GENERATORS OF ALGEBRAIC COVARIANT DERIVATIVE CURVATURE TENSORS AND YOUNG SYMMETRIZERS

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Abstract. We show that the space of algebraic covariant derivative curvature tensors \( R' \) is generated by Young symmetrized product tensors \( T \otimes \hat{T} \) or \( \hat{T} \otimes T \), where \( T \) and \( \hat{T} \) are covariant tensors of order 2 and 3 whose symmetry classes are irreducible and characterized by the following pairs of partitions: \{ (2), (3) \}, \{ (2), (2 1) \} or \{ (1^2), (2 1) \}. Each of the partitions (2), (3) and (1^2) describes exactly one symmetry class, whereas the partition (2 1) characterizes an infinite set \( \mathcal{S} \) of irreducible symmetry classes. This set \( \mathcal{S} \) contains exactly one symmetry class \( S_0 \in \mathcal{S} \) whose elements \( \hat{T} \in S_0 \) can not play the role of generators of tensors \( R' \). The tensors \( \hat{T} \) of all other symmetry classes from \( \mathcal{S} \setminus \{ S_0 \} \) can be used as generators for tensors \( R' \).

Foundation of our investigations is a theorem of S. A. Fulling, R. C. King, B. G. Wybourne and C. J. Cummins about a Young symmetrizer that generates the symmetry class of algebraic covariant derivative curvature tensors. Furthermore we apply ideals and idempotents in group rings \( \mathbb{C}[S_r] \), the Littlewood-Richardson rule and discrete Fourier transforms for symmetric groups \( S_r \). For certain symbolic calculations we used the Mathematica packages Ricci and PERMS.

1. Introduction

In [13] we described constructions of generators of algebraic curvature tensors. The present paper searches for generators of algebraic covariant derivative curvature tensors.

Algebraic curvature tensors are tensors of order 4 which have the same symmetry properties as the Riemann tensor of a Levi-Civita connection in Differential Geometry. Let \( \mathcal{T}_r V \) be the vector space of the \( r \)-times covariant tensors \( T \) over a finite-dimensional \( \mathbb{K} \)-vector space \( V \), \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{K} = \mathbb{C} \). We assume that \( V \) possesses a fundamental tensor \( g \in \mathcal{T}_2 V \) (of arbitrary signature) which can be used for raising and lowering of tensor indices.

Definition 1.1. A tensor \( \mathcal{R} \in \mathcal{T}_4 V \) is called an algebraic curvature tensor iff \( \mathcal{R} \) has the index commutation symmetry

\[
\forall w, x, y, z \in V: \quad \mathcal{R}(w, x, y, z) = -\mathcal{R}(w, x, z, y) = \mathcal{R}(y, z, w, x)
\]

and fulfills the first Bianchi identity

\[
\forall w, x, y, z \in V: \quad \mathcal{R}(w, x, y, z) + \mathcal{R}(w, y, z, x) + \mathcal{R}(w, z, x, y) = 0.
\]

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Definition 1.2. A tensor \( R' \in T_5 V \) is called an algebraic covariant derivative curvature tensor iff \( R' \) has the index commutation symmetry
\[
R'(w, x, y, z, u) = -R'(w, x, z, y, u) = R'(y, z, w, x, u)
\]
and fulfills the first Bianchi identity
\[
R'(w, x, y, z, u) + R'(w, y, z, x, u) + R'(w, z, x, y, u) = 0
\]
and the second Bianchi identity
\[
R'(w, x, y, z, u) + R'(w, x, z, u, y) + R'(w, x, u, y, z) = 0
\]
for all \( u, w, x, y, z \in V \).

The relations (1.5) and (1.6) correspond to the well-known formulas
\[
R_{ijkl} = -R_{ijlk} = R_{klij}
\]
\[
R_{ijkl} + R_{iklj} + R_{iljk} = 0
\]
for the Riemann tensor \( R \) and
\[
R_{ijkm} = -R_{ikjm} = R_{klij}
\]
\[
R_{ijkm} + R_{ikjm} + R_{iljk} = 0
\]
\[
R_{ijkm} + R_{ijlm} = 0
\]
for its first covariant derivative which we present here in terms of tensor coordinates with respect to an arbitrary local coordinate system.

Investigations of tensors of type \( R \) and \( R' \) were carried out by many authors. (See the extensive bibliography in the book [19] by P. B. Gilkey.) A famous problem connected with algebraic curvature tensors is the Osserman conjecture.

Definition 1.3. Let \( \mathcal{R} \in T_4 V \) be an algebraic curvature tensor and \( x \in V \) be a vector with \( |g(x, x)| = 1 \). The Jacobi operator \( J_{\mathcal{R}}(x) \) of \( \mathcal{R} \) and \( x \) is the linear operator \( J_{\mathcal{R}}(x) : V \to V \), \( J_{\mathcal{R}}(x) : y \mapsto J_{\mathcal{R}}(x)y \) that is defined by
\[
\forall w \in V : g(J_{\mathcal{R}}(x)y, w) = \mathcal{R}(y, x, x, w).
\]

Definition 1.4. An algebraic curvature tensor \( \mathcal{R} \) is called spacelike Osserman (resp. timelike Osserman) if the eigenvalues of \( J_{\mathcal{R}}(x) \) are constant on \( S^+(V) := \{ x \in V \mid g(x, x) = +1 \} \) (resp. \( S^-(V) := \{ x \in V \mid g(x, x) = -1 \} \)).

Since the notions ”spacelike Osserman” and ”timelike Osserman” are equivalent one simply says Osserman.

If \( R \) is the Riemann tensor of a Riemannian manifold \((M, g)\) which is locally a rank one symmetric space or flat, then the eigenvalues of \( J_R(x) \) are constant on the unit sphere bundle of \((M, g)\). Osserman [31] wondered if the converse held. This question is known as the Osserman conjecture.

The correctness of the Osserman conjecture has been established for Riemannian manifolds \((M, g)\) in all dimensions \(\neq 8, 16\) (see [1, 30]) and for Lorentzian manifolds \((M, g)\) in all dimensions (see [1, 17]). However Osserman’s question has a negative answer in the case of a pseudo-Riemannian metric with signature \((p, q), p, q \geq 2\) (see
Numerous examples of Osserman algebraic curvature tensors can be constructed by means of operators $\alpha$ and $\gamma$ given below. It turned out that these operators lead to generators for arbitrary algebraic curvature tensors.

