Uniform bounds on locations of zeros of partial theta function

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Abstract

We consider the partial theta function

\[ \theta(q, z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j \]

where \((q, z) \in \mathbb{C}^2, |q| < 1\). We show that for any \(0 < \delta_0 < \delta < 1\), there exists \(n_0 \in \mathbb{N}\) such that for any \(q\) with \(\delta_0 \leq |q| \leq \delta\) and for any \(n \geq n_0\) the function \(\theta\) has exactly \(n\) zeros with modulus \(|q|^{-n-1/2}\) counted with multiplicity.

Keywords: partial theta function; Rouché theorem; spectrum

AMS classification: 26A06

1 Introduction

We consider the bivariate series \(\theta(q, z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j\), where \((q, z) \in \mathbb{C}^2, |q| < 1\). This series defines a partial theta function. The terminology is explained by the fact that the Jacobi theta function is defined by the series \(\sum_{j=-\infty}^{\infty} q^{j^2} z^j\) and the following equality holds true: \(\theta(q^2, z/q) = \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j\). The word “partial” is justified by the summation in \(\theta\) ranging from 0 to \(\infty\) and not from \(-\infty\) to \(\infty\). In what follows we consider \(z\) as a variable and \(q\) as a parameter. For each fixed value of the parameter \(q\) the function \(\theta\) is an entire function in the variable \(z\).

The function \(\theta\) finds applications in various domains, such as statistical physics and combinatorics (see [18]), Ramanujan type \(q\)-series (see [19]), the theory of (mock) modular forms (see [3]), asymptotic analysis (see [2]), and also in problems concerning real polynomials in one variable with all roots real (such polynomials are called hyperbolic, see [4], [5], [16], [15], [6], [14] and [7]). Other facts about \(\theta\) can be found in [1].

The zeros of \(\theta\) depend on the parameter \(q\). For some values of \(q\) (called spectral) confluence of zeros occurs, so it would be correct to regard the zeros as multivalued functions of \(q\); about the spectrum of \(\theta\) see [14], [12] and [13].

Notation 1. We denote by \(D_\rho\) the open disk in the \(q\)-space centered at 0 and of radius \(\rho\), by \(C_\rho\) the corresponding circumference, and by \(A_{\delta_0, \delta}\) the closed annulus \(\{q \in \mathbb{C} | \delta_0 \leq |q| \leq \delta\}\).

In the present paper we prove the following theorem:

Theorem 2. For any couple of numbers \((\delta_0, \delta)\) such that \(0 < \delta_0 < \delta < 1\), there exists \(n_0 \in \mathbb{N}\) such that for any \(q \in A_{\delta_0, \delta}\) and for any \(n \geq n_0\) the function \(\theta\) has exactly \(n\) zeros in \(D_{|q|^{-n-1/2}}\) counted with multiplicity.

Remarks 3. (1) The proof of the theorem is based on a comparison between \(\theta\) and the function

\[ u(q, z) := \prod_{\nu=1}^{\infty} (1 + q^{\nu} z) \]

(1)
We use the equality
\[ u = \sum_{j=0}^{\infty} q^{j(1+1)/2} z^j / (q; q)_j, \]

where \((q; q)_j := (1 - q)(1 - q^2) \cdots (1 - q^j)\) is the \(q\)-Pochhammer symbol; it follows directly from Problem I-50 of [17] (see pages 9 and 186 of [17]). The analog of the above theorem for the deformed exponential function \(\sum_{j=0}^{\infty} q^{j(1+1)/2} z^j / j!\) is proved in a non-published text by A. E. Eremenko using a different method.

(2) For \(q\) close to 0 the zeros of \(\theta\) are of the form \(-q^{-\ell}(1 + o(1)), \ell \in \mathbb{N}\), see more details about this in [9, 10] and [11].

Acknowledgement. The author has discussed (electronically and directly) questions concerning the partial theta and the deformed exponential function with A. Sokal, A. E. Eremenko, B. Z. Shapiro, I. Krasikov and J. Forsgård to all of whom he expresses his most sincere gratitude.

2 Proofs

Proof of Theorem 2. It is shown in [9] that for \(0 < |q| \leq 0.108\) the zeros of \(\theta\) can be expanded in convergent Laurent series. Recall that the function \(u\) (defined by (11)) satisfies equality (2), i.e. the zeros of \(u\) are the numbers \(-q^{-\ell}, \ell \in \mathbb{N}\). We show that for \(n \in \mathbb{N}\) sufficiently large the functions \(u\) and \(\theta\) have one and the same number of zeros in the open disk \(D_{|q|-n-1/2}\). To this end we show that for the restrictions \(u^0\) and \(\theta^0\) of \(u\) and \(\theta\) to the circumference \(C_{|q|-n-1/2}\) one has \(|u^0 - \theta^0/(q; q)_n| < |u^0|\) after which we apply the Rouché theorem.

For \(0 < |q| \leq 0.108\) one can establish a bijection between the zeros of \(\theta\) and \(u\), because their \(\ell\)th zeros are of the form \(-q^{-\ell}(1 + o(1))\) and the moduli of the zeros increase with \(\ell\), see part (2) of Remarks 3.

