Why the lowest Landau level approximation works in strongly type II superconductors.

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Abstract

Higher than the lowest Landau level contributions to magnetization and specific heat of superconductors are calculated using Ginzburg - Landau equations approach. Corrections to the excitation spectrum around solution of these equations (treated perturbatively) are found. Due to symmetries of the problem leading to numerous cancellations the range of validity of the LLL approximation in mean field is much wider than a naive range and extends all the way down to $H = \frac{H_{c2}(T)}{13}$. Moreover the contribution of higher Landau levels is significantly smaller compared to LLL than expected naively. We show that like the LLL part the lattice excitation spectrum at small quasimomenta is softer than that of usual acoustic phonons. This enhances the effect of fluctuations. The mean field calculation extends to third order, while the fluctuation contribution due to HLL is to one loop. This complements the earlier calculation of the LLL part to two loop order.
I. Introduction

Ginzburg - Landau effective description of high $T_c$ superconductors has been remarkably successful in describing various thermodynamical and transport properties. However when fluctuations are of importance, even this effective description becomes very complicated. Some progress can be achieved when certain additional assumptions are made. One of the often made additional assumption is that only the lowest Landau level (LLL) significantly contributes to physical quantities of interest. There is a debate however on how restrictive the LLL approximation actually is. Naively when $H < \frac{H_{c2}(T)}{3}$ (see the dotted line on Fig. 1), even within mean field approximation, one should consider higher Landau levels (HLL) mixing in the Abrikosov vortex lattice solution of the GL equations. When fluctuations are included one can argue using Hartree approximation that the LLL range of validity is even smaller. However direct application of the LLL scaling to magnetization and specific heat on YBCO suggest that the range of applicability is much wider - all the way down to $1 - 3T^{11,8,12}$. It is not clear why HLL do not contribute.

In this paper we explicitly calculate the effects of HLL at low temperatures in the vortex solid or liquid phase and establish the realistic range where the LLL approximation is valid (see the heavy dashed line on Fig.1). We reanalyze the HLL corrections to mean field equations going to higher order then in and find that the expansion converges for $H_{c2} > H > \frac{H_{c2}}{13}$. Importantly within this radius of convergence the LLL contribution constitutes more than 95%. Then we calculate the HLL fluctuation effects to one loop order complementing the LLL calculation to two loops by one of us (later referred to as I).

Ginzburg parameter $G_i$ characterizing importance of thermal fluctuations is much larger in high $T_c$ superconductors than in the low temperature ones. Moreover in the presence of magnetic field the importance of fluctuations in high $T_c$ superconductors is further enhanced. Under these circumstances corrections to various physical quantities like magnetization or specific heat are not negligible even at low temperatures. It is quite straightforward to systematically account for the fluctuations effect on magnetization, specific heat or conductivity perturbatively above the mean field transition line using Ginzburg - Landau description. However in the interesting region below this line it turned out to be extremely difficult to develop a quantitative theory.

Within LLL in order to approach the region below the mean field transition line $T < T_{mf}(H)$ Thouless proposed a perturbative approach around homogeneous (liquid) state was in which all the ”bubble” diagrams are resumed. The series provide accurate results at high temperatures, but for the LLL dimensionless temperature $a_T \equiv \left( \frac{2H^2}{G_i T H_{c1}^2} \right)^{1/3} \frac{T - T_{mf}(H)}{\frac{\pi}{2}} \lesssim -2$ become inapplicable. Generally attempts to extend the theory to lower temperature by Pade extrapolation were not successful. Alternative, more direct approach to low temperature
fluctuations physics is to start from the mean field solution and then take into account perturbatively fluctuations around this inhomogeneous solution. Experimentally it is reasonable since, for example, specific heat at low temperatures is a smooth function and the fluctuations contribution experimentally is quite small. For some time this was in disagreement with theoretical expectations.

Eilenberger calculated spectrum of harmonic excitations of the triangular vortex lattice (see eq.(28) below) and noted that the gapless mode is softer then the usual Goldstone mode expected as a result of spontaneous breaking of translational invariance. The inverse propagator for the ”phase” excitations behaves as $k_x^2 + \text{const}(k_x^4 + k_y^4)$. The influence of this unexpected additional ”softness” apparently goes beyond enhancement of the contribution of fluctuations at leading order. It leads to disastrous infrared divergencies at higher orders rendering the perturbation theory around the vortex state doubtful. One therefore tends to think that nonperturbative effects are so important that such a perturbation theory should be abandoned. However it was shown in I that a closer look at the diagrams reveals that in fact one encounters actually only logarithmic divergencies. This makes the divergencies similar to so called ”spurious” divergencies in the theory of critical phenomena with broken continuous symmetry and they exactly cancel at each order provided we are calculating a symmetric quantity. Qualitatively physics of fluctuating $D = 3$ GL model in magnetic field turns out to be similar to that of spin systems in $D = 2$ possessing a continuous symmetry. In particular, although within perturbation theory in thermodynamic limit the ordered phase (solid) exists only at $T = 0$, at low temperatures liquid differs very little in most aspects from solid. One can effectively use properly modified perturbation theory to quantitatively study various properties of the vortex liquid phase. This perturbative approach agrees very well with the direct Monte Carlo simulation of. The question arises whether one can extend the well controlled perturbative calculation beyond the LLL. Sometimes a hope is expressed that the additional softness is an accidental artifact of LLL approximation. The present work explicitly shows that this is not so. It is a fundamental general phenomenon.

