An Approximation Algorithm for Covering Linear Programs and its Application to Bin-Packing

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Abstract
We give an $\alpha(1 + \varepsilon)$-approximation algorithm for solving covering LPs, assuming the presence of a $(1/\alpha)$-approximation algorithm for a certain optimization problem. Our algorithm is based on a simple modification of the Plotkin-Shmoys-Tardos algorithm \cite{12}. We then apply our algorithm to $\alpha(1 + \varepsilon)$-approximately solve the configuration LP for a large class of bin-packing problems, assuming the presence of a $(1/\alpha)$-approximate algorithm for the corresponding knapsack problem (KS). Previous results give us a PTAS for the configuration LP using a PTAS for KS. Those results don’t extend to the case where KS is poorly approximated. Our algorithm, however, works even for polynomially-large $\alpha$.

1 Introduction

Algorithms for solving linear programs (LPs) have been the cornerstone of operations research. Linear programming also has applications in computer science; Grötschel, Lovász and Schrijver \cite{7} give examples of combinatorial optimization problems that can be solved using linear programming. Linear programs are especially important in the area of approximation algorithms. Many optimization problems can be expressed as integer programs. Rounding-based algorithms first solve the LP relaxation of these integer programs, and then round the relaxed solution to get an approximate solution to the original problem \cite{13, 14, 11}.

We study a large and important class of linear programs, called covering linear programs. Our main result is an approximation algorithm, called \texttt{covLP-solve}, for solving covering LPs.

\textbf{Definition 1.} A linear program is called a covering LP iff it is of the form

$$\min_{x \in \mathbb{R}^N} c^T x \text{ where } Ax \geq b \text{ and } x \geq 0,$$

where $A \in \mathbb{R}^{m \times N}$ (m-by-N matrix over non-negative reals), $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^N$. Denote this covering LP by \texttt{covLP}(A, b, c).

Our motivating application stems from the bin-packing problem. There are multiple ways of representing bin-packing as an integer LP, but probably the most useful of them is the configuration LP (formally defined in Section 2). Rounding the configuration LP was used in the first linear-time APTAS for bin-packing by de la Vega and Lueker \cite{6}. It was later used by Karmarkar and Karp \cite{9} to get an algorithm for bin-packing that uses $\text{OPT} + O(\log^2(\text{OPT}))$ bins, and by Hoberg and Rothvoss \cite{8} for an algorithm that uses $\text{OPT} + O(\log(\text{OPT}))$ bins. Bansal, Caprara and Sviridenko \cite{1} devised the Round-and-Approx (R&A) framework for solving variants of the bin-packing problem. The R&A framework requires an approximate solution
the configuration LP. They used the R&A framework to get approximation algorithms for vector bin-packing and 2-dimensional geometric bin-packing. Improved algorithms were later devised for these bin-packing variants \cite{DBLP:journals/jcss/Szegedy98, DBLP:journals/tcs/Ale入库缚古诺2014}, but those algorithms also use the R&A framework. Thus, solving the configuration LP of (variants of) bin-packing is an important problem. In Section 2, we show how to approximately solve the configuration LP of a large class of bin-packing problems.

An implicit covering LP is one where \( A \) and \( c \) are not given to us explicitly. Instead, we are given an input \( I \), and \( A, b, c \) are defined in terms of \( I \). The configuration LP for bin-packing, for example, is defined implicitly. Such an implicit definition is helpful when \( N \), the number of columns in \( A \), is super-polynomial in the input size \(|I|\). We assume that \( m \), the number of rows in \( A \), is polynomial in \(|I|\) and that \( b \) has already been computed. Since \( A \) and \( c \) are not given to us explicitly, we will assume the presence of certain oracles that can help us indirectly get useful information about \( A \) and \( c \). Our main result is an approximation algorithm \texttt{covLP-solve} (described in Section 5) that solves \texttt{covLP}(\( A, b, c \)) in polynomial time using these oracles.

The main implication of our result is that for any \( \varepsilon > 0 \), we can \( \alpha(1 + \varepsilon) \)-approximately solve the configuration LP of some variants of bin-packing, using a \((1/\alpha)\)-approximation algorithm for the corresponding knapsack problem. Previous results give us a PTAS for the configuration LP using a PTAS for the corresponding knapsack problem (see Section 3 for details). For many variants of knapsack, a PTAS is not known, so previous results cannot be applied. Our algorithm, however, works even for polynomially-large \( \alpha \).

### 1.1 Formal Statement of Our Results

Preliminaries:

- For a non-negative integer \( n \), let \([n] := \{1, 2, \ldots, n\}\).
- Let \( \mathbb{R}_{\geq 0} \) be the set of non-negative real numbers. Let \( \mathbb{R}_{> 0} := \mathbb{R}_{\geq 0} - \{0\} \).
- For a vector \( x \), \( \text{support}(x) := \{j : x_j \neq 0\} \).
- For a vector \( x \), \( x \geq 0 \) means that every coordinate of \( x \) is non-negative.
- For a matrix \( A \), \( A[i, j] \) is the entry in the \( i \)th row and \( j \)th column of \( A \).
- Let \( e_j \) be a vector whose \( j \)th component is 1 and all other components are 0.
- \( \text{poly}(n) \) is the set of functions of \( n \) that are upper-bounded by a polynomial in \( n \).

**Definition 2** (Column oracle). The column oracle for \( A \in \mathbb{R}_{\geq 0}^{m \times N} \) takes \( j \in [N] \) as input and returns the \( j \)th column of \( A \).

**Definition 3** (Cost oracle). The cost oracle for \( c \in \mathbb{R}_{\geq 0}^{N} \) takes \( j \in [N] \) as input and returns \( c_j \).

**Definition 4** (Index-finding oracle). Let \( A \in \mathbb{R}_{\geq 0}^{m \times N} \) and \( c \in \mathbb{R}_{\geq 0}^{N} \) be implicitly defined in terms of input \( I \). For \( j \in [N] \), define the function \( D_j : \mathbb{R}_{\geq 0}^{m} \mapsto \mathbb{R}_{\geq 0} \) as

\[
D_j(y) := y^T A \begin{pmatrix} e_j \\ c_j \end{pmatrix} = \frac{1}{c_j} \sum_{i=1}^{m} y_i A[i, j]
\]

Then for \( \eta \in (0, 1] \), an \( \eta \)-weak index-finding oracle for \( I \), denoted by \texttt{index-find}, is an algorithm that takes as input \( y \in \mathbb{R}_{\geq 0}^{m} \) and returns \( k \in [N] \) such that \( D_k(y) \geq \eta \max_{j=1}^{N} D_j(y) \).

The algorithm \texttt{covLP-solve} takes the following inputs:

- \( I \): the input used to implicitly define \texttt{covLP}(\( A, b, c \)).
- \( q \): an upper-bound on \( \text{opt}(\texttt{covLP}(\( A, b, c \))) \).
- \( \rho \): an upper-bound on \( q \max_{i=1}^{m} \max_{j=1}^{N} A[i, j] / b_i c_j \).
\begin{itemize}
\item $\varepsilon, \eta \in (0, 1]$.
\end{itemize}

covLP-solve is also provided a column oracle for $A$, a cost oracle for $c$, and an $\eta$-weak index-finding oracle.

Theorems 1 and 2 below are our main results, the proofs of which can be found in Section 5.

**Theorem 1.** Let $\text{covLP}(A, b, c)$ be implicitly defined in terms of input $I$. Then $\text{covLP-solve}(I, q, \rho, \varepsilon, \eta)$ returns a $(1 + \varepsilon + \varepsilon^2)/\eta$-approximate solution to $\text{covLP}(A, b, c)$.

**Theorem 2.** Let $\text{covLP}(A, b, c)$ be implicitly defined in terms of input $I$, where $A \in \mathbb{R}_{\geq 0}^{m \times N}$. Let

\[
M := 3 + 2 \log \left( \frac{1}{\varepsilon} + 1 \right) + \log \left( \frac{1}{\eta} \right) + \log \left( \frac{q}{\text{opt}(\text{covLP}(A, b, c))} \right)
\]

\[
U := m + \left\lceil \ln \left( \frac{m}{\eta} \right) \right\rceil \left\lceil \frac{312 m \rho (1 + \varepsilon)}{\eta^3} \right\rceil \ln \left( \frac{12 m}{\varepsilon} \right) \in \tilde{O} \left( \frac{m \rho}{\eta^3} \right)
\]

Then all of the following hold for $\text{covLP-solve}(I, q, \rho, \varepsilon, \eta)$:

\begin{itemize}
\item $\text{covLP-solve}$ makes at most $MU$ calls to the index-finding oracle, at most $MU$ calls to the column oracle, and at most $MU$ calls to the cost oracle.
\item In $\text{covLP-solve}$, the time taken by non-oracle operations is $O(MUm)$.
\item The solution $\hat{x}$ returned by $\text{covLP-solve}$ has $|\text{support}(\hat{x})| \leq U$.
\end{itemize}

**Corollary 2.1.** Let $r^* = \text{opt}(\text{covLP}(A, b, c))$. Then to approximately solve $\text{covLP}(A, b, c)$ in polynomial time using $\text{covLP-solve}$, we need a way to compute $\rho$ and $q$ in polynomial time, $m, 1/\varepsilon, 1/\eta, \log(q/r^*), \rho \in \text{poly}(|I|)$, and the oracles should run in $\text{poly}(|I|)$ time.

1.2 The Fractional Covering Problem

A subsidiary contribution of this paper is an algorithm for the fractional covering problem. We use that algorithm as a subroutine in $\text{covLP-solve}$.

**Definition 5** (Fractional Covering Problem [12]). Let $A \in \mathbb{R}_{\geq 0}^{m \times N}$ be an $m$-by-$N$ matrix and $b \in \mathbb{R}_{\geq 0}^N$ be an $m$-dimensional vector. Let $P \subseteq \mathbb{R}^N$ be a convex polytope such that $Ax \geq 0$ for all $x \in P$. The fractional covering problem on input $(A, b, P)$, denoted as $\text{fcov}(A, b, P)$, requires us to do the following:

\begin{itemize}
\item output a feasible solution $x$, i.e. $x \in P$ such that $Ax \geq b$.
\item claim that $\text{fcov}(A, b, P)$ is unsatisfiable, i.e. $Ax \geq b$ is not satisfied by any $x \in P$.
\end{itemize}

For $\alpha \in (0, 1]$, an algorithm is said to $\alpha$-weakly solve $\text{fcov}(A, b, P)$ iff it does one of the following:

\begin{itemize}
\item output an $\alpha$-approximate solution $x$, i.e. $x \in P$ such that $Ax \geq \alpha b$.
\item claim that $\text{fcov}(A, b, P)$ is unsatisfiable.
\end{itemize}

An implicit fractional covering problem is one where $A$, $b$ and $P$ are not given to us explicitly. Instead, we are given an input $I$, and $A$, $b$, $P$ are defined in terms of $I$.

