Abstract

In this article, we investigate the role of Grothendieck groups of coherent sheaves in the study of D-branes. We show how global bound state construction in topological $K$-theory can be adapted to our context, showing that D-branes wrapping a subvariety are holomorphically classified by a relative $K$-group. By taking the duality between the relative $K$-groups and the $K$-homologies, we show that D-brane charge of type IIB superstrings is properly classified by the $K$-homology.
1 Introduction

In [1], E. Witten showed that the D-brane charge is classified by topological $K$-theory using some ideas of A. Sen [2]. A full description of all states with a D-brane wrapping a closed submanifold of the spacetime can be made in terms of states of a brane-antibrane pair on the spacetime and such a brane-antibrane pair with a Tachyon field naturally defines a $K$-theory class that is trivial at infinity and hence $K$-theory with compact support. The global bound state construction depends on the topology of the normal bundle of the D-brane worldvolume in the spacetime. In the absence of the Neveu-Schwarz B-field, a single D-brane can wrap a submanifold if and only if the normal bundle to the submanifold has a Spin$^c$ structure, or equivalently the submanifold is a Spin$^c$ manifold. This follows from the world-sheet global anomalies in the presence of D-branes studied in [3]. The topological obstruction condition to brane wrapping corresponds to that of Poincaré duality in $K$-theory. It was argued in [4] that the Poincaré duality makes it more natural for D-brane charges to take values in analytic $K$-homology groups, based on the Polchinski's basic covariant operational definition of D-branes [5]. In [6], it was argued that $K$-homology is an appropriate setting for the topological classification of D-brane charges by an extensive use of $KK$-theory. Also, in [7], it was asserted that the stable D-brane configurations can be classified by topological $K$-homology groups.

In the same spirit of [1], we study how $K$-homology can be used to describe type IIB D-branes in the context of algebraic geometry. In this context, the Grothendieck group of locally free sheaves corresponds to the topological $K$-theory, as discussed in [8],[9]. On the other hand, the Grothendieck group of coherent sheaves is referred as the $K$-homology since it corresponds to the topological $K$-homology, [8]. The Poincaré duality is the identification of both Grothendieck groups and it holds for nonsingular projective algebraic varieties. In [9], the $K$-homology has been studied associated with the derived category of coherent sheaves. In there, it was asserted that the only physically relevant part of an object in a derived category of coherent sheaves is its image in the $K$-homology. However, the physical applications of $K$-homologies has been limited to the smooth case where the Poincaré duality holds.

In this paper, we will consider the $K$-theoretic Lefschetz type of duality in algebraic geometry, given in [10]. The duality is the identification of a relative version of the Grothendieck group of locally free sheaves with a $K$-
homology group. We shall argue that the elements in the relative $K$-group can be naturally interpreted as a brane-antibrane system with a tachyon field on a nonsingular variety, in the absence of D-branes outside of a closed subvariety. The duality implies that such brane-antibrane systems can be deformed to a system of branes on a closed subvariety, which corresponds to the $K$-homology groups. A novelty of using $K$-homology instead of the Grothendieck group of locally free sheaves is that we can deal with coherent and locally free sheaves on the same footing. Also, it is more suitable when we are dealing with singular varieties, see [8] for example. Based on this, we shall show that D-brane charges of type IIB superstrings are naturally classified by $K$-homology groups for complex projective algebraic varieties including some singular cases. We discuss how D-branes wrapping a nonsingular variety are classified by the relative version of the Grothendieck group of locally free sheaves which will be referred as the relative $K$-theory.

Throughout this paper we will work in the absence of the Neveu-Schwarz B-field. Also, we will not work with space-filling D-branes.

2 Grothendieck groups of locally free sheaves

In this section we shall review the Grothendieck group of locally free sheaves and the relative $K$-theory. We show how elements in the relative $K$-group are interpreted as a brane-antibrane pair.

Let $X$ be a complex projective algebraic variety (or a compact complex manifold). Denote by $\mathcal{O}_X$ the sheaf of regular functions on $X$ (or the sheaf of holomorphic functions). The sheaf of sections of a holomorphic vector bundle of rank $r$ on a complex manifold $X$ is locally free which is locally isomorphic to $\mathcal{O}_X^r$. Conversely, every locally free sheaf of $\mathcal{O}_X$-modules is the sheaf of sections of a holomorphic vector bundle, which is determined up to isomorphism.

