ON THE SEMISIMPLICITY OF THE CYCLOTOMIC BRAUER ALGEBRAS, II

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Abstract. In this paper, we give a necessary and sufficient condition for the semisimplicity of cyclotomic Brauer algebras \( B_{m,n}(\delta) \) of types \( G(m,1,n) \) with \( m \geq 2 \). This generalizes [11, 1.2–1.3] and [12, 2.5] on Brauer algebras.

Dedicated to Professor Gordon James on the occasion of his 60th birthday

1. Introduction

The cyclotomic Brauer algebras \( B_{m,n}(\delta) \) have been introduced by Hæring-Oldenburg in [10] as classical limits of cyclotomic Birman-Murakami-Wenzl algebras. When \( m = 1 \), they are Brauer algebras \( B_n(\delta) \) [2].

The main purpose of this paper is to give a necessary and sufficient condition for the semisimplicity of \( B_{m,n}(\delta) \) under the assumption \( m \geq 2 \). For \( m = 1 \), such a criterion has been given in [11, 1.2-1.3] and [12, 2.5].

Unless otherwise stated, we assume that \( F \) is a splitting field of \( x^m - 1 \), which contains \( \delta_i, 1 \leq i \leq m \). By assumption, there are \( u_i \in F \) such that \( x^m - 1 = \prod_{i=1}^m (x - u_i) \). Define

\[
e = \begin{cases} +\infty, & \text{if } \text{char } F = 0, \\ \text{char } F, & \text{if } \text{char } F > 0. \end{cases}
\]

(1.1)

Following [12], we define \( \tilde{Z}_{m,n} = \{ ma \mid a \in \tilde{Z}_{m,n} \} \), where \( \tilde{Z}_{m,n} \) is given as follows:

1. \( \tilde{Z}_{2,n} = \tilde{Z}_{1,n} = \{ k \in \mathbb{Z} \mid 3 - n \leq k \leq n - 3 \} \cup \{ 2k - 3 \mid 3 \leq k \leq n, k \in \mathbb{Z} \} \).
2. \( \tilde{Z}_{m,n} = \tilde{Z}_{1,n} \cup \{ 2 - n, n - 2 \} \) if \( m \geq 3 \) and \( n \geq 2 \).

Suppose that \( x_1, x_2, \ldots, x_m \) are indeterminates over \( F \). If \( F \) contains \( \xi \), a primitive \( m \)-th root of unity, then we define

\[
\tau_i = \sum_{j=1}^m x_j \xi^{ji}, \quad 0 \leq i \leq m - 1.
\]

(1.2)

Note that \( F \) contains \( \xi \) if \( e \mid m \) [8, 8.2]. The following is the main result of this paper.

Theorem A. Fix two positive integers \( m, n \) with \( m > 1 \). Let \( \mathcal{B}_{m,n}(\delta) \) be a cyclotomic Brauer algebra over \( F \).

(a) Suppose \( n \geq 2 \). If \( \delta_i \neq 0 \) for some \( i \), \( 0 \leq i \leq m - 1 \), then \( \mathcal{B}_{m,n}(\delta) \) is (split) semisimple if and only if

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(1) \( e \nmid m \cdot n! \),

(2) \( \varepsilon_{i,0} m - \delta_i \not\in \mathbb{Z}_{m,n}, \ 0 \leq i \leq m - 1 \), where \( \varepsilon_{i,0} \) is the Kronecker function.

(b) Suppose \( n \geq 2 \). If \( \delta_i = 0, \ 0 \leq i \leq m - 1 \), then \( \mathcal{B}_{m,n}(0) \) is not (split) semisimple.

(c) \( \mathcal{B}_{m,1}(\delta) \) is (split) semisimple if and only if \( e \nmid m \).

In what follows, we write \( \delta_j = \delta_i \) if \( i, j \in \mathbb{Z} \) and \( i \equiv j \mod m \).

Let \( \mathcal{H}_{i,k} \) be the hyperplane in \( F^m \), which is determined by the linear function \( \varepsilon_{i,0} m - \xi_i = k, \ 0 \leq i \leq m - 1 \) and \( k \in \mathbb{Z}_{m,n} \). Condition (2) in Theorem A(a) is equivalent to the fact that \( (\delta_0, \delta_1, \ldots, \delta_{m-1}) \not\in \bigcup_{0 \leq i \leq m-1, k \in \mathbb{Z}_{m,n}} \mathcal{H}_{i,k} \). When \( m = 1 \), \( \mathcal{H}_{i,k} \) collapses to a point in 1-dimensional \( F \)-space. This result has been proved in [11, 1.2-1.3] and [12, 2.5]. We remark that certain sufficient conditions for semisimplicity of complex Brauer algebras have been given in [3, 4, 14].

Our proof depends on Graham-Lehrer’s theory on cellular algebras [6] and Doran-Wales-Hanlon’s work [4, 3.3-3.4] on Brauer algebras. Let’s explain the idea as follows.

In [6], Graham and Lehrer have introduced the notion of cellular algebra which is defined over a poset \( \Lambda \). Such an algebra has a nice basis, called a cellular basis. For each \( \lambda \in \Lambda \), one can define \( \Delta(\lambda) \), called a cell module. Graham and Lehrer have shown that there is a symmetric, associative bilinear form \( \phi_\lambda \) defined on \( \Delta(\lambda) \). It has been proved in [6, 3.8] that a cellular algebra is (split) semisimple if and only if \( \phi_\lambda \) is non-degenerate for any \( \lambda \in \Lambda \). It is well known that a cellular algebra is split semisimple if and only if it is semisimple. Therefore, one can determine whether a cellular algebra is semisimple by deciding whether all \( \phi_\lambda \) are non-degenerate.

In [6], Graham-Lehrer have proved that a Brauer algebra \( \mathcal{B}_n(\delta) \) over a commutative ring is a cellular algebra over the poset \( \Lambda \) which consists of all pairs \((f, \lambda)\), with \( 0 \leq f \leq \lfloor n/2 \rfloor \) and \( \lambda \) being a partition of \( n-2f \). Here \( \lfloor n/2 \rfloor \) is the maximal integer which is less than \( n/2 \). Therefore, one can study the semisimplicity of \( \mathcal{B}_n(\delta) \) by deciding whether \( \phi_{f,\lambda} \) is non-degenerate or not for any \((f, \lambda) \in \Lambda \). Unfortunately, it is difficult to determine whether \( \phi_{f,\lambda} \) is degenerate or not for a fixed \((f, \lambda) \).

In [11], the first author has proved that the semisimplicity of \( \mathcal{B}_n(\delta) \) is completely determined by \( \phi_{f,\lambda} \) for all partitions \( \lambda \) of \( n-2f \) with \( f = 0, 1 \). Using [4, 3.3-3.4], he has decided whether such \( \phi_{f,\lambda} \)'s are degenerate or not in [11]. This gives a complete solution of the problem of semisimplicity of \( \mathcal{B}_n(\delta) \) over an arbitrary field. This method will be used to study the semisimplicity of \( \mathcal{B}_{m,n}(\delta) \) in the current paper.

The contents of this paper are organized as follows. In section 2, we state some results on cyclotomic Brauer algebras, and complex reflection group \( W_{m,n} \). In section 3, we describe explicitly the zero divisors of the discriminants for certain cell modules. Theorem A will be proved in section 4.

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2. Cyclotomic Brauer algebras

Let $R$ be a commutative ring which contains the identity $1_R$ and $\delta_i, 1 \leq i \leq m$. The cyclotomic Brauer algebra $\mathcal{B}_{m,n}(\delta)$ with parameters $\delta_i, 1 \leq i \leq m$, is the associative $R$-algebra which is free as $R$-module with basis which consists of all labelled Brauer diagrams [10]. $\mathcal{B}_{m,n}(\delta)$ can also be defined as the $R$-algebra generated by $\{s_i, e_i, t_j \mid 1 \leq i < n \text{ and } 1 \leq j \leq n\}$ subject to the relations

- $s_i^2 = 1$, for $1 \leq i < n$.
- $s_is_j = s_js_i$ if $|i-j| > 1$.
- $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $1 \leq i < n - 1$.
- $s_i t_j = t_j s_i$, if $j \neq i, i + 1$.
- $e_i^2 = \delta_i e_i$, for $1 \leq i < n$.
- $s_i e_j = e_j s_i$ if $|i-j| > 1$.
- $e_i e_j = e_j e_i$, if $|i-j| > 1$.
- $e_i t_j = t_j e_i$, if $j \neq i, i + 1$.
- $t_i t_j = t_j t_i$, for $1 \leq i, j \leq n$.
- $s_i t_i = t_i s_i$, for $1 \leq i < n$.
- $t_i^m = 1$, for $1 \leq i \leq n$.

One can prove that the two definitions of $\mathcal{B}_{m,n}(\delta)$ are equivalent by the arguments similar to those for Brauer algebras in [9].

The following result can be proved easily by checking the defining relations of $\mathcal{B}_{m,n}(\delta)$.