We give an algorithm $\text{frac-cover}$ (described in Section 4) that weakly solves $\text{fcov}(A, b, P)$ in polynomial time using certain oracles. $\text{frac-cover}$ is obtained by modifying the algorithm of Plotkin, Shmoys and Tardos [12] for the fractional covering problem. Moreover, we use $\text{frac-cover}$ as a subroutine in $\text{covLP-solve}$.

**Definition 6** (Product oracle). Let $\text{fcov}(A, b, P)$ be a fractional covering problem instance, where $A, b, P$ are defined implicitly in terms of $I$. The product oracle for $I$ takes $x \in P$ as input and returns $Ax$.  

3
Definition 7 (Point-finding oracle). Let $\text{fcov}(A,b,P)$ be a fractional covering problem instance, where $A,b,P$ are defined implicitly in terms of $I$. For $\eta \in (0,1]$, an $\eta$-weak point-finding oracle for $I$, denoted by $\text{point-find}$, is an algorithm that takes as input $y \in \mathbb{R}^{m}_{\geq 0}$ and returns $\hat{x} \in P$ such that $y^T \hat{x} \geq \eta \max_{x \in P} y^T x$.

Definition 8.

$$\text{width}(A,b,P) := \max_{x \in P} \max_{i=1}^{m} \frac{(Ax)_i}{b_i}$$

The algorithm $\text{frac-cover}$ takes the following inputs:

- $I$: the input used to implicitly define $\text{fcov}(A,b,P)$.
- $\rho$: an upper-bound on $\text{width}(A,b,P)$.
- $\varepsilon, \eta \in (0,1]$.

$\text{frac-cover}$ is also provided a product oracle for $A$ and an $\eta$-weak point-finding oracle.

Theorems 3 and 4 below are the main results for this problem, the proofs of which can be found in Section 4.

Theorem 3. Let $\text{fcov}(A,b,P)$ be a fractional covering problem instance where $A,b,P$ are implicitly defined in terms of input $I$. Then $\text{frac-cover}(I,\rho,\varepsilon,\eta)$ will $\eta/(1+\varepsilon)$-weakly solve $\text{fcov}(A,b,P)$, i.e., if it returns null, then $\text{fcov}(A,b,P)$ is unsatisfiable, and if it returns a vector $x$, then $x \in P$ and $Ax \geq (\eta/(1+\varepsilon))b$.

Theorem 4. Let $\text{fcov}(A,b,P)$ be implicitly defined in terms of input $I$, where $A \in \mathbb{R}^{m \times N}_{\geq 0}$. Let $\tau$ be an upper-bound on the support of the output of $\text{point-find}$. Suppose $\text{frac-cover}(I,\rho,\varepsilon,\eta)$ calls the point-finding oracle $T$ times. Then

$$T \leq U := m + \left\lceil \ln \left( \frac{m}{\eta} \right) \right\rceil \left\lceil 312m^3 \varepsilon^3 \ln \left( \frac{12m}{\varepsilon} \right) \right\rceil \in \tilde{O} \left( \frac{m \rho}{\eta \varepsilon^3} \right)$$

Additionally,

- $\text{frac-cover}$ makes at most $T$ calls to the product oracle. For every input $x$ to the product oracle, $|\text{support}(x)| \leq \tau$.
- The running time of $\text{frac-cover}$, excluding the time taken by oracles, is $O(T(m+\tau))$.
- The solution $\hat{x}$ returned by $\text{frac-cover}$ has $|\text{support}(\hat{x})| \leq T\tau$.

1.3 Organization of This Paper

Section 2 defines the bin-packing problem and its corresponding configuration LP, and shows how to use $\text{covLP-solve}$ to approximately solve the configuration LP in polynomial time. Section 3 describes some well-known algorithms for solving LPs and compares them to our algorithm. Section 4 describes the $\text{frac-cover}$ algorithm and proves Theorems 3 and 4. Section 5 describes the $\text{covLP-solve}$ algorithm and proves Theorems 1 and 2. Section 6 describes avenues for further improvement.

2 The Bin-Packing Problem

In this section, we will define the bin-packing problem and its corresponding configuration LP. Then we will see how to apply $\text{covLP-solve}$ to approximately solve the configuration LP in polynomial time.
In the classic bin-packing problem (classic-BP), we are given a set $I$ of $n$ items. Each item $i$ has a size $s_i \in (0, 1]$. We want to partition $I$ such that the sum of sizes of items in each partition is at most 1. Each partition is called a bin, and we want to minimize the number of bins. We want an algorithm for this problem whose worst-case running time is polynomial in $n$. See the survey by Coffman et al. on approximation algorithms for bin-packing [5].

In the classic knapsack problem (classic-KS), we are given a set $I$ of $n$ items. Each item $i$ has a size $s_i \in (0, 1]$ and a profit $p_i$ associated with it. We want to select a subset $J \subseteq I$ of items such that $p(J) := \sum_{i \in J} p_i$ is maximized.

There are many variants of the classic bin-packing problem. In the 2D geometric bin-packing problem (2GBP) [4], we are given a set $I$ of $n$ axis-parallel rectangular items, and we have to place the items into the minimum number of rectangular bins without rotating the items, such that no two items overlap. In the vector bin-packing problem (VBP), we are given a set $I$ of $n$ vectors over $\mathbb{R}^d_{\geq 0}$ that we have to pack into the minimum number of bins such that in each bin, the maximum coordinate of the sum of vectors is at most 1. We can similarly define 2D geometric knapsack (2GKS) and vector knapsack (VKS). Note that in classic-BP and VBP, we only need to partition the items into bins, whereas in 2GBP, we also need to decide the position of the items into the bins.

### 2.1 Abstract Bin-Packing and the Configuration LP

We will now state the bin-packing problem and the knapsack problem abstractly, so that our results hold for a large class of their variants. Let $I$ be a set of $n$ items. A configuration is a packing of some items from $I$ into a bin. Let $C$ be the set of all possible configurations of $I$. In the abstract bin-packing problem (BP), we have to pack the items into the minimum number of bins, such that the packing in each bin is according to some configuration in $C$. The abstract knapsack problem (KS) requires us to choose the max-profit configuration where each item has an associated profit. Note that we can get different variants of BP and KS by defining $C$ appropriately. For example, when $C = \{X : X \subseteq I$ and $\sum_{i \in X} s_i \leq 1\}$, we get classic-BP and classic-KS.

We will now formulate BP as an integer linear program. Let there be $m$ distinct items in the set $I$ of $n$ items. Let $b \in \mathbb{R}_{\geq 0}^m$ be a vector where $b_i$ is the number of items of type $i$. Therefore, $n = \sum_{i=1}^m b_i$. Let $N := |C|$. Let $A$ be an $m$-by-$N$ matrix where $A[i, C]$ is the number of items of type $i$ in configuration $C$. Then $A$ is called the configuration matrix of $I$. Let $1 \in \mathbb{R}^N$ be a vector whose each component is 1.

For every configuration $C$, suppose we pack $x_C$ bins according to $C$. Then the total number of bins used is $1^T x$. The number of items of type $i$ that got packed is $\sum_{C \in C} A[i, C]x_C = (Ax)_i$. Therefore, the optimal solution to BP is given by the optimal integral solution to covLP($A, b, 1$). covLP($A, b, 1$) is called the configuration LP of $I$ (also known as the Gilmore-Gomory LP of $I$).

Finding an approximately optimal (not necessarily integral) solution to the configuration LP of $I$ is also an important problem. The algorithm of Karmarkar and Karp for classic-BP [9] requires a $(1 + 1/n)$-approximate solution to covLP($A, b, 1$). The Round-and-Approx framework of Bansal and Khan [3], which is used to obtain the best-known approximation factor for 2GBP, requires a $(1 + \varepsilon)$-approximate solution to covLP($A, b, 1$).

### 2.2 Solving the Configuration LP using covLP-solve

**Indexing convention:** Instead of using an integer $j \in [N]$ to index the columns in the configuration matrix $A$ and the entries in a feasible solution $x$, we will index them by the
corresponding configuration $C$. Hence, instead of writing $A[i,j]$ and $x_j$, we will write $A[i,C]$ and $x_C$. Similarly, index-find will return a configuration instead of an integer.

**Lemma 5.** Let $\text{covLP}(A,b,1)$ be the configuration LP of a bin-packing instance $I$ having $n$ items. Then $1 \leq \text{opt}(\text{covLP}(A,b,1)) \leq n$.

**Proof.** Configurations that contain only a single item are called singleton configurations. Let $x_C = 0$ when $C$ is not a singleton configuration and $x_C = b_i$ when $C$ is a singleton configuration of item type $i$. Then $x$ is a feasible solution to $\text{covLP}(A,b,1)$ and $1^T x = n$. Therefore, $\text{opt}(\text{covLP}(A,b,1)) \leq n$.

Let $r^* = \text{opt}(\text{covLP}(A,b,1))$. Let $x^*$ be an optimal solution to $\text{covLP}(A,b,1)$. Let $i$ be an arbitrary number in $[m]$ ($m$ is the number of distinct items in $I$). Since $x^*$ is feasible,

$$b_i \leq (Ax^*)_i = \sum_{C \in \mathcal{C}} A[i,C]x^*_C \leq \sum_{C \in \mathcal{C}} b_ix^*_C = b_ir^* \implies 1 \leq r^*$$

To solve $\text{covLP}(A,b,1)$ using $\text{covLP-solve}$, we need to compute $q$, an upper-bound on $\text{opt}(\text{covLP}(A,b,1))$, and $\rho$, an upper-bound on $q \max_{i=1}^m \max_{C \in \mathcal{C}} A[i,C]/(b_i 1_C)$. By Lemma 5, we can select $q := n$. Since $A[i,C] \leq b_i$, we can choose $\rho := n$.

The cost oracle simply outputs 1 for every input. Let $a_C$ be the column of $A$ corresponding to configuration $C$. Then $a_C \in \mathbb{R}^m_\geq$ and the $i$th coordinate of $a_C$ is the number of items of type $i$ in configuration $C$. Therefore, for any configuration $C$, can get $a_C$ in $O(m)$ time.

