INVERSE SPECTRAL PROBLEMS FOR SCHröDINGER AND PSEUDO-DIFFERENTIAL OPERATORS.

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Abstract. Starting from the semi-classical spectrum of Schrödinger operators \(-h^2\Delta + V\) (on \(\mathbb{R}^n\) or on a Riemannian manifold) it is possible to detect critical levels of the potential \(V\). Via micro-local methods one can express spectral statistics in terms of different invariants:

- Geometry of energy surfaces (heat invariant like).
- Classical orbits (wave invariants).
- But also classical equilibria (new wave invariants).

Any critical point of \(V\) with zero momentum is an equilibrium of the flow and generates many singularities in the semi-classical distribution of eigenvalues. Via sharp spectral estimates, this phenomena indicates the presence of a critical energy level and the information contained in this singularity allows to reconstruct partially the local shape of \(V\).

Several generalizations of this approach are also proposed.

Keywords: Spectral analysis, P.D.E., Micro-local analysis; Schrödinger operators; Inverse spectral problems.

1. Introduction.

1.1. Background and basic definitions. In this article we are interested in the inverse spectral problem for partial differential operators and pseudo-differential operators in the semi-classical or high-energy regime. A natural question is to try to understand how the semi-classical spectrum of such an operator can describe the shape of the graph of the (principal) symbol: critical points, extrema and associated local Taylor expansions. Because the spectrum is invariant under translation of the symbol it is in general not possible to obtain more than a qualitative answer. For example there is no hope to locate critical points of the symbol starting only from the spectrum.

The results we would like to present are perhaps of particular interest for \(h\)-quantized Schrödinger operators on \(L^2(\mathbb{R}^n)\):

\[ P_h = -h^2\Delta + V, \]

also called semi-classical Schrödinger operators. Here we will assume that the potentials \(V\) are smooth on \(\mathbb{R}^n\) and bounded from below. For this class of operators, the question is then to understand how certain fluctuations in the semi-classical spectrum can describe the shape of \(V\). But our results will also apply to more general operators like \(h\)-pseudo-differential, \(h\)-admissible...
operators (see definitions below) or Schrödinger operators on a Riemannian manifold $M$ (simply replace the Laplace operator by a Laplace-Beltrami operator $\Delta_M$, see section 5). Most of these modifications are possible because our methods are micro-local and do not use global results on these operators.

Notations. Before entering into the details, let us give some definitions and recall some basic facts about the spectral theory for $P_h$. By a standard result, see [3], when $V$ is bounded from below and with tempered growth, $P_h$ has a self-adjoint realization on a dense subset of $L^2(\mathbb{R}^n)$. As usually, to this quantum operator $P_h$ we can associate a classical counterpart with the Hamiltonian function $p(x, \xi) = \xi^2 + V(x)$, or total energy, on the phase space $\mathbb{R}^n \times \mathbb{R}^n$. In what follows, we note $\Phi_t$ the flow of the Hamiltonian vector field:

$$H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi.$$  

A classical energy surface is:

$$\Sigma_E = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n / \xi^2 + V(x) = E\}$$

and for a classical energy $E \in \mathbb{R}$ we will say that:

- $E$ is regular when $dp \neq 0$ everywhere on $\Sigma_E$.
- $E = E_c$ is critical if $dp = 0$ somewhere on $\Sigma_{E_c}$.

In the present article we are very precisely interested in a relation between the asymptotic properties as $h \to 0^+$ of eigenvalues $\lambda_j(h)$ of $P_h$:

$$P_h \varphi_j(x, h) = \lambda_j(h) \varphi_j(x, h), \quad \varphi_j \in L^2(\mathbb{R}^n),$$

and the set of fixed point for the map $\Phi_t$ (viewed as a map on $\mathbb{R}_t \times T^*\mathbb{R}^n$). We recall in section 2, a sufficient condition to get discrete spectrum. It is well known, see [11], that this semi-classical problem and the high-energy limit for the spectrum of the Laplace operator $\Delta_M > 0$ on a compact Riemannian manifold $M$ are related. This relation can be viewed by quantizing $1/h = \sqrt{\lambda_j}$ where $\lambda_j \to +\infty$ is the increasing sequence of eigenvalues of $\Delta_M$. But in this article we will mainly consider the semi-classical problem and associated micro-local methods.

Duality between the quantum and classical worlds. In geometry spectrum and periodic orbits can be related, in a very explicit way, by means of the Selberg (e.g. see [22]) and Duistermaat-Guillemin [14] trace formulae. It is in general in the most atypical situations, compact surfaces of constant negative curvatures or at the opposite completely integrable systems (e.g. like the free Laplacian on a flat torus) that the most explicit and exact results can be obtained. In quantum mechanics, the existence of such a relation is strongly suggested by the correspondence principle which asserts that, in the semiclassical regime $h \to 0$, many properties of $P_h$ can be related to integral curves of $\Phi_t$ and many invariant attached to the flow around these curves (see Eq.(1) below). For a general Hamiltonian $H$, not necessarily of the form kinetic energy plus potential, this correspondence principle is
Inverse Spectral Problems. 3

also true under very reasonable assumptions on the symbol $h(x,\xi)$ of $H$, a function on the phase space.

In physics a more precise formulation of this principle appeared in the
works of Balian&Bloch [2] and Gutzwiller [18]. The Gutzwiller formula is
usually written as a trace formula for the resolvent of $P_h$ at a given
energy $E$:

\begin{equation}
\sum_{j\in\mathbb{N}} \frac{1}{\lambda_j(h) - E} = \frac{\text{Lvol}(\Sigma_E)}{(2\pi h)^n} + \frac{1}{ih} \sum_{\gamma\in\Sigma_E} A_\gamma e^{\frac{i}{h}S_\gamma},
\end{equation}

where in the r.h.s the sum concerns the closed orbits $\gamma$ inside $\Sigma_E$. Here Lvol$(\Sigma_E)$ is the Riemannian volume of $\Sigma_E$ (defined w.r.t. the invariant Liouville measure), $S_\gamma = \int_\gamma \xi dx$ and $A_\gamma$ are respectively the classical action and the stability factor (including the Maslov phase) of the curve $\gamma$. Recall that the Liouville-volume Lvol$(\Sigma_E)$ satisfies the co-area formula:

$$\int_{[a,b]} \text{Lvol}(\Sigma_E) dE = \int_{\{(x,\xi) : a < p(x,\xi) < b\}} dx d\xi = \text{Vol}_{\mathbb{R}^2n}(p^{-1}([a,b]),$$

where the measure on the r.h.s. is the Lebesgue measure of the pull-back.

In mathematics and in physics, such a relation between spectrum and
periodic orbits provides a powerful tool of analysis and computation. See e.g. [25] concerning the asymptotic behavior of eigenvectors $\varphi_j(x,h)$ and [19] for various applications in quantum chaos. See also [11] for a nice overview and applications in Riemannian geometry.

Mathematical problems. For a Schrödinger operator on $\mathbb{R}^n$, it is easy to check that two different type of divergence generally occur in Eq.(1):

1) The sum over the spectrum is divergent when the resolvent is not a trace-class operator. In particular this is the case if $V$ does not go fast enough to $\infty$ when $|x| \to \infty$. When the sum appears to be convergent it can also have a divergent behavior when $h \to 0$. Worst, it can be that both sides of Eq. (1) do not fit in the regime $h \to 0$.

2) The sum over closed orbits is generally divergent. This is the case if $|A_\gamma|$ does not decrease fast enough or if the number of periodic orbits of period smaller than $T$ is exponentially growing with $T$.

For example, if $n = 1$, using scaling and the asymptotic properties of the spectrum (here simply given by some Bohr-Sommerfeld quantization conditions, see [3]) it is easy to check that the trace of the resolvent of:

$$Q_h = -h^2 \frac{d^2}{dx^2} + |x|^\alpha, \quad h > 0, \quad \alpha > 0,$$

exists if and only if $\alpha > 2$. Here, the harmonic oscillator, obtained for $\alpha = 2$ for which $\lambda_j(h) = h(2j + 1)$, is the limit case and the series diverges like the harmonic series.
1.2. Mathematical approach of the Gutzwiller formula.

As seen above, the question to remove divergences has many important implications and we explain now a mathematical way to solve this problem via a smoothing of the so-called spectral density. To begin the discussion, simply assume that:

For some $E \in \mathbb{R}$, the spectrum of $P_h$ is discrete, with finite multiplicities, in the interval $[E - \varepsilon, E + \varepsilon], \varepsilon > 0$.

A sufficient condition to obtain this property, uniformly w.r.t. $E$, is given in section 2. A well-posed problem is to study the asymptotic behavior of the spectral distributions:

\[
\Upsilon(E, h, \varphi) = \sum_{|\lambda_j(h) - E| \leq \varepsilon} \varphi \left( \frac{\lambda_j(h) - E}{h} \right), \text{ as } h \to 0,
\]

where $\varphi$ is a test function chosen to remove the divergences. We can justify this terminology if we observe that the truncated spectral distribution:

\[
T_{E, \varepsilon}(x) = \sum_{|\lambda_j(h) - E| \leq \varepsilon} \delta_{\lambda_j(h)}(x), \langle \delta_{x_0}, f \rangle = f(x_0),
\]

acting on a function $\varphi$ shifted by $E$ and scaled w.r.t. $h$ provides:

\[
\Upsilon(E, h, \varphi) = \left\langle T_{E, \varepsilon}(x), \varphi \left( \frac{x - E}{h} \right) \right\rangle.
\]

In reality this scaling w.r.t. $h$ is very important and is used to get parametrics involving the classical dynamics. Also it is not very hard to verify that when $\varphi \in S(\mathbb{R})$ the size of the truncation (materialized here as $\varepsilon$) is irrelevant on a scale of size $O(h^\infty)$ as long as $\varepsilon$ is strictly positive. I refer to section 3 for these points but I simply recall that $O(h^\infty)$ is the class of functions of $h$ being in $O(h^k)$ for every $k \in \mathbb{N}$ near $h = 0$.

**Statistical quantum mechanics.** In general, apart in some very specific situations, it is not possible to compute explicitly the spectrum of $P_h$ and a motivation to do semi-classical or high-energy estimates is to derive statistics about eigenvalues and their distribution. For example, in Eq.(2) the formal choice of $\varphi$ as the characteristic function of $[-\eta, \eta], 0 < \eta < \varepsilon$, determines the number $N(E, h)$ of bound states in $[E - \eta h, E + \eta h]$.

This formal correspondence between $\Upsilon$ and the micro-local counting function $N$ has a mathematically rigorous formulation in term of Tauberian-theorems, see e.g. [4]. Under certain (generic) conditions on the symbol $p$ of $P_h$, it can be proven that $N(E, h)$ is proportional to $h^{1-n}$ times the Liouville-volume of the energy shell $\Sigma_E$:

\[
N(E, h) \sim \frac{1}{(2\pi h)^{n-1}} Lvol(\Sigma_E) \text{ as } h \to 0^+.
\]

\[\text{1Many important questions concerning the range of trace formulae (e.g., their validity beyond the Ehrenfest-time) are still open. We do not discuss these questions in this article.} \]

\[\text{2In particular the condition that } E \text{ is non-critical for } p, \text{ see below.} \]
This is a micro-local formulation of the Weyl-law. A fortiori when \( n \geq 2 \) this implies that the finite sum defining \( \Upsilon(E, h, \varphi) \) will involve a large number of eigenvalues in the regime \( h \to 0^+ \). In general the formula for \( N(E, h) \) can be formally integrated to obtain a formula for counting eigenvalues in a compact interval:

\[
N([a, b], h) = \# \{ j \in \mathbb{N} : \lambda_j(h) \in [a, b] \} \sim \frac{1}{(2\pi h)^h} \text{Vol}_{\mathbb{R}^{2n}} p^{-1}([a, b]).
\]

Of course if the operator is bounded from below you can also use \( N(x, h) := N([-\infty, x], h) \). The question to estimate the remainder function for \( N \) is in general a relatively complicated problem and requires to use the properties of the underlying classical dynamics inside \( p^{-1}([a, b]) \). Several other related problems, Riesz-moments or Lieb-Thirring inequalities, can be formulated in terms of \( N(E, h) \) and these problems are important in the 'stability of matter' problem. See [26] for an overview and references. Finally, I mention that certain Schrödinger operators with very singular critical sets (e.g. see [4, 8]) or non-confining potentials (e.g. see [30]) can lead to some very different kind of 'Weyl-asymptotics' for \( N \) or \( N \).