**Lemma 2.1.** Let $\mathcal{B}_{m,n}(\delta)$ be a cyclotomic Brauer algebra over $R$. There is an $R$-linear anti-involution $*: \mathcal{B}_{m,n}(\delta) \rightarrow \mathcal{B}_{m,n}(\delta)$ such that $h^* = h$ for all $h \in \{e_i, s_i, t_j \mid 1 \leq i < n, 1 \leq j \leq n\}$.

Recall that $F$ is a splitting field of $x^m - 1$. In the remaining part of this section, we assume $e \not| m \cdot n!$. By [8, 8.2], $F$ contains $\xi$, a primitive $m$-th root of unity.

We will decompose an $FW_{m,n}$-module in Proposition 2.5 where $W_{m,n}$ is the complex reflection group of type $G(m,1,n)$. Note that $W_{m,n}$ is generated by $s_i, t_i$ satisfying the relations

- $s_i^2 = t_i^m = 1$ for $1 \leq i \leq n - 1$.
- $s_is_js_i = s_js_is_j$, if $|i-j| = 1$.
- $s_is_j = s_js_i$, if $|i-j| > 1$.
- $s_it_i = t_is_i$ if $1 < i \leq n - 1$.
- $s_is_it_t = t_is_it_is_i$.

The order of $W_{m,n}$ is $m^n \cdot n!$. By Maschke’s theorem, the group algebra $FW_{m,n}$ is (split) semisimple.

Let $\Lambda_m^+(n)$ be the set of $m$-partitions of $n$. When $m = 1$, we use $\Lambda^+(n)$ instead of $\Lambda_1^+(n)$. For any $\lambda \in \Lambda_m^+(n)$, let $S^\lambda$ be the classical Specht module with respect to $\lambda$ (see [5, 2.1]).
For any \( \lambda \in \Lambda^+(n) \), let \( \mu = (\mu_1, \mu_2, \cdots) \) with \( \mu_i = \# \{ j \mid \lambda_j \geq i \} \). Then \( \mu \), which will be denoted by \( \lambda' \), is called the dual partition of \( \lambda \). If \( \lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(m)}) \in \Lambda^+_m(n) \), we write \( \lambda' = (\lambda^{(m)}, \lambda^{(m-1)}, \cdots, \lambda^{(1)}) \) and call \( \lambda' \) the dual partition of \( \lambda \).

**Remark 2.2.** All modules considered in this paper are left modules. I.e. \( S^\lambda = FW_{m,n} y_{\lambda'} x_{\lambda} \) if we keep the notation in [5]. In [13], we have assumed \( u_i = \xi^i, 1 \leq i \leq m \). In this paper, we keep this assumption in order to use results in [13] directly.

Since \( FW_{m,n} \) is the Ariki-Koike algebra with \( q = 1 \) and \( x_1^m - 1 = \prod_{i=1}^m (x_1 - u_i) \), the following result is a special case of the result in [5].

**Lemma 2.3.** The set \( \{ S^\lambda \mid \lambda \in \Lambda^+_m(n) \} \) is a complete set of pairwise non-isomorphic irreducible \( FW_{m,n} \)-modules.

**Definition 2.4.** Let \( m \) be a positive integer. If \( m \) is even, we define \( \varphi_m(2) = \{ \eta \mid \frac{m+1}{2} \leq i \leq m \} \), where \[
\eta_i = \begin{cases} 
(0, \cdots, 0, 2), & \text{if } i = m, \\
(0, \cdots, 0, \frac{1}{\varpi_{m-i}}, 0, \cdots, 0), & \text{if } i = \frac{m}{2}, \\
(0, \cdots, 0, \frac{1}{(m-1)-th}, 0, \cdots, 0, 1-th, 0, \cdots, 0), & \text{if } \frac{m}{2} < i \leq m - 1.
\end{cases}
\]

If \( m \) is odd, we define \( \varphi_m(2) = \{ \eta \mid \frac{m+1}{2} \leq i \leq m \} \), where \[
\eta_i = \begin{cases} 
(0, \cdots, 0, 2), & \text{if } i = m, \\
(0, \cdots, 0, \frac{1}{(m-1)-th}, 0, \cdots, 0, 1-th, 0, \cdots, 0), & \text{if } \frac{m+1}{2} \leq i \leq m - 1.
\end{cases}
\]

**Proposition 2.5.** Let \( \mathbb{Z}_m \wr B_1 \) be the subgroup of \( W_{m,2} \) generated by \( s_1, t_1 t_2 \). As \( FW_{m,2} \)-modules, \( \text{Ind}^{W_{m,2}}_{\mathbb{Z}_m \wr B_1} 1 \cong \bigoplus_{\eta \in \varphi_m(2)} S^n_\eta \).

**Proof.** Since \( \{ 1, t_1, \cdots, t_1^{m-1} \} \) is a complete set of left coset representatives of \( \mathbb{Z}_m \wr B_1 \) in \( W_{m,2} \), \( \{ t_1 \sum_{i=0}^{m-1} (t_1 t_2)^i (1 + s_1) \mid 0 \leq k \leq m - 1 \} \) is an \( F \)-basis of \( \text{Ind}^{W_{m,2}}_{\mathbb{Z}_m \wr B_1} 1 \). By assumption, \( F \) contains a primitive \( m \)-th root of unity, say \( \xi \). Since we are assuming that \( u_i = \xi^i, 1 \leq i \leq m \), \( \text{Ind}^{W_{m,2}}_{\mathbb{Z}_m \wr B_1} 1 \) has a basis \( \{ w_i \mid 1 \leq i \leq m \} \), where \[
w_i = \prod_{j \neq i, 1 \leq j \leq m} (t_1 - u_j) \sum_{l=0}^{m-1} (t_1 t_2)^l (1 + s_1).
\]

Since \( \prod_{j=1}^m (t_1 - \xi^i) = 0 \), \[
w_i = \prod_{j \neq i} (t_1 - u_j) \prod_{1 \leq j \leq m-1} (u_i t_2 - u_j) (1 + s_1).
\]

By rescaling the above elements, \( \{ v_i \mid 1 \leq i \leq m \} \) is a basis of \( \text{Ind}^{W_{m,2}}_{\mathbb{Z}_m \wr B_1} 1 \), where \[
v_i = \prod_{j \neq i} (t_1 - u_j) \prod_{j \neq m-i} (t_2 - u_j) (1 + s_1).
\]

We have:

- \( F v_m \) is an \( FW_{m,2} \)-module with \( s_1 v_m = t_1 v_m = v_m \). By [5, 2.1], \( F v_m \cong S^n_\eta \).
• Suppose 2 \nmid m. If \( \frac{m+1}{2} \leq i \leq m-1 \), then \( \xi^i \neq \xi^{m-i} \). The subspace 
\( Fv_i \oplus Fv_{m-i} \) is an \( FW_{m,2} \)-module such that 
\( t_1 v_j = u_j v_j \) for \( j = i, m-i \), and 
\( s_1 v_i = v_{m-i} \). Therefore, 
\( Fv_i \oplus Fv_{m-i} \cong S^0 \), \( \frac{m+1}{2} \leq i \leq m-1 \).

• Suppose \( 2 \mid m \). If \( \frac{m}{2} < i \leq m-1 \), then 
\( Fv_i \oplus Fv_{m-i} \) is an \( FW_{m,2} \)-module such that 
\( t_1 v_j = u_j v_j \) for \( j = i, m-i \), and 
\( s_1 v_i = v_{m-i} \). Therefore, 
\( Fv_i \oplus Fv_{m-i} \cong S^0 \), \( \frac{m}{2} < i \leq m-1 \).

• Suppose \( i = \frac{m}{2} \). Then \( Fv_i \) is an \( FW_{m,2} \)-module such that 
\( s_1 v_i = v_i \) and 
\( t_1 v_i = u_i v_i \). Therefore, \( Fv_i \cong S^0 \).

Consequently, \( \text{Ind}_{\mathbb{Z}}^{Z_m(\mathbb{Q})} 1 \cong \bigoplus_{\gamma \in \mathbb{Z}_m(\mathbb{Q})} S^0 \) no matter whether \( m \) is even or odd. \( \square \)

**Remark 2.6.** Proposition 2.5 is a special case of [13, (4.4)]. The decomposition given there involves certain \( m \)-partitions \( \eta \). In fact, we have to put more restrictions on \( \eta \). The reason is that \( \sum_{i=0}^{m-1} t_i w_{i,\alpha} \) may be equal to zero for general \( \alpha \) (Here, we keep the notation in [13]). Therefore, the first equality in [13, (4.3)] is not true in general. If we denote by \( c_\eta \) the multiplicity of \( S^0 \) in \( \text{Ind}_{\mathbb{Z}}^{Z_m(\mathbb{Q})} 1 \), [13, (4.1), 6.2] are still true although we do not know the explicit description of \( c_\eta \). Proposition 2.5 gives us the explicit information for \( \eta \) and \( c_\eta \) when \( k = 1 \).

