Homogenization of the higher-order Schrödinger-type equations with periodic coefficients

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Abstract

In $L^2(\mathbb{R}^d; \mathbb{C}^n)$, we consider a matrix strongly elliptic differential operator $A_\varepsilon$ of order $2p$, $p \geq 2$. The operator $A_\varepsilon$ is given by $A_\varepsilon = b(D)^* g(x/\varepsilon) b(D)$, $\varepsilon > 0$, where $g(x)$ is a periodic, bounded, and positive definite matrix-valued function, and $b(D)$ is a homogeneous differential operator of order $p$. We prove that, for fixed $\tau \in \mathbb{R}$ and $\varepsilon \to 0$, the operator exponential $e^{-i\tau A_\varepsilon}$ converges to $e^{-i\tau A^0}$ in the norm of operators acting from the Sobolev space $H^s(\mathbb{R}^d; \mathbb{C}^n)$ (with a suitable $s$) into $L^2(\mathbb{R}^d; \mathbb{C}^n)$. Here $A^0$ is the effective operator. Sharp-order error estimate is obtained. The results are applied to homogenization of the Cauchy problem for the Schrödinger-type equation $i\partial_\tau u_\varepsilon = A_\varepsilon u_\varepsilon + F$, $u_\varepsilon|_{\tau=0} = \phi$.

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Introduction

The paper concerns homogenization theory of periodic differential operators (DOs). First of all, we mention the books [1], [20].

0.1 Operator error estimates

In a series of papers [2, 3, 4] by Birman and Suslina, an operator-theoretic (spectral) approach to homogenization problems was developed. In $L^2(\mathbb{R}^d; \mathbb{C}^n)$, a wide
class of matrix strongly elliptic second order DOs $A_\varepsilon$ was studied. The operator $A_\varepsilon$ is given by $A_\varepsilon = b(D)^* g(x/\varepsilon) b(D)$, $\varepsilon > 0$, where $g(x)$ is a bounded and positive definite $(m \times m)$-matrix-valued function periodic with respect to some lattice $\Gamma \subset \mathbb{R}^d$, and $b(D) = \sum_{l=1}^d b_l D_l$ is a first order DO. Here $b_l$ are constant $(m \times n)$-matrices. It is assumed that $m \geq n$ and the symbol $b(\xi)$ has maximal rank.

In [2], it was shown that the resolvent $(A_\varepsilon + I)^{-1}$ converges in the operator norm in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to the resolvent of an effective operator $A^0$, and

$$
\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C \varepsilon. \quad (0.1)
$$

The effective operator is given by $A^0 = b(D)^* g^0 b(D)$, where $g^0$ is a constant positive matrix called the effective matrix. In [16], a similar result was obtained for the parabolic semigroup:

$$
\|e^{-\tau A_\varepsilon} - e^{-\tau A^0}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(\tau) \varepsilon, \quad \tau > 0. \quad (0.2)
$$

Estimates (0.1) and (0.2) are order-sharp. Such inequalities are called operator error estimates in homogenization theory.

A different approach to operator error estimates (the shift method) was developed by Zhikov and Pastukhova. In [19, 21, 22], estimates (0.1), (0.2) were obtained for the operators of acoustics and elasticity. Further results were discussed in a survey [23].

The operator error estimates for the Schrödinger-type and hyperbolic equations were studied in [5] and in the recent works [6, 7, 8, 11, 17]. In operator terms, the behavior of the operator-valued functions $e^{-i\tau A_\varepsilon}$, $\cos(\tau A_\varepsilon^{1/2})$, $A_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2})$, $\tau \in \mathbb{R}$, was investigated. It turned out that the nature of the results differed from the case of elliptic and parabolic equations: the type of the operator norm must be changed.

Let us dwell on the case of the operator exponential $e^{-i\tau A_\varepsilon}$. In [5], the following sharp-order estimate was proved:

$$
\|e^{-i\tau A_\varepsilon} - e^{-i\tau A^0}\|_{H^3(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |\tau|) \varepsilon. \quad (0.3)
$$

In [17, 6], it was shown that in the general case the result (0.3) is sharp both regarding the type of the operator norm and regarding the dependence of the estimate on $\tau$ (it is impossible to replace $(1 + |\tau|)$ on the right by $(1 + |\tau|)^{\alpha}$ with $\alpha < 1$). On the other hand, under some additional assumptions the result admits improvement:

$$
\|e^{-i\tau A_\varepsilon} - e^{-i\tau A^0}\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)^{1/2} \varepsilon. \quad (0.4)
$$

The operator-theoretic approach was applied to the higher-order operators $A_\varepsilon$ in [18, 10]. It was assumed that the operator $A_\varepsilon$ is given by

$$
A_\varepsilon = b(D)^* g(x/\varepsilon) b(D), \quad \text{ord} b(D) = p \geq 2, \quad \varepsilon > 0, \quad (0.5)
$$

where $g(x)$ is a periodic, bounded, and positive definite $(m \times m)$-matrix-valued function, and $b(D) = \sum_{\beta} b_\beta D^\beta$. Here $b_\beta$ are constant $(m \times n)$-matrices. It is assumed
that \( m \geq n \) and the symbol \( b(\xi) \) has maximal rank. In [18, 10], an estimate of the form (0.1) for such operators \( A_\varepsilon \) was obtained. A more accurate approximation for the resolvent of \( A_\varepsilon \) was found recently in [14, 15]. The shift method was applied to homogenization of the elliptic higher-order operators in the papers [12], [13] by Pastukhova.

### 0.2 Main results

In the present paper, the behavior of the operator exponential \( e^{-i\tau A_\varepsilon} \) for the operator \( A_\varepsilon \) of order \( 2p \) given by (0.5) is studied. Our main result is the following estimate:

\[
\|e^{-i\tau A_\varepsilon} - e^{-i\tau A^0}\|_{H^{2p+1}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon. \tag{0.6}
\]

Here \( A^0 = b(D)^* g(\xi) b(D) \) is the effective operator. By the interpolation with the obvious estimate \( \|e^{-i\tau A_\varepsilon} - e^{-i\tau A^0}\|_{L_2 \rightarrow L_2} \leq 2 \), we also obtain “intermediate” results:

\[
\|e^{-i\tau A_\varepsilon} - e^{-i\tau A^0}\|_{H^s(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(s) \left(1 + |\tau|\right)^{(s)/(2p+1)} \varepsilon^{s/(2p+1)}, \quad 0 \leq s \leq 2p + 1.
\]

Under some additional assumptions formulated in terms of the spectral characteristics of \( A = b(D)^* g(x) b(D) \) near the bottom of the spectrum, it is proved that

\[
\|e^{-i\tau A_\varepsilon} - e^{-i\tau A^0}\|_{H^{2p+2}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon^2. \tag{0.7}
\]

This means that the difference \( e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \) is of order \( O(\varepsilon^2) \) in a suitable norm. It should be noted that the imposed additional assumptions are valid automatically for a scalar operator \( A_\varepsilon \) (i.e., \( n = 1 \)) with real-valued coefficients. By the interpolation, we deduce

\[
\|e^{-i\tau A_\varepsilon} - e^{-i\tau A^0}\|_{H^s(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(s) (1 + |\tau|)^{(s)/(2p+2)} \varepsilon^{s/(p+1)}, \quad 0 \leq s \leq 2p + 2.
\]

In particular, for \( s = p + 1 \) we have

\[
\|e^{-i\tau A_\varepsilon} - e^{-i\tau A^0}\|_{H^{p+1}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)^{1/2} \varepsilon. \tag{0.8}
\]

This improves (0.6) regarding both the type of the norm and the dependence on \( \tau \).

We stress that, for the second order operators \( A_\varepsilon \), there is an analog of (0.8) (cf. (0.4)), but there is no analog of (0.7).

