ASSOCIATIVITY AND THE COSMASH PRODUCT
IN OPERADIC VARIETIES OF ALGEBRAS

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Abstract. In this article, we characterise the operadic variety of commutative associative algebras over a field via a (categorical) condition: the associativity of the so-called cosmash product. This condition, which is closely related to commutator theory, is quite strong: for example, groups do not satisfy it. However, in the case of commutative associative algebras, the cosmash product is nothing more than the tensor product; which explains why in this case it is associative. We prove that in the setting of operadic varieties of algebras over a field, it is the only example. Further examples in the non-operadic case are also discussed.

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1. Introduction

The original question we set out to answer at the onset of this work was to characterise, by means of some universal construction, when the objects in a variety of algebras over a field $K$ have a multiplication that is associative. Note that what we here call an algebra is a $K$-vector space $A$ equipped with a bilinear multiplication $· : A \times A \to A$, which is not necessarily associative or unitary. We let $\text{Alg}_K$ denote the category of such $K$-algebras, where morphisms are linear maps that preserve the multiplication. A variety of $K$-algebras is an equational class of algebras over $K$: any subvariety of $\text{Alg}_K$, as for instance the variety of associative algebras, which satisfy the identity $x(yz) = (xy)z$, or the variety of Lie algebras, which satisfy the Jacobi identity $(x(yz) + y(zx) + z(xy) = 0)$ and anti-commutativity $(xy = -yx)$.

In previous work, characterisations of the variety $\text{Lie}_K$ of Lie algebras over an infinite field $K$ (such that $\text{char}(K) \neq 2$) as a subvariety of $\text{Alg}_K$ were obtained, in essentially two different ways: in [10, 11], it is shown that $\text{Lie}_K$ is the only non-abelian locally algebraically cartesian closed [12] subvariety of $\text{Alg}_K$; and in [9], the variety $\text{Lie}_K$ is shown to be the only subvariety of $\text{Alg}_K$ whose actions are representable [3]. So we have two independent categorical descriptions of the variety $\text{Lie}_K$, and thus of the Jacobi identity (however, only for anti-commutative algebras).

This naturally led to our present question: How to characterise associative algebras? The answer we found—and this is the subject of our present article—is a categorical description of the variety of commutative associative algebras over $K$ amongst all so-called operadic varieties over $K$, where a set of multilinear identities suffices to describe the algebras in the variety. As it turns out, the variety of commutative associative algebras over $K$ is the only operadic variety whose so-called cosmash products are not just symmetric, but also associative in the sense of Carboni and Janelidze—see Theorem 9.2. Thus, associativity of an algebra (a “microcosmic” property at object level) is characterised by associativity on the categorical level (a “macrocosmic” property). This is an instance of what seems to be a common phenomenon, comparable, for instance, with the definition of monoids in a monoidal category (which, however, fits a slightly different pattern in that there, a correspondence between two types of structure is apparent).

1.1. The cosmash product. In [5], A. Carboni and G. Janelidze extend the definition of the classical smash product from pointed topological spaces to pointed objects in suitable categories. On the way, they study a condition they call smash associativity, which is basically the requirement that any ternary smash product $X \land Y \land Z$ is canonically isomorphic to either one of the repeated binary smash products $(X \land Y) \land Z$ and $X \land (Y \land Z)$. Depending on the surrounding category, this may or may not happen.

In the present article, working in the context of algebras over a field, we interest ourselves in the dual notion: cosmash associativity. Indeed, binary cosmash products are well-known to be useful in the development of commutator theory in general algebraic contexts [23, 14, 13], essentially because they may be seen as “objects of formal commutators”. For example, in the case of groups, the binary cosmash product $X \circ Y$ of two groups $X$ and $Y$ is generated by commutator words of the form $xyx^{-1}y^{-1}$ (see Section 3). The binary definition of the cosmash product is the $n = 2$ case of a more general $n$-ary definition [5, 14].
In general, iterating the binary cosmash product does not give the desired higher-dimensional notion of n-fold cosmash product. For example, letting $X$, $Y$, $Z$ be three objects in a (suitable) category, we write $X \bowtie Y \bowtie Z$ for the ternary cosmash product ($n = 3$) of $X$, $Y$ and $Z$. This object need not be isomorphic to either $(X \bowtie Y) \bowtie Z$ or $X \bowtie (Y \bowtie Z)$—in Section 5 we explain why this isomorphism almost never holds for groups or Lie algebras. Cosmash associativity is the condition that these three objects are canonically isomorphic. The goal of this paper is to investigate this categorical condition in the context of varieties of algebras over a field. Our hunch was that it must be a relatively strong condition, since even Lie algebras do not satisfy it. In fact, our main results, which are Theorem 8.14 and Theorem 9.2, say that this condition is so strong that, in the context of homogeneous varieties of non-associative algebras, it characterises when the multiplication is both commutative and associative.

1.2. Structure of the text. The paper is organised as follows. Section 2 gives a short survey of the mathematical context: non-associative algebras and their varieties. Special attention goes to homogeneous and operadic varieties of algebras over a field, since our main theorems appear in those precise settings. Section 3 defines cosmash products in their binary, ternary and general form, and gives an overview of some basic properties and examples. Section 4 recalls how cosmash products are related to (Higgins) commutators. Section 5 discusses the definition of cosmash associativity: certain canonical comparison morphisms are isomorphisms. We study examples and have a first look at some computational techniques we use later on. Section 6 recalls that in the variety of commutative associative algebras, the cosmash product is the tensor product; it follows that this variety is cosmash associative.

In order to properly understand the statements of our two main theorems, reading Sections 2, 3 and 5 should be sufficient. The details of their proofs and the techniques involved are explained in Sections 7, 8 and 9.

Section 7 looks at the consequences of surjectivity of the comparison maps. Proposition 7.1 shows that it is related to the validity of certain degree-three equations, such as for instance the equation $(yx)z = \beta_1 x(yz) + \beta_2 x(zy) + \beta_3 (yz)x + \beta_4 (zy)x$ which allows us to “pull the $x$ out of the parentheses”. Section 8 is devoted to the injectivity of the comparison maps. First we give examples of varieties where these are not injective. We then use computer algebra to prove in Proposition 8.12 that a homogeneous variety which is cosmash associative should satisfy an identity of degree less than or equal to 2. From this we may deduce our first main result: Theorem 8.14, which says that if a homogeneous variety of $K$-algebras is cosmash associative, then it is a subvariety of the variety of commutative associative $K$-algebras. Section 9 interprets this in the particular setting of operadic varieties (for example, when the characteristic of the field is zero). In this context, we understand with Theorem 9.2 that in a non-abelian cosmash associative operadic variety, no non-trivial identities can be added to commutativity and associativity. In other words, the variety of commutative associative algebras is characterised by cosmash associativity. Section 10 is an attempt to extend our results to the case of non-operadic varieties. We study some examples of varieties which could not be considered in the setting of the main theorems. The final Section 11 discusses some ideas for further development of the theory.
2. (Operadic) Varieties of Algebras over a Field

In this section, we introduce the algebraic setting where our main results are expressed: the context of varieties of non-associative \( \mathbb{k} \)-algebras. Those are basically collections of algebras over \( \mathbb{k} \) satisfying a chosen set of equations of a certain type, i.e., an equational class. Unless otherwise stated, \( \mathbb{k} \) will denote a field. Some of our results hold when \( \mathbb{k} \) is an integral domain, or even just a commutative ring with unit.

Let us recall some well-known concepts in order to present things in a consistent way. The interested reader may find a more exhaustive presentation of this material in \cite{28}. By a \textbf{(non-associative) \( \mathbb{k} \)-algebra} we mean a \( \mathbb{k} \)-vector space equipped with a bilinear operation called the \textit{multiplication}. We do not assume the existence of a unit element, or place any assumptions on the multiplication besides bilinearity.

We let \( \text{Alg}_\mathbb{k} \) denote the category of not necessarily associative \( \mathbb{k} \)-algebras. In this category, the morphisms are the \( \mathbb{k} \)-linear maps which preserve the multiplication. In \cite{30}, this type of algebra is called a \textit{magma algebra}.

We consider the functor \( \text{Set} \to \text{Alg}_\mathbb{k} \) which associates to a set \( X \) the free \( \mathbb{k} \)-algebra generated by elements of \( X \). This functor is left adjoint to the forgetful functor and factorises through the free magma functor \( M: \text{Set} \to \text{Mag} \) and the \textit{magma algebra functor} \( \mathbb{k}[-]: \text{Mag} \to \text{Alg}_\mathbb{k} \). In particular, for a set \( X \), \( M(X) \) is the magma of non-empty (non-associative) words in \( X \), which are sometimes represented by means of \textit{binary trees} (which are rooted and proper, with leaves labelled by elements in \( X \)).

By a \textbf{non-associative polynomial} \( \varphi \) on a set \( X \) we mean an element of \( \mathbb{k}[M(X)] \). Such a polynomial is said to be a \textbf{monomial} if it is a scalar multiple of an element in \( M(X) \). For example, if \( X = \{x, y, z\} \), then \( xy + yx \), \( x^2 + yz \) and \( x(yz) \) are polynomials in \( X \) and only the last one is a monomial. The \textbf{type} of a monomial \( \varphi \) on a set \( \{x_1, \ldots, x_n\} \) is an \( n \)-tuple \( (k_1, \ldots, k_n) \in \mathbb{N}^n \) where \( k_i \) is the number of times \( x_i \) appears in \( \varphi \). The \textbf{degree} of \( \varphi \) is \( k_1 + \cdots + k_n \). A polynomial is said to be \textbf{homogeneous} if all its monomials are of the same type. A homogeneous polynomial is said to be \textbf{multilinear} if it is of type \((1, \ldots, 1)\).

**Definition 2.1.** A non-associative polynomial \( \varphi = \varphi(x_1, \ldots, x_n) \) is called an \textit{identity} of an algebra \( A \) if \( \varphi(a_1, \ldots, a_n) = 0 \) for all \( a_1, \ldots, a_n \in A \). We also say that \( A \) \textbf{satisfies} the identity \( \varphi \).

Let \( X \) be a set of variables and \( I \) be a subset of \( \mathbb{k}[M(X)] \). The class of all \( \mathbb{k} \)-algebras that satisfy all identities in \( I \) is called the \textbf{variety of \( \mathbb{k} \)-algebras} determined by \( I \). Given such an \( I \), we say that a variety \textbf{satisfies the identities in} \( I \) if all algebras in this variety satisfy the given identities. In particular, if the variety is determined by a set of homogeneous polynomials, then we say that the variety is \textbf{homogeneous}, and if the variety is determined by a set of multilinear polynomials, then we say that it is \textbf{operadic}.

For the sake of simplicity, we may sometimes not write an identity as a polynomial but rather as an equality. For example, for commutativity, in order to express that the polynomial \( xy - yx \) is an identity, we may write that \( xy = yx \) (or \( xy - yx = 0 \)) is satisfied in the algebra under consideration. This abuse of terminology will make the reading easier.

**Remark 2.2.** If the characteristic of the field \( \mathbb{k} \) is zero, then by a multilinearisation process, any variety of non-associative algebras over \( \mathbb{k} \) is operadic (see \cite{24} Corollary 3.7]). For instance, the identity \( xx = 0 \) for alternating algebras implies
algebras are anti-commutative; in characteristic zero, this multilinear identity determines the class of alternating algebras, which may thus be viewed as an operadic variety.

The reason we call the varieties determined by multilinear polynomials “operadic” lies in Proposition 5.7.3 of the reference book [20] about algebraic operads in Proposition 5.7.3.

Similarly, we say that any type of algebras (over a field of characteristic 0) in the sense of operads whose identities are multilinear always determines an operad whose algebras form the given variety.

