Non-asymptotic control of the cumulative distribution function of Lévy processes

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Abstract

We propose non-asymptotic controls of the cumulative distribution function $P(|X_t| \geq \varepsilon)$, for any $t > 0$, $\varepsilon > 0$ and any Lévy process $X$ such that its Lévy density is bounded from above by the density of an $\alpha$-stable type Lévy process in a neighborhood of the origin. The results presented are non-asymptotic and optimal, they apply to a large class of Lévy processes.

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1 Introduction and motivations

The law of any Lévy process $X$ is the convolution between a Gaussian process, the martingale $M$ describing its small jumps and a compound Poisson process. However, for most Lévy processes a closed form expression for the law of their increments is not known. The core of the problem lies in computing the distribution of the small jumps. This technical limitation makes both inference and simulations difficult for Lévy processes. To cope with this shortcoming it is usual to approximate a general Lévy process $X$ with a family of compound Poisson processes by ignoring the jumps smaller than some level $\varepsilon$. Also, when the Lévy measure is of infinite variation, solutions that consist in approximating the law of $M_t$ with a Gaussian distribution are motivated by Gnedenko and Kolmogorov (1954) (see also Cont and Tankov (2004), Cohen et al. (2007) or Carpentier et al. (2018)). This type of approximations are of interest because both Gaussian and compound Poisson processes are nowadays well understood, both in terms of continuous and discrete observations. The same cannot be said for the small jumps which remain complex objects, difficult to manipulate.

In order to quantify the precision of such approximations it becomes of crucial importance to have a sharp control of quantities such as $P(|X_t| > \varepsilon)$ and $P(|M_t| > \varepsilon)$. The issue, besides being interesting in itself, are sometimes required, for example to study non-asymptotic risk bounds for estimators of the Lévy density from discrete observations of $X$ (see Figueroa-López and Houdré (2009) or Duval and Mariucci (2017)). This has important consequences in various fields of application where Lévy processes are commonly used to describe real life phenomena. The literature on the applications of Lévy processes is extensive, ranging from financial, biology, geophysics and neuroscience, to name but a few. In this respect, we will limit ourselves to mention Barndorff-Nielsen et al. (2012) and the references therein.

Formally, a Lévy process $X$ is characterized by its Lévy triplet $(b, \Sigma^2, \nu)$ where $b \in \mathbb{R}$, $\Sigma \geq 0$ and $\nu$ is a Borel measure on $\mathbb{R}$ such that

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} (y^2 \wedge 1) \nu(dy) < \infty.$$
The Lévy-Itô decomposition (see Bertoin (1996)) allows to write a Lévy process $X$ of Lévy triplet $(b, \Sigma^2, \nu)$ as the sum of four independent Lévy processes, for all $t \geq 0$,

$$X_t = tb + \Sigma W_t + \lim_{\eta \to 0} \left( \sum_{s \leq t} \Delta X_s 1_{(\eta, 1)}(\vert \Delta X_s \vert) - t \int_{\eta < \vert x \vert \leq 1} x \nu(dx) \right) + \sum_{s \leq t} \Delta X_s 1_{(1, \infty)}(\vert \Delta X_s \vert)$$

$$=: tb + \Sigma W_t + M_t + Z_t,$$  

(1)

where $\Delta X_r$ denotes the jump at time $r$ of the cdlg process $X$: $\Delta X_r = X_r - \lim_{s \uparrow r} X_s$. The first term is a deterministic drift, $W$ is a standard Brownian motion which is path-wise continuous and $M$ and $Z$ compose the discontinuous jump part of $X$. The process $M$ is a centered martingale gathering the small jumps i.e. the jumps of size smaller than 1 and it has Lévy measure $1_{|\varepsilon| \leq 1} \nu$. The process $Z$ instead, is a compound Poisson process gathering jumps larger than 1 in absolute value, it has Lévy measure $1_{|\varepsilon| > 1} \nu$. In the sequel we make $(b, \Sigma) = (\gamma, 0)$ with

$$\gamma : \begin{cases} \int_{|x| \leq 1} x \nu(dx) & \text{if } \int_{|x| \leq 1} |x| \nu(dx) < \infty \\ 0 & \text{if } \int_{|x| \leq 1} |x| \nu(dx) = \infty, \end{cases}$$

(see Section 2.5 for a discussion in the general case) and rewrite (1) as

$$X_t = tb(\varepsilon) + M_t(\varepsilon) + Z_t(\varepsilon), \quad \forall \varepsilon > 0,$$  

(2)

where,

$$b(\varepsilon) : \begin{cases} \int_{|x| \leq \varepsilon} x \nu(dx) & \text{if } \int_{|x| \leq 1} |x| \nu(dx) < \infty \\ - \int_{|x| \leq \varepsilon} x \nu(dx) & \text{if } \int_{|x| \leq 1} |x| \nu(dx) = \infty, \end{cases}$$

$$M(\varepsilon) = (M_t(\varepsilon))_{t \geq 0}$$

is a Lévy process accounting for the centered jumps of $X$ with size smaller than $\varepsilon$:

$$M_t(\varepsilon) = \lim_{\eta \to 0} \left( \sum_{s \leq t} \Delta X_s 1_{\eta < \vert \Delta X_s \vert \leq \varepsilon} - t \int_{\eta < \vert x \vert \leq \varepsilon} x \nu(dx) \right),$$

and $Z(\varepsilon) = (Z_t(\varepsilon))_{t \geq 0}$ is a compound Poisson process of the form $Z_t(\varepsilon) := \sum_{i=1}^{N_t(\varepsilon)} Y_i(\varepsilon)$, where

$$N(\varepsilon) = (N_t(\varepsilon))_{t \geq 0}$$

is a Poisson process of intensity $\lambda_\varepsilon := \int_{|x| > \varepsilon} \nu(dx)$ independent of the sequence of i.i.d. random variables $(Y_i(\varepsilon))_{i \geq 1}$ with common law $\nu_{|x| > \varepsilon}/\lambda_\varepsilon$. In the sequel we use the notations $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

A first well known result (see e.g. Bertoin (1996) Section I.5 or Corollary 3 in Rüschendorf and Woerner (2002)) relates the Lévy measure to the limit of $\mathbb{P}(|X_t| \geq \varepsilon)$ as $t \to 0$ as follows.

**Lemma 1.** Let $X$ be a Lévy process with Lévy measure $\nu$. For all $\varepsilon > 0$ it holds that

$$\lim_{t \to 0} \frac{\mathbb{P}(|X_t| \geq \varepsilon)}{t} = \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \nu(dy).$$

In particular, it leads to

$$\lim_{t \to 0} \frac{\mathbb{P}(|M_t(\varepsilon)| \geq \varepsilon)}{t} = 0 \quad \text{and} \quad \lim_{t \to 0} \frac{\mathbb{P}(|tb(\varepsilon) + M_t(\varepsilon)| \geq \varepsilon)}{t} = 0.$$

Lemma 1 suggests that $\mathbb{P}(|X_t| \geq \varepsilon) \approx \lambda_\varepsilon t$ “for $t$ small enough”, however it gives no information on how small $t$ should be, nor on the size of the error term $\mathbb{P}(|X_t| \geq \varepsilon) - \lambda_\varepsilon t$ nor on what happens if $\varepsilon$ gets small. Of course, $\mathbb{P}(|X_t| \geq \varepsilon)$ and $\mathbb{P}(|M_t(\varepsilon)| \geq \varepsilon)$ can be controlled with elementary inequalities, such as the Markov inequality, but this often leads to sub-optimal results. Indeed, the Markov inequality gives $\mathbb{P}(M_t(\varepsilon) > \varepsilon) \leq t \sigma^2(\varepsilon) \varepsilon^{-2}$, if we denote by $\sigma^2(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} x^2 \nu(dx)$, the variance of $M_t(\varepsilon)$, whereas a sharper result, can be achieved using the Chernov inequality as follows.
Lemma 2. For any $\varepsilon \in (0, 1]$, $t > 0$ and $x > 0$, it holds:

$$
\mathbb{P}(|M_t(\varepsilon)| > x) \leq 2e^2\left(\frac{t\alpha^2(\varepsilon)}{x + t\alpha^2(\varepsilon)}\right)^{\frac{x + t\alpha^2(\varepsilon)}{2}}.
$$

Moreover, if $t\alpha^2(\varepsilon)x^{-2} \leq 1$, it leads to

$$
\mathbb{P}(M_t(\varepsilon) > x) \leq \left(\frac{x\alpha^2(\varepsilon)}{e^2}\right)^{\frac{\alpha}{2}}e^{-1}x^{\frac{\alpha}{2}} \quad \text{and} \quad \mathbb{P}(M_t(\varepsilon) \leq -x) \leq \left(\frac{x\alpha^2(\varepsilon)}{e^2}\right)^{\frac{\alpha}{2}}e^{-1}x^{\frac{\alpha}{2}}.
$$

Lemma 2 is a modification of Remark 3.1 in Figueroa-López and Houdré (2009). A similar result can also be obtained using martingale arguments (see Dzhaparidze and Van Zanten (2001), Theorem 4.1). Again Lemma 2 is suboptimal as it does not allow to derive that $\lim_{t \to 0} \mathbb{P}(M_t(\varepsilon) \geq \varepsilon)/t = 0$. If we want to be more precise about the behavior for $t \to 0$ we need additional assumptions.

Studying the behavior in small times of the transition density of Lévy process goes back to Léandre (1987) (see also Ishikawa (1994)) and is carried in the real case in Picard (1997) which is also interested in the behavior of the supremum of this quantity and its derivatives. For the cumulative distribution function, expansions of order 2 for $\mathbb{P}(X_t \geq y)$, for fixed $y$ and $t$ going to 0, are given in Marchal (2009) in the particular cases where $X$ is the sum of a compound Poisson process and either a Brownian motion or an $\alpha$-stable process.

The most complete results can be found in Figueroa-López and Houdré (2009), which, for general Lévy processes, establishes asymptotic expansions at any order of $\mathbb{P}(X_t \geq y)$, for fixed $y$ bounded away from 0 and $t \to 0$. They prove that

$$
\mathbb{P}(X_t \geq y) = e^{-\lambda_0} \sum_{j=1}^{n} c_j t^j + O(n)(t^{n+1}),
$$

where $n \geq 1$, $0 < \eta < 1 \wedge (y/(n + 1))$, the Lévy density has $2n + 1$ bounded derivatives away from the origin, $y \geq 0$ and $0 < t < t_0$, for some $y$ and $t_0$. No bounds on either $y$ or $t_0$ are provided.

In the case $n = 1$, they further prove that

$$
d_2(y) = \lim_{t \to 0} \frac{1}{t} \left(\frac{1}{t} \mathbb{P}(X_t > y) - \nu((y, \infty))\right)
$$

exists, when the Lévy density $f$ is bounded outside the interval $[-\eta, \eta]$, $0 < \eta < y/2 \wedge 1$, and either $f$ is $C^1$ in a neighborhood of $y$, or $f$ is continuous in a neighborhood of $y$, of bounded variation and $\Sigma = 0$ (defined as in (1)). This is again an asymptotic result; therefore, it provides no information on how small $t$ should be for the approximation of $\mathbb{P}(X_t > y) - \nu((y, \infty))$ by $d_2(y)t^2$ to be accurate. Moreover, even though they give an explicit characterization of $d_2(y)$, this does not translate in a readily understandable dependency on $y$.

Our main contribution is a non-asymptotic control of $\mathbb{P}(|X_t| \geq \varepsilon)$, which is valid for any $\varepsilon > 0$ and any $0 < t < t_0(\varepsilon)$. A lot of effort has been made to make the dependency on $\varepsilon$ explicit, both in $t_0(\varepsilon)$ and in the final bound. Concerning the hypotheses on the Lévy density $f$, in the finite variation case we do not require any continuity, but only that it is bounded from above by an $\alpha$-stable like density in a neighborhood of 0, see the definition on the class $\mathcal{L}_{M,\alpha}$ below. In the setting of infinite variation, we distinguish two cases: when $f$ is also Lipschitz continuous in a neighborhood of $\varepsilon$ (a similar condition to that of Figueroa-López and Houdré (2009)), we find a non-asymptotic bound of the order of $t^2$. We also analyze the case where the continuity hypothesis on $f$ is dropped. Then, the order in $t$ of the non-asymptotic bound deteriorates to $t^{1+1/\alpha}$, $1 \leq \alpha < 2$. This is not an artifact of the proof, as an example in Marchal (2009) indicates.
The case of the small jumps is treated separately as an intermediate step to the general case (see Theorems 1 and 3). We think that these results are of independent interest and provide a new insight on the process of the small jumps. Finally, our proofs are elementary and self-contained and they do not rely on the use of the infinitesimal generator.

Next Section 2 gathers the main results of the paper. We begin with defining the classes of Lévy densities that we consider. On these classes we provide a non-asymptotic control of $\mathbb{P}(|M_t(\varepsilon)| \geq \varepsilon)$ and $\mathbb{P}(|X_t| \geq \varepsilon)$. We consider separately finite variation Lévy processes and infinite variation Lévy processes, for which we only detail the symmetric case. In both cases our results permit to recover Lemma 1. We compare our results to examples for which the quantity $\mathbb{P}(|X_t| \geq \varepsilon)$ is known. Section 2 ends with a discussion on the validity of the results in presence of a Brownian component. Section 3 gathers the proofs of the main results whereas in Appendix A all auxiliary results are established and the computations of the examples are carried out.

2 Non-asymptotic expansions

Consider $\alpha \in (0,2)$ and $M$ be positive constants, define the classes of functions

$$\mathcal{L}_{M,\alpha} := \left\{ f : f(x) \leq \frac{M}{|x|^{\alpha}}, \quad \forall |x| \leq 2 \right\}, \quad \mathcal{L}_M := \left\{ f : \sup_{|x| \geq 1} |f(x)| \leq M \right\}.$$

A Lévy density $f$ belongs to the class $\mathcal{L}_M$, $M > 0$, if it is bounded outside a neighborhood of the origin. It belongs to $\mathcal{L}_{M,\alpha}$, $M > 0$ and $\alpha > 0$, if $\sup_{x \in [2,2]} |f(x)|^{1+\alpha} \leq M$. In particular $\mathcal{L}_{M,\alpha}$ contains any $\tilde{\alpha}$-stable Lévy density such that $\tilde{\alpha} \leq \alpha$. Also any finite variation Lévy process is in the class $\mathcal{L}_{M,1}$, for some positive $M$. We stress that no lower bound condition is required for the Lévy density.

2.1 Finite variation Lévy processes

We state two non-asymptotic results offering a control of the distribution function of a finite variation Lévy process.

**Theorem 1.** Let $\nu$ be a Lévy measure absolutely continuous with respect to the Lebesgue measure and denote by $f = \frac{d\nu}{dx}$. Let $\varepsilon \in (0,1], \alpha \in (0,1)$, $M > 0$ and $f \in \mathcal{L}_{M,\alpha}$. Then, there exists a constant $C_1 > 0$, only depending on $\alpha$, such that

$$\mathbb{P}(|t \varepsilon + M_t(\varepsilon)| \geq \varepsilon) \leq 2t^2 M^2 C_1 \varepsilon^{-2\alpha}, \quad \forall 0 < t \leq \frac{(1 - \alpha)\varepsilon^\alpha}{M^{1+\alpha}}.$$

If, in addition, $f$ is a symmetric function, then there exists a constant $C_2 > 0$, only depending on $\alpha$, such that

$$\mathbb{P}(|M_t(\varepsilon)| \geq \varepsilon) \leq 2t^2 M^2 C_2 \varepsilon^{-2\alpha}, \quad \forall 0 < t \leq \frac{\varepsilon^\alpha(2 - \alpha)}{M^{2\alpha+1}}.$$

Explicit formulas for the constants $C_1$ and $C_2$ are given in (21) and (22), respectively.

Theorem 1 highlights how likely the process of the jumps smaller than $\varepsilon$ is to present excursions larger than their size $\varepsilon$ in a time interval of length $t$. When dealing with a discretized trajectory of a Lévy process, this provides relevant information on the contribution of the small jumps to the value of the observed increment. The following result generalizes Theorem 1 to any Lévy process with a Lévy density in $\mathcal{L}_{M,\alpha}$, $\alpha \in (0,1)$ or in $\mathcal{L}_{M,\alpha} \cap \mathcal{L}_M$ if $\varepsilon > 1$. In particular it permits to derive an order of the rate of convergence in Lemma 1.
\textbf{Theorem 2.} Let $X_t = \sum_{s \leq t} \Delta X_s$ be a finite variation Lévy process with Lévy measure $\nu$ absolutely continuous with respect to the Lebesgue measure and denote by $f = \frac{d\nu}{dx}$.

- If $\varepsilon \in (0, 1]$ and $f \in \mathcal{L}_{M, \alpha}$ for some $\alpha \in (0, 1)$ and $M > 0$, then for all $0 < t < (1 - \alpha)M^{-\varepsilon}4^{-1(1+\alpha)}$ it holds
  \[ |P(|X_t| > \varepsilon) - \lambda_\varepsilon t| \leq t^2 M^2 \varepsilon^{-2\alpha} (2C_1 + D_1) + t^2 M \lambda_\varepsilon \varepsilon^{-\alpha} D_2 + 2t^2 \lambda_\varepsilon^2, \]
  where $C_1$, $D_1$, and $D_2$ only depend on $\alpha$ and are defined in (21) and (41).

