TIGHT BOUNDS ON THE PROBABILISTIC ZERO FORCING ON HYPERCUBES AND GRIDS

NATALIE C. BEHAGUE, TRENT MARBACH, AND PAWEŁ PRALAT

Abstract. Zero forcing is a deterministic iterative graph colouring process in which vertices are coloured either blue or white, and in every round, any blue vertices that have a single white neighbour force these white vertices to become blue. Here we study probabilistic zero forcing, where blue vertices have a non-zero probability of forcing each white neighbour to become blue.

In this paper we explore the propagation time for probabilistic zero forcing on hypercubes. This is a preliminary work. We are currently working on grids and some other families of graphs. Stay tuned.

1. Introduction

1.1. Definition of Zero Forcing. Zero forcing is an iterative graph colouring procedure which can model certain real-world propagation and search processes such as rumor spreading. Given a graph $G$ and a set of marked, or blue, vertices $Z \subseteq G$, the process of zero forcing involves the application of the zero forcing colour change rule in which a blue vertex $u$ forces a non-blue (white) vertex $v$ to become blue if $N(u) \setminus Z = \{v\}$, that is, $u$ forces $v$ to become blue if $v$ is the only white neighbour of $u$.

We say that $Z$ is a zero forcing set if when starting with $Z$ as the set of initially blue vertices, after iteratively applying the zero forcing colour change rule until no more vertices can be forced blue, the entire vertex set of $G$ becomes blue. Note that the order in which forces happen is arbitrary since if $u$ is in a position in which it can force $v$, this property will not be destroyed if other vertices are turned blue. As a result, we may process vertices sequentially (in any order) or all vertices that are ready to turn blue can do so simultaneously. The zero forcing number, denoted $z(G)$, is the cardinality of the smallest zero forcing set of $G$.

Zero forcing has sparked a lot of interest recently. Some work has been done on calculating or bounding the zero forcing number for specific structures such as graph products [10], graphs with large girth [6] and random graphs [11][14], while others have studied variants of zero forcing such as connected zero forcing [3] or positive semi-definite zero forcing [2].

While zero forcing is a relatively new concept (first introduced in [10]), the problem has many applications to other branches of mathematics and physics. For example, zero forcing can give insight into linear and quantum controllability for systems that stem from networks. More precisely, in [4], it was shown that for both classical and...
quantum control systems that can be modelled by networks, the existence of certain zero forcing sets guarantees controllability of the system.

Another area closely related to zero forcing is power domination [17]. Designed to model the situation where an electric company needs to continually monitor their power network, one method that is used is to place phase measurement units (PMUs) periodically through their network. To reduce the cost associated with this, one asks for the least number of PMUs necessary to observe a specific network fully. To be more specific, given a network modelled with a simple graph \( G \), a PMU placed at a vertex will be able to observe every adjacent vertex. Furthermore, if an observed vertex has exactly one unobserved neighbour, this observed vertex can observe this neighbour. In this way, power domination involves an observation rule compatible with the zero forcing colour change rule.

In the present paper we are concerned with a parameter associated with zero forcing known as the propagation time, which is the fewest number of rounds necessary for a zero forcing set of size \( z(G) \) to turn the entire graph blue. More formally, given a graph \( G \) and a zero forcing set \( Z \), we generate a finite sequence of sets \( Z_0 \subset Z_1 \subset \ldots \subset Z_t \), where \( Z_0 = Z \), \( Z_t = V(G) \), and given \( Z_i \), we define \( Z_{i+1} = Z_i \cup Y_i \), where \( Y_i \subseteq V(G) \setminus Z_i \) is the set of white vertices that can be forced in the next round if \( Z_i \) is the set of the blue vertices. Then the propagation time of \( Z \), denoted \( pt(G, Z) \), is defined to be \( t \). The propagation time of the graph \( G \) is then given by \( pt(G) = \min_Z pt(G, Z) \), where the minimum is taken over all zero forcing sets \( Z \) of cardinality \( z(G) \). The propagation time for zero forcing has been studied in [11].

