Capacities, Green function, and Bergman functions

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Abstract
Using the logarithmic capacity, we give quantitative estimates of the Green function, as well as precise lower bounds of the Bergman kernel for bounded pseudo-convex domains in \( \mathbb{C}^n \) and the Bergman distance for bounded planar domains. In particular, it is shown that the Bergman kernel satisfies the sharpest estimate \( K_\Omega(z) \gtrsim \delta_\Omega(z)^{-2} \) if \( \Omega \) is a bounded pseudoconvex domain with \( C^0 \)-boundary.

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1 | INTRODUCTION

The famous Wiener’s criterion states that a boundary point \( a \) of a bounded planar domain \( \Omega \) is regular if and only if

\[
\sum_{k=1}^{\infty} \frac{k}{\log[1/C_l(A_k(a) \cap \Omega^c)]} = \infty,
\]

where \( A_k(a) := \{ z \in \mathbb{C} : \frac{1}{2^{k+1}} \leq |z - a| \leq \frac{1}{2^k} \} \) and \( C_l(\cdot) \) stands for the standard logarithmic capacity (relative to \( \infty \)). It is also known that the regularity of \( a \) implies that the Bergman kernel is exhaustive at \( a \) (cf. [26]) and the Bergman metric is complete at \( a \) (cf. [4, 20]). Zwonek [36] showed that the Bergman kernel is exhaustive at \( a \) if and only if

\[
\sum_{k=1}^{\infty} \frac{2^{2k}}{\log[1/C_l(A_k(z) \cap \Omega^c)]} \to \infty \quad (z \to a).
\]
On the other hand, a similar characterization for the Bergman completeness at \( a \) is still missing, although some partial results exist (cf. [22, 28]). Only recently, interesting lower and upper bounds of \( K_\Omega(z) \) for planar domains were obtained in terms of the logarithmic capacity of \( \Omega^c \) relative to \( z \) (cf. [3, 5]).

The goal of this paper is to present some quantitative estimates of the Bergman kernel and the Bergman distance in terms of the boundary distance \( \delta_\Omega \) through the logarithmic capacity. We first give a useful result as follows.

**Theorem 1.1.** Let \( \Omega \) be a bounded pseudoconvex domain in \( \mathbb{C}^n \). Suppose there are constants \( \alpha > 1 \) and \( \varepsilon > 0 \) such that for any \( z \in \Omega \), there exists a complex line \( L_z \ni z \) such that

\[
C_1 \left( L_z \cap \overline{B(z, \alpha \delta_\Omega(z))} \cap \Omega^c \right) \geq \varepsilon \delta_\Omega(z),
\]

where \( B(z, r) = \{ w \in \mathbb{C}^n : |w - z| < r \} \). Then

\[
K_\Omega(z) \gtrsim \delta_\Omega(z)^{-2}, \quad z \in \Omega.
\]

A well-known result implicitly contained in Ohsawa–Takegoshi [27] and explicitly formulated in Fu [16]† shows that (1.4) holds for any bounded pseudoconvex domain with \( C^2 \)-boundary. Surprisingly, this result can be extended to bounded pseudoconvex domains with \( C^0 \)-boundaries in view of Theorem 1.1. More precisely, we have the following main result.

**Theorem 1.2.** Let \( \Omega \) be a bounded pseudoconvex domain in \( \mathbb{C}^n \). Then (1.3) (hence (1.4)) holds under one of the following conditions.

(a) \( \partial \Omega \) is \( C^0 \), that is, it can be written locally as the graph of a continuous function.

(b) There are constants \( \varepsilon, r_0 > 0 \) such that

\[
|B(\zeta, r) \cap \Omega^c| \geq \varepsilon |B(\zeta, r)|, \quad \zeta \in \partial \Omega, \quad 0 < r \leq r_0,
\]

where \( |\cdot| \) stands for the volume.

(c) \( \Omega \) is fat (i.e., \( \overline{\Omega} = \Omega \)) and \( \overline{\Omega} \) is \( \mathcal{O}(U) \)-convex, where \( U \) is a neighborhood of \( \overline{\Omega} \).

(d) \( \Omega \) is strongly hyperconvex.