For any configuration $C$, define the function $D_C : \mathbb{R}^m_\geq \rightarrow \mathbb{R}_\geq$ as

$$D_C(y) := y^T A e_C = \sum_{i=1}^m y_i A[i,C]$$

Then for $\eta \in (0,1]$, an $\eta$-weak index-finding oracle for $I$ is an algorithm that takes as input $y \in \mathbb{R}^m_\geq$ and returns $\tilde{C} \in \mathcal{C}$ such that $D_C(y) \geq \eta \max_{C \in \mathcal{C}} D_C(y)$. Note that if we assign profit $y_i$ to items of type $i$, then $D_C(y)$ is the profit of configuration $C$. Therefore, an $\eta$-weak index-finding oracle is an $\eta$-approximation algorithm for KS.

Now that we have the oracles ready, we can call $\text{covLP-solve}(I,n,n,\varepsilon,\eta)$ to get a $(1+\varepsilon+\varepsilon^2)/\eta$-approximate solution to $\text{covLP}(A,b,1)$. Let us now look at the time complexity of this solution.

**Theorem 6.** Let $I$ be a set of $n$ items, of which there are $m$ distinct items. Assume we have an $\eta$-approximate algorithm for KS that runs in time $O(T(m,n))$, for some function $T$ where $T(m,n) \geq m$. Then $\text{covLP-solve}(I,n,n,\varepsilon,\eta)$ runs in time $O(M \text{UT}(m,n))$. Here $M \in O(\log(n/(\varepsilon \eta)))$ and $U \in \tilde{O}(mn/(\eta \varepsilon^3))$ (as defined in Theorem 2).

**Proof.** By Theorem 2, the time taken by non-oracle operations is $O(MUm)$, the time taken by the product oracle is $O(MUm)$, and the time taken by calls to the algorithm for KS is $O(\text{MUT}(m,n))$.

Therefore, if $T(m,n) \in \text{poly}(n)$, then $\text{covLP-solve}$ gives us a polynomial-time algorithm for solving the configuration LP of $I$.

Note that $\eta$ can be very small here, i.e., this algorithm works even if the approximation factor of KS is very bad. As far as we know, all previous algorithms for approximately solving the configuration LP of BP assumed a PTAS for KS. For many variants of KS, no PTAS is known.
3 Comparison with Prior Work

Many algorithms for solving general and special LPs exist. In this section, we will look at the algorithms that have been used in the past to solve implicitly-defined covering LPs, especially the configuration LP of some variants of BP, and why they cannot be used for other variants of BP.

3.1 Ellipsoid Algorithm

The Ellipsoid algorithm by Khachiyan [10], in addition to being the first polynomial-time algorithm for linear programming, can solve LPs that are implicitly defined. Specifically, it uses a separation oracle, which takes a vector as input, and either claims that is feasible or outputs a constraint of the LP that is violated by . This is useful for solving LPs where the number of constraints is super-polynomial in the input size (Grötschel, Lovász and Schrijver [7] give many examples of this).

Let us see how the ellipsoid algorithm may be used for solving the configuration LP of a bin-packing instance. Let be implicitly defined in terms of input . has constraints, where , but the number of variables, , can be super-polynomial. We therefore compute the dual of , that has variables and constraints. We will solve using the Ellipsoid algorithm and then use that solution of to obtain a solution to . This is what looks like:

\[
\max_{y \in \mathbb{R}^m} b^T y \text{ where } A^T y \leq 1 \text{ and } y \geq 0
\]

The separation oracle for takes a vector as input and checks if .

\[
A^T y \leq 1 \iff \max_{C \in C}(A^T y)_C \leq 1 \iff \max_{C \in C} \sum_{i=1}^{m} y_i A[i, C] \leq 1
\]

If we interpret as the profit of item , then is the profit of configuration . Therefore, the separation oracle is the decision version of the knapsack problem. Specifically, the separation oracle should either claim that the optimal profit is at most 1, or it should output a configuration of profit more than 1. Since the decision version of the knapsack problem is known to be NP-complete, we cannot design a polynomial-time separation oracle.

Grötschel, Lovász and Schrijver [7] gave a variant of the Ellipsoid algorithm (which we will hereafter refer to as the GLS algorithm) that can approximately solve an LP using an approximate separation oracle. Karmarkar and Karp [9] modified the GLS algorithm to solve the dual of the configuration LP of classic-BP, and described how to obtain a solution to the configuration LP using a solution to the dual. Their algorithm, however, requires an FPTAS for classic-KS. Our algorithm doesn’t have such strict requirements, and can work with very poorly-approximated algorithms for KS.

3.2 Plotkin-Shmoys-Tardos Algorithm

Plotkin, Shmoys and Tardos [12] gave algorithms for solving the fractional covering problem (see Definition 5) and the fractional packing problem. Our algorithm frac-cover is obtained by slightly modifying their algorithm. The following theorem is their most relevant result to us:
Theorem 7 (Theorem 3.10 in [12]). For $0 < \varepsilon < 1$, given a $(1 - \varepsilon/2)$-weak point-finding oracle, the algorithm of [12] $(1 - \varepsilon)$-weakly solves the fractional covering problem.

The above result holds only for a sufficiently small $\varepsilon$. [12] doesn’t explicitly state how small $\varepsilon$ should be, but even the optimistic case of $\varepsilon < 1$ tells us that for an $\eta$-weak point-find, we require $\eta > 1/2$. However, we are interested in the case where $\eta$ can be very small.

Moreover, there is a large gap between $(1 - \varepsilon/2)$ and $(1 - \varepsilon)$ when $\varepsilon$ is large enough. For example, for $\varepsilon = 1/3$, point-find is $5/6$-weak, but their algorithm will only give us a $2/3$-weak solution to the fractional covering problem. Our modified algorithm frac-cover, on the other hand, outputs a solution that is roughly $5/6$-weak for this example.

We did not focus on optimizing the running time of our algorithm; instead, we focused on getting as small an approximation factor as possible. Our algorithm is, therefore, slower than that of [12].

4 The Fractional Covering Problem

Recall that in the problem $\text{fcov}(A, b, P)$, we need to find $x \in P$ such that $Ax \geq b$ or claim that no such $x$ exists. Also, $\rho \geq \text{width}(A, b, P)$.

4.1 Optimization Version of $\text{fcov}$

Let us try to frame $\text{fcov}(A, b, P)$ as an optimization problem.

Definition 9. For the problem $\text{fcov}(A, b, P)$, let $\lambda(x) := \max_{\lambda}(Ax \geq \lambda b)$. The problem $\text{ofcov}(A, b, P)$ is defined as

$$\argmax_{x \in P} \lambda(x)$$

Let $x^*$ be the optimal solution to $\text{ofcov}(A, b, P)$ and let $\lambda^* := \lambda(x^*)$. Then $x \in P$ is said to be $\varepsilon$-optimal for $\text{ofcov}(A, b, P)$ iff $\lambda(x) \geq (1 - \varepsilon)\lambda(x^*)$.

Claim 8.

$$\lambda(x) = \min_{i \in [m]} \frac{(Ax)_i}{b_i}$$

So $\lambda(x)$ can be computed using the product oracle.

Note that $\text{fcov}(A, b, P)$ is unsatisfiable iff $\lambda(x^*) < 1$, and otherwise $x^*$ is a solution to $\text{fcov}(A, b, P)$. However, we can’t directly use this fact to solve $\text{fcov}(A, b, P)$, since it may be very hard to compute $x^*$. So instead, we’ll compute an $\varepsilon$-optimal solution $\hat{x}$ to $\text{ofcov}(A, b, P)$. Then $\lambda(\hat{x})$ is an approximation to $\lambda(x^*)$, since $(1 - \varepsilon)\lambda(x^*) \leq \lambda(\hat{x}) \leq \lambda(x^*)$.

Claim 9. If $x$ is $\varepsilon$-optimal for $\text{ofcov}(A, b, P)$, then

$$\lambda(x) < 1 - \varepsilon \implies \lambda^* \leq \frac{\lambda(x)}{1 - \varepsilon} < 1 \implies \text{fcov}(A, b, P) \text{ has no solution}$$

$$\lambda(x) \geq 1 - \varepsilon \implies Ax \geq (1 - \varepsilon)b \implies x \text{ is } (1 - \varepsilon)\text{-approx for } \text{fcov}(A, b, P)$$

We’ll now focus on finding an $\varepsilon$-optimal solution to $\text{ofcov}(A, b, P)$.
4.2 Weak Duality

**Definition 10.** For the problem \( \text{ofcov}(A,b,P) \), define

\[
C(y) := \max_{x \in P} y^T Ax.
\]

Define the problem \( \text{dfcov}(A,b,P) \) as

\[
\arg\min_{y \in \mathbb{R}_m} y^T b
\]

We call it the dual problem of \( \text{ofcov} \).

**Lemma 10 (Weak duality).** Let \( \hat{x} \in P \). Then

\[
\lambda(\hat{x}) y^T b \leq y^T A\hat{x} \leq C(y)
\]

**Proof.**

\[
\lambda(\hat{x})(y^T b) = y^T (\lambda(\hat{x}) b) \\
\leq y^T A\hat{x} \quad \text{ (y \geq 0 and } A\hat{x} \geq \lambda(\hat{x}) b) \\
\leq \max_{x \in P} y^T Ax = C(y)
\]

4.3 Relaxed Optimality Conditions

If we could find \( x \in P \) and \( y \geq 0 \) such that \( \lambda(x)y^T b \geq y^T Ax \geq C(y) \), then weak duality would imply that \( x \) is optimal for \( \text{ofcov} \) and \( y \) is optimal for \( \text{dfcov} \). To find approximate optima, we slightly relax these conditions.

\[
(1 + \varepsilon_1)\lambda(x)y^T b \geq y^T Ax \quad \text{(condition } C_1(\varepsilon_1))
\]

\[
C(y) - y^T Ax \leq \varepsilon_2 C(y) + \varepsilon_3 \lambda(x)y^T b \quad \text{(condition } C_2(\varepsilon_2, \varepsilon_3))
\]

Condition \( C_2(\varepsilon_2, \varepsilon_3) \) can equivalently be written as

\[
y^T Ax \geq (1 - \varepsilon_2)C(y) - \varepsilon_3 \lambda(x)y^T b
\]

**Lemma 11.** Suppose \( x \in P \) and \( y \geq 0 \) satisfy conditions \( C_1(\varepsilon_1) \) and \( C_2(\varepsilon_2, \varepsilon_3) \), where \( 0 < \varepsilon_1, \varepsilon_2, \varepsilon_3 < 1 \). Let \( \varepsilon' := (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/(1 + \varepsilon_2 + \varepsilon_3) \). Then \( x \) is \( \varepsilon' \)-optimal for \( \text{ofcov}(A,b,P) \).

