Abstract

We call a multidimensional distribution to be decomposable with respect to a partition of two sets of coordinates if the original distribution is the product of the marginal distributions associated with these two sets. We focus on the stationary distribution of a multidimensional semimartingale reflecting Brownian motion (SRBM) on a non-negative orthant. An SRBM is uniquely determined (in distribution) by its data that consists of a covariance matrix, a drift vector, and a reflection matrix. Assume that the stationary distribution of an SRBM exists. We first characterize two marginal distributions under the decomposability assumption. We prove that they are the stationary distributions of some lower dimensional SRBMs. We also identify the data for these lower dimensional SRBMs. Thus, under the decomposability assumption, we can obtain the stationary distribution of the original SRBM by computing those of the lower dimensional ones. However, this characterization of the marginal distributions is not sufficient for the decomposability. So, we next consider necessary and sufficient conditions for the decomposability. We obtain those conditions for several classes of SRBMs. These classes include SRBMs arising from Brownian models of queueing networks that have two sets of stations with feed-forward routing between these two sets. This work is motivated by applications of SRBMs and geometric interpretations of the product form stationary distributions.

Keywords: Semimartingale reflecting Brownian motion, stationary distribution, decomposability, marginal distribution, product form approximation, completely-$S$ matrix;

1 Introduction

We are concerned with a $d$-dimensional semimartingale reflecting Brownian motion (SRBM) that lives on the nonnegative orthant $\mathbb{R}_+^d$, where $\mathbb{R}_+$ is the set of all nonnegative real numbers. The SRBM is specified by a $d \times d$ covariance matrix $\Sigma$, a drift vector $\mu \in \mathbb{R}^d$, and a...
A $d \times d$ reflection matrix $R$. Namely, $(\Sigma, \mu, R)$ is the modeling primitives of the SRBM on $\mathbb{R}^d_+$. As usual, we assume that $\Sigma$ is positive definite and $R$ is completely-$S$ (see Appendix A for the definition of such a matrix). For the complete definition of an SRBM $Z = \{Z(t): t \geq 0\}$, we refer to Section A.1 of [7] (see [21, 22] for more details). A $(\Sigma, \mu, R)$-SRBM $Z$ has the following semimartingale representation:

$$
Z(t) = Z(0) + X(t) +RY(t) \in \mathbb{R}^d_+, \quad t \geq 0,
$$

(1.1)

$$
X = \{X(t), t \geq 0\} \text{ is a } (\Sigma, \mu)\text{-Brownian motion,}
$$

(1.2)

$$
Y(0) = 0, Y(\cdot) \text{ is nondecreasing,}
$$

(1.3)

$$
\int_0^\infty Z_i(t)dY_i(t) = 0 \text{ for } i = 1, \ldots, d.
$$

(1.4)

Our focus is on the stationary distribution of the $d$-dimensional SRBM. Throughout this paper, we assume that the stationary distribution exists. As a consequence, the primitive data satisfies the following condition:

$$
R \text{ is nonsingular, and } R^{-1}\mu < 0.
$$

(1.5)

If $R$ is either a $\mathcal{P}$-matrix for $d = 2$ or an $\mathcal{M}$-matrix for an arbitrary $d \geq 3$, then this condition is known to be sufficient, but generally not for $d \geq 3$ (see, e.g., [3]).

Let $J \equiv \{1, 2, \ldots, d\}$. A pair $(K, L)$ is said to be a partition of $J$ if $K \cup L = J$ and $K \cap L = \emptyset$. We consider conditions for the stationary distribution of an SRBM in $\mathbb{R}^d_+$ to be the product of two marginal distributions associated with a partition $(K, L)$ of the set $J$. Such a stationary distribution is said to be decomposable with respect to $K$ and $L$. We have two major contributions for the decomposability of the stationary distribution.

We first characterize, in Theorem 1, two marginal distributions associated with a partition $(K, L)$ under the decomposability assumption. We prove that they are the stationary distributions of some $|K|$- and $|L|$-dimensional SRBMs, where $|U|$ denotes the cardinality of a set $U$. We also identify the data for these lower dimensional SRBMs. Thus, under the decomposability assumption, we can obtain the stationary distribution of the original SRBM by computing those of the lower dimensional ones.

However, this characterization of the marginal distributions is not sufficient for the decomposability. So, we next consider necessary and sufficient conditions for the decomposability. We obtain those conditions for several classes of SRBMs (Theorem 2 and Corollary 2). These classes include diffusion limits of tandem queues and of queueing networks that have two sets of nodes with feed-forward routing between these two sets. Note that the decomposability does not mean a complete separation of such a network into two subnetworks. We illustrate a tandem queue next.

Consider a $d$-station generalized Jackson network in series, which is referred to as a tandem queue. In this tandem queue, the interarrival times to station 1 are assumed to be iid with mean $1/\beta_0$ and squared coefficient of variation (SCV) $c_0$. The service times at station $i$ are assumed to be iid with mean $1/\beta_i$ and SCV $c_i$, $i \in J$ (see Figure 1). The diffusion limit of this tandem queue is known to be the $d$-dimensional SRBM with the following reflection matrix $R$, the covariance matrix $\Sigma$, and the drift vector $\mu$.

$$
R_{ij} = \begin{cases} 
1, & i = j \text{ for } i = 1, 2, \ldots, d, \\
-1, & j = i - 1 \text{ for } i = 2, \ldots, d, \\
0, & \text{otherwise},
\end{cases}
$$

(1.6)
Figure 1: The $d$-station tandem queue partitioned into two blocks

\[ \Sigma_{ij} = \begin{cases} 
  c_{i-1} + c_i, & i = j \text{ for } i = 1, 2, \ldots, d, \\
  -c_{i-1}, & j = i - 1 \text{ for } i = 2, \ldots, d, \\
  -c_i, & j = i + 1 \text{ for } i = 1, \ldots, d - 1, \\
  0, & \text{otherwise},
\end{cases} \]

(1.7)

\[ \mu_i = \beta_{i-1} - \beta_i \text{ for } i = 1, \ldots, d. \]

(1.8)

For example, when $d = 3$, $R$ and $\Sigma$ are given by

\[
R = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} c_0 + c_1 & -c_1 & 0 \\ -c_1 & c_1 + c_2 & -c_2 \\ 0 & -c_2 & c_2 + c_3 \end{pmatrix}.
\]

We assume that $\Sigma$ is nonsingular and condition (1.5) is satisfied, which is equivalent to $\beta_0 < \beta_i$ for $i = 1, 2, \ldots, d$. It follows from [12] that the SRBM $Z$ has a unique stationary distribution $\pi$. For each $k \in \{1, 2, \ldots, d - 1\}$, let

\[ K = \{1, 2, \ldots, k\}, \quad L = \{k + 1, \ldots, d\}. \]

See Figure 1 for an illustration of these sets. It will be shown in Corollary 2 that if

\[ c_0 = c_i \quad \text{for } i = 1, \ldots, k, \]

(1.9)

then the $d$-dimensional stationary distribution is decomposable with respect to $K$ and $L$.

We are motivated by applications of SRBMs and our recent work [9]. Multidimensional SRBMs have been widely used for queueing applications [23] and some areas including mathematical finance [20]. For these applications, it is important to obtain the stationary distribution in a tractable form. However, this is a very hard problem even for $d = 2$ unless the $d$-dimensional SRBM has a product form stationary distribution. A multidimensional distribution is said to have a product form if it is the product of one dimensional marginal distributions.

It is shown in [13] that this product form holds for the stationary distribution of an SRBM $Z$ if and only if the following skew symmetry condition

\[ 2\Sigma = R\text{diag}(R)^{-1}\text{diag}(\Sigma) + \text{diag}(\Sigma)\text{diag}(R)^{-1}R^T \]  

(1.10)

is satisfied, where for a matrix $A$, $\text{diag}(A)$ denotes the diagonal matrix whose entries are diagonals of $A$, and $A^T$ denotes the transpose of $A$. Furthermore, under (1.10), the one-dimensional marginal stationary distribution in the $i$th coordinate has the exponential distribution with mean $1/\alpha_i$, where column vector $\alpha \equiv (\alpha_1, \ldots, \alpha_d)^T$ is given by

\[ \alpha = -2\text{diag}(\Sigma)^{-1}\text{diag}(R)R^{-1}\mu. \]

(1.11)
Thus, the stationary distribution of $Z$ is explicitly obtained under the skew symmetry condition (1.10). However, the condition (1.10) may be too strong in some applications. In particular, (1.10) is independent of covariances $\Sigma_{ij}$ for $i \neq j \in J \equiv \{1, 2, \ldots, d\}$. Still, product form based approximation is often used even though its accuracy cannot be assessed when condition (1.10) is not satisfied; see, for example, [4].

