Flexible Bayesian modeling of counts: constructing penalized complexity priors

May 19, 2021

Mahsa Nadifar*,‡, Hossein Baghishani*, Thomas Kneib‡, Afshin Fallah**

* Department of Statistics, Faculty of Mathematical Sciences, Shahrood University of Technology, Iran
‡ Georg-August-University Göttingen, Germany
** Department of Statistics, Imam Khomeini International University, Qazvin, Iran

Abstract

Many of data, particularly in medicine and disease mapping, are count. Indeed, under- or over-dispersion problem in count data distrusts the performance of the classical Poisson model. For taking into account this problem, in this paper, we introduce a new Bayesian structured additive regression model, called gamma count, with enough flexibility in modelling dispersion. Setting convenient prior distributions on the model parameters is a momentous issue in Bayesian statistics that characterize the nature of our uncertainty parameters. Relying on a recently proposed class of penalized complexity priors, motivated from a general set of construction principles, we derive the prior
structure. The model can be formulated as a latent Gaussian model, and consequently, we can carry out the fast computation by using the integrated nested Laplace approximation method. We investigate the proposed methodology simulation study. Different expropriate prior distribution are examined in order to provide a reasonable sensitivity analysis. To explain the applicability of the proposed model, we analyzed two real word data sets related to larynx mortality cancer in Germany and handball champions league.

**Keywords:** Structured additive regression model, over-dispersion, Penalized Complexity prior, count, under-dispersion, Gamma Count, INLA.

**Mathematics Subject Classification (2010):** 62 – 06, 62J12, 62F15, 62H11.

1 Introduction

The Poisson regression is the prevalent model for analyzing count data, especially when provided by a structured additive regression (STAR; Fahrmeir et al., 2004; Kneib et al., 2009). STAR modeling allows to simultaneously incorporate model components for non-linear, temporal, and spatial effects. Although the Poisson regression model is widespread, it limits the conditional variance to be equal to the conditional mean, which leads to an equi-dispersion situation. Such a case is seldom observed in many practical data analyses. Indeed, in real-life applications, count data can exhibit different aspects, specifically under-dispersion (the mean exceeding the variance) and over-dispersion (the variance exceeding the mean).

Over the years, several extended count regression models have been developed for dealing with non-equivalent dispersion. Some approaches, such as adopting a generalized linear mixed model (GLMM; Breslow and Clayton, 1993), embedded a random
effect in the model to consider the over-dispersion of counts. A convenient proposal, in this direction, is a Poisson model with gamma-distributed random effects yielding to a negative binomial (NB) model. However, the NB model is not a good candidate for analyzing under-dispersed counts (Cameron and Trivedi, 2013). Two alternative classes of models for accounting unobserved heterogeneity are finite mixture models (Pearson, 1894) and hurdle models (see, for example, Cameron and Trivedi, 2013; Baetschmann and Winkelmann, 2017). Indeed, we can use the hurdle model for accounting both over-dispersion and under-dispersion in the counts. Other methods incorporate weighting the Poisson distribution (Ridout and Besbeas, 2004), the COM-Poisson distribution (Lord \textit{et al.}, 2010; Lord \textit{et al.}, 2008), and the generalized Poisson inverse Gaussian family (Zhu and Joe, 2009) to name a few. Generally, analysis of under-dispersed counts has received less attention (Zeviani \textit{et al.}, 2014).

Winkelmann (1995) proposed a new methodology for over- or under-dispersed counts based on the renewal theory (Cox, 1962) that relates non-exponential durations (inter-arrival/waiting times) between events and the distribution of the counts. He connected the models for counts and models for durations relaxing the assumption of equi-dispersion at the cost of an extra parameter. Winkelmann (1995) replaced independently and identically exponentially distributed waiting times (which would lead to the Poisson distribution for counts) by a less restrictive non-negative distribution with a non-constant hazard function. The hazard function is the crucial quantity when studying renewal processes; it completely characterizes the distribution of inter-arrival times and describes the type of dispersion observed in the corresponding count data. Winkelmann (1995) showed that if the hazard function is monotonic, increasing (decreasing) hazard corresponds to count data with under-dispersion (over-dispersion). Therefore, providing a more flexible hazard function results in a more flexible counting process able to support over-dispersed and under-dispersed, as well as equi-dispersed data. Several researchers
have introduced some models considering this methodology, including a gamma-count (GC) model (Winkelmann, 1995; Toft et al., 2006; Zeviani et al., 2014), Weibull-count model (McShane et al., 2008), and lognormal-count model (Gonzales-Barron and Butler, 2011). Nadifar et al. (2019) extended the GC model to analyze spatially correlated count data in a Bayesian framework using integrated nested Laplace approximation (INLA; Rue et al., 2009). They used the GC spatial regression model for the analysis of groundwater quality data.

In this article, we aim to further generalize the suggested model in Nadifar et al. (2019) to a gamma-count structured additive (GCSA) regression model by developing a new class of penalized complexity (PC) prior distributions, introduced by Simpson et al. (2017), for the dispersion parameter as well as precision parameters of Gaussian priors considered for various effects in the predictor. PC priors are informative and have desirable aspects, such as invariance under reparameterizations, connection to Jeffreys' prior, and enjoying robustness properties. As Simpson et al. (2017) have noticed, their proposed approach uses the natural nested structure of many model components, which represents the model component as a flexible extension of a base model. Therefore, deviations from the base model increase the complexity of the model. For example, the t-distribution model can be regarded as a generalization of the normal distribution, as the base model, for the limiting case of increasing degrees of freedom. An interesting base model is obtained in STAR models when, e.g. for a non-linear effect, setting the smoothing variance to zero.

In short, Simpson et al. (2017) developed the PC prior by the following steps: 1) The increased complexity between the more flexible model and the base model is measured using the Kullback-Leibler divergence, 2) The deviation from the base model is penalized with a constant decay rate. The base model would be favored until there is enough support for a more complex model. Therefore, an exponential prior is assigned...
to the Kullback–Leibler distance such that the mode of the prior corresponds to the base model. 3) The decay rate is determined by controlling the prior mass in the tail. In STAR models, the rate can be extracted based on prior assumptions about the scaling of the model components. That is why Klein and Kneib (2016) referred to the priors as scale-dependent priors. They extended scale-dependent priors for the variance parameters in structured additive distributional regression models and compared them by some accepted alternative prior distributions. Klein and Kneib (2016) developed Markov chain Monte Carlo simulation inference with scale-dependent priors by constructing proposal densities based on the idea of iteratively weighted least squares (Gamerman, 1997; Brezger and Lang, 2006).

The proposed inferential framework of Klein and Kneib (2016) could be comprehensive, as it incorporates scale-dependent priors not only in mean regression models for responses from the exponential family but rather considers the general framework of distributional regression (Klein, Kneib and Lang, 2015; Klein, Kneib, Lang and Sohn, 2015) where further moments or general shape parameters of the conditional response distribution can be linked to a predictor. Despite these advantages, we prefer to use the integrated nested Laplace approximations (INLA) method introduced by Rue et al. (2009) due to the wide range of problems of MCMC algorithms regarding convergence and computational time when applied to complex models like our proposed model. To overcome the difficulties associated with the MCMC methods, Rue et al. (2009) introduced the INLA, a very fast, non-sampling-based approximate Bayesian methodology. This method combines Laplace approximations and numerical integration efficiently for a particular class of models, the so-called latent Gaussian models. INLA substitutes MCMC simulations with accurate, deterministic approximations to posterior marginal distributions. The R package R-INLA is free for download from [http://www.r-inla.org/](http://www.r-inla.org/) and fairly easy to use. Many examples of applications in several fields have also appeared in
the recent literature (Bakka et al., 2018; Rue et al., 2017; Sørbye et al., 2018; Simpson et al., 2016; Muff et al., 2015; Blangiardo et al., 2013; Martins et al., 2013; Schrödle and Held, 2011; Paul et al., 2010; Martino and Rue, 2010).