Here are a few remarks. Recall that the \( \mathcal{O}(U) \)-convex hull of a compact set \( E \subset U \) is given by

\[
\hat{E}_{\mathcal{O}(U)} := \left\{ z \in U : |f(z)| \leq \sup_{E} |f| \text{ for all } f \in \mathcal{O}(U) \right\}.
\]

We say that \( E \) is \( \mathcal{O}(U) \)-convex if \( E = \hat{E}_{\mathcal{O}(U)} \); in particular, if \( U = \mathbb{C}^n \) then \( E \) is called polynomially convex. Recall also that a bounded domain \( \Omega \subset \mathbb{C}^n \) is called strongly hyperconvex if there exists a continuous plurisubharmonic (psh) function \( \rho \) defined in a neighborhood \( U \) of \( \overline{\Omega} \) such that \( \Omega = \{ z \in U : \rho(z) < 0 \} \). Based on the work of Zwonek [36], Pflug–Zwonek [28] proved that for any \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon > 0 \) such that \( K_\Omega(z) \geq C_\varepsilon \delta_\Omega(z)^{-2+\varepsilon} \) when \( \Omega \) is strongly hyperconvex. Finally, we point out that the condition (1.5) was first introduced in Wiener [33], which are popular

†The author is indebted to Professor Siqi Fu for bringing his attention to this reference.
in second order elliptic boundary problems (cf. [24, chapter 1, section 1]), and spectrum theory of the Laplacian (cf. [13, 25]).

We also have the following analogous result that does not follow from Theorem 1.1.

**Proposition 1.3.** If \( \Omega \subset \mathbb{C}^n \) is a bounded pseudoconvex Runge domain, then (1.4) holds.

**Remark.** It is known that every star domain in \( \mathbb{C}^n \) is a Runge domain (cf. [1] or [23]).

All domains in Theorem 1.2 or Proposition 1.3 are expected to be hyperconvex. On the other hand, there are bounded hyperconvex domains on which (1.4) do not hold (see, e.g., [35, Theorem 1.1] and the subsequent remark).

It is worthwhile to point out that the argument in Zwonek [36] actually yields the following result closely related to Theorem 1.1.

**Proposition 1.4.** Let \( \Omega \) be a bounded pseudoconvex domain in \( \mathbb{C}^n \). Suppose there are constants \( \alpha > 1, \beta > 1 \) and \( \varepsilon > 0 \) such that for any \( z \in \Omega \), there exists a complex line \( \mathcal{L}_z \ni z \) such that

\[
C_1 \left( \mathcal{L}_z \cap \overline{B(z, \alpha \delta_{\Omega}(z))} \cap \Omega^c \right) \geq \varepsilon \delta_{\Omega}(z)^\beta.
\]

(1.6)

Then

\[
K_\Omega(z) \gtrsim \delta_{\Omega}(z)^{-2} |\log \delta_{\Omega}(z)|^{-1}
\]

(1.7)

for all \( z \) sufficiently close to \( \partial \Omega \).

For the Bergman kernel, the celebrated Ohsawa–Takegoshi extension theorem [27] serves as a bridge passing from 1-dimensional estimates to high dimensions. Unfortunately, such a powerful tool is not available for estimating other objects like the (pluricomplex) Green function or the Bergman metric/distance. Thus, we have to focus on bounded planar domains in the sequel.

We begin with the following concepts arising from the work of Carleson–Totik [7].

**Definition 1.1.** Let \( \varepsilon > 0 \) and \( 0 < \lambda < 1 \) be fixed. For every \( a \in \partial \Omega \) we set

\[
K_1(a) := \overline{D_1(a)} - \Omega; \quad D_1(a) := \{ z : |z - a| < t \}
\]

\[
\mathcal{N}_a(\varepsilon, \lambda) := \{ n \in \mathbb{Z}^+ : C_1(K_{\lambda^n}(a)) \geq \varepsilon \lambda^n \}
\]

\[
\mathcal{N}_a^n(\varepsilon, \lambda) := \mathcal{N}_a(\varepsilon, \lambda) \cap \{1, 2, ..., n\}.
\]

The \((\varepsilon, \lambda)\)-capacity density of \( \partial \Omega \) at \( a \) is defined by

\[
D_a(\varepsilon, \lambda) := \liminf_{n \to \infty} \inf_{a \in \partial \Omega} \frac{|\mathcal{N}_a^n(\varepsilon, \lambda)|}{n},
\]

where \(| \cdot |\) stands for the cardinality. The weak and strong \((\varepsilon, \lambda)\)-capacity density of \( \partial \Omega \) are defined by

\[
D_W(\varepsilon, \lambda) := \liminf_{n \to \infty} \inf_{a \in \partial \Omega} \frac{\inf_{a \in \partial \Omega} |\mathcal{N}_a^n(\varepsilon, \lambda)|}{n}
\]
and

\[ D_S(\varepsilon, \lambda) := \liminf_{n \to \infty} \frac{1}{n} \left| \bigcap_{a \in \partial \Omega} \mathcal{N}_a^n(\varepsilon, \lambda) \right|, \]

respectively.

