S-matrix for the massive and massless modes of the AdS$_2 \times S^2$ superstring

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Abstract

We derive the exact S-matrix for the integrable system associated to light-cone gauge string theory on AdS$_2 \times S^2 \times T^6$. The S-matrix for the massive modes consists of two copies of a centrally-extended $\mathfrak{su}(1|1)$ invariant S-matrix and is in agreement with the tree-level result following from perturbation theory. Although the overall factor is left unfixed, the constraints following from crossing symmetry and unitarity are given. The scattering involves long representations of the symmetry algebra, and the relevant representation theory is studied in detail. We also discuss Yangian symmetry and find it has a standard form in the massless case, for which we specify all S-matrices explicitly. Finally, some brief comments are given on the Bethe ansatz.
1 Introduction

The remarkable successes of integrability techniques in the study of the $AdS_5 \times S^5$ superstring [1] motivates the application of these methods to other integrable string backgrounds with less supersymmetry [2]. In this work we investigate the $AdS_2 \times S^2 \times T^6$ background supported by Ramond-Ramond fluxes in Type II superstring theory, which preserves a quarter of the supersymmetries. These can be found as the near-horizon limit of various intersecting brane solutions of Type IIB supergravity, which are related by T-duality [3]. The dual [4] should be a one-dimensional CFT, and is understood to either be a superconformal quantum-mechanical system or a chiral two-dimensional CFT [5].

The $AdS_2 \times S^2$ part of the background can be written as a Metsaev-Tseytlin [6] type supercoset model [7] for $PSU(1,1|2)/SO(1,1) \times SO(2)$. The algebra $\mathfrak{psu}(1,1|2)$ has a $Z_4$ automorphism and hence the supercoset model is classically integrable via the same construction as for the $AdS_5 \times S^5$ case [8]. While there exists a classical truncation of the Green-Schwarz action [9] for the $AdS_2 \times S^2 \times T^6$ geometry to the supercoset degrees of freedom, there is no $\kappa$-symmetry gauge choice which decouples them from the remaining fermions [10]. The integrability of the Green-Schwarz action for the complete background has been demonstrated to quadratic order in fermions [10, 11].
The aim of this paper is to use symmetries and integrability to construct exact S-matrices for the scattering of the worldsheet excitations of the decompactified light-cone gauge [12] $AdS_2 \times S^2 \times T^6$ superstring. These S-matrices describe the scattering above the BMN vacuum [13], a point-like string moving at the speed of light on a great circle of the two-sphere. The light-cone gauge-fixed Lagrangian [14, 15] is in general rather complicated with the interaction terms breaking two-dimensional Lorentz invariance. The quadratic action is however Lorentz invariant and describes $2 + 2$ (bosons+fermions) massive modes, the bosons of which are associated to the transverse directions in $AdS_2 \times S^2$, and $6 + 6$ massless modes, associated to the $T^6$.

In the $AdS_5 \times S^5$ light-cone gauge-fixed theory all of the excitations have equal non-vanishing mass and furthermore the symmetries completely fix the S-matrix up to an overall phase [16, 17]. Here the situation is more similar to $AdS_3 \times S^3 \times T^4$ for which there are $4 + 4$ massive and $4 + 4$ massless excitations. In this case the symmetries of the supercoset leaving the BMN string invariant can be used to conjecture an exact S-matrix for the scattering of the massive modes [18, 19]. Following a similar approach we observe that the subalgebra of the $\mathfrak{psu}(1,1|2)$ symmetry of the $AdS_2 \times S^2$ supercoset preserved by the BMN string is given by $\mathfrak{psu}(1|1)^2 \ltimes \mathbb{R}$. Relaxing the level-matching condition we extend this algebra by two additional central extensions and conjecture the exact S-matrix for the scattering of massive modes up to an overall factor.

The resulting S-matrix satisfies crossing symmetry [20] and is unitarity so long as the overall factor satisfies the relevant identities. Here the setup is more similar to the $AdS_3 \times S^3 \times T^4$ case as opposed to the $AdS_3 \times S^3 \times T^4$ case for which there were multiple phases related by crossing transformations [21]. It was observed in [15] that the one-loop logarithms in the massive S-matrix for $AdS_2 \times S^2 \times T^6$ are consistent with part of this overall factor being given by the BES phase [22]. Assuming this to be true we find the remaining rational piece still has to satisfy somewhat complicated relations. Finally, the near-BMN expansion of the exact result is consistent with perturbative computations [14, 15, 23].

While many features of the construction are similar to the $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times T^4$ cases, there are some important differences. In particular, unlike for the $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times T^4$ superstrings, the representations we are scattering turn out to be long and hence there is no shortening condition to be interpreted as the dispersion relation. An additional consequence is that the symmetries do not completely fix the S-matrix up to a single overall factor, rather there is an additional undetermined function that can be found by demanding the Yang-Baxter equation is satisfied. These properties are reminiscent of similar features seen for the scattering of long representations of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ [24] and also in the Pohlmeyer reduction of strings on $AdS_2 \times S^2$ [25].

The S-matrix has an accidental $U(1)$ symmetry under which the fermions are charged, while the bosons are not. From the perspective of the complete $AdS_2 \times S^2 \times T^6$ superstring this $U(1)$ originates from the $T^6$ compact space [15]. Furthermore, its presence appears to be important to have any hope of applying a Bethe ansatz construction as it allows one to define a pseudovacuum. A conjecture for a set of asymptotic Bethe ansatz equations was given in [10], however, due to the somewhat involved structure of the S-matrix it is not clear how to derive them.

It is not currently known how the massless modes transform under the symmetry group of the light-cone gauge-fixed theory, and therefore it is not possible to completely determine the corresponding S-matrices. Furthermore, they may depend on the choice of Type II background [3] – in the decompactified
light-cone gauge-fixed theory the $T^6$ formally has an $SO(6)$ symmetry, however, this will be broken by
the presence of Ramond-Ramond fluxes. Different choices related by T-duality naively lead to different
subgroups. Therefore, we take an alternative approach motivated by the recent explicit computation of
the light-cone gauge symmetry algebra for the $AdS_3 \times S^3 \times T^4$ superstring [26], the $AdS_5 \times S^5$
version of which was constructed in [27]. Assuming a similar outcome occurs for the $AdS_2 \times S^2 \times T^6$
superstring one may expect the massless modes to transform in representations of $psu(1|1)^2 \ltimes \mathbb{R}^3$, and hence the
$S$-matrices describing their scattering should be built from the massless limits (one massless and one
massive or two massless particles) of the massive $S$-matrix.

The structure of the paper is as follows. In section 2 we describe the near-BMN symmetry algebra
and investigate its representation theory. This symmetry is then used in section 3 to determine the
exact $S$-matrix up to an overall phase. We determine the constraints that the phase should satisfy for
crossing symmetry and unitarity and compare with perturbation theory. In section 4 we discuss when
this symmetry can be extended to a Yangian, finding that it can be done in the standard form for the
massless case. Using this Yangian symmetry in section 5 we then construct the massless version of the
$S$-matrix and briefly explore the notions of crossing symmetry and unitarity in this limit. In section 6
we give some initial considerations of the algebraic Bethe ansatz, noting in particular the existence of a
pseudovacuum, and we conclude in section 7 with some comments.

2 Symmetry for massive modes of $AdS_2 \times S^2$

The BMN light-cone gauge $AdS_2 \times S^2 \times T^6$ superstring action describes $2 + 2$ massive and $6 + 6$ massless
modes. The algebra underlying the scattering of the massive modes is expected to be $psu(1|1)^2 \ltimes \mathbb{R}^3$, which is found by considering the subalgebra of $psu(1,1|2)$ that is preserved by the BMN geodesic. The
two additional central extensions appear, by analogy with the $AdS_5 \times S^5$ case, in the decompactification
limit and relaxing the level-matching condition.

Let us denote the massive boson associated to the transverse direction of $S^2$ as $y$ and the corresponding
boson for $AdS_2$ as $z$. The two massive fermions will be represented as two real Grassmann fields $\zeta$ and
$\chi$. We can then formally define the following tensor product states

$$
|y\rangle = |\phi\rangle \otimes |\phi\rangle , \quad |z\rangle = |\psi\rangle \otimes |\psi\rangle ,
$$

$$
|\zeta\rangle = |\phi\rangle \otimes |\psi\rangle , \quad |\chi\rangle = |\psi\rangle \otimes |\phi\rangle ,
$$

where $\phi$ is bosonic and $\psi$ is fermionic, such that we expect one of the factors of $psu(1|1)$ to act on
each of the two entries. Furthermore, as a consequence of the form of the symmetry algebra and the
integrability of the theory [10, 14] we expect that the $S$-matrix for $y$, $z$, $\zeta$ and $\chi$ can be constructed
as a graded tensor product of an $S$-matrix for $\phi$ and $\psi$, with each factor $S$-matrix invariant under the
symmetry $psu(1|1) \ltimes \mathbb{R}^3$.

In this section we will construct the relevant massive representation of $psu(1|1) \ltimes \mathbb{R}^3$. This representation
has an obvious massless limit, and, by analogy with the construction for $AdS_3 \times S^3 \times T^4$ [26], one
may expect the massless modes to also transform in representations of $psu(1|1) \ltimes \mathbb{R}^3$ in the light-cone
gauge-fixed theory. The massless limit is discussed in detail in section 5.

Let us also briefly mention that there is an additional $U(1)$ outer automorphism symmetry [15] of the
$S$-matrix (3.2), under which the $psu(1|1)$ factors transform in the vector representation. The origin of
this $U(1)$ symmetry is the $T^6$ compact space that is required for a consistent 10-d superstring theory.
Under this symmetry \((\zeta, \chi)^T\) also transforms as a vector, while the bosons are uncharged. It is worth noting that taking the tensor product of two copies of any \(S\)-matrix for \(\phi\) and \(\psi\) preserving fermion number we find that the \(U(1)\) symmetry is present so long as a certain quadratic relation between the parametrizing functions is satisfied (see appendix \(A\)). In the case of interest, this quadratic identity turns out to be true just from demanding invariance under the \(\mathfrak{psu}(1|1) \ltimes \mathbb{R}^3\) symmetry and satisfaction of the Yang-Baxter equation. The \(U(1)\) does not act in a well-defined way on the individual factor \(S\)-matrices and hence for now we will ignore it. We will reconsider it in section 6, where it will play a role in defining a pseudovacuum, an important first step in the algebraic Bethe ansatz.

2.1 The \(\mathfrak{gl}(1|1)\) Lie superalgebra and its representations

Let us start by summarizing the relevant information from \([28]\) regarding the Lie superalgebra \(\mathfrak{gl}(1|1)\) and its representations. There are two bosonic generators \(\mathfrak{R}\) and \(\mathfrak{C}\), with \(\mathfrak{C}\) central, and two fermionic generators \(\Omega\) and \(\Theta\). The commutation relations read

\[
[\mathfrak{R}, \Omega] = -\Omega, \quad [\mathfrak{R}, \Theta] = \Theta, \quad \{\Omega, \Theta\} = 2\mathfrak{C}.
\] (2.2)

The typical (long) irreps are the 2-dimensional Kac modules \(\langle C, \nu \rangle\), defined by the following non-zero entries on a boson-fermion \((|\phi\rangle, |\psi\rangle)\) pair of states:

\[
\begin{align*}
\mathfrak{R} |\phi\rangle &= |\psi\rangle, \quad \Theta |\phi\rangle = 2C |\phi\rangle, \quad \mathfrak{N} |\phi\rangle &= (\nu - 1) |\phi\rangle, \quad \mathfrak{N} |\psi\rangle &= (\nu - 1) |\psi\rangle, \\
\mathfrak{C} |\Phi\rangle &= C |\Phi\rangle \quad \forall \ |\Phi\rangle \in \{|\phi\rangle, |\psi\rangle\}, \quad C, \nu \in \mathbb{C}, \quad C \neq 0.
\end{align*}
\] (2.3)

As long as \(C \neq 0\), this module is isomorphic to the anti-Kac module \(\langle -C, \nu \rangle\)

\[
\begin{align*}
\mathfrak{R} |\psi\rangle &= 2C |\phi\rangle, \quad \Theta |\phi\rangle = |\psi\rangle, \quad \mathfrak{N} |\phi\rangle &= (\nu - 1) |\phi\rangle, \quad \mathfrak{N} |\psi\rangle &= \nu |\psi\rangle, \\
\mathfrak{C} |\Phi\rangle &= C |\Phi\rangle \quad \forall \ |\Phi\rangle \in \{|\phi\rangle, |\psi\rangle\}, \quad C, \nu \in \mathbb{C}, \quad C \neq 0.
\end{align*}
\] (2.4)

However, if \(C = 0\), the two modules are not isomorphic and they are no longer irreducible. Rather they become reducible but indecomposable.

To elucidate further we introduce the 1-dimensional modules \(\langle \mu \rangle\), which form the atypical (short) irreps of \(\mathfrak{gl}(1|1)\). These irreps are characterized by the vanishing of all generators except \(\mathfrak{R}\), which acts with eigenvalue \(\mu\). We then see that for the Kac module, \(\langle 0, \nu \rangle\), the fermion \(|\psi\rangle\) spans a sub-representation \(\langle \nu - 1 \rangle\), and the indecomposable is denoted as

\[
\langle \nu - 1 \rangle \leftarrow \langle \nu \rangle.
\] (2.5)

The anti-Kac module \(\langle 0, \nu \rangle\) is also reducible but indecomposable and is denoted as

\[
\langle \nu - 1 \rangle \rightarrow \langle \nu \rangle,
\] (2.6)

with the fermion \(|\psi\rangle\) once again spanning the sub-representation \(\langle \nu \rangle\). This indecomposable is not isomorphic to \(\langle 0, \nu \rangle\). Let us mention that modding out the indecomposable representations by their sub-representations one obtains the factor representations, which in this case are isomorphic to the short 1-dimensional \(\langle \mu \rangle\) modules and are spanned by the boson \(|\phi\rangle\).

If we take the tensor product of two typical modules, we get

\[
\begin{align*}
\langle C_1, \nu_1 \rangle \otimes \langle C_2, \nu_2 \rangle &= \langle C_1 + C_2, \nu_1 + \nu_2 - 1 \rangle \oplus \langle C_1 + C_2, \nu_1 + \nu_2 \rangle \quad \text{if } C_1 + C_2 \neq 0, \\
\langle C_1, \nu_1 \rangle \otimes \langle -C_1, \nu_2 \rangle &= P_{\nu_1 + \nu_2},
\end{align*}
\] (2.7)
where $P_\nu$ is the so-called projective module

$$\langle \nu \rangle \rightarrow \langle \nu + 1 \rangle \oplus \langle \nu - 1 \rangle \rightarrow \langle \nu \rangle , \quad (2.8)$$

on which $\mathcal{C}$ acts identically as zero. The rightmost 1-dimensional short sub-module $\langle \nu \rangle$ is known as the socle of $P_\nu$.