This product form based approximation may be improved by the decomposability. For example, let us consider the SRBM for the tandem queue depicted in Figure 1, and assume the decomposability condition (1.11). Then, its $|K|$-dimensional marginal is of product-form, which can be computed easily. Its $|L|$-dimensional marginal is the stationary distribution of an $|L|$-dimensional SRBM. When $|L|$ is small, say $\leq 4$, the algorithm in [3] can be used to compute this marginal distribution quickly. Therefore, the original $d$-dimensional stationary distribution can also be computed quickly. On the other hand, if we apply the algorithm in [3] directly to the $d$-dimensional SRBM when $c_{k+1} \neq c_0$, there is no computer that can compute the $d$-dimensional SRBM stationary distribution at all when $|K|$ is large.

In recent years, for $d = 2$, the tail asymptotics of the stationary distribution including decay rates have been well studied (e.g., see [1, 2, 3]) even though condition (1.10) is not satisfied. Those decay rates may be used for better approximations of a two-dimensional stationary distribution as recently shown for a two dimensional reflecting random walk in [2]. We hope that such a two-dimensional approximation can be used to develop better approximations for the stationary distribution of a high dimensional SRBM. Thus, weaker conditions than the product form will be useful to facilitate an approximation for a higher dimensional SRBM. A decomposability condition that is checkable by modeling primitives will considerably widen the applicability of a multidimensional SRBM.

We are also inspired by geometric interpretations in [1] for the product form characterization. That work focuses on the product form, but considers characterizations that are different from the skew symmetric condition (1.10). Among them, Corollary 2 of [1] shows that, for each pair of $i \neq j \in J$, the corresponding two-dimensional, marginal distribution is equal to the stationary distribution of some two-dimensional SRBM if the $d$-dimensional stationary distribution has a product form, that is, condition (1.10) is satisfied. This motivates us to consider the lower dimensional SRBMs corresponding to the marginal distributions under the decomposability.

This paper consists of four sections. We present our results, Theorems 1 and 2 and their corollaries, in Section 2. We discuss basic facts and preliminary results in Section 3.1. Theorems 1 and 2 are proved in Sections 3.2 and 3.3, respectively. We finally remark some future work in Section 4.

We will use the following notation unless otherwise stated.

For example, for $i \in U$, the $i$-th entry of $A^{(U,V)}_{x^V}$ is $\sum_{j \in V} [A^{(U,V)}]_{ij} x^V_j$, where both $T_{ij}$ and $[T]_{ij}$ denote the $(i, j)$-entry of a matrix $T$.

## 2 Main results

To state our main results, we need the following notation. Let

$$Q = R^{-1}.$$
A summary of basic notation

\[ J \{1, 2, \ldots, d\}, \text{ and, for } U, V \subset J \]
\[ A^{(U,V)} \text{ submatrix of a } d \times d \text{-dimensional matrix } A. \]
\[ x^U \text{ } \{U\}-\text{dimensional vector with } x^U_i = x_i \text{ for } i \in U, \]
where \( x^U_i \) is the \( i \)-th entry of \( x^U \).
\[ \uparrow x^U \text{ } d \times d \text{-dimensional vector } x \text{ with } x_i = x^U_i \text{ for } i \in U \]
and \( x_i = 0 \) for \( i \in J \setminus U \).
\[ f^U_\ast(x^U) \text{ } f(\uparrow x^U) \text{ for function } f \text{ from } \mathbb{R}^d \text{ to } \mathbb{R} \]
\[ \langle x, y \rangle \text{ } \sum_{i=1}^d x_iy_i \text{ for } x, y \in \mathbb{R}^d \]

Table 1: A summary of basic notation

For a non-empty set \( U \subset J \) such that \( Q^{(U,U)} \) is invertible, define

\[ \tilde{\Sigma}(U) = (Q^{(U,U)})^{-1}(Q\Sigma Q^T)^{(U,U)}((Q^{(U,U)})^{-1})^T, \] \hspace{1cm} (2.1)
\[ \tilde{\mu}(U) = (Q^{(U,U)})^{-1}(Q\mu)^U, \] \hspace{1cm} (2.2)
\[ \tilde{R}(U) = (Q^{(U,U)})^{-1}. \] \hspace{1cm} (2.3)

**Theorem 1.** Assume that \( R \) is completely-\( S \) and that the \( d \)-dimensional SRBM \( Z = \{Z(t), t \geq 0\} \) has a stationary distribution \( \pi \). Let \( K \) and \( L \) be a non-empty partition of \( J \). Assume that \( Z^K(0) \) and \( Z^L(0) \) are independent under the stationary distribution \( \pi \). Then, for \( U = K \) and \( U = L \),

(a) \( Q^{(U,U)} \) is invertible, and its inverse matrix \( \tilde{R}(U) \) is completely-\( S \).

(b) The \( |U| \)-dimensional \( (\tilde{\Sigma}(U), \tilde{\mu}(U), \tilde{R}(U)) \)-SRBM has a stationary distribution that is equal to the distribution of \( Z^U(0) \) under \( \pi \).

**Remark 1.** If \( R \) is a \( \mathcal{P} \)-matrix, then \( R \) is invertible and \( R^{-1} \) is also a \( \mathcal{P} \)-matrix by Lemma 6 of [1]. Therefore part (a) is immediate when \( R \) is a \( \mathcal{P} \)-matrix since each \( \mathcal{P} \)-matrix is completely-\( S \). However, even under condition (2.4), the completely-\( S \) property of \( R \) does not imply that \( R^{-1} \) has the same property as shown by the following example:

\[ R = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \text{ implies } R^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}, \]

where \( R \) is completely-\( S \), but \( R^{-1} \) is not. Thus, part (a) of the theorem is not obvious at all.

**Theorem 1** will be proved in Section 3.3, and the corollary below is proved in Appendix B.

**Corollary 1.** Under the same assumptions of Theorem 1, for \( i \in J \), assume that \( Z^{(i)}(0) \) and \( Z^{J \setminus \{i\}}(0) \) are independent under the stationary distribution \( \pi \). Then, \( Z^{(i)}(0) \) has the exponential distribution with mean \( 1/\lambda_i \), where

\[ \lambda_i = \Delta_i Q_{i1}, \] \hspace{1cm} (2.4)
\[ \Delta_i = -\frac{2(\mu_i, (Q^T)^{(i)})}{\langle (Q^T)^{(i)}, \Sigma(Q^T)^{(i)} \rangle} > 0, \] \hspace{1cm} (2.5)

and \( (Q^T)^{(i)} \) is the \( i \)-th column of \( Q^T \).
Remark 2. The parameter \( \lambda_i \) in (2.4) has a geometric interpretation; see (B.1) in the proof of Corollary [4]. Note that \( \lambda_i \) uses information on covariance \( \Sigma_{ij} \) in general, so it may be different from \( \alpha_i \) of (2.4), although we must have \( \alpha_i = \lambda_i \) if the stationary distribution has a product form. Since the exponential distribution in Corollary [4] is obtained under the weaker condition than the product form condition (1.10) for \( d \geq 3 \), it is intuitively clear that one should use \( \lambda_i \) instead of \( \alpha_i \) in the product form approximation of a stationary distribution. We will further discuss approximations in Section 4.

In general, (a) and (b) of Theorem [4] are not sufficient for \( Z^K(0) \) and \( Z^L(0) \) to be independent under the stationary distribution \( \pi \). For example, if \( J = \{1, 2\} \), then the marginal exponential distributions are determined by the mean \( 1/\lambda_i \) for \( i = 1, 2 \) by Corollary [4], but these marginals are not sufficient for the skew symmetric condition, which is equivalent to that \( Z^{(1)}(0) \) and \( Z^{(2)}(0) \) are independent. This is because a condition weaker than the decomposability condition is used in the proof of Theorem [4] and therefore of Corollary [4]. This fact will be detailed in Section 4. Thus, we require extra conditions for necessary and sufficient conditions for the decomposability. However, to identify these extra conditions is generally a hard problem, so we consider a relatively simple situation. For this, we consider SRBMs arising from queueing networks that have two sets of stations with feed-forward routing between these two sets.

