An adaptive Douglas-Rachford dynamic system for finding the zero of sum of two operators

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Abstract The Douglas-Rachford splitting method is a popular method for finding a zero of the sum of two operators, and has been well studied as a numerical method. Nevertheless, the convergence theory for its continuous versions is still in infancy. In this paper, we develop an adaptive Douglas-Rachford dynamic system with perturbations or computational errors for finding a zero of the sum of two operators, one of which is strongly monotone while the other one is weakly monotone, in a real Hilbert space. With proper tuning the parameters, we demonstrate that the trajectory of the adaptive dynamic system converges weakly to a fixed point of the adaptive Douglas-Rachford operator. Even if there is a perturbation, we show that its shadow trajectory strongly converges to a solution of the original problem, and the rate of asymptotic regularity of the adaptive Douglas-Rachford operator is \( O\left(\frac{1}{\sqrt{t}}\right) \). Furthermore, global exponential-type rates of the trajectory of the adaptive dynamic system are achieved under a metric subregularity or Lipschitz assumption.

Keywords Operator inclusions · Dynamic system · Douglas-Rachford algorithm · Quasi-nonexpansive · Computational errors · Exponential-type convergence

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1 Introduction

Let $H$ be a real Hilbert space endowed with the scalar $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$. In this paper, we consider the following inclusion problem:

$$\text{find } x \in H \text{ such that } 0 \in A(x) + B(x),$$

where $A, B : H \rightrightarrows H$ be two operators. This problem models a variety of tasks in diverse applied fields such as signal processing, machine learning and statistics [12, 21, 32, 58]. A popular and powerful method for solving problem (1) is the Douglas-Rachford (DR) algorithm [30, 31, 47, 40, 11]. In its formulation the operator is decomposed into simpler individuals which are then processed separately in the subproblems, hence, DR algorithm is often referred to as a splitting algorithm. In 1956, this splitting algorithm was introduced originally by Douglas and Rachford [30] to solve heat conduction flow problems in a finite dimensional space. The original splitting scheme is

$$\begin{align*}
\frac{1}{\lambda} \left( u_{k+\frac{1}{2}} - u_k \right) + A \left( u_{k+\frac{1}{2}} \right) + B (u_k) &= 0 \\
\frac{1}{\lambda} \left( u_{k+1} - u^{k+\frac{1}{2}} \right) + B (u_{k+1}) - B (u_k) &= 0,
\end{align*}$$

where $A$ and $B$ are single-valued linear monotone operators. Eliminating $u^{k+\frac{1}{2}}$, and defining $z_k = (J_{\lambda B})^{-1} u_k$, one can rewrite the above scheme as

$$z_{k+1} = J_{\lambda A} \left( 2 J_{\lambda B} - I \right) z_k + (I - J_{\lambda B}) z_k,$$

where $J_{\lambda A}$ and $J_{\lambda B}$ are the resolvent operators of $A$ and $B$ respectively, and $I$ is the identity operator in $H$. The algorithm (2) is just the classic DR algorithm model. In 1979, Lions and Mercier [47] made the algorithm applicable to the problem with $A$ and $B$ being set-valued nonlinear operators. They proved in a Hilbert space that the algorithm converges weakly to a point which can be used to solve the problem. Later, Svaiter [56] revealed that the shadow sequence associated with DR algorithm is weakly convergent to a solution. In 1992, Eckstein and Bertsekas [31] further analyzed the DR algorithm with summable errors as well as with over/under relaxation. Moreover, interpretation of DR algorithm as a proximal point method and an alternating direction method of multipliers (ADMM) can be dated back to [44] and [34], respectively. We refer to [46] for a good overview on the DR algorithm and its applications.

Compared with the rich literature for DR algorithms with the involved operators being monotone or strongly monotone, the convergence theory for weakly monotone settings is far from being complete. When $A$ and $B$ are the subdifferentials of a strongly convex function and a weakly convex function respectively, some nice works on convergence analysis for DR algorithms can
be found in [11, 37, 36]. Recently, Dao and Phan [24] proposed the following adaptive DR algorithm for problem (1):

\[ z_{k+1} = \tilde{T}z_k := (1 - \epsilon)z_k + \epsilon R_{\delta B}^\mu R_{\gamma A}^\lambda z_k, \]  

where \( \epsilon \in (0, 1), \lambda, \mu, \gamma, \delta > 0 \) and

\[ J_{\gamma A} = ((1+\gamma A)^{-1}, \quad R_{\gamma A}^\lambda = (1 - \lambda)I + \lambda J_{\gamma A}, \]  

\[ J_{\delta B} = ((1+\delta B)^{-1}, \quad R_{\delta B}^\mu = (1 - \mu)I + \mu J_{\delta B}. \]

As a more general scheme than (2), algorithm (3) has a nice flexibility thanks to its adjustable parameters. This plays an important role in its convergence analysis. With proper tuning the parameters, Dao and Phan [24] proved the convergence of the sequence generated by algorithm (3) under the conditions that \( A \) is strongly monotone while \( B \) is weakly monotone. They further demonstrated that the algorithm enjoys global linear convergence under the additional condition that \( A \) or \( B \) is Lipschitz continuous. Dao and Phan’s results generalized and improved several contemporary works such as [11, 37, 36].

In this paper, motivated by algorithm (3), we take into account a continuous time dynamic system for solving problem (1). Historically, the study of continuous time dynamic systems for solving optimization problem can be traced back at least to 1950s [5]. Later on, the interplay between continuous and discrete dynamic systems has been studied by many authors, e.g., [3, 4, 23, 33, 13, 7, 1, 2, 51, 15, 18, 16, 17]. In [3], Antipin proposed the following dynamic system:

\[ \frac{dx}{dt} = P_H(x - \epsilon \nabla h(x)) - x, \]  

where \( P_H \) is the metric projection operator onto a nonempty closed convex set \( \Omega \subset \mathbb{R}^n \), \( \epsilon > 0 \) and \( \nabla h(u) \) is the gradient of a convex and continuously differentiable function \( h : \mathbb{R}^n \to \mathbb{R} \). Under certain conditions, the trajectory of (4) converges to a solution of the following convex program problem

\[ \min_{x \in \Omega} h(x). \]  

In [15], Bot and Csetnek proposed a dynamic system governed by a nonexpansive operator, which can be regarded as a continuous version of the Krasnosel’ski˘i-Mann algorithm. By relying on Lyapunov analysis, Bot and Csetnek [15] proved the weak convergence of the trajectory to a fixed point of the operator and showed also an order of convergence of \( O(\sqrt{1/t}) \) for the fixed point residual of the trajectory of the dynamical system. In [7], Attouch and Svaiter developed the dynamic system

\[ \begin{cases} v(t) \in A(x(t)) \\ \lambda(t) \frac{dv}{dt} + \frac{dv}{dt} + v(t) = 0, \end{cases} \]

for approaching problem (1) with \( B \) vanishing. Its explicit time discretization with step size \( \Delta t_k > 0 \) is nothing but the Levenberg-Marquardt algorithm

\[ A(x_k) + (\lambda_k I + A'(x_k)) \left( \frac{x_{k+1} - x_k}{\Delta t_k} \right) = 0. \]
In [60], Zhu et al. proposed dynamic systems for solving the minimum value problem of the sum of a strongly convex function and a weakly convex function, which can be cast as problem (1). The systems proposed in [60] can be regarded as special cases of the following dynamic system

$$\begin{align*}
\dot{z}(t) &= \theta [J_{AA} (2JAB - I) z(t) - JAB(z(t))] \\
z(t_0) &= z_0 \in \mathbb{H},
\end{align*}$$

(5)

whose an explicit discretization with $\theta = 1$ is nothing but the DR algorithm (2). Very recently, Csetnek et al. [23] proposed and analyzed a new iterative algorithm

$$z_{k+1} = J_{AA} (z_k - \lambda B (z_k)) - \lambda [B (z_k) - B (z_{k-1})]$$

(6)

for problem (1) with the involved operators being monotone. As pointed out by Csetnek et al. [23], algorithm (6) can also be regarded as a non-standard discretization of dynamic system (5). Motivated by the above works, we endow with the continuous behavior with time for DR algorithm (3), and propose the following dynamic system with a perturbation for problem (1):

$$\begin{align*}
\frac{du}{dt} + \theta(t) [u(t) - R^\mu_{\delta B} R^\lambda_{\gamma A}(u(t))] &= f(t), \\
u(t_0) &= u_0 \in \mathbb{H},
\end{align*}$$

(7)

where $t_0 > 0$, $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally integrable function and $f : \mathbb{R}_+ \to \mathbb{H}$ is a locally integrable operator as a perturbation or computational error. The explicit discretization of the above dynamic system with respect to the time variable $t$, with step size $\Delta t_k > 0$, yields the following iterative scheme

$$\frac{u_{k+1} - u_k}{\Delta t_k} = \theta_k [R^\mu_{\delta B} R^\lambda_{\gamma A}(u_k) - u_k] + f_k, \quad k \in \mathbb{N}. $$

(8)

After transposition and setting $\epsilon_k = \theta_k \Delta t_k$, iterative scheme (8) reduces to

$$x_{k+1} = (1 - \epsilon_k) x_k + \epsilon_k R^\mu_{\delta B} R^\lambda_{\gamma A}(x_k) + \Delta t_k f_k, \quad k \in \mathbb{N},$$

which is a relaxed version of DR algorithm (3) with computing errors. This paper is devoted to the convergence analysis of dynamic system (7). Prior to this, we study an abstract dynamic system governed by a quasi-nonexpansive operator. Main contributions of this paper can be summarized as follows.

a) We show that the trajectory of the abstract dynamic system governed by a quasi-nonexpansive operator converges weakly to a fixed point of the operator (see Theorem 1). A global exponential-type convergence rate is achieved under a metric subregularity (see Theorem 8).

b) We endow with the continuous behavior with time for the adaptive DR algorithm (3), and propose an adaptive Douglas-Rachford dynamic system with perturbations or computational errors (see dynamic system (7)). We tune the parameters properly so that the adaptive Douglas-Rachford operator is quasi-nonexpansive, and the corresponding results come into being
An adaptive Douglas-Rachford dynamic system (see Theorem 4, Theorem 5 and Theorem 9). Meanwhile, it is also shown that the shadow trajectory of the adaptive system strongly converges to a solution of problem (1) (see Theorem 4).

c) With appropriately chosen parameters, the rate $O\left(\frac{1}{\sqrt{t}}\right)$ of asymptotic regularity of $R^\mu_{\delta B} R^\lambda_{\gamma A}$ is achieved even if there is a perturbation (see Theorem 7). Moreover, the global exponential-type convergence rate of the trajectory of the adaptive system can be achieved under a Lipschitz assumption instead of the metric subregularity (see Theorem 11 and Theorem 12).

In comparison with existing algorithms considered in [24], [37] and [36], our dynamic system has more mild convergence conditions (see Remark 5 and Remark 13). The exponential-type rate is dependant on the design of $\theta$ and the construction of $f$. This allows the proposed dynamical systems to enjoy varied convergence rates (see Remark 7). The obtained results can be applied to deal with the minimum value problem of the sum of two functions, one of which is strongly convex while the other one is weakly convex (see Sect. 5).

The rest of this paper is organized as follows. In Section 2, we recall some notions and results for further analysis. Then we analyze the global weak convergence of the proposed dynamical systems in Section 3. The exponential-type convergence rate of the proposed dynamic systems is established in Section 4. Finally, dynamic system (7) is applied to solve a class of minimum value problems.

2 Preliminaries

In this section, we recall some notions and results that are useful in this paper. Throughout, we denote the set of real numbers by $\mathbb{R}$, the set of nonnegative real numbers by $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$, and the set of positive real numbers by $\mathbb{R}_+^+ := \{x \in \mathbb{R} \mid x > 0\}$. We use $w - \lim_{t \to +\infty} x(t) = x^*$ to indicate that the operator $x(t)$ converges weakly to $x^*$ as $t \to +\infty$.

Definition 1 (See, e.g., [8, Definition 4.1] or [53, Definition 9.1]) A function $h : \Omega \subset \mathbb{H} \to \mathbb{H}$ is said to be

(i) Lipschitz continuous with constant $L > 0$ on $\Omega$ if

$$\|h(y) - h(x)\| \leq L\|y - x\|, \quad \forall x, y \in \Omega;$$

(ii) nonexpansive on $\Omega$ if it is Lipschitz continuous with constant 1 on $\Omega$, i.e.,

$$\|h(y) - h(x)\| \leq \|y - x\|, \quad \forall x, y \in \Omega;$$

(iii) quasi-nonexpansive on $\Omega$ if

$$\|h(y) - x\| \leq \|y - x\|, \quad \forall y \in \Omega, \forall x \in \text{Fix } h := \{x \in \Omega \mid h(x) = x\};$$
(iv) uniformly continuous on $\Omega$ if for every real number $\varepsilon > 0$ there exists $\eta > 0$ such that for every $x, y \in \Omega$ with $\|x - y\| < \eta$, we have that $\|h(x) - h(y)\| < \varepsilon$.

