Conformal Transformations and Weak Field Limit of Scalar-Tensor Gravity

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The weak field limit of scalar tensor theories of gravity is discussed in view of conformal transformations. Specifically, we consider how physical quantities, like gravitational potentials derived in the Newtonian approximation for the same scalar-tensor theory, behave in the Jordan and in the Einstein frame. The approach allows to discriminate features that are invariant under conformal transformations and gives contributions in the debate of selecting the true physical frame. As a particular example, the case of $f(R)$ gravity is considered.

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I. INTRODUCTION

The current cosmological observations point out a spatially flat model with a bulk of dark matter and dark energy related to a large negative pressure necessary to explain the observed accelerating expansion of the Hubble fluid and the large scale structure [1–11]. Despite of the observational evidences, the nature and the origin of dark sector remain a non-solved puzzle of theoretical physics that give rise to a plethora of alternative cosmological scenarios. Most of them are based either on the existence of new fields (aimed to address the ”dark” problem at fundamental level) or on extensions and modifications of General Relativity. In this latter picture, the accelerating behavior and the amount of dark matter can be seen as different geometric effects [12–44].

From a genuine theoretical viewpoint, a straightforward way to study dynamics is to look for conformally related models in order to disentangle further degrees of freedom. Such new degrees are not present in the standard view where only the Hilbert-Einstein action and perfect fluid matter are taken into account. In particular, conformally equivalent theories can be used to select viable cosmological models [45]. This point has to be discussed in some detail. In fact, by conformally transforming cosmological models can happen that some features as couplings and potentials can be directly related to the cosmological observables. The ”selection” means that in a given conformal frame, some observational features are more evident. In [45], examples in this sense are given. In particular, it is shown that several non-minimally coupled models, if conformally transformed, give rise to an effective cosmological constant and then can be directly matched with observations in the $\Lambda$CDM framework.

On the other hand, further scalar fields (degrees of freedom) into the gravitational Lagrangian give rise to two separate classes of theories: minimally and non-minimally coupled theories. In general, also higher-order theories of gravity can be reduced to the non-minimally coupled standard (see [15] for details).

In the first case, the gravitational coupling is the Newton constant. The scalar fields are added to the Ricci scalar $R$ in the gravitational Lagrangian. In this case, we are dealing with the so-called Einstein frame.

In the second case, the gravitational coupling is a function of space and time and it is dynamically related to the scalar fields. The paradigm is the Brans-Dicke gravity, formerly deduced by Jordan which is closely related to what in later times got the name ”Brans-Dicke gravity” [46,47]. It consists of a scalar field $\phi$ non-minimally coupled to $R$ and a kinetic term for the scalar field into the gravitational action. As a result, the coupling is non-minimal and the gravitational interaction changes with distance and time according to the Mach principle. The straightforward generalization is to take into account theories where also a self-interacting potential or more scalar fields are present.

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Furthermore, gravitational theories non-linear in the Ricci scalar $R$ or containing other curvature invariants can be reduced to scalar-tensor ones. In general, when we take into account non-minimal couplings or higher-order terms, we are dealing with the **Jordan frame**.

The Einstein and Jordan frames are related by geometrical maps that are the **conformal transformations** and the question is whether such frames are only mathematically equivalent or also physically equivalent. The problem of identifying the physical frame has been long debated and nowadays strongly emerges in order to address the problem of "dark sector" either from a geometrical or a material viewpoint. [49]

An important example is related to the geodesic motion. In the Jordan frame, in vacuum, neutral massive test particles fall along time-like geodesics. This is not true in the Einstein frame where they deviate from geodesic motion due to a force coming from the conformal scalar field gradient. As a consequence, from conformal transformations point of view, the Equivalence Principle holds only in the Jordan frame. It is important to stress that such a Principle is the basic foundation of relativistic theories of gravity. Then, a representation-independent formulation should physically discriminate between frames. No final result holds in this sense and the violation of the Equivalence Principle (in the Einstein frame) could be interpreted as the fact that frames are not physically equivalent. On the other hand, if the Equivalence Principle holds in a given frame and not in any frame means that it is not a covariant feature but only a kinematical one. In other words, Equivalence Principle is not sufficient to discriminate between conformal frames.

However, the vacuum interpretation has to be discussed. It have two different meanings: If the energy-momentum tensor is $T_{\mu\nu} = 0$, the scalar field belongs to the gravitational field sector then it is a part of geometry. On the other hand, if $T_{\mu\nu} + T^\phi_{\mu\nu} = 0$, the sum of matter fluid and scalar field contributions is zero. In this case, the scalar field can be considered as a matter field. In the Jordan frame, both interpretations have the same meaning as soon as the scalar field gradient is zero. In other words, the contracted Bianchi identities must hold. However, the meaning of vacuum is different and then also the motion of test particles moving along geodesics is different. This fact has to be carefully considered if one wants to discriminate between Einstein and Jordan frames.

Furthermore, there are results where exact cosmological solutions accelerate in one frame but not in the other. This fact could mean that, for an astronomer attempting to fit observations, the two frames are not physically equivalent. [50, 51]. In these situations, one must state precisely what the physical equivalence is and the concept is not obvious at all. In a naive formulation, such an equivalence could be related to the fact that it should be possible to select a set of physically invariant quantities that can be conformally transformed.