1.2. Definition of Probabilistic Zero Forcing. Zero forcing was initially formulated to bound a problem in linear algebra known as the min-rank problem [10]. In addition to this application to mathematics, zero forcing also models many real-world propagation processes. One specific application of zero forcing could be to rumor spreading, but the deterministic nature of zero forcing may not fit the chaotic nature of real-life situations. As such, probabilistic zero forcing has also been proposed and studied where blue vertices have a non-zero probability of forcing white neighbours, even if there is more than one white neighbour. More specifically, given a graph \( G \), a set of blue vertices \( Z \), and vertices \( u \in Z \) and \( v \in V(G) \setminus Z \) such that \( uv \in E(G) \), in a given time step, vertex \( u \) will force vertex \( v \) to become blue with probability

\[
P(u \text{ forces } v) = \frac{|N[u] \cap Z|}{\deg(u)},
\]

where \( N[u] \) is the closed neighbourhood of \( u \).

In a given round, each blue vertex will attempt to force each white neighbour independently. If this happens, we may say that the edge \( uv \) is forced. A vertex becomes blue as long as it is forced by at least one blue neighbour, or in other words if at least one edge incident with it is forced. Note that if \( v \) is the only white neighbour of \( u \), then with probability 1, \( u \) forces \( v \), so given an initial set of blue vertices, the set of vertices forced via probabilistic zero forcing is always a superset of the set of vertices forced by traditional zero forcing. In this sense, probabilistic zero forcing and traditional zero forcing can be coupled. In the context of rumor spreading, the probabilistic colour
change rule captures the idea that someone is more likely to spread a rumor if many of their friends have already heard the rumor.

Under probabilistic zero forcing, given a connected graph, it is clear that starting with any non-empty subset of blue vertices will with probability 1 eventually turn the entire graph blue, so the zero forcing number of a graph is not an interesting parameter to study for probabilistic zero forcing. Initially in [15], the authors studied a parameter that quantifies how likely it is for a subset of vertices to become a traditional zero forcing set the first time-step that it theoretically could under probabilistic zero forcing.

Instead, in this paper, we will be concerned with a parameter that generalizes the zero forcing propagation time. This generalization was first introduced in [9]. Given a graph $G$, and a set $Z \subseteq V(G)$, let $pt_{pzf}(G,Z)$ be the random variable that outputs the propagation time when probabilistic zero forcing is run with the initial blue set $Z$. For ease of notation, we will write $pt_{pzf}(G,v) = pt_{pzf}(G,\{v\})$. The propagation time for the graph $G$ will be defined as the random variable $pt_{pzf}(G) = \min_{v \in V(G)} pt_{pzf}(G,v)$. More specifically, $pt_{pzf}(G)$ is a random variable for the experiment in which $n$ iterations of probabilistic zero forcing are performed independently, one for each vertex of $G$, then the minimum is taken over the propagation times for these $n$ independent iterations.

1.3. Asymptotic Notation. Our results are asymptotic in nature, that is, we will assume that $n \to \infty$. Formally, we consider a sequence of graphs $G_n = (V_n, E_n)$ (for example, $G_n$ is an $n$-dimensional hypercube or an $n$ grid) and we are interested in events that hold asymptotically almost surely (a.a.s.), that is, events that hold with probability tending to 1 as $n \to \infty$.

Given two functions $f = f(n)$ and $g = g(n)$, we will write $f(n) = O(g(n))$ if there exists an absolute constant $c \in \mathbb{R}_+$ such that $|f(n)| \leq c|g(n)|$ for all $n$, $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$, $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, and we write $f(n) = o(g(n))$ or $f(n) \ll g(n)$ if $\lim_{n \to \infty} f(n)/g(n) = 0$. In addition, we write $f(n) \gg g(n)$ if $g(n) = o(f(n))$ and we write $f(n) \sim g(n)$ if $f(n) = (1 + o(1))g(n)$, that is, $\lim_{n \to \infty} f(n)/g(n) = 1$.

Finally, as typical in the field of random graphs, for expressions that clearly have to be an integer, we round up or down but do not specify which: the choice of which does not affect the argument.

1.4. Results on Probabilistic Zero Forcing. In [9], the authors studied probabilistic zero forcing, and more specifically the expected propagation time for many specific structures. A summary of this work is provided in the following theorem.