**Definition 2** (See, e.g. [7, Definition 2.1]) A function $F : [0, b] \rightarrow \mathbb{H}$ (where $b > 0$) is said to be absolutely continuous if one of the following equivalent properties holds:

(i) there exists an integrable function $G : [0, b] \rightarrow \mathbb{H}$ such that

$$F(t) = F(0) + \int_0^t G(s) ds, \quad \forall t \in [0, b];$$

(ii) $F$ is continuous and its distributional derivative is Lebesgue integrable on $[0, b]$;

(iii) for every $\varepsilon > 0$, there exists $\eta > 0$ such that for any finite family of intervals $I_k = (a_k, b_k) \subset [0, b]$ we have the implication:

$$\left( I_k \cap I_j = \emptyset \text{ and } \sum_k |b_k - a_k| < \eta \right) \Rightarrow \sum_k \|F(b_k) - F(a_k)\| < \varepsilon.$$

**Remark 1** Let us recall some basic nature of absolutely continuous functions:

(a) In the light of the definitions above, Lipschitz continuity implies absolute continuity, which gives rise to uniform continuity. An absolutely continuous function is differentiable almost everywhere.

(b) If $F_1 : [0, b] \rightarrow \mathbb{H}$ is absolutely continuous and $F_2 : \mathbb{H} \rightarrow \mathbb{H}$ is $L$-Lipschitz continuous, then their composition function $F = F_2 \circ F_1$ is absolutely continuous. Moreover, $F$ is almost everywhere differentiable and the inequality $\|F'()\| \leq L \|F_2'()\|$ holds almost everywhere. (See, e.g. [15, Remark 1]).

**Definition 3** (See, e.g., [24, Definition 3.1]) A set-valued operator $A : \mathbb{H} \rightrightarrows \mathbb{H}$ is said to be $\alpha$-monotone with constant $\alpha \in \mathbb{R}$ if,

$$\langle x - y, u - v \rangle \geq \alpha \|x - y\|^2, \quad \forall (x, u), (y, v) \in \text{gra} A,$$

where

$$\text{gra} A := \{(x, u) : u \in Ax\}.$$

Moreover, we also say that $A$ is maximally $\alpha$-monotone if it is $\alpha$-monotone and there is no $\alpha$-monotone operator whose graph strictly contains $\text{gra} A$.

Apparently, $A$ is (resp. maximally) $\alpha$-monotone if and only if $A - \alpha I$ is (resp. maximally) monotone. We also note that if $\alpha > 0$, $\alpha = 0$, $\alpha < 0$, then $\alpha$-monotonicity can be referred to as strong monotonicity, monotonicity and weak monotonicity, respectively. In [53, Example 12.28], the weak monotonicity is also called hypomonotonicity. It was shown in [24] that $\alpha$-monotonicity of a single-valued operator along with continuity leads to maximal $\alpha$-monotonicity. For detailed discussions on maximal monotonicity and its variants as well as the connection to optimization problems, we refer the reader to [8, 14].

The following lemma comes from [24, Proposition 3.4, Corollary 3.11, Corollary 3.12].
Lemma 1 Let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be a maximally $\alpha$-monotone operator and $R := (1 - \nu)I + \nu J_{\gamma A}$ with $\nu, \gamma > 0$. Suppose that $1 + \gamma \alpha > 0$. Then the following statements are true:

(i) $J_{\gamma A}$ is single-valued and Lipschitz continuous (with constant $\frac{1}{1 + \gamma \alpha}$).

(ii) If $\nu \geq 1$, then, for all $x, y \in \text{dom } J_{\gamma A}$,

$$\|Rx - Ry\|^2 \leq (1 - \nu)^2\|x - y\|^2 + \nu[(1 - \nu)(2 + 2\gamma \alpha) + \nu]\|J_{\gamma A} - J_{\gamma A}\|^2.$$ 

(iii) If $A$ is Lipschitz continuous with constant $l$ and $\nu(1 + 2\gamma \alpha) - 2(1 + \gamma \alpha) \geq 0$, then $R$ is Lipschitz continuous with constant

$$\sqrt{(\nu - 1)^2 - \frac{\nu((\nu - 1)(2 + 2\gamma \alpha) - \nu)}{1 + 2\gamma \alpha + \gamma^2 l^2}}.$$ 

Lemma 2 ([20, Lemme A.5]) Let $F : [a, b] \to [0, +\infty)$ be an integrable function and $G : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that $c \geq 0$ and

$$\frac{1}{2} G^2(t) \leq \frac{1}{2} t^2 + \int_a^b F(s)G(s)ds$$

for all $t \in [a, b]$. Then $|G(t)| \leq c + \int_a^b F(s)ds$ for all $t \in [a, b]$.

Lemma 3 ([1, Lemma 5.1]) Let $F : [t_0, +\infty) \to \mathbb{R}$ be a locally absolutely continuous, bounded below function, and $G \in L^1([t_0, +\infty))$. Suppose that

$$\frac{d}{dt} F(t) \leq G(t)$$

for almost all $t$. Then there exists $\lim_{t \to +\infty} F(t) \in \mathbb{R}$.

Lemma 4 (See, e.g., [38, Lemma 1.2.2]) Let $t_0 \in \mathbb{R}$ and $h : [t_0, +\infty) \to \mathbb{H}$ be a uniformly continuous function. If $\int_{t_0}^{+\infty} \|h(s)\|ds < +\infty$, then

$$\lim_{t \to +\infty} h(t) = 0.$$ 

Lemma 5 (See, e.g., [1, Lemma 5.3]) Let $\Omega \subseteq \mathbb{H}$ be a nonempty set and $x : [t_0, +\infty) \to \mathbb{H}$ be a given map. Suppose

(i) for every $z \in \Omega$, $\lim_{t \to +\infty} \|x(t) - z\| = 0$ exists;

(ii) every weak sequential cluster point of the map $x$ belongs to $\Omega$.

Then there exists $x_\infty \in \Omega$ such that $w - \lim_{t \to +\infty} x(t) = x_\infty$. 

Definition 4 (See [29, p.183]) A set-valued operator $F : \mathbb{H} \rightrightarrows \mathbb{H}$ is said to be metrically subregular at $z^*$ for $y^*$ if $y^* \in F(z^*)$ and there exists $\kappa \geq 0$, along with an open ball $B(z^*, r) := \{ y \in \mathbb{H} \mid ||y - z^*|| < r \}$, such that

$$\text{dist}(z, F^{-1}(y^*)) \leq \kappa \cdot \text{dist}(y^*, F(z)), \quad \forall z \in B(z^*, r),$$

where dist($z, \Omega$) = $\min_{y \in \Omega} ||z - y||$.

Remark 2 For a set-valued operator $F$ and a vector $y^*$, as Dontchev and Rockafellar pointed out in [29], metric subregularity gives an estimate for how far a point $z$ is from being a solution to the generalized equation $F(z) \ni y^*$ in terms of the “residual” dist($y^*, F(z)$). The constant $\kappa$ measures the stability under perturbations of inclusion $y^* \in F(z)$. We refer the reader to [29] for more details.

Before we finish this section, let us review a useful identity, which will be used several times in the following sections. Its proof is straightforward and so we omit it. For all $x, y \in \mathbb{H}$ and all $\epsilon, \rho \in \mathbb{R}$,

$$||\epsilon x + \rho y||^2 = \epsilon(\epsilon + \rho)||x||^2 + \rho(\epsilon + \rho)||y||^2 - \epsilon\rho||x - y||^2. \quad (9)$$

3 Asymptotic analysis

In this section, we analyze the global convergence for the proposed dynamic system. To this end, we first consider an abstract dynamic system:

$$\begin{cases}
\frac{du}{dt} = \theta(t)[T(u(t)) - u(t)] + f(t) \\
u(t_0) = u_0 \in \mathbb{H},
\end{cases} \quad (10)$$

where $\theta(t), f(t)$ are same as in (7), and $T$ is an operator from $\mathbb{H}$ to $\mathbb{H}$.

Remark 3 The abstract dynamic system (10) with $T$ being nonexpansive and $\theta(t) \equiv 1$ is just the nonautonomous evolution equation considered in [19, Section 5]. When $T$ is nonexpansive and $f(t) \equiv 0$, (10) reduces to the dynamic system studied in [15]. The dynamic system (10) can also be regarded as a continuous version of the inexact Krasnosel’ski-Mann algorithm considered in [45,19]. Indeed, the explicit discretization of dynamic system (10) with respect to the time variable $t$, with step size $\Delta t_k > 0$, yields the inexact Krasnosel’ski-Mann algorithm

$$u_{k+1} = \Theta_k T u_k + (1 - \Theta_k) u_k + \xi_k, \quad k \in \mathbb{N},$$

where $\Theta_k = \theta_k \Delta t_k$ and $\xi_k = f_k \Delta t_k$. The convergence analysis for the inexact Krasnosel’ski-Mann algorithm with $T$ being nonexpansive were discussed in [45,19] where the former focuses on $\mathbb{H}$ and the latter on a Banach space.

We say that $u : [t_0, +\infty) \to \mathbb{H}$ is a strong global solution of dynamic system (10) if and only if the following properties are satisfied:
(i) \( u : \mathbb{R}_+ \rightarrow \mathbb{H} \) is absolutely continuous on each interval \([t_0, b], t_0 < b < +\infty\);

(ii) \( \frac{du}{dt} = \theta(t)[T(u(t)) - u(t)] + f(t) \) for almost every \( t \geq t_0 \);

(iii) \( u(t_0) = u_0 \).

**Definition 5** (see, e.g., [8, Definition 4.26]) An operator \( F : \mathbb{H} \rightarrow \mathbb{H} \) is said to be demiclosed at 0 (or that \( F \) satisfies the demiclosedness principle), if one has \( F(x^*) = 0 \) for any sequence \( \{x_n\} \) such that \( x_n \) converges weakly to \( x^* \) and \( F(x_n) \) converges strongly to 0.

**Remark 4** According to the Browder’s demiclosedness principle (see, e.g., [8, Theorem 4.27]), \( I - F \) is demiclosed at every point in \( \mathbb{H} \) when \( F \) is nonexpansive. It is worth mentioning that \( I - F \) need not to be demiclosed when \( F \) is only quasi-nonexpansive.

3.1 Abstract convergence

Unless otherwise stated, throughout this paper, we always assume that \( \theta : [t_0, +\infty) \rightarrow \mathbb{R}_+ \) is locally integrable and \( f \in L^1([t_0, +\infty)) \). Moreover, the following two assumptions about \( \theta \) are used in the rest of the paper and refer to them when appropriate.

\[
\text{(A1)} : \inf_{t \in [t_0, +\infty)} \theta(t) > 0; \quad \text{(A2)} : \int_{t_0}^{+\infty} \theta(s)ds = +\infty.
\]

It is clear that (A1) implies (A2).

**Theorem 1** Let \( \mathcal{T} : \mathbb{H} \rightarrow \mathbb{H} \) be quasi-nonexpansive and \( \text{Fix}(\mathcal{T}) \neq \emptyset \). Suppose that there exists a strong global solution \( u(t) \) of dynamic system (10). Then the following statements are true:

(i) \( \int_{t_0}^{+\infty} \theta(s)\|T(u(s)) - u(s)\|^2ds < +\infty \).

(ii) If \( \mathcal{T} \) is uniformly continuous on \( \mathbb{H} \) and assumption (A1) holds, then

\[
\lim_{t \to +\infty} \|T(u(t)) - u(t)\| = 0. \quad (11)
\]

(iii) If the equation (11) holds and \( \mathcal{T} - I \) satisfies the demiclosedness principle, then there exists \( \hat{u} \in \text{Fix}(\mathcal{T}) \) such that \( w - \lim_{t \to +\infty} u(t) = \hat{u} \).