The above results are applied to homogenization of the Cauchy problem for the Schrödinger-type equation

\[
i \partial_\tau u_\varepsilon(x, \tau) = A_\varepsilon u_\varepsilon(x, \tau) + F(x, \tau), \quad u_\varepsilon(x, 0) = \phi(x).
\]
0.3 Method

We rely on the operator-theoretic approach. By the scaling transformation, the problem is reduced to the study of the operator $e^{-i\varepsilon A\tau}$. Next, using the Floquet-Bloch theory, we expand $A$ in the direct integral of the operators $A(k)$ acting in $L_2(\Omega; \mathbb{C}^n)$ and given by $b(D + k)^*g(x)b(D + k)$ with periodic boundary conditions; here $\Omega$ is the cell of the lattice $\Gamma$. Since $A(k)$ is an analytic operator family with compact resolvent, it can be studied by means of the analytic perturbation theory with respect to the one-dimensional parameter $t = |k|$. It turns out that only the spectral characteristics of $A(k)$ near the bottom of the spectrum are responsible for homogenization. It is convenient to study the family $A(k)$ in the framework of an abstract operator-theoretic scheme.

0.4 Plan of the paper

The paper consists of five sections. In Section 1, the abstract operator-theoretic method is developed. In Section 2, we introduce the class of operators $A$ acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and describe the direct integral expansion for $A$. Section 3 is devoted to application of the abstract results to the operator family $A(k)$. In Section 4, using the results of Section 3, we obtain approximations for the operator exponential of $A$. Section 5 is devoted to homogenization problems: we deduce approximations for the exponential $e^{-i\varepsilon A\tau}$ from the results of Section 4 and apply them to find approximations for the solutions of the Cauchy problem.

0.5 Notation

Let $H$ and $H_\ast$ be complex separable Hilbert spaces. By $(\cdot, \cdot)_H$ and $\| \cdot \|_H$ we denote the inner product and the norm in $H$, respectively; the symbol $\| \cdot \|_{H_\ast \to H_\ast}$ denotes the norm of a linear continuous operator acting from $H_\ast$ to $H$. Sometimes, we omit the indices. If $\mathcal{N}$ is a subspace of $H$, then $\mathcal{N}^\perp$ denotes its orthogonal complement. If $A$ is a closed linear operator in $H$, its domain and kernel are denoted by $\text{Dom}A$ and $\text{Ker}A$, respectively; $\sigma(A)$ stands for the spectrum of $A$.

The inner product and the norm in $\mathbb{C}^n$ are denoted by $(\cdot, \cdot)$ and $| \cdot |$, respectively, $1_n = 1$ is the unit $(n \times n)$-matrix. We denote $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $iD_j = \partial / \partial x_j$, $j = 1, \ldots, d$, $D = -i\nabla = (D_1, \ldots, D_d)$.

The class $L_2$ of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O} \subset \mathbb{R}^d$ is denoted by $L_2(\mathcal{O}; \mathbb{C}^n)$. The Sobolev classes of $\mathbb{C}^n$-valued functions in a domain $\mathcal{O}$ are denoted by $H^s(\mathcal{O}; \mathbb{C}^n)$. For $n = 1$, we write simply $L_2(\mathcal{O}), H^s(\mathcal{O})$, but sometimes we use such simple notation also for the spaces of vector-valued or matrix-valued functions.
1 Abstract operator-theoretic scheme

1.1 Polynomial nonnegative operator pencils

Let $\mathcal{H}$ and $\mathcal{H}_s$ be complex separable Hilbert spaces. Let $X(t)$ be a family of operators (a polynomial pencil) of the form

$$X(t) = \sum_{j=0}^{p} X_j t^j, \quad t \in \mathbb{R}, \quad p \in \mathbb{N}, \quad p \geq 2.$$ 

The operators $X(t), X_j$ act from $\mathcal{H}$ into $\mathcal{H}_s$. It is assumed that the operator $X_0$ is densely defined and closed, while the operator $X_p$ is defined on the whole $\mathcal{H}$ and bounded. In addition, we impose the following conditions.

**Condition 1.1.** For any $j = 0, \ldots, p$ and $t \in \mathbb{R}$, we have

$$\text{Dom} X(t) = \text{Dom} X_0 \subset \text{Dom} X_j \subset \text{Dom} X_p = \mathcal{H}.$$ 

**Condition 1.2.** For any $j = 0, \ldots, p - 1$ and $u \in \text{Dom} X_0$ we have

$$\|X_j u\|_{\mathcal{H}_s} \leq C_0 \|X_0 u\|_{\mathcal{H}_s},$$

where a constant $C_0 \geq 1$ is independent of $j$ and $u$.

Under the above assumptions, the operator $X(t)$ is closed for $|t| \leq (2(p - 1)C_0)^{-1}$.

Our main object is the following family of nonnegative selfadjoint operators in $\mathcal{H}$:

$$A(t) = X(t)^* X(t), \quad t \in \mathbb{R}, \quad |t| \leq (2(p - 1)C_0)^{-1}.$$ 

Denote $A(0) = X_0^* X_0 =: A_0$, $\mathcal{N} := \text{Ker} A_0 = \text{Ker} X_0$, and $\mathcal{N}_s := \text{Ker} X_0^*$. Let $P$ be the orthogonal projection of $\mathcal{H}$ onto the subspace $\mathcal{N}$, and let $P_s$ be the orthogonal projection of $\mathcal{H}_s$ onto the subspace $\mathcal{N}_s$.

**Condition 1.3.** Suppose that the point $\lambda_0 = 0$ is an isolated point of the spectrum of $A_0$, and $n := \dim \mathcal{N} < \infty$, $n \leq n_s := \dim \mathcal{N}_s \leq \infty$.

By $d^0$ we denote the distance from the point $\lambda_0 = 0$ to $\sigma(A_0) \setminus \{\lambda_0\}$. Let $F(t, h)$ be the spectral projection of the operator $A(t)$ corresponding to the interval $[0, h]$. We fix a positive number $\delta \leq \min\{d^0/36, 1/4\}$ and choose a number $t_0 > 0$ such that

$$t_0 \leq \delta^{1/2} C_1^{-1}, \quad \text{where} \quad C_1 = \max\{(p - 1)C_0, \|X_p\|\}. \quad (1.1)$$

Note that $t_0 \leq 1/2$. The operator $X(t)$ is automatically closed for $|t| \leq t_0$, because $t_0 \leq (2(p - 1)C_0)^{-1}$. According to [18, Proposition 3.10], for $|t| \leq t_0$ we have

$$F(t, \delta) = F(t, 3\delta), \quad \text{rank} F(t, \delta) = n.$$ 

This means that, for $|t| \leq t_0$, the operator $A(t)$ has exactly $n$ eigenvalues counting their multiplicities on the interval $[0, \delta]$, and the interval $(\delta, 3\delta)$ is free of the spectrum. We write $F(t) := F(t, \delta).$
1.2 Operators $Z, R, \text{ and } S$

Let $\mathcal{D} = \text{Dom} \ X_0 \cap \mathcal{N}^\perp$. Obviously, $\mathcal{D}$ is a Hilbert space with the inner product $(f_1, f_2)_\mathcal{D} = (X_0 f_1, X_0 f_2)_{\mathcal{S}_\gamma}$, $f_1, f_2 \in \mathcal{D}$.

Let $u \in \mathcal{S}_\gamma$. Consider the equation $X_0^*(X_0 \psi - u) = 0$ for $\psi \in \mathcal{D}$, which is understood in the weak sense:

$$
  (X_0 \psi, X_0 \zeta)_{\mathcal{S}_\gamma} = (u, X_0 \zeta)_{\mathcal{S}_\gamma}, \quad \forall \zeta \in \mathcal{D}.
$$

(1.2)

The right-hand side of (1.2) is an antilinear continuous functional of $\zeta \in \mathcal{D}$. Hence, by the Riesz theorem, there exists a unique solution $\psi \in \mathcal{D}$, and $\|X_0 \psi\|_{\mathcal{S}_\gamma} \leq \|u\|_{\mathcal{S}_\gamma}$.

Now, let $\omega \in \mathcal{N}$ and $u = -X_p \omega$. In this case, the solution of equation (1.2) is denoted by $\psi(\omega)$. We define a bounded linear operator $Z : \mathcal{S}_\gamma \to \mathcal{D}$ putting

$$
  Z \omega = \psi(\omega), \quad \omega \in \mathcal{N}; \quad Zv = 0, \quad v \in \mathcal{N}^\perp.
$$

Next, we define the operator $R : \mathcal{N} \to \mathcal{N}_+$ by the relation $R \omega = X_0 \psi(\omega) + X_p \omega$, $\omega \in \mathcal{N}$. Another representation for $R$ is given by $R = P X_p|_{\mathcal{N}_+}$. The selfadjoint operator $S = R^* R : \mathcal{N} \to \mathcal{N}$ is called the spectral germ of the operator family $A(t)$ at $t = 0$. The germ $S$ is called non-degenerate if $\text{Ker} S = \{0\}$.

