Unitary vertex operator algebras

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Abstract
Unitary vertex operator algebras are introduced and studied. It is proved that most well-known rational vertex operator algebras are unitary. The classification of unitary vertex operator algebras with central charge $c \leq 1$ is also discussed.

1 Introduction

Both bilinear form and Hermitian form are power tools in the study of general algebras and their representations. Based on a symmetric contravariant bilinear form with respect to a Cartan involution in [B] for vertex algebras associated to even lattices, notions of invariant bilinear form for an vertex operator algebra and contragradient module were introduced and studied in [FHL]. The space of invariant bilinear form for an arbitrary vertex operator algebra was determined in [Li]. In this paper, we study the invariant Hermitian form on vertex operator algebras and their modules, and use the Hermitian form to define and investigate the unitary vertex operator algebras.

One important motivation for studying the unitary vertex operator algebra comes from the unitary representations of infinite dimensional Lie algebras such as Virasoro algebras and Kac-Moody algebras. The unitary highest weight representations of these algebras produce fundamental families of rational vertex operator algebras. It is well known today that the theory of vertex operator algebra unifies representation theory of many infinite dimensional Lie algebra via locality. So it is natural to have a notion of unitary vertex operator algebra so that in the case of Virasoro and affine Kac-Moody algebras, these two unitarities are equivalent. While the Hermitian form in the the theory of infinite dimensional Lie algebras is defined to be the Hermitian form which is contravariant under some anti-linear anti-involution of corresponding universal enveloping algebras, the Hermitian forms on vertex operator algebras and their modules are defined
to be contravariant under some anti-linear involutions. In the case of Virasoro and affine vertex operator algebras, the anti-linear involutions induce anti-linear anti-involutions which are exactly the given ones of the infinite dimensional Lie algebras which are required for the unitary representations. The unitary vertex operator algebra is defined to be a vertex operator algebra associated with a positive definite Hermitian form. We will prove that the vertex operator algebras associated to the unitary highest weight representations for the Heisenberg algebra, Virasoro algebra and affine Kac-Moody algebras are indeed the unitary vertex operator algebras. We also discuss the unitarity of irreducible modules for these vertex operator algebras.

The positive definite Hermitian form already appeared in the theories of lattice vertex operator algebras [B], [FLM], and their \( \mathbb{Z}_2 \)-orbifold vertex operator algebras [DGM]. We show that the lattice vertex operator algebras associated to positive definite even lattices are unitary vertex operator algebras. We also establish the unitarity for the irreducible modules and the irreducible \( \theta \)-twisted modules for lattice vertex operator algebras where \( \theta \) is the automorphism of lattice vertex operator algebras induced from the \( -1 \) isometry of the lattices. These results are then used to show that the moonshine vertex operator algebra \( \mathcal{V} \) is also unitary.

Another motivation comes from connection between algebraic and analytic approaches to 2-dimensional conformal field theory. While the algebraic approach uses vertex operator algebras, the analytic approach uses conformal nets. It has been expected that these two approaches are equivalent in the following sense: one can construct conformal nets and vertex operator algebras from each other. Although it is not clear how this can be achieved, one can see the similarity of these two approaches in many examples. The basic object in the theory of operator algebras is Hilbert space. So it is desirable to have a positive definite Hermitian form on vertex operator algebra whose completion gives rise to a Hilbert space. From this point of view, studying the unitary vertex operator algebra is the first step in constructing conformal nets from vertex operator algebras.

This paper is organized as follows. In Section 2, we introduce the notion of unitary vertex operator algebra, and give some elementary facts about unitary vertex operator algebras. In Section 3, we show that the unitary rational and \( C_2 \)-cofinite vertex operator algebras could be extended to a unitary vertex operator algebra by a simple current under some assumption. In Section 4, we prove that some well-known vertex operator algebras are unitary. In Section 5, we give some results about the classification of unitary vertex operator algebras with central charge \( c \leq 1 \).
2 Preliminaries

We assume that the readers are familiar with the notion of vertex operator algebra and the basic facts about vertex operator algebra as presented in [FLM], [FHL], [DLM1], [DLM2], [LL] and [Z]. In this paper, we only consider the vertex operator algebra \((V, Y, 1, \omega)\) of CFT-type, i.e. \(V_n = 0, n < 0\) and \(V_0 = \mathbb{C}1\).

**Definition 2.1.** Let \((V, Y, 1, \omega)\) be a vertex operator algebra. An anti-linear automorphism \(\phi\) of \(V\) is an anti-linear isomorphism (as anti-linear map) \(\phi : V \rightarrow V\) such that \(\phi(1) = 1, \phi(\omega) = \omega\) and \(\phi(u_n v) = \phi(u)_n \phi(v)\) for any \(u, v \in V\) and \(n \in \mathbb{Z}\).

**Definition 2.2.** Let \((V, Y, 1, \omega)\) be a vertex operator algebra and \(\phi : V \rightarrow V\) be an anti-linear involution, i.e. an anti-linear automorphism of order 2. The \((V, \phi)\) is called unitary if there exists a positive definite Hermitian form \((,\) : \(V \times V \rightarrow \mathbb{C}\) which is \(\mathbb{C}\)-linear on the first vector and anti-\(\mathbb{C}\)-linear on the second vector such that the following invariant property holds: for any \(a, u, v \in V\)

\[
(Y(e^{zL(1)}(-z^{-2})L(0)a, z^{-1})u, v) = (u, Y(\phi(a), z)v)
\]

where \(L(n)\) is defined by \(Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}\).

**Remark 2.3.** For a unitary vertex operator algebra \((V, \phi)\), the positive definite Hermitian form \((,\) : \(V \times V \rightarrow \mathbb{C}\) is uniquely determined by the value \((1, 1)\). In fact, for any \(u, v \in V_n\) there exists a complex number \(\lambda \in \mathbb{C}\) such that

\[
(u, v) = (u_{-1}1, v) = (Y(u, z)1, v) = (1, Y(e^{zL(1)}(-z^{-2})L(0)\phi(u), z^{-1})v) = (1, Res_zz^{-1}Y(e^{zL(1)}(-z^{-2})L(0)\phi(u), z^{-1})v) = (1, \lambda 1) = \bar{\lambda}(1, 1)
\]

We will normalize the Hermitian form \((,\) on \(V\) such that \((1, 1) = 1\).

**Definition 2.4.** Let \((V, Y, 1, \omega)\) be a vertex operator algebra and \(\phi\) an anti-linear involution of \(V\), and \(g\) a finite order automorphism of \(V\). An ordinary \(g\)-twisted \(V\)-module \((M, Y_M)\) [DLM2] is called a unitary \(g\)-twisted \(V\)-module if there exists a positive definite Hermitian form \((,)_M : M \times M \rightarrow \mathbb{C}\) which is \(\mathbb{C}\)-linear on the first vector and anti-\(\mathbb{C}\)-linear on the second vector such that the following invariant property:

\[
(Y_M(e^{zL(1)}(-z^{-2})L(0)a, z^{-1})w_1, w_2)_M = (w_1, Y_M(\phi(a), z)w_2)_M
\]

holds for \(a \in V\) and \(w_1, w_2 \in M\).
Note that if \((V, \phi)\) is a unitary vertex operator algebra, then \(V\) is a unitary \(V\)-module.

**Lemma 2.5.** Let \(V\) be a vertex operator algebra and \(\phi\) an anti-linear involution of \(V\), and \(g\) a finite order automorphism of \(V\). Then

1. Any unitary \(g\)-twisted \(V\)-module \(M\) is completely reducible.
2. Any unitary \(g\)-twisted \(V\)-module \(M\) is a completely reducible module for the Virasoro algebra.

**Proof:** The proof of (1) is fairly standard using the invariant property. For (2) notice that the invariant property also implies that \((L(n)u, v)_M = (u, L(-n)v)_M\) for \(w_1, w_2 \in M\) and \(n \in \mathbb{Z}\). As a result, \(M\) is a unitary representation of the Virasoro algebra and the result follows immediately. \(\Box\)

In the following we construct unitary vertex operator algebras from the given unitary vertex operator algebras. Recall that a vertex operator subalgebra \(U = (U, Y, 1, \omega)\) of \((V, Y, 1, \omega)\) is a vector subspace \(U\) of \(V\) such that the restriction of \(Y\) to \(U\) gives a structure of vertex operator subalgebra on \(U\). The following proposition is immediate.

**Proposition 2.6.** Let \((V, \phi)\) be a unitary vertex operator algebra and \(U\) be a vertex operator subalgebra of \(V\) such that the Virasoro element of \(U\) is the same as that of \(V\) and \(\phi(U) = U\). Then \((U, \phi|_U)\) is a unitary vertex operator algebra.

Let \(V\) be a vertex operator algebra and \(g\) be a finite order automorphism of \(V\). Then the fixed point subspace \(V^g = \{a \in V|g(a) = a\}\) is a vertex operator subalgebra of \(V\).

**Corollary 2.7.** Let \((V, \phi)\) be a unitary vertex operator algebra and \(g\) be a finite order automorphism of \(V\) which commutes with \(\phi\). Then \((V^g, \phi|_{V^g})\) is a unitary vertex operator algebra.

**Proof:** If \(a\) lies in \(V^G\) we have \(g \phi(a) = \phi g(a) = \phi(a)\). That is, \(\phi(V^g) = V^g\). Then \((V^g, \phi|_{V^g})\) is unitary from Proposition 2.6. \(\Box\)

Let \((V, Y, 1, \omega)\) be a vertex operator algebra and \((U, Y, 1, \omega)\) is a vertex operator subalgebra of \(V\) such that \(\omega' \in V_2\) and \(L(1)\omega' = 0\). Then \((U^c, Y, 1, \omega - \omega')\) is a vertex operator subalgebra of \(V\) where \(U^c = \{v \in V|L'(-1)v = 0\}\) [FZ].

**Corollary 2.8.** Let \((V, Y, 1, \omega), (U, Y, 1, \omega')\) be vertex operator algebras satisfying the conditions above. Assume that \((V, \phi)\) is unitary and \(\phi(\omega') = \omega'\). Then \((U^c, \phi|_{U^c})\) is a unitary vertex operator algebra.

**Proof:** For \(a \in U^c\), i.e. \(L'(-1)a = 0\), we have \(L'(-1)\phi(a) = \omega_0' \phi(a) = \phi(\omega_0' a) = 0\). Thus \(\phi(U^c) \subset U^c\). Since \(\phi(\omega') = \omega'\), we have \(\phi|_{U^c}\) is an anti-linear involution of \(U^c\). Since

\[
(Y(e^{2L(1)}(-z^{-2})L(0)a, z^{-1})u, v) = (u, Y(\phi(a), z)v)
\]
for $a, u, v \in U^c$, we have

$$(Y(\mathcal{L}''(1))(-z^{-2})\mathcal{L}''(0)a, z^{-1})u, v) = (u, Y(\phi(a), z)v)$$

holds for $a, u, v \in U^c$. Then $(U^c, \phi|_{U^c})$ is unitary.

Now we recall some facts about the tensor product vertex operator algebra [FHL]. Let $(V^1, Y_{V^1}, \omega^1), \ldots, (V^p, Y_{V^p}, \omega^p)$ be vertex operator algebras. The tensor product of vertex operator algebras $V^1, \ldots, V^p$ is constructed on the tensor product vector space

$$V = V^1 \otimes \cdots \otimes V^p$$

where the vertex operator $Y_V$ is defined by

$$Y_V(u^1 \otimes \cdots \otimes u^p, z) = Y_{V^1}(u^1, z) \otimes \cdots \otimes Y_{V^p}(u^p, z)$$

for $u^i \in V^i$ $(1 \leq i \leq p)$, the vacuum vector is

$$1 = 1 \otimes \cdots \otimes 1$$

and the Virasoro element is

$$\omega = \omega^1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes \omega^p.$$ 

Then $(V, Y_V, 1, \omega)$ is a vertex operator algebra (see [FHL], [LL]).

