Quantum Hall Droplets on Disc and Effective Wess-Zumino-Witten Action for Edge States

Mohammed Daoud\textsuperscript{a} and Ahmed Jellal\textsuperscript{b}

\textsuperscript{a} Max Planck Institute for The Physics of Complex Systems, Nöthnitzer Str. 38, D-01187 Dresden, Germany
\textsuperscript{b} Theoretical Physics Group, Laboratory of Condensed Matter Physics, Faculty of Sciences, Chouaïb Doukkali University, P.O. Box 4056, 24000 El Jadida, Morocco

Abstract

We algebraically analyze the quantum Hall effect of a system of particles living on the disc $B^1$ in the presence of an uniform magnetic field $B$. For this, we identify the non-compact disc with the coset space $SU(1,1)/U(1)$. This allows us to use the geometric quantization in order to get the wavefunctions as the Wigner $D$-functions satisfying a suitable constraint. We show that the corresponding Hamiltonian coincides with the Maass Laplacian. Restricting to the lowest Landau level, we introduce the noncommutative geometry through the star product. Also we discuss the state density behavior as well as the excitation potential of the quantum Hall droplet. We show that the edge excitations are described by an effective Wess-Zumino-Witten action for a strong magnetic field and discuss their nature. We finally show that LLL wavefunctions are intelligent states.

\textsuperscript{1} Permanent address : Physics Department, Faculty of Sciences, University Ibn Zohr, Agadir, Morocco.
e-mail : m_daoud@hotmail.com
\textsuperscript{2} e-mail : ajellal@ictp.it – jellal@ucd.ac.ma
1 Introduction

Quantum Hall effect (QHE) [1] has been realized on different two-dimensional manifolds. For instance, in 1983, Haldane [2] proposed an approach to overcome the symmetry problem that brought by the Laughlin theory [3] for the fractional QHE at the filling factor $\nu = \frac{1}{m}$ with $m$ is odd integer. By considering particles living on two-sphere $\mathbb{CP}^1$ in a magnetic monopole, Haldane formulated a theory that possess all symmetries and generalizes the Laughlin proposal. Very recently, Karabali and Nair [4] elaborated an elegant algebraic analysis that supports the Haldane statement and gives a more general results.

Karabali and Nair [4] analyzed the Landau problem on the complex projective space $\mathbb{CP}^k$ from a theory group point of view. This analysis is based on the fact that $\mathbb{CP}^k$ can be seen as the coset space $SU(k+1)/U(k)$. More precisely, an Hamiltonian has been written in terms of the $SU(k+1)$ generators and its spectrum has been given. Also a link to the effective Wess-Zumino-Witten (WZW) action for the edge states is well established. This work has been done on the compact manifolds and in particular two-sphere $\mathbb{CP}^1$. It is natural to ask, can we do the same analysis on non-compact manifolds like the ball $B^k$? The present paper will partially answer the last question and the general case will be examined separately [5]. Our motivation is based on [4] and the analytic method used by one [6] of the authors to deal with QHE on $B^k$.

We algebraically investigate a system of particles living on the disc $B^1$ in the presence of an uniform magnetic field $B$. After realizing $B^1$ as the coset space $SU(1,1)/U(1)$, we construct the wavefunctions as the Wigner $D$-functions verifying a suitable constraint. The corresponding Hamiltonian $H$ can be written in terms of the $SU(1,1)$ generators. This will be used to define $H$ as a second order differential operator, in the complex coordinates, which coincides with the Maass Laplacian [7] on $B^1$, and get the energy levels. We introduce an excitation potential to remove the degeneracy of the ground state. For this, we consider a potential expressed in terms of the $SU(1,1)$ left actions. For a strong magnetic field, we show that the excitations of the lowest landau level (LLL) are governed by an effective Wess-Zumino-Witten (WZW) action. It turn out that this action coincides with one-chiral bosonic action for QHE at the filling factor $\nu = 1$ [11, 12]. We show that the field describing the edge excitations is a superposition of oscillating on the boundary $S^1$ of the quantum Hall droplet. Finally, we discuss the squeezing property of $SU(1,1)$ raising and lowering operators in the lowest Landau levels and we show that the squeezing disappear in presence of high magnetic field.

The present paper is organized as follows. In section 2, we present a group theory approach to analysis the Landau problem on the disc. We build the wavefunctions and give the corresponding Hamiltonian as well as its energy levels. In section 3, we restrict our attention to LLL to write down the corresponding star product and defining the relevant density matrix. Also we consider the excitation potential and get the associate symbol to examine the excited states. We determine the effective WZW action for the edge states for a strong magnetic field and discuss the nature of the edge states in section 4. Section 5 is devoted to analyzing the squeezing property of $SU(1,1)$
Weyl generators in LLL. For this, we show that the Robertson-Schrödinger uncertainty relation [13] is minimized and according to the literature [14] LLL wavefunctions are intelligent. For a large magnetic background, one recovers the Heisenberg uncertainty relation and the LLL states reduces to harmonic oscillator coherent states. We conclude and give some perspectives in the last section.

2 Landau problem on the disc

We analyze the basic features of a particle living on the disc \( B^1 \) in the presence of an uniform magnetic field \( B \). To do this, we realize the disc as the coset space \( SU(1,1)/U(1) \) and write down the appropriate Hamiltonian in terms of the Casimir operator corresponding to \( SU(1,1) \). We show that the wavefunctions in LLL, which are obtained to be the Wigner \( D \)-functions, can be seen as the coherent states of \( SU(1,1) \). These materials and related matters will be clarified in the present section.

