}(ξ) = e^{B(x, y, ξ)}
\]

The space \( \mathcal{M}(\partial X, ρ_{x}) \) is independent of the choice of \( x \in X \), and we will denote it by \( \mathcal{M}(\partial X) \). The map \( i_{X} : X \rightarrow \mathcal{M}(\partial X), x \mapsto ρ_{x} \), is an isometric embedding, and the image is \( \frac{1}{2} \log 2 \)-dense in \( \mathcal{M}(\partial X) \).

For \( x \in X \) and a subset \( B \subset X \), we define the shadow of the set \( B \) as seen from \( x \) to be the subset of \( \partial X \) defined by

\[
\mathcal{O}(x, B) := \{ ξ \in \partial X \mid [x, ξ] \cap B \neq \emptyset \}
\]

The following lemma will be useful:
Lemma 2.2. Let $x_0 \in X$ and $R > 0$. For $x \in X$, the diameter of the shadow $O(x, B(x_0, R))$ with respect to the visual metric $\rho_x$ tends to 0 as $x \to \infty$. More precisely, for all $\xi, \eta \in O(x, B(x_0, R))$,
\[
\rho_x(\xi, \eta) \leq e^{2R-d(x, x_0)}
\]

Proof: Given $x \in X$ and $\xi \in O(x, B(x_0, R))$, by definition there exists $z \in [x, \xi] \cap B(x_0, R)$. Then we have
\[
B(x, x_0, \xi) = B(x, z, \xi) + B(z, x_0, \xi) \\
\geq d(x, z) - d(z, x_0) \\
\geq d(x, x_0) - 2R
\]

Thus for $\xi, \eta \in O(x, B(x_0, R))$ we have
\[
\rho_x(\xi, \eta)^2 = \frac{d\rho_x(\xi)}{d\rho_x(\eta)} \rho_x(\xi, \eta)^2 \]
\[
= e^{B(x, x_0, \xi)} e^{B(x, z, \xi)} \rho_x(\xi, \eta)^2 \\
\leq e^{2R-d(x, x_0)} e^{2R-d(x, x_0)}
\]

and so
\[
\rho_x(\xi, \eta) \leq e^{2R-d(x, x_0)}
\]

The space of geodesics $GX$ of $X$ is defined to be the space $GX := \{ \gamma : \mathbb{R} \to X \mid \gamma \text{ is an isometric embedding} \}$ equipped with the topology of uniform convergence on compacts. We define continuous maps $\pi : GX \to X$ and $p : GX \to \partial X$ by $\pi(\gamma) = \gamma(0) \in X$ and $p(\gamma) = \gamma(+\infty) \in \partial X$, and for $x \in X$, we define $T^1_+X := \pi^{-1}(x) \subset GX$. The geodesic flow of the CAT(-1) space $X$ is the one-parameter group of homeomorphisms $(\phi_t : GX \to GX)_{t \in \mathbb{R}}$ defined by $(\phi_t(\gamma))(s) := \gamma(s + t)$. When $X$ is a simply connected, complete Riemannian manifold of negative sectional curvature $K \leq -1$, then the map $GX \to T^1_+X, \gamma \mapsto \gamma(0)$ is a homeomorphism conjugating the geodesic flow on $GX$ to the usual geodesic flow on $T^1_+X$.

Let $Y$ be another proper, geodesically complete CAT(-1) space, and suppose there is a M"obius homeomorphism $f : \partial X \to \partial Y$. The M"obius map $f$ induces a homeomorphism $\phi : GX \to GY$ conjugating the geodesic flows, which is defined as follows: given $\gamma \in GX$, let $x = \gamma(0), \xi = \gamma(+\infty), \eta = \gamma(-\infty)$, then $\phi(\gamma)$ is defined to be the unique $\tilde{\gamma} \in GY$ such that $\tilde{\gamma}(+\infty) = f(\xi), \tilde{\gamma}(-\infty) = f(\eta)$, and $\tilde{\gamma}(0) = y$, where $y$ is the unique point in the bi-infinite geodesic $(f(\eta), f(\xi)) \subset Y$ such that $\frac{d\tilde{\gamma}}{dt}(\xi) = 1$.