The paper is organized as follows. Model is described and perturbative mean field solution developed in section II. The expansion parameter will be the distance from the mean field critical line $a_h \equiv \frac{1}{2}(1 - \frac{T}{T_c} - \frac{H}{H_c})$. Range of validity of the expansion and of the LLL approximation is discussed. Then in section III we derive the spectrum of excitations to leading order and to the next to leading order in $a_h$. The free energy to one loop is calculated in section IV. Section V contains expressions for magnetization and specific heat and discussion of validity range of the fluctuation contributions calculation. Finally we summarize the results in section VI. Details of the mean field calculation can be found in Appendix A, while details of the HLL spectrum calculation can be found in Appendix B.
II. Model and the perturbative mean field solution

A. Model

Our starting point is the GL free energy:

$$ F = \int d^3x \frac{\hbar^2}{2m_{ab}} |(\vec{\nabla} - ie^* \hbar c^{-1} \vec{A})\psi|^2 + \frac{\hbar^2}{2m_c} |\partial_z \psi|^2 + a|\psi|^2 + \frac{b'}{2}|\psi|^4 $$

(1)

Here \( \vec{A} = (-By, 0) \) describes a nonfluctuating constant magnetic field. For strongly type II superconductors \( (\kappa \sim 100) \) far from \( H_{c1} \) (this is the range of interest in this paper) magnetic field is homogeneous to a high degree due to superposition from many vortices. For simplicity we assume \( a = \alpha(1 - t) \), \( t \equiv T/T_c \) although this dependence can be easily modified to better describe the experimental coherence length.

Throughout most of the paper will use the following units. Unit of length is \( \xi = \sqrt{\hbar^2 / (2m_{ab}\alpha T_c)} \) and unit of magnetic field is \( H_{c2} \), so that dimensionless magnetic field is \( b \equiv B/H_{c2} \). The dimensionless Boltzmann factor in these units is (the order parameter field is rescaled as \( \psi^2 \rightarrow 2\alpha T_c b' \psi^2 \)):

$$ F/T = \frac{1}{\omega} \int d^3x \frac{1}{2} |D\psi|^2 + \frac{1}{2} |\partial_z \psi|^2 - \frac{1}{2} t |\psi|^2 + \frac{1}{2} |\psi|^4, $$

(2)

The dimensionless coefficient is

$$ \omega = \sqrt{2Gi\pi^2 t}. $$

(3)

where the Ginzburg number is defined by \( Gi \equiv \frac{1}{2} \left( \frac{32\pi e^2 \kappa^2 \xi^{1/2} T_c}{c^2 \hbar^2} \right)^2 \) and \( \gamma \equiv m_c/m_{ab} \) is an anisotropy parameter. This coefficient determines the strength of fluctuations, but is irrelevant as far as mean field solutions are concerned.

B. Mean field solution by expansion in \( a_h \)

Now we turn to a perturbative solution of the Ginzburg-Landau equations near the mixed state - normal phase transition line. This has been done before to the second order, however the range of applicability and precision of the LLL approximation at large \( \kappa \) has not been fully explored. The \( z \) direction dependence of the solutions is trivial and will not be mentioned until fluctuations will be discussed. The expansion parameter is

$$ a_h \equiv \frac{1 - t - b}{2}. $$

(4)
Rewriting the quadratic part in terms of operator ("Hamiltonian") \( \mathcal{H} \equiv \frac{1}{2}(-D^2 - b) \) whose spectrum starts from zero, one obtains the following free energy density over \( T \)

\[
\frac{F}{T} \equiv \frac{f}{\omega} = \frac{1}{\omega} \int d^2x \left( \psi^* \mathcal{H} \psi - a_h |\psi|^2 + \frac{1}{2} |\psi|^4 \right).
\] (5)

The equation of motion is therefore

\[
\mathcal{H} \psi - a_h \psi + \psi |\psi|^2 = 0
\] (6)

This equation is solved perturbatively in \( a_h \) by assuming

\[
\Phi = (a_h)^{1/2} [\Phi_0 + a_h \Phi_1 + ...]
\] (7)

It is convenient to represent \( \Phi_0, \Phi_1, ... \) in the basis of eigenfunctions of \( \mathcal{H} \), \( \mathcal{H}\varphi^n = nb\varphi^n \), normalized to unit "Cooper pairs density" \( \langle |\varphi^n|^2 \rangle \equiv \int_{\text{cell}} d^2x |\varphi^n|^2 \frac{b}{2\pi} = 1 \), where "cell" is a primitive cell of the vortex lattice. Assuming hexagonal lattice symmetry one explicitly has;

\[
\varphi^n = \sqrt{\frac{2\pi}{\sqrt{\pi}2^n n! a}} \sum_{l=-\infty}^{\infty} H_n(y\sqrt{b} - \frac{2\pi}{a} l) \exp \left\{ i \left[ \pi l (l - 1) \right] + \frac{2\pi \sqrt{b}}{a} lx \right\} - \frac{1}{2} (y\sqrt{b} - \frac{2\pi}{a})^2
\]

where \( a_{\sqrt{b}} = \sqrt{\frac{4\pi}{\sqrt{3}b}} \) is the lattice spacing.

To order zero,

\[
\mathcal{H} \Phi_0 = 0
\] (8)

and \( \Phi_0 \) is proportional to the Abrikosov vortex lattice solution \( \varphi \) which is the expression eq.(6) for \( n = 0 \):

\[
\Phi_0 = g_0 \varphi.
\]

To order \( k \), one expands

\[
\Phi_i = g_i \varphi + \sum_{n=1}^{\infty} g_i^n \varphi^n
\] (9)

Inserting into eq.(6) one obtains to order \( a_h^3 \).
\[ \mathcal{H}\Phi_1 = g_0\varphi - g_0 |g_0|^2 |\varphi|^2 \]  

(10)

Taking the inner product with \( \varphi \) one finds that

\[ g_0 = \frac{1}{\sqrt{\beta_A}}, \]  

(11)

where the Abrikosov’s constant is the following average over primitive cell: \( \beta = \beta_A \equiv < |\varphi|^4 > \approx 1.16 \). Inner product with \( \varphi^n \) determines \( g_1^n \):

\[ g_{1,n} = -\frac{\beta^n}{nb^{3/2}}, \]  

(12)

where \( \beta^n \equiv < |\varphi|^2 \varphi^n \varphi^* > \). To find \( g_1 \) we need in addition also the order \( a_h^{5/2} \) equation:

\[ \mathcal{H}\Phi_2 = \Phi_1 - (g_0)^2(2\Phi_1 |\varphi|^2 + \Phi_1^* \varphi^2) \]  

