Hyperbolic geometry in general education: comprehending the incomprehensible

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Abstract. This paper is a self-independent continuation of my article on Comparative geometry between the plane and the sphere that was presented at the previous edition of ICon-MaSTEd Conference 2020. Below I discuss the possibility of adding a third geometry to the plane and the sphere, namely, the hyperbolic geometry on the hemisphere. I describe my own path to the subject, then the content of the syllabus which contains basic concepts of hyperbolic geometry for future preschool, elementary school, and secondary school teachers. Finally, I give reasons to introduce the subject into primary and secondary schools, not just for the “talented” but also for the “average” students.

1. Introduction

In the 2020 edition of ICon-MaSTEd Conference (XII International Conference on Mathematics, Science and Technology Education) I described the Comparative Geometry project, its underlying ideas and my experiences applying it in distance education [1].

The fundamental idea of the project is to teach and learn two (or later more) different systems of geometry simultaneously for a wide range of age groups. Students compare and contrast concepts and theorems in two or more different worlds of geometry, as suggested, for example, in the book of Henderson and Taimina for university students [2], or my book for the upper elementary and secondary school [3], or science popularization books like Van Brummelen’s work [4].

Instead of a Euclidean monologue, we use a drama-play between different geometric systems. It is the drama, the comparison and contrast that make geometry interesting. It gives the student the opportunity to build his own geometry and live through the torment and beauty of creation.

In last year’s article [1], the two systems I compared were the geometry of the planar surface and the spherical surface. The essence of the present article is to deal with the basics of a third system in the classroom, but only after the introduction of the first two systems. The third system is the hyperbolic, Gauss-Bolyai-Lobachevskian geometry. In the conclusions I summarize my reasons for this proposal. Detailed description of comparing the three geometries in education is found in the book of Rybak and Lénárt [5].

I do not describe a planned and analytically evaluated experiment, but an action research, a set of experiences I gained during decades of teaching comparative geometry.
As I mentioned in my presentation last year (and repeat here to make the background and conditions clear) I have been giving college and university courses in comparative geometry since 1990, mainly for prospective kindergarten teachers, elementary school teachers, and high school teachers, occasionally future mathematicians and geographers too.

The mathematical knowledge of the target audience has been extremely diverse. For quite a few students, basic concepts of plane geometry are only (vaguely) memorized but not interiorized. In contrast, other students have profound competence up to calculus and higher algebra. Spherical geometry was practically unknown for most of them. Several times I have come across students who are familiar with calculus or set theory but do not know how to draw a straight line on the sphere.

Hyperbolic geometry was completely unknown to the vast majority of students. Even the few who heard of the subject remembered it with sacred horror much more than enthusiasm.

In the first decade of my teaching about comparative geometry, I only dealt with plane geometry versus spherical geometry in the courses. For a long time, I didn’t consider hyperbolic geometry as part of the material. It seemed much more difficult and formidable than spherical geometry. Besides, I did not feel my own pedagogical and demonstrational tools or even my mathematical competences adequate for the task.

It was not until around 2000 that I started incorporating hyperbolic geometry into the syllabus. Some groups completed the original syllabus earlier than others. There were still a few hours left to deal with a new topic, a third system of geometry.

However, it was not at all easy, even for high-performing groups, to keep up with the new concepts. Major goals like creating interest or removing inferiority complexes have often been hampered by the acceptance of a very new, very different approach to geometry.

For that reason, I gradually changed my strategy. The new material about hyperbolic geometry was sporadically inserted into the main syllabus about the plane and the sphere. From time to time I hinted at the third option beyond plane geometry and spherical geometry. This method proved to be more effective, less tiring, and more suitable for arousing interest in the students.

When we still have time at the end of the semester and the students are curious about the topic, we sketch the basic concepts of hyperbolic geometry in 3-4 consecutive lessons, referring to the related concepts of planar geometry and spherical geometry all the time.

### 2. My way to hyperbolic geometry

I was fully empathetic to my students’ problems because my own path to hyperbolic geometry was also far from smooth.

My first attempt to get acquainted with the subject was the booklet “Scientia Spatii” (The Science of Space) [6], the classical work of the Hungarian János Bolyai, one of the discoverers of hyperbolic geometry. To my great disappointment, I found it incomprehensible at first reading. Whenever I came across the word “obvious”, I suspected that a statement would follow which was far beyond my competence.

