SOME DEGENERATIONS OF KAZHDAN-LUSZTIG IDEALS AND MULTIPlicITIES OF SCHUBERT VARIETIES

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Abstract. We study Hilbert-Samuel multiplicity for points of Schubert varieties in the complete flag variety, by Gröbner degenerations of the Kazhdan-Lusztig ideal. In the covexillary case, we give a positive combinatorial rule for multiplicity by establishing (with a Gröbner basis) a reduced and equidimensional limit whose Stanley-Reisner simplicial complex is homeomorphic to a shellable ball or sphere. We show that multiplicity counts the number of facets of this complex. We also obtain a formula for the Hilbert series of the local ring. In particular, our work gives a multiplicity rule for Grassmannian Schubert varieties, providing alternative statements and proofs to formulae of [Lakshmibai-Weyman ’90], [Rosenthal-Zelevinsky ’01], [Krattenthaler ’01], [Kreiman-Lakshmibai ’04] and [Woo-Yong ’09]. We suggest extensions of our methodology to the general case.

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References

1. Introduction

1.1. Overview. Let Flags($\mathbb{C}^n$) denote the variety of complete flags in $\mathbb{C}^n$. Its Schubert subvarieties $X_w$ are indexed by permutations $w$ in the symmetric group $S_n$. There has been substantial interest in understanding the singularity structure of Schubert varieties. While the singular loci have been determined, and fundamental properties that hold for all Schubert varieties have been long established, many mysteries remain about measures of singularities; see, e.g., [BilLak01, Bri03, WooYon08]. This paper treats a classical example of such a measure, the (Hilbert–Samuel) multiplicity of a point $p$ in a scheme $X$, denoted $\text{mult}_p(X)$. This positive integer is the degree of the projectivized tangent cone $\text{Proj}(\text{gr}_{m_p}O_{p,X})$ as a subvariety of the projectivized tangent space $\text{Proj}(\text{Sym}^m m_p/m_p^2)$, where $(O_{p,X}, m_p)$ is the local ring associated to $p \in X$. Equivalently, if the Hilbert–Samuel polynomial of $O_{p,X}$ is $a_dx^d + a_{d-1}x^{d-1} + \ldots + a_0$ ($a_d \neq 0$) then $\text{mult}_p(X) = d!a_d$. In particular, $\text{mult}_p(X) = 1$ if and only if $X$ is smooth at $p$.

It is an open problem to give a positive combinatorial rule for the multiplicity of a Schubert variety $X_w$ at its torus fixed points $e_v \in X_w$ (the problem for arbitrary $p \in X_w$ reduces to this case). The analogous problem for Grassmannians has been solved; see, e.g., [RosZel01, Kra01, KreLak04, Kre08, WooYon09] and the references therein. There has also been related work on multiplicities of (co)minuscule Grassmannians and for determinantal varieties; a sampling includes [LakWey90, HerTru92, GhoRag06, IkeNar07, RagUpa07].

The thesis of this paper is as follows. A neighbourhood of $e_v \in X_w$ is encoded by the Kazhdan-Lusztig variety $N_{v,w}$ with explicit coordinates and equations given in [WooYon08]. We propose to study a choice of term orders $\prec_{v,w,\pi}$ that depends on $v, w$ and a shuffling (total ordering) of variables $\pi$. The corresponding Gröbner degenerations break $N_{v,w}$, and its projectivized tangent cone, into an initial scheme $\text{init}_{\prec_{v,w,\pi}}N_{v,w}$ whose reduced scheme structure is of a union of coordinate subspaces. By construction, multiplicity is the degree of this monomial ideal. However, more seems conjecturally true: first, there exists $\pi$ such that $\text{init}_{\prec_{v,w,\pi}}N_{v,w}$ is both reduced and equidimensional; and second, one can furthermore choose $\pi$ so that the corresponding Stanley-Reisner simplicial complex is homeomorphic to a shellable ball or sphere. These conjectures assert multiplicity reduces to the combinatorics of counting the number of facets of a desirable simplicial complex. We label facets by $\pi$-shuffled tableaux that assign +’s to the $n \times n$ grid, using $\pi$ and the corresponding prime component of the initial ideal.
This paper further formulates the above thesis and collects some evidence for its efficacy towards the multiplicity problem.

Our main theorems prove the above conjectures for **covexillary Schubert varieties**, i.e., those $X_w$ where $w$ avoids the pattern 3412. We obtain the first multiplicity rule in this case, which is presently the most general one available in type A. Actually, these Schubert varieties have attracted significant attention in the study of Schubert geometry and combinatorics; see, e.g., [LakSan90, Mac91, Ful92, Las95, Man01, KnuMilYon05] and the references therein. For comparison, A. Lascoux [Las95] studied a different measure of singularities of Schubert varieties. He gave a combinatorial rule for the Kazhdan-Lusztig polynomials at singular points of covexillary $X_w$, extending work of A. Lascoux and M.-P. Schützenberger [LasSch81] for Grassmannian Schubert varieties. Similarly, our rule also specializes to the Grassmannian case.

For covexillary Schubert varieties, our key observation is that one can pick $\pi$ (depending on $v, w$) so that the limit scheme is (after $\pi$-shuffling the coordinates and crossing by affine space) the limit scheme of a matrix Schubert variety [KnuMilYon05] for a different covexillary permutation. We deduce an explicit Gröbner basis, with squarefree initial terms, for the Kazhdan-Lusztig ideal under $\prec_{v,w,\pi}$, extending the Gröbner basis theorem of that earlier paper. The limit is reduced and equidimensional. Using the results of [KnuMilYon05], we prime decompose the initial ideal and show that the $\pi$-shuffled tableaux are in an easy bijection with flagged semistandard Young tableaux (thus providing some justification for the nomenclature). Hence, the number of the stated tableaux counts the desired multiplicity, and as in [WooYon09], a well-known generalization of the Jacobi-Trudi identity yields a simple proof of a determinantal formula. Also, the Stanley-Reisner complex homeomorphic to a vertex decomposable and hence shellable ball or sphere. This feature allows us to prove an “alternating-sign” formula for a richer invariant than multiplicity, the Hilbert series of $O_{e_v X_w}$.

We remark, that although we work over $\mathbb{C}$, since our Gröbner basis involves only coefficients $\pm 1$, it follows that our formulae are valid over any characteristic. To our best knowledge, independence of characteristic for multiplicities was not known for general $e_v \in X_w$ (and not even in the covexillary case).

Summarizing, the results in the covexillary case provide some “proof of concept” for our thesis.

### 1.2. Some related work.

Gröbner degeneration has been exploited in a number of related settings in recent years, and in particular has been applied to the multiplicity problem. We now discuss some earlier results in type A to provide context for our specific treatment.

V. Lakshmibai and J. Weyman [LakWey90] and V. Kreiman and V. Lakshmibai [KreLak04] utilized standard monomial theory to determine multiplicity rules for Grassmannians (actually, [LakWey90] deduces a recursive rule valid for any minuscule $G/P$).

A. Woo and the second author [WooYon09] explain how the Kazhdan-Lusztig ideals of [WooYon08] are compatible with the Schubert polynomial combinatorics of A. Lascoux and M.-P. Schützenberger [LasSch82a, LasSch82b]. Moreover, a Gröbner basis theorem for arbitrary Kazhdan-Lusztig ideals was obtained, generalizing work on Schubert determinantal ideals due to [KnuMil05]. The squarefree initial ideal is equidimensional, and the Stanley-Reisner simplicial complex is homeomorphic to a shellable ball or sphere; more precisely, it is a **subword complex** as defined by A. Knutson and E. Miller [KnuMil04]. For special cases of Kazhdan-Lusztig varieties, and choices of $\pi$, the $\pi$-shuffled tableaux are the pipe dreams of
S. Fomin and A. N. Kirillov [FomKir94], and our thesis subsumes the geometric explanation for these pipe dreams from [KnuMil05]. Similar results to [KnuMil05], used in this paper, were obtained for covexillary Schubert determinantal ideals in [KnuMilYon05].

As an application of [WooYon09], formulae for the multigraded Hilbert series of Kazhdan-Lusztig ideals were geometrically proved, where the multigrading comes from the torus action of the invertible diagonal matrices \( T \subseteq \text{GL}_n \). While this theorem is actually used in a crucial way in the present paper, in general this Hilbert series does not help to directly compute multiplicity, because this torus action is not compatible with the dilation action. However, if a Kazhdan-Lusztig ideal happens to already be homogeneous with respect to the standard grading that assigns each variable degree one, then it is automatic that it is also the ideal for its projectivized tangent cone, and one can deduce a formula for multiplicity from this Hilbert series (homogeneity is guaranteed if \( w_0 v \) is 321-avoiding; see [Knu09, pg. 25]). Moreover, it was explained that for the Grassmannian cases, one can always use the trick of parabolic moving to reduce to the homogeneous case. This gives an easy solution to the Grassmannian multiplicity problem, using Kazhdan-Lusztig ideals. Unfortunately, even for covexillary Schubert varieties, parabolic moving is ineffective for even some small examples. The approach of this paper avoids this issue, by using more direct arguments.

While this paper focuses on type A, our results should have analogues for other Lie types. Recent papers of A. Knutson [Knu08, Knu09] point the way towards coordinates and equations for Kazhdan-Lusztig varieties. His papers also explain how to iteratively degenerate these varieties, although the degenerations he considers are not directly applicable in general to the multiplicity problem, since they do not degenerate the projectivized tangent cone. Finally, we remark that the notion of covexillary for type B has already been examined in a paper by S. Billey and T. K. Lam [BilLam98].

1.3. **Organization and summary of results.** In Section 2 we recall necessary preliminaries about flag, Schubert and Kazhdan-Lusztig varieties. In Section 3 we rigorously formulate the our approach towards multiplicities. This is encapsulated in our initial theorem (Theorem 3.1). In Sections 4–6 we turn to the covexillary setting and state our main theorems. We begin by stating our Gröbner basis theorem (Theorem 4.1) in Section 4. In Section 5, we state our prime decomposition theorem (Theorem 5.5) for the initial ideal of the Kazhdan-Lusztig ideal in terms of flagged tableaux and their bijectively equivalent pipe dreams. Section 6 exploits these results to obtain combinatorial and determinantal rules for the multiplicity and the Hilbert series of the projectivized tangent cone (Theorems 6.1, 6.2 and 6.6 respectively). Section 7 is devoted to the proofs of the theorems of Sections 4–6. Finally, in Section 8 we return to the general case and state our conjectures.

2. **Preliminaries**

We recall some notions about the varieties discussed in this article. Our conventions agree with the ones used in [WooYon08, WooYon09].

2.1. **Flag and Schubert varieties.** Let \( G = \text{GL}_n(\mathbb{C}) \), \( B \) be the Borel subgroup of strictly upper triangular matrices, \( T \subseteq B \) the maximal torus of diagonal matrices, and \( B_- \) the corresponding opposite Borel subgroup of strictly lower triangular matrices. The complete flag variety is \( \text{Flags}(\mathbb{C}^n) := G/B \). The fixed points of \( \text{Flags}(\mathbb{C}^n) \) under the left action of \( T \)
are naturally indexed by the symmetric group $S_n$ thanks to its role as the Weyl group of $G$; we denote these points $e_v$ for $v \in S_n$. One has the **Bruhat decomposition**

$$G/B = \coprod_{w \in S_n} Be_w B/B.$$  

The **Schubert cell** is the $B$-orbit $X^o_w := Be_w B/B$, and its closure $X_w := \overline{X^o_w}$ is the **Schubert variety**. It is a subvariety of dimension $\ell(w)$, where $\ell(w)$ is the length of any reduced word of $w$. Each Schubert variety $X_w$ is a union of Schubert cells. The **Bruhat order** is the partial order on $S_n$ defined by declaring that $v \leq w$ if $X^o_v \subseteq X_w$.

Since every point on $X_w$ is in the $B$-orbit of some $e_v$ (for $v \leq w$ in Bruhat order), the study of local questions on Schubert varieties reduces to the case of these fixed points. An affine neighbourhood of $e_v$ is given by $v \Omega^o_\text{id}$, where in general $\Omega^o_v := B - u B/B$ is the opposite **Schubert cell**. Hence to study $X_w$ locally at $e_v$ one only needs to understand $X_w \cap v \Omega_\text{id}^o$. However, by [KazLus79, Lemma A.4], one has the isomorphism

$$X_w \cap v \Omega^o_\text{id} \cong (X_w \cap \Omega_v^o) \times \mathbb{A}^{\ell(v)}.$$  

(2.1)

Hence, we study the (reduced and irreducible) **Kazhdan–Lusztig variety**

$$\mathcal{N}_{v,w} = X_w \cap \Omega_v^o,$$

harmlessly dropping the factor of affine space.

2.2. **Kazhdan-Lusztig ideals.** We now recall coordinates on $\Omega^o_v$, and the **Kazhdan–Lusztig ideal** $I_{v,w}$ in these coordinates [WooYon08].