It is easy to see that \( D_W(\varepsilon, \lambda) \geq D_S(\varepsilon, \lambda) \), and \( C_f(K_t(a)) \geq \varepsilon t, \forall t > 0 \), implies \( D_0(\varepsilon, \lambda) = 1 \). Recall that \( \partial \Omega \) is said to be uniformly perfect if \( \inf_{a \in \partial \Omega} C_f(K_t(a)) \geq \varepsilon t, \forall t > 0 \) (cf. [30]). Thus, if \( \partial \Omega \) is uniformly perfect then \( D_W(\varepsilon, \lambda) = D_S(\varepsilon, \lambda) = 1 \) for some \( \varepsilon > 0 \). On the other hand, the domain

\[ \Omega := \mathbb{D} - \{0\} - \bigcup_{k=1}^{\infty} \left[ 2^{-2^{k+1}}, 2^{-2^k} \right] \]

satisfies \( D_W(\varepsilon, 1/2) > 0 \) for some \( \varepsilon > 0 \) while \( \partial \Omega \) is nonuniformly perfect (cf. [7]). Actually, the argument shows that \( D_S(\varepsilon, 1/2) > 0 \).

Let \( z_0 \) be a fixed point in \( \Omega \). Carleson and Totik proved the following quantitative analogue of the sufficiency part of Wiener’s criterion.

**Theorem 1.5** (cf. [7]). If \( D_W(\varepsilon, \lambda) > 0 \) for some \( \varepsilon, \lambda \), then there exists \( \beta > 0 \) such that the Green function \( g_\Omega \) satisfies

\[ -g_\Omega(z, z_0) \lesssim \delta_\Omega(z)^{\beta} \quad (1.8) \]

for all \( z \) sufficiently close to \( \partial \Omega \).

**Remark.** It is remarkable that the converse of Theorem 1.5 holds under the additional condition that \( \Omega \) contains a fixed size cone with vertex at any \( a \in \partial \Omega \) (cf. [7]).

As the original proof in [7] is rather technical, we take this opportunity to present a self-contained proof of this result by using a quantitative estimate of the capacity potential given in Theorem 3.2, which can be applied in more general context.

**Definition 1.2.** For \( \varepsilon > 0, 0 < \lambda < 1 \) and \( \gamma > 1 \) we set

\[ \mathcal{N}_a(\varepsilon, \lambda, \gamma) := \{ n \in \mathbb{Z}^+ : C_f(K_{\lambda^n}(a)) \geq \varepsilon \lambda^{\gamma n} \} \]

\[ \mathcal{N}_a^n(\varepsilon, \lambda, \gamma) := \mathcal{N}_a(\varepsilon, \lambda, \gamma) \cap \{1, 2, \ldots, n\}. \]

We define the \((\varepsilon, \lambda, \gamma)\)-capacity density of \( \partial \Omega \) at \( a \) by

\[ D_a(\varepsilon, \lambda, \gamma) := \liminf_{n \to \infty} \frac{1}{\log n} \sum_{k \in \mathcal{N}_a^n(\varepsilon, \lambda, \gamma)} \frac{k^{-1}}{\log n} \]

\[ \tag{†} \]

Zhiyuan Zheng informed the author that a similar construction yields an example with \( D_W(\varepsilon, \lambda) > D_S(\varepsilon, \lambda) \) for some \( \varepsilon, \lambda \).
and the weak and strong \((\epsilon, \lambda, \gamma)\)-capacity densities of \(\partial \Omega\) by

\[
D_W(\epsilon, \lambda, \gamma) := \liminf_{n \to \infty} \inf_{\alpha \in \partial \Omega} \frac{\sum_{k \in \mathcal{N}_n^\alpha(\epsilon, \lambda, \gamma)} k^{-1} \log n}{},
\]

and

\[
D_S(\epsilon, \lambda, \gamma) := \liminf_{n \to \infty} \frac{\sum_{k \in \bigcap_{\alpha \in \partial \Omega} \mathcal{N}_n^\alpha(\epsilon, \lambda, \gamma)} k^{-1} \log n}{},
\]

respectively.

Note that \(D_W(\epsilon, \lambda, \gamma) \geq D_S(\epsilon, \lambda, \gamma)\). If \(C_i(K_t(a)) \geq \epsilon t^\gamma\), \(\forall t > 0\), for some \(\epsilon > 0\) and \(\gamma > 1\), then \(D_a(\epsilon, \lambda, \gamma) = 1\) in view of the following well-known formula,

\[
\lim_{n \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) = \text{Euler constant}.
\]

**Theorem 1.6.**

1. If \(D_W(\epsilon, \lambda, \gamma) > 0\) for some \(\epsilon, \lambda, \gamma\), then there exists \(\beta > 0\) such that

\[
-g_\Omega(z, z_0) \leq (-\log \delta_\Omega(z))^{-\beta}
\]

for all \(z\) sufficiently close to \(\partial \Omega\).

2. Suppose \(\inf_{\alpha \in \partial \Omega} C_i(K_t(a)) \geq \epsilon t^\gamma\) for all \(0 < t \ll 1\) and some \(\epsilon > 0\) and \(\gamma > 1\). For every \(\tau < \frac{1}{\gamma - 1}\), there exists \(C > 0\) such that

\[
-g_\Omega(z, z_0) \leq C(-\log \delta_\Omega(z))^{-\tau}
\]

for all \(z\) sufficiently close to \(\partial \Omega\).