I add that Bolyai’s work became understandable and fascinating later on, by the hemispherical model. I could even use some parts of it with my students in the classroom.

I read excellent references on the subject including Greenberg [7], Szőkefalvi-Nagy [8], and others who applied some versions of the flat disc model, or the half-plane model of Poincaré [9]. I understood more or less what they were saying but I didn’t like the subject.

Some other sources discussed the topic on a purely algebraic basis which I could grasp, but could not imagine how a human being came to axioms so far from my perception and common sense.

Later on, I looked into outstanding software materials that also dealt with the flat disk model of hyperbolic geometry, like Cinderella [10], dynamic geometry of Szilassi on Geogebra [11] or
Non-Euclid [12]. These software materials were very useful, but, again, only after grasping the basic concepts on the hemispherical model, or when I tried to prove or refute the validity of an assumption in hyperbolic geometry.

It was Poincaré’s hemispherical model in Horváth’s book [13] that opened up the path of hyperbolic geometry. Concepts that could not be understood on other models became clear and illustrative in this way. My previous experience with spherical geometry has been of great help in exploring new hyperbolic concepts.

Even so, I kept on drawing and constructing on palpable hemispheres. The hemispherical model gave the inspiration and self-confidence to teach the material and even start my own research on hyperbolic geometry.

3. Summary of the content of the syllabus
Below I describe the main topics, illustrations and wordings by which I tried to make the basic concepts accessible to my students.

In order to illustrate the constructions, I use round-shaped goods (fruits, ball, globes, etc.) and spherical construction materials [14, 15]. Round-shaped commodities come in handy in distance education.

3.1. Surface and basic elements:
Planar geometry is built on the surface of the infinite plane, and spherical geometry on the finite sphere. What kind of surface could be used to create a third type of geometry that admits points, straight lines, polygons, circles, etc., but differs in many respects from plane geometry or spherical geometry?

There are several models on different surfaces on which the new geometry can be built. In what follows, I mainly use the surface of an open hemisphere whose equator does not belong to the model.

The simplest element of this geometry is the point, just as in plane geometry or spherical geometry.

![Figure 1. Points on the hemisphere.](image1)

![Figure 2. Intersecting vs. non-intersecting curves.](image2)

Figure 1 shows points as basic elements of hyperbolic geometry on the open hemisphere. Curves a and b on figure 2 cross in a hyperbolic point, but curves a and c do not, since the point of intersection is on the bordering equator which does not belong to the model.

What to choose for the simplest line, the hyperbolic straight line? The Euclidean straight line is out of question, because it has only one point in common with the hemisphere. The spherical
great circle fits on the surface, but this choice would take back to the well-known system of spherical geometry.

Instead, choose the spherical semi-circles which are perpendicular to the omitted equator.

Figure 3. Straight line on the plane. Figure 4. Straight line on the sphere. Figure 5. Straight line on the hemisphere.

The straight lines on different surfaces can be illustrated in several ways, as with a planar ruler on figure 3, the cutting line of a halved apple on 4, or of a sliced onion on 5.

My experience is that the most difficult transition in comparative geometry is the change from the planar straight line to the spherical straight line or great circle and especially from the planar and spherical straight lines to their hyperbolic counterpart.

For that reason, it is very useful to invent exercises for distinguishing hyperbolic straight lines from other hyperbolic lines, especially those lines which correspond to certain spherical circles.

- All lines on figure 6 are hyperbolic straight lines.
- No line is hyperbolic straight line on figure 7.
- Only the longitudes are hyperbolic straight lines or segments on figure 8.
- On figure 9 all latitudes are hyperbolic straight lines, plus one longitude through the pole point at the top of the hemisphere.

Following are some fundamental properties of the hyperbolic straight line which are extremely hard to grasp from the Euclidean or spherical perspective:

- Each straight line is infinite, because the endpoints are on the omitted equator, that is, the straight line has no endpoint within the open hemisphere.
- Any two are always congruent with each other, even if they seem ‘large’ or ‘small’ circles in spherical geometry.
- Each straight line divides the surface into two congruent halves, even if the two regions seem very different in size from the spherical perspective.