**Proof.** By weak duality (Lemma 10), we get \( \lambda^* \leq C(y)/b^Ty \).

Conditions \( C_1(\varepsilon_1) \) and \( C_2(\varepsilon_2, \varepsilon_3) \) give us

\[
(1 + \varepsilon_1)\lambda(x)y^T b \geq y^T Ax \geq (1 - \varepsilon_2)C(y) - \varepsilon_3 \lambda(x)y^T b \\
\implies (1 + \varepsilon_1 + \varepsilon_3)\lambda(x)y^T b \geq (1 - \varepsilon_2)C(y) \\
\implies \lambda(x) \geq \frac{1 - \varepsilon_2}{1 + \varepsilon_1 + \varepsilon_3} \lambda^* = (1 - \varepsilon')\lambda^* \\
\implies x \text{ is } \varepsilon' \text{-optimal}
\]

We’ll now focus on finding \( x \in P \) and \( y \geq 0 \) such that conditions \( C_1(\varepsilon_1) \) and \( C_2(\varepsilon_2, \varepsilon_3) \) are satisfied.
### 4.4 Dual Fitting

**Definition 11.** For some $\alpha > 0$, define the vector $y_\alpha(x) \in \mathbb{R}^m_{>0}$ as  
\[
y_\alpha(x)_i := \frac{1}{b_i} \exp \left( -\frac{\alpha(Ax)_i}{b_i} \right)
\]

**Lemma 12.**  
\[
\exp(-\alpha \lambda(x)) \leq b^T y_\alpha(x) \leq m \exp(-\alpha \lambda(x))
\]

**Proof.**  
\[
b^T y_\alpha(x) = \sum_{i=1}^m \exp \left( -\frac{\alpha(Ax)_i}{b_i} \right)
\]
\[
\in [1,m] \max_{i=1}^m \exp \left( -\frac{\alpha(Ax)_i}{b_i} \right)
\]
\[
= [1,m] \exp \left( -\alpha \min_{i=1}^m \frac{(Ax)_i}{b_i} \right) = [1,m] \exp(-\alpha \lambda(x)) \quad \square
\]

**Lemma 13.** Let $0 < \varepsilon < 1$ and let  
\[
\beta := \frac{2}{\varepsilon} \ln \left( \frac{4m}{\varepsilon} \right)
\]
(note that $\beta \geq 1$). Then  
\[
\alpha \lambda(x) \geq \beta \implies (x,y_\alpha(x)) \text{ satisfies } C_1(\varepsilon)
\]

**Proof.** See lemma 3.2 in [12].  

**Lemma 14.** width($A,b,P$) = 0 $\implies$ fcov($A,b,P$) is unsatisfiable.

**Proof.** We know that width($A,b,P$) $\geq 0$. Suppose width($A,b,P$) = 0. Then $\forall x \in P, Ax = 0$, so $Ax \geq b$ cannot be true.  

Lemma 14 means that if $\rho = 0$, then we know that fcov($A,b,P$) is unsatisfiable. So we’ll assume from now on that $\rho > 0$.

**Theorem 15.** Let $\varepsilon_\sigma, \varepsilon_\alpha, \varepsilon_2, \varepsilon_3 \in (0,1)$ be constants. Suppose $x \in P, \tilde{x} \in P, \alpha \in \mathbb{R}_{>0}, \eta \in (0,1]$ and $\sigma \in \mathbb{R}_{>0}$ satisfy the following properties:  
- $\lambda(x) > 0$.  
- $(x,y_\alpha(x))$ doesn’t satisfy condition $C_2(\varepsilon_2, \varepsilon_3)$. Denote $y_\alpha(x)$ by $y$ for simplicity.  
- $y^T A \tilde{x} \geq \eta C(y)$.  
- $\sigma \leq \varepsilon_\sigma / (\alpha \rho)$. Let $\hat{x} = (1 - \sigma)x + \sigma \tilde{x}$. Denote $y_\alpha(\hat{x})$ by $\hat{y}$ for simplicity.  
- $\alpha \geq \frac{2}{\lambda(x) \varepsilon_\alpha} \ln \left( \frac{4m}{\varepsilon_\alpha} \right)$.  
- $\eta \geq (1 - \varepsilon_2) \left( \frac{1 + \varepsilon_\sigma}{1 - \varepsilon_\sigma} \right)$.

Then $\hat{x} \in P$ and  
\[
\frac{b^T y - b^T \hat{y}}{b^T y} > \alpha \sigma \lambda(x) \varepsilon_3 (1 - \varepsilon_\sigma) \frac{\eta}{1 - \varepsilon_2}
\]
Intuitively, this means that under certain conditions, a point \( x \in P \) can be replaced by a different point in \( P \) such that \( b^T y_\alpha(x) \) decreases by a significant factor. The reasons for choosing such conditions will be apparent on reading Section 4.5.

**Proof.**

\[
\sigma \leq \frac{\varepsilon_\sigma}{\alpha \rho} \leq \frac{\varepsilon_\sigma}{\rho} \frac{\lambda(x) \varepsilon_\alpha}{2 \ln(4m/\varepsilon_\alpha)} \quad (\alpha \geq \frac{2}{\lambda(x) \varepsilon_\alpha} \ln \left( \frac{4m}{\varepsilon_\alpha} \right)) \\
< \frac{\varepsilon_\sigma \varepsilon_\alpha}{2 \ln(4m/\varepsilon_\alpha)} \leq \frac{1}{2} \quad (\lambda(x) \leq \rho) \\
< \frac{1}{2 \ln(4m)} \leq \frac{1}{2} \quad (\varepsilon_\sigma, \varepsilon_\alpha < 1)
\]

Therefore, \( \sigma \leq 1 \), so \( \tilde{x} \) is a convex combination of \( x \) and \( \tilde{x} \). Therefore, \( \tilde{x} \in P \).

Let \( \lambda_i := (Ax)_i/b_i, \tilde{\lambda}_i := (A\tilde{x})_i/b_i \) and \( \hat{\lambda}_i := (A\hat{x})_i/b_i \). Then

\[
\hat{x} - x = \sigma(\hat{x} - x) \quad \hat{\lambda}_i = (1 - \sigma)\lambda_i + \sigma \tilde{\lambda}_i \quad \tilde{\lambda}_i - \lambda_i = \sigma(\hat{\lambda}_i - \lambda_i) \\
y_i = \frac{1}{b_i} e^{-\alpha \lambda_i} \quad \hat{y}_i = \frac{1}{b_i} e^{-\alpha \hat{\lambda}_i} \\
\frac{\hat{y}_i}{y_i} = \exp(-\alpha(\hat{\lambda}_i - \lambda_i)) = \exp(-\alpha \sigma(\hat{\lambda}_i - \lambda_i))
\]

Let \( \delta := -\alpha \sigma(\hat{\lambda}_i - \lambda_i) \). By definition of \( \rho \) and width, \( \lambda_i \leq \rho \) and \( \tilde{\lambda}_i \leq \rho \). Therefore, \( |\hat{\lambda}_i - \lambda_i| \leq \rho \). This gives us \( |\delta| = \alpha \sigma |\hat{\lambda}_i - \lambda_i| \leq \alpha \sigma \rho \leq \varepsilon_\sigma < 1 \).

For \( |\delta| \leq 1 \), we get \( e^\delta < 1 + \delta + \delta^2 \leq 1 + \delta + |\delta| \varepsilon_\sigma \).

\[
\begin{align*}
b_i \hat{y}_i &= b_i y_i e^\delta \\
&< b_i y_i (1 + \delta + \varepsilon_\sigma |\delta|) \\
&= b_i y_i - \alpha \sigma y_i b_i (\hat{\lambda}_i - \lambda_i) + \alpha \sigma \varepsilon_\sigma |y_i b_i (\hat{\lambda}_i - \lambda_i)| \\
&= b_i y_i - \alpha \sigma (y_i (A\tilde{x})_i - y_i (Ax)_i) + \alpha \sigma \varepsilon_\sigma |y_i (A\tilde{x})_i - y_i (Ax)_i| \\
\Rightarrow \quad \frac{b_i y_i - b_i \hat{y}_i}{\alpha \sigma} &> (y_i (A\tilde{x})_i - y_i (Ax)_i) - \varepsilon_\sigma |y_i (A\tilde{x})_i - y_i (Ax)_i| \\
&\geq (y_i (A\tilde{x})_i - y_i (Ax)_i) - \varepsilon_\sigma (y_i (A\tilde{x})_i - y_i (Ax)_i) + y_i (Ax)_i) \\
&= (1 - \varepsilon_\sigma)y_i (A\tilde{x})_i - (1 + \varepsilon_\sigma)y_i (Ax)_i \\
\Rightarrow \quad \frac{b^T y - b^T \hat{y}}{\alpha \sigma} &> (1 - \varepsilon_\sigma)y^T A\tilde{x} - (1 + \varepsilon_\sigma)y^T Ax \\
&\geq (1 - \varepsilon_\sigma)\eta C(y) - (1 + \varepsilon_\sigma)y^T Ax
\end{align*}
\]

Since condition \( C_2(\varepsilon_2, \varepsilon_3) \) is not satisfied, we get

\[
(1 - \varepsilon_2)C(y) > y^T Ax + \varepsilon_3 \lambda(x)b^T y
\]

\[
\Rightarrow \quad \frac{b^T y - b^T \hat{y}}{\alpha \sigma} > (1 - \varepsilon_\sigma)\eta C(y) - (1 + \varepsilon_\sigma)y^T Ax \\
> (1 - \varepsilon_\sigma)\eta \frac{y^T Ax + \varepsilon_3 \lambda(x)b^T y}{1 - \varepsilon_2} - (1 + \varepsilon_\sigma)y^T Ax \\
= \left( 1 - \varepsilon_\sigma \right) \frac{\eta}{1 - \varepsilon_2} - (1 + \varepsilon_\sigma) \frac{\eta \varepsilon_3}{1 - \varepsilon_2} \lambda(x)b^T y
\]
\[ \eta \geq (1 - \varepsilon_2) \frac{1 + \varepsilon_\sigma}{1 - \varepsilon_2} \implies (1 - \varepsilon_\sigma) \frac{\eta}{1 - \varepsilon_2} - (1 + \varepsilon_\sigma) \geq 0 \]

Therefore,
\[
\frac{b^T y - b^T \hat{y}}{\alpha \sigma} > (1 - \varepsilon_\sigma) \frac{\eta \varepsilon_3}{1 - \varepsilon_2} \lambda(x) b^T y
\]
\[
\implies \frac{b^T y - b^T \hat{y}}{b^T y} \geq \alpha \sigma \lambda(x) \varepsilon_3 (1 - \varepsilon_\sigma) \frac{\eta}{1 - \varepsilon_2}
\]

4.5 Algorithm improve-cover

We’ll now start building an algorithm for fcov(A, b, P), where A, b, P are implicitly defined by input I.