As we said, conformal transformations allow to disentangle the further gravitational degrees of freedom coming from general actions [13, 52]. The idea is to perform a conformal rescaling of the space-time metric $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu}$ and a redefinition of the scalar field $\phi$ as $\phi \rightarrow \tilde{\phi}$. New dynamical variables $\{\tilde{g}_{\mu\nu}, \tilde{\phi}\}$ are thus obtained. The scalar field redefinition allows, for example, to cast the kinetic energy density of this field in a canonical form. The new set of variables $\{\tilde{g}_{\mu\nu}, \tilde{\phi}\}$ defines the **Einstein conformal frame**, while $\{g_{\mu\nu}, \phi\}$ constitutes the **Jordan frame**. When a scalar degree of freedom $\phi$ is present in the theory, as in scalar tensor or $f(R)$ gravity, it generates the transformation to the Einstein frame in the sense that the rescaling is completely determined by a function of $\phi$. In principle, infinite conformal frames could be introduced, giving rise to many representations of the theory.

Let the pair $\{\mathcal{M}, g_{\mu\nu}\}$ be a space-time, with $\mathcal{M}$ a smooth manifold of dimension $n \geq 2$ and $g_{\mu\nu}$ a (pseudo)-Riemannian metric on $\mathcal{M}$. The point-dependent rescaling of the metric tensor

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

where $\Omega = \Omega(x)$ is a nowhere vanishing, regular function, called a **Weyl** or **conformal** transformation. Obviously the transformation rule for the contorvariant metric tensor is $\tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}$.

Due to this metric rescaling, the lengths of space-like and time-like intervals and the norms of space-like and time-like vectors change, while null vectors and null intervals of the metric $g_{\mu\nu}$ remain null in the rescaled metric $\tilde{g}_{\mu\nu}$ (in this sense, they are conformally invariant quantities). The light cones are left unchanged by the transformation [11] and the space-times $\{\mathcal{M}, g_{\mu\nu}\}$ and $\{\mathcal{M}, \tilde{g}_{\mu\nu}\}$ exhibit the same causal structure; the converse is also true. A vector that is time-like, space-like, or null with respect to the metric $g_{\mu\nu}$ has the same character with respect to $\tilde{g}_{\mu\nu}$, and **vice-versa**.

In wide sense, conformal invariance corresponds to the absence of characteristic lengths and masses. In general, the effective potential of scalar field $V(\phi)$ coming from conformal transformations contains dimensional parameters (such as masses, that are further "characteristic gravitational lengths"). This means that the further degrees of freedom coming from extended or alternative gravities give rise to features that could play a fundamental role in the dynamics of astrophysical structures, from the "infrared" side, and in quantum gravity, from the "ultraviolet" side.

However, an important remark is necessary here. Typically, the absence of characteristic lengths and masses is called **scale-invariance**, and even for scale-invariance, several different interpretations exist. This means that conformal invariance and scale-invariance must be precisely distinguished. As discussed, for example, in [52], a class of isotropic
cosmologies in fourth-order gravity with Lagrangians of the form $L = FR + KG$, where $R$ and $G$ are the Ricci and Gauss-Bonnet scalars respectively, can be made scale-invariant. It is important to stress that such theories can be also conformally transformed. This is a typical case where the two invariances can be clearly distinguished. In general, scale invariance means that physical systems do not change if scales of length, energy, or other variables, are multiplied by some factor. Technically this transformation is a dilation. Such a feature can be part of a larger conformal symmetry where angles are preserved.

In this paper, we want to address the problem of how conformally transformed models behave in the weak field limit approximation. This issue could be extremely relevant in order to select conformally invariant physical quantities.

This point deserves some discussion. In general, a gauge theory is a type of field theory where the Lagrangian is invariant under a continuous group of local transformations. In particular, graviations is a field theory on a principal frame bundle whose gauge symmetries are covariant transformations \[55\]. In this case, the term "gauge" refers to redundant degrees of freedom in the Lagrangian. The transformations between possible gauges, called gauge transformations, form a Lie group which is the symmetry group or the gauge group of the theory.

On the other hand, the conformal group is the group of transformations from a space to itself that preserve all angles within the space. More formally, it is the group of transformations that preserve the conformal geometry of the space. These definitions immediately point out that the gauge and conformal groups do not coincide and then breaking gauge invariance could be not related to conformal invariance.

Furthermore, gauge invariance is broken in the weak field limit approximation and redundant degrees of freedom can be gauged away by this procedure. Comparing two conformally related models in the weak field limit could be a procedure to select physically invariant quantities once the behavior of gauges in the two frames is determined and their conformal transformations derived.

With these considerations in mind, we will take into account the weak field limit of scalar-tensor gravity in the Jordan frame (Sec. II) and compare it with the analogous in the Einstein frame (Sec. III). The particular case of $f(R)$ gravity will be considered in Sec. IV. Discussion and conclusions are drawn in Sec.V.

### II. SCALAR TENSOR GRAVITY IN THE JORDAN FRAME

The action of a scalar tensor field theory of gravity in 4 dimensions is

$$A^{JF} = \int d^{4}x \sqrt{-g} \left[ \phi R + V(\phi) + \omega(\phi) \phi_{,\alpha} \phi^{\alpha} + \mathcal{X} \mathcal{L}_m \right]$$  \hspace{1cm} (2)$$

where $R$ is the Ricci scalar and $\mathcal{X} = 8\pi G$ where we assumed $c = 1$. The convention for the Ricci tensor is $R_{\mu\nu} = R^\sigma_{\mu\sigma\nu}$, while for the Riemann tensor $R_{\alpha\beta\mu\nu} = \Gamma^\sigma_{\beta\mu,\nu} + \ldots$. The affinities are the standard Christoffel symbols of the metric: $\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\alpha}(g_{\alpha,\beta} + g_{\beta,\alpha} - g_{\alpha\beta})$. The adopted signature is $(+ - - -)$ while the coordinates $x^\mu = (t, x^1, x^2, x^3) = (t, \mathbf{x})$ are the isotropic coordinates. The Greek index runs from 0 to 3; the Latin index runs from 1 to 3. Let us note that the action

$$A^{JF} = \int d^{4}x \sqrt{-g} \left[ F(\phi) R + V(\phi) + \omega(\phi) \phi_{,\alpha} \phi^{\alpha} + \mathcal{X} \mathcal{L}_m \right]$$  \hspace{1cm} (3)$$

is apparently more general than \[2\]. In fact by substituting $F(\phi) \rightarrow \phi$, we obtain only a new definition of functions $\omega(\phi)$ and $V(\phi)$ so the two formulations are essentially equivalent.