Let $(M^i, Y_{M^i})$ be an ordinary $V^i$-module for $i = 1, \ldots, p$. We may construct the tensor product ordinary module $M^1 \otimes \cdots \otimes M^p$ for the tensor product vertex operator algebra $V^1 \otimes \cdots \otimes V^p$ by

$$Y_{M^1 \otimes \cdots \otimes M^p}(u^1 \otimes \cdots \otimes u^p, z) = Y_{M^1}(u^1, z) \otimes \cdots \otimes Y_{M^p}(u^p, z).$$

Then $(M^1 \otimes \cdots \otimes M^p, Y_{M^1 \otimes \cdots \otimes M^p})$ is an ordinary $V^1 \otimes \cdots \otimes V^p$-module.

Now let $(V^1, \phi^1), \ldots, (V^p, \phi^p)$ be unitary vertex operator algebras and $(,)_i$ be the corresponding Hermitian form on $V^i$. We define a Hermitian form on $V^1 \otimes \cdots \otimes V^p$ as follow: for any $u_1 \otimes \cdots \otimes u_p, v_1 \otimes \cdots \otimes v_p \in V^1 \otimes \cdots \otimes V^p$,

$$(,)_i : V^1 \otimes \cdots \otimes V^p \times V^1 \otimes \cdots \otimes V^p \to \mathbb{C}$$

$$(u_1 \otimes \cdots \otimes u_p, v_1 \otimes \cdots \otimes v_p) \mapsto (u_1, v_1)_i \cdots (u_p, v_p)_i.$$ 

We also define an anti-linear map $\phi$ by:

$$\phi : V^1 \otimes \cdots \otimes V^p \to V^1 \otimes \cdots \otimes V^p$$

$$\phi(u_1 \otimes \cdots \otimes u_p) \mapsto \phi^1(u_1) \otimes \cdots \otimes \phi^p(u_p).$$

Obviously, $(,)_i$ is a positive definite Hermitian form on $V^1 \otimes \cdots \otimes V^p$ and $\phi$ is an anti-linear involution of $V^1 \otimes \cdots \otimes V^p$. Now we have the following:
Proposition 2.9. Let \((V^1, \phi_1), \ldots, (V^p, \phi_p)\) be unitary vertex operator algebras and \(\phi\) be the anti-linear involution of \(V^1 \otimes \cdots \otimes V^p\) defined above. Then \((V^1 \otimes \cdots \otimes V^p, \phi)\) is a unitary vertex operator algebra.

Proof: It is good enough to check that the invariant property

\[
(Y_V(e^{zL(1)}(-z^{-2})L(0)a_1 \otimes \cdots \otimes a_p, z^{-1})u_1 \otimes \cdots \otimes u_p, v_1 \otimes \cdots \otimes v_p)
\]

holds for \(a_1 \otimes \cdots \otimes a_p, u_1 \otimes \cdots \otimes u_p, v_1 \otimes \cdots \otimes v_p \in V = V^1 \otimes \cdots \otimes V^p\).

In fact, we have

\[
(Y_V(e^{zL(1)}(-z^{-2})L(0)a_1 \otimes \cdots \otimes a_p, z^{-1})u_1 \otimes \cdots \otimes u_p, v_1 \otimes \cdots \otimes v_p)
\]

\[
= (Y_V(e^{zL(1)}(-z^{-2})L(0)a_1, z^{-1})u_1 \otimes \cdots \otimes u_p, v_1 \otimes \cdots \otimes v_p)
\]

\[
= (Y_V(e^{zL(1)}(-z^{-2})L(0)a_1, z^{-1})u_1, v_1)_{1 \cdots v_p}
\]

\[
= (u_1, Y_V(\phi_1(a_1), z)v_1)_{1 \cdots v_p}
\]

\[
= (u_1 \otimes \cdots \otimes u_p, Y_V(\phi(a_1 \otimes \cdots \otimes a_p), z)v_1 \otimes \cdots \otimes v_p).
\]

Then \((V^1 \otimes \cdots \otimes V^p, \phi)\) is a unitary vertex operator algebra.

We could obtain the following proposition by the similar discussion as Proposition 2.9.

Proposition 2.10. Let \(V^1, \ldots, V^p\) be vertex operator algebras and \(\phi_i\) be an anti-linear involution of \(V^i\) \((i = 1, \ldots, p)\). Assume that \(M^i\) is a unitary module of \(V^i\) \((i = 1, \ldots, p)\), then \(M^1 \otimes \cdots \otimes M^p\) is a unitary \(V^1 \otimes \cdots \otimes V^p\)-module.

The following proposition is useful to prove the unitarity of vertex operator algebra.

Proposition 2.11. Let \(V\) be a vertex operator algebra equipped with a positive definite Hermitian form \(,\) : \(V \times V \to \mathbb{C}\) and \(\phi\) be an anti-linear involution of \(V\). Assume that \(V\) is generated by the subset \(S \subset V\), i.e.

\[
V = \text{span}\{u_{n_1} \cdots u_{n_k} | k \in \mathbb{N}, u^1, \ldots, u^k \in S\}
\]

and the invariant property

\[
(Y(e^{zL(1)}(-z^{-2})L(0)a, z^{-1})u, v) = (u, Y(\phi(a), z)v)
\]

holds for \(a \in S, u, v \in V\). Then \((V, \phi)\) is a unitary vertex operator algebra.
Proof: Let $U$ be the subset of $V$ defined as follow:

$$ U = \{ a \in V \mid (Y(e^{zL(1)}(-z^{-2})L(0)a, z^{-1})u, v) = (u, Y(\phi(a), z)v), \ \forall u, v \in V \}. $$

It is easy to prove that $1 \in U$. Now we prove that if $a, b \in U$, then $a_nb \in U$ for any $n \in \mathbb{Z}$. First, we have the following identity which was proved in Theorem 5.2.1 of [FHL],

$$ -z_0^{-1} \delta(-z_0)Y(e^{zL(1)}(-z_1^{-2})L(0), z_1^{-1})Y(e^{z_2L(1)}(-z_0^{-2})L(0)b, z_0^{-1}) $$

$$ +z_0^{-1} \delta(-z_0)Y(e^{z_2L(1)}(-z_2^{-2})L(0)b, z_2^{-1})Y(e^{z_1L(1)}(-z_1^{-2})L(0)a, z_1^{-1}) $$

$$ = z_1^{-1} \delta(-z_1)Y(e^{z_2L(1)}(-z_2^{-2})L(0)Y(a, z_0)b, z_2^{-1}). $$

By this identity, we have

$$ Y(e^{z_2L(1)}(-z_2^{-2})L(0)a_nb, z_2^{-1}) $$

$$ = Res_{z_1} \{ (-(-z_2 + z_1)^nY(e^{z_1L(1)}(-z_1^{-2})L(0)a, z_1^{-1})Y(e^{z_2L(1)}(-z_0^{-2})L(0)b, z_0^{-1}) $$

$$ +(z_1 - z_2)^nY(e^{z_2L(1)}(-z_0^{-2})L(0)b, z_2^{-1})Y(e^{z_1L(1)}(-z_1^{-2})L(0)a, z_1^{-1}) \}. $$

Then

$$ (Y(e^{z_2L(1)}(-z_2^{-2})L(0)a_nb, z_2^{-1})u, v) $$

$$ = Res_{z_1} \{ (-(-z_2 + z_1)^nY(e^{z_1L(1)}(-z_1^{-2})L(0)a, z_1^{-1})Y(e^{z_2L(1)}(-z_2^{-2})L(0)b, z_2^{-1}) $$

$$ +(z_1 - z_2)^nY(e^{z_2L(1)}(-z_0^{-2})L(0)b, z_2^{-1})Y(e^{z_1L(1)}(-z_1^{-2})L(0)a, z_1^{-1}) $$

$$ = Res_{z_1} \{ ((u, (-(-z_2 + z_1)^nY(\phi(b), z_2)Y(\phi(a), z_1)v) $$

$$ +(u, (z_1 - z_2)^nY(\phi(a), z_1)Y(\phi(b), z_2)v) \}. $$

On the other hand, we have

$$ Y(\phi(a), \phi(b), z_2) $$

$$ = Res_{z_1} \{ ((z_1 - z_2)^nY(\phi(a), z_1)Y(\phi(b), z_2) - (-z_2 + z_1)^nY(\phi(b), z_2)Y(\phi(a), z_1)) \}. $$

Thus we have

$$ (Y(e^{z_2L(1)}(-z_2^{-2})L(0)a_nb, z_2^{-1})u, v) $$

$$ = (u, Y(\phi(a), \phi(b), z_2)v) $$

$$ = (u, Y(\phi(a), \phi(b), z_2)v). $$

Then we have $a_nb \in U$ for any $n \in \mathbb{Z}$ if $a, b \in U$. Since $S \subset U$ and $S$ generates $V$, we have $U = V$. So $(V, \phi)$ is a unitary vertex operator algebra. \qed

The following proposition could be proved by the similar discussion as in Proposition 2.11.
Proposition 2.12. Let \( g \) be a finite order automorphism of \( V \) and \( M \) be an ordinary \( g \)-twisted \( V \)-module equipped with a positive definite Hermitian form \( (,)_M : M \times M \to \mathbb{C} \). Assume that \( V \) is a vertex operator algebra which has an anti-linear involution \( \phi \) and that \( V \) is generated by the subset \( S \subset V \), i.e.

\[
V = \text{span}\{u_{n_1}^1 \cdots u_{n_k}^k | k \in \mathbb{N}, u^1, \ldots, u^k \in S\}
\]

and the invariant property

\[
(Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})w_1, w_2)_M = (w_1, Y(a, z)w_2)_M
\]

holds for \( a \in S, w_1, w_2 \in M \). Then \( M \) is a unitary \( g \)-twisted \( V \)-module.

3 Extension vertex operator algebra by irreducible unitary simple current module

In this section, we assume that \( (V, \phi) \) is a unitary vertex operator algebra. For a unitary irreducible \( V \)-module \( M \), we construct an intertwining operator of type \( \left( \begin{array}{cc} M_3 & M \end{array} \right)^{M^1 M^2} \) by using the positive definite Hermitian forms \( (,)_V, (,)_M \) on \( V, M \). We then use intertwining operator to give a unitary vertex operator algebra structure on \( V \oplus M \) if we further assume that \( V \) is rational and \( C_2 \)-cofinite, and \( M \) is a simple current. These results will be useful in Section 4.

We first recall the notion of intertwining operators from [FHL].

Definition 3.1. Let \( M^1, M^2, M^3 \) be weak \( V \)-modules. An intertwining operator \( \mathcal{Y}(\cdot, z) \) of type \( \left( \begin{array}{cc} M^3 & M^1 \\ M^2 & M^3 \end{array} \right) \) is a linear map

\[
\mathcal{Y}(\cdot, z) : M^1 \to \text{Hom}(M^2, M^3)\{z\}
\]

\[
v^1 \mapsto \mathcal{Y}(v^1, z) = \sum_{n \in \mathbb{C}} v^1_n z^{-n-1}
\]

satisfying the following conditions:

(1) For any \( v^1 \in M^1, v^2 \in M^2 \) and \( \lambda \in \mathbb{C}, v^1_{n+\lambda} v^2 = 0 \) for \( n \in \mathbb{Z} \) sufficiently large.

