A Note on Flagg and Friedman’s Epistemic and Intuitionistic Formal Systems

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Abstract
We report our findings on the properties of Flagg and Friedman’s translation from Epistemic into Intuitionistic logic, which was proposed as the basis of a comprehensive proof method for the faithfulness of the Gödel translation. We focus on the propositional case and raise the issue of the admissibility of the translated necessitation rule. Then, we contribute to Flagg and Friedman’s program by giving an explicit proof of the soundness of their translation.

Keywords: Proof theory; Intuitionistic Logic; Modal Logic.

1. Introduction
In their work Epistemic and Intuitionistic Formal Systems (1986), Flagg and Friedman gave a new proof–as an alternative to Goodman’s [1]–for the faithfulness of the Gödel translation (·)T from intuitionistic logic into epistemic modal logic S4 once (·)T is applied to Heyting arithmetic. They aimed at a comprehensive proof method, to be deployed not only for arithmetic but for various formal systems whose intuitionistic and epistemic versions are connected by a faithful translation:

\[ \vdash_{\text{Epi}} A^T \implies \vdash_{\text{Int}} A, \]

for any intuitionistic formula A. To this end, they proposed a class of embeddings of epistemic into intuitionistic logic, which are, in a certain sense, inverse to (·)T.

In order to prove the faithfulness of the Gödel translation, Flagg and Friedman claim that their translation (\( \cdot \))\( ^{(E)} \) is sound in both the propositional and the predicate calculus, and in arithmetic: i.e. for any epistemic formula B

\[ \vdash_{\text{Epi}} B \implies \vdash_{\text{Int}} B_{\text{Epi}}^{(E)}, \]

\footnote{Notation and conventions will adhere as much as possible to Flagg and Friedman’s [2].}
where $\Gamma$ is an arbitrary set of intuitionistic formulae and $E$ is a chosen formula from $\Gamma$.

In this article we discuss some issues about the proof of soundness of the generic Flagg-and-Friedman translation $(\cdot)_{\Gamma}^{(E)}$ from Epistemic Propositional logic, shortly $\text{EP}$ (i.e. the modal logic $\text{S4}$), into Intuitionistic Propositional logic, $\text{IP}$.

Let $\Gamma \neq \emptyset$ be a finite set of $\text{IP}$-formulae and $E \in \Gamma$. Then, for each formula $A$ in $\text{EP}$, Flagg and Friedman define the formula $A_{\Gamma}^{(E)}$ of $\text{IP}$ (simply denoted by $A^{(E)}$ when this is not ambiguous) as follows, recursively on the structure $\mathcal{B}$ of $A$:

\[
A_{\Gamma}^{(E)} = \neg E \neg E A, \text{ if } A \text{ is atomic};
\]

\[
(B_0 \land B_1)^{E}_{\Gamma} = (B_0^{E}_{\Gamma} \land B_1^{E}_{\Gamma});
\]

\[
(B_0 \lor B_1)^{E}_{\Gamma} = \neg E \neg E (B_0^{E}_{\Gamma} \lor B_1^{E}_{\Gamma});
\]

\[
(\Box B)^{E}_{\Gamma} = \neg E \neg E \bigwedge_{C \in \Gamma} B^{C}_{\Gamma}.
\]

Their basic result is the following theorem (Th. 1.8 in [2]), which states the soundness of their translation in propositional calculus.

**Theorem 1.** Let $A_1, \ldots, A_n, A$ be $\text{EP}$-formulae (possibly, $n = 0$). If

\[
A_1, \ldots, A_n \models_{\text{EP}} A,
\]

then, for any finite set $\Gamma$ of $\text{IP}$-formulae and for any $E \in \Gamma$,

\[
A_{1 \Gamma}^{(E)}, \ldots, A_{n \Gamma}^{(E)} \models_{\text{IP}} A_{\Gamma}^{(E)}.
\]

The authors suggest to prove Theorem 1 by induction on the length of derivations, but do not give enough details for a straightforward proof. In fact, a plain and standard induction on the length of derivations would presuppose that the generic $(\cdot)_{\Gamma}^{(E)}$ preserved the admissibility of all epistemic inference rules. Unfortunately that allows counter-examples to the preservation of the necessitation rule, as shown in Section 2 by means of an algebraic counter-model [4]. Thus, to obtain a successful proof, we define in Section 3 a more structured application of induction which is not based on preservation of the necessitation rule and succeeds in proving soundness in the propositional case.

2. Inadmissibility of the necessitation rule

If we were to prove Th. 1 above by a plain induction on length of derivation, then, according to a standard procedure, we would expect that, for any finite
\( \Gamma \subseteq \mathcal{L}_{\text{IP}} \) and any \( E \in \Gamma \), the translation \( (\cdot)^{(E)}_{\Gamma} \) would transform any \( \text{EP} \)-primitive inference rule into an \( \text{IP} \)-admissible inference rule. It can be easily proved for all \( \text{EP} \)-primitive rules but for the \( \Box \)-introduction rule (denoted by \( \Box \text{I} \)):

\[
\Box \text{I} \quad \frac{A}{\Box A} \quad \text{if all open assumptions are of the form} \quad \Box B.
\]

In fact, there is at least one counter-example to the admissibility of every \( (\Box \text{I})^{(E)}_{\Gamma} \) in \( \text{IP} \), as we show now.

