Simple nonparametric inference for first-price auctions via bid spacings

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Abstract

In a classic model of the first-price auction, we propose a nonparametric estimator of the quantile function of bidders’ valuations, based on weighted bid spacings. We derive the Bahadur-Kiefer expansion of this estimator with a pivotal influence function and an explicit uniform remainder rate. This expansion allows us to develop a simple algorithm for the associated uniform confidence bands that does not rely on bootstrap. Monte Carlo experiments show satisfactory statistical and computational performance of the estimator and the confidence bands. Estimation and inference for related functionals is also considered.

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Keywords: first-price auction, uniform inference, quantile density, spacings, Bahadur-Kiefer expansion, strong approximation

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1 Introduction

In the empirical studies of first-price auctions, a structural approach to estimation and inference is often used. This approach exploits restrictions derived from economic theory to assess bidders’ preferences (valuations) from the observed bids. With these valuations at hand, the researcher can make predictions about the effects of changes in auction rules or composition of bidders. Various methods for recovering the valuations, in both parametric and nonparametric frameworks, have been developed, see, e.g., Paarsch et al. (2006), Athey and Haile (2007), and Perrigne and Vuong (2019) for an overview.

In this paper we focus on a special class of these methods that exploits weighted sums of the first differences of ordered bids, often referred to as bid spacings. Since spacings are increments of the empirical quantile function, our work is related to the rapidly emerging literature on quantile methods in first-price auctions, see Marmer and Shneyerov (2012), Ma et al. (2019) for kernel-based estimators, Luo and Wan (2018) for isotone regression-based estimators, and Guerre and Sabbah (2012), Gimenes and Guerre (2021) for local polynomial estimators. More generally, spacings have been used for collusion detection in Ingraham (2005), for set identification of bidders’ rents in Paul and Gutierrez (2004), Marra (2020), and in the prior-free clock auction design in Loertscher and Marx (2020).

Our primary contribution is the analysis of a novel nonparametric estimator \( \hat{v}_h(u) \) of the quantile function \( v(u) \) of the distribution of valuations,

\[
\hat{v}_h(u) = \hat{Q}(u) + u \cdot \hat{q}_h(u)/(m - 1),
\]

\[
\hat{q}_h(u) = \int_0^1 K_h(u - z) \, d\hat{Q}(z), \quad K_h(z) = K(z/h)/h.
\]

Here \( K \) is a compactly supported kernel, \( h > 0 \) is a bandwidth, \( \hat{Q}(u) \) is the empirical quantile function of observed bids, and \( m \) is the number of bidders in each auction in the sample. At the center of this approach is \( \hat{q}_h(u) \), a kernel estimator of the bid quantile density \( q(u) = Q'(u) \), which takes the form of a weighted sum of bid spacings,

\[
\hat{q}_h(u) = \frac{1}{n} \sum_{i=1}^{n} K_h(u - i/n) \left( b_{(i)} - b_{(i-1)} \right),
\]

where \( b_{(i)} \) is the \( i \)-th smallest bid, \( b_{(0)} = 0 \) is the theoretical lower bound of bids, and \( n \) is the total number of bids in the sample. This estimator was previously studied by Siddiqui (1960), Bloch and Gastwirth (1968) for the case of rectangular kernel and by Csörgő et al. (1991), Falk (1986), Welsh (1988) and Jones (1992) for a general kernel. However, to the best of our knowledge, we are the first to provide its complete first-order asymptotic analysis.
and use it in the context of first-price auctions.

In particular, we construct uniform confidence bands around the functional estimand \( v(u) \). For this purpose, we derive the uniform asymptotically linear (Bahadur-Kiefer, or BK for short) expansion of our estimator \( \hat{v}_h \),

\[
\sqrt{n h} \hat{v}_h(u) - v(u) = -\frac{1}{\sqrt{n h}} \sum_{i=1}^{n} \left[ K\left( \frac{u - F(b_i)}{h} \right) - \mathbb{E}K\left( \frac{u - F(b_i)}{h} \right) \right] + R_n(u),
\]

where \( F \) is the CDF of bids, and \( R_n(u) \) converges to zero uniformly with an explicit rate. The main (linear) term of this expansion is the kernel density estimator on IID uniform data \( F(b_1) \ldots, F(b_n) \), with the same kernel and bandwidth as \( \hat{q}_h \). This fact has two major implications: (1) the estimator \( \hat{v}_h(\cdot) \), despite converging to a Gaussian distribution pointwise, does not converge uniformly; and (2) the linear term is known and pivotal, for any given \( n \). To deal with lack of uniform convergence (1), we use the Gaussian coupling of Rio (1994) along with the anti-concentration theory of Chernozhukov et al. (2013) to show that the quantiles of the linear term are good approximations of the true quantiles of \( \hat{v}_h(\cdot) \), see Theorem 3. The implication (2) then allows us to develop the following simulation-based algorithm for the construction of the uniform confidence bands.

First, compute \( \hat{q}_h(u) \) and \( \hat{v}_h(u) \) with a compact kernel \( K \) and bandwidth \( h \) of your choice. Second, draw \( N \) independent samples of size \( n \), as in the original data, of uniformly distributed observations (pseudo-bids). Third, for each sample \( k = 1, \ldots, N \), compute

\[
W_{n,U}^k := \sup_{u} \frac{|\hat{q}_h^k(u) - 1|}{\sqrt{n h}},
\]

where \( \hat{q}_h^k(u) \) is the quantile density estimator, calculated using the pseudo-bids, with the same kernel and bandwidth as the original \( \hat{q}_h(u) \). Fourth, for a desired confidence level \( 1 - \alpha \), set the critical value \( \hat{c}_{1-\alpha/2} \) equal to the \( (1 - \alpha/2) \)-th percentile of \( \{W_{n,U}^k\}_{k=1}^{N} \). Finally, a valid confidence band with the confidence level \( 1 - \alpha \) is

\[
v(u) = \hat{v}_h(u) \pm \frac{\hat{q}_h(u)\hat{c}_{1-\alpha/2}}{\sqrt{n h}}.
\]

Note that this confidence band uses the original choice of kernel and bandwidth and only depends on the data through \( \hat{q}_h(u) \). Our Monte Carlo simulations confirm that the simulated coverage converges to the nominal one, for realistic bid distributions and various sample sizes, see Section 5.

An obvious bottleneck in constructing the estimator \( \hat{v}_h(u) \), both conceptually and programmatically, is computation of the estimator of the bid quantile density \( \hat{q}_h(u) \). We bench-
mark \( \hat{q}_h(\cdot) \) against its main competitor, the reciprocal of the kernel estimator \( \hat{f}_l(\cdot) \) of the bid density function \( f(b) = F'(b) \), as in the first step of the procedure of Guerre et al. (2000),

\[
\hat{f}_l(b) = \int_{-\infty}^{\infty} K_l(b-z)d\hat{F}(z) = \frac{1}{nl} \sum_{i=1}^{n} K\left(\frac{b-b_i}{l}\right),
\]

where \( \hat{F}(b) \) is the empirical CDF of the bids and \( l \) is the bandwidth. To make the comparison fair, we pick a standard Silverman rule-of-thumb bandwidth \( l^{\text{rot}} \) for the bid density, and match it with a rule-of-thumb bandwidth \( h^{\text{rot}} \) for the bid quantile density,

\[
h^{\text{rot}} := l^{\text{rot}}/(b_{(n)} - b_{(0)}),
\]

motivated by the scale match-up bandwidth of Jones (1992). With these bandwidths, we argue that the reciprocal kernel density and the quantile density estimators have similar performance in terms of integrated mean square error (IMSE), but the latter is superior in terms of running time, see Section 5.

Our paper is most closely related to Luo and Wan (2018) in that we focus on the value quantile function, instead of the value density function; the latter was historically in the spotlight, following the seminal works by Guerre et al. (2000) and Li et al. (2000). However, it is up to debate, which of these two objects is an appropriate target of structural estimation. In fact, the researcher is more likely to be interested in a certain scalar metric, such as bidder’s expected utility or total expected surplus, rather than the value distribution itself. This motivates the following companion result.

