An HDG method with orthogonal projections in facet integrals

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Abstract We propose and analyze a new hybridizable discontinuous Galerkin (HDG) method for second-order elliptic problems. Our method is obtained by inserting the $L^2$-orthogonal projection onto the approximate space for a numerical trace into all facet integrals in the usual HDG formulation. The orders of convergence for all variables are optimal if we use polynomials of degree $k + l, k + 1$ and $k$, where $k$ and $l$ are any non-negative integers, to approximate the vector, scalar and trace variables, which implies that our method can achieve superconvergence for the scalar variable without postprocessing. Numerical results are presented to verify the theoretical results.

Keywords Discontinuous Galerkin · Hybridization · Superconvergence

Mathematics Subject Classification (2000) 65N12 · 65N30

1 Introduction

In this paper, we propose a new hybridizable discontinuous Galerkin (HDG) method for second-order elliptic problems. For simplicity, the following diffusion problem is considered:

\begin{align}
q + \nabla u &= 0 \quad \text{in } \Omega, \\
\nabla \cdot q &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align}

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where $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is a bounded and convex polygonal or polyhedral domain and $f$ is a given $L^2$-function.

To begin with, let us define notations for the description of the standard HDG method. Let $T_h$ be a mesh of $\Omega$, which consists of polygons or polyhedrons, where $h$ stands for the mesh size. Let $\mathcal{E}_h$ denote the set of faces of all elements in $T_h$. A family of meshes $\{T_h\}$ is assumed to satisfy the chunkiness condition [2], under which the trace and inverse inequalities hold. We use the usual notation of the Sobolev spaces $[1]$, such as $H^m(D)$, $\|w\|_{m,D} := \|w\|_{H^m(D)}$, $\|w|_{m,D} := \|w|_{H^m(D)}$ for an integer $m$ and a domain $D \subset \mathbb{R}^d$. When $D = \Omega$ or $m = 0$, we omit the domain or the index and simply write $\|w\|_m = \|w\|_{m,\Omega}$, $\|w|_m = \|w|_{m,\Omega}$ and $\|w\| = \|w\|_{0,\Omega}$. The piecewise or broken Sobolev space of order $m$ is defined by $H^m(T_h) := \{v \in L^2(\Omega) : v|_K \in H^m(K) \forall K \in T_h\}$. We denote by $L^2(\mathcal{E}_h)$ the $L^2$ space on the union of all faces of $\mathcal{E}_h$ and by $P_k(T_h)$ the space of piecewise polynomials of degree $k$. The piecewise inner products are written as

$$(u,v)_{T_h} = \sum_{K \in T_h} \int_K uvdx, \quad \langle u,v \rangle_{\partial T_h} = \sum_{K \in T_h} \int_{\partial K} uvds.$$ 

The induced piecewise norm are denoted as $\|v\|_{T_h} = (\langle v,v \rangle_{T_h})^{1/2}$ and $\|v\|_{\partial T_h} = (\langle v,v \rangle_{\partial T_h})^{1/2}$, and the piecewise Sobolev seminorm is defined by $|v|_{1,T_h} = (\sum_{K \in T_h} \|v|_K\|^2)^{1/2}$.

Throughout the paper, we will use the symbol $C$ to denote generic constants independent of $h$. Vector variables and function spaces are displayed in boldface, such as $P_k(T_h) = P_k(T_h)^d$.

We define finite element spaces for $q$, $u$ and the trace of $u$ by

$$V_h = \{v \in L^2(\Omega) : v|_K \in V(K) \forall K \in T_h\},$$
$$W_h = \{w \in L^2(\Omega) : w|_K \in P_{k+1}(K) \forall K \in T_h\},$$
$$M_h = \{\mu \in L^2(\mathcal{E}_h) : \mu|_F \in P_k(F) \forall F \in \mathcal{E}_h\},$$

respectively, where $V(K)$ is a finite-dimensional spaces satisfying $P_k(K) \subset V(K)$. The $L^2$-orthogonal projections onto $V_h$, $W_h$ and $M_h$ are denoted by $P_V$, $P_W$ and $P_M$, respectively. We simply write $P_M w = P_M(w|_{\mathcal{E}_h})$ for $w \in H^2(T_h)$. Note that $P_M w$ may not belong to $M_h$ since it is double-valued in general.

