Abstract

What is the most efficient search strategy for the random located target sites subject to the physical and biological constraints? Previous results suggested the Lévy flight is the best option to characterize this optimal problem, however, which ignores the understanding and learning abilities of the searcher agents. In this paper we propose the Continuous Time Random Walk (CTRW) optimal search framework and find the optimum for both of search length’s and waiting time’s distributions. Based on fractional calculus technique, we further derive its master equation to show the mechanism of such complex fractional dynamics. Numerous simulations are provided to illustrate the non-destructive and destructive cases.

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1. Introduction

Over the recent years the accumulating experimental evidences show that the moving organisms are ubiquitous. For instance, the foraging behavior of the wandering albatross (Diomedea exulans) on the ocean surface was found to obey a power-law distribution [19]; the foraging patterns of free-ranging spider monkeys (Ateles geoffroyi) in the forests of the Yucatan Peninsula was also found to be a power law tailed distribution of steps.
consistent with a Lévy walk [12, 2]. More experimental findings can be found in [20, Part II]. A number of foundational but important questions arise naturally: How to model these organisms’ movement trajectories? What factors determine the shape and the statistical properties of such trajectories? How to optimize the efficiency to search of randomly located targets? These questions have been studied from many different points of view. For example, Lévy flight search was claimed to be an optimal strategy in sparsely target site with an inverse square power-law distribution of flight lengths [13]. Then composite Brownian walk searches were found to be more efficient than any Lévy flight when searching is non-destructive and when the Lévy walks are not responsive to conditions found in the search [1]. In particular, the movement patterns have scale-free and super-diffusive characteristics. So the fractional Brownian motions and fractional Lévy motions are possible to account for the movement patterns [13]. However, the above strategies ignore one important factor, waiting time between the successive movement steps, since the search agents need some time to understand the visited target sites. The relation between waiting time and flight length for efficient search was discussed [5].

This fact inspires us to propose a potentially transformative framework for optimal random search based on continuous time random walk (CTRW). The CTRW strategy is composed of the flight lengths of a movement step with a random direction, as well as the waiting time elapsing between two successive movement steps, both of which are independent random variables, identically, distributed according to their probability densities. In addition, CTRW is also the stochastic solution of non-integer order diffusion equation based on the fractional calculus [21], which is a part of Calculus dealing with derivatives and integration of arbitrary order, e.g. [11]. Different from the analytical results on linear integer-order differential equations, which are represented by the combination of exponential functions, the analytical results on the linear fractional order differential equations are represented by the Mittag-Leffler function, which exhibits a power-law asymptotic behavior, [9]. Therefore, fractional calculus is being widely used to analyze the random signals with power-law distributions or power-law decay of correlations, [15]. By choosing the flight lengths subject to heavy-tailed distribution and finite characteristic waiting time, the CTRW encompasses the Lévy flight as a special case. Therefore, in this paper we propose the CTRW optimal search framework, which may provide new insights into the optimal random search in unpredictable environments.

The paper is organized of as follows: In Section 2 we review the Lévy flight optimal random search strategy. Then we formulate the CTRW optimal random search framework and find the optimum for both of search
length’s and waiting time’s distributions in Section 3. Based on fractional calculus technique, we derive the corresponding master equation to CTRW search strategy in Section 4. Finally, we give some concluding remarks and close this paper in Section 5.

2. Lévy flight optimal random search

In this section, we review the basic idea of Lévy flight optimal random search and reproduce the main results, which can help us better to understand the proposed CTRW strategy. In [18, 16, 17], the authors assumed the search length distribution

\[ p(l_j) \sim l_j^{-\mu} , \]  

with \( 1 < \mu < 3 \). Then they defined the search efficiency function \( \eta(\mu) \) to be the ratio of the number of target sites visited to the total distance traversed by the forager as following

\[ \eta(\mu) = \frac{1}{N\langle L \rangle} , \]  

in which \( N \) denotes the mean number of flights taken by a Lévy forager in order to travel between two successive target sites and \( \langle L \rangle \) denotes the mean flight distance. For the case of destructive foraging, they found that the mean efficiency \( \mu \) has no maximum, with lower values of \( \mu \) leading to more efficient foraging. For the case of nondestructive foraging, they found that an optimal strategy for a forager is to choose \( \mu^* = 2 \) when \( \lambda \) is large but not exactly known. The main results were reproduced in the cases of destructive foraging and nondestructive foraging, shown in Figure 2.1.

