Plane wave analysis of the second post-Newtonian hydrodynamic equations

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The second post-Newtonian hydrodynamic equations are analyzed within the framework of a plane wave solution. The hydrodynamic equations for the mass and momentum density are coupled with six Poisson equations for the Newtonian and post-Newtonian gravitational potentials. Perturbations of the basic fields and gravitational potentials from a background state by assuming plane wave representations lead to a dispersion relation where the Jeans instability condition emerges. The influence of the first and second post-Newtonian approximations on the Jeans mass is determined. It was shown that the relative difference of the first post-Newtonian and the Newtonian Jeans masses is negative while the one of the second post-Newtonian approximation is positive. The two contributions imply a smaller mass needed for an overdensity to initiate the gravitational collapse than the one given by the Newtonian theory.

I. INTRODUCTION

The determination of self-gravitating fluid instabilities from the hydrodynamic equations coupled with the Newtonian Poisson equation is an old subject in the literature which goes back to the pioneer work of Jeans in 1922 [1]. The analysis of the instabilities is based on a dispersion relation where one can infer a wavelength cutoff, known as the Jeans wavelength. Two distinct behaviors follow from the dispersion relation, one refers to wavelength perturbations that are smaller than the Jeans wavelength and the perturbations propagate as harmonic waves in time, while in the other the wavelength perturbations are greater than the Jeans wavelength and the perturbations grow or decay in time. The Jeans instability [2–4] refers to the gravitational collapse of self-gravitating interstellar gas clouds which are associated with the mass density perturbations which grow exponentially with time. A simple physical model is that the collapse of a mass density inhomogeneity happens if the inwards gravitational force is bigger than the outwards internal pressure force of the self-gravitating interstellar gas cloud.

The first post-Newtonian hydrodynamic equations for Eulerian fluids from Einstein’s field equations where all terms in the $\mathcal{O}(c^{-2})$ order are considered were derived by Chandrasekhar [5]. Apart from the Newtonian gravitational potential there exist a scalar and a vector gravitational potentials which are coupled with the hydrodynamic equations through Poisson equations. The second post-Newtonian hydrodynamic equations for Eulerian fluids were derived by Chandrasekhar and Nutku [6] where all terms in the $\mathcal{O}(c^{-4})$ order are taken into account. In this approximation the hydrodynamic equations are coupled with six Poisson equations, since there appear – additionally to the gravitational potentials of the first post-Newtonian approximation – a scalar, a vector and a tensor gravitational potentials.

The Jeans instability was recently analysed within the framework of the first post-Newtonian theory by two different methodologies. In [7, 8] the hydrodynamic equations for the mass and momentum densities are coupled with the Poisson equations, while in [9] the first post-Newtonian expression for the collisionless Boltzmann equation [10–12] is coupled with the Poisson equations. It was shown that in the first post-Newtonian analysis the mass necessary for an overdensity to begin the gravitational collapse is smaller than the one in the Newtonian theory.

The Jeans instability was also analysed within the framework of $f(R)$ gravity or modified gravity theories [14–16] where a modified dispersion relation along with a new kind of an unstable mode follow.

The aim of this work is to analyse the equations that follow from the second post-Newtonian approximation of Einstein’s field equations. In the second post-Newtonian theory the hydrodynamic equations for the mass and momentum densities are coupled with six Poisson equations for the Newtonian and post-Newtonian gravitational potentials. Here we analyse the perturbations from a background state of the basic fields and gravitational potentials by assuming plane wave representations which lead to a system of algebraic equations whose solution is a dispersion relation. From the dispersion relation the condition related with the Jean instability comes out and as a consequence it is possible to determine the influence of the second post-Newtonian approximation in the Jeans mass, which is related with the minimum mass necessary for an overdensity to initiate the gravitational collapse.

The paper is structured as follows. In Section III the hydrodynamic equations for the mass and momentum densities and the six Poisson equations which follow from the second post-Newtonian approximation are introduced. The subject of Section III is to determine the hydrodynamic and Poisson equations when perturbations in the background...
state of the fields and and gravitational potentials are considered. In Section IV a dispersion relation is obtained from a plane wave representation of the perturbations where the post-Newtonian influence in the Jeans mass is analyzed. The conclusions of the work are stated in the last section.

