Self-modulation of a relativistic charged-particle beam as thermal matter wave envelope

R Fedele¹,⁴, D Jovanović²,⁴, S De Nicola³,⁴, A Mannan⁴,⁵ and F Tanjia⁵

¹ Dipartimento di Fisica, Università di Napoli “Federico II”, Napoli, Italy
² Institute of Physics, University of Belgrade, Belgrade, Serbia
³ CNR-SPIN Napoli, Italy
⁴ Dipartimento di Matematica e Fisica, Seconda Università degli Studi di Napoli, Sede di Caserta, Italy
⁵ INFN Sezione di Napoli, Italy

E-mail: renato.fedele@na.infn.it

Abstract. The self-modulation, resulting from its interaction with the surrounding medium, of a relativistic charged-particle beam traveling through an overdense plasma, is investigated theoretically. The description of the transverse nonlinear and collective beam dynamics of an electron (or positron) beam in a plasma-based accelerator is provided in terms of a thermal matter wave envelope propagation. This is done using the quantum-like description provided by the thermal wave model. It is shown that the charged-particle beam dynamics is governed by a Zakharov-type system of equations, comprising a nonlinear Schrödinger equation that is governing the spatiotemporal evolution of the thermal matter wave envelope and a Poisson-like equation for the wake potential that is generated by the bunch itself.

1. Introduction

Since its early development, the study of the charged particle motion in electric and magnetic fields has played a special role in the construction of the analogy between optics and mechanics. The problems of charged particle motion, such as those occurring in electrostatic and magnetostatic lenses (dipole, quadrupole and other multipoles), magnetic mirrors, magnetic bottles, mass spectrometers and many others [1], have been used as classic examples to show the correspondence between the geometrical optics and classical mechanics. The phenomenological wealth exhibited in this context has generated a separate discipline known as electron optics. Remarkably, within the context of electron optics, important scientific and technological developments in electron microscopy [1, 2, 3, 4, 5, 6] and in particle accelerators [7], respectively, have been registered.

Thanks to those developments, electron optics has been expanded mainly in two directions that characterized the analogy between optics and mechanics in two different ways, also at the level of the wave description.

One of them concerned the behavior of the single particle, or the systems of relatively few particles, in which the extension from the geometrical to the wave-like description coincided with the development of the quantum-mechanical description of electron optics, such as in the contexts in which the quantum behavior of the single particle prevails over the collective behavior, and the concept of temperature does play a role. Valuable examples are encountered
in the recent studies of vortex states (angular orbital momentum effects and spin) with integer or fractional topological charge that are associated with the single charged-particle matter wave in the electron microscope [8]; or in studies of the transport of relativistic charged-particle beams traveling in a high-energy accelerating machines based on the quantum mechanical behavior of the single-particle [9, 10, 11].

The other branch of the electron optics is concerned, instead, with the situations in which the behavior of a beam, constituted by an extremely large number of charged particles, is affected by the electromagnetic interactions that are established within such a system, while the effects of the temperature cannot be ignored due to the large number of particles. Therefore, in general, the behavior of such a system is expected to be collective and affected by the thermal spreading among the particles (thermal regimes). In this regime, electron optics has received a strong development within both conventional and non-conventional accelerator physics [7, 12]. Hereafter, we refer to this branch to as electron optics in thermal regime (EOTR). A peculiar aspect of the paraxial charged-particle beam propagation, as described by the electron optics in thermal regime, is the mixing of the electron rays (charged-particle trajectories) that characterizes the motion of non laminar beams [7]. In fact, due to the thermal spreading among the particles, the electron rays in a beam, that is traveling in vacuo with a relativistic speed, deviate with respect to the propagation direction with small and random slopes, whose r.m.s. deviation \( \sim v_T/c \ll 1 \) (\( v_T \) and \( c \) are the transverse thermal velocity and the light speed, respectively).

In this paper, we deepen the description of the spatiotemporal evolution of a relativistic non laminar charged particle beam. This is done within the context of the quantum-like formalism provided by the thermal wave model (TWM) that attributes a complex function to the beam as a whole, called the beam wave function (BWF). Its squared modulus is proportional to the beam density. In this way, TWM attributes to the beam a sort of a quantum-like matter wave, whose dispersion properties are related to the thermal spreading of the beam particles (thermal matter waves). As an example, we describe the self-modulation of a long non laminar electron (or positron) beam in an unmagnetized overdense plasma as the modulation of a thermal matter wave envelope, whose spatiotemporal evolution is provided by a pair of Zakharov-like equations. The latter take into account the plasma wake field (PWF) that is excited by the beam and experienced as a feedback by the beam itself (self-interaction).

