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RECURRENCE RATE IN RAPIDLY MIXING DYNAMICAL SYSTEMS

BENOIT SAUSSOL

ABSTRACT. For measure preserving dynamical systems on metric spaces we study the time
needed by a typical orbit to return back close to its starting point. We prove that when the
decay of correlation is super-polynomial the recurrence rates and the pointwise dimensions are
equal. This gives a broad class of systems for which the recurrence rate equals the Hausdorff
dimension of the invariant measure.

1. INTRODUCTION

1.1. Decay of correlations. Let \((X, f, \mu)\) be a measure preserving dynamical system. Recall
that the system is said to be mixing if for any functions \(\varphi, \psi\) in \(L^2\) the covariance
\[
\text{Cov}(\varphi \circ f^n, \psi) := \int \varphi \circ f^n \psi d\mu - \int \varphi d\mu \int \psi d\mu \to 0 \quad \text{as } n \to \infty.
\]
The decay of the correlation function is, in general, arbitrarily slow. The notion of rapid
mixing needs a little more structure.

Assume that \(X\) is a metric space with metric \(d\), and consider the space \(\text{Lip}(X)\) of real Lipschitz
functions on \(X\). For many dynamical systems an upper bound for (1) of the form \(\|\varphi\|\|\psi\| \theta_n\) has
been computed, where \(\theta_n \to 0\) with some rate, and \(\|\cdot\|\) is a norm on a space of functions with
some regularity. Without loss of generality we are considering in this paper the rate of decay of
correlations for Lipschitz observables\(^1\).

A broad class of systems enjoy exponential decay of correlations. The main result of the paper
(Theorem 3) applies to systems with super-polynomial decay of correlation. This includes for
example Axiom A systems with equilibrium states, hyperbolic systems with singularities with
their SBR measures such as those considered by Chernov in [2], many systems with a Young
tower \([10, 17]\), expanding maps with singularities such as in \([4]\), some non-uniformly expanding
maps \([7]\), etc. The main reference for these questions is certainly the book by Baladi [2]. The
reader will also find in the review by Luzzatto [11] an exposition of the recent methods for
non-uniformly expanding systems and an extensive bibliography on this active field.

1.2. Recurrence rate and dimensions. The return time of a point \(x \in X\) under the map \(f\)
in its \(r\)-neighborhood is
\[
\tau_r(x) = \inf \{n \geq 1 : d(f^n x, x) < r\}.
\]
We are interested in the behavior as \(r \to 0\) of the return time. We define the recurrence rate as the limits
\[
\underline{R}(x) = \liminf_{r \to 0} \frac{\log \tau_r(x)}{\log(1/r)} \quad \text{and} \quad \overline{R}(x) = \limsup_{r \to 0} \frac{\log \tau_r(x)}{\log(1/r)}.
\]

\(^1\)For example an immediate approximation argument allows easily to go from Holder or class \(C^k\) to Lipschitz.
Whenever \( R(x) = \overline{R}(x) \) we denote by \( R(x) \) the value of the limit.

From now on we assume that \( X \) is a finite dimensional Euclidean space. Denote by \( HD(Y) \) the Hausdorff dimension of a set \( Y \subset X \). We define the Hausdorff dimension of a probability measure \( \mu \) by

\[
HD(\mu) = \inf \{ HD(Y) : \mu(Y) = 1 \}
\]

We also define a local version of the dimension, namely

\[
d_\mu(x) = \lim \inf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \lim \sup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}
\]

It is well known that the Hausdorff dimension satisfies the relation

\[
HD(\mu) = \text{ess-sup} \ d_\mu.
\]

Barreira and Saussol established in [4] the following relation

Proposition 1. Let \( f \) be a measurable map and \( \mu \) be an invariant measure for \( f \). The recurrence rates are bounded from above by the pointwise dimensions :

\[
\underline{R} \leq d_\mu \quad \text{and} \quad \overline{R} \leq \overline{d}_\mu \mu \text{-a.e.}
\]

We refer to the works by Boshernitzan [6] and Ornstein and Weiss [12] for pioneering related results.

