SUPERTROPICAL ALGEBRA

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Abstract. We develop the algebraic polynomial theory for “supertropical algebra,” as initiated earlier over the real numbers by the first author. The main innovation there was the introduction of “ghost elements,” which also play the key role in our structure theory. Here, we work somewhat more generally over an ordered monoid, and develop a theory which contains the analogs of several basic theorems of classical commutative algebra. This structure enables one to develop a Zariski-type algebraic geometric approach to tropical geometry, viewing tropical varieties as sets of roots of (supertropical) polynomials, leading to an analog of the Hilbert Nullstellensatz.

Particular attention is paid to factorization of polynomials. In one indeterminate, any polynomial can be factored into linear and quadratic factors, and unique factorization holds in a certain sense. On the other hand, the failure of unique factorization in several indeterminates is explained by geometric phenomena described in the paper.

1. Introduction

One of the goals of algebra is to find the “correct” algebraic structure with which to frame some mathematical theory. The underlying motivation of this paper is to provide a direct algebraic approach to the rapidly developing theory of tropical mathematics. Tropical geometry has been the subject of intensive recent research, including some remarkable applications in various areas of mathematics, such as combinatorics, polynomials (Newton’s polytopes), linear algebra, and algebraic geometry; cf. [12] and [31]. Before bringing in our structure, let us review briefly how one passes from “classical” algebraic geometry to tropical geometry.

For any complex affine variety \( W = \{(z_1, \ldots, z_n) : z_i \in \mathbb{C}\} \subset \mathbb{C}^n \), and any small \( t \), one could define its amoeba, cf. [8],

\[ A(W) = \{(\log |z_1|, \ldots, \log |z_n|) : (z_1, \ldots, z_n) \in W \} \subset \mathbb{R}^n, \]

where \( \mathbb{R}_{-\infty} := \mathbb{R} \cup \{-\infty\} \). Note that \( \log_t |z_1z_2| = \log_t |z_1| + \log_t |z_2| \), and the limiting case \( t \to 0 \) degenerates to a polyhedral complex, i.e., non-Archimedean amoeba, in \( \mathbb{R}^n_{-\infty} \) where now \( \mathbb{R}_{-\infty} \) is given the structure of the max-plus algebra, for which the new addition is defined as the maximum, multiplication is taken to be the original addition in \( \mathbb{R} \), and the zero element is \( -\infty \). Passing from the original algebraic variety to this “tropical variety” preserves various geometric invariants involving intersections, and has been used to simplify proofs of deep results from algebraic geometry. As developed in [1, 2, 3, 4, 6, 7, 21, 24, 27, 28, 29, 30], the max-plus algebra (or dually, the min-plus algebra) lies at the foundation of “tropical algebra” and “tropical geometry.” A survey can be found in [20], and [5] provides a fine explanation of how one arrives at tropical geometry defined over the max-plus algebra.

Although many ideas of tropical geometry can be found in the pq-webs of [11], researchers in tropical geometry have focused on definitions of tropical varieties arising from complex analysis and symplectic geometry. In the simplicial geometric approach of [22], a finite polyhedral complex is said to be of pure dimension \( k \) if each of its faces of dimension \( < k \) is contained in a \( k \)-dimensional face – called a top-dimensional face. A \( k \)-dimensional tropical variety \( X \subset \mathbb{R}^n \) is a finite rational polyhedral complex.
of pure dimension $k$ whose top-dimensional faces $\delta$ are equipped with positive integral weights $m(\delta)$ such that, for each face $\sigma$ of codimension 1 in $X$, the following condition is satisfied, called the balancing condition:

$$\sum_{\sigma \in \delta} n(\delta)n_{\sigma}(\delta) = 0,$$

where $\delta$ runs over all $k$-dimensional faces of $X$ containing $\sigma$, and $n_{\sigma}(\delta)$ is the primitive unit vector normal to $\sigma$ lying in the cone centered at $\sigma$ and directed by $\delta$. Accordingly, a tropical hypersurface, i.e., an $(n - 1)$-dimensional tropical variety in $\mathbb{R}^n$, must have (topological) dimension $n - 1$.

An alternative approach, more algebraic in nature, is to consider tropical polynomials as piecewise linear functions $f_\sigma : \mathbb{R}^n \rightarrow \mathbb{R}$; then the corner locus, denoted $\text{Cor}(f_\sigma)$, is defined as the domain of non-differentiability of $f_\sigma$, or, in other words, the set of points on which the evaluation of $f_\sigma$ is attained by at least two of its monomials. Yet, this notion has no pure algebraic framework over the max-plus algebra $\mathbb{R}_{\infty}$, and our structure aims for such a framework.

There is a direct passage from (classical) affine algebraic geometry to tropical geometry, in which algebraic varieties are transformed to polyhedral complexes. Namely, the max-plus algebra appears as the target of a non-Archimedean valuation $\text{val} : K \rightarrow \mathbb{R}_{\infty}$ of the field $K$ of locally convergent Puiseux series of the form $p(t) = \sum_{r \in T} c_r t^r$, where $c_r \in \mathbb{C}$ and $T \subseteq \mathbb{Q}$ is bounded from below, where

$$\text{val}(p(t)) := \begin{cases} \min\{r \in R : c_r \neq 0\}, & p(t) \in K^X, \\ -\infty, & p(t) = 0. \end{cases}$$

Given a polynomial $f_\sigma = \sum_{i \in \Omega} p_i \lambda_1^{i_1} \cdots \lambda_n^{i_n}$, for $p_i = p_i(t)$, over $K$ with zero set $\mathcal{Z}(f_\sigma) \subseteq K^{(n)}$, the non-Archimedean amoeba $\mathcal{A}(f_\sigma) \subseteq \mathbb{R}^n$ is now defined be the closure $\text{val}(\mathcal{Z}(f_\sigma))$ of $\text{val}(\mathcal{Z}(f_\sigma))$, where the valuation is taken coordinate-wise.

**Theorem 1.1** (Kapranov, [2]). $\mathcal{A}(f_\sigma)$ is contained in the corner locus of the tropical function

$$f_\sigma(a) = \max_{i \in \Omega} (\langle i, a \rangle + \text{val}(p_i)), \quad a = (a_1, \ldots, a_n) \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ stands for the standard scalar product. (Note that the term $a_1^{i_1} \cdots a_n^{i_n}$ is evaluated as $\langle i, a \rangle$ in the max-plus algebra.) Equality holds when $\text{val}$ is onto.

Kapranov’s theorem implies not only that every non-Archimedean amoeba is a corner locus of a tropical polynomial, but also that any corner locus of a tropical polynomial $f_\sigma$ is a non-Archimedean amoeba. In [25], tropical varieties were defined as non-Archimedean amoebas $\text{val}(\mathcal{Z}(I))$, where $I \triangleleft K[\lambda_1, \ldots, \lambda_n]$. (However, there exist balanced polyhedral complexes of codimension $> 1$ that cannot be described as non-Archimedean amoebas.)

From a categorical perspective, one would like to study these tropical varieties directly, in terms of the underlying algebraic structure.

**Definition 1.2.** A semiring $(R, +, \cdot, 0_R, 1_R)$ is a set $R$ endowed with binary operations $+$ and $\cdot$ and distinguished elements $0_R$ and $1_R$, such that $(R, +, 1_R)$ and $(R, +, 0_R)$ are monoids satisfying distributivity of multiplication over addition on both sides, and such that $0_R \cdot r = r \cdot 0_R = 0_R$ for every $r \in R$.

Semirings have attracted interest because of their impact on computer science, and we use [9] as a general reference. Occasionally we need the more general notion of a semiring without zero, which satisfies all the axioms of semiring except those involving the element $0_R$. For any semiring without zero, we obtain a semiring by formally adjoining the element $0_R$ and stipulating that $a + 0_R = 0_R + a = a$ and $a \cdot 0_R = 0_R \cdot a = 0_R$ for each $a \in R$.

The algebraic structure of the max-plus algebra is that of a semiring without zero, which becomes a semiring when we formally adjoin the element $-\infty$. The complications in utilizing the max-plus algebra as the underlying structure in tropical geometry crop up almost immediately. Unfortunately, the max-plus algebra has no additive inverse (even after one adjoins $-\infty$), and thus its algebraic structure as a semiring is handicapped.

Consequently, the direct algebraic-geometric development of the category of tropical varieties has lagged behind. For example, one could define the algebraic set of a polynomial $f$ to be the corner
locus of the function \( R_{\infty}^{(n)} \to R_{-\infty} \) determined by \( f \). This formulation is more cumbersome than the classical formulation in algebraic geometry that \( f(a) = 0, a = (a_1, \ldots, a_n) \), and its awkwardness becomes apparent the moment one starts to work with algebraic sets. The alternative definition used in \([25]\) for the algebraic set of \( f \), namely the set of points on which \( f \) is not differentiable, works well from the perspective of differential geometry, but is even more difficult to apply in various algebraic situations. Consequently, much current research relies heavily on passing back and forth frequently from “classical” algebraic geometry to tropical geometry.

Furthermore, although any non-Archimedean valuation \( \text{val} \) satisfies \( \text{val}(pq) = \text{val}(p) + \text{val}(q) \) as well as\
\[
\text{val}(p + q) = \max\{\text{val}(p), \text{val}(q)\} \quad \text{if} \quad \text{val}(p) \neq \text{val}(q),
\]
one does not know \( \text{val}(p+q) \) in the case that \( \text{val}(p) = \text{val}(q) \). Thus, from the point of view of Kapranov’s Theorem, not only is the max-plus algebra a difficult structure to study, but in some sense it may not even be the right structure.

The first author \([13]\) had addressed these issues by introducing extended tropical arithmetic \( \mathbb{T} \), the disjoint union of two copies of \( \mathbb{R} \), denoted respectively as \( \mathbb{R} \) and \( \mathbb{R}^\nu = \{a^\nu : a \in \mathbb{R}\} \), together with a formal element \(-\infty\). One defines the map \( \nu : \mathbb{T} \to R_{\nu,\infty} \) to be the identity on \( R_{\nu,\infty} := \mathbb{R}^\nu \cup \{-\infty\} \), and to satisfy \( \nu(a) = a^\nu \) for each \( a \in \mathbb{R} \). (As presently defined, \( \nu \) is 1:1.)

\( \mathbb{T} \) is also endowed with the two operations \( \oplus \) and \( \odot \), satisfying the following axioms (using the generic notation that \( a, b \in \mathbb{R}, x, y \in \mathbb{T} \)):

1. \(-\infty \oplus x = x \oplus -\infty = x;\
2. \( x \oplus y = \max\{x, y\} \) unless \( \nu(x) = \nu(y) \);
3. \( a \oplus a = a^\nu \oplus a^\nu = a^\nu \oplus a = a^\nu \);
4. \(-\infty \odot x = x \odot -\infty = -\infty;\
5. \( a \odot b = a + b \) for all \( a, b \in \mathbb{R} \);
6. \( a^\nu \odot b = a \odot b^\nu = a^\nu \odot b^\nu = (a + b)^\nu \).

\( (\mathbb{T}, \oplus, \odot, -\infty, 0) \) is seen in \([13]\) to have the structure of a (non-idempotent) commutative semiring. The verification is a special case of Lemma \([22]\) below. Our motivating example is \( \mathbb{T} \) with this notation, which we call logarithmic notation, where the zero element \( 0_\mathbb{T} = -\infty \).

**Definition 1.3.** A semiring homomorphism \( \nu : \mathbb{R} \to \mathbb{R} \) is idempotent if \( \nu^2 = \nu \).

Note that \( \mathbb{R}^\nu \) (with the max-plus operations) is a sub-semiring without zero of \( \mathbb{T} \) isomorphic to the usual max-plus algebra, and the map \( \nu : \mathbb{T} \to \mathbb{R}_{\nu,\infty} \) is an idempotent semiring homomorphism. Moreover, \( \mathbb{R}_{-\infty} \) is a semiring ideal of \( \mathbb{T} \). In this sense, \( \mathbb{T} \) is a “cover” of the max-plus algebra (and its role is similar to that of a covering space). In applying \( \mathbb{T} \) to tropical geometry, one focuses on the first copy of \( \mathbb{R} \), which we call the set of tangible elements, while elements of \( \mathbb{R}_{-\infty} \) are called ghost elements; \( \mathbb{R}_{-\infty} \) is called the ghost ideal.

The lack of additive inverses is bypassed by identifying all ghost elements in some sense as “zero”; this leads to a much more malleable structure theory, which is also compatible with tropical geometry. The intuition here is that the second component \( \mathbb{R}^\nu \) is a “shadow” of the tangible component \( \mathbb{R} \), with respect to which a ghost element \( a^\nu \) could be interpreted as the interval from \(-\infty\) to \( a \), in the sense that there is an uncertainty and one does not know which element in this interval to choose. Thus, its elements often act as “noise,” especially with regard to multiplication, and one is led to treat this ghost component the same way that one would customary treat the zero element in commutative algebra.

It is surprising how well the use of the ghost ideal enables one to overcome the shortcomings of the general structure theory of semirings. Also, as we shall see in this paper, non-tangible elements also have their own special properties of independent interest.

Polynomials over \( \mathbb{T} \) are defined as formal sums
\[
\bigoplus_{i \geq 0} \alpha_i \lambda^i
\]
where almost all \( \alpha_i = 0_\mathbb{T} \); addition (denoted \( \oplus \)) and multiplication (denoted \( \odot \)) of polynomials are defined in the usual manner. In order to simplify the notation, we write polynomials in the usual notation,
We shall cope with this difficulty shortly.

Multiplicative monoid $(\mathbb{C}^\times)$ is rewritten as

$$(\lambda + 7) \circ (\lambda + 3) = (\lambda \cdot \lambda) \oplus (7 \cdot 3) \circ \lambda \oplus (7 \cdot 3) = (\lambda \circ \lambda) \oplus 7 \cdot \lambda \oplus 10$$

is rewritten as

$$(\lambda + 7)(\lambda + 3) = \lambda^2 + 7\lambda + 10.$$  

Note that the polynomial semiring $\mathbb{T}[\lambda]$ is not a max-plus algebra since, for example,

$$(\lambda + 2) + (2\lambda + 1) = 2\lambda + 2.$$  

We shall cope with this difficulty shortly.

In this paper we generalize the structure of $\mathbb{T}$ to the more abstract setting of a supertropical semiring $R = (R, \mathcal{G}_0, \nu)$, in which $\mathcal{G}_0 := \mathcal{G} \cup \{0 \_\}$ is an ideal, called the ghost ideal, and $\nu : R \rightarrow \mathcal{G}_0$ is an idempotent semiring homomorphism. The “supertropical” structure defined in §3.2 gives an axiomatic description of the extended tropical arithmetic $\mathbb{T}$. Our overall objective is to cover the max-plus algebra by an algebraic structure that we call the supertropical semiring, which has a more reasonable structure, and in whose language many basic concepts of tropical geometry can be described more intrinsically. The main structures for us are the supertropical domain (Definition 3.9) in which $\mathcal{T} = R \setminus \mathcal{G}_0$ is a monoid comprising the tangible elements (which provide the link to tropical geometry), and especially the special case of a supertropical semifield (Definition 3.13) in which $\mathcal{G}$ is an ordered Abelian group.

A few words about our terminology supertropical and its interpretation. Usually “super” in mathematics means graded by the additive group $(\mathbb{Z}_2, +)$. However, here our structure is “graded” by the multiplicative monoid $(\mathbb{Z}_2, \cdot)$ (viewing the tangibles as the 1-component and the ghosts as the 0-component), since the product of elements of degree $i$ and $j$ is an element of degree $ij$. Our focus is on the tangible elements, which provide the link the usual tropical theory. Nevertheless, at times it is useful to view the supertropical semiring as a “cover” of the max-plus algebra, via the ghost map $\nu$.

As noted earlier, the structure of a polynomial semiring over a supertropical semiring is no longer supertropical, so, in order to study polynomials over a supertropical semiring, we introduce a somewhat weaker algebraic structure, that of a semiring with ghosts, which also enables us to handle matrices. Viewing the algebraic theory from this perspective, one can carry over much of the classical theory of commutative algebra and linear algebra.

The roots of a polynomial $f \in R[\lambda_1, \ldots, \lambda_n]$ over a supertropical semiring $R$ are defined as those $n$-tuples $a = (a_1, \ldots, a_n) \in R^{(n)}$ such that $f(a)$ is a ghost element. (We call them roots even when $n > 1$, since the more customary terminology “zeroes” seems misleading in this context.) The geometric object of interest to us is the set of tangible roots of a supertropical polynomial, denoted as $\mathcal{Z}_{\text{tan}}(f)$. This definition encompasses other formulations in tropical geometry, as we see in §6.2 and is considerably weaker than the customary definition of tropical root described above; especially when one needs to add and multiply polynomials.

This definition permits us to describe tropical varieties as in classical algebraic geometry. The tropical variety $\mathcal{A}(f_a)$ arising from the original algebraic variety $W = \mathcal{Z}(f_a)$ should be written as the set of roots of a standard $f_a \in \mathbb{T}[\lambda_1, \ldots, \lambda_n]$, suitably interpreted in our new structure, as to be made explicit in §3 below. This approach provides a clear-cut categorical framework for a direct algebraic study of tropical varieties, without constantly referring back to classical algebraic geometry, much in the spirit that one can study the category of Lie algebras without always referring back to Lie groups. Our approach also yields the extra dividend of providing new, previously inaccessible, examples in tropical geometry, such as subvarieties having the same dimension as the original variety (as exemplified in Figure 4, also cf. Example 6.12).

Our main result in this paper is a tropical version of the Hilbert Nullstellensatz (Theorem 7.17). This part of the theory is rather delicate, because the connection between algebra and geometry is more subtle than in the classical case – here, radical semiring ideals correspond to components of the complements of root sets.

One needs to study factorization of polynomials to facilitate the computation of roots, but this is a delicate matter. Much of the difficulty in factorizing of polynomials arises from the fact that polynomials that look quite different may behave as the same function from $R^{(n)}$ to $R$. Thus, strictly speaking, we should study the natural image of the polynomial semiring in the semiring of functions from $R^{(n)}$ to $R$. This leads to equivalence classes of polynomials which we call $e$-equivalent, and representatives of a specific form, which we call full polynomials.
Let $\mathbb{N}$ denote the positive natural numbers. It is not difficult to show when the supertropical semifield $F$ is $\mathbb{N}$-divisible, that every polynomial that is not a monomial has a tangible root.

Since factorizations of polynomials respect the roots, we consider factorization of polynomials (up to $e$-equivalence). In the case of one indeterminate, one already has the analog \[4\] of the fundamental theorem of algebra, that every tangible polynomial can be factored (as a function) uniquely as a product of linear tangible polynomials, stated in the context of supertropical algebras as Propositions \[5.9\] and \[5.17\] In general, we have a full description of factorization of a polynomial $f$ in one indeterminate (as a product of linear and quadratic factors) in terms of the tangible roots of $f$; cf. Theorems \[8.21\] and \[8.43\] and Proposition \[8.46\].

Although something like unique factorization holds in one indeterminate, it fails miserably in several indeterminates. However, its failure should be interpreted geometrically as the ability to partition a tropical variety in different ways as a union of irreducible subvarieties. All non-unique factorizations that we know are consequences of such geometric ambiguities. From a more positive viewpoint, every polynomial divides a product of binomials (Theorem \[8.53\]), which has the geometric consequence that every algebraic set is embedded naturally into a finite union of hyperplanes; also, there is a way to obtain the minimal such product, as illustrated in Example \[8.63\]. This latter result is best understood in terms of Laurent polynomials (whose root sets match those of polynomials), since we can extend the natural algebraic duality between the max-plus algebra and the min-plus algebra (given by sending an element to its inverse) to the Laurent polynomial semiring without zero, thereby yielding a geometric duality.

One bonus of viewing polynomials (and Laurent polynomials) as functions is the surprising result reminiscent of the Frobenius automorphism (Corollary \[5.28\]):

$$\left( \sum f_i \right)^n = \sum f_i^n$$

for any natural number $n$.

Tangible polynomials provide the (affine) varieties familiar from tropical geometry; yet, nontangible polynomials yield new and interesting examples of varieties. Consequently, our theorems about polynomials often are stated for arbitrary supertropical polynomials, even though the formulations and proofs may be shorter in the tangible case. Also, there is a polyhedron which yields the correspondence of supertropical polynomials with Newton polytopes (Proposition \[8.7\]; this is analogous to the \textbf{grid} in \[2\].

Although the algebraic definitions given in this paper can be generalized even further, permitting different “layers” of ghosts, we feel that the rich theory described above justifies the presentation of the structure theory at the current level of generality. This theory also is useful in describing matrices and solutions to equations. In subsequent papers including \[17\], we develop the matrix theory, including the description of nonsingular matrices in terms of the tropical determinant (which is really the permanent), and a supertropical version of the Hamilton-Cayley theorem. Resultants of supertropical polynomials are studied in \[18\].
2. Valued monoids

In [43] we define our main algebraic structure: Supertropical domains, and supertropical semifields. Since their definitions could seem technical at first, we motivate them with a preliminary structure that provides our major example, as well as a transition to the supertropical theory.

A monoid \( M \) is **ordered** if \( M \) as a set has a total order \( \leq \) such that

\[
ab \leq ac \text{ and } ba \leq ca \quad \text{for all } b \leq c \text{ and } a \text{ in } M.
\]

Given any ordered monoid \((G, +)\), one may adjoin the formal element \(-\infty\) to \(G\) by declaring

\[-\infty < g, \quad \forall g \in G,
\]

and define \((-\infty) + g = g + (-\infty) = -\infty, \forall g \in G\). We denote this new ordered monoid \(G_{-\infty} := G \cup \{-\infty\}\), declaring \(-\infty + -\infty = -\infty\). Of course \(G_{-\infty}\) is not a group, even if \(G\) is a group.

Recall that a monoid homomorphism from \((M, \cdot)\) to \((G, +)\) is a function

\[
\varphi : M \rightarrow G
\]

such that \(\varphi(1_M) = 0_G\) (the neutral element of \(G\)) and \(\varphi(ab) = \varphi(a) + \varphi(b), \forall a, b \in M\).

**Definition 2.1.** A monoid \((M, \cdot)\) is **valued with respect to an ordered cancellative monoid** \((G, +)\) if there is a monoid homomorphism \(\nu : M \rightarrow G\). We notate this set-up as the **triple** \((M, G, \nu)\); \(\nu\) is called the **value function** for \((M, G, \nu)\).

Given any triple \((M, G, \nu)\), where \(G = (G, +)\), we define the **extended semiring** \(T(M, G, \nu)\) to be the triple \((T(M), G_{-\infty}, \nu)\) where \(T(M)\) is the disjoint union \(M \cup G_{-\infty}\), whose value function

\[
\nu : T(M) \rightarrow G_{-\infty}
\]

extends the original value function \(\nu\) by putting \(\nu(g) = g, \forall g \in G_{-\infty}\). Furthermore, \(T(M)\) is made into a semiring, where multiplication is defined by incorporating the given monoid operations of \(M\) and \(G_{-\infty}\), and also defining

\[
(1) \quad (-\infty) \odot x = x \odot (-\infty) = -\infty \quad \text{for all } x \in T(M);
\]

\[
(2) \quad a \odot g = \nu(a) + g \text{ and } g \odot a = g + \nu(a) \quad \text{for all } a \in M, \ g \in G_{-\infty};
\]

addition \(\oplus\) on \(T(M)\) is defined as follows, for \(x, y \in T(M)\):

\[
x \oplus y = \begin{cases} x & \text{if } \nu(x) > \nu(y); \\ y & \text{if } \nu(x) < \nu(y); \\ \nu(x) & \text{if } \nu(x) = \nu(y). \end{cases} \tag{2.1}
\]

**Lemma 2.2.** \(T(M)\) is a semiring.

**Proof.** The operation \(\oplus\) is clearly commutative; to check that \(\odot\) is distributive over \(\oplus\), one wants to verify that

\[
x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z). \tag{2.2}
\]

This is clear if one of the entries is \(-\infty\). If \(\nu(y) \neq \nu(z)\), then by hypothesis \(\nu(x \odot y) \neq \nu(x \odot z)\), so again \((2.2)\) holds.

So assume that \(\nu(y) = \nu(z)\). Then

\[
x \odot (y \oplus z) = x \odot \nu(y) = \nu(x) + \nu(y) = \nu(x \odot y) = (x \odot y) \oplus (x \odot z),
\]

as desired. Associativity of addition is checked in a similar fashion. Associativity of multiplication is clear.

Note that the zero element of \(T(M)\) is \(-\infty\), whereas the original unit element \(1_M\) of \(M\) is also the unit element of \(T(M)\), in view of the following verifications:

\[
1_M \odot (-\infty) = (-\infty) \odot 1_M = -\infty, \quad 1_M \odot a = a \odot 1_M = a
\]

for all \(a \in M\), by definition, whereas, for all \(g \in G_{-\infty}\),

\[
1_M \odot g = \nu(1_M) + g = (-\infty) + g = g.
\]
Remark 2.3. The monoid \((T(M), \odot)\) is valued in \(\mathcal{G}_{-\infty}\), with respect to the value function \(\nu\). Indeed, let us check that
\[
\nu(x \odot y) = \nu(x) + \nu(y), \quad \forall x, y \in T(M).
\]
If \(x\) and \(y\) are both in \(M\) or in \(\mathcal{G}_{-\infty}\), then this is true by definition, so suppose \(x \in M\) and \(y \in \mathcal{G}_{-\infty}\). Then
\[
\nu(x \odot y) = \nu(x) + \nu(y) = \nu(x) + \nu(x) + \nu(y) = \nu(x) + \nu(y).
\]

Example 2.4. Here are some examples of extended semirings (of valued monoids).

\begin{enumerate}[(i)]
\item For any ordered monoid \((\mathcal{G}, +)\), we have the extended semiring \(D(\mathcal{G}) := T(\mathcal{G}, \mathbb{G}, 1_\mathcal{G})\), where \(M = \mathcal{G}\) and \(1_\mathcal{G}\) is the identity map. The semiring \(D(\mathcal{G})\) is the extended tropical arithmetic, in the sense of \cite{13}. Note that in \(D(\mathcal{G})\), \(\nu\) restricts to a 1:1 map \(\nu_M : M \to \mathcal{G}\).

