The BF Formalism for QCD and Quark Confinement

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Abstract

Using the BF version of pure Yang-Mills, it is possible to find a covariant representation of the ’t Hooft magnetic flux operator. In this framework, ’t Hooft’s pioneering work on confinement finds an explicit realization in the continuum. Employing the Abelian projection gauge we compute the expectation value of the magnetic variable and find the expected perimeter law. We also check the area law behaviour for the Wilson loop average and compute the string tension which turns out to be of the right order of magnitude.

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1 Introduction

The problem of quark confinement in continuum QCD is still unsolved. Confinement of electric charges is observed if one is willing to use the lattice as a regulator but direct computations in the continuum theory are not easily performed due to the large scale nature of the problem at hand.

Among the different explanations put forward in literature, we find that the so called dual Meissner effect is the most appealing and complete: if the QCD vacuum behaves as a magnetic superconductor color electric flux lines will form long and narrow tubes with color electric charges at its ends. The story of this idea is quite long and has received contributions from many authors \[1, 2, 3, 4\]. In this paper we will stick to the version of ’t Hooft in which all these ideas are brought to a high degree of definitiveness \[5, 6\].

More in detail, the whole idea of the dual superconductor can be phrased in terms of ’t Hooft order-disorder variables. These variables are non-local quantum operators, \(W(C), M(C')\), creating thin color electric and magnetic fluxes, along the curves \(C, C'\), in the QCD vacuum. In their turn, the curves \(C, C'\) border two surfaces \(\Sigma, \Sigma'\). In the absence of massless particles, these surfaces may become dynamical and we may have linkage when one of these curves (say \(C'\) for example) intersects the surface (say \(\Sigma\)) spanned by the other curve (\(C\)). The linkage costs energy per unity of area and, roughly speaking, the associated effective action goes as the area, \(A\), of the dynamical surface: in our previous example it will be \(< W(C) > \simeq e^{const. A(\Sigma)}\). All these properties are summarized by the commutation relation of \(W(C), M(C')\) which is of braiding type and which contains the entire physics of the problem. Which surface is intersecting which curve is not known in this picture which supports all possibilities: Higgs phase, Coulomb phase, partial Higgs with confinement and confinement phase \[5\]. To support the picture of the dual superconductor the QCD vacuum must be filled by magnetic vortex lines with non-trivial topology, generated by means of a true or effective dual Higgs dynamics.

Despite the appeal of this qualitative picture one may still be left with a feeling of uneasiness since quantitative computation in the continuum theory have not yet been
performed.

In order to expose the topological content of the theory, one of us has proposed to study pure Yang-Mills theory using the first order formalism (BF-YM). The main advantage of this formulation is that a color magnetic operator can be easily guessed in the continuum \[7\]. In this formalism we show that our magnetic operator is the dual 't Hooft observable \[5\] and compute its vev, employing the Abelian projection gauge \[6, 8\], finding the expected perimeter law.

The first order form of pure Yang-Mills is described by the action functional

\[
S_{BF-YM} = \int \text{Tr}(iB \wedge F + \frac{g^2}{4} B \wedge *B) \quad \quad (1.1)
\]

where

\[
F \equiv \frac{1}{2} F_{\mu}^a T^a dx^\mu \wedge dx^\nu \equiv dA + i[A, A] \quad ,
\]

\[
D \equiv d + i[A, \cdot] \quad \text{and} \quad B \quad \text{is a Lie valued 2-form. The generators of the SU}(N) \quad \text{Lie algebra in the fundamental representation are normalized as Tr}\ T^a T^b = \frac{\delta_{ab}}{2} \quad \text{and the * product (Hodge duality) for a p form in d dimension is defined as * = }e^{i1 \ldots id}/(d-p)!. \quad \text{The classical gauge invariance of (1.1) is given by}
\]

\[
\delta A = D\Lambda_0 \quad ,
\]

\[
\delta B = -i[A_0, B] \quad .
\]

Using the field equations (which is equivalent in the path integral to integrate over B)

\[
F = \frac{ig^2}{2} * B \quad ,
\]

\[
DB = 0 \quad ,
\]

we get the standard YM action

\[
S_{YM} = \frac{1}{g^2} \int \text{Tr}(F \wedge *F) \quad .
\]

Note that the relation between the formulations (1.1) and (1.5) may be also undertood as a duality map; however off shell B does not satisfy a Bianchi identity and this fact will be connected with the introduction in the theory of magnetic vortex lines. We remark
that the short distance quantum behaviour of (1.1) and (1.3) are the same as it has been explicitely checked [12].

We remark that the action functional in (1.1) defined by

\[ S_{BF} = i \int \text{Tr}(B \wedge F) \]  

is known as the 4D pure bosonic BF-theory and defines a topological quantum field theory [9, 10]. The form of (1.1) was understood in [7] as an indication that the bosonic YM theory can be viewed as a perturbative expansion in the coupling \( g \) around the topological pure BF theory (1.6). This procedure has been called topological embedding [11]. Using this perturbative picture around BF-theory we check in this paper the area law behaviour for the Wilson operator.

The plan of the paper is the following: in section 2 we introduce an explicit analytic realization of the ’t Hooft magnetic variable in terms of the \( A, B \) fields and check its properties at the operator level using the canonical quantization. In section 3 we compute its expectation value in the abelian projection gauge and verify the perimeter law. In section 4 we verify the area law for the Wilson loop operator. At last, in section 5 we draw our conclusions.

## 2 A Color Magnetic Operator

In this section we define a gauge invariant non local operator \( M(C) \equiv M(\Sigma, C) \) associated with a fixed orientable surface \( \Sigma \) in \( M^4 \) (our base manifold) and with a suitable choice of a closed contour \( C \) on \( \Sigma \), which gives an explicit realization of the ’t Hooft loop. Part of this section was already investigated in [7] by one of the present authors.

The presence of the Lie-algebra valued two-form \( B \) field in (1.1) allows the definition of the observable gauge invariant operator

\[ M(\Sigma, C) \equiv \text{Tr exp}\{ik \int_{\Sigma} d^2 y \text{Hol}_{\bar{x}}^y(\gamma) B(y) \text{Hol}^\bar{x}_y(\gamma')\} \]  

where \( k \) is an arbitrary expansion parameter, \( \bar{x} \) is a fixed point over the orientable surface \( \Sigma \subset M^4 \) (we do not integrate over \( \bar{x} \)) and the relation between the assigned paths \( \gamma, \gamma' \)
over $\Sigma$ and the closed contour $C$ is the following: let $\hat{\Sigma}$ to be a piecewise linear (PL) approximation of $\Sigma$ by plaquettes (any two dimensional topological variety admits a PL decomposition). Then the closed path $C$ is given by a succession of subpaths formed by the 1-skeleton of the 2-PL manifold $\hat{\Sigma}$. In other words $C$ starts from the fixed point $\bar{x}$, connects a point $y \in \Sigma$ by the open path $\gamma_{\bar{x}y}$ and then returns back to the neighborhood of $\bar{x}$ by $\gamma'_{\bar{x}y}$, (which is not restricted to coincide with the inverse $(\gamma_{\bar{x}y})^{-1} = \gamma_{y\bar{x}}$). From the neighborhood of $\bar{x}$ the path starts again to connect another point $y' \in \Sigma$. Then it returns back to the neighborhood of $\bar{x}$ and so on until all points on $\Sigma$ are connected. The path $C_{\bar{x}} = \{ \gamma \cup \gamma' \}$ is generic and we do not require any particular ordering prescription as it is done in similar constructions devoted to obtain a non abelian Stokes theorem [13].

In figure 1 we show a typical path $C$ connecting four points $x_1, x_2, x_3, x_4$ on a surface $\Sigma$ given by a plane.

![Figure 1](image)

Of course the quantity (2.1) is path dependent and our strategy is to regard it as a loop variable once the surface $\Sigma$ is given. In Eq. (2.1) $\text{Hol}_y^x(\gamma)$ denotes the usual holonomy along the open path $\gamma \equiv \gamma_{\bar{x}y}$ with initial and final point $\bar{x}$ and $y$ respectively,

$$\text{Hol}_y^x(\gamma) \equiv P \exp(i \int_\bar{x}^y dx^\mu A_\mu(x)) \ .$$  (2.2)

Given the finite local gauge transformations

$$\text{Hol}_y^x \to g^{-1}(\bar{x})\text{Hol}_y^x g(y) \ ,$$  \hspace{1cm} (2.3)

$$B(y) \to g^{-1}(y)B(y)g(y) \ ,$$  \hspace{1cm} (2.4)

it is immediate to prove the gauge invariance of (2.1). A comment on the geometrical
meaning of the operator $M$ is in order. If, for example, we take the surface $\Sigma$ to be a torus we may define $M(\Sigma, C)$ in terms of the parallel transport operator of loops [4] given by $P \exp \{ i k \oint_{C_1} \tilde{B}_\mu(C_1) dx^\mu \}$, where $\tilde{B}_\mu(C_1) \equiv \oint_{C_2} (\text{Hol} B_{\mu \nu} \text{Hol}'') dx^\nu$ is a connection in the loop space and $C$ is a linear combination of the two fundamental cycles $C_1$ and $C_2$ of the torus.