**Theorem 2.** Let \( B \) and \( C \) be arbitrary atomic formulae (including \( \bot \)) and let \( E \) be a propositional letter distinct from \( B \) and \( C \); moreover, let \( \Gamma = \{ C, E \} \). If \( A = (E \rightarrow B) \), then

\[
\vdash_{\text{IP}} A^{(E)}_{\Gamma},
\]

while

\[
\not\vdash_{\text{IP}} (\Box A)^{(E)}_{\Gamma}.
\]

**Proof 1.** Clearly \( A^{(E)}_{\Gamma} \vdash_{\text{IP}} E \rightarrow \neg E \neg B \), hence \( A^{(E)}_{\Gamma} \) is an \( \text{IP} \)-theorem. Let us then prove that, in contrast, \( (\Box A)^{(E)}_{\Gamma} \) is not. Since \( (\Box A)^{(E)}_{\Gamma} = \neg E \neg B \neg (A^{(C)} \land A^{(E)}) \), thanks to Lemma 1.6/(v) of [2] and the \( \land \)-elimination rule it suffices to prove that \( \not\vdash_{\text{IP}} \neg E \neg B \neg (A^{(C)} \rightarrow \neg C \neg B) \).

Thanks to the soundness of \( \text{IP} \) wrt. the algebraic semantics, we obtain the result.

In fact, let \( \mathfrak{H} = (H, \leq) \) be a totally-ordered Heyting algebra\(^5\) (i.e. a chain) such that \( b, c, e \in H \) and \( 0 \leq b \leq c < e < 1 \). Now, let \( v \) be a valuation in \( H \) (i.e. a function from the set of propositional letters to \( H \)) such that, denoting by \( v_{\mathfrak{H}} \) the standard extension of \( v \) to the \( \text{IP} \)-formulae w.r.t. \( \mathfrak{H} \), we obtain \( v_{\mathfrak{H}}(B) = b \), \( v_{\mathfrak{H}}(C) = c \), and \( v(E) = e \). Then

\[
v_{\mathfrak{H}}(\neg E \neg B \neg (A^{(C)} \rightarrow \neg C \neg B)) = (e \triangleright c \triangleright c) \triangleright (b \triangleright c \triangleright c) \triangleright e \triangleright e = e \neq 1.
\]

Hence \( v_{\mathfrak{H}}(\neg E \neg B \neg (A^{(C)})) \neq 1 \). As a result, \( \not\vdash_{\text{IP}} \neg E \neg B \neg (A^{(C)}) \).

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\(^5\)A Heyting algebra is a bounded lattice \( \mathfrak{H} = (G; \leq; \cup, \cap; 0, 1) \) (where \( G \) is the subjacent set of \( \mathfrak{H} \), the relation \( \leq \) is the characteristic order of the lattice, \( \cup \) and \( \cap \) are the usual supremum and infimum operators respectively, \( 0 \) and \( 1 \) the top and the bottom of \( \mathfrak{H} \) respectively) such that, for any \( x, y \in G \), there exists the relative pseudo-complement of \( x \) w.r.t. \( y \), denoted by \( x \triangleright y \), i.e. there exists a \( z \in G \) such that, for any \( w \in G \), we have: \( w \trianglelefteq z \iff w \cap x \leq y \).
3. A non-trivial proof of soundness

We shall propose a new, explicit proof for Th. \( \Box \) with a focus on the treatment of the necesitation rule.

**Proof 2.** Let \( \delta \) be an EP-derivation, in one arbitrary step of which a formula \( A \) is asserted under the assumptions \( A_1, \ldots, A_n \). Also, let \( \Gamma \) be an arbitrary finite set of IP-formulae. We suppose, as induction hypothesis, that for each \( C \in \Gamma \) the asserted formula in any previous step of \( \delta \), once translated w.r.t. \( C \) and \( \Gamma \), is provable in IP from the translations of the open assumptions concerning that step of \( \delta \).

Consider the case that the assertion of \( A \) is obtained—through the \( \Box \) rule—from an earlier step of \( \delta \). Let then \( B_1, \ldots, B_n, B \) be such that \( A_1 = \Box B_1, \ldots, A_n = \Box B, A = \Box B \). We claim that, for any \( E \in \Gamma \),

\[
\neg E \neg E \bigwedge_{C \in \Gamma} B_1^{(C)}, \ldots, \neg E \neg E \bigwedge_{C \in \Gamma} B_n^{(C)} \vdash_{\text{IP}} \neg E \neg E \bigwedge_{C \in \Gamma} B^{(C)}.
\]

To prove it, thanks to Lemma 1.6/iv of [2] it suffices to prove, by fixing an arbitrary formula \( D \in \Gamma \), that \( \neg E \neg E B^{(D)} \) is provable from the above-mentioned assumptions. By the induction hypothesis we have (let us omit \( \Gamma \) from now on)

\[
\neg D \neg D \bigwedge_{C} B_1^{(C)}, \ldots, \neg D \neg D \bigwedge_{C} B_n^{(C)} \vdash_{\text{IP}} B^{(D)},
\]

from which, by Lemma 1.6/(vii) we obtain:

\[
\neg E \neg E \neg D \neg D \bigwedge_{C} B_1^{(C)}, \ldots, \neg E \neg E \neg D \neg D \bigwedge_{C} B_n^{(C)} \vdash_{\text{IP}} \neg E \neg E B^{(D)}.
\]

Now, for all \( i \) such that \( 1 \leq i \leq n \), by Lemma 1.6/(i), (vii) we have the following:

\[
\neg E \neg E \bigwedge_{C} B_i^{(C)} \vdash_{\text{IP}} \neg E \neg E \neg D \neg D \bigwedge_{C} B_i^{(C)}.
\]

Finally, by composition we get to:

\[
\neg E \neg E \bigwedge_{C} B_1^{(C)}, \ldots, \neg E \neg E \bigwedge_{C} B_n^{(C)} \vdash_{\text{IP}} \neg E \neg E B^{(D)}.
\]

\( \Box \)
References

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