A method to obtain the all order quantum corrected Bose–Einstein distribution from the Wigner equation

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Abstract. A method has been introduced to derive the all order Bose–Einstein distribution as the solution of the Wigner equation. The process is a perturbative one where the Bose–Einstein distribution has been taken as the unperturbed solution. This solution has been applied to calculate the number density of the bosons at finite temperature. The study may be important to investigate the properties of bosons at finite temperature. This process can also be applied to obtain the all order Fermi distribution.

Keywords: quantum gases
1. Introduction

In contrast to the standard approach of the wave function in the Schrodinger picture, the phase space formulation of quantum mechanics [1–7] treats the position and momentum variables on equal footing and the quantum state is described by a quasiprobability distribution function. In the existing literature, there are several approaches to find out the quasiprobability function [8–13]. Among them, perhaps the most popular distribution was discovered by Wigner [3] in 1932 in the context of many body system where he calculated the quantum correction terms to the Gibbs–Boltzmann distribution function. In later years this method has been utilized in equilibrium and non-equilibrium quantum statistical mechanics. In addition to that, it has also been applied to pure quantum mechanical problems [14–18]. For example, It has been applied to study the quantum stochastic problems [15]. The pair distribution function of liquid neon [19] has been worked out and published. It has also been applied to study the quantum systems with Hamiltonian quadratic in the coordinates and momentas [14, 16]. Wigner distribution has turned out to be instrumental to determine the quantum corrections to simple molecular fluids [20]. It has been used to obtain the approximate solution of nonlinear Schrodinger equation [21]. It has been employed in the derivation of quantum corrections to the one component plasmas for two and three dimensional systems [22, 23]. Wigner equation is also relevant in the context of subjects like quantum chemistry and quantum optics [24].

Recently, an extensive review on recent advances in Wigner function approaches has been published and it seems to give a complete account of the past and recent scientific works involving Wigner function [25]. I would like to mention a few of them. Wigner function is useful for studying decoherence processes. A formulation of quantum mechanics in which the central notion is that of a quantum-mechanical history—a sequence of events at a succession of times has been studied. The primary aim is to identify sets of decoherent histories for the system [26]. The quantum to classical transition has been presented by comparing the dynamics of several nonclassicality indicators, such as the Wigner function interference fringe, the negativity of the Wigner
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function [27]. Open quantum dots provide a natural system in which to study both classical and quantum features of transport. Wigner function has been used to investigate the quantum-classical transition in open quantum dots [28].

A recent phase-space formulation of quantum mechanics in terms of the Glauber coherent states is applied to study the interaction of a one-dimensional harmonic oscillator with an arbitrary time-dependent force and wave functions of the simultaneous values of position q and momentum p and general expressions for the cross-Wigner functions are deduced [29].

It is shown that the Gaussian correlation functions derived from the Wigner functions of the oscillator are distorted by the driving electric field [30].

The harmonic oscillator coupled to a heat bath has been studied when the external force is a kicked impulsive force and the Wigner functions have been calculated, where the system approaches a quasistationary cyclic evolution. Subsequently, the thermodynamic properties of a nonintegrable, quantum chaotic system in contact with a heat bath at finite temperature has been studied [31].

The quantum interference between two well-separated trapped-ion mechanical oscillator wave packets has been observed. Reconstruction of the Wigner function of these states has been worked out and published [32].

The ability of the truncated Wigner phase-space method to reproduce the non-Gaussian statistics of the single-mode anharmonic oscillator has been studied [33].

The truncated Wigner approximation has been employed to analyze conservation laws and dynamics of a one-dimensional Bose gas [34].

