We discuss the formal structure of a functional measure for Gauge Theories preserving the Slavnov-Taylor identity in the presence of Gribov horizons. Our construction defines a gauge-fixed measure in the framework of the lattice regularization by dividing the configuration space into patches with different gauge-fixing prescriptions. Taking into account the bounds described by Dell’Antonio and Zwanziger we discuss the behaviour of the measure in the continuum limit for finite space-time volume.

The purpose of this note is to define a suitable framework to study the continuum-weak-coupling limit of lattice gauge theories; that is the limit in which locally the gauge fields fluctuations are small in comparison with the inverse product of the bare coupling $g_0$ times the lattice spacing. In this limit the non-perturbative Wilson lattice theory is believed to approach in some sense the short-distance covariant perturbation theory.

The considered limit is formally obtained from Wilson’s theory making the lattice spacing to vanish with respect to a given scale $\mu^{-1}$ and keeping the product $g_0^2 \log \frac{1}{a\mu}$ constant. To avoid the overwhelming difficulties connected with the infra-red limit we keep the space-time volume of the lattice constant and smaller than $\mu^{-D}$ in $D$ space-time dimensions. The topological structure of the configuration space is reasonably well known at finite volume\[1\], while it is expected to develop catastrophic singularities in the infinite volume limit.

To limit the field fluctuations that are longitudinal with respect to the gauge orbits we introduce a suitable gauge-fixing condition. Only those conditions that are complete and translational invariant induce a sufficient suppression of the longitudinal field fluctuations. For example the temporal gauge: $A_0 = 0$ is not suitable for our purposes.

Our strategy is based on the introduction of gauge-fixing terms, together with the corresponding Faddeev-Popov determinants, already in the lattice
functional measure in order to have a uniform limitation of the field fluctuations in the continuum limit. However in this limit the situation is made cumbersome by the appearance of Gribov horizons \(2\); the Faddeev-Popov determinant is expected to become degenerate for a non-trivial subset of configurations. In this case the gauge-fixing procedure would encounter a singularity and hence the functional measure would appear ill-defined. It is for this reason that it has been speculated \(3\) that the Gribov phenomenon could induce a breakdown of the Slavnov-Taylor identity that ensures the consistency of the Faddeev-Popov quantization procedure. In perturbation theory this breakdown would not appear since also the Gribov horizons are absent.

We want to show, in contrast with these speculations, that there exists a lattice gauge-fixed functional measure based on a decomposition of the configuration space into patches equipped with different gauge-fixing prescriptions for which the Slavnov-Taylor identity is satisfied. For a suitable choice of the patch decomposition this measure should avoid the Gribov horizons \(4\).

Requiring the functional measure to obey the Slavnov-Taylor identity ensures two main results. First, whenever a theory leads to a space of asymptotic spaces this identity guarantees the unitarity of the restriction of the \(S\)-matrix to a suitable physical subspace \(5\); more generally it ensures the invariance of the correlators of physical operators under deformations of the gauge-fixing prescriptions \(6\). This implies in particular the independence of physical correlators on the choice of the patch decomposition of the configuration space. Notice that this does not exclude the existence of physical effects induced by the very presence of Gribov horizons; indeed the atlas of patches can be modified but not reduced to a single patch, at least in the continuum limit.

For simplicity we choose in our study the Landau background gauge prescription. Given two gauge configurations \(\{U_l\}\) and \(\{V_l\}\), where \(l\) labels the lattice links, we define the relative distance as:

\[
d^2 (U, V) = \sum_l \text{Tr} \left[ (U_l - V_l) \left( U_l^† - V_l^† \right) \right].
\]  

(1)

This defines the lattice \(L^2\) norm which tends in the continuum limit to the norm:

\[
d^2 (A, B) = \int dx \text{Tr} \left[ (A^\mu - B^\mu) \left( A^\dagger_\mu - B^\dagger_\mu \right) \right],
\]  

(2)

where \(A\) and \(B\) are the matrix valued gauge fields corresponding to \(U\) and \(V\) respectively. The Landau gauge-fixing prescription associates every gauge orbit with the configuration on the orbit for which the distance from the background has an absolute minimum. The neighborhood of the background for
which this minimum is unique is called its "fundamental modular region". We associate a sphere with radius \( R \) in the norm defined above to each background gauge configuration. The radius \( R \) of the sphere corresponding to every background is such that in the continuum limit the sphere does not intersect the horizon of the Gribov domain corresponding to the Landau choice. Indeed it is known that in the continuum limit and at finite space-time volume the fundamental modular region contains a sphere whose radius is independent of the background.