Let $M_n$ be the set of all $n \times n$ matrices with entries in $\mathbb{C}$, with coordinate ring $\mathbb{C}[z]$ where $z = (z_{ij})_{i,j=1}^n$ are the coordinate functions on the entries of a generic matrix $Z$. We index the matrix so that $z_{ij}$ is in the $i$-th row from the bottom of the matrix and $j$-th column from the left. Concretely realizing $G$, $B$, $B_\text{+}$, and $T$ as invertible, upper triangular, lower triangular, and diagonal matrices respectively, as explained in [Ful97], we can think of the opposite Schubert cell $\Omega^o_v$ as an affine subspace of $M_n$. Specifically, a matrix is in (our realization of) $\Omega^o_v$ if, for all $i$,

$$z_{n-v(i)+1,i} = 1, \text{ and } z_{n-v(i)+1,b} = 0 \text{ and } z_{b,i} = 0 \text{ for } a > i \text{ and } b > n - v(i) + 1.$$  

Let $z^{(v)} \subseteq z$ denote the remaining unspecialized variables, and $Z^{(v)}$ the specialized generic matrix representing a generic element of $\Omega^o_v$.

Let $Z_{ab}^{(v)}$ denote the southwest $a \times b$ submatrix of $Z^{(v)}$. Also let

$$R^w = [z_{ij}^{w}]_{i,j=1}^n$$  

be the **rank matrix** (which we index similarly) defined by

$$r^w_{ij} = \# \{ k \mid w(k) \geq n - i + 1, k \leq j \}.$$  

Define the **Kazhdan–Lusztig ideal**

$$I_{v,w} \subseteq \mathbb{C}[z^{(v)}] \cong \text{Fun}[\Omega^o_v]$$

to be the ideal generated by all of the size $1 + r^w_{ij}$ minors of $Z_{ij}^{(v)}$ for all $i$ and $j$. 

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2.3. Schubert determinantal ideals. The Schubert determinantal ideal $I_w$ is generated by all size $1 + r_{ij}^n$ determinants of the southwest $i \times j$ submatrix $Z_{ij}$ of $Z$, for all $i, j$. It is known that $I_w$ is generated by the smaller set of essential determinants which is the subset of the above generators coming from only $(i, j)$ in the essential set of $w$ (we recall the definition of the essential set in Section 4.1). The matrix Schubert variety $\overline{X}_w$ is the (reduced and irreducible) variety in $M_n$ defined by $I_w$. Matrix Schubert varieties were introduced in [Ful92]. In fact, matrix Schubert varieties can be realized as special cases of Kazhdan-Lusztig varieties, as seen in [Ful92] and recapitulated in [WooYon09, Section 2.3].

2.4. Torus actions. The action of $T \cong (\mathbb{C}^*)^n$ on Flags($\mathbb{C}^n$) induces the usual action. This action is the left action of diagonal matrices on $B$-cosets of $G$ written in our coordinates. The action rescales rows independently and rescales columns dependently, as upon rescaling a row one must rescale a corresponding column to ensure there is a 1 in position $(n - v(j) + 1, j)$ (as read with our upside-down matrix coordinates). Applying the usual convention that the homomorphism picking out the $i$-th diagonal entry is the weight $t_i$ and writing weights additively, this action gives the matrix entry at $(i, j)$ the weight $t_{n-i+1} - t_{v(j)}$. The variable $z_{ij}$ is the coordinate function on this matrix entry and therefore (the torus action on the variable) has weight

$$\text{wt}(z_{ij}) = t_{v(j)} - t_{n-i+1}.$$ 

Let us call this the usual action grading; it is a fact that this is a positive grading (cf. Section 7.3). The Kazhdan-Lusztig ideal $I_{v,w}$ is homogeneous with respect to the usual action grading, since one can easily check that each defining determinant is homogeneous.

3. Gröbner degeneration and multiplicity

Let $\pi$ be a shuffling, i.e., an ordering of the variables of $\mathbb{C}[z^{(v)}]$ by reading the rows of $Z^{(v)}$ from left to right and bottom to top, each of the $\ell(wv)!$ orderings of the variables can be identified with a permutation $\pi$ in the symmetric group $S_{\ell(wv)}$. Let $\prec_{v,w,\pi}$ be the local term order (i.e., one where $z_{ij} \prec_{v,w,\pi} 1$) that favors monomials of lowest total degree first, and then breaks ties lexicographically according to $\pi$.

Rather than using $\prec_{v,w,\pi}$ directly, we find it more convenient to study a different term order $\prec_{v,w,\pi}$ on monomials in $\mathbb{C}[z^{(v)}]_1$, defined as follows. For each $t_i$, define $\phi(t_i) = n + 1 - i$. Define the non-standard degree $\text{deg}$ of $z_{ij}$ to be

$$\text{deg}(z_{ij}) = \phi(t_{v(j)}) - \phi(t_{n+i-1}) = n + 1 - i - v(j).$$

Also, define the standard degree $\text{deg}'$ by $\text{deg}'(z_{ij}) = 1$. As usual, extend these definitions to monomials $m = c \prod_{ij} z_{ij}^{d_{ij}}$ (where $c \in \mathbb{C}^*$) by

$$\text{deg}(m) = \sum_{ij} a_{ij} \text{deg}(z_{ij})$$

e tc., and where $\text{deg}(c) = \text{deg}'(c) = 0$. Note that $\text{deg}(m)$ is a $\mathbb{Z}$-graded coarsening of the usual-action grading of Section 2.4.

Let $m_1$ and $m_2$ be two monomials in $\mathbb{C}[z^{(v)}]$. Define $m_1 \prec_{v,w,\pi} m_2$ if

(a) $\text{deg}(m_1) < \text{deg}(m_2)$, or if
(b) $\text{deg}(m_1) = \text{deg}(m_2)$ and $m_1 \prec_{v,w,\pi}' m_2$. 

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The statement of the result below also requires the **Stanley-Reisner correspondence**. This bijectively associates a squarefree monomial ideal \( I \subseteq \mathbb{C}[z_1, \ldots, z_N] \) with a simplicial complex \( \Delta_I \) whose vertex set is \( \{1, 2, \ldots, N\} \) and whose faces correspond naturally to monomials *not* in \( I \). Conversely, to each such simplicial complex \( \Delta \), there is an associated ideal \( I_{\Delta} \subseteq \mathbb{C}[z_1, \ldots, z_N] \) and **face ring** \( \mathbb{C}[\Delta] = \mathbb{C}[z_1, \ldots, z_N]/I_{\Delta} \). Our resource for facts about combinatorial commutative algebra is the textbook by E. Miller and B. Sturmfels [MilStu05]; cf. Section 7.3.

We have:

**Theorem 3.1.** Let \( \pi \in S_{\ell(w_0)} \) be a shuffling for \( \mathbb{C}[z^{(v)}] \). Then the following holds:

1. \( \prec_{v,w,\pi} \) is a global term order (i.e., one where \( 1 \prec_{v,w,\pi} z_{ij} \)) such that if \( f \in \mathbb{C}[z^{(v)}] \) is homogeneous with respect to usual action grading, then \( \text{init}_{\prec_{v,w,\pi}}(f) = \text{init}_{\prec'_{v,w,\pi}}(f) \).
2. \( \text{init}_{\prec_{v,w,\pi}} I_{v,w} = \text{init}_{\prec'_{v,w,\pi}} I_{v,w} = \text{init}_{\prec_{v,w,\pi}} T_{v,w} \), where
   \[
   T_{v,w} = \langle \hat{f} : \hat{f} \text{ is the lowest standard degree component of } f \rangle
   \]
defines the ideal of the projectivized tangent cone of \( N_{v,w} \); \( T_{v,w} \) is homogeneous with respect to both standard and usual action gradings.
3. \( \text{mult}_{e_v}(X_w) = \text{degree}(T_{v,w}) = \text{degree(init}_{\prec_{v,w,\pi}} I_{v,w}) \)
4. Under the usual action grading, the Hilbert series for \( \mathbb{C}[z^{(v)}]/I_{v,w} \) equals the Hilbert series of \( \mathbb{C}[z^{(v)}]/\text{init}_{\prec_{v,w,\pi}} I_{v,w} \).
5. If \( \text{init}_{\prec_{v,w,\pi}} I_{v,w} \) is reduced and equidimensional, then \( \text{mult}_{e_v}(X_w) \) equals the number of irreducible components of \( \text{init}_{\prec_{v,w,\pi}} I_{v,w} \), or alternatively, equals the number of facets of the Stanley-Reisner simplicial complex \( \Delta_{v,w,\pi} \) associated to \( \text{init}_{\prec_{v,w,\pi}} I_{v,w} \).
6. If in addition to the hypothesis of (IV), \( \Delta_{v,w,\pi} \) is homeomorphic to a ball or sphere, then the \( \mathbb{Z} \)-graded Hilbert series for \( \mathcal{O}_{e_v X_w} \) is given by
   \[
   G_{v,w}(t) = \sum_{i \geq 0} \dim(m_{e_v}^i \mathfrak{m}_{e_v}^{i+1}) t^i = G_{v,w}(t)/(1-t)^{\binom{n}{2}}
   \]
   where
   \[
   G_{v,w}(t) = \sum_{k \geq 0} (-1)^k(1-t)^{\ell(w_0) + k} \times \#\{\text{interior faces of } \Delta_{v,w,\pi} \text{ of codimension } k\}.
   \]

**Proof.** For (I), to check that \( \prec_{v,w,\pi} \) is a term order, first, we need to show that it is a total ordering on monomials; and second, that it is multiplicative, meaning that for monomials \( m_1, m_2, m_3 \), if \( m_1 \prec_{v,w,\pi} m_2 \) then \( m_1 m_3 \prec_{v,w,\pi} m_2 m_3 \); and third, that it is Artinian, meaning \( 1 \prec_{v,w,\pi} m \) for all nonunit monomials \( m \). Clearly \( \prec_{v,w,\pi} \) is a total order. It is also straightforward to check that \( \prec_{v,w,\pi} \) is multiplicative by considering cases (a) and (b) separately. To see that \( \prec_{v,w,\pi} \) is Artinian, it suffices to show that \( \mathrm{deg}(1) < \mathrm{deg}(m) \) for any nonunit monomial \( m \), hence \( 1 \prec_{v,w,\pi} m \) by (a). Indeed, note that \( \prec_{v,w,\pi} \) is a positive weighting on monomials: if \( z_{ij} \) appears in \( z^{(v)} \) then we must have \( i < n+1-v(j) \) by construction. Hence \( \mathrm{deg}(z_{ij}) = n+1-v(j)-i > 0 \). Finally, if \( f \) is homogeneous with respect to the usual action grading, then the comparison of terms of \( f \) falls into case (b) of the definition of \( \prec_{v,w,\pi} \) and hence we pick the initial term according to \( \prec'_{v,w,\pi} \).

For (II), the equality \( \text{init}_{\prec_{v,w,\pi}} I_{v,w} = \text{init}_{\prec'_{v,w,\pi}} I_{v,w} \) follows from (I) and the fact that \( I_{v,w} \) is an homogeneous ideal with respect to the non-standard degree \( \mathrm{deg} \) (cf. Section 2.4). The remaining equality and claim about \( T_{v,w} \) holds similarly.
For (III), the degree of the projectivized tangent cone of $e_v$ in $X_w$ as a subscheme of the projectivized tangent space equals the degree of $T_{v,w}$. Hence we have $\text{mult}_{e_v}(X_w) = \text{degree } T_{v,w}$. That the latter degree equals $\text{init}_{\prec_{v,w}, \pi} I_{v,w}$ is an application of Mora’s tangent cone algorithm [MorPhTr92]. Then apply (II).

(IV) holds since the usual action grading is a positive grading on monomials in $\mathbb{C}[z^{(v)}]$ and it is a general fact that Hilbert series for positively graded modules are preserved under Gröbner degeneration, see, e.g., [MilStu05].

For (V), note that by (II) and (III), $\text{mult}_{e_v}(X_w) = \text{degree } \text{init}_{\prec_{v,w}, \pi} I_{v,w}$. Hence the first claim follows from the hypothesis and additivity of degrees. The second half of (V) is a standard translation concerning Stanley-Reisner simplicial complexes.

