STABILITY OF EXPONENTIAL BASES ON 
\( d \)-DIMENSIONAL DOMAINS

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Abstract. We find explicit stability bounds for exponential Riesz bases on domains of \( \mathbb{R}^d \). Our results generalize Kadec theorem and other stability theorems in the literature.

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1. Introduction

We are concerned with the stability properties of exponential bases on domains of \( \mathbb{R}^d \). A domain is a bounded and measurable set of finite Lebesgue measure. An exponential basis on \( D \) is a Riesz basis on \( L^2(D) \) in the form of \( \{e^{2\pi i \langle \lambda_n, x \rangle}\}_{n \in \mathbb{Z}^d} \), with \( \lambda_n = (\lambda_1^n, \ldots, \lambda_d^n) \in \mathbb{R}^d \).

We let \( e(\lambda) = e^{2\pi i \langle \lambda, x \rangle} \) and \( E(\Lambda) = \{e(\lambda)\}_{\lambda \in \Lambda} \), with \( \Lambda \subset \mathbb{R}^d \). We will use the same notation to define frames of exponentials on \( L^2(D) \).

We will assume, often without saying, that \( \Lambda \) is a sequence denoted by \( \{\lambda_n\}_{n \in \mathbb{Z}^d} \); we will make similar assumptions about other sets of exponents.

Exponential Riesz bases are stable, in the sense that a small perturbation of a Riesz basis produces a Riesz basis. It is proved by Paley and Wiener (see e.g. [20] or [14]) that if \( E(\Lambda) \) is an exponential basis on \( D \), then the same is true of \( E(\Lambda + \Delta) \) whenever \( \sup_{n \in \mathbb{Z}^d} |||\delta_n|||_2 = \sup_{n \in \mathbb{Z}^d} ((\delta_{1n})^2 + \ldots + (\delta_{dn})^2)^{\frac{1}{2}} < \eta \) for a sufficiently small \( \eta > 0 \). Here and throughout the paper, \( \Lambda + \Delta \) denotes the sequence \( \{\lambda_n + \delta_n\}_{n \in \mathbb{Z}^d} \). Exponential frames are stable in the same sense.

With a slight abuse of notation, we let \( |||\Delta|||_\infty = \sup_{\delta_n \in \Delta} \max_{1 \leq j \leq d} |\delta_{jn}| \), and \( |||\Delta|||_p = \sup_{\delta_n \in \Delta} (|\delta_{1n}|^p + \ldots + |\delta_{dn}|^p)^{\frac{1}{p}} \) for \( 1 \leq p < \infty \). Note that \( d^{-\frac{d}{2}} |||\Delta|||_p \leq |||\Delta|||_\infty \leq |||\Delta|||_p \), and \( |||\Delta|||_p = |||\Delta|||_\infty = \sup_{\delta_n \in \Delta} |\delta_n| \) when \( d = 1 \).

We say that \( \Delta \) is an admissible perturbation for a Riesz basis (or frame) \( E(\Lambda) \) on \( L^2(D) \) if \( E(\Lambda + \Delta) \) is a Riesz basis (or frame) of \( L^2(D) \).
We say that $T$ is a *stability bound* for $E(\Lambda)$ if sequences $\Delta$ for which $|||\Delta|||_2 < T$ are admissible perturbations for $E(\Lambda)$. Sometimes it is convenient to use $|||\Delta|||_p$ instead of $|||\Delta|||_2$ and define $\ell^p$ stability bounds for $E(\Lambda)$ in a similar fashion.

The proof of the Paley-Wiener theorem does not provide an explicit stability bound. The celebrated theorem by M. I. Kadec shows that $\frac{1}{4}$ is a stability bound for the exponential basis $E(\mathbb{Z})$ on $(0, 1)$. An example by Ingham shows that $\frac{1}{4}$ cannot be replaced by a larger constant. Proofs of these results are in \[20\].

Kadec theorem has been extensively generalized (see \[1\], \[7\], \[17\], \[18\], just to cite a few) but to the best of our knowledge, explicit stability bounds for exponential bases on higher dimensional domains have been obtained only when $D$ is a Cartesian product of intervals of $\mathbb{R}$.

In this paper we find explicit stability bounds for exponential bases or frames on domains of $\mathbb{R}^d$. We will express our results using the function

$$K(t) = \frac{1}{4} - \frac{1}{\pi} \arcsin \left( \frac{1}{\sqrt{2}} (1 - \sqrt{t}) \right), \quad 0 < t \leq 1. \quad (1.1)$$

The function $K$ appears also in \[17\] and \[2\]. Note that $K(1) = \frac{1}{4}$. Our main result is the following

**Theorem 1.1.** Let $E(\Lambda)$ be a Riesz basis on $L^2(D)$ with frame bounds $0 < A \leq B$. Let $K = K(AB^{-1})$ be as in (1.1). Let $\bar{x} \in \mathbb{R}^d$, and let $\Delta \subset \mathbb{R}^d$ be a sequence that satisfies

$$L = \sup_{x \in D} \sup_{\delta_n \in \Delta} |\langle \delta_n, x - \bar{x} \rangle| < \frac{K}{2}. \quad (1.2)$$

Then, $E(\Lambda + \Delta)$ is a Riesz basis on $L^2(D)$ with bounds

$$B' = B(2 - \cos(2\pi L) + \sin(2\pi L))^2;$$

$$A' = \left( \sqrt{A} - \sqrt{B}(1 - \cos(2\pi L) + \sin(2\pi L)) \right)^2. \quad (1.3)$$

Theorem \[11\] and most of the results in this paper are valid also for exponential frames on $L^2(D)$. When $E(\Lambda)$ is a frame on $D$, we can argue as in \[19\] to show that $\frac{K}{2}$ on the right hand side of (1.2) can be replaced by a a larger constant. We leave this generalization to the reader.

When $d = 1$ we can restate Theorem \[11\] as follows.