2. Wave envelope representation of a charged particle beam

The thermal spreading introduces an uncertainty in the electron ray positions in the transverse plane at any longitudinal position. Then, the picture that comes from the envelope of the above electron ray mixing resembles the one that is exhibited by the paraxial ray diffraction in the light rays of an electromagnetic radiation beam. It is well known that from this comparison one can conclude that the statistical behavior of the charged particles in a paraxial beams, which is a fully classical process, simulates the paraxial diffraction exhibited by the electromagnetic radiation beams. Its experimental evidence is manifested in all the processes that are relevant to EOTR [7] and is supported by theoretical kinetic descriptions (Boltzmann/Vlasov equation). To deepen this aspect, let us consider a charged-particle beam travelling in vacuo along the \( z \)-axis with speed \( \beta c \) (\( \beta \approx 1 \)). The assumption of paraxial beam implies that the slopes of the electron rays are very small with respect to \( z \). Consequently, if at each longitudinal location \( z \) the pair \( (x(z), y(z)) \) represents the transverse location of an electron ray (\( z \) plays the role of a clock coordinate), the paraxial approximation can be expressed by the conditions \( p_x \equiv dx/dz \ll 1 \), and \( p_y \equiv dy/dz \ll 1 \). Note that \( (x, p_x) \) and \( (y, p_y) \) constitute two pairs of canonical conjugate variables [1, 7] and the beam transverse motion, affected by the thermal spreading, should be in principle described statistically by the first- and the second-order moments of the classical phase-space distribution function \( \rho(x, y, p_x, p_y, z) \) [13]. This function represents the probability density of finding an electron ray at the transverse phase space location \( (x, y, p_x, p_y) \) and time
t = z/βc.

3. Is the thermal spreading of a non laminar beam a diffusion process?

One is naturally lead to consider the transverse beam thermal spreading as a diffusion process in the real space. For the sake of simplicity, let us now restrict our analysis to the (1+1)D case where, for instance, the transverse motion takes place along the x-axis only (z being still the time-like variable). Then ρ = ρ(x,p,z), where we have replaced px with p = dx/dz. We introduce the second-order moments of the latter as follows: σx(z) ≡ ⟨(x − 〈x〉)2⟩1/2 (r.m.s. dispersion in x electron ray position), σp(z) ≡ ⟨(p − 〈p〉)2⟩1/2 (r.m.s. dispersion in p electron ray slope), and σxp ≡ ⟨(x − 〈x〉)(p − 〈p〉)⟩1/2 (electron ray correlation term), where < ... > stands for the phase space average. Then the diffusion coefficient can be defined as [14]:

\[ \frac{d^2 \sigma_x}{dz^2} - \frac{\epsilon^2}{4\sigma_x^2} = 0 \]  \( (1) \)

Equation (1), is in full agreement with the experimental observations concerning with the beam propagation in vacuo (e.g., in the final stage of a linear collider) [15, 16]. Since we are trying to describe the thermal spreading of the beam as a diffusion process, it is natural (but not obvious!) to assume the existence of a real function f(x,z), connected somehow with the phase space distribution function satisfying the following diffusion equation [17]:

\[ \frac{\partial f}{\partial z} = \frac{\epsilon}{2} \frac{\partial^2 f}{\partial x^2} . \]  \( (2) \)

Assuming that initially (z = z0) the beam has a Gaussian transverse shape, a Gaussian non-stationary normalized solution of Eq. (2) can be found:

\[ f(x, z) = \frac{1}{\sqrt{2\pi}\sigma_f(z)} \exp \left[ -\frac{x^2}{2\sigma_f^2(z)} \right] , \]  \( (3) \)

where the r.m.s \( \sigma_f(z) \) satisfies to the following equation

\[ \frac{d^2 \sigma_f}{dz^2} + \frac{\epsilon^2}{4\sigma_f^2} = 0 . \]  \( (4) \)

Note that (4) does not formally coincide with (1). Thus, taking for Eqs. (1) and (4) the same initial conditions, namely: \( \sigma_f(z_0) = \sigma_x(z_0) \equiv \sigma_0, \quad (d\sigma_f/dz)_{z=z_0} = (d\sigma_x/dz)_{z=z_0} \equiv \sigma'_0, \) they do not give the same solution. In fact, the envelope equation (1) gives the following solution:

\[ \sigma_x(z) = \sqrt{2\epsilon (z - z_0)^2 + \sigma_0^2}, \]  

where \( 2\epsilon \equiv \epsilon^2/4\sigma_0^2 + (\sigma'_0)^2 > 0, \) whilst the envelope equation (4) gives \( \sigma_f(z) = \sqrt{\sigma_0^2 + \epsilon(z - z_0)}. \) These solutions differ each other substantially. In particular, the former cannot reduce to zero for any values of z, whilst the latter vanishes for a finite value of this variable. In other words, the former preserves the non laminar beam from the focusing in a single point, in full agreement with the experimental evidence, whilst the latter does not preserve such a circumstance. We can conclude that Eq. (4) does not describe the envelope transverse non laminar beam motion.
4. The Schrödinger-like equation for non laminar beams