We also obtain the following lower bounds for the Bergman distance \(d_B\).

**Theorem 1.7.**

1. If \(D_S(\epsilon, \lambda) > 0\) for some \(\epsilon, \lambda\), then

\[
d_B(z_0, z) \geq |\log \delta_\Omega(z)|
\]

for all \(z\) sufficiently close to \(\partial \Omega\).

2. If \(D_W(\epsilon, \lambda) > 0\) for some \(\epsilon, \lambda\), then

\[
d_B(z_0, z) \geq \frac{|\log \delta_\Omega(z)|}{\log |\log \delta_\Omega(z)|}
\]

for all \(z\) sufficiently close to \(\partial \Omega\).
If \( D_W(\varepsilon, \lambda, \gamma) > 0 \) for some \( \varepsilon, \lambda, \gamma \), then

\[
d_B(z_0, z) \gtrsim \log \log |\log \delta_{\Omega}(z)|
\]

(1.13)

for all \( z \) sufficiently close to \( \partial \Omega \).

Remark.

(1) In [10], (1.11) was verified by a different method in case \( \partial \Omega \) is uniformly perfect.

(2) Estimate of type (1.12) was first obtained by Błocki [2] for bounded pseudoconvex domains with Lipschitz boundaries in \( \mathbb{C}^n \) (see also [11] and [15] for related results).

(3) We conjecture that (1.13) can be improved to \( d_B(z_0, z) \gtrsim \log |\log \delta_{\Omega}(z)| \).

By now it becomes a standard method for obtaining lower bounds of Bergman functions by estimating the sublevel set \( \{ g_{\Omega}(\cdot, z) \leq -c \} \) for \( c > 0 \), where \( g_{\Omega}(\cdot, z) \) stands for the (pluricomplex) Green function with pole at \( z \). Namely, one has

\[
K_{\Omega}(z) \gtrsim_c \{ g_{\Omega}(\cdot, z) \leq -c \}^{-1},
\]

(1.14)

\[
d_B(z, z') \gtrsim_c 1 \text{ whenever } \{ g_{\Omega}(\cdot, z) \leq -c \} \cap \{ g_{\Omega}(\cdot, z') \leq -c \} = \emptyset,
\]

(1.15)

where \( A \gtrsim_c B \) means \( A \geq CB \) for some constant \( C = C(c) > 0 \) (cf. [2, 4, 5, 8–11, 15, 20, 26]). In the proofs of Theorems 1.1 and 1.7, we shall use (1.14) and (1.15), respectively.

### 2 | CAPACITIES

In this section, we shall review different notions of capacities and present some basic properties of them. Let \( \Omega \) be a bounded domain in \( \mathbb{C} \) and \( K \subset \Omega \) a compact (nonpolar) set in \( \Omega \). The Dirichlet capacity \( C_d(K, \Omega) \) of \( K \) relative to \( \Omega \) is given by

\[
C_d(K, \Omega) = \inf_{\phi \in \mathcal{L}(K, \Omega)} \int_{\Omega} |\nabla \phi|^2,
\]

(2.1)

where \( \mathcal{L}(K, \Omega) \) is the set of all locally Lipschitz functions \( \phi \) on \( \Omega \) with a compact support in \( \overline{\Omega} \) such that \( 0 \leq \phi \leq 1 \) and \( \phi|_K = 1 \). If \( \Omega = \mathbb{C} \), then we write \( C_d(K) \) for \( C_d(K, \Omega) \). From the definition, we immediately get

\[
K_1 \subseteq K_2 \text{ and } \Omega_1 \supseteq \Omega_2 \Rightarrow C_d(K_1, \Omega_1) \leq C_d(K_2, \Omega_2).
\]

(2.2)

In view of Dirichlet’s principle, the infimum in (2.1) is attained at the function \( \phi_{\min} \) that is exactly the Perron solution to the following (generalized) Dirichlet problem in \( \Omega \setminus K \):

\[
\Delta u = 0; \ u = 0 \text{ n.e. on } \partial \Omega; \ u = 1 \text{ n.e. on } \partial K.
\]

(2.3)
We call $\phi_{\min}$ the capacity potential of $K$ relative to $\Omega$. In case $\partial \Omega$ and $\partial K$ are both $C^2$-smooth, integration by parts gives

$$C_d(K, \Omega) = \int_{\Omega} |\nabla \phi_{\min}|^2 = \int_{\Omega \setminus K} |\nabla \phi_{\min}|^2$$

$$= -\int_{\Omega \setminus K} \phi_{\min} \Delta \phi_{\min} + \int_{\partial(\Omega \setminus K)} \phi_{\min} \frac{\partial \phi_{\min}}{\partial \nu} d\sigma$$

$$= \int_{\partial K} \frac{\partial \phi_{\min}}{\partial \nu} d\sigma =: -\text{flux}_{\partial K} \phi_{\min}, \quad (2.4)$$

where $\nu$ is the outward unit normal vector fields on $\partial(\Omega \setminus K)$. Note that $\frac{\partial \phi_{\min}}{\partial \nu} \geq 0$ holds on $\partial K$ in view of the Hopf lemma.