This article deals with the Wigner distribution function of indistinguishable particles. The first step in this direction was taken by Uhlhenbeck and Gropper in 1932 [35]. They did not take into account the explicit spin effect while calculating the equation of state of nonideal Bose and Fermi gas. In the next stage, Green [7] worked out the connection between the density matrix obtained on the basis of classical statistics and the corresponding matrices for the bosons and fermions. An expression for the Wigner distribution function valid for systems of bosons or fermions is obtained by making use of correspondence relations between classical quantities and quantum mechanical operators [36]. Exchange quantum corrections in the case of one component plasma has been published [23, 37]. Recently, there has been a renewed interest in the application of the Wigner phase-space approach in the context of density-functional theory [38, 39]. In phase-space, the semiclassical expansion has been applied for the inclusion of gradient corrections to the phase-space distribution for the spatially inhomogeneous problems to incorporate quantum corrections beyond the local density approximation [40]. Another, simple closed form expression for the exact Wigner function of an ideal gas of harmonically trapped fermions or bosons at arbitrary temperature and dimensionality has been derived [41]. An introduction to the theory along with a Monte Carlo method for the simulation of time-dependent quantum systems of fermions evolving in a phase-space has been presented [43]. In a previous work [45] of the author the Wigner equation has been solved perturbatively for any arbitrary order without taking into account the quantum statistics. In this article quantum statistics is also incorporated to ensure the completeness of the problem. Moreover, in that previous article [45] the derivation of the solution of a particular order requires the exact solution of the previous order. If the solution of a particular order is correct up to that order but not exact.
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then the solution of the next order becomes incomplete. This limitation is also lifted in this work. We have calculated perturbatively the higher order corrections to the distribution function of the non condensate bosons. In this process we have chosen the Bose–Einstein distribution function as our unperturbed distribution. Finally, we have applied the result to calculate the number density of the bosons by taking the proper moments of the distribution function obtained in this article. This work may be useful to deal with the properties of the bosons at finite temperature. It may be important for the construction of fluid models and determination of bulk properties of the bosons.

2. Solution of the complete Wigner equation

The Wigner equation is

\[
\frac{\partial f}{\partial t} + p \frac{\partial f}{\partial x} - \frac{\partial \phi}{\partial x} \frac{\partial f}{\partial p} + \sum_{j=1}^{\infty} (-1)^{j+1} C_j \hbar^{2j} \frac{\partial^{2j+1} \phi}{\partial x^{2j+1}} \frac{\partial^{2j+1} f}{\partial p^{2j+1}} = 0
\]  

\( C_j = 1/(2)^{2j}(2j+1)! \)

\( f(x, p, t) \) is the single particle quasi-distribution function and \( \phi \) is the potential energy. The bosons are considered to be non interacting [42]. Hence, \( \phi \) is the external potential energy (it may be the trapping potential). This condition is justified as long as the temperature of the system is greater or around the degeneracy temperature. On the contrary, interaction is important for the Bose condensed state. Therefore, the distribution is meaningful for non condensate bosons only.

Equation (1) is written in a normalized form with the following normalized variables.

\[ t \sim \frac{t}{l \sqrt{m \beta}} \]
\[ x \sim \frac{x}{l} \]
\[ p \sim \frac{p \sqrt{\beta}}{\sqrt{m}} \]

\( l \) is the length scale of the system and \( \beta = 1/k_B T \). \( k_B \) is the Boltzmann constant and \( T \) is the temperature of the system. The normalized equation is then

\[
\frac{\partial f}{\partial t} + p \frac{\partial f}{\partial x} - \frac{\partial \phi}{\partial x} \frac{\partial f}{\partial p} + \sum_{j=1}^{\infty} (-1)^{j+1} C_j \hbar^{2j} \frac{\partial^{2j+1} \phi}{\partial x^{2j+1}} \frac{\partial^{2j+1} f}{\partial p^{2j+1}} = 0
\]

\( \Lambda = \sqrt{\frac{\hbar^2 \beta}{ml^2}} \)
A method to obtain the all order quantum corrected Bose–Einstein distribution from the Wigner equation and this equation is legitimate only if it is possible to develop the potential energy $\phi$ in a Taylor series.

The semi classical equilibrium solution is determined in a perturbative way. The starting point is the equation (2) by setting $\partial_t f$ to zero (to obtain the steady state solution) and the first term of the infinite series of the normalized equation is retained. The corresponding phase space distribution function is denoted by $f_2$

$$\frac{p}{\partial x} \frac{\partial f_2}{\partial x} - \frac{\phi}{\partial x} \frac{\partial f_2}{\partial p} + C_1 \Lambda^2 \frac{\partial^3 \phi}{\partial x^3} \frac{\partial f_2}{\partial p^3} = 0.$$  

(3)

If the $\Lambda^2$ order term is neglected, the following expression may be obtained as the solution of the steady state Vlasov equation for the Bose particles.

$$f_v = \frac{1}{\exp(-a_{01} + a_{11} \frac{p^2}{2}) - 1}$$  

(4)

where $a_{ij}$ are functions of $x$ but not $p$. $a_{01} = -\phi + \mu$ and $a_{11} = 1$. $\mu$ is the chemical potential and normalized as $\beta \mu$.