Since Eq. (4) is a consequence of the parabolic diffusion equation (2), the latter cannot be taken to describe the beam transport in a vacuum in the presence of finite emittance, as well. It turns out that we need to find an evolution equation which recovers Eq. (1) and takes also into account the information related to the slopes of the particle trajectories. To this end, we observe that (4) transforms into (1) by means of the formal substitution:

$$\epsilon \rightarrow i\epsilon$$  (5)

where \(i\) is the imaginary unity. Correspondingly, (2) transforms into the following Schrödinger-like equation for the free motion:

$$i\frac{\partial \Psi}{\partial z} = -\frac{\epsilon^2}{2} \frac{\partial^2 \Psi}{\partial x^2} ,$$  (6)

where now, instead of \(f\), we have the function \(\Psi\) which in principle may be complex. Taking into account also the other transverse component, \(y\), of the beam motion, Eq. (6) can be easily generalized to the following (2+1)D Schrödinger-like equation

$$i\frac{\partial \Psi}{\partial z} = -\frac{\epsilon^2}{2} \nabla^2_{\perp} \Psi ,$$  (7)

where we have assumed, for simplicity, the same emittance for each transverse direction. Here \(\nabla_{\perp} = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\) denotes the transverse gradient.

Let us represent \(\Psi\) in the following form:

$$\Psi(x, y, z) = \sqrt{n(x, y, z)} \exp \left[ \frac{i}{\epsilon} \theta(x, y, z) \right] ,$$  (8)

with \(\theta(x, y, z)\) being a real function and \(n(x, y, z)\) a positive and real function, satisfying the following condition: \(\int_{-\infty}^{\infty} |\Psi(x, y, z)|^2 \, dx \, dy = \int_{-\infty}^{\infty} n(x, y, z) \, dx \, dy = N\) (\(N\) being the total number of particles of the beam. It is immediately clear, following the language of quantum mechanics, that we have now two suitable information: the transverse probability density of the beam particles, i.e. \(n(x, y, z) = |\Psi(x, y, z)|^2\), and the transverse current velocity, i.e. \(V(x, y, z) = \nabla_{\perp} \theta(x, y, z)\).

It is easy to see that Eq. (7) admits the following Gaussian solution:

$$\Psi(x, y, z) = \frac{1}{\sqrt{2\pi \sigma_x(z) \sigma_y(z)}} \exp \left[ -\frac{x^2}{2\sigma_x^2(z)} - \frac{y^2}{2\sigma_y^2(z)} \right] \exp \left[ \frac{i}{\epsilon} \theta(x, y, z) \right] ,$$  (9)

where \(\sigma_j \ (j = x, y)\) satisfies Eq. (1) and \(\theta\), is given by:

$$\theta(x, y, z) = \frac{x^2}{2R_x(z)} + \frac{y^2}{2R_y(z)} + \phi_x(z) + \phi_y(z)$$

with

$$\frac{1}{R_j(z)} = \frac{1}{\sigma_j(z)} \frac{d\sigma_j(z)}{dz} \quad \text{and} \quad \frac{d\phi_j(z)}{dz} = -\frac{\epsilon}{4\sigma_j(z)} .$$

Note that, like in the electromagnetic beam optics, \(R_x(z)\) and \(R_y(z)\) describe the local curvature of the wave front associated with the eikonal \(\theta(x, y, z)\) and they are directly connected with the slope of the particle trajectories. Since the gradient of \(\theta(x, y, z)\) defines the current velocity \(V(x, y, z)\) and, on the other hand, is directly connected with the particle trajectory slopes, we can also conclude that the transverse free motion of the beam is governed by a Schrödinger-like equation for a complex function, called beam wave function (BWF) which carries two
information pertinent to the fluid picture: the beam density, proportional to its squared modulus 
\( n(x, y, z) = |\Psi(x, y, z)|^2 \), and the beam current velocity, corresponding to the gradient of its
phase \( \mathbf{V}(x, y, z) = \nabla_{\perp}\theta(x, y, z) \).

