STABLE RANK THREE VECTOR BUNDLES WITHOUT THETA DIVISORS OVER BIELLIPTIC CURVES

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Abstract. Raynaud has shown that over a general curve of genus \( g \geq 2 \), every semistable bundle of rank three and integral slope admits a theta divisor. We show that this can fail for special curves: Over any bielliptic curve of genus \( g \geq 5 \), we construct a stable rank three bundle of trivial determinant with no theta divisor. This gives a partial answer to a question of Beauville.

1. Introduction

Let \( X \) be a complex projective smooth curve of genus \( g \geq 2 \). We write \( J^d_X \) for the Jacobian variety parametrising line bundles of degree \( d \) over \( X \). Given a vector bundle \( V \to X \) of rank \( r \) and trivial determinant, we consider the set

\[
\{ L \in J^{g-1}_X : h^0(X, L \otimes V) > 0 \}.
\]

If \( V \) is semistable and generic, then (1.1) is the support of a divisor \( D(V) \) on \( J^{g-1}_X \) linearly equivalent to \( r \Theta \), where \( \Theta \) is the Riemann theta divisor.

We write \( SU_r \) for the moduli space of semistable bundles over \( X \) of rank \( r \) and trivial determinant. The assignment \( V \mapsto D(V) \) defines a rational map \( SU_r \to |r \Theta| \), called the theta map, and hence a line bundle \( D^*O(1) =: L \). By Drezet–Narasimhan [8], the group Pic \( (SU_r) \) is infinite cyclic, and \( L \) is the ample generator.

If \( h^0(X, L \otimes V) > 0 \) for all \( L \) of degree \( g-1 \), then we say that \( V \) has no theta divisor. Such bundles were first studied in the 1980s by Raynaud [19]. Bundles in \( SU_r \) without theta divisors define base points of the linear series \( |L| \), and are thus of relevance for questions on projective models and intersection theory of \( SU_r \). For a comprehensive survey of results on the theta maps and linear series, see Beauville [3]. Examples of bundles in \( SU_r \) with no theta divisor have been given for various \( r \) and \( g \) by Raynaud [19], Popa [18] and Pauly [16] (see also [11]). The dimension of the base locus of \( |L| \) has also been studied in various cases by Schneider [20], Hein [9] and Pauly [17].

In the present work we are primarily interested in bundles of low rank. Raynaud [19] showed that in the following situations, every bundle in \( SU_r \) admits a theta divisor:

- \( r = 2 \) and all \( g \geq 2 \)
- \( r = 3 \) and \( g = 2 \)
- \( r = 3 \) and \( g \geq 3 \) for a general curve \( X \)

More recently, Beauville [2] showed that every \( V \in SU_3 \) admits a theta divisor if \( g = 3 \) without the assumption of generality on \( X \), and conjectured [3, Conjecture 6.2] that the
same holds in higher genus. In the present note, we give counterexamples to this conjecture for \( g \geq 5 \):

**Theorem 1.1.** Let \( X \) be a bielliptic curve of genus \( g \geq 5 \). Then there exists a stable bundle of rank three over \( X \) with no theta divisor.

Moreover, following [3], we write \( r(X) \) for the least integer \( r \) such that there exists a semistable bundle \( V \to X \) of rank \( r \) not admitting a theta divisor. In [3, Question 6.4 b)], Beauville asks the following:

Put \( r(g) := \min\{r(X) : X \text{ a curve of genus } g\} \). Is \( r(g) \) an increasing function of \( g \)?

As Raynaud showed that any bundle in \( SU_2 \) has a theta divisor, we have \( r(g) \geq 3 \) for all \( g \). Therefore, by Theorem 1.1 we obtain:

**Corollary 1.2.** For all \( g \geq 5 \), we have \( r(g) = 3 \).

Note that Pauly [16, §2.2] has already constructed bundles over hyperelliptic curves showing that \( r(g) \leq 4 \) for all \( g \geq 2 \).

Regarding the low genus cases: \( r(2) \geq 4 \) and \( r(3) \geq 4 \) by Raynaud [19] and Beauville [2] respectively, and we have equality in view of Pauly’s construction [16]. It remains unresolved at this point whether \( r(4) = 3 \) or 4. See Question 5.2 for more discussion.

Let us motivate the construction of the bundles referred to in the theorem. Some time ago, Raynaud found a stable rank two bundle with reducible theta divisor over a bielliptic curve of genus three. Building on Lange–Narasimhan [14], he considered a stable bundle \( V\to X \) of rank two and trivial determinant, with a one-parameter family of maximal line subbundles of degree \(-1\). The construction generalises to higher genus, and the theta divisor \( D(V) \subset J^{g-1}_X \) contains the irreducible component

\[
D_1 := \{M^{-1} \otimes O_X(D) : M \subset V \text{ a maximal subbundle; } D \in \text{Sym}^{g-2}X\}.
\]

Moreover, it emerges that there is a nonempty residual component \( D_2 \). For details and proofs, see the appendix by C. Pauly to [13]. We note that \( V \) has minimal Segre invariant for a stable bundle; in other words, the maximal line subbundles \( M \) are of the largest possible degree given that \( V \) is stable.

With this in mind, the following approach suggests itself: If one could produce a stable bundle \( W \) of rank three and integral slope admitting a two-parameter family of maximal subbundles of largest possible degree, then one could hope to obtain a locus analogous to (1.2) having dimension \( g \) instead of \( g - 1 \). The bundle \( W \) would thus have no theta divisor.