Let $K^0_{\text{alg}}(X)$ be the Grothendieck group of holomorphic vector bundles on $X$. It is the free abelian group on the isomorphism classes of holomorphic vector bundles on $X$, modulo the subgroup generated by elements of the form $E - E' - E''$ whenever there is an exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

of holomorphic vector bundles on $X$. Equivalently, $K^0_{\text{alg}}(X)$ can be considered as the Grothendieck group of locally free sheaves on $X$ as discussed in [11].
and \([3]\). The elements in \(K^0_{\text{alg}}(X)\) will be denoted by \([E]\) represented by a holomorphic vector bundle \(E\).

We also introduce the relative \(K\)-groups of holomorphic vector bundles following \([10]\). Let \(X\) be a complex projective algebraic variety and thus \(X\) may have singularities. Choose a holomorphic embedding \(i : X \hookrightarrow Y\), where \(Y\) is a nonsingular complex manifold. Consider a complex \(E_\bullet\) of holomorphic vector bundles on \(Y\)

\[
E_\bullet : 0 \rightarrow E_r \rightarrow E_{r-1} \rightarrow \cdots \rightarrow E_0 \rightarrow 0
\]

which are exact on \(Y - X\). Let \(K^0_{\text{alg}}(Y, Y - X)\) be the Grothendieck group on the isomorphism classes of such complexes of bundles on \(Y\). It is the free abelian group on such complexes modulo elements \(E_\bullet - E_\bullet' - E_\bullet''\) where

\[
0 \rightarrow E_\bullet' \rightarrow E_\bullet \rightarrow E_\bullet'' \rightarrow 0
\]

is an exact sequence of complexes. The elements in \(K^0_{\text{alg}}(Y, Y - X)\) will be denoted by \([E_\bullet]\), represented by a complex \(E_\bullet\) of holomorphic vector bundles on \(Y\). If a complex \(E_\bullet\) is exact on all of \(Y\), then it defines zero element in \(K^0_{\text{alg}}(Y, Y - X)\). In particular, if \(X = Y\), then the relative \(K\)-group \(K^0_{\text{alg}}(Y, \emptyset)\) is identified with the group \(K^0_{\text{alg}}(Y)\) and the identification is given by \([E_\bullet] \mapsto \sum_i (-1)^i [E_i]\).

Now, one can describe the complex \(E_\bullet\), which defines a class in \(K^0_{\text{alg}}(Y, Y - X)\), as brane-antibrane pairs on \(Y\). To be more precise, for a complex \(E_\bullet:\)

\[
0 \rightarrow E_r \xrightarrow{d_r} E_{r-1} \xrightarrow{d_{r-1}} \cdots \rightarrow E_0 \rightarrow 0
\]

of holomorphic vector bundles on \(Y\) which is exact on \(Y - X\), choose isomorphisms \(E_i \cong \text{Ker} \, d_i \oplus \text{Ker} \, d_{i-1}\) on \(Y - X\). This gives isomorphisms on \(Y - X\)

\[
\sigma_1 : E_{\text{ev}} \rightarrow \sum_i \text{Ker} \, d_i
\]

and

\[
\sigma_2 : E_{\text{odd}} \rightarrow \sum_i \text{Ker} \, d_i,
\]

where \(E_{\text{ev}} = \sum_k E_{2k}\) and \(E_{\text{odd}} = \sum_k E_{2k+1}\). The composition \(T = \sigma_1^{-1} \circ \sigma_2\) gives a complex

\[
0 \rightarrow E_{\text{odd}} \xrightarrow{T} E_{\text{ev}} \rightarrow 0
\]
which is exact on $Y - X$. The map $T$ is a holomorphic section of the bundle $E_{\text{odd}}^* \otimes E_{\text{ev}}$ and it has the adjoint $\bar{T}$ which is a section of $E_{\text{odd}} \otimes E_{\text{ev}}^*$ such that $T \bar{T} T = T$. In other words, the map $T$ is a tachyon field on the brane-antibrane pair $(E_{\text{ev}}, E_{\text{odd}})$ on $Y$. With this choice of tachyon field $T$, the system of the brane-antibrane system is in a vacuum state on $Y - X$. Thus we are considering a system of brane-antibrane pair in the absence of D-branes outside of $X$. We can also interpret the tachyon field as a holomorphic map $T : Y - X \to U(N)$, for some large $N$ and the whole content of D-brane bound state is captured by such functions, (cf. [12], [13]). In fact, this brane-antibrane system is deformed to a system of D-branes on $X$, as we will see in the next section.