**Definition 1.5.**

(1) Let $S \in \mathcal{T}_2 V$ be a symmetric tensor of order 2, i.e. the coordinates of $S$ satisfy $S_{ij} = S_{ji}$. We define a tensor $\gamma(S) \in \mathcal{T}_4 V$ by

$$\gamma(S)_{ijkl} := \frac{1}{3} (S_{il}S_{jk} - S_{ik}S_{jl}) .$$

(1.12)

(2) Let $A \in \mathcal{T}_2 V$ be a skew-symmetric tensor of order 2, i.e. the coordinates of $A$ satisfy $A_{ji} = -A_{ij}$. We define a tensor $\alpha(A) \in \mathcal{T}_4 V$ by

$$\alpha(A)_{ijkl} := \frac{1}{3} (2A_{ij}A_{kl} + A_{ik}A_{jl} - A_{il}A_{jk}) .$$

(1.13)

Now we can construct an example of an Osserman algebraic curvature tensor in the following way. Let $g \in \mathcal{T}_2 V$ be a positive definite metric and $\{C_i\}_{i=1}^r$ be a finite set of real, skew-symmetric $(\dim V \times \dim V)$-matrices that satisfy the Clifford commutation relations

$$C_i \cdot C_j + C_j \cdot C_i = -2 \delta_{ij} .$$

(1.14)

If we form skew-symmetric tensors $A_i \in \mathcal{T}_2 V$ by $A_i(x,y) := g(C_i \cdot x, y)$ ($x,y \in V$), then

$$\mathcal{R} = \lambda_0 \gamma(g) + \sum_{i=1}^r \lambda_i \alpha(C_i) , \quad \lambda_0, \lambda_i = \text{const.}$$

(1.15)

is an Osserman algebraic curvature tensor (see [13]). Further examples which allow also indefinite metrics can be found in [19, pp.191-193]. (See also [13, Sec.6].)

The operators $\alpha$ and $\gamma$ can be used to form generators for arbitrary algebraic curvature tensors. P. Gilkey [19, pp.41-44] and B. Fiedler [13] gave different proofs for

**Theorem 1.6.** Each of the sets of tensors

(1) $\{\gamma(S) \mid S \in \mathcal{T}_2 V \text{ symmetric}\}$

(2) $\{\alpha(A) \mid A \in \mathcal{T}_2 V \text{ skew-symmetric}\}$

generate the vector space of all algebraic curvature tensors $\mathcal{R}$ on $V$.

Note that the tensors $\gamma(S)$ and $\alpha(A)$ are expressions which arise from $S \otimes S$ or $A \otimes A$ by a symmetrization

$$\gamma(S) = \frac{1}{12} y_t(S \otimes S) , \quad \alpha(A) = \frac{1}{12} y_t(A \otimes A)$$

(1.16)

where $y_t$ is the Young symmetrizer of the Young tableau

$$t = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

(1.17)

(see [13]).

In the present paper we search for similar generators of algebraic covariant derivative curvature tensors. We use Boerner’s definition of symmetry classes for
tensors $T \in \mathcal{T}_r V$ by right ideals $r \subseteq \mathbb{K}[S_r]$ of the group $\mathbb{K}[S_r]$ of the symmetric group $S_r$ (see Section 2 and [2, 3, 9, 12]). On this basis we investigate the following

**Problem 1.7.** We search for generators of algebraic covariant derivative curvature tensors which can be formed by a suitable symmetry operator from tensors

$$T \otimes \hat{T} \text{ or } \hat{T} \otimes T, \quad T \in \mathcal{T}_2 V, \hat{T} \in \mathcal{T}_3 V$$

where $T$ and $\hat{T}$ belongs to symmetry classes of $\mathcal{T}_2 V$ and $\mathcal{T}_3 V$ which are defined by minimal right ideals $r \subset \mathbb{K}[S_2]$ and $\hat{r} \subset \mathbb{K}[S_3]$, respectively.

We will see that all such generators can be gained by means of the *Young symmetrizer* $y_{t'}$ of the *Young tableau*

$$t' = \begin{array}{cccc}
1 & 3 & 5 \\
2 & 4 \\
\end{array}$$

(1.19)

We obtain the following results:

**Theorem 1.8.** A solution of Problem 1.7 can be constructed at most from such pairs of tensors (1.18) whose symmetry classes are characterized by the following partitions $\lambda \vdash 2$, $\hat{\lambda} \vdash 3$:

|   | $\lambda$ for $T$ | $\hat{\lambda}$ for $\hat{T}$ |
|---|-------------------|-------------------------------|
| (a) | (2)               | (3)                           |
| (b) | (2)               | (2 1)                         |
| (c) | (1 2)             | (2 1)                         |

The case (a) of Theorem 1.8 is specified by

**Theorem 1.9.** Let us denote by $S \in \mathcal{T}_2 V$ and $\hat{S} \in \mathcal{T}_3 V$ symmetric tensors of order 2 and 3, respectively. Then the set of all tensors which belong to exactly one of the following tensor types

$$\tau : \quad y_{t'}^*(S \otimes \hat{S}), \quad y_{t'}^*(\hat{S} \otimes S),$$

(1.20)

generates the vector space of all algebraic covariant derivative curvature tensors $\mathfrak{R}' \in \mathcal{T}_5 V$.

Moreover, the tensors (1.20) coincide and their coordinates fulfill

$$\hat{\gamma}(S, \hat{S})_{ijkl} := (y_{t'}^*(S \otimes \hat{S}))_{ijkl} = (y_{t'}^*(\hat{S} \otimes S))_{ijkl} = 4 \left\{ S_{ilj}\hat{S}_{jkl} - S_{jkl}\hat{S}_{ilj} + S_{jlk}\hat{S}_{jkl} - S_{kjl}\hat{S}_{jkl} \right\}$$

(1.21)

The operator $\hat{\gamma}$ plays the same role for the generators of algebraic covariant derivative curvature tensors considered in Theorem 1.9 as the operators $\alpha$ and $\gamma$ play for the generators of algebraic curvature tensors. A first proof that the expressions (1.21) are generators for $\mathfrak{R}'$ was given by P. B. Gilkey [19, p.236].

The cases (b) and (c) of Theorem 1.8 lead to
Theorem 1.10. Let us denote by $S, A \in T^2 V$ symmetric or alternating tensors of order 2 and by $U \in T^3 V$ covariant tensors of order 3 whose symmetry class $T_r$ is defined by a fixed minimal right ideal $r \subset K[S_3]$ from the equivalence class characterized by the partition $(2,1) \vdash 3$. We consider the following types $\tau$ of tensors

$$ (1.22) \quad \tau : \quad y^*_r(S \otimes U) , \quad y^*_r(U \otimes S) , \quad y^*_r(A \otimes U) , \quad y^*_r(U \otimes A) . $$

Then for each of the above types $\tau$ the following assertions are equivalent:

1. The vector space of algebraic covariant derivative curvature tensors $R' \in T^5 V$ is the set of all finite sums of tensors of the type $\tau$ considered.

2. The right ideal $r$ is different from the right ideal $r_0 := f \cdot K[S_3]$ with generating idempotent

$$ (1.23) \quad f := \left\{ \frac{1}{2} (\text{id} - (13)) - \frac{1}{6} y \right\} , \quad y := \sum_{p \in S_3} \text{sign}(p) p . $$

In the situation of Theorem 1.10 we can also determine operators of the type $\alpha, \gamma, \hat{\gamma}$ which describe the generators of the algebraic covariant derivative curvature tensors $R'$ considered. However, these operators depend on the right ideal $r$ (or its generating idempotent $e$) that defines the symmetry class of $U$. And they yield no short expressions of 2, 3, or 4 terms but longer expressions between 10 and 20 terms of length. We describe their determination in a forthcoming paper.