To prove (VI), we use the following formula established in [KnuMil04, Theorem 4.1]: if $\Delta$ is a ball or a sphere and $S$ is its Reisner-Stanley ring, then the $K$-polynomial is given by

$$K(S, t) = \sum_F (-1)^{\dim \Delta - \dim F} \prod_{i \in F} (1 - t),$$

where sum over all interior faces of $F$ of $\Delta$. We now apply this formula to

$$S = \mathbb{C}[z^{(v)}]/\text{init}_{\prec_{v,w}, \pi} I_{v,w} = \mathbb{C}[z^{(v)}]/\text{init}_{\prec_{v,w}, \pi} T_{v,w}.$$ 

Now,

$$\#\{i \mid i \notin F\} = \#\{\text{variables in the ring } \mathbb{C}[z^{(v)}]\} - \dim F - 1 = \ell(w_0 v) - \dim F - 1.$$ 

Using the $Z$-grading, the denominator of the $Z$-graded Hilbert series for $S$ is

$$(1 - t)^{\#\{\text{variables in the ring } \mathbb{C}[z^{(v)}]\}} = (1 - t)^{\ell(w_0 v)}.$$ 

Then

$$\text{Hilb}(S, t) = \sum_F (-1)^{\text{codim } F} (1 - t)^{\ell(w_0 v) - \dim F - 1} (1 - t)^{\ell(w_0 v)} = \sum_F (-1)^k (1 - t)^{\binom{n}{2} - \dim F - 1} (1 - t)^{\binom{n}{2}}$$

where the sum over the interior faces $F$ and where $k = \dim \Delta - \dim F$ is the codimension of a face $F$. Since $\dim \Delta = \ell(w_0 v) - \ell(w_0 w) - 1 = \ell(w) - \ell(v) - 1$, we have $\binom{n}{2} - \dim F - 1 = \ell(v) + \ell(w_0 w) + k$.

By (2.1), $e_v$ has a neighborhood in $X_w$ that is isomorphic to $N_{v,w} \times \mathbb{C}^{\ell(v)}$. Under this isomorphism, $e_v$ maps to the point $(0, \vec{0}) \in N_{v,w} \times \mathbb{C}^{\ell(v)}$, where $0 \in N_{v,w}$ and $\vec{0} \in \mathbb{C}^{\ell(v)}$. So we have

$$\text{Hilb}(O_{e_v X_w}, t) = \text{Hilb}(O_{0, N_{v,w}}, t) \cdot \frac{1}{(1 - t)^{\ell(v)}}.$$ 

Meanwhile, the tangent cone of $N_{v,w}$ at $0$ is $\text{Spec} (\mathbb{C}[z^{(v)}]/T_{v,w})$, so

$$\text{Hilb}(O_{0, N_{v,w}}, t) = \text{Hilb}(\mathbb{C}[z^{(v)}]/T_{v,w}, t)$$

and therefore

$$\text{Hilb}(O_{e_v X_w}, t) = \text{Hilb}(\mathbb{C}[z^{(v)}]/T_{v,w}, t) \cdot \frac{1}{(1 - t)^{\ell(v)}}.$$ 

Combining these facts, the Hilbert series of the local ring $O_{e_v X_w}$ is

$$\sum_F (-1)^k (1 - t)^{\ell(v) + \ell(w_0 w) + k} (1 - t)^{\ell(v)} = \sum_F (-1)^k (1 - t)^{\ell(w_0 w) + k} (1 - t)^{\binom{n}{2}}.$$
Now (VI) immediately follows.

Since by (II), $T_{v,w}$ is homogeneous with respect to the standard and usual action grading, we remark it is not hard to compute the multigraded Hilbert series of $O_{v,w}$, for the combined multigrading, with a similar argument as in the proof of (VI) (replacing the $\#\{\text{interior faces of } \Delta_{v,w,\pi} \text{ of codimension } k\}$ by a Laurent polynomial in $t_1, \ldots, t_n$).

We need a few more definitions for future reference: We are mainly interested when $\text{init}_{v,w,\pi} I_{v,w}$ defines a reduced and equidimensional scheme, at which point we consider its prime decomposition

$$\text{init}_{v,w,\pi} I_{v,w} = \bigcap J_i$$

where each $J_i = \langle z_{a_1,b_1}, \ldots, z_{a_m,b_m} \rangle$. Define the shuffled generic matrix $\widetilde{Z}^{(v)}$ by starting with $Z^{(v)}$ and reading the rows left to right and bottom to top, replacing the $k$-th variable in this reading by the $k$ variable of $\pi$. Now define the $\pi$-shuffled tableau associated to $J_i$ to be a filling of the $n \times n$ grid where a $+$ is placed in the positions of $z_{a_1,b_1}, \ldots z_{a_m,b_m}$ of $\widetilde{Z}^{(v)}$. These tableaux are closely related to (and in fact generalize for special choices of $v, w, \pi$) the pipe dreams of [FomKir94] as geometrically interpreted by [KnuMil05], and as we will see, they also generalize (flagged) semistandard Young tableaux.

Two remarks about $\pi$-shuffled tableaux are in order. First, strictly speaking, there is no need to shuffle the coordinates to write down some combinatorial object which labels a prime component of $\text{init}_{v,w,\pi} I_{v,w}$. However, in the covexillary case, as well as what we surmise about [KnuMil05, KnuMilYon05, WooYon09], it seems that the $\pi$-shuffling converts otherwise weird subsets of $n \times n$ into coherent combinatorics. It is for this reason that we propose using this transformation in general. Second, in view of the connection to pipe dreams, it is also plausible to call these objects “$\pi$-shuffled pipe dreams”. However, at present we do not know of any way in general to add elbows to the positions of $n \times n$ not filled by $+$’s that would generate reasonable strand diagrams as in [FomKir94] that would justify the “pipe dream” name (as first introduced in [KnuMil05]).

Theorem 3.1 is most likely combinatorially useful when the limit is reduced and equidimensional. Conjecturally, there is some term order $\prec_{v,w,\pi}$ such that this is true. Therefore, multiplicity would be counted by the inherently combinatorial object $\Delta_{v,w,\pi}$. With this in mind, the choice of coordinates and equations for the Kazhdan-Lusztig variety is not arbitrary. Indeed, whether a variety can be Gröbner degenerated to a reduced scheme is embedding dependent. For example, the only two Gröbner degenerations of $\text{Spec} (\mathbb{C}[x,y]/(x^2 - y^2))$ give multiplicity 2 lines. However, after the linear change of coordinates $u = x - y, v = x + y$, we arrive at the $\text{Spec} (\mathbb{C}[u,v]/(uv))$ which is already a reduced union of coordinate subspaces and hence equal to any of its Gröbner limits.

We will discuss the aforementioned conjecture in more specific detail in Section 8. In the interim, we prove this conjecture in the covexillary case.

4. A Gröbner basis for covexillary Kazhdan-Lusztig ideals

We now begin our application of Theorem 3.1 to covexillary Schubert varieties. In this section, we pick $\prec_{v,w,\pi}$ so that the hypotheses of (V) and (VI) of the theorem hold. We then prove a Gröbner basis theorem for this term order that explicates the degeneration.
Proposition 4.1. The rank matrix \( R \) is uniquely determined by its diagram and the restriction of the rank matrix \( R^w = [r^w_{ij}]_{i,j=1}^n \) to its essential set.

Proposition 4.2. Permutations \( v, w \in S_n \) satisfy \( v \leq w \) (in Bruhat order) if and only if \( r^v_{ij} \leq r^w_{ij} \) for all \((i, j) \in n \times n\).

Definition–Theorem 4.3. The following are equivalent for a permutation \( w \in S_n \):

(i) \( w \) is covexillary\(^1\).
(ii) \( w \) is 3412-avoiding, i.e., there do not exist \( 1 \leq i_1 < i_2 < i_3 < i_4 \leq n \) such that \( w(i_2) < w(i_1) < w(i_3) < w(i_2) \);
(iii) the boxes of the essential set of \( w \) lie on a piecewise linear curve oriented weakly southeast to northwest; and
(iv) the diagram \( D(w) \), up to a permutation of the rows and the columns gives a Young diagram.

If (iv) holds, then in fact the Young diagram \( \lambda = \lambda(w) \) is unique, and we will refer to this as the shape of the covexillary permutation \( w \).

4.2. The Gröbner basis theorem. We now give our central definition, the ordering of variables \( \pi \in S_{\ell(w,v)} \) that we use in the main results of this section and the next.

We say that the box \((x, y)\) is dominated by \((w, z)\) if \( x \leq w \) and \( y \leq z \), i.e., if \((x, y)\) lies in the rectangular region with \((w, z)\) and \((1, 1)\) as its northeast and southwest corners, respectively.

Let \( \lambda = \lambda(w) = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0) \) be as in Section 4.1. For \( 1 \leq i \leq \ell \), let

\[
\left\{ (\alpha^{(i)}_1, \beta^{(i)}_1), \ldots, (\alpha^{(i)}_k, \beta^{(i)}_k) \right\}
\]

\(^1\)In [Man01] (and other sources) one instead considers vexillary permutations, which are equal to \( w_0 w \) where \( w \) is covexillary and \( w_0 \) is the longest length element of \( S_n \). The results we use therefore only differ by a change in conventions.
be the coordinates of those 1’s in $Z^{(v)}$ that are dominated by $(b_i, \lambda_i - i + b_1)$. Here $b_i$ is defined as follows: Let $B(w)$ be the smallest Young diagram (drawn in French notation) with corner in position $(1,1)$ that contains all of $E(w)$. Then set
\[ b_i = \max_{m} \{ B(w)_m \geq \lambda(w)_i + m - i \}. \]

Observe that
\[(4.4) \quad b_1 \leq b_2 \leq \cdots \leq b_t \text{ and } \lambda_1 - 1 + b_1 \geq \lambda_2 - 2 + b_2 \geq \cdots \geq \lambda_t - \ell + b_t. \]

(In Section 5.2, $B(w)$ and $b_i$ will be pictorially motivated and utilized.)

By definition, $\alpha_j^{(i)} = n + 1 - v(\beta_j^{(i)})$ for $1 \leq j \leq k_i$. Define
\[ R_i = \{ 1, 2, \ldots, b_i \} \setminus \{ \alpha_1^{(i)}, \ldots, \alpha_{k_i}^{(i)} \}, \]
\[ C_i = \{ 1, 2, \ldots, \lambda_i - i + b_i \} \setminus \{ \beta_1^{(i)}, \ldots, \beta_{k_i}^{(i)} \}, \]
and set $R_0 = \emptyset$, $R_{t+1} = \{ 1, \ldots, n \}$, $C_0 = \{ 1, \ldots, n \}$, $C_{t+1} = \emptyset$. From (4.4) we have the filtrations of $\{ 1, 2, \ldots, n \}$:
\[ R_0 \subseteq R_1 \subseteq \cdots \subseteq R_{t+1} \text{ and } C_t \supseteq C_t \supseteq \cdots \supseteq C_0. \]

For $0 \leq i \leq \ell$, set
\[ R_{i+1} - R_i = \{ r_1^{(i)} < \cdots < r_p^{(i)} \} \]
and thus we can define $\rho \in S_n$ to be the following permutation (written in one-line notation):
\[ \rho := r_1^{(0)} \cdots r_p^{(0)} \quad r_1^{(1)} \cdots r_p^{(1)} \cdots \quad r_1^{(\ell)} \cdots r_p^{(\ell)} \in S_n. \]

Similarly, for $0 \leq i \leq \ell$, set
\[ C_i - C_{i+1} = \{ c_1^{(i)} < \cdots < c_{q_i}^{(i)} \} \]
and let $\chi \in S_n$ be the following permutation:
\[ \chi := c_1^{(\ell)} \cdots c_{q_{\ell}}^{(\ell)} \quad c_1^{(\ell-1)} \cdots c_{q_{\ell-1}}^{(\ell-1)} \cdots \quad c_1^{(0)} \cdots c_{q_0}^{(0)} \in S_n. \]

Let $Z$ be the shuffled generic matrix obtained by reordering the rows of the generic matrix by $\rho$ and the columns by $\chi$ (cf. Section 3).

Let $\prec_{v,w,\pi}$ be the term order defined in Section 3, using the ordering of variables $\pi$ obtained by reading the rows of $Z$ left to right, and from bottom to top. Strictly speaking, we have defined $\prec_{v,w,\pi}$ as a term order on all monomials in $\mathbb{C}[z]$, which we restrict, in the obvious way, to one for monomials in $\mathbb{C}[z^{(v)}]$.

The ideal $I_{v,w}$ is known to be generated by a smaller set of generators, i.e., the essential minors which are the $r_{i,j}^{w} + 1$ minors of $Z_{ij}^{(w)}$ for all $(i,j) \in E(w)$, see [WooYon08] and the references therein.

Our main result is:

**Theorem 4.4.** The essential minors of $I_{v,w}$ form a Gröbner basis with respect to the term order $\prec_{v,w,\pi}$.

