Intersection homology $\mathcal{D}$-Module and Bernstein polynomials associated with a complete intersection

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Abstract. Let $X$ be a complex analytic manifold. Given a closed subspace $Y \subset X$ of pure codimension $p \geq 1$, we consider the sheaf of local algebraic cohomology $H^p_Y(\mathcal{O}_X)$, and $\mathcal{L}(Y, X) \subset H^p_Y(\mathcal{O}_X)$ the intersection homology $\mathcal{D}_X$-Module of Brylinski-Kashiwara. We give here an algebraic characterization of the spaces $Y$ such that $\mathcal{L}(Y, X)$ coincides with $H^p_Y(\mathcal{O}_X)$, in terms of Bernstein-Sato functional equations.

1 Introduction

Let $X$ be a complex analytic manifold of dimension $n \geq 2$, $\mathcal{O}_X$ be the sheaf of holomorphic functions on $X$ and $\mathcal{D}_X$ the sheaf of differential operators with holomorphic coefficients. At a point $x \in X$, we identify the stalk $\mathcal{O}_{X,x}$ (resp. $\mathcal{D}_{X,x}$) with the ring $\mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$ (resp. $\mathcal{D} = \mathcal{O} \langle \partial/\partial x_1, \ldots, \partial/\partial x_n \rangle$).

Given a closed subspace $Y \subset X$ of pure codimension $p \geq 1$, we denote by $H^p_Y(\mathcal{O}_X)$ the sheaf of local algebraic cohomology with support in $Y$. Let $\mathcal{L}(Y, X) \subset H^p_Y(\mathcal{O}_X)$ be the intersection homology $\mathcal{D}_X$-Module of Brylinski-Kashiwara ([5]). This is the smallest $\mathcal{D}_X$-submodule of $H^p_Y(\mathcal{O}_X)$ which coincides with $H^p_Y(\mathcal{O}_X)$ at the generic points of $Y$ ([5], [3]).

A natural problem is to characterize the subspaces $Y$ such that $\mathcal{L}(Y, X)$ coincides with $H^p_Y(\mathcal{O}_X)$. We prove here that it may be done locally using Bernstein functional equations. This supplements a work of D. Massey ([15]), who studies the analogous problem with a topological viewpoint. Indeed, from the Riemann-Hilbert correspondence of Kashiwara-Mebkhout ([11], [17]), the regular holonomic $\mathcal{D}_X$-Module $H^p_Y(\mathcal{O}_X)$ corresponds to the perverse sheaf $\mathcal{C}_Y[n-p]$ ([7], [9], [16]) where as $\mathcal{L}(Y, X)$ corresponds to the intersection complex $IC^*_Y$ ([5]). By this way, this condition $\mathcal{L}(Y, X) = H^p_Y(\mathcal{O}_X)$ is equivalent to the following one: the real link of $Y$ at a point $x \in Y$ is a rational homology sphere.

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First of all, we have an explicit local description of $L(Y, X)$. This comes from the following result, due to D. Barlet and M. Kashiwara.

**Theorem 1.1** ([3]) *The fundamental class $C^Y_X \in H^p_{[Y]}(\mathcal{O}_X) \otimes \Omega^p_X$ of $Y$ in $X$ belongs to $L(Y, X) \otimes \Omega^p_X$.***

For more details about $C^Y_X$, see [1]. In particular, if $h$ is an analytic morphism $(h_1, \ldots, h_p) : (X, x) \rightarrow (\mathbb{C}^p, 0)$ which defines the complete intersection $(Y, x)$ - reduced or not -, then the inclusion $L(Y, X)_x \subset H^p_{[Y]}(\mathcal{O}_X)_x$ may be identified with:

$$L_h = \sum_{1 \leq k_1 < \cdots < k_p \leq n} D \cdot \frac{m_{k_1, \ldots, k_p}(h)}{h_1 \cdots h_p} \subset R_h = \frac{\mathcal{O}[1/h_1 \cdots h_p]}{\prod_{i=1}^p \mathcal{O}[1/h_1 \cdots h_i \cdots h_p]}$$

where $m_{k_1, \ldots, k_p}(h) \in \mathcal{O}$ is the determinant of the columns $k_1, \ldots, k_p$ of the Jacobian matrix of $h$. In the following, $\mathcal{J}_h \subset \mathcal{O}$ denotes the ideal generated by the $m_{k_1, \ldots, k_p}(h)$, and $\delta_h \in R_h$ the section defined by $1/h_1 \cdots h_p$.

When $Y$ is a hypersurface, we have the following characterization.

**Theorem 1.2** *Let $Y \subset X$ be a hypersurface and $h \in \mathcal{O}_{X,x}$ denote a local equation of $Y$ at a point $x \in Y$. The following conditions are equivalent:

1. $L(Y, X)_x$ coincides with $H^p_{[Y]}(\mathcal{O}_X)_x$.
2. The reduced Bernstein polynomial of $h$ has no integral root.
3. 1 is not an eigenvalue of the monodromy acting on the reduced cohomology of the fibers of the Milnor fibrations of $h$ around any singular points of $Y$ contained in some open neighborhood of $x$ in $X$.*

Let us recall that the *Bernstein polynomial* $b_f(s)$ of a nonzero germ $f \in \mathcal{O}$ is the monic generator of the ideal of the polynomials $b(s) \in \mathbb{C}[s]$ such that:

$$b(s)f^s = P(s) \cdot f^{s+1}$$

in $\mathcal{O}[1/f, s]f^s$, where $P(s) \in \mathcal{D}[s] = \mathcal{D} \otimes \mathbb{C}[s]$. The existence of such a nontrivial equation was proved by M. Kashiwara ([8]). When $f$ is not a unit, it is easy to check that $-1$ is a root of $b_f(s)$. The quotient of $b_f(s)$ by $(s+1)$ is the so-called *reduced Bernstein polynomial* of $f$, denoted $\tilde{b}_f(s)$. Let us recall that their roots are rational negative numbers in $]-n, 0[$ (see [20] for the general case, [26] for the isolated singularity case).
Example 1.3 Let \( f = x_1^2 + \cdots + x_n^2 \). It is easy to prove that \( b_f(s) \) is equal to \((s + 1)(s + n/2)\), by using the identity:

\[
\left[ \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right] \cdot f^{s+1} = (2s + 2)(2s + n)f^s
\]

In particular, \( R_f \) coincides with \( L_f \) if and only if \( n \) is odd.