Here is a brief outline to the present paper. In Section 2 we give a summary of basic facts about symmetry classes, in particular about the connection between product tensors and Littlewood-Richardson products of corresponding representations. Such Littlewood-Richardson products are used in Subsection 3.1 to prove Theorem 1.8. In the Subsections 3.2 and 3.3 we prove the Theorems 1.9 and 1.10 by a method which we already applied in [13]. Idempotents which define the symmetry class of the tensors $U$ are determined by means of diskrete Fourier transforms.

2. Basic facts about symmetry classes

The vector spaces of algebraic curvature tensors or algebraic covariant derivative tensors over $V$ are symmetry classes in the sense of H. Boerner [2, p.127]. We denote by $K[S_r]$ the group ring of a symmetric group $S_r$ over the field $K$. Every group ring element $a = \sum_{p \in S_r} a(p) p \in K[S_r]$ acts as so-called symmetry operator on tensors $T \in T^r V$ according to the definition

$$ (2.1) \quad (aT)(v_1, \ldots, v_r) := \sum_{p \in S_r} a(p) T(v_{p(1)}, \ldots, v_{p(r)}) , \quad v_i \in V . $$

Equation (2.1) is equivalent to

$$ (2.2) \quad (aT)_{i_1 \ldots i_r} = \sum_{p \in S_r} a(p) T_{i_{p(1)} \ldots i_{p(r)}} . $$
Definition 2.1. Let \( r \subseteq \mathbb{K}[S_r] \) be a right ideal of \( \mathbb{K}[S_r] \) for which an \( a \in r \) and a \( T \in T_r V \) exist such that \( aT \neq 0 \). Then the tensor set
\[
T_r := \{ aT \mid a \in r, T \in T_r V \}
\]
is called the symmetry class of tensors defined by \( r \).

Since \( \mathbb{K}[S_r] \) is semisimple for \( \mathbb{K} = \mathbb{R}, \mathbb{C} \), every right ideal \( r \subseteq \mathbb{K}[S_r] \) possesses a generating idempotent \( e \), i.e. \( r \) fulfills \( r = e \cdot \mathbb{K}[S_r] \). It holds (see e.g. \[13\] or \[23\])

Lemma 2.2. If \( e \) is a generating idempotent of \( r \), then a tensor \( T \in T_r V \) belongs to \( T_r \) iff
\[
eT = T.
\]
Thus we have
\[
T_r = \{ eT \mid T \in T_r V \}.
\]

Now we summarize tools from our Habilitationsschrift \[11\] (see also its summary \[12\]). We make use of the following connection between \( r \)-times covariant tensors \( T \in T_r V \) and elements of the group ring \( \mathbb{K}[S_r] \).

Definition 2.3. Any tensor \( T \in T_r V \) and any \( r \)-tuple \( b := (v_1, \ldots, v_r) \in V^r \) of \( r \) vectors from \( V \) induce a function \( S_r : V^r \rightarrow \mathbb{K} \) according to the rule
\[
S_r(v_1, \ldots, v_r) := T(v_p(1), \ldots, v_p(r)), \quad p \in S_r.
\]

We identify this function with the group ring element \( T_b := \sum_{p \in S_r} T_b(p) p \in \mathbb{K}[S_r] \).

Obviously, two tensors \( S, T \in T_r V \) fulfill \( S = T \) iff \( S_b = T_b \) for all \( b \in V^r \). We denote by \( * \) the mapping \( * : a = \sum_{p \in S_r} a(p)p \mapsto a^* := \sum_{p \in S_r} a(p)p^{-1} \). Then the following important formula\(^1\) holds
\[
\forall T \in T_r V, \ a \in \mathbb{K}[S_r], \ b \in V^r : \ (aT)_b = T_b \cdot a^*.
\]

Now it can be shown that all \( T_b \) of tensors \( T \) of a given symmetry class lie in a certain left ideal of \( \mathbb{K}[S_r] \).

Proposition 2.4.\(^2\) Let \( e \in \mathbb{K}[S_r] \) be an idempotent. Then a \( T \in T_r V \) fulfills the condition \( eT = T \) iff \( T_b \in I := \mathbb{K}[S_r] \cdot e^* \) for all \( b \in V^r \), i.e. all \( T_b \) of \( T \) lie in the left ideal \( I \) generated by \( e^* \).

The proof follows easily from (2.7). Since a right ideal \( r \) defining a symmetry class and the left ideal \( I \) from Proposition 2.4 satisfy \( r = I^* \), we denote symmetry classes also by \( T_r \). A further result is

Proposition 2.5.\(^3\) If \( \dim V \geq r \), then every left ideal \( I \subseteq \mathbb{K}[S_r] \) fulfills \( I = \mathcal{L}_K \{ T_b \mid T \in T_r, b \in V^r \} \). (Here \( \mathcal{L}_K \) denotes the forming of the linear closure.)

\(^1\)See B. Fiedler \[9\] Sec.III.1] and B. Fiedler \[11\].
\(^2\)See B. Fiedler \[11\] or B. Fiedler \[9\] Prop. III.2.5, III.3.1, III.3.4].
\(^3\)See B. Fiedler \[11\] or B. Fiedler \[9\] Prop. III.2.6].
If \( \dim V < r \), then the \( T_b \) of the tensors from \( T_r \) will span only a linear subspace of \( l \) in general.

Important special symmetry operators are Young symmetrizers, which are defined by means of Young tableaux.

A Young tableau \( t \) of \( r \in \mathbb{N} \) is an arrangement of \( r \) boxes such that

1. the numbers \( \lambda_i \) of boxes in the rows \( i = 1, \ldots, l \) form a decreasing sequence \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0 \) with \( \lambda_1 + \ldots + \lambda_l = r \),
2. the boxes are fulfilled by the numbers 1, 2, \ldots, \( r \) in any order.

For instance, the following graphics shows a Young tableau of \( r = 16 \).

\[
\begin{array}{cccccccc}
\lambda_1 = 5 & \overset{11}{12} & 2 & 5 & 4 & 12 \\
\lambda_2 = 4 & 9 & 6 & 16 & 15 \\
\lambda_3 = 4 & 8 & 14 & 1 & 7 \\
\lambda_4 = 2 & 13 & 3 \\
\lambda_5 = 1 & 10 \\
\end{array}
\]

\( = t \).

Obviously, the unfilled arrangement of boxes, the Young frame, is characterized by a partition \( \lambda = (\lambda_1, \ldots, \lambda_l) \vdash r \) of \( r \).