By adding suitable bosonic vector and ghosts fields to the BF-YM theory, the “topological” (we will clarify later what we use quotes here) symmetry of the pure BF action (1.6), given by

$$
\delta A(y) = 0 , \quad \delta B(y) = D\Lambda_1(y) + i[A(y), \Lambda_1(y) , \text{Hol}) \text{Hol}' \text{Hol}''),
$$

could be extended to the BF-YM theory. Then by using the well-known identity [14]

$$
d_{(y)} (\text{Hol}^y(g) \Lambda_1(y)) = \text{Hol}^y(g) \{ d(y) \Lambda_1(y) + i[A(y), \Lambda_1(y)] \} \text{Hol}^y(g')
$$

one may prove the invariance of (2.1) under the transformations (2.5) up to boundary terms for non-compact $\Sigma$. However these boundary terms may be neglected in the renormalized theory if mixed Dirichlet-Neumann boundary conditions for the quantum fields at the boundary $\partial (M^4 \setminus \Sigma) = -\partial \Sigma$ are used, in a way similar to the quantization of systems over a non complete space time (e.g. Casimir effect).

Let us give a geometrical interpretation of the “topological” symmetry (2.5). The holonomy dressed field $\text{Hol}B\text{Hol}'$ can be interpreted as a colored charged field that transforms covariantly under the global colour symmetry $g(x)$. Accordingly the transformation (2.5) represents a local change (with parameter $\text{Hol}\Lambda_1\text{Hol}'$) of the colour charge [15].

In the following the path dependence of (2.1) from the closed contour $C$ will be crucial. Indeed, we shall see below that different choices of $C$ for fixed surface $\Sigma$, we call it a framing $(C, \Sigma)$, give different color magnetic fluxes in QCD.

The classical BF-YM action (1.1) is linear in time derivatives, and hence it is easily cast into its canonical form [9]. Let $M^4 = N^3 \times \mathbb{R}$ and let $t$ be a coordinate labelling the different $N^3$ surfaces. Let then $\xi^\mu$ be any vector satisfying $\xi^\mu \partial_\mu t = 1$. The appropriate
notion of time derivative is then given by the Lie derivative

\[ L_\xi A^a_\mu = \partial_\mu (A^a_\nu \xi^\nu) + \xi^\nu \partial_\nu A^a_\mu - \xi^\nu \partial_\mu A^a_\nu = D_\mu (A^a_\nu \xi^\nu) + \xi^\nu F_{\nu\mu} \]  

(2.7)

For \( N^3 = \mathbb{R}^3 \) and \( \xi_\mu = (1, 0, 0, 0) \) one recovers the usual definition of time derivative. Time derivatives act only on the space components of \( A^a_\mu \). Therefore the conjugate momenta of \( A^a_0, B^a_0 \) are zero and these fields appear as Lagrangian multipliers since they may be eliminated through the classical constraint equations. In order to have first class constraints in the Hamiltonian formalism, the field content of the action (1.1) must be enlarged in such a way that the resulting action enjoys the same topological symmetry (2.5) of the action (1.6). We thus introduce a Lie valued vector field \( \eta \) which transforms under the symmetry (2.5) as \( \delta \eta = \Lambda_1 \) and enters the action (1.1) by substituting \( B^2 \to (B - D\eta)^2 \) [16].

The 3+1-action becomes

\[
S_{BF-YM} = \frac{1}{2} \int dt \int_{N^3} d^3x \left\{ 2i\epsilon^{ijk} B^a_{ij} \dot{A}^a_k + \frac{g^2}{2} [B^a_{0i} - (D_0 \eta^a_i - D_i \eta^a_0)]^2 \right. \\
- \left. i\epsilon^{i\ell k} B^a_{0\ell} F^a_{i\ell} + iA^a_0 \epsilon^{ijk} (D_i B^a_{jk}) = \frac{g^2}{4} [B^a_{ij} - (D_i \eta^a_j - D_j \eta^a_i)]^2 \right\} 
\]

(2.8)

and the classical constraints read

\[
C_1[\tau(\vec{x})] \equiv \frac{1}{2} \int_{N^3} d^3x \ i\epsilon^{ijk} (D_i B^a_{jk}) = 0 , \\
C_2[v_i(\vec{x})] \equiv \frac{1}{2} \int_{N^3} d^3x \ v^a_i (\Pi^a_i - i\epsilon^{ikl} F^a_{kl}) = 0 ,
\]

(2.9, 10)

where \( a \) is the Lie algebra index, \( i, j, k = 1, 2, 3 \) and \( \tau^a, v^a_i \) are arbitrary Lie algebra valued fields on \( N^3 \). \( \Pi^a_i \) is the canonical momentum conjugate to \( \eta \)

\[
\Pi^a_i = \frac{g^2}{2} \left\{ B^a_{0i} - \partial_0 \eta^a_i + i[A_0, \eta_i]^a - i[A_i, \eta_0]^a \right\} ,
\]

(2.11)

and

\[
\chi^a_i = i\epsilon_{ijk} B^a_{jk}
\]

(2.12)

is the canonical momentum conjugate to \( A^a_i \).

The appropriate gauge for the Hamiltonian formalism is the temporal gauge, which in this contest reads

\[
A^a_0 = 0 ,
\]

(2.13)
\[ B_{0i}^a = 0 \quad (2.14) \]

(2.13) fixes the conventional gauge symmetry and (2.14) fixes the topological symmetry.

The Poisson brackets are defined by
\[
\{ A_i^a(\vec{x}), \chi_j^b(\vec{y}) \}_{x_0=y_0} = i\delta^{ab}\delta_{ij}\delta^3(\vec{x}-\vec{y}),
\]
\[
\{ \eta_i^a(\vec{x}), \Pi_j^b(\vec{y}) \} = \delta^{ab}\delta_{ij}\delta^3(\vec{x}-\vec{y}),
\] (2.15)
and yield
\[
\delta A_i^a(\vec{x}) = \{ C_1, A_i^a(\vec{x}) \} + \{ C_2, A_i^a(\vec{x}) \} = -2iD_i\tau^a(\vec{x})\,,
\]
\[
\delta B_{ij}^a(\vec{x}) = \{ C_1, B_{ij}^a(\vec{x}) \} + \{ C_2, B_{ij}^a(\vec{x}) \} = -2i[\tau(\vec{x}), B_{ij}^a(\vec{x})]^a
\]
\[
-2i(D_i v_i^a(\vec{x}) - D_j v_j^a(\vec{x}))\,,
\]
\[
\delta \eta_i^a = \{ C_1, \eta_i^a(\vec{x}) \} + \{ C_2, \eta_i^a(\vec{x}) \} = -i[\tau(\vec{x}), \eta_i^a(\vec{x})]^a - v_i^a(\vec{x})\,. \quad (2.16)
\]

The constraint \( C_1 \) generates the usual non abelian gauge transformations for \( A_i^a, \eta_i^a \) and \( B_{ij}^a \), and \( C_2 \) generates the classical topological symmetry (2.5). The canonical quantization of (2.8) is obtained by replacing the Poisson bracket \( \{ \cdot, \cdot \} \) in (2.15) with the commutator
\[-i\hbar[\cdot, \cdot] \] and promoting the classical constraints \( C_1, C_2 \) to operator valued ones
\[
\hat{C}_1 | A, B, \eta > = 0 \quad ,
\]
\[
\hat{C}_2 | A, B, \eta > = 0 \quad ,
\] (2.17)
where \( | A, B, \eta > \) is called a physical state if it satisfies the above quantum constraints.

We now show that \( M(C) \equiv M(C ; \Sigma = S^2) \) generates a local singular (or equivalently a multivalued regular) gauge transformation, \( \Omega_C(\vec{x}) \), along \( C \): this is precisely the defining property of the ‘t Hooft color magnetic variable [5].

We start from the classical quantity associated to (2.1) and assume that \( \Sigma \), and hence \( C \), is small, so that one may consider a Taylor expansion of (2.1) around \( k = 0 \). By using the Gauss theorem, the gauge (2.13), (2.14) and the identity (2.6), which in our case becomes
\[
\text{Tr}\int_{S^2} d\sigma^{ij}(\text{Hol}B_{ij}\text{Hol}') = \text{Tr}\int_{M^3} d^3x e^{ijk}\partial_l(\text{Hol}B_{jk}\text{Hol}') =
\]
\[
= \text{Tr}\int_{M^3} d^3x(\text{Hol} e^{ijk}D_i B_{jk}\text{Hol}') = \text{Tr}\int_{M^3} d^3x\text{Hol}^{-1}(C_x)e^{ijk}D_i B_{jk} \quad ,
\] (2.18)
one gets at lowest order in $k$

$$M(C) \simeq \text{Tr}\left\{1 + ik \int_{N^3} d^3 x [\text{Hol}^{-1}(C_x)^{\epsilon^j k} D_i B_{jk}(x)]\right\} ,$$

(2.19)

where $M^3$ is a small 3-ball with boundary $S^2$ : $\partial M^3 = S^2$ and $C_x$ is defined as $C_x \equiv \gamma^x_\bar{x} \cup \gamma^{\bar{x} x}$. We now expand also Hol$^{-1}$ at this order

$$\text{Hol}^{-1}(C_x) \simeq (1 + i\varphi_C(x)) \quad , \quad \varphi_C(x) \equiv \oint_{C_x} dy^i A_i .$$

(2.20)

Substituting (2.20) in (2.19) and using the definition of the constraint $C_1$, Eq. (2.9), we see that $M(C)$ becomes the generator of the classical infinitesimal local gauge transformations with parameter $k\varphi_C(x)$:

$$M(C) \simeq \text{Tr}\left\{1 + k\tilde{C}_1 [\varphi_C(\vec{x})]\right\} ,$$

(2.21)

where $C_1 = 2\text{Tr}[\tilde{C}_1]$ in the temporal gauge $\tilde{g}$.