**Corollary 1.2.** Let $D \subset (a, b)$. If $\Delta \subset \mathbb{R}$ satisfies

$$2L = |||\Delta|||_{\infty} (b - a) < K \quad (1.4)$$
the conclusions of Theorem 1.1 hold.

Corollary 1.2 implies Kadec’s theorem and the main theorem in [17].

From (1.2) one can easily obtain stability bounds for exponential bases or frames $E(\Lambda)$ on $L^2(D)$. As in Theorem 1.1, we let $\bar{x} \in \mathbb{R}^d$ and $K = K(AB^{-1})$.

**Corollary 1.3.** Every $\Delta \subset \mathbb{R}^d$ that satisfies

$$\|\|\Delta\|\|_2 < \frac{K}{2 \sup_{x \in D} \|x - \bar{x}\|_2}$$

(1.5)

it is an admissible perturbation for $E(\Lambda)$.

We let $\text{diam}(D)$ be the diameter of $D$, i.e., $\sup_{x, y \in D} \|x - y\|_2$. We can choose $\bar{x} \in D$ so that $\sup_{x \in D} \|x - \bar{x}\|_2 = \frac{1}{2} \text{diam}(D)$. So, it follows from (1.5) that $\Delta$ is an admissible perturbation for $E(\Lambda)$ if

$$\|\|\Delta\|\|_2 < \frac{K}{\text{diam}(D)}.$$

(1.6)

It is worth remarking that there may be admissible perturbations with norm larger than the right-hand side of (1.6), especially when $\Delta$ is in a proper subspace of $\mathbb{R}^d$. For example, consider $R = [-1, 1] \times [-3, 3]$, $\bar{x} = (0, 0)$ and $\Delta = \{(d_n, 0)\}_{n \in \mathbb{Z}}$; we can see from (1.2) that $\Delta$ is an admissible perturbation of $L^2(E)$ if $\sup_{n \in \mathbb{Z}} |d_n| \leq \frac{1}{2} K$, while (1.6) only gives $\sup_{n \in \mathbb{Z}} |d_n| \leq \frac{1}{2 \sqrt{10}} K$.

A natural question arises: how can we chose $\bar{x}$ so that $\sup_{x \in D} \|x - \bar{x}\|_2$ is as small as possible? That is, how to find $x^* \in \mathbb{R}^n$ so that

$$\sup_{x \in D} \|x - x^*\|_2 = \min_{x \in \mathbb{R}^n} \sup_{x \in D} \|x - \bar{x}\|_2.$$ 

When $D$ is convex, $x^*$ is the center of the largest sphere contained in $D$ and it is called Chebyshev center. The Chebyshev center can be also defined for subsets of infinite-dimensional Banach spaces and it is very relevant in optimization and other applied problems (see [3], [21] and the references cited there). The Chebyshev center of $D$ does not necessarily coincide with its centroid - which we recall is the unique point $\zeta \in D$ for which

$$\int_D \|x - \zeta\|^2 dx = \min_{\bar{x} \in \mathbb{R}^d} \int_D \|x - \bar{x}\|^2 dx.$$ 

To the best of our knowledge, the relation between centroid and Chebyshev center is known only for convex symmetric domains of $\mathbb{R}^d$. 
Theorem 1.1 can be improved when only one components of $\lambda_n$ and $\delta_n$ change with the corresponding components of $n = (n_1, \ldots, n_d)$.

**Theorem 1.4.** Let $D = D_1 \times \ldots \times D_d$, with $D_j \subset \mathbb{R}$; Let $E(\Lambda)$ be an exponential basis of $L^2(D)$, with $\Lambda = \{(\lambda^1_{n_1}, \ldots, \lambda^d_{n_d})\}_{n_j \in \mathbb{Z}}$. Then, $\Delta = \{((\delta^1_{n_1}, \ldots, \delta^d_{n_d}))_{n_j \in \mathbb{Z}}\}$ is an admissible perturbation for $E(\Lambda)$ if, for some $\bar{x} \in \mathbb{R}^d$,

$$\sup_{\substack{n_j \in \mathbb{Z} \\ j=1,\ldots,d}} \sup_{x \in D_j} |\delta_{n_j}(x_j - \bar{x}_j)| < \frac{1}{2} K. \tag{1.7}$$

In particular, (1.7) is verified if $|||\Delta|||_\infty < K \frac{1}{2} \sup_{x \in D} ||x - \bar{x}||_\infty$.

The second part of Theorem 1.4 generalizes Theorem 1.2 in [17].

When $D$ is not connected, we can improve Theorem 1.1 and its corollaries as follows.

**Theorem 1.5.** Suppose that $D = D_1 \cup \ldots \cup D_m$, where the $D_j$’s are disjoint; If, for some $\bar{x}_1, \ldots, \bar{x}_m \in \mathbb{R}^d$, $\Delta$ satisfies

$$L = \sup_{x \in D_1} |\langle \delta_n, x - \bar{x}_1 \rangle| + \ldots + \sup_{x \in D_m} |\langle \delta_n, x - \bar{x}_m \rangle| < \frac{K}{2} \tag{1.8}$$

then the conclusions of Theorem 1.1 hold.

It has been recently proved by Kozma and Nitzan [10] that every finite union of intervals in $\mathbb{R}$ has an exponential basis. The following corollary of Theorem 1.5 provides a stability bound for such bases that depend the total length of the intervals, but not on the gaps between them.

**Corollary 1.6.** Let $D$ be a finite union of disjoint intervals in $\mathbb{R}$. If $E(\Lambda)$ is an exponential basis on $D$ with bounds $A$ and $B$, and $\Delta \subset \mathbb{R}$ satisfies

$$2L = |||\Delta|||_\infty |D| < K \tag{1.9}$$

the conclusions of Theorem 1.1 hold.

Let us apply our theorems to some simple but important examples.