Consider a functional

\[
T(\varphi) = \int_{0}^{1} \varphi(u)dQ(u),
\]

where \( \varphi(u) \) is a smooth weighting function. For example, the bidders’ expected surplus corresponds to \( \varphi(u) = \frac{mu^m}{m-1} \) and the total expected surplus corresponds to \( \varphi(u) = 1 + \frac{u^m}{m-1} \), see Proposition 1. We propose a plug-in estimator of \( T(\varphi) \),

\[
\hat{T}(\varphi) = \int_{0}^{1} \varphi(u) d\hat{Q}(u) = \sum_{i=1}^{n} \varphi(i/n) \left( b_{(i)} - b_{(i-1)} \right)
\]

and show that it is asymptotically normal with zero mean and variance

\[
V(T(\varphi)) = \text{Var} \int_{0}^{1} \varphi'(u)dQ(u),
\]
where $U$ is a uniform[0,1] random variable, see Proposition 2. The asymptotic variance $V(T(\varphi))$ admits a consistent plug-in\(^1\) estimator

$$
\hat{V}(T(\varphi)) = \frac{1}{n} \sum_{i=1}^{n} \left[ \sum_{j=i}^{n} \varphi'(j/n) (b(j) - b(j-1)) - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=i}^{n} \varphi'(j/n) (b(j) - b(j-1)) \right]^2,
$$

see Proposition 3.

The rest of the paper is organized as follows. In Section 2 we briefly overview the equilibrium analysis of first price auctions. In Sections 3 and 4 we derive the theoretical results needed for inference on the value quantile function $v(u)$, and benchmark its performance in Section 5. Section 6 covers estimation and inference on functionals. Section 7 concludes the paper. Proofs of theoretical results and codes in Python are provided in the Appendix.

## 2 Framework

Our setting is a sealed-bid first-price auction with independent private values. In such an auction, there are $m$ ex-ante identical, risk-neutral bidders. A bidder’s latent valuation $v$ of the object being sold is drawn from the CDF $G(v)$, with the associated density $g(v)$ supported on the interval $[0, \bar{b}]$. In the unique symmetric Bayes-Nash equilibrium, the bidder finds it optimal to submit a bid $b = b(v)$, where $b(v)$ is the bidding strategy

$$b(v) = v - \int_{0}^{v} G(z)dz / G(v).
$$

On the other hand, if $F(b)$ is the equilibrium bid distribution with the associated density $f(b)$, the equilibrium can be characterized by the inverse bidding strategy

$$v(b) = b + \frac{F(b)}{(m-1)f(b)}, \tag{1}
$$

allowing to recover the latent valuations from the observed bids. This suggests a nonparametric estimation approach that was popularized, most notably, by Guerre et al. (2000) and Li et al. (2000).

An alternative approach relies on reformulating the first order conditions in terms of quantiles of bids. Denoting $\tilde{v}(u) = v(Q(u))$, we obtain

$$
\tilde{v}(u) = Q(u) + q(u) \cdot u / (m-1), \tag{2}
$$

\(^1\)See Remark 4 for intuition behind this estimator.
where \( Q(u) \) is the **bid quantile function** and \( q(u) := \frac{dQ}{du}(u) \) is the associated **bid quantile density**. Formula (2) follows from formula (1) due to the identities \( F(Q(u)) = u \) and \( f(Q(u))q(u) = 1 \). The advantage of this approach was demonstrated, among others, by Marmer and Shneyerov (2012), Ma et al. (2019) and Gimenes and Guerre (2021).

While the functions \( \tilde{v}(u), v(b), b(v), G(v) \) and \( g(v) \) are all natural targets for estimation, in practice the focus is often on certain scalar metrics of auction performance, such as bidders’ expected surplus and total expected surplus. The following simple proposition shows that these metrics can be rewritten in the quantile form, when there is no reserve price.\(^2\)

**Proposition 1** (Quantile versions of bidders’ and total expected surplus).

\[
\text{bidder’s expected surplus} = \int \left( \frac{m^u}{m-1} \right) dQ(u), \tag{3}
\]

\[
\text{total expected surplus} = \int \left( 1 + \frac{u^m}{m-1} \right) dQ(u). \tag{4}
\]

We will return to estimation and inference for the integral functionals of this type in Section 6.

The observed data consists of bids \( b_1, \ldots, b_n \) from \( k \) independent auctions of \( m \) bidders, where \( n = mk \). We assume that there is no auction heterogeneity. Throughout the rest of the paper, we write \( v(u) \) instead of \( \tilde{v}(u) \), with a convenient abuse of notation.

### 3 Bahadur-Kiefer representation

In this section we derive the Bahadur-Kiefer (i.e. uniform asymptotically linear) representation of our estimator of the form

\[
\sqrt{n}h(\hat{v}_h(u) - v(u)) = \frac{1}{n} \sum_{i=1}^{n} [\varphi_n(b_i; u) - \mathbb{E}\varphi_n(b_i; u)] + R_n(u), \tag{5}
\]

where \( \varphi_n \) is a measurable function (called **influence function**) and the remainder \( R_n(u) \) converges to zero a.s. with an explicit uniform rate.

This representation serves two purposes. First, it directly implies the asymptotic distribution of \( \hat{v}_h(u) \) at any given point \( u \in (0, 1) \), as well as rates of pointwise and uniform consistency.\(^3\) Second, it is a first step in establishing a simple procedure for computing uni-

\(^2\)For a positive binding reserve price, similar formulas can be obtained, see Krishna (2009).

\(^3\)We do not explicitly write out the rate of uniform consistency of our estimator. Our Bahadur-Kiefer expansion implies that this rate is equal to that of the kernel density estimator, which is readily available from the literature (e.g., Silverman, 1978).
form confidence bands around \( v(\cdot) \). The crucial feature of the influence function \( \varphi_n(b;u) \) is that it only depends on \( b_i \) through \( F(b_i) \), which is a uniform \([0,1]\) random variable; in other words, the distribution of the linear term in (5) is pivotal. We use this feature, along with existing results on strong approximations and anti-concentration inequalities for Gaussian processes, to establish validity of confidence bands in the next section.

We make the following assumptions.

**Assumption 1** (Kernel function).

1. \( K : \mathbb{R} \rightarrow \mathbb{R} \) is a nonnegative function such that
   \[
   \int_{\mathbb{R}} K(z) \, dz = 1, \quad R_K = \int_{\mathbb{R}} K(z)^2 \, dz < \infty. \tag{6}
   \]

2. \( K \) is a Lipschitz function supported on the interval \([-1,1]\).

**Assumption 2** (Distribution of bids).

1. The distribution of bids is supported on a bounded interval \([0,\bar{b}]\) and admits a density \( f \) that is bounded away from zero on \([0,\bar{b}]\).

Assumption 1.1 states that \( K \) is a valid PDF. Assumption 1.2 is standard in the literature on strong approximations of local empirical processes (see, e.g., Rio, 1994). In particular, it implies that \( K \) is a function of bounded variation, which is crucial in our derivation of the BK expansion. Assumption 2.1 allows us to avoid boundary issues; we believe it be dispensed with to a certain extent at a cost of boundary bias correction and more cumbersome notation.

In our proof, we rely on the classic Bahadur-Kiefer expansion of the quantile function (Bahadur, 1966; Kiefer, 1967),

\[
\hat{Q}_n(u) - Q(u) = -q(u) \left( \hat{F}(Q(u)) - u \right) + R_n(u), \tag{7}
\]

where \( R_n(u) = O_{a.s.} \left( n^{-3/4} l(n) \right) \) uniformly in \( u \in (0,1) \). \tag{8}

Here \( l(n) = (\log n)^{1/2}(\log \log n)^{1/4} \) is a logarithmic offset factor that arises due to the uniform nature of the approximation and may often be disregarded in practice. Note that the BK expansion represents a nonlinear estimator \( \hat{Q}_n(u) \) as a sum of a linear estimator \( \hat{F}_n(Q(u)) \) and a remainder \( R_n(u) \) that converges to zero a.s. uniformly at a nonparametric (slow) rate \( n^{-3/4} \).
Denote the (bandwidth-dependent) population analogs of \(\hat{q}_h\) and \(\hat{v}_h\) by

\[
q_h(u) = \int_0^1 K_h(u - z) dQ(z),
\]

\[
v_h(u) = Q(u) + u \cdot q_h(u)/(m - 1).
\]

(9) (10)

**Theorem 1** (Bahadur-Kiefer expansion for estimator of valuation function). Under Assumptions 1 and 2, the estimator \(\hat{v}_h(u)\) has the representation

\[
Z_n(u) = Z_n^*(u) + R_n(u),
\]

(11)

where

\[
Z_n(u) = \sqrt{nh} \frac{\hat{v}_h(u) - v_h(u)}{q(u)},
\]

\[
Z_n^*(u) = -\frac{1}{\sqrt{nh}} \sum_{i=1}^n \left[ K \left( \frac{u - F(b_i)}{h} \right) - \mathbb{E} K \left( \frac{u - F(b_i)}{h} \right) \right],
\]

(12) (13)

\[
R_n(u) = O_{a.s.} \left( \sqrt{h} + h \log h + \frac{1}{2} h^{1/4} l(n) \right) \text{ uniformly in } u \in [0, 1].
\]

(14)

**Remark 1** (BK expansion for quantile density). The statement of the theorem remains true if the centered estimator \(\hat{v}_h(u) - v_h(u)\) on left-hand side of (11) is replaced by the centered quantile density \(\hat{q}_h(u) - q_h(u)\).

