Left Truncated of Mixture Topp-Leone and Exponential Distribution with Estimation by Maximum likelihood

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Abstract
In this research, which contains three parameters, where a truncated state was taken from the left and a probability density function was found, as well as the moment function and the survival function, using the estimation method is the maximum likelihood estimation (MLE). And draw the figure of some of the mentioned functions using the Matlab program as well as estimating the values for the parameters.

Keywords: LT-TLGE, PDF, Moment function, Survival function, estimation, MLE.

1. Introduction

In studying statistical models that depend on analyzing and interpreting data there are some data that may be restricted within a specific area dependent on the nature of that restriction so that that data can be classified as truncated data and it is that in which certain values or a specific population group are excluded where it has excluding those values or groups is more accurate data analysis. Amputated samples were first discovered by Francis Galton in the year (1897) in connection with the analysis of the recorded speeds of American jogging horses as indicated in detail [1]. And the reference of many researchers on this topic. When truncation occurs on one side, it is called singly and if it is from two sides it is called doubly, in this paper, the truncation is on the
left of mixture Topp Leone and Exponential distribution and we symbolize it LT-TLGE

2. Topp-Leone and exponential distribution TLG-E

Display the formula for the cumulative distribution function and the probability density function for this distribution, which we found with some mathematical and statistical properties, which is the result of combining two distributions Topp-Leone and exponential.

\[
F(\rho) = \left\{ \left( 1 - e^{-\lambda \rho} \right)^{\alpha} \left[ 2 - \left( 1 - e^{-\lambda \rho} \right)^{\alpha} \right] \right\}^{\beta} \quad \ldots \ldots \ldots (1)
\]

\[
f(\rho) = 2 \beta \alpha \lambda e^{-\lambda \rho} \left( 1 - e^{-\lambda \rho} \right)^{\beta \alpha - 1} \left[ 2 - \left( 1 - e^{-\lambda \rho} \right)^{\alpha} \right]^{\beta - 1} \left( 1 - (1 - e^{-\lambda \rho})^{\alpha} \right) \quad \ldots \ldots \ldots \ldots (2)
\]

3. Left Truncated of TLG-E distribution

Let \( \rho \) be a random variable of truncated TLG-E distribution on support \( a \leq \rho < \infty \). In this case it is truncated on the left side, the CDF is the follows

\[
F_t(x; a \leq \rho < \infty) = \frac{\int_a^x f(t) \, dt}{F(\infty) - F(a)} = \frac{F(\rho) - F(a)}{1 - F(a)}
\]

\[
F_t(x; a \leq \rho < \infty) = \frac{F(\rho) - F(a)}{\delta(a)}
\]

\[
F_t(x; a \leq x < \infty) = \frac{\left\{ (1 - e^{-\lambda \rho})^\theta \left[ 2 - (1 - e^{-\lambda \rho})^\theta \right] \right\}^\beta - \left\{ (1 - e^{-\lambda \alpha})^\theta \left[ 2 - (1 - e^{-\lambda \alpha})^\theta \right] \right\}^\beta}{1 - \left\{ (1 - e^{-\lambda \alpha})^\theta \left[ 2 - (1 - e^{-\lambda \alpha})^\theta \right] \right\}^\beta} \quad \ldots \ldots \ldots (3)
\]

The part \( 2 - (1 - e^{-\lambda \rho})^\theta \) can be represented as follows

\[
2 - (1 - e^{-\lambda \rho})^\theta = \sum_{k=0}^\infty (\frac{\theta}{k}) \, \lambda^k \quad 2^{\beta - k} \, (1 - e^{-\lambda \rho})^\theta^k \quad \ldots \ldots \ldots (4)
\]
And also put the symbol for the fixed value \( \left( \frac{\beta}{k} \right) 2^{\beta-k} = \eta_k \)

\[ F_I(x; \alpha \leq \rho < \infty) = \frac{\sum_{k=0}^{\infty} \eta_k \left( 1 - e^{-\lambda \rho} \right)^{\theta(\beta+k)} - \sum_{k=0}^{\infty} \eta_k \left( 1 - e^{-\lambda \alpha} \right)^{\theta(\beta+k)}}{1 - \sum_{k=0}^{\infty} \eta_k \left( 1 - e^{-\lambda \alpha} \right)^{\theta(\beta+k)}} \] .... (5)

Obviously, the magnitude that is in the denominator is a function of survival for constant \( \alpha \)

\[ S(\alpha) = 1 - \sum_{k=0}^{\infty} \eta_k \left( 1 - e^{-\lambda \alpha} \right)^{\theta(\beta+k)} \] .... (6).