(13)

Inner product with \( \varphi \) gives:

\[ g_1 = \frac{3}{2} \sum_{n=1}^{\infty} \frac{(\beta_n)^2}{nb^{3/2}}. \]  

(14)

C. Mean field result for free energy. Orders \( a_h^2 \) and \( a_h^3 \)

The mean field expression for the free energy to order \( a_h^2 \) is well known. Inserting the next correction eq.(7) into eq.(5) one obtains the free energy density:

\[ \frac{F_{mf}}{T} = \frac{1}{\omega} \left[ -\frac{a_h^2}{2\beta} - \frac{a_h^3}{\beta^3 b} \sum_{n=1}^{\infty} \frac{(\beta^n)^2}{n} \right] = \frac{1}{\omega} \left[ -0.43a_h^2 - 0.0078a_h^3 \right] \]  

(15)

It is interesting to note that \( \beta_n \neq 0 \) only when \( n = 6j \), where \( j \) is an integer. This is due to hexagonal symmetry of the vortex lattice.\(^{13}\) For \( n = 6j \) it decreases very fast with \( j \): \( \beta_6 = -0.2787, \beta_{12} = 0.0249 \). Because of this the coefficient of the next to leading order is very small (additional factor of 6 in the denominator). We might preliminarily conclude therefore that the perturbation theory in \( a_h \) works much better that might be naively anticipated (see dashed line on Fig.1) and can be used very far from transition line. If we demand that the correction is smaller then the main contribution the corresponding line on the phase diagram will be \( b = 0.015 \cdot (1 - t) \). For example the LLL melting line corresponds to \( a_h \sim 1 \). This overly optimistic conclusion is however incorrect as calculation of the following term in Appendix A shows.
D. Range of applicability of the expansion. How precise is LLL?

Now we discuss in what region of the parameter space the expansion outlined above can be applied. First of all note that all the contributions to $\Phi_1$ are proportional to $\frac{1}{b}$. This is a general feature: the actual expansion parameter is $\frac{a_h}{b}$. One can as whether the expansion is convergent and, if yes, what is its radius of convergence. Looking just at the leading correction and comparing it to the LLL one gets a very optimistic estimate. For this purpose we calculated higher orders coefficients in Appendix A. The results for the $\Phi_2$ are following:

\[ g_2^n = \frac{1}{nb} \left[ g_1^n - \frac{1}{\beta} \sum_{i=0}^{\infty} g_1^i (2 < n, 0 | i, 0 > + < 0, 0 | i, n >) \right] \]  

(16)

and

\[ g_2 = -\frac{3}{2\beta} \frac{1}{2\sqrt{\beta}} \sum_{i,j=0}^{\infty} g_1^i g_1^j (< 0, 0 | i, j > + 2 < j, 0 | i, 0 >) \]  

(17)

where $< i_1, i_2 | j_1, j_2 > \equiv < \varphi^{i_1} \varphi^{i_2} \varphi^{j_1} \varphi^{j_2} >$ and $g_j^i$ when $i = 0$ is defined to be equal to $g_j \varphi^i = \varphi$ when $i = 0$.

We already can see that $g_2^n$ and $g_2$ are proportional to $g_1^n$ and in addition there is a factor of $1/n$. Since, due to hexagonal lattice symmetry all the $g_1^n$ , $n \neq 6j$ vanish, so do $g_2^n$. We checked that there is no more small parameters, so we conclude that the leading order coefficient is much larger than first (factor $6 \cdot 5$), but the second is only 6 times larger than the third.

The correction to free energy is

\[ \frac{F_{mf}}{T} = \frac{1}{\omega} \cdot \frac{0.056 a_h^4}{6^2 b^2} \]  

(18)

Accidental smallness by factor $1/6$ of the coefficients in the $\frac{a_h}{b}$ expansion due to symmetry means that the range of validity of this expansion is roughly $a_h < 6b$ or $H < \frac{H_{c2}}{b}$. Moreover additional smallness of all the HLL corrections compared to the LLL means that they constitute just several percent of the correct result inside the region of applicability. To illustrate this point we plot on the Fig.2 the perturbatively calculated solution for $h = .1, t = .5$. One can see that although the leading LLL function has very thick vortices (Fig. 2a), the first nonzero correction makes them of order of the coherence length (Fig. 2b). Following correction of the order $(\frac{a_h}{b})^2$ makes it practically indistinguishable from the numerical solution. Amazingly the order parameter between the vortices approaches its vacuum value. Paradoxically starting from the region close to $H_{c2}$
the perturbation theory knows to correct the order parameter so that it looks very similar to the London approximation (valid only close to $H_{c1}$) result of well separated vortices.

We conclude therefore that the expansion in $a_h/b$ works in the mean field better that one can naively expect. In the next section we investigate whether the same is true for the fluctuation contribution.

III. Fluctuations spectrum

A. Fluctuations to leading order in $a$

To find an excitation spectrum in harmonic approximation one expands free energy functional around the solution found in the previous section. Within the LLL approximation this has been done by. We generalize it to the case of all the Landau levels when perturbations due to nonlinear term are included. The fluctuating order parameter field $\psi$ should be divided into a nonfluctuating (mean field) part and a small fluctuation

$$\psi(x) = \Phi(x) + \chi(x).$$

The energy eq.(5) is then expanded in $\chi$ retaining only quadratic terms

$$f_2 \equiv \int d^2x \left[ \chi^* \mathcal{H} \chi - a_h |\chi|^2 + 2|\Phi|^2 |\chi|^2 + \frac{1}{2}(\Phi^* \chi^2 + \Phi^2 \chi^* 2) \right]$$