The plan for the rest of this paper is as follows. In Section 2 we demonstrate the GCSA regression model and the derivation of the new prior structures. The fundamental methodology for Bayesian analysis of GCSA regression is described in Section 3. The performance of the proposed approach is examined in simulation studies under various scenarios. The basic results are briefly summarized in Section 4. In contrast, all simulations are documented in more detail in the supplementary materials. Section 5 applies the methodology to two real datasets: Larynx cancer mortality counts in Germany, as an under-dispersed data example; and handball match data, as an over-dispersed data example. Finally, we discussed the results in section 6.

2 Penalized Complexity Priors for the GCSA Model

Our goal is to construct a PC prior for the dispersion parameter and scale-dependent hyperpriors for the precision parameters of the Gaussian priors considered for various effects in the structured additive predictor of a GCSA regression model. In the proposed model, the dispersion is treated as a fixed parameter.

2.1 Observational Model

In this section, we provide a brief background on the definition and properties of the GC distribution and its underlying classical regression model for uncorrelated data.

As Winkelmann (2008) has noticed, the count and the duration view are just two different representations of the same underlying stochastic process. From a statistical viewpoint, the distribution of cumulative waiting times uniquely determines the distri-
bution of counts and vice versa. This relationship can be employed to derive new count data distributions (Winkelmann, 1995; McShane et al., 2008, Gonzales-Barron and Butler, 2011; Ong et al., 2015). For example, the Poisson distribution corresponds to the exponential inter-arrival times between events. The GC distribution has been proposed based on gamma distributed inter-arrival times by Winkelmann (1995).

Let \( u_k \) be the waiting time between the \((k - 1)\)th and \(k\)th events. Therefore, the arrival time of the \(n\)th event is given by

\[
S_n = \sum_{k=1}^{n} u_k, \quad n = 1, 2, \ldots.
\]

Let \( Y(t) \) denote the total number of events that have occurred in the interval \((0, t)\). Hence, \( \{Y(t), \ t > 0\} \) is a counting process and for a fixed \(t\), \( Y(t) \) is a count variable. The stochastic properties of the counting process (and consequently of the count variable) are entirely determined once we know the joint distribution function of the waiting times, \( \{u_k, \ k \geq 1\} \). In particular, \( Y(t) < n \) if and only if \( S_n > t \). Therefore,

\[
P(Y(t) < n) = P(S_n > t) = 1 - F_n(t),
\]

where \( F_n(t) \) is the cumulative distribution function of \( S_n \). Moreover,

\[
P(Y(t) = n) = F_n(t) - F_{n+1}(t).
\]

Generally, \( F_n(t) \) is a complicated convolution of the underlying densities of \( u_k \)'s, which makes it analytically intractable. However, by using the theory of renewal processes, a significant simplification arises if \( u_k \)'s are independently and identically distributed with a standard distribution.

We assume that \( \{u_k, \ k \geq 1\} \) is a sequence of independently and identically gamma distributed variables, \( Gamma(\alpha, \gamma) \), with mean \( E(u_k) = \alpha/\gamma \) and variance \( Var(u_k) = \alpha/\gamma^2 \). It can be shown that if \( Y(t) \) denotes the number of events within \((0, t)\) interval, it
follows a GC distribution with parameters $\alpha$ and $\gamma$, denoted by $Y(t) \sim GC(\alpha, \gamma)$. Here, $\alpha$ is the dispersion parameter of distribution and controls the amount of dispersion in the counts. The probability mass function of $Y(t)$ is given by

$$P(Y(t) = y) = G(y\alpha, \gamma t) - G((y + 1)\alpha, \gamma t), \quad y = 0, 1, 2, \cdots,$$

where

$$G(n\alpha, \gamma t) = \frac{1}{\Gamma(n\alpha)} \int_0^{\gamma t} x^{n\alpha - 1} e^{-x} dx, \quad n = 1, 2, \cdots.$$  \hspace{1cm} (1)

For non-integer $\alpha$, no closed form expression is available for $G(y\alpha, \gamma t)$ and thus for $P(Y(t) = y)$. Also, for $\alpha = 1$, the distribution of $u_k$ reduces to the exponential, and (1) simplifies to the Poisson distribution. Therefore, the GC distribution could be considered as a flexible extension of the Poisson distribution, as a base model, when setting the dispersion parameter to one. More importantly, when $0 < \alpha < 1$ ($\alpha > 1$) the GC distribution is over-dispersed (under-dispersed). Moreover, the mean of GC distribution is given by

$$E(Y(t)) = \sum_{k=1}^{\infty} G(k\alpha, \gamma t).$$  \hspace{1cm} (2)

We refer readers to Winkelmann (1995) for more details about properties of the GC distribution.

### 2.1.1 Regression Model based on GC Distribution

We assume that GC distribution for count observations $y_i$ as well as covariates information $x_i = (1, x_{i1}, \ldots, x_{ip})'$ have been collected for individuals $i = 1, \ldots, n$. The mean of GC distribution in (2) has no closed form. Therefore, extending a regression model based on the mean is not straightforward. Assuming that the length of the time interval is the same for all observations, we can set $t$ to unity, without loss of generality. This
results in the following regression model (Zeviani et al., 2014):

$$\log \left( \frac{t \gamma \left( \mathbf{x}_i \right)}{\gamma \left( \mathbf{x}_i \right)} \right) = -\left( \beta_0 + x_1 \beta_1 + \ldots + x_p \beta_p \right) = -\eta_i, \quad i = 1, \ldots, n, \quad (3)$$

where $\beta = (\beta_0, \beta_1, \ldots, \beta_p)' \in \mathbb{R}^{p+1}$ is the vector of regression coefficients. We should notice that the regression model is defined on the waiting times $u_{k_i}$ instead of $Y_i$, where $u_{k_i}$ is the generic representation of waiting times for the $i$th observation. Its origin is for failure to establish the equality $E(Y_i | \mathbf{x}_i) = (E(u_{k_i} | \mathbf{x}_i))^{-1}$ unless for $\alpha = 1$. Indeed, given the inverse relationship between gaps and the number of occurrences, the minus sign behind $\beta$ is due to the reverse effect of covariates on waiting times instead of counts; the longer the expectation of time interval, the fewer the number of occurrences. Since $\{u_{k_i}, \ k \geq 1\}$ are independently and identically distributed random variables, we drop the index $k$ without loss of generality. For given covariates $\mathbf{x}_i$, the responses $y_i$ are conditionally independent where their conditional mean $E(Y_i | \mathbf{x}_i)$ can be evaluated via equation (2). From (3), one can write

$$\gamma(\mathbf{x}_i) = \alpha \exp(\eta_i), \quad i = 1, \ldots, n.$$ 

Therefore, given a sample of independent observations $\{(y_i, \mathbf{x}_i), i = 1, \ldots, n\}$, the GC regression model can be written as $y_i | \alpha, \eta_i \sim GC(\alpha, \alpha \exp(\eta_i)), i = 1, \ldots, n$. The likelihood function is given by

$$L(\alpha, \eta | \mathbf{y}) = \prod_{i=1}^{n} \left\{ G(\alpha y_i, \alpha \exp(\eta_i)) - G(\alpha y_i + \alpha, \alpha \exp(\eta_i)) \right\},$$

where $\mathbf{y} = (y_1, \ldots, y_n)'$ and $\eta = (\eta_1, \ldots, \eta_n)'$ are the vector of observed counts and linear predictor, respectively. There are no explicit forms for the maximum likelihood estimators of the parameters, due to the complexity of the likelihood function. Therefore, numerical optimization is needed for estimating the model parameters (Winkelmann, 1995; Zeviani et al., 2014).
2.2 The Proposed Model

For constructing the GC structured additive regression model, we assume that the conditional distribution of response variable $y_i, i = 1, \ldots, n,$ has density

$$y_i | \alpha, \eta_i \sim GC(\alpha, \alpha \exp(\eta_i)), \quad i = 1, \ldots, n,$$  \hspace{1cm} (4)

where $\eta_i$ is the structured additive predictor as

$$\eta_i = \beta_0 + \sum_{j=1}^{J} f_j(x_i)$$  \hspace{1cm} (5)

in which $\beta_0$ is an intercept term representing the overall level of the predictor. The functions $\{f_j\}$ show different functional effects, including the time effect, the spatial effect, and the non-linear effect of a continuous covariate. Each function $f_j(x)$ depending on (different subsets of) $x$, is represented by a linear combination of basis functions. Ignoring the index $j$, a typical function $f(x)$ is then specified as

$$f(x) = \sum_{d=1}^{D} \beta_d B_d(x),$$

where $B_d(x), d = 1, \ldots, D,$ is a set of appropriate basis functions and $\beta = (\beta_1, \ldots, \beta_D)'$ is the vector of corresponding basis coefficients. Indeed, to ensure identifiability, the functional effects are centered about zero. Using this approach, we can express all vectors $f_j = (f_j(x_1), \ldots, f_j(x_n))'$, $j = 1, \ldots, J$, as the matrix product of an appropriately defined design matrix $Z_j$ and a vector $\beta_j$ of unknown parameters, i.e. $f_j = Z_j \beta_j$. See Fahrmeir et al. (2013) and Klein and Kneib (2016) for more details.

