Persistent homotopy theory

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Basic setup I

$X \subset Z$ finite subset, $Z$ a metric space. $D(Z) =$ poset of finite subsets of $Z$. $s \geq 0$.

- $P_s(X) =$ poset of subsets $\sigma \subset X$ such that $d(x, y) \leq s$ for all $x, y \in \sigma$.

$P_s(X)$ is the poset of non-degenerate simplices of the Vietoris-Rips complex $V_s(X)$. $BP_s(X)$ is barycentric subdivision of $V_s(X)$.

We have poset inclusions

$$\sigma : P_s(X) \subset P_t(X), \ s \leq t,$$

$P_0(X) = X$, and $P_t(X) = \mathcal{P}(X)$ (all subsets of $X$) for $t$ suff large.

- $k \geq 0$: $P_{s,k}(X) \subset P_s(X)$ subposet of simplices $\sigma$ such that each element $x \in \sigma$ has at least $k$ neighbours $y$ such that $d(x, y) \leq s$.

$P_{s,k}(X)$ is the poset of non-degenerate simplices of the degree Rips complex $L_{s,k}(X)$. 

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Basic setup II

The usual inclusions: $s \leq t$

\[
\begin{array}{c}
\text{Ps}(X) \xrightarrow{\sigma} \text{Pt}(X) \\
\uparrow \hspace{1cm} \uparrow \\
\text{Ps,k}(X) \xrightarrow{\sigma} \text{Pt,k}(X) \\
\uparrow \hspace{1cm} \uparrow \\
P_{s,k+1}(X) \xrightarrow{\sigma} P_{t,k+1}(X)
\end{array}
\]

Also

- $P_{s,0}(X) = P_s(X)$ for all $s$,
- $P_{s,k}(X) = \emptyset$ for $k$ suff. large.

**Initial impression**: $BP_s(X)$ is a huge model for $V_s(X)$, because all simplices of $V_s(X)$ are vertices of $BP_s(X)$. 

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Fundamental groupoid

\[ x_0, \ldots, x_k : \text{list of elements of } X \text{ such that } d(x_i, x_j) \leq s \text{ (may have repeats)}. \]

\[ [x_0, \ldots, x_k] = \{x_0\} \cup \cdots \cup \{x_k\}. \]

**Graph** \( Gr_s(X) \): vertices are elements of \( X \), there is an edge \( x \to y \) if \([x, y] \in P_s(X)\).

There is an edge \([x, y] : x \to y \) if and only if there is an edge \([y, x] : y \to x \). There is an edge \([x, x] : x \to x \).

\( \Gamma_s(X) \) is **category** generated by \( Gr_s(X) \), subject to relations defined by simplices \([x_0, x_1, x_2]\).

**Lemma 1.**

\( \Gamma_s(X) \) is a groupoid, and \( \Gamma_s(X) \simeq G(P_s(X)) \simeq \pi V_s(X) \).

\( \pi V_s(X) \) is the fundamental groupoid of \( V_s(X) \), \( G(P_s(X)) \) is the free groupoid on the poset \( P_s(X) \).
$D(Z)$ is the poset of finite subsets of $Z$ (all data sets in $Z$), with Hausdorff metric $d_H$.

**Hausdorff metric:**

$r > 0$: Given $X \subset Y$ in $D(Z)$, $d_H(X, Y) < r$ if for all $y \in Y$ there is an $x \in X$ such that $d(y, x) < r$.

For arbitrary $X, Y \in D(Z)$: $d_H(X, Y) < r$ if and only if (equivalently)

1) $d_H(X, X \cup Y) < r$ and $d_H(Y, X \cup Y) < r$.

