Bicat is not triequivalent to Gray

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Abstract

Bicat is the tricategory of bicategories, homomorphisms, pseudonatural transformations, and modifications. Gray is the subtricategory of 2-categories, 2-functors, pseudonatural transformations, and modifications. We show that these two tricategories are not triequivalent.

Weakening the notion of 2-category by replacing all equations between 1-cells by suitably coherent isomorphisms gives the notion of bicategory [1]. The analogous weakening of a 2-functor is called a homomorphism of bicategories, and the weakening of a 2-natural transformation is a pseudonatural transformation. There are also modifications between 2-natural or pseudonatural transformations, but this notion does not need to be weakened. The bicategories, homomorphisms, pseudonatural transformations, and modifications form a tricategory (a weak 3-category) called Bicat.

The subtricategory of Bicat containing only the 2-categories as objects, and only the 2-functors as 1-cells, but with all 2-cells and 3-cells between them, is called Gray. As well as being a particular tricategory, there is another important point of view on Gray. The category 2-Cat of 2-categories and 2-functors is cartesian closed, but it also has a different symmetric monoidal closed structure [3], for which the internal hom [A, B] is the 2-category of 2-functors, pseudonatural transformations, and modifications between A and B. A category enriched over 2-Cat with respect to this closed structure is called a Gray-category. A Gray-category has 2-categories as hom-objects, so is a 3-dimensional categorical structure, and it can be seen as a particular sort of tricategory. The closed structure of 2-Cat gives it a canonical enrichment over itself and the resulting Gray-category is just Gray. Gray is also sometimes used as a name for 2-Cat with this monoidal structure.

A homomorphism of bicategories T : A → C is called a biequivalence if it induces equivalences TA,B : A(A, B) → C(TA, TB) of hom-categories for

∗The support of the Australian Research Council and DETYA is gratefully acknowledged.
all objects $A, B \in \mathcal{C}$ ($T$ is locally an equivalence), and every object $C \in \mathcal{C}$ is equivalent in $\mathcal{C}$ to one of the form $TA$ ($T$ is biessentially surjective on objects).

We then write $A \sim B$. Every bicategory is equivalent to a 2-category [4].

A trihomomorphism of tricategories $T : \mathcal{A} \to \mathcal{C}$ is called a triequivalence if it induces biequivalences $T_{A,B} : \mathcal{A}(A,B) \to \mathcal{C}(TA, TB)$ of hom-bicategories for all objects $A, B \in \mathcal{A}$ ($T$ is locally a biequivalence), and every object $C \in \mathcal{C}$ is biequivalent in $\mathcal{C}$ to one of the form $TA$ ($T$ is triessentially surjective on objects).

It is not the case that every tricategory is triequivalent to a 3-category, but every tricategory is triequivalent to a Gray-category [2].

Perhaps since a Gray-category is a category enriched in the monoidal category Gray, and a tricategory can be seen as some sort of “weak Bicat-category”, it has been suggested that Bicat might be triequivalent to Gray, and indeed Section 5.6 of [2] states that this is the case. We prove that it is not. First we prove:

**Lemma 1** The inclusion $\text{Gray} \to \text{Bicat}$ is not a triequivalence.

**Proof:** If it were then each inclusion $\text{Gray}(\mathcal{A}, \mathcal{B}) \to \text{Bicat}(\mathcal{A}, \mathcal{B})$ would be a biequivalence, and so each homomorphism (pseudofunctor) between 2-categories would be pseudonaturally equivalent to a 2-functor. This is not the case. For example (see [4 Example 3.1]), let $\mathcal{A}$ be the 2-category with a single object $*$, a single non-identity morphism $f : * \to *$ satisfying $f^2 = 1$, and no non-identity 2-cells (the group of order 2 seen as a one-object 2-category); and let $\mathcal{B}$ be the 2-category with a single object $*$, a morphism $n : * \to *$ for each integer $n$, composed via addition, and an isomorphism $n \cong m$ if and only if $n - m$ is even (the “pseudo-quotient of $\mathbb{Z}$ by $2\mathbb{Z}$”). There is a homomorphism $\mathcal{A} \to \mathcal{B}$ sending $f$ to 1; but the only 2-functor $\mathcal{A} \to \mathcal{B}$ sends $f$ to 0, so this homomorphism is not pseudonaturally equivalent to a 2-functor. □

**Theorem 2** $\text{Gray}$ is not triequivalent to $\text{Bicat}$.

**Proof:** Suppose there were a triequivalence $\Phi : \text{Gray} \to \text{Bicat}$. We show that $\Phi$ would be biequivalent to the inclusion, so that the inclusion itself would be a triequivalence; but by the lemma this is impossible.

The terminal 2-category 1 is a terminal object in Gray, so must be sent to a “triterminal object” $\Phi 1$ in Bicat; in other words, Bicat($\mathcal{B}, \Phi 1$) must be biequivalent to 1 for any bicategory $\mathcal{B}$. For any 2-category $\mathcal{A}$, we have biequivalences

$$\mathcal{A} \sim \text{Gray}(1, \mathcal{A}) \sim \text{Bicat}(\Phi 1, \Phi \mathcal{A}) \sim \text{Bicat}(1, \Phi \mathcal{A}) \sim \Phi \mathcal{A}$$

where the first is the isomorphism coming from the monoidal structure on Gray, the second is the biequivalence on hom-bicategories given by $\Phi$, the third is given by composition with the biequivalence $\Phi 1 \sim 1$, and the last is a special case of the biequivalence Bicat($1, \mathcal{B}$) $\sim \mathcal{B}$ for any bicategory, given by evaluation at the unique object $*$ of 1. All of these biequivalences are “natural” in a suitably weak tricategorical sense, and so $\Phi$ is indeed biequivalent to the inclusion. □
Remark 3 The most suitable weak tricategorical transformation is called a tritransformation. The axioms are rather daunting, but really the coherence conditions are not needed here. We only need the obvious fact that for any 2-functor $T : \mathcal{A} \to \mathcal{B}$, the square

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\sim} & \Phi \mathcal{A} \\
T \downarrow & & \downarrow \Phi T \\
\mathcal{B} & \xrightarrow{\sim} & \Phi \mathcal{B}
\end{array}
$$

commutes up to equivalence.

The fact that every bicategory is biequivalent to a 2-category is precisely the statement that the inclusion $\text{Gray} \to \text{Bicat}$ is triessentially surjective on objects, but as we saw in the lemma, it is not locally a biequivalence. On the other hand Gordon, Power, and Street construct in [2] a trihomomorphism $\text{st} : \text{Bicat} \to \text{Gray}$ which is locally a biequivalence (it induces a biequivalence on the hom-bicategories). They do this by appeal to their Section 3.6, but this does not imply that $\text{st}$ is a triequivalence, as they claim, and by our theorem it cannot be one. In fact Section 5.6 is not used in the proof of the main theorem of [2], it is only used to construct the tricategory $\text{Bicat}$ itself, and this does not need $\text{st}$ to be a triequivalence.

By the coherence result of [2], $\text{Bicat}$ is triequivalent to some Gray-category; and by the fact that $\text{st}$ is locally a biequivalence, $\text{Bicat}$ is triequivalent to a full sub-Gray-category of $\text{Gray}$, but it is not triequivalent to $\text{Gray}$ itself.

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