Theorem 1.1. [9] Let $n > 2$. Then

$$
\begin{align*}
\min_{v \in V(P_n)} \mathbb{E}(pt_{pzf}(P_n,v)) &= \begin{cases} 
n/2 + 2/3 & \text{if } n \text{ is even} \\
n/2 + 1/2 & \text{if } n \text{ is odd}, \end{cases} \\
\min_{v \in V(C_n)} \mathbb{E}(pt_{pzf}(C_n,v)) &= \begin{cases} 
n/2 + 1/3 & \text{if } n \text{ is even} \\
n/2 + 1/2 & \text{if } n \text{ is odd}, \end{cases} \\
\min_{v \in V(K_{1,n})} \mathbb{E}(pt_{pzf}(K_{1,n},v)) &= \Theta(\log n), \\
\Omega(\log \log n) &= \min_{v \in V(K_n)} \mathbb{E}(pt_{pzf}(K_n,v)) = O(\log n).
\end{align*}
$$


In [5], the authors used tools developed for Markov chains to analyze the expected propagation time for many small graphs. The authors also showed, in addition to other things, that \(\min_{v \in V(K_n)} \mathbb{E}(pt_{pzf}(K_n, v)) = \Theta(\log \log n)\) and for any connected graph \(G\), \(\min_{v \in V(G)} \mathbb{E}(pt_{pzf}(G, v)) = O(n)\). This result was then improved in [16], where the authors showed that

\[
\log_2 \log_2(n) \leq \min_{v \in V(G)} \mathbb{E}(pt_{pzf}(G, v)) \leq \frac{n}{2} + o(n)
\]

for general connected graphs \(G\). In the same paper, the authors also showed that

\[
\min_{v \in V(G)} \mathbb{E}(pt_{pzf}(G, v)) = O(n^{2 + o(1)}), \tag{1.1}
\]

where \(r \geq 1\) denotes the radius of the connected graph \(G\). Moreover, they provided a class of graphs for which the bounds of the theorem are tight.

In addition to the results mentioned above, the authors of [9] also considered the binomial random graph \(G(n, p)\), proving the following theorem.

**Theorem 1.2.** [9] Let \(0 < p < 1\) be constant. Then a.a.s. we have that

\[
\min_{v \in V(G(n,p))} \mathbb{E}(pt_{pzf}(G(n,p), v)) = O((\log n)^2).
\]

However, the authors in [16] conjectured that for the random graph, a.a.s.

\[
\min_{v \in V(G(n,p))} \mathbb{E}(pt_{pzf}(G(n,p), v)) = (1 + o(1)) \log \log n.
\]

Of course, Theorem 1.2 can be improved immediately via (1.1) and the fact that for \(0 < p < 1\) constant, the radius of \(G(n, p)\) is a.a.s. 2 (see e.g. [8]), but even with this improvement, the bound is still far from the conjectured value. In [7], the authors explored probabilistic zero forcing on \(G(n, p)\) in more detail and, in particular, proved the above conjecture. Their results can be summarized in the following theorem that shows that probabilistic zero forcing occurs much faster in \(G(n, p)\) than in a general graph \(G\), as evidenced by the bounds of (1.1).

**Theorem 1.3.** [7] Let \(v \in V(G(n,p))\) be any vertex of \(G(n, p)\). If \(p = \log^{-o(1)} n\) (in particular, if \(p\) is a constant), then a.a.s.

\[
pt_{pzf}(G(n,p), v) \sim \log_2 \log_2 n.
\]

On the other hand, if \(\log n/n \ll p = \log^{-\Omega(1)} n\), then a.a.s.

\[
pt_{pzf}(G(n,p), v) = \Theta(\log(1/p)).
\]

The results of most interest to us are from [12]. The authors of that paper are concerned with hypercubes \(Q_n\) and grids \(G_{m \times n}\), and prove the following result.

**Theorem 1.4.** [12] The following bounds hold:

- \(\min_{v \in V(Q_n)} \mathbb{E}(pt_{pzf}(Q_n, v)) = O(n \log n)\),
- \((1/2 - o(1))(m + n) \leq \min_{v \in V(G_{m \times n})} \mathbb{E}(pt_{pzf}(G_{m \times n}, v)) \leq (4 + o(1))(m + n)\).
1.5. **Our Results.** In this paper, we improve both results from [12]. Instead of considering the expectation of the propagation time, we will calculate bounds on the propagation time that a.a.s. hold.

**Theorem 1.5.** The following bounds hold a.a.s.:

(a) \( pt_{pzf}(Q_n) \sim n \),

(b) \( pt_{pzf}(G_{m \times n}) \sim ? \) (stay tuned).