1.3 Analytic branches of eigenvalues and eigenvectors of $A(t)$

According to the analytic perturbation theory (see [9] and also [18], [10]), for $|t| \leq t_0$ there exist real-analytic functions $\lambda_j(t)$ (the branches of eigenvalues) and real-analytic $\mathcal{S}_\gamma$-valued functions $\varphi_j(t)$ (the branches of eigenvectors) such that

$$
  A(t) \varphi_j(t) = \lambda_j(t) \varphi_j(t), \quad j = 1, \ldots, n, \quad |t| \leq t_0,
$$

and the set $\{ \varphi_j(t) \}_{j=1}^n$ forms an orthonormal basis in the space $F(t) \mathcal{S}_\gamma$ for $|t| \leq t_0$. For sufficiently small $t_* \in (0, t_0]$ we have the following convergent power series expansions (see [18, Theorem 3.15])

$$
  \lambda_j(t) = \gamma_j t^{2p} + \mu_j t^{2p+1} + \ldots, \quad j = 1, \ldots, n, \quad |t| \leq t_*; \quad (1.3)
$$

$$
  \varphi_j(t) = \omega_j + t \varphi_j^{(1)} + \ldots, \quad j = 1, \ldots, n, \quad |t| \leq t_*.
$$

(1.4)

We have $\gamma_j \geq 0$, $\mu_j \in \mathbb{R}$. The set $\omega_1, \ldots, \omega_n$ forms an orthonormal basis in $\mathcal{N}$. The numbers $\gamma_j$ and the vectors $\omega_j$ are eigenvalues and eigenvectors of the spectral germ: $S \omega_j = \gamma_j \omega_j$, $j = 1, \ldots, n$.

1.4 Threshold approximations

The following statement was proved in [18, 10]. Below different constants depending only on $p$ are denoted by $C(p)$. 
Proposition 1.4. Suppose that Conditions 1.1, 1.2, and 1.3 are satisfied. Then for $|t| \leq t_0$ we have
\[
\|F(t) - P\| \leq C_2|t|, \quad C_2 = C(p)C_T, \tag{1.5}
\]
\[
\|A(t)F(t) - t^{2p}SP\| \leq C_3|t|^{2p+1}, \quad C_3 = C(p)C_T^{2p+1}. \tag{1.6}
\]
Here $C_T = pC_0^2 + \|X_p\|^2 \delta^{-1}$.

More accurate threshold approximations were found in the recent paper [15, Theorem 3.2].

Proposition 1.5. Suppose that Conditions 1.1, 1.2, and 1.3 are satisfied. Let
\[
G := (RP)^*X_1Z + (X_1Z)^*RP. \tag{1.7}
\]
In terms of the expansions (1.3), (1.4),
\[
G = \sum_{j=1}^{n} \mu_j(\cdot, \omega_j)S_j \omega_j + \sum_{j=1}^{n} \gamma_j \left( (\cdot, \phi_j^{(1)})S_j \omega_j + (\cdot, \omega_j)S_j \phi_j^{(1)} \right).
\]
Then for $|t| \leq t_0$ we have
\[
\|F(t) - P\| \leq C_4|t|^p, \quad C_4 = C(p)C_T^p, \tag{1.8}
\]
\[
\|A(t)F(t) - t^{2p}SP - t^{2p+1}G\| \leq C_5t^{2p+2}, \quad C_5 = C(p)C_T^{2p+2}. \tag{1.9}
\]

1.5 Approximation for $e^{-itA(t)}$

Proposition 1.6. Denote
\[
J(t, \tau) := \left( e^{-it\tau A(t)} - e^{-it\tau^{2p}SP} \right) P. \tag{1.10}
\]
For $\tau \in \mathbb{R}$ and $|t| \leq t_0$ we have
\[
\|J(t, \tau)\| \leq 2C_2|t| + C_3|\tau|^{2p+1}. \tag{1.11}
\]
Proof. We put $E(t, \tau) := e^{-it\tau A(t)}F(t) - e^{-it\tau^{2p}SP}P,$
\[
\Sigma(t, \tau) := e^{it\tau^{2p}SP}E(t, \tau) = e^{it\tau^{2p}SP}F(t)e^{-itA(t)} - P.
\]
Obviously,
\[
\|J(t, \tau)\| \leq \|E(t, \tau)\| + \|F(t) - P\|. \tag{1.12}
\]
We have $\Sigma(t, 0) = F(t) - P$ and
\[
\Sigma'(t, \tau) := \frac{d\Sigma(t, \tau)}{d\tau} = ie^{it\tau^{2p}SP} \left( t^{2p}SP - A(t)F(t) \right)F(t)e^{-itA(t)}. \tag{1.13}
\]
Since $\Sigma(t, \tau) = \Sigma(t, 0) + \int_0^\tau \Sigma'(t, \rho) \, d\rho$, then
\[
\|E(t, \tau)\| = \|\Sigma(t, \tau)\| \leq \|F(t) - P\| + |\tau|\|t^{2p}SP - A(t)F(t)\|. \tag{1.13}
\]
Combining this with (1.5), (1.6), and (1.12), we arrive at the required estimate (1.11).\qed
In the case where $G = 0$, the result can be improved.

**Proposition 1.7.** Let $G$ be the operator (1.7). Suppose that $G = 0$. Let $J(t, \tau)$ be the operator (1.10). Then for $\tau \in \mathbb{R}$ and $|t| \leq t_0$ we have

$$
\|J(t, \tau)\| \leq 2C_4|t|^p + C_5|\tau|^{2p+2}. \quad (1.14)
$$

**Proof.** Estimate (1.14) follows from (1.8), (1.9), (1.12), (1.13) and the condition $G = 0$. \qed

### 1.6 Approximation for the operator $\exp(-i\tau \varepsilon^{-2p}A(t))$

Let $\varepsilon > 0$. We study the behavior of the operator $\exp(-i\tau \varepsilon^{-2p}A(t))$ for $\tau \in \mathbb{R}$ and small $\varepsilon$. Let us estimate the operator $J(t, \tau \varepsilon^{-2p})$ multiplied by the “smoothing factor” $\varepsilon^s(t^2 + \varepsilon^2)^{-s/2}$ with $s = 2p + 1$. (In applications to DOs, such multiplying turns into smoothing.)

**Theorem 1.8.** Let $J(t, \tau)$ be the operator (1.10). For $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $|t| \leq t_0$ we have

$$
\|J(t, \tau \varepsilon^{-2p})\| \frac{\varepsilon^{2p+1}}{(t^2 + \varepsilon^2)^{p+1/2}} \leq (C_2 + C_3|\tau|)\varepsilon. \quad (1.15)
$$

**Proof.** From (1.11) with $\tau$ replaced by $\tau \varepsilon^{-2p}$ it follows that

$$
\|J(t, \tau \varepsilon^{-2p})\| \frac{\varepsilon^{2p+1}}{(t^2 + \varepsilon^2)^{p+1/2}} \leq (2C_2|t| + C_3|\tau|\varepsilon^{-2p}|t|^{2p+1}) \frac{\varepsilon^{2p+1}}{(t^2 + \varepsilon^2)^{p+1/2}}
\leq (C_2 + C_3|\tau|)\varepsilon.
$$

\qed

In the case where $G = 0$, this result can be improved.