In a CAT(-1) space, any bounded set $B \subset X$ has a unique circumcenter $c(B) \in X$, i.e. the unique point minimizing the function $x \in X \mapsto \sup_{y \in B} d(x, y)$. For a compact set $K \subset GX$ such that $p(K) \subset \partial X$ has at least two points, the limit of the circumcenters $c(\pi(\phi_t(K)))$ exists as $t \to +\infty$, we call the limit the asymptotic circumcenter of the set $K$ and denote it by $c_{\infty}(K) \in X$. The geodesic conjugacy
\( \phi : \mathcal{G}X \to \mathcal{G}Y \) induced by a Moebius map \( f : \partial X \to \partial Y \) then allows us to define an extension \( F : X \to Y \) of \( f \), called the circumcenter extension of \( f \), by

\[
F(x) := c_\infty(\phi(T^1_x X)) \in Y
\]

The circumcenter extension is a \((1, \log 2)\)-quasi-isometry and is locally \(1/2\)-Holder. For \( x \in X \), the point \( F(x) \in Y \) can be characterized as the unique point in \( Y \) minimizing the function \( y \in Y \mapsto d_M(f_*\rho_x, \rho_y) \) (where \( f_*\rho_x \in \mathcal{M}(\partial Y) \) is the push-forward of \( \rho_x \in \mathcal{M}(\partial X) \) by the Moebius map \( f \)).

3. Some properties of the circumcenter extension

Throughout this section, \( X, Y \) will denote two complete, simply connected manifolds with pinched negative curvature \(-b^2 \leq K \leq -1\). Suppose there is a Moebius homeomorphism \( f : \partial X \to \partial Y \) with inverse \( g : \partial Y \to \partial X \), and let \( F : X \to Y \) and \( G : Y \to X \) be the circumcenter extensions of \( f \) and \( g \) respectively. Then from \cite{Bis17b}, we have that \( F \) and \( G \) are \( \sqrt{b}\)-bi-Lipschitz homeomorphisms which are inverses of each other. Define a function \( r : X \to \mathbb{R} \) by

\[
r(x) := d_M(f_*\rho_x, \rho_{F(x)}) = \sup_{\xi \in \partial X} \log \left( \frac{df_*\rho_x}{d\rho_{F(x)}}(f(\xi)) \right)
\]

In the following, we identify \( \mathcal{G}X, \mathcal{G}Y \) with \( T^1 X, T^1 Y \) respectively, and we identify the geodesic conjugacy \( \phi : \mathcal{G}X \to \mathcal{G}Y \) with a geodesic conjugacy \( \phi : T^1 X \to T^1 Y \). We also identify the maps \( \pi : \mathcal{G}X \to X, p : \mathcal{G}X \to \partial X \) with maps \( \pi : T^1 X \to X, p : T^1 X \to \partial X \) respectively (and similarly for the corresponding maps for \( Y \)). For \( x \in X, \xi \in \partial X \) we denote by \( \overrightarrow{x\xi} \in T^1_x X \) the unit tangent vector at \( x \) given by \( \gamma'(0) \), where \( \gamma \) is the unique geodesic satisfying \( \gamma(0) = x, \gamma(\infty) = \xi \). The flip \( T^1_x X \to T^1_x X, v \mapsto -v, \) induces a continuous involution \( i_x : \partial X \to \partial X, \) defined by requiring that \( \overrightarrow{x\xi} = -\overrightarrow{x\xi} \) for all \( \xi \in \partial X \). Similarly for \( y \in Y \) we have an involution \( i_y : \partial Y \to \partial Y \). The following lemma follows from Lemma 4.13 of \cite{Bis17b}:

**Lemma 3.1.** For \( x \in X, y \in Y, \xi \in \partial X \), we have

\[
\log \left( \frac{df_*\rho_x}{d\rho_y}(f(\xi)) \right) = B(y, \pi(\phi(\overrightarrow{x\xi})), f(\xi))
\]

In particular,

\[
r(x) = \sup_{\xi \in \partial X} B(F(x), \pi(\phi(\overrightarrow{x\xi})), f(\xi))
\]

**Lemma 3.2.** The function \( r : X \to \mathbb{R} \) is 1-Lipschitz.

**Proof:** Let \( x, y \in X \). Since \( \phi : T^1 X \to T^1 Y \) conjugates the geodesic flows, we have, for any \( \xi \in \partial X \),

\[
B(\pi(\phi(\overrightarrow{x\xi})), \pi(\phi(\overrightarrow{y\xi})), f(\xi)) = B(x, y, \xi)
\]
We then have, using Lemma 3.1 above,
\[
r(x) = d_M(f_*\rho_x, \rho_{F(x)}) \leq d_M(f_*\rho_x, \rho_{F(y)})
\]
\[
= \sup_{\xi \in \partial X} \log \frac{df_*\rho_x}{d\rho_{F(y)}}(f(\xi))
\]
\[
= \sup_{\xi \in \partial X} B(F(y), \pi(\phi(\overrightarrow{x\xi})), f(\xi))
\]
\[
= \sup_{\xi \in \partial X} B(F(y), \pi(\phi(\overrightarrow{y\xi})), f(\xi)) + B(\pi(\phi(\overrightarrow{x\xi})), \pi(\phi(\overrightarrow{y\xi})), f(\xi))
\]
\[
\leq \sup_{\xi \in \partial X} B(F(y), \pi(\phi(\overrightarrow{y\xi})), f(\xi)) + d(x, y)
\]
\[
= r(y) + d(x, y)
\]