To extend (2.21) to the quantum level we substitute the Poisson brackets with the quantum mechanical commutators and rescale the geometrical fields to the physical ones,

$$\hat{A}^a_i(\vec{y}) = g \hat{A}^a_i(\vec{y}) \quad ,$$

(2.22)

$$\left[\hat{A}^a_i(\vec{y}), \hat{B}^b_{rs}(\vec{x})\right]_{x_0 = y_0} = \frac{2i}{g} \delta^{ab} \epsilon_{irs} \delta^{(3)}(\vec{y} - \vec{x}) .$$

(2.23)

Owing to the cyclicity properties of the trace, (2.19) must be understood as symmetrized in $DB$ and Hol. Then, when (2.21) becomes operator valued, the ordering procedure required by the quantization and enforced by (2.17), implies that we can substitute the product of fields with their canonical commutator.

Therefore using the Gauss theorem of (2.18) in (2.19), we express the divergence of the $B$ field as the flux through the surface $S^2$ and obtain

$$\hat{M}(C)|A > \simeq \text{Tr}\left\{1 + 2ik \oint_C dy^i \int_{\Sigma \sim S^2} d\sigma^{rs}_{(x)} \epsilon_{irs} \delta^{(3)}(\vec{y} - \vec{x}) \delta^{ab} T_a T_b\right\}|A > .$$

(2.24)
Notice now that
\[
\oint_C dy^i \int \Sigma d\sigma^r s(\vec{y} - \vec{x}) = -\oint_C dy^i \int \Sigma \frac{d\sigma^r s}{4\pi} \epsilon_{ir} \frac{\partial}{\partial x^l} \left( \frac{1}{|\vec{y} - \vec{x}|} \right)
\]
\[
\simeq \frac{1}{4\pi} \oint_C dy^i \oint_{C'} dx^r \epsilon_{ir} \frac{\partial}{\partial x^l} \left( \frac{1}{|\vec{y} - \vec{x}|} \right) = \text{Link}(C, C') ,
\]
(2.25)
where \( \Sigma = \Sigma' \cup \Sigma'' \), \( \partial \Sigma' = C' \) and \( C' \) encloses the singularity of (2.23). Here we have used the formulae
\[
\frac{\partial}{\partial x^l} \frac{\partial}{\partial x^l} \left( \frac{1}{|\vec{y} - \vec{x}|} \right) = -4\pi \delta^{(3)}(\vec{y} - \vec{x}) ,
\]
\[
\int_\Sigma d\sigma^r s(x) \simeq \delta \sigma^r s(\Sigma') = \oint_{C'} dx^r x^s ,
\]
(2.26)
and the definition of the so called linking number between two closed loops,
\[
\text{Link}(C, C') = \frac{1}{4\pi} \oint_C dy^i \oint_{C'} dx^r \epsilon_{ir} \frac{\partial}{\partial x^l} \left( \frac{1}{|\vec{y} - \vec{x}|} \right)
\]
\[
= \frac{1}{4\pi} \oint_C dy^i \oint_{C'} dx^r \epsilon_{ir} \frac{(y^i - x^i)}{|\vec{y} - \vec{x}|^3} ,
\]
(2.27)
where \( |\vec{y} - \vec{x}| \) is the usual Euclidean distance between two points. Notice that in our construction the closed contour \( C' \) is a framing contour of \( C \), and hence the above linking number is the so-called self-linking number of \( C \) \[20\],
\[
\text{sLink}(C) \equiv \text{Link}(C, C')_{\epsilon \to 0} ,
\]
(2.28)
\[
C \equiv \{ x^i(t) \} ,
\]
\[
C' \equiv \{ y^i(t) = x^i(t) + \epsilon n^i(t) \) ; \( \epsilon > 0 , |n^i| = 1 \} ,
\]
where \( n^i \) is a vector field orthogonal to \( C \). The integral (2.28) is well defined and finite, takes integer values and equals the number of windings of \( C' \) around \( C \).

Putting the above formulae in (2.24), we find
\[
\hat{M}(C)|A > = |A + D\varphi_C > = \quad (2.29)
\]
\[
\simeq \text{Tr}\{ \mathbb{1}(1 + 2ikc_2(t)s\text{Link}(C))\}|A > ,
\]
where
\[
\delta^{ab}T_a T_b = c_2(t)\mathbb{1} ,
\]
(2.30)
and
\[ c_2(t) = \frac{N^2 - 1}{2N} \] (2.31)
for the fundamental representation of \( SU(N) \). Eq. (2.29) implies that \( \hat{M}(C) \) generates an infinitesimal multivalued gauge transformation \( \varphi_C(\vec{x}) \). Whenever \( \vec{x} \equiv \{ x^i \} \) traces a framing contour \( C' \) of the closed curve \( C \) that winds \( n \equiv s\text{Link}(C) \) times around \( C \), \( \hat{M}(C) \) creates a magnetic flux \[ \Phi_C \equiv \frac{k(N^2 - 1)}{gN} s\text{Link}(C) \] . (2.32)

At this order of approximation the finite multivalued gauge transformation \( \Omega_C[\vec{x}] \) generated by the action of \( \hat{M}(C) \) over some state functional is given by
\[ \hat{M}(C)|A(\vec{x})> = |\Omega_C^{-1}[\vec{x}](A(\vec{x}) + id\vec{x})\Omega_C[\vec{x}]> \simeq \text{Tr}\{e^{ig\Phi_C} \mathbb{1}\}|A(\vec{x})> . \] (2.33)

Because of the multi-valued nature of \( \Omega_C[\vec{x}] \), one has that
\[ \Omega_C[\vec{x}_f] = Z_{\Phi_C} \Omega_C[\vec{x}_i] \] , (2.34)
where \( x^i_f \equiv x^i(t = 1) = x^i(t = 0) \equiv x^i_1 \) is the base point of the loop \( C \) parametrized by \( C \equiv \{ x^i(t) : 0 \leq t \leq 1 \} \) and \( Z_{\Phi_C} \) is
\[ Z_{\Phi_C} = e^{(ig\Phi_C)} \mathbb{1} = e^{ik(N^2 - 1)/N s\text{Link}(C)} \mathbb{1} \] . (2.35)

Since \( A^{\Omega_C} \equiv \Omega_C^{-1}(A + d)\Omega_C \) should always be single valued, \( Z_{\Phi_C} \) must be in the center of \( SU(N) \). To recover the standard form of the center, we normalize the free expansion parameter as \( k = 2\pi/(N^2 - 1) \) and require a special framing for \( C \), namely that \( n = s\text{Link}(C) \in [0, \ldots, N-1] \). With these normalizations the form of the color magnetic flux is given by
\[ \Phi_C = \frac{2\pi n}{N g} \] . (2.36)

Given the properties of \( \hat{M}(C) \), its commutation relations with the Wilson line operator \( \hat{W}(C_1) \) are easily deduced \[3, 7, 21\]:
\[ \hat{M}(C_1)\hat{W}(C_2) = e^{ig\text{Link}(C_1,C_2)\Phi_C} \hat{W}(C_1)\hat{M}(C_2) \] . (2.37)
The previous equation is the well known ‘t Hooft algebra. The non trivial commutation relation between the operators $\hat{M}$ and $\hat{W}$ set their duality correspondence in the sense of the order and disorder operators in statistical mechanics.

