INFINITELY MANY MULTI-PULSES NEAR A BIFOCAL CYCLE

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ABSTRACT. The entire dynamics in a neighbourhood of a reversible heteroclinic cycle involving a bifocus is far from being understood. In this article, using the well known theory of reversing symmetries, we prove that there are infinitely many pulses near a cycle involving two symmetric equilibria, a real saddle and a bifocus, giving rise to a complex network. We also conjecture that suspended blenders might appear in the neighbourhood of the network.

1. Introduction. In three-dimensions, heteroclinic cycles and networks involving saddle-foci have been considered by several authors in different settings [8, 11, 12, 16, 17, 19]. Nowadays, the big challenge is the analysis of heteroclinic cycles involving a bifocus in four dimensions. These cycles are usually known as bifocal cycles. Shilnikov [18, 19] was the first who studied the dynamics associated to them and proved the existence of a countable set of periodic solutions in their neighbourhood. Subsequent works [5, 13] considered the formation and bifurcations of these solutions by studying the Poincaré map associated to a cross-section in a neighbourhood of the bifocus. In any transverse section to the bifocal cycle, for any $N \in \mathbb{N}$, there is a compact invariant hyperbolic set on which the Poincaré map is topologically conjugate to the Bernoulli shift on $N$ symbols [9].

The existence of periodic solutions near heteroclinic cycles in reversible systems has been explored in [4, 20]. A bifurcation analysis of cycles in Hamiltonian systems can be found in [15] – here, sequences of parameter values have been detected for which homoclinic fold bifurcations occur. Note that the symbolic dynamics, which has been proved to occur near the cycle in the Hamiltonian case, are not expected in the reversible case because of the lack of level sets in the latter case.

In [10], the authors studied the dynamics around a heteroclinic cycle associated to two saddles, in four dimensions. In the unfolding of the cycle, using the Lin’s method, they showed that homoclinic snaking occurs if and only if at least one of the saddles is a bifocus.

In the present paper, in the spirit of the works of [1, 3, 7], we study the behaviour of a four dimensional reversible vector field whose flow has a heteroclinic cycle involving a bifocus and a saddle, for which we give a description in Section 3. Without breaking the bifocal cycle, we prove the existence of infinitely many periodic solutions and homoclinic cycles coexisting with the original cycle, which themselves are accompanied by complex dynamics. At the end of this article, we conjecture about the existence of heterodimensional cycles and blenders near the bifocal cycle. An example of a bifocal cycle associated to a saddle and a bifocus has been found in [10, Ch. 3], in the context of Bussinesq equations.

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In the present work, we use a Tresser type analysis [21]: we assume the existence of a heteroclinic cycle with some properties and then we $C^1$–linearize the vector field in a small neighbourhood of each saddle. We assume that a bifocus (resp. saddle) is an equilibrium point of a differential equation whose leading eigenvalues associated to its linearization are complex non-real (resp. real). As far as we know, this type of analysis applied to a reversible heteroclinic cycle involving a saddle and a bifocus is new. We stress that our results hold for reversible vector fields that are not necessarily divergence-free.

**Framework of the paper.** This paper is organised as follows. Section 3 sets up the context of our problem, preceded by definitions and preliminary results in Section 2. In Section 4, applying a $C^1$–coordinate changes, we linearize the vector field around the equilibria. In Section 5, we study a way to find different reversible pulses and then we introduce some notation that will be used in the rest of the article. We obtain a geometrical description of the way the flow transforms a 2–dimensional disk of initial conditions lying across the stable manifold of the bifocus. In Section 7, the main result is presented: we put together all the informations of the previous sections to show the existence of infinitely many multipulses near the reversible bifocal cycle. We end this article with a short discussion of the results and related open problems.

2. **Preliminaries.** Let $f$ be a $C^2$ vector field on $\mathbb{R}^n$ with flow given by the unique solution $x(t) = \varphi(t, x_0) \in \mathbb{R}^n$ of the Initial Value Problem (IVP):

$$\dot{x} = f(x) \quad \text{and} \quad x(0) = x_0.$$

2.1. **Heteroclinic cycle.** Given two equilibria $p_1$ and $p_2$, a heteroclinic connection from $p_1$ to $p_2$ is a solution contained in $W^u(p_1) \cap W^s(p_2)$. There may be more than one trajectory connecting $p_1$ and $p_2$. Let $\mathcal{S} = \{p_j : j \in \{1, \ldots, k\}\}$ be a finite ordered set of mutually disjoint invariant saddles. We say that there is a heteroclinic cycle associated to $\mathcal{S}$ if

$$\forall j \in \{1, \ldots, k\}, W^u(p_j) \cap W^s(p_{j+1}) \neq \emptyset \quad (\text{mod } k).$$

When $k = 1$, we say that the invariant set is a homoclinic cycle. Sometimes we refer to the equilibria defining the heteroclinic cycle as nodes. A heteroclinic network is a connected union of heteroclinic cycles.

2.2. **Reversibility.** Let $R \in GL(\mathbb{R}^n)$. The differential equation (IVP) is $R$–reversible if $R$ is an involution on $\mathbb{R}^n$ (i.e. $R^2 = Id$) and $R \circ f = -f \circ R$. In particular, $R(x(-t))$ is a solution of the (IVP) if and only if $x(t)$ also is. More details about reversible vector fields are postponed to Section 3.3 for the case $n = 4$.

Let $Fix(R)$ the set of fixed points of $R$, i.e. $Fix(R) = \{x \in \mathbb{R}^n : R(x) = x\}$. For a periodic solution $\mathcal{C}$, note that if $p \in \mathcal{C} \cap Fix(R)$ and $f(p) \neq \emptyset$, then $f(p)$ cannot be tangent to $Fix(R)$ in $p$. Otherwise, the reversibility property would not be respected.

Let $\gamma$ be a solution of the (IVP). We say that $\gamma$ is $R$–symmetric if $R(\gamma) = \gamma$ ie if $\gamma$ is invariant under the involution $R$. We will focus our concentration on these solutions.