\item Recall that \(\mathbb{R}^\times\) denotes \(\mathbb{R} \setminus \{0\}\), and \(\mathbb{R}^+\) denotes the positive real numbers. The group \((\mathbb{R}^\times, \cdot)\), with its absolute value, yields the triple \((\mathbb{R}^\times, \mathbb{R}^+, | |)\). We often refer back to this example for intuition in the case that \(v\) is not 1:1. Likewise, for any ordered field \(F\), we have the valued monoid \((F^\times, F^+, | |)\).

\item If \(F\) is a field with valuation \(v : F \to \mathcal{G}\), then \(T(F^\times, \mathcal{G}, v)\) is an extended semiring. (In particular, we could take \(F\) to be the field of Puiseux series.)

\item Algebraic groups over ordered fields, or over fields with valuation, can be valued by means of the determinant.
\end{enumerate}

Remark 2.5. Given a triple \((M, \mathcal{G}, v)\), one can also define the dual extended semiring \(T^\dual(M, \mathcal{G}, v)\), where addition is defined by reversing the order in the formula \((\ref{eq:3.5})\); namely \(x \oplus y\) equals \(y\) if \(\nu(x) > \nu(y)\), and equals \(x\) if \(\nu(x) < \nu(y)\). As before, we formally adjoin the element \(-\infty\). This duality will be explained algebraically in \cite{13}.

3. Supertropical semirings

We start this section by laying out the basic algebraic notion of a supertropical domain, showing how it is just a reformulation of a valued monoid. Then, having made the transition to semirings, we can then bring in related semirings such as the semiring of functions of \([3.5]\) and the polynomial semiring. In this paper, we assume throughout that all of our semirings are commutative (under multiplication as well as addition), although in \cite{17} we need to drop this assumption in order to deal with matrices.

Having already constructed our main object \(R = T(M)\), together with the operations \(\oplus\) and \(\odot\), let us first describe it more intrinsically in the language of semirings.

Note 3.1. In line with the customary algebraic notation for semirings, the zero element \(0_R\) of \(R\) replaces what we originally called \(-\infty\). Under this notation, we write \(\mathcal{G}_0\) instead of \(\mathcal{G}_{-\infty}\). Likewise, multiplication in \(R\) is taken to subsume the original monoid operation of \(\mathcal{G}\), so \(\nu(1_R)\) is the neutral element of \(\mathcal{G}\).

But in order to emphasize the tropical aspect, we often revert to what we have called logarithmic notation when discussing our motivating example \(R = T(\mathbb{R})\); in these instances we retain the usage of \(-\infty\) for the zero element and 0 for the multiplicative unit.

3.1. Semirings with a designated ghost ideal. All of our structures fit into the framework of a semiring \(R\) with a designated ideal \(\mathcal{G}_0 := \mathcal{G} \cup \{0_R\}\), called throughout the ghost ideal. Recall from \cite{9} that an ideal of a semiring \(R\), denoted \(A \triangleleft R\), is a submonoid \(A\) of the monoid \((R, +)\) such that \(ra \in A\) for all \(r \in R\) and \(a \in A\).

In the following definition we consider \(\mathcal{G}_0\) as a semiring in its own right, with neutral element \(\nu(1_R)\).

Definition 3.2. A semiring with ghosts \((R, \mathcal{G}_0, \nu)\) is a semiring \(R\) (with zero element denoted as \(0_R\)) together with a semiring ideal \(\mathcal{G}_0\), called the ghost ideal, and an idempotent semiring homomorphism \(\nu : R \to \mathcal{G}_0\), called the ghost map, satisfying
\[
a + a = \nu(a), \quad \forall a \in R.
\] (3.1)

From now on we formulate tropical concepts in the language of supertropical semirings, in order to draw from the structure theory of semirings (together with its parallels in ring theory).
Remark 3.3. The notion of ideal is standard in semiring theory. [9, Proposition 9.10] shows that an ideal $A$ of $R$ is a kernel of a suitable homomorphism iff $A$ is subtractive, which means that for every $a, b \in R$ such that $a \in A$ and $a + b \in A$, we must have $b \in A$. Whereas one often goes on to define a congruence and a quotient structure, cf. [9, p. 68], this approach is not relevant to our theory here, specifically for $G_0$. Indeed, for any element $a \in R$, we have both $2a = a + a \in G_0$ and $2a + a \in G_0$, so from this point of view, there is only one coset of $G_0$, which is all of $R$. The ghost ideal $G_0$ is far from subtractive, and we must abandon this aspect of classical semiring theory; the main feature of this research is an alternative structure theory utilizing the ghost ideal in a fundamental role.

We are finally ready for the main definition of this paper.

Definition 3.4. A supertropical semiring is a semiring with ghosts $(R, G_0, \nu)$, satisfying the extra properties, where we write $a^\nu$ for $\nu(a)$:

(a) (Bipotence) $a + b \in \{a, b\}$, $\forall a, b \in R$ such that $a^\nu \neq b^\nu$;

(b) (Supertropicality) $a + b = a^\nu$ if $a^\nu = b^\nu$.

Note that Equation (3.1), a special case of supertropicality, implies that the ghost map $\nu$ is given by $\nu(a) = a + a$.

Remark 3.5.

(i) It follows from Equation (3.1) that $(ab)^\nu = ab + ab = (a + a)b = (a^\nu)b$, and likewise $(ab)^\nu = a(b^\nu)$. Thus, $(ab)^\nu = (a^\nu b^\nu) = a^\nu b^\nu$. In particular, $b^\nu = 1^\nu b$.

(ii) If $a^\nu = 0_R$, then $a = a + 0_R = 0_R^\nu = 0_R$. It follows that $\nu(R \setminus \{0_R\}) \subseteq G$.

(iii) The fact that $G_0$ is an ideal of the supertropical semiring $R$ is a formal consequence of the properties of the map $\nu$. Indeed, if $a \in R$ and $b \in G_0$, then $ab = a(b^\nu) = (ab)^\nu \in G_0$, and likewise $ba \in G_0$. ($G_0$ is closed under addition, by bipotence.)

(iv) If $a + b = 0_R$, then $a = b = 0_R$. Indeed, if $a^\nu \neq b^\nu$, then $a + b \notin \{a, b\}$; let us assume that $a + b = a$. Then $a = a + 0 = 0$, so $a + b = b$, and $a = b = 0$.

We have shown that $a^\nu = b^\nu$; but then $a^\nu = a + b = 0_R$, implying $a = 0_R$, by (ii).

Strictly speaking, one could have $G_0 = R$, with $\nu$ the identity map. In this case $R$ is an additively idempotent semiring, such as the usual max-plus algebra. However, we view this case as degenerate, and are much more interested in the case where $G_0$ is a proper ideal of $R$.

Remark 3.6 (Universal characteristic). Supertropicality implies that $G_0 \supseteq \{nr : r \in R\}$, (where $nr = r + \cdots + r$ repeated $n$ times), for every natural number $n > 1$; more precisely, $a + a = a + a + a = \cdots = a^\nu \in G_0$, $\forall a \in R$. Thus, $R$ might be expected simultaneously to have properties of rings of every positive characteristic.

This leads to a surprising fact.

Proposition 3.7. If $R$ is a supertropical semiring and $a, b \in R$, then

$$(a + b)^m = a^m + b^m, \quad \forall m \in \mathbb{N}.$$  \hspace{7cm} (3.2)

Proof. We need to show that the only terms needed to compute $(a + b)^m$ are $a^m$ and $b^m$. Write

$$(a + b)^m = a^m + \sum_{k=1}^{m-1} R^k a^{m-k-1} b + \cdots + \sum R^m a^m b^{m-1} + b^m;$$

then (3.2) is clear if $a \cong b$, since each side of Equation (3.2) is then $(a^m)^\nu$, so we may assume that $a > \nu b$. Then $a^m > \nu a^i b^{m-i}$ whenever $i + j = m$. This means that the single dominating term in the expansion of $(a + b)^m$ is $a^m$; i.e.,

$$(a + b)^m = a^m = a^m + b^m,$$

as desired. \hfill \Box

Remark 3.8. A suggestive way of viewing this proposition is to note that for any $m$ there is a semiring endomorphism $R \to R$ given by $a \mapsto a^m$, strongly reminiscent of the Frobenius automorphism in classical algebra. This plays an important role in our theory.
3.2. Supertropical domains and semifields. Suppose \((R, \mathcal{G}_0, \nu)\) is a supertropical semiring.

**Definition 3.9.** A **supertropical domain** \((R, \mathcal{G}_0, \nu)\) is a supertropical semiring for which the following extra properties hold:

(a) \(T := R \setminus \mathcal{G}_0\) is a (multiplicative) Abelian monoid; i.e., is closed under multiplication.

(b) The restriction \(\nu_T\) of \(\nu\) to \(T\) is onto; in other words, every element of \(\mathcal{G}\) has the form \(a^\nu\) for some \(a \in T\).

Note that the ghost ideal \(\mathcal{G}_0\) of a supertropical domain \(R = (R, \mathcal{G}_0, \nu)\) determines both \(T = R \setminus \mathcal{G}_0\) and \(\mathcal{G} = \mathcal{G}_0 \setminus \{0_R\}\).

**Remark 3.10.** One can obtain condition (b) by replacing \(R\) by \(T \cup \nu(T) \cup \{0_R\}\).

We call \(T\) the set of **tangible elements**; these comprise one of our main focuses, since they lead us back to tropical geometry. Ironically, for supertropical semirings in general, the tangible elements are more complicated to define than the ghost elements. In an arbitrary semiring with ghosts, the definition of \(T\) is much subtler, but we do not consider that issue in this paper. Two elements of \(R\) have the same **parity** if they are both ghosts or both tangible.

**Notation 3.11.** Viewing \(\mathcal{G}_0\) as an ordered monoid, we write \(a >_\nu b\) (resp. \(a \geq_\nu b\)) to denote \(a^\nu > b^\nu\) (resp. \(a^\nu \geq b^\nu\)); we write \(a \equiv_\nu b\) to denote that \(a^\nu = b^\nu\). We say \(a\) is \(\nu\)-maximal in \(S \subset R\) if \(a \equiv_\nu s\) for all \(s \in S\).

A major question in algebra is when two elements are equal. Normally in a ring one determines whether \(a = b\) by checking if \(a - b = 0\). This simple procedure is no longer available in general semirings, but in our supertropical setting we note for tangible \(a, b \in R\) that \(a \equiv_\nu b\) if \(a + b \in \mathcal{G}\). This point of view provides the motivation for the supertropical theory. We also define an equivalence \(\equiv\) by the rule \(a \equiv b\) iff \(a, b\) have the same parity with \(a \equiv_\nu b\). This means that either \(a = b\) or \(a, b \in T\) with \(a \equiv_\nu b\).

(Thus, when \(\nu\) is 1:1, this reduces to equality.) Equivalent elements are interchangeable in the sense that if \(a \equiv b\) then \(ac \equiv bc\) and \(a + c \equiv b + c\) for all elements \(c\).

Much of the theory can be carried out for \(T\) not necessarily Abelian, but this assumption is useful when we consider factorization of polynomials (and is even more crucial for studying matrices later on).

The mild condition that \(T\) is a multiplicative monoid has some impressive consequences.

**Remark 3.12.** Suppose that \(R\) is a supertropical domain.

(i) \(R\) is \(\nu\)-cancellative, in the sense that \(ca \equiv_\nu cb\) for \(c \neq 0_R\) implies \(a \equiv_\nu b\). Indeed, since \(\nu_T\) is onto, we may assume that \(a, b, c \in T\). But then \(c(a + b) = ca + cb \in \mathcal{G}\) by supertropicality, which contradicts the fact that \(T\) is a monoid unless \(a + b \in \mathcal{G}\); i.e., \(a \equiv_\nu b\).

In particular, the monoid \(\mathcal{G}\) is cancellative.

(ii) If \(ca \equiv_\nu db\) and \(c \equiv_\nu d\) \(\neq 0_R\), then \(a \equiv_\nu b\). Indeed, \((ca)^\nu = (db)^\nu = d^\nu b^\nu = c^\nu b^\nu\), so we are done by (i).

(iii) If \(ca >_\nu cb\), then \(a >_\nu b\). Indeed, again we may assume that \(a, b, c \in T\). If \(a \equiv_\nu b\), then \(a + b \in \mathcal{G}\), implying \(ca + cb = c(a + b) \in \mathcal{G}\), contradicting the fact that \(ca + cb = ca \in T\).

(iv) The same argument shows that \(R\) has cancellation over \(\mathbb{N}\), in the sense that \(a^n \equiv_\nu b^n\) implies \(a \equiv_\nu b\). Indeed, again we may assume that \(a, b \in T\). But Proposition 3.13 implies that \((a + b)^n = a^n + b^n \in \mathcal{G}\), implying \(a + b \in \mathcal{G}\); i.e., \(a \equiv_\nu b\).

(v) It follows from (i) and (iv) (by applying \(\nu\)) that the monoid \(\mathcal{G}\) is cancellative and also has cancellation over \(\mathbb{N}\).

(vi) \(R\) is a commutative semiring. Indeed, any two elements of \(T\) commute, by definition; hence,

\[
a(b^\nu) = (ab)^\nu = (ba)^\nu = (b^\nu)a; \quad a^\nu b^\nu = (ab)^\nu = (ba)^\nu = b^\nu a^\nu.
\]

Let us tie supertropical domains to the preliminary notions of the previous section.

**Remark 3.13.** Any valued Abelian monoid \((M, \mathfrak{G}, \nu)\) (as in Definition 3.4) gives rise to the supertropical domain \((T(M), \mathcal{G}_0, \nu)\); cf. Remark 2.3.
Conversely, given a supertropical domain \((R, \mathcal{G}_0, \nu)\), we can recover the valued monoid \(M = T = R \setminus \mathcal{G}_0\), and \(v = \nu_T : T \to \mathcal{G}\) provides the value function. We view \(\mathcal{G}_0\) as a cancellative ordered monoid (but now with its operation being written as multiplication), under the following order:

\[
g \geq h \text{ in } \mathcal{G}_0 \iff g + h = g \text{ in } R.
\]

To verify that “≥” defines an order, note that the properties of identity and antisymmetry are immediate; to check transitivity, we note that if \(g_1 \geq g_2\) and \(g_2 \geq g_3\), then the following equalities hold in \(R\):

\[
g_1 + g_3 = (g_1 + g_2) + g_3 = g_1 + (g_2 + g_3) = g_1 + g_2 = g_1.
\]

We want to identify \(\langle T(T), \mathcal{G}_0, \nu \rangle\) with \((R, \mathcal{G}_0, \nu)\). Definition 3.14 gives as Definition 2.1, and Remark 2.4 is seen case by case, assuming \(v(a) > v(b)\):

(a) \(a^\nu + b^\nu = a^\nu\), so

\[
(a + b)^\nu = a^\nu + b^\nu = a^\nu.
\]

Also \(a + b = a\), for if \(a + b = b\), then \(v(a) = a^\nu = (a + b)^\nu = b^\nu = v(b)\), contrary to assumption.

(b) Likewise, \(a + b^\nu = a\), for if \(a + b^\nu = b^\nu\), the same argument would show that

\[
v(a) = a^\nu + b^\nu = v(a + b) = v(b),
\]

contrary to assumption.

The following computation is useful in later sections.

**Lemma 3.14.**

(i) If \(b^2 = ac\) in a supertropical semiring, then \(a + c = a + b + c\).

(ii) More generally (but over a supertropical domain), if \(bc \equiv \nu ad\), then \(ac + bd = (a + b)(c + d)\).

**Proof.** (i) If \(a \geq \nu b\) or \(c \geq \nu b\), then there is nothing to prove, so we may assume that \(a \leq \nu b\) and \(c \leq \nu b\). But then if \(a \leq \nu b\) or \(c \leq \nu b\), we would have \(ac \leq \nu b^2\) by Remark 3.12(ii), contrary to hypothesis. Hence, \(a \equiv \nu b \equiv \nu c\), implying

\[
a + b + c = a^\nu = a + c.
\]

(ii) By symmetry we may assume that \(ad \geq \nu bc\); we are done unless \(ad \geq \nu ac\) and \(ad \geq \nu bd\), implying by Remark 3.12 that \(d \equiv \nu c\) and \(a \equiv \nu b\); hence

\[
ac + bd = (ac)^\nu = a^\nu c^\nu = (a + b)(c + d).
\]

\(\square\)

**Definition 3.15.** A supertropical domain \((R, \mathcal{G}_0, \nu)\) is called a supertropical semifield if \(T\) is a (multiplicative) Abelian group, i.e., if every tangible element is invertible.

Supertropical semifields play a basic role in our theory, analogous to the role of fields in linear algebra and algebraic geometry.

**Remark 3.16.** For any supertropical semifield \((R, \mathcal{G}_0, \nu)\), \(\mathcal{G}\) is a group with neutral element \(1_{R^\nu}\). Indeed, if \(a^\nu \in \mathcal{G}\) for \(a \in T\), then taking \(b \in T\) such that \(ab = 1_R\), we have \(a^\nu b^\nu = 1_{R^\nu}\); thus \(a^\nu\) is invertible in \(\mathcal{G}\), as desired.

We usually designate a supertropical semifield as \(F = (F, \mathcal{G}_0, \nu)\), still denoting the ghost ideal as \(\mathcal{G}_0\).

**Example 3.17.** The extended semirings of Example 2.4 all are supertropical semifields.

### 3.3. Supertropical duality

For any supertropical domain \(R = (R, \mathcal{G}_0, \nu)\), the set \(R^+ = R \setminus \{0_R\}\) is a semiring without zero, and one can define the dual semiring without zero \(R^\nu_+\) to have the same underlying set as \(R^+\) with the same tangibles and ghosts, the same ghost map \(\nu\), and the same multiplication, but with addition defined by putting \(a + b = b\) in \(R^\nu\) iff \(a + b = a\) in \(R^+\), and \(a + b = a^\nu\) in \(R^\nu\) whenever \(a \equiv \nu b\). (This is well-defined in view of supertropicality.)

Formally adjoining a zero element \(0_{R^\nu}\) to \(R^\nu_+\) yields a semiring which we call the supertropical dual \(R^\nu\). The zero element \(0_R\) of \(R\) has been treated separately since if we formally included \(0_R\) in \(R^\nu\), it would behave like \(\infty\) rather than like the zero element. \(R^\nu\) is a supertropical domain, seen by combining Remarks 2.3 and 2.5 and in fact is the supertropical domain that matches the min-plus algebra in [30].
Lemma 3.18. When $R$ is a supertropical semifield, so is its supertropical dual $R^\wedge$, and moreover there is a semiring isomorphism $\Phi : R \to R^\wedge$ given by

$$0_R \mapsto 0_{R^\wedge}, \quad a \mapsto a^{-1}, \quad a' \mapsto (a^{-1})'. \quad \text{for all } a \in T.$$

Proof. Take $a, b \in T$. If $a + b = a$, then $a \geq b$, so $a^{-1} \leq b^{-1}$, implying

$$\Phi(a + b) = \Phi(a) = a^{-1} = a^{-1} + b^{-1} = \Phi(a) + \Phi(b)$$

in $R^\wedge$. The same argument works with nonzero ghosts. For $a \in T$, $b = 0_R$, we have

$$\Phi(a + b) = a^{-1} = a^{-1} + 0_{R^\wedge} = \Phi(a) + \Phi(b).$$

$\square$

This duality provides the reversals of polyhedral complexes in tropical geometry, as described in [16].

3.4. The divisible closure of a supertropical domain. Given any ordered monoid $(\mathcal{M}, +)$ with cancellation over $\mathbb{N}$, one can form an $\mathbb{N}$-divisible ordered monoid

$$\mathcal{M} = \left\{ \frac{a}{m} : a \in \mathcal{M}, \ m \in \mathbb{N} \right\},$$

called the divisible closure of $\mathcal{M}$; here $\frac{a}{m} \leq \frac{b}{n}$ iff $na \leq mb$. The canonical map $\mathcal{M} \to \mathcal{M}$ given by $a \mapsto \frac{a}{1}$ is 1:1. When $\mathcal{M}$ is a group, $\mathcal{M}$ is then a group which can be viewed as containing $\mathcal{M}$.

We want to perform the same procedure for a supertropical domain $R$, but now proceed using the semiring notation. We first note by Remark [3.12] that the ghost set $\mathcal{G}_0$ has cancellation over $\mathbb{N}$. Viewing the ghost ideal $\mathcal{G}$ as an ordered monoid as in Remark [3.13] (with respect to multiplication), we form its divisible closure $\mathcal{G}$, which we notate

$$\mathcal{G} = \left\{ \sqrt[n]{a} : a \in \mathcal{G}, \ m \in \mathbb{N} \right\}.$$

We formally define the $\mathbb{N}$-localization

$$\mathcal{R} = \left\{ \sqrt[n]{a} : a \in R, \ m \in \mathbb{N} \right\};$$

here $\sqrt[n]{a} = \sqrt[n]{b}$ when $a^m = b^m$ for some $n \in \mathbb{N}$. (In the tropical examples, using logarithmic notation, one would write $\frac{a}{m}$ instead of $\sqrt[n]{a}$.)

Multiplication is defined by

$$\sqrt[n]{a} \cdot \sqrt[m]{b} = \sqrt[n \cdot m]{a^m b^m},$$

and addition by

$$\sqrt[n]{a} + \sqrt[m]{b} = \sqrt[n \cdot m]{a^m + b^m}.$$

We extend $\nu : R \to \mathcal{G}$ to a map $\mathcal{R} \to \mathcal{G}$ by putting $\nu(\sqrt[n]{a}) = \sqrt[n]{a'}$, and call $\mathcal{R}$ the divisible closure of $R$.

We say that $R$ is divisibly closed if $\mathcal{R} = R$. For example, $D(\mathbb{Q})$ is divisibly closed.

Proposition 3.19. If $(R, \mathcal{G}, \nu)$ is a supertropical domain, then $(\mathcal{R}, \mathcal{G}, \nu)$ is also a supertropical domain, and there is a semiring homomorphism $R \to \mathcal{R}$ given by $a \mapsto a$ (identifying $\sqrt[n]{a}$ with $a$) which is 1:1 on equivalence classes with respect to our equivalence relation $\equiv$. When $R$ is a supertropical semifield, $\mathcal{R}$ is also a divisibly closed supertropical semifield.

Proof. The operations are clearly well-defined. For example, if $\sqrt[n]{a} = \sqrt[n]{a'}$ and $\sqrt[m]{b} = \sqrt[m]{b'}$, then for some numbers $k, \ell$ we have $a^m = a'^m$ and $b^n = b'^n$, so

$$(ab)^{m'k} = (a'b')^{m'k},$$

implying $m\sqrt[n]{ab} = m\sqrt[n]{a'b'}$. Clearly $\mathcal{R}$ is the disjoint union of $\mathcal{T} = \{ \sqrt[n]{a} : a \in T, \ m \in \mathbb{N} \}$ and $\mathcal{G}_0$, and Proposition [3.17] shows that $\mathcal{R}$ is a divisibly closed supertropical domain.

It remains to show that if $\sqrt[n]{a} \equiv \sqrt[n]{b}$ then $a \equiv b$. By definition $a^n \equiv b^n$ for some $n$, implying $a \equiv b$ by Remark [3.12](iv), and clearly $a, b$ have the same parity, so $a \equiv b$. $\square$

The reason for passing to the divisible closure is to enrich the structure by means of the following observation:
Remark 3.20. If $R$ is divisibly closed, then $a^{m/n}$ is defined in $R$ for any $a \in R$ and any rational number $\frac{m}{n}$.

Remark 3.21. By the same token as in Proposition 3.23, in view of Remark 3.13, one can formally localize the ghost elements of a supertropical domain to obtain a supertropical semifield, under which process equivalence classes are preserved. This trick enables one to extend many of the results about supertropical semifields to supertropical domains.

3.5. The semiring of continuous functions. It is useful to introduce the following topology on $R$, obtained from the order topology on $\mathbb{G}$:

Definition 3.22. Suppose $(R, \mathbb{G}, \nu)$ is a supertropical domain. Viewing $\mathbb{G}$ as an ordered monoid, as in Remark 3.18, we define the $\nu$-topology on $R$, whose open sets have a base comprised of the open intervals

$$W_{\alpha, \beta} = \{a \in R : \alpha < a^\nu < \beta\}; \quad W_{\alpha, \beta;T} = \{a \in T : \alpha < a^\nu < \beta\}, \quad \alpha, \beta \in \mathbb{G}.$$ 

We also define $[\alpha, \beta] = \{a \in T : \alpha \leq a^\nu \leq \beta\}$. In other words, $[\alpha, \beta]$ is the intersection of $T$ with the closure of $W_{\alpha, \beta;T}$, and we call it a tangible closed interval.

Remark 3.23. If $R$ is divisibly closed, then $T$ is dense in $R$ in the sense that each nonempty open interval contains a tangible element. (Indeed, if $\alpha, \beta \in T$ with $\alpha < \nu \beta$, then $\alpha^{1/2} \beta^{1/2} \in W_{\alpha, \beta}$, in view of Remark 3.22.)

Here is an important semiring construction, given in [10].

Definition 3.24. Given any set $S$ and semiring $R$, we define the semiring $\text{Fun}(S, R)$ of functions from $S$ to $R$, under pointwise addition and multiplication. The zero function $0_{\text{Fun}}$ is given by $0_{\text{Fun}}(a) = 0_R$ for all $a \in S$.

Remark 3.25. The map $f \mapsto f(a)$ is a semiring homomorphism $\text{Fun}(S, R) \to R$, for any fixed $a \in S$.

Remark 3.26. When $(R, \mathbb{G}, \nu)$ is a semiring with ghosts, then the semiring $\text{Fun}(S, R)$ also is viewed as a semiring with ghosts, where a function $f \in \text{Fun}(S, R)$ is said to be ghost if

$$f(a) \in \mathbb{G} \quad \text{for every } a \in S.$$ 

The ghost ideal $\text{Fun}(S, \mathbb{G})$ of $\text{Fun}(S, R)$ is the set of ghost functions, and the ghost map $\nu$ is defined by $f \mapsto f^\nu$, where by definition

$$f^\nu(a) := f(a)^\nu, \quad \forall a \in S.$$ 

Proposition 3.27. If $f, g \in \text{Fun}(S, R)$, then $(f + g)^m = f^m + g^m$ for any positive $m \in \mathbb{N}$.