Here is a summary of the present article. In [2] we recall some well-known facts about vector bundles over elliptic and bielliptic curves. In [3] we develop an important technical tool for the construction: a geometric criterion for the existence of certain subsheaves of a rank three extension of vector bundles. This generalises Lange–Narasimhan [14, Proposition 1.1].
In §4, we show that over a bielliptic curve, the construction of Raynaud and Lange–Narasimhan can indeed be adapted to produce a rank three bundle $W$ with a two-parameter family of maximal subbundles as suggested above (Proposition 4.1). The bundle $W$ so obtained does not have integral slope. However, using the existence of certain linearisations on $W$ and its components (Lemma 4.5), we show that there exists an elementary transformation $W$ of $W$ which preserves the family of maximal subbundles, in a suitable sense, and which has integral slope. Here we make use of Kempf’s descent lemma.

Tensoring $W^*$ by a cube root of det $W$, we obtain a bundle $V$ with trivial determinant. An intersection-theoretic argument using a calculation by Pauly [13, Appendix] then shows that if $g \geq 5$, then $D(V)$ cannot be a well-defined divisor (Proposition 4.10). We conclude the proof of Theorem 1.1 by showing that $V$ is a stable vector bundle (Proposition 4.11).

In the last section, we discuss the construction in the low genus cases, and end with some questions.

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2. Bundles over elliptic and bielliptic curves

Here we quote some results of Atiyah [1], and make some computations which will be needed for the construction.

Let $Z$ be an elliptic curve over $\mathbb{C}$. By Atiyah [1, Theorem 7] and the corollary following it, the moduli space $\mathcal{U}_Z(r,d)$ of indecomposable vector bundles of rank $r$ and degree $d$ over $Z$ may be identified with $Z$ in such a way that the following diagram commutes:

$$
\begin{array}{cccc}
\mathcal{U}_Z(r,d) & \overset{\text{det}}{\longrightarrow} & J_Z^d \\
\downarrow i & & \downarrow i \\
Z & \overset{h}{\longrightarrow} & Z
\end{array}
$$

where $h = \gcd(r,d)$. The case of interest to us here is $r = 2$ and $d = 2g - 1$, where $g \geq 3$ is an integer. Here $h = 1$, and the determinant gives an isomorphism $\mathcal{U}_Z(2,2g - 1) \iso J_Z^{2g-1}$.

Now let $X$ be a curve of genus $g \geq 3$ admitting a double covering $f : X \to Z$.

**Proposition 2.1.** Let $E \to Z$ be an indecomposable vector bundle of rank two and degree $2g - 1$, and $f^*E \to X$ its pullback.

1. We have $h^0(Z,E) = 2g - 1$.
2. Every $N \in J_Z^{2g-1}$ is a maximal subbundle of $E$.
3. We have $h^0(X,f^*E) = 2g$.

**Proof.** (1) and (2) are straightforward to check, using Riemann–Roch and Serre duality. As for (3): Let $N \in J_Z^{2g-1}$ be general, and consider the sequence

$$0 \to f^*N \to f^*E \to f^*(E/N) \to 0.$$
Since by part (2) we may assume \( f^*N \neq K_X \), we have \( h^1(X, f^*N) = 0 \). Then it is easy to see that \( h^1(X, f^*E) = 0 \), whence \( h^0(X, f^*E) = 2g \) by Riemann–Roch.

3. A geometric criterion for lifting in extensions

In this section we give a criterion analogous to Lange–Narasimhan [14, Proposition 1.1] for liftings in certain extensions of vector bundles. It should be noted that similar statements have already been obtained by T. Johnsen and I. Choe together with the present author in [12] and [5, 6, 7] respectively. As the statements we require here are slightly different, we give detailed proofs, but the ideas are all present in the aforementioned works.

Let \( X \) be a curve and \( V \to X \) a vector bundle with \( h^1(X, V) \geq 1 \). Then \( V \) has a flasque resolution \( 0 \to V \to \text{Rat}(V) \to \text{Prin}(V) \to 0 \) by rational sections and principal parts.

Taking global sections, we obtain
\[
0 \to H^0(X, V) \to \text{Rat}(V) \to \text{Prin}(V) \to H^1(X, V) \to 0.
\]
Now write \( \pi: \mathbb{P}V \to X \) for the projection. We have a sequence of identifications (compare with [6, §2.3])
\[
H^1(X, V) \cong H^0(X, K_X \otimes V^*)^* \cong H^0(\mathbb{P}V, \pi^* K_X \otimes \mathcal{O}_{\mathbb{P}V}(1))^*.
\]
by Serre duality, the projection formula, and the definition of direct image. By standard algebraic geometry, we obtain a map
\[
(3.1) \quad \psi: \mathbb{P}V \to \mathbb{P}H^1(X, V) \cong |\pi^* K_X \otimes \mathcal{O}_{\mathbb{P}V}(1)|^*.
\]
The map \( \psi \) may be realised on a fibre \( V|_x \) as the projectivised coboundary map in
\[
(3.2) \quad \cdots \to H^0(X, V(x)) \to V(x)|_x \to H^1(X, V) \to \cdots
\]
The middle term can be identified with the space of principal parts with values in \( V \) and with at most simple poles supported at \( x \).

Now we apply this to extensions. Henceforth we suppose \( V = \text{Hom}(F, K_X) = K_X \otimes F^* \) where \( F \) is any vector bundle. If \( 0 \to K_X \to W \to F \to 0 \) is an extension, we write \([W]\) for the corresponding class in \( H^1(X, K_X \otimes F^*) \). We now adapt some results from [6] and [10].

**Lemma 3.1.** Let \( W \) be an extension of \( F \) by \( K_X \), and suppose \( h^0(X, K_X \otimes F^*) = 0 \). Then there is a bijection between the sets of

- principal parts in \( \text{Prin}(K_X \otimes F^*) \) defining the extension class \([W]\); and
- elementary transformations of \( F \) lifting to subbundles of \( W \).

