Unbounded Fredholm modules and double operator integrals

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Abstract

In noncommutative geometry one is interested in invariants such as the Fredholm index or spectral flow and their calculation using cyclic cocycles. A variety of formulae have been established under side conditions called summability constraints. These can be formulated in two ways, either for spectral triples or for bounded Fredholm modules. We study the relationship between these by proving various properties of the map on unbounded self adjoint operators \( D \) given by \( f(D) = D(1 + D^2)^{-1/2} \). In particular we prove commutator estimates which are needed for the bounded case. In fact our methods work in the setting of semifinite noncommutative geometry where one has \( D \) as an unbounded self adjoint linear operator affiliated with a semi-finite von Neumann algebra \( M \). More precisely we show that for a pair \( D, D_0 \) of such operators with \( D - D_0 \) a bounded self-adjoint linear operator from \( M \) and \( (1 + D_0^2)^{-1/2} \in \mathcal{E} \), where \( \mathcal{E} \) is a noncommutative symmetric space associated with \( M \), then

\[
\|f(D) - f(D_0)\|_\mathcal{E} \leq C \cdot \|D - D_0\|_M.
\]

This result is further used to show continuous differentiability of the mapping between an odd \( \mathcal{E} \)-summable spectral triple and its bounded counterpart.

1 Introduction

This paper concerns questions arising in noncommutative geometry in general and the study of spectral flow in particular. The basic issues were first exposed in A. Connes [14, 15]. The object of study is a spectral triple which consists of a separable Hilbert space \( \mathcal{H} \), a densely defined unbounded self-adjoint linear operator \( D \) and a *-algebra of bounded operators on \( \mathcal{H} \) such that \([D,a]\) extends to a bounded operator on \( \mathcal{H} \) for all \( a \in A \). If there is a grading operator \( \gamma \) (that is \( \gamma \) is self adjoint and \( \gamma^2 = 1 \)) which anticommutes with \( D \) then the spectral triple is said to be even and otherwise it is odd. As the grading operator will not play a role in what we do here we will ignore it in the sequel. We note however that spectral flow, which will form a major application of our results, is non-trivial only in the odd case.

In order to construct formulae for spectral flow or the Fredholm index one employs explicit cyclic cocycles whose existence requires ‘summability conditions’ on \( D \). These take the form of specifying some symmetrically normed ideal \( \mathcal{E} \) of compact operators on \( \mathcal{H} \) and requiring \( (1 + D^2)^{-1/2} \in \mathcal{E} \). In [15] three
cases arise naturally namely the Schatten ideals \( L^p \) (the \( p \)-summable case), the ideal \( Li \) which is relevant to so-called theta summable spectral triples and the ideal \( L^{p,\infty} \) which is naturally associated to the Dixmier trace.

In constructing formulae for cyclic cocycles one is faced with deciding when a given cocycle is in the cohomology class of the Chern character [15]. This Chern character is defined not for spectral triples but for bounded Fredholm modules for \( \mathcal{A} \). The passage from unbounded to bounded requires us to study the map \( D \to F_D = D(1 + D^2)^{-1/2} \). The definition of an \( \mathcal{E} \)-summable bounded Fredholm module requires the commutator \([F_D,a] \) to be in \( \mathcal{E} \) so that we want to know when this follows from the assumption \((1 + D^2)^{-1/2} \in \mathcal{E} \). This explains the need for methods to prove commutator estimates which generalize earlier work.

In the setting of Schatten-von Neumann ideals, it was established in [5] (respectively, in [36]) that if \((1 + D_0^2)^{-\frac{1}{2}} \in L^q, \ q < p \) (respectively, \((1 + D^2)^{-\frac{1}{2}} \in Li^\beta, \ \beta > \alpha \)), then we have the Lipschitz estimates in \( L^p \) (respectively \( Li^\alpha \)). The sharp commutator estimate in this setting was proved in [33].

Beginning in [5] and continuing in [1, 2, 4, 6–11] an extension is made to the framework of noncommutative geometry described in [15]. This extension is to the situation where we take \( \mathcal{M} \) be a semi-finite von Neumann algebra acting on \( \mathcal{H} \) with normal self-adjoint faithful trace \( \tau \), we let \( D \) be an unbounded self-adjoint linear operator affiliated with \( \mathcal{M} \) and take \( \mathcal{E} \) to be a noncommutative symmetric space of \( \tau \)-measurable operators (all these notions are explained in the next Section). Summability in this setting means \((1 + D^2)^{-1/2} \in \mathcal{E} \) however now the passage from the unbounded to bounded picture is a much more complex issue. Our systematic approach to these questions results in a general approach which we illustrate in Theorems 11 and 18. We establish in particular that for the case of a general semifinite von Neumann algebra from the condition \((1 + D^2)^{-1/2} \in \mathcal{E} \) there follows the bound

\[
\| [F_D,a] \|_\mathcal{E} \leq c \| [D,a] \|.
\]

A related question arises in [5,6,29] where the notion of spectral flow along a path of unbounded self-adjoint operators affiliated to \( \mathcal{M} \) is studied and analytic formulae to calculate spectral flow are given. In [9] and [10] a local index formula for spectral flow (analogous to the formula of Connes-Moscovici [13] for the case where \( \mathcal{M} \) is the bounded operators on \( \mathcal{H} \)) is given and its relation to the Chern character of a ‘Breuer-Fredholm module’ studied. In all of this work the properties of the function

\[
(1.1) \quad f(t) = \frac{t}{(1 + t^2)^{3/2}}, \quad t \in \mathbb{R}
\]

defined on unbounded self-adjoint Breuer-Fredholm operators plays an essential role. In particular the question of operator differentiability of \( f \) arises. Until now results about this function have been proved in an ad hoc fashion and are restricted to particular choices of the ideal \( \mathcal{E} \). More general ideals do need to be studied as they arise in a very natural way once one begins a deeper study of noncommutative geometry in this setting, see for example [12].

The principal result of our paper in this direction is that if \((\mathcal{M}, D)\) is an odd \( \mathcal{E} \)-summable semifinite spectral triple, then \((\mathcal{M}, f(D))\) is an odd bounded
\( (M, D_0) \mapsto (M, f(D)) \)

is Lipschitz continuous and continuously differentiable on the affine space of bounded self-adjoint perturbations of \( D \) (where the perturbation comes from \( M \)). The need for such a result is noted in [37].

Consider the following example. Let \( \mathcal{H} = L^2[0,1] \) and let \( M = B(\mathcal{H}) \) (i.e., the algebra of all bounded linear operators on \( \mathcal{H} \)). Consider the operator \( D_0 = i \frac{d}{dt} \) with \( \text{Dom} \ D_0 \) given by the class of all absolutely continuous functions \( \xi \) on \([0,1]\) such that \( \xi(0) = \xi(1) \). It is well-known that \( \sigma(D_0) = \mathbb{Z} \) and

\[
(1 + D_0^2)^{-\frac{1}{2}} \in C^{1,\infty},
\]

where \( C^{1,\infty} \) is the weak \( L^1 \) ideal (see Section 5 below). Taking a path of operators

\[
s \mapsto D_s = i \frac{d}{dt} + V(t, s),
\]

where the path of potentials \( s \mapsto V(t, s) \) is continuously differentiable in \( L^\infty[0,1] \), Theorem 23 implies that the path of operators \( s \mapsto f(D_s) \) is continuously differentiable in \( C^{1,\infty} \).

Partial results of this nature were earlier obtained in [4–6, 33, 36] however the methods employed in those papers are not adequate to determine the sense in which this mapping \( f \) is smooth on operators. This suggested to the authors that there was a need for a more powerful method that could answer this question in full generality. The technique we describe here is partly based on an approach to the calculus of functions of operators known as the theory of ‘double operator integrals’. It has only recently been developed for the general semi-finite von Neumann algebras in [16, 17] and its applications to Lipschitz and commutator estimates of operator functions begun in [18, 30, 31].

The organization of the paper is as follows. We briefly outline the theory of double operator integrals in Section 3 where we also prove a number of technical results needed to analyze the behavior of the operator function \( f \) in our context. Section 4 contains the main results concerning Lipschitz and commutator estimates for the function \( f \) (and some other operator functions) which occur in noncommutative geometry and the theory of spectral flow. Section 5 contains a specialization of the results given in Section 4 to important applications to the case of weak \( L_p \)-spaces and answers a question raised by A. Carey in the context of applications to spectral flow. Finally, the last section 6 explains how results presented in Section 4 can be further refined to prove the differentiability of the mapping \( (M, D_0) \mapsto (M, f(D_0)) \). We also indicate there important implications for the theory of spectral flow which are motivated by the papers [1, 5, 6, 37]. The strategy of our proof is straightforward and applies equally well to von Neumann algebras of type I and II.

2 Preliminaries

Let \( M \) be a semi-finite von Neumann algebra acting on a Hilbert space \( \mathcal{H} \) and equipped with normal semi-finite faithful trace \( \tau \). The identity in \( M \) is denoted by 1 and \( \| \cdot \| \) stands for the operator norm. An operator \( D : \mathcal{D}(D) \mapsto \mathcal{H}, \)
with domain \( \mathcal{D}(D) \subseteq \mathcal{H} \), is called affiliated with \( \mathcal{M} \) if and only if, for every unitary \( u \in \mathcal{M}', u^* Du = D \), i.e. (i) \( u(\mathcal{D}(D)) \subseteq \mathcal{D}(D) \) and (ii) \( u^* Du(\xi) = D(\xi) \), for every \( \xi \in \mathcal{D}(D) \). We use the standard notation \( D\mathcal{M} \) to indicate that the operator \( D \) is affiliated with \( \mathcal{M} \). A closed and densely defined operator \( D\mathcal{M} \) is called \( \tau \)-measurable if for every \( \epsilon > 0 \) there exists an orthogonal projection \( p \in \mathcal{M} \) such the \( p(\mathcal{H}) \subseteq \mathcal{D}(D) \) and \( \tau(1 - p) < \epsilon \). The set of all \( \tau \)-measurable operators is denoted \( \mathcal{M} \). The set \( \mathcal{M} \) is a \(*\)-algebra (with respect to strong multiplication and addition) complete in the measure topology. We refer the reader to [22, 26, 35] for more details.