**Remark 2** (Bandwidth rate selection). The above expansion is centered around \(v_h\), a natural population analog of \(\hat{v}_h\). However, we are more interested in performing inference on \(v\) rather than \(v_h\). The standard way to deal with this issue is to undersmooth, i.e. choose a (suboptimally) small bandwidth to eliminate the asymptotic difference between the two estimands. Namely, since \(q_h(u) - q(u) = O(h)\), we need \(\sqrt{nh} \cdot h = o(1)\) or \(h = o(n^{-1/3})\) to achieve undersmoothing. Hence, as long as this condition is satisfied, \(v_h\) can be replaced by \(v\) in the expansion (11). Conversely, if the rate is larger than \(n^{-1/3}\), as in the case of Silverman’s rule-based bandwidth \(h^{\text{rot}} = O\left(n^{-1/5}\right)\), the confidence bands will be centered at \(v_h\) rather than \(v\). This conflict between optimal estimation and uniform inference is a feature of most nonparametric estimators, see Horowitz (2001) and Hall (2013).

**Remark 3.** Since the kernel \(K\) is supported on \([-1,1]\) and \(F(b_i) \sim \text{Uniform}[0,1]\), we have

\[
\mathbb{E} K \left( \frac{u - F(b_i)}{h} \right) = 1 \text{ on the interval } u \in [h, 1 - h].
\]

An unusual and important feature of this representation is that the linear term is *pivotal*. Indeed, it only depends on the data through \(F(b_1), \ldots, F(b_n)\), which are independent
uniform[0,1] random variables. Moreover, the influence function does not depend on any unknown quantities. These two properties will allow us to develop a simple simulation procedure to construct confidence intervals and uniform confidence bands in the next section.

4 Confidence intervals and bands

Theorem 1 allows us to construct pointwise confidence intervals and uniform confidence bands for the value quantile function. In particular, the following theorem establishes the pointwise asymptotic distribution of the value quantile function.

**Theorem 2.** If \( h \to 0 \) and \( nh \to \infty \), then, for every \( u \in (0, 1) \),

\[
Z_n(u) = \sqrt{nh} \left( \hat{v}_h(u) - q(u) \right) \rightsquigarrow N(0, R_K),
\]

where \( R_K = \int_{-\infty}^{\infty} K^2(x) \, dx \).

Special cases of this theorem (in the abstract framework of quantile density estimation) were derived by Siddiqui (1960) and Bloch and Gastwirth (1968). The theorem implies that a confidence interval of nominal confidence level \( (1 - \alpha) \) for \( v_h(u) \) can be constructed as

\[
\left[ \hat{v}_h(u) - q(u) \sqrt{nh} z_{1 - \frac{\alpha}{2}}, \hat{v}_h(u) + q(u) \sqrt{nh} z_{1 - \frac{\alpha}{2}} \right],
\]

where \( z_{1 - \frac{\alpha}{2}} \) is the standard normal quantile of level \( 1 - \frac{\alpha}{2} \).

We now turn to the problem of uniform inference on \( v(\cdot) \).

If the process \( Z_n \) converged weakly in \( L^\infty \) (i.e. uniformly) to a known or estimable process, this would have enabled a standard way to construct asymptotically valid confidence bands. Unfortunately, although \( Z_n(u) \) is asymptotically Gaussian at each point \( u \in (0, 1) \), it does not converge uniformly. This follows from the fact that the main term in the BK expansion is the normalized kernel density estimator, which is known to lack uniform convergence (see, e.g., Rio, 1994).\(^4\) In such a case, there are two common ways to circumvent this problem and derive valid confidence intervals.

First, one could try to derive the asymptotic distribution of \( W_n = \sup_u Z_n(u) \) using extreme value theory – an approach pioneered by Smirnov (1950) and Bickel and Rosenblatt

\(^4\)For an example of a sequence of stochastic processes that weakly converges pointwise, but not uniformly, consider \( X_n(u) = \frac{B(u + h_n) - B(u)}{\sqrt{h_n}} \), where \( B \) is the Brownian motion, \( h_n \to 0 \) and \( u \in [0, 1] \). Clearly, \( X_n(u) \rightsquigarrow N(0, 1) \) for all \( u \), but, by Lévy’s modulus of continuity theorem, \( \sup_u |X_n(u)| \to \infty \) a.s., and so there is no uniform convergence.
(1973) in the case of histogram and kernel density estimators. However, convergence to the asymptotic Gumbel approximation is very slow, leading to the coverage error of the resulting confidence band to be $O(1/\log n)$, as shown by Hall (1991). Therefore, we do not pursue this approach in this paper.

The other approach would be to derive a coupling of the process $Z_n$ with a simpler (typically Gaussian) process, and then show that bootstrapping or simulation of this process yields valid critical values. This is the approach we take in this paper. Moreover, although our results imply bootstrap validity, we show that bootstrap can be replaced by simple simulation from the linear term $Z_n^*$ of the BK expansion, which is feasible because of its pivotality.

Denote $W_n = \sup_u Z_n(u)$, $W_n^* = \sup_u Z_n^*(u)$ and observe that Theorem 1 implies

$$W_n = W_n^* + r_n, \tag{18}$$

where $r_n$ tends to zero a.s. at a known rate. Unfortunately, such a coupling does not generally imply that quantiles of $W_n^*$ are a good approximation to the quantiles of $W_n$, i.e. Kolmogorov convergence

$$\sup_{t \in \mathbb{R}} |P(W_n \leq t) - P(W_n^* \leq t)| \to 0. \tag{19}$$

As an illustration, consider an extreme case $W_n = n^{-1}U$, $W_n^* = n^{-1}(U - 1)$, where $U \sim \text{Uniform}[0, 1]$. Then $r_n = n^{-1}$, but

$$P(W_n \leq 0) - P(W_n^* \leq 0) = 0 - 1 = -1 \not\to 0,$$

and so (19) does not hold. On the other hand, if $W_n^*$ has an absolutely continuous asymptotic distribution $\mathcal{D}$, then the CDF of $W_n$ converges to the CDF of $\mathcal{D}$ pointwise, and hence we can use the quantiles of $\mathcal{D}$ as valid critical values.

Therefore, intuitively, a certain degree of anti-concentration of $W_n^*$ is needed to guarantee that the coupling (18) implies Kolmogorov convergence (19) and hence validity of simulated critical values. The anti-concentration literature mainly focuses on Gaussian processes, while our approximating process $Z_n^*$ is not Gaussian. Fortunately, $Z_n^*$ is just the centralized kernel density estimator for uniform data, which is a well-studied process. In particular, Rio (1994) established the strong approximation of $W_n^*$ with a Gaussian process, while Chernozhukov et al. (2013) proved that this process exhibits sufficient anti-concentration. Using this theory

\footnote{For a nonasymptotic version of Smirnov-Bickel-Rosenblatt extreme value CLT, see Theorem 1.2 in Rio (1994).}
along with our BK representation allows us to obtain the following result.

**Theorem 3.** Suppose $\log^{3/2} n/\sqrt{nh} \to 0$, $h \log n \to 0$ and Assumptions 1 and 2 hold. Then

$$\sup_{x \in \mathbb{R}} |\mathbb{P}(W_n \leq x) - \mathbb{P}(W_n^* \leq x)| \to 0,$$

(20)

and hence quantiles of $W_n^*$ are valid critical values.

Since the $W_n^*$ has an explicit formula and only depends on the data through IID uniform[0,1] random variables $F(b_1), \ldots, F(b_n)$, quantiles of $W_n^*$ can be obtained by simulation. Denote $\hat{c}_{1-\alpha/2}$ the $(1 - \alpha/2)$-quantile of $W_n^*$. Then the event

$$v(u) \in \left[ \hat{v}_h(u) + \frac{\hat{q}_h(u)\hat{c}_{1-\alpha/2}}{\sqrt{nh}}, \hat{v}_h(u) - \frac{\hat{q}_h(u)\hat{c}_{1-\alpha/2}}{\sqrt{nh}} \right] \text{ for all } u \in [0,1]$$

(21)

occurs with probability $1 - \alpha + o(1)$, i.e. (21) is a valid uniform confidence band for $v(\cdot)$ with nominal confidence level $1 - \alpha$. Note that the confidence band is nonempty by construction since $\hat{q}_h(u) > 0$ a.s.