**Proof** Taking arbitrarily a \( u^* \in \text{Fix}(\mathcal{T}) \), we consider the following auxiliary function:

\[
V(t) = \|u(t) - u^*\|^2.
\]

Computing the time derivative of \( V(t) \), we have

\[
\frac{dV}{dt} = 2 \left( u(t) - u^* \frac{du}{dt} \right)
\]

\[
= 2\theta(t) \langle u(t) - u^*, T(u(t)) - u(t) \rangle + 2\langle u(t) - u^*, f(t) \rangle
\]

\[
= \theta(t) \left( \|T(u(t)) - u^*\|^2 - \|T(u(t)) - u(t)\|^2 - \|u(t) - u^*\|^2 \right)
\]

\[
+ 2\langle u(t) - u^*, f(t) \rangle
\]

\[
\leq -\theta(t) \|T(u(t)) - u(t)\|^2 + 2\|u(t) - u^*\| \|f(t)\|
\]

\[
\leq -\theta(t) \|T(u(t)) - u(t)\|^2 + 2\|u(t) - u^*\| \|f(t)\|. \quad (12)
\]
where the first inequality obtains by the quasi-nonexpansiveness of \( \mathcal{T} \). Integrating this inequality from \( t_0 \) to \( t \), we have

\[
V(t) - V(t_0) \leq - \int_{t_0}^{t} \theta(s) \| \mathcal{T}(u(s)) - u(s) \|^2 ds + 2 \int_{t_0}^{t} \| u(s) - u^* \| \| f(s) \| ds,
\]

(13)

which gives

\[
V(t) \leq V(t_0) + 2 \int_{t_0}^{t} \| u(s) - u^* \| \| f(s) \| ds.
\]

(14)

According to Lemma 2, we obtain from (14) that

\[
\| u(t) - u^* \| \leq \| u_0 - u^* \| + \int_{t_0}^{t} \| f(s) \| ds,
\]

which implies that the trajectory \( u(t) \) is bounded on \([t_0, +\infty)\) on account of \( f \in L^1([t_0, +\infty)) \). Hence, there exists \( M \in \mathbb{R}_{++} \) such that for all \( t \)

\[
\| u(t) - u^* \| \leq M.
\]

(15)

In the rest of the proof, we verify the statements (i)-(iii) in order.

a) It is from (13) and (15) that

\[
\int_{t_0}^{t} \theta(s) \| \mathcal{T}(u(s)) - u(s) \|^2 ds \leq V(u_0) + 2M \int_{t_0}^{t} \| f(s) \| ds.
\]

Taking \( t \to +\infty \) and noting that \( f \in L^1([t_0, +\infty)) \), we get (i).

b) Suppose that \( \mathcal{T} \) is uniformly continuous on \( \mathbb{H} \) and assumption \((A1)\) holds. Then, the latter along with (i) yields

\[
\int_{t_0}^{+\infty} \| \mathcal{T}u(s) - u(s) \|^2 ds < +\infty.
\]

On the other hand, by the absolute continuity of the trajectory \( u(t) \), \( \mathcal{T}u(t) \) is uniformly continuous with respect to \( t \), so is \( \mathcal{T}u(t) - u(t) \). Consequently, according to Lemma 4, we deduce (11) directly, and so (ii) holds.

c) We verify the last assertion via Lemma 5. Firstly, suppose that the equation (11) holds, and that \( \mathcal{T} - I \) satisfies the demiclosedness principle. From (12) and (15), we have

\[
\frac{d}{dt} \| u(t) - u^* \|^2 \leq 2M \| f(t) \|,
\]

which gives rise to the existence of \( \lim_{n \to +\infty} \| u(t) - u^* \| \) by Lemma 3. Since \( u^* \in \text{Fix}(\mathcal{T}) \) has been chosen arbitrary, the first assumption in Lemma 5 is fulfilled. On the other hand, in view of the boundedness of the trajectory \( u(t) \), let \( \hat{u} \in \mathbb{H} \) be a weak sequential cluster point of \( u(t) \), that is, there
exists a sequence \( t_n \to +\infty \) (as \( n \to +\infty \)) such that \( w - \lim_{n \to +\infty} u(t_n) = \hat{u} \).

Applying the demiclosedness principle of \( \mathcal{T} - I \) and (11), we have \( \hat{u} \in \text{Fix}\mathcal{T} \), and the assertion (iii) follows from Lemma 5. The proof is complete.

Now, we shall discuss an another abstract convergence for dynamic system (10).

**Theorem 2** Let \( u(t) \) be a strong global solution of dynamic system (10) and let \( \mathcal{T}_1, \mathcal{T}_2 : \mathbb{H} \to \mathbb{H} \) be two operators. Suppose that assumption \( \text{(A1)} \) holds and the following conditions are satisfied:

\( \text{(H1)} \) \( \mathcal{T}_1, \mathcal{T}_2 \) and \( \mathcal{T} \) are uniformly continuous, single-valued and have full domain.

\( \text{(H2)} \) \( I - \mathcal{T} = \mu (\mathcal{T}_1 - \mathcal{T}_2), \mu \neq 0. \)

\( \text{(H3)} \) \( \text{Fix}\mathcal{T} \neq \emptyset. \)

\( \text{(H4)} \) For all \( t \in \mathcal{T}, x^+ \in \text{Fix}\mathcal{T}, \)

\[
\|\mathcal{T}x - x^+\|^2 \leq \|x - x^+\|^2 - \omega_1 \|\mathcal{T}_1x - \mathcal{T}_1x^+\|^2 - \omega_2 \|\mathcal{T}_2x - \mathcal{T}_2x^+\|^2, \tag{16}
\]

where \( w_i \in \mathbb{R}, i = 1, 2, \) subject to

either \( \{\omega_1 = \omega_2 = 0\}; \)
or \( \left\{ \omega_1 + \omega_2 > 0 \text{ and } \frac{\omega_1\omega_2}{\mu(\omega_1 + \omega_2)} > -1 \right\} \).

\( \tag{17} \)

Then for any \( u^* \in \text{Fix}\mathcal{T} \), the following statements hold:

(i) \( \int_0^{+\infty} \|\mathcal{T}u(s) - u(s)\|^2ds < +\infty. \)

(ii) \( \lim_{t \to +\infty} \|\mathcal{T}u(t) - u(t)\| = 0. \)

(iii) If \( \mathcal{T} - I \) satisfies the demiclosedness principle, then there exists \( \hat{u} \in \text{Fix}\mathcal{T} \) such that \( w = \lim_{t \to +\infty} u(t) = \hat{u} \).

(iv) If \( \omega_1 + \omega_2 > 0, \) then

\[
\int_0^{+\infty} \|\omega_1 (\mathcal{T}_1u(s) - \mathcal{T}_1u^*) + \omega_2 (\mathcal{T}_2u(s) - \mathcal{T}_2u^*))\|^2ds < +\infty,
\]

\[
\lim_{t \to +\infty} \|\omega_1 (\mathcal{T}_1u(t) - \mathcal{T}_1u^*) + \omega_2 (\mathcal{T}_2u(t) - \mathcal{T}_2u^*))\| = 0,
\]

and \( \lim_{t \to +\infty} \mathcal{T}_1u(t) = \mathcal{T}_1u^* = \lim_{t \to +\infty} \mathcal{T}_2u(t) = \mathcal{T}_2u^*. \)

**Proof Set**

\[
\omega'_1 := \begin{cases} \frac{\omega_1\omega_2}{\omega_1 + \omega_2} & \text{if } \omega_1 + \omega_2 > 0, \\ 0 & \text{if } \omega_1 = \omega_2 = 0 \end{cases} \quad \text{and} \quad \omega'_2 := \begin{cases} \frac{1}{\omega_1 + \omega_2} & \text{if } \omega_1 + \omega_2 > 0 \\ 0 & \text{if } \omega_1 = \omega_2 = 0 \end{cases}.
\]

Then, by (17),

\[
1 + \frac{\omega'_1}{\mu^2} > 0 \quad \text{and} \quad \omega'_2 > 0. \tag{18}
\]
For all \( x, y \in \mathbb{H} \), we derive from (9) and the assumption \((H_2)\) that
\[
\begin{align*}
\omega_1 \| T_1 x - T_1 y \|^2 + \omega_2 \| T_2 x - T_2 y \|^2 \\
= \omega_1 \| (T_1 - T_2) x - (T_1 - T_2) y \|^2 \\
+ \omega_2 \| (T_1 x - T_1 y) + \omega_2 (T_2 x - T_2 y) \|^2 \\
= \omega_1^2 \| (I - T)x - (I - T)y \|^2 \\
+ \omega_2 \| (T_1 x - T_1 y) + \omega_2 (T_2 x - T_2 y) \|^2.
\end{align*}
\]
This combines with (16) and (18) implies that \( T \) is quasi-nonexpansive. So (i),(ii) and (iii) hold by Theorem 1.

Next, to verify (iv), we consider the auxiliary function: \( V(t) = \| u(t) - u^* \|^2 \).
Computing the time derivative of \( V(t) \), we have
\[
\frac{dV}{dt} = 2 \left< u(t) - u^*, \frac{du}{dt} \right> \\
= 2 \langle u(t) - u^*, \theta(t) (Tu(t) - u(t)) \rangle + 2 \langle u(t) - u^*, f(t) \rangle \\
= \| u(t) - u^* + \theta(t) (Tu(t) - u(t)) \|^2 - \| u(t) - u^* \|^2 \\
- \theta(t) \| Tu(t) - u(t) \|^2 + 2 \langle u(t) - u^*, f(t) \rangle \\
= \| \theta(t) (Tu(t) - u^*) + (1 - \theta(t)) (u(t) - u^*) \|^2 - \| u(t) - u^* \|^2 \\
- \theta(t) \| Tu(t) - u(t) \|^2 + 2 \langle u(t) - u^*, f(t) \rangle \\
\leq \theta(t) \| Tu(t) - u^* \|^2 - \theta(t) \| u(t) - u^* \|^2 \\
- \theta(t) \| Tu(t) - u(t) \|^2 + 2 \| u(t) - u^* \| \| f(t) \|.
\]
By (16) and (19), it ensues that
\[
\frac{dV}{dt} \leq -\omega_1 \theta(t) \| T_1 u(t) - T_1 u^* \|^2 - \omega_2 \theta(t) \| T_2 u(t) - T_2 u^* \|^2 \\
- \theta(t) \| Tu(t) - u(t) \|^2 + 2 \| u(t) - u^* \| \| f(t) \| \\
\leq -\theta(t) \left( \omega_1 \| T_1 u(t) - T_1 u^* \|^2 + \omega_2 \| T_2 u(t) - T_2 u^* \|^2 \right) \\
- \theta(t) \| Tu(t) - u(t) \|^2 + 2 \| u(t) - u^* \| \| f(t) \| \\
= -\frac{\omega_1 \theta(t)}{\mu^2} \| (I - T)u(t) - (I - T)u^* \|^2 \\
- \theta(t) \omega_1 \| T_1 u(t) - T_1 u^* \|^2 + \omega_2 \| T_2 u(t) - T_2 u^* \|^2 \\
- \theta(t) \| Tu(t) - u(t) \|^2 + 2 \| u(t) - u^* \| \| f(t) \| \\
= -\frac{\omega_1 \theta(t)}{\mu^2} \| (I - T)u(t) - (I - T)u^* \|^2 \\
- \theta(t) \omega_1 \| T_1 u(t) - T_1 u^* \|^2 + \omega_2 \| T_2 u(t) - T_2 u^* \|^2 \\
- \theta(t) \| Tu(t) - u(t) \|^2 + 2 \| u(t) - u^* \| \| f(t) \|. 
\]
If $\omega_1 + \omega_2 > 0$, then $\omega_2' = \frac{\omega_2}{\omega_1 + \omega_2} > 0$. Similar to the proof of Theorem 1 (i) and Theorem 1 (ii), one gets the first two conclusions of the assertion (iv). Let us verify the last one. Observer that $T_1 u^* = T_2 u^*$ thanks to the assumption $(H_2)$ and $u^* \in \text{Fix}(T)$. It is from (ii) and the assumption $(H_2)$ that

$$
\lim_{t \to +\infty} |T_1 u(t) - T_1 u^* - (T_2 u(t) - T_2 u^*)| = \lim_{t \to +\infty} \frac{1}{\mu} (T u(t) - u(t)) = 0.
$$

Together with the second conclusion in (iv) and noting that $\omega_1 + \omega_2 > 0$, we derive $\lim_{t \to +\infty} T_1 u(t) = T_1 u^*$ and $\lim_{t \to +\infty} T_2 u(t) = T_2 u^*$. So (iv) follows. The proof is complete.

In the theorems above, we have to assume that the strong global solution $u(t)$ of dynamic system (10) exists, and that the assumptions $(H_1)$-$(H_4)$ hold. These can be guaranteed when $T = \mathcal{R}_{\mu B}^{\mu} \mathcal{R}_{\gamma A}^{\lambda}$ with the appropriate choice of parameters. This will be seen in the following subsection.

### 3.2 Existence and uniqueness of the trajectory of dynamic system (7)

Recall

$$
\begin{align*}
J_{\gamma A} &= (I + \gamma A)^{-1}, \quad R_{\gamma A}^{\lambda} = (1 - \lambda)I + \lambda J_{\gamma A}, \\
J_{\delta B} &= (I + \delta B)^{-1}, \quad R_{\delta B}^{\mu} = (1 - \mu)I + \mu J_{\delta B}.
\end{align*}
$$

(22)

With proper tuning the parameters, the next proposition presents certain nice property of $R_{\delta B}^{\mu} R_{\gamma A}^{\lambda}$. Let us mention that the first assertion of the proposition follows from [24, Proposition 3.4], (ii) and (iii) from [24, Lemma 4.1], and the last one from [24, Proposition 4.3].

**Proposition 1** Let $A : \mathcal{H} \rightrightarrows \mathcal{H}$ and $B : \mathcal{H} \rightrightarrows \mathcal{H}$ be respectively maximally $\alpha$- and $\beta$-monotone. Suppose that the parameters $\gamma, \delta, \lambda, \mu$ in (22) satisfy

$$
\begin{align*}
\min \{1 + \gamma \alpha, 1 + \delta \beta\} &> 0, \\
(\lambda - 1)(\mu - 1) &> 0, \quad \text{and} \quad \delta = (\lambda - 1)\gamma.
\end{align*}
$$

Then the following statements are true:

(i) $J_{\gamma A}, J_{\delta B},$ and $R_{\delta B}^{\mu} R_{\gamma A}^{\lambda}$ are single-valued and have full domain.

(ii) $1 - R_{\delta B}^{\mu} R_{\gamma A}^{\lambda} = \mu (J_{\gamma A} - J_{\delta B} R_{\gamma A}^{\lambda}).$

(iii) $J_{\gamma A} (\text{Fix}(R_{\delta B}^{\mu} R_{\gamma A}^{\lambda})) = J_{\delta B} R_{\gamma A}^{\lambda} (\text{Fix}(R_{\delta B}^{\mu} R_{\gamma A}^{\lambda})) = \text{zer}(A + B)$.