**Theorem 2.** Assume that \( R \) is completely-S and that the \( d \)-dimensional SRBM \( Z = \{ Z(t), t \geq 0 \} \) has a stationary distribution \( \pi \). Let \( K \) and \( L \) be a non-empty partition of \( J \), and assume that

\[
R^{(K,L)} = 0. \tag{2.6}
\]

If \( Z^K(0) \) and \( Z^L(0) \) are independent under \( \pi \) and if \( Z^K(0) \) is of product form under \( \pi \), then

\[
2\Sigma^{(K,K)} = R^{(K,K)} \text{diag}(R^{(K,K)})^{-1} \text{diag}(\Sigma^{(K,K)}) + \text{diag}(\Sigma^{(K,K)}) \text{diag}(R^{(K,K)})^{-1} (R^{(K,K)})^\top, \tag{2.7}
\]

\[
2\Sigma^{(L,K)} = R^{(L,K)} \text{diag}(\Sigma^{(K,K)}) \text{diag}(R^{(K,K)})^{-1}. \tag{2.8}
\]

Conversely, if \( \Sigma \) and \( R \) satisfy (2.7) and (2.8), and if the \( |L| \)-dimensional \( (\Sigma^{(L,L)}, \tilde{\mu}(L), R^{(L,L)}) \)-SRBM has a stationary distribution, then \( Z^K(0) \) and \( Z^L(0) \) are independent under \( \pi \) and \( Z^K(0) \) is of product form under \( \pi \).

**Remark 3.** After an appropriate recording of the coordinates, the condition (2.7) can be written as

\[
R = \begin{pmatrix}
R^{(K,K)} & 0 \\
R^{(L,K)} & R^{(L,L)}
\end{pmatrix}.
\]

In this case, the covariance matrix \( \Sigma \) and the drift vector \( \mu \) are partitioned as

\[
\Sigma = \begin{pmatrix}
\Sigma^{(K,K)} \\
\Sigma^{(L,K)} \\
\Sigma^{(L,L)}
\end{pmatrix}, \quad \mu = (\mu^K, \mu^L)^\top.
\]
This theorem is proved in Section 3.3. The next corollary is for an SRBM arising from the $d$ station tandem queue, which was discussed in Section 1 (see Figure 1). We omit its proof because it is an immediate consequence of Theorem 2.

**Corollary 2.** Assume that the $(\Sigma, \mu, R)$-SRBM has a stationary distribution $\pi$, where the reflection matrix $R$, the covariance matrix $\Sigma$, and the drift vector $\mu$ are given by (1.1), (1.2) and (1.3), respectively. For each positive integer $k \leq d - 1$, set $K = \{1, \cdots, k\}$ and $L = J \setminus K$. Then $Z^K(0)$ and $Z^L(0)$ are independent under $\pi$ if $c_0 = c_1 = \cdots = c_k$. Furthermore, for $k = 1$, $c_0 = c_1$ is also necessary for this decomposability.

# 3 Proofs of main results

We will prove Theorems 1 and 2. For this, we first discuss about equations to characterize the stationary distribution and some basic facts obtained from the decomposability.

## 3.1 The stationary distribution

Assume the SRBM has a stationary distribution. The stationary distribution must be unique [5]. Our first tool is the basic adjoint relationship (BAR) that characterizes the stationary distribution. For this, we first introduce the boundary measures for a distribution on $(\mathbb{R}^d_+, \mathcal{B}(\mathbb{R}^d_+))$, where $\mathcal{B}(\mathbb{R}^d_+)$ is the Borel $\sigma$-field on $\mathbb{R}^d_+$. They are defined as

$$\nu_i(B) = \mathbb{E}_\pi \left[ \int_0^1 1\{Z(t) \in B\} dY_i(t) \right], \quad B \in \mathcal{B}(\mathbb{R}^d_+), \ i \in J.$$

Our BAR is in terms of moment generating functions, which are defined as

$$\varphi_i(\theta) = \mathbb{E}_\pi[e^{\theta Z(0)}], \quad \varphi_i(\theta) = \mathbb{E}_\pi \left[ \int_0^1 e^{\theta Z(t)} dY_i(t) \right], \quad i \in J,$$

where $\mathbb{E}_\pi$ is the expectation operator when $Z(0)$ has the distribution $\pi$.

Because for each $i \in J$, $Y_i$ increases only when $Z_i(t) = 0$, one has $\varphi_i(\theta)$ depends on $\theta^J \setminus \{i\}$ only. Therefore,

$$\varphi_i(\theta) = \varphi_i(\theta^J \setminus \{i\}).$$

For a $(\Sigma, \mu, R)$-SRBM, its data can be alternatively described in terms of $d$-dimensional polynomials, which are defined as

$$\gamma(\theta) = -\frac{1}{2} \langle \theta, \Sigma \theta \rangle - \langle \mu, \theta \rangle, \quad \theta \in \mathbb{R}^d,$$

$$\gamma_i(\theta) = \langle R^{(i)} \theta \rangle, \quad \theta \in \mathbb{R}^d, \quad i \in J,$$

where $R^{(i)}$ is the $i$th column of the reflection matrix $R$. Obviously, those polynomials uniquely determine the primitive data, $\Sigma$, $\mu$ and $R$. Thus, we can use those polynomials to discuss everything about the SRBM instead of the primitive data themselves.

The following lemma is critical in our analysis. Equation (3.1) below is the moment generating function version of the standard basic adjoint relationship. We still refer to it as BAR.

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Lemma 1. (a) Assume \( \pi \) is the stationary distribution of a \((\Sigma, \mu, R)\)-SRBM. For \( \theta \in \mathbb{R}^d \), \( \varphi(\theta) < \infty \) implies \( \varphi_i(\theta) < \infty \) for \( i \in J \). Furthermore,

\[
\gamma(\theta) \varphi(\theta) = \sum_{i=1}^{d} \gamma_i(\theta) \varphi_i(\theta) \tag{3.1}
\]

holds for \( \theta \in \mathbb{R}^d \) such that \( \varphi(\theta) < \infty \). (b) Assume that \( \pi \) is a probability measure on \( \mathbb{R}^d_+ \) and that \( \nu_i \) is a positive finite measure whose support is contained in \( \{ x \in \mathbb{R}^d_+ : x_i = 0 \} \) for \( i \in J \). Let \( \varphi \) and \( \varphi_i \) be the moment generating functions of \( \pi \) and \( \nu_i \), respectively. If \( \varphi \), \( \varphi_1, \ldots, \varphi_d \) satisfy (3.1) for each \( \theta \in \mathbb{R}^d \) with \( \theta \leq 0 \), then \( \pi \) is the stationary distribution and \( \nu_i \) is the corresponding boundary measure on \( \{ x \in \mathbb{R}^d_+ : x_i = 0 \} \).

For (a), the fact that (3.1) holds for \( \theta = 0 \) is a special case of the standard basic adjoint relationship (BAR); see, e.g., equation (7) of [5]. When some components of \( \theta \) are allowed to be positive in (3.1), readers are referred to the proof of Lemma 4.1 (a) in [7] to see how to rigorously derive the relationship. For (b), we refer to Theorem 1.2 of [6] (see also [15] for a more general class of reflecting processes). In [6], BAR (7) is given using differential operators; see, for example, again equation (7) of [5]. Under the condition of our lemma, that BAR (7) is satisfied for all functions \( f \) of the form

\[
f(x) = \sum_{i=1}^{n} a_i e^{(\theta^i, x)} \text{ for } x \in \mathbb{R}^d_+ \tag{3.2}
\]

where \( n \) is any positive integer, \( a_i \in \mathbb{R} \) and \( \theta^i \in \mathbb{R}^d \) with \( \theta^i \leq 0 \). Using an argument similar to that in Section 4 of [18], that BAR (7) continues to hold for functions \( f \) when \( \theta^i \)'s are replaced by \( (z^i_1, \ldots, z^i_d) \) where each \( z^i_j \) is a complex variable with \( \Re z^i_j \leq 0 \). This means that BAR is satisfied for all \( f \) that are a finite Fourier series. Since a continuous function with a compact support is uniformly approximated by a sequence of finite Fourier series (see, e.g., Sections 0.42 and 1.14 of [24]), one can argue that BAR (7) in [5] holds for all \( C^2 \) functions with compact support, which is sufficient for BAR (7) to hold for all \( C^2 \) functions whose first- and second-order derivatives are bounded.

In the rest of this paper, whenever we write \( \varphi(\theta) \), we implicitly assume it is finite.

Let \( K \) and \( L \) be a non empty partition of \( J \). In this paper, we consider conditions for \( Z^K(0) \) to be independent of \( Z^L(0) \) under the stationary distribution \( \pi \). The independence is equivalent to

\[
\varphi(\theta) = \varphi^K(\theta^K) \varphi^L(\theta^L), \tag{3.3}
\]

where, for \( U \subset J \),

\[
\varphi^U(\theta^U) = \varphi(\theta^U).
\]

The next lemma shows how the boundary measure is decomposed under (3.3).