**Algorithm 1** improve-cover(I, x, ε_σ, ε_1, ε_2, ε_3, ρ):  
Requires \( x \in P \), \( \rho > 0 \), \( \lambda(x) > 0 \), \( 0 < \varepsilon_\sigma, \varepsilon_1, \varepsilon_2, \varepsilon_3 < 1 \) and \( (1 - \varepsilon_2) \frac{1 + \varepsilon_\sigma}{1 - \varepsilon_2} \leq \eta \leq 1 \).

```plaintext
1: \( \lambda_0 = \lambda(x) \quad \alpha := \frac{4}{\lambda_0 \varepsilon_1} \ln \left( \frac{4m}{\varepsilon_1} \right) \quad \sigma := \frac{\varepsilon_\sigma}{\alpha \rho} \\
2: \textbf{while } \lambda(x) \leq 2 \lambda_0 \text{ and } (x, y_\alpha(x)) \text{ doesn’t satisfy condition } C_2(\varepsilon_2, \varepsilon_3) \text{ do} \\
3: \quad \hat{x} = \text{point-find}(y_\alpha(x)) \quad // \text{now } y_\alpha(x)^T A \hat{x} \geq \eta C(y_\alpha(x)). \\
4: \quad x = (1 - \sigma)x + \sigma \hat{x} \\
5: \textbf{end while} \\
6: \text{success} = \text{true} \text{ if } (x, y_\alpha(x)) \text{ satisfies condition } C_2(\varepsilon_2, \varepsilon_3) \text{ else false} \\
7: \text{return } (x, \text{success})
```

**Lemma 16.** Let \( x^{(0)} \) be the initial value of \( x \). Throughout improve-cover, the following conditions are satisfied:

- \( x \in P \)
- \( b^T y_\alpha(x) \leq b^T y_\alpha(x^{(0)}) \)
- \( \lambda(x) \geq \frac{3}{4} \lambda_0 \)
- \( \alpha \geq \frac{3}{\lambda(x) \varepsilon_1} \ln \left( \frac{4m}{\varepsilon_1} \right) \)

**Proof.** These conditions are satisfied at the beginning of the algorithm.

Assume that the conditions are satisfied at the beginning of the while loop body. Then the conditions for Theorem 15 are satisfied, so \( \hat{x} = (1 - \sigma)x + \sigma \hat{x} \in P \) and \( b^T y_\alpha(\hat{x}) \leq b^T y_\alpha(x) \leq b^T y_\alpha(x^{(0)}) \).

Let \( \hat{\lambda} = \lambda(\hat{x}) \). By Lemma 12, we get
\[
e^{-\alpha \hat{\lambda}} \leq b^T y_\alpha(\hat{x}) \leq b^T y_\alpha(x^{(0)}) \leq me^{-\alpha \lambda_0} \]
\[
\implies me^{\alpha \hat{\lambda}} \geq e^{\alpha \lambda_0} \\
\implies \lambda_0 - \hat{\lambda} \leq \frac{\ln m}{\alpha} = \ln m \frac{\lambda_0 \varepsilon_1}{4 \ln \left( \frac{4m}{\varepsilon_1} \right)} \leq \frac{\lambda_0}{4} \\
\implies \frac{3 \lambda_0}{4} \leq \hat{\lambda}
\]
\[ \alpha = \frac{4}{\lambda_0 \varepsilon_1} \ln \left( \frac{4m}{\varepsilon_1} \right) \geq \frac{3}{\lambda \varepsilon_1} \ln \left( \frac{4m}{\varepsilon_1} \right) \]

Therefore, the conditions are also satisfied after the \textit{while} loop.

\textbf{Corollary 16.1.} Throughout improve-cover, \((x, y(x))\) satisfies condition \(C_1(\varepsilon_1)\).

\textit{Proof.} Use \(\alpha \geq \frac{3}{\lambda(x) \varepsilon_1} \ln \left( \frac{4m}{\varepsilon_1} \right)\) from Lemma 16 and Lemma 13. \hfill \Box

\textbf{Lemma 17.} Let \(\varepsilon' = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/(1 + \varepsilon_1 + \varepsilon_3)\). If improve-cover terminates and success is \textit{true}, then \(x\) is \(\varepsilon'-\text{optimal}\) for ofcov\((A, b, P)\). If improve-cover terminates and success is \textit{false}, then \(\lambda(x) > 2\lambda_0\).

\textit{Proof.} By Corollary 16.1, \(x\) satisfies condition \(C_1(\varepsilon_1)\). If success is \textit{true}, then \(x\) satisfies condition \(C_2(\varepsilon_2, \varepsilon_3)\). So by Lemma 11, \(x\) is \(\varepsilon'-\text{optimal}\) for ofcov\((A, b, P)\).

If success is \textit{false}, then \(x\) doesn’t satisfy condition \(C_2(\varepsilon_2, \varepsilon_3)\). Since the \textit{while} loop ended, \(\lambda(x) > 2\lambda_0\). \hfill \Box

\textbf{Claim 18.} \(\forall x \in \mathbb{R}_{>0}, \ln x > (x - 1)/x\).

\textbf{Theorem 19.} Let \(T\) be the number of times improve-cover calls \textit{point-find}. Then \(T\) is finite and

\[ T \leq \left\lceil \frac{4p}{3 \varepsilon_2 \varepsilon_3 \lambda_0} \left( \ln m + \frac{4}{\varepsilon_1} \ln \left( \frac{4m}{\varepsilon_1} \right) \right) \right\rceil. \]

\textit{Proof.} Let \(x^{[t]}\) be the value of \(x\) after \(t\) runs of the \textit{while} loop. Let \(\psi(t) := b^T y(x^{[t]})\). So \(\lambda(x^{[t]}) \leq 2\lambda_0\) for all \(t < T\). By Lemmas 12 and 16, we get

\[ e^{-\alpha(2\lambda_0)} \leq e^{-\alpha \lambda(x^{[t]})} \leq b^T y(x^{[t]}) \leq b^T y(x^{[0]}) \leq m e^{-\alpha \lambda_0} \]

\[ \Rightarrow \frac{\psi(0)}{\psi(t)} \leq \frac{me^{-\alpha \lambda_0}}{e^{-\alpha(2\lambda_0)}} = me^{\alpha \lambda_0} \]

Define

\[ \beta := \frac{3}{4} \alpha \sigma \lambda_0 \varepsilon_3 (1 - \varepsilon_2) \frac{\eta}{1 - \varepsilon_2} \]

By Lemma 16, the conditions for Theorem 15 are satisfied for \(x = x^{[j]}\) for all \(j < T\). By Theorem 15,

\[ \forall j < T, \frac{\psi(j + 1)}{\psi(j)} < 1 - \beta \Rightarrow \frac{\psi(0)}{\psi(t)} > (1 - \beta)^{-t} \]

Therefore,

\[ (1 - \beta)^{-t} < \frac{\psi(0)}{\psi(t)} \leq me^{\alpha \lambda_0} \Rightarrow t < \frac{\ln m + \alpha \lambda_0}{\ln(1/(1 - \beta))} \iff t \leq \left\lceil \frac{\ln m + \alpha \lambda_0}{\ln(1/(1 - \beta))} \right\rceil - 1 \]

Since this is true for all \(t < T\), we get that \(T\) is finite and

\[ T \leq \left\lceil \frac{\ln m + \alpha \lambda_0}{\ln(1/(1 - \beta))} \right\rceil. \]
By Claim 18, \( \ln((1 - \beta)^{-1}) \geq \beta \).

\[
\beta = \frac{3}{4} \alpha \sigma \lambda_0 (1 - \varepsilon_\sigma) \frac{\eta \varepsilon_3}{1 - \varepsilon_2} = \frac{3 \varepsilon_\sigma \lambda_0}{4 \rho} (1 - \varepsilon_\sigma) \frac{\eta \varepsilon_3}{1 - \varepsilon_2} \quad \text{(since } \sigma := \varepsilon_\sigma / (\alpha \rho)) \]

\[
\geq \frac{3 \varepsilon_3 \varepsilon_\sigma \lambda_0}{4 \rho} (1 + \varepsilon_\sigma) > \frac{3 \varepsilon_3 \varepsilon_\sigma \lambda_0}{4 \rho} \quad \text{(}\eta \geq (1 - \varepsilon_2) \frac{1 + \varepsilon_\sigma}{1 - \varepsilon_\sigma}\text{)}
\]

Therefore,

\[
T \leq \left\lceil \frac{\ln m + \alpha \lambda_0}{\ln((1 - \beta)^{-1})} \right\rceil \leq \left\lceil \frac{1}{\beta} \ln m + \frac{4}{\varepsilon_1} \ln \left(\frac{4m}{\varepsilon_1}\right) \right\rceil \leq \left\lceil \frac{4 \rho}{3 \varepsilon_3 \varepsilon_\sigma \lambda_0} \left(\ln m + \frac{4}{\varepsilon_1} \ln \left(\frac{4m}{\varepsilon_1}\right)\right) \right\rceil \]

\[\square\]

4.6 Algorithm frac-cover

4.6.1 Starting with Good \( \lambda(x) \)

The number of iterations within improve-cover depends inversely on \( \lambda_0 \). Therefore, we have to ensure that \( \lambda_0 \) isn’t too small. Let \( e_i \) be the \( i \)th standard basis vector for \( \mathbb{R}^m \). So \( e_i^T A x = (Ax)_i \).

**Lemma 20.** Let \( x^{(i)} := \text{point-find}(e_i) \). Let \( \bar{x} := \frac{1}{m} \sum_{i=1}^{m} x^{(i)} \). If \( (Ax^{(i)})_i < \eta b_i \) for some \( i \in [m] \), then \( \text{fcov}(A, b, P) \) is unsatisfiable. Otherwise, \( \lambda(\bar{x}) \geq \eta/m \).

**Proof.** Since point-find is \( \eta \)-weak, \( (Ax^{(i)})_i \geq \eta \max_{x \in P} (Ax)_i \).

\[
(Ax^{(i)})_i < \eta b_i \implies \max_{x \in P} (Ax)_i < b_i \implies \forall x \in P, Ax \not\geq b
\]

\[
\implies \text{fcov}(A, b, P) \text{ is unsatisfiable.}
\]

Suppose \( (Ax^{(i)})_i \geq \eta b_i \) for all \( i \in [m] \). Since each \( x^{(i)} \in P \), we get \( \bar{x} \in P \). Also,

\[
(A\bar{x})_i = \frac{1}{m} \sum_{j=1}^{m} (Ax^{(j)})_i \geq \frac{1}{m} (Ax^{(i)})_i \geq \frac{\eta}{m} b_i \implies \lambda(\bar{x}) \geq \frac{\eta}{m}
\]

**\square**

**Algorithm 2** gives us Algorithm 2 (get-seed).

**Algorithm 2 get-seed(I, \eta)**

1: for \( i \in [m] \) do
2: \( x^{(i)} = \text{point-find}(e_i) \quad \text{// so } e_i^T A x^{(i)} \geq \eta \max_{x \in P} e_i^T A x. \)
3: \[ \text{if } (Ax^{(i)})_i < \eta b_i \text{ then} \]
4: \[ \quad \text{return null} \]
5: \[ \text{end if} \]
6: \[ \text{end for} \]
7: \[ \text{return } \bar{x} := \frac{1}{m} \sum_{i=1}^{m} x^{(i)} \]

14
4.6.2 Iteratively Running improve-cover

Algorithm 3 frac-cover($I, \rho, \varepsilon, \eta$): Returns either $x \in P$ or null.
Requires $\rho \geq \text{width}(A, b, P)$ and $\varepsilon, \eta \in (0, 1]$.