Thus \(r(x) - r(y) \leq d(x, y)\). Interchanging \(x\) and \(y\) the same argument as above gives \(r(y) - r(x) \leq d(x, y)\), hence \(|r(x) - r(y)| \leq d(x, y)\). \(\diamondsuit\)

We say that a probability measure \(\mu\) on \(\partial X\) is balanced at a point \(x \in X\) if the vector-valued integral \(\int_{\partial X} \overrightarrow{v\xi} d\mu(\xi) \in T_xX\) equals \(0 \in T_xX\), or equivalently if \(\int_{\partial X} \langle v, \overrightarrow{v\xi} \rangle > d\mu(\xi) = 0\) for all \(v \in T_xX\). If the compact \(K \subset \partial X\) denotes the support of \(\mu\), then it is shown in [Bis17b] that \(\mu\) is balanced at \(x\) if and only if the convex hull in \(T_xX\) of the compact set \(\{x\xi : \xi \in K\}\) contains the origin of \(T_xX\).

For \(x \in X\), let \(K_x \subset \partial X\) denote the set on which the function \(\xi \in \partial X \mapsto \frac{d_f \rho_{F(x)}}{d\rho_{F(x)}}(f(\xi))\) attains its maximum value. In [Bis17b], it is shown that for any \(x \in X\), there exists a probability measure \(\mu_x\) on \(\partial X\) with support contained in \(K_x\) such that the measure \(\mu_x\) is balanced at \(x\), and such that the measure \(f_*\mu_x\) on \(\partial Y\) is balanced at \(F(x) \in Y\) (with a similar definition of balanced measures for measures on \(\partial Y\) and points of \(Y\)).

The main result of this section is the following:

**Theorem 3.3.** The function \(r\) is constant.

**Proof:** Since the function \(r\) and the circumcenter map \(F\) are both Lipschitz, they are differentiable almost everywhere, so the set of points \(D \subset X\) at which both \(r\) and \(F\) are differentiable has full measure. Let \(x \in D\) and let \(\xi \in K_x\). Then for any \(y \in X\),
\[
r(y) \geq B(F(y), \pi(\phi(\overrightarrow{y\xi})), f(\xi))
\]
\[
= B(F(y), F(x), f(\xi)) + B(F(x), \pi(\phi(\overrightarrow{x\xi})), f(\xi)) + B(\pi(\phi(\overrightarrow{x\xi})), \pi(\phi(\overrightarrow{y\xi})), f(\xi))
\]
\[
= B(F(y), F(x), f(\xi)) + r(x) + B(x, y, \xi)
\]
so
\[
(1) \quad r(y) - r(x) \geq B(F(y), F(x), f(\xi)) + B(x, y, \xi)
\]
for all \( y \in X, \xi \in K_x \). It is well-known that the gradient at \( x \) of the function \( y \in X \mapsto B(x, y, \xi) \) is given by the vector \( \overrightarrow{\xi} \), while the gradient at \( F(x) \) of the function \( z \in Y \mapsto B(z, F(x), f(\xi)) \) is given by the vector \( -\overrightarrow{F(x)f(\xi)} \). Let \( v \in T_xX \) and \( t > 0 \), and let \( y = \exp_v tv \in X \). Then as \( t \to 0 \), using the fact that \( r \) and \( F \) are differentiable at \( x \), equation \( \text{(1)} \) above gives
\[
\frac{dr_x(tv) + o(t)}{t} \geq -\langle dF_x(tv), \overrightarrow{F(x)f(\xi)} \rangle + \langle tv, \overrightarrow{\xi} \rangle + o(t)
\]
so dividing by \( t \) above and letting \( t \) tend to 0 gives
\[
\text{(2)} \quad \frac{dr_x(v)}{v} \geq \langle v, \overrightarrow{\xi} \rangle - \langle dF_x(v), \overrightarrow{F(x)f(\xi)} \rangle
\]
for all \( v \in T_xX, \xi \in K_x \). Integrating both sides of inequality \( \text{(2)} \) above over the set \( K_x \) with respect to the probability measure \( \mu_x \), and using the facts that the support of \( \mu_x \) is contained in \( K_x \), the measure \( \mu_x \) is balanced at \( x \) and the measure \( f_*\mu_x \) is balanced at \( F(x) \), we obtain
\[
\text{dr}_x(v) = \int_{\partial X} \text{dr}_x(v) d\mu_x(\xi) = \int_{K_x} \text{dr}_x(v) d\mu_x(\xi)
\]
\[
\geq \int_{K_x} \langle v, \overrightarrow{\xi} \rangle d\mu_x(\xi) - \int_{K_x} \langle dF_x(v), \overrightarrow{F(x)f(\xi)} \rangle d\mu_x(\xi)
\]
\[
= \int_{\partial X} \langle v, \overrightarrow{\xi} \rangle d\mu_x(\xi) - \int_{\partial X} \langle dF_x(v), \overrightarrow{F(x)f(\xi)} \rangle d\mu_x(\xi)
\]
\[
= \int_{\partial X} \langle v, \overrightarrow{\xi} \rangle d\mu_x(\xi) - \int_{\partial Y} \langle dF_x(v), \overrightarrow{F(x)f(\xi)} \rangle d\mu_x(\eta)
\]
\[
= 0
\]
Thus \( \text{dr}_x(v) \geq 0 \) for all \( v \in T_xX \), replacing \( v \) by \( -v \) gives \( \text{dr}_x(-v) \geq 0 \) so \( \text{dr}_x(v) \leq 0 \) for all \( v \in T_xX \), and hence \( \text{dr}_x(v) = 0 \) for all \( v \in T_xX \). Since \( r \) is Lipschitz and \( dr_x = 0 \) for \( x \) in the full measure set \( D \), it follows that \( r \) is constant.