Proof. In view of Proposition 3.21

$$(f + g)^m(a) = (f(a) + g(a))^m = f(a)^m + g(a)^m = f^m(a) + g^m(a) \quad (3.1)$$

for each $a \in S$. \hfill $\Box$

Corollary 3.28. If $f_1, \ldots, f_k \in \text{Fun}(S, R)$, then

$$\left( \sum_{i=1}^{k} f_i \right)^m = \sum_{i=1}^{k} f_i^m,$$

for any positive $m \in \mathbb{N}$.

We also have duality:

Remark 3.29. The isomorphism $\Phi : F \to F^\wedge$ of Remark 3.18 extends to an isomorphism $\Phi_{\text{Fun}} : \text{Fun}(S, F) \to \text{Fun}(S, F^\wedge)$ given by $f \mapsto f^\wedge$, where $f^\wedge(a) = \Phi(f(a))$. (Indeed, for $f(a), g(a) \in T$,

$$(f^\wedge + g^\wedge)(a) = f(a)^{-1} + g(a)^{-1} = (f + g)^\wedge(a),$$

where “+” is taken in the appropriate semiring; the other verifications are analogous.)
Definition 3.35. If \( A \in R \), define

\[
\Phi_a : \text{Fun}(R^{(n)}, R) \to \text{Fun}(R^{(n-1)}, R)
\]

by sending \( f \mapsto f_a \), where

\[
f_a(a_1, \ldots, a_{n-1}) = f(a_1, \ldots, a_{n-1}, a).
\]

Then \( \Phi_a \) is a semiring homomorphism.

Proof. Write \( a = (a_1, \ldots, a_{n-1}) \) and \( a(a) = (a_1, \ldots, a_{n-1}, a) \). Then

\[
\Phi_a(f + g)(a) = (f + g)(a(a)) = f(a(a)) + g(a(a)) = \Phi_a(f)(a) + \Phi_a(g)(a),
\]
yielding \( \Phi_a(f + g) = \Phi_a(f) + \Phi_a(g) \); the verification for multiplication is analogous, and \( \Phi_a(0_R) = 0_R \). \( \Box \)

Later on, we consider

\[
\ker \Phi_a = \{ f \in \text{Fun}(R^{(n)}, R) : f(a_1, \ldots, a_{n-1}, a) = 0_R : \forall a_i \in R \}.
\]

This is a rather restrictive view of the kernel, and is to be weakened in subsequent research.

Let us bring in the \( \nu \)-topology.

Definition 3.31. \( \text{CFun}(R^{(n)}, R) \) is the sub-semiring with ghosts, comprised of functions in the semiring \( \text{Fun}(R^{(n)}, R) \) which are continuous with respect to the \( \nu \)-topology of Definition 3.26.

\( \text{CFun}(R^{(n)}, R) \) plays a very important role in this paper.

Definition 3.32. Given \( a = (a_1, \ldots, a_n) \in R^{(n)} \) and \( \beta_1, \ldots, \beta_n \in T \) with each \( \beta_i \geq_\nu 1_R \), the closed \( \mathbf{a} \)-box is defined as the product of closed tangible intervals (cf. Definition 3.26)

\[
\left[ \frac{a_1}{\beta_1}, \beta_1a_1 \right] \times \left[ \frac{a_2}{\beta_2}, \beta_2a_2 \right] \times \cdots \times \left[ \frac{a_n}{\beta_n}, \beta_na_n \right] \subset T^{(n)}.
\]

Definition 3.33. The semifield \( F \) is archimedean, if for every \( a >_\nu 1_F \) and \( b \) in \( F \), there is suitable \( m \) such that \( a^m >_\nu b \).

This guarantees that \( F \) (and thus \( T \)) has “large enough” and “small enough” elements.

Remark 3.34. If \( (R, G_0, \nu) \) is a supertropical domain, then \( R^{(n)} \) is endowed with the usual product topology (obtained from the \( \nu \)-topology on \( R \)). When \( (R, G_0, \nu) \) is an archimedean supertropical semifield, the closed boxes comprise a sub-base for the closed sets of the relative topology on \( T^{(n)} \).

3.6. Radical ideals and prime ideals of semirings.

Definition 3.35. Suppose \( A \subset R \). The radical \( \sqrt{A} \) is defined as \( \{ a \in R : a^k \in A \text{ for some } k \} \). An ideal \( A \) of \( R \) is radical if \( A = \sqrt{A} \).

Remark 3.36. If \( A \) is an ideal of a commutative semiring \( R \), then \( \sqrt{A} \subset R \), by the usual ring-theoretic argument. More surprisingly, if \( R \) is a commutative supertropical semiring and \( A \) is a sub-semiring of \( R \), then \( \sqrt{A} \) is also a sub-semiring of \( R \), by Proposition 3.17; by the same reasoning, if \( W \) is a sub-semiring of \( \text{Fun}(R^{(n)}, R) \), then \( \sqrt{W} \) is also a sub-semiring of \( \text{Fun}(R^{(n)}, R) \), by Corollary 3.28.

The following definition is also lifted from ring theory.

Definition 3.37. An ideal \( P \) of a semiring \( R \) is prime if it satisfies the following condition:

\[
AB \subset P \text{ for } A, B \in R \text{ implies } A \subset P \text{ or } B \subset P.
\]

Proposition 3.38. Every radical ideal \( A \) of a commutative semiring \( R \) is the intersection of prime ideals.

Proof. We copy the standard argument from commutative algebra. For any element \( b \notin A \), take an ideal of \( P \) maximal with respect to \( b^k \notin P \), for each \( k \in \mathbb{N} \). Then \( P \) is a prime ideal, since if \( a_1a_2 \in P \) with \( a_1, a_2 \notin P \), then, for \( i = 1, 2 \) the ideal \( P + Ra_i \) properly contains \( P \), and thus contains a power \( b^{k_i} \) of \( b \); Letting \( k = k_1 + k_2 \) we see that \( (P + Ra_1)(P + Ra_2) \subset P \) contains \( b^k \), contradiction. \( \Box \)
3.7. Ghost-closed ideals.

**Definition 3.39.** A ghost-closed ideal of a semiring \( R = (R, \mathcal{G}_0, \nu) \) with ghosts is a semiring ideal containing the ghost ideal \( \mathcal{G}_0 \).

Clearly, a supertropical domain \((R, \mathcal{G}_0, \nu)\) is a supertropical semifield iff it has no proper ghost-closed ideals other than \( \mathcal{G}_0 \). This is one reason why we focus on ghost-closed ideals.

**Example 3.40.** The ghost ideal \( \text{Fun}(F(n), \mathcal{G}_0) \) is itself a radical ideal of the supertropical semiring \( \text{Fun}(R(n), R) \). Indeed, if \( f^m(a) \in \mathcal{G}_0 \), then \( f(a) \in \mathcal{G}_0 \). By Proposition 3.38, \( \text{Fun}(F(n), \mathcal{G}_0) \) is the intersection of prime ideals, each of which clearly is ghost-closed.

**Definition 3.41.** The ghost-closed ideal \( \langle S \rangle \) (classically generated) by a set \( S \) is the intersection of all ghost-closed ideals containing \( S \) (or, in other words, the ideal generated by \( S \) and \( \mathcal{G} \)).

There is a weaker version of this definition, called “tropical generation,” which although more appropriate to the tropical theory is more technical; in this paper we focus on classical generation in order to obtain more precise information about the ideals in question.

3.8. Supertropical divisibility and the supertropical radical. We say that \( a = b + \text{ghost} \) in a semiring \( R \) with ghosts when \( a = b + c \) for some ghost element \( c \in \mathcal{G}_0 \); in this case, we write \( a \equiv b \). This relation arises naturally in many supertropical contexts, including the following.

**Definition 3.42.** In any semiring \( R \), an element \( b \in R \) divides \( a \in R \) if \( a = qb \) for some \( q \in R \). For \( R \) a semiring with ghosts, an element \( b \in R \) supertropically divides \( a \in R \) if \( a \equiv qb \) for some \( q \in R \).

**Definition 3.43.** Suppose \( A \subset R \). The supertropical radical \( \sqrt{\text{trop}} A \) is defined as the set

\[
\{ a \in R : \text{some power } a^k \text{ is supertropically divisible by an element of } A \}.
\]

An ideal \( A \) of \( R \) is supertropically radical if \( A = \sqrt{\text{trop}} A \).

**Remark 3.44.** If \( A \) is an ideal of a commutative semiring \( R \) with ghosts, then \( \sqrt{\text{trop}} A = \sqrt{A + \mathcal{G}_0} \triangleleft R \). It follows at once that every supertropically radical ideal of a commutative semiring \( R \) with ghosts is the intersection of ghost-closed prime ideals of \( R \) (and vice versa).

By the same sort of argument, in analogy with Remark 3.44, if \( R \) is a commutative supertropical semiring and \( A \) is a sub-semiring of \( \text{Fun}(R(n), R) \), then \( \sqrt{\text{trop}} A \) is also a sub-semiring of \( \text{Fun}(R(n), R) \).

4. Polynomials

**Definition 4.1.** Given any semiring \((R, \mathcal{G}_0, \nu)\) with ghosts, we define the semiring \((R[\lambda], \mathcal{G}[\lambda], \nu)\) of polynomials

\[
\left\{ \sum_{i \in \mathbb{N}} \alpha_i \lambda^i : \alpha_i \in R, \text{ almost all } \alpha_i = 0_R, \right\}
\]

where we define polynomial addition and multiplication in the familiar way:

\[
\left( \sum_i \alpha_i \lambda^i \right) \left( \sum_j \beta_j \lambda^j \right) = \sum_k \left( \sum_{i+j=k} \alpha_i \beta_j \right) \lambda^k.
\]

We have denoted the semiring of polynomials as \( R[\lambda] \) rather than by the familiar notation \( R[\lambda] \). The reason is that, as we shall see, different polynomials can take on identically the same values as functions, and we want to reserve the notation \( R[\lambda] \) for the equivalence classes of polynomials (with respect to taking on the same values as functions). But before discussing this issue, let us develop some more notions.

We write a polynomial \( f = \sum_{i=t}^{t} \alpha_i \lambda^i \) as a sum of monomials \( \alpha_i \lambda^i \), where \( \alpha_i \neq 0_R \) and \( \alpha_i = 0_R \) for all \( i > t \), and define its degree, \( \text{deg } f \), to be \( t \). By analogy, we sometimes write \( \lambda^\nu \) for \( 1_R^{\nu} \lambda \). A polynomial is monic if its leading coefficient is \( 1_R \). A polynomial \( f \) is said to be tangible if its coefficients are all tangible. We identify \( \alpha_0 \lambda^0 \) with \( \alpha_0 \), for each \( \alpha_0 \in R \). Thus, we may view \( R \subset R[\lambda] \). Often we use logarithmic notation for the coefficients of polynomials over \( \mathbb{T} \); \( \lambda \) then means \( 0\lambda + (-\infty) \).
Since the polynomial semiring was defined over an arbitrary semiring, we can define inductively $R[\lambda_1, \ldots, \lambda_n] = R[\lambda_1, \ldots, \lambda_{n-1}] / [\lambda_n]$. Often we write $\Lambda$ for $\{\lambda_1, \ldots, \lambda_n\}$.

**Definition 4.2.** In particular, we define the polynomial semiring $R[\Lambda] = R[\lambda_1, \ldots, \lambda_n]$ in $n$ indeterminates over a supertropical semiring $R$. Any such polynomial can be written uniquely as a sum
\[
f = \sum_{i_1, \ldots, i_n} \alpha_{i_1, \ldots, i_n} \lambda_1^{i_1} \cdots \lambda_n^{i_n},
\]
which we denote more concisely as $\sum_i \alpha_i \Lambda^i$, where $i$ denotes the $n$-tuple $(i_1, \ldots, i_n)$ and $\Lambda^i$ stands for $\lambda_1^{i_1} \cdots \lambda_n^{i_n}$. We write $\deg_k \alpha_i \Lambda^i = i_k$ for $1 \leq k \leq n$. The support of $f$ is
\[
\text{supp}(f) = \{i : \alpha_i \neq 0_R\}.
\]

A binomial is the sum of two monomials.

Binomials play the key role in this theory, because, as we shall see, tangible roots often can be defined in terms of binomials.

We sometimes write $f(\lambda_1, \ldots, \lambda_n)$ for a polynomial $f \in R[\Lambda]$, indicating that it involves the $n$ indeterminates $\lambda_1, \ldots, \lambda_n$.

**Remark 4.3.** If $F$ is a supertropical semifield, then $\{f \in F[\Lambda] : f \text{ is not a tangible constant}\}$ is the unique maximal ideal of $F[\lambda_1, \ldots, \lambda_n]$, comprised of all the noninvertible elements, and it is a ghost-closed prime ideal.

4.1. The polynomial semiring (as functions). A more concise way of viewing polynomials is inside the larger semiring $\text{CFun}(R^{(n)}, R)$ of §3.5.

**Remark 4.4.** There is a natural semiring homomorphism
\[
\Psi : R[\Lambda] \to \text{CFun}(R^{(n)}, R),
\]

obtained by viewing any polynomial $f \in R[\Lambda]$ as the (continuous) function sending $(a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n)$.

In classical commutative algebra, when $R$ contains an infinite field, $\Psi$ is $1:1$, by the easy part of the fundamental theorem of algebra. Thus, one always can “make” $\Psi$ $1:1$ by enlarging $R$ suitably. But in our supertropical setting, different tropical polynomials could always represent the same function, i.e., take on the same values at each element of $R$.

For example, take elements $\alpha, \beta$ in a supertropical semifield $R$, for which $\beta >_\nu \alpha^2$. The polynomials $\lambda^2 + \alpha \lambda + \beta$ and $\lambda^2 + \beta$ define the same function. Indeed, otherwise there is $a \in R$ such that $aa$ has $\nu$-value at least both that of $a^2$ and $\beta$. But $a^2 \leq_\nu aa$ implies $a \leq_\nu \alpha$, and thus
\[
\beta \leq_\nu \alpha a \leq_\nu \alpha^2,
\]
contrary to hypothesis. This argument did not depend on any other properties of $R$, and thus shows that $\Psi$ is not $1:1$ over any semifield containing $R$, as opposed to the classical situation.

From now on, we work with
\[
R[\Lambda] := \Psi(R[\Lambda]);
\]
i.e., in $\text{CFun}(R^{(n)}, R)$, whose ghost ideal (as observed in §3.5) is $\text{CFun}(R^{(n)}, \mathcal{G}_0)$. Thus, $R[\Lambda]$ can be viewed as a semiring with ghost ideal consisting of all polynomials which as functions take on only ghost values.

4.2. Equivalence of polynomials, and essential polynomials. As noted above, the semiring of polynomials over a supertropical semifield is not supertropical, and, even worse, the tangible polynomials are not closed under multiplication; for example $(\lambda + 3)^2 = \lambda^2 + 3^\nu \lambda + 6$, which has a ghost term. (Recall that our examples are computed in logarithmic notation.) Accordingly, we need another definition to enable us to consider polynomials over $R$ in $\text{CFun}(R^{(n)}, R)$, i.e., as continuous functions from $R^{(n)}$ to $R$. 

Definition 4.5. Two polynomials $f, g \in R[A]$ are $e$-equivalent, denoted as $f \sim g$, if $f(a) = g(a)$ for any tuple $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. (In other words, polynomials $f$ and $g$ are $e$-equivalent if $\Psi(f) = \Psi(g)$.)

Two polynomials $f, g \in R[A]$ are weakly $(\nu, e)$-equivalent, denoted as $f \sim^\nu g$, if they identically take on $\nu$-equivalent values, i.e., $f^\nu = g^\nu$, or, explicitly, $f(a) \equiv^\nu g(a)$ for any $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Weakly $(\nu, e)$-equivalent polynomials $f, g \in R[A]$ are $(\nu, e)$-equivalent if $f(a)$ and $g(a)$ have the same parity, for all $a \in \mathbb{R}^n$.

Note 4.6.

(i) Polynomials of different degree over a supertropical semifield cannot identically take on $\nu$-equivalent values. Thus, $(\nu, e)$-equivalent polynomials (and, a fortiori, $e$-equivalent polynomials) have the same degree.

(ii) The difference (for tangible polynomials) between $e$-equivalent and $(\nu, e)$-equivalent only arises when the restriction $\nu_T$ of $\nu$ to $T$ is not 1:1. Since $\nu_T$ is 1:1 in the "standard" tropical example $D(G)$, this distinction only exists in the more unusual examples, such as $(\mathbb{R}^1, \mathbb{R}^+, \nu)$ where $\nu$ is the absolute value; here $\lambda + 2$ and $\lambda + (-2)$ are $(\nu, e)$-equivalent but not $e$-equivalent. We may resort to this example when $(\nu, e)$-equivalence comes up, but we focus on $e$-equivalence whenever possible, indicating how the theory simplifies when $\nu_T$ is 1:1.

(iii) One can reduce an arbitrary supertropical domain $R$ to the case when $\nu_T$ is 1:1. Namely, $\equiv^\nu$ restricts to an equivalence $\sim$ on $T_0 = R \setminus \mathcal{G}$; Then $(T_0/\sim) \cup \mathcal{G}$ becomes a supertropical domain under the natural operations of the equivalence classes (and this can be identified with $D(G)$).

Example 4.7. The following facts hold for all $a, b \in T$, $a \neq b$:

(i) $\lambda + a \sim^\nu \lambda + a'$, although $\lambda + a \sim \lambda + a'$;

(ii) $\lambda + a \sim \lambda + b$;

(iii) $\lambda + a^2 \sim \lambda^2 + a^2$ (a special case of Proposition 3.27).

Let us introduce a natural representative for each $e$-equivalence class.

Definition 4.8. The function $f \in \text{Fun}(\mathbb{R}^n, R)$ dominates $g$ if $f(a) \geq^\nu g(a)$ for all $a \in \mathbb{R}^n$; $f \in \text{Fun}(\mathbb{R}^n, R)$ strictly dominates $g$ if $f(a) >^\nu g(a)$ for all $a \in \mathbb{R}^n$.

(Thus, when $f$ dominates $g$, $f(a) + g(a) \notin \{f(a), f(a)\}$ for all $a \in \mathbb{R}^n$; when $f$ strictly dominates $g$, $f + g = f$. )

Definition 4.9. Suppose $f = \sum \alpha_i A^i$, $h = \alpha_j A^j$ is a monomial of $f$, and write $f_h = \sum_{i \neq j} \alpha_i A^i$. The monomial $h$ is inessential in $f$ if $f_h$ dominates $h$. The essential part $f^{es}$ of a polynomial $f = \sum \alpha_i A^i$ is the sum of those monomials $\alpha_i A^i$ that are essential, while its inessential part $f^{in}$ consists of the sum of all inessential monomials $\alpha_i A^i$. The polynomial $f$ is said to be an essential polynomial when $f = f^{es}$.

The following equivalent formulation indicates the direction we wish to take:

Remark 4.10.

(i) A monomial $h$ is essential in a polynomial $f$ iff $h(a) >^\nu f_h(a)$ for some $a$ and thus for all $a'$ in an open set $W_a$ of $a$.

(ii) Any monomial $h$ of $f^{es}$ is essential in $f^{es}$. Indeed, by definition, $f_h(a) + h(a) >^\nu f_h(a)$ for some $a \in \mathbb{R}^n$, implying $h(a) >^\nu f_h(a)$. A fortiori, this implies $h(a) >^\nu f_h^{es}(a)$.

We want the essential part of a polynomial $f$ to be $e$-equivalent to $f$. Towards this end, we turn to the divisible closure.

Remark 4.11. Any archimedean supertropical semifield $F$ (in the sense of Definition 3.5) satisfies the following property:

For any nonconstant monomials $g_1, g_2, h_1, \ldots, h_m$ and $a \in F^n$ with $g_2(a) \equiv^\nu g_1(a) >^\nu h_i(a)$, $1 \leq i \leq m$, and any open set $W_a$ of $F^n$ containing $a$, there exists $a' \in W_a$ with $g_2(a') >^\nu g_1(a') >^\nu h_i(a')$, $1 \leq i \leq m.$
Lemma 4.12. Suppose the supertropical semifield $F$ is archimedean. For any monomials $g_1, \ldots, g_ℓ, \ h_1, \ldots, h_m$ and $a \in F^{(n)}$ with
\[
g_1(a) \cong_ν g_2(a) \cong_ν \cdots \cong_ν g_ℓ(a) >_ν h_i(a), \quad 1 \leq i \leq m,
\]
there exists $a' \in F^{(n)}$ and $1 < j \leq ℓ$ such that
\[
g_j(a') >_ν g_i(a') \quad \forall i \neq j; \quad g_j(a') >_ν h_i(a'), \quad 1 \leq i \leq m.
\]
Proof. By Remark 4.11, we have $a' \in F^{(n)}$ such that
\[
g_2(a') >_ν g_1(a') >_ν h_i(a'), \quad 1 \leq i \leq m.
\]
Take $j$ such that $g_j(a')$ is $ν$-maximal, and expand the $h_i$ to include all $g_i$ such that $g_j(a') >_ν g_i(a')$. Then we have the same hypothesis as before, but with smaller $ℓ$. □

Proposition 4.13. If $F$ is a archimedean supertropical semifield, then $f^{es} \sim f$ for any $f \in F[Λ]$.

Proof. Given any $a \in F^{(n)}$, there is a monomial $g_1$ such that $f(a) \cong_ν g_1(a)$. We need to show that $f^{es}(a) \cong_ν g_1(a)$. This is clear unless $g_1(a) \cong_ν g_2(a) \cong_ν \cdots \cong_ν g_ℓ(a) >_ν f^{es}(a)$ for some other monomial(s) $g_2, \ldots, g_ℓ$ of $f$ that are inessential. But then, by the lemma, we may find $a'$ such that $g_j(a') >_ν g_i(a')$ takes on the single largest $ν$-value of the monomials of $f$, for some $2 \leq j \leq ℓ$, contrary to $g_j$ being inessential in $f$. □

Now we have a new way of viewing the polynomial semiring $F[Λ]$.

Remark 4.14. For $F$ archimedean, the supertropical polynomial semiring $F[Λ]$ can be viewed as the collection of essential polynomials, viewed as a semiring whereby we perform the usual operations in $F[Λ]$ and then take the essential part. The ghost ideal is comprised of those essential polynomials whose coefficients are all in $G_0$.

If $f_1$ dominates $f_2$, then obviously $f_1 + g$ dominates $f_2 + g$ and $f_1 g$ dominates $f_2 g$, for any polynomial $g$. Accordingly, one can discard the inessential monomials at any stage of the computation, which shows that our new operations of addition and multiplication in $F[Λ]$ remain associative and distributive.

We write $f \models g$ when $f \sim g + h$ for $h \in G_0[Λ]$. Occasionally we only need $f \sim g + h$ as functions on a given subset $S \subseteq R^{(n)}$; then we say $f \models g$ on $S$.

The following definition is easily seen to be a special case of Definition 3.26. Since we are viewing polynomials as functions, we consider only essential polynomials.

Definition 4.15. The tangible part $f^{tan}$ of an essential polynomial $f = \sum α_i Λ^i$ is defined as the sum of those $α_i Λ^i$ for which $α_i$ is tangible; the ghost part $f^{ghost}$ of $f$ is the sum of those $α_i Λ^i$ for which $α_i \in G_i$.

Thus, any essential polynomial $f$ is written uniquely as the sum of its tangible part $f^{tan}$ plus its ghost part $f^{ghost}$. We say that a polynomial is essential-tangible if its essential part is tangible.

Proposition 4.16. If $R = R$, then the product $q = fg$ of two essential-tangible polynomials $f, g$ in $R[Λ]$ is essential-tangible.

Proof. Assume $q = fg$ is the product of two essential-tangible polynomials $f = \sum α_i Λ^i$ and $g = \sum β_j Λ^j$. Write $f = f^{es} + f^{in}$, and $g = g^{es} + g^{in}$; clearly, $f^{in}g^{in}, f^{in}g$ and $fg^{in}$ belong to $q^{in}$, and $q^{es} \sim f^{es}g^{es}$. Thus, a ghost monomial $h$ of $q^{es}$, if it existed, would be obtained from some two (or more) identical products
\[
α_i Λ^i β_j Λ^j = α_ℏ Λ^h β_k Λ^k;
\]
but these are dominated by $α_i Λ^i β_k Λ^k + α_ℏ Λ^h β_j Λ^j$, in view of Lemma 3.14(ii), so $h$ is inessential. □
5. Roots of polynomials

As in classical algebra, our main interest in polynomials lies in their roots, which are to be defined in the tropical sense. As mentioned earlier, in our philosophy, ghost elements are to be treated like zero.

**Definition 5.1.** (Compare with [25]) Suppose $R = (R, \mathcal{G}, \nu)$ is a supertropical domain. An element $a \in R^{[n]}$ is a (tropical) root of a polynomial $f \in R[\lambda_1, \ldots, \lambda_n]$ if $f(a) \in \mathcal{G}$; in this case we also say $f$ satisfies $a$. The root $a = (a_1, \ldots, a_n)$ is tangible if each $a_i$ is tangible or $0_R$; $a = (a_1, \ldots, a_n)$ is strictly tangible if each $a_i$ is tangible.

For example, any ghost $a'$ is a root of the monomial $\lambda$, and $\lambda$ has no strictly tangible roots; any tangible constant $\neq 0_R$ has no roots. On the other hand, every element of $R$ is a root of all ghost polynomials.

**Note 5.2.** Of course, a tangible polynomial could take on some non-tangible values. For example, the tangible polynomial $f = \lambda + 1$ satisfies $f(1) = 1' \in \mathcal{G}$. This is precisely the idea behind roots of a tangible polynomial.

Of course, if $f \in R[\lambda]$ and $f(a) = 0_R$, then $a$ is a root of $f$. Although this is usually much too special to be of use, it does help us keep track of monomials. Note by Remark 3.5(iv) that $f(a) = 0_R$ iff $a = 0_R$ and $\lambda | f$. (Otherwise, some monomial of $f$ would take a nonzero value.) Let us generalize this observation.