If a Young tableau \( t \) of a partition \( \lambda \vdash r \) is given, then the Young symmetrizer \( y_t \) of \( t \) is defined by

\[
y_t := \sum_{p \in \mathcal{H}_t} \sum_{q \in \mathcal{V}_t} \text{sign}(q) p \circ q
\]

where \( \mathcal{H}_t \), \( \mathcal{V}_t \) are the groups of the horizontal or vertical permutations of \( t \) which only permute numbers within rows or columns of \( t \), respectively. The Young symmetrizers of \( \mathbb{K}[S_r] \) are essentially idempotent and define decompositions

\[
\mathbb{K}[S_r] = \bigoplus_{\lambda \vdash r} \bigoplus_{t \in \mathcal{S}T_\lambda} \mathbb{K}[S_r] \cdot y_t \quad , \quad \mathbb{K}[S_r] = \bigoplus_{\lambda \vdash r} \bigoplus_{t \in \mathcal{S}T_\lambda} y_t \cdot \mathbb{K}[S_r]
\]

of \( \mathbb{K}[S_r] \) into minimal left or right ideals. In (2.9), the symbol \( \mathcal{S}T_\lambda \) denotes the set of all standard tableaux of the partition \( \lambda \). Standard tableaux are Young tableaux in which the entries of every row and every column form an increasing number sequence.

S.A. Fulling, R.C. King, B.G. Wybourne and C.J. Cummins showed in [13] that the symmetry classes of the Riemannian curvature tensor \( R \) and its symmetrized covariant derivatives

\[
(\nabla^{(u)} R)_{ijkl s_1 \ldots s_u} := \nabla_{s_1} \nabla_{s_2} \ldots \nabla_{s_u} R_{ijkl} = R_{ijkl; (s_1 \ldots s_u)}
\]

are generated by special Young symmetrizers.

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4 We use the convention \((p \circ q)(i) := p(q(i))\) for the product of two permutations \( p, q \).

5 About Young symmetrizers and Young tableaux see for instance [2, 3, 13, 16, 20, 21, 26, 27, 28, 29, 32, 33]. In particular, properties of Young symmetrizers in the case \( \mathbb{K} \neq \mathbb{C} \) are described in [28].

6 \( (\ldots) \) denotes the symmetrization with respect to the indices \( s_1, \ldots, s_u \).
Theorem 2.6. Consider the Levi-Civita connection $\nabla$ of a pseudo-Riemannian metric $g$. For $u \geq 0$ the Riemann tensor and its symmetrized covariant derivatives $\nabla^{(u)} R$ fulfil

$$e^*_t \nabla^{(u)} R = \nabla^{(u)} R$$

(2.11)

where $e_t := y_t(u+1)/(2 \cdot (u+3)!)$ is an idempotent which is formed from the Young symmetrizer $y_t$ of the standard tableau

$$t = \begin{array}{cccc}
1 & 3 & 5 & \ldots \ldots \\
2 & 4 & & \\
\end{array} (u+4)$$

(2.12)

A proof of this result of [14] can be found in [8, Sec.6], too. The proof needs only the symmetry properties (1.1) or (1.6), the identities Bianchi I and Bianchi II and the symmetry with respect to $s_1, \ldots, s_u$. Thus Theorem 2.6 is a statement about algebraic curvature tensors and algebraic covariant derivative curvature tensors. We can specify this in the following way:

Definition 2.7. A tensor $\mathcal{R}^{(u)} \in T_{4+u} V, u \geq 0$, is called a symmetric algebraic covariant derivative curvature tensor of order $u$ iff $\mathcal{R}^{(u)}(w, y, z, x, a_1, \ldots, a_u)$ is symmetric with respect to $a_1, \ldots, a_u$ and fulfills

$$\mathcal{R}^{(u)}(w, x, y, z, a_1, \ldots, a_u) = -\mathcal{R}^{(u)}(w, x, y, z, a_1, \ldots, a_u) = \mathcal{R}^{(u)}(y, z, w, x, a_1, \ldots, a_u)$$

for all $a_1, \ldots, a_u, w, x, y, z \in V$.

Now symmetric algebraic covariant derivative curvature tensors can be characterized by means of the Young symmetrizer of the tableau (2.12).

Proposition 2.8. A tensor $T \in T_{4+u} V, u \geq 0$, is a symmetric algebraic covariant derivative curvature tensor of order $u$ iff $T$ satisfies

$$e^*_t T = T$$

(2.16)

where $e_t$ is the idempotent from Theorem 2.6.

A proof of Proposition 2.8 is given in the proof of [8, Prop.6.1]. If we consider now the values $u = 0$ and $u = 1$, we obtain

Corollary 2.9. (1) A tensor $T \in T_4 V$ is an algebraic curvature tensor iff $T$ satisfies

$$y^*_t T = 12 T$$

(2.17)
where \( y_t \) is the Young symmetrizer of the standard tableau

\[
(2.18) \quad t = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.
\]

(2) A tensor \( \tilde{T} \in T_5 V \) is an algebraic covariant derivative curvature tensor iff \( \tilde{T} \) satisfies

\[
(2.19) \quad y^t \tilde{T} = 24 \tilde{T}
\]

where \( y^t \) is the Young symmetrizer of the standard tableau

\[
(2.20) \quad t' = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 \end{bmatrix}.
\]

Now we describe the left ideal \( \mathfrak{l} \) which defines the symmetry class \( T_\mathfrak{l} \) of a product tensor. We consider products \( T^{(1)} \otimes \ldots \otimes T^{(m)} \) with possibly different \( T^{(i)} \).

**Proposition 2.10.** \(^7\) Let \( \mathfrak{l}_i \subseteq \mathbb{K}[S_{r_i}] \) (\( i = 1, \ldots, m \)) be left ideals and \( T^{(i)} \in T^{(i)}_\mathfrak{l}_i \subseteq T_{r_i} V \) be \( r_i \)-times covariant tensors from the symmetry classes characterized by the \( \mathfrak{l}_i \). Consider the product

\[
(2.21) \quad T := T^{(1)} \otimes \ldots \otimes T^{(m)} \in T_r V, \quad r := r_1 + \ldots + r_m.
\]

For every \( i \) we define an embedding

\[
(2.22) \quad \iota_i : S_{r_i} \rightarrow S_r, \quad (\iota_i(s))(k) := \begin{cases} \Delta_i + s(k - \Delta_i) & \text{if } r_{i-1} < k \leq r_i \\ k & \text{else} \end{cases}
\]

where \( \Delta_i := r_0 + \ldots + r_{i-1} \) and \( r_0 := 0 \). Then the \( T_b \) of the tensor (2.21) fulfill

\[
(2.23) \quad \forall b \in V^r : T_b \in \mathfrak{l} := \mathbb{K}[S_r] \cdot \mathcal{L}\{\overline{\mathfrak{l}_1} \cdot \ldots \cdot \overline{\mathfrak{l}_m}\} = \mathbb{K}[S_r] \cdot (\iota_1 \otimes \ldots \otimes \iota_m)
\]

where \( \overline{\mathfrak{l}_i} := \iota_i(\mathfrak{l}_i) \) are the embeddings of the \( \mathfrak{l}_i \) into \( \mathbb{K}[S_r] \) induced by the \( \iota_i \). If \( \dim V \geq r \), then the above left ideal \( \mathfrak{l} \) is generated by all \( T_b \in \mathbb{K}[S_r] \) which are formed from tensor products (2.21) of arbitrary tensors \( T^{(i)} \in T^{(i)}_\mathfrak{l}_i \).