Let $z \in \Omega \setminus K$ be given. As $\Delta g_\Omega(\cdot, z) = 2\pi \delta_z$, where $\delta_z$ stands for the Dirac measure at $z$, we infer from Green’s formula that

$$2\pi \phi_{\min}(z) = \int_{\Omega \setminus K} \phi_{\min} \Delta g_\Omega(\cdot, z) = \int_{\Omega \setminus K} g_\Omega(\cdot, z) \Delta \phi_{\min}$$

$$+ \int_{\partial(\Omega \setminus K)} \phi_{\min} \frac{\partial g_\Omega(\cdot, z)}{\partial \nu} d\sigma - \int_{\partial(\Omega \setminus K)} g_\Omega(\cdot, z) \frac{\partial \phi_{\min}}{\partial \nu} d\sigma$$

$$= \int_{\partial K} \frac{\partial g_\Omega(\cdot, z)}{\partial \nu} d\sigma - \int_{\partial K} g_\Omega(\cdot, z) \frac{\partial \phi_{\min}}{\partial \nu} d\sigma$$

$$= -\int_{\partial K} g_\Omega(\cdot, z) \frac{\partial \phi_{\min}}{\partial \nu} d\sigma \quad (2.5)$$

because $g_\Omega(\cdot, z)$ is harmonic on $K$. This equality combined with (2.4) gives the following fundamental inequality that connects the capacity, Green’s function and the capacity potential:

$$\frac{C_d(K, \Omega)}{2\pi \inf_{\partial K} (-g_\Omega(\cdot, z))} \leq \phi_{\min}(z) \leq \frac{C_d(K, \Omega)}{2\pi \sup_{\partial K} (-g_\Omega(\cdot, z))}, \quad z \in \Omega \setminus K. \quad (2.6)$$

As $\Omega \setminus K$ can be exhausted by bounded domains with smooth boundaries, we conclude by passing to a standard limit process that the same inequality holds for every compact set $K$.

Given a finite Borel measure $\mu$ on $\mathbb{C}$ whose support is contained in $K$, define the Green potential of $\mu$ relative to $\Omega$ by

$$p_\mu(z) = \int_{\Omega} g_\Omega(z, w) d\mu(w), \quad z \in \Omega.$$

Clearly, $p_\mu$ is negative, subharmonic on $\Omega$, harmonic on $\Omega \setminus K$, and satisfies $p_\mu(z) = 0$ nearly everywhere (n.e.) on $\partial \Omega$. Moreover,

$$\Delta p_\mu = 2\pi \mu \quad (2.7)$$

holds in the sense of distributions. Recall that the Green energy $I(\mu)$ of $\mu$ is given by

$$I(\mu) := \int_{\Omega} p_\mu d\mu = \int_{\Omega} \int_{\Omega} g_\Omega(z, w) d\mu(z) d\mu(w).$$
By (2.7), we have

\[ I(\mu) = \frac{1}{2\pi} \int_{\Omega} p_\mu \Delta p_\mu = -\frac{1}{2\pi} \int_{\Omega} |\nabla p_\mu|^2. \] (2.8)

Every compact set $K$ admits an equilibrium measure $\mu_{\max}$, which maximizes $I(\mu)$ among all Borel probability measures $\mu$ on $K$. A fundamental theorem of Frostman states that

1. $p_{\mu_{\max}} \geq I(\mu_{\max})$ on $\Omega$;
2. $p_{\mu_{\max}} = I(\mu_{\max})$ on $K \setminus E$ for some $F_\sigma$ polar set $E \subset \partial K$.

By the uniqueness of the solution of the (generalized) Dirichlet problem, we have

\[ \phi_{\min} = p_{\mu_{\max}} / I(\mu_{\max}). \] (2.9)

Now define the Green capacity $C_g(K, \Omega)$ of $K$ relative to $\Omega$ by

\[ C_g(K, \Omega) := e^{I(\mu_{\max})}. \]

It follows from (2.8) and (2.9) that

\[ \frac{C_g(K, \Omega)}{2\pi} = -\frac{1}{\log C_g(K, \Omega)}. \] (2.10)

Analogously, the logarithmic capacity $C_l(K)$ of $K$ is defined by

\[ \log C_l(K) := \sup_{\mu} \int_{\mathbb{C}} \int_{\mathbb{C}} \log |z - w| d\mu(z) d\mu(w), \]

where the supremum is taken over all Borel probability measures $\mu$ on $\mathbb{C}$ whose support is contained in $K$. Let $R$ be the diameter of $\Omega$ and set $d = d(K, \Omega)$. As

\[ \log |z - w| / R \leq g_\Omega(z, w) \leq \log |z - w| / d, \quad z, w \in K, \]

we have

\[ \log C_l(K) - \log R \leq \log C_g(K, \Omega) \leq \log C_l(K) - \log d. \] (2.11)

We collect a few basic properties of the logarithmic capacity that will be used in the sequel.