3 $K$-homology in algebraic geometry and duality

Let $X$ be a projective algebraic variety. A coherent sheaf $\mathcal{F}$ on $X$ admits a local presentation as an exact sequence $\mathcal{O}_X^p \to \mathcal{O}_X^q \to \mathcal{F} \to 0$ and thus it has an exact sequence of sheaves

$$\mathcal{E}_r \to \mathcal{E}_{r-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

where the $\mathcal{E}_i$ are locally free sheaves on $X$. Note that the sequence

$$\mathcal{E}_r \to \mathcal{E}_{r-1} \to \cdots \to \mathcal{E}_0 \to 0$$

is a complex of locally free sheaves on $X$ and the complex $\mathcal{E}_\bullet$ has the homology $H_0(\mathcal{E}_\bullet) = \mathcal{F}$ and all others are zero by exactness. A typical example of a non-locally free sheaf is a skyscraper sheaf which is a coherent sheaf supported at a point.

The Grothendieck group of coherent sheaves on $X$ is defined by the free abelian group on the isomorphism classes of coherent sheaves on $X$, modulo the subgroup generated by elements of the form $\mathcal{F} \to \mathcal{F}' - \mathcal{F}''$ whenever there is an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of coherent sheaves on $X$. As given in [3], the Grothendieck group is called the $K$-homology of $X$ because of its relation with topological $K$-homology.
The group is denoted by $K_{alg}^0(X)$ and the elements in $K_{alg}^0(X)$ will be denoted by $[\mathcal{F}]$ represented by a coherent sheaf $\mathcal{F}$ on $X$.

If $X$ is nonsingular, two groups $K_{alg}^0(X)$ and $K_{alg}^0(X)$ are isomorphic and is called the Poincaré duality. Following [10], we study here the more general type of duality which includes the case of singular varieties. For any $[\mathcal{E}_\bullet] \in K_{alg}^0(Y, Y - X)$, the homology sheaves $H_i(\mathcal{E}_\bullet)$ of the complex of locally free sheaves on $Y$ are coherent sheaves on $Y$ which are supported on $X$ and thus determine a class $[H_i(\mathcal{E}_\bullet)]$ in $K_{alg}^0(X)$. We define the homology map

$$h : K_{alg}^0(Y, Y - X) \longrightarrow K_{alg}^0(X)$$

by

$$h([\mathcal{E}_\bullet]) = \sum_i (-1)^i [H_i(\mathcal{E}_\bullet)].$$

The homology map is an isomorphism if $Y$ is nonsingular. To see this, we define the inverse map $h^{-1}$. Let $[\mathcal{F}] \in K_{alg}^0(X)$, represented by a coherent sheaf $\mathcal{F}$ on $X$. The sheaf $\mathcal{F}$ on $X$ can be extended to all of $Y$ by zero and defines a coherent sheaf on $Y$, which will be denoted by $i_\ast \mathcal{F}$. Since $Y$ is nonsingular, there is a finite locally free resolution of $i_\ast \mathcal{F}$:

$$0 \longrightarrow \mathcal{E}_r \longrightarrow \mathcal{E}_{r-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_0 \longrightarrow i_\ast \mathcal{F} \longrightarrow 0.$$ 