Relation with the classical dynamics. In reality the quantity \( \Upsilon(E, h, \varphi) \) contains many interesting information (a priori more than the counting functions) since the asymptotic expansion of \( \Upsilon(E, h, \varphi) \) when \( h \to 0^+ \) involves explicitly the classical dynamics on \( \Sigma_E \) and in particular the set of fixed point of the flow inside the energy surface:

\[
\text{Fix}_E = \{ (T, x, \xi) \in \mathbb{R} \times \Sigma_E : \Phi_T(x, \xi) = (x, \xi) \}.
\]

We recall that \( E \) is regular if \( \nabla p(x, \xi) \neq 0 \) on \( \Sigma_E \) and critical otherwise. Every critical point \( (x_0, \xi_0) \in \Sigma_{E_c} \) of \( p \) is a fixed point of our flow \( \Phi_t \) since \( H_p(x_0, \xi_0) = 0 \). In this situation we have \( \mathbb{R} \times \{ (x_0, \xi_0) \} \subset \text{Fix}_{E_c} \).

When \( E \) is not critical and the periodic orbits satisfy a condition of non-degeneracy, the asymptotics behavior of Eq.(2) is well determined by the closed orbits of \( \Phi_t \) on \( \Sigma_E \) and the geometry of \( \Sigma_E \). For the full treatment of this problem, and a complete formulation of the asymptotic expansion, we refer to [5, 27].

Removing divergences. We explain now shortly why the problem stated in Eq.(2) leads to a mathematically rigorous version of the Gutzwiller formula. First, for each \( h > 0 \) the sum is finite and a fortiori convergent. A convenient choice of \( \varphi \) also ensures that this quantity has an asymptotic expansion when \( h \to 0 \) independently from the choice of \( \varepsilon > 0 \) up to corrections of order \( O(h^\infty) \) as long as \( \varepsilon \) stays strictly positive. Such a difference of size \( O(h^\infty) \) plays no rôle since the discussion will be based on some finite order asymptotics w.r.t. \( h \).
On the other side, only the periods of $\Phi_t$ inside $\text{supp}(\hat{\varphi})$, the support of the Fourier transform:

$$\hat{\varphi}(t) = \int_{\mathbb{R}} e^{itx} \varphi(x) dx,$$

contribute in the asymptotic expansion. This principle is useful since when $\text{supp}(\hat{\varphi})$ is compact then finitely many closed orbits of $\Sigma_E$ contribute and the second divergence is solved. Hence if $\varphi \in C_0^\infty(\mathbb{R})$, the space of smooth functions with compact support, $\varphi$ is in the Schwartz space $\mathcal{S}(\mathbb{R})$. Since elements of $\mathcal{S}(\mathbb{R})$ are smooth with exponential decay at infinity, no divergence occurs and the size of $\varepsilon$ is irrelevant, up to an $O(h^\infty)$-error.

Finally, in Eq.(2) the scaling w.r.t. $h$ is important. With this choice and via Fourier transform considerations, we can use the semi-classical propagator $U_h(t) = \exp(itP_h/h)$, solution of the Schrödinger equation:

$$-i\hbar \partial_t U_h(t) = P_h U_h(t),$$

to obtain a precise control w.r.t. $h$. Roughly, $U_h(t)$ can be expanded w.r.t. $h$ via a so-called WKB approximation. This expansion also provides the explicit relation with the classical dynamics. The precise technical justifications are given in section 3.

1.3. Critical values and contributions of equilibria. In the previous section we heuristically outlined a relation valid when the semi-classical parameter tends to 0:

$$\lim_{h \to 0} \Upsilon(E, h, \varphi) \equiv \text{Fix}_E = \{(t, x, \xi) \in \mathbb{R} \times \Sigma_E / \Phi_t(x, \xi) = (x, \xi)\}.$$

Meaning that the asymptotic behavior of the left hand side can be expressed in terms of distributions generated by fixed point of the flow.

In the r.h.s any point $(x, \xi)$ of a periodic orbit $\gamma$ appears only at times $kT^*_{\gamma}$, $k \in \mathbb{Z}^*$, where $T^*_{\gamma}$ is the primitive period of $\gamma$ orbit. Also any point of the energy surface contributes for $t = 0$ since the flow is the identity at $t = 0$. But an equilibrium $(x_0, \xi_0)$ satisfies $\Phi_t(x_0, \xi_0) = (x_0, \xi_0)$ for all $t$. Hence when $E$ is no more a regular value the nature of the set of fixed point changes and some new contributions appear in the asymptotic expansion. These new contributions can be qualified of new wave invariants (see below) and are extremely important for the inverse spectral problem.

When $E = E_c$ is a critical value of the principal symbol $p$, the asymptotic behavior of Eq.(2) is more complicated and is closely related to the geometry of the flow inside $\Sigma_{E_c}$. The presence of classical equilibria inside $\Sigma_{E_c}$ and the stability of the flow near the critical set affects strongly the nature of the asymptotic expansions. For a non-degenerate critical point, i.e. when $d^2p(x_0, \xi_0)$ is an invertible matrix when $dp(x_0, \xi_0) = 0$, the reader can consult [4]. The problem is treated there for quite general operators, also including the case of a manifold of critical points, but for $\text{supp}(\hat{\varphi})$ small around the origin. For Schrödinger operators on $\mathbb{R}^n$ and $\text{supp}(\hat{\varphi})$ compact but arbitrary, the results of [4] are improved in [24].
Two important problems occur in presence of critical points. First, at every point where $dp = 0$ the surface $\Sigma_{E}$ and the metric of $\Sigma_{E}$ are not smooth. Next, the determination of the asymptotic expansion w.r.t. $h$ can be very difficult. The point is that $\Upsilon(E, h, \varphi)$ can be expressed in terms of oscillatory integrals:

$$I(h) = \int_{\mathbb{R} \times T^{*} \mathbb{R}^{n}} a(t, x; \xi) e^{\frac{i}{h} \psi(t, x, \xi)} dt dx d\xi, \ h \to 0^{+}.$$ 

The oscillating coefficient $h^{-1}$ is precisely imposed by the scaling w.r.t. $h$ in Eq.(2) and plays an important role since $I(h)$ oscillates fast in the semi-classical regime. Via the WKB approximation, the phase $\psi$ is related to the flow so that the asymptotic behavior of $I(h)$ is determined by the closed orbits. The technical problem is that, in presence of an equilibrium, $\psi$ has some degenerate critical points. The stationary phase method cannot be applied and the asymptotic expansion of $I(h)$ is radically different: e.g. some terms $h^{\alpha}$, $\alpha \in \mathbb{Q}$ and powers of $\log(h)$ generally appear in this setting (see below). Also the nature of these new terms can be very different since for example they can be associated to some distributions acting on $\varphi$ with a continuous support (e.g. the full set of real numbers or a half-line).

**Wave-invariants.** A classical approach (used in section 3) is to study the asymptotic behavior, as $h \to 0^{+}$, of the localized trace:

$$\Omega(E, h, t) = \text{Tr} \left( \Theta(P_{h}) e^{-\frac{i}{h} (P_{h} - E)} \right), \Theta \in C_{0}^{\infty}. \$$

I will follow now the terminology used in [23]. Under certain assumptions (see section 3 and 4), and for $E$ regular it is well known that $\Omega$ admits an asymptotic expansion of the form:

$$\Omega(E, h, t) \sim \sum_{j=-n}^{\infty} a_{j}(E, t) h^{j}, \text{ as } h \to 0^{+}. \$$

The coefficients $a_{j}(E, t)$ are some tempered distributions on the line $\mathbb{R}_{t}$ and are called wave invariants of $P_{h}$. These distributions have a different expression when $E$ varies and many of them are continuous functions of $E$ as long as we do not cross critical levels of the energy function $p$.

When $E \to E_{c}$ ($E_{c}$ stands for critical levels) one can observe a discontinuity in the asymptotic expansion. Also some new coefficients generally appear, since we have the asymptotics:

$$\Omega(E_{c}, h, t) \sim \sum_{k=0}^{n-1} \sum_{j=-m_{0}}^{\infty} a_{j,k}(E_{c}, t) h^{j} \log(h)^{k}, \text{ as } h \to 0^{+}, \quad \text{for some } p \in \mathbb{N}^{*}, \text{ see [4, 6, 8, 9, 24] for details and examples. These new coefficients are called new wave invariants and the top order coefficient of the expansion in Eq.(3) contains many information on the shape of the symbol.}$$
1.4. Results and strategy. Our first objective is to relate some variations in the discrete spectrum of $P_h$ with the presence of fixed points for the classical system: this principle detects the presence of new wave invariants and a fortiori critical energy levels. Secondly, we establish that the precise knowledge of such a spectral fluctuation can describe the singularity of the potential (or of the symbol for general operators). In theory, such a determination is possible since the contributions of equilibriums are highly sensitive to the local shape of $V$ and are extremely persistent when the test function $\varphi$ varies.

For Schrödinger operators, we will consider the case of a potential $V$ with finitely many critical points $x^j_0$ attached to local homogeneous extremum of $V$. An immediate consequence is that $p$ admits, locally, a unique critical point $(x^j_0,0)$ on the surface $\Sigma_{E^j_c} = \{ (x,\xi) \in \mathbb{R}^{2n} / \xi^2 + V(x) = V(x^j_0) \}$. A typical example is a polynomial double well in dimension 1 where 3 critical points occur at the 2 minima and at the maximum of $V$ (see figure 1).

![Figure 1. Non-symmetric and symmetric double well.](image)

In fact using certain generalizations of stationary phase methods, necessary if the phase has some degenerate critical points one can derive a very precise relation between the spectrum and the set of fixed points of the flow also including the new wave invariants. Once this relation is established in the form of asymptotic expansion, the main results follow since:

- Equilibriums have a continuous contribution w.r.t. the time $t$.
- Shrinking $\text{supp}(\hat{\varphi})$ erases all other contributions: Weyl-terms and periodic orbits.
- The remaining contribution, given by new wave invariants, displays some nice information on $V$.

The first assertion simply means that a fixed point generally contributes to the asymptotic expansion of $\Upsilon(E_c^j, h, \varphi)$ in the form $h^\alpha \log(h)^\beta \langle T_{\alpha,\beta}, \hat{\varphi} \rangle$ where $T_{\alpha,\beta}$ is a distribution such that $\text{supp}(T_{\alpha,\beta}) = \mathbb{R}$, $[a, \infty]$ or $[\infty, a]$. Contrary to standard periodic orbits whom contributions are supported in the set of periods, such a term supported on the line cannot be erased just by shrinking the support of $\hat{\varphi}$.

