Minimal hulls of compact sets in $\mathbb{R}^3$

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Geometric Methods of Complex Analysis

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We characterize the **minimal surface hull** of a compact set $K$ in $\mathbb{R}^3$ by sequences of conformal minimal discs whose boundaries converge to $K$ in the measure theoretic sense. The analogous result is obtained for the **null hull** of a compact subset of $\mathbb{C}^3$. These are inspired by Poletsky’s characterization of the polynomial hull of a compact set in $\mathbb{C}^n$ by sequences of holomorphic discs.
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We also describe the minimal hull by 2-dimensional minimal currents which are limits of Green currents supported by conformal minimal discs, in the spirit of the Duval-Sibony-Wold characterization of polynomial hulls by positive \((1,1)\)-currents.
Abstract

(Joint work with Barbara Drinovec Drnovšek, University of Ljubljana.)

- We characterize the **minimal surface hull** of a compact set $K$ in $\mathbb{R}^3$ by sequences of conformal minimal discs whose boundaries converge to $K$ in the measure theoretic sense. The analogous result is obtained for the **null hull** of a compact subset of $\mathbb{C}^3$. These are inspired by Poletsky’s characterization of the polynomial hull of a compact set in $\mathbb{C}^n$ by sequences of holomorphic discs.

- We also describe the minimal hull by 2-dimensional **minimal currents** which are limits of Green currents supported by conformal minimal discs, in the spirit of the Duval-Sibony-Wold characterization of polynomial hulls by positive $(1, 1)$-currents.

- We obtain a **polynomial hull version of Bochner’s tube theorem**.

B. Drinovec Drnovšek; F. Forstnerič: Minimal hulls of compact sets in $\mathbb{R}^3$. Preprint (2014). arxiv.org/abs/1409.6906
Hulls of compact sets

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Given a set $\mathcal{P}$ of real functions on a manifold $X$, the $\mathcal{P}$-hull of a compact subset $K \subset X$ is defined by

$$\hat{K}_\mathcal{P} = \{ x \in X : f(x) \leq \sup_K f \quad \forall f \in \mathcal{P} \}.$$

Suppose that $G$ is a class of geometric objects in $X$ (submanifolds, subvarieties, minimal surfaces,...) such that the restriction $f|_C$ satisfies the maximum principle for every $f \in \mathcal{P}$ and $C \in G$. Then $C \subset \hat{K}_\mathcal{P}$ for every $C \in G$ with boundary $bC \subset K$.

The main question: How closely is the hull $\hat{K}_\mathcal{P}$ described by the objects in $G$ with boundaries in $K$ (in a suitably general sense)?
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**The Main question:** How closely is the hull $\hat{K}_\mathcal{P}$ described by the objects in $\mathcal{G}$ with boundaries in $K$ (in a suitably general sense)?
A classical example is the **polynomial hull**, $\mathring{K}$, of a compact set $K \subset \mathbb{C}^n$. This is the hull $\mathring{K}_P$ for $P = \{|f| : f \in \mathcal{O}(\mathbb{C}^n)\}$, or $P = \text{Psh}(\mathbb{C}^n)$ (plurisubharmonic functions).
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**Wermer (1958), Stolzenberg (1966), Alexander (1971):** If \( K \) is a compact sets of finite linear measure then \( \hat{K} \setminus K \) is a (possibly empty) one dimensional closed complex subvariety of \( \mathbb{C}^n \setminus K \).
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**Theorem (Poletsky 1991, 1993)**

Let \( K \) be a compact set in \( \mathbb{C}^n \), and let \( B \subset \mathbb{C}^n \) be a ball containing \( K \). A point \( p \in B \) belongs to \( \hat{K} \) iff there exists a sequence of holomorphic discs \( f_j : \overline{D} \to B \) satisfying the following for every \( j = 1, 2, \ldots \):

\[
f_j(0) = p \quad \text{and} \quad \left| \left\{ t \in [0, 2\pi] : \text{dist}(f(e^{it}), K) < 1/j \right\} \right| \geq 2\pi - 1/j.
\]
Definition

A smoothly immersed surface $f: M \to \mathbb{R}^n$ is a **minimal surface** if its **mean curvature vector field** $H: M \to \mathbb{R}^n$ is identically zero: $H = 0$. When $n = 3$, we have $H = H \cdot N$ where $N$ is the Gauss map and $H = \kappa_1 + \kappa_2 + \ldots$ the mean curvature function.

In local isothermal coordinates $(x, y)$ on $M$ we have:

$$\Delta f = 2|\nabla f_x|^2 = |\nabla f_y|^2.$$

**Lemma (Classical; proof in Appendix A)**
The following are equivalent for a smooth conformal immersion $f: M \to \mathbb{R}^n$ from an open Riemann surface $M$ to $\mathbb{R}^n$:

- $f$ is minimal (a stationary point of the area functional).
- $f$ has vanishing mean curvature vector: $H = 0$.
- $f$ is harmonic: $\Delta f = 0$. 
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- \( f \) is harmonic: \( \triangle f = 0 \).
In the sequel $M$ denotes an open or a bordered Riemann surface.

**Definition**

A holomorphic immersion

$$F = (F_1, F_2, \ldots, F_n): M \to \mathbb{C}^n, \quad n \geq 3$$

is a **null curve** if the derivative $F' = (F'_1, F'_2, \ldots, F'_n)$ with respect to any local holomorphic coordinate $\zeta = x + iy$ on $M$ satisfies

$$(F'_1)^2 + (F'_2)^2 + \ldots + (F'_n)^2 = 0.$$
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The nullity condition is equivalent to $F'(\zeta) \in A_* = A \setminus \{0\}$ where

$$A = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} z_j^2 = 0 \} \ldots \text{the null quadric.}$$
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It is easily seen that $A_*$ is **elliptic** in the sense of Gromov, and hence an **Oka manifold**.
The Oka principle for null curves

There exist plenty of null curves $M \rightarrow \mathbb{C}^n$ from any open Riemann surface; we have the Runge and the Mergelyan approximation theorem, and also the following Oka principle.

Theorem
Let $\vartheta$ be a nowhere vanishing holomorphic 1-form on an open Riemann surface $M$. Then every continuous map $h_0 : M \rightarrow A^*$ is homotopic to a holomorphic map $h : M \rightarrow A^*$ such that $h \vartheta$ has vanishing periods, and hence $F(x) = F(p) + \int_{x}^{p} h \vartheta$, $x \in M$ is a null holomorphic immersion $M \rightarrow \mathbb{C}^n$. $F$ can be chosen proper. The same holds if $A \subset \mathbb{C}^n$ is an irreducible cone with the only singularity at $0 \in \mathbb{C}^n$ such that $A^* = A \{0\}$ is an Oka manifold.

[A. Alarcón, F. Forstnerič: Null curves and directed immersions of open Riemann surfaces. Inventiones Math. 196 (2014), 733–771]
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$$F(x) = F(p) + \int_p^x h\theta, \quad x \in M$$

is a null holomorphic immersion $M \to \mathbb{C}^n$. $F$ can be chosen proper.
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If $F = f + ig: M \to \mathbb{C}^n$ is a holomorphic null curve, then

$$f = \Re F: M \to \mathbb{R}^n, \quad g = \Im F: M \to \mathbb{R}^n$$

are conformal harmonic (hence minimal) immersions into $\mathbb{R}^n$. 
Connection between null curves and minimal surfaces

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**Proof:** Let \( F = f + ig = (F^1, \ldots, F^n) : M \to \mathbb{C}^n \) be a holomorphic null curve and \( \zeta = x + iy \) a local holomorphic coordinate on \( M \). Then

\[
0 = \sum_{j=1}^{n} (F^j_x)^2 = \sum_{j=1}^{n} \left( f^j_x + i g^j_x \right)^2 = \sum_{j=1}^{n} \left( (f^j_x)^2 - (g^j_x)^2 \right) + 2i \sum_{j=1}^{n} f^j_x g^j_x.
\]

Since \( g_x = -f_y \) by the Cauchy-Riemann equations, the above reads

\[
0 = \|f_x\|^2 - \|f_y\|^2 - 2i f_x \cdot f_y \iff \|f_x\| = \|f_y\|, \ f_x \cdot f_y = 0.
\]

It follows that \( f \) is a conformal minimal immersion (CMI).
Conversely,

A conformal minimal immersion (CMI) \(f : \mathbb{D} \to \mathbb{R}^n\) of the disc \(\mathbb{D} = \{|\zeta| < 1\}\) is the real part of a null disc \(F : \mathbb{D} \to \mathbb{C}^n\).
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**Proof:** In any local holomorphic coordinate \( \zeta = x + iy \) on \( M \) we have

\[ ||f_x|| = ||f_y|| > 0, \quad f_x \cdot f_y = 0 \quad \text{(conformality)} \]

Equivalently, \( f_x \pm if_y \in A_* \) are null vectors. From \( 2\partial f = (f_x - if_y)d\zeta \) we infer that \( f \) is conformal if and only if

\[ (\partial f^1)^2 + (\partial f^2)^2 + \cdots + (\partial f^n)^2 = 0, \]

and is conformal harmonic (=conformal minimal) iff

\[ \partial f = (\partial f^1, \ldots, \partial f^n) \] is a holomorphic 1-form with values in \( A_* \).