Field $\chi$ can be expanded in a basis of quasimomentum $\vec{k}$ eigenfunctions:

$$\varphi^a_{n\vec{k}} = \sqrt{\frac{2\pi}{\sqrt{\pi}2^n n! a}} \sum_{l=-\infty}^{\infty} H_n(y\sqrt{b} + k_x - \frac{2\pi}{a} l) \exp \left\{ i \left[ \frac{\pi l(l-1)}{2} + \frac{2\pi(\sqrt{b} x - k_y)}{a} l - \sqrt{b} k_x \right] - \frac{1}{2}(y\sqrt{b} + k_x - \frac{2\pi}{a} l)^2 \right\}$$

In addition instead of complex field $\chi^a_{\vec{k}}$ we will use two "real" fields $O^a_{\vec{k}}$ and $A^a_{\vec{k}}$ satisfying $O^a_{\vec{k}} = O^a_{-\vec{k}}, A^a_{\vec{k}} = A^a_{-\vec{k}}$:

$$\chi(x) = \int_{\vec{k}} \sum_{n=0}^{\infty} \varphi^a_{n\vec{k}}(x) (O^a_{\vec{k}} + iA^a_{\vec{k}})$$

$$\chi^*(x) = \int_{\vec{k}} \sum_{n=0}^{\infty} \varphi^a_{n\vec{k}}(x) (O^a_{-\vec{k}} - iA^a_{-\vec{k}})$$
In terms of these fields representing "optical" and "acoustic" phonons eq. (20) takes a form:

\[
f_2 = \int_\mathbf{k} \sum_{n=1}^{\infty} 2(nh - a_h)(O_k^nO_{-k}^n + A_k^nA_{-k}^n) - 2a_h(O_kO_{-k} + A_kA_{-k}) \\
+ 2 \sum_{i,j=0}^{\infty} A_k^iA_{-k}^jK_k^{i,j} + O_k^iA_{-k}^jL_k^{i,j} + O_{-k}^iA_{-k}^jM_k^{i,j} + O_k^iO_{-k}^jN_k^{i,j}
\]  

(22)

where elements of the matrix

\[
K_k^{i,j} = <\Phi|^2(\varphi^i_{-k}\varphi^j_{-k} + \varphi^j_{-k}\varphi^i_{-k}) - \frac{1}{2}(\Phi^*\varphi^i_{-k}\varphi^j_{-k} + \varphi^j_{-k}\varphi^i_{-k} > \\
L_k^{i,j} = <\Phi|^2(\varphi^i_{-k}\varphi^j_{-k} - \varphi^j_{-k}\varphi^i_{-k}) + \frac{1}{2}(\Phi^*\varphi^i_{-k}\varphi^j_{-k} - \varphi^j_{-k}\varphi^i_{-k} > \\
M_k^{i,j} = <\Phi|^2(\varphi^i_{-k}\varphi^j_{-k} - \varphi^i_{-k}\varphi^j_{-k}) + \frac{1}{2}(\Phi^*\varphi^i_{-k}\varphi^j_{-k} - \varphi^i_{-k}\varphi^j_{-k} > \\
N_k^{i,j} = <\Phi|^2(\varphi^i_{-k}\varphi^j_{-k}) + \frac{1}{2}(\Phi^*\varphi^i_{-k}\varphi^j_{-k} > \\
\]

(23)

We expand \( f_2 \) in \( a_h \). The order \( a_h \) term is

\[
\int_\mathbf{k} \sum_{i=0}^{\infty} -a_h(O_k^iO_{-k}^i + A_k^iA_{-k}^i) + \\
a_h \sum_{i,j}^{\infty} [A_k^iA_{-k}^jK_k^{i,j}(1) + O_k^iA_{-k}^jL_k^{i,j}(1) + O_{-k}^iA_{-k}^jM_k^{i,j}(1) + O_k^iO_{-k}^jN_k^{i,j}(1)]
\]  

(24)

We will use the degenerate perturbation theory similar to one used in Quantum Mechanics to calculate the correction to the eigenvalues of matrix the LLL states (A, O states) to order \( a_h^2 \). The matrix \( \hat{H} \equiv \begin{pmatrix} K & L \\ M & N \end{pmatrix} \) is analogous to "Hamiltonian" while \( \begin{pmatrix} A \\ O \end{pmatrix} \) is analogous to eigenvector. Explicitly the matrix elements are:
\[ K_{k}^{i,j}(1) = \frac{1}{\beta} \langle |\phi|^{2}(\phi^{*}_{-k}\phi_{j}^{j} + \phi^{*}_{k}\phi_{j}^{j}) - \frac{1}{2}(\phi^{*2}_{-k}\phi^{2}_{j} + \phi^{*2}_{k}\phi^{2}_{j}) \rangle > \]
\[ N_{k}^{i,j}(1) = \frac{1}{\beta} \langle |\phi|^{2}(\phi^{*}_{-k}\phi_{j}^{j} + \phi^{*}_{k}\phi_{j}^{j}) + \frac{1}{2}(\phi^{*2}_{-k}\phi^{2}_{j} + \phi^{*2}_{k}\phi^{2}_{j}) \rangle > \]
\[ L_{k}^{i,j}(1) = \frac{1}{\beta} \langle |\phi|^{2}(\phi^{*}_{-k}\phi_{j}^{j} - \phi^{*}_{k}\phi_{j}^{j}) + \frac{1}{2}(\phi^{*2}_{-k}\phi^{2}_{j} - \phi^{*2}_{k}\phi^{2}_{j}) \rangle > \]
\[ M_{k}^{i,j}(1) = \frac{1}{\beta} \langle |\phi|^{2}(\phi^{*}_{-k}\phi_{j}^{j} - \phi^{*}_{k}\phi_{j}^{j}) + \frac{1}{2}(\phi^{*2}_{-k}\phi^{2}_{j} - \phi^{*2}_{k}\phi^{2}_{j}) \rangle > \] (25)

We will use definitions

\[ \beta^{n}_{k} = \langle |\phi|^{2}\phi^{*}_{k}\phi^{*}_{k} > \]
\[ \beta^{n}_{k} = \langle \phi^{*}\phi^{*}_{k}\phi^{*}_{k} \rangle \]
\[ \gamma^{n}_{k} = \langle (\phi^{*})^{2}\phi_{-k}\phi_{-k} \rangle \]
\[ \gamma^{n}_{k} = \langle \phi^{*}\phi^{*}_{k}\phi^{*}_{k} \rangle \] (26)

When an index is zero we drop it throughout the paper. For example when \( n = 0 \), we call \( \beta^{n}_{k} \) as \( \beta_{k} \), and when \( k = 0 \), \( \beta^{n}_{k} = \beta^{n} \) etc..)

