Logically automorphically equivalent knowledge bases

July 5, 2017

E. Aladova, T. Plotkin
Bar Ilan University,
5290002, Ramat Gan, Israel
E-mail address: aladoval (at) mail.ru
plot (at) macs.biu.ac.il

Abstract
Knowledge bases theory provide an important example of the field where applications of universal algebra and algebraic logic look very natural, and their interaction with practical problems arising in computer science might be very productive.

In this paper we study the equivalence problem for knowledge bases. Our interest is to find out how the informational equivalence is related to the logical description of knowledge. The main objective of this paper is logically automorphically equivalent knowledge bases. As we will see this notion gives us a good enough characterization of knowledge bases.

1 Motivation
Research work on databases was began at the end of the sixties. The classical work of Codd [12] gave the theoretical foundation of databases. From this time advances in database theory are closely related to mathematical logic, theory of algorithms and general algebra.

Knowledge base systems go far beyond the relational database model. They require complex data processing, which may include rules. Knowledge base systems combine database features and artificial intelligence techniques.

Investigations in database theory led to the formal definitions for various types of databases, whereas knowledge bases are often defined informally.
There are different views on knowledge bases which are discussed, in particular, in the information technology, strategic management, and organizational theory literature. The informal representation of a knowledge base, for example, does not allow identifying the duplicate information represented in different formats by different knowledge base implementations. The formal mathematical model of a knowledge base allows to get formal solutions for various problems arise in knowledge bases theory, in particular, for equivalence problem for knowledge bases.

Plotkin in [30] proposed a mathematical model of a database and gave a formal definition of the databases equivalence concept. Researches in this area give rise to the algebraic model of a knowledge base which was introduced and developed in [26], [29], [30]. The main peculiarity of this approach is that a database and a knowledge base is considered as a certain algebraic structure. The mathematical model of a knowledge base (or database), which is viewed as an algebraic structure helps us to understand the nature of a concrete real knowledge base (database), and it enables to solve various problems in knowledge bases (databases) theory. The model of a knowledge base involves various ideas of universal algebra, algebraic logic and algebraic geometry. Such a model is useful for many reasons. There are a lot of specialized knowledge bases and it is desirable to determine their characteristic properties without referring to their complicated structure and without studying their detailed architecture. Mathematical model of a knowledge base allows to distinguish some invariants of knowledge bases which rigidly determine them.

In this paper we discuss the equivalence problem for knowledge bases. This problem goes back to the similar one for databases. It was first posed by Aho, Sagiv, Ullman in [11] and Beeri, Mendelzon, Sagiv, Ullman in [9] and gave rise to the notion of databases schemes equivalence. They propose an approach to databases schemes equivalence based on the notion of a fixed point. In this setting two relational database schemes are equivalent if their sets of fixed points coincide. Correspondingly, two relational databases are equivalent if their sets of all fixed points intersected with the sets of feasible instances coincide. This and other approaches to the database equivalence problem had been studied in numerous papers (see [6], [7], [8], [10], [18], [35], [37], etc.).

We are interested in a special kind of equivalence, namely, informational equivalence. Informally, one can say that two knowledge bases are informationally equivalent if and only if all information that can be retrieved from the one knowledge base can be also obtained from the other one and vice versa. The formal mathematical model of a knowledge base allows to solve formally the informational equivalence problem. Various solutions for the
knowledge bases equivalence problem based on algebraic geometry approach were obtained in [19], [25], [29], [30], [33], [34], [36]. This paper continues the research of the knowledge bases equivalence problem based on logical geometry approach which was started in [2].

The paper is organized as follows. In Section 2 we give a brief review of basic notions and notations from universal algebraic geometry. In particular, we define Halmos categories and construct the Galois correspondence which is very important in our considerations. The material of this section can be found in the papers of B. Plotkin ([26], [27], [29], see also [3] for some detailed proofs). In Section 3 we introduce a knowledge base model under consideration. Section 4 deals with various equivalences of knowledge bases and connections between them. In particular, we give the formal definition of informationally equivalent knowledge bases. In Section 4.2 we introduce one more equivalence for knowledge bases, namely, logically automorphical equivalence, and present the main result of the paper which state that logically automorphically equivalent knowledge bases are informationally equivalent (Theorem 4.14).

2 Preliminaries: Mathematical apparatus

2.1 Basic notions and notations

Let \( X^0 = \{x_1, \ldots, x_n, \ldots \} \) be an infinite set of variables. Denote by \( \Gamma \) the collection of all finite subsets \( X \) of \( X^0 \).

Let \( \Theta \) be a variety of algebras, that is a class of algebras satisfying a set of identities (see, for instance, [21], [30]). We denote by \( \text{Var}(H) \) the variety generated by the algebra \( H \).

Denote by \( W(X) \) the free algebra in the variety \( \Theta \) with free generating set \( X, X \in \Gamma \). All free algebras \( W(X) \in \Theta \), form a category of free algebras \( \Theta^0 \) with homomorphisms \( s : W(X) \rightarrow W(Y) \) as morphisms, \( X, Y \in \Gamma \).

By a model \( H \) we mean a triple \((H, \Psi, f)\), where \( H \) is an algebra from \( \Theta \), \( \Psi \) is a set of relation symbols \( \varphi \), \( f \) is an interpretation of all \( \varphi \) in \( H \) (see, for instance, [11], [22], [30]).

Take an algebra \( H \) in \( \Theta \). A point \((a_1, \ldots, a_n)\) from \( n \)-th Cartesian power of \( H \) can be represented as a map \( \mu : X \rightarrow H \) such that \( a_i = \mu(x_i) \). This map can be extended up to homomorphism of algebras \( \mu : W(X) \rightarrow H \). Thus, the point \((a_1, \ldots, a_n)\) can be also viewed as a homomorphism \( \mu : W(X) \rightarrow H \).

Denote by \( \text{Hom}(W(X), H) \) the set of all homomorphism from \( W(X) \) to \( H \). We will regard \( \text{Hom}(W(X), H) \) as an affine space.
All affine spaces $Hom(W(X), H)$ with various $X \in \Gamma$ constitute the category $\Theta^0(H)$ of affine spaces with morphisms

$$\tilde{s} : Hom(W(X), H) \to Hom(W(Y), H),$$

for each homomorphism of free algebras $s : W(Y) \to W(X)$. The map $\tilde{s}$ is defined as $\tilde{s}(\mu) = \mu s$, where $\mu : W(X) \to H$, $\tilde{s}(\mu) : W(Y) \to H$.

The categories $\Theta^0$ and $\Theta^0(H)$ are very important for further considerations. Moreover, the following theorem takes place.

**Theorem 2.1** ([23]). The categories $\Theta^0$ and $\Theta^0(H)$ are dual if and only if $Var(H) = \Theta$.

### 2.2 Halmos categories

Halmos categories were introduced in papers of B.I. Plotkin [26], [29]. Halmos categories are related to the first-order logic in a way analogous to the relationship between boolean algebras and propositional logic. Such an approach allows us to use technics and structures of algebraic logic (see [16], [30]). The immediate advantage of this phenomenon is that we can view queries to a knowledge base and replies to these queries as objects of the same nature, i.e., objects of Halmos categories. Then the transition query-reply can be treated as a functor (for details see Section 3).

We start from the notion of an existential quantifier on a boolean algebra. Let $B$ be a boolean algebra. Existential quantifier on $B$ is a unary operation $\exists : B \to B$ such that the following conditions hold:

1. $\exists 0 = 0$,
2. $a \leq \exists a$,
3. $\exists(a \wedge \exists b) = \exists a \wedge \exists b$.

Universal quantifier $\forall : B \to B$ is dual to $\exists : B \to B$, they are related by $\forall a = \neg(\exists(\neg a))$.

**Definition 2.2.** Let a set of variables $X = \{x_1, \ldots, x_n\}$ and a set of relations $\Psi$ be given. A boolean algebra $B$ is called an extended boolean algebra over $W(X)$ relative to $\Psi$, if

1. the existential quantifier $\exists x$ is defined on $B$ for all $x \in X$, and $\exists x \exists y = \exists y \exists x$ for all $x, y \in X$;
2. to every relation symbol $\varphi \in \Psi$ of arity $n_\varphi$ and a collection of elements $w_1, \ldots, w_{n_\varphi}$ from $W(X)$ there corresponds a nullary operation (a constant) of the form $\varphi(w_1, \ldots, w_{n_\varphi})$ in $B$. 