It remains to choose a suitable set of background configurations such that every orbit intersects at least one sphere of radius \( R \) centered in an element of the set. This can be done on the lattice owing to the existence of global gauge-fixings. Indeed one can choose, for example, the configurations for which \( U_l \) is constant along every lattice link parallel to the first coordinate axis and also constant along those links parallel to the second axis which belong to a given hyperplane orthogonal to the first axis. Furthermore \( U_l \) can be taken constant along the links parallel to the third coordinate axis and belonging to a plane orthogonal to the first two axis. Finally the \( U_l \) are chosen constant along a single lattice line parallel to the fourth axis. This is a complete axial gauge prescription that, as mentioned above, does not induce the wanted limitation of the longitudinal field fluctuations in the continuum limit thus being unsuited for studying the continuum limit. However we can use this gauge-fixing to select the wanted background configurations. Indeed since these gauge-fixed configurations define a compact manifold \( M_L \), we can choose a finite lattice of configurations on this manifold such that the distance between two neighboring elements is smaller than \( \sqrt{2}R \) and such that the corresponding set of spherical patches of radius \( R \) centered in the elements of the configuration lattice covers \( M_L \) completely.

We now come to a formal construction of the lattice measure. We consider a periodic hypercubic space-time lattice with lattice spacing \( a \) and period \( L a \). We label by \( x \) a generic lattice site whose coordinates are: \( x^\mu = an^\mu \), where \( n \) is a four-vector with integer components. Let \( e_\mu \) be the vector with components: \( e_\nu^\mu = \delta_\nu^\mu \). Then \( x + e_\mu \) is the nearest neighboring site to \( x \) in the direction of the \( \mu \) axis. We limit our study to the gauge group \( SU(N) \). A lattice configuration corresponds to the set of unitary, unimodular, \( N \)-dimensional matrices: \( \{ U_{x,\mu} \} \) where the index \( x,\mu \) labels the link joining the sites \( x \) and \( x + e_\mu \). A gauge transformation is given by the set of unitary, unimodular, \( N \)-dimensional matrices \( \{ g_x \} \) satisfying the constraint \( g_o = I \), where \( o \) labels the space-time origin and \( I \) is the unity matrix. This constraint distinguishes the gauge transformations from the \( SU(N) \) rigid symmetry of the model. The action of a gauge transformation on the configuration \( U \) is
Figure 1. The circles represent the patches and the dotted lines the local sections.

given by:

\[ U \rightarrow U^{(g)}_{x,\mu} = g_{x}U_{x,\mu}g_{x+e_{\mu}}^{-1}. \tag{3} \]

Given a background configuration \( V \), the gauge-fixed configuration \( U^{(g)}_{(\bar{g})} \) corresponding to the orbit \( U^{(g)} \) is the one for which \( d^2 (U^{(g)}, V) \) has an absolute minimum.

In general only a subset of the orbits admits a unique gauge-fixed configuration corresponding to a given background. Indeed it is possible that the same orbit contains two or more configurations minimizing the distance, or that \( V \) coincides with the center of an osculator circle to the orbit. However it is clear that the minimizing configuration is unique for the orbit \( V^{(g)} \) itself, and thus is also unique for all the orbits crossing a small ball around \( V \). The work of Dell’Antonio and Zwanziger \( \textsuperscript{7} \) implies that the radius of this ball can be chosen to be independent of \( V \) and of the number of lattice points for a given space-time volume.