**Example 1.** Let $D$ be the disk in $\mathbb{R}^2$ centred at the origin; it is not known whether $D$ has an exponential Riesz basis or not, but we can apply Theorem 1.1 to exponential frames. For a fixed $\delta = (\delta_1, \delta_2) \neq (0,0)$,
the function $f(x) = \langle x, \delta \rangle = x_1\delta_1 + x_2\delta_2$ attain its maximum and minimum on the boundary of $D$, i.e., on the circumference of equation $x_1^2 + x_2^2 = 1$. We can easily verify (using e.g. Lagrange multipliers theorem) that the maximum and minimum of $f$ are attained at $(\pm \|\delta\|_1, \pm \|\delta\|_2)$, where we have let $\|\delta\| = \|\delta\|_2 = \sqrt{\delta_1^2 + \delta_2^2}$. Thus,

$$f(x) \leq \frac{\delta_1^2}{\|\delta\|} + \frac{\delta_2^2}{\|\delta\|} = \|\delta\|.$$

So by Theorem 1.1, $E(\Lambda + \Delta)$ is an exponential frame on $L^2(D)$ if $\|\Delta\|_2 < \frac{1}{2}K$.

**Example 2.** Let $P$ be a polyhedron in $\mathbb{R}^d$. It is proved in [13] and [8] that $P$ has an exponential basis if it is sufficiently symmetric and regular. Let $E(\Lambda)$ be an exponential basis or frame of $L^2(P)$ with bounds $A$ and $B$; let $\Delta = \{\delta_n\}_{n \in \mathbb{Z}^d}$ and let $f(x) = f(n, x) = \langle x - \bar{x}, \delta_n \rangle$, where $\bar{x}$ is the Chebyshev center of $P$. By (1.2), $E(\Lambda + \Delta)$ is a Riesz basis or frame of $L^2(P)$ if $|f(x)| < \frac{1}{2}K$ Since $f$ is linear, it attains its maximum and minimum at vertices of $P$. So, (1.2) is equivalent to

$$\sup_{\delta_n \in \Delta} |\langle \zeta_j - \bar{x}, \delta_n \rangle| < \frac{1}{2}K \quad \text{(1.10)}$$

where $\zeta_1, ..., \zeta_N$ are the vertices of $P$. From (1.10) follows that

$$\|\Delta\|_2 < \frac{K}{2 \max_j \|\zeta_j - \bar{x}\|_2}. \quad \text{(1.11)}$$

When $P = (-1, 1)^d$, the stability bound on the right-hand side of (1.11) improves that of Theorem 1.3 in [17] (see Section 2.2).

$E(\mathbb{Z}^d)$, the standard orthonormal basis of $L^2([0, 1]^d)$, is a frame on every domain $D \subset [0, 1]^d$ (see e.g. Proposition 2.1 in [7]). Although frames can be viewed as over-complete Riesz bases, it may be very difficult, and sometimes impossible, to extract Riesz bases from them. That may depend on the domain $D$, but also on the frame itself. For example, K. Seip proved in [15] that $L^2(a, b)$ has an exponential frame that does not contain an exponential basis; however, it is proved in [16] that if $[a, b] \subset [0, 1]$, then $E(\mathbb{Z})$ contains an exponential basis of $L^2(a, b)$, and in [10] that $E(\mathbb{Z})$ contains a Riesz basis of $L^2(D)$ also if $D$ is a finite union of intervals in $[0, 1]$. Riesz bases extracted from $E(\mathbb{Z}^d)$ are especially useful in the applications, and can be considered "canonical" in some sense.
We pose the following question: if a domain \( D \subset [0,1]^d \) has an exponential basis, does it also have an exponential basis extracted from \( E(Z^d) \)?

Our Theorem 1.7 provides a partial answer to this question.

**Theorem 1.7.** Let \( L \) be an \( \ell_\infty \) stability bound for an exponential basis on \( D \). For every fraction \( 0 < \frac{p}{q} < L \), there exists \( \Gamma \subset (\frac{\ell}{q}Z)^d \) such that \( E(\Gamma) \) is a Riesz basis of \( L^2(D) \).

From (1.6) follows that we can choose \( \frac{p}{q} < \frac{K}{\sqrt{d \text{diam}(D)}} \). So, an exponential basis on \( D \subset [0,1]^d \) can be extracted from the standard orthogonal basis of the hyper-cubes \( [0, \frac{4}{p}]^d \), with \( \frac{4}{p} > \frac{4}{K} \). If \( L^2(D) \) has an orthogonal exponential basis, we can choose \( \frac{p}{q} > 4d \).

We prove Theorem 1.7 and its corollaries in Section 4. In Section 3 we prove other theorems stated in this section, and in Section 2 we collect some preliminary results and definitions. Section 5 contains open problems and conclusive remarks.

2. PRELIMINARIES

2.1. **Frames and Riesz bases.** A sequence of vectors \( B = \{v_n\}_{n \in \mathbb{N}} \) in a Hilbert space \( H \) is a frame if there exist constants \( A, B > 0 \) such that for every \( w \in H \),

\[
A \|w\|^2 \leq \sum_{j=1}^{\infty} |\langle w, v_j \rangle|^2 \leq B \|w\|^2.
\]  

(2.1)

Here, \( \| \| \) and \( \langle \, \rangle \) are the norm and the inner product in \( H \). We say that \( B \) is a tight frame if \( A = B \); \( B \) is a Riesz basis if it is an exact frame, i.e., if it ceases to be a frame when any of its elements is removed. Equivalently, \( B \) is a Riesz basis if it is complete and there exist constants \( 0 < A \leq B \) such that, for every finite set \( \{c_j\}_{j \leq n} \subset \mathbb{C} \),

\[
A \sum_{j=1}^{n} |c_j|^2 \left\| \sum_{j=1}^{n} c_j v_j \right\|^2 \leq B \sum_{j=1}^{n} |c_j|^2.
\]

We refer to the textbooks [6] and [20] and to the excellent paper [4] for a survey on bases and frames in Hilbert spaces.