$f_v$ can be expressed as

$$f_v = \sum_{n=1}^{\infty} \exp[n(a_{01} - a_{11} \frac{p^2}{2})].$$  

(5)

If the $\Lambda^2$ order correction term is added to the Vlasov equation, the following solution is obtained [44].

$$f_2 = \sum_{n=1}^{\infty} W_{2n} = \sum_{n=1}^{\infty} \exp(a_{01n} - a_{11n} \frac{np^2}{2}).$$  

(6)

If, terms up to $\Lambda^2$ order are retained

$$a_{01n} = -n\phi + n\mu + \frac{n^3 \Lambda^2}{24} \frac{d\phi}{dx} \frac{d\phi}{dx} - \frac{n^2 \Lambda^2}{8} \frac{d^2 \phi}{dx^2}$$  

(7)

$$a_{11n} = 1 - \frac{n^2 \Lambda^2}{12} \frac{d^2 \phi}{dx^2}$$  

(8)

In the next stage, the first two terms of the series of equation (2) are taken into account and setting the time derivative to zero (to obtain the steady state solution)

$$\frac{p}{\partial x} \frac{\partial f_4}{\partial x} - \frac{\phi}{\partial x} \frac{\partial f_4}{\partial p} + C_1 \Lambda^2 \frac{\partial^3 \phi}{\partial x^3} \frac{\partial f_4}{\partial p^3} - C_2 \Lambda^4 \frac{\partial^5 \phi}{\partial x^5} \frac{\partial f_4}{\partial p^5} = 0$$  

(9)

and a solution of equation (9) is sought of the following form

$$f_4 = \sum_{n=1}^{\infty} W_{2n} W_{4n}$$

$$W_{4n} = \exp[\Lambda^4(a_{02n} + a_{12n} \frac{np^2}{2} + a_{22n} \frac{np^2}{2})].$$  

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In the case of \( W_{4n} \), a \( \Lambda^4 \) order polynomial of even power of \( p \) has been chosen as the argument of the exponential so that if the \( \Lambda^4 \) order correction is neglected the value of the exponential is unity and the previous lower order solution is found. In addition to that the exponential form is necessary as the solution is multiplicative in nature with the inclusion of the higher order terms of the Wigner equation. In \( W_{4n} \) the polynomial has been truncated at \( p^4 \) term because inclusion of \( p^6 \), \( p^8 \) etc produce unfeasible solution because the associated coefficients are constants.

Inserting \( W_{2n} \) and \( W_{4n} \) from equations (6) and (10) respectively and collecting the \( \Lambda^4 \) order coefficients of different powers of \( p \) and separately equating them to zero, a set of first order differential equations emerge [45].

These equations are given in the appendix and solved to derive the \( \Lambda^4 \) order solution which is given by

\[
f_4 = (1 - C_{11} \partial^4 - C_{21} \partial^2 - C_{31} \partial^3 - D_1 \partial^5 + E_1 \partial^4 + F_1 \partial^6 - G_1 \partial^3 + H_1 \partial^4 + I_1 \partial^6 - J_1 \partial^5 \\
- K_1 \partial^5 + L_1 \partial^4 + M_1 \partial^6 - N_1 \partial^6) f_n.
\]

This process as described in the appendix and in [45] may be continued and the solution of the complete Wigner equation is

\[
f_n = \sum_{n=1}^{\infty} \prod_{j=1}^{\infty} W_{2jn}.
\]

For \( j = 1 \), we have already derived \( W_{2n} \)

For \( j > 1 \), one can propose

\[
W_{2jn} = \exp(U_{2jn})
\]

\[
U_{2jn} = \sum_{i=0}^{j} \Lambda^{2j} a_{ijn} (\frac{p\sqrt{\Pi}}{\sqrt{2}})^{2i}
\]

where

\[
\frac{1}{2} \frac{d a_{ijn}}{d x} = \frac{n + i + 1}{2} \frac{1}{2} \frac{d}{d x} a_{(i+1)jn} + b_{ij} n^{i+1} + g_{ijn}
\]

the last term is the coefficient of \( p^{i+1} \) of

\[
\sum_{i=1}^{j-1} (-1)^{i+1} C_i \Lambda^{2i} \frac{\partial^{2i+1} \phi}{\partial x^{2i+1}} \frac{\partial^{2i+1} f_{2j-2n}}{\partial p^{2i+1}} W_{2jn}
\]

and

\[
b_{ij} = - \frac{\partial^{2i+1} \phi}{\partial x^{2i+1}} D_i C_j
\]

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in which, \( D_{i} = \text{coefficient of } p^{2i+1} \) of the Hermite polynomial \( H_{2j+1}(\frac{\sqrt{\rho}}{\sqrt{2}}) \).