3. **Description of the Problem.** Let $f$ be a $C^2$–vector field on a three-dimensional differential manifold $\mathbb{R}^3$ with flow given by the unique solution $x(t) = \varphi(t, x_0) \in \mathbb{R}^4$ of (1). We assume that the flow is complete, ie, the flow is defined for all $t \in \mathbb{R}$. We denote by $O$ the point $(0, 0, 0, 0) \in \mathbb{R}^4$ and by $S$ the saddle equilibrium point far from $O$, say $(1, 1, 1, 1)$.

3.1. **The hypotheses.** Our object of study is the dynamics around a reversible (not necessarily conservative) heteroclinic cycle associated to a bifocus and a saddle, for which we give a rigorous description here. Specifically, we study a $C^2$–vector field:

$$f : \mathbb{R}^4 \to \mathbb{R}^4$$

satisfying the following properties:
(P1) $f$ is $R$–reversible that is $f(Rx) = -f(x)$.\(^1\)

(P2) The points $O$ and $S$ are equilibria of (1) where $O, S \in Fix(R)$.

(P3) The eigenvalues of $df_X$ are:
(a) $\pm \alpha \pm i\omega$ where $\alpha > 0$ and $\omega > 0$, for $X = O$;
(b) $\pm \lambda_1$ and $\pm \lambda_2$ where $0 < \lambda_1 < \lambda_2$, $\lambda_2 \neq 2\lambda_1$, for $X = S$.

(P4) There are two heteroclinic connections: $\gamma_1$ from $O$ to $S$ and $\gamma_2$ from $S$ to $O$. The heteroclinic cycle associated to $S$ and $O$ will be denoted by $\Gamma$.

(P5) \(\dim(T_{\gamma_1(t)}W^u(O) \cap T_{\gamma_1(t)}W^s(S)) = \dim(T_{\gamma_2(t)}W^u(S) \cap T_{\gamma_2(t)}W^s(O)) = 1\).

(P6) No orbit-flip at $S$:

\[
\lim_{t \to +\infty} \frac{T_{\gamma_1(t)}/\gamma_1}{|T_{\gamma_1(t)}/\gamma_1|} \notin T_S W^{ss}(S) \quad \text{and} \quad \lim_{t \to -\infty} \frac{T_{\gamma_2(t)}/\gamma_2}{|T_{\gamma_2(t)}/\gamma_2|} \notin T_S W^{uu}(S).
\]

Hypothese have been stated as (P1)–(P6) to make the paper readable. We point out some remarks about the hypotheses emphasizing that they may be relaxed.

3.2. A tour around the hypotheses. First of all, note that we do not need to specify all the eigenvalues of $df_O$. One eigenvalue would be enough; hypothesis (P1) and the fact the non-real eigenvalues “come” in pairs (see Lemma 3.3 below) allow us to conclude (P3a). The same holds for $S$ and (P3b).

According to [7], hypothesis (P5) implies that $W^s(O)$ intersect $Fix(R)$ transversally and so does $W^u(O)$. The dimension of both $W^s(X) \subset R^4$ and $Fix(R) \subset R^4$ is 2, where $X \in \{O, S\}$.

3.3. Auxiliary results about reversibility. In this subsection, under hypotheses (P1)–(P6), we prove that $\dim Fix(R)$ must be 2 and that if $\lambda$ is an eigenvalue of $df_X$, then $-\lambda$ is also an eigenvalue. These results are well known in the literature – see for instance [14]. We restrict the proofs to the case under consideration. Hereafter, we assume that $X$ is a saddle equilibrium of (1), where $X \in \{O, S\}$.

Lemma 3.1. If $f$ is $R$–reversible and $X \in Fix(R)$, then the linearization $df_X : R^4 \to R^4$ is $R$–reversible.

Proof. Since $f$ is a $R$–reversible vector field then $R \circ f = -f \circ R$. Differentiating both sides of the equality and using the chain rule, we get $dR_{f(x)} \circ df_X = -df_R_{f(x)} \circ dR_X$. Since $R$ is linear and $X \in Fix(R)$, it follows that $dR \equiv R$ and $R(X) = X$ and therefore $R \circ df_X = -df_X \circ R$, that is, $df_X$ is a $R$–reversible vector field. \(\square\)

Lemma 3.2. If the saddle $X$ is a symmetric hyperbolic equilibrium then $\dim Fix(R) = 2$.

Proof. As in [20], we may assume, without loss of generality that $R^4 = X^+ \oplus X^-$, where $X^+ = Fix(R) = \{x \in R^4 : Rx = x\}$ and $X^- = \{x \in R^4 : Rx = -x\}$.

By Lemma 3.1, since $X$ is symmetric then $df_X \circ R = -R \circ df_X$. Hence:

\[
df_X(X^+) \subset X^- \quad \text{and} \quad df_X(X^-) \subset X^+.
\]

Therefore, $df_X \in GL(R^4)$ if and only if $\dim X^+ = \dim X^-$. An operator which does not satisfy $\dim X^+ = \dim X^-$ would be singular (and $X$ would not be hyperbolic). Thus $\dim X^- + \dim X^+ = \dim R^4$ and the result follows. \(\square\)

The symmetric equilibrium $X$ persists under small $R$–reversible perturbations of $f$. In fact, the set of $R$–reversible vector fields which possesses hyperbolic symmetric equilibria is non-empty and open in the $C^1$–Whitney topology [20, pp. 300].

\(^1\)Throughout this paper, as in [3], an explicit expression for $R$ is not needed. Nevertheless, without loss of generality, we may assume that $R(x_1, x_2, x_3, x_4) = (x_3, x_4, x_1, x_2)$.\]
Lemma 3.3. If \( X \in \text{Fix}(R) \) and \( \lambda \) is an eigenvalue of \( df_X \), then \(-\lambda\) is also an eigenvalue of \( df_X \). Moreover, if \( 0 \) is an eigenvalue of \( df_X \), then it must occur with multiplicity (at least) \( 2 \).