Eq. (7) is the starting point of the TWM [18], which has been formulated almost 22 years ago.
In its early formulation, TWM has extended Eq (7) to the case of an arbitrary potential energy,
say \( U \), such that:
\[
\frac{i}{\hbar} \frac{\partial \Psi}{\partial z} = -\frac{\epsilon}{2} \nabla_{\perp}^2 \Psi + U \Psi .
\]  
(10)

With the use of TWM, a number of linear and nonlinear problems in both conventional and plasma-based particle acceleration were successfully described [19, 20, 21, 22, 23, 24, 25]. In particular, TWM has been applied to the Gaussian particle-beam optics and dynamics for a quadrupole-like device [18], to luminosity estimates in final focusing stages of linear colliders in the presence of small aberrations [26], whilst the TWM predictions have been compared with tracking-code simulations and a fair agreement has been demonstrated (the analysis has been carried out in both configuration space and phase space [23, 27, 28, 29]). Remarkably, a self consistent theory of the interaction between a relativistic electron (positron) beam and a cold unmagnetized plasma has been also developed [19]. Recently, the TWM description of the self-consistent beam-plasma interaction has been also developed in strongly magnetized plasmas, where the collective vortex beam states (orbital angular momentum states) has been predicted [30, 31]. The approach was also successfully applied to the longitudinal beam dynamics. For instance, it was useful to predict soliton-like states of the charged-particle beams or to provide a wave key of reading for the nonlinear and collective effects exhibited by the coherent instabilities in high-energy accelerating machines, showing that it can be formulated as the deterministic or the statistical approach to modulational instability, where a Landau-type damping plays a basic role [22, 32, 33, 34].

5. Self-modulation of a thermal matter wave envelope in the strongly nonlocal regime

We want to apply the above description to the case of relativistic non laminar electron (or positron) beam which is travelling through an overdense plasma, of unperturbed density \( n_0 \), with spot size \( \sigma_{\perp} \). It is also assumed that the beam is sufficiently long, i.e., its longitudinal extent, say \( \sigma_z \), is much greater than a plasma length, say \( \lambda_p \). The electron (or positron) beam is launched into the plasma in the presence of a preformed 3D large amplitude electrostatic plasma wave traveling along the \( z \)-axis at the phase velocity \( \beta c \), where \( \beta \approx 1 \). The beam is launched along the \( z \)-axis with velocity \( \beta c \), as well. We assume that the electric field associated with the plasma wave has the form
\[
\mathbf{E}_p(r, \xi) = \mathbf{E}_{\perp 0}(r) \cos k_p \xi + \hat{z} E_{z0}(r) \sin k_p \xi ,
\]  
(11)

where \( r \) is the radial cylindrical coordinate, \( \xi = z - \beta ct \) and \( k_p = 2\pi/\lambda_p \) is the plasma wavenumber. here, \( E_{\perp 0}(r) \) and \( E_{z0}(r) \) account for the radial profile of the the transverse and the longitudinal components, respectively, of the electric field \( \mathbf{E}_p \). To obey the electrostatic condition \( \nabla \times \mathbf{E}_p = 0 \), \( E_{\perp 0}(r) \) and \( E_{z0}(r) \) must be related each other. In particular, if we assume that \( E_{z0} \) has the following Gaussian radial profile, viz.,
\[
E_{z0}(r) = A_0 \exp \left( -r^2/r_{\perp}^2 \right) ,
\]  
(12)

where \( A_0 \) and \( r_{\perp} \) are the constant amplitude and the constant width, respectively, then we easily get
\[
E_{\perp 0}(r) = \frac{1}{k_p} \frac{dE_{z0}(r)}{dr} = \frac{2r/r_{\perp}}{k_p r_{\perp}} E_{z0}(r) .
\]  
(13)
Since the particle beam is assumed much longer than a plasma wavelength, there are two effects experienced by the beam particles that are manifested at time scales much longer than $2\pi/c k_p = 2\pi/\omega_p$ ($\omega_p$ being the electron’s plasma frequency). They are the ponderomotive effect produced by the plasma wave and the PWF excitation. For the time being, let us analyze them separately.