2.2. **Stability of Riesz bases.** Riesz bases are stable, in the sense that a small perturbation of a Riesz basis produces a Riesz basis. Let us recall the Paley-Wiener stability theorem, and the celebrated Kadec theorem. The proof of both theorems can be found in [20]. Kadec theorem was originally proved in [9].
Theorem 2.1. (Paley-Wiener) Let \( \{v_n\}_{n \in \mathbb{N}} \) be a Riesz basis for a Hilbert space \( H \). Suppose that \( \{w_n\}_{n \in \mathbb{N}} \) is a sequence of elements of \( H \) for which there exists a constant \( 0 < \lambda < 1 \) such that, for every finite set of constants \( \{c_j\} \subset \mathbb{C} \)

\[
\left\| \sum_{j=1}^{n} c_j(v_j - w_j) \right\| \leq \lambda \left\| \sum_{j=1}^{n} c_j v_j \right\| .
\]

(2.2)

Then \( \{w_n\}_{n \in \mathbb{N}} \) is a Riesz basis for \( H \).

Note that if \( \{v_n\}_{n \in \mathbb{N}} \) is an orthonormal basis of \( H \), and \( \sum |c_j|^2 = 1 \), the right hand side of (2.2) equals to \( \lambda \).

Theorem 2.2. (Kadec) Let \( \{\alpha_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) be such that

\[
\sup_{n \in \mathbb{Z}} |\alpha_n - n| < \frac{1}{4} .
\]

(2.3)

Then, \( B = \{e^{2\pi i \alpha_n x}\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( L^2(0, 1) \). The constant \( \frac{1}{4} \) cannot be replaced by any larger number.

The next theorem is Theorem 1.3 in [17]. We have rewritten its statement using slightly different normalization to better compare this result with ours.

Theorem 2.3. Let \( \Gamma = \{\gamma_n\}_{n \in \mathbb{Z}^d} \) and \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}^d} \). Suppose that \( E(\Lambda) \) is a Riesz basis (or a frame) for \( L^2([0, 1]^d) \) with bounds \( A \) and \( B \). If \( 0 < L < \frac{1}{4} \),

\[
p(L) := \left( 1 - \cos(\pi L) + \sin(\pi L) + \frac{\sin(\pi L)}{\pi L} \right)^d - \left( \frac{\sin(\pi L)}{\pi L} \right)^d < \frac{A}{B},
\]

and

\[|||\Lambda - \Gamma|||_\infty \leq L,\]

then \( E(\Gamma) \) is a Riesz basis (or a frame) for \( L^2([0, 1]^d) \) with bounds \([\sqrt{A} - \sqrt{Bp(L)}]^2 \) and \( B[1 + p(L)]^2 \).

When \( A = B \), Corollary 1.3 (see also (1.6)) applied with \( D = (-\frac{1}{4}, \frac{1}{4})^d \) and \( K = \frac{1}{4} \) implies that if \( |||\Lambda - \Gamma|||_\infty < \frac{1}{4d} \), then \( E(\Gamma) \) is an exponential basis of \( L^2(D) \).

The table below shows a numerical comparison between our stability bounds \( L = \frac{1}{4d} \), (DP) and the approximated \( L \)'s that satisfy \( p(L) = 1 \) in Theorem 2.3 (SZ). A similar table can be produced also when \( \frac{A}{B} < 1 \).
**Table 1. Comparison stability bounds \((A = B)\)**

| d | SZ | DP | d | SZ | DP |
|---|----|----|---|----|----|
| 1 | 0.25 | 0.25 | 5 | 0.044783 | 0.05 |
| 2 | 0.11565 | 0.125 | 4 | 0.056237 | 0.0625 |
| 3 | 0.075618 | 0.083 | 6 | 0.037211 | 0.0416 |

3. Most of the proofs

In this section we prove the theorems stated in the introduction, with the exception of Theorem 1.7 that will be proved in Section 4. To prove Theorem 1.1 we need the following

**Lemma 3.1.** Let \(\delta \in \mathbb{R}\), and let

\[
A_0 = 1 - \frac{\sin(\pi \delta)}{\pi \delta}, \quad A_m = \frac{(-1)^m 2 \delta \sin(\pi \delta)}{\pi (m^2 - \delta^2)},
\]

\[
B_m = i \frac{(-1)^m 2 \delta \cos(\pi \delta)}{\pi ((m - \frac{1}{2})^2 - \delta^2)}.
\] (3.1)

The sequence

\[
S_N(t) = A_0 + \sum_{m=1}^{N} A_m \cos(\pi mt) + \sum_{m=1}^{N} B_m \sin(\pi t(m - \frac{1}{2}))
\] (3.2)

converges to \(1 - e^{\pi i \delta t}\) for every \(t \in [-1, 1]\).

**Proof.** The proof of this lemma is perhaps in the literature, but we sketch it here for the convenience of the reader. Let \(\psi(t) = 1 - e^{\pi i \delta t}\). When \(t = \pm 1\) we can verify, using the Taylor expansion of \(\tan(\pi \delta t)\) and \(\cot(\pi \delta t)\) (see also the proof of Theorem 1.1) that \(S_N[t]\) converges to \(\psi(t)\).

When \(t \in (-1, 1)\), \(S_n(t)\) in (3.2) is the partial expansion of \(\psi(t)\) in terms of the complete orthonormal system \(\left\{ \frac{1}{\sqrt{2}}, \cos(\pi mt), \sin(\pi (m - \frac{1}{2})t) \right\}\) of \(L^2(-1, 1)\). That is, \(B_m = \int_{-1}^{1} \sin(\pi y(m - \frac{1}{2})) (1 - e^{\pi i dy}) dy\), and similarly for the other coefficients. The \(A_m\)'s are the standard Fourier coefficients of \(\text{Re}(\psi(t)) = 1 - \cos(\delta t)\). The Fourier series of a differentiable function converges pointwise (see e.g. [22]) and so

\[
\lim_{N \to \infty} \left( A_0 + \sum_{m=1}^{N} A_m \cos(\pi mt) \right) = 1 - \cos(\delta \pi t)
\]

for every \(t \in (-1, 1)\).