If \( g \) is any local harmonic conjugate of \( f \), then CR equations imply \( \partial(f + ig) = 2\partial f = 2i\partial g \), so \( F = f + ig \) is a null holomorphic immersion.
Weierstrass representation of conformal minimal immersions

Hence every conformal minimal immersion \( f: M \rightarrow \mathbb{R}^n \) is of the form

\[
f(x) = f(p) + \Re \int_p^x \phi \quad (p, x \in M)
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where \( \phi = (\phi_1, \ldots, \phi_n) \) is a \( \mathbb{C}^n \)-valued holomorphic 1-form on \( M \) without zeros satisfying the nullity condition

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Every conformal minimal immersion $M \to \mathbb{R}^n$ is homotopic through conformal minimal immersions to the real part of a holomorphic null curve $M \to \mathbb{C}^n$.

[A. Alarcón, F. Forstnerič, Preprint (2014), arxiv.org/abs/1408.5315]
Null plurisubharmonic functions

Definition

An upper semicontinuous function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ on a domain $\Omega \subset \mathbb{C}^n$ is null plurisubharmonic if the restriction of $u$ to any affine complex line $L \subset \mathbb{C}^n$ directed by a null vector $\theta \in A^*$ is subharmonic on $L \cap \Omega$. The class of all such functions is denoted $\mathcal{MPsh}(\Omega)$. 

Clearly $\mathcal{Psh}(\Omega) \subset \mathcal{MPsh}(\Omega)$. The inclusion is proper: the function $u(z_1, z_2, z_3) = |z_1|^2 + |z_2|^2 - |z_3|^2$ is null plurisubharmonic on $\mathbb{C}^3$, but is not plurisubharmonic.

Null plurisubharmonic functions satisfy most of the standard properties of plurisubharmonic functions. The main point: if $F : M \to \Omega$ is a holomorphic null curve and $u \in \mathcal{MPsh}(\Omega)$ then $u \circ F$ is subharmonic on $M$. 
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Definition (Harvey & Lawson; Drinovec Drnovšek & F.)

An upper semicontinuous function $u$ on a domain $\omega \subset \mathbb{R}^n$ ($n \geq 3$) is minimal plurisubharmonic (M-psh) if the restriction of $u$ to any affine 2-dimensional plane $L \subset \mathbb{R}^n$ is subharmonic on $L \cap \omega$ (in isothermal coordinates). The set of all such functions is denoted $\mathcal{MPsh}(\omega)$. 
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- Every $\text{MPsh}$ function can be approximated by smooth $\text{MPsh}$ functions.
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- Every $\mathcal{MPsh}$ function can be approximated by smooth $\mathcal{MPsh}$ functions.
- A $C^2$ function $u$ is $\mathcal{MPsh}$ if and only if the sum of the two smallest eigenvalues of $\text{Hess } u$ is nonnegative at every point.
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- A $C^2$ function $u$ is $\text{MPsh}$ if and only if the sum of the two smallest eigenvalues of $\text{Hess } u$ is nonnegative at every point.
- The restriction of a $\text{MPsh}$ function to an immersed minimal surface is subharmonic (in isothermal coordinates on the surface).
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- If $u \in \mathcal{MPsh}(\omega)$ then the function $U(x + iy) = u(x)$ ($x + iy \in T_\omega$) is null plurisubharmonic on the tube $T_\omega$.

**Proof.**

Recall that the real and the imaginary part of a holomorphic null disc $f \in \mathcal{N}(D, \mathbb{C}^n)$ are conformal minimal discs in $\mathbb{R}^n$; conversely, every conformal minimal disc in $\mathbb{R}^n$ is the real part of a holomorphic null disc in $\mathbb{C}^n$. Since $U \circ f = u \circ \Re f$ for all $f \in \mathcal{NPsh}(D, T_\omega)$, the lemma follows.
Lemma

Let $\omega \subset \mathbb{R}^n$ and let $\mathcal{T}_\omega = \omega \times i\mathbb{R}^n \subset \mathbb{C}^n$ be the tube over $\omega$.

- If $u \in \mathcal{MPsh}(\omega)$ then the function $U(x + iy) = u(x)$ ($x + iy \in \mathcal{T}_\omega$) is null plurisubharmonic on the tube $\mathcal{T}_\omega$.
- Conversely, if $U \in \mathcal{NPsh}(\mathcal{T}_\omega)$ is independent of $y = \Im z$ then $u(x) = U(x + i0)$ is minimal plurisubharmonic on $\omega$. 

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Connection between $\mathcal{M}Psh(\omega)$ and $\mathcal{N}Psh(\mathcal{T}_\omega)$

Lemma

Let $\omega \subset \mathbb{R}^n$ and let $\mathcal{T}_\omega = \omega \times i\mathbb{R}^n \subset \mathbb{C}^n$ be the tube over $\omega$.

- If $u \in \mathcal{M}Psh(\omega)$ then the function $U(x + iy) = u(x)$ ($x + iy \in \mathcal{T}_\omega$) is null plurisubharmonic on the tube $\mathcal{T}_\omega$.
- Conversely, if $U \in \mathcal{N}Psh(\mathcal{T}_\omega)$ is independent of $y = \Re z$ then $u(x) = U(x + i0)$ is minimal plurisubharmonic on $\omega$.

Proof.

Recall that

- the real and the imaginary part of a holomorphic null disc $f \in \mathcal{N}(\mathbb{D}, \mathbb{C}^n)$ are conformal minimal discs in $\mathbb{R}^n$;
Connection between $\mathcal{MPsh}(\omega)$ and $\mathcal{NPsh}(\mathcal{T}_\omega)$

**Lemma**

Let $\omega \subset \mathbb{R}^n$ and let $\mathcal{T}_\omega = \omega \times i\mathbb{R}^n \subset \mathbb{C}^n$ be the tube over $\omega$.

- **If** $u \in \mathcal{MPsh}(\omega)$ **then** the function $U(x + iy) = u(x)$ $(x + iy \in \mathcal{T}_\omega)$ **is** null plurisubharmonic on the tube $\mathcal{T}_\omega$.

- **Conversely,** if $U \in \mathcal{NPsh}(\mathcal{T}_\omega)$ **is** independent of $y = \mathfrak{S}z$ **then** $u(x) = U(x + i0)$ **is** minimal plurisubharmonic on $\omega$.

**Proof.**

Recall that

- the real and the imaginary part of a holomorphic null disc $f \in \mathcal{N}(\mathbb{D}, \mathbb{C}^n)$ **are** conformal minimal discs in $\mathbb{R}^n$;

- conversely, every conformal minimal disc in $\mathbb{R}^n$ **is** the real part of a holomorphic null disc in $\mathbb{C}^n$.

Since $U \circ f = u \circ \Re f$ for all $f \in \mathcal{N}(\mathbb{D}, \mathcal{T}_\omega)$, the lemma follows.
The following is our main result. It is analogous to Poletsky’s theorem on plurisubharmonic minorants (with generalisations by Lárusson and Sigurdsson, Rosay, B.D.-F., Kuzman,...).

$\text{CMI}(\overline{D}, \omega) \ldots$ conformal minimal immersions $\overline{D} \to \omega \subset \mathbb{R}^3$. 
Disc formulas for biggest $\mathcal{M}_{\text{Psh}}$ and $\mathcal{N}_{\text{Psh}}$ minorants

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Theorem (B. Drinovec Drnovšek & F., 2014)

(a) If $\omega$ is a domain in $\mathbb{R}^3$ and $\phi: \omega \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function, then the function $u: \omega \to \mathbb{R} \cup \{-\infty\}$,

$$u(x) = \inf \left\{ \int_0^{2\pi} \phi(f(e^{it})) \frac{dt}{2\pi} : f \in \text{CMI}(\overline{D}, \omega), f(0) = x \right\}$$ (1)

is minimal plurisubharmonic or identically $-\infty$; moreover, $u$ is the supremum of functions in $\mathcal{M}_{\text{Psh}}(\omega)$ which are bounded above by $\phi$. 
Disc formulas for biggest $\mathcal{M}\text{Psh}$ and $\mathcal{N}\text{Psh}$ minorants

The following is our main result. It is analogous to Poletsky’s theorem on plurisubharmonic minorants (with generalisations by Lárusson and Sigurdsson, Rosay, B.D.-F., Kuzman, ...).

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is minimal plurisubharmonic or identically $-\infty$; moreover, $u$ is the supremum of functions in $\mathcal{M}\text{Psh}(\omega)$ which are bounded above by $\phi$.