We can also exploit the pivotality of $W_n^*$ in a different way.

Note that a quantile of $W_n(u) = \sup_u Z_n(u)$ under any distribution of the bids (satisfying the assumptions of Theorem 3) is a valid critical value. When the empirical distribution of bids is used for this purpose, this constitutes *nonparametric bootstrap*. A more convenient choice would be the uniform distribution, since in this case the bid quantile function is equal to 1, and thus does not require estimation.

Formally, let $W_{n,U}(u) = \sup_u Z_{n,U}(u)$, where $Z_{n,U}(u)$ is the process $Z_n(u)$ computed using uniform data. Then $W_{n,U}$ approximates $W_n^*$ in the sense of Theorem 3, which is, in turn, approximated by $W_n$. This leads to the following simple corollary.

**Corollary 1.** Suppose $\log^{3/2} n/\sqrt{nh} \to 0$, $h \log n \to 0$ and Assumptions 1 and 2 hold. Then

$$\sup_{x \in \mathbb{R}} |\mathbb{P}(W_n \leq x) - \mathbb{P}(W_{n,U} \leq x)| \to 0,$$

(22)

and hence quantiles of $W_{n,U}$ are valid critical values.

This corollary underlies the algorithm for the construction of the uniform confidence bands described in Section 1.
5 Benchmarking and MC simulations

In this section we benchmark the performance of our estimator $\hat{q}_h(u)$ of the bid quantile density $q(u)$, which is the crucial part of our estimator of $v(u)$, against the reciprocal of the kernel estimator of the bid density $\hat{f}_h(\cdot)$, evaluated at the empirical quantile function $\hat{Q}(u)$. Performance will be measured by two factors: integrated mean squared error (MSE), and computational speed.

We will consider two different choices of the kernel function. The first choice will be a commonly used triweight kernel $K(u) = \frac{35}{32} \cdot (1 - u^2)^3 \cdot \mathbb{I}(u < 1)$; we will refer to the quantile density estimator based on this kernel as smooth. The second one will have a rectangular kernel $K(u) = \mathbb{I}(u < \frac{1}{2})$; we will refer to the corresponding estimator as ragged. Note that the latter does not satisfy Assumption 1, since it is not Lipschitz; however, this distinction will be effectively lost during numerical integration, for which both kernels will be discretized.

We find that, with matching bandwidths as in (24), the two estimators are very similar in how they approximate the true quantile function, see Figure 1 for illustration with just $n = 200$ bids, distributed according to power law $f(b) = Ab^{A-1}$.

In the following sections we will assess the quality of the approximation in terms of integrated MSE and the computational speed of each of the three estimators.

Figure 1: True quantile density and its estimates with $n = 200$ observations and Silverman’s bandwidth with scale match-up. The DGP is the power distribution with parameter $A = 2$. 
5.1 Integrated MSE

In this section, we argue that it is not possible to unambiguously rank the two estimators in terms of integrated MSE.

From a theoretical perspective, this phenomenon was first discovered by Jones (1992). Building upon the work of Falk (1986), Sheather (1987) and Hall et al. (1989), he noticed that the variance component of the MSE in both estimators can be matched if a natural scale match-up of the bandwidths

\[ h_{nu}(u) := l(Q(u)) \cdot f(Q(u)) \]

is used, for any choice of (potentially variable) bandwidth \( l(b) \) for the estimator of \( f(b) \). Consequently, the difference in MSE of the two estimators equals the difference of their biases, and takes a particularly simple form

\[
MSE(\hat{q}_h(u)) - MSE(1/\hat{f}_l(\hat{Q}(u))) \approx \frac{h^4}{4} \cdot \frac{3(q'(u))^2}{q(u)} \cdot \left(2q''(u) - 3\frac{(q'(u))^2}{q(u)}\right) \cdot \left(\int z^2 K(z)dz\right)^2.
\]

With this approach, at any given point \( u \in (0, 1) \), the MSE (or bias) of \( \hat{q}_h(u) \) is only less than or equal to that of \( 1/\hat{f}_l(\hat{Q}(u)) \) when the following condition is satisfied: \( q(u)q''(u) \leq 1.5q'(u)^2 \), or, equivalently,

\[
f(b)f''(b) \geq 1.5f'(b)^2. \tag{23}
\]

Therefore, the inverse kernel density should perform better close to the center of the distribution, while the kernel quantile density is preferable at the tails.

To put this into perspective, consider the power distribution \( f(b) = Ab^{A-1}, b > 0 \), where \( A > 0 \) is the parameter. Observe that condition (23) becomes \( A(A-2) \geq 1.5(A-1) \), or equivalently \( (A-0.5)(A-3) \geq 0 \). This means that there exist distributions such that one estimator asymptotically dominates the other on the whole range of \( u \in [0, 1] \), and vice versa, and thus they can not possibly be ranked in terms of MSE or bias.

To reassess this claim in small samples, we conduct a large number of Monte Marlo experiments. For the comparison of all the three estimators to be fair, several precautions have to be made. First, a reasonably regular family of distributions is chosen, viz. the power distribution \( f(b) = Ab^{A-1} \), with the parameter \( A \) close to 1. Second, we pick bandwidths that are reasonable and comparable across estimators.

For the inverse kernel density estimator, we pick a constant rule-of-thumb bandwidth \( l^{rot} \) via the standard Silverman’s rule. Since we need a constant bandwidth for the quantile
estimator, we cannot use the scale match-up bandwidth $h_{\text{mu}}(u)$ of Jones (1992), and we use a constant rule-of-thumb bandwidth $h_{\text{rot}}$ instead,

$$h_{\text{rot}} := l_{\text{rot}} / (b(n) - b(0)),$$

$$l_{\text{rot}} := 1.06 \cdot \hat{\sigma}^2 \cdot n^{-1/5},$$

where $\hat{\sigma}^2$ is the standard deviation of bids. The idea behind this choice of bandwidth is that, if the distribution was uniform, it would coincide with an estimate of the scale match-up bandwidth $h_{\text{mu}}(u)$. For alternative approaches using variable bandwidths, see Jones (1990), Cheng et al. (2006) and Prendergast and Staudte (2016).

With only 100 bids drawn from the power distribution with parameter $A$, we find that, on average, the kernel quantile density (either smooth or ragged) outperforms the inverse kernel density for all values of $A < 1$ and $A > 2$, see Figure 2. For larger samples, the pattern is similar.

5.2 Algorithmic complexity and running time

In this section, we argue that the algorithms associated with $\hat{f}_l(\hat{Q}(u))$, the smooth (with a triweight kernel) and ragged (with a rectangular kernel) versions of $\hat{q}_h(u)$, have fundamentally different complexity, which can be heuristically shown to be $O(n^2)$, $O(n \log n)$ and $O(n)$, respectively. This means that the first stage of Guerre et al. (2000) is the slowest of the three algorithms. Efficient implementations of each algorithm can be found in Appendix E.

While there exist multiple computational models to assess complexity of algorithms (see, e.g., Cormen et al., 2009), it is most commonly understood as the average number of elementary operations, performed by the machine to compute the value of the estimator using generic input data. Put differently, it is the running time as a function of input size. We are, of course, interested in its rate of growth, for which the big $O$ notation is convenient.

In the context of structural estimation, we suggest the following analysis. For a sample of $n$ ordered bids, denote the vector of ordered bids as $B = (b(i))_{i=1}^n$, and the vector of bid spacings as $\delta B = (b(i) - b(i-1))_{i=1}^n$. We are interested in obtaining three vectors: $V_f = \left(\hat{f}_l(b_i)\right)_{i=1}^n$ for the bid density estimator, $V_{q,\text{smooth}} = \left(\hat{q}_h(\frac{i}{n})\right)_{i=1}^n$ for an estimator of $q(u)$ with a smooth kernel and $V_{q,\text{ragged}}$ with a rectangular kernel. All the three vectors are linear transformations of either $B$ or $\delta B$, and hence we can think of each estimator as a certain linear operator, for which the computational complexity can be heuristically derived.

For $V_{q,\text{ragged}}$, there is a trivial $O(n)$ implementation $q_{\text{ragged}}$, and no algorithm can have lower complexity than the size of the input. For $V_{q,\text{smooth}}$, we can refer to a well-known fact that a convolution of two vectors of size $n$, implemented using the Fast Fourier Transform,
Figure 2: Left: integrated MSE for the two smooth estimators, with \( n = 200 \) bids drawn from a power law distribution with parameter \( A \). Right: running time for the smooth and ragged estimators.

has algorithmic complexity \( O(n \log n) \) see, e.g., Cochran et al. (1967) and Smith et al. (1997).

