ON IDEAL CLASS GROUPS OF NUMBER FIELDS AND SELMER GROUPS OF MODULAR GALOIS REPRESENTATIONS

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Abstract. Let $p$ be an odd prime number and $M_{f,D}$ a quadratic twist by a square-free integer $D$ of an $\mathbb{F}_p$-valued representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ attached to a modular form $f$. Putting $K_{f,D}$ as a number field cut out by $M_{f,D}$, we study the ideal class group $\mathrm{Cl}_{K_{f,D}}$ of $K_{f,D}$. We give a condition that $\mathrm{Cl}_{K_{f,D}} \otimes \mathbb{F}_p$ has $M_{f,D}$ as its quotient $\mathrm{Gal}(K_{f,D}/\mathbb{Q})$-module in terms of Bloch-Kato’s Selmer group of $M_{f,D}$. This is a generalization of the result of Prasad and Shekhar on elliptic curves for modular forms of higher weight.

1. Introduction

The ideal class groups of number fields, the Tate-Shafarevich groups and the Selmer groups of elliptic curves are central objects to study in number theory. Various relations between them have been studied by many people. For example in [23], Washington considered a specific elliptic curve defined by the equation of the simplest cubic, and studied a relation between its 2-Selmer group and the class group of its 2-division field. In [14], Nekovář studied a relation between the ideal class groups of certain quadratic fields and the Tate-Shafarevich groups of twists of the cubic Fermat curve. We note here that they studied the ideal class groups of abelian number fields over $\mathbb{Q}$. Recently, non-abelian number fields over $\mathbb{Q}$ have been also studied well. Let $E$ be an elliptic curve over $\mathbb{Q}$ and $p$ an odd prime number. Hiranouchi [10] studied the class number of the $p^n$-th division field $\mathbb{Q}(E[p^n])$ in terms of the Morell-Weil group $E(\mathbb{Q})$, where the extension $\mathbb{Q}(E[p^n])/\mathbb{Q}$ is non-abelian in general. Ohshita [17] further generalize Hiranouchi’s result for a number field $F$ cut out by a general $p$-adic Galois representation. He studied the class number of $F$ using the Selmer group of the $p$-adic representation.

On the other hand, Prasad and Shekhar [19] studied the structure of the ideal class group of the $p$-th division field $\mathbb{Q}(E[p])$ as a $\mathrm{Gal}(\mathbb{Q}(E[p])/\mathbb{Q})$-module rather than the class number. Here we briefly introduce their result.

Theorem (Prasad-Shekhar). Let $\bar{\rho}_{E,p} : \mathbb{G}_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F}_p)$ be a $\mathbb{F}_p$-valued Galois representation associated to $E[p]$ where $\mathbb{G}_\mathbb{Q}$ denotes the absolute Galois group of $\mathbb{Q}$. Suppose that the following conditions on $E$ hold:

1. $E$ has good reduction at $p$.
2. The $p$-adic Tate module $T_p(E)$ is crystalline.
3. The $p$-adic Hasse invariant of $E$ is zero.

Then $\bar{\rho}_{E,p}$ is potentially semistable at $p$. In particular, the following conditions hold:

1. The $p$-adic Igusa zeta function of $E$ is $1$.
2. The $p$-adic $L$-function of $E$ is $1$.
3. The $p$-adic BSD conjecture is true for $E$.

Moreover, if $E$ is an elliptic curve over $\mathbb{Q}$, then $\bar{\rho}_{E,p}$ is irreducible.
(1) $E$ has good reduction at $p$.

(2) In the case that $E$ has good ordinary reduction at $p$, $a_p(E) \equiv 1 \pmod{p}$, and $E$ has no CM over an extension of $\mathbb{Q}_p$, then $\rho_{E,p}$ is wildly ramified at $p$.

(3) For every finite prime $\ell \neq p$, the Tamagawa number $c_\ell(E/\mathbb{Q}_\ell)$ of $E/\mathbb{Q}_\ell$ is a $p$-adic unit.

(4) The $\mathbb{F}_p$-representation $\bar{\rho}_{E,p}$ of $G_\mathbb{Q}$ is irreducible.

Then $\dim_{\mathbb{F}_p}(\text{Sel}(\mathbb{Q}, E[p])) \geq 2$ implies that the $\mathbb{F}_p$-representation $\text{Cl}_{\mathbb{Q}(E[p])} \otimes \mathbb{F}_p$ of $\text{Gal}(\mathbb{Q}(E[p])/\mathbb{Q})$ has $E[p]$ as its quotient representation.

In this article, we study a higher weight analogue of the above theorem of Prasad and Shekhar. In other words, we consider $\mathbb{F}_p$-valued representations attached to modular forms rather than those associated to elliptic curves. Here we describe the details. Let $f(z) = \sum_{n=0}^{\infty} a_n q^n$ be a normalized Hecke eigen newform of even weight $k \geq 2$ and level $\Gamma_0(N)$ where $q := e^{2\pi\sqrt{-1}z}$ and the parameter $z$ is in the complex upper half plane. We fix $p$ again as an odd prime number and assume $p$ splits completely in the Hecke field of $f$. Then there is an associated $p$-adic representation $\rho_f^0 : G_\mathbb{Q} \to \text{Aut}_{\mathbb{Q}_p}(V_f^0) \cong \text{GL}_2(\mathbb{Q}_p)$ where $V_f^0$ denotes its representation space. Note that we take $V_f^0$ as the one with the Hodge-Tate weight $\{0, k-1\}$. Fixing a Galois stable $\mathbb{Z}_p$-lattice $T_f^0$ of $V_f^0$, we put $A_f^0 := V_f^0/T_f^0$, $M_f^0 := T_f^0/pT_f^0$ and $\bar{\rho}_f^0 : G_\mathbb{Q} \to \text{Aut}_{\mathbb{F}_p}(M_f^0) \cong \text{GL}_2(\mathbb{F}_p)$ as a group homomorphism corresponding to $M_f^0$. We take twists of these representations to make them self-dual. Let $\chi_{\text{cyc}}$ (resp. $\omega_{\text{cyc}}$) denotes the $p$-adic (resp. mod $p$) cyclotomic character. We define $\rho_f := \rho_f^0 \otimes \chi_{\text{cyc}}^{-1}, \bar{\rho}_f := \bar{\rho}_f^0 \otimes \omega_{\text{cyc}}^{-1}$. Next we consider various quadratic twists of $\rho_f$ and $\bar{\rho}_f$. Taking a quadratic discriminant $D$ and corresponding quadratic character $\chi_D$ of $G_\mathbb{Q}$, we put $\rho_{f,D} := \rho_f \otimes \chi_D, \bar{\rho}_{f,D} := \bar{\rho}_f \otimes \chi_D$ and write $V_{f,D}, M_{f,D}$ as their representation spaces. We take $T_{f,D}$ as the Galois stable $\mathbb{Z}_p$-lattice of $V_{f,D}$ which is the same as $T_f^0$ as a $\mathbb{Z}_p$-module and put $A_{f,D} := V_{f,D}/T_{f,D}$. Let $K_{f,D}$ be the Galois extension of $\mathbb{Q}$ cut out by $\bar{\rho}_{f,D}$. Note that we consider the representation $\bar{\rho}_{f,D}$ and the number field $K_{f,D}$ as analogues of $\rho_{E,p}$ and $\mathbb{Q}(E[p])$ in the theorem of Prasad and Shekhar respectively.

The main theorem of this article is the following.

**Theorem 1.1.** Under the above setting, we further assume the following conditions are satisfied.

1. $p \nmid N$.
2. If $f$ is supersingular at $p$, then $k \leq p + 1$.
   
   If $f$ is ordinary at $p$, then $p - 1 \nmid k - 1$ and the conditions in Proposition 5.11 do not occur.
3. $\text{Im}(\bar{\rho}_f^0) \supset \text{SL}_2(\mathbb{F}_p)$
4. $c(\mathbb{Q}_\ell, A_{f,D}) = 1$ for all prime $\ell | N$. 


Then \( \dim_{\mathbb{F}_p}(H^1_f(\mathbb{Q}, M_{f,D})) \geq 2 \) implies that the \( \mathbb{F}_p \)-representation \( \text{Cl}_{K_{f,D}} \otimes \mathbb{F}_p \) of \( \text{Gal}(K_{f,D}/\mathbb{Q}) \) has \( M_{f,D} \) as its quotient representation. Here \( c(\mathbb{Q}_\ell, A_{f,D}) \) denotes the Tamagawa factor of \( A_{f,D} \) at \( \ell \) and \( H^1_f(\mathbb{Q}, M_{f,D}) \) is Bloch-Kato’s Selmer group of \( M_{f,D} \).

**Remark 1.2.** Due to Bloch and Kato’s conjecture, the order of \( \text{III}^{BK}_p(\mathbb{Q}, A_{f,D}) \) is related to the central value \( L(k/2, \chi_D, f) \) of twisted \( L \) function of \( f \) if it does not vanish. Here \( \text{III}^{BK}_p(\mathbb{Q}, A_{f,D}) \) denotes the \( p \)-part of Bloch-Kato’s Tate-Shafarevich group of \( A_{f,D} \). Especially, if \( p \) divides the algebraic part \( L^{\text{alg}}(k/2, \chi_D, f) \) of \( L(k/2, \chi_D, f) \), then we have \( \text{III}^{BK}_p(\mathbb{Q}, A_{f,D}) \neq 0 \) assuming the conjecture. On the other hand, we know that if \( \text{III}^{BK}_p(\mathbb{Q}, A_{f,D}) \neq 0 \), then we have \( \dim_{\mathbb{F}_p}(H^1_f(\mathbb{Q}, M_{f,D})) \geq 2 \) because of the existence of the generalized Cassels-Tate pairing for \( A_{f,D} \). Hence, under Bloch and Kato’s conjecture, if \( p \) divides \( L^{\text{alg}}(k/2, \chi_D, f) \), we can see that the class group \( \text{Cl}_{K_{f,D}} \) has the representation \( M_{f,D} \) as its quotient from our main result. Thus we conjecturally obtain an Herbrand-Ribet type phenomenon for a representation associated to a modular form.

At the end of this section, we write the outline of this paper. In section 2, we recall the definition of Bloch-Kato’s Selmer groups and Tate-Shafarevich groups for \( p \)-adic representations and write a sketch of the proof of Theorem 1.1 dividing it into 3 steps. In section 3, we prove the first step of the proof. In section 4, the basic notions of Tamagawa factor are explained and the second step is proved. In section 5, we prove the third step and complete the proof. Finally in section 6, we introduce two numerical examples of Theorem 1.1.