Proof. Since \( f \) is \( R \)-reversible, then \( df_X \circ R = -R \circ df_X \) by Lemma 3.1. Therefore:
\[
\det[df_X - \lambda Id] = \det[-(R(df_X)^{-1}R) - \lambda Id] = (-1)^4 \det[df_X + \lambda Id].
\]
and then the lemma follows. In particular, if \( \lambda \) is a zero of the characteristic polynomial, then \(-\lambda\) is also a zero of the same polynomial.

The proof of Lemma 3.3 can be used to show that if \( \gamma \) is a multiplier of a \( R \)-reversible diffeomorphism at a symmetric closed orbit, then \( \frac{1}{\gamma} \) is also a multiplier. More details in [4, 14]. Due to the reversibility, the spectrum of \( df_X \) is always symmetric with respect to 0.

Let \( V \) a tubular neighbourhood of \( \Gamma \). According to [20], if there is \( t^* \in \mathbb{R} \) such that \( X(t^*) \in \text{Fix}(R) \) and \( X \) is a \( R \)-reversible solution that stays in \( V \) for all time, then one of the following situations occurs:

(A) \( \text{graph}(X) \subset \text{Fix}(R) \) and \( X \) consists of an equilibrium point;
(B) \( \text{graph}(X) \cap \text{Fix}(R) \) contains exactly one point and then \( X \) is not a closed curve;
(C) \( \text{graph}(X) \cap \text{Fix}(R) \) contains exactly two points and then \( X \) is a periodic solution.

Therefore, to find symmetric periodic solutions of (1), we only need to look at intersections of \( \text{Fix}(R) \) and their images under the first return map. This note will be used in Section 5.

4. Local dynamics. We perform a similar analysis to that of [21] adapted to our purposes. The flow in a neighbourhood of \( \Gamma \) consists of two parts: first, when a trajectory is near the equilibria its behaviour is essentially governed by the linearized vector field. Far from the equilibria, we use the Tubular Flow Theorem.

4.1. Linearization near \( O \) and \( S \). Hypothesis (P3) means that there is a resonance in the set of eigenvalues of \( O \). This means that the Sternberg classic \( C^1 \)-linearization results cannot be applied to this case. Nevertheless, as shown by Harterich [7], one may construct a linearization for the flow from the linearization of the time 1–map.

Lemma 4.1. If \( X = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \), system (1) is locally \( C^1 \)-orbitally equivalent to the following systems:
\[
\dot{X} = \begin{pmatrix} -\alpha & -\omega & 0 & 0 \\ \omega & -\alpha & 0 & 0 \\ 0 & 0 & \alpha & \omega \\ 0 & 0 & -\omega & \alpha \end{pmatrix} X \quad \text{and} \quad \dot{X} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 \\ 0 & 0 & 0 & -\lambda_2 \end{pmatrix} X,
\] near \( O \) and \( S \), respectively.

Hereafter, we are assuming that the neighbourhoods \( V_O \) and \( V_S \) in which the flow can be linearized near \( O \) and \( S \) are a “hypercylinder” and a “hypercube”, respectively. Rescaling the coordinates, the height of the hypercylinder is 2 and the side of the hypercube is 1.

Analogously to [4, 7], we construct \( R \)-invariant local codimension one sets containing \( O \) and \( S \), \( \Sigma^0 \) and \( \Sigma^S \), intersecting the subspace \( \text{Fix}(R) \) as subdisks of dimension 2. We make use of the reversibility in the construction of the global first return map. Notice that \( \text{Fix}(R) \cap \Sigma^X \) may be seen as a “slice” of \( \Sigma^X \). Recall also that \( S \) is far from \( O \), \( \dim \Sigma^S = \dim \Sigma^0 = 3 \) and that \( \Sigma^0 \) and \( \Sigma^S \) are not cross sections because they contain the equilibria – see Figure 1.
4.2. Local coordinates near $O$ – bipolar coordinates. Using (3), we define new coordinates near the bifocus $O$. Bipolar coordinates $(r_s, \phi_s, r_u, \phi_u)$ will be more convenient:

$$x_1 = r_s \cos(\phi_s), \quad x_2 = r_s \sin(\phi_s), \quad x_3 = r_u \cos(\phi_u), \quad x_4 = r_u \sin(\phi_u)$$

The local invariant manifolds are given by:

$$W^s_{loc}(O) = \{ r_u = 0 \} \quad \text{and} \quad W^u_{loc}(O) = \{ r_s = 0 \}.$$ 

Near $O$, in bipolar coordinates $(r_s, \phi_s, r_u, \phi_u)$, the dynamics is governed by the differential equations:

$$\dot{r}_s = -\alpha r_s, \quad \dot{\phi}_s = \omega, \quad \dot{r}_u = \alpha r_u, \quad \dot{\phi}_u = -\omega$$

whose solutions may be written as:

$$r_s(t) = r_s(0)e^{-\alpha t} \quad \phi_s(t) = \phi_s(0) + \omega t \quad r_u(t) = r_u(0)e^{\alpha t} \quad \phi_u(t) = \phi_u(0) - \omega t.$$ 

4.3. Cross sections near $O$. In order to construct the Poincaré map, we consider three-dimensional return surfaces near the origin, $\Sigma^\text{in}_O$ and $\Sigma^\text{out}_O$, which are solid tori defined by:

(i) $\Sigma^\text{in}_O = \{ r_s = 1 \}$ parametrized by the coordinates $(\phi^\text{in}_s, r^\text{in}_u, \phi^\text{in}_u)$ and (ii) $\Sigma^\text{out}_O = \{ r_u = 1 \}$ parametrized by the coordinates $(\phi^\text{out}_s, \phi^\text{out}_u, \phi^\text{out}_u)$. Note that $R(\Sigma^\text{in}_O) = \Sigma^\text{out}_O$. Trajectories starting at interior points of $\Sigma^\text{in}_O$ go inside $V_O$ in positive time. Trajectories starting at interior points of $\Sigma^\text{out}_O$ go outside $V_O$ in positive time. Moreover, the set $\Sigma^0 = \{ r_s = r_u \leq 1 \}$ is $R$–invariant, contains $O$ and $\Sigma^\text{in}_O = R(\Sigma^\text{out}_O)$. It is worth to observe that $R$–reversible periodic solutions in a tubular neighbourhood of $\Gamma$ intersect $Fix(R)$ either in $\Sigma^0 \cap Fix(R)$ or in $\Sigma^S \cap Fix(R)$ or in both.