Let \( \tilde{\omega}_G : G \rightarrow GL(\mathbb{K}[G]) \) denote the regular representation of a finite group \( G \) defined by \( \tilde{\omega}_g(f) := g \cdot f, g \in G, f \in \mathbb{K}[G] \). If we use the above left ideals \( \mathfrak{l}_i \), \( \mathfrak{l} \) to define subrepresentations \( \alpha_i := \tilde{\omega}_{S_{r_i}}|_{\mathfrak{l}_i}, \beta := \tilde{\omega}_{S_r}|_{\mathfrak{l}} \), then the representation \( \beta \) is equivalent to a Littlewood-Richardson product (see B. Fiedler \[9, Sec.III.3.2\]):

\[
(2.24) \quad \mathfrak{l} \text{ according to } (2.23) \quad \Rightarrow \quad \beta \sim \alpha_1 \# \ldots \# \alpha_m | S_r
\]

(‘\#’ denotes the outer tensor product of the above representations.) This result corresponds to statements of S.A. Fulling et al. \[14\]. Relation (2.23) allows us to determine information about the structure of the left ideal (2.23) by means of the Littlewood-Richardson rule.\(^8\)

\(^7\)See B. Fiedler \[9, Sec.III.3.2\] and B. Fiedler \[11\].

\(^8\)See the references \[21, 22, 24, 26, 27, 14, 15\] for the Littlewood-Richardson rule.
In this section we prove the Theorems 1.8, 1.9 and 1.10.

3.1. Use of the Littlewood-Richardson rule (Proof of Theorem 1.8). We consider the Problem 1.7. Theorem 2.6 and Corollary 2.9 tell us that a tensor \( W \in T_5V \) is an algebraic covariant derivative curvature tensor iff a tensor \( \tilde{W} \in T_5V \) exists such that

\[
W = \frac{1}{24} \gamma t' \tilde{W}
\]

where \( \gamma t' \) is the Young symmetrizer of the tableau (2.20).

It is well-known that every tensor \( \tilde{W} \in T_5V \) can be represented as a finite sum of tensors \( W'' \otimes W''' \) where \( W'' \in T_2V \), \( W''' \in T_3V \). Thus we can find a finite subset \( P \subset T_2V \times T_3V \) for every \( \tilde{W} \in T_5V \) such that

\[
\tilde{W} = \sum_{(W'',W''') \in P} W'' \otimes W'''.
\] (3.1)

Assume that we know defining idempotents \( f'' \in K[S_2], f''' \in K[S_3] \) of the symmetry classes of such tensors \( W'', W''' \). If we determine decompositions of those idempotents into pairwise orthogonal, primitive idempotents, e.g.

\[
f''' = \sum_i e_i, \quad e_i \cdot e_j = \delta_{ij} e_i, \quad e_i \in K[S_3] \text{ primitive}
\] (3.2)

then we obtain decompositions of \( W'', W''' \) into tensors whose symmetry classes are defined by minimal right ideals (e.g. \( W''' = f'''W''' = \sum_i e_iW''' \)). Consequently we can construct such a representation (3.1) for \( \tilde{W} \) in which all \( W'', W''' \) belong to symmetry classes which are defined by minimal right ideals \( r'', r''' \). We determine the structure of the symmetry class of such a product tensor \( W'' \otimes W''' \).

Because of Proposition 2.4 all group ring elements \( W'' b', W''' b'' \) according to Definition 2.3 lie in minimal left ideals \( l'' = (e'')^*, l''' = (e''')^* \). From Proposition 2.10 follows that the group ring elements \( (W'' \otimes W''')_b \) belong to a left ideal \( l \subset K[S_5] \) which is the representation space of a representation \( \beta \sim \alpha'' \# \alpha''' \uparrow S_5 \) where \( \alpha'' := \tilde{\omega}_{S_2}|v, \alpha''' := \tilde{\omega}_{S_3}|v \). Since \( l'' \), \( l''' \) are minimal the representations \( \alpha'', \alpha''' \) are irreducible. Their equivalence classes are characterized by certain partitions, i.e. there are partitions \( \lambda'' \vdash 2, \lambda''' \vdash 3 \), such that \( \alpha'' \sim [\lambda''], \alpha''' \sim [\lambda'''] \).

Now we can determine information about the decomposition of \( \beta \) into irreducible representations by means of the Littlewood-Richardson rule. There are the following possibilities to form Littlewood-Richardson products:

\[
\begin{align*}
[3][2] & \sim [5] + [3 2] + [4 1] \\
[3][1^2] & \sim [4 1] + [3 1^2] \\
[2 1][2] & \sim [3 2] + [4 1] + [2^2 1] + [3 1^2] \\
[2 1][1^3] & \sim [3 2] + [2^2 1] + [3 1^2] + [2 1^3] \\
[1^3][2] & \sim [3 1^2] + [2 1^3] \\
[1^3][1^2] & \sim [2^2 1] + [2 1^3] + [1^5]
\end{align*}
\]
Every irreducible representation of the right-hand sides of these relations has a representation space which is a minimal left ideal \( \hat{1} = \mathbb{K}[S_5] \cdot \hat{f} \) generated by a primitive idempotent \( \hat{f} \in \mathbb{K}[S_5] \). A tensor \( \hat{W} \in \mathcal{T}_\hat{r} \) of the symmetry class \( \mathcal{T}_\hat{r} \) defined by \( \hat{r} = \hat{1}^* \) satisfies \( \hat{W} = \hat{f}^* \hat{W} \) and the symmetrization of \( \hat{W} \) by means of \( y_r^\ast \) yields \( y_r^\ast \hat{W} = y_r^\ast \hat{f}^* \hat{W} = (f \cdot y_r)^* \hat{W} \). But since the Young symmetrizer \( y_r^\ast \) generates a minimal left ideal which lies in the equivalence class of minimal left ideals characterized by \( (3.2) \vdash 5 \), we obtain \( f \cdot y_r \neq 0 \) at most if \( \hat{1} \) belongs to the class of \( (3.2) \vdash 5 \), too. This happens only in the cases \( (3.3), (3.4), (3.5) \). Thus Theorem \( 1.8 \) is valid for tensor products \( (1.18) \) of type \( T \otimes \hat{T} \). Tensor products \( T \otimes T \) can be handled in the same way.

3.2. Proof of Theorem \( 1.9 \). The proof of Theorem \( 1.9 \) uses a method which we applied already in the proof of \( 1.8 \) Theorem 5.1.