2. **Preliminaries**

2.1. **Concentration Inequalities.** Let us first state a specific instance of Chernoff’s bound that we will find useful. Let \( X \in \text{Bin}(n, p) \) be a random variable with the binomial distribution with parameters \( n \) and \( p \). Then, a consequence of Chernoff’s bound (see e.g. [13, Corollary 2.3]) is that

\[
\Pr(|X - \mathbb{E}X| \geq \varepsilon \mathbb{E}X) \leq 2 \exp \left( -\frac{\varepsilon^2 \mathbb{E}X}{3} \right)
\]

for \( 0 < \varepsilon < 3/2 \). Moreover, let us mention that the bound holds for the general case in which \( X = \sum_{i=1}^{n} X_i \) and \( X_i \in \text{Bernoulli}(p_i) \) with (possibly) different \( p_i \) (again, e.g. see [13] for more details).

2.2. **Useful Coupling.** We will be using the following obvious coupling to simplify both our upper and lower bounds. Indeed, for lower bounds, it might be convenient to make some white vertices blue at some point of the process. Similarly, for upper bounds, one might want to make some blue vertices white. Given a graph \( G, S, T \subseteq V(G) \), and \( \ell \in \mathbb{N} \), let \( A(S, T, \ell) \) be the event that starting with blue set \( S \), after \( \ell \) rounds every vertex in \( T \) is blue. The following simple observation was proved in [7].

**Lemma 2.1 ([7]).** For all sets \( S_1 \subseteq S_2 \subseteq V(G) \), \( T \subseteq V(G) \), and \( \ell \in \mathbb{N} \),

\[
\Pr(A(S_1, T, \ell)) \leq \Pr(A(S_2, T, \ell)).
\]

3. **Hypercubes**

The whole section is devoted to prove part (a) of Theorem 1.5.

Let us start with a formal definition of the hypercube. The \( n \)-dimensional hypercube \( Q_n \) has vertex set consisting of all binary strings of length \( n \) and there is an edge between two vertices if and only if their binary strings differ in exactly one bit. For \( 0 \leq k \leq n \), level \( k \) of the hypercube \( Q_n \) is defined to be the set of all vertices whose binary strings contain exactly \( k \) ones. Note that each vertex in level \( k \) has \( k \) neighbours in level \( k - 1 \) and \( n - k \) neighbours in level \( k + 1 \).

**Proof of Theorem 1.5(a).** Trivially, for any vertex \( v \in V(Q_n) \), we deterministically have that \( pt_{pzf}(Q_n, v) \geq n \). Hence, in order to show that \( pt_{pzf}(Q_n) \sim n \), it is enough to show that a.a.s. \( pt_{pzf}(Q_n, v) \leq n + o(n) \) for some vertex \( v \in V(Q_n) \). Let \( v \) be the vertex \((0, 0, \ldots, 0)\) on level 0; in fact, since \( Q_n \) is a vertex-transitive graph \( v \) can be any vertex.

Suppose that for some integer \( k \), \( 0 \leq k < n/\log n \), the following property \( \mathcal{P}(k) \) holds at some point of the process: all vertices at levels up to \( k \) are blue (including level \( k \)).
We will show that with probability $1-o(n^{-1})$ property $P(k+1)$ holds after an additional $O(\ln n)$ rounds. Since trivially $P(0)$ holds, by the union bound (over $n/\ln^2 n$ possible values of $k$) we will conclude that a.a.s. $P(n/\ln^2 n)$ holds after $O(n/\ln n) = o(n)$ rounds.

Suppose that property $P(k)$ holds from some $k$, $0 \leq k < n/\ln^2 n$. By Lemma 2.1 we may assume that all vertices at level $k+1$ are white. It will be convenient to independently consider 3 phases after which property $P(k+1)$ will hold with probability $1 - o(n^{-1})$. The first phase lasts $320 \ln n$ rounds. The probability that a given vertex at level $k+1$ stays white during this phase is at most

$$\left(1 - \frac{k+1}{n}\right)^{320 \ln n} \leq \exp\left(-\frac{320(k+1) \ln n}{n}\right) \leq 1 - \frac{160(k+1) \ln n}{n} =: p,$$

since $k \leq n/\ln^2 n$. Hence, for a given vertex $w$ at level $k$, the number of neighbours at level $k+1$ that turned blue during this phase is stochastically lower bounded by

$$\text{Bin}(n-k,1-p) \geq \text{Bin}(n/2,1-p) \ni X \text{ with } E[X] = (1-p)n/2 = 80(k+1) \ln n.$$

It follows from Chernoff’s bound (2.1) (applied with $\varepsilon = 1/2$) that $w$ has at most $40(k+1) \ln n$ blue neighbours at level $k+1$ with probability at most

$$2\exp\left(-\frac{E[X]}{12}\right) = \frac{2}{n^{(80/12)(k+1)}} = o\left(\frac{1}{n^{k+1}}\right).$$

By the union bound (over at most $n^k$ vertices at level $k$), with probability $1-o(n^{-1})$, each vertex at level $k$ has at least $40(k+1) \ln n$ blue neighbours at level $k+1$ at the end of the first phase. As we aim for a statement that holds with probability $1 - o(n^{-1})$, we may assume that this property holds once we enter the second phase.