![Fig. 2.1: The product of the search efficiency \( \eta \) and the mean free path \( \lambda \) VS the parameter \( \mu \) for different \( \lambda \): (a) the destructive case; (b) the nondestructive case.](image)
3. CTRW optimal random search

Now we state the main idea of CTRW search strategy by using the probability distribution functions of search length and waiting time. Specifically, the former is given by

$$w(l_j) \sim l_j^{-(\alpha+1)},$$

(3.1)

where $0 < \alpha < 2$, and $l_j$ is the search length at the $j$th step. The latter is characterized by

$$\psi(t_j) \sim t_j^{-(\beta+1)},$$

(3.2)

where $0 < \beta < 1$, and $t_j$ is the waiting time length before starting the $j$th step. Then the CTRW search strategy is described by the following two simple rules: 1) If a target site is located within a visible and finite distance $r_v$, then the search agent moves on a straight line to it without learning; (2) if there is no target site within a finite distance $r_v$, then the agent spends some waiting time $t_j$, which is also can be characterized by a power-law function (3.2), to understand what is detected, and chooses a random direction and a random distance $l_j$ following another power-law distribution function (3.1); otherwise, it proceeds to the target as in first rule.

The schematic idea of CTRW search strategy is drawn in Figure 3.1.

![Fig. 3.1: The CTRW search strategy](image)

Fig. 3.1: The CTRW search strategy: (a) if the target site is located within a visible range $r_v$, then the search agent moves on a straight line to it; (b) if there is no target site within a distance $r_v$, the search agent stop and wait for $t_j$ (the height of the cylinder) and then chooses a random direction and a random distance $l_j$ until it finds the target site.
Following the similar idea of Lévy flight [18, 16, 17], we can define the search efficiency function \( \eta(\alpha, \beta) \) to be the ratio of the number of target sites visited to the total distance traversed by the search agent as following

\[
\eta(\alpha, \beta) = \frac{1}{N\langle LT \rangle},
\]

(3.3)

where \( \langle LT \rangle \) is given by

\[
\langle LT \rangle \sim \frac{\int_0^\lambda l^{-\alpha} dl \int_0^T t^{-\beta} dt + \lambda \int_0^\infty l^{-\alpha-1} dl \int_0^\infty t^{-\beta-1} dt}{\int_0^\infty l^{-\alpha-1} dl \int_0^\infty t^{-\beta-1} dt} = \frac{\alpha\beta T}{(1-\alpha)(1-\beta)} \left( \lambda^{1-\alpha} r_\nu^\alpha - r_\nu \right) + T \lambda^{1-\alpha} r_\nu^\alpha,
\]

(3.4)

\( \lambda \) is the mean free path and \( T \) is the mean wait time. Moreover, the mean number of search \( N \) between two successive target sites for the destructive foraging case

\[
N_d = \left( \frac{\lambda}{r_\nu} \right)^\alpha,
\]

(3.5)

and for the nondestructive foraging case

\[
N_n = \left( \frac{\lambda}{r_\nu} \right)^{\alpha/2},
\]

(3.6)

respectively. Substituting equations (3.4) and (3.5) into (3.3), we find the mean efficiency \( \eta(\alpha, \beta) \) has no maximum, with lower values of \( \alpha \) and \( \beta \) leading to more efficient search. For the nondestructive case, substituting (3.4) and (3.6) into (3.3), we find that the efficiency \( \eta(\alpha, \beta) \) is optimum at

\[
\alpha = 1 - \delta_1, \quad \beta = 0.5 - \delta_2,
\]

(3.7)

where \( \delta = \max(\delta_1, \delta_2) \ll 1 \).

Next we test the above theoretical results with numerical simulations. Let \( r_\nu = 1, \ T = 5, \ \lambda = 10, \ 10^3, \ 10^5, \ 10^7 \). For the destructive case, the relation of the product of mean free path and search efficiency and the parameters \( \alpha \) and \( \beta \) is performed in three dimensional space, shown in Figure 3.2. To better verify the result, we just choose the 2D projection relationship of parameters \( \alpha \) and \( \beta \), shown in Figure 3.3, implying that lower values of \( \alpha \) and \( \beta \) leading to more efficient search, which agrees with the analytical results.
Fig. 3.2: The product of the search efficiency $\eta$ and the mean free path $\lambda$ VS the parameter $\alpha$ and $\beta$ for different $\lambda$: the destructive case.

Fig. 3.3: 2D projection of Fig. 3.2 with parameter $\alpha$ and $\beta$ for different $\lambda$. 
For the nondestructive case, we perform the results by using the same parameters, shown in Figure 3.4 and Figure 3.5, from which we find that the optimal values $\alpha = 1$ and $\beta = 0.5$ when $\lambda \to \infty$. This fact shows that the analytical results are reliable.

Fig. 3.4: The product of the search efficiency $\eta$ and the mean free path $\lambda$ VS the parameter $\alpha$ and $\beta$ for different $\lambda$: the nondestructive case.

Fig. 3.5: 2D projection of Fig. 3.2 with parameter $\alpha$ and $\beta$ for different $\lambda$. 

Two-dimensional path of CTRW search for $\alpha = 1$ and $\beta = 0.5$ is shown in Figure 3.6.

