FRACTIONAL CALCULUS AND THE ESR TEST

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Abstract
We consider a partial differential equation associated with a mathematical model describing the concentration of nutrients in blood which interferes directly on the erythrocyte sedimentation rate in the case of an average fluid velocity equal to zero. Introducing the fractional derivative in the Caputo sense, we propose a time-fractional mathematical model which contains, as a particular case, the model proposed by Sharma et al. [1]. Our main purpose is to obtain an analytic solution of this time-fractional partial differential equation in terms of the Mittag-Leffler function and Wright function.

Keywords: ESR, Mittag-Leffler functions, time-fractional PDE, Wright function

1 Introduction
In 1897, the Polish physician E. F. Biernacki introduced a blood test that helped in diagnosing the acute phase of inflammatory diseases and in following up body inflammation itself, known as Erythrocyte Sedimentation Rate (ESR) [2, 3, 4]. The discovery was announced in two articles: the first, in Polish, in Lekarska Gazeta, the second, in German, in Deutsche Medizinische Wochenschrift [5, 6]. On the other hand, at the beginning of
nineteenth century, Robin Fahraeus and A. Westergren, when performing pregnancy and tuberculosis tests, developed a test similar to ESR known as the Fahraeus-Westergren test [7, 8, 9, 10].

Nowadays, due to the discovery of new and more accurate methods, ESR is little used despite it being a quick and low cost test. Nevertheless, the test is still recommended for patients with suspected giant cell arthritis, rheumatics polymyalgia and rheumatoid arthritis, among others [11]. However, as ESR is not quite specific, it is necessary to conduct further tests to confirm the result obtained by ESR in order to avoid false-positive and false-negative results, which are likely to occur in the presence of some factor whose influence on blood properties affects the test results [12, 13, 14], e.g.:

- Analytic factors such as an inclined tube and ambient temperature, which would respectively increase and decrease ESR [15]. Other factors which affect the results are the presence of external vibration and tube deformation [16].

- Physiological and pathological factors such as anemia due to low erythrocyte concentration, pregnancy and old age, resulting in increased ESR; polycythemia and increased leukocyte counting, resulting in decreased ESR.

The concentration of nutrients in blood also plays a role in the analysis of ESR results [17]. Moreover, Nayha [18] noted that people who drink coffee and smoke present higher values of ESR. The use of some types of anticoagulants such as Sodium citrate, oxalate or $K_3$ EDTA is also responsible for influencing the test results [19, 20, 21, 22].

Huang et al. [23] published a work in which they measured the concentration of red cells in blood at different times in samples of 5 male people. In the same year, Huang et al. [24] developed a mathematical model to describe the behavior of the concentration of blood cells, giving importance to mobile boundary problems. Another notable work in ESR context was written by Sartory [25], whose objective was to study the prediction of erythrocyte sedimentation profiles. Moved by Huang’s 1971 work, in 1990 Reuben and Shannon [26] discussed some problems in the mathematical modeling of concentration of red blood cells. However, the authors of those studies did not take into account the transfer of nutrients from capillaries to tissues. Due to this fact, Sharma et al. [1] established a mathematical model taking into account the transfer of nutrients, making it a more precise model.

Thus the ESR test can be used to obtain several clinical diagnoses and may be studied as a particular type of transport phenomena [27]. It is worth mentioning that there are several transport phenomena whose fractional versions provide better descriptions than the corresponding classical models [28, 29, 30].

Those models are generally constructed with the help of nonlinear partial differential equations whose solutions require numerical methods to be discussed. On the other
hand, the corresponding linear partial differential equations can be solved by means of some analytic method, but the solutions sometimes do not actually describe a particular phenomenon.

The main motivation for the study of concentration $C(x, t)$ by means of fractional calculus is the discrete mathematical model proposed by Sharma et al. [1]. In this article we present an application of fractional calculus, specifically of fractional derivatives in the Caputo sense, to study the profile of $C(x, t)$. Event though we have the same purpose as Sharma et al. [1], the equivalence, in an appropriate limit, between our results and the integer order case is not immediate, because the way they develop their calculations with integer order derivatives has different directions from the calculations using fractional derivatives in the Caputo sense, and each step has to be checked.

As we are seeking the same goal, it is useful to make some comparisons in order to highlight the developments brought by this work. Indeed, it is not an easy task, and much less trivial, to find the analytic solution of the proposed fractional model and to present its behavior graphically. The main differences and similarities with regard to the integer case are:

- We assume an average speed $U = 0$, as in Sharma et al. model [1], thus restricting ourselves to the diffusion model. We use this model to introduce the basic concepts of fractional calculus and to present our fractional mathematical model.

- We propose a model with fractional derivatives in the Caputo sense with a time derivative of order $0 < \mu \leq 1$. Consequently, the solution is dependent on the parameter $\mu$. In the limit $\mu \to 1$ we recover the solution of the Sharma et al. [1] as a particular case.

- What is expected is that the solutions of each model are distinct, and that is what actually happens. Indeed, the analytic solution obtained by Sharma et al. [1] is expressed as a product of the exponential function $\exp(\cdot)$ and the complementary error function $\text{erfc}(\cdot)$. On the other hand, the solution obtained for the fractional mathematical model is given in terms of the Mittag-Leffler function and the Wright function.