Considering to order \( a_{h} \) above matrix element \( i, j = 0 \) of \( K, L, M, N \) is:

\[ K_{1} = \frac{2}{\beta} \beta_{k} - \frac{1}{2\beta} \text{Re} \gamma_{k} \]
\[ N_{1} = \frac{2}{\beta} \beta_{k} + \frac{1}{2\beta} \text{Re} \gamma_{k} \]
\[ L_{1} = M_{k} = -\frac{1}{\beta} \text{Im} \gamma_{k} \] (27)

In deriving this we have used a property \( \beta_{k} = \beta_{-k} \).

We now diagonalize the matrix to find the eigenstates of LLL (which is of order \( a_{h} \)). Eigenvalues are

\[ \epsilon_{A} = a_{h} \left( -1 + \frac{2}{\beta} \beta_{k} - \frac{1}{\beta} |\gamma_{k}| \right) \] (28)
\[ \epsilon_{O} = a_{h} \left( -1 + \frac{2}{\beta} \beta_{k} + \frac{1}{\beta} |\gamma_{k}| \right) \]
as was found originally by Eilenberger.\textsuperscript{3} The ”acoustic” branch is shown on Fig.3a. The rotation transforming in to these eigenstates is:

\[
\tilde{A}_k = \cos \frac{\theta_k}{2} A_k + \sin \frac{\theta_k}{2} O_k
\]

\[
\tilde{O}_k = -\sin \frac{\theta_k}{2} A_k + \cos \frac{\theta_k}{2} O_k
\]

where \(\gamma_k \equiv |\gamma_k| \exp[i\theta_k]\). Similar calculation for \(n^{th}\) Landau level gives the spectrum:

\[
\varepsilon_{A,O}^n = a_h \left(-1 + \frac{2}{\beta} \langle |\varphi|^2 \varphi_k^* \varphi_k^n > + \frac{1}{\beta} | < (\varphi^* \varphi_k^n \varphi_{-k}^* > \right)
\]

**B. Spectrum of fluctuations beyond leading order in \(a_h\)**

In this subsection we calculate the correction of eigenvalues of LLL to order \(a_h^2\). The ”Hamiltonian” \(\hat{H}\) in addition to has the \(a_h\) part \(\hat{H}_1\) given in eq.(25) also have the \(a_h^2\) part \(\hat{H}_2\). As will be explained in the next section, we will need only correction to the LLL to the \(a_h^2\) order, not the HLL. Therefore we will need only the \(i,j = 0\) matrix element of \(\hat{H}_2\):

\[
K_2 = \sum_{n=1}^{\infty} \frac{1}{nb \beta^2} \left[ \frac{3}{\beta} \beta_n^2 (2\beta_k - \Re \gamma_k) - 2\beta_n (\Re \beta_n + \Re \beta_{-n} - \Re \gamma_k) \right]
\]

\[
N_2 = \sum_{n=1}^{\infty} \frac{1}{nb \beta^2} \left[ \frac{3}{\beta} \beta_n^2 (2\beta_k + \Re \gamma_k) - 2\beta_n (\Re \beta_n + \Re \beta_{-n} + \Re \gamma_k) \right] \quad (31)
\]

\[
L_2 + M_2 = \sum_{n=1}^{\infty} \frac{1}{nb \beta^2} \left[ -\frac{6}{\beta} \beta_n \Im \gamma_k + \frac{4}{\beta} \beta_n \Im \gamma_k \right]
\]

Note that we do not show \(L_2\) and \(M_2\) separately as our result will depend only on \(L_2 + M_2\). According to the degenerate perturbation theory we need to diagonalize \(\hat{H}_1\) which already has been done in the previous subsection and then use the resulting states \(\tilde{A}_k\) and \(\tilde{O}_k\) to calculate the second order correction to the eigenvalue: \(\varepsilon_k^{(2)} = a_h^2 (E_{\text{diag}} + E_{\text{offdiag}})\). The diagonal contribution is

\[
E_{\text{diag}} \equiv \langle \tilde{A}_k | \hat{H}_2 | \tilde{A}_k \rangle = (\cos \frac{\theta_k}{2})^2 K_2 + N_2 (\sin \frac{\theta_k}{2})^2 + (L_2 + M_2) \sin \frac{\theta_k}{2} \cos \frac{\theta_k}{2}
\]

Substituting the matrix elements eq.(31) we obtain

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In the off diagonal contribution,

\[ E_{\text{off diag}} = -\sum_{n=1}^{\infty} \frac{1}{nb} \left\{ \beta_n \{ 3\beta_n (2\beta_k - |\gamma_k|) - 2 \left[ \text{Re} \beta_n^* + \text{Re} \beta_{-n}^* - \cos \theta_k \text{Re} \gamma_n^* - \sin \theta_k \text{Im} \gamma_n^* \right] \right\} \]  

(33)

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(34)

Details of calculation of these matrix elements can be found in Appendix B together with definitions of quantities \( F \). The result is:

\[ E_{\text{off diag}} = -\frac{1}{b} \sum_n \frac{1}{n} \left\{ \beta_n \{ 3\beta_n (2\beta_k - |\gamma_k|) - 2 \left[ \text{Re} \beta_n^* + \text{Re} \beta_{-n}^* - \cos \theta_k \text{Re} \gamma_n^* - \sin \theta_k \text{Im} \gamma_n^* \right] \right\} \]  

(35)

Similar result for \( O_k \) can be obtained from the above formula by changing the sign of \( \cos \theta_k \) and the sign of \( \sin \theta_k \) in the formula above.

