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Transfer Report

Developing Theoretical Foundations for Runtime Enforcement

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Declaration

I, the undersigned, declare that the dissertation entitled:

Developing Theoretical Foundations for Runtime Enforcement

submitted is my work, except where acknowledged and referenced.

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Abstract

The ubiquitous reliance on software systems increases the need for ensuring that systems behave correctly and are well protected against security risks. Runtime enforcement is a dynamic analysis technique that utilizes software monitors to check the runtime behaviour of a software system with respect to a correctness specification. Whenever the runtime behaviour of the monitored system is about to deviate from the specification (either due to a programming bug or a security hijack attack), the monitors apply enforcement techniques to prevent this deviation.

Current Runtime Enforcement techniques require that the correctness specification defines the behaviour of the enforcement monitor itself. This burdens the specifier with not only having to define property that needs to be enforced, but also with having to specify how this should be enforced at runtime; we thus relieve the specifier from this burden by resorting to a highly expressive logic. Using a logic we allow the specifier to define the correctness specification as a logic formula from which we can automatically synthesise the appropriate enforcement monitor.

Highly expressive logics can, however, permit for defining a wide variety of formulae, some of which cannot actually be enforced correctly at runtime. We thus study the enforceability of Hennessy Milner Logic with Recursion (\(\mu\text{HML}\)) for which we identify a subset that allows for defining enforceable formulae. This allows us to define a synthesis function that translates enforceable formulae into enforcement monitors. As our monitors are meant to ensure the correct behaviour of the monitored system, it is imperative that they work correctly themselves. We thus study formal definitions that allow us to ensure that our enforcement monitors behave correctly.
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1. Introduction

Modern society is becoming more dependent on software solutions, thus increasing the need for software systems to behave correctly. In an ideal world, the correctness of software systems should be entirely verified pre-deployment using static verification techniques (e.g., Theorem Proving or Model Checking). These techniques can statically determine whether a system is well-behaved, or not, as specified by a correctness property, which is often expressed in terms of an abstract logic.

However, as software systems become increasingly larger and more complex, pre-deployment verification becomes exponentially harder due to the state explosion problems inherent to static verification techniques [34, 5]. As a compromise, correctness properties can be decomposed [67, 60, 7] into smaller parts and verified using a combination of static (pre-deployment) and dynamic (post-deployment) verification techniques. Runtime Monitoring is a lightweight, dynamic verification technique in which the correctness of a program is assessed by only analysing the current execution wrt. some correctness property.

In most monitoring settings [15, 48, 25, 49], the correctness property is generally specified as a formula in a logic with precise formal semantics, from which a monitor is then automatically synthesised. This monitor is essentially the executable software which observes and analyses the runtime execution of a program in relation to the given property, and reacts accordingly. Monitoring is currently gaining interest in verification [54, 81] since it provides a mechanism for verifying...
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and (in certain cases) ensuring correct system behaviour after a system is deployed.

Runtime Enforcement (RE) [45, 54, 64] is a monitoring technique which ensures that the system behaviour is *always* in agreement with the correctness specification. The monitor (a.k.a. the *enforcer*) should therefore be capable of anticipating incorrect behaviour and counteracting it before it actually happens. Hence, enforcers are typically designed to *act as an intermediary* which wraps around the system and scrutinises its interactions. During analysis, the enforcers are thus able to *transform* incorrect executions into correct ones by either *suppressing* incorrect events (interactions) exhibited by the system, or by *inserting* events by executing actions on behalf of the system [54, 64].

The execution transformation capabilities of action suppression and insertion were first introduced in [54] in terms of special finite state automata called *Edit-Automata*. However, specifying correctness properties directly in terms of Edit-Automata, burdens the specifier with having to *manually identify* the points in which the enforcer must suppress or insert a specific system action. As identified in earlier work by Bielova *et. al* [21, 18], in most RE approaches there does *not* exist a distinction between the specification and the enforcer, which means that the specifiers must not only reason about *what* property they want to enforce, but also about *how* they should enforce it. Hence, an *algorithmic* enforcer should ideally be automatically derived from a *declarative* correctness property that is expressed as a *logic formula*.

Being able to derive enforcers from logic formulae allows for integrating Runtime Enforcement within a multi-pronged verification approach that combines static and dynamic verification. Such an approach can therefore benefit from the dynamic nature of runtime enforcement to completely ensure correct system behaviour, while minimizing the possibilities of incurring state explosion when applying pre-deployment verification in cases where enforcement might not be possible. This, however, requires understanding the boundaries of *enforceability*, *i.e.*, determining which types of properties can actually be enforced at runtime or not.
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When used in conjunction to other verification techniques, Runtime Enforcement raises a number of issues related to:

(i) the expressiveness of the logic used for defining correctness specifications,

(ii) the correctness of the enforcers themselves, and

(iii) the implementability and feasibility of enforcing a property at runtime.

In the case of (i), making use of a highly expressive logic is important since the more expressive the logic is, the more types of correctness properties one can express. Although some parts of the logic might not be enforceable, when used in combination with other verification techniques, one can employ standard techniques \([67, 7, 60]\) to decompose a large (possibly non-enforceable) property into a collection of smaller properties, such that the enforceable ones can be enforced at runtime, while the others are statically verified. Identifying which parts of the logic are enforceable is therefore crucial.

The second issue, i.e., (ii), stipulates that ensuring a degree of correctness about the enforcers is essential, especially since enforcers are often treated as part of the trusted computing base. Prior work \([54, 20, 44, 76]\) suggests that enforcers must at least be both Sound and Transparent. A sound enforcer is one which always manages to enforce the given property, while a transparent one only applies enforcement when necessary i.e., if a system is well-behaved the enforcers should not modify its runtime behaviour, and if they do, the modified behaviour should be somehow equivalent to the original one.

Finally, issue (iii) concerns the fact that most work carried out so far on runtime enforcement has either focussed entirely on its theoretical aspects (e.g., \([20, 44, 54, 64, 81]\)), or else on the implementation aspect (e.g., \([22, 23, 29, 75]\)). To our knowledge, little to no work has been conducted to study the enforceability of a logic, assess whether it is possible to synthesise sound and transparent enforcers that are also implementable, and if so develop an actual implementation that is based on these provably correct enforcers. Assessing whether these enforcement
mechanisms can actually be implemented thus enables the understanding of the potential of runtime enforcement \textit{wrt.} to real world constraints.

Despite these issues, software development and maintenance can however benefit from Runtime Enforcement in various ways. For instance, enforcement techniques provide an excellent way of ensuring the correct functionality of critical systems, \textit{i.e.}, systems which do not afford to misbehave. Such techniques can also be used as a means of protecting systems against security attacks that attempt to hijack the control flow of the enforced system, \textit{e.g.}, enforcers can shield the system by suppressing harmful external stimuli [20], or steer the execution of the system to a more stable state from where the system can be controlled using safe and well-understood procedures [30].

1.1 Aims and Objectives

In this report we investigate ways of enabling correctness properties to first be specified using a highly-expressive logic, which is independent of the verification or enforcement technique, and then be automatically converted into an enforcer capable of enforcing the specified property by inserting or suppressing specific system actions as necessary.

We thus follow the line of research investigated in [49, 50, 31, 2, 6] where they establish a correspondence between a declarative model of correctness, \textit{i.e.}, the logic, and an operational model of correctness, \textit{i.e.}, the monitors. We aim to apply this methodology to runtime enforcement, and in turn develop a notion of enforceability, \textit{i.e.}, a relation between the meaning of a property expressed as a logic formula, and the ability to enforce it at runtime. Based on this notion, we identify a maximally expressive subset of our logic that allows for defining enforceable properties.

More concretely, we aim to study the enforceability of properties defined in terms of \textit{Hennessy Milner Logic with recursion} ($\mu$HML) [5, 61, 49] — a well stud-
ied branching time logic. Due to its high expressivity, $\mu$HML can allow for defining a wide variety of properties, some of which might not be enforceable. It is, however, an ideal logic to use in a multi-pronged verification approach, by which a large property can be rewritten into smaller parts and verified (or enforced) using multiple techniques. Moreover, since $\mu$HML is one of the most expressive logics, it embeds other widely used logics and formalisms, such as LTL [77, 15, 14], CTL and CTL* [33]. By conducting this study in the context of highly expressive logics, we permit for less expressive logics to also benefit from some results obtained for this logic.

To our knowledge, the enforceability of properties expressible via a highly expressive logic such as $\mu$HML, has never been studied in depth since no one has yet presented a formal relation between a logic and the existing enforcement mechanisms. To fulfil our aims we therefore subdivide our work in the following 4 objectives:

O1. Defining Enforceability and Abstract Enforcers: To address issue (i) (expressivity), we investigate how $\mu$HML specifications can be synthesised into enforcers. Enforcers will first be defined using automata-based abstractions as this permits the study of runtime enforcement without having to deal with the complexities of a full implementation. This should lead to defining a notion of enforceability wrt. which we can identify the maximally expressive enforceable subset of $\mu$HML specifications.

O2. A Formal Evaluation for Abstract Enforcers: We aim to prove a number of correctness guarantees about our enforcement enforcers, such as soundness and transparency [54], in order to address issue (ii) (correctness), i.e., that of guaranteeing that the abstract enforcers O1 exhibit a level of correctness.

O3. Defining Implementable Enforcers: In preparation for tackling issue (iii) (implementability and feasibility), we intend to develop another synthesis function that converts enforceable $\mu$HML properties into implementable enforcers. To completely address research problem (ii) (i.e., correctness) we
must ensure that these enforcers follow the formal guarantees proven for the abstract enforcers in $O2$. We thus intend to prove correspondence, ascertaining that the behaviour of our implementable enforcers is equivalent to that described by the abstract enforcers of $O2$. This allows the former to inherit every guarantee proven for the latter.

O4. Tool Development and Evaluation: Finally, we intend to completely address feasibility by developing a RE prototype tool which implements the synthesis of implementable enforcers introduced in $O3$. Having an actual implementation will permit us to analyse and assess the performance overheads that the enforcers impose upon the enforced system during runtime.

1.2 Report Structure

In this report we discuss the initial investigations addressing our first two objectives $O1$ and $O2$; this is presented in Chapters 3 and 4. We outline the rest of the objectives as part of our future work in Chapter 6. We structure our document as follows:

- In Chapter 2 we provide the necessary preliminary material required for understanding our novel contributions; in this chapter we thus explain the chosen logic ($\mu$HML), labelled transition systems, detection monitors and monitorability.

- In Chapter 3 we define a formal runtime enforcement model capable of transforming system events, with the aim of converting invalid system executions into valid ones. We also present novel definitions by which we formally define the meaning of enforceability, i.e., we define the criteria required for a $\mu$HML formula to be enforceable.

- In Chapter 4 we identify a subset of $\mu$HML formulae that are enforceable via suppressions and establish a synthesis function that converts formulae from the identified enforceable subset into the resp. suppression enforcers. To
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ensure that the synthesised enforcers behave deterministically, we first apply a *normalization algorithm* that converts the given formula into a semantically equivalent normalized formula from which we generate the required enforcer. As means to assess the correctness of our synthesis function, we prove that the synthesised enforcers are deterministic, sound and transparent.

- Finally, in Chapter 5 we compare other related research, and then we conclude in Chapter 6 with a summary of our contributions and future work.

- The Appendix Chapters A to C *resp.* provide: additional background material for better understanding the concept of bisimilarity; proofs for lemmas required for the normalization algorithm presented in Chapter 4; and further proofs for lemmas required for proving that the synthesised enforcers are deterministic, sound and transparent as specified in Chapter 4.
2. Preliminaries

In this section we overview preliminary material that is required for understanding the novel work that we will be presenting in the forthcoming chapters.

2.1 Runtime Monitoring

In general, monitoring can be seen as the empirical observation of the behaviour of some dynamic entity; when applied to software verification, the dynamic entity is the software system being verified, while its behaviour is its runtime execution. In software verification, monitoring is therefore a lightweight compromise for automatically assessing the correctness of a system by observing and analysing its current execution. Runtime monitoring constitutes the basis of several other techniques including Runtime Verification, Adaptation and Enforcement.

In Runtime verification (RV) [62, 48] monitors adopt a passive role [15, 8] and are exclusively concerned with receiving system events, analysing them, and detecting (flagging) violations (or satisfactions) of their respective correctness properties; this is illustrated in Figure 2.1a. Hence, RV monitors are capable of recognising a (valid or invalid) execution and produce a verdict accordingly, while refraining from directly modifying the system’s behaviour in any way.

By contrast, monitors in Runtime Adaptation (RA) [26, 56, 25, 53] break this passivity by executing adaptation actions after analysing a particular sequence of
Figure 2.1: Distinguishing between Runtime Verification, Adaptation and Enforcement

system events. As shown in Figure 2.1b, rather than flagging violations, RA monitors can execute adaptation actions upon recognising a specific execution sequence, e.g., one which denotes incorrect behaviour. The adaptation actions executed by the RA monitor do not necessarily correct or revert the detected misbehaviour [66, 56]; instead they attempt to mitigate its effect by changing certain aspects of the system as it executes, with the aim of preventing either future occurrences of the same error, or of other errors that may potentially occur as a side-effect of the detected violation. RA may also be used to optimise [1, 56] the system’s behaviour based on the information collected by the monitor, e.g., switching off redundant processes when under a small load, or increasing processes and load balancing when under a heavy load.

In Runtime Enforcement (RE) [45, 54, 64] the system behaviour is kept in line with the correctness requirement by anticipating incorrect behaviour and countering it before it actually happens. In RE the monitor (a.k.a. the enforcer) is typically designed to act as a proxy which wraps around the system and analyses its external interactions (see the dotted-line in Figure 2.1c). The allows the enforcer to transform incorrect executions into correct ones by either suppressing incorrect events exhibited by the system, or by inserting events by executing actions
on behalf of the system [54, 64]. This contrasts with runtime adaptation, where the monitors may allow violations to occur but then execute remedial actions to mitigate the effects of the violation.

## 2.2 Concrete Events and System Actions

Monitors have the task of observing and analysing the behaviour of a given system. System behaviour is generally represented as a stream of observable discrete operations that can be performed by the system. We thus represent these operations as atomic \textit{concrete events}. Concrete events, $\alpha, \beta \in \text{ConcEvt}$, are used to explicitly represent and identify a single, specific system operation, e.g., $i?3$ denotes an atomic \textit{input} operation where a process with identifier $i$ inputs the value 3, while $i!4$ denotes an \textit{output} operation where process $i$ outputs the value 4. Given that concrete events describe only actual values (e.g., $i?3$ describes an actual process id $i$ and value 3), these events can \textit{easily be distinguished} depending on the type of operation they describe, and the \textit{concrete data values} they specify. We thus say that two concrete events are \textit{disjoint} from one another whenever they are not \textit{syntactically equal}, i.e., as formally defined below.

**Definition 2.1 (Disjoint Concrete Events).** Two Concrete Events $\alpha$ and $\beta$ are disjoint ($\#$) whenever they are distinct, i.e.,

$$\alpha \# \beta \equiv \alpha \neq \beta$$

We abuse notation and define a list of disjoint concrete events as follows:

$$\#_{i \in Q} \alpha_i = \forall i, j \in Q : \alpha_i \neq \alpha_j$$

where $Q$ is a set of indices, i.e., $Q = \{1, \ldots, n\}$.

**Example 2.1 (Disjoint Concrete Events).** Consider the following input events $i?3$, $i?4$ and output event $i!3$. Even though the data specified by events $i?3$ and $i!3$ is the same, namely id $i$ and value 3, these two events differ since they describe a
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different operation, i.e., input (i?3) vs. output (i!3). The input events i?3 and i?4 differ since they define different values, i.e., since 3 ≠ 4.

System Actions. Systems in general can act either in a verbose manner by executing operations that are perceivable by external entities, or else, silently by executing internal unobservable actions. We thus represent the set of actions that a system can perform as \( \mu \in \text{ACT} = \text{CONCEVT} \cup \{\tau\} \) which include all the system’s observable concrete events, \( \alpha, \beta \in \text{CONCEVT} \), along with a distinguished silent action, \( \tau \notin \text{CONCEVT} \), denoting unobservable internal system operations.

2.3 Pattern Matching, Data-binding and Symbolic Events

Pattern matching allows for open values a.k.a. patterns, \( o, l \in \text{PAT} \), defining data variables, to be compared to closed values such as concrete events. As defined below pattern \( o \) serves to define the operation type (e.g., input or output operation), along with concrete values (including process identifiers, \( i, j \in \text{PID} \), or generic data, \( v, u \in \text{VAL} \)), or variables, \( d, f, g \in \text{VAR} \).

Definition 2.2 (Patterns).

\[
\begin{align*}
o, l & \in \text{PAT} \quad ::= \delta ? \gamma \text{ (Input)} \mid \delta ! \gamma \text{ (Output)} \\
\delta & \in \text{SYMID} \quad ::= \text{VAR} \mid \text{PID} \\
\gamma & \in \text{DATA} \quad ::= \text{VAR} \mid \text{VAL}
\end{align*}
\]

For instance, pattern \( o = i?d \) describes the set of all input operations that can be performed by a process \( i \), i.e., pattern \( i?d \) describes the following set of concrete events \( \{\ldots, i?1, i?2, \ldots\} \); hence this pattern specifies that a concrete process \( i \) may input any value \( d \), where \( d \) is a variable. When referring to a symbolic pattern, \( o \), defining an arbitrary system operation (i.e., input or output) ranging over variables \( x_0 \ldots x_n \), we abuse notation and use \( o(x_0 \ldots x_n) \).
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2.3.1 Pattern Matching

We follow the standard way of representing pattern matching functionality in terms of the function $\text{mtch}(o, \alpha)$. As stated in Definition 2.3, this function matches a (possibly) open pattern (i.e., defining data variables) with a concrete event $\alpha$, and upon a successful match returns substitution environment $\sigma$, where $\sigma$ defines a bijective function which maps the variables, $d, f \in \text{Vars}$, defined in the pattern $o$ to the respective values, $v, u \in \text{Vals}$, defined in the matching concrete event $\alpha$.

**Definition 2.3** (Pattern Matching). Given a pattern $o$ and a concrete event $\alpha$,

$$\text{mtch}(o, \alpha) = \sigma \text{ such that } o\sigma = \alpha$$

**Example 2.2** (Pattern Matching). Consider the following pattern matching applications:

$$\begin{align*}
\text{mtch}(d?f, i?3) &= \{d \mapsto i, f \mapsto 3\} \\
\text{mtch}(i?3, i?3) &= \{\} \\
\text{mtch}(i?f, j?3) &= \text{undef} \\
\text{mtch}(d?f, i!3) &= \text{undef}
\end{align*}$$

In (2.1) the input pattern $d?f$ is successfully matched with the concrete input event $i?3$, where $d$ and $f$ are pattern matched with the values $i$ and $3$ resp. In (2.2) the two concrete events are matched (exactly), returning the empty substitution. The mismatch in (2.3) is due to mismatching identifiers of the input events i.e., $i$ vs. $j$, whereas the mismatch in (2.4) is because the input pattern $d?f$ cannot be matched with actions defining a different operation, e.g., output event $i!3$.

**Pattern Variants.** A pattern is said to be fully closed if it only defines concrete values, e.g., $i?3$, while a fully open pattern does not define any concrete value, i.e., defines data variables only, e.g., $d?f$. 
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**Pattern Equivalence.** We identify patterns up to the consistent renaming of their variables, i.e., two patterns \( \eta_1 \) and \( \eta_2 \) are equivalent if there exists a bijective relation \( \sigma : \text{Vars} \mapsto \text{Vars} \), such that \( \eta_1 = \sigma \eta_2 \), e.g., \( o(d_0, \ldots, d_n) \) and \( o(f_0, \ldots, f_n) \) are equivalent since \( o(f_0, \ldots, f_n)[d_0/f_0, \ldots, d_n/f_n] \) becomes syntactically equal to \( o(d_0, \ldots, d_n) \). Hence, two equivalent patterns can match the same set of concrete events and can thus define the exact same set of concrete events as shown in the definition below.

**Definition 2.4** (Equivalence). Patterns \( o_1 \) and \( o_2 \) are equivalent (\( \equiv \)) whenever

\[
o_1 \equiv o_2 \equiv [o_1] = [o_2]
\]

2.3.2 Symbolic Events

A generalization of concrete events can be attained through Symbolic Events, \( \eta \in \text{SYM_EVT} \), denoting a set of concrete events specified by a fully open pattern, \( o \in \text{PAT} \), (as defined earlier) and a filtering condition \( c \in \text{COND} \); a symbolic event \( \eta \) is thus defined as \( \eta = \{o, c\} \).

Filtering conditions specified in symbolic events represent a decidable predicate, \( c \), ranging over the variables, \( d_0, \ldots, d_n \), defined in the respective pattern \( o(d_0, \ldots, d_n) \). Once again we abuse notation and use \( c(d_0 \ldots d_n) \) to denote any condition that analyses the values bound to variables \( d_0 \ldots d_n \). Finally, we define condition evaluation as follows

**Definition 2.5** (Condition Evaluation). Given a closed, decidable condition \( c \),

- \( c \downarrow t \) iff \( c \) evaluates to true.
- \( c \downarrow f \) iff \( c \) evaluates to false.

For example, by using these filtering conditions we can restrict the range of input operations described by \( o = d?f \) via condition \( c = (d = i \land f \geq 10 \land f \leq 15) \) such that the resultant symbolic event \( \{o, c\} \), defines a set containing every concrete input event (as stated by pattern \( d?f \)) that is performed by a process with identifier \( i \).
2. Preliminaries

in which the input value is between 10 and 15 (as specified by conditions \(d=i\) and \(f \geq 10 \land f \leq 15\) resp.), such that the resultant set is \(\{i?10, \ldots, i?15\}\).

As specified by the denotational semantics given in Definition 2.6 (below), a symbolic event \(\eta = \{o, c\}\) thus defines a set containing every concrete event which matches pattern \(o\) and satisfies condition \(c\) as a result of the pattern match.

**Definition 2.6** (Denotational Semantics for Symbolic Events). For an arbitrary symbolic event \(\eta = \{o, c\}\),

\[
[[o, c]] \equiv \{ \alpha \mid \forall \alpha \cdot \text{mtch}(o, \alpha) = \sigma \text{ and } c_{\sigma} \Downarrow t \}
\]

We therefore say that a concrete event \(\alpha\) is an element of a symbolic event \(\eta = \{o, c\}\) whenever \(\alpha \in [[\eta]]\), i.e., when \(\alpha\) pattern matches \(o\) creating substitution \(\sigma\) as a result (i.e., \(\text{mtch}(o, \alpha) = \sigma\)), such that when \(\sigma\) is applied to the filtering condition \(c\), this evaluates to true (i.e., \(c_{\sigma} \Downarrow t\)).

Alternatively, we use notation \(\eta(\alpha)\) (as defined below) whenever we need to be aware about the substitution environment, \(\sigma\), that is obtained when \(\alpha\) matches the pattern \(o\) of \(\eta\) and satisfies the associated filtering condition \(c\).

\[
\eta(\alpha) \equiv \begin{cases} 
\sigma & \text{when } \text{mtch}(o, \alpha) = \sigma \land c_{\sigma} \Downarrow t \\
\text{undef} & \text{otherwise}
\end{cases}
\]

**Shorthand Notations** Although we assume that symbolic events define fully opened patterns, i.e., \(o(d_0, \ldots, d_n)\), we adopt a shorthand notation and write \(\{o(v_0, \ldots, v_n), t\}\) in lieu of \(\{o(d_0, \ldots, d_n), d_0=v_0 \land \ldots d_n=v_n\}\).

**Example 2.3.** Consider event \(\{d?f, d=i \land f=\text{req}\}\), as a shorthand we can represent this as \(\{i?\text{req}, t\}\), since conditions \(d=i\) and \(f=\text{req}\) evaluate to false in case \(d\) and \(f\) are matched with some other value.

Moreover, whenever the filtering condition \(c\) evaluates to true, we simply write \(o\) instead of \(\{o, t\}\). We will be using these shorthand notations interchangeably throughout this report.
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Singleton Symbolic Events  As specified in Definition 2.6, a symbolic event \( \{o,c\} \) denotes a set of 0 or more concrete events. A symbolic event is said to be singleton whenever it denotes a set containing a single concrete event.

Example 2.4 (Singleton Symbolic Event). Recall from Example 2.3 that \( i?req \) is a shorthand for symbolic event \( \{d?f, d=i \land f=req\} \). This symbolic event is a singleton event since \( \{d?f, d=i \land f=req\} = \{i?req\} \) where the resultant set of the denotation only contains concrete event \( i?req \).

Distinguishing between Symbolic Events. Contrary to concrete events, distinguishing between symbolic events is, however, not quite straightforward. As an example, consider events \( \{d?3, d\neq j\} \) and \( \{i?f, f>2\} \), although these events are syntactically different, concrete events such as \( i?3 \) is an element of both \( \{d?3, d\neq j\} \) and \( \{i?f, f>2\} \), since it defines an output action (?) and the concrete values \( i \) and 3 which match and satisfy the patterns and conditions of both symbolic events. By Definition 2.6 we thus know that the sets of concrete events denoted by these two symbolic events intersect with one another. Hence, as defined by Definition 2.7, symbolic events are said to be disjoint whenever the sets of concrete events they denote are disjoint. This happens whenever they define, either:

(i) a different operation, e.g., \( \{d_1?f_1, c_1\} \) and \( \{d_2?f_2, c_2\} \);

(ii) conflicting concrete data, e.g., \( \{i?f_1, c_1\} \) and \( \{j?f_1, c_1\} \), where \( i \neq j \); or

(iii) conditions which contradict each other, e.g., \( \{i?f, f>10\} \) and \( \{i?f, y\leq10\} \).

Definition 2.7 (Disjoint Symbolic Events). Two Symbolic Events \( \eta_1 \) and \( \eta_2 \) are disjoint whenever:

\[
\eta_1 \# \eta_2 \overset{\text{def}}{=} [\eta_1] \cap [\eta_2] = \emptyset
\]

Once more we abuse this notation to define a list of disjoint symbolic events as follows:

\[
\#_{i \in Q} \eta_i \overset{\text{def}}{=} \bigcap_{i \in Q} [\eta_i] = \emptyset
\]
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where \( Q \) is a set of indices, i.e., \( Q = \{1, \ldots, n\} \).

\[ \]

**Example 2.5** (Disjoint Symbolic Events). Consider events \{\( d \neq j \), \( i?f, f > 2 \)\} and \{\( i?g, g \leq 2 \)\}. Notice that the sets denoted by the first two events, \{\( d \neq j \), \( i?f, f > 2 \)\} and \{\( i?g, g \leq 2 \)\}, intersect with each other on event \( i?3 \) as shown by (2.5), meaning that they are **not** disjoint.

\[
\begin{align*}
\{[d?3, d \neq j] \cap \{i?f, f > 2\} = \{i?3\} & \quad (2.5) \\
\{[d?3, d \neq j] \cap \{i?g, g \leq 2\} = \emptyset & \quad (2.6) \\
\{i?f, f > 2\} \cap \{i?g, g \leq 2\} = \emptyset & \quad (2.7)
\end{align*}
\]

By contrast, (2.6) shows that even though patterns \( d?3 \) and \( i?f \) can match the **same** concrete event \( i?3 \), symbolic event \{\( d \neq j \), \( i?f, f > 2 \)\} is **disjoint** from \{\( i?g, g \leq 2 \)\} (and vice-versa) since this concrete event does not satisfy the filtering condition of \{\( i?g, g \leq 2 \)\}, i.e., \( (g \leq 2) \{3/g\} \equiv 3 \leq 2 = \text{false} \). Similarly, (2.7) states that events \{\( i?f, f > 2 \)\} and \{\( i?g, g \leq 2 \)\} are disjoint from each other even though their patterns can match the **exact same** set of concrete events. They are, however, guaranteed to be disjoint as they define **contradicting** filtering conditions, i.e., \( \exists n \cdot n > 2 \wedge n \leq 2 \).

\[ \]

2.4 Labelled Transition Systems

Labelled Transition Systems (LTSs) provide a convenient framework for defining an operational description of the behaviour of a system. An LTS, is a triple \( \langle \text{PROC}, \text{ACT}, \rightarrow \rangle \) which is composed from:

\( (i) \) a set of states, \( p, q, r \in \text{PROC} \), corresponding to processes;

\( (ii) \) a set of actions, \( \mu, \gamma \in \text{ACT} \), which include all the system’s observable (\( \alpha, \beta \in \text{CONC_EVT} \)) and internal \( \tau \not\in \text{EVT} \) actions; and

\( (iii) \) a transition relation, \( \rightarrow \subseteq (\text{PROC} \times \text{ACT} \times \text{PROC}) \) that relates the state on the RHS, to the other state on the LHS of the transition via a unidirectional reduction from left to right over a specific action, e.g., \( (p, \alpha, p') \in \rightarrow \) describes a unidirectional reduction from state \( p \) to \( p' \) over action \( \alpha \).
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Syntax

\[ p, q, r \in \text{Proc} ::= \text{nil} \quad \text{(inaction)} \quad | \mu.p \quad \text{(prefixing)} \quad | \sum_{i \in Q} p_i \quad \text{(choice)} \quad | \text{rec}\,x.p \quad \text{(recursion)} \quad | x \quad \text{(rec. variable)} \]

Dynamics

\[
\begin{array}{c}
\text{Act} \quad \mu.p \xrightarrow{\mu} p \\
\text{Sel} \quad \sum_{i \in Q} p_i \xrightarrow{\mu} q_j \quad j \in Q \\
\text{Rec} \quad p\{\text{rec}\,x.p/x\} \xrightarrow{\mu} p' \\
\end{array}
\]

Figure 2.2: A Model for describing Systems

For convenience, we define processes, Proc, using the regular fragment of CCS [70] as defined by the syntax in Figure 2.2. Assuming a specific set of (visible) concrete events, \( \alpha \) and a denumerable set of (recursion) variables \( x, y, z \in \text{Vars} \), processes are defined as either the inactive process nil, an action-prefixed process \( \mu.p \) i.e., prefixed by an action \( \mu \) where \( \mu \) is either a visible \( (\alpha) \) or a silent \( (\tau) \) action, a mutually-exclusive choice amongst processes where \( \sum_{i \in Q} p_i \) sums up the processes identified by the unique indices in \( Q \) (such that \( \sum_{i \in Q} p_i \) represents \( p_1 + \ldots + p_n \), where \( 1, \ldots, n \in Q \)), or a recursive process where \( \text{rec}\,x.p \) acts as a binder for \( x \) in \( p \). We work up to alpha-conversion of bound recursion variables and assume that all recursive processes are guarded, meaning that all occurrences of bound recursion variables occur under an action prefix (either directly or indirectly). Closed terms are processes where all occurrences of recursion variables are bound.

When describing the dynamic behaviour of processes, we use the more intuitive notation \( p \xrightarrow{\mu} p' \) in lieu of \( (p, \mu, p') \in \rightarrow \), along with notation \( p \xrightarrow{\mu} \) to denote \( \neg(\exists p' \cdot p \xrightarrow{\mu} p') \). The rules in Figure 2.2 are standard. Rule Act allows a \( \mu \)-prefixed process \( \mu.p \) to reduce over action \( \mu \) to the derivative \( p \), i.e., \( \mu.p \xrightarrow{\mu} p \), while by rule Sel, a choice process, \( \sum_{i \in Q} p_i \) reduces to \( q_j \) whenever there exists a process identified by some index \( j \in Q \) in this summation, i.e., \( p_j \) which performs a \( \mu \)-transition, i.e., \( p_j \xrightarrow{\mu} q_j \) (resp. \( q \xrightarrow{\mu} q' \)). By rule Rec, a recursive process \( \text{rec}\,x.p \) can reduce to \( p' \) over action \( \mu \), i.e., \( \text{rec}\,x.p \xrightarrow{\mu} p' \), when its unfolded version, \( p\{\text{rec}\,x.p/x\} \), reduces to \( p' \) over \( \mu \), i.e., \( p\{\text{rec}\,x.p/x\} \xrightarrow{\mu} p' \).
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We also employ the usual notation $p \Rightarrow p'$ and $p \overset{\alpha}{\Rightarrow} p'$ to denote weak transitions representing $p(\overset{\tau}{\Rightarrow})^* p'$ and $p \Rightarrow \cdot \overset{\alpha}{\Rightarrow} \cdot \Rightarrow p'$ resp., referring to $p'$ as a $\mu$-derivative of $p$. We also employ notation $p \overset{\hat{\alpha}}{\Rightarrow} p'$ to collectively refer to $p \Rightarrow p'$ and $p \overset{\alpha}{\Rightarrow} p'$. Sequences of visible actions are expressed as traces $t, u \in \text{ConCEvt}^*$, such that sequences of transitions are defined as $p \overset{\alpha_1}{\Rightarrow} \ldots \overset{\alpha_n}{\Rightarrow} p_n$ as $p \overset{\alpha}{\Rightarrow} p_n$, where a trace $t = \alpha_1, \ldots, \alpha_n$. For more details, consult standard texts such as [70, 5].

**Example 2.6.** Consider a (reactive) system that acts as a server that is identified with process id $i$, and which repeatedly accepts requests and subsequently responds by outputting an answer, with the possibility of terminating through the special close request (cls). Such a system may be expressed as the following process, $p_1$.

$$p_1 = \text{rec } x.(i?\text{req}.i!\text{ans}.x + i?\text{cls.nil})$$

We can outright notice that process $p_1$ is designed to output an answer ($i!\text{ans}$) for every input request ($i?\text{req}$). The same behaviour can also be represented by processes $r_1$ and $s_1$ (below), which differ from $p_1$ since $r_1$ is an unfolded version of
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\( r_1 \), while \( s_1 \) performs an internal \( \tau \) action before inputting a request.

\[
\begin{align*}
  r_1 & \equiv \text{rec } x.(i?\text{req}.i!\text{ans}.(i?\text{req}.i!\text{ans}.x + i?\text{cls.nil}) + i?\text{cls.nil}) \\
  s_1 & \equiv \text{rec } x.(&\tau.i?\text{req}.i!\text{ans}.x + i?\text{cls.nil})
\end{align*}
\]

By contrast, process \( q_1 \) denotes a server that, although similar to \( p_1 \), can also non-deterministically refuse to answer for a given request.

\[ q_1 = \text{rec } x.(i?\text{req}.i!\text{ans}.x + i?\text{req}.x + i?\text{cls.nil}) \]

Pictorially, the resp. LTSs denoted by processes \( p_1, q_1, r_1 \) and \( s_1 \) may be represented by the graphs in Figure 2.3, where the nodes correspond to processes and the arcs correspond to transitions, \( \mu \rightarrow \).

**Bisimilarity of LTSs**  Intuitively, a bisimulation [5, 70] is a binary relation that associates the behaviour of two Labelled Transition Systems that exhibit the same behaviour, i.e., in a way that one system simulates the other and vice versa. Variants of bisimulation includes Strong and Weak Bisimulation (see Definitions 2.8 and 2.9, below).

**Definition 2.8** (Strong Bisimulation). A binary relation \( \mathcal{R} \) is a Strong Bisimulation relation whenever \((p, q) \in \mathcal{R}, \) such that

(a) if \( p \xrightarrow{\mu} p' \) then there exists a strong transition \( q \xrightarrow{\mu} q' \) such that \((p', q') \in \mathcal{R} \)

(b) if \( q \xrightarrow{\mu} q' \) then there exists a strong transition \( p \xrightarrow{\mu} p' \) such that \((p', q') \in \mathcal{R} \)  ■

Hence, for a pair of processes, \((p, q)\), to be in a Strong Bisimulation relation, it is required that each process is able to simulate both visible (\( \alpha \)) and internal (\( \tau \)) actions of the other process, e.g., if \( p \xrightarrow{\alpha} p' \) then \( q \xrightarrow{\alpha} q' \) and if \( q \xrightarrow{\tau} q' \) then \( p \xrightarrow{\tau} p' \). By contrast, Weak Bisimulation abstracts away from internal (\( \tau \)) actions, i.e., each process is only required to simulate the visible actions of the other process, and can thus simulate visible actions after performing a zero or more internal (\( \tau \)) transitions, e.g., if \( p \xrightarrow{\alpha} p' \) then \( q \xrightarrow{\tau^*} \xrightarrow{\alpha} \xrightarrow{\tau^*} q' \equiv q \xrightarrow{\alpha} q' \) and if \( q \xrightarrow{\tau} q' \) then \( p \xrightarrow{\tau^0} p' \).
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Definition 2.9 (Weak Bisimulation). A binary relation $\mathcal{R}$ is a bisimulation relation whenever $(p, q) \in \mathcal{R}$, such that

(a) if $p \xrightarrow{\mu} p'$ then there exists a weak transition $q \xrightarrow{\hat{\mu}} q'$ such that $(p', q') \in \mathcal{R}$

(b) if $q \xrightarrow{\mu} q'$ then there exists a weak transition $p \xrightarrow{\hat{\mu}} p'$ such that $(p', q') \in \mathcal{R}$ ■

Proving Bisimilarity of LTS processes Since Strong (resp. Weak) Bisimilarity is the largest Strong (resp. Weak) Bisimulation relation, proving that two LTS processes $p$ and $q$ are Strong (resp. Weak) Bisimilar, i.e., $p \sim q$ (resp. $p \approx q$), only requires showing that there exists a Strong (resp. Weak) bisimulation relation $\mathcal{R}$ that can relate them as stated by Definitions 2.8 and 2.9. Note that since Strong Bisimilarity is stricter than its Weak counterpart, two Strong bisimilar processes are inherently Weak bisimilar as well.

However, proving the contrary i.e., $p \not\sim q$ (resp. $p \not\approx q$) is not as straightforward. In fact, to show that a process $p$ is not Strong (resp. Weak) bisimilar to $q$, we must show that every binary relation that exists between $p$ and $q$ do not satisfy Definition 2.8 (resp. Definition 2.9), and hence do not constitute towards valid bisimulation relations. Given the lack of practicality of this exhaustive approach, a game characterization for Strong (resp. Weak) bisimulation [5, 82] is generally employed to prove that two LTS processes are either bisimilar ($p \sim q$) or not ($p \not\sim q$); in the former case, a relation $\mathcal{R}$ can be deduced from the game derivation. The definition for a bisimulation game is given below. For more information regarding the bisimulation game characterisation one may consult Appendix Section A.1 or standard texts such as [5, 82].

Example 2.7 (Proving Strong Bisimilarity). To prove that $p_1 \sim r_1$, consider the following relation $\mathcal{R}$,

$$\mathcal{R} \doteq \{(p_1, r_1), (p_2, r_2), (p_1, r_3), (p_2, r_4), (p_3, r_5)\}$$

Since $p_1 \sim r_1$ is the largest strong bisimulation relation relating processes $p_1$ and $r_1$, it thus suffices showing that relation $\mathcal{R}$ is a Strong Bisimulation Relation as
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stated by Definition 2.8; we prove that this is true as follows:

**Proof** $p_1 \sim r_1$. To prove that the pair $(p_1, r_1) \in \mathcal{R}$ satisfies Definition 2.8, we show that $p_1 \xrightarrow{i?\text{req}} p_2$ can be strongly simulated by $r_1 \xrightarrow{i?\text{req}} r_2$ and vice-versa, such that the resultant process pair $(p_2, r_2) \in \mathcal{R}$ is true. In addition, we show that similarly reduction $p_1 \xrightarrow{i?\text{cls}} p_3$ can be simulated by $r_1 \xrightarrow{i?\text{cls}} r_5$ and vice-versa, such that $(p_3, r_5) \in \mathcal{R}$ is also true.

The same argument applies for pairs $(p_2, r_2), (p_1, r_3)$ and $(p_2, r_4)$, i.e.,

\[
(p_2, r_2) : \quad p_2 \xrightarrow{i\text{ans}} p_1 \text{ bisimulates } r_2 \xrightarrow{i\text{ans}} r_3 \text{ such that } (p_1, r_3) \in \mathcal{R} \text{ is true.} \\
(p_1, r_3) : \quad p_1 \xrightarrow{i?\text{req}} p_2 \text{ bisimulates } r_3 \xrightarrow{i?\text{req}} r_4 \text{ such that } (p_2, r_4) \in \mathcal{R} \text{ is true.} \quad \text{and} \quad p_1 \xrightarrow{i?\text{cls}} p_3 \text{ bisimulates } r_3 \xrightarrow{i\text{ans}} r_5 \text{ such that } (p_3, r_5) \in \mathcal{R} \text{ is true.} \\
(p_2, r_4) : \quad p_2 \xrightarrow{i\text{ans}} p_1 \text{ bisimulates } r_4 \xrightarrow{i\text{ans}} r_1 \text{ such that } (p_1, r_1) \in \mathcal{R} \text{ is true.}
\]

Finally, since both processes in the pair $(p_3, r_5)$ are unable to perform any further reductions, i.e., $p_3 \not\xrightarrow{\mu}$ and $r_5 \not\xrightarrow{\mu}$, they still satisfy the constraints of Definition 2.8.

--- End of Proof. ---

Hence, with the above proof we can conclude that relation $\mathcal{R}$ is a **Strong Bisimulation relation**, such that since $(p_1, r_1) \in \mathcal{R}$, we can also conclude that $p_1 \sim r_1$.

Similarly, to prove that $p_1 \sim s_1$, we must once again find a binary relation and prove that it is a Strong Bisimulation Relation as defined by Definition 2.8. Hence, we consider relation $\mathcal{R}'$,

\[
\mathcal{R}' \equiv \{ (p_1, s_1), (p_1, s_2), (p_2, s_3), (p_3, s_4) \}
\]

From the case of $(p_1, s_1) \in \mathcal{R}'$, we can immediately notice that relation $\mathcal{R}'$ is **not** a Strong Bisimulation relation since reduction $p_1 \xrightarrow{i?\text{req}} p_2$ cannot be strongly simulated by process $s_1$, i.e., $s_1 \not\xrightarrow{i?\text{req}}$. Hence, from relation $\mathcal{R}'$ we are **unable to conclude** whether $p_1 \sim s_1$ or not.

However, as proven below, by abstracting over internal ($\tau$) transitions, by Definition 2.9 we can conclude that relation $\mathcal{R}'$ is a **Weak Bisimulation relation**.

**Proof** $p_1 \approx s_1$. To prove that the pair $(p_1, s_1) \in \mathcal{R}'$ satisfies Definition 2.9,
we show that transitions $p_1 \xrightarrow{i?\text{req}} p_2$ and $p_1 \xrightarrow{i?\text{cls}} p_3$ can be weakly simulated as $s_1 \xrightarrow{\tau} s_3$ and $s_1 \xrightarrow{i?\text{cls}} s_4$ resp., such that $(p_2, s_3) \in \mathcal{R}'$ and $(p_3, s_4) \in \mathcal{R}'$ are both true; while transition $s_1 \xrightarrow{\tau} s_2$ can also be weakly simulated as $p_1 \xrightarrow{\tau} p_1$, where $(p_1, s_2) \in \mathcal{R}'$ is true.

In the case of $(p_1, s_2) \in \mathcal{R}'$, we show that transitions $p_1 \xrightarrow{i?\text{req}} p_2$ and $p_1 \xrightarrow{i?\text{cls}} p_3$ can be simulated by $s_2 \xrightarrow{i?\text{req}} s_3$ and resp. by $s_2 \xrightarrow{i?\text{cls}} s_4$, such that $(p_2, s_3) \in \mathcal{R}'$ is true and $(p_3, s_4) \in \mathcal{R}'$ is true as well; dually, $s_2 \xrightarrow{i?\text{req}} p_3$ and $s_2 \xrightarrow{i?\text{cls}} s_4$ can also be simulated by $p_1 \xrightarrow{i?\text{req}} p_2$ and resp. by $p_1 \xrightarrow{i?\text{cls}} p_3$.

Finally, we know that the pair $(p_3, s_4) \in \mathcal{R}'$ satisfies the constraints of Definition 2.9 since neither of the two processes can perform a $\mu$-reduction.

--- End of Proof. ---

Since $\mathcal{R}'$ is a Weak Bisimulation relating processes $p_1$ and $s_1$, we can therefore conclude that $p_1 \approx s_1$.

One may already notice that processes $p_1$ and $q_1$ from Figure 2.3 appear to behave differently even when observed as a black box from an external perspective; this is because $p_1$ is always obliged to provide an answer to an external request, while $q_1$ may occasionally refuse to do so.

In general, proving that $p_1 \not\approx q_1$ is an exhaustive technique that requires proving that all binary relations $\mathcal{R}$ that can relate processes $p_1$ and $p_2$ are not a Strong Bisimulation; this inherently proves that there does not exist some Strong Bisimulation relation $\mathcal{R}$, such that $(p_1, q_1) \in \mathcal{R}$. Alternatively, the bisimulation game characterisation provides a more practical alternative to formally prove that $p_1 \not\approx p_2$ (see Appendix Section A.1 for more details).

### 2.5 Linear vs Branching Time Logics

As advocated by [49, 14, 15], in runtime monitoring and enforcement, correctness properties can be defined in a wide variety of logics. These logics can be categorized...
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into two major classes [34, 71], namely, Linear Time and Branching Time Logics.

Linear Time Logics, [34, 71] treat time as if each moment has a unique possible future. Hence, formulas expressed in linear time logics are regarded as specifying the behaviour of a single program computation, as they are interpreted over linear sequences of system actions. By contrast, in Branching Time Logics [34, 71], each moment in time may split into several possible futures. Hence, branching time formulas describe properties of both finite and infinite computation trees, each of which describes the behaviour of the possible computations of a non-deterministic program.

Several researchers [71, 38, 32, 17, 59, 39, 40] have been discussing the relative advantages of linear versus branching time logics wrt. system specification and verification, since the 1980s. Certain discussions [32, 59, 39] led to the conclusion that linear and branching time logics are expressively incomparable. However in [71], the authors discuss that branching time logics are more expressive in the context of algorithmic verification.

Due to their high expressiveness, branching time logics such as $\mu$HML [5] and CTL*, thus allow for specifying a wide variety of properties which can be verified using various verification techniques. However, since dynamic verification techniques, such as monitoring, are incapable of verifying every expressible property [50], these logics are instead often favoured to be used with static analysis techniques such as model checking [5].

Multi-pronged verification approaches that combine static and dynamic verification [10], provide a practical compromise for verifying such properties by employing dynamic verification as much as possible and only use static verification for those parts that cannot be verified statically. This helps minimize the state explosion problems that are inherent to classic static verification techniques such as model checking [34, 5].
2.6 The Logic

Hennessy-Milner Logic with recursion (µHML) \cite{HennessyMilner92,Hoare85} is a highly expressive, branching-time logic that allows for defining correctness properties as logical formulae. µHML assumes a countable set of logical variables \( X, Y \in \text{LVars} \), and is defined as the set of \textit{closed} formulae generated by the grammar of Figure 2.4. The logic is equipped with the standard constructs for \textit{truth}, \textit{falsehood}, \textit{conjunction} and \textit{disjunction}, where \( \bigwedge_{\varphi_i \in \mathcal{Q}} \varphi_i \) (resp. \( \bigvee_{\varphi_i \in \mathcal{Q}} \varphi_i \)) describes a \textit{compound} conjunction, \( \varphi_1 \wedge \ldots \wedge \varphi_n \), (resp. disjunction, \( \varphi_1 \vee \ldots \vee \varphi_n \)) where 1\ldots n \in \mathcal{Q}.