We recall from [22] the notion of generalized singular value function. Given a self-adjoint operator \( T \) in \( \widetilde{\mathcal{M}} \), we denote by \( E^T(\cdot) \) the spectral measure of \( T \). For every \( T \in \widetilde{\mathcal{M}} \), \( E^{|T|}(B) \in \mathcal{M} \) for all Borel sets \( B \subseteq \mathbb{R} \), and there exists \( s > 0 \) such that \( \tau(E^{|T|}(s, \infty)) < \infty \). For \( t > 0 \), we define

\[
\mu_t(T) = \inf\{s \geq 0 : \tau(E^{|T|}(s, \infty)) \leq t\}.
\]

The function \( \mu(T) : [0, \infty) \to [0, \infty] \) is called the generalized singular value function (or decreasing rearrangement) of \( T \); note that \( \mu(\cdot)(T) \in L_\infty \) if and only if \( T \in \mathcal{M} \).

Throughout the text let \( E = E(0, \infty) \) be a symmetric Banach function space, i.e. \( E = E(0, \infty) \) is a rearrangement invariant Banach function space on \([0, \infty)\) (see [25]). We use the notation \( g \ll f \) to denote submajorization in the sense of Hardy, Littlewood and Polya, i.e.

\[
\int_0^t \mu_s(g) \, ds \leq \int_0^t \mu_s(f) \, ds, \quad t > 0.
\]

We will always require \( E \) to have the additional property that \( f, g \in E \) and \( g \ll f \) implies that \( \|g\|_E \leq \|f\|_E \).

There is associated to each such space \( E \) a noncommutative symmetric space \( \mathcal{E} = E(\mathcal{M}, \tau) \) defined by

\[
\mathcal{E} = \{T \in \widetilde{\mathcal{M}} : \mu_{(\cdot)}(T) \in E\} \quad \text{with} \quad \|T\|_{\mathcal{E}} = \|\mu_{(\cdot)}(T)\|_E.
\]

If \( E = L^p \), \( 1 \leq p \leq \infty \), then \( L^p \) is the noncommutative \( L^p \)-space. For the sake of brevity, we shall denote the norm in the space \( L^p \) by \( \|\cdot\|_p \). Note, that the spaces \( L^\infty \) and \( L^1 \) coincide with the algebra \( \mathcal{M} \) and the predual \( \mathcal{M}_* \), respectively, and that \( \|\cdot\|_\infty \) is the operator norm \( \|\cdot\| \). We refer the reader to [17, 20, 21] for more information on noncommutative symmetric spaces.

The Köthe dual \( \mathcal{E}^\times \) of a symmetric space \( E \) is the symmetric space given by

\[
\mathcal{E}^\times = \{T \in \widetilde{\mathcal{M}} : TS \in L^1, \text{ whenever } S \in \mathcal{E}\}
\]

and

\[
\|T\|_{\mathcal{E}^\times} := \sup_{S \in L^1 \cap L^\infty, \|S\|_{\mathcal{E}} \leq 1} \tau(TS) < \infty,
\]

see, for example, [19]. It is a subspace of the dual space \( \mathcal{E}^* \) (the norms \( \|\cdot\|_{\mathcal{E}^\times} \) and \( \|\cdot\|_{\mathcal{E}^*} \) coincide on \( \mathcal{E}^\times \)) and \( \mathcal{E}^\times = \mathcal{E}^* \) if and only if the space \( E \) is separable. It is known that \( (L^p)^\times, 1 \leq p \leq \infty \) coincides with \( L^{p'} \), where \( p' \) is the conjugate exponent, i.e. \( p^{-1} + p'^{-1} = 1 \).
In the present text, we prove a number of results concerning perturbation and commutator estimates in noncommutative symmetric spaces which are relevant to noncommutative geometry. In this context, the main interest lies with symmetric spaces $E \subseteq L^\infty(0,\infty)$, that is $E$ can be thought of as a unitarily invariant ideal of $\mathcal{M}$ equipped with a unitarily invariant norm. If $\mathcal{M}$ is a type $I$ factor, then such ideals are customarily called symmetrically normed ideals (of compact operators), see e.g. [23].

Let $D_0, D_1 \eta \mathcal{M}$ be self-adjoint linear operators and let $a \in \mathcal{M}$. We adopt the following definition, see [3] (see also [31]). We shall say that the operator $D_0 a - a D_1$ is well defined and bounded (equivalently $D_0 a - a D_1 \in L^\infty$) if and only if (i) $a(\mathcal{D}(D_1)) \subseteq \mathcal{D}(D_0)$; (ii) the operator $D_0 a - a D_1$, initially defined on $\mathcal{D}(D_1)$, is closable; (iii) the closure $\overline{D_0 a - a D_1}$ is bounded. In this case, the symbol $D_0 a - a D_1$ also stands for the closure $\overline{D_0 a - a D_1}$. In the special case $D_0 = D_1 = D$, we shall write $[D, a] \in L^\infty$.

3 Double Operator Integrals

Let $X, Y$ be a normed spaces. Recall that $B(X, Y)$ stands for the normed space of all bounded linear operators $T : X \to Y$. If $X = Y$, then we shall write $B(X)$.

Throughout this paper we will let $D_0, D_1$ denote self-adjoint unbounded operators affiliated with $\mathcal{M}$ and let $dE^0_\lambda, dE^1_\mu$ be the corresponding spectral measures. Recall that

$$
\tau(x dE^0_\lambda y dE^1_\mu), \ \lambda, \mu \in \mathbb{R}
$$

is a $\sigma$-additive complex-valued measure on the plane $\mathbb{R}^2$ with the total variation bounded by $\|x\|_2 \|y\|_2$, for every $x, y \in L^2$, see [17, Remark 3.1].

Let $\phi = \phi(\lambda, \mu)$ be a bounded Borel function on $\mathbb{R}^2$. We call the function $\phi$ $dE^0 \otimes dE^1$-integrable in the space $E$, $1 \leq p \leq \infty$ if and only if there is a linear operator $T_\phi = T_\phi(D_0, D_1) \in B(E)$ such that

$$
\tau(x T_\phi(y)) = \int_{\mathbb{R}^2} \phi(\lambda, \mu) \tau(x dE^0_\lambda y dE^1_\mu),
$$

for every $x \in L^2 \cap \mathcal{E}^\times$ and $y \in L^2 \cap \mathcal{E}$.

If the operator $T_\phi(D_0, D_1)$ exists, then it is unique, [17, Definition 2.9]. The latter definition is in fact a special case of [17, Definition 2.9]. See also [17, Proposition 2.12] and the discussion there on pages 81–82. The operator $T_\phi$ is called the Double Operator Integral.

We shall write $\phi \in \Phi(E)$ if and only if the function $\phi$ is $dE^0 \otimes dE^1$-integrable in the space $E$ for any measures $dE^0$ and $dE^1$. The following result is used throughout the text.

**Theorem 1** ([16,17]). Let $D_0, D_1 \eta \mathcal{M}$. The mapping

$$
\phi \mapsto T_\phi = T_\phi(D_0, D_1) \in B(E), \ \phi \in \Phi(E)
$$

is a $\ast$-homomorphism. Moreover, if $\phi(\lambda, \mu) = \alpha(\lambda)$ (resp. $\phi(\lambda, \mu) = \beta(\mu)$), $\lambda, \mu \in \mathbb{R}$, then

$$
T_\phi(x) = \alpha(D_0) x \ (\text{resp. } T_\phi(x) = x \beta(D_1)), \ x \in E,
$$

where $\alpha, \beta : \mathbb{R} \mapsto \mathbb{C}$ are bounded Borel functions.
The latter result allows the construction of a sufficiently large class of functions in $\Phi(\mathcal{E})$, $1 \leq p \leq \infty$. Indeed, let us consider the class $\mathcal{A}_0$ which consists of all bounded Borel functions $\phi(\lambda, \mu)$, $\lambda, \mu \in \mathbb{R}$ admitting the representation

\[(3.2) \quad \phi(\lambda, \mu) = \int_S \alpha_s(\lambda) \beta_s(\mu) \, d\nu(s)\]

such that

$$\int_S \|\alpha_s\|_\infty \|\beta_s\|_\infty \, d\nu(s) < \infty,$$

where $(S, d\nu)$ is a measure space, $\alpha_s, \beta_s : \mathbb{R} \rightarrow \mathbb{C}$ are bounded Borel functions, for every $s \in S$ and $\| \cdot \|_\infty$ is the uniform norm. The space $\mathcal{A}_0$ is endowed with the norm

$$\|\phi\|_{\mathcal{A}_0} := \inf \int_S \|\alpha_s\|_\infty \|\beta_s\|_\infty \, d\nu(s),$$

where the minimum runs over all possible representations (3.2). The space $\mathcal{A}_0$ together with the norm $\| \cdot \|_{\mathcal{A}_0}$ is a Banach algebra, see [16] for details. The following result is a straightforward corollary of Theorem 1.

**Corollary 2 ([16, Proposition 4.7]).** Every $\phi \in \mathcal{A}_0$ is $d\mathcal{E}^0 \otimes d\mathcal{E}^1$-integrable in the space $\mathcal{E}$ for any measures $d\mathcal{E}^0$, $d\mathcal{E}^1$, i.e. $\mathcal{A}_0 \subseteq \Phi(\mathcal{E})$. Moreover, if $T_\phi = T_\phi(D_0, D_1)$, for some self-adjoint operators $D_0, D_1 \in \mathcal{M}$, then

$$\|T_\phi\|_{B(\mathcal{E})} \leq \|\phi\|_{\mathcal{A}_0},$$

for every $\phi \in \mathcal{A}_0$.

The following result explains the connection between Double Operator Integrals and Lipschitz and commutator estimates, see also [18].

**Theorem 3 ([31, Theorem 3.1]).** Let $D_0, D_1 \in \mathcal{M}$ be self-adjoint linear operators, let $a \in \mathcal{M}$ and let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a $C^1$-function with bounded derivative. Let

$$\psi_f(\lambda, \mu) = \frac{f(\lambda) - f(\mu)}{\lambda - \mu}, \quad \text{if } \lambda \neq \mu$$

and $\psi_f(\lambda, \lambda) = f'(\lambda)$. If $D_0 a - a D_1 \in \mathcal{L}^\infty$ and $\psi_f \in \Phi(\mathcal{L}^\infty)$, then $f(D_0) a - a f(D_1) \in \mathcal{L}^\infty$ and

$$f(D_0) a - a f(D_1) = T_{\psi_f}(D_0 a - a D_1),$$

where $T_{\psi_f} = T_{\psi_f}(D_0, D_1)$.

The result above stated and proved in [31, Theorem 3.1] under the additional assumption that $\mathcal{M}$ is taken in its left regular representation. As shown in [30, Theorem 2.4.3] this assumption is redundant.