II. HYDRODYNAMIC AND POISSON EQUATIONS

The components of the metric tensor in the second post-Newtonian approximation were derived by Chandrasekhar and Nutku [6] from Einstein’s field equations and read

\[ g_{00} = 1 - \frac{2U}{c^2} + \frac{2}{c^4} \left( U^2 - 2\Phi \right) + \frac{\Psi_{00}}{c^6} + \mathcal{O}(c^{-8}), \]  
\[ g_{0i} = \Pi_{0i} + \frac{\Psi_{0i}}{c^2} + \mathcal{O}(c^{-7}), \]  
\[ g_{ij} = - \left( 1 + \frac{2U}{c^2} \right) \delta_{ij} + \frac{\Psi_{ij}}{c^4} + \mathcal{O}(c^{-6}), \]

where the Newtonian \( U \) and the post-Newtonian \( \Phi, \Pi_i, \Psi_{ij}, \Psi_{0i} \) and \( \Psi_{00} \) gravitational potentials satisfy the Poisson equations

\[ \nabla^2 U = -4\pi G\rho, \quad \nabla^2 \Phi = -4\pi G\rho \left( V^2 + \frac{\varepsilon}{2} + \frac{3p}{2\rho} \right), \]
\[ \nabla^2 \Pi_i = -16\pi G\rho V_i + \frac{\partial^2 U}{\partial t \partial x^i}, \]
\[ \nabla^2 \Psi_{kk} = 32\pi G\rho \left( V^2 + 4U + \varepsilon \right) - 12 \left( \frac{\partial U}{\partial x^j} \right)^2, \]
\[ \nabla^2 \Psi_{0i} = -16\pi G\rho \left[ V_i \left( V^2 + \varepsilon + \frac{p}{\rho} + 4U \right) - \Pi_i \right] - 10 \frac{\partial U}{\partial t} \frac{\partial \Pi_i}{\partial x^i} - 2 \frac{\partial U}{\partial x^i} \frac{\partial \Pi_j}{\partial x^j} + 2\Pi_i \frac{\partial^2 U}{\partial x^i \partial x^j}, \]
\[ \nabla^2 \Psi_{00} = 16\pi G\rho \left[ V^2 \left( V^2 + \varepsilon + \frac{p}{\rho} + 4U \right) - U^2 - 2\Phi \right] + 2 \frac{\partial U}{\partial x^i} \frac{\partial \Pi_i}{\partial t} - 6 \left( \frac{\partial U}{\partial t} \right)^2 \]
\[ + 12 \frac{\partial U}{\partial x^i} \frac{\partial \Phi}{\partial x^i} + \frac{\partial \Phi}{\partial x^i} \left( \frac{\partial \Pi_i}{\partial x^i} - \frac{\partial \Pi_j}{\partial x^j} \right) - 12U \left( \frac{\partial U}{\partial x^i} \right)^2 + 2\Psi_{ij} \frac{\partial^2 U}{\partial x^i \partial x^j}. \]

In the above Poisson equations \( G \) is the universal gravitational constant and their expressions were written by considering the gauge [6]

\[ 3 \frac{\partial U}{\partial t} + \frac{\partial \Pi_i}{\partial x^i} + \frac{1}{c^4} \left[ \frac{\partial \Psi_{0j}}{\partial x^j} - \frac{1}{2} \frac{\partial \Psi_{jk}}{\partial x^j} \right] = 0. \]

Here we follow [11, 12] and denote the potentials of [6] as: \( \Pi_i \equiv P_i, \Psi_{ij} \equiv Q_{ij}, \Psi_{0i} \equiv Q_{0i} \) and \( \Psi_{00} \equiv Q_{00}. \) Furthermore, in [11] \( \rho \) denotes the mass density, \( p \) the hydrostatic pressure, \( V_i \) the hydrodynamic velocity and \( \varepsilon \) the specific internal energy of the fluid.