The standard HDG method reads as follows: Find $(q_h, u_h, \hat{u}_h) \in V_h \times W_h \times M_h$ such that

$$(q_h, v)_{T_h} - \langle u_h, \nabla \cdot v \rangle_{T_h} + (\hat{u}_h, v \cdot n)_{\partial T_h} = 0 \quad \forall v \in V_h, \quad (2a)$$

$$-(q_h, \nabla w)_{T_h} + (\hat{q}_h \cdot n, w)_{\partial T_h} = (f, w) \quad \forall w \in W_h, \quad (2b)$$

$$(\hat{q}_h \cdot n, \mu)_{\partial T_h} = 0 \quad \forall \mu \in M_h, \quad (2c)$$

where $\hat{q}_h$ is the numerical flux defined by

$$\hat{q}_h \cdot n = q_h \cdot n + \tau (u_h - \hat{u}_h). \quad (3)$$

Here, $\tau$ is a positive parameter and is set to be of order $O(h^{-1})$ in the paper.
The so-called Lehrenfeld-Schöberl (LS) numerical flux \cite{8} is obtained by inserting $P_M$ into the stabilization part of the numerical flux:

$$
\hat{q}_h^{LS} \cdot n = q_h \cdot n + \tau (P_M u_h - \hat{u}_h).
$$

In \cite{10}, it was proved that the HDG method using the LS numerical flux (the HDG-LS method) achieves optimal-order convergence for all variables if we use polynomials of degree $k$, $k+1$ and $k$ for $V_h$, $W_h$ and $M_h$, respectively. It can be said that the HDG-LS method is superconvergent for the scalar variable $u$ without postprocessing. Another more elaborate flux was introduced in the hybrid higher-order (HHO) method \cite{6,5}, which was recently linked to the HDG method in \cite{4}. The LS numerical flux approach has been applied to various problems; linear elasticity \cite{12}, convection-diffusion problems \cite{13}, Stokes equations \cite{11}, Navier-Stokes equations \cite{14,9} and Maxwell’s equations \cite{3}.

Let us here point out that the superconvergence of the HDG-LS method is sensitive to the choice of $V_h$. For example, the superconvergence property is no longer maintained if $V_h$ is taken to be $P_{k+1}(T_h)$ instead of $P_k(T_h)$. We now demonstrate that by numerical experiments for the test problem \cite{19} which will be provided in Section 4. The numerical results are shown in Table 1. In the case of $V_h = P_1(T_h)$, the orders of convergence are optimal for both variables $q$ and $u$. On the other hand, all the orders become sub-optimal for $V_h = P_2(T_h)$.

### Table 1

Convergence history of the HDG-LS method for $V_h \times W_h \times M_h = P_l(T_h) \times P_2(T_h) \times P_1(E_h)$

| $l$ | $1/h$  | $\|q - q_h\|$ Error Order | $\|u - u_h\|$ Error Order | $\|h^{-1/2}(P_M u_h - \hat{u}_h)\|_{\partial T_h}$ Error Order |
|-----|-------|-----------------------------|-----------------------------|----------------------------------------------------------|
| 1   | 10    | 1.236E-02 –                  | 7.400E-04 –                 | 2.719E-02 –                                              |
|     | 20    | 3.083E-03 2.00               | 9.085E-05 3.03             | 6.676E-03 2.03                                           |
|     | 40    | 7.655E-04 2.01               | 1.140E-05 2.99             | 1.662E-03 2.01                                           |
|     | 80    | 1.915E-04 2.00               | 1.414E-06 3.01             | 4.113E-04 2.01                                           |
| 2   | 10    | 1.464E-01 –                  | 2.738E-03 –                 | 1.064E-02 –                                              |
|     | 20    | 7.177E-02 1.03               | 6.517E-04 2.07             | 4.999E-03 1.09                                           |
|     | 40    | 3.543E-02 1.02               | 1.568E-04 2.06             | 2.363E-03 1.08                                           |
|     | 80    | 1.744E-02 1.02               | 3.815E-05 2.04             | 1.180E-03 1.00                                           |

The aim of the paper is to recover the superconvergence property for such cases. The key idea is to insert the orthogonal projection $P_M$ into the facet integrals in the usual HDG formulation. The resulting method can achieve optimal convergence in $q$ and superconvergence in $u$ without postprocessing if $V_h$ contains $P_k(T_h)$, see Theorems 1 and 2.

The rest of the paper is organized as follows. In Section 2, we introduce a new HDG method. In Section 3, error estimates for both variables $u$ and $q$ are provided. Numerical results are presented to verify our theoretical results in Section 4.
2 An HDG method with orthogonal projections

We begin by introducing our method: Find \((q_h, u_h, \tilde{u}_h) \in V_h \times W_h \times M_h\) such that

\[
(q_h + \nabla u_h, v)_{\mathcal{T}_h} - (P_M u_h - \tilde{u}_h, v \cdot n)_{\partial \mathcal{T}_h} = 0, \quad \forall v \in V_h, \tag{4a}
\]

\[-(q_h, \nabla w)_{\mathcal{T}_h} + (\tilde{q}_h \cdot n, P_M w)_{\partial \mathcal{T}_h} = (f, w), \quad \forall w \in W_h, \tag{4b}
\]

\[(\tilde{q}_h \cdot n, \mu)_{\partial \mathcal{T}_h} = 0, \quad \forall \mu \in M_h, \tag{4c}
\]

where \(\tilde{q}_h\) is the standard numerical flux defined by \(\mathbf{3}\). The derivation of our method is simple. Integrating by parts in \((2a)\) and replacing \(u_h\) by \(P_M u_h\), we get \((1a)\). The second equation \((1a)\) is obtained by replacing \(w\) by \(P_M w\) in \((2a)\). The third equation \((1a)\) is just the same as \((2c)\). Since \(\mu = P_M \mu\) in \((1c)\) (and \((2c)\)), we can also consider that our method is obtained by inserting the orthogonal projection \(P_M\) in all facet integrals in the standard HDG method.