The bijection is given by \( p \leftrightarrow \text{Ker}(p: F \to \text{Prin}(K_X)) \).

**Proof.** This is a special case of [10, Theorem 3.3 (i)]. \( \square \)

Let \( \phi_1, \ldots, \phi_k \) be elements of \( F^* \) lying respectively over points \( x_1, \ldots, x_k \) of \( X \). For simplicity, we assume the points are distinct. For each \( i \), let \( z_i \) be a uniformiser at \( x_i \). In
view of (3.2), the point \( \phi_i \) determines a principal part in \( (K_X \otimes F^*)|_{x_i} \), uniquely up to scalar. Abusing notation, we denote this principal part by

\[
\frac{dz_i \otimes \phi_i}{z_i}.
\]

**Proposition 3.2.** Let \( W \) be an extension of \( F \) by \( K_X \). The elementary transformation \( \tilde{F} \) of \( F \) determined by \( \phi_1, \ldots, \phi_k \in \mathbb{P}F^* \) lifts to \( W \) if and only if \( [W] \) belongs to the secant in \( \mathbb{P}H^1(X, K_X \otimes F^*) \) spanned by \( \psi(dz_1 \otimes \phi_1), \ldots, \psi(dz_k \otimes \phi_k) \).

**Proof.** (Similar to [6, Lemma 2.10 (i)]) Suppose \( \tilde{F} \) lifts to a subsheaf of \( W \). By Lemma 3.1, the class \( [W] \) may be defined by a principal part \( p \) satisfying \( \tilde{F} \subseteq \text{Ker}(p) \). Since the \( x_i \) are distinct, clearly \( p \) must be a linear combination

\[
\sum_{i=1}^{k} \alpha_i \left( \frac{dz_i \otimes \phi_i}{z_i} \right) \in \bigoplus_{i=1}^{k} K_X \otimes F^*(x_i)|_{x_i}.
\]

By the interpretation of the map \( \psi \) coming from the sequence (3.2), the class \( [W] \) lies on the secant spanned by the points \( \psi(dz_i \otimes \phi_i) \).

The converse implication can be proven by reversing the above argument. \( \square \)

**Remark 3.3.** Note that the proposition is valid even if \( \psi \) is not base point free.

It will be convenient to state the lifting criterion in a slightly different form. Since \( K_X \) is a line bundle, there is a canonical identification \( \mathbb{P}(K_X \otimes F^*) \xrightarrow{\sim} \mathbb{P}F^* \).

**Notation.** For \( \phi \in F^*|_x \), write \( \text{ev}_\phi \) for the evaluation map \( H^0(X, F) \to \mathbb{C} \) defined by \( \text{ev}_\phi(s) := \phi(s(x)) \). We write \([p]\) for the cohomology class of \( p \in \text{Prin}(K_X \otimes F^*) \).

**Lemma 3.4.** There is a commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}(K_X \otimes F^*) & \xrightarrow{\psi} & \mathbb{P}H^1(X, K_X \otimes F^*) \\
\cong & & \cong \\
\mathbb{P}F^* & \xrightarrow{\phi \mapsto \text{ev}_\phi} & \mathbb{P}H^0(X, F)^* \\
\end{array}
\]

where the righthand vertical map is the Serre duality isomorphism.

**Proof.** Suppose \( \phi \in F^*|_x \), and let \( s \) be a section of \( F \). Then \( \text{ev}_\phi(s) = 0 \) if and only if

\[
\frac{dz \otimes \phi(s(x))}{z}
\]

is regular at \( x \). This is equivalent to

\[
\left\langle \frac{dz \otimes \phi}{z} | s \right\rangle = \left\langle \frac{dz \otimes \phi}{z} \right\rangle \cup s
\]

being zero in \( H^1(X, K_X) = \mathbb{C} \), because any \( K_X \)-valued principal part with a single pole of order one is cohomologically nontrivial. Thus the linear forms

\[
\text{ev}_\phi : H^0(X, F) \to \mathbb{C} \quad \text{and} \quad \left[ \frac{dz \otimes \phi}{z} \right] : H^0(X, F) \to H^1(X, K_X)
\]
have the same kernel. This proves the statement. □

In view of Lemma 3.4 we also denote the map \( \mathbb{P}F^* \to \mathbb{P}H^0(X,F)^* \) by \( \psi \). Now via Serre duality, an extension \( 0 \to K_X \to W \to F \to 0 \) defines a point in \( \mathbb{P}H^0(X,F)^* \).

Using Lemma 3.4 we may rephrase the lifting criterion directly in terms of the image of \( \mathbb{P}F^* \to \mathbb{P}H^0(X,F)^* \):

**Corollary 3.5.** The elementary transformation determined as above by the points \( \phi_1, \ldots, \phi_k \in F^* \) lifts to \( W \) if and only if the class \( [W] \in \mathbb{P}H^1(X,F)^* \) lies in the image of the secant to \( \mathbb{P}F^* \) spanned by \( \psi(\phi_1), \ldots, \psi(\phi_k) \). □

The following is straightforward to check:

**Lemma 3.6.** If \( h^1(X,F(-D)) = 0 \) for all degree two divisors \( D \) on \( X \), then the map \( \mathbb{P}F^* \to \mathbb{P}H^0(X,F)^* \) is an embedding. □

4. The construction

In this section we will prove Theorem 1.1 by an explicit construction. Let \( X \) be a bielliptic curve of genus \( g \geq 3 \), and \( f: X \to Z \) a double covering of an elliptic curve \( Z \). If \( g = 3 \), we also assume that \( X \) is not hyperelliptic; for \( g \geq 4 \), the Castelnuovo-Severi inequality implies that this is always the case. We write \( \iota \) for the bielliptic involution of \( X \).