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2. A sketch of the proof

In this section, we describe a sketch of the proof of Theorem 1.1. We mainly follow the strategy used in [3] in which we gave a condition that \( \text{Cl}_{\mathbb{Q}(E[p])} \otimes \mathbb{F}_p \) has other irreducible \( \text{Gal}(\mathbb{Q}(E[p])/\mathbb{Q}) \)-representation than \( E[p] \) as its quotient representation in the same setting as [19]. In [19], they used the classical \( p \)-Selmer group \( \text{Sel}_p(E/\mathbb{Q}) \) for an elliptic curve but, to treat representations coming from modular forms, we have to deal with Bloch-Kato’s Selmer group \( H^1_f \) which we first recall.
2.1. **Bloch-Kato’s Selmer groups and Tate-Shafarevich groups.** For a field $F$, $G_F$ denotes its absolute Galois group $\text{Gal}(\overline{F}/F)$. For every prime number $\ell$ and the $p$-adic representation $V_{f,D}$ which we define in Section 1, we fix a local condition $H^1_f(\mathbb{Q}_\ell, V_{f,D})$ in $H^1(\mathbb{Q}_\ell, V_{f,D})$ as

$$H^1_f(\mathbb{Q}_\ell, V_{f,D}) := H^1(\mathbb{Q}_\ell, V_{f,D}) := \ker (H^1(\mathbb{Q}_\ell, V_{f,D}) \to H^1(\mathbb{Q}_{ur, \ell}, V_{f,D})) \quad (\ell \neq p),$$

$$H^1_f(\mathbb{Q}_p, V_{f,D}) := \ker (H^1(\mathbb{Q}_p, V_{f,D}) \to H^1(\mathbb{Q}_p, V_{f,D} \otimes B_{crys})) \quad (\ell = p).$$

Here $\mathbb{Q}_{ur, \ell}$ is the maximal unramified extension of $\mathbb{Q}_\ell$ and $B_{crys}$ denotes Fontaine’s crystalline period ring which is defined in [II, Section 1]. We also define $H^1_f(\mathbb{Q}_\ell, A_{f,D}) := \pi (H^1_f(\mathbb{Q}_\ell, V_{f,D}))$ for every $\ell$, where $\pi : H^1(\mathbb{Q}_\ell, V_{f,D}) \to H^1(\mathbb{Q}_\ell, A_{f,D})$ is a homomorphism induced by a natural map $\pi : V_{f,D} \to A_{f,D}$.

**Definition 2.1.** For $V_{f,D}$ and $A_{f,D}$, we define Bloch-Kato’s Selmer groups as

$$H^1_f(\mathbb{Q}, V_{f,D}) := \ker \left( H^1(\mathbb{Q}, V_{f,D}) \xrightarrow{\prod \text{Loc}_\ell} \prod \frac{H^1(\mathbb{Q}_\ell, V_{f,D})}{H^1_f(\mathbb{Q}_\ell, V_{f,D})} \right),$$

$$H^1_f(\mathbb{Q}, A_{f,D}) := \ker \left( H^1(\mathbb{Q}, A_{f,D}) \xrightarrow{\prod \text{Loc}_\ell} \prod \frac{H^1(\mathbb{Q}_\ell, A_{f,D})}{H^1_f(\mathbb{Q}_\ell, A_{f,D})} \right),$$

where for every prime number $\ell$, $\text{Loc}_\ell$ denotes the restriction of cohomology classes to the decomposition group at $\ell$ and the products run over all prime numbers.

The $p$-part of Bloch-Kato’s Tate-Shafarevich group for $A_{f,D}$ which we write as $\prod_p^{BK}(\mathbb{Q}, A_{f,D})$ is defined in [II, Section 5] as follows.

**Definition 2.2.** We define the $p$-part of Bloch-Kato’s Tate-Shafarevich group for $A_{f,D}$ as

$$\prod_p^{BK}(\mathbb{Q}, A_{f,D}) := \frac{H^1_f(\mathbb{Q}, A_{f,D})}{\pi(H^1_f(\mathbb{Q}, V_{f,D}))},$$

where $\pi : H^1(\mathbb{Q}, V_{f,D}) \to H^1(\mathbb{Q}, A_{f,D})$ is a homomorphism induced by a natural map $\pi : V_{f,D} \to A_{f,D}$. In other words, $\prod_p^{BK}(\mathbb{Q}, A_{f,D})$ is defined by an exact sequence

$$0 \to \pi(H^1_f(\mathbb{Q}, V_{f,D})) \to H^1_f(\mathbb{Q}, A_{f,D}) \to \prod_p^{BK}(\mathbb{Q}, A_{f,D}) \to 0.$$

We note that $\pi(H^1_f(\mathbb{Q}, V_{f,D}))$ is the maximal divisible subgroup of $H^1_f(\mathbb{Q}, A_{f,D})$, so their quotient $\prod_p^{BK}(\mathbb{Q}, A_{f,D})$ is always finite which is a well-known conjecture for classical Tate-Shafarevich groups for elliptic curves.

We also define Bloch-Kato’s Selmer group for the finite module $M_{f,D}$. We have an exact sequence of $G_{\mathbb{Q}}$-modules

$$0 \to M_{f,D} \xrightarrow{1} A_{f,D} \xrightarrow{\times p} A_{f,D} \to 0.$$
from which we obtain an exact sequence
\[ 0 \to (A_{f,D})^G \otimes \mathbb{F}_p \to H^1(\mathbb{Q}, M_{f,D}) \xrightarrow{\iota} H^1(\mathbb{Q}, A_{f,D})[p] \to 0, \]
where the map \( \iota \) in the first exact sequence denotes the inclusion and the second denotes the one which induced by the first one.

**Definition 2.3.** We define Bloch-Kato’s Selmer group for \( M_{f,D} \) as
\[ H^1_f(\mathbb{Q}, M_{f,D}) := \iota^{-1}(H^1_f(\mathbb{Q}, A_{f,D})[p]). \]

The Selmer group \( H^1_f(\mathbb{Q}, M_{f,D}) \) can be also defined by using local conditions. In fact, we define local conditions for \( M_{f,D} \) at \( \ell \) as
\[ H^1_f(\mathbb{Q}_\ell, M_{f,D}) := \iota^{-1}(H^1_f(\mathbb{Q}_\ell, A_{f,D})). \]
Then we have
\[ H^1_f(\mathbb{Q}, M_{f,D}) = \text{Ker} \left( H^1_f(\mathbb{Q}, M_{f,D}) \xrightarrow{\prod_{\ell} H^1(\mathbb{Q}_\ell, M_{f,D})} \prod_{\ell} H^1_f(\mathbb{Q}_\ell, M_{f,D}) \right) \]
as in Definition 2.1.

**2.2. A sketch of the proof of Theorem 1.1.** Now we describe a sketch of the proof of Theorem 1.1. In the following, we fix a modular form \( f \) and a quadratic discriminant \( D \) and omit the suffixes of \( V_{f,D}, T_{f,D}, A_{f,D}, M_{f,D}, K_{f,D} \) as \( V, T, A, M, K \) when no confusion occurs.

**(Step 1)** We show that a restriction map
\[ \text{Res}_{K/\mathbb{Q}} : H^1(\mathbb{Q}, M) \to H^1(K, M)^{\text{Gal}(K/\mathbb{Q})} \]
is injective under the assumption (3) in Theorem 1.1.

Let \( F \) be a number field or a local field and \( N \) a \( G_F \)-module. We define the unramified cohomology group \( H^1_{\text{ur}}(F, N) \) as a subgroup of cohomology classes in \( H^1(F, N) \) which are trivial on the inertia subgroup at every place of \( F \). Assuming the claim in (Step 1), the restriction map \( \text{Res}_{K/\mathbb{Q}} \) induces an injective homomorphism between unramified cohomology groups
\[ \text{Res}_{K/\mathbb{Q}} : H^1_{\text{ur}}(\mathbb{Q}, M) \hookrightarrow H^1_{\text{ur}}(K, M)^{\text{Gal}(K/\mathbb{Q})}. \]
Using class field theory, we have \( H^1_{\text{ur}}(K, M)^{\text{Gal}(K/\mathbb{Q})} = \text{Hom}_{\text{Gal}(K/\mathbb{Q})}(\text{Cl}_K \otimes \mathbb{F}_p, M) \). Every nontrivial homomorphism in \( \text{Hom}_{\text{Gal}(K/\mathbb{Q})}(\text{Cl}_K \otimes \mathbb{F}_p, M) \) is surjective since we assume the condition (3) in Theorem 1.1 which implies that \( M^0_f \) and so \( M = M_{f,D} \) are irreducible. Hence every non-trivial homomorphism in \( \text{Hom}_{\text{Gal}(K/\mathbb{Q})}(\text{Cl}_K \otimes \mathbb{F}_p, M) \) realizes \( M \) as a quotient \( \text{Gal}(K/\mathbb{Q}) \)-module of \( \text{Cl}_K \otimes \mathbb{F}_p \). From this observation and the above injection between unramified cohomology groups, we obtain an implication,
\[ H^1_{\text{ur}}(\mathbb{Q}, M) \neq 0 \Rightarrow \text{Cl}_K \otimes \mathbb{F}_p \text{ has } M \text{ as its quotient } \text{Gal}(K/\mathbb{Q}) \text{-module}. \]
We will show the existence of non-trivial elements in $H^1_{ur}(\mathbb{Q}, M)$ using Bloch-Kato’s Selmer group of $M$ in the succeeding steps.

**Step 2** Under the assumption (4) in Theorem 1.4, we show that the image of $H^1_f(\mathbb{Q}, M)$ in $H^1(\mathbb{Q}_{ur}, M)$ is zero for any $\ell \neq p$. In other words, we show that elements in $H^1_f(\mathbb{Q}, M)$ are unramified outside $p$.

Here $H^1_f(\mathbb{Q}, M)$ is the Selmer group of $M$ defined in Definition 2.3. Assuming the claim in (Step 2), for a restriction map $\text{Res}_{ur} : H^1_f(\mathbb{Q}, M) \to H^1(\mathbb{Q}_{ur}, M)$, we have $\text{Ker}(\text{Res}_{ur}) \subset H^1_{ur}(\mathbb{Q}, M)$. Thus it suffices to show that $\text{Ker}(\text{Res}_{ur}) \neq 0$ to get the main theorem.

**Step 3** We study the image of $\text{Res}_{ur}$ and show that $\dim_{\mathbb{F}_p}(\text{Im}(\text{Res}_{ur})) \leq 1$.

This completes the proof since we assume $\dim_{\mathbb{F}_p}(H^1_f(\mathbb{Q}, M)) \geq 2$.

3. Injectivity of the restriction map

In this section, we prove the claim in (Step 1).