**Example 4.5.** Let $w = 7531462$, $v = 5123746$ (in one line notation). Then $w$ is covexillary, $\lambda(w) = (4, 2, 1)$, the Rothe diagram $D(w)$ and the matrix of variables $Z^{(v)}$ are given by the following figure.
The essential set consists of 3 boxes $e_1 = (2, 5)$, $e_2 = (4, 4)$, $e_3 = (6, 4)$. The Kazhdan-Lusztig ideal is generated by all $2 \times 2$ minors of $Z^{(v)}_{e_1}$, all $3 \times 3$ minors of $Z^{(v)}_{e_1}$, and all $4 \times 4$ minors of $Z^{(v)}_{e_1}$.

$$I_{5123746,7531462} = \left\langle \begin{pmatrix} z_{21} & z_{22} \\ z_{11} & z_{12} \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 & 0 \\ z_{21} & z_{22} & z_{23} \end{pmatrix}, \ldots \right\rangle = \left\langle \begin{pmatrix} 0 & z_{42} & z_{43} & z_{44} \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} z_{21} & z_{22} & z_{23} & z_{24} \\ z_{11} & z_{12} & z_{13} & z_{14} \end{pmatrix} \right\rangle.$$

In this example, $R_1 = \{2\}$, $R_2 = \{1, 2, 4\}$, $R_3 = \{1, 2, 4\}$, therefore $\rho = 2143567 \in S_7$. Similarly, $C_1 = \{1, 2, 3, 4\}$, $C_2 = \{2, 3, 4\}$, $C_3 = \{2\}$, hence $\chi = 2341567 \in S_7$. Thus we have the shuffled generic matrix

$$\tilde{Z} = \begin{pmatrix} \tilde{z}_{71} & \tilde{z}_{72} & \tilde{z}_{73} & \tilde{z}_{74} & \tilde{z}_{75} & \tilde{z}_{76} & \tilde{z}_{77} \\ \tilde{z}_{61} & \tilde{z}_{62} & \tilde{z}_{63} & \tilde{z}_{64} & \tilde{z}_{65} & \tilde{z}_{66} & \tilde{z}_{67} \\ \tilde{z}_{51} & \tilde{z}_{52} & \tilde{z}_{53} & \tilde{z}_{54} & \tilde{z}_{55} & \tilde{z}_{56} & \tilde{z}_{57} \\ \tilde{z}_{41} & \tilde{z}_{42} & \tilde{z}_{43} & \tilde{z}_{44} & \tilde{z}_{45} & \tilde{z}_{46} & \tilde{z}_{47} \\ \tilde{z}_{31} & \tilde{z}_{32} & \tilde{z}_{33} & \tilde{z}_{34} & \tilde{z}_{35} & \tilde{z}_{36} & \tilde{z}_{37} \\ \tilde{z}_{21} & \tilde{z}_{22} & \tilde{z}_{23} & \tilde{z}_{24} & \tilde{z}_{25} & \tilde{z}_{26} & \tilde{z}_{27} \\ \tilde{z}_{11} & \tilde{z}_{12} & \tilde{z}_{13} & \tilde{z}_{14} & \tilde{z}_{15} & \tilde{z}_{16} & \tilde{z}_{17} \end{pmatrix} = \begin{pmatrix} z_{72} & z_{73} & z_{74} & z_{75} & z_{76} & z_{77} \\ z_{62} & z_{63} & z_{64} & z_{65} & z_{66} & z_{67} \\ z_{52} & z_{53} & z_{54} & z_{55} & z_{56} & z_{57} \\ z_{42} & z_{43} & z_{44} & z_{45} & z_{46} & z_{47} \\ z_{32} & z_{33} & z_{34} & z_{35} & z_{36} & z_{37} \\ z_{22} & z_{23} & z_{24} & z_{25} & z_{26} & z_{27} \\ z_{12} & z_{13} & z_{14} & z_{15} & z_{16} & z_{17} \end{pmatrix}$$

satisfying $z_{ij} = z_{p(i),\pi(j)}$. Hence a w, w, w ordering is defined by the ordering of variables

$$\tilde{z}_{11} > \tilde{z}_{12} > \cdots > \tilde{z}_{17} > \tilde{z}_{21} > \tilde{z}_{22} > \cdots > \tilde{z}_{27} > \tilde{z}_{31} > \cdots$$

$$\tilde{z}_{22} > \tilde{z}_{23} > \cdots > \tilde{z}_{27} > \tilde{z}_{12} > \tilde{z}_{13} > \cdots > \tilde{z}_{17} > \tilde{z}_{42} > \cdots ,$$

by reading the rows left to right, and bottom to top. Restricting to the variables actually used in $Z^{(v)}$ gives the ordering $\pi$ to be

$$\pi : \tilde{z}_{22} > \tilde{z}_{23} > \tilde{z}_{24} > \tilde{z}_{21} > \tilde{z}_{26} > \tilde{z}_{12} > \tilde{z}_{13} > \tilde{z}_{14} > \tilde{z}_{11} > \tilde{z}_{42} > \tilde{z}_{43} > \tilde{z}_{44} > \tilde{z}_{52} > \tilde{z}_{53} > \tilde{z}_{62}.$$ 

Thus, the given generators form a Gröbner basis with respect to $\prec_{w, w, w}$ for this choice of $\pi$. \hfill \Box

We record the fact below for future reference. The proof is immediate from the above definitions:

**Lemma 4.6.** Let $1 \leq i \leq \ell$ and define $b_i' = b_i - k_i$, where, as above

$$k_i = \# \{ 1 \text{'s dominated by } (b_i, \lambda_i - i + b_i) \}. $$

Then

$$\{1, 2, \ldots, b_i\} \backslash \{ a_1^{(i)}, \ldots, a_{k_i}^{(i)} \} = R_i = \{ r_1, \ldots, r_{b_i'} \}$$

and

$$\{1, 2, \ldots, \lambda_i - i + b_i\} \backslash \{ b_1^{(i)}, \ldots, b_{k_i}^{(i)} \} = C_i = \{ c_1, \ldots, c_{\lambda_i - i + b_i'} \}. $$

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5. The prime decomposition theorem

5.1. The covexillary permutation \( \Theta_{v,w} \). We now associate to a covexillary \( w \) and a permutation \( v \leq w \) a new covexillary permutation \( \Theta_{v,w} \).

**Definition–Lemma 5.1.** Given \( v \leq w \) and \( w \) covexillary, there is a unique covexillary permutation \( \Theta_{v,w} \in S_n \) such that \( \lambda(w) = \lambda(\Theta_{v,w}) \), and
\[
E(\Theta_{v,w}) = \{ e' : e' \text{ is obtained by moving an } e \in E(w) \text{ diagonally southwest by } r_e w - r_e v \text{ units} \}
\]
where \( r_{\Theta_{v,w}} e = r_e w - r_e v \), for each \( e \in E(w) \).

Although the proof of Definition-Lemma 5.1 actually describes an iterative algorithm for constructing \( \Theta_{v,w} \), we emphasize that for the main theorems of this section and the next, it is sufficient to know just \( E(\Theta_{v,w}) \), which can be handily computed from \( v \) and \( w \).

The proof is delayed until Section 5.3, where we collect some related facts.

**Example 5.2.** Continuing Example 4.5, the reader can check that \( \Theta_{512746, 7531462} = 4635721 \) is the unique permutation satisfying the conditions of Definition-Lemma 5.1.

5.2. From pipe dreams to flagged tableaux. Given
\[
\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0)
\]
and a vector of nonnegative integers
\[
b = (b_1, \ldots, b_\ell)
\]
define a semistandard Young tableau \( T \) of shape \( \lambda \) to be flagged by \( b \) if the labels of \( T \) in row \( i \) are at most \( b_i \).

Associated to each covexillary permutation \( w \in S_n \), there is a flagging \( b = b(w) \): As in Section 4.2, consider the smallest French notation Young diagram (i.e., where the i-th row from the bottom is of length \( \lambda_i \)) \( B(w) \subseteq n \times n \) that contains all the boxes of \( E(w) \) as well as the box at \((1,1)\). A pipe dream consists of a placement of +’s in a subset of the boxes of \( B(w) \). The initial pipe dream for \( w \) places +’s in each box of the French Young diagram \( \lambda(w) \subseteq B(w) \) with its southwest corner is at \((1,1)\) (the fact that one has “⊆” is well-known, and follows, e.g., from the discussion of Section 5.3). Iteratively define all other pipe dreams for \( w \) by using the following local transformation in any \( 2 \times 2 \) square in \( B(w) \):
\[
\begin{array}{c}
. & . \\
+ & .
\end{array} \quad \rightarrow \quad 
\begin{array}{c}
. & . \\
. & +
\end{array}
\]
Each + in the initial pipe dream for \( w \) is in obvious one to one correspondence with the box of \( \lambda(w) \) that it sits in. More generally, this extends inductively to every other pipe dream of \( w \). Thus, we can construct a tableau of shape \( \lambda(w) \) by recording in each box the row that its + lies in. Again by induction, using the transformations above, it is easy to verify that this tableau is semistandard.

**Example 5.3.** Continuing the previous example, the reader can check that the pipe dreams for \( \Theta_{v,w} = 4635721 \) are given in Figure [1] below, where the left pipe dream is the initial pipe dream for \( \Theta_{v,w} \). We have also drawn in \( B(\Theta_{v,w}) = (4, 3, 3) \). (Alternatively, starting directly from \( v \) and \( w \) one can quickly determine \( E(\Theta_{v,w}) \) and thus \( B(\Theta_{v,w}) \), without knowing \( \Theta_{v,w} \) itself, and then write down the pipe dreams.)
Not every semistandard tableau of shape $\lambda(w)$ can be obtained this way. The maximum entry of row $i$ of such a tableau $T$ is bounded from above by how far north the rightmost $+$ in the $i$-th row of the starting pipe dream can travel diagonally (not taking into account any other $+$’s) and remain inside $B(w)$. Let $b_i$ denote this row number. Actually, this gives the same $b_i$ as defined in Section 4.2, which we recall:

$$b_i = \max_m \{B(w)_m \geq \lambda(w)_i + m - i\}.$$ 

**Example 5.4.** The corresponding tableaux to the above pipe dreams are:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 \\
3
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 3 \\
3
\end{array}
\]

and here $b(\Theta_{v,w}) = (1, 3, 3)$.

**Theorem 5.5.** We have

$$\text{init}_{\leq_{v,w,\pi}} I_{v,w} = \bigcap_{P} \{\tilde{z}_{ij} : (i, j) \in P\}$$

where $\tilde{z}_{ij} = z_{p(i)\chi(j)}$ (cf. Section 4.2 and Example 4.5). Here the intersection is over all pipe dreams for $\Theta_{v,w}$.

The associated Stanley Reisner complex $\Delta_{v,w,\pi}$ is homeomorphic to a vertex decomposable ball or sphere. In particular, the limit defines an equidimensional scheme.

The irreducible components, or equivalently, the facets of $\Delta_{v,w,\pi}$ are in bijection with semi-standard Young tableaux of shape $\lambda(w)$ and flagged by $b(\Theta_{v,w})$.

**Example 5.6.** We have the following prime decomposition

$$\text{init}_{\leq_{v,w,\pi}} I_{5123746,7531462} = \langle z_{12}, z_{21}, z_{22}, z_{23}, z_{24}, z_{42}, z_{13}z_{44} \rangle$$

$$= \langle z_{12}, z_{13}, z_{21}, z_{22}, z_{23}, z_{24}, z_{42} \rangle \cap \langle z_{12}, z_{21}, z_{22}, z_{23}, z_{24}, z_{42}, z_{44} \rangle.$$ 

We can associate a $\pi$-shuffled tableaux to each component by placing a $+$ in the position of $z_{ab}$ in the shuffled generic matrix $\tilde{Z}$ whenever $z_{ab}$ appears as a generator of the prime ideal for that component (and $\cdot$’s everywhere else). The result are precisely the pipe dreams given in Example 5.3, which are themselves in an easy bijection with the semistandard Young tableaux of Example 5.4. This accounts for the use of $\tilde{z}_{ij}$ in Theorem 5.5, and provides some rationale for our introduction of $\pi$-shuffled tableaux in general in Section 3.
5.3. **Proof of Definition-Lemma 5.1** and some properties of b and B(w). Suppose w is covexillary and 
\[ c = (i_0, j_0) \in \mathcal{E}(w) \text{ where } r_{i_0,j_0}^w > 0. \]
Define the **transitioned permutation** w’ as follows. Let \((i_1, j_1)\) be the northeast most dot in D(w) that is dominated by \((i_0, j_0)\). Such a dot exists because of the assumption \(r_{i_0,j_0}^w > 0\).
By the condition that w is covexillary, there is at one such choice. Let \((i_2, j_2)\) be the dot that is in the same column as \((i_0, j_0)\), and \((i_3, j_3)\) be the dot that is in the same row as \((i_0, j_0)\).
Hence 
\[ i_2 = i_0 + 1, j_2 = j_0 \quad \text{and} \quad i_3 = i_0, j_3 = j_0 + 1. \]
Then define w’ by letting
\[
\begin{align*}
w'(j_1) &= n + 1 - i_3 = n + 1 - i_0, \\
w'(j_2) &= n + 1 - i_1 = w(j_1), \\
w'(j_3) &= n + 1 - i_2 = w(j_0), \\
\text{and} \quad w'(j) &= w(j) \text{ for } j \neq j_1, j_2, j_3.
\end{align*}
\]

The figure below illustrates this description of w’:

![Figure 2](image)

**Figure 2.** Going from D(w) to D(w’) in Definition-Lemma 5.1

The proof of Definition-Lemma 5.1 is based on the following fact, whose proof is straightforward and omitted (cf. [KnuMilYon05, Lemma 3.5]).