In the second post-Newtonian approximation the continuity equation becomes [6, 11, 12]

\[ \frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \bar{\rho}V_i}{\partial x^i} = 0, \]

where the mass density \( \bar{\rho} \) is given by

\[ \bar{\rho} = \rho \left[ 1 + \frac{1}{c^2} \left( \frac{V^2}{2} + 3U \right) + \frac{1}{c^4} \left( \frac{3}{8} \nabla^4 + \frac{7}{2} UV^2 + \frac{3}{2} U^2 - \frac{1}{2} \Psi_{kk} - \Pi_i V_i \right) \right]. \]

The hydrodynamic equation for the momentum density in the second post-Newtonian approximation can be written as [6, 11, 12]

\[ \frac{\partial \rho \bar{V}_i}{\partial t} + \frac{\partial \rho \bar{V}_j V_j}{\partial x^i} + \frac{\partial \rho}{\partial x^i} \left[ 1 + \frac{2U}{c^2} - \frac{1}{c^4} \left( U^2 + 2\Phi + \frac{\Psi_{kk}}{2} \right) \right] - \rho \frac{\partial U}{\partial x^i} \left[ 1 + \frac{2}{c^2} \left( \frac{V^2}{2} + U + \varepsilon + \frac{p}{2\rho} \right) \right] + \rho \frac{\partial \bar{V}_j}{\partial x^j} \left[ 1 + \frac{1}{c^2} \left( V^2 + 4U + \varepsilon + \frac{p}{\rho} \right) \right] + \rho \frac{\partial \bar{V}_0}{\partial x^i} + \frac{\partial \bar{V}_j}{\partial x^j} + \frac{\partial \Psi_{00}}{\partial x^i} + 2V_j \frac{\partial \Psi_{0j}}{\partial x^j} + V_k \frac{\partial \Psi_{jk}}{\partial x^i} = 0. \]
where the following abbreviation for the momentum density was introduced

\[
\rho \mathcal{V}_i = \rho V_i \left(1 + \frac{1}{c^2} \left(V^2 + 6U + \varepsilon + \frac{p}{\rho}\right) + \frac{1}{c^2} \left[V^4 + 10V^2U + 2\Phi - 2\Pi_i V_i - \frac{\Psi_{kk}}{2} + 13U^2 + (V^2 + 6U) \left(\varepsilon + \frac{p}{\rho}\right)\right]\right) - \frac{p}{c^2} \Pi_i \left[1 + \frac{1}{c^2} \left(V^2 + 4U + \varepsilon + \frac{p}{\rho}\right)\right] - \frac{p}{c^4} (\Psi_{0i} + \Psi_{ij} V_j).
\]

(13)

### III. FIELD PERTURBATIONS

We consider that the fields are perturbed from a background state where the mass density \( \rho \), hydrostatic pressure \( p \), specific internal energy \( \varepsilon \) and Newtonian gravitational potential \( U \) assume constant values, while the background hydrodynamic velocity \( V_i \) and the post-Newtonian gravitational potentials \( \Phi, \Pi_i, \Psi_{kk}, \Psi_{0i} \) and \( \Psi_{00} \) vanish. The background and perturbed fields are denoted by the sub- and super-scripts 0 and 1, respectively, and the perturbed fields are considered to be small so that only the linear perturbed terms will be considered in the analysis of the present work. The representation of the fields are written as

\[
\begin{align*}
\rho(x, t) &= \rho_0 + \rho_1(x, t), \\
V_i(x, t) &= V_i^1(x, t), \\
U(x, t) &= U_0 + U_1(x, t), \\
\varepsilon(x, t) &= \varepsilon_0 + \varepsilon_1(x, t), \\
\Phi(x, t) &= \Phi_1(x, t), \\
\Psi_{0i}(x, t) &= \Psi_{01}(x, t), \\
\Psi_{00}(x, t) &= \Psi_{001}(x, t), \\
\Psi_{kk}(x, t) &= \Psi_{k1}(x, t).
\end{align*}
\]