In a Bayesian model specification, priors for $f_j$ are specified through appropriate design matrices and priors for the random effects $\beta_j$. See Fahrmeir and Echavarria (2006) for details on defining the appropriate design matrices for crucial predictor components, including P-splines and Markov random fields. Priors for $\beta_j$ are (partially improper) multivariate Gaussian

$$p(\beta_j | \sigma_j^2) \propto (\sigma_j^2)^{-\frac{\nu}{2}} \exp \left(-\frac{1}{2\sigma_j^2} \beta_j' K_j \beta_j \right), \quad j = 1, \ldots, J,$$  \hspace{1cm} (6)
with fixed positive (semi-)definite precision matrix $K_j$, and $rk(\cdot)$ denoting the rank of matrix. The matrix $K_j$ is chosen to penalize roughness of the function. Its structure depends on the type of covariate and the smoothness of the function. The variance parameter $\sigma^2_j$ quantifies our prior uncertainty about the properties enforced by $K_j$. It corresponds to a smoothing parameter in a frequentist setting (Fahrmeir and Echavarria, 2006). For intercept $\beta_0$, we typically assume a zero-mean Gaussian prior with the precision equal to 0.01.

For completing the model specification, a suitable prior for dispersion parameter, $\alpha$, and hyperpriors for the variance parameters, $\sigma^2_j$, have to be provided. Constructing a suitable prior for $\alpha$ has been less noticed until now. Nadifar et al. (2019) considered the $\text{Gamma}(1, 0.005)$ prior and it was fixed as prior assumptions in the R-INLA package. Since the Poisson distribution is a nested model of the GC distribution, in this paper, we apply the approach of Simpson et al. (2017) to define a PC prior for $\alpha$ as well. To this end, we employ the distributional structure of inter-arrival times. We compare the proposed PC prior with the gamma prior as well.

Given a comprehensive debate on the appropriateness of the gamma distribution as a natural conjugate hyperprior for $\sigma^2_j$, Klein and Kneib (2016) developed scale-dependent priors by extending the principled approach of Simpson et al. (2017) in structured additive distributional regression models. We will use their results to specify the scale-dependent hyperpriors given the dispersion parameter of the GCSA model. We also consider generalised beta prime, and inverse gamma priors for the variances that have been suggested as hyperpriors in Bayesian mixed-effects models (Gelman, 2005, 2006; Hodges, 2013; Klein and Kneib, 2016) and compare their performances. One of the advantages of our proposed model is ensuring that the posterior would be proper. It would be essential because priors for the vectors of regression coefficients for several effect types, such as Markov random fields, in the structured additive regression are
2.3 Derivation of the Prior Structure

We will now explain how to construct the PC prior for dispersion parameter and scale-dependent hyperpriors for the variance parameters. Let $\xi$ be the parameter of interest we want to define a prior. According to Simpson et al. (2017), principles for constructing a PC prior distribution are as follows:

**Principle 1: Occam’s razor.** Based on the principle of parsimony, to build a prior, a simple base model for each effect is supported unless the data give enough evidence for a more complex modeling alternative.

**Principle 2: Measure of complexity.** The increased complexity between the base model represented by density $p_b$ and the alternative complex model represented by density $p$ is measured by the unidirectional measure

$$d := d(p||p_b) = \sqrt{2\text{KLD}(p||p_b)}$$

where $\text{KLD}(p||p_b)$ is the Kullback–Leibler divergence given by

$$\text{KLD}(p||p_b) = \int p(x) \log \left( \frac{p(x)}{p_b(x)} \right) dx.$$

**Principle 3: Constant rate penalization.** This principle implies an exponential prior $p(d) = \lambda \exp(-\lambda d)$, with a constant decay-rate $r = \exp(-\lambda)$, on the distance scale $d$ obtained from Principle 2. The mode of the prior is at $d = 0$, i.e. the base model is in favor. Using the change of variable theorem, the prior on the original space is as follows:

$$p(\xi) = \lambda e^{-\lambda d(\xi)} |\frac{\partial d(\xi)}{\partial \xi}|.$$
Principle 4: User-defined scaling. Using this principle, we can select \( \lambda \) by controlling the prior mass in the tail. This condition is of the form

\[
P(q(\xi) \leq u) = 1 - a
\]  

(7)

where \( q(\cdot) \) is an interpretable transformation of \( \xi \) and \( u > 0 \) and \( a \in (0, 1) \) are some user-defined values. The probability in (7) depends on the intensity \( \lambda \) through the density of \( q(\xi) \) such that solving the expression for \( \lambda \) yields the exact prior specification for \( \xi \). It also allows the user to prescribe how informative the resulting PC prior is. We will discuss about the choice of \( q(\cdot) \), \( u \) and \( a \) in the following.

A significant advantage of PC priors is the reparameterization-invariance property since the prior is defined on the distance \( d \), which is then transformed to the corresponding prior for \( \xi \). We first define the PC prior for \( \alpha \). Let \( GC(\alpha, \beta) \), be the GC distribution with the dispersion parameter \( \alpha \). When \( \alpha = 1 \), the GC distribution reduces to the Poisson, the simple base model here. Considering dual modeling of counting and timing processes, we define flexible and base models based on the inter-arrival distribution.

**Theorem 2.1.** Let Gamma\((\alpha, \beta_1)\) be the flexible model, and Gamma\((1, \beta_2)\) is the base model. The PC prior for \( \alpha \) is defined as follows

\[
p(\alpha) = \lambda e^{-\lambda \sqrt{-2 \log \Gamma(\alpha) + 2(\alpha - 1)\psi(\alpha) + 2 \log \frac{\beta_1}{\beta_2} - 2\alpha(1 - \frac{\beta_2}{\beta_1})}} \left| \frac{(\alpha - 1) \psi'(\alpha) - (1 - \frac{\beta_2}{\beta_1})}{\sqrt{-2 \log \Gamma(\alpha) + 2(\alpha - 1)\psi(\alpha) + 2 \log \frac{\beta_1}{\beta_2} - 2\alpha(1 - \frac{\beta_2}{\beta_1})}} \right|.
\]  

(8)

To prove the theorem, we have to adjust the Kullback-Leibler divergence between two gamma distributions, which is simple to measure (Abdulrahman et al., 2015). The details are provided in the supplementary file.

**Remark 2.2.** According to the definition of GC distribution, provided \( f_j \)s are given, the rate parameters of base and flexible models should be equal, i.e. \( \beta_1 = \beta_2 = \beta \). Therefore,
the proposed PC prior density for $\alpha$ reduces to a simplified form as follows

$$
\begin{align*}
p(\alpha) &= \lambda e^{-\lambda \sqrt{-2 \log \Gamma(\alpha) + 2(\alpha-1)\psi(\alpha)}} \\
&\quad \left| \frac{(\alpha - 1) \psi'(\alpha)}{\sqrt{-2 \log \Gamma(\alpha) + 2(\alpha-1)\psi(\alpha)}} \right|.
\end{align*}
$$

(9)

**Remark 2.3. User-defined scaling.** The shrinkage parameter, $\lambda$, has to be determined by solving $P(q(\alpha) > u) = a$. By considering $q(\alpha) = d(\alpha)$, we have

$$
P(q(\alpha) < u) = \int_0^u \lambda e^{-\lambda \sqrt{-2 \log \Gamma(\alpha) + 2(\alpha-1)\psi(\alpha)}} d\alpha = 1 - a.
$$

(10)

Equation (10) could be easily solved using numerical integration. We can also determine hyperparameters based on our beliefs (Simpson et al., 2017). Considering the main feature of the GC distribution, i.e. modeling under- (over-)dispersed data, we can select some optimal values for the hyperparameters based on a simulation study. We describe it in Section 3.2. The scale-dependent hyperprior for the generic variance parameter $\sigma^2$ is given in the following theorem.