\[ \diamond \]

A corollary of the proof of the above theorem is the following:

**Proposition 3.4.** Let \( x \in X \) be a point of differentiability of \( F \). Then for all \( \xi \in K_x, v \in T_xX \) we have
\[
\langle dF_x(v), \overrightarrow{F(x)f(\xi)} \rangle = \langle v, \overrightarrow{\xi} \rangle
\]
Equivalently,
\[
dF_x^\ast(\overrightarrow{F(x)f(\xi)}) = \overrightarrow{\xi}
\]
for all \( \xi \in K_x \).

**Proof:** By the previous theorem the function \( r \) is constant, so the set \( D \) in the proof of the previous theorem is just the set of points of differentiability of \( F \). Let \( x \in D \), and \( \xi \in K_x \). Since \( r \) is constant, equation \( \text{(2)} \) above gives
\[
0 \geq \langle v, \overrightarrow{\xi} \rangle - \langle dF_x(v), \overrightarrow{F(x)f(\xi)} \rangle
\]
for all \( v \in T_xX \). Replacing \( v \) by \( -v \) in the above equation gives
\[
0 \leq \langle v, \overrightarrow{\xi} \rangle - \langle dF_x(v), \overrightarrow{F(x)f(\xi)} \rangle
\]
for all \( v \in T_x X \). Combining the two gives \( <DF_x(v), \overrightarrow{F(x)\xi}> = <v, \overrightarrow{x\xi}> \) for all \( v \in T_x X \).

**Lemma 3.5.** Let \( M \geq 0 \) denote the constant value of the function \( r \). Then the circumcenter map \( F : X \to Y \) is a \((1,2M)\)-quasi-isometry, i.e.

\[
d(x,y) - 2M \leq d(F(x),F(y)) \leq d(x,y) + 2M
\]

for all \( x, y \in X \).

**Proof:** Note that push-forward of metrics by \( f \) gives an isometry \( f_* : \mathcal{M}(\partial X) \to \mathcal{M}(\partial Y) \). So for \( x, y \in X \), we have

\[
d(x,y) = d_M(p_x,p_y)
\]

\[
= d_M(f_*p_x,f_*p_y)
\]

\[
\leq d_M(f_*p_x,\rho_{F(x)}) + d_M(\rho_{F(x)},\rho_{F(y)}) + d_M(\rho_{F(y)},f_*p_y)
\]

\[
= M + d(F(x),F(y)) + M
\]

Similarly,

\[
d(F(x),F(y)) = d_M(\rho_{F(x)},\rho_{F(y)})
\]

\[
\leq d_M(\rho_{F(x)},f_*p_x) + d_M(f_*p_x,f_*p_y) + d_M(f_*p_y,\rho_{F(y)})
\]

\[
= M + d(x,y) + M
\]

thus

\[
d(x,y) - 2M \leq d(F(x),F(y)) \leq d(x,y) + 2M
\]

The following lemma is a straightforward consequence of Lemma 2.1

**Lemma 3.6.** Let \( x \in X \) and \( y \in Y \). Then:

(i) The function \( \frac{d\rho_x}{d\xi} \) attains its maximum at \( \xi \in \partial X \) if and only if it attains its minimum at \( i_x(\xi) \). Moreover in this case \( f(i_x(\xi)) = i_y(f(\xi)) \), so \( y \) lies on the bi-infinite geodesic \( (f(\xi),f(i_x(\xi))) \).

(ii) If \( \xi \in \partial X \) is a maximum of \( \frac{d\rho_x}{d\xi} \) then the point \( z = \pi(\overrightarrow{\xi y}) \in Y \) is the unique point on the geodesic ray \( [y,f(\xi)] \subset Y \) at a distance \( d_M(f_*p_x,p_y) \) from \( y \).