Lemma 2. Let \( K \) and \( L \) be a nonempty partition of \( J \). Assume that \( Z^K(0) \) and \( Z^L(0) \) are independent under the stationary distribution \( \pi \). Assume \( \varphi(\theta) < \infty \). Then

\[
\varphi_j(\theta) = \varphi^K_j(\theta^K) \varphi^L(\theta^L), \quad j \in K, \tag{3.4}
\]
where, for $U \subset J$,
\[
\phi^U_j(\theta^U) = \phi_j(\theta^U).
\]

**Proof.** We first prove that for $i \neq j$,
\[
\lim_{\theta_j \downarrow -\infty} \phi_i(\theta) = 0.
\] (3.5)

By the monotone convergence theorem, we have
\[
\lim_{\theta_j \downarrow -\infty} \phi_i(\theta) = \mathbb{E}_\pi \left[ \int_0^1 e^{\langle \theta \setminus \{i, j\}, Z^{\setminus\{i,j\}}(t) \rangle} 1(Z_j(t) = 0) dY_i(t) \right].
\]

By (1.4), $Y_i(t) = \int_0^t 1(Z_i(s) = 0) dY_i(s)$ for all $t \geq 0$. Thus,
\[
\int_0^1 e^{\langle \theta \setminus \{i, j\}, Z^{\setminus\{i,j\}}(t) \rangle} 1(Z_j(t) = 0) dY_i(t)
= \int_0^1 e^{\langle \theta \setminus \{i, j\}, Z^{\setminus\{i,j\}}(t) \rangle} 1(Z_j(t) = 0, Z_i(t) = 0) dY_i(t),
\]
which equals to zero almost surely by Theorem 1 of [19]. Therefore, we have proved (3.5).

Hence, for each $j \in J$ and $\theta \leq 0$, dividing both sides of (3.1) by $\theta_j$ and letting $\theta_j \downarrow -\infty$, we have
\[
- \lim_{\theta_j \downarrow -\infty} \left( \frac{1}{2} \Sigma_{jj} \theta_j + \mu_j \right) \phi(\theta) = R_{jj} \phi_j(\theta) + \sum_{i \neq j} R_{ji} \lim_{\theta_j \downarrow -\infty} \phi_i(\theta)
= R_{jj} \phi_j(\theta).
\] (3.6)

Let $\theta = \uparrow \theta^K$ for $j \in K$ in this equation, we have
\[
- \lim_{\theta_j \downarrow -\infty} \left( \frac{1}{2} \Sigma_{jj} \theta_j + \mu_j \right) \phi^K(\theta^K) = R_{jj} \phi^K_j(\theta^K).
\] (3.7)

By the independence assumption, we can write
\[
\phi(\theta) = \varphi^K(\theta^K) \varphi^L(\theta^L).
\]

Hence, multiplying $\phi^L(\theta^L)$ to both sides of (3.6), then (3.8) yields (3.7).

### 3.2 Proof of Theorem [19]

Because the SRBM $Z$ has a stationary distribution, (3.6) is satisfied. Thus, $Q \equiv R^{-1}$ exists. Let $V(t) = QZ(t)$ for $t \geq 0$. It follows from (3.4) that
\[
V(t) = QZ(0) + QX(t) + Y(t), \quad t \geq 0.
\] (3.8)

Note that $QX$ is still Brownian motion with drift vector $Q\mu$ and covariance matrix $Q\Sigma Q^T$. For an SRBM $Z$ arising from open multiclass queueing networks, the process $V$ is known as the total workload process [19]. The key idea of our proof is to use (3.4) instead of (3.1), and (3.8) allows us to easily separate the entries of $Y(t)$ among coordinates in a partition $K$ and $L$.

The most difficult part is in the proof of part (a) of the theorem. We prove the first half of part (a) in the following lemma.


Lemma 3. $Q^{(K,K)}$ is invertible.

Proof. In this proof, we apply truncation arguments similarly to those in [8]. To this end, we introduce the following sequences of functions. For each positive integer $n$, let

$$ g_n(s) = \begin{cases} 
\frac{1}{2}(s + n + 2)^2, & -(n + 2) < s \leq -(n + 1), \\
1 - \frac{1}{2}(s + n)^2, & -(n + 1) < s \leq -n, \\
1, & -n < s \leq n, \\
1 - \frac{1}{2}(s - n)^2, & n < s \leq n + 1, \\
\frac{1}{2}(n + 2 - s)^2, & n + 1 < s \leq n + 2, \\
0, & s \leq -(n + 2) \text{ or } s > n + 2.
\end{cases} $$

and let

$$ f_n(u) = \begin{cases} 
\int_0^u g_n(s) ds, & u \geq 0, \\
\int_0^0 g_n(s) ds, & u < 0.
\end{cases} $$

Clearly, for each fixed $n$, $f_n(u)$ is bounded, twice continuously differentiable, and its derivatives $f'_n(u)$ and $f''_n(u)$ are bounded by 1 in absolute values. Furthermore, for each $u \in \mathbb{R}$, $f'_n(u) = g_n(u)$ is monotone in $n$, and for each $u \in \mathbb{R}$,

$$ \lim_{n \to \infty} f_n(u) = u, \quad \lim_{n \to \infty} f'_n(u) = 1, \quad \lim_{n \to \infty} f''_n(u) = 0. \quad (3.9) $$

For each $i \in J$ and $t \geq 0$, we apply Itô’s integration formula for $f_n([QZ(t)]_i)$ for each fixed $n$, then it follows from (3.8) that

$$ f_n([V(t)]_i) - f_n([V(0)]_i) = \int_0^t f'_n([V(u)]_i) d([QX(u)]_i + Y_i(u)) \right.
\left. + \frac{1}{2} \int_0^t f''_n([V(u)]_i) [Q\Sigma Q^T]_{ii} du. \quad (3.10) $$

Because $f_n$, $f'_n$ and $f''_n$ are all bounded, we take the expectation $E_\pi$ on both sides of (3.10) with $t = 1$, and obtain

$$ 0 = \int_{\mathbb{R}_+^4} f'_n([Qx]_i)[Q\mu]_i \pi(dx) + \int_{\mathbb{R}_+^4} f'_n([Qx]_i)\nu_i(dx) \right.
\left. + \frac{1}{2} [Q\Sigma Q^T]_{ii} \int_{\mathbb{R}_+^4} f''_n([Qx]_i) \pi(dx). $$

Applying the dominated convergence theorem on the $f''_n$ term and the monotone convergence theorem on two $f'_n$ terms, by letting $n \to \infty$, we have

$$ (Q\mu)_i + \nu_i(\mathbb{R}_+^4) = 0 \quad i \in J. \quad (3.11) $$

We now assume that $Q^{(K,K)}$ is singular. Then, there exists a non-zero $|K|$-dimensional row vector $\eta$ such that

$$ \eta Q^{(K,K)} = 0. \quad (3.12) $$
We will prove that (3.12) implies
\[ \eta Q^{(K,L)} = 0. \] (3.13)

Assuming (3.13), we now show that it leads to a contradiction, thus proving the lemma. To see this, it follows from (3.8) that
\[ V^K(t) = V^K(0) + (QX)^K(t) + Y^K(t). \] (3.14)

Since \( V(t) = QZ(t) \), (3.12) and (3.13) imply that \( \eta V^K(t) = 0 \) for all \( t \geq 0 \). Similarly, (3.12) and (3.13) imply that \( \eta (QX(t))^K = 0 \) for all \( t \geq 0 \). From (3.8), we now have \( \eta Y^K(t) = 0 \) for all \( t \geq 0 \). Namely,
\[ \sum_{i \in K} \eta_i Y^K_i(t) = 0, \quad t \geq 0. \]

Assume \( \eta_j \neq 0 \) for some \( j \in K \). Since \( Y^K_i(t) = Y_i(t) \), we have
\[ \sum_{i \in K} \eta_i \int_0^t 1(Z_j(u) = 0) dY_i(u) = 0. \]

By Theorem 1 of [19] (see also Theorem 4.2 of [6]), we have
\[ \int_0^1 1(Z_j(u) = 0) dY_j(t) = 0 \text{ almost surely for each pair } i \neq j. \]

This yields
\[ \eta_j \int_0^1 1(Z_j(u) = 0) dY_j(t) = 0 \text{ almost surely,} \]
which contradicts the fact that \( \eta_j \neq 0 \) and that
\[ E \left( \int_0^1 1(Z_j(u) = 0) dY_j(t) \right) = \nu_j(\mathbb{R}^d_+) = -(Q \mu)_j > 0. \]

Now we prove (3.12). Note that \((QX)^K\) in (3.12) is a \(|K|-\text{dimensional Brownian motion with drift } Q^{(K,J)} \mu \text{ and covariance matrix} \)
\[ Q^{(K,J)} \Sigma Q^{(J,K)} = (Q \Sigma Q^T)^{(K,K)}. \]

We apply Itô’s integration formula to \( h_n(V^K(t)) \), where \( h_n(x) = e^{-f_n(\eta,x)} \) for \( x \in \mathbb{R}^{|K|} \).