(2) For any \( a \in V, v^1 \in M^1 \),

\[
z_0^{-1} \delta(\frac{z_1 - z_2}{z_0})Y_{M^3}(a, z_1)\mathcal{Y}(v^1, z_2) - z_0^{-1} \delta(\frac{z_2 - z_1}{z_0})\mathcal{Y}(v^1, z_2)Y_{M^3}(a, z_1)
\]

\[
= z_2^{-1} \delta(\frac{z_1 - z_0}{z_2})\mathcal{Y}(Y_{M^1}(a, z_0)v^1, z_2).
\]

(3) For \( v^1 \in M^1 \),

\[
\frac{d}{dz} \mathcal{Y}(v^1, z) = \mathcal{Y}(L(-1)v^1, z).
\]
In the following we assume that $M$ is a unitary irreducible $V$-module which has an anti-linear map $\psi$ such that $\psi(v, w) = \phi(v) \psi(w)$ for $v \in V, w \in M, n \in \mathbb{Z}$ and $\psi^2 = id$. Note that $Y_M(\cdot, z)$ is an intertwining operator of type $\left( \begin{array}{c} M \\ M V M \end{array} \right)$. Define an operator

$$\mathcal{Y}^*(\cdot, z) : M \to Hom(V, M) \{z\}$$

by the formula: for $v \in V, w \in M$,

$$\mathcal{Y}^*(w, z)v = e^{zL(-1)}Y_M(v, -z)w.$$ 

It is well-known that the operator $\mathcal{Y}^*(\cdot, z)$ is an intertwining operator of type $\left( \begin{array}{c} M \\ M V M \end{array} \right)$. We also define an operator

$$\mathcal{Y}'(\cdot, z) : M \to Hom(M, V) \{z\}$$

by the formula: for $v \in V, w_1, w_2 \in M$,

$$(\mathcal{Y}'(w_1, z)w_2, v)_V = (w_2, \mathcal{Y}^*(e^{zL(1)}(-z^{-2})L(0)\psi(w_1), z^{-1})v)_M.$$ 

**Proposition 3.2.** $\mathcal{Y}'(\cdot, z)$ is an intertwining operator of type $\left( \begin{array}{c} V \\ M M \end{array} \right)$.

**Proof:** Recall the following identities from [FHL]: for $f(z) \in \mathbb{C}[[z]]$,

$$L(-1)e^{f(z)L(0)} = e^{f(z)L(0)}L(-1)e^{-f(z)},$$

$$L(1)e^{f(z)L(0)} = e^{f(z)L(0)}L(1)e^{f(z)},$$

$$L(-1)e^{f(z)L(1)} = e^{f(z)L(1)}L(-1) - 2f(z)L(0)e^{f(z)L(1)} - f(z)^2L(1)e^{f(z)L(1)} = e^{f(z)L(1)}L(-1) - 2f(z)e^{f(z)L(1)}L(0) + f(z)^2e^{f(z)L(1)}L(1).$$

**Claim:** $\frac{d}{dz}\mathcal{Y}'(w_1, z) = \mathcal{Y}'(L(-1)w_1, z)$ for $w_1 \in M$.

For any $v \in V, w_1, w_2 \in M$, we have

$$\left(\frac{d}{dz}\mathcal{Y}'(w_1, z)w_2, v\right)_V = \frac{d}{dz}(w_2, \mathcal{Y}^*(e^{zL(1)}(-z^{-2})L(0)\psi(w_1), z^{-1})v)_M$$

$$= (w_2, \mathcal{Y}^*(\frac{d}{dz}e^{zL(1)}(-z^{-2})L(0)\psi(w_1), z^{-1})v)_M$$

$$+ (w_2, \frac{d}{dz}\mathcal{Y}^*(w, z^{-1})|_{w=e^{zL(1)}(-z^{-2})L(0)\psi(w_1)}v)_M.$$ 

Since $\mathcal{Y}^*(\cdot, z)$ is an intertwining operator, we prove the following identity by the similar discussion as in Theorem 5.2.1 of [FHL]:

$$\frac{d}{dz}\mathcal{Y}^*(w, z^{-1})|_{w=e^{zL(1)}(-z^{-2})L(0)\psi(w_1)} = \mathcal{Y}^*(e^{zL(1)}(-z^{-2})L(0)L(-1)\psi(w_1), z^{-1}) + \mathcal{Y}^*(2z^{-1}e^{zL(1)}L(0)(-z^{-2})L(0)\psi(w_1), z^{-1})$$

$$- \mathcal{Y}^*(L(1)e^{zL(1)}(-z^{-2})L(0)\psi(w_1), z^{-1}).$$
Using the following identity (see [FHL]):

\[
g \frac{d}{dz} e^{zL(1)}(-z^{-2})L(0)
= L(1)e^{zL(1)}(-z^{-2})L(0) - 2z^{-1}e^{zL(1)}L(0)(-z^{-2})L(0),
\]
gives:

\[
\left( \frac{d}{dz} \gamma'(w_1, z)w_2, v \right)_V = (w_2, \gamma^v(L(1)e^{zL(1)}(-z^{-2})L(0)\psi(w_1), z^{-1})v)_M
\]
\[
-(w_2, \gamma^v(2z^{-1}e^{zL(1)}L(0)(-z^{-2})L(0)\psi(w_1), z^{-1})v)_M
\]
\[
+(w_2, \gamma^v(e^{zL(1)}(-z^{-2})L(0)L(-1)\psi(w_1), z^{-1})v)_M
\]
\[
+(w_2, \gamma^v(2z^{-1}e^{zL(1)}L(0)(-z^{-2})L(0)\psi(w_1), z^{-1})v)_M
\]
\[
-(w_2, \gamma^v(L(1)e^{zL(1)}(-z^{-2})L(0)\psi(w_1), z^{-1})v)_M
\]
\[
= (w_2, \gamma^v(e^{zL(1)}(-z^{-2})L(0)L(-1)\psi(w_1), z^{-1})v)_M
\]
\[
= (\gamma^v(L(-1)w_1, z)w_2, v)_V.
\]

**Claim:** For any \( v \in V \) and \( w_1, w_2 \in M \), we have

\[
z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y(v, z_1)\gamma'(w_1, z_2)w_2 - z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right)\gamma'(w_1, z_2)Y_M(v, z_1)w_2
\]
\[
= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\gamma'(Y_M(v, z_0)w_1, z_2)w_2.
\]

For any \( v_1 \in V \), we have the following identity which was essentially proved in Theorem 5.2.1 of [FHL],

\[
-z_0^{-1}\delta\left(\frac{z_2 - z_1}{z_0}\right)Y_M(e^{z_1L(1)}(-z_1^{-2})L(0)v_1, z_1^{-1})\gamma^v(e^{z_2L(1)}(-z_2^{-2})L(0)v_1, z_2^{-1})v_1
\]
\[
+z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\gamma^v(e^{z_1L(1)}(-z_2^{-2})L(0)v_1, z_2^{-1})Y(e^{z_1L(1)}(-z_1^{-2})L(0)v_1, z_1^{-1})v_1
\]
\[
= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\gamma^v(e^{z_2L(1)}(-z_2^{-2})L(0)Y_M(v, z_0)w_1, z_2^{-1})v_1.
\]
Since $M$ is a unitary $V$-module, we have

\[
(z^{-1}_0 \delta(\frac{z_1 - z_2}{z_0})Y(v, z_1)Y'(w_1, z_2)w_2, v_1)_V \\
-(z^{-1}_0 \delta(\frac{z_2 - z_1}{-z_0})Y'(w_1, z_2)Y_M(v, z_1)w_2, v_1)_V \\
= (w_2, z^{-1}_0 \delta(\frac{z_1 - z_2}{z_0})Y^*(e^{z_2L(1)}(-z^{-2}_2L(0))\psi(w_1), z^{-1}_2)_M \\
\cdot Y(e^{z_1L(1)}(-z^{-2}_1L(0))\phi(v), z^{-1}_1)v_1)_M \\
-(w_2, z^{-1}_0 \delta(\frac{z_2 - z_1}{-z_0})Y_M(e^{z_1L(1)}(-z^{-2}_1L(0))\phi(v), z^{-1}_1)_M \\
\cdot Y^*(e^{z_2L(1)}(-z^{-2}_2L(0))\psi(w_1), z^{-1}_2)v_1)_M \\
= (w_2, z^{-1}_2 \delta(\frac{z_1 - z_0}{z_2})Y^*(e^{z_2L(1)}(-z^{-2}_2L(0))Y_M(\phi(v), \phi(z_0)\psi(w_1), z^{-1}_2)v_1)_M \\
= (w_2, z^{-1}_2 \delta(\frac{z_1 - z_0}{z_2})Y^*(e^{z_2L(1)}(-z^{-2}_2L(0))\psi(Y_M(v, z_0)w_1), z^{-1}_2)v_1)_M \\
= (z^{-1}_2 \delta(\frac{z_1 - z_0}{z_2})Y^*(Y_M(v, z_0)w_1, z_2), v_1)_V.
\]

This completes the proof. \qed

We now assume that $(V, \phi)$ is a rational and $C_2$-cofinite unitary vertex operator algebra. Recall that an irreducible $V$-module $M$ is called simple current if the tensor product $M \otimes M_1$ is an irreducible $V$-module for any irreducible $V$-module $M_1$. It was proved in [LY] that there exists a vertex operator algebra structure on $U = V \oplus M$ if $M$ is a simple current $V$-module satisfying some additional conditions. By using the intertwining operator constructed above, we construct a unitary vertex operator algebra structure on $U = V \oplus M$ in the following way.

**Theorem 3.3.** Let $(V, \phi)$ be a rational and $C_2$-cofinite unitary self-dual vertex operator algebra and $M$ be a simple current irreducible $V$-module having integral weights. Assume that $M$ has an anti-linear map $\psi$ such that $\psi(v, w) = \phi(v)\psi(w)$ and $\psi^2 = id$, $(\psi(w_1), \psi(w_2))_M = (w_1, w_2)_M$ and the Hermitian form $(,)_V$ on $V$ has the property that $(\phi(v_1), \phi(v_2))_V = (v_1, v_2)_V$. Then $(U, \phi_U)$ has a unique unitary vertex operator algebra structure, where $\phi_U : U \rightarrow U$ is the anti-linear involution defined by $\phi_U(v, w) = (\phi(v), \psi(w))$, for $v \in V, w \in M$. Furthermore, $U$ is rational and $C_2$-cofinite.

**Proof:** If $U$ has a vertex operator algebra structure, then the vertex operator algebra structure is unique [DM]. Moreover, $U$ is rational and $C_2$-cofinite [Y]. So it is good enough to construct a unitary vertex operator algebra structure on $U$. Let $(,)_U : U \times U \rightarrow \mathbb{C}$ be the Hermitian form on $U$ defined by $(v_1, v_1)_U = (v_1, v_2)_V$, $(v_1, w_1) = 0$ and $(w_1, w_2)_U = (w_1, w_2)_M$ for any $v_1, v_2 \in V, w_1, w_2 \in M$. It is obvious that this Hermitian form is positive definite.
Define a linear operator $Y_U(., z) : U \rightarrow End(U)[[z^{-1}, z]]$ by

$$Y_U(v, z) = \begin{pmatrix} Y(v, z) & 0 \\ 0 & Y_M(v, z) \end{pmatrix}$$

$$Y_U(w, z) = \begin{pmatrix} 0 & \mathcal{Y}(w, z) \\ \mathcal{Y}^*(w, z) & 0 \end{pmatrix}$$

for any $v \in V$ and $w \in M$.

**Claim:** The operator $Y_U(., z)$ satisfies the skew-symmetry property, i.e. for any $u, v \in U$, we have

$$Y_U(u, z)v = e^{zL(-1)}Y_U(v, -z)u.$$

We need the following identities:

$$(-z^2)^L(0)e^{zL(1)}(-z^2)^{-L(0)} = e^{-z^{-1}L(1)},$$

$$(-z^2)^{-L(0)}e^{zL(-1)}(-z^2)L(0) = e^{-z^{-1}L(-1)}.$$

Here is a proof of the second identity and the proof of the first identity is similar. It is enough to show

$$(-z^2)^{-L(0)}zL(-1)(-z^2)L(0) = -z^{-1}L(-1)$$

or

$$z^{-L(0)}L(-1)z^{L(0)} = z^{-1}L(-1)$$

which is clear.