4
Thus, the signature $L_X$ of an extended boolean algebra consists of the boolean connectives, existential quantifiers $\exists x$ and constants $\varphi(w_1, \ldots, w_n)$:

$$L_X = \{ \lor, \land, \neg, \exists x, M_X \},$$

where $M_X$ is a set of all $\varphi(w_1, \ldots, w_n)$.

There are two important examples of extended boolean algebras.

**Example 2.3.** Let a model $H = (H, \Psi, f)$ be given. Let $\text{Bool}(W(X), H)$ be the boolean algebra of all subsets in $\text{Hom}(W(X), H)$. One can equip the boolean algebra $\text{Bool}(W(X), H)$ with the structure of an extended boolean algebra ([2], [3], [30]). We denote this extended boolean algebra by $\text{Hal}_X^H(H)$.

**Example 2.4.** Another important example of an extended boolean algebra is presented by the algebra of formulas $\Phi_0(X) = L_X/\tau_X$, where $L_X$ is the absolutely free algebra in the signature $L_X$ over the set $M_X$, $\tau_X$ is a congruence relation on $L_X$ defined by the rule: $u \tau_X v$ if and only if $\vdash (u \rightarrow v) \land (v \rightarrow u)$, $u, v \in L_X$. Boolean operations and quantifiers on $\Phi_0(X)$ are naturally inherited from $L_X$. For more details see [2], [3], [26], [30], [31].

Now we define a Halmos category which plays a very important role in further considerations.

**Definition 2.5.** A category $\mathbb{H}$ is a Halmos category if:

1. Every its object has the form $\mathbb{H}(X)$, where $\mathbb{H}(X)$ is an extended boolean algebra over $W(X)$.

2. The morphisms in $\mathbb{H}$ correspond to morphisms in the category $\Theta^0$. To every morphism $s : W(X) \rightarrow W(Y)$ in $\Theta^0$ it corresponds a morphism $s_* : \mathbb{H}(X) \rightarrow \mathbb{H}(Y)$ in $\mathbb{H}$ such a way that

   (a) the transitions $W(X) \rightarrow \mathbb{H}(X)$ and $s \rightarrow s_*$ determine a covariant functor from $\Theta^0$ to $\mathbb{H}$.

   (b) $s_* : \mathbb{H}(X) \rightarrow \mathbb{H}(Y)$ is a homomorphism of corresponding boolean algebras.

3. There are special identities controlling the interaction of morphisms with quantifiers and constant (for details see [20], [31], [32]).

Next two examples of Halmos categories are based on examples of extended boolean algebras above.
Example 2.6. Category $\text{Hal}_{0}(\mathcal{H})$. Objects of this category are extended boolean algebras $\text{Hal}_{0}(\mathcal{H})$ from Example 2.3 for various $X \in \Gamma$. Morphisms $s_{*} : \text{Hal}^{X}_{0}(\mathcal{H}) \to \text{Hal}^{Y}_{0}(\mathcal{H})$, are defined as follows:

$$\mu \in s_{*}A \iff \mu s \in A,$$

where $\mu : W(Y) \to H$, $A \subset \text{Hom}(W(X), H)$, $s : W(X) \to W(Y)$.

**Remark 2.7.** A homomorphism $s : W(X) \to W(Y)$ gives rise a map

$$\tilde{s} : \text{Hom}(W(Y), H) \to \text{Hom}(W(X), H),$$

by the rule $\tilde{s}(\mu) = \mu s$, which is a morphism in the category of affine spaces $\Theta^{0}(H)$ (see Section 2.7). Let a subset $A$ from $\text{Hom}(W(X), H)$ be given. Then $s_{*}A$ is the full pre-image of $A$ under $\tilde{s}$.

Example 2.8. Category $\tilde{\Phi}$. Objects of the category $\tilde{\Phi}$ are constructed using extended boolean algebras $\Phi_{0}(X)$, $X \in \Gamma$. Denote by $[\varphi(w_{1}, \ldots, w_{n_{\varphi}})]_{\tau_{X}}$ the image of the element $\varphi(w_{1}, \ldots, w_{n_{\varphi}}) \in M_{X}$ under the homomorphism $\mathcal{L}_{X} \to \Phi_{0}(X) = \mathcal{L}_{X}/\tau_{X}$. Let $[M_{X}]_{\tau_{X}}$ be a set of all $[\varphi(w_{1}, \ldots, w_{n_{\varphi}})]_{\tau_{X}}$, $\varphi \in \Psi$.

A homomorphism of free algebras $s : W(X) \to W(Y)$ induces the map $s_{*} : [M_{X}]_{\tau_{X}} \to [M_{Y}]_{\tau_{Y}}$, by the rule

$$s_{*}([\varphi(w_{1}, \ldots, w_{n_{\varphi}})]_{\tau_{X}}) = [\varphi(sw_{1}, \ldots, sw_{n_{\varphi}})]_{\tau_{Y}}.$$  

This map can be extended up to homomorphism of boolean algebras $s_{*} : \Phi_{0}(X) \to \Phi_{0}(Y)$.

Note that morphism of a Halmos category $s_{*}$ correlates with quantifiers under the certain rules (see [26], [31]) and they are not homomorphisms of extended boolean algebras. Thus, the extended boolean algebras $\Phi_{0}(X)$ cannot be an object of the category $\tilde{\Phi}$. We should to add to each $\Phi_{0}(X)$ all formulas of the form $s_{*}u$, $u \in \Phi_{0}(X)$. Denote objects of the category $\tilde{\Phi}$ by $\Phi(X)$. So, the category $\tilde{\Phi}$ is a category with objects of the form $\Phi(X)$ and morphisms $s_{*} : \Phi(X) \to \Phi(Y)$, $X, Y \in \Gamma$.

The next remark is connected with Remark 2.7.

**Remark 2.9.** Let a homomorphism $s : W(X) \to W(Y)$ be given. In parallel to the map $\tilde{s} : \text{Hom}(W(Y), H) \to \text{Hom}(W(X), H)$, we define a map from the set of subsets in $\Phi(Y)$ to the set of subsets in $\Phi(X)$. We denote it the same symbol $\tilde{s}$ and define as

$$\tilde{s}T = \{u \in \Phi(X) \mid s_{*}u \in T\},$$

where $T \subset \Phi(Y)$. Then $\tilde{s}T$ is the full pre-image of $T$ under $s_{*}$.
Halmos categories $\tilde{\Phi}$ and $Hal_\Theta(H)$ are tightly connected via homomorphism of extended boolean algebras

$$Val_X^H : \Phi(X) \rightarrow \text{Bool}(W(X), H).$$

Intuitively, the image of a formula $u \in \Phi(X)$ under the homomorphism $Val_X^H$ is a value of $u$ in the algebra $H$, i.e. $Val_X^Hu$ is a set of point in $Hom(W(X), H)$ satisfied $u$. For details see [26], [29], [31].

Let $\mu$ be a point from the affine space $Hom(W(X), H)$.

**Definition 2.10.** The logical kernel $LKer(\mu)$ of a point $\mu$ is the set of all formulas $u \in \Phi(X)$ which hold true on the point $\mu$, that is

$$LKer(\mu) = \{ u \in \Phi(X) \mid \mu \in Val_X^H(u) \}$$

Note that the logical kernel $LKer(\mu)$ of a point $\mu$ is a boolean ultrafilter (maximal filter) in the algebra $\Phi(X)$ (see [28]).

### 2.3 Galois correspondence

Now we define a correspondence between sets of formulas in the algebra $\Phi(X)$ and subsets of points from the affine space $Hom(W(X), H)$.

Let $T$ be a set of formulas from $\Phi(X)$. We define a set of points $T^L_H$ in $Hom(W(X), H)$ as

$$T^L_H = \{ \mu : W(X) \rightarrow H \mid T \subset LKer(\mu) \}.$$  

That is, $T^L_H$ is a set of all points $\mu \in Hom(W(X), H)$ satisfying all formulas from $T \subset \Phi(X)$. The set $T^L_H$ can be written as follows:

$$T^L_H = \bigcap_{u \in T} Val_X^H(u).$$

Take a set of points $A \subset Hom(W(X), H))$ and define a set of formulas $A^L_H$ in $\Phi(X)$:

$$A^L_H = \{ u \in \Phi(X) \mid A \subset Val_X^H(u) \}.$$  

The set $A^L_H$ is the set of all formulas $u \in \Phi(X)$ hold true at all points from $A$. One can present the set $A^L_H$ as follows:

$$A^L_H = \bigcap_{\mu \in A} LKer(\mu).$$

The defined above correspondence between sets of formulas and sets of points is the Galois correspondence (see [20]). In the case of the Galois
correspondence one can speak about Galois closures. In particular, subsets $T^L_H \subset \text{Hom}(W(X), H)$ and $A^L_H \subset \Phi(X)$ are Galois-closed.