(b) If $\Omega$ is a domain in $\mathbb{C}^3$, $\phi: \Omega \to \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function, and we use null holomorphic discs $f: \overline{D} \to \Omega$ in (1), then the resulting function $u$ is the biggest null plurisubharmonic function on $\Omega$ which is bounded above by $\phi$. 
A lemma of Edgar and Bu-Schachermayer

Part (a) follows from part (b) in view of the previous lemma. The proof of (a) is based on the following EBS-lemma (Edgar 1985; Bu-Schachermayer 1992) adapted to null plurisubharmonic functions.
A lemma of Edgar and Bu-Schachermayer

Part (a) follows from part (b) in view of the previous lemma. The proof of (a) is based on the following **EBS-lemma** (*Edgar 1985; Bu-Schachermayer 1992*) adapted to null plurisubharmonic functions.

**Lemma**

Let $\Omega$ be a domain in $\mathbb{C}^n$ ($n \geq 3$) and $\phi: \Omega \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function. Define $u_1 = \phi$ and for $j > 1$

$$u_j(z) = \inf \left\{ \int_0^{2\pi} \left( u_{j-1}(z + e^{it}\theta) \frac{dt}{2\pi} \right) \right\}, \quad z \in \Omega,$$

where the infimum is taken over all null vectors $\theta \in A_*$ such that

$$\{z + \zeta\theta : |\zeta| \leq 1\} \subset \Omega.$$

Then the functions $u_j$ are upper semicontinuous and decrease pointwise to the largest null plurisubharmonic function $u_\phi$ on $\Omega$ with $u_\phi \leq \phi$. 
Proof of EBS lemma

**Proof:** We first show by induction that the functions $u_j$ are upper semicontinuous (u.s.c.); then $u_\phi$ is also u.s.c.
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Assume that $u_{j-1}$ is u.s.c.; this holds when $j = 2$. Given $z_0 \in \Omega$ and $\epsilon > 0$, there exists a null vector $\theta \in A^*$ such that

$$u_j(z_0) \leq \int_0^{2\pi} u_{j-1}(z_0 + e^{it}\theta) \frac{dt}{2\pi} < u_j(z_0) + \epsilon.$$
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For $z$ close enough to $z_0$ we have $u_{j-1}(z + e^{it}\theta) < u_{j-1}(z_0 + e^{it}\theta) + \epsilon$ for all $t$ (since $u_{j-1}$ is u.s.c.). Hence the average increases by at most $\epsilon$, so we get $u_j(z) < u_j(z_0) + 2\epsilon$. 
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Next we show that $u_\phi \in \mathcal{NPsh}(\Omega)$. Pick a point $z \in \Omega$ and a null vector $\theta \in A^*$ such that $z + \overline{D}\theta \subset \Omega$. By the monotone convergence theorem:

$$u_\phi(z) = \lim_{j \to \infty} u_j(z) \leq \lim_{j \to \infty} \int_0^{2\pi} u_{j-1}(z + e^{it}\theta) \frac{dt}{2\pi} = \int_0^{2\pi} u_\phi(z + e^{it}\theta) \frac{dt}{2\pi}.$$
Proof of null plurisubharmonicity

It remains to prove that $u_\phi$ is the largest null plurisubharmonic function dominated by $\phi$. 

Choose a null plurisubharmonic function $v \leq \phi = u_1$. We show by induction that $v \leq u_j$ for every $j \in \mathbb{N}$; clearly this will imply that $v \leq u_\phi$.

Suppose that $v \leq u_{j-1}$ for some $j$. Then for every $z \in \Omega$ and for every null vector $\theta \in A^*$ such that $z + D\theta \subset \Omega$ we have

$$v(z) \leq \int_{\pi}^\pi v(z + e^{i\theta}) dt \leq \int_{\pi}^\pi u_{j-1}(z + e^{i\theta}) dt.$$ 

Taking infimum over all $\theta$ we get $v(z) \leq u_j(z)$ which concludes the inductive step. This proves the EBS lemma.
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\[
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This proves the EBS lemma.
Riemann-Hilbert problem for null discs

We also use the following RH lemma. (Proof in Appendix B.)

Lemma (Alarcón & F. (2013), arxiv.org/abs/1308.0903)

Let $f : \overline{D} \to \mathbb{C}^3$ be a null holomorphic immersion, let $\theta \in A_*$ be a null vector, and let $\mu : \mathbb{T} = bD \to [0, \infty)$ be a continuous function. Set

$$g : \mathbb{T} \times \overline{D} \to \mathbb{C}^3, \quad g(\zeta, z) = f(\zeta) + \mu(\zeta)z\theta.$$
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Given $\epsilon > 0$ and $r \in (0, 1)$, there exist $r' \in [r, 1)$ and a null holomorphic disc $h: \overline{D} \to \mathbb{C}^3$ with $h(0) = f(0)$ satisfying the following properties:
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\textbf{Lemma (Alarcón \& F. (2013), arxiv.org/abs/1308.0903)}

Let $f: \overline{D} \rightarrow \mathbb{C}^3$ be a null holomorphic immersion, let $\theta \in A_*$ be a null vector, and let $\mu: \Gamma = bD \rightarrow [0, \infty)$ be a continuous function. Set

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Given $\epsilon > 0$ and $r \in (0, 1)$, there exist $r' \in [r, 1)$ and a null holomorphic disc $h: \overline{D} \rightarrow \mathbb{C}^3$ with $h(0) = f(0)$ satisfying the following properties:

(i) \dist(h(\zeta), g(\zeta, \Gamma)) < \epsilon \text{ for all } \zeta \in \Gamma.$
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(i) \( \text{dist}(h(\zeta), g(\zeta, \mathbb{T})) < \epsilon \) for all \( \zeta \in \mathbb{T} \).

(ii) \( \text{dist}(h(\rho\zeta), g(\zeta, \mathbb{D})) < \epsilon \) for all \( \zeta \in \mathbb{T} \) and \( \rho \in [r', 1) \).
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(iii) \( h \) is \( \epsilon \)-close to \( f \) on \( \{ \zeta \in \mathbb{C} : |\zeta| \leq r' \} \).
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Given $\epsilon > 0$ and $r \in (0, 1)$, there exist $r' \in [r, 1)$ and a null holomorphic disc $h : \overline{D} \to \mathbb{C}^3$ with $h(0) = f(0)$ satisfying the following properties:

(i) $\text{dist}(h(\zeta), g(\zeta, \mathbb{T})) < \epsilon$ for all $\zeta \in \mathbb{T}$.

(ii) $\text{dist}(h(\rho\zeta), g(\zeta, \mathbb{D})) < \epsilon$ for all $\zeta \in \mathbb{T}$ and $\rho \in [r', 1)$.

(iii) $h$ is $\epsilon$-close to $f$ on $\{\zeta \in \mathbb{C} : |\zeta| \leq r'\}$.

Furthermore, given an upper semicontinuous function $u : \mathbb{C}^3 \to \mathbb{R} \cup \{-\infty\}$ and an arc $I \subset \mathbb{T}$, we may achieve that

$$\int_I u(h(e^{i\phi})) \frac{ds}{2\pi} \leq \int_0^{2\pi} \int_{s \in I} u(g(e^{i\phi}, e^{it})) \frac{ds}{2\pi} \frac{dt}{2\pi} + \epsilon.$$
Proof of the disc formula for the $\mathcal{NPsh}$ minorant

The EBS-lemma furnishes a decreasing sequence of upper semicontinuous functions $u_n$ on $\Omega$ which converges pointwise to the largest null plurisubharmonic function $u_\phi$ on $\Omega$ with $u_\phi \leq \phi$. 

Let $N(D, \Omega, z)$ denote the set of null discs $f : D \to \Omega$ with $f(0) = z$. To conclude the proof, we need to show that for every $z \in \Omega$ 

$$u_\phi(z) = u_\phi(f(0)) \leq \int_0^{2\pi} u_\phi(f(e^{i\theta})) \, d\theta \leq \int_0^{2\pi} \phi(f(e^{i\theta})) \, d\theta.$$ 

Since $u_\phi \in \mathcal{NPsh}(\Omega)$, $u_\phi \circ f$ is subharmonic for any $f \in N(D, \Omega, z)$, and hence we get by the submeanvalue property (using also $u_\phi \leq \phi$) that 

$$u_\phi(z) = u_\phi(f(0)) \leq \int_0^{2\pi} u_\phi(f(e^{i\theta})) \, d\theta \leq \int_0^{2\pi} \phi(f(e^{i\theta})) \, d\theta.$$ 

Taking the infimum over all such $f$ we obtain $u_\phi \leq u$ on $\Omega$. 
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Let \( \mathcal{N}(\mathbb{D}, \Omega, z) \) denote the set of null discs \( f: \overline{\mathbb{D}} \to \Omega \) with \( f(0) = z \).