The term $\mathcal{L}_m$ is the minimally coupled ordinary matter contribution considered as a perfect fluid; $\omega(\phi)$ is a function of the scalar field and $V(\phi)$ is its potential which specifies the dynamics. Actually if $\omega(\phi) = \pm 1, 0$ the nature and the dynamics of the scalar field is fixed. It can be a canonical scalar field, a phantom field or a field without dynamics (see e.g. \[56, 57\] for details). In the metric approach, the field equations are obtained by varying the action \[2\] with respect to $g_{\mu\nu}$ and $\phi$. The field equations are

$$\phi R_{\mu\nu} - \frac{\phi R + V(\phi) + \omega(\phi) \phi_{,\alpha} \phi^{\alpha}}{2} g_{\mu\nu} + \omega(\phi) \phi_{,\mu} \phi_{,\nu} - \phi_{,\mu\nu} + g_{\mu\nu} \Box \phi = \mathcal{X} T_{\mu\nu}$$  \hspace{1cm} (4)$$

$$2 \omega(\phi) \Box \phi + \omega_{,\phi}(\phi) \phi_{,\alpha} \phi^{\alpha} - R - V_{,\phi}(\phi) = 0$$

and the trace equation is

$$\phi R + 2V(\phi) + \omega(\phi) \phi_{,\alpha} \phi^{\alpha} - 3 \Box \phi = -\mathcal{X} T$$  \hspace{1cm} (5)$$
Here we introduced, respectively, the energy-momentum tensor of matter and the d’Alembert operator

\[ T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}}, \quad \Box(\cdot) = \partial_\sigma(\sqrt{-g} g^{\sigma\tau} \partial_\tau(\cdot)) \]  \hspace{1cm} (6)

\[ T = T^\sigma_\sigma \] is the trace of energy-momentum tensor and \( V_\phi = \frac{dV}{d\phi} \), \( \omega_\phi(\phi) = \frac{d\omega}{d\phi} \). If we assume that the Lagrangian density \( L_m \) of matter depends only on the metric components \( g_{\mu\nu} \) and not on its derivatives, we obtain \( T_{\mu\nu} = 1/2 L_m g_{\mu\nu} - \delta L_m / \delta g^{\mu\nu} \). Let us consider a source with mass \( M \). The energy-momentum tensor is

\[ T_{\mu\nu} = \rho u_\mu u_\nu, \quad T = \rho \] \hspace{1cm} (7)

where \( \rho \) is the mass density, \( u_\mu \) satisfies the condition \( g^{00} u_0 u_i = 1 \), and \( u_i = 0 \). Here, we are not interested to the internal structure. It is useful to get the expression of \( L_m \). In fact from the definition (6), we have

\[ \delta \int d^4x \sqrt{-g} L_m = -\int d^4x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu} = -\int d^4x \sqrt{-g} \rho u_\mu u_\nu \delta g^{\mu\nu} \] \hspace{1cm} (8)

From the mathematical properties of metric tensor we have

\[ \delta(\sqrt{-g} \rho) = 1/2 \sqrt{-g} \rho u^\mu u^\nu \delta g_{\mu\nu} = -1/2 \sqrt{-g} \rho u_\mu u_\nu \delta g^{\mu\nu} \] \hspace{1cm} (9)

then we find

\[ L_m = 2 \rho \] \hspace{1cm} (10)

The variation of density is given by

\[ \delta \rho = \frac{\rho}{2} (g_{\mu\nu} - u_\mu u_\nu) \delta g^{\mu\nu} \] \hspace{1cm} (11)

order to deal with standard self-gravitating systems, any theory of gravity has to be developed in its Newtonian or post-Newtonian limit depending on the order of approximation in terms of squared velocity \( v^2 \) \[58, 59\]. The Newtonian limit starts from developing the metric tensor (and other additional quantities in the theory) with respect to the dimensionless velocity \( v \) of the moving massive bodies embedded in the gravitational potential. The perturbative development takes only first term of \((0, 0)\)- and \((i, j)\)-component of metric tensor \( g_{\mu\nu} \) (for details, see \[59, 60\]). The metric assumes the form

\[ ds^2 = (1 + 2\Phi) dt^2 - (1 - 2\Psi) \delta_{ij} dx^i dx^j \] \hspace{1cm} (12)

where the gravitational potentials \( \Phi, \Psi < 1 \) are proportional to \( v^2 \). The Ricci scalar is approximated as \( R = R^{(1)} + R^{(2)} + \ldots \) where \( R^{(1)} \) is proportional to \( \Phi \), and \( \Psi \), while \( R^{(2)} \) is proportional to \( \Phi^2, \Psi^2 \) and \( \Phi \Psi \). In this context, also the scalar field \( \phi \) is approximated as the Ricci scalar. In particular we get \( \phi = \phi^{(0)} + \phi^{(1)} + \ldots \) while the functions \( V(\phi) \) and \( \omega(\phi) \) can be substituted by their corresponding Taylor series.