**Theorem 1.9.** Let $J(t, \tau)$ be the operator (1.10). Let $G$ be the operator (1.7). Suppose that $G = 0$. Then for $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $|t| \leq t_0$ we have

$$
\|J(t, \tau \varepsilon^{-2p})\| \frac{\varepsilon^{2p+2}}{(t^2 + \varepsilon^2)^{p+1}} \leq \left(2C_4t_0^{p-2} + C_5|\tau|\right)\varepsilon^2. \quad (1.16)
$$

**Proof.** Estimate (1.16) follows from (1.14) with $\tau$ replaced by $\tau \varepsilon^{-2p}$:

$$
\|J(t, \tau \varepsilon^{-2p})\| \frac{\varepsilon^{2p+2}}{(t^2 + \varepsilon^2)^{p+1}} \leq \left(2C_4|t|^p + C_5|\tau|\varepsilon^{-2p}|t|^{2p+2}\right) \frac{\varepsilon^{2p+2}}{(t^2 + \varepsilon^2)^{p+1}}
\leq 2C_4t_0^{p-2}\varepsilon^2 + C_5|\tau|\varepsilon^2.
$$

We took into account that $p \geq 2$ and $|t| \leq t_0$. \qed
2 Periodic differential operators in $L_2(\mathbb{R}^d; \mathbb{C}^n)$

2.1 Lattices. The Gelfand transformation

Let $a_1, \ldots, a_d$ be a basis in $\mathbb{R}^d$ generating the lattice $\Gamma$:

$$\Gamma = \left\{ a \in \mathbb{R}^d : a = \sum_{j=1}^d l_j a_j, l_j \in \mathbb{Z} \right\},$$

and let $\Omega \subset \mathbb{R}^d$ be the elementary cell of $\Gamma$:

$$\Omega = \left\{ x \in \mathbb{R}^d : x = \sum_{j=1}^d \xi_j a_j, 0 < \xi_j < 1 \right\}.$$ 

The basis $b_1, \ldots, b_d$ in $\mathbb{R}^d$ dual to $a_1, \ldots, a_d$ is defined by the relations $\langle b_i, a_j \rangle = 2\pi \delta_{ij}$. This basis generates the lattice $\Gamma$ dual to $\Gamma$. Let $\hat{\Omega}$ be the central Brillouin zone of $\Gamma$ given by

$$\hat{\Omega} = \left\{ k \in \mathbb{R}^d : |k| < |k - b|, 0 \neq b \in \Gamma \right\}.$$ 

We use the notation $|\Omega| = \text{meas} \Omega$, $|\hat{\Omega}| = \text{meas} \hat{\Omega}$. Note that $|\Omega| |\hat{\Omega}| = (2\pi)^d$.

Let $r_0$ be the radius of the ball inscribed in $\text{clos} \Omega$. We have $2r_0 = \min_{0 \neq b \in \Gamma} |b|$.

Below, $H^s(\Omega)$ stands for the subspace of all functions $f \in H^s(\Omega)$ such that the $\Gamma$-periodic extension of $f$ to $\mathbb{R}^d$ belongs to $H^s_{\text{loc}}(\mathbb{R}^d)$.

Initially, the Gelfand transformation $\mathcal{U}$ is defined on the functions $v$ belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^d; \mathbb{C}^n)$ by the formula

$$\mathcal{U}v(k, x) = (\mathcal{U}v)(k, x) = |\hat{\Omega}|^{-1/2} \sum_{a \in \Gamma} e^{-i(k, x + a)} v(x + a), \quad x \in \Omega, \quad k \in \hat{\Omega}.$$

Then $\mathcal{U}$ extends by continuity up to a unitary mapping

$$\mathcal{U} : L_2(\mathbb{R}^d; \mathbb{C}^n) \to \int_{\hat{\Omega}} \oplus L_2(\Omega; \mathbb{C}^n) dk =: \mathcal{H}. \quad (2.1)$$

The relation $v \in H^p(\mathbb{R}^d; \mathbb{C}^n)$ is equivalent to $\mathcal{U}v \in L_2(\hat{\Omega}; \mathcal{H}^p(\Omega; \mathbb{C}^n))$. Under the transformation $\mathcal{U}$, the operator of multiplication by a bounded $\Gamma$-periodic function in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ turns into multiplication by the same function on the fibers of the direct integral $\mathcal{H}$ (see (2.1)). The linear DO $b(D)$ of order $p$ applied to $v \in H^p(\mathbb{R}^d; \mathbb{C}^n)$ turns into the operator $b(D + k)$ applied to $\mathcal{U}v(k, \cdot) \in \mathcal{H}^p(\Omega; \mathbb{C}^n)$.

2.2 Factorized operators of order $2p$

In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider an operator $A$ formally given by the differential expression

$$A = b(D)^* g(x) b(D). \quad (2.2)$$
Here $g(x)$ is a Hermitian $(m \times m)$-matrix-valued function, in general, with complex entries. It is assumed that $g(x)$ is $\Gamma$-periodic, bounded, and positive definite:

$$g, g^{-1} \in L_\infty(\mathbb{R}^d); \quad g(x) > 0. \quad (2.3)$$

The operator $b(D)$ is given by $b(D) = \sum_{|\beta|=p} b_\beta D^\beta$, where $b_\beta$ are constant $(m \times n)$-matrices, in general, with complex entries. It is assumed that $m \geq n$ and the symbol $b(\xi) = \sum_{|\beta|=p} b_\beta \xi^\beta$ satisfies $\text{rank} b(\xi) = n$ for $0 \neq \xi \in \mathbb{R}^d$. This condition is equivalent to the estimates

$$\alpha_0 1_n \leq b(\theta)^* b(\theta) \leq \alpha_1 1_n, \quad \theta \in \mathbb{R}^{d-1}, \quad 0 < \alpha_0 \leq \alpha_1 < \infty, \quad (2.4)$$

with some constants $\alpha_0, \alpha_1$.

The precise definition of the operator $A$ is given in terms of the quadratic form

$$a[u, u] = \int_{\mathbb{R}^d} \langle g(x)b(D)u(x), b(D)u(x) \rangle \, dx, \quad u \in H^p(\mathbb{R}^d; \mathbb{C}^n). \quad (2.5)$$

Using the Fourier transform and (2.3), (2.4), it is easy to check that

$$c_0 \int_{\mathbb{R}^d} |D^\beta u(x)|^2 \, dx \leq a[u, u] \leq c_1 \int_{\mathbb{R}^d} |D^\beta u(x)|^2 \, dx, \quad u \in H^p(\mathbb{R}^d; \mathbb{C}^n). \quad (2.6)$$

Here $|D^\beta u(x)|^2 := \sum_{|\beta|=p} |D^\beta u(x)|^2$. The constants $c_0, c_1$ are given by

$$c_0 = C(p) \alpha_0 \|g^{-1}\|_{L_\infty}^{-1}, \quad c_1 = C(p) \alpha_1 \|g\|_{L_\infty}. \quad (2.7)$$

Hence, the form (2.5) is closed and nonnegative. By definition, $A$ is a selfadjoint operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ generated by this form.

Note that the operator $A$ can be written as $A = X^*X$, where $X : L_2(\mathbb{R}^d; \mathbb{C}^n) \to L_2(\mathbb{R}^d; \mathbb{C}^m)$ is a closed operator defined by

$$X = g^{1/2} b(D), \quad \text{Dom} \, X = H^p(\mathbb{R}^d; \mathbb{C}^n).$$

### 2.3 Operators $A(k)$ in $L_2(\Omega; \mathbb{C}^n)$

Let $k \in \mathbb{R}^d$. In $L_2(\Omega; \mathbb{C}^n)$, we consider the quadratic form

$$a(k)[u, u] = \int_{\Omega} \langle g(x)(D+k)u(x), (D+k)u(x) \rangle \, dx, \quad u \in H^p(\Omega; \mathbb{C}^n). \quad (2.8)$$

Using the discrete Fourier transform and (2.3), (2.4), it is easy to check that

$$c_0 \int_{\Omega} |(D+k)^p u(x)|^2 \, dx \leq a(k)[u, u] \leq c_1 \int_{\Omega} |(D+k)^p u(x)|^2 \, dx, \quad u \in H^p(\Omega; \mathbb{C}^n).$$

Here $c_0, c_1$ are the same constants as in (2.6); see (2.7). Hence, the form (2.8) is closed and nonnegative. A selfadjoint operator in $L_2(\Omega; \mathbb{C}^n)$ corresponding to this form is denoted by $A(k)$. Formally, we have $A(k) = b(D+k)^* g(x) b(D+k)$.