3.2. Lines through two points

- On the plane, two points share one straight line that passes through them.
- On the sphere, if two points are not opposite, they have exactly one straight line / great circle through them. If they are opposite, there are infinitely many straight lines through them.
- On the hemisphere, any two points have always one straight line passing through them, just as with the plane. (At this initial stage, I am not confusing students with saying that one or even both points could be on the omitted equator. In other words, the statement remains valid if we include ideal points on the equator.)
3.3. Common points of two lines

- On the plane, two straight lines share one common point if they are intersecting, or no common point, if they are parallel.
- On the sphere, two straight lines / great circles always intersect in two opposite points. Parallel straight lines / great circles do not exist on the sphere.
- On the hemisphere, the picture is different from the plane or the sphere. Two straight lines may or may not intersect. If they are intersecting, they have one point in common. If they are not intersecting, they may or may not meet at the same point of the omitted equator. We agree that we call two hyperbolic straight lines parallel lines (also called asymptotic lines) if they meet at a point of the omitted equator, that is, in an ideal point. In all other cases of non-intersecting hyperbolic straight lines, we call them skew straight lines. This latter case cannot happen on the plane or on the sphere where skew straight lines do not exist.
- Figure 10 shows two intersecting lines.
- Figure 11 shows two parallel (asymptotic) straight lines.
- Figure 12 shows two skew lines.
- Figure 13 also shows skew straight lines. One is tempted to call them parallels from the spherical perspective, but these lines do not meet at the same ideal point.

3.4. Pencils of straight lines

On the plane, there are two types of pencils of straight lines:

- All straight lines passing through a point.
- All straight lines parallel with each other.
Figure 10. Intersecting hyperbolic straight lines.

Figure 11. Parallel hyperbolic straight lines.

Figure 12. Skew hyperbolic straight lines.

Figure 13. Skew hyperbolic straight lines.

On the sphere, there is only one type of pencils:

- All great circles through two opposite points, like the longitudes through the North and South Poles.

On the hemisphere, there are in fact three types of pencils, but we only consider two types, because the third one is too alien for the beginner:

- All the straight lines passing through a point.
- All the straight lines parallel with each other, that is, passing through the same point of the omitted equator.

An example of a hyperbolic pencil passing through a point is all the longitudes on the Northern Hemisphere passing through the North Pole. However, if we mark a point not on the top of the hemisphere, we get a very unusual shape of a pencil, reminding of a spider’s web. In the second case, we get a pencil of parallel lines, which, in turn, reminds me of a fountain. (These childish analogies make easier to grasp and remember the concept.)

These two types of pencils are apparently different from the spherical perspective, but are indistinguishable from the hyperbolic point of view. This is another trait that is very far from what we are used to on the plane or on the sphere, so I do not bother my students with it.
3.5. The Parallel Postulate
Consider a pencil of straight lines, and another straight line $L$ that does not belong to the pencil.

How many straight lines of the pencil do not intersect straight line $L$?

- On the plane, there are infinitely many intersecting straight lines, and only one non-intersecting line in the pencil.
- On the sphere, there are infinitely many intersecting straight lines, and no non-intersecting line in the pencil.
- On the hemisphere, the picture is far more complex. There are infinitely many intersecting straight lines, and also infinitely many non-intersecting straight lines in the pencil. The two sets of lines are separated from each other by two “guards”, two non-intersecting lines which are called parallel or asymptotic lines.

It is interesting to mention how Bolyai himself came up with the basic idea. He was unaware of the hyperbolic models described in this article, so he imagined the scene on a flat sheet as the lines of the pencil were rotating around the common point of intersection. One of the rotating lines in a certain position bounces off the outer line, and the rotation continues until another rotating line intersects the outer line again. On the Euclidean plane, the bouncing-off
line is the same as the bouncing-back line, but what if we assume that there are infinitely many non-intersecting lines between the two?

These detours are very important and instructive. The learner’s main concern is how he or she would have come to the same idea. The spark of thought that led Bolyai to his discovery makes the seemingly superhuman achievement understandable for the learner.

3.6. Measuring hyperbolic distance
I usually omit this topic, because it is too complex for the beginner. I only try to explain the gist of the problem when students specifically ask for an explanation. The following is a summary of my interpretation when it is explicitly required by the students.