Since $\mathfrak{N}$ does not appear on the r.h.s. of the commutation relations, the algebra $\mathfrak{gl}(1|1)$ has a non-trivial ideal generated by $\mathfrak{N}$, $\mathfrak{S}$ and $\mathfrak{C}$. This ideal is the superalgebra $\mathfrak{sl}(1|1)$. Furthermore, this algebra is also not simple, as $\mathfrak{C}$, being central, is a non-trivial ideal. Additionally modding out $\mathfrak{C}$ gives the algebra $\mathfrak{psl}(1|1)$, which is still not simple, as the two remaining anti-commuting fermionic generators each form a separate ideal. The fact that $\mathfrak{psl}(1|1)$ is not simple sets this algebra outside the classification of the possible central extensions of basic classical Lie superalgebras presented in [29].

2.2 The centrally-extended $\mathfrak{psu}(1|1)$ Lie superalgebra

We are now ready to introduce the centrally-extended version of the algebra we discussed in the previous section, which is relevant for the scattering of the massive modes of the $AdS_2 \times S^2 \times T^6$ superstring. The algebra $\mathfrak{psu}(1|1) \ltimes \mathbb{R}^3$ is defined by the commutation relations

$$\{ Q, Q \} = 2P , \quad \{ S, S \} = 2K , \quad \{ Q, S \} = 2C . \quad (2.9)$$

The states $|\phi\rangle$ and $|\psi\rangle$, introduced in (2.1), then transform in the following representation:

$$\begin{align*}
\mathfrak{N} |\phi\rangle &= a |\psi\rangle , & \mathfrak{N} |\psi\rangle &= b |\phi\rangle , & \mathfrak{S} |\phi\rangle &= c |\psi\rangle , & \mathfrak{S} |\psi\rangle &= d |\phi\rangle , \\
\mathfrak{C} |\Phi\rangle &= C |\Phi\rangle , & \mathfrak{P} |\Phi\rangle &= P |\Phi\rangle , & \mathfrak{R} |\Phi\rangle &= K |\Phi\rangle .
\end{align*} \quad (2.10)$$

Here $a$, $b$, $c$, $d$, $C$, $P$ and $K$ are the representation parameters that will eventually be functions of the energy and momentum of the states. For the supersymmetry algebra to close the following conditions should be satisfied

$$ab = P , \quad cd = K , \quad ad + bc = 2C . \quad (2.11)$$

This representation corresponds to the typical (long) Kac module $(C, \nu)$ discussed in the previous section. We will be interested in a particular real form of the algebra (2.9), which is given by

$$\begin{align*}
\mathfrak{N}^\dagger &= \mathfrak{S} , & \mathfrak{P}^\dagger &= \mathfrak{R} , & \mathfrak{C}^\dagger &= \mathfrak{C} .
\end{align*} \quad (2.12)$$

These relations further constrain the representation parameters as follows

$$a^* = d , \quad b^* = c , \quad C^* = C , \quad P^* = K . \quad (2.13)$$

The closure conditions (2.11) imply that

$$C^2 = \frac{(ad - bc)^2}{4} + PK . \quad (2.14)$$

Unlike the $AdS_5 \times S^5$ case, with the larger symmetry algebra $\mathfrak{psu}(2|2) \times \mathbb{R}^3$, here we are scattering long representations and hence there is no shortening condition – that is, $ad - bc$ is free to take any value, which we will denote as $m$. From the reality conditions (2.13) we see that $m$ is real. Furthermore, we will choose the branch of the square root such that $m$ is positive

$$m = ad - bc = 2\sqrt{C^2 - PK} , \quad C^2 > PK . \quad (2.15)$$
Later it will be useful to solve the set of equations (2.11) for \( a, b, c \) and \( d \) in terms of \( m, C, P \) and \( K \):

\[
\begin{align*}
a &= \alpha e^{-\frac{\pi}{4} (C + \frac{m}{2})^2}, \\
b &= \alpha^{-1} e^{\frac{\pi}{4} (C + \frac{m}{2})^2} P, \\
c &= \alpha e^{-\frac{\pi}{4} (C + \frac{m}{2})^2} K, \\
d &= \alpha^{-1} e^{\frac{\pi}{4} (C + \frac{m}{2})^2}.
\end{align*}
\] (2.16)

Here \( \alpha \) is a phase parametrizing the normalization of the fermionic states with respect to the bosonic states and can be a function of the central extensions.

To define the action of this symmetry on the two-particle states we need to introduce the coproduct

\[
\begin{align*}
\Delta(\Omega) &= \Omega \otimes 1 + \mathbb{U} \otimes \Omega, \\
\Delta(\mathbb{S}) &= \mathbb{S} \otimes 1 + U^{-1} \otimes \mathbb{S}, \\
\Delta(\mathbb{P}) &= \mathbb{P} \otimes 1 + U^2 \otimes \mathbb{P}, \\
\Delta(\mathbb{C}) &= \mathbb{C} \otimes 1 + 1 \otimes \mathbb{C}, \\
\Delta(\mathbb{K}) &= \mathbb{K} \otimes 1 + U^{-2} \otimes \mathbb{K},
\end{align*}
\] (2.17)

and the opposite coproduct, defined as

\[
\Delta^\text{op}(J) = \mathcal{P} \Delta(J),
\] (2.18)

where \( \mathcal{J} \) is an arbitrary abstract generator (prior to considering a representation), and \( \mathcal{P} \) defines the graded permutation of the tensor product.

The coproduct differs from the trivial one by the introduction of a new abelian generator \( \mathbb{U} \), with \( \Delta(\mathbb{U}) = \mathbb{U} \otimes \mathbb{U} \) [30]. This is done according to a \( \mathbb{Z} \)-grading of the algebra, whereby the charges \( -2, -1, 1 \) and \( 2 \) are associated to the generators \( \mathbb{R}, \mathbb{S}, \mathbb{Q} \) and \( \mathbb{P} \) respectively, while \( \mathbb{C} \) remains uncharged. The action of \( \mathbb{U} \) on the single-particle states is given by

\[
\begin{align*}
\mathbb{U} \ket{\phi} &= U \ket{\phi}, \\
\mathbb{U} \ket{\psi} &= U \ket{\psi}.
\end{align*}
\] (2.19)

This braiding allows for the existence of a non-trivial S-matrix.

One important consequence of the non-trivial braiding (2.17) is that it leads to a constraint between \( U \) and the eigenvalues of the central charges. This follows from the requirement that, to admit an S-matrix, the coproduct of any central element should be equal to its opposite.\(^1\) This implies

\[
\begin{align*}
\mathbb{P} &\propto (1 - U^2), \\
\mathbb{K} &\propto (1 - U^{-2}).
\end{align*}
\] (2.20)

We fix the normalization of \( \mathbb{P} \) relative to \( \mathbb{K} \) by taking both constants of proportionality to be equal to \( \frac{1}{2} \hbar \) where the reality conditions (2.13) require that \( \hbar \) is real.\(^2\) The parameter \( \hbar \) is a coupling constant and eventually should be fixed in terms of the string tension, which we will return to in section 3.2. Acting on the single-particle states then gives us the relations

\[
\begin{align*}
P &= \frac{\hbar}{2} (1 - U^2), \\
K &= \frac{\hbar}{2} (1 - U^{-2}),
\end{align*}
\] (2.21)

where \( U \) should satisfy, as a consequence of (2.13), the following reality condition

\[
U^* = U^{-1}.
\] (2.22)

\(^1\)If \( \Delta(\mathcal{C}) \) is central, then

\[
\Delta^\text{op}(\mathcal{C}) R = R \Delta(\mathcal{C}) = \Delta(\mathcal{C}) R,
\]

which, for an invertible R-matrix, necessarily implies \( \Delta^\text{op}(\mathcal{C}) = \Delta(\mathcal{C}) \). This is expressed by saying that the coproduct of \( \mathcal{C} \) is co-commutative.

\(^2\)The reality conditions (2.13) do allow for the introduction of an additional phase into the constants of proportionality, i.e. \( \frac{1}{2} \hbar e^{i\varphi} \) and \( \frac{1}{2} \hbar e^{-i\varphi} \). However, this phase does not appear in the S-matrix and thus we set \( \varphi = 0 \).
The relation (2.14) in terms of $C$, $U$ and $m$ is then given by

$$C^2 = \frac{m^2 - \hbar^2(U - U^{-1})^2}{4}. \tag{2.23}$$

While this is a single equation for three undetermined parameters, we will later still attempt to interpret it as a dispersion relation with $C$, $U$ and $m$ defined in terms of just two kinematic variables, the energy and momentum. These precise definitions are not fixed by symmetry considerations, and hence should be found from direct string computations.

It is now useful to introduce the Zhukovsky variables $x^\pm$, in terms of which we will write the S-matrix, in place of the central extensions, $C$ and $U$. These are defined as \[ U^2 = x^+ x^-, \quad 2C + m = i\hbar(x^- - x^+) \tag{2.24}\]

In these variables the dispersion relation (2.23) takes the following familiar form

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2im}{\hbar} \tag{2.25}$$

The representation parameters $a$, $b$, $c$ and $d$ in (2.16) and (2.31) are then given by

$$a = \alpha e^{i \frac{\pi}{4}} \sqrt{\frac{x^+}{x^-}} \sqrt{\frac{\hbar}{2} \eta}, \quad b = \alpha^{-1} e^{i \frac{\pi}{4}} \sqrt{\frac{x^-}{x^+}} \sqrt{\frac{\hbar}{2} \frac{\eta}{x^+}}, \quad c = \alpha e^{i \frac{\pi}{4}} \sqrt{\frac{x^+}{x^-}} \sqrt{\frac{h}{2} \frac{\eta}{x^-}}, \quad d = \alpha^{-1} e^{i \frac{\pi}{4}} \sqrt{\frac{x^-}{x^+}} \sqrt{\frac{h}{2} \frac{\eta}{x^+}}, \tag{2.26}$$

where

$$\eta \equiv \sqrt{i(x^- - x^+)}. \tag{2.27}$$

Here we clearly see that the advantage of these variables is that the parameters $a$, $b$, $c$ and $d$ do not depend on $m$ and hence, written as a function of $x^\pm$ and $m$, neither will the S-matrix. Finally, let us note that for the reality conditions (2.13) we have the usual $(x^\pm)^* = x^\mp$.

We could also eliminate the central extensions, $C$ and $U$, in terms of two variables that will later be identified with the energy and momentum. Motivated by the $AdS_5 \times S^5$ case we write

$$C = \frac{e}{2}, \quad U = e^{i p}, \tag{2.28}$$

where $e$ is the energy and $p$ is the spatial momentum. Solving for $x^\pm$ in terms of $e$ and $p$ we find

$$x^\pm = r U^{\pm 1}, \quad r = \frac{e + m}{2h \sin \frac{p}{2}} = \frac{2h \sin \frac{p}{2}}{e - m}, \quad U = e^{i p}. \tag{2.29}$$

Using (2.21) and (2.28) we can substitute in for $C$, $P$ and $K$ in terms of the energy and the momentum in (2.23) to find the following familiar dispersion relation

$$e^2 = m^2 + 4h^2 \sin^2 \frac{p}{2}. \tag{2.30}$$

It is important to emphasize that here $m$ is algebraically a free parameter. However, for (2.30) to really be interpreted as a dispersion relation $m$ should be fixed by the spectral analysis of the theory. In terms of the energy and the momentum the representation parameters $a$, $b$, $c$ and $d$ (2.16) are then given by

$$a = \frac{\alpha e^{i \frac{\pi}{4}}}{\sqrt{2}} \sqrt{e + m}, \quad b = \frac{\alpha^{-1} e^{-i \frac{\pi}{4}}}{\sqrt{2}} \frac{h(1 - e^{ip})}{\sqrt{e + m}}, \quad c = \frac{\alpha e^{-i \frac{\pi}{4}}}{\sqrt{2}} \frac{h(1 - e^{-ip})}{\sqrt{e + m}}, \quad d = \frac{\alpha^{-1} e^{i \frac{\pi}{4}}}{\sqrt{2}} \sqrt{e + m}. \tag{2.31}$$
In the $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times M^4$ models, the choice of the phase factor $\alpha$ that is appropriate for the light-cone gauge-fixed string theory is

$$\alpha = 1.$$  \hspace{1cm} (2.32)

As we will see, this is also a natural choice for $\alpha$ in the $AdS_2 \times S^2$ theory.

2.3 Tensor product of irreps and scattering theory

In this section we consider the tensor product of two of the irreps we discussed in the previous section, with the aim of constructing the relevant scattering theory. In particular, we want to investigate the persistence of the phenomenon observed for $gl(1|1)$ modules in section 2.1, namely complete reducibility of the tensor product of two 2-dimensional irreps, for generic values of the momenta, into two 2-dimensional irreps of the same type.