**Proof:** (i) We first note that since \( X \) is a simply connected manifold of negative curvature, for \( \xi, \eta \in \partial X \) we have \( \rho_x(\xi,\eta) = 1 \) if and only if \( \eta = i_x(\xi) \). Let \( \xi, \eta \in \partial X \) be a maximum of \( \frac{d\rho_x}{d\xi} \). Let \( \eta = f^{-1}(i_y(f(\xi))) \in \partial X \), then \( f^*\rho_y(\xi,\eta) = \rho_y(f(\xi),f(\eta)) = \rho_y(f(\xi),i_y(f(\xi))) = 1 \), hence by Lemma 2.1 we have that \( \eta \) is a minimum of \( \frac{d\rho_x}{d\xi} \). Moreover, by Lemma 2.1 \( \rho_x(\xi,\eta) = 1 \), thus \( \eta = i_x(\xi) \), so \( \frac{d\rho_x}{d\xi} \) attains its minimum at \( i_x(\xi) \), and \( f(i_x(\xi)) = i_y(f(\xi)) \).

For the converse, suppose that \( i_x(\xi) \in \partial X \) is a minimum of \( \frac{d\rho_x}{d\xi} \). Then \( \rho_x(i_x(\xi),\xi) = 1 \) implies by Lemma 2.1 that \( \frac{d\rho_x}{d\xi} \) attains its maximum at \( \xi \). Moreover, by Lemma 2.1 \( f^*\rho_y(\xi,i_x(\xi)) = 1 \), so \( \rho_y(f(\xi),f(i_x(\xi))) = 1 \), hence \( f(i_x(\xi)) = i_y(f(\xi)) \).
(ii) Let $\xi$ be a maximum of $\left.\frac{d\rho}{df}\right|_{y}$. By definition of the geodesic conjugacy $\phi$, the point $z = \pi(\phi(x, \xi)) \in Y$ lies on the bi-infinite geodesic $(f(\xi), f(i_{z}(\xi))) \subset Y$. By (i) above, the point $y$ also lies on the bi-infinite geodesic $(f(\xi), f(i_{x}(\xi)))$. Since $\xi$ is a maximum of $\left.\frac{d\rho}{df}\right|_{y}$, it follows that $\log \left.\frac{d\rho}{df}\right|_{y}(\xi) = d_{M}(\rho_{x}, f^{*}\rho_{y}) = d_{M}(f_{*}\rho_{x}, \rho_{y})$ (note that push-forward of metrics by $f$ gives an isometry $f_{*} : M(\partial X) \to M(\partial Y)$).

Thus by Lemma 3.1 we have

$$B(y, z, f(\xi)) = \log \left.\frac{d\rho}{df}\right|_{y}(f(\xi))$$

$$= \log \left.\frac{d\rho}{df}\right|_{y}(\xi)$$

$$= d_{M}(f_{*}\rho_{x}, \rho_{y})$$

Since $y, z$ both lie on the geodesic $(f(\xi), f(i_{x}(\xi)))$, it follows that $z$ is the unique point on the geodesic ray $[y, f(\xi)]$ at a distance $d_{M}(f_{*}\rho_{x}, \rho_{y})$ from $y$.

Finally, we need a lemma about Riemannian angles and comparison angles from [Bis17a]. For $x \in X$ and $\xi, \eta \in \partial X$, let $\angle x_{y} \in [0, \pi]$ denote the Riemannian angle between the geodesic rays $[x, \xi]$ and $[x, \eta]$ at the point $x$. Then the following holds (this is Lemma 6.6 of [Bis17a]):

**Lemma 3.7.** For all $x \in X$ and $\xi, \eta \in \partial X$ we have

$$\rho_{x}(\xi, \eta)^{b} \leq \sin \left(\frac{1}{2} \angle x_{y}\right) \leq \rho_{x}(\xi, \eta)$$

4. Proof of main theorem

Let $(X, g_{0})$ be a complete, simply connected manifold of pinched negative curvature $-b^{2} \leq K_{g_{0}} \leq -1$. Let $g_{1}$ be a metric on $X$ such that $g_{1} = g_{0}$ outside a compact in $X$, and suppose $g_{1}$ is negatively curved, $K_{g_{1}} \leq -1$. Then the metrics $g_{0}, g_{1}$ are bi-Lipschitz, so the identity map $id : (X, g_{0}) \to (X, g_{1})$ induces a homeomorphism between boundaries which we denote by $f : \partial g_{0}X \to \partial g_{1}X$. Suppose the map $f$ is Mobius. Let $F : (X, g_{0}) \to (X, g_{1})$ be the circumcenter extension of the Mobius map $f$. Note that both metrics $g_{0}, g_{1}$ have pinched negative curvature (since $g_{0}$ does, and $g_{1} = g_{0}$ outside a compact), so the results of the previous section apply to $F$. In particular, by Theorem 3.3, the function $r(x) = d_{M}(f_{*}\rho_{x}, \rho_{F(x)})$ is constant, let $M \geq 0$ denote its constant value. By Lemma 3.5 to show that the circumcenter map $F$ is an isometry, it suffices to show that $M = 0$.