**Proposition 2.1** (cf. [31]). Let $K, K_1, K_2, \ldots$ be compact sets in $\mathbb{C}$.

1. If $K_1 \subset K_2$, then $C_l(K_1) \leq C_l(K_2)$.
2. If $T : K \to \mathbb{C}$ is a map satisfying $|T(z) - T(w)| \leq A|z - w|^\alpha$, then $C_l(T(K)) \leq AC_l(K)^\alpha$.
3. If $K$ is a subset of the real axis of Lebesgue measure $m$, then $C_l(K) \geq m / 4$.
4. If $\text{diam}(\bigcup_j K_j) \leq d$, then

\[ \frac{1}{\log d / C_l(\bigcup_j K_j)} \leq \sum_j \frac{1}{\log d / C_l(K_j)}. \]
3 \ | \ ESTIMATES OF THE GREEN FUNCTION

We first give a basic lemma as follows.

**Lemma 3.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{C} \) with \( 0 \in \partial \Omega \). Let \( 0 \leq h \leq 1 \) be a harmonic function on \( \Omega \) such that \( h = 0 \) n.e. on \( \partial \Omega \cap D_{r_0} \) for some \( r_0 < 1 \). For all \( 0 < \alpha < 1/16 \) and \( r \leq \alpha r_0 \), we have

\[
\sup_{\Omega \cap \mathbb{D}_r} h \leq \exp \left[ - \frac{\log 1/(16\alpha)}{\log 1/\alpha} \int_r^{\alpha r_0} \left( t \log \frac{t/\alpha}{2c(K_t)} \right)^{-1} dt \right], \tag{3.1}
\]

where \( K_t := \mathbb{D}_t - \Omega \).

**Proof.** The idea of the proof comes from [19] (see also [12]). Let \( \mathbb{D} \) be the unit disc. For \( t < r_0 \) and \( |z| = t \), we have

\[
\sup_{\partial K_{\alpha t}} (-g_\mathbb{D} (\cdot, z)) \leq \log 2 + \sup_{\partial K_{\alpha t}} (-\log |\cdot - z|) \leq \log 2 - \log |t - \alpha t| \leq \log \frac{4}{t}. \tag{3.2}
\]

Let \( \phi_{\alpha t} \) be the capacity potential of \( K_{\alpha t} \) relative to \( \mathbb{D} \). By (2.6) and (3.2), we have

\[
\phi_{\alpha t}(z) \leq \log \frac{4}{t} \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi} \quad \text{for } |z| = t.
\]

It follows that for \( z \in \Omega \cap \partial \mathbb{D}_t \),

\[
(1 - \phi_{\alpha t}(z)) \sup_{\Omega \cap \mathbb{D}_t} h \geq \left[ 1 - \log \frac{4}{t} \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi} \right] h(z), \tag{3.3}
\]

while the same inequality holds for \( z \in \partial \Omega \cap \partial \mathbb{D}_t \), because \( \lim_{z \to \zeta} h = 0 \) for n.e. \( \zeta \in \partial \Omega \cap \partial \mathbb{D}_t \). By the (generalized) maximum principle, (3.3) holds on \( \Omega \cap \mathbb{D}_t \). On the other hand, as for \( |z| = \alpha t \) we have

\[
\inf_{\partial K_{\alpha t}} (-g_\mathbb{D} (\cdot, z)) \geq \log 1/2 + \inf_{\partial K_{\alpha t}} (-\log |\cdot - z|) \geq \log 1/2 - \log(2\alpha t) = \log \frac{1}{4\alpha t}, \tag{3.4}
\]

it follows from (2.6) that

\[
\phi_{\alpha t}(z) \geq \log \frac{1}{4\alpha t} \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi} \quad \text{for } |z| = \alpha t. \tag{3.5}
\]