(iv) If $\lambda, \mu \geq 1$, then for all $x, y \in \mathcal{H}$,

$$
\begin{align*}
\|R_{\delta B}^{\mu} R_{\gamma A}^{\lambda} x - R_{\delta B}^{\mu} R_{\gamma A}^{\lambda} y\|^2 \\
&\leq \|x - y\|^2 - \mu(2 + 2\gamma \alpha - \mu) \|J_{\gamma A} x - J_{\gamma A} y\|^2 \\
&\quad - \mu(\mu - (2 - 2\gamma \beta)) \|J_{\delta B} R_{\gamma A}^{\lambda} x - J_{\delta B} R_{\gamma A}^{\lambda} y\|^2.
\end{align*}
$$

(24)
According to Lemma 1 and Proposition 1, the operator $R^\mu_B R^\lambda_A$ is Lipschitz continuous if the parameters $\gamma, \delta, \lambda, \mu$ are subject to (23), and such a property makes for a guarantee to bring about existence and uniqueness of a strong global solution of dynamic system (7). Indeed, dynamic system (7) can be written as
\[
\begin{aligned}
dt u &= F(t, u) \\
ut(t_0) &= u_0 \in \mathbb{H},
\end{aligned}
\]
where $F : [t_0, +\infty) \times \mathbb{H} \to \mathbb{H}$ is defined by $F(t, u) = \theta(t)(Tu - u) + f(t)$. Applying the Lipschitz continuity of $T$, the local integrability of $\theta(\cdot)$ and $f \in L^1([t_0, +\infty))$, one can easily verify that the conditions of the Cauchy-Lipschitz theorem (see e.g. [39], Proposition 6.2.1, [55], Theorem 54, [57], Corollary 2.6) are satisfied. In this way, we get a strong global solution of dynamic system (7). In addition, the solution is a classical solution of class $C^1$ if the functions $\theta(t)$ and $f(t)$ are continuous.

In view of the discussion above, an immediate conclusion follows:

**Theorem 3** Let $A : \mathbb{H} \ni x \mapsto \mathbb{H}$ and $B : \mathbb{H} \ni \mathbb{H}$ be respectively maximally $\alpha$- and $\beta$-monotone. Suppose that (23) holds. Then for each initial point $u_0 \in \mathbb{H}$, there exists a unique strong global solution (trajectory) $u(t)$ of dynamic system (7) in the global time interval $[t_0, +\infty)$.

### 3.3 Convergence of the trajectories

We note that $R^\mu_B R^\lambda_A$ is nonexpansive on $\mathbb{H}$, provided that all parameters occurring in (24) cater for
\[
\mu(2 + 2\gamma\alpha - \mu) \geq 0 \quad \text{and} \quad \mu(\mu - (2 - 2\gamma\beta)) \geq 0.
\]

(25)

Now we are going to discuss that how the parameters play a role in the convergence analysis of dynamic system (7). Consider the following two parametric options:

**(C1)** $\mu = \lambda = 2$, $\gamma = \delta \in \mathbb{R}^+$, and $\alpha + \beta > 0$ and $\frac{\gamma\alpha\beta}{\alpha + \beta} > -1$.

(26)

**(C2)** $\alpha + \beta \in \mathbb{R}_+$ and $(\gamma, \delta, \lambda, \mu) \in \mathbb{R}_+^2 \times [1, +\infty)^2$ satisfy
\[
\begin{aligned}
&1 + 2\gamma\alpha > 0, \\
&\mu \in [2 - 2\gamma\beta, 2 + 2\gamma\alpha], \\
&\lambda - 1)(\mu - 1) = 1, \quad \text{and} \quad \delta = (\lambda - 1)\gamma.
\end{aligned}
\]

(27)

It is clear that (27) implies (25), and so $R^\mu_B R^\lambda_A$ is nonexpansive in case (C2). However, case (C1) alone is not sufficient for such a property to be guaranteed. Indeed, in this case, the expression (24) reduces to
\[
\|R^\mu_B R^\lambda_A x - R^\mu_B R^\lambda_A y\|^2 \leq \|x - y\|^2 - 4\gamma\Phi(\alpha, \beta, x, y), \quad \forall x, y \in \mathbb{H},
\]
where

\[ \Phi(\alpha, \beta, x, y) = \alpha \|J_{\gamma A}x - J_{\gamma A}y\|^2 + \beta \|J_{\delta B}R^\lambda_{\gamma A}x - J_{\delta B}R^\lambda_{\gamma A}y\|^2. \]

Note that \( \Phi(\alpha, \beta, x, y) \) is not necessarily nonnegative even if \( \alpha + \beta > 0 \) (a similar discussion can be found in [37]). Thus, some existing results depending on the nonexpansiveness of an operator are not applicable in case \((C1)\). Fortunately, we notice that if \( y \) is confined to \( \text{Fix}(R^\mu_{\delta B}R^\lambda_{\gamma A}) \), then \( \Phi(\alpha, \beta, x, y) \) is always nonnegative by a parallel derivation of \((19)\), that is, \( R^\mu_{\delta B}R^\lambda_{\gamma A} \) is quasi-nonexpansive in case \((C1)\). This allows us to fall back on results of Subsect. 3.1.

**Proposition 2** Suppose that the parameters \( \gamma, \delta, \lambda, \mu \) in \((22)\) satisfy either \((C1)\) or \((C2)\). Then \((23)\) holds.

**Proof** Observe that \((\lambda - 1)(\mu - 1) = 1 \) and \( \delta = (\lambda - 1)\gamma \) are evident in both case, and that \((C2)\) implies \( \min \{1 + \gamma \alpha, 1 + \delta \beta\} > 0 \) by [24, Lemma 4.4]. So, we just need to verify the inequality in case \((C1)\). It is from \((26)\) that

\[ 1 + \gamma \alpha = \left( 1 + \frac{\gamma \alpha \beta}{\alpha + \beta} \right) + \frac{\gamma \alpha^2}{\alpha + \beta} > 0 \]

and that

\[ 1 + \delta \beta = 1 + \gamma \beta = \left( 1 + \frac{\gamma \alpha \beta}{\alpha + \beta} \right) + \frac{\gamma \beta^2}{\alpha + \beta} > 0. \]

These yield the desired result. The proof is complete.

We are now in position to establish the weak convergence of dynamic system \((7)\) in case \((C1)\) and \((C2)\). Note that the sum \( A + B \) is strong convex in case \((C1)\) due to \( \alpha + \beta > 0 \), which gives rise to that problem \((1)\) has a unique solution. We then learn from Lemma 1 (i), Proposition 1 (iii) and Proposition 2 that \( \text{Fix}(R^\mu_{\delta B}R^\lambda_{\gamma A}) \neq \emptyset \) in case \((C1)\).

**Theorem 4** Let \( A : \mathbb{H} \rightrightarrows \mathbb{H} \) and \( B : \mathbb{H} \rightrightarrows \mathbb{H} \) be respectively maximally \( \alpha \)- and \( \beta \)-monotone. Suppose that the parameters \( \alpha, \beta, \gamma, \delta, \lambda, \mu \) satisfy \((C1)\), and that assumption \((A1)\) holds. Let \( u(t) \) be the trajectory of dynamic system \((7)\), and let \( u^* \in \text{Fix}(R^\mu_{\delta B}R^\lambda_{\gamma A}) \). Then the following statements are true:

(i) \( \int_t^{+\infty} \|R^\mu_{\delta B}R^\lambda_{\gamma A}u(s) - u(s)\|^2 ds < +\infty \).

(ii) \( \lim_{t \to +\infty} \|R^\mu_{\delta B}R^\lambda_{\gamma A}u(t) - u(t)\| = 0 \).

(iii) If \( R^\mu_{\delta B}R^\lambda_{\gamma A} - I \) satisfies the demiclosedness principle, then there exists \( \hat{u} \in \text{Fix}(R^\mu_{\delta B}R^\lambda_{\gamma A}) \) such that \( w - \lim_{t \to +\infty} u(t) = \hat{u} \).

(iv) \( \int_t^{+\infty} \|\alpha (J_{\gamma A}u(s) - J_{\gamma A}u^*) + \beta (J_{\delta B}R^\lambda_{\gamma A}u(s) - J_{\delta B}R^\lambda_{\gamma A}u^*)\|^2 ds < +\infty \).

(v) \( \lim_{t \to +\infty} \|\alpha (J_{\gamma A}u(t) - J_{\gamma A}u^*) + \beta (J_{\delta B}R^\lambda_{\gamma A}u(t) - J_{\delta B}R^\lambda_{\gamma A}u^*)\| = 0 \).

(vi) \( \lim_{t \to +\infty} J_{\gamma A}u(t) = J_{\gamma A}u^* = \lim_{t \to +\infty} J_{\delta B}R^\lambda_{\gamma A}u(t) = J_{\delta B}R^\lambda_{\gamma A}u^* = \text{zer}(A + B) \).
Proof We prove this theorem by virtue of Theorem 2 with $T_1 = J_{\gamma A}$, $T_2 = J_{\delta B} R_{\gamma A}^\lambda$ and $T = R_{\delta B}^\mu R_{\gamma A}^\lambda$. Let us set $w_1 = 4\gamma \alpha$ and $w_2 = 4\gamma \beta$. Then both obey (17) by (26). In order to fulfil the assumptions (H1)-(H4) of Theorem 2 it suffices to show that (C1) implies (23). This is immediate by Proposition 2. The proof is complete.

Remark 5 It turns out from Theorem 4 (vi) that $J_{\gamma A} u(t)$ and $J_{\delta B} R_{\gamma A}^\lambda u(t)$ globally strongly converge to the unique solution of problem (1). Moreover, for the case:

$$\lambda = \mu = 2, \quad \gamma = \delta \in \mathbb{R}_{++} \quad \text{and} \quad \alpha + \beta > 0,$$

the convergence of the DR algorithm (3) researched in [24] requires the following constraint:

$$\frac{\gamma \alpha \beta}{\alpha + \beta} > \epsilon - 1, \quad 0 < \epsilon < 1.$$

In contrast, a more mild condition

$$\frac{\gamma \alpha \beta}{\alpha + \beta} > -1$$

is required in Theorem 4.

In what follows, we turn our attention to case (C2) in which $R_{\delta B}^\mu R_{\gamma A}^\lambda$ is nonexpansive. We learn from Remark 4 that $I - R_{\delta B}^\mu R_{\gamma A}^\lambda$ is demiclosed at 0. Let us set $w_1 = \mu(2 + 2\gamma \alpha - \mu) \geq 0$ and $w_2 = \mu(\mu - (2 - 2\gamma \beta)) \geq 0$. Then $w_1$ and $w_2$ cater for (17). On the other hand, it follows from Proposition 2 that (C2) implies (23). Thus, the assumptions (H1)-(H4) of Theorem 2 with $T_1 = J_{\gamma A}$, $T_2 = J_{\delta B} R_{\gamma A}^\lambda$ and $T = R_{\delta B}^\mu R_{\gamma A}^\lambda$ are fulfilled in case (C2), provided that $\text{zer}(A + B) \neq \emptyset$. By the analysis above, we derive the following theorem immediately.

Theorem 5 Let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ and $B : \mathbb{H} \rightrightarrows \mathbb{H}$ be respectively maximally $\alpha$- and $\beta$-monotone. Suppose that the parameters $\alpha, \beta, \gamma, \delta, \lambda, \mu$ satisfy (C2), $\text{zer}(A + B) \neq \emptyset$, and that assumption (A1) holds. Let $u(t)$ be the trajectory of dynamic system (7). Then, for any $u^* \in \text{Fix}(R_{\delta B}^\mu R_{\gamma A}^\lambda)$, the following statements are true:

(i) $\int_{t_0}^{+\infty} \| R_{\delta B}^\mu R_{\gamma A}^\lambda u(s) - u(s) \|^2 ds < +\infty.$

(ii) $\lim_{t \to +\infty} \| R_{\delta B}^\mu R_{\gamma A}^\lambda u(t) - u(t) \| = 0.$

(iii) there exists $\dot{u} \in \text{Fix}(R_{\delta B}^\mu R_{\gamma A}^\lambda)$ such that $w - \lim_{t \to +\infty} u(t) = \dot{u}$.

(iv) If $\alpha + \beta > 0$, then

$$\int_{t_0}^{+\infty} \left\| w_1 (J_{\gamma A} u(s) - J_{\gamma A} u^*) + w_2 (J_{\delta B} R_{\gamma A}^\lambda u(s) - J_{\delta B} R_{\gamma A}^\lambda u^*) \right\|^2 ds < +\infty,$$

$$\lim_{t \to +\infty} \left\| w_1 (J_{\gamma A} u(t) - J_{\gamma A} u^*) + w_2 (J_{\delta B} R_{\gamma A}^\lambda u(t) - J_{\delta B} R_{\gamma A}^\lambda u^*) \right\| = 0,$$

and

$$\lim_{t \to +\infty} J_{\gamma A} u(t) = J_{\gamma A} u^* = \lim_{t \to +\infty} J_{\delta B} R_{\gamma A}^\lambda u(t) = J_{\delta B} R_{\gamma A}^\lambda u^* = \text{zer}(A + B).$$
The next theorem serves to show that the system still converges when assumption (A1) is weakened to (A2), i.e., \( \inf_{t \in [t_0, +\infty)} \theta(t) > 0 \) is replaced by \( \int_{t_0}^{+\infty} \theta(s) ds = +\infty \). Note that \( R^\mu_{SB} R_{\gamma A}^\lambda \) is nonexpansive in case (C2).