4.1. An extension with many maximal subbundles. Let \( E \to Z \) be an indecomposable bundle of degree \( 2g - 1 \), and \( f^*E \) its pullback to \( X \). Since \( f^*E \) is \( \iota \)-invariant, we can lift \( \iota \) to a linearisation \( \bar{\iota} \) of \( f^*E \). The assignment \( \phi \mapsto \phi \circ \bar{\iota} \) defines a linearisation on \( f^*E^* \), which we denote \( \iota^* \phi \). Then \( \iota \) induces an involution on \( H^0(X,f^*E) \) by \( s \mapsto \bar{\iota} \circ s \circ \iota \), and hence a decomposition into \( \pm 1 \) eigenspaces \( H^0(X,f^*E)_\pm \). The invariant subspace \( H^0(X,f^*E)_+ \) is canonically isomorphic to \( H^0(Z,E) \), so is a hyperplane by Proposition 2.4(1). We write \( w \) for the point \( \mathbb{P}H^0(X,f^*E)_+ \), which is the centre of the projection \( \mathbb{P}H^0(X,f^*E)^* \to \mathbb{P}H^0(X,f^*E)_+^* \).

As in the previous section, \( w \) defines an extension \( 0 \to K_X \to W \to f^*E \to 0 \) by Serre duality.

Now a pair of points of the form \( \{ \phi, \iota(\phi) \} \) of \( \mathbb{P} (f^*E^*) \) determines an elementary transformation of \( f^*E \), which we denote \( F_\phi \). For general \( \phi \), we have \( \deg(F_\phi) = \deg f^*E - 2 = 4g - 4 \).

(Note that \( F_\phi = F_{\iota(\phi)} \).

**Proposition 4.1.** For general \( \phi \in \mathbb{P} (f^*E^*) \), the elementary transformation \( F_\phi \) of \( f^*E \) lifts to a subbundle of \( W \). Thus \( W \) admits a two-parameter family of rank two subbundles of degree \( 4g - 4 \), parametrised by an open subset of the ruled surface \( \mathbb{P}E^* \) over \( Z \).

**Proof.** We recall the map \( \psi: \mathbb{P} (f^*E^*) \to \mathbb{P}H^0(X,f^*E)^* \) defined in 3. Suppose \( x \) is not a fixed point of \( \iota \), and let \( \phi \in \mathbb{P} (f^*E^*) |_{x} \). For all equivariant sections \( s \in H^0(X,f^*E)_+ \) we have

\[
\text{ev}_\phi(s) = \phi(s(x)) = \phi(\iota \circ s \circ \iota(x)) = \iota^* \phi(s(\iota(x))) = \text{ev}_{\iota^* \phi}(s).
\]
Therefore, the composed map \( \mathbb{P}(f^*E^*) -\to \mathbb{P}H^0(X, f^*E)^* -\to \mathbb{P}H^0(X, f^*E)_+^{\ast} \) factorises via the quotient of \( \mathbb{P}(f^*E^*) \) by \( \langle t^i \rangle \), which is \( \mathbb{P}E^* \). Hence there is a commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}(f^*E^*) & \xrightarrow{\psi} & \mathbb{P}H^0(X, f^*E)^* \\
\downarrow & & \downarrow \\
\mathbb{P}E^* & \xrightarrow{\phi} & \mathbb{P}H^0(X, f^*E)_+^{\ast} \\
& \xrightarrow{\theta} & \mathbb{P}H^0(Z, E)^*.
\end{array}
\]

Next, we claim that the map \( \mathbb{P}E^* \to \mathbb{P}H^0(Z, E)^* \) is an embedding. By Lemma \( 3.6 \) it suffices to show that \( h^1(Z, E(-D)) = 0 \) for all \( D \in \text{Sym}^2 X \). By Serre duality, this is equivalent to \( h^0(Z, E^*(D)) = 0 \). But \( \mu(E^*(D)) = 2 - \frac{2g-2}{2} = \frac{3}{2} - g \), which is negative since \( g \geq 3 \). Since \( E \) is stable, therefore, \( h^0(Z, E^*(D)) = 0 \).

Hence in view of (4.1), we deduce:

- The map \( \psi: \mathbb{P}(f^*E^*) \to \mathbb{P}H^0(X, f^*E)^* \) is base point free.
- The centre \( w \) of the projection does not lie on the image of \( \mathbb{P}(f^*E^*) \).
- The only pairs of points that can be contracted by \( \psi \) are of the form \( \{ \phi, \iota \iota(\phi) \} \).

Moreover, as not all sections of \( f^*E \) are \( \iota \)-equivariant, \( \psi \) does not contract a general pair \( \{ \phi, \iota \iota(\phi) \} \). Hence a general such pair spans a secant line in \( \mathbb{P}H^0(X, f^*E)^* \). Since all such pairs are identified by the projection from \( w \), \textit{all these secant lines pass through the point} \( w \).

By Corollary \( 3.5 \), for general \( \phi \in \mathbb{P}(f^*E^*) \), the elementary transformation of \( f^*E \) defined by \( \phi \) and \( \iota \iota(\phi) \) lifts to a subsheaf of the extension \( W \) defined by \( w \) as above. Since \( w \) does not lie on \( \psi(\mathbb{P}(f^*E^*)) \), by Corollary \( 3.5 \) we deduce that no subsheaf strictly containing \( F_\phi \) lifts to \( W \). Hence \( F_\phi \) is a subbundle. The statement follows. \( \square \)

**Remark 4.2.** The bundle \( W \) above is a rank three analogue of the rank two bundles over bielliptic curves described in Lange–Narasimhan \( 13, \S 5 \), admitting one-parameter families of maximal line subbundles of maximal nondestabilising degree. In both cases, the existence of a “large” family of maximal subbundles is due to the existence of a point in a certain extension space lying on a one- or two-dimensional family of secant lines to a scroll; and in both cases the existence of this point is a consequence of the biellipticity of the curve.