**Proposition 5.3.** Define $\Phi_a : R[\lambda_1, \ldots, \lambda_n] \to R[\lambda_1, \ldots, \lambda_n-1]$ as in Proposition 3.30 sending $f \mapsto f_a$, where

$$f_a(\lambda_1, \ldots, \lambda_{n-1}) = f(\lambda_1, \ldots, \lambda_{n-1}, a).$$

If $f \in \ker \Phi_a$, then $\lambda_n$ divides $f$.

**Proof.** We are given $f(a_1, \ldots, a_{n-1}, a) = 0_R$ for all $a_1, \ldots, a_{n-1} \in R$. But writing

$$f(\lambda_1, \ldots, \lambda_n) = \sum h_i$$

as a sum of monomials, we can view

$$f_a(\lambda_1, \ldots, \lambda_{n-1}) = \sum h_i(\lambda_1, \ldots, \lambda_{n-1}, a)$$

as a sum of monomials, yielding

$$\sum_i h_i(a_1, \ldots, a_{n-1}, a) = 0_R, \quad \forall a_j \in R$$

which by Remark 3.5(iv) implies that each $h_i(a_1, \ldots, a_{n-1}, a) = 0_R$ for all $a_j$; i.e., each $\Phi_a(h_i) = 0_R$. In other words, $\lambda_n | h_i$ for each $i$, implying $\lambda_n | f$. □

**Lemma 5.4.** If $\lambda_j$ divides a polynomial $f = \sum h_i$, where the $h_i$ are monomials, then $\lambda_j | h_i$ for each $i$.

**Proof.** Write $-$ for the specialization $\lambda_j \mapsto 0_R$. Then

$$0_R = f = \sum h_i,$$

by Proposition 3.30. Applying Remark 3.5(iv) yields each $\overline{h_i} = 0_R$, so $\lambda_j$ divides each $h_i$. □

**Proposition 5.5.** If $\lambda_j$ divides $\sum g_i$, a sum of polynomials, then $\lambda_j$ divides each $g_i$.

**Proof.** Write each $g_i$ as a sum of monomials; by the lemma, $\lambda_j$ divides each of these monomials, and thus divides each $g_i$. □
5.1. The fundamental theorem of supertropical algebra. We return to our general considerations about roots.

Remark 5.6. Obviously, any $e$-equivalent polynomials have precisely the same roots.

Remark 5.7. Any root $a \in R^{(n)}$ of $f^{\tan}$ is a root of $f$. Indeed, $f(a) = f^{\tan}(a) + \operatorname{ghost} \in G_0$.

One classical result, the **Fundamental Theorem of Algebra**, has a very easy analog here. We work over a divisibly closed supertropical semifield $F = (F, G_0, \nu)$; i.e., $F = \bar{F}$.

Lemma 5.8. Suppose $F = \bar{F}$. Then for any nonconstant polynomial $f \in F[\lambda]$ and for any $a^\nu \notin F$ in $G$, there exists tangible $r \in F$ with $f(r) \cong \nu a$.

Proof. Write $f = \sum \alpha_i \lambda^i$. For each $k > 0$, there is some tangible $r_k$ such that

$$r_k \cong \nu \sqrt[k]{\frac{\alpha}{\alpha_k}};$$

thus, $\alpha_k r_k^k \cong \nu a$. Take $r$ such that $r^\nu$ is minimal among these $r_k^k$. Then $f(r) \cong \nu a$. \qed

Proposition 5.9. Over any divisibly closed supertropical semifield $(F, G_0, \nu)$, every polynomial $f \in F[\lambda]$, which is not a tangible monomial, has a strictly tangible root.

Proof. Suppose $f(\lambda) = \sum_{i=1}^m \alpha_i \lambda^i$, where $\alpha_u \notin 0_R$. Replacing $f$ by $\sum_{i=1}^m \alpha_i \lambda^i - u$, and renumbering the coefficients, we may assume that $\alpha_0 \neq 0_F$. Write $g(\lambda) = \sum_{i=1}^m \alpha_i \lambda^i$, so $f(\lambda) = g(\lambda) + \alpha_0$. If $\alpha_0 \in G_0$, then we could erase $\alpha_0$ and divide by $\lambda$, and conclude by induction on $\deg f$. Thus, we may assume that $\alpha_0 \notin G_0$. By the lemma, there is some tangible $r$ such that $g(r) \cong \nu \alpha_0$, implying $f(r) = \alpha_0^\nu + \alpha_0 \in G$. \qed

This proposition, whose analog for the max-plus algebra was proved in the sense of polynomial factorization [4], was included here to give a quick positive result, but its proper formulation in this theory is more sophisticated; cf. Proposition 5.17 below.

5.2. Different kinds of roots. Varieties of tropical geometry come up in our theory as tangible roots of tangible polynomials. However, since we have the ghost structure at our disposal, we might as well consider roots of non-tangible polynomials as well, thereby enriching the geometry and also adding insight to factorization.

Note that ghost elements are automatically roots when $f$ lacks a constant term. Thus, our main interest is in tangible roots. A bit of thought shows that, unlike the classical situation where the tangible roots of a polynomial in one indeterminate are topologically isolated, here we can have a continuum of tangible roots. For example, every number less than 1 is a root of $\lambda + 1^\nu$. Thus, we need to investigate roots of polynomials more carefully.

Remark 5.10. Consider an arbitrary nonzero polynomial $f = \sum_i \alpha_i \lambda^i \in R[\lambda]$, over a supertropical domain $R = (R, G_0, \nu)$. For any essential monomial $\alpha_i \lambda^i$ of $f$, and $a \in R^{(n)}$, let us write $c_i = \alpha_i a^\nu$, and

$$S(a) = \{c_i : i \in \operatorname{supp}(f^{es})\}.$$ 

Write

$$c(a)^\nu = \max\{c_i^\nu : c_i \in S(a)\}, \quad \text{and} \quad S(a) = \{c_i \in S(a) : c_i^\nu = c(a)^\nu\}.$$ 

In evaluating $f(a)$, we may discard all $c_i$ which are not $\nu$-maximal. There are two cases:

**Case I:** $S(a)$ has at least two elements.

**Case II:** $S(a)$ has a unique element $c_j$.

In Case I, we call $a$ a **corner root**. These are the familiar roots in tropical geometry, i.e., those arising in tropical geometry in the corner locus of polynomials over the max-plus algebra. Note that any corner root $a$ is also a root of the binomial consisting of the sum of any two monomials $\alpha_i \lambda^i$ of $f$ for which $c_j \in S(a)$; this hints at the key role to be assumed by binomials in tropical geometry. A tangible corner root $a$ is called **ordinary** if, under the notation above, $S(a) \subset T$, i.e., if the monomials determining the root are tangible.
In Case II, \(a\) is a root of \(f\) iff \(c_j \in G_0\); we call this a \textit{cluster root}. This is a new phenomenon which arises from the supertropical structure, and does not occur in the familiar theory of corner loci for tropical geometry based on the max-plus algebra.

**Example 5.11.** Consider the polynomial \(\lambda^4 + 3\nu \lambda^3 + 5\nu^2 \lambda^2 + 6\lambda + 6\) in \(D(\mathbb{R})[\lambda]\). The tangible roots are 0 and all \(a\) such that \(1 \leq a \leq 3\). The corner roots are 0, (which is ordinary), and 1, 2, and 3 (which are not ordinary). All the other tangible roots are cluster roots.

**Remark 5.12.** Some immediate consequences, for \(f \in R[\lambda]:

(i) If \(a^\nu \in G\) is “large enough” (for example, for \(f = \lambda^2 + \alpha_1 \lambda + \alpha_0\), if \(a \geq \alpha_1\) and \(a_0 \geq \alpha_2\)), then \(a^\nu\) is a root of \(f\).

(ii) Likewise, if the leading coefficient of \(f\) is ghost and \(a^\nu\) is “large enough,” then \(a\) is a root of \(f\).

(iii) If \(a_0 \in G\) and \(a^\nu\) is “small enough,” then \(a\) is a root of \(f\). For example, in \(D(\mathbb{R})[\lambda]\), every \(a \leq 7\) is a root of \(\lambda^2 + 8\lambda + 15\).

(iv) More generally, suppose \(a \in T\) is a root of \(f = \sum a_i \lambda^i\); let \(c_i = a_i a_i\), and (notation as in Remark 5.10), take

\[
\hat{S}(a) = \{c_j \text{ is } \nu \text{-maximal in } S(a)\}.
\]

If \(\hat{S}(a)\) has only a single element \(c_j\), i.e., \(a\) is a cluster root, then this \(c_j\) must be a ghost, and thus there is an open set containing \(a\), all of whose elements of which are roots of \(f\). (This could be viewed as a form of Krasner’s Lemma from valuation theory.)

(v) On the other hand, notation as in (iv), if all \(c_j \in S(a)\) are tangible, then taking \(b \in T\) “close” to \(a\), but not equal, will make all the \(c_j : c_j \in S(b)\) distinct, and thus \(b\) is not a root of \(f\).

(vi) If \(a \equiv \nu b\) and \(a\) is a corner root of \(f\), then \(b\) is also a corner root. Thus, even when \(a \notin T\), taking \(b \in T\) for which \(b \equiv \nu a\) yields a tangible root.

(vii) If \(a \in T\) is a root of \(f\) and \(b \models a\), then \(b\) is a root of \(f\) as well.

A similar situation occurs for polynomials in \(n\) indeterminates. Nevertheless, ordinary roots also involve extra subtleties, as seen in Example 6.12 differing considerably from the situation in classical algebraic geometry.

Remark 5.12 shows that if \(\hat{f}\) is tangible, then all of the tangible roots of \(f\) are ordinary.

### 5.3. Laurent polynomials and rational Laurent functions

Often it is convenient to consider a slightly larger semiring than the semiring of polynomials. Let \(F^\times := F \setminus \{0_F\}\), for a supertropical semifield \(F\). As before, we write \(\Lambda = \{\lambda_1, \ldots, \lambda_n\}\) and \(\Lambda^{-1}\) for \(\{\lambda_1^{-1}, \ldots, \lambda_n^{-1}\}\). We want to consider functions such as \(\lambda^{-1}\). Since these are not defined at \(0_F\), we must be more careful, and define \(\text{Fun}^\times(F^{(n)}, F)\) to be the semiring of functions \((F^\times)^{(n)} \to F\) and \(\text{CFun}((F^\times)^{(n)}, F)\) to be the sub-semiring of continuous functions from \(\text{Fun}^\times(F^{(n)}, F)\). \((F^\times)^{(n)}\) could be called the supertropical torus.

**Definition 5.13.** The semiring \(F[\Lambda, \Lambda^{-1}]\) of Laurent polynomials is the sub-semiring of \(\text{CFun}((F^\times)^{(n)}, F)\) generated by the \textit{Laurent monomials} \(\alpha_i \lambda^i\), where \(\alpha_i \in F\) and

\[
\Lambda^i = \lambda_1^{i_1} \cdots \lambda_n^{i_n}, \quad i_1, i_2, \ldots, i_n \in \mathbb{Z}.
\]

Strictly speaking, first we embed the semiring \(F[\Lambda]\) into \(F[\Lambda, \Lambda^{-1}]\), and then pass to \(F[\Lambda, \Lambda^{-1}]\) inside \(\text{CFun}((F^\times)^{(n)}, F)\). Explicitly, we have:

**Proposition 5.14.** There is a canonical 1:1 semiring homomorphism

\[
\Psi : F[\lambda_1, \ldots, \lambda_n] \to F[\lambda_1, \lambda_1^{-1}, \ldots, \lambda_n, \lambda_n^{-1}],
\]

given by \(f \mapsto f\). Every element of \(F[\lambda_1, \lambda_1^{-1}, \ldots, \lambda_n, \lambda_n^{-1}]\) can thereby be written in the form \(h\), where \(h\) is a suitable monomial in \(F[\lambda_1, \ldots, \lambda_n]\).

**Proof.** Clearly \(\Psi\) is a semiring homomorphism, which is 1:1 since \((F^\times)^{(n)}\) is dense in \(F^{(n)}\) in the \(\nu\)-topology. (Any two continuous functions that agree on a dense subset are equal.) The last assertion is seen by clearing denominators.

\(\square\)
Remark 5.15. If \( a \in R^{(n)} \) is a tangible root of \( \alpha_i \lambda^i f \), where \( \alpha_i \) is tangible, then \( a \) is also a root of \( f \).

Since multiplying by a tangible monomial does not affect the roots of a polynomial, it is convenient to be able to have monomials invertible, especially when \( n \geq 2 \). Thus, \( F[\Lambda, \Lambda^{-1}] \) has the following important advantage over \( F[\Lambda] \):

Remark 5.16. Every tangible Laurent monomial is invertible. Consequently, we may always reduce Laurent polynomials to Laurent polynomials having constant term \( 1_F \). For example, the roots of \( \lambda_1 + \lambda_2 \) in \( (\Lambda^\times)^{(2)} \) are the same as those of \( \lambda_1 \lambda_2^{-1} + 1_F \).

In fact, an element of \( F[\Lambda, \Lambda^{-1}] \) is invertible in \( F[\Lambda, \Lambda^{-1}] \) iff it is a tangible Laurent monomial. (This is seen by a degree argument: When multiplying Laurent polynomials that are not Laurent monomials, one gets a highest-order term and a lowest-order term, and so the product cannot be a Laurent monomial.)

Here is an example of a proof made easier by passing to \( F[\Lambda, \Lambda^{-1}] \).

Proposition 5.17. Suppose \( F = F \), and \( f \in F[\lambda_1, \ldots, \lambda_n] \) is not a tangible monomial. Then \( f \) has a tangible root.

Proof. The case when \( f \) is a ghost monomial is clear. Take an essential monomial \( f_i = \alpha_i \lambda^i \) of \( f \). Passing to \( F[\Lambda, \Lambda^{-1}] \) and dividing by \( f_i \), we may assume that \( 1_F \) is a monomial of \( f \) (where now \( f \) is a sum of Laurent monomials). Write \( f = g + 1_F \). By an argument analogous to Lemma 5.8 there exists tangible \( r \) for which \( g(r) \cong_F 1_F \). Thus \( r \) is a root of \( f \).

The Laurent polynomial semiring also permits us to focus the duality of Remark 3.20.

Remark 5.18. If \( i = (i_1, \ldots, i_n) \), write \( -i \) for \((-i_1, \ldots, -i_n)\). The isomorphism

\[
\Phi_{\text{Fun}} : \text{Fun}^\times(F^{(n)}, F) \longrightarrow \text{Fun}^\times((F^\Lambda)^{(n)}, F^\Lambda)
\]

of Remark 3.20 extends to an isomorphism

\[
\Phi_{\text{poly}} : F[\Lambda, \Lambda^{-1}] \longrightarrow F[\Lambda, \Lambda^{-1}]
\]

given by \( \sum a_i \lambda^i \longmapsto \sum a_i^{-1} \lambda^{-i} \).

5.4. Convex sets. Suppose \( F \) is a dually closed supertropical semifield. In this case, \( a^t \) is defined for each \( a \in F \) and \( t \in \mathbb{Q} \), by Remark 3.20. Given \( a = (a_1, \ldots, a_n) \), we define \( a^t = (a_1^t, \ldots, a_n^t) \).

Definition 5.19. Suppose \( F \) is dually closed. We define the path \( \gamma_{a,b} \) joining points \( a \) and \( b \) in \( F^{(n)} \) to be the set

\[
\gamma_{a,b} := \{ a^t b^{1-t} : t \in [0,1] \cap \mathbb{Q} \}.
\]

A subset \( S \subseteq F^{(n)} \) is convex if whenever \( a, b \in S \) then \( \gamma_{a,b} \) lies in \( S \).

The left ray (resp. right ray) joining points \( a \) and \( b \) in \( F^{(n)} \) is the set

\[
\gamma_{a,b} := \{ a^t b^{1-t} : t \in (-\infty, 1] \cap \mathbb{Q} \}
\]

(resp. \( \gamma_{a,b} := \{ a^t b^{1-t} : t \in [0, \infty) \cap \mathbb{Q} \} \)). By (closed) ray we mean left ray or right ray. We define open (left, right) rays analogously. We define a two-sided ray to be a set of the form \( \gamma_{a,b} := \{ a^t b^{1-t} : t \in \mathbb{Q} \} \).

A function \( \phi \in \text{Fun}(F^{(n)}, F) \) is called linear if, for any \( a, b \in F^{(n)} \),

\[
\phi(a)^t \phi(b)^{1-t} = \phi(a^t b^{1-t}), \quad \forall t \in \mathbb{Q}.
\]

Remark 5.20. Any Laurent monomial \( h = \lambda_1^{i_1} \cdots \lambda_n^{i_n} \) is linear; indeed write

\[
h(a^t b^{1-t}) = \alpha_i (a_1^{i_1} b_1^{1-t})^{i_1} \cdots (a_n^{i_n} b_n^{1-t})^{i_n} = (\alpha_1 a_1^{i_1} \cdots a_n^{i_n})^t (\alpha_1 b_1^{1-t} \cdots b_n^{1-t})^{1-t} = h(a)^t h(b)^{1-t}.
\]

Consequently, we have:

Lemma 5.21. Given \( a, b \in F^{(n)} \), for all \( c \neq a, b \) in the path \( \gamma_{a,b} \) joining \( a \) and \( b \):

(i) If Laurent monomials \( f_i(a) \geq_F f_j(a) \) and \( f_i(b) \geq_F f_j(b) \), then \( f_i(c) \geq_F f_j(c) \);
(ii) If Laurent monomials \( f_i(a) \geq_F f_j(a) \) and \( f_i(b) \geq_F f_j(b) \), then \( f_i(c) \geq_F f_j(c) \);
(iii) If \( f_i(a) \geq_F f_j(a) \) and \( f_i(b) \geq_F f_j(b) \), then \( f_i(c) \geq_F f_j(c) \).
Proof. (i) and (ii): Write $c = a^t b^{1-t}$ for $0 \leq t \leq 1$. Then
\[ f_i(c) = f_i(a^t b^{1-t}) = f_i(a)^t f_i(b)^{1-t} >_\nu f_j(a)^t f_j(b)^{1-t} = f_j(c). \]

The proof for (iii) is analogous. \qed

6. Supertropical geometry

Having the basic concepts of polynomials under our belt, we are ready to apply them to tropical geometry. For convenience, we treat affine geometry; the analogous discussion using homogeneous polynomials would yield the parallel results for projective geometry. Although the following definitions could be formulated over an arbitrary semiring with ghosts, we work throughout over a supertropical semifield $F = (F, \mathcal{G}_0, \nu)$.

6.1. Supertropical root sets.

**Definition 6.1.** For a set $S \subseteq F[\Lambda]$, we define its root set
\[ Z(S) = \{ a = (a_1, \ldots, a_n) \in F^n : f(a) \in \mathcal{G}_0, \forall f \in S \} \subseteq F^n. \]
The complement of $Z(S)$ is $F^n \setminus Z(S)$.

The tangible root set, denoted $Z_{\text{tan}}(S)$, is $Z(S) \cap \mathcal{T}_n(0)$. The tangible complement of $Z_{\text{tan}}(S)$ is $\mathcal{T}_n(0) \setminus Z_{\text{tan}}(S)$. When $S = \{ f \}$ consists of only one polynomial, we write $Z_{\text{tan}}(f)$ for $Z_{\text{tan}}(S)$, which is called a supertropical hypersurface; we call $Z_{\text{tan}}(f)$ a tangible primitive when $f$ is a tangible binomial.

**Lemma 6.2.** If two points lie in a tangible primitive, then the two-sided ray containing them also lies in this tangible primitive.

**Proof.** Suppose $f = h_1 + h_2$ is a sum of monomials, with $f(a), f(b) \in \mathcal{G}_0$. Then, in view of Remark 5.20
\[ h_1(a^t b^{1-t}) = h_1(a)^t h_1(b)^{1-t}. \]

(6.1) since also $h_1(a)$ and $h_2(a)$ are tangible, we have $h_1(a) \cong \nu h_2(a)$, and likewise $h_1(b) \cong \nu h_2(b)$; thus, Equation (6.1) yields $h_1(a^t b^{1-t}) + h_2(a^t b^{1-t}) \in \mathcal{G}_0$ since also $h_1(b) + h_2(b) \in \mathcal{G}_0$. \qed

(Note that this argument does not work for ghosts: If $h_1(a), h_1(b) \in \mathcal{G}_0$ with $h_1(a) \geq \nu h_2(a)$ and $h_1(b) \geq \nu h_2(b)$, then clearly
\[ h_1(a^t b^{1-t}) + h_2(a^t b^{1-t}) = h_1(a)^t h_1(b)^{1-t} \in \mathcal{G}_0 \]
for $0 \leq t \leq 1$, but for small $t$ or large $t$, one could have $h_2$ dominating, and thus the sum may be tangible.)

The tangible root sets are the geometric objects that we would like to view as supertropical varieties. One advantage of this approach is that elementary considerations yield the usual correspondence between varieties and ideals of polynomials, whose analogous formulation under other definitions (involving domains of non-differentiability) might fail:

**Remark 6.3.**

(i) $Z_{\text{tan}}(S_1) \cap Z_{\text{tan}}(S_2) = Z_{\text{tan}}(S_1 \cup S_2)$. Thus, the intersection of tangible root sets is a tangible root set.

(ii) $Z_{\text{tan}}(f) \cup Z_{\text{tan}}(g) = Z_{\text{tan}}(fg)$. Thus, the union of finitely many supertropical hypersurfaces is a supertropical hypersurface.

Nevertheless we continue to use the terminology “(tangible) root set” to avoid confusion with other definitions of tropical varieties.

**Remark 6.4.** For any $S \subseteq F[\Lambda]$, the root set $Z(S)$ of $S$ is closed in the $\nu$-topology; its tangible complement (as well as its complement) is open in the $\nu$-topology.

Some examples of tangible root sets are given in Examples 8.20, 8.22 and 5.63.

For an appetizer, let us start with a sample result, reminiscent of a “weak Nullstellensatz.”
Remark 6.5. Any finite set \( S \) of non-constant polynomials has common roots. In fact, one can just take ghosts “large enough” so that they outweigh the constant terms in the polynomials. On the other hand, \( S \) could have no common tangible roots; for example, \( \lambda + 2 \) and \( \lambda + 3 \) have no common tangible root.

Definition 6.6. The ideal \( \mathcal{I}(Z) \) of a set \( Z \subset F^{(n)} \) is defined to be
\[
\mathcal{I}(Z) = \{ f \in F[\lambda_1, \ldots, \lambda_n] : f(a) \in \mathcal{G}_0, \forall a \in Z \}.
\]
We call \( \mathcal{I}(Z) \) the ideal of polynomials satisfying \( Z \).

Proposition 6.7. For any set \( Z \subset R^{(n)} \), the ideal \( \mathcal{I}(Z) \) is a ghost-closed radical ideal of \( F[\Lambda] \).

Proof. To check that \( \mathcal{I}(Z) \) is an ideal, note that if \( f(a) \in \mathcal{G}_0 \) and \( g(a) \in \mathcal{G}_0 \) for polynomials \( f \) and \( g \), then \( (f + g)(a) = f(a) + g(a) \in \mathcal{G}_0 \); likewise if \( f(a) \in \mathcal{G}_0 \) and \( g \) is any other polynomial, then \( f(a)g(a) \in \mathcal{G}_0 \).

Furthermore, \( f(a) \in \mathcal{G}_0 \) iff \( f(a)^k \in \mathcal{G}_0 \) (for any number \( k \geq 1 \)), implying \( f \in \mathcal{I}(Z) \) iff \( f^k \in \mathcal{I}(Z) \); hence \( \mathcal{I}(Z) \) is a radical ideal. Finally, every element is a root of each ghost polynomial, so \( \mathcal{I}(Z) \) contains all the ghost polynomials.

This leads us to try to identify root sets with the ghost-closed radical ideals of the polynomial semiring, which we study further when considering the Nullstellensatz in the next section.

Remark 6.8. Any ghost-closed ideal \( P \) of \( F[\Lambda] \) contains
\[
(\lambda + \alpha')(\lambda' + \alpha) = \nu(\lambda^2 + \alpha\lambda + \alpha^2) \in \mathcal{G}[\lambda].
\]
It follows that the ghost ideal \( \mathcal{G}_0[\Lambda] \) is not prime.

As soon as one tries to dig deeper, one encounters many potential pitfalls, which we illustrate with a few examples in one indeterminate.

Example 6.9. Suppose \( F = D(R) \). Thus, \( T = R \).

(i) No ideal of \( F[\Lambda] \) defined by a set of tangible roots contains both \( \lambda^2 + \lambda + 2 \) and \( \lambda^2 + 3\lambda + 1 \), since the latter has tangible roots 3 and –2, whereas the former only has the tangible root 1.

(ii) Consider the ideal \( A \) of polynomials having 2 as a root. The polynomial \( f = \lambda + 3\nu \in A \), since \( f(2) = 2 + 3\nu = 3\nu \). Also \( \lambda + 2 \in A \) and \( f + (\lambda + 2) = \lambda' + 3\nu \), a ghost. On the other hand, by the same token, \( f + (\lambda + 3) \) is the same ghost, although \( \lambda + 3 \notin A \). (Actually, every real number \( \leq 3 \) is a tangible root of \( f \).)

(iii) Besides being a root of \( \lambda + 2 \), the number 2 is the maximal tangible root of \( \lambda + 2\nu \), and the minimal tangible root of \( \lambda' + 2 \). Every element of \( F \) is a root of \( \lambda + 2\nu \) or \( \lambda' + 2 \) (cf. Remark 6.8).

(iv) We would like the ideal of polynomials having 1, 2 as roots to be generated by \( (\lambda + 1)(\lambda + 2) = \lambda^2 + 2\lambda + 3 \). But \( \lambda + 3\nu \) is in this ideal, and its degree is too small!

(v) 0 is a root both of \( 3\lambda + 3 \) and \( \lambda^2 + 3\lambda + 3 \), but not of the tangible part of their sum \( \lambda^2 + 3\nu\lambda + 3\nu \), which is \( \lambda^2 \).