Note that the operator $A(k)$ can be written as $A(k) = X(k)^* X(k)$, where $X(k) : L_2(\Omega; \mathbb{C}^n) \to L_2(\Omega; \mathbb{C}^m)$ is a closed operator defined by

$$X(k) = g^{1/2} b(D+k), \quad \text{Dom} \, X(k) = H^p(\Omega; \mathbb{C}^n).$$
2.4 Direct integral expansion for the operator $A$

Using the Gelfand transform $\mathcal{U}$ defined in Subsection 2.1, we expand the operator $A$ in the direct integral of the operators $A(k)$. Let $v \in H^p(\mathbb{R}^d; \mathbb{C}^n)$ and let $\tilde{v}(k, x) = (\mathcal{U} v)(k, x)$. Then $\tilde{v} \in L_2(\Omega; H^p(\Omega; \mathbb{C}^n))$ and

$$a[v, v] = \int_{\Omega} a(k)[\tilde{v}(k, \cdot), \tilde{v}(k, \cdot)] dk. \quad (2.9)$$

Conversely, if $\tilde{v} \in L_2(\Omega; H^p(\Omega; \mathbb{C}^n))$, then $v = \mathcal{U}^{-1} \tilde{v} \in H^p(\mathbb{R}^d; \mathbb{C}^n)$ and identity (2.9) is fulfilled. This means that

$$\mathcal{U} A \mathcal{U}^{-1} = \int_{\Omega} \oplus A(k) dk. \quad (2.10)$$

3 Application of the abstract results to $A(k)$

3.1 Incorporation of the operators $A(k)$ in the abstract scheme

We shall apply the scheme of Section 1, putting $\mathcal{S}_t = L_2(\Omega; \mathbb{C}^n)$ and $\mathcal{S}_t^* = L_2(\Omega; \mathbb{C}^n)$.

We write $k$ as $k = t \theta$, where $t = |k|$ and $\theta \in S^{d-1}$. The roles of $X(t)$ and $A(t)$ are played by the operators $X(k) =: X(t, \theta)$ and $A(k) =: A(t, \theta)$. They depend on the one-dimensional parameter $t$ and the additional parameter $\theta$, which was absent in the abstract scheme. He have to take care about this and to prove estimates uniform in $\theta$.

Let us check that all the assumptions of Section 1 are fulfilled. We have

$$X(k) = g^{1/2} \sum_{|\beta|=p} b_\beta (D + k)^\beta = g^{1/2} \sum_{|\beta|=p} b_\beta \sum_{\gamma \leq \beta} C_\beta^\gamma t^{\beta - \gamma} \theta^{\beta - \gamma} D^\gamma.$$

Hence, the operator $X(k) =: X(t, \theta)$ can be written as $X(t, \theta) = X_0 + \sum_{j=1}^p t^j X_j(t, \theta)$, where the operator $X_0 = g^{1/2} b(D)$, $\text{Dom}X_0 = H^p(\Omega; \mathbb{C}^n)$, is closed, the operators $X_1(\theta), \ldots, X_{p-1}(\theta)$ are given by

$$X_j(\theta) = g^{1/2} \sum_{|\beta|=p} b_\beta \sum_{\gamma \leq \beta; |\gamma| = p-j} C_\beta^\gamma \theta^{\beta - \gamma} D^\gamma, \quad \text{Dom}X_j(\theta) = H^{p-j}(\Omega; \mathbb{C}^n), \quad (3.1)$$

and the operator $X_p(\theta) = g^{1/2} b(\theta)$ is bounded from $\mathcal{S}_t$ to $\mathcal{S}_t^*$.

Obviously, Condition 1.1 is satisfied. Condition 1.2 is also satisfied with

$$C_0 = C(d, p) \alpha_1^{1/2} \alpha_0^{-1/2} \|g\|_{L^\infty}^{1/2} \|g^{-1}\|_{L^\infty}^{1/2} (1 + r_0^{-1})^{p-1}, \quad (3.2)$$

where $C(d, p)$ depends only on $d$ and $p$; see [10, Proposition 5.2].

By (2.4), we obtain the uniform bound for the norm of $X_p(\theta)$:

$$\|X_p(\theta)\| \leq \alpha_1^{1/2} \|g\|_{L^\infty}^{1/2}, \quad \theta \in S^{d-1}. \quad (3.3)$$
Let \( \mathcal{N} = \text{Ker} A(0) = \text{Ker} X_0 \). It is easy to check that \( \mathcal{N} \) consists of constant vector-valued functions (see [10, Proposition 5.1]):

\[
\mathcal{N} = \{ u \in L_2(\Omega; \mathbb{C}^n) : u(x) = c \in \mathbb{C}^n \}.
\] (3.4)

So, \( \dim \mathcal{N} = n \). The orthogonal projection of \( L_2(\Omega; \mathbb{C}^n) \) onto \( \mathcal{N} \) is the operator of the averaging over the cell:

\[
P u = |\Omega|^{-1} \int_{\Omega} u(x) \, dx, \quad u \in L_2(\Omega; \mathbb{C}^n).
\] (3.5)

Let \( \mathcal{N}_s = \text{Ker} X_0^s \) and \( n_s = \dim \mathcal{N}_s \). The condition \( m \geq n \) ensures that \( n \leq n_s \). Moreover, either \( n_s = \infty \) (if \( m > n \)), or \( n_s = n \) (if \( m = n \)). See [10, Section 5.1] for details.

Since the embedding of \( H^p(\Omega; \mathbb{C}^n) \) into \( L_2(\Omega; \mathbb{C}^n) \) is compact, the spectrum of the operator \( A(0) \) is discrete. The point \( \lambda_0 = 0 \) is an isolated eigenvalue of \( A(0) \) of multiplicity \( n \); the corresponding eigenspace \( \mathcal{N} \) is given by (3.4). Thus, Condition 1.3 is satisfied.

Let \( d^0 \) be the distance from the point \( \lambda_0 = 0 \) to the rest of the spectrum of \( A(0) \). According to [10, (5.17)],

\[
d^0 \geq \alpha_0 \| g^{-1} \|_{L_\infty}^{-1} (2r_0)^{2p}.
\] (3.6)

In Subsection 1.1 it was required to fix a positive number \( \delta \leq \min\{d^0/36, 1/4\} \). Using (3.6), we choose \( \delta \) as follows:

\[
\delta = \min\{\alpha_0 \| g^{-1} \|_{L_\infty}^{-1} (2r_0)^{2p}/36, 1/4\}.
\] (3.7)

Next, the constant \( C_1(\theta) = \max\{(p - 1)C_0, \| X_p(\theta) \| \} \) now depends on \( \theta \) (see (1.1)). Using (3.3), we see that \( C_1(\theta) \leq C_1 \), where

\[
C_1 = \max\{(p - 1)C_0, \alpha_1^{1/2} \| g \|_{L_\infty}^{1/2} \}.
\] (3.8)

Here \( C_0 \) is given by (3.2). According to (1.1), we fix a number \( t_0 \leq \delta^{1/2} C_1(\theta)^{-1} \) as follows:

\[
t_0 = \delta^{1/2} C_1^{-1},
\] (3.9)

where \( \delta \) and \( C_1 \) are defined by (3.7) and (3.8), respectively.

### 3.2 The operators \( Z(\theta), R(\theta), \) and \( S(\theta) \)

For the operator family \( A(t, \theta) \), the operators \( Z, R, \) and \( S \) defined in Subsection 1.2 in the abstract setting depend on the parameter \( \theta \).

To describe these operators, we introduce the \( (n \times m) \)-matrix-valued function \( \Lambda(x) \) which is a \( \Gamma \)-periodic solution of the following problem:

\[
b(D)^* g(x) (b(D) \Lambda(x) + 1_m) = 0, \quad \int_\Omega \Lambda(x) \, dx = 0.
\] (3.10)
The equation is understood in the weak sense: for each $C \in \mathbb{C}^m$ we have $\Lambda C \in \widetilde{H}^p(\Omega; \mathbb{C}^n)$ and
\[
\int_{\Omega} \langle g(x)(b(D)\Lambda(x)C + C), b(D)\eta(x) \rangle \, dx = 0, \quad \eta \in \widetilde{H}^p(\Omega; \mathbb{C}^n).
\]
Then (cf. [10, Section 5.3])
\[
Z(\theta) = [\Lambda]b(\theta)P, \tag{3.11}
\]
where $[\Lambda]$ denotes the operator of multiplication by the matrix-valued function $\Lambda(x)$.