- The analytic behavior of the solution obtained with fractional derivatives allows a more detailed analysis, because we have an extra degree of freedom in parameter $\mu$ ($0 < \mu \leq 1$), which permits better fitting of experimental data on nutrient concentration in blood.

We should point out that the methodology to discuss a problem involving a particular fractional derivative is growing in all fields of the knowledge, specifically the integral transform methodology.
This paper is organized as follows: In section two we introduce the so-called fractional mathematical model associated with ESR, a generalization of the model proposed by Sharma et al. [1], which will be recovered through a convenient limit process. Section three, our main result, is dedicated to obtain the analytic solution of our model, which is found using the methodology of Laplace transform and is expressed in terms of the Mittag-Leffler function and the Wright function. We also present a graphical analysis of the solution. In section four we recover as a special case, through a convenient limit process, the solution found by Sharma et al. [1].

2 Time-fractional partial differential equation

In this section we present a fractional version of the linear partial differential equation (PDE) associated with the mathematical model of Sharma et al. [1] used to describe the behavior of the concentration of nutrients in blood cells, a factor that directly affects ESR. We assume that the average fluid velocity is equal to zero. Our model can be considered a generalization of the Sharma et al. [1] model, in the sense that it recovers the latter as a special case, as we shall see in section four.

The blood concentration of nutrients \( C(x, t) \) satisfies the following non-homogeneous time-fractional PDE,

\[
D_L D^2_\mu C(x, t) - D^\mu D_t C(x, t) = \phi(x, t),
\]

with \( 0 < \mu \leq 1 \), where \( D_L \) is a positive constant and \( \phi(x, t) \) is a function describing the nutrient transfer rate and which satisfies the partial differential equation:

\[
D^2 \phi(x, t) - k \phi(x, t) - D_t \phi(x, t) = 0,
\]

with \( D \) and \( k \) both positive constants.

The initial and boundary conditions imposed here are given by

\[
\begin{align*}
\phi(x, 0) &= \exp \left( -\sqrt{\frac{k-a}{D}}x \right), \quad k \geq a, D > 0, \\
\phi(0, t) &= \exp (-at), \quad t > 0, \\
\phi(\infty, t) &= 0, \quad t > 0.
\end{align*}
\]

The solutions of the Eq. (2) can be written as

\[
\phi(x, t) = \exp \left( -(at + bx) \right),
\]

where \( b^2 = \frac{(k-a)}{D} > 0 \) and \( a \) is a constant to be conveniently chosen from a known value of \( \phi(x, t) \).

For the fractional mathematical model proposed, we assume that \( 0 < \mu \leq 1 \) and the
fractional derivative of order \( \mu \) is considered in the Caputo sense \[31, 32, 33\], defined as follows:

\[
D^\mu_t C(x,t) = \frac{\partial^\mu C(x,t)}{\partial t^\mu} := \begin{cases}
\frac{1}{\Gamma(n-\mu)} \int_0^t C^{(n)}(\tau,t) (t-\tau)^{n-\mu-1} d\tau, & n-1 < \mu < n \\
C^{(n)}(x,t), & \mu = n,
\end{cases}
\]

where \( C^{(n)}(x,t) \) is the usual derivative of order \( n \) with respect to \( t \). The particular case \( \mu = 1 \), recovers the result obtained by Sharma et al. \[1\]. Furthermore, we must impose the following initial and boundary conditions for Eq.(1):

\[
\begin{align*}
C(x,0) = 0, & \quad x \geq 0 \\
C(0,t) = 1, & \quad t > 0 \\
C(\infty,t) = 0, & \quad t > 0.
\end{align*}
\] (3)

From these considerations, it follows that the time-fractional mathematical model to be addressed is composed of a non-homogeneous fractional PDE

\[
D_L D^2 x C(x,t) - D^\mu_t C(x,t) = \exp (- (at + bx)) , \quad a, b \in \mathbb{R},
\] (4)

with initial and boundary conditions given by Eq.(3).

3 Analytic Solution

To solve this problem, we employ the methodology of Laplace transform to convert the non-homogeneous fractional PDE into a non-homogeneous linear ordinary differential equation.

Then, applying the Laplace transform \[34, 35\] in relation to the time variable \( t \) on both sides of Eq.(4), we have

\[
D_L \frac{d^2}{dx^2} C(x,s) - s^\mu C(x,s) + s^{\mu-1} C(x,0) = \frac{\exp (-bx)}{s + a}.
\]

Using the initial condition \( C(x,0) = 0 \) we can rewrite this equation as

\[
D_L \frac{d^2}{dx^2} C(x,s) - s^\mu C(x,s) = \frac{\exp (-bx)}{s + a},
\] (5)

where \( 0 < \mu \leq 1, D_L > 0 \) and

\[
C(x,s) = \mathcal{L}\{C(x,t)\} =: \int_0^\infty e^{-st} C(x,t) \, dt
\]
is the Laplace transform of $C(x, t)$ with parameter $s$, $\text{Re}(s) > 0$.