In some sense, “operadic” simply means “expressible in a (symmetric) monoidal category”, as is the viewpoint in the categorical references [21, 19] on this subject. Following their terminology, our varieties determined by multilinear polynomials might instead be called PROP varieties of $K$-algebras.

**Examples 2.3.** A $K$-algebra $A$ is said to be **associative** if the polynomial $x(yz) - (xy)z$ is an identity of $A$, or equivalently if $a(bc) = (ab)c$ for all $a, b, c \in A$. Similarly, we say that $A$ is **anti-associative** if it satisfies the identity $x(yz) + (xy)z$.

**Commutative** if $xy - yx = 0$, **anti-commutative** if $xy + yx = 0$, **abelian** if $xy = 0$ and **trivial** if $x = 0$.

The variety of commutative associative $K$-algebras will be denoted by $\text{CA}_K$. We write $\text{Anti}_K$ for the variety of anti-associative and anti-commutative $K$-algebras. The subvariety $\text{Ab}_K$ of $\text{Alg}_K$ determined by the abelian algebras is isomorphic to the variety $\text{Vect}_K$ of $K$-vector spaces. An algebra is trivial if and only if it is a singleton; the variety of such algebras is denoted $\text{Triv}_K$.

We write $\text{Lie}_K$ for the variety of $\text{Lie algebras}$ over $K$, which are anti-commutative and satisfy the **Jacobi identity** $x(yz) + yzx + zxy$.

All the above-mentioned examples are operadic varieties; let us give some non-operadic examples. The non-homogeneous variety $\text{Bool}_K$ of Boolean $K$-algebras is the variety determined by associativity, commutativity and the identity $xx - x$. For a field $K$ of prime characteristic $p$ we denote by $\text{CA}_K^p$ the homogeneous variety of commutative associative $K$-algebras satisfying the additional identity $x^p = 0$. Non-operadic varieties are the subject of Section 4.

**Remark 2.4.** Since a variety of non-associative algebras is always a variety of $\Omega$-groups in the sense of [18], it is a semi-abelian category [18]. This will be useful in Section 10.

The commutative and anti-commutative cases are of critical importance, because of the following lemma:

**Lemma 2.5.** Let $K$ be a field. If $V$ is a variety of $K$-algebras satisfying a non-trivial homogeneous identity of degree 2, then it must satisfy at least one of the identities $xy - yx, xy + yx$ or $xy$. In particular, if $V$ is non-abelian, then it is either a subvariety of the variety of commutative algebras, or a subvariety of the variety of anti-commutative algebras.

**Proof.** A homogenous identity of degree 2 is necessarily an element $\varphi(x, y)$ of the free $K$-algebra on two generators $x$ and $y$, so a polynomial of the form

$$\lambda_1xx + \lambda_2xy + \lambda_3yx + \lambda_4yy.$$ 

Homogeneity implies that either $\lambda_1xx = 0$ or $\lambda_2xy + \lambda_3yx = 0$. Note that the former implies the latter. If now either $\lambda_2 = 0$ or $\lambda_3 = 0$, then we see that $yx = 0$ or $xy = 0$.
and $\mathcal{V}$ is abelian. Otherwise, we write $\lambda = -\lambda_3/\lambda_2$ and $xy = -\lambda yx = \lambda^2 xy$, so that $(1 - \lambda^2)xy = 0$. Hence either $\mathcal{V}$ is abelian or $\lambda \in \{-1, 1\}$.

3. Cosmash products

In order to define cosmash associativity, which is the central concept of this paper, in Section 4 we need to explain what is a cosmash product. In this section, we start with the binary cosmash product, ask ourselves how we can extend it to a ternary one, and conclude that those two are specific cases (for $n \in \{2, 3\}$) of an $n$-ary definition. In Section 4 we shall see how, in the context of a so-called semi-abelian category, this leads to a categorical approach to the Higgins commutator. In this section, and throughout the rest of the paper unless otherwise stated, we work in a pointed category ($0 = 1$) with finite sums, finite products, and kernels.

**Notation 3.1.** Let $\mathcal{C}$ be a pointed category with finite sums and finite products and $X, Y$ two objects of $\mathcal{C}$. Then there exists a canonical morphism

$$
\Sigma_{X,Y} := \begin{pmatrix} 1_Y & 0 \\ 0 & 1_Y \end{pmatrix} : X + Y \to X \times Y.
$$

**Definition 3.2.** The **binary cosmash product** $X \circ Y$ of two objects $X$ and $Y$ in a pointed category with finite sums, finite products, and kernels is defined as the kernel

$$
0 \longrightarrow X \circ Y \xrightarrow{i_{X,Y}} X + Y \xrightarrow{\Sigma_{X,Y}} X \times Y
$$

of the morphism $\Sigma_{X,Y}$.

This definition was first given (dually) by Carboni and Janelidze in [5] where they define the (binary) smash product as the cokernel of $\Sigma_{X,Y}$. Independently, in [23], Mantovani and Metere used the binary cosmash product as a so-called **formal commutator** in a categorical approach to the Higgins commutator of [16]. Later, in [14], Hartl and Van der Linden recovered it as a special case of a cross-effect, using higher cosmash products—i.e., $n$-ary cosmashes for $n \geq 2$, see Definition 3.9—to capture the higher commutators of [10] as in Section 4 below. See also [2] for a complementary point of view.

**Examples 3.3.**

1. If the category $\mathcal{C}$ is additive, then any cosmash product $X \circ Y$ is trivial, because each comparison morphism $\Sigma_{X,Y}$ is an isomorphism. (We shall not worry here about potential non-existence of kernels in additive categories, since we are only ever taking kernels of monomorphisms.)

2. In the category $\text{Gp}$ of groups, it is well known that the cosmash product of two objects $X$ and $Y$ is the subgroup of $X + Y$ generated by words of the form $xyx^{-1}y^{-1}$ and $yxy^{-1}x^{-1}$, where $x \in X$ and $y \in Y$. We recall the argument: any element $w$ of in $X + Y$ can be written as

$$
w = x_1y_1 \cdots x_ny_n
$$

where $x_i \in X$ and $y_i \in Y$ for all $i \in \{1, \ldots, n\}$. If now $w$ is $x, y, xy, yx, yx'$ or $yxy'$, then its image under $\Sigma_{X,Y}$ is respectively $(x, 1)$, $(1, y)$, $(x, y)$, $(xx', y)$ or $(x, yy')$. For such a $w$, belonging to the kernel $X \circ Y$ of $\Sigma_{X,Y}$ simply means that $w$ is trivial. If, however, $w = xyx'y'$ or $w = yxy'x'$, then $\Sigma_{X,Y}$ sends it to $(xx', yy')$, so that it belongs to the kernel of $\Sigma_{X,Y}$ precisely when $x' = x^{-1}$ and $y' = y^{-1}$. Now, a generic word $w = x_1y_1 \cdots x_ny_n$ of length $2n$ can be rewritten as

$$
w = (x_1y_1x_1^{-1}y_1^{-1})y_1x_1x_2y_2 \cdots x_ny_n,$$
where \( x_1x_2 \) is another element \( x' \) of \( X \). We can observe that the word \( y_1x'y' \cdots x_ny_n \) is of length \( 2n - 1 \). Therefore, by an induction argument, we may write
\[
w = (x_1y_1x_1^{-1}y_1^{-1})(y_1x'_1y_1^{-1}x'^{-1}_1) \cdots (x'^n_ny_nx_ny_n)
\]
which is sent to \((x''_n,y''_n)\) by \( \Sigma_{X,Y} \). Hence if \( x \in X \circ Y \), then \( x'' = x_n^{-1} \) and \( y'' = y_n^{-1} \), so that \( w \) is a product of elements of the form \( xyx^{-1}y^{-1} \) and \( yxy^{-1}x^{-1} \) with \( x \in X \) and \( y \in Y \).

(3) In a variety \( V \) over a field \( \mathbb{k} \), it is easy to see that for two algebras \( X \) and \( Y \) in \( V \), the coproduct \( X + Y \) is the set of (equivalence classes with respect to the identities of \( V \) and those of \( X \) and \( Y \)) of the (non-associative) polynomials with variables in \( X \) and \( Y \). Hence the cosmash product \( X \circ Y \) is the subalgebra of \( X + Y \) determined by such polynomials where each monomial contains at least one variable in \( X \) and one in \( Y \). Indeed, those that are not exactly polynomials in the elements of \( X \) and \( Y \) but \( x^2y + y^2 \) is not.

(4) Section \( 6 \) explains why in the variety \( \mathcal{CA} \), the cosmash product \( X \circ Y \) is the tensor product \( X \otimes Y \). (This is not to be confused with the well-known fact that in the category of unitary commutative associative algebras, the coproduct is already the tensor product: see, for instance, [22].)

Next, we want to extend our definitions to \( n \)-ary cosmash products for any natural number \( n \geq 2 \). A first idea might be to define the ternary cosmash product of three objects \( X \), \( Y \), \( Z \) in \( \mathcal{C} \) by iterating the binary cosmash product, which gives us \( X \circ (Y \circ Z) \) or \((X \circ Y) \circ Z\). The problem with this approach is that those objects are not isomorphic in general, as shown for instance in Examples \( 7 \) and Examples \( 8 \). Here, we may already notice that in the category \( \text{Alg}_K \), for some \( x \in X \), \( y \in Y \) and \( z \in Z \), the element \( x(yz) \) lies in \( X \circ (Y \circ Z) \), yet it cannot be seen as an element of \((X \circ Y) \circ Z\), since \( x \) is outside the brackets. Actually, the elements of \( X \circ (Y \circ Z) \) are not exactly polynomials in the elements of \( X \), \( Y \) and \( Z \), but rather polynomials with variables in \( X \) and in \( Y \circ Z \).

In general, there is not even a canonical morphism from \( X \circ (Y \circ Z) \) to \((X \circ Y) \circ Z\). The idea is now to consider an unbiased ternary cosmash product \( X \circ Y \circ Z \) which may act as a “common receptacle” for both, as in the diagram

\[
\begin{array}{ccc}
X \circ Y \circ Z & \xrightarrow{\Phi_{X,Y,Z}} & (X \circ Y) \circ Z \\
\Phi_{X,Y,Z} & \xrightarrow{\Psi_{X,Y,Z}} & X \circ (Y \circ Z)
\end{array}
\]

Asking that both \( \Phi_{X,Y,Z} \) and \( \Psi_{X,Y,Z} \) are isomorphisms then provides us with a way to express that \( X \circ (Y \circ Z) \) and \((X \circ Y) \circ Z\) (and \( X \circ Y \circ Z \)) are isomorphic.

A first idea towards a definition of a ternary cosmash product \( X \circ Y \circ Z \) might be to take the kernel of the morphism
\[
\begin{pmatrix}
1_x & 0 & 0 \\
0 & 1_y & 0 \\
0 & 0 & 1_z
\end{pmatrix} : X + Y + Z \to X \times Y \times Z.
\]
However, in the category of groups, for example, the word \( xyx^{-1}y^{-1} \) will be an element of this kernel. This does not agree with what a commutator word in \( X \), \( Y \) and \( Z \) is supposed to be, since it contains no elements of \( Z \). Instead, we follow the approach of [5] [14] [2]:
Proof. The upper row in Figure 1 represents \( W \cong X \circ Y \circ Z \) as cross-effect.