To conclude the proof, we need to show that for every \( z \in \Omega \)

\[
    u_\phi(z) = u(z) := \inf \left\{ \int_0^{2\pi} \phi(f(e^{it})) \frac{dt}{2\pi} : f \in \mathcal{N}(\mathbb{D}, \Omega, z) \right\}.
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Since \( u_\phi \in \mathcal{NPsh}(\Omega) \), \( u_\phi \circ f \) is subharmonic for any \( f \in \mathcal{N}(\mathbb{D}, \Omega, z) \), and hence we get by the submeanvalue property (using also \( u_\phi \leq \phi \)) that

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Since $u_\phi \in \mathcal{NPsh}(\Omega)$, $u_\phi \circ f$ is subharmonic for any $f \in \mathcal{N}(\mathbb{D}, \Omega, z)$, and hence we get by the submeanvalue property (using also $u_\phi \leq \phi$) that

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Taking the infimum over all such $f$ we obtain $u_\phi \leq u$ on $\Omega$. 
Proof of the reverse inequality $u \leq u_\phi$:

Fix a point $z \in \Omega$ and a number $\epsilon > 0$. Pick $n \in \mathbb{N}$ such that $u_\phi(z) \leq u_n(z) < u_\phi(z) + \epsilon$. (2)

By the definition of $u_n$ there exists a null vector $\theta \in A^*$ such that the linear null disc $D \ni \zeta \mapsto f_n - 1(\zeta) = z + \zeta \theta \in \Omega$ satisfies $u_n(z) \leq \int_{2\pi}^0 u_n - 1(f_n - 1(\epsilon + it \theta)) ds < u_n(z) + \epsilon$. (3)

Fix a point $e^{i s_0} \in T$. By the definition of $u_n - 1$ there exists a null vector $\theta s_0 \in A^*$ such that the null disc $D \ni \zeta \mapsto f_n - 1(e^{i s_0}) + \zeta \theta s_0 \in \Omega$ satisfies $\int_{2\pi}^0 u_n - 2(f_n - 1(e^{i s_0}) + e^{i t \theta s_0}) dt \leq u_n - 1(e^{i s_0}) + \epsilon$. (4)
Proof of the reverse inequality \( u \leq u_\phi \):

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\[
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\]

By the definition of \( u_n \) there exists a null vector \( \theta \in A_* \) such that the linear null disc 

\[
\overline{D} \ni z \mapsto f_{n-1}(z) := z + \zeta \theta \in \Omega
\]

satisfies 

\[
u_n(z) \leq \int_0^{2\pi} u_{n-1}\left(f_{n-1}(e^{i\phi})\right) \frac{ds}{2\pi} < u_n(z) + \frac{\epsilon}{2n}. \tag{3}
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By the definition of $u_n$ there exists a null vector $\theta \in A_\ast$ such that the linear null disc

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satisfies

$$u_n(z) \leq \int_0^{2\pi} u_{n-1} \left( f_{n-1}(e^{i s}) \right) \frac{ds}{2\pi} < u_n(z) + \frac{\epsilon}{2n}. \quad (3)$$

Fix a point $e^{i s_0} \in \mathbb{T}$. By the definition of $u_{n-1}$ there exists a null vector $\theta_{s_0} \in A_\ast$ such that the null disc $\overline{D} \ni \zeta \mapsto f_{n-1}(e^{i s_0}) + \zeta \theta_{s_0} \in \Omega$ satisfies

$$\int_0^{2\pi} u_{n-2} \left( f_{n-1}(e^{i s_0}) + e^{i t} \theta_{s_0} \right) \frac{dt}{2\pi} \leq u_{n-1}(f_{n-1}(e^{i s_0})) + \frac{\epsilon}{4n}. \quad (4)$$
Proof of the disc formula - page 3

Setting
\[ g_{n-1}(e^{is}, \zeta) = f_{n-1}(e^{is}) + \zeta \theta t_0, \]
it follows from (4) that there is an arc \( I \subset \mathbb{T} \) around \( e^{is_0} \) such that
\[
\int_0^{2\pi} \int_{s \in I} u_{n-2}(g_{n-1}(e^{is}, e^{it})) \frac{ds \, dt}{2\pi \, 2\pi} \leq \int_I u_{n-1}(f_{n-1}(e^{is})) \frac{ds}{2\pi} + \frac{|I|}{2\pi} \frac{\epsilon}{3n}.
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it follows from (4) that there is an arc \( I \subset \mathbb{T} \) around \( e^{is_0} \) such that
\[ \int_{0}^{2\pi} \int_{s \in I} u_{n-2}(g_{n-1}(e^{is}, e^{it})) \frac{ds}{2\pi} \frac{dt}{2\pi} \leq \int_{I} u_{n-1}(f_{n-1}(e^{is})) \frac{ds}{2\pi} + \frac{|I|}{2\pi} \frac{\epsilon}{3n}. \]

We may work in a compact subset of \( \Omega \), hence \( u_{n-2} \leq M \) for some \( M \in \mathbb{R} \). Repeating this construction at other points of \( \mathbb{T} \) we find pairwise disjoint arcs \( I_1, \ldots, I_\ell \subset \mathbb{T} \) such that the set \( E = \mathbb{T} \setminus \bigcup_{j=1}^{\ell} I_j \) has measure
\[ |E| < \frac{\epsilon}{3nM} \]
and for each \( j = 1, \ldots, \ell \) we have
\[ \int_{0}^{2\pi} \int_{s \in I_j} u_{n-2}(g_{n-1}(e^{is}, e^{it})) \frac{ds}{2\pi} \frac{dt}{2\pi} \leq \int_{I_j} u_{n-1}(f_{n-1}(e^{is})) \frac{ds}{2\pi} + \frac{|I_j|}{2\pi} \frac{\epsilon}{3n}. \]
Let $\chi: \mathbb{T} \rightarrow [0, 1]$ be a smooth function such that $\chi \equiv 1$ on $\bigcup_{j=1}^{\ell} l_j$ and $\chi \equiv 0$ on $\mathbb{T} \setminus \bigcup_{j=1}^{\ell} l'_j$, where $l'_j \supset l_j$ are bigger pairwise disjoint arcs.
Let $\chi: \mathbb{T} \rightarrow [0, 1]$ be a smooth function such that $\chi \equiv 1$ on $\bigcup_{j=1}^{\ell} I_j$ and $\chi \equiv 0$ on $\mathbb{T} \setminus \bigcup_{j=1}^{\ell} I'_j$, where $I'_j \supset I_j$ are bigger pairwise disjoint arcs. Define the map $h_{n-1}: \mathbb{T} \times \overline{D} \rightarrow \mathbb{C}^3$ by

$$h_{n-1}(\zeta, \xi) = g_{n-1}(\zeta, \chi(\zeta)\xi), \quad (\zeta, \xi) \in \mathbb{T} \times \overline{D}.$$
Let $\chi : \mathbb{T} \to [0, 1]$ be a smooth function such that $\chi \equiv 1$ on $\bigcup_{j=1}^{\ell} I_j$ and $\chi \equiv 0$ on $\mathbb{T} \setminus \bigcup_{j=1}^{\ell} I'_j$, where $I'_j \supset I_j$ are bigger pairwise disjoint arcs. Define the map $h_{n-1} : \mathbb{T} \times \overline{\mathbb{D}} \to \mathbb{C}^3$ by

$$h_{n-1}(\zeta, \bar{\zeta}) = g_{n-1}(\zeta, \chi(\zeta)\bar{\zeta}), \quad (\zeta, \bar{\zeta}) \in \mathbb{T} \times \overline{\mathbb{D}}.$$ 

By the RH lemma for null discs we get $f_{n-2} \in \mathcal{M}(\mathbb{D}, \Omega, z)$ satisfying

$$\int_0^{2\pi} u_{n-2} \left( f_{n-2}(e^{is}) \right) \frac{ds}{2\pi} \leq \int_0^{2\pi} \int_0^{2\pi} u_{n-2} \left( h_{n-1}(e^{is}, e^{it}) \right) \frac{ds}{2\pi} \frac{dt}{2\pi} + \frac{\epsilon}{3n}$$

$$\leq \int_0^{2\pi} u_{n-1} \left( f_{n-1}(e^{it}) \right) \frac{dt}{2\pi} + \frac{\epsilon}{n}$$

$$\leq u_n(z) + \frac{2\epsilon}{n}.$$ 

(We apply the RH-lemma $\ell$ times, once for each of the segments $I_1, \ldots, I_\ell$.)
Repeating this procedure we get null discs $f_1, f_2, \ldots, f_{n-1} \in \mathcal{M}(\mathbb{D}, \Omega, z)$ such that for every $j = 1, \ldots, n - 2$ we have that

\[
\int_0^{2\pi} u_j \left( f_j(e^{is}) \right) \frac{ds}{2\pi} \leq \int_0^{2\pi} u_{j+1} (f_{j+1}(e^{is})) \frac{ds}{2\pi} + \frac{\epsilon}{n}.
\]
Repeating this procedure we get null discs \( f_1, f_2, \ldots, f_{n-1} \in \mathcal{H}(\mathbb{D}, \Omega, z) \) such that for every \( j = 1, \ldots, n - 2 \) we have that

\[
\int_0^{2\pi} u_j \left( f_j(e^{is}) \right) \frac{ds}{2\pi} \leq \int_0^{2\pi} u_{j+1} \left( f_{j+1}(e^{is}) \right) \frac{ds}{2\pi} + \frac{\epsilon}{n}.
\]

Since \( u_1 = \phi \), we get by (2) and (3) that

\[
u(z) \leq \int_0^{2\pi} \phi \left( f_1(e^{is}) \right) \frac{ds}{2\pi} \leq \int_0^{2\pi} u_{n-1} \left( f_{n-1}(e^{is}) \right) \frac{ds}{2\pi} + \frac{(n-2)\epsilon}{n} < u_n(z) + \epsilon < u_\phi(z) + 2\epsilon.
\]
Repeating this procedure we get null discs $f_1, f_2, \ldots, f_{n-1} \in \mathcal{N}(\mathbb{D}, \Omega, z)$ such that for every $j = 1, \ldots, n-2$ we have that

$$
\int_0^{2\pi} u_j \left( f_j(e^{i\theta}) \right) \frac{ds}{2\pi} \leq \int_0^{2\pi} u_{j+1} \left( f_{j+1}(e^{i\theta}) \right) \frac{ds}{2\pi} + \frac{\epsilon}{n}.
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Since this holds for any $\epsilon > 0$, we get $u(z) \leq u_\phi(z)$ as claimed.