Then, we have
\[ h_n(V^K(t)) - h_n(V^K(0)) = \sum_{i \in K} \int_0^t \frac{\partial h_n(x)}{\partial x_i} \bigg|_{x=V^K(u)} \, d(QX)^K_i(u) \]
\[ + \frac{1}{2} \sum_{i,j \in K} \int_0^t \frac{\partial^2 h_n(x)}{\partial x_i \partial x_j} \bigg|_{x=V^K(u)} (Q \Sigma Q^T)^{(K,K)}_{ij} du \]
\[ + \sum_{i \in K} \int_0^t \frac{\partial h_n(x)}{\partial x_i} \bigg|_{x=V^K(u)} dY_i(u), \] (3.15)
where
\[
\frac{\partial h_n(x)}{\partial x_i} = -\eta_t g_n(\langle \eta, x \rangle) h_n(x),
\]
\[
\frac{\partial^2 h_n(x)}{\partial x_i \partial x_j} = \eta_t \eta_j (g_n^2(\langle \eta, x \rangle) + g_n(\langle \eta, x \rangle)) h_n(x).
\]

Setting \( t = 1 \) and taking expectation \( \mathbb{E}_\pi \) on both side, we have
\[
0 = \sum_{i \in K} \mathbb{E}_\pi \left[ \frac{\partial h_n(x)}{\partial x_i} \right]_{x=V^K(0)} (Q\mu)_i + \frac{1}{2} \sum_{i,j \in K} (Q\Sigma Q^T)_{ij} \mathbb{E}_\pi \left[ \frac{\partial^2 h_n(x)}{\partial x_i \partial x_j} \right]_{x=V^K(0)}
\]
\[
+ \sum_{i \in K} \mathbb{E}_\pi \left[ \int_0^1 \frac{\partial h_n(x)}{\partial x_i} \right]_{x=M^K(u)} dY_i(u) \]
\[
= \nu_t(\mathbb{R}^d) \mathbb{E}_\pi \left[ \frac{\partial h_n(x)}{\partial x_i} \right]_{x=M^K(0)},
\]
(3.16)

Recall that \( V^K(t) = Q^{(K,K)}Z^K(t) + Q^{K,L}Z^K(t) \). Let \( M^K(t) = Q^{K,L}Z^K(t) \).

Because \( \eta Q^{(K,K)} = 0 \), we have \( \eta V^K(t) = \eta M^K(t) \), and therefore, for \( i \in K \),
\[
\mathbb{E}_\pi \left[ \int_0^1 \frac{\partial h_n(x)}{\partial x_i} \right]_{x=V^K(u)} dY_i(u) = \mathbb{E}_\pi \left[ \int_0^1 \frac{\partial h_n(x)}{\partial x_i} \right]_{x=M^K(u)} dY_i(u)
\]
\[
= \nu_t(\mathbb{R}^d) \mathbb{E}_\pi \left[ \frac{\partial h_n(x)}{\partial x_i} \right]_{x=M^K(0)},
\]
(3.17)

where in the second equality, we have used that fact that \( M^K(t) \) is a functions of \( Z^K(t) \) and Lemma 3. It follows from (3.16), (3.17), and (3.18) that
\[
\frac{1}{2} \sum_{i,j \in K} (Q\Sigma Q^T)_{ij} \mathbb{E}_\pi \left[ \frac{\partial^2 h_n(x)}{\partial x_i \partial x_j} \right]_{x=V^K(0)} = 0,
\]
or equivalently
\[
\frac{1}{2} \sum_{i,j \in K} \eta_i \eta_j (Q\Sigma Q^T)_{ij}
\]
\[
\times \mathbb{E}_\pi \left[ g_n^2(\langle \eta, V^K(0) \rangle) + g_n'(\langle \eta, V^K(0) \rangle) \right] = 0.
\]

By the construction of functions \( g_n \) and \( f_n \), \( g_n'(u) \geq 0 \) except for \( u \in (n, n + 2) \), in which \( g_{n+1}'(u) \in [-1, 0) \), and \( e^{-f_n(u)} \) monotonically converges to \( e^{-u} \) as \( n \to \infty \) and is bounded by 1 for \( u \geq 0 \). Furthermore, \( g_n(u) \) and \( g_{n+1}'(u) \) are bounded by 1 for all \( u \) and \( n \). We have
\[
\mathbb{E}_\pi \left[ g_n^2(\langle \eta, V^K(0) \rangle) + g_n'(\langle \eta, V^K(0) \rangle) \right] = \mathbb{E}_\pi \left[ g_n^2(\langle \eta, V^K(0) \rangle) + g_n'(\langle \eta, V^K(0) \rangle) \right]_{0} \geq 0
\]
for each \( n \geq 1 \). By dominated converge theorem,
\[
\lim_{n \to \infty} \mathbb{E}_\pi \left[ \left( g_n^2(\langle \eta, V^K(0) \rangle) + g_n^\prime(\langle \eta, V^K(0) \rangle) \right) 1_{\langle \eta, V^K(0) \rangle > 0} h_n(V^K(0)) \right] = \mathbb{E}_\pi \left[ 1_{\langle \eta, V^K(0) \rangle > 0} e^{-\langle \eta, V^K(0) \rangle} \right] > 0,
\]
where the strict inequality follows from the fact that the Lebesgue measure of set \( \{ z \in \mathbb{R}^d : \eta z^K > 0 \} \) is positive and the fact that \( \pi(A) > 0 \) for every measurable set \( A \) that has positive Lebesgue measure \( \mathbb{I} \). Therefore, we can find a large enough \( n \) such that
\[
\mathbb{E}_\pi \left[ (g_n^2(\langle \eta, V^K(0) \rangle) + g_n^\prime(\langle \eta, V^K(0) \rangle)) h_n(V^K(0)) \right] > 0.
\]
Thus, we arrive at
\[
\frac{1}{2} \sum_{i,j \in K} \eta_i \eta_j (Q \Sigma Q^T)^{(K,K)}_{ij} = 0.
\]
Namely,
\[
\frac{1}{2} \eta Q^{(K,J)} \Sigma (Q^T)^{(J,K)} \eta^T = 0.
\]
Since \( \Sigma \) is positive definite, for this to be true, we must have
\[
\eta Q^{(K,J)} = 0,
\]
thus proving (3.18).

We now return to the proof of Theorem 1. Because \( V^K(t) = Q^{(K,K)} Z^K(t) + Q^{K,L} Z^L(t) \) and \( Q^{(K,K)} \) is invertible by Lemma 3, we have
\[
Z^K(t) + W^K(t) = (Q^{(K,K)})^{-1} V^K(t) = (Q^{(K,K)})^{-1} V^K(0) + (Q^{(K,K)})^{-1} (Q X)^K(t) + (Q^{(K,K)})^{-1} Y^K(t),
\]
where
\[
W^K(t) = (Q^{(K,K)})^{-1} Q^{(K,L)} Z^L(t),
\]
and the second equality follows from (3.18). Note that
\[
(Q^{(K,K)})^{-1} (Q X)^K(t)
\]
is a \(|K|\)-dimensional Brownian motion with drift vector \( \tilde{\mu}(K) = (Q^{(K,K)})^{-1} (Q \mu)^K \) and covariance matrix
\[
\tilde{\Sigma}(K) = (Q^{(K,K)})^{-1} (Q \Sigma Q^T)^{(K,K)} ((Q^{(K,K)})^{-1})^T.
\]
We now apply Itô’s integral formula to \( f(Z^K(t) + W^K(t)) \), where \( f(x) \equiv e^{i(t \theta^K x)} \) with \( x \in \mathbb{R}^{|K|} \) for each fixed \( \theta^K \in \mathbb{R}^{|K|} \) and \( i = \sqrt{-1} \) is the imaginary unit of a complex number.
(We really apply the Itô formula twice, one for \( \cos(\langle \theta^K, x \rangle) \) and one for \( \sin(\langle \theta^K, x \rangle) \).) We have

\[
\begin{align*}
&f(Z^K(t) + W^K(t)) - f(Z^K(0) + W^K(0)) \\
&= -\frac{1}{2} \int_0^t \sum_{i,j \in K} \theta_i \theta_j \Sigma(K)_{ij} f(Z^K(u) + W^K(u)) du \\
&\quad + \epsilon \int_0^t \sum_{i \in K} \theta_i f(Z^K(u) + W^K(u)) d[(Q^{(K,K)})^{-1}((QX)^K)], \\
&\quad + \epsilon \int_0^t \sum_{i \in K} \theta_i f(Z^K(u) + W^K(u)) \sum_{j \in K} \tilde{R}(K)_{ij} dY_j(u),
\end{align*}
\]

where

\[
\tilde{R}(K) = (Q^{(K,K)})^{-1}.
\]