For \( V_f \), we focus on the \( n \) by \( n \) matrix that transforms \( B \) into \( V_f \). The complexity of the algorithm is bounded from below by the number of generically unique elements in this matrix, as the program will have to perform at least that many independent calculations. Here we have at least \( (n - 1)n/2 \) generically unique elements, viz. \( K(b_i - b_j) \), for all \( i, j = 1, \ldots, n \).

For generic input and kernel, we therefore obtain the \( O(n^2) \) lower bound on complexity. This bound is trivially attained by a nested-loop implementation of the \( \hat{f}_i(\hat{Q}(u)) \) estimator, see the \texttt{f_naive} routine. If the kernel is compact, it is possible to improve the complexity of the algorithm to \( O(n^2h) \), while taking advantage of sorted bids, see the \texttt{f_fast} routine. However, since the bandwidth is proportional to \( n^{-1/5} \), the rate of growth is effectively the same.

To assess the computational time in a realistic environment, we measure the timings of each of the four algorithms in Monte Carlo simulations. As expected, the \texttt{f_naive} is the slowest of the four algorithms, and \texttt{q_ragged} is the fastest. The difference is, in fact, so large that we had to use the logarithm of running time to put the measurements into perspective, see Figure 2.

It is clear from Figure 2 that the algorithms have very different computational performance. For example, for \( n = 1500 \), the fast algorithm \texttt{f_fast}, is approximately 100 times slower than \texttt{q_smooth}, which, in turn, is approximately 10 times slower than \texttt{q_ragged}. While the exact timings depend on the hardware and implementation details, we can safely say that our estimator of the quantile density is of orders of magnitude faster than its competitors.
In our analysis, we glossed over the running time associated with sorting the original sequence of bids. There are three main reasons for this omission. First, complexity of the sorting routine is well known to be $O(n \log n)$, and hence it can only possibly change the asymptotics of $q_{\text{ragged}}$. Second, both $q_{\text{smooth}}$ and $f_{\text{fast}}$ use ordered bids as their inputs, so their ranking cannot change. Third, in a computationally intensive exercise, such as, e.g., data driven bandwidth selection, sorting is likely to be performed outside of the main loop, in which case its running time will be mostly irrelevant.

Finally, it is worth mentioning that one could try the Fast Fourier Transform in conjunction with numerical integration, to formally bring the complexity of $\hat{f}_l(\hat{Q}(u))$ down to $O(n \log n)$, see Silverman (1982) and Jones and Lotwick (1984). However, difficulties associated with integration in the frequency domain, rounding errors and often wiggly shape of the estimate make it an unpopular choice.

### 5.3 Coverage of the confidence bands

While we showed in Theorem 3 that the confidence bands (21) are asymptotically valid, it may happen that sample sizes at which the result kicks in are very large. In this section, we evaluate the performance of our approach with realistic data, kernels and bandwidths. Here and throughout, in calculations of our estimators and statistics, we use the uniform grid $u = i/n$, where $i = 1, \ldots, n$. From this grid we exclude the values of $i$ for which the support of the kernel $K_h(\cdot - i/n)$ is not entirely contained in $[0, 1]$. In the literature, this is known as trimming and is needed to avoid boundary effects. For samples greater than $n = 1000$, we use 5% trimming on both ends, and for samples between $n = 100$ and $n = 1000$ we use 10% trimming.

For the model data generating process, we pick the Beta(5,2) distribution, since it has fairly generic “shoulder” and “tail” areas, from which we calculate $N$ statistics $W^k_n$,

$$W^k_n = \frac{\sup_i |\hat{q}^k_{\theta}(i/n)/q(\cdot) - 1|}{\sqrt{nh}}, \quad k = 1, \ldots, N,$$

(26)

where $q(u)$ is the true quantile density function.

To assess the quality of inference, we calculate $N$ statistics

$$W^k_{n,U} = \frac{\sup_i |\hat{q}^k_{\theta}(i/n) - 1|}{\sqrt{nh}}$$

(27)

using the IID uniform[0,1] data. With the simulated arrays $\{W^k_n\}_{k=1}^N$ and $\{W^k_{n,U}\}_{k=1}^N$ at hand,

---

6See Hickman and Hubbard (2015) for an overview of boundary correction methods.
Figure 3: Left: a simulated approximation of the quantile function of $W_n$ by the quantile function of $W_{n,U}$, for the smooth and ragged estimators, with $n = 100$. Right: simulated and nominal coverage for the smooth estimator. The number of simulations is $N = 1000$ and the bid DGP is Beta(5, 2).

one can assess whether the simulated coverage converges to the nominal one, as $n, N \to \infty$. In fact, by sorting these two arrays in an ascending order, we obtain an estimate of the whole quantile functions of $W^k_n$ and $W^k_{n,U}$, see Appendix E for a Python implementation.

We find that, for a smooth estimator (triweight kernel and rule-of-thumb bandwidth), the simulated coverage typically falls within 2 percentage points from the nominal coverage for $N = 1000$ simulated draws, see Figure 3; within 1 percentage point for $N = 10,000$, see Figure 4; finally, the two are virtually indistinguishable for $N = 100,000$, see Figure 5. Moreover, these figures show that, even for sample sizes as small as $n = 100$, the quantiles of $W_{n,U}$ approximate the quantiles of $W_n$ uniformly well. The horizontal distance between the two curves represents the gap between the simulated and nominal coverages, for a given confidence level. Notably, the quantile functions of $W^k_n$ statistics associated with smooth and ragged estimators are visibly different, which follows from the fact that they have different linear terms in their respective BK expansions.

6 Estimation and inference for functionals

As was demonstrated in Section 2, some important metrics of auction performance can be expressed as integral functionals of the form

$$T(\varphi) = \int_0^1 \varphi(u) \, dQ(u),$$  \hspace{1cm} (28)
where $\varphi : [0, 1] \to \mathbb{R}$ is a smooth function. In this section, we propose simple estimators for such functionals and show that such estimators admit standard asymptotic inference.

A natural estimator of $T(\varphi)$ is the plug-in estimator

$$
\hat{T}(\varphi) = \int_0^1 \varphi(u) \, d\hat{Q}(u) = \sum_{i=1}^n \varphi \left( \frac{i}{n} \right) \left( b(i) - b(i-1) \right) .
$$

(29)

The following proposition establishes asymptotic normality of this estimator.

**Proposition 2.** Suppose $\varphi$ is continuously differentiable. Then

$$
\sqrt{n} \left( \hat{T}(\varphi) - T(\varphi) \right) \rightsquigarrow N(0, V(T(\varphi))),
$$

(30)

$$
V(T(\varphi)) = \text{Var} \left[ \int_U \varphi'(u) \, dQ(u) \right], \quad U \sim \text{Uniform}[0, 1].
$$

(31)
To conduct inference on $T(\phi)$, we need a consistent estimator of the asymptotic variance. The following proposition exhibits such an estimator.

**Proposition 3.** Suppose $\phi$ is continuously differentiable and let

$$
\hat{V}(T(\phi)) = \frac{1}{n} \sum_{i=1}^{n} \left[ \sum_{j=i}^{n} \phi'(j/n) (b_{(j)} - b_{(j-1)}) - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=i}^{n} \phi'(j/n) (b_{(j)} - b_{(j-1)}) \right]^2.
$$

Then $\hat{V}(T(\phi)) \xrightarrow{p} V(T(\phi))$.

**Remark 4.** Given the formula (31) for the asymptotic variance, a more intuitive expression for its estimator $\hat{V}(T(\phi))$ is

$$
\hat{V}(T(\phi)) = \hat{\text{Var}} \left[ \int_U \phi'(u) d\hat{Q}(u) \right],
$$

where $\hat{\text{Var}}$ denotes the conditional variance with respect to the uniform $[0,1]$ random variable $U$ that is independent of the data. The equivalence between (32) and (33) follows from the observation that $u \mapsto \int_u^1 \phi'(u) d\hat{Q}(u) = \sum_{j=1+\lfloor nu \rfloor}^{n} \phi'(j/n) (b_{(j)} - b_{(j-1)})$ is a step function.

## 7 Conclusion

In this paper we suggest a new approach to estimation and inference on bidders’ preferences (captured by the quantile function of bidders’ private values), as well as certain associated functionals (28), using bid spacings – the differences between consecutive order statistics of the observed bids.