This completes the proof of the disc formula for $\mathcal{N}$Psh minorants (part (b) of our main theorem). As said before, part (a) follows from (b).
Minimal hulls and null hulls

Definition

- The **minimal hull** of a compact set $K \subset \mathbb{R}^n$ ($n \geq 3$) is the set

  $$\hat{K}_M = \{ x \in \mathbb{R}^n : u(x) \leq \sup_{K} u \quad \forall u \in \mathcal{M}_{Psh}(\mathbb{R}^n) \}.$$
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- By the maximum principle for subharmonic functions, every bounded minimal surface $M \subset \mathbb{R}^n$ with boundary $bM \subset K$ lies in $\hat{K}_M$.

- Clearly $K \subset \hat{K}_M \subset \text{Co}(K) = \text{the convex hull of } K$, and $K \subset \hat{K}_N \subset \hat{K}_M$. All inclusions are proper in general.
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If $K$ is a compact set in $\mathbb{C}^n$ then every bounded holomorphic null curve $M \subset \mathbb{C}^n$ with $bM \subset K$ lies in the null hull $\hat{K}_N$ of $K$. 

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Clearly \( K \subset \hat{K}_M \subset \text{Co}(K) = \) the convex hull of \( K \), and
\( K \subset \hat{K}_N \subset \hat{K} \). All inclusions are proper in general.
Let $\pi: \mathbb{C}^n \to \mathbb{R}^n$ be the projection $\pi(x + iy) = x$. 

Recall that a minimal plurisubharmonic function $u$ on $\mathbb{R}^n$ lifts to a null plurisubharmonic function $u \circ \pi$ on $\mathbb{C}^n$. This implies that for any compact set $L \subset \mathbb{C}^n$ we have the inclusion $\hat{\pi}(\hat{L}) \subset \hat{\pi}(L)$. This inclusion may be strict: take $L \subset \mathbb{C}^3$ to be a smooth embedded Jordan curve such that $K = \pi(L) \subset \mathbb{R}^3$ is also a smooth Jordan curve. By the solution of the Plateau problem such $K$ bounds a minimal surface $M$ which is therefore contained in $\hat{K} \subset M$. However, we have $L = \hat{L}$ for most curves $L$. 

Relationship between minimal hulls and null hulls
Let $\pi : \mathbb{C}^n \to \mathbb{R}^n$ be the projection $\pi(x + iy) = x$.

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$$\pi(\hat{L}_M) \subset \pi(L)_{\mathfrak{M}}.$$

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$$\pi(\hat{L}_\mathbb{M}) \subset \pi(L)_\mathbb{M}.$$ 

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Take $L \subset \mathbb{C}^3$ to be a smooth embedded Jordan curve such that $K = \pi(L) \subset \mathbb{R}^3$ is also a smooth Jordan curve. By the solution of the Plateau problem such $K$ bounds a minimal surface $M$ which is therefore contained in $\hat{K}_\mathbb{M}$. However, we have $L = \hat{L}_\mathbb{M} = \hat{L}$ for most curves $L$. 
Characterization of minimal hull and null hull

The following characterisation of the minimal hull and the null hull is a corollary to our disc formula for $\mathcal{MPsh}$ and $\mathcal{NPsh}$ functions.
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**Corollary**

(a) Let $K$ be a compact set in $\mathbb{R}^3$ and let $\omega \subset \mathbb{R}^3$ be a bounded open convex set containing $K$. A point $p \in \omega$ belongs to the minimal hull $\hat{K}_{\text{MH}}$ if and only if there exists a sequence of conformal minimal discs $f_j: \overline{D} \to \omega$ such that for all $j = 1, 2, \ldots$ we have $f_j(0) = p$ and

$$\left| \left\{ t \in [0, 2\pi] : \text{dist}(f_j(e^{it}), K) < 1/j \right\} \right| \geq 2\pi - 1/j. \quad (5)$$
Characterization of minimal hull and null hull

The following characterisation of the minimal hull and the null hull is a corollary to our disc formula for $M_{Psh}$ and $N_{Psh}$ functions.

Corollary

(a) Let $K$ be a compact set in $\mathbb{R}^3$ and let $\omega \subset \mathbb{R}^3$ be a bounded open convex set containing $K$. A point $p \in \omega$ belongs to the minimal hull $\hat{K}_{\mathbb{R}}$ if and only if there exists a sequence of conformal minimal discs $f_j: \overline{D} \rightarrow \omega$ such that for all $j = 1, 2, \ldots$ we have $f_j(0) = p$ and

$$\left| \{ t \in [0, 2\pi] : \text{dist}(f_j(e^{it}), K) < 1/j \} \right| \geq 2\pi - 1/j.$$  \hspace{2cm} (5)

(b) Let $K$ be a compact set in $\mathbb{C}^3$, and let $\Omega \subset \mathbb{C}^3$ be a bounded pseudoconvex Runge domain containing $K$ (hence $\hat{K}_{\mathbb{C}} \subset \hat{K} \subset \Omega$). A point $p \in \Omega$ belongs to the null hull $\hat{K}_{\mathbb{C}}$ of $K$ if and only if there exists a sequence of null holomorphic discs $f_j: \overline{D} \rightarrow \Omega$ such that for all $j = 1, 2, \ldots$ we have $f_j(0) = p$ and the estimate (5).
Proof

We prove (b); (a) is similar. Assume that for some \( p \in \Omega \) there exists a sequence \( f_j \in \mathcal{M}(\mathbb{D}, \Omega, p) \) satisfying (5). Pick \( u \in \mathcal{M}_{Psh}(\mathbb{C}^3) \). Let \( U_j = \{ z \in \mathbb{C}^3 : \text{dist}(z, K) < 1/j \} \), \( M_j = \sup_{U_j} u \), \( M = \sup_{\Omega} u \), and \( E_j = \{ t \in [0, 2\pi] : f_j(e^{it}) \notin U_j \} \). Then \( |E_j| \leq 1/j \) by (5) and

\[
\begin{align*}
  u(p) &= u(f_j(0)) \leq \int_{E_j} + \int_{[0,2\pi]\setminus E_j} u(f_j(e^{it})) \frac{dt}{2\pi} \\
  &\leq M/j + M_j.
\end{align*}
\]

Passing to the limit gives \( u(p) \leq \sup_K u \); hence \( p \in \hat{K}_\mathcal{M} \).
Proof

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\[
 u(p) = u(f_j(0)) \leq \int_{E_j} + \int_{[0,2\pi]\setminus E_j} u(f_j(e^{it})) \frac{dt}{2\pi} \leq M/j + M_j.
\]

Passing to the limit gives \( u(p) \leq \sup K u \); hence \( p \in \hat{K}_{\mathcal{M}} \).

To prove the converse, pick an open set \( U \) in \( \mathbb{C}^3 \) with \( K \subset U \subset \Omega \). The function \( \phi: \Omega \to \mathbb{R} \), which equals \(-1\) on \( U \) and \( 0 \) on \( \Omega \setminus U \), is upper semicontinuous. Let \( u \in \mathcal{M}_{Psh}(\Omega) \) be the associated extremal function. Then \( -1 \leq u \leq 0 \) on \( \Omega \) and \( u = -1 \) on \( K \), whence \( u = -1 \) on \( \hat{K}_{\mathcal{M}} \). Fix a point \( p \in \hat{K}_{\mathcal{M}} \) and a number \( \epsilon > 0 \). By the disc formula there is a null disc \( f \in \mathcal{M}(\mathbb{D}, \Omega, p) \) such that \( \int_0^{2\pi} \phi(f(e^{it})) \frac{dt}{2\pi} < -1 + \epsilon/2\pi \). This implies \( |\{ t \in [0, 2\pi] : f(e^{it}) \in U \}| \geq 2\pi - \epsilon \). Apply this with the sequences \( U_j = \{ z \in \mathbb{C}^3 : \text{dist}(z, K) < 1/j \} \) and \( \epsilon_j = 1/j \).
We also characterize the null hull and the minimal hull by currents.
Null positive forms and currents

We also characterize the null hull and the minimal hull by currents.