(14) - (18)

We begin by determining the linearized expression for the ratio \( p/\rho \), yielding

\[
\frac{p}{\rho} = \frac{\rho_0}{\rho_0} \left(1 + \frac{p_1}{\rho_0} \right) \approx \frac{\rho_0}{\rho_0} \left(1 + \frac{p_1}{\rho_0} - \frac{p_1}{\rho_0}\right).
\]

(19)

Next we have to evaluate the perturbed specific internal energy and for that end we make use of the following result which comes out from the kinetic theory of relativistic monatomic gases for the specific internal energy \( \varepsilon \) (see e.g. [13]):

\[
\varepsilon = \frac{3kT}{2m} \left(1 + \frac{5kT}{4mc^2}\right) = \frac{1}{\gamma - 1} \rho \left(1 + \frac{5}{6(\gamma - 1)} \frac{p}{c^2}\right),
\]

(20)

thanks to the relationship \( \varepsilon = p/(\gamma - 1)\rho \) where \( \gamma = 5/3 \) is the ratio of the specific heats at constant pressure and constant volume for a monatomic gas.

By taking into account the expression for the sound speed \( c_s^2 = dp/d\rho \) the background and perturbed hydrostatic pressure and specific internal energy become

\[
\begin{align*}
p_0 &= \frac{c_s^2}{\gamma} \rho_0, & \varepsilon_0 &= \frac{c_s^2}{\gamma - 1} \left(1 + \frac{5}{6\gamma(\gamma - 1)} \frac{c_s^2}{c^2}\right), \\
p_1 &= c_s^2 \rho_1, & \varepsilon_1 &= \frac{c_s^2}{\gamma} \left(1 + \frac{5}{6\gamma(\gamma - 1)} \frac{c_s^2}{c^2}\right) \frac{\rho_1}{\rho_0}.
\end{align*}
\]

(21) - (22)

The linearized mass density [11] and its balance equation [10] by considering the representations [14] - [18] read

\[
\begin{align*}
\tilde{\rho} &= (\rho_0 + \rho_1) \left[1 + \frac{3U_0}{c^2} + \frac{3U_1^2}{2c^4}\right] + \rho_0 \left[1 + \frac{U_0}{c^2} \right] \partial_{V_1} - \rho_0 \frac{\Psi_{kk}}{2c^4}, \\
&\left[1 + \frac{3U_0}{c^2} \left(1 + \frac{U_0}{2c^2}\right) \right] \left[\partial_{\rho_1} + \rho_0 \frac{\partial V_1}{\partial x^1}\right] + 3\rho_0 \left(1 + \frac{U_0}{c^2}\right) \partial U_1 - \rho_0 \frac{\partial \Psi_{kk}}{2c^4} = 0.
\end{align*}
\]

(23) - (24)

Equation (24) can be rewritten as

\[
\partial_{\rho_1} + \rho_0 \frac{\partial V_1}{\partial x^1} + 3\rho_0 \left(1 - \frac{2U_0}{c^2}\right) \partial U_1 - \rho_0 \frac{\partial \Psi_{kk}}{2c^4} = 0,
\]

(25)

if we multiply it by the first expression within the brackets and keep terms up to the \( 1/c^4 \)-order.
Following the same methodology as above we multiply (27) by the first expression within the brackets and keep terms up to the $1/c^4$-order, yielding

\[
\rho_0 \left\{ \frac{1}{c^2} \left( 6U_0 + \varepsilon_0 + \frac{p_0}{\rho_0} \right) + \frac{1}{c^4} \left[ 13U_0^2 + 6U_0 \left( \varepsilon_0 + \frac{p_0}{\rho_0} \right) \right] \right\} \frac{\partial V_i^1}{\partial t} + \left( \frac{1}{c^2} \left( 2U_0 + \varepsilon_0 + \frac{3p_0}{\rho_0} \right) + \frac{1}{c^4} \left[ 2U_0 \left( \varepsilon_0 + \frac{p_0}{\rho_0} \right) - 3U_0^2 \right] \right) \frac{\partial U_i}{\partial x^j} + \rho_0 \left( \frac{1}{c^2} \left( \frac{1}{2} \frac{\partial \Psi^1_{0i}}{\partial t} - \frac{\partial \Psi^1_{i0}}{\partial t} \right) \right) = 0.
\]

