Online Stochastic Optimization with Wasserstein-Based Non-stationarity

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Abstract: We consider a general online stochastic optimization problem with multiple budget constraints over a horizon of finite time periods. In each time period, a reward function and multiple cost functions are revealed, and the decision maker needs to specify an action from a convex and compact action set to collect the reward and consume the budget. Each cost function corresponds to the consumption of one budget. In each time period, the reward function and the cost functions are drawn from an unknown distribution, which is non-stationary across time. The objective of the decision maker is to maximize the cumulative reward subject to the budget constraints. This formulation captures a wide range of applications including online linear programming and network revenue management, among others. In this paper, we consider two settings: (i) a data-driven setting where the true distribution is unknown but a prior estimate (possibly inaccurate) is available; (ii) an uninformative setting where the true distribution is completely unknown. We propose a unified Wasserstein-distance based measure to quantify the inaccuracy of the prior estimate in setting (i) and the non-stationarity of the system in setting (ii). We show that the proposed measure leads to a necessary and sufficient condition for the attainability of a sublinear regret in both settings. For setting (i), we propose an informative gradient descent algorithm. The algorithm takes a primal-dual perspective and it integrates the prior information of the underlying distributions into an online gradient descent procedure in the dual space. The algorithm also naturally extends to the uninformative setting (ii). Under both settings, we show the corresponding algorithm achieves a regret of optimal order. In numerical experiments, we demonstrate how the proposed algorithms can be naturally integrated with the re-solving technique to further boost the empirical performance.

1. Introduction

In this paper, we study a general online stochastic optimization problem with \( m \) budgets, each with an initial capacity, over a horizon of finite discrete time periods. At each time \( t \), a reward function \( f_t : \mathcal{X} \to \mathbb{R} \) and a cost function \( g_t : \mathcal{X} \to \mathbb{R}^m \) are drawn independently from a distribution. Then the decision maker should specify a decision \( x_t \in \mathcal{X} \), where \( \mathcal{X} \) is assumed to be a convex and compact set. Accordingly, a reward \( f(x_t) \) is generated, and each budget \( i \in \{1, 2, \ldots, m\} \) is consumed by an amount of \( g_{it}(x_t) \), where \( g_t(x) = (g_{1t}(x), \ldots, g_{mt}(x))^\top \). The decision maker’s objective is to maximize the total generated reward subject to the budget capacity constraints.

Our formulation generalizes several existing problems studied in the literature. When \( f_t \) and \( g_t \) are linear functions for each \( t \), our formulation reduces to the online linear programming (OLP)
problem (Buchbinder and Naor, 2009). Our formulation could also be applied to network revenue management (NRM) problem (Talluri and Van Ryzin, 2006), including the quantity-based model, price-based model and choice-based model (Talluri and Van Ryzin, 2004) (See detailed discussions in Section 2.2). Note that in the OLP problem, the reward function and cost functions are assumed to be drawn from an unknown distribution which is stationary across time (Li and Ye, 2019), while the NRM literature (Talluri and Van Ryzin, 2006) always assumes a precise knowledge of the true distribution though it can be nonidentical across time. In this paper, our focus is an unknown non-stationary setting where the functions $f_t$ and $g_t$ are drawn from a distribution $\mathcal{P}_t$ that is nonidentical over time and unknown to the decision maker. Knowledge-wise, we consider two settings: (i) a data-driven setting where there exists an available prior estimate (possibly inaccurate) for the true distribution $\mathcal{P}_t$ at each time period and (ii) an uninformative setting where the true distribution $\mathcal{P}_t$ is completely unknown. Specifically, when the prior estimates are identical to the true distributions, the data-driven setting reduces to the known non-stationary setting considered in the NRM literature. When the distribution is identical over time, the uninformative setting reduces to the unknown stationary setting considered in the OLP literature.

For both settings, we assume that the true distribution falls into an uncertainty set, which controls the inaccuracy of the prior estimate in the data-driven setting and the non-stationarity of the distributions in the uninformative setting. Our goal is to derive near-optimal policies for both settings, which perform well over the entire uncertainty set. We compare the performances of our policies to the “offline” optimization problem which maximizes the total reward with full information/knowledge of all the $f_t$’s and $g_t$’s. We use regret as the performance metric, which is defined as worst-case additive optimality gap (over the uncertainty set) between the total reward generated by our policy and the offline optimal objective value. The formal definitions will be provided in the next section after introducing the notations and formulations.

1.1. Main Results and Contributions

For the data-driven setting (in Section 4), we assume the availability of a prior estimate of $\mathcal{P}_t$ for each $t$. We propose a Wasserstein-based deviation measure, Wasserstein-based deviation budget (WBDB), to quantify the deviation of the prior estimate from the true distribution. Based on WBDB, we introduce an uncertainty set driven by the notion of WBDB and a parameter $W_T$ (called deviation budget), and the uncertainty set encapsulates all the distributions $\mathcal{P}_t$’s that have WBDB no greater than $W_T$. We illustrate the sharpness of WBDB by showing that if the variation budget $W_T$ is linear in $T$, sublinear regret could not be achieved by any policy. Next, we develop a new Informative Gradient Descent algorithm with prior estimates (IGDP), which adaptively combines the distribution knowledge with an update in the dual space. Our algorithm is motivated
by the traditional online gradient descent (OGD) algorithm (Hazan, 2016). The OGD algorithm applies a linear update rule according to the gradient information at the current period and has been shown to work well in the stationary setting, even when the distribution is unknown (Lu et al., 2020; Sun et al., 2020; Li et al., 2020). However, the update in OGD only involves historical information and for the non-stationary setting, we have to incorporate the prior estimates of the future time periods. Specifically, based on a primal-dual convex relaxation of the underlying offline problem, we obtain a prescribed allocation of the budgets over the entire horizon from the prior estimates. Then, IGDP uses this allocation to adjust the gradient descent direction. This idea is new in the literature as the IGDP descent direction at each period does not simply come from the historical observations, but it is also informed by the distribution knowledge of the entire horizon. We show that IGDP achieves a regret bound $O(\max\{\sqrt{T}, W_T\})$, which is of optimal order.

Note that even for a special case where $W_T = 0$, i.e., the prior estimate is identical to the true distribution at each time period, our $O(\sqrt{T})$ regret bound turns out to be new. Similar result for this setting is only known in (Devanur et al., 2019) for a $1 - O(1/\sqrt{c})$ competitive ratio, where $c$ denotes the minimal capacity of the budget constraints. The paper assumes the reward function and cost function are all linear functions, and the resultant competitive ratio does not translate to $O(\sqrt{T})$ bound on regret. It is also not clear how to generalize the method in (Devanur et al., 2019) to the setting where the distribution knowledge is inaccurate or absent. Importantly, our analysis are totally different. Specifically, the analysis in (Devanur et al., 2019) is based on the concentration property of the distribution and applying Chernoff-type inequalities to derive high probability bounds. In contrast, our analysis is based on showing that the adjusted gradient descent step actually balance the budget consumption. We show that the budget consumption on every sample path can be represented by the dual variables and our update rule ensures that these dual variables are bounded almost surely. In this way, we provide a new methodology to analyze the online optimization problem in a non-stationary environment.

For the uninformative setting, we assume no prior knowledge on the true distribution. This setting is consistent with the no prior knowledge assumption in the literature of OLP problem (Molinaro and Ravi, 2013; Agrawal et al., 2014; Gupta and Molinaro, 2014) and the setting of blind NRM (Besbes and Zeevi, 2012; Jasin, 2015). We modify the WBDB by replacing the prior estimate of each distribution with their uniform mixture distribution to propose a new measure called Wasserstein-based non-stationarity measure (WBNB). By its definition, the WBNB captures the cumulative deviation for all the distribution $P_t$ from their centric distribution, and it thus reflects the intensity of the non-stationarity associated with $P_t$’s. In this sense, the WBNB concerns the global change of the distributions, whereas the previous non-stationarity measures (Besbes et al., 2014, 2015; Cheung et al., 2019) in an unconstrained setting characterize the local and temporal
change of the distributions over time. In Section 5.1, we illustrate by a simple example that such
temporal change measures actually fail in a constrained setting thus address the necessity of such
a global measure and, in particular, reveal the interaction between the existence of constraints
and the non-stationarity. Note that a simultaneous and independent work (Balseiro et al., 2020)
also uses global change of the distributions to derive a measure of non-stationarity. However, their
measure is based on the total variation metric between distributions. With the same example, we
illustrate the advantage of using Wasserstein distance instead of total variation distance or KL-
divergence. Specifically, the Wasserstein distance compares both the support and densities between
two distributions, while total variation distance or KL-divergence compares only the densities. In
this way, we show that our measure is sharper and we justify the suitability of the Wasserstein-
base measures. Then we formulate the uncertainty set accordingly with a variation budget and
we propose *Uninformative Gradient Descent* Algorithm (UGD) as a natural reduction of the IGD
algorithm in the uninformative setting. We prove that UGD algorithm achieves a regret bound of
optimal order.

As a probability distance metric, the Wasserstein distance has been widely used as a measure
of the deviation between estimate and true distribution in the distributionally robust optimization
literature (e.g. Esfahani and Kuhn (2018)) to represent confidence set and it has demonstrated
good performance both theoretically and empirically. To the best of our knowledge, we are the first
to use the Wasserstein distance in an online optimization/learning context. From a modeling per-
spective, the two proposed measures WBDB and WBNB contribute to the study of non-stationary
environment for online optimization/learning problem. Specifically, the data-driven setting relaxes
the common assumption adopted in the NRM literature that the true distributions are known to
the decision maker by allowing the prior estimates to deviate from the true distributions. This
deviation can be interpreted as an estimation or model misspecification error, and WBDB estab-
lishes a connection between the deviation and algorithmic performance. The uninformative setting
generalizes a stream of online learning literature (e.g. Besbes et al. (2015)), which mainly concerned
with the unconstrained settings and includes bandits problem (Garivier and Moulines, 2008; Besbes
et al., 2014) and reinforcement learning problem (Cheung et al., 2019; Lecarpentier and Rachelson,
2019) as special cases. WBDB adds to the current dictionary of non-stationarity definitions and it
specializes for a characterization of the constrained setting.

1.2. Other Related Literature

Our formulation of the online stochastic optimization problem roots in two major applications:
the online linear programming (LP) problem and the network revenue management problem. We
briefly review these two streams of literature as follows.
The online LP problem (Molinaro and Ravi, 2013; Agrawal et al., 2014; Gupta and Molinaro, 2014) covers a wide range of applications through different ways of specifying the underlying LP, including secretary problem (Ferguson et al., 1989), online knapsack problem (Arlotto and Xie, 2020; Jiang and Zhang, 2020), resource allocation problem (Vanderbei et al., 2015; Asadpour et al., 2020), quantity-based network revenue management (NRM) problem (Jasin, 2015), generalized assignment problem (Conforti et al., 2014), network routing problem (Buchbinder and Naor, 2009), matching problem (Mehta et al., 2005), etc. Notably, the problem has been studied under either the stochastic input model where the coefficient in the objective function, together with the corresponding column in the constraint matrix is drawn from an unknown distribution $\mathcal{P}$, or the random permutation model where they arrive in a random order. As noted in the paper (Li et al., 2020), the random permutation model exhibits similar concentration behavior as the stochastic input model. The non-stationary setting of our paper relaxes the i.i.d. structure and it can be viewed as a third paradigm for analyzing the online LP problem.

The network revenue management (NRM) problem has been extensively studied in the literature and a main focus is to propose near-optimal policies with strong theoretical guarantees. One popular way is to construct a deterministic linear program as an upper bound of the optimal revenue and use its optimal solution to derive heuristic policies. Specifically, (Talluri and Van Ryzin, 1998) proposes a static bid-price policy based on the dual variable of the linear programming upper bound and proves that the revenue loss is $O(\sqrt{k})$ when each period is repeated $k$ times and the capacities are scaled by $k$. Subsequently, (Reiman and Wang, 2008) shows that by re-solving the linear programming upper bound once, one can obtain an $o(\sqrt{k})$ upper bound on the revenue loss. Then, (Jasin and Kumar, 2012) shows that under a so-called “non-degeneracy” assumption, a policy which re-solves the linear programming upper bound at each time period will lead to an $O(1)$ revenue loss, which is independent of the scaling factor $k$. The relationship between the performances of the control policies and the number of times of re-solving the linear programming upper bound is further discussed in their later paper (Jasin and Kumar, 2013). Recently, Bumpensanti and Wang (2020) propose an infrequent re-solving policy and show that their policy achieves an $O(1)$ upper bound of the revenue loss even without the “non-degeneracy” assumption. With a different approach, Vera and Banerjee (2020) prove the same $O(1)$ upper bound for the NRM problem and their approach is further generalized in (Vera et al., 2019) for other online decision making problems, including online stochastic knapsack, online probing, and dynamic pricing. Note that all the approaches mentioned above are developed for the stochastic/stationary setting. When the arrival process of customers is non-stationary over time, Adelman (2007) develops a strong heuristic based on a novel approximate dynamic programming (DP) approach. This approach is further investigated under various settings in the literature (for example (Zhang and Adelman,
2009; Kunnumkal and Talluri, 2016)). Remarkably, although the approximate DP heuristic is one of the strongest heuristics in practice, it does not feature for a theoretical bound. Finally, by using non-linear basis functions to approximate the value of the DP, Ma et al. (2020) develop a novel approximate DP policy and derive a constant competitiveness ratio dependent on the problem parameters.

2. Problem Formulation

Consider the following convex optimization problem

\[
\max \sum_{t=1}^{T} f_t(x_t) \quad \text{(CP)}
\]

\[
\text{s.t. } \sum_{t=1}^{T} g_{it}(x_t) \leq c_i, \quad i = 1, \ldots, m,
\]

\[
x_t \in \mathcal{X}, \quad t = 1, \ldots, T,
\]

where the decision variables are \(x_t \in \mathbb{R}^k\) for \(t = 1, \ldots, T\). Here \(\mathcal{X}\) is a compact convex set in \(\mathbb{R}^k\). The function \(f_t\)'s are functions in the space \(\mathcal{F} = \mathcal{F}(\mathcal{X})\) of concave continuous functions and \(g_{it}\)'s are functions in the space \(\mathcal{G} = \mathcal{G}(\mathcal{X})\) of convex continuous functions, both of which are supported on \(\mathcal{X}\).

Compactly, we define the vector-value function \(g_t(x) = (g_{1t}(x), \ldots, g_{mt}(x))^\top: \mathbb{R}^k \to \mathbb{R}^m\). Throughout the paper, we use \(i\) to index the constraint and \(t\) (or sometimes \(j\)) to index the decision variables, and we use bold symbols to denote vectors/matrices and normal symbols for scalars.

In this paper, we study the online stochastic optimization problem where the functions in (CP) are revealed in an online fashion and one needs to determine the value of decision variables sequentially. Specifically, at each time \(t\), the functions \((f_t, g_t)\) are revealed, and we need to decide the value of \(x_t\) instantly. Different from the offline setting, at time \(t\), we do not have the information of the future part of the optimization problem. Given the history \(H_{t-1} = \{f_j, g_j, x_j\}_{j=1}^{t-1}\), the decision of \(x_t\) can be expressed as a policy function of the history and the observation at the current time period. That is,

\[
x_t = \pi_t(f_t, g_t, H_{t-1}).
\]

The policy function \(\pi_t\) can be time-dependent and we denote policy \(\pi = (\pi_1, \ldots, \pi_n)\). The decision variable \(x_t\) must conform to the constraints in (CP) throughout the procedure, and the objective is aligned with the maximization objective for the offline problem (CP).

2.1. Parameterized Form, Probability Space, and Assumptions

Consider a parametric form of the underlying problem (CP) where the functions \((f_t, g_t)\) are parameterized by a parameter \(\theta_t \in \Theta \subset \mathbb{R}^l\). Specifically,

\[
f_t(x_t) := f(x_t; \theta_t), \quad g_{it}(x_t; \theta_t) := g_i(x_t; \theta_t)
\]
for each $i = 1, \ldots, m$ and $t = 1, \ldots, T$. The function $f$ is concave in its first argument, while
the function $g_i$ is convex in its first argument. We define the vector-value function $g(x; \theta) = (g_1(x; \theta), \ldots, g_m(x; \theta))^\top : \mathcal{X} \to \mathbb{R}^m$. Then the problem (CP) can be rewritten as the following parameterized convex program

$$\begin{align*}
\max & \quad \sum_{t=1}^T f(x_t, \theta_t) \\
\text{s.t.} & \quad \sum_{t=1}^T g_i(x_t, \theta_t) \leq c_i, \quad i = 1, \ldots, m, \\
& \quad x_t \in \mathcal{X}, \quad t = 1, \ldots, T,
\end{align*}$$

(PCP)

where the decision variables are $(x_1, \ldots, x_T)$. We note that this parametric form (PCP) is introduced mainly for presentation purpose, since it avoids the complication of defining probability measure in function space, and also it does not change the nature of the problem. We assume the knowledge of $f$ and $g$ a priori. Here and hereafter, we will use (PCP) as the underlying form of the online stochastic optimization problem.