**Definition 3.4.** Let \( X, Y \) and \( Z \) be three objects of \( \mathcal{C} \) and consider the canonical morphism

\[
\Sigma_{X,Y,Z} := \begin{pmatrix} \iota_X & \iota_Y & 0 \\ 0 & 0 & \iota_Z \end{pmatrix} : X + Y + Z \to (X + Y) \times (X + Z) \times (Y + Z).
\]

The **ternary cosmash product** \( X \circ Y \circ Z \) of \( X, Y \) and \( Z \) is defined as the kernel of \( \Sigma_{X,Y,Z} \). The canonical inclusion is denoted \( \iota_{X,Y,Z} : X \circ Y \circ Z \to X + Y + Z \).

With this definition, in the case of groups we cannot have elements such as \( x y z^{-1} \) in the ternary cosmash product. However, we have elements such as \( x y z^{-1} x y z^{-1} \) in \( X \circ Y \circ Z \) for some \( x \in X, y \in Y \) and \( z \in Z \). In the category \( \text{Alg}_{\mathcal{K}} \), elements of \( X \circ Y \circ Z \) will be polynomials where each monomial contains variables in \( X, Y \) and \( Z \). The same holds for any subvariety of \( \text{Alg}_{\mathcal{K}} \). In any additive category, the ternary cosmash product is the zero object, since there \( \Sigma_{X,Y,Z} \) is a monomorphism.

3.5. **Another point of view.** The following well-known lemma will be useful for us; its proof follows immediately from the universal properties involved. It allows us to obtain the alternative construction of the ternary cosmash product of Lemma 3.7.

**Lemma 3.6.** For some \( n \geq 2 \), consider objects \( Y_1, \ldots, Y_n \) and a morphism \( (x_1, \ldots, x_n) : X \to Y_1 \times \cdots \times Y_n \). Then the kernel of \( (x_1, \ldots, x_n) \) is the intersection \( \bigwedge_{i=1}^{n} \ker(x_i) \) of the kernels of all the \( x_i \).

**Lemma 3.7.** The ternary cosmash product can be obtained out of the binary cosmash as a so-called cross-effect [11][13]. In fact, \( X \circ Y \circ Z \) is a kernel of

\[
\begin{pmatrix} \langle 1_X,0 \rangle \otimes Z \\ \langle 0,1_Y \rangle \otimes Z \end{pmatrix} : (X \circ Y) \circ Z \to (X \circ Z) \times (Y \circ Z).
\]

**Proof.** The upper row in Figure 1 represents \( W \) as a kernel of the morphism \( \begin{pmatrix} \langle 1_X,0 \rangle \otimes Z \\ \langle 0,1_Y \rangle \otimes Z \end{pmatrix} \); indeed, by Lemma 3.6 the object \( W \) is the intersection of the kernels of \( \langle 1_X, \iota_Y, 0 \rangle, \langle \iota_X, 0, \iota_Z \rangle, \langle 0, \iota_Y, \iota_Z \rangle \) and \( \langle 0, 0, 1_Z \rangle \), which is the same as the kernel of \( \Sigma_{X,Y,Z} \) because the morphism \( \langle 0, 0, 1_Z \rangle \) factors through \( \langle 1_X, \iota_Y, 0 \rangle \). ![a diagram](https://via.placeholder.com/150)

See also Definition 3.11 and Remark 5.5 below. This viewpoint will be helpful in Section 6 when we calculate the cosmash product in the variety \( \text{CA}_{\mathcal{K}} \), and also in Section 8 where it will allow us to obtain Lemma 8.8.
3.8. Higher cosmash products. The binary and ternary cosmash products are instances of a general definition:

**Definition 3.9.** [5][14] For \( n \geq 2 \), let \( X_1, \ldots, X_n \) be objects, and consider the canonical morphism

\[
\Sigma_{X_1, \ldots, X_n} : X_1 + \cdots + X_n \to \prod_{k=1}^{n} \prod_{j \neq k} X_j
\]
determined by

\[
\pi_{1 \neq k} X_j \circ \Sigma_{X_1, \ldots, X_n} \circ \iota_{X_i} = \begin{cases} \iota_{X_i} & \text{if } l \neq k \\ 0 & \text{if } l = k. \end{cases}
\]

The \( n \)-ary cosmash product \( X_1 \circ \cdots \circ X_n \) of \( X_1, \ldots, X_n \) is the kernel of \( \Sigma_{X_1, \ldots, X_n} \). We write \( \iota_{X_1, \ldots, X_n} : X_1 \circ \cdots \circ X_n \to X_1 + \cdots + X_n \) for the canonical inclusion.

**Remark 3.10.** Using that products and coproducts are symmetric, it is easy to see that also Definition 3.9 is symmetric in the variables \( X_i \) and thus, for any permutation \( \sigma \in \mathfrak{S}_n \), we have that \( X_1 \circ \cdots \circ X_n \cong X_{\sigma(1)} \circ \cdots \circ X_{\sigma(n)} \). In some sense, it is precisely this fundamental feature of the cosmash product which will limit the scope of our main result to the context of (anti-)commutative algebras.

Also higher-order cosmash products admit a cross-effect interpretation (as in Lemma 3.7). In order to be sufficiently precise for the use we make of this in Section 8 we recall the following definition from [14].

**Definition 3.11.** Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor from a pointed category with finite sums \( \mathcal{C} \) to a pointed category with finite products and kernels \( \mathcal{D} \). The **second cross-effect of** \( F \) is the functor \( \text{Cr}_2(F) : \mathcal{C} \times \mathcal{C} \to \mathcal{D} \) defined as follows: it sends a pair of objects \( (X, Y) \) in \( \mathcal{C} \times \mathcal{C} \) to the object \( \text{Cr}_2(F)(X, Y) \) which is the kernel of

\[
(F((1_X, 0_Y))) : F(X + Y) \to F(X) \times F(Y)
\]

— which extends to morphisms in the obvious manner.

**Examples 3.12.** We let \( \mathcal{C} \) be a pointed category with finite coproducts, products and kernels.

1. It is immediately clear that when \( F = 1_\mathcal{C} \) we regain Definition 3.2 so that \( X \circ Y = \text{Cr}_2(1_\mathcal{C})(X, Y) \).
2. Lemma 3.7 tells us that \( X \circ Y \circ Z \cong \text{Cr}_2((-) \circ Z)(X, Y) \); in fact, the functor \( \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) which sends a couple of objects \( (X, Y) \) to \( X \circ Y \circ Z \) is the second cross-effect \( \text{Cr}_2((-) \circ Z) \) of the functor \((-) \circ Z : \mathcal{C} \to \mathcal{C} \). By symmetry, also \( X \circ Y \circ Z \cong \text{Cr}_2(X \circ (\cdot))(Y, Z) \).
3. Lemma 2.20 in the first arXiv version of [14] implies that for any \( X, Y, Z \) and \( W \),

\[
\text{Cr}_2(X \circ Y \circ (-))(Z, W) \cong X \circ Y \circ Z \circ W \cong \text{Cr}_2((-) \circ Z \circ W)(X, Y).
\]

The proof is a straightforward variation on Lemma 3.7; see [27] for further details.

**Remark 3.13.** Note that the cross-effect operation \( \text{Cr}_2 \) may be seen as a functor \( \text{Fun}(\mathcal{C}, \mathcal{D}) \to \text{Fun}(\mathcal{C}^2, \mathcal{D}) \) from \( \text{Fun}(\mathcal{C}, \mathcal{D}) \), the category of functors from \( \mathcal{C} \) to \( \mathcal{D} \), to \( \text{Fun}(\mathcal{C}^2, \mathcal{D}) \), the category of functors from \( \mathcal{C}^2 \) to \( \mathcal{D} \). It follows immediately from its construction in Definition 3.11 that \( \text{Cr}_2 \) preserves natural monomorphisms.
4. The Higgins commutator

In this section we work in a semi-abelian category [18]: a pointed, Barr-exact, Bourn-protomodular category with binary coproducts. From these axioms it follows that any semi-abelian category is finitely (co)complete. Non-expert readers need not to know too much about semi-abelian categories to understand the basic use we make of them here, and can just think of the context as being weaker than abelian, yet including categories such as the category $\mathbf{Grp}$ of groups, the category $\mathbf{Rng}$ of rings without unit, or any variety of non-associative $K$-algebras (as defined in Section 2).

If the objects considered in Examples 3.3 are subobjects $K$ and $L$ of a common object $X$ in the category $\mathcal{C}$, then the cosmash product $K \diamond L$ consists of “formal commutator words” which lie in the coproduct $K + L$, but are not concrete elements of $X$. By taking a suitable image factorisation, this can be used to define the commutator of $K$ and $L$ as a subobject of $X$.

Definition 4.1. [23, 14] For an object $X$, consider subobjects $K, L \subseteq X$ respectively represented by monomorphisms $k : K \rightarrow X$ and $l : L \rightarrow X$. We define the Higgins commutator $[K, L]$ of $K$ and $L$ as the regular image of the composite $\langle k, l \rangle \circ \iota_{K,L}$ as in the diagram

$$
\begin{array}{c}
K \diamond L \xrightarrow{\iota_{K,L}} K + L \\
\downarrow \\
[K, L] > X.
\end{array}
$$

We observe that, in the case of groups, this gives the usual commutator of group theory. In the case of non-associative algebras, one can be tempted to expect that elements such as $kl - lk$ lie in $[K, L]$—which they do—but not elements such as $kl$. However, the “universal” abelian subcategory of any variety of non-associative $K$-algebras is the one defined by $kl = 0$, the variety $\mathbf{Ab}_K$ which, as already explained in Section 2, is isomorphic to the category of $K$-vector spaces. So here, rather than characterising commutativity, it characterises abelianness.

As before, one can define the $n$-ary Higgins commutator $[K_1, \ldots, K_n]$ of a finite collection $K_i \subseteq X$ of subobjects of an object $X$ represented by monomorphisms $(k_i : K_i \rightarrow X)_{1 \leq i \leq n}$ for $n \geq 2$ as the regular image

$$
\begin{array}{c}
K_1 \diamond \cdots \diamond K_n \xrightarrow{\iota_{K_1,\ldots,K_n}} K_1 + \cdots + K_n \\
\downarrow \\
[K_1, \ldots, K_n] > X
\end{array}
$$

of the composite $\langle k_1, \ldots, k_n \rangle \circ \iota_{K_1,\ldots,K_n}$; this is the approach taken in [14] towards a categorical version of the definition in [16].

5. Cosmash associativity

Thanks to Section 3 we are now able to explain what cosmash associativity is, and see why this condition does not hold for some “classical” algebraic objects. Section 6 deals with a specific variety of algebras which does satisfy the cosmash associativity condition.
We again let \( \mathscr{C} \) be a pointed category with finite sums, finite products, and kernels. Let us observe that for any \( X, Y, Z \in \mathscr{C} \), there exists a canonical comparison map from \( X \circ (Y \circ Z) \) (or from \( (X \circ Y) \circ Z \)) to the ternary smash product \( X \circ Y \circ Z \).

This can be seen through the diagram

\[
\begin{array}{c}
0 \to X \circ (Y \circ Z) \xrightarrow{\Phi_{X,Y,Z}} X + (Y \circ Z) \xrightarrow{\Sigma_{X,Y,Z}} X \times (Y \circ Z) \\
\end{array}
\]

whose rows are exact sequences. Since the right hand square of the diagram commutes, by the universal property of kernels there exists a unique morphism \( \Phi_{X,Y,Z} \) from \( X \circ (Y \circ Z) \) to \( X \circ Y \circ Z \) making the left hand square commute. Likewise, a canonical morphism \( \Psi_{X,Y,Z} : (X \circ Y) \circ Z \to X \circ Y \circ Z \) may be constructed. Remark 5.5 below provides alternate constructions for these comparison maps. As we shall explain, they need not be isomorphisms in general—whence the following definition.