Substituting (3.5) into (3.3), we have

\[
h(z) \leq \sup_{\Omega \cap \mathbb{D}_t} h \cdot \frac{1 - \log \frac{1}{4\alpha t} \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi}}{1 - \log \frac{4}{t} \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi}} \leq \sup_{\Omega \cap \mathbb{D}_t} h \left( 1 + \frac{\log(16\alpha) \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi}}{1 - \log \frac{4}{t} \cdot \frac{C_d(K_{\alpha t}, \mathbb{D})}{2\pi}} \right), \tag{3.6}
\]
for \( z \in \Omega \cap D_{\alpha t} \). Set \( M(t) := \sup_{\Omega \cap D_t} h \). It follows from (3.6) and (2.10) that
\[
\frac{\log M(t)}{t} - \frac{\log M(\alpha t)}{t} \geq \log \frac{1}{16\alpha} \left( t \log \frac{t}{4C_{\gamma}(K_{\alpha t}, \Omega)} \right)^{-1}
\]
\[
\geq \log \frac{1}{16\alpha} \left( t \log \frac{t}{2C_{\gamma}(K_{\alpha t})} \right)^{-1} \quad \text{(by (2.11))}.
\]
Integration from \( r/\alpha \) to \( r_0 \) gives
\[
\log \frac{1}{16\alpha} \int_{r/\alpha}^{r_0} \left[ t \log \frac{t}{2C_{\gamma}(K_{\alpha t})} \right]^{-1} dt
\]
\[
\leq \int_{r/\alpha}^{r_0} \frac{\log M(t)}{t} dt - \int_{r_\alpha}^{r_0} \frac{\log M(t)}{t} dt
\]
\[
\leq \int_{r_\alpha}^{r_0} \frac{\log M(t)}{t} dt - \int_{r}^{r_\alpha} \frac{\log M(t)}{t} dt
\]
\[
\leq (\log M(r_0) - \log M(r)) \log \frac{1}{\alpha},
\]
because \( M(t) \) is nondecreasing. Thus (3.1) holds because \( M(r_0) \leq 1 \).

**Theorem 3.2.** Fix a compact set \( E \) in \( \Omega \) with \( C_1(E) > 0 \). Let \( \phi_E \) be the capacity potential of \( E \) relative to \( \Omega \). Set \( d = d(E, \partial \Omega) \). Then for all \( 0 < \alpha < 1/16 \) and \( r > 0 \)
\[
\sup_{\Omega \cap D_r} \phi_E \leq \exp \left[ -\frac{\log 1/(16\alpha)}{\log 1/\alpha} \int_r^{\alpha d} \left( t \log \frac{t/\alpha}{2C_{\gamma}(K_{t})} \right)^{-1} dt \right]. \tag{3.7}
\]

**Proof.** The solution of the (generalized) Dirichlet problem gives \( \lim_{z \to \zeta} \phi_E(z) = 0 \) for n.e. \( \zeta \in \partial \Omega \), and the (generalized) maximum principle gives \( 0 \leq \phi_E \leq 1 \). Thus, Lemma 3.1 applies.

It is interesting to remark that in Wiener’s original paper [34], only the following criterion is proved: \( 0 \in \partial \Omega \) is regular if and only if
\[
\sum_{k=1}^{\infty} \frac{2^k}{\log 1/C_{\gamma}(J_k \setminus \Omega)} = \infty, \tag{3.8}
\]
where \( J_k = \{ z \in \mathbb{C} : 2^{-2^k} \leq |z| \leq 2^{-2^k-1} \} \). We claim that the sufficiency part of this criterion follows quickly from Theorem 3.2. To see this, simply note that
\[
\int_0^1 \frac{dt}{t \log 1/C_{\gamma}(K_t)} = \log 2 \int_0^{\infty} \frac{ds}{\log 1/C_{\gamma}(K_{2^{-s}})}
\]
\[
\geq \sum_{k=1}^{\infty} \int_{2^{k-1}}^{2^k} \frac{ds}{\log 1/C_{\gamma}(K_{2^{-s}})}
\]
\[
\geq \sum_{k=1}^{\infty} \frac{2^k}{\log 1/C_{\gamma}(J_{k+1} \setminus \Omega)},
\]
1940

while

\[ \int_0^{2\alpha} \frac{dt}{t \log \frac{t/\alpha}{2C_i(K_i)}} \geq \int_0^{2\alpha} \frac{dt}{t \log 1/C_i(K_i)}, \]

thus Theorem 3.2 applies. On the other hand, it turns out that (3.8) is actually equivalent to (1.1) (compare [17, p. 116]).

**Proof of Theorem 1.5.** As \( D_W(\varepsilon, \lambda) > 0 \), there exist \( c > 0 \) and \( n_0 \in \mathbb{Z}^+ \) such that

\[ |N^n_a(\varepsilon, \lambda)| \geq cn, \quad \forall \, n \geq n_0 \quad \text{and} \quad a \in \partial \Omega. \]

