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Two Equivalent Realizations of Trigonometric Dynamical Affine Quantum Group $U_{q,\lambda}(\hat{sl}_2) = U_{q,x}(\hat{sl}_2)$, Drinfeld Currents and Hopf Algebroid Structures

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Abstract

Two new realizations, denoted $U_{q,x}(\hat{gl}_2)$ and $U(R_{q,x}(\hat{gl}_2))$ of the dynamical quantum affine algebra $U_{q,\lambda}(\hat{gl}_2)$ are proposed, based on Drinfeld-currents and RLL relations respectively, along with a Heisenberg algebra $\{P,Q\}$, with $x = q^{2P}$. Here $P$ plays the role of the dynamical variable $\lambda$ and $Q = \frac{\partial}{\partial P}$. An explicit isomorphism from $U_{q,x}(\hat{gl}_2)$ to $U(R_{q,x}(\hat{gl}_2))$ is established, which is a dynamical extension of the Ding-Frenkel isomorphism of $U_q(\hat{gl}_2)$ with $U(R_q(\hat{gl}_2))$ between the Drinfeld realization and the Reshetikhin-Tian-Shanksy construction of quantum affine algebras. Hopf algebroid structures and an affine dynamical determinant element are introduced and it is shown that $U_{q,x}(\hat{sl}_2)$ is isomorphic to $U(R_{q,x}(\hat{sl}_2))$. The dynamical construction is based on the degeneration of the elliptic quantum algebra $U_{q,p}(\hat{sl}_2)$ of Jimbo, Konno et al. as the elliptic variable $p \to 0$.

1 Introduction

The elliptic affine quantum group $U_{q,p}(\hat{sl}_2)$, where $q$ denotes the quantum variable and $p$ the elliptic, is studied in detail by M. Jimbo, H. Konno et al. in [22,26] using elliptic deformations (i.e. twists) of the quantum Drinfeld currents and a Heisenberg algebra $\mathcal{H}$ containing the dynamical variable $P$, to construct an operator $L^+(u)$ which obeys the
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$RLL$ relations:

$$R^{+(12)}(u, P + h)L^{+(1)}(u_1)L^{+(2)}(u_2) = L^{+(2)}(u_2)L^{+(1)}(u_1)R^{+(12)}(u, P). \quad (1.1)$$

associated to the elliptic $R$-matrix:

$$R^+(u, P) = \rho^+(u) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{[P+1][P-1][u]}{[P]^2[1+u]} & \frac{[1][P+u]}{[P][1+u]} & 0 \\ 0 & \frac{[1][P-u]}{[P][1+u]} & \frac{u}{[1+u]} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1.2)$$

where $\rho^+(u)$ is a suitably chosen coefficient, $R^{++}(u, P) = R^+(u, P)|_{r \to r^*}$ with $r^* = r - c$ and the Jacobi theta function is given by

$$[u] = \frac{q^{2u^2 - u}}{(p;p)^3_\infty} \Theta_p(z), \quad p = q^{2r}, \quad z = q^{2u}, \quad \Theta_p(z) = (z;p)_\infty(p/z;p)_\infty(p;p)_\infty, \quad (z;p)_\infty = \prod_{n=0}^{\infty} (1 - zp^n).$$

Most of the degenerations of this $R$-matrix, denoted $R_{q,p}(x)$, yield well-known $R$-matrices (see Figure 1.1).

![Diagram](image.png)

Figure 1.1: $R$-matrix degenerations

The top entry is the quantum affine $R$-matrix $R_q(z) = R_q(\hat{gl}_2)$ (2.2) considered by Reshetikhin and Semenov-Tian-Shansky in [29]. The lowest entry is the quantum...
dynamical $R$-matrix $R_{q,x}(\mathfrak{sl}_2)$ obtained by a twist construction using the quantum $6j$-symbols \[3,11,24]. The left-most $R$-matrix $R_q$ corresponds to the standard Drinfeld-Jimbo quantum group. The second one from the right, $R_{qx}(z) = R_{q,x}(u, P)$ (see \[4,1\]) is the $R$-matrix considered in this paper. Here $x = q^{2\, P}$, where $P$ is a generator of the Heisenberg algebra $\mathcal{H} = \{Q, P\}$ and plays the role of the dynamical variable $\lambda$ in the more standard formulations \[10, 12, 13, 24], while $Q = \frac{\partial}{\partial P}$. The exact relation of $R_{qx}$ to the $R$-matrix $R(\lambda)$ in \[24\] is given by $R(\lambda) = \lim_{z \to 0} R_{21}(u, P) |_{q = q^{-1}}$.

Based on each of these $R$-matrices, one can define a pair of algebras $(U(R), A(R))$, each of which carries a Hopf-type structure and can be considered as appropriately defined twists (or deformations) of the underlying universal enveloping algebra $U(\mathfrak{sl}_2)$ and coordinate function algebra $\mathbb{C}[SL_2]$, respectively.

The algebra of primary concern in this article will be $U(R)$ for $R = R_{q,x}(u, P)$, located in the center of Figure 1.2.

Jimbo, Konno et al. \[22\] use elliptic Drinfeld currents to define the algebra $\hat{U}_{q,p} = U_{q,p}(\hat{\mathfrak{sl}}_2)$ which can be viewed as a tensor product of the underlying quantum affine algebra, $\hat{U}_q = U_q(\hat{\mathfrak{sl}}_2)$ and a Heisenberg algebra which includes the elliptic variable $r$. They use only positive half-currents to define an $L$-operator that satisfies the RLL relation \[4,6\].

There is another construction of the elliptic affine algebra, denoted $E_{r,\eta}(\hat{\mathfrak{sl}}_2)$, by Enriquez and Felder \[10\], using both positive and negative half-currents (for the precise relation to $U_{q,p}(\hat{\mathfrak{sl}}_2)$ see Section (6.2) in \[22\]). At $c = 0$, we can identify $U_{q,p}(\hat{\mathfrak{sl}}_2)$ with Felder’s original elliptic quantum group $E_{r,\eta} = E_{r,\eta}[SL_2]$ (the topmost box) \[15\]. A vertex-type non-dynamical elliptic quantum algebra $A_{q,p} = A_{q,p}(\hat{\mathfrak{sl}}_2)$ is investigated in \[16\].

J. Ding and I. Frenkel \[8\] prove an isomorphism between the two presentations of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ - in terms of Drinfeld currents \[9\] and via RLL-relations $U(R_q(\hat{\mathfrak{sl}}_2))$, due to Reshetikhin and Semenov Tian-Shansky \[29\]. The non-elliptic limit of $A_{q,p}(\hat{\mathfrak{sl}}_2)$, denoted $A_{q,0}(\hat{\mathfrak{sl}}_2)$, is the box $A_{q,0}$ in the Figure 1.2. This algebra is also considered in \[16\] and it is speculated there that it must be the same as the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ in \[8\], which is indicated by the two arrows connecting the lowest two boxes. Actually, the results of this paper confirm their speculation, although somewhat indirectly. The dynamical quantum group $F_{q,\lambda} = F_{q,\lambda}[SL_2]$ is discussed in \[12, 24\], based on an FRST-construction. Farthest on the left sits $U_q = U_q(\mathfrak{sl}_2)$, the usual Drinfeld-Jimbo
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Figure 1.2: Dynamical Algebra Degenerations for $sl_2$

quantum group.

Clearly, only the box that is in the center of Figure 1.2 remains to be investigated. Since it is only one link away from four other boxes, it is natural to expect that it has all the desirable characteristics of those algebras. The purpose of this article is to confirm this expectation by defining two dynamical quantum algebras $U_{q,x}(\hat{gl}_2)$ and $U(R_{q,x}(\hat{gl}_2))$, endowing them with suitable H-bialgebroid structures and proving that they are isomorphic as $H$-algebras. Thus we extend the corresponding result of Ding-Frenkel to the dynamical case. We also confirm that the generators and relations for the degenerations in the non-dynamical and non-affine directions are consistent with the known models. As mentioned earlier, one recovers $U_{q,x}(\hat{sl}_2)$ in the limit as $p \to 0$ of the elliptic algebra.
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$U_{q,p}(\hat{sl}_2)$ \cite{23,27}, allowing us to identify $U_{q,x}(\hat{sl}_2)$ with $U_{q,p}(\hat{sl}_2)/p \cdot U_{q,p}(\hat{sl}_2)$. Our algebra is similar, but not identical, to the construction by P. Xu in \cite{31} of the dynamical quantum groupoid for finite-dimensional Lie algebras.

The key to transit most efficiently from the quantum to dynamical quantum world is the introduction, by H. Konno \cite{26} of the Heisenberg algebra $H = \{P, Q\}$. $U_{q,x}(\hat{sl}_2)$ can be viewed as a semidirect product (smash product) algebra $(U_q(\hat{sl}_2) \otimes M_{H^*}) \otimes \mathbb{C}[H^*_0]$. An important difference from the elliptic case is the observation that, at $p = 0$, the positive half currents $\{E^+(u), F^+(u), K_1^+(u), K_2^+(u)\}$ contain only non-positive powers of $z$. Therefore, to recover the total algebra, i.e. non-negative powers, one must include the negative half-currents $\{E^-(u), F^-(u), K_1^-(u), K_2^-(u)\}$ as well and the full set of RLL relations:

$$R^{\pm(12)}(u, P + h)L^{\pm(1)}(u_1) L^{\pm(2)}(u_2) = L^{\pm(2)}(u_2) L^{\pm(1)}(u_1) R^{\pm(12)}(u, P),$$

$$R^{\pm(12)}\left(u \pm \frac{c}{2}, P + h\right) L^{\pm(1)}(u_1) L^{\mp(2)}(u_2) = L^{\mp(2)}(u_2) L^{\pm(1)}(u_1) R^{\pm(12)}\left(u \mp \frac{c}{2}, P\right),$$

(1.3)

where $R^+(u, P)$ and $R^-(u, P)$ differ by the coefficient $\rho^\pm(z)$ \cite{4,2}. The $L^\pm(u)$ can be viewed as parts of a single L-operator which is a doubly infinite series \cite{16} with a single RLL relation, but we will not pursue this approach here. We will also consider the algebras $U_{q,x}^\pm(\hat{gl}_2)$ (resp. $U_{q,x}^-(\hat{gl}_2)$) defined by the equations containing $L^+$ (resp $L^-$) only, consisting of non-positive (resp. non-negative) powers of $z$ only.

The main results proven in this article are:

(i) $U_{q,x}(\hat{gl}_2) \simeq U(R_{q,x}(\hat{gl}_2))$ as $H$-algebras.

(ii) The subalgebras $U_{q,x}^\pm(\hat{gl}_2)$ and $U(R_{q,x}^\pm(\hat{gl}_2))$ are $H$-Hopf algebroids.

(iii) The total algebras $U_{q,x}(\hat{gl}_2)$ and $U(R_{q,x}(\hat{gl}_2))$ are $H$-Hopf algebroids.

(iv) The results (i) - (iii) remain true upon replacing $\hat{gl}_2$ by $\hat{sl}_2$.

(see Theorems \cite{4,5.7,5.12 and 5.13}).

Let us mention some motivation for the current investigation. In \cite{2}, Arnaudon et al. illustrate the various degenerations and their inverses (twists) of the elliptic algebras and double Yangians. They derive the $R$-matrix and write the basic RLL relations for
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\( U_{q,\lambda}(\hat{sl}_2) \) but do not perform the Lax expansion. They also stress the physical importance, relevant to the Calegoro-Moser systems, and point out that the ”dynamical” variable \( \lambda \) (which is elevated by Jimbo, Konno et al. in \([23, 26]\)) to an actual generator \( P \) for the algebra \( U_{q,p}(\hat{sl}_2) \) as being identified with the momentum of the system. The results in the current article provide a positive step towards confirming their expectation that similar genuinely dynamical structures exist for all formal limits (or twists) of the quantum algebras described there, and play an important role in solving the models where such algebras arise (see the Conclusion in \([2]\)). This article creates the framework to develop the representation theory of \( U_{q,x}(\hat{sl}_2) \). The companion article \([28]\) examines the finite dimensional representations of \( U_{q,x}(\hat{sl}_2) \) and its relation to representations of \( U_{q,p}(\hat{sl}_2) \) and hypergeometric series. The main algebra introduced in this paper, and its representations, can be considered part of the algebraic analysis framework of solving statistical models by using the representation theory of infinite dimensional algebras, due to M. Jimbo. Specifically, we are in the Andrews-Baxter-Forrester regime, studying (degenerations of) solutions of the DQYBE for the 8-vertex and RSOS models. A deep result that the DQYBE is equivalent to the Star-Triangle relation was shown by Felder in \([15]\). The relation of \( U_{q,x}(\hat{sl}_2) \) with the Universal vertex-IRF transformation is given by Buffenoir et al. in \([5]\). Further, the scaling limit as \( q \to 1 \) of the elliptic quantum algebra is studied by Hou and Yang \([19]\).

Outline of the Article. Section 2 describes the Ding-Frenkel construction of the quantum affine algebra isomorphism \( U_q(\hat{gl}_2) \simeq U(R_q(\hat{gl}_2)) \). In Section 3, the \( H \)-algebra \( U_{q,x}(\hat{gl}_2) \) is defined using Drinfeld currents and a Heisenberg algebra. Further, the dynamical half-currents are defined and their commutation relations are proven. The \( H \)-algebra \( U(R_{q,x}(\hat{gl}_2)) \) is defined in Section 4 where we prove our main result, that \( U_{q,x}(\hat{gl}_2) \simeq U(R_{q,x}(\hat{gl}_2)) \). Section 5 is devoted to \( H \)-Hopf-algebroid structures, the dynamical determinant element and the subalgebra \( U_{q,x}(\hat{sl}_2) \). The basic definitions are included at the beginning of the section. The expansion of the RLL relations is in Appendix A.
2 The Quantum Affine Algebra $U_q(\hat{\mathfrak{g}l}_2)$

In this section we review the definition of the Drinfeld realization of the quantum affine algebra $U_q(\hat{\mathfrak{g}l}_2)$, as given in [8]. The quantum affine algebra $U_q(\hat{\mathfrak{g}l}_2)$ has been defined in terms of Chevalley generators (Kac-Moody algebras), Drinfeld currents (Yangians and quantum loop algebras), and Reshetikhin-Tian-Shansky’s $RLL$-algebra (FRT-type construction). We consider the last two models here.