**Proposition 3.1.** Suppose that $\text{Im}(\bar{\rho}_f^0)$ contains $\text{SL}_2(\mathbb{F}_p)$ (the assumption (3) in Theorem 1.4). Then the restriction map

$$\text{Res}_{K/\mathbb{Q}} : H^1(\mathbb{Q}, M) \to H^1(K, M)^{\text{Gal}(K/\mathbb{Q})}$$

is injective.

(Proof of Proposition 3.1)

It suffices to show that $H^1(\text{Gal}(K/\mathbb{Q}), M) = 0$. We use the following lemma.

**Lemma 3.2.** Let $G$ be a finite group and $N$ a finite dimensional representation of $G$ over $\mathbb{F}_p$. If there is a normal subgroup $H$ of $G$ such that

1. $\#H$ is prime to $p$
2. $N^H = 0$

then $H^i(G, N) = 0$ for all $i \geq 0$.

For a proof of this lemma, see [3, Lemma 3.2]. We divide a proof of Proposition 3.1 into two cases.

(Case 1: $p \geq 5$)

Let $L := \mathbb{Q}(\zeta_p, \sqrt{D})$ where $\zeta_p$ is a primitive $p$-th root of unity. First we show that the image of $\bar{\rho}_f^0 : G_L \to \text{GL}_2(\mathbb{F}_p)$ still contains $\text{SL}_2(\mathbb{F}_p)$ when restricted to $G_L$. Let $F$ and $F_L$ be the fields corresponding to the kernel of $\bar{\rho}_f^0$ and $\bar{\rho}_f^0|_{G_L}$ in Galois theory respectively. So we have $\text{Gal}(F/\mathbb{Q}) \cong \text{Im}(\bar{\rho}_f^0) \supset \text{SL}_2(\mathbb{F}_p)$ and $\text{Gal}(F_L/L) \cong \ldots$
Gal(F/F \cap L) \cong \text{Im}(\bar{\rho}_f|_{G_L}).  Let F' be the intermediate field of the Galois extension \( F/Q \) corresponding to SL_2(\mathbb{F}_p). Then F' \cdot (F \cap L)/F' \) is an abelian extension as the extension F \cap L/Q is. Since we assume p \geq 5, SL_2(\mathbb{F}_p) is a perfect group. In other words, SL_2(\mathbb{F}_p) has no non-trivial abelian quotients. So we have F' \cdot (F \cap L) = F' and hence Gal(F/F \cap L) \cap SL_2(\mathbb{F}_p) = SL_2(\mathbb{F}_p) from Galois theory. Thus \text{Im}(\bar{\rho}_f|_{G_L}) \cong Gal(F/F \cap L) \supset SL_2(\mathbb{F}_p). Since \bar{\rho}_{f_D}|_{G_L} = \bar{\rho}_{f_D}|_{G_L}, the image of \bar{\rho}_{f_D} also contains SL_2(\mathbb{F}_p) which implies that \(-I \in \text{Im}(\bar{\rho}_{f_D}) \cong Gal(K/Q), \) where I denotes the unit matrix in SL_2(\mathbb{F}_p). Let H \subset Gal(K/Q) be the order 2 subgroup generated by \(-I. \) Then H satisfies the conditions (1), (2) in Lemma 3.2 and we have \( H^i(Gal(K/Q), M) = 0 \) for all \( i \geq 0. \)

(Case 2: \( p = 3 \))

In this case, we know \( #SL_2(\mathbb{F}_3) = 8 \) and the elements in SL_2(\mathbb{F}_3) are the following:

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}, \begin{pmatrix}
1 & 1 \\
1 & -1 \\
\end{pmatrix}, \begin{pmatrix}
-1 & 1 \\
1 & 1 \\
\end{pmatrix}, \begin{pmatrix}
0 & -1 \\
1 & 0 \\
\end{pmatrix}, \begin{pmatrix}
-1 & 0 \\
0 & -1 \\
\end{pmatrix}, \begin{pmatrix}
1 & -1 \\
-1 & 1 \\
\end{pmatrix}, \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}, \begin{pmatrix}
-1 & 1 \\
1 & -1 \\
\end{pmatrix}, \begin{pmatrix}
-1 & -1 \\
-1 & 1 \\
\end{pmatrix}, \begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix}
\]

We put

\[ A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}. \]

Since we assume that \text{Im}(\bar{\rho}_f|_{G_L}) \supset SL_2(\mathbb{F}_3), \) we can take \( \sigma \in G_Q \) such that \( \bar{\rho}_f(\sigma) = A_1. \) Since both mod 3 cyclotomic characters \( \omega_{cyc} \) and \( \chi_D \) are order 2, we have \( \bar{\rho}_{f_D}(\sigma) = A_1 \) or \( A_2. \) We can show that there are no non-trivial elements in M which fixed by \( A_1 \) or \( A_2 \) by direct computation. So if \( A_1 = \bar{\rho}_{f_D}(\sigma) \in Gal(K/Q) \) (resp. \( A_2 \in Gal(K/Q) \)), then the subgroup of Gal(K/Q) generated by \( A_1 \) (resp. \( A_2 \)) satisfies the conditions (1), (2) in Lemma 3.2 since every subgroup of SL_2(\mathbb{F}_3) is normal and 2-group. Thus we have \( H^i(Gal(K/Q), M) = 0 \) for all \( i \geq 0 \) from Lemma 3.2 \( \square \)

Thus the restriction map \( \text{Res}_{K/Q} : H^1(Q, M) \to H^1(K, M)_{\text{Gal}(K/Q)} \) is in fact an isomorphism although its injectivity is enough for the proof of the Theorem 4.1

4. Unramifiedness of \( H^1_f(Q, M) \) outside \( p \)

4.1. Tamagawa factor of \( A_{f,D} \) at \( \ell(\neq p) \). First, we introduce some basic notions on the Tamagawa factor of A. For a prime number \( \ell \neq p, \) let \( I_\ell \) denotes the inertia subgroup of \( G_{Q_\ell} \) and \( A := A_{\ell}/(A_{\ell})_{\text{div}}, \) where \( (A_{\ell})_{\text{div}} \) denotes the maximal divisible subgroup of \( A_{\ell}. \) So \( A \) is a finite group.

Definition 4.1. We define the p-part of the Tamagawa factor of A at \( \ell \) as

\[ c(Q_\ell, A) := \#A/(\text{Frob}_\ell - 1)A = \#A_{\text{Frob}_\ell = 1}. \]
Here Frobenius element in $\text{Gal}(\mathbb{Q}_\ell^u/\mathbb{Q}_\ell)$. Note that for an endo-
morphism $g$ of $\mathcal{A}$, the orders of $\ker(g)$ and $\coker(g)$ are the same since $\mathcal{A}$ is finite,
hence the second equality in the definition holds. Roughly speaking, this Tamagawa
factor of $A$ can be used to measure the difference $H^1_{ur}(\mathbb{Q}_\ell, M)$ and the local condition
$H^1_I(\mathbb{Q}_\ell, M)$.

**Proposition 4.2.** If $c(\mathbb{Q}_\ell, A) = 1$, then $H^1_I(\mathbb{Q}_\ell, M) = H^1_{ur}(\mathbb{Q}_\ell, M)$.

(Proof of Proposition 4.2)

We first consider the local condition with the coefficient in $A$.

$$H^1_{ur}(\mathbb{Q}_\ell, A)/H^1_I(\mathbb{Q}_\ell, A) \cong \ker(H^1_I(\mathbb{Q}_\ell, M) \to H^1_{ur}(\mathbb{Q}_\ell, M))$$

$$\cong \ker(V^I_{\ell}/(Frob_{\ell} - 1)(V^I_{\ell}) \to A^I/(Frob_{\ell} - 1)(A^I))$$

$$\cong \mathcal{A}/(Frob_{\ell} - 1)\mathcal{A}.$$  

Here we use the following identification in the second isomorphism above.

$$H^1_{ur}(\mathbb{Q}_\ell, V) = H^1(\mathbb{Q}_\ell^u/\mathbb{Q}_\ell, V^I_{\ell}) \cong V^I_{\ell}/(Frob_{\ell} - 1)(V^I_{\ell})$$

where the isomorphism $ev$ is given by evaluating every 1-cocycle with Frobenius. Thus
if $c(\mathbb{Q}_\ell, A) = \#\mathcal{A}/(Frob_{\ell} - 1)\mathcal{A} = 1$, we have $H^1_{ur}(\mathbb{Q}_\ell, A) = H^1_I(\mathbb{Q}_\ell, A)$. On the
other hand, since $H^1_I(\mathbb{Q}_\ell, M)$ is the inverse image of $H^1_I(\mathbb{Q}_\ell, A) = H^1_{ur}(\mathbb{Q}_\ell, A)$ under
$\iota : H^1_I(\mathbb{Q}_\ell, M) \to H^1_I(\mathbb{Q}_\ell, A)$, we have

$$H^1_I(\mathbb{Q}_\ell, M) = \ker(H^1(\mathbb{Q}_\ell, M) \to H^1(\mathbb{Q}_\ell, A) \to H^1_I(\mathbb{Q}_\ell^u, A)_{Frob_{\ell}=1})$$

$$= \ker(H^1(\mathbb{Q}_\ell, M) \to H^1(\mathbb{Q}_\ell^u, M)_{Frob_{\ell}=1} \to H^1_I(\mathbb{Q}_\ell^u, A)_{Frob_{\ell}=1}).$$

Here the homomorphism $g$ is injective. In fact, from an exact sequence of $I_\ell$-modules

$$0 \to M \to A \to 0,$$

we have an exact sequence

$$0 \to (A^I_{I_\ell} \otimes \mathbb{F}_p)^{Frob_{\ell}=1} \to H^1(\mathbb{Q}_\ell^u, M)^{Frob_{\ell}=1} \to H^1(\mathbb{Q}_\ell^u, A)^{Frob_{\ell}=1}.$$

Since $A^I_{I_\ell} \otimes \mathbb{F}_p = A \otimes \mathbb{F}_p$, we obtain

$$\ker(g) = (A^I_{I_\ell} \otimes \mathbb{F}_p)^{Frob_{\ell}=1} = (A \otimes \mathbb{F}_p)^{Frob_{\ell}=1} = A^{Frob_{\ell}=1} \otimes \mathbb{F}_p = 0.$$

In the third equality above, we use the assumption $c(\mathbb{Q}_\ell, A) = \#A^{Frob_{\ell}=1} = 1$ to get
$H^1(\mathbb{Q}_\ell^u/\mathbb{Q}_\ell, A) = 0$. Thus the homomorphism $g$ is injective and we get

$$H^1_I(\mathbb{Q}_\ell, M) = \ker(H^1(\mathbb{Q}_\ell, M) \to H^1(\mathbb{Q}_\ell^u, M)^{Frob_{\ell}=1}) = H^1_{ur}(\mathbb{Q}_\ell, M).$$

□
4.2. Unramifiedness of $H_1^f$ outside $p$. Now we prove the following proposition which is the claim in (Step 2).