Remark 1: If \(v \cdot n|_F \in P_k(F)\) for any \(F \in E_h\), then our method is identical to the HDG-LS method since

\[
(P_M u_h, v \cdot n)_{\partial \mathcal{T}_h} = (u_h, v \cdot n)_{\partial \mathcal{T}_h} \quad \text{in } \mathbf{1a},
\]

\[
(\tilde{q}_h \cdot n, P_M w)_{\partial \mathcal{T}_h} = (P_M (\tilde{q}_h \cdot n), w)_{\partial \mathcal{T}_h} = (\tilde{q}^{LS}_h \cdot n, w)_{\partial \mathcal{T}_h} \quad \text{in } \mathbf{1b},
\]

\[
(\tilde{q}_h \cdot n, \mu)_{\partial \mathcal{T}_h} = (P_M (\tilde{q}_h \cdot n), \mu)_{\partial \mathcal{T}_h} = (\tilde{q}^{LS}_h \cdot n, \mu)_{\partial \mathcal{T}_h} \quad \text{in } \mathbf{1c}.
\]

3 Error analysis

In this section, we provide the optimal-order error estimates of our method. We are going to use the following approximation properties:

\[
\|v - P_M v\| \leq Ch^j |v|_j, \quad 1 \leq j \leq k + 1,
\]

\[
\|v - P_M v\|_{\partial \mathcal{T}_h} \leq Ch^{j-1/2} |v|_j, \quad 1 \leq j \leq k + 1,
\]

\[
\|\nabla (w - P_M w)\|_{\mathcal{T}_h} \leq Ch^{j-1} |w|_j, \quad 1 \leq j \leq k + 2,
\]

\[
\|w - P_M w\|_{\mathcal{T}_h} \leq Ch^{j} |w|_j, \quad 1 \leq j \leq k + 2,
\]

\[
\|w - P_M w\|_{\partial \mathcal{T}_h} \leq Ch^{j-1/2} |w|_j, \quad 1 \leq j \leq k + 2,
\]

\[
\|w - P_M w\|_{\partial \mathcal{T}_h} \leq Ch^{j-1/2} |w|_j, \quad 1 \leq j \leq k + 1,
\]

for any \(v \in H^j(\Omega)\) and \(w \in H^j(\Omega)\). For the piecewise Sobolev spaces, the following hold:

\[
\|w - P_M w\|_{\partial \mathcal{T}_h} \leq Ch^{j/2} |w|_{1, \mathcal{T}_h}, \quad \forall w \in H^1(\mathcal{T}_h), \tag{5}
\]

\[
\|v \cdot n - P_M (v \cdot n)\|_{\partial \mathcal{T}_h} \leq Ch^{j/2} |v|_{1, \mathcal{T}_h}, \quad \forall v \in H^1(\mathcal{T}_h). \tag{6}
\]

Let \(\Pi_k\) be the orthogonal projection from \(H^1(\mathcal{T}_h)\) onto \(P_k(\mathcal{T}_h)\), which satisfies

\[
\Pi_k v \cdot n \mid_{K \subset \partial \mathcal{T}_h} \subset P_k(\partial K) \quad \forall K \in \mathcal{T}_h, \tag{7}
\]

\[
\|v \cdot n - \Pi_k v \cdot n\|_{\partial \mathcal{T}_h} \leq Ch^{j-1/2} |v|_j \quad \text{for } v \in H^j(\Omega), 1 \leq j \leq k + 1. \tag{8}
\]
The insertion of $P_M$ in (11) gives rise to some terms in the form

$$R(v, w) := \langle (I - P_M)v \cdot n, w \rangle_{\partial T_h}$$

in error analysis. We show the bound of $R(\cdot, \cdot)$ by the properties (7) and (8).

**Lemma 1** For all $v \in H^{k+1}(\Omega)$ and $w \in H^1(T_h)$, we have

$$|R(v, w)| \leq Ch^{k+1}|v|_{k+1}|w|_{1, T_h}.$$ 

**Proof** By (7), (8) and (5), we have

$$|R(v, w)| = |\langle (I - P_M)(v - \Pi_k v) \cdot n, (I - P_M)w \rangle_{\partial T_h}|$$

$$\leq \|v \cdot n - \Pi_k v \cdot n\|_{\partial T_h} \| (I - P_M)w \|_{\partial T_h}$$

$$\leq Ch^{k+1}|q|_{k+1}|w|_{1, T_h}.$$ 

This completes the proof. \hfill \Box

### 3.1 Error equations

As a lemma, we show the error equations in terms of the projections of the errors:

$$e_q = P_V q - q_h, \quad e_u = P_W u - u_h, \quad e_u = P_M u - \tilde{u}_h.$$ 

The approximation errors are denoted as

$$\delta_V q = q - P_V q, \quad \delta_W u = u - P_W u, \quad \delta_M u = u - P_M u.$$ 