It is possible to represent the \( F_I(\rho) \) as

\[ F_I(x; \alpha \leq \rho < \infty) = \frac{S(\alpha) - S(\rho)}{S(\alpha)} \] ............... (7)

Figure (1)

In Figure (1) it shows that the (CDF) is truncated from the left such that \( 0.5 \leq \rho \leq \infty \), the parameter values are \( \lambda = 1.5, 1.1 \) and \( \theta = 1.6, 3 \) in the two curves with value stability \( \beta = 4 \).
In Figure (2) it shows that the (CDF) is truncated from the left such that $2 \leq \rho \leq \infty$, the parameter values are $\lambda = 1.5, 1.1$ and $\theta = 1.6, 3$ in the two curves with value stability $\beta = 203$.

The PDF for Truncated Left of TLGE distribution and we symbolize it $T_l(\rho)$

$$T_l(\rho) = \frac{\sum_{k=0}^{\infty} \eta_k \theta(\beta + k)\lambda e^{-\lambda \rho} (1 - e^{-\lambda \rho})^{\theta(\beta+k)-1}}{1 - \sum_{k=0}^{\infty} \eta_k (1 - e^{-\lambda \rho})^{\theta(\beta+k)}} \ldots \ldots \ldots (8)$$

The numerator in (5) is a PDF of TLGE

$$f(\rho) = \sum_{k=0}^{\infty} \eta_k \theta(\beta + k)\lambda e^{-\lambda \rho} (1 - e^{-\lambda \rho})^{\theta(\beta+k)-1}.$$  

And thus becomes a PDF for Truncated Left

$$T_l(\rho) = \frac{f(\rho)}{S(\alpha)} \ldots \ldots \ldots (9)$$
In Figure (3) it shows that the (PDF) is truncated from the left such that $0.5 \leq \rho \leq \infty$, the parameter values are $\beta = 3, 4$ and $\theta = 4, 3$ in the two curves with value stability $\lambda = 1.5$.

In Figure (4) it shows that the (PDF) is truncated from the left such that
1 ≤ ρ ≤ ∞, the parameter values are λ = 1.1, 1.5 and θ = 4, 6 in the two curves with value stability β = 1.5.

4. Moment of Left Truncated

\[ \mu'_s = \int_{a}^{\infty} \rho^s T_1(\rho) d\rho \]

By substituting the relationship of (5) in the moment function

\[ \mu'_s = \int_{a}^{\infty} \rho^s \sum_{k=0}^{\infty} \eta_k \theta(\beta + k) \lambda e^{-\lambda \rho} \left( 1 - e^{-\lambda \rho} \right)^{\theta(\beta+k)-1} \]

\[ = \frac{\sum_{k=0}^{\infty} \eta_k}{\delta(a)} \int_{a}^{\infty} \rho^s \theta(\beta + k) \lambda e^{-\lambda \rho} \left( 1 - e^{-\lambda \rho} \right)^{\theta(\beta+k)-1} d\rho \]

Using Integral by part

\[ = \frac{\sum_{k=0}^{\infty} \eta_k}{\delta(a)} \left\{ -a^s \left( 1 - e^{-\lambda a} \right)^{\theta(\beta+k)} - s \int_{a}^{\infty} \rho^{s-1} \left( 1 - e^{-\lambda \rho} \right)^{\theta(\beta+k)} d\rho \right\} \]

\[ (1 - e^{-\lambda \rho})^{\theta(\beta+k)} = \sum_{h=0}^{\infty} \left( \frac{\theta(\beta+k)}{h} \right)(-1)^{\theta(\beta+k)-h} e^{-\lambda h \rho} \ldots \ldots \ldots \ldots \ldots \ldots \ast \]