3 Computation of $< M(\Sigma, C) >$

The purpose of this section is to compute the average of the BF-observable $M(\Sigma, C)$ defined in Sec. 3. Namely we define the dual loop functional by the normalized connected expectation value

$$< M(C) >_{\text{conn}} \equiv \frac{< M(C) >}{<1>},$$

(3.1)

where

$$M(C) \equiv M(\Sigma, C) \equiv \text{Tr}[\exp\{2\pi i q g \int_{\Sigma} d^2 y \text{Hol}_{\Sigma}^y(\gamma) B(y) \text{Hol}_{\Sigma}^y(\gamma')\}] ,$$

(3.2)

$$<1> \equiv \int \mathcal{D}A \mathcal{D}B \exp\{-\int \text{Tr}[iB \wedge F + \frac{g^2}{4} B \wedge \ast B]\} .$$

(3.3)

With respect to the notation employed in the previous section, we have defined the expansion parameter $k$ in units of the bare color charge $g$ with a suitable normalization: $k = 2\pi q g. $

Due to its $SU(N)$ gauge invariance, (3.1) must be gauge fixed. Following Ref.[6] we choose the abelian projection gauge. Before using this gauge we would like to make some comments. Magnetic monopoles are believed to play a major role in confinement. They appear as classical solutions of the Yang-Mills theory in the Georgi-Glashow model coupled with scalar fields transforming in the adjoint representation. Their classical mass is calculable, but turns out to be too big to allow for a magnetic Higgs mechanism. The Abelian projection gauge is supposed to generate monopoles of low or zero mass without introducing the scalar Higgs field. As monopoles transform as $U(1)$ gauge fields, the idea is to isolate the $U(1)^{N-1}$ degrees of freedom from those transforming in the coset $SU(N)/U(1)^{N-1}$ with an appropriate partial gauge fixing leaving the $U(1)^{N-1}$ maximal

\[^4\text{For the sake of generality we do not ask any flux quantization and, as a consequence, any fixed } k \text{ value. After gauge fixing and a saddle point evaluation of the functional integral this quantization will naturally emerge.}\]
abelian torus unbroken. The residual $U(1)^{N-1}$ gauge invariance does not protect charged degrees of freedom from getting a mass. Since confinement is a large scale phenomenon, in the crudest of approximations, these degrees of freedom can be discarded. Therefore the effective theory should contain only “photons” and monopoles.

To implement this gauge we need a microscopic field or a composite of it, $X$, to transform in the adjoint representation of the gauge group $SU(N)$ so that a gauge transformation can diagonalize it,

$$\tilde{X} = g^{-1}Xg = \text{diag}(\lambda_1, \ldots, \lambda_N) .$$

(3.4)

The eigenvalues of such matrix can be naturally ordered $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. When two eigenvalues coincide we have a line of singularities representing the monopole world-line. By analogy with the interpolating gauges used in massive Yang-Mills, having chosen (for example) $X = F_{12}$ and $Y_{ij} = (F_{12})_{ij}/(\lambda_i - \lambda_j)$, $i \neq j = 1, \ldots, N$, the proposed gauge condition is

$$Y^{ch} + \xi D^0 * A^{ch} = 0 ,$$

(3.5)

where the superscripts $ch$, $0$ stand for the off-diagonal and diagonal part of the matrix $A = A^0 + A^{ch}$ and $D^0$ is the covariant derivative with respect to the diagonal part of the gauge field. Interpolating gauges are such that for short distances the relevant gauge is $D^0 * A^{ch} = 0$ and the theory is renormalizable, while for large distances the gauge is $Y^{ch} = 0$. For this reason the field $Y$ needs to be adimensional and $\xi$ has the dimension of the inverse of a mass squared. In standard YM theory, $X$ can only be a composite of the $F_{\mu \nu}$ (and its covariant derivatives) in order to transform according to the adjoint representation. But as the dependence of $F_{\mu \nu}$ on the momentum $p_\mu$ is the same as the second term in the r.h.s. of (3.3) the dominance argument is only possible if $X$ is made adimensional. This makes this gauge difficult to apply to standard quantum Yang-Mills theory because $Y$ is non-polynomial outside the tube around the monopole string and divergent inside it.

---

5 This mass does not have to be physical and it can also be an infrared regulator. The decoupling of charged degrees of freedom is insensitive to the nature of the mass.
Quite remarkably, these problems are not present in the $BF - YM$ theory due to the presence of the microscopic $B$ field. The interpolating gauge can now be easily implemented choosing, for instance, $X = B_{12}$:

$$B_{12}^{\text{ch}} + D^0 * A^\text{ch} = 0$$

(3.6)

Equivalently in our formalism we can diagonalize the two-form $B$ on the surface $\Sigma$

$$\tilde{B} = V^{-1} B V = \text{diag}(\beta_1, ..., \beta_N), \quad \beta_1 \geq \beta_2 \geq ... \geq \beta_N$$

$$\tilde{A} = V^{-1} (A + d)V, \quad \tilde{F} = V^{-1} F V$$

(3.7)

and then use the background gauge condition $D^0 * A^\text{ch} = 0$ in the renormalization program.

According to the previous observations, in the computation of the expectation value of the observable $M(C, \Sigma)$, employing the abelian projection gauge, in the large scale region we may neglect the massive off-diagonal degrees of freedom $B^{\text{ch}}, A^{\text{ch}}$. This approximation is often called Abelian dominance ([6, 8]).

We would like now to interpret this approximation from a different point of view which will turn out to be very useful in the following. The existence of monopoles in the Abelian projection gauge is due to the compactness of the $U(1)^{N-1}$ group and it is related to the existence of non trivial topological objects for the entire $SU(N)$ theory. So, in this contest, Abelian dominance can also be seen as the prescription of having reducible gauge connection which, as we will find later, will also be singular. In the simple $SU(2)$ case this leads to a connection

$$A = \frac{1}{2} \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$$

(3.8)

The reducibility of the $SU(2)$ gauge connection, implies that the gauge bundle is split and thus requires the existence of a positive definite first Pontrjagin class and intersection number (which we will define later on). This fact, in its turn, implies the absence of anti-self-dual harmonic (closed) two forms [23]. Later on in this chapter, we will comment on the possible relation between this split bundle and the moduli space of instantonic classical solutions.

In the general $SU(N)$ case we now rewrite the functional integral in terms of the
variables
\[ \alpha_i = [A^0]_{ii}, \quad i = 1, \ldots, N, \quad \sum_i \alpha_i = 0, \quad (3.9) \]
and
\[ f_i \equiv [F^0]_{ii} = d[A^0]_{ii}, \quad i, j = 1, \ldots, N. \quad (3.10) \]

It is then convenient to rescale the previous geometrical fields \( A^0, B^0 \) to the physical ones \( A^0 \to gA^0, B^0 \to \frac{1}{g}B^0 \). Furthermore, we replace the surface integral in Eq. (3.2) with the integration over the so-called Poincarè dual form, \( \omega_\Sigma \), of (the homology class of) the surface \( \Sigma \) \([7\]\. By definition, \( \omega_\Sigma \) is closed. Moreover, choosing an orientation of the four manifold and remembering the absence of anti-self-dual harmonic two forms (which is imposed by requiring the existence of a gauge connection of the type of (3.8)) \( \omega_\Sigma \) is chosen to be self-dual \( i.e. \) \( \ast \omega_\Sigma = \omega_\Sigma \) and with the property that (up to gauge one-forms)

for a generic two-form \( t \)
\[ \int_\Sigma t \simeq \int_{M^4} \omega_\Sigma \land t. \quad (3.11) \]

In a local system of coordinates \((x, y, u, v)\) on \( \Sigma \), so that \( \Sigma \) is given by the equations \( x = y = 0 \), the dual form \( \omega_\Sigma \) can be taken as \( \omega_\Sigma \simeq \delta^{(2)}(x, y)dx \land dy \) and normalized as
\[ \int_{N(\Sigma)} \omega_\Sigma \simeq \int \delta^{(2)}(x, y)dx \land dy = 1, \quad (3.12) \]
where \( N(\Sigma) \) is the transversal tubular neighbourhood on the surface \( \Sigma \).

Let us now compute the average of the ‘t Hooft loop operator \( M(C) \) in the above abelian projection scheme.

The functional integrations over the \( \alpha_i \)'s should be constrained by \( \sum_i \alpha_i = 0 \). In order to perform the calculations without imposing the previous constraint, we extend the \( SU(N) \) gauge group of the BF-YM theory to \( U(N) \) \([24\]\. More precisely we identify the fields \( \alpha_i \) with the Cartan generators of \( U(N) \). Then we can recover the original \( SU(N) \) BF-YM theory by gauging the spurious \( U(1) \) symmetry, generated by \( \alpha_N \), associated to the standard embedding \( SU(N) \to U(N)/SU(N) = U(1) \). Finally the abelian gauge fields \( \alpha_1, \ldots, \alpha_{N-1} \) associated to the Cartan subalgebra of \( SU(N) \) should be gauged to remove the remaining \( U(1) \) gauge degrees of freedom.
Using the Abelian dominance approximation, the 't Hooft loop operator $M(C)$ becomes
\begin{equation}
M(C) = \text{Tr}\{O_{ij}(C)\} \quad ,
\end{equation}
\begin{equation}
O_{ij}(C) = \delta_{ij} \exp\{i2\pi q \int_{M^4} \omega_{\Sigma} \wedge \beta_j [\cos(g \oint_C \alpha_j) + i \sin(g \oint_C \alpha_j)]\} \quad .
\end{equation}

As we have already observed, the gauge transformation in (3.7) is singular [6], and the singularities occur if two consecutive eigenvalues of $B$ coincide. If $\beta_i = \beta_{i+1}$, we shall label such a point by $x^{(i)}$. Moreover $\beta_i$ is a two-form and hence its support is a two-cycle in the four manifold. Thus, in order to avoid the previous singularities $x^{(i)}$, we must remove a ball $S^2_r(x^{(i)})$ of radius $\epsilon \to 0^+$ at each point $x^{(i)}$ of the base manifold. As a consequence of that, we get a non trivial Stokes theorem ($g\alpha_i$ is the geometrical $U(1)$ connection):
\begin{equation}
g \oint_C \alpha_i = g \int_S : \partial_S = C d\alpha_i = g \int_{S^2_r(x^{(i)})} d\alpha_i \equiv 2\pi g q_i
\end{equation}
(S = $S' \cup S^2_r(x^{(i)}))$, since the two-cycle $S$ is not homotopically trivial (it contains $S^2_r(x^{(i)})$).