5.1. Ponderomotive effect of the plasma wave

Due to the expression of the electric field given by (11), on the time scales much longer than $T_p = 2\pi/\omega_p$, each particle of the beam experiences a second-order nonlinear force, say $f_{NL}$, called ponderomotive force [35], viz.,

$$f_{NL}(r) = -\frac{1}{2} \frac{e^2}{m \gamma_0 \omega_p} \nabla \langle \mathbf{E}_p^2(r, \xi) \rangle = \frac{1}{4} \frac{e^2}{m \gamma_0 \omega_p} \nabla_{\perp} \left[ E_{\perp 0}^2(r) + E_{z0}^2(r) \right] = -\nabla_{\perp} \Omega_{NL}(r) \quad (14)$$

where $\langle \ldots \rangle$ denotes the average over a plasma period $T_p$; $m$, $e$ and $\gamma_0$ are rest mass, electric charge and initial longitudinal relativistic gamma factor of the electron, respectively. It is easy to see that the dimensionless ponderomotive potential energy $U_{NL}(r) \equiv \Omega_{NL}(r)/m \gamma_0 e^2$ can be cast in the form

$$U_{NL}(r) = \left( \frac{e A_0}{2 m \gamma_0 \omega_p e} \right)^2 \left[ 1 + \left( \frac{2r / r_{\perp}}{k_p r_{\perp}} \right)^2 \right] \exp \left( -2r^2 / r_{\perp}^2 \right). \quad (15)$$

5.2. PWF excitation in the long beam limit

According to the PWF theory [36, 37] for long beams in overdense regime, the plasma provides an adiabatic shielding of the beam. In fact, the perturbation of the plasma charge density (cylindrical coodinates) $en_1(r, \varphi, \xi)$ that is originated by the insertion of the long beam in the plasma tends to compensate the local beam charge density $q\rho_{b}(r, \varphi, \xi)$, i.e., $en_1(r, \varphi, \xi) \approx q\rho_{b}(r, \varphi, \xi)$. For cylindrically symmetric beams, $\rho_0(r, \varphi, \xi) = \rho_b(r, \xi)$ becomes the source of the plasma wake potential energy, i.e., $U_w(r, \xi)$, through the following Poisson-like equation [36, 37]

$$(\nabla_{\perp}^2 - k_p^2) U_w = k_p^2 \rho_b / m \gamma_0. \quad (16)$$

In turn, the PWF, i.e., $\mathbf{W} \propto \nabla U_w$ acts on the particles of the beam itself (self-interaction). The simultaneous effect of $U_{NL}$ and $U_w$ characterizes the transverse dynamics of the long beam. Note that, thanks to the long beam limit, no longitudinal dynamics is practically involved in our analysis. In fact, due to this assumption: (i) the effects of the plasma wave come into play only through the ponderomotive potential that manifests on the time scales longer than a plasma period (therefore the ponderomotive force has a transverse component, only); (ii) the longitudinal component of the plasma wake field, i.e., $W_z \propto \partial U_w / \partial \xi$, is much smaller than the transverse one, i.e., $W_{\perp} \propto \nabla_{\perp} U_w$. In these conditions, the longitudinal beam dynamics can be ignored, so that our analysis becomes 2D.

Within the context of TWM [18], the latter is governed by Eq (10), where $U = U_{NL} + U_w$, i.e., the following 2D Schrödinger-like equation (here given in cylindrical coodinates)

$$i c e \frac{\partial \psi}{\partial \xi} = -\frac{\hbar^2}{2} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2} \right] + (U_{NL} + U_w) \psi. \quad (17)$$

Here, we cast the normalization of $\psi$ in such a way that $\rho_b(r, \varphi, \xi) = (N/\sigma_z) |\psi(r, \varphi, \xi)|^2$ and $\int_0^{2\pi} d\varphi \int_0^\infty r \, dr |\psi(r, \varphi, \xi)|^2 = 1$. Note that Eq. (17) describes the feedback that the beam
produces on itself through $U_w$. On the basis of the above interpretation of $\psi$, Eqs. (16) and (17) can be easily cast, in cylindrical symmetry, as

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U_w}{\partial r} \right) - k_p^2 U_w = \frac{k_p^2 N}{n_0 \gamma_0 \sigma z} |\psi_m|^2.$$  

(18)

$$i e \frac{\partial \psi_m}{\partial \xi} = - \frac{e^2}{2} \frac{1}{r} \frac{\partial r}{\partial r} \left( r \frac{\partial \psi_m}{\partial r} \right) + \left( \frac{e^2 m^2}{2r^2} + U_{NL} + U_w \right) \psi_m,$$  

(19)

where we have cast the BWF as $\psi(r, \varphi, \xi) = \psi_m(r, \xi) e^{im\varphi}$, with $m$ arbitrary integer. Pair of Eqs. (18) and (19) describes the self-consistent transverse spatiotemporal evolution on time scales longer than a plasma period of a charged-particle beam while experiencing its PWF interaction in the long beam limit in the presence of a pre-formed 3D large amplitude plasma wave. It was earlier derived by Fedele and Shukla in 1992 for the case of $m = 0$, and $U_{NL} = 0$ [19]. It can be regarded as a sort of Zakharov pair of equations.