Substitution (*) in the step that preceded it then

\[ \mu'_s = \frac{\sum_{k=0}^{\infty} \eta_k}{\sum_{h=0}^{\infty} \left( \frac{\theta(\beta+k)}{h} \right)(-1)^{\theta(\beta+k)-h} \left\{ -a^s e^{-\lambda a h} - s \int_{a}^{\infty} \rho^{s-1} e^{-\lambda h x} dx \right\} } \]

Using the upper incomplete gamma function for integral above

\[ \mu'_s = \sum_{k=0}^{\infty} \eta_k \sum_{h=0}^{\infty} \left( \frac{\theta(\beta+k)}{h} \right)(-1)^{\theta(\beta+k)-h+1} \frac{1}{\delta(a)} \left\{ a^s e^{-\lambda a h} + \frac{s}{\lambda h} \Gamma(s, a) \right\} \]

4.1. Moment about mean of Left Truncated

\[ \mu_s = E(X - \mu)^s = \sum_{j=0}^{s} \binom{s}{j} (-1)^j (\mu'_s - \mu)^j \]
Remarks: From the formula (11) we getting of the following:

(1) \( \mu_0 = 1 \), \( \mu_1 = 0 \).

(2) \( \mu_2 = \text{Variance}(\sigma_p^2) = \mu'_2 - \mu'_1^2 \).

(3) \( \mu_3 = \text{Skewness}(\text{Sk}) = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3 \).

(4) \( \mu_4 = \text{Kurtosis}(\text{Ku}) = \mu'_4 - 4\mu'_1\mu'_1 + 6\mu'_1\mu'_1 - 3\mu'_1^4 \).

4.2. Mean

To find the mean of (7) is when \( s = 1 \) That is \( \mu'_1 = E(x) \)

\[
\mu'_1 = E(\rho) = \sum_{k=0}^{\infty} \eta_k \sum_{h=0}^{\infty} \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\beta+k-h} - h + 1 \frac{1}{\delta(a)} \left\{ a e^{-\lambda a h} + \frac{1}{\lambda h} \Gamma(1, a) \right\}
\]

\[\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (10A)\]

Likewise, it is \( \mu'_2 = E(\rho^2) \) such that

\[
\mu'_2 = E(\rho^2) = \sum_{k=0}^{\infty} \eta_k \sum_{h=0}^{\infty} \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\beta+k-h} - h + 1 \frac{2}{\lambda h} \left\{ a^2 e^{-\lambda a h} + \frac{2}{(\lambda h)^2} \Gamma(2, a) \right\}
\]

\[\ldots \ldots \ldots \ldots (10B)\]

4.3. Variance

\[
\text{Var}(\rho) = E(\rho^2) - (E(\rho))^2
\]

\[
\text{Var}(\rho) = \sum_{k=0}^{\infty} \eta_k \sum_{h=0}^{\infty} \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\beta+k-h} - h + 1 \frac{2}{\lambda h} \left\{ a^2 e^{-\lambda a h} + \frac{2}{(\lambda h)^2} \Gamma(2, a) \right\} - \sum_{k=0}^{\infty} \eta_k \sum_{h=0}^{\infty} \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\beta+k-h} - h + 1 \frac{1}{R(a)} \left\{ a e^{-\lambda a h} + \frac{1}{(\lambda h)^r} \Gamma(1, a) \right\}
\]
\[
\frac{1}{\lambda h} \Gamma(1, a) \right) \right) * \sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a e^{-\lambda ah} + \frac{1}{\lambda h} \Gamma(1, a) \right\} + \\
\frac{1}{\lambda h} \Gamma(1, a) \right) \right) \\
= \sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a^2 e^{-\lambda ah} + \frac{2}{(\lambda h)^2} \Gamma(2, a) \right\} - \\
\sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a e^{-\lambda ah} + \frac{1}{\lambda h} \Gamma(1, a) \right\} \right)^2 \\
\] 

\[= \sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a^2 e^{-\lambda ah} + \frac{3}{(\lambda h)^3} \Gamma(3, a) \right\} - \\
3 \sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a e^{-\lambda ah} + \frac{2}{(\lambda h)^2} \Gamma(2, a) \right\} * \\
\sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a e^{-\lambda ah} + \frac{1}{\lambda h} \Gamma(1, a) \right\} + \\
2(\sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a e^{-\lambda ah} + \frac{1}{\lambda h} \Gamma(1, a) \right\} \\
\] 

\[\ldots \ldots (12)\]