In (3.14) the magnetic charge is given by the first Chern class, $c_1(L)$, of the line bundle $L$ with fiber $S^2_r(x^{(i)})$. Since we have in principle $(N-1)$ connections $U(1)$ whose curvatures are supported near the points $x^{(i)}$ of $\Sigma$ ($x^{(i)}$ is identified with $x^{(N)}$) and the total first Chern class is an integer ($c_1(L \otimes (N-1)) = (N-1)c_1(L) \in \mathbb{Z}$), we obtain that $gg_i$ satisfies the Dirac quantization condition rule
\begin{equation}
g q_i = \frac{n_i}{N-1} \quad , \quad n_i = \pm 1, \pm 2, \ldots \quad , \quad \forall i \in [1, N] \quad .
\end{equation}

In other words the appropriate homotopy group is $\Pi_2[SU(N)/U(1)^{N-1}] = \mathbb{Z}^{N-1}$, as expected.

In the following we shall show that the effective field equations associated to the strong coupling limit of $\Gamma(C) \equiv - \ln < \tilde{M}(C) >_{\text{conn}}$ admit as critical points magnetic monopole strings - which we shall construct explicitely - with winding numbers $n_i \in \mathbb{Z}$ at the singular points $x^{(i)}$. The key point will be the identification of the arbitrary expansion parameter $q$ in (3.2) with the magnetic charges $q_i$, i.e. we will set
\begin{equation}
q_i \propto q \propto \frac{1}{g} \quad .
\end{equation}
We start now the computation of the magnetic order parameter (3.1) in the $q \to 0$, which will be identified by (3.16) as the strong coupling limit $g \to \infty$. In this limit the operator (3.13) can be approximated by

$$M(C) = \sum_i O_{ii}(C) \simeq N \exp \left\{ \frac{i2\pi q}{N} \int \omega_\Sigma \wedge \sum_i \beta_i \cos(g \oint C \alpha_i) \right\} .$$  (3.17)

Eq. (3.17) is obtained by taking into account that, according to our previous observations, there are two types of monopole configurations $\alpha_i$ for each $i$, depending on the sign of the magnetic charges $q_i$. Indeed we have that $\{q_i\} \equiv \{q_i^s\} \equiv \{q_i^+ > 0, q_i^- < 0\}$. Hence it is resonable to define the sum over the field configurations as $\sum_i \equiv \sum_{i,s}$, where $i \in [1, N]$ and $s = \pm$. Notice that $< M(C) >_{conm}$ in the strong coupling limit is the partition function of $N$ compact QED’s or 4D Villain models (e.g. see [4]).

With all these approximations taken into account the form of the magnetic order parameter in the strong coupling region becomes

$$< M(C) >_{conm} \simeq N \int D\alpha_i D\beta_i \exp \left\{ -\frac{1}{2} \sum_i \left( \frac{1}{4} \beta_i \wedge * \beta_i + i \beta_i \wedge [d\alpha_i + \frac{4\pi q}{N} \omega_\Sigma \cos(g \oint C \bar{\alpha}_i)] \right) \right\} .$$  (3.18)

where the square of a form $t$ means $t \wedge *t$. We now split the gauge fields as $\alpha_i = \bar{\alpha}_i + \hbar Q_i$, where the quantum fluctuations $Q_i$ must be gauged (e.g. by a covariant gauge condition) and the $\bar{\alpha}_i$ are singular classical configurations. Postponing for a while the discussion of quantum fluctuations, we concentrate on the semi-classical contribution to the path integral which is

$$< M(C) >_{conm} \simeq N \exp \left\{ -\frac{8\pi^2 q^2}{N^2} \sum_i \int \omega_\Sigma \wedge \omega_\Sigma' \cos(g \oint C \bar{\alpha}_i) \cos(g \oint C' \bar{\alpha}_i') \right\} .$$  (3.19)

In the derivation of the above equation, we have exploited the following facts:

• partial integration is not allowed on the $\bar{\alpha}_i$ due to their singular behaviour [24],

6We note that the appearence of the effective expansion parameter $q/N$ in (3.17) suggests that the same conclusions of this section can also be obtained in the large $N$-limit (holding $q$ fixed) in which the $SU(N)$ and the $U(N)$ gauge theory are the same [23]. Thus, in the $q \to 0$ limit, we can consistently drop the constraint $\sum_i \alpha_i = 0$.  

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\[ * d\tilde{\alpha}_i = \frac{4\pi q}{N} \omega_\Sigma \cos(g \oint_C \tilde{\alpha}_i), \quad (3.20) \]

- \( \tilde{\alpha}_i \) obeys the (monopole) equation

- the absence of electric currents in the model,

- the self-duality and closedness of \( \omega_\Sigma \). The above properties imply the absence of terms linear in \( Q_i \).

Equations of the type \((3.20)\), appeared already in the study of the duality properties of gauge theories and 4D manifold invariants \([26, 27]\).

The r.h.s. of \((3.20)\) can be written in terms of spinor fields for a direct comparison with Ref.\([27]\). If \( M^4 \) is a spin manifold any self dual two-form \( B^+ \), satisfying the conditions (in spinorial notation) \( B^+_{AB \wedge B^+_{CD}} = 0 \) and \( B^+_{AB} \wedge B^+_{AB} \neq 0 \), can be represented in the form \( (B^+)_{AB} = \frac{1}{2}(M_A \bar{M}_B + M_B \bar{M}_A) \), where \( M \) is a two component spinor field. Furthermore \( B^+_{\mu\nu} \eta_{\mu\nu} \frac{1}{\sqrt{2}} \sigma^a_{AB} = B^+_{AB} \), where \( \eta \) are the \('t\)Hooft symbols and \( \sigma \) the Pauli matrices. In terms of the space-time indices \( B^+_{\mu\nu} = \bar{M} \Gamma_{\mu\nu} M \), with \( \Gamma_{\mu\nu} = \frac{1}{2} [\gamma_\mu, \gamma_\nu] \). Using the equation of motion for the \( B \) field, \( M \) is seen to obey the Dirac equation relative to the case of manifolds which are Kähler and spin as discussed in section four of Ref.\([27]\). This connection with Ref.\([27]\), in which the infrared limit of the supersymmetric \( N = 2 \) theory is described in terms of abelian gauge fields and monopoles, should not be surprising. In fact, if we take the attitude (that will be developed in the next section) that the term \( g^2/4B \wedge *B \) is a perturbation of the “pure BF” theory \((1.6)\), this perturbation theory will be performed around the vacuum of the theory which coincides with that of \((1.6)\). This requires to gauge fix the local symmetries \((2.3), (1.3)\) of the action \((1.6)\) by introducing appropriate ghosts \([10]\).

We would now like to explain in which respect the pure BF theory (with action \((1.6)\)) deserves to be called “topological”. In first place the action \((1.6)\) is metric independent. Moreover, after some work, a BRST charge, \( s \) (nilpotent on-shell) and a vector charge, \( \delta_\mu \), operator can be defined for the quantized theory \([10]\). The commutator \( \{s, \delta_\mu\} \) closes on translations on-shell and this shows that the stress energy tensor of the theory can be written as a BRST commutator. Introducing external sources coupled to the BRST
variations of the microscopic fields a new linearized BRST charge, $\Omega$ (nilpotent off-shell), can be defined. Thus the BF theory is proved to be “cohomological” \[29\] with respect to the $\Omega$ operator but it is not clearly equivalent to the topological theory defined in \[28\] since the BRST charges of the two theories are different.

Independently from these considerations, the symmetries \[2.5\] must be anyway gauge fixed, which can be done using a covariant gauge for the field $B$.