We are left to prove that \(S_N(t) = \sum_{m=1}^{N} B_m \sin(\pi t(m - \frac{1}{2}))\) converges pointwise to \(i \text{Im}(\psi(t)) = i \sin(\delta \pi t)\).
Recalling the expression for $B_m$ and the fact that $\sin(\delta \pi t)$ is odd,

$$S'_N(t) = \sum_{m=1}^{N} \int_{-1}^{1} \sin(\pi \delta y) \sin(\pi y(m - \frac{1}{2})) \sin(\pi t(m - \frac{1}{2})) dy$$

$$= i \int_{-1}^{1} \sin(\pi \delta y) \left( \sum_{m=1}^{N} \cos(\pi(y - t)(m - \frac{1}{2})) \right) dy.$$

It is not too difficult to verify that the sum in parenthesis equals

$$\widetilde{D}_N(t) = \frac{\sin(\pi N(t - y))}{2 \sin(\frac{\pi(t - y)}{2})}. \text{ The Dirichlet kernel is } D_N(\pi y) = \frac{\sin(\pi(N + \frac{1}{2})(t - y))}{2 \sin(\frac{\pi(t - y)}{2})}.$$

We let $I_N(t) = \int_{-1}^{1} \sin(\delta \pi y) D_N(\pi(t - y)) dy$ and $S'_N(t) - I_N(t) = J_N(t)$.

Since $\lim_{N \to \infty} I_N(t) = i \sin(\delta \pi t)$ for every $t \in (-1, 1)$, to conclude the proof we are left to show that $\lim_{N \to \infty} J_N(t) = 0$. Using standard trigonometric identities, we can show that

$$J_N(t) = \frac{i}{2} \cos((\pi(N + \frac{1}{2})t) \int_{-1}^{1} \sin(\delta \pi y) \sin((\pi(N + \frac{1}{2})y) dy.$$

The integral above is the Fourier transform of $-i \sin(\delta \pi t) \chi_{(-1,1)}(t)$ evaluated at $\frac{1}{2}(N + \frac{1}{2})$. The Fourier transforms of a $L^1$ functions goes to zero at infinity, and so $\lim_{N \to \infty} S'_N(t) = i \sin(\pi \delta t)$.

\[\square\]

**Proof of Theorem 1.1.** Since $E(\Lambda)$ is a Riesz basis of $L^2(D)$ with bounds $0 < A \leq B < \infty$, then, for all finite sets $\{c_n\} \subset \mathbb{C}$ for which $\sum |c_n|^2 = 1$,

$$A \leq \left\| \sum c_n e^{2\pi i \langle \lambda_n, x \rangle} \right\|^2_{L^2(D)} = \left\| \sum c'_n e^{2\pi i \langle \lambda_n, x - \bar{x} \rangle} \right\|^2_{L^2(D)} \leq B \quad (3.3)$$

where $c'_n = c_n e^{2\pi i \langle \lambda_n, \bar{x} \rangle}$. Note that $\sum_j |c'_n|^2 = 1$. By Paley-Wiener theorem, if

$$\left\| \sum c'_n \left( e^{2\pi i \langle \lambda_n, x - \bar{x} \rangle} - e^{2\pi i \langle \lambda_n + \delta_n, x - \bar{x} \rangle} \right) \right\|_{L^2(D)}$$

$$= \left\| \sum c'_n e^{2\pi i \langle \lambda_n, x - \bar{x} \rangle} (1 - e^{2\pi i \langle \delta_n, x - \bar{x} \rangle}) \right\|_{L^2(D)} \leq \lambda \sqrt{A} \quad (3.4)$$

for some $0 < \lambda < 1$, the sequence $E(\Lambda + \Delta)$ is a Riesz basis for $L^2(D)$. Without loss of generality, we can let $\bar{x} = 0$ and $c'_n = c_n$. 


We let $L = \sup_{\delta_n \in \Delta} |\langle x, \delta_n \rangle|$, and $t = t(x, n) = \frac{1}{L} \langle x, \delta_n \rangle$. By definition, $t \in [-1, 1]$. By Lemma 3.1,

$$1 - e^{2\pi iLt} = A_0 + \sum_{m=1}^{\infty} A_m \cos(\pi mt) + \sum_{m=1}^{\infty} B_m \sin(\pi t(m - \frac{1}{2})) \quad (3.5)$$

where the $A_j$’s and the $B_j$’s are defined as in (3.1), with $2L$ replacing $\delta$. By the triangle inequality, $\| \sum c_n e^{2\pi i(\lambda_n, x)} (1 - e^{2\pi iLt}) \|_{L^2(D)} \leq S_0 + S + T$, where

$$S_0 = \left\| \sum c_n e^{2\pi i(\lambda_n, x)} \left( 1 - \frac{\sin(2\pi L)}{2\pi L} \right) \right\|_{L^2(D)}$$

$$S = \left\| \sum c_n e^{2\pi i(\lambda_n, x)} \sum_{m=1}^{\infty} \frac{(-1)^m 4L \sin(2\pi L)}{\pi (m^2 - 4L^2)} \cos(\pi mt) \right\|_{L^2(D)}$$

$$T = \left\| \sum c_n e^{2\pi i(\lambda_n, x)} \sum_{m=1}^{\infty} \frac{(-1)^m 4L \cos(2\pi L)}{\pi (m^2 - 4L^2)} \sin(\pi (m - \frac{1}{2})t) \right\|_{L^2(D)} .$$

By (3.3), $S_0 \leq \sqrt{B} \left( 1 - \frac{\sin(2\pi L)}{2\pi L} \right)$.