Using the methods of characteristic equation and undetermined coefficients in Eq. (5) we obtain the general solution, given by

$$C(x, s) = \left( \frac{1}{s} + \frac{1}{(s + a)(s^\mu - \frac{\beta^2}{\alpha^2})} \right) \exp\left(-\alpha xs^{\mu/2}\right) + \frac{\exp(-bx)}{(s + a)(\frac{\beta^2}{\alpha^2} - s^\mu)}, \quad (6)$$

where $\alpha^2 = \frac{1}{D_L}$ and $D_L > 0$.

In order to recover the solution in the time variable we take the inverse Laplace transform on both sides of Eq. (6), obtaining

$$C(x, t) = \mathcal{L}^{-1}\{C(x, s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} \exp\left(-\alpha xs^{\mu/2}\right)\right\} +$$

$$+ \mathcal{L}^{-1}\left\{\frac{1}{(s + a)(s^\mu - \frac{\beta^2}{\alpha^2})} \exp\left(-\alpha xs^{\mu/2}\right)\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s + a)(s^\mu - \frac{\beta^2}{\alpha^2})} \exp(-bx)\right\}, \quad (7)$$

where

$$C(x, t) = \mathcal{L}^{-1}\{C(x, s)\} =: \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} C(x, s) \, ds$$

is the inverse Laplace transform, and the integral is performed in the complex plane with the singularities $C(x, s)$ on the left side of $\gamma = \text{Re}(s)$.

Introducing the change $\beta^2 = b^2 D_L$, we rewrite Eq. (7) as

$$C(x, t) = C_1(x, t) + C_2(x, t) - \exp(-bx)C_3(x, t),$$

with

$$C_1(x, t) = \mathcal{L}^{-1}\left\{\frac{\exp\left(-\alpha xs^{\mu/2}\right)}{s}\right\};$$

$$C_2(x, t) = \mathcal{L}^{-1}\left\{\frac{\exp\left(-\alpha xs^{\mu/2}\right)}{(s + a)(s^\mu - \beta^2)}\right\};$$

$$C_3(x, t) = \lim_{x \to 0} C_2(x, t).$$

In order to proceed, we calculate each inverse Laplace transform separately. To calculate $C_1(x, t)$ we introduce the MacLaurin series associated with the exponential function; choosing $f^{(k)}(0) = 1$ in the series, we have
\[
\frac{1}{s} \exp \left( -\alpha x s^{\mu/2} \right) = \sum_{k=0}^{\infty} \frac{(-\alpha x)^k}{k!} s^{\mu k/2}.
\] 

(8)

Applying the inverse Laplace transform on both sides of Eq.(8), we obtain

\[
C_1 (x, t) = \mathcal{L}^{-1} \left\{ \frac{1}{s} \exp \left( -\alpha x s^{\mu/2} \right) \right\} = \sum_{k=0}^{\infty} \frac{(-\alpha x)^k}{k!} s^{\mu k/2} \mathcal{L}^{-1} \left\{ s^{\mu k/2} \right\}.
\]

(9)

Using the result

\[
\mathcal{L}^{-1} \left\{ s^{-q} \right\} = \frac{\mu^{q-1}}{\Gamma (q)},
\]

with \( \text{Re}(q) > 0, q = 1 - \mu k/2 \), we can rewrite Eq.(9) as follows:

\[
C_1 (x, t) = \sum_{k=0}^{\infty} \frac{(-\alpha x/t)^k}{k! \Gamma \left( 1 - \frac{\mu k}{2} \right)}.
\]

(10)

Moreover, considering \( \beta = 1, \alpha = -\mu/2 \) and \( z = -\frac{\alpha x}{t^{\mu/2}} \) in the definition of the Wright function [29], we can write

\[
\mathcal{W} \left( -\frac{\mu}{2}, 1; z \right) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma \left( -\frac{\mu k}{2} + 1 \right)}.
\]

(11)

Then, from Eq.(10) and Eq.(11) we obtain

\[
C_1 (x, t) = \mathcal{W} \left( -\mu/2, 1; -\frac{\alpha x}{t^{\mu/2}} \right).
\]

(12)

Now we evaluate the second inverse Laplace transform,

\[
C_2 (x, t) = \mathcal{L}^{-1} \left\{ \frac{1}{(s + a) (s^{\mu} - \beta^2)} \exp \left( -\alpha x s^{\mu/2} \right) \right\}.
\]

(13)

As with \( C_1(x, t) \), we also write the exponential function in terms of its MacLaurin series. So, we have

\[
\frac{1}{(s + a) (s^{\mu} - \beta^2)} \exp \left( -\alpha x s^{\mu/2} \right) = \sum_{m=0}^{\infty} \frac{(-\alpha x)^m}{m!} s^{\mu m/2} \frac{1}{(s + a) (s^{\mu} - \beta^2)}.
\]

(14)

Once more, applying the inverse Laplace transform on both sides of Eq.(14), we can write

\[
\mathcal{L}^{-1} \left\{ \frac{1}{(s + a) (s^{\mu} - \beta^2)} \exp \left( -\alpha x s^{\mu/2} \right) \right\} = \sum_{m=0}^{\infty} \frac{(-\alpha x)^m}{m!} \mathcal{L}^{-1} \left\{ \frac{s^{\mu m/2}}{(s + a) (s^{\mu} - \beta^2)} \right\}.
\]

(15)

In order to evaluate this inverse Laplace transform, we consider the following expres-
sion \[36\]:

\[\Omega = \frac{s^\sigma}{s^\alpha + \tilde{a}s^\delta + bs^\gamma + cs^\mu + d}\]

with \(\tilde{a}, b, c, d \in \mathbb{R}\) and \(\alpha, \delta, \gamma, \mu \in \mathbb{R}\) such that \(\tilde{a} \neq 0\) and \(\alpha > \delta > \gamma > \mu\).