Because \(|f(Z^K(u) + W^K(u))| \leq 1\), setting \( t = 1 \), we can take expectation \( \mathbb{E}_\pi \) on both sides of this equation for \( \theta \in \mathbb{R}^d \), we have

\[
-\frac{1}{2} \sum_{i,j \in K} \theta_i \theta_j \Sigma(K)_{ij} \mathbb{E}_\pi(e^{i\langle \theta^K, Z^K(0) \rangle + i\langle \theta^K, W^K(0) \rangle})
\]

\[
+ \epsilon \sum_{i \in K} \theta_i \mathbb{E}_\pi(e^{i\langle \theta^K, Z^K(0) \rangle + i\langle \theta^K, W^K(0) \rangle})
\]

\[
+ \epsilon \sum_{i,j \in K} \theta_i \tilde{R}(K)_{ij} \mathbb{E}_\pi \left( \int_0^1 e^{i\langle \theta^K, Z^K(u) \rangle + i\langle \theta^K, W^K(u) \rangle} dY_j(u) \right) = 0. \tag{3.20}
\]

We now use the assumption that \( Z^K(0) \) and \( Z^L(0) \) are independent under \( \pi \). Since \( W^K(0) \) is a function of \( Z^L(0) \), \( Z^K(0) \) and \( W^K(0) \) are independent. Thus, applying Lemma 4 to (3.20), we have

\[
\left( -\frac{1}{2} \sum_{i,j \in K} \theta_i \theta_j \Sigma(K)_{ij} + \epsilon \sum_{i \in K} \theta_i \mathbb{E}_\pi(e^{i\langle \theta^K, Z^K(0) \rangle}) \right)
\]

\[
+ \epsilon \sum_{i,j \in K} \theta_i \tilde{R}(K)_{ij} \mathbb{E}_\pi \left( \int_0^1 e^{i\langle \theta^K, Z^K(u) \rangle} dY_j(u) \right) = 0. \tag{3.21}
\]

Denote the left-hand side of this equation by \( g(i\theta) \) as a function of \( i\theta \) for \( \theta \in \mathbb{R}^K \). If we replace \( i\theta \) by a complex vector \( z = (z_1, z_2, \ldots, z_{|K|}) \) such that \( \Re z_i \leq 0 \) for all \( i \in K \), where \( \Re z_i \) is the real part of \( z_i \). Obviously, \( g(z_1, z_2, \ldots, z_{|K|}) \) is analytic in each \( z_i \) such that \( \Re z_i < 0 \) when \( z_j \) for \( j \neq i \) is fixed satisfying \( \Re z_j \leq 0 \). Let \( i = 1 \) and fix an arbitrary \( \theta \in \mathbb{R}^{|K|} \). Since \( g(z_1, i\theta_2, \ldots, i\theta_{|K|}) \) converges to \( g(i\theta) \equiv 0 \) as \( z_1 \) with \( \Re z_1 < 0 \) continuously moves to \( i\theta_1 \), we must have

\[
g(z_1, i\theta_2, \ldots, i\theta_{|K|}) = 0 \text{ for } \Re z_1 \leq 0
\]
by the so called boundary uniqueness theorem (e.g., see page 371 of Volume I of [1]). We then inductively replace \( \vartheta_i \) by \( z_i \) with \( \Re z_i \leq 0 \) for \( i = 1, 2, \ldots, |K| \), and we have
\[
g(z_1, z_2, \ldots, z_{|K|}) = 0, \quad \Re z_i \leq 0 \quad \text{for} \quad i = 1, 2, \ldots, |K|.
\]
In particular, letting \( z_i = \theta_i \) for real \( \theta_i \leq 0 \) for \( i = 1, 2, \ldots, |K| \), we have
\[
\left( \frac{1}{2} \sum_{i,j \in K} \theta_i \theta_j [\Sigma(K)]_{ij} + \sum_{i \in K} \theta_i [\tilde{\mu}(K)]_i \right) \varphi^K(\theta^K) + \sum_{i,j \in K} \theta_i [\tilde{R}(K)]_{ij} \varphi^K_j(\theta^K) = 0. \tag{3.22}
\]
We are now ready to prove the remaining part of (a) in the following lemma.

**Lemma 4.** Under the assumptions of Theorem [1], \( \tilde{R}(K) \) and \( \tilde{R}(L) \) are completely-\( S \) matrices.

**Proof.** For an arbitrarily fixed \( \ell \in K \), let \( \theta^K_j = 0 \) in \( \theta^K \) for \( j \neq \ell \). We denote this vector \( \theta^K \) as \( (\theta^K, \ell) \). For \( j \neq \ell \), let \( \theta_j = 0 \) in (3.22) and divide the resulting formula by \( \theta_{\ell} < 0 \), then we have
\[
\frac{1}{2} \theta_{\ell} [\Sigma(K)]_{\ell \ell} + [\tilde{\mu}(K)]_{\ell} \varphi^K((\theta^K)_\ell) + \sum_{j \in K} [\tilde{R}(K)]_{\ell j} \varphi^K_j(\theta^K) = 0. \tag{3.23}
\]
Similarly to (3.22),
\[
- \lim_{\theta_{\ell} \to -\infty} \left( \frac{1}{2} \theta_{\ell} [\Sigma(K)]_{\ell \ell} + [\tilde{\mu}(K)]_{\ell} \right) \varphi^K((\theta^K)_\ell) = [\tilde{R}(K)]_{\ell \ell} \varphi^K_\ell(0^K).
\]
Since the left-hand side of this formula is positive by (3.22), its right-hand side must be positive. Hence, \( [\tilde{R}(K)]_{\ell \ell} > 0 \). Furthermore, we can take sufficiently small \( \theta_{\ell} < 0 \) such that
\[
\frac{1}{2} \theta_{\ell} [\Sigma(K)]_{\ell \ell} + [\tilde{\mu}(K)]_{\ell} \varphi^K((\theta^K)_\ell) < 0.
\]
By (3.22), we have, for this \( \theta_{\ell} \),
\[
\sum_{j \in K} [\tilde{R}(K)]_{\ell j} \varphi^K_j((\theta^K)_\ell) > 0. \tag{3.24}
\]
Since \( \varphi^K_j((\theta^K)_\ell) > 0 \) for all \( j \in K \), \( \tilde{R}(K) \) is an \( S \)-matrix. Let \( U \) be a subset of \( K \) such that \( \ell \in U \) and \( U \neq K \), then we can choose \( \theta_{\ell} \) such that \( \varphi^K_j((\theta^K)_\ell) \) is sufficiently small for \( j \in K \setminus U \). This yields that \( (\tilde{R}(K))^{(U,U)} \) is an \( S \)-matrix, and therefore we have proved that \( \tilde{R}(K) \) is a completely-\( S \) matrix.