µHML also provides the \textit{possibility} and \textit{necessity} modal operators, together with recursive formulae expressing \textit{least or greatest fixpoints} denoted by formulae \( \text{min} X. \varphi \) and \( \text{max} X. \varphi \) resp. Fixpoints bind free instances of the logical variable \( X \) in \( \varphi \), inducing the usual notions of open/closed formulae and formula equality up to alpha-conversion. Modal operators allow for defining \textit{symbolic events} \( \eta = \{o, c\} \), consisting in a pattern \( o \) and filtering condition \( c \). As defined in Section 2.2, the pattern may contain \textit{data variables} \( d, f, g \in \text{Var} \) that \textit{bind to system data} from a \textit{matching} concrete system event and can be used to evaluate the associated filtering condition.

Formulae are interpreted over the process powerset domain where \( S \in \mathcal{P}(\text{Proc}) \). The semantic definition of Figure 2.4 is given for \textit{both} open and closed formulae and employs a valuation (\textit{i.e.}, a map) from logical variables to sets of processes, \( \rho \in (\text{LVars} \rightarrow \mathcal{P}(\text{Proc})) \), where \( \rho'[X \mapsto S] \) denotes a valuation such that \( \rho'(X) = S \) and \( \rho'(Y) = \rho(Y) \) for all other \( Y \neq X \). This permits an inductive definition for \( [\varphi, \rho] \), the set of processes satisfying the formula \( \varphi \) \textit{wrt.} an environment \( \rho \), based on the structure of the formula.

For instance, in Figure 2.4, the semantic meaning of a variable \( X \) in relation to a map \( \rho \) is the mapping \( \rho(X) \). The semantics of truth, falsehood, conjunction and disjunction are standard, \textit{i.e.}, \( \lor \) and \( \land \) are interpreted as set-theoretic union and intersection. Possibility formulae \( \langle \eta \rangle \varphi \) describe processes that can perform an action \( \alpha \), where \( \eta(\alpha) = \sigma \) (see Definition 2.3), such that \textit{at least one} \( \alpha \)-derivative
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Syntax
\[ \varphi, \psi \in \mu_{HML} ::= tt \quad (\text{truth}) \quad | \quad ff \quad (\text{falsehood}) \]
\[ | \bigvee_{i \in Q} \varphi_i \quad (\text{disjunction}) \quad | \quad \bigwedge_{i \in Q} \varphi_i \quad (\text{conjunction}) \]
\[ | \langle \eta \rangle \varphi \quad (\text{possibility}) \quad | \quad \lceil \eta \rceil \varphi \quad (\text{necessity}) \]
\[ | \min X. \varphi \quad (\text{min. fixpoint}) \quad | \quad \max X. \varphi \quad (\text{max. fixpoint}) \]
\[ | \, X \quad (\text{rec. variable}) \]

Semantics
\[ [\tt, \rho] \overset{=} \procd \quad [\ff, \rho] \overset{=} \emptyset \quad [X, \rho] \overset{=} \rho(X) \]
\[ [\bigwedge_{i \in Q} \varphi_i, \rho] \overset{=} \bigcap_{i \in Q} [\varphi_i, \rho] \]
\[ [\bigvee_{i \in Q} \varphi_i, \rho] \overset{=} \bigcup_{i \in Q} [\varphi_i, \rho] \]
\[ [\min X. \varphi, \rho] \overset{=} \bigcap \{S | [\varphi, \rho[X \mapsto S]] \subseteq S\} \]
\[ [\max X. \varphi, \rho] \overset{=} \bigcup \{S | S \subseteq [\varphi, \rho[X \mapsto S]]\} \]
\[ [[\eta] \varphi, \rho] \overset{=} \{p | (\forall \alpha, q \cdot p \overset{\alpha}{\rightarrow} q \text{ and } \eta(\alpha) = \sigma) \text{ implies } q \in [\varphi, \rho]\} \]
\[ [[\langle \eta \rangle \varphi, \rho] \overset{=} \{p | \exists \alpha, q \cdot p \overset{\alpha}{\rightarrow} q \text{ and } \eta(\alpha) = \sigma \text{ and } q \in [\varphi, \rho]\} \]

Figure 2.4: \(\mu_{HML}\) Syntax and Semantics

satisfies \(\varphi \sigma\). By contrast, necessity formulae \([\eta] \varphi\) describe processes capable of performing a compliant action \(\alpha\), where \(\eta(\alpha) = \sigma\), such that all of their \(\alpha\)-derivatives (possibly none) satisfy \(\varphi \sigma\).

The powerset domain \(\mathcal{P}(\text{PROC})\) is a complete lattice wrt. set-inclusion, \(\subseteq\), which guarantees the existence of least and largest solutions for the recursive formulae of the logic — these are defined resp. as the intersection of all the pre-fixpoint solutions and the union of all post-fixpoint solutions [5]. Since the interpretation of closed formulae is independent of the environment \(\rho\), we write \([\varphi]\) in lieu of \([\varphi, \rho]\).

Example 2.8. Recall processes \(p_1\) and \(q_1\) from Example 2.6, using \(\mu_{HML}\) we can formally define that two consecutive requests indicate invalid behaviour, as the desired behaviour entails that every request must be provided with an answer. This safety property can be defined as formula \(\varphi_0\):

\[ \varphi_0 \overset{=} \max X. [i?\text{req}] ([i!\text{ans}].X \land [i?\text{req}] ff) \]

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Formula $\varphi_0$ describes a recursive ($\max X \ldots$) property requiring that whenever a process identified by process id $i$, inputs a first request ($i?\text{req}$), then it cannot input a subsequent request (i.e., $[i?\text{req}]\text{ff}$), unless it outputs an answer beforehand, in which case the formula recurses (i.e., $[i!\text{ans}]X$).

With formula $\varphi_1$ (given below), we generalize formula $\varphi_0$ to range over any process id $d$, as opposed to a specific process id $i$. We however assume that this formula should not apply for the process identified by id $j$, and hence we add this restriction as the filtering condition $d \neq j$ of the symbolic event defined in the first necessity of $\varphi_1$.

\[
\varphi_1 \equiv \max X.([d?\text{req}, d \neq j])/([d!\text{ans}, t])X \wedge ([d?\text{req}, t])\text{ff}
\]

Although $\varphi_0$ and $\varphi_1$ both describe the same system behaviour, they differ in terms of the LTS they define. For instance, formula $\varphi_0$ specifies a finite LTS describing the finite set of actions of a server process that is specifically identified by id $i$, namely actions $i?\text{req}$ and $i!\text{ans}$. By contrast, since $\varphi_1$ specifies an infinite LTS since it describes the actions that can be preformed by an infinite number of server processes each identified by some process id $d \neq j$.

We say that a formula $\varphi$ is satisfiable (i.e., $\varphi \in \mathsf{SAT}$) whenever there exists some process $p$ such that $p \in \llbracket \varphi \rrbracket$, i.e., as formally defined below. Hence, in order to find whether $\varphi_1$ is satisfiable or not it suffices finding a single process $p$ which satisfies the formula.

**Definition 2.10 (Satisfiable Formulae).**

\[
\varphi \in \mathsf{SAT} \iff \exists p \cdot p \in \llbracket \varphi \rrbracket
\]

Satisfiable formulae can also be used to differentiate between processes. As formally specified in Theorem 2.1, the Hennessy-Milner theorem [5, 52] dictates that given an image-finite LTS (i.e., an LTS in which all of its states have a finite number of outgoing transitions), two LTS states (processes) are bisimilar if they both satisfy the same (non-recursive) $\mu$HML formulae (and vice-versa).
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**Theorem 2.1** (The Hennessy-Milner Theorem). Given an image finite Labelled Transition System, $\langle \text{PROC}, \text{ACT}, \rightarrow \rangle$, assuming two states $p, q \in \text{PROC}$,

$$p \sim q \iff \forall \varphi \cdot p \in \llbracket \varphi \rrbracket \iff q \in \llbracket \varphi \rrbracket$$

**Example 2.9.** Using formula $\varphi_1$ we can thus distinguish between processes $p_1$ and $q_1$ by showing that process $p_1$ satisfies $\varphi_1$ (i.e., $p_1 \in \llbracket \varphi_1 \rrbracket$) and that $q_1$ does not (i.e., $q_1 \notin \llbracket \varphi_1 \rrbracket$). We prove result using Tarski’s Fixpoint Algorithm [5, 85] in the Appendix Section A.2.

Hence, since process $q_1$ is unable to satisfy all $\mu\text{HML}$ properties that can be satisfied by $p_1$ (this is proven to be true by $\varphi_1$), by the Hennessy-Milner theorem [5, 52] (recited in Theorem 2.1), we can also conclude that $p_1 \not\sim q_1$.

2.7 A model for Detection Monitors

Runtime Verification provides an alternative mechanism for checking whether a program exhibits the expected behaviour or not. In RV, this is achieved via detection monitors, which analyse the current execution trace of the system so as to determine whether this behaves correctly or not as specified by some correctness property. Detection monitors are said to recognise a good (resp. bad) trace whenever they are able to conclude that the program satisfies (resp. violates) the given correctness property, by just analysing the trace of events generated by the executing program.

In [49, 47], Francalanza et. al defined the structure and dynamic behaviour of detection monitors in terms of the LTS syntax and semantics provided in Figure 2.5. Assuming a specific set of symbolic events, $\text{SYM_EVT}$, and a denumerable set of (recursion) variables $x, y \in \text{VARS}$, detection monitors are defined as either a verdict $v$, a prefixed process by a symbolic action $\eta = \{o, c\}$, a mutually-exclusive choice amongst two monitors, or a recursive monitor.

A monitor can issue one of three verdicts, namely, yes, no and end, resp. denoting acceptance, rejection and termination (i.e., an inconclusive outcome). Verdicts
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Syntax

\[ m, n \in \text{MON} ::= v \mid \eta.m \mid \sum_{i \in Q} m_i \mid \text{rec} x.m \mid x \]

\[ v, u \in \text{VERD} ::= \text{end} \mid \text{no} \mid \text{yes} \]

Dynamics

\[ \text{mAct} \quad \eta(\alpha) = \sigma \quad c\sigma \downarrow t \quad \eta.m \xrightarrow{\alpha} m \]

\[ \text{mRec} \quad m\{ \text{rec} x.m / x \} \xrightarrow{\alpha} n \quad \text{rec} x.m \xrightarrow{\alpha} n \]

\[ \text{mSel} \quad m_j \xrightarrow{\mu} n_j \quad \sum_{i \in Q} m_i \xrightarrow{\mu} n_j \quad j \in Q \]

\[ \text{mVer} \quad v \xrightarrow{\alpha} v \]

Instrumentation

\[ \text{iMon} \quad p \overset{\alpha}{\rightarrow} p' \quad m \overset{\alpha}{\rightarrow} m' \quad \text{mSel} \quad \sum_{i \in Q} m_i \overset{\mu}{\rightarrow} m' \quad \text{iAsyP} \quad p \overset{\tau}{\rightarrow} p' \quad m \overset{\tau}{\rightarrow} m \]

\[ \text{iTer} \quad p \overset{\alpha}{\rightarrow} p' \quad m \overset{\gamma}{\rightarrow} m \quad \text{end} \overset{\alpha}{\rightarrow} \text{end} \]

Figure 2.5: Monitors and Instrumentation

in detection monitors are irrevocable as specified by rule \text{mVer}, which states that a verdict may transition with any system action \( \alpha \in \text{Act} \) and go back to the same state. The monitor \( \text{rec} X.m \) acts as a \textit{binder} for recursion variable \( x \) in \( m \) where, by rule \text{mRec} a recursive monitor can reduce over \( \alpha \), i.e., \( \text{rec} x.m \xrightarrow{\alpha} n \), whenever its unfolded version reduces over \( \alpha \), i.e., \( m\{ \text{rec} x.m / x \} \xrightarrow{\alpha} n \). All recursive monitors are assumed to be guarded, meaning that all occurrences of bound recursive variables occur under an action prefix (either directly or indirectly).

As specified by rule \text{mAct}, a prefix monitor \( \{o,c\}.m \) binds in \( m \) any concrete event \( \alpha \) that is within the constraints of the symbolic event prefix, i.e., \( \{o,c\}.m \overset{\alpha}{\rightarrow} m\sigma \) whenever \( \alpha \) matches the pattern \( o \) and satisfies condition \( c \) via \( \{o,c\}(\alpha) = \sigma \). Substitution environment \( \sigma \) is used to bind in the derived monitor \( m \), any data variable, \( d \in \text{VAR} \), (defined in pattern \( o \)), to the resp. concrete values provided by the concrete system event \( \alpha \). For example, if \( \{i?d, d > 5\}(i?6) = \{6/d\} \) then
The behaviour of a mutually exclusive choice is specified by rule mSel, which states that \( \sum_{i \in Q} m_i \xrightarrow{\alpha} n_j \) if there exists an index \( j \in Q \) such that \( m_j \xrightarrow{\alpha} n_j \).

The rules we have seen so far, specify the runtime behaviour of monitors in isolation, without describing any notion of interaction between the monitor and the process under scrutiny. Hence, Figure 2.5 also describes an instrumentation relation, connecting the behaviour of a process \( p \) with that of a monitor \( m \) such that the configuration \( m \prec p \) denotes a monitored system.

In an instrumentation, the process leads the (visible) behaviour of a monitored system (i.e., if the process cannot \( \alpha \)-transition, then the monitored system will not either) while the monitor passively follows, transitioning accordingly. Specifically, rule IMON states that if a process can transition with action \( \alpha \) and the assigned monitor can follow this by transitioning with the same action, then in an instrumented monitored system they transition in lockstep.

However, if the monitor is unable to perform such a transition, i.e., \( m \not\xrightarrow{\alpha} \), even after any number of internal actions, i.e., \( m \not\xrightarrow{\tau} \), the instrumentation rule ITER forces it to terminate with an inconclusive verdict, end, while the process is allowed to proceed unhindered. Also, note that since end can follow any visible system event, future transitions by \( p \) are still allowed while the terminated monitor maintains its state, using the rule IMON. Finally, rule IAsyp allows processes to transition independently from the monitor wrt. internal moves, thus reducing the coupling between the process and the monitor.

**Example 2.10.** By using detection monitors we can analyse the runtime execution of processes \( p_1 \) and \( q_1 \) from Example 2.6 (restated below) in order to recognise witness traces that testify for negative behaviour as specified by \( \varphi_1 \) (given in Example 2.8), i.e., when monitoring for \( \varphi_1 \), the monitor should be able to detect the cases where processes do not always provide an answer for a given request.

\[
\begin{align*}
p_1 &= \text{rec } x. (i?req.i!ans.x + i?cls.nil) \\
q_1 &= \text{rec } x. (i?req.i!ans.x + i?req.x + i?cls.nil)
\end{align*}
\]
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To detect such instances consider monitor \( m_1 \) (defined below) that reaches a violation verdict, \( \texttt{no} \), after observing that a process that is identified by \( \text{any} \) process id \( d \), has executed two consecutive request actions (i.e., \( \{d?\text{req},d\neq j\}\{d?\text{req},t\} \cdot \texttt{no} \)), but recurses after observing that the request was serviced by an answer output action, i.e., \( \{d?\text{req},d\neq j\}\{d!\text{ans},t\} \cdot x \).

\[
m_1 \overset{\text{rec}}{=} \text{rec} \cdot x \cdot (d?\text{req} \cdot (d!\text{ans} \cdot x + d?\text{req} \cdot \texttt{no}))
\]

When instrumented with process \( q_1 \) from Example 2.6, we observe the following behaviour for the monitored system whereby on line (\( * \)) the monitor preserves the \( \texttt{no} \) verdict for all remaining transitions.

\[
m_1 \triangleright q_1 \overset{i?\text{req}}{\rightarrow} (\{d!\text{ans},t\},m_1 + \{d?\text{req},t\} \cdot \texttt{no}) \overset{i/d}{\triangleright} q_1
\]

\[
\equiv i!\text{ans} \cdot m_1 + i?\text{req} \cdot \texttt{no} \overset{i?\text{req}}{\rightarrow} \texttt{no} \overset{\text{MSEL}+\text{MACT}}{\triangleright} q_2 \equiv \texttt{no} \overset{\text{IMON}}{\triangleright} q_2
\]

\[
\equiv' \texttt{no} \overset{\text{MVER}}{\triangleright} q_1 \overset{t'}{\rightarrow} \texttt{no} \overset{\text{IMON}}{\triangleright} q_1
\]

Note how the monitor’s runtime analyses abstracts over the system’s data via variables defined in the prefixing symbolic events, e.g., \( \{d?\text{req},d\neq j\} \) and \( \{d!\text{ans},t\} \). This allows the monitor to detect the specified behaviour regardless of the data associated to the system’s action, e.g., if the process id of \( q_1 \) changes from \( i \) to \( j \), monitor \( m_1 \) would still be able to analyse the behaviour this modified version of \( q_1 \), since variable \( d \) can match with \( \text{any} \) process id.

Also, note that monitor \( m_1 \) was only able to detect a property violation since \( q_1 \) has actually executed two consecutive request, such that \( t = i?\text{req} \cdot i?\text{req} \cdot t' \). However, given the non-deterministic behaviour of \( q_1 \), it is possible that while executing, this process provides an answer for every request and never executes two consecutive requests. In this way, the monitor would never be able to conclude any verdict about \( q_1 \), despite the fact that \( q_1 \) actually violates \( \varphi_1 \), i.e., \( q_1 \notin [\varphi_1] \) (see Example 2.9). Hence, this shows that the monitor’s verdicts are limited to the behaviour exhibited by the process at runtime, and can only be issued when the
system actually executes the specified behaviour.

Similarly, in the case of process $p_1$, the monitors are unable to issue a negative verdict, $\text{no}$, since the process is designed to always issue an answer for any given request; they are also incapable of producing a positive verdict, $\text{yes}$, since the monitor keeps on recursing upon perceiving a request-answer sequence.

\section*{2.8 The monitorability of $\mu$HML}

One can immediately start to notice the structural resemblance that exists between monitor $m_1$, defined in Example 2.10, and formula $\varphi_1$, defined in Example 2.8 (both restated below). Intuitively, the maximal fixpoint and fixpoint variable, the conjunction operation and the necessity operations in $\varphi_1$, resp. map to the recursive constructs, the choice operation and the action prefixes in $m_1$.

\begin{align*}
  m_1 & \equiv \text{rec } x \cdot (d?\text{req} \cdot (d!\text{ans} \cdot x + d?\text{req} \cdot \text{no} )) \\
  \varphi_1 & \equiv \max X. [(d?\text{req}, d\not= j)][(d!\text{ans}, t)]X \land [(d?\text{req}, t)][\text{ff}]
\end{align*}

In [49], Francalanza \textit{et. al} proved the existence a formal correspondence between monitors and logic formulae. In their work they present a notion of \textit{monitorability} as a property of a correctness specification describing the ability to be adequately analysed at runtime. This definition is fundamentally dependent on the monitoring setup assumed and the conditions that constitute an \textit{adequate runtime analysis}. The authors start by distinguishing between \textit{acceptance} and \textit{rejection} monitors ($i.e.$, monitors that yield a $\text{yes}$ and $\text{no}$ verdicts resp.), by defining the predicates in Definition 2.11.

\begin{definition}[Acceptance and Rejection Predicates] \ \ \\
  1. $\text{acc}(p, m) \equiv \exists t, p' \cdot m \triangleleft p \Rightarrow \text{yes} \triangleleft p'$ \\
  2. $\text{rej}(p, m) \equiv \exists t, p' \cdot m \triangleleft p \Rightarrow \text{no} \triangleleft p'$
\end{definition}

The acceptance predicate, $\text{acc}(p, m)$, states that a program $p$ is \textit{accepted} by a monitor $m$ whenever the process is able to generate a trace of execution $t$ from which
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the monitor can deduce a positive verdict, \textit{yes}. Similarly, the rejection predicate, \textit{rej}(p,m), states that a process \textit{p} is \textit{rejected} by a monitor \textit{m} whenever the process is capable of generating an execution trace \textit{t} from which the monitor can conclude a negative verdict, \textit{no}.

**Example 2.11** (Rejecting a Process). Recall process \textit{q\textsubscript{1}} defined in Example 2.6, and monitor \textit{m\textsubscript{1}} defined in Example 2.10 (both restated below).

\[ q\textsubscript{1} \overset{\equiv}{=} \text{rec}\ x. (i?\text{req}.i!\text{ans}.x + i?\text{req}.x + i?\text{cls}.\text{nil}) \]
\[ m\textsubscript{1} \overset{\equiv}{=} \text{rec}\ x. (d?\text{req}.(d!\text{ans}.x + d?\text{req}.\text{no})) \]

As shown in Example 2.10, monitor \textit{m\textsubscript{1}} issues a \textit{no} verdict when process \textit{q\textsubscript{1}} executes trace \textit{t} = \textit{i?req}.\textit{i?req}.\textit{t}'. Hence, by the definition of \textit{rej}(p,m), we can conclude that monitor \textit{m\textsubscript{1}} is capable of \textit{rejecting} process \textit{q\textsubscript{1}}, \textit{i.e.}, \textit{rej}(q\textsubscript{1},m\textsubscript{1}).

Based on the predicates discussed in Definition 2.11, the authors define the criteria expected of a monitor \textit{m} when it \textit{monitors soundly for a property} \textit{\varphi} as \textit{smon}(m,\varphi) in Definition 2.12.

**Definition 2.12** (Sound Monitoring). A monitor \textit{m} monitors soundly for the property represented by the formula \textit{\varphi}, denoted as \textit{smon}(m,\varphi), whenever for all processes \textit{p} \in \text{Proc} the following hold:

(i). \textit{acc}(p,m) implies \textit{p} \in \llbracket \varphi \rrbracket \\
(ii). \textit{rej}(p,m) implies \textit{p} \notin \llbracket \varphi \rrbracket \\

Sound monitoring thus \textit{relates} a logic formula \textit{\varphi} to a monitor \textit{m} in such a way that whenever the monitor is able to accept (resp. reject) any process \textit{p} \in \text{Proc}, then the related formula must be \textit{satisfied} (resp. violated) by the same set of processes, \textit{i.e.}, \text{Proc}.

**Example 2.12** (Sound Monitoring). In general it is very difficult to prove that a monitor \textit{m} soundly monitors for a formula \textit{\varphi}, since as stated by Definition 2.12, this requires proving that predicates (i) and (ii) hold for \textit{every} possible process.
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$p \in \text{PROC}$ (where $\text{PROC}$ can be an infinite set). However, to better explain the concept of soundness, in this example we limit ourselves to processes $p_1$ and $q_1$ (i.e., we assume that $\text{PROC}$ is restricted to $\{p_1, q_1\}$). Therefore, using the result from Example 2.11 and the result from Example 2.9, i.e., that $q_1 \not\in [\varphi_1]$, we can conclude

\[ \text{rej}(q_1, m_1) \implies q_1 \not\in [\varphi_1] \tag{2.1} \]

Since $m_1$ is incapable of producing a positive verdict, yes, none of the processes in $\text{PROC}$ can ever generate a trace that can be accepted by $m_1$. Since this premise is false, we can conclude

\[ \text{acc}(q_1, m_1) \implies q_1 \in [\varphi_1] \tag{2.2} \]
\[ \text{acc}(p_1, m_1) \implies p_1 \in [\varphi_1] \tag{2.3} \]

On the other hand, from the LTS given in Figure 2.3, we can deduce that process $p_1$ is incapable of producing a trace that can be rejected by monitor $m_1$, which allows us to conclude

\[ \text{rej}(p_1, m_1) \implies p_1 \not\in [\varphi_1] \tag{2.4} \]

Hence by (2.1), (2.4) and (2.2), (2.3), and since we assume that $\text{PROC}$ is restricted to just $\{p_1, q_1\}$, we can deduce

\[ \forall p \in \text{PROC} \cdot \text{rej}(p, m_1) \implies p \not\in [\varphi_1] \tag{2.5} \]
\[ \forall p \in \text{PROC} \cdot \text{acc}(p, m_1) \implies p \in [\varphi_1] \tag{2.6} \]

Finally, by (2.5), (2.6) and Definition 2.12 we can conclude that monitor $m$ is able to soundly monitor processes $p_1$ and $q_1$, wrt. formula $\varphi$, i.e., we can conclude that $\text{smon}(m_1, \varphi_1)$. \hfill \blacksquare

In addition, the authors also define relate formula satisfactions and violation to monitor detections in the opposite direction, by defining the notion of partially complete monitoring in Definition 2.13. This notion entails that monitoring should be either satisfaction or violation complete, i.e., if a formula $\varphi$ is satisfiable (resp.
unsatisfiable), the monitor should accept (resp. reject) it at runtime.

**Definition 2.13** (Satisfaction, Violation, and Partially-Complete Monitoring).

\[
\begin{align*}
\text{scmon}(m, \varphi) & \equiv \forall p \cdot p \in [\varphi] \text{ implies acc}(p, m) \quad \text{(satisfaction complete)} \\
\text{vcmon}(m, \varphi) & \equiv \forall p \cdot p \notin [\varphi] \text{ implies rej}(p, m) \quad \text{(violation complete)} \\
\text{cmon}(m, \varphi) & \equiv \text{scmon}(m, \varphi) \text{ or vcmon}(m, \varphi) \quad \text{(partially complete)}
\end{align*}
\]

**Example 2.13** (Partially-Complete Monitoring). Same as per Sound Monitoring, in general, proving that a monitor \(m\) monitors for a formula \(\varphi\) in a Partially Complete manner is a very hard task, since as stated by Definition 2.13, this requires proving satisfaction and violation completeness for every possible process \(p \in \text{PROC}\) (where \(\text{PROC}\) can be an infinite set). Once again, in this example we simplify the proof by limiting ourselves to just processes \(p_1\) and \(q_1\) (i.e., we assume that \(\text{PROC}\) is limited to just \(\{p_1, q_1\}\)). Hence, using the result from Example 2.9, i.e., that \(q_1 \notin [\varphi_1]\), and the result from Example 2.11 i.e., that \(\text{rej}(q_1, m_1)\), we can conclude

\[
q_1 \notin [\varphi_1] \text{ implies rej}(q_1, m_1) \tag{2.7}
\]

Although from Example 2.9 we know that \(p_1 \in [\varphi_1]\), since monitor \(m_1\) is unable to produce positive verdicts, we can conclude

\[
(p_1 \in [\varphi_1] \text{ implies acc}(p_1, m_1)) \text{ is false} \tag{2.8}
\]

Hence, the result in (2.8) prevents us from immediately deducing partial completeness, however, we also know that \(p_1 \in [\varphi_1]\), which inherently means that \(p_1 \notin [\varphi_1]\) is a false statement. Therefore, given that a *false statement can imply any result*, we can simply conclude

\[
p_1 \notin [\varphi_1] \text{ implies rej}(p_1, m_1) \text{ is true} \tag{2.9}
\]

Therefore, from (2.7), (2.9) and the definition of \(\text{vcmon}(m, \varphi)\) we can conclude that monitor \(m\) can monitor processes \(p_1\) and \(q_1\) wrt. formula \(\varphi\) in a violation-complete manner, i.e., we know \(\text{vcmon}(m_1, \varphi_1)\). Finally, by the definition of \(\text{cmon}(m, \varphi)\) we can conclude that there exists a partially-complete monitoring relation between
monitor $m_1$ and formula $\varphi_1$, i.e., we can conclude $\text{cmon}(m_1, \varphi_1)$.

Finally, using Definition 2.12 and Definition 2.13, the authors thus define a monitor-formula correspondence relation as stated by Definition 2.14. This relation states that a monitor $m$ is said to monitor for a formula $\varphi$, whenever it can do so in a sound and partially-complete manner.

**Definition 2.14 (Monitor-Formula Correspondence).**

\[
\text{mon}(m, \varphi) \iff \text{smon}(m, \varphi) \text{ and } \text{cmon}(m, \varphi)
\]

**Example 2.14 (Establishing a Monitor-Formula Correspondence).** The results of Examples 2.12 and 2.13, i.e., $\text{smon}(m_1, \varphi_1)$ and $\text{cmon}(m_1, \varphi_1)$, along with Definition 2.14, allow us to conclude that there exists a correspondence relation between monitor $m$ and formula $\varphi_1$, meaning that $m_1$ is able to monitor for $\varphi_1$. Hence, we can conclude $\text{mon}(m_1, \varphi_1)$.

Based on the correspondence that was established between monitors and logic formulae in Definition 2.14, the authors define the meaning of a monitorable formula and a monitorable language subset as monitorability (stated by Definition 2.15, below).

**Definition 2.15 (Monitorability).** Formula $\varphi$ is monitorable iff there exists a monitor $m$ such that $\text{mon}(m, \varphi)$. A logical language $L \subseteq \mu\text{HML}$ is monitorable iff every $\varphi \in L$ is monitorable.

**Example 2.15 (Monitorability).** Using the result obtained in Example 2.14, i.e., $\text{mon}(m_1, \varphi_1)$, we know that since monitor $m_1$ is able to monitor for formula $\varphi_1$, then this $\mu\text{HML}$ formula is monitorable.

Francalanza et. al also showed that not all logical formulae are monitorable and as a result, they identified a syntactic subset of $\mu\text{HML}$ formulae called mHML, for which they proved it is a monitorable subset. The language subset consists of the safe and co-safe syntactic subsets of $\mu\text{HML}$, denoted as sHML and cHML resp. in Definition 2.16.
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**Definition 2.16** (Monitorable Logic). \( \varphi, \psi \in \text{MHML} \equiv \text{sHML} \cup \text{cHML} \) where:

\[
\begin{align*}
\theta, \vartheta \in \text{sHML} & \equiv \text{tt} \mid \text{ff} \mid [\eta] \theta \mid \theta \land \vartheta \mid \text{max} X.\theta \mid X \\
\pi, \varpi \in \text{cHML} & \equiv \text{tt} \mid \text{ff} \mid \langle \eta \rangle \pi \mid \pi \lor \varpi \mid \text{min} X.\pi \mid X
\end{align*}
\]

**Example 2.16** (Monitorable Formulae). Reconsider formula \( \varphi_1 \), from Example 2.8 (restated below), along with formulae \( \varphi_2 \) and \( \varphi_3 \) (defined below).

\[
\begin{align*}
\varphi_1 & \equiv \text{max} X.\left[\left\{d\text{\ req}, d \neq j\right\}\left(\left\{d\text{\ ans}, t\right\}\left[X \land \left[\left\{d\text{\ req}, t\right\}\text{ff}\right]\right]\right)\right]
\end{align*}
\]

\[
\begin{align*}
\varphi_2 & \equiv \langle\left\{d\text{\ req}, t\right\}\left\{d\text{\ ans}, t\right\}\rangle\text{tt}
\end{align*}
\]

\[
\begin{align*}
\varphi_3 & \equiv \varphi_1 \lor \varphi_2
\end{align*}
\]

The syntactic restrictions given in Definition 2.16 allow us to conclude that formula \( \varphi_1 \in \text{sHML} \), while \( \varphi_2 \in \text{cHML} \), thus meaning that both are monitorable according to Definitions 2.15 and 2.16. However, since \( \varphi_3 \) can neither be defined in terms of \( \text{sHML} \) nor \( \text{cHML} \), we are unable to draw any conclusion regarding its monitorability, i.e., we cannot say whether it is monitorable or not.

To show that \( \text{MHML} \) is monitorable as implied by Definition 2.15, the authors define the synthesis function \( \{ - \} \) (defined in Figure 2.6) that generates a detection monitor for each \( \varphi \in \text{MHML} \). They also show that \( \{ \varphi \} \) is able to generate the witness monitor required by Definition 2.15 to demonstrate the monitorability of \( \varphi \), i.e., they show that the synthesis generates sound and partially-complete detection monitors.

Their synthesis (restated in Figure 2.6) converts the logic falsehood \( \text{ff} \in \text{MHML} \) and truth \( \text{tt} \in \text{MHML} \) into monitor verdicts \( \text{no}, \text{yes} \in \text{VERD} \) resp., and logical variables \( X \in \text{MHML} \) into the corresponding recursive variables \( x \in \text{MON} \). Logic necessities, \( [\eta] \varphi \in \text{sHML} \), and possibilities, \( \langle \eta \rangle \varphi \in \text{cHML} \), are both mapped to monitor actions, \( \eta, \{ \varphi \} \in \text{MON} \) in the general case, yet in certain cases, the synthesis simplifies the monitor based on logical equivalences, e.g., since \( [\eta]\text{tt} \equiv \text{tt}, \{ [\eta]\text{tt} \} \) synthesises monitor \( \text{yes} \) instead of \( \eta, \text{yes} \), similarly since \( \langle \eta \rangle \text{ff} \equiv \text{ff}, \{ \langle \eta \rangle \text{ff} \} \) synthesises monitor \( \text{no} \).
2. Preliminaries

Conjunctions, $\bigwedge_{i \in Q} \varphi_i \in \text{SHML}$, and disjunctions, $\bigvee_{i \in Q} \varphi_i \in \text{CHML}$, are mapped to a monitor summation, $\sum_{i \in Q} \{\varphi_i\}$ unless they can be optimized. One optimization for conjunctions and \textit{resp.} disjunctions consists in synthesising a \textit{single no} (\textit{resp. yes}) monitor when there exists at least one formula in the conjunction (\textit{resp.} disjunction) which yields a \textit{no} (\textit{resp. yes}) monitor, \textit{e.g.}, since $\text{ff} \land \varphi = \text{ff}$ and $\text{tt} \lor \varphi = \text{tt}$, then $\{\text{ff} \land \varphi\}$ and $\{\text{tt} \lor \varphi\}$ \textit{resp.} yield monitors \textit{no} and \textit{yes}, rather than $\text{no} + \{\varphi\}$ and $\text{yes} + \{\varphi\}$. Another optimization serves to yield smaller summations $\sum_{i \in Q'} \{\varphi_i\}$ (where $Q' \subseteq Q$), whenever there exists a \textit{disjoint subset of indices} $Q''$, \textit{i.e.}, $Q = Q' \cup Q''$, such that every formula identified by the indices in $Q''$ yield \textit{yes} for conjunctions, and \textit{no} for disjunctions, \textit{e.g.}, since $\text{ff} \lor \varphi \equiv \varphi$ and $\text{tt} \land \varphi \equiv \varphi$, then $\{\text{ff} \lor \varphi\}$ and $\{\text{tt} \land \varphi\}$ both yield $\{\varphi\}$ instead of $\text{no} + \{\text{ff} \lor \varphi\}$ and $\text{yes} + \{\text{ff} \lor \varphi\}$ \textit{resp.}.

In the general case, the fixpoint binders, $\max X.\varphi \in \text{SHML}$ and $\min X.\varphi \in \text{CHML}$, are both mapped to the recursive construct $\text{rec } x.\{\varphi\} \in \text{MON}$. However, in certain cases the synthesis also optimizes the synthesised monitors, \textit{e.g.}, since $\max X.\texttt{tt} \equiv \texttt{tt}$, $\{\max X.\texttt{tt}\}$ synthesises a monitor \textit{yes} instead of $\text{rec } x.\text{yes}$, since $\{\texttt{tt}\} = \text{yes}$; same applies for $\min X.\texttt{ff} \equiv \texttt{ff}$, where $\{\min X.\texttt{ff}\}$ synthesises monitor $\text{no}$ instead of $\text{rec } x.\text{no}$.

Although the synthesis covers both \text{SHML} and \text{CHML}, the syntactic con-
strains of Definition 2.16 implicitly infer that the synthesis for a formula \( \varphi \) uses at most the first row and then either the first column (in the case of \( s\text{HML} \)) or the second column (in case of \( c\text{HML} \)).

**Example 2.17** (Synthesising Detection Monitors). Recall formula \( \varphi \) defined in Example 2.8, from this formula we can synthesise a detection monitor using the synthesis function defined in Figure 2.6.

\[
\begin{align*}
\{ \max X. & \{[d?req, d\ne j]\}([d?req, t]\text{ff} \land [d!ans, t]\text{X}) \} \\
= & \ \text{rec} \ x. (\{ \{[d?req, d\ne j]\}([d?req, t]\text{ff} \land [d!ans, t]\text{X}) \}) \\
= & \ \text{rec} \ x. (\{[d?req, d\ne j]\}([d?req, t]\text{ff} \land [d!ans, t]\text{X}) \}) \\
= & \ \text{rec} \ x. (\{[d?req, d\ne j]\}([d?req, t]\text{ff} \land [d!ans, t]\text{X}) \}) \\
= & \ \text{rec} \ x. (\{[d?req, d\ne j]\}.([d?req, t]\text{ff} \land [d!ans, t]\text{X}) \}) \\
\end{align*}
\]

Resultant Monitor:

\[
\text{rec} \ x. ([d?req, d\ne j].([d?req, t]\text{ff} \land [d!ans, t]\text{X}))
\]

Notice how the resultant monitor is *identical* to monitor \( m_1 \) (defined earlier in Example 2.10). Hence, this derivation demonstrates an automated way of constructing monitors directly from a formula, such that the derived monitor corresponds (in the sense of Definition 2.14) to the formula it was derived from.

Finally, the authors also proved that \( s\text{HML} \) and \( c\text{HML} \) are *maximally expressive wrt. safety and co-safety properties* i.e., every safety (resp. co-safety) property that can be defined in \( \mu\text{HML} \) can be converted into a *semantically equivalent* property expressed in \( s\text{HML} \) (resp. \( c\text{HML} \)).

**Example 2.18** (Maximally expressive subsets). Consider the following safety property \([a]\text{ff} \lor [a]\text{ff} \notin s\text{HML} \). This property can be redefined into the semantically equivalent \( s\text{HML} \) property \([a]\text{ff} \in s\text{HML} \).
3. A Framework for Runtime Enforcement

Enforcement monitors (a.k.a. enforcers) implement mechanisms which ensure that the runtime behaviour of some process is kept in line with some correctness property. Hence, unlike detection monitors, enforcers are not only capable of recognising execution traces by detecting whether they satisfy or violate some property, but are also able to transform invalid executions into valid ones, thereby enforcing the behaviour dictated by the said property upon the process under scrutiny.

Chapter Overview We open this chapter by giving a brief background in relation to runtime enforcement in Section 3.1. Following this, in Section 3.2 we define a formal mechanism for transforming concrete system events which are then employed by the runtime enforcement framework, presented in Section 3.3, to transform invalid system executions into valid ones.

In Section 3.4, we present novel definitions by which we formally define the meaning of enforceability, i.e., we define the criteria required for a $\mu$HML formula to be enforceable. Finally, we conclude this chapter in Section 3.4.1 with a summary of the presented content, in which we highlight the main contributions of this chapter.
3. A Framework for Runtime Enforcement

3.1 A brief account of Runtime Enforcement

Runtime Enforcement (RE) [44, 54, 64] is a monitoring technique that aims to ensure that a given system always behaves in accordance to a given correctness property. Enforcement monitors are thus used to keep track of the system’s behaviour at runtime, and if necessary, modify the system’s dynamic behaviour to keep it in line with the given property.

Runtime Enforcement is used in areas such as software security [81, 54], since enforcement monitors provide an excellent mechanism to counter malicious attacks that hijack the control flow of the enforced system. Runtime Enforcement, however, raises a number of issues relating to the expressiveness of the logic used for defining specifications, the correctness of the enforcers themselves, and the performance overheads imposed by the enforcers.

In general, the more expressive the logic is, the more types of correctness properties one can express. Despite the high expressiveness of the logic, it may allow for defining properties that cannot be enforced at runtime by an enforcer — identifying which parts of the logic are enforceable is therefore crucial. Moreover, ensuring that the derived enforcers behave correctly is also essential as they must ensure correct system behaviour, meaning that if they behave erroneously, they might corrupt well-behaved systems. Furthermore, designing the enforcers to be as efficient as possible is necessary since the enforcers need to execute alongside the system under scrutiny. Hence, an enforced system needs to make use of additional hardware resources (CPU time, memory, etc.) when executing, since it must also execute the enforcer’s code; this is often seen as an overhead which must be minimized so to avoid rendering the enforced system unusable in practice.

The notion of Runtime Enforcement was first introduced by Schneider et. al [41, 81] as Security Automata — later renamed to Truncation Automata by Ligatti et. al in [54]. These automata, however, are limited wrt. the type of properties that they can enforce, since these automata can only prevent the specified bad behaviour from occurring by terminating the system just before the property is violated — this
means that Truncation automata can only enforce safety properties. Truncation Automata are sequence recognizers, i.e., they are only able to read an action from a trace of system events, and transition from one state to another without altering the trace in any way.

Ligatti et. al [16, 54] sought to widen the set of enforceable properties by defining Suppression and Insertion automata. These two enforcement mechanisms differ from Truncation automata as they are based on sequence transformation, rather than recognition. Sequence transformers are automata that are not only capable of reading an action from the system’s trace and of transitioning from one state to another accordingly (as in a standard automaton), but are also capable of modifying the trace as a result of the applied transition.

A Suppression automaton is able to enforce properties by suppressing specific program actions; this allows for enforcing safety properties by suppressing violating actions rather than by terminating the program outright. Insertion automata seek to enforce properties by inserting (executing) a sequence of one or more actions on behalf of the enforced program; this allows for enforcing more expressive types of properties such as co-safety and infinite-renewal properties, amongst others. Automata defining both suppressions and insertions are known as Edit Automata [54, 64].

Ligatti et. al realised the importance of providing correctness guarantees about enforcement automata, and thus proposed that enforcement automata should at least guarantee Soundness and Transparency [54, 64]. Soundness requires that a system that is being enforced by a monitor must never violate the enforced property, while Transparency states that valid executions should not be altered in any way by the enforcement automaton.

Bielova [21, 18] however stressed that until now there is no distinction between the specification and the enforcement monitor, which burdens the specifier with having to specify the enforcers themselves in terms of edit-automata. This means that the specifier must manually identify the points in which the monitor must
3. A Framework for Runtime Enforcement

suppress or insert a system action.

3.2 From Symbolic Events to Transformations

As described in Section 2.3, symbolic events, \( \eta = \{o, c\} \), provide a neat way for describing a set of concrete events, thereby enabling monitors to recognize a wider range of execution traces. However, by themselves symbolic events are not capable of modifying the events that they describe. We thus introduce Symbolic Transformations, \( \{o, c, o'\} \), which extend symbolic events by allowing for replacing concrete events which match pattern \( o \) and satisfy condition \( c \sigma \), with the concrete event \( o' \sigma \), where \( \sigma \) is obtained as a result of successful pattern matching. This behaviour is formally defined below.

**Definition 3.1** (Denotational Semantics for Symbolic Transformations). For an arbitrary Symbolic Transformation \( \{o, c, o'\} \),

\[
\llbracket\{o, c, o'\}\rrbracket \triangleq \lambda x. \begin{cases} (o' \sigma, \sigma) & \text{when } \{o, c\}(x) = \sigma \\ \bot & \text{otherwise} \end{cases}
\]

A symbolic transformation, \( \{o, c, o'\} \), thus denotes a function which accepts a concrete event \( \alpha \) via argument \( x \), and returns a pair containing a (possibly) different concrete event \( o' \sigma \) along with the substitution environment, whenever the input concrete event \( \alpha \) satisfies the symbolic event \( \{o, c\} \), i.e., when \( \{o, c\}(x) = \sigma \) (see Definition 2.6).

As a result of this function application, i.e., \( \llbracket\{o, c, o'\}\rrbracket(\alpha) \), the input concrete event \( \alpha \) is transformed into another concrete event \( o' \sigma = \beta \), where the transformation pattern \( o' \) is a closed symbolic pattern, i.e., any data variables present in \( o' \) must also be defined in \( o \), such that when substitution \( \sigma \) is applied on \( o' \), this yields a concrete event \( \beta = o' \sigma \).

We abuse notation and say that two symbolic transformations are disjoint whenever the symbolic events they define are disjoint, i.e., \( \{o_1, c_1, o'_1\}\#\{o_2, c_2, o'_2\} \) whenever \( \{o_1, c_1\}\#\{o_2, c_2\} \). Finally, we adopt the shorthand notation \( \{o, c, o'\}(\alpha) \) to
3. A Framework for Runtime Enforcement

Example 3.1 (Symbolic Transformations). Symbolic Transformations thus allow for replacing a concrete event with another. For instance, consider event $i!3$ and transformation $\{i!d, d>2, i!\text{err}(d)\}$, where the latter transformation replaces input $i!3$ into an error report $i!\text{err}(3)$, since value 3 was matched with variable $d$, such that $(d > 2)\{3/d\}$, i.e., $3 > 2$ evaluates to true.

As $\tau$-transitions are not perceivable by external observers, the replacement mechanism provided by the Symbolic Transformations, can also be used to replace visible concrete events into unobservable silent $\tau$-actions. For instance, by using $\{i!d, d>2, \tau\}$ we can suppress events such as $i!3$ by transforming them into a $\tau$-action, thereby making them invisible to external observers.

By omitting transformation, an identity transformation such as $\{i!d, d>2, i!d\}$ can simulate action recognition by outputting the original input event, and hence $\{i!d, d>2, i!d\}(i!3)$ produces the pair $(i!3, \{3/d\})$.

Symbolic Transformations will therefore serve to provide the bases for action transformation in our enforcement framework.

3.3 The Framework

We model enforcers in terms of LTSs, through the syntax of Figure 3.1. The syntax allows for defining: the identity enforcer ($\text{id}$), enforcers that are prefixed by symbolic transformations ($\{\alpha, c, o', e\}$), recursive enforcers ($\text{rec}\ x. e$), and selections ($\sum_{i \in Q} e_i$ where $Q$ is a set of indices such that $\sum_{i \in Q} e_i$ represents $e_1 + \ldots + e_n$, where $1, \ldots, n \in Q$). The structure of enforcers is thus very similar to that of detection monitors, with the exception that enforcers cannot issue verdicts yes and no, and action recognition via symbolic events, is extended to action transformation via symbolic transformations.