**Definition**

A 2-form $\alpha$ on a domain $\Omega \subset \mathbb{C}^n$ is **null positive** if for every point $z \in \Omega$ and null vector $\nu \in A_\ast$ we have

$$\langle \alpha(z), \nu \wedge J\nu \rangle \geq 0.$$ 

(We identify $\nu$ with a tangent vector in $T_z\mathbb{C}^n$, and $J$ denotes the complex structure operator.)
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- A current $T$ on $\mathbb{C}^n$ of bidimension $(1, 1)$ is null positive if $T(\alpha) \geq 0$ for every null positive 2-form $\alpha$ with compact support.
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- A current $T$ on $\mathbb{C}^n$ of bidimension $(1, 1)$ is null positive if $T(\alpha) \geq 0$ for every null positive 2-form $\alpha$ with compact support.

Note that a $C^2$ function $u$ is null plurisubharmonic if and only if $dd^c u$ is null positive.
Null hulls in terms of null positive currents

The following result is analogous to the characterization of the polynomial hull due to Duval and Sibony (1995) and Wold (2011).

**Theorem**

Let $K$ be a compact set in $\mathbb{C}^3$. A point $p \in \mathbb{C}^3$ belongs to the null hull $\hat{K}_\mathcal{M}$ of $K$ if and only if there exists a null positive $(1, 1)$-current $T$ with compact support on $\mathbb{C}^3$ satisfying

$$dd^c T = \mu - \delta_p,$$

where $\mu$ is a probability measure on $K$ such that

$$u(p) \leq \int_K u \, d\mu \quad \forall u \in \mathcal{MPsh}(\mathbb{C}^3).$$

The support of any such current $T$ is contained in the null hull $\hat{K}_\mathcal{M}$. (Here $\delta_p$ denotes the point evaluation at $p$.)
Proof

Let $\zeta = x + iy$ be the coordinate on $\mathbb{C} \cong \mathbb{R}^2$. The Green current on the disc $\overline{D}$ is defined on any 2-form $\alpha = adx \wedge dy$ with $a \in \mathcal{C}(\overline{D})$ by

$$G(\alpha) = -\frac{1}{2\pi} \int_{\overline{D}} \log|\zeta| \cdot a(\zeta) dx \wedge dy.$$
Proof

Let $\zeta = x + iy$ be the coordinate on $C \cong \mathbb{R}^2$. The Green current on the disc $\overline{D}$ is defined on any 2-form $\alpha = adx \wedge dy$ with $a \in C(\overline{D})$ by

$$G(\alpha) = -\frac{1}{2\pi} \int_{\overline{D}} \log |\zeta| \cdot a(\zeta) dx \wedge dy.$$ 

$G$ is a positive current of bidimension $(1, 1)$ satisfying $dd^c G = \sigma - \delta_0$, where $\sigma$ is the normalized Lebesgue measure on $bD = \mathbb{T}$. 

(Proof in Appendix C)
Let $\zeta = x + iy$ be the coordinate on $\mathbb{C} \cong \mathbb{R}^2$. The Green current on the disc $\overline{D}$ is defined on any 2-form $\alpha = adx \wedge dy$ with $a \in \mathcal{C}(\overline{D})$ by

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If $f : \overline{D} \to \mathbb{C}^n$ is a (null) holomorphic map then $f_* G$ is a (null) positive $(1, 1)$ current on $\mathbb{C}^n$ satisfying $dd^c(f_* G) = f_* \sigma - \delta_{f(0)}$. Its mass equals

$$M(f_* G) = \frac{1}{4} \left( \int_{\mathbb{T}} |f|^2 d\sigma - |f(0)|^2 \right). \quad \text{(Proof in Appendix C)}$$
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Assume now that $p \in \hat{K}_N$ and $f_j$ is a bounded sequence of null discs in $\mathbb{C}^3$ with centers at $p$ whose boundaries converge to $K$. The Green currents $T_j = (f_j)_* G$ have bounded masses, and hence there is a weakly convergent subsequence $(T_{j_k})_{k \in \mathbb{N}}$. The limit $T = \lim_{k \to \infty} T_{j_k}$ is a null positive current satisfying the conclusion of the theorem.
Bochner’s tube theorem for polynomial hulls

Let $\pi: \mathbb{C}^n \to \mathbb{R}^n$ denote the projection $\pi(x + iy) = x$.

**Theorem**

Let $K$ be a connected compact set in $\mathbb{R}^n$. For every point $z_0 = p + iq \in \mathbb{C}^n$ with $p \in \text{Co}(K)$ there exists a positive $(1, 1)$ current $T$ on $\mathbb{C}^n$ with finite mass satisfying $\text{supp } T \subset \text{Co}(K) \times i\mathbb{R}^n$ and $dd^c T = \mu - \delta_{z_0}$, where $\mu$ is a probability measure on $\mathcal{T}_K$. 
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Conversely, let $T$ be a positive $(1, 1)$ current on $\mathbb{C}^n$ with finite mass such that $\pi(\text{supp } T)$ is a bounded set in $\mathbb{R}^n$. If $dd^c T \leq 0$ on $\mathbb{C}^n \setminus \overline{T}_K$, then $\text{supp } T \subset \text{Co}(K) \times i\mathbb{R}^n$. 

Proof: Let $p \in \text{Co}(K)$. It is classical that there exists a sequence of holomorphic discs $f_j = g_j + ih_j : D \rightarrow \mathbb{C}^n$ with $f_j(0) = p$ such that the sequence $g_j$ is uniformly bounded and the boundaries $g_j(T)$ converge to $K$. It follows that the $L^2$-norms of $g_j$, and hence also of $f_j$, are uniformly bounded. Hence the Green currents $T_j = (f_j)^*G$ have uniformly bounded masses. A subsequence converges weakly to a current $T$ satisfying the conclusion of the theorem. The converse is standard.
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**Theorem**

Let $K$ be a connected compact set in $\mathbb{R}^n$. For every point $z_0 = p + iq \in \mathbb{C}^n$ with $p \in \text{Co}(K)$ there exists a positive $(1, 1)$ current $T$ on $\mathbb{C}^n$ with finite mass satisfying $\text{supp} \ T \subset \text{Co}(K) \times i\mathbb{R}^n$ and $dd^c T = \mu - \delta_{z_0}$, where $\mu$ is a probability measure on $\mathcal{T}_K$.

Conversely, let $T$ be a positive $(1, 1)$ current on $\mathbb{C}^n$ with finite mass such that $\pi(\text{supp} \ T)$ is a bounded set in $\mathbb{R}^n$. If $dd^c T \leq 0$ on $\mathbb{C}^n \setminus \mathcal{T}_K$, then $\text{supp} \ T \subset \text{Co}(K) \times i\mathbb{R}^n$.

**Proof:** Let $p \in \text{Co}(K)$. It is classical that there exists a sequence of holomorphic discs $f_j = g_j + ih_j : \overline{D} \to \mathbb{C}^n$ with $f_j(0) = p$ such that the sequence $g_j$ is uniformly bounded and the boundaries $g_j(\overline{T})$ converge to $K$. It follows that the $L^2$-norms of $g_j$, and hence also of $f_j$, are uniformly bounded. Hence the Green currents $T_j = (f_j)_* G$ have uniformly bounded masses. A subsequence converges weakly to a current $T$ satisfying the conclusion of the theorem. The converse is standard.
The same proof, together with the connection between conformal minimal discs in a domain $\omega \subset \mathbb{R}^3$ and null discs in the tube $T_\omega = \omega \times i\mathbb{R}^3 \subset \mathbb{C}^3$ over $\omega$, gives the following analogous result, characterizing minimal hulls of compact sets in $\mathbb{R}^3$ by projections of supports of certain null currents in $\mathbb{C}^3$. 

Theorem

Let $K$ be a compact set in $\mathbb{R}^3$. A point $p \in \mathbb{R}^3$ belongs to the minimal hull $\hat{K}_M$ if and only if there exists a null positive current $T$ on $\mathbb{C}^3$ of finite mass such that $\pi(\text{supp} T) \subset \mathbb{R}^3$ is a bounded set and $dd^c T = \mu - \delta_p$, where $\mu$ is a probability measure on the tube $T_K = K \times i\mathbb{R}^3$. 
Minimal hulls in terms of currents

The same proof, together with the connection between conformal minimal discs in a domain $\omega \subset \mathbb{R}^3$ and null discs in the tube $\mathcal{T}_\omega = \omega \times i\mathbb{R}^3 \subset \mathbb{C}^3$ over $\omega$, gives the following analogous result, characterizing minimal hulls of compact sets in $\mathbb{R}^3$ by projections of supports of certain null currents in $\mathbb{C}^3$.

**Theorem**

Let $K$ be a compact set in $\mathbb{R}^3$. A point $p \in \mathbb{R}^3$ belongs to the minimal hull $\hat{K}_{\mathfrak{m}}$ if and only if there exists a null positive current $T$ on $\mathbb{C}^3$ of finite mass such that $\pi(\text{supp } T) \subset \mathbb{R}^3$ is a bounded set and

$$dd^c T = \mu - \delta_p,$$

where $\mu$ is a probability measure on the tube $\mathcal{T}_K = K \times i\mathbb{R}^3$. 