\footnote{Some discrete contributions can also sometimes occur as pointed out in [4] or [6]. But, to attain our objectives, we can avoid to include them in the spectral estimates.}
For example, if \( \text{supp}(\hat{\varphi}) \) contains no period of the flow our analysis follows if we view \( \Upsilon(E, h, \varphi) \) as a function of \( E \):

- The order w.r.t \( h \) of \( \Upsilon(E, h, \varphi) \) changes when \( E \to E_c^j \) (Prop. 5).
- This discontinuity at \( E_c^j \) describes the shape of \( V \).

This indicates the presence of an equilibrium for \( \Phi_t \), a fortiori of a critical point for \( V \).

**Remark 1.** For a degenerate singularity the information is more difficult to interpret compared to a non-degenerate singularity (see section 2).

2. Hypotheses and main result.

Let \( p(x, \xi) = \xi^2 + V(x) \), where the potential \( V \) is real valued and smooth on \( \mathbb{R}^n \). To this Hamiltonian is attached the operator \( P_h = -h^2 \Delta + V(x) \) and by a classical result, see [3], \( P_h \) is essentially self-adjoint starting from a dense domain of \( L^2(\mathbb{R}^n) \) when \( V \) is bounded from below and with tempered growth.

**Remark 2.** In this section, we are here mainly interested in the case of Schrödinger operators but generalizations to an \( h \)-admissible operator (e.g. in the sense of [29]) are given in section 5.

First, to obtain a well defined spectral problem, we use:

\[
(H_1) \quad V \in C^\infty(\mathbb{R}^n). \quad \text{There exists } \ C \in \mathbb{R} \text{ such that } \liminf_{\infty} V > C.
\]

Note that \((H_1)\) is always satisfied if \( V \) goes to infinity at infinity. Now, consider an energy interval \( J = [E_1, E_2] \) with \( E_2 < \liminf_{\infty} V \). In the following we note:

\[
J(\varepsilon) = [E_1 - \varepsilon, E_2 + \varepsilon].
\]

For \( \varepsilon < \varepsilon_0 \) the set \( p^{-1}(J(\varepsilon)) \) is compact. By Theorem 3.13 of [29] the spectrum \( \sigma(P_h) \cap J(\varepsilon) \) is discrete and consists in a sequence:

\[
\lambda_1(h) \leq \lambda_2(h) \leq \ldots \leq \lambda_j(h),
\]

of eigenvalues of finite multiplicities, if \( \varepsilon \) and \( h \) are small enough. In general such a condition that the pullback of \( J \) or \( J(\varepsilon) \) by \( p \) is compact is sufficient to obtain a discrete spectrum. This is not necessary as shows the non-confining potential \( x^2y^2 \) on \( \mathbb{R}^2 \), see [30] where precise spectral estimates are given for such potentials.

The central object of study is the spectral distribution:

\[
\Upsilon(E, h, \varphi) = \sum_{\lambda_j(h) \in J(\varepsilon)} \varphi\left(\frac{\lambda_j(h) - E}{h}\right),
\]

and, more precisely, the asymptotic information contained in this object as \( h \to 0^+ \). To avoid any problem of convergence we impose the condition:

\[
(H_2) \quad \text{We have } \hat{\varphi} \in C^\infty_0(\mathbb{R}) \text{ with a sufficiently small support near the origin.}
\]
Remark 3. \((\mathcal{H}_2)\) is used to erase contributions of non-trivial closed orbits and can be relaxed to \(\hat{\phi} \in C_0^\infty(\mathbb{R})\) with a weaker result. A more precise description of \(\text{supp}(\hat{\phi})\) is given in Lemma 16. For a non-degenerate minimum, it is more comfortable to assume that \(\text{supp}(\hat{\phi})\) contains no period of \(d\Phi_t(z_0)\). Some singularities, not directly relevant here, are generated by these periods and we refer to [4, 24] for a detailed study of these contributions.

To simplify notations we write \(z = (x, \xi) \in \mathbb{R}^{2n}\) and \(\Sigma_E = p^{-1}(\{E\})\) and we use the subscript \(E_c\) to distinguish out critical values of \(p\). Of course one can also work with \(T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n\). In \(J\) there is finitely many critical values \(E_1^c, ..., E_l^c\) and in \(p^{-1}(J)\) finitely many fixed points \(z_0^1, ..., z_0^m, m \geq 1\).

We impose now the type of singularity:

\((\mathcal{H}_3)\) On each \(\Sigma_{E_j^c}\) the symbol \(p\) has isolated critical points \(x_j^0 = (x_j^0, 0)\). These critical points can be degenerate but are associated to a local extremum of \(V\):

\[
V(x) = E_c + V_{2k}(x) + \mathcal{O}(|x - x_j^0|^{2k+1}), \quad k \in \mathbb{N}^*,
\]

where \(V_{2k}\), homogeneous of degree \(2k\), is definite positive or negative.

Remark 4. For non-degenerate singularities we can apply the results of [4, 6, 24] and the extremum condition is not really necessary. We will recall and use these results in the next section. But for a degenerate critical point of \(V\) the extremum condition is required since, to our knowledge, the contribution of such a singularity to the wave expansion is unknown.

The next assumption, erases the mean values, i.e. the heat-like invariants or so-called Weyl-terms, in the trace formula:

\((\mathcal{H}_4)\) \(\hat{\phi}\) is flat at 0, i.e. \(\hat{\phi}^{(j)}(0) = 0, \forall j \in \mathbb{N}\).

We can weaken condition \((\mathcal{H}_4)\) to \(\hat{\phi}^{(j)}(0) = 0, \forall j \leq k_0, \) where \(k_0 \in \mathbb{N}^*\) depends only on the degree of the singularities of \(V\) (see section 4) without essential change. Such a function \(\varphi\) exists and is easy to construct. Pick \(\phi \in C_0^\infty(\mathbb{R}), \text{supp}(\phi) \subset [-M, M]\), then \(\hat{\varphi}(t) = t^{2k_0} \phi(t)\) satisfies our hypotheses. In this case, we can chose the function \(\phi\) even so that \(\varphi\) is real.

Finally, to relax a bit \((\mathcal{H}_2)\) we need a control on the contribution of closed orbits. To do so, we impose the classical condition :

\((\mathcal{H}_5)\) All periodic trajectories of the flow are non-degenerate.

Non-degenerate closed orbits are those whose Poincaré map does not admit 1 as eigenvalue and are isolated. The non-degeneracy condition on orbits is not a central argument in this work and is only used to control the order w.r.t. \(h\) of the contribution of closed orbits (i.e. the order of the usual wave-invariant). One could also impose a condition of ‘clean-flow’ to consider families/submanifolds or bunches of closed orbits of positive dimension. These conditions on the classical dynamics can be simply discarded as soon as we have a strictly positive lower-bound on periods of closed-orbits like in Lemma 16.
Detecting critical levels. The first result shows how to detect critical energy levels by revealing a singularity in the spectral estimates:

**Proposition 5 (Spectral variation).**
Assume that conditions \((H_1)\) to \((H_4)\) are satisfied. As \(h\) tends to \(0^+\), we have:

\[
\Upsilon(E,h,\varphi) = \begin{cases} O(h^{\infty}) & \text{if } E \in [E_1, E_2] \setminus \{E_1^c, \ldots, E_l^c\}, \\ O(f_j(h)) & \text{if } E = E_j^c, j \in \{1, \ldots, l\}, \end{cases}
\]

where each \(f_j(h)\) has a finite order w.r.t. \(h\).

For a non-degenerate critical point the coefficients \(f_j\) can be determined explicitly. It is in general possible to predict a full asymptotic expansion but to obtain an invariant formulation of all the coefficients can be difficult since for degenerate critical points the method you have to use is more complicated than the usual stationary-phase formula.

For example, if \(E = E_j^c\) and the surface \(\Sigma_{E_j^c}\) carries a single minimum of degree \(2k\), \(k > 1\), using the main result of [8] we obtain:

\[
f_j(h) = D(n, k, \varphi) h^{\frac{n}{2} + \frac{1}{k} - n}.
\]

In the same situation, but for a local maximum of \(V\), using the results of [9], we can obtain a logarithm of \(h\):

\[
f_j(h) = D(n, k, \varphi) h^{\frac{n}{2} + \frac{1}{k} - n} \log(h)^j, \quad j = 0 \text{ or } 1.
\]

In Eq.\((7)\) and Eq.\((8)\) the coefficient \(D\) is a tempered distributions acting \(\varphi\) characteristic from the nature of the critical point. In fact if the critical surface carries more than one critical point then \(f_j\) is the sum of their respective contributions. Note that for \(n = 1\) and \(k > 1\) the singular term has negative order w.r.t. \(h\). A detailed formulation of the coefficients \(f_j(h)\) is given in Propositions 18, 19 and 20.

**Inverse spectral results.** An interesting property is that in the singularity of \(\Upsilon(E,h,\varphi)\) when \(E \to E_c^d\) the order w.r.t. \(h\) but also the constants of the top order-coefficients, describes partially the shape of \(V\):

**Theorem 6 (Inverse result for Morse-critical points).**
Assume that \(\Sigma_{E_c}^d\) carries exactly one critical point associated to a non-degenerate critical point \(x_0\) of \(V\). Then the discontinuity of \(\Upsilon(s,h,\varphi)\) at \(s = E_c^d\) determines the spectrum of \(d^2V(x_0)\).

This result follows from the special form of the Duistermaat-Guillemin-UrIBE density at a non-degenerate critical point. Observe that in particular we retrieve the Morse index of \(V\) at \(x_0\) (number of positive eigenvalues minus number of negative eigenvalues of \(d^2V(x_0)\)). Once more, because of the invariance under coordinates permutations, or under a rotation of the potential around the critical point \(x_0\), it is in general not possible to retrieve the quadratic form \(d^2V(x_0)\) in a given system of coordinates \(x\).

For a degenerate homogeneous singularity we have also a nice result:
Theorem 7 (Inverse result. Degenerate critical points of $V$.)
Assume that there is exactly one critical point $(x_0, 0)$ on the singular energy surface $\Sigma_{E_c}$. Assume that $x_0$ is attached to an homogeneous maximum or minimum of the potential of degree $2k$. Then the discontinuity of $\Upsilon(E, h, \varphi)$ at $E = E_c$ determines:

- The degree $2k$ of the critical point of $V$.
- The spherical mean-value of the germ of $V$ in $x_0$: 
  $$A(V) = \int_{S^{n-1}} |V_{2k}(\theta)|^{-\frac{n}{2}} d\theta.$$  

Observe that $A(V)$ itself is invariant under rotation and translation of $V$. Both results of Theorems 6 and 7 are limited in presence of multiple equilibriums on the same surface since the sum of contributions of each critical point could lead to a compensation or to several contributions of exactly same nature and order. In general if $V$ is not a Morse function, or if a surface of energy $\Sigma_{E_c}$ carries more than one critical point, eigenfunction-estimates seems to be required to have a well-posed inverse problem. Finally, at the end of the article we will show up some interesting invariants for pseudo-differential operators with homogeneous singularities attached to extremum of the symbol.

Remark 8. I would like to emphasize that a maximum is more difficult to detect contrary to a local minimum which is an isolated point of the energy surface (locally $\Sigma_{E_c}$ is just a point). A similar result holds for an operator of the form $T(\xi) + V(x)$ (kinetic plus potential energy) where $T$ is convex near the origin and $V$ has a local minimum at $x_0$. Moreover a maximum, attached to an unstable critical point of the flow, is much more complicated to treat with semi-classical methods. See [9] for a detailed study.