The behaviour of the enforcers is also similar to that of detection monitors for the common constructs such as recursion ($\text{eRec}$) and selections ($\text{eSel}$). In fact,
3. A Framework for Runtime Enforcement

Syntax
\[ e, f \in \text{ENF} \quad ::= \quad \{o, c, o\}'e \quad | \quad \sum_{i \in Q} e_i \quad | \quad \text{rec}x.e \quad | \quad x \quad | \quad \text{id} \]

Dynamics

\[
\begin{align*}
\text{EID} & \quad \quad e_{\alpha} \quad \Rightarrow \quad \text{id} \quad \frac{e_{\alpha}}{e_{\mu}} \quad \text{id} \quad \frac{\text{id}}{e'_{\sigma}} \\
\text{ESel} & \quad \quad \sum_{i \in Q} e_i_{\alpha} \quad \Rightarrow \quad e'_{j} \quad \quad j \in Q \\
\text{ERE} & \quad \quad e_{\text{rec}x.e/x}_{\alpha} \quad \Rightarrow \quad e'_{\mu} \\
\text{ETRNS} & \quad \quad \{o, c, o\}'e_{\alpha} \quad \Rightarrow \quad e_{\sigma} \\
\end{align*}
\]

Instrumentation

\[
\begin{align*}
\text{ITER} & \quad \quad p_{\alpha} \quad \Rightarrow \quad p' \quad e_{\tau} \quad \Rightarrow \quad e[p]_{\alpha} \quad \Rightarrow \quad \text{id}[p'] \\
\text{IASYP} & \quad \quad p_{\tau} \quad \Rightarrow \quad p' \quad e_{\tau} \quad \Rightarrow \quad e[p]_{\tau} \quad \Rightarrow \quad e'[p'] \\
\text{IENF} & \quad \quad p_{\alpha} \quad \Rightarrow \quad p' \quad e_{\alpha} \quad \Rightarrow \quad e'[\sigma] \\
\end{align*}
\]

Figure 3.1: A model for Enforcement monitors

for recursion we use a \(\tau\)-free semantics via rule \(\text{ERE}\), which allows a recursive enforcer, \(\text{rec}x.e\), to reduce to some \(e'\) whenever its unfolded version, \(e\{\text{rec}x.e/x\}\), reduces to \(e'\). For selections, rule \(\text{ESel}\) states that whenever a single enforcer \(e_j\) (where \(e_j\) is part of the summation \(\sum_{i \in Q} e_i\), i.e., \(j \in Q\)) is able to transform some action \(\alpha\) and reduce to \(e'_{j}\), i.e., \(e_j_{\alpha} \Rightarrow e'_{j}\), then the entire summation reduces to \(e'_{j}\) i.e., \(\sum_{i \in Q} e_i_{\alpha} \Rightarrow e'_{j}\).

As described by the transition rules in Figure 3.1, enforcer transitions range over action transformation rather than action recognition, i.e., the enforcer is able to modify an input action by outputting a different one, \(\alpha_{\ast} \mu\) (where \(\alpha\) might differ from \(\mu\)). Our enforcers thus achieve action transformation as described by rule \(\text{ETRNS}\). This rule states that at runtime an enforcer prefixed by a symbolic transformation, \(\{o, c, o\}'e\), can replace an action \(\alpha\) into a (possibly) different concrete action \(\mu\), thereby reducing into its derivative \(e_{\sigma}\), where \(\mu\) and \(\sigma\) are obtained by applying the symbolic transformer to \(\alpha\), i.e., via \(\{o, c, o\}'(\alpha) = (\mu, \sigma)\).

Figure 3.1 also describes an instrumentation relation for enforcement monitors which relates the behaviour of an LTS process \(p\) with that of an enforcer \(e\) where the
resultant LTS process $e[p]$ denotes the enforced system. As in the case of detection monitors, in an enforcement instrumentation, the process leads the behaviour of an enforced system: whenever the process cannot perform a transition, the enforced system will not either. However, unlike detection monitors, this type of instrumentation also allows the enforcer to determine the visible actions of the enforced system, i.e., this instrumentation allows the monitor to change an $\alpha$-transition into a $\mu$-transition, where $\mu$ can either be the same as the input (i.e., remains $\alpha$), be changed into a different concrete event $\beta$, or else be suppressed into a $\tau$-transition.

Specifically, rule $\text{iEnf}$ states that if a process can transition with action $\alpha$ and the resp. enforcer can transform this action into $\mu$, then in an instrumented system, $e[p]$, the enforcer, $e$, and the system, $p$, transition in lockstep over the enforcer’s output action $\mu$. However, if the enforcer cannot perform such a reduction, $e \not\xrightarrow{\alpha\mu}$, i.e., $\not\exists \mu, e'$ such that $e \xrightarrow{\alpha\mu} e'$, the instrumentation forces it to terminate by reducing to the identity enforcer, $id$, while the process is allowed to proceed unaffected as shown by rule $\text{iTer}$. Same as per the instrumentation for detection monitors, rule $\text{iAsyP}$ allows processes to transition independently wrt. internal transitions. However, the instrumentation does not allow enforcers to transition independently over $\tau$-transitions, given that our enforcers are $\tau$-free, i.e., do not perform silent actions themselves.

We use the notation $e \xrightarrow{t \uparrow u} e'$, to denote a sequence of transformations, e.g., $t \uparrow u = \alpha_1 \uparrow \mu_1, \ldots, \alpha_n \uparrow \mu_n$, performed by enforcer $e$, where $t$ denotes the input trace, e.g., $t = \alpha_1, \ldots, \alpha_n$, while $u$ denotes the output trace, e.g., $t = \mu_1, \ldots, \mu_n$. Since the instrumented system, $e[p]$ is a standard LTS, we use standard notation $e[p] \xRightarrow{u} e'[p']$ where $u$ denotes the sequence of enforced events, i.e., $u$ is the output trace generated by the enforcer $e$.

Example 3.2. Consider processes $p_1$ and $q_1$ (defined in Example 2.6). Assume that this time we want to enforce the property stated in Example 2.10, i.e., that every request is followed by an answer, as formally specified by formula $\varphi_1$ (see Example 2.8). One drastic way to enforce this behaviour is by suppressing every
request sent to the server which can be done by transforming every request action matching pattern \(d?req\), into a silent \(\tau\)-action, when \(d \neq j\), as shown below.

\[
e_1 = \text{rec } x. \{[d?req, d \neq j, \tau], x + \{d!ans, t, d!ans\}, x\}
\]

From an external point of view, by suppressing every request, the server can never execute two requests in a row, and so it can never be the case where a request is not followed by an answer. For instance, given the non-determinism inherent to process \(q_1\), a violating trace, \(i?req, i?req, i!ans\), may be exhibited. As shown in the derivation below, enforcer \(e\) neutralises this invalid behaviour by suppressing every request via transformation \([d?req, d \neq j, \tau]\) such that the output trace is \(\tau.\tau.i!ans\), which is equivalent to \(i!ans\).

\[
e_1[q_1] \xrightarrow{\tau} e_1[q_1] \quad \text{when } q_1 \xrightarrow{i?req} q_1, \text{ since by IEnf, } e_1 \xrightarrow{i?req>\tau} e_1
\]

\[
e_1[q_2] \xrightarrow{\tau} e_1[q_2] \quad \text{when } q_1 \xrightarrow{i?req} q_2 \text{ (same)}
\]

\[
e_1[q_1] \xrightarrow{i!ans} e_1[q_1] \quad \text{when } q_2 \xrightarrow{i!ans} q_1, \text{ since by IEnf, } e_1 \xrightarrow{i!ans>i!ans} e_1
\]

\[
\ldots
\]

Although \(e_1\) enforces the desired behaviour as required, it employs a premature enforcement stance by which it may unnecessarily edit correct executions as well. For instance, the derivation below shows how \(e_1\) needlessly modifies the behaviour of \(q_1\) such that when it produces a correct execution trace \(i?req, i!ans, i?req\), this is transformed into, \(\tau.\tau.i!ans.\tau\).

\[
e_1[q_1] \xrightarrow{\tau} e_1[q_2] \quad \text{when } q_1 \xrightarrow{i?req} q_2, \text{ since by IEnf, } e_1 \xrightarrow{i?req>\tau} e_1
\]

\[
e_1[q_1] \xrightarrow{i!ans} e_1[q_1] \quad \text{when } q_2 \xrightarrow{i!ans} q_1, \text{ since by IEnf, } e_1 \xrightarrow{i!ans>i!ans} e_1
\]

\[
e_1[q_2] \xrightarrow{\tau} e_1[q_1] \quad \text{when } q_1 \xrightarrow{i?req} q_2, \text{ since by IEnf, } e_1 \xrightarrow{i?req>\tau} e_1
\]

\[
\ldots
\]

An alternative enforcer can be defined such that enforcement is delayed to the very end, \(i.e.,\), until the enforcer is sure that the desired behaviour will definitely
be compromised; for instance consider $e_2$ (defined below).

$$e_2 \equiv \text{rec } x.\{d?\text{req}, d \neq j, d?\text{req}\}.e'_2$$

$$e'_2 \equiv \text{rec } y.\{d!\text{ans}, t, d!\text{ans}\}.x + \{d?\text{req}, t, \tau\}.y$$

Enforcer, $e_2$, prevents the violation by using an additional recursive construct, $\text{rec } y.\ldots$, to continuously suppress every request matching pattern $d?\text{req}$ such that $d \neq j$, succeeding the primary occurrence of the request action. In this way, whenever the enforced process executes two (or more) consecutive requests matching $d?\text{req}$ such that $d \neq j$, the enforcer keeps suppressing requests succeeding an unanswered request until an answer matching $d!\text{ans}$ is produced by the enforced process.

The following derivation thus shows how enforcer $e_2$ modifies the invalid execution trace $i?\text{req}.i?\text{req}.i!\text{ans}$, produced by $q_1$, into the corresponding valid trace $i?\text{req}.\tau.i!\text{ans}$, i.e., $i?\text{req}.i!\text{ans}$.

$$e_2[q_1] \xrightarrow{i?\text{req}} e'_2\{e_2/x\}[q_1] \quad \text{when } q_1 \xrightarrow{i?\text{req}} q_1, \text{ since by } \text{iEnf}, e_2 \xrightarrow{i?\text{req}} e'_2\{e_2/x\} \xrightarrow{\tau} e'_2\{e_2/x\}[q_2] \quad \text{when } q_1 \xrightarrow{i?\text{req}} q_2, \text{ since by } \text{iEnf}, e'_2\{e_2/x\} \xrightarrow{i?\text{req}} y\{e'_2\{e_2/x\}/y\} \xrightarrow{i!\text{ans}} e_2[q_1] \quad \text{when } q_2 \xrightarrow{i!\text{ans}} q_1, \text{ since by } \text{iEnf}, e'_2\{e_2/x\} \xrightarrow{i!\text{ans}} x\{e_2/x\} \ldots$$

In contrast to enforcer $e_1$, the derivation below demonstrates how the late enforcement methodology adopted by $e_2$ helps to preserve correct execution traces such as $i?\text{req}.i!\text{ans}.i?\text{req}$ by leaving them unchanged.

$$e_2[q_1] \xrightarrow{i?\text{req}} e'_2\{e_2/x\}[q_2] \quad \text{when } q_1 \xrightarrow{i?\text{req}} q_2, \text{ since by } \text{iEnf}, e_2 \xrightarrow{i?\text{req}} e'_2\{e_2/x\} \xrightarrow{i!\text{ans}} e_2[q_1] \quad \text{when } q_2 \xrightarrow{i!\text{ans}} q_1, \text{ since by } \text{iEnf}, e'_2\{e_2/x\} \xrightarrow{i!\text{ans}} x\{e_2/x\} \xrightarrow{i?\text{req}} e'_2\{e_2/x\}[q_2] \quad \text{when } q_1 \xrightarrow{i?\text{req}} q_2, \text{ since by } \text{iEnf}, e_2 \xrightarrow{i?\text{req}} e'_2\{e_2/x\} \ldots$$
3.4 Defining Enforceability of the Logic

In this section we shift back to the logic and investigate what it means for an enforcer to enforce a property. As specified below, we define enforceability of a logic as being the relationship between the meaning of a property, expressed in the said logic, and the ability to enforce it at runtime upon a specific system.

**Definition 3.2 (Enforceability).** A $\mu$HML formula $\varphi \in \text{Sat}$ is enforceable whenever

\[ \exists e \in \text{Enf} \cdot e \text{ enforces } \varphi \]

Intuitively, for a property $\varphi$ to be enforceable, there must exist an enforcer $e$ which is capable of modifying the dynamic behaviour of any process $p$ in order to keep it in line with the behaviour specified in $\varphi$. However, as discussed in Example 3.2, an enforcer may adopt different approaches in order to enforce the behaviour dictated by the given property, e.g., in Example 3.2 although $e_1$ actually prevents the violation of $\varphi_1$, it unnecessarily modifies correct behaviours, whereas $e_2$ does not.

As stated by Ligatti et. al in [54, 44, 18] (see Section 3.1), for an enforcer to adequately enforce a property, it must at least ensure that the enforced process always executes correctly wrt. $\varphi$, i.e., by either preventing it from executing violating runtime behaviour, or by ensuring the execution of runtime behaviour satisfying $\varphi$. An enforcer capable of doing so is said to be Sound. In our case, since we represent the enforced system, $e[p]$, as an LTS in itself, we can formally specify Enforcement Soundness as follows:

**Definition 3.3 (Sound Enforcement).** We say that enforcer $e$ soundly enforces a formula $\varphi$, denoted as $\text{senf}(e, \varphi)$, iff $\forall p \in \text{PROC} \cdot e[p] \in \text{[}[\varphi]\text{]}$.

The above definition specifies that enforcer $e$ soundly enforces a formula $\varphi$ if $e$ can enforce any process $p$ such that the resultant enforced LTS, $e[p]$, always satisfies $\varphi$.

Enforcement soundness on its own is, however, a relatively weak constraint as it does not regulate the extent of the applied enforcement. For instance, Example 3.2
demonstrates that enforcer although $e_1$ manages to keep the execution of process $q_1$ in line with property $\varphi_1$, it however adopts a conservative enforcement approach which needlessly modifies the correct behaviour of $q_1$ as well. Intuitively, this example indicates\(^1\) that enforcer $e_1$ *soundly enforces* property $\varphi_1$, yet lacks an element of *Transparency* [54, 44, 20], since the enforcer can unnecessarily modify valid system behaviour. Similarly, to enforce a simple property $\varphi = [a][b]$ on a process $p$, we can use an enforcer that *suppresses every action* produced by $p$ thereby suppressing the entire execution: in doing so $p$ will surely never violate $\varphi$, but will neither exhibit any kind of valid behaviour.

*Transparency* [54, 44, 20], thus dictates that whenever a process $p$ already satisfies the property $\varphi$, the assigned enforcement monitor $e$ must refrain from altering the runtime behaviour of $p$ as this would not be necessary. Transparency therefore aims to preserve the original behaviour of the process as much as possible by imposing the least number of enforcement actions, *i.e.*, enforcement is only applied when necessary. For instance, enforcer $e_2$ in Example 3.2, applies a lazy enforcement approach whereby the behaviour of process $q_1$ is only modified when the runtime behaviour of $q_1$ violates $\varphi_1$. This ensures that valid behaviour is never modified unnecessarily. Hence, once again Example 3.2 demonstrates that enforcer $e_2$ is more transparent then $e_1$ when enforcing $\varphi_1$ on $q_1$, as unlike $e_1$, it does not affect the valid behaviour of $q_1$. Hence, we formally define *enforcement transparency* in Definition 3.4 below.

**Definition 3.4** (Transparent Enforcement). An enforcer $e$ is transparent when enforcing a formula $\varphi$, denoted as $\text{tenf}(e, \varphi)$, iff

$$\forall p \in \text{Proc} \cdot p \in \llbracket \varphi \rrbracket \text{ implies } e[p] \sim p$$

This definition states that an enforcer $e$ enforces a formula $\varphi$ in a transparent manner, if for every LTS process $p$ that satisfies $\varphi$, the enforced LTS $e[p]$ is bisimilar \(^1\)

---

\(^1\)In general proving enforcement soundness is very hard as it requires proving that $e[p] \in \llbracket \varphi \rrbracket$, for *every possible process* $p$. However, Example 3.2 only shows that $e_1$ soundly enforces property $\varphi_1$ on $q_1$ (it might not be the case other processes).
to the original process \( p \). This ensures that although the resultant enforced LTS, \( e[p] \), is structurally different from the original LTS \( p \), its behaviour is still perceived to be the same as that of \( p \). Based on Definitions 3.3 and 3.4 we can thus define a stronger notion of enforcement as defined below.

**Definition 3.5 (Strong Enforcement).** We say that an enforcer \( e \) strongly enforces formula \( \varphi \), denoted as \( \text{enf}(e, \varphi) \), when

- \( \text{senf}(e, \varphi) \), i.e., \( e \) soundly enforces \( \varphi \); and
- \( \text{tenf}(e, \varphi) \), i.e., \( e \) is transparent when enforcing \( \varphi \).

Strong enforcement thus requires an enforcer \( e \) to soundly and transparently enforce a property \( \varphi \).

**Remark 3.1 (Novelty).** It is important to note that unlike existing work [54, 64, 43, 44, 18], we define the resultant enforced system as the LTS \( e[p] \). This allows is to give stronger definitions for enforcement soundness and transparency (i.e., Definitions 3.3 and 3.4), as these are given in terms of the original LTS process \( p \) and the enforced LTS process \( e[p] \), as opposed to the more classical definitions that present them in terms of an input and output (enforced) trace.

In fact, the classic definitions for soundness (e.g., [54, 64, 43, 44, 18]) require that every enforced execution trace that can be produced by an enforcement automaton (monitor), \( e \), should satisfy the enforced property \( \varphi \). By contrast, in our definition of soundness, we require that for every process \( p \), the resultant enforced LTS process \( e[p] \) must always satisfy property \( \varphi \).

Similarly, unlike the classic definitions of transparency (e.g., [54, 64, 43, 44, 18]), our definition does not only require trace equivalence between \( e[p] \) and \( p \) (as per classic definitions), but instead imposes a stronger equivalence criterion, i.e., that \( e[p] \sim p \) (see [5] for Trace Equivalence vs Bisimilarity).
3. A Framework for Runtime Enforcement

3.4.1 Concluding Remarks

In this chapter we have presented a novel framework describing the runtime behaviour of enforcers. The main novel contributions of this chapter include:

Symbolic Transformations, which formally define a mapping mechanism for transforming a concrete system event into a (possibly) different one as specified the transformation pattern (see Section 3.2).

An LTS semantics for Enforcers, which formalise the structure and dynamic behaviour of enforcers, along with the interaction between the enforcer, $e$ and the process under scrutiny, $p$, in the form of the instrumented LTS, $e[p]$ (see Section 3.3); and

A Formal Definition for Enforceability, defining the relationship between the meaning of a $\mu$HML property and its ability to be adequately enforced at runtime by an enforcer. With this definition we establish that an enforcer $e$ strongly enforces a formula $\varphi$ whenever it is able to do it in a sound and transparent manner. By defining the enforced system as an LTS, $e[p]$, we were able to provide novel definitions Soundness and Transparency which are stronger than the classic definitions (see Section 3.4).

Discussion. So far we have defined the meaning of enforceability in relation to our logic, however, we still have to explore which of the properties, expressible via $\mu$HML, can actually be enforced. Several works in RE [54, 19, 44] have already established that suppression enforcers are ideal for detecting potentially violating executions and suppressing parts of them to prevent the violation of safety properties. Similarly, it was established that insertion enforcers can also be used to enforce other types of properties such as co-safety.

We thus explore the enforceability of $\mu$HML properties in an incremental manner. We will first start by exploring the enforceability of $\mu$HML properties, wrt. suppression enforcement, and later on we aim to explore it wrt. insertion enforce-
3. A Framework for Runtime Enforcement

ment.
As stated in other work [54, 64, 18, 44], suppression enforcement is ideal to prevent the violation of safety properties by stopping erroneous events from occurring. In this chapter we limit ourselves to identifying a subset of $\mu$HML formulae that are enforceable via suppressions and establish a synthesis function that converts formulae from the identified enforceable subset into the resp. suppression enforcers.

Particularly, we investigate the enforceability of safety properties expressed in terms of sHML since, in [49], this syntactic subset was proven to be maximally expressive wrt. safety properties, i.e., any safety property that can be defined in $\mu$HML can be expressed in terms of a semantically equivalent sHML formula.

In Figure 4.1 we recall the syntax for sHML. The logic is restricted to truth and falsehood (tt and ff), conjunctions ($\varphi \land \psi$), and necessity modalities ($[\eta]\varphi$), while recursion may only be expressed through maximal fixpoints ($\max X.\varphi$). The semantics for these constructs follows from that of Figure 2.4.

**Example 4.1** (Enforcing sHML formulae). Consider the following recursive formula $\varphi_2$,

$$\varphi_2 \equiv \max X.([i?req][i!ans].X) \land ([i?req][i?req][i?req]ff)$$

Formula $\varphi_2$ defines the same invariant property as $\varphi_0$ (defined in Example 2.8),
which holds when a request is not immediately followed by a subsequent request after an arbitrary number of answered requests. The formula thus specifies that a process is incorrect when it performs two consecutive requests, \( [i\text{?req}] [i\text{?req}] \text{ff} \), but recurses whenever an answer is produced following a request, \( [i\text{?req}] [i\text{!ans}] X \).

One way how to enforce \( \varphi_2 \) is by generating a suppression enforcer such as \( e_0 \) that prevents a process from performing two or more subsequent requests.

\[
e_0 \overset{\text{def}}{=} \text{rec } x. (\{d\text{?req}, d = i, d\text{?req}\}, \text{rec } y. (\{d\text{?req}, t, \tau\}, y + \{d\text{!ans}, t, d\text{!ans}\}, x))
\]

Enforcer \( e_0 \) enforces formula \( \varphi_0 \) by suppressing every request action, \( i\text{?req}, \) that is performed after an unanswered request, until an answer \( i\text{!ans} \) is produced, in which case the enforcer recurses. This ensures that the invariant property is enforceable after an arbitrary number of requests as defined by the property.

In Example 3.2 we had also shown that this invariant property (that is formalized by both \( \varphi_0 \) and \( \varphi_2 \)) can also be enforced via enforcers \( e_1 \) and \( e_2 \) (restated below).

\[
e_1 \overset{\text{def}}{=} \text{rec } x. (\{d\text{?req}, d \neq j, \tau\}, x + \{d\text{!ans}, t, d\text{!ans}\}, x)
\]

\[
e_2 \overset{\text{def}}{=} \text{rec } x. (\{d\text{?req}, d \neq j, d\text{?req}\}, \text{rec } y. (\{d\text{?req}, t, \tau\}, y + \{d\text{!ans}, t, d\text{!ans}\}, x))
\]

Notice how enforcer \( e_0 \) is very similar to \( e_2 \). Their main difference is that \( e_0 \) only enforces the property when the requesting process is identified by \( i \) (as defined by \( \varphi_0 \)). Enforcer \( e_2 \) is, however, more generic as it applies enforcement for any requesting process that is identified by any identifier (including \( i \)) except for the process identified by \( j \).

In general, a property can be enforced either deterministically or non-deterministically as defined by Definition 4.1.

**Definition 4.1** (Deterministic Enforcement). An enforcer \( e \) behaves deterministi-
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Informally, an enforcer $e$ is deterministic whenever it is unable to react differently for the same input trace of events $t$, i.e., it always reduces to the same state, i.e., $e' = e''$, and always produces the same output (enforced) trace, i.e., $t' = t''$. In general, an enforcer behaves non-deterministically whenever it defines a selection of two (or more) non-disjoint symbolic transformations, e.g., enforcer $\{d\text{?req}, d \neq j, d\text{?req}\} . e' + \{d\text{?req}, d = i, d\text{?req}\} . e''$ is non-deterministic since the symbolic transformations that are guarding the summation branches are not disjoint, i.e., $\{d\text{?req}, d \neq j, d\text{?req}\} \# \{d\text{?req}, d = i, d\text{?req}\}$ is false.

Determining statically whether an enforcer is deterministic or not, is not a straightforward task. In fact, concluding that two (or more) symbolic transformations are disjoint cannot be done via a simple syntactic check, instead it requires making sure that there does not exist some concrete event $\alpha$ that can be transformed by multiple transformations guarding different branches in the summation.

Example 4.2 (Deterministic Enforcement). In Example 4.1 we claimed that enforcers $e_1$ and $e_2$ behave deterministically. Intuitively, this is due to the fact that the selections (+) in both $e_1$ and $e_2$ are guarded by disjoint symbolic transformations, i.e., in both cases $\{d\text{?req}, d \neq j, \tau\} \# \{d\text{!ans}, t, d\text{!ans}\}$. This ensures that when instrumented with a process $p$, the enforcers consistently make the same selections upon analysing the same concrete events generated by $p$, and thus the enforcers always behave in the same way.

In fact, enforcer $e_1$ is bound to always choose: the left branch, i.e., $\{d\text{?req}, d \neq j, \tau\} . x$, upon analysing a request event i.e., $i\text{?req}, \text{the right branch, i.e., } \{d\text{!ans}, t, d\text{!ans}\} . x$, upon an answer event i.e., $i\text{!ans}$, and reduces to the identity enforcer, $id$, upon a close event, i.e., $i\text{?cls}$; a similar argument applies for the selection applied in $e_2$. 

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\[
e \xrightarrow{t \cdot t'} e' \quad \text{and} \quad e \xrightarrow{t \cdot t''} e'' \quad \text{implies} \quad e' = e'' \quad \text{and} \quad t' = t''
\]

where $t \cdot t'$ and $t \cdot t''$ represent sequences of transformations performed by enforcer $e$ over the same input trace $t$, that may result in different output (enforced) traces. ■
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In this way, $e_1$ and $e_2$ can never reach a point in which they are able to (non-deterministically) choose between two or more branches, and so they are always bound to react in the same way for the same input events, thereby producing the same output (enforced) events.

Deterministic enforcers are appealing due to their predictable runtime behaviour, i.e., it is easier to predict how a deterministic enforcer will transform a given concrete input event. By contrast, non-determinism introduces subtleties that can lead to harmful unpredictable behaviour, in which the enforcer may sometimes (non-deterministically) select a branch which does not adequately enforce the given property.

Example 4.3 (Harmful Unpredictable Enforcement). Consider the non-deterministic enforcer $e_3$ (defined below) obtained by applying a “naive” synthesis function on $\varphi_2$, which informally converts the maximal fixpoint in $\varphi_2$ into a recursive construct, the modal necessities that are immediately followed by falsehood, into a suppression transformation, e.g., $\max X.([i?req][!ans]X)$ into $\text{rec } x.([d?req,t,\tau].x)$, and other necessities into the identity transformations, e.g., $\max X.([i?req][ff])$ into $\text{rec } x.([d?req,t,\tau].x)$.

$$\varphi_2 \equiv \max X.([i?req][!ans]X) \land ([i?req][i?req][ff])$$
$$e_3 \equiv \text{rec } x. \left( \{d?req,t,d?req\}, \{d!ans,t,d!ans\}.x \right) + \{d?req,d=i,d?req\}, \{d?req,t,\tau\}.x \right)$$

Now, consider a very simple process $r$ which only issues two consecutive request actions and terminates.

$$r \equiv i?req.r'$$
$$r' \equiv i?req.nil$$

Since the transformations guarding the selections in $e_3$, i.e., $\{d?req,t,d?req\}$ and $\{d?req,d=i,d?req\}$, are not disjoint (since $i?req \in \{d?req,t\} \cap \{d?req,d=i\}$), enforcer $e_3$ can make a non-deterministic selection (using rule ESEL) whenever process $r$ makes an initial request. However, based on this choice, $e_3$ might not always en-
force the required behaviour. In the derivation below we can see that $e_3$ manages to enforce the required behaviour by non-deterministically choosing the second branch.

$$e_3[r] \xrightarrow{i\text{req}} \{d\text{?req}, t, \tau\}.e_3[r']$$

By $\text{IENF}$, since $r \xrightarrow{i\text{req}} r'$ and $e_3 \xrightarrow{i\text{req},i\text{req}} \{d\text{?req}, t, \tau\}.e_3$

However, being non-deterministic, $e_3$ can also choose the other branch which allows for the occurrence of violating runtime behaviour as shown below.

$$e_3[r] \xrightarrow{i\text{req}} \{d\text{!ans}, t, d\text{!ans}\}.e_3[r']$$

By $\text{IENF}$, since $r \xrightarrow{i\text{req}} r'$ and $e_3 \xrightarrow{i\text{req},i\text{req}} \{d\text{!ans}, t, d\text{!ans}\}.e_3$

Notice how in the above derivation, the monitored process still manages to issue two subsequent request actions.

If we want to adhere to the notion of enforceability from Section 3.4 (Definition 3.5), our enforcers must, first and foremost, guarantee soundness, i.e., that a process $p$ monitored by a sound enforcer $e$ must always satisfy a given $\mu\text{HML}$ formula $\varphi$. This implies that the outcome of the applied transformations must always prevent the property from being violated.

From Example 4.3 we can thus notice that enforcer $e_3$ is not always capable of enforcing $\varphi_2$, and is therefore unsound. In general, however, despite exhibiting unpredictable behaviour, non-deterministic enforcers are not necessarily unsound.

**Example 4.4** (Harmless Non-Deterministic Enforcement). Recall process $q_1$, from Example 2.6, and consider the non-deterministic enforcer $e_4$ defined below.

$$q_1 \overset{\mu}{=} \text{rec } x.(i\text{?req}.i\text{!ans}.x + i\text{?req}.x + i\text{?cls.nil})$$

$$e_4 \overset{\mu}{=} e_1 + e_2$$

Enforcer $e_4$ enforces property $\varphi_2$ by first making a non-deterministic selection (using
4. Enforcing Safety Properties via Suppressions

rule ESEL) upon the occurrence of an initial request action $i?\text{req}$. Based on this non-deterministic choice, it either enforces the property by *eagerly* suppressing each and every request action via $e_1$ or else on a *by-need-basis*, by suppressing requests that occur after an initial unanswered request following an initial request via $e_2$ (see Example 3.2 for the resp. derivations). Despite being non-deterministic, enforcer $e_4$ still *soundly enforces* $\varphi_2$, as it always prevent $q_1$ from violating $\varphi_2$, even though it does not always apply the same enforcement strategy.

However, recall from Definition 3.5 that soundness is not the only criterion that an enforcer must abide by in order to *strongly enforce* a property; an enforcer is also required to be *transparent*. Non-determinism can, once again, introduce intricate behaviour that may breach the transparency criterion.

**Example 4.5** (Non-Determinism and Transparency). Despite being sound, enforcer $e_4$ is *not transparent* as it may occasionally select enforcer $e_1$ which employs an eager enforcement strategy causing it to unnecessarily modify valid system behaviour (as shown in Example 3.2). Hence, enforcer $e_4$ *fails* to strongly enforce formula $\varphi_2$.

A correct synthesis function must therefore be aware of the subtleties introduced by non-deterministic behaviour. Since the runtime behaviour of deterministic enforcers is more *predictable* compared to non-deterministic ones, they are generally less subtle and thus easier to understand and debug. For instance, if a deterministic enforcer makes a mistake while enforcing a property, it is easier (or rather more intuitive) to backtrack to the point where the mistake was made, since the enforcer is always forced to react in the same way for the same input. By contrast, understanding the behaviour of a non-deterministic enforcer is harder as one needs to take into consideration the selections that the enforcer chose during its execution, thereby making it harder to understand and debug.

We thus develop a synthesis function which only yields *deterministic enforcers* (as defined by Definition 4.1) from a given sHML formula. In Section 4.1 we
present a normalization algorithm for converting a given $\text{shml}$ formula into a semantically equivalent formula that is in a normal form, from which we can easily synthesise deterministic enforcers. In Section 4.2, we then present a novel synthesis algorithm which converts normalized $\text{shml}$ formulae into deterministic enforcers for which we prove that the synthesised enforcers always behave deterministically and are guaranteed to strongly enforce the formula they were derived from, i.e., the synthesised enforcers are guaranteed to adhere to soundness and transparency (see Definition 3.5).

### 4.1 Towards Synthesising Deterministic Enforcers through Normalization

In order to obtain deterministic monitors from a given logic formula, we follow the approach presented in [3] and adapt it to our setting; the approach works in two phases. The first phase converts the given $\text{shml}$ (resp. $\text{chml}$) formula into a semantically equivalent formula that is in an intermediary format which can then be easily converted (by the second phase) into the required monitor. Since this phase works at the level of the logic, namely wrt. the $\text{shml}$ and $\text{chml}$ subsets, it is thus independent of the type of monitor (detection or enforcement) being synthesised by the second phase.

The second phase thus employs a synthesis function that converts the result obtained from the previous phase, into the required deterministic monitors; different synthesis algorithms may be created depending on the desired type of resultant monitor, e.g., the synthesis function in [49] can be used to obtain deterministic detection monitors, while in our case, we must define a synthesis function that converts the output of the first phase into a deterministic enforcement monitor.

In this section we focus on explaining the first phase of this approach wrt. $\text{shml}$ formulae, as applying the same approach for $\text{chml}$ formula only requires minimal (syntactic) changes. We thus subdivide this section as follows: in Sec-
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In Section 4.1.1 we provide some preliminary material for understanding normalization, then in Section 4.1.2 we present a normalization algorithm which only works for sHML formulae defining *singleton symbolic events*. We then *extend* this algorithm in Section 4.1.3 which enables for normalizing sHML (resp. cHML) formulae defining symbolic events which are *not necessarily singleton*.

### 4.1.1 Preliminaries for Normalization

Normalization represents a conversion process which translates a given sHML (resp. cHML) formula into an intermediary form known as the *normal form*.

**Definition 4.2 (Normal Form).** A formula is in normal form when every conjunction branch is guarded by a disjoint necessity modality, which denotes a set of concrete events that does not intersect with the set denoted by any other symbolic event defined in the necessities guarding the other branches, i.e., a concrete event can only match one of the symbolic events.

**Example 4.6 (Normal Form Formulae).** Recall formulae $\varphi_0$ from Example 2.8, and $\varphi_2$ from Example 4.1 (both restated below).

\[
\varphi_0 \overset{\text{def}}{=} \max X. ([i \text{ ? req}]X \land [i \text{ ? req}][\text{ff}])
\]

\[
\varphi_2 \overset{\text{def}}{=} \max X. ([\text{req}]i\text{ ? ans}.X) \land ([\text{req}]i\text{ ? req})[\text{ff}])
\]

Notice how the conjunct branches in $\varphi_0$ are guarded by necessities defining *disjoint* concrete events $i\text{ ? req}$ and $i\text{ ? ans}$, while the branches in $\varphi_2$ are both guarded by the same concrete event $i\text{ ? req}$. Hence, by Definition 4.2, we can deduce that $\varphi_0$ is in normal form, while $\varphi_2$ is not.

Based on Definition 4.2, in Figure 4.2 we restrict the syntax of our sHML subset into sHML$_{nf}$. With this restricted syntax one can only define *normalized* sHML formulae, i.e., sHML formulae that adhere to Definition 4.2. Concretely, in

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1In [3] this is also referred to as the deterministic form.
4. Enforcing Safety Properties via Suppressions

The Normalized Syntax

\[ \varphi, \psi \in \text{sHML}_{nf} \ ::= \text{tt} \mid \text{ff} \mid X \mid \text{max} \varphi \mid \bigwedge_{i \in Q} [\eta_i] \varphi_i \] where \( \# \eta_i \)

where \( i \in Q \) is an index that identifies a branch in a conjunction.

Figure 4.2: The syntax of normal-form formulae.

sHML_{nf} we introduce a syntactic restriction which combines the conjunction operator (\( \bigwedge_{i \in Q} \varphi_i \)) with the necessity operator ([\( \eta \] \( \varphi \)) into \( \bigwedge_{i \in Q} [\eta_i] \varphi_i \) as shown in Figure 4.2. One can immediately notice that this restriction forbids from defining sHML formulae such as \( \varphi_2 \), since the conjunct branches (\([i?\text{req}][i!\text{ans}], X \land [i?\text{req}][i?\text{req}][\text{ff}]\)) do not define disjoint events.

We aim to prove that despite the syntactic restrictions, sHML_{nf} is still as expressive as the unrestricted sHML subset\(^2\). To obtain this result, we devise a set of conversion algorithms and prove Theorem 4.1, i.e., that any formula \( \varphi \in \text{sHML} \) can be converted into a semantically equivalent normalized formula \( \psi \in \text{sHML}_{nf} \).

**Theorem 4.1 (Semantic Equivalence).**

\[ \forall \varphi \in \text{sHML}, \exists \psi \in \text{sHML}_{nf} : [\varphi] = [\psi] \]

For instance, through the normalization algorithms we should be able to convert formula \( \varphi_2 \) into (an unfolded version of) \( \varphi_0 \), since these two formulae are semantically equivalent to each other.

### 4.1.2 Reconstructing sHML into sHML_{nf} wrt. Singleton Symbolic Events

Inspired from [3], we define the normalization algorithm for singleton sHML formulae in terms of the four constructions given below; each construction is accompanied by a proof guaranteeing semantic preservation, i.e., that the result of each translation is equivalent to its input. The construction sequence is as follows:

\(^2\)In [3] the authors prove that this result holds in relation to a version of the logic which only allows for defining concrete events, we thus follow up on their proofs and extend them to the version that includes symbolic events.
§1. Standardization of sHML: This step serves to convert a given sHML formula into a semantically equivalent formula that is an intermediate form known as the Standard Form; this is discussed in Section 4.1.2.1.

§2. Equation Form Conversion: As explained in Section 4.1.2.2, the standard form formula is then reformulated into a system of equations which makes it easier to manipulate in later stages.

§3. Normalization of Equations: The normalization procedure reviewed in Section 4.1.2.3, restructures the obtained system of equations into an equivalent system that is in the required normal form.

§4. sHML Form Conversion: Finally, the normal form system of equations is converted back into an sHML formula that is in normal form; this conversion is described in Section 4.1.2.4.

4.1.2.1 Standardization of sHML

The first step towards achieving the required normal form requires converting the given sHML formula into a semantically equivalent sHMLsf formula, i.e., an sHML formula that satisfies the structural constraints of standard formed formulae as defined in Definition 4.3 below.

Definition 4.3 (Standard Form Formulae). According to [3], a formula $\varphi \in \text{sHML}$ is in standard form if all free and unguarded recursion variables $X_i$ in $\varphi$, are at the topmost level, i.e., if $\varphi = \psi \land \bigwedge_{i \in Q} X_i$ where $\psi$ does not contain any free and unguarded recursion variables and $Q$ is a finite set of indices.

Example 4.7 (Standard Form Formulae). Formula $(\max Y.([i?3]Y \land X)) \land [i?3]\text{ff}$ is not in standard form since the free logical variable $X$ is not scoped under the topmost conjunction, and instead it is scoped under the maximal fixpoint $\max Y.\ldots$. This formula can thus be easily standardized by elevating $X$ to the topmost conjunction and thus obtain $((\max Y.([i?3]Y)) \land [i?3]\text{ff} \land X) \in \text{sHML}_{\text{sf}}$.  ■
Construction

\[
\llbracket \varphi \rrbracket_1 = \begin{cases}
\psi \{ \text{max } X_j. \psi / X_j \} \land \bigwedge_{i \in f(\varphi) \setminus \{j\}} X_i & \text{if } \varphi = \text{max } X_j. \varphi' \\
(\psi_1 \land \psi_2) \land \bigwedge_{i \in f(\varphi_1)} X_i \land \bigwedge_{i \in f(\varphi_2)} X_i & \text{if } \varphi = (\varphi_1 \land \varphi_2) \\
\varphi & \text{otherwise}
\end{cases}
\]

where \( f(\varphi) = \{ i \mid \text{if } X_i \text{ occurs free and unguarded in } \varphi \} \)

Figure 4.3: The Standardization Algorithm

In Figure 4.3 we present the construction algorithm \( \llbracket - \rrbracket_1 : \text{sHML} \mapsto \text{sHML}_{\text{sf}} \). This construction reformulates a given formula \( \varphi \) such that every occurrence of a free and unguarded logical (recursion) variable, \( X_i \) (i.e., \( i \in f(\varphi) \)), is elevated to the topmost conjunction to obtain the required standard form, and in the process, any bound logical variable \( X_j \notin f(\varphi) \) is also unfolded. The unfolding is required to ensure that the resultant conjunction branches are always guarded by a necessity operation. For example, the free logical variable \( X \) in formula \( (\text{max } Y.[i?3] Y \land X) \land [i?3] \text{ff} \) is elevated to the topmost conjunction, while the bound variable \( Y \) is unfolded so to obtain \( ([i?3] \text{max } Y.[i?3] Y \land [i?3] \text{ff}) \land X \).

More specifically, when analysing a conjunction, i.e., \( \llbracket \varphi_1 \land \varphi_2 \rrbracket_1 \), the construction is reapplied on the individual branches, i.e., \( \llbracket \varphi_1 \rrbracket_1 \) and \( \llbracket \varphi_2 \rrbracket_1 \), in order to obtain \( \psi_1 \land \bigwedge_{i \in f(\varphi_1)} X_i \land \psi_2 \land \bigwedge_{i \in f(\varphi_2)} X_i \) resp. Conjunctions \( \bigwedge_{i \in f(\varphi_1)} X_i \) and \( \bigwedge_{i \in f(\varphi_2)} X_i \) represent every free and unguarded logical variable defined in \( \varphi_1 \) and \( \varphi_2 \) resp. These free variables are added at the topmost conjunction such that the result of \( \llbracket \varphi_1 \land \varphi_2 \rrbracket_1 \) is \( (\psi_1 \land \psi_2) \land \bigwedge_{i \in f(\varphi_1)} X_i \land \bigwedge_{i \in f(\varphi_2)} X_i \).

When analysing a maximal fixpoint, i.e., \( \llbracket \text{max } X_j. \varphi' \rrbracket_1 \), the construction is immediately reapplied on \( \varphi' \), such that \( \llbracket \varphi' \rrbracket_1 \) returns \( \psi \land \bigwedge_{i \in f(\varphi') \setminus \{j\}} X_i \); the construction then uses this result to construct \( \psi \{ \text{max } X_j. \psi / X_j \} \land \bigwedge_{i \in f(\varphi') \setminus \{j\}} X_i \). Notice that the first part of the reconstructed formula, i.e., \( \psi \{ \text{max } X_j. \psi / X_j \} \) defines a substitution.
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which unfolds the formula. The second part, i.e., \( \bigwedge_{i \in f(\varphi) \setminus \{ j \}} X_i \) then serves to ensure that all the free and unguarded variables defined in \( \varphi \), except \( X_j \), are grouped and added to the topmost conjunction layer; variable \( X_j \) is not included as this is now bound under the maximal fixpoint, i.e., \( \max X_j \ldots \).

**Example 4.8** (Standardization of \( sHML \)). Recall \( sHML \) formula \( \varphi_2 \) from Example 4.1 (restated below).

\[
\varphi_2 \equiv \max X_0. \varphi'_2
\]
\[
\varphi'_2 \equiv ([i?req][i!ans].X_0) \land ([i?req][i?req][i?req][ff])
\]

We recursively apply \( \langle - \rangle_1 \) on \( \varphi_2 \) to obtain the result from the following derivation:

**Derivation 1.** Since \( \varphi_2 = \max X_0. \varphi'_2 \) we know

\[
\langle \max X_0. \varphi'_2 \rangle_1 = \psi \{ \max X_0. \varphi'/X_0 \} \land \text{tt}
\]

where \( \langle \varphi'_2 \rangle_1 = \psi_2 \land \bigwedge X_j \) and since \( f(\varphi'_2) \setminus \{ 0 \} = \emptyset \), then \( \bigwedge_{j \in f(\varphi'_2) \setminus \{ 0 \}} X_i = \text{tt} \).

Since \( \varphi'_2 = ([i?req][i!ans].X_0 \land [i?req][i?req][i?req][ff]) \) we know

\[
\langle [i?req][i!ans].X_0 \land [i?req][i?req][ff] \rangle_1 = (\psi'_2 \land \psi''_2) \land \text{tt} \land \text{tt}
\]

where \( \psi'_2 = \langle [i?req][i!ans].X_0 \rangle_1 = [i?req][i!ans].X_0 \land \text{tt} \)

and \( \psi''_2 = \langle [i?req][i?req][ff] \rangle_1 = [i?req][i?req][ff] \land \text{tt} \)

and since \( f(\varphi''_2) = \emptyset = f(\varphi''_2) \) then \( \bigwedge \bigwedge_{j \in f(\varphi''_2)} X_j = \text{tt} = \bigwedge_{j \in f(\varphi''_2)} X_j \).

Therefore, the resulting formula is the following:

\[
\langle \varphi_2 \rangle_1 = \left( [i?req][i!ans].X_0 \land [i?req][i?req][ff] \right) \{ \max X_0. [i?req][i!ans].X_0 \land [i?req][i?req][ff]/X_0 \}
\]

\[
\quad \{ \text{By applying the substitution we obtain the following formula} \}
\]

\[
= ([i?req][i!ans].\max X_0. [i?req][i!ans].X_0 \land [i?req][i?req][ff]) \land ([i?req][i?req][ff])
\]

\[
= \varphi'_2
\]

**Proving Semantic Preservation for \( \langle - \rangle_1 \).** To prove that standardization construction \( \langle - \rangle_1 \) preserves the original semantics of the given \( sHML \) formula, we
must prove that the following criterion holds:

\[ \forall \varphi \in \text{SHML} \cdot \langle\langle \varphi \rangle\rangle_1 \equiv \varphi \text{ where } \varphi \in \text{SHML}_{\text{sf}} \]

**Proof.** We refer to Lemma 8 from [3] in order to prove that construction \( \lla - \rra_1 \) preserves the semantics of the given formula \( \varphi \) and thus creates a semantically equivalent standardized formula \( \varphi_{\text{sf}} \). Although Lemma 8 is proven wrt. a version of SHML that only allows for defining concrete events, the proof of this lemma still applies to our setting.

In fact, Lemma 8 shows that semantics are preserved when moving the free and unguarded logical variables to the topmost conjunction and when unfolding the formula, *i.e.*, as done by construction \( \lla - \rra_1 \), and pays no regard to the type of events described in the necessities; adapting the proof for our setting thus only requires minor syntactic changes, *i.e.*, changing \( \alpha \) into \( \eta \).

--- End of Proof. ---

4.1.2.2 Equation Form Conversion

The second construction reformulates \( \text{SHML}_{\text{sf}} \) formulae into an equivalent *system of equations*. As defined in Definition 4.4 (below), systems of equations provide an alternative way of defining \( \mu \text{HML} \) formulae in terms of a set of equations between a logical variable and a formula.

**Definition 4.4 (System of Equations).** A system of equations \( \text{SYS} \) is defined as a triple \((\text{Eq}, X, \mathcal{Y})\), where \( X \) represents the principle logical variable which identifies the starting equation, \( \mathcal{Y} \) is a finite set of free logical variables, and \( \text{Eq} \) is an \( n \)-tuple of equations, *i.e.*, \[ \{X_1 = \varphi_1, X_2 = \varphi_2, \ldots, X_n = \varphi_n\} \], where for \( 1 \leq i < j \leq n \), \( X_i \) is different from \( X_j \), and each \( \varphi_i \) is a (possibly open) \( \text{SHML} \) expression.

Maximal fixpoints within a system of equations are denoted by referring to a priorly defined recursion variable. We sometimes abuse the notation of \( \text{Eq} \) and use it as a map from logical variables to the equated expression, *i.e.*, we denote
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\[ Eq(X_i) = \varphi_i \text{ in lieu of } X_i = \varphi_i \in Eq. \]

The equation form construction also needs to ensure that the constructed equations are also in standard form as defined by Definition 4.5.