The second phase lasts $\log_{5/4} n$ rounds. As before, let us concentrate on a given vertex $w$ at level $k$. Suppose that $w$ has $\ell$ blue neighbours for some integer $\ell$ such that $40(k+1) \ln n \leq \ell \leq n/2$ ($w$ has only $k$ neighbours at level $k-1$ so, of course, it includes neighbours at level $k+1$). The number of white neighbours of $w$ (at level $k+1$) that turned blue in one round of the process is stochastically lower bounded by

$$\text{Bin}(n-\ell, (\ell+1)/n) \geq \text{Bin}(n/2, (\ell+1)/n) \ni Y \text{ with } E[Y] = (\ell+1)/2 \geq 20(k+1) \ln n.$$

We get from Chernoff’s bound (2.1) (applied with $\varepsilon = 1/2$) that $Y \leq 10(k+1) \ln n$ with probability at most

$$2\exp\left(-\frac{E[Y]}{12}\right) = \frac{2}{n^{(20/12)(k+1)}} = o\left(\frac{1}{n^{k+1} \ln n}\right).$$

By the union bound (over at most $n^k$ vertices at level $k$ and at most $\log_{5/4} n$ rounds), with probability $1-o(n^{-1})$, each vertex at level $k$ increases the number of blue neighbours by a multiplicative factor of $5/4$ each round, reaching at least $n/2$ blue neighbours by the end of the second phase. We may assume that this property holds once we enter the third phase.

The third (and last) phase lasts $3 \log_2 n$ rounds. This time, let us concentrate on a given white vertex $w$ at level $k+1$. This vertex has $k+1$ neighbours at level $k$, each of which has at least $n/2$ blue neighbours. Hence, vertex $w$ stays white by the end of
this phase with probability at most
\[
\left(\frac{1}{2}\right)^{k+1} \cdot 3\log_2 n = 2^{-3k+2} \cdot \frac{1}{n^{k+1}} = o\left(\frac{1}{n^{k+2}}\right).
\]

By the union bound (over at most \(n^{k+1}\) vertices at level \(k+1\)), with probability \(1 - o(n^{-1})\), all vertices at level \(k+1\) turn blue by the end of this phase and so property \(P(k+1)\) holds.

Since we aim for a statement that holds a.a.s., we may assume that \(P(n/\ln^2 n)\) holds after \(o(n)\) rounds, and continue the process from there. We say that a vertex at layer \(k\) is \textit{happy} if all but at most \(\ell = \ell(n) = \ln^5 n\) neighbours at layer \(k-1\) are blue. Similarly, a vertex at layer \(k\) is \textit{very happy} if not only it is happy but also all but at most \(\ell\) neighbours at layer \(k+1\) are blue. (Note that happy or even very happy vertex might still be white.) Trivially, all vertices at levels up to \(n/\ln^2 n + 1\) are happy (including level \(n/\ln^2 n + 1\)), and all vertices at levels before level \(n/\ln^2 n\) are very happy.

Suppose that for some integer \(k\), \(n/\ln^2 n \leq k \leq n-1\), all vertices at levels up to \(k\) are happy (including level \(k\)). We will show that after one single round all vertices at layer \(k+1\) are going to be happy and all vertices at layer \(k-1\) are going to be very happy with probability \(1 - o(n^{-1})\). By the union bound (over all possible values of \(k\)), we will get that a.a.s. after less than \(n\) rounds all vertices of the hypercube are going to be very happy.