Then the complex of locally free sheaves $\mathcal{E}_\bullet$ on $Y$ is exact on $Y - i(X)$ and hence defines an element $[\mathcal{E}_\bullet] \in K_{alg}^0(Y, Y - X)$. We define a map $\bar{h} : K_{alg}^0(X) \longrightarrow K_{alg}^0(Y, Y - X)$ by

$$\bar{h}([\mathcal{F}]) = [\mathcal{E}_\bullet].$$

The construction of $\bar{h}$ does not depend on the choice of locally free resolution of $i_\ast \mathcal{F}$, because any two such resolutions are dominated by the third. Hence the map $\bar{h}$ is well-defined. Furthermore, it is straightforward to show that $\bar{h} = h^{-1}$. Thus we have

$$h : K_{alg}^0(Y, Y - X) \cong K_{alg}^0(X),$$

when $Y$ is nonsingular variety. In particular, when $X = Y$ and $Y$ is nonsingular, the homology map is the Poincaré duality map and we have

$$PD : K_{alg}^0(Y) \cong K_{alg}^0(Y).$$
In Section 3, we have shown that an element in $K^0_{\text{alg}}(Y, Y - X)$ can be understood as a brane-antibrane pair with a tachyon field which is a unitary map outside of $X$. When $X$ is nonsingular, we have an isomorphism $h : K^0_{\text{alg}}(Y, Y - X) \cong K^0_{\text{alg}}(X)$, via the Poincaré duality. This suggests that a system of brane-antibrane pair supported on $X$ can be deformed to a system of brane-antibrane wrapped on $X$ since $K^0_{\text{alg}}(X) = K^0_{\text{alg}}(X, \emptyset)$, in the absence of $D$-branes on $Y - X$. Furthermore, since the duality holds for a singular variety $X$, the system on $Y$ is deformed to a system of branes on $X$.

4 The global bound state construction

In [1], E. Witten showed that D-brane charge of Type II superstrings are classified by $K$-theory with compact support. In the same spirit of [1], we will show the analogous statement using the relative $K$-theory and $K$-homology. It has been argued in [14] that certain singularities are allowed in the study of D-branes and those are nonsingular in string theory. So, one can have more general coherent sheaves as D-brane bound states, as suggested in [15]. In this sense, it is more natural to use $K$-homology rather than the Grothendieck group of locally free sheaves to classify D-brane bound states. Also, it is suitable to describe a configuration of D-branes on a singular variety.

First we describe a simple modification of the Witten’s argument, given in [1]. Let $X$ be a codimension 2, closed subvariety of a nonsingular variety $Y$. We will build a $p$-brane wrapping $X$ from a $p + 2$ brane-antibrane pair on $Y$, where $p + 1 = \dim_{\mathbb{R}} X$. One can define a holomorphic line bundle $L$ on $Y$ which is trivialized on $Y - X$. Let $\mathcal{L}$ be the associated locally free sheaf on $Y$. Consider the complex of $\mathcal{O}_Y$-modules:

$$0 \rightarrow \mathcal{O}_Y \xrightarrow{T} \mathcal{L} \rightarrow 0 \quad (2)$$

where $T$ is the zero map on $X$ and is a unitary map on $Y - X$. The complex (2) can be interpreted as a $(p + 2)$-brane-antibrane pair $(\mathcal{L}, \mathcal{O}_Y)$ with the tachyon field $T$. Note that the tachyon field is a holomorphic section of $L$ which vanishes along $X$ and hence it has charges $(1, -1)$ under the $\text{U}(1) \times \text{U}(1)$ that live on the brane and antibrane. Here the $\text{U}(1)$ gauge field on the brane is coming from the holomorphic connection on $L$, with the same $p$-brane charge as that of a $p$-brane wrapped on $X$ which is associated to the restriction of the bundle $L$ to $X$. Also, from the definition of the tachyon field $T$, we have the trivial gauge field on the antibrane, with vanishing $p$-brane charge. Now by
taking the homology of the complex (2), we obtain two sheaves $\mathcal{L}_X$, with $p$-brane charge and $\mathcal{O}_X$, with 0-brane charge, where $\mathcal{L}_X$ is the locally free sheaf associated to the restriction bundle of $L$ to $X$. Thus we see that the system is deformed to a system consisting of a $p$-brane wrapped on $X$. In the above discussion of brane-antibrane annihilation, we have started from a rather strong condition for the choice of a holomorphic line bundle on $Y$. With a suitable choice of a tachyon field, we get a system of a brane-antibrane pair wrapping $Y$, which is in vacuum state along $Y - X$. This is a basic assumption made in \([1]\) for brane-antibrane annihilation and the assumption exactly agrees with that of complexes giving a class in $K^0_{\text{alg}}(Y, Y - X)$. The duality between the relative $K$-group and $K$-homology, discussed in Section 3, shows that only such brane-antibrane systems on $Y$ can be deformed to a system of branes on $X$. In other words, a brane-antibrane system on $Y$ in the absence of D-branes in $Y - X$ is deformed to a system of branes on $X$.