2.1 Definition of $U_q(\hat{\mathfrak{g}l}_2)$

Let us recall the results of [8] which are relevant to this treatise. Our definition of the Drinfeld realization of the quantum affine algebra $U_q(\hat{\mathfrak{g}l}_2)$ is adapted from Ding-Frenkel’s definition \[8,17\] of $U_q(\mathfrak{g}l_N)$\footnote{For the relation with the more standard presentation of quantum affine $\mathfrak{sl}_2$, see Section (5.2)}. Let $q$ be a complex number $q \neq 0$ such that $|q| < 1$.

**Definition 2.1.** \[8\] For a field $\mathbb{K} \supseteq \mathbb{C}$, the quantum affine algebra $\mathbb{K}[U_q(\hat{\mathfrak{g}l}_2)]$ in the Drinfeld realization is an associative algebra over $\mathbb{K}$ generated by the generators $q^h$, $q^c$, $k_1^\pm(z)$, $k_2^\pm(z)$, $e_n$, $f_n (n \in \mathbb{Z})$. The defining relations are given as follows.

$$
\begin{align*}
q^c &: \text{ central,} \\
q^h q^{-h} &= q^{-h} q^h = 1, \\
k_1^+(z) k_1^-(w) &= k_1^-(w) k_1^+(z), \quad k_2^+(z) k_2^-(w) = k_2^-(w) k_2^+(z), \\
k_1^+(z) k_1^+(w) &= k_1^+(w) k_1^+(z), \quad k_2^+(z) k_2^+(w) = k_2^+(w) k_2^+(z), \\
\frac{zq^{-\frac{c}{2}} - wq^{\frac{c}{2}}}{zq^{-\frac{1}{2}} - wq^{\frac{1}{2} + 1}} k_1^+(z) k_2^-(w) &= k_2^-(w) k_1^+(z) - \frac{zq^{\frac{c}{2}} - wq^{-\frac{c}{2}}}{zq^{-\frac{1}{2}} - wq^{\frac{1}{2} + 1}}, \\
\frac{zq^{\frac{c}{2}} - wq^{-\frac{c}{2}}}{zq^{\frac{1}{2} + 1} - wq^{-\frac{1}{2} - 1}} k_2^+(z) k_1^-(w) &= k_1^-(w) k_2^+(z) - \frac{zq^{\frac{c}{2}} - wq^{-\frac{c}{2}}}{zq^{-\frac{1}{2} + 1} - wq^{\frac{1}{2} - 1}}, \\
k_1^+(z) e(w) k_1^-(z)^{-1} &= \frac{zq^{\frac{c}{2}} - w}{zq^{\frac{1}{2} - 1} - wq} e(w), \quad k_2^+(z) e(w) k_2^-(z)^{-1} = \frac{zq^{-\frac{c}{2}} - w}{zq^{\frac{1}{2} + 1} - wq^{-1}} e(w), \\
k_1^+(z)^{-1} f(w) k_1^+(z) &= \frac{zq^{\frac{c}{2}} - w}{zq^{\frac{1}{2} - 1} - wq} f(w), \quad k_2^+(z)^{-1} f(w) k_2^+(z) = \frac{zq^{-\frac{c}{2}} - w}{zq^{\frac{1}{2} + 1} - wq^{-1}} f(w), \\
(zq^{-1} - wq) e(z) e(w) &= (zq - wq^{-1}) e(w) e(z), \quad (zq^{-1} - wq^{-1}) f(z) f(w) = (zq - wq^{-1}) f(w) f(z), \\
[e(z), f(w)] &= \frac{1}{q - q^{-1}} \left( \delta(q^{-c} w z) \psi(q^{-c/2} w) - \delta(q^c w z) \varphi(q^{c/2} w) \right),
\end{align*}
$$

\footnote{For the relation with the more standard presentation of quantum affine $\mathfrak{sl}_2$, see Section (5.2).}
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with

\[ e(z) = \sum_{n \in \mathbb{Z}} e_n z^{-n}, \quad f(z) = \sum_{n \in \mathbb{Z}} f_n z^{-n}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \]

\[ \psi(z) = \sum_{n \geq 0} \psi_n z^{-n}, \quad \varphi(z) = \sum_{n \geq 0} \varphi_n z^n, \quad \delta(z) = \sum_{n \in \mathbb{Z}} z^n, \]

\[ \psi(z) = k_1^+(z) k_2^+(z)^{-1}, \quad \varphi(z) = k_1^-(z) k_2^-(z)^{-1}. \]

Note that $\psi(0) = q^h$, $\varphi(0) = q^{-h}$ and the final defining relation in Definition 2.1 can be equivalently stated in terms of the generating modes as:

\[ [e_m, f_n] = \frac{1}{q - q^{-1}} \left( q^{c(m-n)} \psi_{m+n} - q^{c(m-n)} \varphi_{m+n} \right) \quad m, n \in \mathbb{Z}. \tag{2.1} \]

2.2 Ding-Frenkel’s Equivalence

We summarize the results in [8] on the quantum affine algebra $U_q(\hat{gl}_n)$ at $n = 2$. For $0 < q < 1$, consider the quantum $R$-matrix:

\[ R_q(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q(1-z)}{1-q^2 z} & \frac{1-q^2}{1-q^2} & 0 \\ 0 & \frac{1-q^2 z}{1-q^2} & \frac{q(1-z)}{1-q^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

Reshetikhin and Semenov-Tian-Shanksy [29] define the quantum affine algebra $U(R_q)$ as:

**Definition 2.2.** $U(R_q)$ is an associative algebra over $\mathbb{C}$ with central element $c$, generated by $L_\pm(z) = (L_{ab}(z))_{a,b=1}^2$, with $L_{ab}(z) = \sum_{n=0}^\infty L_{a,b,n} z^n$ such that $L_{aa,o} L_{aa,o} = \sum_{n=0}^\infty L_{a,n} z^n$, and the affine RLL-relations:

\[ R_q \left( \frac{z}{w} \right) L_1^+(z) L_2^-(w) = L_2^+(w) L_1^-(z) R_q \left( \frac{z}{w} \right), \]

\[ R_q \left( \frac{q^c z}{w} \right) L_1^+(z) L_2^-(w) = L_2^+(w) L_1^-(z) R_q \left( \frac{q^{-c} z}{w} \right), \]

where we denote

\[ L_1^+(z) = L^+(z) \otimes 1 \quad \text{and} \quad L_2^+(z) = 1 \otimes L^+(z). \]

Some extra relations involving an auxillary operator $\tilde{L}(z)$, appearing in the original definition [29] are absorbed into Ding-Frenkel’s by imposing that $\tilde{L}(z)$ satisfies the following natural condition (see eq(3.21) in [8]):

\[ \tilde{L}(z) = ((L^+(z))^4)^{-1}. \]
It is known \(^{2}\) that \(L^\pm(z)\) have the following unique Gauss decompositions:

\[
L^\pm(z) = \begin{pmatrix}
1 & f^\pm(z) \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
k_1^\pm(z) & 0 \\
0 & k_2^\pm(z)
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
e^\pm(z) & 0
\end{pmatrix},
\]

where \(e^\pm(z), f^\pm(z), k_1^\pm(z)\) and \(k_2^\pm(z)\) are elements in \(U(R_q)\), with \(k_i^\pm(z)\) invertible. Ding-Frenkel’s main result guarantees that the Drinfeld realization is isomorphic to Reshetikhin and Semenov-Tian-Shanksy’s presentation \(^{8}\):

**Theorem 2.3.** There exists an algebra isomorphism

\[\phi : U_q(\widehat{\mathfrak{gl}_2}) \to U(R_q),\]

\[e(z) \mapsto e^+ (q^{-c/2}z) - e^- (q^{c/2}z),\]

\[f(z) \mapsto f^+ (q^{c/2}z) - f^- (q^{-c/2}z),\]

\[k_i^\pm(z) \mapsto k_i^\pm(z)\text{ and } q^\pm c \mapsto q^\pm c.\]

### 3 The \(H\)-algebra \(U_{q,x}(\widehat{\mathfrak{g}_2})\)

#### 3.1 \(H\)-algebras

We will need with the notions of \(H\)-algebras \((A, H, \mu_l, \mu_r)\), \(H\)-bialgebroids \((A, H, \mu_l, \mu_r, \Delta, \varepsilon)\) and \(H\)-Hopf algebroids \((A, H, \mu_l, \mu_r, \Delta, \varepsilon, S)\). We also need their dynamical tensor products \(A \hat{\otimes} B\). The definitions are adapted from \(^{4,27,30}\). It is worthwhile to observe that, when \(H = 0\), an \(H\)-Hopf algebroid becomes an ordinary Hopf algebra.

**Definition 3.1 (\(H\)-Algebra).** An associative algebra \(A\) over \(\mathbb{C}\) is an \(H\)-algebra if it has an \(H\)-bigrading such that \(A = \bigoplus_{\alpha,\beta \in H^*} A_{\alpha,\beta}\), along with the left and right moment maps, \(\mu_l, \mu_r : M_{H^*} \to A_{00}\) satisfying:

\[\mu_l(\hat{f}) x = x \mu_l(T_\alpha \hat{f}), \quad \mu_r(\hat{f}) x = x \mu_r(T_\beta \hat{f}), \quad a \in A_{\alpha,\beta}, \quad \hat{f} \in M_{H^*},\]

where \(T_\alpha\) denotes the automorphism \((T_\alpha \hat{f})(\lambda) = \hat{f}(\lambda + \alpha)\) of \(M_{H^*}\).

Consider two \(H\)-algebras \(A\) and \(B\). An \(H\)-algebra homomorphism \(\Psi\) between \(A\) and \(B\) is an algebra homomorphism which preserves the bigrading and moment maps:

\[\Psi(A_{\alpha,\beta}) \subseteq B_{\alpha,\beta}\text{ for all } \alpha, \beta \in H^*\text{ and }\]

\[\Psi(\mu_l^A(\hat{f})) = \mu_l^B(\hat{f}), \quad \Psi(\mu_r^A(\hat{f})) = \mu_r^B(\hat{f}).\]

\(^{2}\)For the exact correspondence to \(^{8}\) use \(L^{-1}, (k_1^\pm)^{-1}\) and \(-c\) in place of \(L, k_i^\pm\) and \(c\).
3.2 The Construction of $U_{q,x}(\hat{gl}_2)$

In the dynamical case, there are two main constructions known as the face and vertex models which are referred to in the literature as the $B$ and $A$ models, respectively. These algebras are both $U_q(\hat{sl}_2)$ as algebras, obtained via quasi-Hopf twists (twistors) as described explicitly in [23]. We will employ the Heisenberg algebra and trigonometric Drinfeld currents to construct the dynamical half-currents and define the dynamical quantum affine algebra, $U(R_{q,x}(\hat{gl}_2))$, which is of face type.

3.2.1 Heisenberg Algebra $\mathcal{H}$

Let us define a Heisenberg algebra $\mathcal{H}$ with generators $P$ and $Q$ such that $[P, Q] = -1$. Denote $H = \mathbb{C}P$, $H^* = \mathbb{C}Q$, $\mathcal{H} = H \oplus H^*$ with the pairing given by $<Q, P> = 1$ and $<x, y> = 0$ for all other $x, y$ (for example, we can choose $Q = \frac{\partial}{\partial P}$). Let $\bar{H}^* = \mathbb{Z}Q$. Consider the isomorphism $\Phi : \mathcal{Q} \rightarrow \bar{H}^*, e_{\alpha_1} \mapsto e^\alpha$. We will identify $\bar{H}^*$ with its group algebra $\mathbb{C}[\bar{H}^*]$ by $\alpha \mapsto e^\alpha$.

Just like in the elliptic case considered in [27], we identify $\hat{f} = f(P) \in \mathbb{C}[H]$ and meromorphic functions on $H^*$ by

$$\hat{f}(\mu) = f <\mu, P>, \quad \mu \in H^*$$

and consider the field of meromorphic functions $M_{H^*}$ on $H^*$

$$M_{H^*} = \left\{ \hat{f} : H^* \rightarrow \mathbb{C} \mid \hat{f} = f(P) \in \mathbb{C}[H] \right\}.$$

3.2.2 Definition of the $H$-algebra $U_{q,x}(\hat{gl}_2)$

Let $\mathbb{K} := \mathbb{C}[H]$ and define the $H$-algebra $U_{q,x}(\hat{gl}_2) := \mathbb{K}[U_q(\hat{gl}_2)] \otimes \mathbb{C}[\bar{H}^*]$. The moment maps are given by:

$$\mu_l(\hat{f}) = f(P + h), \quad \mu_r(\hat{f}) = f(P). \quad (3.1)$$

The $H$-bigrading is:

$$U_{q,x}(\hat{gl}_2) = \bigoplus_{\alpha, \beta \in H} U_{q,x}(\hat{gl}_2)_{\alpha, \beta},$$

$$U_{q,x}(\hat{gl}_2)_{\alpha, \beta} = \left\{ x \in U_{q,x}(\hat{gl}_2) \mid \begin{array}{l} q^{p+h}xq^{-\alpha, p+h} = q^{\alpha, p}>x, \\ q^p xq^{-\beta} = q^{\beta, p}>x \end{array} \right\}.$$
For $a, b$ in $\mathbb{C}[U_q(\mathfrak{sl}_2)]$, the multiplication in $U_{q,x}(\mathfrak{sl}_2)$ is defined through the expression

$$(f(P)a \otimes e^\alpha) \cdot (g(P)b \otimes e^\beta) = f(P)g(P + \alpha, P >)ab \otimes e^{\alpha + \beta}.$$  (3.2)

**Definition 3.2** (The Dynamical Currents of $U_{q,x}(\mathfrak{sl}_2)$).

$$E(u) = e(z)e^{2Q}, \quad F(u) = f(z),$$

$$K_1^i(u) = k_1^i(z)e^Q, \quad K_2^i(u) = k_2^i(z)e^{-Q},$$

$$H^\pm(u) = K_1^\pm(u)K_2^\pm(u)^{-1}.$$ (3.3)