These equations are true for \( i = 0 \) to \( i = j - 1 \)

For \( i = j \),

\[
\frac{1}{2^{i}} \frac{\partial a_{ijn}}{\partial x} = -n^{j+1} \frac{\partial^{2j+1} \phi}{\partial x^{2j+1}} c_{j} D_{j}.
\]  

(14)

This is a first order equation and can be easily solved to obtain the value of \( a_{ijn} \).

Using this result, all the equations of this group can be successively solved from the recursion relation as given by equation (12) to find out the remaining coefficients. Finally, \( a_{ijn} \) can be expressed in the following compact form.

\[
a_{ijn} = \sum_{k=0}^{j-i} c_{ikj} n^{j+k+1}
\]  

(15)

c_{ikj} can be identified from the expression of \( a_{ijn} \).

Therefore, the distribution up to the \( \Lambda^{2j'} \) order is

\[
\exp \left[ \sum_{j=1}^{j'} \sum_{i=0}^{j} \sum_{k=0}^{j-i} \Lambda^{2j} \frac{p^{2i}}{2^{i}} c_{ikj} \phi^{i+k+j+1} \right] f_{v}
\]

\[
f_{v} = \sum_{n=1} \exp \left[ -n \left( \frac{p^{2}}{2} + \phi - \mu \right) \right] = \frac{1}{\exp \left[ \left( \frac{p^{2}}{2} + \phi - \mu \right) \right] - 1}.
\]

In [44, 45] the same perturbative approach was considered. Therefore, it is necessary to clarify the differences between this work with [44, 45]. In [44] the classical Gibbs–Boltzmann distribution was chosen as the zeroth order solution of the Vlasov equation and the exact solution for the \( \Lambda^{2} \) order was obtained. In [45] the same classical Gibbs–Boltzmann distribution was chosen as the zeroth order solution of the Vlasov equation and the solution for any arbitrary order was attempted. Consequently, the particles in that system were considered to be distinguishable. On the other hand, in this article the indistinguishability factor is taken into account because the zeroth order solution is chosen as the Bose–Einstein distribution and that extends the applicability of this technique to explore systems that have a lower temperature and higher density than the previous systems analysed in [44, 45].

The difference between the distribution functions will be noticed if the \( \Lambda^{4} \) order solutions of [45] and this work are compared.

3. Discussion

In this work the all order solution of the Wigner equation for the bosons has been derived in the presence of external potential. In the original work of Wigner he calculated the second order quantum correction to the Gibbs–Boltzmann distribution and applied it to calculate the thermodynamic properties of the many-body system. It is evident that as we increase the density and decrease the temperature the system will approach the situation where quantum effect is more important and to describe
A method to obtain the all order quantum corrected Bose–Einstein distribution from the Wigner equation. In this condition the indistinguishibility factor of the particles should be incorporated for the completeness of the problem and we have taken care of that factor in our calculation.

At finite temperature we can calculate physical quantities like pressure by taking the proper moments of the distribution function derived in this article. It is observed that the higher order terms of the Wigner equation contain higher derivatives of the potential function. Therefore, the potential should be smooth enough so that the convergence of the problem is achieved.

It is important to compare the results obtained in this article with the ones already present in the literature. It can be seen that the solution derived in this article is identical with the one dimensional form of the $\Lambda^2$ order solution derived in [40].

We can apply the present formalism to obtain the distribution function of the non interacting bosons in the case of harmonic potential and for simplicity we shall seek the distribution function corrected up to the $\hbar^2$ order. In the next stage one can proceed step by step to derive the distribution function corrected up to any desired order. The distribution function can simply follow from equation (A.9) if terms up to the $\hbar^2$ are retained.

\[ f_{ho} = (1 - C_{11}\partial^3 - C_{21}\partial^2 - C_{31}\partial^3) f_v \]

(16)

