Explicit and Implicit Dynamic Coloring of Graphs with Bounded Arboricity

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Abstract

Graph coloring is a fundamental problem in computer science. We study the fully dynamic version of the problem in which the graph is undergoing edge insertions and deletions and we wish to maintain a vertex-coloring with small update time after each insertion and deletion.

We show how to maintain an $O(\alpha \log n)$-coloring with polylogarithmic update time, where $n$ is the number of vertices in the graph and $\alpha$ is the current arboricity of the graph. This improves upon a result by Solomon and Wein (ESA’18) who maintained an $O(\alpha_{\text{max}} \log^2 n)$-coloring, where $\alpha_{\text{max}}$ is the maximum arboricity of the graph over all updates.

Furthermore, motivated by a lower bound by Barba et al. (Algorithmica’19), we initiate the study of implicit dynamic colorings. Barba et al. showed that dynamic algorithms with polylogarithmic update time cannot maintain an $f(\alpha)$-coloring for any function $f$ when the vertex colors are stored explicitly, i.e., for each vertex the color is stored explicitly in the memory. Previously, all dynamic algorithms maintained explicit colorings. Therefore, we propose to study implicit colorings, i.e., the data structure only needs to offer an efficient query procedure to return the color of a vertex (instead of storing its color explicitly). We provide an algorithm which breaks the lower bound and maintains an implicit $2^{O(\alpha)}$-coloring with polylogarithmic update time. In particular, this yields the first dynamic $O(1)$-coloring for graphs with constant arboricity such as planar graphs or graphs with bounded tree-width, which is impossible using explicit colorings.

To obtain our implicit coloring result we show how to dynamically maintain a partition of the graph’s edges into $O(\alpha)$ forests with polylogarithmic update time. We believe this data structure is of independent interest and might have more applications in the future.

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# 1 Introduction

Graph coloring is one of the most fundamental and well-studied problems in computer science. Given a graph $G = (V, E)$ with $n$ vertices, a $C$-coloring assigns a color from $\{1, \ldots, C\}$ to each vertex. The coloring is proper if all adjacent vertices have different colors. The smallest $C$ for which there exists a proper $C$-coloring is called the chromatic number of $G$. Unfortunately, it is NP-hard to approximate the chromatic number within a factor of $n^{1-\epsilon}$ for all $\epsilon > 0$ [21, 32]. Hence, graph coloring is usually studied w.r.t. certain graph parameters such as the maximum degree $\Delta$ of any vertex or the arboricity $\alpha$, which is the minimum number of forests into which the edges of $G$ can be partitioned. It is well-known that proper $(\Delta + 1)$-colorings and proper $O(\alpha)$-colorings can be computed in polynomial time.

In the dynamic version of the problem, the graph is undergoing edge insertions and deletions and a data structure needs to maintain a proper coloring with small update time. More concretely, suppose there are $m$ update operations each inserting or deleting a single edge. This implies an sequence of graphs $G_0, G_1, \ldots, G_m$ such that $G_i$ and $G_{i+1}$ differ by exactly one edge. Then for each $G_i$ the dynamic algorithm must maintain a proper coloring.

When studying dynamic algorithms w.r.t. graph parameters such as the maximum degree $\Delta$ or the arboricity $\alpha$ it is important that the dynamic algorithms are adaptive to the parameter. That is, during a sequence of edge insertions and deletions, the values of parameters such as $\Delta$ and $\alpha$ might change over time. For example, suppose $\alpha(G_i)$ is the arboricity of $G_i$ and let $\alpha_{\text{max}} = \max_i \alpha(G_i)$ denote the maximum arboricity of all graphs. Then ideally we would like that after the $i$'th update the number of colors used by a dynamic algorithm depends on $\alpha(G_i)$ and not on $\alpha_{\text{max}}$ because it might be that $\alpha(G_i) \ll \alpha_{\text{max}}$.

Bhattacharya et al. [8] studied the dynamic coloring problem and showed how to maintain a $(\Delta + 1)$-coloring with polylogarithmic update time and their algorithm is adaptive to the current maximum degree of the graph. In follow-up work [9, 19] the update time was improved to $O(1)$.

Later, Solomon and Wein [30] provided a dynamic $O(\alpha_{\text{max}} \log^2 n)$-coloring algorithm with polylog($\log n$) update time. Note that the number of colors used by [30] depends on maximum arboricity $\alpha_{\text{max}}$ over all graphs $G_i$. Hence, we ask the following question.

**Question 1.** Are there dynamic coloring algorithms with polylogarithmic update time which maintain a coloring that is adaptive to the current arboricity of the graph?

Another interesting question concerns limitations of dynamic coloring algorithms. A lower bound of Barba et al. [3] shows that there exist dynamic graphs which are 2-colorable but any dynamic algorithm maintaining a $c$-coloring must recolor $\Omega\left(n^{\frac{2}{c^2 - 1}}\right)$ vertices after each update. The lower bound holds even for forests, i.e., for graphs with arboricity $\alpha = 1$. This implies that any dynamic algorithm maintaining an $f(\alpha)$-coloring for any function $f$ must recolor $n^{\Omega(1)}$ vertices after each update. Note that this rules out dynamic $O(1)$-colorings for forests and, more generally, planar graphs with polylogarithmic update times.

However, the lower bound only applies to dynamic algorithm that are maintaining explicit colorings. That is, a coloring is explicit if after each update the data structure stores an array $C$ of length $n$ such that $C[u]$ stores the color of vertex $u$. Thus, the color of each vertex can be determined with a single memory access. All of the previously mentioned dynamic coloring algorithms maintain explicit colorings but are allowed to use more than a constant number of colors.

In the light of the above lower bound, it is natural to ask whether it can be bypassed by implicit colorings. That is, a coloring is implicit if the data structure offers a query routine
query(v) which after some computation returns the color of a vertex v. In particular, we require the following consistency requirement for the query operation:

- Consider any sequence of consecutive query operations \( \text{query}(v_1), \ldots, \text{query}(v_k) \) which are not interrupted by an update. Then if vertices \( v_i \) and \( v_j \), \( i \neq j \), are adjacent, we have that \( \text{query}(v_i) \neq \text{query}(v_j) \).

Note that in the above definition we only consider consecutive query operations which are not interrupted by an update. This is because after an update potentially a lot of vertex colors may change (due to the lower bound). Furthermore, observe that the definition implies that if we query all vertices of the graph consecutively, then we obtain a proper coloring.

Observe that an explicit coloring always implies an implicit coloring: when queried for a vertex \( u \), the data structure simply returns \( C[u] \). However, implicit colorings are much more versatile than explicit colorings: when the colors of many vertices change, this does not affect the implicit coloring because it does not have to update the array \( C \). Hence, we ask the following natural question.

▶ Question 2. Can we break the lower bound of Barba et al. [3] with algorithms maintaining implicit colorings?

## 1.1 Our Contributions

We answer both questions affirmatively.

### Adaptive explicit colorings.
First, we show that there exists a randomized algorithm which maintains an explicit and adaptive \( O(\alpha \log n) \)-coloring with polylogarithmic update time. This answers Question 1 affirmatively.

▶ Theorem 1. There is a randomized data structure that maintains an explicit and adaptive \( O(\alpha \log n) \)-coloring on a graph with \( n \) vertices and arboricity \( \alpha \) with expected amortized update time \( O(\log^2 n) \).