From the lowest order of field Eqs. (4) we have

\[ V(\phi^{(0)}) = 0, \quad V_\phi(\phi^{(0)}) = 0 \] \hspace{1cm} (13)

and also in the scalar tensor gravity a missing cosmological component in the action (1) implies that the space-time is asymptotically Minkowskian; moreover the ground value of scalar field \( \phi \) must be a stationary point of potential. In the Newtonian limit, we have

\[ 1 \] The velocity \( v \) is here expressed in light speed units.
$$\triangle \left[ \Phi - \frac{\phi^{(1)}}{\phi^{(0)}} \right] - \frac{R^{(1)}}{2} = \frac{\mathcal{X} \rho}{\phi^{(0)}}$$

$$\left\{ \triangle \left[ \Psi + \frac{\phi^{(1)}}{\phi^{(0)}} \right] + \frac{R^{(1)}}{2} \right\}_{ij} \delta_{ij} + \left\{ \Psi - \Phi - \frac{\phi^{(1)}}{\phi^{(0)}} \right\}_{,ij} = 0$$

$$\triangle \phi^{(1)} + \frac{V_{\phi\phi}(\phi^{(0)})}{2\omega(\phi^{(0)})} \phi^{(1)} + \frac{R^{(1)}}{2\omega(\phi^{(0)})} = 0$$

$$R^{(1)} + 3 \frac{\triangle \phi^{(1)}}{\phi^{(0)}} = -\frac{\mathcal{X} \rho}{\phi^{(0)}}$$

where $\triangle$ is the Laplacian in the flat space. These equations are not simply the merging of field equations of GR and a further massive scalar field, but come out to the fact that the scalar tensor gravity generates a coupled system of equations with respect to Ricci scalar $R$ and scalar field $\phi$. The gravitational potentials $\Phi$, $\Psi$ and the Ricci scalar $R^{(1)}$ are given by

$$\Phi(x) = -\frac{\mathcal{X}}{4\pi \phi^{(0)}} \int d^3x' \frac{\rho(x')}{|x-x'|} - \frac{1}{8\pi} \int d^3x' \frac{R^{(1)}(x')}{|x-x'|} + \frac{\phi^{(1)}(x)}{\phi^{(0)}}$$

$$\Psi(x) = \Phi(x) + \frac{\phi^{(1)}(x)}{\phi^{(0)}}$$

$$R^{(1)}(x) = -\frac{\mathcal{X} \rho(x)}{\phi^{(0)}} - 3 \frac{\triangle \phi^{(1)}(x)}{\phi^{(0)}}$$

and supposing that $2\omega(\phi^{(0)}) \phi^{(0)} - 3 \neq 0$ we find for the scalar field $\phi^{(1)}$ the Yukawa-like field equation

$$\left[ \triangle - m_\phi^2 \right] \phi^{(1)} = \frac{\mathcal{X} \rho}{2\omega(\phi^{(0)}) \phi^{(0)} - 3}$$

where we introduced the mass definition

$$m_\phi^2 \doteq -\frac{\phi^{(0)} V_{\phi\phi}(\phi^{(0)})}{2\omega(\phi^{(0)}) \phi^{(0)} - 3}.$$ 

It is important to stress that the potential $\Psi$ can be found also as

$$\Psi(x) = \frac{1}{8\pi} \int d^3x' \frac{R^{(1)}(x')}{|x-x'|} - \frac{\phi^{(1)}(x)}{\phi^{(0)}}$$

see for example [61].

By using the Fourier transformation, the solution of Eq. (16) has the following form

$$\phi^{(1)}(x) = -\frac{\mathcal{X}}{2\omega(\phi^{(0)}) \phi^{(0)} - 3} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{\hat{\rho}(k) e^{ik \cdot x}}{k^2 + m_\phi^2}$$

The expressions (15) and (19) represent the most general solution of any scalar-tensor gravity in the Newtonian limit. Since the superposition principle is yet valid (the field Eqs. (14) are linear), it is sufficient to consider the solutions generated by a point-like source with mass $M$. Then if we consider $\rho = M \delta(x)$ the solutions are [59, 61].
\begin{align*}
\phi^{(1)}(x) &= -\frac{1}{2\omega(\phi(0))} \frac{r_g}{|x|} e^{-m_\phi |x|} \\
R^{(1)}(x) &= -\frac{4\pi r_g}{\phi(0)} \delta(x) + \frac{3 m_\phi^2}{2\omega(\phi(0)) \phi(0) - 3 |x|} \frac{r_g}{|x|} e^{-m_\phi |x|} \\
\Phi(x) &= -\frac{GM}{\phi(0)|x|} \left\{ 1 - \frac{e^{-m_\phi |x|}}{2\omega(\phi(0)) \phi(0) - 3} \right\} \\
\Psi(x) &= -\frac{GM}{\phi(0)|x|} \left\{ 1 + \frac{e^{-m_\phi |x|}}{2\omega(\phi(0)) \phi(0) - 3} \right\}
\end{align*}
where \( r_g = 2GM \) is the Schwarzschild radius. In the case \( V(\phi) = 0 \), the scalar field is massless and \( \omega(\phi) = -\omega_0/\phi \), we obtain

\begin{align*}
\Phi(x) &= \Phi_{BD}(x) = -\frac{GM}{\phi(0)|x|} \left[ \frac{2(2 + \omega_0)}{2\omega_0 + 3} \right] = -\frac{G^* M}{|x|} \\
\Psi(x) &= \Psi_{BD}(x) = -\frac{G^* M}{|x|} \left( \frac{1 + \omega_0}{2 + \omega_0} \right)
\end{align*}
the well-known Brans-Dicke solutions \[46\] with Eddington’s parameter \( \gamma = \frac{1+\omega_0}{2+\omega_0} \)[62] where the gravitational constant is defined as \( G \rightarrow G^* = \frac{G}{\phi(0)^2} \frac{2(2+\omega_0)}{2\omega_0+3}. \)

### III. SCALAR TENSOR GRAVITY IN THE EINSTEIN FRAME

Let us now introduce the conformal transformation \([11]\) to show that scalar-tensor theories are, in general, conformally equivalent to the Einstein theory plus minimally coupled scalar fields. However if standard matter is present, the conformal transformation generates the non-minimal coupling between the matter component and the scalar field.