We now can see that (3.22) is nothing but the BAR for the \( |K| \)-dimensional SRBM with data \( (\tilde{\Sigma}(K), [\tilde{\mu}(K), \tilde{R}(K))] \) because \( \tilde{R}(K) \) is completely-\( S \) by Lemma [1]. By part (b) of Lemma [1], the \( |K| \)-dimensional \((\tilde{\Sigma}(K), \tilde{\mu}(K), \tilde{R}(K))\)-SRBM has a stationary distribution that is equal to the distribution of \( Z^K(0) \) under \( \pi \). By the symmetric roles of \( K \) and \( L \), the \( |L| \)-dimensional \((\tilde{\Sigma}(L), \tilde{\mu}(L), \tilde{R}(L))\)-SRBM has a stationary distribution that is equal to the distribution of \( Z^L(0) \) under \( \pi \).
3.3 Proof of Theorem 2

First of all, we will prove if $Z^K(0)$ and $Z^L(0)$ are independent and $Z^K(0)$ is of product form under $\pi$, then (24) and (25) hold. According to Theorem 2, the distribution of $Z^{(i)}(0)$ under $\pi$ is equal to the stationary distribution of $(\tilde{\Sigma}(|i\rangle), \tilde{\mu}(|i\rangle), \tilde{R}(|i\rangle))$-SRBM for $i \in K$. The distribution of $Z^L(0)$ under $\pi$ is equal to the stationary distribution of $(\bar{\Sigma}(L), \bar{\mu}(L), \bar{R}(L))$-SRBM. Then by Lemma 1, for $\theta \in \mathbb{R}^d$ with $\theta \leq 0$,

$$- \left( \frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j \tilde{\Sigma}(L)_{ij} + \sum_{i \in L} \theta_i [\bar{\mu}(L)]_i \right) \varphi_L^1(\theta) = 1, \quad i \in K, \quad (3.25)$$

$$- \left( \frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j \tilde{\Sigma}(L)_{ij} + \sum_{i \in L} \theta_i [\bar{\mu}(L)]_i \right) \varphi_L^{(1)}(\theta) = \sum_{i,j \in L} \theta_i [\bar{R}(L)]_{ij} \varphi_L^1(\theta). \quad (3.26)$$

By the definition of $\tilde{\Sigma}(L), \bar{\mu}(L)$ and $\bar{R}(L)$, we can find they are

$$\tilde{\Sigma}(L) = \Sigma^{(L,L)} + R^{(L,K)}(R^{(K,K)} \Sigma^{(K,K)} R^{(K,K)})^{-1} \Sigma^{(K,K)} \Sigma^{(K,K)} (R^{(K,K)} \Sigma^{(K,K)} R^{(K,K)})^{-1} \Sigma^{(K,K)},$$

$$\bar{\mu}(L) = \mu^L - R^{(L,K)}(R^{(K,K)} \mu^L - R^{(K,K)} \mu^L) - 1 \Sigma^{(L,K)} \Sigma^{(L,K)},$$

$$\bar{R}(L) = R^{(L,L)}.$$

By the independence assumptions, we have, for $i \in K$,

$$\varphi(\theta) = \varphi_i^{(1)}(\theta_i) \varphi_j^{(1)}(\theta_j^\perp(i)) = \varphi_K(\theta^K) \varphi_L(\theta^L).$$

By Lemma 1, we get

$$\varphi_i(\theta) = \varphi_i^{(1)}(\theta_i), \quad i \in K, \quad \varphi_j(\theta) = \varphi_j^{(1)}(\theta_j^\perp(i)) = \varphi_j^{(1)}(\theta_j^\perp), \quad j \in L.$$

Using the fact that $\varphi_i^{(1)}(0) = 1$, we can rewrite (3.24) and (3.25) as

$$- \left( \frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j \tilde{\Sigma}(L)_{ij} + \sum_{i \in L} \theta_i [\bar{\mu}(L)]_i \right) \varphi(\theta) = \varphi_i(\theta), \quad i \in K, \quad (3.27)$$

$$- \left( \frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j \tilde{\Sigma}(L)_{ij} + \sum_{i \in L} \theta_i [\bar{\mu}(L)]_i \right) \varphi(\theta) = \sum_{i,j \in L} \theta_i [\bar{R}(L)]_{ij} \varphi_i(\theta). \quad (3.28)$$

We further modify (3.27) into

$$- \left( \frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j \tilde{\Sigma}(L)_{ij} + \sum_{i \in L} \theta_i [\bar{\mu}(L)]_i \right) \varphi(\theta) = \gamma_i(\theta) \varphi_i(\theta). \quad (3.29)$$

Then adding (3.28) for $i \in K$ and (3.29), we have

$$- \left( \sum_{i \in K} \frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j \tilde{\Sigma}(L)_{ij} + \sum_{i \in L} \theta_i [\bar{\mu}(L)]_i \right) \gamma_i(\theta)$$

$$+ \frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j \tilde{\Sigma}(L)_{ij} + \sum_{i \in L} \theta_i [\bar{\mu}(L)]_i \right) \varphi(\theta) = \sum_{i=1}^d \gamma_i(\theta) \varphi_i(\theta). \quad (3.30)$$
So according to Lemma 1, we have
\[
\gamma(\theta) = -\left( \sum_{i \in K} \left( \frac{1}{2} \Sigma_{ii} \theta_i + \left( (R^{(K,K)})^{-1} \mu^{K}\right)_i \right) \gamma_i(\theta) \right) + \frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j \Sigma(L)_{ij} + \sum_{i \in L} \theta_i [\mu(L)]_i \right).
\] (3.31)

Comparing the coefficients of \(\theta_i \theta_j\) and coefficients of \(\theta_i\) of both sides, we can get
\[
-\frac{1}{2} (\Sigma_{ij} + \Sigma_{ji}) = -\frac{\Sigma_{ii}}{2R_{ii}} R_{ij}, \quad i, j \in K, \quad (3.32)
\]
\[
-\frac{1}{2} (\Sigma_{ij} + \Sigma_{ji}) = -\frac{\Sigma_{ij}}{2R_{ii}} R_{ij}, \quad i \in K, \quad j \in L, \quad (3.33)
\]
\[
R^{(L,K)}(R^{(K,K)})^{-1} \Sigma^{(K,K)}(R^{(K,K)})^{-1}_\top (R^{(L,K)})_\top - \Sigma^{(L,K)}((R^{(K,K)})^{-1})(R^{(L,K)})_\top - R^{(L,K)}(R^{(K,K)})^{-1}(\Sigma^{(L,K)})_\top = 0. \quad (3.34)
\]

Observe that (3.29) is equivalent to (3.26), and (3.28) is equivalent to (3.25). Under (3.26) and (3.25), (3.31) automatically holds. So we have proved if \(Z^K(0)\) and \(Z^L(0)\) are independent and \(Z^K(0)\) is of product form under \(\pi\), then (3.26) and (3.25) hold.

Next we will prove that if \(\Sigma\) and \(R\) satisfy (3.25) and (3.26), and the \(|L|\)-dimensional \((\Sigma^{(L,L)}, \hat{\mu}(L), R^{(L,L)})\)-SRBM has a stationary distribution, then \(Z^K(0)\) and \(Z^L(0)\) are independent and \(Z^K(0)\) is of product form under \(\pi\). First observe that if (3.26) and (3.25) holds, then (3.24), (3.23) and (3.22) hold. Therefore, (3.24) holds.

Because the skew symmetry condition (3.22) holds, the \((\Sigma^{(K,K)}, \mu^{(K,K)}, R^{(K,K)})\)-SRBM has a product form stationary distribution [26]. Let \(\tilde{\varphi}^K(\theta^K)\) and \(\varphi^K_i(\theta^K)\) be the moment generating functions of the stationary distribution and 1th boundary measure for this \((\Sigma(K), \mu(K), R(K))\)-SRBM. Then
\[
-\left( \frac{1}{2} \Sigma_{ii} \right) \gamma_i(\theta) + \left( (R^{(K,K)})^{-1} \mu^{K}\right) \gamma_i(\theta) = 1, \quad i \in K.
\]

Because the \(|L|\)-dimensional \((\Sigma^{(L,L)}, \hat{\mu}(L), R^{(L,L)})\)-SRBM has a stationary distribution, \(\Sigma(L) = \Sigma^{(L,L)}\) and \(\hat{R}(L) = R^{(L,L)}\), we have
\[
-\left( \frac{1}{2} \sum_{i,j \in L} \theta_i \theta_j \Sigma(L)_{ij} + \sum_{i \in L} \theta_i [\hat{\mu}(L)]_i \right) \varphi^L(\theta^L) = \sum_{i,j \in L} \theta_i [\hat{R}(L)]_{ij} \tilde{\varphi}^L_j(\theta^L).
\]

where \(\tilde{\varphi}^L(\theta^L)\) and \(\varphi^L_j(\theta^L)\) are the moment generating functions of the stationary distribution and 1th boundary measure for \((\Sigma^{(L,L)}, \hat{\mu}(L), R^{(L,L)})\)-SRBM.