**Theorem 2.4.** Let $\beta \sim \mathcal{N}_{\text{rn}}(K)(0, \frac{1}{\sigma^2}K^{-1})$. Assume $\frac{1}{\sigma^2}K$ be the precision matrix of the flexible model for a vector of parameters $\beta$ and $\frac{1}{\sigma^2_b}K$ the precision matrix of the base model where $\sigma^2_b \rightarrow 0$. Furthermore, let $p(\sigma^2)$ be the prior for $\sigma^2$ depending on a hyperparameter $\epsilon$. Then

$$
p(\sigma^2) = \frac{1}{2\epsilon} \left( \frac{\sigma^2}{\epsilon} \right)^{-1/2} \exp \left( - \left( \frac{\sigma^2}{\epsilon} \right)^{1/2} \right),
$$

(11)

which is a Weibull prior with shape parameter $a = 1/2$ and scale parameter $\epsilon$.

Proof of this theorem is conveniently taken from the proof in Appendix A2 by Simpson et al. (2017) by changing variable. Klein and Kneib (2016) provide the details of proof in Supplement A.1. Following Principle 4, we can infer $\epsilon$ from a notion of scale
by considering $P(\sigma^2 > u) = a$. It would assume that large standard deviations are less likely. Consequently, we have $\epsilon = -\ln(a)/u$ (Simpson et al., 2017).

As indicated by Simpson et al. (2017), the invariance property of the prior is guaranteed. For example, type-2 Gumbel distribution is obtained as the prior for the precision parameter $\xi = 1/\sigma^2$.

### 2.4 Competing Priors

To analyze the sensitivity of the various priors of parameters, we consider some alternative priors following previous researches and based on simulation studies. A brief explanation of the parameters are in Table 1:

| Name       | Prior                                                                 | Information                                                                 |
|------------|----------------------------------------------------------------------|----------------------------------------------------------------------------|
| PC($\lambda, R$) | $p(\alpha) \propto \lambda e^{-\lambda \sqrt{-2 \log \Gamma(\alpha) + 2(\alpha - 1) \psi(\alpha) - 2\alpha(1 - \frac{1}{R})}}$ | PC prior for $\alpha$ with parameter $\lambda$ and $R = \frac{\beta_1}{\beta_2}$ |
| Gamma($\theta$) | $p(\alpha) \propto \exp(-\theta \alpha)$                             | flat prior for $\alpha$ for small $\theta$                                |
| Gamma($\theta, \theta$) | $p(\alpha) \propto \alpha^{\theta - 1} \exp(-\theta \alpha)$       | prior for $\alpha$ with thick and thin tails ($\theta \in \{10, 100\}$)     |
| SD($\epsilon$) | $p(\sigma^2) \propto \left(\frac{\sigma^2}{\epsilon}\right)^{-1/2} \exp \left(-\left(\frac{\sigma^2}{\epsilon}\right)^{1/2}\right)$ | scale-dependent prior for $\sigma^2$                                       |
| GBP($\epsilon$) | $p(\sigma^2) \propto \left(1 + \frac{\sigma^2}{\epsilon^2}\right)^{-1} \left(\frac{\sigma^2}{\epsilon^2}\right)^{-1/2}$ | generalised beta prime prior for $\sigma^2$ or Half-Cauchy prior for $\sigma$ |
| G($1/2, 2\epsilon^2$) | $p(\sigma^2) \propto (\sigma^2)^{1/2-1} \exp(-\sigma^2/2\epsilon^2)$ | gamma prior for $\sigma^2$ or half-normal for $\sigma$                     |
| IG($1, \epsilon$) | $p(\sigma^2) \propto (\sigma^2)^{-1} \exp(-\epsilon/\sigma^2)$     | flat prior for $\sigma^2$ for $\epsilon$ is small. $\epsilon = 0.005$        |
| IG($\epsilon, \epsilon$) | $p(\sigma^2) \propto (\sigma^2)^{-\epsilon-1} \exp(-\epsilon/\sigma^2)$ | flat prior for $\sigma^2$ on log-scale for $\epsilon \to 0$. $\epsilon \in (0.01, 0.001)$ |

**Dispersion parameter.** Since there is rare Bayesian research about GC distribution, we consider some priors based on its property. In case of flat prior, we consider $\text{Gamma}(1, \theta)$ with small $\theta$ (0.05), that was fixed in r-INLA package. The $\text{Gamma}(\theta, \theta)$ with $\theta = \{1, 10, 100\}$ are another alternative priors for $\alpha$ with differ-
ent tails (thick and thin). Finally, we determine hyperprior for achieved PC prior in (8). It has three hyperparameters which should be assigned. Unfortunately, principle 4 is not applicable when the rate parameter of base and flexible models are different. Therefore, we specify them due to evaluation studies. For the shrinkage parameter, $\lambda$, we consider three values as $\lambda = 1, 3, 5$. Due to (8), we need the deviation of rate parameters of base and flexible models. Consequently, we consider some values based on its ratio as follows: $\frac{\beta_1}{\beta_2} < 1$, $\frac{\beta_1}{\beta_2} = 1$ and $\frac{\beta_1}{\beta_2} > 1$. Figure 1 panel (a) shows the density of priors.

Scale parameter. In this case, we characterize hyperparameter of scale prior, $\tau$. Scale dependent prior which is obtained in (11), is a type-2 Gumbel distribution with scale parameter $\theta$. To infer $\theta$ by using principle 4, $Pr(1/\sqrt{\tau} > u) = a$ that leads to $\theta = -\ln(a)/u$. Thus hyperparameters $(a, u)$ should be assigned. In this literature we consider $(a = 1, u = 0.01)$ (Simpson et al., 2017). As it can be seen from figure 1, PC prior is heavy tail and concave. A Gamma prior with shape parameter one and scale parameter $\theta$ small as a flat prior for precision (Klein and Kneib, 2016; Gelman, 2006), is one choice and another priors are $\text{Gamma}(\theta, \theta)$ with small $\theta$. Half Cauchy as a weakly informative with the heavy tail (Gelman, 2006; Palson and Scott, 2012; Simpson et al., 2017), and as a final alternative prior, we consider proper uniform prior, $\tau \sim U(0, \theta)$ where $\theta$ is large. Figure 1 panel (b) shows the plot of density priors of $\tau$. 
3 Developing a Bayesian Framework

3.1 Inference

Posterior distribution can be derived using Bayes’ theorem with the likelihood from (4) and the prior assumptions in section 2.4. We complete Bayesian model formulation by the following conditional independence assumptions:

1. \( \eta_i, i = 1,\ldots,n_i \) is the structured additive predictor of model.