(vi) The ideal of \( F[\Lambda] \) generated by all \( \{\lambda + \alpha : \alpha \in R\} \) is not finitely generated in the classical sense.
(For any \( S = \{\lambda + \alpha_1, \ldots, \lambda + \alpha_m\} \), just take \( \alpha < \min\{\alpha_1, \ldots, \alpha_m\} \), and \( \lambda + \alpha \) is not generated by \( S \).)

As we continue, we need to pick our way through these various examples. One also wants to go in the other direction, from root sets to polynomials. Different polynomials could define the same root sets, and the same idea used in classical algebraic geometry is applied here.

6.2. Tropical varieties versus super tropical root sets. Our definition of root set encompasses the corner locus defined in the introduction. In fact, one can apply this idea to any field \( K \) with non-Archimedean valuation \( \text{val} : K \to T \). Namely, given any affine hypersurface defined as the roots of the polynomial \( f_\nu \in K[\Lambda] \), one can pass to the tangible root set of the polynomial \( f_\nu \) defined over the supertropical semifield \( D(\text{val}(K)) \). (For example, one can take \( K \) to be the field of Puiseux series.)

To incorporate the ideas of tropical geometry into the supertropical theory, we recall the special case of tangible polynomials over the supertropical semifield \( T := D(R) \), in which we recall \( T = R \), \( \mathcal{G} = R^\nu \),
Consider the polynomial 

\[ f = \lambda_1^2\lambda_2 + \alpha\lambda_1\lambda_2 + \beta \]

with \( \alpha \) and \( \beta \) having tangible or ghost values. The ghost vertices are indicated by dashed lines in the left diagrams. The corresponding Newton polytopes are shown in the right diagrams. The one-dimensional faces of the polytope come from binomials \( \lambda_1^j\lambda_2^k \). The one-dimensional faces in \( \text{Cor}(f) \) are lattice edges of \( \Delta(f) \) whose integral length provides the weight \( m(\delta) \) for \( \delta \). The one-dimensional faces of the polytope come from binomials \( \alpha_1\lambda^j + \alpha_j\lambda^j \) appearing in \( f \), whose zero sets satisfy \( \frac{\alpha_1}{\alpha_j}\lambda^{j-i} = 1 \), and thus has normal vector of slope \( j - i \).

**Example 6.10.** Consider the polynomial \( f = \lambda_1^2\lambda_2 + \lambda_1\lambda_2 + \alpha\lambda_1 + \lambda_2 + 0 \) in \( \mathbb{R}_{-\infty}[\lambda_1, \lambda_2] \). The corner locus \( \text{Cor}(f) \) consists of three line segments and three rays; these are exactly the tangible root set \( \mathcal{Z}_{\text{tan}}(f) \) as viewed in \( \mathbb{T}[\lambda_1, \lambda_2] \). (See Figure 2(a)).

Supertropical geometry permits a wider scope for the definition of polyhedral complexes, since we also have non-tangible polynomials at our disposal, which enable us to describe \( n \)-dimensional polyhedral complexes within \( n \)-dimensional spaces (for example polytopes); a supertropical hypersurface \( \mathcal{Z}_{\text{tan}}(f) \subset \mathbb{R}^{(n)} \) may have (topological) dimension \( n \) when \( f \) has an essential ghost monomial. For example, the tangible root set of \( \lambda + 1^n \) in \( F[\lambda] \) is a ray. Here is a more interesting illustration of supertropical varieties that previously were not available.

**Example 6.11.** Consider the essential polynomial

\[ f = \lambda_1^2\lambda_2 + \lambda_1\lambda_2 + \alpha\lambda_1 + \lambda_2 + \beta \]
Figure 3. (a) The root set of the supertropical tangible line $f = \lambda_1 + \lambda_2 + a$, $a \in T$.
(b) The root set of the supertropical conic $f = \lambda_1^2 + \lambda_2^2 + a'\lambda_1 \lambda_2 + b$, $a, b \in T$.

and the tangible root set $Z_{\text{tan}}(f)$ of $f$. When $f$ is tangible, $Z_{\text{tan}}(f)$ is just a standard tropical curve of genus 1 (see Figure 2(a)). When $\alpha$ is a ghost, $Z_{\text{tan}}(f)$ is of dimension 2 and has genus 0 (see Figure 2(b)). For $\beta$ ghost, $Z_{\text{tan}}(f)$ has genus 1 and dimension 2 (see Figure 2(c)).

The Newton polytope $\Delta(f)$ of a supertropical polynomial $f$ is defined in the same manner, where vertices that correspond to ghost monomials are designated as ghosts. The duality between the tangible root set of a polynomial and the subdivision $S(f)$ of its Newton polytope is preserved; here, a ghost vertex of $\Delta(f)$ corresponds to an $n$-dimensional face of $Z_{\text{tan}}(f)$ (see for example Figure 2).

The ghost roots provide a new dimension to the geometry, as illustrated in the following examples of root sets of polynomials in two indeterminates. We display both the tangible and ghost parts of the root sets, by taking each axis to represent tangible elements in one direction (from $-\infty$) and ghost elements in the other direction. In other words, each axis looks like:

\[ \ldots \quad \overset{\text{tangibles in increasing $\nu$-order}}{\sim} \overset{\text{ghosts in decreasing $\nu$-order}}{\sim} -\infty \ldots \]

Example 6.12.
(i) Supertropical hypersurfaces: The tangible roots of $f = \lambda_1 + \lambda_2 + a$ in $F[\lambda]$ are

\[
(a, b) \text{ for } b \leq_\nu a;
\]

\[
(b, a) \text{ for } b \leq_\nu a;
\]

\[
(b, b) \text{ for } b \geq_\nu a.
\]

Thus, $Z_{\text{tan}}(f)$ is comprised of three rays, all emanating from $(a, a)$, and its tangible complement has three components; cf. Figure 2(a).

(ii) The tangible complement of the supertropical conic, given by $f = \lambda_1^2 + \lambda_2^2 + a'\lambda_1 \lambda_2 + b$; cf. Figure 2(b), is comprised of three components. This conic is of dimension 2.

(iii) Tangible root sets of dimension 2 in $F(2)$; cf. Figure 2.

(a) The unbounded strip $Z_{\text{tan}}(\{\lambda_1 + a'\lambda_2, b'\lambda_1 + \lambda_2\}, b >_\nu a^{-1}$;

(b) The supertropical hypersurface consisting of two half spaces, $Z_{\text{tan}}(\{a'\lambda_1^2 + b\lambda_1 \lambda_2 + a'\lambda_2^2\}, b >_\nu a$;

(c) A convex bounded tangible root set $Z_{\text{tan}}(\{\lambda_1 + a', \lambda_2 + a'', (\lambda_1 \lambda_2)^\nu + a\})$; for $a, b, c \in T$. 

7. The Nullstellensatz

One very basic goal (and perhaps the main result of this paper) is to find an analog of Hilbert’s Nullstellensatz, in order to provide an algebraic foundation for tropical geometry. Unfortunately, the naive tropical formulation just does not work, even over a supertropical semifield.

The “naive tropical Nullstellensatz” would be that for any divisibly closed, archimedean supertropical semifield $F$ and any ideal $A$ of $F[\Lambda] = F[\lambda_1, \ldots, \lambda_n]$, a polynomial $f$ satisfies all common roots of $A$ iff $f \in \sqrt{A}$. Unfortunately, there are many counterexamples to this assertion, some of which are given in Example 6.2. This leaves us with a dilemma: Do we want to hold on the notion of ideal and move our focus away from root sets, or do we want to stay with root sets and modify our definition of ideal in the tropical sense? We deal with the first approach, since it turns out to be more straightforward and quite natural. It also turns out that the proofs are most easily expressed in terms of the Laurent polynomial semiring $F[\Lambda, \Lambda^{-1}]$.

In this discussion, we view a supertropical semifield $F = (F, G, \nu)$, with $T_0 = F \setminus G$, endowed with the $\nu$-topology described in Definition 3.22 and assume that $F$ is divisibly closed and archimedean.

**Definition 7.1.** Given a supertropical semifield $F = (F, G, \nu)$ and a polynomial $f \in F[\lambda_1, \ldots, \lambda_n]$, we define the set

$$D_f := \{ a = (a_1, \ldots, a_n) \in T_0^{(n)} : f(a) \in T_0 \} = T_0^{(n)} \setminus Z_{\text{tan}}(f).$$

Refining this definition, writing $f = \sum f_i$, a sum of monomials, define $D_{f,i}$ to be

$$D_{f,i} := \{ a \in T_0^{(n)} : f(a) = f_i(a) \in T_0 \}.$$

We call the $D_{f,i}$ the (tangible) components of $f$; we call $f_i$ the dominant monomial of $f$ on the component $D_{f,i}$.

Likewise, we define the closed components of $f$ to be

$$\overline{D}_{f,i} := \{ a \in T_0^{(n)} : f(a) \equiv_\nu f_i(a) \}.$$

Note that $f$ has finitely many components, since $f$ is a sum of finitely many monomials.

Components and closed components are defined the same way for Laurent polynomials, although here we only consider points in $T^{(n)}$ (since a Laurent polynomial need not be defined at $0_F$). Namely, given a Laurent polynomial $f \in F[\lambda_1, \lambda_1^{-1}, \ldots, \lambda_n, \lambda_n^{-1}]$, we define the set

$$D_f = \{ a = (a_1, \ldots, a_n) \in T^{(n)} : f(a) \in T \};$$

thus $T \setminus D_f$ is the set of tangible roots of $f$ in $T^{(n)}$.

Again, writing $f = \sum f_i$, a sum of Laurent monomials, define $D_{f,i}$ to be

$$D_{f,i} = \{ a \in T_0^{(n)} : f(a) = f_i(a) \in T \}.$$

We call the $D_{f,i}$ the (tangible) components of $f$; we call $f_i$ the dominant Laurent monomial of $f$ on the component $D_{f,i}$.

By definition, $a \in D_{f,i}$ iff $f_i(a)$ is tangible and $f_i(a) >_\nu f_j(a)$ for all $j \neq i$; hence

$$D_f = \bigcup_i D_{f,i}.$$

On the other hand, when $f$ is tangible, $\bigcup_i \overline{D}_{f,i} = T_0^{(n)}$, and one obtains a chain complex by taking the intersections of closed components.

**Remark 7.2.** $\overline{D}_{f,i_1} \cap \overline{D}_{f,i_2} \cap \cdots \cap \overline{D}_{f,i_k} = \{ a \in T_0^{(n)} : f(a) \equiv_\nu f_{i_u}(a), \, 1 \leq u \leq k \}$.

We call any such nonempty set a $k$-border, and we call $\{ \overline{D}_{f,i_1}, \ldots, \overline{D}_{f,i_k} \}$ its bordering components. Note that $k$ does not necessarily describe the codimension, since many components of the same dimension could meet at a common border. Nevertheless, we can call a $k$-border extremal if it does not have any other bordering components. This means in the terminology of Remark 7.2 that there is no $a \in \cap_{i=1}^k \overline{D}_{f,i_u}$ for which $f(a) \equiv_\nu f_i(a)$ for some $i \neq i_1, \ldots, i_k$.

**Remark 7.3.** Since any $k$-border is determined by the monomials of the components bordering it, there are at most $\binom{m}{k}$ $k$-borders, where $m$ is the number of monomials of $f$. 


The extremal points $a$ and $b$ are 4-border and 3-border, respectively. The line segment $s$ is 5-border, while $s \setminus \{a, b\}$ borders only the two components $D_2$ and $D_5$.

The border between two components $D_{f,i}$ and $D_{f,i'}$ is defined as
$$D_{f,i} \cap D_{f,i'} \setminus \{\cup_{j \neq i,i'} D_{f,j}\},$$
in other words, the 2-border after we remove all 3-borders (which include all the $k$-borders, $k > 3$). Two components having a nonempty border are called neighbors. (Note that the border is then dense in the 2-border.) See example in Figure 4.

Keeping Corollary 3.28 in mind, we have

**Remark 7.4.** For any $m \in \mathbb{N}$, $D_f = D_{f,m}$, and $D_{f,i} = D_{f,m,i}$.

Clearly, the $D_{f,i}$ are open sets (with respect to the $\nu$-topology). Furthermore, they are convex, in the sense of Definition 5.19.

**Proposition 7.5.** Each component $D_{f,i}$ is convex, and every closed component is convex.

**Proof.** Follows immediately from Lemma 5.21, applied to the dominant monomial in the component. □

**Corollary 7.6.** Every $k$-border is convex.

In other words, the common boundaries of the tangible components are convex. Thus, we can apply convexity arguments in our proofs, even without any extra topological assumptions on the supertropical semifield $F$.

**Definition 7.7.** A closed component $\overline{D}_{f,i}$ is 2-sided unbounded if it contains a two-sided ray; otherwise, it is partially bounded. The closed component $\overline{D}_{f,i}$ is bounded if it contains no one-sided ray.

**Lemma 7.8.** Any extremal border of a partially bounded component $\overline{D}_{f,i}$ must be a point.

**Proof.** Otherwise this extremal border contains two points, and thus the path connecting them. But then the two-sided ray through this path must also intersect some other border of $\overline{D}_{f,i}$, contrary to the extremal hypothesis. □

In view of this observation, we define an extremal point of a partially bounded component to be an extremal border.

**Note 7.9.** The same sort of argument shows that any extremal border must be the intersection of primitives, but we do not need that fact.

**Proposition 7.10.** Any Laurent monomial $h$ takes on $\nu$-maximal and $\nu$-minimal values in any bounded closed component $\overline{D}_{f,i}$, and furthermore, for $h$ non-constant on $\overline{D}_{f,i}$, the maximal and minimal $\nu$-values are taken at extremal points.
Proof. By symmetry, we need only look for the $\nu$-minimal values on $D_{f,i}$. We claim that for any $a$ in $D_{f,i}$, there is a point $a'$ on some border of $D_{f,i}$ such that $h(a') <_\nu h(a)$. Indeed, take any point $b \in D_{f,i}$ with $h(b) <_\nu h(a)$. The path connecting $a$ and $b$ lies in the same component, and continuing further along the ray until the border produces a smaller value of $h$.

We continue with the same argument applied to this border, showing that if $a$ lies in a $k$-border, there is a point $a'$ in a $k+1$-border with $h(b) <_\nu h(a)$, unless the $k$-border is extremal. Thus, we reach an extremal border (since the number of times we can apply this argument is at most the number of monomials in $f$). By definition, this cannot contain a ray, so is an extremal point, and since there are only finitely many extremal points, the $\nu$-minimal value of $h$ on all of these must be the $\nu$-minimal value on $D_{f,i}$.

Since Remark 7.13 shows there are only finitely many extremal points, we now see that the minimal and maximal $\nu$-values of a monomial on the various bounded components are all obtained at finitely many points. This is the key to our proof of the Nullstellensatz below.

Definition 7.11. Notation as above, we write $f \leq_{D_{f,i}} g$ if $g(D_{f,i}) \subseteq T$; we write $f \leq_{\text{comp}} S$ for $S \subseteq F(\lambda_1, \ldots, \lambda_n)$, if for every essential monomial $f_i$ of $f$ there is some $g \in S$ (depending on $D_{f,i}$) with $f \leq_{D_{f,i}} g$.

Remark 7.12.

1. The use of components is more precise than merely considering tangible root sets. For example, the tangible root set of the ideal $A$ of $F[\lambda]$ generated by $\lambda+2$ and $\lambda+3$ is $\{2\} \cap \{3\} = \emptyset$. However, the constant polynomial 1 has no tangible roots, but does not belong to $A$. On the other hand, the component of 1 is all of $T$, and is not contained in the component of any element of $A$, so $1 \not\in_{\text{comp}} A$.

2. Sometimes polynomials behave differently when viewed as Laurent polynomials. Consider the polynomials $f = \lambda_1 + 1$ and $g = (\lambda_1 + 1)\lambda_2$ in $F[\lambda_1, \lambda_2]$. Then $g(a, 0_F) = 0_F$ whereas $f(a, 0_F) = a + 1 \neq 0_F$, and $f \not\in_{D_{f,i}} g$. But as Laurent polynomials, $f$ and $g$ take on precisely the same tangible roots, so $f \leq_{D_{f,i}} g$.

Nevertheless, Laurent polynomials are useful in proving the Nullstellensatz for ordinary polynomials.

Let us record some information about the components of a polynomial.

Remark 7.13. For any polynomials $g, h$, any component of $g$ contains a component of $gh$, since $Z(g) \subseteq Z(gh)$.

Remark 7.14. Any polynomial $f$ has a unique dominant monomial on any component $D$ of $f$. Otherwise, there are two dominant monomials $g$ and $h$ of $f$ on part of $D$, and $\{a \in T : g(a) = h(a)\}$ is contained in $D \cap Z(f) = \emptyset$, a contradiction.

Proposition 7.15. If $f = \sum g_j$ for polynomials $g_j$, then any component $D$ of $f$ is contained in some component of some $g_j$; i.e., $f \leq_{D} g_j$.

Proof. Suppose $a \in D$, and $h$ is the dominant monomial of $f$ on $D$. Then $h$ is a monomial of some $g_j$. By Remark 7.13, $h$ is the dominant monomial of $f$ on all of $D$, and thus is the dominant monomial of $g_j$ on all of $D$. Hence the component of $g_j$ contains $D$.

Corollary 7.16. If, for some $k \geq 1$, the polynomial $f^k$ belongs to the ideal $A$ generated by the set of polynomials $S = \{g_i : i \in I\}$, then $f \leq_{\text{comp}} S$.

Proof. Since $f$ and $f^k$ have the same components, we may assume that $k = 1$. Write $f = \sum g_i g_j$. At each component $D$ of $f$ we have suitable $j$ such that $f \leq_{D} g_j$, so $f \leq_{D} g_j$; we conclude that $f \leq_{\text{comp}} S$.

Our goal is to prove the following theorem:
Theorem 7.17. (Supertropical Nullstellensatz) Suppose $F$ is a divisibly closed, archimedian, supertropical semifield, $A \preceq F[\Lambda]$, and $f \in F[\Lambda]$. Then
\[ f \preceq_{\text{comp}} A \iff f \in \sqrt[\text{comp}]{A}. \]

The rest of this section is devoted to the proof of the Theorem. Since multiplying a polynomial $f$ by any element of $A$ does not affect the root set $\mathcal{Z}(f)$, and also does not affect whether or not $f$ belongs to a given ideal, we often will replace a polynomial by a scalar multiple.

7.1. Proof of Theorem 7.17. In view of Proposition 5.14, we can embed $F[\Lambda] = F[\lambda_1, \ldots, \lambda_n]$ into $F[\Lambda, \Lambda^{-1}] = F[\lambda_1, \lambda_1^{-1}, \ldots, \lambda_n, \lambda_n^{-1}]$. The interplay between polynomials and Laurent polynomials is quite useful, since it enables us to divide by monomials. The proof of the Nullstellensatz is attained according to the following sequence of steps, writing $f = \sum f_i$, a sum of monomials:

Step 1. Take polynomials $g_{D_i} \in A$ such that $f \preceq_{D_i} g_{D_i}$, and modify them such that $f_i^m = g_{D_i}$ on $D_i$, for each $i$.

Step 2. Increasing $m$ if necessary, one may assume for every component $i$ and all large enough $m$ that $f_i^m$ strictly dominates $g_{D_i} f_i^{m-1}$ on $D_i$, for each component $D_i$ neighboring $D_i$.

Step 3. Increasing $m$ if necessary, one may assume for $f$ tangible and all large enough $m$ that $f_i^m$ also strictly dominates $g_{D_i} f_i^{m-1}$ on all components $D_i$ not neighboring $D_i$ (with respect to the same $m$).

Step 4. Steps 2 and 3 imply for $f$ tangible that $f^m \models \sum g_{D_i} f_i^{m-1}$, where the sum is taken over the components of the tangible essential monomials of $f$. This means $f \in \sqrt[\text{comp}]{\sum D_i F[\Lambda] g_{D_i}}$.

Step 5. For general $f$, one may assume that $f_i^m \models g_{D_i} f_i^{m-1}$ on all components $D_i$ of the essential-tangible part of $f$.

Step 6. Step 5 implies $f^m \models \sum g_{D_i} f_i^{m-1}$, where the sum is taken over the tangibles of the component essential monomials of $f$. This means $f \in \sqrt[\text{comp}]{\sum D_i F[\Lambda] g_{D_i}}$.

There is a version of the Nullstellensatz, with somewhat easier proof, using the Laurent polynomial semiring. Although the Laurent version is a bit different from the polynomial version, as seen in Example 7.12, we end this section with Proposition 7.21 which could be used to link the two versions.

Step 1 of the proof. The idea is to match some power $f^m$ at each component with some polynomial of the ideal $A$.

Lemma 7.18. Suppose $D = D_{f,i}$, $f \preceq_{D} g$ in $F[\Lambda]$, and $\lambda_j$ divides $g$. Then $\lambda_j$ divides $f_i$.

Proof. Suppose $\lambda_j$ does not divide $f_i$, and $a = (a_1, \ldots, a_n) \in D$. Let $\mathbf{b}$ be the point obtained by specializing $a_j \mapsto 0_F$. Then $f(\mathbf{b}) = f_i(\mathbf{a})$, but $f_i(\mathbf{b}) = f_i(\mathbf{a})$, implying $\mathbf{b} \in D$. On the other hand, $g(\mathbf{b}) = 0_F$, contrary to hypothesis. \qed

For each component $D_{f,i}$ we choose a polynomial $g_{D_{f,i}} \in A$ for which $f \preceq_{D_{f,i}} g_{D_{f,i}}$. Write $h_i$ for the dominant monomial of $g_{D_{f,i}}$ on $D_{f,i}$. By the lemma, taking $m$ large enough, we have $\deg_j f_i^m \geq \deg_j g$ for each $j$, so $f_i^m = q_i h_i$ for a suitable monomial $q_i$. Replacing $g_{D_{f,i}}$ by $q_i g_{D_{f,i}}$, we may assume that $q_i = 1$, and $f_i^m = g_{D_{f,i}}$. This achieves Step 1.

Step 2 of the proof. Before proceeding to Step 2 – Step 5, we note that Step 4 follows formally from Steps 2 and 3, whereas Step 6 follows formally from Step 5. Since we may replace $f$ by $f^m = \sum f_i^m$ without affecting the outcome of the theorem, we thus may assume that $f_i = h_i$ (notation as in the proof of Step 1); in other words, the leading monomials of $f$ and $g_{D_{f,i}}$ on $D_{f,i}$ are the same. It is convenient at this stage to move to the Laurent polynomial ring, replacing $f, g_{D_{f,i}}$ respectively by
\[ \frac{f}{f_i}, \frac{g_{D_{f,i}}}{f_i}. \]

Thus, we assume that $f_i = h_i = 1_F$.

We aim to verify Steps 2 and 5 for all sufficiently large $m$; since there are only finitely many components, this means that we need only prove these steps for a given single component $i$, which we fix for the
remainder of the proof. This simplifies the notation, since we can write \( D = D_f, \) and \( g = g_D. \) For each component \( D', \) we write \( f_{D'} \) for the leading component of \( f \) and \( h_{D'} \) for the leading component of \( g_{D'}. \) Thus \( f_D = h_D = 1_F, \) which strictly dominates each \( f_{D'} \) and each \( h_{D'} \) on \( D. \)

We need \( m_0 \) such that \( f_{D'}^m \) strictly dominates \( h_{D'} \) on \( D' \) for all \( m \geq m_0, \) for each component \( D' \) neighboring \( D. \) We pick a point \( a \) on the border, and take a small enough closed \( a \)-box \( B, \) cf. Definition 3.32, such that its extremal points lie in \( D' \cup D. \) Hence, some extremal point \( b \) of the box \( B \) lies in \( D'. \) By definition,

\[
f_{D'}(a) = f_D(a) = 1_F = h_D(a) \geq h_{D'}(a),
\]

whereas

\[
f_{D'}(b) > h_{D'}(b) = 1_F.
\]

Thus, since \( F \) is archimedean, there is \( m_0 \) such that

\[
f_{D'}(b)^m > h_{D'}(b) \tag{7.1}
\]

for all \( m \geq m_0. \) Since there are only finitely many extremal points on the box \( B, \) we may assume that \( (7.1) \) holds for all extremal points of \( B \) and thus, by Lemma 5.24, for all points in \( B \cap D'. \)

But now for any point \( b' \in D', \) the path from \( a \) to \( b' \) passes through \( B \cap D'. \) It starts at \( a, \) where \( f_{D'}(a)^m = 1_F = h_{D'}(a), \) and then passes through some point \( c \) where \( f_{D'}(c)^m > h_{D'}(c) \) (see Figure 5), so applying Lemma 5.24 to \( f_{D'}^m \), we also have \( (7.2) \) at the point \( b', \) as desired.

**Step 3 of the proof.** This is the subtler part of the proof. Pick a point \( a \in D. \) Then for every monomial \( f_V \neq f_1, \)

\[
f_V(a) < f_D(a) = 1_F,
\]

so picking \( m \) large enough, we have

\[
f_V^m(a) < h(a) \tag{7.2}
\]

for every monomial \( h \) of \( g. \) Furthermore, since there are only finitely many components, we may assume that \( (7.2) \) holds for every monomial \( f_V \) of \( f \) (other than \( f_1)). \)

Now we fix \( m \) for the remainder of the proof, and consider the polynomial \( f = f^m + g. \) We take the components with respect to \( f; \) this is just a subdivision of the components with respect to \( f. \) We call a component \( D_V \) good if \( f_V^m \) strictly dominates the polynomial \( g_{D_V} f_V^{m-1} \) on \( D_V. \) Otherwise, we say that \( D_V \) is bad, which means \( g_{D_V} f_V^{m-1} \) dominates \( f_V^m \) on \( D_V. \)

Our aim is to prove that all components are good; we assume that some component \( D' \) is bad and reach a contradiction. Let \( L \) be the set of good components. Take a point \( a \) in \( D, \) and a point \( b \) in \( D; \) adjusting them if necessary, we may assume that the path \( p \) from \( a \) to \( b \) passes only from neighbor to neighbor. (In other words, any point on \( p \) not lying inside a component lies on the common border of two neighbors.)