In the remaining part of this section, we recall the result in [13], which says that 
\( \mathcal{B}_{m,n}(\delta) \) is a cellular algebra in the sense of [6]. We also prove Theorem 2.9 which will play the key role in the proof of Theorem 3.

Recall that a dotted Brauer diagram \( D \) with \( k \) horizontal arcs is determined by a pair of labelled \( (n,k) \)-parenthesis diagrams \( \alpha, \beta \) and \( w \in W_{m,n-2k} \), and vice versa [13]. In this situation, we write \( D = \alpha \otimes w \otimes \beta \) if

- \( \alpha \) (resp. \( \beta \)) is the top (resp. bottom) row of \( D \).
- \( w \) corresponds to the dotted Brauer diagram (or braid diagram) which is obtained from \( D \) by removing the horizontal arcs at top and bottom rows of \( D \).

We denote by \( P(n,k) \) the set of all labelled \( (n,k) \)-parenthesis diagrams.

Suppose \( \lambda \in \Lambda_m^+(n) \). A \( \lambda \)-tableau is a bijection \( t = (t_1, \ldots, t_{m-1}, t_m) : (Y(\lambda^{(1)}), \ldots, Y(\lambda^{(m-1)}), Y(\lambda^{(m)})) \to \{1, 2, \ldots, n\} \). If the entries in each \( t_i \), \( 1 \leq i \leq m \) increase from left to right in each row and from top to bottom in each column, then \( t \) is called a standard \( \lambda \)-tableau. Let \( T^s(\lambda) \) be the set of all standard \( \lambda \)-tableaux. Let \( \{y_{st}^\lambda \mid \lambda \in \Lambda_m^+(n), s, t \in T^s(\lambda)\} \) be the Murphy basis for \( FW_{m,n} \) [5, 2.8]. Define

\[
C^{(k,\lambda)}_{(\alpha, \beta), (\gamma, \delta)} = \alpha \otimes y_{st}^\lambda \otimes \beta, \quad \alpha, \beta \in P(n,k), s, t \in T^s(\lambda)
\]

Recall that \( R \) is a commutative ring containing the identity \( 1 \) and \( \delta_1, \ldots, \delta_m \).

**Theorem 2.8.** [13, 5.11] Suppose \( R \) contains \( u_1, \ldots, u_m \) such that \( x^m - 1 = (x - u_1)(x - u_2) \cdots (x - u_m) \). Let \( \Lambda = \{(f, \lambda) \mid 0 \leq f \leq \lfloor n/2 \rfloor, \lambda \in \Lambda_m^+(m - 2f)\} \). Then

\[
\{C^{(k,\lambda)}_{(\alpha, \beta), (\gamma, \delta)} \mid \alpha, \beta \in P(n,k), s, t \in T^s(\lambda), (k, \lambda) \in \Lambda\}
\]
is a cellular basis of $R_{m,n}(\delta)$. The R-linear anti-involution defined on $R_{m,n}(\delta)$ is that defined in Lemma 2.7.

Following [6, 2.1], we have the cell modules for $R_{m,n}(\delta)$ with respect to the cellular basis provided in Theorem 2.8. Let $\Delta(k, \lambda)$ be the cell module for $R_{m,n}(\delta)$ with respect to $(k, \lambda) \in \Lambda$. Let $\Delta(\lambda)$ be the cell module for $FW_{m,n}$ with respect to the cellular basis \{\eta^\lambda_m | \lambda \in \Lambda^+_{m,n} \}.

It has been proved in [5, 2.7] that $\Delta(\lambda) \cong S^{\lambda'}$, where $\lambda'$ is the dual partition of $\lambda$. By [6, 2.1], $\Delta(k, \lambda)$ is spanned by $\alpha \otimes v_j \otimes a_0 \mod R_{m,n}(\delta)^{> (k, \lambda)}$, where $v_j$ ranges over the basis elements of $S^{\lambda'}$.

Suppose $\lambda \in \Lambda^+_{m,n}(n)$ and $\mu \in \Lambda^+_{m,n}(n-1)$. If there is a pair $(i,j)$ such that $\lambda_i^{(j)} = \mu_i^{(j)} + 1$ and $\lambda_{i+1}^{(k)} = \mu_{i+1}^{(k)}$ for any $(k,l) \neq (i,j)$, then we write $\mu \rightarrow \lambda$ and say that $\mu$ is obtained from $\lambda$ by removing a box. In this situation, we also say that $\lambda$ can be obtained from $\mu$ by adding a box.

**Theorem 2.9.** Let $R_{m,n}(\delta)$ be a cyclotomic Brauer algebra over $F$. If $\mu \in \Lambda^+_{m,n}(n-2)$ and $\lambda \in \Lambda^+_{m,n}(n)$, then either $[\Delta(1,\mu') : \Delta(\lambda')] = 0$ or $[\Delta(1,\mu') : \Delta(\lambda')] = 1$. Furthermore, $[\Delta(1,\mu') : \Delta(\lambda')] = 1$ if one of the following conditions holds true.

1. $\lambda^{(j)} = \mu^{(j)}$, $j \neq m$ and two boxes in the skew Young diagram $Y(\lambda^{(m)}/\mu^{(m)})$ are not in the same column.
2. Suppose that $m$ is odd. There is an $i$ with $\frac{m+1}{2} \leq i < m$ such that $\mu^{(i)} \rightarrow \lambda^{(i)}$ and $\mu^{(m-i)} \rightarrow \lambda^{(m-i)}$, and $\lambda^{(j)} = \mu^{(j)}$ for $j \neq i, m-i$.
3. Suppose that $m$ is even. There is an $i$ with $\frac{m}{2} < i < m$ such that $\mu^{(i)} \rightarrow \lambda^{(i)}$ and $\mu^{(m-i)} \rightarrow \lambda^{(m-i)}$, and $\lambda^{(j)} = \mu^{(j)}$ for $j \neq i, m-i$.
4. Suppose that $m$ is even. $\lambda^{(j)} = \mu^{(j)}$, $j \neq m/2$ and two boxes in the skew Young diagram $Y(\lambda^{(m)/\mu^{(m)})}$ are not in the same column.

**Proof.** Since we are assuming that $F$ is a splitting field of $x^{m-1}$ such that $e \upharpoonright m \cdot n$, both $FW_{m,n}$ and $FS_k$ are (split) semisimple for $k \leq n$.

For $\lambda \in \Lambda^+(n_1 + n_2), \mu \in \Lambda^+(n_1), \eta \in \Lambda^+(n_2)$, let $L_{\eta,\mu}^{\lambda}$ be the corresponding Littlewood-Richardson coefficient for symmetric groups. If $\lambda \in \Lambda^+_{m,n}(n), \mu \in \Lambda^+_{m,n}(n-2)$ and $\eta \in \Lambda^+_{m,n}(2)$, the Littlewood-Richardson coefficient $L_{\eta,\mu}^{\lambda}$ for complex reflection groups is $\prod_{i=1}^m L_{\eta^{(i)},\mu^{(i)}}^{\lambda^{(i)}}$ [3, §4]. Let $c_{\eta}$ be the multiplicity of $S^\eta$ in $\text{Ind}_{BF_{m+2}}^{FW_{m+2}} W_{m+2}$. By [13, 6.2], $[\Delta(1,\mu') : \Delta(\lambda')] = m_{\mu,\lambda}$, where $m_{\mu,\lambda} = \sum_{\eta \in \varphi_{m}(2)} c_{\eta} L_{\eta,\mu}^{\lambda}$. Note that $L_{\eta,\mu}^{\lambda} \neq 0$ if and only if $L_{\eta^{(i)},\mu^{(i)}}^{\lambda^{(i)}} \neq 0$ for $1 \leq i \leq m$. Consequently, if $L_{\eta,\mu}^{\lambda} \neq 0$ for $\eta \in \varphi_{m}(2)$, then there is a unique $\eta \in \varphi_{m}(2)$. Suppose $\lambda$ and $\mu$ are partitions. It is known that two boxes in the skew Young diagram $Y(\lambda/\mu)$ are not in the same column if $L_{\eta,\mu}^{\lambda} \neq 0$ (see, e.g. [4, §3]). In this situation, $L_{(2),\mu}^{\lambda} = 1$, and $\lambda \supset \mu$. We use classical branching rule for symmetric groups if either $\eta \notin \{\eta_m, \eta_{m+2}\}, 2 \mid m$ or $\eta \notin \eta_m, 2 \upharpoonright m$. In any case, we have $m_{\mu,\lambda} = 1$, if one of the conditions in (1)-(4) holds true. \qed

**Definition 2.10.** Suppose $\mu \in \Lambda^+_{m,n}(n-2)$ and $\lambda \in \Lambda^+_{m,n}(n)$. $\lambda$ is called $\mu$-admissible if one of the conditions in Theorem 2.9 (1)-(4) holds true. Let $\mathcal{A}(\mu)$ be the set of all $\mu$-admissible $m$-partitions.
3. Zero divisors of certain discriminants

In this section, we assume \( \delta_i \in F \) for \( 1 \leq i \leq m \), where \( F \) is a splitting field of \( x^m - 1 \) and \( e \nmid m \cdot n! \). The main purpose of this section is to prove Theorem 3.9, which will give all zero divisors of the discriminants of the Gram matrices \( G_{1, \mu'} \) with respect to the cell modules \( \Delta(1, \mu') \), \( \mu \in \Lambda^+_{m}(n - 2) \).