The problem of online stochastic optimization, as its name refers, involves stochasticity on the functions for the underlying optimization problem. The parametric form (PCP) reduces the randomness from the function to the parameters $\theta_t$’s, and therefore the probability measure can be defined in the parameter space of $\Theta$. First, we consider the following distance function between two parameters $\theta, \theta' \in \Theta$,

$$\rho(\theta, \theta') := \sup_{x \in \mathcal{X}} \| (f(x, \theta), g(x, \theta)) - (f(x, \theta'), g(x, \theta')) \|_{\infty}$$

(2)

where $\| \cdot \|_{\infty}$ is the $L_\infty$ norm in $\mathbb{R}^{m+1}$. Without loss of generality, let $\Theta$ be a set of class representatives, that is, for any $\theta \neq \theta' \in \Theta$, $\rho(\theta, \theta') > 0$. In this way, the parameter space $\Theta$ can be viewed as a metric space equipped with metric $\rho(\cdot, \cdot)$. Also, note that we define the metric $\rho$ based on the vector-valued function $(f, g) : \mathcal{X} \to \mathbb{R}^{m+1}$, instead of a metric in the parameter space $\Theta$ (or $\mathbb{R}^l$). This is because the main focus is on the effect of different parameter on the function value rather than the original Euclidean difference in the parameter space. Let $\mathcal{B}_{\Theta}$ be the smallest $\sigma$-algebra in $\Theta$ that contains all open subsets (under metric $\rho$) of $\Theta$. We denote the distribution of $\theta_t$ as $\mathcal{P}_t$ and $\mathcal{P}_t$ can thus be viewed as a probability measure on $(\Theta, \mathcal{B}_{\Theta})$.

Throughout the paper, we make the following assumptions. Assumption 1 (a) and (b) impose boundedness on function $f$ and $g_i$’s. Assumption 1 (c) states the ratio between $f$ and $g_i$ is uniformly bounded by $q$ for all $x$ and $\theta$. Intuitively, it tells that for each unit consumption of resource, the maximum amount of revenue earned is upper bounded by $q$. In Assumption 1 (d), we assume $\mathcal{P}_t$’s are independent of each other but we do not assume the exact knowledge of them. Also, there can be dependence between components in the vector-value functions $(f(\cdot, \theta_t), g(\cdot, \theta_t))$. 
Assumption 1 (Boundedness and Independence) We assume

(a) \(|f(x, \theta)| \leq 1\) for all \(x \in \mathcal{X}, \theta \in \Theta\).

(b) \(g_i(x, \theta) \in [0, 1]\) for all \(x \in \mathcal{X}, \theta \in \Theta\) and \(i = 1, \ldots, m\). In particular, \(g_i(0, \theta) = 0\) for all \(\theta \in \Theta\).

(c) There exists a positive constant \(q\) such that for any \(\theta \in \Theta\) and each \(i\), we have that \(f(x, \theta) \leq q \cdot g_i(x, \theta)\) holds for any \(x \in \mathcal{X}\) as long as \(g_i(x, \theta) > 0\).

(d) \(\theta_t \sim P_t\) and \(P_t\)'s are independent with each others.

In the following, we illustrate the online formulation through two application contexts: online linear programming and online network revenue management. We choose the more general convex formulation (PCP) with the aim of uncovering the key mathematical structures for this online optimization problem, but we will occasionally return to these two examples to generate intuitions throughout the paper.

2.2. Examples

Online linear programming (LP): The online LP problem (Molinaro and Ravi, 2013; Agrawal et al., 2014; Gupta and Molinaro, 2014) can be viewed as an example of the online stochastic optimization formulation of (CP). Specifically, the decision variable \(x_t \in \mathcal{X} = [0, 1]\), the functions \(f\) and \(g_i\) are linear functions, and the parameter \(\theta_t = (r_t, a_t)\) where \(a_t = (a_{1t}, \ldots, a_{mt})\). Specifically, \(f(x_t; \theta_t) = r_t x_t\) and \(g_i(x_t; \theta_t) = a_{it} x_t\). At each time \(t = 1, \ldots, T\), the coefficient in the objective \(r_t\) together with the corresponding column in the constraint matrix \(a_t\) is revealed and one needs to determine the value of \(x_t\) immediately.

Price-based network revenue management (NRM): In the price-based NRM problem (Gallego and Van Ryzin, 1994), a firm is selling a given stock of products over a finite time horizons by posting a price at each time. The demand is price-sensitive and the firm’s objective is to maximize the total collected revenue. This problem could be cast in the formulation (PCP). Specifically, the parameter \(\theta_t\) refers to the type of the \(t\)-th arriving customer, and the decision variable \(x_t\) represents to the price posted by the decision maker at time \(t\). Accordingly, \(g(x_t; \theta_t)\) denotes the resource consumption under the price \(x_t\) and \(f(x_t; \theta_t)\) denotes the collected revenue.

Choice-based network revenue management: In the choice-based NRM problem (Talluri and Van Ryzin, 2004), the seller offers an assortment of the products to the customer arriving in each time period and the customer chooses a product from the assortment to purchase according to a given choice model. The formulation (PCP) can model the choice-based NRM problem as a special case by assuming that given each \(\theta\) and \(x\), \(f(x; \theta)\) and \(g(x; \theta)\) are all random variables. Specifically, for each \(t\), \(x_t \in \mathcal{X} \subset \{0, 1\}^m\) refers to the assortment offered at time \(t\) and \(\theta_t\) denotes the customer type. Then \(f(x_t; \theta_t)\) denotes the revenue collected by offering assortment \(x_t\), and \(g(x_t; \theta_t)\)
denotes the according resource consumption, where \( f(x_t; \theta_t) \) and \( g(x_t; \theta_t) \) are both stochastic and their distribution follows the choice model of the customer with type \( \theta_t \). Note that although in the following sections we only analyze the case where for each \( \theta \) and \( x \), \( f(x; \theta) \) and \( g(x; \theta) \) are deterministic, our analysis and results could be generalized directly to the case where \( f(x; \theta) \) and \( g(x; \theta) \) are random and follow known distributions.

2.3. Performance Measure

We denote the offline optimal solution of optimization problem (CP) as \( x^* = (x^*_1, ..., x^*_n) \), and the offline (online) objective value as \( R^*_n \) (\( R_n \)). Specifically,

\[
R^*_T := \sum_{t=1}^{T} f_t(x^*_t)
\]

\[
R_T(\pi) := \sum_{t=1}^{T} f_t(x_t).
\]

in which online objective value depends on the policy \( \pi \). Aligned with general online learning/optimization problems, we focus on minimizing the gap between the online and offline objective values. Specifically, the optimality gap is defined as follows:

\[
\text{Reg}_T(H, \pi) := R^*_T - R_T(\pi)
\]

where the problem profile \( H \) encapsulates a random realization of the parameters, i.e., \( H := (\theta_1, ..., \theta_T) \). Note that \( R^*_T, R_T(\pi), x^*_t \) and \( x_t \) are all dependent on the problem profile \( H \) as well, but we omit \( H \) in these terms for notation simplicity when there is no ambiguity. We define the performance measure of the online stochastic optimization problem formally as regret

\[
\text{Reg}_T(\pi) := \max_{P \in \Xi} \mathbb{E}_{H \sim P}[\text{Reg}_T(H, \pi)]
\]

(3)

where \( P = (P_1, ..., P_T) \) denotes the probability measure of all time periods and the expectation is taken with respect to the parameter \( \theta_t \sim P_t \); compactly, we write the problem profile \( H \sim P \). We consider the worst-case regret for all the distribution \( P \) in a certain set \( \Xi \) where the set \( \Xi \) will be specified in later sections.

We conclude this section with a few comments on our formulation of the online stochastic optimization problem. Generally speaking, the problem of online learning/optimization with constraints falls into two categories: (i) first-observe-then-decide and (ii) first-decide-then-observe. Our formulation belongs to the first category in that at each time \( t \), the decision maker first observes the parameter \( \theta_t \) and hence functions \( (f(x; \theta_t), g(x; \theta_t)) \), and then determines the value of \( x_t \). In many application contexts of operations research and operations management, the observations
constitute the meaning of customers/orders arriving sequentially to the system, and the decision variables capture accordingly the acceptance/rejection/pricing decisions of the customers. The problems discussed earlier, such as matching, resource allocation, network revenue management, all fall into this category. For the second category, the representative problems are bandits with knapsacks (Badanidiyuru et al., 2013) and online convex optimization (Hazan, 2016), where the decision is made first and the observation arrives after the decision. For example, in the classic bandits problem, the decision of which arm to play will affect the observation, and in the online convex optimization (or more generally two-player game setting (Cesa-Bianchi and Lugosi, 2006)), the “nature” may even choose the function $f_t$ against our made decision $x_t$ in an adversarial manner. There is a line of literature on online convex optimization with constraints, namely, the OCOwC problem (Mahdavi et al., 2012; Yu et al., 2017; Yuan and Lamperski, 2018). While the same underlying optimization problem (CP) is used in our formulation and the OCOwC problem, a key distinction is which of the decision or the observation is made first. Our formulation allows to observe before making the decision, and it thus enables us to adopt a stronger benchmark (as the definition of $R^*_T$), that is, a dynamic oracle which permits different value over different time periods. In contrast, the OCOwC problem requires to make decision before observe the functions $(f_t, g_t)$ and thus it considers a weaker benchmark which requires the decision variables take the same value over different time periods.

In the formulation, we have not yet discussed much about the conditions on the distributions $\mathcal{P} = (\mathcal{P}_1, ..., \mathcal{P}_T)$ except for independence. Importantly, this is one of the main themes of our paper. The canonical setting of online stochastic learning problem refers to the case when all the distributions are the same, i.e., $\mathcal{P}_t = \mathcal{P}_0$ for $t = 1, ..., T$. On the other extreme, the adversarial setting of online learning problem refers to the case when $\mathcal{P}_t$’s are adversarially chosen. Our work aims to bridge these two ends of the spectrum with a novel notion of non-stationarity, and we aim to relate the regret of the problem with structural property on $\mathcal{P} = (\mathcal{P}_1, ..., \mathcal{P}_T)$. In the same spirit, the work on non-stationary stochastic optimization (Besbes et al., 2015) proposes an elegant notion of non-stationarity called variation budget. Subsequent works consider similar notions in the settings of bandits (Besbes et al., 2014; Russac et al., 2019) and reinforcement learning (Cheung et al., 2019). To the best of our knowledge, all the previous works along this line consider unconstrained setting and thus our work contributes to this line of work in illustrating how the constraints interact with the non-stationarity. We will return to the point later in the paper.

3. Known Distribution and Informative Gradient Descent

In this section, we consider the case when the distributions $\mathcal{P}_t$’s are all known at the beginning of the horizon. We use this case to motivate and present our main algorithm – informative gradient descent algorithm which incorporates the prior information of $\mathcal{P}_t$’s with the online gradient
descent algorithm. In the following sections, we will discuss the case when the distributions \( P_t \)'s are unknown and analyze the algorithm accordingly.

### 3.1. Benchmark Upper Bounds

For starters, we analyze the “offline” optimum \( E[R_T^*] \) and derive three upper bounds for it to inspire our algorithm design. The derivation of the first upper bound is standard in online decision making problems and it is also known as the deterministic upper bound (for example, see (Gallego and Van Ryzin, 1994)). The reason for proposing this deterministic upper bound is that the offline optimum \( E[R_T^*] \) (obtained by solving (PCP)) often preserves complex structure, and thus is very hard to analyze. Comparatively, the proposed upper bound features for better analytical tractability. We introduce the following notation for a function \( u(x; \theta) : \mathcal{X} \rightarrow \mathbb{R} \) and a probability measure \( \mathcal{P} \) in the parameter space \( \Theta \),

\[
\mathcal{P} u(x(\theta)) := \int_{\theta \in \Theta} u(x(\theta); \theta) d\mathcal{P}(\theta)
\]

where \( x(\theta) : \Theta \rightarrow \mathbb{R} \) is a measurable function. Thus \( \mathcal{P} u(\cdot) \) can be viewed as a deterministic functional that maps function \( x(\theta) \) to a real value and it is obtained by taking expectation with respect to the parameter \( \theta \sim \mathcal{P} \).

Consider the following optimization problem

\[
R_T^{UB} = \max_{T} \sum_{t=1}^{T} \mathcal{P}_t f(x_t(\theta))
\]

s.t.

\[
\sum_{t=1}^{T} \mathcal{P}_t g_i(x_t(\theta)) \leq c_i, \quad i = 1, \ldots, m,
\]

\[
x_t(\theta) : \Theta \rightarrow \mathcal{X} \text{ is a measurable function for } t = 1, \ldots, T.
\]

The optimization problem (4) can be viewed as a convex relaxation of (PCP) where the objective/constraints are all replaced with their expected counterparts. Here \( x_{1:T}(\theta) = (x_1(\theta), \ldots, x_T(\theta)) \) encapsulates all the primal decision variables. The primal variables are expressed in a function form because for each different value of \( \theta \), we allow a different choice of the primal variables. At time \( t \), the parameter \( \theta \) follows the distribution \( \mathcal{P}_t \). In the following, Lemma 1 shows the optimal objective value \( R_T^{UB} \) works as an upper bound for \( E[R_T^*] \). Thus it formally establishes \( R_T^{UB} \) as a surrogate benchmark for \( E[R_T^*] \) when analyzing the regret.

**Lemma 1** It holds that \( R_T^{UB} \geq E[R_T^*] \).

Now we seek for a second upper bound by considering the Lagrangian function of (4),

\[
L(p, x_{1:T}(\theta)) = c^T p + \sum_{t=1}^{T} \mathcal{P}_t (f(x_t(\theta)) - p^T g(x_t(\theta)))
\]
where the (Lagrangian multiplier) vector $p = (p_1, \ldots, p_m)\top$ conveys a meaning of dual price for each budget where $p_i \geq 0$ is the multiplier/dual variable associated with the $i$-th constraint. It follows from weak duality that

$$R_T^{UB} \leq \min_{p \geq 0} \max_{x_{1:T}(\theta)} L(p, x_{1:T}(\theta))$$

where the maximum is taken with respect to all measurable functions that maps $\Theta$ to $X$. In fact, the inner maximization with respect to $x_{1:T}$ can be achieved in a point-wise manner by defining the following function for each $\theta$

$$h(p; \theta) := \max_{x \in X} \{ f(x; \theta) - p\top g(x; \theta) \}$$

where $h$ is a function of the dual variable $p$ and it is also parameterized by $\theta$. This also echoes the “first-observe-then-decide” setting where at each time $t$, the decision maker first observes the parameter $\theta_t$ and then decides the value of $x_t$. Moreover, let

$$L(p) := c\top p + \sum_{t=1}^{T} P_t h(p, \theta)$$

and it holds that $L(p) = \max_{x_{1:T}(\theta)} L(p, x_{1:T}(\theta))$. Thus,

$$R_T^{UB} \leq \min_{p \geq 0} L(p).$$

where the right-hand-side serves as the second upper bound of the problem. The above discussions are summarized in Lemma 2. The advantage of the function $L(p)$ is that it only involves the dual variable $p$, and the dual variable is not time-dependent.

**Lemma 2** It holds that

$$\min_{p \geq 0} \max_{x \in X} L(p, x) = \min_{p \geq 0} L(p).$$

Consequently, we have the following upper bound of $E[R_T^*]$,

$$E[R_T^*] \leq \min_{p \geq 0} L(p).$$

Now we derive our last upper bound of $E[R_T^*]$, which will directly translate to our main algorithm. Let $p^*$ denote an optimal solution of the upper bound in Lemma 2,

$$p^* \in \operatorname{argmin}_{p \geq 0} L(p) \quad (5)$$

and for each $t$, define

$$\gamma_t := P_t g(x^*(\theta); \theta) \quad \text{where } x^*(\theta) = \operatorname{argmax}_{x \in X} \{ f(x; \theta) - (p^*)\top g(x; \theta) \}. \quad (6)$$
Here, $x^*(\theta)$ is the associated primal optimal solution under the dual solution $p^*$, and $\gamma_t$ can be interpreted as the expected budget consumption in the $t$-th time period under the primal-dual pair $(x^*(\theta), p^*)$. Accordingly, for each $t$, we define

$$L_t(p) := \gamma_t^\top p + \mathcal{P}_t h(p; \theta).$$

The following lemma states the relation between $L(\cdot)$ and $L_t(\cdot)$.