A decomposition of $\psi_f$ for the function $f$ from (1.1) in the form (3.2) and further analysis of this decomposition given in this paper show that $T_{\psi_f} \in B(\mathcal{L}^\infty, \mathcal{E})$ for every symmetric space $\mathcal{E}$ and this result underlies the applications of double operator integration theory to noncommutative geometry given in Section 6. In the rest of this section, we collect some preliminary material for this analysis.
Recall that $\Lambda_\alpha$, $0 \leq \alpha \leq 1$ stands for the class of all Hölder functions, i.e. the functions $f : \mathbb{R} \mapsto \mathbb{C}$ such that

$$
\|f\|_{\Lambda_\alpha} := \sup_{t_1, t_2} \frac{|f(t_1) - f(t_2)|}{|t_1 - t_2|^\alpha} < +\infty.
$$

**Theorem 4.** Let $f : \mathbb{R} \to \mathbb{C}$. If $\|f\|_{\Lambda_\theta}$, $\|f'\|_\infty$, $\|f'\|_{\Lambda_\epsilon} < \infty$, for some $0 \leq \theta < 1$ and $0 < \epsilon \leq 1$, then $\psi_f \in \mathcal{A}_0$. Moreover, there is a constant $c = c(\epsilon, \theta) > 0$ such that

$$
\|\psi_f\|_{\mathcal{A}_0} \leq c (\|f\|_{\Lambda_\theta} + \|f'\|_\infty + \|f'\|_{\Lambda_\epsilon}).
$$

**Proof.** We let symbol $c$ stand for a positive constant which may vary from line to line. The proof is based on the following result due to V.Peller [28], see also [16]. If $f' \in L^\infty$ and $f \in \dot{B}_{\infty, 1}^1$, then $\psi_f \in \mathcal{A}_0$ and

$$
\|\psi_f\|_{\mathcal{A}_0} \leq c (\|f'\|_\infty + \|f\|_{\dot{B}_{\infty, 1}^1}),
$$

where $\dot{B}_{\infty, 1}$ is the homogeneous Besov class, see [27,34]. To finish the proof, we shall show that

$$
\|f\|_{\dot{B}_{\infty, 1}^1} \leq c (\|f\|_{\Lambda_\theta} + \|f'\|_{\Lambda_\epsilon}).
$$

The argument is rather standard. Let $f(t)$ be a function and let $u(t, s)$, $s \geq 0$ be the Poisson integral of the function $f$, i.e.

$$
u(t, s) = f \ast P_s(t) = \int_{\mathbb{R}} f(\tau) P_s(t - \tau) d\tau, \quad t \in \mathbb{R}, \ s > 0,
$$

where $P_s(t)$ is the Poisson kernel, i.e.

$$
P_s(t) = \frac{1}{\pi} \frac{s}{t^2 + s^2}.
$$

Let $u'_s$ and $u''_s$ stand for the derivatives $\frac{du_s}{ds}$ and $\frac{d^2u_s}{ds^2}$, respectively. Recall that, for every $0 \leq \alpha \leq 1$, there is a numerical constant $c_\alpha > 0$ such that (see [34, Ch. V, Section 4.2])

$$
\sup_{s > 0} s^{1-\alpha} \|u'_s\|_\infty \leq c_\alpha \|f\|_{\Lambda_\alpha}.
$$

Recall also that (see [27,34])

$$
\|f\|_{\dot{B}_{\infty, 1}^1} \sim \int_0^\infty \|u''_s\|_\infty ds
$$

with equivalence up to a positive numerical constant.

Fix $f$ such that $f \in \Lambda_\theta$, $0 \leq \theta < 1$ and $f' \in \Lambda_\epsilon$, $0 < \epsilon \leq 1$. It now follows from (5.3) that

$$
\|u'_s\|_\infty \leq c_\theta \frac{\|f\|_{\Lambda_\theta}}{s^{1-\theta}} \quad \text{and} \quad \|u''_s\|_\infty \leq c_\epsilon \frac{\|f'\|_{\Lambda_\epsilon}}{s^{1-\epsilon}}.
$$

The Poisson kernel $P_s$ possesses the group property $P_{s_1} \ast P_{s_2} = P_{s_1 + s_2}$, $s_1, s_2 > 0$. Consequently,

$$
u(s_1 + s_2, t) = u(s_1, \cdot) \ast P_{s_2}(t).
$$

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Taking $\frac{\partial^2}{\partial s_1 \partial s_2}$ and then letting $s_1 = s_2 = \frac{t}{2}$ yields
\[ u''_{ss}(s, t) = u'(s/2, \cdot) * \frac{\partial P_{s/2}}{\partial s}(t). \]

The latter implies
\[
\|u''_s\|_\infty \leq \|u'_s\|_\infty \left\| \frac{\partial P_{s/2}}{\partial s} \right\|_1 \leq c_0 \frac{\|u'_s\|_\infty}{s}, \quad s > 0,
\]
where $c_0$ is given by
\[ c_0 = s \left\| \frac{\partial P_s}{\partial s} \right\|_1 > 0. \]

Combining (3.7) with the first estimate in (3.6) yields
\[
\|u''_s\|_\infty \leq c \|f\|_{\Lambda_{\theta}}.
\]

Now the last inequality together with the second estimate in (3.6) gives
\[
\left\| f \right\|_{B_{s,1}} \leq c \int_0^\infty \|u''_{ss}\|_\infty ds
\]
\[
= c \int_0^1 \|u''_{ss}\|_\infty ds + c \int_1^\infty \|u''_{ss}\|_\infty ds
\]
\[
\leq c \|f'\|_{\Lambda_{s}} \int_0^1 \frac{ds}{s^{1-\epsilon}} + c \|f\|_{\Lambda_{s}} \int_1^\infty \frac{ds}{s^{2-\theta}}
\]
\[
\leq c \left( \|f'\|_{\Lambda_{s}} + \|f\|_{\Lambda_{s}} \right).
\]

The latter finishes the proof of (3.3). The theorem is proved.

Remark 5. Theorem 4 is stated in a rather restrictive form since the requirement $\|f'\|_\infty < \infty$ is redundant. Indeed, it may be shown that
\[
\|f'\|_\infty \leq c(\theta, \epsilon) (\|f\|_{\Lambda_{s}} + \|f'\|_{\Lambda_{s}}), \quad 0 \leq \theta \leq 1, \quad 0 \leq \epsilon \leq 1.
\]

On the other hand, for all applications of Theorem 4 in the text below the requirement $\|f'\|_\infty < \infty$ clearly holds.

The following is a well-known criterion to verify boundedness of the operator $T_\phi = T_\phi(D_0, D_1)$. We supply a simple proof for convenience of the reader.

Lemma 6. Let $D_0, D_1, M$ be self-adjoint linear operators, let $f : \mathbb{R} \mapsto \mathbb{C}$ and let $f$ be the Fourier transform of $f$. If $f$ is integrable, i.e. $\hat{f} \in L^1(\mathbb{R})$, then $T_\phi = T_\phi(D_0, D_1) \in B(L^\infty)$, where $\phi(\lambda, \mu) = f(\lambda - \mu)$, $\lambda, \mu \in \mathbb{R}$ and
\[
\|T_\phi\|_{B(L^\infty)} \leq \frac{1}{\sqrt{2\pi}} \|\hat{f}\|_{L^1}.
\]

Proof. The proof is straightforward. For the function $\phi(\lambda, \mu)$ we have the representation
\[
\phi(\lambda, \mu) = f(\lambda - \mu) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(s) e^{-is(\lambda - \mu)} ds.
\]
Since $\hat{f}$ is integrable, we readily obtain that $\phi \in \mathfrak{A}_0$ and $\|\phi\|_{\mathfrak{A}_0} \leq \frac{1}{\sqrt{2\pi}} \|\hat{f}\|_{L^1}$. The claim of the lemma now follows from Corollary 2. \qed
Let $T_0 = T_0(D_0, D_1)$ in the latter lemma does not depend on the operators $D_0$ and $D_1$. Next we shall give a simple criterion (from [3]) for a Borel function $f : \mathbb{R} \to \mathbb{C}$ to be such that $f \in L^1(\mathbb{R})$. We shall present the proof for convenience of the reader.

**Lemma 7.** If $f : \mathbb{R} \to \mathbb{C}$ is an absolutely continuous function with $f, f' \in L^2(\mathbb{R})$, then $f \in L^1(\mathbb{R})$ and

$$
\|f\|_{L^1} \leq \sqrt{2} (\|f\|_{L^2} + \|f'\|_{L^2}).
$$

**Proof.** The proof is a combination of the Hölder inequality and the Plancherel identity

$$
\int_{\mathbb{R}} |f(t)| \, dt = \int_{t \in [-1,1]} |\hat{f}(t)| \, dt + \int_{t \notin [-1,1]} |t|^{-1} |\hat{f}(t)| \, dt
$$

$$
\leq \sqrt{2} \left[ \int_{t \in [-1,1]} |\hat{f}(t)|^2 \, dt \right]^{\frac{1}{2}}
$$

$$
+ \left[ \int_{t \notin [-1,1]} |t|^{-2} \, dt \right]^{\frac{1}{2}} \cdot \left[ \int_{t \notin [-1,1]} |t|\hat{f}(t)|^2 \, dt \right]^{\frac{1}{2}}
$$

$$
\leq \sqrt{2} (\|f\|_{L^2} + \|f'\|_{L^2})
$$

Let $(S, \mathfrak{F})$ and $(S', \mathfrak{F}')$ be two measure spaces and let $\nu$ be a measure on $(S, \mathfrak{F})$. If $\omega : (S, \mathfrak{F}) \to (S', \mathfrak{F}')$ is a measurable mapping, i.e. $\omega : S \to S'$ and $\omega^{-1}(A) \in \mathfrak{F}'$ for every $A \in \mathfrak{F}'$, then the mapping $\omega$ induces the measure $\nu \circ \omega^{-1}$ on the space $(S', \mathfrak{F}')$ by the assigning

$$
\nu \circ \omega^{-1}(A) := \nu(\omega^{-1}(A)), \quad A \in \mathfrak{F}'.
$$

If $f : S' \to \mathbb{C}$ is a $\mathfrak{F}'$-measurable function, then $f \circ \omega$ is $\mathfrak{F}$-measurable and

$$
\int_{S} f \circ \omega \, d\nu = \int_{S'} f \, d\nu \circ \omega^{-1},
$$

provided either of the of the integrals exists. The following lemma extends this relation to the setting of double operator integrals.