**Proposition 3.3.** Let $\eta(a) := 1 - q^{2a}$, then the generators of $U_{q,x}(\mathfrak{sl}_2)$ satisfy:

$q^c$ : central,

$$[q^h, E(u)] = q^2E(u), \quad [q^h, F(u)] = q^{-2}F(u),$$

$$K_1^\pm(u)K_1^\mp(v) = K_1^\pm(v)K_1^\mp(u), \quad K_2^\pm(u)K_2^\mp(v) = K_2^\pm(v)K_2^\mp(u), \quad K_1^\pm(u)K_2^\pm(v) = K_2^\pm(v)K_1^\pm(u),$$

$$K_1^+(u)K_1^-(v) = \frac{\rho^+(u - v - \frac{3}{2})}{\rho^+(u - v + \frac{3}{2})}K_1^-(v)K_1^+(u), \quad K_2^+(u)K_2^-(v) = \frac{\rho^+(u - v - \frac{3}{2})}{\rho^+(u - v + \frac{3}{2})}K_2^-(v)K_2^+(u),$$

$$K_1^+(u)K_2^-(v) = \frac{\rho^-(u - v - \frac{3}{2})}{\rho^-(u - v + \frac{3}{2})}\eta(u - v - \frac{3}{2})\eta(u - v + \frac{3}{2} + 1)K_2^-(v)K_1^+(u),$$

$$K_2^+(u)K_1^-(v) = \frac{\rho^+(u - v + \frac{3}{2})}{\rho^+(u - v - \frac{3}{2})}\eta(u - v - \frac{3}{2})\eta(u - v + \frac{3}{2} + 1)K_1^-(v)K_2^+(u),$$

$$K_1^+(u)^{-1}E(v)K_1^+(u) = \frac{\eta(u - v + \frac{c}{4} - 1)}{\eta(u - v + \frac{c}{4})}E(v), \quad K_1^+(u)F(v)K_1^+(u)^{-1} = \frac{\eta(u - v + \frac{c}{4} - 1)}{\eta(u - v + \frac{c}{4})}F(v),$$

$$K_2^+(u)^{-1}E(v)K_2^+(u) = \frac{\eta(u - v + \frac{c}{4} + 1)}{\eta(u - v + \frac{c}{4})}E(v), \quad K_2^+(u)F(v)K_2^+(u)^{-1} = \frac{\eta(u - v + \frac{c}{4} + 1)}{\eta(u - v + \frac{c}{4})}F(v),$$

$$E(u)E(v) = \frac{q^{-2}\eta(u - v + 1)}{\eta(u - v - 1)}E(v)E(u), \quad F(u)F(v) = \frac{q^2\eta(u - v - 1)}{\eta(u - v + 1)}F(v)F(u),$$

$$[E(u), F(v)] = \frac{1}{q - q^{-1}}(\delta(q^{-c}\frac{z}{w})H^+(q^2w) - \delta(q^{-c}\frac{z}{w})H^-(q^2w)).$$

**Proof.** Straightforward. Using the definitions, one can easily verify the defining relations for $U_{q,x}(\mathfrak{sl}_2)$ given in the proposition.  

As suggested in Section (5) of [27], the elliptic half currents survive the degeneration $p \to 0$. For $U_{q,x}(\mathfrak{sl}_2)$, our definition of the dynamical trigonometric (total) currents in Definition 3.2 and the half-currents in 3.4 and 3.5 is based on this observation.
3.2.3 Half Currents

The elements $e^\pm(z)$ and $f^\pm(z)$ are not given explicitly in the definition of $U_q(\hat{\mathfrak{sl}}_2)$ in [8]. In the dynamical case, they can be conveniently expressed with the aid of the Heisenberg algebra, either as contour integrals of the total currents 3.2 or as Laurent series in $z$ with a modification in the zero Fourier modes $E_0$ and $F_0$. We will now give their definitions along with their RLL-type commutation relations.

**Definition 3.4. Positive Half-Currents as Integrals**

$$E^+(u) = q^{-1}a_1 \oint_{C_1} E(u') \frac{q^{2(P-1)}\eta(u-u'-c/4-P+1)}{\eta(u-u'-c/4)(P-1)} dz' \frac{1}{2\pi iz'},$$

$$F^+(u) = q^{-1}a_2 \oint_{C_2} F(u') \frac{\eta(u-u'+c/4+P+h-1)}{\eta(u-u'+c/4)\eta(P+h-1)} dz' \frac{1}{2\pi iz'}.$$

*Here the contours are chosen such that*

$$C_1 : 0 < |z'| < |q^{c/2}z|, \quad C_2 : 0 < |z'| < |q^{-c/2}z|,$$

*and the constants $a_1, a_2$ are chosen to satisfy $\frac{(a_1a_2)\eta(1)}{q^{-q^{-1}}} = 1$.*

For our purposes, it will sometimes be more convenient to do calculations with the series versions of these definitions. Having access to the individual modes gives us more flexibility and makes the dynamical structure more transparent. Note that whereas the total currents do not depend on $P$, the half-currents do.

The key is the Laurent expansion valid in the domain $|z| > 1$:

$$\frac{\eta(u+s)}{\eta(u)\eta(s)} = -\sum_{n \in \mathbb{Z}_{\geq 0}} \frac{1}{1-q^{-2s}\delta_{0,n}} z^{-n}. \quad (3.4)$$

An important consequence is the following relation:

$$\frac{\eta(u+s)}{\eta(u)\eta(s)} + \frac{\eta(-u-s)}{\eta(-u)\eta(-s)} = -\delta(q^{2u}), \quad (3.5)$$

with the first summand expanded around $z = \infty$ and the second around $z = 0$.

Using the first equation (3.4), Definitions 3.4 of the positive half-currents are easily reformulated and extended to their negative counterparts.
Definition 3.5. The Series Representation of Positive and Negative Half-Currents.

\[ E^+(u) = e^{2Q} q^{-1} a_1 \eta(1) \left( e_0 \frac{1}{1 - q^{-2(1-P)}} + \sum_{n>0} e_n (q^{-\xi} z)^{-n} \right), \]

\[ E^-(u) = -e^{2Q} q^{-1} a_1 \eta(1) \left( e_0 \frac{1}{1 - q^{2(1-P)}} + \sum_{n<0} e_n (q^{\xi} z)^{-n} \right), \]

\[ F^+(u) = -q^{-1} a_2 \eta(1) \left( f_0 \frac{1}{1 - q^{2(P+h-1)}} + \sum_{n>0} f_n (q^{\xi} z)^{-n} \right), \]

\[ F^-(u) = q^{-1} a_2 \eta(1) \left( f_0 \frac{1}{1 - q^{2(P+h-1)}} + \sum_{n<0} f_n (q^{-\xi} z)^{-n} \right), \]

here the positive (resp. negative) currents are expanded around \( \infty \) (resp. 0).

This yields the following decomposition of the total currents

\[ a_1 \eta(1) E(u) = E^+ \left( u + \frac{c}{4} \right) - E^- \left( u - \frac{c}{4} \right), \]

\[ -a_2 \eta(1) F(u) = F^+ \left( u - \frac{c}{4} \right) - F^- \left( u + \frac{c}{4} \right). \] (3.6)

Proposition 3.6 (Commutation relations for Positive Half-Currents).

\[ K_i^+(u_1) K_j^+(u_2) = K_j^+(u_2) K_i^+(u_1), \quad (i, j = 1, 2) \] (3.7)

\[ K_i^+(u_1)^{-1} E^+(u_2) K_1^+(u_1) = E^+(u_2) \frac{q \eta(u-1)}{\eta(u)} + E^+(u_1) \frac{q^{-1} \eta(1) \eta(P + u - 2)}{\eta(P-2) \eta(u)}, \] (3.8)

\[ K_i^+(u_1) F^+(u_2) K_1^+(u_1)^{-1} = F^+(u_2) \frac{q \eta(u-1)}{\eta(u)} + F^+(u_1) \frac{q^{2u-1} \eta(1) \eta(P + h - u)}{\eta(P + h) \eta(u)}, \] (3.9)

\[ K_2^+(u_1)^{-1} E^+(u_2) K_2^+(u_1) = E^+(u_2) \frac{q^{-1} \eta(u + 1)}{\eta(u)} - E^+(u_1) \frac{q^{-1} \eta(1) \eta(P + u)}{\eta(P) \eta(u)}, \] (3.10)

\[ K_2^+(u_1) F^+(u_2) K_2^+(u_1)^{-1} = F^+(u_2) \frac{q^{-1} \eta(u + 1)}{\eta(u)} - F^+(u_1) \frac{q^{2u-1} \eta(1) \eta(P + h - u - 2)}{\eta(P + h - 2) \eta(u)}, \] (3.11)

\[ \frac{q^{2u} \eta(1-u)}{\eta(u)} \left( E^+(u_1) E^+(u_2) + \frac{\eta(1+u)}{q \eta(u)} E^+(u_2) E^+(u_1) \right) = E^+(u_1)^2 \frac{\eta(1) \eta(P + u - 2)}{q \eta(u) \eta(P - 2)} + E^+(u_2)^2 \frac{q^{2u-1} \eta(1) \eta(P - u - 2)}{q \eta(u) \eta(P - 2)}, \] (3.12)

\[ \frac{q^{-1} \eta(1+u)}{\eta(u)} \left( F^+(u_1) F^+(u_2) + \frac{q^{2u-1} \eta(1-u)}{\eta(u)} F^+(u_2) F^+(u_1) \right) = F^+(u_1)^2 \frac{q^{2u-1} \eta(1) \eta(P + h - u - 2)}{\eta(P + h - 2) \eta(u)} + F^+(u_2)^2 \frac{q^{-1} \eta(1) \eta(P + h + u - 2)}{\eta(P + h - 2) \eta(u)}, \] (3.13)

\[ \left[ E^+(u_1), F^+(u_2) \right] = q^{2u} \left( q^{-1} - q \right) \left( K_2^+(u_2) K_1^+(u_2) \frac{\eta(P - u - 1)}{\eta(u) \eta(P - 1)} - K_2^+(u_1) K_1^+(u_1) \frac{\eta(P + h - u - 1)}{\eta(u) \eta(P + h - 1)} \right). \] (3.14)
Proof.

The relations (3.7) are direct consequences of the definitions of \( K_i^+(u) \), for \( i=1,2 \). The remaining equations can be grouped into three types:

(i) (3.8) - (3.11). By using the defining relations in Proposition 3.3 along with the contour integral definitions of the half-currents 3.4 and the following identity, one can verify the commutation relations for \( E^+(u_2) \) or \( F^+(u_2) \) with \( K_i^+(u_1) \):

\[
\frac{\eta(u_1 - u_2 + t)\eta(u_2 + s)}{\eta(u_1)\eta(u_2)\eta(s)} = \frac{\eta(u_1 - u_2 + t)\eta(u_2 + s + t)}{\eta(u_1 - u_2)\eta(u_2)\eta(s + t)} + \frac{\eta(u_2 - u_1 + s)\eta(u_1 + s + t)\eta(t)}{\eta(u_2 - u_1)\eta(u_1)\eta(s)\eta(s + t)}.
\]

(ii) Relations (3.12) and (3.13). Symmetrize the integration variables in the definition of the half-currents and use the fact (eqn.(4.16) in [22]) that the following expression remains unchanged on interchanging \( u_1 \) and \( u_2 \) only:

\[
\frac{\eta(u_1 - u_2 - t)\eta(u_1 - u'_1 + s + t)\eta(t)\eta(u_2 - u'_2 + s - t)\eta(t)}{\eta(u_1 - u_2)\eta(u_1 - u'_1)\eta(s + t)\eta(u_2 - u'_2)\eta(s - t)}
\]

\[
-\frac{\eta(u_2 - u_1 + s + t)\eta(t)\eta(u_1 - u'_1 + s + t)\eta(t)\eta(u_2 - u'_1 + s - t)\eta(t)}{\eta(u_2 - u_1)\eta(u_1 - u'_1)\eta(s + t)\eta(u_2 - u'_2)\eta(s - t)}
\]

\[
+\frac{\eta(u_1 - u_2 - t)\eta(u_1 - u'_2 + s + t)\eta(t)\eta(u_2 - u'_1 + s - t)\eta(t)}{\eta(u_1 - u_2)\eta(u_1 - u'_2)\eta(s + t)\eta(u_2 - u'_1)\eta(s - t)}
\]

\[
-\frac{\eta(u_2 - u_1 + s + t)\eta(t)\eta(u_1 - u'_2 + s + t)\eta(t)\eta(u_2 - u'_1 + s - t)\eta(t)\eta(u'_2 - u'_1 + t)}{\eta(u_2 - u_1)\eta(u_1 - u'_2)\eta(s + t)\eta(u_1 - u'_1)\eta(s - t)\eta(u'_2 - u'_1 - t)}.
\]

(iii). Relation (3.14). Finally, use Definition 3.5 for the positive half-currents series to obtain the \([E^+(u_1), F^+(u_2)]\) relation. Write the left hand side as

\[
[E^+(u_1), F^+(u_2)] = e^{2q} a_1 a_2 \eta[1] \left( e_0 \frac{q^{2(p-1)}}{1 - q^{2(p-1)}} + \sum_{n=0}^{\infty} e_n \frac{q^{2(P+h-1)}}{1 - q^{2(P+h-1)}} + \sum_{n=1}^{\infty} f_n (q^2 z_2)^{-n} \right).
\]

Expand the four terms on the right side, using the commutation relation between \( e_n \).
and \( f_n \) in the defining relation \([2.1]\):

\[
\begin{align*}
\sum_{n=0}^{\infty} e_n(q^{\frac{\xi}{2}}z_1)^{-n}, \sum_{n=1}^{\infty} f_n(q^{\frac{\xi}{2}}z_2)^{-n} &= \frac{1}{q-q^{-1}} \frac{1}{1-q^{2(u_2-u_1)}} (\psi(z_2) - \psi(z_1)) ,
\end{align*}
\]

\[
\begin{align*}
\sum_{n=0}^{\infty} e_n(q^{-\frac{\xi}{2}}z_1)^{-n}, f_0 q^{-2(P+h-1)} & = \frac{1}{q-q^{-1}} \frac{q^{2(P-1)}}{1-q^{2(P-1)}} \frac{1}{1-q^{-2(P+h-1)}} (q^h - q^{-h}) ,
\end{align*}
\]

\[
\begin{align*}
\sum_{n=0}^{\infty} e_n(q^{-\frac{\xi}{2}}z_1)^{-n}, f_0 q^{-2(P+h-1)} & = \frac{1}{q-q^{-1}} \frac{1}{1-q^{-2(P+h-1)}} (\psi(z_1) - q^{-h}) ,
\end{align*}
\]

\[
\begin{align*}
\sum_{n=1}^{\infty} e_n(q^{\xi}z_2)^{-n}, f_0 q^{2(P-1)} & = \frac{1}{q-q^{-1}} \frac{q^{2(P-1)}}{1-q^{2(P-1)}} (\psi(z_2) - q^h) .
\end{align*}
\]

Now use these four expressions in Equation \((3.16)\), along with the formula:

\[
\frac{1}{\eta(x)} - \frac{1}{\eta(y)} = \frac{\eta(x-y)}{\eta(x)\eta(-y)} . \tag{3.17}
\]

\[\square\]

**Remark.** We do not solve the commutation relations for the negative half-currents because we will not explicitly need them.