By Proposition \( 4.1 \), the bundle \( W \), which has slope \( \frac{6g-4}{3} \), has a two-parameter family of subbundles of maximal nondestabilising slope \( 2g - 2 \). According to the heuristic sketched in the introduction, \( W \) ought to be a candidate for a stable bundle of rank three with no theta divisor. However, the slope of \( W \) is never an integer. We now proceed to show that a certain elementary transformation of \( W \) has integral slope and satisfies the properties we are interested in. We begin by showing that \( W \) admits a linearisation with some useful properties.

### 4.2. Linearisations on \( W \) and its components.

By the Riemann–Hurwitz formula and since \( K_Z \) is trivial, we have \( K_X = \mathcal{O}_X(R) \) where \( R \) is the ramification divisor of \( f \). In
Proof. Since such that the following diagram commutes:

\[\begin{array}{c}
\square \\
\end{array}\]

without this hypothesis. The bundle \(\text{Lemma 4.5.}\)

\(-\)

\(M\) has the hypothesis that \(f\) acts on the extension space \(H^1(X, \text{Hom}(f^*E, K_X))\) sending \([W]\) to \([\iota^*W]\).

Proof. By the identification (3.1), the map \([W] \mapsto [\iota^*W]\) is identified with the transpose of the action of \(\iota\) on \(H^0(X, f^*E)\) discussed in [4, 2].

Lemma 4.3. Via Serre duality, the map \([W] \mapsto [\iota^*W]\) is identified with the transpose of the action of \(\iota\) on \(H^0(X, f^*E)\) is nondegenerate. Thus we may choose a basis of \(H^1(X, K_X \otimes F^*)\) consisting of points of the form

\(\begin{bmatrix}
\frac{dz \otimes \phi}{z}
\end{bmatrix}\)

where \(\phi\) is a section of \(F^*\) near some \(x \in X\) and \(z\) a local coordinate at \(x\). For simplicity we assume \(x\) is not a fixed point of \(\iota\). By Lemma 3.4 this class may be identified (up to nonzero scalar multiple) with the linear form \(\text{ev}_\phi\). One checks that pullback by \(\iota\) on a class of this form coincides with \(\text{ev}_\phi \mapsto \text{ev}_{\iota^*\phi}\). It is then easy to verify that this is the transpose of the action \(s \mapsto \iota \circ s \circ \iota\) on \(H^0(X, f^*E)\). The statement follows for all points of \(H^1(X, K_X \otimes f^*E^*)\) by linearity of the \(\iota\)-action.

We recall now the descent lemma of Kempf, stated in the case we will need it:

Lemma 4.4. Let \(F \to X\) be a vector bundle admitting a linearisation \(\iota_0\) of the action of \(\iota\) on \(X\). Then \(F\) can be descended to \(Z\) if and only if \(\iota_0\) acts trivially on \(F|_y\) for each fixed point \(y\) of \(\iota\).

Proof. This a special case of Drezet–Narasimhan [3, Théorème 2.3]. Note that this theorem has the hypothesis that \(M = X/G\) is a good quotient, which is not the case here, since \(f_*O_X \neq O_Z\) (Beauville [2, §1]). However, the proof for the statement we require is valid without this hypothesis.

Lemma 4.5. The bundle \(W\) admits a linearisation \(\iota_1\) which for each fixed point \(y\) of \(\iota\) acts as \(-\text{Id}\) on \(K_X|_y\) and induces the identity on \(f^*E|_y\).

Proof. Since \(f^*E\) and \(K_X\) both descend to \(Z\), there are vector bundle isomorphisms

\(\alpha: \iota^*K_X \xrightarrow{\sim} K_X\) and \(\beta: \iota^*f^*E \xrightarrow{\sim} f^*E\)

such that the following diagram commutes:

\[\begin{array}{cccccccc}
0 & \longrightarrow & K_X & \longrightarrow & W & \longrightarrow & f^*E & \longrightarrow & 0 & [W] \\
\downarrow{a} & & \downarrow{c} & & \downarrow{b} & & & \\
0 & \longrightarrow & \iota^*K_X & \longrightarrow & \iota^*W & \longrightarrow & \iota^*f^*E & \longrightarrow & 0 \\
\downarrow{a} & & = & & \downarrow{\beta} & & & \\
0 & \longrightarrow & K_X & \longrightarrow & \iota^*W & \longrightarrow & f^*E & \longrightarrow & 0 & [\iota^*W] \\
\end{array}\]

Here the maps \(a, b\) and \(c\) are induced by pullback, and are maps of varieties, but not maps of vector bundles. The composed maps \(\alpha \circ a\) and \(\beta \circ b\) are the linearisations of \(K_X\) and
f^*E \text{ respectively. Since both } K_X \text{ and } f^*E \text{ are pullbacks from } Z, \text{ by Lemma 4.4 we may assume that these linearisations act trivially on the fibres } K_X|_y \text{ and } f^*E|_y \text{ respectively for each fixed point } y \text{ of } \iota.