An isolated minimizing configuration satisfies the local gauge condition:

\[ \sum_{\mu} \left( V_{x+e_{\mu},\mu}U_{x+e_{\mu},\mu} - U_{x+e_{\mu},\mu}^{-1}V_{x+e_{\mu},\mu} \right) = \sum_{\mu} \left( U_{x,\mu}V_{x,\mu}^{-1} - V_{x+e_{\mu},\mu}U_{x+e_{\mu},\mu}^{-1} \right), \tag{4} \]

that in the continuum becomes:

\[ D_{\mu}^{(V)} [A^{\mu} - V^{\mu}] = 0, \tag{5} \]

where \( D_{\mu}^{(V)} \) is the covariant derivative in the background \( V \). This is the well known Landau background gauge condition. \( \textsuperscript{4} \) and \( \textsuperscript{5} \) define local sections \( \Sigma_{V} \) of the gauge bundle whose total space is the configuration space.

We select, as described above, a collection of background configurations \( \{ V^{(a)} \} \) that is finite on the lattice, but becomes infinite in the continuum.
limit. We also define the characteristic functions of the corresponding patches in the following way. We call “Gauge Strip” the union of the spherical patches around the background configurations $V^{(a)}$; its characteristic function is:

$$
\chi_{\text{strip}}(U) \equiv \Theta \left( R - \inf_a d(U, V^{(a)}) \right),
$$

(6)

where $\Theta$ is a smoothened Heavyside function. Then, the characteristic function of the patch around $V^{(a)}$ is:

$$
\chi_a(U) \equiv \chi_{\text{strip}}(U) \frac{\Theta (R - d(U, V^a))}{\sum_b \Theta (R - d(U, V^b))}.
$$

(7)

Thus

$$
\sum_a \chi_a(U) = \chi_{\text{strip}}(U).
$$

(8)

We assume without proof that the configurations in the intersection of the Landau local sections $\Sigma_a$ with the corresponding patches are internal to the Gauge Strip. The meaning of this assumption is illustrated in the figure.

The proof of the above assumption can be given following the study of the covariant Laplacian presented in [7].

To construct the wanted functional measure we introduce the exterior (BRS) derivative $s$:

$$
\begin{align*}
    sU_{x,\mu} &= i \left( \omega_x U_{x,\mu} - U_{x,\mu} \omega_x + e_{x,\mu} \right), \\
    s\omega_x &= i \omega_x^2, \\
    s\bar{\omega}_x &= i b_x, \\
    sV_{x,\mu} &= 0.
\end{align*}
$$

(9)

where $b_x$ is the Lagrange-Nakanishi-Lautrup multiplier implementing the gauge-fixing constraint. We also introduce the anti-derivative:

$$
\begin{align*}
    \bar{s}U_{x,\mu} &= i \left( \bar{\omega}_x U_{x,\mu} - U_{x,\mu} \bar{\omega}_x + e_{x,\mu} \right), \\
    \bar{s}V_{x,\mu} &= 0.
\end{align*}
$$

(10)

The standard Faddeev-Popov measure corresponding to the background $V^a$ is:

$$
d\mu_{\nu} \prod_x d\omega_x d\bar{\omega}_x db_x e^{-\bar{s}d^2(U, V)} \equiv d\mu e^{-s\Psi_a},
$$

(11)

where $d\mu_{\nu}$ is the lattice gauge invariant measure and we have identified the gauge-fixing fermionic operator $\Psi_a$ with $\bar{s}d^2(U, V)$. 

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Naively, the measure corresponding to the chosen patch decomposition of the configuration space would be:

$$d\mu_{\text{naive}} = \sum_a \chi_a d\mu e^{-s\Psi_a}.$$  

(12)

This measure, however, does not satisfy the Slavnov-Taylor identity, stating that

$$\int d\mu_{\text{naive}} s\Omega = 0$$  

(13)

for any field functional $\Omega$, which is required for the physical reasons we discussed above.

One has instead:

$$\int d\mu_{\text{naive}} s\Omega = -\sum_a \int d\mu_s \chi_a e^{-s\Psi_a} \Omega,$$  

(14)

whose right-hand side does not vanish since the characteristic functions of the patches are not gauge invariant.