2.1 Wavefunctions

The disc is a two-dimensional non-compact surface \( B^1 = \{ z \in \mathbb{C}, \bar{z} \cdot z < 1 \} \). The manifold \( B^1 \) can be viewed as the coset space \( SU(1,1)/U(1) \) generated by the \( g \) elements, which are 2 \( \times \) 2 matrices of a fundamental representation of the group \( SU(1,1) \). They satisfy the relation

\[
\det g = 1, \quad \eta g^\dagger \eta = g^{-1}
\]  

(1)

where \( \eta = \text{diag}(1,-1) \). An adequate parametrization can be written as

\[
g = \begin{pmatrix} \bar{u}_2 & u_1 \\ \bar{u}_1 & u_2 \end{pmatrix}
\]

(2)

where \( u_1 \) and \( u_2 \) are the global coordinates of \( B^1 \), such as

\[
u_1 = \frac{z}{\sqrt{1 - \bar{z} \cdot z}}, \quad u_2 = \frac{1}{\sqrt{1 - \bar{z} \cdot z}}.
\]

(3)

To generate the gauge potential, we introduce the Maurer-Cartan one-form \( g^{-1}dg \). A straightforward calculation gives

\[
g^{-1}dg = -i \, t_+ \, e_+ \, dz - i \, t_- \, e_- \, d\bar{z} - 2i \, \theta \, t_3
\]

(4)

where the one-orthonormal forms \( e_+ \) and \( e_- \) are

\[
e_+ = -\frac{i}{1 - \bar{z} \cdot z}, \quad e_- = \frac{i}{1 - \bar{z} \cdot z}
\]

(5)

and the \( U(1) \) symplectic one-form, i.e. the \( U(1) \) connection, is

\[
\theta = i \, \text{Tr} \left( t_3 g^{-1}dg \right) = \frac{i}{2} \, \frac{\bar{z} \cdot dz - z \cdot d\bar{z}}{1 - \bar{z} \cdot z}.
\]

(6)

In (4), \( t_+, t_- \) and \( t_3 \) are the \( SU(1,1) \) generators in the fundamental representation. They can be written in terms of the matrices \( (E_{ij})_{kl} = \delta_{ik}\delta_{jl} \) of the algebra \( gl(2) \) as

\[
t_+ = -E_{12}, \quad t_- = E_{21}, \quad t_3 = \frac{1}{2} (E_{11} - E_{22}).
\]

(7)
It will be clear later that (4) will be used to define the covariant derivatives in order to get a diagonalized Hamiltonian describing the quantum system.

With the above realization, the disc is equipped with the Kähler-Bergman metric
\[ ds^2 = \frac{1}{(1 - \bar{z} \cdot z)^2} dz \cdot d\bar{z} \]  
(8)
as well as a symplectic closed two-form
\[ \omega = \frac{i}{(1 - \bar{z} \cdot z)^2} dz \wedge d\bar{z}. \]  
(9)
One can check that the two-form is related to \( \theta \) as \( \omega = d\theta \). Further, \( \theta \) will be linked to the gauge potential of the magnetic field. The Poisson bracket on \( \mathbb{B}^1 \) is given by
\[ \{ f_1, f_2 \} = i(1 - \bar{z} \cdot z)^2 \left( \frac{\partial f_1}{\partial z} \frac{\partial f_2}{\partial \bar{z}} - \frac{\partial f_1}{\partial \bar{z}} \frac{\partial f_2}{\partial z} \right) \]  
(10)
where \( f_1 \) and \( f_2 \) are function on \( SU(1,1) \). They can be expanded as
\[ f(g) = \sum f_k^m D_{m',m}^k(g) \]  
(11)
where the Wigner \( D \)-functions \( D_{m',m}^k(g) \) on \( SU(1,1) \) are
\[ D_{m',m}^k(g) = \langle k, m' | g | k, m \rangle. \]  
(12)
k is labeling the discrete \( SU(1,1) \) irreducible representation. We choose \( k \) to be integer because we are interested only to the discrete part of the Landau system on the disk. We denote the positive discrete representation of \( SU(1,1) \) by \( D_k^+ \) for \( 2k \in \mathbb{N} \) and \( k > \frac{1}{2} \). For a given \( k \), the representation space is spanned by the basis \( \{| k, m \rangle, m \in \mathbb{N} \} \). The \( SU(1,1) \) generators
\[ [t_3, t_\pm] = \pm t_\pm, \quad [t_-, t_+] = 2t_3 \]  
(13)
act on the vectors basis as
\[ t_\pm |k,m\rangle = \sqrt{\left(m + \frac{1}{2} \pm \frac{1}{2}\right)} \left(2k + m - \frac{1}{2} \pm \frac{1}{2}\right) |k,m\pm 1\rangle, \quad t_3 |k,m\rangle = (k + m) |k,m\rangle. \]  
(14)
The associated second order Casimir operator is given by
\[ C_2 = t_3^2 - \frac{1}{2} (t_- t_+ + t_+ t_-) \]  
(15)
and its eigenvalue is \( k(k - 1) \).

To characterize the admissible (physical) states in analyzing the Landau problem on \( \mathbb{B}^1 \), we introduce the generators of the right \( R_a \) and left \( L_a \) translations of \( g \)
\[ R_a g = g t_a, \quad L_a g = t_a g \]  
(16)
where \( a \) runs for \( +, -, 3 \). They act on the Wigner \( D \)-functions as
\[ R_a D_{m',m}^k(g) = D_{m',m}^k(g t_a), \quad L_a D_{m',m}^k(g) = D_{m',m}^k(t_a g). \]  
(17)
To obtain the Hilbert space corresponding to the quantum system living on $B^1$, we should reduce the degrees of freedom on the manifold $SU(1, 1)$ to the coset space $SU(1, 1)/U(1)$. It will be clear soon that this reduction can be formulated in terms of a suitable constraint on the Wigner $D$-functions. Note that, the present system is submitted to the magnetic strength

$$F = dA$$

(18)

where the $U(1)$ gauge field potential is given by

$$A = n \frac{i}{2} \frac{\bar{z} \cdot dz - z \cdot d\bar{z}}{1 - \bar{z} \cdot z}. $$

(19)

where $n$ is a real number. It is obvious that $A$ is proportional to the one-form $\theta$, i.e. $A = n\theta$. Since we have a closed two-form $\omega = d\theta$, the components of the magnetic field expressed in terms of the frame fields defined by the metric are constants. Hereafter, we set

$$n = \frac{B}{2}. $$

(20)