Note that this improves upon the results in [30] in two ways: It makes the coloring adaptive and it shaves a \( \log n \)-factor in the number of colors used by the algorithm. To obtain our result, we use a similar approach as the one used in [30]. In [30], the vertices were assigned to \( O(\log n) \) levels and the vertices on each level were colored using \( O(\alpha_{\max} \log n) \) colors. In our result, we assign the vertices to \( O(\log^2 n) \) levels and partition the levels into groups of \( O(\log n) \) consecutive levels each. We then make sure that for coloring the \( \ell \)'th group we use only \( O(2^\ell \log n) \) colors and that the levels of groups with \( \ell > \Omega(\log \alpha) \) are empty. Then a geometric sum argument implies that we use \( O(\alpha \log n) \) colors in total.

### Adaptive implicit colorings.
Furthermore, we provide two algorithms maintaining implicit colorings. Both of these algorithms are also adaptive. We first provide an algorithm which maintains an adaptive implicit \( 2^{O(\alpha)} \)-coloring with polylogarithmic update time and query time \( O(\alpha \log n) \). This improves upon the coloring of Theorem 1 for \( \alpha = o(\log \log n) \).

▶ Theorem 2. There is a deterministic data structure that maintains an adaptive implicit \( 2^{O(\alpha)} \)-coloring, with update time \( O(\log^3 n) \) and query time \( O(\alpha \log n) \), where \( \alpha \) is the current arboricity of the graph.

Note that Theorem 2 implies that for graphs with constant arboricity we can maintain \( O(1) \)-colorings with polylogarithmic update and query times. This class of graphs contains

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1 As usual in the study of randomized dynamic algorithms we assume that the adversary is oblivious, i.e., that the sequence of edge insertions and deletions is fixed before the algorithm runs.
trees, planar graphs, graphs with bounded tree-width, and all minor-free graphs. In particular, this breaks the lower bound of Barba et al. \[3\] and answers Question 2 affirmatively.

**Corollary 3.** There is a deterministic data structure that for dynamic graphs with constant arboricity maintains an implicit $O(1)$-coloring, with update time $O(\log^3 n)$ and query time $O(\log n)$.

Next, we improve upon the results of Theorem 1 and Theorem 2 in the parameter regime $\Omega(\log \log n) \leq \alpha \leq \log^{o(1)} n$. More concretely, we obtain the following result.

**Theorem 4.** There is a deterministic data structure that maintains an adaptive implicit $O(\alpha \log n \cdot \min\{1, \log \alpha/\log \log n\})$-coloring with an amortized update time of $O(\log^2 n)$ and a query time of $O(\log n)$.

**Dynamic arboricity decomposition.** To derive the results of Theorem 2, we introduce a data structure which maintains an adaptive arboricity decomposition of a dynamic graph. That is, it explicitly maintains a partition of the edges of the dynamic graph into $O(\alpha)$ undirected forests. This data structure might be of independent interest and might be useful in future applications.

To obtain the result we assume that we have black box access to an algorithm maintaining a low-outdegree orientation of the graph. More concretely, a $D$-outdegree edge-orientation for an undirected graph $G = (V, E)$ assigns a direction to each edge and ensures that each vertex has outdegree at most $D$. We then provide a reduction showing that any data structure maintaining a $D$-outdegree orientation of a graph can be turned into a data structure for maintaining an arboricity decomposition.

**Theorem 5.** Let $G$ be a dynamic graph. Suppose there exists a data structure with (amortized or worst-case update) update time $T$ maintaining a $D$-outdegree orientation of $G$. Then there exists a data structure that maintains an arboricity decomposition of $G$ with $2D$ forests and with (amortized or worst-case, resp.) update time $O(T)$.

Theorem 5 yields the following corollary which we obtain by showing that a data structure of Bhattacharya et al. [10] can be extended to maintain an adaptive $O(\alpha)$-outdegree orientation of an undirected graph (see Section 2).

**Corollary 6.** There exists a deterministic adaptive data structure that maintains a partition of the edges into $O(\alpha)$ forests with amortized update time $O(\log^2 n)$, where $\alpha$ is the current arboricity of the graph.

The corollary complements a result by Banerjee et al. [2] who presented an algorithm for dynamically maintaining an arboricity decomposition consisting of exactly $\alpha$ forests with $O(m \cdot \text{poly}(\log n))$ update time, where $m$ is the number of edges currently in the graph. Thus the result in the corollary obtains an exponentially faster update time by increasing the number of forests by a constant factor.

See Appendix A for more discussions on known data structures for maintaining low outdegree edge-orientations and also further related work.

## Level Data Structure

In this section we introduce a version of the data structure presented in [10], which we will refer to as *level data structure*. The data structure dynamically maintains an adaptive $O(\alpha)$-outdegree orientation of a dynamic graph, where $\alpha$ is the current arboricity of the graph.
More precisely, the level data structure maintains an undirected graph with \( n \) vertices and provides an update operation for inserting and deleting edges. It maintains an orientation of the edges of the graph such that each vertex has outdegree at most \( O(\alpha) \). We emphasize that, unlike the data structure presented in [10], our level data structure does not require that \( \alpha \) is an upper bound on the maximum arboricity of the graph over the whole sequence of edge insertions and deletions.

For the rest of the paper, we will write \( \{u, v\} \) to denote undirected edges and \( (u, v) \) to denote directed edges.

**Levels, Groups and Invariants.** Internally, the data structure maintains a partition of the of the vertices into \( k = O(\log^2 n) \) levels which we call hierarchy. For each \( i = 1, \ldots, k \), we let \( V_i \) denote the set of vertices that are currently assigned to level \( i \). Furthermore, we partition the levels into groups \( G_1, \ldots, G_{\lceil \log n \rceil} \) such that each group contains \( L = 2 + \lceil \log n \rceil \) consecutive levels. More precisely, for each \( \ell \in \mathbb{N}_0 \), we set \( G_\ell = \{\ell L + 1, \ldots, (\ell + 1)L\} \). Note that \( k = L \cdot \lceil \log n \rceil \) and that neither the total number of level \( k \) nor the number of levels per group \( L \) depend on the arboricity.

The data structure maintains the following invariants for each vertex \( v \):

1. If \( v \in V_i \), \( i < k \) and \( i \in G_\ell \), then \( v \) has at most \( 5 \cdot 2^\ell \) neighbors in \( \bigcup_{j \geq i} V_j \). That is, each vertex \( v \) has at most \( 5 \cdot 2^\ell \) neighbors at its own or higher levels.
2. If \( v \in V_i \), \( i > 1 \) and \( i \in G_\ell \), then \( v \) has at least \( 2^{\ell'} \) neighbors in levels \( \bigcup_{j \geq i-1} V_j \), where \( \ell' \) is such that \( i - 1 \in G_{\ell'} \). That is, each vertex \( v \) has at least \( 2^{\ell'} \) neighbors at levels \( i - 1 \) and above.

Due to edge insertions and deletions the above invariants might get violated. If a vertex \( v \) does not satisfy the invariants, we call it dirty. Otherwise, we say that \( v \) satisfies the degree-property.

Note that the above partitioning of the vertices implies an edge orientation: For an (undirected) edge \( \{u, v\} \) such that \( u \in V_i \) and \( v \in V_{i'} \), we assign the orientations as follows: \( (u, v) \) if \( i < i' \), \( (v, u) \) if \( i > i' \) and an arbitrary orientation if \( i = i' \). This corresponds to directing an edge from the vertex of lower level towards the vertex of higher level in the hierarchy. Note that due to Invariant 1, each vertex at level \( i \in G_\ell \) has outdegree at most \( 5 \cdot 2^\ell \).