Set \(P_k(|q|) := \prod_{\ell=0}^{k} (1 - |q|^{\ell+1/2}), k \in \mathbb{N} \cup \infty\). For \(|u^0|\) one obtains the estimation
\[ |u^0| \geq |q|^{-n^2/2} P_{n-1}(|q|) P_\infty(|q|) > |q|^{-n^2/2} (P_\infty(|q|))^2 \geq |q|^{-n^2/2} (P_\infty(\delta))^2. \]

Indeed, for \(|z| = |q|^{-n-1/2}\) one can set \(z := |q|^{-n-1/2} \omega, |\omega| = 1\). For \(1 \leq \nu \leq n\) (resp. for \(\nu > n\), the factor \((1 + q^{\nu} z)\) in (11) is of the form \((1 - |q|^{-\ell-1/2} \omega_\ell)\), where \(\ell = n - \nu\) and \(|\omega_\ell| = 1\) (resp. of the form \((1 - |q|^{\ell+1/2} \omega_\ell^*)\), where \(\ell = \nu - n + 1\) and \(|\omega_\ell^*| = 1\). Thus
\[ u(q, |q|^{-n-1/2} \omega^{-n-1/2}) = \prod_{\ell=0}^{n-1} (1 - |q|^{-\ell-1/2} \omega_\ell) \prod_{\ell=0}^{\infty} (1 - |q|^{\ell+1/2} \omega_\ell^*). \]

The first of the factors in the right-hand side can be represented in the form \(|q|^{-n^2/2} \tilde{\omega} \prod_{\ell=0}^{n-1} (1 - |q|^{\ell+1/2} \omega_\ell^*)\) with \(|\tilde{\omega}| = |\omega_\ell| = 1\). Therefore
\[ u(q, |q|^{-n-1/2} \omega^{-n-1/2}) = |q|^{-n^2/2} \tilde{\omega} \prod_{\ell=0}^{n-1} (1 - |q|^{\ell+1/2} \omega_\ell^*) \prod_{\ell=0}^{\infty} (1 - |q|^{\ell+1/2} \omega_\ell^*). \]

The modulus of the right-hand side is minimal for \(\omega_\ell^* = \omega_\ell^** = 1\) in which case one obtains the lefmost inequality in (3).

Consider the monomial \(\beta_j := \alpha_j z^j\) in the series \(u - \theta/(q; q)_n\). Hence for \(j = n\) it vanishes and for \(j > n\) one has
\( \alpha_j = q^{(j+1)/2}(1/(q;q)_j - 1/(q;q)_n) = q^{(j+1)/2}U_{j,n} \), where

\[
U_{j,n} := (1 - \prod_{\ell=n+1}^{j} (1 - q^\ell)) / (q;q)_j,
\]

so for \( |z| = |q|^{-n-1/2} \) one has \( |\beta_j| = |q|^{-n^2/2 + (j-n)^2/2} |U_{j,n}| \). One can observe that \( U_{j,n} = q^{n+1} + O(q^{n+2}) \). Set

\[
U_{j,n} := \sum_{\nu \geq n+1} u_{j,n;\nu} q^\nu \quad \text{and} \quad U := ((\prod_{\ell=1}^{\infty} (1 + q^{\ell})) - 1) / (q;q)_\infty = \sum_{\nu=1}^{\infty} u_{\nu} q^\nu.
\]

The Taylor series of \( U \) converges for \( |q| < 1 \) because the infinite products defining \( U \) converge. Clearly \( u_{j,n;\nu} \in \mathbb{Z} \), \( u_{\nu} \in \mathbb{N} \) (because all coefficients of the series \( 1/(q;q)_j \) and \( 1/(q;q)_\infty \) are positive integers) and \( u_{j,n;\nu+1} = u_{j,n;\nu} \).

The following lemma explains in what sense the series \( U \) majorizes the series \( U_{j,n} \).

**Lemma 4.** One has \( |u_{j,n;\nu}| \leq u_{\nu}, \nu \in \mathbb{N} \).

Before proving Lemma 4 (the proof is given at the end of the paper) we continue the proof of Theorem 2.

Set \( R(|q|) := \sum_{j>n} |q|^{(j-n)^2/2} \). The following inequality results immediately from the lemma:

\[
Z_1 := \sum_{j>n} |\beta_j| \leq |q|^{-n^2/2} |q|^n U(|q|) R(|q|) \leq |q|^{-n^2/2} \delta^n U(\delta) R(\delta).
\]

The first condition which we impose on the choice of \( n \) is the following inequality to be fulfilled:

\[
\delta^n U(\delta) R(\delta) < (P_\infty(\delta))^2 / 4.
\]

For \( j < n \) and \( |z| = |q|^{-n-1/2} \) one has \( |\beta_j| = |q|^{-n^2/2 + (j-n)^2/2} |\tilde{U}_{j,n}| \), where

\[
\tilde{U}_{j,n} := (\prod_{\ell=j+1}^{n} (1 - q^\ell)) / (q;q)_n.
\]