Now recall that the class } [W] \text{ belongs to } H^0(X, f^*E^*)_*. \text{ Thus in view of Lemma 4.3 we have } [\iota^*W] = -[W] \text{ in } H^1(X, K_X \otimes f^*E^*). \text{ Therefore, there exists a vector bundle isomorphism } \gamma: \iota^*W \cong W \text{ making the following diagram commute:}

\[
\begin{array}{ccc}
0 & \rightarrow & K_X & \rightarrow & \iota^*W & \rightarrow & f^*E & \rightarrow & 0 \\
| & & \downarrow{\text{Id}} & & | & & \downarrow{\gamma} & & | & & \downarrow{\text{Id}} \\
0 & \rightarrow & K_X & \rightarrow & W & \rightarrow & f^*E & \rightarrow & 0
\end{array}
\]

Then } \tilde{\iota}_1 \equiv \gamma \circ c \text{ is a linearisation of } W \text{ which has the required properties.} \qed

Next, we require an observation about the structure of the subbundles } F_{\phi}.

**Lemma 4.6.** For general } \phi \in \mathbb{P}(f^*E^*), \text{ there exist mutually nonisomorphic } N, N' \in J^*_Z \text{ such that } F_{\phi} = f^*N \oplus f^*N'.

**Proof.** Let } \phi \text{ be a general point of } \mathbb{P}(f^*E^*) \text{ lying over } x \in X. \text{ We denote by } \overline{\phi} \text{ the image of } \phi \text{ in } \mathbb{P}E^*, \text{ and write } z = f(x). \text{ There are strings of identifications}

\[
E^*_x|_z = f^*E^*|_x = f^*E^*|_{f(x)} \text{ and } E|_z = f^*E|_x = f^*E|_{f(x)}.
\]

Under the first of these, } \overline{\phi} \text{ is proportional to } \phi \text{ and } \iota^*\overline{\phi}. \text{ Thus } \text{Ker}(\overline{\phi}) \text{ coincides with } \text{Ker}(\phi) \text{ and } \text{Ker}(\iota^*\overline{\phi}) \text{ via the second set of identifications.}

Now } \overline{\phi} \text{ determines an elementary transformation } 0 \rightarrow E_{\overline{\phi}} \rightarrow E \rightarrow \mathbb{C}_z \rightarrow 0 \text{ over } Z. \text{ By the last paragraph, the pullback of this sequence to } X \text{ coincides with}

\[
0 \rightarrow f^*(E_{\overline{\phi}}) \rightarrow f^*E \xrightarrow{(\phi, \iota^*\overline{\phi})} \mathbb{C}_x \oplus \mathbb{C}_{f(x)} \rightarrow 0,
\]

and so } F_{\phi} = f^*E_{\overline{\phi}}.

Next, by Lemma 2.1 (2), every line bundle } N \text{ of degree } g - 1 \text{ over } Z \text{ is a subbundle of } E. \text{ Since } \deg E_{\overline{\phi}} = 2(g - 1), \text{ at most two such } N \subset E \text{ belong to } E_{\overline{\phi}}. \text{ For a general } z \in Z, \text{ consider the map } r: J^*_Z \rightarrow \mathbb{P}E|_z \text{ given by } N \mapsto N|_z. \text{ Then } N \in J^*_Z \text{ belongs to } E_{\overline{\phi}} \text{ if and only if } r(N) = N|_z = \text{Ker}(\overline{\phi}).

Since } J^*_Z \text{ is isomorphic to } Z, \text{ the map } r: Z \rightarrow \mathbb{P}^1 \text{ is a covering of degree } \geq 2. \text{ By the last paragraph, } r \text{ must have degree exactly two. Hence for general } \phi \in \mathbb{P}(f^*E^*)|_x \text{ there exist distinct } N, N' \in J^*_Z \text{ such that } E_{\overline{\phi}} = N \oplus N'. \text{ Thus } F_{\phi} = f^*(E_{\overline{\phi}}) = f^*N \oplus f^*N'. \qed

**Lemma 4.7.** For general } \phi \in f^*E^*, \text{ the linearisation } \iota_1 \text{ on } W \text{ restricts to a linearisation on the subbundle } F_{\phi} \text{ which acts trivially on the fibres } F_{\phi}|_y \text{ for each fixed point } y \text{ of } \iota.

**Proof.** By Lemma 4.6 the bundle } F_{\phi} \text{ is } \iota\text{-invariant. Thus, to see that } F_{\phi} \text{ is invariant under the linearisation } \iota_1 \text{ on } W, \text{ it suffices to show that, up to automorphisms of } F_{\phi}, \text{ there is only one injective map } F_{\phi} \rightarrow W; \text{ more precisely, that there is only one point of the Quot scheme}
of subsheaves of $W$ of rank two and degree $4g - 4$ corresponding to a subsheaf isomorphic to $F_\phi$.

To achieve this, by Lemma 4.6 it suffices to show that $h^0(X, \text{Hom}(f^*N, f^*E)) = 1$ for each $N \in J_{Z}^{g-1}$. For any $N_1 \in J_{Z}^{g-1}\{N\}$, consider the exact sequence

$$0 \to f^*N_1 \to f^*E \to f^*(E/N_1) \to 0.$$ 

Since $\deg f^*N = \deg f^*N_1$, all maps $f^*N \to f^*E$ lift from elementary transformations $0 \to f^*N \to f^*(E/N_1) \to T \to 0$ where $T$ is torsion of degree two. Since $X$ is nonhyperelliptic, there is exactly one possibility for $T$. Therefore, we must have $h^0(X, \text{Hom}(f^*N, f^*E)) = 1$.

Thus, up to automorphisms of $f^*N \oplus f^*N'$, there is only one copy of $f^*N \oplus f^*N'$ in $W$, and so the linearisation on $W$ induces a linearisation on $F_\phi$.