2. Priors \( \pi(\beta_j|\tau_j) \), \( j = 0,\ldots,J \), are conditionally independent.

3. Priors \( \pi(\alpha) \) and hyperpriors \( \pi(\tau_j) \) are mutually independent.
Consequently, the joint posterior distribution of the model parameters is obtained as follows

$$
\pi(\beta_0, \beta, \tau, \alpha | y) = \prod_{i=1}^{n} p(y_i | \alpha, \eta_i) \pi(\alpha) \times \prod_{j=0}^{J} \pi(\beta_j | \tau_j) \pi(\tau_j),
$$

where $\psi = (\beta'_1, \ldots, \beta'_J)'$, $\tau = (\tau_0, \ldots, \tau_J)'$ and $y = (y_1, \ldots, y_n)'$. The conventional approach to inference for the model (12) is based on MCMC. It is well known, however that MCMC methods have serious problems, regarding both convergence and computational time, when applied to such models (Rue et al., 2009). Particularly, the complexity of the proposed model could lead to several hours or even days of computing time to implement Bayesian inference via MCMC algorithms. To overcome this issue, Rue et al. (2009) introduced the INLA method that is a deterministic algorithm and provides accurate results in seconds or minutes. INLA combines Laplace approximations (Tierney and Kadane, 1986) and numerical integration in a very efficient manner to approximate posterior marginal distributions. Let $\theta = (\alpha, \beta_0, \tau_1, \ldots, \tau_J)'$ denotes the hyperparameters of the model (12). Let also $\psi$ denotes the $pJ \times 1$ vector of latent variables. In practice, the primary interest lies in the marginal posterior distributions for each element of the latent variables vector

$$
\pi(\psi_j | y) = \int \pi(\psi_j, \theta | y) d\theta = \int \pi(\psi_j | \theta, y) \pi(\theta | y) d\theta, \quad j = 1, \ldots, s,
$$

and for each element of the hyperparameter vector

$$
\pi(\theta_k | y) = \int \pi(\theta | y) d\theta_{-k}, \quad k = 1, 2,
$$

where $\theta_{-k}$ is equal to $\theta$ with eliminated $k$th element. The essential feature of INLA is to use this form to construct nested approximations

$$
\tilde{\pi}(\psi_j | y) = \int \tilde{\pi}(\psi_j | \theta, y) \tilde{\pi}(\theta | y) d\theta,
$$
$$
\tilde{\pi}(\theta_k | y) = \int \tilde{\pi}(\theta | y) d\theta_{-k},
$$
where Laplace approximation is applied to carry out the integrations required for evaluation of \( \tilde{\pi}(\psi_j|\theta, y) \). A crucial success of INLA is its ability to compute model comparison criteria, such as deviance information criterion (DIC; Spiegelhalter et al., 2002) and Watanabe–Akaike information criterion (WAIC; Watanabe, 2010; Gelman et al. 2014), and various predictive measures, e.g., conditional predictive ordinate (CPO; Pettit, 1990) and probability integral transform (PIT; Dawid, 1984), to compare different possible models. Our proposed GC model has already implemented in the \texttt{R-INLA} package as a family argument with the name "\texttt{gammacount}".

### 3.2 Propriety of Posterior

One of the most important questions is whether the joint posterior distribution is proper. Propriety of the posterior in distributional regression can be ensured when combining the assumptions considered by Klein, Kneib and Lang (2015). They found sufficient conditions for the propriety in the general framework of structured additive distributional regression model under inverse Gamma distribution for the smoothing variance. Their results are based on the work of Sun \textit{et al.} (2001) who derived several upper and lower bounds for the required integrals. Also, Klein and Kneib (2016) established adapted bounds that generalized the results of Klein, Kneib and Lang (2015) to distributional regression with scale-dependent hyperpriors. They introduced some conditions for scale-dependent hyperpriors to be proper. Furthermore, Simpson \textit{et al.} (2017) showed that posterior propriety with PC prior is automatically guaranteed. Therefore, the posterior distribution in (12) is proper forever (For more details, see Klein and Kneib, 2016; Simpson \textit{et al.}, 2017).
4 Experimental Assessment

In this section we performed several simulations in which we assessed the performance of the PC and scale dependent priors for proposed Bayesian GCSA model in comparison to other alternative priors discussed in Section 2.4. We consider three scenarios of under-, equal- and over-dispersion for analyzing different models.

4.1 Simulation Settings

In all the scenarios, we keep simulation settings fixed to allow for a consistent comparison:

- Sample size $n$ are chosen from $\{50, 100, 500\}$.
- Covariates are simulated from $U(-3, 3)$ and have been centralized.
- Simulated functional effects shown in Figure 2:
  1: $f_1(x) = \sin(x)$.
  2: $f_2(x) = \exp(-\exp(5x))$.
  3: $f_3(x) = -0.5 \sinh^{-1}(1.25\pi x)$.
- Responses are simulated as
  
  $y|x \sim GC(\alpha, \alpha \exp(\beta + f(x)))$

- Hyperprior specifications

  - $\alpha$ - PC with equal rate parameter of base and flexible models and parameter of exponential family, $\lambda \in \{1, 3, 5\}$.
  - PC with rate parameter of base model equal 0.1 and flexible model equal 0.2 and parameter of exponential family, $\lambda \in \{1, 3, 5\}$.
Figure 2: Experimental assess: Simulated function effects $f(x)$.

- PC with rate parameter of base model equal 0.2 and flexible model equal 0.1 and parameter of exponential family, $\lambda \in \{1, 3, 5\}$.
- $(\epsilon, \epsilon)$-gamma priors with $\epsilon \in \{1, 10\}$.
- $\text{Gamma}(1, 0.005)$.

$\tau$ - scale dependent prior with PC(1,0.01).

- $(\epsilon, \epsilon)$-gamma priors with $\epsilon \in \{0.01, 0.001\}$.
- $(1, \epsilon)$-gamma prior with $\epsilon = 0.005$ as used frequently in the literature.
- Flat prior, $\text{Uniform}(1, 1000)$.
- Half-Cauchy, $\text{HC}(0.022)$.

• Comparison criteria
We define two criteria $Q_1$ and $Q_2$ for functional effects and $\alpha$:

\begin{align*}
Q_1(f) &= \frac{1}{n} \sum_{i=1}^{n} (\hat{f}_i - f_i)^2 \\
Q_2(\alpha) &= (\hat{\alpha} - \alpha)^2
\end{align*}

where, $n$ denotes the number of observations.

For each scenario, the number of replications, $R$, is fixed to be 500, which seems large enough to take into account the uncertainty in the sampling procedure. Also, we computed CPO as they promote the computation of the cross-validated logarithmic score. For each observation, we have

$$CPO_i = \pi(y_i|\mathbf{y}_{-i}) \quad i = 1, \ldots, n,$$

where $\mathbf{y}_{-i}$ is the observations vector $\mathbf{y}$ with the $i$th component removed. It denotes the posterior probability of observing the value of $y_i$ when the model is fitted to all data without $i$th observation. By using the values of $\{CPO_i\}_{i=1}^{n}$, we can calculate the logarithmic score as follows:

$$\text{Log.score} = - \sum_{i} \log(CPO_i).$$

A smaller value of the logarithmic score indicates a better prediction quality of the model. Finally, we computed the DIC criterion for which lower values imply better model properties. Therefore, we draw boxplot of the values of $Q_1(f)$, $Q_2(\alpha)$ and $\alpha$’s estimates.

In this section, firstly we apply PC prior in equation (8) for sample size 50. According to the extraction results of this sample size, we reclaim that PC prior to different values for the rate parameters of the base and flexible model is not suitable. Then we apply equation (9) for $\alpha$ under other sample sizes, 100 and 500. Since for there are so many scenarios and models to consider, we have included a few figures here and the rest are in the supplementary.
4.2 Scenario 1: Under dispersion

We observe due to the figure 3, except priors $\text{Gamma}(1, 0.005)$ and $\text{Gamma}(1, 1)$ for parameter $\alpha$, other priors have the same performance under sample size 50. Although, by increasing sample size, the likelihood dominates priors and therefore it is clear that the models have no special diversity as it can be seen from the second row of figure 3. Figure 4 shows that the performance of the model corresponding to the $\tau$’s priors has the same cycle. Although, $\text{Gamma}(0.01, 0.01)$ has the best efficiency among other $\tau$’s prior when the sample size is 50. As the figure 4 exhibits amongst $\alpha$’s priors, $PC(5)$ has the best performance. After that, $PC(3)$ and $PC(1)$ are better than others, respectively. For sample size 100, this result is established, though the differences are slight. Because of $\alpha$’s estimation values in figure 5, the median of the estimation values of $\alpha$ under $PC(1)$, is more near the real value of $\alpha$ than $PC(5)$ or $PC(3)$.

4.3 Scenario 2: Equal dispersion

As it is clear from figure 6, all priors for parameters $\tau$ and $\alpha$ have almost the same performance, and there are little differences between the results for all sample sizes. Figures 7 and 8 present amongst $\alpha$’s priors, indicate that $PC(5)$ has the best performance. After that, $PC(3)$ and $PC(1)$ are better than others. By monitoring these figures, we can recognize when $\tau$ has the scale-dependent prior, the results are the best.