**Initialization and Data Structures.** The initialization of the data structure is implemented as follows. We assume that at the beginning the data structure is given a graph with \( n \) vertices and no edges. We initialize the sets \( V_i \) by setting \( V_1 := V \) and \( V_i := \emptyset \) for all \( i = 2, \ldots, k \). The groups \( G_\ell \) are defined as above and do not depend on the edges of the graph.

Furthermore, for each vertex \( v \) with \( v \in V_i \), we maintain the following data structures. For each level \( i' < i \), we maintain a doubly-linked list \( \text{neighbors}(v, i') \) containing all neighbors of \( v \) in \( V_{i'} \). Furthermore, there is a doubly-linked list \( \text{neighbors}(v, i) \) containing all neighbors of \( v \) in \( \bigcup_{j \geq i} V_j \). Additionally, for each edge \( \{u, v\} \) we store a pointer to the position of \( v \) in \( \text{neighbors}(u, \cdot) \) and vice versa. Note that by additionally maintaining for each list \( \text{neighbors}(v, \cdot) \) the number of vertices stored in the list, we can check in time \( O(1) \) whether one of the invariants is violated for \( v \).

**Updates.** Now suppose that an edge \( e = \{u, v\} \) is inserted or deleted. Then one of the vertices might get dirty and we have to recover the degree-property. While there exists a dirty vertex \( v \) with \( v \in V_i \), we proceed as follows. If \( v \) violates Invariant 1, we move \( v \) to level \( i + 1 \). If \( v \) violates Invariant 2, we move \( v \) to level \( i - 1 \). Note that during the above process, the algorithm might change the levels of vertices \( v' \) with \( v' \notin \{u, v\} \).

Observe that when a vertex \( v \) changes its level due to one of these operations, it is
We present an algorithm that maintains an $O(\alpha \log n)$-coloring using the level data structure from Section 2. To obtain our coloring, we will assign disjoint color palettes to all levels of the data structure. Our main observation is that since the level data structure guarantees that each vertex at level $i \in G_\ell$ has at most $O(2^\ell)$ neighbors at its own level, it suffices to use $O(2^\ell)$ colors for level $i$. Then a geometric sum argument yields that we only use $O(\alpha \log n)$ colors in total. As before, we do not require an upper bound on $\alpha$ in advance, but the number of colors only depends on the current arboricity of the graph.

Initialization. Again, assume that when the data structure is initialized, we are given a graph with $n$ vertices and no edges. For this graph, we build the level data structure from Section 2. Furthermore, to each level $i$ in some group $\ell$ we assign a new palette of $(K + \epsilon) \cdot 2^\ell$ colors, where $K$ is as in Lemma 7 and $\epsilon = 1/10$. At the very beginning, we assign a random color to each vertex $v \in V$.

Note that the above choice of the color palettes implies that for any two levels $i \neq i'$ their color palettes are disjoint.

Updates. Now suppose that an edge $\{u, v\}$ is inserted or deleted. We process this update using the update procedure of the level data structure. Whenever a vertex $w$ changes its level in the level data structure, we say that $w$ is affected. We now provide a re-coloring routine for affected vertices and for the vertices $u$ and $v$.

For $u$ and $v$ we proceed as follows. If $u$ and $v$ are in different levels, then we do not have to recolor any of them (because the color palettes of different levels are disjoint). If $u$ and $v$
are on the same level and of different colors, we do nothing. If \( u \) and \( v \) are on the same level \( i \in G_\ell \) and have the same color, then suppose that w.l.o.g. \( u \) received its current color before \( v \) was last recolored. Now we scan the list \( \text{neighbors}(u, \geq i) \) for the colors of all neighbors of \( u \) in \( V_i \). By Lemma 7 there are at most \( K \cdot 2^\ell \) such neighbors and, hence, they use at most \( K \cdot 2^\ell \) different colors. Thus, there must be at least \( c 2^\ell \) available colors for \( u \) in the palette of level \( i \), i.e., colors that are not used by any of the neighbors of \( u \) in level \( i \). From these available colors, we pick one uniformly at random and assign it to \( u \). Note that \( v \) is not recolored.

Whenever an affected vertex \( w \) changes its level, we recolor \( w \) as follows. Suppose that \( w \) is moved to level \( i \in G_\ell \). We consider the colors of the vertices in \( \text{neighbors}(w, \geq i) \cap V_i \) by simply scanning the list \( \text{neighbors}(w, \geq i) \). As before, this yields at least \( c 2^\ell \) available colors. We assign \( w \) a random color among these available colors.

**Analysis.** We start by analyzing the update time of algorithm.

**Lemma 8.** The expected amortized update time of the algorithm is \( O(\log^2 n) \).

**Proof.** By Lemma 7, the amortized update time for the level data structure is \( O(\log^2 n) \). Now observe that the work for recoloring affected vertices can be charged to the work done by the level data structure: When the level data structure moves an affected vertex \( w \) from level \( i \) to a new level \( i' \in \{i - 1, i + 1\} \), then it has to scan all neighbors of \( w \) in the lists \( \text{neighbors}(w, i) \) and \( \text{neighbors}(w, i') \). When the data structure performs these operations, we can keep track of the colors of the neighbors of \( w \) at the new level \( i' \) as described above. Thus, the cost for recoloring affected vertices can be charged to the running time analysis of the level data structure.

We are left to analyze the recoloring routine for vertices \( u \) and \( v \) which are on the same level \( i \in G_\ell \). Note that for recoloring \( u \), the algorithm spends time \( O(2^\ell) \) because the list \( \text{neighbors}(u, \geq i) \) has size at most \( K \cdot 2^\ell \) by Invariant. Now suppose that \( u \) is recolored and stays on its level \( i \). We show that in expectation it takes \( c 2^\ell \) edge insertions to vertices on the same level until \( v \) needs to be recolored again: Indeed, suppose that a new edge \( \{u, v\} \) is inserted with \( v \in V_i \). When \( v \) received its color, it randomly picked one of at least \( c 2^\ell \) colors and thus it picked the same color as \( u \) with probability at most \( \frac{1}{c 2^\ell} \). Now let \( X \) be the random variable which counts how many such edges from \( u \) to vertices on the same level as \( u \) are inserted until \( u \) needs to be recolored. Observe that \( X \) is geometrically distributed. Thus, we have that \( \mathbb{E}[X] = c 2^\ell \). This proves the claim. By charging \( O(1/\epsilon) \) to each update operation, this gives that this recoloring step has an amortized update time of \( O(1/\epsilon) \). This running time is subsumed by the update time for maintaining the level data structure.