**Lemma 3** For each $t = 1, \ldots, T$, it holds that

$$p^* \in \arg\min_{p \geq 0} L_t(p)$$

where $p^*$ is defined in (5) as the minimizer of the function $L(\cdot)$. Moreover, it holds that

$$L(p^*) = \sum_{t=1}^T L_t(p^*).$$

With the definition of $L_t(\cdot)$, we find a way to decompose the function $L(\cdot)$ into a summation of $T$ functions. More importantly, as stated in Lemma 3, the way how we construct $\gamma_t$’s ensures that all the $T$ functions share the same optimal solution $p^*$. To conclude, we establish an upper bound for the offline optimum with the functions $L_t(\cdot)$.

**Proposition 1** We have the following upper bound of $\mathbb{E}[R_T^*]$:

$$\mathbb{E}[R_T^*] \leq \min_{p \geq 0} \sum_{t=1}^T L_t(p) = \sum_{t=1}^T \min_{p_t \geq 0} L_t(p_t).$$

The significance of Proposition 1 and the introduction of the functions $L_t(\cdot)$ goes beyond an analytical purpose of serving as a deterministic benchmark for regret analysis. Note that Proposition 1 represents the offline optimal objective $R_T^*$ (which is originally defined in primal space) with functions defined in the dual space. The advantage of this dual-based representation is that the dual optimal solution for each time period (with respect to the function $L_t(\cdot)$) is the same, whereas the primal optimal solution $x_t$ to (PCP) may be different from each $t$. In the literature of online learning/online optimization, there are two types of benchmarks in defining the regret: static oracle and dynamic oracle. The static oracle refers to the case where we compare to an offline decision maker adopting a common optimal solution throughout the entire horizon, and the dynamic oracle allows the offline decision maker to take an individual optimal solution for each time period. For our online stochastic optimization problem (PCP), we consider the dynamic oracle in defining $R_T^*$, while the online convex optimization (OCO) literature (Hazan, 2016) adopts the static oracle. However, Proposition 1 tells that in our setting, the static oracle and the dynamic oracle coincide in the dual space with a careful construction of $L_t(\cdot)$. This connection makes it possible to apply the techniques from OCO literature for our setting in the dual space. Besbes et al. (2015) derive a similar argument to connect the dynamic oracle and static oracle for the unconstrained setting as a backbone for the algorithm design and regret analysis therein.
3.2. Main Algorithm and Regret Analysis

Now we present our main algorithm – *Informative Gradient Descent Algorithm* – fully described in Algorithm 1. The algorithm is motivated from the dual-based representation in Proposition 1. Specifically, it maintains a dual vector/price $p_t$, and performs a stochastic gradient descent update for $p_t$ with respect to the function $L_t(\cdot)$ at each time period; the step size of gradient descent is set to be $\frac{1}{\sqrt{T}}$. To see that the expectation of the dual gradient update (7) is the gradient with respect to the function $L_t(\cdot)$ evaluated at $p_t$,

$$
E[g(\tilde{x}_t; \theta_t) - \gamma_t] = -\gamma_t + \mathcal{P}_t g(\tilde{x}_t; \theta_t) = -\frac{\partial}{\partial p} \left( \gamma_t^\top p + \mathcal{P}_t h(p, \theta) \right) \bigg|_{p=p_t}.
$$

Here the first line comes from taking expectation with respect to $\theta_t$ and the second line comes from the definition of $\tilde{x}_t$ in Algorithm 1. In our IGD algorithm, the primal decision variable $x_t$ is then determined based on the dual price $p_t$ and the observation $\theta_t$, in the same manner as the definition of the function $h(p; \theta)$. Throughout the paper, we assume the optimization problem in defining $h(p; \theta)$ can be solved efficiently (See further discussion in Section A3).

We now provide another perspective to interpret Algorithm 1. Note that by its definition, $\gamma_t$’s represent the “optimal” way to allocate the budget over time according to the dual problem in Proposition 1. Specifically, a larger (resp. smaller) value of $\gamma_{i,t}$, where $\gamma_{i,t}$ denotes the $i$-th component of $\gamma_t$, indicates that more (resp. less) budget should be allocated to time period $t$ for constraint $i$. In Algorithm 1, from the update rule (7) of the dual variable at time period $t$, we know that if the budget consumption of constraint $i$ is larger (resp. smaller) than $\gamma_{i,t}$, i.e., $g_i(\tilde{x}_t; \theta_t) > \gamma_{i,t}$ (resp. $g_i(\tilde{x}_t; \theta_t) < \gamma_{i,t}$), then we have that $p_{i,t+1} \leq p_{i,t}$ (resp. $p_{i,t+1} > p_{i,t}$), where $p_{i,t}$ denotes the $i$-th component of $p_t$. If more (resp. less) budget is consumed in the earlier periods, then the dual price will be likely to increase (resp. decrease), and consequently, less (resp. more) budget will be consumed in the future periods. In this sense, the dual variable $p_t$ balances the process of the budget consumption. In addition, Lemma 4 shows that the dual vector $p_t$ remains bounded during the process of Algorithm 1. As a result, it implies that for each $t$, the cumulative budget consumption during Algorithm 1 for the first $t$ time periods always stay “close” to $\sum_{j=1}^{t} \gamma_j$. We remark that the proof of Lemma 4 largely relies on Assumption 1 (c), and conversely, the main usage of Assumption 1 (c) throughout our analysis is to ensure the boundedness of the dual vector.

**Lemma 4** Under Assumption 1, the dual price vector satisfies $\|p_t\|_\infty \leq q + 1$ for $t = 1, 2, \ldots, T$. Here $p_t$ is specified by (7) in Algorithm 1 and the constant $q$ is defined in Assumption 1 (c).
Algorithm 1 Informative Gradient Descent Algorithm (IGD)

1: Initialize the initial dual price $p_1 = 0$ and initial constraint capacity $c_1 = c$
2: Obtain parameters $\gamma_t$ for each $t$ from (6).
3: for $t = 1, \ldots, T$ do
4: Observe $\theta_t$ and solve

$$\tilde{x}_t = \arg\max_{x \in X} \left\{ f(x; \theta_t) - p_t^\top g(x; \theta_t) \right\}$$

where $g(x, \theta_t) = (g_1(x, \theta_t), \ldots, g_m(x, \theta_t))^\top$
5: Set

$$x_t = \begin{cases} \tilde{x}_t, & \text{if } c_t \text{ permits a consumption of } g(\tilde{x}_t; \theta_t) \\ 0, & \text{otherwise} \end{cases}$$
6: Update the dual price

$$p_{t+1} = \left( p_t + \frac{1}{\sqrt{T}} (g(\tilde{x}_t; \theta_t) - \gamma_t) \right) \lor 0$$

where the element-wise maximum operator $u \lor v = \max\{v, u\}$
7: Update the remaining capacity

$$c_{t+1} = c_t - g(x_t; \theta_t)$$
8: end for
9: Output: $x = (x_1, \ldots, x_T)$

The following theorem builds upon Proposition 1 and Lemma 4 and it states that the regret of Algorithm 1 is upper bounded by $O(\sqrt{T})$. Note that from Lemma 1 in (Arlotto and Gurvich, 2019), even for a stationary setting where $P_t$ for each $t$ is identical to each other, the lower bound of any online policy is $\Omega(\sqrt{T})$ when there is no additional assumption on $P_t$. Thus, the $O(\sqrt{T})$ upper bound of Algorithm 1 is of optimal order.

Theorem 1 Under Assumption 1, if we consider the set $\Xi = \{\mathcal{P} : \mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_T), \forall \mathcal{P}_1, \ldots, \forall \mathcal{P}_T\}$, then the regret of Algorithm 1 has the following upper bound

$$\text{Reg}_T(\pi_{IGD}) \leq O(\sqrt{T})$$

where $\pi_{IGD}$ stands for the policy specified by Algorithm 1.

Theorem 1 discusses the setting when no further restriction is put on the distribution $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_T)$ other than Assumption 1 but the distribution $\mathcal{P}$ is known in priori. The $O(\sqrt{T})$ regret
upper bound of Algorithm 1 under the non-stationary environment is new in the literature. Specifically, for the NRM problem, Talluri and Van Ryzin (1998) derives a $O(\sqrt{k})$ regret bound when the system size is scaled by $k$ times, i.e., each period in the original problem is split into $k$ statistically independent and identical periods and the capacities are scaled up $k$ times. Following works subsequently improves this bound to $O(1)$ (e.g. (Reiman and Wang, 2008; Jasin and Kumar, 2012; Bumpensanti and Wang, 2020; Vera and Banerjee, 2020)). However, these methods are developed for a stationary setting and none of them account for the achievability of a sublinear regret under a non-stationary setting where the distribution at each period could be arbitrarily different from each other. To the best of our knowledge, the only similar result to ours is from (Devanur et al., 2019) where the authors derives a $1 - 1/\sqrt{c_{\text{min}}}$ competitive ratio under the non-stationary setting, where $c_{\text{min}} = \min\{c_1, c_2, \ldots, c_m\}$ is the minimal budget and they assume that $f_t(\cdot)$ and $g_t(\cdot)$ are all linear functions for each $t$. However, the competitive ratio result could not be translated into a regret bound in our setting since we do not assume any relationship between $T$ and $c$. Moreover, Algorithm 1 and the corresponding analysis are totally different from theirs. Specifically, their method is based on showing that the arrivals possess a concentration property and applying Chernoff-type inequality to derive high probability bounds on the event that all the constraints are not violated. In contrast, our method is based on using the dual variable $p_t$ to balance the budget consumption on every sample path and showing that the dual variable is bounded almost surely according to our update rule (7). In this sense, we provide a new methodology that is more intuitive and easily to implement. In the following sections, we continue our pursuit and investigate how the IDG algorithm rolls out in a non-stationary environment when the true distribution is unknown.

4. Non-stationary Environment with Prior Estimate: Wasserstein Based Ambiguity and Analysis

In this section, we consider a data-driven setting where the true distribution is unknown, but a prior estimate of the true distribution is available. Mathematically, the setting relaxes the assumption on the knowledge of the true distribution in the last section. Practically, the availability of the prior estimate characterizes the predictable patterns of the non-stationarity in various application contexts. For example, the decision maker may not be able to foresee the future demand (distribution), but (s)he can form an estimate based on history data or domain expertise for demand seasonality, the day-of-week effect, and demand surge due to pre-scheduled promotions or shopping festivals. When the prior estimate is accurate (the same as the true distribution), the setting here reduces to our discussion in the last section. However, when the prior estimate deviates from the true distribution, as often the case in reality, then two natural questions are: (i) how to measure the inaccuracy of the prior estimate from the true distribution, (ii) how to design and analyze algorithm under an inaccurate prior estimate. We answer these two questions in this section.
4.1. Wasserstein-Based Measure of Deviation

Consider the decision maker has a prior estimate/prediction $\hat{P}_t$ for the true distribution $P_t$ for each $t$, and all the predictions $\{\hat{P}_t\}_{t=1}^T$ are made available at the very beginning of the procedure. We use the Wasserstein distance between $\hat{P}_t$ and $P_t$ to measure the deviation of the prior estimate from the true distribution. In following, we first formalize the definition and then discuss the suitability of the proposed Wasserstein-based measure.

The Wasserstein distance, also known as Kantorovich-Rubinstein metric or optimal transport distance (Villani, 2008; Galichon, 2018), is a distance function defined between probability distributions on a metric space. Its notion has a long history dating back over decades ago and gains increasingly popularity in recent years with a wide range of applications including generative modeling (Arjovsky et al., 2017), robust optimization (Esfahani and Kuhn, 2018), statistical estimation (Blanchet et al., 2019), etc. In our context, the Wasserstein distance for two probability distributions $Q_1$ and $Q_2$ on the metric parameter space $(\Theta, B)$ is defined as follows,

$$W(Q_1, Q_2) := \inf_{Q_{1,2} \in J(Q_1, Q_2)} \int \rho(\theta_1, \theta_2) dQ_{1,2}(\theta_1, \theta_2)$$

(8)

where $J(Q_1, Q_2)$ denotes all the joint distributions $Q_{1,2}$ for $(\theta_1, \theta_2)$ that have marginals $Q_1$ and $Q_2$. The distance function $\rho(\cdot, \cdot)$ is defined earlier in (2).

We define the following Wasserstein-based deviation budget (WBDB) to measure the cumulative deviation of the prior estimate,

$$W_T(P, \hat{P}) := \sum_{t=1}^T W(P_t, \hat{P}_t)$$

where $P = (P_1, ..., P_T)$ denotes the true distribution and $\hat{P} = (\hat{P}_1, ..., \hat{P}_T)$ denotes the prior estimate.

Based on the notion of WBDB, we define a set of distributions

$$\Xi_P(W_T) := \{P : W_T(P, \hat{P}) \leq W_T, P = (P_1, ..., P_T)\}$$

for a non-negative constant $W_T$, which is called deviation budget. In this section, we consider a regret based on the set $\Xi_P$ as defined in (3). In this way, the regret characterizes a “worst-case” performance of a certain algorithm for all the distributions $P = (P_1, ..., P_T)$ in the set $\Xi_P$. The deviation budget $W_T$ defines the set $\Xi_P$ by inducing an upper bound for the deviation of prior estimate. Our next theorem provides an intuitive result that $W_T$ is an inevitable cost (in terms of the algorithm regret) paid for the inaccuracy of the prior estimate.

**Theorem 2** Under Assumption 1, if we consider the set $\Xi_P(W_T) := \{P : W_T(P, \hat{P}) \leq W_T, P = (P_1, ..., P_T)\}$, there is no algorithm that can achieve a regret better than $\Omega(\max\{\sqrt{T}, W_T\})$. 

Theorem 2 states that the lower bound of the regret is \( \Omega(\max\{\sqrt{T}, W_T\}) \). The theorem characterizes the best achievable algorithm performance under an inaccurate prior estimate, and precisely, the lower bound is linear in respect with the deviation of the prior estimate from the true distribution. The \( \Omega(\sqrt{T}) \) part captures the intrinsic uncertainty of the underlying stochastic process over a time horizon \( T \), and its proof is due to Lemma 1 in (Arlotto and Gurvich, 2019). Comparatively, the \( \Omega(W_T) \) part captures the uncertainty arising from the inaccurate prior estimate, and its proof can be established from a simple example (for which we defer to the later sections and the appendix).

4.2. Informative Gradient Descent Algorithm with Prior Estimate

Now we extend our informative gradient descent algorithm to the setting of prior estimate. A natural thought is to pretend that the prior estimate \( \hat{P}_t \) is indeed the true distribution \( P_t \). In parallel with derivation of Algorithm 1, we define

\[
\hat{L}(p) = c^\top p + \sum_{t=1}^{T} \hat{P}_t h(p, \theta)
\]

where the true distribution \( P_t \) is replaced by its estimate \( \hat{P}_t \) for each component in function \( L(\cdot) \). Thus it can be viewed as an approximation for the true dual function \( L(\cdot) \) based on prior estimate. Let \( \hat{p}^* \) denote an optimal solution to \( \hat{L}(\cdot) \),

\[
\hat{p}^* \in \arg\min_{p \geq 0} \hat{L}(p)
\]

and for each \( t \), define

\[
\hat{\gamma}_t := \hat{P}_t g(\hat{x}(\theta); \theta) \quad \text{where} \quad \hat{x}(\theta) = \arg\max_{x \in X} \{ f(x; \theta) - (\hat{p}^*)^\top \cdot g(x; \theta) \}.
\]

Here, \( \hat{\gamma}_t \) denotes the “optimal” expected budget consumption in the \( t \)-th time under the prior estimate. In the setting of prior estimate, we do not have the exact knowledge of the true distributions \( P_t \)'s and therefore \( \gamma_t \)'s, so we alternatively use \( \hat{\gamma}_t \) as a substitute. Algorithm 2 summarizes the idea and its only difference from Algorithm 1 lies in the replacement of \( \gamma_t \) with \( \hat{\gamma}_t \).