Now we prove the claim. By definition we need to show that

$$Y_U(w_1, z)w_2 = e^{zL(-1)}Y_U(w_2, -z)w_1$$

for $w_1, w_2 \in M$. For any $v_1 \in V$, we have

$$(Y_U(w_1, z)w_2, v_1)_V$$

$$= (w_2, \mathcal{Y}^*(e^{zL(1)}(-z^{-2})L(0)\psi(w_1), z^{-1})v_1)_M$$

$$= (w_2, e^{z^{-1}L(-1)}Y_M(v_1, -z^{-1})e^{zL(1)}(-z^{-2})L(0)\psi(w_1))_M$$
and
\[(e^{zL(-1)}Y_U(w_2, -z)w_1, v_1) = (w_1, \mathcal{Y}^*(e^{-zL(1)}(-z^{-2})L(0)\psi(w_2), -z^{-1})e^{zL(1)}v_1)_M\]
\[= (w_1, e^{-(z^{-1}L(-1))}Y_M(e^{zL(1)}v_1, z^{-1})e^{-zL(1)}(-z^{-2})L(0)\psi(w_2))_M\]
\[= (Y_M(e^{zL(1)}(-z^{-2})L(0)\phi(e^{zL(1)}v_1), z)e^{-zL(1)}w_1, e^{-zL(1)}(-z^{-2})L(0)\psi(w_2))_M\]
\[= (Y_M((-z^{-2})L(0)e^{-zL(-1)}Y_M((-z^{-2})L(0)\phi(v_1), z)e^{-zL(1)}w_1, \psi(w_2))_M\]
\[= (e^{zL(-1)}((-z^{-2})L(0)Y_M((-z^{-2})L(0)\phi(v_1), z)e^{-zL(1)}w_1, \psi(w_2))_M\]
\[= (e^{zL(-1)}Y_M(\phi(v_1), -z^{-1}(-z^{-2})L(0)e^{-zL(1)}w_1, \psi(w_2))_M\]
\[= (e^{zL(-1)}Y_M(\phi(v_1), -z^{-1}e^{zL(1)}(-z^{-2})L(0)w_1, \psi(w_2))_M\]
\[= (\psi(e^{zL(-1)}Y_M(\phi(v_1), -z^{-1}e^{zL(1)}(-z^{-2})L(0)w_1), w_2)_M\]
\[= (w_2, e^{-zL(-1)}Y_M(v_1, -z^{-1}e^{zL(1)}(-z^{-2})L(0)\psi(w_1), w_2)_M\]
\[= (Y_U(w_1, z)w_2, v_1)_V.\]

So the claim is established.

We can now prove that \((U, Y_U(\cdot, z))\) is a vertex operator algebra. Since the skew symmetry holds, it is enough to have the locality. By Theorem 5.6.2 of [FHL], we only need to prove the locality for three elements in \(M\). But this follows a similar discussion in Proposition 3 of [LY].

To prove that \((U, \phi_U)\) is a unitary vertex operator algebra, we first prove that \(\phi_U\) is an anti-linear involution of \(U\). Obviously, the order of \(\phi_U\) is 2. So it suffices to prove that \(\phi_U(u_nv) = \phi_U(u)_n\phi_U(v)\) for any \(u, v \in U\). We now verify this property case by case. If \(u, v \in V\) this is obvious. If \(u \in V\) and \(v \in M\), we have \(\phi_U(u_nv) = \phi(u)_n\psi(v) = \phi_U(u)_n\phi_U(v)\). If \(u \in M, v \in V\), we have \(\phi_U(Y_U(u, z)v) = \phi_U(e^{zL(-1)}Y(v, -z)u) = e^{zL(-1)}Y(\phi(v), -z)\psi(u) = Y_U(\phi_U(u), z)\phi_U(v)\), this implies \(\phi_U(u_nv) = \phi_U(u)_n\phi_U(v)\). If \(u \in M, v \in M\), for any \(v_1 \in V\) we have:

\[\phi_U(Y_U(u, z)v, \phi(v_1)) = (Y_U(u, z)v, v_1)\]
\[= (Y_U(u, z)v, \psi(v_1)) = (\psi(v), \psi(Y^*(e^{zL(0)}(-z^{-2})L(0)\psi(u), z^{-1})v_1))\]
\[= (\psi(v), Y^*(e^{zL(0)}(-z^{-2})L(0)u, z^{-1})\phi(v_1))\]
\[= (Y_U(\psi(u), z)\psi(v), \phi(v_1)).\]

Thus we have \(\phi_U(u_nv) = \phi_U(u)_n\phi_U(v)\), then \(\phi_U\) is an anti-linear involution of \(U\).

It remains to prove the invariant property

\[(Y(e^{zL(1)}(-z^{-2})L(0)a, z^{-1})u, v) = (u, Y(\phi_U(x), z)v)\]
holds for any $a, u, v \in U$. This is obvious from the definition of $Y_U(., z)$. The proof of theorem is complete. 

4 Examples of unitary vertex operator algebras

In this section we prove that most of the well-known rational and $C_2$-cofinite vertex operator algebras are unitary vertex operator algebras. However, there exist some unitary vertex operator algebras which are neither rational nor $C_2$-cofinite.

4.1 Unitary Virasoro vertex operator algebras

In this subsection we construct unitary vertex operator algebras associated to the Virasoro algebra. First, we recall some facts about Virasoro vertex operator algebras [FZ], [W]. We denote the Virasoro algebra by $L = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_n \oplus \mathbb{C} C$ with the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0} C,$$

$$[L_m, C] = 0.$$

Set $b = (\bigoplus_{n \geq 1} \mathbb{C} L_n) \oplus (\mathbb{C} L_0 \oplus \mathbb{C} C)$, then we know that $b$ is a subalgebra of $L$. For any two complex numbers $(c, h) \in \mathbb{C}$, let $\mathbb{C}$ be a 1-dimensional $b$-module defined as follows:

$$L_n \cdot 1 = 0, n \geq 1,$$

$$L_0 \cdot 1 = h \cdot 1,$$

$$C \cdot 1 = c \cdot 1.$$

Set

$$V(c, h) = U(L) \otimes_{U(b)} \mathbb{C}$$

where $U(.)$ denotes the universal enveloping algebra. Then $V(c, h)$ is a highest weight module of the Virasoro algebra of highest weight $(c, h)$, which is called the Verma module of Virasoro algebra, and $V(c, h)$ has a unique maximal proper submodule $J(c,h)$. Let $L(c,h)$ be the unique irreducible quotient module of $V(c,h)$. Set

$$\overline{V(c,0)} = V(c,0)/(U(L)L_{-1}1 \otimes 1),$$

it is well-known that $\overline{V(c,0)}$ has a vertex operator algebra structure with Virasoro element $\omega = L_{-2}1$ and $L(c,0)$ is the unique irreducible quotient vertex operator algebra of $\overline{V(c,0)}$ [FZ].

For $m \geq 2$, set

$$c_m = 1 - \frac{6}{m(m+1)}.$$
\[ h_{r,s}^m = \frac{(r(m + 1) - sm)^2 - 1}{4m(m + 1)}, \quad (1 \leq s \leq r \leq m - 1). \]

It was proved in [W], [DLM3] that \( L(c_m, 0) \) \((m \geq 2)\) are rational and \( C_2 \)-cofinite vertex operator algebra and \( L(c_m, h_{r,s}^m) \) are the complete list of irreducible \( L(c_m, 0) \)-modules.

Now we recall some fact about the Hermitian form on \( L(c, h) \). For \((c, h) \in \mathbb{R}\), it was proved in Proposition 3.4 of [KR] that there is a unique Hermitian form \( (, ) \) such that
\[ (v_{c,h}, v_{c,h}) = 1, \]
\[ (L_n u, v) = (u, L_{-n} v), \]
for any \( u, v \in L(c, h) \), where \( v_{c,h} \) denotes the highest weight vector of \( L(c, h) \). It is well-known that \( L(c, h) \) is unitary, i.e. the Hermitian form \( (, ) \) on \( L(c, h) \) is positive definite, if and only if \( c \geq 1, h \geq 0 \) or \( c = c_m, h = h_{r,s}^m \) [KR].

For any real number \( c \), define an anti-linear map \( \overline{\phi} \) of \( V(c, 0) \) as follow:
\[ \overline{\phi} : \overline{V(c, 0)} \rightarrow \overline{V(c, 0)} \]
\[ L_{-n_1} \cdots L_{-n_k} \cdot 1 \mapsto L_{-n_1} \cdots L_{-n_k} \cdot 1, n_1 \geq \cdots \geq n_k \geq 2. \]

**Lemma 4.1.** Assume that \( c \in \mathbb{R} \), and let \( \overline{\phi} \) be the anti-linear map defined above. Then \( \overline{\phi} \) is an anti-linear involution of vertex operator algebra \( \overline{V(c, 0)} \). Furthermore, \( \overline{\phi} \) induces an anti-linear involution \( \phi \) of \( L(c, 0) \).

**Proof:** Since we have \( \overline{\phi}^2 = id \), it is good enough to prove that \( \overline{\phi} \) is an anti-linear automorphism. Let \( U \) be the subspace of \( \overline{V(c, 0)} \) which is defined by
\[ U = \{ u \in \overline{V(c, 0)} | \overline{\phi}(u_n v) = \overline{\phi}(u_n) \overline{\phi}(v), \forall v \in \overline{V(c, 0)}, n \in \mathbb{Z} \}. \]
It is easy to prove that if \( a, b \in U \), then \( a m b \in U \) for any \( m \in \mathbb{Z} \). Note that \( 1 \in U \) and \( \omega = L_{-2} \cdot 1 \in U \). Thus \( U = \overline{V(c, 0)} \) as \( \overline{V(c, 0)} \) is generated by \( \omega \). This implies that \( \overline{\phi} \) is an anti-linear involution of \( \overline{V(c, 0)} \).

Let \( \overline{J(c, 0)} \) be the maximal proper \( L \)-submodule of \( \overline{V(c, 0)} \), we have \( \overline{\phi(\overline{J(c, 0)})} \) is a proper \( L \)-submodule of \( \overline{V(c, 0)} \), this implies \( \overline{\phi(\overline{J(c, 0)})} \subset \overline{J(c, 0)} \). Thus \( \overline{\phi} \) induces an anti-linear involution \( \phi \) of \( L(c, 0) \). □

Now we have the main result in the subsection.

**Theorem 4.2.** Assume that \( c \in \mathbb{R} \) and let \( \phi \) be the anti-linear involution of \( L(c_m, 0) \) defined above. Then \( (L(c, 0), \phi) \) is a unitary vertex operator algebra if and only if \( c \geq 1 \) or \( c = c_m \) for some integer \( m \geq 2 \).

**Proof:** Assume that \( c \geq 1 \) or \( c = c_m \) for some integer \( m \geq 2 \). Then the Hermitian form \( (, ) \) defined above is positive definite. We now prove the invariant property in the definition of the unitary vertex operator algebra. Recall from [FZ],
[W] that \( L(n)u = L_n u \) for any \( u \in L(c, 0) \), then we have \((L(n)u, v) = (u, L(-n)v)\) for any \( u, v \in L(c, 0) \). Thus we have

\[
(Y(e^{zL(1)}(-z^{-2})L(0)\omega, z^{-1})u, v) = z^{-4}(Y(\omega, z^{-1})u, v)
= \sum_{n \in \mathbb{Z}} (\omega_{n+1}u, v)z^{n-2}
= \sum_{n \in \mathbb{Z}} (L(n)u, v)z^{n-2}
= \sum_{n \in \mathbb{Z}} (u, L(-n)v)z^{n-2}
= \sum_{n \in \mathbb{Z}} (u, \omega_{-n+1}v)z^{n-2}
= (u, Y(\omega, z)v)
= (u, Y(\phi(\omega), z)v).
\]

Since \( L(c, 0) \) is generated by \( \omega \), \( (L(c, 0), \phi) \) is a unitary vertex operator algebra by Proposition 2.11.

We now assume that \( (L(c, 0), \phi) \) is a unitary vertex operator algebra. By the Lemma 2.5, \( L(c, 0) \) is unitary as the module of the Virasoro algebra, thus \( c \geq 1 \) or \( c = cm \) for some integer \( m \geq 2 \).

**Remark 4.3.** Note that if \( c \geq 1 \), \( (L(c, 0), \phi) \) is a unitary vertex operator algebra, although \( L(c, 0) \) is neither rational nor \( C_2 \)-cofinite.

We also have the following proposition by the similar discussion as in Theorem 4.2.