Let $T_{1}X_{g_{0}} \subset TX$ and $T_{1}X_{g_{1}} \subset TX$ denote the unit tangent bundles with respect to the metrics $g_{0}, g_{1}$ respectively, and let $\phi : T_{1}X_{g_{0}} \to T_{1}X_{g_{1}}$ denote the geodesic conjugacy induced by the Mobius map $f$. For $x \in X$, let $\rho_{x}^{g_{0}}$ and $\rho_{x}^{g_{1}}$ denote the visual metrics based at $x$ on the boundaries $\partial g_{0}X$ and $\partial g_{1}X$ of $(X, g_{0})$ and $(X, g_{1})$ respectively. For $x \in X$ and $\xi, \eta \in \partial g_{1}X$, let $(\xi, \eta)_{i} \subset X$ denote the bi-infinite $g_{i}$-geodesic with endpoints $\xi, \eta$, and let $[x, \xi]_{i} \subset X$ denote the $g_{i}$-geodesic ray joining $x$ to $\xi$, and let $x_{1}^{\xi} \in T_{1}X_{g_{1}}$ denote the $g_{i}$-unit tangent vector to the $g_{i}$-geodesic ray $[x, \xi]_{i}$ at the point $x$, where $i = 0, 1$. For $x \in X$ and a compact $K \subset X$, let $\Omega(x, K) \subset \partial g_{1}X$ denote the shadow of the set $K$ as seen from the
point $x$ with respect to the metric $g_i$, where $i = 0, 1$. For $i = 0, 1$ and $x \in X$, let $i_{\xi}^0: \partial_g X \to \partial_g X$ denote the involution of the boundary of $(X, g_i)$ as defined in the previous section.

**Lemma 4.1.** Let $K = \text{supp}(g_1 - g_0)$ denote the support of the symmetric 2-tensor $g_1 - g_0$. Let $x \in X - K$. If $\xi, \eta \in \partial_g X$ is such that $\xi, i_{\xi}^0(\xi) \in \partial_g X - O_0(x, K)$, then $\frac{\partial}{\partial x} = x f(\xi)^1 \in T_x X_{g_0} \cap T_x X_{g_1}$, and $\phi(x f(\xi)^1) = x f(\xi)^1 = x f(\xi)^0$.

**Proof:** The hypothesis on $\xi$ implies that the $g_0$-geodesics $[x, \xi)_0$ and $[x, i_{\xi}^0(\xi))_0$ are disjoint from $K$, hence so is the bi-infinite $g_0$-geodesic $(\xi, i_{\xi}^0(\xi))_0$, thus it is also a $g_1$-geodesic, hence $(\xi, i_{\xi}^0(\xi))_0$ equals the bi-infinite $g_1$-geodesic $(f(\xi), f(i_{\xi}^0(\xi)))_1$. In particular $\frac{\partial}{\partial x} f(\xi)^1 \in T_x X_{g_0} \cap T_x X_{g_1}$, and $\phi(x f(\xi)^1)$ is tangent to $(\xi, i_{\xi}^0(\xi))_0$, so $\pi(\phi(x f(\xi)^1))$ lies on $(\xi, i_{\xi}^0(\xi))_0$. Now we can choose a neighbourhood $U$ of $\xi$ in $\partial_g X$ which is disjoint from $O_0(x, K)$, and such that for any $\eta \in U$, the $g_0$-geodesic $(\xi, \eta)_0$ is disjoint from $K$ (by choosing $U$ small enough). Then for $\eta \in U$, the $g_0$-geodesics $[x, \xi)_1, [x, \eta)_1$, $(\xi, \eta)_0$ are disjoint from $K$, hence they are $g_1$-geodesics as well, and it follows that $\rho^0(\xi, \eta) = \rho^1(\xi, \eta)$ for all $\eta \in U$. Hence

$$\frac{d\rho^0}{d\rho^1}(\xi, \eta) = \lim_{\eta \to \xi} \frac{\rho^0(\xi, \eta)}{\rho^1(\xi, \eta)} = 1$$

so it follows from the definition of $\phi$ that $\pi(\phi(x f(\xi)^1)) = x$, thus $\phi(x f(\xi)^1) = x f(\xi)^1 = x f(\xi)^0$.