## 4 Definition of \( U(R_{q,x}(\hat{q}_2)) \) and the Main Theorem

### 4.1 Definition of \( U(R) \)

We will use the following presentation of the dynamical affine \( R \)-matrix, obtained as the degeneration of the elliptic \( R \)-matrix as \( p \to 0 \) (see [20], [27], also eq(5.9) and the remark below it, in [2]):

\[
R^\pm(u, P) = \rho^\pm(z) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\eta(P+1)\eta(P-1)\eta(u)}{\eta(\eta(P))} & \frac{\eta(1)\eta(P+u)}{\eta(u+1)\eta(P)} & 0 \\
0 & \frac{\eta(1)\eta(P-u)\eta(2u)}{\eta(u+1)\eta(P)} & \frac{\eta(u)\eta(P)}{\eta(u+1)} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
= \rho^\pm(z) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1-q^{2(P-1)}}{1-q^{2(P+1)}} & \frac{1-q^{2(P+1)}}{1-q^{2(P-u)}} & 0 \\
0 & \frac{1-q^{2(P-u)}}{1-q^{2(P+1)}} & \frac{1-q^{2(P+1)}}{1-q^{2(u+1)}} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} . \tag{4.1}
\]
Here, the coefficient is given by
\[
\rho^+(z) = \frac{\sqrt{q}(z^{-1}; q^4 \infty)(q^4 z^{-1}; q^4 \infty)}{(q^2 z^{-1}; q^4 \infty)}, \quad (a; x)_{\infty} = \prod_{k=0}^{\infty}(1 - ax^k)
\] (4.2)
and \(\rho^- (z) = (\rho^+(z^{-1}))^{-1}\), which expands as
\[
\rho^- (z) = \frac{q^{-\frac{1}{2}}(q^2 z; q^4)_{\infty}^2}{(z; q^4)_{\infty} (q^4 z; q^4)_{\infty}}.
\] (4.3)

Observe that the quantum affine \(R\)-matrix
\(R_q(z)\) in Section 2.2 is the degeneration of \(R^+(u, P)\) as \(P \to \infty\). It is easily verified that the \textit{unitarity} condition (eq(4.9) in [8]) continues to hold in the dynamical case:
\[
R^+_2 (u, P)^{-1} = R^- (-u, P),
\]
where the inverted \(R\)-matrix is
\[
R^+ (u, P)^{-1} = \rho^+(z)^{-1} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\eta(u)}{q\eta(u-1)} & -\frac{\eta(1)\eta(P+u)}{q^2\eta(u-1)\eta(P)} & 0 \\
0 & \frac{\eta(1)\eta(P-u)}{\eta(1-u)\eta(P)} & \frac{\eta(u)\eta(P-1)\eta(P+1)}{q\eta(u-1)\eta(P)} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\] (4.4)

As mentioned in the introduction, Arnaoudon et al. [2] suggest a definition of \(U_{q,\lambda}(\hat{sl}_2)\) based on \(RLL\) relations. Besides being motivated by those considerations, the definition presented in this treatise is natural, from the location of \(R_q(z)\) (resp. \(U_{q,x}(\hat{sl}_2)\)) in the cladistics of quantum \(R\)-matrices (resp. quantum algebras) as illustrated in Figure 1.1) (resp. Figure 1.2). We can extend the Heisenberg algebra for \(U_{q,x}(\hat{sl}_2)\) to include the element \(h\) as follows:

**Definition 4.1.** Heisenberg algebra for \(U(R)\). Let
\[
H = \mathbb{C}P + \mathbb{C}(P + h), \quad H_Q^* = \sum Q, \quad H^* = \eta^* + H_Q^*
\]
We identify \(\hat{f} = f(P, P + h) \in \mathbb{C}[H]\) and meromorphic functions on \(H^*\) via
\[
f(P, P + h)(\xi) = f(< P, \xi >, < P + h, \xi >), \quad \xi \in H^*
\] (4.5)

\[\text{The factors } \rho^\pm(z) \text{ are arbitrary and are set to 1 in } R_q(z)\].
Definition 4.2. $U(R) := U(R_{q,x}(\widehat{gl}_2))$ is the algebra over $M_{H^*}$ with generators $L^\pm(u)$ given by $L^\pm(u) = (L^\pm_{ab}(u))_{a,b=1}^2$, with $L^\pm_{ab}(u) = \sum_{n=0}^\infty L_{ab,\pm n}q^{\mp 2un}$ and the dynamical affine RLL-relations:

$$R^{\pm(12)}(u, P + h)L^{\pm(1)}(u_1) L^{\pm(2)}(u_2) = L^{\pm(2)}(u_2) L^{\pm(1)}(u_1) R^{\pm(12)}(u, P),$$

$$R^{\pm(12)}\left(u \pm \frac{c}{2}, P + h\right) L^{\pm(1)}(u_1) L^{\pm(2)}(u_2) = L^{\pm(2)}(u_2) L^{\pm(1)}(u_1) R^{\pm(12)}\left(u + \frac{c}{2}, P\right).$$

(4.6)

Lemma 4.3. $U(R)$ is an $H$-algebra with the $H$-bigrading defined by

$$U(R) = \bigoplus_{\alpha, \beta \in H} U(R)_{\alpha, \beta} \quad \text{with} \quad U(R)_{\alpha, \beta} = \left\{ x \in U(R) \mid q^{P+h}xq^{-(P+h)} = q^{<\alpha,P>x}, q^P x q^{-P} = q^{<\beta,P>x} \right\},$$

and moment maps

$$\mu_t(f) = f(P + h), \quad \mu_r(f) = f(P), \quad f \in M_{H^*} \quad (4.7)$$

Proof. A straightforward verification.

Remark. The action on the generators is given by

$$f(P + h)L^\pm_{ab}(u) = L^\pm_{ab}(u)f(P + h - w(a)),$$

$$f(P)L^\pm_{ab}(u) = L^\pm_{ab}(u)f(P - w(b)).$$

(4.8)

where the weight function $w : \{1, 2\} \rightarrow \{\pm 1\}$ is given by identifying

$$L^\pm(u) \rightarrow \begin{pmatrix} L_{++} & L_{+-} \\ L_{-+} & L_{--} \end{pmatrix}$$

(4.9)

We also define the subalgebras $U(R^\pm) := U(R^{\pm}_{q,x}(\widehat{gl}_2))$. The expansion of the RLL-relations [4.6] is given in Appendix A.

For proving our main result it will be convenient to summarize the left action of $P$ and $P + h$ on the half-currents:

|   | $e^Q$ | $E(u), E^\pm(u)$ | $F(u), F^\pm(u)$ | $K_1^\pm(u)$ | $K_2^\pm(u)$ | $H^+(z)$ | $H^-(z)$ |
|---|---|---|---|---|---|---|---|
| $P$ | $P + h - 1$ | $P + h$ | $P + h - 2$ | $P + h - 1$ | $P + h + 1$ | $P + h - 2$ | $P + h - 2$ |
| $P + h$ | $P + h$ | $P + h - 2$ | $P + h - 1$ | $P + h + 1$ | $P + h - 2$ | $P + h - 2$ |

For example, the third column yields $f(P)E(u) = E(u)f(P - 2)$, $f(P + h)E(u) = E(u)f(P + h)$, $f(P)E^\pm(u) = E^\pm(u)f(P - 2)$ and $f(P + h)E^\pm(u) = E^\pm(u)f(P + h)$. 

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4.2 Isomorphism of \( H \)-Algebras \( U_{q,x}(\widehat{\mathfrak{gl}}_2) \simeq U(R_{q,x}(\widehat{\mathfrak{gl}}_2)) \).

We now prove our main result, an extension of the Ding-Frenkel type isomorphism (see Subsection 2.2) to the dynamical case.

**Theorem 4.4** (Main Theorem). There exists a unique Gauss decomposition of \( L^\pm(u) \):

\[
L^\pm(u) = \begin{pmatrix}
1 & F^\pm(u) \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
K^\pm_1(u) & 0 \\
0 & K^\pm_2(u)
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
E^\pm(u) & 0
\end{pmatrix}.
\]

(4.10)

\[
= \begin{pmatrix}
K^\pm_1(u) + F^\pm(u)K^\pm_2(u)E^\pm(u) & F^\pm(u)K^\pm_2(u) \\
K^\pm_2(u)E^\pm(u) & K^\pm_2(u)
\end{pmatrix},
\]

(4.11)

which yields an an isomorphism of \( M_{H^*} \)-algebras

\[
\Phi : U_{q,x}(\widehat{\mathfrak{gl}}_2) \longrightarrow U(R_{q,x}(\widehat{\mathfrak{gl}}_2)), \quad \text{defined by}
\]

\[
E(u) \mapsto E^+\left(u + \frac{c}{4}\right) - E^-\left(u - \frac{c}{4}\right),
\]

\[
F(u) \mapsto F^+\left(u - \frac{c}{4}\right) - F^-\left(u + \frac{c}{4}\right),
\]

\[
K^\pm_i(u) \mapsto K^\pm_i(u), \quad q^\pm c \mapsto q^\pm c.
\]

**Proof.** The \( H \)-algebra structure is defined the same way for both algebras. It is easy to see that the \( H \)-bigrading and moment maps are preserved by \( \Phi \), using the Gauss decomposition of \( L^\pm(u) \) (4.11), Lemma 4.3 and the table following Lemma 4.3. Hence \( \Phi \) is an \( H \)-algebra homomorphism.

We will need the inverse of the \( L \)-operator, which is obtained using the Gauss decomposition, as:

\[
(L^\pm(u))^{-1} = \begin{pmatrix}
K^{-\pm}_1(u)^{-1} & -K^\pm_1(u)^{-1}F^\pm(u) \\
-E^\pm(u)K^\pm_1(u)^{-1} & K^\pm_2(u)^{-1} + E^\pm(u)K^\pm_1(u)^{-1}F^\pm(u)
\end{pmatrix}.
\]

(4.12)

The verification of surjectivity reduces to checking that the \( RLL \) relations (4.6) imply the defining relations (3.3). Two key differences from Theorem 2.3 are that \( K^+_i(u) \) and \( K^-_i(u) \) do not commute, for \( i = 1 \) and \( 2 \), and that the entries in the \( L^+(u) \)-matrix (the half-currents) do not commute with the entries of the \( R \)-matrix.
The inverted $RLL$-relations $4.6$ become:

\[
L^{\pm(1)}(u_1)^{-1} R^{+(12)}(u_1 - u_2, P + h)^{-1} L^{\pm(2)}(u_2) = L^{\pm(2)}(u_2) R^{+(12)}(u_1 - u_2, P)^{-1} L^{\pm(1)}(u_1)^{-1}, 
\]

(4.13)

\[
L^{-(1)}(u_1)^{-1} R^{-(12)}\left(u - \frac{c}{2}, P + h\right)^{-1} L^{+(2)}(u_2) = L^{+(2)}(u_2) R^{-(12)}\left(u + \frac{c}{2}, P\right)^{-1} L^{-(1)}(u_1)^{-1}, 
\]

(4.14)

\[
L^{+(1)}(u_1)^{-1} R^{+(12)}\left(u + \frac{c}{2}, P + h\right)^{-1} L^{-(2)}(u_2) = L^{-(2)}(u_2) R^{+(12)}\left(u - \frac{c}{2}, P\right)^{-1} L^{+(1)}(u_1)^{-1}, 
\]

(4.15)

\[
R^{+(12)}\left(u + \frac{c}{2}, P + h\right)^{-1} L^{-(2)}(u_2) L^{+(1)}(u_1) = L^{+(1)}(u_1) L^{-(2)}(u_2) R^{+(12)}\left(u - \frac{c}{2}, P\right)^{-1}, 
\]

(4.16)

\[
L^{\pm(2)}(u_2)^{-1} L^{\pm(1)}(u_1)^{-1} R^{+(12)}(u, P + h)^{-1} = R^{+(12)}(u, P)^{-1} L^{\pm(1)}(u_1)^{-1} L^{\pm(2)}(u_2)^{-1}, 
\]

(4.17)

\[
L^{+(2)}(u_2)^{-1} L^{-(1)}(u_1)^{-1} R^{-(12)}\left(u - \frac{c}{2}, P + h\right)^{-1} = R^{-(12)}\left(u + \frac{c}{2}, P\right)^{-1} L^{-(1)}(u_1)^{-1} L^{+(2)}(u_2)^{-1}. 
\]

(4.18)

(i). The formulae between the $K_i^\pm(u)$ are easily obtained by directly equating the required matrix entries and using the definition of $\rho^z(u)$ in (4.2).