By definition, \( D \) and its neighboring components lie in \( L; \) we take the first bad component traversed by our path \( p, \) and clearly may replace this component by \( D'. \) Thus, we may assume that \( p \) contains
a point $b'$ on the border of $D'$ with some component of $L$, which we call $D''$. Say $f_\nu$ is the dominant monomial of $f$ on $D'$, and $f_\nu'$ is the dominant monomial of $f$ on $D''$. Hence $f_\nu(b') = f_\nu'(b')$ since it lies on the border.

We know that

$$f_\nu^m(a) \leq g_{D_i}(a)f_\nu^{m-1}(a).$$

By hypothesis, since $D'$ is bad,

$$f_\nu^m(b) \leq g_{D_i}(a)f_\nu^{m-1}(a).$$

On the other hand, since $D''$ is good,

$$f_\nu^m(b') = f_\nu^m(b') \geq g_{D_i}(a)f_\nu^{m-1}(a).$$

Since $b'$ lies between $a$ and $b$ on the path $p$, we have a contradiction to Lemma 5.21.

Step 5 of the proof. Let $\tilde{f}$ denote the tangible polynomial having the same $\nu$-value as $f$. We relabel the components of $\tilde{f}$ as $D_1, \ldots, D_q$. Some of these remain as components of $Z_{\text{tan}}(f)$; we call these components “true”. Other components are in the root set of $f$ (because of its extra ghost coefficients) and thus belong to $Z_{\text{tan}}(f)$; and we call these components “fictitious.” For each true component $D_j$, take a polynomial $g_{D_j} \in A$ with a component containing $D_j$, and let $\tilde{f}$ be the (tangible) sum of these monomials $f_j$, from the true components. Applying Step 3 to $\tilde{f}$, we see that $f_\nu^m \mid g_{D_j}f_\nu^{m-1}$ on all true components $D_j$.

Step 6 of the proof. This is formal: Clearly $f_\nu^m \mid \tilde{f}^m = \sum g_{D_j}f_\nu^{m-1}$ (summed on the true components). This concludes the proof of the Nullstellensatz.

Example 7.19. Let $R = F[\lambda]$, and consider the polynomial $f = \lambda^2 + 6\nu\lambda + 7$, whose tangible root set is the interval $[1, 6]$.

(i) If $g = \lambda + 4$, whose tangible root set is $\{4\}$, then

$$f = (\lambda + 3)g,$$

so $f \mid g$.

(ii) $g = \lambda^2 + 4\nu\lambda + 6$, whose tangible root set is the interval $[2, 4]$, then

$$f^2 = \lambda^4 + 6\nu\lambda^3 + 12\nu^2\lambda^2 + 13\nu\lambda + 14$$

and $(\lambda^2 + 8)g = \lambda^4 + 4\nu^2\lambda^3 + 8\lambda^2 + 12\nu^2\lambda + 14$, implying

$$f^2 \mid g.$$

Example 7.20. Generalizing Example 7.19, suppose $f = \lambda^2 + a_2\lambda + a_1a_2$, for $a_1, a_2$ tangible. Then, for a tangible, $\lambda + a$ supertropically divides $f$ iff $a_1 \leq \nu a \leq a_2$. Indeed, suppose $f \mid (\lambda + a)q$. Then $q = \lambda + b$, where, comparing constant terms, we see that $ab \leq \nu a_1a_2$. Now matching the coefficients of $\lambda$ shows that $\max\{b^\nu, a^\nu\} \leq a_2^\nu$, and thus $\min\{b^\nu, a^\nu\} \geq a_1^\nu$.

7.1.1. An explicit connection to the Laurent polynomial semiring. The following result links the Nullstellensatz to Laurent polynomials, and could be used to provide an alternate proof.

Proposition 7.21. Suppose $hf^m = g = \sum g_i$, for some monomial $h$ and some $m$, where $g_i$ are polynomials such that $f \leq g_i$ in $F[\lambda]$, for each $i$. Then there is some $m' > m$ for which $f^{m'} \in \sum g_i F[\lambda]g_i$.

Proof. Write $f = \sum f_i$ as a finite sum of monomials, and write $h = \prod_{u=1}^n \lambda_{j_u}^{k_u}$. We proceed by induction on $\deg h$. Let

$$U = \{ u : k_u \neq 0 \} \subseteq \{1, \ldots, n\}.$$

Pick $u \in U$, and write $\lambda = \lambda_{j_u}$, and $k = k_u$. By Proposition 5.23 $\lambda$ divides $h_i g_i$ for each $i$.

Let $J_u = \{ i : \lambda \divides h_i \}$. If $i \in J_u$, then we can write $h_i = \lambda h'_i$, for some monomial $h'_i$, and $f^{mk} = h'^k g_i^k$. 

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If \( i \notin J_u \), i.e., \( \lambda \) does not divide \( h_i \), then \( \lambda \) divides \( g_i \); hence, by Lemma 8.18, \( \lambda \) divides \( f_1 \), and \( \lambda^k \) divides \( f_1^k \). Writing \( f_1^k = \lambda^k f_1^k \), we then have \( f_1^{m+k} = \lambda^k f_1^m f_1^k \). Applying this for each \( u \) and taking \( \tilde{k} = \sum k_u \) yields

\[
 f_1^{m+k} = h_1 f_1^{\tilde{k}} g_1.
\]

Hence, taking \( m' \) to be the maximum of \( \{ m + \sum k_u, \ m \prod k_u : u \in U \} \), we see that each \( f_1^{m'} \in F[\Lambda]g_1 \), implying by Corollary 8.28 that

\[
 f^{m'} = \sum f_1^{m'} \in \sum F[\Lambda]g_1.
\]

\[\square\]

8. Factorization of Polynomials

One way of determining the roots of a polynomial is by factoring it first into irreducible factors, which is the main theme of this section. Unfortunately, factorization over the max-plus algebra is quite cumbersome. As noted in the introduction, the polynomial \( \lambda^2 + 4 \) is irreducible even though 2 is a root. One can bypass this difficulty by factoring up to \( e \)-equivalence. Since a polynomial \( f \) has precisely the same roots as its essential part, we always study divisibility and factorization in the sense of \( e \)-equivalence and \( (\nu, e) \)-equivalence, cf. Definition 4.5.

Difficulties are still encountered when studying factorization, especially if one wants to understand polynomials having cluster roots, so let us start this section with a brief guide to its results. We embark on a thorough investigation of factorization of polynomials over a supertropical semifield \( F \), with emphasis on the factorization of a polynomial \( f(\lambda) \) in one indeterminate \( \lambda \). This requires finding the appropriate representative of \( f \) in \( F[\Lambda] \). Although one could work with essential polynomials, the computations do not match so well, as we see in Example 8.3 (iii), and we look for a more convenient representative. The answer comes from a description of the polytope of a polynomial in \( \mathbb{S}2.1 \) which leads us to the notion of a full polynomial (Definition 8.11).

Tangible full polynomials behave quite like polynomials in classical algebra, having unique factorization into linear factors; cf. Theorem 8.21. However, nontangible polynomials behave more poorly, and unique factorization is violated in Example 8.51. However, this also can be understood in terms of root sets, and by rewriting factorizations in terms of binomials, cf. Theorem 8.53 which has the geometric interpretation that every root set can be embedded naturally in a union of hyperplanes. The remainder of this section contains examples which clarify the geometric content of this theorem.

8.1. General observations about factorization.

**Definition 8.1.** A polynomial \( g \in R[\Lambda] \) \( e \)-divides \( f \), written \( g \mid_{e} f \), if \( f \sim qg \) for some polynomial \( q \). (In other words, the image of \( g \) in \( R[\Lambda] \) divides the image of \( f \).) A polynomial \( f \) is said to be \( e \)-reducible if \( f \sim gh \) for some \( g, h \in R[\Lambda] \) each not \( e \)-equivalent to a nonconstant; otherwise \( f \) called is \( e \)-irreducible.

The product \( f \sim q_1 \cdots q_s \) is called a factorization of \( f \) into irreducibles if each of the \( q_i \)’s is \( e \)-irreducible.

**Remark 8.2.** For \( f, g \in R[\Lambda] \), if \( f \mid_{e} g \) and \( g \mid_{e} h \), then \( f \mid_{e} h \).

**Example 8.3.** (Logarithmic notation)

1. \((\lambda + 1) \mid_{e} (\lambda^2 + 2\lambda + 3) \), since \( \lambda^2 + 2\lambda + 3 = (\lambda + 1)(\lambda + 2) \);
2. \((\lambda + 1) \mid_{e} (\lambda^2 + 2) \), in view of Example 4.7 (iii);
3. \(\sum f_i^k \mid_{e} \sum f_i^m \) for each \( m \geq k \geq 1 \), in view of Corollary 8.28.

**Proposition 8.4.** The polynomial \( g \mid_{e} f \), iff the essential part of \( qg \) equals the essential part of \( f \) for some polynomial \( q \).
Proof. For each condition, the essential parts have to be $e$-equivalent, and thus equal, monomial for monomial.

**Corollary 8.5.** The polynomial $g | f$, iff the essential part of $g$ $e$-divides the essential part of $f$ with respect to the multiplication of $R[\Lambda]$.

**Proof.** By Proposition 8.3.

**Example 8.6.** The following is a list of all $e$-irreducible polynomials in $F[\lambda]$, together with their tangible root sets and their tangible complements. We normalize, to assume that the leading coefficient is $1_F$ or $1_{F''}$. For convenience, we also assume throughout that $\nu_T: T \to G$ is $1:1$ and $a, b \in T$.

**Type I.a:** $f = \lambda; \mathcal{Z}(f) = \mathcal{G}_{0}$, whereas $\mathcal{Z}_{\text{tan}}(f) = \{0_F\}$. The tangible complement is all of $T$.

**Type I.b:** $f = \lambda + a; \mathcal{Z}_{\text{tan}}(f) = \{a\}$. The tangible complement is the union of two open rays.

**Type II (right ghost):** $f = \lambda + a^\nu; \mathcal{Z}_{\text{tan}}(f) = \{b \in T : b \leq \nu a\}$, the closed left ray up to $a$. The tangible complement is the open right ray from $a$.

**Type III (left ghost):** $f = \nu^\lambda + a; \mathcal{Z}_{\text{tan}}(f) = \{b \in T : b \geq \nu a\}$, the closed right ray from $a$. The tangible complement is the open left ray to $a$.

**Type IV:** $f = \lambda^2 + b\lambda + ab$ for $a <_\nu b; \mathcal{Z}_{\text{tan}}(f) = \{d \in T : a \leq d \leq \nu b\}$, the closed interval from $a$ to $b$. The tangible complement is comprised of two open rays, one open left to $a$ and the other open right from $b$.

### 8.2. The geometry of polynomials

Important as they are to our theory, essential polynomials miss the mark when computing factorizations, since we have to continue to take essential parts when computing products. We want a different representative inside $F[\Lambda]$ that will more accurately reflect this product. In order to put the algebraic theory into perspective, we turn to a key geometric interpretation of polynomials, which enables us to overcome this difficulty.

#### 8.2.1. The polyhedron of a polynomial

We identify each monomial $\alpha_i \lambda^i$ (for $i = (i_1, \ldots, i_n)$) with the point

$$(i, \alpha_i^\nu) = (i_1, \ldots, i_n, \alpha_i^\nu) \in \mathbb{N}^{(n)} \times \mathcal{G} \subset \mathbb{R}^{(n)} \times \mathcal{G},$$

where $\mathcal{G}$ is the divisible closure of $\mathcal{G}$. For any polynomial $f = \sum_i \alpha_i \lambda^i \in R[\Lambda]$, we define the polytope $C_f$ determined by the convex hull of the points

$$\{(i, \alpha_i^\nu) : i \in \text{supp}(f)\}.$$  

The upper part of $C_f$ is called the **essential polyhedron of** $f$, and is denoted $\overline{C}_f$, whose vertices we call the **upper vertices** of $C_f$. The points of $\overline{C}_f$ of the form $\{(i, \alpha_i^\nu) : i \in \mathbb{N}^{(n)}\}$ are called **lattice points** of $f$. For example, when $f = \lambda^2 + 2$, its lattice points are $(2, 0^\nu)$, $(1, 1^\nu)$, and $(0, 2^\nu)$. A vertex $(i, \alpha_i^\nu)$ of $\overline{C}_f$ is called a **tangible vertex** if $\alpha_i$ is tangible; otherwise the vertex is called a **ghost vertex**.

(The essential polyhedron of $f$ should not be confused with the graph of $f$ itself, which is in a sense dual; in the graph of $f$, the vertices themselves correspond to ordinary roots of $f$.

The structure described above can be stated in the context of the Newton polytope as described in [6.2]. In this sense the convex hull, $\Delta(f)$, of the $i$’s in supp($f$) describes the Newton polytope of $f$ and, by taking the projection, by deleting the last coordinates, of the non-smooth part of $\overline{C}_f$ (that is a polyhedral complex) on $\Delta(f)$, the induced polyhedral subdivision $S(f)$ of $\Delta(f)$ is obtained. A dual geometric object having combinatorial properties is thereby produced. This object plays a major role in the classical tropical theory; cf. [11, 21, 23, 26, 30].

The following result shows how the roots of a polynomial correspond to its essential polyhedron. As mentioned earlier, when studying the polyhedron, we use the additive (logarithmic) notation for $\mathcal{G}$.

**Proposition 8.7.** Over a divisibly closed supertropical domain $R$, any polynomial $f$ is weakly $(\nu, \nu e)$-equivalent to the polynomial corresponding to $C_f$, and $C_{f^{\nu e}} = C_f$.  

the essential polyhedron of $\mathcal{G}_f$. Geometrically, the full closure $\overline{\mathcal{C}}_f$ of a monomial $f$ is the smallest cone $\mathcal{C}_f$ containing all the ordinary roots of $f$. The essential polyhedron of $\mathcal{C}_f$ is the set of all monomials that are ordinary for $f$.

**Definition 8.11.** A polynomial $f \in R[\Lambda]$ is called full if every lattice point lying on $\overline{\mathcal{C}}_f$ corresponds to a monomial of $f$ that is either essential or quasi-essential, and furthermore, the coefficient of each quasi-essential monomial of $f$ is a ghost; a full polynomial $f$ is tangibly-full if $f$ is also essential-tangible. The full closure $\overline{\mathcal{C}}_f$ of $f$ is the sum of $f^{es}$ with all the quasi-essential ghost monomials interpolated from the polyhedron $\overline{\mathcal{C}}_f$.

In this paper, we only consider full polynomials in the case that $n = 1$; i.e., $f = \sum_{i=0}^{m} r_i \Lambda^i$. Here $f(0) = r_0$, which thus is essential in $f$. Whenever $r_0 \neq 0$, and likewise the monomial $r_m \Lambda^m$ is essential in $f$. The polynomial $f$ is full iff the intermediate monomials $f_i \Lambda^i$ are essential or quasi-essential for all $0 < i < m$.

**Remark 8.12.** Geometrically, the full closure $\overline{\mathcal{C}}_f$ has a monomial corresponding to each lattice point of the essential polyhedron of $f$. However, one needs to take care: The full closure is only defined over $\mathcal{G}$. For example, if $F = D(\mathcal{G})$ where $\mathcal{G} = (\mathbb{Z}, +)$, then the essential polynomial $\Lambda^2 + 1$ is defined over $F$ but its full closure, $\Lambda^2 + \frac{1}{2} \nu \Lambda + 1$, is defined not over $F$, but over $\overline{\mathcal{G}}$. 

**Remark 8.9.** An inessential monomial is quasi-essential if any (arbitrarily small) increase of the $\nu$-value of its coefficient makes it essential.

**Remark 8.10.** Given a polynomial $f = \sum_i \alpha_i \Lambda^i$, and assume that $h_i = \alpha_i \Lambda^i$ is a monomial of $f$ for which $(i, \alpha_i^\nu) = \sum_i t_u (h_i, \alpha_i^\nu)$ for some $h_1, \ldots, h_n$, where $t_u \in \mathbb{Q}^+$ and $\sum_u t_u = 1$. Then $h_i$ is inessential for $f$; when all the corresponding $h_i$ are essential, then $h_i$ is quasi-essential. This means that $(i, \alpha_i^\nu)$ lies on $\overline{\mathcal{C}}_f$ (or under) but is not a vertex. (The proof is as in Equation (5.5).)

8.2. Full polynomials. Having the geometric interpretation in hand, we are ready for our main class of polynomials. Essential polynomials slightly miss the mark, since the polyhedron of an essential polynomial may lack interior lattice points.

**Definition 8.11.** A polynomial $f \in R[\Lambda]$ is called full if every lattice point lying on $\overline{\mathcal{C}}_f$ corresponds to a monomial of $f$ that is either essential or quasi-essential, and furthermore, the coefficient of each quasi-essential monomial of $f$ is a ghost; a full polynomial $f$ is tangibly-full if $f$ is also essential-tangible. The full closure $\overline{\mathcal{C}}_f$ of $f$ is the sum of $f^{es}$ with all the quasi-essential ghost monomials interpolated from the polyhedron $\overline{\mathcal{C}}_f$. 

In this paper, we only consider full polynomials in the case that $n = 1$; i.e., $f = \sum_{i=0}^{m} r_i \Lambda^i$. Here $f(0) = r_0$, which thus is essential in $f$. Whenever $r_0 \neq 0$, and likewise the monomial $r_m \Lambda^m$ is essential in $f$. The polynomial $f$ is full iff the intermediate monomials $f_i \Lambda^i$ are essential or quasi-essential for all $0 < i < m$.

**Remark 8.12.** Geometrically, the full closure $\overline{\mathcal{C}}_f$ has a monomial corresponding to each lattice point of the essential polyhedron of $f$. However, one needs to take care: The full closure is only defined over $\mathcal{G}$. For example, if $F = D(\mathcal{G})$ where $\mathcal{G} = (\mathbb{Z}, +)$, then the essential polynomial $\Lambda^2 + 1$ is defined over $F$ but its full closure, $\Lambda^2 + \frac{1}{2} \nu \Lambda + 1$, is defined not over $F$, but over $\overline{\mathcal{G}}$.
Thus, by definition, the full closure of a tangible polynomial is tangible-full.

**Example 8.13.** The polynomials $\lambda^2 + 2\nu \lambda + 4$, $\lambda^2 + 2\nu \lambda + 4\nu$, and $0\nu \lambda^2 + 2\nu \lambda + 4\nu$ are full. However, the polynomial $f = \lambda^2 + 2\lambda + 4$ is not full, since the middle term is not essential but is tangible; the monomial $2\lambda$ is quasi-essential for $f$, and the full closure of $f$ is $\lambda^2 + 2\nu \lambda + 4$, which is tangible-full.

The polynomial $\lambda^2 + 3\nu \lambda + 4$ is full, and essential, but not tangibly-full.

**Remark 8.14.** The full closure $\tilde{f}$ is $e$-equivalent to $f$, for any polynomial $f$. Conversely, different full polynomials cannot be $e$-equivalent. Thus, any class of polynomials in $R[\lambda_1, \ldots, \lambda_n]$ has a unique full representative $\tilde{f}$, and we can view $R[\lambda_1, \ldots, \lambda_n]$ as the set of full polynomials, under the operations

$$f + g = \tilde{f} \oplus \tilde{g}, \quad fg = \tilde{f} \otimes \tilde{g}.$$ 

Thus, we have identified another canonical representative for each $e$-equivalence class in $R[\lambda_1, \ldots, \lambda_n]$, cf. Remark 8.8.

### 8.3. The essential graph of coefficients

We utilize the results of the previous subsection, for the case $n = 1$. For a polynomial $f = \sum_{i=0}^{t} \alpha_i \lambda^i \in F[\lambda]$, take the sequence $\alpha''_0, \ldots, \alpha''_t$, and the graph $G_f$ whose vertices are the points

$$(0, \alpha''_0), (1, \alpha''_1), \ldots, (t, \alpha''_t). \quad (8.1)$$

In the case of the polynomial semiring over a supertropical semifield, any polynomial of degree $t$ is determined by the graph $G_f$ (having at most $t$ edges). Note that $C_f$ is the convex hull of $G_f$, cf. [8.2.1]

The essential graph of coefficients, $\overline{C}_f$, is constructed as the top edges of $C_f$. (This is the essential polyhedron of a polynomial in one indeterminate.) When the polynomial $f$ is full, the graph of coefficients is already essential. As we shall see, these edges correspond to ordinary roots of $f$.

**Remark 8.15.** The slopes of the edges of the graph $\overline{C}_f$ of $f = \sum_i f_i$ decrease as we move to the right, since $\overline{C}_f$ is convex.

**Example 8.16.** $f = (\lambda + 1)^2(\lambda + 2) = (\lambda^2 + 1\nu \lambda + 2)(\lambda + 2) = \lambda^3 + 2\lambda^2 + 3\nu \lambda + 4$. Then the graph of coefficients has upper vertices $(0, 4\nu), (1, 3\nu), (2, 2\nu), (3, 0\nu)$, and the convex hull is determined by the upper vertices $(0, 4\nu), (2, 2\nu), (3, 0\nu)$, thereby corresponding to the polynomial $\lambda^3 + 2\lambda^2 + 4$, the essential part of $f$.

Note that $\overline{C}_f$ may contain lattice points not corresponding to monomials of the original polynomial $f$. For instance, in Example 8.16 the point $(1, 3\nu)$ lies on an edge of $\overline{C}_f$, although it is not a vertex.

**Proposition 8.17.** For $f \in F[\lambda]$, the $\nu$-equivalence classes of ordinary roots correspond to the negations of the slopes of the edges of $\overline{C}_f$, as to be described in the proof. Such roots exist whenever $F$ is divisibly closed.

**Proof.** For any ordinary root $a$ of $f$, we need $i < j$ for which

$$\alpha_j a^{j-i} \equiv_{\nu} \alpha_i, \quad (8.2)$$

i.e., in logarithmic notation,

$$(j - i)a \equiv_{\nu} \alpha_i - \alpha_j.$$ 

This means $a$ must satisfy

$$a \equiv_{\nu} \frac{\alpha_i - \alpha_j}{j - i} \equiv_{\nu} \frac{\alpha_j - \alpha_i}{j - i},$$

the negation of the slope of an edge of the graph of coefficients; conversely, any such tangible root $a$ is ordinary. \qed
8.3.1. Factoring tangible polynomials in one indeterminate. Assume that \( f \in R[\lambda] \), for a supertropical domain \( R \).

The tropical theory of polynomials in one indeterminate is rather close to the classical theory, when we work with tangibly-full polynomials.

**Remark 8.18.**

(i) Suppose \( f = pq \) for \( p, q \in R[\lambda] \). Then \( a \) is a root of \( f \) iff \( a \) is a root of \( p \) or \( q \). (Indeed \( f(a) = p(a)q(a) \), which is in \( G \) iff one of the factors is in \( G \).)

(ii) As a special case of (i), if \( f = (\lambda + a)q \) for \( f, q \in R[\lambda] \), then \( a \) is a root of \( f \).

To start a theory of factorization, we need a converse for Remark 8.18: Given a tangible root \( a \) of \( f \), we would like to divide \( f \). This issue is surprisingly tricky, and also leads us to the question of “multiple roots,” so the following calculation will be useful.

**Example 8.19.** Write \( \alpha^2 \) for \( \alpha \alpha \) (which is computed in logarithmic notation as \( 2\alpha \)); likewise \( \alpha^3 = \alpha \alpha \alpha \).

(i) \((\lambda + \alpha)^2 = \lambda^2 + \alpha \lambda + \alpha \lambda + \alpha^2 = \lambda^2 + \alpha^e \lambda + \alpha^2 \).

(ii) By Proposition 3.27, \((f + g)^m \simeq f^m + g^m \); in particular, \((\lambda + \alpha)^m \simeq \lambda^m + \alpha^m \).

**Lemma 8.20.** Suppose \( R = (R, G, \nu) \) is a supertropical domain. If \( a \) is an ordinary root of \( f \in R[\lambda] \), then \( \lambda + a \) e-divides \( bf \) for some \( b \in T \). In particular, when \( R \) is a supertropical semifield, \( \lambda + a \) e-divides \( f \).

**Proof.** Write \( f = \sum_{i=0}^{t} \alpha_i \lambda^i \). By Proposition 8.17 explicitly Equation (8.2), there are \( j < k \) such that, in semiring notation,

\[
\alpha_j \simeq \nu \alpha_k a^{k-j}.
\]

where \( b \simeq \nu \alpha_i \lambda^{t-j} \) and \( g = \sum_{i=0}^{j-1} \alpha_i \lambda^i \). Note that \( b \) is tangible since the root \( a \) is ordinary. One computes that

\[
\alpha_i (\lambda + a)^{t-j} (b \lambda^j + g) \simeq (\alpha_i \lambda^{t-j} + b)(b \lambda^j + g) \simeq b(\alpha_i \lambda^i + b \lambda^j + g) + \alpha_i \lambda^{t-j} g = bf + \alpha_i \lambda^{t-j} g.
\]

so we need only show that \( \alpha_i \lambda^{t-j} g \) is inessential in the right hand side. When \( c < \nu a \),

\[
\alpha_i c^{t-j} g(c) < \nu \alpha_i a^{t-j} g(c) \simeq \nu b g(c).
\]

When \( c \simeq \nu a \), then \( g(c) \leq \nu b c^j \) (since the monomial \( b \lambda^j \) dominates \( g \) for all substitutions to elements of \( \nu \)-value greater than the largest root), so

\[
\alpha_i c^{t-j} g(c) \leq \nu \alpha_i a^{t-j} b c^j \leq \nu b a \nu c^j.
\]

We conclude in each case that \( \alpha_i \lambda^{t-j} g \) is inessential.

For \( k < t \), Theorem 8.17 implies \( a \) is a root of \( f_1 = \sum_{i=0}^{t-1} \alpha_i \lambda^i \), (since \( a \) is the negation of the slope of some other edge of \( \overline{C_f} \), so by induction on degree, there is \( g(\lambda) \) of degree \( t - 2 \) such that \( \lambda + a \) has essential part \( b f_1 \). But then \( (\lambda + a)(\alpha_i b \lambda^{t-1} + g) \) has the same essential part as \( \alpha_i b \lambda^t + (\lambda + a)g \), which has the same essential part as \( b f \).