**Theorem 4.4.** Assume that \( c \in \mathbb{R} \) and let \( \phi \) be the anti-linear involution of \( L(c, 0) \) defined above. Then \( L(c, h) \) is a unitary module of \( L(c, 0) \) if and only if \( c \geq 1, h \geq 0 \) or \( c = cm, h = km \).

### 4.2 Unitary affine vertex operator algebras

In this subsection we construct unitary vertex operator algebra associated to the affine Kac-Moody algebras. First, we recall some facts about affine vertex operator algebras [FZ]. Let \( \mathfrak{g} \) be a finite dimensional simple Lie algebra and \( \mathfrak{h} \) a Cartan subalgebra. Fix a non-degenerate symmetric invariant bilinear form \( (, ) \) on \( \mathfrak{g} \) so that \( (\theta, \theta) = 2 \) where \( \theta \) is the maximal root of \( \mathfrak{g} \). Consider the affine Lie algebra \( \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \) with the commutation relations

\[
[x(m), y(n)] = [x, y](m + n) + Km(x, y)\delta_{m+n,0},
[\hat{\mathfrak{g}}, K] = 0,
\]

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where \( x(m) = x \otimes t^n \).

For complex number \( k \in \mathbb{C} \), set

\[
V_g(k) = U(\hat{g})/J_k,
\]

where \( J_k \) is the left ideal of \( U(\hat{g}) \) generated by \( x(n) \) and \( K - k \) for \( x \in g \) and \( n \geq 0 \). It is well-known that \( V_g(k) \) has a vertex operator algebra structure if \( k \neq -h^\vee \) where \( h^\vee \) is the dual Coxeter number of \( g \). Moreover, \( V_g(k) \) has a unique maximal proper \( \hat{g} \)-submodule \( J(k) \) and \( L_g(k, 0) = V_g(k)/J(k) \) is a simple vertex operator algebra. As usual we denote the corresponding irreducible highest weight module for \( \hat{g} \) associated to a highest weight \( \lambda \in \mathfrak{h}^* \) of \( g \) by \( L_g(k, \lambda) \). It was proved in [FZ], [DLM3] that \( L_g(k, 0) \) is a simple rational and \( C_2 \)-cofinite vertex operator algebra if \( k \in \mathbb{Z}^+ \) and

\[
\{ L_g(k, \lambda) | (\lambda, \theta) \leq k, \lambda \in \mathfrak{h}^* \text{ is integral dominant} \}
\]

are the complete list of inequivalent irreducible \( L_g(k, 0) \)-modules.

Let \( \omega_0 \) be the compact involution [K] of \( g \) which is the anti-linear automorphism determined by:

\[
\omega_0(e_i) = -f_i, \quad \omega_0(f_i) = -e_i, \quad \omega_0(h_i) = -h_i,
\]

where \( \{h_i, e_i, f_i\} \) are the Chevalley generators of \( g \).

**Lemma 4.5.** For any \( x, y \in g \), we have \( \langle \omega_0(x), \omega_0(y) \rangle = \langle x, y \rangle \).

**Proof:** Note that it is good enough to prove that \( \kappa(\omega_0(x), \omega_0(y)) = \kappa(x, y) \), where \( \kappa(, ) \) is the Killing form of \( g \). Recall that \( \kappa(x, y) = \text{tr}(adxady) \), let \( z_1, \ldots, z_k \) be the bases of \( g \) such that \( adxady \) is a upper-triangular matrix, then we have \( adxady(z_i) = \lambda_i z_i + w_i \) for \( 1 \leq i \leq k \), where \( w_i \) is some linearly combine of \( z_1, \ldots, z_{i-1} \). On the other hand, we have \( [\omega_0(x), [\omega_0(y), \omega_0(z)]] = \omega_0([x, [y, z]]) = \omega_0(adxady(z)) = \lambda_i \omega_0(z_i) + v_i \) for \( 1 \leq i \leq k \), where \( v_i \) is some linearly combine of \( \omega_0(z_1), \ldots, \omega_0(z_{i-1}) \). This implies that \( \kappa(\omega_0(x), \omega_0(y)) = \kappa(x, y) \). \( \square \)

Let \( \hat{\omega}_0 \) be the compact involution [K] of \( \hat{g} \) which is the anti-linear automorphism determined by:

\[
\hat{\omega}_0(x \otimes t^m) = \omega_0(x) \otimes t^{-m}, \hat{\omega}_0(K) = -K.
\]

It was proved in Theorem 11.7 of [K] that if \( k \in \mathbb{Z}^+ \) and \( \lambda \) is an integral dominant weight such that \( (\lambda, \theta) \leq k \), then \( L_g(k, \lambda) \) has a unique positive definite Hermitian form \( (, ) \) such that

\[
(1, 1) = 1,
\]

\[
(xu, v) = -(u, \hat{\omega}_0(x)v)
\]

for \( x \in \hat{g}, u, v \in L_g(k, \lambda) \) where \( 1 \) is a fixed highest weight vector of \( L_g(k, \lambda) \).
We also introduce a linear map $\omega_k : \hat{\mathfrak{g}} \rightarrow \hat{\mathfrak{g}}$ such that $\omega_\hat{\mathfrak{g}}(x(n)) = \omega_0(x)(n)$ and $\omega_\hat{\mathfrak{g}}(K) = K$. Let $\hat{T}(\hat{\mathfrak{g}})$ be the tensor product algebra of the affine Lie algebra $\hat{\mathfrak{g}}$. Define an anti-linear map $\Phi_T$ of $\hat{T}(\hat{\mathfrak{g}})$ as follow: for any $y_1, \ldots, y_k \in \hat{\mathfrak{g}}$

$$\Phi_T : \hat{T}(\hat{\mathfrak{g}}) \rightarrow \hat{T}(\hat{\mathfrak{g}})$$

$$y_1 \otimes \cdots \otimes y_k \mapsto \omega_\hat{\mathfrak{g}}(y_1) \otimes \cdots \otimes \omega_\hat{\mathfrak{g}}(y_k).$$

It is easy to prove that $\Phi_T$ is an anti-linear involution such that $\Phi_T x(m) \Phi_T^{-1} = \omega_0(x)(m)$. Recall that the universal enveloping algebra $U(\hat{\mathfrak{g}})$ is defined to be the quotient $\hat{T}(\hat{\mathfrak{g}})/I$, where $I$ is the ideal of $\hat{T}(\hat{\mathfrak{g}})$ generated by $a \otimes b - b \otimes a - [a, b]$, $a, b \in \hat{\mathfrak{g}}$. By Lemma 4.5, we have

$$\Phi_T(x(m) \otimes y(n) - y(n) \otimes x(m) - [x(m), y(n)]) = \omega_0(x)(m) \otimes \omega_0(y)(n) - \omega_0(y)(n) \otimes \omega_0(x)(m)$$

$$- [\omega_0(x), \omega_0(y)](m + n) - Km(x, y)\delta_{m+n,0}$$

$$\Phi_T(x(m) \otimes y(n) - y(n) \otimes x(m) - [x(m), y(n)]) = \omega_0(x)(m) \otimes \omega_0(y)(n) - \omega_0(y)(n) \otimes \omega_0(x)(m)$$

$$- [\omega_0(x), \omega_0(y)](m + n) - Km(0, x, y)\omega_0(0)\delta_{m+n,0}$$

$$= \omega_0(x)(m) \otimes \omega_0(y)(n) - \omega_0(y)(n) \otimes \omega_0(x)(m) - [\omega_0(x)(m), \omega_0(y)(n)].$$

This implies that $\Phi_T(I) \subset I$, then $\Phi_T$ induces an anti-linear involution $\Phi_U$ of $U(\hat{\mathfrak{g}})$ such that $\Phi_U x(m) \Phi_U^{-1} = \omega_0(x)(m)$. Note that if $k \in \mathbb{R}$, we have $\Phi_U(J_k) \subset J_k$, then $\Phi_U$ induces an anti-linear map $\Phi$ of $V_\mathfrak{g}(k)$ such that $\Phi x(m) \Phi^{-1} = \omega_0(x)(m)$ and $\Phi^2 = id$.

**Lemma 4.6.** Assume that $k \in \mathbb{R}$ and $k \neq h^\vee$, and let $\Phi$ be the anti-linear map defined above. Then $\Phi$ is an anti-linear involution of vertex operator algebra $V_\mathfrak{g}(k)$. Furthermore, $\Phi$ induces an anti-linear involution $\phi$ of $L_\mathfrak{g}(k, 0)$.

**Proof:** Since we have $\Phi^2 = id$, it is good enough to prove that $\Phi$ is an anti-linear automorphism. Let $U$ be the subspace of $V_\mathfrak{g}(k)$ which is defined by

$$U = \{ u \in V_\mathfrak{g}(k) \mid \Phi(u_n v) = \Phi(u)_n \Phi(v), \forall v \in V_\mathfrak{g}(k), n \in \mathbb{Z} \}.$$

It is easy to prove that if $u, v \in U$, then $u_m v \in U$ for any $m \in \mathbb{Z}$. Note that $1 \in U$ and $x(-1)1 \in U$ for any $x \in \mathfrak{g}$. Thus we have $U = V_\mathfrak{g}(k)$, since $V_\mathfrak{g}(k)$ is generated by $x(-1)1, x \in \mathfrak{g}$. Thus $\Phi$ is an anti-linear involution of $V_\mathfrak{g}(k)$.

Note that $\Phi(J(k))$ is a proper $\hat{\mathfrak{g}}$-submodule of $V_\mathfrak{g}(k)$. This forces $\Phi(J(k)) \subset J(k)$. So $\Phi$ induces an anti-linear involution $\phi$ of $L_\mathfrak{g}(k, 0)$.

Now we can prove the main result in this subsection.

**Theorem 4.7.** Assume that $k \in \mathbb{R}$ and $k \neq h^\vee$, and let $\phi$ be the anti-linear automorphism of $L_\mathfrak{g}(k, 0)$ defined as above. Then $(L_\mathfrak{g}(k, 0), \phi)$ is a unitary vertex operator algebra if and only if $k \in \mathbb{Z}^+$. 

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Proof: Assume that $k \in \mathbb{Z}^+$. Then the Hermitian form $(,)$ on $L_g(k, 0)$ is positive definite. We now prove the invariant property. Recall from [FZ] that $Y(x(-1)1, z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1}$ for $x \in \mathfrak{g}$. Note that

$$(x(n)u, v) = -(u, \omega_0(x)(-n)v)$$

for $x \in \hat{\mathfrak{g}}$ and $u, v \in L_g(k, 0)$. Thus we have:

$$
\begin{align*}
(Y(e^{zL(1)}(-z^{-2})L(0)x(-1)1, z^{-1})u, v) &= (Y(-z^{-2}x(-1)1, z^{-1})u, v) \\
&= \sum_{n \in \mathbb{Z}} -z^{-2}(x(n)u, v)z^{n+1} \\
&= \sum_{n \in \mathbb{Z}} (u, \omega_0(x)(-n)v)z^{n-1} \\
&= (u, Y(\phi(x(-1)1), z)v).
\end{align*}
$$

Since $L_g(k, 0)$ is generated by $x(-1)1$, we have $(L_g(k, 0), \phi)$ is a unitary vertex operator algebra by Proposition 2.11.

Conversely, assume that $(L_g(k, 0), \phi)$ is a unitary vertex operator algebra. By the definition and the calculation as above, we have a positive definite Hermitian form $(,)$ on $L_g(k, 0)$ such that for any $x \in \mathfrak{g}$

$$(x(n)u, v) = -(u, \omega_0(x)(-n)v).$$

This implies that $L_g(k, 0)$ is unitary as $\hat{\mathfrak{g}}$-module and $k \in \mathbb{Z}^+$ by Theorem 11.7 of [K].

The following result is immediate by the similar discussion as in Theorem 4.7.

**Theorem 4.8.** Assume that $k \in \mathbb{Z}^+$ and let $\phi$ be the anti-linear automorphism of $L_g(k, 0)$ defined as above. Assume that $\lambda \in \mathfrak{h}^*$. Then $L_g(k, \lambda)$ is unitary $L_g(k, 0)$-modules if and only if $\lambda$ is integral dominant and $(\lambda, \theta) \leq k$.