The operator $R(\theta)$ is given by
\[
R(\theta) = [g^{1/2}(b(D)\Lambda + 1_m)]b(\theta)|_{\gamma\Omega}. \tag{3.12}
\]

Then (cf. [10, Section 5.3]), the spectral germ $S(\theta) = R(\theta)^*R(\theta)$ acts in the subspace $\gamma\Omega$ (see (3.4)) and is represented as
\[
S(\theta) = b(\theta)^*g^0b(\theta), \quad \theta \in \mathbb{S}^{d-1}. \tag{3.13}
\]

Here $g^0$ is the so called effective matrix (of size $m \times m$) given by
\[
g^0 = |\Omega|^{-1}\int_{\Omega} \widetilde{g}(x) \, dx, \quad \widetilde{g}(x) := g(x)(b(D)\Lambda(x) + 1_m).
\]

It turns out that the effective matrix $g^0$ is positive definite. So, the germ $S(\theta)$ is non-degenerate. We mention some properties of $g^0$; see [10, Propositions 5.3, 5.4].

**Proposition 3.1.** Denote
\[
\overline{g} = |\Omega|^{-1}\int_{\Omega} g(x) \, dx, \quad \underline{g} = \left(\Omega^{-1}\int_{\Omega} g(x)^{-1} \, dx\right)^{-1}.
\]

The effective matrix $g^0$ satisfies the following estimates (the Voigt–Reuss bracketing): $\underline{g} \leq g^0 \leq \overline{g}$. In the case where $m = n$, we have $g^0 = \underline{g}$.

**Proposition 3.2.**

1. Let $g_k(x), \ k = 1, \ldots, m$, be the columns of the matrix $g(x)$. The relation $g^0 = \overline{g}$ is equivalent to the identities
\[
b(D)^*g_k(x) = 0, \quad k = 1, \ldots, m. \tag{3.14}
\]

2. Let $l_k(x), \ k = 1, \ldots, m$, be the columns of the matrix $g(x)^{-1}$. The relation $g^0 = \underline{g}$ is equivalent to the representations
\[
l_k(x) = l_k^0 + b(D)v_k(x), \quad l_k^0 \in \mathbb{C}^m, \quad v_k \in \widetilde{H}^p(\Omega; \mathbb{C}^n); \quad k = 1, \ldots, m. \tag{3.15}
\]

**Remark.** In the case where $g^0 = \underline{g}$, the matrix $\widetilde{g}(x)$ is constant: $\widetilde{g}(x) = g^0 = \underline{g}$. 
3.3 The effective operator

By (3.13) and the homogeneity of the symbol \(b(k)\), we have

\[ S(k) := t^{2p}S(\theta) = b(k)^*g^0b(k), \quad k \in \mathbb{R}^d. \]  

Expression (3.16) is the symbol of the DO

\[ A^0 = b(D)^*g^0b(D), \quad \text{Dom}A^0 = H^{2p}(\mathbb{R}^d; \mathbb{C}^n), \]  

which is called the effective operator for \(A\).

Let \(A^0(k)\) be the operator family in \(L_2(\Omega; \mathbb{C}^n)\) corresponding to the operator \(A^0\). Then \(A^0(k)\) is given by the differential expression

\[ b(D + k)^*g^0b(D + k) \]  

on the domain \(\tilde{H}^{2p}(\Omega; \mathbb{C}^n)\).

By (3.15) and (3.16), we have

\[ S(k)P = A^0(k)P. \]  

3.4 The operator \(G(\theta)\)

For \(A(t, \theta)\), the operator \(G\) defined by (1.7) in the abstract setting depends on \(\theta\):

\[ G(\theta) = (R(\theta)P)^*X_1(\theta)Z(\theta) + (X_1(\theta)Z(\theta))^*R(\theta)P. \]

Let \(B_1(\theta; D)\) be the DO of order \(p - 1\) such that \(X_1(\theta) = g^{1/2}B_1(\theta; D)\) (see (3.1)). Then

\[ B_1(\theta; D) = \sum_{|\beta| = p} b_\beta \sum_{\gamma \leq \beta, |\gamma| = p - 1} C_\gamma^{\gamma} \theta^{\beta - \gamma} D^\gamma. \]

Using (3.11) and (3.12), we obtain

\[ G(\theta) = b(\theta)^*g^{(1)}(\theta)b(\theta)P, \]  

where \(g^{(1)}(\theta)\) is a Hermitian \((m \times m)\)-matrix given by

\[ g^{(1)}(\theta) = |\Omega|^{-1} \int_\Omega (\tilde{g}(x)^*B_1(\theta; D)\Lambda(x) + (B_1(\theta; D)\Lambda(x))^*\tilde{g}(x)) \, dx. \]  

We distinguish some cases where the operator (3.19) is equal to zero.

**Proposition 3.3.** 1°. Suppose that relations (3.14) are satisfied. Then \(\Lambda(x) = 0\), whence \(g^{(1)}(\theta) = 0\) and \(G(\theta) = 0\).

2°. Suppose that relations (3.15) are satisfied. Then \(g^{(1)}(\theta) = 0\) and \(G(\theta) = 0\).

3°. Suppose that \(n = 1\) and the matrices \(g(x), b_\beta, |\beta| = p, \) have real entries. Then \(G(\theta) = 0\) for any \(\theta \in S^{d-1}\).
Proof. Obviously, if relations (3.14) are satisfied, then the solution $\Lambda(x)$ of problem (3.10) is equal to zero. From (3.20) it follows that $g^{(1)}(\theta) = 0$. Then, by (3.19), $G(\theta) = 0$.

If relations (3.15) are satisfied, then $\tilde{g}(x) = g^0 = \underline{g}$. Since the integral over the cell of the derivatives of a periodic function is equal to zero, then

$$
\int_{\Omega} B_1(\theta; \mathbf{D}) \Lambda(x) \, dx = 0
$$

and hence, we have $g^{(1)}(\theta) = 0$. Consequently, by (3.19), $G(\theta) = 0$.

Now, suppose that $n = 1$ and the matrices $g(x), b_\beta, |\beta| = p$, have real entries. Then, for $p$ even, the solution $\Lambda(x)$ of problem (3.10) is a $(1 \times m)$-matrix with real entries. Hence, the matrix $\tilde{g}(x) = g(x)(b(\mathbf{D})\Lambda(x) + \mathbf{1}_m)$ is an $(m \times m)$-matrix with real entries. Next, $B_1(\theta; \mathbf{D})\Lambda(x)$ is a Hermitian $(m \times m)$-matrix with imaginary entries. By (3.20), $g^{(1)}(\theta)$ is equal to zero, as a Hermitian imaginary $(1 \times 1)$-matrix.

For $p$ odd, the solution $\Lambda(x)$ of problem (3.10) is a $(1 \times m)$-matrix with imaginary entries. The matrix $\tilde{g}(x)$ has real entries. Next, $B_1(\theta; \mathbf{D})\Lambda(x)$ is an $(m \times m)$-matrix with imaginary entries. Hence, $g^{(1)}(\theta)$ is a Hermitian $(m \times m)$-matrix with imaginary entries. Again, $b(\theta)^*g^{(1)}(\theta)b(\theta)$ is equal to zero, as a Hermitian imaginary $(1 \times 1)$-matrix.

Remark. In the general case, the operator $G(\theta)$ may be non-zero. In particular, it is easy to give examples of the scalar operator $A = b(\mathbf{D})^*g(x)b(\mathbf{D})$ (i.e., $n = 1$), where $g(x)$ is a Hermitian matrix with complex entries, such that the corresponding operator $G(\theta)$ is not zero.