A natural question may arise that what if we simply use the “offline” optimal dual solution \( p^* \) or \( \hat{p}^* \) to form a static decision rule throughout the procedure? As the renowned fixed bid-price policy (Talluri and Van Ryzin, 1998) for the network revenue management problem. In Section B7, we show that such static policy can result in linear regret even when the WBDB is arbitrarily small. This comparison highlights the necessity of the dynamic dual updating rule in Algorithm 2.
Algorithm 2 Informative Gradient Descent Algorithm with Prior Estimate (IGDP)

1: Initialize the initial dual price $p_1 = 0$ and initial constraint capacity $c_1 = c$.
2: Solve the optimization problem $\min_{p \geq 0} \hat{L}(p)$ and compute $\hat{\gamma}_t$ from (10) for each $t$
3: for $t = 1, ..., T$ do
4: Observe $\theta_t$ and solve
$$\tilde{x}_t = \arg\max_{x \in X} \{f(x; \theta_t) - p_t^\top g(x; \theta_t)\}$$
where $g(x, \theta_t) = (g_1(x, \theta_t), ..., g_m(x, \theta_t))^\top$
5: Set
$$x_t = \begin{cases} \tilde{x}_t, & \text{if } c_t \text{ permits a consumption of } g(\tilde{x}_t; \theta_t) \\ 0, & \text{otherwise} \end{cases}$$
6: Update the dual price
$$p_{t+1} = \left( p_t + \frac{1}{\sqrt{T}} (g(\tilde{x}_t; \theta_t) - \hat{\gamma}_t) \right) \vee 0$$
where the element-wise maximum operator $u \vee v = \max\{v, u\}$
7: Update the remaining capacity
$$c_{t+1} = c_t - g(x_t; \theta_t)$$
8: end for
9: Output: $x = (x_1, ..., x_T)$

4.3. Regret Analysis

The analysis of Algorithm 2 is slightly more complicated than that of Algorithm 1 because the algorithm is built upon the function $\hat{L}(\cdot)$ defined by the prior estimate instead of the true distribution. So we first study how the deviation between prior estimates and true distributions will differentiate between the function $\hat{L}(\cdot)$ and $L(\cdot)$. For a probability measure $Q$ over the metric parameter space $(\Theta, \mathcal{B}_\Theta)$, we define
$$L_Q(p) := Qh(p; \theta).$$
Then the function $\hat{L}(\cdot)$ and $L(\cdot)$ can be expressed as
$$\hat{L}(p) = p^\top c + \sum_{t=1}^{T} L_{\hat{P}_t}(p) \quad \text{and} \quad L(p) = p^\top c + \sum_{t=1}^{T} L_{P_t}(p)$$
Lemma 5 states that the function $L_Q(p)$ has certain “Lipschitz continuity” in regard with the underlying distribution $Q$. Specifically, the supremum norm between two functions $L_{Q_1}(p)$ and
$L_{Q_2}(p)$ is bounded by the Wasserstein distance between two distributions $Q_1$ and $Q_2$ up to a constant dependent on the dimension and the boundedness of the function’s argument.

**Lemma 5** For two probability measures $Q_1$ and $Q_2$ over the metric parameter space $(\Theta, B_\Theta)$, we have that

$$\sup_{p \in \Omega_p} |L_{Q_1}(p) - L_{Q_2}(p)| \leq \max\{1, \bar{p}\} \cdot (m + 1) W(Q_1, Q_2)$$

where $\Omega_p = [0, \bar{p}]^m$ and $\bar{p}$ is an arbitrary positive constant.

Note that the Lipschitz constant in Lemma 5 involves an upper bound of the function argument $p$. The following lemma provides such an upper bound for the dual price $p_t$’s in Algorithm 1. The derivation is essentially the same as Lemma 4.

**Lemma 6** For each $t = 1, 2, \ldots, T$, we have that $\|p_t\|_\infty \leq q + 1$ with probability 1, where $p_t$ is specified by (11) in Algorithm 2.

The rest of Algorithm 2 is similar to that of Algorithm 1 in Theorem 1. The regret of Algorithm 2 is formally stated in Theorem 3. Notably, the algorithm’s regret matches the lower bound of the setting and thus it establishes the optimality of the algorithm.

**Theorem 3** Under Assumption 1, suppose a prior estimate $\hat{P}$ is available and the regret is defined based on the set $\Xi_P(W_T)$, then the regret of Algorithm 2 has the following upper bound

$$\text{Reg}_T(\pi_{IGDP}) \leq O(\max\{\sqrt{T}, W_T\})$$

where $\pi_{IGDP}$ stands for the policy specified by Algorithm 2.

We remark Algorithm 2 does not depend on or utilize the knowledge of the quantity $W_T$. On the upside, this avoids the assumption on the prior knowledge of $W_T$ (as the knowledge of variation budget $V_T$ (Besbes et al., 2015)). On the downside, there is nothing the algorithm can do even when it knows a priori $W_T$ is small or large. Technically, it means for Algorithm 2, the WBNB contributes nothing in the dimension of algorithm design, and it will only influence the algorithm analysis. In particular, if we compare Theorem 3 with Theorem 1, the extra term $W_T$ captures how the deviation of the prior estimate from the true distribution will deteriorate the performance of the gradient-based algorithm. The good news is that when $W_T$ is small, the $O(\sqrt{T})$ will be dominant and we do not need to worry about the deviation because its effect on the regret is marginal. In this light, the regret result illuminates the effect of model misspecification/estimation error on the algorithm’s performance in a non-stationary environment.
5. Non-stationary Environment Without Prior Estimate

In this section, we consider an uninformative setting where the true distribution is entirely concealed from the decision maker. To one end, the discussion in this section can be viewed as a reduction of the results in the last section to a setting with “uninformative” prior estimate. To the other, the uninformative setting draws an interesting comparison with the literature on (unconstrained) online learning/optimization in non-stationary environment (Besbes et al., 2014, 2015; Cheung et al., 2019) and exemplifies the interaction between the constraints and the non-stationarity.

5.1. Revisiting WBDB

Before we formally introduce the uninformative version of WBDB, we first illustrate how the non-stationarity over $\{P_t\}_{t=1}^T$ interplays with the constraints, through the following example adapted from (Golrezaei et al., 2014). The original usage of the example in their paper is to stress the importance of balancing resource consumption in an online context. Specifically, consider the following two linear programs as the underlying problem (PCP) for two online stochastic optimization problems,

$$\begin{align*}
\max & \quad x_1 + \ldots + x_c + (1 + \kappa)x_{c+1} + \ldots + (1 + \kappa)x_T \\
\text{s.t.} & \quad x_1 + \ldots + x_c + x_{c+1} + \ldots + x_T \leq c \\
& \quad 0 \leq x_t \leq 1 \quad \text{for} \quad t = 1, \ldots, T.
\end{align*}$$

$$\begin{align*}
\max & \quad x_1 + \ldots + x_c + (1 - \kappa)x_{c+1} + \ldots + (1 - \kappa)x_T \\
\text{s.t.} & \quad x_1 + \ldots + x_c + x_{c+1} + \ldots + x_T \leq c \\
& \quad 0 \leq x_t \leq 1 \quad \text{for} \quad t = 1, \ldots, T.
\end{align*}$$

where $\kappa \in (0, 1)$, $c = \frac{T}{2}$ and the true distributions for both scenarios are one-point distributions. Without loss of generality, we assume $c$ is an integer. Note that for the first LP (12), the optimal solution is to wait and accept the later half of the orders while for the second LP (13), the optimal solution is to accept the first half of the orders and deplete the resource at half time. The contrast between the two LPs (two scenarios of whether the first half or the second half is more profitable) creates difficulty for the online decision making. Without knowledge of the future orders, there is no way we can obtain a sub-linear regret in both scenarios simultaneously. Because if we exhaust too much resource in the first half of the time, then for the first scenario (12), we do not have enough capacity to accept all the relatively profitable orders in the second half. On the contrary, if we have too much remaining resource at the half way, then for the second scenario (13), those relatively profitable orders that we miss in the first half are irrevocable.
An equivalent view of these two examples is to assume the existence of an adversary: the adversary is aware of the policy of the decision maker at the very beginning and then chooses the distribution $\mathcal{P}_t$ in an adversarial manner. Specifically, the adversary’s purpose is against us and it is to maximize the optimality gap between the offline optimal objective value and the online objective value. For example, in (12) and (13), the adversary can make a decision of which scenario for us to enter for the second half of the time based on our remaining inventory at the half way. The adversary view augments our previous interpretation of $W_T$ as the maximal derivation (of the prior estimate from the true distribution): the regret definition based on $\Xi_{\mathcal{P}}$ in the last section can be viewed as a partially adversarial setting where the adversary chooses the true distribution against our will in a sequential manner but his choice of the distributions $\mathcal{P}_t$ is subject to the set $\Xi_{\mathcal{P}}$. Then the parameter $W_T$ in defining the set $\Xi_{\mathcal{P}}$ serves as a measure of both the estimation error and the intensity of adversity.

The above discussions are summarized in Proposition 2.

**Proposition 2** The worst-case regret of constrained online stochastic optimization in adversarial setting is $\Omega(T)$.

Proposition 2 states that the achievability of a sub-linear regret does not permit a fully adversarial setting where $\mathcal{P}_t$ can change arbitrarily over $t$. The same observation is also made in the literature (Besbes et al., 2014, 2015; Cheung et al., 2019) for unconstrained online learning problems where there is no function $g(\cdot)$ and the decision $x_t$ is made before the revealing of $f(\cdot)$. Specifically, Besbes et al. (2015) propose a novel measure to control the non-stationarity, which can be stated as follows (in the language of our paper),

$$V_T := \sum_{t=1}^{T-1} TV(\mathcal{P}_t, \mathcal{P}_{t+1})$$

where $TV(\cdot, \cdot)$ denotes the total variation distance between two distributions. The quantity $V_T$ represents the cumulative temporal change of the distributions by comparing $\mathcal{P}_t$ and $\mathcal{P}_{t+1}$. Unfortunately, such temporal measure fails in the constrained setting. Note that for both (12) and (13), there is only one change point throughout the whole procedure thus the non-stationarity; their temporal change measure is $O(1)$ but a sub-linear regret is still unattainable.

Now, we propose the definition of the Wasserstein-based non-stationarity budget (WBNB) as

$$W_T(\mathcal{P}) := \sum_{t=1}^{T} W(\mathcal{P}_t, \bar{\mathcal{P}}_T)$$

where $\mathcal{P} = (\mathcal{P}_1, ..., \mathcal{P}_T)$ and $\bar{\mathcal{P}}_T$ is defined to be the uniform mixture distribution of $\{\mathcal{P}_t\}_{t=1}^{T}$, i.e.,

$$\bar{\mathcal{P}}_T := \frac{1}{T} \sum_{t=1}^{T} \mathcal{P}_t.$$ 

The non-stationarity measure WBNB can be viewed as a degeneration of our
previous deviation measure WBDB in that WBNB replaces all the prior estimates $\hat{P}_t$’s with the uniform mixture $\bar{P}_T$. The caveat is that in the uninformative setting, no distribution knowledge is assumed, so the decision maker does not have access to $\bar{P}_T$ unlike the prior estimate in the last section. However, as we will see shortly, the knowledge of $\bar{P}_T$ does not affect anything from the algorithm design to analysis.

Based on the notion of WBNB, we define a set of distributions

$$\Xi_U(W_T) = \{\mathcal{P} : W_T(\mathcal{P}) \leq W_T, \mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_T)\}$$

for a non-negative constant $W_T$, which we name as the variation budget. Throughout this section, we consider a regret based on the set $\Xi_U$ as defined in (3) in aim to characterize a “worst-case” performance of certain policy/algorithm for all the distributions in the set $\Xi_U$.

The variation budget $W_T$ defines the uncertainty set $\Xi_U$ by providing an upper bound on the non-stationarity of the distributions. Our next theorem states that it is impossible to get rid of $W_T$ in the regret bound of any algorithm, which illustrates the sharpness of our definition of WBNB. Intuitively, it means that apart from the intrinsic stochasticity term $O(\sqrt{T})$, the (unknown) non-stationarity of the underlying distributions defined by WBNB appears to be a bottleneck for algorithm performance.

**Theorem 4** Under Assumption 1, if we consider the set $\Xi_U(W_T) = \{\mathcal{P} : W_T(\mathcal{P}) \leq W_T, \mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_T)\}$, there is no algorithm that can achieve a regret better than $\Omega(\max\{\sqrt{T}, W_T\})$.

**Why Wasserstein Distance?**

We detour shortly from our main discussion and provide an explanation for why we adopt the Wasserstein distance of the total variation distance or the KL-divergence in defining both WBDB and WBNB. If we revisit the examples (12) and (13), a smaller value of $\kappa \in (0, 1)$ should indicate a smaller variation/non-stationarity between the first half and the second half of observations in both examples. However, the total variation distance fails to characterize this subtlety in that for any non-zero value of $\kappa$, the total variation distance between $\mathcal{P}_t$ and $\mathcal{P}_{t'}$ for $t \leq \frac{T}{2} < t'$ is always 1 (since $\mathcal{P}_t$ and $\mathcal{P}_{t'}$ have different supports). In other words, if we replace the Wasserstein distance with the total variation distance in our definition of WBNB, then the quantity will always be $\frac{T}{2}$ for all $\kappa \in (0, 1)$. The KL-divergence also only concerns about the difference between densities thus fails to characterize this point. In this light, the Wasserstein distance is a “smoother” representation of the distance between two distributions than the total variation distance or the KL-divergence. Interestingly, this coincides with the intuitions in the literature of generative adversarial network (GAN) where Arjovsky et al. (2017) replace the KL-divergence with the Wasserstein distance in
training GANs. Simultaneously and independently, Balseiro et al. (2020) analyze the dual mirror descent algorithm under a similar setting as our results in this section. The paper focuses only on the uninformative setting, but not the known true distribution setting and the prior estimate setting in the previous sections. For the uninformative setting, their definition of non-stationarity is parallel to WBNB and thus can be viewed as a reduction from the more general WBDB. The key difference is that Balseiro et al. (2020) consider the total variation distance in defining WBNB, which inherits the definition of variation functional from (Besbes et al., 2015). In comparison, the Wasserstein distance is smoother in measuring the difference between distributions and thus it provides sharper regret upper bounds in both the setting with prior estimate in the last section and the uninformative setting in this section.

5.2. Algorithm and Regret Analysis

One pillar of designing Algorithm 1 and Algorithm 2 is the budget allocation plan $\gamma_t$’s (or $\hat{\gamma}_t$) prescribed by either the true distribution or the prior estimate. In the uninformative setting, the most straightforward (and probably the best) plan is to allocate the budget evenly over the entire horizon. Algorithm 3 – uninformative gradient descent algorithm (UGD) – implements the intuition by letting $\hat{\gamma}_t = \frac{c}{T}$ for all $t = 1,...,T$. Returning to the point mentioned earlier on the knowledge of the centric distribution $\bar{P}_T$, it does not matter we know it or not; because as long as all the prior estimate distributions $\hat{P}_t$ are the same over time, we always have the same budget allocation plan. One step further, when all the $P_t$’s are the same, which means the variation budget $W_T = 0$, Algorithm 3 and its analysis collapse into several recent studies on the gradient-based online algorithm under a stationary environment (Lu et al., 2020; Li et al., 2020).

**Theorem 5** Under Assumption 1, if we consider the set $\Xi_{WP}(W_T) = \{P : W_T(P) \leq W_T, P = (P_1,...,P_T)\}$, then the regret of Algorithm 3 has the following upper bound

$$\text{Reg}_T(\pi_{UGD}) \leq O(\max\{\sqrt{T},W_T\})$$

where $\pi_{UGD}$ stands for the policy specified by Algorithm 3.

Theorem 5 states the upper bound of Algorithm 3, which matches the regret lower bound in Theorem 4. Remarkably, the factors on $T$ and $W_T$ are additive in the regret upper bound of Algorithm 1. In comparison, the factor on $T$ and the variation budget $V_T$ are usually multiplicative in the regret upper bounds in the line of works that adopts the temporal change variation budget as nonstationary measure (Besbes et al., 2014, 2015; Cheung et al., 2019). The price of such an advantage for WBNB is that the WBNB is a more restrictive notion than the variation budget; recall that in (12) and (13), the temporal change variational budget is $O(1)$, but the WBNB is
Algorithm 3 Uninformative Gradient Descent Algorithm (UGD)

1: Initialize the initial dual price $p_1 = 0$ and initial constraint capacity $c_1 = c$.
2: for $t = 1, ..., T$ do
3: Observe $\theta_t$ and solve
4: $\tilde{x}_t = \arg\max_{x \in X} \{ f(x; \theta_t) - p_t^\top g(x; \theta_t) \}$
5: where $g(x, \theta_t) = (g_1(x, \theta_t), ..., g_m(x, \theta_t))^\top$
6: Set $x_t = \begin{cases} \tilde{x}_t, & \text{if } c_t \text{ permits a consumption of } g(\tilde{x}_t; \theta_t) \\ 0, & \text{otherwise} \end{cases}$
7: Update the dual price
8: $p_{t+1} = \left( p_t + \frac{1}{\sqrt{T}} \left( g(\tilde{x}_t; \theta_t) - \frac{c}{T} \right) \right) \lor 0$ \hspace{1cm} (14)
9: where the element-wise maximum operator $u \lor v = \max\{v, u\}$
10: Update the remaining capacity
11: $c_{t+1} = c_t - g(x_t; \theta_t)$
12: end for
13: Output: $x = (x_1, ..., x_T)$

$O(T)$. Again, as the setting with prior estimate, the knowledge of the quantity $W_T$ does not affect the algorithm design. By putting together Theorem 4 and Theorem 5, we assert that the knowledge of $W_T$ is not useful in further improving the algorithm performance.