**Definition 4.5 (Standard Form Equations).** Similar to Definition 4.3, we say that an equation \( X_i = \varphi_i \) is in standard form if \( \varphi_i \in s\text{HML}_{eq} \), where \( s\text{HML}_{eq} \) is defined as follows:

\[
\varphi \in s\text{HML}_{eq} ::= \text{ff} \mid \bigwedge_{j \in Q} [\eta_j]X_j \land \bigwedge_{k \in Q'} Y_k
\]

for some finite sets of indices \( Q \) and \( Q' \). A system \( \text{SYS}_sf \) is in standard form if every equation in the system is in standard form, i.e., \( \text{SYS}_sf = (Eq_{sf}, X, Y) \) such that \( \forall X=\varphi \in Eq \cdot \varphi \in s\text{HML}_{eq} \).

**Example 4.9 (System of Equations vs. Formulae).** Intuitively, a recursive formula such as \( \varphi = \max X_0.[i?3]([i!4]X_0 \land [i!4]\text{ff}) \) can be represented in standard form via the following system of 3 equations:

\[
Eq \equiv \{ X_0=[i?3]X_1, \ X_1=[i!4]X_0 \land [i!5]X_2, \ X_2 = \text{ff} \}
\]

Notice how recursion is represented by referring to \( X_0 \) in the second equation. Also, notice that since \( \varphi \) starts with a maximal fixpoint defining \( X_0 \), we know that \( X_0 \) is also the principle logical variable of this system of equations. Moreover, we know that \( Y=\emptyset \) since all the logical variables defined in the system are bound, i.e., variables \( X_0, X_1 \) and \( X_2 \) are all equated to some \( s\text{HML}_{eq} \) formula. Hence, the resultant system of equations is \( (Eq, X_0, \emptyset) \).

The construction \( 
\langle - \rangle_2 : s\text{HML}_{sf} \mapsto (Eq_{sf}, \text{VAR}, \mathcal{P}(\text{VAR})) \), defined in Figure 4.4, compositionally inspects a given standard form formula \( \varphi \) and translates it into an equivalent system of equations in standard form. For instance, \text{truth}, \( \text{tt}, \) and \text{falsehood}, \( \text{ff}, \) are respectively translated into equations \( X = \text{tt} \) and \( X = \text{ff} \), with \( X \) being the principle variable of the resultant system of equations. Similarly, a logical variable \( Y \) is translated into equation \( X = Y \), with the addition that \( Y \)
Figure 4.4: The conversion algorithm from a sHML$_{sf}$ formula to a Standard Form System of equations.

is marked as being a free logical variable in $\mathcal{Y}$ to indicate that $Y$ is not currently bound to some fixpoint.

Maximal fixpoints are converted into equation $Y = Eq(X_1)$, where $X_1$ is the principle variable of the system of equations obtained from the recursive application on the continuation $\varphi'$ i.e., $\langle\langle \varphi' \rangle\rangle_2 = (Eq, X, \mathcal{Y})$, and $Eq(X_1)$ refers to the formula $F_1$ equated to variable $X_1$ in $Eq$, i.e., if $X_1 = F_1$ then $Eq(X_1) = F_1$. Upon analysing a maximal fixpoint $\max Y.\varphi'$, variable $Y$ is removed from $\mathcal{Y}$, thus denoting that although $Y$ is free in $\varphi'$, this is no longer the case in $\varphi = \max Y.\varphi'$.

In the case of conjunctions, $\varphi_1 \wedge \varphi_2$, these are reconstructed into a system of equations consisting in the systems of equations obtained from analysing $\varphi_1$ and $\varphi_2$, i.e., $Eq_1$ and $Eq_2$, along with equation $X_0 = Eq_1(X_1) \wedge Eq_2(X_2)$, where $X_1$ and $X_2$ are the principle variables of $\varphi_1$ and $\varphi_2$ resp. Finally, necessity modalities, $[\eta]\varphi'$, are translated into a system consisting in the equations, $Eq$, obtained from $\langle\langle \varphi' \rangle\rangle_2$, along with equation $X_0 = [\eta]X_1$, where $X_1$ is the principle variable obtained from $\langle\langle \varphi' \rangle\rangle_2$.

**Example 4.10.** Recall the standard form, singleton sHML formula $\varphi_{sf}^2$ obtained
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in Example 4.8:

\[
\begin{align*}
\phi_{sf}^{2} & = \phi_{sf}^{2a} \land \phi_{sf}^{2b} \\
\phi_{sf}^{2a} & = \neg i?req \land i!ans \phi_{sf}^{2c} \\
\phi_{sf}^{2b} & = [i?req][i!ans] X \\
\phi_{sf}^{2d} & = \neg i?req \land i!ans \phi_{sf}^{2f} \\
\phi_{sf}^{2c} & = \max X \phi_{sf}^{2d}
\end{align*}
\]

We recursively apply \(\langle\langle \cdot \rangle\rangle_2\) to obtain the resultant system of equations from the following derivation:

**Derivation 2.** Since \(\phi_{sf}^{2} = \phi_{sf}^{2a} \land \phi_{sf}^{2b}\) we know

\[
\langle\langle \phi_{sf}^{2a} \land \phi_{sf}^{2b} \rangle\rangle_2 = \left( \{ X_0 = Eq_1(X_1) \land Eq_1(X_2) \} \cup Eq_1 \cup Eq_2, X_0, \emptyset \right)
\]

where \(\langle\langle \phi_{sf}^{2a} \rangle\rangle_2 = (Eq_1, X_1, \emptyset)\) and \(\langle\langle \phi_{sf}^{2b} \rangle\rangle_2 = (Eq_2, X_2, \emptyset)\).

We now consider \(\langle\langle \phi_{sf}^{2a} \rangle\rangle_2\) (LHS) and \(\langle\langle \phi_{sf}^{2b} \rangle\rangle_2\) (RHS) separately.

**LHS**

Since \(\phi_{sf}^{2a} = [i?req][i!ans] \phi_{sf}^{2a}\) we know

\[
\langle\langle [i?req][i!ans] \phi_{sf}^{2a} \rangle\rangle_2 = \left( \{ X_1 = [i?req]X_3, X_3 = [i!ans]X \} \cup Eq_1', X_1, \emptyset \right)
\]

where \(\langle\langle \phi_{sf}^{2a} \rangle\rangle_2 = (Eq_1', X_1, \emptyset)\).

Since \(\phi_{sf}^{2c} = \max X \phi_{sf}^{2d}\) we know

\[
\langle\langle \max X \phi_{sf}^{2d} \rangle\rangle_1 = \left( \{ X = Eq_3'(X_4) \} \cup Eq_3', X, \emptyset \right)
\]

where \(\langle\langle \phi_{sf}^{2d} \rangle\rangle_2 = (Eq_3', X_4, \emptyset)\).

Since \(\phi_{sf}^{2d} = \phi_{sf}^{2a} \land \phi_{sf}^{2f}\) we know

\[
\langle\langle \phi_{sf}^{2a} \land \phi_{sf}^{2f} \rangle\rangle_1 = \left( \{ X_4 = Eq_3(X_5) \land Eq_4(X_6) \} \cup Eq_3 \cup Eq_4, X_4, \{ X \} \right)
\]

where \(\langle\langle \phi_{sf}^{2a} \rangle\rangle_2 = (Eq_3, X_5, \{ X \})\) and \(\langle\langle \phi_{sf}^{2f} \rangle\rangle_2 = (Eq_4, X_6, \emptyset)\).
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Since we have $\langle\langle \varphi_3 \rangle\rangle_2$ (LHS) and $\langle\langle \varphi_4 \rangle\rangle_2$ (RHS) we consider them separately.

**LHS**

Since $\varphi_{sf}^{\#_e} = [i?req][i!ans]X$ we know

$\langle\langle [i?req][i!ans]X \rangle\rangle_2 = (\{ X_5 = [i?req]X_7, X_7 = [i!ans]X_8, X_8 = ff \}, X_5, \{ X \})$

**RHS**

Since $\varphi_{sf}^{\#_f} = [i?req][i!req]ff$ we know

$\langle\langle [i?req][i!req]ff \rangle\rangle_2 = (\{ X_6 = [i?req]X_9, X_9 = [i?req]X_{10}, X_{10} = ff \}, X_6, \emptyset)$

**RHS**

Since $\varphi_{sf}^{\#_b} = [i?req][i!req]ff$ we know

$\langle\langle [i?req][i!req]ff \rangle\rangle_2 = (\{ X_2 = [i?req]X_{11}, X_{11} = [i?req]X_{12}, X_{12} = ff \}, X_2, \emptyset)$

Hence, the result of this derivation is $\langle\langle \varphi \rangle\rangle_2 = (Eq, X_0, \emptyset)$ where

$$Eq = \begin{cases} 
X_0 = [i?req]X_3 \land [i?req]X_{11}, & X_1 = [i?req]X_3, X_3 = [i!ans]X, \\
X = [i?req]X_7 \land [i?req]X_9, & X_4 = [i?req]X_7 \land [i?req]X_9, X_5 = [i?req]X_7, \\
X_7 = [i!ans]X_8, X_8 = X, & X_6 = [i?req]X_9, X_9 = [i?req]X_{10}, \\
X_{10} = ff, & X_2 = [i?req]X_{11}, X_{11} = [i?req]X_{12}, X_{12} = ff 
\end{cases}$$

Note that the greyed formulae were are redundant since they are not reachable from the principle logical variable $X_0$; for conciseness we will ignore them in forthcoming examples.

**Proving Semantic Preservation for $\langle\langle - \rangle\rangle_2$.** To prove that construction $\langle\langle - \rangle\rangle_2$ preserves the semantics of the given $sHML_{sf}$ formula, we must prove that the following criterion holds:

$$\forall \varphi \in sHML_{sf} : \langle\langle \varphi \rangle\rangle_2 \equiv SYS \text{ where } SYS \text{ is in Standard Form.}$$

**Proof.** The proof guaranteeing that the resultant system of equations $SYS$, constructed via $\langle\langle - \rangle\rangle_2$, is semantically equivalent to the given standard form formula
\( \varphi_{sf} \) follows from Lemma 10 given in [3]. Although this lemma is proven in relation to formulae that define concrete events, this lemma still applies for formulae defining symbolic events, since the construction is independent of the type of event described in the modal necessities.

--- End of Proof. ---

4.1.2.3 Normalization of Equations

The third construction performs the actual normalization procedure as it converts systems of equations that are in standard form, into an equivalent systems of equations that are in normal form as defined in Definition 4.6.

Definition 4.6 (Normalized System of Equations). An equation \( X_i = \varphi_i \) is in normal form if the subformulae of a conjunction are either free and unguarded logical variables, or else guarded by a disjoint necessity. This ensures that at most only one necessity guarding a branch in a conjunction can match a system action, i.e., \( \varphi_i \) has the form \( \text{ff} \) or \( \bigwedge_{i \in Q} [\eta_i] \varphi_i \land \bigwedge_{j \in Q} Y_j \) where \( \#_{i \in Q} \eta_i \). A system of equations \( \text{SYS}_{nf} \) is in normal form when all of its equations, \( \text{Eq}_{nf} \), are in normal form.

Construction

\[
\langle(Eq, X_i, Y)\rangle_3 \equiv (Eq_{nf}, X_{\{i\}}, Y)
\]

\[
E_{nf} \equiv \begin{cases} X_Q = \bigwedge_{\eta \in S(Q)} [\eta] X_{D(Q, \eta)} \land \bigwedge_{j \in E(Q)} Y_j \text{ and } Q \subseteq I(Eq) \text{ and } \#_{i \in Q} \eta_i \text{ is not a subformula of Eq}(X_Q) \\ X_Q = \text{ff} \text{ and } \#_{i \in Q} \eta_i \text{ is a subformula of Eq}(X_Q) \end{cases}
\]

where

\[
S(Q) \equiv \bigcup_{i \in Q} \{ \eta \mid [\eta] X_j \text{ is a subformula in } \varphi_i \}
\]

\[
D(Q, \eta) \equiv \bigcup_{i \in Q} \{ r \mid [\eta] X_r \text{ is a subformula in } \varphi_i \}
\]

\[
E(Q) \equiv \bigcup_{i \in Q} \{ r \mid Y_r \text{ is unguarded in } \varphi_i \}
\]

\[
I(Eq) \equiv \{ i \mid X_i = \varphi_i \in Eq \}
\]

Figure 4.5: The Normalization Algorithm for Systems of Equations
Figure 4.5 presents the normalization algorithm in terms of the construction function $\langle\langle-\rangle\rangle_3 : (Eq, VAR, P(VAR)) \mapsto (Eq, VAR, P(VAR))$. This construction generates a new system of equations which contains the powerset combinations of the equations from the original system of equations. Intuitively, the construction takes two or more equations and combines the equated formulae with a conjunction.

Example 4.11. Consider the system of equations $(Eq, X_0, \emptyset)$, where $Eq$ contains 3 equations $X_0=\varphi_0$, $X_1=\varphi_1$ and $X_2=\varphi_2$. The construction thus takes all the combinations and creates a new system, namely, $(Eq_{nf}, X_{\{0\}}, \emptyset)$ where

$$Eq = \left\{ \begin{array}{l}
X_{\{0\}} = \varphi_0; X_{\{0,1\}} = \varphi_0 \land \varphi_1; X_{\{0,1,2\}} = \varphi_0 \land \varphi_1 \land \varphi_2; X_{\{1\}} = \varphi_1, \\
X_{\{1,2\}} = \varphi_1 \land \varphi_2; X_{\{2\}} = \varphi_2; X_{\{0,2\}} = \varphi_0 \land \varphi_2
\end{array} \right\}$$

As in the original system of equations, the principle variable $X_0$ equates to $\varphi_0$, the new principle variable for the reconstructed equation is therefore variable $X_{\{0\}}$, as this also equates to $\varphi_0$.

While combining the equated formulae, the construction also normalizes the combined equated formulae. The first part of the construction, i.e., $\bigwedge_{\eta \in S(Q)} [\eta]X_D(Q,\alpha)$, thus makes sure that whenever the conjunction branches are guarded by necessities specifying a set of disjoint events $S(Q)$, then the conjunction is normalized by merging together the syntactically equal necessities in the conjunction. This merger provides a resultant conjunction that has branches which are guarded by disjoint (syntactically different) necessities, e.g., equation $X_0 = [\eta]X_1 \land [\eta]X_2 \land [\eta']X_3$ can be normalized into $X_{\{0\}} = [\eta]X_{\{1,2\}} \land [\eta']X_{\{3\}}$ by merging the first two branches, i.e., $[\eta]X_1$ and $[\eta]X_2$ into $[\eta]X_{\{1,2\}}$.

Note how only the branches which guarded by syntactically equal necessities are being merged together; this is done by taking a subset $Q$ of the powerset combinations of the indices provided by $I(Eq)$, i.e., $Q \subseteq I(Eq)$, where $I(Eq)$ returns all the indices specified in the set of equations $Eq$. The merged indices are obtained using $D(Q,\eta)$ which returns the index $r$ of every logical variable $X_r$ that is guarded by the same symbolic event $\eta$.

Finally, the second part of the construction, i.e., $\bigwedge_{j \in E(Q)} Y_j$, ensures that every un-
guarded logical variable $Y_j$ defined in the equation, is kept at the topmost level of the conjunction, therefore retaining the equation in standard form as defined by Definition 4.6.

Example 4.12. Recall the standard form system of equations obtained in Example 4.10, i.e., $\langle\langle Eq_{sf}, X_0, \emptyset \rangle\rangle$ where

$$\begin{align*}
E_{qsf} &= \{ X_0 = [i?req]X_3 \land [i?req]X_{11}, X_3 = [i!ans]X, X = [i?req]X_7 \land [i?req]X_9, \\
X_7 &= [i!ans]X_8, X_8 = X, X_9 = [i?req]X_{10}, X_{10} = ff, X_{11} = [i?req]X_{12}, \\
X_{12} &= ff \} 
\end{align*}$$

When the construction rule is applied, it generates every possible combination and merges the modal necessities where necessary. By definition of $\langle\langle (Eq, X_0, \emptyset) \rangle\rangle_3$ we therefore obtain $(E_{qsf}, X_{\{0\}}, \emptyset)$ where

$$E_{qsf} = \{ X_{\{0\}} = [i?req]X_{\{3,11\}} \} \cup E_{qsf}'$$

For instance, note how this construction collapses $X_0 = [i?req]X_3 \land [i?req]X_{11}$ into $X_{\{0\}} = [i?req]X_{\{3,11\}}$, where continuations $X_3$ and $X_{11}$ were combined into a single conjunct continuation $X_{\{3,11\}}$, in this way every event specified by a necessity in the conjunction becomes disjoint, i.e., none of the reconstructed singleton symbolic events can denote a set containing the same concrete event as another symbolic event residing in the same conjunction. Similarly, the algorithm also merges $X = [i?req]X_7 \land [i?req]X_9$ into $X = [i?req]X_{7,9}$. The combined formulae for variables $X_{\{3,11\}}$ and $X_{\{7,9\}}$ are also constructed by the normalization algorithm as defined in $E_{qsf}'$ below:

$$E_{qsf}' = \{ X_{\{3,11\}} = [i!ans]X \land [i?req]X_{\{12\}}, X = [i?req]X_{\{7,9\}}, \\
X_{\{7,9\}} = [i!ans]X_8 \land [i?req]X_{\{10\}}, X_{\{8\}} = X, \\
X_{\{10\}} = ff, X_{\{11\}} = [i?req]X_{\{12\}}, X_{\{12\}} = ff, \ldots \}$$

For conciseness, some combinations have been ignored from the resultant set as they are not reachable from the principle variable $X_{\{0\}}$ and are thus redundant. \[\square\]

Proving Semantic Preservation for $\langle\langle - \rangle\rangle_3$. To prove that construction $\langle\langle - \rangle\rangle_3$ preserves the semantics of the given standardized system of equations, we must
prove that the following holds:

SYS is in Standard Form implies \( \langle \langle SYS \rangle \rangle_3 \equiv SYS \) where \( \langle \langle SYS \rangle \rangle_3 \) is in Normal Form.

**Proof.** In order to prove that construction \( \langle \langle - \rangle \rangle_3 \) produces a normalized system of equations SYS\(_{nf}\) that is semantically equivalent to the given system of equations SYS, we refer to Lemma 11 in [3].

However, Lemma 11 holds wrt. a version of our construction which requires that the necessities define concrete events; this is necessary since the construction must merge together the conjunct branches that are prefixed by syntactically equal modal necessities, e.g., \( X_0=\langle [i?3]X_1 \wedge \{i?3\}X_2 \rangle \) becomes \( X_0=\langle [i?3]X_{\{1,2\}} \rangle \) since both branches are prefixed by the same concrete necessity \([i?3]\).

Lemma 11, however, still holds for our construction since \( \langle \langle - \rangle \rangle_3 \) requires condition \#S(Q) to hold. This condition states that all the symbolic events guarding a conjunction must be disjoint unless syntactically equal. In this way, if two or more symbolic events are syntactically equal, they would be merged by the construction, while the other (non-syntactically equal) necessities are guaranteed to be disjoint. Hence, a concrete system event can only satisfy at most one symbolic modal necessity, i.e., in the same way as per normalized conjunctions defining concrete events.

--- End of Proof. ---

### 4.1.2.4 \textit{sHML} Form Conversion

The final step for obtaining the final normalized formula only requires regenerating the normalized formula, \( \psi \in \text{sHML}_{nf} \), from a given normalized system of equations, \( SYS_{nf} = (Eq_{nf}, X, Y) \).

Figure 4.6 presents the final construction, \( \langle \langle - \rangle \rangle_4 : (Eq_{nf}, VAR, P(VAR)) \mapsto \text{sHML}_{nf} \), which converts a normalized system of equations into a semantically equivalent \text{sHML}_{nf} formula. The algorithm internally employs function \( \sigma_{shml} :: (\text{sHML}_{nf} \times Eq) \mapsto \text{sHML}_{nf} \) to create a normalized \text{sHML}_{nf} formula that is semantically equivalent to the given system of equations. This function starts by
Construction

\[ \langle\langle Eq, X_i, \mathcal{Y} \rangle\rangle_4 \equiv \sigma_{\text{shml}}(X_i, Eq) \]

\[ \sigma_{\text{shml}}(\varphi, Eq) \equiv \begin{cases} 
\varphi & \text{if } \text{fv}(\varphi) = \emptyset \\
\sigma_{\text{shml}}(\varphi \sigma, Eq) & \text{if } \text{fv}(\varphi) = \mathcal{Y} \text{ then } \sigma = \begin{cases} 
\max X_0.\varphi_0 / X_0 & \text{if } X_0 \in Eq \text{ and } X_0 \in \mathcal{Y} 
\end{cases} 
\end{cases} \]

Figure 4.6: Converting a normalized system of equations into an \( \text{shML}_{\text{nf}} \) formula.

taking as input the principle variable \( X_i \) along with the set of equations \( Eq \), it then searches for the equation \( X_i = \varphi_i \) in \( Eq \) and converts it into a substitution environment which substitutes variable \( X_i \) with \( \max X_i.\varphi_i \), i.e., \( \{\max X_i.\varphi_i / X_i\} \).

This substitution is then applied to \( X_i \) and the function recurses with the substituted value, i.e., \( \sigma_{\text{shml}}(\max X_i.\varphi_i, Eq) \); recursion stops when the resultant formula \( \varphi \) becomes closed, in which case it is returned as the normalized \( \text{shML}_{\text{nf}} \) formula.

Example 4.13. Consider the following normalized system of equations obtained in Example 4.12, \( (Eq, X_{\{0\}}, \emptyset) \) where

\[ Eq_{\text{nf}} = \begin{cases} 
X_{\{0\}} = [i?\text{req}]X_{\{3,11\}}, X_{\{3,11\}} = [i!\text{ans}]X \land [i?\text{req}]X_{\{12\}}, \\
X = [i?\text{req}]X_{\{7,9\}}, X_{\{7,9\}} = [i!\text{ans}]X_{\{8\}} \land [i?\text{req}]X_{\{10\}}, X_{\{8\}} = X, \\
X_{\{10\}} = \text{ff}, X_{\{11\}} = [i?\text{req}]X_{\{12\}}, X_{\{12\}} = \text{ff} 
\end{cases} \]

By \( \langle\langle Eq, X_{\{0\}}, \emptyset \rangle\rangle_4 \) we obtain \( \varphi \in \text{shML}_{\text{nf}} \) where

\[ \psi_2 \equiv \sigma_{\text{shml}}(X_{\{0\}}, Eq_{\text{nf}}) \]

\[ \equiv \max X_{\{0\}}. \left( \begin{array}{c} [i?\text{req}]\max X_{\{3,11\}} \\
[i!\text{ans}]\max X_{\{8\}} \land [i?\text{req}]\max X_{\{12\}}.\text{ff} \\
\land [i?\text{req}]\max X_{\{10\}}.\text{ff} \end{array} \right) \]

The above formula can be optimized by removing redundant maximal fixpoint declarations such as \( \max X_{\{0\}} \) and \( \max X_{\{3,11\}} \), i.e., fixpoints that declare a variable which is never referenced to throughout the rest of the formula. Hence we can optimize our formula as follows:

\[ \psi_2 \equiv [i?\text{req}]([i!\text{ans}]\max X.[i?\text{req}]([i!\text{ans}]X \land [i?\text{req}]\text{ff}) \land [i?\text{req}]\text{ff}) \in \text{shML}_{\text{nf}} \]
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Notice that the obtained optimized formula, i.e., $\psi_2$ is in fact an unfolded version of formula $\varphi_0$, and that both $\varphi_0$ and also $\psi_2$ are in normal form, i.e., $\varphi_0, \psi_2 \in \text{sHML}_{nf}$.

$$\varphi_0 \overset{\text{def}}{=} \max X. [i?\text{req}] ([i!\text{ans}]X \land [i?\text{req}]\text{ff})$$

{ By unfolding $\max X_0$, we obtain }

$$= [i?\text{req}]([i!\text{ans}]\max X.[i?\text{req}]([i!\text{ans}]X \land [i?\text{req}]\text{ff}) \land [i?\text{req}]\text{ff})$$

$$= \psi_2 \in \text{sHML}_{nf} \blacksquare$$

**Proving Semantic Preservation for $\langle\langle - \rangle\rangle_4$.** To prove that construction $\langle\langle - \rangle\rangle_4$ preserves the semantics of the given normalized system of equations, we must prove that the following holds:

SYS is in Normal Form implies $\langle\langle \text{SYS} \rangle\rangle_4 \equiv \text{SYS}$ where $\langle\langle \text{SYS} \rangle\rangle_4 \in \text{sHML}_{nf}$

**Proof.** Since construction $\langle\langle - \rangle\rangle_4$ is independent of the type of events defined in the modal necessities of the given system of equations, we refer to Lemma 12 as proof that this construction preserves the semantics of the given system of equations $\text{SYS}_{nf}$, such that it produces a semantically equivalent formula $\varphi \in \text{sHML}_{nf}$.

--- End of Proof. ---

4.1.3 Reconstructing sHML into sHML$_{nf}$ wrt. Any Symbolic Event

Up until now we have only considered normalizing SHML formulae defining singleton symbolic events, as these events are easy to statically differentiate from each other as required for merging conjunct branches in Item 3 (see Section 4.1.2.3). However, the necessities in the logic we consider can also describe symbolic events which denote wider sets of concrete events e.g., $[[d?\text{req}, d\neq j]]$ where $[[d?\text{req}, d\neq j]] = \{\ldots, i?\text{req}, k?\text{req}, \ldots\}$.

One major difference between singleton and other symbolic events is that the former can easily be distinguished from one another by a simple syntactic check e.g., singleton events $\{d_1?f_1, d_1=i \land f_1=\text{req}\}$ and $\{d_2?f_2, d_2=i \land f_2=\text{cls}\}$ can be repre-
sent as \( i?\text{req} \) and \( i?\text{cls} \) resp. and can thus be easily distinguished since \( i?\text{req} \neq i?\text{cls} \). In general, however, this distinction is not always possible with symbolic events.

For instance, consider \( d?5 \) and \( i?f \), these two events, although syntactically different, they define intersecting sets of input events, i.e., \([d?5] \cap [i?f] \), meaning that both symbolic events can match the same concrete system event \( i?5 \). Hence, this makes it harder to statically differentiate and distinguish between symbolic events, which is crucial when applying the normalization construction step \( \S 3 \).

As shown in Example 4.14, normalizing a non-singleton symbolic formula using the algorithm we presented so far, may sometimes fail to produce an equivalent formula that is in normal form.

**Example 4.14 (Normalizing Symbolic Formulae).** Consider the non-singleton symbolic formula \( \varphi_3 \) given below.

\[
\varphi_3 = \max X. ([d?\text{req}, d\neq h])[d!\text{ans}, t] X \land ([f?\text{req}, f\neq j])[f?\text{req}, t] f f
\]

By applying \( \S 1 \) on \( \varphi_3 \) we obtain the following standard form formula,

\[
\varphi_3^f = \langle \varphi_3 \rangle_1 = ([d!\text{ans}, t])[d!\text{ans}, t](\varphi_3) \land ([f?\text{req}, f\neq j])[f?\text{req}, t] f f
\]

By \( \S 2 \) we then obtain the following system of equations which we can then normalize via \( \S 3 \) as shown below.

\[
\begin{align*}
\text{SYS}_3^f &= \langle \varphi_3^f \rangle_1 = (E_{d3}^f, X_0, \emptyset) \\
E_{d3}^f &= \left\{ \\
X_0 &= [[d?\text{req}, d\neq h]]X_3 \land [[f?\text{req}, f\neq j]]X_11, X_3 = [[d!\text{ans}, t]]X, \\
X &= [[d?\text{req}, d\neq h]]X_7 \land [[f?\text{req}, f\neq j]]X_9, X_7 = [[d!\text{ans}, t]]X_8, X_8 = X, \\
X_9 &= [[f?\text{req}, t]]X_10, X_10 = ff, X_11 = [[f?\text{req}, t]]X_12, X_12 = ff
\right\}
\end{align*}
\]

However, when applying \( \S 3 \) on \( \text{SYS}_3^f \), the algorithm fails to combine symbolic events \( \{d?\text{req}, d\neq h\} \) and \( \{f?\text{req}, f\neq j\} \) as despite not being disjoint, they are neither syntactically equal. Hence, equation \( X_0 = [[d?\text{req}, d\neq h]]X_3 \land [[f?\text{req}, f\neq j]]X_11 \) is not merged into \( X_0 = [[d?\text{req}, d\neq h \land d\neq j]]X_{\{3,11\}} \), but simply remains the same as shown below.

\[
\begin{align*}
\text{SYS}_3^f &= \langle \text{SYS}_3^f \rangle_2 = (E_{d3}^f, X_{\{0\}}, \emptyset) \\
E_{d3}^f &= \left\{ \\
X_0 &= [[d?\text{req}, d\neq h \land d\neq j]]X_{\{3,11\}}
\right\}
\end{align*}
\]
Due to this, when applying §4, we end up with $\psi_3$ (stated below), which despite being semantically equivalent to the original formula $\varphi_3$, it is still not in normal form (i.e., $\psi_3 \not\in \text{SHML}_{nr}$), since its conjunctions are not guarded by necessities defining disjoint events.

$$\psi_3 = \langle \langle \varphi_3 \rangle \rangle = (\langle [d!\text{ans}, t] \rangle [d!\text{ans}, t] (\varphi_3)) \land (\langle [f?\text{req}, f \neq j] \rangle [f?\text{req}, t]) \text{ff}$$

Even though formula $\varphi_3$ is a generic version of $\varphi_2$ (i.e., $\varphi_3$ is less restrictive than $\varphi_2$), our normalization algorithm fails to produce an equivalent normalized formula since $\psi_3 \not\in \text{SHML}_{nr}$.

Therefore, to be able to use the existing normalization procedure introduced in [3], we must add extra steps to make sure that we can clearly distinguish between symbolic events, e.g., $d?5$ and $i?f$ can be replaced by the following guarded patterns:

- $d?5$ can be encoded as $\{d?f, f=5 \land d=i\}$ or $\{d?f, f=5 \land x \neq i\}$; while
- $i?f$ can be encoded as $\{d?f, f=5 \land d=i\}$ or $\{d?f, f \neq 5 \land x=i\}$.

Using this technique we are able to statically tell whether one encoded symbolic event is equal to another event or not. For instance, the reconstructed symbolic events $\{d?f, d=5 \land x = i\}$ and $\{d?f, f \neq 5 \land d = i\}$ can now be statically distinguished by using a simple syntactic check, since their contradicting conditions, i.e., $f=5$ and $f \neq 5$ resp., guarantee that the reconstructed events are also disjoint, i.e., $\langle [d?f, f=5 \land d=i] \rangle \cap \langle [d?f, f \neq 5 \land d=i] \rangle = \emptyset$.

In the following section we thus formalize and present the additional steps that must be performed when working wrt. symbolic events in order to ensure that the obtained conjunct branches are unable to match the same concrete event.
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4.1.3.1 Additional Steps for Normalizing Necessities defining Symbolic Events

We formally define three additional construction rules that must be applied between steps §2 and §3. These new constructions convert conjunctions that are guarded by necessities defining non-disjoint symbolic events, into equivalent conjunctions guarded by disjoint necessities, i.e., necessities describing symbolic events that are both syntactically and semantically different such that the sets of concrete events they denote do not intersect. The additional steps are the following:

§i. Uniformity of Symbolic Events: In this step, we inspect conjunct modal necessities and substitute their data variables with the same fresh variable whenever they define pattern equivalent symbolic events, e.g., we convert \([d_1 ? d_2, c(d_1, d_2)] \land [f_1 ? f_2, c(f_1, f_2)] \land [g_1 ? g_2, c(g_1, g_2)]\) into \([g_1 ? g_2, c(g_1, g_2)] \land [g_1 ? g_2, c(g_1, g_2)] \land [g_1 ? g_2, c(g_1, g_2)]\); we discuss this in detail in Section 4.1.3.2.

§ii. Condition Reformulation of Conjunct Symbolic Events: Once uniformed, the conjunctions are recomposed such that the reconstructed conjunctions are guaranteed to define necessities that define disjoint symbolic events; the details on how this is done are discussed in Section 4.1.3.3.

Internally, the constructions presented in Sections 4.1.3.2 and 4.1.3.3 both make use of the traverse function in order to process the given set of equations in a tree-like manner as defined in Figure 4.7.

The function \(\text{traverse} : (Eq \times P(\text{INDEX}) \times \text{FUN} \times \text{ACC}) \rightarrow \text{ACC}\) is a higher order function which takes as input: a set of equations \(Eq\), a set of indices \(Q\) an arbitrary projection function \(\lambda\), and an accumulator argument \(\delta\).

(i) Performing a Traversal. The traverse function is generally used to conduct a breath first traversal on the given equation set, starting from the equation that equates to the principle variable as the root of the tree traversal, e.g., as shown
Traversing Functions.

\[
\text{traverse}(Eq, Q, \lambda, \delta) = \begin{cases} \\
\text{traverse}(Eq', Q', \lambda, \delta') & \text{if } Eq \neq \emptyset \text{ and } Q \neq \emptyset \\
\text{then } \delta' = \lambda(Eq, Q, \delta) \\
\text{and } Eq' = Eq \setminus Eq_{/Q} \\
\text{and } Q' = \bigcup_{j \in Q} \text{child}(Eq, j) & \text{otherwise} \\
\end{cases}
\]

\[
\text{child}(Eq, i) = \{ j \mid Eq(X_i) = \bigwedge_{j \in Q} [\eta_j] X_j \land \varphi \text{ and } j \neq i \text{ and } X_j \in \text{dom}(Eq) \}
\]

\[
Eq_{/Q} = \{ X_i = \varphi_i \mid (X_i = \varphi_i) \in Eq \text{ and } i \in Q \}
\]

**Figure 4.7:** The Breath First Traversal Algorithm.

![Traversal Algorithm Diagram](image)

**Figure 4.8:** A pictorial view of an example equation set traversal.

In Figure 4.8, for system \((Eq, X_0, \mathcal{V})\), equation \(X_0 = [\eta_1]X_1 \land [\eta_2]X_2 \land [\eta_3]X_3\) is the root of the traversal since \(X_0\) is the principle variable.

The children of the root are calculated via the \(\text{child} : (Eq \times \text{INDEX}) \rightarrow \mathcal{P}(	ext{INDEX})\) function. This function takes as input a set of equations \(Eq\) along with the index of the parent equation, e.g., index 0 for equation \(X_0 = [\eta_1]X_1 \land [\eta_2]X_2 \land [\eta_3]X_3\). It then scans the equated formula and returns the set containing the indices of every branch, defined in the equated formula, which is prefixed by a modal necessity, e.g., in Figure 4.7, the children of \(X_0 = [\eta_1]X_1 \land [\eta_2]X_2 \land [\eta_3]X_3\) are \(\{1, 2, 3\}\), such that branches \([\eta_1]X_1\), \([\eta_2]X_2\) and \([\eta_3]X_3\) are siblings as they are defined at the same conjunction layer.

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Cycles in the traversal are avoided since the child function is always executed wrt. a restricted set of equations, i.e., one which does not include the parent equation. For instance, while analysing equation $X_1 = [\eta_1]X_0$ (see Figure 4.7), traverse is evaluated wrt. $Eq'\backslash Eq_{\backslash \{0\}}$ which does not include the parent equation, i.e., since $Eq' = Eq \backslash Eq_{\backslash \{0\}} = \{X_0 = [\eta_1]X_1 \land [\eta_2]X_2 \land [\eta_3]X_3\}$. In this way, when computing the children of $X_1$ (via $\text{child}(Eq', 1)$) index 0 is not added to the resultant set of child indices, since $X_0 \notin \text{dom}(Eq')$; this avoids cycling back to some (grand) parent equation.

Additionally, the child function avoids cycling back to the (immediate) parent by removing the parent’s index form the returned set of child indices, e.g., when evaluating $\text{child}(Eq', 3)$ to retrieve the child indices of equation $X_3 = [\eta_6]X_3$, index 3 is removed thus avoiding the creation of a loop in the traversal.

(ii) Applying the projection function during traversal. While traversing the equation set, the traverse function can apply an arbitrary projection function $\lambda$. Despite being an arbitrary function, $\lambda$ must adhere to the following type: $\lambda :: (Eq \times \mathcal{P}(\text{INDEX}) \times \text{Acc}) \rightarrow \text{Acc}$, i.e., $\lambda$ must be a function which takes three inputs, including the current set of equations $Eq$, a set of indices $Q$ and an accumulator value $\delta$; as a result $\lambda$ must return an updated version of the accumulator, i.e., $\delta'$.

(iii) Traversal Termination and Return Value. The traversal terminates when either all the equations in $Eq$ have been process such that the traverse function is applied wrt. $Eq = \emptyset$, or whenever no further children can be visited since none of the branches have any valid children, i.e., for every branch $i$, $\text{child}(Eq, i) = 0$. The latter is an optimization which omits the redundant processing of equations that are not reachable from the principle equation.

**Example 4.15.** Consider the system of equations $(Eq, X_0, \emptyset)$ where

\[
Eq = \{X_0 = [\eta]X_1, X_1 = \text{ff}, X_2 = \text{ff}\}
\]

The traversal starts from equation $X_0 = [\eta]X_1$ as the root, followed by $X_1 = \text{ff}$ as its
immediate child, however equation \( X_2 = \text{ff} \) is the child of neither equation and is thus ignored by the traversal.

The latest version of the accumulator value is returned once the traversal is complete.

### 4.1.3.2 Uniformity of Symbolic Events

Intuitively, this part of the normalization algorithm renames the data variables of the pattern equivalent symbolic events (i.e., symbolic events defining equivalent patterns; see Section 2.3.1) defined in necessity operations guarding branches within the same conjunction level, to the same variable names to obtain a uniform system of equations as defined in Definition 4.7.

**Definition 4.7 (Uniform System of Equations).** An equation is uniform when it is in standard form and when every pattern equivalent event (see Section 2.3) defined by sibling necessities within a conjunction defines the exact same data variable names. A system of equations is uniform when all of its equations are uniform.

**Example 4.16.** The symbolic necessity events defined in \( X_0 = [[d_1 ? d_2, c_1 (d_1, d_2)]]X_1 \land [[f_1 ? f_2, c_2 (f_1, f_2)]]X_2 \) are both pattern equivalent, however they are not uniform since they do not define the same variables. This formula can however be easily made uniform by applying \( \langle \langle - \rangle \rangle_0 \) which renames \( d_1 \) and \( f_1 \) to the same \( g_1 \) and similarly \( d_2 \) and \( f_2 \) to a fresh variable \( g_2 \), so to obtain \( X_0 = [[g_1 ? g_2, c_1 (g_1, g_2)]]X_1 \land [[g_1 ? g_2, c_2 (g_1, g_2)]]X_2 \).

In Figure 4.9, we present the construction \( \langle \langle - \rangle \rangle_1 : (\text{Eq}_{\text{stf}}, \text{VAR}, \mathcal{P}(\text{VAR})) \mapsto (\text{Eq}_{\text{uni}}, \text{VAR}, \mathcal{P}(\text{VAR})) \). This construction internally uses the \text{uni} function to create the required uniform set of equations \( \text{Eq}_{\text{uni}} \) from the given standardized equation set \( \text{Eq}_{\text{stf}} \). More specifically \text{uni} reconstructs the equation set by performing a linear scan during which it converts equations of the form \( X_j = \bigwedge_{j \in Q} [\eta_j (j)]X_j \land \varphi \)
Construction

\[ \langle [\langle Eq, X_0, \mathcal{Y} \rangle \rangle] \rangle \overset{\mathcal{I}}{=} (\text{uni}(Eq, \zeta), X_0, \mathcal{Y}) \]

where \( \zeta = \text{traverse}(Eq, \{0\}, \text{partition}, \emptyset) \)

\[
\text{uni}(Eq, \zeta) \equiv \left\{ X_i = \bigwedge_{j \in Q} [\eta_j(\zeta(j))]X_j \land \varphi \bigg| X_i = \bigwedge_{j \in Q} [\eta_j]X_j \land \varphi \in Eq \right\}
\]

\[
\text{partition}(Eq,Q,\zeta) \equiv \left\{ \begin{array}{l}
\quad j \mapsto \zeta(j) \cup \{g^n/d^n\} \\
\quad k \mapsto \zeta(k) \cup \{g^n/f^n\}
\end{array} \right. \quad \forall i, l \in Q \cdot Eq(i) = \bigwedge_{j \in Q} [\eta_j(d^n)]X_j \land \varphi \\
\quad \text{and } Eq(l) = \bigwedge_{k \in Q^*} [\eta_k(f^n)]X_k \land \varphi \text{ s.t.} \\
\quad \text{if } \eta_j(d^n) \text{ is pattern equivalent to } \eta_k(f^n), \text{ then we assign the same} \\
\quad \text{set of fresh variables } g^n. \\
\right\} \cup \zeta
\]

Figure 4.9: The Uniformity Algorithm for Symbolic Events

where \( \zeta : \text{INDEX} \rightarrow \sigma \) is a map that provides a substitution environment \( \sigma \) for a given index \( i \).

Intuitively, a well-formed \( \zeta \) map should provide substitutions that uniformly rename the data variables of pattern equivalent modal necessities that are defined as siblings within a tree of conjunctions, i.e., the data variables defined in such necessities are renamed to the same fresh set of variable names, such that the patterns of the renamed necessities become syntactically equal.

**Definition 4.8 (A Well-Formed \( \zeta \) Map).** We say that \( \zeta \) is a well-formed map for a set of equations \( Eq \), whenever it provides a set of mappings which allow for

(i) renaming the data variables of each pattern equivalent sibling necessity, defined in \( Eq \), to the same set of fresh variables, and for

(ii) renaming any reference to a data variable that is bound by a renamed parent necessity defined in \( Eq \).

\[ \begin{IEEEeqnarray*}{c}
\text{We assume that when an index } i \text{ is not in the domain of the } \zeta \text{ map (i.e., } i \notin \text{dom}(\zeta)) \text{ then } \zeta(i) = \emptyset.
\end{IEEEeqnarray*} \]
Example 4.17. Consider the following system of equations (Eq, X₀, ∅) where

\[
\begin{align*}
X₀ &= [[d^1 ? d^2, d^1 \neq i] \zeta(1)] X₁ \land [[d^3 ? d^4, d^4 \neq 3] \zeta(2)] X₂ \\
X₁ &= [[f^1 ? f^2, f^1 \neq i] X₁ \land [[f^1 ? f^2, f^2 \neq 3] X₂ \\
X₂ &= [[d^5 ! d^6, t] \zeta(3)] X₃ \\
X₃ &= [f^3 ! f^4, t] X₃ \\
X₂ &= [[d^7 ! d^8, d^7 = d^8] \zeta(4)] X₄ \\
X₄ &= [f^3 ! f^4, f^3 = f^1] X₄
\end{align*}
\]

For convenience, we also represent these equations as a tree starting from equation \(X₀=[[d^1 ? d^2, d^1 \neq i]]X₁ \land [[d^3 ? d^4, d^4 \neq 3]]X₂\) as the root of the tree. We also assume the knowledge of a well-formed \(\zeta\) map that has the following form, i.e.,

\[
\zeta = \left\{ \begin{array}{l}
1 \mapsto \{d^1 / f^1, d^2 / f^2\}, \\
2 \mapsto \{d^3 / f^3, d^4 / f^4\}, \\
3 \mapsto \zeta(1) \cup \{d^5 / f^3, d^6 / f^4\}, \\
4 \mapsto \zeta(2) \cup \{d^7 / f^3, d^8 / f^4\}
\end{array} \right\}
\]

As shown by the tree representation in Figure 4.10, necessities \([d^1 ? d^2, d^1 \neq i]\) and \([d^3 ? d^4, d^4 \neq 3]\) are pattern equivalent siblings within the conjunction defined by equation \(X₀\); in order to be uniformed, the substitution map \(\zeta\) projects indices 1 and 2 onto environments \(\{d^1 / f^1, d^2 / f^2\}\) and \(\{d^3 / f^1, d^4 / f^2\}\) resp. Once the substitution is applied to both necessities we obtain \([f^1 ? f^2, f^1 \neq i]\) and \([f^1 ? f^2, f^2 \neq 3]\), in lieu of \([d^1 ? d^2, d^1 \neq i]\) and \([d^3 ? d^4, d^4 \neq 3]\). Notice how the patterns in both of the resultant necessities are now syntactically equal, meaning that the resultant equation \(X₀=[[f^1 ? f^2, f^1 \neq i]]X₁ \land [[f^1 ? f^2, f^2 \neq 3]]X₂\) is now uniform.

As illustrated in the equation tree above, necessities \([d^5 ! d^6, t]\) and \([d^7 ! d^8, d^7 = d^8]\) are also pattern equivalent siblings within the conjunction defined in \(X₀\). In order to make them uniform, \(\zeta\) provides the following mappings, namely, \(3 \mapsto \zeta(1) \cup \)
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\{d^5/f^3, d^6/f^4\} and \(4 \mapsto \zeta(2) \cup \{d^7/f^3, d^8/f^4\}\), where enable for renaming the aforementioned necessities into \([f^3 \uparrow f^4, t]\) and \([f^3 \uparrow f^3, f^3=f^4]\).

Notice how the filtering condition \(d^7=d^3\) in \([d^7 \uparrow d^6, d^7=d^3]\) was also renamed to \(f^3=f^1\) as variable \(d^7\) is substituted by \(f^3\) when its binding necessity \([d^3 \uparrow d^4, d^4 \neq 3]\) is uniformed into \([f^1 \uparrow f^2, f^2 \neq 3]\). This substitution was made possible since mapping \(\zeta(4)\) includes the substitutions returned by the parent’s index, i.e., \(\zeta(2)\); this allows applying the substitutions performed upon the parent, to its children, which is important to keep the equation closed wrt. the data variables.

(i) Creating a Well-Formed \(\zeta\) Map. Up until now we have been assuming the existence of a well-formed \(\zeta\) map which provides all the necessary information, without having any knowledge as to how it is created.

The \(\zeta\) map is created as a result of conducting a breath first traversal, via the traverse function, on the given equation set, using the partition function as the \(\lambda\) projection function required by traverse. The function \(\text{partition} :: (\text{Eq} \times \mathcal{P}(\text{Index}) \times \text{Acc}) \rightarrow \text{Acc}\) follows the format dictated by \(\lambda\), i.e., it takes as input a set of equations \(\text{Eq}\), a set of indices \(Q\) and an accumulator, in this case \(\zeta\); it returns an updated version of \(\zeta\) as a result.

In order to update \(\zeta\), the partition function inspects the sibling equations denoted by the indices in \(Q\) and as a result it creates a substitution environment which renames the variables of each pattern equivalent sibling necessity, to the same fresh set of variables.