1: if $\rho == 0$ then
2: return null
3: end if
4: $x = \text{get-seed}(I, \eta)$
5: if $x == \text{null}$ then
6: return null
7: end if
8: Let $\varepsilon_{\sigma} := \varepsilon / (6 + 5 \varepsilon)$ and $\varepsilon_{1} := \varepsilon_{3} := \varepsilon / 3$.
9: Let $\varepsilon_{2} := 1 - \eta \frac{1 - \varepsilon_{\sigma}}{1 + \varepsilon_{\sigma}}$ and $\varepsilon' := \frac{\varepsilon_{1} + \varepsilon_{2} + \varepsilon_{3}}{1 + \varepsilon_{1} + \varepsilon_{3}}$.
10: while true do
11: if $\lambda(x) \geq 1$ then
12: return $x$
13: end if
14: $(x, \text{success}) = \text{improve-cover}(x, \varepsilon_{\sigma}, \varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}, \rho)$.
15: if success then
16: return $x$ if $\lambda(x) \geq 1 - \varepsilon'$ else null
17: end if
18: end while

Lemma 21. In frac-cover, $1 - \varepsilon' = \eta / (1 + \varepsilon)$.

Proof.

$1 - \varepsilon' = \frac{1 - \varepsilon_{2}}{1 + \varepsilon_{1} + \varepsilon_{3}} = \frac{\eta - \varepsilon_{\sigma}}{\gamma + \varepsilon_{\sigma}} \frac{1}{1 + \varepsilon_{1} + \varepsilon_{3}} = \frac{1 - \varepsilon / (6 + 5 \varepsilon)}{1 + \varepsilon / (6 + 5 \varepsilon)} \frac{1 + 2 \varepsilon / 3}{1 + \varepsilon} = \frac{\eta}{1 + \varepsilon}$

Theorem 3. Let $\text{fcov}(A, b, P)$ be a fractional covering problem instance where $A, b, P$ are implicitly defined in terms of input $I$. Then frac-cover($I, \rho, \varepsilon, \eta$) will $\eta / (1 + \varepsilon)$-weakly solve $\text{fcov}(A, b, P)$, i.e., if it returns null, then $\text{fcov}(A, b, P)$ is unsatisfiable, and if it returns a vector $x$, then $x \in P$ and $Ax \geq (\eta / (1 + \varepsilon))b$.

Proof. When frac-cover returns null at Line 2, then $\text{fcov}(A, b, P)$ is unsatisfiable by Lemma 14. get-seed and improve-cover have output in $P$. Therefore, throughout the execution of frac-cover, $x \in P$.

When frac-cover returns null at Line 6, it does so because get-seed returned null. Then as per Lemma 20, $\text{fcov}(A, b, P)$ is unsatisfiable.

When frac-cover returns $x$ at Line 12, then $\lambda(x) \geq 1 \implies Ax \geq b$.

frac-cover returns at Line 16 when success is true. Then by Lemma 17, $x$ would be $\varepsilon'$-optimal. By Claim 9, we get that if frac-cover returns null, then $\text{fcov}(A, b, P)$ is unsatisfiable. Otherwise, $x$ is a $(1 - \varepsilon')$-approximate solution to $Ax \geq b$, i.e. $Ax \geq (1 - \varepsilon')b$. By Lemma 21, $Ax \geq (\eta / (1 + \varepsilon))b$. 

15
Lemma 22. Let $\text{fcov}(A,b,P)$ be implicitly defined by input $I$, where $A \in \mathbb{R}^{m \times N}_{\geq 0}$. Suppose $\text{frac-cover}(I, \rho, \varepsilon, \eta)$ calls point-find $T$ times. Then

$$T \leq U := m + \left[ \ln \left( \frac{m}{\eta} \right) \right] \left[ \frac{312m\rho(1+\varepsilon)}{\eta \varepsilon^3} \ln \left( \frac{12m}{\varepsilon} \right) \right] \in \tilde{O} \left( \frac{m\rho}{\eta^3} \right)$$

Proof. get-seed calls point-find $m$ times. The other $T - m$ calls to point-find occur inside improve-cover.

For $j \geq 0$, let $x^{[j]}$ be the value of $x$ given as input to the $(j+1)^{\text{th}}$ run of improve-cover. Suppose improve-cover is called at least $t$ times. Then $\lambda(x^{[t-1]}) < 1$ (see Line 11). Also, success was false for the first $t-1$ calls to improve-cover, so by Lemma 17, $\forall j \in [t-1], \lambda(x^{[j]}) > 2\lambda(x^{[t-1]})$. Therefore, $2^{t-1}\lambda(x^{[0]}) \leq \lambda(x^{[t-1]}) < 1$.

By Lemma 20, $\lambda(x^{[0]}) \geq \eta/m$. Therefore,

$$2^{t-1} < \frac{m}{\eta} \implies t - 1 < \lg \left( \frac{m}{\eta} \right) \iff t \leq \left\lfloor \lg \left( \frac{m}{\eta} \right) \right\rfloor$$

Therefore, improve-cover is called at most $\lfloor \lg(m/\eta) \rfloor$ times.

For each input $x$ to improve-cover, $\lambda(x) \geq \eta/m$. So, by Theorem 19, the number of times point-find is called in each call to improve-cover is at most

$$\left\lfloor \frac{4\rho}{3\varepsilon_0 \varepsilon_3 \lambda_0} \left( \ln m + \frac{4}{\varepsilon_1} \ln \left( \frac{4m}{\varepsilon_1} \right) \right) \right\rfloor \leq \left\lfloor \frac{4m\rho}{3\eta \varepsilon_0 \varepsilon_3} \left( \ln m + \frac{4}{\varepsilon_1} \ln \left( \frac{4m}{\varepsilon_1} \right) \right) \right\rfloor \leq \left\lfloor \frac{312m\rho(1+\varepsilon)}{\eta \varepsilon^3} \ln \left( \frac{12m}{\varepsilon} \right) \right\rfloor$$

This gives us

$$T \leq m + \left[ \ln \left( \frac{m}{\eta} \right) \right] \left[ \frac{312m\rho(1+\varepsilon)}{\eta \varepsilon^3} \ln \left( \frac{12m}{\varepsilon} \right) \right]$$

Lemma 23. Let $\text{fcov}(A,b,P)$ be implicitly defined in terms of input $I$, where $A \in \mathbb{R}^{m \times N}_{\geq 0}$. Let $\tau$ be an upper-bound on the support of the output of point-find. Suppose $\text{frac-cover}(I, \rho, \varepsilon, \eta)$ calls point-find $T$ times. Then,

- **frac-cover** makes at most $T$ calls to the product oracle. For every input $x$ to the product oracle, $|\text{support}(x)| \leq \tau$.
- The running time of frac-cover, excluding the time taken by oracles, is $O(T(m + \tau))$.
- The solution $\hat{x}$ returned by frac-cover has $|\text{support}(\hat{x})| \leq \tau T$.

Proof. get-seed calls point-find $m$ times. The other $T - m$ calls to point-find occur inside improve-cover. Let $x^{[0]}$ be the output of get-seed. Assume $x^{[0]}$ is not null (since otherwise the proof is trivial). For $t \in [T - m]$, in the $t^{\text{th}}$ run of line 4 in improve-cover, let $\sigma_i$ be the value of variable $\sigma$, let $\tilde{x}^{[t]}$ be the output of point-find, and let $x^{[t]}$ be the new value of variable $x$. Therefore, $x^{[t]} = (1 - \sigma_t)x^{[t-1]} + \sigma_t\tilde{x}^{[t]}$, and the output of frac-cover is $x^{[T-m]}$.

In get-seed, the product oracle is called $m$ times — to compute $(Ax^{[i]})_i$ for $i \in [m]$. The inputs to these calls have a support of size at most $\tau$. We can take the mean of these product oracle outputs to get $Ax^{[0]}$.

After get-seed, the product oracle is never called directly — it is only needed to compute $\lambda(x^{[t]})$ and $y_0(x^{[t]})$ and to check condition $C_2(\varepsilon_2, \varepsilon_3)$ for all $0 \leq t \leq T - m$. To compute $Ax^{[t]}$, we won’t call the product oracle on $x^{[t]}$, since the support of $x^{[t]}$ can be too large. Instead, we’ll
compute it indirectly from \(x^{[t-1]}\) and \(\tilde{x}^{[t]}\) using \(Ax^{[t]} = (1 - \sigma_t)(Ax^{[t-1]}) + \sigma_t(A\tilde{x}^{[t]})\). Note that we already know \(Ax^{[0]}\) and \(|\text{support}(\tilde{x}^{[t]})| \leq \tau\). Therefore, we need to call the product oracle \(T\) times in \texttt{frac-cover}, and each input to the product oracle has support of size at most \(\tau\).

\(|\text{support}(x^{[0]})| \leq m\tau\), since \(x^{[0]}\) is the mean of \(m\) outputs of \texttt{point-find}. Each modification of \(x\) increases \(|\text{support}(x)|\) by at most \(\tau\). Therefore, \(|\text{support}(x^{[T-m]})| \leq T\tau\).

A crucial observation for reducing the running time of the algorithm is that we don’t really need to keep track of intermediate values of \(x\); we only need the final value \(x^{[T-m]}\). This is because \(x\) isn’t used directly anywhere in the algorithm. We only need it indirectly in the form of \(\lambda(x)\) and \(y_\alpha(x)\) and for checking if \((x, y_\alpha(x))\) satisfies condition \(C_3(\varepsilon_2, \varepsilon_3)\). But for these purposes, knowing \(Ax\) is enough. Actually computing \(x^{[t]}\) for all \(t\), as line 4 in \texttt{improve-cover} suggests, can be wasteful and costly, since \(x^{[t]}\) can have a large support and \(T\) can be large. We will, therefore, compute \(x^{[T-m]}\) directly without first computing the intermediate values of \(x\).