We call the subset $T^L_H \subset \text{Hom}(W(X), H)$ definable set presented by the set of formulas $T$. The set $A^L_H$ is a boolean filter in the algebra $\Phi(X)$, as an intersection of boolean filters $L\text{Ker}\mu$. It is called $H$-closed filter.

The constructed above Galois correspondence give us a bijection between definable sets in $\text{Hom}(W(X), H)$ and $H$-closed filters in the extended boolean algebra $\Phi(X)$.

It is known the following proposition.

**Proposition 2.11 (28).** The intersection of $H$-closed filters is an $H$-closed filter. □

The next proposition describes one more property of $H$-closed filters (3, 26).

**Proposition 2.12.** Let a homomorphism of free algebras $s : W(X) \to W(Y)$ be given. If $T$ is an $H$-closed filter in $\Phi(Y)$, then $\tilde{s}T$ is an $H$-closed filter in $\Phi(X)$.

**Corollary 2.13.** Let $A$ be a set of points in $\text{Hom}(W(Y), H)$ and $s : W(X) \to W(Y)$ be a homomorphism of free algebras. Then

$$\tilde{s}(A^L_H) = (\tilde{s}A)^L_H.$$ 

The next proposition describes the relation between the Galois correspondence and morphisms in the categories $H\alpha(H)$ and $\tilde{\Phi}$ (see 3, 26).

**Proposition 2.14.** Let $T$ be a set of formulas from $\Phi(X)$, $A$ be a set of points in $\text{Hom}(W(X), H)$ and $s : W(X) \to W(Y)$ be a homomorphism of free algebras. Then

1. $(s^*T)^L_H = s^*T^L_H,$
2. $s^*A^L_H \subseteq (s^*A)^L_H.$

**Corollary 2.15.** If $A$ is a definable set in $\text{Hom}(W(X), H)$ then $s^*A$ is also a definable set.

The similar relation takes place between definable sets, $H$-closed filters and maps $\tilde{s}$ (3, 26, 29).

**Proposition 2.16.** Let $T$ be a set of formulas from $\Phi(Y)$, $A$ be a set of points in $\text{Hom}(W(Y), H)$. Let a homomorphism of free algebras $s : W(X) \to W(Y)$ be given. Then

1. $\tilde{s}(A^L_H) = (\tilde{s}A)^L_H,$
2. $\tilde{s}(T^L_H) \subseteq (\tilde{s}T)^L_H.$
3 Knowledge base model

In this section we will introduce the concept of a knowledge base model under consideration. But let us start with discussion what is knowledge.

3.1 What is knowledge?

Although knowledge is one of the most familiar concept, the fundamental question about it: “What is it?”. Rarely this question has been answered directly. Numerous papers introduce one or another definition of knowledge, depending on needs of a particular research and field of interest (see [5], [13], [14], [15], [17], [24], etc.).

Speaking about knowledge we proceed from its representation in three components.

1. Subject area of knowledge,
2. Description of knowledge,
3. Content of knowledge.

Let us describe these component in more details.

Subject area of knowledge is presented by a model $\mathcal{H} = (H, \Psi, f)$, where

- $H$ is an algebra in fixed variety of algebras $\Theta$.
- $\Psi$ is a set of relation symbols $\varphi$.
- $f$ is an interpretation of each symbol $\varphi$ in $H$.

Description of knowledge presents a syntactical component of knowledge. From algebraic viewpoint description of knowledge is a set of formulas $T$, more precise, it is an $\mathcal{H}$-filter in the algebra of formulas $\Phi(X)$, $X = \{x_1, \ldots, x_n\}$.

Content of knowledge is a subset in $H^n$, where $H^n$ is the Cartesian power of $H$. Each content of a knowledge $A$ corresponds to the description of a knowledge $T \subset \Phi(X)$, $|X| = n$. If we regard $H^n$ as an affine space then this correspondence can be treated geometrically via Galois correspondence.

In order to describe the dynamic nature of a knowledge base two categories and a functor are introduced: the category of knowledge description $F_{\Theta}(\mathcal{H})$, the category of knowledge content $D_{\Theta}(\mathcal{H})$ and the knowledge functor $C_{\mathcal{H}}$.

An object $F_{\Theta}^X(\mathcal{H})$ of the category of knowledge description $F_{\Theta}(\mathcal{H})$ is the lattice of all $\mathcal{H}$-closed filters in the algebra $\Phi(X)$, $X \in \Gamma$. 
Remark 3.1. We cannot say that the usual set-theoretical union of $H$-closed filters is an $H$-closed filter. To constitute a lattice of $H$-closed filters in $\Phi(X)$ there was introduced a new operation
\[ T_1 \cup T_2 = (T_1 \cup T_2)^{LL}_H. \]
Then all $H$-closed filters in $\Phi(X)$ form a lattice with the operation $\cup$ and $\cap$ (for details see [3], [20], [23]).

### 3.2 Category of knowledge description $F_\Theta(H)$

An object $F^X_\Theta(H)$ of the category $F_\Theta(H)$ is the lattice of all $H$-closed filters in the algebra $\Phi(X)$, $X \in \Gamma$.

Let a homomorphism $s : W(X) \rightarrow W(Y)$ and $H$-closed filters $T_1 \in \Phi(X)$ and $T_2 \in \Phi(Y)$ be given. We say that a map $[s_*] : T_1 \rightarrow T_2$ is admissible, if $s_*T_1 \subseteq T_2$. Remind that $s_*$ is a map between $H$-closed filters in $\Phi(X)$ and $\Phi(Y)$ induced by the corresponding morphism of the category $\tilde{\Phi}$ (see Section 2.2).

A morphism between objects $F^X_\Theta(H)$ and $F^Y_\Theta(H)$
\[ [s_*] : F^X_\Theta(H) \rightarrow F^Y_\Theta(H) \]
is defined, if $[s_*] : T_1 \rightarrow T_2$ is admissible for every $T_1 \in F^X_\Theta(H)$.

We define a composition of morphisms $[s^1_*] : F^X_\Theta(H) \rightarrow F^Y_\Theta(H)$ and $[s^2_*] : F^Y_\Theta(H) \rightarrow F^Z_\Theta(H)$ as follows
\[ [s^2_*] \circ [s^1_*] = [s^2_*s^1_*]. \]
This definition is correct. Indeed, if $[s^1_*] : T_1 \rightarrow T_2$ and $[s^2_*] : T_2 \rightarrow T_3$, then $s^1_*T_1 \subseteq T_2$ and $s^2_*T_2 \subseteq T_3$. This means that $s^2_*s^1_*T_1 \subseteq T_3$ and $[s^2_*s^1_*]$ is admissible for $T_1$ and $T_3$.

### 3.3 Category of knowledge content $D_\Theta(H)$

An object $D^X_\Theta(H)$ of the category $D_\Theta(H)$ is the lattice of all definable sets in the affine space $\text{Hom}(W(X), H)$, $X \in \Gamma$.

Let a homomorphism $s : W(X) \rightarrow W(Y)$ and definable sets $A_2 \in \text{Hom}(W(Y), H)$ and $A_1 \in \text{Hom}(W(X), H)$ be given. We say that a map $[\tilde{s}] : A_2 \rightarrow A_1$ is admissible, if $\tilde{s}A_2 \subseteq A_1$. Remind that $\tilde{s}$ is a map between definable sets in $\text{Hom}(W(Y), H)$ and $\text{Hom}(W(X), H)$ induced by the corresponding morphism of the category of affine spaces $\Theta^0(H)$ (see Section 2.1).
A morphism between objects $D^Y_G(H)$ and $D^X_G(H)$

$$[\tilde{s}] : D^Y_G(H) \to D^X_G(H)$$

is defined, if $[\tilde{s}] : A_2 \to A_1$ is admissible for every $A_2 \in D^Y_G(H)$.

We define a composition of morphisms $[\tilde{s}^1] : F^Y_G(H) \to D^Y_G(H)$ and $[\tilde{s}^2] : D^Y_G(H) \to D^X_G(H)$ as follows

$$[\tilde{s}^2] \circ [\tilde{s}^1] = [\tilde{s}^2 \tilde{s}^1].$$

This definition is correct. Indeed, if $[\tilde{s}^1] : A_3 \to A_2$ and $[\tilde{s}^2] : A_2 \to A_1$, then $\tilde{s}^1 A_3 \subseteq A_2$ and $\tilde{s}^2 A_2 \subseteq A_1$. This means that $s^2 \tilde{s}^1 A_3 \subseteq A_1$ and $[\tilde{s}^2 \tilde{s}^1]$ is admissible for $A_3$ and $A_1$.