4.4 Scenario 3: Over dispersion

We can observe in figure 9, there are no significant differences amongst priors, and all priors has the same efficiency. As regards the sample size 100, there is a strange jump which $\log(MSE(f))$ values under $\text{Half – Cauchy}$ prior differ significantly from others. In the real example of handball matches you can see the same. Why?!!!
Figure 3: Experimental Assessment: The boxplot of $Q_1(f)$ values under different replications for $f = \sin(x)$ and Scenario 1.
Figure 4: Experimental Assessment: The boxplot of $Q_2(\alpha)$ values under different replications for $f = \sin(x)$ and Scenario 1.
Figure 5: Experimental Assessment: The boxplot of estimated values of $\alpha$ under different replications for $f = \sin(x)$ and Scenario 1.
Figure 6: Experimental Assessment: The boxplot of $Q_1(f)$ values under different replications for $f = -0.5 \sinh^{-1}(1.25\pi x)$ and Scenario 2.
Figure 7: Experimental Assessment: The boxplot of $Q_2(\alpha)$ values under different replications for $f = -0.5 \sinh^{-1}(1.25\pi x)$ and Scenario 2.
Figure 8: Experimental Assessment: The boxplot of estimated values of $\alpha$ under different replications for $f = -0.5 \sinh^{-1}(1.25\pi x)$ and Scenario 2.
shows us the performance of the model corresponding to the $\tau$'s priors has the same cycle. Although, $\text{Gamma}(0.01, 0.01)$ has the best efficiency among other $\tau$'s prior when sample size is 50. As the figure 10 exhibits amongst $\alpha$'s priors, $PC(5)$ has the best performance. After that, $PC(3)$ and $PC(1)$ are better than others, respectively. There are the same results for the sample size 100, though the differences are slight. Due to the $\alpha$’s estimate values in figure 11, the median of the estimate values of $\alpha$ under $PC(1)$, is more near the real value of $\alpha$ than $PC(5)$ or $PC(3)$.
Figure 9: Experimental Assessment: The boxplot of $Q_1(f)$ values under different replications for $f = \exp(-\exp(5x))$ and Scenario 3.
Figure 10: Experimental Assessment: The boxplot of $Q_2(\alpha)$ values under different replications for $f = \exp(-\exp(5x))$ and Scenario 3.
Figure 11: Experimental Assessment: The boxplot of estimated values of $\alpha$ under different replications for $f = \exp(-\exp(5x))$ and Scenario 3.
5 Real Applications

In this section, we briefly describe the two count data set. The first data set is related to the World Men’s Handball Championships 2011 – 2017 (Groll and et al., 2019). As a spatial example, we consider larynx cancer mortality data in Germany, during the years of 1986 to 1990 (Natário and Knorr-Held, 2003). These data sets are both under dispersed; therefore, the Poisson and the Negative Binomial (NB) models do not seem adequate for analyzing them. At first, we compare the different competitor models based on the WAIC and log score (LS) criteria. Finally, we fit the best performing model to the data and utilized to prediction. Prior distributions and their corresponding symbols are presented in the Table 2.

Table 2: Prior distributions and their corresponding symbols.

| Parameter | \( \alpha \) | \( \tau \) |
|-----------|-----------------|-----------------|
| Prior | Symbol | Prior | Symbol |
| PC prior with considering \( \lambda = 1 \) | PC(l1) | PC(1,0.01) | SD |
| PC prior with considering \( \lambda = 3 \) | PC(l3) | Half Cauchy (0.022) | HC |
| PC prior with considering \( \lambda = 5 \) | PC(l5) | unif(1,1000) | Flat |
| Gamma(1,1) | G(1,1) | Gamma(1,0.005) | G(1,0.005) |
| Gamma(1,0.005) | G(1,0.005) | Gamma(0.01,0.01) | G(0.01,0.01) |
| Gamma(10,10) | G(10,10) | Gamma(0.001,0.001) | G(0.001,0.001) |

5.1 Handball Data

In this section, we consider a dataset covering all matches of the four preceding IHF World Men’s Handball Championships 2011 – 2017 together with several potential in-
fluence variables. This data set are first used by Groll et al. (2019). We choose the three covariates which have more measure in Groll et al. (2019). They are observed either for the year of the respective World Cup or shortly before the start of it for each participating team, therefore, vary from one World Cup to another. The variables that we use in this paper are as follows. The response variable is the number of goals for each of team during the World cups. Both of the covariates are related to sportive factors as ODDSET probability and IHF\textsuperscript{1} ranking. ODDSET probability reflects the probability for each team to win the respective World Cup. Also, the IHF ranking is a ranking table of national handball federations published by the IHF (source: http://ihf.info/en-us/thegame/rankingtable). Another variable relates to factors of the team structure, average height. The height average of a team logically affects on the team’s power because tall players might have superiority over short players, as they can release a shot on goal above a defender more. As already stated in previous literature, each score from a match of two handball teams is treated as a count observation. Accordingly, for n teams, the respective model has the form

\[ y_{ijk} \sim GC(\alpha, \alpha \eta_{ij}), \]
\[ \log(\eta_{ijk}) = f(\text{odds}_{ij}) + f(\text{rank}_{ij}) + f(\text{height}_{ij}), \] (13)

where \( y_{ijk} \) denotes the number of goal of team \( i \) versus team \( j \) in tournament \( k \) with \( i \neq j \in \{1, \ldots, n\} \). Tables\[\text{3}\] and \[\text{4}\] provide LS and WAIC criteria for model corresponding to the different priors.

\textsuperscript{1}International Handball Federation (IHF)
Table 3: LS criterion corresponding different models and priors for handball data

| Model | Priors          | \( \alpha \) | \( \tau \) |
|-------|-----------------|--------------|------------|
|       | SD HC Flat G(1,0.005) G(0.01,0.01) G(0.001,0.001) | | |
| PC(11) | 496.94 | 496.08 | 497.22 | 503.99 | 506.62 | 501.80 |
| PC(13) | 496.95 | 496.14 | 497.21 | 503.92 | 506.52 | 501.78 |
| GC | PC(l5) | 497.03 | 496.31 | 497.28 | 503.94 | 506.49 | 501.84 |
| G(10,10) | 496.82 | 496.95 | 497.07 | 503.81 | 506.39 | 501.62 |
| G(1,0.005) | 496.97 | 496.13 | 497.27 | 504.06 | 506.78 | 501.91 |
| G(1,1) | 496.93 | 496.13 | 497.25 | 503.98 | 506.68 | 501.83 |
| NB | – | 499.27 | 498.85 | 499.68 | 505.70 | 508.24 | 503.96 |
| Poisson | – | 498.64 | 498.23 | 499.08 | 505.13 | 507.65 | 503.40 |

Table 4: WAIC criterion corresponding different models and priors for handball data

| Model | Priors          | \( \alpha \) | \( \tau \) |
|-------|-----------------|--------------|------------|
|       | SD HC Flat G(1,0.005) G(0.01,0.01) G(0.001,0.001) | | |
| PC(11) | 993.71 | 991.92 | 993.97 | 1006.12 | 1009.57 | 1001.90 |
| PC(13) | 993.75 | 992.07 | 994.01 | 1006.09 | 1009.57 | 1001.94 |
| GC | PC(l5) | 993.9172 | 992.41 | 994.19 | 1006.24 | 1009.73 | 1002.16 |
| G(10,10) | 993.49 | 991.96 | 993.72 | 1005.84 | 1009.28 | 1001.61 |
| G(1,0.005) | 993.76 | 992.01 | 994.05 | 1006.20 | 1009.69 | 1002.03 |
| G(1,1) | 993.68 | 991.97 | 994.02 | 1006.08 | 1009.60 | 1001.92 |
| NB | – | 998.45 | 997.60 | 999.14 | 1010.26 | 1014.27 | 1006.91 |
| Poisson | – | 997.18 | 996.36 | 997.93 | 1009.05 | 1012.99 | 1005.74 |

We can conclude some points according Tables 3 and 4. GC model with priors PC(l1) for \( \alpha \) and HC for \( \tau \) represents a better fit than others in terms of WAIC and LS (logarithmic score) criteria.
5.2 Bym: Larynx Cancer

