THE FEYNMAN-DE BROGLIE-BOHM PROPAGATOR FOR A SEMICLASSICAL FORMULATION OF THE GROSS-PITAEVSKII EQUATION

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Abstract: In this paper we present the Feynman-de Broglie-Bohm propagator for a semiclassical formulation of the Gross-Pitaevskii equation.

1. Introduction

In the present work we investigate the Feynman-de Broglie-Bohm propagator for a semiclassical formulation of the Gross-Pitaevskii equation with the potential $V(x, t)$ given by:

$$V(x, t) = \frac{1}{2} m \omega^2(t) x^2,$$  \hspace{1cm} (1.1)

which is the time dependent harmonic oscillator potential.

2. Gross-Pitaevskii Equation

Em 1961[1,2], E. P. Gross and, independently, L. P. Pitaevskii proposed a non-linear Schrödinger equation to represent time dependent physical systems, given by:

$$i \hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + \frac{1}{2} m \omega^2(t) x^2 \psi(x, t) + g|\psi(x, t)|^2 \psi(x, t),$$  \hspace{1cm} (2.1)

where $\psi(x, t)$ is a wavefunction and $g$ is a constant.

Writing the wavefunction $\psi(x, t)$ in the polar form, defined by the Madelung-Bohm transformation[3,4], we get:
\[ \psi(x, t) = \phi(x, t) e^{i S(x, t)}, \quad (2.2) \]

where \( S(x, t) \) is the classical action and \( \phi(x, t) \) will be defined in what follows.

Substituting Eq.(2.2) into Eq.(2.1) and taking the real and imaginary parts of the resulting equation, we get[5]:

\[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_{qu})}{\partial x} = 0, \quad (2.3) \]

\[ \hbar \frac{\partial S}{\partial t} + \frac{1}{2} m v_{qu}^2(t) + \frac{1}{2} m \omega^2(t) x^2 + V_{qu} + V_{GP} = 0, \quad (2.4) \]

\[ \frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} + \omega^2(t) x = - \frac{1}{m} \frac{\partial}{\partial x} (V_{qu} + V_{GP}), \quad (2.5) \]

where:

\[ \rho(x, t) = \phi^2(x, t), \quad (2.6) \quad \text{(quantum mass density)} \]

\[ v_{qu}(x, t) = \frac{\hbar}{m} \frac{\partial S(x, t)}{\partial x}, \quad (2.7) \quad \text{(quantum velocity)} \]

\[ V_{qu}(x, t) = - \frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} = - \frac{\hbar^2}{2m \phi} \frac{\partial^2 \phi}{\partial x^2}, \quad (2.8a,b) \quad \text{(Bohm quantum potential)} \]

and

\[ V_{GP} = g \rho. \quad (2.9) \quad \text{(Gross-Pitaevskii potential)} \]

3. Feynman Propagator

In 1948 [6], R. P. Feynman formulated the following principle of minimum action for the quantum mechanics:

*The transition amplitude between the states \( |a> \) and \( |b> \) of a quantum-mechanical system is given by the sum of the elementary contributions, one for each trajectory passing by \( |a> \) at the time \( t_a \) and by \( |b> \) at the time \( t_b \). Each one of these contributions have the same modulus, but its phase is the classical action \( S_{ct} \) for each trajectory.*

This principle is represented by the following expression known as the "Feynman propagator":

\[ K(b, a) = \int_a^b e^{i \int_{x(t)}^t S_{ct}(b, a)} D x(t), \quad (3.1) \]

with:
\[ S_{cl}(b, a) = \int_{t_a}^{t_b} L(x, \dot{x}, t) \, dt, \quad (3.2) \]

where \( L(x, \dot{x}, t) \) is the Lagrangean and \( D x(t) \) is the Feynman’s Measurement. It indicates that we must perform the integration taking into account all the ways connecting the states \( |a> \) and \( |b> \).