### 3.4 The knowledge functor $Ct_H$

The category of knowledge description $F_G(H)$ and the category of knowledge content $D_G(H)$ are related by the knowledge functor (for details see [3]).

$$Ct_H : F_G(H) \to D_G(H),$$

which is defined on objects by

$$Ct_H(F^X_G(H)) = D^X_G(H),$$

and on morphisms by

$$Ct_H([s^*]) = [\tilde{s}],$$

where $s : W(X) \to W(Y)$ is a given homomorphism of free algebras. Moreover, if $[s^*] : F^X_G(H) \to F^Y_G(H)$ is a morphism in $F_G(H)$, such that

$$[s^*] : T_1 \to T_2,$$

then $[\tilde{s}] : D^Y_G(H) \to D^X_G(H)$ is a morphism in $D_G(H)$ defined by the rule

$$[\tilde{s}] : (T_2)_H^L \to (T_1)_H^L.$$

Now we are at the point to give a definition of knowledge base model. Let a model $H = (H, \Psi, f)$ be given.

**Definition 3.2.** A knowledge base $KB = KB(H, \Psi, f)$ is a triple $(F_G(H), D_G(H), Ct_H)$, where $F_G(H)$ is the category of knowledge description, $D_G(H)$ is the category of knowledge content, and

$$Ct_H : F_G(H) \to D_G(H)$$

is the contravariant functor.
Remark 3.3. We will use the term “a knowledge base” instead of a more precise “a knowledge base model”.

One can say that defined knowledge base model is a sort of automaton (see [30]), where queries are objects of the category of knowledge descriptions $F_Θ(\mathcal{H})$, replies are objects of the category of knowledge content $D_Θ(\mathcal{H})$. To be such automaton a knowledge base also presupposes a connection with a particular data (information). This information is held in the subject area presented by the model $\mathcal{H} = (H, \Psi, f)$.

The knowledge functor $Ct_\mathcal{H}$ gives a dynamical passage between queries and replies, namely, between categories $F_Θ(\mathcal{H})$ and $D_Θ(\mathcal{H})$. Moreover, this passage is one-to-one correspondence.

Theorem 3.4 ([4]). The knowledge functor $Ct_\mathcal{H}$ gives rise to the dual isomorphism between the category of knowledge description $F_Θ(\mathcal{H})$ and the category of knowledge content $D_Θ(\mathcal{H})$.

4 Knowledge bases equivalences

4.1 An overview

In this section we give a short review of our previous results about various knowledge bases equivalences and connections between them. We start with the most strong equivalence from algebraic viewpoint, namely, with isomorphic knowledge bases.

Fix a variety of algebras $Θ$, algebras $H_1$ and $H_2$ from $Θ$ and a set of relation symbols $Ψ$. Let two models $\mathcal{H}_1 = (H_1, \Psi, f_1)$ and $\mathcal{H}_2 = (H_2, \Psi, f_2)$ be given.

Definition 4.1. Two knowledge bases $KB(\mathcal{H}_1)$ and $KB(\mathcal{H}_2)$ are called isomorphic if the corresponding models $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic.

The notion of isomorphic knowledge bases is very strong. It presuppose an isomorphism of subject areas of knowledge bases, which automatically implies an isomorphism of categories of knowledge description of corresponding knowledge bases and, according to Theorem 3.4, an isomorphism of categories of knowledge content of corresponding knowledge bases.

For practical needs there is more appropriate and not too strong notion of informationally equivalent knowledge bases.

Let two models $\mathcal{H}_1 = (H_1, \Psi, f_1)$ and $\mathcal{H}_2 = (H_2, \Psi, f_2)$ be given. Take the corresponding knowledge bases $KB(\mathcal{H}_1)$ and $KB(\mathcal{H}_2)$.
Definition 4.2. Knowledge base $KB(H_1)$ and $KB(H_2)$ are called informationally equivalent, if the categories of knowledge description $F_{\Theta}(H_1)$ and $F_{\Theta}(H_2)$ are isomorphic.

Remark 4.3. In view of Theorem 3.4, the categories of knowledge description $F_{\Theta}(H_1)$ and $F_{\Theta}(H_2)$ are isomorphic if and only if the categories of knowledge content $D_{\Theta}(H_1)$ and $D_{\Theta}(H_2)$ are isomorphism. Thus, one can formulate Definition 4.2 in terms of isomorphism of the categories of knowledge content.

In plain words, the informational equivalence of knowledge bases means that everything that can be asked from one knowledge base can be asked from the other and to conclude that this is the same information.

Our main interest is to find out how the informational equivalence is related to the logical description of knowledge bases. In this concern, there were defined elementarily equivalent, logically-geometrical equivalent, LG-isotypic knowledge bases and others.

The notion of LG-equivalence (logically-geometrical equivalence) of knowledge bases is based on geometrical approach, whereas LG-isotypic knowledge bases are defined using logical tools. But these notions give us the same description of knowledge bases:

Theorem 4.4 ([2]). Logically-geometrical equivalent (or LG-isotypic) knowledge bases are informationally equivalent.

In the next section we will deal with one more equivalence for knowledge bases, which is defined using category theory tools.

4.2 Logically automorphically equivalent knowledge bases

As we have seen the notion of LG-equivalent and LG-isotypic knowledge bases is good enough to distinguish two knowledge bases. That is, LG-equivalent and LG-isotypic knowledge bases are informationally equivalent.

In this section we introduce logically automorphical equivalence of knowledge bases. We will see that this notion also gives a good characterization of knowledge bases.

Let us start with some preliminary constructions.

4.2.1 Functor $Cl_{\mathcal{H}}$

The functor $Cl_{\mathcal{H}}$ presents a connection between the category $\tilde{\Phi}$ and the category $F_{\Theta}$ of lattices of all closed filters, for various models $\mathcal{H}_i = (H_i, \Psi, f_i)$ with algebras $H_i$ from a variety $\Theta$ defined as follows.
An object $F_X^X(\mathcal{H}_i)$ of the category $F_E$ is the lattice of all $\mathcal{H}_i$-closed filters in $\Phi(X)$, $X \in \Gamma$.

Morphisms of the category $F_E$ are maps of lattice of all closed filters, which preserve partial order on corresponding objects, but they not to be necessarily homomorphisms of lattices.

**Remark 4.5.** In Section 3 we have defined the category $F_E(\mathcal{H}_i)$ of lattices of all $\mathcal{H}_i$-closed filters (the category of knowledge description), that is, this is the category over a fixed model $\mathcal{H}_i$.

Thus, $F_E(\mathcal{H}_i)$ is a full subcategory of $F_E$ and, hence, morphisms of the category $F_E(\mathcal{H}_i)$ are morphisms of $F_E$. But there are other morphisms in $F_E$, we will do not specify them. For example, there are morphism between objects $F_X^X(\mathcal{H}_1)$ and $F_X^X(\mathcal{H}_2)$, where $\mathcal{H}_1 = (H_1, \Psi, f_1)$, $\mathcal{H}_2 = (H_2, \Psi, f_2)$ are models with different algebras $H_1$ and $H_2$ from $\Theta$.

One can define a correspondence $Cl_H : \tilde{\Phi} \to F_E$, on objects as follows:

$$Cl_H(\Phi(X)) = F_X^X(\mathcal{H})$$

and if $s_* : \Phi(X) \to \Phi(Y)$ is a morphism in $\tilde{\Phi}$, then

$$Cl_H(s_*) = [s_*]^0 : F_X^X(\mathcal{H}) \to F_Y^Y(\mathcal{H})$$

is a morphism in $F_E$, such that

$$[s_*]^0 : (T)^{\mathcal{H}} \to (s_* T)^{\mathcal{H}}$$

where $T \subset \Phi(X)$, $(s_* T)^{\mathcal{H}}$ is an $\mathcal{H}$-closed filter in $F_X^X(\mathcal{H})$, $(s_* T)^{\mathcal{H}}$ an $\mathcal{H}$-closed filter in $F_Y^Y(\mathcal{H})$.