Recall that \( P(n, k) \) is the set of labelled parenthesis Brauer diagrams with \( k \) horizontal arcs. In what follows, we assume \( \alpha_0 = \text{top}(e_{n-1}) \in P(n, 1) \), the top row of \( e_{n-1} \). Define \( M_1 \) and \( M_2 \) by setting

- \( M_1 = \{ \alpha \otimes w \otimes \alpha_0 \mid \alpha \in P(n, 1), w \in W_{m, n-2} \} \).
- \( M_2 = \{ \alpha \otimes w \otimes \beta \mid \alpha, \beta \in P(n, k), w \in W_{m, n-2k}, 2 \leq k \leq [\frac{n}{2}] \} \).

We consider the quotient \( F \)-subspace \( V = V_1/V_2 \), where \( V_1 \) (resp. \( V_2 \)) is spanned by \( M_1 \cup M_2 \) (resp. \( M_2 \)). For convenience, we use \( \alpha \otimes w \otimes \alpha_0 \) instead of \( \alpha \otimes w \otimes \alpha_0 + V_2 \).

Recall that any dotted Brauer diagram can be written as \( \alpha \otimes w \otimes \beta \) where \( \alpha, \beta \in P(n, k) \) and \( w \in W_{m, n-2k} \). Let \( \tilde{\alpha} \in P(n, k) \) be such that

a) \( \alpha \) and \( \tilde{\alpha} \) have the same horizontal arcs.

b) There are \( m - i \) dots on a horizontal arc in \( \tilde{\alpha} \) if and only if there are \( i \) dots on the corresponding horizontal arc in \( \alpha \).

Define an \( R \)-linear isomorphism \( \iota : \mathcal{B}_{m,n}(\delta) \to \mathcal{B}_{m,n}(\delta) \) by declaring that

\[
\iota(\alpha \otimes w \otimes \beta) = \tilde{\beta} \otimes w^{-1} \otimes \tilde{\alpha}.
\]

We remark that \( \iota \) is not an algebraic (anti-)homomorphism since \( \iota(e_i^t e_i) = \delta_k e_i \neq \delta_{m-k} e_i \) in general. However, by straightforward computation, we have

\[
\iota(w(\alpha \otimes w_1 \otimes \beta)) = \iota(\alpha \otimes w_1 \otimes \beta)w^{-1},
\]

for any \( \alpha, \beta \in P(n, k), w \in W_{m,n}, w_1 \in W_{m,n-2k} \).

Following [7], we have the following definition.

**Definition 3.3.** Suppose \( \alpha_0 \otimes w \otimes \alpha_0 \in V \) for \( i = 1, 2 \). Let \( \langle \alpha_1 \otimes w_1 \otimes \alpha_0, \alpha_2 \otimes w_2 \otimes \alpha_0 \rangle \) be the coefficient of \( e_{n-1} \) in the expression of \( \iota(\alpha_1 \otimes w_1 \otimes \alpha_0) \cdot (\alpha_2 \otimes w_2 \otimes \alpha_0) \), where \( \iota \) is defined in (3.1). Let \( G_{m,n}(\delta) \) be the \( f \times f \)-matrix with \( f = \dim V \) such that the entry in \( (\alpha_1 \otimes w_1 \otimes \alpha_0) \)-th row, \( (\alpha_2 \otimes w_2 \otimes \alpha_0) \)-th column is \( \langle \alpha_1 \otimes w_1 \otimes \alpha_0, \alpha_2 \otimes w_2 \otimes \alpha_0 \rangle \).

If either \( h_1 \in M_2 \) or \( h_2 \in M_2 \), then \( \iota(h_1)h_2 \in V_2 \). Since \( e_{n-1} \not\in V_2 \), \( \langle h_1, h_2 \rangle = 0 \). Hence, \( \langle , , \rangle : V \times V \to F \) is a well-defined \( F \)-bilinear form on \( V \).

The following lemma can be verified easily.

**Lemma 3.4.** \( G_{m,n}(\delta) = (g_{ij}) \) is an \( f \times f \) matrix such that \( g_{ii} = \delta_0, 1 \leq i \leq f \) and \( g_{ij} \in \{ 0, \delta_1, \cdots, \delta_{m-1} \} \) if \( i \neq j \).

**Lemma 3.5.** \( G_{m,n}(\delta) : V \to V \) is a left \( FW_{m,n} \)-homomorphism and a right \( FW_{m,n-2} \) homomorphism

**Proof.** We consider \( G_{m,n}(\delta) \) as the \( F \)-linear endomorphism on \( V \) such that

\[
G_{m,n}(\delta)(\alpha_1 \otimes w_1 \otimes \alpha_0) = \sum_{\alpha \in P(n,1), w \in W_{m,n-2}} \langle \alpha \otimes w \otimes \alpha_0, \alpha_1 \otimes w_1 \otimes \alpha_0 \rangle \alpha \otimes w \otimes \alpha_0.
\]
By (3.2),
\[ \langle w(\alpha_1 \otimes w_1 \otimes \alpha_0), w(\alpha_2 \otimes w_2 \otimes \alpha_0) \rangle = \langle \alpha_1 \otimes w_1 \otimes \alpha_0, \alpha_2 \otimes w_2 \otimes \alpha_0 \rangle. \]

In other words, \( G_{m,n}(\delta) : V \to V \) is a left \( FW_{m,n} \)-homomorphism.

On the other hand, since \( (\alpha_1 \otimes w_1 \otimes \alpha_0) y = \alpha_1 \otimes w_1 y \otimes \alpha_0 \) for any \( y \in W_{m,n-2} \), \( e_{n-1} \) appears in \( y^{-1}(\alpha_0 \otimes w_1 \otimes \alpha_0)y \) with non-zero coefficient if and only if \( w_1 = 1 \).

Therefore,
\[ \langle (\alpha_1 \otimes w_1 \otimes \alpha_0)y, (\alpha_2 \otimes w_2 \otimes \alpha_0)y \rangle = \langle \alpha_1 \otimes w_1 \otimes \alpha_0, \alpha_2 \otimes w_2 \otimes \alpha_0 \rangle. \]

Consequently, \( G_{m,n}(\delta) : V \to V \) is a right \( FW_{m,n-2} \)-homomorphism. \( \square \)

Since we are assuming that \( F \) is a splitting field of \( x^m - 1 \) and \( e \nmid m \cdot n! \), \( FW_{m,k} \) is (split) semisimple for any \( k, 1 \leq k \leq n \). Assume that \( \lambda \in \Lambda_m^+(k) \). The classical Specht module \( S^\lambda \) is a direct summand of \( FW_{m,k} \). Consequently, \( \Delta(1, \lambda') \) can be realized as a submodule of \( V \), which is spanned by \( \alpha \otimes v_j \otimes \alpha_0 \) (mod \( V_2 \)), where \( v_j \) ranges over the basis elements of \( S^\lambda \). Note that \( G_{m,n}(\delta) \) is a right \( FW_{m,n-2} \)-module. For any \( \lambda \in \Lambda_m^+(n - 2) \), the restriction of \( G_{m,n}(\delta) \) on \( \Delta(1, \lambda') \) induces a linear endomorphism on \( \Delta(1, \lambda') \).

**Definition 3.6.** For \( \mu \in \Lambda_m^+(n - 2) \), define \( g_{\mu} = \prod_{\lambda \in \mathcal{P}(\mu)} g_{\lambda, \mu} \), where
\[
(3.7) \quad g_{\lambda, \mu} = (\delta_0 - m + m \sum_{p \in Y(\lambda/\mu)} c(p)) \prod_{i=1}^{m-1} (\delta_i + m \sum_{p \in Y(\lambda/\mu)} c(p)).
\]

It follows from [6, 2.3] that there is an invariant symmetric bilinear form defined on each cell module \( \Delta(k, \lambda) \). Via such a bilinear form, one can define a Gram matrix \( G_{k,\lambda} \). Let \( \mathcal{G}_{k,\lambda} \) be the determinant of \( G_{k,\lambda} \). The following result follows from [6, 3.8] and Theorem 2.8 immediately.

**Lemma 3.8.** \( \mathcal{G}_{m,n}(\delta) \) is (split) semisimple over \( F \) if and only if \( \det \mathcal{G}_{k,\lambda} \neq 0 \) for all \( (k, \lambda) \in \Lambda \).

In general, it is difficult to compute \( \det \mathcal{G}_{k,\lambda} \). Assume \( \delta_i \neq 0 \) for some \( 1 \leq i \leq m \). The following result describes all the zero divisors of \( \det \mathcal{G}_{1,\lambda}, \lambda \in \Lambda_m^+(n - 2) \). Fortunately, it completely determines \( \mathcal{B}_{m,n}(\delta) \) being (split) semisimple.