Let us proceed by constructing a 4-dimensional representation of the algebra (2.9). To do this we start with the bosonic state

$$|w_0\rangle.$$  \hspace{1cm} (2.33)

Let us assume that the action of the central elements on this state is given by

$$(P, K, C)|w_0\rangle = (P, K, C)|w_0\rangle.$$  \hspace{1cm} (2.34)

This assumption will be justified by the concrete example we will consider later in our treatment of the scattering theory. We can then construct two more states by considering the action of $Q$ and $S$

$$|w_1\rangle \equiv Q|w_0\rangle, \quad |\tilde{w}_1\rangle \equiv S|w_0\rangle.$$  \hspace{1cm} (2.35)

The action of the central elements on these new states is then easily seen to be given by

$$(P, K, C)|w_1\rangle = (P, K, C)|w_1\rangle, \quad (P, K, C)|\tilde{w}_1\rangle = (P, K, C)|\tilde{w}_1\rangle.$$  \hspace{1cm} (2.36)

We can then look at the action of $Q$ and $S$ on $|w_1\rangle$ and $|\tilde{w}_1\rangle$

$$\mathcal{Q}|w_1\rangle = P|w_0\rangle, \quad \mathcal{Q}|\tilde{w}_1\rangle = C|w_0\rangle + \frac{1}{2} |\mathcal{Q}, \mathcal{S}|w_0\rangle, \quad \mathcal{S}|\tilde{w}_1\rangle = K|w_0\rangle, \quad \mathcal{S}|w_1\rangle = C|w_0\rangle - \frac{1}{2} |\mathcal{Q}, \mathcal{S}|w_0\rangle.$$  \hspace{1cm} (2.37)

Here we see that we have generated one additional new state

$$|\tilde{w}_0\rangle \equiv \frac{1}{M} |\mathcal{Q}, \mathcal{S}|w_0\rangle,$$  \hspace{1cm} (2.38)

where we have chosen a normalization depending on

$$M \equiv 2\sqrt{C^2 - PK}.$$  \hspace{1cm} (2.39)

Given the real form we are interested in, see eq. (2.12), and the assumption that $C^2 > PK$, or equivalently that $M$ is real and non-zero (we will briefly discuss the case when $M$ vanishes at the end of this section), the above normalization implies that $|\tilde{w}_0\rangle$ has the same norm as $|w_0\rangle$. Therefore, the action of $\mathcal{Q}$ and $\mathcal{S}$ on $|w_1\rangle$ and $|\tilde{w}_1\rangle$ is given by

$$\mathcal{Q}|w_1\rangle = P|w_0\rangle, \quad \mathcal{Q}|\tilde{w}_1\rangle = C|w_0\rangle + \frac{M}{2} |\tilde{w}_0\rangle, \quad \mathcal{S}|\tilde{w}_1\rangle = K|w_0\rangle, \quad \mathcal{S}|w_1\rangle = C|w_0\rangle - \frac{M}{2} |\tilde{w}_0\rangle.$$  \hspace{1cm} (2.40)
Again it is clear that the action of the central elements on $|\tilde{w}_0\rangle$ is given by

$$(\mathfrak{P}, \mathfrak{R}, \mathfrak{C})|\tilde{w}_0\rangle = (P, K, C)|\tilde{w}_0\rangle.$$  

Finally, the action of $\Omega$ and $\mathfrak{S}$ on $|\tilde{w}_0\rangle$ is given by

$$\Omega|\tilde{w}_0\rangle \equiv \frac{2P}{M}|\tilde{w}_1\rangle - \frac{2C}{M}|w_1\rangle, \quad \mathfrak{S}|\tilde{w}_0\rangle = -\frac{2K}{M}|w_1\rangle + \frac{2C}{M}|\tilde{w}_1\rangle.$$  

Therefore, in summary, we have constructed the following 4-dimensional representation:

$$(\mathfrak{P}, \mathfrak{R}, \mathfrak{C})|\Phi\rangle = (P, K, C)|\Phi\rangle, \quad \forall |\Phi\rangle \in \{|w_0\rangle, |w_1\rangle, |\tilde{w}_1\rangle, |\tilde{w}_0\rangle\}.$$  

$$\Omega|w_0\rangle = |w_1\rangle, \quad \mathfrak{S}|w_0\rangle = |\tilde{w}_1\rangle, \quad \mathfrak{S}|\tilde{w}_0\rangle = K|w_0\rangle,$$

$$\Omega|\tilde{w}_1\rangle = C|w_0\rangle + \frac{M}{2}|\tilde{w}_0\rangle, \quad \mathfrak{S}|w_1\rangle = C|w_0\rangle - \frac{M}{2}|\tilde{w}_0\rangle,$$

$$\Omega|\tilde{w}_0\rangle = \frac{2P}{M}|\tilde{w}_1\rangle - \frac{2C}{M}|w_1\rangle, \quad \mathfrak{S}|\tilde{w}_0\rangle = -\frac{2K}{M}|w_1\rangle + \frac{2C}{M}|\tilde{w}_1\rangle.$$  

However, using the fact that

$$\Omega\mathfrak{S}|\tilde{w}_0\rangle = C|\tilde{w}_0\rangle + \frac{M}{2}|w_0\rangle, \quad \Omega\mathfrak{S}|w_0\rangle = C|w_0\rangle + \frac{M}{2}|\tilde{w}_0\rangle,$$

$$\mathfrak{S}\Omega|\tilde{w}_0\rangle = C|\tilde{w}_0\rangle - \frac{M}{2}|w_0\rangle, \quad \mathfrak{S}\Omega|w_0\rangle = C|w_0\rangle - \frac{M}{2}|\tilde{w}_0\rangle,$$

we see that defining the linear combinations

$$|\Phi_{\pm}\rangle = |w_0\rangle \pm |\tilde{w}_0\rangle,$$

implies

$$\Omega\mathfrak{S}|\Phi_{\pm}\rangle = (C + \frac{M}{2})|\Phi_{\pm}\rangle, \quad \mathfrak{S}\Omega|\Phi_{\pm}\rangle = (C - \frac{M}{2})|\Phi_{\pm}\rangle.$$  

Furthermore,

$$\Omega|\Phi_{\pm}\rangle = \pm \frac{2C \mp M}{M}|w_1\rangle \pm \frac{2P}{M}|\tilde{w}_1\rangle, \quad \mathfrak{S}|\Phi_{\pm}\rangle = \pm \frac{2C \pm M}{M}|\tilde{w}_1\rangle \mp \frac{2K}{M}|w_1\rangle.$$  

Using the definition of $M$ (2.39) one can easily see that

$$\Omega|\Phi_{\pm}\rangle \propto \mathfrak{S}|\Phi_{\pm}\rangle \propto |\Psi_{\pm}\rangle,$$

and hence the 4-dimensional representation we constructed is actually reducible and is formed of two 2-dimensional representations

$$\{|\Phi_{\pm}\rangle, |\Psi_{\pm}\rangle\}.$$  

To conclude, let us briefly mention orthogonality. Here we will make use of the real form of the algebra given in eq. (2.12), and the assumption that $M$ is real. We then have

$$\langle \Phi_{\pm}|\Phi_{\pm}\rangle = \langle w_0|\left(1 + \frac{1}{M}[[\Omega, \mathfrak{S}] - [\Omega, \mathfrak{S}]^\dagger]\right) - \frac{1}{M^2}[[\Omega, \mathfrak{S}], [\Omega, \mathfrak{S}]^\dagger]|w_0\rangle.$$  

Using the conjugation relations we find that $[\Omega, \mathfrak{S}]^\dagger = [\Omega, \mathfrak{S}]$. Furthermore, as $[\Omega, \mathfrak{S}] = 2\mathfrak{C} - 2\mathfrak{S}\Omega = -2\mathfrak{C} + 2\Omega\mathfrak{S}$ we find

$$\langle \Phi_{\pm}|\Phi_{\pm}\rangle = \langle w_0|\left(1 + \frac{1}{M^2}(2\mathfrak{C} - 2\mathfrak{S}\Omega)(2\mathfrak{C} - 2\Omega\mathfrak{S})\right)|w_0\rangle = \langle w_0|\left(1 - \frac{4}{M^2}(\mathfrak{C}^2 - \mathfrak{P}\mathfrak{K})\right)|w_0\rangle = 0.$$  

(2.51)
Therefore, the two representations are orthogonal.

We will now apply the above construction to the tensor product of two of the 2-dimensional representations of section 2.2. This will be relevant to the scattering theory discussed in section 3. In this case we have four states that are acted on as follows by the generators of the algebra

\[
\begin{align*}
\mathfrak{C}(\mathfrak{P}, \mathfrak{R})|\phi\psi\rangle &= (C, P, K)|\phi\psi\rangle, & \mathfrak{C}(\mathfrak{P}, \mathfrak{R})|\phi\psi\rangle &= (C, P, K)|\phi\psi\rangle \\
\mathfrak{Q}|\phi\psi\rangle &= a_i|\psi\rangle + \tilde{a}_i|\phi\psi\rangle, & \mathfrak{Q}|\phi\psi\rangle &= a_i|\psi\rangle + \tilde{b}_i|\phi\psi\rangle \\
\mathfrak{S}|\phi\psi\rangle &= c_i|\psi\rangle + \tilde{c}_i|\phi\psi\rangle, & \mathfrak{S}|\phi\psi\rangle &= c_i|\psi\rangle + \tilde{d}_i|\phi\psi\rangle
\end{align*}
\]

where the labels 1, 2 refer to the first and second entry in the tensor product and we recall that the action on the tensor product is given by the coproduct (2.17), so that

\[
\begin{align*}
\tilde{a}_2 &= a_2U_1, & \tilde{b}_2 &= b_2U_1, & \tilde{c}_2 &= c_2U_1^{-1}, & \tilde{d}_2 &= d_2U_1^{-1}, \\
2C &= 2C_1 + 2C_2 = a_1d_1 + b_1c_1 + \tilde{a}_2\tilde{d}_2 + \tilde{b}_2\tilde{c}_2 = a_1d_1 + b_1c_1 + a_2d_2 + b_2c_2, \\
P &= P_1 + U_1^2P_2 = a_1b_1 + \tilde{a}_2\tilde{b}_2 = a_1b_1 + U_1^2a_2b_2, \\
K &= K_1 + U_1^{-2}K_2 = c_1d_1 + \tilde{c}_2\tilde{d}_2 = c_1d_1 + U_1^{-2}c_2d_2.
\end{align*}
\]

These relations imply

\[
M^2 = 4(C^2 - PK) = (a_1d_1 + b_1c_2 + \tilde{a}_2\tilde{d}_2 + \tilde{b}_2\tilde{c}_2)^2 - 4(a_1b_1 + \tilde{a}_2\tilde{b}_2)(a_1d_1 + \tilde{c}_2\tilde{d}_2) = (a_1d_1 - b_1c_2 + \tilde{a}_2\tilde{d}_2 - \tilde{b}_2\tilde{c}_2)^2 - 4(a_1\tilde{c}_2 - c_1\tilde{a}_2)(b_1\tilde{d}_2 - d_1\tilde{b}_2) = M_b^2 - \ell_{ac}\ell_{bd},
\]

where

\[
M_b \equiv (a_1d_1 - b_1c_2 + \tilde{a}_2\tilde{d}_2 - \tilde{b}_2\tilde{c}_2), & \ell_{ac} = 2(a_1\tilde{c}_2 - c_1\tilde{a}_2), & \ell_{bd} = 2(b_1\tilde{d}_2 - d_1\tilde{b}_2).
\]

It is then clear that the bound-state points occur when either \(\ell_{ac} = 0\) or \(\ell_{bd} = 0\). Furthermore, for the scattering of two physical states, \(i.e.\) when the following reality conditions are satisfied

\[
a_i^* = d_i, & b_i^* = c_i, & U_i^* = U_i^{-1},
\]

we find

\[
C^* = C, & P^* = K, & M^* = M, & M_b^* = M_b, & \ell_{ac}^* = -\ell_{bd}.
\]

To explicitly find the decomposition into two irreps, let us start by taking

\[
|w_0\rangle = |\phi\psi\rangle, & |\tilde{w}_0\rangle \equiv \frac{1}{M}[\mathfrak{Q}, \mathfrak{S}]|w_0\rangle = \frac{1}{M}\left[-M_b|\phi\psi\rangle + \ell_{ac}|\psi\phi\rangle\right].
\]

It then follows that

\[
|\Phi_\pm\rangle = \frac{1}{M}\left[(M \mp M_b)|\phi\psi\rangle \pm \ell_{ac}|\psi\phi\rangle\right].
\]

Alternatively we could have started by taking

\[
|w_0\rangle = |\psi\phi\rangle,
\]
in which case we end up with

\[ |\Phi_\pm\rangle = \frac{1}{M} [(M \pm M_b)|\psi\rangle \mp \ell_{bd}|\phi\rangle] . \quad (2.62) \]

It is easy to see that these states are proportional to each other from the identity

\[ (M \pm M_b)M \mp M_b + \ell_{ac}\ell_{bd} = 0 . \quad (2.63) \]

This same identity, along with the reality conditions, can be used to see that

\[ \langle \Phi_\mp|\Phi_{\pm}\rangle = 0 . \quad (2.64) \]

Working with the state (2.60) we can apply the fermionic generators to find

\[ \Omega|\Phi_{\pm}\rangle = \frac{1}{2M}(2C \mp M) [(M \mp M_b)|\phi\rangle \pm \ell_{ac}|\psi\rangle] , \]

\[ \Sigma|\Phi_{\pm}\rangle = \frac{1}{2M}(2C \mp M) [(M \mp M_b)|\phi\rangle \pm \ell_{ac}|\psi\rangle] . \quad (2.65) \]

One can then check\(^3\) that

\[ \Omega|\Phi_{\pm}\rangle \propto \Sigma|\Phi_{\pm}\rangle \propto |\Psi_{\pm}\rangle . \quad (2.66) \]

Furthermore,\(^4\)

\[ \Omega \Sigma|\Phi_{\pm}\rangle = \frac{1}{2M}(2C \mp M) [(M \mp M_b)|\phi\rangle \pm \ell_{ac}|\psi\rangle] , \]

\[ \Sigma \Omega|\Phi_{\pm}\rangle = \frac{1}{2M}(2C \mp M) [(M \mp M_b)|\phi\rangle \pm \ell_{ac}|\psi\rangle] , \quad (2.67) \]

so that using (2.63) it is clear that \( \Omega \Sigma|\Phi_{\pm}\rangle \propto |\Phi_{\pm}\rangle \) and \( \Sigma \Omega|\Phi_{\pm}\rangle \propto |\Phi_{\pm}\rangle \) and hence this shows explicitly that \( \{|\Phi_{\pm}\rangle, |\Psi_{\pm}\rangle\} \) form two 2-dimensional irreps.

At the bound-state points \( \ell_{ac} = 0 \) or \( \ell_{bd} = 0 \) one of the irreducible blocks contains \( |\phi\rangle \), as either \( |\Phi_{+}\rangle \) or \( |\Phi_{-}\rangle \) aligns to this state. Therefore, one can focus on the \( \phi \phi \rightarrow \phi \phi \) entry of the S-matrix (supplemented by the appropriate dressing phase) to ascertain whether this corresponds to a pole in the s-channel in the physical region.