3.5 Approximation for the operator exponential of $A(k)$

In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, consider the operator $H_0 := -\Delta$. Let $H_0(k)$ be the operator family in $L_2(\Omega; \mathbb{C}^n)$ corresponding to the operator $H_0$. Then $H_0(k)$ is given by the differential expression $|\mathbf{D} + \mathbf{k}|^2$ on the domain $H^2(\Omega; \mathbb{C}^n)$. Denote

$$
R_0(k, \varepsilon) := \varepsilon^2 (H_0(k) + \varepsilon^2 I)^{-1}.
$$

Clearly, we have

$$
R_0(k, \varepsilon)^{s/2}P := \varepsilon^s (t^2 + \varepsilon^2)^{-s/2}P, \quad t = |k|, \quad s > 0.
$$

We apply Theorem 1.8 to the operator family $A(t, \theta) = A(k)$. Note that, by (3.16), (3.18), $e^{-it\delta^2 s(\theta)}P = e^{-it\delta^0(k)}P$. Thus, the operator (1.10) turns into

$$
J(k, \tau) := \left( e^{-it\delta^0(k)} - e^{-it\delta^0(k)} \right) P.
$$

It remains to implement the values of the constants in estimates. The constants $\delta$ and $t_0$ are given by (3.7) and (3.9), respectively; they do not depend on $\theta$. The constant $C_0$ is
given by (3.2). Using (3.3), we can replace the constant \( C_T(\theta) = pC_0^2 + \|X_p(\theta)\|_2^2 \delta^{-1} \) depending now on \( \theta \) by \( C_T = pC_0^2 + \|X_p(\theta)\|_2^2 \delta^{-1} \). According to (1.5), (1.6), we put \( C_2 = C(p)C_T, C_3 = C(p)C_T^{p+1} \).

Now, applying (1.15) and taking (3.22) into account, we obtain

\[
\left\| \left( J(k, \tau \varepsilon^{-2p}) R_0(k, \varepsilon) \right)^{p+1/2} \right\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq (C_2 + C_3 |\tau|) \varepsilon, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0, \quad |k| \leq t_0.
\]  

(3.23)

For \( k \in \tilde{\Omega}, |k| > t_0 \), estimates are trivial. Obviously, \( \|J(k, \tau)\| \leq 2 \) and \( \|R_0(k, \varepsilon)\| \leq 1 \).

By (3.22),

\[
\left\| R_0(k, \varepsilon)^{1/2} \right\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq t_0^{-1} \varepsilon, \quad k \in \tilde{\Omega}, \quad |k| > t_0, \quad \varepsilon > 0.
\]

Hence,

\[
\left\| \left( J(k, \tau \varepsilon^{-2p}) R_0(k, \varepsilon) \right)^{p+1/2} \right\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq 2t_0^{-1} \varepsilon, \quad \tau \in \mathbb{R}, \quad k \in \tilde{\Omega}, \quad |k| > t_0, \quad \varepsilon > 0.
\]  

(3.24)

Finally, let us show that, within the margin of error, the projection \( P \) in estimates (3.23) and (3.24) can be removed. Indeed, using the discrete Fourier transform, we have

\[
\left\| R_0(k, \varepsilon)^{1/2} (I - P) \right\|_{L_2(\Omega) \rightarrow L_2(\Omega)} = \max_{0 \neq b \in \Gamma} \varepsilon (|b| + |k|^2 + \varepsilon^2)^{-1/2} \leq r_0^{-1} \varepsilon, \quad k \in \tilde{\Omega}, \quad \varepsilon > 0.
\]  

(3.25)

Consequently,

\[
\left\| \left( e^{-i\tau \varepsilon^{-2p}A(k)} - e^{-i\tau \varepsilon^{-2p}A_0(k)} \right) R_0(k, \varepsilon)^{p+1/2} (I - P) \right\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq 2r_0^{-1} \varepsilon, \quad \tau \in \mathbb{R}, \quad k \in \tilde{\Omega}, \quad \varepsilon > 0.
\]  

(3.26)

Combining (3.23), (3.24), and (3.26), we arrive at the following result.

**Theorem 3.4.** For \( \tau \in \mathbb{R}, \varepsilon > 0, \) and \( k \in \tilde{\Omega} \) we have

\[
\left\| \left( e^{-i\tau \varepsilon^{-2p}A(k)} - e^{-i\tau \varepsilon^{-2p}A_0(k)} \right) R_0(k, \varepsilon)^{p+1/2} \right\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \mathcal{C}_1 (1 + |\tau|) \varepsilon.
\]

The constant \( \mathcal{C}_1 \) depends only on \( d, p, \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \) and the parameters of the lattice \( \Gamma \).

Note that, by (3.22),

\[
\|R_0(k, \varepsilon)P\| \leq t_0^{-2} \varepsilon^2, \quad k \in \tilde{\Omega}, \quad |k| > t_0, \quad \varepsilon > 0.
\]  

(3.27)

Similarly to (3.25), we have

\[
\|R_0(k, \varepsilon)(I - P)\| = \max_{0 \neq b \in \Gamma} \varepsilon^2 (|b| + |k|^2 + \varepsilon^2)^{-1} \leq r_0^{-2} \varepsilon^2, \quad k \in \tilde{\Omega}, \quad \varepsilon > 0.
\]  

(3.28)

Applying Theorem 1.9 and using (3.27), (3.28), we arrive at the following result.
Theorem 3.5. Let $G(\theta)$ be the operator given by (3.19), (3.20). Suppose that $G(\theta) = 0$ for any $\theta \in \mathbb{S}^{d-1}$. Then for $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $k \in \bar{\Omega}$ we have

$$\left\| \left( e^{-i\tau \varepsilon^{-2p}A(k)} - e^{-i\tau \varepsilon^{-2p}A^0(k)} \right) R_0(k, \varepsilon)^{p+1} \right\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_2(1 + |\tau|)^2 \varepsilon^2.$$ 

The constant $C_2$ depends only on $d$, $p$, $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice $\Gamma$.

4 Approximation for the operator exponential of $A$

Let $A$ be the operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ given by $A = b(D)^* g(x) b(D)$; see (2.2). Let $A^0 = b(D)^* g^0 b(D)$ be the effective operator (3.17). Recall the notation $H_0 = -\Delta$ and denote

$$R_0(\varepsilon) := \varepsilon^2 (H_0 + \varepsilon^2 I)^{-1}. \quad (4.1)$$

From expansion (2.10) it follows that

$$e^{-i\tau \varepsilon^{-2p}A} = \mathcal{U}^{-1} \left( \int_{\bar{\Omega}} \bigoplus e^{-i\tau \varepsilon^{-2p}A(k)} \, dk \right) \mathcal{U}.$$ 

The operator $e^{-i\tau \varepsilon^{-2p}A^0}$ admits a similar expansion. The operator (4.1) is decomposed into the direct integral of the operators (3.21):

$$R_0(\varepsilon) = \mathcal{U}^{-1} \left( \int_{\bar{\Omega}} \bigoplus R_0(k, \varepsilon) \, dk \right) \mathcal{U}.$$ 

From these direct integral expansions, taking into account that $\mathcal{U}$ is unitary, we obtain

$$\left\| \left( e^{-i\tau \varepsilon^{-2p}A} - e^{-i\tau \varepsilon^{-2p}A^0} \right) R_0(\varepsilon)^{s/2} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = \sup_{k \in \bar{\Omega}} \left\| \left( e^{-i\tau \varepsilon^{-2p}A(k)} - e^{-i\tau \varepsilon^{-2p}A^0(k)} \right) R_0(k, \varepsilon)^{s/2} \right\|_{L_2(\bar{\Omega}) \to L_2(\bar{\Omega})}.$$ 

Combining this with Theorem 3.4, we arrive at the following result.

Theorem 4.1. For $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\left\| \left( e^{-i\tau \varepsilon^{-2p}A} - e^{-i\tau \varepsilon^{-2p}A^0} \right) R_0(\varepsilon)^{p+1/2} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_1(1 + |\tau|) \varepsilon. \quad (4.2)$$

Similarly, Theorem 3.5 implies the following result.

Theorem 4.2. Let $G(\theta)$ be the operator given by (3.19), (3.20). Suppose that $G(\theta) = 0$ for any $\theta \in \mathbb{S}^{d-1}$. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\left\| \left( e^{-i\tau \varepsilon^{-2p}A} - e^{-i\tau \varepsilon^{-2p}A^0} \right) R_0(\varepsilon)^{p+1} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_2(1 + |\tau|)^2 \varepsilon^2.$$
5 Homogenization of the Schrödinger-type equation

5.1 The operator $A_\varepsilon$. The scaling transformation

We use the notation $g^\varepsilon(x) := g(\varepsilon^{-1}x)$, $\varepsilon > 0$. Our main object is the operator $A_\varepsilon$ acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and formally given by

$$A_\varepsilon = b(D)^* g^\varepsilon(x) b(D). \quad (5.1)$$

The precise definition of $A_\varepsilon$ is given in terms of the corresponding quadratic form; cf. Subsection 2.2. Our goal is to approximate the operator exponential $e^{-i\tau A_\varepsilon}$ for small $\varepsilon$.