It is crucial to see whether there is an \( k^2 \) term in higher orders for the "acoustic" branch \( A \). We calculated numerically the contributions to the spectrum till \( n = 8 \). All the \( k^2 \) contributions to any of them cancel. Moreover even all the \( k^4 \) contributions for odd \( n \) cancel although the even \( n \) give negative contribution to the rotationally symmetric combination \( (k_x^2 + k_y^2)^2 \). Numerically the coefficients are: \( 2.2 \cdot 10^{-6}, 5.0 \cdot 10^{-5}, -6.3 \cdot 10^{-6}, 4.7 \cdot 10^{-7} \) for \( n = 2, 4, 6, 8 \) correspondingly. The resulting correction to the spectrum of "acoustic" branch due to the \( n = 2 \) level is shown on Fig.3b.

After we have established the spectrum of elementary excitations of the Abrikosov lattice, we are ready to calculate the fluctuations contributions to various physical quantities.
IV. Fluctuation contributions to free energy, magnetization and specific heat

A. Higher Landau levels contributions to free energy

The thermal fluctuation part is

\[ -T \log[Z] \equiv F_{mf} + F_{fluc}; \]

\[ F_{fluc} = \frac{T}{2} \sum_{n=0}^{\infty} \left\{ Tr \ln \left[ \varepsilon_A^n(k) + \frac{k_z^2}{2} \right] + Tr \ln \left[ \varepsilon_O^n(k) + \frac{k_z^2}{2} \right] \right\} \]

(36)

\[ = TL_x^2 L_z \sigma b \sum_{n=0}^{\infty} \left[ < \sqrt{\varepsilon_A^n(k)} > + < \sqrt{\varepsilon_O^n(k)} > \right] \]

where \( \sigma \equiv \frac{1}{\sqrt{22\pi}} \) and we performed integration over \( k_z \).

The LLL contribution to order \( \sqrt{a_h} \) in 2D has been calculated by Eilenberger.\(^3\) The 3D result is\(^9\):

\[ \frac{F_{fluc}^{(1/2)}}{T} = \sigma b a_h^{1/2} \left[ < \sqrt{\varepsilon_A^{(1)}(k)} > + < \sqrt{\varepsilon_O^{(1)}(k)} > \right] = 3.16\sigma b a_h^{1/2} \]

(37)

We calculated it’s higher \( a_h \) correction which is of order \( a_h^{3/2} \) using eq.(33) and eq.(35)

\[ \frac{F_{fluc}^{(3/2)}}{T} = \frac{1}{2} \sigma b a_h^{3/2} \left[ < \frac{\varepsilon_A^{(2)}(k)}{\sqrt{\varepsilon_A^{(1)}(k)}} > + < \frac{\varepsilon_O^{(2)}(k)}{\sqrt{\varepsilon_O^{(1)}(k)}} > \right] = -0.445\sigma b a_h^{3/2} \]

(38)

As noted below \( \varepsilon_A^{(2)}(k) \) and \( \varepsilon_O^{(2)}(k) \) given in the last section contain contributions from mixing with all the HLL. Table 1 details contributions to this term from levels till \( n = 8 \). The contributions are negative for all \( n \neq 6j \) where \( j \) is an integer and positive otherwise.

**Table 1.**

Contributions to free energy of mixing of the LLL with HLL.
Given in units of \( \frac{1}{2}\sigma b a_h^{3/2} \).

| level n | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|
| A mode  | -.253| -.082| -.053| -.063| -.063| .247| -.017| -.005|
| O mode  | -.230| -.086| -.051| -.023| -.012| .018| -.003| -.001|
The contribution of HLL is:

\[
\frac{\mathcal{F}_{\text{fluc}}}{T} = \sigma b \sum_{n=1}^{\infty} \left[ < \sqrt{nb + a_h \varepsilon^n_A(k)} > + < \sqrt{nb + a_h \varepsilon^n_O(k)} > \right] \approx \\
2\sigma b^{3/2} \sum_{n=1}^{\infty} \sqrt{n} + \frac{1}{2} \sigma b^{1/2} a_h \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[ < \varepsilon^n_A(k) > + < \varepsilon^n_O(k) > \right] + O(a_h^2) \quad (39)
\]

\[
= \frac{1}{T} (\mathcal{F}^{(0)}_{\text{fluc}} + \mathcal{F}^{(1)}_{\text{fluc}})
\]

The first divergent (as powers $3/2$ and $1/2$ in the ultraviolet) term renormalizes energy. However it has a finite magnetic field dependant part which should be calculated by subtracting the $b = 0$ value of the free energy. The proper regularization is made by restricting the number of Landau levels and then showing that after regularization the answer does not depend on it. The calculation is the same as one done in the normal phase (obviously UV divergencies are insensitive to the phase in which they are calculated), see for details and discussion. The result is:

\[
\frac{\mathcal{F}^{(0)}_{\text{fluc}}}{T} = .526 \sigma b^{3/2}
\]

This exhibits the diamagnetic nature of the bosonic field. The second term is proportional to $\sum \frac{1}{\sqrt{n}}$ and also diverges but only as power $1/2$ in the ultraviolet and renormalizes $a_h$. To see this we calculate the sum

\[
< \varepsilon^n_A(k) + \varepsilon^n_O(k) >_k = -2 + \frac{4}{\beta} << |\varphi|^2 \varphi_k^* \varphi_k^n >_x >_k = -2 + \frac{4}{\beta}
\]

The last equality follows from the curious property of $\varphi_k^n(x) \varphi_k^n(x)$ that it depends only on $x, \sqrt{b} - \varepsilon ij k_j$. We see that apart of renormalizations there is a finite correction:

\[
\frac{\mathcal{F}^{(1)}_{\text{fluc}}}{T} = 1.459 \left( -1 + \frac{2}{\beta} \right) \sigma b^{1/2} a_h
\]

The following one is of the order $a_h^2$ and will not be calculated here.