On solving the recurrence \(x^{[t]} = (1 - \sigma_t)x^{[t-1]} + \sigma_t\tilde{x}^{[t]}\), we get

\[
x^{[T-m]} = \gamma^{T-m} \left(x^{[0]} + \sum_{t=1}^{T-m} \frac{\sigma_t\tilde{x}^{[t]}}{\gamma_t} \right)
\]

where \(\gamma_t := \prod_{j=1}^{t} (1 - \sigma_j)\). This way, \(x^{[T-m]}\) can be computed in \(O(T\tau)\) time.

Other than oracle calls and computing \(x^{[T-m]}\), the time taken by \texttt{frac-cover} is \(O(Tm)\): \(O(m)\) in \texttt{get-seed} and \(O(m)\) in each iteration of the \texttt{while} loop in \texttt{improve-cover}.

\[\Box\]

**Proof of Theorem 4.** Follows from Lemmas 22 and 23.

\[\Box\]

## 5 Covering LPs

Recall that \(\text{covLP}(A, b, c)\) is this linear program:

\[
\min_{x \in \mathbb{R}^N} c^T x \text{ where } Ax \geq b \text{ and } x \geq 0
\]

where \(A \in \mathbb{R}^{m \times N}_{\geq 0}\), \(b \in \mathbb{R}^m\) and \(c \in \mathbb{R}^N\). \(A, b, c\) are defined implicitly by an input \(I\).

Before we try to approximately solve \(\text{covLP}(A, b, c)\), let us see why an optimal solution always exists.

**Lemma 24.** A covering LP is feasible and has bounded objective value.

**Proof.** Consider \(\text{covLP}(A, b, c)\). Let \(a_i^T\) be the \(i\)th row of \(A\). For any feasible solution \(x\), we have \((Ax)_i = a_i^T x \geq b_i > 0\). Hence, \(a_i \neq 0\). Let

\[
\hat{x} := \sum_{i=1}^{m} \frac{b_i}{\|a_i\|^2} a_i > 0.
\]

For all \(i, j \in [m]\), we have \(a_i^T a_j \geq 0\). This gives us

\[
(A\hat{x})_i = a_i^T \hat{x} = a_i^T \left(\sum_{j=1}^{m} \frac{b_j}{\|a_j\|^2} a_j\right) \geq a_i^T \left(\frac{b_i}{\|a_i\|^2} a_i\right) = b_i
\]

Therefore, \(\hat{x}\) is feasible for \(\text{covLP}(A, b, c)\).

For any feasible solution \(x\), we have \(c^T x \geq 0\) because \(c > 0\) and \(x \geq 0\). Therefore, \(\text{covLP}(A, b, c)\) is bounded. 

\[\Box\]
We will try to solve \(\text{covLP}(A, b, c)\) by binary searching on the objective value \(c^T x\).

Given \(r \in \mathbb{R}_{\geq 0}\), we want to either find a feasible solution to \(\text{covLP}(A, b, c)\) of objective value \(r\), or claim that no solution exists of objective value \(r\). This is equivalent to \(\text{fcov}(A, b, P_r)\), where \(P_r := \{x : c^T x = r \text{ and } x \geq 0\}\).

Let \(r^* = \text{opt}(\text{covLP}(A, b, c))\). Then \(\text{fcov}(A, b, P_r)\) has a solution iff \(r \geq r^*\). If we could exactly solve \(\text{fcov}\), then finding a \((1 + \varepsilon)\)-approximate solution to \(\text{covLP}(A, b, c)\) is straightforward: use binary search to find \(r\) such that \(r^* \leq r \leq (1 + \varepsilon)r^*\), and then solve \(\text{fcov}(A, b, P_r)\) to get a feasible solution \(x\) such that \(c^T x = r\). However, we can’t do this because we can’t solve \(\text{fcov}(A, b, P_r)\) exactly. Nevertheless, a similar approach can be used to approximately solve \(\text{covLP}(A, b, c)\) if we can weakly solve \(\text{fcov}(A, b, P_r)\).

### 5.1 Solving the Fractional Covering Problem

\(\text{covLP}\)-\(\text{solve}\) receives the following inputs:

- \(I\): the input used to implicitly define \(\text{covLP}(A, b, c)\).
- \(q\): an upper-bound on \(\text{opt}(\text{covLP}(A, b, c))\).
- \(\rho\): an upper-bound on \(\frac{m}{\max_{i=1}^{N} \sum_{j=1}^{N} |b_i c_j|}\).
- \(\varepsilon \in (0, 1]\).
- \(\eta \in (0, 1]\).

\(\text{covLP}\)-\(\text{solve}\) is also provided a column oracle for \(A\), a cost oracle for \(c\), and an \(\eta\)-weak index-finding oracle. We will now design an algorithm \(\text{frac-cover-2}(I, r, \rho, \varepsilon, \eta)\) that weakly solves \(\text{fcov}(A, b, P_r)\) for any \(r \in [0, q]\) using these inputs and oracles.

As per Theorem 3, \(\text{frac-cover}\) can \(\eta/(1 + \varepsilon)\)-weakly solve \(\text{fcov}(A, b, P_r)\) if we can provide it the following values and oracles:

- an upper-bound on \(\text{width}(A, b, P_r)\).
- \(\varepsilon, \eta \in (0, 1]\).
- A product oracle.
- An \(\eta\)-weak point-finding oracle.

Providing \(\eta\) and \(\varepsilon\) is trivial, since they are inputs to \(\text{covLP}\)-\(\text{solve}\). We will now prove that \(\rho\) is an upper-bound on \(\text{width}(A, b, P_r)\) and see how to implement \(\text{point-find}\) using \(\text{index-find}\) and the cost oracle, and how to implement the product oracle using the column oracle.

#### 5.1.1 Upper-Bounding Width

**Lemma 25.** Let \(P_r := \{x : c^T x = r \text{ and } x \geq 0\}\). Then \(P_r\) is bounded, and the extreme points of \(P_r\) are

\[
S := \left\{ \frac{r}{c_j} e_j : j \in [N] \right\}
\]

Here \(e_j\) is the \(j^{th}\) standard basis vector of \(\mathbb{R}^N\).

**Proof.** Let \(j \in [N]\). Then \(c^T x = r \implies c_j x_j \leq r \implies x_j \in [0, r/c_j]\). Therefore, \(P_r\) is bounded. Let \(x \in P_r\). Then

\[
x = \sum_{j=1}^{N} \left( \frac{r c_j}{r} \right) \left( \frac{r e_j}{c_j} \right)
\]
Therefore, all points in $P_r$ can be represented as convex combinations of $S$. Hence, the set of extreme points of $P_r$ is a subset of $S$. Since $S$ is a basis of $\mathbb{R}^N$, no point in $S$ can be represented as a linear combination of the other points in $S$. Therefore, $S$ is the set of extreme points of $P_r$. 

Claim 26. Let $h : \mathbb{R}^N \rightarrow \mathbb{R}$ be a linear function and $P \subseteq \mathbb{R}^N$ be a polytope. Then there exists an extreme point $\tilde{x}$ of $P$ such that $h(\tilde{x}) = \max_{x \in P} h(x)$.

Lemma 27. Let $P_r := \{x : c^T x = r \text{ and } x \geq 0\}$. Then

$$\text{width}(A, b, P_r) = r \max_{i=1}^m \max_{j=1}^N A[i,j] b_i c_j$$

Proof. Let $a_i^T$ be the $i$th row of $A$. Since a linear function $(a_i^T x / b_i)$ over polytope $P_r$ is maximized at its extreme points (see Claim 26 and Lemma 25), we get

$$\text{width}(A, b, P_r) = \max_{i=1}^m \max_{x \in P_r} a_i^T x / b_i = \max_{i=1}^m \max_{j=1}^N \frac{1}{b_i} \left( \frac{r}{c_j} \right) = r \max_{i=1}^m \max_{j=1}^N \frac{A[i,j]}{b_i c_j}$$

Since $r \leq q$, we get that $\rho \geq \text{width}(A, b, P_r)$.

5.1.2 Implementing the Point-Finding Oracle

Lemma 28. Let $P_r := \{x : c^T x = r \text{ and } x \geq 0\}$. Then

$$\max_{x \in P_r} y^T Ax = r \max_{j=1}^N y^T A \left( \frac{e_j}{c_j} \right)$$

Proof sketch. Use Claim 26 with $h(x) = y^T Ax$ and use Lemma 25.

Corollary 28.1. Let $\text{index\-find}$ be an $\eta$-weak for $\text{fcov}(A, b, c)$. If we set $\text{point\-find}(y) = (r/c_k) e_k$, where $k := \text{index\-find}(y)$, then $\text{point\-find}$ is $\eta$-weak for $\text{fcov}(A, b, P_r)$, where $P_r := \{x : c^T x = r \text{ and } x \geq 0\}$. Here $c_k$ is obtained as $\text{cost\-oracle}(k)$.

Proof. Let $\tilde{x} = \text{point\-find}(y)$. Then by Lemma 28, we get

$$y^T A \tilde{x} = r y^T A \left( \frac{e_k}{c_k} \right) \geq r \eta \max_{j=1}^N y^T A \left( \frac{e_j}{c_j} \right) = \eta \max_{x \in P_r} y^T Ax$$

5.1.3 Implementing the Product Oracle

Let $a_j$ be the $j$th column of $A$. To compute $Ax$, simply use

$$Ax = \sum_{j=1}^N x_j a_j$$

Therefore, we can implement the product oracle over $A$ using $|\text{support}(x)|$ calls to the column oracle.
5.1.4 Summary

The description of frac-cover-2 is now complete, and we get the following result:

**Theorem 29.** Let covLP($A, b, c$) be defined implicitly in terms of $I$. Let $q$ be an upper-bound on opt(covLP($A, b, c$)) and $\rho$ be an upper-bound on $q \max_{i=1}^{m} \max_{j=1}^{N} A[i, j]/(b_i c_j)$. Let $P_r := \{x : c^T x = r \text{ and } x \geq 0\}$. Let index-find be an $\eta$-weak index-finding oracle.

Then we can implement an $\eta$-weak point-find for $fcov(A, b, P_r)$ using a single call to index-find and the cost oracle. For the product oracle, we can compute $Ax$ using $|\text{support}(x)|$ calls to the column oracle. Also, for every $\hat{x}$ output by point-find, $|\text{support}(\hat{x})| \leq 1$.

Furthermore, frac-cover-2($I, r, \rho, \varepsilon, \eta$) will $\eta/(1 + \varepsilon)$-weakly solve $fcov(A, b, P_r)$, and frac-cover-2($I, r, \rho, \varepsilon, \eta$) works by returning the output of frac-cover-((I), $\rho, \varepsilon, \eta$).