The local invariant manifolds of $O$ in the cross sections are circles that may be parametrized as follows:

$$W^s_{loc}(O) \cap \Sigma^\text{in}_O = \{ r_u = 0, r_s = 1, \phi_s \mod 2\pi \leq 2\pi \}$$

and

$$W^u_{loc}(O) \cap \Sigma^\text{out}_O = \{ r_s = 0, r_u = 1, \phi_u \mod 2\pi \leq 2\pi \}.$$ 

As depicted in Figure 1, label the two points $\Sigma^\text{in}_O \cap \Gamma$ and $\Sigma^\text{out}_O \cap \Gamma$ by $q_s(O)$ and $q_u(O)$, respectively.
4.4. Local flow near $O$. The time of flight inside $V_O$ of a trajectory with initial condition $(\phi_s^{in}, r_u^{in}, \phi_u^{in}) \in \Sigma_O^{in} \setminus W^s(O)$ is given by:

$$\frac{1}{\alpha} \ln \left( \frac{1}{r_u^{in}} \right) = -\frac{\ln(r_u^{in})}{\alpha}.$$ Integrating (4) we have: $\Pi_O : \Sigma_O^{in} \setminus W^s(O) \to \Sigma_O^{out}$ defined by:

$$(\phi_s^{in}, r_u^{in}, \phi_u^{in}) \mapsto \left( r_u^{in}, \phi_s^{in} + \frac{\omega}{\alpha} \ln(r_u^{in}), \phi_u^{in} - \frac{\omega}{\alpha} \ln(r_u^{in}) \right) =: (r_u^{out}, \phi_s^{out}, \phi_u^{out}). \quad (7)$$

We refine the characterization of $\Pi_O$, by defining the diffeomorphisms:

$$\Pi^S_O : \Sigma_O^{in} \setminus W^s(O) \to \Sigma^0 \setminus \{O\} \quad \text{and} \quad \Pi^O_O : \Sigma^0 \setminus \{O\} \to \Sigma_O^{out}$$

linked by the equality:

$$\Pi^S_O = R \circ (\Pi_O)^{-1} \circ R. \quad (8)$$

In particular we have $\Pi_O = \Pi^S_O \circ \Pi^O_O$. In what follows, we define similar cross sections near the saddle $S$. The ideas are similar to those used for the bifocus $O$.

4.5. Local Coordinates near $S$. Using (3), we define new coordinates near the saddle $S$. Rectangular coordinates $(y_1, y_2, y_3, y_4)$ will be more convenient. The local invariant manifolds can be parametrized as:

$$W^s_{loc}(S) = \{y_1 = y_2 = 0\} \quad \text{and} \quad W^u_{loc}(S) = \{y_3 = y_4 = 0\}.$$ Near $S$, in rectangular coordinates $(y_1, y_2, y_3, y_4)$, the vector field $f$ can be linearized as:

$$\dot{y}_1 = \lambda_1 y_1, \quad \dot{y}_2 = \lambda_2 y_2, \quad \dot{y}_3 = -\lambda_3 y_3 \quad \text{and} \quad \dot{y}_4 = -\lambda_4 y_4. \quad (9)$$

whose explicit solutions are:

$$y_1(t) = y_1(0) e^{\lambda_1 t}, \quad y_2(t) = y_2(0) e^{\lambda_2 t}, \quad y_3(t) = y_3(0) e^{-\lambda_3 t}, \quad y_4(t) = y_4(0) e^{-\lambda_4 t}.$$  

4.6. Cross Sections near $S$. In order to construct the Poincaré map, we consider three-dimensional return surfaces near $S$, $\Sigma_S^{in}$ and $\Sigma_S^{out}$, which are solid cubes defined by: $\Sigma_S^{in} = \{y_3 = \pm 1\}$ parametrized by the coordinates $(y_1, y_2, y_4) \in [-1, 1]^3$ and $\Sigma_S^{out} = \{y_1 = \pm 1\}$ parametrized by the coordinates $(y_2, y_3, y_4) \in [-1, 1]^3$. Due to the non-degeneracy condition (P6), orbit-flip does not hold. Hence almost all solutions starting at $\Sigma_S^{in}$ leave $V_S$ by the direction associated to the strongest unstable direction. The local invariant manifolds in the cross sections are lines that may be parametrized as follows:

$$W^s_{loc}(S) \cap \Sigma_S^{in} = \{y_3 = 1, y_1 = 0, y_2 = 0, |y_3| \leq 1\} \quad (10)$$

and

$$W^u_{loc}(S) \cap \Sigma_S^{out} = \{y_1 = 1, y_3 = 0, y_4 = 0, |y_2| \leq 1\}. \quad (11)$$

As depicted in Figure 1, label the two points $\Sigma_S^{in} \cap \Gamma$ and $\Sigma_S^{out} \cap \Gamma$ by $q_\infty(S)$ and $q_u(S)$, respectively.