**Theorem 3.9.** Suppose \( \delta_i \neq 0 \) for some \( i, 1 \leq i \leq m \). \( \det \mathcal{G}_{1,\mu'} = 0 \) if and only if \( g_{\mu} = 0 \).

**Proof.** (\( \Rightarrow \)) If \( \det \mathcal{G}_{1,\mu'} = 0 \), then we can find an irreducible \( \mathcal{B}_{m,n}(\delta) \)-module \( M \subset \text{Rad} \mathcal{G}_{1,\mu'} \), where \( \text{Rad} \mathcal{G}_{1,\mu'} = \{ v \in \Delta(1, \mu') | \mathcal{G}_{1,\mu'}(v) = 0 \} \). It follows from [6, 2.6, 3.4] that any irreducible module of a cellular algebra must be the simple head of a cell module, say \( \Delta(k, \lambda') \). Hence, there is a non-zero homomorphism from \( \Delta(k, \lambda') \) to \( \Delta(1, \mu') \) with \( (k, \lambda') < (1, \mu') \). Therefore, either \( k = 1 \) or \( k = 0 \).
Assume that $k = 1$. We use [13, 7.4] \footnote{[13, 7.4] is for $\mathcal{B}_{m,n}(\delta)$ over the complex field. However, it is still true if we use $F$, a splitting field of $x^m - 1$ with $e \mid m \cdot n!$, instead of $C$. See [13, 8.8].} to get a non-zero homomorphism from $\Delta(0, \lambda')$ to $\Delta(0, \mu')$. Notice that, as $FW_{m,n}$-modules, $\Delta(0, \lambda') \cong S^\lambda$. Since $FW_{m,n}$ is (split) semisimple, we have $\lambda = \mu$, a contradiction since $(1, \lambda') < (1, \mu')$.

If $k = 0$, then there is a non-zero $\mathcal{B}_{m,n}(\delta)$-homomorphism from $\Delta(0, \lambda')$ to $\Delta(1, \mu')$, forcing $\lambda \in \mathcal{A}(\mu)$. By [13, 8.6, 8.8], $g_{\lambda, \mu} = 0$. We have $g_{\mu, \mu} = 0$ as required.

($\Leftarrow$) Suppose $g_{\mu, \mu} = 0$. Then there is a $\lambda \in \mathcal{A}(\mu)$ such that $g_{\lambda, \mu} = 0$. Since $\lambda \in \mathcal{A}(\mu)$, by Theorem 2.9, $[\Delta(1, \mu') : S^\lambda] = 1$. Hence, there is a unique $FW_{m,n}$-submodule $M$ of $\Delta(1, \mu')$ which is isomorphic to $S^\lambda$. Recall that $G_{m,n}(\delta)|_{\Delta(1, \mu')}$ is a linear endomorphism on $\Delta(1, \mu')$. For simplicity, we use $G_{m,n}(\delta)$ instead of $G_{m,n}(\delta)|_{\Delta(1, \mu')}$ if there is no confusion.

Since $G_{m,n}(\delta)$ is an $FW_{m,n}$-homomorphism, and $[\Delta(1, \mu') : S^\lambda] = 1$, $G_{m,n}(\delta)(M) \subset M$. By Schur’s Lemma, $G_{m,n}(\delta)|_M = f(\delta)I$, where $I$ is the identity matrix and $f(\delta) := f(\delta_0, \delta_1, \cdots, \delta_{m-1})$ is a polynomial in $\delta_i$, $0 \leq i \leq m - 1$.

Take a basis of $M$ and extend it to get a basis of $V$ via the elements $\alpha \otimes w \otimes \alpha_0$. Then $G_{m,n}(\delta)$ is conjugate to
\[
\begin{pmatrix}
 f(\delta)I & 0 \\
 0 & B
\end{pmatrix},
\]
where any entry in the diagonal of $B$ is $\delta_0$, and the term of the entry of $B$ elsewhere does not contain $\delta_0$. Since the degree of $\delta_0$ in det $G_{m,n}(\delta)$ is dim $V$ (see Lemma 3.2), the degree of $\delta_0$ in $f(\delta)$ must be 1. In particular, $f(\delta)$ is not a constant number. Take the parameters $\delta_0, \delta_1, \cdots, \delta_{m-1}$ such that $f(\delta) = 0$. Then $G_{m,n}(\delta)|_M = 0$.

We claim $e_{n-1}v = 0$ for any $v \in M$. Write $v = \sum_{a^s} a_{a^s} \cdot w \otimes \alpha_0$, where there are $s$ dots at the left endpoint of the unique arc in $\alpha^s$. We divide $P(n, 1)$ into three disjoint subsets $P_1, P_2, P_3$ as follows. Recall that a point in $\alpha^s$ is called a fixed point if it is an endpoint of a horizontal arc of $\alpha^s$. Otherwise, it is called a free point.

- $P_1$ consists of all $\alpha^s \in P(n, 1)$ such that $(n - 1, n)$ is a unique arc of $\alpha^s$. Then $e_{n-1}(\alpha^s \otimes w \otimes \alpha_0) = 0$.
- $P_2$ consists of all $\alpha^s \in P(n, 1)$ such that both $n - 1$ and $n$ are free points in $\alpha$. Then $e_{n-1} \alpha^s \otimes w \otimes \alpha_0 = 0$.
- $P_3$ consists of all $\alpha^s \in P(n, 1)$ such that either $n - 1$ or $n$ is a fixed point.

Let $i$ be the left endpoint of the unique arc in $\alpha^s$. By assumption, there are $s$ dots at the endpoint $i$. We define $w_{\alpha^s} \in \mathfrak{S}_{n-2}$ by setting
\[
w_{\alpha^s} = \begin{pmatrix}
i & i + 1 & i + 2 & \cdots & n - 3 & n - 2 
 n - 2 & i & i + 1 & \cdots & n - 4 & n - 3
\end{pmatrix}
\]
Define $y_{\alpha^s} := t^i_{\alpha^s}w_{\alpha^s}$. Then $e_{n-1} \cdot (\alpha^s \otimes 1 \otimes \alpha_0) = \alpha_0 \otimes y_{\alpha^s} \otimes \alpha_0$. Therefore, the coefficient of $\alpha_0 \otimes w_1 \otimes \alpha_0$ in $e_{n-1}v$ is $\sum_{\alpha^s \in P_2} a_{\alpha^s, y_{\alpha^s}, w_1} + \sum_{s=0}^{m-1} \delta_s a_{\alpha^s, w_1}$. 

\[\]
On the other hand, by direct computation, the coefficient of $\alpha_0 \otimes w_1 \otimes w_0$ in $G_{m,n}(\delta)v$ is $\sum_{\alpha^* \in P(n,1)\cup \{0\},w \in W_{m,n-2}} a_{\alpha^*,w}\langle \alpha_0 \otimes w_1 \otimes \alpha_0, \alpha^* \otimes w \otimes \alpha_0 \rangle$. We have

$$\langle \alpha_0 \otimes w_1 \otimes \alpha_0, \alpha^* \otimes w \otimes \alpha_0 \rangle = \begin{cases} \delta_s, & \text{if } \alpha^* \in P_1, w = w_1, \\ 0, & \text{if } \alpha^* \in P_1, w \neq w_1, \\ 0, & \text{if } \alpha^* \in P_2, \\ 1, & \text{if } \alpha^* \in P_3 \text{ and } w = y_0^{-1}w_1, \\ 0, & \text{if } \alpha^* \in P_3 \text{ and } w \neq y_0^{-1}w_1. \end{cases}$$

Since $G_{m,n}(\delta)v = 0$, the coefficient of $\alpha_0 \otimes w_1 \otimes \alpha_0$ in $G_{m,n}(\delta)v$ is zero. Therefore,

$$\sum_{\alpha^* \in P(n,1)\cup \{0\},w \in W_{m,n-2}} a_{\alpha^*,w}\langle \alpha_0 \otimes w_1 \otimes \alpha_0, \alpha^* \otimes w \otimes \alpha_0 \rangle = \sum_{\alpha^* \in P_3} a_{\alpha^*,y_0^{-1}w_1} + \sum_{s=0}^{m-1} \delta_s a_{\alpha^*,w_1} = 0,$$

forcing the coefficient of $\alpha_0 \otimes w_1 \otimes \alpha_0$ in $e_{n-1}v$ to be zero for all $w_1 \in W_{m,n-2}$. This completes the proof of the claim.

Therefore, as $\mathcal{B}_{m,n}(\delta)$-module, $M \cong \Delta(0, \lambda')$. We obtain a non-zero $\mathcal{B}_{m,n}(\delta)$-homomorphism from $\Delta(0, \lambda')$ to $\Delta(1, \mu')$. In particular, $\det G_{1,\mu'} = 0$. By [13, 8.6, 8.8] the parameters $\delta_i$’s must satisfy the equation $g_{\lambda,\mu} = 0$, the condition we have assumed.

\[\square\]

4. Proof of Theorem [A]

In this section, we prove Theorem [A] the main result of this paper. Unless otherwise stated, we assume that $F$ is a splitting field of $x^m - 1$, which contains $\delta_i, 1 \leq i \leq m$. Assume $m > 1$.