Longer range estimates. In $(H_2)$ the condition that $\text{supp}(\hat{\varphi})$ is small implies a very accurate spectral estimate (e.g. by a Paley-Wiener estimates for the decay of $\varphi$). It is possible to relax this assumption but the result is a bit weaker:

Corollary 9. Assume that conditions $(H_1)$, $(H_3)$, $(H_4)$ and $(H_5)$ are satisfied and that $\hat{\varphi} \in C_0^\infty(\mathbb{R})$, then we obtain:
$$\Upsilon(E, h, \varphi) = O(1) \text{ if } E \in [E_1, E_2] \backslash \{E_1^1, ..., E_\ell^1\}.$$  
For critical values of $p$, estimates are the same as in Proposition 5.

The justification, see section 4, is that in this case the asymptotics is given by a finite sum over periodic orbits of energy $s$. This result is weak if the singularity of $V$ is non-degenerate since the equilibrium has a contribution of degree $0$ w.r.t. $h$, see Propositions 18, 19, 20 or section 3 of [4].

In theory there is always a variation when $E \to E_c$ but of course this effect can be harder to detect if there is no change in the order w.r.t. $h$. In
that situation there is only a discontinuity in the top-order coefficient w.r.t. $h$ so that the result can be qualified of 'weaker'.

3. Oscillatory representation.

The construction below is more or less classical and will be sketchy. The only change with the usual construction, around a single energy level, is that we use a more global localization around $J = [E_1, E_2]$. Strictly speaking, with $(H_1)$, we could also consider $]-\infty, E_2]$ since there is no eigenvalue below a fixed energy level $E_0$ given by the minimum of the quadratic form attached to $P_h$. Let be $\varphi \in S(\mathbb{R})$ with $\hat{\varphi} \in C_0^\infty(\mathbb{R})$, we recall that:

$$\Upsilon(E, h, \varphi) = \sum_{\lambda_j(h) \in J(\varepsilon)} \varphi\left(\frac{\lambda_j(h) - E}{h}\right), \quad J(\varepsilon) = [E_1 - \varepsilon, E_2 + \varepsilon],$$

with $p^{-1}(J(\varepsilon))$ compact in $T^*\mathbb{R}^n$. For $\varepsilon > 0$ small enough, we localize around $J$ with a cut-off $\Theta \in C_0^\infty(]-E_1, E_2+[)$, such that $\Theta = 1$ on $J$ and $0 \leq \Theta \leq 1$ on $\mathbb{R}$. We accordingly split-up our spectral distribution as:

$$\Upsilon(E, h, \varphi) = \Upsilon_1(E, h, \varphi) + \Upsilon_2(E, h, \varphi),$$

with:

$$\Upsilon_1(E, h, \varphi) = \sum_{\lambda_j(h) \in J(\varepsilon)} (1 - \Theta)(\lambda_j(h))\varphi\left(\frac{\lambda_j(h) - E}{h}\right),$$

$$\Upsilon_2(E, h, \varphi) = \sum_{\lambda_j(h) \in J(\varepsilon)} \Theta(\lambda_j(h))\varphi\left(\frac{\lambda_j(h) - E}{h}\right).$$

Since $\varphi \in S(\mathbb{R})$ a classical estimate, see e.g. Lemma 1 of [7], is:

$$(9) \quad \Upsilon_1(E, h, \varphi) = O(h^\infty), \quad as \quad h \to 0^+.$$  

By inversion of the Fourier transform we have:

$$\Theta(P_h)\varphi\left(\frac{P_h - E}{h}\right) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\frac{P_h}{h}} \varphi(t) \exp\left(-\frac{it}{h}P_h\right) \Theta(P_h) dt.$$  

The trace of the left hand-side is $\Upsilon_2(E, h, \varphi)$ and Eq.(9) provides :

$$\Upsilon(E, h, \varphi) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} e^{itE} \varphi(t) \exp\left(-\frac{it}{h}P_h\right) \Theta(P_h) dt + O(h^\infty).$$  

Eq.(10) is very close to the classical Poisson summation formula on $S^1$, see [31] for a discussion and an interpretation, since the r.h.s. is expressed below in term of the classical dynamics. This asymptotic relation justifies the terminology of trace formula.

Moreover, the formulation in Eq.(10) shows that the scaling w.r.t. $h$, imposed in the initial definition of $\Upsilon(E, h, \varphi)$, is the best one since we will solve now the semi-classical propagator homogeneously w.r.t. $h$. Let $U_h(t) = \exp\left(-\frac{it}{h}P_h\right)$ be the quantum propagator. We approximate $U_h(t)\Theta(P_h)$ by a
Fourier integral operator (FIO) depending on $h$. Let $\Lambda$ be the Lagrangian manifold associated to the flow of $p$:

$$\Lambda = \{(t, \tau, x, \xi, y, \eta) \in T^*\mathbb{R} \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n : \tau = p(x, \xi), (x, \xi) = \Phi_t(y, \eta)\},$$

and $I(\mathbb{R}^{2n+1}, \Lambda)$ the class of oscillatory integrals based on $\mathbb{R}^{2n+1}$ and whose Lagrangian manifold is $\Lambda$. The next result is a semi-classical version of a well known result on the propagator, see e.g. Duistermaat [13].

**Theorem 10.** The operator $U_h(t)\Theta(P_h)$ is an $h$-FIO associated to $\Lambda$. For each $N \in \mathbb{N}$ there exists $U^{(N)}_{\Theta_h}(t)$ with integral kernel in Hörmander’s class $I(\mathbb{R}^{2n+1}, \Lambda)$ and $R_h^{(N)}(t)$ bounded, with a $L^2$-norm uniformly bounded for $0 < h \leq 1$ and $t$ in a compact subset of $\mathbb{R}$, such that:

$$U_h(t)\Theta(P_h) = U^{(N)}_{\Theta_h}(t) + h^N R_h^{(N)}(t).$$

This result provides the existence of an asymptotic expansion in power of $h$ with a remainder that can be controlled since $\text{supp}(\hat{\varphi})$ is a compact. After perhaps a reduction of $\varepsilon$, this remainder $R_h^{(N)}(t)$ is estimated via:

**Corollary 11.** Let $\Theta_1 \in C^\infty_0(\mathbb{R})$, with $\Theta_1 = 1$ on $\text{supp}(\Theta)$ and $\text{supp}(\Theta_1) \subset [-E_1 - 2\varepsilon, E_2 + 2\varepsilon]$, then $\forall N \in \mathbb{N}$:

$$\text{Tr}(\Theta(P_h)\varphi(\frac{P_h - E}{h})) = \frac{1}{2\pi} \text{Tr} \int_\mathbb{R} \hat{\varphi}(t)e^{\frac{it}{h}E}U^{(N)}_{\Theta_h}(t)\Theta_1(t)dt + O(h^{N-n}).$$

For a proof of this result, based on the cyclicity of the trace and a priori estimates on the spectral projectors (see [29]), we refer to [7]. For the particular case of a Schrödinger operator the BKW ansatz shows that the integral kernel of $U^{(N)}_{\Theta_h}(t)$ can be recursively constructed as:

$$R_h^{(N)}(t, x, y) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} b_h^{(N)}(t, x, y, \xi) e^{\frac{i}{h}(S(t,x,\xi)-\langle y,\xi \rangle)} d\xi,$$

where $S$ satisfies the Hamilton-Jacobi equation:

$$p(x, \partial_x S(t, x, \xi)) + \partial_t S(t, x, \xi) = 0,$$

with initial condition $S(0, x, \xi) = \langle x, \xi \rangle$. In particular we obtain that:

$$\{(t, \partial_t S(t, x, \xi), x, \partial_x S(t, x, \xi), \partial_\xi S(t, x, \eta), -\eta) \subset \Lambda,$$

and that the function $S$ is a generating function of the flow, i.e.:

$$(11) \quad \Phi_t(\partial_\eta S(t, x, \eta), \eta) = (x, \partial_\xi S(t, x, \eta)).$$

We insert this approximation in Eq. (10), we set $x = y$ and we integrate w.r.t. $x$. Modulo an error $O(h^{N-n})$, we obtain that $Y(E, h, \varphi)$ equals:

$$\begin{equation}
\frac{1}{(2\pi h)^n} \int_{\mathbb{R} \times T^*\mathbb{R}^n} e^{\frac{i}{h}(S(t,x,\xi)-\langle x,\xi \rangle + tE)} a_h^{(N)}(t, x, \xi) \hat{\varphi}(t)dtdxd\xi,
\end{equation}$$
where $a_h^{(N)}(t, x, \eta) = b_h^{(N)}(t, x, x, \eta)$.

**Remark 12.** By Theorem 3.11 & Remark 3.14 of [29], $\Theta(P_h)$ is $h$-admissible. Moreover, the symbol is compactly supported in $p^{-1}(E_1 - \epsilon, E_2 + \epsilon)$. This global result w.r.t. $E \in [E_1, E_2]$ allows to consider below only oscillatory integrals with compact support for the evaluation of the spectral distributions.

**Micro-localization of the trace.**

If $\psi \in C_0^\infty(T^*\mathbb{R}^n)$, we recall that $\psi^w(x, hD_x)$ is the linear operator obtained by Weyl-quantization of $\psi$ and semi-classical quantization of $\xi$. This means:

\[
\psi^w(x, hD_x)f(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{i\frac{x-y}{2}\xi} \psi(x+y/2, \xi) f(y) dy d\xi.
\]

Mainly, the contribution of an equilibrium $z_0 \in \Sigma_{E_c}$ can be reached via:

\[
\Upsilon_{z_0}(E_c, h, \varphi) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} e^{\frac{iE_c}{h}\varphi(t)} \psi^w(x, hD_x) \exp\left(-\frac{i}{h} tP_h\right) \Theta(P_h) dt,
\]

where $\psi \in C_0^\infty(T^*\mathbb{R}^n)$ is equal to 1 near $z_0$. Since the trace is a cyclic operation, we can use the symbolic-calculus and to insert an $L^2$-bounded observable (here a cut-off in the phase space) is in general not expensive in the FIO construction. This construction with smooth cut-off allows to work with pseudo-differential partition of unity. This approach is useful to obtain a weak generalization of our results in presence of multiple equilibriums.

We recall some basic results on the symbolic calculus with FIO. Hörmander’s class of distributions with Lagrangian manifold $\Lambda$ over $\mathbb{R}^n$ is noted $I(\mathbb{R}^n, \Lambda)$. If $(x_0, \xi_0) \in \Lambda$ and $\varphi(x, \theta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N)$ parameterizes $\Lambda$ in a sufficiently small neighborhood $U$ of $(x_0, \xi_0)$, then for each $u_h \in I(\mathbb{R}^n, \Lambda)$ and $\chi \in C_0^\infty(T^*\mathbb{R}^n)$, supp$(\chi) \subset U$, there exists a sequence of amplitudes $c_j(x, \theta) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^N)$ such that for all $L \in \mathbb{N}$:

\[
\chi^w(x, hD_x)u_h = \sum_{-d \leq j < L} h^j I(c_j e^{\frac{i}{h}\varphi}) + O(h^L).
\]

Hence, for each $N \in \mathbb{N}^*$ and modulo an error $O(h^{N-d})$, the localized trace $\Upsilon_{z_0}(E_c, h, \varphi)$ of Eq. (13) can be written as:

\[
(2\pi h)^{-d} \int_{\mathbb{R} \times \mathbb{R}^{2n}} e^{\frac{i}{h}(S(t,x,\xi) - (x,\xi)+tE_c)} a_h^{(N)}(t, x, \xi) \hat{\varphi}(t) dt dx d\xi.
\]

To obtain the exact power $-d$ of $h$, we apply results of Duistermaat [13] on the order of FIO. Since:

- $h$-pseudo-differential operators $\psi^w(x, hD_x)$ are of order 0 w.r.t. $1/h$.
- the order of $U_h(t)\Theta(P_h)$ is $-\frac{1}{4}$.  

if we identify the operator with its distributional-kernel we have:
\[
\psi^\omega(x, hD_x)U_h(t)(P_h) \sim (2\pi h)^{-n} \int_{\mathbb{R}^n} \tilde{a}_h^{(N)}(t, x, y, \eta)e^{\frac{i}{h}(S(t,x,y)-(y,\eta))}d\eta.
\]

Multiplying by \(\hat{\psi}(t)e^{\frac{i}{h}tE_c}\) and passing to the trace we find Eq.(14) with \(d = n\) and we write again \(\tilde{a}_h^{(N)}(t, x, \eta)\) for the diagonal evaluation \(\tilde{a}_h^{(N)}(t, x, x, \eta)\).