**Lemma 2** The following equations hold:

\begin{align*}
(e_q, v)_{T_h} + (\nabla e_u, v)_{T_h} - (P_M e_u - e_u, v \cdot n)_{\partial T_h} &= F_1(v) \quad \forall v \in V_h, \quad (9a) \\
-(e_q, \nabla w) + \langle \tilde{e}_q \cdot n, P_M w \rangle_{\partial T_h} &= F_2(w) \quad \forall w \in W_h, \quad (9b) \\
\langle \tilde{e}_q \cdot n, \mu \rangle_{\partial T_h} &= F_3(\mu) \quad \forall \mu \in M_h, \quad (9c)
\end{align*}

where $\tilde{e}_q \cdot n = e_q \cdot n + \tau (P_M e_u - e_u)$ and

\begin{align*}
F_1(v) &= -\langle \nabla \delta_W u, v \rangle_{T_h} + \langle P_M \delta_W u, v \cdot n \rangle_{\partial T_h}, \\
F_2(w) &= -R(q, w) - \langle \delta V q \cdot n - \tau \delta_W u, P_M w \rangle_{\partial T_h}, \\
F_3(\mu) &= -\langle \delta V q \cdot n - \tau \delta_W u, \mu \rangle_{\partial T_h}.
\end{align*}

**Proof** We easily see that the exact solution satisfies

\begin{align*}
(q, v)_{T_h} + (\nabla u, v)_{T_h} &= 0 \quad \forall v \in V_h, \quad (10a) \\
-(q, \nabla w)_{T_h} + \langle q \cdot n, w \rangle_{\partial T_h} &= (f, w) \quad \forall w \in W_h, \quad (10b) \\
\langle q \cdot n, \mu \rangle_{\partial T_h} &= 0 \quad \forall \mu \in M_h. \quad (10c)
\end{align*}
Each term in (11) is rewritten in terms of $P_V q, P_W u$ and $P_M u$ as follows:

\[
\begin{align*}
(q, v)_{\mathcal{T}_h} &= (P_V q, v)_{\mathcal{T}_h}, \\
(\nabla u, v)_{\mathcal{T}_h} &= (\nabla P_W u, v)_{\mathcal{T}_h} + (\nabla \delta W u, v)_{\mathcal{T}_h}, \\
(q, \nabla w)_{\mathcal{T}_h} &= (P_V q, \nabla w)_{\mathcal{T}_h}, \\
(q \cdot n, w)_{\partial \mathcal{T}_h} &= \langle q \cdot n, P_M w \rangle_{\partial \mathcal{T}_h} + R(q, w) \\
&= (P_V q \cdot n, P_M w)_{\partial \mathcal{T}_h} + \langle \delta V q \cdot n, P_M w \rangle_{\partial \mathcal{T}_h} + R(q, w), \\
(q \cdot n, \mu)_{\partial \mathcal{T}_h} &= (P_V q \cdot n, \mu)_{\partial \mathcal{T}_h} + \langle \delta V q \cdot n, \mu \rangle_{\partial \mathcal{T}_h}.
\end{align*}
\]

Taking the stabilization terms into account, we have

\[
\begin{align*}
(P_V q + \nabla P_W u, v)_{\mathcal{T}_h} - \langle P_M (P_W u) - P_M u, v \cdot n \rangle_{\partial \mathcal{T}_h} &= F_1(v) & \forall v \in V_h, \\
-(P_V q, \nabla w)_{\mathcal{T}_h} + (P_V q \cdot n, P_M w)_{\partial \mathcal{T}_h} &= (f, w) + F_2(w) & \forall w \in W_h, \\
(P_V q \cdot n, \mu)_{\partial \mathcal{T}_h} &= F_3(\mu) & \forall \mu \in M_h,
\end{align*}
\]

where $\hat{P_V q} \cdot n := P_V q \cdot n + \tau (P_M u - P_W u).$ Subtracting (4) from (11), we obtain the required equations. \hfill \Box

From Lemma 2 the below inequalities follow.