4.4. Skewness

\[
(Sk) = \\
= \sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a^2 e^{-\lambda ah} + \frac{3}{(\lambda h)^3} \Gamma(3, a) \right\} - \\
3 \sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a e^{-\lambda ah} + \frac{2}{(\lambda h)^2} \Gamma(2, a) \right\} * \\
\sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a e^{-\lambda ah} + \frac{1}{\lambda h} \Gamma(1, a) \right\} + \\
2(\sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a e^{-\lambda ah} + \frac{1}{\lambda h} \Gamma(1, a) \right\} \\
\] 

\[\ldots \ldots (13)\]

4.5. Kurtosis

\[
(Ku) = \\
= \sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a^4 e^{-\lambda ah} + \frac{4}{(\lambda h)^4} \Gamma(4, a) \right\} - \\
4 \sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a e^{-\lambda ah} + \frac{1}{\lambda h} \Gamma(1, a) \right\} * \\
\sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a^3 e^{-\lambda ah} + \frac{3}{(\lambda h)^3} \Gamma(3, a) \right\} + \\
6(\sum_{k=0}^\infty \eta_k \sum_{h=0}^\infty \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a e^{-\lambda ah} + \frac{1}{\lambda h} \Gamma(1, a) \right\} \right)^2 * \\
\]
\[ \sum_{k=0}^{\infty} \eta_k \sum_{h=0}^{\infty} \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a^s e^{-\lambda a h} + \frac{1}{(\lambda h)^2} \Gamma(2, a) \right\} - \\
3 \left( \sum_{k=0}^{\infty} \eta_k \sum_{h=0}^{\infty} \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a^s e^{-\lambda a h} + \frac{1}{\lambda h} \Gamma(1, a) \right\} \right)^4 \\
\ldots \ldots \ldots \ldots (14) \]

### 4.6. Moment Generating Function of left truncated

The moment generating function (m.g.f) of random variable \( x \) for left truncated distribution TLGE

\[
M_{x}(t) = E(e^{t\beta}) = \sum_{s=0}^{\infty} \frac{t^s}{s!} \mu'_{s} \\
M_{x}(t) = \sum_{s=0}^{\infty} \frac{t^s}{s!} \sum_{k=0}^{\infty} \eta_k \sum_{h=0}^{\infty} \left( \frac{\theta(\beta+k)}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a^s e^{-\lambda a h} + \frac{1}{(\lambda h)^2} \Gamma(2, a) \right\} - \\
\frac{s}{(\lambda h)^s} \Gamma(s, a) \right\} \\
\ldots \ldots \ldots \ldots (15) \]

### 5. Survival of truncated left

The survival function of truncated left can be represented as follows

\[
S_{i}(t) = 1 - F_{i}(t) \quad \text{from (3) getting to} \\
S_{i}(t) = 1 - \frac{\sum_{k=0}^{\infty} \eta_k (1-e^{-\lambda t})^{\theta(\beta+k)} - \sum_{k=0}^{\infty} \eta_k (1-e^{-\lambda a})^{\theta(\beta+k)}}{1-\sum_{k=0}^{\infty} \eta_k (1-e^{-\lambda a})^{\theta(\beta+k)}} \\
\ldots \ldots \ldots \ldots (16) \]

From (4) it is possible to deduce a formula for the survival function in general when the truncated left

\[
S_{i}(t) = 1 - \frac{s(a)-s(t)}{s(a)} \implies S_{i}(t) = \frac{s(t)}{s(a)} \\
\ldots \ldots \ldots \ldots (17) \]
In Figure (5) it shows that the (survival function) is truncated from the left such that $0.5 \leq \rho \leq \infty$, the parameter values are $\lambda = 1.5,1.1$ and $\theta = 1.4,3$ in the two curves with value stability $\beta = 5$.