Assuming the condition \(\frac{bs^\gamma + cs^\mu + d}{s^\alpha + \tilde{a}s^\delta} < 1\) and using the geometric series we have

\[\sum_{k=0}^{\infty} (-1)^{k} s^{\sigma-\delta-k} \frac{(bs^\gamma + cs^\mu + d)^k}{(s^\alpha + \tilde{a})^k + 1} = \frac{s^\sigma}{s^\alpha + s^\delta\tilde{a}} \left(1 + \frac{bs^\gamma + cs^\mu + d}{s^\alpha + \tilde{a}s^\delta}\right) = \frac{bs^\gamma + cs^\mu + d + s^\alpha + \tilde{a}s^\delta}{s^\alpha + \tilde{a}s^\delta}.\] \hspace{1cm} (16)

The binomial theorem and the definition of binomial coefficients \[37\] allow us to write in a convenient way Eq.(16) as

\[\Omega = \sum_{k=0}^{\infty} (-1)^{k} \sum_{l=0}^{k} \left(\frac{k}{l}\right) \frac{k!}{l!(k-l)!} \frac{(bs^\gamma + cs^\mu)^{k-l}}{(s^\alpha - \tilde{a})^{k+1}} \frac{s^{\sigma-\delta-k}}{(s^\alpha + \tilde{a})^{k+1}} \]

\[= \sum_{k=0}^{\infty} (-1)^{k} \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} \frac{(bs^\gamma)^{k-l-j} (cs^\mu)^{j}}{(s^\alpha - \tilde{a})^{k+1}} \frac{s^{\sigma-\delta-k}}{(s^\alpha + \tilde{a})^{k+1}} \]

\[= \sum_{k=0}^{\infty} (-1)^{k} b^k k! \sum_{l=0}^{k} \frac{(d/b)^l}{l!} \sum_{j=0}^{k-l} \frac{(c/b)^j}{j!(k-l-j)!} \Lambda_\sigma, \] \hspace{1cm} (17)

where \(\Lambda_\sigma = \frac{s^{\sigma-\delta(1+k)+\mu+j+(k-l-j)}}{(s^\alpha + \tilde{a})^{k+1}}\).

Taking the inverse Laplace transform on both sides of Eq.(17) and using the result

\[\mathcal{L}^{-1}\{\Lambda_\sigma\} = \mathcal{L}^{-1}\left\{\frac{s^{\sigma-\delta(1+k)+\mu+j+(k-l-j)}}{(s^\alpha - \tilde{a})^{k+1}}\right\} = t^{\xi-1} E^{k+1}_{\alpha-\delta,\xi} (-\tilde{a}t^{\alpha-\delta}),\]

we get

\[\mathcal{L}^{-1}\{\Omega\} = \sum_{k=0}^{\infty} (-1)^{k} b^k k! \sum_{l=0}^{k} \frac{(d/b)^l}{l!} \sum_{j=0}^{k-l} \frac{(c/b)^j}{j!(k-l-j)!} t^{\xi-1} E^{k+1}_{\alpha-\delta,\xi} (-\tilde{a}t^{\alpha-\delta}),\] \hspace{1cm} (19)

with \(\xi = -\sigma + \alpha + (\alpha - \gamma) k + \gamma l - (\mu - \gamma) j\) and where \(E^{k+1}_{\alpha-\delta,\xi} (\cdot)\) is the three parameters Mittag-Leffler function \[36 \[38\].

In particular, considering \(c = 0\) in Eq.(19), we have that \(j = 0\) is the only term contributing to the sum and we conclude that
\[ \mathcal{L}^{-1} \left\{ \frac{s^\alpha}{s^\alpha + \tilde{a}s^\delta + bs^\gamma + d} \right\} = \sum_{k=0}^{\infty} (-1)^k b^k k! \sum_{l=0}^{k} \frac{(d/b)^l}{l!} \left( \frac{\Gamma(k+1)}{\alpha-\delta,\xi} \right) (\tilde{a}t^{\alpha-\delta}) , \]

where \( \xi = -\sigma + \alpha + (\alpha - \gamma) k + \gamma l \) and \( \alpha > \delta > \gamma \).