**Theorem 6** Let \( A : \mathbb{H} \rightrightarrows \mathbb{H} \) and \( B : \mathbb{H} \rightrightarrows \mathbb{H} \) be respectively maximally \( \alpha \)- and \( \beta \)-monotone. Suppose that the parameters \( \alpha, \beta, \gamma, \delta, \lambda, \mu \) satisfy (C2), \( \text{zer}(A + B) \neq \emptyset \), and that assumption (A2) holds. Let \( u(t) \) be the trajectory of dynamic system (7). Then the following statements are true:

(i) \( \int_{t_0}^{+\infty} \theta(s) \| R^\mu_{SB} R_{\gamma A}^\lambda u(s) - u(s) \|^2 ds < +\infty \).

(ii) \( \lim_{t \to +\infty} \| R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \| = 0 \).

(iii) There exists \( \hat{u} \in \text{Fix}(R^\mu_{SB} R_{\gamma A}^\lambda) \) such that \( w - \lim_{t \to +\infty} u(t) = \hat{u} \).

**Proof** Observe that \( R^\mu_{SB} R_{\gamma A}^\lambda \) is single-valued, and has full domain in that (27) satisfies (23). Similar to the proof of Theorem 1 (i), we get the assertion (i) and the boundedness of \( u(t) \). Owing to the nonexpansiveness of \( R^\mu_{SB} R_{\gamma A}^\lambda \), \( R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \) is bounded on \([t_0, +\infty)\). We learn from Remark 1 (b) that the function \( t \mapsto R^\mu_{SB} R_{\gamma A}^\lambda u(t) \) is almost everywhere differentiable and \( \left\| \frac{d}{dt} R^\mu_{SB} R_{\gamma A}^\lambda u(t) \right\| \leq \left\| \frac{da(t)}{dt} \right\| \) holds for almost all \( t \geq 0 \). Thus we deduce that

\[
\frac{d}{dt} \left( \frac{1}{2} \left\| R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \right\|^2 \right) = \left\langle \frac{d}{dt} R^\mu_{SB} R_{\gamma A}^\lambda u(t) - \frac{du(t)}{dt}, R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \right\rangle \\
= - \left\langle \frac{du(t)}{dt}, R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \right\rangle + \left\langle \frac{d}{dt} R^\mu_{SB} R_{\gamma A}^\lambda u(t), R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \right\rangle \\
= - \theta(t) \left\| R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \right\|^2 - \left\langle f(t), R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \right\rangle \\
+ \left\langle \frac{d}{dt} R^\mu_{SB} R_{\gamma A}^\lambda u(t), R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \right\rangle \\
\leq - \theta(t) \left\| R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \right\|^2 - \left\langle f(t), R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \right\rangle \\
+ \left\langle \frac{du(t)}{dt}, \| R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \| \right\rangle \\
\leq - \theta(t) \left\| R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \right\|^2 - \left\langle f(t), R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \right\rangle \\
+ \left\langle \frac{du(t)}{dt}, \| R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \| \right\rangle \\
\leq 2 \left\| f(t) \right\| \cdot \left\| R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \right\|. 
\]

Note that the right side is integrable owing to the boundedness of \( R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \) as well as \( f \in \mathbb{L}^1([t_0, +\infty)) \). Therefore, it follows from Lemma 3 that \( \lim_{n \to +\infty} \| R^\mu_{SB} R_{\gamma A}^\lambda u(t) - u(t) \| \) exists, and the assertion (ii) holds by assumption (A2) and (i). The last assertion is a straightforward consequence of Theorem 1 (iii) with \( T = R^\mu_{SB} R_{\gamma A}^\lambda \). The proof is complete.
4 Exponential-type rate

First let us look back some historical aspects concerning convergence rates of discrete DR algorithms: The DR algorithms have long been known to converge under mild assumptions, however, there are a few results on their convergence rate until the recent past. The linear convergence rate for discrete DR algorithms can be dated to \[47\], which was, to the best of our knowledge, the sole linear convergence rate result for a long period of time for the methods. Only very recently have many works shown their linear convergence rates in different settings such as \[35, 28, 9, 27, 43, 50, 52, 42, 25\]. Among them, \[28, 43, 52, 42\] concern local linear convergence and the others focus on global linear convergence or both. Moreover, the sublinear convergence rates such as \(O(\frac{1}{\sqrt{k}})\), \(O(\frac{1}{k})\) and \(O(\frac{1}{k^2})\) can be found in, e.g., \[40, 26, 28, 41\].

In this section, we are interested in the convergence rate of dynamic system (7), a continuous time version with perturbation of DR algorithm (3). Considerations based upon the adaptability of \(\lambda_\delta B R_\gamma A\) present that for suitable choice of \(\theta\) and \(f\), dynamic system (7) may enjoy various convergence rates such as sublinear convergence rate of asymptotic regularity and global exponential-type convergence rate. Our purpose here is to establish this fact.

4.1 Rate of asymptotic regularity

**Theorem 7** Let \(A : \mathbb{H} \Rightarrow \mathbb{H}\) and \(B : \mathbb{H} \Rightarrow \mathbb{H}\) be respectively maximally \(\alpha\)- and \(\beta\)-monotone. Suppose that the parameters \(\alpha, \beta, \gamma, \delta, \lambda, \mu\) satisfy (C2), \(\text{zer}(A + B) \neq \emptyset\), and that assumption (A1) holds. Let \(u(t)\) be the trajectory of dynamic system (7). If \(\theta\) and \(f\) are subject to

\[
\int_{t_0}^{+\infty} \theta(s) \int_s^{+\infty} \|f(\tau)\| d\tau ds < +\infty, \tag{29}
\]

then

\[
\|R_\delta^{\mu} R_\gamma^{\lambda} u(t) - u(t)\| = O\left(\frac{1}{\sqrt{t}}\right). \tag{30}
\]

In particular, if \(f(t) \equiv 0\), then the convergence rate above can be improved to \(o\left(\frac{1}{\sqrt{t}}\right)\).

**Proof** Observe that all the conditions in Theorem 6 are satisfied. Taking arbitrarily a \(u^* \in \text{Fix}(R_\delta^{\mu} R_\gamma^{\lambda})\), it is from the proof of Theorem 6 that \(\|R_\delta^{\mu} R_\gamma^{\lambda} u(t) - u(t)\|\) and \(\|u(t) - u^*\|\) are bounded on \([t_0, +\infty)\). To lighten the notion, let

\[
\mathcal{M}_\infty = 2 \int_{t_0}^{+\infty} \|u(s) - u^*\| \|f(s)\| ds < +\infty \tag{31}
\]

and

\[
\mathcal{M}_{RR} = \sup_{t \in (t_0, +\infty)} \|R_\delta^{\mu} R_\gamma^{\lambda} u(t) - u(t)\| < +\infty.
\]
Consider the auxiliary function:

\[ V(t) = \|u(t) - u^*\|^2. \]

Then, with a parallel deducing of (13), we get

\[
\int_{t_0}^{t} \theta(s)\|R_{\delta B}^{\mu_{\ell}} R_{\gamma_A}^{\lambda} u(s) - u(s)\|^2 ds \leq V(t_0) + 2 \int_{t_0}^{t} \|u(s) - u^*\|\|f(s)\| ds
\]
\[
\leq V(t_0) + M_\infty. \tag{32}
\]

On the other hand, it follows from (28) that for any \( t_0 \leq s \leq t, \)

\[
\|R_{\delta B}^{\mu_{\ell}} R_{\gamma_A}^{\lambda} u(t) - u(t)\|^2 \leq \|R_{\delta B}^{\mu_{\ell}} R_{\gamma_A}^{\lambda} u(s) - u(s)\|^2 + 2M_{RR} \int_{s}^{t} \|f(\tau)\|d\tau. \tag{33}
\]

Consequently, we have

\[
\left( \inf_{t \in [t_0, +\infty)} \theta(t) \right) \int_{t_0}^{t} \|R_{\delta B}^{\mu_{\ell}} R_{\gamma_A}^{\lambda} u(t) - u(t)\|^2 ds
\]
\[
\leq \int_{t_0}^{t} \theta(s)\|R_{\delta B}^{\mu_{\ell}} R_{\gamma_A}^{\lambda} u(t) - u(t)\|^2 ds
\]
\[
= \int_{t_0}^{t} \theta(s)\|R_{\delta B}^{\mu_{\ell}} R_{\gamma_A}^{\lambda} u(t) - u(t)\|^2 ds
\]
\[
\leq \int_{t_0}^{t} \theta(s) \left( \|R_{\delta B}^{\mu_{\ell}} R_{\gamma_A}^{\lambda} u(s) - u(s)\|^2 + 2M_{RR} \int_{s}^{t} \|f(\tau)\|d\tau \right) ds
\]
\[
\leq V(u_0) + M_\infty + 2M_{RR} \int_{t_0}^{t} \theta(s) \int_{s}^{t} \|f(\tau)\|d\tau ds,
\]

namely,

\[
\|R_{\delta B}^{\mu_{\ell}} R_{\gamma_A}^{\lambda} u(t) - u(t)\|^2 \leq \frac{\mathcal{K}}{\inf_{t \in [t_0, +\infty)} \theta(t) (t - t_0)}, \tag{34}
\]

where \( \mathcal{K} = V(u_0) + M_\infty + 2M_{RR} \int_{t_0}^{+\infty} \theta(s) \int_{s}^{+\infty} \|f(\tau)\|d\tau ds < +\infty \) by (29).

So, (30) is verified. When the perturbation \( f \) is vanishing, noting that \( R_{\delta B}^{\mu_{\ell}} R_{\gamma_A}^{\lambda} \) is nonexpansive in case (C2), we get the last conclusion by a parallel proof of [15, Theorem 11]. The proof is complete.

Remark 6 Let us mention that the condition (29) is not restrictive. In fact, by virtue of the Fubini’s Theorem, (29) is implied by the following condition:

\[
\sup_{t \in [t_0, +\infty)} \theta(t) < +\infty \quad \text{and} \quad \int_{t_0}^{+\infty} s\|f(s)\| ds < +\infty. \tag{35}
\]

A straightforward example of (35) is that \( \theta(t) \equiv M > 0 \), and \( f(t) = \frac{1}{tp} \) with \( p > 2 \). It is worth mentioning that a discretization version of (35) has been used in [45, Theorem 1] for a convergence rate analysis of an inexact fixed-point iteration of a nonexpansive operator. Of course, the condition (29) holds automatically when \( f(t) \equiv 0 \).
4.2 Exponential-type convergence under metric subregularity assumption

Next, we establish the global exponential-type convergence of dynamic system (7). To this end, an abstract convergence rate is presented in the first place. Given a strong global solution \( u(t) \) of dynamic system (10), notice that under suitable conditions, \( \lim_{n \to +\infty} \| u(t) - u^* \| \) exists for any \( u^* \in \text{Fix}(T) \) and \( u(t) \) converges weakly to a fixed point of \( T \)(see Theorem 1). Then we have the following theorem.