**Lemma 8.** Let $\phi \in \mathcal{A}_0$ and let $f_j : \mathbb{R} \to \mathbb{R}$, $j = 0, 1$ be Borel functions. We have that

$$
T_{\phi'}(D_0, D_1) = T_{\phi'}(D_0', D_1'),
$$

where $\phi' \in \mathcal{A}_0$, $\phi'(\lambda, \mu) := \phi(f_0(\lambda), f_1(\mu))$, $\lambda, \mu \in \mathbb{R}$ and $D_j' := f_j(D_j) \eta \mathcal{M}$, $j = 0, 1$.

**Proof.** We fix $x, y \in \mathcal{L}_2$ and set

$$
d\nu = d\nu_{\lambda, \mu} = d\nu_{\lambda, \mu}(x, y, D_0, D_1) := \tau(x \, dE^0_{\lambda} \, y \, dE^1_{\mu}).
$$

Let $T_{\phi'} = T_{\phi'}(D_0, D_1)$. Consider the mapping $\omega : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$(\lambda, \mu) \mapsto (f_0(\lambda), f_1(\mu)).$$
Note that $\phi' = \phi \circ \omega$. Applying identity (3.8) and (3.1), we now have

$$
\tau(x T_{\phi'}(y)) = \int_{\mathbb{R}^2} \phi' \, d\nu = \int_{\mathbb{R}^2} \phi \circ \omega \, d\nu = \int_{\mathbb{R}^2} \phi \, d\nu \circ \omega^{-1}.
$$

The measure $\nu \circ \omega^{-1}$ is given by

$$
\nu \circ \omega^{-1} = \tau(x E^0 y E^1) \circ \omega^{-1} = \tau(x (E^0 \circ f_0^{-1}) y (E^1 \circ f_1^{-1})),
$$

where the spectral measure $E^j \circ f_j^{-1}$, $j = 0, 1$ is defined by

$$
E^j \circ f_j^{-1}(A) := E(f^{-1}(A)), \quad j = 0, 1,
$$

for Borel set $A \subseteq \mathbb{R}$. Let us note that, applying (3.8) again (see also [32, Section 13.28]), we see that

$$
\int_{\mathbb{R}} \lambda (E^j \circ f_j^{-1}) \lambda = \int_{\mathbb{R}} f_j(\lambda) \, dE^j_\lambda = f_j(D_j), \quad j = 0, 1.
$$

Thus, the measure $dF^j := d(E^j \circ f_j^{-1})$ is the spectral measure of the operator $f_j(D_j)$, $j = 0, 1$. Consequently, combining (3.10) and (3.11), we readily obtain that

$$
\tau(x T_{\phi'}(y)) = \int_{\mathbb{R}^2} \phi(\lambda, \mu) \, d\tau(x \, dF^0 \, y \, dF^1) = \tau(x T_{\phi}(y)),
$$

where $T_{\phi} = T_{\phi}(D'_0, D'_1)$. The latter identity, together with uniqueness of the operator $T_{\phi}$ satisfying (3.1), completes the proof of the lemma.

The identity (3.9) together with Lemma 6 yield

**Lemma 9.** Let $D_0, D_1 \eta M$ be positive linear operators, let $f : \mathbb{R} \mapsto \mathbb{C}$ be Borel and let $g(t) = f(e^t)$, $t \in \mathbb{R}$. If $\hat{g} \in L^1(\mathbb{R})$, then $T_{\phi}(D_0, D_1) \in B(\mathcal{L}^\infty)$, where

$$
\phi(\lambda, \mu) = f \left( \frac{\lambda}{\mu} \right), \quad \lambda, \mu > 0
$$

and

$$
\|T_{\phi}\|_{B(\mathcal{L}^\infty)} \leq \frac{1}{\sqrt{2\pi}} \|\hat{g}\|_{L^1}.
$$

Furthermore, the decomposition (3.2) for the function $\phi(\lambda, \mu)$ is given by

$$
\phi(\lambda, \mu) = \int_{\mathbb{R}} \hat{g}(s) \lambda^{is} \mu^{-is} \, ds, \quad \lambda, \mu > 0.
$$

**Proof.** Let us introduce the operator $D'_j := \log D_j$, $j = 0, 1$ and the function $\phi'(\lambda, \mu) = \phi(e^\lambda, e^\mu) = f(e^{\lambda-\mu}) = g(\lambda - \mu), \quad \lambda, \mu \in \mathbb{R}$. Note, that $D'_j \eta M$, $j = 0, 1$. Since $\hat{g} \in L^1(\mathbb{R})$, it readily follows from Lemma 6 that $T_{\phi'}(D'_0, D'_1) \in B(\mathcal{L}^\infty)$. On the other hand, from (3.9), we obtain that $T_{\phi}(D_0, D_1) = T_{\phi'}(D'_0, D'_1) \in B(\mathcal{L}^\infty)$. Furthermore, it follows from Lemma 6 that the function $\phi'$ has the decomposition

$$
\phi(e^\lambda, e^\mu) = \phi'(\lambda, \mu) = \int_{\mathbb{R}} \hat{g}(s) e^{is(\lambda-\mu)} \, ds.
$$

Making the back substitution finishes the proof of the lemma.
At the end of the section we prove the following lemma which is implicit in literature and frequently used in the following section.

**Lemma 10.** If $A \in \mathcal{M}$ and if $B_0, B_1 \in \mathcal{E}$ and $B_0, B_1$ are positive, then $B_0^{1-\theta} AB_1^\theta \in \mathcal{E}$, for every $0 \leq \theta \leq 1$ and

$$\|B_0^{1-\theta} AB_1^\theta\|_{\mathcal{E}} \leq \|B_0\|_{\mathcal{E}}^{1-\theta}\|A\|\|B_1\|_{\mathcal{E}}^\theta.$$  

**Proof.** Consider the following holomorphic function with values in $\mathcal{E}$

$$f(z) = \|B_0\|_{\mathcal{E}}^{-1}B_1\|_{\mathcal{E}}^{-z}B_0^{1-z}AB_1^z, \quad z \in \mathbb{C}.$$  

Clearly, we have

$$\sup_{t \in \mathbb{R}} \|f(it)\|_{\mathcal{E}}, \sup_{t \in \mathbb{R}} \|f(1+it)\|_{\mathcal{E}} \leq \|A\|.$$  

Since the function $f$ is holomorphic, the claim of the lemma follows from the maximum principle applied to the strip $0 \leq \Re z \leq 1$.  

**4 Lipschitz and commutator estimates.**

The objective of this Section is to establish a general approach to proving the kind of commutator estimates that arise in noncommutative geometry. In the present section we fix self-adjoint linear operators $D_0, D \eta M$ and, for every $\alpha \geq 0$, we set

$$\Delta_\alpha := (\alpha^2 + D^2)^{1/2} \quad \text{and} \quad \Delta_{\alpha, \alpha} := (\alpha^2 + D_0^2)^{1/2}$$

and

$$\Delta := (1 + D^2)^{1/2} \quad \text{and} \quad \Delta_0 := (1 + D_0^2)^{1/2}.$$  

**Theorem 11.** Let $\alpha > 0$, $\Delta_\alpha^{-1} \in \mathcal{E}$ and $a \in \mathcal{M}$. If $[D, a] \in \mathcal{L}\infty$, then $[D\Delta_\alpha^{-1}, a] \in \mathcal{E}$ and there is a constant $c > 0$ independent of $\alpha$ such that

$$\|[D\Delta_\alpha^{-1}, a]\|_{\mathcal{E}} \leq c \|\Delta_\alpha^{-1}\|_{\mathcal{E}}\|[D, a]\|.$$  

**Proof.** Let us consider the functions

$$f_\alpha(t) = \frac{t}{(\alpha^2 + t^2)^{1/2}}, \quad t \in \mathbb{R} \quad \text{and} \quad \psi_\alpha = \psi_{f_\alpha}.$$  

By Theorem\[^4\] and Corollary\[^2\] we have $\psi_\alpha = \psi_{f_\alpha} \in \Phi(\mathcal{L}\infty)$, and therefore $T_\alpha = T_{\psi_\alpha}(D, D) \in B(\mathcal{L}\infty)$. Consequently, it follows from Theorem\[^3\] that

$$[D\Delta_\alpha^{-1}, a] = T_\alpha([D, a]).$$

We shall show that

$$T_\alpha \in B(\mathcal{L}\infty, \mathcal{E}).$$  

\[\]
Consider the following representation of the function $\psi_\alpha$.

$$
\psi_\alpha(\lambda, \mu) = \frac{\lambda(\alpha^2 + \lambda^2) - \frac{1}{2} - \mu(\alpha^2 + \mu^2)^{-\frac{1}{2}}}{\lambda - \mu}
= \frac{(\lambda + \mu)(\lambda(\alpha^2 + \lambda^2) - \frac{1}{2} - \mu(\alpha^2 + \mu^2)^{-\frac{1}{2}})}{(\alpha^2 + \lambda^2) - (\alpha^2 + \mu^2)}
= \frac{(\alpha^2 + \lambda^2)^{\frac{1}{2}} + (\lambda - \alpha^2)(\alpha^2 + \lambda^2)^{-\frac{1}{2}}}{(\alpha^2 + \lambda^2) - (\alpha^2 + \mu^2)}
- \frac{(\alpha^2 + \mu^2)^{\frac{1}{2}} + (\lambda - \alpha^2)(\alpha^2 + \mu^2)^{-\frac{1}{2}}}{(\alpha^2 + \lambda^2) - (\alpha^2 + \mu^2)}
= \psi'_\alpha(\lambda, \mu) \left( 1 + \frac{\lambda^2 - \lambda \mu}{(\alpha^2 + \lambda^2)^{\frac{1}{2}}(\alpha^2 + \mu^2)^{\frac{1}{2}}} \right),
$$

(4.3)

where

$$
\psi'_\alpha(\lambda, \mu) := \frac{1}{(\alpha^2 + \lambda^2)^{\frac{1}{2}} + (\alpha^2 + \mu^2)^{\frac{1}{2}}}.
$$

(4.4)

The corresponding resolution for $T_\alpha$ is given by

$$
T_\alpha(x) = T'_\alpha(x) + \alpha^2 \Delta_{-1}^{-1} T'_\alpha(x) \Delta_{-1}^{-1} - D \Delta_{-1}^{-1} T'_\alpha(x) D \Delta_{-1}^{-1}, \quad x \in L^\infty,
$$

(4.5)

where we have put for brevity $T'_\alpha = T'_{\psi_\alpha}(D, D)$. Note that $\alpha \|\Delta_{-1}^{-1}\|, \|D \Delta_{-1}^{-1}\| \leq 1$ for every $\alpha > 0$. Thus, the claim (4.2) will readily follow as soon as we establish that

$$
T'_\alpha \in B(L^\infty, E).
$$

(4.6)

