A Note on the Maximum Number of Minimal Connected Dominating Sets in a Graph

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Abstract

We prove constructively that the maximum possible number of minimal connected dominating sets in a connected undirected graph of order $n$ is in $\Omega(1.489^n)$. This improves the previously known lower bound of $\Omega(1.4422^n)$ and reduces the gap between lower and upper bounds for input-sensitive enumeration of minimal connected dominating sets in general graphs as well as some special graph classes.

1 Introduction

A connected dominating set in a graph $G = (V, E)$ is a set of vertices whose closed neighborhood is $V$ that induces a connected subgraph. A connected dominating set is inclusion minimal if it does not contain another connected dominating set as a proper subset.

Enumerating all minimal connected dominating sets in a given graph can be trivially performed in $O(2^n)$. Whether a better enumeration algorithm exists was one of the most important open problems posed in the first workshop on enumeration (Lorentz Center, Netherlands, 2015) [2]. The problem has been subsequently addressed in [5] where an algorithm that runs in $O((2 - 10^{-50})^n)$ was presented. This slightly improves the upper bound on the number of minimal connected dominating sets in a (general) graph.

On the other hand, the maximum number of minimal connected dominating sets in a graph was shown to be in $\Omega(3^{\sqrt{n}})$ [3]. This lower bound is obviously very low compared to the upper bound and to the running time of the current asymptotically-fastest exact algorithm, which is in $O(1.862^n)$
The gap between upper and lower bounds is narrower when it comes to special graph classes. On chordal graphs, for example, the upper bound has been recently improved to $O(1.4736^n)$ [4]. Other improved lower/upper bounds have been obtained for AT-free, strongly chordal, distance-hereditary graphs, and cographs in [3]. Further improved bounds for split graphs and cobipartite graphs have been obtained in [6].

In this note we report an improved lower bound on the maximum number of minimal connected dominating sets in a graph. This is related to the enumeration of all the minimal connected dominating sets since it also gives a lower bound on the asymptotic performance of any input-sensitive enumeration algorithm.

2 Graphs with Large Minimal Connected Dominating Sets

Given arbitrary positive integers $k, t$, we construct a graph $G^k_t$ of order $n = k(2t + 1) + 1$ as follows.

The main building blocks of $G^k_t$ consist of $k$ copies of a base-graph $G_t$, of order $2t - 1$. The vertex set of $G_t$ consists of three layers. The first layer is a set $X = \{x_1 \ldots x_t\}$ that induces a clique. The second is an independent set $Y = \{y_1 \ldots y_t\}$, while the third layer consists of a singleton $\{z\}$. Each vertex $x_i \in X$ has exactly $t - 1$ neighbors in $Y$: $N(x_i) = \{y_j \in Y : i \neq j\}$. In other words, the base-graph $G_t$ has a maximum anti-matching $\{\{x_j, y_j\} : 1 \leq j \leq t\}$. In fact, $X \cup Y$ induces a copy of $K_{t,t}$ minus a perfect matching. Finally the vertex $z$ is adjacent to all the $t$ vertices in $Y$. Figure [1] below shows the graph $G_t$ for $t = 4$.

Lemma 1. For each $t > 0$, the graph $G_t$ has exactly $\frac{t^3 + t^2}{2} - t$ minimal connected dominating sets that have non-empty intersection with the set $X$.

Proof. The set $X$ cannot have more than two vertices in common with any minimal connected dominating set, since any two elements of $X$ dominate $X \cup Y$. Any minimal connected dominating set that contains exactly one vertex $x_i$ of $X$ must contain the vertex $z$, to dominate $y_i$, and one of the $t - 1$ neighbors of $x_i$ (to be connected). There are $t(t - 1)$ sets of this type. Moreover, each pair of elements of $X$ dominates $Y$. So a minimal connected dominating set can be formed by (any) two elements of $X$ and any of the elements of $Y$ (to dominate $z$). There are $\frac{t(t-1)}{2}$ such sets. \[\square\]

\[1\text{An anti-matching in } G \text{ is a collection of disjoint non-adjacent pairs of its vertices.}\]
The hub-vertex \( s \) in \( G^k_t \) must be in any connected dominating set, being a cut-vertex. Therefore, there is no need for the set \( X \) in \( G_t \) to induce a clique (in \( G^k_t \)), being always dominated by \( s \). In other words, the counting used in the above proof still holds if each copy of \( G_t \) is replaced by \( G_t - E(X) \) in \( G^k_t \). Here \( E(X) \) denotes the set of edges connecting pairs of vertices in \( X \). The below figure shows \( G^3_3 \) without the edges between pairs of element of \( X \) in each copy of \( G_3 \).
Theorem 1. The maximum number of minimal connected dominating sets in a simple undirected connected graph of order \( n \) is in \( \Omega(1.489^n) \).

Proof. By Lemma 1, each copy of the graph \( G_t \) has \( \frac{t^3 + t^2}{2} - t \) minimal connected dominating sets. There are \( k \) such graphs in \( G_t^k \), in addition to the vertex \( s \) that connects them all. Every minimal connected dominating set must contain \( s \) and at least one element from \( N(s) \) in each \( G_t \). Therefore, the total number of minimal connected dominating sets in \( G_t^k \) is \( \left( \frac{t^3 + t^2}{2} - t \right)^k = \left( \frac{t^3 + t^2}{2} - t \right)^{\frac{n-1}{2t+1}} \). The claimed lower bound is achieved when \( t = 4 \), which gives a total of \( 36 \frac{n-1}{9} \in \Omega(1.489^n) \).

We note that \( G_t^k \) is a \( t \)-degenerate graph that is also bipartite (since the set \( X \) in each copy of \( G_t \) can be an independent set). Furthermore, we observe that \( G_3^k \) is planar. To see this, simply re-order the elements of \( Y \) in each copy of \( G_3 \) as shown in Figure 3 below.

![Figure 3: A plane drawing of \( G_3^3 \)](image)

Therefore, we can obtain an improved lower bound for \( 3 \)-degenerate, bipartite and planar graphs. We conclude with the following corollary.

Corollary 1. The maximum number of minimal connected dominating sets in a \( 3 \)-degenerate bipartite planar graph of order \( n \) is in \( \Omega(1.472^n) \).
3 Conclusion

The method we adopted for constructing asymptotic worst-case examples for enumerating minimal connected dominating sets consists of combining copies of a certain base-graph having a particular subset of vertices that must contain elements of any minimal connected dominating set, being linked to a main hub-vertex. For example, the graph $G_4$ has 36 minimal connected dominating sets, each of which must contain elements of the set $X$, which in turn is linked to the hub-vertex $s$ in $G_k$.

The main question at this stage is: can we do better? We believe it is very difficult to find a base-graph of order eight or less that can be used to achieve a higher lower-bound since it would have to have at least 25 minimal connected dominating sets. Moreover, any better example that contains more than 9 vertices must have a much larger number of minimal connected dominating sets. For example, to achieve a better lower bound with a base-graph of order 10 (or 11), such a graph must have at least 54 (respectively 80) minimal connected dominating sets. It would be challenging to obtain such a construction, which is hereby posed as an open problem.

References

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