Example 4.18. Recall the system of equations defined earlier in Example 4.17, i.e., \((\text{Eq}, X_0, \emptyset)\) where

\[
\begin{align*}
\text{Eq} = \begin{cases} 
X_0 = &\{[d^1 \uparrow d^2, d^1 \neq i]X_1 \land [d^3 \uparrow d^4, d^4 \neq 3]X_2, \\
\text{where } X_1 = &\{[d^5 \uparrow d^6, t]X_3, \\
X_2 = &\{[d^7 \uparrow d^8, d^7=d^3]X_4, \\
X_3 = &\text{ff}, \quad X_4 = \text{ff} \}
\end{cases}
\end{align*}
\]

Figure 4.11 depicts the breath first traversal performed by the traverse function in which the projection function partition was applied on each set of siblings. Notice that when partition is applied on the root equation, the initially empty
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\[
\text{partition}(\{ X_0=[[d^1?d^2, d^1\neq i]]X_1 \land [[[d^3?d^4, d^1\neq 3]]X_2] \cup Eq', \{0\}, \emptyset) = \zeta
\]
\[
\text{child}(Eq, 0) = \{1,2\}
\]

\[
\text{partition}(\{ X_1=[[d^5!d^6, t]]X_3, \cup Eq'', \{1,2\}, \zeta
\]
\[
\text{child}(Eq', 1) = \{3\}
\]

\[
\text{partition}(X_3=ff, X_4=ff, \{5\}, \zeta') = \zeta'
\]

Figure 4.11: A breath first traversal using \text{partition} to obtain \zeta.

map gets extended by 2 entries, namely \(\zeta = \emptyset \cup \{1 \mapsto \emptyset \cup \{ f^1/d^1, f^2/d^2 \}, 2 \mapsto \emptyset \cup \{ f^1/d^3, f^2/d^4 \} \} \); as shown in Example 4.17, this allows for the sibling necessities defined in \(X_0\) to be uniformed.

The \(\zeta\) map is further extended into \(\zeta' = \zeta \cup \{ 3 \mapsto \zeta(1) \cup \{ f^3/d^5, f^4/d^6 \}, 4 \mapsto \zeta(2) \cup \{ f^3/d^7, f^4/d^8 \} \} \), since the partition function recognises that the sibling necessities \([[d^5!d^6, t]]\) and \([[d^7!d^8, d^7=d^3]]\) are also pattern equivalent; it therefore maps variables \(d^5\) and \(d^7\) to the same fresh variable \(f^3\), and \(d^6\) and \(d^8\) to \(f^4\).

\[\Box\]

**Example 4.19.** We now recall the standardized system of equations \(SYS_{sf}^{\text{sf}}\) obtained from Example 4.14 (restated below); for convenience, we will stop using the shorthand notation and we will instead use guarded symbolic event notation in which the patterns are fully opened and guarded by a filtering equation, *e.g.*, we write \([d?d^1, d^1=\text{req} \land d\neq h] \) instead of \([d?\text{req}, d\neq h] \).

\[
SYS_{sf}^{\text{sf}} = (Eq_{sf}^d, X_0, \emptyset)
\]

where
4. Enforcing Safety Properties via Suppressions

\[ E_d^{s_3} = \left\{ X_0 = \land \left[ [d ? d^1, d^1 = \text{req} \land d \neq h] \right] X_3 \right\} \cup E_d^{s_3} \]

\[ E_d^{s_3'} = \left\{ X_3 = [[d^2 \land d^3 = \text{ans}]] X_{13}, \right. \]  
\[ \left. X_{11} = [[f^2 \land f^3 = \text{req}]] X_{12} \right\} \cup E_d^{s_3'} \]

\[ E_d^{s_3''} = \left\{ \begin{array}{l}
X_{13} = \land \left[ [d^4 ? d^6, d^6 = d \land d^5 = \text{req} \land d^4 \neq h] \right] X_7 \\
X_7 = [[d^6 \land d^7, d^6 = \text{ans}]] X_8, \\
X_9 = [[f^6 \land f^7, f^6 = f \land f^7 = \text{req}]] X_{10}, \\
X_8 = X_{13}, \quad X_{10} = \text{ff}, \quad X_{12} = \text{ff}
\end{array} \right\} \]

To attain the uniform equivalent of \( \text{SYS}_d^{s} \) via construction \( \langle - \rangle \), we must first create a well-formed \( \zeta \) map using \( \text{traverse}(E_d^{s_3}, \{0\}, \text{partition}, \emptyset) \). The traversal is initiated with the initial set of indices \( Q \), as being equal to \( \{0\} \), this is required since our formula starts from the principal logical variable \( X_0 \), \( i.e. \), the \( \text{traverse} \) function starts by inspecting equation \( X_0 = \emptyset \).

Since the sibling necessities define equivalent patterns \( d ? d^1 \) and \( f ? f^1 \), once applied the \( \text{traverse} \) function applies \( \text{partition}(E_d^{s_3'}, \{3, 11\}, \emptyset) \) over these sibling necessities, and creates an initial \( \zeta \) map, where

\[ \zeta = \{3 \mapsto \{g^1/d, g^2/d^1\}, 11 \mapsto \{g^1/f, g^2/f^1\}\}. \]

Hence, this map will later on permit the \( \text{uni} \) function to replace \( d \) and \( f \) with the same fresh variable \( g^1 \) by applying \( \{g^1/d\} \) and \( \{g^1/f\} \), and similarly \( f \) and \( f^1 \) with \( g^2 \) via \( \{g^2/d^1\} \) and \( \{g^2/f^1\} \) accordingly.

The \( \text{traverse} \) function is subsequently applied \( \text{wrt.} \) the children of \( X_0 \) \( i.e. \), \( Q' = \{3, 11\} \), during which it applies the \( \text{partition} \) function on equations \( X_3 = [[d^2 \land d^3, d^2 = d \land d^3 = \text{ans}]] X_{13} \), and \( X_{11} = [[f^2 \land f^3, f^2 = f \land f^3 = \text{req}]] X_{12} \). Since patterns \( d^2 \land d^3 \) and \( f^2 \land f^3 \) are not equivalent, they do not require renaming as they are already uniform, hence \( \text{partition}(E_d^{s_3'}, \{13, 12\}, \zeta) \) (where \( \{13, 12\} \) are the children of \( X_3 \) and \( X_{11} \)) returns

\[ \zeta' = \zeta \cup \{13 \mapsto \zeta(3), 12 \mapsto \zeta(11)\}. \]
Although none of the variables declared in these patterns require renaming, the
substitution map still includes mappings $13 \mapsto \zeta(3)$ and $12 \mapsto \zeta(11)$ which allows
for renaming variable references $d$ and $f$ (which are bound by a parent necessity
defined in $X_0$), into $g^1$, thus keeping the system of equations closed wrt. data
variables.

The traverse function keeps on performing the breath first traversal until finally
it creates the required map, i.e.,

$$\zeta'' = \zeta' \cup \{7 \mapsto \zeta'(13) \cup \{g^3/d^4, g^4/d^5\}, 9 \mapsto \zeta'(13) \cup \{g^3/f^4, g^4/f^5\}\}$$

The traverse function keeps on performing the breath first traversal until finally
it creates the required map, i.e.,

$$\zeta'' = \zeta' \cup \{7 \mapsto \zeta''(7), 10 \mapsto \zeta''(9)\}$$

Finally, when we apply the uni function using the mappings provided in $\zeta''$, we
obtain the following system of equations, i.e., $\text{SYS}^{\text{uni}}_3 = (\text{Eq}^{\text{uni}}_3, X_0, \emptyset)$ where

$$\text{Eq}^{\text{uni}}_3 = \left\{ \begin{array}{c}
X_0 = [[g^1?g^2, g^2=\text{req}\wedge g^1\neq h]]X_3 \land [[g^1?g^2, g^2=\text{req}\wedge g^1\neq j]]X_{11} \\
X_3 = [[d^2\!\equiv d^3, d^2=g^1 \land d^3=\text{ans}]]X_{13}, \\
X_{11} = [[f^2?f^3, f^2=g^1 \land f^3=\text{req}]]X_{12} \\
X_{13} = \land [[g^3?g^4, g^3=g^1 \land g^4=\text{req}\wedge g^1\neq h]]X_7 \\
\qquad [[g^3?g^4, g^3=g^1 \land g^4=\text{req}\wedge g^1\neq j]]X_9, \\
X_7 = [[d^6\!\equiv d^7, d^6=g^1 \land d^7=\text{ans}]]X_8, \\
X_9 = [[f^6?f^7, f^6=g^1 \land f^7=\text{req}]]X_{10}, \\
X_8 = X_{13}, \quad X_{10} = \text{ff}, \quad X_{12} = \text{ff} \\
\end{array} \right\}$$

(ii) Proving Semantic Preservation for $\langle\langle - \rangle\rangle_\emptyset$. To prove that construction
$\langle\langle - \rangle\rangle_\emptyset(i)$ preserves the semantics of the given standardized system of equations, we
must prove that the following criterion holds:

$$\text{SYS}$$ is in Standard Form implies $\langle\langle \text{SYS} \rangle\rangle_\emptyset(i) \equiv \text{SYS}$$ where $\langle\langle \text{SYS} \rangle\rangle_\emptyset(i)$ is Uniform.

In the proof given below, we make use of the following lemmas:

Lemma 4.1. traverse($\text{Eq}, \{0\}$, partition, $\emptyset$)$=\zeta$ implies
$\zeta$ is a well-formed map for $\text{Eq}$.

Lemma 4.2. $\forall (X_j=\varphi_j) \in \text{Eq}$ equation $X_j=\varphi_j$ is in Standard form, and $\zeta$ is a well-
formed map for $\text{Eq}$ implies $\text{uni} (\text{Eq}, \zeta) \equiv \text{Eq}$ and $\forall (X_k=\psi_k) \in \text{uni} (\text{Eq}, \zeta)$.

equation $X_k=\psi_k$ is Uniform.
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Lemma 4.1 dictates that whenever a $\zeta$ map is obtained by conducting a breath first traversal on an equation set $Eq$ using the partition projection function, then the resultant map is well-formed wrt. $Eq$.

Lemma 4.2 builds on the result of the previous lemma by stating that upon obtaining a well-formed map for $Eq$ and when all the equations in $Eq$ are in Standard Form, a semantically equivalent, Uniform equation set can be obtained by applying the uni function on $Eq$ using the well-formed $\zeta$ map to obtain the required uniformity.

The proofs for both of these lemmas are provided in Appendix Section B.1.

\textbf{Proof.} Initially we know

\[ SYS \text{ is in Standard Form} \quad (4.1) \]

By (4.1) and definition of $SYS$ we also know

\[ (Eq, X_0, Y) \text{ is in Standard Form} \quad (4.2) \]

because

\[ \forall (X_j = \varphi_j) \in Eq \cdot \text{equation } X_j = \varphi_j \text{ is in Standard Form} \quad (4.3) \]

We create a $\zeta$ map for our equation set $Eq$ by using a breath first traversal that applies the partition projection function, such that we know

\[ \text{traverse}(Eq, \{0\}, \text{partition}, \emptyset) = \zeta \quad (4.4) \]

By (4.4) and Lemma 4.1, we know

\[ \zeta \text{ is a well-formed map for } Eq \quad (4.5) \]
By (4.3), (4.5) and Lemma 4.2, we know

\[ \text{uni}(Eq, \zeta) \equiv Eq \]  
\[ \forall (X_k=\psi_k) \in \text{uni}(Eq, \zeta) \cdot \text{equation } (X_k=\psi_k) \text{ is Uniform} \]

By (4.6), (4.7) and the definition of \langle\langle SYS\rangle\rangle (i) we can conclude

\[ \langle\langle SYS\rangle\rangle (i) \equiv SYS \text{ where } \langle\langle SYS\rangle\rangle (i) \text{ is Uniform}. \]

--- End of Proof. ---

4.1.3.3 Condition Reformulation of Conjunct Symbolic Events

Reformulating the filtering conditions of conjunct symbolic events involves reconstructing a uniform system of equations into a semantically equivalent equi-disjoint conjunction as defined in Definition 4.9 below.

**Definition 4.9 (System of Equi-Disjoint Equations).** An equation is equi-disjoint when it is uniform, and when multiple necessities defined at the top-level of the same conjunction i.e., \( \bigwedge_{j \in Q} [\eta_j]X_j \), are unable to be matched by the same concrete system action \( \alpha \), unless they are syntactically equal; formally defined as:

\[ \text{if } X_i = \bigwedge_{j \in Q} [\eta_j]X_j \text{ then } \forall k, l \in Q \cdot [\eta_k] \cap [\eta_l] \neq \emptyset \implies \eta_k = \eta_l \]

A system of equations is equi-disjoint when all of its equations are equi-disjoint.

**Example 4.20.** As defined by Definition 4.9, we can deduce that equation \( X_0 = ([\{d?f, f>5\}]X_1) \land ([\{d?f, f>5\}]X_2) \land (\{d?f, f\leq 5\}]X_3) \) is equi-disjoint since there does not exist a system action that is able to satisfy both \( \{d?f, f>5\} \) and \( \{d?f, f\leq 5\} \), i.e., \( \{d?f, f\leq 5\} \cap \{d?f, f>5\} = \emptyset \).

The only two branches that are satisfiable by the same system actions are \( \{d?f, f>5\}X_1 \) and \( \{d?f, f>5\}X_2 \) but they are both prefixed by syntactically equal necessities i.e., \( \{d?f, f>5\} \cap \{d?f, f>5\} \neq \emptyset \) since \( \{d?f, f>5\} = \{d?f, f>5\} \).

However, equation \( X_1 = \{d_1?f_1, t\}X_4 \land \{d_1?f_1, f_1 \neq 5\}]X_5 \) is not equi-disjoint since \( \{d_1?f_1, t\} \cap \{d_1?f_1, f_1 \neq 5\] \neq \emptyset \) but \( \{d_1?f_1, t\} \neq \{d_1?f_1, f_1 \neq 5\} \).
Construction

\[\llangle (Eq, X_0, \mathcal{Y}) \rrangle_{(ii)} = (\text{traverse}(Eq, \{0\}, \text{cond\_comb}, \varnothing), X, \mathcal{Y})\]

\[\text{cond\_comb}(Eq, Q, \omega) = \left\{ \begin{array}{ll}
X_i = \bigwedge_{c_j \in \{0, c_k\}} X_j \land \varphi \quad & (X_i = \bigwedge_{c_j \in \{0, c_k\}} X_j \land \varphi) \in Eq_{i, Q} \\
\text{and } Q' = \bigcup_{l \in Q} \text{child}(Eq, l) \\
such that Q'' \subseteq Q'
\end{array} \right\} \cup \omega\]

\[\mathcal{C}(i, Q) = \left\{ \begin{array}{l}
\begin{cases}
c_i \land c_j \ldots \land c_k, \\
c_i \land \neg c_j \ldots \land c_k,
\end{cases}
\end{array} \right\} \quad \forall j \ldots n \in Q \text{ where } i \neq j \neq \ldots \neq n \quad \text{such that } o_i = o_j = \ldots = o_k\]

Figure 4.12: The Conjunction Reformulation Algorithm.

Remark 4.1. Note that normalization construction §3 actually checks that the branches are Equi-Disjoint by using #S(Q). This is since S(Q) returns the set of symbolic events defined by the branches in the guarded conjunction that is being analysed, i.e., \(\bigcup_{i \in Q} \{\eta_j \mid \eta_j \text{ is a subformula in } \varphi_i\}\). Since sets do not contain repeated (syntactically equal) values, the predicate #S(Q) can only return false when two or more events in S(Q) are syntactically different yet still non-disjoint.

Figure 4.12 presents the construction function, \(\llangle - \rrangle_{(ii)} : (Eq_{\text{uni}}, \text{VAR}, \mathcal{P}(\text{VAR}))\) for recomposing a uniform system of equation into an equi-disjoint one. Internally, this construction uses the \text{traverse} function to perform a breadth first traversal on the given uniform equation set, \(Eq_{\text{uni}}\), starting from the equation that equates to the principle variable, i.e., with \(Q=\{0\}\). While conducting the traversal, this construction applies the \text{cond\_comb} function in order to reconstruct the uniform conjunctions, i.e., \(\bigwedge_{j \in Q} [o, c_j] X_j\) defined in \((X_i = \varphi_i) \in Eq_{\text{uni}}\), into equi-disjoint ones, thereby producing an \textit{equi-disjoint} equation set \(Eq_{\text{comb}}\) at the end of the traversal.

The function \text{cond\_comb} : \((Eq_{\text{uni}} \times \mathcal{P}(\text{INDEX}) \times \text{ACC}) \rightarrow \text{ACC}\) is a projection function that takes as input a uniform equation set \(Eq_{\text{uni}}\), a set of indices \(Q\), and an accumulator \(\omega\). The accumulator \(\omega\) contains a partial equi-disjoint set of equations which is first initialized to \(\varnothing\) and is constantly extended by repeated \text{cond\_comb} applications until the traversal is complete, in which case \(\omega\) is returned as the
resultant equi-disjoint equation set.

In order to update \( \omega \), the \texttt{cond_comb} function inspects the sibling equations denoted by the indices in \( Q \), i.e., \( (X_i=\varphi_i) \in \text{Eq}_{\neg\neg Q} \), and computes the truth combinations of the filtering conditions defined by the sibling symbolic necessities (specified in these equations) which define the same (syntactically equal) patterns.

To compute these truth combinations, the \texttt{cond_comb} function starts by computing the child indices of the current sibling equations, denoted by \( Q \), by using the \texttt{child} function, i.e., \( Q' = \bigcup_{l \in Q} \text{child}(\text{Eq}, l) \). Following this, it inspects the conjunctions defined in the selected equations, i.e., \( \bigwedge_{j \in Q'} \bigl[[o_j, c_j]\bigr]X_j \land \varphi \), and reconstructs them into \( \bigwedge_{c_k \in C(j, Q') \bigl[[o_j, c_k]\bigr]} X_j \land \varphi \). Notice that \( c_k \) is a truth combination of all the filtering conditions that are defined by symbolic necessities that specify syntactically equal patterns and which are defined by the branches identified by the indices in \( Q' \), e.g., if \( Q' = \{1, 2, 3\} \), then one possible truth combination \( c_k \) is \( c_1 \land \neg c_2 \land c_3 \).

The truth combinations, such as \( c_k \), are generated through the combinatorial function \( C :: (\text{INDEX} \times \mathcal{P}(\text{INDEX})) \). This function takes as input the index of the branch that is being analysed, i.e., the one identified by index \( j \), along with the indices of all the sibling branches specified in \( Q' \). As a result, \( C(j, Q') \) returns the truth combinations in the which filtering condition, \( c_j \), of the branch that is currently being reconstructed, i.e., \( [[o_j, c_j]]X_j \), is true, i.e., \( C(1, \{1, 2, 3\}) = \{(c_1 \land c_2 \land c_3), (c_1 \land c_2 \land \neg c_3), (c_1 \land \neg c_2 \land c_3), (c_1 \land \neg c_2 \land \neg c_3)\} \). These truth combinations are then used to reconstruct the existing branch into a collection of equi-disjoint branches.

**Example 4.21.** Consider equation \( X_0 = [[o, c_1]]X_1 \land [[o, c_2]]X_2 \land [[o, c_3]]X_3 \), using the truth combinations provided by \( C(1, \{1, 2, 3\}) \) we can reconstruct branch \( [[o, c_1]]X_1 \) into:

\[
[[o, c_1 \land c_2 \land c_3]]X_1 \land [[o, c_1 \land c_2 \land \neg c_3]]X_1 \land [[o, c_1 \land \neg c_2 \land c_3]]X_1 \land [[o, c_1 \land \neg c_2 \land \neg c_3]]X_1.
\]

Similarly, with \( C(2, \{1, 2, 3\}) \) and \( C(3, \{1, 2, 3\}) \), we can reconstruct branches \( [[o, c_2]]X_2 \)
and \( [[o, c_3]]X_3 \) in the same way such that the resultant equation is:

\[
X_0 = \frac{[\{o, c_1 \land c_2 \land c_3\}]X_1 \land [\{o, c_1 \land c_2 \land \neg c_3\}]X_1 \land [\{o, c_1 \land \neg c_2 \land c_3\}]X_1 \land [\{o, c_1 \land \neg c_2 \land \neg c_3\}]X_1 \land [\{o, c_1 \land c_2 \land c_3\}]X_2 \land [\{o, c_1 \land c_2 \land \neg c_3\}]X_2 \land [\{o, \neg c_1 \land c_2 \land c_3\}]X_3 \land [\{o, \neg c_1 \land c_2 \land \neg c_3\}]X_3 \land [\{o, c_1 \land \neg c_2 \land c_3\}]X_3 \land [\{o, c_1 \land \neg c_2 \land \neg c_3\}]X_3}{\} \]

Note that the resultant reconstructed equations (case in point \( X_0 \) in Example 4.21) are equi-disjoint as the truth combination conditions ensure that a concrete system event \( \alpha \) can never satisfy multiple symbolic necessities in the reconstructed branches, unless these are syntactically equal.

Moreover, note that the truth combinations generated by function \( C(j, Q') \) do not include the cases where \( c_j \) is false. This is essential to ensure that none of the reconstructed branches can be satisfied when the original condition \( c_j \) is true, thereby preserving the semantics of the original branch.

**Example 4.22.** Recall equation \( X_0 = [[o, c_1]]X_1 \land [[o, c_2]]X_2 \land [[o, c_3]]X_3 \) from Example 4.21. Note that logical variables \( X_1, X_2 \) and \( X_3 \) can only be evaluated when their prefixing necessities are satisfied by some concrete system event, meaning that continuation \( X_1 \) is reachable when \( c_1 \) is true, and \( \text{resp.} \) \( X_2 \) and \( X_3 \) when \( c_2 \) and \( c_3 \) are true. Hence, in the reconstructed equation, the conditions are never negated when prefixing \( \text{resp.} \) logical variable as highlighted below:

\[
X_0 = \frac{[[o, c_1 \land c_2 \land c_3]]X_1 \land [[o, c_1 \land c_2 \land \neg c_3]]X_1 \land [[o, c_1 \land \neg c_2 \land c_3]]X_1 \land [[o, c_1 \land \neg c_2 \land \neg c_3]]X_1 \land [[o, c_1 \land c_2 \land c_3]]X_2 \land [[o, c_1 \land c_2 \land \neg c_3]]X_2 \land [[o, \neg c_1 \land c_2 \land c_3]]X_3 \land [[o, \neg c_1 \land c_2 \land \neg c_3]]X_3 \land [[o, c_1 \land \neg c_2 \land c_3]]X_3 \land [[o, c_1 \land \neg c_2 \land \neg c_3]]X_3}{\} \]

Once the traversal completes, the construction outputs the final accumulator value \( \omega \) containing the required equi-disjoint equation set.
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Example 4.23 (Applying the Construction). Recall the uniform system of equations $\text{SYS}^\text{uni}_3$ obtained from Example 4.19 (restated below):

$$\text{SYS}^\text{uni}_3 = (\text{Eq}^\text{uni}_3, X_0, \emptyset)$$

where

$$\begin{align*}
\text{Eq}^\text{uni}_3 &= \left\{ X_0 = \land \left\{ \left[ [g^1 ? g^2, g^2 = \text{req} \land g^1 \neq h] \right] X_3 \right\} \cup \text{Eq}^\text{uni}_3, X_1 = \land \left\{ \left[ [g^1 ? g^2, g^2 = \text{req} \land g^1 \neq j] \right] X_1 \right\} \cup \text{Eq}^\text{uni}_3, X_3 = \left\{ d^2 \downarrow d^3, d^2 = g^1 \land d^3 = \text{ans} \right\} X_13, X_11 = \left\{ f^2 \downarrow f^3, f^2 = g^1 \land f^3 = \text{req} \right\} X_12, X_13 = \land \left\{ \left[ [g^3 ? g^4, g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq h] \right] X_7 \right\}, X_7 = \left\{ d^6 \downarrow d^7, d^6 = g^1 \land d^7 = \text{ans} \right\} X_8, X_9 = \left\{ f^6 \downarrow f^7, f^6 = g^1 \land f^7 = \text{req} \right\} X_10, X_8 = X_{13}, X_{10} = \text{ff}, X_{12} = \text{ff} \right\}
\end{align*}$$

The construction initiates the traversal from the principle equation, i.e., $X_0 = \varphi_0$, by invoking $\text{traverse}(\text{Eq}^\text{uni}_3, \{0\}, \text{cond_comb}, \emptyset)$, which inspects the conjunct necessities defined in the root equation. Since events \{g^1 ? g^2, g^2 = \text{req} \land g^1 \neq h\} and \{g^1 ? g^2, g^2 = \text{req} \land g^1 \neq j\} define the same pattern $g^1 ? g^2$, by applying the cond_comb projection function this construction generates all the truth combinations of the filtering conditions defined in the aforementioned necessities of the formula, i.e., ($g^2 = \text{req} \land g^1 \neq h$) and ($g^2 = \text{req} \land g^1 \neq j$), by using:

$$\begin{align*}
\mathbb{C}(3, \{3, 11\}) &= \left( (g^2 = \text{req} \land g^1 = \text{h}) \land (g^2 = \text{req} \land g^1 \neq j) \right), \\
\mathbb{C}(11, \{3, 11\}) &= \left( (g^2 = \text{req} \land g^1 = \text{h}) \land (g^2 = \text{req} \land g^1 \neq j) \right)
\end{align*}$$

and

$$\begin{align*}
\mathbb{C}(3, \{3, 11\}) &= \left( (g^2 = \text{req} \land g^1 \neq h) \land (g^2 = \text{req} \land g^1 \neq j) \right), \\
\mathbb{C}(11, \{3, 11\}) &= \left( (g^2 = \text{req} \land g^1 \neq h) \land (g^2 = \text{req} \land g^1 \neq j) \right)
\end{align*}$$

Notice that the results of both $\mathbb{C}(3, \{3, 11\})$ and $\mathbb{C}(11, \{3, 11\})$ do not include combinations in which the original condition (i.e., $c_3 = (g^2 = \text{req} \land g^1 \neq h)$ and $c_{11} = (g^2 = \text{req} \land g^1 \neq j)$ resp.) is negated, i.e., truth combinations such as $(g^2 = \text{req} \land g^1 \neq h) \land (g^2 = \text{req} \land g^1 \neq j)$ and $(g^2 = \text{req} \land g^1 \neq h) \land (g^2 = \text{req} \land g^1 \neq j)$ are not included in the resultant set returned by $\mathbb{C}(3, \{3, 11\})$, since they negate parts of the original condition i.e., $c_3$. 

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A branch is thus added to the reconstructed formula whenever the guard of the original necessity is \textit{not negated} in the generated truth combination, \textit{e.g.}, branch
\([\{g^1 \land g^2 \land g^3 \land d^1 \land d^2 \land d^3 \land \text{ans}\}] X_3\) must only be replaced by
\([\{g^1 \land g^2 \land (g^2 \land g^1 \neq h) \land (g^2 \land g^1 \neq j)\}] X_3\)
and
\([\{g^1 \land g^2 \land (g^2 \land g^1 \neq h) \land \neg(g^2 \land g^1 \neq j)\}] X_3\)
but not
\([\{g^1 \land g^2 \land \neg(g^2 \land g^1 \neq h) \land (g^2 \land g^1 \neq j)\}] X_3\)
and
\([\{g^1 \land g^2 \land \neg(g^2 \land g^1 \neq h) \land \neg(g^2 \land g^1 \neq j)\}] X_3\)
since the original condition is negated in these last two necessities, and hence, function \(\text{cond\_comb}(Eq^{\text{uni}}, \{0\}, \emptyset)\) evaluates to \(\omega\), where
\[
\omega = \left\{ \begin{array}{l}
X_0 = \left\{ \begin{array}{l}
\{g^1 \land g^2 \land (g^2 \land g^1 \neq h) \land (g^2 \land g^1 \neq j)\} X_3 \\
\land \{g^1 \land g^2 \land (g^2 \land g^1 \neq h) \land \neg(g^2 \land g^1 \neq j)\} X_3 \\
\land \{g^1 \land g^2 \land (g^2 \land g^1 \neq h) \land (g^2 \land g^1 \neq j)\} X_{11} \\
\land \{g^1 \land g^2 \land \neg(g^2 \land g^1 \neq h) \land (g^2 \land g^1 \neq j)\} X_{11} \\
\end{array} \right. \\
\end{array} \right\}
\]

The traversal proceeds by computing the children of \(X_0\) via \(\bigcup_{l=0} \text{child}(Eq^{\text{uni}}, l)\)
which returns \(\{3, 11\}\) as the set of child indices; it then it recurses \textit{wrt.} these indices, \textit{i.e.}, via \(\text{traverse}(Eq^{\text{uni}}, \{3, 11\}, \text{cond\_comb}, \omega)\). Once again the traversal applies \(\text{cond\_comb}\) which attempts to generate the new truth combinations for \(\{\text{d}^2 \land \text{d}^3, \text{d}^2 = \text{g}^1 \land \text{d}^3 = \text{ans}\}\) and \(\{\text{f}^2 \land \text{f}^3, \text{f}^2 = \text{g}^1 \land \text{f}^3 = \text{req}\}\), defined in \(X_3\) and \(X_{11}\) \textit{resp.}
via \(C(13, Q'')\) and \(C(12, Q'')\) where \(Q'' = \bigcup_{l=12, 13} \text{child}(Eq^{\text{uni}}, l) = \{12, 13\}\). However, since these two symbolic events \textit{do not} define syntactically equal patterns, \textit{i.e.}, \(\text{d}^2 \land \text{d}^3 \land \text{f}^2 \land \text{f}^3\), their filtering condition remains intact, \textit{i.e.}, \(C(13, Q'') = \{\text{d}^2 = \text{g}^1 \land \text{d}^3 = \text{ans}\}\) and \(C(12, Q'') = \{\text{f}^2 = \text{g}^1 \land \text{f}^3 = \text{req}\}\). Hence, \(\text{cond\_comb}(Eq^{\text{uni}}, \{3, 11\}, \omega)\) evaluates to \(\omega'\), where
\[
\omega' = \omega \uplus \left\{ \begin{array}{l}
X_3 = \left\{\text{d}^2 \land \text{d}^3, \text{d}^2 = \text{g}^1 \land \text{d}^3 = \text{ans}\right\} X_{13}, \\
X_{11} = \left\{\text{f}^2 \land \text{f}^3, \text{f}^2 = \text{g}^1 \land \text{f}^3 = \text{req}\right\} X_{12} \\
\end{array} \right\}
\]

The breath first traversal continues until all the remaining equations have been
analysed and reconstructed into:

\[\omega'' = \omega' \cup \]

\[X_{13} = \begin{cases} \\
\{([g^3 \land g^4, (g^2 = g^1 \land g^4 = \text{req} \land g^1 \neq h) \land (g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq j)] \} X_7 \\
\{([g^3 \land g^4, (g^2 = g^1 \land g^4 = \text{req} \land g^1 \neq h) \land \neg(g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq j)] \} X_7 \\
\{([g^3 \land g^4, (g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq h) \land (g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq j)] \} X_9 \\
\{([g^3 \land g^4, \neg(g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq h) \land (g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq j)] \} X_9 \\
\} \\
\end{cases} \]

Hence, the resultant system of equations is \(\text{SYS}^\text{comb} = (E_{d_3}^\text{comb}, X_0, \emptyset)\) where \(E_{d_3}^\text{comb} = \omega\).

Since all the necessities guarding conjunctions are now \(\text{equi-disjoint}\), \(\text{i.e.}\), they can only be satisfied by the same action iff they are syntactically equal, the resultant equation can now be converted into \textit{normal form} by continuing from step §3 of the normalization algorithm presented Section 4.1.2. Hence, by applying §3 on \(\text{SYS}^\text{comb}\), we obtain \(\text{SYS}^\text{nf}_3 = (E_{d_3}^\text{nf}, X_0, \emptyset)\) where \(E_{d_3}^\text{nf}\) is defined as:

\[X_{10} = \begin{cases} \\
\{([g^1 \land g^2, (g^2 = \text{req} \land g^1 \neq h) \land (g^3 = \text{req} \land g^1 \neq j)] \} X_{13,11} \\
\{([g^1 \land g^2, (g^2 = \text{req} \land g^1 \neq h) \land \neg(g^3 = \text{req} \land g^1 \neq j)] \} X_{13,11} \\
\{([g^1 \land g^2, \neg(g^3 = \text{req} \land g^1 \neq h) \land (g^3 = \text{req} \land g^1 \neq j)] \} X_{11,11} \\
\} \\
\end{cases} \]

\[X_{13,11} = \{([d^6 \land d^3, d^2 = g^1 \land d^3 = \text{ans}] X_{13}, \ [([f^6 \land f^3, f^2 = g^1 \land f^3 = \text{req}] X_{12}, \\
X_{13} = \{([d^6 \land d^3, d^2 = g^1 \land d^3 = \text{ans}] X_{13}, \ X_{11} = \{([f^6 \land f^3, f^2 = g^1 \land f^3 = \text{req}] X_{12}, \\
X_{13} = \{([g^3 \land g^4, (g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq h) \land (g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq j)] \} X_{7,9} \\
\{([g^3 \land g^4, (g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq h) \land \neg(g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq j)] \} X_{7,9} \\
\{([g^3 \land g^4, \neg(g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq h) \land (g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq j)] \} X_{9} \\
\} \\
\end{cases} \]

\[X_{7,9} = \{([d^6 \land d^7, d^6 = g^1 \land d^7 = \text{ans}] X_{8}, \ [([f^6 \land f^7, f^6 = g^1 \land f^7 = \text{req}] X_{10}, \\
X_{7} = \{([d^6 \land d^7, d^6 = g^1 \land d^7 = \text{ans}] X_{8}, \ X_{9} = \{([f^6 \land f^7, f^6 = g^1 \land f^7 = \text{req}] X_{10}, \\
X_{8} = X_{13}, \ X_{10} = \text{ff}, \ X_{12} = \text{ff} \]
such that by §4 we finally obtain the required normalized formula \( \psi_3 \in \text{SHML}_{nf} \):

\[
\psi_3 = \max X_{13}, \left( \begin{array}{c}
\left[ \left[ g^3 \equiv g^1 \land g^4 = \text{req} \land g^1 \neq h \right] \land (g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq j) \right] \right)
\wedge
\left( \begin{array}{c}
\left[ \left[ d^0 \equiv d^7, d^6 = g^1 \land d^7 = \text{ans} \right] X_{13} \land \left[ \left[ f^0 \equiv f^7, f^6 = g^1 \land f^7 = \text{req} \right] \right] \right] \right)
\end{array} \right)
\]

\[
\psi'_3 = \max X_{13}, \left( \begin{array}{c}
\left[ \left[ g^3 \equiv g^1 \land g^4 = \text{req} \land g^1 \neq h \right] \land (g^3 = g^1 \land g^4 = \text{req} \land g^1 \neq j) \right] \right)
\wedge
\left( \begin{array}{c}
\left[ \left[ d^0 \equiv d^7, d^6 = g^1 \land d^7 = \text{ans} \right] X_{13} \land \left[ \left[ f^0 \equiv f^7, f^6 = g^1 \land f^7 = \text{req} \right] \right] \right] \right)
\end{array} \right)
\]

**Proving Semantic Preservation for \( \langle - \rangle_{(ii)} \).** To prove that construction \( \langle - \rangle_{(ii)} \) preserves the semantics of the given uniform system of equations, we must prove that the following criterion holds:

\( \text{SYS is Uniform implies } \langle \langle \text{SYS} \rangle \rangle_{(ii)} \equiv \text{SYS where } \langle \langle \text{SYS} \rangle \rangle_{(ii)} \text{ is Equi-Disjoint.} \)

In the proof given below, we make use of the following lemma:

**Lemma 4.3.** Lemma 4.3. \( \forall (X_j = \varphi_j) \in \text{Eq} \text{ equation } X_j = \varphi_j \text{ is Uniform implies } \text{Eq} \equiv \text{traverse(} \text{Eq}, \{0\}, \text{cond_comb}, \emptyset \text{) and } \forall (X_k = \psi_k) \in \text{traverse(} \text{Eq}, \{0\}, \text{cond_comb}, \emptyset \text{)} \text{ equation } (X_k = \psi_k) \text{ is Equi-Disjoint.} \)

This lemma states that whenever the equations in an equation set \( \text{Eq} \) are all Uniform, then a semantically equivalent, equi-disjoint equation set can be obtained by performing a breadth first traversal upon \( \text{Eq} \) using the \text{cond_comb} projection.
function. The proof for this lemma is provided in Appendix Section B.2.

**Proof.** Initially we know

\[ SYS \text{ is Uniform} \quad (4.1) \]

By (4.1) and definition of SYS we also know

\[ (Eq, X_0, \mathcal{Y}) \text{ is Uniform} \quad (4.2) \]

because

\[ \forall (X_j = \varphi_j) \in Eq \cdot \text{equation } X_j = \varphi_j \text{ is Uniform} \quad (4.3) \]

By applying \( \langle\langle - \rangle\rangle_{(ii)} \) on \((Eq, X_0, \mathcal{Y})\) we know

\[ \langle\langle (Eq, X_0, \mathcal{Y}) \rangle\rangle_{(ii)} = (\text{traverse}(Eq, \{0\}, \text{cond}_\text{comb}, \emptyset), X_0, \mathcal{Y}) \quad (4.4) \]

By (4.3) and Lemma 4.3, we know

\[ Eq \equiv \text{traverse}(Eq, \{0\}, \text{cond}_\text{comb}, \emptyset) \quad (4.5) \]

\[ \forall (X_k = \psi_k) \in \text{traverse}(Eq, \{0\}, \text{cond}_\text{comb}, \emptyset) \cdot \text{equation } (X_k = \psi_k) \text{ is Equi-Disjoint} \quad (4.6) \]

Hence, by (4.4), (4.5) and (4.6) we can conclude

\[ \langle\langle SYS \rangle\rangle_{(ii)} \equiv SYS \text{ where } \langle\langle SYS \rangle\rangle_{(ii)} \text{ is Equi-Disjoint}. \]

--- End of Proof. ---

4.2 Synthesising Deterministic Enforcers

In the previous section we have presented an algorithm which demonstrates that for every formula \( \varphi \in sHML \) we can find a semantically equivalent normalized formula \( \psi \in sHML_{nf} \). By working on normalized formulae we are able to provide
a more intuitive synthesis function which maps formulae to enforcement monitors in a similar way as was done in [49] for detection monitors (see Figure 2.6).

### Optimization Function

\[
\text{opt}(\varphi) = \begin{cases} 
\varphi & \text{if } \varphi \in \{\text{ff}, \text{tt}\} \\
\text{opt}(\varphi') & \text{if } \varphi = \max X_0.\varphi' \text{ and } X_0 \notin \text{fv}(\varphi') \\
\max X_0.\text{opt}(\varphi') & \text{if } \varphi = \max X_0.\varphi' \text{ and } X_0 \in \text{fv}(\varphi') \\
\bigwedge_{i \in Q} [\eta_i]\text{opt}(\varphi_i) & \text{if } \varphi = \bigwedge_{i \in Q}[\eta_i]\varphi_i
\end{cases}
\]

### Synthesis Function

\[
\llbracket \varphi \rrbracket \triangleq \llbracket \text{opt}(\varphi) \rrbracket_{\bot}
\]

where

\[
\llbracket X \rrbracket_\rho \triangleq x \\
\llbracket \text{ff} \rrbracket_\rho \triangleq y \quad \text{when } \rho = y \\
\llbracket \text{tt} \rrbracket_\rho \triangleq \text{id} \\
\llbracket \max X.\varphi \rrbracket_\rho \triangleq \text{rec } x.(\llbracket \varphi \rrbracket_\rho) \\
\llbracket \bigwedge_{i \in Q}[\{o, c\}_i]\varphi_i \rrbracket_\rho \triangleq \text{rec } y.\sum_{i \in Q} \left\{\{o, c, \tau\}_i.\llbracket \varphi_i \rrbracket_y \right\} \quad \text{if } \varphi_i = \text{ff}
\]

Figure 4.13: Synthesis of Enforcement Monitors from \text{SHML}_nf formulæ.

The synthesis function, \(\llbracket \varphi \rrbracket : \text{SHML}_nf \mapsto \text{ENF} \), presented in Figure 4.13 compositionally analyses a given formula \(\varphi \in \text{SHML}_nf\) in order to produce the required enforcement monitor. Before initiating the synthesis, function \(\llbracket \varphi \rrbracket \) optimizes the given normalized formula \(\varphi\) via the function \(\text{opt} : \text{SHML}_nf \mapsto \text{SHML}_nf\). This function compositionally inspects a given formula \(\varphi\) and removes maximal fixpoint declarations whenever their variable is never referenced in \(\varphi\). This ensures the removal of any redundant fixpoint declarations that might have been added during normalization (see Section 4.1).

Once the formula is optimized, function \(\llbracket \varphi \rrbracket \) initiates the synthesis, for obtaining the required enforcer, by invoking function \(\llbracket - \rrbracket : (\text{SHML}_nf \times \text{STATE}) \mapsto \text{ENF} \). This recursive function produces the required enforcer by compositionally analysing the given formula \(\varphi \in \text{SHML}_nf\). During analysis, it also maintains a state \(\rho\) which can either be \(\bot\), or may contain some recursive variable, \(y\); initially, \(\rho\) is set to \(\bot\),
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but is then modified accordingly during the synthesis derivation.

The synthesis converts truth formulae, \( \tt \in \text{sHML}_{nf} \), into the identity enforcer \( \id \), and logical variables, \( X \in \text{sHML}_{nf} \), into the corresponding recursion enforcement variable, whereas maximal fixpoints, \( \max X.\varphi \in \text{sHML}_{nf} \), are converted into a recursive enforcer; these three conversions are performed irrespective of the contents of \( \rho \).

Normalized conjunctions, \( \bigwedge_{i \in Q} \varphi_i \), are synthesised into a recursive summation of enforcers, i.e., \( \rec y.\sum_{i \in Q} e.\langle \varphi_i \rangle_y \), where \( e \) can be either an identity transformation, \( \{o,c,o\}_i \), or a suppression transformation, \( \{o,c,\tau\}_i \), whenever the continuation \( \varphi_i \) is a falsehood, i.e., \( \varphi_i = \false \). The latter ensures that the synthesised enforcer is capable of suppressing any system action \( \alpha \) that satisfies necessities that lead to a falsehood (e.g., \( \{o,c\}(\alpha) = \sigma \) such that \( \alpha \) satisfies necessity \( \{o,c\}\false \)), thereby preventing the system from violating the property.

Also, notice how the fresh recursive variable, \( y \), introduced by the synthesised recursive construct, \( \rec y. \), is passed on to the subsequent applications of the synthesis function, i.e., \( \langle \varphi_i \rangle_y \). This is required to enable converting falsehood, \( \false \), into the newly introduced recursive variable. In this way, an action is only suppressed whenever it satisfies a necessity that is followed by a falsehood, e.g., \( \{i?x, x=5\}\false \) becomes \( \rec y.\{i?x, x=5, \tau\}_y \). This ensures that any action leading to a violation is continuously suppressed until a non-matching action occurs, thereby preventing the violation from occurring.

**Example 4.24.** Recall formula \( \varphi_1 \in \text{sHML}_{nf} \) from Example 2.8:

\[
\varphi_1 = \max X.\langle i?\text{req}\rangle([i!\text{ans}]X \land [i?\text{req}]\false)
\]

Using the synthesis function defined in Figure 4.13 we can automatically generate enforcer \( e_2 \) (defined in Example 3.2) as shown by the derivation below.

\[
\langle \max X.\langle i?\text{req}\rangle([i!\text{ans}]X \land [i?\text{req}]\false) \rangle_\bot = \langle \rec x.\langle \langle i?\text{req}\rangle([i!\text{ans}]X \land [i?\text{req}]\false) \rangle_\bot \rangle_\bot
\]

\[
\langle \rec x.\rec z.i?\text{req}.\langle ([i!\text{ans}]X \land [i?\text{req}]\false) \rangle_z \rangle_\bot
\]

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Since \([i\text{req}]\) is followed by \( ff \), we need to suppress every \( i\text{req} \) action.

\[
\begin{align*}
= & \text{rec } x. \text{rec } z. \{i\text{req}, t, i\text{req}\}. \text{rec } y. (\{i \text{ans}, t, i \text{ans}\}. \ll X \gg_y + \{i\text{req}, t, \tau\}. \ll ff \gg_y) \\
= & \text{rec } x. \text{rec } z. \{i\text{req}, t, i\text{req}\}. \text{rec } y. (\{i \text{ans}, t, i \text{ans}\}. X + \{i\text{req}, t, \tau\}. y)
\end{align*}
\]

In practice, the resultant monitor can be further optimized by removing any redundant recursive constructs such as \( \text{rec } z. \), thereby obtaining:

\[
\text{rec } x. \{i\text{req}, t, i\text{req}\}. \text{rec } y. (\{i \text{ans}, t, i \text{ans}\}. X + \{i\text{req}, t, \tau\}. y)
\]

4.2.1 Providing Formal Guarantees

Now that we have established a synthesis function for normalised formulas, we proceed to prove that our synthesis actually generates deterministic monitors (as defined by Definition 4.1) that strongly enforce \( \text{sHML} \_\text{nf} \) formulas (as defined by Definition 3.5).

4.2.1.1 Ensuring Deterministic Behaviour of the Synthesized Enforcers

We ensure the deterministic behaviour (as defined by Definition 4.1) of any enforcer that can be synthesised by the synthesis algorithm provided in Figure 4.13, by proving Theorem 4.2.

**Theorem 4.2** (Deterministic Behaviour). For every enforcer, \( \ll \varphi \gg \), that can be synthesised from a normalized formula \( \varphi \in \text{sHML} \_\text{nf} \),

\[
\ll \varphi \gg \xrightarrow{\bullet^{e'}} e' \text{ and } \ll \varphi \gg \xrightarrow{\bullet^{e''}} e'' \text{ implies } e' = e'' \text{ and } t' = t''
\]

In order to prove Theorem 4.2, we make use of Lemma 4.4 which states that our synthesis function, \( \ll \varphi \gg = e \), always produces well-formed enforcers, i.e., \( e \in \text{ENF}_{\text{wf}} \).

**Lemma 4.4** (Enforcer Well-formedness).

\[
\forall \varphi \in \text{sHML} \_\text{nf} \cdot \ll \varphi \gg = e \text{ implies } e \in \text{ENF}_{\text{wf}} \quad \text{where}
\]

\[
e \in \text{ENF}_{\text{wf}} := \text{id } x \mid \text{rec } x. e \mid \sum_{i \in Q} \{o_i, c_i, o'_i\}. e'_i \quad \text{where } \# \{o_i, c_i\}
\]

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Enforcer well-formedness is a syntactic restriction that requires that every branch in a summation is prefixed by a *disjoint* symbolic transducer\(^3\).

Proving Theorem 4.2 also requires proving *Single Step Determinism* (Lemma 4.5), which states that whenever a *well-formed* enforcer \(e\) is able to perform two different reductions for the *same input event* \(\alpha\), i.e., \(\{\alpha\} \overset{t\rightarrow t'}{\longrightarrow} e'\) and \(\{\alpha\} \overset{t\rightarrow t''}{\longrightarrow} e''\), then both reductions should process and modify \(\alpha\) in the same way such that the output actions, \(\mu'\) and \(\mu''\), and the resultant continuation enforcers, \(e'\) and \(e''\), are both well-formed and syntactically equal.