Our estimator of the value quantile function relies on a kernel estimator of the quantile density (derivative of the quantile function) of the bids. This functional estimator does not possess an asymptotic distribution, which prevents us from using standard techniques for performing uniform inference. To overcome this difficulty, we derive a Bahadur-Kiefer expansion of the estimator, which allows us to show that valid confidence bands may be constructed using the studentized estimator calculated on an auxiliary sample of pseudo-bids from the uniform distribution. In a companion result, we also show how to conduct estimation and inference on integral functionals of the form (28), of which bidders’ expected surplus and total expected surplus are special cases.

Avenues for further research include incorporating variable number of bidders and auction heterogeneity, developing procedures for data-driven bandwidth selection, shape-restricted
estimation of the valuation quantile function and inference for the valuation density estimator associated with our valuation quantile function estimator.

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Appendix

A Proof of Proposition 1

Denote bidders’ expected surplus as $T_1$ and total expected surplus as $T_2$. To derive the first formula, observe that a single bidders’ interim surplus is measured by $(v(b) - b)F^{m-1}(b)$. Taking expectation over $F$ and multiplying by $m$ gives us

$$T_1 = m \int (v(b) - b)F^{m-1}(b)dF(b) = m \int (\tilde{v}(u) - Q(u))u^{m-1}du.$$ 

Then using the identifying equation (2) we obtain the first formula

$$T_1 = m \int \frac{q(u)u}{(m-1)}u^{m-1}du = \int \left( \frac{mu^m}{m-1} \right) dQ(u).$$

To derive the second formula, observe that the total surplus is the highest among $m$ valuations, distributed with $G^m(v)$. Plugging in $v = \tilde{v}(u)$ and $G(\tilde{v}(u)) = u$ gives us

$$T_2 = \int vdG^m(v) = m \int vG^{m-1}(v)dG(v) = m \int \tilde{v}(u)u^{m-1}du.$$ 

Then using the identifying equation (2) we obtain

$$T_2 = m \int (Q(u) + q(u) \cdot \frac{u}{m-1})u^{m-1}du.$$ 

Note that the expected total surplus is greater than expected bidders’ surplus by an additional term that depends on the quantile function $Q(u)$:

$$T_2 - T_1 = m \int Q(u)u^{m-1}du = \int Q(u)d(u^m - 1) = \int (1 - u^m)dQ(u),$$

since $Q(0) = 0$, by assumption that the theoretical lower bound of bids $b_{(0)} = 0$.

B Proof of Theorem 1

First, we need the following two lemmas concerning expressions that appear further in the proof.
Lemma 1. Suppose $K$ is a continuous function of bounded variation. Then
\[
\int_0^1 K_h(u - z) \, d\left(\hat{Q}(z) - Q(z)\right) = -\int_0^1 \left(\hat{Q}(z) - Q(z)\right) \, dK_h(u - z) + R^I_n(u),
\] (34)
where $\sup_u |R^I_n(u)| = O_{a.s.} \left(\frac{1}{nh}\right)$.

Proof. Denote $\hat{\psi}(z) = \hat{Q}(z) - Q(z)$ and note that $\hat{\psi}$ is a function of bounded variation a.s. Using integration by part for the Riemann-Stieltjes integral (see e.g. Stroock, 1998, Theorem 1.2.7), we have
\[
\int_0^1 K_h(u - z) \, d\hat{\psi}(z) = -\int_0^1 \hat{\psi}(z) \, dK_h(u - z) + K_h(u - 1)\hat{\psi}(1) - K_h(u)\hat{\psi}(0)\quad (35)
\]
Note that $\hat{\psi}(1) = b(u) - \bar{b} = O_{a.s.}(n^{-1})$, $\hat{\psi}(0) = b(1) - b = O_{a.s.}(n^{-1})$ and $|K_h(u - 1)| \leq h^{-1}K(0)$, $|K_h(u)| \leq h^{-1}K(0)$.

REMARK: this is the worst-case bound on $R^I_n$, i.e. when $u$ is very close to 0 or 1. If $K$ has compact support, then $K_h(u) = K_h(u - 1) = 0$ if $u$ is outside of $O(h)$-neighborhoods of 0 and 1.

Lemma 2. Suppose $K$ is a continuous function of bounded variation. Then, for every $u \in [0, 1]$,
\[
\int_0^1 (\hat{F}(Q(z)) - z) \, dK_h(u - z) = -\frac{1}{n} \sum_{i=1}^n \left[K_h(u - F(b_i)) - \mathbb{E}K_h(u - F(b_i))\right]\quad (36)
\]
\[
= -\left(\hat{f}_h(u) - f_h(u)\right),
\] (37)
where $\hat{f}_h$ denotes the kernel density estimator using the iid uniform data $F(b_1), \ldots, F(b_n)$ and $f_h = \mathbb{E}\hat{f}_h(u)$.

Proof. Using integration by part for the Riemann-Stieltjes integral (see e.g. Stroock, 1998, Theorem 1.2.7), we have
\[
\int_0^1 (\hat{F}(Q(z)) - z) \, dK_h(u - z) = -\int_0^1 K_h(u - z) \, d\left[\hat{F}(Q(z)) - z\right] + K_h(u - 1)\left[\hat{F}(\bar{b}) - 1\right] + K_h(u)\hat{F}(0)
\]
\[
= -\int_0^1 K_h(u - z) \, d\left[\hat{F}(Q(z)) - z\right],
\]
where we used the fact that \( \hat{F}(\bar{b}) = 1 \) a.s. and \( \hat{F}(0) = 0 \) a.s. We further write

\[
\int_0^1 (\hat{F}(Q(z)) - z) dK_h(u - z) = - \int_0^1 K_h(u - z) d[\hat{F}(Q(z)) - z]
\]

\[
= - \int_0^b K_h(u - F(x)) d[\hat{F}(x) - F(x)]
\]

\[
= - \frac{1}{n} \sum_{i=1}^n [K_h(u - F(b_i)) - E K_h(u - F(b_i))] =: - \left( \hat{f}_h(u) - f_h(u) \right),
\]

where in the second equality we used the change of variables \( x = Q(z) \).

We now proceed with the proof of Theorem 1.

Plug in the BK expansion (7) and use Lemma 1 to obtain

\[
\hat{q}_h(u) - q_h(u) = \int_0^1 K_h(u - z) d \hat{Q}(z) - Q(z) \tag{38}
\]

\[
= \int_0^1 [\hat{Q}(z) - Q(z)] dK_h(u - z) + R^I_n(u) \tag{39}
\]

\[
= \int_0^1 q(z)(\hat{F}(Q(z)) - z) dK_h(u - z) + \int_0^1 R^{BK}_n(z) dK_h(u - z) + R^I_n(u). \tag{40}
\]

**First term in (40).**

Since \( f \) is bounded away from zero, \( |q'| \leq M < \infty \) for some constant \( M \), and hence \( |q(z) - q(u)| \leq M|z - u| \). The first term in (40) can then be rewritten as

\[
\int_0^1 q(z)(\hat{F}(Q(z)) - z) dK_h(u - z) = q(u) \int_0^1 (\hat{F}(Q(z)) - z) dK_h(u - z) + R^{II}_n(u), \tag{41}
\]

where

\[
|R^{II}_n(u)| = \left| \int_0^1 (q(z) - q(u))(\hat{F}(Q(z)) - z) dK_h(u - z) \right| \tag{42}
\]

\[
\leq Mh \left| \int_0^1 (\hat{F}(Q(z)) - z) dK_h(u - z) \right|. \tag{43}
\]

By Lemma 2, \( Z^*_n(u) = \int_0^1 (\hat{F}(Q(z)) - z) dK_h(u - z) \) is minus the centralized kernel density estimator for uniform data. The process \( Z^*_n(\cdot) \) has the strong uniform convergence rate \( |\log h|/\sqrt{nh} \) (see e.g. Silverman, 1978; Stute, 1984), and hence

\[
R^{II}_n(u) = O_{a.s.} \left( \frac{h|\log h|}{\sqrt{nh}} \right) \text{ uniformly over } u. \tag{44}
\]
Applying Lemma 2 to the first term in (41) allows us to rewrite
\[ \int_0^1 q(z)(\hat{F}(Q(z)) - z)\,dK_h(u - z) = -q(u)\left(\hat{f}_h(u) - f_h(u)\right) + O_{a.s.}\left(\frac{h|\log h|}{\sqrt{nh}}\right). \]  \hspace{1cm} (45)

**Second term in (40).**

This term can be upper bounded as follows,
\[ \sup_u \left| \int_0^1 R_n(z)\,dK_h(u - z) \right| \leq \sup_u \int_0^1 |R_n(z)|\,|dK_h(u - z)| \leq \sup_z |R_n(z)|TV(K_h) \]  \hspace{1cm} (46)
\[ = O_{a.s.}\left(n^{-3/4} l(n)\right) h^{-1} TV(K) = O_{a.s.}\left(h^{-1} n^{-3/4} l(n)\right), \]  \hspace{1cm} (47)

where we used the properties of total variation in the first inequality and in the second equality.