Next diagram illustrates the correspondence $Cl_H$:

\[
\begin{array}{ccc}
\Phi(X) & \xrightarrow{s_*} & F_X^X(\mathcal{H}) \\
\phi^I \downarrow & & \downarrow [s_*]^0 \quad \cdots \quad \downarrow [s_*]^I \\
\Phi(Y) & \xrightarrow{} & F_Y^Y(\mathcal{H})
\end{array}
\]

where $[s_*]^I$ are some other morphisms between objects $F_X^X(\mathcal{H})$ and $F_Y^Y(\mathcal{H})$ associated with morphism $s_*$ of the category $\tilde{\Phi}$.

The following proposition takes place.
Proposition 4.6. The correspondence \( \Phi : \tilde{\Phi} \to F_{\emptyset} \) is a covariant functor.

Proof. If \( s_* = id_{\Phi}(X) : \Phi(X) \to \Phi(X) \) is the identity morphism of the object \( \Phi(X) \), then \( s_*T = T \) for every \( T \subset \Phi(X) \). Thus, \( Cl_{\mathcal{H}}(id_{\Phi}(X)) = [id_{\Phi}(X)]^0 = id_{\tilde{\Phi}(\mathcal{H})} \) is the identity morphism of the object \( F_{\emptyset}(\mathcal{H}) \).

Let \( s_*^1 : \Phi(X) \to \Phi(Y) \), \( s_*^2 : \Phi(Y) \to \Phi(Z) \) be morphisms in \( \tilde{\Phi} \). Take a subset \( T \) from \( \Phi(X) \). Then

\[
s_*^1 : T \to s_*^1 T, \\
s_*^2 : s_*^1 T \to s_*^2(s_*^1 T)
\]

and

\[
Cl_{\mathcal{H}}(s_*^2 \circ s_*^1) : T^{\mathcal{L}}_{\mathcal{H}} \to (s_*^2 s_*^1 T)^{\mathcal{L}}_{\mathcal{H}}. \tag{1}
\]

From the other hand,

\[
Cl_{\mathcal{H}}(s_*^1) = [s_*^1]^0 : T^{\mathcal{L}}_{\mathcal{H}} \to (s_*^1 T)^{\mathcal{L}}_{\mathcal{H}}
\]

and

\[
Cl_{\mathcal{H}}(s_*^2) = [s_*^2]^0 : (s_*^1 T)^{\mathcal{L}}_{\mathcal{H}} \to (s_*^2(s_*^1 T)^{\mathcal{L}}_{\mathcal{H}})^{\mathcal{L}}_{\mathcal{H}}.
\]

Therefore,

\[
Cl_{\mathcal{H}}(s_*^1) \circ Cl_{\mathcal{H}}(s_*^2) = [s_*^1]^0 \circ [s_*^2]^0 : T^{\mathcal{L}}_{\mathcal{H}} \to (s_*^2(s_*^1 T)^{\mathcal{L}}_{\mathcal{H}})^{\mathcal{L}}_{\mathcal{H}}.
\]

Let us simplify the right part of the last equation. By Proposition 2.14, the equality \( (s_*T)^L_{\mathcal{H}} = s_* T^L_{\mathcal{H}} \) takes place. Thus,

\[
(s_*^2(s_*^1 T)^{\mathcal{L}}_{\mathcal{H}})^{\mathcal{L}}_{\mathcal{H}} = (s_*^2((s_*^1 T)^{\mathcal{L}}_{\mathcal{H}})^{\mathcal{L}})_{\mathcal{H}}.
\]

Using the property of the Galois correspondence, namely, \( T^{\mathcal{L}}_{\mathcal{H}} = T^L_{\mathcal{H}} \), we have

\[
(s_*^2((s_*^1 T)^{\mathcal{L}}_{\mathcal{H}})^{\mathcal{L}})_{\mathcal{H}} = (s_*^2(s_*^1 T)^{\mathcal{L}}_{\mathcal{H}})^{\mathcal{L}}_{\mathcal{H}}.
\]

Applying again the equality \( (s_*T)^L_{\mathcal{H}} = s_* T^L_{\mathcal{H}} \), we get

\[
(s_*^2(s_*^1 T)^{\mathcal{L}}_{\mathcal{H}})^{\mathcal{L}}_{\mathcal{H}} = (s_*^2 s_*^1 T)^{\mathcal{L}}_{\mathcal{H}}.
\]

Thus,

\[
Cl_{\mathcal{H}}(s_*^1) \circ Cl_{\mathcal{H}}(s_*^2) = [s_*^1]^0 \circ [s_*^2]^0 : T^{\mathcal{L}}_{\mathcal{H}} \to (s_*^2 s_*^1 T)^{\mathcal{L}}_{\mathcal{H}}. \tag{2}
\]

Comparing equations (1) and (2), we conclude that

\[
Cl_{\mathcal{H}}(s_*^2 \circ s_*^1) = Cl_{\mathcal{H}}(s_*^2) \circ Cl_{\mathcal{H}}(s_*^1),
\]

and \( Cl_{\mathcal{H}} \) is a covariant functor. \( \square \)
4.2.2 Definition of logically automorphically equivalence

We will use the notion of isomorphism of two functors (natural isomorphism in terms of [20]).

**Definition 4.7.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be functors from a category $\mathcal{C}_1$ to a category $\mathcal{C}_2$. An isomorphism $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$ of functors $\mathcal{F}_1$ and $\mathcal{F}_2$ is a function which assigns to each object $C$ in $\mathcal{C}_1$ a two-sided morphism $\alpha(C) : \mathcal{F}_1(C) \leftrightarrow \mathcal{F}_2(C)$ in the category $\mathcal{C}_2$ in such a way that for every morphism $\nu : C \to C'$ of the category $\mathcal{C}_1$ the diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{F}_1(C) & \xrightarrow{\alpha(C)} & \mathcal{F}_2(C') \\
\mathcal{F}_1(\nu) & \downarrow & \mathcal{F}_2(\nu) \\
\mathcal{F}_1(C') & \xrightarrow{\alpha(C')} & \mathcal{F}_2(C')
\end{array}
$$

In Section 4.2.1 we construct the covariant functor $Cl_H : \tilde{\Phi} \to F_\Theta$, where $\mathcal{H} = (H, \Psi, f)$ is a model, $\tilde{\Phi}$ is the category of algebras of formulas, $F_\Theta$ is the category of lattices of closed filters. Using this functor we define a notion of logically automorphical equivalence for models.

Let two models $\mathcal{H}_1 = (H_1, \Psi, f_1)$ and $\mathcal{H}_2 = (H_2, \Psi, f_2)$ be given and let $\varphi$ be an automorphism of the category $\tilde{\Phi}$.

**Definition 4.8.** Models $\mathcal{H}_1 = (H_1, \Psi, f_1)$ and $\mathcal{H}_2 = (H_2, \Psi, f_2)$ are called logically automorphically equivalent if for some automorphism $\varphi$ of the category $\tilde{\Phi}$ there is the functor isomorphism

$$
\alpha_{\varphi} : Cl_{\mathcal{H}_1} \to Cl_{\mathcal{H}_2} \cdot \varphi.
$$

This definition gives rise to the notion of logically automorphically equivalent knowledge bases. Let two models $\mathcal{H}_1 = (H_1, \Psi, f_1)$, $\mathcal{H}_2 = (H_2, \Psi, f_2)$ and the corresponding knowledge bases $KB(\mathcal{H}_1)$ and $KB(\mathcal{H}_2)$ be given.

**Definition 4.9.** Knowledge bases $KB(\mathcal{H}_1)$ and $KB(\mathcal{H}_2)$ are called logically automorphically equivalent if the corresponding models $\mathcal{H}_1$ and $\mathcal{H}_2$ are logically automorphically equivalent.

4.2.3 Auxiliary constructions

In this section we present results which we will use to prove the main result about logically automorphically equivalent knowledge bases.
Let two logically automorphically equivalent models \( \mathcal{H}_1 = (H_1, \Psi, f_1) \) and \( \mathcal{H}_2 = (H_2, \Psi, f_2) \) be given.