Note that the integral which defines \( K(b, a) \) is called ”path integral” or ”Feynman integral” and that the Schrödinger wavefunction \( \Psi(x, t) \) of any physical system is determined using the expression (we indicate the initial position and initial time by \( x_o \) and \( t_o \), respectively)[7]:

\[ \Psi(x, t) = \int_{-\infty}^{+\infty} K(x, x_o, t, t_o) \, \Psi(x_o, t_o) \, dx_o, \quad (3.3) \]

with the quantum causality condition:

\[ \lim_{t, t_o \to 0} K(x, x_o, t, t_o) = \delta(x - x_o). \quad (3.4) \]

4. Calculation of the Feynman-de Broglie-Bohm Propagator for a semiclassical formulation of the Gross-Pitaevskii equation

The wavefunction \( \psi(x, t) \) for the non-linear Gross-Pitaevskii is given by [8]:

\[
\begin{align*}
\psi(x, t) & = [\pi \sigma(t)]^{-1/4} e^{\frac{i}{\hbar} \int_0^t dt' \left\{ \frac{1}{2} m \dot{q}^2(t') - \frac{1}{2} m \omega^2(t') q^2(t') - \frac{\hbar^2}{2 m \sigma(t')^2} - g \rho(x, t') \right\}} \\
& \times \exp \left\{ \frac{i m}{\hbar} \left[ \frac{1}{2} m \dot{q}^2(t') - \frac{1}{2} m \omega^2(t') q^2(t') - \frac{\hbar^2}{2 m \sigma(t')^2} - g \rho(x, t') \right] \right\}, \quad (4.1)
\end{align*}
\]

where:

\[ \ddot{q} + \omega^2(t) q = 0, \quad (4.2) \]

\[ \frac{\dot{\sigma}(t)}{2 \sigma(t)} - \frac{\dot{\sigma}(t)^2}{4 \sigma^2(t)} + \omega(t)^2 + \frac{2 g \rho(x, t)}{m \sigma(t)} = \frac{\hbar^2}{m^2 \sigma^2(t)}, \quad (4.3) \]

\[ \rho(x, t) = [\pi \sigma(t)]^{-1/2} e^{-\frac{|x - q(t)|^2}{\sigma(t)}} \left( 1 + \frac{|x - q(t)|^2}{\sigma(t)} \right), \quad (4.4) \]

with the following initial conditions are obeyed:

\[ q(0) = x_o, \quad \dot{q}(0) = v_o, \quad \sigma(0) = a_o, \quad \dot{\sigma}(0) = b_o, \quad (4.5a-d) \]
Therefore, considering (4.1), the looked for Feynman-de Broglie-Bohm propagator will be calculated using the expression (3.3), in which we will put with no loss of generality, \( t_o = 0 \).

Thus:

\[
\Psi(x, t) = \int_{-\infty}^{+\infty} K(x, x_o, t) \Psi(x_o, 0) \, dx_o . \quad (4.6)
\]

Initially let us define the normalized quantity:

\[
\Phi(v_o, x, t) = (2\pi a_o^2)^{1/4} \Psi(v_o, x, t) , \quad (4.7)
\]

which satisfies the following completeness relation [9]:

\[
f_{-\infty}^{+\infty} dv_o \Phi^*(v_o, x, t) \Phi(v_o, x', t) = \left( \frac{2\pi \hbar}{m} \right) \delta(x - x') . \quad (4.8)
\]

Considering the eqs. (2.2), (2.6) and (4.7), we get:

\[
\Phi^*(v_o, x, t) \Psi(v_o, x, t) =
\]

\[
= (2\pi a_o^2)^{1/4} \Psi^*(v_o, x, t) \Psi(v_o, x, t) = (2\pi a_o^2)^{1/4} \rho(v_o, x, t) \rightarrow
\]

\[
\rho(v_o, x, t) = (2\pi a_o^2)^{-1/4} \Phi^*(v_o, x, t) \Psi(v_o, x, t) . \quad (4.9)
\]