In this paper we are giving conditions under which the opposite inequalities will hold, establishing the equalities

\[
\underline{R} = d_\mu \quad \text{and} \quad \overline{R} = \overline{d}_\mu \mu \text{-a.e.}
\]

1.3. Statement of the results.

Definition 2. We say that \((X, f, \mu)\) has super-polynomial decay of correlations if we have

\[
\left| \int \varphi \circ f^n \psi d\mu - \int \varphi d\mu \int \psi d\mu \right| \leq \| \varphi \| \| \psi \| \theta_n
\]

with \( \lim_n \theta_n/n^p = 0 \) for all \( p > 0 \), where \( \| \cdot \| \) is the Lipschitz norm.

We say that the local decay of correlations is super-polynomial if there exists a partition (modulo \( \mu \)) into open sets \( V_i \) and sequences \( \theta_n^i \) such that (5) holds whenever \( \text{supp} \varphi \subset V_i \) and \( \text{supp} \psi \subset V_i \), where \( \lim_n \theta_n^i/n^p = 0 \) for all \( p > 0 \).

The main result of the paper is the following.

Theorem 3. Let \((X, f, \mu)\) be a measure preserving dynamical system. If the entropy \( h_\mu(f) > 0 \), \( f \) is Lipschitz (or piecewise Lipschitz with finite average Lipschitz exponent ; see Definition 15) and the (local) decay of correlation is super-polynomial then

\[
\underline{R} = d_\mu \quad \text{and} \quad \overline{R} = \overline{d}_\mu \mu \text{-a.e.}
\]

We postpone the proof at the end of Section 3. This extends some results by Barreira and Saussol in [4, 5], including the case of Axiom A systems with equilibrium states. The theorem also applies to loosely Markov dynamical systems and we recover Urbanski’s result in [15]. The hypotheses in Theorem 3 are satisfied in a number of systems such as those already quoted in the introduction. All these systems have in common some hyperbolic behavior. We now give an example of a relatively different nature, due to the possibility of zero Lyapunov exponents, where one can still apply Theorem 3.
Example 4 (Ergodic toral automorphisms). Recall that any matrix $A \in SL(k, \mathbb{Z})$ (i.e. the entries of $A$ are in $\mathbb{Z}$ and $|\det A| = 1$) gives rise to an automorphism $f$ of the torus $\mathbb{T}^k$ by $f(x) = Ax \mod \mathbb{Z}^k$ which preserves the Lebesgue measure. The map $f$ is ergodic if and only if the matrix $A$ has no eigenvalue root of unity. Lind’s established [10] the exponential decay of correlations (using the algebraic nature and Fourier transform) which is more than enough to apply Theorem 3 and get

$$R(x) = k \quad \text{for Lebesgue a.e. } x \in \mathbb{T}^k.$$ 

for any ergodic automorphism of the torus, even non-hyperbolic.

Let $f$ be a diffeomorphism of a compact manifold $M$ and $\mu$ be an invariant measure. By Oseledec’s multiplicative ergodic Theorem the Lyapunov exponents

$$\lambda(x, v) = \lim \frac{1}{n} \log |d_x f^n v|$$

are well defined for all nonzero $v \in T_x M$ for a.e. $x \in M$. Recall that a measure $\mu$ is said to be hyperbolic if none of its Lyapunov exponents are zero. Barreira, Pesin and Schmeling [3] prove the following.

Proposition 5. Let $f$ be a diffeomorphism of a compact manifold and $\mu$ be an ergodic hyperbolic measure. Then we have

$$d_\mu = \overline{d}_\mu = HD(\mu) \quad \mu\text{-a.e.}$$

The case of an hyperbolic measure with zero entropy is completely understood.

Proposition 6. let $f$ be a diffeomorphism of a compact manifold and $\mu$ be an hyperbolic invariant measure. If $h_\mu(f) = 0$ then $R = 0 = HD(\mu) \quad \mu\text{-a.e.}$

Proof. Barreira and Saussol established in [4] the inequality $\overline{R} \leq \overline{d}_\mu \mu\text{-a.e.}$ and it follows from Ledrappier and Young’s work [8] that $HD(\mu) = 0$ if $h_\mu(f) = 0$, which allows to conclude by Proposition 3. \qed

Corollary 7. Let $f$ be a diffeomorphism of a compact manifold and $\mu$ be an hyperbolic measure with super-polynomial rate of decay of correlation. Then we have

$$R = HD(\mu) \quad \mu\text{-a.e.}$$

Proof. If the entropy is zero then this is the content of Proposition 6. If the entropy is non-zero then this is the content of Theorem 3. \qed

We point out that in the case of interval maps with nonzero Lyapunov exponent, Saussol, Troubetzkoy and Vaienti prove that $R = HD(\mu) \mu\text{-a.e.}$ for ergodic measures, under very weak regularity conditions [14]. See Remark 17-(i) for related results.