(ii). Let us prove the relation between $K_1^+(u_1)$ and $E(u_2)$, for $i = 1, 2$. Use the relation between $K_1^+(u_1)$ and $K_2^-(u_2)$ in the following relation $4.16$.

\[
K_1^+(u_1)^{-1} q^{-1} \eta(u + \frac{c}{2})(\rho^+)^{-1}(u + \frac{c}{2}) K_2^-(u_2) E^-(u_2) = K_2^-(u_2)^{-1} q^{-2} \eta(u - \frac{c}{2}) \eta(P) \eta(u + \frac{c}{2} - 1) E^+(u_1) K_1^+(u_1)^{-1}
\]

\[
+ K_2^-(u_2)^{-1} E^-(u_2)(\rho^+)^{-1}(u - \frac{c}{2}) K_1^+(u_1)^{-1},
\]

which implies that

\[
K_1^+(u_1)^{-1} E^-(u_2) K_1^+(u_1) = \frac{q^{-1} \eta(u + \frac{c}{2}) \eta(P + u - \frac{c}{2})}{\eta(P) \eta(u + \frac{c}{2} - 1)} E^+(u_1) + \frac{q \eta(u - \frac{c}{2})}{\eta(u - \frac{c}{2} - 1)} E^-(u_2).
\]

\[
\text{The positioning of the } R\text{-matrix elements is important because they do not commute with the half currents.}
\]
Similarly, from the first set of relations in Proposition (A.1), it follows that

$$-E^+(u_2)K_1^+(u_2)^{-1}K_1^+(u_1)^{-1} = \frac{q^{-2} \eta(1) \eta(P + u)}{\eta(P) \eta(u - 1)} E^+(u_1)K_1^+(u_1)^{-1}K_1^+(u_2)^{-1} - \frac{q^{-1} \eta(u)}{\eta(u - 1)} K_1^+(u_1)^{-1}E^+(u_2)K_1^+(u_2)^{-1},$$

from which we get

$$K_1^+(u_1)^{-1}E^+(u_2)K_1^+(u_1) = \frac{q^{-1}(1) \eta(P + u)}{\eta(P) \eta(u)} E^+(u_1) + \frac{q \eta(u - 1)}{\eta(u)} E^+(u_2).$$

Hence we conclude that

$$K_1^+(u_1)^{-1}E(u_2)K_1^+(u_1) = \frac{q \eta(u - \frac{c}{4} - 1)}{\eta(u - \frac{c}{4})} E(u_2),$$

as required. Start with the following expression which we obtain from (A.14)

$$K_2^+(u_1)K_2^+(u_2)E^+(u_2) = K_2^+(u_2)E^+(u_2)K_2^+(u_1) \frac{q \eta(u)}{\eta(u + 1)} + K_2^+(u_2)K_2^+(u_1)E^+(u_1) \frac{\eta(1) \eta(P + u)}{\eta(P) \eta(u + 1)}$$

$$\Rightarrow K_2^+(u_1)^{-1}E^+(u_2)K_2^+(u_1) = -E^+(u_1) \frac{\eta(1) \eta(P + u)}{q \eta(P) \eta(u)} + E^+(u_2) \frac{\eta(u + 1)}{\eta(u)}.$$

The corresponding entry in the second equation from (4.6) reads (A.29):

$$\rho^+(u + \frac{c}{2})K_2^+(u_1)K_2^-(u_2)E^-(u_2) = K_2^-(u_2)E^-(u_2)K_2^+(u_1) \frac{q \eta(u - \frac{c}{2} + P)}{\eta(u - \frac{c}{2} + 1)} +$$

$$K_2^-(u_2)K_2^+(u_1)E^+(u_1) \eta(1) \eta(u - \frac{c}{2} + P) \frac{\rho^+(u - \frac{c}{2})}{\eta(u)}$$

$$\Rightarrow K_2^+(u_1)^{-1}E^-(u_2)K_2^+(u_1) = -E^+(u_1) \frac{\eta(1) \eta(P + u - \frac{c}{2})}{q \eta(P) \eta(u - \frac{c}{2})} + E^-(u_2) \frac{\eta(u - \frac{c}{2} + 1)}{\eta(u - \frac{c}{2})}.$$

Hence, we conclude that

$$K_2^+(u_1)^{-1}E(u_2)K_2^+(u_1) = q^{-1} \eta(u - \frac{c}{4} + 1) \frac{\eta(u - \frac{c}{4})}{\eta(u - \frac{c}{4})} E(u_2),$$

as required.

Since the RLL relations are symmetric in c, the relations for $K_i^-(u)$ with $E(u)$ can be obtained by choosing the ”negative” counterparts of the expressions in the preceding proof for $K_i^+(u)$. This reduces to just replacing c by $-c$. Similarly, for the proof of $F(u)$
with $K_2^\pm(u)$, use the following expressions:

$$\begin{align*}
K_2^\pm(u_1)F^\pm(u_2)K_2^\pm(u_1)^{-1} &= -\frac{q^{2u-1}\eta(1)\eta(P + h - u)}{\eta(P + h)\eta(u)}F^\pm(u_1) + \frac{q^{-1}\eta(u + 1)}{\eta(u)}F^\pm(u_2), \\
K_2^\pm(u_1)F^\mp(u_2)K_2^\pm(u_1)^{-1} &= -\frac{q^{2u+1}\eta(1)\eta(P + h - u + \frac{c}{2})}{\eta(P + h)\eta(u + \frac{c}{2})}F^\mp(u_1) + \frac{q^{-1}\eta(u + \frac{c}{2} + 1)}{\eta(u + \frac{c}{2})}F^\pm(u_2)
\end{align*}$$

(4.19)

To obtain the required relation. The verification of $F(u)$ with $K_1^\pm(u)$ is the same.

(iii). We prove the relation between $F(u_1)$ and $F(u_2)$. The proof for $E(u_1)$ with $E(u_2)$ is similar. Start with the 2 pairs of equalities from (4.6):

$$\begin{align*}
F^\pm(u_1)K_2^\pm(u_1)F^\pm(u_2)K_2^\pm(u_2) &= F^\pm(u_2)K_2^\pm(u_2)F^\pm(u_1)K_2^\pm(u_1), \\
F^\pm(u_2)K_2^\pm(u_2)F^\mp(u_1)K_2^\mp(u_1) &= F^\mp(u_1)K_2^\mp(u_1)F^\pm(u_2)K_2^\pm(u_2).
\end{align*}$$

(4.20)

Simplifying these using the four expressions (4.19) yields 4 equations:

$$\begin{align*}
&\frac{q^{-1}\eta(1 + u)}{\eta(u)} F^\pm(u_1) F^\pm(u_2) + \frac{q^{2u-1}\eta(1 - u)}{\eta(u)} F^\pm(u_2) F^\pm(u_1) \\
&= F^\pm(u_1)^2 \frac{q^{2u-1}\eta(1)\eta(P + h - u - 2)}{\eta(P + h - 2)\eta(u)} + F^\pm(u_2)^2 \frac{q^{-1}\eta(1)\eta(P + h + u - 2)}{\eta(P + h - 2)\eta(u)}
\end{align*}$$

$$\begin{align*}
&\frac{q^{-1}\rho(u + \frac{c}{2})\eta(u + \frac{c}{2} + 1)}{\rho(u + \frac{c}{2})\eta(u + \frac{c}{2})} F^\pm(u_1) F^\pm(u_2) - \frac{q^{-1}\eta(1 - u + \frac{c}{2})}{\eta(-u + \frac{c}{2})} F^\pm(u_2) F^\pm(u_1) \\
&= F^\pm(u_1)^2 \frac{\rho^+(u + \frac{c}{2})q^{2u+1}\eta(1)\eta(P + h - u + \frac{c}{2} - 2)}{\rho^+(u + \frac{c}{2})\eta(P + h - 2)\eta(u + \frac{c}{2})} \\
&\quad - F^\pm(u_2)^2 \frac{q^{-2u+1}\eta(1)\eta(P + h + u + \frac{c}{2} - 2)}{\eta(P + h - 2)\eta(u + \frac{c}{2})}.
\end{align*}$$

Use these four in the expansion of

$$\begin{align*}
[F^+(u_1 - \frac{C}{4}) - F^-(u_1 + \frac{C}{4}), F^+(u_2 - \frac{C}{4}) - F^-(u_2 + \frac{C}{4})],
\end{align*}$$

to yield that $[F(u_1), F(u_2)]$ satisfies the required result.

(iv). Finally, recalling the identity given in (3.5), the $[E(u_1), F(u_2)]$ relation is proven by using the two relations obtained from the first set in Proposition A.1

$$\begin{align*}
[E^\pm(u_1), F^\pm(u_2)] &= q^{2u}(q^{-1} - q) \left( K_2^\pm(u_2)^{-1}K_1^\pm(u_2) \frac{\eta(P + h - u - 1)}{\eta(u)\eta(P - 1)} \\
&\quad - K_2^\pm(u_1)^{-1}K_1^\pm(u_1) \frac{\eta(P + h - u - 1)}{\eta(u)\eta(P + h - 1)} \right),
\end{align*}$$

(4.21)
and the following two expressions:

\[
[E^+(u_1), F^-(u_2)] = K_2^-(u_2)^{-1}K_1^-(u_2) \frac{q^{2(u-\frac{c}{2})-1}\eta(1)\eta(P-u+\frac{c}{2}-1)}{\eta(u-\frac{c}{2})\eta(P-1)}
\]

\[-K_2^+(u_1)^{-1}K_1^+(u_1) \frac{q^{2(u+\frac{c}{2})-1}\eta(1)\eta(P-\frac{c}{2}+h-u-1)}{\eta(u+\frac{c}{2})\eta(P+h-1)}, (4.22)\]

\[
[E^-(u_1), F^+(u_2)] = K_2^-(u_2)^{-1}K_1^-(u_2) \frac{q^{2(u+\frac{c}{2})-1}\eta(1)\eta(P-u-\frac{c}{2}-1)}{\eta(u+\frac{c}{2})\eta(P-1)}
\]

\[-K_2^+(u_1)^{-1}K_1^+(u_1) \frac{q^{2(u-\frac{c}{2})-1}\eta(1)\eta(P+\frac{c}{2}+h-u-1)}{\eta(u-\frac{c}{2})\eta(P+h-1)}. (4.23)\]

This completes the verification that the map \( \Phi \) in the theorem is surjective.

The proof of the injectivity of \( \Phi \) is analogous to \([8]\) for \( \phi: U_q(\widehat{su}_2) \rightarrow U(\mathfrak{g}) \). We view \( U_q(\widehat{su}_2) \) and \( U(R_q(\widehat{su}_2)) \) as the degenerations as \( x \rightarrow 0 \) of \( U_q,x(\widehat{su}_2) \) and \( U(R) \), respectively. Consider the representation \( (\pi_{\lambda,c}^x, V_{\lambda,c}) \) of \( U_q,x(\widehat{su}_2) \) induced from an integrable representation \( (\pi_{\lambda,c}^q, V_{\lambda,c}) \) of \( U_q(\widehat{su}_2) \), with level \( c \) (we can identify \( U_q(\widehat{su}_2) \otimes \mathbb{C}[[\hat{H}^*]] \) with \( U_q(\widehat{su}_2) \) and \( U(R_q) \otimes \mathbb{C}[[\hat{H}^*]] \) with \( U(R_q) \), because \( P \)-independence implies that \( Q \) is central). Since \( \Phi \) is a homomorphism of \( H \)-algebras, we get the commutative diagram:

\[
\begin{array}{cccc}
U_q,x/xU_q,x & \xrightarrow{\Phi} & U(R)/xU(R) \\
\cong & & \cong \\
U_q \otimes \mathbb{C}[[\hat{H}^*]] & \xrightarrow{\phi \otimes Id} & U(R_q) \otimes \mathbb{C}[[\hat{H}^*]] \\
\cong & & \cong \\
U_q & \xrightarrow{\phi} & U(R_q) & \xrightarrow{\pi_{\lambda,c}^q} & \text{End}(V_{\lambda,c})
\end{array}
\]

Hence \( \ker \Phi \subseteq xU_q,x(\widehat{su}_2) \). Clearly, \( \ker \Phi \subseteq \bigcap_{\lambda} \ker \pi_{\lambda,c}^x \circ \Phi \). Now, it is well-known from the representation theory of \( U_q(\widehat{sl}_n) \), that \( \bigcap_{\lambda} \ker \pi_{\lambda,c}^q = \{0\} \) (see eq(5.73) in \([8]\)). Since \( \ker \pi_{\lambda,c}^x \subseteq \ker \pi_{\lambda,c}^q \), it follows that \( \ker \Phi = \{0\} \), which completes the verification that \( \Phi \) is injective.

**Remark.** The pair of relations (4.22) and (4.23) can be directly verified (the proof is similar to 3.14). This indicates their consistency with the Definition 3.5 of negative half-currents.
5 \textit{H-Hopf Algebroid Structures}

Let $A$ be a $H$-algebra \cite{3.1}. We now recall the definition of $H$-bialgebroid and $H$-Hopf algebroid structures on $A$. Start with the dynamical tensor product:

**Definition 5.1** (Tensor Product of $H$-algebras $A \tilde{\otimes} B$). The tensor product of $A$ and $B$ denoted by $A \tilde{\otimes} B$ is the $H^*$ bigraded vector space with

$$(A \tilde{\otimes} B)_{\alpha \beta} = \bigoplus_{\gamma \in H^*} (A_{\alpha \gamma} \otimes_{M_{H^*}} B_{\gamma \beta}),$$

where $\otimes_{M_{H^*}}$ is the usual tensor product $\otimes$ modulo the relation:

$$\mu_r(f) a \tilde{\otimes} b = a \tilde{\otimes} \mu_l(f) b.$$ (5.1)

We will need a certain $H$-algebra of automorphisms that plays the role of the unit object:

**Definition 5.2** (Algebra of Shift Operators $D$). Let $D = \{ \sum \hat{f}_k T_{\beta_k} | \hat{f}_k \in M_{H^*}, \beta_k \in H^* \}$.

The bigrading and moment maps on $D$ are given as:

$$D_{\alpha \alpha} = \left\{ \sum \hat{f} T_{\alpha} | \hat{f} \in M_{H^*}, \alpha \in H^* \right\}, \quad D_{\alpha \beta} = 0 \text{ if } \alpha \neq \beta$$

$$\mu_l(\hat{f}) = \mu_r(\hat{f}) = \hat{f} T_{00}.$$

**Remark.** Note that $a \cong a \tilde{\otimes} T_{-\beta} \cong T_{-\alpha} \tilde{\otimes} a$ for all $a \in A_{\alpha \beta}$, proving the canonical isomorphism

$$D \tilde{\otimes} A \simeq D \simeq A \tilde{\otimes} D,$$ (5.2)

where the tensor product $\tilde{\otimes}$ is the usual $\otimes$ modulo the relation

$$f(u, P) a \tilde{\otimes} b = a \tilde{\otimes} f(u, P + h) b.$$ (5.3)

We will write the $R$ matrix elements in $M_{H^*}$ as $(\hat{R}^+ )_{ab}^{xy} \equiv R^+(u, P)_{ab}^{xy}$, where $a, b, x, y \in \{1, 2\}$ (the inner subscript $(u)$ will be omitted when the context is clear). We then have

$$\mu_l((R^+ )_{ab}^{xy}) = R^+(u, P + h)_{ab}^{xy}, \quad \mu_r((R^+ )_{ab}^{xy}) = R^+(u, P)_{ab}^{xy}$$ (5.4)
Definition 5.3 (H-Bialgebroid). An H-Hopf Algebroid \((A, H, \Delta, \varepsilon)\) is an H-algebra with the comultiplication and counit maps:

\[ \Delta : A \to A \tilde{\otimes} A, \quad \varepsilon : A \to \mathcal{D}, \]

which are required to satisfy the compatibility conditions:

\[ (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta, \]

\[ (\varepsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varepsilon) \circ \Delta = \text{id}. \] (5.5)

Definition 5.4 (H-Hopf algebroid). An H-bialgebroid \(A\) is an H-Hopf algebroid \((A, H, \Delta, \varepsilon, S)\) if there is an algebra antihomomorphism \(S : A \to A\) satisfying the compatibility conditions:

\[ S(\mu_l(\hat{f})) = \mu_l(\hat{f}), \quad S(\mu_r(\hat{f})) = \mu_r(\hat{f}), \]

\[ m \circ (\text{id} \otimes S) \circ \Delta(x) = \mu_l(\varepsilon(x)1), \quad \forall x \in A, \] (5.6)

\[ m \circ (S \otimes \text{id}) \circ \Delta(x) = \mu_r(T_\alpha(\varepsilon(x)1)), \quad \forall x \in A_{\alpha\beta}. \] (5.7)

5.1 \(H\)-Hopf Algebroid Structure on \(U(R)\)

Let us define the \(H\)-bialgebroid maps for the \(H\)-algebra \(U(R)\).