**Definition 5.1.** A pointed category with finite sums, finite products, and kernels \( \mathscr{C} \) is said to be **cosmash associative**, or satisfies cosmash associativity, whenever for every three objects \( X, Y, Z \) in \( \mathscr{C} \) the two canonical comparison maps

\[
\begin{array}{c}
X \circ Y \circ Z \\
\Phi_{X,Y,Z} \\
\Psi_{X,Y,Z}
\end{array}
\]

are invertible, which yields the dotted comparison isomorphism.

This definition is in some sense redundant. Indeed, suppose that for any \( X, Y, Z \in \mathscr{C} \), the comparison map \( \Phi_{X,Y,Z} : X \circ (Y \circ Z) \to X \circ Y \circ Z \) is an isomorphism. Since by Remark 3.10 we have that \( X \circ (Y \circ Z) \cong (Y \circ Z) \circ X \) and \( X \circ Y \circ Z \cong Y \circ Z \circ X \), also the comparison map \( \Psi_{X,Y,Z} : (X \circ Y) \circ Z \to Y \circ Z \circ X \) is an isomorphism. This shows that in order to prove the cosmash associativity of a category \( \mathscr{C} \), it suffices to check the property for any one of the two canonical comparison maps. Note that we are already using this redundancy, in some sense, since we never mention the third canonical comparison map \( Y \circ (X \circ Z) \to X \circ Y \circ Z \).

**Examples 5.2.**

1. Any additive category is cosmash associative, since all cosmash products are trivial. In particular, the varieties \( \text{Ab}_K \) and \( \text{Triv}_K \) are so.
2. The variety \( \text{CA}_K \) of associative and commutative \( K \)-algebras is cosmash associative, since the cosmash product corresponds to the tensor product. See Section 6 for further details.
3. In [5], examples are given of (non-algebraic) categories which satisfy the dual condition: the so-called smash product is associative.
4. In the category \( \text{Gp} \) of groups, cosmash products are not associative in general, essentially because commutators are not associative. Indeed, if they were, it would be possible to deduce for any triple of subgroups \( K, L, M \subseteq X \) that \([K, [L, M]] = [K, L, M] \) as subobjects of \( X \). This does, however, contradict the fact that \([K, [L, M]] \) need not be contained in
\([[[K, L], M]]\). We may, for instance, take \(K = L = Z\) viewed as included on the left in \(X = M = Z + Z\). Here the latter commutator \([[[Z, Z], Z + Z]]\) vanishes since \(Z\) is abelian, while the former \([[[Z, Z], Z + Z]]\) is highly non-trivial.

(5) A similar reasoning works to prove that smash products of Lie algebras need not be associative. A more direct argument will be given below (Example 7.3), where the result follows from an analysis of the comparison morphism occurring in Definition 5.1.

Remark 5.3. Proposition 2.21 in [14] tells us that removing brackets in a commutator enlarges the object, which can be written as

\([[[K_1, \ldots, K_i], K_{i+1}, \ldots, K_n]] < [K_1, \ldots, K_n]\)

for each \(2 \leq i \leq n - 1\), where the \(K_j\) are subobjects of an object \(X\). This does not hold for smash products. In particular, the comparison maps in Definition 5.1 need not be injective in general—see Section 8. In fact, they need not be surjections either, as explained in Section 7. Both conditions make sense on their own, and will be explored in some detail in what follows.

Remark 5.4. One might be tempted to define smash associativity as the condition that \(X \circ (Y \circ Z)\) and \((X \circ Y) \circ Z\) are isomorphic objects. This idea does not seem as fruitful as the approach taken here, essentially because there exists no canonical map between these two objects. Furthermore, our methodology is compatible with the underlying so-called lax-monoidal structure of the smash product, which we plan to further investigate in subsequent work [27].

Remark 5.5. Lemma 3.7 gives us an alternative view on the construction of the comparison maps: for instance, \(\Psi_{X,Y,Z}\) is the unique dotted factorisation in the diagram

\[
\begin{array}{c}
\Psi_{X,Y,Z} \\
\downarrow \\
0 \\
\longrightarrow X \circ Y \circ Z \\
\longrightarrow (X + Y) \circ Z \\
\longrightarrow (X \circ Z) \times (Y \circ Z).
\end{array}
\]

Cosmash associativity may thus be viewed as a kind of left exactness property of the functor \((-) \circ Z : \mathcal{C} \to \mathcal{C}\), since it means that the morphism \(\iota_{X,Y} \circ Z\) is the intersection of the kernels of \(\langle 1_X, 0 \rangle \circ Z\) and \(\langle 0, 1_Y \rangle \circ Z\).

5.6. A linear independence result. We now take the time to analyse a rather technical consequence of smash associativity, a key aspect of the structure of the smash product of three free algebras which turns out to be important in what follows. Let \(\mathcal{V}\) be a variety of \(\mathbb{k}\)-algebras which satisfies no non-trivial identities of degree smaller than or equal to 2. We assume that \(A, B\) and \(C\) are free \(\mathcal{V}\)-algebras on a single generator written \(a, b\) and \(c\), respectively.

We start by studying the structure of the smash product \(A \circ B\). We notice that, since \(\mathcal{V}\) has no identities of degree lower than 3, as a vector space, the algebra \(A \circ B\) may be written as a direct sum \(\mathbb{k}[[\{ab, ba\}]] \oplus V\) where \(\mathbb{k}[[\{ab, ba\}]]\) denotes the free vector space on the monomials \(ab\) and \(ba\), and all monomials in \(V\) have strictly larger degree. Since the multiplication on \(A \circ B\) has an image which is contained in \(V\), the identities of \(\mathcal{V}\) leave the space \(\mathbb{k}[[\{ab, ba\}]]\) untouched.
Likewise, the free algebra \( C \) may be written as a direct sum \( \mathbb{k}[[c]] \oplus W \) where \( \mathbb{k}[[c]] \) denotes the free vector space on the monomial \( c \), and all monomials in \( W \) have strictly larger degree.

Item (3) of Examples 3.3 essentially tells us that the coproduct \((A \circ B) + C\) has, for its underlying vector space, a sum of tensor products

\[ (A \circ B) \oplus C \oplus (A \circ B) \oplus (C \circ (A \circ B)) \oplus (C \circ (A \circ B) \circ C) \oplus \cdots \]

quotiented by the identities of \( \mathcal{V} \). Now

\[ \left((A \circ B) \otimes C \right) \oplus \left(C \otimes (A \circ B)\right) \]

\[ \cong \left((\mathbb{k}[[ab, ba]] \otimes \mathbb{k}[[c]]) \oplus \mathbb{k}[[c]] \oplus \mathbb{k}[[ab]]\right) \]

where we may observe that those elements of \( D \) which lie in the image of the algebra multiplication arise from multiplying an element of \( \mathbb{k}[[c]] \) with an element of \( \mathbb{k}[[ab]] \). Hence, the fact that \( \mathcal{V} \) has no equations of degree 2 ensures that these terms are left untouched by the above-mentioned quotient. We see that the underlying vector space of \( (A \circ B) + C \) is \( D \oplus X \) where \( X \) is a quotient of the direct sum of

\[ (A \circ B) \oplus C \oplus (C \circ (A \circ B) \circ C) \oplus \cdots \]

with

\[ (\mathbb{k}[[ab, ba]] \otimes W) \oplus (V \otimes \mathbb{k}[[c]]) \oplus (V \otimes W) \]

\[ \oplus (\mathbb{k}[[c]] \otimes V) \oplus (W \otimes \mathbb{k}[[ab]]) \oplus (W \otimes V). \]

In particular, \( D \) is a subspace of \( (A \circ B) + C \), which proves the independence in \( (A \circ B) + C \) of the set of polynomials \{\( (ab)c, (ba)c, c(ab), c(ba)\}\}. Since \( (A \circ B) \circ C \) is a subobject of \( (A \circ B) + C \), this gives us:

**Lemma 5.7.** Let \( \mathcal{V} \) be a variety of \( \mathbb{k} \)-algebras which satisfies no non-trivial identities of degree smaller than or equal to 2. Let \( A \), \( B \) and \( C \) be free algebras on a single generator written \( a \), \( b \) and \( c \), respectively. Then all the “degree 3 monomials” in \( (A \circ B) \circ C \) are linearly independent: if

\[ \lambda_1(ab)c + \lambda_2(ba)c + \lambda_3(c(ab)) + \lambda_4(c(ba)) = 0 \]

in \( (A \circ B) \circ C \), then \( \lambda_i = 0 \) for all \( 1 \leq i \leq 4 \). \( \square \)

**Remark 5.8.** The above procedure is easily modified to yield strengthenings of, and variations on, Lemma 5.7. For instance, the set of monomials

\[ \{\omega(c) - (ab)c, (ba)c, \omega(c)(ab), \omega(c)(ba)\} \]

where \( \omega(c) \) is any non-zero element of \( \mathbb{k}[[c]] \) is still linearly independent in the algebra \( (A \circ B) \circ C \). As a consequence, if \( E = \mathbb{k}[[e_1, \ldots, e_n]] \) is free on the set of generators \{\( e_1, \ldots, e_n \)\}, then the surjective algebra morphism \( f: E \rightarrow C: e_i \mapsto c \) induces the algebra morphism

\[ (1_A \circ 1_B) \circ f: (A \circ B) \circ E \rightarrow (A \circ B) \circ C. \]

If now

\[ \lambda_1(ab)\phi(e_1, \ldots, e_n) + \lambda_2(ba)\phi(e_1, \ldots, e_n) + \lambda_3\phi(e_1, \ldots, e_n)(ab) + \lambda_4\phi(e_1, \ldots, e_n)(ba) \]

assumptions.
is zero in \((A \circ B) \circ E\), then
\[ \lambda_1(ab)\phi(c, \ldots, c) + \lambda_2(ba)\phi(c, \ldots, c) + \lambda_3\phi(c, \ldots, c)(ab) + \lambda_4\phi(c, \ldots, c)(ba) = 0 \]
in \((A \circ B) \circ C\), so that \(\lambda_i = 0\) for all \(1 \leq i \leq 4\). In particular, the monomials \((ab)\phi(e_1, \ldots, e_n)\), \((ba)\phi(e_1, \ldots, e_n)\), \(\phi(e_1, \ldots, e_n)(ab)\) and \(\phi(e_1, \ldots, e_n)(ba)\) are linearly independent in \((A \circ B) \circ E\), for any non-zero element \(\phi(e_1, \ldots, e_n)\) of \(E\).

This may be further generalised by considering arbitrary non-zero elements of \(k\{ab, ba\}\) instead of just \(ab\) or \(ba\).

6. Cosmash and Tensor

The aim of this short section is to recall from [5] that in the operadic variety \(\mathcal{C}A_k\), the cosmash product is the tensor product over \(k\). It follows that here, cosmash products are associative. We prove the result in full detail, so that it may serve as a basis for similar results in Section [10].

In the variety \(\mathcal{C}A_k\), the product of two algebras \(A\) and \(B\) is just the cartesian product \(A \oplus B\) of the underlying vector spaces, equipped with the pointwise multiplication. It is also well known that the coproduct of \(A\) and \(B\) has
\[ A \oplus (A \otimes B) \oplus B \]
for its underlying vector space, equipped with the multiplication determined by
\[ (a, a' \otimes b', b)(c, c' \otimes d', d) = (ac, ac' \otimes d' + a \otimes d + a'c \otimes b' + a'c' \otimes b'd + a' \otimes b'd + c \otimes b + c' \otimes bd', bd) . \]

Indeed, given any two morphisms \(f: A \rightarrow C\) and \(g: B \rightarrow C\), the map
\[ A \oplus (A \otimes B) \oplus B \rightarrow C: \left( a, \sum_i a_i \otimes b_i, b \right) \mapsto f(a) + \sum_i f(a_i)g(b_i) + g(b) \]
is the unique algebra morphism which restricts to \(f\) and \(g\) via the inclusion of \(A\) on the left and of \(B\) on the right. Using the commutativity of the algebras in \(\mathcal{C}A_k\), it is easily seen that this agrees with the description in item (3) of Examples [3.3]. In particular, the map sending \(a \in A\) to \((a, 0, 0)\), \(b \in B\) to \((0, 0, b)\) and \(ab\) to \((0, a \otimes b, 0)\) determines an algebra isomorphism from the coproduct as described in Examples [3.3] to \(A \oplus (A \otimes B) \oplus B\) with the given multiplication.