\(^3\)This is seen explicitly from the following algebra:

\[ (M \mp M_b)a_1 \mp \ell_{ac}\tilde{b}_2)((M \mp M_b)\tilde{c}_2 \pm \ell_{ac}d_1) - ((M \mp M_b)\tilde{a}_2 \pm \ell_{ac}b_1)((M \mp M_b)c_1 \mp \ell_{ac}d_2) \]

\[ = (M \mp M_b)^2(a_1\tilde{c}_2 - \tilde{a}_2c_1) + (M \pm M_b)\ell_{ac}(a_1d_1 - b_1c_1 - \tilde{a}_2d_2 - \tilde{b}_2\tilde{c}_2) + \ell_{ac}^2(b_2d_2 - d_1\tilde{b}_2) \]

\[ = \frac{1}{2}(M \pm M_b)^2\ell_{ac} \pm M M_b\ell_{ac} + \frac{1}{2} \ell_{ac}^2 \ell_{bd} \]

\[ = \frac{1}{2}(M^2 + M_b^2)\ell_{ac} \pm M M_b\ell_{ac} + M^2\ell_{ac} \pm M M_b\ell_{ac} + \frac{1}{2}(M_b^2 - M^2)\ell_{ac} = 0 . \]

\(^4\)The explicit derivation is

\[ \Omega \Sigma|\Phi_{\pm}\rangle = \frac{1}{M} \left[ [(M \mp M_b)(a_1d_1 + \tilde{b}_2d_2) \pm \ell_{ac}(b_1d_2 - d_1\tilde{b}_2)]|\phi\rangle + ([M \mp M_b)(a_1\tilde{c}_2 - c_1\tilde{a}_2) \pm \ell_{ac}(a_1d_1 - \tilde{a}_2d_2)]|\psi\rangle \right] \]

\[ = \frac{1}{2M} [(M \mp M_b)(2C \mp M_b) \mp (M^2_b - M^2)]|\phi\rangle + (M \pm c)\ell_{ac}|\psi\rangle] , \]

\[ \Sigma \Omega|\Phi_{\pm}\rangle = \frac{1}{M} \left[ [(M \mp M_b)(a_1d_1 + \tilde{b}_2d_2) \pm \ell_{ac}(b_1d_2 - d_1\tilde{b}_2)]|\phi\rangle - ([M \mp M_b)(a_1\tilde{c}_2 - c_1\tilde{a}_2) \pm \ell_{ac}(a_1d_1 - \tilde{a}_2d_2)]|\psi\rangle \right] \]

\[ = \frac{1}{2M} [(M \mp M_b)(2C \mp M_b) \pm (M^2_b - M^2)]|\phi\rangle - (M \mp c)\ell_{ac}|\psi\rangle] , \]

\[ = \frac{1}{2M}(2C \mp M) [(M \mp M_b)|\phi\rangle \pm \ell_{ac}|\psi\rangle] . \]
Let us finally make the important observation that the arguments of this section cannot be applied for the $M = 0$ case (such as, for instance, the scattering of two massless particles with the momenta taken at the bound-state point\(^5\)). In this case what we find is the analog of the projective indecomposable representation of section 2.1. In particular, one can check that, at $M = 0$, the state $|\tilde{w}_0^{(0)}\rangle \equiv [\mathcal{O}, \mathcal{G}]|w_0\rangle$ is such that
\begin{align}
\mathcal{O}\mathcal{S} |\tilde{w}_0^{(0)}\rangle = \mathcal{S}\mathcal{O} |\tilde{w}_0^{(0)}\rangle = C |\tilde{w}_0^{(0)}\rangle, \\
\mathcal{O} |\tilde{w}_0^{(0)}\rangle \propto \mathcal{S} |\tilde{w}_0^{(0)}\rangle,
\end{align}
where we have used $M^2 = 4(C^2 - PK) = 0$ to derive the last proportionality statement. However, this is the only state which satisfies these properties, meaning we do not have two solutions to these conditions (as we did in the $M \neq 0$ case above). Therefore, there is only one irreducible 2-dimensional block, containing the states $\{ |\tilde{w}_0^{(0)}\rangle, \mathcal{O}|\tilde{w}_0^{(0)}\rangle \}$, and the 4-dimensional representation is reducible but not fully reducible (i.e. it is indecomposable).

### 3 S-matrix for massive modes of AdS$_2 \times S^2$

In this section we study the S-matrix for the massive modes of the light-cone gauge $AdS_2 \times S^2 \times T^6$ superstring. As mentioned in section 2 from the structure of the symmetry algebra and the integrability of the theory we expect the S-matrix for the massive fields $y$, $z$, $\zeta$ and $\chi$ to be constructed from the graded tensor product of two copies of an S-matrix describing the scattering of $1 + 1$ massive modes, $\phi$ and $\psi$. The former are defined in terms of the latter in (2.1).

The excitations $\phi$ and $\psi$ should transform in the massive representation of $\mathfrak{psu}(1|1) \times \mathbb{R}^3$ discussed in section 2.2. Their S-matrix is then fixed by demanding invariance under this symmetry
\begin{equation}
\Delta^{op}(J) S = S \Delta(J).
\end{equation}
Accounting for conservation of fermion number, the most general form for the S-matrix is
\begin{align}
\mathbb{S} |\phi\phi'\rangle &= S_1 |\phi\phi'\rangle + Q_1 |\psi\psi'\rangle, & \mathbb{S} |\psi\psi'\rangle &= S_2 |\psi\psi'\rangle + Q_2 |\phi\phi'\rangle, \\
\mathbb{S} |\phi\psi'\rangle &= T_1 |\phi\psi'\rangle + R_1 |\psi\phi'\rangle, & \mathbb{S} |\psi\phi'\rangle &= T_2 |\psi\phi'\rangle + R_2 |\phi\psi'\rangle,
\end{align}
where $x^{\pm}$, $m$ are the kinematic variables associated to the first particle and $x'^{\pm}$, $m'$ to the second particle, that is
\begin{align}
x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} &= \frac{2im}{\hbar}, & x'^+ + \frac{1}{x'^+} - x'^- - \frac{1}{x'^-} &= \frac{2im'}{\hbar}.
\end{align}
As a consequence of the discussion in section 2.3 this symmetry will only fix the S-matrix up to two arbitrary functions. One of these functions can be found by requiring the S-matrix also satisfies the Yang-Baxter equation along with additional physical requirements. There are four solutions to the Yang-Baxter equation, two of which we ignore as they violate crossing symmetry. The other two are related by a sign. To fix the sign, we demand that in the BMN limit (for details see section 3.2) the S-matrix reduces to the identity operator. The functions parametrizing the exact S-matrix (3.2) are then
\(^5\)Here by bound-state point we simply mean the value of momenta such that $\Delta^2(C) - \Delta(P)\Delta(K) = (m_1 + m_2)^2 = 0$, namely $\ell_{ac} = 0$ or $\ell_{ad} = 0$. In fact, it is not clear if there is a meaning of bound states for massless scattering [32].
given by\textsuperscript{6,7}

\[
S_1 = \sqrt{\frac{x^+ x^+ - x^+ x^- + 1 + s_1}{2}} \tilde{\mathcal{P}}_0, \quad S_2 = \frac{1 + s_2}{2} \tilde{\mathcal{P}}_0, \\
T_1 = \sqrt{\frac{x^+ x^- - x^+ x^- + 1 + t_1}{2}} \tilde{\mathcal{P}}_0, \quad T_2 = \sqrt{\frac{x^+ x^- - x^+ x^- + 1 + t_2}{2}} \tilde{\mathcal{P}}_0, \\
\frac{Q_2}{\alpha \alpha'} = \alpha' Q_2 = -i \sqrt{\frac{x^+ x^+ - x^+ x^- + f}{x^+ x^+ - x^+ x^-}} \tilde{\mathcal{P}}_0, \quad \frac{\alpha'}{\alpha} R_1 = \frac{\alpha'}{\alpha} R_2 = -i \sqrt{\frac{x^+ x^+ - x^+ x^- + f}{x^+ x^+ - x^+ x^-}} \tilde{\mathcal{P}}_0,
\]

where

\[
f = \sqrt{\frac{x^+ x^-}{x^+ x^-}} - \sqrt{\frac{x^+ x^-}{x^+ x^-}} \frac{x^+ x^-}{x^+ x^-} \frac{x^+ x^-}{x^+ x^-}, \quad s_1 = 1 - \frac{1}{x^+ x^-} f, \quad s_2 = 1 - \frac{1}{x^+ x^-} f, \\
t_1 = 1 - \frac{1}{x^+ x^-} f, \quad t_2 = 1 - \frac{1}{x^+ x^-} f.
\]

\(\tilde{\mathcal{P}}_0\) is an overall factor that sits outside the matrix structure and is not fixed by symmetries or the Yang-Baxter equation. Let us emphasize that, as discussed beneath eq. (2.26), when written in these variables the S-matrix is independent of \(m\) and \(m'\), which can take any value. The limits \(m \to 0\) and \(m' \to 0\) are subtle however, and will be discussed in detail in section 5. Let us also note that if we take \(\alpha\) to be given by (2.32), which is the choice suitable for string theory, then \(Q_1 = Q_2\) and \(R_1 = R_2\). From now on we will take \(\alpha\) to be given by this value.

This S-matrix can be thought of as a 4 \(\times\) 4 block diagonal matrix

\[
\begin{pmatrix}
S_1 & Q_1 & 0 & 0 \\
Q_2 & S_2 & 0 & 0 \\
0 & 0 & T_1 & R_1 \\
0 & 0 & T_2 & R_2
\end{pmatrix}.
\]

One can then check that each of the two 2 \(\times\) 2 blocks have equal trace and determinant,

\[
S_1 + S_2 = T_1 + T_2, \quad S_1 S_2 - Q_1 Q_2 = T_1 T_2 - R_1 R_2.
\]

The second of these equations is particularly important as it implies the tensor product of two copies of the S-matrix possesses an additional \(U(1)\) symmetry, which will be discussed further in section 6 and appendix A.

\textsuperscript{6}Note that here we are choosing the branch so that \((\frac{x^+}{x^-})^\# = (\frac{x^-}{x^+})^{-\#}\) for \(\# = \frac{1}{2}, \frac{1}{4}\) and similarly for \(x^{\pm}x^{\pm}\). For \(p \in [-\pi, \pi]\) this corresponds to taking the branch cut on the negative real axis.

\textsuperscript{7}The solutions that violate crossing symmetry are given by \(f = 0\) and \(f \to \infty\). (For the latter one should first rescale \(\tilde{\mathcal{P}}_0\) by \(f^{-1}\) and then take \(f \to \infty\.) As \(\phi\) and \(\psi\) are real, the two processes

\[
\phi \phi \to \psi \psi \quad \text{and} \quad \phi \psi \to \psi \phi,
\]

should be related by a crossing transformation. However, if \(f\) vanishes then so does the amplitude for the first of these processes, but not for the second. Similarly, if \(f \to \infty\) then the amplitude for the second process vanishes, but not for the first. Consequently, in both cases the two processes cannot be related by a crossing transformation and hence there is a violation of crossing symmetry as claimed. It is interesting to note that taking \(f = 0\) and \(f \to \infty\) we recover the massive S-matrices of the AdS\(_{2}\) \(\times S^3 \times T^4\) light-cone gauge superstring \cite{18, 19}. The symmetry is enhanced accordingly from \(\text{psu}(1|1) \times \mathbb{R}^3\) to \[\mathfrak{u}(1) \oplus \text{psu}(1|1)^2 \times \mathfrak{u}(1) \times \mathbb{R}^3\]. For the AdS\(_{3}\) \(\times S^3 \times T^4\) light-cone gauge-fixed theory there is no issue with crossing symmetry as the fields are complex. Therefore, the individual S-matrices do not map to themselves under the crossing transformation, rather to a different S-matrix with the crossed particle replaced by its antiparticle. Finally let us also point out that the S-matrix relevant for the AdS\(_{2}\) \(\times S^3 \times T^9\) light-cone gauge superstring, see eqs. (3.2) and (3.4), is a linear combination, with coefficients depending on \(x^+\) and \(x^{\pm}\), of the \(f = 0\) and \(f \to \infty\) S-matrices. It is non-trivial that such a combination exists with unitarity, crossing symmetry and the Yang-Baxter equation all satisfied.
Finally, let us remark that a significant difference with respect to the $AdS_5 \times S^5$ S-matrix of [16] is the presence of scattering processes sending two bosons into two fermions (and vice versa) in (3.2). This makes the formal embedding of (3.2) into the $AdS_5 \times S^5$ S-matrix hard to implement. Embedding the algebra $\mathfrak{psu}(1|1) \times \mathbb{R}^3$ into $\mathfrak{psu}(2|2) \times \mathbb{R}^3$ is possible.\(^8\) To embed the S-matrix, one possibility would be that two bosons of the same type scatter to two fermions of the same type in $AdS_5 \times S^5$ (for instance, $\phi^1 \phi^{i} \rightarrow \psi^3 \psi^{4 \prime}$, in the notation of [16]). A scattering process like this one is not present in the $AdS_5 \times S^5$ S-matrix. It is prohibited by the extra $\mathfrak{su}(2)^2$ symmetry which is present in $\mathfrak{psu}(2|2) \times \mathbb{R}^3$ but not in $\mathfrak{psu}(1|1) \times \mathbb{R}^3$.

An alternative would be to allow different $AdS_5 \times S^5$ states for the two particles being scattered, for instance $(\phi^1, \psi^3)$ and $(\phi^{2 \prime}, \psi^{4 \prime})$. Reducing the S-matrix of [16] according to this choice would leave us with many terms, but not the one involving

\[ \phi^1 \psi^{4 \prime} \rightarrow \psi^3 \phi^{2 \prime} . \] (3.9)

This process is not present before reduction, yet it would be necessary to reproduce the $AdS_2$ S-matrix.

### 3.1 The overall factor and crossing symmetry

As currently written the factor $\tilde{P}_a$ is neither a phase factor or antisymmetric. Indeed, given the reality conditions $(x^\pm)^* = x^\mp$ and $(x'^\pm)^* = x'^\mp$, the functions $f$, $s_{1,2}$ and $t_{1,2}$ satisfy the following relations\(^9\)

\[ f^* = f , \quad s_{1,2}^* = s_{2,1} , \quad t_{1,2}^* = t_{2,1} , \] (3.10)

\[ f(x', x) = -f(x', x) , \quad s_{1,2}(x', x) = s_{2,1}(x, x') , \quad t_{1,2}(x', x) = t_{1,2}(x, x') , \] (3.11)

and hence, as a consequence of braiding and QFT unitarity the overall factor should satisfy\(^10\)

\[ \tilde{P}_a \tilde{P}_a^* = \tilde{P}_a(x, x') \tilde{P}_a(x', x) = \frac{4(x^- - x'^+)(x^+ - x^-)}{(x^+ - x^-)(x^- - x'^+)(1 + t_1)(1 + t_2)(1 + t_1) - (x^+ - x^-)(x'^+ - x'^-)} . \] (3.12)

To isolate an antisymmetric phase factor, we can define $P_o$ as follows:

\[ P_o = \sqrt{\det \begin{pmatrix} S_1 & Q_1 \\ Q_2 & S_2 \end{pmatrix}} = \sqrt{\det \begin{pmatrix} T_1 & R_1 \\ R_2 & T_2 \end{pmatrix}} . \] (3.13)

The second equality follows from eq. (3.8). As claimed the unitarity conditions for $P_o$ are then

\[ P_o P_o^* = P_o(x, x') P_o(x', x) = 1 . \] (3.14)
Crossing symmetry provides an additional constraint on the overall factor \( \tilde{P}_0 \), which takes the form

\[
(\tilde{P}_0)_c = s_2 \tilde{P}_0 .
\]  

(3.15)

Here the label \( c \) denotes that the corresponding arguments are taken as \((\bar{x}' \pm, x \pm)\) instead of the original \((x \pm, x' \pm)\) where the “crossed” Zhukovsky variables \( \bar{x} \pm \) are, as usual, given by

\[
\bar{x} \pm = \frac{1}{x} x' ,
\]

(3.16)

corresponding to \( \bar{e} = - e \) and \( \bar{p} = - p \). It is useful to note that we have the following identities

\[
s^c_{1,2} = s^{-1}_{1,2} \quad \quad t^c_{1,2} = t^{-1}_{2,1} \quad \quad f^c = - \frac{x^+ x^-}{f} .
\]

(3.17)

The relation (3.15) translates to the following constraint for \( P_0 \)

\[
\left( \frac{S_1 S_2 + R_1 R_2}{S_1 S_2 - Q_1 Q_2} \right) P_0 = \left( \frac{T_1 T_2 + Q_1 Q_2}{T_1 T_2 - R_1 R_2} \right) P_0 ,
\]

(3.18)

and hence it appears that we either have a simple crossing relation or simple unitarity relations.