4.7. Local flow near $S$. The time of flight inside $V_S$ of a trajectory with initial condition $(y_1, y_2, y_4) \in \Sigma_S^{in} \setminus W^s(S)$ is given by:

$$\frac{1}{\lambda_3} \ln \left( \frac{1}{y_1} \right) = -\frac{\ln(y_1)}{\lambda_1}.$$ and the local map near $S$, $\Pi_S : \Sigma_S^{in} \setminus W^s(S) \to \Sigma_S^{in}$ can be written as:

$$(y_1, y_2, y_4) \mapsto (y_2 y_1^{-\delta_S}, y_3 y_1, y_4 y_1^{-\delta_S}) =: (y_2^{out}, y_3^{out}, y_4^{out}),$$

where $\delta_S = \frac{\lambda_2}{\lambda_1}$. 
4.8. The transitions and the global map. We obtain a global map using the linearizations constructed around $V_O$ and $V_S$. The diffeomorphisms $\Psi_{OS}: \Sigma_O^\text{out} \to \Sigma_S^\text{in}$ and $\Psi_{SO}: \Sigma_S^\text{out} \to \Sigma_O^\text{in}$ are defined by flow box fashion. The following maps are induced by the flow along $\Gamma$:

$$
\Pi^{OS}: \Sigma^0 \setminus \{O\} \to \Sigma^S \quad \text{and} \quad \Pi^{SO}: \Sigma^S \setminus \{S\} \to \Sigma^0.
$$

Also by reversibility, we have:

$$
\Pi^{OS} = R \circ (\Pi^{SO})^{-1} \circ R 
$$

and thus we define $\Pi_1: \Sigma^0 \setminus \{O\} \to \Sigma^0$ as $\Pi^{SO} \circ \Pi^{OS}$ and $\Pi_2: \Sigma^S \setminus \{S\} \to \Sigma^S$ as $\Pi^{OS} \circ \Pi^{SO}$ at all points where they are well defined. Reversibility is induced to the global Poincaré map $\Pi_i: \Sigma^X \setminus \{X\} \to \Sigma^N$. In other words, we have $R \circ \Pi_i = \Pi_i^{-1} \circ R$, for $i \in \{1, 2\}$ and $X \in \{O, S\}$.

5. Catching $R$–reversible pulses. The next two definitions constitute the core of the present paper. If $N \in \mathbb{N}$, a $N$–pulse is a $N$–periodic solution or a $N$–homoclinic cycle contained in a small tubular neighbourhood of $\Gamma \subset \mathbb{R}^4$. Formal definitions follow.

**Definition 5.1.** A $k$–homoclinic cycle associated to $X \in \{O, S\}$ is a homoclinic cycle of (1) that is contained in a tubular neighbourhood of $\Gamma$ hitting a cross section to $\Gamma$ exactly $k$ times.

**Definition 5.2.** A $k$–periodic solution is a periodic solution of (1) that is contained in a tubular neighbourhood of $\Gamma$ hitting a cross section to $\Gamma$ exactly $k$ times before closing up at the $(k + 1)^{th}$ hit.

For $i \in \{1, 2\}$, in this section, we study a way to “catch” the different types of pulses through the reversibility of the Poincaré global map $\Pi_i$. The solutions differ on the number of revolutions inside a tubular neighbourhood of the original cycle $\Gamma$. This is what we call the order of a pulse.

**Lemma 5.3.** With respect to the solutions of (1), the following assertions hold:

1. if $x \in \Sigma^S \cap \text{Fix}(R) \cap \Pi^{OS}(\text{Fix}(R))$, then $x(t)$ is a 1–periodic solution.
2. if $x \in \Sigma^S \cap W^s(O) \cap \Pi^{OS}(\text{Fix}(R))$, then $x(t)$ is a 2–homoclinic cycle of $O$.
3. if $x \in \Sigma^0 \cap \text{Fix}(R) \cap \Pi_1(\text{Fix}(R))$, then $x(t)$ is a 2–periodic solution.
4. if $x \in \Sigma^S \cap \text{Fix}(R) \cap \cap W^s(O) \cap \Pi_2(\text{Fix}(R))$, then $x(t)$ is a 3–homoclinic cycle of $O$.
5. if $x \in \Sigma^S \cap \text{Fix}(R) \cap W^s(S)$, then $x(t)$ is a 1–homoclinic cycle of $S$.

**Proof.** 1. Let $x \in \Sigma^S \cap \text{Fix}(R) \cap \Pi^{OS}(\text{Fix}(R))$. Since $x \in \Pi^{OS}(\text{Fix}(R))$, then there exists $y \in \Sigma^0 \cap \text{Fix}(R)$ such that $R(y) = y$ and $\Pi^{OS}(y) = x$. By (12), we have:

$$
(\Pi^{OS})^{-1}(y) = (R \circ \Pi^{OS} \circ R)(y) = (R \circ \Pi^{OS})(y) = R(x) = x
$$

Therefore $\Pi^{SO}(x) = y$ and then $\Pi_2(x) = (\Pi^{OS} \circ \Pi^{SO})(x) = \Pi^{OS}(y) = x$. We conclude that $\Pi_2(x) = x$ and we get the result.

2. Let $x \in \Sigma^S \cap \text{Fix}(R) \cap W^s(O) \cap \Pi^{OS}(\text{Fix}(R))$. Since $x \in \Pi^{OS}(\text{Fix}(R))$, then there exists $y \in \Sigma^S \cap \text{Fix}(R)$ such that $R(y) = y$ and $\Pi^{OS}(y) = x \in W^s(O)$. Thus, we may write:

$$
(\Pi^{SO})^{-1}(y) = (\Pi^{SO})^{-1} \circ (\Pi^{OS})^{-1}(x) \in W^u(O)
$$

This means that $x(t)$ turns twice around $\Gamma$, i.e., $x$ belongs to a 2–homoclinic cycle to $O$.

3. Let $x \in \Sigma^S \cap \text{Fix}(R) \cap \Pi_1(\text{Fix}(R))$. Then there exists $y \in \text{Fix}(R)$ such that $\Pi_1(y) = x$ (or $\Pi^{-1}(x) = y$). Since the map $\Pi_1$ is $R$–reversible map and $R$ is an involution (i.e., $R^{-1} = R$), it follows that: $\Pi_1^{-1}(y) = R \circ \Pi_1 \circ R(y) = R \circ \Pi_1(y) = R(x) = x$. We conclude that $\Pi_1(x) = y$ and $\Pi_1 \circ \Pi_1(x) = \Pi_1(y) = x$. 