### 4.3 Unitary Heisenberg vertex operator algebras

In this subsection we prove that the Heisenberg vertex operator algebras are unitary. First, we recall some facts about Heisenberg vertex operator algebras from [FLM], [LL]. Let $\mathfrak{h}$ be a finite dimensional vector space of dimension $d$ which has a non-degenerate symmetric bilinear form $(, )$. Consider the affine algebra

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

with the commutation relations: for $\alpha, \beta \in \mathfrak{h}$,

$$[\alpha(m), \beta(n)] = Km(\alpha, \beta)\delta_{m+n,0}.$$
\[ [\mathfrak{h}, K] = 0, \]

where \( \alpha(n) = \alpha \otimes t^n \).

For any \( \lambda \in \mathfrak{h} \), set

\[ M_\mathfrak{h}(1, \lambda) = U(\mathfrak{h})/J_\lambda, \]

where \( J_\lambda \) is the left ideal of \( U(\mathfrak{h}) \) generated by \( \alpha(n) \), \( n \geq 1 \), \( \alpha(0) - (\alpha, \lambda) \) and \( K - 1 \). Set \( e^\lambda = 1 + J_\lambda \), we know that \( M_\mathfrak{h}(1, \lambda) \) is spanned by \( \alpha_1(-n_1) \cdots \alpha_k(-n_k)e^\lambda \), \( n_1 \geq \cdots \geq n_k \geq 1 \). Let \( \alpha_1, \cdots, \alpha_d \) be an orthonormal basis of \( \mathfrak{h} \). Set \( \omega = \frac{1}{2} \sum_{1 \leq i \leq d} \alpha_i(-1)^21 \). It is well-known that \( M_\mathfrak{h}(1, 0) \) is a vertex operator algebra such that \( 1 = 1 + J_0 \) is the vacuum vector and \( \omega \) is the Virasoro element [LL]. Furthermore, \( M_\mathfrak{h}(1, \lambda) \) is an irreducible ordinary \( M_\mathfrak{h}(1, 0) \)-module.

In the following we assume that \( \mathfrak{h} \) is of dimension 1, i.e. \( \mathfrak{h} = \mathbb{C} \alpha \), and that \( (\alpha, \alpha) = 1 \). In this case, we will denote \( M_\mathfrak{h}(1, 0) \) and \( M_\mathfrak{h}(1, \lambda) \) by \( M(1, 0) \) and \( M(1, \lambda) \), respectively. It was proved in [KR] that if \( (\alpha, \lambda) \geq 0 \) there exists a unique positive definite Hermitian form on \( M(1, \lambda) \) such that

\[ (e^\lambda, e^\lambda) = 1, \]

\[ (\alpha(n)u, v) = (u, \alpha(-n)v) \]

for \( u, v \in M(1, \lambda) \).

Let \( \phi \) be an anti-linear map \( \phi : M(1, 0) \to M(1, 0) \) such that

\[ \phi(\alpha(-n_1) \cdots \alpha(-n_k)) = (-1)^k \alpha(-n_1) \cdots \alpha(-n_k). \]

Note that \( \phi \alpha(n)\phi^{-1} = -\alpha(n) \) for \( n \in \mathbb{Z} \). Using a proof similar that of Lemmas 4.1, 4.6 shows that \( \phi \) is an anti-linear involution of \( M(1, 0) \).

**Proposition 4.9.** Let \( \phi \) be the anti-linear involution of \( M(1, 0) \) defined above. Then \( (M(1, 0), \phi) \) is a unitary vertex operator algebra and \( M(1, \lambda) \) is a unitary irreducible \( M(1, 0) \)-module if \( (\alpha, \lambda) \geq 0 \).

**Proof:** By the discussion above, we only need to prove the invariant property. Since \( M(1, 0) \) is generated by \( \alpha(-1) \), by Proposition 2.12 it is enough to prove that

\[ (Y(e^{zL(1)}(-z^{-2})^{L(0)}\alpha(-1), z^{-1})u, v) = (u, Y(\phi(\alpha(-1)), z)v) \]

for \( u, v \in M(1, \lambda) \). A straightforward computation gives

\[
(Y(e^{zL(1)}(-z^{-2})^{L(0)}\alpha(-1), z^{-1})u, v) \\
= (Y(-z^{-2}\alpha(-1), z^{-1})u, v) \\
= \sum_{n \in \mathbb{Z}} -z^{-2}(\alpha(n)u, v)z^{n+1} \\
= \sum_{n \in \mathbb{Z}} (u, -\alpha(-n)v)z^{n-1} \\
= (u, Y(\phi(\alpha(-1)), z)v).
\]
Then $M(1, \lambda)$ is a unitary module for $M(1, 0)$. In particular, $M(1, 0)$ is a unitary vertex operator algebra. 

Note that if $\mathfrak{h}$ is a finite dimensional vector space of dimension $d$ and assume that $\alpha_1, \ldots, \alpha_d$ is an orthonormal basis of $\mathfrak{h}$ with respect $(, )$. Then we have $M_\mathfrak{h}(1, 0) \cong M_{\mathbb{C} \alpha_1}(1, 0) \otimes \cdots \otimes M_{\mathbb{C} \alpha_d}(1, 0)$, by Propositions 2.9, 2.10 we have the following result for general Heisenberg vertex operator algebra $M_\mathfrak{h}(1, 0)$.

**Proposition 4.10.** Let $\mathfrak{h}$ be a finite dimensional vector space of dimension $d$ which has a non-degenerate symmetric bilinear form $(, )$ and $\alpha_1, \ldots, \alpha_d$ be an orthonormal basis of $\mathfrak{h}$ with respect $(, )$. Then $M_\mathfrak{h}(1, 0)$ is a unitary vertex operator algebra. Furthermore, if $(\alpha_i, \lambda) \geq 0$, $1 \leq i \leq d$, then $M_\mathfrak{h}(1, \lambda)$ is a unitary irreducible $M_\mathfrak{h}(1, 0)$-module.

### 4.4 Unitary lattice vertex operator algebras

In this subsection we prove that the lattice vertex operator algebras associated to positive definite even lattices are unitary. First, we recall from [FLM], [D1] some facts about lattice vertex operator algebras. Let $L$ be a positive definite even lattice and $\hat{L}$ be the canonical central extension of $L$ by the cyclic group $\langle \kappa \rangle$ of order 2:

$$1 \rightarrow \langle \kappa \rangle \rightarrow \hat{L} \rightarrow L \rightarrow 1,$$

with the commutator map $c(\alpha, \beta) = \kappa^{(\alpha, \beta)}$. Let $e : L \rightarrow \hat{L}$ be a section such that $e_0 = 1$ and $e_0 : L \times L \rightarrow \langle \kappa \rangle$ be the corresponding 2-cocycle. Then $e_0(\alpha, \beta)e_0(\beta, \alpha) = \kappa^{(\alpha, \beta)}$ and $e_0 e_\beta = e_0(\alpha, \beta)e_{\alpha + \beta}$ for $\alpha, \beta \in L$. Let $\nu : \langle \kappa \rangle \rightarrow \langle \pm 1 \rangle$ be the isomorphism such that $\nu(\kappa) = -1$ and set

$$\epsilon = \nu \circ e_0 : L \times L \rightarrow \langle \pm 1 \rangle.$$

Consider the induced $\hat{L}$-module:

$$\mathbb{C}\{L\} = \mathbb{C}[\hat{L}] \otimes_{\langle \kappa \rangle} \mathbb{C} \cong \mathbb{C}[L] \text{ (linearly)},$$

where $\mathbb{C}[.]$ denotes the group algebra and $\kappa$ acts on $\mathbb{C}$ as multiplication by $-1$. Then $\mathbb{C}[L]$ becomes a $\hat{L}$-module such that $e_\alpha \cdot e_\beta = \epsilon(\alpha, \beta)e_{\alpha + \beta}$ and $\kappa \cdot e_\beta = -e_\beta$.

We also define an action $h(0)$ on $\mathbb{C}[L]$ by $h(0) \cdot e_\alpha = (h, \alpha)e_\alpha$ for $h \in \mathfrak{h}, \alpha \in L$ and an action $z^h$ on $\mathbb{C}[L]$ by $z^h \cdot e_\alpha = z^{(h, \alpha)}e_\alpha$.

Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$, and consider the corresponding Heisenberg vertex operator algebra $M_\mathfrak{h}(1, 0)$, which is denoted by $M(1)$ in the following. The untwisted Fock space associated with $L$ is defined to be

$$V_L = M(1) \otimes_{\mathbb{C}} \mathbb{C}\{L\} \cong M(1) \otimes_{\mathbb{C}} \mathbb{C}[L] \text{ (linearly)}.$$

Then $\hat{L}$, $h(n)(n \neq 0)$, $h(0)$ and $z^{h(0)}$ act naturally on $V_L$ by acting on either $M(1)$ or $\mathbb{C}[L]$ as indicated above. It was proved in [B] and [FLM] that $V_L$ has a vertex
operator algebra structure which is determined by

\[ Y(h(-1)1,z) = h(z) = \sum_{n \in \mathbb{Z}} h(n)z^{-n-1} \quad (h \in \mathfrak{h}), \]

\[ Y(e^\alpha, z) = E^\alpha(-\alpha, z)E^\beta(-\alpha, z)e_\alpha z^\alpha, \]

where

\[ E^\alpha(-\alpha, z) = \exp(\sum_{n<0} \frac{\alpha(n)}{n} z^{-n}), \]

\[ E^\beta(-\alpha, z) = \exp(\sum_{n>0} \frac{\alpha(n)}{n} z^{-n}). \]

Recall that \( L^* = \{ \lambda \in \mathfrak{h} | (\alpha, \lambda) \in \mathbb{Z} \} \) is the dual lattice of \( L \). There is an \( \hat{L} \)-module structure on \( \mathbb{C}[L^*] = \bigoplus_{\lambda \in \mathfrak{L}} \mathbb{C}e^\lambda \) such that \( \kappa \) acts as \(-1\) (see [DL]). Let \( L^* = \cup_{i \in \mathbb{L}/L}(L+\lambda_i) \) be the coset decomposition such that \((\lambda_i, \lambda_i)\) is minimal among all \((\lambda, \lambda)\) for \( \lambda \in L+\lambda_i \). In particular, \( \lambda_0 = 0 \). Set \( \mathbb{C}[L+\lambda_i] = \bigoplus_{\alpha \in L} \mathbb{C}e^{\alpha+\lambda_i} \). Then \( \mathbb{C}[L^*] = \bigoplus_{i \in \mathbb{L}/L} \mathbb{C}[L+\lambda_i] \) and each \( \mathbb{C}[L+\lambda_i] \) is an \( \hat{L} \)-submodule of \( \mathbb{C}[L^*] \).

The action of \( \hat{L} \) on \( \mathbb{C}[L+\lambda_i] \) is defined as follow:

\[ e_\alpha e^{\beta+\lambda_i} = e(\alpha, \beta)e^{\alpha+\beta+\lambda_i} \]

for \( \alpha, \beta \in L \). On the surface, the module structure on each \( \mathbb{C}[L+\lambda_i] \) depends on the choice of \( \lambda_i \) in \( L+\lambda_i \). It is easy to prove that different choices of \( \lambda_i \) give isomorphic \( \hat{L} \)-modules.

Set \( \mathbb{C}[M] = \bigoplus_{\lambda \in \mathfrak{M}} \mathbb{C}e^\lambda \) for a subset \( M \) of \( L^* \), and define \( V_M = M(1) \otimes \mathbb{C}[M] \). Then \( V_{L+\lambda_i} \) for \( i \in \mathbb{L}/L \) are the irreducible modules for \( V_L \) (see [B], [FLM], [D1]).