Logically automorphical equivalence of the models \( \mathcal{H}_1 = (H_1, \Psi, f_1) \) and \( \mathcal{H}_2 = (H_2, \Psi, f_2) \) means that there exists an automorphism \( \varphi \) of the category \( \tilde{\Phi} \), such that the functors \( Cl_{\mathcal{H}_1} \) and \( Cl_{\mathcal{H}_2} \cdot \varphi \) are isomorphic. This fact implies that there is the commutative diagram (see Definition 4.7):

\[
\begin{align*}
Cl_{\mathcal{H}_1}(\Phi(X)) & \xrightarrow{\alpha_{\varphi}(\Phi(X))} Cl_{\mathcal{H}_2} \cdot \varphi(\Phi(X)) \\
Cl_{\mathcal{H}_1}(s_*) & \downarrow \quad \downarrow Cl_{\mathcal{H}_2} \cdot \varphi(s_*) \\
Cl_{\mathcal{H}_1}(\Phi(Y)) & \xrightarrow{\alpha_{\varphi}(\Phi(Y))} Cl_{\mathcal{H}_2} \cdot \varphi(\Phi(Y)),
\end{align*}
\]

where \( \alpha_{\varphi} : Cl_{\mathcal{H}_1} \rightarrow Cl_{\mathcal{H}_2} \cdot \varphi \) is an isomorphism of functors, \( \Phi(X) \) and \( \Phi(Y) \) are objects of the category \( \tilde{\Phi} \) and \( s_* : \Phi(X) \rightarrow \Phi(Y) \) is a morphism in \( \tilde{\Phi} \).

Recall that

\[
Cl_{\mathcal{H}_i}(\Phi(X)) = F^X_\Theta(H_i),
\]

and if \( s_* : \Phi(X) \rightarrow \Phi(Y) \), then

\[
Cl_{\mathcal{H}_i}(s_*) = [s_*]_{H_i}^0 : F^X_\Theta(H_i) \rightarrow F^Y_\Theta(H_i),
\]

such that

\[
[s_*]_{H_i}^0 : T_{H_i}^{LL} \rightarrow (s_* T)_{H_i}^{LL},
\]

where \( T \subset \Phi(X) \), \( T_{H_i}^{LL} \) is an \( H_i \)-closed filter in \( F^X_\Theta(H_i) \), \( (s_* T)_{H_i}^{LL} \) an \( H_i \)-closed filter in \( F^Y_\Theta(H_i) \), for more details see Section 4.2.1.

**Remark 4.10.** We add subscribe index \( H_i \) for morphism \([s_*]^0\) in order to distinguish morphisms in \( F_\Theta(\mathcal{H}_1) \) and \( F_\Theta(\mathcal{H}_2) \).

Let \( \varphi \) be an automorphism of \( \tilde{\Phi} \), such that

\[
\varphi(\Phi(X)) = \Phi(X') \text{ and } \varphi(\Phi(Y)) = \Phi(Y'),
\]

where \( X, Y, X', Y' \in \Gamma \). In particular, this means that if \( s_* : \Phi(X) \rightarrow \Phi(Y) \), then \( \varphi(s_*) : \Phi(X') \rightarrow \Phi(Y') \).

Using the settings above, we have

\[
\begin{align*}
Cl_{\mathcal{H}_2} \cdot \varphi(\Phi(X)) &= Cl_{\mathcal{H}_2}(\Phi(X')) = F^X_\Theta(H_2), \\
Cl_{\mathcal{H}_2} \cdot \varphi(\Phi(Y)) &= Cl_{\mathcal{H}_2}(\Phi(Y')) = F^Y_\Theta(H_2), \\
Cl_{\mathcal{H}_2} \cdot \varphi(s_*) &= Cl_{\mathcal{H}_2}(\varphi(s_*)) = [\varphi(s_*)]_{H_2}^0.
\end{align*}
\]

17
We can rewrite diagram (3) as follows

\[
\begin{array}{c}
F_{\Theta}^{X}(H_1) \xrightarrow{\alpha_{\varphi}} F_{\Theta}^{X'}(H_2) \\
\downarrow \quad \downarrow \quad \downarrow \\
F_{\Theta}^{Y}(H_1) \xrightarrow{\alpha_{\varphi}} F_{\Theta}^{Y'}(H_2).
\end{array}
\]

(4)

**Remark 4.11.** Here and later on we will write simply \(\alpha_{\varphi}\) instead of \(\alpha_{\varphi}(\Phi(X))\) or \(\alpha_{\varphi}(\Phi(Y))\).

The next proposition provides a connection between morphism in categories \(F_{\Theta}(H_1)\) and \(F_{\Theta}(H_2)\) over logically automorphically equivalent models \(H_1\) and \(H_2\).

**Proposition 4.12.** Let two logically automorphically equivalent models \(H_1 = (H_1, \Psi, f_1)\) and \(H_2 = (H_2, \Psi, f_2)\) be given. The map \([s_\ast]_{H_1} : T_1 \to T_2\) is admissible if and only if the map \([\varphi(s_\ast)]_{H_2} : \alpha_{\varphi}T_1 \to \alpha_{\varphi}T_2\) is admissible, where \(T_1 \in F_{\Theta}^{X}(H_1)\), \(T_2 \in F_{\Theta}^{Y}(H_1)\).

**Proof.** Diagram (3) gives rises to the following diagram:

\[
\begin{array}{ccc}
T_1 & \xrightarrow{\alpha_{\varphi}} & \alpha_{\varphi}T_1 \\
\downarrow \quad \downarrow \quad \downarrow & & \downarrow \quad \downarrow \quad \downarrow \\
[s_\ast]_{H_1} & \xrightarrow{[\varphi(s_\ast)]_{H_2}} & [\varphi(s_\ast)]_{H_2} \\
(s_{\ast}T_1)_{H_1} & \xrightarrow{\alpha_{\varphi}} & (\varphi(s_{\ast})(\alpha_{\varphi}T_1))_{H_2}.
\end{array}
\]

By the definition, the map \([s_\ast]_{H_1} : T_1 \to T_2\) is admissible if and only if \(s_{\ast}T_1 \subseteq T_2\). Moreover, \(T_2\) is an \(H_1\)-closed filter, hence \((s_{\ast}T_1)_{H_1} \subseteq T_2\). Thus, we can extend the diagram above as follows:

\[
\begin{array}{ccc}
T_1 & \xrightarrow{\alpha_{\varphi}} & \alpha_{\varphi}T_1 \\
\downarrow \quad \downarrow \quad \downarrow & & \downarrow \quad \downarrow \quad \downarrow \\
[s_\ast]_{H_1} & \xrightarrow{[\varphi(s_\ast)]_{H_2}} & [\varphi(s_\ast)]_{H_2} \\
(s_{\ast}T_1)_{H_1} & \xrightarrow{\alpha_{\varphi}} & (\varphi(s_{\ast})(\alpha_{\varphi}T_1))_{H_2} \\
\cap & & \cap \\
T_2 & \xrightarrow{\alpha_{\varphi}} & \alpha_{\varphi}T_2.
\end{array}
\]

From the diagram follows that

\((\varphi(s_{\ast})(\alpha_{\varphi}T_1))_{H_2} \subseteq \alpha_{\varphi}T_2\).
Moreover, $\varphi(s_*)(\alpha_{\varphi} T_1) \subseteq (\varphi(s_*)(\alpha_{\varphi} T_1))^{LL}_{\mathcal{H}_2}$ and the map $[\varphi(s_*)]_{\mathcal{H}_2} : \alpha_{\varphi} T_1 \to \alpha_{\varphi} T_2$ is admissible.

Thus, the map $[s_*]_{\mathcal{H}_1} : T_1 \to T_2$ is admissible if and only if the map $[\varphi(s_*)]_{\mathcal{H}_2} : \alpha_{\varphi} T_1 \to \alpha_{\varphi} T_2$ is admissible. The following diagram gives the illustration:

4.2.4 The main result

In this section we present the main result of this paper, namely, we will show that logically automorphically equivalent knowledge bases are informationally equivalent.

The following preliminary result describes relation between logically automorphically equivalent models and categories of lattices of $\mathcal{H}$-closed filters over corresponding models.

In this section we will use notations from the previous section.

**Theorem 4.13.** If models $\mathcal{H}_1 = (H_1, \Psi, f_1)$ and $\mathcal{H}_2 = (H_2, \Psi, f_2)$ are logically automorphically equivalent, then the categories $F_{\Theta}(\mathcal{H}_1)$ and $F_{\Theta}(\mathcal{H}_2)$ are isomorphic.

**Proof.** To prove the theorem we will construct a correspondence (functor) $\mathcal{F} : F_{\Theta}(\mathcal{H}_1) \to F_{\Theta}(\mathcal{H}_2)$ and show that it gives rise to an isomorphism of the given categories.

For an object $F_\Theta^X(\mathcal{H}_1)$ from $F_{\Theta}(\mathcal{H}_1)$ we set

$$\mathcal{F}(F_\Theta^X(\mathcal{H}_1)) = F_\Theta^{X'}(\mathcal{H}_2),$$

where $X$ and $X'$ are correlated by the given automorphism $\varphi$, that is, $\varphi(\Phi(X)) = \Phi(X')$ (see Section 4.2.3).