$f_v$ is the Bose–Einstein distribution. The harmonic potential $\phi$ is given by $\frac{1}{2}m\omega_0^2 x^2$ in one dimension. Therefore, the values of $C_{11}$, $C_{21}$ and $C_{31}$ are

\[
C_{11} = \frac{1}{24} (\beta \hbar \omega_0)^2 \frac{m \omega_0^2 x^2}{k_B T} \\
C_{21} = \frac{1}{8} (\beta \hbar \omega_0)^2 \\
C_{31} = \frac{1}{24} (\beta \hbar \omega_0)^2 \frac{p^2}{m k_B T}.
\]

It is observed that two other terms have been added to the usual Bose–Einstein distribution term and we can combine them to obtain the distribution function in the following compact form

\[ f_{ho} = (1 - \frac{E}{12\beta^2} (\beta \hbar \omega_0)^2 \partial^3 - \frac{1}{8\beta^2} (\beta \hbar \omega_0)^2 \partial^2) f_v \]

(17)

where $E = \frac{1}{2}m\omega_0^2 x^2 + \frac{1}{2}p^2$, the partial derivative may be with respect to $E$ and $f_v$ is equal to $\frac{1}{e^{E} - 1}$. In the limit $e^{\beta(E-\mu)} \gg 1$ the above expression may be further simplified to

\[
\frac{\beta \hbar \omega_0}{\hbar} [1 - \frac{(\beta \hbar \omega_0)^2}{12} + \frac{(\beta \hbar \omega_0)^2}{12} \beta E e^{-\beta E}]
+ \frac{3}{2\hbar} (\beta \hbar \omega_0)^2 [1 - \frac{(\beta \hbar \omega_0)^2}{12} + \frac{(\beta \hbar \omega_0)^2}{12} \beta E e^{-\beta E}]
+ \frac{(\beta \hbar \omega_0)^2}{\hbar} [1 + 3\beta \hbar \omega_0 + \frac{11(\beta \hbar \omega_0)^2}{6} + 2\frac{(\beta \hbar \omega_0)^2}{3} \beta E e^{-2\beta E}].
\]

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The terms of the first third bracket represent the contribution of the inclusion of the first quantum correction term to the Vlasov equation and that part is in exact agreement with the $\hbar^2$ order solution extracted from the following exact Wigner distribution function

$$W_{\text{SHO}} = \frac{2 \tanh(\hbar \omega \beta / 2)}{\hbar} \exp\left[-\frac{2}{\hbar \omega} \tanh(\hbar \omega \beta / 2) E\right]$$

as given in [12] (without exchange effect) in the case of harmonic potential. The remaining terms are the result of the indistinguishibility factor.

In figure 1 $h_{fho}$ has been plotted against the normalized energy $\beta E$. The dashed line represents the contribution up to the $\hbar^2$ order correction term of the Wigner equation without the exchange effect. On the other hand, the solid line represents the contribution up to the $\hbar^2$ order correction term of the Wigner equation with the exchange effect. As expected, one can notice the difference between the two results in the low $\beta E$ limit and that diminishes gradually with the increase of the $\beta E$. The above observation may justify the importance of the inclusion of the exchange effect in the distribution function.

The interesting feature about the harmonic potential is the presentation of the distribution function in the above form (equation (17)) which may not be possible for other kind of potential. For example, in the case of quartic potential the distribution function can not be presented in the above form as in the case of the harmonic potential.

4. Conclusion

A formalism has been developed to determine the distribution function which incorporates the effect of higher order correction terms in the Wigner equation in addition to
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the indistinguishibility factor which is absolutely necessary as we probe more and more the high density and the low temperature regime.

The present formalism is applicable to a system of non interacting particles placed in external potentials of wide variety as long as the infinite series of the Wigner equation is convergent. Among those, the case of harmonic potential is discussed here and the distribution function is found to incorporate both the effects of the higher order correction terms to the Vlasov equation as well as the indistinguishibility factor.

The formalism may be applied even in the case of interacting bosons and the evaluation of the second virial coefficient including exchange effect will be the subject of the next article.

Another example of systems which may be treated by the same formalism is the case of Bose gases [46] in confining traps. The main point is how the present formalism describes both the quantum effects—one of which is related to the differential quotients of the potential of the concerned systems and the other is due to the exchange effect. At the end, it can be concluded that this article extends the phase space formulation of Wigner to the system of bosons with all higher order terms and may be applied to probe the properties of the bosons at finite temperature in the presence of external potential.