These polynomials are famous because of the link of their roots with the monodromy of the Milnor fibration associated with \( f \). This was established by B. Malgrange [14] and M. Kashiwara [10]. More generally, by using the algebraic microlocalization, M. Saito [20] prove that \( \{ e^{-2i\pi \alpha} \mid \alpha \text{ root of } \hat{\beta}_f(s) \} \) coincides with the set of the eigenvalues of the monodromy acting on the Grothendieck-Deligne vanishing cycle sheaf \( \phi_f C_{X(x)} \) (where \( X(x) \subset X \) is a sufficiently small neighborhood of \( x \)). Thus the equivalence \( 2 \iff 3 \) is an easy consequence of this deep fact.

We give a direct proof of \( 1 \iff 2 \) in part 4.

Remark 1.4 In [2], D. Barlet gives a characterization of \( 3 \) in terms of the meromorphic continuation of the current \( \int_{X(x)} f^{\lambda} \square \).

Remark 1.5 The equivalence \( 1 \iff 3 \) for the isolated singularity case may be due to J. Milnor [18] using the Wang sequence relating the cohomology of the link with the Milnor cohomology. In general, this equivalence is well-known to specialists. It can be proved by using a formalism of weights and by reducing it to the assertion that the \( N \)-primitive part of the middle graded piece of the monodromy weight filtration on the nearby cycle sheaf is the intersection complex (this last assertion is proved in [19] (4.5.8) for instance). It would be quite interesting if one can prove the equivalence between \( 1 \) and \( 3 \) by using only the theory of \( D \)-modules.

In the case of hypersurfaces, it is well known that condition \( 1 \) requires a strong kind of irreducibility. This may be refined in terms of Bernstein polynomial.

Proposition 1.6 Let \( f \in \mathcal{O} \) be a nonzero germ such that \( f(0) = 0 \). Assume that the origin belongs to the closure of the points where \( f \) is locally reducible. Then \(-1\) is a root of the reduced Bernstein polynomial of \( f \).

Example 1.7 If \( f = x_1^2 + x_3x_2^2 \), then \((s + 1)^2\) divides \( b_f(s) \) because \( f^{-1}\{0\} \subset \mathbb{C}^3 \) is reducible at any \((0, 0, \lambda), \lambda \neq 0\) (in fact, we have: \( b_f(s) = (s + 1)^2(s + 3/2) \)).
What may be done in higher codimensions? If \( f \in \mathcal{O} \) is such that \((h,f)\)
defines a complete intersection, we can consider the Bernstein polynomial \(b_f(\delta_h, s)\) of \(f\) associated with \(\delta_h \in \mathcal{R}_h\). Indeed, we again have nontrivial functional equations:

\[
b(s)\delta_h f^s = P(s) \cdot \delta_h f^{s+1}
\]

with \(P(s) \in \mathcal{D}[s]\) (see part 2). This polynomial \(b_f(\delta_h, s)\) is again a multiple of \((s + 1)\), and we can define a reduced Bernstein polynomial \(\tilde{b}_f(\delta_h, s)\) as above. Meanwhile, in order to generalize Theorem 1.2, the suitable Bernstein polynomial is neither \(\tilde{b}_f(\delta_h, s)\) nor \(b_f(\delta_h, s)\), but a third one trapped between these two.

**Notation 1.8** Given a morphism \((h, f) = (h_1, \ldots, h_p, f) : (\mathbb{C}^n, 0) \to (\mathbb{C}^{p+1}, 0)\) defining a complete intersection, we denote by \(b'_f(h, s)\) the monic generator of the ideal of polynomials \(b(s) \in \mathbb{C}[s]\) such that:

\[
b(s)\delta_h f^s \in \mathcal{D}[s](\mathcal{J}_{h,f}, f)\delta_h f^s.
\]

**Lemma 1.9** The polynomial \(b'_f(h, s)\) divides \(b_f(\delta_h, s)\), and \(b_f(\delta_h, s)\) divides \((s + 1)b'_f(h, s)\). In other words, \(b'_f(h, s)\) is either \(b_f(\delta_h, s)\) or \(b_f(\delta_h, s)/(s + 1)\).

The first assertion is clear since \(\mathcal{D}[s]\delta_h f^{s+1} \subset \mathcal{D}[s](\mathcal{J}_{h,f}, f)\delta_h f^s\). The second relation uses the identities:

\[
(s + 1)m_{k_1, \ldots, k_{p+1}}(h, f)\delta_h f^s = \sum_{i=1}^{p+1} (-1)^{p+i+1}m_{k_1, \ldots, k_i, \ldots, k_{p+1}}(h) \frac{\partial}{\partial x_{k_i}} \cdot \delta_h f^{s+1}
\]

for \(1 \leq k_1 < \cdots < k_{p+1} \leq n\), where the vector field \(\Delta^h_{k_1, \ldots, k_{p+1}}\) annihilates \(\delta_h\). In particular, we have: \((s + 1)b'_f(h, s)\delta_h f^s \in \mathcal{D}[s]\delta_h f^{s+1}\), and the assertion follows.