By Lemma 8, we have

$$
T'_\alpha = T'_{\psi_\alpha}(D, D) = T_{\psi_0}(\Delta_{-1}, \Delta_{-1}) =: T_0,
$$

(4.7)

where

$$
\psi_0(\lambda, \mu) = \frac{1}{\lambda + \mu}, \quad \lambda, \mu > 0.
$$

Therefore, it is sufficient to show that

$$
T_0 \in B(L^\infty, E).
$$

(4.8)

Representing the function $\psi_0$ as

$$
\psi_0(\lambda, \mu) = \frac{1}{\lambda + \mu} = \lambda^{-\frac{1}{2}} \mu^{-\frac{1}{2}} \left( \frac{1}{\lambda} \right)^{\frac{1}{2}} + \left( \frac{1}{\mu} \right)^{\frac{1}{2}} =: \lambda^{-\frac{1}{2}} \mu^{-\frac{1}{2}} \psi(\lambda, \mu),
$$

(4.9)

and setting $T_\psi := T_{\psi}(\Delta_{-1}, \Delta_{-1})$, we obtain

$$
T_0(x) = \Delta_{-1}^{-\frac{1}{2}} T_\psi(x) \Delta_{-1}^{-\frac{1}{2}}, \quad x \in L^\infty.
$$

(4.10)

Note that in (4.10), we have used Theorem 1 which is applicable here since the functions $\lambda^{-\frac{1}{2}}$ and $\mu^{-\frac{1}{2}}$ are bounded on the spectrum of the operator $\Delta_{-1}$. By the assumption, $\Delta_{-1}^{-1} \in E$ and therefore, (4.8) follows from (4.10) via Lemma 10.
provided we know that \( T_\psi \in B(\mathcal{L}^\infty) \). The definition of the function \( \psi \) (see (4.9)) and Lemma 9 guarantee the latter embedding provided that \( \tilde{g} \in L^1 \), where the function \( g \) is given by

\[
(4.11) \quad g(t) = \frac{1}{e^{\frac{t^2}{2}} + e^{-\frac{t^2}{2}}}, \quad t \in \mathbb{R}.
\]

To see that \( \tilde{g} \in L^1 \) it is sufficient to observe that \( g, g' \in L^2 \) and apply Lemma 7.

**Remark 12.** For future use, we note that by Lemma 9, the functions \( \psi \) and \( g \) defined in (4.9) and (4.11) respectively satisfy the equality

\[
(4.12) \quad \psi(\lambda, \mu) = \int_{\mathbb{R}} \tilde{g}(s) \lambda^{is} \mu^{-is} ds, \quad \lambda, \mu > 0.
\]

Since the estimate in Theorem 11 is uniform with respect to \( \alpha > 0 \), letting \( \alpha \to 0 \) and noting that the left hand side of the estimate in Theorem 11 tends to \( [\text{sgn } D, a] \) in the weak operator topology proves Corollary 13.

**Corollary 13.** Let \( 1 \leq p \leq \infty \), \( |D|^{-1} \in \mathcal{L}^p \) and \( a \in \mathcal{M} \). If \( [D, a] \in \mathcal{L}^\infty \), then \( [\text{sgn } D, a] \in \mathcal{L}^p \) and there is a constant \( c > 0 \) such that

\[
||[\text{sgn } D, a]||_p \leq c |||D|^{-1}||_p ||[D, a]||.
\]

The latter result was proved for the setting of \( \mathcal{M} = B(\mathcal{H}) \) by elementary (and different) reasoning in [33] (see also the exposition of this result in [24, Lemma 10.18]).

Let us note that the argument in the proof of Theorem 11 also works for Lipschitz estimates. The precise statement follows.

**Theorem 14.** Let \( \alpha \geq 0 \), \( \Delta^{-1}_{0, \alpha}, \Delta^{-1}_{\alpha} \in \mathcal{E} \). If \( D - D_0 \in \mathcal{L}^\infty \), then \( D\Delta^{-1}_{0, \alpha} - D_0\Delta^{-1}_{\alpha} \in \mathcal{E} \) and there is a constant \( c > 0 \) independent of \( \alpha \) such that, for every \( 0 < \theta < 1 \),

\[
\|D\Delta^{-1}_{\alpha} - D_0\Delta^{-1}_{0, \alpha}\|_\mathcal{E} \leq c_\theta \|\Delta^{-1}_{0, \alpha}\|_\mathcal{E} \theta \|\Delta^{-1}_{\alpha}\|_\mathcal{E} \theta \|D - D_0\|,
\]

where

\[
(4.13) \quad c_\theta \leq c \max\left\{ \theta^{-\frac{1}{2}}, (1 - \theta)^{-\frac{1}{2}} \right\}.
\]

**Proof.** The proof is a repetition of that of Theorem 11. The only place which requires additional reasoning is the estimate (4.13). To this end, we shall estimate the norm of the operator \( T_0 \) from (4.7) differently. We slightly modify representation (4.9)

\[
\psi_0(\lambda, \mu) = \frac{1}{\lambda + \mu} = \lambda^{\theta - 1} \mu^{-\theta} \left( \frac{\lambda}{\mu} \right)^{\theta - 1} \left( \frac{\mu}{\lambda} \right)^{-\theta - 1} = : \lambda^{\theta - 1} \mu^{-\theta} \psi_0(\lambda, \mu).
\]

Note that

\[
T_0(x) = \Delta^{-1}_{0, \alpha} T_\psi(x) \Delta^{-\theta}_{\alpha}.
\]
where $T_\theta = T_{\psi}(\Delta_{0,\alpha}, \Delta_{\alpha})$. Suppose, that we know that $T_\theta \in B(\mathcal{L}_\infty)$, then Lemma 10 yields the implication
\[ T_0 \in B(\mathcal{L}_\infty, \mathcal{E}) \iff T_\theta \in B(\mathcal{L}_\infty) \]
and
\[ c_\theta \leq \|T_\theta\|_{B(\mathcal{L}_\infty)}. \]

Setting
\[ g(t) = \frac{1}{e^{\theta t} + e^{(\theta-1)t}}, \quad t \in \mathbb{R} \]
we have
\[ \|g\|_{L^2} + \|g'\|_{L^2} \leq c \max \left\{ \frac{\theta}{2}, (1 - \theta)^{-\frac{1}{2}} \right\}, \]
for some numerical constant $c > 0$, and therefore, by Lemmas 9 and 7 we indeed have $T_\theta \in B(\mathcal{L}_\infty)$ and
\[ \|T_\theta\|_{B(\mathcal{L}_\infty)} \leq c \sqrt{\pi} \max \left\{ \frac{\theta}{2}, (1 - \theta)^{-\frac{1}{2}} \right\}, \]
which yields (4.13).

Results given in Theorems 11 and 14 are based on the analysis of the function $\psi f_\alpha$, where $f_\alpha$ is given in (4.1). A similar analysis can be also performed for the function $\psi h_\alpha$, where
\[ h_\alpha(t) = \frac{1}{(\alpha^2 + t^2)^{\frac{1}{2}}}, \quad t \in \mathbb{R}. \]

**Theorem 15.** Let $\alpha > 0$, $\Delta_0^{-1}$, $\Delta_\alpha^{-1} \in \mathcal{E}$. If $D - D_0 \in \mathcal{L}_\infty$, then $\Delta_0^{-1} - \Delta_\alpha^{-1} \in \mathcal{E}$ and, for every $0 < \theta < 1$,
\[ \|\Delta_0^{-1} - \Delta_\alpha^{-1}\|_{\mathcal{E}} \leq c_\theta \alpha^{-1} \|\Delta_0^{-1}\|_{\mathcal{E}}^{1-\theta} \|\Delta_\alpha^{-1}\|_{\mathcal{E}}^{\theta} \|D - D_0\|, \]
where $c_\theta$ satisfies (4.13) with some constant $c > 0$ independent of $\alpha$ and $\theta$.

**Proof.** We have
\[ \psi_{h_\alpha}(\lambda, \mu) = \frac{\lambda + \mu}{(\alpha^2 + \lambda^2)^{\frac{1}{2}} (\alpha^2 + \mu^2)^{\frac{1}{2}}} \psi'_\alpha(\lambda, \mu), \]
where $\psi'_\alpha$ is given in (4.4). Note that
\[ \left| \frac{\lambda + \mu}{(\alpha^2 + \lambda^2)^{\frac{1}{2}} (\alpha^2 + \mu^2)^{\frac{1}{2}}} \right| \leq \alpha^{-1}, \quad \lambda, \mu \in \mathbb{R} \]
and therefore, by Theorem 1
\[ \|T_{\psi_{h_\alpha}}\|_{B(\mathcal{L}_\infty, \mathcal{E})} \leq \alpha^{-1} \|T_{\psi'_\alpha}\|_{B(\mathcal{L}_\infty, \mathcal{E})}. \]
The claim now follows from (4.6) and the proof of Theorem 14.

Theorems 14 and 15 require that $\|\Delta_0^{-1}\|_{\mathcal{E}}, \|\Delta_\alpha^{-1}\|_{\mathcal{E}} < +\infty$. We shall next relax this hypothesis.
Theorem 16. Let \( \alpha > 0, \Delta_{0,\alpha}^{-1} \in \mathcal{E} \). If \( D - D_0 \in L^\infty \), then \( \Delta_{\alpha}^{-1} - \Delta_{0,\alpha}^{-1} \in \mathcal{E} \) and there is a constant \( c > 0 \) independent of \( \alpha \) such that
\[
\| \Delta_{\alpha}^{-1} - \Delta_{0,\alpha}^{-1} \| \leq c \max\{1, \alpha^{-1}\} \| \Delta_{0,\alpha}^{-1} \| \| D - D_0 \|.
\]

Proof. Let us first assume that \( \| D - D_0 \| \leq 1 \). We set
\[
A := \frac{\| \Delta_{\alpha}^{-1} - \Delta_{0,\alpha}^{-1} \| \| \mathcal{E} \|}{\| \Delta_{0,\alpha}^{-1} \| \| \mathcal{E} \|}.
\]
It follows from Theorem 15 that
\[
(4.15) \quad \| \Delta_{\alpha}^{-1} - \Delta_{0,\alpha}^{-1} \| \leq c^\theta \alpha^{-1} \| \Delta_{0,\alpha}^{-1} \| \theta \| \Delta_{\alpha}^{-1} \| \theta \| D - D_0 \|.
\]
On the other hand, it follows from triangle inequality that
\[
(4.16) \quad \| \Delta_{\alpha}^{-1} \| \leq \| \Delta_{0,\alpha}^{-1} \| + \| \Delta_{\alpha}^{-1} - \Delta_{0,\alpha}^{-1} \|.
\]
Replacing \( \| \Delta_{\alpha}^{-1} \| \) on the right in (4.15) with the right-hand side of (4.16) and applying the following standard inequality
\[
(1 + x)^\theta \leq 1 + \theta x, \quad \theta \leq 1, \quad x \geq 0
\]
yields that
\[
(4.17) \quad A \leq c^\theta \alpha^{-1} \| D - D_0 \| (1 + \theta A).
\]
Fix \( \theta = \min \left\{ \frac{1}{4}, \frac{\alpha^2}{4} \right\} \). Since \( \| D - D_0 \| \leq 1 \) it readily follows from (4.15) that
\[
c^\theta \alpha^{-1} \| D - D_0 \| \leq \frac{1}{2}.
\]
We let \( c_\alpha = \frac{c^\theta}{2} \). It follows from (4.15) that
\[
c_\alpha \leq c \max\{1, \alpha^{-1}\}, \quad \text{for some } c > 0.
\]
It is now clear that (4.17) implies
\[
\frac{\| \Delta_{\alpha}^{-1} - \Delta_{0,\alpha}^{-1} \| \| \mathcal{E} \|}{\| \Delta_{0,\alpha}^{-1} \| \| \mathcal{E} \|} = A \leq c_\alpha \| D - D_0 \|.
\]
The latter inequality finishes the proof of the theorem in the case \( \| D - D_0 \| \leq 1 \).