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Appendix. Derivation of the \( \Lambda^4 \) order stationary Wigner distribution

In order to determine the \( \Lambda^4 \) order stationary Wigner distribution we should insert \( W_{2n} \) and \( W_{4n} \) from equation (6) and equation (10) respectively in equation (9) and collecting the \( \Lambda^4 \) order coefficients of powers of \( p^5 \), \( p^3 \) and \( p \) and separately equating them to zero, the following three first order differential equations emerge [45].

\[
\frac{\partial a_{22n}}{\partial x} = -4n^3 C_2 \frac{\partial^5 \phi}{\partial x^5}
\]

(A.1)

\[
\frac{\partial a_{12n}}{\partial x} = 20n^3 C_2 \frac{\partial^5 \phi}{\partial x^5} + 2n \frac{\partial \phi}{\partial x} a_{22n} - C_1 n^4 \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^3 \phi}{\partial x^3}
\]

(A.2)

\[
\frac{\partial a_{02n}}{\partial x} = n \frac{\partial \phi}{\partial x} a_{12n} - 15 n^3 C_2 \frac{\partial^5 \phi}{\partial x^5} + C_1 n^4 \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^3 \phi}{\partial x^3}
\]

(A.3)

These first order equations are solved to obtain

\[
a_{22n} = -4n^3 C_2 \frac{\partial^4 \phi}{\partial x^4}
\]

(A.4)
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\[a_{12n} = 20n^3C_2 \frac{\partial^4\phi}{\partial x^4} - 8n^4C_2 \left( \frac{\partial\phi}{\partial x} \frac{\partial^3\phi}{\partial x^3} - \frac{1}{2} \frac{\partial^2\phi}{\partial x^2} \frac{\partial^2\phi}{\partial x^2} \right) - C_1 \frac{n^4}{4} \frac{\partial^2\phi}{\partial x^2} \frac{\partial^2\phi}{\partial x^2} \tag{A.5}\]

\[a_{02n} = -15n^3C_2 \frac{\partial^4\phi}{\partial x^4} + 20C_2n^4 \left( \frac{\partial\phi}{\partial x} \frac{\partial^3\phi}{\partial x^3} - \frac{1}{2} \frac{\partial^2\phi}{\partial x^2} \frac{\partial^2\phi}{\partial x^2} \right) - 8C_2n^5 \left( \frac{\partial\phi}{\partial x} \right)^2 \frac{\partial^2\phi}{\partial x^2} + \frac{C_1}{4} \left( \frac{\partial^2\phi}{\partial x^2} \frac{\partial^2\phi}{\partial x^2} \right). \tag{A.6}\]

These values of \(a_{02n}, a_{12n}\) and \(a_{22n}\) are substituted in the expression of \(W_4n\) in equation (10). Finally, all the exponential factors of \(W_2n\) and \(W_4n\) except the zeroth order term are expanded in the expression

\[f_4 = \sum_{n=1}^{\infty} W_{2n} W_{4n}\]

and the \(\Lambda^4\) order solution is given by

\[f_4 = \sum_{n=1}^{\infty} (1 + n^3C_{11} - n^2C_{21} + n^3C_{31} + n^5D_1 + n^4E_1 + n^6F_1 + n^3G_1 + n^4H_1 + n^6I_1 + n^5J_1 + n^5K_1 + n^4L_1 + n^6M_1 + n^5N_1) e^{n(a_0 - a_1 \frac{v^2}{2})}. \tag{A.7}\]

We find terms with different power of \(n\) in the above expression and that may be further simplified by the introduction of the partial derivative with respect to \(E\) (\(E\) is equal to \(\phi + p^2/2\)) which absorbs \(n\) according to the following identify

\[\sum_{n=1}^{\infty} n^k e^{n(a_0 - a_1 \frac{v^2}{2})} = (-1)^k \partial^k f_v \tag{A.8}\]

\((f_v\) is given by equation (5) of the manuscript).

Hence, the final form of the \(\Lambda^4\) order stationary Wigner distribution simplifies to

\[f_4 = (1 - C_{11} \partial^3 - C_{21} \partial^2 - C_{31} \partial^3 - D_1 \partial^5 + E_1 \partial^4 + F_1 \partial^6 - G_1 \partial^3 + H_1 \partial^4 + I_1 \partial^6 - J_1 \partial^5 - K_1 \partial^5 + L_1 \partial^4 + M_1 \partial^6 - N_1 \partial^5) f_v. \tag{A.9}\]

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