Then, putting \( \sigma = \mu m/2, d = -a\beta^2, \alpha = \mu + 1, \gamma = \mu, \delta = 1, b = a \) and \( \tilde{a} = -\beta^2 \) in Eq.\((20)\) and going back to Eq.\((15)\), we can write

\[ C_2(x, t) = t^\mu \sum_{m=0}^{\infty} \left( \frac{-\alpha x t - \mu/2}{m!} \right)^m \sum_{k=0}^{\infty} (-at)^k \frac{(-\beta^2 t^2)^l}{l!} \left( \frac{\Gamma(k+1)}{\mu,\theta} \right) (\beta^2 t^\mu) , \]

where \( \theta = -\mu m/2 + \mu + 1 + k + \mu l \).

With the aim of obtaining the solution of the PDE in terms of the two-parameters Mittag-Leffler function, we evaluated the sum on \( l \) appearing in the last expression in order to find a relationship between two- and three-parameters Mittag-Leffler functions. Using the identity

\[ \sum_{j=0}^{k} \frac{(z)^j}{j! (k-j)!} \mathbb{E}^{\rho}_{\lambda,\lambda j+\delta} (-z) = \sum_{j=0}^{k} \sum_{l=0}^{\infty} \frac{(z)^j}{j! (k-j)!} \frac{(\rho)_l (-z)^l}{l! \Gamma (\lambda j + \lambda j + \delta)} , \]

where \( (\rho)_l = \rho (\rho + 1) \ldots (\rho + l - 1) \), together with the definition and properties of the binomial coefficients in Eq.\((22)\), we can write \[30\]

\[ \sum_{j=0}^{k} \frac{(z)^j}{j! (k-j)!} \mathbb{E}^{\rho}_{\lambda,\lambda j+\delta} (-z) = \sum_{i=0}^{\infty} \frac{(-z)^i}{\Gamma (\lambda i + \delta)} \sum_{j=0}^{k} \frac{( -1)^j k!}{j! (k-j)!} \left( \frac{i-j+\rho-1}{\rho-1} \right) \]

\[ = \sum_{i=0}^{\infty} \frac{(-z)^i}{\Gamma (\lambda i + \delta)} \frac{1}{i!} \frac{(\rho-k)_i}{i!} = \frac{1}{k!} \mathbb{E}^{\rho-k}_{\lambda,\delta} (-z) . \]

Choosing \( z = -\beta^2 t^\mu, \rho = k + 1, \lambda = \mu, j = l \) and \( \lambda = k + \mu + 1 - \mu m/2 \) in Eq.\((23)\) and substituting the result into Eq.\((21)\), we conclude that

\[ C_2(x, t) = t^\mu \sum_{m=0}^{\infty} \frac{(-\alpha x t - \mu/2)^m}{m!} \sum_{k=0}^{\infty} (-at)^k \mathbb{E}_{\mu,\mu+k+1-\mu m/2} (\beta^2 t^\mu) , \]

where \( \mathbb{E}_{\alpha,\beta} (\cdot) \) is the two-parameters Mittag-Leffler function.

The last inverse Laplace transform, \( C_3(x, t) \), is obtained by means of a convenient limit, i.e., we consider \( x \to 0 \) in Eq.\((24)\). The only term that contributes in this limit is \( m = 0 \), i.e., we get

\[ C_3(x, t) = t^\mu \sum_{k=0}^{\infty} (-at)^k \mathbb{E}_{\mu,\mu+k+1} (\beta^2 t^\mu) . \]
Thus, from the results obtained in Eq. (12), Eq. (24) and Eq. (25), we get the solution associated with our initial problem, i.e., a solution of Eq. (4) satisfying the conditions given by Eq. (3)

\[
C(x, t) = t^\mu \sum_{m=0}^{\infty} \frac{(-\alpha xt^{-\mu/2})^m}{m!} \sum_{k=0}^{\infty} (-at)^k \mathcal{E}_{\mu,\mu+k+1,-\mu m/2} \left( \beta^2 t^\mu \right) + \nonumber
\]

\[
+ \mathbb{W} \left( -\mu/2, 1; -\frac{\alpha x}{t^{\mu/2}} \right) - \exp \left( -bx \right) t^\mu \sum_{k=0}^{\infty} (-at)^k \mathcal{E}_{\mu,\mu+k+1} \left( \beta^2 t^\mu \right), \tag{26}
\]

where the parameters are given by \( \alpha^2 = 1/D_L, \beta^2 = b^2 D_L \) and \( 0 < \mu \leq 1 \).

Let us now perform a graphical analysis. For this sake, we have to choose values for some parameters appearing in the solution given by Eq. (26). We used the following values: axial dispersion coefficient \( D_L = 4.8 \times 10^{-4} \text{cm}^2 \text{s}^{-1} \) \[39\]; diffusivity coefficient of oxygen \( D = 9.8 \times 10^{-5} \text{cm}^2 \text{s}^{-1} \) \[40\]; nutrient transfer coefficient \( k = 1.5 \times 10^{-4} \text{ms}^{-1} \) \[41\]; \( a = -0.005 \times 10^{-4} \text{ms}^{-1} \) \[41\]. We also fix a time \( t = 15 \text{s} \) and we consider a certain interval \( x = [0, 4] \), which can be extended.

Figure 1: Analytic solution of fractional order PDE, Eq. (26).
In figures 1 and 2, the horizontal axis $x$ represents space and the vertical axis $y$ is the normalized concentration of nutrients in blood.