**Lemma 4.5** (Single Step Determinism). For every well-formed enforcer, \(e \in \text{ENF}_{\text{wf}}\),

\[
e \overset{\alpha\vdash \mu'}{\longrightarrow} e' \quad \text{and} \quad e \overset{\alpha\vdash \mu''}{\longrightarrow} e'' \quad \text{implies} \quad e' = e'' \quad \text{and} \quad \mu' = \mu''
\]

The proofs for Lemma 4.4 and Lemma 4.5 are provided in Appendix Section C.1.

**To Prove Theorem 4.2**

\[
\forall \varphi \in \text{sHML}_{\text{nf}} \cdot \{\varphi\} \overset{t\rightarrow t'}{\longrightarrow} e' \quad \text{and} \quad \{\varphi\} \overset{t\rightarrow t''}{\longrightarrow} e'' \quad \text{implies} \quad e' = e'' \quad \text{and} \quad t' = t''
\]

By Lemma 4.4 we can instead prove

\[
\Rightarrow \forall e \in \text{ENF}_{\text{wf}} \cdot e \overset{t\rightarrow t'}{\longrightarrow} e' \quad \text{and} \quad e \overset{t\rightarrow t''}{\longrightarrow} e'' \quad \text{implies} \quad e' = e'' \quad \text{and} \quad t' = t''
\]

**Proof by induction on the structure of the input trace** \(t\).

**Case** \(t=\varepsilon\): We know

\[
e \in \text{ENF}_{\text{wf}}
\]

\[
e \overset{\varepsilon \vdash t'}{\longrightarrow} e'
\]

\[
e \overset{\varepsilon \vdash t''}{\longrightarrow} e''
\]

Since \(t=\varepsilon\), we know that no transitions can be made by enforcer \(e\) in (4.2) and (4.3) since the input trace is *empty*, and so no output trace can be produced in

\(^3\)One can notice the similarities between normalized formulae \(\varphi \in \text{sHML}_{\text{nf}}\), which requires all conjunctions to be prefixed by disjoint necessities, and well-formed enforcers \(e \in \text{ENF}_{\text{wf}}\).
both cases. Hence, we conclude

\[ e' = e = e'' \quad (4.4) \]
\[ t' = \epsilon = t'' \quad (4.5) \]

\[ \text{Case } t = \alpha; u : \]

We know

\[ e \in \text{Enf}_{\text{wf}} \quad (4.1) \]
\[ e \xrightarrow{\alpha \cdot u \cdot t'} e' \quad (4.2) \]
\[ e \xrightarrow{\alpha \cdot u \cdot t''} e'' \quad (4.3) \]

By (4.2) and the definition of \( \xrightarrow{\alpha \cdot u \cdot t'} \), we know

\[ e \xrightarrow{\alpha \cdot \mu'} e''' \quad (4.4) \]
\[ e''' \xrightarrow{\mu \cdot u'} e' \quad (4.5) \]
\[ t' = \mu'; u' \quad (4.6) \]

Similarly, by (4.3) and the definition of \( \xrightarrow{\alpha \cdot u \cdot t''} \), we know

\[ e \xrightarrow{\alpha \cdot \mu''} e''' \quad (4.7) \]
\[ e''' \xrightarrow{\mu \cdot u''} e' \quad (4.8) \]
\[ t'' = \mu''; u'' \quad (4.9) \]

By (4.1), (4.4), (4.7) and Single Step Determinism (Lemma 4.5), we know

\[ \mu' = \mu'' \quad (4.10) \]
\[ e''' = e''' \quad (4.11) \]
\[ e''' \in \text{Enf}_{\text{wf}} \quad (4.12) \]
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By (4.5), (4.8), (4.11), (4.12) and IH we know

\[ e' = e'' \quad (4.13) \]
\[ t' = t'' \quad (4.14) \]

Hence, from (4.6), (4.9), (4.10) and (4.14) we conclude

\[ t' = (\mu'; u') = (\mu''; u'') = t'' \quad (4.15) \]

\[ \therefore \text{ Case holds by (4.13) and (4.15).} \]

\[ \text{−−−−−−− ◦ −−−−−−−} \]

\[ \text{−−−−−−− End of Proof. −−−−−−−} \]

4.2.1.2 Proving Strong Enforceability by the Synthesized Enforcers

In order to prove that our synthesised deterministic enforcers are capable of strongly enforcing the formula they were derived from, we prove Theorem 4.3.

**Theorem 4.3 (Strong Enforcement).** For every satisfiable formula, \( \varphi \in \text{Sat} \), that is also in normal form, i.e., \( \varphi \in \text{sHML}_{\text{nf}} \),

\[ \langle \varphi \rangle \text{ strongly enforces } \varphi \]

In order to prove Theorem 4.3, by the definition of Strong Enforcement (Definition 3.5) we must prove the following lemmas:

**Lemma 4.6 (Enforcement Soundness).**

\[ \forall p \in \text{PROC} : \langle \varphi \rangle [p] \in [\varphi] \]

**Lemma 4.7 (Enforcement Transparency).**

\[ \forall p \in \text{PROC} : p \in [\varphi] \text{ implies } \langle \varphi \rangle [p] \sim p \]

Proving these two lemmas requires making use of Lemma 4.8, thus allowing us to work up to optimized formulae, i.e., \( \psi \in \text{sHML}_{\text{nf}}^{\text{opt}} \).

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Lemma 4.8. For every normalized formula $\varphi \in \text{sHML}_{nf}^{\text{opt}}$

$$\text{opt}(\varphi) = \psi \implies \psi \equiv \varphi \quad \text{and} \quad \psi \in \text{sHML}_{nf}^{\text{opt}}$$

where

$$\varphi, \psi \in \text{sHML}_{nf}^{\text{opt}} \iff X \mid \text{ff} \mid \text{tt} \mid \bigwedge_{i \in Q} [\eta_i] \varphi_i \text{ where } \#_{i \in Q} \eta_i \mid \text{max } X . \varphi \text{ where } X \in \text{fv}(\varphi)$$

Lemma 4.8 states that semantics are preserved when a normalized formula $\varphi \in \text{sHML}_{nf}^{\text{opt}}$ is optimized into $\psi \in \text{sHML}_{nf}^{\text{opt}}$, where $\psi$ is said to be optimized when every fixpoint variable $X$, that is bound to a maximal fixpoint $\text{max } X . \varphi$, is used at least once in the continuation formula $\varphi$. We prove this lemma by structural induction on $\varphi$ in Appendix Section C.2.3.

Moreover, in order to facilitate our proofs we use an alternative satisfaction semantics for sHML as explained below.

**Alternative sHML Semantics**  An alternative semantics for sHML was presented by Aceto et. al in [4, 5] in terms of a satisfaction relation, $\models_s$. When restricted to sHML, $\models_s$ is the largest relation satisfying the implications defined in Figure 4.14.

- $p \models_s \text{tt}$ always
- $p \models_s \text{ff}$ never
- $p \models_s \bigwedge_{i \in Q} \varphi_i$ whenever $p \models_s \varphi_i$ for all $i \in Q$
- $p \models_s [\eta] \varphi$ whenever $(\forall q \cdot p \xrightarrow{\alpha} q$ and $\eta(\alpha) = \sigma)$ implies $q \models_s \varphi$
- $p \models_s \text{max } X . \varphi$ whenever $p \models_s \varphi\{\text{max } X . \varphi/X\}$

Figure 4.14: A Satisfaction Relation for sHML formulae

The satisfaction relation states that truth, tt, is always satisfied, while falsehood, ff, can never be satisfied. Conjunctions, $\bigwedge_{i \in Q} \varphi_i$ are satisfied when all branches are satisfied (i.e., $\forall i \in Q$ such that $p \models_s \varphi_i$), while necessities, $[\eta] \varphi$, are satisfied by a process $p$ when all derivatives $q$ that are reachable over an action $\alpha$ where $\eta(\alpha) = \sigma$ (possibly none), also satisfy $\varphi\sigma$, i.e., $q \models_s \varphi\sigma$. Finally, a process $p$ satisfies a maximal fixpoint $\text{max } X . \varphi$ when it is also able to satisfy an unfolded version of $\varphi$, i.e., $p \models_s \varphi\{\text{max } X . \varphi/X\}$.  

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The authors proved that these satisfaction semantics, \( p \models_s \varphi \), correspond to the denotational semantics of the \( sHML \) subset of \( \muHML \), \( [\varphi] \), presented in Figure 2.4, such that \( p \models_s \varphi \) can be used in lieu of \( p \in [\varphi] \) (see [4, 5] for more detail).

**Proving Enforcement Soundness** In order to prove that our synthesised enforcers are *sound* we must show that Lemma 4.6 holds, i.e., that

\[
\forall p \in \text{Proc}, \varphi \in sHML_{nf} \text{ when } \varphi \in \text{Sat} \cdot ([\varphi]) [p] \in [\varphi]
\]

Since \( p \in [\varphi] \equiv p \models_s \varphi \), and by the definition of \( [\cdot] \), we get

\[
\forall p \in \text{Proc}, \varphi \in sHML_{nf} \text{ when } \varphi \in \text{Sat} \cdot ([\text{opt}(\varphi)])_\perp [p] \models_s \varphi
\]

Since by Lemma 4.8 we know that \( \text{opt}(\varphi) \) implies \( \varphi \equiv \psi \) and \( \psi \in sHML_{\text{opt} \, nf} \), we can thus prove

\[
\forall p \in \text{Proc}, \psi \in sHML_{\text{opt} \, nf} \text{ when } \psi \in \text{Sat} \cdot ([\psi])_\perp \{e/\perp\} [p] \models_s \psi
\]

Since \( \{\cdot\}_\perp \) does not yield a result, by definition of \( [\cdot]_\perp \) we know that \( \perp \) cannot be added to the resultant enforcer and hence substitution \( \{e/\perp\} \) is negligible, meaning that \( ([\varphi])_\perp \{e/\perp\} \) is equivalent to \( ([\varphi])_\perp \). Hence, we can instead prove

\[
\forall p \in \text{Proc}, e \in \text{Enf}_{\text{wf}}, \psi \in sHML_{\text{opt} \, nf} \text{ when } \psi \in \text{Sat} \cdot ([\psi])_\perp \{e/\rho\} [p] \models_s \psi
\]

This, however, allows us to prove a stronger result ranging over all possible \( \rho \) (i.e., where \( \rho = \perp \) is a specific instance), as follows

\[
\forall \rho, p \in \text{Proc}, e \in \text{Enf}_{\text{wf}}, \psi \in sHML_{\text{opt} \, nf} \text{ when } \psi \in \text{Sat} \cdot ([\psi])_\rho \{e/\rho\} = e' \text{ implies } e'[p] \models_s \psi
\]

We prove this result coinductively by showing that there exists a *satisfaction relation* \( \mathcal{R} \) (i.e., a relation that conforms to the rules given in Figure 4.14) such that for every optimized formula \( \varphi \in sHML_{\text{opt} \, nf} \) we cab show that the enforced process \( e'[p] \) and formula \( \psi \) are related by \( \mathcal{R} \) (defined below), meaning that process \( e'[p] \) satisfies \( \psi \).

\[
\mathcal{R} \equiv \{(e'[p], \psi) \mid \psi \in \text{Sat} \text{ and } \forall e \in \text{Enf}_{\text{wf}} \cdot ([\psi])_\rho \{e/\rho\} = e'\}
\]
4. Enforcing Safety Properties via Suppressions

The proof for this Soundness result is provided in Appendix Section C.2.1.

Proving Enforcement Transparency To prove that the synthesised enforcers are also transparent we must prove that Lemma 4.7 holds, i.e., that

$$\forall p \in \text{PROC}, \varphi \in \text{sHML}_{nf} \cdot p \models [\varphi] \implies (\varphi)[p] \sim p$$

Since $$p \models [\varphi] \equiv p \models_s \varphi$$, and by the definition of $$\models_s$$, we get

$$\forall p \in \text{PROC}, \varphi \in \text{sHML}_{nf} \cdot p \models_s \varphi \implies (\text{opt}(\varphi))[p] \sim p$$

Since by Lemma 4.8 we know that $$\text{opt}(\varphi)$$ implies $$\varphi \equiv \psi$$ and $$\psi \in \text{sHML}_{nf}^{\text{opt}}$$, we can instead prove

$$\forall p \in \text{PROC}, \psi \in \text{sHML}_{nf}^{\text{opt}} \cdot p \models_s \psi \implies (\psi)[p] \sim p$$

For all $$e \in \text{Enf}_{wf}$$ we know that $$(\varphi) \perp \{e/\perp\} \equiv (\varphi) \perp$$, since by definition of $$(\perp)$$, we know that $$\perp$$ cannot be added to the resultant enforcer since $$(\text{ff}) \perp$$ does not yield a result, and hence substitution $$\{e/\perp\}$$ is negligible, such that, we can instead prove

$$\forall p \in \text{PROC}, e \in \text{Enf}_{wf}, \psi \in \text{sHML}_{nf}^{\text{opt}} \cdot p \models_s \psi \implies (\psi)[p] \perp \{e/\perp\} \sim p$$

This, however, allows us to prove a stronger result ranging over all possible $$\rho$$ (i.e., where $$\rho = \perp$$ is a specific instance), as follows

$$\forall \rho, p \in \text{PROC}, e \in \text{Enf}_{wf}, \psi \in \text{sHML}_{nf}^{\text{opt}} \cdot p \models_s \psi \text{ and } (\psi)[p] \rho \{e/\rho\} = e' \implies e'[p] \sim p$$

We prove this result in a coinductive manner by showing that there exists a bisimulation relation $$\mathcal{R}$$ (defined below), such that for every optimized formula $$\varphi \in \text{sHML}_{nf}^{\text{opt}}$$, we can show that processes $$e'[p]$$ and $$p$$ are related by relation $$\mathcal{R}$$, and are thus bisimilar.

$$\mathcal{R} = \quad \{ (e'[p], p) \mid p \models_s \psi \text{ and } \forall e \in \text{Enf}_{wf} : (\psi)[p] \rho \{e/\rho\} = e' \}$$

The proof for this Transparency result is provided in Appendix Section C.2.2.
4. Enforcing Safety Properties via Suppressions

4.3 Concluding Remarks

In this chapter we have investigated the enforceability of safety properties via suppressions. Since the sHML subset was proven to be maximally expressive \textit{wrt. safety properties} [49], we have focussed on defining a novel mapping between the sHML subset and suppression enforcers in the form of a synthesis function. To reduce the complexity of synthesising deterministic enforcers, we assume a syntactic restriction that limits the domain of our synthesis function to normalized formulae. Despite this restriction, we have also produced a novel algorithm (based on [3]) capable of converting any formula $\varphi \in \text{sHML}$ into a semantically equivalent formula $\psi$ which is also in \textit{normal form}, \textit{i.e.}, $\psi \in \text{sHML}_{nf}$. This result advocates that our restricted syntax, sHML$_{nf}$, is still as expressive as the full (unrestricted) syntax of sHML.
5. Related Work

In this section we review the state-of-the-art related to runtime enforcement and enforceability and compare it to our work.

5.1 Enforceability

In general, the term enforceability refers to the relationship between the meaning of a property and its ability to be forcibly imposed upon a system at runtime. In our work, we investigate this relationship wrt. properties that are expressed as $\mu$HML formulae, and as a result we identify a syntactic subset of the logic that allows for defining properties that can be enforced at runtime. Through our synthesis function, we also maintain a clear separation between the declarative specification, i.e., the $\mu$HML logic, and the operational enforcement model, i.e., our enforcers.

Various works [81, 54, 44, 67] have presented a wide variety of definitions for enforceability. However, unlike our work, most of these definitions do not distinguish between the specification and the enforcer, thereby requiring the property to be expressed in terms of the enforcement mechanism itself.

For instance, when Schneider first explored enforceability in [81], he stated that a property is enforceable if its violation can be detected by a truncation automaton which in turn prevents it by terminating the system. In this setting, properties thus require to be specified in terms of the enforcement model itself, i.e., as a truncation
automaton. Furthermore, since this enforcement model can only prevent the occurrence of misbehaviour via system termination, the set of enforceable properties is thus limited to just safety properties.

As a continuation of Schneider’s work, Ligatti et al. in [54, 65, 63], sought to widen the set of enforceable properties by introducing edit automata i.e., an enforcement mechanism capable of suppressing and inserting system actions. Based on this enforcement model, Ligatti et al. introduced a new notion of enforceability stating that a property is enforceable if it can be expressed as an edit automaton that is able to transform an invalid system execution into a valid one.

Edit automata were shown to be capable of enforcing instances of safety and liveness properties, along with other properties such as infinite renewal properties, i.e., properties that are satisfied by infinite executions that have an infinite number of valid prefixes, and violated when an infinite execution has only a finite number of valid prefixes.

Based on these enforcement mechanisms, the authors define different notions of enforceability based on the criteria of

(i) enforcement soundness, i.e., an enforcer is sound when it always converts invalid executions into valid ones, and

(ii) transparency, i.e., a strongly transparent enforcer does not modify valid executions at all, while a weakly transparent one can only modify valid traces into semantically equivalent ones (for some notion of equivalence).

Similar to Schneider [81], in this setting, there exists no clear separation between the specification and the enforcement mechanism, such that the properties are required to be expressed in terms of the enforcement model itself, i.e., as edit automata. The authors thus state that a property is strongly enforceable if it can be expressed as an edit-automaton which enforces the said property in a sound and strongly transparent manner; a weaker notion for enforceability is similarly defined in terms of soundness and weak transparency.

The fundamental difference between these definitions for enforceability and ours
is that, we investigate enforceability \textit{wrt.} a logic, \textit{i.e.,} we identify which subsets of our logic, \textit{i.e.,} $\mu$HML, can be enforced \textit{soundly} and \textit{transparently}. Furthermore, since we define enforceability \textit{wrt.} the entire process, and not \textit{wrt.} just a single execution trace, we are able to prove stronger results for soundness and transparency. For instance, in the case of transparency, we can prove that in certain cases, when a process $p$ already satisfies formula $\varphi$, then the enforcer $e$ that is synthesised from $\varphi$, does not change the behaviour of $p$ in any way. We prove this result by showing that the enforced process $e[p]$ is \textit{strongly bisimilar} to the original process $p$. This result is thus stricter than trace equivalence constraint imposed by Ligatti \textit{et. al}’s strong transparency criteria.

Other researchers such as Fong \textit{[46]} and Talhi \textit{et. al} [84], investigated the enforceability of properties \textit{wrt.} enforcement automata with limited resources such as a bounded history of system events. This research shows that although some properties are enforceable in theory via unbounded enforcement automata, in practice it would require an infeasible amount of memory. They thus showed that, in practice, the level of enforceability of an enforcement automaton is relative to the bounds imposed by the available resources. This implies that although some properties may be theoretically enforceable, they might not be so when limited by practical constraints.

In their work, Bielova \textit{et. al} [18, 20, 21] remarked that the transparency constraint, previously introduced by Ligatti \textit{et. al}, only dictates how the monitor should react when analysing a \textit{valid} execution prefix, \textit{i.e.,} the meaning of valid executions should remain intact. However, neither soundness nor transparency specify the extent of the modifications that an enforcer should be able to apply over an invalid prefix.

The authors thus propose a notion of \textit{predictability} that restricts the enforcers from transforming the invalid executions in an arbitrary way. More precisely, an enforcer is \textit{predictable} if one can predict the number of transformations that the enforcer will apply in order to transform an invalid execution into a valid one.
5. Related Work

Predictability thus prevents enforcers from applying unnecessary transformations upon an invalid execution. Based on this notion they thus devise a more stringent notion of enforceability that states that a property is enforceable if there exists a sound, transparent and predictable enforcer.

By synthesising enforcers from a logic, we always produce enforcers that are, in some sense, predictable since the synthesised enforcers are designed to enforce properties of the same type, in the same way. For example, our syntheses converts safety property $[\alpha][\beta]\mathsf{ff}$ into a deterministic enforcer which enforces the property by continuously suppressing events that directly lead to a violation, i.e., in this case every occurrence of event $\beta$ that occurs after event $\alpha$ is thus suppressed. A similar approach is applied to other properties of the same type (i.e., safety) such as $\mathsf{max}X.[\alpha][[\beta]X \land [\gamma]\mathsf{ff}]$, where only the events preceding the violation, i.e., $\gamma$, are required to be suppressed when preceded by an $\alpha$ action and zero or more $\beta$ actions.

5.2 Separating the Property from the Enforcer

As remarked by Bielova in [18], most of the definitions for enforceability do not make a clear separation between the logic behind the property and the resp. enforcement mechanisms. In fact, the enforcement definitions we have seen so far in Section 5.1, do not make a distinction between the enforcement mechanism and the logic for specifying correctness properties, as they assume that properties are specified in terms of the resp. enforcement mechanism, e.g., as truncation automata, edit automata etc..

Various research [18, 44, 74, 67] has been conducted to introduce a synthesis phase which allows for expressing properties in terms of a more abstract model from which the required enforcer is then derived. This allows the specifier to focus on defining what should be enforced and not on how this should be done.

Most of this research, however, was conducted wrt. automata based specifi-
cations, such as Policy automata [18] and Streett automata [44], unlike in our case where we study the enforceability of branching time logic formulae. Mapping automata-based specifications to enforcement automata is arguably easier to attain compared to mapping a logic to an enforcer, mainly since both the specification and the enforcement models are defined as variants of automata.

Bielova in [18] thus proposes an algorithm for automatically constructing enforcement automata from policy automata, i.e., an automata based representation that combines the acceptance states of Büchi Automata and Finite State Automata. This therefore contrasts with our synthesis function which derives the required enforcers directly from branching-time logic formulae.

Similarly, Falcone et. al in [42, 45, 44], propose a way how a variety of properties, defined in terms of Streett automata, can be mapped onto an enforcement automaton that enforces the resp. property. More precisely, Falcone et. al showed that most of the property classes defined within the Safety-Progress hierarchy [79] are enforceable, as they can be encoded as Streett automata and subsequently converted into enforcement automata.

Hence, as opposed to Ligatti et. al, Bielova and Falcone et. al separate the specification of the property from the enforcement mechanism. This was done by providing construction rules that convert properties expressed as Policy and Streett automata into the resp. enforcement automata.

In relation to the work by Falcone et. al, Pinisetty et. al [74, 73] studied the enforceability of safety and co-safety Timed Properties expressed as a variant of Timed Automata. Similar to our work, non-deterministic specifications must first be converted into deterministic ones.

Despite this step being conceptually similar to our normalization phase, since the properties considered are expressed as Timed Automata, the determination of such non-deterministic properties was thus attainable via the conventional determinization construction rules that are standard in automata theory. Conversely, the normalization of $\mu$HML specifications was only recently explored in [3] and
5. Related Work

required being heavily adapted to our setting.

Following the determinization step, the resultant deterministic timed automata are then projected onto the resp. enforcement mechanisms accordingly. Their work was also implemented as a prototype tool, using Python, which allowed for assessing the feasibility of their proposed timed enforcement.

More similar to our work, in [67, 68, 69], Martinelli et. al performed an initial study of the enforceability of a logic. Hence, they partially addressed this issue by devising a synthesis algorithm that converts formulae expressed in modal $\mu$-calculus [57] (a reformulation of $\mu$HML), into either a truncation, suppression, insertion or an edit automaton, as decided by the specifier.

This synthesis algorithm thus burdens the specifier with still having to manually deduce the appropriate enforcement automaton (if any) capable of enforcing the said formula, and with having to manually define a function which specifies the points where the synthesised automaton should perform the required enforcement. By contrast, we identify the subsets of $\mu$HML properties that can be enforced wrt. insertions and suppressions such that the specifier is completely relieved from additional manual work related to the enforcement mechanisms required for enforcing the specified property.

The presented synthesis algorithm is defined in terms of two techniques, namely, partial model checking [7] and satisfiability [83]. The former is used to simplify the formula and modify it according to the function defined by the specifier, while the latter is the main mechanism used for obtaining the required enforcer; this is achieved by finding a process which satisfies the modified formula obtained from partially model checking the original formula; this contrasts with the way we generate the required enforcers via a compositional analysis of the specified $\mu$HML formula.

In a multi-pronged verification approach, our synthesis may, however, benefit from their former technique, i.e., partial model checking, to statically verify the non-enforceable parts of the given formula and reduce it into an enforceable one
which can then be synthesised using our synthesis algorithm.

5.3 Monitorability

Monitorability amounts to the relationship between the meaning of a property and its ability to be recognized at runtime [48, 50]. As both runtime monitoring and enforcement are dynamic analysis techniques capable of observing the behaviour of a system, the differences between these two techniques is sometimes blurred such as in [35, 36, 30, 55].

However, in contrast to enforceability which requires imposing property compliance, a property is said to be monitorable when there exists a detection monitor that can issue a definitive verdict determining whether the system under scrutiny satisfies or violates the property.

Along the years, several variants for monitorability have been defined wrt. various criteria. Viswanathan et. al in [87] defined monitorable properties as a strict subset of safety properties defined over infinite execution traces. This restrictive definition came as the result of limiting the monitor’s detection capabilities to only detecting misbehaviour after observing a finite prefix of a potentially infinite execution trace. As a by-product, this monitoring limitation restricts the domain of monitorable properties to just safety properties.

Pnueli et. al in [78] defined a wider notion of monitorability which states that properties are monitorable only when a positive or a negative verdict can be issued by the monitor after analysing a finite execution trace. Specifically, a monitor should be able to issue a negative (resp. positive) verdict, for a safety (resp. co-safety) property \( \varphi \), only in cases when a finite prefix of an execution trace provides enough information from which the monitor can deduce that any possible continuation of the prefix still violates (resp. satisfies) property \( \varphi \). The authors also determined that once a verdict is issued by the monitor wrt. a finite prefix of the complete execution trace, this implicitly applies to the complete execution,
and hence, the monitoring becomes redundant and can therefore be halted.

Inspired by [78], Bauer et. al in [12, 15] formulated another definition for monitorability in which they adopted the notion of good and bad prefix from the field of model checking [58]. The authors conjecture that it is possible to detect a violation or satisfaction for properties describing infinite behaviour, by only observing a finite prefix of a potentially infinite execution. More specifically, a finite execution prefix is said to be a bad prefix (resp. good prefix) wrt. a property $\varphi$ if it provides enough information to conclude that property $\varphi$ was violated (resp. satisfied). A finite execution prefix is, however, said to be ugly if it does not provide the necessary information to draw either of these conclusions.

Based on these three types of prefixes, the authors define new semantics for Linear Time Logic (LTL) which reasons about finite traces. This variant of the logic was called LTL$_3$, for its ability of producing three verdicts, namely, $true$ ($\top$), $false$ ($\bot$) resp. indicating property satisfaction and violation, along with an inconclusive verdict (?). In this way, an LTL$_3$ property $\varphi$ is monitorable wrt. an execution trace $t$, if there exists a bad (or good) prefix of $t$ that violates (or satisfies) $\varphi$.

Bauer et. al further extend their work in [13, 14] by extending the monitoring semantics of LTL$_3$ in order to refine the inconclusive verdict (i.e., ?), into a more informative verdict. Hence, the authors proposed new LTL semantics defining a 4-valued truth-domain which allows for producing two concrete verdicts, namely $true$ and $false$, along with approximation results denoting presumably true and resp. presumably false. Despite still being inconclusive, the latter two approximation verdicts provide more information compared to the single inconclusive verdict defined in LTL$_3$. This is therefore ideal for providing approximate positive (resp. negative) verdicts for properties that are satisfied (resp. violated) by infinite executions.

For instance, a monitor can easily deduce the violation of a safety property after observing bad prefix of an infinite execution trace, however, determining satisfaction of the same property requires analysing the entire infinite sequence. Using the
new truth domain, monitors are thus able to issue a presumably true verdict for execution prefixes that do not violate a safety property, and conversely, a presumably false for finite execution prefixes that do not satisfy a co-safety property; in LTL3 the same inconclusive verdict (i.e., ?) is issued in both cases.

Falcone et al in [44] generalized the definition by Bauer et al by parameterizing it with various truth-domains. Using this definition they are able to identify cases when a truth domain is not expressive or fine-grained enough to monitor for some specific classes of properties pertaining to the safety-progress hierarchy [79, 28, 43]. Their new definition of monitorability is therefore based on the distinguishability of good and bad execution sequences. This alternative definition is able to better distinguish between inconclusive situations that a monitor might encounter while analysing a finite trace.

The definitions we have seen so far have always been explored wrt. linear-time logics or automata-based specifications, and execution traces. Francalanza et al in [49, 50], studied the monitorability of branching-time properties expressed in terms of Hennessy-Milner Logic with recursion (µHML) wrt. the entire computational structure of the process (i.e., not just traces).

Francalanza et. al thus defined their own definition of monitorability which states that a µHML property is monitorable, if for every process capable of executing a particular trace, there exists a monitor that can issue a positive or negative verdict by only analysing the executed trace. Similar to previous work by Pnueli et. al [78], once a verdict is issued by a monitor, this becomes irrevocable and hence the monitor must consistently provide the same verdict.

In relation to this notion of monitorability, the authors identify the maximally expressive of µHML that syntactically characterises the monitorable µHML formulae, i.e., they identify a syntactic restriction for µHML that allows for defining all the possible µHML formulae that are monitorable according to their notion of monitorability. The identified subset was thus termed as Monitorable HML (mHML) and consists in the union of Safe-HML (sHML) and Co-Safe-HML (cHML).
the name implies, Safe-HML can only be used for specifying safety properties, while Co-Safe-HML only allows for specifying co-safety (liveness) properties.

In this approach, the authors thus preserve the original semantics of the chosen logic, \( \mu \text{HML} \), and instead identify the monitorable subset of this logic thus keeping the meaning of the logic independent from the employed verification technique. This is therefore different from the body of work proposed by Bauer \textit{et. al} [12, 13, 14, 15] where they redefined the semantics of their chosen logic, \( \text{LTL} \), \text{wrt.} finite traces in order to appease the selected verification technique, \text{i.e., runtime monitoring.}

The authors also prove the existence of a relationship between the logic and the monitor’s verdicts. More precisely, the authors prove that for any process \( p \) which satisfies (resp. violates) an arbitrary \( \text{mHML} \) property \( \varphi \), there exists a detection monitor \( m \) that is able to detect the property satisfaction (resp. violation) by analysing a witness trace executed by \( p \), and thus issue an irrevocable positive (resp. negative) verdict as a result. In addition, the authors prove that the converse also holds, \text{i.e.,} that for any process \( p \) which can be monitored by a monitor \( m \) capable of issuing a positive or a negative verdict, there exists a \( \text{mHML} \) formula that can be either satisfied or violated by process \( p \).

On top of this, the authors also provide a synthesis function capable of analysing and monitorable \( \text{mHML} \) formula and deriving a monitor that is able to detect the violation or satisfaction of the property it was derived from, by issuing the appropriate verdict. Their theory is also supported by a runtime verification tool for monitoring Erlang programs called \texttt{detectEr} [51, 9, 27, 24, 8]. This tool is able to synthesise Erlang detection monitors from the monitorable subset of \( \mu \text{HML} \) identified in [49].
5. Related Work

5.4 Supervisory-Control

Supervisory-Control [80, 86] is a static analysis technique that ensures that a system always adheres to a specific correctness property by modifying its behaviour in a pre-deployment phase. More precisely, a synthesis function in the sense of supervisory-control, has the task of taking an existing program and reconstructing it in a way which conforms to the given correctness property [86]. In general, the modifications made to the system’s internal behaviour ensure that the system always conforms to the property by completely removing invalid execution sequences from the resultant modified system.

This differs from runtime enforcement since, instead of modifying the internal system behaviour pre-deployment, enforcers ensure that the system conforms to the property by performing on-the-fly modifications (i.e., event suppression or insertion) while the system executes, and not during synthesis. Hence, the enforcer ensures that the system behaves as specified by the property by wrapping around the system and acting as a proxy for its inputs and outputs. In fact, unlike supervisory-control, in enforcement the system under scrutiny is barely modified during synthesis and is generally treated as being a black box; this makes enforcement ideal in cases where the internal behaviour of the system is unknown pre-deployment.

Van Hulst et. al in [86] studied a variant of $\mu$HML wrt. supervisory-control by establishing a synthesis algorithm that produces a controlled system that complies to the specified property. Their synthesis is therefore designed to produce the controlled system by statically analysing a given system and reformulating its behaviour to keep it in line with some correctness property expressed as a formula of the logic variant. As an evaluation of their work, the authors prove that their synthesis always terminates, and that it adheres to a number of constraints, such as validity, maximality and controllability.

Validity can be seen as the equivalent of soundness in RE, as this constraint serves to ensure that the synthesised controlled system must always satisfy the
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resp. property. On the other hand, maximality is, in some sense, equivalent to transparency, since this criterion dictates that the synthesis should remove the least possible behaviour from the original system, such that any valid behaviour is left intact. In addition, controllability is used to ensure that the synthesis affect other behaviour that was not specified by the property.

Similar work on supervisory-control was also conducted in relation to other logics such as the modal $\mu$-Calculus [11, 72], LTL [88] and CTL* [37] amongst others.
6. Conclusion

In this report we outlined preliminary results towards developing theoretical foundations for understanding the enforceability of properties that are expressed in terms of a highly expressive logic. As stated in Section 1.1, in order to achieve this aim, we intend to address the following objectives:

**O1.** Developing an abstract model for runtime enforcers and define enforceability \( \mu \text{HML} \).

**O2.** Assess the correctness of our abstract enforcers.

**O3.** Assessing the implementability of our enforcers by redefining them into a more implementable version.

**O4.** Developing a prototype implementation for our enforcers and evaluating their feasibility.

Until now, we have conducted an initial investigation of our first two objectives by studying the ability to enforce \( \mu \text{HML} \) properties via event suppression; this allowed us to identify sHML as being the subset of \( \mu \text{HML} \) which is enforceable via sound and transparent suppression enforcers. We summarize our main novel contributions as follows:

- In Chapter 3 we have mainly addressed our first objective by defining:
  - *Symbolic Transformations*, that formally define a mapping mechanism
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for transforming a concrete system event into a (possibly) different one as specified by the transformation pattern;

– An LTS semantics for Enforcers, which formalise the structure and dynamic behaviour of enforcers, along with the interaction between the enforcer, \( e \) and the process under scrutiny, \( p \), in the form of the instrumented LTS, \( e[p] \); and

– A Formal Definition for Enforceability, defining the relationship between the meaning of a \( \mu \text{HML} \) property and its ability to be adequately enforced at runtime by an enforcer. With this definition we establish that an enforcer \( e \) strongly enforces a formula \( \varphi \) whenever it is able to do it in a sound and transparent manner. By defining the enforced system as an LTS, \( e[p] \), we were able to provide novel definitions Soundness and Transparency which are stronger than the classic definitions (see Section 3.4).

• In Chapter 4 we continued addressing objective \( O1 \) by:
  
  – Identifying sHML as being Suppression Enforceable thus denoting that every safety property expressible in \( \mu \text{HML} \) can be enforced via action suppression.
  
  – Defining a Normalization algorithm for reducing a symbolic sHML formula into a semantically equivalent formula that is in normal form, from which we can then synthesise deterministic enforcers; this algorithm was heavily inspired from the work in [3]; and by

  – Developing a Synthesis algorithm for converting normalized symbolic sHML formulae into suppression enforcers.

• In Chapter 4 we also contributed towards objective \( O2 \) by:

  – Guaranteeing semantic preservation for our normalization algorithm by proving that each normalization step preserves the original semantics of
the given formula (or system of equations); in Appendix Chapter B we provide the proofs of the lemmas used when proving semantic equivalence; and by

– Proving that the synthesised enforcers always behave deterministically and can always strongly enforce the formula they were synthesised from; in Appendix Chapter C we provide the proofs for the supporting lemmas used when proving the aforementioned results.

6.1 Future Work

We plan to extend this work along two different avenues, namely, (i) by expanding the work on objectives $O1$ and $O2$ by investigating a wider notion of enforceability, and (ii) by addressing objectives $O3$ and $O4$ by exploring the implementability and feasibility of our enforcers. To tackle (i) would require enlarging the subset of enforceable specifications by investigating the enforceability of $\mu$HML formulae wrt. action insertions. This will potentially require extending the formal model, proofs and other results obtained so far according to the new notion of enforceability.

Meanwhile, in (ii), we envision the development of another synthesis function that converts the subset of enforceable $\mu$HML specifications into enforcers that are defined in terms of a more implementable model. However, to ensure that these implementable enforcers are also well-behaved (i.e., deterministic, sound and transparent), it would require proving correspondence to the enforcers defined by the abstract enforcement model we have so far. Based on this new synthesis we aim to develop a runtime enforcement tool implementation that checks whether a given specification is enforceable or not, when possible, converts it into an equivalent enforceable property, and finally generates the required enforcer automatically. At the time of writing we are unaware of the existence of such a tool.
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A. Supporting Material

This Appendix Chapter contains extra supporting information and examples that may help in better understanding our contributions. Within this section we refer to the following processes which are also pictorially represented in Figure 2.3.

\[ p_1 = \text{rec } x.(i?\text{req}..i!\text{ans}x + i?\text{cls.nil}) \]

\[ q_1 = \text{rec } x.(i?\text{req}..i!\text{ans}.x + i?\text{req}.x + i?\text{cls.nil}) \]

\[ r_1 = \text{rec } x.(i?\text{req}..i!\text{ans}.(i?\text{req}..i!\text{ans}.x + i?\text{cls.nil}) + i?\text{cls.nil}) \]

\[ s_1 = \text{rec } x.(\tau.i?\text{req}..i!\text{ans}.x + i?\text{cls.nil}) \]

A.1 Bisimulation Game Characterisation

A Strong (resp. Weak) bisimulation game for LTS processes \( p \) and \( q \), is a turn-based game between two players, namely an attacker which aims to disprove the Strong (resp. Weak) bisimulation, and a defender which aims to prove the Strong (resp. Weak) bisimulation. The game is played in rounds, each of which considers a pair (a.k.a. configuration) of LTS processes \( (p_n, q_n) \). The game starts its first round with the initial pair being \( (p, q) \), and in each round the players change the current pair according to the following rules:

1. Given a pair \( (p, q) \), the attacker chooses either the first or second element, \textit{i.e.}, \( p \) or \( q \), along with a suitable action \( \mu \). Based on the choice, the attacker
must perform a transition over the selected action $\mu$, i.e., $p \xrightarrow{\mu} p'$ if $p$ is chosen, and $q \blacktriangleleft q'$ when $q$ is chosen.

2. To counter the attack, the defender must reply by making a transition over the same action $\mu$, by using the process which was not selected by the attacker, i.e., if the attacker chose $p$ and attacks with $p \xrightarrow{\mu} p'$, the defender must reply with a strong $\mu$-reduction $q \xrightarrow{\mu} q'$, when proving a Strong Bisimulation, and with a weak $\mu$-reduction $q \xrightarrow{\Leftrightarrow} q'$, when proving a Weak one.

3. After successfully defending an attack $p \xrightarrow{\mu} p'$, with a defensive Strong (resp. Weak) transition, the new pair becomes $(p', q')$. The game continues for another round using the same rules.

A play of the game consists in a maximal sequence of pairs constructed by the players according to the given rules, meaning that during a play an attacker must try every possible attack with the attempt to disprove the bisimulation, while the defender must defend for these attacks to prevent this. Hence, an attacker wins a finite play whenever it is able to issue an attack, e.g., $p \xrightarrow{\mu} p'$, for which the defender is unable to find a counting defensive move, i.e., $p \xrightarrow{\Leftrightarrow} q'$ in a Strong Bisation Game, and $p \xrightarrow{\Leftrightarrow} q'$ in a Weak Game; otherwise the defender wins. In an infinite play (i.e., when comparing infinite LTSs), the defender wins if the play is infinite, i.e., the attacker is bound to keep attacking infinitely with attacks that can be countered by the defender.

Whenever, one of the players, either the attacker or the defender, is able to provide a strategy that can always win the game, regardless of how the opposing player chooses its moves, then we say that the player has a universal winning strategy. Hence, two LTS processes $p$ and $q$ are said to be Strong (resp. Weak) bisimilar iff the defender has a universal winning strategy in the Strong (resp. Weak) bisimulation game which starts with the initial pair $(p, q)$, otherwise the processes are not bisimilar iff the attacker has a universal winning strategy.

The bisimulation games thus allow us to prove whether two processes are Strong (resp. Weak) bisimilar to each other or not. Using a strong bisimulation game we
A. Supporting Material

are able to find a strong bisimulation relation $R$ proving that processes $p_1$ and $r_1$ produce the same runtime behaviour, i.e., they both provide an answer for every request with the possibility of closing while waiting for requests.

**Example A.1.** Recall example Example 2.7, in this example we give an alternative way how to show that processes $p_1$ and $r_1$ from Figure 2.3 are Strongly Bisimilar via the strong bisimulation game characterization. With this game we must therefore obtain a Strong bisimulation relation $R$ that proves that these two processes exhibit the same behaviour.

**Proof for $p_1 \sim r_1$.** To prove that $p_1 \sim r_1$ we use the Strong bisimulation game, starting with the initial pair being $(p_1, r_1)$, to find a relation $R$ that observes Definition 2.8. We denote an attacker’s and defender’s move as $A:$ and $D:$ resp.

**Round 1:** We show $(p_1, r_1) \in R$ as,

\[
\begin{align*}
A: & \quad p_1 \xrightarrow{?\text{req}} p_2 \\
A: & \quad p_1 \xrightarrow{?\text{cls}} p_3 \\
A: & \quad r_1 \xrightarrow{?\text{req}} r_2 \\
A: & \quad r_1 \xrightarrow{?\text{cls}} r_5 \\
D: & \quad r_1 \xrightarrow{?\text{req}} r_2 \\
D: & \quad r_2 \xrightarrow{?\text{cls}} r_5 \\
D: & \quad p_1 \xrightarrow{?\text{req}} p_2 \\
D: & \quad p_1 \xrightarrow{?\text{cls}} p_3
\end{align*}
\]

In the next round we must show $(p_2, r_2) \in R$ and $(p_3, r_5) \in R$. 

\[
\quad - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
\]

**Round 2:** We show $(p_2, r_2) \in R$ as,

\[
\begin{align*}
A: & \quad p_2 \xrightarrow{?\text{ans}} p_1 \\
D: & \quad r_2 \xrightarrow{?\text{ans}} r_3 \\
A: & \quad r_2 \xrightarrow{?\text{ans}} r_3 \\
D: & \quad p_2 \xrightarrow{?\text{ans}} p_1
\end{align*}
\]

The case for $(p_3, r_5) \in R$ holds immediately as neither $p_3$, nor $r_5$ can perform any further reductions. In the next round we must show $(p_1, r_3) \in R$. 

\[
\quad - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
\]
A. Supporting Material

Round 3: We show \((p_1, r_3) \in \mathcal{R}\) as,

\[
\begin{align*}
A: & \quad p_1 \xrightarrow{i?req} p_2 & D: & \quad r_3 \xrightarrow{i?req} r_4 \\
A: & \quad p_1 \xrightarrow{i?cls} p_3 & D: & \quad r_3 \xrightarrow{i?cls} r_5 \\
A: & \quad r_3 \xrightarrow{i?req} r_4 & D: & \quad p_1 \xrightarrow{i?req} p_2 \\
A: & \quad r_3 \xrightarrow{i?cls} r_4 & D: & \quad p_1 \xrightarrow{i?cls} p_3
\end{align*}
\]

In the next round we must show that \((p_2, r_4) \in \mathcal{R}\).

Round 4: We show \((p_2, r_4) \in \mathcal{R}\) as,

\[
\begin{align*}
A: & \quad p_2 \xrightarrow{i?ans} p_1 & D: & \quad r_4 \xrightarrow{i?ans} r_1 \\
A: & \quad r_4 \xrightarrow{i?ans} r_1 & D: & \quad p_2 \xrightarrow{i?ans} p_1
\end{align*}
\]

No further rounds are required for this game since we already know \((p_1, r_1) \in \mathcal{R}\) from Round 1.

Hence, given that the defender was always able to defend with a strong transition for every possible attack, we can conclude that there exists the following strong bisimulation relation \(\mathcal{R}\), such that by the definition of strong bisimilarity we can conclude that \(p_1 \sim r_1\), where:

\[
\mathcal{R} = \{(p_1, r_1), (p_2, r_2), (p_1, r_3), (p_2, r_4), (p_3, r_5)\}
\]

--- End of Proof. ---

Example A.2. In Section 2.4 we had argued that showing \(p_1 \not\sim q_1\) is in general very hard to deduce since we need to prove that all binary relations \(\mathcal{R}\) that can relate \(p_1\) and \(q_1\), are not a Strong Bisimulation relation. A more practical alternative is to resort to the bisimulation games. In the following proof we will use the Weak Bisimulation Games to show that \(p_1 \not\approx q_1\), which inherently implies that \(p_1 \not\sim q_1\).

Proof that \(p_1 \not\approx q_1\). To prove that \(\not\approx\) we use the Weak Bisimulation Game,
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starting with the initial pair being \((p_1, q_1)\).

**Round 1:** We show \((p_1, q_1) \in \mathcal{R}\) as,

\[
\begin{array}{ll}
A: & p_1 \xrightarrow{i?\text{req}} p_2 \\
D: & q_1 \xrightarrow{i?\text{req}} q_2 \\
A: & p_1 \xrightarrow{i?\text{cls}} p_3 \\
D: & q_1 \xrightarrow{i?\text{cls}} q_3 \\
A: & q_1 \xrightarrow{i?\text{req}} q_2 \\
D: & p_1 \xrightarrow{i?\text{req}} p_2 \\
A: & q_1 \xrightarrow{i?\text{cls}} q_3 \\
D: & p_1 \xrightarrow{i?\text{cls}} p_3
\end{array}
\]

In the next round we must show \((p_2, q_2) \in \mathcal{R}, (p_2, q_1) \in \mathcal{R}\) and \((p_3, q_3) \in \mathcal{R}\).

**Round 2:** We show \((p_3, q_3) \in \mathcal{R}\) holds immediately as neither \(p_3\) nor \(q_3\) can perform any further reductions. We also show \((p_2, q_2) \in \mathcal{R}\) as,

\[
\begin{array}{ll}
A: & p_2 \xrightarrow{\text{?\text{ans}}} p_1 \\
D: & q_2 \xrightarrow{\text{?\text{ans}}} q_1 \\
A: & q_2 \xrightarrow{\text{?\text{ans}}} q_1 \\
D: & p_2 \xrightarrow{\text{?\text{ans}}} p_1
\end{array}
\]

However, the defender fails to counter the attacker’s attack when showing that \((p_2, q_1) \in \mathcal{R}\), since

\[
\begin{array}{ll}
A: & p_2 \xrightarrow{\text{?\text{ans}}} p_1 \\
D: & q_1 \xrightarrow{\text{?\text{ans}}}
\end{array}
\]

Hence, given that the defender was unable to defend for one of the attacks, by Definition 2.9 we can conclude that \(p_1 \not\approx p_2\), which inherently implies that \(p_1 \not\sim p_2\).