4. Let \( x \in \Sigma^S \cap Fix(R) \cap \Pi_2(Fix(R)) \cap W^s(O) \). Then there exist \( y \in Fix(R) \) such that 
\[ \Pi_2(y) = x \in W^s(O). \]
In particular the trajectory \( \phi(y, t) \) “turns 1.5” times around \( \Gamma \)
for \( t \geq 0 \). By \( R \)-reversibility, it follows that \( y \in W^s(O) \) and the number of rounds of 
the orbit around \( \Gamma \) is \( 2 \times 1.5 = 3 \).
5. The proof is analogous to the previous, this is why it is left to the reader.

\[ \square \]

6. **Local Geometry near the bifocus.** In this section we describe the spiralling behaviour of 
solutions near \( O \). Recalling that \( \{ q_s(O) \} = \Sigma^m_O \cap \Gamma \), without loss of generality, we will 
assume that the coordinates of \( q_s(O) \in \Sigma^m_O \) are given by \((0,0,0)\).

6.1. **Spiralling sheet.**

**Definition 6.1 ([9]).** Let \( a \in \mathbb{R}, D \) be a disc centered at \( p \in \mathbb{R}^2 \). A spiral on \( D \) around 
the point \( p \) is a smooth curve \( \alpha : [a, +\infty[ \to D \), satisfying \( \lim_{s \to +\infty} \alpha(s) = p \) and such that if 
\( \alpha(s) = (r(s), \theta(s)) \) is its expression in polar coordinates around \( p \) then:

1. the map \( r \) is bounded by two monotonically decreasing maps converging to zero as \( s \to +\infty \);
2. the map \( \theta \) is monotonic for some unbounded subinterval of \([a, +\infty[ \) and \( \lim_{s \to +\infty} |\theta(s)| = +\infty \).

If \( D \) is a disc centered at \( p \in \mathbb{R}^2 \), a double spiral on \( D \) around the point \( p \) is the union of 
two spirals accumulating on \( p \) and a curve connecting the other end points.

**Definition 6.2 ([9]).** A two-dimensional manifold \( \mathcal{H} \) embedded in \( \mathbb{R}^3 \) is called a spiralling sheet 
accumulating on a curve \( \gamma \) if there exist a spiral \( S \) around \((0,0)\), a neighbourhood \( V \subset \mathbb{R}^3 \) of \( \gamma \), a neighbourhood \( W_0 \subset \mathbb{R}^2 \) of the origin, a non-degenerated closed interval \( I \) 
and a diffeomorphism \( \eta : V \to I \times W_0 \) such that:

\[ \eta(\mathcal{H} \cap V) = I \times (S \cap W_0) \quad \text{and} \quad \gamma = \eta^{-1}(I \times \{0\}). \]

6.2. **A spiralling geometry.** The proof of the following results may be found in [7, 9].

**Proposition 1.** Let \( F \) be a two dimensional \( C^1 \)-compact manifold in \( \Sigma^m_O \) intersecting 
\( W^s_{loc}(O) \) transversely. Then \( \Pi_O(F \setminus W^s_{loc}(O)) \) is a spiralling sheet accumulating on 
\( W^s_{loc}(O) \cap \Sigma^m_O \).

The spiralling sheet forced by the existence of complex eigenvalues will be used to show 
that any cross section of \( \Gamma \) accumulating on \( W^s_{loc}(O) \) cannot fail to intersect \( Fix(R) \cap \Sigma^0 \)
along a very special curve and \( W^s_{loc}(S) \) infinitely many times – see Theorem 7.1. For small 
\( \tau > 0 \), let us define the set:

\[ W^u_\tau(O) \cap \Sigma^m_O = \{ (r_s^{out}, \phi_s^{in}, \phi_u^{out}) : r_s^{out} = 0 \ \text{and} \ \phi_u^{out} \in [-\tau, \tau] \}. \]

Geometrically, \( W^u_\tau(O) \cap \Sigma^m_O \) corresponds to a piece of the circle of \( W^s_{loc}(O) \cap \Sigma^m_O \), where 
\( q_u(O) \in W^u_\tau(O) \).

**Corollary 1.** Let \( F \) be a two dimensional \( C^1 \)-compact manifold in \( \Sigma^m_O \) intersecting \( W^s_{loc}(O) \)
transversely. Then there is \( F^* \subset F \) such that \( \Pi_O(F^*) \) is a spiralling sheet accumulating on 
\( W^u_\tau(O) \cap \Sigma^m_O \). The set \( F^* \subset F \) is bounded by a double spiral.

Suppose that \( Fix(R) \cap \Sigma^0 = \{ r_s = r_u \ \text{and} \ \phi_s = \phi_u \} \) is parametrized by \( (r_s^*, \phi_s^*) \). Thus 
\[ \Pi^u_{\| Fix(R) } : Fix(R) \cap \Sigma^0 \to \Sigma^m_O \]
can be explicitly written as:

\[ \Pi^u_{\| Fix(R) } (r_s^* , \phi_s^*) = \left( (r_s^*)^2 , \phi_s^* - \frac{\omega}{\alpha} \ln(r_s^*) , \phi_s^* + \frac{\omega}{\alpha} \ln(r_s^*) \right) = (r_s^{out} , \phi_s^{out} , \phi_u^{out} ), \]

and the proof of Corollary 1 allows us to conclude easily that:
Corollary 2. Under conditions of Corollary 1, there is a subset $G^*$ of $Fix(R) \cap \Sigma^0$ such that $\Pi_O^u(G^*)$ is a spiralling sheet accumulating on $W^*_l(\sigma) \cap \Sigma^0_O$. The set is bounded by a double spiral around $O \in Fix(R) \cap \Sigma^0$.