In Figure (6) it shows that the (survival function) is truncated from the left such that $2 \leq \rho \leq \infty$, the parameter values are $\lambda = 1.5,1.1$ and $\theta = 1.4,3$ in the two curves with value stability $\beta = 203$. 
5.1 Hazard function

\[ H_i(t) = \frac{f_i(t)}{S_i(t)} \]

from (6) & (10) then hazard function

\[ H'_i(t) = \frac{f(t)}{S(t)} \Rightarrow H'_i(t) = \frac{f(t)}{S(t)} \ldots \ldots \ldots (18) \]

It should be noted that most of the statistical properties differ in the case of the truncated left distribution from the original distribution, but in the hazard function it is observed that there is no difference. When the \( H_i(t) \) function is constant for all values of the random variable this means that the model is suitable for lifetime data.

6. Maximum likelihood estimator MLE of Truncated Left

The (MLE) method is considered one of the most important methods used to estimate truncated distributions and refer to it as a group of statisticians. And of them [1], [4], [6] and others we did not refer to them we denote the maximum likelihood of truncated (TLG-E) from left to symbol.

\[ L_i(\sigma, \theta, \beta) = \prod_{i=1}^{n} \frac{f_i(\sigma)}{1-F(a)} \]

\[ = \frac{(2\theta \lambda)^n e^{-\lambda a} \sum_{i=1}^{n} \rho_i \sum_{i=1}^{n} (1-e^{-\lambda \rho_i})^{\beta-1} 2^{-\beta-1}}{[R(a)]^n \left(1-e^{-\lambda \rho_i})^{\beta} \right) \ldots (19) \]

take the \( \ln \) to both sides

\[ \ell_i = \ln L_i(\sigma, \theta, \beta) = n \ln 2 + n \ln \theta + n \ln \beta + n \ln \lambda - \lambda \sum_{i=1}^{n} \rho_i + n(\beta \theta - 1) \ln \sum_{i=1}^{n} (1 - e^{-\lambda \rho_i}) + n(\beta - 1) \ln \left(\sum_{i=1}^{n} (2 - (1 - e^{-\lambda \rho_i})^{\theta}) \right) + \]
\[
\text{nl}(\sum_{i=1}^{n}(1 - (1 - e^{-\lambda \rho_i})^\theta)) - \text{nl}(1 - [(1 - e^{-\lambda \rho_i})^\theta(2 - (1 - e^{-\lambda \rho_i})^\theta)]) = 0
\] ... ... ... (20)

derive the function \( (\ell_i) \) of partial derivation with respect to the parameters \( \beta, \lambda, \theta \) and then the relationships that were derived are equated to zero

\[
\frac{\partial \ell_i}{\partial \beta} = \frac{n}{\beta} + n \theta \ln(\sum_{i=1}^{n}(1 - e^{-\lambda \rho_i})) + n \ln(1 - [(1 - e^{-\lambda \rho_i})^\theta(2 - (1 - e^{-\lambda \rho_i})^\theta)]) + \frac{n \beta(1 - e^{-\lambda \rho_i})^\theta(2 - (1 - e^{-\lambda \rho_i})^\theta)}{1 - (1 - e^{-\lambda \rho_i})^\theta(2 - (1 - e^{-\lambda \rho_i})^\theta)} - \frac{n \theta(1 - e^{-\lambda \rho_i})^\theta}{\sum_{i=1}^{n}(1 - e^{-\lambda \rho_i})^\theta}
\]

\[
0 = ... ... ... (20a)
\]

\[
\frac{\partial \ell_i}{\partial \lambda} = \frac{n}{\lambda} - \frac{n(\beta - 1) \sum_{i=1}^{n}(1 - e^{-\lambda \rho_i})^\theta}{\sum_{i=1}^{n}(1 - e^{-\lambda \rho_i})^\theta} - \frac{n \theta(1 - e^{-\lambda \rho_i})^\theta}{\sum_{i=1}^{n}(1 - e^{-\lambda \rho_i})^\theta} + \frac{n \theta(1 - e^{-\lambda \rho_i})^\theta}{\sum_{i=1}^{n}(1 - e^{-\lambda \rho_i})^\theta} - \frac{n \theta e^{-\lambda \rho_i}(1 - e^{-\lambda \rho_i})^\theta(2 - (1 - e^{-\lambda \rho_i})^\theta)}{1 - (1 - e^{-\lambda \rho_i})^\theta(2 - (1 - e^{-\lambda \rho_i})^\theta)}
\]