2) for all $x \in X$ there is a $y \in Y$ such that $d(x, y) < r$, and for all $y \in Y$ there is an $x \in X$ such that $d(y, x) < r$. 
Stability

$X \subset Y$, $d_H(X, Y) < r$: Construct a function $\theta : Y \to X$ such that

$$\theta(y) = \begin{cases} y & \text{if } y \in X \\ x_y & \text{for some } x_y \in X \text{ with } d(y, x_y) < r. \end{cases}$$

If $\tau \in P_s(Y)$ then $\theta(\tau) \in P_{s+2r}(X)$. Have a diagram of poset morphisms

\[
\begin{array}{c}
P_s(X) \xrightarrow{\sigma} P_{s+2r}(X) \\
i \downarrow \theta \downarrow i \\
P_s(Y) \xrightarrow{\sigma} P_{s+2r}(Y)
\end{array}
\quad
\begin{array}{c}
y_1 \xrightarrow{s} y_2 \\
r \leftarrow \theta(y_1) \xleftarrow{s+2r} \theta(y_2)
\end{array}
\]

such that upper triangle commutes, and lower triangle commutes up to homotopy:

$$\sigma(\tau) \to \sigma(\tau) \cup i(\theta(\tau)) \leftarrow i(\theta(\tau)).$$
Stability results

Theorem 2 (Rips stability).

Suppose $X \subset Y$ in $D(Z)$ such that $d_H(X, Y) < r$. There is a homotopy commutative diagram (homotopy interleaving)

$$
P_s(X) \xrightarrow{\sigma} P_{s+2r}(X) \\
i \downarrow \quad \theta \quad \downarrow i \\
P_s(Y) \xrightarrow{\sigma} P_{s+2r}(Y)
$$

Theorem 3.

Suppose $X \subset Y$ in $D(Z)$ such that $d_H(X_{dis}^{k+1}, Y_{dis}^{k+1}) < r$. There is a homotopy commutative diagram

$$
P_{s,k}(X) \xrightarrow{\sigma} P_{s+2r,k}(X) \\
i \downarrow \quad \theta \quad \downarrow i \\
P_{s,k}(Y) \xrightarrow{\sigma} P_{s+2r,k}(Y)$$
Blumberg-Lesnick Theorem

Theorem 4.

Suppose given $X, Y \subset Z$ are data sets with $d_H(X, Y) < r$. Then there are maps $\phi : P_s(X) \to P_{s+2r}(Y)$ and $\psi : P_s(Y) \to P_{s+2r}(X)$ such that

$$\psi \cdot \phi \simeq \sigma : P_s(X) \to P_{s+4r}(X) \quad \text{and}$$

$$\phi \cdot \psi \simeq \sigma : P_s(Y) \to P_{s+4r}(Y).$$
Proof

Set

\[ U = \{(x, y) \mid x \in X, y \in Y, d(x, y) < r \} \].

\( P_{s,X}(U) \subset \mathcal{P}(U) \): all subsets \( \sigma \) such that \( d(x, x') \leq s \) for all \( (x, y), (x', y') \in \sigma \). Define poset \( P_{s,Y}(U) \) similarly.

1) The maps \( P_{s,X}(U) \to P_s(X) \), \( P_{s,Y}(U) \to P_s(Y) \) are weak equivalences (Quillen Theorem A).

2) There are inclusions

\[ P_{s,X}(U) \subset P_{s+2r,Y}(U), \quad P_{s,Y}(U) \subset P_{s+2r,X}(U), \]

(triangle inequality) and

\[ P_{s,X}(U) \subset P_{s+2r,Y}(U) \subset P_{s+4r,X}(U) \]
\[ P_{s,Y}(U) \subset P_{s+2r,X}(U) \subset P_{s+4r,Y}(U) \]
Suppose that $X \subset Y$ in $D(Z)$ and we have a homotopy interleaving

\[
\begin{align*}
V_s(X) & \xrightarrow{\sigma} V_{s+r}(X) \\
i \downarrow & \quad \theta \quad \downarrow i \\
V_s(Y) & \xrightarrow{\sigma} V_{s+r}(Y)
\end{align*}
\]

(as in stability theorem), where upper triangle commutes and lower triangle commutes up to homotopy fixing $\sigma : V_s(X) \to V_{s+r}(X)$.