- If $\varepsilon > 1$ and $f \in \mathcal{L}_{M, \alpha} \cap \mathcal{L}_M$ for some $\alpha \in (0, 1)$ and $M > 0$, then for all $0 < t < (1 - \alpha)(5M)^{-1}$ it holds
  \[ |P(|X_t| > \varepsilon) - \lambda_\varepsilon t| \leq 2M^2 t^2 (\widetilde{D}_1 + C_1) + 2t^2 \lambda_\varepsilon^2 + 2Mt^2 \left( \frac{4}{2 - \alpha} (\varepsilon - 3/2 - t|b(1)|) 1_{\varepsilon > 3/2 + t|b(1)|} \right) \]
  \[ + Mt^2 \left( 4 \times 5^\alpha 1_{1 < \varepsilon < 1 + 2t|b(1)|} + \frac{8}{5} + \frac{3}{2} \lambda_\varepsilon + \frac{4\lambda_\varepsilon}{2 - \alpha} \right), \]
  where $C_1$ and $\widetilde{D}_1$ only depend on $\alpha$ and are defined in (21) and (49).

If in addition we suppose that $\nu$ is a symmetric measure, then

- If $\varepsilon \in (0, 1]$ and $f \in \mathcal{L}_{M, \alpha}$ for some $\alpha \in (0, 1)$ and $M > 0$, for any $0 < t < \varepsilon^2 (2 - \alpha) M^{-1/2 - \alpha}$ it holds
  \[ |P(|X_t| > \varepsilon) - \lambda_\varepsilon t| \leq 2t^2 M^2 \varepsilon^{-2\alpha} (C_2 + D_3) + \frac{t^2 M}{2(2 - \alpha)} (\lambda_\varepsilon \varepsilon^{-\alpha} + 4\lambda_\varepsilon \varepsilon^{-\alpha}) + 2t^2 \lambda_\varepsilon^2, \]
  where $C_2$ and $D_3$ only depend on $\alpha$ and are defined in (22) and (50).

- If $\varepsilon > 1$ and $f \in \mathcal{L}_{M, \alpha} \cap \mathcal{L}_M$ for some $\alpha \in (0, 1)$ and $M > 0$, for any $0 < t < (2 - \alpha) M^{-1/2 - \alpha}$ it holds
  \[ |P(|X_t| > \varepsilon) - \lambda_\varepsilon t| \leq 2t^2 M^2 C_2 + \frac{t^2 M}{2 - \alpha} \left( \lambda_1 2^{-\alpha} + \frac{4M}{\alpha(1 - \alpha)} + \lambda_1 \varepsilon \right) + 2t^2 \lambda_1^2, \]
  where $C_2$ is defined in (22).

The results of Theorems 1 and 2 are non-asymptotic. If we apply Theorem 2 to a Lévy process $X$ whose Lévy measure $\nu$ is concentrated on $[-\varepsilon, \varepsilon]$, for $\varepsilon \in (0, 1]$, we recover the result of Theorem 1 up to the constant $D_1$ as in that case $\lambda_\varepsilon = 0$. Though, Theorem 1 is not a corollary of Theorem 2 as the proof of the latter results uses Theorem 1.

These results show that for a finite variation Lévy process whose Lévy density lies in $\mathcal{L}_{M, \alpha}$, for some $\alpha \in (0, 1)$ and $M > 0$, the discrepancy between $P(|X_t| > \varepsilon)$ and $\lambda_\varepsilon t$ is in $t^2$. Moreover, as the role of the cutoff $\varepsilon$ is made explicit in the upper bound, it is possible to measure the accuracy of this approximation when $\varepsilon$ gets small. Then, the rate of the upper bound is up to a constant in $t^2 (\varepsilon^{-2\alpha} \vee \lambda_\varepsilon \varepsilon^{-\alpha} \vee \lambda_\varepsilon^2)$. For example for an $\alpha$-stable process with $\alpha \in (0, 1)$ this order simplifies in $t^2 \lambda_\varepsilon^2$.

\section*{2.2 Symmetric infinite variation Lévy processes}

We generalize Theorems 1 and 2 to symmetric infinite variation Lévy processes whose Lévy density lies in $\mathcal{L}_{M, \alpha}$, $\alpha \in (1, 2)$ and $M > 0$. 

\vspace{1cm}

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Theorem 3. Let $\nu$ be a symmetric Lévy measure absolutely continuous with respect to the Lebesgue measure and denote by $f = \frac{d\nu}{d\lambda}$. Let $\varepsilon \in (0, 1]$, $\alpha \in [1, 2)$, $f \in \mathcal{L}_{M, \alpha}$ and $0 < t < (\varepsilon/2)^\alpha(1 \wedge ((2 - \alpha)/2M))$. Then, there exists a constant $E_1 > 0$, only depending on $\alpha$ (see (30)), such that

$$
P(\{M_i(\varepsilon) \geq \varepsilon\}) \leq \frac{\varepsilon^{2+\alpha}}{\varepsilon^{1+\alpha}} \left( 1 + \frac{M}{\alpha(2 - \alpha)(\alpha - 1)} \right) + 2t^2 M^2 E_1 \varepsilon^{-2\alpha}, \quad \alpha \in (1, 2),$$

$$
P(M_i(\varepsilon) \geq \varepsilon) \leq \frac{4t^2 M^2}{\varepsilon^2} \left( \varepsilon^{2+1/\varepsilon} + \frac{37}{9} + \frac{4M^2}{\varepsilon^2} + \frac{16M^2}{\varepsilon^2} t^2 \ln \left( \frac{\varepsilon}{2t} \right), \quad \alpha = 1. \right.$$  

Theorem 4. Let $\nu$ be a symmetric Lévy measure with density $f$ with respect to the Lebesgue measure and $f \in \mathcal{L}_{M, \alpha} \cap \mathcal{L}_M$ for some $\alpha \in [1, 2)$ and $M > 0$. Then, for all $0 < t < (\varepsilon \wedge 1/2)^\alpha(1 \wedge ((2 - \alpha)/2M))$, $\varepsilon > 0$, it holds:

$$
|P(|X_t| > \varepsilon) - \lambda_\varepsilon t| \leq G_1 \left( \frac{t^{1+1/\alpha}}{(\varepsilon \wedge 1)^{1+\alpha}} + G_2 \left( \frac{t^2}{(\varepsilon \wedge 1)^{2+2\alpha}} + \frac{5M}{2 - \alpha} \left( \frac{t^2}{(\varepsilon \wedge 1)^{2+2\alpha}} \right)^2 + \frac{4M^2t^2}{2 - \alpha} \right) \right) \right) \right) + 2M^2 \chi_{\varepsilon \wedge 1} t^2,$$

where $C := (1 \wedge ((2 - \alpha)/2M))^{1/\alpha}$ and $G_1$ and $G_2$ are positive constants, only depending on $M$ and $\alpha$, defined in (31).

Compared to Theorems 1 and 2 the rates of Theorems 3 and 4 are slower as $t^2 \leq t^{1+1/\alpha}$ for $\alpha \in (1, 2)$. Nevertheless, the rate $t^{1+1/\alpha}$ of Theorems 3 and 4 seems optimal. Indeed, as shown in Remark 3.5 of Figueroa-López and Houdré (2009) (see also Marchal (2009)) it is possible to build a discontinuous Lévy measure $f$ as the sum of an $\alpha$-stable Lévy process plus a compound Poisson process presenting a discontinuity at $\varepsilon$ that lies in $\mathcal{L}_{M, \alpha}$ and attains this rate $t^{1+1/\alpha}$. Adding a regularity assumption on $f$ on a neighborhood of $\varepsilon$, it is possible to have a finer bound in $t^2$ as established in the following result.

Theorem 5. Let $\nu$ be a symmetric Lévy measure having a density $f$ with respect to the Lebesgue measure with $f \in \mathcal{L}_{M, \alpha}$ for some $\alpha \in [1, 2)$ and $M > 0$. Let $\varepsilon > 0$ and assume that $f$ is $M(\varepsilon \wedge 1)^{-(2+\alpha)}$. Lipschitz on the interval $((3/4(\varepsilon \wedge 1), 2\varepsilon - 3/4(\varepsilon \wedge 1))$. For all $0 < t \leq \frac{(2 - \alpha)(1 + \alpha)}{2(\varepsilon \wedge 1)^{2+2\alpha}}$, it holds:

$$
P(|X_t| > \varepsilon) - \lambda_\varepsilon t \leq t^2 M^2 \left( (F_1 \varepsilon^{-2\alpha} + \lambda_1 \varepsilon^{-\alpha} F_2) \chi_{0 < \varepsilon \leq 1} + (\varepsilon^2 F_4 + F_4) \chi_{\varepsilon > 1} \right) + 2t^2 \lambda_1^2 + \frac{t^4 M^4 F_5}{(\varepsilon \wedge 1)^{4\alpha}},$$

where $F_1, \ldots, F_5$ are universal positive constants, only depending on $\alpha$, defined in (32).

First, note that any Lévy density $f$ that writes as $L(x)/x^{1+\alpha}$ for $x \in [-2, 2] \setminus \{0\}$, where $L$ is differentiable, bounded, with bounded derivative and $\alpha \in (1, 2)$ satisfies the assumptions of Theorem 5. Moreover, under the latter assumption, Theorem 5 applied to a Lévy process $X$ whose Lévy density $f$ is concentrated on $[-\varepsilon, \varepsilon]$, $\varepsilon \in (0, 1]$, leads to a finer rate than the one of Theorem 3, namely,

$$
P(|M_t(\varepsilon)| > \varepsilon) \leq t^2 M^2 (F_1 \varepsilon^{-2\alpha} + \lambda_1 \varepsilon^{-\alpha} F_2) + 2t^2 \lambda_1^2 + \frac{t^4 M^4 F_5}{(\varepsilon \wedge 1)^{4\alpha}}.$$  

2.3 Discussion

The results of Theorems 1 to 5 are non-asymptotic and show the impact of the cutoff $\varepsilon$ in the constants. In particular they permit to recover, for every fixed $\varepsilon > 0$, on the classes considered, the result of Lemma 1 having $t \to 0$.

Optimality of the results The rates of Theorems 1, 2 and 5 are of the form $t^2(\varepsilon \wedge 1)^{-2\alpha}$, up to a constant depending on $M$ and $\alpha$. This quantity is optimal in $t$ on the considered classes. Indeed, in
next Section 2.4 we show that for compound Poisson processes, for which explicit calculations can be performed and which are included in \( L_{M,\alpha} \) for all \( \alpha \in (0, 2) \), examples can be built attaining this rate. As already highlighted, the rate of Theorem 3 is also optimal. The dependency in \( \varepsilon \) of the constant \( \varepsilon^{-2\alpha} \) also appears to be the right one, since, for an \( \alpha \)-stable process, it holds that \( \lambda_\varepsilon = O(\varepsilon^{-\alpha}) \). Therefore, in general it is not possible to improve the rates derived in these Theorems, even though this might be possible on specific examples (see the Cauchy process in Section 2.4).

**Strategy of the proofs** All the proofs are self-contained, they rely on the decomposition (2), which holds for any Lévy process and any level \( \varepsilon > 0 \), and Lemma 2. More precisely, to establish Theorems 2 and 4, we consider the decomposition (2), and write

\[
\mathbb{P}(|X_t| > \varepsilon) = \mathbb{P}\left(|tb(\varepsilon) + M_t(\varepsilon) + \sum_{i=1}^{N_t(\varepsilon)} Z_i| > \varepsilon\right).
\]

Decomposing on the values of the Poisson process \( N(\varepsilon) \) leads to

\[
\mathbb{P}(|X_t| > \varepsilon - \lambda_\varepsilon t) \leq \mathbb{P}(tb(\varepsilon) + M_t(\varepsilon) + Z_1| > \varepsilon) + \lambda_\varepsilon t|tb(\varepsilon) + M_t(\varepsilon) + Z_1| > \varepsilon)e^{-\lambda_\varepsilon t} - 1 + \mathbb{P}(N_t(\varepsilon) \geq 2).
\]

The last term raises no difficulty as \( \mathbb{P}(N_t(\varepsilon) \geq 2) = O(\lambda_t^2 \varepsilon^2) \). The first term is treated in Theorems 1 and 3 which are established using decomposition (2) at level \( \varepsilon/2 \) and Lemma 2. The proof of Theorem 1 is made particularly technical by the presence of the drift term \( b(\varepsilon) \). This is the reason why, in the infinite variation counterpart Theorem 3 we specialize to the symmetric case, hence \( b(\varepsilon) = 0 \). Finally, to prove Theorems 2 and 5 (resp. Theorem 4) it remains to show: \( \mathbb{P}(tb(\varepsilon) + M_t(\varepsilon) + Z_1| > \varepsilon) = O(t\lambda_\varepsilon) \) (resp. \( O(\varepsilon(1/\alpha)) \)) which corresponds to proving that \( \mathbb{P}(tb(\varepsilon) + M_t(\varepsilon) + Z_1 > \varepsilon)e^{-\lambda_\varepsilon t} - 1 = O(\lambda_\varepsilon t) \) (resp. \( O(\varepsilon(1/\alpha)) \)).

For this term the cases of finite variation (Theorem 2) and infinite variation (Theorems 4 and 5) Lévy processes essentially differ. For finite variation Lévy processes, \( \alpha \in (0, 1) \), the result \( \mathbb{P}(tb(\varepsilon) + M_t(\varepsilon) + Z_1| \leq \varepsilon) = O(t\lambda_\varepsilon) \) holds true and a main difficulty here lies in the management of the drift that can be nonzero. For infinite variation Lévy processes, \( \alpha \in [1, 2] \), this result is not true in general. For instance, consider the case of a Cauchy process \( X \) and fixed \( \varepsilon \). The Cauchy process has a Lévy density \( (\pi x^2)^{-1} 1_{\mathbb{R}\setminus\{0\}} \) and is therefore in \( L_{1/\pi, 1} \cap L_{1/\pi} \) and is \( \pi^{-1/2} 3^{-3/2} (\varepsilon \wedge 1)^{-3/2} \)-Lipschitz on the interval \( ((3/4\varepsilon \wedge 1), 2\varepsilon - 3/4\varepsilon \wedge 1) \) for all \( \varepsilon > 0 \). Theorems 3, 4, and 5 thus apply. For this example direct calculations allow to show that \( \lim_{t \to 0} \frac{\mathbb{P}(tb(\varepsilon) + M_t(\varepsilon) + Z_1| \leq \varepsilon)}{\varepsilon} = \infty \) implying that \( \mathbb{P}(tb(\varepsilon) + M_t(\varepsilon) + Z_1| \leq \varepsilon) = O(t\lambda_\varepsilon) \) cannot hold. Indeed, the Lévy measure being symmetric it leads to \( b(\varepsilon) = 0 \) and

\[
\mathbb{P}(M_t(\varepsilon) + Z_1| \leq \varepsilon) = \frac{1}{\lambda_\varepsilon} \int_{-\varepsilon}^{\varepsilon} \mathbb{P}(M_t(\varepsilon) + z| \leq \varepsilon) \frac{dz}{\pi z^2} + \frac{1}{\lambda_\varepsilon} \int_{-\varepsilon}^{\varepsilon} \mathbb{P}(M_t(\varepsilon) + z| \leq \varepsilon) \frac{dz}{\pi z^2}.
\]

Fatou Lemma, joint with \( \lim_{t \to 0} \frac{\mathbb{P}(M_t(\varepsilon) \in A)}{t} = \nu_\varepsilon(A), \nu_\varepsilon = \nu_{|z| \leq \varepsilon}, \) and \( f \) being symmetric, gives

\[
\lambda_\varepsilon \liminf_{t \to 0} \frac{\mathbb{P}(M_t(\varepsilon) + Z_1| \leq \varepsilon)}{t} \geq \left( \int_{-\varepsilon}^{\varepsilon} + \int_{-\varepsilon}^{\varepsilon} \right) \liminf_{t \to 0} \frac{\mathbb{P}(M_t(\varepsilon) \in (-\varepsilon - z, \varepsilon + z))}{t} \frac{dz}{\pi z^2} \geq \int_{-\varepsilon}^{\varepsilon} \nu_\varepsilon(z - \varepsilon, z + \varepsilon) \nu(dz) = \int_{-\varepsilon}^{\varepsilon} \nu_\varepsilon(z - \varepsilon, \varepsilon) \nu(dz)
\]

\[
= \frac{1}{\pi} \int_{-2\varepsilon}^{2\varepsilon} \frac{2\varepsilon - z}{\varepsilon(z - \varepsilon)^2} \frac{dz}{\varepsilon(z - \varepsilon)} = \infty.
\]

We derive that the decomposition (4) that leads to Theorem 2, \( \alpha \in (0, 1) \), does not permit to obtain optimal results for \( \alpha \in [1, 2] \) such as Theorem 5. This is instead obtained by firstly adding a regularity assumption in a neighborhood of \( \varepsilon \) and secondly modifying the decomposition (4), considering a cutoff level \( \varepsilon' < \varepsilon \), for example \( \varepsilon' = 3\varepsilon/4 \) (see Lemmas 5 and 6 below).

Generalizing the results of Theorems 3, 4 and 5 to non-symmetric Lévy processes is possible at the expense of more cumbersome proofs and modifying the conditions on \( t \).
2.4 Examples

We consider four examples of Lévy processes for which explicit formulas for their laws are available. This permits to conduct direct computations and expansions for the marginal laws and allows to compare them with the previous results. Let us stress that even in these cases where the law of the process is known, we do not know the law of the process corresponding to its small jumps. Besides the compound Poisson process, it is hard to propose examples to compare with Theorems 1 and 3. Finally, we present a non-asymptotic control of the marginal law of α-stable type processes. Proofs are postponed to Section A.7.

1. Let \( X \) be a compound Poisson process. Then, for any \( \varepsilon > 0 \)
\[
|\mathbb{P}(|X_t| > \varepsilon) - \lambda t| = O_{\varepsilon}(t^2) \quad \text{and} \quad \mathbb{P}(|M_t| > \varepsilon) = O_{\varepsilon}(t^2), \quad \text{as } t \to 0.
\]
It is possible to build examples for which these rates are sharp (see Section A.7).