In particular:
\[
\tilde{a}_h^{(0)}(t, x, x, \eta) = \psi(x, \eta)a_0(t, x, x, \eta),
\]
is independent of \(h\) and is compactly supported w.r.t. \((x, \eta)\) since \(\psi \in C_0^\infty(\mathbb{T}^*\mathbb{R}^n)\).

4. Proof of the main result.

Let be \(E_c\) any critical value in \([E_1, E_2]\) and \(z_0\) an equilibrium of \(\Sigma_{E_c}\). We choose a function \(\psi \in C_0^\infty(\mathbb{T}^*\mathbb{R}^n)\), with \(\psi = 1\) near \(z_0\), hence:
\[
\Upsilon_2(E_c,h,\varphi) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} e^{\frac{i}{h}tE_c}\hat{\varphi}(t)\psi^\omega(x, hD_x)\exp(-\frac{i}{h}tP_h)\Theta(P_h)dt
\]
\[
+ \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} e^{\frac{i}{h}tE_c}\hat{\varphi}(t)(1 - \psi^\omega(x, hD_x))\exp(-\frac{i}{h}tP_h)\Theta(P_h)dt.
\]

If there is no other singularity on \(\Sigma_{E_c}\) with \((\mathcal{H}_5)\) the asymptotic expansion of the second term is given by the semi-classical trace formula on a regular level. For finitely many critical point on \(\Sigma_{E_c}\), we can repeat the procedure. The first term is micro-local and precisely generates the singularity in Theorem 5. We note \(\Omega\) the discrete set of critical points \(z_j^0\) in \(p^{-1}(J)\).

Classical dynamics near the equilibrium.

A generic critical points of the phase function of Eq.(12) satisfies the relations:
\[
\begin{align*}
E &= -\partial_t S(t, x, \xi), \\
x &= \partial_\xi S(t, x, \xi), \\
\xi &= \partial_x S(t, x, \xi),
\end{align*}
\Rightarrow \begin{align*}
p(x, \xi) &= E, \\
\Phi_t(x, \xi) &= (x, \xi).
\end{align*}
\]
The right hand side coincide with the set \(\text{Fix}_E\) of fixed points of \(\Phi_t\) defined in section 1. This set generally consist of closed trajectories of the flow inside \(\Sigma_E\), the energy surface (for \(t = 0\) the flow is the identity) and finally equilibria/critical-points. By the non-stationary phase lemma, outside of the critical set \(\text{Fix}_E\) the contribution is of order \(O(h^\infty)\).

To apply the stationary-phase methods it is important to study the nature of the phase function along the critical-set \(\text{Fix}_E\). For the regularity of the Hessian of the phase function, the next lemma is particulary useful. We will often denote points \((x, \xi) \in T^*\mathbb{R}^n\) of the phase space by a single letter \(z\).
Lemma 13. Let us define \( \Psi(t,x,\xi) = S(t,x,\xi) - \langle x, \xi \rangle + tE_c \), then if \( z_0 \) is critical point of \( \Psi \), we have the equivalence:

\[
d^2\Psi(z_0)\delta z = 0 \iff d_z\Phi_t(z_0)\delta z = \delta z, \forall \delta z \in T_{z_0}(T^*\mathbb{R}^n).
\]

In other words, degenerate directions of the phase correspond to fixed points of the linearized flow at \( z_0 \).

The proof is standard and can for example be found in [24, 6]. We recall that the linearized flow \( d\Phi_t \) is the differential of the flow \( \Phi_t \) w.r.t. initial conditions \( z = (x, \xi) \). If we use Lemma 13, we obtain for our phase function

Corollary 14. A critical point \( (T, x_0, \xi_0) \) of \( \Psi(t,x,\xi) \) is degenerate with respect to \( (x,\xi) \) if and only if \( T \) is a period of the linearized flow \( d_{x,\xi}\Phi_t(x_0,\xi_0) \).

The next result is also well known, see, e.g., [1], from classical mechanics and differential geometry:

Lemma 15. If \( \partial_x p(x_0,\xi_0) = \partial_\xi p(x_0,\xi_0) = 0 \) then \( d_{x,\xi}\Phi_t(x_0,\xi_0) \) is the Hamiltonian flow of the quadratic form \( \frac{1}{2}d^2p(x_0,\xi_0) \) on \( T_{x_0,\xi_0}(T^*\mathbb{R}^n) \).

Hence, when \( z_0 \) is a critical point of \( p \), the linear map \( w \mapsto d\Phi_t(z_0)w \) can be interpreted as the Hamiltonian flow of \( u \mapsto Q(u) = \langle \frac{1}{2}d^2p(z_0)u, u \rangle \).

Observe that for a Schrödinger operator a critical point is always of the form \( z_0 = (x_0,0) \) with \( dV(x_0) = 0 \) and the quadratic form is:

\[
Q(u) = Q(u_1, u_2) = ||u_1||^2 + \frac{1}{2}d^2V(x_0)u_2, u_2 || u_1 \in \mathbb{R}^n, u_2 \in \mathbb{R}^n.
\]

A fortiori the map \( d\Phi_t(z_0) \) is governed by a quadratic Schrödinger operator.

**Degenerate critical point.** If the kernel of \( d^2V(x_0) \) is not trivial and contains a non-zero vector \( v \) the linearized-flow in the direction of \( v \) is the flow of the free Laplace operator and we have \( d\Phi_t(z_0)(v,0) = (v,0) \) for all \( t \). Hence it is never possible to apply the Morse-lemma, and a fortiori the stationary phase method (and this for any time \( t \). But if we restrict the study to some homogeneous singularity for \( V \) it is possible to find a local diffeomorphism changing the phase \( \Psi \) into a polynomial function \( R \) (i.e. a local normal-form for \( \Psi \)). For these normal forms it is possible to generalize the stationary-phase method:

\[
\int_{\mathbb{R}^{2n+1}} e^{\frac{i}{\hbar}R(w)}a(w)dw \sim \sum_{j,k} h^{-n_j} \log(h)^k c_{j,k}(a), \quad h \to 0^+.
\]

We will not review all these geometric and analytic results individually. We will refer to [4, 7, 8, 9, 10] for the reduction of the phase function and the asymptotic of the resulting oscillatory integrals. I mention that it is complicated to express all distributional coefficients \( c_{j,k}(a) \) invariantly since \( a \) depends on the local-diffeomorphism \( \chi \) transforming \( \Psi \) into \( R \). The top-order coefficients are relatively easy to express, the next one start to depend on the derivatives of \( \chi \). To express all \( c_{j,k}(a) \) in terms of \( V \) would improve the inverse spectral result to higher derivatives of \( V \) at the critical point.
Non-degenerate critical point of \( V \). When \( d^2 V(x_0) \) is invertible, the have a classical result. It is easy to check that the sign of the quadratic map \( Q \) determines the 'stable' and 'unstable' directions after diagonalizing \( d^2 V \) via an orthogonal linear transformation.

Combining the results of Lemma 15 and Corollary 14 to achieve our goal it will be sufficient to stay below the smallest positive period of the linearized flow. Working in some suitable local coordinates near \( z_0 \) we can assume that the quadratic form attached to the potential is of the form:

\[
V(x) = E_c - \sum_{j=1}^{r} \alpha_j x_j^2 + \sum_{j=r+1}^{n} \alpha_j x_j^2 + O(||x||^3), \quad \alpha_j > 0.
\]

This can be achieved by a translation, a change of linear coordinates (via an orthogonal matrix) and an eventual permutation of coordinates. Observe that all these linear transformations, in particular the action of the orthogonal matrix, leave the Laplace operator invariant. Of course the variable attached to indices \( j = 1, ..., r \) will generate hyperbolic functions (and hence never degenerate in the sense of Corollary 14).

**A lower bound on primitive periods.**

The next result provides a global information on the smallest positive primitive periods of the classical flow. This lemma will be used to extract the new wave invariants appearing at a critical energy level.

**Lemma 16.** There exists \( T > 0 \), depending only on \( V \) and \( J = [E_1, E_2] \), such that \( \Phi_t(z) \neq z \) for all \( z \in p^{-1}(J) \setminus \Omega \) and all \( t \in [-T, 0[\cup]0, T[\).

**Proof.** If \( H_p \) is our hamiltonian vector field and \( z = (x, \xi) \) we have:

\[
||H_p(z_1) - H_p(z_2)||^2 = 4||\xi_1 - \xi_2||^2 + ||\nabla_x V(x_1) - \nabla_x V(x_2)||^2.
\]

When \( z_1 \) and \( z_2 \) are in the compact \( p^{-1}(J) \) there exists \( b > 0 \) such that:

\[
||\nabla_x V(x_1) - \nabla_x V(x_2)|| \leq b||x_1 - x_2||.
\]

Hence, there exists \( a > 0 \) such that:

\[
||H_p(z_1) - H_p(z_2)|| \leq a||z_1 - z_2||, \quad \forall z_1, z_2 \in p^{-1}(J).
\]

The main result of [32] shows that any periodic orbit inside \( p^{-1}(J) \) has a period \( \tau \geq 2\pi/a > 0 \). The lemma follows with \( T := T(V, J) = 2\pi/a \). ■

**Remark 17.** The result of [32] is optimal (for the harmonic oscillator the previous inequality becomes an equality). Note that \( T \) is decreasing if one increase the size of \( J \). Lemma 16 provides a total control on the r.h.s. of the trace formula. If \( \varphi \in C^\infty_0([0, T]) \), the only contribution arises from the set \( \{(t, z_0), \quad t \in \text{supp}(\hat{\varphi})\} \), i.e. from the new wave invariants.

Now, we restrict our attention to the singular contribution generated by one critical point. We check now the condition of non-degeneracy of our phase-function. As it was explained in section 2, for a non-degenerate extremum a minor technical problem could occur. If \( x_0 \) is a maximum of the
potential $d\Phi_t(z_0)$ has no non-zero period which ends immediately the discussion. If $x_0$ is a minimum $d\Phi_t(z_0)$ is elliptic with primitive periods $(T_1, \ldots, T_n)$ generated by the eigenvalues of $d^2V(x_0)$. But the constant $b$ of Lemma 13 is certainly bigger than the spectral radius of $d^2V(x_0)$ and hence we have the inequality $T < \min\{T_1, \ldots, T_n\}$. Following the approach of [4, 6, 24], if $\text{supp}(\hat{\varphi}) \subset [0, T]$ the associated contribution is smooth on $\text{supp}(\hat{\varphi}) \setminus \{0\}$.