Hence \( |\tilde{U}_{j,n}| \leq T(|q|) := (\prod_{\ell=1}^{\infty} (1 + |q|^\ell) + 1) / (|q|; |q|)_\infty \) and

\[
|\beta_j| \leq |q|^{-n^2/2} |q|^{(j-n)^2/2} T(\delta).
\]

Choose \( m \in \mathbb{N} \) such that \( T(\delta) \sum_{\ell=m}^{\infty} \delta^{2\ell/2} \leq (P_\infty(\delta))^2 / 4 \). Inequality 7 implies that

\[
Z_2 := \sum_{j=0}^{n-m} |\beta_j| \leq |q|^{-n^2/2} (P_\infty(\delta))^2 / 4
\]

Notice that for \( n < m \) the above sum is empty and the inequality trivially holds true. The finite sum

\[
Z_3 := \sum_{j=n-m+1}^{n-1} |\beta_j|
\]

is of the form \( |q|^{-n^2/2} O(|q|^n) \). Indeed, consider formula 6. There exists \( M > 0 \) depending only on \( \delta_0 \) and \( \delta \) such that

\[
0 < 1/(q;q)_n \leq 1/(|q|; |q|)_n < 1/(|q|; |q|)_\infty \leq M \quad \text{for} \quad \delta_0 \leq |q| \leq \delta.
\]
Thus
\[ |\tilde{U}_{j,n}| \leq M(\prod_{\ell=j+1}^{n} (1 + |q|^{\ell}) - 1) . \]

The index \( j \) can take only the values \( n - m + 1, \ldots, n - 1 \). In the last product each monomial \( |q|^{\ell} \) can be represented in the form \( |q|^{n}|q|^{\ell-n} \), where \( \ell - n = 2 - m, \ldots, 0 \). The modulus of each factor \( |q|^{\ell-n} \) is not larger than \( 1/\delta_{0}^{\max(0,m-2)} \). Therefore
\[ |\tilde{U}_{j,n}| \leq M((1 + |q|^{n}/\delta_{0}^{\max(0,m-2)})^{m-1} - 1) = O(|q|^{n}) . \]

The sum \( Z_{3} \) (see (8)) can be made less than \( |q|^{-n^{2}/2}(P_{\infty}(\delta))^{2}/4 \) by choosing \( n \) large enough. Thus inequalities (3), (4) and (8) yield
\[ |u^{0} - \theta^{0}/(q;q)_{n}| \leq Z_{1} + Z_{2} + Z_{3} \leq (3/4)|q|^{-n^{2}/2}(P_{\infty}(\delta))^{2} < |q|^{-n^{2}/2}(P_{\infty}(\delta))^{2} \leq |u^{0}| \]
which proves the theorem. \( \square \)

**Proof of Lemma 4** We first compare the coefficients of the series
\[ \prod_{\ell=p}^{r}(1 + q^{\ell}) - 1 = \sum_{\nu \geq p} \gamma_{\nu}^{1}q^{\nu} \quad \text{and} \quad \prod_{\ell=p}^{r}(1 - q^{\ell}) - 1 = \sum_{\nu \geq p} \gamma_{\nu}^{2}q^{\nu}, \quad p \leq r . \]

They are obtained respectively as a sum of the non-negative coefficients of monomials and as a linear combination of the same coefficients some of which are taken with the + and the rest with the − sign. Therefore \( \gamma_{\nu}^{1} \geq |\gamma_{\nu}^{2}|, \nu \geq p \). This means that \( |u_{j,n;\nu}| \leq v_{j,n;\nu} \leq v_{\infty,n;\nu}, \) where
\[ V_{j,n} := \left( \prod_{\ell=n+1}^{j} (1 + q^{\ell}) - 1 / (q;q)_{j} \right) = \sum_{\nu \geq n+1} v_{j,n;\nu}q^{\nu}, \quad V_{\infty,0} = U \quad \text{and} \quad v_{\infty,0;\nu} = u_{\nu} . \]

To prove the lemma it suffices to show that
\[ v_{\infty,n;\nu} \leq v_{\infty,0;\nu} . \] (10)

Consider the series \( S_{r} := \prod_{\ell=r+1}^{\infty}(1 + q^{\ell}) - 1 = \sum_{\nu \geq r+1} s_{r;\nu}q^{\nu} \) for \( r = 0 \) and \( r = n \). Compare the coefficients \( s_{0;\nu} \) and \( s_{n;n+\nu} \). The coefficient \( s_{0;\nu} \) is equal to the number of ways in which \( \nu \) can be represented as a sum of distinct natural numbers forming an increasing sequence whereas \( s_{n;n+\nu} \) is the number of ways in which \( n + \nu \) can be represented as a sum of distinct natural numbers \( \geq n + 1 \) forming an increasing sequence.

Clearly \( s_{n;n+\nu} \leq s_{0;\nu} \). This implies inequality (10) and the lemma, because once has \( V_{\infty,r} = S_{r} / (q;q)_{\infty} \) and the coefficients of the series \( 1/(q;q)_{\infty} \) are all positive. \( \square \)

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