1) $i : \pi_0 V_*(X) \to \pi_0 V_*(Y)$ is an $r$-monomorphism: if $i([x]) = i([y])$ in $\pi_0 V_s(Y)$ then $\sigma[x] = \sigma[y]$ in $\pi_0 V_{s+r}(X)$

2) $i : \pi_0 V_*(X) \to \pi_0 V_*(Y)$ is an $r$-epimorphism: given $[y] \in \pi_0 V_s(Y)$, $\sigma[y] = i[x]$ for some $[x] \in \pi_0 V_{s+r}(X)$.

3) All $i : \pi_n(V_*(X), x) \to \pi_n(V_*(Y), i(x))$ are $r$-isomorphisms.
A system of spaces is a functor $X : [0, \infty) \to s\text{Set}$, aka. a diagram of simplicial sets with index category $[0, \infty)$.

A map of systems $X \to Y$ is a natural transformation of functors defined on $[0, \infty)$.

**Examples**

1) The functors $V_\ast(X), BP_\ast(X), s \mapsto V_s(X), BP_s(X)$ are systems of spaces, for a data set $X \subset Z$.

2) If $X \subset Y \subset Z$ are data sets, the induced maps $P_s(X) \to P_s(Y), V_s(X) \to V_s(Y)$ define maps of systems $P_\ast(X) \to P_\ast(Y)$ and $V_\ast(X) \to V_\ast(Y)$. 

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Persistent homotopy theory
There are many ways to discuss homotopy types of systems. The oldest is the **projective structure** (Bousfield-Kan):

A map \( f : X \to Y \) is a **weak equivalence** (resp. **fibration**) if each map \( X_s \to Y_s \) is a weak equiv. (resp. fibration) of simplicial sets.

A map \( A \to B \) is a projective cofibration if it has the left lifting property with respect all maps which are trivial fibrations.

**Example:** \( L_s(A) \) is the system with \( L_s(A)_t = \emptyset \) for \( t < s \) and \( L_t(A) = A \) for \( t \geq s \). If \( A \subset B \) is an inclusion of simplicial sets, then \( L_s(A) \to L_s(B) \) is a projective cofibration.

**Lemma 5.**

*Suppose that \( X \subset Y \subset Z \) are data sets. Then \( V_*(X) \to V_*(Y) \) is a projective cofibration.*
Suppose that $f : X \to Y$ is a map of systems. Say that $f$ is an $r$-equivalence if

1) the map $f : \pi_0(X) \to \pi_0(Y)$ is an $r$-isomorphism of systems of sets
2) the maps $f : \pi_k(X_s, x) \to \pi_k(Y_s, f(x))$ are $r$-isomorphisms of systems of groups, for all $s \geq 0$, $x \in X_s$.

**Observation:** Suppose given a diagram of systems

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\sim\downarrow & & \downarrow\sim \\
X_2 & \xrightarrow{f_2} & Y_2 \\
\end{array}
\]

Then $f_1$ is an $r$-equivalence iff $f_2$ is an $r$-equivalence.

**Example (stability):** Suppose that $X \subset Y \subset Z$ are data sets, and that $d_H(X, Y) < r$. Then the maps $i : V_*(X) \to V_*(Y)$ and $i : \text{BP}_*(X) \to \text{BP}_*(Y)$ are $2r$-equivalences.
**Lemma 6.**

Suppose given a commutative triangle

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{g} \\
Z & & \\
\end{array}
\]

If one of the maps is an $r$-equivalence, a second is an $s$-equivalence, then the third map is a $(r + s)$-equivalence.

**Proof.**

Suppose $X, Y, Z$ are systems of sets, $h$ is an $r$-isomorphism and $g$ is an $s$-isomorphism. Given $z \in Y_{t}$, $g(z) = h(w)$ for some $w \in X_{t+s}$. Then $g(z) = g(f(w))$ in $Z_{t+s}$ so $z = f(w)$ in $Y_{t+s+r}$. 