Lemma 5.5. The Comultiplication \(\Delta : U(R^\pm) \to U(R^\pm) \tilde{\otimes} U(R^\pm)\) is given by

\[ \Delta(\mu_l(\hat{f})) = \mu_l(\hat{f}) \otimes 1, \quad \Delta(\mu_r(\hat{f})) = 1 \otimes \mu_r(\hat{f}), \] (5.8)

\[ \Delta(e^Q) = e^Q \otimes e^Q, \quad \Delta(L^\pm_{ab}(u)) = \sum_{k=1}^2 L^\pm_{ab}(u) \otimes L^\pm_{kb}(u), \] (5.9)

and the Counit \(\varepsilon : U(R^\pm) \to \mathcal{D}\) is

\[ \varepsilon(\mu_l(\hat{f})) = \varepsilon(\mu_r(\hat{f})) = \hat{f} T_0, \] (5.10)

\[ \varepsilon(e^Q) = e^Q, \quad \varepsilon(L^\pm_{ab}(u)) = \delta_{a,b} T_{w(b)Q}. \] (5.11)

Proof.

The verification of the compatibility conditions [5.5] is straightforward, using the isomorphism [5.2]. We show the \(\Delta\)-invariance of the first pair of relations in Definition [4.2].
Then the first pair of relations in (4.6) are:

\[
R(a) \quad \text{by the identification:}
\]

A similar calculation shows the invariance of the second pair, by using \( c = c \overset{\mp}{\otimes} 1 + 1 \overset{\pm}{\otimes} c \). Let \( R(u, P)(e_a \overset{\mp}{\otimes} e_b) = \sum_{x, y} R(u, P)_{xy}^{ab} e_x \overset{\mp}{\otimes} e_y \) and define the weight function \( w : \{1, 2\} \to \{\pm 1\} \) by the identification:

\[
R(u, P)_{xy}^{ab} \rightarrow \begin{pmatrix}
R_{++} \quad R_{++} \quad R_{+-} \quad R_{+-}
R_{++} \quad R_{+} \quad R_{-+} \quad R_{-}
R_{++} \quad R_{+} \quad R_{-+} \quad R_{-}
R_{++} \quad R_{+} \quad R_{-+} \quad R_{-}
\end{pmatrix}.
\]

Then the first pair of relations in (4.6) are:

\[
\sum_{a, b} R^\pm(u, P + h)_{ab}^{cd} L^\pm_{ad'}(u_1) L^\pm_{cb'}(u_2) = \sum_{c', d'} L^\pm_{dc'}(u_2) L^\pm_{cd'}(u_1) R^\pm(u, P)_{d'c'}^{a'b'}.
\]

Apply \( \Delta \) on both sides.

\[
\Delta(LHS) = \sum_{a, b} \Delta(R^\pm(u, P + h)_{ab}^{cd}) \Delta(L^\pm_{ad'}(u_1)) \Delta(L^\pm_{cb'}(u_2))
\]

\[
= \sum_{a, b, c', d'} R^\pm(u, P + h)_{ab}^{cd} L^\pm_{ad'}(u_1) L^\pm_{bc'}(u_2) \otimes L^\pm_{cd'}(u_1) L^\pm_{cb'}(u_2)
\]

\[
= \sum_{a, b, c', d'} L^\pm_{dc'}(u_2) L^\pm_{da'}(u_1) R^\pm(u, P)_{d'c'}^{a'b'} \otimes L^\pm_{cb'}(u_1) L^\pm_{cb'}(u_2),
\]

\[
\Delta(RHS) = \sum_{c', d'} \Delta(L^\pm_{dc'}(u_2)) \Delta(L^\pm_{cd'}(u_1)) \Delta(R^\pm(u, P)_{d'c'}^{a'b'})
\]

\[
= \sum_{a, b, c', d'} L^\pm_{dc'}(u_2) L^\pm_{da'}(u_1) \otimes L^\pm_{bc'}(u_2) L^\pm_{cb'}(u_1) R^\pm(u, P)_{d'c'}^{a'b'}
\]

\[
= \sum_{a, b, c', d'} L^\pm_{dc'}(u_2) L^\pm_{da'}(u_1) \otimes R^\pm(u, P + h)_{d'c'}^{a'b'} L^\pm_{cb'}(u_1) L^\pm_{cb'}(u_2)
\]

\[
= \sum_{a, b, c', d'} R^\pm(u, P)_{d'c'}^{a'b'} L^\pm_{dc'}(u_2) L^\pm_{da'}(u_1) \otimes L^\pm_{cd'}(u_1) L^\pm_{cb'}(u_2).
\]

The final equality was obtained by using the following formula:

\[
f(u, P) a \overset{\mp}{\otimes} b = a \overset{\mp}{\otimes} f(u, P + h) b \quad a, b \in U_{q, p}, \quad (5.12)
\]

The equality of both expressions \( \Delta(LHS) \) and \( \Delta(RHS) \) follows from the fact:

\[
R^\pm(u, P + w(a) + w(b))_{ab}^{d'c'} = R^\pm(u, P)_{ab}^{d'c'}.
\]
The invariance of the first pair of RLL-relations \((4.6)\) under \(\varepsilon\) follows using \((5.10), (5.11)\) and \((5.4)\).

Let us define the antipodal map using the inverse of \(L(u)\). It coincides with the elliptic case at \(p = 0\).

**Lemma 5.6.** If \(L^\pm(u) = ((L^\pm_{ij}(u)))_{i,j=1,2}\), the antipode \(S : U(R) \to U(R)\) is an algebra antihomomorphism given by

\[
S(e^Q) = e^{-Q}, \quad S(\mu_r(\hat{f})) = \mu_l(\hat{f}), \quad S(\mu_l(\hat{f})) = \mu_r(\hat{f}),
\]

\[
S(L^\pm_{11}(u)) = L^\pm_{22}(u - 1), \quad S(L^\pm_{12}(u)) = -q^{-1} \frac{\eta(P + h + 1)}{\eta(P + h)} L^\pm_{12}(u - 1),
\]

\[
S(L^\pm_{21}(u)) = -q \frac{\eta(P)}{\eta(P + 1)} L^\pm_{21}(u - 1), \quad S(L^\pm_{22}(u)) = \frac{\eta(P + h + 1)\eta(P)}{\eta(P + h)\eta(P + 1)} L^\pm_{11}(u - 1).
\]

**Proof.** For \(i, j = 1, 2\), by definition \(S(L^\pm_{ij}(u)) = (L^\pm(u^{-1}))_{ij}\). Then, the fact that the RLL-relations are invertible implies their \(S\)-invariance. For the calculation that the map \(S\) is compatible with \(\varepsilon\) and \(\Delta\), we use the expansion of the RLL-relations in the appendix. Let us verify \((5.6)\) for \(L^+_{12}(u)\). The relation \((A.3)\) with \(u_1 = u - 1, u_2 = u\) and \(u = -1\) becomes

\[
L^\pm_{11}(u)L^\pm_{12}(u - 1) = \frac{q\eta(P - 1)}{\eta(P)} L^\pm_{12}(u)L^\pm_{11}(u - 1), \tag{5.13}
\]

which yields the final equality in the following calculation:

\[
m \circ (id \otimes S) \circ \Delta(L^\pm_{12}(u))
\]

\[
= -L^\pm_{11}(u) \frac{q^{-1} \eta(P + h + 1)}{\eta(P + h)} L^\pm_{12}(u - 1) + L^\pm_{12}(u) \frac{\eta(P + h + 1)\eta(P)}{\eta(P + h)\eta(P + 1)} L^\pm_{11}(u - 1)
\]

\[
= q^{-1} \frac{\eta(P + h + 2)}{\eta(P + h + 1)} L^\pm_{11}(u)L^\pm_{12}(u - 1) + \frac{\eta(P + h + 2)\eta(P - 1)}{\eta(P + h + 1)\eta(P)} L^\pm_{12}(u)L^\pm_{11}(u - 1)
\]

\[
= 0.
\]

Since \(\varepsilon(L^+_{12}(u)) = 0\), the relation \((5.6)\) is confirmed. The proofs for the remaining generators \(L^\pm_{11}(u), L^\pm_{21}(u)\) and \(L^\pm_{22}(u)\) are similar. \(\square\)

**Theorem 5.7.** The following statements are true:

(i) The \(H\)-algebras \(U(R^\pm) := U(R_{q,x}(\mathfrak{gl}_2))\) are \(H\)-Hopf Algebroids.

(ii) The Total \(H\)-algebra \(U(R)\) is an \(H\) bialgebroid.
Proof. (i). The $H$-bialgebra structure is proven in Lemma \[5.5\], while the antipodal map is established in Lemma \[5.6\]. (ii). Follows from (i) and one checks that the second pair is $\Delta$-invariant using $\Delta(q^c) = q^c \tilde{\otimes} q^c$.

In order to develop the representation theory and to make contact with the more familiar dynamical algebras, we will consider the subalgebra $U_{q,x}(\hat{sl}_2)$ and $U(R_{q,x}(\hat{sl}_2))$. The $H$-Hopf algebroid structures is also more transparent on these subalgebras. We begin by introducing the dynamical determinant element, similar to the finite-dimensional case \[24\] as follows:

**Definition 5.8.** The Dynamical Determinant element is defined as

$$\text{Det}(L^\pm(u)) = L^\pm_{11}(u+1)L^\pm_{22}(u) - q \frac{\eta(P-1)}{\eta(P)} L^\pm_{12}(u+1)L^\pm_{21}(u).$$

The main properties are given below:

**Proposition 5.9.** The element $\text{Det}(L^\pm(u))$ is central in $U(R)$. Further,

$$\Delta(\text{Det}(L^\pm(u))) = \text{Det}(L^\pm(u)) \tilde{\otimes} \text{Det}(L^\pm(u)), \quad \varepsilon(\text{Det}(L^\pm(u))) = 1. \quad (5.14)$$

**Proof.** Using Theorem \[4.4\], we can apply the Gauss decomposition \[4.11\] of $L^\pm(u)$ along with the commutation relation \[3.10\] to obtain

$$\text{Det}(L^\pm(u)) = K^\pm_1(u+1)K^\pm_2(u). \quad (5.15)$$

Then it is straightforward to verify that this element lies in the center of $U(R)$ (resp. $U(R^+)$) by using the defining relations in Proposition \[3.3\] (resp. \[3.6\]) and the next expression which is obtained by expanding \[4.2\]

$$\rho^+(u)\rho^+(u+1) = q^{-1} \frac{\eta(u+1)}{\eta(u)}. \quad (5.16)$$

For the first expression in \[5.14\], expand both the sides using the definition \[5.8\] and the coproduct in Lemma \[5.5\]. Now use the formulae

$$\bar{c}(1,P) = b(1,P) = \frac{q(P-1)}{\eta(P)} \quad (5.17)$$

in relations \[A.3\] and \[A.5\], with $u_1 = u+1$ and $u_2 = u$. It remains to show that

$$L^\pm_{12}(u+1)L^\pm_{21}(u) \tilde{\otimes} D(u) = 0 \quad (5.18)$$
The subalgebra $U(R_{q,x}(\widehat{\mathfrak{sl}_2}))$ of $U(R_{q,x}(\widehat{\mathfrak{gl}_2}))$ is defined by the condition:

$$\text{Det}(L^\pm(z)) = 1.$$ 

We will obtain a similar statement as Theorem (4.4) for $\widehat{\mathfrak{sl}_2}$ in the next subsection.

\section*{5.2 Relation to Standard Drinfeld Realization of $U_q(\widehat{\mathfrak{sl}}_2)$}

Consider a field $\mathbb{K} \supseteq \mathbb{C}$. The following standard presentation is well-known [9].

\begin{definition}
$\mathbb{K}[U_q(\widehat{\mathfrak{sl}_2})]$ is the associative algebra over $\mathbb{K}$ generated by the generators $a_n$ ($n \in \mathbb{Z}_{\neq 0}$), $x_n^\pm$ ($n \in \mathbb{Z}$), $h$, $c$ and $d$. The defining relations are given as follows.

- $c$: central,
- $[h,d] = 0$, $[d,a_n] = na_n$, $[d,x_n^\pm] = nx_n^\pm$,
- $[h,a_n] = 0$, $[h,x_n] = \pm 2x_n$,
- $[a_m,a_n] = [2n]_q [cn]_q \delta_{n+m,0}$,
- $[a_n,x_m^+] = \frac{[2n]_q}{n} q^{\frac{cn}{2}} x_{m+n}^+$,
- $[a_n,x_m^-] = -\frac{[2n]_q}{n} q^{-\frac{cn}{2}} x_{m+n}^-$,
- $x_{m+1}^\pm x_m^\mp - q^{\pm 2} x_m^\pm x_{m+1}^\mp = q^{\pm 2} x_m^\pm x_{m+1}^\pm - x_{m+1}^\pm x_m^\pm$,
- $[x_n^+,x_m^-] = \frac{1}{q - q^{-1}} \left( q^{\frac{c(n-m)}{2}} \psi_{m+n} - q^{\frac{c(n-m)}{2}} \varphi_{m+n} \right)$.