For simplicity, by Lemma 2.1 we may assume that all vertices at level \(k\) are white (despite the fact that they are happy). Let us concentrate on a given vertex \(w\) at level \(k\). Since \(w\) is happy and all of its blue neighbours at level \(k-1\) are happy too, the probability that \(w\) stays white is at most
\[
\left( 1 - \frac{k-\ell}{n} \right)^{k-\ell} \leq \exp\left( -\left(1 + o(1)\right) \frac{k^2}{n} \right) =: p.
\]

Now, the probability that a given vertex at level \(k+1\) is \textit{not} happy is at most
\[
\binom{k+1}{\ell} p^\ell \leq n^\ell \exp\left( -\left(1 + o(1)\right) \frac{k^2}{n} \right)
= \exp\left( \ell \ln n - \left(1 + o(1)\right) \frac{k^2}{n} \right)
\leq \exp\left( \ln^6 n - \left(1 + o(1)\right) n \ln n \right)
= o\left( \frac{1}{2^{\ell/n}} \right).
\]

The desired property holds by the union bound (over at most \(2^n\) vertices at level \(k+1\)). Vertices at levels below level \(n-\ell\) (including level \(n-\ell\)) are trivially very happy. If \(k \leq n-\ell\), then a given vertex at level \(k-1\) is \textit{not} very happy with probability at most
\[
\binom{n-(k-1)}{\ell} p^\ell \leq n^\ell \exp\left( -\left(1 + o(1)\right) \frac{k^2}{n} \right) = o\left( \frac{1}{2^{\ell/n}} \right),
\]
and the desired bound holds by the union bound (over at most $2^n$ vertices at level $k - 1$).

At this point we may assume that all vertices of the hypercube are very happy. It is easy to see that all remaining white vertices will turn blue in one single round. Indeed, since all vertices are very happy, the probability that some vertex stays white is at most

$$2^n \left( 1 - \frac{n - 2\ell}{n} \right)^{n-2\ell} = 2^n \left( \frac{2\ell}{n} \right)^{(1+o(1))n} = \exp (O(n) - (1 + o(1))n \ln n) = o(1).$$

The proof is finished. □

REFERENCES

[1] D. Bal, P. Bennett, S. English, C. MacRury, and P. Prałat. Zero forcing number of random regular graphs. arXiv:1812.06477, 2018.
[2] F. Barioli, W. Barrett, S. Fallat, H. Hall, L. Hogben, B. Shader, P. van den Driessche, and H. van der Holst. Zero forcing parameters and minimum rank problems. Linear Algebra Appl., 433(2):401–411, 2010.
[3] B. Brimkov and I. Hicks. Complexity and computation of connected zero forcing. Discrete Appl. Math., 229:31–45, 2017.
[4] D. Burgarth, D. D’Alessandro, L. Hogben, S. Severini, and M. Young. Zero forcing, linear and quantum controllability for systems evolving on networks. IEEE Trans. Automat. Control, 58(9):2349–2354, 2013.
[5] Y. Chan, E. Curl, J. Geneson, L. Hogben, K. Liu, I. Odegard, and M. Ross. Using markov chains to determine expected propagation time for probabilistic zero forcing. arXiv:1906.11083, 2019.
[6] R. Davila and F. Kenter. Bounds for the zero forcing number of graphs with large girth. Theory Appl. Graphs, 2(2):Art. 1, 8, 2015.
[7] Sean English, Calum MacRury, and Paweł Prałat. Probabilistic zero forcing on random graphs. European J. Combin., 91:103207, 22, 2021.
[8] A. Frieze and M. Karoński. Introduction to random graphs. Cambridge University Press, Cambridge, 2016.
[9] J. Geneson and L. Hogben. Propagation time for probabilistic zero forcing. arXiv:1812.10476, 2018.
[10] AIM Minimum Rank-Special Graphs Work Group. Zero forcing sets and the minimum rank of graphs. Linear Algebra Appl., 428(7):1628–1648, 2008.
[11] L. Hogben, M. Huynh, N. Kingsley, S. Meyer, S. Walker, and M. Young. Propagation time for zero forcing on a graph. Discrete Appl. Math., 160(13-14):1994–2005, 2012.
[12] David Hu and Alec Sun. Probabilistic zero forcing on grid, regular, and hypercube graphs. arXiv:2010.12343, 2020.
[13] S. Janson, T. Łuczak, and A. Rucinski. Random graphs. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.
[14] T. Kalinowski, N. Kamčev, and B. Sudakov. The zero forcing number of graphs. SIAM J. Discrete Math., 33(1):95–115, 2019.
[15] C. Kang and E. Yi. Probabilistic zero forcing in graphs. Bull. Inst. Combin. Appl., 67:9–16, 2013.
[16] S. Narayanan and A. Sun. Bounds on expected propagation time of probabilistic zero forcing. arXiv:1909.04482, 2019.
[17] M. Zhao, L. Kang, and G. Chang. Power domination in graphs. Discrete Math., 306(15):1812–1816, 2006.