As a consequence of this result, \(b'_f(h, s)\) coincides with \(\tilde{b}_f(\delta_h, s)\) when \(-1\) is not a root of \(b'_f(h, s)\); but it is not always true (see part 3). We point out some facts about this polynomial in part 3.

**Theorem 1.10** Let \(Y \subset X\) be a closed subspace of pure codimension \(p + 1 \geq 2\), and \(x \in Y\). Let \((h, f) = (h_1, \ldots, h_p, f) : (X, x) \to (\mathbb{C}^{p+1}, 0)\) be an analytic morphism such that the common zero set of \(h_1, \ldots, h_p, f\) is \(Y\) in a neighbourhood of \(x\). Up to replace \(h_i\) by \(h_i^m\) for some non negative integer \(m \geq 1\), let us assume that \(\mathcal{D}\delta_h = \mathcal{R}_h\). The following conditions are equivalent:
1. $\mathcal{L}(Y, X)_x$ coincides with $H^p_{[Y]}(\mathcal{O}_X)_x$.

2. The polynomial $b'_f(h, s)$ has no strictly negative integral root.

Let us observe that the condition $\mathcal{D}\delta_h = \mathcal{R}_h$ is not at all a constraining condition on $(Y, x)$. Moreover, using the boundaries of the roots of the classical Bernstein polynomial, one can take $m = n - 1$ (since $1/(h_1 \cdots h_p)^{n-1}$ generates the $\mathcal{D}$-module $\mathcal{O}[1/h_1 \cdots h_p]$, using Proposition 4.2 below). Finally, one can observe that this technical condition $\mathcal{D}\delta_h = \mathcal{R}_h$ is difficult to verify in practice. Thus, let us give an inductive criterion.

**Proposition 1.11** Let $h = (h_1, \ldots, h_p) : (X, x) \to (\mathbb{C}^p, 0)$ be an analytic morphism defining a germ of complete intersection of codimension $p \geq 1$. Assume that $-1$ is the only integral root of the Bernstein polynomial $b_{h_1}(s)$. Moreover, if $p \geq 2$, assume that $-1$ is the smallest integral root of $b_{h_{i+1}}(\delta_{h_i}, s)$ with $h_i = (h_1, \ldots, h_i) : (X, x) \to (\mathbb{C}^i, 0)$, for $1 \leq i \leq p - 1$. Then the left $\mathcal{D}$-module $\mathcal{R}_h$ is generated by $\delta_h$.

**Example 1.12** Let $n = 3$, $p = 2$, $h_1 = x_1^2 + x_2^3 + x_3^4$ and $h_2 = x_1^2 - x_2^3 + 2x_3^4$. As $h_1$ defines an isolated singularity and $h = (h_1, h_2)$ defines a weighted-homogeneous complete intersection isolated singularity, we have closed formulas for $b_{h_1}(s)$ and $b_{h_2}(\delta_{h_1}, s)$, see [28], [22]. From the explicit expression of these two polynomials, we see that they have no integral root smaller than $-1$. Thus $\delta_h$ generates $\mathcal{R}_h$.

The proofs of Theorems 1.2 & 1.10 are given in part 4. They are based on a natural generalization of a classical result due to M. Kashiwara which links the roots of $b_f(s)$ to some generators of $\mathcal{O}[1/f]f^\alpha$, $\alpha \in \mathbb{C}$ (Proposition 4.2). The last part is devoted to remarks and comments about Theorem 1.10.

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### 2 Bernstein polynomials associated with a section of a holonomic $\mathcal{D}$-module

In this paragraph, we recall some results about Bernstein polynomials associated with a section of a holonomic $\mathcal{D}_X$-Module.
Given a nonzero germ \( f \in \mathcal{O}_{X,x} \cong \mathcal{O} \) and a local section \( m \in \mathcal{M}_x \) of a holonomic \( \mathcal{D}_X \)-Module \( \mathcal{M} \) without \( f \)-torsion, M. Kashiwara [9] proved that there exists a functional equation:

\[
b(s)mf^s = P(s) \cdot mf^{s+1}
\]

in \((\mathcal{D}m) \otimes \mathcal{O}[1/f, s]f^s\), where \( P(s) \in \mathcal{D}[s] = \mathcal{D} \otimes \mathbb{C}[s] \) and \( b(s) \in \mathbb{C}[s] \) are nonzero. The Bernstein polynomial of \( f \) associated with \( m \), denoted by \( b_f(m, s) \), is the monic generator of the ideal of polynomials \( b(s) \in \mathbb{C}[s] \) which satisfies such an equation. When \( f \) is not a unit, it is easy to check that if \( m \in \mathcal{M}_x - f\mathcal{M}_x \), then \(-1 \) is a root of \( b_f(m, s) \).

Of course, if \( \mathcal{M} = \mathcal{O}_X \) and \( m = 1 \), this is the classical notion recalled in the introduction.

Let us recall that when \( \mathcal{M} \) is a regular holonomic \( \mathcal{D}_X \)-Module, the roots of the polynomials \( b_f(m, s) \) are closely linked to the eigenvalues of the monodromy of the perverse sheaf \( \psi_f(Sol(\mathcal{M})) \) around \( x \), the Grothendieck-Deligne nearby cycle sheaf, see [12] for example. Here \( Sol(\mathcal{M}) \) denotes the complex \( RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \) of holomorphic solutions of \( \mathcal{M} \), and the relation is similar to the one given in the introduction (since \( Sol(\mathcal{O}_X) \cong \mathbb{C}_X \)). This comes from the algebraic construction of vanishing cycles, using Malgrange-Kashiwara \( V \)-filtration [14], [10].