We now give a sketch of the strategy adopted in this paper.

Theorem 3 states that under sufficiently rapid mixing the recurrence rates equal the pointwise dimensions a.e. on the set where $R > 0$. Indeed, mixing implies that $\mu(B \cap f^{-n}B) \to \mu(B)^2$ as $n \to \infty$. Thus we have $\mu(B \cap f^{-n}B) \leq 2\mu(B)^2$ for large $n$. If now we consider the set $B \cap f^{-n}B \cap f^{-n-1}B \cap \cdots \cap f^{-n-\ell}B$ then its measure is bounded by $2\mu(B)^2$. If $\ell \leq \mu(B)^{-1+\epsilon}$ then we get that the proportion of points inside $B$ that never enter in $B$ in the time interval $[n, n + \ell]$ is bounded by $2\mu(B)^{\epsilon}$. Using the decay of correlations we are able to prove that this
last statement is true for \( n \) of the order \( \text{diam}(B)^{-\delta} \) for some small \( \delta > 0 \), whenever \( B \) is a ball. This is what we call the long fly property. A Borel Cantelli argument then shows that typical points do have long flies (see Lemma 9 for precise statement). If in addition we also have \( R > \delta \) then it immediately shows that the return time into small neighborhoods \( B \) cannot be much less (at an exponential scale) than \( \mu(B)^{-1} \), establishing Equation \( (\text{4}) \).

On the other hand, for systems which are not too wild (e.g. finite Lyapunov exponents, see Lemma 13) and with nonzero metric entropy, a symbolic coding (see Lemma 14) allows to use Ornstein-Weiss’ theorem on repetition time of symbolic sequences to prove that the return time of a typical point in a ball \( B \) is not less than \( \text{diam}(B)^{-\delta} \); see Lemma 12.

The structure of the paper is as follows. We state and prove in Section 2 the core result, Theorem 8. In Section 3 we provide some conditions under which the recurrence rate is nonzero.

2. Rapid mixing implies long flies

**Theorem 8.** Assume that the local rate of decay of correlations is super-polynomial. Then on the set \( \{ R > 0 \} \) we have

\[
R = \text{d}_{\mu} \quad \text{and} \quad \overline{R} = \overline{\text{d}}_{\mu} \quad \mu\text{-a.e.}
\]

**Proof.** By Proposition 1 we know that \( R \leq \text{d}_{\mu} \) and \( \overline{R} \leq \overline{\text{d}}_{\mu} \). Furthermore, the first inequality implies that \( \{ R > a \} \subset \{ \text{d}_{\mu} > a \} \) \( \mu\text{-a.e.} \). But on the set \( \{ R > a \} \) we have \( \tau_r(x) \geq r^{-a} \) provided \( r \) is sufficiently small. By Lemma 9 below with \( \delta = a \) and \( \varepsilon > 0 \) we get that \( \tau_r(x) \geq \mu(B(x,r))^{-1+\varepsilon} \) provided \( r \) is sufficiently small, for \( \mu\text{-a.e.} \ x \in \{ R > a \} \). Thus \( \text{d} \geq (1 - \varepsilon)\text{d}_{\mu} \) and \( \overline{R} \geq (1 - \varepsilon)\overline{\text{d}}_{\mu} \) \( \mu\text{-a.e.} \) on \( \{ R > a \} \). The conclusion follows by taking \( \varepsilon > 0 \) arbitrary small. \( \square \)

The following lemma expresses that the orbit of a typical point has the long fly property.