The suitable constraint on the Wigner $D$-functions can be established by considering the $U(1)$ gauge transformation

$$g \rightarrow gh = g \exp(it\phi)$$

(21)

where $\phi$ is the $U(1)$ parameter. (21) leads to the transformation in the gauge field

$$A \rightarrow A + nd\phi. $$

(22)

It follows that the functions (12) transform as

$$D^{k}_{m', m}(gh) = \exp\left(\int \dot{A}dt\right)D^{k}_{m', m}(g) = \exp\left(\frac{n}{2}\phi\right)D^{k}_{m', m}(g).$$

(23)

Therefore, the canonical momentum corresponding to the $U(1)$ direction is $n/2$. Thus, an admissible quantum states $\psi \equiv D^{k}_{m', m}(g)$ must satisfy the constraint

$$R_{3}\psi = \frac{n}{2}\psi. $$

(24)

Equivalently, we have

$$[R_{-}, R_{+}] = n. $$

(25)

This relation is very interesting in many respects. Indeed, the operators $R_{+}$ and $R_{-}$ can be seen as the creation and annihilation operators in analogy with the standard harmonic oscillator involved in the Landau problem on the plane. A similar result was obtained by Karabali and Nair [4] in dealing with the same problem on $\mathbb{C}P^1$. Furthermore, (25) is suggestive to make contact with the noncommutative geometry through the star product. Finally, the physical states constrained by (24) are the Wigner $D$-functions $D^{k}_{m', m}(g)$ with the condition

$$m = \frac{n}{2} - k. $$

(26)
The Lowest Landau condition is

\[ R_- \mathcal{D}^k_{m',m}(g) = 0 \]  \(27\)

which corresponds to \( m = 0 \), i.e. \( k = \frac{n}{2} \). This shows that the index \( k \), labeling the SU(1, 1) irreducible representations, is related to the magnetic field by \( k = \frac{B}{4} \). In the geometric quantization language \(27\) is called the polarization condition. This is an holomorphicity condition, which means that the LLL wavefunctions \( \mathcal{D}^k_{m',0}(g) \)

\[ \psi_{LLL} = \mathcal{D}^k_{m',0}(g) = \langle k, m'|g|k, 0 \rangle \]  \(28\)

are holomorphic in the \( z \) coordinate. More precisely, in the fundamental representation, we can define \( g \) in terms of the generators \( t_\pm \) by

\[ g = \exp(\eta t_+ - t_- \bar{\eta}) \]  \(29\)

where \( \eta \) is related to the local coordinates via

\[ z = \frac{\eta}{|\eta|} \tanh |\eta|. \]  \(30\)

Using \(14\), we end up with the required wavefunctions

\[ \psi_{LLL}(\bar{z}, z) = (1 - \bar{z} \cdot z)^{\frac{n}{2}} \sqrt{\frac{(m' + n - 1)!}{m'! (n - 1)!}} z^{m'}, \quad m' \in \mathbb{N}. \]  \(31\)

Note that, LLL is infinitely degenerated and \( \psi_{LLL} \) are nothing but the SU(1, 1) coherent states. They constitute an over-complete set with respect to the measure

\[ d\mu(\bar{z}, z) = \frac{n - 1}{\pi} \frac{d^2z}{(1 - \bar{z} \cdot z)^2}. \]  \(32\)

The orthogonality relation writes as

\[ \int d\mu(\bar{z}, z) \mathcal{D}^{*k}_{m',0}(g) \mathcal{D}^k_{m',0}(g) = \delta_{m',m'}. \]  \(33\)

Recall that the LLL wavefunctions of a particle living on two-sphere coincide with the SU(2) coherent states \(4\). Note also that for the Landau problem on the plane, the LLL vectors are given by the harmonic oscillator coherent states \(12\).

At this level, it is natural to look for the energy levels corresponding to the wavefunctions \( \mathcal{D}^k_{m',\frac{n}{2}-k}(g) \). These can be obtained by defining the relevant Hamiltonian that describes the quantum system living on the coset space SU(1, 1)/U(1).

### 2.2 Hamiltonian and energy levels

To derive the appropriate Hamiltonian, we start by noting that from above and more precisely relation \(25\), \( R_+ \) and \( R_- \) can be seen, respectively, as raising and lowering operators. This is in analogy with the creation and annihilation operators corresponding to the standard harmonic oscillator. Therefore, the Hamiltonian, describing a system of charged particle living on the disc in the presence of a background field, can be written as

\[ H = \frac{1}{2} (R_- R_+ + R_+ R_-). \]  \(34\)
To write this Hamiltonian in terms of the complex coordinates $z$ and $\bar{z}$, we introduce the $U(1)$ covariant derivatives. These can be obtained from (4), such as

$$D_z = \frac{\partial}{\partial z} - iA_z, \quad D_{\bar{z}} = \frac{\partial}{\partial \bar{z}} - iA_{\bar{z}}$$

(35)

where the components of the gauge potential have the forms

$$A_z = \frac{in}{2} \frac{\bar{z}}{1 - \bar{z}z}, \quad A_{\bar{z}} = -\frac{in}{2} \frac{z}{1 - \bar{z}z}.$$  

(36)

Using (4) and (16), we can map the raising and lowering operators $R_{\pm}$ in terms of $D_z$ and $D_{\bar{z}}$ as

$$R_{+} = -(1 - \bar{z} \cdot z) D_z, \quad R_{-} = (1 - \bar{z} \cdot z) D_{\bar{z}}.$$  

(37)

With these relations, we finally end up with the second order differential realization of the required Hamiltonian. This is

$$H = -(1 - \bar{z} \cdot z) \left\{ (1 - \bar{z} \cdot z) \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} + \frac{n}{2} \left( \bar{z} \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right) \right\} + \frac{n^2}{4} \bar{z} \cdot z.$$  

(38)

This exactly coincides with the Maass Laplacian [7]. It has been investigated at many occasions and generalized to the higher dimensional spaces [8].