Denote $x^\pm(z) = \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n}$, and the auxilliary currents $\psi_n, \varphi_n$, $(n \geq 0)$ by

$$\sum_{n \geq 0} \psi_n z^{-n} = q^h \exp \left( (q - q^{-1}) \sum_{n > 0} a_n z^{-n} \right), \quad \sum_{n \geq 0} \phi_n z^{n} = q^{-h} \exp \left( -(q - q^{-1}) \sum_{n > 0} a_{-n} z^{n} \right).$$

\end{definition}
Choosing $\mathbb{K} = \mathbb{C}[H]$ and defining $\hat{U}_{q,x}(\hat{\mathfrak{sl}}_2) := \mathbb{K}[U_q(\mathfrak{sl}_2)] \otimes \mathbb{C}[\hat{H}^*]$, consider the two operators:

\[ K^+(u) = \exp\left(\sum_{n>0} \frac{[n]_q}{[2n]_q} (q - q^{-1}) a_n q^{-(2n+1)} u^n\right) e^{Q q^{\frac{h}{2}}}, \tag{5.20} \]
\[ K^-(u) = \exp\left(-\sum_{n>0} \frac{[n]_q}{[2n]_q} (q - q^{-1}) a_{-n} q^{(2n+1)} u^n\right) e^{Q q^{-\frac{h}{2}}}. \tag{5.21} \]

The $H$-algebra structure on $\hat{U}_{q,x}(\hat{\mathfrak{sl}}_2)$ is defined in exactly the same way as $U_{q,x}(\mathfrak{sl}_2)$, using the same Heisenberg algebra $\mathcal{H}$ in Subsection 3.2.1. Now we define the dynamical currents $U_{q,x}(\hat{\mathfrak{sl}}_2)$ by the same relations as Definition 5.11 but without the element $d$.

**Corollary 5.12.** Corollary to Theorem 4.4. The isomorphism in Theorem 4.4 restricts to an $H$-subalgebra isomorphism: $\hat{U}_{q,x}(\hat{\mathfrak{sl}}_2)' \simeq U(R_{q,x}(\mathfrak{sl}_2))$.

**Proof.** We identify $\hat{U}_{q,x}(\hat{\mathfrak{sl}}_2)'$ with the $H$-algebra generated by

\[ \{ h, c, \hat{E}(u), \hat{F}(u), \hat{K}^+_1(u), \hat{K}^+_2(u), \hat{K}^-_1(u)^{-1}, \hat{K}^-_2(u)^{-1} \}. \]

We must check that these generators satisfy the defining relations of the algebra $U_{q,x}(\mathfrak{sl}_2)$ in Proposition 3.3. The calculation is using the Baker-Campbell-Hausdorff formula. The relations between the $\hat{K}^\pm_i(u)$ are derived from the relation for $[a_n, a_m]$ and Definition 5.20 using the formula: $[A, B] = k \implies \exp(A) \exp(B) \exp(-A) = e^k \exp(B)$.

The relations between $\hat{K}^\pm_i(u)$ and $\{ \hat{E}(u), \hat{F}(u) \}$ are verified by writing the defining relations $[a_n, x^\pm_m]$ as

\[ [a_n, x^+(z)] = \frac{[2n]_q}{n} q^{\frac{[n]_q}{2}} z^n x^+(z), \quad [a_n, x^-(z)] = -\frac{[2n]_q}{n} q^{-\frac{[n]_q}{2}} z^n x^-(z). \]

$^5K^\pm(u)$ can be obtained as degenerations of the expressions for the elliptic operators $K(z)$ and $H^\pm(u)$ in eqns (3.15, 3.25, 3.29) (without the elliptic shift by the central element: $p \to pq^{-2c}$).
and applying the fact:
\[ [A,X] = kX \implies \exp(A)X\exp(-A) = e^kX. \]

It follows that
\[ q^h x^\pm(z) q^{-h} = q^{\pm 2} x^\pm(z). \]
Finally, the relation for \([\hat{E}(u), \hat{F}(u)]\) is a consequence of the definitions (5.23), since \([x^+_n, x^-_m]\) is exactly (2.1). Thus \(\hat{U}_{q,x}(\hat{sl}_2)'\) is a subalgebra of \(U_{q,x}(\hat{gl}_2)\). The \(L\)-operators constructed by replacing \(\{E^\pm(u), F^\pm(u), K^\pm_i(u)\}\) by \(\{\hat{E}^\pm(u), \hat{F}^\pm(u), \hat{K}^\pm_i(u)\}\) are in \(U(R^\pm_q,\hat{sl}_2))\) because
\[
\det(L^\pm(u)) = \hat{K}^\pm_1(u + 1)\hat{K}^\pm_2(u) = K^\pm(u)K^\pm(u)^{-1} = 1. \quad (5.24)
\]

We will hereafter write \(U_{q,x}(\hat{sl}_2)\) for \(\hat{U}_{q,x}(\hat{sl}_2)'\) and the half-current subalgebras
\[
U^\pm_{q,x}(\hat{sl}_2) := U_{q,x}(\hat{sl}_2) \cap U^\pm_{q,x}(\hat{gl}_2) \subseteq U_{q,x}(\hat{sl}_2).
\]

### 5.3 \(H\)-Hopf algebroid structure on \(U_{q,x}(\hat{sl}_2)\)

**Theorem 5.13.** The following statements are true.

(i) The Half-current Algebras \((U^\pm_{q,x}(\hat{sl}_2), \hat{H}^*, \mu_t, \mu_r, \Delta, \varepsilon, S)\) are \(H\)-Hopf-algebroids.

(ii) The Total Algebra \((U_{q,x}(\hat{sl}_2), \hat{H}^*, \mu_t, \mu_r)\) is an \(H\)-bialgebroid at \(c = 0\).

**Proof.** (i) Since \(U(R^\pm)\) is an \(H\)-Hopf algebroid (Subsection 5.1), we can define the comultiplication, counit and antipodal maps on \(U^\pm_{q,x}(\hat{sl}_2)\), by using Theorem 4.4 and the Gauss decomposition of \(L^\pm(u)\) given in (4.11). The coproduct is given explicitly in Proposition 5.14. Use the expression 4.12 for the inverse of \(L^+(u)\) along with the commutation relations 3.10 and 3.11 (with \(u_1 = u - 1\)), in \(S(L^+(u)) = L^+(u)^{-1}\) to obtain the explicit formula for the image of the half-currents under the antipodal map. (ii) The statement follows from Theorem 5.7. □

Explicit expressions for the comultiplication map of the positive and negative half-currents are available in our case. The elliptic version of the next result appears in Proposition 3.12 of [27].
Proposition 5.14. The coproduct for the half-currents is given as:

\[
\begin{align*}
\Delta(K_1^+(u)) &= K_1^+(u) \tilde{\otimes} K_1^+(u) + \sum_{j=1}^{\infty} (-1)^j K_1^+(u) E^+(u-1)^j \tilde{\otimes} F^+(u-1)^j K_1^+(u), \\
\Delta(K_2^+(u)) &= K_2^+(u) \tilde{\otimes} K_2^+(u) + K_2^+(u) E^+(u) \tilde{\otimes} F^+(u) K_2^+(u), \\
\Delta(E^+(u)) &= 1 \tilde{\otimes} E^+(u) + E^+(u) \tilde{\otimes} K_2^+(u) \tilde{\otimes} F^+(u) K_2^+(u), \\
\Delta(F^+(u)) &= F^+(u) \tilde{\otimes} 1 + K_1^+(u) K_2^+(u) \tilde{\otimes} F^+(u) \\
&\quad + \sum_{j=1}^{\infty} (-1)^j K_1^+(u) E^+(u) \tilde{\otimes} K_2^+(u) \tilde{\otimes} F^+(u) \tilde{\otimes} F^+(u) K_2^+(u), \\
\Delta(H^+(u)) &= H^+(u) \tilde{\otimes} H^+(u) \mod A_{\geq 0} \tilde{\otimes} A_{\leq 0},
\end{align*}
\]

where \( A_{\geq 0} \) (resp. \( A_{\leq 0} \)) is the subalgebra generated by \( \{ E^+(u) \) (resp. \( F^+(u) \)) , \( K_1^+(u) \), \( q^\pm \} \).

Proof. Use the coproduct formula in (5.14) with the Gauss decomposition of \( L^\pm(u) \) in (4.11) for the first four formulae. The last relation is obtained by substituting the first two in the definition of \( H^+(u) \).

As mentioned in [27], the Hopf algebroid structure on the elliptic quantum group \( U_{q,p}(\hat{sl}_2) \) does indeed survive the degeneration as \( p \to 0 \).

5.4 Concluding Remarks and Questions

It is natural to ask whether the results presented in this article can be extended to the higher-rank algebras, since the Ding-Frenkel equivalence holds at least for \( U_q(\hat{sl}_N) \).

After completing this study, there appeared a new definition of elliptic \( U_{q,p}(\hat{g}) \) associated to any untwisted affine Lie algebra [14] for arbitrary values of \( p \), that can be identified at \( p = 0 \) with the definition given here of \( U_{q,x}(\hat{sl}_2) \) (see Theorem 2.2 in [14]). The precise details will be discussed elsewhere.

Finally, it would be interesting to find a relationship of \( U_{q,x}(\hat{sl}_2) \) with the elliptic algebra \( U(2) \) in [25] when \( c=0 \).
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Acknowledgments

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A The RLL relations of $U(R)$

Proposition A.1. The first relation in [(4.6)]

\[ R^{\pm(12)} (u, P + h) L^{\pm(1)} (u_1) L^{\pm(2)} (u_2) = L^{\pm(2)} (u_2) L^{\pm(1)} (u_1) R^{\pm(12)} (u, P), \]

is expanded as

\[ L^{\pm(11)} (u_1) L^{\pm(11)} (u_2) = L^{\pm(11)} (u_2) L^{\pm(11)} (u_1), \quad L^{\pm(12)} (u_1) L^{\pm(12)} (u_2) = L^{\pm(12)} (u_2) L^{\pm(12)} (u_1), \quad (A.1) \]

\[ L^{\pm(21)} (u_1) L^{\pm(21)} (u_2) = L^{\pm(21)} (u_2) L^{\pm(21)} (u_1), \quad L^{\pm(22)} (u_1) L^{\pm(22)} (u_2) = L^{\pm(22)} (u_2) L^{\pm(22)} (u_1), \quad (A.2) \]

\[ L^{\pm(11)} (u_1) L^{\pm(12)} (u_2) = L^{\pm(11)} (u_2) L^{\pm(12)} (u_1) \bar{c}(u, P) + L^{\pm(12)} (u_2) L^{\pm(11)} (u_1) b(u, P), \quad (A.3) \]

\[ L^{\pm(12)} (u_1) L^{\pm(11)} (u_2) = L^{\pm(12)} (u_2) L^{\pm(11)} (u_1) \bar{b}(u, P) + L^{\pm(11)} (u_2) L^{\pm(12)} (u_1) c(u, P), \quad (A.4) \]

\[ L^{\pm(21)} (u_1) L^{\pm(22)} (u_2) = L^{\pm(21)} (u_2) L^{\pm(22)} (u_1) \bar{c}(u, P) + L^{\pm(22)} (u_2) L^{\pm(21)} (u_1) b(u, P), \quad (A.5) \]

\[ L^{\pm(22)} (u_1) L^{\pm(21)} (u_2) = L^{\pm(22)} (u_2) L^{\pm(21)} (u_1) b(u, P) + L^{\pm(21)} (u_2) L^{\pm(22)} (u_1) c(u, P), \quad (A.6) \]

\[ b(u, P + h) L^{\pm(11)} (u_1) L^{\pm(21)} (u_2) + c(u, P + h) L^{\pm(21)} (u_1) L^{\pm(11)} (u_2) = L^{\pm(21)} (u_2) L^{\pm(11)} (u_1), \quad (A.7) \]

\[ b(u, P + h) L^{\pm(12)} (u_1) L^{\pm(22)} (u_2) + c(u, P + h) L^{\pm(22)} (u_1) L^{\pm(12)} (u_2) = L^{\pm(22)} (u_2) L^{\pm(12)} (u_1), \quad (A.8) \]

\[ \bar{b}(u, P + h) L^{\pm(11)} (u_1) L^{\pm(11)} (u_2) + \bar{c}(u, P + h) L^{\pm(12)} (u_1) L^{\pm(12)} (u_2) = L^{\pm(12)} (u_2) L^{\pm(12)} (u_1), \quad (A.9) \]

\[ \bar{b}(u, P + h) L^{\pm(22)} (u_1) L^{\pm(22)} (u_2) + \bar{c}(u, P + h) L^{\pm(21)} (u_1) L^{\pm(21)} (u_2) = L^{\pm(21)} (u_2) L^{\pm(21)} (u_1), \quad (A.10) \]

\[ b(u, P + h) L^{\pm(11)} (u_1) L^{\pm(22)} (u_2) + c(u, P + h) L^{\pm(22)} (u_1) L^{\pm(11)} (u_2) \]
\[ = L^{\pm(22)} (u_2) L^{\pm(11)} (u_1) \bar{c}(u, P) + L^{\pm(11)} (u_2) L^{\pm(22)} (u_1) b(u, P), \quad (A.11) \]

\[ b(u, P + h) L^{\pm(12)} (u_1) L^{\pm(21)} (u_2) + c(u, P + h) L^{\pm(21)} (u_1) L^{\pm(12)} (u_2) \]
\[ = L^{\pm(21)} (u_2) L^{\pm(12)} (u_1) \bar{b}(u, P) + L^{\pm(12)} (u_2) L^{\pm(21)} (u_1) c(u, P), \quad (A.12) \]

\[ \bar{b}(u, P + h) L^{\pm(11)} (u_1) L^{\pm(12)} (u_2) + \bar{c}(u, P + h) L^{\pm(12)} (u_1) L^{\pm(11)} (u_2) \]
\[ = L^{\pm(12)} (u_2) L^{\pm(11)} (u_1) \bar{c}(u, P) + L^{\pm(11)} (u_2) L^{\pm(12)} (u_1) b(u, P), \quad (A.13) \]

\[ \bar{b}(u, P + h) L^{\pm(22)} (u_1) L^{\pm(21)} (u_2) + \bar{c}(u, P + h) L^{\pm(21)} (u_1) L^{\pm(22)} (u_2) \]
\[ = L^{\pm(22)} (u_2) L^{\pm(21)} (u_1) \bar{b}(u, P) + L^{\pm(21)} (u_2) L^{\pm(22)} (u_1) c(u, P). \quad (A.14) \]
The second set

\[ R^{\pm(12)} \left( u \pm \frac{c}{2}, P + h \right) L^{\pm(1)}(u_1) L^{\mp(2)}(u_2) = L^{\mp(2)}(u_2) L^{\pm(1)}(u_1) R^{\pm(12)} \left( u \mp \frac{c}{2}, P \right), \]

is expanded below (we write the R-matrix entries \( c(u, P) \) and \( \bar{c}(u, P) \) as \( c_0(u, P) \) and \( \bar{c}_0(u, P) \) respectively, to distinguish them from the central element \( c \)).