**Theorem 8** Suppose that \( T : \mathbb{H} \to \mathbb{H} \) be a quasi-nonexpansive operator with \( \text{Fix}(T) \neq \emptyset \) and that \( T - I \) satisfies the demiclosedness principle. Let \( u(t) \) be a strong global solution of dynamic system (10) and \( \hat{u} \in \text{Fix}(T) \) such that \( \omega - \lim_{t \to +\infty} u(t) = \hat{u} \in \text{Fix}(T) \). If \( I - T \) is metrically subregular at \( \hat{u} \) for 0 with a ball \( B(\hat{u}, r) \) and modulus \( \kappa \), and \( r > \lim_{t \to +\infty} \| u(t) - \hat{u} \| \), then there exist \( t' \geq t_0 \) and \( M_1 > 0 \) such that for all \( t \geq t' \)

\[
\text{dist}^2(u(t), \text{Fix}(T)) \leq e^{-\frac{r^2}{\kappa^2}} \int_{t'}^t f(s) \|e^{-\frac{\theta(s)}{\kappa^2}}f(x)dx\|ds + \|u(t') - \hat{u}\|^2.
\]  

(36)

**Proof** Owing to \( r > \lim_{t \to +\infty} \| u(t) - \hat{u} \| \), there exists \( t' \geq t_0 \) such that \( u(t) \in B(\hat{u}, r) \) for all \( t \geq t' \). Since \( I - T \) is metrically subregular at \( \hat{u} \in \text{Fix}(T) \) for 0 with the ball \( B(\hat{u}, r) \) and modulus \( \kappa \), noting \( (I - T)^{-1}(0) = \text{Fix}(T) \), we obtain

\[
\text{dist}(u(t), \text{Fix}(T)) \leq \kappa \| u(t) - T(u(t)) \|, \quad \forall t \geq t'.
\]  

(37)

On the other hand, consider the following function:

\[
V_p(t) = \text{dist}^2(u(t), \text{Fix}(T)) = \| u(t) - P_{\text{Fix}(T)}(u(t)) \|^2.
\]

Since \( \text{Fix}(T) \) is closed and convex by Proposition 4.23 in [8], the metric projection \( P_{\text{Fix}(T)}(u(t)) \) is well defined. Note that \( P_{\text{Fix}(T)}(u(t)) \in \text{Fix}(T) \) and

\[
\frac{dV_p}{dt} = 2 \left( u(t) - P_{\text{Fix}(T)}(u(t)), \frac{du}{dt} \right)
\]

by Corollary 12.31 in [8]. Therefore, we can replace \( \hat{u} \) by \( P_{\text{Fix}(T)}(u(t)) \) in (12) to obtain

\[
\frac{dV_p}{dt} \leq -\theta(t)\|T(u(t)) - u(t)\|^2 + 2 \| u(t) - P_{\text{Fix}(T)}(u(t)) \| \| f(t) \|
\]

(37)

\[
\leq -\frac{\theta(t)}{\kappa^2} V_p(t) + 2 \| u(t) - P_{\text{Fix}(T)}(u(t)) \| \| f(t) \|
\]

(15)

\[
\leq -\frac{\theta(t)}{\kappa^2} V_p(t) + 2M\| f(t) \|, \quad \forall t \geq t'.
\]
Multiplying this equation by $e^{\frac{1}{\kappa^2} \int_t^s f'(\theta(s))ds}$, and rearranging the terms, we obtain

$$
\frac{dV_P}{dt} e^{\frac{1}{\kappa^2} \int_t^s f'(\theta(s))ds} + \frac{\theta(t)}{\kappa^2} V_P(t)e^{\frac{1}{\kappa^2} \int_t^s f'(\theta(s))ds} \leq 2M\|f(t)\|e^{\frac{1}{\kappa^2} \int_t^s f'(\theta(s))ds},
$$

that is,

$$
\frac{d}{dt} \left( V_P(t)e^{\frac{1}{\kappa^2} \int_t^s f'(\theta(s))ds} \right) \leq 2M\|f(t)\|e^{\frac{1}{\kappa^2} \int_t^s f'(\theta(s))ds}.
$$

Integrating from $t'$ to $t$, we have

$$
V_P(t) e^{\frac{1}{\kappa^2} \int_t^s f'(\theta(s))ds} \leq 2M \int_{t'}^t \|f(s)\| e^{\frac{1}{\kappa^2} \int_t^s f'(\theta(s))ds} ds + V_P(t'),
$$

which contributes to (36) with $M_1 = 2M$. This completes the proof.

**Remark 7** (i) When $\mathbb{H}$ is finite-dimensional, the condition $r > \lim_{t \to +\infty} \|u(t) - \hat{u}\|$ is satisfied automatically. (ii) By suitable choices of $\theta$ and $f$, we can obtain from (36) various convergence rates:

a) If $\theta(t) \equiv \Theta > 0$ and $f(t) = \frac{1}{\kappa^2} e^{-\frac{t}{\kappa^2} \Theta}$, $p > 1$, then

$$
\text{dist}(u(t), \text{Fix}(\mathcal{T})) = O \left( e^{-\frac{t}{\kappa^2} \Theta} \right),
$$

which is an exponential convergence rate.

b) If $\theta(t) = t^m$ with $m > -1$ and $f(t) = \frac{1}{\kappa^2} e^{-\frac{t^{m+1}}{\kappa^2} \Theta}$ with $p > 1$, then

$$
\text{dist}(u(t), \text{Fix}(\mathcal{T})) = O \left( e^{-\frac{t^{m+1}}{\kappa^2} \Theta} \right).
$$

c) If $\theta(t) = \frac{1}{t}$ and $f(t) = t^{-\left(\kappa^2 + p\right)}$ with $p > 1$, then

$$
\text{dist}(u(t), \text{Fix}(\mathcal{T})) = O \left( t^{-\frac{1}{\kappa^2}} \right).
$$

**Remark 8** Recently, Liang et al. [45] presented a convergence rate analysis for an inexact Krasnosel’skii-Mann iteration algorithm of a nonexpansive operator. Under a metric subregularity assumption, they demonstrated that the inexact Krasnosel’skii-Mann iteration algorithm enjoys a local line convergence rate.

The following result is an immediate corollary of Theorem 8, where we consider $f \equiv 0$.

**Corollary 1** Suppose that $\mathcal{T} : \mathbb{H} \to \mathbb{H}$ be a quasi-nonexpansive operator with $\text{Fix}(\mathcal{T}) \neq \emptyset$ and that $\mathcal{T} - I$ satisfies the demiclosedness principle. Let $u(t)$ be a strong global solution of dynamic system (10) with $f(t) \equiv 0$ and $\hat{u} \in \text{Fix}(\mathcal{T})$ such that $w = \lim_{t \to +\infty} u(t) = \hat{u} \in \text{Fix}(\mathcal{T})$. If $I - \mathcal{T}$ is metrically subregular at $\hat{u}$ for 0 with a ball $B(\hat{u}, r)$ and modulus $\kappa$, and $r > \lim_{t \to +\infty} \|u(t) - \hat{u}\|$, then there exists $t' \geq t_0$ such that for all $t \geq t'$

$$
\text{dist}(u(t), \text{Fix}(\mathcal{T})) \leq \|u(t') - \hat{u}\| e^{-\frac{1}{\kappa^2} \int_{t'}^t f'(\theta(s))ds}.
$$
We are now in position to establish the global exponential-type convergence for dynamic system (7). Note that $R^\mu_{\delta B}R^\lambda_{\gamma A}$ is quasi-nonexpansive in both cases (C1) and (C2). The following result follows directly from Theorem 8.

**Theorem 9** Suppose that all the conditions in Theorem 4, Theorem 5 or Theorem 6 are satisfied. Let $u(t)$ be the trajectory of dynamic system (7), and let $\hat{u} \in \text{Fix}(R^\mu_{\delta B}R^\lambda_{\gamma A})$ such that $w - \lim_{t \to +\infty} u(t) = \hat{u}$. Suppose that $I - R^\mu_{\delta B}R^\lambda_{\gamma A}$ is metrically subregular at $\hat{u}$ for 0 with a ball $B(\hat{u}, r)$ and modulus $\kappa$. If $r > \lim_{t \to +\infty} \|u(t) - \hat{u}\|$, then there exist $t' \geq t_0$ and $M_2 > 0$ such that for all $t \geq t'$

$$\text{dist}^2 (u(t), \text{Fix}(R^\mu_{\delta B}R^\lambda_{\gamma A})) \leq e^{-\frac{\kappa}{2} \int_t^{\infty} \|f(s)\| ds} \left( M_2 \int_{t'}^{\infty} \|f(s)\| ds + \|u(t') - \hat{u}\|^2 \right).$$

Furthermore, if

$$\int_{t_0}^{\infty} \|f(s)\| ds < +\infty,$$  \tag{38}

then

$$\text{dist} (u(t), \text{Fix}(R^\mu_{\delta B}R^\lambda_{\gamma A})) = O \left( e^{-\frac{\kappa}{2} \int_t^{\infty} \|f(s)\| ds} \right).$$

**Remark 9** Condition (38) is mild and it is also satisfied when $f$ and $\theta$ are taken as in Remark 7 (ii). Of course, it is satisfied automatically when $f(t) \equiv 0$.

**Corollary 2** Suppose that all the conditions in Theorem 4 or Theorem 5 are satisfied. Let $u(t)$ be the trajectory of dynamic system (7) with $f(t) \equiv 0$, and let $\hat{u} \in \text{Fix}(R^\mu_{\delta B}R^\lambda_{\gamma A})$ such that $w - \lim_{t \to +\infty} u(t) = \hat{u}$. Suppose that $I - R^\mu_{\delta B}R^\lambda_{\gamma A}$ is metrically subregular at $\hat{u}$ for 0 with a ball $B(\hat{u}, r)$ and modulus $\kappa$. If $r > \lim_{t \to +\infty} \|u(t) - \hat{u}\|$, then there exist $t' \geq t_0$ and $\Theta_0 > 0$ such that

$$\text{dist} (u(t), \text{Fix}(R^\mu_{\delta B}R^\lambda_{\gamma A})) \leq \|u(t') - \hat{u}\| e^{\Theta_0 (t-t')}, \quad t \geq t'.$$

**Proof** Set

$$\Theta_0 = -\frac{\inf_{t \in [t', +\infty)} \theta(t)}{\kappa^2}.$$  

Then $\Theta_0 > 0$ by assumption (A1). With the help of Corollary 1 we obtain the desired results. The proof is complete.

**Theorem 10** Suppose that all the conditions in Theorem 4 or Theorem 5 are satisfied. Let $u(t)$ be the trajectory of dynamic system (7), and let $\hat{u} \in \text{Fix}(R^\mu_{\delta B}R^\lambda_{\gamma A})$ such that $w - \lim_{t \to +\infty} u(t) = \hat{u}$. Suppose that $I - R^\mu_{\delta B}R^\lambda_{\gamma A}$ is metrically subregular at $\hat{u}$ for 0 with a ball $B(\hat{u}, r)$ and modulus $\kappa$. If $r > \lim_{t \to +\infty} \|u(t) - \hat{u}\|$ and (29) holds, then there exists $t' \geq t_0$ such that

$$\text{dist} (u(t), \text{Fix}(R^\mu_{\delta B}R^\lambda_{\gamma A})) = O \left( \frac{1}{\sqrt{t}} \right), \quad t \geq t'.$$
Proof Observe that all the conditions in Theorem 7 are satisfied. Replacing $T$ by $R_\mu^\alpha R_\gamma^A$ in (37) and combining with (30) we get the desired result.

Remark 10 The convergence results above can also be established by using the coercivity condition. According to Lemma 3.1(b) in [42], a set-valued operator $T : D \rightrightarrows \mathbb{H}$ satisfies the coercivity condition, provided that there exists $\vartheta > 0$ such that
\[
\| z - \bar{z} \| \geq \vartheta \text{dist}(z, S) \quad \forall \bar{z} \in T(z), \quad \forall z \in U,
\]
where $D \subseteq \mathbb{H}$, $S \subseteq \text{Fix}(T)$, and $U \subseteq D$. The corresponding proofs are similar to that of Theorem 8, and thus are omitted. We also refer the reader to [42, 6, 24] for a local linear convergence rate of discrete DR algorithms for minimizing the sum of two convex functions under the coercivity condition or metric subregularity.

4.3 Exponential-type convergence under Lipschitz assumption

Next, we use a Lipschitz assumption instead of the metric subregularity to derive another exponential-type convergence of dynamic system (7) in case (C2). Notice that a contraction operator (whose Lipschitz’s constant less than 1) has a unique fixed point on $\mathbb{H}$ (by the Banach-Picard’s Theorem).

Theorem 11 Let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be $\alpha$-monotone and Lipschitz continuous with constant $l$ such that $l \geq |\alpha|$, and $B : \mathbb{H} \rightrightarrows \mathbb{H}$ be maximally $\beta$-monotone. Suppose that the parameters $\alpha, \beta, \gamma, \delta, \lambda, \mu$ satisfy (C2) and $\mu \neq 2 + 2\gamma\alpha$.