**Lemma 5.7.** Let w be covexillary, \((i_0, j_0) \in \mathcal{E}(w)\) with \(r_{i_0,j_0}^w > 0\). Then the transitioned permutation w’ defined by (5.1) has the following properties:

(i) w’ is covexillary;
(ii) \(\lambda(w) = \lambda(w')\);
(iii) \(\mathcal{E}(w') = (\mathcal{E}(w) \setminus \{(i_0, j_0)\}) \cup \{(i_0 - 1, j_0 - 1)\}\); in particular w and w’ have the same number of essential set boxes; and
(iv) \(r_{i_0-1,j_0-1}^{w'} = r_{i_0,j_0}^w - 1 \quad \text{and} \quad r_{\epsilon}^{w'} = r_{\epsilon}^w\) for the remaining (common) essential set boxes \(\epsilon\).

**Proof of Definition-Lemma 5.1:** We algorithmically construct the covexillary permutation \(\Theta_{w,w}\) with the stated essential set and rank conditions. (Once achieved, Proposition 4.1 implies that the permutation is unique, and hence \(\Theta_{e,w}\) is well defined.)

Attach to each essential box \(\epsilon\) the nonnegative integer 
\[ f(\epsilon) = r_{\epsilon}^w - r_{\epsilon}^w, \]
thought of as indicating how many steps the box $e$ should be moved in the southwest direction. Note that $f(e) \geq 0$ follows from Proposition 4.2. Let

$$k = \sum_{e \in E(w)} f(e).$$

We repeat the following process, which decreases exactly one of the $f(e)$ by 1, giving another (intermediate) covexillary permutation. We terminate when all the $f(e)$’s become 0 at which point we output $\Theta_{v,w}$.

Define

$$i_0 = \max\{i \mid (i,j) \text{ is an essential box and } f((i,j)) > 0\},$$

$$j_0 = \max\{j \mid (i_0,j) \text{ is an essential box and } f((i_0,j)) > 0\}.$$

Hence $(i_0,j_0)$ gives the coordinates of the northmost then eastmost essential set box that still needs to be moved (in particular, $r_{i_0,j_0}^w > 0$). Then set $w'$ to be the transitioned permutation for $w$.

By Lemma 5.7 $w'$ is covexillary, and $\mathcal{E}(w')$ and $\mathcal{E}(w)$ are the same except that $(i_0,j_0) \in \mathcal{E}(w)$ has now moved to $(i_0-1,j_0-1) \in \mathcal{E}(w')$. Attach to $(i_0-1,j_0-1)$ the integer $f(i_0,j_0) - 1$ and keep the attached integers unchanged for other essential boxes.

Repeat the algorithm for $w'$. In view of Lemma 5.7(iv) it follows that we can do this process $k$ steps. We obtain a permutation and name it $\Theta_{v,w}$. That $\Theta_{v,w}$ has the desired properties follows from the construction and inductively applying parts (i), (ii) and (iii) of Lemma 5.7.

**Example 5.8.** We now illustrate the algorithm described in the proof of Definition-Lemma 5.1 by computing $\Theta_{5123746,7531462} = 4635721$ in steps:

In Section 7.3 we will need two properties of $b(w)$, whose proofs also follow from Lemma 5.7.

**Lemma 5.9.** Suppose $w$ is covexillary and $\lambda = \lambda(w) = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$. Furthermore, set

$$\{i_1 < i_2 < \cdots < i_\ell\} = \{i \mid 1 \leq i \leq \ell, \lambda_i > \lambda_{i+1}\}.$$

Then

$$\mathcal{E}(w) = \{(b_i,\lambda_i - i + b_i) \mid i = i_1, \ldots, i_\ell\}.$$

In other words, there is a one to one bijection between $\mathcal{E}(w)$ and the righthand corners of $\lambda$ (drawn in French notation).
Proof. Consider a sequence
\[ w = w^{(0)} \mapsto w^{(1)} \mapsto \cdots \mapsto w^{(M)} \]
where \( w^{(i+1)} \) is the transitioned permutation of \( w^{(i)} \), and \( w^{(M)} \) has the property that the rank of each of its essential set boxes is 0 (this permutation is often known as “dominant”). The diagram of \( w^{(M)} \) is a Young diagram (drawn in French notation), with its southwest corner at \((1,1)\). By Lemma 6.1(ii), this Young diagram must be \( \lambda(w) \).

The essential set boxes of \( w^{(M)} \) are precisely boxes at the end of the rows \( \{i_1 < i_2 < \cdots < i_m\} \). By Lemma 5.7(iii), it follows that each such \( \epsilon' \in E(w^{(M)}) \) (say in row \( i \in \{i_1 < i_2 < \cdots < i_m\} \)) corresponds one to one with \( \epsilon \in E(w) \) that lives on the same southwest-northeast diagonal. However, by the definition of \( b_i \), \( \epsilon \) must coordinates \( (b_i, \lambda_i - i + b_i) \), since both are the extremal box of \( B(w) \) on the said diagonal. □

**Lemma 5.10.** Under the same assumptions as Lemma 5.4, we have
\[ b_i = \max(b_{i_k+1} - i_{k+1} + i, b_{i_k}) \]
(\( define \ b_i = 0 = 0 \).

Proof. To see this, consider the following picture:

![Figure 3. Proof of Lemma 5.10](image)

Here, (a) is the case when \( b_i = b_{i_k+1} - i_{k+1} + i \), and (b) is the case when \( b_i = b_{i_k} \). It remains to show that \( (b_i, \lambda_i - i + b_i) \) has to be either on the vertical boundary defined by \( (b_{i_k+1}, \lambda_{i_k+1} - i_{k+1} + b_{i_k+1}) \), which is case (a), or on horizontal boundary defined by \( (b_{i_k}, \lambda_{i_k} - i_k + b_{i_k}) \), which is case (b).

The only concern is if \( (b_i, \lambda_i - i + b_i) \) appears strictly east of the vertical boundary in case (a) or north of the horizontal boundary in case (b). However, this implies that \( E(w) \) contains a box not associated to a corner of \( \lambda \), which contradicts Lemma 5.9. □

6. Combinatorial Formulae for Multiplicity and Hilbert Series of \( \mathcal{O}_{e_v X_w} \)

We now arrive at our formulae for multiplicity of \( e_v \in X_w \) and the Hilbert series for \( \mathcal{O}_{e_v X_w} \) in the case \( w \) is co vexillary.

**Theorem 6.1.** \( \text{mult}_{e_v}(X_w) \) counts the number of flagged semistandard Young tableaux of shape \( \lambda(w) \) whose rows are bounded by \( b(\Theta_{v,w}) \).

Proof. This is immediate from Theorem 4.4, Theorem 5.5 and Theorem 3.1(V). □
The following result generalizes the determinantal formula (with the same proof) from [WooYon09] for cograssmannian permutations:

**Theorem 6.2.** We have the following expression for multiplicity as a determinant of a matrix with binomial coefficient entries:

$$\text{mult}_{\mathbf{e}_v}(X_w) = \det \left( \binom{b_i + \lambda_i - i + j}{\lambda_i - i + j} \right)_{1 \leq i,j \leq \ell(\lambda)},$$

where $\ell(\lambda)$ is the number of nonzero parts of $\lambda$ and $b = b(\Theta_{v,w})$.

The proof of Theorem 6.2 is immediate from Theorem 6.1 once we have discussed the determinantal expression for flagged Schur functions in Section 6.3.

**Example 6.3.** Continuing our example from the previous section, it follows from Example 5.3, Example 5.4 and Theorem 6.1 that the multiplicity of $X_{7531462}$ at $e_{5123746}$ is 2. To illustrate Theorem 6.2, note that since $\lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 1, b_1 = 1, b_2 = 3, b_3 = 3$, Theorem 6.2 asserts that the multiplicity

$$\text{mult}_{\mathbf{e}_v}(X_w) = \left| \begin{array}{ccc} b_1 + \lambda_1 - 1 & b_1 + \lambda_1 & b_1 + \lambda_1 + 1 \\ \lambda_1 & \lambda_1 + 1 & \lambda_1 + 2 \\ b_2 + \lambda_2 - 2 & b_2 + \lambda_2 - 1 & b_2 + \lambda_2 \\ b_2 & b_2 + 1 & b_2 + 2 \\ b_3 + \lambda_3 - 3 & b_3 + \lambda_3 - 2 & b_3 + \lambda_3 \\ \lambda_3 & \lambda_3 & \lambda_3 \\ \end{array} \right| = \left| \begin{array}{ccc} 4 & 5 & 6 \\ 3 & 4 & 5 \\ 1 & 2 & 3 \\ -1 & 0 & 1 \\ 0 & 1 & 3 \\ \end{array} \right| = 2,$$

in agreement with our previous computation.

**Example 6.4.** A. Woo [Woo04] proved that when $w = (n+2)23\ldots(n+1)1$, the multiplicity of the Schubert variety $X_w \subseteq \text{Flags}(\mathbb{C}^{n+2})$ at the most singular point $e_{12}$ is given by the Catalan number $C_n = \frac{1}{n+1}{2n \choose n}$. Moreover, he conjectured that the largest value multiplicity can attain for $v, w \in S_{n+2}$ is this Catalan number. Woo’s permutation is covexillary, and this multiplicity problem is also solved by Theorem 6.1.

A richer invariant than multiplicity is the Hilbert series of $O_{\mathbf{e}_v,X_w}$. In order to state our formula for it, recall the notion of flagged set-valued semistandard tableaux from [KimMilYon05]. A **set-valued, semistandard filling** of $\lambda$ [Buc02] is an assignment of non-empty sets to each box of $\lambda$ so that each entry of a box is weakly smaller than each entry to its right, and strictly smaller than any entry strictly below it. Such a filling is flagged by $b = b(w)$ if each entry in a row $i$ is at most $b_i$.

**Example 6.5.** Continuing Example 5.4, the additional flagged set-valued semistandard tableau for the flagging $b = b(\Theta_{v,w})$ that is not (ordinary) semistandard is

```
1 1 1 1
2 2, 3
3
```

Recall

$$T_{v,w} = \langle \hat{f}; \hat{f} \text{ is the lowest (standard) degree component of } f \in I_{v,w} \rangle$$

is the (homogeneous) ideal of the projectivized tangent cone of $N_{v,w}$.
Theorem 6.6. The $\mathbb{Z}$-graded Hilbert series of $\mathcal{O}_{v,X_{w}}$ and $\mathbb{C}[z^{(v)}/T_{v,w}$ are given respectively by

$$\text{Hilb}(\mathcal{O}_{v,X_{w}}, t) := \sum_{i \geq 0} \dim(m_{e_{i}}^{i}/m_{e_{i}}^{i+1})t^{i} = G_{\lambda}(t)/(1-t)\binom{t}{2}$$

and

$$\text{Hilb}([z^{(v)}/T_{v,w}, t) := \sum_{i \geq 0} \dim([z^{(v)}/T_{v,w}]_{i}t^{i} = G_{\lambda}(t)/(1-t)^{f(w,v)},$$

where

$$G_{\lambda}(t) = \sum_{k \geq |\lambda|} (-1)^{k-|\lambda|}(1-t)^{k} \times \#\text{SetSSYT}(\lambda, b, k)$$

and $\#\text{SetSSYT}(\lambda, b, k)$ equals the number of flagged set-valued semistandard Young tableaux of shape $\lambda$ with flag $b = b(\Theta_{v,w})$ and which uses exactly $k$ entries.

Remark 6.7. As with Theorem 3.1(VI) one can straightforwardly write down multigraded Hilbert series that takes into account both the standard grading and the usual action grading. We leave this as a remark since one requires a bunch of prerequisites about double Grothendieck polynomials (for covexillary permutations) from [KnuMilYon05, WooYon09] that we do not need otherwise in the text. □

Remark 6.8. A permutation $w$ is cograssmannian if it has a unique ascent, at position $d$, i.e., $w(k) < w(k+1)$ if and only if $k = d$. Each cograssmannian $w$ is clearly also covexillary. Moreover, there is a bijective correspondence between $\lambda \subseteq d \times (n-d)$ and these cograssmannian permutations. Under this correspondence, the multiplicity of a Grassmannian Schubert variety $X_{\lambda} \subseteq \text{Gr}(d, \mathbb{C}^{n})$ at a torus fixed point $e_{u}$ can be computed using Theorem 6.1. Geometrically, this follows from the fact that the natural “forgetting subspaces” projection $\pi: \text{Flags}(\mathbb{C}^{n}) \rightarrow \text{Gr}(d, \mathbb{C}^{n})$ restricts to a locally trivial fibration $X_{w} \rightarrow X_{\lambda}$ with fiber $P/B$ where $P$ is the maximal parabolic such that $G/P \cong \text{Gr}(d, \mathbb{C}^{n})$; see [Bri03, Example 1.2.3]. □

Remark 6.9. It is natural to wonder about relations between multiplicity and the Kazhdan-Lusztig polynomial, as might be seen by comparing our formulae with the Kazhdan-Lusztig polynomial formula for covexillary Schubert varieties [Las95]. Small computations contraindicate any simple comparisons.