Recall that there is an automorphism \( \theta \) of \( V_L \) which is defined as follow:

\[ \theta: V_L \to V_L \]

\[ \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes e^\alpha \mapsto (-1)^k \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes e^{-\alpha}. \]

In particular, \( \theta \) induces an automorphism of \( M(1) \). Next we recall a construction of \( \theta \)-twisted modules for \( M(1) \) and \( V_L \) [FLM], [D2]. Denote \( \mathfrak{h}[-1] = \mathfrak{h} \otimes t^{1/2}\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \) the twisted affinization of \( \mathfrak{h} \) defined by the communication relations

\[ [\alpha \otimes t^m, \beta \otimes t^n] = Km(\alpha, \beta)\delta_{m+n,0}, \]

\[ [\mathfrak{h}, K] = 0, \]

for \( m, n \in 1/2 + \mathbb{Z} \). Then the symmetric algebra \( M(1)(\theta) = S(t^{1/2}\mathbb{C}[t^{-1}] \otimes \mathfrak{h}) \) is the unique \( \mathfrak{h}[-1] \)-module such that \( K = 1 \) and \( \alpha \otimes t^n \cdot 1 = 0 \) if \( n > 0 \). It was proved in [FLM] that \( M(1)(\theta) \) is a \( \theta \)-twisted \( M(1) \)-module.

By abuse the notation we also use \( \theta \) to denote the automorphism of \( \hat{L} \) defined by \( \theta(e_\alpha) = e_{-\alpha} \) and \( \theta(\kappa) = \kappa \). Set \( K = \{ \theta(a)a^{-1} | a \in \hat{L} \} \). Let \( \chi \) be a central
character of $\hat{L}/K$ such that $\chi(\kappa K) = -1$ and $T_\chi$ be the irreducible $\hat{L}/K$-module with the character $\chi$. Then $\hat{L}$, $h(n)$ $(n \in 1/2 + \mathbb{Z})$ act naturally on $V_{L_\chi}^{T_\chi} = M(1)(\theta) \otimes T_\chi$ by acting on either $M(1)(\theta)$ or $T_\chi$ as indicated above. It was proved in [FLM] that $V_{L_\chi}^{T_\chi} = M(1)(\theta) \otimes T_\chi$ is an irreducible $\theta$-twisted $V_L$-module such that

$$Y_\theta(\alpha(-1) \cdot 1, z) = \alpha(z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} \alpha(n) z^{-n-1},$$

$$Y_\theta(e^\alpha, z) = 2^{-(\alpha,\alpha)} E^-(\alpha, z) E^+(-\alpha, z) e_\alpha z^{-(\alpha,\alpha)/2},$$

where

$$E^\pm(\alpha, z) = \exp\left( \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{\alpha(n)}{n} z^{-n} \right).$$

We now define a Hermitian form on $V_{L_\circ}$. First, there is a positive definite Hermitian form on $\mathbb{C}[L_\circ]$:

$$(\cdot, \cdot) : \mathbb{C}[L_\circ] \times \mathbb{C}[L_\circ] \to \mathbb{C}$$

determined by the conditions $(e^\alpha, e^\beta) = 0$ if $\alpha \neq \beta$ and $(e^\alpha, e^\beta) = 1$ if $\alpha = \beta$. And there is a unique positive definite Hermitian form $(\cdot, \cdot)$ on $M(1)$ such that for any $h \in \mathfrak{h}$, we have

$$(\mathbf{1}, \mathbf{1}) = 1,$$

$$(h(n)u, v) = (u, h(-n)v)$$

for all $u, v \in M(1)$.

Define a positive definite Hermitian form on $V_{L_\circ}$ as follow: for any $u, v \in M(1)$ and $e^\alpha, e^\beta \in \mathbb{C}[L_\circ]$

$$(u \otimes e^\alpha, v \otimes e^\beta) = (u, v)(e^\alpha, e^\beta).$$

Note that the positive definite Hermitian form on $V_{L_\circ}$ induces a positive definite Hermitian form on $V_{L+\lambda_i}$.

**Lemma 4.11.** Let $(\cdot, \cdot)$ be the positive definite Hermitian form defined above. Then we have: for any $\alpha \in L$ and $w_1, w_2 \in V_{L_\circ}$,

$$(e^\alpha w_1, w_2) = (w_1, (-1)^{(\alpha,\alpha)} e_\alpha w_2),$$

$$(z^\alpha w_1, w_2) = (w_1, z^\alpha w_2).$$

**Proof:** The second identity is obvious. The first identity follows immediately from the fact that $(e^\alpha w_1, e^\alpha w_2) = (w_1, w_2)$ for any $w_1, w_2 \in V_{L_\circ}$. □

Let $\phi : V_L \to V_L$ be an anti-linear map which is determined by:

$$\phi : V_L \to V_L$$
\[ \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes e^\alpha \mapsto (-1)^k \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes e^{-\alpha}. \]

Note that we have
\[ \phi \alpha(n) \phi^{-1} = -\alpha(n) \]
for \( \alpha \in L \) and \( n \in \mathbb{Z} \), and
\[ \phi Y(e^\alpha, z) \phi^{-1} = Y(e^{-\alpha}, z). \]

Again use a similar discussion as in the proof of Lemmas 4.1, 4.6 shows that \( \phi \) is an anti-linear involution of \( V_L \).

**Theorem 4.12.** Let \( L \) be a positive definite even lattice and \( \phi \) be the anti-linear involution of \( V_L \) defined above. Then the lattice vertex operator algebra \( (V_L, \phi) \) is a unitary vertex operator algebra and each \( V_{L+\lambda_i} \) for \( \lambda_i \in L^0/L \) is a unitary module for \( V_L \).

**Proof:** We only give the proof of the unitarity of \( V_L \) here. The proof for \( V_{L+\lambda_i} \) is similar.

From the discussion above, we only need to prove the invariant property. Since the lattice vertex operator algebra \( V_L \) is generated by
\[ \{ \alpha(-1) | \alpha \in L \} \cup \{ e^\alpha | \alpha \in L \}, \]
it is sufficient to prove the following identities
\[ (Y(e^{zL(1)}(-z^{-2})L(0)\alpha(-1) \cdot 1, z^{-1})w_1, w_2) = (w_1, Y(\phi(\alpha(-1) \cdot 1), z)w_2), \] (4.1)
\[ (Y(e^{zL(1)}(-z^{-2})L(0)e^\alpha, z^{-1})w_1, w_2) = (w_1, Y(\phi(e^\alpha), z)w_2) \] (4.2)
for any \( w_1, w_2 \in V_L \) by Proposition 2.11.

Assume that \( w_1 = u \otimes e^{\gamma_1} \), \( w_2 = v \otimes e^{\gamma_2} \) for some \( u, v \in M(1) \) and \( \gamma_1, \gamma_2 \in L \). By the definition of the Hermitian form, we have
\[ (Y(e^{zL(1)}(-z^{-2})L(0)\alpha(-1), z^{-1})w_1, w_2) \]
\[ = -z^{-2} \sum_{n \in \mathbb{Z}} (\alpha(n)w_1, w_2)z^{n+1} \]
\[ = \sum_{n \in \mathbb{Z}} -(w_1, \alpha(-n)w_2)z^{n-1} \]
\[ = (w_1, Y(\phi(\alpha(-1)), z)w_2). \]

This gives (4.1).
To prove the identity (4.2), we assume that \((\alpha, \alpha) = 2k\) and \(\alpha + \gamma_1 = \gamma_2\), then we have

\[
(Y(e^{zL(1)}(-z^{-2})L^{(0)}e^a,z^{-1})u_1,w_2)
= (-z^{-2})^k(Y(e^a,z^{-1})u \otimes e^{\gamma_1},v \otimes e^{\gamma_2})
= (-z^{-2})^k(E^-(\alpha,z^{-1})E^+(\alpha,z)e_{\alpha}(z^{-1})^\alpha u \otimes e^{\gamma_1},v \otimes e^{\gamma_2})
= (-z^{-2})^k(u \otimes e^{\gamma_1},E^-(\alpha,z)E^+(\alpha,z)z^{-1})^\alpha(1)^k e_{-\alpha}v \otimes e^{\gamma_2})
= (z^{-2})^k(u \otimes e^{\gamma_1},E^-(\alpha,z)E^+(\alpha,z)e_{-\alpha}(z^{-1})^\alpha(1)^k e_{-\alpha}v \otimes e^{\gamma_2})
= (u \otimes e^{\gamma_1},E^-(\alpha,z)E^+(\alpha,z)e_{-\alpha}(z^{-1})^\alpha v \otimes e^{\gamma_2})
= (w_1,Y(e^{-\alpha},z)w_2)
= (w_1,Y(\phi(e^a),z)w_2).
\]

Then \((V_L, \phi)\) is a unitary vertex operator algebra. □

We now prove that the \(\theta\)-twisted \(V_L\)-module \(V_L^{T_\chi}\) is unitary. First, recall from [FLM] that there exists a maximal abelian subgroup \(\hat{\Phi}\) of \(\hat{L}\) and a homomorphism \(\psi : \hat{\Phi}/K \to \mathbb{C}\) extending \(\chi\) such that \(T_\chi\) has form \(\text{Ind}_{\hat{\Phi}}^L \mathbb{C}_\psi = \mathbb{C}[\hat{L}] \otimes_{\mathbb{C}[\hat{\Phi}]} \mathbb{C}_\psi\). And there is a positive definite Hermitian form \((,): T_\chi \times T_\chi \to \mathbb{C}\) on \(T_\chi\) defined by the conditions: for any \(a, b \in \hat{L}\), \((t(a), t(b)) = 0\) if \(a \hat{\Phi} \neq b \hat{\Phi}\) and \((t(a), t(b)) = 1\) if \(a \hat{\Phi} = b \hat{\Phi}\), where \(t(a) = a \otimes 1 \in T_\chi\) for \(a \in \hat{L}\). Also recall from [FLM] that there is a positive definite Hermitian form \((,): M(1)(\theta)\) such that

\[(1,1) = 1,\]

\[(h(n) \cdot u, v) = (u, h(-n) \cdot v),\]

for any \(u, v \in M(1)(\theta), h \in \mathfrak{h}, n \in 1/2 + \mathbb{Z}\). Now we define a positive definite Hermitian form on \(V_L^{T_\chi}\) by \((v_1 \otimes w_1, v_2 \otimes w_2) = (v_1, v_2)(w_1, w_2)\) for any \(v_1, v_2 \in M(1)(\theta), w_1, w_2 \in T_\chi\).

**Lemma 4.13.** Let \((,): M(1)(\theta)\) be the positive definite Hermitian form defined above. Then we have: for any \(\alpha \in \hat{L}\) and \(u, v \in V_L^{T_\chi}\),

\[(e_\alpha u, v) = (u, (-1)^{(\alpha,\alpha)}e_{-\alpha}v).\]

**Proof:** The lemma follows immediately from the fact that \((e_\alpha u, e_\alpha v) = (u, v)\) for any \(u, v \in V_L^{T_\chi}\). □

We now give the unitarity of the \(\theta\)-twisted \(V_L\)-module \(V_L^{T_\chi}\).

**Theorem 4.14.** For any central character \(\chi\), \(V_L^{T_\chi}\) is a unitary \(\theta\)-twisted \(V_L\)-module.

**Proof:** As before, we only need to verify the invariant property in the Definition 2.4. By Proposition 2.12, it is sufficient to check

\[
(Y(e^{zL(1)}(-z^{-2})L^{(0)}x,z^{-1})u,v) = (u,Y(\phi(x),z)v)
\]

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for \( x \in \{ \alpha(-1) | \alpha \in L \} \cup \{ e^\alpha | \alpha \in L \} \), \( u, v \in V^T_L \).