5.2 Algorithm Based on Binary Search

Let $r^* := \text{opt}(\text{covLP}(A, b, c))$ and $\mu := \eta/(1 + \varepsilon)$. Note that $r^* > 0$, since every feasible solution is non-zero, and hence has positive objective value.

**Algorithm 4 covLP-solve($I, q, \rho, \varepsilon, \eta$):**

Finds a $(1 + \delta)/\mu$-approximate solution to covLP($A, b, c$), where $A, b, c$ are implicitly defined by $I$. Here $\varepsilon, \eta \in (0, 1]$, $q$ is an upper-bound on $r^* := \text{opt}(\text{covLP}(A, b, c))$, $\rho$ is an upper-bound on $q \max_{i=1}^{m} \max_{j=1}^{N} A[i, j]/(b_i c_j)$, $P_r := \{x : c^T x = r \text{ and } x \geq 0\}$, and index-find is an $\eta$-weak index-finding oracle.

1: $\delta := \varepsilon^2/(1 + \varepsilon)$
2: $\alpha = 0$
3: $\beta = q$
4: $\hat{x} = \text{frac-cover-2}(I, q, \rho, \varepsilon, \eta)$
5: while $\beta > (1 + \delta)\alpha$ do
6: $r = (\alpha + \beta)/2$
7: $\hat{y} = \text{frac-cover-2}(I, r, \rho, \varepsilon, \eta)$
8: if $\hat{y}$ is null then
9: $\alpha = r$
10: else
11: $\beta = r$
12: $\hat{x} = \hat{y}$
13: end if
14: end while
15: return $(\alpha, \beta, \hat{x})$

**Definition 12.** Let $g : [0, q] \rightarrow \{0, 1\}$ be a function where $g(r) = 0$ iff frac-cover-2($I, r, \rho, \varepsilon, \eta$) returns null.

Every call to frac-cover-2 in covLP-solve probes a point $r$ in the interval $[0, q]$ and gives us $g(r)$.

**Lemma 30.** When $r < \mu r^*$, $g(r)$ is always 0. When $r \geq r^*$, $g(r)$ is always 1. (When $\mu r^* \leq r < r^*$, $g(r)$ may be 0 or 1.)

**Proof.** Let $r < \mu r^*$. Assume $g(r) = 1$. This means that frac-cover-2 returned a $\mu$-approximate solution $x$ to $fcov(A, b, P_r)$ (by Theorem 29), i.e., $Ax \geq \mu b$ and $c^T x = r$. Therefore, $x/\mu$ is
feasible for \( \text{covLP}(A, b, c) \) and \( c^T(x/\mu) = r/\mu < r^* \). This is a contradiction, since we found a feasible solution to \( \text{covLP}(A, b, c) \) of objective value less than the optimum. Therefore, \( g(r) = 0 \).

Let \( r \geq r^* \) and let \( x^* \) be an optimal solution to \( \text{covLP}(A, b, c) \). Therefore, \( Ax^* \geq b \) and \( c^T x^* = r^* \). Let \( x = (r/r^*)x^* \). Then \( Ax \geq (r/r^*)b \geq b \) and \( c^T x = r \). Therefore, \( x \) is a feasible solution to \( \text{fcov}(A, b, P_r) \). This means \( \text{fcov}(A, b, P_r) \) is satisfiable, so \( \text{frac-cover-2} \) cannot return \( \text{null} \). Therefore, \( g(r) = 1 \). □

**Lemma 31.** Throughout \( \text{covLP}'s \) execution, \( g(\alpha) = 0, g(\beta) = 1, c^T \hat{x} = \beta \) and \( A\hat{x} \geq \mu b \).

*(assuming the while loop body is executed atomically)*

**Proof.** When \( \hat{y} = \text{null} \), then \( g(r) = 0 \) and \( \alpha = r \). Otherwise, \( g(r) = 1 \) and \( \beta = r \).

\( c^T \hat{x} = \beta \) and \( A\hat{x} \geq \mu b \) follow from \( \hat{x} \neq \text{null} \), \( \hat{x} = \text{frac-cover-2}(I, \beta, \rho, \varepsilon, \eta) \) and the fact that \( \text{frac-cover-2} \) can \( \mu \)-weakly solve \( \text{fcov}(A, \beta, P_r) \) (by Theorem 29). □

**Theorem 1.** Let \( \text{covLP}(A, b, c) \) be implicitly defined in terms of input \( I \). Then \( \text{covLP-solve}(I, q, \rho, \varepsilon, \eta) \) returns a \( (1 + \varepsilon + \varepsilon^2)/\eta \)-approximate solution to \( \text{covLP}(A, b, c) \).

**Proof.** Let \( (\alpha, \beta, \hat{x}) = \text{covLP-solve}(I, q, \rho, \varepsilon, \eta) \). By Lemma 31, \( g(\alpha) = 0, g(\beta) = 1, c^T \hat{x} = \beta \) and \( A\hat{x} \geq \mu b \). By Lemma 30, \( \alpha < r^* \) and \( \beta \geq \mu r^* \). Since the algorithm terminated, \( \beta \leq (1 + \delta)\alpha \).

This gives us \( r^* \leq c^T(\hat{x}/\mu) \leq r^*(1 + \delta)/\mu \) and \( A(\hat{x}/\mu) \geq b \). Therefore, \( \hat{x} \) is a \((1 + \delta)/\mu\)-approximate solution to \( \text{covLP}(A, b, c) \).

\[
\frac{1 + \delta}{\mu} = \left(1 + \frac{\varepsilon^2}{1 + \varepsilon}\right) \frac{1 + \varepsilon}{\eta} = \frac{1 + \varepsilon + \varepsilon^2}{\eta}
\]

□

**Theorem 32.** Suppose the while loop in \( \text{covLP-solve} \) runs \( T \) times. Then

\[
T \leq 2 + \log\left(\frac{q}{r^*}\right) + \log\left(\frac{1}{\eta}\right) + 2\log\left(\frac{1}{\varepsilon} + 1\right)
\]

**Proof.** Let \( \alpha_t \) and \( \beta_t \) denote the values of \( \alpha \) and \( \beta \) after \( t \) runs of the while loop. Then \( \alpha_0 = 0 \) and \( \beta_0 = q \).

Suppose \( \alpha_t = 0 \) for all \( t < p \). Then \( \beta_{p-1} = q/2^{p-1} \). \( g(\beta_{p-1}) = 1 \) by Lemma 31, and \( \beta_{p-1} \geq \mu r^* \) by Lemma 30. Therefore, \( p \leq \log(q/(\mu r^*)) + 1 \). Let \( p \) be the largest possible such that \( \alpha_t = 0 \) for all \( t < p \). Then \( \alpha_p = q/2^p \) and \( \beta_p = \beta_{p-1} = q/2^{p-1} \).

\( \beta_t - \alpha_t \) halves in each iteration. So for any \( t \geq p \), \( \beta_t - \alpha_t = (\beta_p - \alpha_p)/2^{t-p} \). Let \( t = p + \lceil\log(1/\delta)\rceil \).

Assume that \( T > t \). Since the while loop ran the \((t+1)\)th time, \( \beta_t > (1 + \delta)\alpha_t \).

\[
\frac{\beta_t - \alpha_t}{\alpha_t} = \frac{\beta_p - \alpha_p}{2\log(1/\delta)\alpha_t} \leq \frac{q/2^p}{\alpha_p/\delta} = \delta \quad \Rightarrow \quad \beta_t \leq (1 + \delta)\alpha_t
\]

This is a contradiction. Therefore,

\[
T \leq p + \left\lceil\log\left(\frac{1}{\delta}\right)\right\rceil \leq 2 + \log\left(\frac{q}{r^*}\right) + \log\left(\frac{1}{\mu}\right)
\]

\[
= 2 + \log\left(\frac{q}{r^*}\right) + \log\left(\frac{1}{\eta}\right) + 2\log\left(\frac{1}{\varepsilon} + 1\right)
\]

□
Theorem 2. Let $\text{covLP}(A,b,c)$ be implicitly defined in terms of input $I$, where $A \in \mathbb{R}^{m \times N}$.

Let
\[
M := 3 + 2 \lg \left( \frac{1}{\varepsilon} + 1 \right) + \lg \left( \frac{1}{\eta} \right) + \lg \left( \frac{q}{\text{opt}(\text{covLP}(A,b,c))} \right)
\]
\[
U := m + \left\lceil \ln \left( \frac{m}{\eta} \right) \right\rceil \left\lceil \frac{312m\rho(1 + \varepsilon)}{\eta^3} \ln \left( \frac{12m}{\varepsilon} \right) \right\rceil \in \tilde{O} \left( \frac{m\rho}{\eta^3} \right)
\]

Then all of the following hold for $\text{covLP-solve}(I,q,\rho,\varepsilon,\eta)$:

- $\text{covLP-solve}$ makes at most $MU$ calls to the index-finding oracle, at most $MU$ calls to the column oracle, and at most $MU$ calls to the cost oracle.
- In $\text{covLP-solve}$, the time taken by non-oracle operations is $O(MUm)$.
- The solution $\hat{x}$ returned by $\text{covLP-solve}$ has $|\text{support}(\hat{x})| \leq U$.

Proof sketch. By Theorem 32, $\text{covLP-solve}$ calls $\text{frac-cover-2}$ at most $M$ times. By Theorem 29, $\tau = 1$ and every call to $\text{frac-cover-2}$ results in one call to $\text{frac-cover}$, and every call to $\text{point-find}$ results in one call to $\text{index-find}$. The rest follows from Theorem 4. \qed

6 Future Work

$\text{frac-cover}$ is based on a simplified version of the Plotkin-Shmoys-Tardos algorithm [12] for fractional covering. We did not focus on optimizing the running time of $\text{frac-cover}$; instead, we focused on getting as small an approximation factor as possible, even when the point-finding oracle could be very weak. [12] uses many tricks to get a low running time. We didn’t adapt those tricks to our algorithm, so our algorithm is not as fast as theirs. For example, they use different values of $\varepsilon$ for each call to $\text{improve-cover}$, and they have fast randomized versions of their algorithms.

The most important of these tricks, in our opinion, is the one that reduces the dependence on $\rho$. The number of times our algorithm $\text{frac-cover}$ calls the point-finding oracle varies linearly with $\rho$. But for some applications, $\rho$ can be super-polynomial in the input size. Section 4 of [12] explains a possible approach to fix this. The number of times their algorithm calls the point-finding oracle is linear in $\log \rho$.

Another direction of work would be to adapt our techniques to the fractional packing problem. [12] already have an algorithm for this, but their algorithm doesn’t work for very small values of $\eta$ (when using a packing analogue of the $\eta$-weak point-finding oracle).

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