**Proposition 6.1.** In the variety \(\mathcal{C}A_k\), \(A \circ B \cong A \otimes B\) via the isomorphism
\[ A \otimes B \rightarrow A \circ B: a \otimes b \mapsto ab . \]

**Proof.** It suffices to notice that the comparison morphism
\[ \langle (1_A, 0), (0, 1_B) \rangle: A \oplus (A \otimes B) \oplus B \rightarrow A \oplus B \]
sends an element \((a, a' \otimes b', b)\) to \((a, 0) + (a', 0)(0, b') + (0, b) = (a, b)\), so that its kernel is the tensor product \(A \otimes B\). The form of the isomorphism in terms of description in item (3) of Examples [3.3] follows right away from the description of the coproduct recalled above.

**Proposition 6.2.** In the variety \(\mathcal{C}A_k\) we have \(A \circ B \circ C \cong (A \otimes B) \otimes C\) for all algebras \(A, B, C\).
Proof. Via Proposition 6.1, the exact sequence in Lemma 3.7 becomes

\[
0 \rightarrow A \circ B \circ C \rightarrow (A + B) \otimes C \rightarrow (A \otimes C) \times (B \otimes C)
\]

which says that the vector space underlying \( A \circ B \circ C \) is the kernel of the linear map

\[
(A \oplus (A \otimes B) \oplus B) \otimes C \cong (A \otimes C) \oplus ((A \otimes B) \otimes C) \oplus (B \otimes C) \rightarrow (A \otimes C) \oplus (B \otimes C),
\]

which is necessarily isomorphic to \((A \otimes B) \otimes C\). It is clear that the algebra structures agree with this. \(\square\)

Corollary 6.3. The variety \( \mathcal{C} \mathcal{A}_K \) has an associative cosmash product, which is part of a symmetric monoidal structure whose unit is the field \( K \). \(\square\)

Remark 6.4. This works as well when \( K \) is a commutative ring with unit; we regain, for instance, the example of commutative rings (= commutative associative \( \mathbb{Z} \)-algebras) considered in [5].

7. Surjectivity of the comparison maps

Let \( \mathcal{V} \) be a homogeneous variety of algebras over a field \( K \). Let \( A, B \) and \( C \) be free algebras in \( \mathcal{V} \), respectively generated by elements \( a, b \) and \( c \). Suppose the map \( \Phi_{A,B,C} : A \circ (B \circ C) \rightarrow A \circ B \circ C \) is surjective. Then the polynomial \((ab)c\) lies in the image of \( \Phi_{A,B,C} \). So some polynomial

\[
\alpha_1 a(bc) + \alpha_2 a(cb) + \alpha_3 (bc)a + \alpha_4 (cb)a + t(a,b,c)
\]

exists in \( A \circ (B \circ C) \) which is mapped to \((ab)c\) by \( \Phi_{A,B,C} \). Here \( \alpha_1, \ldots, \alpha_4 \in K \) and \( t(a,b,c) \) is a polynomial in \( a, b \) and \( c \) which does not contain any multilinear monomial of degree 3. This implies that in \( A + B + C \), we have

\[
(ab)c = \alpha_1 a(bc) + \alpha_2 a(cb) + \alpha_3 (bc)a + \alpha_4 (cb)a + t(a,b,c).
\]

Since \( A + B + C \) is free, this expression is a linear combination of identities of \( \mathcal{V} \). By the homogeneity of those identities, necessarily also

\[
(ab)c = \alpha_1 a(bc) + \alpha_2 a(cb) + \alpha_3 (bc)a + \alpha_4 (cb)a
\]

is an identity of \( \mathcal{V} \). We can think of this as an identity which “pulls the left factor \( a \) out of the parentheses” in the expression \((ab)c\). Doing the same for the polynomial \((ba)c\) gives us the identity

\[
(ba)c = \beta_1 a(bc) + \beta_2 a(cb) + \beta_3 (bc)a + \beta_4 (cb)a,
\]

which pulls the right factor out of the parentheses. The polynomials \( c(ab) \) and \( c(ba) \) net us the identities

\[
c(ab) = \gamma_1 a(bc) + \gamma_2 a(cb) + \gamma_3 (bc)a + \gamma_4 (cb)a
\]

and

\[
c(ba) = \delta_1 a(bc) + \delta_2 a(cb) + \delta_3 (bc)a + \delta_4 (cb)a,
\]

which allows to pull the factors out when the parentheses are located on the right.

Proposition 7.1. A homogeneous variety of \( \mathbb{K} \)-algebras \( \mathcal{V} \) is such that for all algebras \( X, Y, Z \) in \( \mathcal{V} \) the comparison map

\[
\Phi_{X,Y,Z} : X \circ (Y \circ Z) \rightarrow X \circ Y \circ Z
\]
is surjective, if and only if elements \(\alpha_i, \beta_i, \gamma_i \) and \(\delta_i, i \in \{1, 2, 3, 4\}\) exist in \(K\) for which

\[
\begin{align*}
(xy)z &= \alpha_1 x(yz) + \alpha_2 x(zy) + \alpha_3 (yz)x + \alpha_4 (zy)x \\
(yx)z &= \beta_1 x(yz) + \beta_2 x(zy) + \beta_3 (yz)x + \beta_4 (zy)x \\
z(xy) &= \gamma_1 x(yz) + \gamma_2 x(zy) + \gamma_3 (yz)x + \gamma_4 (zy)x \\
z(yx) &= \delta_1 x(yz) + \delta_2 x(zy) + \delta_3 (yz)x + \delta_4 (zy)x 
\end{align*}
\] (7.1)

are identities of \(\mathcal{V}\).

This immediately excludes certain homogeneous varieties:

**Examples 7.2.** \(\text{Alg}_K\) does not satisfy the identities \((7.1)\), since it does not satisfy any non-trivial equations; more generally, any homogeneous variety of algebras that does not have any degree three identities cannot have surjective comparison maps \(\Phi_{X,Y,Z}\). For instance, the variety of nonassociative commutative algebras—the subvariety of \(\text{Alg}_K\) determined by the identity \(xy - yx = 0\), called **commutative-magmatic** algebras in \([30]\)—is such.

**Example 7.3.** The variety of (non-commutative) associative algebras does not satisfy the second identity in \((7.1)\), so it cannot be cosmash associative—even though it satisfies a degree three equation.

Assuming (anti-)commutativity, what should we add for the \(\Phi_{X,Y,Z}\) to become surjective? We may deduce from the first equation that some \(\lambda \in K\) exists such that

\[(xy)z = \lambda x(yz).
\]

In the commutative case, this implies that

\[x(yz) = (yz)x = \lambda y(xz) = \lambda y(xz) = (xy)z = \lambda x(yz),\]

whence \(\lambda = 1\), meaning the algebras of our variety are associative. The same calculation as above in the anti-commutative case nets \(x(yz) = -\lambda x(yz)\), which implies that \(\lambda = -1\), meaning the algebras of our variety are anti-associative.

**Lemma 7.4.** Let \(\mathcal{V}\) be a homogeneous variety of \(K\)-algebras such that for all algebras \(X, Y, Z\) in \(\mathcal{V}\) the comparison map

\[\Phi_{X,Y,Z} : X \circ (Y \circ Z) \to X \circ Y \circ Z\]

is surjective. If (anti-)commutativity holds in \(\mathcal{V}\) then so does (anti-)associativity.

**Example 7.5.** By the above reasoning, Lie algebras are essentially excluded: indeed, since the validity of the equations makes anti-commutativity imply anti-associativity, from the Jacobi identity we may deduce

\[0 = x(yz) + y(xz) + z(xy)\]

(Jacobi identity),

\[= x(yz) - (yz)x - (xy)z\]

(anti-associativity and anti-commutativity),

\[= x(yz) + x(yz) + x(yz)\]

(anti-commutativity and anti-associativity).

When \(\text{char}(K) \neq 3\), we conclude that \(x(yz) = 0\) and so, since not all Lie algebras are 2-nilpotent, the variety of \(K\)-Lie algebras does not satisfy the equations \((7.1)\) of Proposition 7.4. We come back to the equation \(zxy = 0\) in Example 8.3 and eliminate the characteristic 3 case in Proposition 8.10.

We may now consider some positive examples.
**Example 7.6.** Each of the varieties $CA_k$ and $Ant_k$ is an example of an operadic variety of algebras over a field $K$ in which the equations (7.1) of Proposition 7.1 can be satisfied. For instance, in the first case, we may take

$$\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 1$$

and all other coefficients zero.

**Example 7.7.** We may create non-classical examples by means of a careful choice of coefficients in (7.1). For instance, the subvariety $\mathcal{V}'$ of $\text{Alg}_k$ determined by

\[
\begin{cases}
(xy)z = (yz)x = (zx)y \\
x(yz) = y(zx) = z(xy)
\end{cases}
\]

fits those identities, because we may choose

$$\gamma_1 = \delta_2 = \alpha_3 = \beta_4 = 1$$

and all other coefficients zero.

Note that here, the comparison maps $\Phi_{X,Y,Z}: X \circ (Y \circ Z) \to X \circ Y \circ Z$ need not be injective: if we take $X = A$, $Y = B$ and $Z = C$ the free algebras is this subvariety generated by elements $a$, $b$ and $c$, we may calculate that

$$(a(bc))a = (b(ca))a = ((ca)a)b = ((aa)c)b = (cb)(aa)$$

in $A \circ B \circ C$. However, $(aa)(bc) = (a(bc))a = (cb)(aa)$ need not hold in $A \circ (B \circ C)$ by the reasoning in Remark 5.8.

**Example 7.8.** Another, quite similar example is the operadic variety of so-called perm algebras ([6] and [30] page 235), which are determined by the identities $(xy)z = x(yz) = x(zy)$. Here we may choose

$$\alpha_1 = \beta_3 = \gamma_4 = \delta_4 = 1$$

and all other coefficients zero.

**Remark 7.9.** The identities (7.1) imply an instance of the equations

\[
\begin{cases}
z(xy) = \lambda_1 y(zx) + \lambda_2 x(yz) + \lambda_3 y(xz) + \lambda_4 x(zy) \\
&+ \lambda_5 (zx)y + \lambda_6 (yz)x + \lambda_7 (xz)y + \lambda_8 (zy)x \\
(xy)z = \lambda_9 y(zx) + \lambda_{10} x(yz) + \lambda_{11} y(xz) + \lambda_{12} x(zy) \\
&+ \lambda_{13} (zx)y + \lambda_{14} (yz)x + \lambda_{15} (xz)y + \lambda_{16} (zy)x
\end{cases}
\]

$(\lambda_1, \ldots, \lambda_{16} \in K)$ which characterise [10] Theorem 2.12 those varieties of $K$-algebras that are algebraically coherent in the sense of [7]. Indeed, we may take

$$\lambda_{2k} = \gamma_k, \quad \lambda_{2k+8} = \alpha_k$$

for $1 \leq k \leq 4$ and all other $\lambda_i$ zero. Where the identities (7.1) allow us to “pull the $x$ out of the parentheses”, the identities characterising algebraic coherence allow us to “pull the $z$ into the parentheses”; which is clearly weaker, since to pull out the $x$, we need to pull in the $z$. Hence any homogeneous variety of $K$-algebras with surjective comparison maps $\Phi_{X,Y,Z}: X \circ (Y \circ Z) \to X \circ Y \circ Z$ is an algebraically coherent category. As a consequence, any such category has the convenient categorical-algebraic properties mentioned in [7]. In [25], we further investigate the cosmash associativity condition from a general categorical-algebraic perspective.
On the other hand, the variety $\text{Lie}_K$ is algebraically coherent, but we just saw that it does not satisfy the equations of Proposition 7.1. The same holds for the variety of (non-commutative) associative algebras considered in Example 7.3.