Plugging (45) and (47) into (40) and multiplying by \(\sqrt{nh}\) completes the proof.

**C Proof of Theorem 3**

A key ingredient of the proof is to note that Lemmas 2.3 and 2.4 in Chernozhukov et al. (2013) continue to hold even if their random variable \(Z_n\) does not have the form \(Z_n = \sup_{f \in F_n} G_n f\) for a standard empirical process \(G_n\), but is instead a generic random variable admitting a strong Gaussian approximation with a sufficiently small remainder.

For completeness, we provide the described trivial extensions of these two lemmas here.

Let \(X\) be a random variable with distribution \(P\) taking values in a measurable space \((S, S)\). Let \(F\) be a class of real-valued functions on \(S\). We say that a function \(F : S \to \mathbb{R}\) is an envelope of \(F\) if \(F\) is measurable and \(|f(x)| \leq F(x)\) for all \(f \in F\) and \(x \in S\).

We impose the following assumptions (A1)-(A3) of Chernozhukov et al. (2013).

- **(A1)** The class \(F\) is pointwise measurable, i.e. it contains a countable subset \(G\) such that for every \(f \in F\) there exists a sequence \(g_m \in G\) with \(g_m(x) \to f(x)\) for every \(x \in S\).

- **(A2)** For some \(q \geq 2\), an envelope \(F\) of \(F\) satisfies \(F \in L^q(P)\).

- **(A3)** The class \(F\) is \(P\)-pre-Gaussian, i.e. there exists a tight Gaussian random variable \(G_P\) in \(l^\infty(F)\) with mean zero and covariance function
\[ \mathbb{E}[G_P(f)G_P(g)] = \mathbb{E}[f(X)g(X)] \] for all \(f, g \in F\).
Lemma 3 (An extension of Lemma 2.3 of Chernozhukov et al. (2013)). Suppose that Assumptions (A1)-(A3) are satisfied and that there exist constants \( \sigma, \bar{\sigma} > 0 \) such that \( \sigma^2 \leq P f^2 \leq \bar{\sigma}^2 \) for all \( f \in \mathcal{F} \). Moreover, suppose there exist constants \( r_1, r_2 > 0 \) and a random variable \( \tilde{Z} = \sup_{f \in \mathcal{F}} G_P f \) such that \( P(|Z - \tilde{Z}| > r_1) \leq r_2 \). Then

\[
\sup_{t \in \mathbb{R}} \left| P(Z \leq t) - P(\tilde{Z} \leq t) \right| \leq C_{\sigma} r_1 \left\{ \mathbb{E} \tilde{Z} + \sqrt{1 \lor \log(\sigma/r_1)} \right\} + r_2,
\]

where \( C_{\sigma} \) is a constant depending only on \( \sigma \) and \( \bar{\sigma} \).

Proof. For every \( t \in \mathbb{R} \), we have

\[
P(Z \leq t) = P(\{|Z \leq t\} \cap \{|Z - \tilde{Z}| \leq r_1\}) + P(\{|Z \leq t\} \cap \{|Z - \tilde{Z}| > r_1\}) \leq P(\tilde{Z} \leq t + r_1) + r_2 \leq P(\tilde{Z} \leq t) + C_{\sigma} r_1 \left\{ \mathbb{E} \tilde{Z} + \sqrt{1 \lor \log(\sigma/r_1)} \right\} + r_2,
\]

where Lemma A.1 of Chernozhukov et al. (2013) (an anti-concentration inequality for \( \tilde{Z} \)) is used to deduce the last inequality. A similar argument leads to the reverse inequality, which completes the proof.

Lemma 4 (An extension of Lemma 2.4 of Chernozhukov et al. (2013)). Suppose that there exists a sequence of \( P \)-centered classes \( \mathcal{F}_n \) of measurable functions \( S \rightarrow \mathbb{R} \) satisfying assumptions (A1)-(A3) with \( \mathcal{F} = \mathcal{F}_n \) for each \( n \), where in the assumption (A3) the constants \( \sigma \) and \( \bar{\sigma} \) do not depend on \( n \). Denote by \( B_n \) a Brownian bridge on \( l^\infty(\mathcal{F}_n) \), i.e. a tight Gaussian random variable in \( l^\infty(\mathcal{F}_n) \) with mean zero and covariance function

\[
\mathbb{E}[B_n(f)B_n(g)] = \mathbb{E}[f(X)g(X)] \text{ for all } f, g \in \mathcal{F}_n.
\]

Moreover, suppose that there exists a sequence of random variables \( \tilde{Z}_n = \sup_{f \in \mathcal{F}_n} B_n(f) \) and a sequence of constants \( r_n \rightarrow 0 \) such that \( |Z_n - \tilde{Z}_n| = o_P(r_n) \) and \( r_n \mathbb{E} \tilde{Z}_n \rightarrow 0 \). Then

\[
\sup_{t \in \mathbb{R}} \left| P(Z_n \leq t) - P(\tilde{Z}_n \leq t) \right| \rightarrow 0.
\]

Proof. Take \( \beta_n \rightarrow \infty \) sufficiently slowly such that \( \beta_n r_n (1 \lor \mathbb{E} \tilde{Z}_n) = o(1) \). Then since \( P(|Z_n - \tilde{Z}_n| > \beta_n r_n) = o(1) \), by Lemma 3, we have

\[
\sup_{t \in \mathbb{R}} \left| P(Z_n \leq t) - P(\tilde{Z}_n \leq t) \right| = O \left( r_n (\mathbb{E} \tilde{Z}_n + |\log(\beta_n r_n)|) \right) + o(1) = o(1).
\]

This completes the proof. \( \square \)
Going back to the proof of Theorem 3, let

$$\mathcal{F}_n = \{[0,1] \ni x \mapsto K_{h_n}(u - x), \quad u \in [0,1]\}.$$

Theorem 1.1 of Rio (1994) implies that there exists a Brownian bridge $B_n$ on $\mathcal{F}_n$ such that, for $\tilde{W}_n = \sup_u B_n(u)$,

$$W^*_n = \tilde{W}_n + O_{a.s.} \left( \frac{\log n}{\sqrt{nh}} \right)$$

(see also a discussion in Chernozhukov et al., 2013, Remark 3.1(ii)). Lemma 4 and Remark 3.2 in Chernozhukov et al. (2013) then imply

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(W^*_n \leq t) - \mathbb{P}(\tilde{W}_n \leq t) \right| \to 0. \quad (49)$$

On the other hand, by Theorem 1 we have

$$W_n = W^*_n + O_{a.s.} \left( \sqrt{h} + h \log h + h^{-1/2}n^{-1/4}l(n) \right). \quad (50)$$

Substituting (48) into this equation, we obtain

$$W_n = \tilde{W}_n + O_{a.s.} \left( \frac{\log n}{\sqrt{nh}} + \sqrt{h} + h \log h + h^{-1/2}n^{-1/4}l(n) \right). \quad (51)$$

Under the assumptions on $h$, we have $W_n - \tilde{W}_n = o_p(\log^{-1/2}n)$, and hence Remark 3.2 of Chernozhukov et al. (2013) implies

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P}(W_n \leq t) - \mathbb{P}(\tilde{W}_n \leq t) \right| \to 0. \quad (52)$$

Note that although our $W_n$ does not have a form of a supremum of an empirical process like in CCK, this does not play any role in the proof of their Lemmas 2.3 and 2.4.