Let $T_\varepsilon$ be the scaling transformation defined by $(T_\varepsilon u)(x) = \varepsilon^{d/2} u(\varepsilon x)$. Then $T_\varepsilon$ is unitary in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. We have $A_\varepsilon = \varepsilon^{-2p} T_\varepsilon^* A T_\varepsilon$. Hence, $e^{-i\tau A_\varepsilon} = T_\varepsilon^* e^{-i\varepsilon^{-2p}\varepsilon A} T_\varepsilon$. A similar relation holds for the effective operator $A^0$: $e^{-i\tau A^0} = T_\varepsilon^* e^{-i\varepsilon^{-2p}\varepsilon A} T_\varepsilon$. Applying the scaling transformation to the resolvent of $H_0 = -\Delta$, we obtain

$$(H_0 + I)^{-1} = \varepsilon^2 T_\varepsilon^* (H_0 + \varepsilon^2 I)^{-1} T_\varepsilon = T_\varepsilon^* R_0(\varepsilon) T_\varepsilon.$$ 

Using these relations and taking into account that $T_\varepsilon$ is unitary, we have

$$\left\| \left( e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right) (H_0 + I)^{-s/2} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = \left\| \left( e^{-i\varepsilon^{-2p}A} - e^{-i\varepsilon^{-2p}A^0} \right) R_0(\varepsilon)^{s/2} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} > 0. \quad (5.2)$$

5.2 Approximation for the operator exponential $e^{-i\tau A_\varepsilon}$

Combining (5.2) and (4.2), we see that

$$\left\| \left( e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right) (H_0 + I)^{-(p+1/2)} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_1 (1 + |\tau|)\varepsilon, \quad \tau \in \mathbb{R}, \varepsilon > 0.$$ 

Since $(H_0 + I)^{p+1/2}$ is an isometric isomorphism of the Sobolev space $H^{2p+1}(\mathbb{R}^d; \mathbb{C}^n)$ onto $L_2(\mathbb{R}^d; \mathbb{C}^n)$, this yields the following result.

**Theorem 5.1.** For $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^{2p+1}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_1 (1 + |\tau|)\varepsilon. \quad (5.3)$$

The constant $C_1$ depends only on $d, p, \alpha_0, \alpha_1, \|g\|_{L_{\alpha_0}}, \|g^{-1}\|_{L_{\alpha_1}},$ and the parameters of the lattice $\Gamma$.

Interpolating between the obvious estimate $\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq 2$ and (5.3), we obtain the following corollary.
Corollary 5.2. Let $0 \leq s \leq 2p + 1$. For $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have
\[
\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(s) (1 + |\tau|)^{s/(2p+1)} \varepsilon^{s/(2p+1)}.
\]
Here $C(s) = 2^{1-s/(2p+1)} C_1^{s/(2p+1)}$.

Similarly, by the scaling transformation, we deduce the following result from Theorem 4.2.

Theorem 5.3. Let $G(\theta)$ be the operator given by (3.19), (3.20). Suppose that $G(\theta) = 0$ for any $\theta \in \mathbb{S}^{d-1}$. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have
\[
\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^{2p+2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_2 (1 + |\tau|) \varepsilon^{2}.
\]
The constant $C_2$ depends only on $d$, $p$, $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and the parameters of the lattice $\Gamma$.

Corollary 5.4. Under the assumptions of Theorem 5.3, let $0 \leq s \leq 2p + 2$. For $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have
\[
\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C'(s) (1 + |\tau|)^{s/(2p+2)} \varepsilon^{s/(p+1)}.
\]
Here $C'(s) = 2^{1-s/(2p+2)} C_2^{s/(2p+2)}$. In particular, for $s = p + 1$
\[
\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^{p+1}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C'(p + 1) (1 + |\tau|)^{1/2} \varepsilon. \tag{5.4}
\]

Recall that some sufficient conditions ensuring that $G(\theta) \equiv 0$ are given in Proposition 3.3.

Remark. Theorem 5.3 shows that, if $G(\theta) \equiv 0$, the difference $e^{-i\tau A_\varepsilon} - e^{-i\tau A^0}$ is of order $O(\varepsilon^2)$ in a suitable norm. Estimate (5.4) improves (5.3) regarding both the norm type and the dependence of the estimate on $\tau$. We note that for the second-order operators there is analog of estimate (5.4) (see [6]), but there is no analog of Theorem 5.3.

5.3 Homogenization of the Cauchy problem for the Schrödinger-type equation

Let $u_\varepsilon(x, \tau)$ be the solution of the following Cauchy problem:
\[
\begin{align*}
i \partial_\tau u_\varepsilon(x, \tau) &= b(D)^* g_\varepsilon(x) b(D) u_\varepsilon(x, \tau) + F(x, \tau), & x \in \mathbb{R}^d, \tau \in \mathbb{R}, \\
u_\varepsilon(x, 0) &= \phi(x), & x \in \mathbb{R}^d,
\end{align*}
\tag{5.5}
\]
where $\phi \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $F \in L_{1,\text{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^n))$. The solution of problem (5.5) admits the following representation:

$$u_\varepsilon(\cdot, \tau) = e^{-i\tau A_\varepsilon} \phi - i \int_0^\tau e^{-i(\tau - \tau') A_\varepsilon} F(\cdot, \tau) d\tau.$$  

(5.6)

Let $u_0(x, \tau)$ be the solution of the homogenized Cauchy problem:

$$i \partial_\tau u_0(x, \tau) = (D)^* g^0 b(D) u_0(x, \tau) + F(x, \tau), \quad x \in \mathbb{R}^d, \quad \tau \in \mathbb{R},$$

$$u_0(x, 0) = \phi(x), \quad x \in \mathbb{R}^d.$$  

(5.7)

The solution of problem (5.7) can be represented as

$$u_0(\cdot, \tau) = e^{-i\tau A_0} \phi - i \int_0^\tau e^{-i(\tau - \tau') A_0} F(\cdot, \tau) d\tau.$$  

(5.8)

**Theorem 5.5.** Let $u_\varepsilon(x, \tau)$ be the solution of the Cauchy problem (5.5). Let $u_0(x, \tau)$ be the solution of the homogenized problem (5.7).

1°. Let $0 \leq s \leq 2p + 1$. If $\phi \in H^s(\mathbb{R}^d; \mathbb{C}^n)$ and $F \in L_{1,\text{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; \mathbb{C}^n))$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leq C(s) (1 + |\tau|)^{s/(2p+1)} \varepsilon^{s/(2p+1)}$$

$$\times \left( \|\phi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L_1((0, \tau); H^s(\mathbb{R}^d))} \right).$$  

(5.9)

2°. If $\phi \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $F \in L_{1,\text{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^n))$, then for $\tau \in \mathbb{R}$ we have

$$\lim_{\varepsilon \to 0} \|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} = 0.$$  

**Proof.** Statement 1° follows from Corollary 5.2 and representations (5.6), (5.8).

Estimate (5.9) with $s = 0$ means that the norm $\|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)}$ is uniformly bounded provided that $\phi \in L_2(\mathbb{R}^d; \mathbb{C}^n)$ and $F \in L_{1,\text{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^n))$. Hence, using statement 1° with $s = 2p + 1$ and applying the Banach–Steinhaus theorem, we obtain statement 2°.

Similarly, from Corollary 5.4 we deduce the following result.

**Theorem 5.6.** Let $G(\theta)$ be the operator given by (3.19), (3.20). Suppose that $G(\theta) = 0$ for any $\theta \in \mathbb{S}^{d-1}$. Let $u_\varepsilon(x, \tau)$ be the solution of the Cauchy problem (5.5). Let $u_0(x, \tau)$ be the solution of the homogenized problem (5.7). Let $0 \leq s \leq 2p + 2$. If $\phi \in H^s(\mathbb{R}^d; \mathbb{C}^n)$ and $F \in L_{1,\text{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; \mathbb{C}^n))$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leq C'(s) (1 + |\tau|)^{s/(2p+2)} \varepsilon^{s/(p+1)}$$

$$\times \left( \|\phi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L_1((0, \tau); H^s(\mathbb{R}^d))} \right).$$  

(5.10)

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