8. Injectivity of the comparison maps

8.1. Examples and counterexamples. It is easy to give examples of varieties of $K$-algebras whose comparison morphisms $\Phi_{X,Y,Z} : X \circ (Y \circ Z) \to X \circ Y \circ Z$ fail to be injective.

Examples 8.2. (1) We may, for instance, consider the variety of $K$-algebras that satisfy no other identities besides the identity $x(yz) - x(zy) = 0$: in the situation of Lemma 5.7 an identity of degree two is missing for us to conclude that $a(bc) = a(cb)$ in $A \circ (B \circ C)$, even though this equality holds in $A \circ B \circ C$.

(2) Example 7.7 shows that the comparison maps $\Phi_{X,Y,Z}$ need not be injective even when they are surjective.

Lemma 5.7 provides us with an additional technique to exclude cosmash associativity for certain types of algebras. Indeed, when each morphism

$$\Psi_{X,Y,Z} : (X \circ Y) \circ Z \to X \circ Y \circ Z$$

is an injection, if $A$, $B$ and $C$ are free algebras as in the statement of the lemma, then

$$\lambda_1(ab)c + \lambda_2(ba)c + \lambda_3(c(ab) + \lambda_4(c(ba)) = 0$$

in $A \circ B \circ C$ implies $\lambda_i = 0$ for all $1 \leq i \leq 4$. Hence, for instance:

Example 8.3. An extreme variation on the theme in Examples 8.2 is the subvariety $\text{Nil}_2(\text{Alg}_K)$ of $\text{Alg}_K$ determined by the equations

$$(xy)z = 0 = x(yz).$$

Here all ternary cosmash products vanish, so that each of the comparison maps $\Phi_{X,Y,Z}$ is surjective. The above technique, however, allows us to prove that this variety is not cosmash associative. The reason is, that a repeated binary cosmash product in $\text{Nil}_2(\text{Alg}_K)$ may be non-trivial. For instance, if $A$, $B$ and $C$ are freely generated by $a$, $b$ and $c$ as above, then $(ab)c$ is a non-zero element of $(A \circ B) \circ C$ by Lemma 5.7.

We note that the comparison morphisms $\Phi_{X,Y,Z}$ can be injective without being all surjective:

Example 8.4. The operadic variety $\text{Alg}_K$ itself is such. Indeed, by lack of any non-trivial identities in this variety, any equality between elements of $X \circ (Y \circ Z)$ must either be induced by an equality in $X$, $Y$ or $Z$ or by basic polynomial manipulations. None of this happens more easily in $X \circ Y \circ Z$, so that the map $\Phi_{X,Y,Z}$ is indeed injective. Its non-surjectivity was already commented upon in Examples 7.2.

Remark 8.5. When $A$, $B$ and $C$ are free non-associative algebras, Example 8.4 implies that the elements of $A \circ (B \circ C)$ are polynomials, since they may now be viewed as elements of the free algebra $A + B + C$. Part of this extends to the free algebras of any variety of $K$-algebras $\mathcal{V} \subseteq \text{Alg}_K$, even for those varieties which are not cosmash injective: indeed, their cosmash product—whose elements are thus polynomials modulo identities in $\mathcal{V}$—will be a quotient of the cosmash product.
$A \circ (B \circ C)$ in $\text{Alg}_K$ for some free non-associative algebras $A$, $B$ and $C$. Here we may essentially follow the reasoning of Subsection 2.5 of [10]. This provides a complementary view on 5.6.

Example 8.6. Recall from Examples 7.2 that also the operadic variety of non-associative commutative algebras has non-surjective $\Phi_{X,Y,Z}$. On the other hand, all $\Phi_{X,Y,Z}$ are injective, since any equality between two elements of the cosmash product $X \circ Y \circ Z$ is induced by a finite chain of applications of the commutative law. (For instance, if $x \in X$, $y \in Y$ and $z \in Z$, then $x(yz) + x((yz)z) = x(zy) + x((yz)z)$ is such.) Each such application, however, is also valid in the algebra $X \circ (Y \circ Z)$, as an instance of commutativity either in $X \circ (Y \circ Z)$ or in $Y \circ Z$. It follows that there are no equalities in $X \circ Y \circ Z$ between elements of $X \circ (Y \circ Z)$ which do not already hold in $X \circ (Y \circ Z)$.

The case of commutative algebras is in some sense generic. In what follows, we prove that in a variety with surjective $\Phi_{X,Y,Z}$, these comparison maps can only be injective if an identity of degree two holds (Proposition 8.12). By means of Lemma 2.5, we may then deduce that the algebras are either commutative or anti-commutative.

8.7. Preparatory results. We first need to extend Lemma 5.7 to a result which holds for quaternary cosmash products. In a category $C$ in which all comparison maps $\Phi_{X,Y,Z}$ are monomorphisms, we consider objects $X$, $Y$, $Z$ and $W$. Then the natural transformation

$$\Phi_{-,Z,W} : (-) \circ (Z \circ W) \to (-) \circ Z \circ W : C \to C$$

is a monomorphism. On the other hand, by the second item in Examples 3.12

$$X \circ Y \circ (Z \circ W) \cong \text{Cr}_2((-) \circ (Z \circ W))(X,Y),$$

while by the third item in Examples 3.12

$$X \circ Y \circ Z \circ W \cong \text{Cr}_2((-) \circ Z \circ W)(X,Y).$$

Since by Remark 3.13 the functor $\text{Cr}_2$ preserves natural monomorphisms, we thus find a monomorphism

$$X \circ Y \circ (Z \circ W) \cong \text{Cr}_2((-) \circ (Z \circ W))(X,Y)$$

$$\to \text{Cr}_2((-) \circ Z \circ W)(X,Y)$$

$$\cong X \circ Y \circ Z \circ W,$$

which when composed with the injection

$$\Phi_{X,Y,Z,W} : (X \circ Y) \circ (Z \circ W) \to X \circ Y \circ (Z \circ W)$$

yields an inclusion $(X \circ Y) \circ (Z \circ W) \to X \circ Y \circ Z \circ W$ which happens to be the canonical map. Hence:

Lemma 8.8. The canonical comparison morphism

$$(X \circ Y) \circ (Z \circ W) \to X \circ Y \circ Z \circ W$$

is a monomorphism whenever $\Phi_{-,Z,W} : (-) \circ (Z \circ W) \to (-) \circ Z \circ W : C \to C$ is a natural monomorphism. $\square$
Lemma 8.9. Suppose $\mathcal{V}$ is a variety of $\mathbb{K}$-algebras which satisfies no non-trivial identities of degree smaller than or equal to 2 and in which the comparison map

$$
\Phi_{X,Y,Z} : X \circ (Y \circ Z) \to X \circ Y \circ Z
$$

is always injective. Let $A$, $B$, $C$ and $D$ be free algebras in $\mathcal{V}$ on a single generator written $a$, $b$, $c$ and $d$, respectively. Then the sets of polynomials

$$
\{(ab)c, (ba)c, c(ab), c(ba)\}
$$

and

$$
\{(ab)(cd), (ba)(cd), (ab)(dc), (ba)(dc), (cd)(ab), (cd)(ba), (dc)(ab), (dc)(ba)\}
$$

are linearly independent in the free algebra $A + B + C + D$.

Furthermore, if either commutativity or anti-commutativity is added to the identities of $\mathcal{V}$, then $(ab)c$ and $(ab)(cd)$ are still non-zero in $A + B + C + D$.

Proof. Independence of the first set follows from Lemma 5.7 and the comment immediately following it, once we notice that the coproduct is injective as well. Analogously to the reasoning in Lemma 5.7, we can prove the given polynomials are indeed linearly independent in $A + B + C + D$. The final statement follows from a straightforward variation on the reasoning leading up to Lemma 5.7 in the settings of commutative and anti-commutative algebras, where the structure of the coproduct is simpler.

We may now return to Example 7.6 where we explained that the variety $\text{Anti}_{\mathbb{K}}$ of anti-commutative anti-associative $\mathbb{K}$-algebras has surjective $\Phi_{X,Y,Z}$. It is not cosmash associative, though—unless char$(\mathbb{K}) = 2$, which is just the commutative case again:

Proposition 8.10. If $\mathbb{K}$ is a field of characteristic different from 2, then a variety of anti-commutative anti-associative $\mathbb{K}$-algebras is only cosmash associative when it is abelian.

Proof. This follows from the well-known fact that we may deduce the identity $x y z t = 0$ from anti-associativity. Indeed,

$$
(x y)(z t) = -(x y)z t = (x y z)t = -x((y z) t) = x(y z t) = -(x y)(z t),
$$

whence $(x y)(z t) = 0$ follows—as long as char$(\mathbb{K}) \neq 2$. Letting $A$, $B$, $C$, $D$ be free with respective generators $a$, $b$, $c$, $d$ we have that, in particular, the element $(ab)(cd)$ of $A + B + C + D$ is zero. Lemma 8.9 now tells us that, if the variety in question is non-abelian, then the canonical $\Phi_{X,Y,Z}$ cannot all be injective.

8.11. Result and proof. From this, combined with Lemma 7.4 and Lemma 2.5, we may now deduce that amongst varieties of algebras which satisfy an identity of degree two, only the commutative associative ones are potentially cosmash associative. In what follows, our strategy is to show that injectivity of the comparison maps, together with the identities (7.1), implies the existence of such a degree two identity. Indeed, assuming a variety to be cosmash associative entails some linear dependence conditions (Proposition 7.4), while additionally assuming the non-existence of non-trivial degree $\leq 2$ identities implies some linear independence.
conditions (Lemma 8.9). We will now proceed explain how one can show that these two assumptions contradict each other, allowing us to obtain Proposition 8.12. This mimics the approach followed in the articles [10, 11]: the former uses elementary methods to arrive at a characterisation (of a different property, irrelevant for our present purposes) in the presence of a degree two identity, whereas in the latter a proof by computer is needed to show that such a degree two identity must hold.

**Proposition 8.12.** If a homogeneous variety of \( k \)-algebras \( V \) has an associative cosmash product, then the variety must satisfy at least one non-trivial identity of degree \( \leq 2 \).

Suppose we are given any multilinear monomial \( M \) in \( n \geq 3 \) variables \( x_1, \ldots, x_n \) and any multilinear monomial \( X \) in a subset of the variables \( \{x_1, \ldots, x_n\} \). Observe that there is at most one identity among (7.1) that can be used to pull \( X \) one step out of the parentheses. This allows us to define a linear map \( H_X \) from multilinear polynomials (in the variables \( x_i \)) to multilinear polynomials. For each multilinear monomial \( M \) we define \( H_X M \) as follows:

1. if one of the left hand sides of the identities (7.1), where \( x \) is taken to be \( X \), is applicable to the monomial \( M \), then we let \( H_X M \) be the polynomial we get by substituting the corresponding right hand side of the applicable identity into \( M \);
2. otherwise, we let \( H_X M \) equal \( M \).

By linear extension we can apply the map \( H_X \) to any multilinear polynomial. We note that the case (2) happens precisely when \( X \) does not appear as a submonomial of \( M \), or when \( M = XY \) or \( M = YX \) for some submonomial \( Y \).