The problem is that the points on the equator of the hemispherical model are out of reach of our measurement, because these points do not belong to the hyperbolic surface. As we move forward along a straight line from an inside point to an ideal point (a point on the equator), our steps appear shorter and shorter for an outer observer.

Another problem is that all hyperbolic straight lines are congruent, although they may appear different in size from the spherical perspective. If we want to make a scale of measurement of hyperbolic distance (that is, to create a hyperbolic ruler), it has to fit any hyperbolic straight line.

![Figure 20. On the plane, equal units seem equal for the outer observer.](image1)

![Figure 21. On the sphere, equal units seem equal for the outer observer.](image2)

![Figure 22. On the hemisphere, equal units seem shorter when approaching the equator.](image3)

Still another problem is that we would like to define hyperbolic distance and angle so that the wonderful concord between the sides and angles of the isosceles triangle remains valid on the hemispherical model. Equal sides subtend to equal angles, and conversely (“Pons Asinorum”).

The proper definition of distance involves the concept of cross-ratio. It is not too hard, but too lengthy for the beginner. Figure 23 and figure 24 show a possible scale for the purpose, but, again, I usually omit it altogether from the introduction.

3.7. Measuring hyperbolic angle
The situation is much simpler than measuring distance. If two angles appear to be the same in the hemispherical model, they are in fact the same. In contrast to distance measurement, we can trust our eyes when measuring the angle.

Measuring angle on the plane and on the sphere does not need explanation.

On the hemisphere, construct two tangential spherical straight lines through the vertex of the hyperbolic angle. The spherical angle of the two spherical straight lines can be taken for the measure of hyperbolic angle of the two hyperbolic straight lines.
Figure 23. An option of scaling the hyperbolic ruler.  

Figure 24. Measuring the hyperbolic distance of two points along the scale.  

Figure 25. Measuring angle on the plane.  

Figure 26. Measuring angle on the sphere.  

Figure 27. Measuring angle on the hemisphere.  

A striking consequence of the definition:  

Figure 28. Different cases of arranging two hyperbolic straight lines.  

The angle of two parallel lines as a limit of non-parallel lines can be defined as equal to 0. However, the angle of two skew lines (which do not share real or ideal points) cannot be defined!  

3.8. Sum of interior angles in a hyperbolic triangle  
- On the plane (figure 29): The sum of interior angles of any triangle is 180°.
• On the sphere (figure 30): The smaller the spherical triangle, the closer to the plane. Therefore, the sum of interior angles changes from the smallest possible \(180^\circ\) (the point) to \(540^\circ\) in the greatest possible triangle with three vertices on the equator.

• On the hemisphere (figure 31): The smaller the hyperbolic triangle, the closer to the plane. Therefore, the sum of the interior angles changes from \(180^\circ\) in the smallest possible triangle (the point) to \(0^\circ\) in the largest possible triangle with three ideal points on the omitted equator. This triangle is also known as the triply asymptotic triangle. (Any two sides of this triangle are parallel to each other according to the definition of parallels on the hyperbolic surface. Another notable difference to the plane or the sphere: a triangle with parallel sides!)

3.9. Khayyam-Saccheri quadrilateral

This was one of the first problems leading to the discovery of hyperbolic geometry.

For a given segment, erect two perpendiculars at two endpoints, measure two equal distances on the perpendiculars and connect the new endpoints. Due to the symmetry of the figure, the two top angles are congruent; but how big will they be?

On the plane (figure 32), each will be a right angle; on the sphere (figure 33), always bigger than that; on the hemisphere (figure 34), always smaller than that.
3.10. Lambert quadrilateral
This is another problem that probably would have led Lambert to the discovery of hyperbolic geometry before Gauss, Bolyai and Lobachevsky had he not died at the age of 49: Given a quadrilateral with three right angles, how big is the fourth angle?

On the plane (figure 35) each will be a right angle; on the sphere (figure 36) always bigger than that; on the hemisphere (figure 37) always smaller than that.

![Figure 35. Planar rectangle.](image1)
![Figure 36. Spherical Lambert quadrilateral.](image2)
![Figure 37. Hyperbolic Lambert quadrilateral.](image3)

3.11. Napier shape
Create a chain of five consecutive perpendicular segments. Can the fifth segment be perpendicular to the first one? In other words, can you close the chain of perpendiculars into a cycle?