We now look for an explicit solution of \[3.20\] under the constraint $\cos (g \oint C \bar{\alpha}_i) = 1$ which is consistent with the Dirac quantization rule \[3.15\] if we set $n_i = m(N - 1)$, $m \in \mathbb{Z}$, $q_i = \frac{2}{N} q$. We then get

$$q g = \frac{m N}{2}. \quad (3.21)$$

Given an arbitrary and topologically trivial surface $\Sigma$ the dual Poincaré form can be locally written as

$$\omega^{\Sigma}_{\mu \nu} (x) = \int_{\Sigma} d\sigma_{\mu \nu} (y) \delta^{(4)} (x - y), \quad (3.22)$$

$$d\sigma_{\mu \nu} (y) = \varepsilon^{a b} \frac{\partial y^a (\xi)}{\partial \xi^a} \frac{\partial y^b (\xi)}{\partial \xi^b} d^2 \xi = \varepsilon^{a b} t_\mu^a t_\nu^b d^2 \xi, \quad (3.22)$$

where $\xi_a$ are the coordinates on the surface, $t_\mu^a$ are the normal vectors of the bidimensional surface $\Sigma$ and $a = 1, 2$. Putting \[3.22\] in \[3.20\] we find the classical background solution \[31, 32\]

$$\bar{\alpha}_i^\mu (y) = 4 q_i \int_{\Sigma} d^2 \xi P_+^{\mu \nu a \beta} \varepsilon^{a b} t_\mu^a t_\nu^b \frac{1}{|y - x^{(i)}|^2}, \quad (3.23)$$

where $P^+ = (1 + \ast)/2$ and, as before, $x^{(i)}$ is the location of the singularity at the intersection of the surface $\Sigma$ with the $i$-th monopole world-line and $\Sigma$ locally appears as the “Dirac sheet” \[3\]. The curve $C$ of \[3.20\] clearly winds around $x^{(i)}$. Eq. \[3.23\] is a magnetic vortex-line or Abrikosov-Nielsen-Olesen (ANO) string for each $i \in [1, N]$ \[1\], obtained here without the use of the Higgs fields, but introducing the disorder effect of the $M(C)$ operator. \[7\] We stress that these configurations do not correspond to ‘t Hooft-Polyakov like monopoles but are rather singular Dirac monopoles \[33\].

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\[7\] In Ref.\[30, 1\] the same solution is found in the abelian Higgs model with $\bar{\alpha}_i^\mu = q_i \partial_\mu \chi$, and the Higgs field is parametrized as $\varphi = |\varphi| \exp (i \chi)$.

In Ref.\[32\] these kind of configurations are shown to describe massless monopoles in 4D compact QED, giving rise to a planar model like thermodynamics.
reduction of the solution (3.23) (obtained fixing the surface \(\Sigma\)) coincides with eq.(31) of Ref.[34] (near the singularity) which is a solution of the dimensionally reduced monopole equations of Ref.[27].

Finally, the classical contribution is

\[
S_0 = \frac{2\pi^2 m^2}{g^2} \int \sum_i \omega_\Sigma \wedge \omega_{\Sigma'} \cos(g \oint_C \bar{\alpha}_i) \cos(g \oint_{C'} \bar{\alpha}'_i) \equiv \frac{2\pi^2 m^2 N}{g^2} Q(\Sigma, \Sigma') .
\]  

(3.24)

For (3.24) to make sense, the two cosines must be different from zero which implies that the paths \(C, C'\) must encircle monopole flux tubes. There are now two options:

- \( g \oint_C \bar{\alpha}_i \neq g \oint_{C'} \bar{\alpha}'_i \)
- \( g \oint_C \bar{\alpha}_i = g \oint_{C'} \bar{\alpha}'_i \).

In the first case, \(Q\) is the intersection number \([35]\)

\[
Q(\Sigma, \Sigma') = \int_{M^4} \omega_{\Sigma[C]} \wedge \omega_{\Sigma'[C']} ,
\]  

(3.25)

which is an integer (it is a four dimensional topological invariant) characterizing the self-intersection of \(\Sigma\) with itself. This number is given once the surface \(\Sigma\) and the ambient space \(M^4\) are assigned.

In the second case the closed curve \(C' \equiv \{y^\mu(t)\}\) is a framing contour of the closed curve \(C \equiv \{x^\mu(t)\}\), i.e. if it happens that

\[
y^\mu(t) = x^\mu(t) + \epsilon n^\mu(t) , \quad \epsilon \to 0 , \quad |n^\mu| = 1 ,
\]  

(3.26)

where \(n^\mu(t)\) is a vector field orthogonal to \(C\). In this case \(Q(\Sigma, \Sigma')\) becomes the self-linking number of \(C\) given by (2.28).

Then we get that

\[
Q(\Sigma, \Sigma') = \text{sLink}(C) .
\]  

(3.27)

From a physical point of view, we may define \(\text{sLink}(C)\) in (3.27) as

\[
\text{sLink}(C) \equiv \frac{L(C)}{\rho} ,
\]  

(3.28)
where $\rho$ plays the role of unit of length and $L(C)$ is the perimeter of the loop $C$. We may understand (3.28) by a $n$-vertex polygon discretization $C \rightarrow C_{PL}$ of the loop $C$. Indeed by the very definition of the self-linking number one gets that $\text{sLink}(C_{PL}) = n = L(C_{PL})/l$, where $l$ is the lattice spacing. (3.28) is obtained in the continuum limit $n \rightarrow \infty, l \rightarrow 0$, since in this limit $L(C_{PL})/l \sim L(C)/\rho$. Thus, in this picture, the perimeter law comes out from the arbitrary framing dependence of the path $C$ in the color magnetic operator $M(C, \Sigma)$. In turn, the framing dependence can be seen as a point splitting regularization due to the non-local nature of the operator. In our construction $\text{sLink}(C)$ in 4D cannot be slinked - in general the linking between two arbitrary curves in 4D has no geometrical meaning - since $C$ and $C'$ belong to the same surface.

Let us now discuss quantum fluctuations. If the effective theory for large length scales is a $U(1)$ type theory, for short length scales the charged degrees of freedom cannot be discarded anymore. Let $\Lambda$ be the length scale separating these two regimes and let us divide the gauge field according to a background field prescription: $A^a = \bar{A}^a + Q^a$ where $a$ is a gauge index. Moreover the gauge field can be written as the sum of two parts giving the contribution of the large length scale sector ($> \Lambda$) and the short length scale sector ($< \Lambda$); the functional integration over the gauge field factorizes in a way compatible with this separation. Moreover let, for the sake of simplicity, the gauge group be $SU(2)$. For length scales bigger than $\Lambda$ we take $A^3 = \bar{\alpha} + Q^3$ which is the usual $U(1)$ prescription. For length scales smaller than $\Lambda$ we take $A^a = \delta^{a3}\bar{\alpha} + Q^a$, i.e. we continue the classical solution into the small scales region where the quantum fluctuations coming from the charged degrees of freedom cannot be discarded. The expectation is that the small scales behaviour is insensitive to the classical solution according to the background field method. Performing the functional integration over the quantum fluctuations leads to a double contribution, in complete analogy with the saddle point evaluation around an instanton background [36].

- The first contribution is given by a ratio of determinants given by

$$
\left[ \frac{\text{Det}'(-L_0)}{\text{Det}'(-L)} \right]^{1/2} \left[ \frac{\text{Det}(-\bar{D}^2)}{\text{Det}(-\partial^2)} \right], \quad (3.29)
$$
where $\bar{D} = d - ig\bar{\alpha}$, $\bar{D}^2 \equiv \bar{D}^\mu \bar{D}_\mu$,

\[
L = \bar{D}^2 \delta_{\mu\nu} - (1 - \frac{1}{\xi})\bar{D}_\nu \bar{D}\nu + 2igF^0_{\mu\nu}(x) ,
\]

and $L_0$ is given from $L$ evaluated around the trivial background. $\xi$ is a gauge parameter usually chosen to be one. In (3.29) the determinants are primed to remind the reader that the contribution of zero modes is omitted and that the determinants are regularized.

- The second contribution is given by the Pauli-Villars regularization of the determinants and it amounts to a scale $\mu$ (which is the Pauli-Villars mass) raised to a certain power which is given by the dimension of the moduli space of the classical solution.

Let us now proceed with the evaluation of these two contributions. Using the self-duality property of our classical solution, the ratio of determinants (3.29) can be written as $R = \left[Det(-\partial^2)/Det(-\bar{D}^2)\right]_{30}$. This ratio has been evaluated in Ref.[37] using the heat kernel method in the case of an $SU(N)$ gauge group but it is easy to generalize this result to our case too. We now give a brief outline of this computation to justify the previous statement.

To be mathematically well-defined, the split connection (3.8) must be considered either to be regular in $\mathbb{R}^4$ or to be singular in $S^4$. In our case we found explicitly a singular $U(1)$ connection and we can go from one point of view to the other by removing the singularity locus spanned by the $x^{(i)}$ in (3.23). Indeed, by removing a disk from $S^4$ we in fact obtain $\mathbb{R}^4$.

The computation we want now to perform is intended to probe the region around the singularity by evaluating the operator $R$ obtained by expanding around the classical solution given in (3.20). This computation is carried on a space-time bounded by the length scale $\Lambda$ which can be taken to be the compact manifold $S^4$ after a conformal transformation, to allow a proper definition of the eigenvalue problem and to avoid infrared problems.