Now, if \( Y \subset X \) is a subspace of pure codimension \( p \), then the regular holonomic \( \mathcal{D}_X \)-Module \( H^p_Y(\mathcal{O}_X) \) corresponds to \( \mathbb{C}_Y[n-p] \). Thus the roots of the polynomials \( b_f(\delta, s), \delta \in H^p_Y(\mathcal{O}_X)_x \), are linked to the monodromy associated with \( f : (Y, 0) \to (\mathbb{C}, 0) \). For more results about these polynomials, see [21].

### 3 The polynomials \( b'_f(h, s) \) and \( \tilde{b}_f(\delta_h, s) \)

Let us recall that \( b'_f(h, s) \) is always equal to one of the two polynomials \( b_f(\delta_h, s) \) and \( \tilde{b}_f(\delta_h, s) \). In this paragraph, we point out some facts about these Bernstein polynomials associated with an analytic morphism \( (h, f) = (h_1, \ldots, h_p, f) : (\mathbb{C}^n, 0) \to (\mathbb{C}^{p+1}, 0) \) defining a complete intersection.

First we have a closed formula for \( b'_f(h, s) \) when \( h \) and \( (h, f) \) define weighted-homogeneous isolated complete intersection singularities.

**Proposition 3.1** ([22]) Let \( f, h_1, \ldots, h_p \in \mathbb{C}[x_1, \ldots, x_n], p < n, \) be some weighted-homogeneous of degree 1, \( \rho_1, \ldots, \rho_p \in \mathbb{Q}^+ \) for a system of weights \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Q}^+)^n \). Assume that the morphisms \( h = (h_1, \ldots, h_p) \)
and \((h, f)\) define two germs of isolated complete intersection singularities. Then the polynomial \(b'_f(h, s)\) is equal to:

\[
\prod_{q \in \Pi} (s + |\alpha| - \rho_h + q)
\]

where \(|\alpha| = \sum_{i=1}^{n} \alpha_i\), \(\rho_h = \sum_{i=1}^{p} \rho_i\) and \(\Pi \subset \mathbb{Q}^+\) is the set of the weights of the elements of a weighted-homogeneous basis of \(\mathcal{O}/(f, h_1, \ldots, h_p)\mathcal{O} + J_{h, f}\).

When \(h\) is not reduced, the determination of \(b'_f(h, s)\) is more difficult, even if \((h, f)\) is a homogeneous morphism.

**Example 3.2** Let \(p = 1, f = x_1\) and \(h = (x_1^2 + \cdots + x_n^n)^\ell\) with \(\ell \geq 1\). By using a formula given in [24], Remark 4.12, the polynomial \(\tilde{b}_{x_1}(\delta_h, s)\) is equal to \((s+n-2\ell)\) for any \(\ell \in \mathbb{N}^* = \mathbb{C}^* \cap \mathbb{N}\). For \(\ell \geq n/2\), let us determine \(b'_{x_1}(h, s)\) with the help of Theorem 1.10. From Example 1.3, we have \(R_h = D \delta_h\) if \(\ell \geq n/2\), and \(L_{h, x_1} = \mathcal{R}_{h, x_1}\) if and only if \(n\) is even.

If \(n \leq 2\ell\) is odd, \(b'_{x_1}(h, s) = (s+1)(s+n-2\ell)\) because of Theorem 1.10 (since \(L_{h, x_1} \neq \mathcal{R}_{h, x_1}\)). On the other hand, if \(n \leq 2\ell\) is even, we have \(b'_{x_1}(h, s) = (s+n-2\ell) = \tilde{b}_{x_1}(\delta_h, s)\) by the same arguments.

Let us refine this last fact by a direct calculus.

As \(\tilde{b}_{x_1}(\delta_h, s) = (s+n-2\ell)\) divides \(b'_{x_1}(h, s)\), we just have to check that this polynomial \((s+n-2\ell)\) provides a functional equation for \(\tilde{b}_{x_1}(\delta_h, s)\) when \(n\) is even. First, we observe that

\[
(s+n-2\ell)\delta_h x_i^n = \left[\sum_{i=1}^{n} \frac{\partial}{\partial x_i} x_i^n\right] \cdot \delta_h x_i^n
\]

If \(\ell = 1\), we get the result (since \(J_{h, x_1} = (x_2, \ldots, x_n)\mathcal{O}\) in that case). Now we assume that \(\ell \geq 2\). Let us prove that \(x_i \delta_h x_i^n\) belongs to \(\mathcal{D}(J_{h, x_1}, x_1)\delta_h x_i^n\) for \(2 \leq i \leq n\). We denote by \(g\) the polynomial \(x_1^n + \cdots + x_n^n\) and, for \(0 \leq j \leq \ell-1\), by \(\mathcal{N}_j \subset \mathcal{D}(J_{h, x_1}, x_1)\delta_h x_i^n\) the submodule generated by \(x_1 \delta_h x_i^n, x_2 g^j \delta_h x_i^n, \ldots, x_n g^j \delta_h x_i^n\). In particular, \(h = g^\ell, N_{\ell-1} = \mathcal{D}(J_{h, x_1}, x_1)\delta_h x_1^n\) and \(\mathcal{N}_{j+1} \subset \mathcal{N}_j\) for \(1 \leq j \leq \ell - 2\). To conclude, we have to check that \(\mathcal{N}_0 = \mathcal{N}_{\ell-1}\).