**Lemma 3** If $u \in H^{k+2}(\Omega),$ then we have

\[
\|\nabla e_u\|_{\mathcal{T}_h} \leq C \left( \|e_q\| + h^{-1/2} \|P_M e_u - e_\hat{u}\|_{\partial \mathcal{T}_h} + h^{k+1} \|u\|_{k+2} \right) \tag{12}
\]

and

\[
\|e_q\| \leq C \left( \|\nabla e_u\|_{\mathcal{T}_h} + h^{-1/2} \|P_M e_u - e_\hat{u}\|_{\partial \mathcal{T}_h} + h^{k+1} \|u\|_{k+2} \right). \tag{13}
\]

**Proof** Taking $v = \nabla e_u$ in (11a), we have

\[
\|\nabla e_u\|^2_{\mathcal{T}_h} = -(e_q, \nabla e_u)_{\mathcal{T}_h} + \langle P_M e_u - e_\hat{u}, \nabla e_u \cdot n \rangle_{\partial \mathcal{T}_h} + F_1(\nabla e_u). 
\]

The first two terms on the right-hand side are bounded as

\[
\left| (e_q, \nabla e_u)_{\mathcal{T}_h} \right| \leq \|e_q\| \|\nabla e_u\|_{\mathcal{T}_h}, \\
\left| \langle P_M e_u - e_\hat{u}, \nabla e_u \cdot n \rangle_{\partial \mathcal{T}_h} \right| \leq Ch^{-1/2} \|P_M e_u - e_\hat{u}\|_{\partial \mathcal{T}_h} \|\nabla e_u\|_{\mathcal{T}_h}.
\]

By the inverse inequality, we can estimate the remaining term as

\[
\left| F_1(\nabla e_u) \right| \leq \left| (\nabla \delta W u, \nabla e_u)_{\mathcal{T}_h} \right| + \left| \langle P_M \delta W u, \nabla e_u \cdot n \rangle_{\partial \mathcal{T}_h} \right| \\
\leq Ch^{k+1} \|u\|_{k+2} \|\nabla e_u\|_{\mathcal{T}_h}. \tag{14}
\]

Combining these results yields (12). Similarly, we can prove (13). \hfill \Box
3.2 The estimate of $e_q$

We are now ready to show the main results of the paper.

**Theorem 1** If $u \in H^{k+2}(\Omega)$, then we have

$$\|e_q\| + \|\tau^{1/2}(P_M e_u - e_\tilde{u})\|_{\partial \Omega} \leq C h^{k+1}|u|_{k+2}. $$

**Proof** Substituting $w = e_u$ in (9b) and $\mu = e_\tilde{u}$ in (9c), we have

$$- (e_q, \nabla e_u)_{\partial \Omega} + (e_q \cdot n, P_M e_u - e_\tilde{u})_{\partial \Omega} = F_2(e_u) - F_3(e_\tilde{u}). \quad (15)$$

Taking $v = e_q$ in (9a) and adding it to (15), we get

$$\|e_q\|^2 + \|\tau^{1/2}(P_M e_u - e_\tilde{u})\|^2_{\partial \Omega} = F_1(e_q) + F_2(e_u) - F_3(e_\tilde{u}).$$

In a similar way to (14), we have

$$|F_1(e_q)| \leq C h^{k+1}|u|_{k+2}\|e_q\|.$$ 

The rest terms are written as

$$F_2(e_u) - F_3(e_\tilde{u}) = -R(q, e_u) + (\delta V q \cdot n, \tau P_M \delta W u, P_M e_u - e_\tilde{u})_{\partial \Omega} =: I_1 + I_2.$$ 

By Lemmas 1 and 3, we have

$$|I_1| \leq C h^{k+1}|u|_{k+2}\|\nabla e_u\|_{\partial \Omega} \leq C h^{k+1}|u|_{k+2}\left(\|e_q\| + h^{-1/2}\|P_M e_u - e_\tilde{u}\|_{\partial \Omega} + h^{k+1}|u|_{k+2}\right).$$

The term $I_2$ is bounded as, in view of $\tau = O(h^{-1}),$

$$|I_2| \leq (\|\delta V q\|_{\partial \Omega} + \tau\|\delta W u\|_{\partial \Omega})\|P_M e_u - e_\tilde{u}\|_{\partial \Omega} = C h^{k+1}|u|_{k+2}\cdot \tau^{1/2}\|P_M e_u - e_\tilde{u}\|_{\partial \Omega}.$$ 

Using Young’s inequality and arranging the terms, we obtain

$$\|e_q\|^2 + \|\tau^{1/2}(P_M e_u - e_\tilde{u})\|^2_{\partial \Omega} \leq C h^{2(k+1)}|u|^2_{k+2},$$

which completes the proof. \qed
3.3 The estimate of $e_u$

We show that the order of convergence in the variable $u$ is optimal by the duality argument. To this end, we consider the adjoint problem of (1): Find $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\theta \in H^1(\Omega)$ such that

$$
\begin{align*}
\nabla \psi + \theta &= 0 \quad \text{in } \Omega, \\
\nabla \cdot \theta &= e_u \quad \text{in } \Omega, \\
\psi &= 0 \quad \text{on } \partial \Omega.
\end{align*}
$$

As is well known, the elliptic regularity holds:

$$
\|\theta\|_1 + \|\psi\|_2 \leq C \|e_u\|.
$$

Let us denote the approximation errors of $\psi$ and $\theta$ as follows:

$$
\delta \psi = \theta - P_V \theta, \quad \delta W \psi = \psi - P_W \psi, \quad \delta M \psi = \psi - P_M \psi.
$$