In terms of the perturbed fields the linearized gauge condition (4) is expressed as

\[
3 \frac{\partial U_i}{\partial t} + \frac{\partial \Pi^1_{i}}{\partial x^i} + \frac{1}{c^2} \left( \frac{\partial \Psi^1_{0i}}{\partial x^i} - \frac{1}{2} \frac{\partial \Psi^1_{kk}}{\partial t} \right) = 0.
\]

For the Poisson equations we make use of "Jeans swindle" and assume that (11) – (8) are only valid for the perturbed fields, so that the linearized Poisson equations read

\[
\nabla^2 U_1 = -4\pi G \rho_1, \quad (30)
\]

\[
\nabla^2 \Phi_1 = -4\pi G \rho_0 \left[ U_1 + \frac{\varepsilon_1}{2} + \frac{3p_1}{2\rho_0} + \frac{2U_0 + \varepsilon_0}{2\rho_0} \right], \quad (31)
\]

\[
\nabla^2 \Pi_1^1 = -16\pi G \rho_0 V_i^1 + \frac{\partial^2 U_i}{\partial t \partial x^j}, \quad (32)
\]

\[
\nabla \Psi_{kk} = 32\pi G \rho_0 \left[ 4U_1 + \varepsilon_1 + \frac{4U_0 + \varepsilon_0}{\rho_0} \right], \quad (33)
\]

\[
\nabla^2 \Psi^1_{0i} = -16\pi G \rho_0 \left[ V_i^1 \left( \varepsilon_0 + \frac{p_0}{\rho_0} + 4U_0 \right) - \frac{\Pi^1_{i}}{2} \right], \quad (34)
\]

\[
\nabla^2 \Psi^1_{00} = -32\pi G \rho_0 \left( U_0 U_1 + \Phi_1 + \frac{U_0^2}{2\rho_0} \right). \quad (35)
\]

The time derivative of the perturbed mass density balance equation (26) and the spatial divergence of the perturbed momentum density balance equation (28) lead to

\[
\rho \frac{\partial V_i^1}{\partial t} + \left\{ \frac{1}{c^2} \left( 6U_0 + \varepsilon_0 + \frac{p_0}{\rho_0} \right) + \frac{1}{c^4} \left[ 13U_0^2 + 6U_0 \left( \varepsilon_0 + \frac{p_0}{\rho_0} \right) \right] \right\} \frac{\partial U_i}{\partial x^j} + \rho \left( \frac{1}{c^2} \left( \frac{1}{2} \frac{\partial \Psi^1_{0i}}{\partial t} - \frac{\partial \Psi^1_{i0}}{\partial t} \right) \right) = 0.
\]