6. Numerical Experiments

6.1. Experiment I: Online Linear Programming

We first present a simulation experiment in the setting of online linear programming where both the reward and cost function are linear, i.e., $f_t(x) = r_t x$ and $g_{it}(x) = a_{it} x$ for $i = 1, ..., m$ and $t = 1, ..., T$. Table 1 summarizes how $r_t$’s and $a_{it}$’s are generated where we denote the prior estimate as $\hat{r}_t$ and $\hat{a}_{it}$. Specifically, we consider a setting with both non-stationarity and prior estimate. Both the true distribution and the prior estimate distribution of $a_{ij}$ follow Uniform$[0, 1, 1.1]$ throughout the horizon. The true distribution of $r_t$ follows Uniform$[0, 1]$ for the first half of the time and Uniform$[0, \alpha]$ for the second half, while the prior estimate of $\hat{r}_t$ follows Uniform$[0, 1 + \beta]$ for the first half of the time and Uniform$[0, \alpha + \beta]$ for the second half.
From the setup of the experiment, we know that the parameter $\alpha$ reflects the intensity of the non-stationarity of the problem and the parameter $\beta$ represents the error of the prior estimate. In particular, the WBNB is linear in $\alpha$ under the setup of Section 5 and the WBDB is linear in $\beta$ under the setup of Section 4. Thus, we implement Algorithm 2 (IGDP) and Algorithm 3 (UGD) with different choices of $\alpha$ and $\beta$ to study the performances under both settings in our paper. Besides, we implement the classic fixed bid price control heuristics (FBP) proposed in Talluri and Van Ryzin (1998). The FBP method uses $\hat{p}^*$ computed from the prior estimate (9) as the bid price; it then accepts the order (setting $x_t = 1$) when $r_t \geq a_t^\top \hat{p}^*$ and there is enough resource, and otherwise it rejects the order (setting $x_t = 0$). Table 2 and Figure 1 report the performances of the three algorithms where we set the horizon $T = 1000$, number of constraints $m = 10$, and initial resource capacity $c_i = 200$ for each $i = 1, \ldots, m$.

![Figure 1](image1.png)

**Figure 1** The percentage of the total reward collected by IGDP, UGD and FBP over the upper bound for different $\alpha$ and different $\beta$.

We first discuss the performance of three algorithms for different $\alpha$ when $\beta = 0$. It corresponds to the setting of Section 5 where the value of $\alpha$ indicates the intensity of the non-stationarity for the
underlying distribution \( P_t \)'s. The computation results of the three algorithms are plotted in Figure 1(a). The performance of UGD deteriorates as \( \alpha \) increases, and thus it validates the role of WBNB in characterizing the algorithm performance. IGDP and FBP provide unsurprisingly better and more stable performance because both of them utilize the prior estimate (which is exactly the true distribution when \( \beta = 0 \)). This comparison highlights the effectiveness of WBNB in characterizing the learnability of a non-stationary environment when there is no prior knowledge available (as for UGD), and also it underscores the usefulness of prior knowledge. Alternatively, Figure 1(b) presents a different setting where \( \alpha \) is fixed but \( \beta \) is varying. In the plot, the significant better performance of Algorithm UGD and IGDP over FBP shows the effectiveness of the gradient-based update, and the gap between the curve of UGD and that of IGDP reinforces the importance of utilizing the prior knowledge. Specifically, the performance of FBP drops quickly as the non-stationarity intensity \( \alpha \) increases, while both gradient-based algorithms remain stable. Based on these experiments, we make the following remarks: First, FBP can handle the non-stationarity to some extent (as in Figure 1(a)), however, its performance highly depends on the accuracy of the prior estimates. If the deviation of the prior estimates from the true distributions is large, the performance of FBP could become very poor. Second, the performance of UGD is robust to the deviation of prior estimate since it does not utilize any prior estimate. The downside is that UGD is more sensitive to the intensity of non-stationarity compared to the other two algorithms. Third, the performance of IGDP is relatively more robust to both the non-stationarity and the deviation of the prior estimates. Specifically, IGDP obtains parameters \( \gamma_t \) for each \( t \) to handle the non-stationarity and also utilizes the gradient descent updates to hedge against the deviation. In this experiment, IGDP combines both the advantages of FBP and UGD.

### 6.2. Experiment II: Network Revenue Management with Resolving Heuristics

Our second numerical experiment is adapted from the network revenue management experiment in Jasin (2015). We consider a hub-and-spoke model with 8 cities, 14 connecting flights, and 41 itineraries. The detailed itinerary structure and the normalized capacities in our experiment are referred to Table 1 and Table 2 in Jasin (2015), respectively. To model the non-stationarity, we add random noises to the arrival probabilities of the itineraries. Specifically, we denote \( P_0 \in \mathbb{R}^{41} \) as the arrival probabilities in Jasin (2015) and we set \( \alpha \in [0, 1] \) as a parameter, which will be specified later. For each \( t \), the arrival probabilities of our experiment would be set as

\[
P_t = (P_0 + \alpha \cdot \text{Unif}(41, 1)) / \text{sum}(P_0 + \alpha \cdot \text{Unif}(41, 1))
\]

where \( \text{Unif}(41, 1) \) denotes a 41-dimensional vector of i.i.d. uniformly distributed random variables over \( [0, 1] \) and \( \text{sum}(\cdot) \) denotes the summation of all the components of a vector. For each time \( t \),
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
& \( \alpha = 1 \) & \( \alpha = 1.5 \) & \( \alpha = 2 \) & \( \alpha = 2.5 \) & \( \alpha = 3 \) \\
\hline
Upper Bound & 282.5433 & 363.7044 & 459.7807 & 563.3545 & 670.5960 \\
\hline
\( \beta = 0 \) & IGDP & 270.2411 (96\%) & 349.1769 (96\%) & 441.6677 (96\%) & 543.3373 (96\%) & 645.6582 (96\%) \\
& UGD & 270.3621 (96\%) & 337.3192 (93\%) & 403.7044 (88\%) & 469.7643 (83\%) & 535.0654 (80\%) \\
& FBP & 270.1211 (96\%) & 347.4997 (96\%) & 439.7016 (96\%) & 539.9865 (96\%) & 642.3940 (96\%) \\
\hline
\( \beta = 0.5 \) & IGDP & 270.1595 (96\%) & 347.9148 (96\%) & 439.6166 (96\%) & 539.8179 (96\%) & 643.6777 (96\%) \\
& UGD & 270.3568 (96\%) & 338.6916 (93\%) & 405.7927 (88\%) & 473.6640 (84\%) & 540.0894 (81\%) \\
& FBP & 64.7526 (23\%) & 174.2642 (48\%) & 314.7806 (68\%) & 446.7646 (79\%) & 582.7744 (87\%) \\
\hline
\( \beta = 1 \) & IGDP & 269.8058 (95\%) & 347.1246 (95\%) & 437.6279 (95\%) & 535.3521 (95\%) & 638.8322 (95\%) \\
& UGD & 269.6893 (95\%) & 339.1676 (93\%) & 408.3862 (89\%) & 477.2329 (84\%) & 544.9401 (81\%) \\
& FBP & 4.8549 (2\%) & 53.7881 (15\%) & 188.3271 (41\%) & 340.850 (61\%) & 486.4006 (73\%) \\
\hline
\( \beta = 2 \) & IGDP & 265.1512 (94\%) & 343.7802 (95\%) & 432.2275 (94\%) & 527.4351 (94\%) & 627.7440 (94\%) \\
& UGD & 265.4187 (94\%) & 337.4751 (93\%) & 410.3510 (89\%) & 482.8652 (86\%) & 554.0038 (83\%) \\
& FBP & 0.0171 (0\%) & 1.6210 (0.5\%) & 22.5201 (5\%) & 104.5026 (19\%) & 243.1575 (36\%) \\
\hline
\end{tabular}
\caption{Computation results for Experiment I: The results are reported based on 500 simulation trials. For each entry \( b(c) \) of the table, \( b \) denotes the expected reward collected by the algorithm and \( c \) denotes the percentage of the expected reward of the algorithm over the upper bound.}
\end{table}

\begin{itemize}
\item An independent random noise vector will be generated. In our experiment, the probability vector \( \mathcal{P}_t \) is only generated once at the very beginning, and it will remain the same in all the simulation trials. Intuitively, the value of \( \alpha \) represents the magnitude of adjustment compared to the original probability \( \mathcal{P}_0 \), and thus it reflects the intensity of non-stationarity of the underlying distributions.
\item Similarly, we introduce another parameter \( \beta \) to reflect the deviation of the prior estimates \( \hat{\mathcal{P}}_t \) from the true distribution \( \mathcal{P}_t \). Let
\[ \hat{\mathcal{P}}_t = (\mathcal{P}_t + \beta \cdot \text{Unif}(41, 1))/\text{sum}(\mathcal{P}_t + \beta \cdot \text{Unif}(41, 1)). \]
\item As in the last experiment, we report the algorithm performance for different \( (\alpha, \beta) \) in Table 3.
\item In the following, we present a re-solving version of Algorithm 2 (IGDP) which integrates periodically a re-solving procedure based on the remaining resource and the prior estimate to the dual
\end{itemize}
updates in Algorithm 2.

**Re-solving Heuristic for Algorithm 2:** The re-solving heuristic of Algorithm 2 re-solves the upper bound function $\hat{L}(\cdot)$ based on the current time period $t$ and the remaining budget $c_i$ to obtain an updated $p_t$ on a regular basis. For each time $t \in T \subset \{1, ..., T\}$, we solve the following problem

$$\hat{p}_t^* = \arg\min_{p \geq 0} c_i^T p + \sum_{j=t}^{T} \hat{P}_j h(p, \theta)$$

and use its optimal solution as the dual price for the $t$-th time period. Also, we update $\hat{\gamma}_t$'s accordingly. Here the set $T$ contains the time periods that we will re-solve the problem. For each time $t \in T$, the preceding dual price $p_{t-1}$ will be discarded and after time period $t$, we will continue to implement the gradient-based update as in Algorithm 2 until the next re-solving time. The re-solving method has gain great popularity in both theory and application (Gallego et al., 2019), and in this experiment, we investigate how the technique can be used to further boost the performance of Algorithm 2.

Table 3 reports the performance of Algorithm 2 and its re-solving heuristics with frequency $k = 1, 50, 100, 200$. First, we note that the original version of Algorithm 2 exhibits stably well performance across different combinations of $\alpha$ (the non-stationary intensity) and $\beta$ (the estimation error). Second, the re-solving heuristic further boosts the performance of Algorithm 2. We observe an improved performance even when the frequency $k = 200$ (which means only re-solving for 5 times throughout the horizon), but the performance improvement becomes marginal as we further increase the re-solving frequency. From a computational perspective, the re-solving heuristic for Algorithm 2 provides a good trade-off between computational efficiency and algorithm performance, especially for large-scale system. It naturally blends the computational advantage of Algorithm 2 and the algorithmic adaptivity of the re-solving technique. In recent years, the literature has developed a good understanding of both the advantages and drawbacks of the re-solving technique in a stationary environment (See (Cooper, 2002; Reiman and Wang, 2008; Jasin and Kumar, 2012; Jasin, 2015; Bumpensanti and Wang, 2020) among others). In parallel, the experiment here raises an interesting but challenging future direction of understanding the re-solving technique in a non-stationary environment.

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Appendix. Proofs of Lemmas, Propositions and Theorems

A1. Proof of Lemma 1

Proof: Given the realized parameters \( \mathcal{H} = (\theta_1, \theta_2, \ldots, \theta_T) \), we can denote the offline optimum of (PCP) as a function of \( \mathcal{H} \), namely, \( \{x^*_t(\mathcal{H})\}_{t=1}^T \). Let

\[
\bar{x}_t(\theta) = \mathbb{E}[x^*_t(\mathcal{H})|\theta_t = \theta]
\]

where the conditional expectation is taken with respect to \( P_j \) for \( j \neq t \). We show that \( \bar{x}_t(\theta) \) is a feasible solution to (4). Specifically, note that for each \( i = 1, \ldots, m \),

\[
c_i \geq \mathbb{E} \left[ \sum_{t=1}^T g_i(x^*_t(\mathcal{H}); \theta_t) \right] = \sum_{t=1}^T \mathbb{E}_{\theta_t \sim P_t} [\mathbb{E}[g_i(x^*_t(\mathcal{H}); \theta_t)|\theta_t = \theta]] \\
\geq \sum_{t=1}^T \int_{\theta \in \Theta} g_i(\bar{x}_t(\theta); \theta) dP_t(\theta) = \sum_{t=1}^T P_t g_i(x_t)
\]
where the first inequality comes from the feasibility of the optimal solution \( x^*_t(\mathcal{H}) \) and the second inequality follows from that the function \( g_i(\cdot; \theta_i) \) is a convex function for each \( i \) and \( \theta_i \in \Theta \). Thus, \( \{\tilde{x}_i(\theta)\} \) is a feasible solution to (4). Similarly, we can analyze the objective function

\[
\mathbb{E}[R^*_T] = \mathbb{E} \left[ \sum_{t=1}^{T} f(x^*_t(\mathcal{H}); \theta_t) \right] = \sum_{t=1}^{T} \mathbb{E}_{\theta_t \sim \mathcal{P}_t} [\mathbb{E}[f(x^*_t(\mathcal{H}); \theta_t)|\theta_t = \theta]]
\]

\[
\leq \sum_{t=1}^{T} \int_{\theta \in \Theta} f(\tilde{x}_t(\theta); \theta) d\mathcal{P}_t(\theta) \leq R^*_T
\]

where the first inequality follows from that the function \( f(\cdot; \theta) \) is a concave function for any \( \theta \in \Theta \) and the last inequality comes from the optimality of \( R^*_T \). Thus we complete the proof. □

A2. Proof of Lemma 3

Proof: We first prove that \( p^* \) is an optimal solution for \( L_t \). Note that for each \( t \), \( L_t(p) \) is a convex function over \( p \) and

\[
\nabla L_t(p^*) = \gamma_t + \mathcal{P}_t \nabla h(p^*; \theta) = \gamma_t - \mathcal{P}_t g(x^*(\theta); \theta)
\]

where \( x^*(\theta) = \arg\max_{x \in \mathcal{X}} \{f(x; \theta) - (p^*)^T \cdot g(x; \theta)\} \). With the definition of \( \gamma_t \) in (6), it follows immediately that

\[
\nabla L_t(p^*) = 0
\]

which implies that \( p^* \) is a minimizer of the function \( L(\cdot) \) for each \( t \). We then prove that \( L(p^*) = \sum_{t=1}^{T} L_t(p^*) \). Define the set of binding constraints \( \mathcal{I}_B = \{i : p^*_i > 0, i = 1, ..., m\} \). From the convexity of the function \( L(p) \) over \( p \), for each \( i \in \mathcal{I}_B \), it holds that

\[
0 = \nabla_i L(p^*) = c_i - \sum_{t=1}^{T} \mathcal{P}_t g(x^*(\theta); \theta) = c_i - \sum_{t=1}^{T} \gamma_{t,i}
\]

Thus, we have that

\[
c^T \cdot p^* = \sum_{t=1}^{T} \gamma_t^T \cdot p^*
\]

It follows immediately that \( L(p^*) = \sum_{t=1}^{T} L_t(p^*) \). □

A3. Discussions on the function \( h \)

The definition of the function \( h(p; \theta) \), described as follows for completeness, plays a critical role in deriving our results,

\[
h(p; \theta) = \max_{x \in \mathcal{X}} \left\{ f(x; \theta) - \sum_{i=1}^{n} p_i \cdot g_i(x; \theta) \right\}.
\]

(15)

Throughout the paper, we assume that the optimization problem in (15) can be solved efficiently so as to obtain both its optimal solution and optimal objective value. Here, we justify this assumption
with a discussion of the computational aspect of solving (15). When (PCP) refers to the online linear programming problem or price-based network revenue management (NRM) problem as described in Section 2.2, the optimization problem in (15) is reduced to a simple convex optimization problem and it can indeed be solved in polynomial time by existing methods. We now consider solving (15) when (PCP) represents the choice-based NRM problem.

In the choice-based NRM problem, there are \( n \) products and each product \( i \) is associated with a revenue \( r_i \). The decision variable \( x_t \in \mathcal{X} \subset \{0, 1\}^n \) refers to the assortment provided by the decision maker at time period \( t \) and given this assortment \( x_t \), the customer with the type \( \theta_t \) would choose one product \( i \) to purchase with a probability \( \eta_i(x_t; \theta_t) \), where \( \eta \) is specified by the choice model of the customer and is assumed to be known to the decision maker. Here, the function \( f() \) refers to the expected revenue of the assortment \( x_t \) and the function \( g_i() \) refers to the probability that a product \( i \) is purchased:

\[
f(x_t; \theta_t) = \sum_{j=1}^{n} r_j \cdot \eta_j(x_t; \theta_t) \quad \text{and} \quad g_i(x_t; \theta_t) = \eta_i(x_t; \theta_t) \quad \forall i
\]

Then, the optimization problem in (15) could be rewritten in the following equivalent formulation:

\[
\max_{x_t \in \mathcal{X} \subset \{0,1\}^n} \sum_{i=1}^{n} (r_i - p_i) \cdot \eta_i(x; \theta)
\]  

(16)

With the following choice models to specify the function \( \eta_i \)'s, the optimization problem in (16) could all be solved efficiently.