**Proposition 4.3.** If the Tamagawa factor $c(Q_\ell, A)$ of $A$ at $\ell$ is trivial for every prime $\ell$ with $\ell \mid N$ (the assumption (4) in Theorem 1.1), then the elements in $H_1^f(Q, M)$ are unramified outside $p$.

(Proof of Proposition 4.3)

Since $p \geq 3$, there is nothing to prove for unramifiedness at the infinite place of $Q$. If $\ell \nmid N$, the inertia subgroup $I_\ell$ at $\ell$ acts on $A_0^f = V_0^f/T_0^f$ trivially. So we have

$$A_{f,D}^I = \begin{cases} A_{f,D} & (\sqrt{D} \in Q_\ell^{ur}), \\ 0 & \text{(otherwise)}. \end{cases}$$

Hence we have $c(Q_\ell, A) = 1$ in both cases to get $H_1^f(Q_\ell, M) = H_{ur}^1(Q_\ell, M)$ from Proposition 4.2. While for $\ell \mid N$, we assume that $c(Q_\ell, T) = 1$ and obtain $H_1^f(Q_\ell, M) = H_{ur}^1(Q_\ell, M)$. Thus elements in the Selmer group $H_1^f(Q, M)$ are unramified outside $p$. $\Box$

Finally, we introduce a sufficient condition on $M = A[p]$ for $c(Q_\ell, A) = 1$.

**Proposition 4.4.** For a prime number $\ell \neq p$, if $M^{G_{Q_\ell}} = 0$, then $c(Q_\ell, A) = 1$.

(Proof of Proposition 4.3)

We have an commutative diagram of exact sequences

$$
\begin{array}{cccccccc}
0 & \rightarrow & (A_{f,D})_{div} & \rightarrow & A_0^f & \rightarrow & A & \rightarrow & 0 \\
& & \downarrow \text{Frob}_p-1 & & \downarrow \text{Frob}_p-1 & & \downarrow \text{Frob}_p-1 & \\
0 & \rightarrow & (A_{f,D})_{div} & \rightarrow & A_0^f & \rightarrow & A & \rightarrow & 0.
\end{array}
$$

Since we assume $M^{G_{Q_\ell}} = 0$ and this is equivalent to the condition $H^0(Q_\ell, A) = 0$, the middle vertical arrow is injective and so is the left vertical arrow. While $(A_{f,D})_{div}$ is the direct sum of finite number of $Q_p/Z_p$, the injective left vertical arrow is in fact an isomorphism. So the right vertical arrow is injective by Snake lemma which implies $c(Q_\ell, A) = \#A^{\text{Frob}_p=1} = 1$. $\Box$

5. The image of the restriction map $\text{Res}_p^{ur}$

5.1. Image of the localization at $p$. In this section, we consider the restriction map $\text{Res}_p^{ur} : H_1^f(Q, M) \rightarrow H^1(Q_p^{ur}, M)$ at $p$ and show that its image has at most dimension 1 over $\mathbb{F}_p$ under the assumptions (2) in Theorem 1.1. We decompose the restriction map as

$$\text{Res}_p^{ur} : H_1^f(Q, M) \xrightarrow{\text{Loc}_p} H^1(Q_p, M) \xrightarrow{\text{Res}_p^{ur}/Q_p} H^1(Q_p^{ur}, M),$$
and study the image of \( \text{Loc}_p \) which is the restriction of cohomology classes to the decomposition group at \( p \). The image of \( H^1_f(\mathbb{Q}, M) \) under \( \text{Loc}_p \) is in the local condition \( H^1_f(\mathbb{Q}_p, M) \) at \( p \). By the definition of the local condition and the exact sequence (1), we have an exact sequence

\[
0 \to A^{G_{\mathbb{Q}_p}} \otimes \mathbb{F}_p \to H^1_f(\mathbb{Q}_p, M) \xrightarrow{\iota} H^1_f(\mathbb{Q}_p, A)[p] \to 0.
\]

So we have an inequality

\[
\dim_{\mathbb{F}_p}(\text{Im}(\text{Loc}_p)) \leq \dim_{\mathbb{F}_p}(H^1_f(\mathbb{Q}_p, M)) \leq \dim_{\mathbb{F}_p}(A^{G_{\mathbb{Q}_p}} \otimes \mathbb{F}_p) + \dim_{\mathbb{F}_p}(H^1_f(\mathbb{Q}_p, A)[p]).
\]

The dimension of \( H^1_f(\mathbb{Q}_p, A)[p] \) can be computed by \( p \)-adic Hodge theory. We use the following fact.

**Proposition 5.1** ([1], Corollary 3.8.4). Let \( V \) be a \( p \)-adic representation of \( G_{\mathbb{Q}_p} \), and \( D_{\text{dR}}(V) := (V \otimes \mathbb{B}_{\text{dR}})^{G_{\mathbb{Q}_p}}, D_{\text{dR}}^+(V) := (V \otimes \mathbb{B}_{\text{dR}}^+)^{G_{\mathbb{Q}_p}} \), where \( \mathbb{B}_{\text{dR}} \) is Fontaine’s de Rham period ring and \( \mathbb{B}_{\text{dR}}^+ \) is its valuation ring. If \( V \) is a de Rham representation, then

\[
\dim_{\mathbb{Q}_p}(H^1_f(\mathbb{Q}_p, V)) = \dim_{\mathbb{Q}_p}(D_{\text{dR}}(V)/D_{\text{dR}}^+(V)) + \dim_{\mathbb{Q}_p}H^0(\mathbb{Q}_p, V).
\]

Since we assume \( p \nmid N \), the \( p \)-adic representation \( V^0_f \) of \( G_{\mathbb{Q}_p} \) is crystalline and its Hodge-Tate weight is \( \{0, k-1\} \). So \( V = V_{f,D} \) is also crystalline with Hodge-Tate weight \( \{1-k/2, k/2\} \), hence we obtain \( \dim_{\mathbb{Q}_p}(D_{\text{dR}}(V)/D_{\text{dR}}^+(V)) = 2 - 1 = 1 \). Since \( H^1_f(\mathbb{Q}_p, A) \) is the image of \( H^1_f(\mathbb{Q}_p, V) \) and cofinitely generated as a \( \mathbb{Z}_p \)-module, we have

\[
\dim_{\mathbb{F}_p}(H^1_f(\mathbb{Q}_p, A)[p]) = \dim_{\mathbb{Q}_p}(H^1_f(\mathbb{Q}_p, V)).
\]

Then by (2), we obtain

\[
\dim_{\mathbb{F}_p}(H^1_f(\mathbb{Q}_p, A)[p]) = 1 + \dim_{\mathbb{Q}_p}H^0(\mathbb{Q}_p, V).
\]

From (2), (1), we have

\[
\dim_{\mathbb{F}_p}(\text{Im}(\text{Loc}_p)) \leq \dim_{\mathbb{F}_p}(A^{G_{\mathbb{Q}_p}} \otimes \mathbb{F}_p) + 1 + \dim_{\mathbb{Q}_p}H^0(\mathbb{Q}_p, V).
\]

In the following, we compute the first and the third terms in the above equality.

### 5.2. Supersingular case.

**Proposition 5.2.** If \( f \) is supersingular at \( p \) and \( k \leq p + 1 \), then we have

\[
A^{G_{\mathbb{Q}_p}} \otimes \mathbb{F}_p = V^{G_{\mathbb{Q}_p}} = 0.
\]

We use the following result of Fontaine and Edixhoven.

**Theorem** (Fontaine-Edixhoven).

*If the weight \( k \) of \( f \) satisfies \( 2 \leq k \leq p + 1 \), then \( \bar{\rho}_f^0 \big|_{G_{\mathbb{Q}_p}} \) is irreducible.*

**Remark 5.3.** Note that we cite their result as far as we use it. Their result describe a more precise local behavior of \( \bar{\rho}_f^0 \) at a supersingular prime. See for example [6].
Hence we obtain $M$ reducible as a $G_{Q_p}$-modules. The residual representation of Proposition 5.4. Suppose the ordinary case.

(Proof of Proposition 5.4)

When $D$ splits as a $G_{Q_p}$-module and $p - 1 | k/2$, and $a_p \equiv 1 \pmod{p}$. The equality can not be hold. This is a contradiction and hence we obtain $V^{G_{Q_p}} = 0$. □

5.3. Ordinary case.

Proposition 5.4. Suppose $f$ is ordinary at $p$. Then we have $V^{G_{Q_p}} = 0$.

(Proof of Proposition 5.4)

We fix a basis of $T^0$ over $\mathbb{Z}_p$ and this also yields a basis of $V^0$ over $Q_p$. Taking corresponding bases $\{v_1, v_2\}$ for its twist $V = V_{f,D}$, we know $g \in G_{Q_p}$ acts on it as

$$\begin{pmatrix} \chi_{\text{cyc}}(g)\psi^{-1}(g) & u(g) \\ 0 & \psi(g)\chi_{\text{cyc}}^{1-k/2} \end{pmatrix} : \chi_D(g),$$

where $\psi : G_{Q_p} \to \mathbb{Z}_p^\times$ is an unramified character and $u(g) \in \mathbb{Z}_p$. We have a subspace $W_1 := Q_p v_1$ of $V$ on which $G_{Q_p}$ acts via the character $\chi_{\text{cyc}}^k \chi_D \psi^{-1} : G_{Q_p} \to \mathbb{Z}_p^\times$. If $V^{G_{Q_p}} \neq 0$, in other words, there is a non-trivial subspace $W_0$ of $V$ on which $G_{Q_p}$ acts trivially, then that is 1-dimensional and linearly independent with $W_1$ since $\chi_{\text{cyc}}^k \chi_D \psi^{-1} \neq 1$. Then the matrix (6) is similar to

$$\begin{pmatrix} \chi_{\text{cyc}}^{k/2}(g)\chi_D \psi^{-1}(g) & 0 \\ 0 & 1 \end{pmatrix}.$$

Taking determinants of above matrices, we have

$$\chi_{\text{cyc}}(g) = \chi_{\text{cyc}}^{k/2}(g)\chi_D(g)\psi^{-1}(g)$$

for all $g \in G_{Q_p}$, but this equality can not be hold. This is a contradiction and hence we obtain $V^{G_{Q_p}} = 0$. □

Proposition 5.5. Suppose $f$ is ordinary at $p$ and $p - 1 \nmid k - 1$. Then we have $M^{G_{Q_p}} = 0$ if and only if the following situations depending on $D$ do not happen:

(When $p \nmid D$)

(a) $M$ does not split as a $G_{Q_p}$-module and $p - 1 | k/2$ and $a_p \equiv 1 \pmod{p}$.
(b) $M$ splits as a $G_{Q_p}$-module and $p - 1 | k/2$ or $1 - k/2$ and $a_p \equiv 1 \pmod{p}$.