The parameter values used to plot figure 1 were also used to plot the solution of the integer order PDE, figure 2. The graphics were plotted using MATLAB 7:10 software (R2010a).

Remark that as $x$ (space) increases, the value of $C/C_1$ (concentration of nutrients) decreases, that is, when we move towards the extremity of the artery ($x=0$), the blood concentration of solute decreases. A decrease in solute concentration means that cells are not enough efficient in getting their nutrition, so we conclude that the efficiency of nutrient transport near the artery is greater than at its venous extremity.

With the freedom given to parameter $\mu$ ($0 < \mu \leq 1$), it is possible to describe more accurately the information about the concentration of nutrients near the arterial extremity because, as seen above the fractionalization of the derivative refines the solution. Note that for $\mu = 0.10$ the behavior of the analytic solution remains near the arterial ($x = 0$) for longer time. We can thus see that as $\mu \to 1$, the fractional solution converges to the solution of the integer order PDE.

We supposed that the space variable $x$ lies within the range $[0, 4]$. We might as well examine variable $x$ in the range $[0, 12]$ or any other interval; however the first representative feature is that because for $x \geq 3.8$ the level $C/C_1$ remains below the $x$ axis. So it interesting, in this context, to do analysis only on the $[0, 4]$ range.

4 Particular Case: $\mu \to 1$

In this section, we analyze the solution of the fractional PDE in the limit $\mu \to 1$, in order to recover the result found by Sharma et al. [11].
Since the solution of the fractional PDE Eq.(4) is given by Eq.(26), taking the limit \( \mu \to 1 \), it follows that

\[
C(x, t) = t \sum_{m=0}^{\infty} \left( \frac{-\alpha xt^{-1/2}}{m!} \right) \sum_{k=0}^{\infty} (-at)^{k} E_{1,k+2-m/2} \left( \beta^2 t \right) + \nonumber
\]

\[
+ \mathcal{W} \left( -1/2, 1; -\frac{\alpha x}{t^{1/2}} \right) - \exp \left( -bx \right) t \sum_{k=0}^{\infty} (-at)^{k} E_{1,k+2} \left( \beta^2 t \right). \tag{27}
\]

In the last two terms of the sum in Eq.(27), we can use the following results [42]:

\[
\mathcal{W} \left( -1/2, 1; -x \right) = \text{erfc} \left( \frac{x}{\sqrt{t}} \right)
\]

and

\[
t \sum_{k=0}^{\infty} (-at)^{k} E_{1,2+k} \left( \beta^2 t \right) = \frac{\exp \left( \beta^2 t \right) - \exp \left( -at \right)}{a + \beta^2}. \tag{28}
\]

Thus, Eq.(27) can be written as

\[
C(x, t) = t \sum_{m=0}^{\infty} \left( \frac{-\alpha xt^{-1/2}}{m!} \right) \sum_{k=0}^{\infty} (-at)^{k} E_{1,k+2-m/2} \left( \beta^2 t \right) + \nonumber
\]

\[
+ 1 + \text{erf} \left( -\alpha x/2\sqrt{t} \right) - \exp \left( -bx \right) \frac{\exp \left( \beta^2 t \right) - \exp \left( -at \right)}{a + \beta^2}. \tag{29}
\]

To recover the result presented by Sharma et al. [1], we need to express Eq.(29) in terms of \( \text{erfc}(\cdot) \) and \( \exp(\cdot) \) functions, since their solution of the convection-diffusion equation is given in terms of the product of \( \exp(\cdot) \) and \( \text{erfc}(\cdot) \) functions. With this aim, we evaluate the inverse Laplace transform in Eq.(13) using partial fractions. Taking the limit \( \mu \to 1 \) in Eq.(13), it follows that

\[
C_2(x, t) = \mathcal{L}^{-1} \left\{ \frac{\exp \left( -\alpha x \sqrt{s} \right)}{(s + a)(s - \beta^2)} \right\}. \tag{30}
\]

Using partial fractions, we rewrite \( G_a^3(s) = \frac{1}{(s+a)(s-\beta^2)} \) as follows:

\[
\frac{2 (\beta^2 + a)}{(s + a)(s - \beta^2)} = -\frac{1}{\sqrt{s}(\sqrt{s} - i\sqrt{a})} - \frac{1}{\sqrt{s}(\sqrt{s} + i\sqrt{a})} + \frac{1}{\sqrt{s}(\sqrt{s} - \beta)} + \frac{1}{\sqrt{s}(\sqrt{s} + \beta)}. \tag{31}
\]

Multiplying Eq.(31) by \( \exp \left( -\alpha x \sqrt{s} \right) \) and taking the inverse Laplace transform of both sides, we have
\[2(\beta^2 + a) \mathcal{L}^{-1} \left\{ \frac{\exp(-\alpha x \sqrt{s})}{(s + a)(s - \beta^2)} \right\} = -\mathcal{L}^{-1} \left\{ \frac{\exp(-\alpha x \sqrt{s})}{\sqrt{s}(\sqrt{s} + i\sqrt{a})} \right\} - \mathcal{L}^{-1} \left\{ \frac{\exp(-\alpha x \sqrt{s})}{\sqrt{s}(\sqrt{s} + i\sqrt{a})} \right\} + \mathcal{L}^{-1} \left\{ \frac{\exp(-\alpha x \sqrt{s})}{\sqrt{s}(\sqrt{s} + \beta)} \right\} + \mathcal{L}^{-1} \left\{ \frac{\exp(-\alpha x \sqrt{s})}{\sqrt{s}(\sqrt{s} - \beta)} \right\}. \quad (32)\]