Given (49) and (52), applying the triangle inequality finishes the proof.
D Proofs of Propositions 2 and 3

D.1 Proof of Proposition 2

Integration by parts yields

\[ \int_0^1 \varphi(u) d\left( \hat{Q}(u) - Q(u) \right) = - \int_0^1 \left( \hat{Q}(u) - Q(u) \right) \varphi'(u) + R_n^{IBP}, \]  

where

\[ R_n^{IBP} = \varphi(1) (b(n) - \bar{b}) - \varphi(0) (b(1) - \bar{b}) = O_{a.s.} \left( \frac{1}{n} \right). \]  

By the BK expansion,

\[ \int_0^1 \left( \hat{Q}(u) - Q(u) \right) \varphi'(u) = \int_0^1 q(u) \left( \hat{F}(Q(u)) - u \right) \varphi'(u) + R_n^{BK} \]  

\[ = \frac{1}{n} \sum_{i=1}^{n} \int_0^1 q(u) (I(b_i \leq Q(u)) - u) \varphi'(u) + R_n^{BK} \]  

\[ = \frac{1}{n} \sum_{i=1}^{n} \left[ \int_{F(b_i)}^{1} q(u) \varphi'(u) \left( \int_{F(b_i)}^{1} q(u) \varphi'(u) \right) - \mathbb{E} \int_{F(b_i)}^{1} q(u) \varphi'(u) \right] + R_n^{BK}, \]  

where

\[ R_n^{BK} = \int_0^1 R_n^{BK}(u) d\varphi(u). \]  

The latter integral remainder may be bounded as

\[ \left| R_n^{BK} \right| \leq \int_0^1 \left| R_n^{BK}(u) \right| \left| d\varphi(u) \right| \leq \sup_u \left| R_n^{BK}(u) \right| TV(\varphi) = O_{a.s.} \left( n^{-3/4} l(n) \right). \]  

Therefore,

\[ \sqrt{n} \left( \hat{\theta} - \theta \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \int_{F(b_i)}^{1} \varphi'(u) dQ(u) - \mathbb{E} \int_{F(b_i)}^{1} \varphi'(u) dQ(u) \right] + O_{a.s.} \left( n^{-1/4} l(n) \right), \]  

where we used the fact that a continuously differentiable function \( \varphi \) has finite total variation. Applying the central limit theorem to the first term completes the proof.
D.2 Proof of Proposition 3

Write the asymptotic variance $V$ as a functional of $q \in L^\infty[0, 1]$,

$$V = V(q) = \int_0^1 \left( \int z q(u) d\varphi(u) \right)^2 dz - \left( \int_0^1 \int z q(u) d\varphi(u) dz \right)^2.$$  

(61)

Clearly, $V : L^\infty[0, 1] \to \mathbb{R}$ is a continuous functional (under the uniform norm). Since $\hat{q} \overset{p}{\to} q$ uniformly, the continuous mapping theorem implies $\hat{V} = V(\hat{q}) \overset{p}{\to} V(q) = V$.

E Codes

For the codes below to work, one needs to have a Python interpreter (we use version 3.7) and to be able to import three packages: numpy (1.19.2), scipy (1.5.2) and numba (0.51.2) for efficient random number generation, convolution and looping. The reader should be able to replicate the results with these exact versions, after they eventually get outdated.

We create two contiguous arrays: `sorted_bids` for the sorted bids and `f_density`, where the density $\hat{f}(\hat{Q}(u))$ will be eventually stored. We also compute three constants: `band` – the Silverman rule rule-of-thumb bandwidth for $\hat{f}(b)$, `u_band` – the matching rule-of-thumb bandwidth for the quantile density $\hat{q}(u)$, and `i_band` – an integer analog of `u_band`, equal to the number of observations that fit in the support for the rectangular kernel, or half that number for the triweight kernel.

```python
import numpy as np
import scipy as sp
import numba as nb

sample_size = 200  # or bigger
rv = sp.stats.beta(5, 2)  # or any other distribution
bids = rv.rvs(sample_size)  # this is the simulated data
sorted_bids = np.sort(bids)

scale = np.std(sample_of_bids)
band = 1.06*scale*np.power(sample_size, -0.2)  # Silverman rule
u_band = band/(bids.max() - bids.min())  # match-up rule
i_band = int(u_band*sample_size)
trim = int(.1*sample_size)  # ad-hoc trimming, assuming u_band < .1

f_density = np.zeros(sample_size)
```

Below are the two implementations of the classic nonparametric estimator of bid density
with a triweight kernel. The \texttt{f\_naive} routine uses nested loops to iterate over all pairs of bids, while the \texttt{f\_fast} routine only loops over those pairs of bids that fall within the range of the kernel. The latter is achieved by storing the index of the lowest bid that falls inside the support of a current kernel, and using this information to reduce the size of the inner loop, in the next iteration of the outer loop.

\begin{verbatim}
@nb.jit(nopython=True)
def triweight(u):
    return np.maximum(35*np.power(1 - np.power(u, 2), 3)/32, 0)

@nb.jit(nopython=True)
def rectangular(u):
    return (np.sign(1/2 - np.abs(u)) + 1)/2

@nb.jit(nopython=True)
def f_naive(bids, f_density, kernel = triweight):
    for i in range(trim, sample_size - trim):
        f_density[i] = 0 # safety measure
        current_bid = bids[i]

        for j in range(sample_size):
            distance = current_bid - bids[j]
            f_density[i] += kernel(distance/band)/band

        f_density /= sample_size

@nb.jit(nopython=True)
def f_fast(sorted_bids, f_density, kernel = triweight):
    inner_loop_offset = 0

    for i in range(trim, sample_size - trim):
        f_density[i] = 0 # safety measure
        j = inner_loop_offset
        current_bid = sorted_bids[i]

        while sorted_bids[j] < current_bid - band:
            j += 1

        inner_loop_offset = j

        while sorted_bids[j] <= current_bid + band and j < sample_size:
            distance = current_bid - sorted_bids[j]
            f_density[i] += kernel(distance/band)/band
\end{verbatim}
j += 1

f_density /= sample_size

We further create three contiguous arrays: spacings – that stores bid increments, kernel – that stores the discrete kernel, which is often referred to as filter in the signal processing literature, and q_density – where the quantile density will be stored.

def make_filter(i_band, kernel=triweight):
    return np.array([kernel(j/i_band)/i_band for j in range(-i_band+1, i_band)])

spacings = sorted_bids - np.roll(sorted_bids, 1)
q_density = np.zeros(sample_size)

Below are the implementations of the smooth and ragged estimators of the quantile density. The ragged version is a single loop over the sorted bids, that computes bid increments on the fly. The smooth version convolves the vector of spacings with the discrete kernel using the efficient np.convolve function.

def q_smooth(spacings, kernel=triweight):
    discrete_filter = make_filter(i_band, kernel=kernel)
    q_density = np.convolve(spacings, discrete_filter)[i_band-1:-i_band+1]
    return q_density*sample_size

def q_ragged(sorted_bids, q_density):
    half_band = int(i_band/2)

    for i in range(trim, sample_size - trim):
        left = sorted_bids[i - half_band]
        right = sorted_bids[i + half_band]
        q_density[i] = right - left

    q_density *= sample_size/(2*half_band)

To obtain all four estimators, run the routines: f_naive, f_fast, q_ragged, and q_smooth with the corresponding parameters.

To obtain the critical values associated with the suprema of the studentized versions of \( \hat{q}(u) \) and the linear part of the BK expansion, we first compute and store the true value of \( f(Q(u)) \) in true_fQ and then define functions W_model and W_uniform, that simulate N statistics: \( W_n^k \) and \( W_{nU}^k \), respectively, where N is equal to draws.

rv = sp.stats.beta(5, 2) # dgp - data generating process
x = np.linspace(0, 1, sample_size)
true_fQ = rv.pdf(rv.ppf(x))
draws = 10000

@nb.jit(nopython = True, parallel = True)
def W_model(suprema, kernel = triweight):
    discrete_filter = make_filter(i_band, kernel = kernel)
    for i in nb.prange(draws): # parallel looping
        sorted_bids = np.sort(np.random.beta(5, 2, sample_size)) # dgp
        spacings = sorted_bids - np.roll(sorted_bids, 1)
        out = q_smooth(spacings, discrete_filter)
        suprema[i] = np.max(np.abs((out*true_fQ)[trim:-trim]-1))
    suprema /= np.sqrt(sample_size*band)

@nb.jit(nopython = True, parallel = True)
def W_uniform(suprema, kernel = triweight):
    discrete_filter = make_filter(i_band, kernel = kernel)
    for i in nb.prange(draws): # parallel looping
        sorted_bids = np.sort(np.random.uniform(0, 1, sample_size))
        spacings = sorted_bids - np.roll(sorted_bids, 1)
        out = q_smooth(spacings, discrete_filter)
        suprema[i] = np.max(np.abs(out[trim:-trim]-1))
    suprema /= np.sqrt(sample_size*band)

Finally, fix the nominal coverage nominal and the number of draws draws, and run the code below to compute the simulated coverage coverage. To see the empirical quantile functions of $W^k_n$ and $W^k_{n,t}$, sort and plot model_suprema and uniform_suprema, respectively.

nominal = 99
model_suprema = np.zeros(draws)
W_model(model_suprema)
model_suprema.sort()

uniform_suprema = np.zeros(draws)
W_uniform(uniform_suprema)
uniform_suprema.sort()

critical = np.percentile(uniform_suprema, nominal)
coverage = ((1+np.sign(critical - model_suprema))/2).mean()