**Theorem 2** If $u \in H^{k+2}(\Omega)$, then we have

$$
\|e_u\| \leq C h^{k+2} |u|_{k+2}.
$$

**Proof** Similarly to (11), we deduce

$$
(P_V \theta + \nabla P_W \psi, v)_{T_h} - (P_M(\psi - P_W \psi), v \cdot n)_{\partial T_h} = G_1(v) \quad \forall v \in V_h,
$$

$$
-(P_V \theta, \nabla w)_{T_h} + (P_V \theta \cdot n, P_M w)_{\partial T_h} = (e_u, w) + G_2(w) \quad \forall w \in W_h,
$$

$$
(P_V \theta \cdot n, \mu)_{\partial T_h} = G_3(\mu) \quad \forall \mu \in M_h,
$$

where $P_V \theta \cdot n = P_V \theta \cdot n + \tau (P_W \psi - P_M \psi)$ and

$$
G_1(v) = -(\nabla \delta W \psi, v)_{T_h} + (P_M \delta W \psi, v \cdot n)_{\partial T_h},
$$

$$
G_2(w) = -R(\theta, w) - \langle \delta \psi, \theta \cdot n - \tau P_M \delta W \psi, P_M w \rangle_{\partial T_h},
$$

$$
G_3(\mu) = -\langle \delta \psi, \theta \cdot n - \tau P_M \delta W \psi, \mu \rangle_{\partial T_h}.
$$

Substituting $v = e_q$ in (16a), $w = e_q$ in (16b) and $\mu$ in (16c) yields

$$
(\theta + \nabla P_W \psi, e_q)_{T_h} - (P_M(\psi - P_W \psi), e_q \cdot n)_{\partial T_h} = G_1(e_q),
$$

$$
-(\theta, \nabla e_u)_{T_h} + (P_V \theta \cdot n, P_M e_u - e_q)_{\partial T_h} = \|e_u\|^2 + G_2(e_u) - G_3(e_u).
$$

Taking $v = P_V \theta$, $w = P_W \psi$ and $\mu = P_M \psi$ in the error equations (11), we have

$$
(e_q + \nabla e_u, \theta)_{T_h} - (P_M e_u - e_q, P_V \theta \cdot n)_{\partial T_h} = F_1(P_V \theta),
$$

$$
-(e_q, \nabla P_W \psi) + \langle P_M(e_q \cdot n), P_W \psi - \psi \rangle_{\partial T_h} = F_2(P_W \psi) - F_3(P_M \psi).
$$
Note that $P_M\psi \in M_h$ since $\psi$ is single-valued on $\mathcal{E}_h$. Adding (18b) to (17m) and (18a) to (17b), we have

\[
(\theta, e_q)_{\mathcal{T}_h} - (\tau(P_M e_u - e_{\mathbf{n}}), \delta_W \psi)_{\mathcal{O}_{\mathcal{T}_h}} = G_1(e_q) + F_2(P_W \psi) - F_3(P_M \psi),
\]

\[
(e_q, \theta)_{\mathcal{T}_h} - (\tau(P_M e_u - e_{\mathbf{n}}), \delta_W \psi)_{\mathcal{O}_{\mathcal{T}_h}} = \|e_u\|^2 + F_1(P_V \theta) + G_2(e_u) - G_3(e_{\mathbf{n}}),
\]

respectively. Since the left-hand sides are equal to each other, we obtain

\[
\|e_u\|^2 = G_1(e_q) - G_2(e_u) + G_3(e_{\mathbf{n}}) - (F_1(P_V \theta) - F_2(P_W \psi) + F_3(P_M \psi)).
\]

By the inverse and trace inequalities, we have

\[
|G_1(e_q)| \leq Ch|\psi|_2\|e_q\|.
\]

By Lemma 3 and Theorem 1 we get

\[
|G_2(e_u) - G_3(e_{\mathbf{n}})| \leq |R(\theta, e_u)| + \|(-\delta_V \theta \cdot n + \tau \delta_W \psi, P_M e_u - e_{\mathbf{n}})_{\mathcal{O}_{\mathcal{T}_h}}| + \|\theta\|_{\mathcal{O}_{\mathcal{T}_h}} + \|\delta_W \psi\|_{\mathcal{O}_{\mathcal{T}_h}} + \|P_M e_u - e_{\mathbf{n}}\|_{\mathcal{O}_{\mathcal{T}_h}}
\]

\[
\leq Ch|\theta|_1\|\nabla e_u\|_{\mathcal{T}_h} + Ch(\|\theta\|_{\mathcal{O}_{\mathcal{T}_h}} + \|e_{\mathbf{n}}\|_{\mathcal{O}_{\mathcal{T}_h}} + \|\delta_W \psi\|_{\mathcal{O}_{\mathcal{T}_h}} + h^{k+1}\|e_u\|_{\mathcal{O}_{\mathcal{T}_h}})
\]

\[
\leq Ch^{k+2}\|e_u\|_{\mathcal{T}_h}.
\]