Using Hopf algebra arguments, we have checked that crossing symmetry is present for the representation of interest for any value of \( m \) and \( m' \). Denoting the symmetry algebra as \( A \), the antipode \( \Sigma \) is found from the defining rule

\[
\mu (\Sigma \otimes 1) \Delta = \eta \epsilon ,
\]

(3.19)

where \( \mu \) is the multiplication map, \( \eta : \mathbb{C} \to A \) is the unit and \( \epsilon : A \to \mathbb{C} \) is the counit, which annihilates all generators apart from \( 1 \) and \( e^{ip} \) (acting on which, it returns 1). The antipode being a Lie algebra anti-homomorphism, we simply need to derive

\[
\Sigma(\Omega) = - e^{-i \frac{p}{2}} \Omega , \quad \Sigma(\Omega) = - e^{i \frac{p}{2}} \phi , \quad \Sigma(1) = 1 , \quad \Sigma(e^{ip}) = e^{-ip} .
\]

(3.20)

This map is idempotent and therefore equal to its inverse. We impose

\[
\Sigma(\mathfrak{J}(x^\pm)) = \mathcal{C}^{-1} \left[ \begin{array}{c} 1 \\ \pm \frac{1}{x^\pm} \end{array} \right]^{str} \mathcal{C} ,
\]

(3.21)

where \( \mathcal{C} \) is the charge conjugation matrix

\[
\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} ,
\]

(3.22)

and the label \( str \) denotes supertransposition. The fundamental crossing relation for an R-matrix\(^{11}\) is then given by (cf. [20])

\[
(\Sigma \otimes 1) R = R^{-1} = (1 \otimes \Sigma^{-1}) R ,
\]

(3.23)

which projects into representations as

\[
(\mathcal{C}^{-1} \otimes 1) \mathbb{R}^{str} \left( \frac{1}{x^\pm, x'^\pm} \right) (\mathcal{C} \otimes 1) \mathbb{R} (x^\pm, x'^\pm) = 1 \otimes 1 ,
\]

(3.24)

and an analogous equation for the second factor. Here \( str \) denotes the supertranspose for factor \( i \). The S-matrix (3.2) with parametrizing functions (3.4) satisfies this relation provided the overall factor satisfies the crossing equation given in (3.15).

\(^{11}\)For our purposes, S-matrices will be representations of abstract R-matrices.
To conclude this discussion of the overall factor, we compare with the unitarity and crossing relations of the $AdS_5 \times S^5$ light-cone gauge-fixed theory. The amplitude for the scattering process $Y_{11}Y_{11} \rightarrow Y_{11}Y_{11}$, otherwise known as the $SU(2)$ sector, is given by \[16, 31, 17, 33\]

\[
\frac{x^+x^-x'^+x'^-}{x^-x'^+x^+x^-} 2S_o^2 ,
\] (3.25)

where

\[
S_o^2 = \frac{x^+ - x'^+ 1 - \frac{1}{x^+x^-} e^{2i\theta_{bes}} .}
\] (3.26)

Here $\theta_{bes}$ is the BES phase \[22\]. $S_o$ satisfies the following unitarity and crossing relations \[20\]

\[
S_o S_o^\ast = S_o(x,x')S_o(x',x) = 1 , \quad S_o^c = \frac{S_o}{t_2} ,
\] (3.27)

while $\sigma_{bes} = \exp(i\theta_{bes})$ satisfies the following crossing relation

\[
\sigma_{bes}^c = \frac{x^+ s_o}{x^- t_2} \sigma_{bes} .
\] (3.28)

Therefore, comparing with equations (3.12), (3.14), (3.15) and (3.18) we see that if the BES phase is part of the overall factor $\mathcal{P}_o$, or equivalently $\mathcal{P}_o$, the crossing and unitarity relations for the remaining (rational) piece will still be non-trivial.

### 3.2 Comparison with perturbation theory

Defining the effective string tension

\[
h = \frac{R^2}{2\pi \alpha'} ,
\] (3.29)

the tree-level S-matrix for the scattering of massive modes in the light-cone gauge $AdS_5 \times S^2 \times T^6$ superstring following from near-BMN perturbation theory can be found by suitably truncating the corresponding result for $AdS_5 \times S^5$ or $AdS_3 \times S^3 \times T^4 \ [23]$ (various components were also computed in \[15\]). This gives

\[
S_1 = 1 + \frac{i}{4h}[(1 - 2a)(e'p - ep') + l_1] + O(\frac{1}{h^2}) ,
\]

\[
S_2 = 1 + \frac{i}{4h}[(1 - 2a)(e'p - ep') - l_1] + O(\frac{1}{h^2}) ,
\]

\[
T_1 = 1 + \frac{i}{4h}[(1 - 2a)(e'p - ep') - l_2] + O(\frac{1}{h^2}) ,
\]

\[
T_2 = 1 + \frac{i}{4h}[(1 - 2a)(e'p - ep') + l_2] + O(\frac{1}{h^2}) ,
\]

\[
Q_1 = Q_2 = \frac{i}{2h}l_3 + O(\frac{1}{h^2}) , \quad R_1 = R_2 = \frac{i}{2h}l_4 + O(\frac{1}{h^2}) ,
\] (3.30)

where the functions $l_i$ are defined as

\[
l_1(p,p') = \frac{p^2 + p'^2}{e'p - ep'} , \quad l_2(p,p') = \frac{p^2 - p'^2}{e'p - ep'} ,
\]

\[
l_3(p,p') = -\frac{pp'}{2(e'p - ep')} [\sqrt{(e + p)(e' - p')} - \sqrt{(e - p)(e' + p')} ] ,
\]

\[
l_4(p,p') = -\frac{pp'}{2(e'p - ep')} [\sqrt{(e + p)(e' - p')} + \sqrt{(e - p)(e' + p')} ] .
\]
The parameter \( \alpha \) is the standard gauge-fixing parameter of the uniform light-cone gauge [12]. In [15] it was shown that to one-loop the near-BMN dispersion relation is given by

\[ e^2 = 1 + p^2 + \mathcal{O}(h^{-2}) \, . \]  

(3.31)

The one-loop near-BMN result can be constructed via unitarity methods following [34]. Doing so we find that the S-matrix takes the following form

\[
S_1 = \exp\left\{ \frac{i}{4\hbar}(1-2a)(e'p - ep') \right\} \sigma_{AdS_2} \left[ 1 + \frac{i}{4\hbar}l_1 - \frac{\ell}{32\hbar^2} \right] + \mathcal{O}(\frac{1}{h^3}) ,
\]

\[
S_2 = \exp\left\{ \frac{i}{4\hbar}(1-2a)(e'p - ep') \right\} \sigma_{AdS_2} \left[ 1 - \frac{i}{4\hbar}l_1 - \frac{\ell}{32\hbar^2} \right] + \mathcal{O}(\frac{1}{h^3}) ,
\]

\[
T_1 = \exp\left\{ \frac{i}{4\hbar}(1-2a)(e'p - ep') \right\} \sigma_{AdS_2} \left[ 1 - \frac{i}{4\hbar}l_2 - \frac{\ell}{32\hbar^2} \right] + \mathcal{O}(\frac{1}{h^3}) ,
\]

\[
T_2 = \exp\left\{ \frac{i}{4\hbar}(1-2a)(e'p - ep') \right\} \sigma_{AdS_2} \left[ 1 + \frac{i}{4\hbar}l_2 - \frac{\ell}{32\hbar^2} \right] + \mathcal{O}(\frac{1}{h^3}) ,
\]

\[
Q_1 = Q_2 = \exp\left\{ \frac{i}{4\hbar}(1-2a)(e'p - ep') \right\} \sigma_{AdS_2} \left[ \frac{i}{2h}l_3 \right] + \mathcal{O}(\frac{1}{h^3}) ,
\]

\[
R_1 = R_2 = \exp\left\{ \frac{i}{4\hbar}(1-2a)(e'p - ep') \right\} \sigma_{AdS_2} \left[ -\frac{i}{2h}l_4 \right] + \mathcal{O}(\frac{1}{h^3}) ,
\]

where the expansion of the phase factor \( \sigma_{AdS_2} \) is given by

\[
\sigma_{AdS_2} = \exp\left\{ \frac{i}{8\pi \hbar^2} \frac{p^2p'^2((e'p - ep') - (e'e - pp')) \arcsinh[e'p - ep]'\mid}{(e'p - ep)'} + \mathcal{O}(\frac{1}{h^3}) \right\} ,
\]

while

\[
\ell = \frac{p^4 + p'^4 + 2p^2p'^2(e'e - pp')}{(e'p - ep)'^2} ,
\]

is fixed by the requirement of unitarity. As observed in [15] the logarithms are consistent with the overall phase being related to the BES phase [22].

We define the near-BMN expansion of the exact result as follows

\[ e = e \, , \quad m = \rho_3 + \rho_4 h^{-1} + \mathcal{O}(h^{-2}) \, , \quad p = \frac{p}{h(\rho_5 + \rho_6 h^{-1} + \mathcal{O}(h^{-2}))} \, , \]

\[ h = h(\rho_1 + \rho_2 h^{-1} + \mathcal{O}(h^{-2})) \, , \]

(3.35)

and similarly for \( e' \), \( p' \) and \( m' \). Here for generality we have allowed for various rescalings, however, for simplicity we will assume that the \( \rho_i \) are constants.\(^{12}\) Expanding the exact dispersion relation (2.30) in the near-BMN regime, we recover (3.31) if we take

\[ \rho_5 = \rho_1 \, , \quad \rho_6 = \rho_2 \, , \quad \rho_3 = 1 \, , \quad \rho_4 = 0 \, . \]

(3.36)

Further expanding the exact S-matrix (3.4) in the near-BMN regime, taking \( \alpha \) given by (3.32), and fixing the overall factor \( \hat{P}_a \) such that any one of the eight amplitudes agrees with perturbation theory, we find that, so long as

\[ \rho_1 = 1 \, , \]

(3.37)

the remaining seven also agree with perturbation theory, (3.30) and (3.32).

\(^{12}\)To be completely general, one could in principle let \( e, m \) and \( p \) be arbitrary functions of \( e \) and \( p \). However, naively truncating the classical/tree-level results for \( AdS_5 \times S^5 \) and \( AdS_3 \times S^3 \times T^4 \), for example [27, 26], to the massive sector of \( AdS_2 \times S^2 \times T^6 \) the ansatz (3.35) seems reasonable. Of course to check this claim one should construct the light-cone gauge symmetry algebra explicitly.
4 Yangian symmetry

4.1 Massive case

In this section we would like to discuss the issue of Yangian symmetry. The first observation is that, in the massive case (we can fix \( m = m' = 1 \) for the purposes of this section), we could not apply the same standard Yangian symmetry of the R-matrix which works for the massless case (see section 4.2). The massive representation is a long one (cf. section 2.2), and a similar result was found for long representations of \( \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3 \) [24]. The long representations studied in [24] bear a strong resemblance to the ones in this paper, up to the different dimensionality.

We proceed by postulating the commutation relations of the standard \( \mathfrak{sl}(1|1) \) Yangian in Drinfeld’s second realization [35, 36] (with central extensions)

\[
\{ e_m, f_n \} = -\delta_{m+n} , \quad \{ e_m, e_n \} = p_{m+n} , \quad \{ f_m, f_n \} = p^i_{m+n} , \quad [h_m, \cdot] = [p_m, \cdot] = [p^i_m, \cdot] = 0 .
\]

(4.1)

One can check that the coproducts obtained from

\[
\Delta(e_1) = e_1 \otimes 1 + e^{i\Phi} \otimes e_1 + h_0 e^{i\Phi} \otimes e_0 ,
\]

\[
\Delta(f_1) = f_1 \otimes 1 + e^{-i\Phi} \otimes f_1 + f_0 \otimes h_0 ,
\]

and their opposites satisfy the defining relations (4.1) and hence provide homomorphisms of the Yangian. The antipode \( \Sigma \) can be easily found from (4.2) using the defining property

\[
\mu(\Sigma \otimes 1) \Delta = \eta \epsilon ,
\]

(4.3)

where \( \epsilon \) annihilates all level 1 generators. Combined, this defines the Hopf algebra structure of the standard Yangian.

One can construct a family of representations of the Yangian (4.2) starting from a slightly simpler level-zero (Lie algebra) representation compared to the one we use in section 2.2. Determining the level 1 generators in this representation, we can obtain all the central elements up to and including level 2, together with their coproducts and opposite coproducts. Following the strategy of [24], one can check whether all the central coproducts are co-commutative, as this is a necessary condition for the existence of an R-matrix scattering two such representations (see footnote 1). We found that

\[
\Delta^{op}(p_2) \neq \Delta(p_2) ,
\]

(4.4)

for all members of the family of representations. This implies that at least one representation of the standard Yangian does not admit an R-matrix, excluding the existence of a universal R-matrix.

However, it is likely that the massive R-matrix may admit a coproduct which is not precisely the same as for massless representations, but still of the type found in [37]. Moreover, considerations as in footnote 3 of [24] are likely to apply. We leave this investigation for future work.

4.2 Massless case

The situation is different for the massless limit \( m = m' = 0 \). In this case, in the absence of the central extensions (\( b = c = 0 \), i.e. considering again the \( \mathfrak{gl}(1|1) \) algebra), the representation would

\[\footnote{In the absence of non-central Cartan elements, we cannot mechanically generate the level 2 and higher supercharges and they would have to be guessed. However we do not need them for the sake of this argument.}
become one of the reducible but indecomposable modules of section 2.1. In fact, in that case the condition $m = ad - bc = ad = 0$ would force one of the fermionic generators to be identically zero. The indecomposable would then be made up of short 1-dimensional $\mathfrak{gl}(1|1)$ irreps. This suggests that the Yangian might now be straightforwardly derived from the standard one.

Indeed, this time we construct an evaluation representation of the Yangian (4.2)

$$
\epsilon_1 = u \epsilon_0 = u \Omega , \quad f_1 = u f_0 = u \Phi , \quad u = \frac{i \hbar}{x^-} ,
$$

starting from the level 0 one we consider in section 2.2, specializing to $m = 0$. Due to the additional parameters compared to the $\mathfrak{gl}(1|1)$ case, the representation remains generically irreducible. Nevertheless, the obstruction encountered in the massive case is no longer present, i.e. all central charges we can build are co-commutative and in fact the R-matrix (for $m = m' = 0$) can be shown to be invariant under the standard Yangian. This is reminiscent of the $AdS_5 \times S^5$ case, where the Yangian for short representations does not directly transfer to long ones as it stands [37, 38].