5.3. Strongly nonlocal regime

We confine here our attention to the case of a charged particle beam with a spot size, say $\sigma_\perp$, much smaller than the width, $r_\perp$, of the radial profile of the preformed plasma wave (strongly focussed beam), i.e., $\sigma_\perp \ll r_\perp$. Here, $\sigma_\perp$ is defined as the rms of the positions of the beam particles, i.e., $\sigma_\perp(\xi) = \int_0^{\infty} d\varphi \int_0^{\infty} r^2 |\psi(r, \varphi, \xi)|^2 r \, dr$. Then, if we require that $k_p r_\perp \approx 2\pi$ (i.e., $r_\perp \approx \lambda_p$), we can assume that $k_p \sigma_\perp \ll 1$. Then, Eq. (18) becomes (strongly nonlocal regime)

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial U_w}{\partial r} \right) = \frac{k_p^2 N}{n_0 \gamma_0 \sigma z} |\psi|^2.$$  

(20)

On the other hand, due to the above conditions the beam particles are confined in a region where $r \ll r_\perp$. Therefore, in Eq. (19) it is enough to expand $U_{NL}(r)$ in powers of $r/r_\perp$ up to the second degree. So, apart from an additive constant, we obtain

$$U_{NL}(r) \approx \frac{1}{2} K_{NL} r^2, \quad K_{NL} = \left( \frac{e A_0}{m \gamma_0 \omega_p c r_\perp} \right)^2 \left( \frac{2}{k_p^2 r_\perp^2} - 1 \right).$$  

(21)

Therefore, $K_{NL}$ is positive (negative) for $r_\perp < r_\perp^* (r_\perp > r_\perp^*)$ or vanishes for $r_\perp = r_\perp^*$, where the threshold $r_\perp^* = \lambda_p/2\pi \approx 0.226\lambda_p$. According to the method for aberrationless solution presented in [38, 39], but confining our analysis to the case of $m = 0$, pair of equations (19) and (20) allows to get the following Schrödinger equation

$$i e \frac{\partial \Psi}{\partial \xi} = - \frac{e^2}{2} \frac{1}{r} \frac{\partial r}{\partial r} \left( r \frac{\partial \Psi}{\partial r} \right) + \frac{1}{2} \bar{K}(\xi) r^2 \Psi,$$  

(22)

where

$$\bar{K}(\xi) = K_{NL} + \frac{\eta}{\sigma_\perp^2(\xi)}, \quad \eta = \frac{k_p^2 N}{2\pi \sigma_\perp n_0 \gamma_0}.$$  

(23)

and $\psi_0(r, \xi)$ has been renamed as $\Psi$. Note that the constant $\eta$ is a positive quantity. Here, $\sigma_\perp$ obeys to the following Ermakov-Pinney type equation for $\sigma_\perp(\xi)$ (i.e., ), viz.,

$$\frac{d^2 \sigma_\perp}{d\xi^2} + K_{NL} \sigma_\perp + \frac{\eta}{\sigma_\perp} - \frac{e^2}{\sigma_\perp^3} = 0,$$  

(24)
Figure 1. (a) Qualitative plots (arbitrary units) of $V(\sigma_\perp)$ for $K_{NL} \geq 0$ keeping constant both $\eta$ and $\epsilon$. $V(\sigma_\perp)$ is a trapping potential for any value of the total energy $E$. (b) Qualitative plots (arbitrary units) $V(\sigma_\perp)$ for a fixed negative value of $K_{NL}$ and $\epsilon$. The two different plots represent two different classes of potentials: the solid plot corresponds to the ones that are trapping up to a certain threshold of the total energy $E$, whilst the dashed plot is non trapping for any value of $E$.

Equation (24) is also usually referred to as Sacherer’s envelope equation [7]. Provided that $K$ is positive, Eq. (22) admits bounded states. A complete set of normalized eigenfunctions of Eq. (22) is given by the Laguerre-Gauss modes, viz.,

$$
\Psi_n(r,\xi) = \frac{1}{\sqrt{\pi \sigma_\perp}} \exp \left( -\frac{r^2}{2 \sigma_\perp^2} + \frac{ir^2}{2 \rho} + i\phi_n \right) L_n \left( \frac{r^2}{\sigma_\perp^2} \right),
$$

where $n$ is an arbitrary integer, $L_n$ are the simple Laguerre polynomials, and $\rho = \rho(\xi)$ and $\phi_n = \phi_n(\xi)$ satisfy the following differential equations

$$
\frac{1}{\rho} = 1 \frac{d\sigma_\perp}{d\xi}, \quad -\frac{\sigma_\perp^2}{2\epsilon} \frac{d\phi_n}{d\xi} = n + \frac{1}{2}.
$$