Now, let $X$ be a closed subvariety of $Y$, which is of arbitrary codimension. For any complex of holomorphic bundles on $Y$ which are exact off $X$, it gives a brane-antibrane pair on $Y$ which is in vacuum state along $Y - X$. By taking homology of the complex, we get a system of branes on $X$. In general, homology sheaves are coherent sheaves. This suggests that we may not have a system of single brane wrapped on $X$ when we deform a brane-antibrane system on $Y$ to a system on $X$. On the other hand, it may be possible to have a configuration consisting of several branes wrapped on $X$ that is represented by a coherent sheaf. This happens when $X$ is a singular variety.

A $p$-brane wrapped on $X$ also has in general lower-dimensional brane charges. In order to fully describe all states with a $p$-brane wrapped on $X$ in terms of states of a brane-antibrane pair wrapped on $Y$, we need to choose a holomorphic line bundle $K$ on $X$. Let $\mathcal{K}$ be the locally free sheaf associated to the holomorphic bundle. Extension by zero gives a coherent sheaf $i_*\mathcal{K}$ on $Y$. This sheaf is resolved by locally free sheaves on $Y$ and the associated complex of holomorphic vector bundles on $Y$ is exact off $X$. Thus it defines a class $K^0_{\text{alg}}(Y, Y - X)$. The class associated to $i_*\mathcal{K}$ is independent of the choice of a resolution because any two such resolutions are dominated by the third. This proves that the D-brane charge takes values in the relative $K$-group when a $Dp$-brane is wrapped on $X$. The lower brane charges are depend on the choice of a line bundle on $X$. This shows that D-branes wrapping a nonsingular closed subvariety $X$ of a variety $Y$ are holomorphically classified by the relative $K$-group $K^0_{\text{alg}}(Y, Y - X)$. Since $Y$ is nonsingular, the homology map, given in (1), is an isomorphism. Thus by applying the homology map,
we conclude that the D-brane charge is properly classified by $K^0_{\text{alg}}(X)$. When $X$ is a singular variety, there are no $p$-branes wrapping $X$ due to the anomaly condition in \cite{3}. However, there can be several branes wrapping subvarieties of $X$ and hence we can have a configuration consisting of several different type of branes wrapped on $X$, supporting suitable holomorphic structures on each branes. On the whole, the configuration must not be a holomorphic vector bundle since its transition functions are not holomorphic around the singular loci. Suppose we have a system that the entire set of bound states is represented by a coherent sheaf. Then we can apply the same argument as in the case of nonsingular varieties. To be more precise, let $\mathcal{M}$ be such a coherent sheaf. Since $Y$ is nonsingular, the sheaf $i_* \mathcal{M}$ is resolved by holomorphic vector bundles on $Y$ and hence the associated complex of bundles defines a class in $K^0_{\text{alg}}(Y, Y - X)$. By applying the homology map, we see that D-branes wrapping $X$ are classified by the $K$-homology of $X$.

5 Concluding remarks

We have described the global version of bound state construction in the algebraic geometry context. We showed how D-branes wrapping $X$ are holomorphically classified by the relative $K$-group $K^0_{\text{alg}}(Y, Y - X)$. By taking the homology map discussed in Section 3, we conclude that D-branes wrapping $X$ are properly classified by the $K$-homology group of $X$ including some singular cases.

We can also make a connection with topological $K$-homolgy theory from our consideration. Since $X$ is a complex projective algebraic variety, the underlying topological space of $X$ is a finite simplicial complex. Thus we can study the topological $K$-group and topological $K$-homology of $X$. As shown in \cite{8}, one has the map from the Grothendieck group of coherent sheaves on $X$ to the topological $K$-homology of the underlying topological space of $X$. However, in general, the map is neither injective nor surjective, see examples in \cite{9} and \cite{11}. From our conclusion, we see that D-branes wrapping $X$ take values in the topological $K$-homology $X$. However, those groups do not give us the classification of topological D-branes from our discussion, because the topological $K$-homology and the Grothendieck group of coherent sheaves cannot be identified.
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