V. Results for magnetization and specific heat. Range of applicability of the loop expansion

Here we discuss the nature and range of applicability of the expansions we used for fluctuating superconductors (for which $Gi$ is not negligibly small). There are two small parameters used. The first one is $\frac{a_h}{b}$ which controls the expansion
of the mean field solution and therefore the HLL corrections was already discussed in section IID. The second small parameter controls the fluctuations. We assumed the mean field is the leading order and then expanded the statistical sum around it. Summarizing all the corrections the free energy density is

\[
\frac{\mathcal{F}}{T} = \omega^{-1}a_h^2 \left[ c_2^{(-1)} + c_3^{(-1)} a_h b + c_4^{(-1)} \left( \frac{a_h}{b} \right)^2 \right] + \sigma b a_h^{1/2} \left[ c_0^{(0)} a_h^{-1/2} + c_1^{(0)} a_h^{1/2} + c_3^{(0)} a_h \right] + \omega \sigma^2 b^2 a_h^{-1} \left[ c^{(1)}_{-1} \right]
\]

where the coefficients (upper index is the power of \( \omega \) and the lower index is the power of \( a_h \)) are:

\[
c_2^{(-1)} = -0.434, c_3^{(-1)} = -0.0078, c_0^{(0)} = 0.526, \\
c_1^{(0)} = 316, c_0^{(0)} = 1.06, c_3^{(0)} = -0.445, c_{-1}^{(1)} = 0.118
\]

and \( \omega \) is defined in eq.\( (3) \). The last term is the two loop contribution calculated in I. One clearly see that \( \omega ba_h^{-3/2} \) always appears together with an important "loop factor" \( \sigma = \frac{1}{3} \sqrt{\frac{2}{\pi}} \simeq 0.11 \). The expansion parameter therefore is

\[
\sigma \omega a_h^{-3/2} = \pi \sqrt{2 \xi} \frac{tb}{(1-t-b)^{3/2}} = \frac{1}{\sqrt{2 \pi} |a_T|^{3/2}}.
\]

In the last equation \( a_T \) is the often used dimensionless LLL temperature introduced by Thouless.\(^2\) For \( \xi = 0.01 \) the condition \( \sigma \omega a_h^{-3/2} < 1 \) is represented by the area above the dotted line on Fig.1.

Correspondingly the scaled magnetization is \( m = -\frac{\partial \mathcal{F}}{\partial b} \):

\[
m = \omega^{-1} \left[ c_2^{(-1)} a_h + \frac{3}{2} c_3^{(-1)} b^{-1} a_h^2 + 2 c_4^{(-1)} b^{-2} a_h^3 \right] + \sigma \left( c_3^{(-1)} \omega^{-1} b^{-2} a_h^2 - \frac{3}{2} c_0^{(0)} b^{1/2} - c_{1/2}^{(0)} a_h^{1/2} \right) \\
+ \frac{1}{4} c_1^{(0)} b a_h^{-1/2} - \frac{1}{2} c_1^{(0)} b^{-1/2} a_h \left[ \frac{1}{2} c_1^{(0)} b^{1/2} + \frac{3}{4} c_3^{(0)} b^{1/2} \right] - \omega \sigma^2 \left( 2 b a_h^{-1} + \frac{1}{2} b^2 a_h^{-2} \right) \left[ c_{-1}^{(1)} \right]
\]

while the scaled specific heat is \( c = -t \frac{\partial^2 \mathcal{F}}{\partial t^2} \):

\[
c = t T_w \omega^{-1} \left[ -\frac{1}{2} c_2^{(-1)} \omega^{-1} - 3 c_3^{(-1)} b^{-1} a_h - 9 c_4^{(-1)} b^{-2} a_h^2 \right] + \sigma T_w \left( \frac{1}{2} c_1^{(0)} b a_h^{-1/2} + \frac{1}{16} c_{1/2}^{(0)} b a_h^{-3/2} \right) \\
+ c_1^{(0)} b^{1/2} + 3 c_3^{(0)} a_h^{-1/2} - \frac{3}{16} c_3^{(0)} b a_h^{-1/2} \right] + T_w \omega \sigma^2 b^2 \left( a_h^{-2} - \frac{a_h^{-3}}{2} \right) \left[ c_{-1}^{(1)} \right]
\]
VI. Conclusion.

In this paper we showed why the LLL results are often valid far beyond the naive limit of applicability of the approximation for both the mean field and the fluctuation parts. Our results are valid strictly speaking between long dashed line representing \( H = \frac{H_{c2}(T)}{2} \) and one of the dashed curves indicating the range of validity of the loop expansion for the fluctuation contribution (depends on value of the Ginzburg number \( G_i \)). For nonfluctuating strongly type II superconductors our results can be directly checked by experiments done at low temperature or numerical solution (or even the "London limit approximation") and are in clear agreement. For small, but not very small Ginzburg parameter \( G_i \) one can compare with existing Monte Carlo simulations\(^8\,17\) or experiments. Of course one can use existing high temperature expansion\(^2\) to interpolate to the present expansion range. Results for LLL were presented in I and the HLL do not alter them significantly. The agreement with the MC simulations is very good although obviously the melting transition is not seen. As argued in I it is not expected to exist within the present model. The HLL do not change this conclusion. On the contrary we explicitly showed that the "supersoft" \( A \) mode has a propagator \( 1/(k_z^2 + \text{const}(k_x^4 + k_y^4)) \) beyond the LLL approximation laying to rest a suspicion that this is a fluke due to LLL. This indicates that this unusual "softness" is due to some underlying symmetry which have yet to be explicitly identified.

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Appendix A: Third order correction to the mean field solution and free energy

In this appendix we provide some details of the third order in \( a_h \) calculation of the mean field solution of the GL equations.

To calculate \( g_2^n \), one takes the inner product of \( \varphi^n \) on the two sides of eq.\((13)\) and obtains eq.\((16)\). To calculate \( g_2 \), we need to consider the GL equation to order \( a_h^{7/2} \):

\[
\mathcal{H}\Phi_3 = \Phi_2 - [(\Phi_0)^2\Phi_2^* + (\Phi_1)^2\Phi_0^* + 2|\Phi_0|^2\Phi_2 + 2|\Phi_1|^2\Phi_0^*]
\]
Scalar product with $\varphi$ gives eq. (17).