(When $D = p^a := (-1)^{k-1}p$)

(c) $M$ does not split as a $G_{Q_p}$-module and $p - 1 | \frac{k-p+1}{2}$ and $a_p \equiv 1 \pmod{p}$.
(d) $M$ splits as a $G_{Q_p}$-module and $p - 1 | \frac{k-p+1}{2}$ or $\frac{k+p-3}{2}$ and $a_p \equiv 1 \pmod{p}$. 
(Proof of Proposition 5.5)

We fix a basis of $T_p^f$ over $\mathbb{Z}_p$ as in the proof of Proposition 5.4 and this yields a basis of $M_f^0$ over $\mathbb{F}_p$. Taking a corresponding basis $\{m_1, m_2\}$ for its twist $M = M_{f,D}$ over $\mathbb{F}_p$, we know $g \in G_{\mathbb{Q}_p}$ acts on $M$ as

$$
(7) \quad \begin{pmatrix}
\omega_{\text{cyc}}^{k/2}(g) & \bar{\psi}(g) \\
0 & \omega_{\text{cyc}}^{1-k/2}(g)
\end{pmatrix} \cdot \chi_D(g),
$$

where $\psi$ and $u(g) \in \mathbb{Z}_p$ are as in (3) and $\bar{\psi}$ and $\bar{u}(g)$ are their reduction modulo $p$. We show that $M_{G_{\mathbb{Q}_p}} = 0$ if the situations described in Proposition 5.5 do not occur by a case-by-case computation.

(Case 1 : $p \nmid D$)

In this case we have $\sqrt{D} \in \mathbb{Q}_p^{ur}$. So $g \in I_p$ acts on $M$ via a matrix $$
\begin{pmatrix}
\omega_{\text{cyc}}^{k/2}(g) & \bar{u}(g) \\
0 & \omega_{\text{cyc}}^{1-k/2}(g)
\end{pmatrix}.
$$

First, suppose $M$ splits as a $G_{\mathbb{Q}_p}$-module, in other words $\bar{u} = 0$. For $a, b \in \mathbb{F}_p$, an element $x = am_1 + bm_2 \in M$ and $g \in I_p$,

$$
g(x) = g(av_1 + bv_2) = a\omega_{\text{cyc}}^{k/2}(g)m_1 + b\omega_{\text{cyc}}^{1-k/2}(g)m_2.
$$

Thus $x \in M_{I_p}$ if and only if

$$
(8) \quad a\omega_{\text{cyc}}^{k/2}(g) = a, \ b\omega_{\text{cyc}}^{1-k/2}(g) = b \quad \text{for all} \ g \in I_p.
$$

If $p - 1 \nmid k/2$ and $p - 1 \nmid 1 - k/2$, then conditions in (8) implies $a = b = 0$ and we get $M_{G_{\mathbb{Q}_p}} \subseteq M_{I_p} = 0$. If one of the conditions $p - 1 \mid k/2$ or $p - 1 \mid 1 - k/2$ holds, then $M_{I_p} = \mathbb{F}_pm_1$ or $M_{I_p} = \mathbb{F}_pm_2$ respectively. We know that the Frobenius element $\text{Frob}_p$ acts on $\mathbb{F}_pm_1$ and $\mathbb{F}_pm_2$ via multiplication by $a_p(f)^{-1}$ mod $p$ and $a_p(f)$ mod $p$ respectively, where $a_p(f)$ is the $p$-th Fourier coefficient of $f$. Thus $M_{G_{\mathbb{Q}_p}} = (M_{I_p})^{\text{Frob}_p} = 1 = 0$ if we have $a_p(f) \equiv 1 \bmod p$. If $a_p(f) \equiv 1 \bmod p$, $M_{G_{\mathbb{Q}_p}}$ is a 1-dimensional subspace of $M$.

Next, suppose $M$ does not split as a $G_{\mathbb{Q}_p}$-module, we use the following lemma.

Lemma 5.6. Suppose $M$ does not split as a $G_{\mathbb{Q}_p}$-module. If $p - 1 \nmid k - 1$, then we have $\bar{u}(G_{\mathbb{Q}_p}^{ab}) \neq 0$ for a suitable basis of $M$, where $\mathbb{Q}_p^{ab}$ denotes the maximal abelian extension of $\mathbb{Q}_p$.

We prove this lemma later. From (7), $G_{\mathbb{Q}_p}^{ab}$ acts on $M$ via $\begin{pmatrix} 1 & \bar{u}(g) \\ 0 & 1 \end{pmatrix}$. By a similar argument with which we obtain the condition (8), we can show that an
element \( x = am_1 + bm_2 \in M \) is fixed by \( G_{\mathbb{Q}_p^ab} \) if and only if

\[ a = a + b\bar{u}(g) \quad \text{for all } g \in G_{\mathbb{Q}_p^ab}. \]

From Lemma 5.7 once we retake a suitable basis of \( M \), there exist \( g \in G_{\mathbb{Q}_p^ab} \) such that \( \bar{u}(g) \neq 0 \) which implies \( b = 0 \) and we obtain \( M^{I_p} = (M^{G_{\mathbb{Q}_p^ab}})^{I_p} = (\mathbb{F}_p^* m_1)^{I_p} \).

Since retaking the basis of \( M \) in Lemma 5.7 changes only the upper right component \( \bar{u} \) of (7) from the proof of Lemma 5.7, \( I_p \) acts on \( \mathbb{F}_p^* m_1 \) still via the character \( \omega_{\text{cyc}}^{k/2} \).

Hence if \( p - 1 \nmid k/2 \), then \( M_{G_{\mathbb{Q}_p^a}} \subset M^{I_p} = 0 \). If \( p - 1 \mid k/2 \), then \( M^{I_p} = \mathbb{F}_p m_1 \) and \( \text{Frob}_p \) acts on this space via multiplication by \( a_p(f)^{-1} \mod p \). Thus if \( a_p(f) \neq 1 \mod p \), we have \( M_{G_{\mathbb{Q}_p^a}} = (M^{I_p})_{\text{Frob}_p=1} = 0 \) otherwise \( M_{G_{\mathbb{Q}_p^a}} = \mathbb{F}_p^* m_1 \).

(Case 2 : \( p \mid D \) and \( D \neq p^* \))

In this case, \( \mathbb{Q}_p^{ur}(\sqrt{D}) \) and \( \mathbb{Q}_p^{ur}(\zeta_p) \) are linearly disjoint over \( \mathbb{Q}_p^{ur} \). Hence there exists \( \sigma \in G_{\mathbb{Q}_p^{ur}(\zeta_p)} \) such that \( \chi_D(\sigma) = -1 \). From (7), \( G_{\mathbb{Q}_p^{ur}(\zeta_p)} \) acts on \( M \) via

\[
\begin{pmatrix}
1 & \bar{u}(g) \\
0 & 1
\end{pmatrix} \cdot \chi_D(g).
\]

Thus an element \( x = am_1 + bm_2 \in M \) fixed by \( G_{\mathbb{Q}_p^{ur}(\zeta_p)} \) if and only if

\[
\chi_D(g)(a + b\bar{u}(g)) = a, \quad \chi_D(g)b = b \quad \text{for all } g \in G_{\mathbb{Q}_p^{ur}(\zeta_p)}.
\]

Putting \( g = \sigma \), we obtain \( b = 0 \) from the second equality in (9) since \( p \) is odd and then we get \( a = 0 \) from the first equality. Thus we have \( M_{G_{\mathbb{Q}_p^a}} \subset M_{G_{\mathbb{Q}_p^{ur}(\zeta_p)}} = 0 \).

(Case 3 : \( D = p^* \))

In this case we have an inclusion \( \mathbb{Q}_p^{ur}(\sqrt{D}) \subset \mathbb{Q}_p^{ur}(\zeta_p) \). Suppose \( M \) splits as a \( G_{\mathbb{Q}_p^a} \)-module. As the calculation in (Case 1), an element \( x = am_1 + bm_2 \in M \) fixed by \( I_p \) if and only if

\[
a_\omega^{k/2}(g)\chi_D(g) = a, \quad b_\omega^{1-k/2}(g)\chi_D(g) = b \quad \text{for all } g \in I_p.
\]

This condition is equivalent to the condition that the same equations hold for a generator \( \tau \) of the Galois group \( \text{Gal}(\mathbb{Q}_p^{ur}(\zeta_p)/\mathbb{Q}_p^{ur}) \). We have \( \chi_D(\tau) = -1 \) and \( \omega_{\text{cyc}}(\tau) \) is order \( p - 1 \). Thus (10) is equivalent to the condition

\[
-\omega_{\text{cyc}}^{k/2}(\tau) = a, \quad -b\omega_{\text{cyc}}^{1-k/2}(\tau) = b.
\]

If \( \frac{k-1}{2} \neq k/2 \mod p - 1 \) and \( \frac{k-1}{2} \neq 1 - k/2 \mod p - 1 \), then \( a = b = 0 \) from the above condition and we get \( M_{G_{\mathbb{Q}_p^a}} \subset M^{I_p} = 0 \). If one of the conditions \( \frac{k-1}{2} \equiv k/2 \mod p - 1 \) and \( \frac{k-1}{2} \equiv 1 - k/2 \mod p - 1 \) holds, then \( M^{I_p} = \mathbb{F}_p^* m_1 \) and \( M^{I_p} = \mathbb{F}_p^* m_2 \) respectively. As in the argument in (Case 1), if \( a_p(f) \neq 1 \mod p \), then \( M_{G_{\mathbb{Q}_p^a}} = (M^{I_p})_{\text{Frob}_p=1} = 0 \) otherwise \( M_{G_{\mathbb{Q}_p^a}} \) is a 1-dimensional subspace of \( M \).
Finally, suppose $M$ does not split as a $G_{Q_p}$-module. By the same argument in (Case 1), we can show that an element $x = am_1 + bm_2 \in M$ is fixed by $G_{Q_p}$ if and only if

$$a = a + b\bar{u}(g) \quad \text{for all } g \in G_{Q_p}.$$ 

and this implies $b = 0$ from Lemma 5.7 after retaking a suitable basis of $M$. Then $M_{I_p} = (M_{G_{Q_p}})_{I_p} = (F_{p_m_1})_{I_p}$ and $I_p$ acts on $F_{p_m_1}$ via $\omega_{cyc}\chi_D$. So we can see that $x = am_1 \in M_{G_{Q_p}}$ is fixed by $I_p$ if and only if

$$-a\omega_{cyc}^{k/2}(\tau) = a$$

as we get the condition (11). If $p - 1 \not\equiv k/2 \pmod{p - 1}$, then $a = 0$ from this equality and we obtain $M_{G_{Q_p}} \subset M_{I_p} = 0$. If $p - 1 \equiv k/2 \pmod{p - 1}$, then $M_{I_p} = F_{p_m_1}$ and we have $M_{G_{Q_p}} = (M_{I_p})_{Frob_p=1} = (F_{p_m_1})_{Frob_p=1}$. This is trivial if $a_p(f) \not\equiv 1 \pmod{p}$ otherwise $M_{G_{Q_p}} = F_{p_m_1}$. $\square$

To finish the proof of Proposition 5.3 completely, we prove the Lemma 5.7.