Noting that when $\beta = 0$, the search agent do not need to learn and wait for the next step. In this case, we can ignore the waiting time part and set $\mu = \alpha + 1$, then we can cover the previous results [18, 16, 17]. This also implies that our results include the previous results [18, 16, 17] as the special case.

4. Master equation

In this section, we discuss the master equation of the CTRW search strategy in previous section based on the fractional calculus. Let us introduce the Fourier transform for the coordinate variable $l$ and the Laplace transform for the time variable $t$:

$$W(k) = \int_{-\infty}^{\infty} e^{ikl}w(l)dl, \quad (4.1)$$

and

$$\Psi(s) = \int_{0}^{\infty} e^{-st}\psi(t)dt, \quad (4.2)$$

respectively. Then we can get the Montroll-Weiss equation [10, 8] in Fourier-Laplace space:
\[
P(k, s) = \frac{1 - \Psi(s)}{s} \frac{1}{1 - W(k) \Psi(s)}. \quad (4.3)
\]

Since we assume that \(w(l)\) and \(\psi(t)\) are characterized by equations (3.1) and (3.2), we have (14)
\[
1 - W(k) \sim |k|^\alpha \quad (4.4)
\]
and
\[
\frac{1}{\Psi(s)} \sim 1 + s^\beta. \quad (4.5)
\]
After substituting (4.4) and (4.5) into (4.3), we get
\[
P(k, s) = \frac{s^{\beta - 1}}{s^\beta + |k|^\alpha}. \quad (4.6)
\]

On the other hand, the so-called time-space fractional order diffusion equation is given by
\[
\frac{\partial \Psi(l,t)}{\partial t^\beta} p(l,t) = \frac{\partial \Psi(l,t)}{\partial \zeta^{\alpha}} p(l,t), \quad (4.7)
\]
where \(\frac{\partial \Psi(l,t)}{\partial t^\beta}\) is the time-fractional Caputo derivative of order \(\beta\), and \(\frac{\partial \Psi(l,t)}{\partial \zeta^{\alpha}}\) is the space-fractional Riesz-Feller derivative of order \(\alpha\). In fact, the Caputo fractional derivative of \(f(t)\) is defined as [3, 11]
\[
\frac{\partial \Psi(l,t)}{\partial t^\beta} f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{f(t - \tau)}{(t - \tau)^\alpha} d\tau, \quad (4.8)
\]
with its Laplace transform
\[
\mathcal{L} \left\{ \frac{\partial \Psi(l,t)}{\partial t^\beta} f(t); s \right\} = s^\beta \hat{F}(s) - s^{\beta - 1} f(0^+). \quad (4.9)
\]
The space-fractional Riesz-Feller derivative of \(g(l)\) is defined as [7, 6]
\[
\frac{\partial \Psi(l,t)}{\partial \zeta^{\alpha}} g(l) = \frac{\Gamma(1 + \alpha)}{\pi} \sin \left(\frac{\alpha \pi}{2}\right) \int_0^\infty \frac{g(l + \xi) - 2g(l) + g(l - \xi)}{\xi^{1+\alpha}} d\xi, \quad (4.10)
\]
with its Fourier transform
\[
\mathcal{F} \left\{ \frac{\partial \Psi(l,t)}{\partial t^\beta} g(l); k \right\} = -|k|^\alpha \hat{G}(k). \quad (4.11)
\]
From [3, equation (2.17)], the Laplace-Fourier transform of fractional order equation (4.7) is
\[
P(k, s) = \frac{s^{\beta - 1}}{s^\beta + |k|^\alpha}. \quad (4.12)
\]
which is the same as (4.6). This means the proposed CTRW search strategy obeys a time-space fractional diffusion equation. Therefore, many interesting mathematical and physical contributions (as [23, 22, 15]) on fractional dynamics can guide us to understand the mechanism of the CTRW search strategy.
5. Concluding remarks

In this paper we have proposed the Continuous Time Random Walk (CTRW) optimal search framework by assuming that both of search length’s and waiting time’s distribution satisfy a power-law function. By introducing the efficiency function, we found that the optimum for parameters $\alpha = 1$ and $\beta = 0.5$ when mean free path $\lambda$ tends to infinity. Based on fractional calculus technique, we further derive its master equation as a time-space fractional order diffusion equation. Thus many interesting contributions related to fractional dynamics can guide us to understand the mechanism of CTRW search strategy. Numerous simulations are provided to illustrate the non-destructive and destructive cases, which verify our analytical results are reliable.

The optimal random search is a relatively new field, and considerable efforts are still being made and many challenges are still to be overcome. What about the optimal search for the big data in small-world network? How to specify the probability distributions of search’s length and waiting time in the practical applications? What is the globally optimum search strategy for the moving target sites? We will focus on this topic and hope this paper can stimulate wide discussion and lead to new investigations of these challenging problems.

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