Let $[s_*]_{\mathcal{H}_1} : F_\Theta^X(\mathcal{H}_1) \to F_\Theta^{X'}(\mathcal{H}_1)$ be a morphism in $F_{\Theta}(\mathcal{H}_1)$, such that $[s_*]_{\mathcal{H}_1} : T_1 \to T_2$, and
where \( T_1 \in F^X_\Theta(H_1), T_2 \in F^Y_\Theta(H_1) \) and \( s_* T_1 \subseteq T_2 \).

We determine the morphism
\[
\mathcal{F}([s_*]_{H_1}) : F^X_\Theta(H_2) \to F^Y_\Theta(H_2)
\]
by the rule
\[
\mathcal{F}([s_*]_{H_1}) = [\varphi(s_*)]_{H_2} : \alpha_{\varphi} T_1 \to \alpha_{\varphi} T_2.
\]
By assumption, \( \alpha_{\varphi} \) is a bijection (two-sided morphism) between objects \( F^X_\Theta(H_1) \) and \( F^Y_\Theta(H_2) \), so if \( T \) runs all \( H_1 \)-closed filters from \( F^X_\Theta(H_1) \), then \( \alpha_{\varphi}(T) \) runs all \( H_2 \)-closed filters from \( F^Y_\Theta(H_2) \). Moreover, in view of Proposition 4.12, the morphism \([\varphi(s_*)]_{H_2}\) is defined correctly, that is, \( \varphi(s_*)(\alpha_{\varphi} T_1) \subseteq \alpha_{\varphi} T_2 \).

Let us show that defined in such a way correspondence \( \mathcal{F} \) is, indeed, a functor.

Denote by \( id_{F^X_\Theta(H_1)} \) the identity morphism of the object \( F^X_\Theta(H_1) \). By definition of \( \mathcal{F} \) we have the diagram:

\[
\begin{array}{ccc}
T & \xrightarrow{\alpha_{\varphi}} & \alpha_{\varphi} T \\
\downarrow{id_{F^X_\Theta(H_1)}} & & \downarrow{\mathcal{F}(id_{F^X_\Theta(H_1)})} \\
T & \xleftarrow{\alpha_{\varphi}} & \alpha_{\varphi} T,
\end{array}
\]

where \( T \in F^X_\Theta(H_1) \). Thus, \( \mathcal{F}(id_{F^X_\Theta(H_1)}) \) is the identity morphism of \( F^Y_\Theta(H_2) = \mathcal{F}(F^X_\Theta(H_1)) \).

Let two morphisms of \( F_\Theta(H_1) \) be given:
\[
[s^1_*]_{H_1} : F^X_\Theta(H_1) \to F^Y_\Theta(H_1),
\]
\[
[s^2_*]_{H_1} : F^Y_\Theta(H_1) \to F^Z_\Theta(H_1).
\]

We will check that
\[
\mathcal{F}([s^2_*]_{H_1} \circ [s^1_*]_{H_1})) = \mathcal{F}([s^2_*]_{H_1}) \circ \mathcal{F}([s^1_*]_{H_1}).
\]

Let \( T_1 \) be an \( H_1 \)-closed filter from \( F^X_\Theta(H_1) \) and
\[
[s^1_*]_{H_1} : T_1 \to T_2,
\]
\[
[s^2_*]_{H_1} : T_2 \to T_3.
\]

Thus,
\[
[s^2_*]_{H_1} \circ [s^1_*]_{H_1} : T_1 \to T_3
\]
and
\[ F([s_2^2]_{H_1} \circ [s_1^1]_{H_1}) : \alpha_\varphi(T_1) \to \alpha_\varphi(T_3). \] (5)

From the other hand,
\[ F([s_1^1]_{H_1}) : \alpha_\varphi T_1 \to \alpha_\varphi T_2 \]
and
\[ F([s_2^2]_{H_1}) : \alpha_\varphi T_2 \to \alpha_\varphi T_3. \]

Consequently, the composition of functors \( F([s_1^1]_{H_1}) \) and \( F([s_2^2]_{H_1}) \) works as follows:
\[ F([s_2^2]_{H_1}) \circ F([s_1^1]_{H_1}) : \alpha_\varphi T_1 \to \alpha_\varphi T_3. \] (6)

Summarizing equations (5) and (6), we have
\[ F([s_2^2]_{H_1} \circ [s_1^1]_{H_1}) = F([s_2^2]_{H_1}) \circ F([s_1^1]_{H_1}). \]

Thus, the correspondence \( F : F_\Theta(H_1) \to F_\Theta(H_2) \) is a covariant functor.

Now we will construct the inverse functor \( F' : F_\Theta(H_2) \to F_\Theta(H_1) \).

The functor \( F' \) works on objects as
\[ F'(F_\Theta^X(H_2)) = F_\Theta^X(H_1), \]
where \( X' \) and \( X \) are correlated by the given automorphism \( \varphi \), that is, \( \varphi^{-1}(\Phi(X')) = \Phi(X) \) (see Section [123]).

Take a morphism \( s_* : \Phi(X') \to \Phi(Y') \) in the category \( \Phi \). Let
\[ [s_*]_{H_2} : F_\Theta^{X'}(H_2) \to F_\Theta^{Y'}(H_2) \]
be a morphism in \( F_\Theta(H_2) \), such that
\[ [s_*]_{H_2} : T_1 \to T_2, \]
where \( T_1 \in F_\Theta^X(H_2) \), \( T_2 \in F_\Theta^Y(H_2) \) and \( s_*T_1 \subseteq T_2 \). We determine the morphism
\[ F'([s_*]_{H_2}) : F_\Theta^X(H_1) \to F_\Theta^Y(H_1) \]
as follows
\[ F'([s_*]_{H_2}) = [\varphi^{-1}(s_*)]_{H_1} : \alpha_\varphi^{-1}T_1 \to \alpha_\varphi^{-1}T_2. \] (7)

The map \( \alpha_\varphi^{-1} \) is defined, since \( \alpha_\varphi \) is a two-sided morphism between objects \( F_\Theta^X(H_1) \) and \( F_\Theta^X(H_2) \). Moreover, if \( T \) runs all \( H_2 \)-closed filters from
$F_{\Theta}^{X}(H_2)$, then $\alpha_{\varphi}^{-1}(T)$ runs all $H_1$-closed filters from $F_{\Theta}^{X}(H_1)$. Hence, in view of Proposition 4.12, morphism $[\varphi^{-1}(s_{*})]_{H_1}$ is determined correctly and the diagram takes place:

\[
\begin{array}{c}
T_1 \xrightarrow{\alpha_{\varphi}^{-1}} \alpha_{\varphi}^{-1}T_1 \\
\downarrow{}^\cong \downarrow{}^\cong \\
T_2 \xrightarrow{\alpha_{\varphi}^{-1}} \alpha_{\varphi}^{-1}T_2.
\end{array}
\]

Let us show that defined in such a way correspondence $F'$ is a functor.

If $id_{F_{\Theta}^{X}(H_2)}$ the identity morphism of the object $F_{\Theta}^{X}(H_2)$, then the following diagram takes place:

\[
\begin{array}{c}
T \xrightarrow{\alpha_{\varphi}^{-1}} \alpha_{\varphi}^{-1}T \\
\downarrow{id_{F_{\Theta}^{X}(H_2)}} \downarrow{\mathcal{F}(id_{F_{\Theta}^{X}(H_2)})} \\
T \xrightarrow{\alpha_{\varphi}^{-1}} \alpha_{\varphi}^{-1}T,
\end{array}
\]

where $T \in F_{\Theta}^{X}(H_2)$, $\alpha_{\varphi}^{-1}T \in F_{\Theta}^{X}(H_1)$. Thus, $\mathcal{F}'(id_{F_{\Theta}^{X}(H_2)}) = id_{F_{\Theta}^{X}(H_1)}$.

Take two morphisms in $F_{\Theta}(H_2)$:

\[
[s_{*}]_{H_2} : F_{\Theta}^{X}(H_2) \to F_{\Theta}^{Y}(H_2),
\]

such that

\[
[s_{*}]_{H_2} : T_1 \to T_2,
\]

and

\[
[s_{*}^2]_{H_2} : F_{\Theta}^{Y}(H_2) \to F_{\Theta}^{Z}(H_2),
\]

such that

\[
[s_{*}^2]_{H_2} : T_2 \to T_3,
\]

where $T_1 \in F_{\Theta}^{X}(H_2)$, $T_2 \in F_{\Theta}^{Y}(H_2)$, $T_3 \in F_{\Theta}^{Z}(H_2)$.