Lemma 5.7. Suppose $M$ does not split as a $G_{Q_p}$-module. If $p - 1 \not\equiv k/2 \pmod{p - 1}$, then we have $\bar{u}(G_{Q_p}) \neq 0$ for a suitable basis of $M$, where $Q_p^{ab}$ denotes the maximal abelian extension of $Q_p$.

(Proof of Lemma 5.7)

If $M = M_{f,D}$ does not split as a $G_{Q_p}$-module, then so does $M^0_f$. Recall that $I_p$ acts on $M^0_f$ via a homomorphism $\rho^0_f$ and the image of $g \in I_p$ under this is a matrix

$$\left( \begin{array}{cc} \omega_{cyc}^{-k/2}(g) & \bar{v}(g) \\ 0 & 1 \end{array} \right),$$

where $\bar{v}(g) \in F_p$. We show that $\bar{v}(G_{Q_p}) \neq 0$ which immediately implies $\bar{u}(G_{Q_p}) \neq 0$ since $\bar{v}$ differs from $\bar{u}$ by a character $\omega_{cyc}^{-k/2} \otimes \chi_D$. We use the following fact.
Proposition 5.8 (13, Lemma 2.2). Let \( p > 2 \) be a prime number and \( B \subset \text{GL}_2(\mathbb{F}_p) \) a Borel subgroup such that \( B \) contains a matrix \( g = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \) with \( a \not\equiv c \pmod{p} \).

Let \( B' := h^{-1}Bh \) with \( h = \begin{pmatrix} 1 & b/(c-a) \\ 0 & 1 \end{pmatrix} \).

1. We can decompose \( B' \) which is conjugate to \( B \) as

\[
B' = B'_d \cdot B'_1,
\]

and \( B/[B, B] \cong B'/[B', B'] \). Here groups \( B'_d, B'_1 \) are defined as

\[
B'_d = B' \cap \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mid a, c \in \mathbb{F}_p^\times \right\},
\]

\[
B'_1 = B' \cap \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_p \right\}.
\]

2. \( [B', B'] = B'_1 \).

We put \( D_p := \bar{\rho}_f^0(G_{Q_p}), I_p := \bar{\rho}_f^0(I_p) \). Since we assume \( p - 1 \nmid k - 1 \), \( \omega_{\text{cyc}}^{k-1} \) is not a trivial character. Then there exist a matrix \( A \in I_p \subset D_p \) such that \( A = \begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix} \), and \( a \not\equiv 1 \pmod{p} \). Thus \( D_p \) satisfies the assumptions in Proposition 5.8 and we have a decomposition of \( D_p \) as in Proposition 5.8 for a suitable basis of \( M_0^f \).

\[
D_p = (D_p)_d \cdot (D_p)_1 = \left\{ \begin{pmatrix} \omega_{\text{cyc}}^{-1}(g) \psi^{-1}(g) \\ 0 \\ \psi(g) \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ a \\ b \end{pmatrix} \mid a, b \in \mathbb{F}_p \right\}.
\]

Since we assume that \( M_0^f \) does not split, we have \( (D_p)_1 \neq \{I\} \). From Proposition 5.8 (2), this means the commutator subgroup of \( D_p \) is non-trivial which implies \( D_p \) is non-abelian. Thus we obtain \( \bar{v}(G_{Q_p}) \neq 0 \) in (12).

Thus we get conditions on which we have \( M^G_{Q_p} = (A[p])^G_{Q_p} = 0 \) in Proposition 5.2 and 5.5. Under such conditions, we especially have \( A^G_{Q_p} \otimes \mathbb{F}_p = 0 \). Then from the inequality (5), Proposition 5.2 5.4 and 5.5 we obtain the desired inequality

\[
\dim_{\mathbb{F}_p}(\text{Im}(\text{Res}^d_p)) \leq \dim_{\mathbb{F}_p}(\text{Im}(\text{Loc}_p)) \leq 1
\]

which is the claim of (Step 3) under the assumptions of the propositions.
5.4. CM case. However, in the ordinary case, if \( k = 2 \) and \( T/p^nT \) splits for all \( n \), we still have the above inequality \( \dim_{\mathbb{F}_p}(\text{Im}(\text{Res}^\text{ur}_p)) \leq 1 \) even when the situation (b) in Proposition 5.5 occurs.

**Proposition 5.9.** If \( k = 2 \) and \( T/p^nT \) splits as a \( G_{\mathbb{Q}_p} \)-module for all \( n \), then we have \( \dim_{\mathbb{F}_p}(\text{Im}(\text{Res}^\text{ur}_p)) \leq 1 \).

**(Proof of Proposition 5.9)**

When the situation (b) occurs, we know that \( M^{G_{\mathbb{Q}_p}} \) is 1-dimensional over \( \mathbb{F}_p \) from the proof of Proposition 5.5 which implies \( A^{G_{\mathbb{Q}_p}} \otimes \mathbb{F}_p \) is also 1-dimensional. Thus we have \( \dim_{\mathbb{F}_p}(\text{Im}(\text{Loc}_p)) \leq \dim_{\mathbb{F}_p}(H^1_f(\mathbb{Q}_p, M)) = 1 + 1 = 2 \) from (5).

For every \( n \in \mathbb{Z}_{\geq 1} \), \( I_p \) acts on \( T/p^nT \) via a diagonal matrix

\[
\begin{pmatrix}
\chi^{(n)}_{\text{cyc}}(g) & 0 \\
0 & 1
\end{pmatrix}
\]

since \( k = 2 \) and we assume that \( T/p^nT \) splits as a \( G_{\mathbb{Q}_p} \)-module. Here \( \chi^{(n)}_{\text{cyc}} \) denotes the mod \( p^n \) cyclotomic character. Note that in the situation (b), we assume \( p \nmid D \) and hence \( \chi_D \) is trivial when restricted to the inertia subgroup \( I_p \). So we have \( (A[p^n])^{I_p} \cong (T/p^nT)^{I_p} \cong \mathbb{Z}/p^n\mathbb{Z} \) and \( A^{I_p} \otimes \mathbb{F}_p \cong (\mathbb{Q}_p/\mathbb{Z}_p) \otimes \mathbb{F}_p = 0 \). On the other hand, there is a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & A^{G_{\mathbb{Q}_p}} \otimes \mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z} \rightarrow H^1_f(\mathbb{Q}_p, M) \\
& & \downarrow \text{Res}^\text{ur}/Q_p \\
0 & \rightarrow & A^{I_p} \otimes \mathbb{F}_p = 0 \rightarrow H^1(\mathbb{Q}^\text{ur}_p, M).
\end{array}
\]

Thus the restriction map \( \text{Res}^\text{ur}/Q_p \) is not injective and its kernel has at least dimension 1 over \( \mathbb{F}_p \). Since we have a decomposition \( \text{Res}^\text{ur}_p = \text{Res}^\text{ur}/Q_p \circ \text{Loc}_p \), we get the desired inequality \( \dim_{\mathbb{F}_p}(\text{Im}(\text{Res}^\text{ur}_p)) \leq 2 - 1 = 1 \).

**Remark 5.10.** The splitting condition in Proposition 5.9 corresponds to the condition (2) in the theorem of Prasad and Shekhar. Let \( T \) be an integral \( p \)-adic Tate module of an elliptic curve \( E \) over \( \mathbb{Q} \) and suppose \( E \) has complex multiplication over an extension of \( \mathbb{Q}_p \). Then \( T/p^nT \) splits as a \( G_{\mathbb{Q}_p} \)-module for every \( n \). Thus the splitting condition in Proposition 5.9 holds for \( T \). It is conjectured by Ghate that for a modular form \( f \) which is ordinary at \( p \), \( T_f/p^n \) splits as a \( G_{\mathbb{Q}_p} \)-module for all \( n \) if and only if \( f \) has complex multiplication. For detail, see [9] for example.

To summarize, we obtain the following proposition.

**Proposition 5.11.** We assume the following:
If \( f \) is supersingular at \( p \), then \( k \leq p + 1 \) holds.

If \( f \) is ordinary at \( p \), then \( p - 1 \nmid k - 1 \) and the conditions below do not occur:

\begin{enumerate}
\item\( (\text{If } p \nmid D. \) \( M \) does not split as a \( G_{\mathbb{Q}_p} \)-module and \( p - 1 \mid k/2 \) and \( a_p \equiv 1 \mod p. \) \)
\item\( (a) M \) does not split as a \( G_{\mathbb{Q}_p} \)-module and \( p - 1 \mid k/2 \) or \( 1 - k/2 \) and \( a_p \equiv 1 \mod p. \)
\item\( (b_1) k = 2 \) and \( M \) splits as a \( G_{\mathbb{Q}_p} \)-module and \( T/p^n T \) does not split for some \( n. \)
\item\( (a) M \) does not split as a \( G_{\mathbb{Q}_p} \)-module and \( p - 1 \mid k/2 \) and \( a_p \equiv 1 \mod p. \)
\item\( (b_2) k > 2 \) and \( M \) splits as a \( G_{\mathbb{Q}_p} \)-module and \( T/p^n T \) does not split for some \( n. \)
\item\( M \) splits as a \( G_{\mathbb{Q}_p} \)-module and \( T/p^n T \) does not split for some \( n. \)
\end{enumerate}

Then we have an equality \( \dim_{\mathbb{F}_p} (\text{Im}(\text{Res}_p^ur)) \leq 1. \)

This completes the proof of Theorem 1.1 as we explain in Section 2.