7. Proofs of the main theorems

7.1. Covexillary Schubert determinantal ideals. Let $\prec_{\text{antidiag}}$ denote any term order that picks off the main antidiagonal (i.e., southwest to northeast main diagonal) term of any minor of $Z$. We will use the following result:

Theorem 7.1 ([KnuMilYon05]). Let $w \in S_{n}$ be covexillary. The essential determinants of $I_{w}$ form a Gröbner basis with respect to $\prec_{\text{antidiag}}$. Moreover, the initial ideal init$_{\prec_{\text{antidiag}}}I_{w}$ is reduced and equidimensional, with prime decomposition

$$\text{init}_{\prec_{\text{antidiag}}}I_{w} = \bigcap_{P} \langle z_{ij} : (i, j) \in P \rangle,$$

where $P$ is a pipe dream for $w$.

The Stanley-Reisner simplicial complex is a homeomorphic to a vertex decomposable (and hence shellable) ball or sphere.
The interior faces of the complex are labeled by set-valued semistandard Young tableaux of shape $\lambda$ and flagged by $b(w)$, and the facets are labeled by the subset of ordinary semistandard Young tableaux. The codimension $k$ interior faces are labeled by these tableaux with $|\lambda| + k$ entries.

As is explained in [KnuMilYon05], the irreducible components are in manifest bijection with pipe dreams for $w$: the appearance of a generator $z_{ij}$ indicates the position of $+$'s, using the usual coordinates consistent with our labeling of the generic matrix $Z$. Compare this with Theorem 7.5 where we instead express things in terms of the variables of $\tilde{Z}$.

Although our proof of Theorem 4.4 will use the Gröbner basis theorem of [KnuMilYon05] for Schubert determinantal ideals (recapitulated in Section 7.1), we remark that Theorem 4.4 actually provides a generalization. This is based on the fact that any Schubert determinantal ideal can be realized as a Kazhdan-Lusztig ideal, and this ideal is homogeneous with respect to the standard grading; see [WooYon09, Section 2.3].

### 7.2. Flagged Schur polynomials; proof of Theorem 6.2

The weight generating series for semistandard tableaux with row entries flagged (bounded) by a vector $b$ is called the flagged Schur polynomial. An application of a standard Gessel-Viennot type argument establishes that

\begin{equation}
\det(h_{\lambda_i-i+j}(x_1, \ldots, x_{b_i})) = \sum_{T \in T(\lambda, b)} x^{\text{wt}(T)},
\end{equation}

where $h_k(x_1, \ldots, x_m)$ is the complete homogeneous symmetric function on the variables $x_1, \ldots, x_m$ and the right-hand side of the equality is by definition the flagged Schur polynomial, where the sum runs over all semistandard tableau of shape $\lambda$ and flagged by $b$. See [Man01, Cor 2.6.3].

We are now ready to give our proofs of the determinantal expression for multiplicity:

**Proof of Theorem 6.2.** This is immediate from Theorem 6.1 combined with (7.1) evaluated at $x_i = 1$ for all $i$ and the fact $h_{\lambda_i-i+j}(1, 1, \ldots, 1) = \binom{b_i + \lambda_i - i - j - 1}{\lambda_i - i - j}$. \hfill $\square$

Now suppose $X = (x_i)_{i \in I}$ and $Y = (y_j)_{j \in J}$ are two finite families of indeterminates. Define polynomials $h_k(X - Y)$ by the power series expansion

\[
\sum_{k \in \mathbb{Z}} u^k h_k(X - Y) = \prod_{j \in J}(1 - uy_j)/\prod_{i \in I}(1 - ux_i).
\]

In the literature one finds the nomenclature flagged double Schur function, which is defined by the following determinantal expression [CLL02, Definition 4.1],

\begin{equation}
s_{\lambda, b}(X - Y) = \det \left( h_{\lambda_i-i+j}(X_{b_i} - Y_{\lambda_i+b_i-i}) \right)_{1 \leq i, j \leq t},
\end{equation}

where

\[X_{b_i} = (x_1, x_2, \ldots, x_{b_i}), Y_{\lambda_i+b_i-i} = (y_1, \ldots, y_{\lambda_i+b_i-i}).\]

There is also a tableau expression

\begin{equation}
s_{\lambda, b}(X - Y) = \sum_{T \in T(\lambda, b)} \prod_{\alpha \in T} \left( x_{T(\alpha)} - y_{T(\alpha) + C(\alpha)} \right),
\end{equation}

where $C(\alpha) = c - r$ if $\alpha$ is in the $r$-th row and $c$-th column.
Notice that by comparing the above formula with [KnuMilYon05, Theorem 5.8], the single (respectively, double) Schubert polynomial \( S_{w_0w}(X, Y) \) for a coveyillary \( w \) is the same as the single (respectively double) flagged Schur polynomial of shape \( \lambda(w) \) with flagging \( b(w) \).

7.3. Hilbert series and an identity of flagged Schur polynomials. We now use standard notions from combinatorial commutative algebra, found in the textbook [MilStu05].

Consider a polynomial ring \( S = \mathbb{C}[z_1, \ldots, z_m] \) with a grading such that \( z_i \) has some degree \( a_i \in \mathbb{Z}^N \). A finitely graded \( S \)-module \( M = \bigoplus_{v \in \mathbb{Z}^N} M_v \) has a free resolution

\[
E_\bullet : 0 \leftarrow E_1 \leftarrow E_2 \leftarrow \cdots \leftarrow E_L \leftarrow 0
\]

where \( E_i = \bigoplus_{j=1}^{\beta_i} S(-d_{ij}) \) is graded with the \( j \)-th summand of \( E_i \) generated in degree \( d_{ij} \in \mathbb{Z}^N \).

Then the (\( \mathbb{Z}^N \)-graded) \( K \)-polynomial of \( M \) is

\[
K(M, t) = \sum_j (-1)^j \sum_i t^{d_{ij}}.
\]

In any case where \( S \) is positively graded, meaning that the \( a_i \) generate a pointed cone in \( \mathbb{Z}^N \), \( K(M, t) \) is the numerator of the \( \mathbb{Z}^N \)-graded Hilbert series:

\[
\text{Hilb}(M, t) = K(M, t) \prod_i (1 - t^{a_i}).
\]

The multidegree \( C(M, t) \) is by definition the sum of the lowest degree terms of \( K(M, 1 - t) \). (This means we substitute \( 1 - t_k \) for \( t_k \) for all \( k, 1 < k < N \).) Also, if \( X = \text{Spec}(S/I) \) then let \( C(X, t) := C(S/I, t) \)

Proposition 7.2. Let \( w \) be coveyillary, \( \lambda = \lambda(w) \) and \( b = b(w) \). Set

\[
X = (t_{v(1)}, \ldots, t_{v(n)}), \quad Y = (t_n, \ldots, t_1)
\]

and \( s_{\lambda,b}(X - Y) \) be the associated flagged Schur function. Then the following equality holds:

\[
(7.4) \quad C(N_{v,w}, t_{ij} \mapsto t_{v(j)} - t_{n-i+1}) = (-1)^{|\lambda|} s_{\lambda,b}(Y - X).
\]

Proof.

\[
C(N_{v,w}, t_{ij} \mapsto t_{v(j)} - t_{n-i+1})
= S_{w_0w}(X, Y) \quad \text{(by [WooYon09, Theorem 4.5])}
= \sum_P \prod_{(i,j) \in P} (x_j - y_i) \quad \text{(summing over pipe dreams \( P \) for \( w \), by [KnuMilYon05])}
= (-1)^{|\lambda|} \sum_P \prod_{(i,j) \in P} (y_i - x_j) \quad \text{(summing over the same \( P \)'s as above)}
= (-1)^{|\lambda|} \sum_{T \in T(\lambda, b)} \prod_{\alpha \in T} (y_{T(\alpha)} - x_{T(\alpha) + C(\alpha)}) \quad \text{(under the correspondence of Section 5.2)}
= (-1)^{|\lambda|} s_{\lambda,b}(Y - X) \quad \text{(by the tableau formula (7.3)).}
\]
7.4. Conclusion of the proofs. Let
\[ I_{v,w} = \langle g_1, g_2, \ldots, g_N \rangle \]
where \( \{g_1, \ldots, g_N\} \) are the essential determinants. Also, let
\[ J_{v,w} = \langle \text{init}_{\prec_{v,w,\pi}} g_1, \text{init}_{\prec_{v,w,\pi}} g_2, \ldots, \text{init}_{\prec_{v,w,\pi}} g_N \rangle. \]
It is always true that
\[ J_{v,w} \subseteq \text{init}_{\prec_{v,w,\pi}} I_{v,w}. \]
Equality holds if and only if \( \{g_1, \ldots, g_N\} \) is a Gröbner basis with respect to \( \prec_{v,w,\pi} \). Let \( \tilde{Z} \) be the shuffled generic matrix determined by \( (v \leq w) \), as defined in Section 4.2. Let
\[ \tilde{I}_{\Theta_{v,w}} \subseteq \mathbb{C}[\tilde{Z}] \cong \mathbb{C}[z] \]
be the Schubert determinantal ideal as defined by taking sub-determinants of the shuffled matrix \( \tilde{Z} \), as determined by the rank matrix for \( \Theta_{v,w} \). Let \( \prec_{\text{antidiag}} \) denote a term order that picks off the (southwest to northeast) antidiagonal term of any sub-determinant of \( \tilde{Z} \). Using Theorem 7.1 we immediately conclude that under \( \prec_{\text{antidiag}} \), the essential (or defining) minors of \( \tilde{Z} \) (coming from the rank conditions for \( \Theta_{v,w} \)) are a Gröbner basis for \( \tilde{I}_{\Theta_{v,w}} \) and hence the lead terms generate \( \text{init}_{\prec_{\text{antidiag}}} \tilde{I}_{\Theta_{v,w}} \).

Lemma 7.3. The aforementioned generators of \( \text{init}_{\prec_{\text{antidiag}}} \tilde{I}_{\Theta_{v,w}} \) are a subset of the generators of \( J_{v,w} \).

Proof. Consider an essential determinant \( g \) of \( I_{v,w} \), which is associated to an \( r \times r \) minor \( M \) of the submatrix \( Z^{(v)}_\epsilon \) of \( Z^{(v)} \) associated to \( \epsilon \in \mathcal{E}(w) \); here \( r = r^w_\epsilon \). There are \( r^w_\epsilon \leq r \) many 1’s in \( Z^{(v)}_\epsilon \), by Proposition 4.2. Assume that the minor \( g \) uses all the rows and columns that these 1’s sit in. Note that since \( g \) is homogeneous with respect to the usual action grading, so by Theorem 3.11, \( \prec_{\text{antidiag}} \) will choose the terms of lowest total degree first, and so it will pick out all terms of the determinant that use all these 1’s in their product. Thus, by the definition of \( \prec_{\text{antidiag}} \) given in Section 4.2, the lead term will exactly be the antidiagonal term of the minor of a submatrix of \( \tilde{Z} \), which, when the rows and columns are permuted, is precisely the submatrix \( M^\circ \) of \( M \) that comes from striking out all the rows and columns of \( M \) having 1’s in them. Thus this lead term is a generator of \( \text{init}_{\prec_{\text{antidiag}}} \tilde{I}_{\Theta_{v,w}} \). This generator corresponds to \( \epsilon' \in \mathcal{E}(\Theta_{v,w}) \), as defined in Definition-Lemma 5.1. On the other hand, one can similarly see that all generators of \( \text{init}_{\prec_{\text{antidiag}}} \tilde{I}_{\Theta_{v,w}} \) can be realized in this manner. \( \square \)

Consider the natural projection
\[ \varphi : \mathbb{C}[z] \to \mathbb{C}[z^{(v)}] \]
that sends all variables not in \( z^{(v)} \) to 0. By Lemma 7.3, all generators of \( \text{init}_{\prec_{\text{antidiag}}} \tilde{I}_{\Theta_{v,w}} \) only use variables in \( \mathbb{C}[z^{(v)}] \) so it makes sense to define the ideal
\[ H_{v,w} = \varphi \left( \text{init}_{\prec_{\text{antidiag}}} \tilde{I}_{\Theta_{v,w}} \right) \mathbb{C}[z^{(v)}] \subseteq \mathbb{C}[z^{(v)}]. \]

Lemma 7.4. \( \text{Spec}(\mathbb{C}[z]/H_{v,w}) \) defines an equidimensional and reduced scheme.
Proof. As we have said, Theorem 7.1 implies \( \text{init}_{\prec} \tilde{I}_{\Theta_v,w} \) defines an equidimensional and reduced scheme. Since dividing out an irrelevant factor of affine space does not affect these properties, the claim holds.