**Lemma 9.** Let \( X_a = \{ \text{d}_{\mu} > a \} \) for some \( a > 0 \). For any \( \delta, \varepsilon > 0 \), for \( \mu\text{-a.e.} \ x \in X_a \) there exists \( r(x) > 0 \) such that for any \( r \in (0, r(x)) \) and any integer \( n \in [r^{-\delta}, \mu(B(x,r))^{-1+\varepsilon}] \) we have \( d(f^n x, x) \geq r \).

**Proof.** For clarity we assume that the (global) rate of decay of correlation is super-polynomial. The obvious modifications in the proof would consits essentially in considering separately each sets \( G \cap \{ x \in V_i : d(x, \partial V_i) > \nu \} \) for arbitrarily small \( \nu > 0 \) in place of the unique set \( G \) defined below.

Let \( D = \text{dim}(X) \). Fix \( b > 0 \), \( c = a\varepsilon/3 \) and consider for \( r_0 > 0 \) the set \( G = G_1 \cap G_2 \cap G_3 \) where

\[
G_1 = \{ x \in X_a : \forall r \leq r_0, \mu(B(x,r)) \leq r^a \}
\]

\[
G_2 = \{ x \in X : \forall r \leq r_0, \mu(B(x,r)) \geq r^{D+b} \}
\]

\[
G_3 = \{ x \in X : \forall r \leq r_0, \mu(B(x,4r)) \geq \mu(B(x,4r))r^c \}.
\]

We claim that \( \mu(G) \to \mu(X_a) \) as \( r_0 \to 0 \). Indeed, by definition of the lower pointwise dimension we have \( \mu(G_1) \to \mu(X_a) \). In addition since \( \overline{\text{d}}_{\mu} \leq D \text{ a.e.} \) we have \( \mu(G_2) \to 1 \) and since \( X \) is Euclidean the measure \( \mu \) is weakly diametrically regular (see Lemma 1 in [4]), thus \( \mu(G_3) \to 1 \) as well. Let \( r \leq r_0 \) and define the set

\[
A_\varepsilon(r) = \{ y \in X : \exists n \in [r^{-\delta}, \mu(B(y,3r))^{-1+\varepsilon}], d(f^n y, y) < r \}.
\]
Let $x \in G$. By the triangle inequality we get the inclusions
\[
B(x, r) \cap A_\varepsilon(r) \subset \{ y \in B(x, r) : \exists n \in [r^{-\delta}, \mu(B(x, 2r))^{-1+\varepsilon}], d(f^ny, x) < 2r \}
\subset \bigcup_{r^{-\delta} \leq n \leq \mu(B(x, 2r))^{-1+\varepsilon}} B(x, r) \cap f^{-n}B(x, 2r).
\]

Let $\eta_r : [0, \infty) \to \mathbb{R}$ be the $r^{-1}$-Lipschitz map such that $1_{[0,r]} \leq \eta_r \leq 1_{[0,2r]}$ and set $\varphi_{x,r}(y) = \eta_r(d(x, y))$. Clearly $\varphi_{x,r}$ is also $r^{-1}$-Lipschitz. By the assumption on the decay of correlation we obtain
\[
\mu(B(x, r) \cap f^{-n}B(x, 2r)) \leq \int \varphi_{x,2r} \varphi_{x,2r} \circ f^n \, d\mu \\
\leq \| \varphi_{x,2r} \|^{2} \theta_n + \left( \int \varphi_{x,2r} \, d\mu \right)^2 \\
\leq r^{-2\theta_n} + \mu(B(x, 4r))^2.
\]

Choose $p > 1$ such that $\delta(p - 1) - 2 \geq D + 2b$ and take $r_0$ so small that $n \geq r_0^{-\delta}$ implies $\theta_n \leq (p - 1)n^{-p}$. Since $\sum_{n \geq q} n^{-p} \leq \frac{1}{p-1}q^{1-p}$ we obtain
\[
\mu(B(x, r) \cap A_\varepsilon(r)) \leq r^{\delta(p-1)-2} + \mu(B(x, 2r))^{-1+\varepsilon} \mu(B(x, 4r))^2 \\
\leq \mu(B(x, r/2)) \left( r^b + r^{c\varepsilon-2\varepsilon} \right).
\]