If \( G \subseteq S_r \) is a subgroup of \( S_r \) and \( i_1, \ldots, i_h \) are natural numbers with \( 1 \leq i_1 \leq r \), then we denote by \( G_{i_1, \ldots, i_h} \) the stabilizer subgroup of all such permutations from \( G \) for which \( i_1, \ldots, i_h \) are fixed points.

Now we consider the following group ring elements

\[
(3.6) \quad \sigma_k := y_r^\ast \cdot \xi_k \in \mathbb{K}[S_5] \quad , \quad (k = 1, 2)
\]

where \( y_r^\ast \) is the Young symmetrizer of the Young tableau \( (2,2,0) \) and

\[
(3.7) \quad \xi_1 := \left( \sum_{p \in (S_5)_{1,4,5}} p \right) \cdot \left( \sum_{q \in (S_5)_{1,2}} q \right)
\]

\[
(3.8) \quad \xi_2 := \left( \sum_{q \in (S_5)_{1,2,3}} q \right) \cdot \left( \sum_{p \in (S_5)_{1,4,5}} p \right)
\]

A calculation (by means of PERMS \( 10 \)) shows that \( \sigma_1 \neq 0 \) and \( \sigma_2 \neq 0 \) (see \( 7 \)).

The right ideals \( \sigma_k \cdot \mathbb{K}[S_5] \) are non-vanishing subideals of the right ideal \( \hat{r} = y_r^\ast \cdot \mathbb{K}[S_5] \) which defines the symmetry class \( \mathcal{T}_\hat{r} \) of algebraic covariant derivative curvature tensors. Since \( \hat{r} \) is a minimal right ideal, we obtain \( \hat{r} = \sigma_1 \cdot \mathbb{K}[S_5] = \sigma_2 \cdot \mathbb{K}[S_5] \), i.e. \( \sigma_1 \) and \( \sigma_2 \) are generating elements of \( \hat{r} \), too.

A tensor \( T \in \mathcal{T}_5 V \) is an algebraic covariant derivative curvature tensor iff there exist \( a \in \hat{r} \) and \( T' \in \mathcal{T}_5 V \) such that \( T = aT' \). Since further every \( a \in \hat{r} \) has representations \( a = \sigma_k \cdot x_k, k = 1, 2 \), a tensor \( T \in \mathcal{T}_5 V \) is an algebraic covariant derivative curvature tensor iff there is a tensor \( T' \in \mathcal{T}_5 V \) such that \( T = \sigma_1 T' \) or \( T = \sigma_2 T' \).

Let us consider the case \( k = 1 \). We obtain all algebraic covariant derivative curvature tensors if we form \( T = \sigma_1 T', T' \in \mathcal{T}_5 V \). As in the proof of Theorem \( 1.8 \) we can use a representation

\[
(3.9) \quad T' = \sum_{(M,N) \in \mathcal{P}} M \otimes N
\]
for a $T' \in T_5V$, where $\mathcal{P}$ is a finite set of pairs $(M,N)$ of tensors $M \in T_2V$, $N \in T_3V$. From (3.9) we obtain

\[
\xi_1 T' = \sum_{(S, \hat{S}) \in \hat{\mathcal{P}}} S \otimes \hat{S}
\]

where $\hat{\mathcal{P}}$ is the finite set of pairs $(S, \hat{S})$ of the symmetrized tensors $S = (\sum_{p \in S_2} p) M$, $\hat{S} = (\sum_{q \in S_3} q) N$, $(M, N) \in \mathcal{P}$. Now the application of $y_1^* \cdot \xi_1$ shows, that the algebraic covariant derivative curvature tensor $T$ is a finite sum of tensors

\[
T = \sigma_1 T' = (y_1^* \cdot \xi_1) T' = \sum_{(S, \hat{S}) \in \hat{\mathcal{P}}} y_1^* (S \otimes \hat{S})
\]

The case $k = 2$ can be treated in exactly the same way. We can show that

\[
\xi_2 T' = \sum_{(\hat{S}, S) \in \mathcal{P}} \hat{S} \otimes S
\]

where $\mathcal{P}$ is a finite set of pairs $(\hat{S}, S)$ of symmetric tensors $S \in T_2V$, $\hat{S} \in T_3V$. Then we obtain

\[
T = \sigma_2 T' = (y_1^* \cdot \xi_2) T' = \sum_{(\hat{S}, S) \in \mathcal{P}} y_1^* (\hat{S} \otimes S)
\]

Formula (1.21) can be proved by a computer calculation. We used the Mathematica packages PERMS and Ricci to this end (see [7]).

3.3. Proof of Theorem 1.10. In the situation considered in Theorem 1.10 the symmetry class of the tensors $U$ is not unique. The $(21)$-equivalence class of minimal right ideals $r \subset K[S_3]$ which we use to define symmetry classes for the $U$ is an infinite set. The set of generating idempotents for these right ideals $r$ is infinite, too. We use discrete Fourier transforms to determine the family of all primitive idempotents in the $(21)$-equivalence class of minimal right ideals $r \subset K[S_3]$.

We denote by $K^{d \times d}$ the set of all $d \times d$-matrices of elements of $K$.

**Definition 3.1.** A discrete Fourier transform for $S_r$ is an isomorphism

\[
D : K[S_r] \rightarrow \bigotimes_{\lambda > r} K^{d_\lambda \times d_\lambda}
\]

according to Wedderburn’s theorem which maps the group ring $K[S_r]$ onto an outer direct product $\bigotimes_{\lambda > r} K^{d_\lambda \times d_\lambda}$ of full matrix rings $K^{d_\lambda \times d_\lambda}$. We denote by $D_\lambda$ the natural projections $D_\lambda : K[S_r] \rightarrow K^{d_\lambda \times d_\lambda}$.

\[\text{9See M. Clausen and U. Baum for details about fast discrete Fourier transforms.}\]
A discrete Fourier transform maps every group ring element \( a \in \mathbb{K}[S_r] \) to a block diagonal matrix

\[
D : a = \sum_{p \in S_r} a(p) p \mapsto \begin{pmatrix}
A_{\lambda_1} & & & \\
& A_{\lambda_2} & & 0 \\
& & \ddots & \\
0 & & & A_{\lambda_k}
\end{pmatrix}.
\]

The matrices \( A_{\lambda} \) are numbered by the partitions \( \lambda \vdash r \). The dimension \( d_{\lambda} \) of every matrix \( A_{\lambda} \in \mathbb{K}^{d_{\lambda} \times d_{\lambda}} \) can be calculated from the Young frame belonging to \( \lambda \vdash r \) by means of the hook length formula.