Often, when proving equality of two homogeneous ideals \( A \subseteq B \) in a positively graded ring \( R \), one expects to show that the multigraded Hilbert series of \( R/A \) and \( R/B \) are equal. Fortunately, our arguments will only require equality of multidegrees, thanks to the following:

**Lemma 7.5** (Lemma 1.7.5 of [KnuMil05]). Let \( I' \subseteq \mathbb{k}[z_1, \ldots, z_m] \) be an ideal homogeneous for a positive \( \mathbb{Z}^d \)-grading. Suppose \( H \) is an equidimensional radical ideal contained inside \( I' \). If the zero schemes of \( I' \) and \( H \) have equal multidegrees, then \( I' = H \).

We will apply Lemma 7.5 in the case \( H = H_{v,w} \subseteq I' = \text{init}_{\prec} \tilde{I}_{\Theta_v,w} \).

**Proposition 7.6.** The multidegree of \( \mathcal{N}_{v,w} \) equals the multidegree of \( \mathbb{C}[z]/(\text{init}_{\prec} \tilde{I}_{\Theta_v,w}) \), each with respect to the usual action of \( T \subset \text{GL}_n \) (as defined in Section 2.4).

**Proof.** By Proposition 7.2 the multidegree of \( \mathcal{N}_{v,w} \) is

\[
C(\mathcal{N}_{v,w}, t_{ij} \mapsto t_{v[i]} - t_{n-i+1}) = (-1)^{|\lambda|} s_{\lambda,b}(Y - X),
\]

where \( b = b(w) \).

On the other hand, in [KnuMilYon05], it was proved that, for \( w \) covexillary, the multidegree of the matrix Schubert variety

\[ \Xi_w = \text{Spec}(\mathbb{C}[z]/I_w) \]

is the flagged Schur polynomial

\[ s_{\lambda,b}(Y - X) \] where \( \lambda = \lambda(w) \) and \( b = b(w) \).

However, this multidegree is with respect to the \( 2n \)-dimensional torus action where a vector

\[ (a_1, \ldots, a_n, a'_1, \ldots, a'_n) \in T \times T \]

acts by rescaling row \( i \) (from the top) of a matrix by \( a_i^{-1} \) and rescaling column \( i \) by \( a'_i \). On the other hand there is an embedding of tori

\[
T \hookrightarrow T \times T : (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n; a_{v[1]}, \ldots, a_{v[n]})
\]

that realizes the usual torus action as a subtorus of \( T \times T \). As is explained in [WooYon09], because of this embedding, one can compute the multidegree for \( \Xi_w \) under the usual torus action by the substitutions

\[
X = (x_1, \ldots, x_n) = (t_{v[1]}, \ldots, t_{v[n]}),
Y = (y_1, \ldots, y_n) = (t_{n}, \ldots, t_{1}).
\]

Let \( \rho = r_1 \cdots r_n \in S_n, \chi = c_1 \cdots c_n \in S_n \) be defined as in 4.2. Set

\[
X' = (x'_1, \ldots, x'_n) = (x_{c_1}, \ldots, x_{c_n}) = (t_{v(c_1)}, \ldots, t_{v(c_n)}),
Y' = (y'_1, \ldots, y'_n) = (y_{r_1}, \ldots, y_{r_n}) = (t_{n+1-r_1}, \ldots, t_{n+1-r_n}),
\]

\[ b'' = b(\Theta_{v,w}). \]

There is another embedding of tori

\[
T \times T \hookrightarrow T \times T : (a_1, \ldots, a_n, a'_1, \ldots, a'_n) \mapsto (a_{r_1}, \ldots, a_{r_n}, a'_{c_1}, \ldots, a'_{c_n})
\]
Composing the two tori embeddings \((7.6)\) and \((7.7)\) allows us to twist the usual action grading to one on \(\text{Fun}(\mathcal{Z}) \cong \mathbb{C}[z]\). Putting this together, the multidegree of the matrix Schubert variety \(\mathbb{C}[z]/(\text{init}_{\text{antidiag}} I_{\Theta_{v,w}})\) is
\[
C \left( \mathbb{C}[z]/(\text{init}_{\text{antidiag}} I_{\Theta_{v,w}}), t \right) = (-1)^{|B_{\lambda,b}'}(Y' - X'),
\]
with respect to the grading \(\text{deg}(z_{ij}) = t_{v(j)} - t_{n+1-i}\). Here, \(\lambda = \lambda(w) = \lambda(\Theta_{v,w})\), see Definition-Lemma 5.1.

In order to prove that the two multidegrees \((7.5)\) and \((7.8)\) are equal polynomials, we define an auxiliary flagging
\[
b' = (b'_1, \ldots, b'_\ell), \text{ where for } 1 \leq i \leq \ell, \ b'_i = b_i - k_i
\]
and
\[
k_i = \#(1's \text{ in } Z^{(v)}) \text{ that are dominated by } (b_i, \lambda_i - i + b_i),
\]
cf. Lemma 4.6. We will instead establish
\[
s_{\lambda,b}(Y - X) = s_{\lambda,b'}(Y' - X')
\]
and
\[
s_{\lambda,b'}(Y' - X') = s_{\lambda,b''}(Y' - X'),
\]
from which the equality follows.

We now prove \((7.9)\). By \((7.2)\), it is equivalent to prove
\[
\det \left( h_{\lambda_i-i+j}(Y_{b_i} - X_{\lambda_i+b_i-i}) \right)_{1 \leq i,j \leq \ell} = \det \left( h_{\lambda_i-i+j}(Y'_{b'_i} - X'_{\lambda_i+b'_i-i}) \right)_{1 \leq i,j \leq \ell},
\]
i.e.,
\[
\det \left( \left[ \begin{array}{c} u^{\lambda_i-i+j} \prod_{x \in X_{\lambda_i-i+b_i}} (1 - xu) \\ \prod_{y \in Y_{b_i}} (1 - yu) \end{array} \right] \right)_{1 \leq i,j \leq \ell} = \det \left( \left[ \begin{array}{c} u^{\lambda_i-i+j} \prod_{x \in X'_{\lambda_i-i+b'_i}} (1 - xu) \\ \prod_{y \in Y'_{b'_i}} (1 - yu) \end{array} \right] \right)_{1 \leq i,j \leq \ell}.
\]
In fact, more strongly we show that for every \(1 \leq i \leq \ell\),
\[
\prod_{x \in X_{\lambda_i-i+b_i}} (1 - xu) = \prod_{y \in Y_{b_i}} (1 - yu).
\]
The equality \((7.11)\) is proved as follows. We use the notation as in Section 4.2 Recall \((4.3)\); we now define
\[
A_i = \left\{ t_{v(\beta_i)} \right\}_{1 \leq j \leq k_i} \subset \{t_1, \ldots, t_n\}.
\]
By Lemma 4.6 we have the following equalities of subsets of \(\{t_1, \ldots, t_n\}\):
\[
Y_{b_i} = \{y_1, \ldots, y_{b_i}\}
\]
\[
= \{t_n, \ldots, t_{n+1-b_i}\}
\]
\[
= \{t_{n+1-t_1}, \ldots, t_{n+1-b_i'}\} \cup \{t_{n+1-a_i(1)}^{(1)}, \ldots, t_{n+1-a_i(1)}^{(1)}\}
\]
\[
= \{t_{n+1-t_1}, \ldots, t_{n+1-b_i'}\} \cup \{t_{v(\beta_i)}^{(1)}, \ldots, t_{v(\beta_i)}^{(1)}\}
\]
\[
= \{y'_1, \ldots, y'_{b_i}\} \cup A_i
\]
\[
= Y_{b_i'} \cup A_i.
\]
\[ X_{\lambda_i - i + b_i} = \{x_1, \ldots, x_{\lambda_i - i + b_i}\} \]
\[ = \{t_{v(1)}, \ldots, t_{v(\lambda_i - i + b_i)}\} \]
\[ = \{t_{v(c_1)}, \ldots, t_{v(c_{\lambda_i - i + b_i})}\} \cup \{t_{v(\beta_i^{(1)}), \ldots, t_{v(\beta_i^{(n_i)})}}\} \]
\[ = \{x'_1, \ldots, x'_{\lambda_i - i + b'_i}\} \cup A_i \]
\[ = X'_{\lambda_i - i + b'_i} \cup A_i, \]

Because of (7.12) and (7.13), we can cancel out the factors
\[ \left(1 - t_{v(\beta_i^{(1)})} \right), \text{ for } 1 \leq j \leq k_i, \]
which appear both in the numerator and in the denominator on the left-hand side of (7.11). After the cancellation, we obtain the the right-hand side of (7.11). This proves (7.11).

Next, we prove (7.10) using the tableau formula for flagged double Schur functions. By the discussion of Section 5.2, it suffices to show that the two sets of flagged semistandard Young tableaux of shape \( \lambda \), flagged by \( b' \) and \( b'' \) respectively, are the same.

Let
\[ \{i_1, i_2, \ldots, i_m\} = \{i \mid 1 \leq i \leq \ell, \lambda_i \geq \lambda_{i+1}\} \]

(assume \( \lambda_{i+1} = 0 \) and \( i_1 < i_2 < \cdots < i_m \)). In other words, this is the set of indices of the rows of the Young diagram of \( \lambda \) that have corners on their right ends.

We claim that \( b''_i \leq b'_i \), with equality when \( i = i_1, \ldots, i_m \). By Lemma 5.9 we have
\[ \mathcal{E}(w) = \{(b_i, \lambda_i - i + b_i) \mid i = i_1, \ldots, i_m\}, \]
and
\[ \mathcal{E}(\Theta_{v,w}) = \{(b''_i, \lambda_i - i + b''_i) \mid i = i_1, \ldots, i_m\}. \]

Therefore for \( i = i_1, \ldots, i_m \), we have, from the definition of \( b' = b - (k_1, k_2, \ldots, k_\ell) \) and \( b'' = b(\Theta_{v,w}) \) that
\[ b'_i = b_i - \text{(the number of 1's in } Z^{(v)} \text{ dominated by } (b_i, \lambda_i - i + b_i)) = b''_i. \]

On the other hand, for
\[ i \in \{1, \ldots, \ell\} \setminus \{i_1, \ldots, i_m\}, \]
let \( k \) be the index that
\[ 0 \leq k \leq m - 1 \text{ and } i_k < i < i_{k+1} \]
(declare \( i_0 = 0 \)).

By Lemma 5.10 we have
\[ b_i = \max(b_{i_{k+1}} - i_{k+1} + i, b_{i_k}) \]
(define \( b_{i_0} = 0 \), and
\[ b''_i = \max(b''_{i_{k+1}} - i_{k+1} + i, b''_{i_k}), \]
(where \( b''_{i_0} = 0 \)).

Therefore the claimed inequality
\[ b''_i \leq b'_i, \]
or
\[ b_i - b''_i \geq b_i - b'_i, \]

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is equivalent to

\[(7.15) \quad \max(b_{ik} - i_{k+1} + i, b_{ik}) - \max(b''_{ik} - i_{k+1} + i, b''_{ik}) \geq \{\text{number of } 1's \text{ in } Z^{(v)} \text{ dominated by } (b_t, \lambda_i - i + b_t)\}.\]

This can be checked in two cases.

Case (1): \(b''_{ik} - i_{k+1} + i \leq b''_{ik}.\)

In this case, the left-hand side of (7.15) is

\[b_t - b''_{ik} = (b_t - b_{ik}) + (b_{ik} - b''_{ik}) = (b_t - b_{ik}) + \#\{1's \text{ in } Z^{(v)} \text{ dominated by } (b_{ik}, \lambda_{ik} - i_k + b_{ik})\}\]

\[\geq \#\{1's \text{ in } Z^{(v)} \text{ dominated by } (b_t, \lambda_i - i + b_t)\}\]

where the second equality is by (7.14) and the last inequality is because, in the rows \(b_{ik} + 1, \ldots, b_t\), there are at most \((b_t - b_{ik})\) many 1’s in \(Z^{(v)}\) dominated by \((b_t, \lambda_i - i + b_t)\), and \(\lambda_i - i + b_t \leq \lambda_{ik} - i_k + b_{ik}\) since \(i_k < i\); see (4.4).