**Lemma 9.** The data structure maintains a \( O(\alpha \log n) \)-coloring.

**Proof.** First, recall that for each level \( i \) with \( i \in G_\ell \) we use \( (K + \epsilon)2^\ell \) different colors and that for different levels, the color palettes are disjoint. This implies that for each group \( G_\ell \), we use \( (K + \epsilon)2^\ell \cdot L \) colors. Furthermore, by Lemma 7, each level \( i \) with \( i > L \cdot \ell^* \) where \( \ell^* = \lceil \log \left( 4 \alpha \right) \rceil \) satisfies that \( V_i = \emptyset \). Thus, the geometric sum implies that the total number of colors used is at most

\[
\sum_{\ell=0}^{\ell^*}(K + \epsilon)2^\ell L = (K + \epsilon)L \cdot \frac{1 - 2^{\ell^*+1}}{1 - 2} = O(\alpha \log n).
\]

The above lemmas imply Theorem 1.
4 Dynamic Arboricity Decomposition

In this section, we present a data structure for maintaining an arboricity decomposition, i.e., we maintain a partition of the edges of a dynamic graph into $O(\alpha)$ edge-disjoint (undirected) forests. In particular, we show that any data structure for maintaining an edge orientation can be used to maintain such an arboricity decomposition, where the number of forests will depend on the maximum outdegree, denoted by $D$ in the sequel. We stress that when $D$ depends on some parameter (e.g., the arboricity which might increase/decrease after a sequence of edge insertions/deletions) then so is the number of forests maintained by our data structure. Using the level data structure from Section 2 this yields that we can maintain an arboricity decomposition with $O(\alpha)$ forests if the current graph has arboricity $\alpha$; the update time is polylogarithmic in $n$. We will use this data structure in the next section to give a deterministic implicit coloring algorithm.

For the rest of the section, we assume that we have access to some black box data structure that maintains an orientation of the edges with update time $T$ for some $T$. We will show how to maintain a set of forests $F_0, \ldots, F_{2n}$ such that if the maximum outdegree of a node is bounded by $D$ (where $D$ which might change over time) the forests $F_0, \ldots, F_{2D-1}$ provide an arboricity decomposition of the graph and the forests $F_{2D}, \ldots, F_{2n}$ are empty.

Initialization and Invariants. We assume that at the beginning we are given a graph with $n$ vertices and no edges. For this graph, we build the black box outdegree data structure. We initialize $F_0, \ldots, F_{2n}$ to $2n$ forests such that each of them contains all vertices $V$ and no edges. Furthermore, for each vertex $v \in V$ we store an array $A_v$ storing $n$ bits and initially we set $A_v(i) = 0$ for all $i = 0, \ldots, n-1$.

For a vertex $v$, we let $d(v)$ denote the outdegree of $v$ in the black box data structure. When running the data structure, we make sure that the following invariants hold for each $v \in V$:

1. For each $\ell \in \{0, \ldots, d(v)-1\}$, either forest $F_{2\ell}$ or $F_{2\ell+1}$ but not both contain an out-edge of $v$.
2. No out-edge of $v$ is assigned to a forest $F_j$ with $j \geq 2d(v)$.
3. For all $v \in V$ and $\ell \in \{0, \ldots, n-1\}$, it holds that $A_v(\ell) = 1$ iff one of the out-edges of $v$ is assigned to forest $F_{2\ell}$ or forest $F_{2\ell+1}$.

Observe that when all of the invariants hold, then for each vertex $v$ we have that $A_v(\ell) = 1$ for $\ell = 0, \ldots, d(v)-1$ and $A_v(\ell) = 0$ for $\ell \geq d(v)$. Thus, we have a desired arboricity decomposition. Further note that after the initialization of the data structure, all invariants hold.

Updates. Suppose that an edge $\{u, v\}$ is inserted or deleted from the graph. We start by inserting or deleting, resp., the edge from the black box data structure. Now the black box data structure might either (1) flip the orientation of an existing edge, (2) add a new out-edge to a vertex (due to an edge insertion) or (3) delete an out-edge of a vertex (due to an edge deletion).

Let us start by considering Case (1), i.e., suppose the black box data structure flips the orientation of an edge $\{u', v'\}$. Then we assume w.l.o.g. that the new orientation is $(u', v')$ and proceed as follows.

First, we add $(u', v')$ as an out-edge to $u'$. Let $F^* \in \{F_{2d(v')-2}, F_{2d(v')-1}\}$ denote the forest in which $v'$ has no out-edge (recall that such a forest must exist by Invariant 2). Now we insert the edge $(u', v')$ into $F^*$ and set $A_{u'}(d(u')-1) = 1$. Note that after this procedure, all invariants for $u'$ are satisfied.

Second, we remove the edge $(v', u')$ (with the old orientation) from $v'$. Let $\ell^*$ be such
that the edge \((v', u')\) was stored in \(F_{2\ell'}\) or \(F_{2\ell'+1}\). We remove \((v', u')\) from the corresponding forest and set \(A_{v'}(\ell') = 0\). Note that this might violate the invariants because now \(v'\) has no out-edge in \(F_{2\ell'}\) and \(F_{2\ell'+1}\), but it might have one in \(F_{2d(v')−2}\) or \(F_{2d(v')−1}\) with \(d(v') = 1 > \ell'\), where \(d(v')\) is the outdegree of \(v'\) before \((v', u')\) was deleted. We fix this in the next step.

Third, let \(\ell^*\) be as before and set \(\ell\) to the largest integer such that \(A_{v'}(\ell) = 1\). If \(\ell < \ell^*\) we do nothing (all invariants already hold). Otherwise (\(\ell > \ell^*\)), we will essentially move the edge stored in forest \(F_{2\ell}\) or \(F_{2\ell+1}\) to forest \(F_{2\ell'}\) or \(F_{2\ell'+1}\). More concretely, let \((v', w)\) be the unique out-edge of \(v'\) stored in \(F_{2\ell}\) or \(F_{2\ell+1}\) and remove \((v', w)\) from this forest. Now let \(F^* \in \{F_{2\ell'}, F_{2\ell'+1}\}\) denote the forest in which \(w\) has no out-edge and insert \((v', w)\) into \(F^*\). Additionally, set \(A_{v'}(\ell^*) = 1\) and \(A_{v'}(\ell) = 0\). This restores all invariants for \(v'\).

In Case (2) above, i.e., the black box data structure inserted an out-edge for a vertex, we run the first step described above and nothing else. In Case (3), i.e., the black box data structure deleted an out-edge for a vertex, we run the second and the third step of the above procedure.

**Analysis.** First, we show that forests \(F_0, \ldots, F_{2D−1}\) indeed provide an arboricity decomposition of the dynamic graph.

**Lemma 10.** Let \(D\) be the maximum outdegree of any vertex in the outdegree decomposition maintained by the black box data structure. Then the forests \(F_0, \ldots, F_{2D−1}\) provide an arboricity decomposition of the graph.

**Proof.** Due to Invariant [1], each edge of the graph is stored in some forest. Thus, the union of all forests contains all edges of the graph. Hence, to prove the lemma, it suffices to prove the following two claims: (1) For each \(\ell = 0, \ldots, 2D−1\), \(F_\ell\) does not contain a cycle. (2) If \(\ell \geq 2D\) then \(F_\ell\) does not contain any edges.

We prove Claim (1) by contradiction. Suppose that \(F_\ell\) contains a cycle \(C\) over \(k\) vertices. Since \(C\) is cycle, \(C\) contains exactly \(k\) edges. By Invariant [1] each vertex has at most one out-edge in \(F_\ell\). Hence, \(C\) must correspond to a directed cycle in \(F_\ell\). Now consider the edge \((u', v')\) which closed the cycle when it was added to \(F_\ell\). In the first step of the algorithm, we only added the edge \((u', v')\) to \(F_\ell\) if \(v'\) had no out-edge in \(F_\ell\). This contradicts the fact that \((u', v')\) closes a directed cycle.