Let $B \subset G$ be a maximal $r$-separated set\footnote{that is if $x \neq x' \in B$ then $d(x, x') \geq r$ and maximal in the sense that for any $y \in G$ there exists $x \in B$ such that $d(x, y) < r$.}. Since $(B(x, r))_{x \in B}$ covers $G$ we have
\[
\mu(G \cap A_\varepsilon(r)) \leq \sum_{x \in B} \mu(B(x, r) \cap A_\varepsilon(r)) \\
\leq \sum_{x \in B} \mu(B(x, r/2))(r^b + r^{c\varepsilon-2\varepsilon}) \\
\leq r^b + r^{c\varepsilon-2\varepsilon}.
\]

since by the balls $(B(x, r/2))_{x \in B}$ are disjoints. This implies that
\[
\sum m \mu(A_\varepsilon(e^{-m})) < \infty,
\]
thus by Borel-Cantelli Lemma we obtain that for $\mu$-a.e. $y \in G$ there exists $m(y)$ such for every $m > m(y)$ there exists no $n \in [e^{-\delta m}, \mu(B(y, 3e^{-m}))^{-1+\varepsilon}]$ such that $d(f^ny, y) < e^{-m}$. By weak diametric regularity (and changing slightly if necessary the values of $\varepsilon$ and $\delta$), this proves the lemma. \qed

**Remark 10.** Observe that we only use that the decay of correlation is at least $n^{-p}$ for some $p > \frac{D+2}{\delta} + 1$. If in addition \(\square\) holds with the first norm $\| \varphi \|$ taken to be the $L^1(\mu)$ norm (e.g. expanding maps) then $p > \frac{D+1}{\delta} + 1$ suffices.
3. Non-zero recurrence rate

We proceed now to find conditions under which the recurrence rate does not vanish. Denote by $\xi(x)$ the unique element of a partition $\xi$ containing the point $x$ and by $\xi^n = \xi \cup f^{-1}\xi \cup \ldots \cup f^{-n+1}\xi$ the dynamical partition, for any integer $n$.

3.1. Coding by symbolic systems: partitions with large interior.

**Definition 11.** We say that a partition $\xi$ has large interior if for $\mu$-a.e. $x$ there exists $\chi = \chi(x) < \infty$ such that $B(x, e^{-\chi^n}) \subset \xi^n(x)$ for all $n$ sufficiently large.

Next lemma, which proof is fairly simple, is the key-observation which gives to Theorem 8 all its interest.

**Lemma 12.** If there exists a partition with large interior and nonzero entropy then $\underline{R} > 0$ $\mu$-a.e.

**Proof.** Let $\xi$ be such a partition. Define

$$R_n(x, \xi) = \min\{k > 0 : f^k x \in \xi^n(x)\}.$$  

Ornstein and Weiss [12] prove that if $\xi$ is a finite partition with entropy $h_\mu(f, \xi)$ then

$$\lim_{n \to \infty} \frac{1}{n} \log R_n(x, \xi) = h_\mu(f, \xi) \text{ } \mu\text{-a.e.}$$

Since $\xi$ has large interior, for $\mu$-a.e. $x \in X$ there exists a number $\chi = \chi(x)$ such that $B(x, e^{-\chi^n}) \subset \xi^n(x)$. Thus

$$\underline{R}(x) = \liminf_{n \to \infty} \frac{\log R_n(x, \xi)}{n\chi(x)} \geq \liminf_{n \to \infty} \frac{\log R_n(x, \xi)}{\log \xi^n(x)} = \frac{h_\mu(f, \xi)}{\chi(x)} > 0 \text{ } \mu\text{-a.e.}$$

Combining Lemma [12] and Theorem 8 we get that if we have local super-polynomial decay of correlations and a partition of positive entropy with large interior then $\underline{R} = d_\mu$ and $\overline{R} = d_\mu$. The rest of the section consists in finding sufficient conditions for the existence of such a partition.

3.2. Reasonable dependence on initial condition.

**Definition 13.** We say that a system $(X, f, \mu)$ is reasonably sensitive if for $\mu$-a.e. $x$ there exists $\gamma, \lambda > 0$ such that $f^n$ is $e^{\lambda n}$-Lipschitz on the ball $B(x, e^{-\gamma n})$ for all $n$ sufficiently large.

**Lemma 14.** If the system $(X, f, \mu)$ is reasonably sensitive and the entropy $h_\mu(f) > 0$ then there exists a partition with large interior and nonzero entropy.