\[
\rho \frac{\partial^2 V_i^1}{\partial t^2} + \rho \frac{\partial V_i^1}{\partial x^j} + \frac{1}{c^2} \left( 6U_0 + \varepsilon_0 + \frac{p_0}{\rho_0} \right) + \frac{1}{c^4} \left[ 13U_0^2 + 6U_0 \left( \varepsilon_0 + \frac{p_0}{\rho_0} \right) \right] \left( \frac{1}{c^2} \left( \frac{1}{2} \frac{\partial \Psi^1_{0i}}{\partial t} - \frac{\partial \Psi^1_{i0}}{\partial t} \right) \right) = 0.
\]
Now by eliminating the perturbed hydrodynamic velocity $V_1^1$ from (36) by using (37) we get
\[
\frac{\partial^2 \rho_1}{\partial t^2} - \left\{ 1 - \frac{1}{c_s^2} \left[ 4U_0 + \varepsilon_0 + \frac{\rho_0}{\rho_0} \right] + \frac{1}{c_s^2} \left[ 10U_0^2 + 4U_0 \left( \varepsilon_0 + \frac{\rho_0}{\rho_0} \right) + \left( \varepsilon_0 + \frac{\rho_0}{\rho_0} \right)^2 \right] \right\} c_s^2 \nabla^2 \rho_1
+ \rho_0 \left[ 1 - 4U_0 \frac{c_s^2}{c^2} + 8U_0^2 \frac{c_s^2}{c^4} \right] \nabla^2 U_1 + \frac{2\rho_0}{c^2} \left( 1 - 2U_0 \frac{c_s^2}{c^2} \right) \nabla^2 \Phi_1 - \frac{\rho_0}{2c^2} \nabla^2 \Psi_{100}^1
+ \rho_0 \frac{\partial}{\partial t} \left[ 3 \frac{\partial U_1}{\partial t} + \partial \Psi_{100}^1 \frac{\partial \Phi_{100}^1}{\partial x} \right] + \frac{1}{c_s^2} \left[ 1 - 2 \frac{\partial \Psi_{100}^1}{\partial x} \right] \left( \frac{\partial \Psi_{100}^1}{\partial x} - \frac{\partial \Psi_{100}^1}{\partial t} \right) \right\} = 0. \tag{38}
\]

Note that the underlined terms vanish thanks to the gauge condition (29) in the $O(c^{-2})$ and $O(c^{-4})$ orders.

Equations (30), (31), (35) and (38) represent a system of differential equations for the determination of the perturbations $U_1$, $\Phi_1$, $\Psi_{100}^1$ and $\rho_1$. This system of differential equations will be analysed in the next section by considering a plane wave representation for the perturbed fields.

### IV. PLANE WAVE REPRESENTATIONS

We consider that the perturbed fields $U_1$, $\Phi_1$, $\Psi_{100}^1$ and $\rho_1$ are represented by plane waves of wave number vector $\mathbf{k}$ and angular frequency $\omega$, namely
\[
\rho_1(\mathbf{x}, t) = \rho_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}, \quad U_1(\mathbf{x}, t) = \mathbf{U} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}, \quad \Phi_1(\mathbf{x}, t) = \Phi e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}, \quad \Psi_{100}^1(\mathbf{x}, t) = \Psi_{100} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}, \tag{39}
\]
where $\mathbf{\rho}$, $\mathbf{U}$, $\mathbf{\Phi}$ and $\mathbf{\Psi}_{100}$ are small amplitudes.

Insertion of the plane wave representations (39) and (40) into the equations (30), (31), (35) and (38), yield the following system of algebraic equations for the amplitudes $\mathbf{\rho}$, $\mathbf{U}$, $\mathbf{\Phi}$ and $\mathbf{\Psi}_{100}:
\[
\kappa^2 \mathbf{U} = 4\pi G \mathbf{\rho}, \quad \kappa^2 \mathbf{\Psi}_{100} = 16\pi G \rho_0 \left[ 2 \left( \mathbf{\Phi} + \mathbf{U} \right) \right], \quad \kappa^2 \mathbf{\Psi} = 4\pi G \rho_0 \mathbf{U} + 2\pi G \left\{ 2U_0 + \frac{c_s^2}{\gamma - 1} \left[ 3\gamma - 2 + \frac{5(2\gamma - 1)}{6\gamma^2(\gamma - 1)} \right] \right\} \mathbf{\rho}, \tag{41}
\]
\[
\left\{ \omega^2 - c_s^2 \kappa^2 \left[ 1 - \frac{1}{c_s^2} \left( 4U_0 + \varepsilon_0 + \frac{\rho_0}{\rho_0} \right) + \frac{1}{c_s^2} \left( 10U_0^2 + 4U_0 \left( \varepsilon_0 + \frac{\rho_0}{\rho_0} \right) + \left( \varepsilon_0 + \frac{\rho_0}{\rho_0} \right)^2 \right) \right] \right\} \mathbf{\rho}
+ \rho_0 \kappa^2 \left[ \left( 1 - \frac{1}{c_s^2} \left( 4U_0 + \varepsilon_0 + \frac{\rho_0}{\rho_0} \right) + \frac{1}{c_s^2} \left( 10U_0^2 + 4U_0 \left( \varepsilon_0 + \frac{\rho_0}{\rho_0} \right) + \left( \varepsilon_0 + \frac{\rho_0}{\rho_0} \right)^2 \right) \right) \right] \mathbf{\Phi}
+ \frac{\partial}{\partial t} \left[ \frac{1}{c_s^2} \kappa^2 \left[ \left( 1 - \frac{1}{c_s^2} \left( 4U_0 + \varepsilon_0 + \frac{\rho_0}{\rho_0} \right) + \frac{1}{c_s^2} \left( 10U_0^2 + 4U_0 \left( \varepsilon_0 + \frac{\rho_0}{\rho_0} \right) + \left( \varepsilon_0 + \frac{\rho_0}{\rho_0} \right)^2 \right) \right) \right] \right] \mathbf{\Psi}_{100} = 0. \tag{42}
\]