2. Let \( X \) be a Gamma process of parameter (1, 1), that is a finite variation Lévy process with Lévy density \( f(x) = \frac{e^{-x}}{x} 1_{(0, \infty)}(x) \), \( \lambda \varepsilon = \int_0^\infty \frac{e^{-x}}{x} dx \) and
\[
\mathbb{P}(|X_t| > \varepsilon) = \mathbb{P}(X_t > \varepsilon) = \int_{\varepsilon}^{\infty} x^{t-1} e^{-x} dx, \quad \forall \varepsilon > 0,
\]
where \( \Gamma(t) \) denotes the \( \Gamma \) function, i.e. \( \Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx \). Then,
\[
|\mathbb{P}(X_t > \varepsilon) - \lambda t| = O_{\varepsilon}(t^2), \quad \text{as } t \to 0.
\]

3. Let \( X \) be an inverse Gaussian process of parameter (1, 1), i.e.
\[
f(x) = \frac{e^{-x}}{x^2} 1_{(0, \infty)}(x) \quad \text{and} \quad \mathbb{P}(X_t > \varepsilon) = t e^{2t} \int_\varepsilon^{\infty} e^{-x} \frac{dx}{x^2}, \quad \forall \varepsilon > 0.
\]
Then,
\[
|\mathbb{P}(|X_t| > \varepsilon) - t\lambda| = O_{\varepsilon}(t^2), \quad \text{as } t \to 0. \tag{5}
\]

4. Cauchy processes. Let \( X \) be a 1-stable Lévy process with
\[
f(x) = \frac{1}{\pi x^2} 1_{\mathbb{R}\setminus\{0\}} \quad \text{and} \quad \mathbb{P}(|X_t| > \varepsilon) = 2 \int_\varepsilon^{\infty} \frac{dx}{\pi(x^2 + 1)}, \quad \forall \varepsilon > 0.
\]
Then,
\[
|\mathbb{P}(|X_t| > \varepsilon) - t\lambda| = O_{\varepsilon}(t^3), \quad \text{as } t \to 0. \tag{6}
\]

For this example, the bound of Theorem 5 is suboptimal. However, improving Theorem 5 relying on the same strategy of proof, i.e. using compound Poisson approximations, is hopeless and a different approach should be considered.

5. \( \alpha \)-stable type processes. Results for the cumulative distribution function for \( \alpha \)-stable processes were already known (see e.g. Marchal (2009)). The following result is a generalization to any Lévy process whose Lévy measure behaves as an \( \alpha \)-stable process in a neighborhood of the origin such as a tempered stable Lévy process (see e.g. Cont and Tankov (2004) Section 4.2 or Rosiński (2007)).

**Corollary 1.** Let \( X \) be a symmetric Lévy process with a Lévy measure \( \nu \) absolutely continuous with respect to the Lebesgue measure and denote by \( f = \frac{d\nu}{dx} \). Suppose that there exist \( \alpha \in (0, 2) \), \( M_1 > 0 \) and \( M_2 > 0 \) such that \( M_1 |x|^{-(1+\alpha)} \leq |f(x)| \leq M_2 |x|^{-(1+\alpha)} \), for all \( 0 < |x| \leq 2 \). Let \( \varepsilon \in (0, 1) \) and \( t > 0 \). We have:
• If $\alpha \in (0,1)$ : there exists a constant $A_{M_1,M_2,\alpha} > 0$, only depending on $M_1$, $M_2$ and $\alpha$, such that
\[
\mathbb{P}(|X_t| > \varepsilon - \lambda \varepsilon t) \leq A_{M_1,M_2,\alpha} t^2 \lambda \varepsilon^2, \quad \forall \ t \lambda \varepsilon \leq 2^{-\alpha}(2 - \alpha)\alpha^{-1}.
\]

• If $\alpha \in [1,2)$ and $f \in \mathcal{L}_{M_2}$ : there exist two constants $B_{M_1,M_2,\alpha} > 0$ and $\tilde{B}$, only depending on $M_1$, $M_2$ and $\alpha$, such that $t \lambda \varepsilon \leq 2^{1-\alpha}M_2(1 \land (2 - \alpha)/2M_2)\alpha^{-1}$ it holds
\[
\mathbb{P}(|X_t| > \varepsilon - \lambda \varepsilon t) \leq B_{M_1,M_2,\alpha} t^{1+1/\alpha} \lambda \varepsilon^{1+1/\alpha} \left(1_{\alpha \in (1,2)} + \frac{\tilde{B}}{\lambda \varepsilon} \right) 1_{\alpha = 1}.
\]

• If $\alpha \in [1,2)$ and $f$ is globally $M \varepsilon^{-(2+\alpha)}$-Lipschitz on the interval $((3/4\varepsilon, 2\varepsilon - 3/4\varepsilon)$: there exists a constant $C_{M_1,M_2,\alpha} > 0$, only depending on $M_1$, $M_2$ and $\alpha$, such that
\[
\mathbb{P}(|X_t| > \varepsilon - \lambda \varepsilon t) \leq C_{M_1,M_2,\alpha} t^2 \lambda \varepsilon^2, \quad \forall \ t \lambda \varepsilon \leq 2^{-\alpha}(2 - \alpha)\alpha^{-1}.
\]

This result is a consequence of Theorems 2, 4 and 5 observing that, under the assumptions of Corollary 1,
\[
2M_1 \varepsilon^{-\alpha} / \alpha \lambda \varepsilon,1 \leq 2M_2 \varepsilon^{-\alpha} / \alpha, \quad \varepsilon^{-\alpha} \leq \frac{2M_2}{\alpha \lambda \varepsilon,1} \quad \text{and} \quad \varepsilon^{-\alpha} \leq \frac{\alpha \lambda \varepsilon}{2M_1}.
\]

2.5 Extension

A natural question is whether the above results hold true for general Lévy processes, that is in presence of a Gaussian part, $\Sigma > 0$ in (1). The answer is essentially positive but to avoid cumbersome proofs we chose to have $\Sigma = 0$. If $\Sigma > 0$, proofs can be adapted following the same steps as in Section 3 replacing $M_t(\varepsilon)$ with $\Sigma W_t + M_t(\varepsilon)$, leading to similar results to those presented in Section 2.

More precisely, in order to mimic what is done in Section 3 for pure jump Lévy processes, we need to generalize Lemma 2. Adapting its proof we obtain the following result. For any $\varepsilon \in (0,1]$, $t > 0$ and $x > 0$, it holds:
\[
\mathbb{P}(\Sigma W_t + M_t(\varepsilon) > x) \leq e^{\frac{t}{2} \left( \varepsilon^{-\alpha} \lambda \varepsilon,1 \right)} \exp \left( \frac{\Sigma^2}{2\varepsilon^2} \log^2 \left(1 + \frac{x\varepsilon}{ta^2(\varepsilon)} \right) \right).
\]

In particular, using that $u \to u \log^2(1+1/u)$ is bounded by 1 for $u > 0$, we observe that the additional term $e^{\frac{t}{2} \left( \varepsilon^{-\alpha} \lambda \varepsilon,1 \right)} \leq e^{\frac{\Sigma^2}{2\varepsilon^2} \log^2 (1+1/u)}$ is bounded.

Similarly, it is possible to have a more general drift $b$ in the triplet (see (1)). Proofs can be adapted at the cost of a more stringent condition on $t$. Indeed, the condition on $t$ in the above Theorems ensures that $tb(\varepsilon) \leq \varepsilon/2$, a similar condition should be satisfied in presence of a general drift $b$.

3 Proofs

3.1 Preliminaries

Introduce the following notations. Consider $b \geq a > 0$, denote by $\lambda_a := \int_{|x| > a} f(x) dx$ and $\lambda_{a,b} := \int_{b > |x| > a} f(x) dx$ with the convention $\lambda_{a,a} = 0$. Recall that $\sigma^2(a) := \int_{0 < |x| < a} x^2 f(x) dx$ and for finite variation processes the drift is denoted by $b(a) := \int_{0 < |x| < a} x f(x) dx$. Furthermore, we write $Y^{(a)}$ (resp. $Y^{(a,b)}$) for a random variable with density $f_{\mathbb{1}_{(-a,a)^c}} / \lambda_a$ (resp. $f_{\mathbb{1}_{(-b,a] \cup [a,b]}} / \lambda_{a,b}$). With these
notations, following (2) consider the decomposition which plays an essential role in the sequel, for all \( t > 0 \)

\[
M_t(\varepsilon) = M_t(\eta) + Z_t(\eta, \varepsilon) - t(b(\varepsilon) - b(\eta)), \quad \forall \ 0 < \eta < \varepsilon \leq 1,
\]

(7)

where \( Z_t(\eta, \varepsilon) = \sum_{i=1}^{N_t(\eta, \varepsilon)} Y_i^{(\eta, \varepsilon)} \), \( N(\eta, \varepsilon) \) being a Poisson process of intensity \( \lambda_{\eta, \varepsilon} \) independent of \( (Y_i^{(\eta, \varepsilon)}) \). Therefore, for all \( 0 < \eta < \delta \) and \( t > 0 \) it holds:

\[
\mathbb{P}(N_t(x, \delta) \geq 1) \leq \lambda_{x, \delta} t \quad \text{and} \quad \mathbb{P}(N_t(x, \delta) \geq 2) \leq (\lambda_{x, \delta} t)^2.
\]

(8)

In the sequel we make intensive use of the following inequalities. For any \( 0 < x \leq y \leq 2 \) and \( f \in \mathcal{L}_{M, \alpha} \), it holds

\[
\sigma^2(x) = \frac{\int_x^2 u^2 f(u) du}{x^2} \leq \frac{2M}{2 - \alpha} x^{-\alpha},
\]

(9)

\[\lambda_{x, y} = \int_{y > |u| > x} f(u) du \leq \frac{2M}{\alpha} x^{-\alpha},\]

(10)

\[b(x) = \int_{|u| \leq x} uf(u) du \leq \frac{2M}{1 - \alpha} x^{1-\alpha}.
\]

(11)

### 3.2 Proof of Theorem 1

First, note that

\[
\mathbb{P}(|tb(\varepsilon) + M_t(\varepsilon)| > \varepsilon) = \mathbb{P}(tb(\varepsilon) + M_t(\varepsilon) > \varepsilon) + \mathbb{P}(tb(\varepsilon) + M_t(\varepsilon) < -\varepsilon).
\]

We consider only the term \( \mathbb{P}(tb(\varepsilon) + M_t(\varepsilon) > \varepsilon) \) as \( \mathbb{P}(tb(\varepsilon) + M_t(\varepsilon) < -\varepsilon) \) can be treated analogously. Define

\[
\eta := \inf \left\{ \frac{\varepsilon}{4} \leq u < \varepsilon : u \leq \varepsilon - t \frac{\int_u^\varepsilon x f(x) dx}{2}, t\lambda_{\varepsilon/8, u} < 2 \right\}.
\]

Observe that if \( f \in \mathcal{L}_{M, \alpha} \), \( M > 0 \), \( \alpha \in (0, 1) \), \( \varepsilon \in (0, 1] \) and \( 0 < t \leq (1 - \alpha)M^{-1} (\varepsilon/4)^\alpha \), then the set \( A_{\varepsilon, t} := \left\{ \frac{\varepsilon}{4} \leq u < \varepsilon : u \leq \varepsilon - t \frac{\int_u^\varepsilon x f(x) dx}{2}, t\lambda_{\varepsilon/8, u} < 2 \right\} \) is not empty as \( \varepsilon/4 \in A_{\varepsilon, t} \) noting in particular that \( t\lambda_{\varepsilon/8, \varepsilon/4} \leq 2(1 - \alpha)(2^\alpha - 1)/\alpha \leq 2 \log(2) \).

By means of (7) and the definition of \( b(\cdot) \), we have

\[
\mathbb{P}(tb(\varepsilon) + M_t(\varepsilon) > \varepsilon) = \mathbb{P}(M_t(\eta) + Z_t(\eta, \varepsilon) > \varepsilon - tb(\eta)) + \mathbb{P}(M_t(\eta) + Y_1^{(\eta, \varepsilon)} > \varepsilon - tb(\eta)) + \mathbb{P}(N_t(\eta, \varepsilon) \geq 2),
\]

(12)

where we decomposed on the values of the Poisson process \( N(\eta, \varepsilon) \). Using (8), we have \( \mathbb{P}(N_t(\eta, \varepsilon) \geq 2) \leq (\lambda_{\eta, \varepsilon} t)^2 \). We thus only have to control the first and second addendum in (12). For the first one, we apply Lemma 2, using that \( t \leq (1 - \alpha)M^{-1} \varepsilon^{-1/2} \) implies that \( t\sigma^2(x) \varepsilon^{-2} \leq 1 \) for all \( x \in [\varepsilon/4, \varepsilon] \).

It follows from the definition of \( \eta \) and (3) that

\[
\mathbb{P}(M_t(\eta) > \varepsilon - tb(\eta)) \leq \mathbb{P}(M_t(\eta) > 2\eta) \leq \left( \frac{e\sigma^2(\eta)}{4\eta^2} \right)^2 e^{\varepsilon - 1} t^2.
\]

Hence, using (9) and the fact that \( \eta \geq \varepsilon/4 \) and \( 4^{2\alpha - 1} e^{2 + 1/\varepsilon} (2 - \alpha)^{-2} \leq 16 \), leads to

\[
\mathbb{P}(M_t(\eta) > \varepsilon - tb(\eta)) \leq 16t^2 M^2 \varepsilon^{−2\alpha}.
\]

(13)
For the second term in (12), set $\varepsilon' := \varepsilon - tb(\eta)$ and notice that $\varepsilon' \geq \varepsilon/2$. It holds

$$
\lambda_{\eta, \varepsilon} \mathbb{P}(M_t(\eta) + Y_1^{(\eta, \varepsilon)} > \varepsilon') = \int_{\eta < |y| < \varepsilon} \mathbb{P}(M_t(\eta) > \varepsilon' - y)f(y)dy
$$
\leq \int_{\eta < y < \varepsilon} f(x)dx + \int_{0 < y < \varepsilon} \mathbb{P}(M_t(\eta) > \varepsilon' - y)f(y)dy =: T_1 + T_2.
$$

From $\varepsilon' > 0$ it follows that $\mathbb{P}(M_t(\eta) > \varepsilon' + \eta) \leq \mathbb{P}(M_t(\eta) > \eta)$. The Markov inequality and (9), joined with the fact that $f \in \mathcal{L}_{M, \alpha}$ and $\eta \geq \varepsilon/4$ yield

$$
T_1 \leq 2M^2\eta^{-\alpha}(2 - \alpha)^{-1} \int_{\eta}^{\varepsilon}[x]^{-(1+\alpha)}dx \leq \frac{2M^2}{\alpha(2 - \alpha)} \eta^{-2\alpha}t \leq tM^2\varepsilon^{-2\alpha}C_{1, \alpha},
$$
with

$$
C_{1, \alpha} := \frac{2^{1+4\alpha}}{\alpha(2 - \alpha)}.
$$

To treat the term $T_2$ we suppose that $b(\eta) \geq 0$, the case $b(\eta) < 0$ is handled similarly. After a change of variable, we obtain

$$
T_2 = \int_{-tb(\eta)}^{\varepsilon'-\eta} \mathbb{P}(M_t(\eta) > x)f(\varepsilon' - x)dx
\leq \int_{-tb(\eta)}^{0} f(\varepsilon' - x)dx + \int_{0}^{\eta/2} \mathbb{P}(M_t(\eta) > x)f(\varepsilon' - x)dx
\quad + \int_{\eta/2}^{\eta} \mathbb{P}(M_t(\eta) > x)f(\varepsilon' - x)dx + \int_{\eta}^{\varepsilon'-\eta} \mathbb{P}(M_t(\eta) > x)f(\varepsilon' - x)dx
= : T_{2,1} + T_{2,2} + T_{2,3} + T_{2,4}.
$$

First observe that for $f \in \mathcal{L}_{M, \alpha}$ and $\varepsilon' \geq \varepsilon/2$ we get

$$
f(\varepsilon' - x) \leq \frac{M}{|\varepsilon' - x|^{1+\alpha}} \leq M(\varepsilon')^{-(1+\alpha)} \leq M2^{1+\alpha}\varepsilon^{-(1+\alpha)}, \quad \forall x \in [-tb(\eta), 0].
$$