Let \(\tilde{\varphi}(\theta) = \tilde{\varphi}^K(\theta^K)\tilde{\varphi}^L(\theta^L), \hat{\varphi}_i(\theta) = \varphi^K_i(\theta^K)\varphi^L_i(\theta^L)\) for \(i \in K\) and \(\hat{\varphi}_j(\theta) = \varphi^K_j(\theta^K)\tilde{\varphi}^L_j(\theta^L)\) for \(j \in L\). Then we can see (3.21), (3.20), (3.19) and (3.18) holds with \(\varphi(\theta), \varphi_i(\theta)\) and \(\varphi_j(\theta)\) replaced by \(\tilde{\varphi}(\theta), \hat{\varphi}_i(\theta)\) and \(\hat{\varphi}_j(\theta)\). Furthermore, as (3.17) holds, we conclude
\[
\gamma(\theta)\tilde{\varphi}(\theta) = \sum_{i=1}^d \gamma_i(\theta)\hat{\varphi}_i(\theta)
\]

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By part (b) of Lemma 1, we know \( \varphi(\theta) = \tilde{\varphi}(\theta) \), \( \varphi^K(\theta^K) = \tilde{\varphi}^K(\theta^K) \) and \( \varphi^L(\theta^L) = \tilde{\varphi}^L(\theta^L) \). So \( \varphi(\theta) = \varphi^K(\theta^K)\varphi^L(\theta^L) \), that is, \( Z^K(0) \) and \( Z^L(0) \) are independent under \( \pi \). Furthermore, the distribution of \( Z^K(0) \) under \( \pi \) is equal to the stationary distribution of the \(|K|\)-dimensional \((\Sigma^{(K,K)}, \mu^K, R^{(K,K)})\)-SRBM. By (4.1), the distribution of \( Z^K(0) \) is of product form because the skew symmetry condition (13) in [13] is satisfied.

4 Concluding remarks

There are two directions for future study. We first comment on the marginal distributions. In the proof of Theorem 1, we may only use the following fact to complete the proof. Random vectors \( Z^K(0) \) and \( W^K(0) \equiv (Q^{(K,K)})^{-1}Q^{(K,L)}Z^L(0) \) are “weakly independent through convolution” under the stationary distribution \( \pi \), that is, for all \( \theta \in \mathbb{R}^{|K|} \),

\[
E(e^{i(\theta,Z^K(0)+W^K(0))}) = E(e^{i(\theta,Z^K(0))})E(e^{i(\theta,W^K(0))}),
\]

where \( i = \sqrt{-1} \) in again the imaginary unit. We also can observe from Remark 2 that this is indeed weaker than the independence of \( Z^K(0) \) and \( Z^L(0) \) under \( \pi \). Thus, it may be interesting to consider the following questions.

**Question 1.** What is a class of SRBM satisfying (4.1) ? How can we characterize this class in terms of the modeling primitives ? How much is it larger than the class satisfying the decomposability ?

**Question 2.** Can the stationary distributions of \( Z^K(0) \) and \( Z^L(0) \) serve good approximations for the marginal distributions of the original stationary distribution when (4.1) does not hold ? If not, for what class of SRBM can they provide good approximations ?

Obviously, these two questions are closely related. Question 1 is hard to answer while Question 2 may be studied through numerical experiments. Furthermore, for \( d = 2 \), we know that the tail asymptotics of the one-dimensional marginals and when their tail decay rates are identical with \( \theta_{(i,r)} \) defined in (4.3) (see Theorem 2.2 and 2.3 of [3]). This may suggest the class of SRBM for which the product form approximation using \( \theta_{(i,r)} \) is reasonable. Unfortunately, we do not have any explicit results yet for the tail decay rates for \( d \geq 3 \) except for some special cases. We hope that this tail decay rate problem will be solved sometime in the future, and Question 1 will be better answered then.

Another question is about sufficient conditions for the decomposability. We partially answered this question by Theorem 1. It seems hard to extend the arguments in the proof of this theorem to more general cases. Such an extension is a challenging open problem.

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A Some matrix classes

We give definitions of some matrix classes in our arguments. One can find them in text books for matrices (e.g., see [2]). Let $A$ be an $n$-dimensional square matrix. Then, $A$ is called an $S$-matrix if there is an $n$-dimensional vector $v > 0$ such that $Av > 0$, where, for a vector $u$, $u > 0$ means each component of $u$ is strictly positive. If all principal sub-matrices of $A$ are $S$-matrices, then $A$ is called a completely-$S$ matrix. If all principal minors of $A$ are positive, then $A$ is called a $P$-matrix. Every $P$-matrix is an $S$-matrix (e.g, see Corollary 3.3.5 of [2]). A $P$-matrix is called an $M$-matrix if its off diagonal entries are all non-positive (see Definition 3.11.1 of [2] for a $K$-matrix, which is another name of an $M$-matrix used in [2, Section 3.13.24]).
B Proof of Corollary 1

To prove the corollary, we introduce some geometric objects:

\[ E = \{ \theta \in \mathbb{R}^d; \gamma(\theta) = 0 \}, \]
\[ P^{(i)} = \cap_{k \in J \setminus \{i\}} \{ \theta \in \mathbb{R}^d; \gamma_k(\theta) = 0 \}, \quad i \in J. \]

The object \( E \) is an ellipse in \( \mathbb{R}^d \). Since \( R \) is invertible and \( \theta \in P^{(i)} \) implies that \( (\theta, R^{(k)}) = 0 \) for \( k \neq i \), \( P^{(i)} \) must be a line going through the origin. Clearly, for each \( i \), \( P^{(i)} \) intersects the ellipse \( E \) by at most two points, one of which is the origin. We denote its non-zero intersection by \( (i; r) \) if it exists. Otherwise, let \( (i; r) = 0 \). The following lemma shows that the latter is impossible by giving an explicit formula for \( (i; r) \). Recall that \( (Q^T)^{(i)} \) be the \( i \)th column of \( Q^T \). We refer to the following fact, obtained as Lemma 1 of [9].

Lemma 5. For each \( i \in J \),
\[ \theta^{(i,r)} = \Delta_i (Q^T)^{(i)}, \quad (B.1) \]
where \( \Delta_i > 0 \) is defined in (B.2).

of Corollary 1. It follows from [10] that the stationary distribution of a one-dimensional SRBM with drift \( \mu < 0 \) and variance \( \sigma^2 \) is exponential with mean \( 1/\lambda \), where
\[ \lambda = -\frac{2\mu}{\sigma^2}. \quad (B.2) \]

We apply Theorem 1 to \( K = \{i\} \). According to the theorem, \( Z^{(i)}(0) \) under \( \pi \) is a one-dimensional SRBM with variance \( \Sigma(\{i\}) \) and drift \( \tilde{\mu}(\{i\}) \). Set
\[ \lambda_i = -\frac{2\tilde{\mu}(\{i\})}{\Sigma(\{i\})}. \]

By (B.1) and (B.2), to prove the corollary it suffices to verify that
\[ \lambda_i = \theta^{(i,r)}_i. \quad (B.3) \]

We first compute \( Q \) and \( Q^{(\{i\},\{i\})} \) for this. By Lemma 4, we have
\[ Q = (\Delta_1^{-1} \theta^{(1,r)}, \Delta_2^{-1} \theta^{(2,r)}, \ldots, \Delta_d^{-1} \theta^{(d,r)})^T, \]
and therefore
\[ Q^{(\{i\},\{i\})} = \Delta_i^{-1} \theta^{(i,r)}_i. \]

Hence,
\[ \hat{\Sigma}(\{i\}) = \Delta_i^2 (\theta^{(i,r)}_i)^{-2} \sum_{j,k} [Q]_{ij} \Sigma_{jk} [Q]_{ik} \]
\[ = \Delta_i^2 (\theta^{(i,r)}_i)^{-2} \sum_{j,k} \Delta_j^{-1} \theta^{(j,r)}_j \Sigma_{jk} [Q]_{ik} \Delta_k^{-1} \theta^{(j,r)}_k \]
\[ = (\theta^{(i,r)}_i)^{-2} \langle \theta^{(i,r)}, \Sigma \theta^{(i,r)} \rangle \]
\[ = -2(\theta^{(i,r)}_i)^{-2} \langle \theta^{(i,r)}, \mu \rangle, \]
where the last equality is obtained since $\gamma(\theta^{(i,r)}) = 0$. Similarly, we have
\[
\tilde{\mu}(\{i\}) = (\theta_i^{(i,r)})^{-1} \langle \theta^{(i,r)}, \mu \rangle.
\]
Hence,
\[
\lambda_i = -\frac{2\tilde{\mu}(\{i\})}{\Sigma(\{i\})} = \theta_i^{(i,r)}.
\]
This completes the proof of Corollary $\blacksquare$. $\square$