Claim (2) follows directly from Invariant [2] and the assumption that \(D\) is the maximum outdegree of any vertex. ▶

Note that in the above proof we did not assume that \(D\) is an upper bound on the maximum outdegree over the entire sequence of edge insertions and deletions. Instead, we only need that \(D\) is the maximum outdegree in the current graph. Hence, the number of forests providing the arboricity decomposition will never be more than \(2D\) at any point in time even when \(D\) is changing over time.

**Lemma 11.** If the (amortized or worst-case) update time of the black box data structure is \(T\), then the (amortized or worst-case, resp.) update time of the above algorithm is \(O(T)\).

**Proof.** In each update, the algorithm spends time \(T\) for inserting or deleting, resp., an edge in the black box data structure.

All the other steps can be implemented in \(O(1)\) time by maintaining the following values: (1) For each \(v \in V\), we maintain the maximum index \(\ell\) such that \(A_v(\ell) = 1\) (if no such \(\ell\) exists we set the corresponding index to \(-1\)). (2) For each \(v\) and \(\ell = 0, \ldots, 2n\), we maintain a pointer to the copy of \(v\) in \(F_\ell\). (3) For each edge \((u, v)\), we store a pointer to the forest \(F_\ell\) in which it is currently stored.
Now observe that the first step of the algorithm can be implemented in time $O(1)$ as follows: To find $d(u')$, we use the index from (1). When we need to check whether $v'$ has an out-edge in $F_{2d(u')}$ or $F_{2d(u')+1}$, we can use the pointers from (2) to the copies of $v'$ in $F_{2d(u')}$ and $F_{2d(u')+1}$.

The second and third step of the algorithm can be implemented similarly. When in the second step we have to remove the edge $(v',u')$, we can use the pointer from (3) to find its copy in $O(1)$ time. ▷

The two lemmas above imply Theorem 5. Using the level data structure from Section 2, we obtain Corollary 6.

5 Implicit Coloring with $2^{O(\alpha)}$ Colors

We present a data structure for implicitly maintaining a $2^{O(\alpha)}$-coloring. The data structure has an update time of $O(\alpha \log n)$ and it provides a query operation $\text{query}(u)$ which in time $O(\alpha \log n)$ returns the color of a vertex $u$. For planar graphs (which have arboricity at most 3) this implies that we can maintain an $O(1)$-coloring with update time $O(\log n)$ and query time $O(\log n)$.

Our algorithm maintains the arboricity data structure of Corollary 6 together with a data structure maintaining the forests of the arboricity decomposition. The latter assigns a unique root to each tree in the forests. Our main observation is that for any two adjacent vertices $u$ and $v$, there is a tree such that the distances of $u$ and $v$ to the root of have different parity. Now the query operation for a vertex $u$ picks the color of $u$ based on the parities of $u$’s distances to the roots of the trees.

Initialization. We assume that initially the graph has $n$ vertices and no edges. For this graph, we build the arboricity decomposition presented in Corollary 6. Furthermore, each of the forests maintained by the arboricity data structure is equipped with the data structure from the following lemma.

▶ Lemma 12. There exists a data structure for maintaining a dynamic forest with the following properties:

- Inserting an edge $\{u,v\}$ into the forest can be done in $O(\log n)$ time, where $u$ and $v$ are in different trees before the edge insertion.
- Deleting an edge $\{u,v\}$ from the forest takes $O(\log n)$ time.
- The data structure assigns a unique root to each tree in the forest.
- For a given vertex $u$, the distance of $u$ to the root of the tree containing $u$ can be reported in time $O(\log n)$.

The lemma is a simple application of dynamic trees or top trees [1] and we prove it in Appendix B.2.

Updates. Suppose an edge $\{u,v\}$ is inserted or deleted from the graph. Then we proceed as follows. First, we insert or delete, resp., the edge in the data structure from Corollary 6. Second, whenever the arboricity decomposition inserts or deletes an edge in one of the forests, we insert or delete the edge in the corresponding forest of the data structure from Lemma 12.

Queries. When we receive a query $\text{query}(u)$ for the color of a vertex $u$, we proceed as follows. For each of the $r = O(\alpha)$ forests we identify the tree containing $u$. Let $T_1, \ldots, T_r$ denote these trees. For each $T_j$, we determine whether the distance of $u$ to the root of $T_j$ using the data structure from Lemma 12. Now for each $j = 1, \ldots, r$, we set $p_u(j) := 0$ if the distance has even parity and $p_u(j) := 1$ otherwise. Now let $p_u := (p_u(1), \ldots, p_u(r)) \in \{0,1\}^r$. We define the color of $u$ to be $p_u$. 
Analysis. We start by showing that indeed we obtain a $2^{O(\alpha)}$-coloring.

Lemma 13. The data structure maintains an implicit $2^{O(\alpha)}$-coloring.

Proof. Consider any edge $\{u,v\}$. We show that query procedure returns vectors $p_u$ and $p_v$ such that $p_u \neq p_v$. Indeed, the edge $\{u,v\}$ must be contained in one of the $r = O(\alpha)$ forests maintained by the arboricity decomposition. Therefore, $u$ and $v$ are adjacent in some tree $T_j$ of that forest. Since $T_j$ has a unique root (by Lemma 12), the distances of $u$ and $v$ to the root of $T_j$ must have a different parity. Thus, we obtain that $p_u(j) \neq p_v(j)$ and, hence, $p_u \neq p_v$.

Furthermore, the total number of used colors is $2^r = 2^{O(\alpha)}$ since there are only $2^r$ possibilities for each vector $p_u$.

Lemma 14. The amortized update time of the data structure is $O(\log^3 n)$. The query time of the algorithm is $O(\alpha \log n)$.

Proof. First, note that the amortized update time of the data structure from Corollary 6 is $O(\log^2 n)$. This implies that amortized per update, the arboricity decomposition inserts or deletes at most $O(\log^2 n)$ edges from the forests. For each such inserted or deleted edge it takes time $O(\log n)$ to update the edge in the data structure from Lemma 12. This gives that the total amortized update time is $O(\log^3 n)$.

When answering a query for a vertex $u$, for each of the $O(\alpha)$ trees containing $u$ we need to query the distance of $u$ to the root node of the tree. Each of these queries takes time $O(\log n)$ by Lemma 12. Hence, the total query time is $O(\alpha \log n)$.

The two lemmas above imply Theorem 2.

6 Implicit Coloring with $O(\alpha \log n \cdot \min\{1, \log \alpha / \log \log n\})$ Colors

We present a data structure maintaining an implicit $O(\alpha \log n \cdot \min\{1, \log \alpha / \log \log n\})$-coloring. The data structure has an update time of $O(\log^2 n)$ and a query time of $O(\log n)$.

We will now focus on the case that $\alpha \leq \frac{1}{100} \log n$ and provide an algorithm maintaining a $O(\alpha \log n \log \alpha / \log \log n)$-coloring; we will only come back to the case $\alpha > \frac{1}{100} \log n$ at the very end of the section when we prove Theorem 4. To obtain the result for $\alpha \leq \frac{1}{100} \log n$, we use the level data structure described in Section 2 and an idea similar to that of Section 3. Recall that in Section 3 we used disjoint color palettes of $O(2^\ell)$ colors for each level in group $G_\ell$. Thus, for all levels in $G_\ell$ we used $O(2^\ell \cdot \log n)$ colors in total. Now we improve upon this result by providing a query procedure which only uses $O(2^\ell \cdot \log \alpha \log n / \log \log n)$ colors per group $G_\ell$. More concretely, we will partition the group $G_\ell$ into $O(\log \alpha \log n / \log \log n)$ subgroups $S_{\ell,j}$ such that for each subgroup the query procedure only uses $O(2^\ell)$ colors.