For \( r = 3 \) we have

\[
\begin{array}{cccc}
\lambda & (3) & (21) & (1^3) \\
d_{\lambda} & 1 & 2 & 1
\end{array}
\]

The inverse discrete Fourier transform is given by

\[
\text{Proposition 3.2.} \quad \text{If } D : \mathbb{K}[S_r] \to \bigotimes_{\lambda \vdash r} \mathbb{K}^{d_{\lambda} \times d_{\lambda}} \text{ is a discrete Fourier transform for } \mathbb{K}[S_r], \text{ then we have for every } a \in \mathbb{K}[S_r]
\]

\[
\forall p \in S_r : \quad a(p) = \frac{1}{r!} \sum_{\lambda \vdash r} d_{\lambda} \text{trace} \left\{ D_{\lambda}(p^{-1}) \cdot D_{\lambda}(a) \right\}
\]

\[
= \frac{1}{r!} \sum_{\lambda \vdash r} d_{\lambda} \text{trace} \left\{ D_{\lambda}(p^{-1}) \cdot A_{\lambda} \right\}.
\]

It is easy to determine minimal right ideals and primitive idempotents in the image \( \bigotimes_{\lambda \vdash r} \mathbb{K}^{d_{\lambda} \times d_{\lambda}} \) of a discrete Fourier transform. We denote by \( \mathbb{K}^d \) the vector space of all \( d \)-tuples (rows) of elements of \( \mathbb{K} \).

\[
\text{Lemma 3.3.} \quad \text{We assign to a fixed } d\text{-tuple } a \in \mathbb{K}^d \text{ the set of } d \times d\text{-matrices } r_a := \{ a^t \cdot u \mid u \in \mathbb{K}^d \}. \text{ Then we have:}
\]

1. The sets \( r_a \) are minimal right ideals of \( \mathbb{K}^d \times d \) for every \( a \in \mathbb{K}^d \), \( a \neq 0 \).
2. Let \( r \subset \mathbb{K}^d \times d \) be a minimal right ideal and \( A \in r \), \( A \neq 0 \). Then every non-vanishing column \( a^t \) of \( A \) yields a \( d \)-tuple \( a \in \mathbb{K}^d \) which fulfils \( r = r_a \).

In our considerations we are interested in the matrix ring \( \mathbb{K}^{2 \times 2} \) which corresponds to the \((21)\)-equivalence class of minimal right ideals \( r \subset \mathbb{K}[S_3] \). We can formulate the following assertion about generating idempotents of minimal right ideals of \( \mathbb{K}^{2 \times 2} \).

\[
\text{Proposition 3.4.} \quad \text{Every minimal right ideal } r \subset \mathbb{K}^{2 \times 2} \text{ is generated by exactly one of the following (primitive) idempotents}
\]

\[
X_{\nu} := \begin{pmatrix}
1 & 0 \\
\nu & 0
\end{pmatrix}, \quad \nu \in \mathbb{K},
\]

\[
Y := \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}.
\]

\[^{10}\text{See M. Clausen and U. Baum [3, p.81].}\]

\[^{11}\text{See B. Fiedler [9, p.18, p.20].}\]
Proof. Lemma 3.3 tells us that two right ideals \( r_{a_1} \) and \( r_{a_2} \) are different iff \( a_1 \) and \( a_2 \) are linearly independent. This leads us to the following two cases:

1. Every vector \((x, y) \in \mathbb{K}^2 \) with \( x \neq 0 \) is proportional to one element of the family

\[
(1, \nu) \in \mathbb{K}^2 \quad \nu \in \mathbb{K}.
\]  

Furthermore the elements of the family (3.17) are pairwise linearly independent.

2. Every vector \((x, y) \in \mathbb{K}^2 \) with \( x = 0 \) is proportional to

\[
(0, 1) \in \mathbb{K}^2.
\]

Moreover \((0, 1)\) and \((1, \nu)\) are linearly independent for all \( \nu \in \mathbb{K} \). Thus we obtain every minimal right ideal of \( \mathbb{K}^{2 \times 2} \) exactly once if we consider all right ideals \( r_{a} \) where \( a \) runs through the family (3.17) and the element (3.18). Now we form ansatzes for idempotents of the basis of (3.17), (3.18).

**Case (1):** First we determine a generating idempotent for a minimal right ideal \( r_{a} \) where \( a \) belongs to the family (3.17). We use an ansatz

\[
X_{\nu} := \begin{pmatrix} 1 \\ \nu \end{pmatrix} \cdot (1, y), \quad \nu, y \in \mathbb{K}.
\]

Since \( X_{\nu} = \begin{pmatrix} 1 & y \\ \nu & \nu y \end{pmatrix} \) and \( X_{\nu} \cdot X_{\nu} = \{1 + \nu y\} X_{\nu} \), we obtain

\[
X_{\nu} \cdot X_{\nu} = X_{\nu} \iff 1 + \nu y = 1 \iff y = 0.
\]

In the case \( \nu \neq 0 \) the only solution of \( 1 + \nu y = 1 \) is \( y = 0 \). In the case \( \nu = 0 \) values \( y \neq 0 \) lead to idempotents \( X_{\nu} \), too. However, all these idempotents generate one and the same right ideal since the column \((1 \ \nu)^t\) is fixed. Because we search for exactly one idempotent in every right ideal, the solution \( y = 0 \) is sufficient.

**Case (2):** For an idempotent belonging to (3.18) we make an ansatz

\[
Y := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (x, y), \quad x, y \in \mathbb{K}.
\]

From this ansatz we obtain \( Y = \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \) and \( Y \cdot Y = y Y \). Consequently we have \( y = 1 \). Furthermore we can set \( x = 0 \) because every value of \( x \) yields a generating idempotent of one and the same right ideal. \( \square \)

Now we determine the primitive idempotents of \( \mathbb{K}[S_3] \) which correspond to (3.15) and (3.16).
Proposition 3.5. Let us use Young’s natural representation\(^{12}\) of \(S_3\) as discrete Fourier transform. If we apply the Fourier inversion formula (3.14) to a \(4 \times 4\) block matrix

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & A & 0 \\
0 & 0 & 0 
\end{pmatrix}
\]

(3.19)

where \(A\) is equal to (3.15) or (3.16), then we obtain the following idempotents of \(K[S_3]\):

\[
\begin{align*}
X_\nu & \Rightarrow \xi_\nu := \frac{1}{3} \{[1, 2, 3] + \nu[1, 3, 2] + (1 - \nu)[2, 1, 3] \\
& \quad - \nu[2, 3, 1] + (-1 + \nu)[3, 1, 2] - [3, 2, 1]\} \\
Y & \Rightarrow \eta := \frac{1}{3} \{[1, 2, 3] - [2, 1, 3] - [2, 3, 1] + [3, 2, 1]\}.
\end{align*}
\]

(3.20, 3.21)

Proof. We see from formula (3.13) and the table of the \(d_\lambda\)-values for \(S_3\) that the images of a discrete Fourier transform of \(S_3\) are block diagonal matrices

\[
D : a = \sum_{p \in S_r} a(p) p \mapsto \begin{pmatrix}
A_{(3)} & 0 \\
0 & A_{(21)} \\
0 & A_{(13)}
\end{pmatrix},
\]