For a degenerate critical point $z_0$ as in $(\mathcal{H}_3)$ a surprising result, established in [8, 9], is that the only singularity, for the first new wave invariant, is located at $t = 0$. Hence the condition $\hat{\varphi} \in C_0^\infty(]-T, T[)$ or $\hat{\varphi} \in C_0^\infty(]-T, 0[)$ is sufficient to determine the new-wave invariants generated by $z_0$.  

4.1. Non-degenerate critical points. With the previous considerations, we consider $z_0 \in \Sigma_{E_c}$ a non-degenerate critical point of $p$, a fortiori isolated, and we assume that $\text{supp}(\hat{\varphi}) \subset [0, \cup [0, T]$. In this setting we know that the phase function $\Psi$ (introduced in Lemma 13) has a non degenerate critical point in $z_0$ for all $t \in \text{supp}(\hat{\varphi})$.

In our setting, up to a change of local coordinates, we can assume that $p(x, \xi) = p(x, \xi) + \mathcal{O}(|x|^2)$ (near the origin $z_0 = (0, 0)$), where the quadratic form is:

\begin{equation}
 p_2(x, \xi) = \left( \sum_{j=1}^r (\xi_j^2 - \alpha_j x_j^2) + \sum_{j=r+1}^n (\xi_j^2 + \alpha_j x_j^2) \right).
\end{equation}

The flow of $p_2$, viewed as an element of $\text{End}(T_0(T^*\mathbb{R}^n)) \simeq \text{End}(\mathbb{R}^{2n})$, is

$$
\exp(tH_{p_2})(x, \xi) = A(t) \begin{pmatrix} x \\ \xi \end{pmatrix},
\quad A(t) = \begin{pmatrix} a(t) & 0 & c(t) & 0 \\ 0 & b(t) & 0 & f(t) \\ c(t) & 0 & a(t) & 0 \\ 0 & d(t) & 0 & b(t) \end{pmatrix},
$$

where $(x, \xi) = (x', x'', \xi', \xi'')$, $x', \xi' \in \mathbb{R}^r$, $x'', \xi'' \in \mathbb{R}^{n-r}$, and

\begin{align*}
 a(t) &= \text{diag}(\text{ch}(\alpha_j t)), \\
 b(t) &= \text{diag}(\cos(\alpha_j t)), \\
 c(t) &= \text{diag}(\alpha_j \text{sh}(\alpha_j t)), \\
 d(t) &= \text{diag}(-\alpha_j \text{sin}(\alpha_j t)), \\
 c(t) &= \text{diag}(\frac{1}{\alpha_j} \text{sh}(\alpha_j t)), \\
 f(t) &= \text{diag}(\frac{1}{\alpha_j} \text{sin}(\alpha_j t)),
\end{align*}

and the symbol ”diag” means diagonal matrix. Observe that for $a(t), c(t), e(t)$ the index $j$ varies in $\{1, r\}$, for the others it varies in $\{r+1, n\}$ and that $\det A(t) = 1$ for all $t$.

Local reduction of the phase function of our FIO. Since $S$ is solution of the Hamilton-Jacobi equation we have (locally) the relation:

$$
\Phi_t(\partial_x S(t, x, \xi), \xi) = (x, \partial_x S(t, x, \xi)).
$$
But since $\Phi_t(z_0) = z_0$, writing a Taylor expansion for:

$$S(t, x, \xi) = S_2(t, x, \xi) + O(||(x, \xi)||^3),$$

and $\Phi_t$ (always near the critical point $z_0 = 0$) we get that $S_2$ satisfies the relation:

$$d\Phi_t(0)(\partial_\xi S_2, \xi) + O(||(x, \xi)||^2) = (x, \partial_x S_2) + O(||(x, \xi)||^2),$$

where the linear map $d\Phi_t(0)$ can be determined as above. This linear system is regular exactly when $\det d(\Phi_t(z_0)) \neq 0$. As a consequence, we pick:

$$l = \inf \left( \frac{2\pi}{\alpha_j}, j \in \{r + 1, ..., n\} \right),$$

and an interval $L \subset ]0, l[$. For each $t \in L$, it follows that we can find a (time-dependant) change of coordinates $(t, x, \xi) \mapsto (t, \chi)$, well defined in a small neighborhood of $\text{supp}(\hat{\varphi}) \times \{z_0\}$ such that:

$$\Psi(t, z) \simeq Q(\chi),$$

$$Q(\chi) = ((\chi_1^2 - \chi_2^2) + \cdots + (\chi_{2r-1}^2 - \chi_{2r}^2) + (\chi_{2r+1}^2 + \chi_{2r+2}^2) + \cdots + (\chi_{2n-1}^2 + \chi_{2n}^2)).$$

Applying the stationary phase method for the $\chi$ variables, which is legal since the remaining integration w.r.t. $t$ is of compact support, we get:

$$\Upsilon_{z_0}(E_c, \varphi, h) \sim e^{i\pi \text{sgn}(Q)} \int_{\mathbb{R}} \hat{\varphi}(t) d\mu_t(z_0) + O(h).$$

Here $\text{sgn}(Q) = (n + n - r) - r = 2(n - r)$, next by checking:

- The value of $\chi(t, z_0)$,
- The value of the Jacobian $\frac{D\chi}{Dz}$ at the point $z_0$,

it comes out, $\chi$ being given by $\chi(t, z) = (d\Phi_t(z_0) - \text{Id})z + O(||z||^2)$ at the first order, and as long as $\det(d\Phi_t(z_0) - \text{Id}) \neq 0$, that:

$$(17) \quad d\mu_t(z_0) = |\det(d\Phi_t(z_0) - \text{Id})|^{-\frac{1}{2}}$$

Observe that this coefficient, the Duistermaat-Guillemin-Uribe density, is indeed a smooth function as long as we stay away from any period of the linearized flow at $z_0$. An explicit computation using $A(t)$ in our coordinates, done for example in [24], shows that the density is given by:

$$d\mu_t(z_0) = \frac{1}{| \prod_{j=1}^{r} \sinh(\alpha_j(z_0)t) \prod_{j=r+1}^{n} \sin(\alpha_j(z_0)t)|}.$$ 

On this formula we see that we have $r$-negative and $n-r$ positive eigenvalues at the critical point $x_0$. The desired result follows since $d\mu_t(z_0)$ determines:

- The signature of the Hessian of $V$ at $x_0$.
- Eigenvalues $\alpha_j(x_0)$. 
The second affirmation follows by Taylor expanding the density and evaluating it at different points.

Comments.
Such a density was first introduced by Duistermaat and Guillemin [14]. But it seems that the exploitation of this term to describe contributions of critical points goes back to Guillemin and Uribe [16]. This kind of density can be extended to Morse-Bott singularities for $V$ (see [4, 24]). For a strict minimum of the potential $x_0$ the shape of $d\mu_t(x_0, 0)$ is fundamental to get better inverse spectral results. In particular one has to check the presence of resonant coefficients. See [12] and [17] for improved results near a minimum.

These densities can be continued as meromorphic-distributions with singular support:

- At the origin, such a study is done in [4].
- With singular at a period $T$ of $d\Phi_t(z_0)$, this is done in [24].

For an operator which is not a Schrödinger operator some new terms can generally appear at a period of $d\Phi_t(z_0)$ (see [6]). All these facts strongly suggest that the ‘pike singularity’ of $d\mu_t(z_0)$ as $t \to T$ near a period $T$ of $d\Phi_t(z_0)$ should also describe the symbol. For example, for a Schrödinger operator a double eigenvalue $w > 0$ of $d^2V(x_0)$ will generates a singularity of double magnitude in $d\mu_t(z_0)$ at the point $T = \frac{2\pi}{w}$.

Example. An important toy model is the case of an $n$-dimensional harmonic oscillator:

$$P_h = -\frac{\hbar^2}{2} \Delta + \frac{1}{2} \sum_{k=1}^{n} w_k^2 x_k^2, \quad w_k \neq 0 \forall k.$$  

This model is one of the few Hamiltonians that can be explicitly solved. Then, for $t$ small and $t \notin \mathbb{Z} \frac{\pi}{w_k}$, the kernel of the propagator $U_h(t) = \exp\left(\frac{it}{\hbar} P_h\right)$ can be explicitly computed as:

$$K(t, x, y) = \left(\prod_{k=1}^{n} \frac{w_k}{2i\pi \sin(w_k t)}\right)^{\frac{1}{2}} e^{S(t, x, y)},$$

$$S(t, x, y) = \sum_{k=1}^{n} \frac{w_k}{\sin(w_k t)} \left(\frac{1}{2} \cos(w_k t)(x_k^2 + y_k^2) - x_k y_k\right).$$

This clearly shows that the small $\hbar$ behavior of $U_h(t)$ determines the eigenvalues $w_k$. This important example can be perturbed to use Dyson-expansions and treat general potentials near a minimum of the potential, see [23].

---

4For $t$ large one can use compositions and the stationary phase method. The result obtained is then exact since the phase in quadratic w.r.t. $(x, y)$. 


4.2. Degenerate singularities. As seen in section 2 it is sufficient to study a micro-localized problem:

\[ \Upsilon_{z_0}(E_c, h, \varphi) = \frac{1}{2\pi} \text{Tr} \int_{\mathbb{R}} e^{iEc/h} \varphi(t) \psi^w(x, hD_x) \exp(-it/hP_h)\Theta(P_h) dt. \]

Here \( \psi \in C_0^\infty(T^*\mathbb{R}^n) \) is micro-locally supported near \( z_0 \) (cf section 2). If the support of \( \psi \) is chosen small enough it is relatively easy to obtain a normal form for the phase function of the FIO approximating \( \exp(-it/hP_h)\Theta(P_h) \).

For the convenience of the reader we recall the contributions of equilibriums in the trace formula. We note \( S(S^{n-1}) \) the surface of \( S^{n-1} \) and in the next two propositions it is understood that conditions \((\mathcal{H}_1)\) to \((\mathcal{H}_3)\) are satisfied.

**Proposition 18.** If \( x_0 \) is a local minimum the first new wave invariant attached to \( x_0 \) are given by:

\[ \Upsilon_{z_0}(E_c, h, \varphi) \sim h^{\frac{n}{2} + \frac{1}{n}} \sum_{j,l \in \mathbb{N}^2} h^{\frac{n}{2} + \frac{1}{n}} \Lambda_{j,l}(\varphi), \]

where the \( \Lambda_{j,l} \) are some distributions. The first new wave-invariant, attached to the leading coefficient is:

\[ h^{\frac{n}{2} + \frac{1}{n}} S(S^{n-1}) \left( \frac{2\pi}{n} \right) \int_{S^{n-1}} |V_{2k}(\eta)|^{-\frac{1}{n}} d\eta \int_{\mathbb{R}^+ \times \mathbb{R}^+} \varphi(u^2 + v^{2k}) u^{n-1} v^{n-1} du dv. \]

Observe the simplicity of the distributional-coefficient acting on \( \varphi \) obtained more or less by computing a volume. Up to the spherical-mean, this coefficient is uniquely determined by \( n \) and \( k \). For an unstable critical point the situation is more complicated and we have:

**Proposition 19.** If \( x_0 \) is a local maximum we have:

\[ \Upsilon_{z_0}(E_c, h, \varphi) \sim h^{\frac{n}{2} + \frac{1}{n}} \sum_{j,l \in \mathbb{N}^2} h^{\frac{n}{2} + \frac{1}{n}} \log(h)^m \Lambda_{j,l,m}(\varphi). \]

If \( \frac{n(k+1)}{2k} \notin \mathbb{N} \), the first non-trivial new wave invariant is given by:

\[ h^{\frac{n}{2} + \frac{1}{n}} \langle W_{n,k}, \varphi \rangle S(S^{n-1}) \int_{S^{n-1}} |V_{2k}(\eta)|^{-\frac{1}{n}} d\eta. \]

At first view the result seems to be the same, but the distributions \( W_{n,k} \) of Proposition 19 are respectively given by:

\[ \langle W_{n,k}, \varphi \rangle = \int_{\mathbb{R}} (C_{n,k}^+ |t|^\frac{n+k+1}{2k} - 1 + C_{n,k}^- |t|^\frac{n+k+1}{2k} - 1) \varphi(t) dt, \]

if \( n \) is odd,

\[ \langle W_{n,k}, \varphi \rangle = C_{n,k}^- \int_{\mathbb{R}} |t|^\frac{n+k+1}{2k} - 1 \varphi(t) dt, \]

if \( n \) is even.