7. The hyperchaos.

7.1. Multiplicity of $N$-pulses near the heteroclinic cycle. The next result is the core of this article: homoclinic cycles and the periodic solutions of different type accumulate on $\Gamma$. We put together all the information that appears in the previous sections.

Theorem 7.1. Suppose that $f$ satisfies (P1)–(P6). Then arbitrarily close to $\Gamma$, there are:

1. infinitely many 1–homoclinic cycles associated to $S$;
2. infinitely many 1–periodic solutions;
3. infinitely many 2–homoclinic cycles associated to $O$;
4. infinitely many 2–periodic solutions.

All these solutions are $R$–reversible. The period of each periodic solution goes to $+\infty$ as the initial condition lying in $Fix(R)$ approaches the different homoclinic cycles.

Note that as stated in [10, Sec 4] there are no symmetric 2–homoclinic cycles to $S$ near $\Gamma$. Recall that $\delta_S = \frac{2\pi}{\lambda} > 1$.

Proof. Let $F$ be a two dimensional $C^1$– compact manifold in $\Sigma^u_O$ intersecting $W^*_l(\sigma)$ transversely in $\varphi_\ell(\sigma)$. Lemma 5.3 will be the key of our proof.

1. By Corollary 2, then $\Pi_O^u(G^*)$ is a spiralling sheet $H$ accumulating on $W^*_l(\sigma) \cap \Sigma^0_O$. Since $\Psi_{OS}$ is a diffeomorphism, $\Psi_{OS}(H)$ is a spiralling sheet accumulating on $W^*_l(S) \cap \Sigma^0_S[S]$, where $q_\ell(S) \in W^*_l(S)$. Since $W^*_l(S) \cap \Sigma^0_S[S]$ and $\Psi_{OS}(W^*_l(\sigma) \cap \Sigma^0_O)$ are not tangent (due to the non-degenerate conditions (P5)–(P6)), it implies that $H$ intersects $W^*_l(S) \cap \Sigma^0_S[S]$ at infinitely many points. Since $W^*_l(S)$ is a primary connection (ie it goes directly to the equilibrium), the solution associated to each point is 1–homoclinic to $S$, by Lemma 5.3.

2. Let us define the two dimensional set $\mathcal{M}^2$:

$$\mathcal{M} = \{y_1 = y_3, \quad y_2 = y_4, \quad y_1 \in [-1, 1], \quad |y_2| \leq y_1^{\delta_S} \} \subset Fix(R) \cap \Sigma^S.$$

Let $\mathcal{M}_1 \subset \Sigma^S$ be the pre-image of $\mathcal{M} \subset Fix(R) \cap \Sigma^S$ by the map $\Pi^S$. It is given by:

$$y_1^{in} = y_1^2, \quad y_2^{in} = y_2 y_1^{\delta_S}, \quad y_3^{in} = 1 \quad \text{and} \quad y_4^{in} = y_2(y_1)^{-\delta_S}.$$

The two-dimensional manifold $\mathcal{M}_1$ intersect the plane

$$\mathcal{P} = \{y_2 = 0, \quad y_3 = 1\} \subset \Sigma^S$$

along lines. The set $\mathcal{P}$ is tangent to $W^*_l(S)$ at $q_\ell(S)$ in $\Sigma^S$, then $\Psi_{OS}(W^*_l(\sigma) \cap \Sigma^0_O[S])$ intersects $W^*_l(S)$ infinitely many times accumulating on $q_\ell(S)$. There are two possibilities. Case (A): if the intersection of $\mathcal{P}$ and $\Psi_{OS}(W^*_l(\sigma) \cap \Sigma^0_O[S])$ is transverse, it yields infinitely many curves accumulating on $q_\ell(S)$. Case (B): if the intersection of $\mathcal{P}$ and $\Psi_{OS}(W^*_l(\sigma) \cap \Sigma^0_O[S])$ is not transverse, infinitely many curves also arise. In both cases, it corresponds to infinitely many 1–periodic $R$–reversible solutions.

---

$^2$We omit the subscripts “in” in order to simplify the notation.
3. By Corollary 2, we know that the set $\Psi_{OS}(\Pi^{u}_{O}(G^r)) \subset \Sigma^{in}_{S}$ is a spiralling sheet crossing transversely $W^{s}_{loc}(S) \cap \Sigma^{in}_{S}$. By reversibility, the set 
$$L = R \circ \Psi_{OS}(H) = (\Psi_{SO})^{-1} \circ (\Pi^{S}_{O})^{-1}(H) \subset \Sigma^{out}_{S}$$

is a spiralling sheet crossing transversely $W^{u}_{loc}(S) \cap \Sigma^{out}_{S}$. By other side, $\Pi_{S}(\Psi_{OS}(W^{u}_{r}(O) \cap \Sigma^{out}_{O}))$ is a line $\ell$ on $\Sigma^{out}_{S} \cap W^{u}_{loc}(S)$. Under the non-degeneracy conditions, the set $L \cap \ell$ has infinitely many points corresponding to $R$–reversible 2–homoclinic cycles to $O$, by Lemma 5.3.

4. The sets $(\Psi_{SO})^{-1}(W^{s}(O) \cap \Sigma^{in}_{O})$ and $\Pi_{s}(\Psi_{OS}(W^{u}_{r}(O) \cap \Sigma^{out}_{O}))$ are not tangent at $q_{n}(S)$. Therefore, the sets $\Pi_{s} \circ \Psi_{OS} \circ \Pi^{u}_{O}(G^r)$ and $(\Psi_{SO})^{-1} \circ (\Pi^{S}_{O})^{-1}(G^r)$ intersect infinitely many times, corresponding to $R$–reversible 2–periodic limit cycles.

The time of flight from $Fix(R) \cap \Sigma^{0}$ to $\Sigma^{out}_{O}$ is given by $\frac{1}{2} \ln(r^{*})$. When an initial condition approaches the point $O$, this means that $r^{*}$ approaches to zero. Therefore, the period of each 1–periodic solution goes to $+\infty$.