![Figure 38. Chain of five perpendiculars on the plane.](image4)
![Figure 39. Spherical Napier pentagram.](image5)
![Figure 40. Hyperbolic Napier pentagon.](image6)

On the plane (figure 38) it is not possible, since the fifth segment is parallel with the first one. On the sphere (figure 39) the chain is closed to form a cycle called a Napier pentagram (“Pentagramma Mirificum”). On the hemisphere (figure 40) the chain is closed in a cycle which can be called a Napier pentagon because it is a convex polygon.

The Napier pentagram is 400 years old, but – to the best of my knowledge – the Napier pentagon is a new invention [16].
4. Conclusions

4.1. Concepts

The above material presents the basic concepts of hyperbolic geometry versus the corresponding concepts of planar and spherical geometry in the order and style that I have used in my courses.

There are many other mathematically and historically interesting and important concepts that can be added to this material, such as the equidistant line (the set of points equidistant from a line), the classification of circles and cycles, or the measurement of area.

All these topics do not require trigonometry of any kind which is an extremely important and beautiful topic. However, it requires a different approach, a different mathematical language, so it is not included into this introductory material.

4.2. Why and to whom is it worth teaching?

The real question is, why and to whom is it worth teaching? What goals can a teacher set in teaching this new, unusual material?

At first glance, the idea of including hyperbolic geometry in general education seems too bold, far from the current curriculum. The eminent Russian scholar Alexandrov [17] wrote: “Lobachevskian geometry can hardly be included in secondary school curricula, but it seems essential to give pupils an idea of it and to show them the greatness of the human spirit, capable of creating unimaginable concepts and theories which, in the course of time, proved to be comprehensible and fruitful”.

Alexandrov’s opinion is also supported by the history of science. Plane geometry and spherical geometry evolved more than two thousand years ago. In contrast, the basic works of hyperbolic geometry are only two hundred years old, although preliminary research had begun much earlier.

4.3. Why do I still recommend using hyperbolic geometry as part of comparative geometry in general education?

- If the learner has already dealt with comparative geometry of the plane and the sphere, he has become more or less free from the shackles of purely Euclidean approach. As a result, the transition to a third geometry becomes much easier.

- The hemispherical model of hyperbolic geometry gives the student the opportunity to apply his / her prior knowledge in spherical geometry.

- The hemispherical model makes many new concepts in hyperbolic geometry easier to visualize and understand, such as the concept of straight line, angle measurement, properties of polygons, classification of circles and cycles, etc. Conversely, the hyperbolic model proves to be very helpful in understanding the deeper meaning of many concepts and theorems in plane and spherical geometry, for example, the circle with infinite radius, or the isosceles triangle theorem.

- Hyperbolic geometry is a vital part of modern geometry, mathematics, physics (for example the theory of relativity), or the philosophy of science. It is becoming increasingly important in other disciplines, and even in the fine arts and architecture.

- Hyperbolic geometry often provides a stark counterexample to certain concepts of plane geometry and spherical geometry.

- The futile attempts over two millennia to refute the existence of a third geometry, then the successful construction of hyperbolic geometry two hundred years ago, and finally its general acceptance and utilization belong to the most exciting chapters in the history of science. Ancient Greek geometers, medieval Arabic and Persian scholars, Italian, German, Russian, Swiss, Hungarian, and other European researchers worked on the development of the new discipline. Hence, the history of hyperbolic geometry is extremely challenging and instructive not only from the mathematical point of view, but also in the general history
of human culture. It can readily be used as a bridge between the humanities and natural sciences.

- Beyond all this, I consider another aspect very important. The axioms of hyperbolic geometry are very unusual from the Euclidean perspective. For most learners, this geometry is an archetype of a discipline that is inaccessible for the “average” student. Only the selected few, the geniuses are able to grasp it. Thus, for many people (who have heard of it at all), hyperbolic geometry reveals the limitations of their comprehension. One of the main purposes of the paper and project is to show that the foundations of hyperbolic geometry are not harder than the geometry of the plane or the sphere, just require a different approach which becomes understandable or even enjoyable by appropriate learning and teaching methods and models.

4.4. My students at ELTE University, Faculty of Preschool and Primary Education

Students of any grade could apply to the elective course Ball Geometry. No prerequisite knowledge of any type of non-Euclidean geometry is expected from the applicants.