To estimate $S$ we exchange the order of summation, and we use the triangle inequality and (3.3):

$$S \leq \sum_{m=1}^{\infty} \left\| \sum c_n e^{2\pi i(\lambda_n, x)} \frac{(-1)^m 4L \sin(2\pi L)}{\pi (m^2 - 4L^2)} \right\|_{L^2(D)}$$

$$\leq \sqrt{B} \sum_{m=1}^{\infty} \left( \sum |c_n|^2 \left( \frac{4L \sin(2\pi L)}{\pi (m^2 - 4L^2)} \right)^2 \right)^{\frac{1}{2}} \leq \sqrt{B} \sum_{m=1}^{\infty} \frac{4L \sin(2\pi L)}{\pi (m^2 - 4L^2)} .$$

A similar argument shows that $T \leq \sqrt{B} \sum_{m=1}^{\infty} \frac{4L \cos(2\pi L)}{\pi ((m - \frac{1}{2})^2 - 4L^2)}$.

We recall the partial fraction expansion for $\tan t = \frac{1}{t} - \sum_{k=1}^{\infty} \frac{2t}{(\pi/2 - k\pi)(t^2 - k^2)}$, which are valid for every $t \in \mathbb{R}$ which is not an integer or a half integer. Thus,

$$S \leq \sqrt{B} \sum_{m=1}^{\infty} \frac{4L \sin(2\pi L)}{\pi (m^2 - 4L^2)} = \sqrt{B} \left( \frac{\sin(2\pi L)}{2\pi L} - \cos(2\pi L) \right)$$

$$T \leq \sqrt{B} \sum_{m=1}^{\infty} \frac{4L \cos(2\pi L)}{\pi (m^2 - 4L^2)} = \sqrt{B} \sin(2\pi L) \quad (3.6)$$
and \( S_0 + S + T \leq \sqrt{B} (1 - \cos(2\pi L) + \sin(2\pi L)) \). Thus, we have proved that
\[
\left\| \sum_{n=1}^{\infty} c_n e^{2\pi i (\lambda_n + \delta_n, x)} (1 - e^{2\pi i (\delta_n, x)}) \right\|_{L^2(D)} \leq \sqrt{B} \sigma(L) \tag{3.7}
\]
where we let \( \sigma(L) = 1 - \cos(2\pi L) + \sin(2\pi L) \).

By \( 3.7 \), we have \( 3.4 \) if \( \sqrt{B} \sigma(L) \leq \sqrt{A} \lambda \). So, if we let \( \lambda = \sqrt{BA^{-1}} \sigma(L) \) and we chose \( L \) so that \( \lambda < 1 \), we have concluded the proof.

Using standard trigonometric identities,
\[
\sqrt{BA^{-1}} \sigma(L) = \sqrt{BA^{-1}} (1 + \sqrt{2} \sin(2\pi L - \pi/4)),
\]
and it easy to see that \( 1.2 \) is equivalent to \( \sqrt{BA^{-1}} (1 + \sqrt{2} \sin(2\pi L - \pi/4)) < 1 \).

To prove \( 1.3 \), we observe that by \( 3.7 \),
\[
\left\| \sum_{n=1}^{\infty} c_n e^{2\pi i (\lambda_n + \delta_n, x)} \right\|_{L^2(D)} \leq \left\| \sum_{n=1}^{\infty} c_n (e^{2\pi i (\lambda_n + \delta_n, x)} - e^{2\pi i (\lambda_n, x)}) \right\|_{L^2(D)} + \left\| \sum_{n=1}^{\infty} c_n e^{2\pi i (\lambda_n, x)} \right\|_{L^2(D)} \leq \sqrt{B} \sigma(L) + \sqrt{B}
\]
and similarly that \( \left\| \sum_{n=1}^{\infty} c_n e^{2\pi i (\lambda_n + \delta_n, x)} \right\|_{L^2(D)} \geq \sqrt{A} - \sqrt{B} \sigma(L) \).

\( \square \)

**Proof of Theorem 1.3** Let \( D = D_1 \cup \ldots \cup D_N \); by Paley-Wiener theorem, \( E(\Lambda + \Delta) \) is a Riesz basis of \( L^2(D) \) if for every finite set of constants \( \{c_n\} \subseteq \mathbb{C} \) such that \( \sum |c_n|^2 = 1 \),
\[
\left\| \sum_{n=1}^{\infty} c_n (e^{2\pi i (\lambda_n + \delta_n, x)} - e^{2\pi i (\lambda_n, x)}) \right\|_{L^2(D)}^2 = \sum_{j=1}^{N} \left\| \sum_{n=1}^{\infty} c_n (e^{2\pi i (\lambda_n + \delta_n, x)} - e^{2\pi i (\lambda_n + \delta_n, x)}) \right\|_{L^2(D_j)}^2 < \lambda^2 A \tag{3.8}
\]
with \( 0 < \lambda < 1 \). Since \( E(\Lambda) \) is a Riesz basis for \( L^2(D) \), it is a frame for each \( L^2(D_j) \) with the same frame bounds. The proof of Theorem 1.1 (see \( 3.7 \)) shows that if we let \( c'_n = c_n e^{2\pi i (\lambda_n, \bar{x}_j)} \), \( L_j = \sup_{x \in D_j} |\langle \delta_n, x - \bar{x}_j \rangle| \) and \( \sigma(L_j) = 1 - \cos(2\pi L_j) + \sin(2\pi L_j) \), we get
\[
\left\| \sum_{n=1}^{\infty} c_n (e^{2\pi i (\lambda_n, x)} - e^{2\pi i (\lambda_n + \delta_n, x)}) \right\|_{L^2(D_j)}^2 = \left\| \sum_{n=1}^{\infty} c'_n e^{2\pi i (\lambda_n, x - \bar{x}_j)} (1 - e^{2\pi i (\delta_n, x - \bar{x}_j)}) \right\|_{L^2(D_j)}^2 < B \sigma(L_j)^2.
\]
Therefore, (3.8) follows if \( B \sum_{j=1}^{N} \sigma(L_j)^2 < \lambda^2 A \).