THANK YOU!
Appendix A: Basics on minimal surfaces in $\mathbb{R}^n$

Assume that $D$ is a domain in $\mathbb{R}^2_{(u_1, u_2)}$ and $x = (x_1, \ldots, x_n): D \rightarrow \mathbb{R}^n$ is a $C^2$ embedding. Let $S = x(D) \subset \mathbb{R}^n$, a parametrized surface in $\mathbb{R}^n$.

Every smooth embedded curve in $S$ is of the form

$$\lambda(t) = x(u_1(t), u_2(t)) \in S$$

where $t \mapsto (u_1(t), u_2(t))$ is a smooth embedded curve in $D$.

Let $s = s(t)$ denote the arc length on $\lambda$. The number

$$\kappa(T, N) := \frac{d^2\lambda}{ds^2} \cdot N = \sum_{i,j=1}^{2} \left( x_{u_i u_j} \cdot N \right) \frac{du_i}{ds} \frac{du_j}{ds}$$

is the normal curvature of $S$ at $p = \lambda(t) \in S$ in the tangent direction $T = \lambda'(s) \in T_pS$ with respect to the normal vector $N \in N_pS$.

(It only depends on $T$ and $N$.)
A2: Curvature in terms of fundamental forms

In terms of \( t \)-derivatives we get

\[
\kappa(T, N) = \frac{\sum_{i,j=1}^{2} (x_{u_i u_j} \cdot N) \dot{u}_i \dot{u}_j}{\sum_{i,j=1}^{2} g_{i,j} \dot{u}_i \dot{u}_j} = \frac{\text{second fundamental form}}{\text{first fundamental form}}
\]

Fix a normal vector \( N \in N_p S \) and vary the unit tangent vector \( T \in T_p S \). The \textbf{principal curvatures} of \( S \) at \( p \) in direction \( N \) are the numbers

\[
\kappa_1(N) = \max_T \kappa(T, N), \quad \kappa_2(N) = \min_T \kappa(T, N).
\]

Their average

\[
H(N) = \frac{\kappa_1(N) + \kappa_2(N)}{2} \in \mathbb{R}
\]

is the \textbf{mean curvature} of \( S \) at \( p \) in the normal direction \( N \in N_p S \).
A3: The mean curvature vector

Let $G = (g_{i,j})$ and $h(N) = (h_{i,j}(N)) = (x_{u_i u_j} \cdot N)$ denote the matrices of the 1st and the 2nd fundamental form, respectively.

The extremal values of $\kappa(T, N)$ are roots of the equation

$$\det(h(N) - \mu G) = 0$$

$$\det G \cdot \mu^2 - (g_{2,2} h_{1,1}(N) + g_{1,1} h_{2,2}(N) - 2 g_{1,2} h_{1,2}(N)) \mu + \det h(N) = 0.$$

The Vieta formula gives

$$H(N) = \frac{\kappa_1 + \kappa_2}{2} = \frac{g_{2,2} x_{u_1 u_1} + g_{1,1} x_{u_2 u_2} - 2 g_{1,2} x_{u_1 u_2}}{2 \det G} \cdot N.$$

There is a unique normal vector $H \in N_p S$ such that

$$H(N) = H \cdot N \quad \text{for all } N \in N_p S.$$

This $H$ is the mean curvature vector of the surface $S$ at $p$. 
A4: The mean curvature in isothermal coordinates

The formulas simplify in isothermal coordinates:

\[ G = (g_{i,j}) = \zeta I, \quad \det G = \zeta^2; \quad \zeta = ||x_{u_1}||^2 = ||x_{u_2}||^2, \quad x_{u_1} \cdot x_{u_2} = 0 \]

\[ H(N) = \frac{x_{u_1}u_1 + x_{u_2}u_2}{2\zeta} \cdot N = \frac{\nabla x}{2\zeta} \cdot N. \]

Lemma

Assume that \( D \) is a domain in \( \mathbb{R}^2_{(u_1,u_2)} \) and \( x: D \to \mathbb{R}^n \) is a conformal immersion of class \( \mathcal{C}^2 \) (i.e., \( u = (u_1, u_2) \) are isothermal for \( x \).) Then the Laplacian \( \Delta x = x_{u_1}u_1 + x_{u_2}u_2 \) is orthogonal to \( S = x(D) \) and satisfies

\[ \Delta x = 2\zeta H \]

where \( H \) is the mean curvature vector and \( \zeta = ||x_{u_1}||^2 = ||x_{u_2}||^2. \)
A5: Proof of the lemma

Proof.

It suffices to show that the vector $\triangle x(u)$ is orthogonal to the surface $S$ at the point $x(u)$ for every $u \in D$. If this holds, it follows from the preceding formula that the normal vector $(2\xi)^{-1}\triangle x(u) \in N_{x(u)}S$ fits the definition of the mean curvature vector $H$, so it equals $H$.

Conformality of the immersion $x$ can be written as follows:

$$x_{u_1} \cdot x_{u_1} = x_{u_2} \cdot x_{u_2}, \quad x_{u_1} \cdot x_{u_2} = 0.$$

Differentiating the first identity on $u_1$ and the second one on $u_2$ yields

$$x_{u_1}x_{u_1} \cdot x_{u_1} = x_{u_1}x_{u_2} \cdot x_{u_2} = -x_{u_2}x_{u_2} \cdot x_{u_1},$$

whence $\triangle x \cdot x_{u_1} = 0$. Similarly we get $\triangle x \cdot x_{u_2} = 0$ by differentiating the first identity on $u_2$ and the second one on $u_1$. This proves the claim. $\square$
A6: Lagrange’s formula for the variation of area

The area of an immersed surface $\mathbf{x}: D \to \mathbb{R}^n$ with the 1st fundamental form $G = (g_{i,j})$ equals

$$A(\mathbf{x}) = \int_D \sqrt{\det G} \cdot du_1 du_2.$$ 

Let $\mathbf{N}: D \to \mathbb{R}^n$ be a normal vector field along $\mathbf{x}$ which vanishes on $bD$. Consider the 1-parameter family of maps $\mathbf{x}^t: D \to \mathbb{R}^n$:

$$\mathbf{x}^t(u) = \mathbf{x}(u) + t \mathbf{N}(u), \quad u \in D, \ t \in \mathbb{R}.$$ 

A calculation gives the formula for the first variation of area:

$$\delta A(\mathbf{x}) \mathbf{N} = \frac{d}{dt} \bigg|_{t=0} A(\mathbf{x}^t) = -2 \int_D \mathbf{H} \cdot \mathbf{N} \sqrt{\det G} \cdot du_1 du_2.$$ 

It follows that $\delta A(\mathbf{x}) = 0 \iff \mathbf{H} = 0.$
A7: Conformal minimal surfaces are harmonic

In view of the already established formula

$$\triangle x = 2\xi H$$

which holds for any conformal immersion $x$ we get

**Corollary**

Let $M$ be an open Riemann surface. The following are equivalent for a smooth conformal immersion $x: M \to \mathbb{R}^n$:

- $x$ is minimal (a stationary point of the area functional).
- $x$ has vanishing mean curvature vector: $H = 0$.
- $x$ is harmonic: $\triangle x = 0$.

In the sequel we shall always assume that $x$ is conformal and hence

$$\delta A(x) \iff H = 0 \iff \triangle x = 0.$$
We wish to emphasize the difference between

- **minimal surfaces**: these are stationary (critical) points of the area functional, and are (only) locally area minimizing; and

- **area-minimizing surfaces**: these are surfaces which globally minimize the area among all nearby surfaces with the same boundary.

- Minimal surfaces which are **graphs** are globally area minimizing.

- Recent work on conditions ensuring that a minimal surface is globally area minimizing was done by C. Arezzo.
Lemma (A. Alarcón and F. F., The Calabi-Yau problem, null curves, and Bryant surfaces. arxiv.org/abs/1308.0903)

Let $f: \overline{D} \to \mathbb{C}^3$ be a null holomorphic immersion, let $\theta \in A_*$ be a null vector, and let $\mu: \Gamma = bD \to [0, \infty)$ be a continuous function. Let

$$g: \Gamma \times \overline{D} \to \mathbb{C}^3, \quad g(\zeta, z) = f(\zeta) + \mu(\zeta)z\theta.$$

Given $\epsilon > 0$ and $0 < r < 1$, there exist a number $r' \in [r, 1)$ and a null holomorphic disc $h: \overline{D} \to \mathbb{C}^3$ with $h(0) = f(0)$, satisfying the following:

(i) $\text{dist}(h(\zeta), g(\zeta, \Gamma)) < \epsilon$ for all $\zeta \in \Gamma$,
(ii) $\text{dist}(h(\rho\zeta), g(\zeta, \overline{D})) < \epsilon$ for all $\zeta \in \Gamma$ and all $\rho \in [r', 1)$, and
(iii) $h$ is $\epsilon$-close to $f$ on $\{\zeta \in \mathbb{C}: |\zeta| \leq r'\}$.