Lemma 7.

Suppose that $p : X \to Y$ is a sectionwise fibration of systems of Kan complexes and that $p$ is an $r$-equivalence. Then each lifting problem

\[
\begin{array}{ccc}
\partial \Delta^n & \xrightarrow{\alpha} & X_s \\
\downarrow & \alpha & \downarrow \\
\Delta^n & \xrightarrow{\beta} & Y_s
\end{array}
\quad
\begin{array}{ccc}
& \xrightarrow{\sigma} & X_{s+2r} \\
\theta & \downarrow & \downarrow p \\
& \xrightarrow{\sigma} & Y_{s+2r}
\end{array}
\]

can be solved up to shift $2r$.  

Proof of Lemma 7

The original diagram can be replaced up to homotopy by a diagram

$$\partial \Delta^n(\alpha_0,*,\ldots,*) \to X_s \xrightarrow{\sigma} X_{s+r}$$

$$\Delta^n \xrightarrow{\beta} Y_s \xrightarrow{\sigma} Y_{s+r}$$

(1)

$p_*(\alpha_0) = 0$ in $\pi_{n-1}(Y_s,*)$, so $\sigma_*(\alpha_0) = 0$ in $\pi_{n-1}(X_{s+r},*)$.

The trivializing homotopy for $\sigma(\alpha_0)$ in $X_{s+r}$ defines a homotopy from (1) (outer) to the diagram

$$\partial \Delta^n \xrightarrow{\omega} X_{s+r}$$

$$\Delta^n \xrightarrow{\omega} Y_{s+r}$$

(2)

$\sigma_*(\omega) \in \pi_n(Y_{s+2r},*)$ lifts to an element of $\pi_n(X_{s+2r},*)$ up to homotopy, giving the desired lifting.
Lemma 8.

Suppose that \( p : X \to Y \) is a sectionwise fibration of systems of Kan complexes, and that all lifting problems have solutions up to shift \( r \), in the sense that the dotted arrow exists making the diagram commute. Then the map \( p : X \to Y \) is an \( r \)-equivalence.

Proof.

If \( p_*([\alpha]) = 0 \) for \( [\alpha] \in \pi_{n-1}(X_s,*), \) then there is a diagram on the left above. The existence of \( \theta \) gives \( \sigma_*([\alpha]) = 0 \) in \( \pi_{n-1}(X_{s+r},*) \).
Corollary 9.

Suppose given a pullback diagram

\[
\begin{array}{c}
X' \quad \rightarrow \quad X \\
p' \downarrow \quad \quad \quad \downarrow p \\
Y' \quad \rightarrow \quad Y
\end{array}
\]

where \( p \) is a sectionwise fibration and an \( r \)-equivalence.

Then the map \( p' \) is a sectionwise fibration and a \( 2r \)-equivalence.

**Question:** Is there a dual statement? Do maps which are cofibrations and \( r \) equivalences push out to \( 2r \)-equivalences?
A map $f : A \to B$ of systems of simplicial abelian groups (chain complexes) is an $r$-equivalence if the induced maps $H_k(A) \to H_k(B)$ are $r$-isomorphisms for $k \geq 0$.

**Example:** Suppose that $X \subset Y \subset Z$ are data sets and that $d_H(X, Y) < r$. Then $\mathbb{Z}(X) \to \mathbb{Z}(Y)$ is a $2r$-equivalence (by the interleaving), so that $H_k(X) \to H_k(Y)$ is a $2r$-isomorphism for $k \geq 0$ (all coefficients).

**Lemma 10.**

1) *Suppose that $f : A \to B$ is an $r$-equivalence with homotopy cofibre $p : B \to C$. Then the map $C \to 0$ is a $2r$-equivalence.*

2) *Suppose that $C \to 0$ is an $r$-equivalence. Then $f : A \to B$ is an $r$-equivalence.*

**Warning:** There is no Hurewicz theorem. We can’t say that if $X \to \ast$ is an $r$-equivalence then $H_\ast(X)$ is $r$-equivalent to $H_\ast(\ast)$. 