By a direct computation, we obtain the identity:

\[
\frac{\partial}{\partial x_1} \left[2(\ell - j) g^{j-1} x_1^2 + \sum_{k=2}^{n} \frac{\partial}{\partial x_k} x_k g^j\right] \cdot \delta_h x_i^n = 2(\ell - j)(n+2(\ell - j)-1)x_1 g^{j-1} \delta_h x_1^n
\]

for \(2 \leq i \leq n, j > 0\). As \(n\) is even, we deduce that \(x_i g^{j-1} \delta_h x_1^n\) belongs to \(\mathcal{N}_j\) for \(2 \leq i \leq n\). In other words, \(\mathcal{N}_{j-1} = \mathcal{N}_j\) for \(1 \leq j \leq \ell - 1\); thus \(\mathcal{N}_0 = \mathcal{N}_{\ell-1}\), as it was expected.
As the polynomial $b'_f(h, s)$ plays the role of $\tilde{b}_f(s)$ in Theorem 1.10, a natural question is to compare these polynomials $b'_f(h, s)$ and $\tilde{b}_f(\delta_h, s)$. Of course, when $(s + 1)$ is not a factor of $b'_f(h, s)$, then $b'_f(h, s)$ must coincide with $\tilde{b}_f(\delta_h, s)$; from Theorem 1.10 this sufficient condition is satisfied when $D\delta_h = R_h$ and $R_{h,f} = L_{h,f}$. But in general, all the cases are possible (see Example 3.2); nevertheless, we do not have found an example with $f$ and $h$ reduced and $b'_f(h, s) = b_f(\delta_h, s)$. Is $b'_f(h, s)$ always equal to $\tilde{b}_f(h, s)$ in this context? The question is open. Let us study this problem when $(h, f)$ defines an isolated complete intersection singularity. In that case, let us consider the short exact sequence:

\[
0 \rightarrow \mathcal{K} \hookrightarrow \frac{D[s] \delta_h f^s}{D[s](J_{h,f}, f) \delta_h f^s} \rightarrow (s + 1) \frac{D[s] \delta_h f^s}{D[s] \delta_h f^{s+1}} \rightarrow 0
\]

where the three $\mathcal{D}$-modules are supported by the origin. Thus the polynomial $b'_f(h, s)$ is equal to $\text{l.c.m}(s + 1, \tilde{b}_f(\delta_h, s))$ if $\mathcal{K} \neq 0$ and it coincides with $\tilde{b}_f(\delta_h, s)$ if not. Remark that $\mathcal{K}$ is not very explicit, since there does not exist a general Bernstein functional equation which defines $\tilde{b}_f(\delta_h, s)$ - contrarily to $\tilde{b}_f(s)$, see part 4. In [23], [24], we have investigated some contexts where such a functional equation may be given. In particular, this may be done when the following condition is satisfied:

$\textbf{A}(\delta_h)$: The ideal $\text{Ann}_D \delta_h$ of operators annihilating $\delta_h$ is generated by $\text{Ann}_\mathcal{O} \delta_h$ and operators $Q_1, \ldots, Q_w \in \mathcal{D}$ of order 1.

Indeed, because of the relations: $Q_i \cdot \delta_h f^{s+1} = (s + 1)[Q_i, f] \delta_h f^s$, $1 \leq i \leq w$, we have the following isomorphism:

\[
\frac{D[s] \delta_h f^s}{D[s](J_{h,f}, f) \delta_h f^s} \cong (s + 1) \frac{D[s] \delta_h f^s}{D[s] \delta_h f^{s+1}}
\]

where $J_{h,f} \subset \mathcal{O}$ is generated by the commutators $[Q_i, f] \in \mathcal{O}$, $1 \leq i \leq w$. Thus $\tilde{b}_f(\delta_h, s)$ may also be defined using the functional equation:

\[
b(s) \delta_h f^s \in D[s](J_{h,f}, f) \delta_h f^s
\]

and $\mathcal{K} = D[s](J_{h,f}, f) \delta_h f^s / D[s](J_{h,f}, f) \delta_h f^s$. For more details about this condition $\textbf{A}(\delta_h)$, see [25].
4 The proofs

Let us recall that $\tilde{b}_f(s)$ may be defined as the unitary nonzero polynomial $b(s) \in \mathbb{C}[s]$ of smallest degree such that:

$$b(s)h^s = P(s) \cdot h^{s+1} + \sum_{i=1}^{n} P_i(s) \cdot h_{x_i}' h^s$$  \hspace{1cm} (2)$$

where $P(s), P_1(s), \ldots, P_n(s) \in \mathcal{D}[s]$ (see [13]).

Remark 4.1 The equation (2) is equivalent to the following one:

$$b(s)h^s = \sum_{i=1}^{n} Q_i(s) \cdot h_{x_i}' h^s$$

where $Q_i(s) \in \mathcal{D}[s]$ for $1 \leq i \leq n$. Indeed, one can prove that $h^{s+1} \in \mathcal{D}[s](h_{x_1}', \ldots, h_{x_n}')h^s$ i.e. $h$ belongs to the ideal $I = \mathcal{D}[s](h_{x_1}', \ldots, h_{x_n}') + \text{Ann}_{\mathcal{D}[s]}h^s$. This requires some computations like in [24] 2.1., using that: $h \partial_{x_i} - sh_i' \in I, 1 \leq i \leq n$.