Then $R_\mu^\alpha R_\gamma^A$ is Lipschitz continuous with constant
\[
\zeta = 1 - \frac{\lambda(\mu - 1)^2[(\lambda - 1)(2 + 2\gamma\alpha) - \lambda]}{1 + 2\gamma\alpha + \gamma^2l^2} < 1.
\]

Let $u(t)$ be the trajectory of dynamic system (7), and $\text{Fix}(R_\mu^\alpha R_\gamma^A) = \{u^*\}$. Then there exists $M_3 > 0$ such that
\[
\| u(t) - u^* \|^2 \leq e^{-(1-\zeta)\int_{t_0}^t \| f(s) \| e^{(1-\zeta)\int_{t_0}^s \| f(x) \| dx} ds + \| u_0 - u^* \|^2}. \quad (40)
\]
Furthermore, if assumption (A2) holds and $\int_{t_0}^{+\infty} \| f(s) \| e^{\int_{t_0}^s \| f(x) \| dx} ds < +\infty$, then
\[
\| u(t) - u^* \| = O \left( e^{\frac{-1-\zeta}{2} \int_{t_0}^t \| f'(x) \| dx} \right). \quad (41)
\]

Proof Recall that $R_\gamma^A = (1 - \lambda)\text{Id} + \lambda J_\gamma A$ and $R_\mu^\alpha = (1 - \mu)\text{Id} + \mu J_\delta B$. Let us first show that
\[
(\mu - 1)(2 + 2\delta\beta) - \mu \geq 0 \quad (42)
\]
and

\[(\lambda - 1)(2 + 2\gamma a) - \lambda > 0.\]  \hspace{1cm} (43)

In fact, the condition (27) implies (25). Noting that 
\[(\lambda - 1)(\mu - 1) = (\lambda - 1)\gamma\] and \[\mu \neq 2 + 2\gamma a,\]
we have

\[(\mu - 1)(2 + 2\beta) - \mu = 2(\mu - 1) + 2\gamma \beta - \mu\]
\[= \mu - (2 - 2\gamma \beta) \geq 0,
\]
and

\[(\lambda - 1)(2 + 2\gamma a) - \lambda = (\lambda - 1) \left( (2 + 2\gamma a) - \frac{\lambda}{\lambda - 1} \right)
\[= \frac{\mu(2 + 2\gamma a - \mu)}{\mu - 1} > 0.
\]

We then learn from Lemma 1 that for \(x, y \in H\)

\[
\| R_{\delta B}^\mu x - R_{\delta B}^\mu y \|^2 \leq (\mu - 1)^2 \| x - y \|^2 - \mu [(\mu - 1)(2 + 2\delta) - \mu] \| J_{\delta B} x - J_{\delta B} y \|^2
\]

\[\leq (\mu - 1)^2 \| x - y \|^2,
\]
and

\[
\| R_{\gamma A}^\lambda x - R_{\gamma A}^\lambda y \|^2 \leq \left( (\lambda - 1)^2 - \frac{\lambda [(\lambda - 1)(2 + 2\gamma a) - \lambda]}{1 + 2\gamma a + \gamma^2 l^2} \right) \| x - y \|^2.
\]

Hence,

\[
\| R_{\delta B}^\mu R_{\gamma A}^\lambda x - R_{\delta B}^\mu R_{\gamma A}^\lambda y \|^2 \leq (\mu - 1)^2 \| R_{\gamma A}^\lambda x - R_{\gamma A}^\lambda y \|^2
\]

\[\leq \zeta \| x - y \|^2,
\]
where

\[
\zeta = (\mu - 1)^2 \left( (\lambda - 1)^2 - \frac{\lambda [(\lambda - 1)(2 + 2\gamma a) - \lambda]}{1 + 2\gamma a + \gamma^2 l^2} \right)
\[= 1 \cdot \frac{\lambda [(\mu - 1)(2 + 2\gamma a) - \lambda]}{1 + 2\gamma a + \gamma^2 l^2}
\[< 1.
\]

Next, let us verify (40). Consider the following auxiliary function again:

\[
V(t) = \| u(t) - u^* \|^2.
\]

Similar to the derivation of (20) with \( T = R_{\delta B}^\mu R_{\gamma A}^\lambda \), we have

\[
\frac{dV}{dt} \leq \theta(t) \| R_{\delta B}^\mu R_{\gamma A}^\lambda u(t) - R_{\delta B}^\mu R_{\gamma A}^\lambda (u^*) \|^2 - \theta(t) V(t)
\[\leq - (1 - \zeta) \theta(t) V(t) + 2 \sqrt{V(t)} \| f(t) \|.
\]

(45)
Note that all conditions in Theorem 6 are fulfilled and so \( \sqrt{V(t)} = \|u(t) - u^*\| \) is bounded. Multiplying (45) by \( e^{\int_{t_0}^{t} (1-\zeta)\theta(s)ds} \), and then following the same roadmap of proof as that for Theorem 8 we get (40), and so (41) occurs whenever assumption (A2) holds and \( \int_{t_0}^{\infty} \|f(s)\| e^{\int_{t_0}^{s} \theta(x)dx} ds < +\infty \). The proof is complete.

**Corollary 3** Let \( A : H \ni h \mapsto H \) be \( \alpha \)-monotone and Lipschitz continuous with constant \( l \) such that \( l \geq |\alpha| \), and \( B : H \ni H \) be maximally \( \beta \)-monotone. Suppose that the parameters \( \alpha, \beta, \gamma, \delta, \lambda, \mu \) satisfy case (C2) and \( \mu \neq 2 + 2\gamma \alpha \), and that \( \theta(t) \) satisfies assumption (A1). Let \( u(t) \) be the unique solution of dynamic system (7) with \( f(t) \equiv 0 \), and let \( \text{Fix}(R_{0B}^\mu R_{1A}^\lambda) = \{u^*\} \). Then there exists \( \Theta_1 > 0 \) such that

\[
\|u(t) - u^*\| \leq \|u_0 - u^*\| e^{-\Theta_1(t-t_0)}.
\]

**Proof** It follows from Theorem 11 that \( R_{0B}^\mu R_{1A}^\lambda \) is Lipschitz continuous with a constant \( \zeta \in (0, 1) \). Set

\[
\Theta_1 = \frac{1 - \zeta}{2} \inf_{t \in [t_0, +\infty)} \theta(t).
\]

Then \( \Theta_1 > 0 \) by assumption (A1). With the help of \( f(t) \equiv 0 \) and (40), the conclusion follows. The proof is complete.

In next theorem we will take into account the condition \( f \in L^1([t_0, b]) \) for all \( b \geq t_0 \) instead of \( f \in L^1([t_0, +\infty)) \), and impose a constraint on \( \theta \). Note that, with such modifications, Theorem 3 still holds according to the Cauchy-Lipschitz theorem (see e.g. [39, Proposition 6.2.1],[57, Corollary 2.6]).

**Theorem 12** Let \( A : H \ni h \mapsto H \) be \( \alpha \)-monotone and Lipschitz continuous with constant \( l \) which satisfies \( l \geq |\alpha| \), and \( B : H \ni H \) be maximally \( \beta \)-monotone. Suppose that the parameters \( \alpha, \beta, \gamma, \delta, \lambda, \mu \) satisfy case (C2) and \( \mu \neq 2 + 2\gamma \alpha \). Let \( u(t) \) be the trajectory of dynamic system (7), and \( \text{Fix}(R_{0B}^\mu R_{1A}^\lambda) = \{u^*\} \). Suppose that \( (1 - \zeta)\theta(t) > 1 \) for any \( t \in [t_0, +\infty) \), and that \( f \in L^1([t_0, b]) \) for all \( b \geq t_0 \). Then

\[
\|u(t) - u^*\|^2 \leq e^{-\int_{t_0}^{t} (1-\zeta)\theta(s) - 1ds} \left( \int_{t_0}^{t} \|f(s)\|^2 e^{\int_{t_0}^{s} (1-\zeta)\theta(x) - 1dx} ds + \|u_0 - u^*\|^2 \right),
\]

where \( \zeta \) is defined in (39). Furthermore, if \( \int_{t_0}^{\infty} [(1 - \zeta)\theta(s) - 1]ds = +\infty \) and \( \int_{t_0}^{\infty} \|f(s)\|^2 e^{\int_{t_0}^{s} (1-\zeta)\theta(x) - 1dx} ds < +\infty \), then

\[
\|u(t) - u^*\| = O \left( e^{-\frac{1}{2} \int_{t_0}^{t} (1-\zeta)\theta(s) - 1ds} \right).
\]
Proof It follows from Theorem 11 (i) that $R_{\alpha}^\mu \{ R_{\gamma}^\lambda \}$ is Lipschitz continuous with $\zeta$ defined in (39), and has a unique fixed point $\{ u^* \} = \text{Fix}(R_{\alpha}^\mu \{ R_{\gamma}^\lambda \})$. Consider the following auxiliary function again:

$$V(t) = \| u(t) - u^* \|^2.$$ 

It turns out from (44) and (45) that

$$\frac{dV}{dt} \leq -\frac{\theta(t)}{\kappa^2} V(t) + 2\sqrt{V(t)} \| f(t) \| \leq -((1 - \zeta)\theta(t) - 1)V(t) + \| f(t) \|^2.$$ 

Noticing that $(1 - \zeta)\theta(t) > 1$. Similar to the proofs of (40) and (41) we obtain the desired results. The proof is complete.

Remark 11 The functions

$$\theta_1(t) = \frac{2t + 1}{2t(1 - \zeta)} \quad \text{and} \quad f_1(t) = \left( \frac{1}{t} - f'_0 \frac{1}{t} ds \right) = \frac{\sqrt{t_0}}{t}, \quad t \geq t_0 > 0,$$

verify the conditions $(1 - \zeta)\theta_1(t) - 1 > 0$, $f_1 \in L^1([t_0, b])$ for all $b \geq t_0$, and

$$\int_{t_0}^b \| f_1(s) \|^2 \int_{t_0}^b (1 - \zeta)\theta_1(s) - 1 ds < +\infty, \quad \text{while} \quad \int_{t_0}^\infty \| f_1(s) \|^2 (1 - \zeta)\theta_1(s) ds < +\infty \quad \text{and} \quad f_1 \in L^1([t_0, +\infty)) \text{ fail. On the other hand, the functions}$$

$$\theta_2(t) = \frac{1}{1 - \zeta} \quad \text{and} \quad f_2(t) = \frac{1}{2t^2} e^{\frac{1}{t^2} - t} ds,$$

cater for the conditions $\int_{t_0}^\infty \| f_2(s) \|^2 e^{1 - \zeta} f'_0 \theta_2(s) ds < +\infty$, $f_2 \in L^1([t_0, +\infty))$ and assumption (A2), while $(1 - \zeta)\theta_2(t) - 1 > 0$ and $\int_{t_0}^b f_2(s) (1 - \zeta)\theta_2(s) ds = +\infty$ are not fulfilled. This indicates that the assumptions on $\theta$ and $f$ in Theorem 11 are independent of those in Theorem 12.

5 Application

In this section, we turn our attention to an important application of dynamic system (7) to a minimization problem which can be regarded as a special case of problem (1). To this end, let us recall some necessary concepts and results in convex analysis. Let $\text{dom } h$ denote the effective domain of a proper function $h : \mathbb{H} \to \mathbb{R} \cup +\infty$, i.e., $\text{dom } h := \{ x \in \mathbb{H} \mid h(x) < +\infty \} \neq \emptyset$. Let $\text{Prox}_{\tau h} : \mathbb{H} \ni x \mapsto \text{Fix}(h)$ denote the proximity operator of $h$, i.e.,

$$\text{Prox}_{\tau h}(x) := \arg\min_{z \in \mathbb{H}} \left( h(z) + \frac{1}{2\tau} \| z - x \|^2 \right), \quad \forall x \in \mathbb{H}.$$ 

The function $h$ is said to be $\alpha$-convex (see, e.g., [59, Definition 4.1]) for some $\alpha \in \mathbb{R}$, if $\forall x, y \in \text{dom } h$, $\forall \tau \in (0, 1)$

$$h((1 - \tau)x + \tau y) + \frac{\alpha}{2}(1 - \tau)\|x - y\|^2 \leq (1 - \tau)h(x) + \tau h(y).$$
We say that $h$ is convex, strongly convex and weakly convex if $\alpha = 0$, $\alpha > 0$ and $\alpha < 0$, respectively. We use $\tilde{\partial}h$ denote the Fréchet subdifferential of $h$, which is defined by

\[
\tilde{\partial}h(x) = \left\{ z \in H \mid \lim_{y \to x} \inf h(y) - h(x) - \langle z, y - x \rangle \geq 0 \right\}.
\]

Notice that if $h$ is convex, then

\[
\tilde{\partial}h(x) = \partial h(x) := \left\{ v \in H \mid h(y) \geq h(x) + \langle v, y - x \rangle \forall y \in \text{dom } h \right\},
\]

see, e.g., [48, Theorem 1.93]. The following lemma comes from [24, Lemma 5.2].

Lemma 6 Let $h : H \to \mathbb{R} \cup \{+\infty\}$ be proper, closed, and $\alpha-$convex. Suppose $\gamma \in \mathbb{R}^{++}$ and $1 + \gamma \alpha > 0$. Then the following conclusions hold:

(i) $\tilde{\partial}h(x)$ is maximally $\alpha$-monotone.

(ii) $\text{Prox}_{\gamma h} = J_{\gamma \tilde{\partial}h(x)}$ is single-valued and has full domain.

We are now in position to deal with the minimum value problem

\[
\min_{x \in H} \phi(x) + \varphi(x), \quad (46)
\]

where $\phi, \varphi : H \to \mathbb{R} \cup \{+\infty\}$ be two proper and closed functions, one of which is strongly convex while the other one is weakly convex. Let us mention some applications of problem (46) with weakly convex term. A particular application fitting this problem is the sparse signal recovery, where $\phi$ represents a data-fidelity term and $\varphi$ is a sparsity-driven penalty term whose weak convexity can often reduce bias in nonzero estimates, see [10, 22, 54] for more details. Moreover, in the application of the joint denoising and sharpening image recovery problem [49], the weakly convex function is used for the design of energy functions that describe some desired effects more accurately than purely convex ones.

Since problem (46) is a special case of problem (1) (by setting $A = \tilde{\partial}\phi$, $B = \tilde{\partial}\varphi$), it is easy to apply dynamic system (7) to problem (46):

\[
\begin{cases}
\frac{dx}{dt} + \theta(t) [u(t) - R_2R_1(u(t))] = f(t), \\
u(t_0) = u_0 \in H,
\end{cases} \quad (47)
\]

where

\[
R_1 := (1 - \lambda)\text{Id} + \lambda P_1, \quad R_2 := (1 - \mu)\text{Id} + \mu P_2,
\]

$P_1 := \text{Prox}_{\gamma \phi}$ and $P_2 := \text{Prox}_{\delta \varphi}$.