**Proof.** Claim: For any $x \in X$, $s > 0$ there exists $\rho \in (s, 2s)$ such that

$$\mu(\{y \in X : \rho - 4^{-n}s < d(x, y) < \rho + 4^{-n}s\}) \leq \frac{1}{2^n-1} \mu(B(x, 2s)). \quad (6)$$

Indeed, let $m$ be the measure on the interval $(0, 2)$ defined by $m([0, t)) = \mu(B(x, st))$. We construct a sequence of open intervals $I_n$ starting from $I_0 = (1, 2)$. If $I_n$ is an interval of length $4^{-n}$ we divide it into 4 pieces of equal length and choose $I_{n+1}$ the left of the right central piece of smallest measure. We have $m(I_{n+1}) \leq \frac{1}{2} m(I_n)$. $I_n$ is a decreasing sequence of intervals with $I_{n+1} \subset I_n$ thus $\cap_n I_n$ contains one point, say $\tilde{\rho}$. Since $\tilde{\rho} \in I_n$ we have $\tilde{\rho} \pm 4^{-n} \in I_{n-1}$ thus
Definition 15. If there exists a partition that 

\[ \mu = \frac{1}{m(I_{n-1})} m(I_0) \]. Proving the claim with \( \rho = s \phi \).

Fix \( s > 0 \) so small that any partition made by sets of diameter less than \( 2s \) has nonzero entropy. Choose a maximal \( s \)-separated set \( E \). For any \( x \in E \) take \( \rho_x \in (s, 2s) \) such that in the claim holds. Let \( E = \{x_1, x_2, \ldots \} \) be an enumeration of the (at most) countable set \( E \). Put 

\[ B_i = B(x_i, \rho_{x_i}) \] and define \( Q_1 = B_1, Q_2 = B_2 \setminus Q_1, Q_3 = B_3 \setminus (Q_1 \cup Q_2), \ldots \) By maximality the collection of sets \( \xi = \{Q_1, Q_2, \ldots \} \) is a partition of \( X \) (modulo \( \mu \)) and since \( \partial \xi \subset \cup_i \partial B_i \) we get

\[ \mu(\{x \in X : d(x, \partial \xi) < 4^{-n}s\}) \leq \mu(\bigcup_i \{x \in X : \rho_{x_i} - 4^{-n} < d(x_i, x) < \rho_{x_i} + 4^{-n}\}) \leq \frac{1}{2n-1} \sum_i \mu(B(x_i, 2s)). \]

Since the \( x_i \) are \( s \)-separated and \( X \) is Euclidean there are at most \( c(X) = c(\dim X) \) balls of radius \( 2s \) that can intersect, thus the last sum is bounded by \( c(X) \frac{2^n}{2} \). This proves that for some constants \( a, c > 0 \) and all \( \varepsilon > 0 \)

\[ \mu(x \in X : d(x, \partial \xi) < \varepsilon) < ce^a. \]

Thus for any \( b > 0 \) we have by the invariance of \( \mu \)

\[ \sum_n \mu(\{x \in X : d(f^n x, \xi) < e^{-bn}\}) \leq \sum_n ce^{-abn} < \infty. \]

This implies by Borel-Cantelli Lemma that for \( \mu \)-a.e. \( x \) there exists \( n(x) < \infty \) such that \( d(f^n x, \partial \xi) \geq e^{-bn} \), hence \( B(f^n x, e^{-bn}) \subset \xi(f^n x) \), for any \( n \geq n(x) \). Taking \( c(x) \in (0, 1) \) sufficiently small we have \( B(f^n x, c(x)e^{-bn}) \subset \xi(f^n x) \) for all integer \( n \).

Fix \( x \in X \) where the reasonable sensitivoty condition holds. Without loss of generality, and changing if necessary \( c(x) \) into a smaller constant we assume that \( f^n \) is \( c \)-Lipschitz on the ball \( B(x, c(x)e^{-\gamma n}) \) for all integer \( n \) and that \( \lambda > \gamma + b \).