**Multinomial logit model (NML):** Talluri and Van Ryzin (2004) show that without further constraints, it is optimal to sort the products according to a decreasing order \( r_i - p_i \) and find an optimal revenue-order assortment among \( \{1\}, \{1,2\}, \ldots, \{1,2,\ldots,n\} \). Rusmevichientong et al. (2010) further propose a simple polynomial-time algorithm to compute the optimal assortment where there is a capacity constraint. Davis et al. (2013) consider the assortment problem with totally unimodular constraints and shows that this problem could be solved as an equivalent linear program.

**Nested logit model:** Li et al. (2015) consider the assortment optimization problem under a \( d \)-level nested logit model and proposes an algorithm to compute the optimal assortment in polynomial time. Gallego and Topaloglu (2014) consider the setting with cardinality constraint and shows that the optimal assortment could be obtained efficiently by solving a linear programs. Gallego and Topaloglu (2014) further consider the setting with space constraint and propose an algorithm with constant bound and good empirical performance.

**Markov chain based models:** Blanchet et al. (2016) propose the Markov chain based model and develops polynomial-time solution algorithms. Feldman and Topaloglu (2017) propose a linear
program to obtain the optimal assortment under this model.

**General choice model:** There are also papers considering the assortment problem without exploiting the specific structures of the choice model. Jagabathula (2014) studies a local search heuristic and shows its great empirical performance. Also, Berbeglia and Joret (2020) analyze the performance of the revenue-order assortment heuristic under the general discrete choice model, which could be applied to solve (16) approximately.

**A4. Proof of Lemma 4**

*Proof:* Note that the following two properties are satisfied by the update rule (7):

(i). If \( \|p_t\|_\infty \leq q \), then we must have \( \|p_{t+1}\|_\infty \leq q + 1 \) by noting that for each \( i \), the \( i \)-th component of \( p_t \), denoted as \( p_{t,i} \), is nonnegative and \( g_i(\cdot, \theta^t) \) is normalized within \([0, 1]\).

(ii). If there exists \( i \) such that \( p_{t,i} > q \), then we must have \( p_{t+1,i} < p_{t,i} \). Specifically, when \( p_{t,i} > q \), we must have that \( g_i(\tilde{x}_t; \theta_t) = 0 \), otherwise we would have that

\[
f(\tilde{x}_t; \theta_t) - p_t^\top g(\tilde{x}_t; \theta_t) \leq f(\tilde{x}_t; \theta_t) - p_{t,i} \cdot g_i(\tilde{x}_t; \theta_t) < 0
\]

which contradicts the definition of \( \tilde{x}_t \) in Algorithm 1 since we could always select \( x_t = 0 \) to obtain a zero objective value as per Assumption 1. Then from (7), it holds that \( p_{t+1,i} < p_{t,i} \).

Starting from \( p_1 = 0 \) and iteratively applying the above two property to control the increase of \( p_t \) from \( t = 1 \) to \( T \), we obtain that for the first time that one component of \( p_t \) exceeds the threshold \( q \), it is upper bounded by \( q + 1 \) and this component will continue to decrease until it falls below the threshold \( q \). Thus, it is obvious that we have \( \|p_t\|_\infty \leq q + 1 \) with probability 1 for each \( t \). \( \square \)

**A5. Proof of Theorem 1**

*Proof:* In Algorithm 1, the true action \( x_t \) taken by the decision maker differs from the virtual action \( \tilde{x}_t \) if and only if \( c_t \) cannot fully satisfy \( g(\tilde{x}_t; \theta_t) \). Thus, we have that

\[
f(\tilde{x}_t; \theta_t) - f(x_t; \theta_t) \leq f(\tilde{x}_t; \theta_t) \cdot \mathbb{I}\{\exists i: c_{t,i} < g_i(\tilde{x}_t; \theta_t)\}
\]

where \( c_{t,i} \) denotes the \( i \)-th component of \( c_t \) and \( \mathbb{I}\{\cdot\} \) denotes the indicator function. Moreover, we know

\[
\mathbb{I}\{\exists i: c_{t,i} < g_i(\tilde{x}_t; \theta_t)\} \leq \sum_{i=1}^m \mathbb{I}\left\{\sum_{j=1}^t g_i(\tilde{x}_j; \theta_j) > c_i\right\}.
\]

Recall that the maximum reward generated by consuming per unit of budget of each constraint is upper bounded by \( q \). We have

\[
f(\tilde{x}_t; \theta_t) \cdot \mathbb{I}\{\exists i: c_{t,i} < g_i(\tilde{x}_t; \theta_t)\} \leq q \sum_{i=1}^m g_i(\tilde{x}_t; \theta_t) \cdot \mathbb{I}\left\{\sum_{j=1}^t g_i(\tilde{x}_j; \theta_j) > c_i\right\}
\]
From the fact that $g(\cdot; \theta_t) \in [0, 1]^m$,
\[
\sum_{t=1}^{T} f(\tilde{x}_t; \theta_t) - \sum_{i=1}^{m} f(x_i; \theta_t) \leq q \cdot \sum_{t=1}^{T} \sum_{i=1}^{m} g_i(\tilde{x}_i; \theta_t) \cdot \mathbb{I}\left\{ \sum_{j=1}^{i} g_j(\tilde{x}_j; \theta_j) > c_i \right\} \\
\leq q \cdot \sum_{t=1}^{T} \left[ \sum_{i=1}^{m} g_i(\tilde{x}_i; \theta_t) - (c_i - 1) \right]^+ 
\]
which related the total collected reward by the true action $\{x_t\}_{t=1}^{T}$ and the virtual action $\{\tilde{x}_t\}_{t=1}^{T}$. Further from Proposition 1, we have that
\[
\text{Reg}_T(\pi) \leq \min_{p \geq 0} \sum_{t=1}^{T} L_t(p) - \mathbb{E}\left[ \sum_{t=1}^{T} f(x_t; \theta_t) \right] \leq \sum_{t=1}^{T} \min_{p \geq 0} L_t(p) - \mathbb{E}\left[ \sum_{t=1}^{T} f(\tilde{x}_t; \theta_t) \right] + q \cdot \mathbb{E}\left[ \sum_{i=1}^{m} \left( \sum_{t=1}^{T} g_i(\tilde{x}_i; \theta_t) - (c_i - 1) \right)^+ \right]
\]
We then bound the term $I$ and term $II$ separately to derive our regret bound.

**Bound I**: Note that for each $t$, the distribution of $p_t$ is independent from the distribution of $\theta_t$ for any $\tau \leq t$, then we have that
\[
\min_{p \geq 0} L_t(p) \leq \mathbb{E}_{p_t} \left[ L_t(p_t) \right] = \mathbb{E}_{p_t} \left[ \gamma_t^\top p_t + \mathcal{P}_t h(p_t; \theta_t) \right]
\]
where the expectation is taken with respect to the randomness of the dual price $p_t$. Thus, we have
\[
I \leq \sum_{t=1}^{T} \mathbb{E}_{p_t} \left[ \gamma_t^\top p_t + \mathcal{P}_t \{ h(p_t; \theta_t) - f(\tilde{x}_t; \theta_t) \} \right]
\]
From the definition of $\tilde{x}_t$, we get that $h(p_t; \theta_t) - f(\tilde{x}_t; \theta_t) = -p_t^\top \cdot g(\tilde{x}_t; \theta_t)$, which implies that
\[
I \leq \sum_{t=1}^{T} \mathbb{E}_{p_t} \left[ p_t^\top \cdot (\gamma_t - \mathcal{P}_t g(\tilde{x}_t; \theta_t)) \right]
\]
Note that from the update rule (7), we have that
\[
\|p_{t+1}\|_2^2 \leq \|p_t\|_2^2 + \frac{1}{T} \cdot \|g(\tilde{x}_t; \theta_t) - \gamma_t\|_2^2 - \frac{2}{\sqrt{T}} \cdot p_t^\top \cdot (\gamma_t - g(\tilde{x}_t; \theta_t))
\]
which implies that
\[
\mathbb{E}_{p_t} \left[ p_t^\top \cdot (\gamma_t - \mathcal{P}_t g(\tilde{x}_t; \theta_t)) \right] \leq \frac{\sqrt{T}}{2} \cdot \left( \mathbb{E}[\|p_t\|_2^2] - \mathbb{E}[\|p_{t+1}\|_2^2] \right) + \frac{m}{2\sqrt{T}}
\]
Thus, it holds that
\[
I \leq \frac{m\sqrt{T}}{2} \tag{17}
\]
Bound II: Note that from the update rule (7), we have that
\[ \sqrt{T} \cdot p_{t+1} \geq \sqrt{T} \cdot p_t + g(\hat{x}_t; \theta_t) - \gamma_t \]
which implies that
\[ \sum_{t=1}^{T} g(\hat{x}_t; \theta_t) - c \leq \sum_{t=1}^{T} g(\hat{x}_t; \theta_t) - \sum_{t=1}^{T} \gamma_t \leq \sqrt{T} \cdot p_{T+1} \]
Thus, it holds that
\[ \text{II} = q \cdot E \left[ \sum_{i=1}^{m} \left( \sum_{t=1}^{T} g_i(\hat{x}_t; \theta_t) - (c_i - 1) \right) \right] \leq mq(q+1) \cdot \sqrt{T} + qm \tag{18} \]
We obtain the \( O(\sqrt{T}) \) regret bound immediately by combining (17) and (18).

A6. Proof of Theorem 2

Proof: It follows directly from Lemma 1 in Arlotto and Gurvich (2019) that for any policy \( \pi \), we have \( \text{Reg}_T(\pi) \geq \Omega(\sqrt{T}) \). Thus, it is enough to consider the \( \Omega(W_T) \) part in the lower bound. We consider the following estimated problem, where the true coefficients in (PCP) are replaced by the estimates:

\[
\begin{align*}
\text{max} & \quad x_1 + \ldots + x_c + x_{c+1} + \ldots + x_T \\
\text{s.t.} & \quad x_1 + \ldots + x_c + x_{c+1} + \ldots + x_T \leq c \\
& \quad 0 \leq x_t \leq 1 \quad \text{for } t = 1, \ldots, T.
\end{align*}
\]

where \( c = \frac{T}{2} \) and the prior estimate \( \hat{P}_t \) is simply a one-point distribution for each \( t \). Now we consider the following two possible true problems, the distributions of which are all one-point distributions and belong to the set \( \Xi_P \) with variation budget \( W_T \):

\[
\begin{align*}
\text{max} & \quad x_1 + \ldots + x_c + (1 + \frac{W_T}{T})x_{c+1} + \ldots + (1 + \frac{W_T}{T})x_T \\
\text{s.t.} & \quad x_1 + \ldots + x_c + x_{c+1} + \ldots + x_T \leq c \\
& \quad 0 \leq x_t \leq 1 \quad \text{for } t = 1, \ldots, T. \\
\text{max} & \quad x_1 + \ldots + x_c + (1 - \frac{W_T}{T})x_{c+1} + \ldots + (1 - \frac{W_T}{T})x_T \\
\text{s.t.} & \quad x_1 + \ldots + x_c + x_{c+1} + \ldots + x_T \leq c \\
& \quad 0 \leq x_t \leq 1 \quad \text{for } t = 1, \ldots, T.
\end{align*}
\]

where \( c = \frac{T}{2} \). Denote \( x_1^t(\pi) \) as the decision of any policy \( \pi \) at period \( t \) for scenario (20) and denote \( x_1^2(\pi) \) as the decision of policy \( \pi \) at period \( t \) for scenario (21). Further define \( T_1(\pi) \) (resp. \( T_2(\pi) \))
as the expected capacity consumption of policy \( \pi \) on scenario (20) (resp. scenario (21)) during the first \( \frac{T}{2} \) time periods:

\[
T_1(\pi) = \mathbb{E} \left[ \sum_{i=1}^{T} x_{1i}(\pi) \right] \quad \text{and} \quad T_2(\pi) = \mathbb{E} \left[ \sum_{i=1}^{T} x_{2i}(\pi) \right]
\]

Then, we have that

\[
R_{1T}(\pi) = T + W_T - \frac{W_T}{T} \cdot T_1(\pi) \quad \text{and} \quad R_{2T}(\pi) = T - W_T + \frac{W_T}{T} \cdot T_2(\pi)
\]

where \( R_{1T}(\pi) \) (resp. \( R_{2T}(\pi) \)) denotes the expected reward collected by policy \( \pi \) on scenario (20) (resp. scenario (21)). Thus, the regret of policy \( \pi \) on scenario (20) and (21) are \( W_T \cdot T_1(\pi) \) and \( W_T - \frac{W_T}{T} \cdot T_2(\pi) \) respectively. Further note that since the implementation of policy \( \pi \) at each time period should be only dependent on the historical information and the coefficients in the estimated problem (19), we must have \( T_1(\pi) = T_2(\pi) \). Thus, we have that

\[
\text{Reg}_{T}(\pi) \geq \max \left\{ \frac{W_T}{T} \cdot T_1(\pi), W_T - \frac{W_T}{T} \cdot T_1(\pi) \right\} \geq \frac{W_T}{2} = \Omega(W_T)
\]

which completes our proof. \( \square \)

### B7. Discussion on Bid Price Policy

At first glance of the setting in Section 4, it is tempting to consider the well-known bid price policy (Talluri and Van Ryzin (1998)), where the bid prices are computed based on the prior estimates. Specifically, the bid price policy will make the decision \( x_t \) at each period \( t \) based on the fixed \( \hat{p}^* \) throughout the procedure:

\[
x_t \in \arg\max_{x \in X} f(x, \theta_t) - (\hat{p}^*)^T \cdot g(x, \theta_t).
\]

When there is no deviation between the true distributions and prior estimates, i.e., \( \mathcal{P}_t = \hat{\mathcal{P}}_t \) for each \( t \), it is proved in Talluri and Van Ryzin (1998) that bid price policy is asymptotically optimal if each period is repeated sufficiently large times and the capacity is scaled up accordingly. One may expect that the bid price policy could still have good performance when the deviations between the true distributions and prior estimates are small enough. However, the following example shows that this is not the case.

Consider the following linear programs as the underlying problem (PCP) for the online stochastic optimization problem:

\[
\begin{align*}
\max & \quad x_1 + ... + x_{2t} + \frac{1}{2} x_{2t+1} + ... + \frac{1}{2} x_T \\
\text{s.t.} & \quad x_1 + ... + x_{2t} + x_{2t+1} + ... + x_T \leq c \\
& \quad 0 \leq x_t \leq 1 \quad \text{for} \quad t = 1, ..., T.
\end{align*}
\]

(22)
where \( c = \frac{T}{3} \) and without loss of generality, we assume \( c \) is an integer. Suppose that the prior estimates for the coefficients in the objective function is larger than the true coefficients by \( 2\epsilon \) for the first \( \frac{T}{3} \) time periods and by \( \epsilon \) for the last \( \frac{2T}{3} \) time periods. Then we obtain the following linear programs as the underlying problem (PCP) for the prior estimates.

\[
\begin{align*}
\max & \quad (1 + 2\epsilon)x_1 + \ldots + (1 + 2\epsilon)x_c + (1 + \epsilon)x_{c+1} + \ldots + (1 + \epsilon)x_{2c} + \left(\frac{1}{2} + \epsilon\right)x_{2c+1} + \ldots + \left(\frac{1}{2} + \epsilon\right)x_T \\
\text{s.t.} & \quad x_1 + \ldots + x_{2c} + x_{2c+1} + \ldots + x_T \leq c \\
& \quad 0 \leq x_t \leq 1 \quad \text{for } t = 1, \ldots, T.
\end{align*}
\] (23)

Obviously, the optimal dual variable for (23) could be any value in \((1 + \epsilon, 1 + 2\epsilon)\). However, if we apply the bid price policy to the true problem (22) where the bid price is set to be equal to the optimal dual variable for (23), the policy will set \( x_t = 0 \) for every \( t \) from 1 to \( \frac{2T}{3} \). Thus, the total reward collected by the bid price policy for (22) is at most \( \frac{T}{6} \), while the objective value for (22) is \( \frac{T}{3} \).