The starting point of the computation is the introduction of the Riemann zeta function $\zeta(s)$ built out of the eigenvalues of the operator $-\bar{D}^2$. The logarithm of the ratio of the
determinants of the self-adjoint operators $R$ is given by the variation, under conformal
transformations, of the derivative of the Riemann zeta function $\zeta'(0)$ continued to zero
value of its argument. In turn this function is given by the residue of
\begin{align}
2 \int_0^\infty t^{s-1} \text{Tr}[G(x, x, t) \delta \omega] dt
\end{align}
at $s = 0$ [37]. In (3.31), $\delta \omega$ is an infinitesimal conformal transformation and
\begin{align}
G(x, y, t) = \frac{1}{16\pi^2 t^2} e^{-\frac{(x-y)^2}{4t}} \sum_{n=0}^{\infty} a_n(x, y)t^n
\end{align}
satisfies the heat equation.

Making use of (3.32), (3.31) can be cast in the form
\begin{align}
R = \zeta'(0) = \frac{1}{8\pi^2} \int \text{Tr}_a(x, x) \delta \omega d^4x
\end{align}

Plugging the expansion (3.32) in the heat-kernel differential equation we obtain a set of
recursion relations from which the values of the coefficients $a_n$ are easily extracted. Since
the only manipulation needed in solving such recursion relations is the commutation of
covariant derivatives, the final result is easily generalized to any gauge group
\begin{align}
\ln \frac{\text{Det}(-\partial^2)}{\text{Det}(-D^2)} = \frac{\ln(\mu\Lambda)}{96\pi^2} \int F_{\mu\nu}^0(x) F_{\mu\nu}^0(x) d^4x = \frac{\ln(\mu\Lambda)}{96\pi^2} \frac{1}{4} \int (d\tilde{\alpha})_{\mu\nu}(x) (d\tilde{\alpha})_{\mu\nu}(x) d^4x
\end{align}
The factor 1/4 comes from the normalization of the gauge group generators according to
(3.8).

The contribution coming from the regularization of the zero modes is obtained once
the dimension of the moduli space is computed, according to Ref. [27], to be
\begin{align}
\text{dim} \mathcal{M} = c_1(L)^2 = c_1(L) \wedge c_1(L) = \frac{1}{32\pi^2} \int (d\tilde{\alpha})_{\mu\nu}(x) (d\tilde{\alpha})_{\mu\nu}(x) d^4x
\end{align}

Putting together the classical result (3.24) with the quantum fluctuations, we find that
the bare coupling $g$ can be substituted by its renormalized expression and that (3.19) can
be written as
\begin{align}
\frac{8\pi^2 c_1(L)^2}{g_R^2} = \frac{c_1(L)^2}{8} \left( (8\pi^2 \frac{2}{3}) \ln(\mu\Lambda) \right) \equiv \frac{c_1(L)^2}{8} \left( \frac{8\pi^2}{g^2} - \beta_1 \ln(\mu\Lambda) \right)
\end{align}
where $\beta_1 = \frac{22}{3}$ is the first coefficient of the $SU(2)$ beta function of the non-abelian Yang-
Mills theory.
4 The Average of the Wilson Loop

In this section we shall compute the average of the Wilson loop and find an area law behaviour for its leading part. Furthermore, in our formalism, the area law gets a nice geometrical interpretation: it is the response of the true QCD vacuum to arbitrary deformations of the quark loop $C$.

The starting point here is given by the Wilson loop operator written in terms of the non abelian Stokes theorem (see e.g. [13]):

$$W_R(C) \equiv W_R(\Sigma, C) = Tr_R\{ P_\Sigma \exp[i \int_\Sigma \text{Hol}_x^a(\gamma) F(x) \text{Hol}_x^a(\gamma')] \} \ , \quad (4.1)$$

where $C = \partial \Sigma$, $C = \{ \gamma(x) \cup \gamma'(x) \}$ was defined at the beginning of section 2 and $P_\Sigma$ means surface path ordering. $W_R$ is calculated with respect to some irreducible representation $R \equiv t$ of $SU(N)$. The loop average in the fundamental representation $R \equiv t$ of $SU(N)$ is

$$< W_t(\Sigma, C) >_{conn} \equiv < W_t(\Sigma, C) > < 1 > \equiv \frac{\int DBDA W_t(\Sigma, C)e^{-S_{BF-YM}}}{\int DBDA e^{-S_{BF-YM}}} \ , \quad (4.2)$$

where $S_{BF-YM}$ was defined in (1.1).

Expanding in series the Wilson loop (4.1) we get

$$< W_t(\Sigma, C) > = \sum_n < \frac{1}{n!} Tr_t P_\Sigma \int_{\Sigma_1} \ldots \int_{\Sigma_n} (i \text{Hol}(\gamma)F(x) \text{Hol}^{-1}(\gamma'))^n > \ . \quad (4.3)$$

We then use the identity

$$e^{-\frac{i}{4} \int *B_{\mu\nu}^a F_{\mu\nu}^a(x)} = 4i \frac{\delta}{\delta * B_{\rho\sigma}^a(x)} (e^{-\frac{i}{4} \int *B_{\mu\nu}^a F_{\mu\nu}^a}) \ . \quad (4.4)$$

Performing a partial integration with respect to the functional derivative in (4.4) we can replace, in the path integral,

$$i \text{Hol}(\gamma)e^{-\frac{i}{4} \int *B_{\mu\nu}^a F(x) \text{Hol}^{-1}(\gamma')} \rightarrow 4 \text{Hol}(\gamma)e^{-\frac{i}{4} \int *B_{\mu\nu}^a \left( \frac{\delta}{\delta * B(x)} \right) \text{Hol}^{-1}(\gamma')} \ . \quad (4.5)$$

The functional derivative acts only to its right on the exponential of the mass term $-g^2/16 \int B_{\mu\nu}^a B_{\mu\nu}^a$, since Hol does not contain the $B$ field. We now need the identity

$$V(\frac{\delta}{\delta * B^a(x)_\rho\sigma}) e^{-\frac{g^2}{8} \int B_{\mu\nu}^a B_{\mu\nu}^a} = V(\frac{g^2}{8} * B_{\rho\sigma}^a) e^{-\frac{g^2}{8} \int B_{\mu\nu}^a B_{\mu\nu}^a} \ , \quad (4.6)$$
where $V$ is the functional defined by

$$V(\frac{\delta}{\delta * B^a(x)}) \equiv P\Sigma \int_{\Sigma_1} \ldots \int_{\Sigma_n} (4\text{Hol}(\gamma) (\frac{\delta}{\delta * B^a(x)}) \text{Hol}^{-1}(\gamma'))^n .$$

(4.7)

Resumming the exponential series for the Wilson loop we finally get the “duality” relation

$$<W_t(\mathcal{C})>_{conn} = \frac{<M_t^*(\Sigma, \mathcal{C} = \partial \Sigma)>}{<1>} ,$$

(4.8)

where

$$M_t^*(\Sigma, \mathcal{C} = \partial \Sigma) = Tr[\text{P}_{\Sigma} \exp\{-\frac{g^2}{2} \int_{\Sigma} \text{Hol}_{\Sigma}^x(\gamma) * B(x) \text{Hol}_{\Sigma}^x(\gamma')\}] .$$

(4.9)

$M_t^*(\Sigma, \mathcal{C})$ is the dual (in the sense that $B \rightarrow *B$) of the observable $M_t(\Sigma, \mathcal{C})$ defined in (2.1) with $k$ set to $k = ig^2/2$.

To calculate (4.8) we expand perturbatively in $g$ both the exponential and the holonomies which appear in the exponent of (4.9) [40]. The first relevant contraction encountered at lower level is given in terms of $<A * B>$, which can be computed starting from the off-diagonal propagator $<AB>$. The latter propagator is the same for the BF-YM theory [12] and the pure BF (1.6). Therefore we find

$$<M_t^*(\mathcal{C})>_{conn} = e^{-\frac{g^2}{2} c_2(t) \int_{\Sigma} <A * B> \Delta(\Sigma)} ,$$

(4.10)

where $c_2(t)$ is given in (2.31) and $\Delta(\Sigma)$ depends on higher order integrations over $\Sigma$. In the limit $g \rightarrow 0$ (1.1) reduces to (1.6) which is a topological theory. Making the perturbative expansion in $g$ around the topological theory, $\Delta(\Sigma)$ in (4.10) is found to be a diffeomorphism invariant quantity, i.e. $\Delta(\Sigma)$ depends only on the knotting properties of $\Sigma$ and does not depend on the linking properties of $\Sigma$ (at least when our ambient space-time is $S^4$). If the dynamical surface $\Sigma$ created by the quantum fluctuations of the true QCD vacuum is the unknotted surface $\Sigma_0$, one should find that

$$\Sigma = \Sigma_0 \rightarrow \Delta(\Sigma_0) = 1 .$$

(4.11)