--- **End of Proof.** ---
A.2 An Iterative Fixpoint Derivation Example

Recall formula $\varphi_1$ from Example 2.8 (restated below),

$$
\varphi_1 = \max X.\varphi'_1 \quad \quad \varphi'_1 = [d?\mathrm{req}]\varphi''_1 \\
\varphi''_1 = \varphi'''_1 \land \varphi''''_1 \quad \quad \varphi''''_1 = [d!\mathrm{ans}]X \\
\varphi'''_1 = [d?\mathrm{req}]\mathrm{ff}
$$

We can thus prove that $p_1 \in [\varphi_1]$ as follows:

$$
[\varphi_1] = [\max X.\varphi'_1, \emptyset] = S \subseteq [\varphi'_1, [X \mapsto S]]
$$

To establish the maximal set $S$ of processes which satisfy $\varphi'_1$, we must iteratively evaluate $[[\varphi'_1, [X \mapsto S]]$ starting with $S = \text{PROC} = \{p_1, p_2, p_3, q_1, q_2, q_3\}$ until the resultant set $S$ stops changing with each iteration, therefore denoting that the maximal fixpoint has been reached.

**First Iteration** ($S = \text{PROC}$)

$$
[[\varphi'_1, [X \mapsto S]] = \{ p \mid \left( \forall v \in \text{VAL} \cdot p \xrightarrow{v?\mathrm{req}} p' \quad \text{and} \quad \{d?\mathrm{req}, d \neq j\}(v?\mathrm{req}) = \{v/d\} \right) \quad \text{implies} \quad p' \in [[[\varphi''_1{v/d}], [X \mapsto S]]
\]
$$

$$
[[\varphi''_1{v/d}, [X \mapsto S]] = [[[\varphi''_1{v/d}], [X \mapsto S]] \cap [[[\varphi'''_1{v/d}], [X \mapsto S]]
\]

$$

$$
[[\varphi'''_1{v/d}, [X \mapsto S]] = [[[d!\mathrm{ans}]X{v/d}, [X \mapsto S]] = [[[v!\mathrm{ans}]X, [X \mapsto S]]
\]

$$

= \left\{ p \left( p \xrightarrow{v!\mathrm{ans}} p' \quad \text{and} \quad \{v!\mathrm{ans}, t\}(v!\mathrm{ans}) = \emptyset \right) \quad \text{implies} \quad p' \in [[X \emptyset, [X \mapsto S]]
\]

= \left\{ p \left( p \xrightarrow{v!\mathrm{ans}} p' \quad \text{and} \quad \{v!\mathrm{ans}, t\}(v!\mathrm{ans}) = \emptyset \right) \quad \text{implies} \quad p' \in S \right\}
\]

= S = \text{PROC}
\]
A. Supporting Material

\[
\begin{align*}
\{\varphi'' \{v/d\}, [X \rightarrow S]\} &= \{[d?req] ff \{v/d\}, [X \rightarrow S]\} = \{[d?req] ff, [X \rightarrow S]\} \\
&= \left\{ \begin{array}{l}
p \left( \forall u \in \text{VAL} \cdot p \xrightarrow{v\text{req}} p' \text{ and } [d?req, t](v\text{req}) = \{u/f\} \implies p' \in \{\text{ff}\{u/f\}, [X \rightarrow S]\} \right) \\
&= \left\{ \begin{array}{l}
p \left( \forall u \in \text{VAL} \cdot p \xrightarrow{v\text{req}} p' \text{ and } [d?req, t](v\text{req}) = \{u/f\} \implies p' \in \emptyset \right) \\
&= \{p_2, p_3, q_2, q_3\} \\
\end{array} \right. \\
\{\varphi'' \{v/d\}, [X \rightarrow S]\} &= \text{PROC} \cap \{p_3, p_4, q_3, q_4\} \\
&= \{p_3, p_4, q_3, q_4\} \\
\{\varphi', [X \rightarrow S]\} &= \left\{ \begin{array}{l}
p \left( \forall v \in \text{VAL} \cdot p \xrightarrow{v\text{req}} p' \text{ and } [d?req, d\neq j](v\text{req}) = \{v/d\} \implies p' \in \{p_3, p_4, q_3, q_4\} \right) \\
S = \{p_1, p_2, p_3, q_2, q_3\} \\
\end{array} \right. \\
\end{align*}
\]

Second Iteration \( (S = \{p_1, p_2, p_3, q_2, q_3\}) \)

\[
\begin{align*}
\{\varphi', [X \rightarrow S]\} &= \left\{ \begin{array}{l}
p \left( \forall v \in \text{VAL} \cdot p \xrightarrow{v\text{req}} p' \text{ and } [d?req, d\neq j](v\text{req}) = \{v/d\} \implies p' \in \{\varphi'' \{v/d\}, [X \rightarrow S]\} \right) \\
\end{array} \right. \\
\{\varphi'' \{v/d\}, [X \rightarrow S]\} &= \{\varphi'' \{v/d\}, [X \rightarrow S]\} \cap \{\varphi''' \{v/d\}, [X \rightarrow S]\} \\
\{\varphi''' \{v/d\}, [X \rightarrow S]\} &= \{[d!\text{ans}]X \{v/d\}, [X \rightarrow S]\} = \{[v!\text{ans}]X, [X \rightarrow S]\} \\
&= \left\{ \begin{array}{l}
p \left( \begin{array}{l}
p \xrightarrow{v!\text{ans}} p' \text{ and } [v!\text{ans}, t](v\text{ans}) = \emptyset \end{array} \right) \implies p' \in [X\emptyset, [X \rightarrow S]] \\
&= \left\{ \begin{array}{l}
p \left( \begin{array}{l}
p \xrightarrow{v!\text{ans}} p' \text{ and } [v!\text{ans}, t](v\text{ans}) = \emptyset \end{array} \right) \implies p' \in S \\
S = \{p_1, p_2, p_3, q_2, q_3\} \\
\end{array} \right. \\
\end{align*}
\]
A. Supporting Material

\[ \varphi'''' \{ v/d \}, [X \mapsto S] = \varphi''' \{ v/d \}, [X \mapsto S] = \varphi'' \{ v/d \}, [X \mapsto S] \]

\[ = \left\{ p \mid \left( \forall u \in \text{VAL} \cdot p \xrightarrow{v?req} p' \text{ and } \{d?req,t\}(v?req) = \{u/f\} \right) \text{ implies } p' \in \{f\} \right\} \]

\[ = \left\{ p \mid \left( \forall u \in \text{VAL} \cdot p \xrightarrow{v?req} p' \text{ and } \{d?req,t\}(v?req) = \{u/f\} \right) \text{ implies } p' \in \emptyset \right\} \]

\[ = \{p_2, p_3, q_2, q_3\} \]

\[ \varphi'' \{ v/d \}, [X \mapsto S] = \{p_1, p_2, p_3, q_2, q_3\} \cap \{p_3, p_4, q_3, q_4\} \]

\[ = \{p_3, p_4, q_3, q_4\} \]

\[ \varphi' \{ X \mapsto S \} \]

\[ = \left\{ p \mid \left( \forall v \in \text{VAL} \cdot p \xrightarrow{v?req} p' \text{ and } \{d?req, d \neq j\}(v?req) = \{v/d\} \right) \text{ implies } p' \in \{p_3, p_4, q_3, q_4\} \right\} \]

\[ S = \{p_1, p_2, p_3, q_2, q_3\} \]

Notice how so far in our derivation, the set of processes \( S \) has changed from Proc to \( \text{Proc}\setminus\{q_1\} \) after the first iteration, but has remained the same \( \text{i.e., } \text{Proc}\setminus\{q_1\} \) after the second one. Hence, by the definition of \( \varphi_1 \), we can conclude:

\[ \varphi_1 \equiv \varphi_1 \]

\[ = \{p_1, p_2, p_3, q_2, q_3\} \]
As explained in Section 4.1.2, the algorithm for normalizing any sHML formula (i.e., defining any type of symbolic event and not just singleton events) amounts to the sequential application of the following 6 steps:

Step 1. For each sHML formula \( \varphi \) a *semantically equivalent* formula \( \varphi^{sf} \) that is in *Standard Form* can be obtained using \( \langle\langle \varphi \rangle\rangle_1 \).

- This construction was reviewed and explained wrt. symbolic events in Section 4.1.2.1 along with a proof advocating for its semantic preservation.

Step 2. For each sHML formula \( \varphi \) that is in *Standard Form*, an *equivalent* System of Equations \( SYS^{sf} \) that is in *Standard Form* can be obtained using \( \langle\langle \varphi^{sf} \rangle\rangle_2 \).

- We have discussed this construction wrt. symbolic events in Section 4.1.2.2 accompanied by a proof denoting that the resultant system of equations \( SYS^{sf} \) is semantically equivalent to the input formula \( \varphi^{sf} \).

Step 3. For each System of Equations \( SYS^{sf} \) that is in *Standard Form*, an *equivalent* System of Equations \( SYS^{uni} \) that is *Uniform* can be obtained using
B. Proving Semantic Preservation for Normalization

\[\langle\langle SYS_{\text{nf}}\rangle\rangle_4\].

- This construction was introduced and explained in Section 4.1.3.2, for which we proved that the obtained uniform system of equations \(SYS_{\text{uni}}\) is semantically equivalent to the given standard form system of equations.
- The proof provided makes use of two lemmas, namely, Lemma 4.1 and Lemma 4.2; we prove these lemmas in Section B.1.

Step 4. For each System of Equations \(SYS_{\text{uni}}\) that is \textit{Uniform}, an \textit{equivalent} System of Equations \(SYS_{\text{comb}}\) that is \textit{Equi-Disjoint}, can be obtained using \(\langle\langle SYS_{\text{uni}}\rangle\rangle_{(ii)}\).

- An introduction and explanation of this construction was given in Section 4.1.3.3. In the same section we also prove that our construction is guaranteed to preserve the semantics of the given Uniform system of equations.
- The proof guaranteeing Semantic Preservation for this construction step makes use of Lemma 4.3 which we now prove in Section B.2.

Step 5. For each System of Equations \(SYS\) that is \textit{Equi-Disjoint} and also in \textit{Standard Form}, an \textit{equivalent} System of Equations \(SYS_{\text{comb}}\) that is in \textit{Normal Form}, can be obtained using \(\langle\langle SYS_{\text{comb}}\rangle\rangle_3\).

- This construction step was presented and explained \textit{wrt.} symbolic events in Section 4.1.2.3 accompanied by a proof sketch showing that the semantics of the original system of equations are preserved by the construction.

Step 6. For each System of Equations \(SYS\) that is \textit{Normal Form}, an \textit{equivalent} formula \(\varphi_{\text{nf}}\) that is also in \textit{Normal Form}, can be obtained using \(\langle\langle SYS_{\text{nf}}\rangle\rangle_4\).

- This construction step was explained in relation to symbolic events in Section 4.1.2.4, and a proof guaranteeing semantic preservation was
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also provided.

B.1 Auxiliary Lemmas for Proving Semantic Preservation of Construction \(\langle\langle-\rangle\rangle_0\)

In this section we prove the following lemmas that are required for guaranteeing the semantic preservation of construction \(\langle\langle-\rangle\rangle_0\).

Lemma 4.1. \(\text{traverse}(\text{Eq}, \{0\}, \text{partition}, \emptyset)=\zeta\) implies \(\zeta\) is a well-formed map for Eq.

Lemma 4.2. \(\forall (X_j=\varphi_j) \in \text{Eq} \) equation \(X_j=\varphi_j\) is in Standard form, and \(\zeta\) is a well-formed map for Eq implies \(\text{uni}(\text{Eq}, \zeta)\equiv \text{Eq}\) and \(\forall (X_k=\psi_k) \in \text{uni}(\text{Eq}, \zeta)\) equation \((X_k=\psi_k)\) is Uniform.

B.1.1 Proving Lemma 4.1.

In order to prove Lemma 4.1, we use Lemma B.1. This new lemma states that a well-formed \(\zeta'\) map for Eq is obtained upon performing a partition traversal on a subset, \(\text{Eq}'\), of the given equation set \(\text{Eq}\), using an arbitrary \(\zeta\) map that is well-formed wrt. a subset of \(\text{Eq}\) that is restricted to the indices defined by the domain of \(\zeta\), i.e., \(\text{Eq}//\text{dom}(\zeta)\).

Lemma B.1. \(\forall Q, \zeta \cdot \text{Eq}' \subseteq \text{Eq} \) and \(\text{traverse}(\text{Eq}', Q, \text{partition}, \zeta)=\zeta'\) and \(\zeta\) is a well-formed map for \(\text{Eq}//\text{dom}(\zeta)\) implies \(\zeta'\) is a well-formed map for Eq.

The proof for this lemma is provided at the end of this section.

To Prove Lemma 4.1.

\(\text{traverse}(\text{Eq}, \{0\}, \text{partition}, \emptyset)=\zeta\) implies \(\zeta\) is a well-formed map for Eq.

Proof. Initially we know

\(\text{traverse}(\text{Eq}, \{0\}, \text{partition}, \emptyset)=\zeta\) \hspace{1cm} (B.1)
B. Proving Semantic Preservation for Normalization

By the definition of $\text{Eq}_{//Q}$ we know

$$\text{Eq}_{//\text{dom}(\emptyset)} = \emptyset$$  \hspace{1cm} (B.2)

By (B.2) and the definition of a *well-formed* map we know

$$\emptyset \text{ is a Well-formed map for } \text{Eq}_{//\text{dom}(\emptyset)}$$ \hspace{1cm} (B.3)

By (B.1), (B.3) and Lemma B.1 we know

$$\zeta \text{ is a well-formed map for Eq}$$

--- End of Proof. ---

To Prove Lemma B.1.

$\forall Q, \zeta \cdot \text{Eq}' \subseteq \text{Eq}$ and $\text{traverse}(\text{Eq}', Q, \text{partition}, \zeta) = \zeta'$ and $\zeta$ is a well-formed map for $\text{Eq}_{//\text{dom}(\zeta)}$ implies $\zeta'$ is a well-formed map for $\text{Eq}$.

**Proof.** *By induction on the structure of Eq'.*

**Case Eq' = \emptyset:** Initially we know

$$\text{traverse}(\emptyset, Q, \text{partition}, \zeta) = \zeta'$$ \hspace{1cm} (B.1)

$$\zeta \text{ is a well-formed map for } \text{Eq}_{//\text{dom}(\zeta)}$$ \hspace{1cm} (B.2)

$$\emptyset \subseteq \text{Eq}$$ \hspace{1cm} (B.3)

Since $\text{Eq}' = \emptyset$, by (B.1) and the definition of $\text{traverse}$ we know

$$\zeta = \zeta'$$ \hspace{1cm} (B.4)

By (B.2) and (B.4) we know

$$\zeta' \text{ is a well-formed map for } \text{Eq}_{//\text{dom}(\zeta')}$$ \hspace{1cm} (B.5)
By (B.1) and the definition of traverse, we know that the traversal starts from the full equation set, \( i.e., Eq' = Eq \), using an empty \( \zeta \) map. With every recursive application of traverse, the equation set \( Eq' \) becomes smaller since when traverse recurses it does so \( w.r.t. Eq'' \), \( i.e., \) a smaller version of the current \( Eq' \) which is computed via \( Eq'' = Eq' \setminus Eq_{/Q} \). By contrast, with every recursive application of traverse, the \( \zeta \) accumulator becomes larger as it is updated with new mappings for each index specified by the set of indices \( Q \) \( i.e., \) with the indices of the equations that are removed from \( Eq' \) when creating \( Eq'' \). Hence, when the traverse function is recursively applied \( w.r.t. \) some \( Eq'' = \emptyset \), it means that all the equations specified in \( Eq \) have been analysed by the traversal and their indices were thus added as maps in the resultant \( \zeta' \). Hence, we can deduce

\[
Eq_{/\text{dom}(\zeta')} = Eq \tag{B.6}
\]

Finally, by (B.5) and (B.6) we conclude

\[
\zeta' \text{ is a well-formed map for } Eq. \tag{B.7}
\]

\[\text{---} - - - - - - - - - - -\]

**Case \( Eq' \neq \emptyset \):** Initially we know

\[
\text{traverse}(Eq', Q, \text{partition}, \zeta) = \zeta' \tag{B.8}
\]

\[
\zeta \text{ is a well-formed map for } Eq_{/\text{dom}(\zeta)} \tag{B.9}
\]

\[
Eq' \subseteq Eq \tag{B.10}
\]

We consider two subcases:

- \( Q = \emptyset \): Since \( Q = \emptyset \), by (B.8) and the definition of traverse we know

\[
\zeta = \zeta' \tag{B.11}
\]
By (B.9) and (B.11) we know

\[ \zeta' \text{ is a well-formed map for } Eq_{/\text{dom}(\zeta')} \]  \hspace{1cm} (B.12)

Since \( Q = \emptyset \), this means that the traversal has reached a point where no more children can be computed, which means that all the relevant equations (i.e., those reachable from the principle variable) have been analysed. This means that any other equation in \( Eq \) (that is not in \( Eq_{/\text{dom}(\zeta')} \), if any) is redundant and irrelevant. Hence, since from (B.12) we know that \( \zeta'' \) is a well-formed map for the relevant subset of equations in \( Eq \), i.e., \( Eq_{/\text{dom}(\zeta')} \), then it is also well-formed for the full blown subset of equations \( Eq \) (i.e., including any unreachable, redundant equations). Therefore, we can conclude

\[ \zeta' \text{ is a well-formed map for } Eq. \]

- \( Q \neq \emptyset \) : By (B.8) and the definition of traverse we know

\[ \zeta'' = \text{partition}(Eq', Q, \zeta) \]  \hspace{1cm} (B.13)

\[ Eq'' = Eq' \setminus Eq'_{/Q} \]  \hspace{1cm} (B.14)

\[ Q' = \bigcup_{j \in Q} \text{child}(Eq', j) \]  \hspace{1cm} (B.15)

\[ \text{traverse}(Eq'', Q', \text{partition}, \zeta'' ) = \zeta' \]  \hspace{1cm} (B.16)

From (B.14) and (B.10) we can deduce

\[ Eq'' \subseteq Eq \]  \hspace{1cm} (B.17)

By (B.9) and the definition of a well-formed map we know that \( \zeta \) provides a
set of mappings which allow for:

- renaming the data variables of each pattern equivalent sibling necessity, defined in $Eq_{//\text{dom}(\zeta)}$, to the same set of fresh variables. \hfill (B.18)

- renaming any reference to a data variable that is bound by a renamed parent necessity defined in $Eq_{//\text{dom}(\zeta)}$. \hfill (B.19)

By (B.13) and the definition of partition we know

$$\zeta'' = \zeta \cup \left\{ \begin{array}{ll}
    j \mapsto \zeta(i) \uparrow \{ g^n/d^n \} & \forall i, l \in Q \cdot Eq'(i) = \bigwedge_{j \in Q''} [\eta_j(d^n)] X_j \land \varphi \\
    k \mapsto \zeta(l) \uparrow \{ g^n/f^n \} & \text{and } Eq'(l) = \bigwedge_{k \in Q'''} [\eta_k(f^n)] X_k \land \varphi \text{ s.t. if } \eta_j(d^n) \text{ is pattern equivalent to } \eta_k(f^n), \text{ then we assign the same set of fresh variables } g^n.
\end{array} \right\}$$

From (B.20) we know that $\zeta''$ includes a mapping for each sibling branch that defines a pattern equivalent necessity. The added mappings map the child indices (i.e., $j, k \in Q'$ since by (B.15) we know that $Q''$ and $Q'''$ are subsets of $Q'$) of the conjunction branches, defined by equations identified by the parent indices (i.e., $i \in Q$) specified in $Q$, to a substitution environment which renames the resp. variable names of these conjunct pattern equivalent sibling necessities, to the same fresh set of variable names, thereby making the equivalent sibling patterns, syntactically equal. Hence, by (B.18) we can deduce that $\zeta''$ provides a set of mappings which allow for

- renaming the data variables of each pattern equivalent sibling necessity, defined in $Eq_{//\text{dom}(\zeta) \cup Q'}$, to the same set of fresh variables. \hfill (B.21)

Similarly, from (B.20) we also know that the mappings in $\zeta''$ include the substitutions performed upon the parent necessities, i.e., in each mapping $j \mapsto \sigma_j$, the mapped substitution environment $\sigma_j$ also includes $\zeta(i)$ where
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$i \in Q$ is the parent index of $j \in Q'$. Hence, by (B.19) we can deduce that the mappings provided by $\zeta''$ also allow for

- renaming any reference to a data variable that is bound by a renamed parent necessity defined in $Eq_{//\text{dom}(\zeta) \cup Q'}$.

(B.22)

Hence, by (B.21), (B.22) and the definition of a well-formed map we know

$$\zeta'' \text{ is a well-formed map for } Eq_{//\text{dom}(\zeta) \cup Q'}.$$  

(B.23)

From (B.20) we know that $\zeta''$ includes a mapping for each child branch, identified by $j \in Q''$ and $k \in Q'''$ (where $Q''$ and $Q'''$ are both subsets of $Q'$), that is defined in the equation identified by index $i \in Q$ and which defines a pattern equivalent necessity. Hence, we know that the domain of $\zeta''$ is an extension of the domain of $\zeta$ which additionally contains the child indices defined in $Q'$, such that we can deduce

$$\text{dom}(\zeta'') = \text{dom}(\zeta) \cup Q'$$  

(B.24)

Therefore, from (B.23) and (B.24) we can infer

$$\zeta'' \text{ is a well-formed map for } Eq_{//\text{dom}(\zeta'')}.$$  

(B.25)

Finally, by (B.16), (B.17), (B.25) and IH we can conclude

$$\zeta' \text{ is a well-formed map for } Eq.$$  

---

End of Proof. ---

B.1.2 Proving Lemma 4.2.

$\forall (X_j=\varphi_j) \in Eq$ equation $X_j=\varphi_j$ is in Standard form, and $\zeta$ is a well-formed map for $Eq$ implies $\forall (X_k=\psi_k) \in \text{uni}(Eq, \zeta)$ equation $(X_k=\psi_k)$ is Uniform and
Proof. By induction on the structure of Eq.

Case Eq = ∅: This case holds trivially since Eq = ∅ = uni(∅, ζ).

Case Eq = \{X_i=\bigwedge_{j \in Q} [o_j(d^m_j), c_j(f^m_{<i})]|\varphi_j \land \varphi}\bigcup Eq': Initially we know
\[∀(X_k=\varphi_k) \in \{X_i=\bigwedge_{j \in Q} [o_j(d^m_j), c_j(f^m_{<i})]|\varphi_j \land \varphi}\bigcup Eq'.\]
equation X_k=\varphi_k is in Standard Form. (B.1)

ζ is a well-formed map for Eq (B.2)

Since Eq' ⊂ Eq from (B.1) we know

Equation X_i=\bigwedge_{j \in Q} [o_j(d^m_j), c_j(f^m_{<i})]|\varphi_j \land \varphi is in Standard Form. (B.3)

∀(X_k=\varphi_k) \in Eq' • equation X_k=\varphi_k is in Standard Form. (B.4)

Since Eq' ⊂ Eq from (B.2) we know

ζ is a well-formed map for Eq' (B.5)

Hence by (B.4), (B.5) and IH we know

∀(X_k=\psi_k) ∈ uni(Eq', ζ) • equation (X_k=\psi_k) is Uniform. (B.6)

\[\text{uni}(Eq', ζ)≡Eq'.\] (B.7)

By applying the uni function on Eq and ζ we obtain

\[\text{uni}(\{X_i=\bigwedge_{j \in Q} [o_j(d^m_j), c_j(f^m_{<i})]|\varphi_j \land \varphi\bigcup Eq', \zeta)\]

= \{X_i=\bigwedge_{j \in Q} [o_j(d^m_j), c_j(f^m_{<i})]|\zeta(j)|\varphi_j \land \varphi\bigcup \text{uni}(Eq', \zeta)\} (B.8)

By (B.2) and the definition of a well-formed map we know that ζ provides a set of
mappings which allow for

- renaming the data variables of each pattern equivalent sibling necessity, defined in Eq. to the same set of fresh variables. (B.9)

- renaming any reference to a data variable that is bound by a renamed parent necessity defined in Eq. (B.10)

Hence, (B.9) and (B.10) allow us to deduce that mapping \( \zeta(j) \) in (B.8) produces a substitution environment which renames the data variables \( d^n_j \) (defined by pattern \( o(d^n_j) \)) to some set of fresh variables \( g^n_j \), which is the same for all the other conjunct sibling necessities that are pattern equivalent to \([o_j(d^n_j), c_j(f^m_{<i})]\). Hence, by the definition of a Uniform Equation, we can deduce

\[
\text{Equation } X_i = \bigwedge_{j \in Q} [o_j(d^n_j), c_j(f^m_{<i})] \varphi_j \land \varphi \text{ is Uniform.} \quad (B.11)
\]

By (B.6), (B.8) and (B.11) we can thus conclude

\[
\forall (X_k = \varphi_k) \in \text{uni}(Eq, \zeta) \cdot \text{equation } X_k = \varphi_k \text{ is Uniform.} \quad (B.12)
\]

By (B.9) and (B.10) we can deduce that equation \( X_i = \bigwedge_{j \in Q} [o_j(d^n_j), c_j(f^m_{<i})] \varphi_j \land \varphi \) is semantically equivalent to the equation reconstructed by the uni function in (B.8), i.e., \( X_i = \bigwedge_{j \in Q} [o_j(d^n_j), c_j(f^m_{<i})] \zeta(j) \varphi_j \land \varphi \). This holds since when the substitution environment, returned by \( \zeta(j) \), is applied on the equated formula, it substitutes symbolic event \([o_j(d^n_j), c_j(f^m_{<i})]\) by \([o_j(g^n_j), c_j(g^m_{<i})]\). Notice that pattern \( o_j(g^n_j) \) is equivalent to the original pattern \( o_j(d^n_j) \) since it only varies by the name of the data variables it defines, while condition \( c_j(g^m_{<i}) \) is also equivalent to \( c_j(f^m_{<i}) \) since by (B.10) we know that \( \zeta(j) \) (where \( \zeta(j) \) also contains \( \zeta(i) \) where \( i \) is the parent of \( j \)) substitutes accordingly the references to variables defined by renamed parent necessities that are being made by the filtering condition \( c_j(f^m_{<i}) \), i.e., \( \zeta(j) \) renames \( f^m_{<i} \) to the variable names, \( g^m_{<i} \), assigned to the renamed parent necessities; this preserves the semantics of the equation by keeping it closed wrt. data variables.
Hence, we can deduce

\[
X_i = \bigwedge_{j \in Q} \{ [o_j (d^n_j), c_j (f^m_{<i})] \varphi_j \land \varphi \}
\equiv \bigwedge_{j \in Q} \{ [o_j (g^n_j), c_j (g_{<i})] \varphi_j \land \varphi \}
\equiv \bigwedge_{j \in Q} \{ [o_j (d^n_j), c_j (f^m_{<i})] \varphi_j \land \varphi \}
\]

(B.13)

Finally, by (B.7), (B.8) and (B.13) we can conclude

\[
\{ X_i = \bigwedge_{j \in Q} \{ [o_j (d^n_j), c_j (f^m_{<i})] \varphi_j \land \varphi \} \} \cup Eq' \equiv \{ X_i = \bigwedge_{j \in Q} \{ [o_j (d^n_j), c_j (f^m_{<i})] \varphi_j \land \varphi \} \} \cup \text{uni}(Eq', \zeta)
\]

(B.14)

i.e., \( Eq \equiv \text{uni}(Eq, \zeta) \)

\[\therefore\] This case holds by (B.12) and (B.14).

---

End of Proof. ---

---

B.2 Auxiliary Lemmas for Proving Semantic Preservation of Construction \(\langle - \rangle_{(ii)}\)

In this section we provide the proof for Lemma 4.3 (restated below) which is are required for ensuring the semantic preservation of construction \(\langle - \rangle_{(ii)}\).

**Lemma 4.3** \(\forall (X_j = \varphi_j) \in Eq\) equation \(X_j = \varphi_j\) is Uniform implies \(Eq \equiv \text{traverse}(Eq, \{0\}, \text{cond}_{\text{comb}}, \emptyset)\) and \(\forall (X_k = \psi_k) \in \text{traverse}(Eq, \{0\}, \text{cond}_{\text{comb}}, \emptyset)\) equation \(X_k = \psi_k\) is Equi-Disjoint.

**B.2.1 Proving Lemma 4.3.**

In order to prove Lemma 4.3, we use Lemma B.2. This new lemma states that one can obtain an Equi-disjoint equation set, \(\omega'\), that is semantically equivalent to the
original equation set \( Eq \), by conducting a traversal upon a \textit{Uniform} subset of \( Eq \) \( (i.e., \ Eq' \) ), using an \textit{Equi-disjoint} accumulator equation set \( \omega \), where \( \omega \) must be \textit{semantically equivalent} to a subset of \( Eq \) that is restricted to the indices associated to the logical variables specified by the domain of \( \omega \), \( i.e., \ \omega \equiv Eq_{/\text{dom}(\omega)} \), where

\[
\text{dom}_{\text{ind}}(\omega) \equiv \left\{ i \ \middle| \ X_i \in \text{dom}(\omega) \right\}
\]

**Lemma B.2.** \( \forall Q, \omega \cdot Eq' \subseteq Eq \) and \( \text{traverse}(Eq', Q, \text{cond\_comb}, \omega) = \omega' \) and \( Eq_{/\text{dom}_{\text{ind}}(\omega)} \equiv \omega \) and \( \forall (X_j = \varphi_j) \in Eq' \cdot \text{equation } X_j = \varphi_j \) is \textit{Uniform} and \( \forall (X_k = \psi_k) \in \omega \cdot \text{equation } (X_k = \psi_k) \) is \textit{Equi-Disjoint} implies \( \forall (X_k = \psi_k) \in \omega' \cdot \text{equation } (X_k = \psi_k) \) is \textit{Equi-Disjoint} and \( Eq \equiv \omega' \)

The proof for this lemma is provided at the end of this section.

**To Prove Lemma 4.3.**

\[
\forall (X_j = \varphi_j) \in Eq \cdot \text{equation } X_j = \varphi_j \text{ is Uniform} \quad \text{implies} \quad Eq \equiv \text{traverse}(Eq, \{0\}, \text{cond\_comb}, \emptyset) \quad \text{and} \quad \forall (X_k = \psi_k) \in \text{traverse}(Eq, \{0\}, \text{cond\_comb}, \emptyset) \cdot \text{equation } (X_k = \psi_k) \text{ is Equi-Disjoint}.
\]

**Proof.** Initially we know

\[
\forall (X_j = \varphi_j) \in Eq \cdot \text{equation } X_j = \varphi_j \text{ is Uniform} \quad \text{(B.1)}
\]

By applying the \textit{traverse} function on \( Eq \) starting from \( Q=\{0\} \) and \( \omega=\emptyset \) we know

\[
\text{traverse}(Eq, \{0\}, \text{cond\_comb}, \omega) = \omega' \quad \text{(B.2)}
\]

\[
\omega = \emptyset \quad \text{(B.3)}
\]

By (B.3) and the definition of a \( Eq_{/\text{Q}} \) we know

\[
Eq_{/\text{dom}(\omega)} = \emptyset = \omega \quad \text{(B.4)}
\]
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From (B.3) we can also deduce
\[ \forall (X_k = \psi_k) \in \omega \cdot \text{equation } X_k = \psi_k \text{ is } \text{Equi-Disjoint} \]  
(B.5)

By (B.1), (B.2), (B.4), (B.5) and Lemma B.2 we know

\[ Eq \equiv \omega' \]  
(B.6)

\[ \forall (X_k = \psi_k) \in \omega' \cdot \text{equation } X_k = \psi_k \text{ is } \text{Equi-Disjoint} \]  
(B.7)

\[ \therefore \text{This Lemma holds by (B.6) and (B.7).} \]

\[ \therefore \text{End of Proof.} \]  

To Prove Lemma B.2.

\[ \forall Q, \omega \cdot Eq' \subseteq Eq \text{ and } \text{traverse}(Eq', Q, \text{cond_comb}, \omega) = \omega' \text{ and } Eq/\text{dom}_{\text{ind}}(\omega) \equiv \omega \text{ and } \forall (X_k = \psi_k) \in Eq' \cdot \text{equation } X_k = \psi_k \text{ is } \text{Uniform} \text{ and } \forall (X_k = \psi_k) \in \omega \cdot \text{equation } (X_k = \psi_k) \text{ is } \text{Equi-Disjoint} \text{ implies } \forall (X_k = \psi_k) \in \omega' \cdot \text{equation } (X_k = \psi_k) \text{ is } \text{Equi-Disjoint} \text{ and } Eq \equiv \omega' \]

**Proof. By induction on the structure of Q.**

**Case Q=Ø:** Initially we know

\[ Eq' \subseteq Eq \]  
(B.1)

\[ \text{traverse}(Eq', \varnothing, \text{cond_comb}, \omega) = \omega' \]  
(B.2)

\[ Eq/\text{dom}_{\text{ind}}(\omega) \equiv \omega \]  
(B.3)

\[ \forall (X_j = \varphi_j) \in Eq' \cdot \text{equation } X_j = \varphi_j \text{ is } \text{Uniform} \]  
(B.4)

\[ \forall (X_k = \psi_k) \in \omega \cdot \text{equation } (X_k = \psi_k) \text{ is } \text{Equi-Disjoint} \]  
(B.5)

By (B.2) and the definition of traverse we know

\[ \omega = \omega' \]  
(B.6)
From (B.5) and (B.6) we can deduce
\[ \forall (X_k = \psi_k) \in \omega' \cdot \text{equation } (X_k = \psi_k) \text{ is } \text{Eqi-Disjoint} \] (B.7)

From (B.3) and (B.6) we also know
\[ E_{\text{Eq}/\text{dom}_{\text{ind}}(\omega')} = \omega' \] (B.8)

Since \( Q = \emptyset \), by (B.2) and the definition of \text{traverse} we know the traversal has reached a point where no more children can be computed, which means that all the \textit{relevant equations} (i.e., those reachable from the principle variable) have been analysed. This implies that any other equation in \textit{Eq} (if any) is \textit{redundant} and \textit{irrelevant}. Hence, since from (B.8) we know that the equations in \( \omega' \) are \textit{equivalent to the relevant subset of equations in Eq}, i.e., \( E_{\text{Eq}/\text{dom}_{\text{ind}}(\omega')} \), and hence we can conclude
\[ \omega' \equiv \text{Eq} \] (B.9)

\[ \therefore \text{This subcase holds by (B.7) and (B.9).} \]

\[ - - - - - - o - - - - - - \]

\textbf{Case } \( Q \neq \emptyset \): Initially we know
\[ E_{\text{Eq}'} \subseteq \text{Eq} \] (B.1)
\[ \text{traverse}(E_{\text{Eq}'}, Q, \text{cond\_comb}, \omega) = \omega' \] (B.2)
\[ E_{\text{Eq}/\text{dom}_{\text{ind}}(\omega)} = \omega \] (B.3)
\[ \forall (X_j = \varphi_j) \in E_{\text{Eq}'} \cdot \text{equation } X_j = \varphi_j \text{ is Uniform} \] (B.4)
\[ \forall (X_k = \psi_k) \in \omega \cdot \text{equation } (X_k = \psi_k) \text{ is } \text{Eqi-Disjoint} \] (B.5)

We consider two subcases:
### B. Proving Semantic Preservation for Normalization

- **$\text{Eq}' = \emptyset$**: Since $\text{Eq}' = \emptyset$, by (B.2) and the definition of `traverse` we know

\[
\omega = \omega'
\]  

(B.6)

By (B.3), (B.5) and (B.6) we know

\[
\text{Eq} = \text{dom}_{\text{ind}}(\omega') = \omega'
\]  

(B.7)

\[
\forall (X_k = \psi_k) \in \omega' \cdot \text{equation } (X_k = \psi_k) \text{ is Equi-Disjoint} \tag{B.8}
\]

By (B.2) and the definition of `traverse` we know that the traversal starts from the full equation set, i.e., $\text{Eq}' = \text{Eq}$, using an empty accumulator, i.e., $\omega = \emptyset$, that would eventually contain the resultant Equi-Disjoint equation set. Every recursive application of the `traverse` function is then performed \textit{wrt.}: a \textit{smaller} version $\text{Eq}$, i.e., $\text{Eq}' = \text{Eq} \setminus \text{Eq} = \emptyset$, and a \textit{larger} accumulator $\omega'$ containing the reformulated, Equi-Disjoint equations whose indices are defined in $Q$ (and which where removed from $\text{Eq}'$). Hence, when $\text{Eq}'$ becomes $\emptyset$ it means that

\[
\text{dom}_{\text{ind}}(\omega') = \text{dom}_{\text{ind}}(\text{Eq})\tag{B.9}
\]

Hence, from (B.9) and by the definition of $\text{Eq} = Q$ we can deduce

\[
\text{Eq} = \text{dom}_{\text{ind}}(\omega') = \text{dom}_{\text{ind}}(\text{Eq}) = \text{Eq}\tag{B.10}
\]

Therefore by (B.7) and (B.10) we conclude

\[
\text{Eq} \equiv \omega'	ag{B.11}
\]

\[
\therefore \text{This subcase holds by (B.8) and (B.11).}
\]
B. Proving Semantic Preservation for Normalization

- \( \text{Eq} \neq \emptyset \): By (B.2) and the definition of traverse we know

\[
\text{cond\_comb}(\text{Eq}', Q, \omega) = \omega'' \quad (B.12)
\]

\[
\text{Eq}'' = \text{Eq}' \setminus \text{Eq}'_{/Q} \quad (B.13)
\]

\[
Q' = \bigcup_{l \in Q} \text{child}(\text{Eq}, l) \quad (B.14)
\]

\[
\text{traverse}(\text{Eq}''', Q', \text{cond\_comb}, \omega'') = \omega' \quad (B.15)
\]

From (B.1) and (B.13) we know

\[
\text{Eq}'' \subseteq \text{Eq} \quad (B.16)
\]

By (B.12) and the definition of \( \text{cond\_comb} \) we know

\[
\omega'' = \omega \uplus \left\{ X_i = \bigwedge_{\alpha \in C(j, Q')} \lbrack \alpha, c_k \rbrack X_j \land \varphi \left( = \psi_i \right) \mid \begin{array}{l}
(X_i = \bigwedge_{j \in Q''} \lbrack \alpha, c_j \rbrack X_j \land \varphi) \in \text{Eq}'_{/Q} \\
\text{and } Q' = \bigcup_{l \in Q} \text{child}(\text{Eq}, l) \\
such that } Q'' \subseteq Q'
\end{array} \right\} 
\]

(B.17)

By (B.17) and the definition of \( C(j, Q') \), we know that the conjunctions in the reformulated equations (i.e., in every \( \psi_i \)) now include an additional branch for each condition \( c_k \in C(j, Q') \) where \( c_k \) is a compound condition e.g., \( c_0 \land c_1 \land \ldots \land c_n \) or \( c_0 \land \neg c_1 \land \ldots \land \neg c_n \). These compound conditions consist in a truth combination of the filtering conditions of the sibling symbolic events which specify syntactically equal patterns; this is guaranteed since by (B.4) we know that the equations in \( \text{Eq}' \) are uniform, meaning that all sibling pattern equivalent events are guaranteed to be syntactically equal as well.

Hence, the reconstructed symbolic events in these additional guarded branches are unable to match the same concrete event \( \alpha \) unless they are syntactically equal (i.e., define the same pattern and condition), since despite their pattern being syntactically equal, only one compound filtering condition can at most be satisfied by the matching concrete event \( \alpha \); a
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case in point is when equation $X_0 = ([o, c_1])X_1 \land ([o, c_2])X_2$ is reconstructed
into $X_0 = ([o, c_1 \land c_2])X_1 \land ([o, c_1 \land \neg c_2])X_2 \land ([o, c_2])X_2 \land ([o, c_2])X_2$. Therefore, by (B.17) and the definition of \textit{Equi-Disjoint}, we can deduce that

$$
\forall (X_k = \psi_k) \in \left\{ X_i = \bigwedge_{c_k \in C(j, Q')}([o, c_k])X_j \land \varphi = \psi_i \right\} \quad \text{(B.18)}
$$

Hence, by (B.6), (B.17) and (B.18) we can conclude

$$
\forall (X_k = \psi_k) \in \omega'' \cdot \text{equation } (X_k = \psi_k) \text{ is Equi-Disjoint} \quad \text{(B.19)}
$$

Finally, we argue that the reconstructed equations in (B.17) (i.e., $X_i = \psi_i$) semantically equivalent to the original ones (i.e., $(X_i = \varphi_i) \in \text{Eq}_{/Q}$) since whenever a guarded branch, $([o, c_i])X_i$, is reconstructed into (possibly) multiple branches, $([o, c_i \land c_j \ldots c_k])X_i \land ([o, c_i \land \neg c_j \ldots c_k])X_i \ldots \land ([o, c_i \land \neg c_j \ldots \neg c_k])X_i$, via the truth combination function $\mathbb{C}(i, Q')$, the condition, $c_i$, of the original branch is never negated. This guarantees that continuation $X_i$ can only be reached when the original condition $c_i$ is true, and thus preserves the original semantics of the branch. Therefore, we conclude

$$
\left\{ X_i = \bigwedge_{c_k \in C(j, Q')}([o, c_k])X_j \land \varphi = \psi_i \right\} 
\quad \begin{array}{l}
(X_i = \bigwedge_{j \in Q''}([o, c_j])X_j \land \varphi) \in \text{Eq}_{/Q} \\
\text{and } Q' = \bigcup_{l \in Q} \text{child}(\text{Eq}, l) \\
such that \ Q'' \subseteq Q'
\end{array}
\quad \equiv \text{Eq}_{/Q} \quad \text{(B.20)}
$$

By (B.3), (B.17) and (B.20) we know

$$
\text{Eq}_{/\text{dom}(\omega'')} \equiv \omega'' \quad \text{(B.21)}
$$
By (B.4) and (B.16) we know
\[
\forall (X_j = \varphi_j) \in Eq'' \cdot \text{equation } X_j = \varphi_j \text{ is Uniform} \tag{B.22}
\]

Finally, by (B.15), (B.16), (B.19), (B.21), (B.22) and IH we know
\[
Eq \equiv \omega' \tag{B.23}
\]
\[
\forall (X_k = \psi_k) \in \omega' \cdot \text{equation } (X_k = \psi_k) \text{ is Equi-Disjoint} \tag{B.24}
\]

\[\therefore\] This case holds by (B.23) and (B.24).

--- End of Proof. ---
C. Proving Enforcement Correctness

In this section we present proofs ascertaining the correctness of our enforcers. Particularly, in Section C.1 we prove auxiliary lemmas required for proving Theorem 4.2, i.e., the synthesised enforcers are Deterministic, while in Section C.2, we show that these enforcers can also Strongly enforce the sHML formula they were synthesised from.

C.1 Proving Determinism for the Synthesised Enforcers

To address the issue of determinism we prove Lemma 4.4 which states that the synthesis function always produces well-formed enforcers from a normalized formula, and Lemma 4.5, which states that an enforcer always processes an input action and thus reduces in the same way, and thus behaves deterministically with every reduction.
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C.1.1 Proving Enforcer Well-formedness (Lemma 4.4)

To Prove.

\[ \forall \phi \in sHML_{nf} \cdot (\phi) = e \implies e \in \text{Enf}_{wf} \]

By the definition of \(( - )\) we can instead prove

\[ \forall \phi \in sHML_{nf} \cdot (\text{opt}(\phi))_\perp = e \implies e \in \text{Enf}_{wf} \]

We therefore quantify over all possible \( \rho \) and prove a stronger result, i.e.,

\[ \forall \rho, \phi \in sHML_{nf} \cdot \text{opt}(\phi) = \psi \text{ and } (\psi)_\rho = e \implies e \in \text{Enf}_{wf} \]

Proof by induction on the structure of \( \phi \).

Case \( \phi = \text{tt} \): This case holds trivially since \( \text{opt}(\text{tt}) = \text{tt} \) and \( (\text{tt})_\rho = \text{id} \) where \( \text{id} \in \text{Enf}_{wf} \).

\[ \begin{array}{c}
\end{array} \]

Case \( \phi = \text{ff} \): We know

\[ \begin{align*}
\text{opt}(\text{ff}) &= \text{ff} \\
(\text{ff})_\rho &= e
\end{align*} \]

We must consider two subcases for \( \rho \).

\( \rho = \perp \): Case does not apply since \( (\text{ff})_\perp \) does not produce an enforcer \( e \).

\( \rho = y \): From (C.2) and since \( \rho = y \) we know

\[ (\text{ff})_y = y \]  \hspace{1em} (C.3)

\( \therefore \) Case holds by (C.3) since \( y \in \text{Enf}_{wf} \).

\[ \begin{array}{c}
\end{array} \]
C. Proving Enforcement Correctness

\textbf{Case } \varphi = \bigwedge_{i \in Q} [[o_i, c_i]] \varphi_i : \quad \text{We know}

\begin{equation}
\text{opt} \left( \bigwedge_{i \in Q} [[o_i, c_i]] \varphi_i \right) = \bigwedge_{i \in Q} \text{opt}(\varphi_i)
\end{equation}

\begin{equation}
\left\langle \bigwedge_{i \in Q} [[o_i, c_i]] \text{opt}(\varphi_i) \right\rangle_{[\sigma]} = \text{rec}_{\sum_{i \in Q} \left\{ \left\langle o_i, c_i, \tau \right\rangle, e_i \right\}} \left( \text{if } \left\langle \text{ff}_i \right\rangle_{[\sigma]} = e_i \right) \left( \text{otherwise} \right)
\end{equation}

because by the definition of \( \left\langle - \right\rangle_{[\sigma]} \) we know

\begin{equation}
\forall i \in Q. \left\langle \text{opt}(\varphi_i) \right\rangle_{[\sigma]} = e_i
\end{equation}

By applying the IH on (C.3) we know

\begin{equation}
\forall i \in Q. e_i \in \text{ENF}_{\text{wf}}
\end{equation}

Hence, from (C.2), (C.3), (C.4) and the definition of \text{ENF}_{\text{wf}} we can conclude

\begin{equation}
\left( \text{rec}_{\sum_{i \in Q} \left\{ \left\langle o_i, c_i, \tau \right\rangle, e_i \right\}} \left( \text{if } \left\langle \text{ff}_i \right\rangle_{[\sigma]} = e_i \right) \left( \text{otherwise} \right) \right) \in \text{ENF}_{\text{wf}}
\end{equation}

---

\textbf{Case } \max X. \varphi \text{ and } X \in \text{fv}(\varphi) : 

\begin{equation}
\text{opt}(\max X. \varphi) = \max X. \text{opt}(\varphi)
\end{equation}

\begin{equation}
\left\langle \max X. \text{opt}(\varphi) \right\rangle_{[\sigma]} = \text{rec}_{x.e}
\end{equation}

because by the definition of \( \left\langle - \right\rangle_{[\sigma]} \) we know

\begin{equation}
\left\langle \text{opt}(\varphi) \right\rangle_{[\sigma]} = e
\end{equation}

By applying the IH on (C.3) we know

\begin{equation}
e \in \text{ENF}_{\text{wf}}
\end{equation}

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C. Proving Enforcement Correctness

Hence, from (C.2), (C.4) and the definition of \( \text{ENF}_{\omega f} \) we can conclude

\[
\text{rec } x.e \in \text{ENF}_{\omega f} \tag{C.5}
\]

---

\text{Case max } X.\varphi \text{ and } X \notin \text{fv}(\varphi):\]

\[
\text{opt}(\text{max } X.\varphi) = \text{opt}(\varphi) \tag{C.1}
\]

\[
\langle \text{opt}(\varphi) \rangle_\rho = e \tag{C.2}
\]

By applying the IH on (C.2) we know

\[
e \in \text{ENF}_{\omega f} \tag{C.3}
\]

---

\text{End of Proof.} ---

C.1.2 Proving Single Step Determinism (Lemma 4.5)

\text{To Prove.}

\[
\forall e \in \text{ENF}_{\omega f}: \underbrace{e \xrightarrow{\alpha \mu'}} e' \text{ and } e \xrightarrow{\alpha \mu''} e'' \text{ implies } e' = e'' \text{ and } \mu' = \mu''
\]

\text{Proof by rule induction on } e \xrightarrow{\alpha \mu'} e'.