To establish a relation between the Casimir operator and the Hamiltonian, we may write the eigenvalue equation as

$$H \psi = \frac{1}{2} (t_- t_+ + t_+ t_-) \psi = E \psi.$$  

(39)

Since the wavefunctions $\psi$ are the Wigner $D$-functions $D_{m', \frac{1}{2} - m}(g)$, the Landau energies are given in terms of $C_2$

$$E = \frac{n^2}{4} - C_2$$

(40)

which gives

$$E_m = \frac{n}{2} (2m + 1) - m (m + 1).$$  

(41)

This is similar to that obtained in [6] and references therein. It is clear from (39) that the eigenvalues of $H$ must be positive. This implies the constraint $0 \leq m < \frac{n-1}{2}$ and therefore we have a finite number of Landau levels, each level is infinitely degenerated. $E_m$ can be compared to one particle spectrum on the sphere [4]

$$E_{m, \text{sphere}} = \frac{n}{2} (2m + 1) + m (m + 1), \quad m \in \mathbb{N}$$

(42)

where the degeneracy of the Landau levels is finite. Note that, for large $n$ we get the Landau spectrum on the Euclidean surface

$$E_{m, \text{plane}} = \frac{n}{2} (2m + 1), \quad m \in \mathbb{N}$$

(43)

showing that the landau levels are infinitely degenerated. The energy levels are indexing by the integer $m$ and for $m = 0$ we obtain the LLL energy

$$E_0 = \frac{n}{2} = \frac{B}{4}.$$  

(44)

This coincides with the ground state of the same problem on the plane. It will be investigated carefully to make contact with QHE at LLL on the coset space $SU(1,1)/U(1)$. 

7
3 Lowest Landau level analysis

For our end, we establish some relevant ingredients. These are the star product, density matrix and excitation potential. We will see how these will play a crucial role in determining the effective WZW action describing the edge states for a strong magnetic field and discussing the nature of the edge excitations.

3.1 Star product

To derive the effective action describing the edge excitations, we will replace the commutators of two operators by a non-commutative Moyal bracket. It coincides with the Poisson bracket for a strong magnetic field. Recall that for large $n$ ($B \sim n$), the particles are constrained to be confined in LLL described by (31).

To define the star product in terms of our language, we start by noting that for any operator $A$ acting on $\psi_{LLL}$, we can associate the function

$$A(\bar{z}, z) = \langle z | \hat{A} | z \rangle = \langle 0 | g^\dagger A g | 0 \rangle$$

where $g$ is given by (29) and $| 0 \rangle \equiv | k, 0 \rangle$ is the lowest highest weight state of the discrete $SU(1,1)$ representation. The vector states $| z \rangle = g | 0 \rangle$ are the $SU(1,1)$ coherent states

$$| z \rangle = (1 - \bar{z} \cdot z)^\frac{n}{2} \sum_{m' = 0}^{\infty} \sqrt{(m' + n - 1)! \over m'!(n - 1)!} z^{m'} | k, m' \rangle.$$  \hspace{1cm} (46)

Note that $\psi_{LLL}$ is nothing but the projection of $| z \rangle$ on the state $| k, m' \rangle$

$$\psi_{LLL} = \langle k, m' | z \rangle.$$  \hspace{1cm} (47)

These can be used to define an associative star product of two functions $\mathcal{A}(\bar{z}, z)$ and $\mathcal{B}(\bar{z}, z)$ by

$$\mathcal{A}(\bar{z}, z) \star \mathcal{B}(\bar{z}, z) = \langle z | AB | z \rangle.$$  \hspace{1cm} (48)

With the help of the unitary condition of $g$, i.e. $g^\dagger g = 1$, and the completeness relation

$$\sum_{m = 0}^{\infty} | k, m \rangle \langle k, m | = 1$$

we are able to write

$$\mathcal{A}(\bar{z}, z) \star \mathcal{B}(\bar{z}, z) = \sum_{m = 0}^{\infty} \langle 0 | g^\dagger A g | k, m \rangle \langle k, m | g^\dagger B g | 0 \rangle.$$  \hspace{1cm} (50)

Using (14) and at large $n$, we show that the star product is

$$\mathcal{A}(\bar{z}, z) \star \mathcal{B}(\bar{z}, z) = \mathcal{A}(\bar{z}, z) \mathcal{B}(\bar{z}, z) + {1 \over n} \langle 0 | g^\dagger A g t_+ | 0 \rangle \langle 0 | t_- g^\dagger B g | 0 \rangle + O \left( \frac{1}{n^2} \right).$$  \hspace{1cm} (51)

It is clear that the first term in h.r.s. is the ordinary product of two functions $\mathcal{A}$ and $\mathcal{B}$. While, the non-commutativity is encoded in the second term. This is interesting result, because it will play an important role when we construct the effective action for the edge states.
To obtain the final form of the star product \( (51) \), it is necessary to evaluate the matrix element of type
\[
\langle 0 | g^\dagger A g | 0 \rangle.
\] (52)

To do this task, we can use the coherent states \( (46) \) to write the holomorphicity condition as
\[
R_- \langle k, m | g | 0 \rangle = 0.
\] (53)

Thus, we have
\[
\langle 0 | g^\dagger A g | 0 \rangle = R_+ \langle 0 | g^\dagger A g | 0 \rangle.
\] (54)

Similarly, we obtain
\[
\langle 0 | t_+ g^\dagger B g | 0 \rangle = -R_- \langle 0 | g^\dagger B g | 0 \rangle
\] (55)

where we have used the condition \( R^*_+ = -R_- \). From the above equations and since we are concerned with a \( U(1) \) abelian gauge field, we show that the star product \( (51) \) becomes
\[
\mathcal{A}(\bar{z}, z) \star \mathcal{B}(\bar{z}, z) = \mathcal{A}(\bar{z}, z) \mathcal{B}(\bar{z}, z) - \frac{1}{n} (1 - \bar{z} z)^2 \partial_z \mathcal{A}(\bar{z}, z) \partial_{\bar{z}} \mathcal{B}(\bar{z}, z) + O \left( \frac{1}{n^2} \right).
\] (56)