From Eq. (23), one can easily see that the positivity of $K$ implies

$$
\frac{\eta}{\sigma_\perp^2} > -K_{NL}.
$$

Note that this condition fixes an inequality which depends on the time-like variable $\xi$. In principle, it can be violated for suitable variation of $\sigma_\perp(\xi)$. To predict the behavior of the beam, let us refer to the Sagdeev pseudo potential [35] that it is given by the first integral of Eq. (24), i.e.,

$$
\frac{1}{2} \left( \frac{d\sigma_\perp}{d\xi} \right)^2 + V(\sigma_\perp) = E = \text{const},
$$

where the pseudo-potential $V(\sigma_\perp)$ is

$$
V(\sigma_\perp) = \frac{1}{2} K_{NL} \sigma_\perp^2 + \eta \log \sigma_\perp + \frac{\epsilon^2}{2 \sigma_\perp^2},
$$

and the total conserved energy $E$ is fixed by the initial conditions for $\sigma_\perp$, i.e., $\sigma_0 = \sigma_\perp(\xi = 0)$ and $\sigma_0 = (d\sigma_\perp/d\xi)_{\xi=0}$.

A. Non-negative $K_{NL}$

Since $\eta$ is positive, for $K_{NL} \geq 0$, condition (27) is always fulfilled. Therefore, according to Eq.
(22), the particle beam is localized; in particular, for $n = 0$ (fundamental mode), the transverse density has a peak around $r = 0$. In addition, for $K_{NL} \geq 0$, $V(\sigma_\perp)$ is always confining. Then, any initial condition evolves stably. More specifically, for fixed initial conditions, $\sigma_\perp$ is a periodic function of $\xi$. This appears as a macroscopic sausage-like modulation as a result of the interplay of three concomitant effects: (i) the radial thermal dispersion (spreading), that is associated with the emittance $\epsilon$; (ii) the radial ponderomotive effects of the pre-existing LWF-excited plasma wave, that is associated with the focusing strength $K_{NL}$; (iii) the self-interaction provided by the plasma wake field generated by the beam itself associated with the collective parameter $\eta$. We refer this collective response of the beam due to the above complex interaction with the plasma to as beam self-modulation of the thermal matter wave envelope (25). Figure 1(a) displays qualitative plots describing the attractive character of $V(\sigma_\perp)$ for sets of parameters in which $K_{NL}$ is non-negative.

B. Negative $K_{NL}$
If $K_{NL}$ is negative, condition (27) may be no longer fulfilled. Although this condition is satisfied, the analysis of $V(\sigma_\perp)$ shows that there are sets of the parameters corresponding to a locally trapping $V$ as shown qualitatively in the example given in Figure 1(b). In fact, for suitable value of the energy $E$ (and therefore of the initial conditions), stable oscillations in $\sigma_\perp$ around the (local) minimum are possible (solid plot). Also in this case, they appear as a macroscopic self-modulation of the thermal matter wave envelope driven by the PWF excitation. As $E$ increases solutions transit from periodic to unbounded ones, which correspond to an unstable evolution. Furthermore, there is also another set of parameters for which $V(\sigma_\perp)$ does not correspond to a trapping potential, even locally (dashed plot). In fact, in this other case, $V(\sigma_\perp)$ does not have local minima to make possible oscillations. Then the thermal matter wave envelope is unstable and evolves in a blowup.

6. Conclusions and remarks
In this paper, we have described the self-modulation of a long, non laminar, relativistic electron (or positron) beam in an unmagnetized plasma. The process is driven by both a 3D, preformed, large amplitude plasma wave, and the self consistent plasma wake field excitation. The analysis has been carried out within the framework of the quantum-like description that is provided by TWM, where the beam as whole can be regarded as a thermal matter wave envelope, whose spatiotemporal evolution is governed by a Zakharov-like pair of equations. In the strongly nonlocal regime, we have analyzed the physical conditions that keep stable the self-modulation of the thermal matter wave envelope, in the form of sausage-like transverse oscillations as well as the instability leading to the beam blowup.