By applying the transformation \([11]\), the action in \([2]\) can be reformulated as follows

\[ A^{EF} = \int d^4 x \sqrt{-g} \left[ \Xi \tilde{R} + W(\tilde{\phi}) + \tilde{\omega}(\tilde{\phi}) \tilde{\phi}_\alpha \tilde{\phi}^{\alpha} + \mathcal{L}_m \right] \]

in which \( \tilde{R} \) is the Ricci scalar relative to the metric \( \tilde{g}_{\mu \nu} \) and \( \Xi \) is a generic constant. The two actions \([2]\) and \([22]\) are mathematically equivalent. In fact the conformal transformation is given by imposing the condition

\[ \sqrt{-g} \left[ \phi R + V(\phi) + \omega(\phi) \phi_\alpha \phi^{\alpha} + \mathcal{L}_m \right] = \sqrt{-\tilde{g}} \left[ \Xi \tilde{R} + W(\tilde{\phi}) + \tilde{\omega}(\tilde{\phi}) \tilde{\phi}_\alpha \tilde{\phi}^{\alpha} + \mathcal{L}_m \right] \]

The relations between the quantities in the two frames are

\[ \tilde{\omega}(\tilde{\phi}) d\tilde{\phi}^{-2} = \frac{\Xi}{2} [2 \phi \omega(\phi) - 3] \left( \frac{d\phi}{\phi} \right)^2 \]

\[ W(\tilde{\phi}) = \frac{\Xi^2}{\phi(\tilde{\phi})^2} V(\phi(\tilde{\phi})) \]

\[ \tilde{\mathcal{L}}_m = \frac{\Xi^2}{\phi(\tilde{\phi})^2} \mathcal{L}_m \left( \frac{\Xi \tilde{g}_{\mu \nu}}{\phi(\tilde{\phi})} \right) \]

\[ \phi \Omega^{-2} = \Xi \]
The field equations for the new fields $\tilde{g}_{\mu\nu}$ and $\tilde{\phi}$ are

$$
\Xi \tilde{R}_{\mu
u} - \frac{\Xi \tilde{R} + W(\tilde{\phi}) + \tilde{\omega}(\tilde{\phi}) \tilde{\phi}_{,\alpha} \tilde{\phi}^{,\alpha}}{2} \tilde{g}_{\mu\nu} + \tilde{\omega}(\tilde{\phi}) \tilde{g}_{\mu
u} \tilde{\phi}_{,\nu} = \mathcal{X} \tilde{T}_{\mu\nu}
$$

and

$$
2 \tilde{\omega}(\tilde{\phi}) \tilde{\Box} \tilde{\phi} + \tilde{\omega}_{,\alpha} \tilde{\phi}_{,\alpha} \tilde{\phi}^{,\alpha} - W_{,\phi}(\tilde{\phi}) - \mathcal{X} \tilde{\delta} \tilde{\mathcal{L}}_{\phi} = 0
$$

(25)

where $\tilde{T}_{\mu\nu}$ and $\tilde{\Box}$ are the re-definition of the quantities $[3]$ with respect to the metric $\tilde{g}_{\mu\nu}$. The field Eqs. (25) can be obtained from (4) by substituting all geometrical and physical quantities in terms of conformally transformed ones. In particular we have

$$
R_{\mu\nu} = \tilde{R}_{\mu\nu} + 2 \ln \Omega;_{\mu\nu} + 2 \ln \Omega_{,\mu} \ln \Omega_{,\nu} + [\tilde{\Box} \ln \Omega - 2 \ln \Omega^{\gamma \sigma} \ln \Omega_{,\gamma} \ln \Omega_{,\sigma}] \tilde{g}_{\mu\nu}
$$

$$
R = \Omega^2 \left[ \tilde{R} + 6 \Box \ln \Omega - 3 \ln \Omega^{\gamma \sigma} \ln \Omega_{,\gamma} \ln \Omega_{,\sigma} \right]
$$

$$
\phi_{,\mu\nu} = \tilde{\phi}_{,\mu\nu} + 2 \phi_{,\mu} \phi_{,\nu} - \ln \Omega^{\gamma \sigma} \phi;_{\gamma} \tilde{g}_{\mu\nu}
$$

$$
\Box(\cdot) = \Omega^2 \tilde{\Box}(\cdot) - 2 \ln \Omega^{\gamma \sigma} \partial_{\gamma} \partial_{\sigma}(\cdot)
$$

(26)

The integration of field Eqs. (25) is only formal because we do not know the analytical expression of the coupling function between the matter and the scalar field $\tilde{\phi}$ (see the third line of (24)). We can make some assumptions on the parameter $\Xi$ and the function $\tilde{\omega}(\tilde{\phi})$ in the minimally coupled Lagrangian (22) and on the function $\omega(\phi)$ in the nonminimally coupled Lagrangian (2). If we choose $\tilde{\omega}(\tilde{\phi}) = -1/2$, $\Xi = 1$ and $\omega(\phi) = -\omega_0/\phi$, the transformation between the scalar fields $\phi$ and $\tilde{\phi}$ is given by the first line in (24), that is

$$
\tilde{\phi}(\phi) = \tilde{\phi}_0 + \sqrt{2\omega_0 + 3} \ln \phi
$$

$$
\phi(\tilde{\phi}) = \exp \left( \frac{\tilde{\phi} - \tilde{\phi}_0}{\sqrt{2\omega_0 + 3}} \right)
$$