Therefore, the symbol or function associated to the commutator of two operators \( \mathcal{A} \) and \( \mathcal{B} \) can be written as
\[
\langle z | [\mathcal{A}, \mathcal{B}] | z \rangle = -\frac{1}{n} (1 - \bar{z} z)^2 \{ \partial_z \mathcal{A}(\bar{z}, z) \partial_{\bar{z}} \mathcal{B}(\bar{z}, z) - \partial_{\bar{z}} \mathcal{B}(\bar{z}, z) \partial_z \mathcal{A}(\bar{z}, z) \}.
\] (57)

This implies
\[
\langle z | [\mathcal{A}, \mathcal{B}] | z \rangle = \frac{i}{n} \{ \mathcal{A}(\bar{z}, z), \mathcal{B}(\bar{z}, z) \} \equiv \{ \mathcal{A}(\bar{z}, z), \mathcal{B}(\bar{z}, z) \}_*,
\] (58)

where \( \{, \}_* \) stands for the Poisson bracket \( (10) \) on the disc and \( \{, \}_* \) for the Moyal bracket defined by
\[
\{ \mathcal{A}(\bar{z}, z), \mathcal{B}(\bar{z}, z) \}_* = \mathcal{A}(\bar{z}, z) \star \mathcal{B}(\bar{z}, z) - \mathcal{B}(\bar{z}, z) \star \mathcal{A}(\bar{z}, z).
\] (59)

The advantage of the obtained star product will be seen in the construction of the effective WZW action describing the edge excitations of the the quantum Hall droplet.

### 3.2 Density matrix

Another important ingredient that should be investigated is the density matrix. Note that, in the disc, LLL are infinitely degenerated and one may fill the LLL states with \( M \) particles, \( M \) very large. The corresponding density operator is
\[
\rho_0 = \sum_{m=0}^{M} |k, m \rangle \langle k, m |.
\] (60)

The associated symbol can be written as
\[
\rho_0(\bar{z}, z) = (1 - \bar{z} \cdot z)^n \sum_{m=0}^{M} \frac{(n - 1 + m)!}{(n - 1)!m!} (\bar{z} \cdot z)^{2m}.
\] (61)
To analyze the behavior of $\rho_0(\vec{z}, z)$ for a strong magnetic field, we note that the normalization factor can be expanded as

$$(1 - \vec{z} \cdot z)^{-n} = \sum_{m=0}^{\infty} \frac{(n - 1 + m)!}{(n - 1)!m!} (\vec{z} \cdot z)^{2m}$$

(62)

which gives for large $n$

$$(1 - \vec{z} \cdot z)^{n} = \exp(-n\vec{z} \cdot z).$$

(63)

On the other hand, one can see that the second term occurring in the expression of $\rho_0(\vec{z}, z)$ behaves for large $n$ as the following series

$$\sum_{m=0}^{M} \frac{\vec{z} \cdot z)^{m}}{m!}.$$  

(64)

Combining all, we obtain an approximated density for large $n$

$$\rho_0(\vec{z}, z) \simeq \exp(-n\vec{z} \cdot z) \sum_{m=0}^{M} \frac{(\vec{z} \cdot z)^{m}}{m!} \simeq \Theta(M - n\vec{z} \cdot z).$$

(65)

This expression is valid for a large number $M$ of particles [9]. The mean value of the density operator, in LLL, is a step function for $n \to \infty$ and $M \to \infty$ ($\frac{M}{n}$ fixed). It corresponds to an abelian droplet configuration with boundary defined by

$$n\vec{z} \cdot z = M$$

(66)

and its radius is proportional to $\sqrt{M}$. Furthermore, the derivative of the density $\rho_0(\vec{z}, z)$ tends to a $\delta$-function. This property will be useful in the description of the edge excitations.

### 3.3 Excitation potential

Once we determined the spectrum of LLL where the quantum Hall droplet is specified by the density matrix $\rho_0$, one may ask about the excited states. The answer can be given by describing the excitations in terms of an unitary time evolution operator $U$. It contains information concerning the dynamics of the excitations around $\rho_0$. Therefore the excited states will be characterized by a density operator given by

$$\rho = U\rho_0 U^\dagger.$$ 

(67)

This is basically corresponding to a perturbation of the quantum system. Its relevant Hamiltonian can be written as

$$\mathcal{H} = E_0 + V$$

(68)

where $E_0 = \frac{\hbar^2}{2}$ is the LLL energy and $V$ is the excitation potential. This perturbation will induces a lifting of the LLL degeneracy. Note that, the $SU(1,1)$ left actions commute with the covariant derivatives. They correspond to the magnetic translations on the disc and lead to degeneracy of the Landau levels. Thus, it is natural to assume that $V$ as a function of the magnetic translations $L_3, L_+$ and $L_-$. A simple choice for the potential $V$ is

$$V = \omega \left( L_3 - \frac{n}{2} \right).$$

(69)
The symbol associated to this potential is given by
\[ V(z, z) = \langle z | V | z \rangle = n \omega \frac{\vec{z} \cdot \vec{z}}{1 - \vec{z} \cdot \vec{z}}. \] (70)

It goes essentially to the harmonic oscillator potential for a strong magnetic field. One now can verify that the spectrum of (68) is
\[ \mathcal{H} \psi_{LLL} \equiv \mathcal{H} D^{k}_{\ell \ell',0}(g) = (E_0 + \omega m') D^{k}_{\ell \ell',0}(g). \] (71)
This shows that we have a lifting of the LLL degeneracy.

4 Wess-Zumino-Witten action for edge states

The analysis developed in the previous section is useful to derive the effective WZW action for the edge states. Recall that for a strong magnetic field, the particles are confined in LLL. The required action will basically describe the behavior of the quantum system on LLL.