For $i = 0, 1$, let $d_{g_i}$ denote the distance function of $(X, g_i)$, and for $x \in X$ and $\xi, \eta \in \partial_g X$ let $\angle_{\xi} x \eta$ denote the Riemannian angle between the $g_i$-geodesics $[x, \xi)_1, [x, \eta)_1$ at the point $x$ with respect to the metric $g_i$.

We can now prove the main theorem:

**Proof of Theorem 1.1.** As remarked earlier, it suffices to show that the constant $M = 0$, where $d_M(f, \rho^0, \rho^1) = M$ for all $x \in X$. Fix $\epsilon > 0$, we will show that $M \leq \epsilon$.

Fix a basepoint $x_0 \in X$ and choose $R > 0$ such that the support of $g_1 - g_0$ is contained in the $g_0$-ball of radius $R$ around $x_0$, and let $B$ denote the closed $g_0$-ball of radius $R$ around $x_0$. Fix $\xi_0, \eta_0 \in \partial_g X$ such that $x_0 \in (\xi_0, \eta_0)_0$, let $\gamma: \mathbb{R} \to X$ be the unique unit speed $g_0$-geodesic such that $\gamma(-\infty) = \xi_0, \gamma(0) = x_0, \gamma(+\infty) = \eta_0$. For $t > R$ let $x_t \in X$ denote the point $\gamma(t)$, and define $\epsilon_t > 0$ by

$$\epsilon_t := \sup \{ \angle_{\xi_0} x_t \xi_0 | \xi \in O_0(x_t, B) \}$$

Then it follows from Lemma 2.2 and Lemma 3.1 that $\epsilon_t \to 0$ as $t \to +\infty$.

Let $K_t \subset \partial_g X$ denote the set where the function $\frac{d\rho^0}{d\rho^1}$ attains its maximum value $M$. Let $C_t \subset T_{x_t} X$ denote the cone

$$C_t := \{ v \in T_{x_t} X | \langle v, x_t \xi_0 \rangle \leq g_0(\xi_0, \eta_0) \}$$

and let $D_t := \{ -v \in T_{x_t} X | v \in C_t \}$. Then for $\xi \in \partial_g X$, if $x_t \xi_0 \notin C_t \cup D_t$, then $\xi, i_{\xi}^0(\xi) \notin O_0(x_t, B)$. Moreover, for $v, w \in C_t$ and $\alpha, \beta \geq 0$ we have $\alpha v + \beta w \in C_t$. Now if $\xi, \eta \in \partial_g X$ are such that $x_t \xi_0 \in C_t$ and $x_t \eta_0 \in D_t$, then by the triangle inequality

$$\rho^0(\xi, \eta) \geq 1 - \rho^0(\xi_0, \eta) - \rho^0(\eta_0, \xi_0)$$
and by Lemma 3.7 we have
\[
\rho_{x_t}^\rho(\xi, \xi_0) \leq \sin(\epsilon_t/2), \rho_{x_t}^\eta(\eta, \eta_0) \leq \sin(\epsilon_t/2),
\]
so since \( \epsilon_t \to 0 \) as \( t \to +\infty \), by choosing \( t > R \) large enough we may assume that
\[
\rho_{x_t}^\rho(\xi, \eta) \geq e^{-\epsilon}
\]
whenever \( \xi, \eta \in \partial g_0 X \) are such that \( \overrightarrow{\xi_0} \in C_t \) and \( \overrightarrow{\eta_0} \in D_t \). We fix such a \( t > R \) large enough so that this holds.

As stated in section 3, there exists a probability measure \( \mu \) on \( \partial g_0 X \) with support contained in \( K_t \) such that \( \mu \) is balanced at \( x_t \in (X, g_0) \), equivalently the convex hull in \( T_{x_t} X \) of the compact set \( \{x_t \xi_0 \xi \in K_t\} \) contains the origin of \( T_{x_t} X \). By the classical Caratheodory theorem on convex hulls, it follows that there exist distinct points \( \xi_1, \ldots, \xi_k \in K_t \) and \( \alpha_1, \ldots, \alpha_k > 0 \) such that \( \alpha_1 x_t \xi_1 + \cdots + \alpha_k x_t \xi_k = 0 \) and \( \alpha_1 + \cdots + \alpha_k = 1 \), where \( 1 \leq k \leq n+1 \) (here \( n \) is the dimension of \( X \)). Note that since the vectors \( x_t \xi_i \) are nonzero, we must have \( k \geq 2 \). We now consider various cases:

**Case 1.** \( k = 2 \):

Then since \( x_t \xi_1 \xi_0 \), \( x_t \xi_2 \xi_0 \) are unit vectors, the relation \( \alpha_1 x_t \xi_1 + \alpha_2 x_t \xi_2 = 0 \) implies that \( x_t \xi_1 = -x_t \xi_2 \), hence \( \xi_2 = i_{x_t}^\rho(\xi_1) \). By Lemma 3.6, the function \( \frac{d\rho_{x_t}^\rho}{d^\ast F(x_t)} \) attains its minimum at \( \xi_2 \), so since \( \xi_2 \in K_t \), the maximum and minimum of the function \( \frac{d\rho_{x_t}^\rho}{d^\ast F(x_t)} \) are equal, thus \( e^M = e^{-M} \), and so \( M = 0 \) as required.