The other options are given by:
Proposition 20. If \( \frac{n(k+1)}{2k} \in \mathbb{N} \) and \( n \) is odd then the top-order coefficients are:

\[
C_{n,k} \log(h) h^{\frac{n}{2} + \frac{n-1}{2k}} \int_{\mathbb{S}^{n-1}} |V_{2k}(\eta)|^{-\frac{n}{2k}} d\eta \int_{\mathbb{R}} |t|^{|n+1|_{2k} - 1} \varphi(t) dt.
\]

Finally, if \( \frac{n(k+1)}{2k} \in \mathbb{N} \) and \( n \) is even, we have:

\[
C_{n,k} h^{\frac{n}{2} + \frac{n-1}{2k} - n} \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{S}^{n-1}} |V_{2k}(\eta)|^{-\frac{n}{2k}} d\eta \int_{\mathbb{R}} |t|^{|n+1|_{2k} - 1} \varphi(t) dt.
\]

A careful examination of the proof shows that the last case in Proposition 20 is similar to the first subcase of Proposition 19 (with \( n \) odd) since \( C_{n,k}^+ = C_{n,k}^- \). But we refer to 

Remark 21. To emphasize the consistency of these results we mention that:

- \( C_{n,k}, C_{n,k}^\pm \) are non-zero universal constants depending only on \( n \) and \( k \). See [9] for an analytic formulation.
- Such terms \( h^\alpha \) and \( h^\alpha \log(h) \), \( \alpha \in \mathbb{Q} \) never appear if \( E \) is regular.

In this work we will mainly use the order w.r.t. \( h \) of these coefficients (and the constants appearing in the expansions). For a detailed proof of Proposition 18 see [8] and for Propositions 19 and 20 see [9]. The case \( k = 1 \), i.e., quadratic singularities, can also be retrieved from certain results of [4] with some support restrictions but, again, this is sufficient to attain our objectives.

Asymptotic expansion at a regular energy level. With \((\mathcal{H}_5)\) and when the energy \( E \) is regular, we have:

\[
\Upsilon(E,h,\varphi) \sim h^{1-n} \left( \frac{1}{2\pi} \right)^n \text{Vol}(\Sigma_E) \hat{\varphi}(0) + \sum_{j=1}^{\infty} h^{1-n+j} \hat{c}_j(\hat{\varphi})(0) + \sum_{\rho \in \Sigma_E} e^{i\pi S_\rho} e^{i\pi \mu_\rho/4} \sum_{j=0}^{\infty} D_{\rho,j}(\hat{\varphi})(T_\rho) h^j.
\]

We refer to [27] for a proof. In the r.h.s. the sum concerns periodic orbits \( \rho \) of energy \( E \) and is finite since \( \text{supp}(\hat{\varphi}) \) is compact. Here \( S_\rho, \mu_\rho \) and \( T_\rho \) are resp. the action, the Maslov-index and the period of the closed orbit \( \rho \) and both \( \hat{c}_j, D_{\rho,j} \) are differential operators of order \( j \). If \( \varphi \) satisfies \((\mathcal{H}_4)\) we have \( \hat{c}_j(\hat{\varphi})(0) = 0 \) for all \( j \in \mathbb{N} \) and for each regular value \( E \in [E_1,E_2] \):

\[
(18) \quad \Upsilon(E,h,\varphi) \sim \sum_{\rho \in \Sigma_E} e^{i\pi S_\rho} e^{i\pi \mu_\rho/4} \sum_{j=0}^{\infty} D_{\rho,j}(\hat{\varphi})(T_\rho) h^j.
\]

We accordingly obtain that this term is bounded and a fortiori:

\[
(19) \quad \Upsilon(E,h,\varphi) = O(1), \quad \forall E \in [E_1,E_2] \setminus \{ E_1, ..., E_l \}.
\]

This point will justify Corollary 9.
Next, by Lemma 16, we have $T_\rho \geq T$ uniformly w.r.t. $E \in [E_1, E_2]$. Hence if $E$ is not critical and if in addition $(\mathcal{H}_2)$ is satisfied the sum over the periods of Eq. (18) is simply 0 and in Eq. (19) we obtain in fact a bound $O(h^\infty)$ (a fortiori $(\mathcal{H}_5)$ is not required in that situation).

**Asymptotics at a critical energy level.** For $E = E_{mc}^m$ critical there is always a continuous contribution w.r.t. $t$ in the spectral distribution showing up the presence of a new wave invariant. A fortiori, a choice of $\hat\varphi$ flat at the origin, or with a small compact support, does not erase this term. We have:

$$\Upsilon(E_{mc}, h, \varphi) \sim \sum_{j=1}^{N_m} f_j(h),$$

where $N_m$ is the number of equilibrium points on $\Sigma_{E_{mc}}$ and each $f_j(h)$ is given by the leading term of Propositions 18, 19 and 20. ■

Note that the bottom of a symmetric double well (degenerate or not) gives a similar answer as a single well of same nature. Hence without extra micro-local considerations (e.g. suitably localized eigenfunctions estimates) it is difficult to distinguish these 2 different settings.

**Proof of Theorem 7.** First, the micro-local Weyl-law for regular energies:

$$\Upsilon(E, h, \varphi) \sim (2\pi h)^{1-n} \hat{\varphi}(0) \text{Vol}(\Sigma_{E}),$$

computes the dimension $n$. Now assume given a critical value $E_c$ with a single critical point. The only choice of the spectral function $\varphi$ allows to detect $E_c$ via the singularity $f(h)$ of Theorem 5. The knowledge of $f(h)$ determines the order of the contribution. For example, if:

$$f(h) \sim Ch^\alpha \log(h),$$

the critical point is a maximum and $\alpha$ computes the degree $2k$ of the singularity. With $\hat{\varphi}$, the knowledge of $k$ allows to compute the quantity :

$$\int_{\mathbb{R}} |t|^{\frac{n-k}{2k}-1} \varphi(t)dt.$$ A fortiori $C$ determines the average of $|V_{2k}|^{-\frac{1}{2k}}$ on $\mathbb{S}^{n-1}$. Without $\log(h)$, the nature of the critical point can be detected by a symmetry argument w.r.t. $\varphi$ since we a priori know $n$ and $k$. In view of Propositions 18, 19, 20 we can choose $\varphi$ odd, even, symmetric or non-symmetric w.r.t. the origin to conclude. Note that if $\hat{\varphi}$ is not even $\varphi$ is a priori complex valued. ■

**Remark 22.** The spherical average of $V_{2k}$ is a Jacobian in polar coordinates around $x_0$. For example, by composition with $e^{-|x|}$ we obtain:

$$\int_{\mathbb{R}^n} e^{-|V_{2k}(x)|} dx = \frac{1}{2k} \Gamma\left(\frac{n}{2k}\right) \int_{\mathbb{S}^{n-1}} |V_{2k}(\eta)|^{-\frac{n}{2k}} d\eta.$$ The same result holds by integration of $f(V_{2k}(x))$, if $f \in L^1(\mathbb{R}_+, r^{\frac{n}{2k}-1}dr)$.
Remark 23. Enlarging the list of singularities would provide a bigger ”dictionary”. The case of non-homogeneous singularities for $V$ is still an open problem, in particular because the determination of an explicit asymptotic expansion w.r.t. $h$ can be a difficult analytic problem.

Remark 24. There are certainly many easy generalizations of Theorem 7 to integrable Schrödinger operators, e.g. when $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$.

5. Extensions. Examples

In this section we propose now several generalizations of the main results.

5.1. Operators with sub-principal symbols. We will show, shortly, how to extend the result of Theorem 5 to the case of an $h$-admissible operator. I mention several obvious motivations to problems involving operators with non-zero sub-principal symbols.

Witten-Laplacians. B. Helffer & J. Sjöstrand for Witten Laplacians, see e.g. [21] for an overview and references, have obtained recently many interesting results for these operators. For example, the Witten-Laplacian on zero-forms attached to the measure $e^{-f/h}$ is:

$$\Delta_{f,h}^{(0)} = -h^2 \Delta + \frac{1}{4} |\nabla f(x)|^2 - \frac{h}{2} \Delta f(x), \ f \in C^\infty(\mathbb{R}^n),$$

whose symbol $p(x, \xi) = p_0(x, \xi) + hp_1(x, \xi)$ depends on $h$.

Schrödinger operators on a manifold. A Schrödinger operator attached to a Laplace-Beltrami operator on a Riemannian manifold $M$ and $h$-quantized by exterior multiplication:

$$P_h = -h^2 \Delta_M + V(x), \ V \in C^\infty(M),$$

generally involves a sub-principal symbol. In local coordinates with a metric $G = g^{ij}$, $G^{-1} = g_{ij}$ and $g = \det G$ the operator is:

$$-h^2 \sum_{i,j} \sqrt{g} \frac{\partial}{\partial x_i} \frac{1}{\sqrt{g}} g_{ij} \frac{\partial}{\partial x_j} + V$$

$$= -h^2 \sum_{i,j} g_{ij}(x) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} + V + h^2 \sum_{i,j} \sqrt{g} \frac{\partial}{\partial x_i} \left( \frac{1}{\sqrt{g}} g_{ij}(x) \frac{\partial}{\partial x_j} .

Hence, in the sense of the $h$-calculus, we have $p_h := p_0 + hp_1$ with:

$$p_0(x, \xi) = \sum_{i,j} g_{ij}(x) \xi_j \xi_i + V(x),$$

$$p_1(x, \xi) = \sqrt{g}(x) \sum_{i,j} \left( \frac{\partial}{\partial x_i} \frac{1}{\sqrt{g}} g_{ij} \right)_{|x} \xi_j .$$

Observe that $p_1 = 0$ at every point where $\xi = 0$. 
General case. More generally, it is possible to consider $h$-admissible operators $P_h$ whose symbols are given by asymptotic sums $p_h \sim \sum h^j p_j$ (e.g. interpreted as a Borel sum w.r.t. $h$) with principal symbol $p_0(x, \xi) = \xi^2 + V(x)$ and a subprincipal symbol $p_1 \neq 0$. Of course in the formula for $p_0$ you can also replace $\xi^2$ by the metric at $x$ if you want to do a similar construction on a Riemannian-manifold (non-necessarily compact).