The crossing symmetry transformation reveals an interesting property, related to what was observed in [18] for the case of $AdS_3 \times S^3 \times T^4$, namely the existence of two different Yangian spectral (evaluation) parameters for the particle and the anti-particle representations. Here, the difference is superficial, as the massless condition makes the two spectral parameters coincide. In fact, the antipode obtained from applying (4.3) reads

$$
\Sigma(\epsilon_1) = -e^{-i \frac{\hbar}{p} (\epsilon_1 + \epsilon_0 \theta_0)} , \quad \Sigma(f_1) = -e^{i \frac{\hbar}{p} (f_1 + f_0 \theta_0)} .
$$

(4.6)

This effectively amounts to a shift in the spectral parameter $u$ by one of the central elements. When plugging this into the relation

$$
\Sigma(\epsilon_1(x^\pm)) = \mathcal{O}^{-1} \left[ \epsilon_a \left( \frac{1}{x^\pm} \right) \right]^{str} \mathcal{O} ,
$$

(4.7)

and postulating that the anti-particle representation is also of evaluation type, that is

$$
\epsilon_a = u_a \Omega , \quad f_a = u_a \Phi ,
$$

(4.8)

we see that the conditions (4.7) and (4.6) reduce to the same equation that holds true for the level 0 charges, i.e. (3.21), provided that the anti-particle spectral parameter is chosen to be

$$
u_a = \frac{i \hbar}{x^+} .
$$

(4.9)

For massless particles,

$$u = u_a .
$$

(4.10)

5 S-matrix for massless modes

5.1 Derivation from Yangian invariance

The S-matrix describing the scattering of two massless excitations can be directly obtained by imposing Lie algebra and Yangian invariance for two $m = 0$ representations of section 2.2, or as an $m, m' \to 0$ limit of the massive S-matrix. In the latter case, one has to treat various $\theta$ limiting expressions, which come from the function $f$ in eq. (3.5).14 Taking care when resolving these singular limits we find agreement

---

14This is somehow reminiscent of the relativistic case [39].
with the result from imposing Yangian invariance. In the massless limit the dispersion relation in terms of the Zhukovsky variables takes the form [26]\(^{15}\)

\[
x^+ = \frac{1}{x^-}.
\] (5.1)

In terms of the energy and momenta this translates to

\[
e^2 = 4\hbar^2 \sin^2 \frac{P}{2} \quad \Rightarrow \quad e = 2\hbar \left| \sin \frac{P}{2} \right|,
\] (5.2)

and hence there are two branches of the dispersion relation depending on the sign of \(\sin \frac{P}{2}\) [26]\(^{16}\)

\[
x^+ = \sigma e^{i \frac{\pi}{8}}, \quad x^- = \frac{1}{x^+}, \quad \sigma = \pm 1, \quad x'^+ = \sigma' e^{i \frac{\pi}{8}}, \quad x'^- = \frac{1}{x'^+}, \quad \sigma' = \pm 1.
\] (5.5)

In the following we will use the convention that \(\sigma = +1\) corresponds to a particle moving from left spatial infinity to right spatial infinity, \(i.e.\) right-moving, while \(\sigma = -1\) corresponds to a left-moving particle.

For \(\sigma = \sigma' = +1\), the Yangian invariance fixes the S-matrix up to two undetermined functions \(\chi^{++}_{1,2}\):

\[
S_1 = -S_2 = \frac{1}{\sin \frac{1}{4}(p + p')} \left[ \chi^{++}_{1} \sin \frac{1}{4}(p - p') + \chi^{++}_{2} \sqrt{\sin \frac{P}{2} \sqrt{\sin \frac{P'}{2}}} \right],
\]

\[
T_1 = -T_2 = -\chi^{++}_{1},
\]

\[
Q_1 = Q_2 = \frac{1}{\sin \frac{1}{4}(p + p')} \left[ \chi^{++}_{2} \sin \frac{1}{4}(p - p') - \chi^{++}_{1} \sqrt{\sin \frac{P}{2} \sqrt{\sin \frac{P'}{2}}} \right],
\]

\[
R_1 = R_2 = \chi^{++}_{2}.
\]

We have checked that the Yangian representation with the coproducts taken in the appropriate branches and away from the bound-state point (see footnote 5) – is fully reducible simultaneously at level zero and one, which is consistent with the appearance of two undetermined functions in the scattering matrix.

In order to match the limit from the massive S-matrix, the functions \(\chi^{++}_{1,2}\) should be chosen as follows:

\[
\chi^{++}_{2} = -\frac{\sqrt{\sin \frac{P}{2} \sqrt{\sin \frac{P'}{2}}}}{2 \sin \frac{1}{4}(p + p')} \hat{f}^{++}, \quad \chi^{++}_{1} = \left( \hat{f}^{++} - \frac{\sin \frac{1}{4}(p - p')}{2 \sin \frac{1}{4}(p + p')} \right) \hat{f}^{++},
\] (5.6)

where \(\hat{f}^{++}\) is the limit of \(f\). The limit of \(f\) is not fixed by the comparison with the Yangian S-matrix. However, imposing the Yang-Baxter equation

\[
S_{12}^{++} S_{13}^{++} S_{23}^{++} = S_{23}^{++} S_{13}^{++} S_{12}^{++}.
\] (5.7)

\(^{15}\)There is a second solution \(x^+ = x^-\), however, this corresponds to \(p = 0\) and therefore is not physically sensible.

\(^{16}\)Although the doubly-branched dispersion relation \(e = 2\hbar \sin \frac{P}{2}\) is non-relativistic, there are some similarities with the kinematics of massless relativistic scattering. Following [39], in the relativistic case one has

\[
e = \frac{m_0}{2} e^\lambda, \quad p = \pm \frac{m_0}{2} e^u, \quad m_0, u \in \mathbb{R}.
\] (5.3)

A boost sends the rapidity \(u \rightarrow u + \lambda\), with \(\lambda \in \mathbb{R}\), hence the two branches can never be connected by such a transformation. In the non-relativistic case we have the two branches

\[
\frac{i e}{\hbar} = \left[ x^+ - \frac{1}{x^-} \right], \quad \frac{i e}{\hbar} = -2i \log x^+ \in [0, \pi], \quad \frac{i e}{\hbar} = \left[ x^+ - \frac{1}{x^-} \right], \quad \frac{i e}{\hbar} = -2i \log(-x^+) \in [-\pi, 0],
\] (5.4)

with \(x^+\) a pure phase for real momentum and energy. As the S-matrix is not of difference form there is a priori no notion of boosts and hence it is not clear if the presence of two branches represents an obstruction to interpreting the \(\sigma = \sigma' = \pm 1\) scattering. However, as pointed out in [26], while the small momentum dispersion relation is relativistic, for the exact non-relativistic dispersion relation, the group velocity \(v = \frac{\partial}{\partial p}\) is a non-trivial function of \(p\) and hence one may hope to give a physical interpretation to the \(\sigma = \sigma' = \pm 1\) scattering.
requires that
\[ f^{++} = \pm 1, 0. \] (5.8)

The Yang-Baxter equation for \( \sigma = \sigma' = +1 \) scattering (5.7) does not allow for non-constant limits of the function \( f \). In particular, the condition it imposes reads (we denote \( \lim_{m,n \to 0} f(p_i, p_j) \equiv f_{ij}^{++} \))
\[ f_{13}^{++} - f_{23}^{++} + f_{12}^{++} (f_{13}^{++} f_{23}^{++} - 1) = 0 . \] (5.9)

If \( f_{13}^{++} f_{23}^{++} = 1 \), we immediately get \( f^{++} = \pm 1 \). If \( f_{13}^{++} f_{23}^{++} \neq 1 \), we find
\[ f_{12}^{++} = f_{13}^{++} - f_{23}^{++} \frac{1}{1 - f_{13}^{++} f_{23}^{++}} . \] (5.10)

However, the l.h.s. of (5.10) does not depend on \( p_3 \), and hence we should impose that the derivative of the r.h.s. with respect to \( p_3 \) is zero. Doing so, we find that either once again \( f^{++} = \pm 1 \), or, if \( f^{++} \neq \pm 1 \), then
\[ \frac{\partial f_{13}^{++}}{1 - (f_{13}^{++})^2} = \frac{1}{2} \partial_3 \log \left( \frac{1 - f_{13}^{++}}{1 + f_{13}^{++}} \right) \] (5.11)
should be independent of \( p_1 \). Let us call this function \( \omega(p_3) \). This implies that
\[ f_{13}^{++} = \frac{1 - \omega(p_3) \omega(p_3)}{1 + \omega(p_3) \omega(p_3)}, \quad \omega(p_3) = \exp \left[ -2 \int_{p_3} \omega(p_4) dp_4 \right] . \] (5.12)

Plugging this expression back into (5.10) we find that either \( \omega(p) = 0 \), in which case \( f^{++} = 1 \) and we are done, or \( \omega(p) = \omega^{-1}(p) \). Finally, substituting into (5.9) we find that \( \omega(p) \) is a constant and hence \( f^{++} = 0 \). This then demonstrates that the solutions of (5.9) are \( f^{++} = 1, 0 \).

As in the relativistic case [39], a different situation applies for \( \sigma = +1 \), \( \sigma' = -1 \). The Yangian invariance again fixes the S-matrix up to two undetermined functions \( \chi_{1,2}^{++} \):
\[ S_1 = S_2 = \frac{1}{\cos \frac{\pi}{4}(p + p')} \left[ \chi_{1}^{++} \cos \frac{1}{4}(p - p') + i\chi_{2}^{++} \sqrt{\sin \frac{p}{2} \sqrt{-\sin \frac{p'}{2}}} \right] , \]
\[ T_1 = T_2 = \chi_{1}^{++} , \]
\[ Q_1 = Q_2 = \frac{1}{\cos \frac{\pi}{4}(p + p')} \left[ \chi_{2}^{++} \cos \frac{1}{4}(p - p') + i\chi_{1}^{++} \sqrt{\sin \frac{p}{2} \sqrt{-\sin \frac{p'}{2}}} \right] , \]
\[ R_1 = R_2 = \chi_{2}^{++} . \]

Again one can check that the Yangian representation with the coproducts taken in the appropriate branches – and away from the bound-state point (see footnote 5) – is fully reducible simultaneously at level zero and one, which is as before consistent with the appearance of two undetermined functions in the scattering matrix. In order to match the limit from the massive S-matrix, the functions \( \chi_{1,2}^{++} \) should be chosen as follows:
\[ \chi_{2}^{+-} = -i \sqrt{\sin \frac{p}{2} \sqrt{-\sin \frac{p'}{2}}} \eta^{++}_{a} , \quad \chi_{1}^{+-} = \frac{f^{++}}{2} + \frac{\cos \frac{1}{4}(p - p')}{} \eta^{++}_{a} , \] (5.13)
where \( f^{++} \) is the limit of \( f \). For this mixed case the limit of \( f \) is also not fixed by the comparison with the Yangian S-matrix. Once again, the Yang-Baxter equation fixes this limiting value. In order to write down the Yang-Baxter equation for the mixed case, we need to first calculate the S-matrix for \( \sigma = \sigma' = -1 \), as schematically it is given by
\[ S_{12}^{++} S_{13}^{++} S_{23}^{--} = S_{23}^{--} S_{13}^{++} S_{12}^{--} . \] (5.14)
The Yangian invariance again fixes the $\sigma = \sigma' = -1$ S-matrix up to two undetermined functions $\chi_{1,2}^{-}$:

$$
S_1 = -S_2 = \frac{1}{\sin \frac{1}{4}(p + p')} \left[ \chi_1^{--} \sin \frac{1}{4}(p - p') - \chi_2^{--} \sqrt{-\sin \frac{p}{2} \sqrt{-\sin \frac{p'}{2}}} \right],
$$

$$
T_1 = -T_2 = -\chi_1^{--},
$$

$$
Q_1 = Q_2 = \frac{1}{\sin \frac{1}{4}(p + p')} \left[ -\chi_2^{--} \sin \frac{1}{4}(p - p') - \chi_1^{--} \sqrt{-\sin \frac{p}{2} \sqrt{-\sin \frac{p'}{2}}} \right],
$$

$$
R_1 = R_2 = \chi_2^{--}.
$$

In order to match the limit from the massive S-matrix, the functions $\chi_{1,2}^{-}$ have to be chosen as follows:

$$
\chi_2^{--} = \frac{-\sin \frac{p}{2} \sqrt{-\sin \frac{p'}{2}}}{2 \sin \frac{1}{4}(p + p')} \bar{\Phi}_o, \quad \chi_1^{--} = \left( -\frac{f^{--}}{2} - \frac{\sin \frac{1}{4}(p - p')}{2 \sin \frac{1}{4}(p + p')} \right) \bar{\Phi}_o,
$$

where $f^{--}$ is the limit of $f$. The Yang-Baxter equation

$$
S_{12}^- S_{13}^- S_{23}^- = S_{23}^- S_{13}^- S_{12}^-.
$$

fixes this limiting value to

$$
f^{--} = \pm 1, 0.
$$

(5.17)

Taking this result into account, the mixed Yang-Baxter equation (5.14) fixes $f^{+-} = \pm 1$ if one chooses either $f^{--} = 1$ or $f^{--} = -1$, or $f^{+-}$ to any constant if one chooses $f^{--} = 0$.

To exhaust all possibilities, the $\sigma = -1, \sigma' = +1$ S-matrix is given by

$$
S_1 = S_2 = \frac{1}{\cos \frac{1}{4}(p + p')} \left[ \chi_1^{++} \cos \frac{1}{4}(p - p') - i\chi_2^{++} \sqrt{-\sin \frac{p}{2} \sqrt{-\sin \frac{p'}{2}}} \right],
$$

$$
T_1 = T_2 = \chi_1^{++},
$$

$$
Q_1 = Q_2 = \frac{1}{\cos \frac{1}{4}(p + p')} \left[ -\chi_2^{++} \cos \frac{1}{4}(p - p') + i\chi_1^{++} \sqrt{-\sin \frac{p}{2} \sqrt{-\sin \frac{p'}{2}}} \right],
$$

$$
R_1 = R_2 = \chi_2^{++}.
$$

In order to match the limit from the massive S-matrix, the functions $\chi_{1,2}^{++}$ have to be chosen as follows:

$$
\chi_2^{++} = \frac{\sqrt{-\sin \frac{p}{2} \sqrt{\sin \frac{p'}{2}}}}{2 \cos \frac{1}{4}(p + p')} \bar{\Phi}_o, \quad \chi_1^{++} = \left( -\frac{f^{++}}{2} + \frac{\cos \frac{1}{4}(p - p')} {2 \cos \frac{1}{4}(p + p')} \right) \bar{\Phi}_o,
$$

where $f^{+-}$ is the limit of $f$.