\[
0 = ... ... ... (20b)
\]

\[
\frac{\partial \ell_i}{\partial \theta} = \frac{n}{\theta} + n \beta \ln(\sum_{i=1}^{n}(1 - e^{-\lambda \rho_i})) - \frac{n(\beta - 1) \sum_{i=1}^{n}(1 - e^{-\lambda \rho_i})^\theta}{\sum_{i=1}^{n}(1 - e^{-\lambda \rho_i})^\theta} + \frac{n \beta(1 - e^{-\lambda \rho_i})^\theta(2 - (1 - e^{-\lambda \rho_i})^\theta)}{1 - (1 - e^{-\lambda \rho_i})^\theta(2 - (1 - e^{-\lambda \rho_i})^\theta)}
\]

\[
0 = ... ... ... (20c)
\]

Estimation Parameter of LT-TLGE by MLE
If we want to apply the truncated data from the left to the patients, for example, in the censored data when some of them are missing or a break or death occurs when observing the data for the patients. We tested the data published by the WHO in the Middle East for the Covid 19 pandemic on 18-6-2020 for the cumulative numbers of deceased patients for twenty-two countries, the numbers were as follows [546, 52, 43, 1850, 9272, 773, 9, 308, 32, 10, 213, 119, 3093, 5, 86, 1139, 88, 487, 7, 50, 298, 245]. apply this data with a function (MLE) using the Matlab program and the results are as shown in the table below.

Table (1)

| Iteration | MLE   | λ    | θ    | β    |
|-----------|-------|------|------|------|
| LT-TLGE   | 2881.565 | 31.506 | 0.086 | 33.546 |
|           | 2365.546 | 28.913 | 0.285 | 26.4   |
| W         | 365.383  | 28.913 | 0.285 |       |
|           | 262.914  | 1.585  | 0.003 |       |
| EW        | 80.6     | 3.898  | 0.205 | 6.221  |
|           | 55.139   | 28.913 | 0.285 | 26.4   |

7. Method of Moment Estimation

Let $\rho \sim TLTLGE(\theta, \beta, \lambda)$, the method of moment estimation of TLTLGE distribution is defined as

$$E(\rho^s) = \sum_{j=1}^{n} \frac{1}{n} \rho_j^s$$

Where $E(\rho^r)$ is a $r^{th}$ moment about the origin point from (10) taking of three cases.
\[ \sum_{k=0}^{\infty} \eta_k \sum_{h=0}^{\infty} \left( \frac{\theta^{(\beta+k)}}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a^s e^{-\lambda a} + \frac{s}{\lambda h} \Gamma(s, a) \right\} = \sum_{j=1}^{n} \frac{1}{n} \rho_j^s \quad \ldots \ldots (21) \]

When \( s = 1 \)

\[ \sum_{k=0}^{\infty} \eta_k \sum_{h=0}^{\infty} \left( \frac{\theta^{(\beta+k)}}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a e^{-\lambda a} + \frac{1}{\lambda h} \Gamma(1, a) \right\} = \sum_{j=1}^{n} \frac{1}{n} \rho_j \]

When \( s = 2 \)

\[ \sum_{k=0}^{\infty} \eta_k \sum_{h=0}^{\infty} \left( \frac{\theta^{(\beta+k)}}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a^2 e^{-\lambda a} + \frac{2}{\lambda h} \Gamma(2, a) \right\} = \sum_{j=1}^{n} \frac{1}{n} \rho_j^2 \]

When \( s = 3 \)

\[ \sum_{k=0}^{\infty} \eta_k \sum_{h=0}^{\infty} \left( \frac{\theta^{(\beta+k)}}{h} \right) (-1)^{\theta(\beta+k)-h+1} \frac{1}{R(a)} \left\{ a^3 e^{-\lambda a} + \frac{3}{\lambda h} \Gamma(3, a) \right\} = \sum_{j=1}^{n} \frac{1}{n} \rho_j^3 \]

8. Conclusion

It is concluded from the above that the model (LT-TLGE) is from the distributions of life and can be applied in the fields of health and biology, and we can say that the truncated distribution from the left is a special case of the survival function.

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