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**Persistent homotopy theory**
Question: What does it mean for \( X \to \ast \) to be an \( r \)-equivalence?

Facts: 1) If \( X \to \ast \) is an \( r \)-equivalence, then all Postnikov sections \( P_nX \) and \( n \)-connected covers \( X(n) \) are \( r \)-equivalent to a point.

2) If \( X \to \ast \) is an \( r \)-equivalence, then

\[
\sigma_* = 0 : \pi_k(X_s, \ast) \to \pi_k(X_{s+r}, \ast)
\]

for \( k \geq 1 \). All \([x] \in \pi_0X_s\) map to the same element of \( \pi_0X_{s+r} \).

Example: \( P_1X = B\pi(X) \), so fundamental groupoid \( \pi(X) \) is \( r \)-equivalent to a point. We can discuss systems of groupoids \( G \) such that \( G \to \ast \) are \( r \)-equivalences.

\( P_0G \) has same objects as \( G \), and exactly one morphism \( x \to y \) if \( \text{hom}_G(x, y) \neq \emptyset \). There is a natural functor \( \pi : G \to P_0G \).
Lemma 11.

Suppose that $G \to \ast$ is an $r$-equivalence. Then there is an interleaving

$$
\begin{array}{ccc}
G_s & \xrightarrow{\sigma} & G_{s+r} \\
\downarrow{\pi} & & \downarrow{\pi} \\
P_0G_s & \xrightarrow{\sigma} & P_0G_{s+r}
\end{array}
$$

and all elements of $\pi_0G_s$ map to the same element of $\pi_0G_{s+r}$.

Proof.

Any two morphisms $\alpha, \beta : x \to y$ of $G_s$ map to the same morphism of $G_{s+r}$, so $\theta$ exists.

In effect, $\beta^{-1} \cdot \alpha \in G_s(x, x) = \pi_1(BG_s, x)$. 

A 2-groupoid $H$ is a groupoid enriched in simplicial sets, such that each simplicial set $H(x, y)$ is the nerve of a groupoid.

Each $H$ has a bisimplicial nerve $BH$ which defines a homotopy type.

Every 2-groupoid $H$ has an associated groupoid $P_1H$ with a functorial map $\pi : H \to P_1H$, such that $P_1H(x, y) = P_0(H(x, y))$.

**Fact:** Every space $X$ has a fundamental 2-groupoid $\pi_2X$ such that $B\pi_2(X) \simeq P_2(X)$.

**Lemma 12 (slightly conjectural).**

Suppose that $H$ is a system of 2-groupoids such that $BH \to *$ is an $r$-equivalence. Then $P_1H \to *$ is an $r$-equivalence, and there is an interleaving

$$
\begin{array}{c}
H_s \xrightarrow{\sigma} H_{s+r} \\
\pi \downarrow \quad \theta \quad \downarrow \pi \\
P_1H_s \xrightarrow{\sigma} P_1H_{s+r}
\end{array}
$$
$H$ a system of 2-groupoids s.t. $BH \to *$ is an $r$-equivalence.

0) $P_0H \to *$ is an $r$-isomorphism. $P_0H$ is a system of disjoint unions of trivial groupoids (contractible spaces). $H_0(BP_0H) \to \mathbb{Z}$ is an $r$-isomorphism, and there are no non-trivial higher homology groups.

$H_0(BH) \cong H_0(BP_0H) \to \mathbb{Z}$ is an $r$-isomorphism.

1) $P_1H \to *$ is an $r$-equivalence. The interleaving

\[
P_1H_s \xrightarrow{\sigma} P_1H_{s+r} \\
\pi \downarrow \quad \theta \quad \downarrow \pi \\
P_0H_s \xrightarrow{\sigma} P_0H_{s+r}
\]

forces $H_k(BP_1H_s) \to 0$ to be an $r$-isomorphism for $k \geq 1$, because all higher homology groups of $BP_0H_s$ are trivial.