Proof of Proposition 1.6 By semi-continuity of the Bernstein polynomial, it is enough to prove the assertion for a reducible germ $f$. Let us write $f = f_1 f_2$ where $f_1, f_2 \in \mathcal{O}$ have no common factor. Assume that $-1$ is not a root of $\tilde{b}_f(s)$. Then, by fixing $s = -1$ in (2), we get:

$$\frac{1}{f} \in \sum_{i=1}^{n} \mathcal{D} h_{x_i}' f^{-1} + \mathcal{O} \subset \mathcal{O}[1/f_1] + \mathcal{O}[1/f_2]$$

since $f_{x_i}'/f = f_{x_i}'/f_1 + f_{x_i}'/f_2, 1 \leq i \leq n$. But this is absurd since $1/f_1 f_2$ defines a nonzero element of $\mathcal{O}[1/f_1 f_2]/\mathcal{O}[1/f_1] + \mathcal{O}[1/f_2]$ under our assumption on $f_1, f_2$. Thus $-1$ is a root of $\tilde{b}_f(s)$. □

The proofs of the equivalence between 1 and 2 in Theorem 1.2 and of Theorem 1.10 are based on the following result:

Proposition 4.2 Let $f \in \mathcal{O}$ be a nonzero germ such that $f(0) = 0$. Let $m$ be a section of a holonomic $\mathcal{D}$-module $\mathcal{M}$ without $f$-torsion, and $\ell \in \mathbb{N}^*$. The following conditions are equivalent:

1. The smallest integral root of $b_f(m, s)$ is strictly greater than $-\ell - 1$.

2. The $\mathcal{D}$-module $(\mathcal{D}m)[1/f]$ is generated by $mf^{-\ell}$.
3. The $\mathcal{D}$-module $(\mathcal{D}^m[1/f])/\mathcal{D}^m$ is generated by $mf^{-\ell}$.

4. The following $\mathcal{D}$-linear morphism is an isomorphism:

$$
\pi_{\ell} : \frac{\mathcal{D}[s]mf^s}{(s+\ell)\mathcal{D}[s]mf^s} \longrightarrow (\mathcal{D}^m)[1/f]
$$

$$
P(s) \cdot mf^s \mapsto P(-\ell) \cdot mf^{-\ell}
$$

**Proof.** This is a direct generalization of a well known result due to M. Kashiwara and E. Björk for $m = 1 \in \mathcal{O} = \mathcal{M}$ (Proposition 6.2, Proposition 6.1.18, 6.3.15 & 6.3.16).

Let us prove $1 \Rightarrow 4$. First, we establish that $\pi_{\ell}$ is surjective. It is enough to see that for all $P \in \mathcal{D}$ and $l \in \mathbb{Z}$: $(P \cdot m)f^l \in \mathcal{D}mf^{-\ell}$. By using the following relations:

$$
\left( \left[ \frac{\partial}{\partial x_i} Q \right] \cdot m \right)f^l = \left[ \frac{\partial}{\partial x_i} f - l \frac{\partial f}{\partial x_i} \right] \cdot ((Q \cdot m)f^{l-1})
$$

where $1 \leq i \leq n$, $Q \in \mathcal{D}$ and $l \in \mathbb{Z}$, we obtain that for all $P \in \mathcal{D}$, $l \in \mathbb{Z}$, there exist $Q \in \mathcal{D}$ and $k \in \mathbb{Z}$ such that $(P \cdot m)f^l = Q \cdot mf^k$. Thus, we just have to prove that $mf^k \in \mathcal{D}mf^{-\ell}$ for $k < -\ell$.

Let $R \in \mathcal{D}[s]$ be a differential operator such that:

$$
b_f(m, s)mf^s = R \cdot mf^{s+1}
$$

and let $k \in \mathbb{Z}$ be such that $k < -\ell$. Iterating (3), we get the following identity in $\mathcal{D}m[1/f, s]f^s$:

$$
b_f(m, s - \ell - k - 1) \cdots b_f(m, s + 1)b_f(m, s)mf^s = Q(s) \cdot mf^{s-\ell-k}
$$

(4)

where $Q(s) \in \mathcal{D}[s]$. By assumption on $\ell$, we have: $c(k) \neq 0$. Thus, by fixing $s = k$ in (4), we get $mf^k \in \mathcal{D}mf^{-\ell}$ and $\pi_{\ell}$ is surjective.

Let us prove the injectivity of $\pi_{\ell}$. If we fix $P(s) \in \mathcal{D}[s]$, then we have the following identity in $\mathcal{D}m[1/f, s]f^s$:

$$
P(s) \cdot mf^s = (Q(s) \cdot m)f^{s-l}
$$

where $Q(s) \in \mathcal{D}[s]$ and $l$ is the degree of $P$. Assume that $P(s) \cdot mf^s \in \ker \pi_{\ell}$. Thus there exists a nonnegative integer $j \in \mathbb{N}$ such that $f^jQ(-\ell)$ annihilates $m \in \mathcal{M}$. In particular: $P(s) \cdot mf^s = (s + \ell)(Q' \cdot m)f^{s-l}$, where $Q' \in \mathcal{D}[s]$ is the quotient of the division of $Q$ by $(s + \ell)$. As in the beginning of the proof,
we obtain that $P(s) \cdot mf^s = (s + \ell)\tilde{Q} \cdot mf^{s-k}$ where $\tilde{Q} \in \mathcal{D}[s]$ and $k \in \mathbb{N}^*$. From (3), we get:

$$b_f(m, s - 1) \cdots b_f(m, s - k + 1)b_f(m, s - k) P(s) \cdot mf^s = (s + \ell)\tilde{Q}S \cdot mf^s$$

where $S \in \mathcal{D}[s]$. By division of $d(s)$ by $(s + \ell)$, we obtain the identity:

$$d(-\ell)P(s) \cdot mf^s = (s + \ell)[\tilde{Q}S + e(s)P(s)] \cdot mf^s$$

where $e(s) \in \mathbb{C}[s]$. Remark that $d(-\ell) \neq 0$ by assumption on $\ell$. Thus $P(s) \cdot mf^s \in (s + \ell)\mathcal{D}[s]mf^s$, and $\pi_\ell$ is injective. Hence the condition 1 implies that $\pi_\ell$ is an isomorphism.