Here, we reconsider larynx cancer mortality data of Natário and Knorr-Held (2003) for males, in the 544 districts of Germany, from 1986 to 1990. During this period 7,283 deaths were recorded due to larynx cancer among the male population. Natário and Knorr-Held (2003) used the information on the corresponding lung cancer mortality rates in the same period, as an ecological covariate to account for smoking consumption \( c \). This dataset is available on the INLA website and see Natário and Knorr-Held (2003) for more details. We assume the observed death counts \( y_i \) in district \( i = 1, \ldots, 544 \), are conditionally independent with a count distribution for which \( \eta_i \) is given by

\[
\eta_i = \beta_0 + \beta_1 c_i + f_i(s),
\]

where \( f_i(s) \) is an ICAR spatial effect, as described in Section ???. We fit the GC, Poisson, and NB regression models as the count distribution of the responses under priors described in previous sections. Tables 5 and 6 provide LS and WAIC criteria for model corresponding to the described priors.

| Model    | Priors           | \( \alpha \) | \( \tau \) | SD  | HC  | Flat | G(1,0.005) | G(0.01,0.01) | G(0.001,0.001) |
|----------|------------------|--------------|------------|-----|-----|------|------------|--------------|-----------------|
| PC(11)   |                  | 1936.46      | 2313.67    | 1952.95 | 2043.67 | 1926.24 | 1999.19     |
| PC(13)   |                  | 1961.98      | 1963.85    | 1964.48 | 1964.11 | 1964.47 | 1964.59     |
| GC       |                  | 1977.24      | 1981.80    | 1981.52 | 1980.23 | 1980.66 | 1981.61     |
| PC(15)   |                  | 1966.08      | 1969.26    | 1969.77 | 1969.16 | 1969.59 | 1969.54     |
| G(10,10) |                  | 1977.39      | 1976.86    | 2012.56 | 2221.87 | 2120.75 | 4854.53     |
| G(1,0.005)|                 | 2246.78      | 2103.87    | 1972.53 | 2079.13 | 2438.30 | 4905.43     |
| NB       |                  | 11744.78     | 16920.37   | 16146.39 | 17627.93 | 16262.42 | 16168.92     |
| Poisson  |                  | 1850.74      | 1855.99    | 1856.46 | 1855.41 | 1856.55 | 1856.54     |
Table 6: WAIC criterion corresponding different models and priors for larynx cancer

| Model   | Priors                      | \(\alpha\) | \(\tau\) |
|---------|-----------------------------|-------------|-----------|
|         | SD  | HC  | Flat | G\((1,0.005)\) | G\((0.01,0.01)\) | G\((0.001,0.001)\) |
| PC(11)  | 2793.59 | 17523.51 | 2838.71 | 3027.33 | 2744.78 | 2948.04 |
| PC(13)  | 2854.951 | 2839.25 | 2837.35 | 2840.74 | 2837.53 | 2837.55 |
| GC      | 2905.59 | 2899.97 | 2898.99 | 2899.44 | 2897.50 | 2898.96 |
| GC      | 2876.99 | 2867.71 | 2866.38 | 2868.50 | 2866.54 | 2866.65 |
| G\((10,10)\) | 26471.83 | 2295e+3 | 1126e+7 | 3253.86 | 7046e+6 |                     |
| G\((1,0.005)\) | 2671.83 | 2295e+3 | 1126e+7 | 3253.86 | 7046e+6 |                     |
| G\((1,1)\) | 3916e+9 | 3698e+6 | 2868.80 | 1119e+5 | 1931e+4 | 4354e+6 |
| NB      | –   | 3210.01 | 3170.35 | 3165.34 | 3181.61 | 3165.12 | 3165.19 |
| Poisson | –   | 2997.78 | 2995.40 | 2995.13 | 2995.65 | 2995.19 | 2995.16 |

We can conclude some points according Tables 5 and 6. The GC model, with priors \(PC(l1)\) for \(\alpha\) and \(Gamma(0.01,0.01)\) for \(\tau\), demonstrates a better fit than others in terms of WAIC and LS criteria. Afterwards, GC model with priors \(PC(l1)\) for \(\alpha\) and \(SD\) for \(\tau\) has good fitting. In previous literature, this dataset called as an under dispersed data. Our obtained results confirm this; the GC model represents the best fit than the Poisson and the NB models. Moreover, we prepare posterior inference for model parameters under different models with these priors (\(PC(l1)\) for \(\alpha\) and \(SD\) for \(\tau\)) in Table 7. The estimate of the dispersion parameter \(\alpha\) for the GC model illustrates an underdispersion in the data; however, the corresponding 95% credible interval shows the Poisson model could also be the appropriate alternative for these data. The results of fitting show that smoking consumption has a significant even so slight positive effect on the mortality rate under three models. For the three proposed models, there is a negligible difference between the regression parameter estimates. Further, the estimates of the precision parameter, \(\tau\), for the GC and the Poisson models are the same, with a slightly larger value for the NB model. Figures 12 and 13 represent the observed versus
Table 7: Summary of posterior estimates, mean (2.5%, 97.5%), for the fitted models of larynx data

| Model | $\alpha$ | $\tau_\phi$ | $\beta_0$ | $\beta_1$ | DIC          |
|-------|---------|------------|----------|----------|--------------|
| GC    | 1.882   | 0.673      | 1.777    | 0.008    | 2839.541     |
|       | (0.936, 3.048) | (0.571, 0.775) | (1.494, 2.058) | (0.002, 0.014) |
| Poisson | —      | 0.714      | 1.740    | 0.009    | 3068.31      |
|       | (0.606, 0.826) | (1.447, 2.033) | (0.003, 0.014) |
| NB    | —      | 0.841      | 1.777    | 0.008    | 3216.954     |
|       | (0.688, 0.974) | (1.481, 2.072) | (0.003, 0.014) |

predicted values and fitted values respectively for the hold-out regions. The predicted values are computed according to the mean of the predictive distributions. From figure 12 this is appears that for our 10 hold-out areas the 95% prediction intervals are quite broad producing a 100% empirical coverage for all three models. However, the length of the prediction intervals for the GC model is generally shorter than the other couple models. The results of figure 13 are the same as 12.

Figure 12: Scatter plots of observed versus predicted values for the hold-out districts and 95% credible intervals with 1:1 line under the GC (first column), Poisson (second column) and NB (third column) models.
Figure 13: Scatter plots of observed versus fitted values for the hold-out districts and 95% credible intervals with 1:1 line under the GC (first column), Poisson (second column) and NB (third column) models.

6 Summary and Discussion

The Poisson regression model is a common practice in many applications for modelling counts. However, count data usually have various levels of dispersion and, consequently, this the inherent property of counts should be included in the model in order to achieve more reliable results. In this paper, we applied a gamma-count (GC) model as a consequence of the renewal theory that relates nonexponential waiting times between events and the distribution of the counts. Our proposed model has excellent flexibility due to analyzing data with various dispersion, under- or over-dispersion and is a generalization of the Poisson model, under certain conditions, becomes the Poisson model.

Although, model fitting and inference in a Bayesian GC structured additive (GCSA) regression model can be carried out using MCMC methods, but for the proposed model it comes with severe problems regarding convergence and computational time. The INLA method (Rue et al. (2009)) can be used as an efficient method for deriving estimation posterior inferences. INLA is a user friendly, one merit of the INLA the approach is that there is a package, called R-INLA, that can be used in the free software R. Furthermore, consequently, practitioners have the methodology at their disposal. Priors
have the prominent role in the Bayesian inferences and as Simpson et al. (2017) point out, Regard to Simpson et al. (2017), we get PC prior to the dispersion parameter of the GCSA model. In Bayesian analysis of the structured additive regression models, prior elicitation is, in particular, an issue with choosing hyperpriors for the smoothing variances. As Klein et al. (2016), proposed scale-dependent prior to the smoothing variances, we use scale-dependent prior to precision parameters. Subsequently, we consider some alternative priors for sensitivity analysis. The outcomes of the simulation study and real examples showed that in most cases, \( PC(1) \) prior for dispersion parameter was superior to the others. Besides, scale-dependent prior has a good performance for the smoothing parameter as the same as \( HC(0.022) \) and \( Gamma(0.01, 0.01) \).
References

[1] G. Baetschmann, and R. Winkelmann, A dynamic hurdle model for zero-inflated count data, Communications in Statistics - Theory and Methods. 46 (2017), pp. 7174–7187.