(27)

where obviously $\omega_0 > -3/2$ and $\tilde{\phi}_0$ is an integration constant. The potential $W$ and the matter Lagrangian $\tilde{L}_m$ are

$$
W(\tilde{\phi}) = \exp \left( -\frac{2\tilde{\phi}}{\sqrt{2\omega_0 + 3}} \right) V\left( e^{\sqrt{2\omega_0 + 3}} \right)
$$

$$
\tilde{L}_m = 2 \rho \exp \left( -\frac{2\tilde{\phi}}{\sqrt{2\omega_0 + 3}} \right)
$$

(28)

In both frames, the scalar fields are expressed as perturbative contributions on the cosmological background $(\phi^{(0)}, \phi^{(0)})$ with respect to the dimensionless quantity $v^2$. Then also for the scalar field $\tilde{\phi}$, we can consider the develop $\tilde{\phi} = \tilde{\phi}^{(0)} + \tilde{\phi}^{(1)} + \ldots$. Such a develop can be applied to the transformation rule (27) and we obtain

$$
\tilde{\phi}(\phi) = \sqrt{2\omega_0 + 3} \ln \phi = \sqrt{2\omega_0 + 3} \ln \phi^{(0)} + \frac{\sqrt{2\omega_0 + 3}}{\phi^{(0)}} \phi^{(1)} + \ldots \equiv \tilde{\phi}^{(0)} + \tilde{\phi}^{(1)} + \ldots
$$

$$
\phi(\tilde{\phi}) = e^{\sqrt{2\omega_0 + 3}} = e^{\sqrt{2\omega_0 + 3} \tilde{\phi}^{(0)}} + \frac{e^{\sqrt{2\omega_0 + 3}}}{{\sqrt{2\omega_0 + 3}}} \tilde{\phi}^{(1)} + \ldots \equiv \phi^{(0)} + \phi^{(1)} + \ldots
$$

(29)

2 Without losing generality, we can set $\tilde{\phi}_0 = 0$. 
Since we are interested in the Newtonian limit of field Eqs. (25), we can assume, for the conformally transformed metric \( \tilde{g}_{\mu\nu} \), an expression as (12) but with some differences. In fact from the conformal transformation (1) and from the last line of (24), we have

\[
\tilde{g}_{\mu\nu} = \varphi \, g_{\mu\nu} + \left[ \varphi^{(0)} \, g^{(1)}_{\mu\nu} + \varphi^{(1)} \, \eta_{\mu\nu} \right] + \ldots = \tilde{\eta}_{\mu\nu} + \tilde{g}_{\mu\nu}^{(1)} + \ldots
\]

then the conformally transformed metric becomes

\[
\tilde{g}_{\mu\nu}^{(0)} = (\tilde{\varphi}^{(0)} + 2 \tilde{\Phi}) \, dt^2 - (\tilde{\varphi}^{(0)} - 2 \tilde{\Psi}) \, dx^i \, dx^j
\]

and the relation between the gravitational potentials in the two frames is

\[
\tilde{\Phi} - \varphi^{(0)} \, \Phi = \frac{\varphi^{(1)}}{2}, \quad \tilde{\Psi} - \varphi^{(0)} \, \Psi = -\frac{\varphi^{(1)}}{2}
\]

Then the field Eqs. (25) become

\[
\frac{\Delta \tilde{\Phi}}{\varphi^{(0)}} - \frac{\tilde{R}^{(1)}}{\varphi^{(0)}} = \mathcal{X} \tilde{T}^{(1)}_{00}
\]

\[
\left\{ \frac{\Delta \tilde{\Psi}}{\varphi^{(0)}} + \frac{\tilde{R}^{(1)}}{2 \varphi^{(0)}} \right\} \delta_{ij} + \frac{(\tilde{\Psi} - \tilde{\Phi})_{ij}}{\varphi^{(0)}} = 0
\]

\[
\frac{\Delta \tilde{\phi}^{(1)}}{\varphi^{(0)}} - W_{\tilde{\phi} \tilde{\phi}}(\tilde{\phi}^{(0)}) \, \tilde{\phi}^{(1)} - \mathcal{X} \left[ \delta \tilde{L}_m / \delta \tilde{\phi} \right]^{(1)} = 0
\]

\[
\tilde{R}^{(1)} = -\mathcal{X} \tilde{T}^{(1)}
\]

where also in this case we have \( W(\tilde{\phi}^{(0)}) = 0 \) and \( W_{\tilde{\phi} \tilde{\phi}}(\tilde{\phi}^{(0)}) = 0 \). However these conditions are an obvious consequence of the conformal transformation of conditions \( V(\phi^{(0)}) = 0 \) and \( V_{\phi}(\phi^{(0)}) = 0 \). In fact we can figure out that \( V(\phi) \propto (\phi - \phi^{(0)})^2 \) and then \( W(\tilde{\phi}) \propto \left( e^{\sqrt{2 \omega_0 + 3}} - \phi^{(0)} \right)^2 \) which, by using relations (29), satisfies the above conditions.