We show then by induction that \( B(x, c(x)e^{-\lambda n}) \subset \xi^k(x) \) for any \( k \leq n \). Indeed, this is trivially true for \( k = 1 \), and if this holds for some \( k \leq n - 1 \) then we have

\[ f^k(B(x, c(x)e^{-\lambda -k n}) \subset B(f^k x, c(x)e^{\lambda k - \gamma n}) \subset B(f^k x, e^{-bn}) \subset \xi(f^k x). \]

Hence \( B(x, c(x)e^{-\gamma -n}) \subset \xi^{k+1}(x) \). \( \square \)

We finally provide a sufficient condition for reasonable sensitivity.

Definition 15. If there exists a partition \( A \) (modulo \( \mu \)) into open sets such that on each \( A \in \mathcal{A} \) the map \( f \) is Lipschitz with constant \( L_f(A) \) and the singularity set \( \partial A = \bigcup_{A \in \mathcal{A}} \partial A \) is such that \( \mu(\{x \in X : d(x, \partial A) < \varepsilon\}) \leq \mu \) for some constants \( c > 0 \) and \( a > 0 \) then we say that \( f \) is piecewise Lipschitz with average Lipschitz exponent \( \log L_f = f \log^+ L_f(A(x))d\mu(x) = \sum_{A \in \mathcal{A}} \log^+ L_f(A)\mu(A) \).

Lemma 16. If \( f \) is Lipschitz, or piecewise Lipschitz with finite Lipschitz exponent then \((X, f, \mu)\) is reasonably sensitive.

Proof. We prove the piecewise case, the other one is obvious. Let \( \lambda > \log L_f \). By the Birkhoff Ergodic Theorem, for \( \mu \)-a.e. \( x \) there exists \( m(x) \) such that

\[ L_f(A(x))L_f(A(f(x)) \cdots L_f(A(f^{n-1} x)) \leq e^{\lambda n} \]
for all $n \geq m(x)$. Replacing if necessary the upper bound by $e^{\lambda n}/c(x)$ for some constant $c(x) \geq 1$ the inequality will hold for any integer $n$. Proceeding as in the last part of the proof of Lemma 14 we get that for any $b > 0$, changing $c(x)$ if necessary, we have $B(f^n x, c(x)e^{-bn}) \subset A(f^n x)$ for any integer $n$. We then conclude similarly that $B(x, c(x)^2 e^{-bn} e^{-\lambda n}) \subset A^n(x)$. This concludes the proof taking $\gamma = b + \lambda$. $\square$

The proof of Theorem 3 follows now easily from the preceding results.

Proof of Theorem 3. By Lemma 16 the map is reasonably sensitive. This implies by Lemma 14 the existence of a partition with large interior. By Lemma 12 we find that $R > 0$ a.e. and the conclusion follows from Theorem 8. $\square$

Remark 17. (i) We remark that if $f$ is $C^1$ on a compact manifold, or more generally if $f$ is piecewise $C^{1+\alpha}$ with reasonable singularity set such as in [8], then the exponents $\lambda$ and $\gamma$ in Definition 13 can be taken arbitrarily close to the largest Lyapunov exponent $\lambda_+$. Thus the exponent $\chi$ in Lemma 12 may also be taken arbitrarily close to $\lambda_+$. This readily implies that $\frac{R}{h_\mu} \leq \lambda_+$. This is optimal in dimension one or more generally for conformal maps, where under mild assumptions we have $HD(\mu) = h_\mu/\lambda_\mu$.

(ii) Combining the above observation with Remark 16 shows that the assumption on the super-polynomial decay of correlations in Theorem 8 may be reduced to a decay at a rate $n^{-p}$ for some $p > \frac{D+2}{h_\mu} \lambda_+ + 1$.

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\[\text{3to see this, consider a Lyapunov chart whose local chart at } x \text{ has a diameter } \rho(x), \text{ where } \rho \text{ is } \eta'-\text{slowly varying. A choice like } \lambda = \lambda_+ + 2\eta \text{ and } \gamma = \lambda + \eta \text{ would do the job.}\]
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\textit{E-mail address: benoit.saussol@u-picardie.fr}

\textit{URL: http://lamfa.u-picardie.fr/saussol}

LAMFA - CNRS UMR 6140, Université de Picardie Jules Verne, 33 rue St Leu, 80039 Amiens cedex 1, France