C8. Proof of Lemma 5

Proof: Due to symmetry, it is sufficient to show that for every \( p \) such that \( p \in \Omega_{\bar{p}} \), we have that

\[ L_{Q_2}(p) - L_{Q_1}(p) \leq \max\{1, \bar{p}\} \cdot W(Q_1, Q_2). \]

Denote \( Q_{1,2}^* \) as the optimal coupling of the distribution \( Q_1 \) and \( Q_2 \), i.e., the optimal solution to (8), and denote

\[ x^*(\theta) = \arg\max_{x \in X} \left\{ f(x; \theta) - \sum_{i=1}^{m} p_i \cdot g_i(x; \theta) \right\} \]

Then for each \( \theta_1 \in \Theta \), we define

\[ \hat{x}(\theta_1) = \int_{\theta_2 \in \Theta} x^*(\theta_2) \frac{dQ_{1,2}^*(\theta_1, \theta_2)}{dQ_1(\theta_1)} \]

where \( \frac{dQ_{1,2}^*(\theta_1, \theta_2)}{dQ_1(\theta_1)} \) is the Radon–Nikodym derivative of \( Q_{1,2}^* \) with respect \( Q_1 \) and it can be interpreted as the conditional distribution of \( \theta_2 \) given \( \theta_1 \). Note that from the definition of \( Q_{1,2}^* \), we have that

\[ \int_{\theta_2 \in \Theta} \frac{dQ_{1,2}^*(\theta_1, \theta_2)}{dQ_1(\theta_1)} = 1. \]

Thus, \( \hat{x}(\theta_1) \) is actually a convex combination of \( \{x^*(\theta_2)\}_{\theta_2 \in \Theta} \). Moreover, from the concavity of \( f(\cdot; \theta_1) \) and the convexity of \( g_i(\cdot; \theta_1) \) for each \( i \), we have that

\[ f(\hat{x}(\theta_1); \theta_1) \geq \int_{\theta_2 \in \Theta} f(x^*(\theta_2); \theta_1) \cdot \frac{dQ_{1,2}^*(\theta_1, \theta_2)}{dQ_1(\theta_1)} \]

and

\[ g_i(\hat{x}(\theta_1); \theta_1) \leq \int_{\theta_2 \in \Theta} g_i(x^*(\theta_2); \theta_1) \cdot \frac{dQ_{1,2}^*(\theta_1, \theta_2)}{dQ_1(\theta_1)} \]
Thus, we have that
\[
L_{\mathcal{Q}_1}(p) = \int_{\theta_1 \in \Theta} \max_{x \in X} \left\{ f(x; \theta_1) - \sum_{i=1}^{m} p_i \cdot g_i(x; \theta_1) \right\} d\mathcal{Q}_1(\theta_1)
\]
\[
\geq \int_{\theta_1 \in \Theta} \left\{ f(\hat{x}(\theta_1); \theta_1) - \sum_{i=1}^{m} p_i \cdot g_i(\hat{x}(\theta_1); \theta_1) \right\} d\mathcal{Q}_1(\theta_1)
\]
\[
\geq \int_{\theta_1 \in \Theta} \int_{\theta_2 \in \Theta} \left\{ f(x^*(\theta_2); \theta_1) - \sum_{i=1}^{m} p_i \cdot g_i(x^*(\theta_2); \theta_2) \right\} d\mathcal{Q}_{1,2}^*(\theta_1, \theta_2)
\]

Also, note that for any \( \theta_1, \theta_2 \in \Theta \), it holds that
\[
f(x^*(\theta_2); \theta_1) - \sum_{i=1}^{m} p_i \cdot g_i(x^*(\theta_2); \theta_1) \geq f(x^*(\theta_2); \theta_2) - \sum_{i=1}^{m} p_i \cdot g_i(x^*(\theta_2); \theta_2) - \max\{1, \bar{p}\} \cdot (m+1) \rho(\theta_1, \theta_2)
\]
which follows the definition of \( \rho(\theta_1, \theta_2) \) in (2). Thus, we get that
\[
L_{\mathcal{Q}_1}(p) \geq \int_{\theta_1 \in \Theta} \int_{\theta_2 \in \Theta} \left\{ f(x^*(\theta_2); \theta_2) - \sum_{i=1}^{m} p_i \cdot g_i(x^*(\theta_2); \theta_2) \right\} d\mathcal{Q}_{1,2}^*(\theta_1, \theta_2)
\]
\[
- \max\{1, \bar{p}\} \cdot (m+1) \int_{\theta_1 \in \Theta} \int_{\theta_2 \in \Theta} \rho(\theta_1, \theta_2) d\mathcal{Q}_{1,2}^*(\theta_1, \theta_2)
\]
\[
= \int_{\theta_2 \in \Theta} \left\{ f(x^*(\theta_2); \theta_2) - \sum_{i=1}^{m} p_i \cdot g_i(x^*(\theta_2); \theta_2) \right\} d\mathcal{Q}_2(\theta_2) - \max\{1, \bar{p}\} \cdot (m+1) \mathcal{W}(\mathcal{Q}_1, \mathcal{Q}_2)
\]
\[
= L_{\mathcal{Q}_2}(p) - \max\{1, \bar{p}\} \cdot (m+1) \mathcal{W}(\mathcal{Q}_1, \mathcal{Q}_2)
\]
where the first equality holds by noting that \( \int_{\theta_1 \in \Theta} d\mathcal{Q}_{1,2}^*(\theta_1, \theta_2) = d\mathcal{Q}_2(\theta_2) \). \( \square \)

C9. Proof of Lemma 6

Proof: Note that the following two property is satisfied by the update rule (11):

(i) If \( \|p_t\|_\infty \leq q \), then we must have \( \|p_{t+1}\|_\infty \leq q+1 \) by noting that for each \( i \), the \( i \)-th component of \( p_t \), denoted as \( p_{t,i} \), is nonnegative and \( g_i(\cdot, \theta_t) \) is normalized within \([0, 1]\).

(ii) If there exists \( i \) such that \( p_{t,i} > q \), then we must have \( p_{t+1,i} < p_{t,i} \). Specifically, when \( p_{t,i} > q \), we must have that \( g_i(\bar{x}_t; \theta_t) = 0 \), otherwise we would have that
\[
f(\bar{x}_t; \theta_t) - p_t^T \cdot g(\bar{x}_t; \theta_t) \leq f(\bar{x}_t; \theta_t) - p_{t,i} \cdot g_i(\bar{x}_t; \theta_t) < 0
\]
which contradicts the definition of \( \bar{x}_t \) in Algorithm 2 since we could always select \( \bar{x}_t = 0 \) to obtain 0 in the objective value. Then from the non-negativity of \( \hat{c}_t \), it holds that \( p_{t+1,i} < p_{t,i} \) in (11).

Starting from \( p_1 = 0 \) and iteratively applying the above two property to control the increase of \( p_t \) from \( t = 1 \) to \( T \), we obtain that for the first time that one component of \( p_t \) exceeds the threshold \( q \), it is upper bounded by \( q+1 \) and this component will continue to decrease until it falls below the threshold \( q \). Thus, it is obvious that we have \( \|p_t\|_\infty \leq q+1 \) with probability 1 for each \( t \). \( \square \)
C10. Proof of Theorem 3

Similar to case of known distribution, we define the following function \( \hat{L}_t(\cdot) \), based on the prior estimate \( \hat{P}_t \).

\[
\hat{L}_t(p) := \gamma^T_t p + \hat{P}_t h(p; \theta).
\]  

(24)

and then similar to Lemma 3, we have the following relation between \( \hat{L}(\cdot) \) and \( \hat{L}_t(\cdot) \).

Lemma 7 For each \( t = 1, \ldots, T \), it holds that

\[
\hat{p}^* \in \arg\min_{p \geq 0} \hat{L}_t(p)
\]

(25)

where \( \hat{p}^* \) is defined in (9) as the minimizer of the function \( \hat{L}(\cdot) \). Moreover, it holds that

\[
\hat{L}(\hat{p}^*) = \sum_{t=1}^T \hat{L}_t(\hat{p}^*).
\]

(26)

Proof: We first prove (25). Note that for each \( t \), \( \hat{L}_t(p) \) is a convex function over \( p \) and

\[
\nabla \hat{L}_t(\hat{p}^*) = \gamma_t + \hat{P}_t \nabla h(\hat{p}^*; \theta) = \gamma_t - \hat{P}_t g(x^*(\theta); \theta)
\]

where \( x^*(\theta) = \arg\max_{x \in X} \{ f(x; \theta) - (\hat{p}^*)^T \cdot g(x; \theta) \} \). With the definition of \( \gamma_t \) in (10), it follows immediately that

\[
\nabla \hat{L}_t(\hat{p}^*) = 0
\]

which implies that \( \hat{p}^* \) is a minimizer of the function \( \hat{L}(\cdot) \) for each \( t \). We then prove (26). Define the set of binding constraints \( I_B = \{ i : \hat{p}_{t,i} > 0, i = 1, \ldots, m \} \). From the convexity of the function \( \hat{L}(p) \) over \( p \), for each \( i \in I_B \), it holds that

\[
0 = \nabla_i \hat{L}(\hat{p}^*) = c_i - \sum_{t=1}^T \hat{P}_t g(x^*(\theta); \theta) = c_i - \sum_{t=1}^T \gamma_{t,i}
\]

Thus, we have that

\[
c^T \cdot \hat{p}^* = \sum_{t=1}^T \hat{c}_{t,i}^T \cdot \hat{p}^*
\]

It follows immediately that \( \hat{L}(\hat{p}^*) = \sum_{t=1}^T \hat{L}_t(\hat{p}^*) \). \( \Box \)

Now we proof Theorem 3 and the idea of proof is similar to Theorem 1.

Proof: In Algorithm 2, the true action \( x_t \) taken by the decision maker differs from the virtual action \( \bar{x}_t \) if and only if \( c_t \) cannot fully satisfy \( g(\bar{x}_t; \theta_t) \). Thus, we have that

\[
f(\bar{x}_t; \theta_t) - f(x_t; \theta_t) \leq f(\bar{x}_t; \theta_t) \cdot \mathbb{I}\{\exists \bar{i} : c_{t,i} < g_{i}(\bar{x}_t; \theta_t)\}
\]
where $c_{t,i}$ denotes the $i$-th component of $\mathbf{c}_t$ and $\mathbb{I}\{\cdot\}$ denotes the indicator function. Moreover, we know

$$\mathbb{I}\{\exists i : c_{t,i} < g_i(\tilde{x}_t; \theta_t)\} \leq \sum_{i=1}^{m} \mathbb{I}\left\{ \sum_{j=1}^{t} g_j(\tilde{x}_j; \theta_j) > c_i \right\}.$$  

Recall that the maximum reward generated by consuming per unit of budget of each constraint is upper bounded by $q$. We have

$$f(\tilde{x}_t; \theta_t) \cdot \mathbb{I}\{\exists i : c_{t,i} < g_i(\tilde{x}_t; \theta_t)\} \leq q \sum_{i=1}^{m} g_i(\tilde{x}_t; \theta_t) \cdot \mathbb{I}\left\{ \sum_{j=1}^{t} g_j(\tilde{x}_j; \theta_j) > c_i \right\}$$

From the fact that $g(\cdot; \theta_t) \in [0,1]^{m}$,

$$\sum_{t=1}^{T} f(\tilde{x}_t; \theta_t) - \sum_{t=1}^{T} f(x_t; \theta_t) \leq q \sum_{i=1}^{m} \sum_{t=1}^{T} g_i(\tilde{x}_t; \theta_t) \cdot \mathbb{I}\left\{ \sum_{j=1}^{t} g_j(\tilde{x}_j; \theta_j) > c_i \right\}$$

$$\leq q \sum_{i=1}^{m} \sum_{t=1}^{T} g_i(\tilde{x}_t; \theta_t) - (c_i - 1)^+$$

which related the total collected reward by the true action $\{x_t\}_{t=1}^{T}$ and the virtual action $\{\tilde{x}_t\}_{t=1}^{T}$.

Further from Lemma 2, we have that

$$\text{Reg}_T(\pi) \leq \min_{p \geq 0} L(p) - \mathbb{E}\left[ \sum_{t=1}^{T} f(x_t; \theta_t) \right] \leq \min_{p \geq 0} L(p) - \mathbb{E}\left[ \sum_{t=1}^{T} f(\tilde{x}_t; \theta_t) \right]$$

$$+ q \cdot \mathbb{E}\left[ \sum_{i=1}^{m} \sum_{t=1}^{T} g_i(\tilde{x}_t; \theta_t) - (c_i - 1)^+ \right]$$

We then bound the term I and term II separately to derive our regret bound.

**Bound I:** Note that $\|\hat{p}^*\|_{\infty} \leq q$, it follows directly from Lemma 5 and Lemma 7 that

$$\min_{p \geq 0} L(p) \leq L(\hat{p}^*) \leq \hat{L}(\hat{p}^*) + \max\{q,1\} \cdot (m + 1) \cdot W_T = \sum_{t=1}^{T} \hat{L}_t(\hat{p}^*) + \max\{q,1\} \cdot (m + 1) \cdot W_T$$

Note that from Lemma 6, we have that for each $t$, $\|p_t\|_{\infty} \leq (q + 1)$ with probability 1. Further note that for each $t$, the distribution of $p_t$ is independent from the distribution of $\theta_t$, then from Lemma 5 and Lemma 7, we have that

$$\hat{L}_t(\hat{p}^*) = \min_{p \geq 0} \hat{L}_t(p) \leq \mathbb{E}_{p_t} \left[ \hat{L}_t(p_t) \right] \leq \mathbb{E}_{p_t} \left[ [\hat{\gamma}^T_t p_t + \mathcal{P}_t h(p_t; \theta_t)] + (q + 1)(m + 1) \cdot W(\mathcal{P}_t, \hat{P}) \right]$$

where the expectation is taken with respect to the randomness of the dual price $p_t$. Thus, we have

$$I \leq \sum_{t=1}^{T} \mathbb{E}_{p_t} \left[ [\hat{\gamma}^T_t p_t + \mathcal{P}_t \{h(p_t; \theta_t) - f(\tilde{x}_t; \theta_t)\}] + 2(q + 1)(m + 1) \cdot W_T. $$
From the definition of $\tilde{x}_t$, we get that $h(p_t; \theta_t) - f(\tilde{x}_t; \theta_t) = -p_t^T \cdot g(\tilde{x}_t; \theta_t)$, which implies that

$$I \leq \sum_{t=1}^{T} \mathbb{E}_{p_t} [p_t^T \cdot (\hat{\gamma}_t - \mathcal{P}_t g(\tilde{x}_t; \theta_t))] + 2(q+1)(m+1) \cdot W_T$$

Note that from the update rule (11), we have that

$$\|p_{t+1}\|_2^2 \leq \|p_t\|_2^2 + \frac{1}{T} \|g(\tilde{x}_t; \theta_t) - \hat{\gamma}_t\|_2^2 - \frac{2}{\sqrt{T}} \cdot p_t^T \cdot (\hat{\gamma}_t - g(\tilde{x}_t; \theta_t))$$

which implies that

$$\mathbb{E}_{p_t} [p_t^T \cdot (\hat{\gamma}_t - \mathcal{P}_t g(\tilde{x}_t; \theta_t))] \leq \frac{\sqrt{T}}{2} \cdot \left( \mathbb{E}[\|p_t\|_2^2] - \mathbb{E}[\|p_{t+1}\|_2^2] \right) + \frac{m}{2\sqrt{T}}$$

Thus, it holds that

$$I \leq \frac{m\sqrt{T}}{2} + 2(q+1)(m+1) \cdot W_T \tag{27}$$

**Bound II**: Note that from the update rule (11), we have that

$$\sqrt{T} \cdot p_{t+1} \geq \sqrt{T} \cdot p_t + g(\tilde{x}_t; \theta_t) - \hat{\gamma}_t$$

which implies that

$$\sum_{t=1}^{T} g(\tilde{x}_t; \theta_t) - c \leq \sum_{t=1}^{T} g(\tilde{x}_t; \theta_t) - \sum_{t=1}^{T} \hat{\gamma}_t \leq \sqrt{T} \cdot p_{T+1}$$

Thus, it holds that

$$II = q \cdot \mathbb{E} \left[ \sum_{i=1}^{m} \left( \sum_{t=1}^{T} g_i(\tilde{x}_t; \theta_t) - (c_i - 1) \right) \right]^+ \leq mq(q+1) \cdot \sqrt{T} + qm \tag{28}$$

We obtain the $O(\max\{\sqrt{T}, W_T\})$ regret bound immediately by combining (27) and (28). □