Furthermore, using that $b(\eta) \leq 2M(1 - \alpha)^{-1}\eta^{1-\alpha} \leq 2M(1 - \alpha)^{-1}\varepsilon^{1-\alpha}$, we conclude that

$$
T_{2,1} \leq \frac{2^{2+\alpha}tM^2}{1 - \alpha} \eta^{-2\alpha}.
$$

Next we consider $T_{2,2}$. By (7), for any $\bar{x} \in (0, \eta)$, we write $M_t(\eta) = M_t(\bar{x}) + Z_t(\bar{x}, \eta) - t(b(\eta) - b(\bar{x}))$. Consider $x \in (2M\eta^{1-\alpha}(1 - \alpha)^{-1}, \eta/2)$ and set $\bar{x} := x - 2M\eta^{1-\alpha}(1 - \alpha)^{-1}$. Observe that, as $0 < t \leq (1 - \alpha)M^{-1}\varepsilon^{\alpha}4^{-(1+\alpha)}$ it holds $2M\eta^{1-\alpha}(1 - \alpha)^{-1} \leq \eta/2$. Using that $f \in \mathcal{L}_{M, \alpha}$ we have:

$$
|b(\eta) - b(\bar{x})| = \left| \int_{|u| \in [\bar{x}, \eta]} uf(u)du \right| \leq 2M\eta^{1-\alpha}(1 - \alpha)^{-1}
$$
from which we derive that $\mathbb{P}(M_t(\bar{x}) > x + t(b(\eta) - b(\bar{x}))) \leq \mathbb{P}(M_t(\bar{x}) > \bar{x})$. It follows that for $x \in (2M\eta^{1-\alpha}(1 - \alpha)^{-1}, \eta/2)$ we may write, decomposing on the values of $N(\bar{x}, \eta)$, that

$$
\mathbb{P}(M_t(\eta) > x) = \mathbb{P}(M_t(\bar{x}) + Z_t(\bar{x}, \eta) > x + t(b(\eta) - b(\bar{x})))
\leq \mathbb{P}(M_t(\bar{x}) > \bar{x}) + \mathbb{P}(N_t(\bar{x}, \eta) \geq 1)
\leq t \frac{2M}{2 - \alpha}\eta^{1-\alpha} + t\lambda_{\bar{x}} \leq \frac{2Mt(\bar{x})^{-\alpha}(2 + \alpha)}{\alpha(2 - \alpha)},
$$
where, in the last inequality, we used the Markov inequality and (9). Consequently, using that $\eta \leq \varepsilon$ and noticing that $3/8\varepsilon \leq \varepsilon' - x \leq 1$ for all $x \in (0, \eta/2)$, we derive

\[
T_{2, 2} \leq \int_{0}^{2Mt\varepsilon^{-2\alpha}} f(\varepsilon' - x) dx + \frac{2(2 + \alpha)Mt}{\alpha(2 - \alpha)} \int_{\frac{Mt\varepsilon^{-2\alpha}}{1 - \alpha}}^{\eta/2} f(\varepsilon' - x) dx
\]

\[
\leq \frac{2M^2t\varepsilon^{-2\alpha}}{1 - \alpha} \left( \frac{8}{3} \right)^{1+\alpha} + \frac{2(2 + \alpha)M^2t}{2^{1-\alpha}\alpha(2 - \alpha)(1 - \alpha)} \left( \frac{8}{3} \right)^{1+\alpha} \varepsilon^{-2\alpha}
\]

\[
\leq \frac{2M^2t\varepsilon^{-2\alpha}}{1 - \alpha} \left( \frac{8}{3} \right)^{1+\alpha} \left( 1 + \frac{2 + \alpha}{2^{1-\alpha}\alpha(2 - \alpha)} \right).
\]  

(16)

To treat the term $T_{2, 3}$ we proceed analogously. Let $x \in [\eta/2, \eta]$ and $	ilde{Z}_t(x, \eta)$ be a centered version of $Z_t(x, \eta)$, that is $\tilde{Z}_t(x, \eta) = \sum_{i=1}^{N_t(x, \eta)} (Y_i(x, \eta) - E[Y_i(x, \eta)])$. In particular, by definition of $\eta$, if follows that $t\lambda_{x, \eta} < 2$ and Lemma 7 applies. On the one hand we derive that

\[
|\mathbb{P}(M_t(x) + Z_t(x, \eta) - E[Z_t(x, \eta)] > x) - \mathbb{P}(M_t(x) + \tilde{Z}_t(x, \eta) > x)|
\]

\[
\leq 4t\lambda_{x, \eta}(\mathbb{E}[Y_1(x, \eta)]) \sup_{y \in [x, \eta]} |f(y)/\lambda_{x, \eta}| \leq tM^2(2^{1-\alpha}\eta^{-1+\alpha}) \int_{\eta}^\eta \frac{\eta u}{\lambda(x, \eta)} du \leq 2^{2+\alpha}tM\eta^{-\alpha},
\]

where we used that $\mathbb{E}[Z_t(x, \eta)] = t(b(\eta) - b(x))$. On the other hand, we have that

\[
\mathbb{P}(M_t(x) + \tilde{Z}_t(x, \eta) > x) \leq \mathbb{P}(M_t(x) > x) + \mathbb{P}(N_t(x, \eta) \geq 1) \leq \frac{10Mtx^{-\alpha}}{\alpha(2 - \alpha)} \leq \frac{20Mt\eta^{-\alpha}}{\alpha(2 - \alpha)},
\]

where we used the Markov inequality, (9), (8), (10) and that $x > \eta/2$. Finally, by the triangle inequality and using that $\varepsilon' - \eta \geq \eta \geq \varepsilon/4$, we deduce that

\[
T_{2, 3} \leq \frac{28Mt\eta^{-\alpha}}{\alpha(2 - \alpha)} \int_{\eta}^{\eta/2} f(\varepsilon' - x) dx \leq \frac{28M^2t\eta^{-\alpha}(\varepsilon' - \eta)^{-\alpha}}{\alpha^2(2 - \alpha)} \leq \frac{28 \times 4^{2\alpha}M^2t\varepsilon^{-2\alpha}}{\alpha^2(2 - \alpha)}.
\]  

(17)

Then, for the term $T_{2, 4}$, the Markov inequality and (9), for any $x \in [\eta, \varepsilon' - \eta]$, lead to

\[
\mathbb{P}(M_t(\eta) > x) \leq \frac{2M}{2 - \alpha} \eta^{-\alpha} t.
\]

Therefore, using that $\varepsilon' - \eta \geq \eta \geq \varepsilon/4$, we get

\[
T_{2, 4} \leq \frac{2M}{2 - \alpha} \eta^{-\alpha} t \int_{\eta}^{\varepsilon' - \eta} f(\varepsilon' - x) dx \leq \frac{2M^2}{(2 - \alpha)\alpha} \varepsilon^{-2\alpha} t \leq \frac{21^{1+\alpha}M^2}{\alpha^2(2 - \alpha)} \varepsilon^{-2\alpha} t.
\]  

(18)

Gathering Equations (15), (16), (17) and (18) yield

\[
T_2 \leq tM^2\varepsilon^{-2\alpha} C_{2, \alpha},
\]

(19)

with

\[
C_{2, \alpha} = \left( \frac{2^{2+\alpha}}{1 - \alpha} + \frac{2^{2+\alpha}(2^{1-\alpha}\alpha(2 - \alpha) + 2 + \alpha)}{\alpha(2 - \alpha)(1 - \alpha)3^{1+\alpha}} + \frac{28 \times 4^{2\alpha}}{\alpha^2(2 - \alpha)} + \frac{21^{1+\alpha}}{\alpha^2(2 - \alpha)} \right).
\]

Combining (14) and (19) we conclude that, if $b(\eta) \geq 0$, then

\[
\lambda_{\eta, \varepsilon} t^2 \mathbb{P}(M_t(\eta) + Y_1^{(\eta, \varepsilon)} > \varepsilon') \leq t^2M^2\varepsilon^{-2\alpha}(C_{1, \alpha} + C_{2, \alpha}).
\]  

(20)
The case \( b(\eta) < 0 \) is treated similarly and therefore not detailed here. Injecting in (12) Equations (13), (10) and (20) we conclude that

\[
\mathbb{P}(tb(\varepsilon) + M_t(\varepsilon) > \varepsilon) \leq t^2M^2\varepsilon^{-2\alpha}(16 + 64\alpha^{-2} + C_{1,\alpha} + C_{2,\alpha}) = : t^2M^2\varepsilon^{-2\alpha}C_1, \tag{21}
\]
as desired.

For a symmetric Lévy measure above computations can be simplified. In this case \( b(\varepsilon) = 0 \) and one can directly take \( \eta = \varepsilon/2 \) in the previous lines. More precisely, it holds

\[
\mathbb{P}(M_t(\varepsilon) > \varepsilon) \leq \mathbb{P}(M_t(\varepsilon/2) > \varepsilon) + (t\lambda_{\varepsilon/2,e})^2 + t\lambda_{\varepsilon/2,e}\mathbb{P}(M_t(\varepsilon/2) + Y_1^{(\varepsilon/2,e)} > \varepsilon).
\]

To control the first two addendum use Lemma 2 and (10). To treat the last term we proceed as follows:

\[
\lambda_{\varepsilon/2,e}\mathbb{P}(M_t(\varepsilon/2) + Y_1^{(\varepsilon/2,e)}) = \int_{\varepsilon/2}^{\varepsilon} + \int_{-\varepsilon}^{-\varepsilon/2} \mathbb{P}(M_t(\varepsilon) > \varepsilon - z)f(z)dz
\]
\[
\leq \int_{\varepsilon/2}^{\varepsilon} \left( \mathbb{P}(M_t(\varepsilon - z) > \varepsilon - z) + t\lambda_{\varepsilon/2,e}f(z) + \mathbb{P}(M_t(\varepsilon/2) > 3/2\varepsilon) \right) \frac{\lambda_{\varepsilon}}{2}
\]
\[
\leq t \int_{\varepsilon/2}^{\varepsilon} \left( \frac{\sigma^2(\varepsilon - z)^2 + \lambda_{\varepsilon/2,e}f(z)}{\varepsilon/2} \right)dz + \mathbb{P}(M_t(\varepsilon/2) > 3/2\varepsilon) \frac{\lambda_{\varepsilon/2,e}}{2}
\]
\[
\leq \frac{4^{1+\alpha}tM^2\varepsilon^{-2\alpha}}{\alpha(1-\alpha)(2-\alpha)} + \mathbb{P}(M_t(\varepsilon/2) > 3/2\varepsilon) \frac{\lambda_{\varepsilon/2,e}}{2}.
\]
The term \( \mathbb{P}(M_t(\varepsilon/2) > 3/2\varepsilon) \) is controlled applying Lemma 2 using that \( 4t\sigma^2(\varepsilon/2) \leq \varepsilon^2 \). Collecting all the pieces together, one derives the following result: For all \( t > 0 \) such that \( t \leq \varepsilon^\alpha(2 - \alpha)M^{-1-\alpha-1} \) (implying that \( t\lambda_{\varepsilon/2,e} \leq 1 \)), it holds:

\[
\mathbb{P}(M_t(\varepsilon) > \varepsilon) \leq t^2\varepsilon^{-2\alpha}M^2C_2, \text{ where}
\]

\[
C_2 := \frac{3 \times 2^{2\alpha-1}e^{2+\epsilon\varepsilon}}{(2-\alpha)^2} + \frac{4^{1+\alpha}}{\alpha(1-\alpha)(2-\alpha)} + \frac{4^\alpha}{\alpha^2}. \tag{22}
\]

### 3.3 Proof of Theorem 2

To prove Theorem 2 we first introduce an auxiliary result.

**Lemma 3.** Let \( \nu \) be a Lévy measure with density \( f \) with respect to the Lebesgue measure and \( \varepsilon \) a positive real number. Set \( \rho := \varepsilon \wedge 1 \) and

\[
Q := |\lambda_t\mathbb{P}(|M_t(\rho) + tb(\rho) + Y_1^{(\rho)}| > \varepsilon) - \lambda_t|.
\]

- If \( \varepsilon \in (0,1] \) and \( f \in \mathcal{L}_{M,\alpha} \) for some \( \alpha \in (0,1) \) and \( M > 0 \), then

  \[
  Q \leq t^2(M^2D_1\varepsilon^{-2\alpha} + M\lambda_\varepsilon\varepsilon^{-\alpha}D_2), \quad \forall \ 0 < t < (1-\alpha)M^{-1}\varepsilon^{\alpha}4^{-(1+\alpha)},
  \]

  where \( D_1 \) and \( D_2 \) are defined as in (41).

- If \( \varepsilon > 1 \) and \( f \in \mathcal{L}_{M,\alpha} \cap \mathcal{L}_{M} \) for some \( \alpha \in (0,1) \) and \( M > 0 \), then for all \( 0 < t < (1-\alpha)(5M)^{-1} \) it holds

  \[
  Q \leq 2M^2t^2 \left( \frac{4}{2-\alpha}(\varepsilon - 3/2 - t|b(1)|)\mathbf{1}_{\varepsilon > 3/2 + t|b(1)|} \right)
  + Mt^2 \left( 4 \times 5^\alpha\mathbf{1}_{1<\varepsilon<1+2t|b(1)|} + \frac{8}{5} + 3\lambda_2 + \frac{4\lambda_1}{2-\alpha} \right),
  \]

  where \( \widetilde{D}_1 \) is defined as in (49).
If in addition we suppose that \( \nu \) is a symmetric measure, then

- If \( \varepsilon \in (0,1] \) and \( f \in \mathcal{L}_{M,\alpha} \) for some \( \alpha \in (0,1) \) and \( M > 0 \), it holds

\[
Q \leq \frac{t^2 M}{2(2-\alpha)} (\lambda_{e^{-\alpha}} + 4\lambda_{2e^{-\alpha}}) + 2t^2 M^2 D_3 e^{-2\alpha}, \quad \forall \; t > 0,
\]

where \( D_3 \) is defined as in (50).

- If \( \varepsilon > 1 \) and \( f \in \mathcal{L}_{M,\alpha} \cap \mathcal{L}_M \) for some \( \alpha \in (0,1) \) and \( M > 0 \), it holds

\[
Q \leq \frac{t^2 M}{2-\alpha} \left( \lambda_{12^{-\alpha}} + \frac{4M}{\alpha(1-\alpha)} + \lambda_{1+\varepsilon} \right), \quad \forall \; t > 0.
\]

**Proof of Theorem 2** Using the decomposition \( X_t = M_t(\rho) + tb(\rho) + Z_t(\rho) \), \( \rho = \varepsilon \wedge 1 \), we derive, decomposing on the Poisson process \( N(\rho) \), that

\[
|P(|X_t| > \varepsilon) - \lambda_{\varepsilon} t| = \left| P(|M_t(\rho) + tb(\rho)| > \varepsilon)e^{-\lambda_{\varepsilon}} + \lambda_{\rho} tP(|M_t(\rho) + tb(\rho) + Y_1(\rho)| > \varepsilon)e^{-\lambda_{\varepsilon}}t
\]

\[
- \lambda_{\varepsilon} t + \sum_{n=2}^{\infty} P\left(|M_t(\rho) + tb(\rho) + \sum_{i=1}^{n} Y_i(\rho)| > \varepsilon\right)P(N_\rho(n) = n)
\]

\[
\leq P(|M_t(\rho) + tb(\rho)| > \varepsilon) + \lambda_{\rho} tP(|M_t(\rho) + tb(\rho) + Y_1(\rho)| > \varepsilon) - \lambda_{\varepsilon} t
\]

\[
+ \lambda_{\rho} t(1 - e^{-\lambda_{\rho}t}) + P(N_t(\rho) \geq 2)
\]

\[= I_1 + I_2 + I_3 + I_4.\]

The term \( I_1 \) is controlled with Theorem 1, \( I_2 \) with Lemma 3, for \( I_3 \) use that \( 1 - e^{-x} \leq x \), for all \( x > 0 \) to get \( I_3 \leq \lambda_{\varepsilon}^2 t^2 \) and finally, it follows from (8) that \( I_4 = P(N_t(\rho) \geq 2) \leq \lambda_{\varepsilon}^2 t^2 \) as \( (1 - e^{-x} - xe^{-x} \leq x^2 \), for all \( x > 0 \).

### 3.4 Proof of Theorem 3

As \( \nu \) is symmetric it holds \( P(|M_t(\varepsilon)| \geq \varepsilon) = 2P(M_t(\varepsilon) \geq \varepsilon) \). Using the same reasoning as in the proof of Theorem 1 we get

\[
P(M_t(\varepsilon) \geq \varepsilon) \leq P(M_t(\varepsilon/2) \geq \varepsilon) + t\lambda_{\varepsilon/2} e^2 P(M_t(\varepsilon/2) + Y_1(\varepsilon/2,\varepsilon) \geq \varepsilon) + (t\lambda_{\varepsilon/2,\varepsilon})^2.
\]

By means of Lemma 2 joined with (9), we get that

\[
P(M_t(\varepsilon/2) \geq \varepsilon) \leq t^2 4M^2 e^{2+1/\varepsilon} \frac{1}{(2-\alpha)^2 \varepsilon^{2\alpha}},
\]

and, using (10), that

\[
(t\lambda_{\varepsilon/2,\varepsilon})^2 \leq \frac{t^2 M^2 4^{1+\alpha}}{\alpha^2 \varepsilon^{2\alpha}}.
\]

Finally, using the symmetry of \( \nu \), we have that

\[
\lambda_{\varepsilon/2,\varepsilon} P(M_t(\varepsilon/2) + Y_1(\varepsilon/2,\varepsilon) \geq \varepsilon) = \int_{\varepsilon/2}^{\infty} \left( P(M_t(\varepsilon/2) \geq \varepsilon - z) + P(M_t(\varepsilon/2) \geq \varepsilon + z) \right) f(z) dz
\]

\[
\leq \int_{\varepsilon/2}^{\infty} P(M_t(\varepsilon/2) \geq \varepsilon - z) f(z) dz + \frac{P(M_t(\varepsilon/2) \geq 3/2\varepsilon)}{2} \lambda_{\varepsilon/2,\varepsilon} =: T_1 + T_2.
\]
To control the term $T_1$, observe that
\[
T_1 = \int_0^{t^{1/\alpha}} \mathbb{P}(M_t(\varepsilon/2) \geq z)f(\varepsilon - z)dz + \int_{t^{1/\alpha}}^{\varepsilon/2} \mathbb{P}(M_t(\varepsilon/2) \geq z)f(\varepsilon - z)dz \\
\leq \frac{M t^{1/\alpha}}{(\varepsilon - t^{1/\alpha})^{1+\alpha}} + \frac{2^{1+\alpha} M}{\varepsilon^{1+\alpha}} \int_{t^{1/\alpha}}^{\varepsilon/2} \mathbb{P}(M_t(\varepsilon/2) \geq z)dz.
\]