By imposing the Yang-Baxter equation for all possible remaining sequences of scattering processes\footnote{Note that here we do not include the following two Yang-Baxter equations:}

$$
S_{12}^+ S_{13}^+ S_{23}^- = S_{23}^- S_{13}^- S_{12}^+ , \quad S_{12}^- S_{13}^- S_{23}^+ = S_{23}^+ S_{13}^+ S_{12}^- ,
$$

as they do not correspond to physically realizable scattering processes. If particles 1 and 3 are both right- or left-moving then they have to scatter with each other before scattering with an excitation travelling in the opposite direction. If we formally include them then the possibilities for the limits of $f$ are reduced to

$$
(f^{++}, f^{+-}, f^{+-}, f^{--}) \in \{(1, 1, 1, 1), (-1, -1, -1, 1), (0, \mu, -\mu, \tilde{\mu}), (\tilde{\mu}, \mu, -\mu, 0)\},
$$

with $\mu$ any constant for $\tilde{\mu} = 0, \mu = \pm 1$ for $\tilde{\mu} = 1$, and $\mu = \pm 1$ for $\tilde{\mu} = -1$. 

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we find the following possibilities for the limits of \( f \):

\[
  f^{++} = \pm 1, \quad f^{-+} = \pm 1, \quad f^{+-} = \pm 1, \quad f^{--} = \pm 1, \quad 0,
\]

\[
  f^{++} = 0, \quad f^{-+} = \mu_1, \quad f^{+-} = \mu_2, \quad f^{--} = 0,
\]  

where \( \mu_1 \) and \( \mu_2 \) are arbitrary constants.

The various choices for \( f^{++}, f^{+-}, f^{-+}, f^{--} \) can be further restricted by considering crossing symmetry. Although in the massless case there is no clear physical interpretation of crossing, see, for example, [39], one may nevertheless demand that it is still present. Let us recall that the crossing transformation simultaneously changes the sign of the energy and momentum, therefore the crossing of a \(+ (−)\) particle is still a \(+ (−)\) particle. Consequently in the crossing relation (3.24) we should consider two massless \(S\)-matrices of the same type. Considering the various possible limits of \( f \), we find that the choices \( f^{++} = 0 \) and \( f^{--} = 0 \) are incompatible with crossing. Indeed, before taking the massless limit, the function \( f \) satisfies the following crossing transformation with respect to the first particle:

\[
f \rightarrow \frac{x^+ x'^-}{f},
\]

which is clearly problematic for \( f \rightarrow 0 \). We are then left with the following choices for the limits of \( f \)

\[
  f^{++} = \pm 1, \quad f^{-+} = \pm 1, \quad f^{+-} = \pm 1, \quad f^{--} = \pm 1.
\]  

(5.21)

It is worth noting that for the crossing relation to be satisfied for these choices we should not only consider two massless \(S\)-matrices of the same type, but also with the same limit of \( f \).

Now that we are left with the choices in eq. (5.21), let us recall that in the massive case the sign of \( f \) is not determined by symmetry or the Yang-Baxter equation, rather from comparing with perturbation theory. This is consistent with the residual ambiguity we are finding in this limit.

If we look at the BMN limit (see section 3.2) for the \( \sigma = \sigma' = \pm 1 \) \(S\)-matrices, we don’t necessarily expect to (and indeed we do not) find the identity. This expectation comes from the fact that the quadratic Lagrangian of the light-cone gauge-fixed theory is relativistic and it is not clear how one should perform a perturbative computation for the scattering of two massless relativistic particles on the same branch, or if there should be a perturbative expansion at all.

For the \( \sigma = -\sigma' = \pm 1 \) \(S\)-matrices one may expect the limit to be better behaved as perturbative computations can be carried out. Indeed, assuming that the phase goes like one plus corrections, then for the \( \sigma = -\sigma' = +1 \) case we find that if \( f^{+-} = 1 \) the \(S\)-matrix is the identity at leading order, while for the \( \sigma = -\sigma' = -1 \) case the same is true, but with \( f^{--} = -1 \). Therefore, we end up with the following choices for the limits of \( f \)

\[
  f^{++} = \pm 1, \quad f^{-+} = 1, \quad f^{+-} = -1, \quad f^{--} = \pm 1.
\]  

(5.22)

We may attribute some physical meaning to this result by considering the group velocities

\[
  v = \frac{\partial \epsilon}{\partial p}, \quad v' = \frac{\partial \epsilon'}{\partial p'}.
\]  

(5.23)

For a physically realizable scattering process with \( \sigma = -\sigma' = +1 \) the group velocities satisfy \( v > v' \), while for a scattering process with \( \sigma = -\sigma' = -1 \) we have \( v' > v \). Therefore, we may associate \( \lim_{m, m' \to 0} f \to 1 \) with \( v > v' \) and \( \lim_{m, m' \to 0} f \to -1 \) with \( v < v' \). This is consistent with the crossing symmetry discussed above as the group velocity is invariant under the crossing transformation. Furthermore, one may expect
the $\sigma = -\sigma' = +1$ and $\sigma = -\sigma' = -1$ S-matrices to be related upon interchanging the arguments.
Indeed, so long as $f^{+} = -f^{-}$, the following equation is satisfied for real momenta
\[ S_{ab}(p, p') |_{f \rightarrow \mp 1} = (-1)^{|a||b|} S_{ba}(p, p')^{*} |_{f \rightarrow \mp 1}. \]
(5.24)
The corresponding relation for the $\sigma = \sigma' = \pm 1$ S-matrices is given by
\[ S_{\pm ab}(p, p') |_{f \rightarrow \mp 1} = (-1)^{|a||b|} S_{\pm ba}(p, p')^{*} |_{f \rightarrow \mp 1}. \]
(5.25)
To conclude, let us briefly comment on unitarity. Motivated by the physical interpretation outlined above, one may expect that braiding unitarity for the massless S-matrix will involve one S-matrix with $f \rightarrow 1$ and one with $f \rightarrow -1$, and indeed, one can explicitly check that braiding unitarity relations can be constructed in this way. They are given by
\[ (-1)^{|c||d|} S_{\pm 
abla ab}(p, p') |_{f \rightarrow \mp 1} S_{\pm \nabla cd}(p, p') |_{f \rightarrow \mp 1} \propto \delta_{a}^{c} \delta_{b}^{d}. \]
(5.26)
These relations can also be found by taking the massless limit of the braiding unitarity relation for the massive S-matrix. Finally, one can see that by combining (5.24), (5.25) and (5.26), all the four massless S-matrices are also QFT unitary so long as the overall factors satisfy appropriate constraints.

5.2 Massless limits and symmetry enhancement

Let us now consider taking the various massless limits of the parametrizing functions of the massive S-matrix, i.e. one massless and one massive or two massive particles. Here we work in terms of the variables $x^{\pm}, x^{\mp}$ as it allows us to consider the four cases of section 5.1 at the same time. For convenience we introduce the following notation for the massless Zhukovsky variables
\[ x = x^{+} = \frac{1}{x}, \quad x' = x'^{+} = \frac{1}{x'}. \]
(5.27)
The parametrizing functions are then given by

\[ S_{1} = T_{1} = -\frac{x'}{\sqrt{x'^{2}}} + \frac{x'}{\sqrt{x'^{2}}} \left( x' + x' \right) + \frac{x'}{\sqrt{x'^{2}}} \left( x' - x' \right) \tilde{\varphi}_{0}, \quad S_{2} = T_{2} = \frac{1}{2\left( 1 - x'+ x' \right)} \tilde{\varphi}_{0}, \]
(5.28)

\[ Q_{1} = Q_{2} = i \sqrt{x'} \frac{1}{\sqrt{x'^{2}}} \frac{x'}{\sqrt{x'^{2}}} \frac{x'}{\sqrt{x'^{2}}} \frac{x'}{\sqrt{x'^{2}}} \tilde{\varphi}_{0}, \quad R_{1} = R_{2} = i \sqrt{x'} \frac{1}{\sqrt{x'^{2}}} \frac{x'}{\sqrt{x'^{2}}} \frac{x'}{\sqrt{x'^{2}}} \tilde{\varphi}_{0}. \]

Massless-Massive
\[ f \rightarrow -x'^{-} \sqrt{x'^{-}}, \]
\[ S_{1} = T_{2} = \frac{\sqrt{x'^{-}}}{x} \left( 1 - x x'^{-} \right) + \frac{\sqrt{x'^{-}}}{x} \left( 1 - x x'^{-} \right) \tilde{\varphi}_{0}, \quad S_{2} = T_{1} = \frac{\sqrt{x'^{-}}}{x} \left( 1 - x x'^{-} \right) \tilde{\varphi}_{0}^{-}, \]
(5.29)

\[ Q_{1} = Q_{2} = i \sqrt{x'} \frac{1}{\sqrt{x'^{2}}} \frac{x'}{\sqrt{x'^{2}}} \frac{x'}{\sqrt{x'^{2}}} \frac{x'}{\sqrt{x'^{2}}} \tilde{\varphi}_{0}, \quad R_{1} = R_{2} = -i \sqrt{x'} \frac{1}{\sqrt{x'^{2}}} \frac{x'}{\sqrt{x'^{2}}} \frac{x'}{\sqrt{x'^{2}}} \tilde{\varphi}_{0}. \]

Here we are defining $|\Phi_{\alpha}^{\nu} \rangle = S_{\alpha}^{\nu}(p, p') |\Phi_{\alpha}^{\nu} \rangle$, $\Phi_{0} = 0$, $\Phi_{1} = \psi$ and $|a| = a$. 

\[ 18 \]
Massless-Massless \( f \to \pm 1 \)

\[
S_1 = -\frac{\sqrt{x^2} x'}{x} \frac{1 - xx' \pm (x - x')}{2(1 - xx')} \tilde{p}_o, \\
T_1 = -\frac{x' - x' \pm (1 - xx')}{2(1 - xx')} \tilde{p}_o, \\
Q_1 = Q_2 = \pm i \frac{\sqrt{x^2}}{x} \frac{x'}{x'} \sqrt{\frac{x^2}{x^2}} \frac{x' \eta' y'}{2(1 - xx')} \tilde{p}_o, \\
R_1 = R_2 = i \sqrt{\frac{x^2}{x^2}} \frac{x' \eta' y'}{2(1 - xx')} \tilde{p}_o.
\]

(5.30)

Given that \( \frac{\sqrt{x^2}}{x} \) and \( \frac{x'}{x'} \) are equal to \( \pm 1 \) one can see that the limit of the function \( f \) is well-defined if we just take one of the two masses to zero. In particular, taking \( m \to 0 \) we have \( f \to x^\pm \sqrt{\frac{x^2}{x^2}} \) while for \( m' \to 0 \) we have \( f \to -x'^\pm \sqrt{\frac{x^2}{x^2}} \).

The factors of \( \frac{\sqrt{x^2}}{x} \) and \( \frac{x'}{x'} \) in (5.30) are the origin of the various expressions for the different choices of \( \sigma \) and \( \sigma' \) in section 5.1. For example, to recover the results of section 5.1 we should take \( \frac{\sqrt{x^2}}{x} = 1 \) for \( \sigma = +1 \) and \( \frac{\sqrt{x^2}}{x} = -1 \) for \( \sigma = -1 \), and similarly for \( x' \). For \( p \in [-\pi, \pi] \), this again corresponds to taking the branch cut on the negative real axis.

We may also consider taking the massless limit of the S-matrices for one massive and one massless excitation. Following the same set of rules as above, i.e. setting \( \frac{\sqrt{x^2}}{x} \) equal to 1 for \( \sigma = +1 \) and \( -1 \) for \( \sigma = -1 \), and similarly for \( x' \), the following table gives the expressions we find for the limits of \( f \)

| Before limit | After limit | Limit of \( f \) |
|--------------|-------------|-------------------|
| Massive - Massless \( \sigma' = +1 \) | Massless-Massless \( \sigma = +1, \sigma' = +1 \) | \( f^{++} = 1 \) |
| Massive - Massless \( \sigma' = +1 \) | Massless-Massless \( \sigma = -1, \sigma' = +1 \) | \( f^{-+} = -1 \) |
| Massive - Massless \( \sigma' = -1 \) | Massless-Massless \( \sigma = +1, \sigma' = -1 \) | \( f^{+-} = 1 \) |
| Massive - Massless \( \sigma' = -1 \) | Massless-Massless \( \sigma = -1, \sigma' = -1 \) | \( f^{-} = -1 \) |
| Massless - Massive \( \sigma = +1 \) | Massless-Massless \( \sigma = +1, \sigma' = +1 \) | \( f^{+} = -1 \) |
| Massless - Massive \( \sigma = +1 \) | Massless-Massless \( \sigma = +1, \sigma' = -1 \) | \( f^{-} = 1 \) |
| Massless - Massive \( \sigma = -1 \) | Massless-Massless \( \sigma = -1, \sigma' = +1 \) | \( f^{-} = -1 \) |
| Massless - Massive \( \sigma = -1 \) | Massless-Massless \( \sigma = -1, \sigma' = -1 \) | \( f^{-} = 1 \) |

Therefore we find the same set of possible limits of \( f \) as found from the analysis in section 5.1, the result of which is given in eq. (5.22).

Finally, from eqs. \((5.28)-(5.30)\) we can see that taking the various massless limits results in many of the parametrizing functions (or products thereof) coinciding. It is clear from the expressions in appendix A that there will then be additional \( U(1) \) symmetries of the S-matrix acting on both the bosons and fermions. This is surely required for these S-matrices to describe the scattering of the massless modes of the light-cone gauge \( AdS_2 \times S^2 \times T^6 \) superstring as they (the bosons and fermions) will transform under various \( U(1) \) symmetries originating from the \( T^6 \) compact space \cite{15}. The precise construction of the S-matrices involving massless modes from the building blocks described above requires the knowledge of the full light-cone gauge symmetry algebra and its action on all the states, as was done for \( AdS_3 \times S^3 \times T^4 \) in \cite{26} and \( AdS_5 \times S^5 \) in \cite{27}.