Observe that 4 $\Rightarrow$ 2 and 2 $\iff$ 3 are clear. Thus let us prove 2 $\Rightarrow$ 1. Let $k \in \mathbb{Z}$ denote the smallest integral root of $b_f(m, s)$. Assume that $-\ell > k$. We have the following commutative diagram:

$$
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{D}[s]mf^{s+1} \hookrightarrow \mathcal{D}[s]mf^s \to \mathcal{D}[s]mf^s/\mathcal{D}[s]mf^{s+1} \to 0 \\
\downarrow & & \downarrow
\end{array}
\begin{array}{ccc}
\downarrow & & \\
0 & \to & \mathcal{D}[s]mf^{s+1} \hookrightarrow \mathcal{D}[s]mf^s \to \mathcal{D}[s]mf^s/\mathcal{D}[s]mf^{s+1} \to 0 \\
\downarrow & & \downarrow
\end{array}
\begin{array}{ccc}
\downarrow & & \\
\mathcal{D}mf^{k+1} & \hookrightarrow & \mathcal{D}mf^k \\
\downarrow & & \\
0 & & \\
\end{array}
\begin{array}{ccc}
\downarrow & & \\
\downarrow & & \\
\downarrow & & \\
\end{array}
\begin{array}{ccc}
\downarrow & & \\
\downarrow & & \\
\downarrow & & \\
0 & & \\
\end{array}
$$

where $\nu$ is the left-multiplication by $(s-k)$. Remark that the second column is exact (since 1 $\Rightarrow$ 4), that $u$ is surjective, and that $i$ is an isomorphism (since $mf^{-\ell} \in \mathcal{D}mf^{k+1}$ generates $\mathcal{D}m[1/f]$ by assumption).

After a diagram chasing, one can check that $\nu$ is surjective. Thus the $\mathcal{D}$-module $\mathcal{D}[s]mf^s/\mathcal{D}[s]mf^{s+1}$ is Artinian, as the stalk of a holonomic $\mathcal{D}$-Module [indeed, it is the quotient of two sub-holonomic $\mathcal{D}$-Modules which are isomorphic (see [8, 9])]. As a surjective endomorphism of an Artinian module is also injective, $\nu$ is injective. But this is absurd since $k$ is a root of $b_f(m, s)$. Hence, $-\ell$ is less or equal to the smallest integral root of $f$. $\square$

**Remark 4.3** Obviously, the statement does not work for any $\ell \in \mathbb{Z}$ (take $m = 1 \in \mathcal{O} = \mathcal{M}$ and $\ell = -1$). Nevertheless, it is true for any $\ell \in \mathbb{C}$ such that for all root $q \in \mathbb{Q}$ of $b_f(m, s)$, we have: $-\ell - q \notin \mathbb{N}^* = \mathbb{N} \cap \mathbb{C}^*$. 

11
Obviously, Proposition 1.11 is obtained by iterating this result. Let us prove Theorem 1.10.

Proof of Theorem 1.10. If $b'(h, s)$ has no strictly negative integral root, then $D\delta_{h, f} = R_{h, f}$ (Lemma 1.9 and Proposition 4.2, using that $D\delta_h = R_h$), and we just have to remark that $\delta_{h, f}$ belongs to $L_{h, f}$ when $-1$ is not a root of $b'(h, s)$. Indeed, by fixing $s = -1$ in the defining equation of $b'(h, s)$:

$$b'_f(h, s)\delta_h f^s \in D[s](J_{h, f}, f)\delta_h f^s$$

we get:

$$\delta_h f^{-1} \in \sum_{1 \leq k_1 < \cdots < k_p \leq n} Dm_{k_1, \ldots, k_p}(h, f)\delta_h f^{-1} + D\delta_h \subset R_h[1/f].$$

Thus $\delta_{h, f} \in R_{h, f} \cong R_h[1/f]/R_h$ belongs to $L_{h, f}$.

Now let us assume that $L_{h, f} = R_{h, f}$. As $L_{h, f} \subset D\delta_{h, f}$, we also have $D\delta_{h, f} = R_{h, f}$ i.e. $-1$ is the smallest integral root of $b_f(\delta_h, s)$ (Proposition 4.2, using the assumption $D\delta_h = R_h$). So let us prove that $-1$ is not a root of $b'_f(h, s)$, following the formulation of 27 Lemma 1.3. Since $\delta_{h, f} \in L_{h, f} = \sum_{1 \leq k_1 < \cdots < k_p \leq n} Dm_{k_1, \ldots, k_p}(h, f)\delta_h f^{-1}$, we have: $1 \in D\delta_{h, f} \cap \text{Ann}_D\delta_{h, f}$, or equivalently: $1 \in D(J_{h, f}, f) + \text{Ann}_D\delta_h f^{-1}$ (using that $Df(\delta_h f^{-1}) = R_h$). Moreover, as $-1$ is the smallest integral root of $b_f(\delta_h, s)$, an operator $P$ belongs to $\text{Ann}_D\delta_h \otimes 1/f$ if and only if there exists $Q(s) \in D[s]$ such that $P - (s + 1)Q(s) \in \text{Ann}_D[s]\delta_h f^s$ (Proposition 4.2). Thus we have:

$$D[s] = D[s](s + 1, J_{h, f}, f) + \text{Ann}_D[s]\delta_h f^s.$$  

In particular, if $(s + 1)$ was a factor of $b'_f(h, s)$, we would have:

$$\frac{b'_f(h, s)}{s + 1} \in D[s](b'_f(h, s), J_{h, f}, f) + \text{Ann}_D[s]\delta_h f^s$$