(3.22)

where \(A_{(3)}\) and \(A_{(13)}\) are \(1 \times 1\)-matrices and \(A_{(21)}\) is a \(2 \times 2\)-matrix. The \((21)\)-equivalence class of minimal right ideals of \(K[S_3]\) corresponds to \(A_{(3)} = A_{(13)} = 0\). Starting with block matrices (3.19) and \(A = X_\nu\) or \(A = Y\) we calculated the idempotents \(\xi_\nu\) and \(\eta\) by means of the tool InvFourierTransform of the Mathematica package PERMS [10] (see also [9, Appendix B].) These calculation can be carried out also by the program package SYMMETRICA [23, 24]. \(\square\)

Remark 3.6. It is interesting to clear up the connection of the idempotents \(\xi_\nu\) and \(\eta\) with Young symmetrizers. A simple calculation shows that

\[
\xi_0 = \frac{1}{3} y_{t_1}, \quad \eta = \frac{1}{3} y_{t_2}
\]

(3.23)

where \(y_{t_1}\) and \(y_{t_2}\) are the Young symmetrizers of the tableaux

\[
t_1 = \begin{array}{c}
1 \\
3 \\
2
\end{array}, \quad t_2 = \begin{array}{c}
1 \\
2 \\
3
\end{array}.
\]

Proof of Theorem 1.10: The proof of Theorem 1.10 uses the same method as the proof of Theorem 1.9. To treat expressions \(y^*_\nu(S \otimes U)\) and \(y^*_\nu(A \otimes U)\) form the

\(^{12}\)Three discrete Fourier transforms \(3, 11\) are known for symmetric groups \(S_r\): (1) Young’s natural representation, (2) Young’s seminormal representation and (3) Young’s orthogonal representation. (See [2, 3, 21, 5]. A short description of (1) and (2) can be found in [9, Sec.1.5.2].) All three discrete Fourier transforms are implemented in the program package SYMMETRICA [23, 24]. PERMS [10] uses the natural representation. The fast DFT-algorithm of M. Clausen and U. Baum [5, 4] is based on the seminormal representation.
following group ring elements of $\mathbb{K}[S_5]$:

(3.24) \[ \sigma_{\nu, \epsilon} := y_{\nu}^* \cdot \zeta'_\epsilon \cdot \xi''_\nu, \quad \rho_\epsilon := y_{\nu}^* \cdot \zeta'_\epsilon \cdot \eta''_\nu \]
(3.25) \[ \zeta'_\epsilon := \text{id} + \epsilon (1 \ 2), \quad \epsilon \in \{1, -1\} \]
(3.26) \[ \xi_{\nu} \mapsto \zeta''_\nu \in \mathbb{K}[S_5], \quad \eta \mapsto \eta'' \in \mathbb{K}[S_5]. \]

Formula (3.26) denotes the embedding of the group ring elements $\xi_{\nu}, \eta \in \mathbb{K}[S_3]$ into $\mathbb{K}[S_5]$ which is induced by the mapping

$$S_3 \to S_5, \quad [i_1, i_2, i_3] \mapsto [1, 2, i_1 + 2, i_2 + 2, i_3 + 2].$$

For expressions $y_{\nu}^*(U \otimes S)$ and $y_{\nu}^*(U \otimes A)$ we consider the group ring elements of $\mathbb{K}[S_5]$:

(3.27) \[ \sigma_{\nu, \epsilon} := y_{\nu}^* \cdot \zeta''_\epsilon \cdot \xi'_\nu, \quad \rho_\epsilon := y_{\nu}^* \cdot \zeta''_\epsilon \cdot \eta'_\nu \]
(3.28) \[ \zeta''_\epsilon := \text{id} + \epsilon (4 \ 5), \quad \epsilon \in \{1, -1\} \]
(3.29) \[ \xi_{\nu} \mapsto \zeta''_\nu \in \mathbb{K}[S_5], \quad \eta \mapsto \eta' \in \mathbb{K}[S_5]. \]

Formula (3.29) denotes the embedding of the group ring elements $\xi_{\nu}, \eta \in \mathbb{K}[S_3]$ into $\mathbb{K}[S_5]$ which is induced by the mapping

$$S_3 \to S_5, \quad [i_1, i_2, i_3] \mapsto [i_1, i_2, i_3, 4, 5].$$

Using the Mathematica package PERMS [10] we verified

$$\rho_\epsilon \neq 0 \quad \text{and} \quad \sigma_{\nu, \epsilon} \neq 0 \iff \nu \neq \frac{1}{2},$$

in both cases. Thus the same arguments which we used in the proof of Theorem 1.9 yield now in both cases:

(1) The minimal right ideals $y_{\nu}^* \cdot \mathbb{K}[S_5], \rho_\epsilon \cdot \mathbb{K}[S_5]$ and $\sigma_{\nu, \epsilon} \cdot \mathbb{K}[S_5]$ coincide, if $\nu \neq \frac{1}{2}$.

(2) A tensor $T \in T_2V$ is an algebraic covariant derivative curvature tensor iff a tensor $T' \in T_2V$ exists such that $T = \rho_\epsilon T'$ or $T = \sigma_{\nu, \epsilon} T'$ (if $\nu \neq \frac{1}{2}$).

Now we represent the tensor $T'$ by finite sums of product tensors. We use sums

(3.30) \[ T' = \sum_{(M,N) \in \mathcal{P}} M \otimes N, \quad \mathcal{P} \subset T_2V \times T_3V \text{ finite} \]

in the case (3.24)–(3.26) and sums

(3.31) \[ T' = \sum_{(N,M) \in \mathcal{P}} N \otimes M, \quad \mathcal{P} \subset T_2V \times T_2V \text{ finite} \]

in the case (3.27)–(3.29).

Now the argumentation of the proof of Theorem 1.9 yield also the statement of Theorem 1.10 (if $\nu \neq \frac{1}{2}$). The symmetrizations $\zeta' \cdot \xi''(M \otimes N)$, $\zeta' \cdot \eta''(M \otimes N)$, $\xi'' \cdot \eta''(N \otimes M)$ and $\eta' \cdot \zeta''(N \otimes M)$ lead to product tensors $S \otimes U$, $A \otimes U$, $U \otimes S$ and $U \otimes A$. Note that the value $\epsilon = 1$ ($\epsilon = -1$) produces symmetric tensors $S \in T_2V$ (alternating tensors $A \in T_2V$) from the tensors $M \in T_2V$.

Thus we can write every tensor $\zeta' \cdot \xi''T'$, $\zeta' \cdot \eta''T'$, $\xi'' \cdot \zeta''T'$ and $\eta' \cdot \zeta''T'$ as a finite sum of suitable tensors $S \otimes U$, $A \otimes U$, $U \otimes S$ or $U \otimes A$. An application of $y_{\nu}^*$ shows
that every algebraic covariant derivative curvature tensor $T$ can be expressed by a finite sum of each of the tensor types (1.22).

Finally we can verify by a simple calculation that the idempotent $\xi_\nu$ is equal to the idempotent (1.23) if $\nu = \frac{1}{2}$. This finishes the proof of Theorem 1.10.

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