In effect, \( H_X \) pulls the submonomial \( X \) one step outside of the parentheses in each monomial, whenever possible. For example, if \( p(a, b, c, d) = d(a(bc)) \), then \( H^b(p) \) will be

\[
\gamma_1d(b(ca)) + \gamma_2d(b(ac)) + \gamma_3d((ca)b) + \gamma_4d((ac)b).
\]

On the other hand

\[
H^b(b(a(dc))) = b(a(dc)),
\]

because none of the left hand sides of the identities (7.1) is applicable when \( x = b \), since this \( b \) is already completely outside of the parentheses.

Note that \( H_X \) acts by transforming non-associative polynomials using the identities (7.1) that hold in any cosmash associative variety. Therefore, when viewing polynomials modulo the identities of a cosmash associative variety, \( H_X \) behaves as the identity transformation.

We can now proceed as follows. Starting with a polynomial such as \( p(a, b, c) = (ab)c \), we have

\[
H^b(p) = \beta_1b(ac) + \beta_2b(ca) + \beta_3(ac)b + \beta_4(ca)b
\]

and

\[
H^b(H^a(p)) = (\alpha_2\alpha_3 + \alpha_4\beta_2 + \alpha_1\gamma_2 + \alpha_2\delta_2)b(ac) + (\alpha_1\alpha_3 + \alpha_4\delta_1 + \alpha_1\gamma_1 + \alpha_2\delta_1)b(ca) + (\alpha_3\alpha_4 + \alpha_4\beta_4 + \alpha_1\gamma_4 + \alpha_2\delta_4)(ac)b + (\alpha_3\alpha_3 + \alpha_4\beta_3 + \alpha_1\gamma_3 + \alpha_2\delta_3)(ca)b.
\]
We must have
\[ H^b(H^a(p)) - H^b(p) = 0 \]
modulo the identities of \( \mathcal{V} \), since \( \mathcal{V} \) is assumed to be cosmash associative. Moreover, observe that the monomials of \( H^b(H^a(p)) - H^b(p) \) are scalar multiples of the elements of the set
\[ \{b(ca), b(ac), (ca)b, (ac)b\} \]
of linearly independent polynomials (Lemma 8.9) in the free algebra in \( V \) modulo the identities of \( V \).

We must have
\[ \alpha_2\alpha_3 + \alpha_4\beta_2 + \alpha_1\gamma_2 + \alpha_2\delta_2 - \beta_1 = 0 \]
\[ \alpha_1\alpha_3 + \alpha_4\beta_1 + \alpha_1\gamma_1 + \alpha_2\delta_1 - \beta_2 = 0 \]
\[ \alpha_3\alpha_4 + \alpha_4\beta_4 + \alpha_1\gamma_4 + \alpha_2\delta_4 - \beta_3 = 0 \]
\[ \alpha_3\alpha_3 + \alpha_4\beta_3 + \alpha_1\gamma_3 + \alpha_2\delta_3 - \beta_4 = 0 . \]

Each of these equations expresses an equality involving elements \( \alpha_i, \beta_i, \gamma_i, \delta_i \) of the field \( K \), which allows us to view their left hand sides as associative commutative polynomials over \( Z \) with variables in \( K \). Hence these four equations may be considered as equations in the “ordinary” (= associative commutative unitary) polynomial algebra over the symbols \( \alpha_i, \beta_i, \gamma_i, \delta_i, 1 \leq i \leq 4 \), which we shall here denote \( Z[\alpha_i, \beta_i, \gamma_i, \delta_i] \).

Our end goal is to reach an inconsistent system of equations by varying \( p \) and the applications of \( H^X \), with each choice giving us a set of equations in the variables \( \alpha_i, \beta_i, \gamma_i \) and \( \delta_i \), which need to be satisfied under our assumptions.

We get a total of 32 degree 2 equations by applying \( H^X \) in various ways to \( p = (ab)c \) and \( p = a(bc) \). However, these 32 equations do not form an inconsistent system, so we add another 64 degree 3 equations, which we produce using the same schema as described above, except \( p \) will be a degree 4 monomial and we will be using the independence of the set
\[ \{(ab)(cd), (ba)(cd), (ab)(dc), (ba)(dc), (cd)(ab), (cd)(ba), (dc)(ab), (dc)(ba)\} \]
of monomials, which is stated in Lemma 8.9. The full set of equations is listed in Appendix 8 along with the corresponding choices of \( p \) and the applications of \( H^X \) which generate them. We note that there is a lot of redundancy in that set of equations. By random sampling we have found a subset of 39 equations which is inconsistent as well. However, it is computationally less intensive to prove the inconsistency of the full set of equations.

**Proof of Proposition 8.12** Using Lemma 8.9 repeatedly on several polynomials \( p \), the above process yields a system of equations \( (f_i = 0)_{1 \leq i \leq 96} \) in the algebra \( Z[\alpha_i, \beta_i, \gamma_i, \delta_i] \); see Appendix 8. We show that this system is inconsistent, by providing coefficients \( \mu_i \) in \( Z[\alpha_i, \beta_i, \gamma_i, \delta_i] \) such that \( \sum \mu_i f_i \) is a non-zero integer in \( K \). Clearly then, a common solution for the equations \( (f_i = 0)_{1 \leq i \leq 96} \) cannot exist. Hence, the hypothesis of Lemma 8.9 that \( \mathcal{V} \) does not satisfy any non-trivial equations of degree 2 must be false.

The size of the system makes it impossible to do the calculations by hand. We used the open-source software package SINGULAR [8] both for generating the system of equations and proving its inconsistency (by means of a Gröbner basis calculation). The full code is available as a set of ancillary files to the arXiv version of this article, while the code as well as its output are accessible via [26].
We first find coefficients $\mu_i$ such that $\sum \mu_i f_i$ equals the integer $m$ in Appendix A. This already shows the inconsistency of the system when either $K$ has characteristic zero, or $K$ has a prime characteristic $p$ which does not divide $m$. We exclude all the other prime characteristics as follows. We first redo the calculations with the roles of $\delta_3$ and $\delta_4$ swapped, which yields coefficients $\nu_i$ such that $\sum \nu_i f_i$ is the integer $m'$ in Appendix A. The Euclidean algorithm tells us that the greatest common divisor of $m$ and $m'$ is 2. Hence we only need to check that the system is inconsistent when $K$ has characteristic 2, and a separate calculation shows that this is indeed the case.

\[ \square \]

Remark 8.13. We checked this by means of an independent calculation in the software package MATHEMATICA [17]. The code in [26] generates output—note that the file containing the $f_i$ and $\mu_i$ is round 130 MB large—which may be used to check in MATHEMATICA that indeed $\sum \mu_i f_i = m$ and $\sum \nu_i f_i = m'$ for the coefficients generated by SINGULAR. We preferred to explain how to arrive at a solution using SINGULAR because it is a freely available open source package, which for the present task seems at least as efficient as MATHEMATICA.

Using Lemma 2.5 and Proposition 8.10, from this we may deduce:

**Theorem 8.14.** If a homogeneous variety of $K$-algebras is cosmash associative, then it must satisfy at least one of the identities $xy = yx$, $xy = x$, or $x = 0$. As a consequence, it is a subvariety of $CA_K$.

\[ \square \]

Remark 8.15. The reasoning leading up to Proposition 8.12 stays valid when $K$ is an integral domain. On the other hand, its interpretation in Theorem 8.14 makes use of Lemma 2.5 which is not valid at that level of generality. It seems the best we can do is consider varieties that satisfy a law of the form $\lambda xy + \mu yx = 0$, something we shall not pursue in this article.

### 9. The operadic case

We are now going to show that in the operadic context (where, recall, the varieties may be presented in terms of multilinear identities), the only cosmash associative subvarieties of $CA_K$ are $Ab_K$ and $Triv_K$. By Remark 2.2, the following then applies in particular when the characteristic of $K$ is zero.

**Proposition 9.1.** Let $\mathcal{V}$ be an operadic variety of $K$-algebras with the cosmash associativity property. If $\mathcal{V}$ is a proper subvariety of $CA_K$, then either $\mathcal{V} = Ab_K$ or $\mathcal{V} = Triv_K$.  

**Proof.** We consider a proper subvariety $\mathcal{V}$ of $CA_K$ whose cosmash product is associative. Since it is a proper subvariety, there exists some homogeneous polynomial $\varphi$ characterising $\mathcal{V}$ which cannot be deduced from associativity and commutativity. We prove by induction on the degree of $\varphi$ that its existence combined with the cosmash associativity property implies that either $xy = 0$ or $x = 0$ is an identity of $\mathcal{V}$.

First, if $\deg(\varphi) = 1$, then $x = 0$ trivially holds in $\mathcal{V}$.

Next, if $\deg(\varphi) = 2$, then Lemma 2.5 tells us that either $xy + yx = 0$ or $xy = 0$ holds in $\mathcal{V}$. If the characteristic of $K$ is 2, then $xy + yx = 0$ is an identity of $CA_K$, so $xy = 0$ must hold in the proper subvariety $\mathcal{V}$. Otherwise, the weakest non-trivial equation of degree two we may add to commutativity is anti-commutativity. Then
both commutativity and anti-commutativity are identities of the variety $\mathcal{V}$, which implies that $xy = 0$ holds.

Finally, if $\deg(\varphi) = n \geq 3$, then by using commutativity and associativity the identity can be rewritten as

$$(x_1 \cdots x_{n-2})(x_{n-1}x_n) = 0 \tag{9.1}$$

where by multilinearity all of the $x_i$ are different. Then, by considering $X$, $Y$ and $Z$ to be the free algebras in $\mathcal{V}$ respectively generated by $\{x_1, \ldots, x_{n-2}\}$, $\{x_{n-1}\}$ and $\{x_n\}$, the hypothesis that the comparison map $X \circ (Y \circ Z) \to X \circ Y \circ Z$ is an isomorphism implies that $(x_1 \cdots x_{n-2})(x_{n-1}x_n) = 0$ in the algebra $X \circ (Y \circ Z) \subseteq X + (Y \circ Z)$.

Lemma 8.9 allows us to vary on the reasoning in Remark 5.8 in order to deduce that either the identity $xy = 0$ holds in $\mathcal{V}$, or $x_{n-1}x_n = 0$ in $Y \circ Z$ so that $xy = 0$ holds in $\mathcal{V}$, or $x_1 \cdots x_{n-2} = 0$ in $X$, which gives a simple induction argument leading to the same conclusion. □

**Theorem 9.2.** Let $\mathbb{K}$ be a field. The cosmash product in an operadic variety of $\mathbb{K}$-algebras $\mathcal{V}$ is associative if and only if $\mathcal{V}$ is one of the following:

1. the variety $\text{CA}_\mathbb{K}$ of associative and commutative $\mathbb{K}$-algebras;
2. the variety $\text{Ab}_\mathbb{K}$ of abelian $\mathbb{K}$-algebras, which is isomorphic to $\text{Vect}_\mathbb{K}$;
3. the variety $\text{Triv}_\mathbb{K}$, containing only $\mathbb{K}$-algebras of cardinality one.

**Proof.** That cosmash associativity holds for these varieties is clear from Section 6 and the fact that additive categories have trivial cosmash products (Examples 5.2). The other implication follows from Theorem 8.14 and Proposition 9.1. □

10. Non-operadic varieties

Certain aspects of the theory may be extended to non-operadic varieties of $\mathbb{K}$-algebras, which gives rise to some new examples, as well as a few complications.