Subgroups. Recall from Section 2 that the level data structure contains $O(\log^2 n)$ levels and that group $G_\ell$ contains the $L$ levels $G_\ell = \{\ell L + 1, \ldots, (\ell + 1)L\}$, where $L = 2 + \lceil \log n \rceil$. Now let $\alpha^*$ be an approximation of the arboricity of the graph with $\alpha \leq \alpha^* \leq 10\alpha$. We partition each group $G_\ell$ into subgroups $S_{\ell,j}$ of $J := \lceil \log \log n / \log \alpha^* \rceil$ consecutive levels each. Formally, for each $\ell \in \mathbb{N}$ and for each $j \in \{0, \ldots, J - 1\}$, we define that subgroup $S_{\ell,j}$ contains the levels $\{\ell L + j \cdot J, \ldots, (\ell L + (j + 1) \cdot J - 1\}$. Thus, there are $O(\log n \log \alpha^* / \log \log n)$ subgroups $S_{\ell,j}$ per group $G_\ell$.

Note that $J$ depends on $\alpha^*$ which is an approximation of the current arboricity of the graph. This implies that as the arboricity of the graph changes (due to edge insertions and deletions), the subgroups $S_{\ell,j}$ will also change. However, note that the groups $G_\ell$ are not affected by this. Also, the algorithm will not need to maintain the subgroups $S_{\ell,j}$ explicitly.
Instead, it will be enough if the algorithm can compute $J$ to check for a given level $i$ in which subgroup the level is contained. Later, whenever we need to compute the subgroup of a level $i$, we can assume that we know a suitable value for $\alpha^*$ and, hence, $J$ with the desired properties via Property 5 of Lemma 7.

Furthermore, to each subgroup $S_{\ell,j}$ we assign a new color palette with $(K + \epsilon)2^\ell$ colors, where $K$ is the constant from Lemma 7 and $\epsilon = 1/10$. In particular, the palettes for any two different subgroups are disjoint.

**Initialization.** As before, we assume that initially we are given a graph over $n$ vertices and without any edges. For this graph we build the level data structure from Section 2. We also maintain a counter $t$ which counts the number of edge insertions and deletions processed by the data structure, but $t$ does not count the number of queries processed. Initially, we set $t = 0$. Furthermore, for each vertex $v$ we store a pair $(c,t)$ consisting of its color $c$ as well as the last time $t$ when its color was last updated. We only store the most recent such pair, i.e., when $v$ is assigned a color at time $t$ and there already exists a pair $(c',t')$ for $v$ with $t' < t$, we delete the old pair $(c',t')$. At the beginning, we initialize the pairs of all vertices to $(0,0)$, indicating that we assigned color 0 after having seen 0 updates. If at some time $t$ for a vertex $v$ we store a pair $(c,t')$ with $t' < t$ then we say that $v$ is outdated, otherwise we say that $v$ is fresh.

**Updates.** Suppose that an edge is inserted or deleted. Then we insert or delete, resp., the edge in the level data structure and update it suitably (but do not change the colors stored for the vertices). Also, we increase the counter $t$.

**Queries.** Suppose that the color of a vertex $v$ in a level $i$ is queried at time $t$. Then we set $\ell$ and $j$ such that level $i$ is in group $G_{\ell}$ and subgroup $S_{\ell,j}$. If $v$ is fresh, we output the color stored for $v$. If $v$ is outdated, we recompute the color of $v$ as follows. First, we iterate over $\text{neighbors}(v,i')$ and find subsets $V'$ and $V''$ which are as follows: $V'$ contains the all neighbors of $v$ in some level $i'$ with $i' > i$ and $i' \in S_{\ell,j}$, $V''$ contains all neighbors of $v$ in level $i$. Second, for each $v' \in V'$ that is outdated, we recursively recompute the color for $v'$. Note that after this step all vertices in $V'$ are fresh. Third, let $V'''$ be the set of all vertices in $V''$ that are fresh. Observe that $|V' \cup V'''| \leq K \cdot 2^\ell$ (by Invariant 2 of the level data structure) while the color palette of subgroup $S_{\ell,j}$ has $(K + \epsilon)2^\ell$ colors. Hence, there are at least $\epsilon 2^\ell$ colors which are not used by any vertex in $V' \cup V'''$ and we pick one of those and assign it to $v$. Furthermore, we update the pair $(c,t)$ for vertex $v$.

**Analysis.** In the next lemma we show that the coloring assigned to fresh vertices is proper. Note that it is enough to prove the claim for fresh vertices: the query routine only needs to provide a proper coloring as long as queries are not interrupted by an update, thus we only need to consider fresh vertices since only fresh vertices are assigned pairs with the most recent timestamp.

**Lemma 15.** Let $e = \{u,v\}$ be an edge. Suppose that $u$ and $v$ are fresh and that $u$ has color $c_u$ and $v$ has color $c_v$. Then $c_u \neq c_v$.

**Proof.** Let $i_u$ denote the level of $u$ and let $i_v$ denote the level of $v$. Let $(\ell_u,j_u)$ and $(\ell_v,j_v)$ be such that $i_u \in S_{\ell_u,j_u}$ and $i_v \in S_{\ell_v,j_v}$. If $u$ and $v$ are from different subgroups (i.e., $(\ell_u,j_u) \neq (\ell_v,j_v)$), then we must have that $c_u \neq c_v$ since we used disjoint color palettes for different subgroups. Now suppose that $u$ and $v$ are from the same subgroup (i.e., $(\ell_u,j_u) = (\ell_v,j_v)$). We distinguish two cases. First, suppose that $i_u = i_v$. Then assume w.l.o.g. that $v$ received its color after $u$. Thus, $v$ was in the set $V'''$ when $v$ was colored and, hence, we must have that $c_u \neq c_v$. Second, suppose that $i_u \neq i_v$. W.l.o.g. assume that $i_u > i_v$. Then $u$ was contained in the set $V'$ when $v$ received its color. Hence, we must have that $c_u \neq c_v$. $\blacksquare$
Lemma 16. The algorithm maintains a $O(\alpha \log \alpha \log n / \log \log n)$-coloring.

Proof. We already showed in Lemma 15 that the obtained coloring is proper. It only remains to bound the number of colors used by the algorithm. First, observe that since each group $G_i$ consists of $L = O(\log n)$ levels and each subgroup $S_{i,j}$ contains $O(\log \log n / \log \alpha)$ levels, there are $O(\log n \log \alpha / \log \log n)$ subgroups per group. Additionally, for each subgroup $S_{i,j}$ we assigned a color palette of $(K + \epsilon)2^\ell$ colors. Thus, the total number of colors used per group is $O(2^\ell \cdot \log n \log \alpha / \log \log n)$. Now the same geometric sum argument as used in the proof of Lemma 9 yields that the data structure uses $O(\alpha \log \alpha \log n / \log \log n)$ colors in total.

Next, we analyze the update and query time of the algorithm. We show that it takes $O(\log n)$ time to query the color of a vertex $v$. This includes all necessary recursive computations for outdated vertices.