In the Poisson equations (41) and (42) we have used the relations (21) and (22) and introduced the modulus of wave number vector $\kappa = \sqrt{\mathbf{k} \cdot \mathbf{k}}$.

The elimination from (43) of the amplitudes $\mathbf{U}$, $\mathbf{\Phi}$ and $\mathbf{\Psi}_{100}$ by using the Poisson equations (41) and (42) together with the relations (21) leads to the following dispersion relation
\[
\omega_s^2 = \kappa_s^2 - 1 - \frac{c_s^2}{\kappa_s^2} \left[ \left( 4U_0 \frac{c_s^2}{c^2} + \frac{1}{\gamma - 1} \right) \kappa_s^2 + \frac{2}{c_s^2} \left( 10U_0^2 + \frac{4U_0}{\rho_0} \right) + \frac{\partial}{\partial t} \left( \frac{\partial \Psi_{100}^1}{\partial x} - \frac{\partial \Psi_{100}^1}{\partial t} \right) \right]
- \frac{12U_0}{c_s^2 \kappa_s^2} \left[ \frac{2}{\gamma - 1} \kappa_s^2 \right] + \frac{2U_0^2}{c_s^4} \left( 5(2\gamma - 1) \right) - \frac{4U_0}{c_s^2} \left( 6\gamma^2(\gamma - 1)^2 \right) - 4 \kappa_s^2 \right], \tag{44}
\]
where only the terms up to the $1/c^2$ order were considered. Furthermore, in (44) we have introduce the dimensionless angular frequency $\omega_s$ and the dimensionless wavenumber $\kappa_s$ defined by
\[
\omega_s = \frac{\omega}{\sqrt{4\pi G \rho_0}}, \quad \kappa_s = \frac{\kappa}{\kappa_J}, \quad where \quad \kappa_J = \frac{\sqrt{4\pi G \rho_0}}{c_s}, \tag{45}
\]
is the Jeans number.

From the solution of (44) for the dimensionless angular frequency $\omega_s$ one has two distinct behaviors: if $\kappa_s > 1$, $\omega_s$ assumes real values and the perturbations will propagate as harmonic waves in time, while if $\kappa_s < 1$, $\omega_s$ acquires pure imaginary values and the perturbations will grow or decay in time. The one which grows with time is associated with the Jeans instability. Hence, by considering $\omega_s = 0$ in (44) we can determine the value of $\kappa_s$ where $\omega_s$ changes from
the real value to the pure imaginary value. The real solution of (44) for \( \kappa_* \) when \( \gamma_* = 0 \) by considering only terms up to the \( 1/c^4 \) order is

\[
\kappa_* = 1 + \left[ \frac{5\gamma - 3}{2(\gamma - 1)} + \frac{U_0}{c_s^2} \right] \frac{c_s^2}{c^2} + \left[ \frac{20 - 9\gamma(\gamma - 1)(35\gamma - 27)}{24\gamma(\gamma - 1)^2} + \frac{9 - 7\gamma}{2(\gamma - 1)} \right] \frac{U_0^2}{c_s^2} - \frac{U_0^2}{2c_s^2} \frac{c_s^4}{c^4}
\]

\[
\kappa_* = 1 + \left[ 4 + \frac{U_0}{c_s^2} \right] \frac{c_s^2}{c^2} - \left[ \frac{33}{2} + 2 \frac{U_0}{c_s^2} + \frac{U_0^2}{2c_s^4} \right] \frac{c_s^4}{c^4}.
\] (46)