**C11. Proof of Proposition 2**

*Proof*: We consider the implementation of any online policy $\pi$ on the two scenarios (12) and (13) for $\kappa = 1$, which is replicated as follows for completeness:

\[
\begin{align*}
\max & \quad x_1 + \ldots + x_c + 2x_{c+1} + \ldots + 2x_T \\
\text{s.t.} & \quad x_1 + \ldots + x_c + x_{c+1} + \ldots + x_T \leq c \\
& \qquad 0 \leq x_t \leq 1 \quad \text{for} \ t = 1, \ldots, T. \\
\end{align*}
\]

\[
\begin{align*}
\max & \quad x_1 + \ldots + x_c \\
\text{s.t.} & \quad x_1 + \ldots + x_c + x_{c+1} + \ldots + x_T \leq c \\
& \qquad 0 \leq x_t \leq 1 \quad \text{for} \ t = 1, \ldots, T. \\
\end{align*}
\]
where $c = \frac{T}{2}$. Denote $x_i^1(\pi)$ as the decision of policy $\pi$ at period $t$ for scenario (29) and denote $x_i^2(\pi)$ as the decision of policy $\pi$ at period $t$ for scenario (30). Further define $T_1(\pi)$ (resp. $T_2(\pi)$) as the expected capacity consumption of policy $\pi$ on scenario (29) (resp. scenario (30)) during the first $\frac{T}{2}$ time periods:

$$T_1(\pi) = \mathbb{E}\left[\sum_{t=1}^{T_2} x_i^1(\pi)\right] \quad \text{and} \quad T_2(\pi) = \mathbb{E}\left[\sum_{t=1}^{T_2} x_i^2(\pi)\right]$$

Then, we have that

$$R_1^T(\pi) = 2T - T_1(\pi) \quad \text{and} \quad R_2^T(\pi) = T_2(\pi)$$

where $R_1^T(\pi)$ (resp. $R_2^T(\pi)$) denotes the expected reward collected by policy $\pi$ on scenario (29) (resp. scenario (30)). Thus, the regret of policy $\pi$ on scenario (29) and (30) are $T_1(\pi)$ and $T - T_2(\pi)$ respectively. Further note that since the implementation of policy $\pi$ at each time period should be independent of the future information, we must have $T_1(\pi) = T_2(\pi)$. Thus, we have that

$$\text{Reg}_T(\pi) \geq \max\{T_1(\pi), T - T_1(\pi)\} \geq \frac{T}{2} = \Omega(T)$$

which completes our proof. □

**C12. Proof of Theorem 4**

**Proof:** It follows directly from Lemma 1 in Arlotto and Gurvich (2019) that for any policy $\pi$, we have $\text{Reg}_T(\pi) \geq \Omega(\sqrt{T})$. Thus, it is enough to consider the $\Omega(W_T)$ part in the lower bound. For any online policy $\pi$, we consider implementing $\pi$ on the following two scenarios, both with a WBNB $W_T$:

**Scenario 1**

$$\max x_1 + ... + x_c + (1 + \frac{W_T}{T})x_{c+1} + ... + (1 + \frac{W_T}{T})x_T$$

s.t. $x_1 + ... + x_c + x_{c+1} + ... + x_T \leq c$

$0 \leq x_t \leq 1 \text{ for } t = 1, ..., T.$

**Scenario 2**

$$\max x_1 + ... + x_c + (1 - \frac{W_T}{T})x_{c+1} + ... + (1 - \frac{W_T}{T})x_T$$

s.t. $x_1 + ... + x_c + x_{c+1} + ... + x_T \leq c$

$0 \leq x_t \leq 1 \text{ for } t = 1, ..., T.$

where $c = \frac{T}{2}$. Denote $x_i^1(\pi)$ as the decision of policy $\pi$ at period $t$ for scenario (31) and denote $x_i^2(\pi)$ as the decision of policy $\pi$ at period $t$ for scenario (32). Further define $T_1(\pi)$ (resp. $T_2(\pi)$) as the expected capacity consumption of policy $\pi$ on scenario (31) (resp. scenario (32)) during the first $\frac{T}{2}$ time periods:

$$T_1(\pi) = \mathbb{E}\left[\sum_{t=1}^{T_2} x_i^1(\pi)\right] \quad \text{and} \quad T_2(\pi) = \mathbb{E}\left[\sum_{t=1}^{T_2} x_i^2(\pi)\right]$$
Then, we have that
\[ R_1^T(\pi) = T + W_T^T \cdot T_1(\pi) \quad \text{and} \quad R_2^T(\pi) = T - W_T^T \cdot T_2(\pi) \]
where \( R_1^T(\pi) \) (resp. \( R_2^T(\pi) \)) denotes the expected reward collected by policy \( \pi \) on scenario (31) (resp. scenario (32)). Thus, the regret of policy \( \pi \) on scenario (31) and (32) are \( W_T^T \cdot T_1(\pi) \) and \( W_T^T - W_T^T \cdot T_2(\pi) \) respectively. Further note that since the implementation of policy \( \pi \) at each time period should be independent of the future information, we must have \( T_1(\pi) = T_2(\pi) \). Thus, we have that
\[ \text{Reg}_T(\pi) \geq \max\{ W_T^T \cdot T_1(\pi), W_T^T - W_T^T \cdot T_2(\pi) \} \geq \frac{W_T}{2} \Omega(W_T) \]
which completes our proof. \( \square \)

C13. Proof of Theorem 5

We first prove the following lemma, which implies that the dual variable updated in (14) is always bounded. It derivation is essentially the same as Lemma 4.

Lemma 8 Under Assumption 1, for each \( t = 1, 2, \ldots, T \), the dual price vector satisfies \( \|p_t\|_\infty \leq q + 1 \), where \( p_t \) is specified by (14) in Algorithm 3 and the constant \( q \) is defined in Assumption 1 (c).

Proof: Note that the following two properties are satisfied by the update rule (14):
(i). If \( \|p_t\|_\infty \leq q \), then we must have \( \|p_{t+1}\|_\infty \leq q + 1 \) by noting that for each \( i \), the \( i \)-th component of \( p_t \), denoted as \( p_{t,i} \), is nonnegative and \( g_i(\cdot; \theta_t) \) is normalized within \([0, 1]\).
(ii). If there exists \( i \) such that \( p_{t,i} > q \), then we must have \( p_{t+1,i} < p_{t,i} \). Specifically, when \( p_{t,i} > q \), we must have that \( g_i(\tilde{x}_t; \theta_t) = 0 \), otherwise we would have that
\[ f(\tilde{x}_t; \theta_t) - p_{t,i}^T \cdot g(\tilde{x}_t; \theta_t) \leq f(\tilde{x}_t; \theta_t) - p_{t,i} \cdot g_i(\tilde{x}_t; \theta_t) < 0 \]
which contradicts the definition of \( \tilde{x}_t \) in Algorithm 3 since we could always select \( x_t = 0 \) to obtain a zero objective value as per Assumption 1. Then from (14), it holds that \( p_{t+1,i} < p_{t,i} \).

Starting from \( p_1 = 0 \) and iteratively applying the above two property to control the increase of \( p_t \) from \( t = 1 \) to \( T \), we obtain that for the first time that one component of \( p_t \) exceeds the threshold \( q \), it is upper bounded by \( q + 1 \) and this component will continue to decrease until it falls below the threshold \( q \). Thus, it is obvious that we have \( \|p_t\|_\infty \leq q + 1 \) with probability 1 for each \( t \). \( \square \)

Now we proceed to prove Theorem 5.
Proof: In Algorithm 3, the true action $x_t$ differs from the virtual action $\tilde{x}_t$ if and only if $c_t$ can not (fully) satisfy $g(\tilde{x}_t; \theta_t)$. Thus, we have

$$f(\tilde{x}_t; \theta_t) - f(x_t; \theta_t) \leq f(\tilde{x}_t; \theta_t) \cdot \mathbb{I}\{\exists i: c_{t,i} < g_t(\tilde{x}_t; \theta_t)\}$$

where $c_{t,i}$ denotes the $i$-th component of $c_t$ and $\mathbb{I}\{\cdot\}$ denotes the indicator function. Also, note that

$$\mathbb{I}\{\exists i: c_{t,i} < g_t(\tilde{x}_t; \theta_t)\} \leq \sum_{i=1}^{m} \mathbb{I}\left\{ \sum_{j=1}^{t} g_j(\tilde{x}_j; \theta_j) > c_i \right\}.$$

From Assumption 1 (c), we know that the maximum reward generated by consuming each unit of resource is upper bounded by $q$. So,

$$f(\tilde{x}_t; \theta_t) \cdot \mathbb{I}\{\exists i: c_{t,i} < g_t(\tilde{x}_t; \theta_t)\} \leq q \cdot \sum_{i=1}^{m} g_i(\tilde{x}_t; \theta_t) \cdot \mathbb{I}\left\{ \sum_{j=1}^{t} g_j(\tilde{x}_j; \theta_j) > c_i \right\}.$$

From Assumption 1 (b), we know $g(\cdot; \theta_t) \in [0, 1]^m$. Then,

$$\sum_{t=1}^{T} f(\tilde{x}_t; \theta_t) - \sum_{t=1}^{T} f(x_t; \theta_t) \leq q \cdot \sum_{i=1}^{m} \sum_{t=1}^{T} g_i(\tilde{x}_t; \theta_t) \cdot \mathbb{I}\left\{ \sum_{j=1}^{t} g_j(\tilde{x}_j; \theta_j) > c_i \right\}$$

$$\leq q \cdot \sum_{i=1}^{m} \left[ \sum_{t=1}^{T} g_i(\tilde{x}_t; \theta_t) - (c_i - 1) \right]^+$$

which relates the total collected reward by the true action $\{x_t\}_{t=1}^{T}$ and the virtual action $\{\tilde{x}_t\}_{t=1}^{T}$. Here $[\cdot]^+$ denotes the positive part function. Furthermore, from Lemma 2, we have that

$$\text{Reg}_T(\pi) \leq \min_{p \geq 0} L(p) - \mathbb{E}\left[ \sum_{t=1}^{T} f(x_t; \theta_t) \right] \leq \min_{p \geq 0} L(p) - \mathbb{E}\left[ \sum_{t=1}^{T} f(\tilde{x}_t; \theta_t) \right]$$

$$+ q \cdot \mathbb{E}\left[ \sum_{i=1}^{m} \sum_{t=1}^{T} g_i(\tilde{x}_t; \theta_t) - (c_i - 1) \right]^+$$

Next, we bound the term I and term II separately to derive our regret bound.

**Bound I:** We first define the following function $\tilde{L}(\cdot)$:

$$\tilde{L}(p) := \frac{1}{T} p^\top \mathbf{c} + \mathcal{P}_T h(p; \theta)$$

Note that $\mathcal{P}_T = \frac{1}{T} \sum_{t=1}^{T} \mathcal{P}_t$, it holds that $L(p) = T \cdot \tilde{L}(p)$ for any $p$. From Lemma 8, we know that for each $t$, $\|p_t\|_\infty \leq q + 1$ with probability 1. In addition, for each $t$, the distribution of $p_t$ is independent from the distribution of $\theta_t$, then from Lemma 5, we have that

$$\min_{p \geq 0} \tilde{L}(p) \leq \mathbb{E}_{p_t} \left[ \tilde{L}(p_t) \right] \leq \mathbb{E}_{p_t} \left[ \frac{1}{T} p_t^\top \mathbf{c} + \mathcal{P}_t h(p_t; \theta_t) \right] + (q + 1)(m + 1) \cdot \mathcal{W}(\mathcal{P}_t, \mathcal{P}_T), \quad (33)$$
where the expectation is taken with respect to \( p_t \) in a random realization of the algorithm. Thus, we have the first term

\[
I \leq \sum_{t=1}^{T} \mathbb{E}_{p_t} \left[ \frac{1}{T} c^\top p_t + \mathcal{P}_t \{h(p_t; \theta_t) - f(\hat{x}_i; \theta_i)\} \right] + (q + 1)(m + 1) \cdot W(\mathcal{P}_t, \mathcal{P}_T)
\]

which comes from combining (33) with the relation \( L(p) = T \cdot \tilde{L}(p) \). By the definition of \( \tilde{x}_i \),

\[
h(p_t; \theta_t) - f(\tilde{x}_i; \theta_i) = -p_t^\top g(\tilde{x}_i; \theta_i),
\]

which implies that

\[
I \leq \sum_{t=1}^{T} \mathbb{E}_{p_t} \left[ p_t^\top \left( \frac{c}{T} - \mathcal{P}_t g(\tilde{x}_i; \theta_i) \right) \right] + (q + 1)(m + 1) \cdot W(\mathcal{P}_t, \mathcal{P}_T)
\]

Note that from the update rule (14), we have that

\[
\|p_{t+1}\|_2^2 \leq \|p_t\|_2^2 + \frac{1}{T} \cdot \|g(\tilde{x}_i; \theta_i) - \frac{c}{T}\|_2^2 - \frac{2}{\sqrt{T}} \cdot p_t^\top \left( \frac{c}{T} - g(\tilde{x}_i; \theta_i) \right).
\]

By taking expectation with respect to both sides,

\[
\mathbb{E}_{p_t} \left[ p_t^\top \left( \frac{c}{T} - \mathcal{P}_t g(\tilde{x}_i; \theta_i) \right) \right] \leq \frac{\sqrt{T}}{2} \cdot \left( \mathbb{E}[\|p_t\|_2^2] - \mathbb{E}[\|p_{t+1}\|_2^2] \right) + \frac{m}{2 \sqrt{T}}.
\]

Plugging (35) into (34), we obtain an upper bound on Term I,

\[
I \leq \frac{m \sqrt{T}}{2} + (q + 1)(m + 1) \cdot W_T
\]

**Bound II:** Note that from the update rule (14), we have

\[
\sqrt{T} \cdot p_{t+1} \geq \sqrt{T} \cdot p_t + g(\tilde{x}_i; \theta_i) - \frac{c}{T}
\]

Taking a summation with respect to both sides,

\[
\sum_{t=1}^{T} g(\tilde{x}_i; \theta_i) - c \leq \sqrt{T} \cdot p_{T+1}
\]

Applying Lemma 8 for a bound on \( p_{T+1} \), we obtain the upper bound for Term II,

\[
II = q \cdot \mathbb{E} \left[ \sum_{i=1}^{m} \left( \sum_{t=1}^{T} g_i(\tilde{x}_i; \theta_i) - (c_i - 1) \right) \right]^+ \leq mq(q + 1) \cdot \sqrt{T} + qm
\]

We obtain the desired regret bound by combining (36) and (37). \( \Box \)
Table 3  The numerical experiment based on the network revenue management problem (Jasin (2015)). The results are obtained for $T = 1000$ and 100 trails. Re-solve (a) denotes the frequency of re-solving, namely, re-solving the upper bound problem to update dual variable and $\gamma_t$ for every a periods. For each entry b(c) of the table, b denotes the expected reward collected by the algorithm and c denotes the percentage of the expected reward of the algorithm over the upper bound.

| $\beta$ | IGDP          | Re-solve (200) | Re-solve (100) | Re-solve (50) | Re-solve (1) |
|--------|---------------|---------------|---------------|---------------|--------------|
| $\alpha = 0$ | 520173(96.9%) | 524233(97.6%) | 523685(97.5%) | 523700(97.5%) | 527839(98.3%) |
|        | 528884(96.1%) | 536703(97.6%) | 536427(97.5%) | 537134(97.6%) | 540176(98.2%) |
|        | 535678(95.8%) | 543961(97.3%) | 544318(97.3%) | 544105(97.3%) | 547443(97.9%) |
|        | 542302(94.9%) | 555291(97.1%) | 556507(97.3%) | 555119(97.1%) | 558460(97.7%) |
| $\beta = 0.01$ | 518718(96.6%) | 524267(97.6%) | 524438(97.7%) | 525178(97.8%) | 529752(98.7%) |
|        | 528611(96.1%) | 536168(97.6%) | 537389(97.7%) | 537462(97.7%) | 541583(98.5%) |
|        | 534221(95.6%) | 545057(97.5%) | 545845(97.7%) | 545829(97.5%) | 549673(98.3%) |
|        | 543138(95.0%) | 555260(97.1%) | 556276(97.3%) | 556498(97.4%) | 560308(98.0%) |
| $\beta = 0.02$ | 518287(96.5%) | 523689(97.5%) | 525031(97.8%) | 525356(97.8%) | 529732(98.7%) |
|        | 527937(96.0%) | 534737(97.6%) | 537302(97.7%) | 537465(97.8%) | 541779(98.5%) |
|        | 533538(95.5%) | 543426(97.3%) | 545829(97.7%) | 544992(97.5%) | 550519(98.5%) |
|        | 545654(95.5%) | 556498(97.4%) | 555397(97.3%) | 555949(97.3%) | 560719(98.2%) |
| $\beta = 0.04$ | 518405(96.5%) | 524127(97.6%) | 525391(97.6%) | 525299(97.8%) | 530349(98.8%) |
|        | 528083(96.0%) | 535318(97.4%) | 536770(97.6%) | 536481(97.6%) | 541633(98.5%) |
|        | 536737(96.0%) | 544818(97.5%) | 544874(97.5%) | 545335(97.6%) | 550063(98.4%) |
|        | 545405(95.5%) | 554169(97.0%) | 556560(97.5%) | 556776(97.5%) | 560365(98.1%) |