The case \( \| D - D_0 \| \geq 1 \) is reduced to the setting above by considering the triple
\[
\frac{D}{\| D - D_0 \|}, \quad \frac{D_0}{\| D - D_0 \|}, \quad \frac{\alpha}{\| D - D_0 \|}.
\]
Thus, the theorem is proved.

Theorem 16 considerably improves [5, Appendix B, Proposition 10], where, for the special case \( \mathcal{E} = L^p \), \( 1 < p < \infty \), the authors prove the Hölder estimate, i.e. that
\[
\| \Delta_{\alpha}^{-1} - \Delta_{0,\alpha}^{-1} \| \leq c \| \Delta_{0,\alpha}^{-1} \| \| D - D_0 \| \frac{1}{2},
\]
provided \( \| D - D_0 \| \leq 1 \).

Finally, using Theorem 16, we can improve Theorem 14.
Theorem 17. Let $\alpha > 0$ and $\Delta^{-1}_{0,\alpha} \in \mathcal{E}$. If $\|D - D_0\| \leq 1$, then $D\Delta^{-1}_{\alpha} - D_0\Delta^{-1}_{0,\alpha} \in \mathcal{E}$ and there is a constant $c > 0$ independent of $\alpha$ such that

$$
\|D\Delta^{-1}_{\alpha} - D_0\Delta^{-1}_{0,\alpha}\|_\mathcal{E} \leq c \max\{1, \alpha^{-\frac{1}{2}}\} \|\Delta^{-1}_{0,\alpha}\|_\mathcal{E} \|D - D_0\|.
$$

Proof. It follows from Theorem 14 that

$$
\|D\Delta^{-1}_{\alpha} - D_0\Delta^{-1}_{0,\alpha}\|_\mathcal{E} \leq c' \|\Delta^{-1}_{0,\alpha}\|_\mathcal{E} \|D - D_0\|, \quad c' > 0.
$$

On the other hand, it follows from the assumption $\|D - D_0\| \leq 1$, Theorem 16 and triangle inequality that

$$
\|\Delta^{-1}_{\alpha}\|_\mathcal{E} \leq \|\Delta^{-1}_{0,\alpha}\|_\mathcal{E} (1 + c'' \max\{1, \alpha^{-1}\}), \quad c'' > 0.
$$

Replacing $\|\Delta^{-1}_{\alpha}\|_\mathcal{E}$ on the right in (4.18) with the right-hand side of the latter inequality, we arrive at

$$
\|D\Delta^{-1}_{\alpha} - D_0\Delta^{-1}_{0,\alpha}\|_\mathcal{E} \leq c \max\{1, \alpha^{-\frac{1}{2}}\} \|\Delta^{-1}_{0,\alpha}\|_\mathcal{E} \|D - D_0\|,
$$

for some $c > 0$. The claim of the theorem is proved.

5 An application to the weak $L^p$ spaces

In view of its relevance to the definition of the Dixmier trace the weak $L^p$-space $\mathcal{L}^{p,\infty}$ has come to play an important role in noncommutative geometry. For this reason we describe in this Section some consequences of our methods for this space. We note that by specialising we obtain sharper estimates (it is possible with additional effort to establish weaker analogous results for more general ideals $\mathcal{E}$).

We shall improve Theorem 11 in the special setting of $\mathcal{L}^{p,\infty}$. For the sake of brevity, we shall denote the norm in the latter space by $\|\cdot\|_{p,\infty}, 1 \leq p < \infty$. Recall, that the latter norm is given by

$$
\|x\|_{p,\infty} = \sup_{t \geq 0} t^{\frac{1}{p}} \mu_t(x), \quad x \in \mathcal{L}^{p,\infty}.
$$

We refer for further detailed discussion of properties of the weak $L^p$-spaces to [4,5,36].

Theorem 18. Let $1 \leq p < \infty$, $r > 1$, $\Delta^{-1} \in \mathcal{L}^{p,\infty}$ and $a \in \mathcal{M}$. If $[D, a] \in \mathcal{L}^{\infty}$, then $[D\Delta^{-r}, a] \in \mathcal{L}^p$, and there is a constant independent of $r$ such that

$$
\|[D\Delta^{-r}, a]\|_p \leq c(p, r) \|[D, a]\|,
$$

where

$$
c(p, r) \leq c \max\{1, [p (r - 1)]^{-1/p} \|\Delta^{-1}\|_{p,\infty}^{\frac{r+1}{2}}\}.
$$

Proof. We shall modify the argument given in the proof of Theorem 11.

Fix $r > 1$. We also fix $\epsilon = \frac{1}{2}(r - 1) > 0$. Let us first note that, since the operator $\Delta^{-1}$ is bounded, it readily follows from the definition of $\mathcal{L}^{p,\infty}$ that

$$
\Delta^{-r+\epsilon} \in \mathcal{L}^p \quad \text{and} \quad \|\Delta^{-r+\epsilon}\|_p \leq c \max\{1, [p (r - 1)]^{-1/p} \|\Delta^{-1}\|_{p,\infty}^{r-\epsilon}\},
$$

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for some numerical constant \( c > 0 \). Note the following simple identities

\[
[D\Delta^{-r}, a] = D\Delta^{-\epsilon}[\Delta^{-r+\epsilon}, a] + [D\Delta^{-\epsilon}, a]\Delta^{-r+\epsilon}
\]

and

\[
[\Delta^{1-r}, a] = \Delta^{1-\epsilon}[\Delta^{-r+\epsilon}, a] + [\Delta^{1-\epsilon}, a]\Delta^{-r+\epsilon}.
\]

Combining these two together we arrive at the following equation

\[
[D\Delta^{-r}, a] = [D\Delta^{-\epsilon}, a]\Delta^{-r+\epsilon} + D\Delta^{-1}([\Delta^{1-r}, a] - [\Delta^{1-\epsilon}, a]\Delta^{-r+\epsilon}).
\]

Consequently, the claim of the theorem will follow if we show that

\[
(5.2) \quad [\Delta^{1-r}, a] \in \mathcal{L}_\infty \quad \text{and} \quad [\Delta^{1-\epsilon}, a] \in \mathcal{L}^p.
\]

The first claim in (5.2) follows from Theorem 3 which is applicable here since the functions

\[
f_1(t) = \frac{t}{(1 + t^2)^{\frac{1}{2}}} \quad \text{and} \quad f_2(t) = (1 + t^2)^{\frac{1-r}{2}}
\]

satisfy the assumptions of Theorem 3. To prove that

\[
(5.3) \quad [\Delta^{1-r}, a] \in \mathcal{L}^p, \quad r > 1.
\]

We consider an infinitely smooth function \( \chi_0(t) \) which is 1 when \( t \in [-1, 1] \); and 0 when \( t \notin [-2, 2] \). We set \( \chi_1 = 1 - \chi_0 \). Let \( f_c(t) = t^{1-r}, t \in \mathbb{R} \) and take the representation

\[
\psi_{f_c}(\lambda, \mu) = \chi_0 \left( \log \frac{\lambda}{\mu} \right) \frac{\mu^{1-r} \left( 1 - \left( \frac{\lambda}{\mu} \right)^{1-r} \right)}{\mu \left( 1 - \frac{\lambda}{\mu} \right)}
+ \chi_1 \left( \log \frac{\lambda}{\mu} \right) \frac{\lambda^{1-r} - \mu^{1-r}}{\lambda^{\frac{1}{2}} \mu^{\frac{1}{2}} \left( \frac{1}{\mu} - \frac{1}{\lambda} \right)^{\frac{1}{2}}}
\]

\[
= \mu^{-r} \chi_0 \left( \log \frac{\lambda}{\mu} \right) \frac{1 - \left( \frac{\lambda}{\mu} \right)^{1-r}}{1 - \frac{1}{\mu}}
+ \chi_1 \left( \log \frac{\lambda}{\mu} \right) \frac{\lambda^{-\frac{1}{2}} \mu^{\frac{1}{2} - r}}{\left( \frac{1}{\mu} \right)^{\frac{1}{2}} - \left( \frac{1}{\lambda} \right)^{\frac{1}{2}}}
\]

\[
(5.4)
\]

Noting that the functions

\[
\chi_0(t) \frac{1 - e^{(1-r)t}}{1 - e^t} \quad \text{and} \quad \frac{\chi_1(t)}{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}
\]

and their first derivatives belong to \( L^2(\mathbb{R}) \), we now infer \( 5.3 \) from Lemmas 9, 7 and 10 and Theorem 3.
6 Applications to Fredholm modules and spectral flow

Let $(\mathcal{M}, \tau)$ be a semi-finite von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ with a n.s.f. trace $\tau$ and let $\mathcal{E} = E(\mathcal{M}, \tau)$ be noncommutative symmetric space. Let $\mathcal{A}$ be a unital Banach $\ast$-algebra which is represented in $\mathcal{M}$ via a continuous faithful $\ast$-homomorphism $\pi$. We shall identify the algebra $\mathcal{A}$ with its representation $\pi(\mathcal{A})$. In ‘semifinite noncommutative geometry’ one studies the following objects (see [4–6, 14, 15, 33, 36]).