Lemma 17. The update time of the algorithm is $O(\log^2 n)$. Furthermore, the query time of the algorithm is $O(\log n)$.

Proof. Since the update procedure of our algorithm only updates the level data structure and increases the counter $t$, the update time is $O(\log^2 n)$ by Lemma 7.

Now let us analyze the query time of the algorithm. Consider any subgroup $S_{i,j}$ and let $i_{\text{max}}$ be the largest level in $S_{i,j}$, i.e., $i_{\text{max}} = \ell L + (j + 1) \cdot J - 1$. We prove by induction on the level $i \in S_{i,j}$ that for any vertex at level $i$ the query time is $O(\alpha^{i_{\text{max}} - i + 1})$. As the algorithm only recolors vertices in $S_{i,j}$, we do not have to consider any other levels.

As base case suppose that $i = i_{\text{max}}$. Then we have that $V' = \emptyset$. Furthermore, the computation of $V''$ of $V'''$ can be performed in time $O(2^\ell) = O(\alpha)$ since $\ell \leq \ell^*$ (see Lemma 7).

Next, consider a vertex at level $i \in S_{i,j}$ with $i < i_{\text{max}}$. Then by Invariant 1 of the level data structure, we have that the set $V'$ contains at most $O(2^\ell) = O(\alpha)$ vertices at levels $i + 1, \ldots, i_{\text{max}}$. By induction hypothesis, coloring each of these vertices takes time $O(\alpha^{i_{\text{max}} - (i+1)+1}) = O(\alpha^{i_{\text{max}} - i})$. Thus, coloring all of these vertices takes time $O(\alpha^{i_{\text{max}} - i + 1})$. Computing the colors of vertices at level $i$ by computing the sets $V''$ and $V'''$ takes time $O(\alpha)$ by the same arguments as in the base case. Thus, the total query time for this vertex is $O(\alpha^{i_{\text{max}} - i + 1})$.

Now let us bound the total query time. Note that difference of $i_{\text{max}} - i$ for $i \in S_{i,j}$ is maximized when $i = \ell L + j \cdot J$ and in this case $i_{\text{max}} - i = J$. Thus, the total query time is at most

$$O(\alpha^{i_{\text{max}} - i}) = O(\alpha^J) \leq O(\alpha^{\log \log n / \log \alpha^*}) \leq O(\alpha^{\log \log n / \log \alpha}) = O(2^{\log \log n}) = O(\log n).$$

Proof of Theorem 4. To obtain the data structure claimed in the theorem, we run the above algorithm and the explicit algorithm from Theorem 1 in parallel. After each update, we use Property 3 of the level data structure to obtain an approximation $\alpha^*$ of the arboricity with $\alpha \leq \alpha^* \leq 10\alpha$. If $\alpha^* \leq 1/10 \log n$, then we will use the data structure from this section for subsequent queries. The previous lemmas imply that this provides a $O(\alpha \log \alpha \log n / \log \log n)$-coloring with amortized update time $O(\log^2 n)$ and query time $O(\log n)$. If $\alpha^* > 1/10 \log n$, we use the data structure from Theorem 1 for queries. This provides an $O(\alpha \log n)$-coloring with amortized update time $O(\log^2 n)$ and query time $O(1)$ because the coloring maintained by the data structure is explicit.
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Explicit and Implicit Dynamic Coloring of Graphs with Bounded Arboricity

The first result for dynamic coloring was obtained by Barenboim and Maimon [4] and they showed how to maintain a $O(\Delta)$-coloring with worst-case update time $O(\sqrt{\Delta} \log \log n)$. This result was later improved by the algorithms [10, 8, 19] to obtain $(\Delta + 1)$-colorings with amortized constant update time. Duan et al. [12] provided an algorithm for $(1 + \epsilon)\Delta$-edge-coloring with polylogarithmic update time if $\Delta \geq \Omega((\log n/\epsilon)^2)$. Furthermore, algorithms for dynamic coloring were also studied in practice, e.g., [26, 31, 17].

Computing graph colorings of static graphs has been an active research area in the distributed community over several decades, e.g., [24, 25, 16, 15, 14]. More recently, Parter et al. [29] also studied dynamic coloring algorithms in the distributed setting.

Providing dynamic algorithms for graphs with bounded arboricity has been a fruitful area of research. Such algorithms have been derived for fundamental dynamic problems including shortest paths [13, 23], maximal independent set [28], matching [6, 7, 27] or coloring [30].

Several papers studied the problem of dynamically maintaining low-outdegree edge orientation. The first such result was obtained by Brodal and Fagerberg [11] who obtained an $O(\alpha)$-orientation with amortized update time $O(\alpha \log n)$. He et al. [18] obtained a tradeoff between the outdegree and the update time of the algorithm. Kopelowitz et al. [22] obtained algorithms with worst-case update time and this result was improved by Berglin and Brodal [5]. Kaplan and Solomon [20] showed how to maintaining edge orientations in the distributed setting when the local memory per node is restricted.

## Omitted Proofs

### B.1 Proof of Lemma 7

Before we prove the lemma, let us first review the data structure by Bhattacharya et al. [10]. Since the data structure of [10] was developed for the densest subgraph problem, let us first introduce this problem and discuss its relationship with arboricity.

**Arboricity and densest subgraph.** The density of the densest subgraph is defined as $\frac{|E(S)|}{|S|}$. By the Nash-Williams Theorem, we have that for the arboricity of a graph it holds that $\alpha = \max_{S \subseteq V} \left\lceil \frac{|E(S)|}{|S|-1} \right\rceil$. Thus, we get

$$\alpha = \max_{S \subseteq V} \left[ \frac{|E(S)|}{|S|} \right] \geq \max_{S \subseteq V} \frac{|E(S)|}{|S|-1} \geq \max_{S \subseteq V} \frac{|E(S)|}{|S|} = d^*. \quad (1)$$
The data structure of [10]. Recall from Section 2 that our data structure maintains \( \lfloor \log n \rfloor \cdot L \) levels. Furthermore, there are \( \lfloor \log n \rfloor \) groups \( G_\ell \) consisting of \( L \) consecutive levels each. Each vertex \( v \) at level \( i \in G_\ell \) satisfies the following two invariants: (1) \( v \) at has at most \( 5 \cdot 2^\ell \) neighbors in \( \bigcup_{j \geq i} V_j \) and (2) \( v \) has at least \( 2^{\ell'} \) neighbors in levels \( \bigcup_{j \geq i-1} V_j \), where \( \ell' \) is such that \( i - 1 \in G + \ell' \).

Now the data structure of [10] essentially works by running \( \lfloor \log n \rfloor \) data structures \( D_\ell \) in parallel, one data structure for each group \( G_\ell \). More precisely, each data structure \( D_\ell \) stores all vertices and edges of the graph. Furthermore, \( D_\ell \) assigns each vertex to exactly one of \( L \) levels \( U_1^\ell, \ldots, U_L^\ell \) and the data structure ensures that the following invariants hold: (1) For each \( v \in V \) at level \( i < L \), \( v \) has at most \( 5 \cdot 2^\ell \) neighbors in \( \bigcup_{j \geq i} U_j^\ell \) and (2) for each \( v \in V \) at level \( i > 1 \), \( v \) has at least \( 2^{\ell'} \) neighbors in levels \( \bigcup_{j \geq i-1} U_j^\ell \). The update procedure of the algorithm is the same as described in Section 2; it only takes into account that now there are only \( L \) levels per data structure \( D_\ell \).