Iterating Lemma 8.20 we get

**Theorem 8.21.** Suppose \((F, G_0, \nu)\) is an \( N \)-divisibly closed, supertropical semifield. Then any polynomial \( f \in T[\lambda] \) is \( e \)-equivalent to the tangible part of a product \( \prod_j (\lambda + a_j)^{\nu_i} \), where the \( a_j \) range over ordinary roots of \( f \).

**Corollary 8.22.** If \( F = \overline{F} \), then any essential-tangible polynomial \( f \) can be factored uniquely to a product \( \prod_j (\lambda + a_j)^{\nu_i} \), where the \( a_j \) range over the ordinary roots of \( f \).

**Proof.** The polynomial \( \prod_j (\lambda + a_j)^{\nu_i} \) is full, and uniqueness is clear.

We would like to think of the \( i_j \) as the multiplicities of the roots, but, as usual, care is required. (We only handle essential-tangible polynomials here, since the general case is considerably subtler.)
Example 8.23. Suppose $F = (\mathbb{R}^x, \mathbb{R}^+, \nu)$, where $\nu$ is the usual absolute value on $\mathbb{R}^x$. The polynomial $\lambda^2 + (-4)$ is $e$-equivalent to $\lambda^2 + 2^i \lambda + (-4) = (\lambda + 2)(\lambda + (-2))$, but intuitively, since $(-2)^\nu = 2^\nu$, we should say that the root 2 has multiplicity 2.

Definition 8.24. For $f \sim \prod (\lambda + a_j)^{\nu_i}$, the multiplicity of some root $a$ of $f$ is

$$\sum \{i_j : a_j \cong \nu a\}.$$ 

Remark 8.25. When $\nu_T$ is 1:1, the multiplicities are just the $i_j$.

Corollary 8.26. The multiplicities of the roots of $f$ in Theorem 8.24 (and Corollary 8.22) are precisely the lengths of the edges of the convex hull of the essential graph of coefficients $C_f$ of $f$.

Proof. Just repeat the proof of Lemma 8.20 noting that the result holds for $\lambda^t + \alpha^t$ by Example 8.19. \qed

When $F$ is not divisibly closed, the following reduction is useful.

Proposition 8.27. Suppose $F$ is a supertropical semifield, and $f, g \in F[\Lambda]$. If $f$ divides $g$ in $F[\Lambda]$, then $f$ divides $g$ in $F[\Lambda]$.

Proof. Otherwise write $g = f^\nu h$ and let $\alpha \Lambda^k$ be the lowest order monomial (under the lexicographical order of $N^{(n)}$) of $h$ for which $\alpha \notin T$. Since it is essential, there must be some value $a$ for which $h(a) = \alpha \Lambda^k b$. But then $f(a) = g(a)\alpha \Lambda^k$, implying some monomial of $f$ has the form $g \alpha \Lambda^k$, for a suitable monomial $g_k$ of $g$. Thus, we may assume that $f$ and $g$ are monomials, and we have a contradiction since $G = \nu(T)$ is assumed to be a group. \qed

8.3.2. Factoring tangibly-full polynomials in one indeterminate. Having obtained decisive (albeit easy) results for tangible polynomials in one indeterminate, we turn towards the general case, focusing first on tangibly-full polynomials (such as $\lambda^2 + 2^\nu \lambda + 4$).

We recall that for any full essential polynomial $f$ of degree $t$, we get a sequence of ghost elements $m_1^t \geq \cdots \geq m_t^t$, defined uniquely by the slopes of the series of edges of $C_f$, each determined by the pair $(i-1, \alpha_{i-1}^t)$ and $(i, \alpha_i^t)$ for $1 \leq i \leq t$. Recall that a monomial $h = \alpha_i \lambda^t$ of $f$ is essential iff $(i, \alpha_i^t)$ is a vertex of $C_f$, which is true iff $m_i^t \neq m_{i+1}^t$. We have three possibilities for a monomial $h = \alpha_i \lambda^t$ of $f$:

(a) $h$ is tangible essential;

(b) $h$ is ghost essential; or

(c) $h$ is quasi-essential (at a lattice point which is not a vertex of $C_f$), which is the case iff $m_i^t = m_{i+1}^t$.

The following observation explains how to factor a tangibly-full polynomial.

Lemma 8.28. Assume that $F$ is a supertropical semifield, with $f = \sum_j \alpha_j \lambda^j \in F[\Lambda]$. If $\alpha_i \lambda^t$ is a tangible essential monomial of $f$, then

$$f = (\alpha_t \lambda^{t-i} + \alpha_{t-1} \lambda^{t-i-1} + \cdots + \alpha_{i+1} \lambda^i + \alpha_i)(\lambda^t + \frac{\alpha_{i-1}}{\alpha_i} \lambda^{t-1} + \cdots + \frac{\alpha_0}{\alpha_i}).$$

Proof. Denote the right side by $p(\lambda)$, and let $h_j$ be the monomial of degree $j$ of $p$. We need to show that $h_j = \alpha_j \lambda^j$ for all $j$. Note that each $h_j$ is a tropical sum of monomials, one of which is $\alpha_j \lambda^j$, so we need to check this is always the one (and only) monomial having the largest $\nu$-value. We do this case by case.

For $j = i$, this is clear unless $\alpha_i \lambda^t \leq \nu \alpha_{i+k} \lambda^k \alpha_i \lambda^{-k}$, for some $0 < k \leq i$. But then $\alpha_i \leq \nu \alpha_{i+k} \alpha_i \lambda^{-k}$, and thus $\alpha_i^t \leq \nu \alpha_{i+k} \alpha_i$. But this contradicts the fact that $\alpha_i \lambda^t$ is essential for $f$.

For $j > i$, we are done unless $\alpha_j \lambda^j \leq \nu \alpha_{j+k} \lambda^{j+i+k} \alpha_i \lambda^{-k}$, for some $0 < k \leq j$. Then $\alpha_j \leq \nu \alpha_{j+k} \alpha_i \lambda^{-k}$, implying $\frac{\alpha_i}{\alpha_j} \leq \nu \alpha_{j+k} \alpha_i \lambda^{-k}$. Since $C_f$ is convex, we must have equality, and $\alpha_{i+k}, \alpha_i, \alpha_j$, and $\alpha_{j+k}$ all lie on the same edge; again this contradicts the essentiality of $\alpha_i \lambda^t$.

For $j < i$, we are done unless $\alpha_j \lambda^j \leq \nu \alpha_{i+k} \lambda^{j+i+k} \alpha_i \lambda^{-k}$ for some $0 < k \leq i$, yielding $\frac{\alpha_i}{\alpha_{i+k}} \leq \nu \alpha_{i+k} \alpha_i \lambda^{-k}$, a contradiction by the same consideration as in the previous paragraph. \qed

Corollary 8.29. Suppose $f = \sum_{j=0}^t \alpha_j \lambda^j$ is full, with $\alpha_i$ tangible for some $0 < i < t$. Then $f = g_1 g_2$, where $g_1 = \sum_{j=0}^{t-i} \alpha_{i+j} \lambda^j$ and $g_2 = \sum_{j=0}^i \alpha_i \lambda^j$ are full.
Remark 8.35. Suppose \( f = \tilde{f} \). Any tangibly-full polynomial \( f \in F[\lambda] \) is the product of some power of \( \lambda \) with a product of tangible binomials.

Proposition 8.30. Suppose \( F = \tilde{F} \). Any tangibly-full polynomial \( f \in F[\lambda] \) is the product of some power of \( \lambda \) with a product of tangible binomials.

Note 8.31. One also could prove Proposition 8.30 geometrically, which provides the dividend that the factorization is unique up to \((\nu, \epsilon)\)-equivalence: We subdivide the graph of \( f \) to its lines of different slopes. In other words, if \( f = \sum_{i=0}^{\nu} \alpha_i \lambda^i \) where the slope changes at \( \lambda^i \), then one sees easily that \( f = gh \) where \( g = \sum_{i=0}^{\nu} \alpha_i \lambda^i \) and \( h = \sum_{j=0}^{\nu} \alpha_i \lambda^j \). Different products of binomials clearly produce different graphs, and thus the factorization is unique.

Corollary 8.32. When \( F = \tilde{F} \), any irreducible tangibly-full polynomial in one indeterminate must be a binomial.

8.3.3. Factoring arbitrary full polynomials in one indeterminate. When considering full polynomials that are not necessarily tangibly-full, we must face the fact that not every nonlinear polynomial \( f \) \( \epsilon \)-reducible; for example, one can easily check that \( f = \lambda^2 + 2\lambda + 3 \) is \( \epsilon \)-irreducible. We need an intermediate notion.

Definition 8.33. A polynomial \( f = \sum_{i=0}^{t} \alpha_i \lambda^i \) is \textit{semitangibly-full} if \( f \) is full with \( \alpha_t \) and \( \alpha_0 \) tangible, but \( \alpha_i \) are ghost for all \( 0 < i < t \).

Dividing out by \( \alpha_t \), we may assume that our semitangibly-full polynomial is monic. We have the following observation:

Lemma 8.34. If \( f = \lambda^t + \alpha_0 + \sum_{i=1}^{t-1} \alpha_i \lambda^i \) is monic semitangibly-full for \( t > 2 \) (where \( \alpha_i \) are taken tangible), then taking

\[
\delta = \frac{\alpha_0}{\alpha_1}, \quad \beta_i = \frac{\alpha_i}{\alpha_{i-1}},
\]

(both tangible), we have

\[
f = (\lambda^2 + \alpha_{t-1} \lambda + \delta \alpha_{t-1}) g,
\]

where \( g = \lambda^{t-2} + \beta_1 + \sum_{i=2}^{t-2} \beta_i \lambda^{i-1} \). Thus, we can extract a quadratic factor from any monic semitangibly-full polynomial of degree \( \geq 2 \).

Proof. The verification is along the same lines as Lemma 8.28. Namely, the constant terms match, and in the middle, the term \( (\alpha_{t-1}) \lambda (\beta_{t-1}) \) strictly dominates \( \lambda^2 (\beta_{t-2}) \) and \( (\delta \alpha_{t-1}) (\beta_{t-1}) \), because the slopes of the graph decrease. (Explicitly, we see that \( \alpha_i \) strictly dominates \( \beta_i \) since \( \alpha_{t-1} \lambda > \nu \frac{\alpha_{t-1}}{\alpha_1} \lambda \), and \( \alpha_i \) strictly dominates \( \delta \alpha_{t-1} \beta_{t+1} = \frac{\alpha_0 \alpha_{t+1}}{\alpha_1} \alpha_{t+1} \) since \( \frac{\alpha_{t+1}}{\alpha_i} > \nu \frac{\alpha_0}{\alpha_1} \).

A qualitative way of obtaining Equation 8.33 is by taking the \( \nu \)-equivalent polynomial \( \tilde{f} \) obtained by making each coefficient tangible, taking the product \( \tilde{h} \) of two linear factors of \( \tilde{f} \) (we took the first and the last in descending order of \( \nu \)-values), writing \( \tilde{f} = \tilde{h} \tilde{g} \), and then making the inner coefficients ghosts.

It remains to factor a polynomial to semitangibly-full polynomials, which we do by means of the following observation.

Remark 8.35. Suppose \( f = \sum_{i=0}^{t} \alpha_i \lambda^i \), where \( \alpha_t = 1 \) \( R^\nu \). Then

\[
f = (\lambda^t + \alpha_{t-1}) \sum_{i=0}^{t-1} \frac{\alpha_i}{\alpha_{t-1}} \lambda^i.
\]

We call a linear polynomial \( \lambda^t + a \) a \textit{linear left ghost}. Thus, whenever the leading terms are ghost we can use Remark 8.33 to factor out linear left ghosts until we reach a tangible leading term. But if we do this twice, we observe for tangible \( a, b \) with \( a \geq \nu b \) that

\[(\lambda^t + a)(\lambda^t + b) = 1 R^\nu \lambda^2 + a^\nu \lambda + ab = (\lambda + a)(\lambda^t + b).
\]

Thus, we always can adjust our factorization to have at most one linear left ghost factor \( \lambda^t + b \) for \( b \) tangible, and this is the \( b \) having the minimal \( \nu \)-value for those factors \( \lambda^t + b \) which can appear as linear left ghosts.
This reduces our considerations to the case where \( f \) is monic but with the constant term ghost. Now we define a **linear right ghost** to have the form \( \lambda + a' \). When the constant term is ghost we can factor out some linear right ghost, and arrange for \( f = (\lambda + a')h \), where \( h \) can be factored along tangible vertices to get semitangibly-full factors, and we continue as above.

Putting together Corollary [8.29] with Lemma [8.34], we see that any irreducible full polynomial \( f \) must have no tangible interior vertices, and at most one interior lattice point (whose corresponding vertex must be nontangible); thus, \( f \) must either be linear or quadratic, of the form

\[
\alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0, \tag{8.4}
\]

where \( \alpha_2 \lambda \) is essential. Iterating, we have the following result:

**Proposition 8.36.** Every full polynomial is the product of at most one linear factor of the form \( \lambda + a' \) (namely with \( a' \) maximal possible), at most one linear factor of the form \( \lambda + b' \) (with \( b' \) tangible and \( b' \) minimal possible), tangible linear factors, and semitangibly-full quadratic polynomials.

**Remark 8.37.** Suppose both the leading and constant coefficients of \( f \) are ghosts, so that we have extracted the right ghost \( \lambda + a' \) and left ghost \( \lambda + b' \). When \( a \geq b \), we also have

\[
(\lambda + a')(\lambda + b') = \lambda^2 + a'\lambda + (ab)'.
\]

**8.3.4. Uniqueness** of factorizations of polynomials in one indeterminate. Assume throughout this subsection that \( F = \mathbb{F} \). Having shown that any full polynomial in one indeterminate has a factorization into irreducibles of degree \( \leq 2 \), we turn in earnest to the companion question, of uniqueness of factorization of a (not necessarily full) polynomial into irreducibles. The answer turns out to be quite interesting, involving subtleties that do not exist in the classical theory of polynomials. Although unique factorization fails in \( F[\lambda] \), there is a version of unique factorization “minimal in ghosts,” which is seen to have a natural connection to the set of roots of the polynomial.

We immediately encounter new difficulties.

**Example 8.38.**

(i) The factorization into e-irreducibles need not necessarily be unique, even up to \( \tilde{\sim} \); for example \( \lambda^2 + 2\nu \sim (\lambda + 1)^2 \) and at the same time \( \lambda^2 + 2\nu \sim (\lambda + 1)(\lambda + 1\nu) \), whereas \( \lambda + 1\nu \sim \lambda + 1 \).

(ii) Another violation of unique factorization: for \( a \geq b \), we have \( \lambda + a')(\lambda + b) = \lambda^2 + a'\lambda + (ab)' = (\lambda + a')(\lambda + b') \).

(iii) The previous examples still have unique \((\nu, e)\)-factorization. A more serious violation of unique factorization:

\[
\lambda^4 + 4\nu \lambda^3 + 6\nu \lambda^2 + 5\nu \lambda + 3 = (\lambda^2 + 4\nu \lambda + 2)(\lambda^2 + 2\nu \lambda + 1) = (\lambda^2 + 4\nu \lambda + 2)(\lambda + 2)(\lambda + (-1)) = (\lambda^2 + 4\nu \lambda + 3)(\lambda^2 + 2\nu \lambda + 0),
\]

all of which are factorizations into e-irreducibles.

The last example is an illustration that the factorization procedure of Lemma [8.34] is not unique; we could factor out any two tangible roots of \( \tilde{f} \) to produce the first factor, just so long as their \( \nu \)-values are not both maximal or both minimal (in which case this trick does not work). Since we may permute the factors, we may always assume that the tangible root of highest \( \nu \)-value belongs to the first factor.

**Example 8.39.** This method explains the different factorizations in the polynomial

\[
f = \lambda^4 + 4\nu \lambda^3 + 6\nu \lambda^2 + 5\nu \lambda + 3
\]

of Example [8.38] (iii). Clearly \( f \) is semitangibly-full and has the four corner roots \(-2, -1, 2, \) and \( 4 \), so we can take the first quadratic factor to be \( \lambda^2 + 4\nu \lambda + 2 \) or \( \lambda^2 + 4\nu \lambda + 3 \). In the first case, the second quadratic factor is \( \lambda^2 + 2 \nu \lambda + 1 \), but we could use \( \lambda^2 + 2 \nu \lambda + 1 \) instead, which factors to \( (\lambda + 2)(\lambda + (-1)) \). (This will be explained in Proposition [8.46].)

Had we tried \( \lambda^2 + 4\nu \lambda + 6 \) for the first factor, we would need \( \lambda^2 + (-1)^\nu \lambda + (-3) \) for the second factor, but then the product is

\[
\lambda^4 + 4\nu \lambda^3 + 6\lambda^2 + 5\nu \lambda + 3,
\]

which is not quite \( f \) (since it has a tangible inner coefficient).
Nevertheless, there is a version of unique factorization in one indeterminate. In conjunction with Remark \[8.35\] and Proposition \[8.36\] we have proved the following result concerning unique factorization:

**Theorem 8.40.** Any full polynomial in one indeterminate is the unique product of a full tangible polynomial (which can be factored uniquely into tangible linear factors), a linear left ghost, a linear right ghost, and semitangibly-full polynomials of maximal possible degree.

**Proof.** Just factor at each tangible vertex, and multiply together the full tangible factors. \qed

This brings us back to semitangibly-full polynomials. By Remark \[8.36\] for \( F = \bar{F} \), any semitangibly-full polynomial can be factored into tangible linear and semitangibly-full quadratic factors. Despite Example \[8.38\] we also get uniqueness of a sort here, when we count the number of non-tangible quadratic components having essential ghost terms; this type of factorization turns out to be unique.

**Example 8.41.** In Example \[8.39\] the latter is the factorization of \( f \) which is minimal in ghosts, having only one ghost component.

**Lemma 8.42.** Suppose that \( a_i \lambda^i \) and \( a_{i+1} \lambda^{i+1} \) are essential monomials of \( f \), such that \( \delta a_{i+1} = a_i \) for \( \delta \) tangibles. (This means that \( a_i, a_{i+1} \) are both tangible or both ghost.) Then

\[
f = \left( \alpha_i \lambda^{i-1} + \cdots + a_{i+1} \lambda^i + \frac{\alpha_{i-1}}{\delta} \lambda^{i-1} + \frac{\alpha_{i-2}}{\delta} \lambda^{i-2} + \cdots + \frac{\alpha_0}{\delta} \right) (\lambda + \delta).
\]

**Proof.** Denote the product by \( g \) and let \( h_j \) be its monomial of degree \( j \). Then to see that \( h_j = (a_j \lambda^{j-1}) \lambda \) for \( j > i \), note that if \( h_j \cong (a_{j+1} \lambda^j) \delta \geq (a_j \lambda^{j-1}) \lambda \), then \( \frac{a_{j+1}}{a_j} \cong \frac{\delta}{\lambda} \geq \frac{a_j}{a_{j+1}} \), which contradicts the fact that the sequence of slopes determined by the coefficients is descending.

For \( j = i \), we have \( (a_{i+1} \lambda^i) \delta = (a_{i+1} \lambda^i) \frac{a_i}{a_{i+1}} = a_i \lambda^i \), which strictly dominates \( (\frac{a_{i+1}}{\delta} \lambda^{i-1}) \lambda \) since the slopes of the graph decrease. Hence, \( h_i = a_i \lambda^i \).

When \( j < i \), \( h_j = (\frac{a_j}{\lambda} \lambda^j) \delta \) since otherwise \( h_j \geq (\frac{a_{j+1}}{\delta} \lambda^{j-1}) \lambda \) by the same argument as for \( j > i \), which leads to the analogous contradiction. \qed

Putting all these results together yields:

**Theorem 8.43.** When \( F = \bar{F} \), any full polynomial in one indeterminate has a factorization into tangible linear factors, quadratic semitangibly-full factors, at most one linear left ghost and at most one linear right ghost, and the factorization which is minimal in ghosts is unique.

**Proof.** Just factor at each tangible vertex, then factor inductively at pairs of adjacent ghost vertices, and multiply together the full factors. \qed

Here is another way of viewing Theorem \[8.43\]

**Corollary 8.44.** Any full polynomial \( f \) can be written as the product \( f = f_1 f_m \), where \( f_i \) is tangible and \( f_m \) is the product in Theorem \[8.43\] of (perhaps) a linear left ghost, a linear right ghost, and semitangibly-full polynomials; \( f_m \) has alternating tangible and ghost coefficients, seen by applying Lemmas \[8.28\] and \[8.42\] inductively for pairs of adjacent tangible or ghost monomials that are essential. We call this procedure **extracting a minimal ghost factor**: note the minimality is in essential ghosts. Accordingly, \( f_i \) can be factored into linear components, and the factorization of \( f_m \) has at most two linear components while all the others are quadratic.

We can understand Theorem \[8.43\] better, by considering the tangible roots of a polynomial. In view of Remark \[8.18\] these roots are determined by the tangible roots of its \( e \)-irreducible factors. The case of one indeterminate is given in Example \[8.6\]

**Remark 8.45.** Working backwards in Type IV of Example \[8.6\] given a closed interval (or point) \( W \) in \( T \), one can write the \( e \)-irreducible polynomial of degree \( \leq 2 \) whose set of tangible roots is precisely \( W \).

In general, given a closed subset \( W \) of \( T \), we write \( W \) as a finite union \( W_1 \cup \cdots \cup W_t \) of disjoint closed intervals (or points), and take an \( e \)-irreducible polynomial \( f_k \) of degree \( \leq 2 \) whose set of tangible roots is precisely \( W_k \), for \( 1 \leq k \leq t \). Then taking \( f = f_1 \cdots f_t \), we see that \( T_{\text{tan}}(f) = W \).
Let us apply this process to an arbitrary semitangibly-full polynomial \( f \).

**Proposition 8.46.** Suppose \( f \) is a semitangibly-full polynomial of degree \( t \), and \( \alpha_1, \ldots, \alpha_t \) are corner roots of \( f \), arranged in ascending \( \nu \)-value. Then

\[
f = (\lambda^2 + \alpha_i^* \lambda + \alpha_i \alpha_1) \prod_{k=2}^{t-1} (\lambda + \alpha_k).
\]

**Proof.** Consider the tangibly-full polynomial \( \tilde{f} \) whose coefficients have the same \( \nu \)-value as those of \( f \). Then \( \alpha_1, \ldots, \alpha_t \) are the ordinary roots of \( f \), so

\[
\tilde{f} = \prod_{k=1}^{t} (\lambda + \alpha_k) = (\lambda^2 + \alpha_1 \lambda + \alpha_1 \alpha_1) \prod_{k=2}^{t-1} (\lambda + \alpha_k).
\]

It remains to note that all the interior coefficients of \( (\lambda^2 + \alpha_i^* \lambda + \alpha_i \alpha_1) \prod_{k=2}^{t-1} (\lambda + \alpha_k) \) are ghost, since the coefficient of \( \lambda^j \) is \( \alpha_i^* \alpha_{i-1} \cdots \alpha_{t-j} \).

\[\square\]

Obviously this is the factorization with the minimal number of ghosts (namely, just one). Note that the corner roots \( \alpha_2, \ldots, \alpha_{t-1} \) are interior points in \( Z_{\tan}(f) \), and all appear in tangible linear factors. When \( \deg f > t \), the statement of the result becomes more complicated since one needs to deal with multiple roots, but the proof is analogous, to be treated in another paper.

**Remark 8.47.** Reversing the logic of Proposition 8.46, take an arbitrary semitangibly-full polynomial \( f = \lambda^t + \sum_{i=1}^{t-1} \alpha_i^* \lambda^i + \alpha_0 \alpha_1 \), where each \( \alpha_i \in \mathcal{T} \). The tangible root set of \( f \) is the interval \([\alpha_0, \alpha_{t-1}]\), so we can factor

\[
f = (\lambda^2 + \alpha_{t-1}^* \lambda + \alpha_0 \alpha_{t-1}) g,
\]

where \( g = \sum_{i=0}^{t} \alpha_i^* \lambda^i \) is a tangible polynomial which can thus be factored into linear factors.

**Remark 8.48.** We are now in a position to explain geometrically the various factorizations of a full polynomial \( f \in F[\lambda] \) of degree \( n \). Namely, we take the set \( S = \{a_1, \ldots, a_n\} \) of tangible corner roots of \( f \), and partition \( S \) into \( n_1 \) pairs \((a_{i_1}, a_{i_2})\), \( 1 \leq i \leq n_1 \), and \( n_2 \) single roots, where \( 2n_1 + n_2 = n \), such that

\[
\cup(a_{i_1}, a_{i_2}), a_{i_2}) = Z_{\tan}(f).
\]

(The union need not be disjoint.) Each of the closed intervals \([a_{i_1}, a_{i_2}]\) is the root set of a polynomial

\[
f_i = \lambda^2 + \alpha_{i_2}^* \lambda + \alpha_{i_1} \alpha_{i_2},
\]

whereas each single root \( a_j \) is the root set of the linear polynomial \( \lambda + a_j \), and the product of all of these polynomials can be seen to be \( f \). Each of these subdivisions corresponds to a factorization of \( f \) into irreducibles.

There will be only one such partition in which each interval \([a_{i_1}, a_{i_2}]\) is a connected component of the tangible root set of \( f \), and this is the (unique) “preferred” factorization.

**Example 8.49.** Let us apply Remark 8.47 to explain Example 8.39. Since \( f \) is semitangibly-full of degree 4, and its corner roots are \(-2, -1, 2, 4\), of which \(-1\) and \(-2\) are in the interior of \( Z_{\tan}(f) \), we have the factorization

\[
f = (\lambda^2 + 4^* \lambda + 2)(\lambda + (-1))(\lambda + 2).
\]

This is the “preferred” factorization; the other factorizations are obtained by taking the partitions \([-2, 2], [-1, 4]\) and \([-2, 4], [-1, 2]\). The “near miss” of Example 8.39 comes from \([-2, -1], [2, 4]\) which is not quite a subdivision of \([-2, 4]\).