$H_1(BH) \cong H_1(BP_1H) \to 0$ is an $r$-isomorphism.
2) $P_2H \rightarrow *$ is an $r$-equivalence. The interleaving

$$
\begin{array}{ccc}
P_2H_s & \xrightarrow{\sigma} & P_2H_{s+r} \\
\pi \downarrow & & \downarrow \pi \\
P_1H_s & \xrightarrow{\sigma} & P_1H_{s+r}
\end{array}
$$

forces $H_k(BP_2H) \rightarrow 0$ to be a 2r-isomorphism for $k \geq 1$:

$\pi \cdot \sigma(\alpha) = \sigma \cdot \pi(\alpha) = 0$ for $\alpha \in H_k(BP_2H_s)$ since $H_k(BP_1H_s) \rightarrow 0$ is an $r$-isomorphism.

Then $\sigma \cdot \sigma(\alpha) = \theta \cdot \pi \cdot \sigma(\alpha) = 0$ in $H_k(BP_2H_{s+2r})$.

$H_2(BH) \cong H_2(BP_2H) \rightarrow 0$ is a 2r-isomorphism.
Spaces of data sets

We construct spaces from the poset of data sets $D(Z)$. There are two choices:

1) $D_s(Z) \subset BD(Z)$ consists of strings of simplices

$$\sigma : \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_n$$

such that $d_H(\sigma_0, \sigma_n) \leq s$.

2) $P_s(Z) \subset \mathcal{P}(D(Z))$ is poset consisting of finite subsets $\sigma$ such that $d_H(X, Y) \leq s$ for all $X, Y \in \sigma$.

**Theorem 13.**

*There are weak equivalences*

$$D_s(Z) \overset{\gamma}{\cong} BND_s(Z) \overset{\phi}{\to} BP_s(Z),$$

*where $\phi(\sigma) = \{\sigma_0, \ldots, \sigma_n\}$.*
Proof I

• There is a functor \( f : P_s(Z) \to D(Z) \) with 
  \[ \sigma = \{X_0, \ldots, X_k\} \mapsto X_0 \cup \cdots \cup X_k. \]

\( f : BP_s(Z) \to BD(Z) \) takes simplices of \( BP_s(Z) \) to simplices of \( D_s(Z) \) and induces \( f : BP_s(Z) \to D_s(Z) \).

The following diagram commutes:

\[
\begin{array}{ccc}
BND_s(Z) & \xrightarrow{\phi} & BP_s(Z) \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
D_s(Z) & \xleftarrow{f} & 
\end{array}
\]

• Show that \( f \) is a weak equivalence. Suppose that 
  \( \tau : Y_0 \subset \cdots \subset Y_k \) is a non-degenerate simplex of \( BD_s(Z) \). Show 
  that \( f : f^{-1}(\tau) \to \Delta^k \) is a weak equivalence.
• \( f^{-1}(\tau) \) is the nerve of a poset, with objects \( \{Z_0, \ldots, Z_m\} \) such that \( \cup_i Z_i \) is some \( Y_j \), with morphisms covering inclusions \( Y_j \subset Y_k \).

• Given \( \tau = \{Z_0, \ldots, Z_m\} \) with \( \cup_i Z_i = Y_j \), there are poset morphisms

\[
\{Z_0, \ldots, Z_m\} \to \{Z_0, \ldots, Z_m\} \cup \{Y_0, \ldots, Y_j\} \leftarrow \{Y_0, \ldots, Y_j\}.
\]

• There is a simplicial set map \( \sigma : \Delta^k \to f^{-1}(\tau) \) defined by the string of inclusions

\[
\{Y_0\} \subset \{Y_0, Y_1\} \subset \cdots \subset \{Y_0, \ldots, Y_k\}
\]

The map \( f : f^{-1}(\tau) \to \Delta^k \) is a homotopy equivalence. \( \square \)
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