**Proposition 4.1.** Suppose $n \geq 2$. If $0 \neq \delta_i \in F$ for some $i$, $1 \leq i \leq m$, then $\mathcal{B}_{m,n}(\delta)$ is (split) semisimple if and only if $e \nmid m \cdot n$! and $\det G_{i,\lambda} \neq 0$ for any $\lambda \in \Lambda^+(k-2)$, $2 \leq k \leq n$.

**Proof.** ($\Leftarrow$) Suppose that $\mathcal{B}_{m,n}(\delta)$ is not (split) semisimple. There is a $(k, \lambda) \in \Lambda$ such that $\det G_{k,\mu} = 0$. Since $FW_{m,n}$ is (split) semisimple, $k \neq 0$.

Take an irreducible submodule $M \subset \text{Rad} \Delta(k, \mu)$. By [6, 2.6, 3.4], $M$ must be isomorphic to the simple head of a cell module, say $\Delta(l, \lambda)$, such that $(l, \lambda) < (k, \mu)$. Furthermore, it results in a non-trivial homomorphism from $\Delta(l, \lambda)$ to $\Delta(k, \mu)$.

If $l = k$, we use [13, 7.4] to get $\Delta(0, \lambda) \cong \Delta(0, \mu)$. As $FW_{m,n-2k}$-modules, $\Delta(0, \lambda) \cong \Lambda'$. Since $FW_{m,n-2k}$ is (split) semisimple, $\lambda = \mu$, which contradicts $(l, \lambda) < (k, \mu)$.

Suppose $l < k$. By [13, 7.4, 7.7], there is a non-trivial homomorphism from $\Delta(0, \tilde{\lambda})$ to $\Delta(1, \tilde{\mu})$ for some $\tilde{\mu} \in \Lambda^+(p-2)$ with $p \leq n$. By assumption, $\det G_{i,\tilde{\mu}} \neq 0$. Hence, $\Delta(1, \tilde{\mu}) = D^{(1,\tilde{\mu})} \cong \Delta(0, \tilde{\lambda})$. By [6, 3.4], $(0, \tilde{\lambda}) = (1, \tilde{\mu})$, a contradiction.

($\Rightarrow$) If $\mathcal{B}_{m,n}(\delta)$ is (split) semisimple, then [6, 3.8] implies that $\det G_{k,\lambda} \neq 0$ for all $0 \leq k \leq \lfloor \frac{m}{2} \rfloor$. Therefore, $FW_{m,n}$ is (split) semisimple, forcing $e \nmid m \cdot n$.

\[\frac{m}{2}\]

2 under our assumption, the group algebra $FW_{m,n}$ is (split) semisimple. Since the proof of [13, 8.6, 8.8] depends only on the fact that $\mathcal{C}W_{m,n}$ is (split) semisimple, we can apply these results here.
Suppose $\det G_1,\mu' = 0$ for some $\mu' \in \Lambda_m^+(k-2)$. Then $k < n$. By Theorem 3.9 there is a $\mu$-admissible $m$-partition $\lambda$ such that $g_{\lambda,\mu} = 0$. Equivalently, there is a non-zero $\mathcal{B}_{m,n}(\delta)$-homomorphism from $\Delta(0,\lambda')$ to $\Delta(1,\mu')$.

Since we are assuming that $m \geq 2$, we can find an $i, 1 \leq i \leq m$, such that $\lambda^{(i)} = \mu^{(i)}$. We can add $l$ boxes to $\lambda^{(i)}$ so as to get another partition $\tilde{\lambda}^{(i)} = \tilde{\mu}^{(i)}$. In this situation, $g_{\lambda,\tilde{\mu}} = g_{\lambda,\mu}$, where $\lambda$ (resp. $\tilde{\mu}$) can be obtained from $\lambda$ (resp. $\mu$) by using $\tilde{\lambda}^{(i)}$ instead of $\lambda^{(i)}$ (resp. $\mu^{(i)}$). By definition, $\lambda \in \mathcal{A}(\tilde{\mu})$. If we take $l$ such that $|\lambda| + l = n$, then $\Delta(0,\tilde{\lambda}')$ and $\Delta(1,\tilde{\mu}')$ are $\mathcal{B}_{m,n}(\delta)$-modules. By Theorem 3.9 $\det G_1,\tilde{\mu}' = 0$. However, since $\mathcal{B}_{m,n}(\delta)$ is (split) semisimple, $\det G_1,\tilde{\mu}' \neq 0$, a contradiction. □

**Corollary 4.2.** Let $\mathcal{B}_{m,n}(\delta)$ be a cyclotomic Brauer algebra over $F$, where $F$ contains a non-zero $\delta_i$ for some $i, 1 \leq i \leq m$. $\mathcal{B}_{m,n}(\delta)$ is (split) semisimple if only if $\det G_{k,\lambda} \neq 0$ for all $0 \leq k \leq \left\lceil \frac{n-1}{2} \right\rceil$. In particular, $\det G_{k,\lambda} \neq 0$ with $k = 0, 1$ and $\lambda \in \Lambda_m^+(n-2k)$.

Proof. Suppose $\mathcal{B}_{m,n}(\delta)$ is (split) semisimple. It follows from [6, 3.8] that $\det G_{k,\lambda} \neq 0$ for all $0 \leq k \leq \left\lceil \frac{n-1}{2} \right\rceil$. In particular, $\det G_{k,\lambda} \neq 0$ with $k = 0, 1$ and $\lambda \in \Lambda_m^+(n-2k)$.

Conversely, if $\det G_{k,\lambda} \neq 0$ for all $\lambda \in \Lambda_m^+(n)$, then $FW_{m,n}$ is (split) semisimple. Suppose that $\mathcal{B}_{m,n}(\delta)$ is not (split) semisimple. By Proposition 4.1 there is a $\mu \in \Lambda_m^+(k-2)$ with $k < n$ such that $\det G_{1,\mu} = 0$. From the proof of Proposition 4.1 we can find a $\tilde{\mu} \in \Lambda_m^+(n-2)$ such that $\det G_{1,\tilde{\mu}} = 0$. This contradicts our assumption. □

Corollary 4.2 has been stated as a question in [13, p220]. We remark that corollary 4.2 is not true if $m = 1$. In fact, the first author has proved that a Brauer algebra is (split) semisimple over $F$ if and only if $e \nmid n$! and $\det G_{1,\lambda} \neq 0$ for all $\lambda \in \Lambda^+(k-2), 2 \leq k \leq n$. By [4, 3.3-3.4], Corollary 4.2 is not true if $m = 1$.

**Definition 4.3.** Suppose that $m, n \in \mathbb{N}$ with $n \geq 2$. For $m \geq 2$, define $\rho_{m,n} = \{ ma | a \in \tilde{\rho}_{m,n} \}$, where

$$\tilde{\rho}_{m,n} = \{ k \in \mathbb{Z} | k = \sum_{p \in Y(\lambda/\mu)} c(p) \mid \mu \in \Lambda_m^+(n-2), \lambda \in \mathcal{A}(\mu) \}.$$ 

If $m = 1$, we define

$$\tilde{\rho}_{m,n} = \{ r \in \mathbb{Z} | r = \sum_{p \in Y(\lambda/\mu)} c(p) \mid \mu \in \Lambda^+(k-2), \lambda \in \Lambda^+(k), 2 \leq k \leq n \},$$

where two boxes in $Y(\lambda/\mu)$ are not in the same column.

At the end of this paper, we will prove $\tilde{\rho}_{m,n} = \tilde{Z}_{m,n}$. Hence, $\rho_{m,n} = Z_{m,n}$.

**Theorem 4.4.** Let $\mathcal{B}_{m,n}(\delta)$ be a cyclotomic Brauer algebra over $F$, where $F$ contains a non-zero $\delta_i$ for some $i, 1 \leq i \leq m$. Suppose $n \geq 2$. $\mathcal{B}_{m,n}(\delta)$ is (split) semisimple if and only if

1. $e \nmid m \cdot n!$,
2. $\varepsilon_i 0 m = \varepsilon_i 0 n, 0 \leq i \leq m - 1$, where $\varepsilon_i$ is the Kronecker function.
Proof. The result follows from Theorem 3.9 and Corollary 4.2

In the remaining part of this section, we deal with the case \(\delta_i = 0\) for all \(1 \leq i \leq m\). First, we discuss \(B_{m,2}(0)\).

We want to compute \(\det G_{1,\lambda}\) with \(\lambda = ((1), 0, \cdots, 0)\). Note that we have assumed \(u_i = \xi_i, 1 \leq i \leq m\). In this situation, \(y_{\lambda, W_{\lambda}} x_{\lambda} = \gamma(t_1) = \prod_{i=1}^{m-1} (t_1 - \xi_i)\).