Finally, we note that \( W_{\tilde{\phi} \tilde{\phi}}(\tilde{\phi}^{(0)}) = V_{\phi\phi}(\phi^{(0)}) \), and by the definition of mass \( m_{\phi}^2 \), given in Eq. (17), we obtain \( W_{\tilde{\phi} \tilde{\phi}}(\tilde{\phi}^{(0)}) = m_{\phi}^2 / \phi^{(0)} \). Finally, the energy-momentum tensor \( \tilde{T}_{\mu\nu} \) is given by the following expression

\[
\tilde{T}_{\mu\nu} = \rho \exp \left( -\frac{2 \tilde{\phi}}{\sqrt{2 \omega_0 + 3}} \right) \tilde{u}_\mu \tilde{u}_\nu
\]

where \( \tilde{g}_{\mu\nu} \tilde{u}^\mu \tilde{u}^\nu = 1 \) then \( \tilde{u}_0 = \sqrt{\phi^{(0)} + 2 \tilde{\Phi}} \). In the Newtonian limit, we find \( \tilde{T}^{(1)}_{00} = \rho / \phi^{(0)} \) and \( \tilde{T}^{(1)} = \rho / \phi^{(0)}^2 \). It remains only to calculate the source term \( \delta \tilde{L}_m / \delta \tilde{\phi} \) of the scalar field \( \tilde{\phi}^{(1)} \). From the third line of (24) and, by using the transformation rules (27), we find the coupling between the scalar field and the ordinary matter

---

3 With the assumptions of the metric (31) the Ricci tensor \( \tilde{R}_{\mu\nu} \) in the Newtonian limit has the form \( \frac{\Delta \tilde{\phi}}{\varphi^{(0)}} \) (a similar behaviour for \( \tilde{R}^{(1)}_{ij} \)), where the Ricci scalar is scaled by the factor \( \phi^{(0)}^2 \). The same scaling occurs for the Laplacian: \( \Delta \rightarrow \frac{\Delta}{\phi^{(0)}} \).
gravity. The issue is easily overcome once the correct analogy between these theories can be recovered by a direct analogy with Brans-Dicke gravity simply supposing the Brans-Dicke characteristic parameter \( \omega_0 \) vanishing (see [63] for a discussion).}.

The redefinition of the gravitational constant \( G \) (as performed in the Jordan frame \( G \rightarrow G^* \) in the case of Brans-Dicke theory [46]) is not available when we are interested to compare the outcomes in both frame. In fact the coupling constant between \( R \) and \( \phi \) is vanishing), then we find the same outcomes of General Relativity with ordinary matter. However by supposing the Jordan frame as starting point and coming back via conformal transformation, we find that the gravitational constant is not invariant and depends on the background value of the scalar field in the Einstein frame, that is \( G \rightarrow G_{\text{eff}} \propto e^{-\phi(0)} G \).

### IV. THE CASE OF \( f(R) \)-GRAVITY

Recently, several authors claimed that higher-order theories of gravity and among them, \( f(R) \) gravity, are characterized by an ill defined behavior in the Newtonian regime. In particular, it is discussed that Newtonian corrections of the gravitational potential violate experimental constraints since these quantities can be recovered by a direct analogy with Brans-Dicke gravity simply supposing the Brans-Dicke characteristic parameter \( \omega_0 \) vanishing (see [63] for a discussion). Actually, the calculations of the Newtonian limit of \( f(R) \)-gravity, directly performed in a rigorous manner, have showed that this is not the case \([55 \, 60 \, 64 \, 65]\) and it is possible to discuss also the analogy with Brans-Dicke gravity. The issue is easily overcome once the correct analogy between \( f(R) \)-gravity and the corresponding scalar-tensor framework is taken into account. It is worth noticing that several results already achieved in the Newtonian regime, see e.g.\([68 \, 67]\), are confirmed by the present approach.

In literature, it is shown that \( f(R) \) gravity models can be rewritten in term of a scalar-field Lagrangian non-minimally coupled with gravity but without kinetic term implying that the Brans-Dicke parameter is \( \omega(\phi) = 0 \). This fact is considered the reason for the ill-definition of the weak field limit that should be \( \omega \rightarrow \infty \) inside the Solar System.
Let us deal with the $f(R)$ gravity formalism in order to set correctly the problem. The action is

$$A_{f(R)}^{JF} = \int d^4 x \sqrt{-g} \left[ f(R) + \mathcal{X} \mathcal{L}_m \right]$$  \hspace{1cm} (39)$$

and the field equations are

$$f_R R_{\mu \nu} - \frac{f}{2} g_{\mu \nu} - f_{R;\mu \nu} + g_{\mu \nu} \Box f_R = \mathcal{X} T_{\mu \nu}$$  \hspace{1cm} (40)$$

with the trace

$$3 \Box f' + f_R R - 2f = \mathcal{X} T$$  \hspace{1cm} (41)$$

where $f_R = \frac{df}{dR}$. These equations can be recast in the framework of scalar-tensor gravity as soon as we select a particular expression for the free parameters of the theory. The result is the so-called O’Hanlon theory \[69\] which can be written as

$$A_{OH}^{JF} = \int d^4 x \sqrt{-g} \left[ \phi R + V(\phi) + \mathcal{X} \mathcal{L}_m \right]$$  \hspace{1cm} (42)$$