Now we compute the $a_4^h$ order correction to free energy. Substituting $\Phi_2$ from eqs. (16) and (17) we find:

$$
\frac{1}{2} a_4^3 \left( < \Phi_2 | \mathcal{H} | \Phi_1 > + < \Phi_1 | \mathcal{H} | \Phi_2 > - < \Phi_2 | \Phi_0 > - < \Phi_2 | \Phi > - < \Phi_1 | \Phi_1 > \right)
$$

$$
= - \frac{g_2}{\beta^{1/2} a_4^3} + a_4^3 \sum_{n=0} \left[ n b g_1^n g_2^n - \frac{1}{2} (g_1^n)^2 \right]
$$

(48)

Dominant contributions come from: $< 6, 6 | 0, 0 > = < 0, 0 | 6, 6 > = .80260$
$< 6, 0 | 6, 0 > = .80283$ and those coefficients are real.

**Appendix B: Second order correction to fluctuations spectrum**

In this appendix we list matrix elements of the correction $\widehat{H}_1$ given by eq. (25) between various states used in the calculation of the second order correction to energies of excitations.

$$
< O_k | \widehat{H}_1 | O_n^k > = \frac{1}{\beta} < | \varphi |^2 (\varphi^*_k \varphi^*_n - \varphi^*_n \varphi^*_k) > + \frac{1}{2 \beta} < \varphi^*_k \varphi^*_n - \varphi^*_n \varphi^*_k > + \frac{1}{2 \beta} \left[ \beta_n^k + \beta_n^* \right]
$$

(49)

$$
< A_k | \widehat{H}_1 | O_n^k > = \frac{i}{\beta} < | \varphi |^2 (\varphi^*_k \varphi^*_n - \varphi^*_n \varphi^*_k) > + \frac{i}{2 \beta} < \varphi^*_k \varphi^*_n - \varphi^*_n \varphi^*_k > + \frac{i}{2 \beta} \left[ - \beta_n^k + \beta_n^* \right]
$$

(50)

17
\[<O_k | \hat{H}_1 | A^n_k> = \frac{i}{\beta} \left[ \beta^n_k - \beta^{*n}_k + \frac{1}{2} (\gamma^n_k - \gamma^{*n}_k) \right] \]

\[<A^n_k | \hat{H}_1 | O_k> = \frac{i}{\beta} \left[ \beta^n_k - \beta^{*n}_k + \frac{1}{2} (-\gamma^n_k + \gamma^{*n}_k) \right] \] (51)

\[<A_k | \hat{H}_1 | A^n_k> = \frac{1}{\beta} \left[ \beta^{*n}_k + \beta^n_k + \frac{1}{2} (\gamma^n_k + \gamma^{*n}_k) \right] \]

etc. From those formula, we can show

\[| <A_k | \hat{H}_1 | A^n_k> |^2 + | <A_k | \hat{H}_1 | O^n_k> |^2 = \frac{2}{\beta^2} \left[ |F^n_k(1)|^2 + |F^{*n}_{-k}(1)|^2 \right] \]

\[| <O_k | \hat{H}_1 | O^n_k> |^2 + | <O_k | \hat{H}_1 | A^n_k> |^2 = \frac{2}{\beta^2} \left[ |F^n_k(2)|^2 + |F^{*n}_{-k}(2)|^2 \right] \] (52)

\[<A_k | \hat{H}_1 | A^n_k> <A^n_k | \hat{H}_1 | O_k> + <A_k | \hat{H}_1 | O^n_k> <O^n_k | \hat{H}_1 | O_k> + c.c = \]

\[\frac{2}{\beta^2} [F^n_k(1)F^{*n}_{-k}(2) - F^{*n}_{-k}(1)F^n_k(2)] + c.c, \]

where

\[F^n_k(1) = \beta^n_k - \frac{1}{2} \gamma^n_k \]

\[F^n_k(2) = \beta^n_k + \frac{1}{2} \gamma^n_k \] (53)

Finally we can show that

\[E_{offdiag} = -\frac{1}{b} \sum_n \frac{1}{n} \left\{ \frac{1}{\beta^2} \left[ |F^n_k(1)|^2 + |F^{*n}_{-k}(1)|^2 + |F^n_k(2)|^2 + |F^{*n}_{-k}(2)|^2 \right] \right\} \]

\[+ \frac{\cos \theta_k}{\beta^2} \left[ |F^n_k(1)|^2 + |F^{*n}_{-k}(1)|^2 - |F^n_k(2)|^2 - |F^{*n}_{-k}(2)|^2 \right] \] (54)

\[+ \frac{\sin \theta_k}{\beta^2} 2 \text{Im}[F^n_k(1)F^{*n}_{-k}(2) - F^{*n}_{-k}(1)F^n_k(2)] \} \]
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Figure captions

Fig. 1

The range of the validity of the expansions in $a_h$ and the loop expansion. The region above the dotted line is the naively expected validity range of the LLL approximation. The region above the long dashed line is the actual validity range for the expansion of the mean field equations. The loop expansion applicability range lies below the dashed curves. We plot two curves with different values of Ginzburg number $G_i$, $G_i = 0.1$ and $G_i = 0.01$. The validity combining the mean field expansion and the loop expansion lies therefore between the long dashed line and the dashed curves.

Fig. 2

The density $|\psi^2|$ of the mean field Abrikosov solution for $b = 0.1, t = 0.5$. Fig. 2a is the lowest order approximation (LLL). Fig. 2b is the solution with the next order correction included, while in Fig. 2c the next next order correction is included.

Fig. 3

The shear mode $A$ spectrum. Fig. 3a is the spectrum obtained within the LLL approximation. Fig. 3b is the correction to the spectrum when the HLL mixing effect is considered.