Definition 19. An odd $\mathcal{E}$-summable semifinite spectral triple for $\mathcal{A}$, is given by a triple $(\mathcal{M}, D_0, \mathcal{A})$, where $D_0$ is an unbounded self-adjoint operator affiliated with $\mathcal{M}$ such that

(i) $(1 + D_0^2)^{-\frac{1}{2}}$ belongs to $\mathcal{E}$;

(ii) the subspace $\mathcal{A}_0$ given by

$$\mathcal{A}_0 := \{ a \in \mathcal{A} : [D_0, a] \in \mathcal{M} \}$$

is a dense $\ast$-subalgebra of $\mathcal{A}$.

Definition 20. An odd $\mathcal{E}$-summable bounded pre-Breuer-Fredholm module for $\mathcal{A}$, is given by a triple $(\mathcal{M}, F_0, \mathcal{A})$, where $F_0$ is a bounded self-adjoint operator in $\mathcal{M}$ such that

(i) $|1 - F_0^2|^{\frac{1}{2}}$ belongs to $\mathcal{E}$;

(ii) the subspace $\mathcal{A}_E$ given by

$$\mathcal{A}_E := \{ a \in \mathcal{A} : [F_0, a] \in \mathcal{E} \}$$

is a dense $\ast$-subalgebra of $\mathcal{A}$.

If $1 - F_0^2 = 0$, the prefix “pre-” is dropped.

Corollary 21. If $(\mathcal{M}, D_0, \mathcal{A})$ is an odd semifinite $\mathcal{E}$-summable spectral triple then $(\mathcal{M}, F_0, \mathcal{A})$, where $F_0 = D_0(1 + D_0^2)^{-\frac{1}{2}}$ is an odd bounded $\mathcal{E}$-summable pre-Breuer-Fredholm module. Furthermore, there is a constant $c = c(F_0)$ such that

$$\| F - F_0 \|_\mathcal{E} \leq c \| D - D_0 \|,$$

for all $D - D_0 \in \mathcal{M}, \| D - D_0 \| \leq 1$.

Proof. A straightforward application of Theorem 11 shows that for an arbitrary $\mathcal{A}$, we have $\mathcal{A}_0 \subseteq \mathcal{A}_E$. An application of Theorem 17 implies (6.1).

The first assertion of Corollary was also proved in [4] (see also [36]) in the special case when $\mathcal{E} = L^p, 1 < p < \infty$ (see [4, Theorem 0.3.(i)]) or when $\mathcal{E}$ is an interpolation space for a couple $(L^p, L^q), 1 < p \leq q < \infty$ (see [4, Corollary 0.5]). However, the methods employed in [4], [36] and [5, Sections A and B] do not extend to an arbitrary operator space $\mathcal{E}$ and more importantly, they do not yield the Lipschitz estimate (6.1), but only Hölder estimates. We now show that our methods not only yield the Lipschitz continuity of the mapping $(\mathcal{M}, D_0) \mapsto (\mathcal{M}, F_0)$ but also its differentiability.

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Theorem 22. Let \( \{D_t\}_{t \in \mathbb{R}} \) be a collection of self-adjoint linear operators affiliated with \( \mathcal{M} \). If \((1 + D_t^2)^{-\frac{\theta}{2}}\) belongs to \( \mathcal{E} \) and if \( t \mapsto D_t \) is an operator-norm differentiable path at the point \( t = 0 \), i.e. \( D_t - D_0 \in \mathcal{M}, t \in \mathbb{R} \) and there is an operator \( G \in \mathcal{M} \) such that

\[
\lim_{t \to 0} \left\| \frac{D_t - D_0}{t} - G \right\| = 0,
\]

then the path \( t \mapsto F_t := D_t (1 + D_t^2)^{-\frac{\theta}{2}} \) is \( \mathcal{E} \)-differentiable at the point \( t = 0 \), i.e. \( F_t - F_0 \in \mathcal{E} \) and there is an operator \( H \in \mathcal{E} \) such that

\[
\lim_{t \to 0} \left\| \frac{F_t - F_0}{t} - H \right\| \leq 0.
\]

Moreover, \( H = T_{\psi_f} (G) \), where \( T_{\psi_f} = T_{\psi_f} (D_0, D_0) \) and

\[
f(t) = \frac{t}{(1 + t^2)^{\frac{\theta}{2}}}.
\]

Proof. We shall use the argument from the proof of Theorems 11, 14 and 17 in the special case when \( \alpha = 1 \) and \( \theta = \frac{1}{2} \). We let \( c \) stand for a positive constant which may vary from line to line. We set \( \Delta_t := (1 + D_t^2)^{\frac{\theta}{2}} \). We start again with the identity (see Theorem 3)

\[
F_t - F_0 = T_{\psi_f} (D_t - D_0), \quad t \in \mathbb{R},
\]

where \( T_{\psi_f} = T_{\psi_f} (D_t, D_0) \). An inspection of the proof of Theorem 11 (see formulae (4.5), (4.10) and (4.12)) shows that

\[
T_{\psi_f} (x) = \Delta_t^{-\frac{\theta}{2}} T_0 (x) \Delta_0^{-\frac{\theta}{2}}, \quad x \in \mathcal{L}^\infty,
\]

where

\[
T_0 (x) = T_0' (x) + \Delta_t^{-1} T_0' (x) \Delta_0^{-1} - D_t \Delta_t^{-1} T_0' (x) D_0 \Delta_0^{-1},
\]

and where \( T_0' = T_{\psi_0} (D_0, D_0) \), \( t \in \mathbb{R} \),

\[
\phi'(\lambda, \mu) = \frac{(1 + \lambda^2)^{\frac{\theta}{2}} (1 + \mu^2)^{\frac{\theta}{2}}}{(1 + \lambda^2)^{\frac{\theta}{2}} + (1 + \mu^2)^{\frac{\theta}{2}}} = \int_\mathbb{R} h(s) \left( 1 + \lambda^2 \right)^{\frac{\theta}{2}} \left( 1 + \mu^2 \right)^{-\frac{\theta}{2}} ds,
\]

for some \( h \in L^1 (\mathbb{R}) \). The last equality in (6.7) follows from Lemma 4 (a further inspection shows that \( h = \tilde{g} \), where \( g \) is given (4.11)). It is clear from (6.7) and Corollary 2 that the operator \( T_0' \in B(\mathcal{L}^\infty) \) uniformly for \( t \in \mathbb{R} \). Thus, by \( \text{(6.6)} \) the operator \( T_t \in B(\mathcal{L}^\infty) \) uniformly for \( t \in \mathbb{R} \).

Our first objective is to prove that

\[
\lim_{t \to 0} \| T_t (x) - T_0 (x) \| = 0, \quad x \in \mathcal{L}^\infty.
\]

However, instead of proving \( \text{(6.8)} \) we shall show a stronger result. Let \( \{D'_t\}_{t \in \mathbb{R}} \) be another collection of linear self-adjoint operators affiliated with \( \mathcal{M} \) and let \( T_{\psi_f} = T_{\psi_f} (D_t, D'_t), t \in \mathbb{R} \). If \( T_{t, s} = T_{\psi_0} (D_t, D'_s), t, s \in \mathbb{R} \), then, alongside with \( \text{(6.5)} \) and \( \text{(6.6)} \), we have

\[
T_{\psi_f} (x) = \Delta_t^{-\frac{\theta}{2}} T_{t, s} (x) \Delta_s^{-\frac{\theta}{2}},
\]
where
\begin{align*}
\bar{T}_{t,s}(x) &= \bar{T}_{t,s}(x) + \Delta_t^{-1} \dot{T}_{t,s}(x) \Delta_s^{-1} - D_t \Delta_t^{-1} \dot{T}_{t,s}(x) D'_t \Delta_s^{-1} \\
\Delta'_t &= (1 + (D'_t)^2)^{\frac{1}{2}}.
\end{align*}

We shall show
\begin{equation}
\lim_{t \to 0} \| \bar{T}_{t,s}(x) - \bar{T}_{0,s}(x) \| = 0, \quad x \in \mathcal{L}^\infty
\end{equation}
which in particular implies (6.8). It obviously follows from (6.2) that
\begin{equation}
\lim_{t \to 0} \| D_t - D_0 \| = 0.
\end{equation}

Consequently, it readily follows from Theorems 16 and 17 that
\begin{equation}
\lim_{t \to 0} \| \Delta^{-1}_t - \Delta^{-1}_0 \| = \lim_{t \to 0} \| D_t \Delta^{-1}_t - D_0 \Delta^{-1}_0 \| = 0.
\end{equation}

Combining the latter with (6.10), it is seen that to show (6.11), we need only to prove that
\begin{equation}
\lim_{t \to 0} \| \bar{T}_{t,t}(x) - \bar{T}_{0,t}(x) \| = 0, \quad x \in \mathcal{L}^\infty.
\end{equation}

For the latter, fix \( \epsilon > 0 \). We also fix \( \delta > 0 \) such that \( \| D_t - D_0 \| < \epsilon \) for every \( |t| < \delta \). It is clear that there is \( s_0 > 0 \) and the function
\begin{equation}
\phi''(\lambda, \mu) = \int_{|s| \leq s_0} h(s) (1 + \lambda^2)^{\frac{1}{2}} (1 + \mu^2)^{-\frac{1}{2}} ds
\end{equation}
such that
\begin{equation}
\| \phi' - \phi'' \|_{A_0} < \epsilon.
\end{equation}

Furthermore, for every fixed \( |s| \leq s_0 \), the function
\begin{equation}
f_s(t) = (1 + t^2)^{\frac{1}{2}}
\end{equation}
satisfies Theorem 3 with constants depending only on \( s_0 \) and therefore, by Corollary 2 and Theorem 4
\begin{equation}
\| \Delta'^{is}_t - \Delta'^{is}_0 \| \leq c \| D_t - D_0 \|, \quad |s| \leq s_0.
\end{equation}