For many years, I had only one course of this type in the Hungarian language at the Faculty. In the last twelve years, I opened an English course for Erasmus students with the same syllabus. At the end of the 2017/18 semester, 15 Hungarian students who had already completed Ball Geometry I, asked me to open Ball Geometry II as the continuation of the first semester. Because of administrative reasons, this could not happen in the spring semester, only the next fall semester.

I emphasize again that these students were future kindergarten and elementary school teachers. Their mathematics curriculum did not include non-Euclidean geometries, apart from the formulas of measuring surface, volume and components of the three-dimensional sphere, and the geographic coordinate system on the globe.

I described in the preliminary prospectus for the students that the aim of the course was to compare different geometric systems, to contrast planar geometric concepts with the corresponding concepts of other geometries. I was expecting that this information would scare many students away from the course, yet I did not want to cause any disappointment to them with an unexpectedly difficult material.

To my pleasant surprise, this was not the case, as shown in the table 1.

Table 1. Number of students in the last five years:

| Semester    | Ball Geometry I | Ball Geometry II | Erasmus students |
|-------------|----------------|------------------|------------------|
| 2015/16 Fall | 19             |                  | 1                |
| 2015/16 Spring | 21         |                  | 4                |
| 2016/17 Fall | 20             |                  | 5                |
| 2016/17 Spring | 22         |                  | 1                |
| 2017/19 Fall | 56             |                  | 2                |
| 2017/18 Spring | 30         |                  | 2                |
| 2018/19 Fall | 29             | 8                |                  |
| 2018/19 Spring | 36         | 6                | 2                |
| 2019/20 Fall | 53             | 16               | 13               |
| 2019/20 Spring | 16         | 9                | 5                |
| 2020/21 Fall | 58             | 21               | 4                |
| 2020/21 Spring | 62         | 15               | 5                |
Following the favourable reception of the Ball Geometry course and the support of several cooperating colleagues, certain elements of comparative geometry have become part of the compulsory examination.

At the beginning of each semester, I asked my students why they applied for the course. The answers could be broken down into four main types, with roughly equal proportions:

- An acquaintance who took the course recommended that it was worth attending.
- The title “Ball Geometry” sounded appealing.
- Mathematics in general, and geometry in particular, did not belong to his/her favourites in the secondary school, and he/she expected a positive turn from math phobia.
- In contrast, math memories from secondary school were positive, and he/she was eager for topics and exercises which were more challenging than the material required in kindergarten or elementary school.

Of course, participants in an elective course do not represent the average level of students. Nevertheless, my experiences during three decades of teaching have shown that prospective kindergarten and primary teachers are open and eager for mathematics that arouses their interest and provides a way to develop their competences.

I find it ridiculous and unfair to trace all the problems of secondary and tertiary education back to preschool and primary teachers. Even if such problems do exist, they can be traced back at least as much to the shortcomings of teacher training as to indifference of the students.

4.5. Does comparative geometry help in writing a test paper or an exam?
The answer depends on what the test or exam requires of the student: ready-made knowledge, routine exercises, or problem-solving thinking?

In fact, comparative geometry is more of an obstacle than an advantage in the first two cases. If the student is expected to answer as many questions as possible within the shortest possible time, he/she will lose valuable time pondering too much over an example.

Learning about comparative geometry is of real advantage if the teacher requires independent, problem-solving thinking from the student and rewards this attitude accordingly.

4.6. Direct vs. ICT experimentation
The widest possible use of ICT technology is essential and should be developed by the practicing teacher and educator as well [18]. This is especially true in the historical era of distance education, which, although not caused, was dramatically accelerated by the onset of the pandemic.

With all this in mind, I still consider direct, hands-on experimentation in geometry to be essential for grasping the basic concepts and developing them further. Hands-on experimentation and ICT research are not mutually exclusive but complementary factors for development. To quote a classic, “Men will always deceive themselves by abandoning experience to follow imaginary systems” [19].

For a long time, hyperbolic geometry presented itself as a very abstract mathematical theory, very distant from practical use. In the last hundred years, it has been widely accepted through its application in the natural sciences and has become an important part of mathematical theory and physical experimentation. As such, it well deserves some introductory lessons as part of comparative geometry in general education.

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