We let \( L = L_1 + \ldots + L_N \) and \( L_j = t_j L \), and we show that
\[
\sigma(t_j L) \leq \sqrt{t_j} \sigma(L)
\]
whenever \( L \) is as in (1.8). With (3.9),
\[
B \sum_{j=1}^{N} \sigma(L_j)^2 \leq B \sigma^2(L) \sum_{j=1}^{N} t_j = B \sigma^2(L),
\]
and if \( L < \frac{1}{2} K \), the proof of Theorem 1.1 shows that \( B \sigma^2(L) < A \), as required.

To prove (3.9), we let 
\[
f(t) = \sigma(t L) - \sqrt{t} \sigma(L) = 1 - \cos(2\pi t L) + \sin(2\pi t L) - \sqrt{t} \sigma(L),
\]
and observe that 
\[
f''(t) = \frac{1}{4t^2} \sigma(L) + 4\pi^2 L^2 (\cos(2\pi t L) - \sin(2\pi t L)) > 0
\]
because by (1.8), \( 2\pi t L < 2\pi L < \pi K \leq \frac{\pi}{2} \) and so \( \cos(2\pi t L) - \sin(2\pi t L) \geq 0 \). Thus, \( f \) is convex in \([0, 1]\) and that implies \( f \leq 0 \).

Proof of Corollary 1.6. Let \( D = \bigcup I_j \). We apply Theorem 1.5 with \( D_j = I_j \) and \( \bar{x}_j \) the mid-point of \( I_j \). So, (1.8) reduces to
\[
\sum_{j=1}^{N} \frac{|I_j|}{2} \sup_{n \in \mathbb{Z}} |\delta_n| \leq \frac{1}{2} K
\]
which implies (1.9).

Proof of Theorem 1.4. We argue as in the proof of Theorem 1.2 in [17]. For simplicity, we assume \( d = 2 \) and \( D = D_1 \times D_2 \). We let \( \mu_n = \lambda_n + \delta_n \), with \( n = (n_1, n_2) \in \mathbb{Z}^2 \) and we let \( \{c_n\} \) be a finite set in \( \mathbb{C} \); by (1.3) (with \( d = 1 \)),
\[
\left\| \sum_{n_1, n_2} c_n e^{2\pi i (\mu_{n_1} x_1 + \mu_{n_2} x_2)} \right\|_{L^2(D_1 \times D_2)}^2
\]
\[
= \int_{D_2} \left( \int_{D_1} \left| \sum_{n_1} \left( \sum_{n_2} c_n e^{2\pi i \lambda_{n_1} x_1} \right) e^{2\pi i \mu_{n_2} x_2} \right|^2 \right) dx_1 \right) dx_2
\]
\[
\leq B(1 + \sigma(L))^2 \int_{D_2} \sum_{n_1} \left| \sum_{n_2} c_n e^{2\pi i \mu_{n_2} x_2} \right|^2 dx_2
\]
\[ \leq B(1 + \sigma(L))^2 \sum_{n_1} \int_{D_2} \left| \sum_{n_2} c_n e^{2\pi i \mu_{n_2} x_2} \right|^2 \, dx_2 \]
\[ \leq B^2(1 + \sigma(L))^4 \sum_{n_1,n_2} |c_n|^2 \leq B^2(1 + \sigma(L))^4. \]

as required. The proof of the lower bound inequality is very similar. \( \square \)

4. Proof of Theorem 1.7

We prove first the following easy

**Lemma 4.1.** Let \( q > 1 \) be a positive integer. For every \( \alpha \in \mathbb{R} \) and every \( N > 0 \), there exists integers \( a = a(\alpha, N) \) such that

\[ 0 \leq \alpha - \frac{a}{q^N} \leq \frac{1}{q^N}. \]

**Proof.** Assume that \( \alpha \geq 0 \), since the proof in the other case is very similar. We can write \( \alpha = m + \frac{a_1}{q} + \frac{a_2}{q^2} + \ldots + \frac{a_N}{q^N} + \ldots \), where \( m = \lfloor \alpha \rfloor \) and \( 0 \leq a_i \leq q - 1 \). Thus,

\[ 0 \leq \alpha - \left( m + \frac{a_1}{q} + \ldots + \frac{a_N}{q^N} \right) = \sum_{n=N+1}^{\infty} \frac{a_n}{q^n} \leq \frac{q-1}{q^{N+1}} \sum_{h=0}^{\infty} \frac{1}{q^h} = \frac{1}{q^N}. \]

Therefore, \( \frac{a}{q^N} = m + \frac{a_1}{q} + \ldots + \frac{a_N}{q^N} \) is as required. \( \square \)

**Proof of Theorem 1.7.** We can assume \( d = 1 \), since the proof in the general case is similar. By Lemma 4.1 for each \( \lambda_n \in \Lambda \) we can find \( a_n \in \mathbb{Z} \) so that \( 0 \leq \frac{\lambda_n}{p} - \frac{a_n}{q} \leq \frac{1}{q} \); thus

\[ \lambda_n - \frac{p}{q} a_n \leq \frac{p}{q} < L \]

So, the sequence \( E \left( \left\{ \frac{\lambda_n}{q} a_n \right\}_{n \in \mathbb{Z}} \right) \) is an exponential basis of \( L^2(D) \).

To prove the second part of the theorem, we recall that by the remark after Corollary 1.3 \( \frac{K}{\text{diam}(D)} \) is a \( \ell^2 \) stability bound for \( E(\Lambda) \), and so \( L = \frac{K}{\sqrt{d \text{diam}(D)}} \) is a \( \ell^\infty \) stability bound. Since \( K = \frac{1}{4} \) when \( A = B \), we have concluded the proof. \( \square \)
The next corollary applies when \( D \) is a disjoint union of intervals of \( \mathbb{R} \). Its proof follows directly from Theorem 1.7 and (1.9).