The field equations are obtained by starting from Eqs. (4)

$$\phi R_{\mu \nu} - \frac{\phi R + V(\phi)}{2} g_{\mu \nu} - \phi_{,\mu \nu} + g_{\mu \nu} \Box \phi = \mathcal{X} T_{\mu \nu}$$

$$R + V_\phi(\phi) = 0$$  \hspace{1cm} (43)$$

$$\phi R + 2V(\phi) - 3 \Box \phi = -\mathcal{X} T$$

By supposing that the Jacobian of the transformation $\phi = f_R$ is non-vanishing, the two representations can be mapped one into the other considering the following equivalence

$$\omega(\phi) = 0$$

$$V(\phi) = f - f_R R$$  \hspace{1cm} (44)$$

$$\phi V_\phi(\phi) - 2V(\phi) = f_R R - 2f$$

From the definition of the mass \[17\] we have $\phi V_\phi(\phi) - 2V(\phi) = 3 m_\phi^2 \phi^{(1)}$, then we have also $f_R R - 2f = 3 m_\phi^2 \phi^{(1)}$ and by performing the Newtonian limit on the function $f$ \[59\], we get $f_R(0) R^{(1)} = -3 m_\phi^2 \phi^{(1)}$. The spatial evolution of Ricci scalar is obtained by solving the field Eq. (40)

$$R^{(1)} = -\frac{3 m_\phi^2 \phi^{(1)}}{f_R(0)} = \frac{m_\phi^2 r_g e^{-m_\phi |x|}}{f_R(0) |x|}$$  \hspace{1cm} (45)$$

without using the conformal transformation \[51, 50\]. The solution for the potentials $\Phi, \Psi$ are obtained simply by setting $\omega(\phi) = 0$ in Eqs. \[20\] and $\phi^{(0)} = f_R(0)$. In the case $f(R) \rightarrow R$, from the second line of \[44\], $V(\phi) = 0 \rightarrow m_\phi = 0$ and the solutions \[20\] become the standard Schwarzschild solution in the Newtonian limit.

Finally, we can consider a Taylor expansion \footnote{The terms resulting from $R^n$ with $n \geq 3$ do not contribute at the Newtonian order.} of the form $f = f_R(0) R^{(1)} + \frac{f_{RR}(0)}{2} R^{(1)2}$ so that the associated scalar field reads $\phi = f_R(0) + f_{RR}(0) R^{(1)}$. The relation between $\phi$ and $R^{(1)}$ is $R^{(1)} = \frac{\phi - f_R(0)}{f_{RR}(0)}$ while the self-interaction potential (second line of \[44\]) turns out the be $V(\phi) = -\frac{(\phi - f_R(0))^2}{2 f_{RR}(0)}$ satisfying the conditions $V(f_R(0)) =$
and $V_{\phi}(f_R(0)) = 0$. In relation to the definition of the scalar field, we can opportuneidentify $f_R(0)$ with a constant value $\phi^{(0)} = f_R(0)$ which justifies the previous ansatz for matching solutions in the limit of General Relativity. Furthermore, the mass of the scalar field can be expressed in term of the Lagrangian parameters as:

$$m_{\phi}^2 = \frac{1}{3}\phi^{(0)} V_{\phi\phi}(\phi^{(0)}) = -\frac{f_R^{(0)}}{3f_{RR}(0)}.$$  

Also in this case the value of mass is the same obtained by solving the problem without invoking the scalar tensor analogy [59, 60]. However with this last remark, it is clear the analogy between $f(R)$-gravity and a particular class of scalar tensor theories [69].

V. DISCUSSION AND CONCLUSIONS

The debate of selecting a physical frame by conformal transformations has become pressing in relation to the problem of cosmological dark components. In fact, both material and geometrical origin of such dark effects are today valid and discrimination could be, in some sense, related to the selection of a set of physical quantities that are conformally invariant, a part the discovery of some new ingredient at fundamental level.

Besides this issue, there is the general problem to understand how the gauge group and the conformal group are related in a given theory of gravity. Actually, the gauge invariance breaks in the weak field limit but some conformal quantities could be preserved contributing in the selection of the physical frame.

In this paper, we have taken into account the problem of weak field limit of scalar-tensor theories of gravity showing how the Newtonian limit behaves in the Jordan and in the Einstein frame. The general result is that Newtonian potentials, masses and other physical quantities can be compared in both frames once the perturbative analysis is performed. The main point is that if such an analysis is carefully developed in the same frame, the perturbative process can be controlled step by step leading to coherent results in both frames. In other words, also if the gauge invariance is broken, there is the possibility to control conformal quantities. In particular, it is important to fix the relation between conformally related potentials in order to understand how gravitational coupling and Yukawa-like corrections behave. Specifically, the potentials

$$\Phi(x) = -\frac{GM}{\phi^{(0)}|x|} \left\{ 1 - \frac{e^{-m_{\phi}|x|}}{2\omega(\phi^{(0)}) \phi^{(0)} - 3} \right\} \quad \Psi(x) = -\frac{GM}{\phi^{(0)}|x|} \left\{ 1 + \frac{e^{-m_{\phi}|x|}}{2\omega(\phi^{(0)}) \phi^{(0)} - 3} \right\}$$

achieved in the Jordan frame (see Eqs. (20) can be rigorously compared with their counterparts in the Einstein frame

$$\tilde{\Phi} = -\frac{GM}{|x|}, \quad \tilde{\Psi} = \tilde{\Phi} \quad \tilde{\phi} = \sqrt{2\omega_0 + 3} \ln \phi^{(0)} + \frac{1}{\phi^{(0)} \sqrt{2\omega_0 + 3} |x|} r_\omega e^{-m_{\phi}|x|}$$

see Eqs. (47) when we set $\tilde{\omega}(\tilde{\phi}) = -1/2, \tilde{\Xi} = 1$ and $\omega(\phi) = -\omega_0/\phi$. This is the main result of this paper which, in principle, could constitute a paradigm to compare physical quantities in both frames. In this sense, the observable consequences of conformal transformations can be achieved. In a forthcoming paper, we will discuss how to give experimental constraints to these results.

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