We will check that

$$\mathcal{F}'([s_{*}^2]_{H_2} \circ [s_{*}]_{H_2}) = \mathcal{F}'([s_{*}^2]_{H_2}) \circ \mathcal{F}'([s_{*}]_{H_2}).$$

Indeed,

$$[s_{*}^2]_{H_2} \circ [s_{*}]_{H_2} : T_1 \to T_3$$

and

$$\mathcal{F}'([s_{*}^2]_{H_2} \circ [s_{*}]_{H_2}) : \alpha_{\varphi}^{-1}T_1 \to \alpha_{\varphi}^{-1}T_3.$$

(8)
From the other hand,
\[ F'(\{s_1^*\}_H^2) : \alpha_{\varphi}^{-1}T_1 \to \alpha_{\varphi}^{-1}T_2 \]
and
\[ F'(\{s_2^*\}_H^2) : \alpha_{\varphi}^{-1}T_2 \to \alpha_{\varphi}^{-1}T_3. \]
Thus, the composition of functors \( F'(\{s_1^*\}_H^2) \) and \( F'(\{s_2^*\}_H^2) \) works as follows:
\[ F'(\{s_2^*\}_H^1) \circ F'(\{s_1^*\}_H^2) : \alpha_{\varphi}^{-1}T_1 \to \alpha_{\varphi}^{-1}T_3. \tag{9} \]
Summarizing equations (8) and (9), we get
\[ F'(\{s_2^*\}_H^2 \circ \{s_1^*\}_H^2) = F'(\{s_2^*\}_H^2) \circ F'(\{s_1^*\}_H^2). \]
Thus, the correspondence \( F' : F_\Theta(H^2) \to F_\Theta(H^1) \) is a covariant functor.
Moreover, one can check that \( F' \) is the inverse functor for \( F \) and categories \( F_\Theta(H^1) \) and \( F_\Theta(H^2) \) are isomorphic. Theorem is proved.

The next theorem is the main result concerning logically automorphically equivalent knowledge bases.

**Theorem 4.14.** Logically automorphically equivalent knowledge bases \( KB(H_1) \) and \( KB(H_2) \) are informationally equivalent.

**Proof.** Remind that knowledge base \( KB(H_1) \) and \( KB(H_2) \) are informationally equivalent, if the categories of knowledge description \( F_\Theta(H_1) \) and \( F_\Theta(H_2) \) are isomorphic (see Definition 4.2).

According to Theorem 4.13, logically automorphical equivalence of models \( H_1 \) and \( H_2 \) implies isomorphism of categories of knowledge description \( F_\Theta(H_1) \) and \( F_\Theta(H_2) \). In turn, this means that the knowledge base \( KB(H_1) \) and \( KB(H_2) \) are informationally equivalent.

**References**

[1] A. V. Aho, Y. Sagiv, J. D. Ullman, *Equivalences Among Relational Expressions*, SIAM J. Comput., 8(2) (1979) 218–246.

[2] E. Aladova, E. Plotkin, T. Plotkin, *Isotypeness of models and knowledge bases equivalence*, Math. Comput. Sci., 7(4) (2013) 421–438.

[3] E. Aladova, *Syntax versus semantics in knowledge bases I*, accepted in Internat. J. Algebra Comput.
[4] E. Aladova, T. Plotkin, *Syntax versus semantics in knowledge bases II*, submitted to IMCP series, Contemporary Mathematics AMS.

[5] Alavi, M., Leidner, D. E., *Review: Knowledge management and knowledge management systems: Conceptual foundations and research issues*, MIS Quarterly, 25(1) (2001) 107-136.

[6] P. Atzeni, G. Aussiello, C. Batini, M. Moscarini, *Inclusion and equivalence between relational database schemes*, Theoret. Comput. Sci., 19 (1982) 267–285.

[7] K.H. Baik, L. L. Miller, *Topological Approach for Testing Equivalence in Heterogenous Relational Databases*, The Computer Journal, 33(1) (1990) 2–10.

[8] F. Bancillon, *On the completeness of query language for relational databases*, Lecture Notes in Comput. Sci., 64 (1978) 112–123.

[9] C. Beeri, A. Mendelzon, Y. Sagiv, J. Ullman, *Equivalence of relational database schemes*, In Proc. Eleventh Annual ACM Symp. on Theory of Computing, (1979) 319–329.

[10] E.M. Beniaminov, *Galois theory of complete relational subalgebras of algebras of relations, logical structures, symmetry*, Nauchno-Tekhnicheskaya Informatsiya, 2(1) (1980) 17-25.

[11] C.C. Chang, H.J. Keisler, *Model Theory*, North-Holland Publ. Co., 1973.

[12] E. F. Codd, *A relational model of data for large shared data banks*, Communications of ACM, 13:6 (1970) 377–387.

[13] T. Davenport, L. Prusak, *Working knowledge: How organization manage what they know*, Boston, MA: Harvard Business School Press, 1998.

[14] R. Davis, H. Shrobe, P. Szolovits, *What is a knowledge representation?*, AI Magazine, 14(1) (1993) 17-33.

[15] W. Frawley, G. Piatetsky-Shapiro, C. Matheus, *Knowledge discovery in databases: An overview*, AI Magazine, (1992) 213-228.

[16] P.R. Halmos, *Algebraic logic*, New York, Chelsea Publishing Co., 1969.

[17] H. Helbig, *Knowledge representation and the semantics of natural language*, New York, Springer, 2006.
[18] A. Heuer, *Equivalent schemes in semantic, nested relational, and relational database models*, Lecture Notes in Comput. Sci., 364 (1989) 237–253.

[19] M. Knjazhansky, T. Plotkin, *Knowledge bases over algebraic models: some notes about informational equivalence*, Int. J. Knowledge Management 8(1) (2012) 22–39.

[20] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics, 5, Springer-Verlag, New York-Berlin, 1971.

[21] A.I. Malcev *Algebraic Systems*, Springer-Verlag, 1973.

[22] D. Marker, *Model Theory: An Introduction*, Springer Verlag, (2002).

[23] G. Mashevitzky, B. Plotkin, E. Plotkin, *Automorphisms of the category of free Lie algebras*, J. Algebra, 282(2) (2004) 490–512.

[24] C. McInerney, *Knowledge management and the dynamic nature of knowledge*, Journal of the American Society for Information Science and Technology, 53(12) (2002) 1009–1018.

[25] B.I. Plotkin, *Algebra, Categories and Databases*, Handbook of algebra, v.2, Elsevier, Springer, 1999, 81–148.

[26] B. Plotkin, *Algebraic geometry in First Order Logic*, J. Math. Sci., 137(5) (2006) 5049–5097.

[27] B. Plotkin, *Algebraic logic and logical geometry in arbitrary varieties of algebras*, In: Proceedings of the Conf. on Group Theory, Combinatorics and Computing, AMS Contemporary Math series, 611 (2014) 151–167.

[28] B.I. Plotkin, *Isotyped algebras*, Proc. Steklov Inst. Mathematics, 287 (2012), 91–115.

[29] B. Plotkin, Seven lectures on the universal algebraic geometry, preprint, (2002), arxiv:math, GM/0204245, 87pp.

[30] B. Plotkin, *Universal algebra, algebraic logic and databases*, Kluwer Acad. Publ., 1994.

[31] B. Plotkin, E. Aladova, E. Plotkin, *Algebraic logic and logically-geometric types in varieties of algebras*, J. Algebra Appl., 12(2), paper no. 1250146, (2013) 23 p.
[32] B. Plotkin, E. Plotkin, *Multi-sorted logic and logical geometry: some problems*, Demonstratio Math., **48**(4) (2015) 577–618.

[33] B. Plotkin, T. Plotkin, *An algebraic approach to knowledge bases equivalence*, Acta Appl. Math., **89** (2005) 109–134.

[34] B. Plotkin, T. Plotkin, *Geometrical aspect of databases and knowledge bases*, Algebra Universalis **46** (2001) 131–161.

[35] T. Plotkin, *Relational databases equivalence problem*, Advances of databases and information systems, Springer (1996) 391–404.

[36] T. Plotkin, M. Knyazhansky, *Verification of knowledge bases informational equivalence*, Journal Scientific Israel - Technological Advantages, **6**(1-2) (2004) 113–119.

[37] J. Rissanen, *On the equivalence of database schemes* in: Proc. ACM Symp. Princ. Of Database Systems, **1** (1982) 22–26.