Note that in the data structure of [10] there exist \( \lfloor \log n \rfloor \) copies of each vertex \( v \in V \), while in our data structure each vertex is only stored once.

The data structure by [10] satisfies the following properties:

Lemma 18 (Bhattcharya et al. [10]). The data structure \( D_\ell \) satisfies the following properties:

1. If \( \ell > \log(4\alpha^*), \) then the highest level of \( D_\ell \) does not contain any vertices, i.e., \( U_L^\ell = \emptyset \).
2. The amortized update time for maintaining \( G_\ell \) is \( O(L) = O(\log n) \).

The lemma follows from Theorem 2.6 and Theorem 4.2 in [10].

Note that since the data structure of [10] maintains \( O(\log n) \) data structures \( D_\ell \) in parallel, the total update time of the data structure becomes \( O(\log^2 n) \).

Proof of Lemma 7. Let us now prove Lemma 7.

Property 1 follows immediately from how we defined the edge orientation in Section 2.

The claim about the update time in Property 2 follows from the analysis of the data structures \( D_\ell \) in [10] (the analysis goes through if we assign \( \Theta(\lfloor \log n \rfloor \cdot L) = \Theta(\log^2 n) \) potential to each edge insertion and deletion); the claim for the number of edge flips follows from the fact that with amortized update time \( O(\log^2 n) \) the data structure cannot flip more than \( O(\log^2 n) \) edges per update (amortized).

Property 3 follows from the fact that [10] show that a 5-approximation of the densest subgraph can be maintained with amortized update time \( O(\log^2 n) \). Since the value of the arboricity and the densest subgraph only differ by a factor 2, we can simply run the data structure of [10] in the background to always have access to a value \( \alpha^* \) with the desired property.

Property 4 follows from Property 3 (which we prove below): By Property 3, we have that \( V_i = \emptyset \) for all levels \( i \) with \( i \in G_\ell \) and \( \ell > \ell^* = \lfloor \log(4\alpha^*) \rfloor \). Thus, all vertices with out-edges must be in a level \( i \) with \( i \in G_\ell \) with \( \ell \leq \lfloor \log(4\alpha^*) \rfloor \). By the invariants maintained by the data structure, we obtain that the outdegree of each such vertex is at most

\[
5 \cdot 2^\ell = 5 \cdot 2^{\log(4\alpha^*) + 1} = O(\alpha).
\]

We are left to prove Property 3 and proceed in two steps.

First, recall that the levels in the levels data structure from Section 2 were denoted \( V_i \) and those in the data structures \( D_\ell \) from [10] were denoted \( U_i^\ell \). We prove that for all \( i \geq 1 \) it holds that

\[
\bigcup_{j \geq i} V_j \subseteq \bigcup_{j=i'} U_j^\ell,
\]

(2)
where $\ell \in \{0, \ldots, \lceil \log n \rceil - 1\}$ and $i' \in \{1, \ldots, L\}$ are such that $i = \ell \cdot L + i'$.

Indeed, suppose that $i' = 1$. Then we have that $\bigcup_{j=1}^{\ell} U_j^\ell = V$ since $D_\ell$ only has $L$ levels and stores all vertices in $V$. Thus, the desired subset relationship trivially holds.

Next, consider $1 < i' \leq L$. Observe that by induction hypothesis we have that $X := \bigcup_{j \geq i-1} V_j \subseteq \bigcup_{j=i' - 1} U_j^\ell =: Y$.

This implies for all $v \in V$ it holds that $d_X(v) \leq d_Y(v)$, where $d_X(v)$ and $d_Y(v)$ denote the degree of $v$ induced by the vertices in $X$ and $Y$, respectively. Since both the level data structure and the data structure $D_\ell$ only promote vertices with at least $5 \cdot 2^\ell$ vertices to the next level, any vertex which is promoted from level $i - 1$ to $i$ in the level data structure must also be promoted from level $i' - 1$ to level $i'$ in the data structure $D_\ell$. Thus, the claim from Equation (2) holds.

Second, the first property of Lemma 18 implies that $U_L^\ell = \emptyset$ for $\ell > \log(4d^*)$. By assumption of Property 3 and Equation (1) we have that $\ell > \ell^* = \lceil \log(4\alpha) \rceil \geq \log(4d^*)$.

Together with the claim above we get that for $i = (\ell^* + 1) \cdot L$,

$$\bigcup_{j \geq i} V_j \subseteq U_L^{\ell^*} = \emptyset.$$

Since all levels $i \in G_\ell$ with $\ell > \ell^*$ satisfy $i > (\ell^* + 1) \cdot L$, we obtain that $V_i = \emptyset$. This implies Property 3.

\section*{B.2 Proof of Lemma 12}

To prove Lemma 12 let us first state an application of the top trees data structure by Alstrup et al. [1].

\begin{lemma}[Alstrup et al. [1, Theorem 2.7]]
There exists a data structure maintaining a dynamic forest which offers the following operations in $O(\log n)$ time:

1. \texttt{link}\{\{u, v\}\}, where $u$ and $v$ are in different trees: Insert the edge $\{u, v\}$ into the dynamic forest.
2. \texttt{cut}\{\{u, v\}\}: Remove the edge $\{u, v\}$ from the dynamic forest.
3. \texttt{mark}(u): Mark the vertex $u$.
4. \texttt{unmark}(u): Unmark the vertex $u$.
5. \texttt{nearestMarkedNeighbor}(u): Return the distance of $u$ to its nearest marked neighbor and report the nearest marked neighbor.
\end{lemma}

Now we show how to implement the data structure claimed in Lemma 12. For inserting edges we use the \texttt{link}-operation and for deleting edges we use the \texttt{cut}-operation from Lemma 19. We only need to argue how the roots of the trees are picked and how to distances to the roots can be computed.

In our construction, we follow the convention that marked vertices (as per Lemma 19) will correspond to the roots of the trees of Lemma 12. We make sure that each tree in the dynamic forest contains exactly one marked vertex.

When we initialize the data structure, we build the top tree data structure for a graph with $n$ vertices and without any edges. Furthermore, we mark all vertices in the graph
using the \textbf{mark}(\cdot) operation. Note that initially all trees have exactly one root because all connected components are isolated vertices and all vertices are marked (and, hence, roots).

When an edge \{u, v\} is inserted, we proceed as follows. We use the \texttt{nearestMarkedNeighbor}(u) routine to obtain the nearest marked neighbor of \(u\) and unmark this vertex. Now we use \texttt{link}\{\{u, v\}\} to link the trees of \(u\) and \(v\) in the top trees data structure. Observe that the resulting tree only contains a single marked vertex.

When an edge \{u, v\} is deleted, we use \texttt{cut}\{\{u, v\}\} to remove the edge \{u, v\} from the top tree data structure. Note that now \(u\) and \(v\) are different trees. Now we query \texttt{nearestMarkedNeighbor}(u) and \texttt{nearestMarkedNeighbor}(v). Observe that exactly one of them will not have a marked neighbor in their new tree. Suppose this vertex is \(u\). Then we perform \texttt{mark}(u) in the top trees data structure. This implies that each tree has a unique root after the operation finished.

When the data structure is queried for the distance of a vertex \(u\) to its root, we simply return the distance to \texttt{nearestMarkedNeighbor}(u).