This difficulty can be overcome by adding to the naive measure contributions associated to the intersections of different patches. Consider for simplicity the case in which one has only two patches. On account of our assumption, on the support of the measure $d\mu e^{-s\Psi_a}$ one has: $\chi_1 + \chi_2 = 1$, which implies

$$s\chi_1 = -s\chi_2 = s\chi_1 \chi_2 - s\chi_2 \chi_1.$$  

(15)

Therefore one has:

$$\int d\mu_{\text{naive}} s\Omega = -\int d\mu_s \chi_1 \left( e^{-s\Psi_1} - e^{-s\Psi_2} \right) \Omega = \int d\mu_s \chi_1 \chi_2 (\Psi_1 - \Psi_2) \int_0^1 dt e^{-s(t\Psi_1 + (1-t)\Psi_2)} \Omega = \int_0^1 dt e^{-s(t\Psi_1 + (1-t)\Psi_2)} s\Omega.$$  

(16)

We thus see that in the case of two patches the measure:

$$d\mu \left[ |\chi_1 e^{-s\Psi_1} + \chi_2 e^{-s\Psi_2} - (s\chi_1 \chi_2 - s\chi_2 \chi_1) (\Psi_1 - \Psi_2) \int_0^1 dt e^{-s(t\Psi_1 + (1-t)\Psi_2)} \right] .$$  

(17)

satisfies the Slavnov-Taylor identity. It is clear that the term added to the naive measure is supported in the intersection of the two patches. This result
can be extended to the case of an atlas with a generic number of patches. To this end, we define \( n \)-chart interpolating measure:

\[
e^{-s\Psi_{a_1 \ldots a_n}} \equiv \int_0^\infty \prod_{i=1}^n dt_i \delta\left(\sum_{j=1}^n t_j - 1\right)e^{-s\sum_{k=1}^n t_k \Psi_{a_k}}.
\] (18)

and we introduce the two following notations:

\[
(s\chi_{a_1} \ldots s\chi_{a_{n-1}} \chi_{a_n})_A \equiv \sum_{k=1}^n (-1)^{k-n} \chi_{a_k} s\chi_{a_1} \ldots s\chi_{a_k} \ldots s\chi_{a_n}
\] (19)

and:

\[
\partial_{\Psi} (\Psi_{a_1} \ldots \Psi_{a_n}) \equiv \sum_{l=1}^n (-1)^{l+1} \Psi_{a_1} \ldots \tilde{\Psi}_{a_l} \ldots \Psi_{a_n}.
\] (20)

In (19) and (20) the check mark above a factor means that the corresponding factor should be omitted. Then the functional measure consistent with the Slavnov-Taylor identity is:

\[
d\mu_{ST} = d\mu \sum_{n=1}^\infty \left(\frac{(-1)^{(n-2)(n-1)}}{n}\right)^{1/2}(s\chi_{a_1} \ldots s\chi_{a_n} \chi_{a_n})_A \partial_{\Psi} (\Psi_{a_1} \ldots \Psi_{a_n}) e^{-s\Psi_{a_1 \ldots a_n}}.
\] (21)

This is the main result presented in this communication. It is apparent that the lack of gauge invariance of the characteristic functions of the patches induces new contributions to the measure localized on the patch (regularized) boundaries. One can interpret the factors multiplying the interpolating measure in the patch intersections as the Jacobians relating the bulk measure to the boundary one. Before the continuum limit, the number of patches being finite, the series appearing in (21) reduces to a finite sum. However in the continuum limit our patch decomposition could present intersections of an infinite number of patches. In this case the meaning of our formula would be unclear since we should have to face convergence problem for our series. As a matter of fact, if the space-time volume remains finite, the configuration space in the continuum loses its compactness but remains paracompact even in the \( L^2 \) norm. This means that one can avoid these convergence problems at finite space-time volume.

Of course, in the infinite volume limit, the above measure could be ill defined if the cells accumulated around some singularity of the configuration space. This could perhaps induce instabilities of the BRS symmetry in the sense of. We believe however that, in analogy with the topological case, these
instabilities could be reabsorbed extending the action of the BRS operator on
the moduli space of the infra-red singular configurations.

It is easy to verify directly that the expectation values of "physical func-
tionals", that are annihilated by s, do not vary if the patch decomposition is
deformed continuously, provided one avoids Gribov horizons; this remark sub-
stantiates the above consideration on the physical consequences of the Gribov
phenomenon. It would be interesting to verify which is the relevance of the
patch intersections closest to the trivial vacuum configuration to a physically
significant expectation value.

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