\text{Case EID: } Initially we know

\[
e \xrightarrow{\alpha \mu'} e' \tag{C.1}
\]
C. Proving Enforcement Correctness

where

\[ e = \text{id} \quad \text{(C.2)} \]
\[ e' = \text{id} \quad \text{(C.3)} \]
\[ \mu' = \alpha \quad \text{(C.4)} \]
\[ e \in \text{ENF}_{\text{wf}} \quad \text{(C.5)} \]

and

\[ e \xrightarrow{\alpha \cdot \mu''} e'' \quad \text{(C.6)} \]

By (C.2), (C.6) and $e\text{Id}$ we know

\[ e'' = \text{id} \quad \text{(C.7)} \]
\[ \mu'' = \alpha \quad \text{(C.8)} \]

Hence, from (C.3), (C.7) and (C.4) and (C.8) we conclude

\[ e' = e'' = \text{id} \quad \text{(C.9)} \]
\[ \mu' = \mu'' \quad \text{(C.10)} \]

From (C.2), (C.5) and (C.9) we also know

\[ e', e'' \in \text{ENF}_{\text{wf}} \quad \text{(C.11)} \]

\[ \therefore \text{Case holds by (C.9), (C.10) and (C.11).} \]

---

**Case eRec:** Initially we know

\[ \text{rec } x.e \xrightarrow{\alpha \cdot \mu'} e' \quad \text{(C.1)} \]
because

\[ e\{\text{rec }x.e/x\} \xrightarrow{\alpha \mu'} e' \]  \hspace{1cm} (C.2)

and

\[ \text{rec }x.e \xrightarrow{\alpha \mu''} e'' \]  \hspace{1cm} (C.3)

\[ \text{rec }x.e \in \text{ENF}_{\text{wf}} \]  \hspace{1cm} (C.4)

By (C.3) and EREC we know

\[ e\{\text{rec }x.e/x\} \xrightarrow{\alpha \mu''} e'' \]  \hspace{1cm} (C.5)

Since \( e\{\text{rec }x.e/x\} \) is the unfolded equivalent of \( \text{rec }x.e \), from (C.4) we can deduce

\[ e\{\text{rec }x.e/x\} \in \text{ENF}_{\text{wf}} \]  \hspace{1cm} (C.6)

Hence, by (C.2), (C.5), (C.6) and IH we know

\[ e' = e'' \]  \hspace{1cm} (C.7)

\[ \mu' = \mu'' \]  \hspace{1cm} (C.8)

\[ e', e'' \in \text{ENF}_{\text{wf}} \]  \hspace{1cm} (C.9)

\[ \therefore \text{Case holds by (C.7), (C.8) and (C.9).} \]

---

**Case eSel:** Initially we know

\[ \sum_{i \in Q} e_i \xrightarrow{\alpha \mu'} e'_j \]  \hspace{1cm} (C.1)

because

\[ \exists j \in Q \cdot e_j \xrightarrow{\alpha \mu'} e'_j \]  \hspace{1cm} (C.2)
C. Proving Enforcement Correctness

and

\[ \sum_{i \in Q} e_i \overset{\alpha \mu''}{\longrightarrow} e_k' \]  \hspace{1cm} (C.3)
\[ \sum_{i \in Q} e_i \in \text{Enf}_{\text{wf}} \]  \hspace{1cm} (C.4)

By (C.3) and eSel we know

\[ \exists k \in Q \cdot e_k \overset{\alpha \mu''}{\longrightarrow} e_k' \]  \hspace{1cm} (C.5)

By (C.4) and the definition of \( \text{Enf}_{\text{wf}} \) we know that every branch \( e_i \) is prefixed by disjoint symbolic transformations, such that we know

\[ \forall i \in Q \cdot e_i = \{a_i, c_i, a'_i\}e'_i \]  \hspace{1cm} (C.6)
\[ \# \{a_i, c_i\} \]  \hspace{1cm} (C.7)

By (C.6) and (C.7) we know that only one of the branches in (C.4) can match the concrete system event \( \alpha \), and hence no matter how many times \( e \) is executed \text{wrt.} \( \alpha \), the same branch is always chosen. Therefore, the same branch is chosen in both reductions (C.2) and (C.5), such that we can conclude

\[ j = k \]  \hspace{1cm} (C.8)

Hence, by (C.2), (C.5) and (C.8) we know

\[ e_j \overset{\alpha \mu''}{\longrightarrow} e'_j \]  \hspace{1cm} (C.9)
\[ e_j \overset{\alpha \mu''}{\longrightarrow} e'_k \]  \hspace{1cm} (C.10)

Since \( j \in Q \), from (C.4) we know

\[ e_j \in \text{Enf}_{\text{wf}} \]  \hspace{1cm} (C.11)
Hence, by (C.9), (C.10), (C.11) and IH we know
\[ e'_j = e'_k \]  \hspace{1cm} (C.12)
\[ \mu' = \mu'' \]  \hspace{1cm} (C.13)
\[ e'_j, e'_k \in \text{ENF}_{\text{wf}} \]  \hspace{1cm} (C.14)

\[ \therefore \text{Case holds by (C.12), (C.13) and (C.14).} \]

---

**Case eTrns:** Initially we know
\[ \{o, c, o'\}.e \xrightarrow{\alpha \bullet \mu'} e\sigma \]  \hspace{1cm} (C.1)
because
\[ \{o, c, o'\}(\alpha) = (\mu', \sigma) \]  \hspace{1cm} (C.2)
and
\[ \{o, c, o'\}.e \xrightarrow{\alpha \bullet \mu''} e'' \]  \hspace{1cm} (C.3)
\[ \{o, c, o'\}.e \in \text{ENF}_{\text{wf}} \]  \hspace{1cm} (C.4)

By (C.2), (C.3) and eTrns we know
\[ e'' = e\sigma \]  \hspace{1cm} (C.5)
\[ \mu'' = \mu' \]  \hspace{1cm} (C.6)

By (C.4) and the definition of ENF_{\text{wf}}, we know that \{o, c, o'\}.e is a special case for
\[ \sum_{i \in Q} \{o_i, c_i, o'_i\}.e_i, \ i.e., \text{where Q contains only one index, and hence we know} \]
\[ e \in \text{ENF}_{\text{wf}} \]  \hspace{1cm} (C.7)

Moreover, when \sigma is applied on an enforcer e, this does not modify the structure
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of $e$, but just binds data variables defined in a prefixing symbolic event to the data defined in the matching concrete system event. Hence, from (C.7) we deduce

$$e\sigma \in \text{Enf}_{\text{wf}}$$

(C.8)

\[ \therefore \text{ Case holds by (C.5), (C.6) and (C.8).} \]

\[ \text{--- End of Proof. ---} \]

C.2 Proving Strong Enforceability for the Synthesised Enforcers

We prove Theorem 4.3 (restated below), by proving that the enforcers synthesised by our synthesis function are sound and transparent. We prove these two criteria in Sections C.2.1 and C.2.2 resp. Finally, we prove the supporting lemma, Lemma 4.8, in Section C.2.3.

C.2.1 Proving Soundness

$$\forall \rho, p \in \text{PROC}, e \in \text{Enf}_{\text{wf}}, \varphi \in \text{sHML}_{\text{opt}} \text{ when } \varphi \in \text{SAT} \cdot \langle \varphi \rangle_p \{ e/\rho \} = e' \implies e'[p] \models_s \varphi$$

To prove this theorem we must show that relation $\mathcal{R}$ (below) is a satisfaction relation ($\models_s$) as defined by the rules in Figure 4.14.

$$\mathcal{R} \overset{\text{def}}{=} \left\{ (e'[p], \varphi) \mid \varphi \in \text{SAT} \text{ and } \forall e \in \text{Enf}_{\text{wf}} \cdot \langle \varphi \rangle_p \{ e/\rho \} = e' \right\}$$

**Proof.** By coinduction on the structure of $\varphi$.

*Case $\varphi = X$:* Does not apply since $X$ is an open formula and thus $X \notin \text{SAT}
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**Case** \(\varphi = \text{ff} :\) Does not apply since \(\text{ff} \notin \text{SAT}\)

**Case** \(\varphi = \text{tt} :\) Holds trivially since *any process* satisfies \(\text{tt},\) which confirms that 
\((\text{id}[p], \text{tt}) \in \mathcal{R},\) since \(\langle \text{tt} \rangle_{\rho}\{e/\rho\} = \langle \text{tt} \rangle_{\rho} = \text{id}.\)

**Case** \(\varphi = \max X.\varphi \text{ and } X \in \text{fv}(\varphi) :\) We know

\[
(e'[p], \max X.\varphi) \in \mathcal{R}
\]  
(C.1)

because

\[
\max X.\varphi \in \text{SAT} \tag{C.2}
\]

\[
\langle \max X.\varphi \rangle_{\rho}\{e/\rho\} = e' \tag{C.3}
\]

By (C.3) and the definition of \(\langle - \rangle_{\rho},\) we know

\[
\langle \max X.\varphi \rangle_{\rho}\{e/\rho\} = \text{rec} x.e'' = e' \tag{C.4}
\]

\[
\langle \varphi \rangle_{\rho}\{e/\rho\} = e'' \tag{C.5}
\]

**Remark:** To prove that \(\mathcal{R}\) is a satisfaction relation, we must prove that when 
\(\langle \varphi\{\max X.\varphi/X\} \rangle_{\rho}\{e/\rho\} = e''\) the following holds:

\[
(e''[p], \varphi\{\max X.\varphi/X\}) \in \mathcal{R}
\]

By (C.2) and the definition of SAT we can deduce

\[
\exists q \cdot q \models_s \max X.\varphi \tag{C.6}
\]

By (C.6) and the definition of \(\models_s\) we can know

\[
\exists q \cdot q \models_s \varphi\{\max X.\varphi/X\} \tag{C.7}
\]
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By (C.7) and the definition of $\text{SAT}$ we know

$$\varphi \{\max X.\varphi / X\} \in \text{SAT} \tag{C.8}$$

Since $\varphi \{\max X.\varphi / X\}$ is the unfolded equivalent of $\max X.\varphi$, by (C.4), (C.5) and the definition of $\llceil - \rrceil_\rho$, we know

$$\llceil \varphi \{\max X.\varphi / X\}\rrceil_\rho \{e / \rho\} = e'' \{\text{rec } x. e'' / x\} \tag{C.9}$$

Hence, by (C.8), (C.9) and the definition of $\mathcal{R}$ we know

$$(e'' \{\text{rec } x. e'' / x\}[p], \varphi \{\max X.\varphi / X\}) \in \mathcal{R} \tag{C.10}$$

---

Case $\varphi = \bigwedge_{i \in Q} [[o_i, c_i]] \varphi_i$ where $\#_{i \in Q} \{o_i, c_i\}$:

We know

$$(e'[p], \bigwedge_{i \in Q} [[o_i, c_i]] \varphi_i) \in \mathcal{R} \tag{C.1}$$

because

$$\bigwedge_{i \in Q} [[o_i, c_i]] \varphi_i \in \text{SAT} \tag{C.2}$$

$$\#_{i \in Q} \{o_i, c_i\} \tag{C.3}$$

$$\llceil \bigwedge_{i \in Q} [[o_i, c_i]] \varphi_i \rrceil_\rho \{e / \rho\} = e' \tag{C.4}$$

By (C.4) and the definition of $\llceil - \rrceil_\rho$, we know

$$\llceil \bigwedge_{i \in Q} [[o_i, c_i]] \varphi_i \rrceil_\rho \{e / \rho\} = \left(\text{rec } y. \sum_{i \in Q} \left\{ \{o_i, c_i, \tau\}. e''_i \right\} \right) \tag{C.5}$$

$$\forall i \in Q \cdot e''_i = \llceil \varphi \rrceil_y \{e / \rho\} = \llceil \varphi \rrceil_y \tag{C.6}$$
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**Remark:** To prove that \( R \) is a satisfaction relation, we must prove that whenever 
\[
\{[o_i, c_i] \varphi_i\}_{\rho} \{e/\rho\} = e_i \text{ the following condition holds:}
\]
\[
\forall i \in Q \cdot (e_i[p], \{[o_i, c_i] \varphi_i\}) \in R
\]

But for \( R \) to be a satisfaction relation, by definition of \( \models_s \) we must prove
\[
\forall i \in Q \cdot (\forall q' \cdot e_i[p] \xrightarrow{\alpha} q' \text{ and } \{o_i, c_i\}(\alpha) = \sigma) \text{ implies } (q', \varphi, \sigma) \in R
\]

Also, since by (C.3) we know that guarded conjunctions in \( \text{sHML}_{nf} \) are **disjoint**, we know that the same event \( \alpha \) can match and satisfy the condition of **at most one** necessity guarding a specific branch. Since the case where **none of the branches match** is satisfied trivially, we can simply prove the case where only **one** branch matches \( \alpha \), i.e., we must show that whenever 
\[
\{\bigwedge_{i \in Q} \{[o_i, c_i] \varphi_i\}_{\rho} \{e/\rho\} = e' \text{ then}
\]
\[
\exists j \in Q \cdot (e'[p] \xrightarrow{\alpha} q' \text{ and } \{o_j, c_j\}(\alpha) = \sigma) \text{ implies } (q', \varphi, \sigma) \in R
\]

Hence we start by assuming the knowledge of
\[
e'[p] \xrightarrow{\alpha} q'
\]
\[
\exists j \in Q \cdot \{o_j, c_j\}(\alpha) = \sigma
\]

By (C.5) and (C.7) we know
\[
\left( \text{rec. } \sum_{i \in Q} \begin{cases} \{o_i, c_i, \tau\}, e_i^{\prime \prime} & \text{(if } \{f_i\}_y = e_i^{\prime \prime} \text{)} \\ \{o_i, c_i, o_i\}, e_i^{\prime \prime} & \text{(otherwise)} \end{cases} \right) [p] \xrightarrow{\alpha} q'
\]

By (C.9) and \( \text{iENF} \) we know
\[
q' = e'''[p']
\]

\[
\left( \text{rec. } \sum_{i \in Q} \begin{cases} \{o_i, c_i, \tau\}, e_i^{\prime \prime} & \text{(if } \{f_i\}_y = e_i^{\prime \prime} \text{)} \\ \{o_i, c_i, o_i\}, e_i^{\prime \prime} & \text{(otherwise)} \end{cases} \right) \xrightarrow{\alpha \cdot \alpha} e''
\]

\[
p \xrightarrow{\alpha} p'
\]
By (C.11) and eREC we know
\[
\left( \sum_{i \in Q} \begin{cases} \{o_i, c_i, \tau_i\}e_i''\{e'/y\} & \text{(if } \ll_i f_{y} = e_i''\text{)} \\ \{o_i, c_i, o_i\}e_i''\{e'/y\} & \text{(otherwise)} \end{cases} \right) \xrightarrow{\alpha} e''
\] (C.13)

Since by (C.3) we know that the summands in (C.13) are prefixed by a disjoint transducer, we thus know that only one branch may be satisfied by action \(\alpha\). Hence, by applying rule eSel on (C.13) we know
\[
\exists j \in Q \cdot \{o_j, c_j, \gamma\}e_j''\{e'/y\} \xrightarrow{\alpha} e'' \quad \text{where } \gamma \in \{\tau, o_j\}
\] (C.14)

Since the output action of the \(\alpha \cdot \alpha\) -reduction in (C.14) is \(\alpha \equiv o_j\sigma\), we know that the selected branch cannot be a suppression operation (otherwise the output would have been \(\tau\)), hence we know
\[
\exists j \in Q \cdot \{o_j, c_j, o_j\}\ll_j M_y\{e/y\} \xrightarrow{\alpha} e''
\] (C.15)

By (C.8) and the definition of \([^\{o, c, o'\}\], we know
\[
\exists j \in Q \cdot \{o_j, c_j, o_j\}\ll_j M_y\{e/y\} (\alpha) = (\alpha, \sigma)
\] (C.16)

By (C.15), (C.16) and eTrns we know
\[
e'' = e_j''\sigma\{e'/y\}
\] (C.17)

Since \(j \in Q\), from (C.6) we can deduce
\[
e_j'' = \ll_j \varphi_j
\] (C.18)

Hence, by (C.17), (C.18) and the definition of \(\ll \rho\), we can conclude
\[
\ll_j \varphi_j\{e'/y\} = e_j''\sigma\{e'/y\} = e''
\] (C.19)
Now, by (C.2) and the definition of $\text{Sat}$, we know

$$\exists q \cdot q \models_s \bigwedge_{i \in Q} \{o_i, c_i\}[\varphi_i]$$  \hspace{1cm} (C.20)

By (C.20) and definition of $\models_s$ we know

$$\exists q, \forall q', i \in Q \cdot (q \xrightarrow{\alpha} q' \text{ and } \{o_i, c_i\}(\alpha) = \sigma) \implies q' \models_s \varphi_j \sigma$$  \hspace{1cm} (C.21)

Since by (C.3) we know that all branches are disjoint from each other, we can deduce that (C.21) can be satisfied if there is one branch that matches $\alpha$ (or none at all), such that we know

$$\forall q', \exists q, j \in Q \cdot (q \xrightarrow{\alpha} q' \text{ and } \{o_j, c_j\}(\alpha) = \sigma) \implies q' \models_s \varphi_j \sigma$$  \hspace{1cm} (C.22)

Therefore, by (C.8), (C.12) and (C.22) we know

$$\exists q' \cdot q' \models_s \varphi_j \sigma$$  \hspace{1cm} (C.23)

Hence, by (C.23), the definition of $\text{Sat}$ we know

$$\varphi_j \sigma \in \text{Sat}$$  \hspace{1cm} (C.24)

Finally, by (C.19), (C.24) and the definition of $\mathcal{R}$ we know

$$\exists j \in Q \cdot (e''[p'], \varphi_j \sigma) \in \mathcal{R}$$  \hspace{1cm} (C.25)

Hence by assumptions (C.7), (C.8), (C.10) and deduction (C.25) we conclude

$$\exists j \in Q \cdot (e'[p] \xrightarrow{\alpha} e''\sigma[e'/y][p'] \text{ and } \{o_i, c_i\}(\alpha) = \sigma)$$

implies $(e''\sigma[e'/y][p'], \varphi_j \sigma) \in \mathcal{R}$

---

--- End of Proof. ---
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C.2.2 Proving Transparency

To Prove.

\[ \forall \rho, p \in \text{PROC}, e \in \text{ENF}_{\text{wf}}, \varphi \in \text{SHML}_{\text{opt}}^\text{opt} \cdot p \vDash_s \varphi \quad \text{and} \quad \langle \varphi \rangle \rho \{ e / \rho \} = e' \quad \text{implies} \quad e'[p] \sim p \]

To prove this lemma we must show that relation \( R \) (below) is a strong bisimulation relation.

\[ R \coloneqq \{ (e'[p], p) \mid p \vDash_s \varphi \quad \text{and} \quad \forall e \in \text{ENF}_{\text{wf}} \cdot \langle \varphi \rangle \rho \{ e / \rho \} = e' \} \]

Hence we must show that \( R \) satisfies the following conditions:

(a) if \( p \xrightarrow{\mu} p' \) then \( e'[p] \xrightarrow{\mu} S' \) and \( (p', S') \in R \)

(b) if \( e'[p] \xrightarrow{\mu} S' \) then \( p \xrightarrow{\mu} p' \) and \( (p', S') \in R \)

Proof. By coinduction on the structure of \( \varphi \).

Case \( \varphi = \text{ff} \) : Case does not apply since \( \#p \cdot p \vDash_s \text{ff} \).

---

Case \( \varphi = X \) : Case does not apply since \( X \) is an open-formula and \( \#p \cdot p \vDash_s X \).

---

Case \( \varphi = \text{tt} \) : Initially we know

\[ (p, e'[p]) \in R \quad \text{(C.1)} \]

because

\[ p \vDash_s \text{tt} \quad \text{(C.2)} \]

\[ \langle \text{tt} \rangle \rho \{ e / \rho \} = e' \quad \text{(C.3)} \]

By (C.3) and the definition of \( \langle - \rangle \rho \) we know that function \( \langle - \rangle \rho \) replaces every occurrence of \( \text{ff} \) with \( \rho \), in this case we have no falsehood declarations, and hence
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we know

\[ \forall e \cdot (\langle tt \rangle \rho \{ e/\rho \} = \langle tt \rangle \rho = \text{id} = e' \quad (C.4) \]

By the definition of \( \models_s \) also we know that \( \text{tt} \) is satisfied by any process, hence we deduce

\[ p' \models_s \text{tt} \quad (C.5) \]

Hence by (C.3), (C.5) and the definition of \( R \) we know.

\[ (p', e'[p']) \in R \quad (C.6) \]

– **To Prove (a):** Since \( \mu \in \{ \alpha, \tau \} \), we must consider the following two subcases.

– \( \mu = \tau \): We start by assuming

\[ p \xrightarrow{\tau} p' \quad (C.7) \]

By (C.7) and \text{IASYP} we know

\[ e'[p] \xrightarrow{\tau} e'[p'] \quad (C.8) \]

Hence by assumption (C.7) and deductions (C.6) and (C.8) we know

\[ p \xrightarrow{\tau} p' \text{ implies } e'[p] \xrightarrow{\tau} e'[p'] \text{ and } (p', e'[p']) \in R \]

– \( \mu = \alpha \): We start by assuming

\[ p \xrightarrow{\alpha} p' \quad (C.9) \]

By \text{EId} we know

\[ \text{id} \xrightarrow{\alpha \circ \alpha} \text{id} \quad (C.10) \]
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By (C.9), (C.10) and iEnf we know

\[ \text{id}[p] \xrightarrow{\alpha} \text{id}[p'] \]  \hspace{1cm} (C.11)

By (C.4) and (C.11) we can thus deduce

\[ e'[p] \xrightarrow{\alpha} e'[p'] \]  \hspace{1cm} (C.12)

Hence by assumption (C.9) and deductions (C.6) and (C.12) we know

\[ p \xrightarrow{\alpha} p' \text{ implies } e'[p] \xrightarrow{\alpha} e'[p'] \text{ and } (p', e'[p']) \in \mathcal{R} \]

- **To Prove (b):** Since \( \mu \in \{\alpha, \tau\} \), we must consider the following two sub-cases.

  - \( \mu = \tau \): We start by assuming

    \[ e'[p] \xrightarrow{\tau} q \]  \hspace{1cm} (C.13)

    By (C.13) and iEnf we know

    \[ q = e''[p'] \]  \hspace{1cm} (C.14)

    \[ e' \xrightarrow{\alpha \cdot \tau} e'' \]  \hspace{1cm} (C.15)

    \[ p \xrightarrow{\alpha} p' \]  \hspace{1cm} (C.16)

    From (C.4) and (C.15) we can deduce

    \[ \text{id} \xrightarrow{\alpha \cdot \tau} e'' \]  \hspace{1cm} (C.17)

    Since there does not exist a rule in our model that allows a monitor of the form id to reduce using a \( \alpha \cdot \tau \)-transition, this means that assumption (C.17) *can never occur*, which implies that this case does not apply.

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– \( \mu = \alpha \): We start by assuming

\[
e'[p] \xrightarrow{\alpha} q
\]  \hspace{1cm} (C.18)

By (C.4) and (C.18) we know

\[
id[p] \xrightarrow{\alpha} q
\]  \hspace{1cm} (C.19)

By (C.19) iEnf and EID we know

\[
p \xrightarrow{\alpha} p'
\]  \hspace{1cm} (C.20)

\[
q = \text{id}[p']
\]  \hspace{1cm} (C.21)

By (C.4) and (C.21) we know

\[
q = e'[p']
\]  \hspace{1cm} (C.22)

Hence by assumptions (C.18), (C.22) and deductions (C.6) and (C.20) we know

\[
e'[p] \xrightarrow{\alpha} e'[p'] \text{ implies } p \xrightarrow{\alpha} p' \text{ and } (p', e'[p']) \in \mathcal{R}
\]

Case \( \varphi = \max X.\varphi \text{ where } X \in \text{fv}(\varphi) \): Initially we know

\[
(p, e'[p]) \in \mathcal{R}
\]  \hspace{1cm} (C.1)

because

\[
p \models s \max X.\varphi
\]  \hspace{1cm} (C.2)

\[
\langle \max X.\varphi \rangle_{\rho}\{e/\rho\} = e'
\]  \hspace{1cm} (C.3)
By (C.3) and the definition of \( (-)_{\rho} \) we know

\[
\llbracket \max X.\varphi \rrbracket_{\rho} \{ e/\rho \} = \text{rec } x.e'' = e' \quad \text{(C.4)}
\]

\[
\llbracket \varphi \rrbracket_{\rho} \{ e/\rho \} = e'' \quad \text{(C.5)}
\]

Since \( X \in \text{fv}(\varphi) \), we know that the fixpoint variable \( X \) is defined in the continuation formula \( \varphi \), hence if we apply the synthesis function on the *unfolded version* of \( \max X.\varphi \), i.e., \( \varphi \{ \max X.\varphi /X \} \), from (C.4) we know that the synthesis produces an *unfolded version* of \( e' \), such that we can deduce

\[
\llbracket \varphi \{ \max X.\varphi /X \} \rrbracket_{\rho} \{ e/\rho \} = e'' \{ \text{rec } x.e'' /x \} \quad \text{(C.6)}
\]

By (C.2) and the definition of \( \models_s \) we know

\[
p \models_s \varphi \{ \max X.\varphi /X \} \quad \text{(C.7)}
\]

Hence, by (C.6), (C.7) and the definition of \( \mathcal{R} \) we know

\[
(p, e'' \{ \text{rec } x.e'' /x \}[p]) \in \mathcal{R} \quad \text{(C.8)}
\]

---

**To Prove (a):** We start by assuming

\[
p \xrightarrow{\nu} p' \quad \text{(C.9)}
\]

From (C.8) and IH we know

\[
p \xrightarrow{\nu} p' \text{ implies } e'' \{ \text{rec } x.e'' /x \}[p] \xrightarrow{\nu} q' \text{ and } (p', q') \in \mathcal{R} \quad \text{(C.10)}
\]

By (C.9) and (C.10) we deduce

\[
e'' \{ \text{rec } x.e'' /x \}[p] \xrightarrow{\nu} q' \quad \text{(C.11)}
\]

\[
(p', q') \in \mathcal{R} \quad \text{(C.12)}
\]
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By (C.9), (C.11) and iEnf we know

\[ e'' \{ \text{rec} \cdot x. e'' / x \} \xrightarrow{\alpha \cdot \mu} e''' \]  \hspace{1cm} (C.13)

\[ q' = e'''[p'] \]  \hspace{1cm} (C.14)

By applying rule eRec on (C.13) we know

\[ \text{rec} \cdot x. e'' \xrightarrow{\alpha \cdot \mu} e''' \]  \hspace{1cm} (C.15)

By (C.9), (C.14), (C.15) and iEnf we know

\[ \text{rec} \cdot x. e''[p] \xrightarrow{\mu} q' \]  \hspace{1cm} (C.16)

By (C.4) and (C.16) we deduce

\[ e'[p] \xrightarrow{\mu} q' \]  \hspace{1cm} (C.17)

Hence by assumption (C.9) and deductions (C.12) and (C.17) we conclude

\[ p \xrightarrow{\mu} p' \text{ implies } e'[p] \xrightarrow{\mu} q' \text{ and } (p', q') \in \mathcal{R} \]

- **To Prove (b):** Since \( \mu \in \{\alpha, \tau\} \), we must consider the following two subcases.

  - **\( \mu = \alpha \):** We start by assuming

\[ e'[p] \xrightarrow{\mu} q' \]  \hspace{1cm} (C.18)

By (C.18) and iEnf we know

\[ q' = e'''[p'] \]  \hspace{1cm} (C.19)

\[ e' \xrightarrow{\alpha \cdot \mu} e''' \]  \hspace{1cm} (C.20)

\[ p \xrightarrow{\alpha} p' \]  \hspace{1cm} (C.21)
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By (C.4) and (C.20) we know

$$\text{rec } x.e' \xrightarrow{\alpha \mu} e'''$$  \hfill (C.22)

By applying rule eREC on (C.22) we know

$$e''\{\text{rec } x.e'/x\} \xrightarrow{\alpha \mu} e'''$$  \hfill (C.23)

By (C.19), (C.21), (C.23) and iEnf we know

$$e''\{\text{rec } x.e'/x\}[p] \xrightarrow{\mu} q'$$  \hfill (C.24)

By (C.8) and IH we know

$$e''\{\text{rec } x.e'/x\}[p] \xrightarrow{\mu} q' \text{ implies } p \xrightarrow{\mu} p' \text{ and } (p', q') \in \mathcal{R}$$  \hfill (C.25)

From (C.24) and (C.25) we thus deduce

$$p \xrightarrow{\mu} p'$$  \hfill (C.26)

$$(p', q') \in \mathcal{R}$$  \hfill (C.27)

Hence by assumptions (C.18) and deductions (C.26) and (C.27) we conclude

$$e'[p] \xrightarrow{\mu} q' \text{ implies } p \xrightarrow{\mu} p' \text{ and } (p', q') \in \mathcal{R}$$

---

Case $\varphi = \bigwedge_{i \in \mathcal{Q}} ([\alpha_i, c_i])\varphi_i$:

Initially we know

$$(p, e'[p]) \in \mathcal{R}$$  \hfill (C.1)
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because

\[ p \models_s \bigwedge_{i \in Q} \left[ \{ o_i, c_i \} \right] \varphi_i \] (C.2)

\[ \# \left\{ o_i, c_i \right\} \] (C.3)

\[ e' = \left\langle \bigwedge_{i \in Q} \left[ \{ o_i, c_i \} \right] \varphi_i \right\rangle_{\rho} \{ e/\rho \} \] (C.4)

By (C.4) and the definition of \( \left\langle - \right\rangle_{\rho} \) we know

\[ \left\langle \bigwedge_{i \in Q} \left[ \{ o_i, c_i \} \right] \varphi_i \right\rangle_{\rho} \{ e/\rho \} = \text{recy.} \] (C.5)

\[ \forall i \in Q \cdot e''_i = \left\langle \varphi \right\rangle_y \{ e/\rho \} = \left\langle \varphi \right\rangle_y \] (C.6)

By (C.2) and the definition of \( \models_s \) we know

\[ \forall i \in Q \cdot p \models_s \left[ \{ o_i, c_i \} \right] \varphi_i \] (C.7)

By (C.7) and the definition of \( \models_s \) we know

\[ \forall i \in Q \cdot (\forall p' \cdot p \xrightarrow{\alpha} p' \text{ and } \{ o_j, c_j \}(\alpha) = \sigma) \text{ implies } p' \models_s \varphi_i \sigma \] (C.8)

Since by (C.3) we know that a concrete event \( \alpha \) can match at most one symbolic event defined in the guarding necessities, then we know that at most one branch can be selected at runtime, hence from (C.8) we can deduce

\[ \exists j \in Q \cdot (\forall p' \cdot p \xrightarrow{\alpha} p' \text{ and } \{ o_j, c_j \}(\alpha) = \sigma) \text{ implies } p' \models_s \varphi_j \sigma \] (C.9)

- **To Prove** (a): Since \( \mu \in \{ \alpha, \tau \} \), we must consider the following two subcases.

  - \( \mu = \tau \): To prove this subcase we assume

    \[ p \xrightarrow{\tau} p' \] (C.10)

    This case holds trivially since (C.10) contradicts the assumption of (C.9), thus trivially satisfying the implication in (C.9).
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\( \mu = \alpha \): We start by assuming

\[ p \xrightarrow{\alpha} p' \quad (C.11) \]

We further investigate the following cases:

- \( \forall j \in Q \cdot \text{match}(o_j, \alpha) = \text{undef} \) \( \text{(i.e., no matching branches)} \):
  This case is trivially satisfied by (C.9) since \( \exists j \in Q \cdot \{ o_i, c_i \}(\alpha) = \sigma \).

- \( \exists j \in Q \cdot \{ o_j, c_j \}(\alpha) = \sigma \) \( \text{(i.e., 1 matching branch)} \):
  We know

\[ \exists j \in Q \cdot \{ o_j, c_j \}(\alpha) = \sigma \quad (C.12) \]

By (C.9), (C.11) and (C.12) we know

\[ \exists j \in Q \cdot p' \models_s \varphi \sigma \quad (C.13) \]

By (C.10) and the definition of \([\{o, c, o'\}]\) we know

\[ \exists j \in Q \cdot \{ o, c, o' \}(\alpha) = (o' \sigma, \sigma) \quad (C.14) \]

By (C.14) and rule \( \text{eTrns} \) we know

\[ \exists j \in Q \cdot \{ o_j, c_j, o' \} \cdot e''_j \{ e'/y \} \xrightarrow{\alpha \cdot o' \sigma} e''_j \sigma \{ e'/y \} \quad (C.15) \]

By (C.11), (C.15) and rule \( \text{eSel} \) we know

\[ \exists j \in Q \cdot \left( \sum_{i \in Q \setminus \{ j \}} \begin{cases} \{ o_i, c_i, \tau \}, e''_i \{ e'/y \} & \text{(if } \{ o_i, c_i, o_i \} = e''_i \text{)} \\ \{ o_i, c_i, o_i \}, e''_i \{ e'/y \} & \text{(otherwise)} \end{cases} \right) \xrightarrow{\alpha \cdot o' \sigma} e''_j \sigma \{ e'/y \} \quad (C.16) \]

By (C.11), (C.16) and rules \( \text{eRec + iEnf} \) we know

\[ \exists j \in Q \cdot \text{rec } y. \left( \sum_{i \in Q} \begin{cases} \{ o_i, c_i, \tau \}, e''_i \{ e'/y \} & \text{(if } \{ o_i, c_i, o_i \} = e''_i \text{)} \\ \{ o_i, c_i, o_i \}, e''_i \{ e'/y \} & \text{(otherwise)} \end{cases} \right) \xrightarrow{\alpha' \sigma} e''_j \sigma \{ e'/y \}[p'] \quad (C.17) \]
Since $j \in Q$, from (C.6) we can deduce

$$\langle \varphi_j \rangle_y \{e'/y\} = e''_j \{e'/y\}$$  \hspace{1cm} (C.18)

As we consider optimized formulae, we cannot have a case where $\text{ff}$ is embedded within a maximal fixpoint, e.g., $\max X.\text{ff}$ would have been optimized into $\text{ff}$. Hence, since $\nexists p \cdot p \models_s \text{ff}$, from (C.12) we can deduce

$$\varphi_j \sigma \neq \text{ff}$$  \hspace{1cm} (C.19)

By (C.19) we know that actions satisfying $[[a_j, c_j]] \varphi_j$ in (C.17) will not be suppressed since $\varphi_j \sigma \neq \text{ff}$, which means that $o' = a_j \neq \tau$. Hence, by the definition of $\langle - \rangle_\rho$, we know

$$\exists j \in Q \cdot e'[p] \overset{o_j \sigma}{\longrightarrow} e''_j \sigma \{e'/y\}[p']$$  \hspace{1cm} (C.20)

$$o' = a_j \text{ such that } o_j \sigma = \alpha$$  \hspace{1cm} (C.21)

Hence, by (C.20) and (C.21) we know

$$\exists j \in Q \cdot e'[p] \overset{\alpha}{\longrightarrow} e''_j \sigma \{e'/y\}[p']$$  \hspace{1cm} (C.22)

Since $\varphi_j \sigma$ is the closed equivalent of $\varphi_i$ (wrt. data variables), from (C.18) we can deduce

$$\langle \varphi_j \sigma \rangle_y \{e'/y\} = e''_j \sigma \{e'/y\}$$  \hspace{1cm} (C.23)

By (C.13), (C.23) and the definition of $\mathcal{R}$ we know

$$\exists j \in Q \cdot (p', e''_j \sigma \{e'/y\}[p']) \in \mathcal{R}$$  \hspace{1cm} (C.24)

Hence, by assumption (C.11) and deductions (C.22) and (C.24) we
can conclude
\[ \exists j \in Q \cdot p \xrightarrow{\alpha} p' \text{ implies } e'[p] \xrightarrow{\alpha} e''[\{e'/y\}[p']] \]
and \((p', e''[\{e'/y\}[p']]) \in \mathcal{R}\)

- **To Prove (b):** Since \(\mu \in \{\alpha, \tau\}\), we must consider the following two subcases.

- **\(\mu = \tau\):** We start by assuming
  \[ e'[p] \xrightarrow{\tau} q \]  
(C.25)

By (C.5) and (C.25) we know
\[
\text{rec.} \cdot \left(\sum_{i \in Q} \begin{cases} \{o_i, c_i, \tau\}.e'' & \text{(if } Lff_i \neq e''_i) \\ \{o_i, c_i, o_i\}.e'' & \text{(otherwise)} \end{cases} \right) [p] \xrightarrow{\tau} q
\]
(C.26)

By (C.26) and iENF + eRec we know
\[
\sum_{i \in Q} \begin{cases} \{o_i, c_i, \tau\}.e'' \{e'/y\} & \text{(if } Lff_i \neq e''_i) \\ \{o_i, c_i, o_i\}.e'' \{e'/y\} & \text{(otherwise)} \end{cases} \xrightarrow{\alpha} e'''
\]
(C.27)
\[ p \xrightarrow{\alpha} p' \]  
(C.28)
\[ q = e'''[p'] \]  
(C.29)

By (C.27) and eSel we know
\[ \exists j \in Q \cdot \{o_i, c_i, o'_j\}.e'' \{e'/y\} \xrightarrow{\alpha\tau} e''' \]  
(C.30)

Hence, by (C.30) and eTrns we know
\[ \{o_j, c_j, o'_j\}(\alpha) = (o'\sigma, \sigma) \]  
(C.31)

By (C.31) and the definition of \([\{o, c, o'\}]\) we know
\[ \{o_j, c_j\}(\alpha) = \sigma \]  
(C.32)
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Hence, by (C.9), (C.28) and (C.32) we know

\[ \exists j \in Q \cdot p' \models_s \varphi_j \sigma \]  \hspace{1cm} (C.33)

However, as the reduction in (C.30) is performed over action \( \alpha \triangleright \tau \), this can only be achieved when the matched branch is prefixed by a suppression transducer, i.e., where \( o' = \tau \), hence from (C.27) we know that \( e''_j \{ e'/y \} \) performs a suppression operation when \( \langle \text{ff} \rangle_y = e''_j \{ e'/y \} \), such that we know

\[ \varphi_j = \text{ff} \]  \hspace{1cm} (C.34)

Hence, this case does not apply (and is thus satisfied trivially) since by definition of \( \models_s \) we know that \( \not\exists p \cdot p \models_s \text{ff} \), which contradicts with (C.33) and (C.34).

\[ \mu = \alpha: \] We start by assuming

\[ e'[p] \xrightarrow{\alpha} q \]  \hspace{1cm} (C.35)

By (C.5) and (C.35) we know

\[ \text{rec.} \left( \sum_{i \in Q} \left\{ \begin{array}{l}
\{ o_i, c_i, \tau \}, e''_i \{ e'/y \} \quad \text{(if } \langle \text{ff} \rangle_y = e''_i \text{)} \\
\{ o_i, c_i, o_i \}, e''_i \{ e'/y \} \quad \text{(otherwise)}
\end{array} \right\} [p] \xrightarrow{\alpha} q \right. \]  \hspace{1cm} (C.36)

By (C.36) and \text{iENF + eREC} we know

\[ \sum_{i \in Q} \left\{ \begin{array}{l}
\{ o_i, c_i, \tau \}, e''_i \{ e'/y \} \quad \text{(if } \langle \text{ff} \rangle_y = e''_i \text{)} \\
\{ o_i, c_i, o_i \}, e''_i \{ e'/y \} \quad \text{(otherwise)}
\end{array} \right\} \xrightarrow{\alpha} e''' \]  \hspace{1cm} (C.37)

\[ p \xrightarrow{\alpha} p' \]  \hspace{1cm} (C.38)

\[ q = e'''[p'] \]  \hspace{1cm} (C.39)

By (C.3), (C.37) and \text{eSEL} we know

\[ \exists j \in Q \cdot \{ o_i, c_i, o' \}, e''_j \{ e'/y \} \xrightarrow{\alpha \triangleright} e''' \]  \hspace{1cm} (C.40)
Since the reduction in (C.40) is performed over an $\alpha \bullet \alpha$ action, this can only be achieved when the matched branch is guarded by an identity transducer, such that $o' = o_j$. Hence, we can infer

$$\exists j \in Q \cdot \{o_j, c_j, o_j\}.e''\{e/y\} \xrightarrow{\alpha \bullet \alpha} e'''$$ (C.41)

By (C.41) and $eTrns$ we know

$$e''' = e''_{j\sigma}\{e/y\}$$ (C.42)

$$\{o_j, c_j, o_j\}(\alpha) = (o_j\sigma, \sigma)$$ (C.43)

From (C.43) and the definition of $[[[o, c, o']]$ we know

$$\{o_j, c_j\}(\alpha) = \sigma$$ (C.44)

By (C.9), (C.38) and (C.44) we can deduce

$$p' \vdash_s \varphi_j\sigma$$ (C.45)

Since $j \in Q$ from (C.6) we know

$$\langle \varphi_j \rangle_y = e''_j$$ (C.46)

Hence, from (C.42), (C.46) and the definition of $\langle \_ \rangle_\rho$ we can deduce

$$\langle \varphi_j\sigma \rangle_y\{e/y\} = e''_{j\sigma}\{e/y\} = e'''$$ (C.47)

Therefore, by (C.45), (C.47) and the definition of $\mathcal{R}$, we know

$$(p', e'''[p']) \in \mathcal{R}$$ (C.48)

Hence by assumptions (C.35), (C.39) and deductions (C.38) and
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(C.48) we can finally conclude

\[ e'[p] \xrightarrow{\alpha} e'''[p'] \text{ implies } p \xrightarrow{\alpha} p' \text{ and } (p', e'''[p']) \in \mathcal{R} \]

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C.2.3 Proving Lemma 4.8

To Prove. \( \text{opt}(\varphi) = \psi \) implies \( \varphi \equiv \psi \) and \( \psi \in \text{SHML}_{\text{nf}}^{\text{opt}} \)

Proof. By structural induction on \( \varphi \).

Cases \( \varphi = \psi \) where \( \psi \in \{\text{tt, ff, } X\} \):

Holds trivially since \( \text{opt}(\psi) = \psi \) and \( \psi \in \text{SHML}_{\text{nf}}^{\text{opt}} \).

Case \( \varphi = \max X.\varphi' \): We know

\[
\text{opt}(\max X.\varphi') = \psi
\]

(C.1)

By definition of \( \text{opt} \), we must consider two subcases

– \( X \in \text{fv}(\varphi') \): We know

\[
\psi = \max X.\varphi'
\]

(C.2)

\[
\varphi' = \text{opt}(\varphi')
\]

(C.3)

By (C.3) and IH we know

\[
\varphi' \equiv \psi'
\]

(C.4)

\[
\psi' \in \text{SHML}_{\text{nf}}^{\text{opt}}
\]

(C.5)

By (C.1), (C.2) and (C.4) we can deduce

\[
\max X.\psi' \equiv \max X.\varphi'
\]

(C.6)
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By (C.5) and the definition of $\text{SHML}_{\text{nf}}^{\text{opt}}$ we know

$$\max X.\psi' \in \text{SHML}_{\text{nf}}^{\text{opt}} \quad \text{(C.7)}$$

\therefore This subcase holds by (C.6) and (C.7).

– $X \notin \text{fv}(\varphi')$ : We know

$$\psi = \text{opt}(\varphi') \quad \text{(C.8)}$$

By (C.8) and IH we know

$$\psi \equiv \varphi' \quad \text{(C.9)}$$

$$\psi \in \text{SHML}_{\text{nf}}^{\text{opt}} \quad \text{(C.10)}$$

Since $X \notin \text{fv}(\varphi')$, we know that $X$ is never referenced in $\varphi'$, thus making the maximal fixpoint declaration $\max X.$ redundant, hence from (C.9) we can deduce

$$\max X.\psi' \equiv \varphi' \equiv \psi \quad \text{(C.11)}$$

\therefore This subcase holds by (C.10) and (C.11).

\begin{center}
\begin{sideways}
$\cdots$
\end{sideways}
\end{center}

Case $\varphi = \bigwedge_{i \in Q} [\eta_i] \varphi_i$ : We know

$$\text{opt}(\bigwedge_{i \in Q} [\eta_i] \varphi_i) = \bigwedge_{i \in Q} [\eta_i] \psi_i \quad \text{(C.1)}$$

because

$$\forall i \in Q \cdot \text{opt}(\varphi_i) = \psi_i \quad \text{(C.2)}$$
By (C.2) and IH we know

\[ \forall i \in Q \cdot \psi_i \equiv \varphi_i \quad \text{(C.3)} \]
\[ \forall i \in Q \cdot \psi_i \in \text{sHML}^{\text{opt}}_{\text{nf}} \quad \text{(C.4)} \]

From (C.1), (C.2) and (C.3) we can deduce

\[ \bigwedge_{i \in Q} [\eta_i] \varphi_i \equiv \bigwedge_{i \in Q} [\eta_i] \psi_i \quad \text{(C.5)} \]

From (C.1), (C.2), (C.4) and the definition of \( \text{sHML}^{\text{opt}}_{\text{nf}} \) we know

\[ \bigwedge_{i \in Q} [\eta_i] \psi_i \in \text{sHML}^{\text{opt}}_{\text{nf}} \quad \text{(C.6)} \]

\[ \therefore \text{This subcase holds by (C.5) and (C.6).} \]

--- End of Proof. ---