Borrowing the arguments of [7, Th. 22], by Theorem 7.1 we are able to show that:

**Theorem 7.2.** Suppose that $f$ satisfies (P1)–(P6). Then arbitrarily close to $\Gamma$, there are:

1. a sequence of $2N$–homoclinic orbits to $O$ and $2N$–periodic solutions;
2. a heteroclinic network involving infinitely many heteroclinic connections.

7.2. **Blenders: an open problem.** In three-dimensions, the existence of strange attractors near homoclinic cycles has been shown to occur in [8] – see also references therein. It is natural to consider that the dynamical complexity associated with heteroclinic cycles increases with the dimension. A bifocal cycle in four-dimensions seems to be the scenario for more complicated dynamics.

The non-wandering set associated to $\Gamma$ is contained in the intersection of a scroll (see a formal definition in [5, 13]) and its image under the first return map, which is a another skewed scroll as shown in [9]. The limit of these intersections gives rise to a generalized horseshoe with an expanding, a contracting and a “neutral” direction. The latter direction is the source of robust non-hyperbolic phenomena as we proceed to explain.

Asymmetric hyperbolic periodic solutions should arise pairwise near bifocal cycles. Thus, if $\gamma$ is a non-symmetric periodic solution with Morse index 2, then $R(\gamma)$ is a periodic solution of Morse index 3 implying that two of their invariant manifolds intersect transversally (generically). The return maps $\Pi_{1}$ and $\Pi_{2}$ defined over transverse sections to the bifocal cycle $\Gamma$ are diffeomorphisms in $\mathbb{R}^{3}$ and thus susceptible to exhibit heterodimensional cycles between hyperbolic periodic solutions of different Morse index, for small $C^{1}$–perturbations. We conjecture that generically suspended blenders should exist near $\Gamma$ – details in [2]. They contain the sheets of $N$–pulses and are robust under small perturbations. A complete study of this topic is in preparation.

**REFERENCES**

[1] M. A. D. Aguiar, I. S. Labouriau and A. A. P. Rodrigues, Switching near a heteroclinic network of rotating nodes, *Dyn. Syst.*, 25 (2010), 75–95.
[2] C. Bonatti and L. J. Díaz, Persistent nonhyperbolic transitive diffeomorphisms, *Ann. of Math.*, 143(2) (1996), 357–396.
[3] R. L. Devaney, Homoclinic orbits in Hamiltonian systems, *J. Diff. Equations*, 21 (1976), 431–438.
[4] R. L. Devaney, Blue sky catastrophes in reversible and Hamiltonian systems, *Indiana Univ. Math. J.*, 2 (1977), 247-263.
[5] A.C. Fowler and C. T. Sparrow, Bifocal homoclinic orbits in four dimensions, *Nonlinearity*, 4 (1991), 1159–1182.
[6] P. Gaspard, R. Kapral and G. Nicolis, Bifurcation phenomena near homoclinic systems: a two parameter analysis, *J. Statist. Phys.*, 35, N. 5–6 (1984), 697–727.
[7] J. Harterich, Cascades of reversible homoclinic orbits to a saddle-focus equilibrium, *Phys. D*, **112**, N. 1-2 (1998), 187–200.

[8] A. J. Homburg, Periodic attractors, strange attractors and hyperbolic dynamics near homoclinic orbit to a saddle-focus equilibria, *Nonlinearity*, **15** (2002), 1029–1050.

[9] S. Ibáñez and A.A.P. Rodrigues, On the dynamics near a homoclinic network to a bifocus: switching and horseshoes, *Int. Journal Bif. Chaos*, (2015), Vol. 25(11), 1530030, 19pp.

[10] J. Knobloch and T. Wagenknecht, Homoclinic snaking near a heteroclinic cycle in reversible systems, *SIAM J. Appl. Dyn. Syst.*, **7** (4), (2008), 13971420.

[11] I. S. Labouriau and A. A. P. Rodrigues, Global generic dynamics close to symmetry, *Journal of Differential Equations*, **253** (2012), 2527–2557.

[12] I. S. Labouriau and A. A. P. Rodrigues, Dense heteroclinic tangencies near a Bykov cycle, *Journal of Differential Equations*, **259** (2015), 5875–5902.

[13] C. Laing and P. Glendinning, Bifocal homoclinic bifurcations, *Physica D*, **102** (1997), 1–14.

[14] J. Lamb and J. Roberts, Time-reversal symmetry in dynamical systems: a survey, *Physica D*, **112** (1998), 1–39.

[15] L. M. Lerman, Homo and Heteroclinic orbits, hyperbolic subsets in a one-parameter unfolding of a Hamiltonian system with heteroclinic contour with two saddle-foci, *Regular and Chaotic Dynamical Systems*, **2** (1997), 139–155.

[16] A. A. P. Rodrigues, Persistent switching near a heteroclinic model for the geodynamo problem, *Chaos, Solitons & Fractals*, **47** (2013), 73–86.

[17] A. A. P. Rodrigues, Repelling dynamics near a Bykov cycle, *J. Dynam. Differential Equations*, **25** (2013), 605–625.

[18] L. P. Shilnikov, The existence of a denumerable set of periodic motions in four dimensional space in an extended neighbourhood of a saddle-focus, *Sov. Math. Dokl.*, **172** (1967), 54–57.

[19] L. P. Shilnikov, On the question of the structure of an extended neighborhood of a structurally stable state of equilibrium of saddle-focus type, *Math. USSR Sb.*, **81**, 123, (1970), 92–103.

[20] A. Vanderbauwhede and B. Fiedler, Homoclinic period blow-up in reversible and conservative systems, *Z. Angew. Math. Phys.*, **43**, (1992), 292–318.

[21] C. Tresser, About some theorems by L. P. Shilnikov, *Ann. Inst. H. Poincaré*, **40** (1984), 441–461.

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