Case (2): \(b''_{ik} - i_{k+1} + i > b''_{ik}.\)

In this case, the left-hand side of (7.15) is

\[b_t - (b''_{ik} - i_{k+1} + i) = (b_{ik} - b''_{ik}) + (b_t - b_{ik} + i_{k+1} - i)\]

\[= \#\{1's \text{ in } Z^{(v)} \text{ dominated by } (b_{ik}, \lambda_{ik} - i_k + b_{ik})\} + (b_t - b_{ik} + i_{k+1} - i)\]

\[\geq \#\{1's \text{ in } Z^{(v)} \text{ dominated by } (b_t, \lambda_i - i + b_t)\}\]

where the second equality is by (7.14) and the last inequality is because, in the columns \((\lambda_{ik} - i_{k+1} + b_{ik} + 1), (\lambda_{ik} - i_{k+1} + b_{ik} + 2), \ldots, (\lambda_i - i + b_t)\), there are at most

\[(\lambda_i - i + b_t) - (\lambda_{ik} - i_{k+1} + b_{ik}) = (b_t - b_{ik} + i_{k+1} - i) + (\lambda_i - \lambda_{ik})\]

\[= b_t - b_{ik} + i_{k+1} - i\]

many 1’s in \(Z^{(v)}\) dominated by \((b_t, \lambda_i - i + b_t)\). (We have again applied (4.4).)

Lastly, we show that the flagging \(b'\) and \(b''\) give the same set of Young tableaux. In other words, we need to show that any semistandard Young tableau of shape \(\lambda\) flagged by \(b'\) is also flagged by \(b''\). Since \(b'_{ik} = b''_{ik}\) for all \(k\), it remains to consider \(i\) that satisfies \(i_k < i < i_{k+1}\) for some \(0 \leq k \leq m - 1\) (again define \(i_0 = 0\)). Since \(\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+1}\), the length of rows \(i, i+1, \ldots, i_{k+1}\) of the Young tableau are the same. Denote by \(\text{label}(i,j)\) the entry at the \(i\)-th row and \(j\)-th column of the Young tableau. Then by the definition of semi-standard Young tableaux,

\[\text{label}(i, \lambda_i) < \text{label}(i+1, \lambda_{i+1}) < \text{label}(i+2, \lambda_{i+2}) < \cdots < \text{label}(i_{k+1}, \lambda_{i_{k+1}}) \leq b''_{ik+1},\]

hence

\[\text{label}(i, \lambda_i) \leq b''_{ik+1} - (i_{k+1} - i) \leq \max(b''_{ik+1} - (i_{k+1} - i), b''_{ik}) = b''_i.\]

So the condition \(\text{label}(i, \lambda_i) \leq b'_i\) is equivalent to the condition \(\text{label}(i, \lambda_i) \leq \min(b'_i, b''_i) = b''_i.\) Therefore \(b'\) and \(b''\) give the same set of Young tableaux, as desired. \(\Box\)

**Proof of Theorem 4.4, Theorem 5.5 and Theorem 6.6.** We know that

\[H_{v,w} = I_{v,w} \subseteq \text{init}_{<v,w,\pi} I_{v,w}\]

so \(H_{v,w}\) is an equidimensional, radical ideal contained inside \(\text{init}_{<v,w,\pi} I_{v,w}\) by Lemmas 7.3 and 7.4 The Hilbert series and hence multidegrees of \(\mathbb{C}[z^{(v)}]/\text{init}_{<v,w,\pi} I_{v,w}\) and \(\mathbb{C}[z^{(v)}]/I_{v,w}\)
are equal by Theorem 3.1 IV). On the other hand, these multidegrees are equal to the multidegree of \( \mathbb{C}[z]/H_{v,w} \) by Proposition 7.6 and the fact that multidegrees are unaffected by crossing the scheme by affine space. Hence

\[ H_{v,w} = J_{v,w} = \text{init}_{v,w,\pi} I_{v,w} \]

by Lemma 7.5. This proves the Theorem 4.4.

Moreover, since \( H_{v,w} = \text{init}_{v,w,\pi} I_{v,w} \) and since \( \text{Spec} (\mathbb{C}[z^{(v)}]/H_{v,w}) \) and \( \text{init}_{\text{antidiag}} \Theta_{v,w} \) only differ by crossing by affine space it follows the prime decomposition of \( \text{init}_{v,w,\pi} I_{v,w} \) lifts to a prime decomposition of \( \text{init}_{\text{antidiag}} \Theta_{v,w} \) and so Theorem 5.5 follows from the prime decomposition theorem of [KnuMilYon05] (taking into account the permutation of coordinates).

Finally, Theorem 6.6 follows from Theorem 3.1 VI, Theorem 7.1 and the discussion above. \( \Box \)

8. Conjectures and final remarks

We now present some conjectures that complement Theorem 3.1.

**Conjecture 8.1.** For some \( \pi \), \( \text{init}_{v,w,\pi} I_{v,w} \) defines a reduced and equidimensional scheme, i.e., the hypothesis and hence conclusion of Theorem 3.1 (V) holds.

Specifically, consider the SE-NW shuffling \( \pi \) that orders the variables by reading columns right to left and bottom to top. Based on the results of [KnuMil05] and [WooYon09] as well as some computation (exhaustively for \( n \leq 6 \), as well as many random examples for \( n \leq 10 \)), we believe that this choice always satisfies Conjecture 8.1.

However, we actually desire a choice of \( \pi \) that, in some sense, gives the neatest combinatorics; the reducedness and equidimensionality offered by Conjecture 8.1 merely provides a necessary criterion. Let us call \( \pi \) **generalized antidiagonal** if, after some permutation of the rows, and separately, some permutation of the columns, of \( Z \), then \( \pi \) induces a pure lexicographic ordering on \( \mathbb{C}[z] \) that favors the antidiagonal (southwest-northeast) term of any sub-determinant of the shuffled generic matrix \( \tilde{Z} \). We can extend this definition to shufflings \( \pi \) for the variables of \( Z^{(v)} \) in the obvious way. Now call \( \prec_{v,w,\pi} \) generalized antidiagonal if \( \pi \) is for \( Z^{(v)} \). The **SW-NE shuffling** \( \pi \) that orders variables by reading rows bottom to top and left to right induces such a term order. Also, the same is true for \( \pi \). However, our main motivation for these definitions comes from the fact that the term order of Section 4.2 is also generalized antidiagonal. On the other hand, we have:

**Example 8.2.** In general, not all choices of generalized antidiagonal \( \pi \) satisfy Conjecture 8.1. In particular, if we consider the Schubert determinantal ideal for \( w = 563412 \) using \( \pi \), the limit scheme is not reduced. (As we have said (cf. Section 2.3), Schubert determinantal ideals are special cases of Kazhdan-Lusztig ideals.) However, it is reduced if one uses \( \pi \) (either by direct computation, or by the Gröbner basis theorem of [KnuMil05]). \( \Box \)

Notwithstanding Example 8.2, we expect that among the generalized antidiagonal \( \pi \)’s, there exists a choice that not only satisfies Conjecture 8.1 but whose \( \pi \)-shuffled tableaux exhibit “good” combinatorial features. This assertion is consistent with our covexillary work, as well as the results of [KnuMil05, KnuMilYon05, WooYon09].
Problem 8.3. Find a Gröbner basis with squarefree initial terms for $I_{v,w}$, with respect to (any of) the orders $\prec_{v,w,\pi}$ satisfying Conjecture [8.7].

One cannot always use the defining (or essential) minors of $I_{v,w}$:

Example 8.4. Consider $w = 45231$ and $v = 23451$. Then the defining minors give

$$I_{v,w} = \left\langle z_{11}, \begin{array}{ccc}
  z_{31} & 1 & 0 \\
  z_{21} & z_{22} & 1 \\
  z_{11} & z_{12} & z_{13}
\end{array} \right\rangle = \langle z_{11}, z_{11} - z_{13}z_{21} - z_{12}z_{31} + z_{13}z_{22}z_{31} \rangle.$$

If they formed a Gröbner basis with respect to some $\prec_{v,w,\pi}$, we would have $\text{init}_{\prec_{v,w,\pi}} I_{v,w} = \langle z_{11} \rangle$. However, $-z_{13}z_{21} - z_{12}z_{31} + z_{13}z_{22}z_{31}$ is in $I_{v,w}$ and its initial term is $-z_{13}z_{21} - z_{12}z_{31}$ or $z_{13}z_{22}z_{31}$, none of which are in the ideal $\langle z_{11} \rangle$. So $\text{init}_{\prec_{v,w,\pi}} I_{v,w} \supsetneq \langle z_{11} \rangle$, therefore the two stated generators do not form a Gröbner basis.

If $V$ is a vertex of a simplicial complex $\Delta$ one can speak of the deletion and the link:

$$\text{del}_V(\Delta) = \{F \in \Delta : V \notin F\}, \quad \text{link}_V(\Delta) = \{F \in \text{del}_V(\Delta) : \{V\} \cup F \in \Delta\},$$

as well as the star of $V$, which is the cone of the link from $V$: $\text{star}_V(\Delta) = \{F \in \Delta : \{V\} \cup F \in \Delta\}$. One has the vertex decomposition $\Delta = \text{star}_V(\Delta) \cup \text{del}_V(\Delta)$. L. Billera and S. Provan [BilPro79] defined what it means for $\Delta$ to be vertex decomposable. By definition, every simplex is vertex decomposable, and in general, $\Delta$ is vertex decomposable if and only if it is pure and have a vertex decomposition where the deletion and link are vertex decomposable. They proved that a vertex decomposable complex is shellable (and therefore Cohen-Macaulay). The conjecture below gives one possible feature of a “good” choice of $\pi$:

Conjecture 8.5. There exists a choice of $\pi$ among those satisfying Conjecture [8.7] such that the Stanley-Reisner simplicial complex $\Delta_{v,w}$ associated to $N_{v,w}$ is homeomorphic to a vertex decomposable, and hence shellable, ball or sphere (in particular the hypothesis of Theorem [3.7](VI) holds).

Our faith in Conjecture [8.5] is mainly based on the results of [KnuMil05, KnuMilYon05, WooYon09], and our covexillary results. We also have some limited experimental evidence for the conjecture. We computationally checked implications of the conjecture (Cohen-Macaulayness and connectedness of $\Delta_{v,w,\pi}$, whether $\Delta_{v,w,\pi}$ has the homology of a ball/sphere, nonnegative h-vector, and that each codimension one face is contained in at most two facets), using the shufflings $\pi_{\prec}$ and $\pi_{\succ}$. We computed these exhaustively for $n \leq 5$ and for the majority of $n \leq 6$ (where already the computational demands are high), as well as some larger cases.

Problem 8.6. For which $\pi$ does the conclusion of Conjecture [8.5] hold?

Example 8.7. Even for choices of $\pi$ such that Conjecture [8.1] holds, Conjecture [8.5] is not always satisfied. Looking at the implication “each codimension one face is contained in at most two facets”, if we utilize $\pi_{\prec}$, this holds for $n = 6$ in all cases except when $w = 563412$, and $v = 123456, 123546, 132456$ or $132546$, whence $\Delta_{v,w,\pi_{\prec}}$ cannot always be a ball. On the other hand, if one chooses $\pi_{\succ}$, the implication is satisfied on these examples, but not on others, say $w = 563412$, $v = 123546$. \(\square\)

Our covexillary results also motivate the next two problems, which indicate successive refinements of Conjecture [8.5].
Problem 8.8. When does there exist a choice of $\pi$ such that the Stanley-Reisner complex $\Delta_{v,w}$ associated to $\text{init}_{v,w,\pi} N_{v,w}$ is homeomorphic to a subword complex $\Delta(Q, \sigma)$, as introduced by [KnuMil04]? We refer the reader to [KnuMil04] for the definition of subword complexes, and where it was established that they are vertex decomposable and homeomorphic to balls/spheres. The facets are indexed by subwords of the fixed word $Q = (i_1, i_2, \ldots, i_M)$ of length $\ell(\sigma)$ such that $s_{i_1} \cdots s_{i_\ell} = \sigma$ where $s_k = (k \leftrightarrow k + 1)$ is a simple reflection in a symmetric group. Thus, one hopes for a combinatorial recipe $(v, w) \mapsto (Q, \sigma)$ that solves the multiplicity problem.

Problem 8.9. When does there exist a choice of $\pi$ such that the $\text{init}_{v,w,\pi} N_{v,w}$ is equal, after crossing by an appropriate affine space and permutation of coordinates, to the limit of a matrix Schubert variety, or another Kazhdan-Lusztig variety, under the Gröbner degeneration of [KnuMil05], [KnuMilYon05], [Knu08] and/or [WooYon09]? Since the Stanley-Reisner complexes of the stated Gröbner limits are subword complexes, Problem 8.8 is solved by Problem 8.9.

Finally, our convexillary results suggest an affirmative answer to:

Problem 8.10. Is multiplicity of $e_v \in X_w$ (and/or the Hilbert series of $O_{e_v,X_w}$) independent of characteristic?

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