In the last equality we have introduced the value \( \gamma = 5/3 \). In the above equation the term in \( c_s^2/c^2 \) represents the contribution of the first post-Newtonian approximation to the dimensionless module of the wave number and was obtained in [8, 9]. The underlined term in \( c_s^4/c^4 \) corresponds to the second post-Newtonian contribution and it is interesting to note that this contribution is negative.

Let us analyse the Jeans mass which corresponds to the minimum amount of mass for an overdensity to initiate the gravitational collapse. The Jeans mass represents the mass contained in a sphere of radius equal to the wavelength of the perturbation. Hence, if \( M_{\lambda}^{PN} \) denotes the mass corresponding to the post-Newtonian wavelength and \( M_{\lambda}^N \) the Newtonian one, we can build the ratio by taking into account (46), yielding

\[
\frac{M_{\lambda}^{PN}}{M_{\lambda}^N} = \frac{\lambda^3}{\lambda^3} \frac{\kappa^3}{\kappa^3} \approx 1 - 3 \frac{c_s^2}{c^2} \left[ 4 + \frac{U_0}{c_s^2} \right] + \frac{1}{2} \frac{c_s^4}{c^4} \left[ 291 + 108 \frac{U_0}{c_s^2} + 15 \frac{U_0^2}{c_s^4} \right].
\] (47)

We note from (47) that the mass for an overdensity needed to begin the gravitational collapse has contributions from the first and second post-Newtonian approximations and that the presence of the background Newtonian potential \( U_0 \) has influence on it. If we consider the virial theorem and approximate the square of the sound speed with the background Newtonian gravitational potential \( U_0 \approx c_s^2 \), we get that (47) can be written as

\[
\frac{M_{\lambda}^{PN} - M_{\lambda}^N}{M_{\lambda}^N} = -15 \frac{c_s^2}{c^2} + 207 \frac{c_s^4}{c^4},
\] (48)

which shows the relative difference of the post-Newtonian and Newtonian Jeans masses. While the relative difference of the first post-Newtonian approximation is negative, the one of the second post-Newtonian approximation is positive. The correction of the first post-Newtonian approximation is the preponderant one, indeed if we consider that \( c_s = 5\% \) \( c \) the ratio \( M_{\lambda}^{PN}/M_{\lambda}^N = 0.9625 \) by considering only the first post-Newtonian approximation and \( M_{\lambda}^{PN}/M_{\lambda}^N = 0.9638 \) by considering the first and second post-Newtonian approximations. Hence, we may conclude that the two contributions imply a smaller mass needed for an overdensity to initiate the gravitational collapse with respect to the one given by the Newtonian theory.

V. CONCLUSIONS

To sum up: in this work the equations of the second post-Newtonian approximation to Einstein’s field equations were analysed. The starting point was the hydrodynamic equations for the mass and momentum densities which were coupled with six Poisson equations for the Newtonian and post-Newtonian gravitational potentials. The basic fields and gravitational potentials were perturbed from a background state and plane wave representations for the perturbations imply a system of algebraic equations whose solution is a dispersion relation where the condition related with the Jean instability emerges. The influence of the first and second post-Newtonian approximations in the Jeans mass – which is related with the minimum mass necessary for an overdensity to initiate the gravitational collapse – are determined. It was shown that the relative difference of the first post-Newtonian and Newtonian Jeans masses is negative, while the one of the second post-Newtonian is positive. The contributions of the first and second approximations lead to a smaller Jeans mass in comparison to the one given by the Newtonian theory.

ACKNOWLEDGMENTS

This work was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), grant No. 304054/2019-4.

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