Furthermore, given an upper semicontinuous function $u: \mathbb{C}^3 \to \mathbb{R} \cup \{-\infty\}$ and an arc $l \subset \Gamma$, we may achieve that

$$\int_I u(h(e^{it})) \frac{dt}{2\pi} \leq \int_0^{2\pi} \int_I u(g(e^{it}, e^{is})) \frac{dt}{2\pi} \frac{ds}{2\pi} + \epsilon.$$
Appendix B: Proof - page 1

Consider the following unbranched two-sheeted holomorphic covering (the \textit{spinor representation of the null quadric}):\[
\pi: \mathbb{C}^2 \setminus \{(0, 0)\} \rightarrow A_*, \quad \pi(u, v) = (u^2 - v^2, i(u^2 + v^2), 2uv).
\]

Since $\overline{D}$ is simply connected, the map $f': \overline{D} \rightarrow A_*$ lifts to a map $(u, v): \overline{D} \rightarrow \mathbb{C}^2 \setminus \{(0, 0)\}$. Hence we have

\[
\begin{align*}
f' &= \pi(u, v) = \left(u^2 - v^2, i(u^2 + v^2), 2uv\right) \in A_* \\
\vartheta &= \pi(a, b) = \left(a^2 - b^2, i(a^2 + b^2), 2ab\right) \in A_* \\
\eta &= \sqrt{\mu}: b\overline{D} \rightarrow \mathbb{R}_+ \\
\eta(\zeta) &\approx \tilde{\eta}(\zeta) = \sum_{j=1}^{N} A_j \zeta^{j-m} \quad \text{(rational approximation)} \\
\mu(\zeta) &\approx \tilde{\eta}^2(\zeta) = \sum_{j=1}^{2N} B_j \zeta^{j-2m}.
\end{align*}
\]
For any integer $n \in \mathbb{N}$ we consider the following functions and maps on the closed disc $\overline{D}$:

$$
\begin{align*}
    u_n(z) &= u(z) + \sqrt{2n+1} \tilde{\eta}(z)z^n a, \\
    v_n(z) &= v(z) + \sqrt{2n+1} \tilde{\eta}(z)z^n b, \\
    \Phi_n(z) &= \pi(u_n(z), v_n(z)) = (u_n^2 - v_n^2, i(u_n^2 - v_n^2), 2u_nv_n): \overline{D} \to A_*, \\
    f_n(\zeta) &= f(0) + \int_0^\zeta \Phi_n(z) \, dz, \quad \zeta \in \overline{D}.
\end{align*}
$$

Then $f_n: \overline{D} \to \mathbb{C}^3$ is a null disc of the form

$$
f_n(\zeta) = f(\zeta) + B_n(\zeta) \vartheta + A_n(\zeta).
$$
The \( \mathbb{C} \)-valued term \( B_n \) equals

\[
B_n(\zeta) = (2n + 1) \sum_{j=1}^{2N} \int_{0}^{\zeta} B_j z^{2n+j-2m} \, dz
\]

\[
= \sum_{j=1}^{2N} \frac{2n + 1}{2n + 1 + j - 2m} B_j \zeta^{2n+1+j-2m}.
\]

Since the coefficients \((2n + 1)/(2n + 1 + j - 2m)\) in the sum for \( B_n \) converge to 1 as \( n \to +\infty \), we have that

\[
\sup_{|\zeta| \leq 1} \left| B_n(\zeta) - \zeta^{2n+1} \eta^2(\zeta) \right| \to 0 \quad \text{as} \quad n \to \infty.
\]
The remainder $C^3$-valued term $A_n(\zeta)$ equals

\[
A_n(\zeta) = 2\sqrt{2n+1} \int_0^\zeta \sum_{j=1}^N A_j \ z^{n+j-m} \ (u(z)(a,ia,b) + v(z)(-b,ib,a)) \ dz
\]

\[
|A_n(\zeta)| \leq 2\sqrt{2n+1} C_0 \sum_{j=1}^N |A_j| \int_0^|\zeta| |z|^{n+j-m} \ d|z|
\]

\[
\leq 2C_0 \sum_{j=1}^N \frac{\sqrt{2n+1}}{n+1+j-m} |A_j|.
\]

It follows that $|A_n| \to 0$ uniformly on $\overline{D}$ as $n \to +\infty$. Hence

\[
f_n(\zeta) \approx f(\zeta) + \zeta^{2n+1} \tilde{\mu}(\zeta) \theta, \quad \zeta \in \overline{D}, \ n >> 0.
\]

The map $h = f_n$ for large enough $n \in \mathbb{N}$ solves the problem.
Recall that the Green current on the disc $\overline{D}$ is defined on any 2-form $\alpha = adx \wedge dy$ with $a \in C(\overline{D})$ by

$$G(\alpha) = -\frac{1}{2\pi} \int_D \log |\zeta| \cdot a(\zeta) dx \wedge dy.$$ 

Green’s formula

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) dt + \frac{1}{2\pi} \int_D \log |\zeta| \cdot \triangle u(\zeta) dx \wedge dy,$$

which holds for any $u \in C^2(\overline{D})$, tells us that $dd^c G = \sigma - \delta_0$.

Given a smooth map $f = (f_1, \ldots, f_n): \overline{D} \to \mathbb{R}^n$ we denote by $f_* G$ the 2-dimensional current on $\mathbb{R}^n$ given on $\alpha = \sum_{i,j=1}^n a_{i,j} dx_i \wedge dx_j$ by

$$(f_* G)(\alpha) = G(f^* \alpha) = -\frac{1}{2\pi} \int_D \log |\zeta| \cdot f^* \alpha.$$

We call $f_* G$ the Green current supported by $f$. 

Mass of Green currents

The following lemma is crucial in the proof of the characterisation of minimal hulls by Greens currents, and also in the proof of Bochner’s tube theorem for polynomial hulls.

Lemma

If \( f = (f_1, \ldots, f_n) : \overline{D} \rightarrow \mathbb{R}^n \) is a conformal harmonic immersion of class \( C^2(\overline{D}) \), then the mass of the Green current \( f_* G \) satisfies

\[
M(f_* G) \leq \frac{1}{4} \left( \int_{\mathcal{T}} |f|^2 d\sigma - |f(0)|^2 \right).
\] (8)

If \( f \) is injective outside of a closed set of measure zero in \( \overline{D} \), or if \( f : \overline{D} \rightarrow \mathbb{C}^n \) is a holomorphic disc, then we have equality in (8).

The possible loss of mass for a non-holomorphic disc \( f \) may be caused by the cancellation of parts of the immersed surface \( f(\overline{D}) \) (considered as a current) due to the reversal of the orientation.
Proof of the mass formula

**Proof.** Denote the partial derivatives of \( f : \overline{D} \rightarrow \mathbb{R}^n \) by \( f_x \) and \( f_y \). Write

\[
|f|^2 = \sum_{i=1}^{n} f_i^2, \quad |\nabla f|^2 = \sum_{i=1}^{n} |\nabla f_i|^2 = \sum_{i=1}^{n} \left( f_{i,x}^2 + f_{i,y}^2 \right).
\]

Since \( f \) is conformal, the vector fields \( f_x \) and \( f_y \) are orthogonal and satisfy \( |f_x| = |f_y| \). Let

\[
\vec{T} = \frac{f_x \wedge f_y}{|f_x| \cdot |f_y|} = \frac{f_x \wedge f_y}{|f_x|^2}.
\]

Given a 2-form \( \alpha \) on \( \mathbb{R}^n \), we have

\[
f^* \alpha = \langle \alpha \circ f, f_x \wedge f_y \rangle \, dx \wedge dy = \langle \alpha \circ f, \vec{T} \rangle \cdot |f_x|^2 \, dx \wedge dy. \quad \text{(9)}
\]

The definition of \( T = f_* G \) and the formula (9) imply

\[
T(\alpha) = -\frac{1}{2\pi} \int_{\overline{D}} \log |\zeta| \cdot \langle \alpha \circ f, \vec{T} \rangle \cdot |f_x|^2 \, dx \wedge dy. \quad \text{(10)}
\]
Conclusion of the proof

From the definition of the mass of a current and (10) it follows that

$$M(T) = \sup \left\{ T(\alpha) : |\langle \alpha, \vec{T} \rangle| \leq 1 \right\} \leq -\frac{1}{2\pi} \int_{\partial D} \log |\zeta| \cdot |f_x|^2 \, dx \wedge dy;$$

(11)
equality holds for holomorphic discs. For any harmonic function $v \in C^2(\overline{D})$ we have

$$dd^c v^2 = d(2v \, d^c v) = 2dv \wedge d^c v = 2|\nabla v|^2 dx \wedge dy.$$

Applying this to each component $f_i$ of $f$ we get

$$|\nabla f|^2 \, dx \wedge dy = \sum_{i=1}^n |\nabla f_i|^2 \, dx \wedge dy = \frac{1}{2} \sum_{i=1}^n dd^c f_i^2 = \frac{1}{2} dd^c |f|^2.$$

Inserting the identity $|f_x|^2 \, dx \wedge dy = \frac{1}{2} |\nabla f|^2 \, dx \wedge dy = \frac{1}{4} dd^c |f|^2$ into (11) and applying Green’s identity (6) gives (8) and proves the lemma.