Similarly, we have

\[
F_1(P_V \theta) = -(\nabla \delta_W u, \theta) + (\delta_W u, P_M (P_V \theta \cdot n))_{\mathcal{O}_{\mathcal{T}_h}}
\]

\[
= (\delta_W u, \nabla \theta) - ((I - P_M)\delta_W u, P_V \theta \cdot n)_{\mathcal{O}_{\mathcal{T}_h}}
\]

\[
= T_1 + T_2,
\]

and the terms are bounded as

\[
|T_1| \leq \|\delta_W u\|\|e_u\| \leq Ch^{k+2}\|e_u\|_{\mathcal{T}_h}.
\]

\[
|T_2| = |((I - P_M)\delta_W u, (P_V \theta - \theta - \Pi_\mathbf{n}\theta) \cdot n)_{\mathcal{O}_{\mathcal{T}_h}}| \leq \|\delta_W u\|_{\mathcal{T}_h}\|\delta_W \theta\|_{\mathcal{O}_{\mathcal{T}_h}} + \|\theta - \Pi_\mathbf{n}\theta\|_{\mathcal{O}_{\mathcal{T}_h}}
\]

\[
\leq Ch^{k+3/2}\|e_u\|_{\mathcal{T}_h} + Ch^{1/2}\|\theta\|_{\mathcal{O}_{\mathcal{T}_h}}
\]

\[
\leq Ch^{k+2}\|e_u\|_{\mathcal{T}_h}.
\]

Moreover,

\[
F_2(P_W \psi) - F_3(P_M \psi) = -R(q, P_W \psi) - (\delta_V q \cdot n - \tau P_M \delta_W u, P_M \delta_W \psi)_{\mathcal{O}_{\mathcal{T}_h}}
\]

\[
= T_3 + T_4.
\]

Since both $q$ and $\psi$ are single-valued on $\mathcal{E}_h$, it follows that

\[
T_3 = -((I - P_M)q \cdot n, \delta_W \psi)_{\mathcal{O}_{\mathcal{T}_h}} = R(q, P_W \psi - \psi).
\]

By Lemma 1 we get

\[
|T_3| \leq Ch^{k+1}\|e_u\|_{\mathcal{T}_h} \leq Ch^{k+2}\|e_u\|_{\mathcal{T}_h}
\]

\[
|T_4| \leq Ch^{k+1}\|u\|_{\mathcal{T}_h} \leq Ch^{k+2}\|u\|_{\mathcal{T}_h}.
\]
The other term is bounded as follows:

\[ |T_4| \leq C(\|\delta v q\|\sigma_{\tau_n} + \tau\|\delta w u\|\sigma_{\tau_n}) \|\delta w \psi\|\sigma_{\tau_n} \]
\[ \leq C(h^{k+1/2}|q|_{k+1} + h^{-1}h^{k+3/2}|u|_{k+2}) \cdot Ch^{3/2}|\psi|_2 \]
\[ \leq Ch^{k+2}|u|_{k+2}|\psi|_2. \]

Combining these results and applying Young’s inequality, we have

\[ \|e_u\|^2 \leq Ch^{k+2} (\|e_u\| + |\theta|_1 + |\psi|_2). \]

Thanks to the elliptic regularity, we obtain the required inequality. □

4 Numerical results

In this section, we carry out numerical experiments to verify our theoretical results. The following test problem is considered:

\[ -\Delta u = 2\pi^2 \sin(\pi x) \sin(\pi y) \quad \text{in } \Omega, \quad (19a) \]
\[ u = 0 \quad \text{on } \partial \Omega, \quad (19b) \]

where \( \Omega = (0, 1)^2 \) and the exact solution is \( \sin(\pi x) \sin(\pi y) \). All computations were done with FreeFem++. The meshes we used are unstructured triangular meshes. We set \( V_h = P_{k+1}(T_h), W_h = P_{k+1}(T_h) \) and \( M_h = P_k(E_h) \) for \( 0 \leq k \leq 2 \), varying \( l \) from 0 to 2. The stabilization parameter \( \tau \) is set to be 1/h in all cases.

The history of convergence of our method is displayed in Tables 2–4. From the results, we observe that the orders or convergence in \( q \), \( u \) and the projected jump quantity are \( k+1 \), \( k+2 \) and \( k+1 \), respectively, which is in full agreement with Theorems 1 and 2. Note that, as mentioned in Remark 1, the errors of our method in Table 3 for \( l = 0 \) coincide with those of the HDG-LS method in Table 1.