Next, for $z \in (t^{1/\alpha}, \varepsilon/2)$, the Markov inequality and (9) lead to
\[
\mathbb{P}(M_t(\varepsilon/2) \geq z) \leq \mathbb{P}(M_t(\varepsilon) \geq z) + t\lambda_{x,\varepsilon/2} \leq \frac{t\sigma^2(z)}{z^2} + 2t M \int_z^{\varepsilon/2} \frac{dx}{x^{1+\alpha}} \\
\leq 2Mt\varepsilon^{-\alpha}\left(\frac{1}{2-\alpha} + \frac{1}{\alpha}\right).
\]

Therefore, for any $\alpha \in (1, 2)$
\[
\int_{t^{1/\alpha}}^{\varepsilon/2} \mathbb{P}(M_t(\varepsilon/2) \geq z)dz \leq \frac{4Mt^2}{\alpha(2-\alpha)(\alpha-1)},
\]
then, using that $\varepsilon - t^{1/\alpha} \geq \varepsilon/2$, we derive that
\[
T_1 \leq \frac{2^{1+\alpha} Mt^{1/\alpha}}{\varepsilon^{1+\alpha}} \left(1 + \frac{M}{\alpha(2-\alpha)(\alpha-1)}\right), \quad \alpha \in (1, 2).
\]

If, instead, $\alpha = 1$, we get
\[
T_1 \leq \frac{4Mt}{\varepsilon^2} + \frac{16M^2}{\varepsilon^2} t \ln\left(\frac{\varepsilon}{2t}\right).
\]

To control the term $T_2$ we use once again the Markov inequality joined with (9) to obtain
\[
T_2 \leq \frac{t\sigma^2(\varepsilon/2)}{9(\varepsilon/2)^2} \frac{\lambda_{\varepsilon/2,\varepsilon}}{2} \leq \frac{2^{\alpha+1} M^2 t}{9\alpha(2-\alpha)\varepsilon^{2\alpha}}, \quad \alpha \in [1, 2).
\]

Gathering (26) and (28) we have, for $\alpha \in (1, 2),$
\[
\lambda_{\varepsilon/2,\varepsilon}\mathbb{P}(M_t(\varepsilon/2) + Y_1^{(\varepsilon/2,\varepsilon)} \geq \varepsilon) \leq \frac{2^{1+\alpha} M t^{1/\alpha}}{\varepsilon^{1+\alpha}} \left(1 + \frac{M}{\alpha(2-\alpha)(\alpha-1)}\right) + \frac{2^{\alpha+1} M^2 t}{9\alpha(2-\alpha)\varepsilon^{2\alpha}}.
\]

Combining (23) with (24), (25) and (29) we conclude that for all $\alpha \in (1, 2)$ it holds
\[
\mathbb{P}(M_t(\varepsilon) \geq \varepsilon) \leq \frac{t^2 M^2}{\varepsilon^{2\alpha}} E_1 + \frac{2^{1+\alpha} M t^{1+1/\alpha}}{\varepsilon^{1+\alpha}} \left(1 + \frac{M}{\alpha(2-\alpha)(\alpha-1)}\right),
\]
with
\[
E_1 := \frac{4e^{2+1/\varepsilon}}{(2-\alpha)^2} + \frac{4^{1+\alpha}}{\alpha^2} + \frac{2^{\alpha+1}}{9\alpha(2-\alpha)}.
\]

If, instead, $\alpha = 1$, then using (27)
\[
\mathbb{P}(M_t(\varepsilon) \geq \varepsilon) \leq \frac{4t^2 M^2}{\varepsilon^2} \left(e^{2+1/\varepsilon} + \frac{37}{9}\right) + \frac{4M t^2}{\varepsilon^2} + \frac{16M^2 t^2}{\varepsilon^2} t^2 \ln\left(\frac{\varepsilon}{2t}\right).
\]

This concludes the proof.
3.5 Proof of Theorem 4

Lemma 4. Let $\nu$ be a symmetric Lévy measure with density $f$ with respect to the Lebesgue measure and $f \in \mathcal{L}_{M, \alpha} \cap \mathcal{L}_M$ for some $\alpha \in [1, 2)$ and $M > 0$. Let $\varepsilon > 0$ and set $\rho = \varepsilon \wedge 1$. Then, for all $0 < t < (\varepsilon \wedge 1/2)^{\alpha} (1 \wedge ((2 - \alpha)/2M))$ it holds:

\[
|\lambda_{\rho} t \mathbb{P}(|M_t(\rho) + Y_1^{(\rho)}| > \varepsilon) - \lambda_{\varepsilon} t| \leq L_1 \frac{t^{1+\alpha}}{(\varepsilon \wedge 1)^{1+\alpha}} + \frac{8M^2}{\alpha(2 - \alpha)} \frac{t^2}{(\varepsilon \wedge 1)^{2\alpha}} + \frac{5M}{2 - \alpha} \frac{t^2\lambda_1}{(\varepsilon \wedge 1)^2} + 4M^2t^2 \frac{1}{2 - \alpha} \varepsilon > 2\varepsilon + 12M^2t \lambda_{\alpha=1} \ln \left( \frac{C(1 \wedge \varepsilon \wedge (\varepsilon - 1 \lor 0))}{\varepsilon \wedge 1} \right) \frac{1}{\varepsilon \wedge 1},
\]

where $C := (1 \wedge ((2 - \alpha)/2M))^{1/\alpha}$ and $L_1$ is defined in (51).

Proof of Lemma 4. The result follows from Theorem 3 and Lemma 4 using the decomposition

\[
|\mathbb{P}(|X_t| > \varepsilon) - \lambda_{\varepsilon} t| \leq \mathbb{P}(|M_t(\rho)| > \rho) + |\lambda_{\rho} t \mathbb{P}(|M_t(\rho) + Y_1^{(\rho)}| > \varepsilon) - \lambda_{\varepsilon} t| + 2\lambda_{\rho}^2 t^2
\]

\[
\leq G_1 \frac{t^{1+\alpha}}{(\varepsilon \wedge 1)^{1+\alpha}} + G_2 \frac{t^2}{(\varepsilon \wedge 1)^{2\alpha}} + \frac{5M}{2 - \alpha} \frac{t^2\lambda_1}{(\varepsilon \wedge 1)^2} + 4M^2t^2 \frac{1}{2 - \alpha} \varepsilon > 2\varepsilon + M^2t^2 \lambda_{\alpha=1} \left( \frac{12}{\varepsilon \wedge 1} \ln \left( \frac{C(1 \wedge \varepsilon \wedge (\varepsilon - 1 \lor 0))}{\varepsilon \wedge 1} \right) \right) + \frac{16}{\varepsilon^2} \ln \left( \frac{\varepsilon}{2t} \right) + 2\lambda_{\rho}^2 t^2,
\]

with $\rho := \varepsilon \wedge 1$ and

\[
G_1 = L_1 + \mathbf{1}_{\alpha \in (1,2)} 2^{2+\alpha} M \left( 1 + \frac{M}{\alpha(2 - \alpha)(\alpha - 1)} \right) + \mathbf{1}_{\alpha=1} \left( 4M^2 \left( e^{2+1/\alpha} + 37/9 \right) + 4M \right),
\]

\[
G_2 = \frac{8M^2}{\alpha(2 - \alpha)} + M^2 E_{\alpha \in (1,2)}.
\]

\[\square\]

3.6 Proof of Theorem 5

We first introduce two auxiliary Lemmas whose proof can be found in the appendix.

Lemma 5. Let $\nu$ be a symmetric Lévy measure with density $f$ with respect to the Lebesgue measure and $f \in \mathcal{L}_{M, \alpha}$ for some $\alpha \in [1, 2)$ and $M > 0$. Let $\varepsilon \in (0,1]$, there exist three positive constants $K_1$, $K_2$ and $K_3$, only dependent on $\alpha$, such that for all $0 < t \leq (2 - \alpha)^{\alpha} / \varepsilon$, it holds:

\[
\mathbb{P}(|M_t(3\varepsilon/4)| > \varepsilon) \leq \frac{M^2 t^2 K_1}{\varepsilon^{2\alpha}} + t^2 \varepsilon^{-2\alpha} M^2 K_2 \mathbf{1}_{\alpha \in (1,2)} + \frac{t^4 M^4 K_3}{\varepsilon^{4\alpha}} + \frac{32M^2 t^2}{\varepsilon^2} \ln(2) \mathbf{1}_{\alpha=1}.
\]

For explicit formulas for $K_1$, $K_2$ and $K_3$ see (54) and (59).

Lemma 6. Let $\nu$ be a symmetric Lévy measure with density $f$ with respect to the Lebesgue measure and $f \in \mathcal{L}_{M, \alpha}$ for some $\alpha \in [1, 2)$ and $M > 0$. Let $\varepsilon > 0$, set $\rho = 3\varepsilon(\varepsilon \wedge 1)$ and assume that $f$ is $M(\varepsilon \wedge 1)^{-(2+\alpha)}$-Lipschitz on the interval $((3/4)(\varepsilon \wedge 1), 2\varepsilon - 3/4(\varepsilon \wedge 1))$. Then, for all $t > 0$ it holds:

\[
|\lambda_{\rho} t \mathbb{P}(|M_t(\rho) + Y_1^{(\rho)}| > \varepsilon) - \lambda_{\varepsilon} t| \leq M^2 t^2 (K_4 \varepsilon^{-2\alpha} \mathbf{1}_{0 < \varepsilon \leq 1} + \varepsilon^2 K_5 \mathbf{1}_{\varepsilon > 1}) + K_6 M t^2 \lambda_{\alpha=1} (\varepsilon \wedge 1)^{-\alpha},
\]

where $K_4$, $K_5$ and $K_6$ are positive universal constants, only depending on $\alpha$, defined in (68).
Proof of Theorem 5. Let \( \rho := 3/4(\varepsilon \land 1) \), using (7) at point \( \rho \) and \( \mathbb{P}(|M_t(\rho)| > \varepsilon) \leq \mathbb{P}(|M_t(\rho)| > 1 \land \varepsilon) \), we derive

\[
|\mathbb{P}(|X_t| > \varepsilon) - \lambda_t \varepsilon| \leq \mathbb{P}(|M_t(\rho)| > 1 \land \varepsilon) + |\lambda_t \mathbb{P}(|M_t(\rho) + Y_1^{(\rho)}| > \varepsilon) - \lambda_t \varepsilon| + \lambda_t \mathbb{P}(|M_t(\rho) + Y_1^{(\rho)}| > \varepsilon) + \mathbb{P}(N_t(\rho) \geq 2).
\]

By Lemma 5, Lemma 6, (8) and (10) it follows that

\[
I_1 + I_2 + I_3 + I_4 \leq t^2 M^2 (F_1 \varepsilon^{-2\alpha} + \lambda_1 \varepsilon^{-\alpha} F_2) 1_{0< \varepsilon \leq 1} + (\varepsilon^2 F_3 + F_4) 1_{\varepsilon > 1})
\]

and therefore, using \( t = \varepsilon \) replaced by \( t = \varepsilon / \lambda_1 \), we find that

\[
F_1 := 2K_1 + 2K_2 1_{\alpha \in (1, 2)} + K_4 + 64 \ln(2) 1_{\alpha = 1} + \frac{6}{\alpha^2},
\]

\[
F_4 := 2K_1 + 2K_2 1_{\alpha \in (1, 2)} + 64 \ln(2) 1_{\alpha = 1} + \frac{6}{\alpha^2}.
\] (32)

\[
\square
\]

A Technical lemmas and additional proofs

A.1 Proof of Lemma 2

For any \( u > 0 \) we have that

\[
\mathbb{E}[e^{u M_t(\varepsilon)}] \leq \exp \left( t \int (e^{u|y|} - u|y| - 1) \nu_\varepsilon(dy) \right)
\]

and therefore, using that \( \int |y|^k \nu_\varepsilon(dy) \leq \varepsilon^{k-2} \sigma^2(\varepsilon) \) for all \( k \geq 2 \),

\[
\mathbb{P}(M_t(\varepsilon) > x) \leq \exp \left( -ux + t \int (e^{u|y|} - u|y| - 1) \nu_\varepsilon(dy) \right)
\]

\[
= e^{\frac{u^2 \sigma^2(\varepsilon)}{2} - ux + t \sum_{k=3}^{\infty} \frac{u^k}{k!} \nu_\varepsilon(dy)} \leq e^{\frac{u^2 \sigma^2(\varepsilon)}{2} - ux + t \sigma^2(\varepsilon) \sum_{k=3}^{\infty} \frac{u^k}{k!}}
\]

\[
= e^{-ux + \frac{t \sigma^2(\varepsilon)}{2} (e^u - 1 - ux)}.
\] (33)

Injecting \( u^* = \frac{1}{2} \log (1 + \frac{\varepsilon^2}{t \sigma^2(\varepsilon)}) \) in (33), we find that \( \mathbb{P}(M_t(\varepsilon) > x) \leq e^{\frac{t \sigma^2(\varepsilon)}{2 \varepsilon} (\frac{t \sigma^2(\varepsilon)}{x^2} + \frac{t \sigma^2(\varepsilon)}{\varepsilon^2})} \), as claimed.

To derive (3), we simply use the fact that \( u^{-u} \leq e^{u^{-1}} \) for all \( u > 0 \). Indeed, set \( u = \frac{t \sigma^2(\varepsilon)}{\varepsilon^2} \) and notice that

\[
\left( \frac{t \sigma^2(\varepsilon)}{x^2 + t \sigma^2(\varepsilon)} \right)^{\frac{x + t \sigma^2(\varepsilon)}{t^2}} = \left( \frac{t \sigma^2(\varepsilon)}{\varepsilon^2} \right)^{u} \leq \left( \frac{t \sigma^2(\varepsilon)}{\varepsilon^2} \right)^{u} \left( \frac{t \sigma^2(\varepsilon)}{\varepsilon^2} \right)^{\frac{x + t \sigma^2(\varepsilon)}{x^2}}.
\]

Equation (3) then follows under the assumption \( t \sigma^2(\varepsilon) \varepsilon^{-2} \leq 1 \).

Analogous arguments, with \( M_t(\varepsilon) \) replaced by \( -M_t(\varepsilon) \), allows to deduce that

\[
\mathbb{P}(-M_t(\varepsilon) > x) \leq e^{\frac{x}{2 \varepsilon} (t \sigma^2(\varepsilon) + \frac{t \sigma^2(\varepsilon)}{x^2})}
\]

and hence the inequality

\[
\mathbb{P}(M_t(\varepsilon) \leq -x) \leq e^{\frac{-x}{t \sigma^2(\varepsilon) \varepsilon^{-2}}},
\]

whenever \( t \sigma^2(\varepsilon) \varepsilon^{-2} \leq 1 \).
A.2  Proof of Lemma 3

First, we consider the general case where \( \nu \) is not symmetric. We control the quantity \( J = \lambda_p P(|M_t(\rho) + \nu_t(\rho)| > \varepsilon) - \lambda \varepsilon \) as \( Q = |J| \). It holds that

\[
J = \int_{\rho}^{\infty} \left( P(M_t(\rho) + \nu_t(\rho) < -\varepsilon - z)f(z) + P(M_t(\rho) + \nu_t(\rho) > \varepsilon + z)f(-z) \right) dz
- \int_{\rho}^{\varepsilon} \left( P(M_t(\rho) + \nu_t(\rho) \leq -\varepsilon)f(z) + P(M_t(\rho) + \nu_t(\rho) > z)f(-z) \right) dz
+ \int_{\rho}^{\varepsilon} \left( P(M_t(\rho) + \nu_t(\rho) > -\varepsilon)f(z) + P(M_t(\rho) + \nu_t(\rho) < \varepsilon)f(-z) \right) dz =: R - S + T.
\]

Recall \( \rho = \varepsilon \wedge 1 \), assumptions on \( t \) ensures that \( |b(\rho)| \leq \rho/2 \), thus

\[
R \leq \int_{\rho}^{\infty} \left( P(M_t(\rho) < -\rho)f(z) + P(M_t(\rho) > \rho)f(-z) \right) dz.
\]

By means of Markov inequality and (9) we then derive

\[
|R| \leq \frac{2M}{2 - \alpha} \lambda \rho^{-\alpha}.
\]

To treat the terms \( S \) and \( T \) we distinguish the cases \( \varepsilon \in (0, 1] \) and \( \varepsilon > 1 \). Moreover, we restrict to the case \( b(\rho) \geq 0 \), the case \( b(\rho) < 0 \) can be obtained similarly and leads to the same result. Decompose \( S := S_1 + S_2 \) where

\[
S_1 + S_2 = \int_{-tb(\rho)}^{\infty} P(M_t(\rho) > x)f(-\varepsilon - \nu_t(\rho) - x)dx + \int_{tb(\rho)}^{\infty} P(M_t(\rho) \leq -x)f(x + \varepsilon - \nu_t(\rho))dx.
\]

We only detail the computations for the term \( S_1 \), for the term \( S_2 \) being analogous.