At each fixed point $y$ of $\iota$ we obtain an eigenspace decomposition

$$W|_y = (W|_y)_+ \oplus (W|_y)_-$$

Since $\iota_1$ restricts to a linearisation of $F_\phi$, for each $\phi$ we also obtain

$$F_\phi|_y = (F_\phi|_y)_+ \oplus (F_\phi|_y)_-,$$

where $(F_\phi|_y)_+ = F_\phi|_y \cap (W|_y)_+$. By Lemma 4.5 we have $(W|_y)_- = K_X|_y$. Thus $(F_\phi|_y)_-$ can be nonzero only if $F_\phi|_y$ contains the fibre of the subbundle $K_X|_y$. For general $\phi$ (in particular, if $\phi$ does not belong to $f^*E^*|_y$), this does not happen. For general $\phi$, therefore, $F_\phi|_y = (F_\phi|_y)_+$. This proves the statement.

By Lemma 4.5, the space $(W|_y)_-$ is nonzero for each fixed point $y$. Thus for dimension reasons, by Lemma 4.7 we have:

**Corollary 4.8.** For each fixed point $y$ of $\iota$, the fibres of all the subbundles $F_\phi$ coincide with $(W|_y)_+ = \mathbb{C}^2$.

**4.3. An elementary transformation of $W$.** Choose one ramification point $y$, and let $\ell \subset (W|_y)_+$ be a general line. Consider the elementary transformation

$$0 \to W \to \overline{W} \to \mathbb{C}_y \to 0$$

defined by $\text{Ker} (W|_y \to \overline{W}|_y) = \ell$. The bundle $\overline{W}$ has degree $\deg W + 1 = 6g - 3$, and hence integral slope $2g - 1$.

**Lemma 4.9.** The bundle $\overline{W}$ satisfies $h^0(X, \overline{W}, K_X(x + \iota(x))) \geq 2$ for all $x \in X$.

**Proof.** By Corollary 4.8 for general $\phi \in \mathbb{P} (f^*E^*)$, the line $\ell$ belongs to the fibre $F_\phi|_y$. Thus we have a diagram

$$\begin{array}{ccccccccc}
0 & \to & F_\phi & \to & G_\phi & \to & \mathbb{C}_y & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & W & \to & \overline{W} & \to & \mathbb{C}_y & \to & 0
\end{array}$$

where $G_\phi$ is a bundle of rank two and degree $\deg(F_\phi) + 1 = 4g - 3$. Thus for general $\phi \in \mathbb{P}(f^*E^*)$, the bundle $\mathbb{W}$ is an extension

$$0 \to G_\phi \to \mathbb{W} \to Q_\phi \to 0$$

where $Q_\phi$ is a line bundle isomorphic to $\det \mathbb{W} \otimes (\det G_\phi)^{-1}$.

Let us compute $Q_\phi$. We write $M := \det(f^*E)$. If $\phi$ lies over $x \in X$, we have

$$\det(F_\phi) = \det f^*E \otimes \mathcal{O}_X(-x - \iota(x)) = M(-x - \iota(x)).$$

By the computation in [13, Proposition A.4], we have

$$\mathbb{D} > 4(g-1)(g-1)!$$

where $\mathbb{D}$ is a bundle of rank two and degree $-\theta_\phi + 1$.

Now if $\mathbb{D}$ is defined, then it contains the component $\mathcal{D}_1$ with multiplicity at least two.

**Proposition 4.10.** The bundle $V$ has no theta divisor.

**Proof.** The following construction is practically identical to that in [13] Lemma A.2], substituting $P^{-1}K_X$ for $\zeta^{-1}$:

The map $Z \otimes \text{Sym}^{g-2}X \to J^{g-1}$ defined by

$$(z, D) \mapsto P^{-1}K_X \otimes \mathcal{O}_X(f^*z + D)$$

is birational to a locus $\mathcal{D}_1$ of dimension $g-1$ in $J^{g-1}_X$. By Lemma [1.9] we have $h^0(X, V \otimes L) \geq 2$ for all $L \in \mathcal{D}_1$. Therefore, by Laszlo [15] Proposition V.2 (see also [13] Theorem 1.1]), if $\mathcal{D}(V)$ is defined, then it contains the component $\mathcal{D}_1$ with multiplicity at least two.

Now if $\mathcal{D}(V)$ is defined, then it is linearly equivalent to $|3\Theta|$, where $\Theta$ is the Riemann theta divisor. Hence we have $\mathcal{D}(V) \cdot \Theta^{g-1} = 3g!$. Clearly

$$\mathcal{D}(V) \cdot \Theta^{g-1} \geq 2\mathcal{D}_1 \cdot \Theta^{g-1}.$$
so $\mathcal{D}(V) \cdot \Theta^{g-1} \geq 2\mathcal{D}_1 \cdot \Theta^{g-1} > 3g!$. This means that $\mathcal{D}(V)$ cannot be a divisor. \hfill \Box

**Proposition 4.11.** The bundle $V$ is stable.

*Proof.* We claim firstly that it suffices to show that $V$ is semistable. For, if $V$ is an extension $0 \to V_1 \to V \to V_2 \to 0$ where $\mu(V_1) = \mu(V_2) = \mu(V) = 0$, then

$$h^0(X, V \otimes L) \leq h^0(X, V_1 \otimes L) + h^0(X, V_2 \otimes L)$$

for all $L \in J_X^{g-1}$. Since $\text{rk} V_i \leq 2$, we have $h^0(X, V_i \otimes L) = 0$ for general $L \in J_X^{g-1}$ by Raynaud [19 Proposition 1.6.2]. This contradicts Proposition 4.10. Hence if $V$ is semistable, in fact it is stable. Since $V = \mathcal{W} \otimes P$, it suffices to prove that $\mathcal{W}$ is semistable.