[2] Yusuf, Abdulrahman and Delpha, Claude and Diallo, Demba. (2015). Analytical Model of the KL Divergence for Gamma Distributed Data: Application to Fault Estimation. 2266-2270. 10.1109/EUSIPCO.2015.7362788.

[3] H. Bakka, H. Rue, G. A. Fuglstad, A. Riebler, D. Bolin, J. Illian, E. Krainski, D. Simpson, and F. Lindgren, Spatial modeling with R-INLA: A review, WIREs Computational Statistics. 10 (2018).

[4] N. Breslow, and D.G. Clayton, Approximate inference in generalized linear mixed models, Journal of the American Statistical Association. 88(1993), pp. 9–25.

[5] A. C. Cameron, and P. K. Trivedi, Regression Analysis of Count Data, Second Eddition, NewYork: Cambridge University Press, 2013.

[6] D. R. Cox, Renewal Theory. Methuen, London, 1962.

[7] N. A. C. Cressie, Statistics for Spatial Data, John Wiley and Sons, Inc, 1993.

[ Fahrmeir and Tutz (2001)] L. Fahrmeir, and G. Tutz, Multivariate Statistical Modelling based on Generalized Linear Models, 2nd ed., Springer, New York, 2001.

[8] Gelman, A. (2006). Prior distributions for variance parameters in hierarchical models. Bayesian Analysis 1: 515–533. MR2221284. doi: http://dx.doi.org/10.1214/06-BA107A.
[9] U. Gonzales-Barron, and F. Butler, *Characterisation of within-batch and between-batch variability in microbial counts in foods using Poisson-gamma and Poisson-lognormal regression models*, Food Control. 22(2011), pp. 1268–1278.

[10] Hastie, T. J. and Tibshirani, R. J. (1990). *Generalized Additive Models*. Chapman & Hall/CRC, New York/Boca Raton. MR1082147. 1071

[11] Klein, N. and Kneib, T. (2016). *Scale dependent priors for variance parameters in structured additive distributional regression*. Bayesian Analysis. 11, pp. 1071–1106. doi: http://dx.doi.org/10.124/15-BA983SUPP.

[12] Klein, N. and Kneib, T. (2015). *Scale dependent priors for variance parameters in structured additive distributional regression: Supplement*. Bayesian Analysis. doi: http://dx.doi.org/10.124/15-BA983SUPP.

[13] Klein, N., Kneib, T. and Lang, S. (2015). *Bayesian generalized additive models for location, scale and shape for zero-inflated and overdispersed count data*. Journal of the American Statistical Association 110: 405–419. MR3338512. doi: http://dx.doi.org/10.1080/01621459.2014.912955.

[14] F. Lindgren, H. Rue, and J. Lindstrom, *An explicit link between Gaussian fields and Gaussian Markov random fields: the stochastic partial differential equation approach*, J. R. Stat. Soc. B, 73(2011), pp. 423–498.

[15] D. Lord, S. R. Geedipally, and S. D. Guikema, *Extension of the application of Conway-Maxwell-Poisson models: analyzing traffic crash data exhibiting underdispersion*, Risk analysis: an official publication of the Society for Risk Analysis. 30(2010), pp. 1268–1276.
[16] D. Lord, S. D. Guikema, and S. R. Geedipally, Application of the Conway-Maxwell-Poisson generalized linear model for analyzing motor vehicle crashes, Accident; analysis and prevention. 40(2008), pp. 1123–1134.

[17] S. Martino, H. Rue, Implementing Approximate Bayesian Inference using Integrated Nested Laplace Approximation: a manual for the inla program, 2010. URL http://www.math.ntnu.no/hrue/GMRFsim/manual.pdf

[18] G. Martins, D. Simpson, F. Lindgren, H. Rue, H., Bayesian computation with INLA: new features, Norwegian University of Science and Technology Report, 2012.

[19] B. McShane, M. Adrian, E. T. Bradlow, and P. S. Fader, Count Models Based on Weibull Interarrival Times, Journal of Business and Economic Statistics. 26(2008), pp. 369–378.

[20] Nadifar, M., Baghishani, H., Fallah, A. and Rue, H. (2019). Statistical modeling of groundwater quality assessment in Iran using a flexible Poisson likelihood. https://arxiv.org/abs/1908.02344.

[21] J. A., Nelder, and R. W. M. Wedderburn, Generalized Linear Models, Journal of the Royal Statistical Society, Series A, 135(1972), pp. 370–384.

[22] K. Pearson, Contributions to the theory of mathematical evolution, Philosphical Transitions of the Royal society of London. 185(1994), pp. 71-110.

[23] H. Rue, and L. Held, Gaussian Markov Random Fields: Theory and Applications, London: Chapman & Hall/CRC Press, 2005.

[24] H. Rue, S. Martino, Approximate Bayesian inference for latent Gaussian models by using integrated nested Laplace approximations, Journal of the Royal Statistical Society. 71(2009), 319–392.
[25] D. Simpson, H. Rue, A. Riebler, T. G. Martins, and S. H. Sørbye, *Penalising Model Component Complexity: A Principled, Practical Approach to Constructing Priors*, Statistical Science. 32(2017), 1–28.

[26] S. H. Sørbye, and H. Rue, *Scaling intrinsic Gaussian Markov random field priors in spatial modelling*, Spatial Statistics. 8(2014), pp. 39–51.

[27] S. H. Sørbye, J. B. Illian, D. P. Simpson, D. F. R. P. Burslem, and H. Rue, *Careful prior specification avoids incautious inference for log-Gaussian Cox point processes*, Journal of the royal statistical society series c-Applied statistics, 2018. DOI: 10.1111/rssc.12321.

[28] M. S. Ridout, and P. Besbeas, *An empirical model for underdispersed count data*, *Statistical Modelling*, Journal of Business and Economic Statistics. 4(2004), pp. 77–89.

[29] H. Rue, A. Riebler, S. H. Sørbye, J. B. Illian, D. P. Simpson, F. K. and Lindgren, *Bayesian Computing with INLA: A Review*, Annual Review of Statistics and Its Application. 4(2017), pp. 395–421. Doi: 0.1146/annurev-statistics-060116-054045.

[30] B. Schrödle, L. Held, *Spatio-temporal disease mapping using INLA*, Environmetrics 22 (2011b), pp. 725–734.

[31] N. Toft, G. T. Innocent, D. J. Mellor, and S. W. Reid, *The Gamma-Poisson model as a statistical method to determine if micro-organisms are randomly distributed in a food matrix*, Food Microbiology. 23(2006), pp. 90–94.

[32] L. Tierney, and J. B. Kadane, *Accurate approximations for posterior moments and marginal densities*, Journal of the American Statistical Association. 81(1986), pp. 82–86.
[33] J. Wakefield, *Disease mapping and spatial regression with count data*, Biostatistics. 8(2007), pp. 158–183.

[34] R. Winkelmann, *Duration Dependence and Dispersion in Count-Data Models*, Journal of Business and Economic Statistics. 13(1995), pp. 467–474.

[35] R. Winkelmann, and K. Zimmermann, *Count data models for demographic data*, Mathematical Population Studies, 4(1994), pp. 205–221.

[36] W. M. Zeviani, P. J. Ribeiro Jr, W. H. Bonat, S. E. Shimakura, J. A. and Muniz, *The Gamma-count distribution in the analysis of experimental underdispersed data*, Journal of Applied Statistics. 41(2014), pp. 2616-2626.

[37] R. Zhu, and H. Joe, *The Gamma-count distribution in the analysis of experimental underdispersed data* Modelling heavy-tailed count data using a generalised Poisson-inverse Gaussian family, Statistics and Probability Letters. 79(2009), pp. 1695-1703.

[38] Wood, S. N. (2006). Generalized Additive Models: An Introduction with R, Chapman & Hall/CRC, New York/Boca Raton. MR2206355.