**Case** \( \varepsilon \in (0, 1] \): Then \( \rho = \varepsilon \) and by means of the triangle inequality it holds

\[
|S_1| \leq \int_{-tb(\varepsilon)}^{tb(\varepsilon)} f(-\varepsilon - \nu_t(\varepsilon) - x)dx + \int_{tb(\varepsilon)}^{\varepsilon/2} P(M_t(\varepsilon) > x)f(-\varepsilon - \nu_t(\varepsilon) - x)dx
+ \int_{\varepsilon/2}^{\varepsilon} P(M_t(\varepsilon) > x)f(-\varepsilon - \nu_t(\varepsilon) - x)dx + \int_{\varepsilon}^{\infty} P(M_t(\varepsilon) > x)f(-\varepsilon - \nu_t(\varepsilon) - x)dx
= S_{1,1} + S_{1,2} + S_{1,3} + S_{1,4}.
\]

Using that \( f \in L_{M,\alpha} \) and (11), it follows that

\[
S_{1,1} \leq 2Mt\varepsilon \varepsilon^{-(1+\alpha)} \leq \frac{4M^2t\varepsilon^{-2\alpha}}{1 - \alpha}.
\]

To control the term \( S_{1,2} \) we proceed as for the control of the term \( T_{2,1} \) in the proof of Theorem 1. Observe that \( 0 < t \leq (1 - \alpha)M^{-1}\varepsilon^{-\alpha}4^{-(1+\alpha)} \) implies \( 2Mt\varepsilon^{1+\alpha}(1 - \alpha)^{-1} < \varepsilon/2 \). Let \( x \in (2Mt\varepsilon^{1+\alpha}(1 - \alpha)^{-1}, \varepsilon/2) \) and set \( \tilde{x} := x - 2Mt\varepsilon^{1+\alpha}(1 - \alpha)^{-1} \). In particular we can write \( M_t(\varepsilon) = M_t(\tilde{x}) + Z_t(\tilde{x}, \varepsilon) - t(b(\varepsilon) - b(\tilde{x})) \). From the assumption \( f \in L_{M,\alpha} \) it also follows that \( |b(\varepsilon) - b(\tilde{x})| \leq 2M\varepsilon^{1-\alpha}(1 - \alpha)^{-1} \) and so \( P(M_t(\tilde{x}) > x + t(b(\varepsilon) - b(\tilde{x}))) \leq P(M_t(\tilde{x}) > \tilde{x}) \). Therefore, for all \( x \in (2Mt\varepsilon^{1+\alpha}(1 - \alpha)^{-1}, \varepsilon/2) \), the Markov inequality and (9), lead to

\[
P(M_t(\varepsilon) > x) \leq P(M_t(\tilde{x}) > x + t(b(\varepsilon) - b(\tilde{x}))) + P(N_t(\tilde{x}, \varepsilon) \geq 1)
\leq P(M_t(\tilde{x}) > \tilde{x}) + t\lambda_{\tilde{x},x} \leq \frac{2(2 + \alpha)Mt\tilde{x}^{-\alpha}}{\alpha(2 - \alpha)}.
\]
Furthermore, by means of (11), \( t|b(\varepsilon)| \leq 2Mt\varepsilon^{1-\alpha}(1-\alpha)^{-1} \) and \( f \in \mathcal{L}_{M,\alpha} \) we get
\[
\int_{tb(\varepsilon)}^{2Mt\varepsilon^{1-\alpha}} f(-\varepsilon - tb(\varepsilon) - x) dx \leq \int_0^{2Mt\varepsilon^{1-\alpha}(1-\alpha)^{-1}} f(-\varepsilon - tb(\varepsilon) - x) dx \leq \frac{2M^2}{1-\alpha} t\varepsilon^{-2\alpha}
\]
and
\[
\int_{2Mt\varepsilon^{1-\alpha}/(1-\alpha)}^{\varepsilon/2} \tilde{x}^{-\alpha} f(-\varepsilon - tb(\varepsilon) - x) dx \leq \frac{M}{\varepsilon^{1+\alpha}} \int_{2Mt\varepsilon^{1-\alpha}(1-\alpha)^{-1}}^{\varepsilon/2} (x - 2Mt\varepsilon^{1-\alpha}(1-\alpha)^{-1})^{-\alpha} dx \leq \frac{M}{1-\alpha} \varepsilon^{-2\alpha}.
\]
We derive that
\[
S_{1,2} \leq \frac{2M^2}{1-\alpha} t\varepsilon^{-2\alpha} + \frac{2(2 + \alpha)M^2}{\alpha(2-\alpha)(1-\alpha)} t\varepsilon^{-2\alpha}.
\]
(37)

To treat the term \( S_{1,3} \) we notice that for any \( t \in (0, (1-\alpha)M^{-1}\varepsilon^{\alpha}4^{-1+(1+\alpha)}) \) and \( x \in [\varepsilon/2, \varepsilon] \) we have that \( t\lambda_{x,\varepsilon} \leq 1 \) and hence, by Lemma 7, we derive that for all \( x \in [\varepsilon/2, \varepsilon] \) it holds:
\[
\mathbb{P}(M_t(x) > x) \leq \mathbb{P}(M_t(x) + \tilde{Z}_t(x, \varepsilon) > x) + 2\varepsilon \sup_{y \in [x, \varepsilon]} f(y),
\]
where \( \tilde{Z}_t(x, \varepsilon) := \sum_{i=1}^{N_t(x, \eta)} (Y_i^{(x, \eta)} - \mathbb{E}[Y_i^{(x, \eta)}]) \). Then, using (8), the Markov inequality, (9) and (10) we get
\[
\mathbb{P}(M_t(x) + \tilde{Z}_t(x, \varepsilon) > x) \leq \mathbb{P}(M_t(x) > x) + \mathbb{P}(N_t(x, \varepsilon) \geq 1) \leq \frac{2(2 + \alpha)Mt\varepsilon^{-\alpha}}{(2-\alpha)\alpha}.
\]
Moreover, the fact that \( f \in \mathcal{L}_{M,\alpha} \) implies \( \sup_{y \in [x, \varepsilon]} f(y) \leq Mx^{-(1+\alpha)} \leq M2^{1+\alpha}\varepsilon^{-(1+\alpha)} \) for \( x \in [\varepsilon/2, \varepsilon] \) and so we deduce that
\[
\mathbb{P}(M_t(x) > x) \leq 2^{\alpha+1}Mt\varepsilon^{-\alpha} \left( 2 + \frac{2 + \alpha}{(2-\alpha)\alpha} \right).
\]
Together with \( \int_{\varepsilon/2}^{\varepsilon} f(-\varepsilon - tb(\varepsilon) - x) dx \leq \lambda_{\varepsilon} \), we obtain
\[
S_{1,3} \leq \lambda_{\varepsilon} 2^{\alpha+1}Mt\varepsilon^{-\alpha} \left( 2 + \frac{2 + \alpha}{(2-\alpha)\alpha} \right).
\]
(38)

Finally, for the term \( S_{1,4} \) we have that
\[
S_{1,4} \leq \mathbb{P}(M_t(\varepsilon) > \varepsilon) \int_{-\infty}^{-2\varepsilon - tb(\varepsilon)} f(x) dx \leq \mathbb{P}(M_t(\varepsilon) > \varepsilon) \lambda_{\varepsilon}.
\]
From the Markov inequality and (9) we then derive
\[
S_{1,4} \leq \frac{2M}{2-\alpha} t\lambda_{\varepsilon} \varepsilon^{-\alpha}.
\]
(39)

Combining (36), (37), (38) and (39) yield
\[
|S_1| \leq \frac{2M^2}{1-\alpha} t\varepsilon^{-2\alpha} \left( 3 + \frac{2 + \alpha}{\alpha(2-\alpha)} \right) + 2Mt\varepsilon^{-\alpha} \lambda_{\varepsilon} \left( 1 + 2^{\alpha} \left( 2 + \frac{2 + \alpha}{\alpha(2-\alpha)} \right) \right).
\]
The term $S_2$ can be controlled in a similar way, in particular it holds that
\[ | - S | \leq \frac{4M^2}{1 - \alpha} t \varepsilon^{-2\alpha} \left( 3 + \frac{2 + \alpha}{\alpha(2 - \alpha)} \right) + 4Mt \varepsilon^{-\alpha} \lambda \left( 1 + 2^\alpha \left( 2 + \frac{2 + \alpha}{\alpha(2 - \alpha)} \right) \right). \]  
(40)

Finally, we observe that when $\varepsilon \in (0, 1]$ the term $T$ is identically zero.

Gathering Equations (34), (35) and (40), we conclude that for $\varepsilon \in (0, 1]$
\[ | \lambda t \varepsilon P(|M_t(\varepsilon) + \tilde{b}(\varepsilon) + Y_1^\varepsilon| > \varepsilon) - \lambda_t | \leq t^2 (M^2 \varepsilon^{-2\alpha} + M \lambda \varepsilon^{-\alpha} \varepsilon^2), \]
where
\[ D_1 := \frac{4}{1 - \alpha} \left( 3 + \frac{2 + \alpha}{\alpha(2 - \alpha)} \right) \quad \text{and} \quad D_2 := 4 \left( \frac{1}{2 - \alpha} + 1 + 2^\alpha \left( 2 + \frac{2 + \alpha}{\alpha(2 - \alpha)} \right) \right). \]  
(41)
as claimed.

Case $\varepsilon > 1$: Then $\rho = 1$, using that $f \in L_{\mathcal{M}, \alpha} \cap \mathcal{L}_M$ we readily derive
\[ S_1 \leq 2 Mt \tilde{b}(1) + M \left( \int_{\tilde{b}(1)}^{1/2} + \int_{1/2}^{\infty} \tilde{P}(M_t(1) > x) dx \right) \leq: \tilde{S}_{1.1} + \tilde{S}_{1.2} + \tilde{S}_{1.3}. \]  
(42)
The term $\tilde{S}_{1.2}$ is the analogous of $S_{1.2}$ above. Observe that under the assumptions $0 < t \leq (1 - \alpha)/(5M)^{-1}$ and $f \in L_{\mathcal{M}, \alpha}$, we get $t|\tilde{b}(1)| \leq 1/2$. For any $x \in (2Mt(1 - \alpha)^{-1}, 1/2)$, set $\tilde{x} := x - 2Mt(1 - \alpha)^{-1}$. The same reasoning as for the term $S_{1.2}$ allows to conclude that, for any $x \in (2Mt(1 - \alpha)^{-1}, 1/2)$,
\[ \tilde{P}(M_t(1) > x) \leq \tilde{P}(M_t(\tilde{x}) > \tilde{x}) + t(\lambda_{\tilde{x}, 2} + \lambda_2) \leq \frac{4}{(2 - \alpha)\alpha} M t \tilde{x}^{-\alpha} + t \lambda_2, \]  
(43)
where in the last inequality we used the fact that $f \in L_{\mathcal{M}, \alpha}$ joined with the Markov inequality, (9) and (10). Therefore, from (43) and using again that $f \in L_M$, we get
\[ \tilde{S}_{1.2} \leq Mt \left( \frac{2M}{1 - \alpha} - b(1) \right) + \frac{4M^2t}{\alpha(2 - \alpha)} \int_{2Mt(1 - \alpha)^{-1}}^{1/2} \left( x - \frac{2Mt}{1 - \alpha} \right)^{-\alpha} dx + \frac{tM \lambda_2}{2}. \]  
(44)
Furthermore, by the Markov inequality and (9), we deduce that
\[ \tilde{S}_{1.3} \leq \frac{4M^2t}{2 - \alpha}. \]  
(45)
Gathering (42), (44) and (45) we conclude that
\[ S_1 \leq 2Mt \tilde{b}(1) + Mt \left( \frac{2M}{1 - \alpha} - b(1) \right) + \frac{8M^2t}{\alpha(2 - \alpha)(1 - \alpha)} + \frac{tM \lambda_2}{2}. \]
Thus the term $S$ in (34) can be bounded by
\[ |S| \leq 4Mt \tilde{b}(1) + \frac{8M^2t}{\alpha(2 - \alpha)(1 - \alpha)} + tM \lambda_2. \]  
(46)
By means of (35), the term $R$ in (34) is bounded by
\[ |R| \leq \frac{2Mt \lambda_1}{2 - \alpha}. \]  
(47)
To control $J$ we are left to control the term $T$ in (34). We provide an upper bound for

$$T_1 := \int_{[0, \infty)} \mathbb{P}(M_t(1) + t b(1) > \varepsilon - z) f(z) \, dz = \int_{-t b(1)}^{\varepsilon - 1 - t b(1)} \mathbb{P}(M_t(1) \geq x) f(\varepsilon - x - t b(1)) \, dx,$$

the control of the quantity $\int_{[0, \infty)} \mathbb{P}(M_t(1) + t b(1) < -\varepsilon + z) f(-z) \, dz$ can be treated similarly. We have, using $t |b(1)| \leq 1/2$,

$$T_1 = 1_{\varepsilon \geq 1 + 2 t b(1)} \left( \int_{-t b(1)}^{t b(1)} \mathbb{P}(M_t(1) \geq x) f(\varepsilon - x - t b(1)) \, dx \right) + 1_{\varepsilon \leq 1 + 2 t b(1)} \left( \int_{-t b(1)}^{t b(1)} \mathbb{P}(M_t(1) \geq x) f(\varepsilon - x - t b(1)) \, dx \right).$$

For $f \in \mathcal{L}_M$, recalling the definition of $\tilde{S}_{1,2}$ given in (42) and that we assumed $b(1) \geq 0$, for $\varepsilon \geq 1 + 2 t b(1)$ we write

$$T_{1,1} \leq 2 M t b(1) + M \int_{t b(1)}^{1/(\varepsilon - 1 - t b(1))} \mathbb{P}(M_t(1) > y) \, dy + M \int_{1/(\varepsilon - 1 - t b(1))}^{\varepsilon - 1 - t b(1)} \mathbb{P}(M_t(1) > y) \, dy \leq M \left( 2 t b(1) + \tilde{S}_{1,2} + \mathbb{P}(M_t(1) > 1/2)(\varepsilon - 3/2 - t b(1)) \right) \mathbb{1}_{t b(1)} \left( \varepsilon - 1 - t b(1) \right) \mathbb{1}_{\varepsilon \leq 1 + 2 t b(1)} \right).$$

where we used (44), the Markov inequality and (9). Concerning the term $T_{1,2}$, using that $f \in \mathcal{L}_{M, \alpha}$ joined with (11) and the assumption $t < (1 - \alpha)(5 M)^{-1}$, we get

$$T_{1,2} \leq 2 M t b(1) (\varepsilon - 2 t b(1))^{-1 - \alpha} \mathbb{1}_{\varepsilon \leq 1 + 2 t b(1)} \leq 4 M t 5^\alpha \mathbb{1}_{\varepsilon < 1 + 2 t b(1)}.$$

This entails that

$$T \leq M t \left( \int_{1 + 2 t b(1)}^{\varepsilon \geq 1 + 2 t b(1)} \left( \frac{6 M}{1 - \alpha} + \frac{4 M}{\alpha(2 - \alpha)(1 - \alpha)} + \frac{\lambda_2}{2} + \frac{8 M}{2 - \alpha} \right) \mathbb{1}_{\varepsilon \leq 1 + 2 t b(1)} \right).$$

Combining (34), (46), (47), (48) and using (10), we conclude that for any $\varepsilon > 1, 0 < \lambda < (1 - \alpha)(5 M)^{-1}$ and $f \in \mathcal{L}_{M, \alpha} \cap \mathcal{L}_M$ it holds, using $t |b(1)| \leq 1/2$,

$$J \leq 2 M^2 t \left( \tilde{D}_1 + \frac{4}{2 - \alpha} \mathbb{1}_{\varepsilon \leq 1 + 2 t b(1)} \right) + M t \left( 4 \times 5^\alpha \mathbb{1}_{\varepsilon < 1 + 2 t b(1)} + \frac{8}{5} + 3 \lambda_2 + \frac{4 \lambda_2}{2 - \alpha} \right),$$

where we used the notation

$$\tilde{D}_1 := \frac{5}{1 - \alpha} + \frac{10}{\alpha(2 - \alpha)(1 - \alpha)}.$$

**Case $\nu$ symmetric and $\varepsilon > 0$:** In the case where $\nu$ is symmetric the proof can be simplified. Since $b(\rho) \equiv 0$, $M_t(\rho) = M_t(x) + Z_t(x, \rho)$ for all $x \in (0, \rho), t > 0$ and it holds

$$\lambda_\rho \mathbb{P}(|M_t(\rho) + Y_1^{(\rho)}| > \varepsilon) - \lambda_\varepsilon = 2 \left( \int_{\rho}^{\infty} \mathbb{P}(M_t(\rho) > \varepsilon + z) - \mathbb{P}(M_t(\rho) < \varepsilon - z) \right) f(z) \, dz \leq \lambda_\rho \mathbb{P}(M_t(\rho) > 2 \rho) + 2 \left( \int_{0}^{\rho} \mathbb{P}(M_t(\rho) > x) f(x + \varepsilon) \, dx + \mathbb{P}(M_t(\rho) > \rho) \int_{\rho}^{\infty} f(x + \varepsilon) \, dx \right).$$
With the same arguments as those used to treat the term $S_{1,2}$ above, one finds that for any $x \in (0, \varepsilon)$ and $t > 0$ it holds $\mathbb{P}(M_t(\rho) > x) \leq \frac{2(2t+\alpha)Mtx^{-\alpha}}{\alpha(2-\alpha)}$. Therefore, by the Markov inequality, (9) and using that $f \in \mathcal{L}_{M,\alpha}$, we conclude for all $\varepsilon \in (0,1)$ and $t > 0$ it holds

$$|\lambda_\rho \mathbb{P}(|M_t(\rho) + Y_1^{(\rho)}| > \varepsilon) - \lambda_\varepsilon| \leq \frac{tM}{2(2-\alpha)}(\lambda_\varepsilon\varepsilon^{-\alpha} + 4\lambda_2\varepsilon^{-\alpha}) + 2tM^2D_2\varepsilon^{2-2\alpha},$$

with

$$D_3 := \frac{2(2+\alpha)}{(2-\alpha)\alpha(1-\alpha)}.$$ \hspace{1cm} (50)

If instead $\varepsilon > 1$, assuming in addition that $f \in \mathcal{L}_M$, we derive

$$|\lambda_1 \mathbb{P}(|M_t(1) + Y_1^{(1)}| > \varepsilon) - \lambda_\varepsilon| \leq \frac{tM}{2-\alpha}(\lambda_12^{-\alpha} + \frac{4M}{\alpha(1-\alpha)} + \lambda_{1+\varepsilon}).$$

This concludes the proof.