In this context, what we call a **variety of $\mathbb{K}$-algebras** as in Definition 2.1 is simply a subvariety of $\text{Alg}_\mathbb{K}$ in the sense of universal algebra: a collection of $\mathbb{K}$-algebras that satisfy a set of (not necessarily homogeneous or multilinear, but necessarily polynomial) equations. For instance:

**Example 10.1 (Alternating (anti-)associative algebras).** An algebra is **alternating** if the identity $xx = 0$ holds. We write $\text{Alt}_\mathbb{K}$ for the variety of anti-associative and alternating algebras over $\mathbb{K}$. The variety $\text{Alt}_\mathbb{K}$ only differs from $\text{Ant}_\mathbb{K}$ when $\text{char}(\mathbb{K}) = 2$; then the former is strictly smaller, because anti-commutativity $xy = -yx$ does not imply $xx = 0$ in this case (cf. Remark 2.2). Rather, then the algebras are associative and commutative, so $\text{Alt}_\mathbb{K}$ may be seen as the subvariety of $\text{CA}_\mathbb{K}$ determined by the identity $xx = 0$. Still under the condition that $\text{char}(\mathbb{K}) = 2$, it is easy to check that the coproduct of two alternating (anti-)associative $\mathbb{K}$-algebras in the variety $\text{CA}_\mathbb{K}$ does actually lie in $\text{Alt}_\mathbb{K}$, which makes it the coproduct there as well. Hence the proof of Proposition 6.1 applies, and the variety $\text{Alt}_\mathbb{K}$ has an associative cosmash product, just like $\text{CA}_\mathbb{K}$.

When $\mathbb{K}$ is an infinite field, any variety of $\mathbb{K}$-algebras is determined by its homogeneous identities only, thanks to the following result.
Lemma 10.2 ([29]). If \( \mathcal{V} \) is a variety of algebras over an infinite field, then all of its identities are of the form \( \phi(x_1, \ldots, x_n) = 0 \), where \( \phi(x_1, \ldots, x_n) \) is a polynomial. Moreover, each homogeneous component \( \psi(x_{i_1}, \ldots, x_{i_m}) \) of such an identity again gives rise to an identity \( \psi(x_{i_1}, \ldots, x_{i_m}) = 0 \). □

Hence, while such a variety need not be operadic, the results of Section 7 and Section 8 stay true. Thus we obtain the following:

**Proposition 10.3.** Over an infinite field \( K \), any variety of \( K \)-algebras with associative cosmash products is a variety of associative and commutative \( K \)-algebras. □

On the other hand, Proposition 9.1 becomes more complicated if the varieties are not operadic, because multilinearity of the equations plays a crucial role here. In characteristic zero, Theorem 9.2 stays valid as is because of Remark 2.2. In prime characteristics this is no longer true, since the multilinearisation process may fail. It is exactly this which is exploited in the following example, which generalises Example 10.1.

**Example 10.4** (\( x^p = 0 \) in characteristic \( p \)). The variety \( \text{CA}_K^p \) from Examples 2.3 has an associative cosmash product. To see this, we simply add to the proof of Proposition 6.1 a verification that the coproduct of two algebras that satisfy \( x^p = 0 \) does again satisfy this equation. Indeed, if \( x = (a, a' \otimes b', b) \) where \( a^p = (a')^p = (b')^p = b^p = 0 \), then

\[
x^p = ((a, 0, b) + (0, a' \otimes b', 0))^p = (a, 0, b)^p + (0, a' \otimes b', 0)^p
\]

\[
= ((a, 0, 0) + (0, 0, b))^p + (0, (a')^p \otimes (b')^p, 0)
\]

\[
= (a, 0, 0)^p + (0, 0, b)^p = (a^p, 0, 0) + (0, 0, b^p) = 0
\]

by the binomial theorem (used twice) and the fact that since \( \text{char}(K) = p \), we have \( \binom{p}{k} = 0 \) for each integer \( 0 < k < p \).

Variations on this idea give rise to further examples; our aim is to study those in future work.

When the field \( K \) is finite, Lemma 10.2 no longer applies, and the identities of a variety of \( K \)-algebras may be non-homogeneous. This yields yet another kind of example, as for instance:

**Example 10.5** (Boolean rings). The variety of Boolean rings may be seen as the subvariety \( \text{Bool}_K \) of \( \text{CA}_K \), where \( K \) is the finite field \( \mathbb{Z}_2 \), determined by the equation \( x = xx \). It is easy to see that the coproduct of two \( \mathbb{Z}_2 \)-algebras that satisfy \( x = xx \) does again satisfy this equation. Hence we may extend Proposition 6.1 to see that \( \text{Bool}_{\mathbb{Z}_2} \) has an associative cosmash product. This shows that over finite fields, examples of a very different nature exist.

11. Final remarks

This work gives rise to several questions of quite diverse nature; here we state a few, often of a more categorical flavour, which we find particularly interesting. For some of these questions, we already know an answer—we plan to treat those in subsequent work.
11.1. The lax-monoidal structure of the cosmash product. The cosmash product, together with all of its higher-order versions and the canonical morphisms between them, forms a so-called (symmetric) lax-monoidal structure on the category where it is considered. This point of view helps proving that cosmash associativity induces isomorphisms between all of the higher-order cosmash products as well, so that those become independent of any chosen bracketing. This development is the subject of the article [27]; in the current article, Lemma 8.8 gives one example of where this may lead.

11.2. Categorical-algebraic consequences. The examples—item (4) in [5,2] for instance—indicate a close connection between properties of the cosmash product and properties of the induced (Higgins) commutators. Some general results on this relationship, and on its implications for Categorical Algebra, form the subject of the article [25], in our opinion, treating these in the present paper would make it less focused and too long. For instance, Remark [7,3] indicates a close relationship between cosmash associativity and algebraic coherence, which may well be valid outside the context of varieties of algebras over a field. A similar question makes sense for action accessibility [1], which is known to be equivalent to algebraic coherence in our present context [9]. The fact that there are no non-abelian varieties over a field which are both cosmash associative and locally algebraically cartesian closed [12]—combine the start of the Introduction, Example 7.5 and Example 8.3—may also be an instance of a more general result.

11.3. Further questions. In the non-operadic case, we do not yet have a complete characterisation of the cosmash associative varieties of algebras. On the other hand, algebras over a field are very special, so are there any other examples? Remark [5,15] indicates that beyond the context of fields, certain results are still valid. And, what about cosmash associative varieties of groups, for instance? George Janelidze told us that the variety of differential graded rings is cosmash associative; does this example fit a general pattern?

Finally: cosmash products are intrinsically symmetric. Can we characterise associativity for non-commutative algebras by means of some kind of non-symmetric cosmash product?

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**APPENDIX A. THE NUMBERS \(m\) AND \(m'\) OF PROPOSITION [S.12]**

The number \(m\) in the proof of Proposition [S.12] is

\[
74396795447269527146818376298675787881246066202215924521090766577979741180147
\]

\[
16349646290491456249829127898568186065921784770547297215420392648452734
\]

\[
68539827696959494310311922923769051184609653516190681621082904092692692970834
\]

\[
2731607639698501360759914413992254475787853154441089600739757593479990777
\]

\[
1493875856576997726012476290198687732962162164533815209632505273589789897497
\]

\[
162156327460677374754762421696065210479621394605315959312666458922858575921779
\]

\[
42226354066105335456731188666582265076558449883242615549674152508444592622255
\]

\[
9359215596961980328721310571900352173274713460740371896851365960335158979798115
\]
Appendix B. The system of equations \((f_i = 0)_{1 \leq i < 96}\) of Proposition 8.12

| \(p = (a,b,c)\) | \(H^i(H^j(p)) - H^i(p) = 0\) | \(p = (a,b,c)\) | \(H^i(H^j(p)) - H^i(p) = 0\) |
|-----------------|---------------------------------|-----------------|---------------------------------|
| \(f_1 = a_0 a_1 + a_2 q_1 + a_3 q_2\) | \(f_13 = a_0 a_1 + a_2 q_1 + a_3 q_2\) | \(f_1 = a_0 a_1 + a_2 q_1 + a_3 q_2\) | \(f_13 = a_0 a_1 + a_2 q_1 + a_3 q_2\) |
| \(f_2 = a_0 a_1 + a_2 q_1 + a_3 q_2\) | \(f_14 = a_0 a_1 + a_2 q_1 + a_3 q_2\) | \(f_2 = a_0 a_1 + a_2 q_1 + a_3 q_2\) | \(f_14 = a_0 a_1 + a_2 q_1 + a_3 q_2\) |
| \(f_3 = a_0 a_1 + a_2 q_1 + a_3 q_2\) | \(f_15 = a_0 a_1 + a_2 q_1 + a_3 q_2\) | \(f_3 = a_0 a_1 + a_2 q_1 + a_3 q_2\) | \(f_15 = a_0 a_1 + a_2 q_1 + a_3 q_2\) |
| \(f_4 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(f_16 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(f_4 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(f_16 = a_0 q_1 + a_2 q_1 + a_3 q_2\) |

| \(p = (a,b,c)\) | \(H^i(H^j(p)) - H^i(p) = 0\) | \(p = (a,b,c)\) | \(H^i(H^j(p)) - H^i(p) = 0\) |
|-----------------|---------------------------------|-----------------|---------------------------------|
| \(r_5 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(r_17 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(r_5 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(r_17 = a_0 q_1 + a_2 q_1 + a_3 q_2\) |
| \(r_6 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(r_18 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(r_6 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(r_18 = a_0 q_1 + a_2 q_1 + a_3 q_2\) |
| \(r_7 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(r_19 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(r_7 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(r_19 = a_0 q_1 + a_2 q_1 + a_3 q_2\) |
| \(r_8 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(r_20 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(r_8 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(r_20 = a_0 q_1 + a_2 q_1 + a_3 q_2\) |

| \(p = (a,b,c)\) | \(H^i(H^j(p)) - H^i(p) = 0\) | \(p = (a,b,c)\) | \(H^i(H^j(p)) - H^i(p) = 0\) |
|-----------------|---------------------------------|-----------------|---------------------------------|
| \(s_0 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(s_21 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(s_0 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(s_21 = a_0 q_1 + a_2 q_1 + a_3 q_2\) |
| \(s_1 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(s_22 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(s_1 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(s_22 = a_0 q_1 + a_2 q_1 + a_3 q_2\) |
| \(s_2 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(s_23 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(s_2 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(s_23 = a_0 q_1 + a_2 q_1 + a_3 q_2\) |
| \(s_3 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(s_24 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(s_3 = a_0 q_1 + a_2 q_1 + a_3 q_2\) | \(s_24 = a_0 q_1 + a_2 q_1 + a_3 q_2\) |
ASSOCIATIVITY AND THE COSMASH PRODUCT

| \(|(p_1 \times_2 p_2) \times_3 p_3\) | \(p_1 \times_2 (p_2 \times_3 p_3)\) |
|-------------------------------------|-------------------------------------|
| \(= \cdots \times_2 \times_3 \times_2 \times_3\) | \(\times_2 \times_3 \times_2 \times_3 = \cdots\) |

Where \(\times_i\) denotes the operation at the \(i\)-th level.
$$p = ((a b c) d) \quad H^{bd}(H^{ab}(H^{cd}(H^{bc}))) - H^{bd}(H^{ac}(H^{bd}(H^{cad}))) = 0$$

$$f_{71} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{72} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{73} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{74} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{75} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{76} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{77} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{78} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{79} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{80} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{81} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{82} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{83} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{84} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{85} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{86} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{87} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$

$$f_{88} = -a_3 b_1 d_1 + a_1 a_3 b_3 + a_3 b_1 b_2 + a_1 b_1 b_2 + a_1 b_1 b_3 + a_3 b_1 b_2 + a_1 a_3 b_3 - b_1 b_2 - a_3 b_3$$