6. Numerical examples

We finally introduce some numerical examples of Theorem 1.1. To do this, we consider situations in which the inequality

\[
\dim_{\mathbb{F}_p} (\bigoplus_{p}^{BK}(\mathbb{Q}, A_{f,D})[p]) \geq 2
\]

holds. This inequality implies the condition \( \dim_{\mathbb{F}_p} (H^1(\mathbb{Q}, M_{f,D})) \geq 2 \) in Theorem 1.1. Since the \( p \)-adic representation \( V_{f,D} \) is self-dual, the \( \mathbb{F}_p \)-dimension of \( \bigoplus_{p}^{BK}(\mathbb{Q}, A_{f,D})[p] \) is even due to the existence of the generalized Cassels-Tate pairing in [7]. Hence the inequality (13) holds if and only if \( \bigoplus_{p}^{BK}(\mathbb{Q}, A_{f,D}) \) is non-trivial.

We study when this occurs assuming the Bloch-Kato conjecture.

6.1. Ratios of central \( L \)-values. In the following, we assume \( N = 1 \). For a quadratic discriminant \( D \) and a modular form \( f(\tau) = \sum a_n q^n \), we consider a twisted \( L \)-function of \( f \) by a quadratic character \( \chi_D \)

\[
L(f, \chi_D, s) := \sum_{n=1}^{\infty} \frac{\chi_D(n)a_n}{n^s}
\]

which converges absolutely when the real part of \( s \) is sufficiently large. This complex \( L \)-function can be continued analytically to the whole complex plane and we can consider its central value \( L(f, \chi_D, k/2). \)

Let \( D, D' \in \mathbb{Z} \) be two quadratic discriminants. We think the ratio of central values of twisted \( L \)-functions \( L(f, \chi_D, k/2) \) and \( L(f, \chi_{D'}, k/2) \). Suppose that \( L(f, \chi_D, k/2) \neq 0 \). Then the Bloch-Kato conjecture predicts such a value in terms of arithmetic invariants including the order of Bloch-Kato’s Tate-Shafarevich group \( \bigoplus_{p}^{BK}(\mathbb{Q}, A_{f,D}), \)
and implies an equality of the following $p$-adic valuations:

\[(14) \quad v_p \left( \frac{L(f, \chi_D, k/2)}{\text{vol}_\infty(\chi_D, 1-k/2)} \right) = v_p \left( \frac{\# \text{III}^B_K(Q, A_{f,D})}{(\# \Gamma_Q(A_{f,D}))^2} \right). \]

Here $\text{vol}_\infty(\chi_D, 1-k/2)$ denotes certain transcendental part of the value $L(f, \chi_D, k/2)$ and $\Gamma_Q(A_{f,D}) := H^0(Q, A_{f,D})$. For the precise definitions, see [5, Section2]. Note that the product of Tamagawa factor does not appear in the above equality since we assume $N = 1$.

Suppose $f$ satisfies the conditions in Theorem 1.1 Then $M_{f,D} = A_{f,D}[p]$ is irreducible as a $G_Q$-module which implies $\# \Gamma_Q(A_{f,D}) = 1$. Thus (14) yields

\[(15) \quad v_p \left( \frac{L(f, \chi_D, k/2)}{\text{vol}_\infty(\chi_D, 1-k/2)} \right) = v_p \left( \frac{\# \text{III}^B_K(Q, A_{f,D})}{(\# \Gamma_Q(A_{f,D}))^2} \right). \]

For the transcendental factor $\text{vol}_\infty(\chi_D, 1-k/2)$, we have the following property.

**Lemma 6.1** ([4], Lemma 6.1). For $D > 0$, we have $\text{vol}_\infty(\chi_D, 1-k/2) = \frac{\text{vol}_\infty(1-k/2)}{\sqrt{D}}$. Here $\text{vol}_\infty(1-k/2)$ denotes the transcendental part of the L-value $L(f, 1, k/2)$ for a trivial character 1.

Suppose we take two positive quadratic discriminants $D, D'$. Using Lemma 6.1 and taking a quotient of (15) for $D$ and $D'$, we have

\[(16) \quad v_p \left( \frac{\sqrt{D}}{\sqrt{D'}} \cdot \frac{L(f, \chi_D, k/2)}{L(f, \chi_{D'}, k/2)} \right) = v_p \left( \frac{\# \text{III}^B_K(Q, A_{f,D})}{\# \text{III}^B_K(Q, A_{f,D'})} \right). \]

On the other hand, the twisted $L$-value $L(f, \chi_D, k/2)$ is studied by Kohnen and Zagier in [12] in terms of Shimura’s theory of modular forms of half integral weight. Shimura’s theory gives a correspondence between modular forms of half integral weight and modular forms of even integral weight. Let $S_k(\text{SL}_2(\mathbb{Z}))$ be the space of cusp forms of even weight $k$ on the full modular group $\text{SL}_2(\mathbb{Z})$ and $S^+_{k+1}(\Gamma_0(4))$ the space of cusp forms of weight $k+\frac{3}{2}$ on the congruence subgroup $\Gamma_0(4)$. In [11], Kohnen defined a certain subspace $S^+_{k+1}(\Gamma_0(4))$ of $S^+_{k+1}(\Gamma_0(4))$ and showed Shimura’s correspondence induces an isomorphism $\kappa : S^+_{k+1}(\Gamma_0(4)) \cong S_k(\text{SL}_2(\mathbb{Z}))$. In [12], Kohnen and Zagier gave a formula of the value $L(f, \chi_D, k/2)$ for $f \in S_k(\text{SL}_2(\mathbb{Z}))$ in terms of the $|D|$-th Fourier coefficient of $\kappa^{-1}(f)$.

**Theorem** (Kohnen-Zagier). Let $f \in S_g(\text{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, $g := \kappa^{-1}(f) = \sum_{n=1}^{\infty} c_n q^n \in S^+_{\frac{k+3}{2}}(\Gamma_0(4))$ the inverse image of $f$ under the Kohnen’s
isomorphism $\kappa$. Let $D$ be a quadratic discriminant with $(-1)^{k/2}D > 0$. Then

$$\frac{c_{|D|}^2}{\langle g, g \rangle} = \frac{(k/2 - 1)!}{\pi^{k/2}} |D|^{(k-1)/2} \frac{L(f, D, k/2)}{\langle f, f \rangle},$$

where $\langle \cdot, \cdot \rangle$ denotes the Petersson inner product.

Now we put $f = \Delta(z) := \sum_{n=1}^{\infty} \tau_n q^n$ ($q = e^{2\pi i z}$), Ramanujan’s cusp form of weight $k = 12$ and level $N = 1$. Using the theorem of Kohnen and Zagier for $f = \Delta(z)$, $k = 12$ and the quadratic discriminants $D, D'$ we take before, we have

$$\sqrt{D} \cdot L(\Delta, \chi_D, 6) = c_D^2 \cdot \left( \frac{D'}{D} \right)^5,$$

where $c_D, c_{D'}$ are the $D$ and $D'$-th Fourier coefficients of $\kappa^{-1}(\Delta(z))$ respectively. From (16) we obtain an equality of $p$-adic valuations

$$(17) \quad v_p \left( \frac{\# \text{III}^B_K(Q, A_{f,D})}{\# \text{III}^B_K(Q, A_{f,D'})} \right) = v_p \left( \frac{c_D^2}{c_{D'}^2} \cdot \left( \frac{D'}{D} \right)^5 \right).$$

The main example in [12] provides a formula

$$\kappa^{-1}(\Delta(z)) = \frac{60}{2\pi i} (2G_4(4z)\theta'(z) - G'_4(4z)\theta(z)),$$

where $G_4(z) := \frac{1}{240} + \sum_{n=1}^{\infty} \sigma_3(n)q^n$ ($\sigma_3(n) = \sum_{d|n} d^3$) and $\theta(z) := 1 + 2 \sum_{n=1}^{\infty} q^n$. Thus we can compute the coefficients of $\kappa^{-1}(\Delta(z))$ explicitly and hence the right-hand side of (17).

6.2. Mod $p$ representations attached to elliptic curves with bad reduction at $p$. We set $p = 11$. This is an ordinary prime of $\Delta(z)$. We consider quadratic discriminants which are proper multiples of 11 since we assume no conditions in (2) in Theorem [12] for these cases. From the formula

$$\kappa^{-1}(\Delta(z)) = \frac{60}{2\pi i} (2G_4(4z)\theta'(z) - G'_4(4z)\theta(z)),$$

we compute the 11$^i$-th Fourier coefficient of $\kappa^{-1}(\Delta(z)) = \sum_{n=1}^{\infty} c_n q^n$ for $2 \leq i \leq 7$ as follows:

| $i$ | 11$^i$ | $c_{11^i}$ |
|-----|------|---------|
| 2   | 22   | 0       |
| 3   | 33   | -6480 = -2^4 \cdot 3^4 \cdot 5 |
| 4   | 44   | -43680 = -2^9 \cdot 3 \cdot 5 \cdot 7 \cdot 13 |
| 5   | 55   | 0       |
| 6   | 66   | 0       |

Continued on next page
We take $11 \times 7 = 77$-th and $11 \times 3 = 33$-th Fourier coefficients of $\kappa^{-1}(\Delta(z))$. We have $33, 77 \equiv 1 \pmod{4}$, so they are both quadratic discriminants. We set $D = 77$, $D' = 33$. From (17) for $D = 77, D' = 33, p = 11$, we have

$$v_{11} \left( \frac{\#\Pi_{11}^{BK}(\mathbb{Q}, A_{\Delta,77})}{\#\Pi_{11}^{BK}(\mathbb{Q}, A_{\Delta,33})} \right) = v_{11} \left( \frac{c_{77}^2}{c_{33}^2} \cdot \left( \frac{3}{7} \right)^5 \right).$$

We know $11 \mid c_{77}$ and $11 \nmid c_{33}$ from the above table. Thus we have $11 \mid \#\Pi_{11}^{BK}(\mathbb{Q}, A_{\Delta,77})$ which implies $rk_{F_{11}}(\Pi_{11}^{BK}(\mathbb{Q}, A_{\Delta,77})(11)) \geq 2$. Now we check that $p = 11, f = \Delta, k = 12, D = 77, D' = 33$ satisfies the assumptions of Theorem 1.1. Since we assume $N = 1$, there is nothing to check for the conditions (1), (4) in Theorem 1.1. It is known that the image of the representation $\overline{\rho}_\Delta : G_\mathbb{Q} \to GL_2(\mathbb{F}_p)$ contains $SL_2(\mathbb{F}_p)$ except for the cases $p = 2, 3, 5, 7, 23$ and 691. Thus the assumption (3) is now satisfied. Since our quadratic discriminants $D = 77, D' = 33$ are proper multiples of 11, we are not in the situations described in Proposition 5.11 and this is the assumption (2). Thus we can apply our Theorem 1.1 in this situation and see that the $F_{11}$-representation $\mathrm{Cl}_{K_{\Delta,77}} \otimes F_{11}$ has $M_{\Delta,77}$ as its quotient representation.