**Case 2.** \( k \geq 3 \), and there exist \( 1 \leq i \neq j \leq k \) such that \( x_t \xi_i \xi_0 \), \( x_t \xi_j \xi_0 \in T_{x_t} X \) \( -(C_t \cup D_t) \):

In this case, \( \xi_i, i_{x_t}^\rho(\xi_i), \xi_j, i_{x_t}^\rho(\xi_j) \in \partial g_0 X - \mathcal{O}_0(x_t, B) \). It follows from Lemma 4.1 that the points \( z_i := \pi(\phi(x_t \xi_i)), z_j := \pi(\phi(x_t \xi_j)) \) satisfy \( z_i = x_t = z_j \). Thus the \( g_1 \)-geodesics \( (f(\xi_i), f(i_{x_t}^\rho(\xi_i))) \) and \( (f(\xi_j), f(i_{x_t}^\rho(\xi_j))) \) intersect at the point \( x_t \). On the other hand, by Lemma 3.6 the geodesics \( (f(\xi_i), f(i_{x_t}^\rho(\xi_i))) \) and \( (f(\xi_j), f(i_{x_t}^\rho(\xi_j))) \) intersect at the point \( F(x_t) \). If \( \xi_j \neq i_{x_t}^\rho(\xi_i) \), then the geodesics \( (f(\xi_i), f(i_{x_t}^\rho(\xi_i))) \) and \( (f(\xi_j), f(i_{x_t}^\rho(\xi_j))) \) have a unique point of intersection, thus \( x_t = F(x_t) \), and by Lemma 3.6 we have
\[
M = d_M(f, \rho_{x_t}^\rho, \rho_{F(x_t)}^\rho) = d_{g_1}(z_i, F(x_t)) = d_{g_1}(x_t, x_t) = 0
\]
If on the other hand \( \xi_j = i_{x_t}^\rho(\xi_i) \), then the same argument as in Case 1 above shows that \( M = 0 \). Thus in either case \( M = 0 \).

**Case 3.** \( k \geq 3 \), \( x_t \xi_0 \in T_{x_t} X \) \( -(C_t \cup D_t) \) for at most one \( i \in \{1, \ldots, k\} \):

Then relabelling the \( \xi_i \)'s if necessary, we may assume that \( x_t \xi_1 \xi_0, \ldots, x_t \xi_k \xi_0 \in C_t \cup D_t \). Now if \( x_t \xi_1 \xi_0, \ldots, x_t \xi_k \xi_0 \in C_t \), then \( \alpha_1 x_t \xi_1 + \cdots + \alpha_k x_t \xi_k = 0 \) \( C_t \) and it follows that \( x_t \xi_k \in D_t \). Similarly if \( x_t \xi_1 \xi_0, \ldots, x_t \xi_k \xi_0 \in D_t \), then we must have \( x_t \xi_0 \in C_t \). Thus either way, there exist \( 1 \leq i \neq j \leq k \) such that \( x_t \xi_i \xi_0 \in C_t \) and \( x_t \xi_j \xi_0 \in D_t \). Let \( \eta = i_{x_t}^\rho(\xi_i), \eta' = i_{x_t}^\rho(\xi_j) \), then \( x_t \eta \xi_0 \in D_t \) and \( x_t \eta' \xi_0 \in C_t \), and
by Lemma 3.6 the function $\frac{d\rho_0}{df^*\rho_{F(x_t)}}$ attains its minimum value $e^{-M}$ at the points $\eta, \eta'$. Now by our hypothesis on $t$ we have

$$\rho_0^{\eta_1}(\eta, \eta') \geq e^{-\epsilon}.$$ 

We then have

$$e^{-2\epsilon} \leq \rho_0^{\eta_1}(\eta, \eta')^2 = \frac{d\rho_0^{\eta_1}}{df^*\rho_{F(x_t)}}(\eta) \frac{d\rho_0^{\eta_1}}{df^*\rho_{F(x_t)}}(\eta') f^*\rho_{F(x_t)}(\xi, \xi')^2 \leq e^{-M} \cdot e^{-M} \cdot 1$$

thus $e^{-2\epsilon} \leq e^{-2M}$, hence $M \leq \epsilon$.

Since Cases 1, 2, 3 above exhaust all possibilities, it follows that $M \leq \epsilon$ for any given $\epsilon > 0$, thus $M = 0$ as required. \(\diamondsuit\)

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