On the other side substituting the relation (4.9) in the expression (2.3), integrating the result and using the expressions (4.4) and (4.7) results [remembering that we have: \( \Psi^* \Psi(\pm \infty) \rightarrow 0 \)\[7\]]:

\[
\frac{\partial(\Phi^* \Psi)}{\partial t} + \frac{\partial(\Phi^* \Psi v_{qu})}{\partial x} = 0 \rightarrow
\]

\[
\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx \, \Phi^* \Psi + \int_{-\infty}^{+\infty} \frac{\partial(\Phi^* \Psi v_{qu})}{\partial x} \, dx = 0 \rightarrow
\]

\[
\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx \, \Phi^* \Psi + (\Phi^* \Psi v_{qu})|_{-\infty}^{+\infty} =
\]

\[
= \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx \, \Phi^* \Psi + (2\pi a_o^2)^{1/4} (\Psi^* \Psi v_{qu})|_{-\infty}^{+\infty} = 0 \rightarrow
\]

\[
\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} dx \, \Phi^* \Psi = 0 . \quad (4.10)
\]

The relation (4.10) shows that the integration is time independent. Consequently:

\[
f_{-\infty}^{+\infty} dx' \Phi^*(v_o, x', t) \Psi(x', t) = \int_{-\infty}^{+\infty} dx_o \Phi^*(v_o, x_o, 0) \Psi(x_o, 0) . \quad (4.11)
\]
Multiplying the relation shown in eq.(4.11) by $\Phi(v_o, x, t)$ and integrating over $v_o$ and using the expression (4.8), we will obtain [remembering that $\int_{-\infty}^{+\infty} dx' f(x') \delta(x' - x) = f(x)$]:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_o dx' \Phi(v_o, x, t) \Phi^*(v_o, x', t) \Psi(x', t) =$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_o dx_o \Phi(v_o, x, t) \Phi^*(v_o, x_o, 0) \Psi(x_o, 0) \rightarrow$$

$$\int_{-\infty}^{+\infty} dx' \left( \frac{2 \pi \hbar}{m} \right) \delta(x' - x) \Psi(x', t) = \left( \frac{2 \pi \hbar}{m} \right) \Psi(x, t) =$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_o dx_o \Phi(v_o, x, t) \Phi^*(v_o, x_o, 0) \Psi(x_o, 0) \rightarrow$$

$$\Psi(x, t) = \int_{-\infty}^{+\infty} \left[ \left( \frac{m}{2 \pi \hbar} \right) \int_{-\infty}^{+\infty} dv_o \Phi(v_o, x, t) \times \right.$$  

$$\left. \times \Phi^*(v_o, x_o, 0) \right] \Psi(x_o, 0) \, dx_o \, . \quad (4.12)$$

Comparing the relations (4.6) and (4.12), we have:

$$K(x, x_o, t) = \frac{m}{2 \pi \hbar} \int_{-\infty}^{+\infty} dv_o \Phi(v_o, x, t) \Phi^*(v_o, x_o, 0) \, . \quad (4.13)$$

Substituting the relations (4.1) and (4.7) in the equation (4.13), we obtain the Feynman-de Broglie-Bohm Propagator for a semiclassical formulation of the Gross-Pitaevskii equation that we were looking for, that is [remembering that $\Phi^*(v_o, x_o, 0) = \exp\left( -\frac{i m}{\hbar} v_o x_o \right)$]:

$$K(x, x_o; t) = \frac{m}{2 \pi \hbar} \int_{-\infty}^{+\infty} dv_o \sqrt{\frac{m}{\iota(t)}} \times$$

$$\times \exp \left[ \left( \frac{i m \dot{q}(t)}{2 \hbar \sigma(t)} - \frac{1}{4 \sigma(t)} \right) [x - q(t)]^2 + \frac{i m \dot{q}(t)}{\hbar} [x - q(t)] \right] \times$$

$$\times \exp \left[ \frac{i}{\hbar} \int_{0}^{t} dt' \left[ \frac{1}{2} m \dot{q}^2(t') - \frac{1}{2} m \omega^2(t') q^2(t') - \frac{\hbar^2}{2 m \sigma(t')} - g \rho(x, t') \right] \right] \, . \quad (4.14)$$

where $q(t)$ and $\sigma(t)$ are solutions of the differential equations given by the (4.2-4).

In conclusion, we observe that the equations (4.1) and (4.14) we show that when $g = 0$, then we obtains, respectively, the equations (3.3.2.25) and (4.2.2.13) of the Reference [5], if $\sigma(t) = 2 \, a^2(t)$, $q(t) = X(t)$ and $\frac{1}{2} m \omega^2(t) q^2(t) = V[X(t)]$. 

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