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An HDG method with orthogonal projections in facet integrals

### Table 2
Convergence history for $k = 0$

| $l$ | $1/h$  | $\|q - q_h\|$ Error | $\|u - u_h\|$ Error | $\|h^{-1/2}(P_M u_h - \hat{u}_h)\|_{\partial T_h}$ Error |
|-----|--------|----------------------|----------------------|--------------------------------------------------|
| 0   | 10     | 2.643E-01            | 1.842E-02            | 3.873E-01                                        |
|     | 20     | 1.306E-01 1.02       | 4.543E-03 2.02       | 1.920E-01 1.01                                   |
|     | 40     | 6.612E-02 0.98       | 1.176E-03 1.95       | 9.775E-02 0.97                                   |
|     | 80     | 3.313E-02 1.00       | 2.928E-04 2.01       | 4.853E-02 1.01                                   |
| 1   | 10     | 2.152E-01            | 6.793E-03            | 5.545E-02                                        |
|     | 20     | 1.069E-01 1.01       | 1.686E-03 2.01       | 2.673E-02 1.05                                   |
|     | 40     | 5.407E-02 0.98       | 4.339E-04 1.96       | 1.366E-02 0.97                                   |
|     | 80     | 2.726E-02 0.99       | 1.113E-04 1.96       | 6.748E-03 1.02                                   |
| 2   | 10     | 2.533E-01            | 6.078E-03            | 1.502E-02                                        |
|     | 20     | 1.254E-01 1.01       | 1.498E-03 2.02       | 7.143E-03 1.07                                   |
|     | 40     | 6.348E-02 0.98       | 3.847E-04 1.96       | 3.654E-03 0.97                                   |
|     | 80     | 3.184E-02 1.00       | 9.910E-05 1.96       | 1.802E-03 1.02                                   |

### Table 3
Convergence history for $k = 1$

| $l$ | $1/h$  | $\|q - q_h\|$ Error | $\|u - u_h\|$ Error | $\|h^{-1/2}(P_M u_h - \hat{u}_h)\|_{\partial T_h}$ Error |
|-----|--------|----------------------|----------------------|--------------------------------------------------|
| 0   | 10     | 1.236E-02            | 7.400E-04            | 2.719E-02                                        |
|     | 20     | 3.085E-03 2.00       | 9.085E-05 3.03       | 6.676E-03 2.03                                   |
|     | 40     | 7.655E-04 2.01       | 1.140E-05 2.99       | 1.662E-03 2.01                                   |
|     | 80     | 1.915E-04 2.00       | 1.414E-06 3.01       | 4.113E-04 2.01                                   |
| 1   | 10     | 7.212E-03            | 1.023E-04            | 3.157E-03                                        |
|     | 20     | 1.744E-03 2.05       | 1.194E-05 3.10       | 7.704E-04 2.03                                   |
|     | 40     | 4.468E-04 1.96       | 1.544E-06 2.95       | 1.863E-04 2.05                                   |
|     | 80     | 1.127E-04 1.99       | 1.922E-07 3.01       | 4.525E-05 2.04                                   |
| 2   | 10     | 8.936E-03            | 1.059E-04            | 1.763E-03                                        |
|     | 20     | 2.187E-03 2.03       | 1.254E-05 3.08       | 4.422E-04 2.00                                   |
|     | 40     | 5.529E-04 1.98       | 1.616E-06 2.96       | 1.062E-04 2.06                                   |
|     | 80     | 1.387E-04 1.99       | 2.019E-07 3.00       | 2.594E-05 2.03                                   |

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Table 4 Convergence history for $k = 2$

| $l$ | $1/h$ | $\|q - q_h\|$ Error | Order | $\|u - u_h\|$ Error | Order | $\|h^{-1/2}(PM_u - \hat{u}_h)\|_{\partial \Omega_h}$ Error | Order |
|-----|-------|-----------------------|-------|-----------------------|-------|----------------------------------------------------------|-------|
| 0   | 10    | 2.365E-04             | –     | 2.705E-05             | –     | 1.270E-03                                               | –     |
|     | 20    | 2.718E-05             | 3.12  | 1.573E-06             | 4.10  | 1.480E-04                                               | 3.10  |
|     | 40    | 3.509E-06             | 2.95  | 1.049E-07             | 3.91  | 1.919E-05                                               | 2.95  |
|     | 80    | 4.350E-07             | 3.01  | 6.381E-09             | 4.04  | 2.346E-06                                               | 3.03  |
| 1   | 10    | 1.172E-04             | –     | 3.673E-06             | –     | 1.291E-04                                               | –     |
|     | 20    | 1.338E-05             | 3.13  | 1.973E-07             | 4.22  | 1.449E-05                                               | 3.15  |
|     | 40    | 1.791E-06             | 2.90  | 1.278E-08             | 3.95  | 1.832E-06                                               | 2.98  |
|     | 80    | 2.199E-07             | 3.03  | 7.694E-10             | 4.05  | 2.194E-07                                               | 3.06  |
| 2   | 10    | 1.775E-04             | –     | 4.812E-06             | –     | 8.311E-05                                               | –     |
|     | 20    | 2.018E-05             | 3.14  | 2.584E-07             | 4.22  | 9.561E-06                                               | 3.12  |
|     | 40    | 2.638E-06             | 2.94  | 1.671E-08             | 3.95  | 1.204E-06                                               | 2.99  |
|     | 80    | 3.146E-07             | 3.07  | 1.002E-09             | 4.06  | 1.445E-07                                               | 3.06  |