6 Bethe Ansatz

As discussed at the beginning of section 2 the tensor product of two copies of any S-matrix of the form (3.2) satisfying (3.8) possesses an additional \( U(1) \) symmetry, which does not have a well-defined action
on the individual factor S-matrices. This symmetry is expected from string theory as a consequence of the additional compact space $T^{6}$ required for a consistent 10-d superstring theory.\textsuperscript{19} Under this symmetry the bosons $y$ and $z$ are uncharged, while the fermions $(\zeta, \chi)^{T}$ form an $SO(2)$ vector. Furthermore $(\Omega_{2}, \Omega_{1})^{T}$ and $(\Theta_{2}, \Theta_{1})^{T}$ are also charged as $SO(2)$ vectors under the symmetry.\textsuperscript{20}

Here we will summarize the relevant details of this symmetry. Explicit details (including the expansion of the tensor product) are given in appendix A. Defining

\[ |\theta_{\pm}\rangle = \frac{1}{\sqrt{2}}(|\zeta\rangle \pm i|\chi\rangle), \quad \Theta_{q,\pm} = \frac{1}{\sqrt{2}} (\Theta_{2} \pm i \Theta_{1}) , \quad \Theta_{s,\pm} = \frac{1}{\sqrt{2}} (\Theta_{2} \pm i \Theta_{1}) , \]

and their conjugates, we have the following actions of the $U(1)$ generator, $J_{U(1)}$:

\[ J_{U(1)}|\theta_{\pm}\rangle = \pm i |\theta_{\pm}\rangle, \quad [J_{U(1)}, \Theta_{q,\pm}] = \pm i \Theta_{q,\pm} . \]

To proceed with the algebraic Bethe ansatz (ABA) technique one constructs the monodromy matrix as a string of R-matrices acting on an auxiliary space $a$ and on $N$ physical spaces

\[ T_{a}(\lambda) = R_{a,1} \cdot \ldots \cdot R_{a,N} = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} , \]

where $\cdot$ denotes multiplication in the auxiliary space. $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ are operators on $N$-particle physical space, while the $2 \times 2$ matrix acts on the auxiliary space. As a consequence of the Yang-Baxter equation one has

\[ R_{a_{1},a_{2}}(\lambda_{1} - \lambda_{2}) T_{a_{1}}(\lambda_{1}) T_{a_{2}}(\lambda_{1}) = T_{a_{2}}(\lambda_{1}) T_{a_{1}}(\lambda_{1}) R_{a_{1},a_{2}}(\lambda_{1} - \lambda_{2}) . \]

Taking the trace $tr_{a_{1}} \otimes tr_{a_{2}}$ on both sides of (6.4), one finds that the transfer matrix $T(\lambda) \equiv tr T_{a}(\lambda) = A(\lambda) + D(\lambda)$ satisfies:

\[ [T(\lambda), T(\lambda')] = 0 . \]

As $T(\lambda)$ is an $N$th order polynomial in $\lambda$ (with the highest-power coefficient chosen equal to 1), we see that (6.5) implies that $T(\lambda)$ generates $N$ non-trivial independent commuting operators.

To find the simultaneous eigenvectors of all the commuting charges (which include the Hamiltonian), one assumes that $B(\lambda)$ is a creation operator acting on a pseudo-vacuum $|\text{vac}\rangle$, which is annihilated by $C(\lambda)$:

\[ |\Psi(\lambda_{1},...,\lambda_{M})\rangle = B(\lambda_{1})...B(\lambda_{M}) |\text{vac}\rangle . \]

The pseudo-vacuum should be a highest-weight $T(\lambda)$-eigenstate, whether or not that is the true ground state of the Hamiltonian. The vectors (6.6) are not immediately eigenstates of $T(\lambda)$ because of unwanted terms obtained when acting with $T(\lambda)$. These unwanted terms are cancelled by imposing the Bethe equations, providing the quantization condition for the momenta of excitations.

Let us now give some initial observations on applying the ABA procedure to the S-matrix for the light-cone gauge $AdS_{2} \times S^{2} \times T^{6}$ superstring. We can immediately remark that a single copy of the

\textsuperscript{19}We are grateful to O. Ohlsson Sax and P. Sundin for pointing out to us the existence of this symmetry in the superstring theory.

\textsuperscript{20}Here the subscripts on the supercharges $Q$ and $\Theta$ refer to the two copies of $\mathfrak{psu}(1|1)$ in the full symmetry algebra. In particular the charges with the label 1 act on the first entry in the tensor product (2.1), while the charges with the label 2 on the second entry.
centrally-extended S-matrix does not seem to admit a pseudovacuum on which to construct the ABA procedure. However, when we take the tensor product of two copies there is a pseudovacuum. This is given by a uniform sequence of either all $|\theta_+\rangle$ states or, alternatively, $|\theta_-\rangle$. In fact, thanks to the conservation of the additional $U(1)$ charge discussed above and in appendix A, these states are the only ones with maximal (minimal) such charge, and therefore have to be eigenvalues of the transfer matrix. By a similar logic they are also annihilated by some of the lower-corner entries of the (now 4-dimensional) transfer matrix. This in principle could allow the ABA procedure to be applied. However, this still remains technically challenging given the complexity of the parametrizing functions of the S-matrix.

7 Comments

In this paper we have constructed the S-matrix describing the scattering of massive modes of the $AdS_2 \times S^2 \times T^6$ light-cone gauge superstring. A significant difference with the $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times T^4$ light-cone gauge superstrings is that the massive excitations transform in long representations of the symmetry algebra $\mathfrak{psu}(1|1)^2 \ltimes \mathbb{R}^3$. Consequently there is no shortening condition and the dispersion relation is not fixed by symmetry. Furthermore, the symmetry only fixes the S-matrix up to an overall phase, for which we have given the crossing and unitarity relations, which appear to be more complicated than those in the $AdS_3 \times S^5$ case. The exact form of both the dispersion relation and the phase remain to be determined.

The massless limits (one massive and one massless or two massless particles) of the massive S-matrix have been studied in detail. The resulting expressions should play the role of building blocks for the S-matrices of the massless modes of the $AdS_2 \times S^2 \times T^6$ superstring. As for the $AdS_3 \times S^3 \times T^4$ case [26], the precise nature of this construction requires the knowledge of how all the states transform under the full light-cone gauge symmetry algebra including any additional bosonic symmetries originating from the $T^6$ compact directions.

In the massless limit the light-cone gauge symmetry $\mathfrak{psu}(1|1)^2 \ltimes \mathbb{R}^3$ can be extended to a Yangian of the standard form. However this does not generalize in an obvious way to the massive S-matrix. It would be interesting to see if there exists a non-standard Yangian in this case. We are also currently investigating the presence of the secret symmetry [40, 41] and the RTT realization of the symmetry algebra [42, 43]. Finally, we gave some initial considerations regarding the Bethe ansatz for the massive S-matrix, in particular highlighting the existence of a pseudovacuum. Due to the complexity of the parametrizing functions of the S-matrix and the fact that we are considering long representations of the symmetry algebra the completion of the algebraic Bethe ansatz remains an open problem.

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Appendix A: Expansion of tensor product and U(1) symmetry

In this appendix we will write explicitly the full expression for the tensor product of two copies of the S-matrix given in (3.2). This will allow us to demonstrate the existence of the U(1) symmetry that was important in section 6 for the Bethe ansatz.

Boson-Boson
\[ S|yy'\rangle = S^2_1|yy'\rangle - Q^2_1|zz'\rangle + S_1 Q_1 (|\zeta\zeta'\rangle + |\chi\chi'\rangle) \]
\[ S|zz'\rangle = S^2_2|zz'\rangle - Q^2_2|yy'\rangle - S_2 Q_2 (|\zeta\zeta'\rangle + |\chi\chi'\rangle) \]
\[ S|y'z\rangle = T^2_1|y'z\rangle + R^2_1|zy'\rangle - T_1 R_1 (|\zeta\zeta'\rangle - |\chi\chi'\rangle) \]
\[ S|z'y\rangle = T^2_2|z'y\rangle + R^2_2|yz'\rangle - T_2 R_2 (|\zeta\zeta'\rangle - |\chi\chi'\rangle) \]

Boson-Fermion
\[ S|yc'\rangle = S_1 T_1 |y_c'\rangle - Q_1 R_1 |z\chi'\rangle + S_1 R_1 |c'\chi'\rangle + T_1 Q_1 |c\zeta'\rangle \]
\[ S|\chi'c\rangle = S_1 T_1 |y\chi'\rangle + Q_1 R_1 |z\zeta'\rangle + S_1 R_1 |c\chi'\rangle - T_1 Q_1 |c\zeta'\rangle \]
\[ S|\zeta'c\rangle = S_2 T_2 |y\zeta'\rangle + Q_2 R_2 |y\chi'\rangle - S_2 R_2 |\zeta\chi'\rangle + T_2 Q_2 |y\chi'\rangle \]
\[ S|x\zeta'\rangle = S_2 T_2 |x\chi'\rangle - Q_2 R_2 |x\zeta'\rangle - S_2 R_2 |x\zeta'\rangle - T_2 Q_2 |x\chi'\rangle \]

Fermion-Boson
\[ S|\zeta c'\rangle = S_1 S_2 |\zeta c'\rangle + Q_1 Q_2 |\chi\chi'\rangle + S_1 Q_2 |y\gamma'\rangle - S_2 Q_1 |z\zeta'\rangle \]
\[ S|\chi c'\rangle = S_1 S_2 |\chi c'\rangle + Q_1 Q_2 |\zeta\zeta'\rangle + S_1 Q_2 |y\chi'\rangle - S_2 Q_1 |y\chi'\rangle \]
\[ S|\chi' c\rangle = T_1 T_2 |\chi' c\rangle - R_1 R_2 |\zeta\chi'\rangle - T_1 R_2 |z\zeta'\rangle - T_2 R_1 |y\chi'\rangle \]
\[ S|\chi' c\rangle = T_1 T_2 |\chi' c\rangle - R_1 R_2 |\zeta\chi'\rangle + T_1 R_2 |y\chi'\rangle + T_2 R_1 |y\chi'\rangle \]

(A.1)

Let us now perform a change of basis for the fermionic states
\[ |\theta_\pm\rangle = \frac{1}{\sqrt{2}} (|\zeta\rangle \pm i|\chi\rangle) , \]
(A.2)

such that in this basis the S-matrix has the form

Boson-Boson
\[ S|yy'\rangle = S^2_1|yy'\rangle - Q^2_1|zz'\rangle + S_1 Q_1 (|\theta_+\theta_-\rangle + |\theta_-\theta_+\rangle) \]
\[ S|zz'\rangle = S^2_2|zz'\rangle - Q^2_2|yy'\rangle - S_2 Q_2 (|\theta_+\theta_-\rangle + |\theta_-\theta_+\rangle) \]
\[ S|y'z\rangle = T^2_1|y'z\rangle + R^2_1|zy'\rangle - i T_1 R_1 (|\theta_+\theta_-\rangle - |\theta_-\theta_+\rangle) \]
\[ S|z'y\rangle = T^2_2|z'y\rangle + R^2_2|yz'\rangle - i T_2 R_2 (|\theta_+\theta_-\rangle - |\theta_-\theta_+\rangle) \]

Boson-Fermion
\[ S|y\theta'\rangle = S_1 T_1 |y\theta'\rangle \pm i Q_1 R_1 |z\theta'\rangle + S_1 R_1 |\theta_+\gamma'\rangle \mp i T_1 Q_1 |\theta_\pm\gamma'\rangle \]
\[ S|z\theta'\rangle = S_2 T_2 |z\theta'\rangle \mp i Q_2 R_2 |y\theta'\rangle - S_2 R_2 |\theta_\pm\zeta'\rangle \pm i T_2 Q_2 |\theta_\pm\zeta'\rangle \]
Fermion-Boson
\[ S(\theta_\pm y') = S_1 T_2 (\theta_\pm y') + i Q_1 R_2 |\theta_\pm z'\rangle + S_1 R_2 |y\theta_\pm'\rangle \pm i T_1 Q_2 |z\theta_\pm'\rangle \]
\[ S(\theta_\pm z') = S_2 T_1 (\theta_\pm z') + i Q_2 R_1 |\theta_\pm y'\rangle - S_2 R_1 |z\theta_\pm'\rangle \pm i T_2 Q_1 |y\theta_\pm'\rangle \]

Fermion-Fermion
\[ S(\theta_\pm \theta_\mp') = \frac{1}{2} (S_1 S_2 + Q_1 Q_2 + T_1 T_2 + R_1 R_2) |\theta_\pm \theta_\mp'\rangle + \frac{1}{2} (S_1 S_2 - Q_1 Q_2 - T_1 T_2 - R_1 R_2) |\theta_\mp \theta_\pm'\rangle \]
\[ + S_1 Q_2 |y\theta_\pm'\rangle - S_2 Q_1 |z\theta_\mp'\rangle \pm i T_1 R_2 |y\theta_\mp'\rangle + i T_2 R_1 |z\theta_\pm'\rangle \]
\[ S(\theta_\mp \theta_\pm') = \frac{1}{2} (S_1 S_2 - Q_1 Q_2 + T_1 T_2 - R_1 R_2) |\theta_\mp \theta_\pm'\rangle + \frac{1}{2} (S_1 S_2 - Q_1 Q_2 - T_1 T_2 + R_1 R_2) |\theta_\pm \theta_\mp'\rangle \]
(A.3)

Provided that
\[ S_1 S_2 - Q_1 Q_2 = T_1 T_2 - R_1 R_2 , \] (A.4)
which was indeed the case for the S-matrix under consideration in the main text (3.8), it is clear that this S-matrix commutes with a \( U(1) \) symmetry acting on the states as follows
\[ \mathcal{G}_{U(1)} |y\rangle = 0 , \quad \mathcal{G}_{U(1)} |z\rangle = 0 , \quad \mathcal{G}_{U(1)} |\theta_\pm\rangle = \pm i |\theta_\pm\rangle . \] (A.5)

Finally for completeness we give the commutation relations of the full algebra under which the S-matrix is invariant. First let us define
\[ \mathcal{G}_{q\pm} = \frac{1}{\sqrt{2}} (\Omega_2 \pm i \Omega_1) , \quad \mathcal{G}_{s\pm} = \frac{1}{\sqrt{2}} (\Theta_2 \pm i \Theta_1) , \] (A.6)
where the subscripts on the supercharges \( \Omega \) and \( \Theta \) refer to the two copies of \( \mathfrak{psu}(1|1) \) in the full symmetry algebra. In particular the charges with the label 1 act on the first entry in the tensor product (2.1), while the charges with the label 2 on the second entry.

The full set of non-vanishing (anti-)commutation relations are then given by \(^{21}\)
\[ [\mathcal{G}_{U(1)}, \Omega_i] = \epsilon_{ij} \Omega_j , \quad [\mathcal{G}_{U(1)}, \Theta_i] = \epsilon_{ij} \Theta_j , \]
\[ \{ \Omega_i, \Omega_j \} = 2 \delta_{ij} \mathcal{P} , \quad \{ \Theta_i, \Theta_j \} = 2 \delta_{ij} \mathcal{R} , \quad \{ \Omega_i, \Theta_j \} = 2 \delta_{ij} \mathcal{C} , \] (A.7)
or alternatively in the complex basis
\[ [\mathcal{G}_{U(1)}, \mathcal{G}_{q,s\pm}] = \pm i \mathcal{G}_{q,s\pm} , \quad \{ \mathcal{G}_{q\pm}, \mathcal{G}_{q\mp} \} = 2 \mathcal{P} , \quad \{ \mathcal{G}_{s\pm}, \mathcal{G}_{s\mp} \} = 2 \mathcal{R} , \quad \{ \mathcal{G}_{q\pm}, \mathcal{G}_{s\mp} \} = 2 \mathcal{C} . \] (A.8)

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\(^{21}\)Here \( \epsilon_{12} = 1 = - \epsilon_{21} \) is the usual antisymmetric tensor.
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