\[
\rho^{\pm}(u \pm \frac{c}{2}) L^{\pm}_{11}(u_1)L^{\mp}_{11}(u_2) = L^{\mp}_{11}(u_2)L^{\pm}_{11}(u_1) \rho^{\pm}(u \mp \frac{c}{2}), \tag{A.15}
\]

\[
\rho^{\pm}(u \pm \frac{c}{2}) L^{\pm}_{12}(u_1)L^{\mp}_{12}(u_2)
= \left( L^{\mp}_{12}(u_2)L^{\pm}_{12}(u_1)\bar{c}_0(u \mp \frac{c}{2}, P) + L^{\pm}_{12}(u_2)L^{\mp}_{11}(u_1)b(u \mp \frac{c}{2}, P) \right) \rho^{\pm}(u \mp \frac{c}{2}), \tag{A.16}
\]

\[
\rho^{\pm}(u \pm \frac{c}{2}) L^{\pm}_{12}(u_1)L^{\mp}_{11}(u_2)
= \left( L^{\mp}_{11}(u_2)L^{\pm}_{12}(u_1)b(u \mp \frac{c}{2}, P) + L^{\pm}_{12}(u_2)L^{\mp}_{11}(u_1)c_0(u \mp \frac{c}{2}, P) \right) \rho^{\pm}(u \mp \frac{c}{2}), \tag{A.17}
\]

\[
\rho^{\pm}(u \pm \frac{c}{2}) L^{\pm}_{12}(u_1)L^{\mp}_{21}(u_2)
= L^{\mp}_{12}(u_2)L^{\pm}_{21}(u_1)\rho^{\pm}(u \mp \frac{c}{2}), \tag{A.18}
\]

\[
\rho^{\pm}(u \pm \frac{c}{2}) \left( b(u \pm \frac{c}{2}, P + h)L^{\mp}_{11}(u_1)L^{\pm}_{21}(u_2) + c_0(u \pm \frac{c}{2}, P + h)L^{\pm}_{21}(u_1)L^{\mp}_{11}(u_2) \right)
= L^{\mp}_{21}(u_2)L^{\pm}_{11}(u_1)\rho^{\pm}(u \mp \frac{c}{2}), \tag{A.19}
\]

\[
\rho^{\pm}(u \pm \frac{c}{2}) \left( b(u \pm \frac{c}{2}, P + h)L^{\mp}_{11}(u_1)L^{\pm}_{22}(u_2) + c_0(u \pm \frac{c}{2}, P + h)L^{\pm}_{22}(u_1)L^{\mp}_{11}(u_2) \right)
= \left( L^{\mp}_{22}(u_2)L^{\pm}_{11}(u_1)b(u \mp \frac{c}{2}, P) + L^{\pm}_{21}(u_2)L^{\mp}_{12}(u_1)c_0(u \mp \frac{c}{2}, P) \right) \rho^{\pm}(u \mp \frac{c}{2}), \tag{A.20}
\]

\[
\rho^{\pm}(u \pm \frac{c}{2}) \left( b(u \pm \frac{c}{2}, P + h)L^{\mp}_{12}(u_1)L^{\pm}_{22}(u_2) + c_0(u \pm \frac{c}{2}, P + h)L^{\pm}_{22}(u_1)L^{\mp}_{12}(u_2) \right)
= \left( L^{\mp}_{22}(u_2)L^{\pm}_{12}(u_1)c_0(u \mp \frac{c}{2}, P) + L^{\pm}_{21}(u_2)L^{\mp}_{12}(u_1)b(u \mp \frac{c}{2}, P) \right) \rho^{\pm}(u \mp \frac{c}{2}), \tag{A.21}
\]

\[
\rho^{\pm}(u \pm \frac{c}{2}) \left( b(u \pm \frac{c}{2}, P + h)L^{\mp}_{12}(u_1)L^{\pm}_{22}(u_2) + c_0(u \pm \frac{c}{2}, P + h)L^{\pm}_{22}(u_1)L^{\mp}_{12}(u_2) \right)
= L^{\mp}_{22}(u_2)L^{\pm}_{12}(u_1)\rho^{\pm}(u \mp \frac{c}{2}), \tag{A.22}
\]

\[
\rho^{\pm}(u \pm \frac{c}{2}) \left( \bar{b}(u \pm \frac{c}{2}, P + h)L^{\pm}_{21}(u_1)L^{\mp}_{11}(u_2) + \bar{c}(u \pm \frac{c}{2}, P + h)L^{\pm}_{11}(u_1)L^{\mp}_{21}(u_2) \right)
= L^{\mp}_{11}(u_2)L^{\pm}_{21}(u_1)\rho^{\pm}(u \mp \frac{c}{2}), \tag{A.23}
\]
\[
\rho^{\pm}(u \pm \frac{c}{2}) \left( \bar{b}(u \pm \frac{c}{2}, P + h) L_{21}^{\pm}(u_1) L_{12}^{\mp}(u_2) + c_0(u \pm \frac{c}{2}, P + h)L_{11}^{\pm}(u_1)L_{22}^{\mp}(u_2) \right) \\
= \left( L_{11}^{\pm}(u_2)L_{22}^{\pm}(u_1)c_0(u \pm \frac{c}{2}, P) + L_{12}^{\mp}(u_2)L_{21}^{\mp}(u_1)b(u \pm \frac{c}{2}, P) \right) \rho^{\pm}(u \pm \frac{c}{2}),
\]
(A.24)

\[
\rho^{\pm}(u \pm \frac{c}{2}) \left( \bar{b}(u \pm \frac{c}{2}, P + h) L_{22}^{\pm}(u_1) L_{11}^{\mp}(u_2) + c_0(u \pm \frac{c}{2}, P + h)L_{12}^{\pm}(u_1)L_{21}^{\mp}(u_2) \right) \\
= \left( L_{11}^{\mp}(u_2)L_{22}^{\mp}(u_1)\bar{b}(u \pm \frac{c}{2}, P) + L_{12}^{\mp}(u_2)L_{21}^{\mp}(u_1)c_0(u \pm \frac{c}{2}, P) \right) \rho^{\pm}(u \pm \frac{c}{2}),
\]
(A.25)

\[
\rho^{\pm}(u \pm \frac{c}{2}) \left( \bar{b}(u \pm \frac{c}{2}, P + h) L_{22}^{\pm}(u_1) L_{21}^{\mp}(u_2) + c_0(u \pm \frac{c}{2}, P + h)L_{22}^{\pm}(u_1)L_{21}^{\mp}(u_2) \right) \\
= \left( L_{21}^{\mp}(u_2)L_{22}^{\mp}(u_1)\bar{b}(u \pm \frac{c}{2}, P) + L_{21}^{\mp}(u_2)L_{22}^{\mp}(u_1)c_0(u \pm \frac{c}{2}, P) \right) \rho^{\pm}(u \pm \frac{c}{2}),
\]
(A.26)

\[
\rho^{\pm}(u \pm \frac{c}{2}) L_{21}^{\pm}(u_1) L_{21}^{\mp}(u_2) = L_{21}^{\mp}(u_2)L_{21}^{\mp}(u_1)\rho^{\pm}(u \pm \frac{c}{2}),
\]
(A.27)

\[
\rho^{\pm}(u \pm \frac{c}{2}) L_{21}^{\pm}(u_1) L_{22}^{\mp}(u_2) \\
= \left( L_{21}^{\mp}(u_2)L_{21}^{\mp}(u_1)\bar{b}(u \pm \frac{c}{2}, P) + L_{21}^{\mp}(u_2)L_{22}^{\mp}(u_1)c_0(u \pm \frac{c}{2}, P) \right) \rho^{\pm}(u \pm \frac{c}{2}),
\]
(A.28)

\[
\rho^{\pm}(u \pm \frac{c}{2}) L_{22}^{\pm}(u_1) L_{21}^{\mp}(u_2) \\
= \left( L_{22}^{\pm}(u_2)L_{21}^{\pm}(u_1)c_0(u \pm \frac{c}{2}, P) + L_{22}^{\pm}(u_2)L_{22}^{\pm}(u_1)\bar{b}(u \pm \frac{c}{2}, P) \right) \rho^{\pm}(u \pm \frac{c}{2}),
\]
(A.29)

\[
\rho^{\pm}(u \pm \frac{c}{2}) L_{22}^{\pm}(u_1) L_{22}^{\mp}(u_2) = L_{22}^{\pm}(u_2)L_{22}^{\pm}(u_1)\rho^{\pm}(u \pm \frac{c}{2}).
\]
(A.30)

References

[1] G. E. Andrews, R. J. Baxter and P. J. Forrester  Eight-vertex SOS model and generalized Rogers-Ramanujan type identities, *J. Statist. Phys.*, 35,1984, 193–266.

[2] D. Arnaudon, J. Avan, L. Frappat, E. Ragoucy and M. Rossi, Towards a cladistics of double Yangians and elliptic algebras. *J. Phys. A* 33, 2000, 6279–6309.

[3] O. Babelon, D. Bernard, and E. Billey. A quasi-Hopf algebra interpretation of quantum 3j- and 6j-symbols and difference equations. *Phys. Lett. B*, 375, 1996, 89–97.

[4] G. Bohm, Hopf Algebroids, *Handbook of Algebra* Vol 6, edited by M. Hazewinkel Elsevier 2009, 173–236.
DYNAMICAL AFFINE QUANTUM GROUP $U_{q,x}(\hat{gl}_2)$

[5] E. Buffenoir, P. Roche, V. Terras. Universal vertex irf transformation for quantum affine algebras. *J. Math. Phys.*, **53**, 103515 (2012)

[6] V. Chari and A. Pressley, Quantum Affine Algebras, *Comm. Math. Phys.*, **142**, 1991, 261-283.

[7] E. Date, M. Jimbo, T. Miwa and M. Okado, Fusion of the Eight-Vertex SOS Model, *Lett. Math. Phys.*, **12**, 1986, 209–215.

[8] N. Ding, I. Frenkel, Isomorphism of Two Realizations of Quantum Affine Algebra $U_q(\hat{gl}_2)$, *Comm. Math. Phys.*, **156**, 1993, 277-300.

[9] V. Drinfeld, A New Realization of Yangians and Quantized Affine Algebras, *Soviet Math. Dokl.*, **36**, 1988, 212-216.

[10] B. Enriquez and G. Felder, Elliptic Quantum Groups $E_{\tau,\eta}(\hat{sl}_2)$ and Quasi-Hopf Algebra, *Comm. Math. Phys.*, **195**, 1998, 651–689.

[11] P. Etingof, T. Schedler and O. Schiffmann, Explicit Quantization of Dynamical $r$ matrices for Finite Dimensional Semisimple Lie Algebras, *J. Amer. Math. Soc.*, **13**, 2000, 595–609.

[12] P. Etingof and A. Varchenko, Solutions of the Quantum Dynamical Yang-Baxter Equation and Dynamical Quantum Groups, *Comm. Math. Phys.*, **196**, 1998, 591–640.

[13] P. Etingof and A. Varchenko, Exchange Dynamical Quantum Groups, *Comm. Math. Phys.*, **205**, 1999, 19–52.

[14] R. Farghly, H. Konno, K. Oshima, Elliptic Algebra $U_{q,p}(\hat{g})$ and Quantum $Z$-algebras http://arxiv.org/pdf/1404.1738.pdf.

[15] G. Felder, Elliptic Quantum Groups, *Proc. ICMP Paris-1994*, 1995, 211–218.

[16] O. Foda, K. Iohara, M. Jimbo, R. Kedem, T. Miwa and H. Yan, An elliptic quantum algebra for $\hat{sl}_2$, *Lett. Math. Phys.*, **32**, 1994, 259-268.

[17] I. Frenkel, N. Jing, Vertex Representation of Quantum Affine Algebras, *Proc. Natl. Acad. Sci. USA* **85**, 1988, 9373–9377

[18] G. Gasper and M. Rahman, Basic Hypergeometric Series, 2nd ed., *Encyclopedia of Mathematics and its Applications*, **96**, 2004, Cambridge Univ. Press.

[19] B. Hou and W. Yang, Dynamically twisted algebra $A_{q,p;x}(\hat{sl}_2)$ as current algebra generalizing screening currents of $q$-deformed Virasoro algebra, *J. Phys. A: Math. Gen.* **31** 5349, 1998

[20] M. Idzumi, K. Iohara, M. Jimbo, T. Miwa, T. Nakashima and T. Tokihiro, Quantum affine symmetry in vertex models, *Int. J. Mod. Phys.*, **A8**, 1993, pp. 1479.
[21] M. Jimbo, A \( q \)-analogue of \( U_q(gl(N + 1)) \), Hecke Algebra and the Yang-Baxter Equation, *Lett. Math. Phys.*, 11, 1986, 247–252.

[22] M. Jimbo, H. Konno, S. Odake and J. Shiraishi, Elliptic algebra \( U_{q,p}(\hat{\mathfrak{sl}}_2) \): Drinfeld currents and vertex operators, *Comm. Math. Phys.*, 199, 1999, 605–647.

[23] M. Jimbo, H. Konno, S. Odake and J. Shiraishi, Quasi-Hopf Twistors for Elliptic Quantum Groups, *Transformation Groups*, 4, 1999, 303–327.

[24] E. Koelink and H. Rosengren, Harmonic Analysis on the \( SU(2) \) Dynamical Quantum Group, *Acta. Appl. Math.*, 69, 2001, 163–220.

[25] E. Koelink, Y. van Norden and H. Rosengren, Elliptic \( U(2) \) Quantum Group and Elliptic Hypergeometric Series, *Comm. Math. Phys.*, 245, 2004, 519–537.

[26] H. Konno, An Elliptic Algebra \( U_{q,p}(\hat{\mathfrak{sl}}_2) \) and the Fusion RSOS Models, *Comm. Math. Phys.*, 195, 1998, 373–403.

[27] H. Konno, Elliptic Quantum Group \( U_{q,p}(\hat{\mathfrak{sl}}_2) \), Hopf Algebroid Structure and Elliptic Hypergeometric Series, *Jour. Geom. Phys.*, 59-11, 2009, 1485-1511.

[28] B. Narayanan, Representations of the Dynamical Affine Quantum Group \( U_{q,x}(\hat{\mathfrak{sl}}_2) = U_{q,\lambda}(\hat{\mathfrak{sl}}_2) \) and Hypergeometric Series, *To appear*.

[29] N. Reshetikhin and M. Semenov-Tian-Shansky, Central Extensions of Quantum Current Groups, *Lett. Math. Phys.*, 19, 1990, 133–142.

[30] H. Rosengren, Duality and Self-duality for Dynamical Quantum Groups, *Algebr. Represent. Theory*, 7, 2004, 363-393.

[31] P. Xu, Quantum Groupoids, *Comm. Math. Phys.*, 216, 2001, 539-581.