Now $\mathcal{W}$ is an elementary transformation

$$0 \to W \to \mathcal{W} \to \mathcal{C}_y \to 0$$

where $\text{Ker} \left( W|_y \to \mathcal{W}|_y \right) = \ell$. Thus $\mathcal{W}$ contains no line subbundles of degree $> 2g - 1$ if and only if for all line subbundles $N \subset W$, we have

$$\deg N \leq \begin{cases} 2g - 1 & \text{if } N|_y \neq \ell \\ 2g - 2 & \text{if } N|_y = \ell. \end{cases}$$

Clearly, any line subbundle of $W$ with degree $\geq 2g - 1$ must lift from a subbundle of $f^*E$.

By Proposition 2.1 (2), for any $B \in J_Z^g$, the bundle $f^*E$ is an extension

$$0 \to f^*A \to f^*E \to f^*B \to 0$$

for some $A \in J_Z^{g-1}$. As $\deg f^*B = 2g$, if $f^*E$ had a line subbundle of degree $\geq 2g$, then for some $B \in J_Z^g$ the pullback exact sequence $f^*E \to f^*B$ would split. But then $f^*E = f^*A \oplus f^*B$, contradicting the fact that by Lemma 2.1 (2), there is an everywhere surjective vector bundle map $f^*E \to f^*B'$ for all $B'$ of degree $g$ over $Z$. This shows that all line subbundles of $f^*E$, and hence of $W$, are of degree $\leq 2g - 1$.

Since $\ell$ is chosen generally in $\mathbb{P}(W|_y) = \mathbb{P}^1$, it will suffice to show that $f^*E$ admits at most a finite number of line subbundles of degree $2g - 1$. Since $\mu(f^*E) = 2g - 1$, we need only show that $f^*E$ is not of the form $S \oplus S$ where $S$ is a line bundle of degree $2g - 1$.

To see this, note that for any $A \in J_Z^{-1}$, we have a vector bundle injection $A \to E$, and hence a vector bundle injection $f^*A \to f^*E$. As $f^*A$ has degree $2g - 2$, if $f^*E$ were of the form $S \oplus S$, then $f^*A$ would have to be of the form $S(-x)$ for some $x \in X$. But then any map $f^*A \to f^*E$ vanishes at $x$, contradicting the fact that $f^*A$ is a subbundle of $f^*E$. Thus $f^*E$ is not of the form $S \oplus S$, and therefore has at most finitely many line subbundles of degree $2g - 1$. Therefore $W$ can have at most finitely many such subbundles, and so we may assume $\ell$ does not coincide with the fibre of any of these at $y$.

It remains to check for desemistabilising subbundles of rank two. As above, we see that $\mathcal{W}$ contains no rank two subbundles of slope $> 2g - 1$ if and only if for all rank two
subbundles $H \subset W$ we have

$$\deg H \leq \begin{cases} 
4g - 2 & \text{if } \ell \not\subset H|_y \\
4g - 3 & \text{if } \ell \subset H|_y 
\end{cases}$$

For any rank two subbundle $H \subset W$, we have a diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & K_X & \rightarrow & W & \rightarrow & f^*E & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \downarrow \\
0 & \rightarrow & H_1 & \rightarrow & H & \rightarrow & H_2 & \rightarrow & 0
\end{array}
$$

where $H_1 = 0$ or $K_X$, and $H_2$ is a subsheaf of $f^*E$. In the first case, $H_2$ is an invertible subsheaf of $f^*E$. By the last paragraph, $\deg H_2 \leq 2g - 1$. Therefore $\deg H \leq 2g - 2 + 2g - 1 = 4g - 3$. If $H_1 = 0$ then $H = H_2$ is an elementary transformation of $f^*E$. Since $W$ is a nontrivial extension, $\deg H_2 \leq \deg f^*E - 1 = 4g - 3$, as required. This completes the proof that $W$ is semistable, and hence stable. \qed

Theorem 1.1 follows from the construction of $V$ and Propositions 4.10 and 4.11.

5. Remarks and questions

Remark 5.1. If $g = 3$ then one knows that $V$ has a well-defined theta divisor $D(V)$, by Beauville’s result [2]. By Lemma 4.9, $D(V) = 2 \cdot D_1 + D_2$. By (4.3), we have

$$2 \cdot D_1 \cdot \Theta^2 = 4 \cdot 2 \cdot 2! = 16.$$ 

On the other hand, $3 \cdot \Theta^3 = 3 \cdot 3! = 18$, so the residual component $D_2$ satisfies $D_2 \cdot \Theta^2 = 2$. In particular, it is nonempty.

Thus in this case the construction yields a stable bundle of rank three with reducible and nonreduced theta divisor. In [13], examples were constructed showing the existence of bundles of rank $r \geq 5$ with reducible and nonreduced theta divisor over curves of genus $g \geq 5$. The construction in §4 shows that at least in genus 3, such bundles exist already in rank 3 if the curve is bielliptic.

Question 5.2. If $g = 4$ then Pauly’s calculation gives the equality

$$2 \cdot D_1 \cdot \Theta^3 = 4 \cdot 3 \cdot 3! = 3 \cdot 4! = 3 \cdot \Theta^4.$$ 

Therefore, the construction gives a stable rank three bundle $V$ either with no theta divisor or with totally nonreduced theta divisor $D(V) = 2D_1$. This is for the moment unresolved.

Question 5.3. A more ambitious question is to ask whether every stable rank three bundle with no theta divisor is isomorphic to some $V$ of the form constructed in §4; and so in particular, whether such a bundle can only exist over a bielliptic curve. From Raynaud’s work [19] it follows that the curve must indeed have some nongeneric feature, but it is not clear to the present author what the appropriate conjecture should be. This will be a subject of continued study.
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