Let us show the identity
\begin{align*}
\bar{T}_{t,t}(x) - \bar{T}_{0,t}(x) &= (\bar{T}_{t,t}(x) - T''_{t,t}(x)) + (T''_{0,t}(x) - T''_{0,t}(x)) \\
&\quad + \int_{|s| \leq s_0} h(s) (\Delta'^{is}_t - \Delta'^{is}_0) x (\Delta'_t)^{is} ds,
\end{align*}
where \( T''_{t,s} = T''_{\phi''}(D_t, D'_s) \), \( t, s \in \mathbb{R} \). Fix \( x \in \mathcal{L}^2 \cap \mathcal{E} \) and \( y \in \mathcal{L}^2 \cap \mathcal{E}' \). We set
\begin{equation}
d\nu_{t,s} = \tau(y dE_{t,\lambda} x dE_{s,\mu}), \quad t, s \in \mathbb{R},
\end{equation}
where
where $dE_{t, \lambda}$ and $dE'_{s, \mu}$ are the spectral measure of the operator $D_t$ and $D'_s$, respectively. By (3.11) we have

$$\tau(y T'_{t,t}(x)) - \tau(y T''_{t,t}(x)) = \int_{\mathbb{R}^2} \phi' \, d\nu_{t,t} - \int_{\mathbb{R}^2} \phi' \, d\nu_{0,t}$$

$$= \int_{\mathbb{R}^2} (\phi' - \phi'') \, d\nu_{t,t} + \int_{\mathbb{R}^2} (\phi' - \phi'') \, d\nu_{0,t}$$

$$\quad + \int_{\mathbb{R}^2} \phi'' \, d\nu_{t,t} - \int_{\mathbb{R}^2} \phi'' \, d\nu_{0,t}.$$  

(6.18)

Replacing $\phi''$ with (6.14) and Fubini’s theorem yield for the last term

$$\int_{\mathbb{R}^2} \phi'' \, d\nu_{t,t} - \int_{\mathbb{R}^2} \phi'' \, d\nu_{0,t}$$

$$= \int_{|s| \leq s_0} h(s) \, ds \left[ \int_{\mathbb{R}^2} (1 + \lambda^2) \frac{d\phi''}{2} (1 + \mu^2)^{-\frac{1}{2}} \, d(\nu_{t,t} - \nu_{0,t}) \right]$$

$$= \int_{|s| \leq s_0} h(s) \, ds \left[ \tau(y \Delta^i_\lambda x (\Delta'_t)^{-i}) - \tau(y \Delta^i_\lambda x (\Delta'_0)^{-i}) \right].$$

Here, we used the spectral theorem as follows

$$\int_{\mathbb{R}^2} (1 + \lambda^2) \frac{d\phi''}{2} (1 + \mu^2)^{-\frac{1}{2}} \, d\nu_{t,s}$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} (1 + \lambda^2) \frac{d\phi''}{2} (1 + \mu^2)^{-\frac{1}{2}} \, \tau(y \, dE_{t,\lambda} x \, dE'_{s,\mu})$$

$$= \tau \left[ y \int_{\mathbb{R}} (1 + \lambda^2) \frac{d\phi''}{2} \, dE_{t,\lambda} x \int_{\mathbb{R}} (1 + \mu^2)^{-\frac{1}{2}} \, dE'_{s,\mu} \right] = \tau(y \Delta^i_\lambda x (\Delta'_t)^{-i}).$$

Hence, combining (6.19) with (6.18) yields (6.17).

Estimating the first two terms in (6.17) with (6.15) and the last one with (6.16) yields

$$\| T'_{t,t}(x) - T''_{t,t}(x) \| \leq c (2 + c \| h \|_{L^1}) \| x \|,$$

provided $|t| < \delta$. The latter finishes the proof of (6.13).

Our next objective is to prove that

$$\lim_{t \to 0} \| \Delta^{\frac{1}{2}}_t - \Delta^{\frac{1}{2}}_0 \|_{\mathcal{E}(x)} = 0,$$

(6.20)

where $\mathcal{E}(2)$ is the 2-convexification of $\mathcal{E}$ (see e.g. [25]), i.e.

$$\mathcal{E}(2) := \{ x \in \mathcal{M} : \| x \|^2 \in \mathcal{E} \} \text{ and } \| x \|_{\mathcal{E}(2)} := \| \| x \|^2 \|_{\mathcal{E}}^{\frac{1}{2}}.$$
Here, the first inequality follows from the simple observation that
\[ \Delta_t^{-\frac{1}{2}} - \Delta_0^{-\frac{1}{2}} = -\Delta_t^{-\frac{1}{2}} (\Delta_t^{\frac{1}{2}} - \Delta_0^{\frac{1}{2}}) \Delta_0^{\frac{1}{2}}. \]
The last inequality follows from Theorem 3 and the fact that the function
\[ h(t) = (1 + t^2)^{\frac{1}{2}} \]
satisfies Theorem 3.

Now we can finish the proof of the theorem. We set \( H = T_{\psi t}(G) = \Delta_0^{-\frac{1}{2}} T_0(G) \Delta_0^{\frac{1}{2}} \). It follows from (6.4), that
\[
\frac{F_t - F_0}{t} - H = \Delta_t^{-\frac{1}{2}} T_t \left( \frac{D_t - D_0}{t} \right) \Delta_0^{\frac{1}{2}} - \Delta_0^{-\frac{1}{2}} T_0(G) \Delta_0^{\frac{1}{2}}
= (\Delta_t^{-\frac{1}{2}} - \Delta_0^{-\frac{1}{2}}) T_t \left( \frac{D_t - D_0}{t} \right) \Delta_0^{\frac{1}{2}}
+ \Delta_0^{-\frac{1}{2}} T_t \left( \frac{D_t - D_0}{t} - G \right) \Delta_0^{\frac{1}{2}}
+ \Delta_0^{-\frac{1}{2}} (T_t(G) - T_0(G)) \Delta_0^{\frac{1}{2}}.
\]
Note that the operators \( T_t \) and \( \frac{D_t - D_0}{t} \) are uniformly bounded for every \( t \in \mathbb{R} \) in the spaces \( B(\mathcal{E}^\infty) \) and \( \mathcal{E}^\infty \), respectively (see remarks following (6.7)). Therefore, when \( t \to 0 \), the first term vanishes in \( \mathcal{E} \) due to (6.20) and generalized H"older inequality \( \|x y\| \leq \|x\| \|y\| \) \( x, y \in \mathcal{E}(^2) \); the second term vanishes in \( \mathcal{E} \) due to (6.22) and Lemma 10 and the last one does the same thanks to (6.8) and Lemma 10. The theorem is proved.

The result above is of importance for the spectral flow theory, for which we refer to [1, 5, 6]. In that theory, given an odd \( \mathcal{E} \)-summable spectral triple (respectively, bounded \( \mathcal{E} \)-summable pre-Breuer-Fredholm module, \((M, F_0, A)) \) one introduces an associated affine space \( \Phi_\mathcal{E} := \{ D = D_0 + A \mid A = A^* \in M \} \) (respectively, \( \mathfrak{M}_\mathcal{E} := \{ F = F_0 + A \mid A = A^* \in \mathcal{E} \} \)). To compare the spectral flow along paths of self-adjoint Breuer-Fredholm operators in \( \mathfrak{M}_\mathcal{E} \) and self-adjoint bounded operators in \( \Phi_\mathcal{E} \) it is important to know that the transformation from spectral triples \((M, D_0, A)\) to bounded \( \mathcal{E} \)-summable modules \((M, F_0, A)\) via the map \( F_D = D(1 + D^2)^{-1/2} \) carries \( C^1 \) paths to \( C^1 \) paths (see e.g. [6, Section 6]). In concrete examples, proving the smoothness of this map is difficult and such a difficulty has led to extra technical assumptions imposed in [5, 6] on the triple \((M, D_0, A)\). The result below removes all such assumptions.

**Theorem 23.** Let \( \{D_t\}_{t \in \mathbb{R}} \) be a collection of self-adjoint linear operators affiliated with \( \mathcal{M} \) such that \((1 + D_0^2)^{-\frac{1}{2}} \) belongs to \( \mathcal{E} \). Let \( F_t := D_t (1 + D_t^2)^{-\frac{1}{2}} \), \( t \in \mathbb{R} \). If \( D_t - D_{t_0} \in \mathcal{E}^\infty \), \( t, t_0 \in \mathbb{R} \), the limit
\[
\frac{dD_t}{dt}(t_0) := \| \cdot \| \lim_{t \to t_0} \frac{D_t - D_{t_0}}{t - t_0}, \; t_0 \in \mathbb{R}
\]
exists and the mapping \( t \mapsto \frac{dD_t}{dt}(t_0) \) is operator norm continuous, then \( F_t - F_{t_0} \in \mathcal{E} \), the limit
\[
\frac{dF_t}{dt}(t_0) := \| \cdot \| \lim_{t \to t_0} \frac{F_t - F_{t_0}}{t - t_0}
\]
exists and the mapping \( t \mapsto \frac{dF_t}{dt}(t_0) \) is \( \mathcal{E} \)-continuous.
Proof. The existence of the derivative \( \frac{dF_t}{dt} \) follows from Theorem 22. We need only to show that the derivative \( \frac{dF_t}{dt} \) is \( \mathcal{E} \)-continuous. Clearly, it is sufficient to show the continuity at \( t = 0 \), i.e. we need to show that
\[
\lim_{s \to 0} \left\| \frac{dF_t}{dt}(s) - \frac{dF_t}{dt}(0) \right\|_{\mathcal{E}} = 0.
\]
It follows from Theorem 22 that
\[
\frac{dF_t}{dt}(s) = T_{\psi_f}(D_s, D_s) \frac{dD_t}{dt}(s), \quad s \in \mathbb{R},
\]
where \( f \) from (6.3). Consequently, we have
\[
\frac{dF_t}{dt}(s) - \frac{dF_t}{dt}(0) = T_{\psi_f}(D_s, D_s) \left[ \frac{dD_t}{dt}(s) - \frac{dD_t}{dt}(0) \right]
\]
\[
+ T_{\psi_f}(D_s, D_s) \frac{dD_t}{dt}(0) - T_{\psi_f}(D_s, D_0) \frac{dD_t}{dt}(0)
\]
\[
\frac{dF_t}{dt}(s) - \frac{dF_t}{dt}(0) = T_{\psi_f}(D_s, D_0) \frac{dD_t}{dt}(0) - T_{\psi_f}(D_0, D_0) \frac{dD_t}{dt}(0).
\]
(6.21)

By (6.12) and Theorem 11 the operator \( T_{\psi_f}(D_s, D_s) \) is bounded in \( B(\mathcal{L}_\infty, \mathcal{E}) \) and there are constants \( c > 0 \) and \( \delta > 0 \), such that
\[
\| T_{\psi_f}(D_s, D_s) \|_{B(\mathcal{L}_\infty, \mathcal{E})} \leq c, \quad |s| < \delta.
\]
Thus, the first term in (6.21) vanishes in \( \mathcal{E} \) as \( t \to 0 \) since \( \frac{dD_t}{dt} \) is operator norm continuous. On the other hand, the last two terms in (6.21) vanish in \( \mathcal{E} \) when \( t \to 0 \) thanks to (6.3), (6.11), (6.12), and (6.20).

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