Write \(v_1^{(0)} = \text{top}(e_1), v_2^{(0)} = \text{top}(s_1 e_2)\) and \(v_3^{(0)} = \text{top}(e_2)\). Let \(v_i^{(k)}\) be obtained from \(v_i^{(0)}\) by putting \(k\) dots at the left endpoint of the unique horizontal arc in \(v_i^{(0)}\). Then \(\Delta(1, \lambda)\) can be considered as a free \(F\)-module with basis \(\{v_i^{(k)} \otimes g(t_1) \otimes v_j^{(0)} | 1 \leq i \leq 3, 0 \leq k \leq m - 1\}\). Let \(a = \prod_{i=1}^{m-1} (1 - \xi_i)\). The Gram matrix with respect to this basis is

\[
G_{1,\lambda} = \begin{pmatrix} 0 & A & A \\ A & 0 & A \\ A & A & 0 \end{pmatrix},
\]

where \(A = (a_{ij})\) is the \(m \times m\) matrix with \(a_{ij} = a, 1 \leq i, j \leq m\). Since we are assuming that \(m > 1\), \(\det G_{1,\lambda} = 0\). In other words, \(\text{Rad} \Delta(1, \lambda) \neq 0\). Take an irreducible submodule \(D\) of \(\text{Rad} \Delta(1, \lambda)\). Note that any irreducible module must be the simple head of a cell module, say \(\Delta(1, \lambda)\). Therefore, there is a non-trivial homomorphism from \(\Delta(k, \mu)\) to \(\Delta(1, \lambda)\). By [6, 2.6], \((k, \mu) < (1, \lambda)\). This proves the following lemma.

**Lemma 4.5.** Suppose \(\lambda = ((1), 0, \cdots, 0)\). There is a cell module \(\Delta(k, \mu)\) of \(B_{m,2}(0)\) with \((k, \mu) < (1, \lambda)\) such that there is a non-trivial homomorphism from \(\Delta(k, \mu)\) to \(\Delta(1, \lambda)\).

Let \(J_{m,n}(0)\) be the left ideal of \(B_{m,n}(0)\) spanned by the dotted Brauer diagrams \(D\) such that \{(n-1,n)\} is a horizontal arc at the bottom row of \(D\). It is clear that \(J_{m,n}(0) = \text{Rad} B_{m,n}(0) e_{n-1} \).

Following [13], let \(I_{m,n}^{\geq l} \) (resp. \(I_{m,n}^{> l}\)) be the vector space generated by \((n,l)\)-dotted Brauer diagrams with \(l \geq k\) (resp. \(l > k\)). Let \(J_{m,n}(0) = I_{m,n}^{\geq k}/I_{m,n}^{> k}\). Then \(J_{m,n}(0)\) is a \(B_{m,n}(0)\)-module. Let \(I_{m,n}(\alpha)\) be the subspace of \(I_{m,n}^{\geq k}(0)\) generated by \(\{\alpha \otimes w \otimes \beta_0 | \alpha \in P(n,k), w \in W_{n-2k}\}\), where \(\beta_0 = \text{top}(e_{n-2k+1} \cdots e_{n-3} e_{n-1})\). Let \(B_{m,n}(0)\)-mod be the category of the left \(B_{m,n}(0)\)-modules. Let

\[
G : B_{m,n-2}(0)\text{-mod} \longrightarrow B_{m,n}(0)\text{-mod}
\]

be the tensor functor defined by declaring that \(G(M) = J_{m,n}(0) \otimes B_{m,n-2}(0) M\), for any \(B_{m,n-2}(0)\)-mod \(M\).

**Proposition 4.6.** Suppose \(\lambda \in \Lambda_m^+(n-2k)\).

(a) The functor \(G\) sends non-zero \(B_{m,n-2}(0)\)-homomorphisms to non-zero ones.

(b) \(G(\Delta(k-1, \lambda)) = \Delta(k, \lambda)\).

**Proof.** Suppose \(\phi : M_1 \rightarrow M_2\) is a \(B_{m,n-2}(0)\)-module homomorphism. Write \(\phi_\lambda = G(\phi)\). For any \(D_1 \in B_{m,n}(0), D \in J_{m,n}(0)\) and \(m \in M_1\),

\[
\phi_\lambda(D_1 (D \otimes m)) = \phi_\lambda(D_1 D \otimes m) = (D_1 D) \otimes \phi(m) = D_1(D \otimes \phi(m)) = D_1 \phi_\lambda(D \otimes m)
\]
Therefore, $\phi_\ast$ is a $\mathcal{B}_{m,n}(0)$-homomorphism. For any $\mathcal{B}_{m,n-2}(0)$-module $M$, define an $F$-linear map $\alpha: J_{m,n}(0) \otimes \mathcal{B}_{m,n-2}(0) \to M$ by setting $\alpha(D \otimes m) = (e_{n-1}D)0_m$, where $(e_{n-1}D)_0$ is obtained from $e_{n-1}D$ by removing the horizontal arcs $\{n-1, n\}$ at the top and bottom rows of $e_{n-1}D$.

Suppose $D^* = s_{m-2e_{n-1}} \in J_{m,n}(0)$. Then $\alpha(D^* \otimes m) = m$. If $\phi \neq 0$, then there is an $m_1 \in M_1$ such that $\phi(m_1) = m_2 \neq 0$. Consequently, $\alpha(D^* \otimes m_2) = m_2 \neq 0$. We have $\phi_\ast \neq 0$ since $\phi_\ast(D^* \otimes m_1) = D^* \otimes m_2 \neq 0$. This completes the proof of (a).

(b) can be proved similarly as [13, 7.2]. We include a proof as follows. First, we claim as $(\mathcal{B}_{m,n}(0), W_{m,n-2\cdot k})$-modules

$$I^k_{m,n}(0) \cong J_{m,n}(0) \otimes \mathcal{B}_{m,n-2}(0) \otimes^{k-1'} \mathcal{B}_{m,n-2k}(0).$$

For the simplification in exposition and notation, we omit $\mathcal{B}_{m,n-2}(0)$ in what follows.

Suppose $D_1 \otimes D_2 \in J_{m,n}(0) \otimes I^k_{m,n-2}(0)$. Let $e_{i,j} = \alpha \otimes 1 \otimes \alpha$, where $\alpha \in P(n,1)$ contains a unique horizontal arc $\{i, j\}$. Define $e_{i,j}^s := t^s_i e_{i,j} t^s_j$. We claim that there is a dotted Brauer diagram $D'_1$ in $I^k_{m,n}(0)$ such that $D_1 \otimes D_2 = D'_1 \otimes e_{i,j}^s \ldots e_{i-k+1,j-k+1} D_2$, where $e_{i,j}^s \in \mathcal{B}_{m,n-2}(0)$, $1 \leq l \leq k-1$.

In fact, if the bottom row of $D_1$ contains a horizontal arc $\{i, j\}$, which is different from $\{n-1, n\}$ and if there are $t$ dots at the left endpoint $i$ of $\{i, j\}$, then we can find another horizontal arc $\{i', j'\}$ at the top row of $D_1$ such that there are $s$ dots at the left endpoint $i'$ of $\{i', j'\}$. Using vertical arcs $\{i, i'\}$ and $\{j, j'\}$ instead of the horizontal arcs $\{i, j\}$ and $\{i', j'\}$ in $D_1$, we get another dotted Brauer diagram $D'_1$. We have $D_1 = D'_1 e_{i,j}^s$. Note that the number of horizontal arcs in $top(D'_1)$ is $k-1$ if the number of horizontal arcs in $top(D_1)$ is $k$. Using this method repeatedly, we have $D_1 \otimes D_2 = D'_1 \otimes e_{i_1,j_1}^s \ldots e_{i_k,j_k}^s D_2$.

Since $D_2 \in I^{k-1}_{m,n-2}(0)$, the number of the horizontal arcs in the top row of the composite of $e_{i_1,j_1}^s \ldots e_{i_{k-1},j_{k-1}}^s$ and $D_2$ is at least $k-1$. If it is bigger than $k$, then $e_{i_1,j_1}^s \ldots e_{i_{k-1},j_{k-1}}^s D_2 = 0$ in $I^{k-1}_{m,n-2}(0)$. If one loop occurs in the composite of $e_{i_1,j_1}^s \ldots e_{i_{k-1},j_{k-1}}^s$ and $D_2$, $e_{i_1,j_1}^s \ldots e_{i_{k-1},j_{k-1}}^s D_2 = 0$ since $\delta_1 = 0$, $0 \leq i \leq m-1$. We have $D_1 \otimes D_2 = 0$. In the remaining case, $e_{i_1,j_1}^s \ldots e_{i_{k-1},j_{k-1}}^s D_2$ is $w \cdot e_n e_{n-5} \ldots e_{n-2 k+1}$ for some $w \in W_{m,n-2}$. Note that $e_n w = w e_{n-1}$.

$$D_1 \otimes D_2 = D'_1 \otimes e_{i_1,j_1}^s \ldots e_{i_{k-1},j_{k-1}}^s D_2 = D'_1 w \otimes e_n e_{n-5} \ldots e_{n-2 k+1}.$$

Since $\{n-1, n\}$ is the unique horizontal arc at the bottom row of $D'_1$, $D'_1 w = w_1 e_{n-1}$ for some $w_1 \in W_{m,n}$. Hence, $D_1 \otimes D_2 = w_1 e_{n-1} \otimes e_n e_{n-5} \ldots e_{n-2 k+1}$. We can identify $D_1 \otimes D_2$ with $w_1 e_{n-1} e_{n-3} \ldots e_{n-2 k+1} \in I^{k'}_{m,n}(0)$ and vice versa. This proves $dim_F U_0 = dim_F I^{k'}_{m,n}(0)$, where $U_0 = J_{m,n}(0) \otimes \mathcal{B}_{m,n-2}(0) \otimes I^{k-1'}_{m,n-2}(0)$.