References
[1] Sturrock P 1955 Static and Dynamic Electron Optics: An Account of Focusing in Lens, Deflector and Accelerator (Cambridge University Press, London)
[2] Zworykin V, Morton G, Ramberg E, Hillier J and Vance A 1945 Electron optics and the electron microscope (John Wiley, New York)
[3] Glaser P 1952 Grundlagen der Elektronenoptik (Springer Verlag, Vienna)
[4] Klemperer O and Barnett M 1971 Electron optics (Cambridge University Press, London)
[5] Grivet P and Septier A 1972 Electron optics (Pergamon, Oxford)
[6] Hawkes P 1972 Quadrupole optics. Springer tracts in modern physics (Springer Verlag, Berlin) volume 42
[7] Lawson J D 1988 The physics of charged-particle beams (Clarendon, Oxford)
[8] McMorran B, Agrawal A, Anderson I, Herzing A and Lezec, H. 2011 Science 331 192
[9] Jagannathan R, Simon R, Sudarshan E and Mukunda N 1989 Phys. Lett. A 134 457
[10] Jagannathan R 1990 Phys. Rev. A 42 6674
[11] Khan S A and Jagannathan R 1995 Phys. Rev. E 51 2510
[12] Chao A and Tigner M 1998 Handbook of accelerator physics and engineering (World Scientific, Singapore)
[13] Fedele R and Man’ko V I 1998 Phys. Rev. E 58 992
[14] Lapostolle P M 1971 IEEE Trans. Nucl. Sc. 18 1101; Sacherer F J 1971 IEEE Trans. Nucl. Sc. 18 1105
[15] Su J J, Katsouleas T, Dawson J M and Fedele R 1990 Phys. Rev. A 41 3321
[16] Fedele R and Wilson E J N 1990 Nuovo Cim. D 12 1497
[17] Fedele R and Jovanović D 2004 AIP Conf. Proc. 740 430-445
[18] Fedele R and Miele G 1991 Nuovo Cim. D 13 1527-1544
[19] Fedele R and Shukla P K 1992 Phys. Rev. A 45 4045-4049
[20] Fedele R and Shukla P K 1992 A novel approach to the interaction between the plasma wakefield and the driving relativistic electron beam, Proc. Int. Conf. on Plasma Physics, Innsbruck, Austria, 29 June - 3 July 1992, Vol. II, p.1293-1296, European Physical Society.
[21] De Nicola S, Fedele R, Man’ko V I and Miele G 1995 Phys. Scr. 52 191
[22] Fedele R, Miele G, Palumbo L and Vaccaro V G 1993 Phys. Lett. A 179 407
[23] Fedele R, Galluccio F and Miele G 1994 Phys. Lett. A 185 93
[24] Fedele R, Miele, G and Palumbo L 1994 Phys. Lett. A 194 113
[25] Fedele R, Shukla P K and Vaccaro V G, 1995 A quantum-like approach to the interaction of relativistic charged particle beams with plasmas, J. Physique 5, Colloque 6, Suppl. JP II, n.10 p. c6-119 c6-130
[26] Fedele R and Miele G 1992 Phys. Rev. A 46 6634
[27] Fedele R, Galluccio G, Miele G and Man’ko V I 1995 Phys. Lett. A 209 263
[28] Jang J, Cho Y and Kwon H 2007 Phys. Lett. A 366 246
[29] Jang J, Cho Y and Kwon H 2010 Nucl.Instrum.Meth. A 624 578
[30] Fedele R, Tanjia F, De Nicola S, Shukla P K and Jovanović D 2011 Proc. 38th EPS Conf. on Plasma Physics, Strasbourg, France 35G, P5.006, ISBN 2-914771-68-1
[31] Tanjia F, De Nicola S, Fedele R, Shukla P K and Jovanović D 2011 Proc. 38th EPS Conf. on Plasma Physics, Strasbourg, France 35G, P5.021, ISBN 2-914771-68-1
[32] Anderson D, Fedele R, Vaccaro V, Lisak M, Berntson A and Johansson S 1999 Phys. Lett. A 258 244
[33] Fedele R and Anderson D 2000 J. Opt. B: Quant. Semicl. Opt. 2 207
[34] Johannisson P et al. 2004 Phys. Rev. E 69 066501
[35] Chen F F 1984 Introduction to plasma physics (Springer, Berlin) second edition
[36] Chen P et al. 1985 Phys. Rev. Lett. 54 693-696
[37] Chen P 1987 Part. Accel. 20 171-182
[38] Fedele R, Tanjia F, De Nicola S, Jovanović D and Ronsivalle C 2013 J. Plasma Phys. to appear
[39] Jovanović D, Fedele R, Tanjia F, De Nicola S and Belić M 2012 Eur. Phys. Lett. 100 5502-p1 - 5502-p6
[40] Fedele R, Tanjia F, De Nicola S and Jovanović D 2013 NIMA Proc. 1st European Advanced Accelerator Concepts Workshop EAAC 2013, Isola d’Elba, 2-7 June 2013, submitted