Here is a satisfactory numerical algorithm for factoring a polynomial \( f \) into e-irreducibles: First factor out the linear left ghost and/or right ghost if necessary, then factor \( f \) into a product of \( m \) semitangibly-full factors, and then apply Proposition 8.46 (or Remark 8.47) to obtain the factorization minimal in ghosts (one ghost for each of the \( m \) semitangibly-full factors).
8.4. Binomial factorization in several indeterminates. We turn to factorization in $F[\lambda_1, \ldots, \lambda_n]$ for $n > 1$. Although the thrust of this subsection is an analysis of how unique factorization fails, first we note a positive cancellation result as consolation. Recall that we write $F[\Lambda]$ for $F[\lambda_1, \ldots, \lambda_n]$ and $F[\Lambda, \Lambda^{-1}]$ for $F[\lambda_1, \lambda_1^{-1}, \ldots, \lambda_n, \lambda_n^{-1}]$.

Remark 8.50. If $f \in F[\Lambda, \Lambda^{-1}]$ is tangible and $fg = fh$, then $g = h$. (Indeed, $g$ and $h$ take on the same values on a dense subset of $F(n)$ in the $\nu$-topology, so are identically equal.)

Note that this topological argument could fail when $f$ is not tangible, even for one indeterminate, as evidenced by the factorizations
\[
\begin{align*}
\lambda^2 + 1^\nu \lambda + 1^\nu &= (\lambda + 1^\nu)(\lambda + 0^\nu) = (\lambda + 1^\nu)(\lambda + 0); \\
\lambda^4 + 4^\nu \lambda^3 + 6^\nu \lambda^2 + 5^\nu \lambda + 3 &= (\lambda^2 + 4^\nu \lambda + 2)(\lambda^2 + 2^\nu \lambda + 1) \\
&= (\lambda^2 + 4^\nu \lambda + 2)(\lambda^2 + 2 + \lambda + 1).
\end{align*}
\]

For several variables, we confront a most severe violation of unique factorization.

Remark 8.51. Suppose $f_1, f_2, f_3 \in F[\Lambda]$.

(i) $f_1 + f_2 + f_3$ is a factor of $(f_1 + f_2)(f_1 + f_3)(f_2 + f_3)$. Indeed,
\[
(f_1 + f_2 + f_3)(f_1f_2 + f_1f_3 + f_2f_3) = \\
(f_1 + f_2)(f_1 + f_3)(f_2 + f_3) + \nu(f_1f_2f_3) = \\
(f_1 + f_2)(f_1 + f_3)(f_2 + f_3),
\]

Note that the polynomial $\nu(f_1f_2f_3)$ is inessential. Thus, full-tangible polynomials need not have unique factorization.

Two variants of (i), for later use, which one checks by matching the tangible parts:

(ii) $(f_1 + f_2 + f_3')(f_1 + f_2 + f_3f_3') = (f_1 + f_2)(f_1 + f_3')(f_2 + f_3);

(iii) $(f_1 + f_2' + f_3')(f_1f_2 + f_1f_3 + f_2f_3) = (f_1 + f_2')(f_1 + f_3)(f_2 + f_3)$.

Example 8.52.
\[
(0 + \lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + \lambda_1 \lambda_2) = \\
\lambda_1 + \lambda_2 + \lambda_1^2 + \lambda_2^2 + \nu(\lambda_1 \lambda_2) + \lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2 = \\
(0 + \lambda_1)(0 + \lambda_2)(\lambda_1 + \lambda_2).
\]

In fact, any polynomial $f = \sum_{i=1}^{m} f_i$ $e$-divides $\prod_{i \neq j} (f_i + f_j)$, which leads us to the main theorem of this section.

Theorem 8.53. Suppose $f = \sum_{i=1}^{m} f_i \in \text{Fun}(R^{(n)}, R)$, for $m \geq 2$. Then
\[
\prod_{i < j}(f_i + f_j) = g_1 \cdots g_{m-1} \quad (8.1)
\]

where $g_1 = f = \sum_i f_i$, $g_2 = \sum_{i < j} f_if_j$, \ldots, and $g_{m-1} = \sum_{i \neq j} f_j$.

Our applications of this theorem are for the sub-semiring $R[\Lambda, \Lambda^{-1}]$ of $\text{Fun}(F(n), F)$, in which this result could be viewed as the utter collapse of unique factorization, since every polynomial $f$ which is a sum of at least three distinct monomials is part of a factorization that is not unique. Specifically, if $f_i$ are the monomials of $f$, then $f$ is a factor of the product $\prod_{i \neq j} (f_i + f_j)$. However, Theorem 8.53 casts considerable light on the geometry, and has a positive geometric interpretation:

Remark 8.54. Every tropical variety $X$ can be “completed” to a variety $\mathcal{P}(X)$ comprised of various hyperplanes, which in turn can be decomposed into a union $X_1 \cup \cdots \cup X_{m-1}$, where $X_i = Z_{\tan}(g_i)$ for $1 \leq i \leq m - 1$. After proving Theorem 8.53, we shall see how such geometric decompositions provide an assortment of factorizations.

The factorization in Theorem 8.53 involves many inessential terms, so, to avoid excessive computation in the proof of the theorem, we consider how inessential terms often arise.

Lemma 8.55. If $h_2^3 = h_1h_3 \in \text{Fun}(R^{(n)}, R)$, then $h_2$ is inessential for $h_1 + h_2 + h_3$. 


We can compute the permanent in two ways:

For the next lemma, we let $\mathcal{I}_m \subset \mathbb{N}^{(m)}$ denote the set of all $m$-tuples $i = (i_1, \ldots, i_m)$ for which each $0 \leq i_a < m$ and $\sum_{a=1}^m i_a = (m^2)/2$. Such $m$-tuples include $(0,1,\ldots,m-1)$ or any permutation of the components. For any $i = (i_1, \ldots, i_m) \in \mathcal{I}_m$ and $0 \leq j \leq m - 1$, we define the $j$-index $i_j(i)$ to be the number of $i_a$'s that equal $j$; define $\iota(i) = (i_{m-1}(i), \ldots, i_0(i))$.

Let $S_m$ denote the set of permutations of $(0,1,\ldots,m-1)$. Thus, $i \in S_m$ iff $\iota(i) = (1,1,\ldots,1)$. We say $i$ is admissible if for each number $k$, the sum of the largest $k$ components of $i$ is at most $(m-1) + \cdots + (m-k) = km - k(k+1)/2$. Thus, all $i \in S_m$ are admissible.

**Lemma 8.56.** Lexicographically, $\iota(i) \leq (1,1,\ldots,1)$ for each admissible $i \in \mathcal{I}_m$.

**Proof.** For any admissible $i$, the sum of the largest component is at most $2m - 3$, which means that at most one component is $m - 1$, so $\iota_{m-1}(i) \leq 1$. We are done unless $\iota_{m-1}(i) = 1$, and conclude by induction on $m$.

Given $f_1, \ldots, f_m \in \text{Fun}(R^n, R)$, for each $i \in \mathcal{I}_m$, we define the function

$$h_i = f_1^{i_1} \cdots f_m^{i_m}.$$ 

For any permutation $\sigma \in S_m$, we denote

$$h_\sigma = f_1^{\sigma(0)} \cdots f_m^{\sigma(m-1)}.$$

**Lemma 8.57.** $\sum_{\sigma \in S_m} h_\sigma = \sum_{i \in \mathcal{I}_m} h_i$.

**Proof.** Let $p = \sum_{i \in \mathcal{I}_m} h_i$. We need to show that $h_i$ is inessential in $p$ whenever $\iota(i) < (1,1,\ldots,1)$. The proof is by reverse induction on the lexicographic order of $\iota(i)$. Since $\iota(i) < (1,1,\ldots,1)$, some $j$-index $i_j(i)$ is 0, and we take the largest such $j$. Then for some $j' < j$, the $j'$-index $i_{j'}(i) \geq 2$; in other words, $i$ has components $i_s = i_t = j'$ for suitable $s \neq t$.

Take $i' = (i_1', \ldots, i_m')$ to be the $m$-tuple in which $i_s' = j' + 1$ and $i_t' = j' - 1$ (with all other components the same as for $i$), and likewise let $i''$ be the $m$-tuple in which $i_s'' = j' - 1$ and $i_t'' = j' + 1$. By Lemma 8.56, $h_i$ is inessential in $h_{i'} + h_{i''}$. We claim that $i'$ and $i''$ are admissible and $\leq (1,1,\ldots,1)$. Indeed, this is clear when $j' < j - 1$, since then $i_j(i') = i_j(i'') = 0$. Thus, we may assume that $j' = j - 1$. Clearly $\iota(i'') = \iota(i')$, since the roles of $s$ and $t$ are interchanged, so it suffices to prove the claim for $i'$.

First assume that $i_{j-1}(i) \geq 3$. Then $j \geq 2$, and the sum of the largest $k = m - j + 2$ components of $i$ is equal to

$$(m-1) + \cdots + (j+1) + 0 + 3(j-1),$$

which is greater than $km - k(k+1)/2$ unless $3(j-1) = j + (j-1)$, which implies $j = 2$.

Thus, we are done unless $i_{j-1}(i) = 2$. Since $i'$ increases the component $i_s$ from $j - 1$ to $j$, and decreases the component $i_t$ from $j - 1$ to $j - 2$, we see that $i_j(i') = 1$ and $i_{j-1}(i') = 0$, proving $\iota(i') < (1,1,\ldots,1)$, as desired.

Clearly, $\iota(i'') = \iota(i')$ is of higher lexicographic order than $\iota(i)$ (since $i_{j'}+1(i') = i_{j'}+1(i) + 1$), so, by reverse induction, either $i' \in S_m$ or $h_{i'}$ is inessential in $p$, and likewise for $i''$, implying that $h_i$ is inessential in $p$.

Our main tool in proving Theorem 8.53 is the tropical Vandermonde matrix $V_f$ of $f = \sum_{i=1}^m f_i$. Define $V_f$ to the $m \times m$ matrix with entries $v_{i,j} = f_i^{j-1}$. Since the signed determinant is not available in tropical algebra (because it involves negative signs), one substitutes the permanent, which we still notate as

$$|V_f| = \sum_{\sigma \in S_m} f_1^{\sigma(0)} \cdots f_m^{\sigma(m-1)}.$$

We can compute the permanent in two ways:

**Lemma 8.58.** If $V_f = (f_i^{j-1})$ is the $m \times m$ Vandermonde matrix for $f = \sum f_i$, then

1. $|V_f| = \prod_{i<j} (f_i + f_j)$ and,
(2) \(|V_f| = (\sum_i f_i)(\sum_{i<j} f_if_j) \cdots (\sum_i \prod_{j \neq i} f_j)\).

**Proof.** Let \(p = |V_f| = \sum_{\sigma \in S_m} h_\sigma\), the function of Lemma 8.54, which says that \(p = \sum_{\sigma \in X_m} h_\sigma\).

But it is easy to see that each summand of \(q_1 = \prod_{i<j} (f_i + f_j)\) has the form \(h_1\) where \(i\) is admissible, and thus \(h_1\) is dominated by \(p\), by Lemma 8.57. Since each summand of \(p\) appears in \(q_1\), we get \(p = q_1\). Likewise, expanding \(q_2 = (\sum_i f_i)(\sum_{i<j} f_if_j) \cdots (\sum_i \prod_{j \neq i} f_j)\) clearly each term has the form \(h_1\) where \(i\) is admissible, and each summand of \(p\) appears in \(q_2\), implying \(p = q_2\).

The proof of Theorem 8.53 now becomes quite transparent:

**Proof of Theorem 8.53** By parts (i) and (ii) of Lemma 8.58.

Algebraically, Theorem 8.53 shows that the factorization of \(|V_f| \in R[\lambda]\) into irreducible polynomials is not unique.

**Example 8.59.** Suppose \(f = \lambda_1^3 + \lambda_2^2 + \alpha\), with \(\alpha \in R\). Then

\[
V_f = \begin{pmatrix}
0 & \alpha & \alpha^2 \\
0 & \lambda_1^3 & \lambda_2^2 \\
0 & \lambda_2^2 & \lambda_2^3
\end{pmatrix}
\]

\(|V_f| \approx (\lambda_1^3 + \lambda_2^2 + \alpha)(\alpha\lambda_1^2 + \alpha\lambda_2^2 + \lambda_1^3\lambda_2^2) \approx (\lambda_1^3 + \lambda_2^2)(\lambda_1^2 + \alpha)(\lambda_2^2 + \alpha)\).

This yields two different tropical factorizations of \(|V_f|\) into irreducible polynomials. (The right factorization is a binomial factorization.)

Thus, in this version of supertropical algebra, perhaps “unique factorization” is the wrong emphasis, but rather we should emphasize factorization of \(|V_f| \subset R[\lambda_1, \ldots, \lambda_n]\) into binomials.

**Remark 8.60.** Lemma 8.55 is clearly self-dual in the sense of Remark 3.29, hence Theorem 8.53 also holds over the dual supertropical semifield \(F^\wedge\). Explicitly, suppose \(f = \sum f_i\), written as a sum of monomials. Taking the isomorphism \(\Phi_{Fun}\) of Remark 3.29 and putting \(f = g_1 \cdots g_{m-1}\), we have \(\Phi_{Fun}(g_i) = (f)^{-1}g_{m-i}\) for \(1 \leq i \leq m-1\).

Thus, \(\Phi_{Fun}\), also induces an action \(\hat{\Phi}\) on tangible root sets, given by

\[
\hat{\Phi}(X_i) = (Z_{\tan}(\Phi_{Fun}(g_i))) = (Z_{\tan}(\Phi_{Fun}(g_{m-i}))) = X_{m-i}.
\]

Likewise,

\[
\Phi_{Fun}(f_i + f_j) = f_i^{-1} + f_j^{-1} = (f_i f_j)^{-1}(f_i + f_j),
\]

so \(\hat{\Phi}\) preserves binomials in the sense that

\[
\hat{\Phi}(Z_{\tan}(f_i + f_j)) = \hat{\Phi}(Z_{\tan}((f_i f_j)^{-1}(f_i + f_j))) = Z_{\tan}(f_i + f_j).
\]

Thus, in Remark 3.57, \(\hat{\Phi}\) induces a partition into dual pairs of root sets.

In this way, the algebraic structure again is reflected in the geometry.

**Remark 8.61.** We also can go in the other direction, and illustrate the Nullstellensatz. If \(f = \sum_{i=1}^m f_i\) then, arguing as before (since \(f_if_j\) is dominated by \(f_i^2 + f_j^2\)),

\[
f^2 \approx \sum f_i^2 \approx f_m(f_m + f_1) + \sum_{i=1}^{m-1} f_i(f_i + f_{i+1}),
\]

which is in the ideal \((f_m + f_1, f_i + f_{i+1} : 1 \leq i \leq m-1)\).

**Example 8.62.** (Illustrating Theorem 8.53) Let \(f = \lambda_1^3 + \lambda_2 + \lambda_1 + 0\) (see Fig. 2), a polynomial over \(D(R)\). Then, notation as in Theorem 8.53, \(g_1 = f\) and \(g_2 = \lambda_1(\lambda_1^2 + \lambda_1^2 + 0\). Defining the binomials \(q_1 = \lambda_1^3 + \lambda_2 + \lambda_1, \ q_2 = \lambda_1^2\lambda_2 + 0\), and \(q_3 = \lambda_1 + 0\), we have the equality

\[
g_1 g_2 = q_1 q_2 q_3 = (\lambda_1 + \lambda_1^2 + \lambda_1^2 + 0)v^1\lambda_1^3 + \lambda_1^4 + \lambda_1^2 + \lambda_1^2 \lambda_2 + \lambda_1^2 \lambda_2 + \lambda_1^2 \lambda_2.
\]

\(X_2 = Z_{\tan}(g_2)\) can be viewed as the complement of \(X = Z_{\tan}(f)\) along \((0,0)\).

In the next example, we can “improve” the factorization of Theorem 8.53.
Example 8.63. Let \( f = \lambda_1^2 + \lambda_2^2 + \alpha \lambda_1 \lambda_2 + 0 \) (see Fig. 7) be a polynomial over \( D(R) \), where \( \alpha > 0 \). Note here that \( \mathcal{Z}_{\text{tan}}(\lambda_1^2 + \lambda_2^2) \) does not affect \( \mathcal{Z}_{\text{tan}}(f) \), since whenever \( (\lambda_1^2)^\nu = (\lambda_2^2)^\nu \), these are both less than \((\alpha \lambda_1 \lambda_2)^\nu\).

Let \( f_i = \lambda_i^2 + \alpha \lambda_1 \lambda_2 + 0, \) for \( i = 1, 2 \). Also, define the binomials \( q_1 = \alpha \lambda_1 \lambda_2 + 0, q_2 = \lambda_1^2 + 0, q_3 = \lambda_2^2 + 0, q_4 = \lambda_1 + \alpha \lambda_2, \) and \( q_5 = \alpha \lambda_1 + \lambda_2 \). Algebraically, Theorem 8.53 applied to \( f_1 \) and \( f_2 \) in turn yields

\[
\begin{align*}
1^f g_1 &= q_1 q_3 q_5; \\
2^f g_2 &= q_1 q_2 q_4,
\end{align*}
\]

where \( g_i = \lambda_2 + \alpha \lambda_1 \lambda_2 + \alpha \lambda_1 \) and \( g_2 = \lambda_1 + \alpha \lambda_1 \lambda_2 + \alpha \lambda_2 \). Furthermore,

\[
q_1 f = \alpha \lambda_1^2 \lambda_2 + \alpha \lambda_1 \lambda_2^3 + \alpha^2 \lambda_1^2 \lambda_2^2 + \alpha^5 \lambda_1 \lambda_2 + \lambda_1^2 + \lambda_2^2 + 0 = f_1 f_2,
\]

since \( \lambda_1^2 \lambda_2^2 \) is strictly dominated by \( \alpha \lambda_1^2 \lambda_2^2 \). Consequently,

\[
q_1 f_1 g_2 = f_1 q_2 f_2 g_2 = q_1^2 q_2 q_3 q_4 q_5,
\]

implying \( f g_1 g_2 = q_1 q_2 q_3 q_4 q_5 \), which is actually an improvement of Theorem 8.53.

Taking tangible root sets, we have

\[
\mathcal{Z}_{\text{tan}}(q_1) \cup \mathcal{Z}_{\text{tan}}(f) = \mathcal{Z}_{\text{tan}}(f_1 f_2) = \mathcal{Z}_{\text{tan}}(f_1) \cup \mathcal{Z}_{\text{tan}}(f_2).
\]

Geometrically, \( \mathcal{Z}_{\text{tan}}(f) \) is contained in the five lines which are respectively the tangible root sets of \( q_1, q_2, q_3, q_4, \) and \( q_5 \). The tangible root sets of \( g_1 \) and \( g_2 \) are the complements of \( f \) along the respective vertices \((-\alpha, 0)\) and \((0, -\alpha)\).

The explanation of Example 8.63 is that one of the binomials of \( f \) (namely \( \lambda_1^2 + \lambda_2^2 \sim (\lambda_1 + \lambda_2)^2 \)) is “fictitious,” since its tangible set does not exist in the graph (as we showed above). Thus we can separate \( f \) into two polynomials whose factorizations do not involve the fictitious binomial. Continuing this process inductively, one can find a factorization that displays \( f \) as a divisor of a product of \( m \) binomials, where \( m \) is the minimal number of hyperplanes whose union contains the graph of \( f \). The precise description of an algorithm for this process seems to involve an investigation of the Newton polytope, which we do not pursue here.

9. Prime ideals of polynomial semirings

Since the Nullstellensatz translates supertropical geometry to radical ideals, and every radical ideal is the intersection of prime ideals, we would like to classify the prime ideals of the polynomial semiring \( F[\Lambda] \) over a supertropical semifield \( F \). The factorization in Theorem 8.33 clearly affects prime ideals.

Example 9.1. The ghost-closed ideal \( \langle \lambda_1 + \lambda_2 + 0 \rangle \) of \( F[\lambda_1, \lambda_2] \) is not prime! Indeed, if \( A \) were prime, Example 8.52 would imply that \( A \) contains one of \( 0 + \lambda_1, 0 + \lambda_2, \) and \( \lambda_1 + \lambda_2 \), which is absurd.

Likewise, reading Example 8.52 from the other direction shows that the ghost-closed ideal \( \langle \lambda_1 + \lambda_2 \rangle \) of \( F[\lambda_1, \lambda_2] \) is not prime.
Given a polynomial \( f = \sum_i f_i \) written as a sum of monomials, we define the set of **binomials of** \( f \) to be \( \{ f_i + f_j : i, j \in \text{supp}(f), i \neq j \} \). The role of binomials is found in the following key observation.

**Remark 9.2.** It follows at once from Theorem [8.53] that if \( P \) is a prime ideal of \( F[\Lambda] \) and \( f \in P \), then some binomial of \( f \) belongs to \( P \).

The key to binomials is found in the following observation, which is a converse to Example [8.3], we already treated the case \( n = 1 \) (twice) in Proposition [8.30].

**Proposition 9.3.** If \( F = \bar{F} \) and \( f \in F[\Lambda] \) is a binomial, then \( f \) can be factored as a product of a monomial times a power of an irreducible binomial.

**Proof.** Let us write \( f^\text{res} = \alpha\Lambda^i + \beta\Lambda^j \). Factoring out \( \beta \), we may assume that \( \beta = \mathbb{1}_F \). It is convenient to work in \( F[\Lambda, \Lambda^{-1}] \), since then we may divide by \( \Lambda^j \) and assume that \( f^\text{res} \) has the form \( \alpha\Lambda^i + \mathbb{1}_F \). We are done unless the full closure of \( f^\text{res} \) has some monomial on the line connecting \( i \) to \( (0, \ldots, 0) \). In other words, \( f \) has some monomial \( \gamma\Lambda^k \), where \( i = mk \) for suitable \( m \). But then \( \frac{n}{m}\Lambda^k \) is a monomial of \( f \), which is the \( m \)-th power of \( h = \frac{n}{m}\Lambda^k + \mathbb{1}_F \). We are done if \( h \) is \( e \)-irreducible, and continue by induction if \( h \) is \( e \)-reducible. (One has to check that the factorization in \( F[\Lambda, \Lambda^{-1}] \) matches a factorization in \( F[\lambda_1, \ldots, \lambda_n] \), by clearing denominators.)

9.1. **Ghost-closed prime ideals of polynomials in one indeterminate.** The classification of ghost-closed prime ideals is difficult even for the case \( n = 1 \), because there are many more of them than in classical ring theory. In this paper we content ourselves with the result that the tangible part of any \( f.g. \) prime ideal of \( F[\lambda] \) is generated by at most two polynomials. We start with the list of \( e \)-irreducible polynomials given in Example [8.6].

**Example 9.4.** Suppose \( F \) is a supertropical semifield, and \( \alpha \in F \) and \( \beta \in T \) with \( \alpha <_\nu \beta \).

(i) The ghost-closed ideal generated by \( \lambda + \alpha \) contains \( \lambda + \beta^\nu = (\lambda + \alpha) + \beta^\nu \).

(ii) Any ideal containing \( f_1 = \lambda + \alpha \) and \( f_2 = \lambda + \beta \) also contains \( \lambda + \gamma \) for all \( \gamma \in T \) with \( \alpha^\nu < \gamma^\nu < \beta^\nu \), since

\[
\lambda + \gamma = (\lambda + \alpha) + \frac{\gamma}{\beta}(\lambda + \beta).
\]

(iii) Any ideal containing \( f_1 = \lambda + \alpha \) and \( f_2 = \lambda^\nu + \beta \) also contains \( \lambda + \gamma \) for all \( \gamma \in T \) with \( \alpha^\nu < \gamma^\nu < \beta^\nu \), by the same computation as in (ii).

(iv) If \( \alpha^\nu \leq \alpha^\nu_1 \leq \beta^\nu \leq \beta^\nu_2 \), then the polynomial \( \lambda^2 + \beta^\nu_1\lambda + \alpha_1\beta_2 \) is contained in the ghost-closed radical ideal generated by \( \lambda^2 + \beta^\nu_1\lambda + \alpha_1\beta_1 \) and \( \lambda^2 + \beta^\nu_2\lambda + \alpha_2\beta_2 \), as seen by the Nullstellensatz.
Proposition 9.7. These generate all other linear factors.

□

Proof. In view of Theorem 8.21, the tangible part of $P$ is generated by linear polynomials, and so we conclude by the lemma.

□

Proposition 9.8. Any f.g. prime ghost-closed ideal of $F[\lambda]$ is supertropically generated by at most two linear polynomials.

Proof. Suppose $P$ is a prime ideal of $F[\lambda]$. By Proposition 8.36, any polynomial in $P$ factors as a product of linear and irreducible quadratic factors $\lambda^2 + \gamma\nu\lambda + \gamma\delta$, where $\gamma, \delta \in T$ with $\delta^\nu < \gamma^\nu$. $P$ is tropically generated by at most four polynomials. In view of Lemma 9.6, the linear polynomials in $P$ are bounded by $\lambda + \alpha$ and $\lambda + \beta$ (or $\lambda^\nu + \beta$), and it remains to consider irreducible quadratic factors.

One could continue using direct computations, but it is easier to apply the Nullstellensatz. Each quadratic polynomial has a tangible root set consisting of a tangible interval, and the nonempty intersections of these intervals give us root sets corresponding to quadratic polynomials in $P$. Thus, the quadratic factors of $P$ are generated by finitely many quadratic factors whose roots sets are $\nu$-disjoint. We can discard $f = \lambda^2 + \gamma\nu\lambda + \gamma\delta$ unless $\gamma^\nu \geq \alpha^\nu$ or $\beta^\nu \leq \delta^\nu$, as seen by checking the tangible complements. But then we replace $\lambda + \alpha$ by the leftmost such quadratic factor, and $\lambda^\nu + \beta$ (or $\lambda^\nu + \beta^\nu$ by the rightmost such quadratic factor), where appropriate, and these generate an ideal whose radical is $P$.

□

Unfortunately, this kind of argument shows that any chain of prime ideals is contained in a chain of infinite length. Indeed, the prime ideal generated by $\{\lambda + \alpha, \lambda + \beta\}$ (for $\alpha^\nu < \beta^\nu$) is contained in the prime ideal generated by $\{\lambda + \alpha_1, \lambda + \beta_1\}$, whenever $\alpha_1^\nu \leq \alpha^\nu$ and $\beta^\nu \leq \beta_1^\nu$.

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