Modification of the first transport equation. In the previous example the sub-principal symbol $p_1$ was non-zero. This requires a light correction. Starting from the results of section 3 we proceed as follows. To each element $u_h$ of $I(R^n, \Lambda)$ we can associate canonically a principal symbol $e^{iS} \sigma_{\text{princ}}(u_h)$, where $S$ is a function on $\Lambda$ such that $\xi dx = dS$ on $\Lambda$. In fact, if $u_h$ can locally be represented by an oscillatory integral with amplitude $a$ and phase $\varphi$, then we have $S = S_{\varphi} = \varphi \circ i_{\varphi}^{-1}$ and $\sigma_{\text{princ}}(u_h)$ is a section of $|\Lambda|^{1/2} \otimes M(\Lambda)$, where:

- $M(\Lambda)$ is the Maslov vector-bundle of $\Lambda$.
- $|\Lambda|^{1/2}$ is the bundle of half-densities on $\Lambda$.

When $p_1 \neq 0$, in the global coordinates $(t, y, \eta)$ on $\Lambda$, the half-density of $U_h(t)$ is given by:

\[
\nu(t, y, \eta) = \exp(i \int_0^t p_1(\Phi_s(y, -\eta)) ds) |dt dy d\eta|^{1/2}.
\]

For this expression, related to the resolution of the first transport equation for the propagator, we refer to Duistermaat and Hörmander [15]. Accordingly, the F.I.O. approximating the propagator has the amplitude:

\[
\tilde{a}(t, z) = a(t, z) \exp(i \int_0^t p_1(\Phi_s(z)) ds).
\]

Since $z_0$ is an equilibrium we have $p_1(\Phi_s(z_0)) = p_1(z_0), \forall s$, and:

\[
\tilde{a}(t, z_0) = \hat{\varphi}(t) e^{\xi_{p_1(z_0)}}.
\]

a) If $p_1(z_0) = 0$. This happens in many interesting situations (in particular for a Laplace-Beltrami operator, see above). Here the top order coefficients in the trace formula remains the same, also for the new wave-invariants at a critical energy level.

b) If $p_1(z_0) \neq 0$. By Fourier inversion formula we simply replace $\varphi(t)$ by $\varphi(t + p_1(z_0))$ in all integral formulae of Propositions 18,19 and 20. Note that when using $(\mathcal{H}_4)$, this has absolutely no effect for the mean values and hence on the detection of the critical energy levels.

5.2. Eigenfunction estimates approach. We inspect now the case of an energy surface supporting more than one critical point. The method we use here is in reality much more restrictive (physically and also from the point of view of spectral theory) since it implicitly use eigenfunctions estimates
via a $C_0^\infty(T^*M)$-observable. These observable are bounded operators on $L^2(M)$ (e.g. via a Calderon-Vaillancourt estimates) and can be inserted in the trace, trace class operators being an ideal.

Since everything below is local we can freely assume that $M = \mathbb{R}^n$, if not we can use local coordinates given by the exponential map. Let $K = p^{-1}(J) \subset T^*\mathbb{R}^n$ be compact and:

$$d_0 = \frac{1}{2} \inf_{i \neq j} d(z_i, z_j),$$

where $d$ is any distance on $T^*\mathbb{R}^n$. By construction, each open ball $B(z, d_0) \subset T^*\mathbb{R}^n$ contains at most 1 critical point for each $z \in K$. Clearly, we can cover a compact neighborhood of $K$ by a finite number of balls $B(z, d_0)$. With a partition of unity adapted to this covering we obtain:

$$\sum_{j=1}^N \psi^w_j(x, hD_x) = \text{Id}, \text{ on } C_0^\infty(K).$$

For each energy $E \in J$, we obtain:

$$\text{Tr} \int_\mathbb{R} \hat{\varphi}(t)\Theta(P_h)e^{\frac{it}{h}(P_h - E)}dt = \sum_{j=1}^N \text{Tr} \int_\mathbb{R} \hat{\varphi}(t)\psi^w_j(x, hD_x)\Theta(P_h)e^{\frac{it}{h}(P_h - E)}dt.$$  

Note that the r.h.s. is studied in section 2. By the same argument as before, if $\Sigma_s \cap \text{supp}(\psi_j)$ contains no critical point we obtain:

$$\text{Tr} \int_\mathbb{R} \hat{\varphi}(t)\psi^w_j(x, hD_x)\Theta(P_h)e^{\frac{it}{h}(P_h - E)}dt = O(h^\infty).$$

And if there is exactly one critical point $z_0 \in \Sigma_{E_c}$ in $\text{supp}(\psi_j)$ we have:

$$\text{Tr} \int_\mathbb{R} \hat{\varphi}(t)\psi^w_j(x, hD_x)\Theta(P_h)e^{\frac{it}{h}(P_h - E_c)}dt = \psi_j(z_0)f_j(h),$$

and by construction no cancellation can occur since $\psi_j(z_0) > 0$.

**Remark 25.** In Corollary 9 we have considered $(H_5)$ for the flow. A similar result holds for a chaotic dynamics and an isolated degenerate closed orbit can be treated as in [28]. Finally, using the results of [27] one can extend Corollary 9 to the case of families of periodic orbits of dimension $d \leq n$.

It is important to notice that to put a pseudo-differential operator in the spectral estimates means that the inverse spectral problem is now implicitly expressed in terms of some $L^2$-expectation:

$$\mu^h_k(\psi_j) = \langle \varphi^h_k, \psi^w_j(x, hD_x)\varphi^h_k \rangle,$$

$$P_h\varphi^h_k = \lambda_k(h)\varphi^h_k, \lambda_k(h) \in [E - ch, E + ch],$$
attached to eigenvectors $\varphi_k^h$ of $P_h$, see, e.g., the section 'eigenvector estimates' of [4]. Recall that, combining Egorov’s theorem and Calderon-Vaillancourt estimates, in the regime $h \to 0^+$ the measure $\mu_k^h$ becomes more and invariant under the Hamiltonian flow. It is also sometimes possible to obtain a measure concentrated on the critical-set (see [4]). There is no paradox here since the flow is constant on the critical set of the principal symbol of $P_h$.

For a Schrödinger operator whose potential is not a Morse function the inverse spectral problem seems to be ill-defined and requires eigenvectors estimates (which are indeed much stronger estimates than those based only the spectrum). Also in the previous construction an interesting problem is to get an a priori lower bound for the number $d_0$ without doing any iteration on the size of $\text{supp}(\psi_j)$.

5.3. **Pseudo-differential operators.** We can also apply the previous strategy for an $h$-pseudo-differential operator with an isolated homogenous singularity as this was considered in [10]. In fact, we will stick to the simpler case of a local extremum as considered in [7]. Assume that $p \sim p_0 + hp_1 + O(h^2)$ and that $p_0$ has a unique critical point $z_0 = (x_0, \xi_0)$ on the critical energy surface $\Sigma_{E_c}$. Also near $z_0$ we have a conical (homogeneous) singularity:

$$p_0(z) = E_c + \sum_{j=k}^N p_j(z) + O(||(z - z_0)||^{N+1}), \quad k > 2,$$

where the functions $p_j$ are homogeneous of degree $j$ w.r.t. $z - z_0$. Furthermore, assume that $z_0$ is a local extremum of $p_k$. This implies that the first non-zero homogeneous component $p_k$ is even and is positive or negative definite. A fortiori $z_0$ is isolated on $\Sigma_{E_c}$. An elementary example in dimension 1 is:

$$p(x, \xi) = \pm(\xi^4 + x^4) + R(x, \xi),$$

where $R(x, \xi) = O(||(x, \xi)||^5)$ and $R$ is chosen so that $p$ is confining and with tempered growth. Applying the results of [7] or [20] we can retrieve 2 invariants of $p$:

**Proposition 26.** The new wave invariants at $E = E_c$ determines:

$$A(p_0) = \frac{1}{(2\pi)^n} \int_{S^{2n-1}} |p_k(\theta)|^{-\frac{m}{n}} d\theta,$$

and the degree of homogeneity $k$ of $p_k$.

**Proof.** The method of proof is the same as before, just select $\hat{\varphi}$ such that all the usual wave-invariants:

- Energy-surface distributions,
- Distribution supported by periodic-orbits,
disappear from the spectral estimates \( \Upsilon(E, h, \varphi) \). For the new-wave invariants, following the construction of \([7]\) or \([20]\), it is possible to transform locally the phase \( \Psi \) of our FIO into \(-tp_k\). The new asymptotic problem has the form:

\[
\Upsilon_{z_0}(E_c, h, \varphi) \sim \frac{1}{(2\pi h)^n} \int_{\mathbb{R} \times T^* \mathbb{R}^n} e^{-\frac{i}{h} p_k(x, \xi)} w(t, x, \xi) dt dx d\xi, \quad w \in C_0^\infty.
\]

After integration w.r.t. \( t \) the becomes elliptic and a little discussion concerning the asymptotic behavior of oscillatory integrals show that:

\[
\Upsilon_{z_0}(E_c, h, \varphi) = h^{2n-2k} \Lambda_{0,k}(\varphi) + O(h^{2n-2k-1}).
\]

This result determines the even integer \( k \). It is possible to give a full asymptotic expansion (in powers of \( h \)) but the first new wave-invariant at \( E = E_c \) is explicitly given by the distribution:

\[
\Lambda_{0,k}(\varphi) = \frac{1}{k} \left\langle \varphi(t + p_1(z_0)), t^{2n-2k} \right\rangle \frac{1}{(2\pi)^n} \int_{\mathbb{S}^{2n-1}} |p_k(\theta)|^{-\frac{2n}{2k}} d\theta,
\]

with \( t_{z_0} = \max(t, 0) \) if \( z_0 \) is a minimum and \( t_{z_0} = \max(-t, 0) \) for a maximum. The 1st coefficient depends on \( \varphi \), the dimension \( n \), the degree \( k \in 2\mathbb{N}^* \) of the singularity and the nature of the extremum. The 2nd coefficient, independent of \( \varphi \), determines the spherical mean of \( p_k \).

Two different spectral-estimates, with too functions \( \varphi_1 \) and \( \varphi_2 \) determine the ratio:

\[
r(\varphi_1, \varphi_2) = \frac{\left\langle \varphi_1(t + p_1(z_0)), t^{2n-2k} \right\rangle}{\left\langle \varphi_2(t + p_1(z_0)), t^{2n-2k} \right\rangle},
\]

so that the spherical mean on the sphere can generally be determined after 2 spectral estimates.

Note that many symbols would give the same value. In particular symbols conjugated by a rotation around the critical point cannot be distinguished from this new-wave invariant.

**An invariance under re-scaling.** Now when \( z_0 \) is a minimum, we observe that for any \( j \in \mathbb{N}^* \) we can re-scale our operator via:

\[
Q_{h,j} = (P_h - E_c)^j,
\]

so that \( z_0 \) is still a minimum of the principal symbol \( q_0 = (p_0 - E_c)^j \) of \( Q_{h,j} \). Since:

\[
q_0(z) = (p_k)^j(z) + O(||z - z_0||^{kj+1}),
\]

by a new application of Proposition \([26]\) at the critical value zero, we get:

\[
A(q_0) = \frac{1}{(2\pi)^n} \int_{\mathbb{S}^{2n-1}} (p_k(\theta))^j \cdot \frac{2n}{2k} d\theta = A(p_0).
\]
It follows that $A(p_0)$ is also invariant under re-scaling (this argument does not apply for a local maximum).

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As a final remark, I mention that there is a lot of information to retrieve from the spectral data of certain particular Schrödinger operators on compact surfaces or manifolds. Certain potentials like smoothed geodesic distances (distance functions are singular at conjugate points) or height functions might describe nicely the manifold. We plan to investigate these kind of inverse spectral problems in a future article.

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