Assume that \( u = v_1 \otimes t(a) \) and \( v = v_2 \otimes t(b) \) for some \( v_1, v_2 \in M(1)(\theta) \), \( a, b \in L \). Then
\[
(\alpha(n)u, v) = (u, \alpha(-n)v)
\]
for any \( \alpha \in L \) and \( n \in 1/2 + \mathbb{Z} \). Thus for \( x = \alpha(-1) \), we have
\[
(Y(e^{zL(1)}(-z^{-2})^{L(0)}\alpha(-1), z^{-1})u, v)
= -z^{-2}(Y(\alpha(-1), z^{-1})v_1 \otimes t(a), v_2 \otimes t(b))
= -z^{-2} \sum_{n \in \mathbb{Z} + \frac{1}{2}} (\alpha(n)v_1, v_2)(t(a), t(b))z^{n+1}
= - \sum_{n \in \mathbb{Z} + \frac{1}{2}} (v_1, \alpha(-n)v_2)(t(a), t(b))z^{n-1}
= (u, Y(\phi(\alpha(-1)), z)v).
\]

Now take \( x = e^\alpha \) and \( (\alpha, \alpha) = 2k \). Then by Lemma 4.13 we have
\[
(Y(e^{zL(1)}(-z^{-2})^{L(0)}e^\alpha, z^{-1})u, v)
= (Y(e^{zL(1)}(-z^{-2})^{L(0)}e^\alpha, z^{-1})v_1 \otimes t(a), v_2 \otimes t(b))
= (-z^{-2})^k(2^{-2k}E^-(\alpha, z^{-1})E^+(\alpha, z^{-1})e_\alpha z^k v_1 \otimes t(a), v_2 \otimes t(b))
= (-z^{-2})^k(v_1 \otimes t(a), 2^{-2k}E^-(\alpha, z)E^+(\alpha, z)(-1)^k e_\alpha z^k v_2 \otimes t(b))
= (v_1 \otimes t(a), 2^{-2k}E^-(\alpha, z)E^+(\alpha, z)e_\alpha z^{-k}v_2 \otimes t(b))
= (v_1 \otimes t(a), Y(\phi(e^\alpha), z)v_2 \otimes t(b)).
\]

Thus \( V^T_L \) is a unitary \( \theta \)-twisted \( V_L \)-module. \( \square \)

### 4.5 Moonshine vertex operator algebra is unitary

In this subsection we prove that the famous Moonshine vertex operator algebra \( V^z \) [FLM] is a unitary vertex operator algebra. First, we recall some facts about Moonshine vertex operator algebra. Let \( \Lambda \) be the Leech lattice which is the unique even unimodular lattice with rank 24 such that \( \Lambda_2 = \emptyset \). Let \( V_\Lambda \) be the lattice vertex operator algebra associated to \( \Lambda \) and \( \theta \) be the automorphism of \( V_\Lambda \) defined as above. Let \( V_\Lambda^+ = V_\Lambda^\theta \) be the vertex operator subalgebra of \( V_\Lambda \). It was proved in [D3] and [DLM3] that \( V_\Lambda^+ \) is a rational and \( C_2 \)-cofinite vertex operator algebra. It is well-known that \( V_\Lambda^T \) is the unique irreducible \( \theta \)-twisted \( V_\Lambda \)-module \( V_\Lambda^T \) [D2] and there is an isomorphism \( \theta \) of \( V_\Lambda^T \) such that \( \theta Y(a, z)\theta^{-1} = Y(\theta(a), z) \) for \( a \in V_\Lambda \) where \( Y(a, z) \) is the twisted vertex operator acting on \( V_\Lambda^T \) [FLM]. Let \((V_\Lambda^T)^+ = (V_\Lambda^T)^\theta\), then \((V_\Lambda^T)^+\) is an irreducible \( V_\Lambda^+ \)-module. It was proved in [FLM] that \( V^z = V_\Lambda^+ \oplus (V_\Lambda^T)^+ \) is a vertex operator algebra. Furthermore, by the fusion rule of \( V_\Lambda^+ \) [ADL], we know that \((V_\Lambda^T)^+\) is a simple current. By Theorems 4.12, 4.14 we define a positive definite Hermitian form on \( V^z \) by Theorem 3.3.
Define an anti-linear map $\Psi : V^T_\Lambda \to V^T_\Lambda$ as follow:

$$\Psi : V^T_\Lambda \to V^T_\Lambda$$

$$\alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes t(a) \mapsto (-1)^{k+1} \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes t(a).$$

By the definition, $\Psi$ is an anti-linear isomorphism of $V^T_\Lambda$ satisfying

$$\Psi \alpha(n) \Psi^{-1} = -\alpha(n),$$

$$\Psi Y(e^\alpha, z) \Psi^{-1} = Y(e^{-\alpha}, z)$$

for $\alpha \in \Lambda$. That is,

$$\Psi Y(\alpha(-1), z) \Psi^{-1} = Y(\phi(\alpha(-1)), z),$$

$$\Psi Y(e^\alpha, z) \Psi^{-1} = Y(\phi(e^\alpha), z).$$

This implies

$$\Psi Y(v, z) \Psi^{-1} = Y(\phi(v), z)$$

for all $v \in V_\Lambda$. Note that $\Psi$ commutes with the automorphism $\theta$ of $V^T_\Lambda$ from the definition of $\theta$ [FLM]. As a result, $\Psi((V^T_\Lambda)^+) \subseteq (V^T_\Lambda)^+$ and $\Psi$ induces an anti-linear isomorphism $\psi$ of $(V^T_\Lambda)^+$ such that

$$\psi Y(v, z) \psi^{-1} = Y(\phi(v), z)$$

for $v \in V_\Lambda^+$. It is easy to prove that $(\Psi(u), \Psi(v)) = (u, v)$ for any $u, v \in V^T_\Lambda$. Similarly, for the anti-linear involution $\phi$ of $V_\Lambda$ we could prove that $\phi$ induces an anti-linear involution $\phi_{V_\Lambda^+}$ of $V_\Lambda^+$ and $(\phi(x), \phi(y)) = (x, y)$ for any $x, y \in V_\Lambda$. Then by Theorem 3.3 and Corollary 2.7, we define an anti-linear involution $\phi_{V^z}$ of $V^z$. In fact, we have proved the following:

**Theorem 4.15.** The Moonshine vertex operator algebra $(V^z, \phi_{V^z})$ is a unitary vertex operator algebra.

## 5 Classification of unitary vertex operator algebras with central charge $c \leq 1$

In this section we consider the classification of unitary vertex operator algebras with central charge $c \leq 1$. First, we have the following results about unitary vertex operator algebra with the central charge $c < 1$. The similar results about the classification of rational and $C_2$-cofinite vertex operator algebra with central charge $c < 1$ were obtained in [DZ], [M]. A classification of local conformal nets with $c < 1$ was given in [KL].
Proposition 5.1. Let \((V, \phi)\) be a unitary vertex operator algebra with central charge \(c < 1\). Then the vertex operator subalgebra \(< \omega >\) which is generated by \(\omega\) is isomorphic to the vertex operator algebra \(L(c_m, 0)\) for some integer \(m \geq 2\). In particular, \(V\) is an extension vertex operator algebra of \(L(c_m, 0)\).

Proof: By Lemma 2.5, we have \(V\) is a unitary module of Virasoro algebra. Since the central charge \(c < 1\), we have \(c = c_m\), for some integer \(m \geq 2\). By Lemma 2.5, \(V\) is a completely reducible module for the Virasoro algebra. This implies that the vertex operator subalgebra of \(V\) generated by \(\omega\) is isomorphic to \(L(c_m, 0)\) and \(V\) is an extension of \(L(c_m, 0)\). \(\square\)

We now consider the unitary vertex operator algebra with central charge \(c = 1\).

Recall from [DM] that for a vertex operator algebra \(V\) of CFT-type, the weight one subspace is a Lie algebra under the bracket operation \([a, b] = a_0b\).

Proposition 5.2. Let \((V, \phi)\) be a unitary simple vertex operator algebra of CFT-type with central charge \(c = 1\), and \(\mathfrak{h} = \mathbb{C}\alpha\) be a one dimensional abelian subalgebra of \(V_1\) such that \(\phi(\alpha) \in \mathbb{R}\alpha\). Then \(V\) is isomorphic to \(M_{\mathfrak{h}}(1, 0)\) or to \(V_L\) for some positive definite even lattice \(L\) with rank 1.

Proof: We normalize the Hermitian form \((,\) so that \((1, 1) = 1\), then we have \((u, v) = (u_{-1}1, v) = -(1, \phi(u_1)v)\), for any \(u, v \in V_1\).

From the assumption we know that \(\phi(\alpha) = \lambda\alpha\) for some real number \(\lambda\). Since \(\phi\) is an anti-linear involution we see that \(\lambda = \pm 1\). Replacing \(\alpha\) by \(i\alpha\) is necessary, we can assume that \(\phi(\alpha) = -\alpha\). We can also assume that \((\alpha, \alpha) = 1\). Then

\[[\alpha_m, \alpha_n] = m\delta_{m+n,0}.

By the Stone-von-Veummann theorem, we have

\[V = M_{\mathfrak{h}}(1, 0) \otimes \Omega_V,

where \(\Omega_V = \{v \in V|\alpha_m v = 0, m > 0\}\). Note that the Hermitian form \((,\)|\(V_n\) is non-degenerate and \((\alpha_0 u, v) = (u, \alpha_0 v)\) for \(u, v \in V_n\). This implies that the eigenvalues of \(\alpha_0\) on \(V_n\) are real. We claim that \(\alpha_0\) acts semisimply on \(V\). Assume \(v \in V_n\) is an generalized eigenvector of \(\alpha_0\) with eigenvalue \(\lambda\) but not an eigenvector. Then there exists \(n \geq 1\) such that \((\alpha_0 - \lambda)^n v \neq 0\) and \((\alpha_0 - \lambda)^m v = 0\) for \(m > n\). This gives

\[0 < ((\alpha_0 - \lambda)^n v, (\alpha_0 - \lambda)^n v) = (v, (\alpha_0 - \lambda)^{2n} v) = 0,

a contradiction. In particular, \(\alpha_0\) acts semisimply on \(\Omega_V\). For \(\lambda \in \mathfrak{h}\).

Let \(\Omega_V(\lambda) = \{u \in V|\alpha_0 u = (\alpha, \lambda)u, \}\). Then

\[\Omega_V = \oplus_{\lambda \in \mathfrak{h}} \Omega_V(\lambda)\]
and
\[ V = \oplus_{\lambda \in \mathfrak{h}} M_{\mathfrak{h}}(1,0) \otimes \Omega_V(\lambda). \]
Let \( L = \{ \lambda \in \mathfrak{h} | \Omega_V(\lambda) \neq 0 \} \), since \( V \) is a simple vertex operator algebra, by the similar proof as in Theorem 2 of [DM] \( L \) is an additive subgroup of \( \mathfrak{h} \). If \( L = 0 \), then we have \( V \) is isomorphic to \( M_{\mathfrak{h}}(1,0) \). We now assume that \( L \neq 0 \) and will prove that \( L \) is a positive definite lattice.

Set
\[ \omega' = \frac{1}{2}(\alpha_{-1})^2 1, \]
and \( L'(n) = \omega'_{n+1} \). Then
\[ [L'(m), L'(n)] = (m - n)L'(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0} \]
and
\[ (L'(n)u, v) = (u, L'(-n)v). \]
In particular, \( \omega' \) is a Virasoro vector with central charge 1. Let \( \omega'' = \omega - \omega' \), then \( \omega'' \) is a Virasoro vector with central charge 0. Now we prove that \( \omega'' = 0 \), i.e. \( \omega = \omega' \). Let \( L''(n) = \omega''_{n+1} \), we have
\[
(\omega'', \omega'') = (\omega' - \omega', \omega' - \omega') \\
= ((L(-2) - L'(-2))1, (L(-2) - L'(-2))1) \\
= (1, (L(2) - L'(2))(L(-2) - L'(-2))1) \\
= (1, L''(2)L''(-2))1 \\
= 0.
\]
Since the Hermitian form is positive definite, we have \( \omega'' = 0 \), i.e. \( \omega = \omega' \). In particular, \( L(0) = L'(0) \). This implies that \( \frac{(\lambda \lambda)}{2} \in \mathbb{Z}^+ \) for \( \lambda \in L \). Thus \( L \) is a positive definite even lattice. Since the rank of \( L \) is 1, Corollary 5.4 of [DM] then asserts that \( V \cong V_L \). The proof is complete. \( \Box \)

It is not surprised that \( M_{\mathfrak{h}}(1,0) \) is a possibility in Proposition 5.2 as we do not assume that \( V \) is rational. A classification of conformal nets of central charge 1 has been given in [X] under some assumptions.

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