In evaluating the inverse Laplace transforms, we use the following result \[43\]:

\[\mathcal{L}^{-1} \left\{ \frac{\exp(-k \sqrt{s})}{\sqrt{s}(\sqrt{s} + b)} \right\} = \exp(bk) \exp(b^2 t) \text{erfc} \left( b\sqrt{t} + \frac{k}{2\sqrt{t}} \right), \quad (33)\]

with \(k \geq 0, b \in \mathbb{C}\) and \(\text{erfc}(x)\) the complementary error function.

Thus, applying Eq.(33) in each term of Eq.(32), we have

\[2(\beta^2 + a) \mathcal{L}^{-1} \left\{ \frac{\exp(-\alpha x \sqrt{s})}{(s + a)(s - \beta^2)} \right\} = \exp(\beta \alpha x) \exp(\beta^2 t) \text{erfc} \left( \beta\sqrt{t} + \frac{\alpha x}{2\sqrt{t}} \right) \quad (34)\]

\[+ \exp(-\beta \alpha x) \exp(\beta^2 t) \text{erfc} \left( -\beta\sqrt{t} + \frac{\alpha x}{2\sqrt{t}} \right) \]

\[- \exp(i\alpha \sqrt{ax}) \exp(-at) \text{erfc} \left( i\sqrt{at} + \frac{\alpha x}{2\sqrt{t}} \right) \]

\[+ \exp(-i\alpha \sqrt{ax}) \exp(-at) \text{erfc} \left( -i\sqrt{at} + \frac{\alpha x}{2\sqrt{t}} \right). \]

Moreover, considering the same particular case \(x = 0\) in Eq.(34) we find the same result found for the inverse Laplace transform \(C_3(x,t)\), i.e.,

\[2(\beta^2 + a) \mathcal{L}^{-1} \left\{ \frac{1}{(s + a)(s - \beta^2)} \right\} = \exp(\beta^2 t) \left( \text{erfc} \left( \beta\sqrt{t} \right) + \text{erfc} \left( -\beta\sqrt{t} \right) \right) - \exp(-at) \left( \text{erfc} \left( i\sqrt{at} \right) + \text{erfc} \left( -i\sqrt{at} \right) \right). \quad (35)\]

Analyzing the error functions in Eq.(35), we conclude that

\[\mathcal{L}^{-1} \left\{ \frac{1}{(s + a)(s - \beta^2)} \right\} = \frac{\exp(\beta^2 t) - \exp(-at)}{\beta^2 + a}. \quad (36)\]

As we evaluated the inverse Laplace transform of \(C_2(x,t)\) in Eq.(13) in the case \(\mu = 1\) using two different procedures, i.e. Eq.(27) and Eq.(34), involving respectively a Mittag-Leffler function and error functions, we can write the interesting identity
\[ 2(a + \beta^2) t \sum_{m=0}^{\infty} \frac{(-\alpha x t^{-1/2})^m}{m!} \sum_{k=0}^{\infty} (-at)^k E_{1,2+k-m/2} (\beta^2 t) = \]

\[ = e^{\alpha x}e^{\beta^2 t} \text{erfc}(\beta \sqrt{t} + \frac{\alpha x}{2\sqrt{t}}) + e^{-\beta^2 t} \text{erfc}(\beta \sqrt{t} + \frac{\alpha x}{2\sqrt{t}}) - \]

\[ -e^{ia \sqrt{x}} e^{-at} \text{erfc}(i \sqrt{at} + \frac{\alpha x}{2\sqrt{t}}) - e^{-ia \sqrt{x}} e^{-at} \text{erfc}(-i \sqrt{at} + \frac{\alpha x}{2\sqrt{t}}). \]

(37)

Further, considering \( \alpha = 0 \) in Eq.(37), which means that only \( m = 0 \) contributes to the first sum, we obtain

\[ 2(a + \beta^2) t \sum_{k=0}^{\infty} (-at)^k E_{1,2+k} (\beta^2 t) = 2(\alpha x t^{-1/2}) \sum_{k=0}^{\infty} (-at)^k E_{1,2+k} (\beta^2 t) = \]

\[ = 2 \left( e^{\beta^2 t} \text{erfc}(\beta \sqrt{t}) + e^{\beta^2 t} \text{erfc}(\beta \sqrt{t}) \right) - \]

\[ -2 \left( e^{-at} \text{erfc}(i \sqrt{at}) - e^{-at} \text{erfc}(-i \sqrt{at}) \right) = 2(e^{\beta^2 t} - e^{-at}), \]

or, rearranging,

\[ t \sum_{k=0}^{\infty} (-at)^k E_{1,2+k} (\beta^2 t) = \frac{e^{\beta^2 t} - e^{-at}}{(a + \beta^2)}, \]

which, in fact, justifies the result given by Eq.(28). Consequently, Eq.(37) can be interpreted as a generalization of Eq.(28). Also, considering \( a = 0 \) in the previous equation, we have

\[ \beta^2 t E_{1,2} (\beta^2 t) = e^{\beta^2 t} - 1, \]

which is an identity involving the Mittag-Leffler function [38].