We note that this $F_{11}$-representation $M_{\Delta,77}$ of $G_\mathbb{Q}$ comes from an elliptic curve over $\mathbb{Q}$. Let $E$ be the modular curve $X_0(11)$ and $f_E = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(11))$ the corresponding cusp form. For $\Delta(z) = \sum_{n=1}^{\infty} \tau_n q^n$ and $f_E$, we have a congruence of coefficients $\tau_n \equiv a_n \pmod{11}$ which induces an isomorphism between $F_{11}$-representations $M_0^\Delta$ and $E[11]$. Thus we have an isomorphism $M_{\Delta,77} \cong E_{77}[11] \otimes \omega^5_{\text{cyc}}$ as $G_\mathbb{Q}$-modules where $E_{77}$ denotes the quadratic twist of $E$ by 77, and $M_{\Delta,77}$ comes from an elliptic curve. However, we can not treat $M_{\Delta,77}$ by the theorem of Prasad and Shekhar or its generalization in [3] since $E_{77}$ is bad at 11 and their theorem and ours in [3] can be used for only good primes.

6.3. Mod $p$ representations attached to modular forms. In the second example, we put $p = 67$. This is also an ordinary prime of $\Delta(z)$ as the first example and we consider quadratic discriminants which are proper multiples of 67. For $i \in \mathbb{Z}, 2 \leq i \leq 43$, the $67i$-th Fourier coefficient of $\kappa^{-1}(\Delta(z)) = \sum_{n=1}^{\infty} c_n q^n$ is computed as follows:
| $i$ | $c_{67i}$ |
|-----|---------|
| 2   | -2686320 = $-2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 41$ |
| 3   | -4016160 = $-2^9 \cdot 3^3 \cdot 5 \cdot 2789$ |
| 4   | 0 |
| 5   | 0 |
| 6   | 0 |
| 7   | -32215680 = $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 17 \cdot 47$ |
| 8   | 24612000 = $2^9 \cdot 3 \cdot 5^3 \cdot 7 \cdot 293$ |
| 9   | 0 |
| 10  | 0 |
| 11  | 52764720 = $2^4 \cdot 3 \cdot 5 \cdot 109 \cdot 2017$ |
| 12  | 150433920 = $2^9 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 41$ |
| 13  | 0 |
| 14  | 0 |
| 15  | 380298240 = $2^{10} \cdot 3^4 \cdot 5 \cdot 7 \cdot 131$ |
| 16  | 96387840 = $2^8 \cdot 3^3 \cdot 5 \cdot 2789$ |
| 17  | 0 |
| 18  | 0 |
| 19  | 293666160 = $2^4 \cdot 3 \cdot 5 \cdot 17 \cdot 167 \cdot 431$ |
| 20  | -197892480 = $-2^8 \cdot 3 \cdot 5 \cdot 103069$ |
| 21  | 0 |
| 22  | 0 |
| 23  | 1340143920 = $2^4 \cdot 3^4 \cdot 5 \cdot 620437$ |
| 24  | 122186880 = $2^9 \cdot 3^3 \cdot 5 \cdot 2357$ |
| 25  | 0 |
| 26  | 0 |
| 27  | -67695264 = $-2^9 \cdot 3^3 \cdot 7^2 \cdot 13 \cdot 41$ |
| 28  | -257725440 = $-2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 17 \cdot 47$ |
| 29  | 0 |
| 30  | 0 |
| 31  | -1106652480 = $-2^9 \cdot 3 \cdot 5 \cdot 1152763$ |
| 32  | -590688000 = $-2^8 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 293$ |
| 33  | 0 |
| 34  | 0 |
| 35  | -743484000 = $-2^9 \cdot 3 \cdot 5^4 \cdot 7 \cdot 53 \cdot 167$ |
| 36  | -36145440 = $-2^9 \cdot 3^4 \cdot 5 \cdot 2789$ |
| 37  | 0 |

*Continued on next page*
We take $67 \times 3 = 201$-th and $67 \times 43 = 2881$-th Fourier coefficients of $\kappa^{-1}(\Delta(z))$. Since $201, 2881 \equiv 1 \pmod{4}$, they are both quadratic discriminants. We set $D = 2881, D' = 201, p = 67$, we have

$$v_{67} \left( \frac{\#\mathcal{III}_6^{BK}(\mathbb{Q}, A_{\Delta,2881})}{\#\mathcal{III}_6^{BK}(\mathbb{Q}, A_{\Delta,201})} \right) = v_{67} \left( \frac{c_{2881}^2}{c_{201}^2} \left( \frac{3}{43} \right)^5 \right).$$

From the above table for $i = 43, 3$, we know $67 \mid c_{2881}$ and $67 \nmid c_{201}$ to get $67 \mid \#\mathcal{III}_6^{BK}(\mathbb{Q}, A_{\Delta,2881})$. Thus $\text{rk}_{67}(\mathcal{III}_{67}^{BK}(\mathbb{Q}, A_{\Delta,2881})[67]) \geq 2$. Now we check that the assumptions in Theorem 1.1 satisfied for $p = 67, f = \Delta(z), k = 12, D = 2881, D' = 201, N = 1$. As in the first example, we see that the conditions (1), (3), (4) are satisfied. The assumption (2) is also satisfied since $\Delta(z)$ is good ordinary at $67$ and our quadratic discriminants $2881, 201$ are proper multiples of $67$. Hence we can apply our main theorem and we can see that the $\mathbb{F}_{67}$-representation $\text{Cl}_{K_{\Delta,2881}} \otimes \mathbb{F}_{67}$ has $M_{\Delta,2881}$ as its quotient representation.

In this case, unlike the first example, we can show that the $\mathbb{F}_{67}$-representation $M_{\Delta,2881}$ never comes from elliptic curve over $\mathbb{Q}$. In other words, $M_{\Delta,2881}$ can not be isomorphic to $E[67] \otimes \omega_{\text{cyc}}^i$ for some elliptic curve $E$ over $\mathbb{Q}$ and $i \in \mathbb{Z}$ with $0 \leq i \leq p - 2$. We prove this when $i = 0$ and the other cases can be proved similarly. Since $\Delta$ is good ordinary at $67$, elements of the inertia subgroup $g \in I_{67}$ at $67$ acts on $M_{\Delta,2881}$ as

$$\begin{pmatrix}
\omega_{\text{cyc}}^6(g) & \bar{u}(g) \\
0 & \omega_{\text{cyc}}^{-5}
\end{pmatrix} \cdot \chi_{2881}(g).$$

Hence $M_{\Delta,2881}$ has a one dimensional subspace $N_1$ on which $I_{67}$ acts via $\omega_{\text{cyc}}^6 \cdot \chi_{2881}$. Suppose this representation $M_{\Delta,2881}$ comes from a group of $67$-torsion points of an elliptic curve $E$ over $\mathbb{Q}$. Suppose $E$ has good reduction at $67$. Then the representation $E[67]$ has a 1-dimensional subrepresentation when we see it as a $G_{\mathbb{Q}_{67}}$-module. So $E$ is good ordinary at $67$ and the above matrix (18) is similar to $\begin{pmatrix}
\omega_{\text{cyc}}^6(g) & \bar{v}(g) \\
0 & 1
\end{pmatrix}$ for all

| $i$ | $67i$ | $c_{67i}$ |
|-----|------|---------|
| 38  | 2546 | 0       |
| 39  | 2613 | $-981246240 = -2^a \cdot 3^2 \cdot 5 \cdot 13 \cdot 23 \cdot 43 \cdot 53$ |
| 40  | 2680 | $3359129280 = 2^b \cdot 3 \cdot 5 \cdot 13 \cdot 17 \cdot 71 \cdot 223$ |
| 41  | 2747 | 0       |
| 42  | 2814 | 0       |
| 43  | 2881 | $-2622438960 = -2^4 \cdot 3 \cdot 5 \cdot 67 \cdot 71 \cdot 2297$ |
$g \in I_{67}$ where $\bar{v}(g) \in F_{67}$. Then $M_{\Delta,2881}$ has a 1-dimensional subspace $N_2$ on which $I_{67}$ acts via $\omega_{\text{cyc}}$. Since it is known that $M_{\Delta,2881}$ does not split as a $G_{Q_{67}}$-representation, we have $N_1 = N_2$ which implies $\omega_{\text{cyc}}^6 \cdot \chi_{2881} = \omega_{\text{cyc}} \iff \omega_{\text{cyc}}^5 \cdot \chi_{2881} = 1$. However, $\omega_{\text{cyc}}$ and $\chi_{2881}$ have orders 66 and 2 respectively and their product is never trivial. This is a contradiction. Next we suppose $E$ has bad reduction at 67. First we assume the reduction is potentially multiplicative. Then the theory of the Tate curve says that $g \in I_{67}$ acts on $E[67]$ via

$$\begin{pmatrix}
\omega_{\text{cyc}}(g) & \bar{w}(g) \\
0 & 1
\end{pmatrix},$$

where $\bar{w}(g) \in F_{67}$. Hence we get the same conclusion as in the case $E$ is good ordinary at 67. Finally we assume that $E$ has bad and potentially good reduction at 67. It is a well-known fact that for an elliptic curve $E$ over $Q_{\ell}$ which has potentially good reduction at $\ell$, $E$ acquires good reduction over a totally ramified extension of degree 4 or 6 over $Q_{\ell}$. Let $L$ be such an extension. Then $g \in G_{Q_{67}^\text{ur}} \cdot L$ acts on $E[67]$ via a matrix

$$\begin{pmatrix}
\omega_{\text{cyc}}(g) & \bar{x}(g) \\
0 & 1
\end{pmatrix},$$

where $\bar{x}(g) \in F_{67}$. Hence this matrix and (18) are similar for all $g \in G_{Q_{67}^\text{ur}} \cdot L$ and this yields a equation $\omega_{\text{cyc}}^5 \cdot \chi_{2881} = 1$ on $G_{Q_{67}^\text{ur}} \cdot L$ as in the above argument. Then, putting $F$ as the Galois extension of $Q_{67}^\text{ur}$ cut out by the character $\omega_{\text{cyc}} \cdot \chi_{2881}$ of $I_{67}$, we have $F \subset L$. However we know $[F : Q_{67}^\text{ur}] = 66$ since $Q_{67}^\text{ur}(\zeta_{67})$ and $Q_{67}^\text{ur}(\sqrt{2881})$ are linearly disjoint over $Q_{67}^\text{ur}$. This is a contradiction and hence the representation $M_{\Delta,2881}$ never comes from an elliptic curve over $Q$.

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