**Corollary 4.2.** Under the assumptions of Corollary 1.6, for every \( \frac{p}{q} < \frac{K}{|D|} \) there exists \( \Gamma \subset \frac{p}{q}Z \) so that \( E(\frac{p}{q}Z) \) is a Riesz basis of \( L^2(D) \). If \( L^2(D) \) has an orthogonal exponential basis, we can chose \( \frac{p}{q} > 4|D| \).

5. Remarks and open problems

We have considered domains \( D \) that have exponential bases, and we have found stability results that depend on the ratio \( \frac{A}{B} \) of the frame bounds. We do not claim that all our stability bounds are sharp, but our results seems to indicate that bases for which \( \frac{A}{B} \) is largest are the most stable. We are wondering what are the supremum and infimum of \( \frac{A}{B} \) for all exponential bases on \( D \). We know that \( \frac{A}{B} = 1 \) only for orthogonal bases (see Proposition 3.3.9 in [5]), but for general domains we not know how close \( \frac{A}{B} \) can be to one or to zero.

Our Theorem 1.7 shows that an exponential basis on \( D \subset [0,1]^d \) can be extracted from the standard orthonormal basis of a "large" hyper-cube. We are wondering whether an exponential basis of \( D \) can extracted from one of \([0,1]^d\), and in general, if an exponential basis of \( D \subset P \) can be extracted from one of \( P \). This is true when \( D \) and \( P \) are hyper-cubes in \( \mathbb{R}^d \), but to the best of our knowledge it is not known for other domains.

We are also wondering about the size of the spectral gaps of exponential bases on \( D \). That is, for a given basis \( E(\Lambda) \) we consider \( \delta = \delta_\Lambda = \inf_{k \neq h} ||\lambda_k - \lambda_h||_\infty \) and we wonder how small or how large the \( \delta_\Lambda \)'s can be. To the best of our knowledge, the answer to this question is not known even when \( D \) is a segment of \( \mathbb{R} \). In [12] the Authors found an upper bound for \( \delta_\Lambda \), but not an universal upper bound that depends only on \( D \). Exponential bases are separated (see e.g. [20] or [10]) and so \( \delta_\Lambda > 0 \), but can we find bases \( E(\Lambda) \) with \( \delta_\Lambda \) arbitrarily small? Our Theorem 1.7 seems to indicate that it is possible to construct exponential bases on \( D \) with arbitrarily small spectral gaps, but we do not have enough elements to conjecture that that is indeed the case.

**References**

[1] Avdonin, S. A. *On the question of Riesz bases of exponential functions in \( L^2 \).* (translated into English in: Vestnik Leningrad Univ. Ser. Mat., 13, pp 203–211 (1979)

[2] Balan, R. *Stability theorems for Fourier frames and wavelet Riesz bases,* J. Four Anal Appl., 3, (1997) 499–504.
[3] Botkin, N.D. and Turova-Botkina, V. L. An algorithm for finding the Chebyshev center of a convex polyhedron, Appl. Math. Optim 29 (1994), 211–222.
[4] Casazza, P. The art of frame theory. Taiwanese J. Math. 4 (2000), no. 2, 129–201.
[5] Christensen, O. A Paley-Wiener theorem for frames. Proc. Amer. Math. Soc. 123 (1995), no. 7, 2199-2201.
[6] Christensen, O. An Introduction to Frames and Riesz Bases, Birkhauser, Boston, 2003.
[7] De Carli, L. and Kumar, A. Exponential bases of two dimensional trapezoids, to appear in Proc. AMS, (2013).
[8] Grepstad, S. and Lev, N. Multi-tiling and Riesz bases, arXiv:1212.4679 (2012)
[9] Kadec, M.I. The exact value of the Paley-Wiener constant, Sov. Math. doklady, 5, 559-561, (1964).
[10] Kozma, G. and Nitzan, S. Combining Riesz bases arXiv:1210.6383 (2012)
[11] Lev, N. Riesz bases of exponentials on multiband spectra. Proc. Amer. Math. Soc. 140 (2012), no. 9, 3127–3132.
[12] Iosevich, A. and Pedersen, S. How large are the spectral gaps?, Pac. J. Math. Vol. 192, (2000) No. 2, 307–314.
[13] Lyubarskii, Y. and Rashkovskii A., Complete interpolating sequences for Fourier transforms supported by convex symmetric polygons, Ark. Mat., 38 (2000), 139–170.
[14] Paley, R. and Wiener, N. Fourier transforms in the complex domain. Amer. Math. Soc. Colloquium Publications vol. 19. Amer. Math. Soc., New York, (1934).
[15] Seip, K. On the connection between exponential bases and certain related sequences in $L^2(−\pi, \pi)$. J. Funct. An. 130 (1995) no. 1, 131–160.
[16] Seip, K. A simple construction of exponential bases in $L^2$ of the union of several intervals. Proc. Edinburgh Math. Soc. (2) 38 (1995) no. 1, 171–177.
[17] Sun, W. and Zhou, X. On the stability of multivariate trigonometric systems. J. Math Anal. Appl. 235 (1999), 159–167.
[18] Sun, W. and Zhou, X. On Kadec’s $\frac{1}{4}$-Theorem and the Stability of Gabor Frames. Applied and Comput. Harm. An. 7 (1999) no. 2, 239–242.
[19] Su, W. and Zhou, X. A Sharper Stability Bound of Fourier Frames J. of Fourier Analysis and Appl. Vol. 5, (1), (1999) 67–71.
[20] Young, R. M. An introduction to nonharmonic Fourier series. Revised first edition. Academic Press, Inc., San Diego, CA, 2001.
[21] Wu, D.I; Zhou, J and Hu, A. A new approximate algorithm for the Chebyshev center. Automatica J. IFAC 49 (2013), no. 8, 2483–2488.
[22] Zygmund, A. Trigonometric series, Cambridge University Press, 1988.

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