**A.3 Proof of Lemma 4**

Decomposition (60) as in the proof of Lemma 6 in $\lambda_\rho \mathbb{P}(|M_t(\rho) + Y_1^{(\rho)}| > \varepsilon) - \lambda_\varepsilon t =: 2t(R_1 + R_2)$ still holds with

$$|R_2| \leq tM_0 \mathbb{1}_{\varepsilon \leq 1} \left( \frac{M\varepsilon^{-2\alpha}}{(\alpha(2-\alpha)} + \frac{\varepsilon^{-\alpha} \lambda_1}{2(2-\alpha)} \right) + tM \mathbb{1}_{\varepsilon > 1} \lambda_1 \frac{1}{(2-\alpha)(\varepsilon + 1)^2}.$$

Set $C = \left(1 \wedge \left((2-\alpha)/2M\right)\right)^{1/\alpha}$ and note that $C(\varepsilon \wedge 1)/2 > t^{1/\alpha}$. Using the symmetry of $f$ we get

$$|R_1| \leq \int_0^{t^{1/\alpha}/C} \left( f(y + \varepsilon) + \mathbb{1}_{\varepsilon > 1} f(\varepsilon - y) \right) dy + \int_{t^{1/\alpha}/C}^{\infty} \mathbb{P}(M_t(y) > y) + t\lambda_{y,\rho} f(\varepsilon + y) dy$$

$$+ \mathbb{P}(M_t(\rho) > \rho) \int_{t^{1/\alpha}/C}^{\infty} f(y + \varepsilon) dy$$

$$+ \mathbb{1}_{\varepsilon > 1} \left( \int_{t^{1/\alpha}/C}^{\varepsilon-1} \mathbb{P}(M_t(y) > y) + t\lambda_{y,1} f(\varepsilon - y) dy + \mathbb{P}(M_t(1) > 1) \int_{1 \wedge (\varepsilon-1)}^{\varepsilon-1} f(\varepsilon - y) dy \right) .$$

Next as $f \in \mathcal{L}_{M,\alpha} \cap \mathcal{L}_M$, it follows from Equations (7), (8), (9), (10) and the Markov inequality, that

$$|R_1| \leq \frac{M^{1/\alpha}}{C} \left( \frac{1}{(\varepsilon \wedge 1)^{1+\alpha}} + \frac{\mathbb{1}_{\varepsilon > 1}}{(\varepsilon - t^{1/\alpha}/C) \wedge 1} \right)$$

$$+ \frac{2M}{2-\alpha} \left( \rho^{-\alpha} \left( M\alpha^{-1}(\rho + \varepsilon)^{-\alpha} + \frac{\lambda_1}{2} \right) + M \mathbb{1}_{\varepsilon > 2}(\varepsilon - 2) \right)$$

$$+ \frac{4M^2t^{1/\alpha}C^{-1}}{\alpha(\varepsilon - 1)} \mathbb{1}_{\varepsilon \in (1,2)} \left( \frac{1}{(\varepsilon - t^{1/\alpha}/C) \wedge 1} + \frac{\mathbb{1}_{\varepsilon > 1}}{(\varepsilon - t^{1/\alpha}/C) \wedge 1} \right)$$

$$+ 4M^2t \mathbb{1}_{\varepsilon = 1} \left( \ln \left( \frac{C(\varepsilon \wedge |\varepsilon - 1|)}{t} \right) \frac{1}{\varepsilon - t/C \wedge 1} + \ln \left( \frac{C\rho}{t} \right) \frac{1}{\varepsilon + t/C \wedge 1} \right)$$

$$\leq \frac{M^{1/\alpha}}{C} \left( \frac{1}{(\varepsilon \wedge 1)^{1+\alpha}} + \frac{2M}{\varepsilon \wedge 1} \left( M\alpha^{-1}(\varepsilon \wedge 1)^{-2\alpha} + \frac{\lambda_1(\varepsilon \wedge 1)^{-\alpha}}{2} + M \mathbb{1}_{\varepsilon > 2}\varepsilon \right) \right)$$

$$+ \frac{12M^2t^{1/\alpha}C^{-1}}{\alpha(\varepsilon - 1)} \mathbb{1}_{\varepsilon \in (1,2)} \frac{1}{\varepsilon \wedge 1} + 12M^2t \mathbb{1}_{\varepsilon = 1} \ln \left( \frac{C(\varepsilon \wedge |\varepsilon - 1 \vee 0|)}{t} \right) \frac{1}{\varepsilon \wedge 1} .$$
with the convention that $0 \ln 0 = 0$. Therefore,
\[
|\lambda \rho t \mathbb{P}(|M_\rho(t) + Y_1^{(\rho)}| > \varepsilon) - \lambda_t t| 
\leq L_1 \frac{t^{1+1/\alpha}}{(\varepsilon \wedge 1)^{1+\alpha}} + \frac{8M^2}{(2-\alpha) (\varepsilon \wedge 1)^{2\alpha}} + \frac{5M}{2 - \alpha} \frac{t^2\lambda_1}{(\varepsilon \wedge 1)^2} + \frac{4M^2t^2}{2 - \alpha} 1_{\varepsilon > 2\varepsilon} 
\]
\[
+ 12M^2t^2 1_{\alpha=1} \ln \left( \frac{C(1 \wedge \varepsilon \wedge (\varepsilon - 1 \vee 0))}{t} \right) \frac{1}{\varepsilon \wedge 1},
\]
where
\[
L_1 = \frac{2M}{C^2} + \frac{4M}{C^2} + \frac{24M^2C^{\alpha-1}1_{\alpha \in (1,2)}}{\alpha(2-\alpha)(\alpha - 1)},
\]
as desired.

### A.4 Proof of Lemma 5

First, using the symmetry of $\nu$ it holds $\mathbb{P}(|M_\rho(t)| > \varepsilon) = 2\mathbb{P}(M_\rho(t) > \varepsilon)$ where we write $\rho := 3\varepsilon/4$. Since $\varepsilon/2 < \rho < \varepsilon$ together with (7) and (8), we obtain
\[
\mathbb{P}(M_\rho(t) > \varepsilon) \leq \mathbb{P}(M_\rho(\varepsilon/2) > \varepsilon) + t\lambda_{\varepsilon/2,\rho} \mathbb{P}(M_\rho(\varepsilon/2) + Y_1^{(\varepsilon/2,\rho)} > \varepsilon) + (t\lambda_{\varepsilon/2,\rho})^2. \tag{52}
\]
Applying Lemma 2 and using (9) we derive
\[
\mathbb{P}(M_\rho(\varepsilon/2) > \varepsilon) + (t\lambda_{\varepsilon/2,\rho})^2 \leq M^2t^2\varepsilon^{-2\alpha}K_1, \tag{53}
\]
with
\[
K_1 := 4^{1+\alpha}\left( \frac{\varepsilon^{2+1/\alpha}}{(2-\alpha)^2} + \frac{1}{\alpha^2} \right). \tag{54}
\]
Using again the symmetry of $\nu$ we can establish
\[
\lambda_{\varepsilon/2,\rho} \mathbb{P}(M_\rho(\varepsilon/2) + Y_1^{(\varepsilon/2,\rho)} > \varepsilon) = \int_{\varepsilon/4}^{\varepsilon/2} \mathbb{P}(M_\rho(\varepsilon/2) > y)f(y)(\varepsilon - y)dy + \int_{\varepsilon/2}^{\varepsilon} \mathbb{P}(M_\rho(\varepsilon/2) > \varepsilon + y)f(y)dy 
\leq \int_{\varepsilon/4}^{\varepsilon/2} \mathbb{P}(M_\rho(\varepsilon/2) > y)f(y)(\varepsilon - y)dy + \mathbb{P}(M_\rho(\varepsilon/2) > 3/2\varepsilon)\lambda_{\varepsilon/2,\rho} \frac{\varepsilon_{\varepsilon/2,\rho}}{2} =: T_1 + T_2. \tag{55}
\]
Applying (7), (8), the Markov inequality and (9), for any $y \in (\varepsilon/4, \varepsilon/2)$ we have
\[
\mathbb{P}(M_\rho(\varepsilon/2) > y) \leq \mathbb{P}(M_\rho(y) > y) + t\lambda_{y,\varepsilon/2} \leq \frac{4Mt\varepsilon^{1/\alpha}}{\alpha(2-\alpha)}. \tag{56}
\]
It follows that
\[
T_1 \leq \frac{4Mt}{\alpha(2-\alpha)} \int_{\varepsilon/4}^{\varepsilon/2} y^{-\alpha}f(y)(\varepsilon - y)dy \leq \frac{2^{1+\alpha}M^2t}{\alpha(2-\alpha)\varepsilon^{1+\alpha}} \left( \frac{(\varepsilon/4)^{1-\alpha}}{\alpha - 1} 1_{\alpha \in (1,2)} + \ln(2) 1_{\alpha = 1} \right). \tag{56}
\]
Furthermore, note that Lemma 2 applies as $t \leq \frac{(3-\alpha)^2}{2\alpha^2 - \alpha M}$ implies $4t\sigma^2(\varepsilon/2)\varepsilon^{-2} \leq 1$, together with (9), it gives
\[
T_2 \leq \frac{\varepsilon^{1+1/\alpha}3^3+1/\alpha M^4}{\alpha(2-\alpha)^3\varepsilon^{4\alpha}}. \tag{57}
\]
Finally, gathering (52), (53) and (58), we derive
\[ \mathbb{P}(M_t(\rho) > \varepsilon) \leq M^2 t^2 \varepsilon^{-2^{\alpha}} K_1 + t^2 \varepsilon^{-2^{\alpha}} M^2 K_2 1_{\alpha \in (1,2)} + \frac{t^4 M^4 K_3}{\varepsilon^{4 \alpha}} + \frac{16 M^2 t^2}{\varepsilon^2} \ln(2) 1_{\alpha = 1}. \]

### A.5 Proof of Lemma 6

First, since \( \nu \) is symmetric, it holds
\[ \lambda_\rho \mathbb{P}(|M_t(\rho) + Y_1^{(\rho)}| > \varepsilon) = 2 \int_{\rho}^{\infty} \left( \mathbb{P}(M_t(\rho) > \varepsilon - z) + \mathbb{P}(M_t(\rho) > \varepsilon + z) \right) \nu(dz). \]

Moreover, since \( \rho < \varepsilon \), and using again the symmetry, we obtain
\[ \begin{align*}
\lambda_\rho t \mathbb{P}(|M_t(\rho) + Y_1^{(\rho)}| > \varepsilon) - \lambda_\varepsilon t \\
= 2t \left[ \int_{\rho}^{\varepsilon} \mathbb{P}(M_t(\rho) > \varepsilon - z) \nu(dz) - \int_{\varepsilon}^{\infty} \mathbb{P}(M_t(\rho) \leq \varepsilon - z) \nu(dz) \right] \\
+ 2t \int_{\rho}^{\infty} \mathbb{P}(M_t(\rho) > \varepsilon + z) \nu(dz) =: 2t(R_1 + R_2).
\end{align*} \]

We begin by controlling the term \( R_1 \). Recalling that \( \rho = 3/4(\varepsilon \wedge 1) \) and setting \( \eta := \varepsilon - 3/4(\varepsilon \wedge 1) \), we have:
\[ \begin{align*}
\int_{\rho}^{\varepsilon} \mathbb{P}(M_t(\rho) > \varepsilon - z) \nu(dz) &= \int_{0}^{\eta} \mathbb{P}(M_t(\rho) > x) f(\varepsilon - x) dx, \\
\int_{\varepsilon}^{\infty} \mathbb{P}(M_t(\rho) \leq \varepsilon - z) \nu(dz) &= \int_{\varepsilon}^{\infty} \mathbb{P}(M_t(\rho) > z - \varepsilon) \nu(dz) = \int_{0}^{\infty} \mathbb{P}(M_t(\rho) > x) f(\varepsilon + x) dx,
\end{align*} \]
where we used the symmetry of \( \nu \) in the second line. The triangle inequality gives
\[ |R_1| \leq \left| \int_{0}^{\eta} \mathbb{P}(M_t(\rho) > x) (f(\varepsilon - x) - f(\varepsilon + x)) dx \right| + \left| \int_{\eta}^{\infty} \mathbb{P}(M_t(\rho) > x) f(\varepsilon + x) dx \right| =: R_{1,1} + R_{1,2}. \]

Therefore, by means of (7), (8), the Markov inequality, (9), and that \( f \) is \( M(\varepsilon \wedge 1)^{-2(\alpha)} \)-Lipschitz on the interval \( (3/4(\varepsilon \wedge 1), 2\varepsilon - 3/4(\varepsilon \wedge 1)) \), it follows that
\[ R_{1,1} \leq 2 M(\varepsilon \wedge 1)^{-2(\alpha)} \left[ 1_{0 < \varepsilon \leq 1} \int_{0}^{\varepsilon/4} (\mathbb{P}(M_t(x) > x) + t\lambda_{x,3/4}) dx + 1_{\varepsilon > 1} \int_{0}^{(\varepsilon - 3/4)/64/3} (\mathbb{P}(M_t(x) > x) + t\lambda_{x,3/4}) dx + 1_{\varepsilon > 1} \mathbb{P}(M_t(3/4) > 3/4) \int_{(\varepsilon - 3/4)/64/3}^{x - 3/4} dx \right] \]
\[ \leq 8 t M^2 (\varepsilon \wedge 1)^{-2(\alpha)} \left[ 1_{0 < \varepsilon \leq 1} \int_{0}^{\varepsilon/4} \frac{dx}{x^{3/4}} + 1_{\varepsilon > 1} \int_{0}^{(\varepsilon - 3/4)/64/3} \frac{dx}{x^{3/4}} \right] + 1_{\varepsilon \geq 3/2} \frac{4^{1+\alpha} 3^{-\alpha} M^2 t^2}{2 - \alpha} \]
\[ \leq \frac{2^{2\alpha - 1}}{\alpha(2 - \alpha)^2} M^2 t^2 \varepsilon^{-2\alpha} 1_{0 < \varepsilon \leq 1} + \frac{8 t M^2}{\alpha(2 - \alpha)^2} (\varepsilon - 3/4)^{2-\alpha} 1_{1 < \varepsilon \leq 3/2} + \frac{4^{1+\alpha} e^2 M^2 t}{3^\alpha (2 - \alpha)^1} 1_{\varepsilon > 3/2}. \]
Concerning the term $R_{1,2}$ we have:

$$R_{1,2} \leq 1_{0<\epsilon\leq 1} \left( \int_{\epsilon/4}^{3/4} \mathbb{P}(M_{t}(3/4\epsilon) > x)f(x+\epsilon)dx + \mathbb{P}(M_{t}(3/4\epsilon) > 3/4\epsilon) \int_{3/4\epsilon}^{\infty} f(x+\epsilon)dx \right)$$

$$+ 1_{1<\epsilon<3/2} \left( \int_{\epsilon-3/4}^{3/4} \mathbb{P}(M_{t}(3/4) > x)f(x+\epsilon)dx + \mathbb{P}(M_{t}(3/4) > 3/4) \int_{3/4}^{\infty} f(x+\epsilon)dx \right)$$

$$+ 1_{\epsilon\geq 3/2} \mathbb{P}(M_{t}(3/4) > 3/4) \int_{\epsilon-3/4}^{\infty} f(x+\epsilon)dx. \quad (63)$$

Using (7), (8), the Markov inequality, (9) and (10), we get

$$\mathbb{P}(M_{t}(3/4\epsilon) > x) \leq \mathbb{P}(M_{t}(x/2) > x) + t\lambda_{x/2,3/4\epsilon} \leq \frac{2^{2+\alpha} M t x^{-\alpha}}{\alpha(2-\alpha)}, \quad \forall x \leq \frac{3\epsilon}{2}, \ \epsilon < 1,$$

$$\mathbb{P}(M_{t}(3/4) > 3/4(e \land 1)) \leq t M \frac{2^{2\alpha+1}}{3^{\alpha}(2-\alpha)} (e \land 1)^{-\alpha}. \quad (64)$$

Therefore, from (63), (64) and (10) we derive using that $f \in \mathcal{L}_{M} \cap \mathcal{L}_{M}$:

$$R_{1,2} \leq 1_{0<\epsilon\leq 1} \left( t M \epsilon^{-2\alpha} \left( \frac{2^{2\alpha}}{\alpha(\alpha-1)(2-\alpha)} + \frac{2^{4\alpha+1}}{21^\alpha \alpha(2-\alpha)} \right) + t M \epsilon^{-\alpha} \lambda_{1} \frac{2^{2\alpha+1}}{3^{\alpha}(2-\alpha)} \right)$$

$$+ 1_{1<\epsilon<3/2} t M \left( \frac{2^{3\alpha} M}{\alpha(\alpha-1)(2-\alpha)} + \frac{2^{4\alpha+1}}{21^\alpha \alpha(2-\alpha)} \lambda_{7/4} \right) + 1_{\epsilon\geq 3/2} t M \lambda_{9/4} \frac{2(4/3)^{\alpha}}{2-\alpha}. \quad (65)$$