We can learn from Theorem 3 that for each initial point $u_0$, there exists a unique absolutely continuous trajectory $u(t)$ of dynamic system (47) in the global time interval $[t_0, +\infty)$. Based on such a fact, Lemma 6 allows us to get the parallel results from the previous sections. Note that $\text{Fix}(R_2R_1) = \text{zer}(\tilde{\partial}\phi + \tilde{\partial}\varphi) \subseteq \text{arg min}(\phi + \varphi)$ by [24, Lemma 5.3], and that $\phi + \varphi$ is strong convex in the case (C1) due to $\alpha + \beta > 0$, which leads to the fact that problem (46) has a unique solution.
Theorem 13 Let \( \phi : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\} \) and \( \varphi : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\} \) be proper, closed, and respectively maximally \( \alpha \)- and \( \beta \)-convex. Suppose that the parameters \( \alpha, \beta, \gamma, \delta, \lambda, \mu \) satisfy \( (C1) \), and that assumption \( (A1) \) holds. Let \( u(t) \) be the trajectory of dynamic system \( (47) \). Then, for any \( u^* \in \text{Fix}(R_2R_1) \), the following statements are true:

(i) \( \int_0^\infty \| R_2R_1 u(s) - u(s) \|^2 ds < +\infty \).

(ii) \( \lim_{t \rightarrow +\infty} \| R_2R_1 u(t) - u(t) \| = 0 \).

(iii) If \( R_2R_1 - I \) satisfies the demiclosedness principle, then there exists \( \hat{u} \in \text{Fix}(R_2R_1) \) such that \( w - \lim_{t \rightarrow +\infty} u(t) = \hat{u} \).

(iv) \( \int_0^\infty \| \alpha (P_1 u(s) - P_1 u^*) + \beta (P_2R_1 u(s) - P_2R_1 u^*) \|^2 ds < +\infty \).

(v) \( \lim_{t \rightarrow +\infty} \| \alpha (P_1 u(t) - P_1 u^*) + \beta (P_2R_1 u(t) - P_2R_1 u^*) \| = 0 \).

(vi) \( \lim_{t \rightarrow +\infty} P_1 u(t) = P_1 u^* = \lim_{t \rightarrow +\infty} P_2R_1 u(t) = P_2R_1 u^* = \text{zer}(A + B) \).

Remark 12 Some convergence results corresponding to the case \( \theta(t) \equiv \Theta > 0 \), \( f(t) \equiv 0 \) and \( \mathbb{H} = \mathbb{R}^n \) have been investigated in \([60, \text{Theorem 3.3 and Theorem 3.7}]) \), which are covered by Theorem 13.

Theorem 14 Let \( \phi : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\} \) and \( \varphi : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\} \) be proper, closed, and respectively maximally \( \alpha \)- and \( \beta \)-convex. Suppose that the parameters \( \alpha, \beta, \gamma, \delta, \lambda, \mu \) satisfy \( (C2) \), \( \text{zer}(\hat{\partial} \phi + \hat{\partial} \varphi) \neq \emptyset \), and that assumption \( (A1) \) holds. Let \( u(t) \) be the trajectory of dynamic system \( (47) \). Then, for any \( u^* \in \text{Fix}(R_2R_1) \), the following statements are true:

(i) \( \int_0^\infty \| R_2R_1 u(s) - u(s) \|^2 ds < +\infty \).

(ii) \( \lim_{t \rightarrow +\infty} \| R_2R_1 u(t) - u(t) \| = 0 \).

(iii) There exists \( \hat{u} \in \text{Fix}(R_2R_1) \) such that \( w - \lim_{t \rightarrow +\infty} u(t) = \hat{u} \).

(iv) If \( \alpha + \beta > 0 \), then

\[
\int_0^\infty \| w_1 (P_1 u(s) - P_1 u^*) + w_2 (P_2R_1 u(s) - P_2R_1 u^*) \|^2 ds < +\infty,
\]

\[
\lim_{t \rightarrow +\infty} \| w_1 (P_1 u(t) - P_1 u^*) + w_2 (P_2R_1 u(t) - P_2R_1 u^*) \| = 0, \text{ and}
\]

\[
\lim_{t \rightarrow +\infty} P_1 u(t) = P_1 u^* = \lim_{t \rightarrow +\infty} P_2R_1 u(t) = P_2R_1 u^* = \text{zer}(A + B).
\]

Remark 13 Guo et al. \([37]\) presented a convergence analysis for a DR algorithm solving problem \( (46) \) in an Euclidean space. This algorithm can be regard as a special case of a discretization of dynamic system \( (47) \). The convergence results in \([37]\) require that the strong convexity of the objective function strictly outweighs the weak counterpart, that is, \( \alpha + \beta > 0 \). Convergence of the same DR algorithm, only for the case \( \alpha + \beta = 0 \), has also been considered in \([36]\) under the condition that one function is strongly convex with Lipschitz continuous gradient. In contrast, we assume \( \alpha + \beta \geq 0 \), and the convergence is still guaranteed without any differentiability assumption; see Theorem 13 and Theorem 14.

Theorem 15 Let \( \phi : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\} \) and \( \varphi : \mathbb{H} \rightarrow \mathbb{R} \cup \{+\infty\} \) be proper, closed, and respectively maximally \( \alpha \)- and \( \beta \)-convex. Suppose that the parameters \( \alpha, \beta, \gamma, \delta, \lambda, \mu \) satisfy \( (C2) \), \( \text{zer}(\hat{\partial} \phi + \hat{\partial} \varphi) \neq \emptyset \), and that assumption \( (A1) \)
holds. Let \( u(t) \) be the trajectory of dynamic system (47). If \( \theta \) and \( f \) are subject to
\[
\int_{t_0}^{+\infty} \theta(s) \int_{s}^{+\infty} \|f(\tau)\|d\tau ds < +\infty,
\]
then \( \|R_2 R_1 u(t) - u(t)\| = O\left(\frac{1}{\sqrt{t}}\right) \). In particular, if \( f(t) \equiv 0 \), then the convergence rate above can be improved to \( o\left(\frac{1}{\sqrt{t}}\right) \).

**Theorem 16** Suppose that all the conditions in Theorem 13 or Theorem 14 are satisfied. Let \( \bar{u} \in \text{Fix}(R_2 R_1) \) such that \( w - \lim_{t \to +\infty} u(t) = \bar{u} \). Suppose that \( I - R_2 R_1 \) is metrically subregular at \( \bar{u} \) for 0 with a ball \( B(\bar{u}, r) \) and modulus \( \kappa \). If \( r > \lim_{t \to +\infty} \|u(t) - \bar{u}\|, \) then there exist \( t' \geq t_0 \) and \( M_4 > 0 \) such that for all \( t \geq t' \)
\[
\text{dist}(u(t), \text{Fix}(R_2 R_1)) \leq e^{-\frac{t}{1+2\gamma}} \int_{t_0}^{t} \|f(x, t\theta(x))\|d\tau ds + \|u(t') - \bar{u}\|.
\]
Furthermore, if \( \int_{t_0}^{+\infty} \|f(s)\|e^{-\frac{s}{1+2\gamma}} \int_{t_0}^{s} \|f(x, s\theta(x))\|d\tau ds < +\infty \), then
\[
\text{dist}(u(t), \text{Fix}(R_2 R_1)) = O\left(e^{-\frac{t}{1+2\gamma}} \int_{t_0}^{t} \|f(x, \theta(x))\| d\tau ds\right).
\]

**Remark 14** Theorem 16 covers [60, Theorem 4.1 and Theorem 4.3] where \( \theta(t) \equiv \Theta > 0, f(t) \equiv 0 \) and \( H = \mathbb{R}^n \). It is worth mentioning that a local linear convergence rate of the DR algorithm for problem (46) was derived in [37] under a metric subregularity condition.

**Theorem 17** Let \( \phi : H \to \mathbb{R} \cup \{+\infty\} \) be \( \alpha \)-convex and \( \hat{\phi} \) be Lipschitz continuous with constant \( l \) such that \( l \geq |\alpha| \). Let \( \varphi : H \to \mathbb{R} \cup \{+\infty\} \) be \( \beta \)-convex. Suppose that the parameters \( \alpha, \beta, \gamma, \delta, \lambda, \mu \) satisfy (C2) and \( \mu \neq 2 + 2\gamma \alpha \). Then \( R_2 R_1 \) is Lipschitz continuous with \( \zeta = 1 - \frac{\lambda (\mu - 1)^2 (\lambda - 1)^2 (2 + 2\gamma \alpha) - \lambda}{2 + 2\gamma \alpha} < 1 \). Let \( u(t) \) be the trajectory of dynamic system (47), and \( \text{Fix}(R_2 R_1) = \{u^*\} \). Then there exists \( M_5 > 0 \) such that
\[
\|u(t) - u^*\|^2 \leq e^{-(1-\zeta)} \int_{t_0}^{t} \|f(s)\|e^{(1-\zeta)} \int_{t_0}^{s} \|f(x, s\theta(x))\| d\tau ds + \|u_0 - u^*\|^2.
\]
Furthermore, if assumption (A2) holds and \( \int_{t_0}^{+\infty} \|f(s)\|e^{(1-\zeta)} \int_{t_0}^{s} \|f(x, s\theta(x))\| d\tau ds < +\infty \), then
\[
\|u(t) - u^*\| = O\left(e^{-\frac{t}{1+2\gamma}} \int_{t_0}^{t} \|f(x, \theta(x))\| d\tau ds\right).
\]
Theorem 18 Let \( \phi : H \to \mathbb{R} \cup \{+\infty\} \) be \( \alpha \)-convex and \( \partial \phi \) be Lipschitz continuous with constant \( l \) which satisfies \( |\alpha| \leq l \). Let \( \varphi : H \to \mathbb{R} \cup \{+\infty\} \) be \( \beta \)-convex. Suppose that the parameters \( \alpha, \beta, \gamma, \delta, \lambda, \mu \) satisfy \((C2)\) and \( \mu \neq 2 + 2\gamma\alpha \). Let \( u(t) \) be the trajectory of dynamic system \((47)\), and \( \{u^*\} = \text{Fix}(R_2R_1) \). Suppose that \( (1 - \zeta)\theta(t) > 1 \) for any \( t \in [t_0, +\infty) \), and that \( f \in L^1([t_0, b]) \) for all \( b \geq t_0 \) instead of \( f \in L^1([t_0, +\infty)) \). Then

\[
\|u(t) - u^*\|^2 \leq e^{-\int_{t_0}^t (1 - \zeta)\theta(s) - 1ds} \left( \int_{t_0}^t \|f(s)\|^2 e^{\int_{t_0}^s (1 - \zeta)\theta(x) - 1dx} ds + \|u_0 - u^*\|^2 \right),
\]

where \( \zeta \) is defined in \((39)\). Furthermore, if \( \int_{t_0}^{t_0} (1 - \zeta)\theta(s) - 1ds = +\infty \) and \( \int_{t_0}^{+\infty} \|f(s)\|^2 e^{\int_{t_0}^s (1 - \zeta)\theta(x) - 1dx} ds < +\infty \), then

\[
\|u(t) - u^*\| = O \left( e^{-\frac{1}{2} \int_{t_0}^t (1 - \zeta)\theta(s) - 1ds} \right).
\]

Remark 15 A global linear convergence rate of the DR algorithm for problem \((46)\) was proved in \([35]\) where it requires that \( \phi \) is \( \alpha \)-convex with \( \alpha > 0 \) such that \( \nabla \phi \) is Lipschitz continuous with constant \( l \geq \alpha \), and \( \varphi \) is \( \beta \)-convex functions with \( \beta \geq 0 \). In contrast, with proper tuning the parameters, we prove that dynamic system \((47)\) enjoys a global exponential-type convergence rate under the conditions \( \phi \) is \( \alpha \)-convex such that \( \partial \phi \) is Lipschitz continuous with constant \( l \geq |\alpha| \), \( \varphi \) is \( \beta \)-convex functions and \( \alpha + \beta \geq 0 \); see Theorem 17 and Theorem 18.

6 Conclusion

We proposed an adaptive Douglas-Rachford dynamic system with perturbations or computational errors for finding a zero of the sum of two operators, one of which is strongly monotone while the other one is weakly monotone, in a real Hilbert space. Prior to carry out the convergence analysis of the adaptive system, we investigated an abstract dynamic system built from quasi-nonexpansive operators, which encompasses several dynamical systems from the literature as special cases. We demonstrated that the trajectory of the abstract system converges weakly to a fixed point of the quasi-nonexpansive operator if the demiclosedness principle is satisfied. The global exponential-type convergence rate of the trajectory is achieved under a metric subregularity assumption. The results of the abstract system were valid for the adaptive system in that the adaptive Douglas-Rachford operator was quasi-nonexpansive with proper tuning the parameters. In addition, it is also shown that the shadow trajectory of the adaptive dynamic system strongly converges to a solution of problem \((1)\), and the rate of asymptotic regularity of the adaptive Douglas-Rachford operator is \( O \left( \frac{1}{\sqrt{t}} \right) \). Under a Lipschitz assumption instead of the metric subregularity, the global exponential-type convergence rate of the
An adaptive Douglas-Rachford dynamic system can be achieved. We applied the obtained results to deal with the minimum value problem of the sum of two functions, one of which is strongly convex while the other one is weakly convex, and derived the corresponding results.

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