The explicit calculation of $\Delta(\Sigma)$ is an open problem, but for the purpose of showing the area law behaviour its knowledge is not essential.
Consider now the integral in the exponent of (4.10),
\[
\oint_C \int_{\Sigma} \mathcal{A}^\dagger \mathcal{B} = \langle \oint_C A \int_{\Sigma} \mathcal{A}^\dagger \mathcal{B} \rangle_{\mu\nu} = \oint_C \int_{\Sigma} (\ast d\sigma)^{\mu\nu}(x) \mathcal{A}^\dagger (y) \mathcal{B} \rangle_{\mu\nu},
\]
where $\ast d\sigma(x)$ is the infinitesimal surface element of the plane $\Sigma_x^*$ dual to $\Sigma$ at the point $x \in \Sigma$. We may rewrite
\[
\oint_C \int_{\Sigma} \mathcal{A}^\dagger \mathcal{B} = \langle \oint_C A \int_{\Sigma} \mathcal{A}^\dagger \mathcal{B} \rangle_{\mu\nu} = \oint_C \int_{\Sigma} (\ast d\sigma)^{\mu\nu}(x) \mathcal{A}^\dagger (y) \mathcal{B} \rangle_{\mu\nu},
\]
repeating that $\omega_{\Sigma}$ is locally given by a bump function with support on $\Sigma$. Eq. (4.13) is by definition the linking number between the curve $C$ and the dual plan $\Sigma_x^*$ in $x$ to $\Sigma$. Indeed the linking $\text{Link}(C, \Sigma)$, with arbitrary $C$ and $\Sigma$, is defined by
\[
\text{Link}(C, \Sigma) = \frac{1}{8\pi^2} \oint_C dx^a \int_{\Sigma} d\sigma^\beta \gamma(y) \epsilon_{\alpha\beta\gamma \delta} \frac{(x - y)^\delta}{|x - y|^4} = 4 \oint_C dx^a \int_{\Sigma} d\sigma^\beta \gamma(y) \langle \mathcal{A}^\alpha(x) \mathcal{B}^{\beta \gamma}(y) \rangle.
\]
In our case, by construction, $\text{Link}(C, \Sigma_x^*) \neq 0$. The residual integration over $\Sigma$ in (4.13) spans all the dual $\Sigma^*$ to $\Sigma$, yielding a contribution proportional to the area of $\Sigma$,
\[
\int_{\Sigma} \text{Link}(C, \Sigma_x^*) \sim A(\Sigma),
\]
where $A(\Sigma)$ is the minimal area bounded by the quark loop $C$ in the fundamental representation and $a$ is the lattice spacing. Passing to the continuum limit $a \to 0$, $N_P = A(\Sigma_{PL})/a^2$ becomes $A(\Sigma)/l^2$ with $l$ a typical scale in QCD which may be choosen as $\Lambda_{QCD}^{-1}$. Therefore one may rewrite (4.13) as
\[
\langle W_t(C) \rangle_{\text{conn}} \sim \exp\left[-g^2(N^2 - 1) \frac{1}{16N} \int_{\Sigma} \text{Link}(C, \Sigma_x^*) \right],
\]
where $A(\Sigma_{PL})$ is the minimal area bounded by the quark loop $C$ in the fundamental representation and $a$ is the lattice spacing. Passing to the continuum limit $a \to 0$, $N_P = A(\Sigma_{PL})/a^2$ becomes $A(\Sigma)/l^2$ with $l$ a typical scale in QCD which may be choosen as $\Lambda_{QCD}^{-1}$. Therefore one may rewrite (4.13) as
\[
\langle W_t(C) \rangle_{\text{conn}} \sim \exp\left[-\sigma(\Lambda_{QCD}) A(\Sigma) \right],
\]
25
where $\sigma(\Lambda_{QCD})$ is the string tension defined by

$$\sigma(\Lambda_{QCD}) \equiv g_R^2 \left( \frac{N^2 - 1}{16N} \right) \Lambda_{QCD}^2 ,$$

(4.18)

with $\Lambda_{QCD} \equiv 1/l$. Here we have replaced, as required by the renormalization of the theory, the bare coupling constant $g$ with the running coupling constant $g_R$ at a QCD energy scale $\Lambda_{QCD}$. Of course we may change the physical scale, but it is well known that this corresponds to a different choice of the renormalization scheme, the difference being only a finite renormalization, leading to the same physical results.

Existing data corresponding to energy scales between 4 and 100(Gev)$^2$ can be fitted with $\Lambda_{QCD} \sim 0.5$ Gev, corresponding to $[g_R^2(\Lambda_{QCD})]/4\pi$ between 0.2 and 0.4 for $N_f = 4$. Inserting these data in (4.18) we find that a rough estimate of the physical value of the string tension can be given by

$$\sqrt{\sigma}(\Lambda_{QCD} \sim 0.5\text{Gev}) \sim 316 - 440 \text{MeV} ,$$

(4.19)

which is of the same order of magnitude of the experimental value.

Before concluding this chapter let us comment on the result we have found. It might seem puzzling that the area law behaviour comes out of a perturbative calculation. In reality our perturbative expansion is an expansion around the gauge fixed topological BF theory whose vacuum contains instantons. It is thus resonable to imagine that the non-perturbative information comes out of the non-trivial vacuum structure of the quantum BF theory. Furthermore we remark that the computations of this section could also have been done along the lines of the chapter three, given the similarity between $M(\Sigma, C)$ and $M^*(\Sigma, C)$. As a consequence the invariant $Q(\Sigma, \Sigma')$ in (3.27) is replaced by Link($\Sigma^*$, $C$), in agreement with (4.13), and abelian dominance translates into (4.11), i.e. lack of interesting knotting properties of $\Sigma$.

## 5 Conclusions

In this paper we have shown that in the framework of the BF formulation of Yang-Mills theory, ’t Hooft views on confinement can be explicitly realized in the continuum formul-
lation. This has been possible because a non-abelian color magnetic operator has been identified quite naturally in terms of a non local observable (2.1) constructed out of the microscopic fields $A, B$. Then we have computed the average of this operator in terms of a saddle point approximation having used the abelian projection gauge. The implementation of this gauge has consisted in performing the computation in terms of the large scale effective lagrangian obtained neglecting massive charged degrees of freedom. This point clearly awaits further clarification and an estimate of the contribution of the discarded sector remains to be evaluated. However, we do not believe that the perimeter law behaviour in (3.34), coming from the evaluation of the classical action, can be disrupted by quantum corrections.

In any case the ratio of two operators of the type (2.1) with different paths $C, C'$ still exhibits the perimeter law since any possible divergence comes from shrinking to zero the size of the magnetic flux tube (ANO string).

In the framework we have investigated, the perimeter law stems from the breaking of the BF topological theory (1.6) due to the $g^2 \text{Tr}(B \wedge \ast B)/4$ term in (1.1). In [40] the expectation value of $M(C, \Sigma)$ was computed in the pure BF theory looking for two-knot invariants i.e. for four dimensional topological invariants of knotting of surfaces. In that computation a classical action piece like (3.24) did not appear because of the missing $\text{Tr}(B \wedge \ast B)$ term (which allows for gaussian integration) and only functions of the four dimensional topological invariant (3.25) were obtained. It is thus reasonable that in the BF Yang-Mills appears a homotopy invariant of the curve $C$ as sLink($C$) which represent the effect of the winding of $C$ around the monopole flux tube (which we can think closes at infinity). The curve $C$ in (2.1) is connected to the holonomy of the gauge connection and contains the disks $S^2(x_i) \subset \Sigma$. These disks become the observable Dirac surfaces of [5] and are responsible for the perimeter law. An argument similar has been developed in [43] for a BF Yang-Mills theory in two dimensions where the two form $B$ becomes a scalar.

Let us now remark on some properties of (3.34) which agree with common wisdom: the appearence of $g_R$ has been confirmed in lattice simulations of $SU(N)$ QCD [38]. The
“photons” $Q_i$ interacting with a plasma of condensed monopoles reproduce the same $SU(N)$ running coupling constant required by the renormalization group. From (3.24) we have a perimeter law behaviour for $m \neq 0$ and finite $N$ as expected for the confining phase of QCD in the absence of massless particles.

As far as the average of the Wilson loop is concerned, (4.15) is the basic result. Indeed it implies that the area law for the Wilson loop average is a consequence of the response of the QCD vacuum to arbitrary path deformations of the loop $C$. This property is evident in our formalism, where the loop $C = \partial \Sigma$ may be understood as the deformation of the loop $C$. The existence of $\text{Link}(\Sigma^*, C)$ means that the expectation value $< W_R(C) >$ is not invariant, as reasonably expected, under the change of the path-prescription $C$ in the non-abelian Stokes theorem given by (4.1). In other words, (4.15) is a non-perturbative Ward identity associated to the path deformation of the static quark loop $C$. It is well-known that while the fundamental representation of the gauge group leads to confinement, the adjoint representation does not. This does not show up in (4.15) due to presence of the coupling $g$ which should be absorbed by the presence of a mass gap (from monopole condensation) in the propagator for the $A$ and $B$ field.

Finally, observe that the results obtained in this paper are due to the non-abelian nature of the model. If we had started with a pure abelian BF-YM theory, the operator $M$ would have been defined without the Hol terms and, therefore, without the possibility to have a non-trivial monopole equation (3.20). In the abelian case an additional Higgs field is needed in order to have a non-trivial magnetic variable $M$.

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