Gathering Equations (61), (62) and (65), we get

$$R_{1} \leq 1_{0<\epsilon\leq 1} \left( t M \epsilon^{-2\alpha} \left( \frac{2^{2\alpha-1}}{\alpha(2-\alpha)^{2}} + \frac{2^{3\alpha}}{\alpha(\alpha-1)(2-\alpha)} + \frac{2^{4\alpha+1}}{21^\alpha \alpha(2-\alpha)} \right) + t M \epsilon^{-\alpha} \lambda_{1} \frac{2^{2\alpha+1}}{3^{\alpha}(2-\alpha)} \right)$$

$$+ 1_{1<\epsilon<3/2} t M \left( \frac{8 M}{\alpha(2-\alpha)^{2}} (\epsilon - 3/4)^{2-\alpha} - \frac{2^{3\alpha} M}{\alpha(\alpha-1)(2-\alpha)} + \frac{2^{2\alpha+1}}{3^{\alpha}(2-\alpha)} \lambda_{7/4} \right)$$

$$+ 1_{\epsilon\geq 3/2} \left( \frac{4^{1+\alpha} \epsilon^{2} M t^{2}}{3^{\alpha}(2-\alpha)} + t M \lambda_{9/4} \frac{2(4/3)^{\alpha}}{2-\alpha} \right). \quad (66)$$

To complete the proof we are left to control the term $R_{2}$ in (60). The Markov inequality, (9), the symmetry of $\nu$ and the fact that $\rho > 1/2(e \land 1)$ yield

$$R_{2} \leq \frac{\mathbb{P}(M_{t}(\rho) > \rho)}{2} (\lambda_{1} + 1_{1}) \leq \frac{2^{\alpha} M t (1 \land \epsilon)^{-\alpha}}{2-\alpha} (2^{\alpha+1} M (1 \land \epsilon)^{-\alpha} \alpha^{-1} + 1_{1}). \quad (67)$$

Therefore, from (60), (66) and (67) we conclude that

$$|\lambda_{t} | \mathbb{P}(|M_{t}(\rho) + Y_{t}^{(\rho)}| > \epsilon) - \lambda_{t} t| \leq M t^{2} \left( K_{4} \epsilon^{-2\alpha} 1_{0<\epsilon<1} + \epsilon^{2} K_{5} 1_{1<\epsilon<1} \right) + K_{6} M t^{2} \lambda_{1} \epsilon \land 1 \land -\alpha,$$

where $K_{4}$, $K_{5}$ and $K_{6}$ are positive universal constants, only depending on $\alpha$, defined as follows:

$$K_{4} := \frac{2^{2\alpha-1}}{\alpha(2-\alpha)^{2}} + \frac{2^{3\alpha}}{\alpha(\alpha-1)(2-\alpha)} + \frac{2^{4\alpha+1}}{21^\alpha \alpha(2-\alpha)} + \frac{2^{2\alpha+2}}{\alpha(2-\alpha)},$$

$$K_{5} := \left( \frac{8(3/4)^{2-\alpha}}{\alpha(2-\alpha)^{2}} + \frac{2^{3\alpha}}{\alpha(\alpha-1)(2-\alpha)} \right) 1_{1<\epsilon<3/2} + \frac{4^{1+\alpha}}{3^{\alpha}(2-\alpha)} 1_{\epsilon\geq 3/2}, \quad (68)$$

$$K_{6} := 1_{0<\epsilon<1} \frac{2^{2\alpha+1}}{3^{\alpha}(2-\alpha)} + 1_{1<\epsilon<3/2} \frac{2^{2\alpha+1}}{3^{\alpha}(2-\alpha)} + 1_{\epsilon\geq 3/2} \frac{2(4/3)^{\alpha}}{2-\alpha}.$$
A.6 A result for compound Poisson processes

**Lemma 7.** Let $N$ a Poisson random variable with mean $0 < \lambda \leq 2$ and $(Y_i)_{i \geq 0}$ a sequence of i.i.d. random variables independent of $N$ with bounded density $g$ (with respect to the Lebesgue measure). Furthermore, let $Z$ be any random variable independent of $(N, (Y_i)_{i \geq 0})$. Then, for all $x \in \mathbb{R}$,

$$
\left| \mathbb{P}\left(Z + \sum_{i=1}^{N} Y_i - \mathbb{E}\left[\sum_{i=1}^{N} Y_i\right] > x\right) - \mathbb{P}\left(Z + \sum_{i=1}^{N} (Y_i - \mathbb{E}[Y_i]) > x\right) \right| \leq 2\lambda e^{-\lambda}||g||_{\infty}.
$$

If, instead, $1 < \lambda < 2$, then for all $x \in \mathbb{R}$,

$$
\left| \mathbb{P}\left(Z + \sum_{i=1}^{N} Y_i - \mathbb{E}\left[\sum_{i=1}^{N} Y_i\right] > x\right) - \mathbb{P}\left(Z + \sum_{i=1}^{N} (Y_i - \mathbb{E}[Y_i]) > x\right) \right| \leq 2\lambda^2 e^{-\lambda}||g||_{\infty}.
$$

**Proof.** First, note that

$$
\left| \mathbb{P}\left(\sum_{i=1}^{N} Y_i - \mathbb{E}\left[\sum_{i=1}^{N} Y_i\right] > x\right) - \mathbb{P}\left(\sum_{i=1}^{N} (Y_i - \mathbb{E}[Y_i]) > x\right) \right|
\leq \sum_{n=0}^{\infty} \left| \mathbb{P}\left(\sum_{i=1}^{n} Y_i > x + \lambda \mathbb{E}[Y_1]\right) - \mathbb{P}\left(\sum_{i=1}^{n} Y_i > x + n \mathbb{E}[Y_1]\right) \right| \mathbb{P}(N = n)
\leq \sum_{n=0}^{\infty} \mathbb{P}(N = n)||\mathbb{E}[Y_1]||n - \lambda||g^n||_{\infty} \leq ||g||_{\infty}||\mathbb{E}[Y_1]|| \sum_{n=0}^{\infty} \mathbb{P}(N = n)|n - \lambda|.
$$

Finally we observe that, since $\lambda \leq 1$, it holds

$$
\sum_{n=0}^{\infty} \mathbb{P}(N = n)|n - \lambda| = \lambda \mathbb{P}(N = 0) + \mathbb{E}[N] - \lambda \mathbb{P}(N \geq 1) = 2\lambda e^{-\lambda}.
$$

If, instead, $1 < \lambda < 2$, then

$$
\sum_{n=0}^{\infty} \mathbb{P}(N = n)|n - \lambda| = \lambda \mathbb{P}(N = 0) + (\lambda - 1)\mathbb{P}(N = 1) + \sum_{n=2}^{\infty} (n - \lambda)\mathbb{P}(N = n) = 2\lambda^2 e^{-\lambda}.
$$

We conclude the proof by observing that for any real random variable $Z_1$ independent of $Z_2$ and $Z_3$ and any $z \in \mathbb{R}$ it holds

$$
\left| \mathbb{P}(Z_1 + Z_2 > z) - \mathbb{P}(Z_1 + Z_3 > z) \right| = \int_{\mathbb{R}} \left( \mathbb{P}(Z_2 > z - y) - \mathbb{P}(Z_1 > z - y) \right) \mu(dy)
\leq \sup_{x \in \mathbb{R}} \left| \mathbb{P}(Z_2 > x) - \mathbb{P}(Z_3 > x) \right|,
$$

where $\mu$ is the law of $Z_1$. 

A.7 Proofs of the Examples

1. **Compound Poisson processes.** Let $X$ be a compound Poisson process with intensity $\lambda = \nu(\mathbb{R}) < \infty$ and jump density $f/\lambda$. Write $X_t = \sum_{i=0}^{N_t} Z_i$, for any $\varepsilon > 0$, it holds

$$
\mathbb{P}(|X_t| > \varepsilon) = t\lambda e^{-\lambda t} + \sum_{n=2}^{\infty} \mathbb{P}\left(\left| \sum_{i=1}^{n} Z_i \right| > \varepsilon \right) \mathbb{P}(N_t = n).
$$
Using \( \mathbb{P}(N_t \geq 2) = O(t^2) \) we obtain \( |\mathbb{P}(|X_t| > \varepsilon) - t\lambda \varepsilon| = O(t^2) \), as \( t \to 0 \). For \( f \) a Lévy density such that \( f = f 1_{[\varepsilon, \infty)} \), it holds \( \lambda = \lambda_\varepsilon \) and later computations simplify in

\[
\mathbb{P}(|X_t| > \varepsilon) = \mathbb{P}(N_t \geq 1) = 1 - e^{-\lambda_\varepsilon t} = \lambda_\varepsilon t - t^2 \sum_{k=2}^{\infty} \frac{(-\lambda_\varepsilon)^k}{k!}.
\]

In that case, the rate is exactly of the order of \( t^2 \). Next considering the small jumps, it holds for \( \varepsilon \in (0,1) \) that \( tb(\varepsilon) + M_t(\varepsilon) = \sum_{i=1}^{N_{\varepsilon}^{(0,e)}} Y_{i}^{(0,e)} \) and using (8)

\[
\mathbb{P}(|tb(\varepsilon) + M_t(\varepsilon)| \geq \varepsilon) = \sum_{k=2}^{\infty} \mathbb{P}(N_{\varepsilon}^{(0,e)} = k) \mathbb{P}(|\sum_{i=0}^{k} Y_{i}^{(0,e)}| \geq \varepsilon) \leq t^2(\lambda - \lambda_\varepsilon)^2.
\]

It is exactly of order \( t^2 \) for any Lévy density such that \( f = f 1_{[\varepsilon/4, \infty)} \).

2. Gamma processes. Set \( \Gamma(t, \varepsilon) = \int_{\varepsilon}^{\infty} x^{t-1} e^{-x} dx \), such that \( \Gamma(t, 0) = \Gamma(t) \). Using that \( \Gamma(t, \varepsilon) \) is analytic we can write

\[
\left| \lambda_\varepsilon - \frac{\mathbb{P}(X_t > \varepsilon)}{t} \right| = \frac{1}{\Delta \Gamma(t)} \left| \Delta \Gamma(t,0) \Gamma(0,\varepsilon) - \sum_{k=0}^{\infty} \frac{\Delta^k}{k!} \left( \frac{\partial^k}{\partial \varepsilon^k} \Gamma(t, \varepsilon) \right)_{\varepsilon=0} \right|
\leq \Gamma(0,\varepsilon) \left| 1 - t \frac{\Delta \Gamma(t,0)}{\Delta \Gamma(t)} \right| + \left| \frac{1}{t \Gamma(t)} \sum_{k=1}^{\infty} t^k \left( \frac{\partial^k}{\partial \varepsilon^k} \Gamma(t, \varepsilon) \right)_{\varepsilon=0} \right|.
\]

As \( \Gamma(t,0) \) is a meromorphic function with a simple pole in 0 and residue 1, there exists a sequence \( (a_k)_{k \geq 0} \) such that \( \Gamma(t) = \frac{1}{t} + \sum_{k=0}^{\infty} a_k t^k \). Therefore,

\[
1 - \frac{t \Gamma(t,0)}{\Gamma(t)} = t \sum_{k=0}^{\infty} a_k t^k,
\]

and

\[
1 - \frac{t \Gamma(t)}{\Gamma(t)} = \frac{t \sum_{k=0}^{\infty} a_k t^k}{1 + t \sum_{k=0}^{\infty} a_k t^k} = O(t), \quad \text{as } t \to 0.
\]

Let us now study the term \( \sum_{k=1}^{\infty} t^k \frac{\partial^k}{\partial \varepsilon^k} \Gamma(t, \varepsilon) \) at \( \varepsilon = 0 \). We have:

\[
\left| \frac{\partial^k}{\partial \varepsilon^k} \Gamma(t, \varepsilon) \right|_{\varepsilon=0} \leq e^{-1} \int_{x}^{1} x^{-1} (\log(x))^k dx + \int_{1}^{\infty} e^{-x} (\log(x))^k dx
\leq e^{-1} \frac{\log(x)}{k} + \int_{1}^{\infty} e^{-x} (\log(x))^k dx.
\]

Let \( x_0 \) be the largest real number such that \( e^{\frac{x_0}{2}} = (\log(x_0))^k \). This equation has two solutions if and only if \( k \geq 6 \). If no such point exists, take \( x_0 = 1 \). Then,

\[
\int_{1}^{\infty} e^{-x} (\log(x))^k dx \leq \int_{1}^{x_0} e^{-x} (\log(x))^k dx + \int_{x_0}^{\infty} e^{-x} dx \leq (\log(x_0))^k (e^{-1} - e^{-x_0}) + 2e^{-\frac{x_0}{2}}
\leq e^{\frac{x_0}{2}} - e^{-\frac{x_0}{2}} \leq k^k + 1,
\]

where we used the inequality \( x_0 < 2k \log k \), for each integer \( k \). Summing up, we get

\[
\sum_{k=1}^{\infty} \frac{t^k}{k!} \left| \frac{\partial^k}{\partial \varepsilon^k} \Gamma(t, \varepsilon) \right|_{\varepsilon=0} \leq e^{-1} \sum_{k=1}^{\infty} \frac{t^k}{k!} \frac{\log(x)}{k} + \sum_{k=1}^{5} 2e^{-\frac{x_0}{2}} \frac{t^k}{k!} + \sum_{k=6}^{\infty} \frac{t^k}{k!} (k^k + 1)
\leq |\log(x)| \left( \frac{e^{t|\log(x)|} - 1}{e} \right) + \sum_{k=6}^{\infty} \frac{t^k}{k!} \left( \frac{k}{\varepsilon} \right)^k + O(t) \leq (\log(x))^2 t + O(t).
\]

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In the last two steps, we have used first that \( t < e^{-2} \) and then the Stirling approximation formula to deduce that the last remaining sum is \( O(t^3) \). Clearly, the factor \( \frac{1}{\Gamma(t)} \sim 1 \), as \( t \to 0 \), in (69) does not change the asymptotic. Finally we derive that

\[
|t\lambda - \mathbb{P}(X_t > \varepsilon)| = O(t^2), \quad \text{as } t \to 0,
\]

as desired.

3. **Inverse Gaussian processes.** To show Equation (5) we write

\[
\left| \frac{\mathbb{P}(X_t > \varepsilon)}{t} - \lambda \right| \leq e^{2t\sqrt{\pi}} \int_\varepsilon^\infty e^{-x\left(e^{-\frac{x^2}{2t}}-1\right)} dx + e^{2t\sqrt{\pi}} \int_\varepsilon^\infty e^{-x} \frac{x^2}{\pi^2} dx =: I + II.
\]

After writing the exponential \( e^{-\frac{x^2}{2t}} \) as an infinite sum, we get \( I = O(t^2) \) if \( t \to 0 \). Expanding \( e^{2t\sqrt{\pi}} \) one finds that, under the same hypothesis, \( II = O(t) \).

4. **Cauchy processes.** Observe that \( \lambda = \frac{2}{\pi} \varepsilon \) and \( \mathbb{P}(\|X_t\| > \varepsilon) = \frac{2}{\pi} \left( \frac{\pi}{2} - \arctan \left( \frac{\varepsilon}{t} \right) \right) \). Hence, in order to prove (6), it is enough to show that

\[
\lim_{t \to 0} \frac{2}{\pi} \frac{e^{3}}{t^3} \left( \frac{\pi}{2} - \arctan \left( \frac{\varepsilon}{t} \right) \right) - \frac{\varepsilon^2}{t^2} < \infty.
\]

Set \( y = \frac{t}{\varepsilon} \) and we compute the limit in (70) by means of de l’Hôpital rule:

\[
\frac{2}{\pi} \lim_{y \to 0} \frac{1}{y^3} \left( \frac{\pi}{2} - \arctan \left( \frac{1}{y} \right) \right) - \frac{1}{y^2} = \frac{2}{\pi} \lim_{y \to 0} \left| \frac{\frac{\pi}{2} - \arctan \left( \frac{1}{y} \right) - y}{y^3} \right| = \lim_{y \to 0} \frac{y^2}{(1 + y^2)3\pi y^2} < \infty.
\]

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**References**

Barndorff-Nielsen, O. E., Mikosch, T., and Resnick, S. I. (2012). *Lévy processes: theory and applications*. Springer Science & Business Media.

Bertoin, J. (1996). *Lévy processes*, volume 121. Cambridge University Press, Cambridge.

Carpentier, A., Duval, C., and Mariucci, E. (2018). Total variation distance for discretely observed Lévy processes: a gaussian approximation of the small jumps. *arXiv preprint arXiv:1810.02998*.

Cohen, S., Rosinski, J., et al. (2007). Gaussian approximation of multivariate Lévy processes with applications to simulation of tempered stable processes. *Bernoulli*, 13(1):195–210.

Cont, R. and Tankov, P. (2004). *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series.
Duval, C. and Mariucci, E. (2017). Compound poisson approximation to estimate the Lévy density. arXiv:1702.08787.

Dzhaparidze, K. and Van Zanten, J. (2001). On Bernstein-type inequalities for martingales. Stochastic processes and their applications, 93(1):109–117.

Figueroa-López, J. E. and Houdré, C. (2009). Small-time expansions for the transition distributions of Lévy processes. Stochastic Process. Appl., 119(11):3862–3889.

Gnedenko, B. and Kolmogorov, A. (1954). Limit distributions for sums of independent random variables. Am. J. Math, 105.

Ishikawa, Y. (1994). Asymptotic behavior of the transition density for jump type processes in small time. Tohoku Mathematical Journal, Second Series, 46(4):443–456.

Léandre, R. (1987). Densité en temps petit d’un processus de sauts. In Séminaire de probabilités XXI, pages 81–99. Springer.

Marchal, P. (2009). Small time expansions for transition probabilities of some Lévy processes. Electronic communications in probability, 14:132–142.

Picard, J. (1997). Density in small time for Lévy processes. ESAIM: Probability and Statistics, 1:357–389.

Rosiński, J. (2007). Tempering stable processes. Stochastic processes and their applications, 117(6):677–707.

Rüschendorf, L. and Woerner, J. H. C. (2002). Expansion of transition distributions of Lévy processes in small time. Bernoulli, 8(1):81–96.