4.1 Effective action

As mentioned above, the dynamical information related to the degrees of freedom of the edge states, is contained in the unitary operator \( U \) (67). The action, describing these excitations, in the Hartree-Fock approximation, can be written as [10]
\[ S = \int dt \, \text{Tr} \left( i \rho_0 U^\dagger \partial_t U - \rho_0 U^\dagger \mathcal{H} U \right) \] (72)
where \( \mathcal{H} \) is given by (68). For a strong magnetic field, i.e. large \( n \), the different quantities occurring in the action can be evaluated as classical functions. To do this, we adopt a method similar to that used in [4]. This is mainly based on the strategy elaborated by Sakita [10] in dealing with a bosonized theory for fermions.

To determine the effective action, we start by calculating the first term in r.h.s. of (72). This can be done by setting
\[ U = e^{+i\Phi}, \quad \Phi^\dagger = \Phi. \] (73)
This suggests to write \( dU \) as
\[ dU = \sum_{n=1}^{\infty} (i)^n \frac{n!}{n!} \sum_{p=0}^{n-1} \Phi^p \, d\Phi \, \Phi^{n-1-p} \] (74)
as well as the operator \( U^\dagger dU \)
\[ U^\dagger dU = i \int_0^1 d\alpha \, e^{-i\alpha \Phi} \, d\Phi \, e^{+i\alpha \Phi}. \] (75)
This leads to the relation
\[ e^{-i\Phi} \partial_t e^{+i\Phi} = i \int_0^1 d\alpha \, e^{-i\alpha \Phi} \, \partial_t \Phi \, e^{+i\alpha \Phi}. \] (76)
We show that the first term in r.h.s. of (72) is
\[
\int dt \, \text{Tr} \left( \rho_0 U^\dagger \partial_t U \right) = -\sum_{n=0}^{\infty} \frac{(i)^n}{(n+1)!} \text{Tr} \left( \Phi \cdots [\Phi, \rho_0] \cdots ] \partial_t \Phi \right).
\]
(77)

Due to the completeness of LLL, the trace of any operator \( A \) is defined by
\[
\text{Tr} \, A = \int d\mu(z, \bar{z}) \, \langle z | A | z \rangle
\]
(78)
where the measure \( d\mu(z, \bar{z}) \) is given by (32). It follows that (77) can be written as
\[
i \int dt \, \text{Tr} \left( \rho_0 U^\dagger \partial_t U \right) = -\int d\mu(z, \bar{z}) \sum_{n=0}^{\infty} \frac{(i)^n}{(n+1)!} \{ \phi, \cdots \{ \phi, \rho_0 \} \cdots \} \ast \partial_t \phi
\]
(79)
with \( \phi = \langle z | \Phi | z \rangle \). This form is more suggestive for our purpose. Indeed, using the relations (56-58), it is easy to see that (79) rewrites as
\[
i \int dt \, \text{Tr} \left( \rho_0 U^\dagger \partial_t U \right) \approx -\frac{1}{2} \int d\mu(z, \bar{z}) \{ \rho_0, \phi \} \ast \partial_t \phi
\]
(80)
where we have dropped the terms in \( \frac{1}{n^2} \) as well as the total time derivative. The Poisson bracket can be calculated to get
\[
\{ \phi, \rho_0 \} = (L \phi) \frac{\partial \rho_0}{\partial (z \cdot z)}
\]
(81)
and the first order differential operator \( L \) is given by
\[
L = i \left( 1 - \bar{z} \cdot z \right)^2 \left( z \cdot \frac{\partial}{\partial z} - \bar{z} \cdot \frac{\partial}{\partial \bar{z}} \right).
\]
(82)
This is the angular momenta mapped in terms of the local coordinates of the disc. Recall that, for large \( n \), the density (65) is a step function. Its derivative is a \( \delta \)-function with a support on the boundary \( \partial \mathcal{D} = S^1 \) of the quantum Hall droplet \( \mathcal{D} \) defined by (66). By setting \( z = re^{i\alpha} \), we show that (82) reduces to \( L = \partial_{\alpha} \) for large \( n \). Therefore, the equation (82) takes the form
\[
i \int dt \, \text{Tr} \left( \rho_0 U^\dagger \partial_t U \right) \approx -\frac{1}{2} \int_{S^1 \times \mathbb{R}^+} dt \, (\partial_{\alpha} \phi) (\partial_t \phi).
\]
(83)
To achieve the derivation of edge states action, it remains to evaluate the second term in r.h.s. of (72). By a straightforward calculation, we find
\[
\text{Tr} \left( \rho_0 U^\dagger V U \right) = \text{Tr} (\rho_0 V) + i \text{Tr} ([\rho_0, V] \Phi) + \frac{1}{2} \text{Tr} ([\rho_0, \Phi] [V, \Phi]).
\]
(84)
The first term in r.h.s of (84) is \( \Phi \)-independent. We can drop it because does not contain any information about the dynamics of the edge excitations. While, the second term can be written in term of the Moyal bracket as
\[
i \text{Tr} ([\rho_0, V] \Phi) \approx i \int d\mu(\bar{z}, z) \, \{ \rho_0, V \} \ast \phi.
\]
(85)
Using (70), one can see that
\[
i \text{Tr} ([\rho_0, V] \Phi) \rightarrow 0
\]
(86)
The last term in r.h.s of (84) can be evaluated in a similar way to get (81). Therefore, adding different terms, we obtain
\[
\int dt \ Tr \left( \rho_0 U^\dagger H U \right) = -\frac{1}{2n^2} \int d\mu(\bar{z}, z) \ (L\phi) \ \frac{\partial \rho_0}{\partial (\bar{z} \cdot z)} \ (L\phi) \ \frac{\partial V}{\partial (\bar{z} \cdot z)}. \tag{87}
\]
Note that, we have eliminated a term containing the ground state energy \(E_0\), because does not contribute to the edge dynamics. For large \(n\), from (70), we notice that
\[
\frac{\partial V}{\partial (\bar{z} \cdot z)} \to n\omega. \tag{88}
\]
Using the spatial shape of density \(\rho_0\), we finally obtain
\[
\int dt \ Tr(\rho_0 U^\dagger H U) = \frac{\omega}{2} \int_{S^1 \times \mathbb{R}^+} dt \ (\partial_\alpha \phi)^2. \tag{89}
\]
Combining (83) and (89), we find the appropriate effective action
\[
S \approx -\frac{1}{2} \int_{S^1 \times \mathbb{R}^+} dt \ \left\{ (\partial_\alpha \phi) (\partial_t \phi) + \omega (\partial_\alpha \phi)^2 \right\}. \tag{90}
\]
This action is actually describing the edge excitations of the quantum Hall droplet. It involves only the time derivative \(\partial_t \phi\) and the tangential derivative \(\partial_\alpha \phi\). The action (90) coincides with the well-known one-dimensional chiral bosonic action describing the edge excitations for QHE at the filling factor \(\nu = 1\) \cite{11, 12}. This is one of the most important results derived in the present paper.