On the other hand, for any $\alpha \in I^{k-1'}_{m,n-2}(0)$, let $\alpha_2^0$ be obtained from $\alpha_2$ by adding two vertical arcs $\{n-1, n-1\}$ and $\{n, n\}$. The $F$-linear map $\phi: U_0 \to I^{k'}_{m,n}(0)$ sending $\alpha_1 \otimes \alpha_2$ to $\alpha_1 \cdot \alpha_2^0$ is surjective. Since $dim_F U_0 = dim_F I^{k'}_{m,n}(0)$, it must be injective. By the definition of the product of two dotted Brauer diagrams in [13],
In this situation, \( n \) any integer between 0 and \( m, n \). Therefore, the second equality in (1) follows.

Suppose \( m, n \) are not in the same column. By [12, 2.4], we can verify that \( \rho_{m,n} = \tilde{\rho}_{m,n} \) is not (split) semisimple over \( F \).

**Proposition 4.9.** Suppose \( m, n \in \mathbb{N} \) with \( n \geq 2 \).

1. \( \tilde{\rho}_{2,n} = \tilde{\rho}_{1,n} = \{ k \in \mathbb{Z} \mid 3 - n \leq k \leq n - 3 \} \cup \{ 2k - 3 \mid 3 \leq k \leq n, k \in \mathbb{Z} \} \).

2. \( \tilde{\rho}_{m,n} = \tilde{\rho}_{1,n} \cup \{ 2 - n, n - 2 \} \) if \( m \geq 3 \).

**Proof.** First, we assume \( m = 2 \). If \( \mu \in \Lambda^+(n-2) \) and \( \lambda \in \mathcal{A}(\mu) \), then either \( \lambda^{(1)} = \mu^{(1)} \), \( \lambda^{(2)} \in \mathcal{A}(\mu^{(2)}) \) or \( \lambda^{(2)} = \mu^{(2)} \), \( \lambda^{(1)} \in \mathcal{A}(\mu^{(1)}) \). We can assume \( \lambda^{(1)} \in \mathcal{A}(\mu^{(1)}) \) without loss of generality. Suppose \( |\mu^{(1)}| = k \). Then \( k \) can be any integer between 0 and \( n - 2 \). If \( r = \sum_{p \in Y(\lambda/\mu)} c(p) \), then \( r \in \tilde{\rho}_{1,n} \), forcing \( \tilde{\rho}_{2,n} \subset \tilde{\rho}_{1,n} \). Identifying \( \lambda \in \Lambda^+(k) \) with bipartition \( (\lambda, n-k) \), we have \( \tilde{\rho}_{2,n} \supset \tilde{\rho}_{1,n} \).

This proves the first equality in (1). Following [11], we define

\[
\mathcal{Z}(n) = \left\{ r \in \mathbb{Z} \mid r = 1 - \sum_{p \in Y(\lambda/\mu)} c(p), \lambda \in \Lambda^+(k), \mu \in \Lambda^+(k-2), 2 \leq k \leq n \right\},
\]

where two boxes in \( Y(\lambda/\mu) \) are not in the same column. By [12, 2.4],

\[
\mathcal{Z}(n) = \{ i \in \mathbb{Z} \mid 4 - 2n \leq i \leq n - 2 \} \setminus \{ i \in \mathbb{Z} \mid 4 - 2n < i < 3 - n, 2 \nu i \}.
\]

Therefore, the second equality in (1) follows.

Suppose \( m \geq 3 \). If \( \mu \) and \( \lambda \) satisfy one of the conditions in Theorem 2.8 (1) and Theorem 2.8 (4), then \( \sum_{p \in Y(\lambda/\mu)} c(p) \in \tilde{\rho}_{1,n} \). If \( \mu \) and \( \lambda \) satisfy the conditions (2) or (3) in Theorem 2.8, then there is an \( i \), such that \( \mu^{(i)} \rightarrow \lambda^{(i)} \) and \( \mu^{(m-i)} \rightarrow \lambda^{(m-i)} \).

In this situation, \( \sum_{p \in Y(\lambda/\mu)} c(p) \in \mathfrak{I}_a + \mathfrak{I}_b \) with \( |\mu^{(i)}| = a \) and \( |\mu^{(m-i)}| = b \), where

- \( \mathfrak{I}_a = \{ \sum_{p \in Y(\lambda/\mu)} c(p) \mid \mu \in \Lambda^+(\alpha), \mu \rightarrow \lambda \} \), and
- \( \mathfrak{I}_a + \mathfrak{I}_b = \{ i \mid i = x + y, x \in \mathfrak{I}_a, y \in \mathfrak{I}_b \} \).
Note that we can choose a suitable $\mu$ such that $a + b = i$ for all $i, 0 \leq i \leq n - 2$. We claim

$$\mathcal{I}_n = \begin{cases} \{0\}, & \text{if } a = 0, \\ \{i \in \mathbb{Z} \mid -a \leq i \leq a\} \setminus \{0\}, & \text{if } a = 1, 2, \\ \{i \in \mathbb{Z} \mid -a \leq i \leq a\}, & \text{otherwise} \end{cases} \tag{1}$$

In fact, one can verify the above result directly when $a \in \{0, 1, 2, 3\}$.

Suppose $\mu \in \Lambda^+(k + 1)$ and $\mu \rightarrow \lambda$. If $\lambda$ has at least two removable nodes, then we can find a box $q$ which is a removable node for both $\lambda$ and $\mu$. Let $\tilde{\lambda}$ (resp. $\tilde{\mu}$) be obtained from $\lambda$ (resp. $\mu$) by removing $q$. Then

$$\sum_{p \in Y(\lambda/\mu)} c(p) = \sum_{p \in Y(\lambda/\bar{\mu})} c(p) \in \mathcal{I}_k = \{ -k \leq i \leq k \}, \tag{2}$$

the last equality follows from the induction assumption.

If $\lambda$ has a unique removable node, then $\lambda = (\lambda_1, \cdots, \lambda_r)$ with $\lambda_i = \lambda_j$, $1 \leq i, j \leq r$. We have $\sum_{p \in Y(\lambda/\mu)} c(p) = \lambda_1 - r$. Note that $-1 - k \leq \lambda_1 - r \leq k + 1$. In any case, we have $\mathcal{I}_{k+1} \subset \{i \in \mathbb{Z} \mid -1 - k \leq i \leq k + 1 \}$.

Conversely, by the induction assumption, we can write $i = \sum_{p \in Y(\lambda/\mu)} c(p)$, for some $\lambda \in \Lambda^+(k + 1)$ and $\mu \rightarrow \lambda$ if $-k \leq i \leq k$. Since any Young diagram of a partition has at least two addable nodes, we can choose an addable node $q$ for both $\lambda$ and $\mu$ such that $q$ and $\lambda/\mu$ are not in the same row. In other words, $i \in \mathcal{I}_{k+1}$.

We have

$$\sum_{p \in Y(\lambda/\mu)} c(p) = -(k+1) \text{ if } \lambda = (1, \cdots, 1) \in \Lambda^+(k+2) \text{ and } \mu = (1, \cdots, 1) \in \Lambda^+(k+1).$$

$$\sum_{p \in Y(\lambda/\mu)} c(p) = k + 1 \text{ if } \lambda = (k + 2) \text{ and } \mu = (k + 1).$$

Consequently, $\mathcal{I}_{k+1} \supset \{i \in \mathbb{Z} \mid -k - 1 \leq i \leq k + 1 \}$. This completes the proof of the claim. Therefore,

$$\bigcup_{0 \leq a + b \leq n - 2} T_a + T_b = \{i \in \mathbb{Z} \mid 2 - n \leq i \leq n - 2 \}. \tag{3}$$

Note that $i \in \mathcal{I}_{n+1}$ if $3 - n \leq i \leq n - 3$. (2) follows immediately. \hfill $\square$

**Proof of Theorem A(a) and (c):** Theorem A(a) follows from Theorem 4.4 and Proposition 4.9. Theorem A(c) follows from Maschke’s theorem.

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