But from the identity (1), we have:

$$b'_f(h, s) \in D[s](J_{h, f}, f) + \text{Ann}_D[s]\delta_h f^s$$

and this is a defining equation of $b'_f(h, s)$. Thus:

$$\frac{b'_f(h, s)}{s + 1} \in D[s](J_{h, f}, f) + \text{Ann}_D[s]\delta_h f^s$$

In particular, $b'_f(h, s)$ divides $b'_f(h, s)/(s + 1)$, which is absurd. Therefore $-1$ is not a root of $b'_f(h, s)$, and this ends the proof. □
Proof of the equivalence between 1 and 2 in Theorem 1.2. Up to notational changes, the proof is the very same than the previous one. Assume that \( \tilde{b}_h(s) \) has no integral root. On one hand, \( D\delta_{h} \) coincides with \( R_h = O[1/h]/O \) by Proposition 4.2 (take \( m = 1 \) and \( M = O \)). On the other hand, by fixing \( s = -1 \) in (2), we get

\[
\frac{1}{h} \in \sum_{i=1}^{n} D \frac{h' x_i}{h} + O \subset O[1/h]
\]

and \( \delta_h \in L_h \). Hence \( L_h = R_h \).

Now let us assume that \( L_h = R_h \). As \( L_h \subset D\delta_{h} \subset R_h \), \( \delta_h \) generates \( R_h \). In particular, \(-1\) is the only integral root of \( b_h(s) \) by using Proposition 4.2 (since the roots of \( b_h(s) \) are strictly negative). By the same arguments as in the proof of Theorem 1.10 one can prove that \(-1\) is not a root of \( \tilde{b}_h(s) \). Thus \( \tilde{b}_h(s) \) has no integral root, as it was expected □

Remark 4.4 Under the assumption \( D\delta_{h,f} = R_{h,f} \), we show in the proof of Theorem 1.10 that if \( \delta_{h,f} \) belongs to \( L_{h,f} \) then \(-1\) is not a root of \( b'_{f_1}(h,s) \). As the reverse relation is obvious, a natural question is to know if this assumption is necessary. In terms of reduced Bernstein polynomial, does the condition: \(-1\) is not a root of \( \tilde{b}_h(s) \) caraceterize the membership of \( \delta_{h} \) in \( L_h \)?

5 Some remarks

Let us point out some facts about Theorem 1.10:

- The assumption \( D\delta_{h} = R_{h} \) is necessary. This appears clearly in the following examples.

Example 5.1 Let \( p = 1 \) and \( h = x_1^2 + \cdots + x_4^2 \). As \( b_h(s) = (s + 1)(s + 2) \), we have \( D\delta_{h} \neq R_{h} \) (Proposition 1.2). If \( f_1 = x_1 \), then \( b'_{f_1}(h,s) = (s + 2) \) by using Proposition 3.1 where as \( L_{h,f_1} = R_{h,f_1} \) (Example 1.3 or because \( D\delta_{f_1} = R_{f_1} \) and \( b'_{f_1}(h,s) = (s + 3/2) \)).

Now if we take \( f_2 = x_5 \), we have \( L_{h,f_2} \neq R_{h,f_2} \) and \( b'_{f_2}(h,s) = 1 \) since:

\[
\frac{2}{x_1^2 + \cdots + x_4^2} x_5^s = \left[ \sum_{i=1}^{4} \frac{\partial}{\partial x_i} x_i \right] \cdot \frac{1}{x_1^2 + \cdots + x_4^2} x_5^s.
\]

- If \( p = 1 \), this condition \( D\delta_{h} = R_{h} \) just means that the only integral root of \( b_h(s) \) is \(-1\) (Proposition 4.2).
- The condition $\mathcal{L}_h = \mathcal{R}_h$ clearly implies $\mathcal{D}\delta_h = \mathcal{R}_h$, but it is not necessary; see Example 1.7 for instance. An other example with $p = 1$ is given by $h = x_1x_2(x_1 + x_2)(x_1 + x_2x_3)$ since $b_h(s) = (s + 5/4)(s + 1/2)(s + 3/4)(s + 1)^3$.

- Contrarily to the classical Bernstein polynomial, it may happen that an integral root of $b'_f(h,s)$ is positive or zero (see Example 3.2 with $f = x_1$, $h = (x_1^2 + \cdots + x_2^2)^\ell$ and $\ell \geq 2$). In particular, 1 is an eigenvalue of the monodromy acting on $\phi_f\mathcal{C}_{h^{-1}(0)}$. For that reason, we do not have here the analogue of condition 3, Theorem 1.2.

- In [6], the authors introduce a notion of Bernstein polynomial for an arbitrary variety $Z$. In the case of hypersurfaces, this polynomial $b_Z(s)$ coincides with the classical Bernstein-Sato polynomial. But it does not seem to us that its integral roots are linked to the condition $\mathcal{L}_{h,f} = \mathcal{R}_{h,f}$. For instance, if $h = x_1^2 + x_2^2 + x_3^2$ and $f = x_1^2 + x_2^2 + x_3^2$ then one can check that $b'_f(h,s) = b(f^s,s) = (s + 3/2)$; in particular $\mathcal{L}_{h,f} = \mathcal{R}_{h,f}$. Meanwhile, by using [6], Theorem 5, we get $b_Z(s) = (s + 3)(s + 5/2)(s + 2)$ if $Z = V(h, f) \subset \mathbb{C}^6$.

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