4.2 Nature of edge excitations

Starting from the action (90), we discuss the nature of the edge excitations. This can be done by solving the equation of motion for the field \(\phi\)
\[
\partial_\alpha (\partial_t \phi + \omega \partial_\alpha \phi) = 0. \tag{91}
\]
The general solutions are
\[
\phi(\alpha, t) = \phi_1(\alpha - \omega t) + \phi_2(t), \tag{92}
\]
which look like the right-moving waves, but in addition there is a hidden gauge symmetry encoded in the term \(\phi_2(t)\). It corresponds to the invariance of the action (90) under the change
\[
\phi \to \phi + \lambda(t). \tag{93}
\]
This takes its origin from the invariance under the \(U(1)\) transformation, such as
\[
U \to \exp[i\lambda(t)] U. \tag{94}
\]
As far as the coset space \(SU(1,1)/U(1)\) is concerned, \(\phi_2(t)\) does not represent any physical degree of freedom. It can be removed by imposing the gauge constraint
\[
(\partial_t + \omega \partial_\alpha) \phi = 0. \tag{95}
\]
Since the edge excitations action (90) is defined on the boundary $S^1$, we also impose the boundary condition

$$\phi(2\pi, t) - \phi(0, t) = -2\pi q$$  \hspace{1cm} (96)$$

and $q$ is a time independent constant. The general form of the field $\phi(\alpha, t)$ is then given by

$$\phi(\alpha - \omega t) = p - q(\alpha - \omega t) + i \sum_{n \neq 0}^{\infty} \frac{\alpha_n}{n} e^{in(\alpha - \omega t)}$$  \hspace{1cm} (97)$$

where $p$ is the canonical momentum and we have set $\alpha_{-n} = \alpha^*_n$. The canonical momentum corresponding to the field $\phi$ is

$$\pi(\alpha, t) = q + \sum_{n \neq 0}^{\infty} \alpha_n e^{in(\alpha - \omega t)}.$$  \hspace{1cm} (98)$$

The quantization of the theory can be performed by imposing the equal time canonical commutation relation, such as

$$[\pi(\alpha, t), \phi(\alpha', t)] = i\delta(\alpha - \alpha').$$  \hspace{1cm} (99)$$

This implies that $p$, $q$ and $\alpha_n$ must satisfy the algebra

$$[\alpha_n, \alpha_m] = \delta_{n+m,0}, \quad [q, p] = i$$  \hspace{1cm} (100)$$

and other commutators vanish. This algebra is describing an infinite set of uncoupled oscillators. It shows that the field $\phi$ is a superposition of oscillating modes on the boundary $S^1$. In other words, the edge excitations constitute the low-lying excitations about the incompressible Hall droplet. In this case, the Hilbert space is the product of the oscillator Fock spaces. The Hamiltonian of the edge excitations is given, with an appropriate normal ordering, by

$$H_e = \frac{1}{4\pi} \int_{S^1} : (\partial_\alpha \phi)^2 :.$$  \hspace{1cm} (101)$$

It can also be written as

$$H_e = \frac{1}{2} \alpha_0^2 + \sum_{n>0} \alpha_n \alpha_{-n}$$  \hspace{1cm} (102)$$

where we have $\alpha_0 = q$. At this level, it is interesting to note that the charge operators $L_0 = H_e$ and

$$L_n = \frac{1}{4\pi} \int_0^{2\pi} : (\partial_\alpha \phi)^2 : e^{-in(\alpha - \omega t)} = \frac{1}{2} \sum_{l=-\infty}^{+\infty} \alpha_{n-l} \alpha_n, \quad n \neq 0$$  \hspace{1cm} (103)$$

satisfy the Virasoro algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{1}{12} (n^3 - n) \delta_{n+m,0}$$  \hspace{1cm} (104)$$

of central charge $c = 1$. This show that the dynamics of the edge excitations are governed by $(1 + 1)$ conformal field theory. This link were studied in [15, 16] in order to describe the excitations on the boundary of the quantum Hall droplets. Although, the result obtained in [16] concerned the disc geometry, it is mainly based on the analysis of the LLL wavefunctions of a non-relativistic particle
living on the plane, which are the harmonic oscillator coherent states. In our case, the relation between
the edge dynamics and the non-relativistic particles on $B^1$ in LLL is shown up by deriving the WZW
action. This derivation based on the $SU(1,1)$ coherent states and the notion of star product to evaluate
the transition amplitude under the confining potential $V$ given by (69).