Finally, as we calculated the inverse Laplace transform in the case \( \mu = 1 \) by two different ways, Eq.(27) and Eq.(34), we can write the main relation we need to recover the solution proposed by Sharma et al. [1], that is, by Eq.(29) and Eq.(37):
\[ C(x, t) = \sum_{m=0}^{\infty} \frac{(-\alpha x)^{-1/2}}{m!} \sum_{k=0}^{\infty} \frac{(-at)^k E_{1, k+2-m/2}}{(\beta^2 t)^{k}} + \]
\[ + 1 + \text{erf} \left( -\alpha x/2\sqrt{t} \right) - \text{exp} \left( -bx \right) \frac{\exp (\beta^2 t) - \exp (-at)}{a + \beta^2} \]
\[ = \text{erf} \left( \frac{\alpha x}{2\sqrt{t}} \right) - \frac{\exp (-bx)}{a + \beta^2} \left( \exp (\beta^2 t) - \exp (-at) \right) \]
\[ + \frac{\exp (\beta^2 t)}{2(a + \beta^2)} \left[ \exp (\beta \alpha x \text{erfc} \left( \frac{\sqrt{t} + \alpha x}{2\sqrt{t}} \right)) + \exp (-\beta \alpha x \text{erfc} \left( -\sqrt{t} + \alpha x \right)) \right] - \]
\[ - \frac{\exp (-\alpha t)}{2(a + \beta^2)} \left[ \exp (i\alpha \sqrt{ax} \text{erfc} \left( i\sqrt{at} + \alpha x \right)) + \exp (-i\alpha \sqrt{ax} \text{erfc} \left( -i\sqrt{at} + \alpha x \right)) \right]. \quad (38) \]

We emphasize that parameters \( D \) and \( D_L \) are positive constants and \( k \geq a \), as conveniently imposed in both models, the one proposed by Sharma et al. \cite{1} and our fractional version, discussed in Section 2. Moreover, returning to the original parameters \( \beta = \sqrt{\frac{(k-a)}{D} D_L}, b = \sqrt{\frac{k-a}{D}} \), from Eq. (38), we conclude that

\[ C(x, t) = \text{erf} \left( \frac{x}{2\sqrt{D_L} t} \right) - \frac{\exp \left( -\sqrt{\frac{k-a}{D}} x \right)}{k \left( \frac{D}{D_L} \right) + a \left( 1 - \frac{D}{D_L} \right)} \left( \exp \left( \frac{(k-a)}{D} D_L t \right) - \exp (-at) \right) + \]
\[ + \frac{\exp \left( \frac{(k-a)}{D} D_L t \right)}{2 \left[ k \left( \frac{D}{D_L} \right) + a \left( 1 - \frac{D}{D_L} \right) \right]} \left[ \exp \left( \sqrt{\frac{k-a}{D}} x \right) \text{erfc} \left( \frac{x + 2\sqrt{D_L} t \sqrt{\frac{k-a}{D}}}{2\sqrt{D_L} t} \right) \right] - \]
\[ - \frac{\exp (-at)}{2 \left[ k \left( \frac{D}{D_L} \right) + a \left( 1 - \frac{D}{D_L} \right) \right]} \left[ \exp \left( \frac{i\sqrt{ax}}{\sqrt{D_L}} \text{erfc} \left( \frac{x + 2i\sqrt{D_L} t \sqrt{\frac{k-a}{D}}}{2\sqrt{D_L} t} \right) \right) + \exp \left( -i\sqrt{ax} \text{erfc} \left( \frac{x - 2i\sqrt{D_L} t \sqrt{\frac{k-a}{D}}}{2\sqrt{D_L} t} \right) \right) \right], \quad (39) \]

which is exactly the result obtained in \cite{1}.

5 Concluding Remarks

After a brief introduction to the study of the concentration of nutrients in blood, a factor that interferes with ESR, by means of a fractional mathematical model employing fractional derivatives in the Caputo sense, we obtained its analytic solution in terms of the Mittag-Leffler function and the Wright function using the methodology of Laplace transform in the time variable. We should point out that one of the greatest challenges of fractional calculus, in the study of differential equations, is to propose a fractional differential equation whose corresponding analytic solution recovers the integer order.
case in a convenient limit. Here, it was possible to recover the solution of the integer case applying the limit $\mu \to 1$ to the analytic solution, Eq. (26), of the fractional PDE, Eq. (4).

As for what was expected about the relation between the fractional mathematical model and the integer order model of [1], we can say that our fractional model provides more accurate information about the concentration of nutrients in blood.

A natural continuation of this work is to confront our fractional model with laboratory data, in order to be able to make predictions using the ESR test. Studies in this direction are being done [44] and will be published in the near future.

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