The space, generated by the zero modes, plies an important role in QHE, especially when the
filling factor is fractional. Indeed, the edge states for the fractional Hall effect at the filling factor
$\nu = \frac{1}{2m+1}$, with $m$ integer, can be described by the obtained WZW action. It can be simply obtained
by substituting

$$p \rightarrow \frac{1}{\sqrt{2m+1}}p, \quad q \rightarrow \sqrt{2m+1}q$$

in the expression of $\phi(\alpha - \omega t)$ (97). The corresponding Hamiltonian takes the form

$$H_e = \frac{1}{2}(2m+1)\alpha_0^2 + \sum_{n>0} \alpha_n \alpha_{-n}. \quad (106)$$

This is exactly the Hamiltonian analyzed by Wen [15] in describing the edge excitations. It is inter-
esting to note that, in a strong magnetic field, (106) provides a description of a system of anyons,
whose statistical parameter is $2m$ [17]. It follows that the obtained WZW action can also be used to
describe particles with intermediate statistics. This is essentially due to the fact that QHE as well as
anyon systems in two-dimensions are involved the $(2+1)$ Chern-Simons interaction.

5 Uncertainty relation in LLL

We show that the LLL wavefunctions are intelligent, i.e. they minimize the Robertson-Schrödinger
(RS) uncertainty relation [13], for instance see also [14]. We also show that the correlation between the
left $SU(1,1)$ Weyl generators vanishes in for a strong magnetic field and the RS uncertainty relation
reduces to Heisenberg one. In this case, the LLL wavefunctions behave like those corresponding to the
standard harmonic oscillators and the underlying dynamical algebra reduces to the Weyl-Heisenberg
one. The latter provides us with a realization of fuzzy disc in the LLL. It seems that the fact of
approximating the algebra of functions on the disc by a finite dimensional matrix model is related to
the absence of correlation between the raising and lowering operators on LLL.

To show that the the states (46) saturate the RS uncertainty relation, we start by evaluating the
mean values of the generators $L_{\pm} = L_1 \pm iL_2$ and $L_3$. These are

$$\langle L_- \rangle = \langle L_+ \rangle = n \frac{z}{1 - \bar{z}z}, \quad \langle L_3 \rangle = \frac{n}{2} + \mathcal{V}(\bar{z}, z) \quad (107)$$

where $\mathcal{V}(\bar{z}, z)$ is the function associated to excitation potential given by (70). It follows that the
dispersion of the generators $L_j$ writes as

$$\sigma_j^2 = \langle L_j^2 \rangle - \langle L_j \rangle^2 = \frac{n}{4} \frac{|1 - (-)^j z^2|^2}{(1 - \bar{z}z)^2}, \quad j = 1, 2. \quad (108)$$

The correlation of $L_1$ and $L_2$ is given by

$$\sigma_{12} = \frac{1}{2} \langle L_1L_2 + L_2L_1 \rangle - \langle L_1 \rangle \langle L_2 \rangle = i \frac{n}{4} \frac{z^2 - \bar{z}^2}{(1 - \bar{z}z)^2} \quad (109)$$
It is now easy to check that the LLL wavefunctions minimize the RS uncertainty relation

\[ \sigma_1^2 \sigma_2^2 = \frac{1}{4} \langle L_3 \rangle^2 + \sigma_{12}^2 \]  

(110)

and therefore they are intelligent. For a large magnetic field, this relation reduces to Heisenberg one. We verify that the correlation

\[ \sigma_{12} \sim O \left( \frac{1}{n} \right) \]  

(111)

vanishes for \( n \) large and the RS uncertainty relation gives the Heisenberg one. This is

\[ \sigma_1^2 \sigma_2^2 \sim \frac{n}{4}. \]  

(112)

It is clear that the absence of correlation between lowering and raising operators leads to the Heisenberg uncertainty relation. Note that, one can verify that a similar result holds for the Landau system on two-sphere in the presence of a strong magnetic field. Furthermore, in this limit, one can obtain a two-dimensional non-commutative plane and think about the fuzzy spaces. It seems that there is a hidden relation between the absence of the quantum correlations and the fuzzy structures. In the case under consideration, LLL provides us with a realization of the so-called fuzzy disc \[18\]. Indeed, the relation (112) suggests that the operators \( L_+ \) and \( L_- \) can be represented as the harmonic oscillator creation and annihilation operators. According to \[18\], a fuzzy disc can be defined by some adequate projection on LLL. The projection operator in \[18\] coincides with the density operator given by (60). The algebra of functions on \( \mathbf{B}^1 \) reduces to a non-commutative subalgebra, which is isomorphic to the algebra of \( M \times M \) matrices. The parameter of non-commutativity is proportional to the strength of the magnetic field. The fuzzy disc is endowed with the Voros star product as expected since for a strong magnetic field the coherent states (46) go to those for the harmonic oscillator and are eigenstates of destruction operator.

6 Conclusion

The quantum mechanics of a charged particles living on the disc \( \mathbf{B}^1 \) is analyzed from group theory point of view. This is achieved by realizing the disc as the non-compact coset space \( SU(1,1)/U(1) \). This realization makes the derivation of the Landau levels and the corresponding wavefunctions obvious. The wavefunctions are identified to the Wigner \( D \)-functions \( D_{m',m}^k(g) \) with the condition \( m = \frac{r}{2} - k \). The index \( k \), labeling the unitary irreducible representation of the group \( SU(1,1) \), is related to the strength of the magnetic field. It is remarkable that the LLL wavefunctions coincide with the \( SU(1,1) \) coherent states. The spectrum of the Landau problem on the disc is generated from the \( SU(1,1) \) Casimir operator.

Restricting to LLL, we have derived the effective WZW action that describes the quantum Hall droplet of radius proportional to \( \sqrt{M} \), with \( M \) is the number of particles in LLL. To obtain the action of the boundary excitations, we have defined the star product and density of states. Also we have introduced the perturbation potential responsible of the degeneracy lifting in terms of the \( L_3 \).
left generator of $SU(1,1)$. We have analyzed the nature of the edge excitations. Finally, we have shown that the LLL wavefunctions minimize the RS uncertainty relation and for a strong magnetic field reduce to the harmonic oscillators coherent states. As by product one can define the fuzzy space equipped with the non-commutative Voros star product which, emerges in this case since the coherent state is eigenvector of the annihilation operator.

Of course still some questions to be answered. It is natural to ask about the nature of edge excitations in higher dimensional spaces. On the other hand, is there some way to deal with QHE on the flag spaces. These different issues are under consideration.

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