As \( N^n_a(\varepsilon, \lambda) \) is decreasing in \( \varepsilon \), we may assume that \( \varepsilon \) is as small as we want. Note that for \( n \gg N \gg 1 \)

\[ \int_{\lambda^n}^{\lambda^N} \left( t \log \frac{t/\alpha}{2C_i(K_i(a))} \right)^{-1} dt \geq \sum_{k \in \mathcal{N}^n_a(\varepsilon, \lambda) \setminus \mathcal{N}^N_a(\varepsilon, \lambda)} \int_{\lambda^k}^{\lambda^{k-1}} \left( t \log \frac{t/\alpha}{2C_i(K_i(a))} \right)^{-1} dt \]

\[ \geq \sum_{k \in \mathcal{N}^n_a(\varepsilon, \lambda) \setminus \mathcal{N}^N_a(\varepsilon, \lambda)} \left( \log \frac{1}{2\lambda\varepsilon\alpha} \right)^{-1} \int_{\lambda^k}^{\lambda^{k-1}} \frac{dt}{t} \]

\[ = \log \frac{1}{\lambda} \cdot \left( \log \frac{1}{2\lambda\varepsilon\alpha} \right)^{-1} \cdot \frac{cn}{2} \]

\[ = \frac{c}{2} \cdot \left( \log \frac{1}{2\lambda\varepsilon\alpha} \right)^{-1} \cdot \log \frac{1}{\lambda^n}. \quad (3.9) \]

As for every \( z \) there exists \( n \in \mathbb{Z}^+ \) such that \( \lambda^n \leq |z - a| \leq \lambda^{n-1} \), it follows from (3.7) and (3.9) that for any compact subset \( E \subset \Omega \)

\[ \phi_E(z) \lessapprox |z - a|^\beta \]

for suitable constant \( \beta > 0 \) that is independent of \( a \). As \( -g_k(z, z_0) \approx \phi_E(z) \) for all \( z \) sufficiently close to \( \partial \Omega \), we conclude that (1.8) holds. \( \square \)

**Proof of Theorem 1.6.** (1) As \( D_W(\varepsilon, \lambda, \gamma) > 0 \), there exist \( c > 0 \) and \( n_0 \in \mathbb{Z}^+ \) such that

\[ \sum_{k \in \mathcal{N}^n_a(\varepsilon, \lambda, \gamma)} k^{-1} \geq c \log n, \quad \forall \, n \geq n_0 \quad \text{and} \quad a \in \partial \Omega. \]

Note that for \( n \gg N \gg 1 \),

\[ \int_{\lambda^n}^{\lambda^N} \left( t \log \frac{t/\alpha}{2C_i(K_i(a))} \right)^{-1} dt \geq \sum_{k \in \mathcal{N}^n_a(\varepsilon, \lambda, \gamma) \setminus \mathcal{N}^N_a(\varepsilon, \lambda, \gamma)} \int_{\lambda^k}^{\lambda^{k-1}} \left( t \log \frac{t/\alpha}{2C_i(K_i(a))} \right)^{-1} dt \]

\[ \geq \sum_{k \in \mathcal{N}^n_a(\varepsilon, \lambda, \gamma) \setminus \mathcal{N}^N_a(\varepsilon, \lambda, \gamma)} \left( \log \frac{1}{2\varepsilon\alpha} \right)^{-1} \int_{\lambda^k}^{\lambda^{k-1}} \frac{dt}{t} \]

\[ = \frac{c}{2} \cdot \left( \log \frac{1}{2\varepsilon\alpha} \right)^{-1} \cdot \log \frac{1}{\lambda^n}. \]

As for every \( z \) there exists \( n \in \mathbb{Z}^+ \) such that \( \lambda^n \leq |z - a| \leq \lambda^{n-1} \), it follows from (3.7) and (3.9) that for any compact subset \( E \subset \Omega \)

\[ \phi_E(z) \lessapprox |z - a|^\beta \]

for suitable constant \( \beta > 0 \) that is independent of \( a \). As \( -g_k(z, z_0) \approx \phi_E(z) \) for all \( z \) sufficiently close to \( \partial \Omega \), we conclude that (1.8) holds.
\[ \sum_{k \in \mathcal{N}^n_{\alpha}(\varepsilon, \lambda, \gamma) \setminus \mathcal{N}^n_{\alpha}(\varepsilon, \lambda, \gamma^\prime)} k^{-1} \geq \log n, \]

where the implicit constants are independent of \( a \). This combined with (3.7) gives for any compact subset \( E \subset \Omega \)

\[ \phi_E(z) \lesssim (-\log |z-a|)^{-\beta} \]

for some constant \( \beta > 0 \) independent of \( a \), which in turn implies (1.9).

(2) By (3.7), we have for every \( a \in \partial \Omega \)

\[ \sup_{\Omega \cap D_r(a)} \phi_E \leq \exp \left[ -\log \frac{1}{16\alpha} \int_0^{\frac{2d}{\alpha}} \frac{dt}{t \log \frac{t/\alpha}{2C(K_t(a))}} \right] \]

\[ \leq \exp \left[ -\log \frac{1}{16\alpha} \int_0^{\frac{2d}{\alpha}} \frac{dt}{t((1-1) \log 1 + \log 1/(2\alpha \varepsilon))} \right] \]

\[ \leq \text{const.} (-\log r)^\beta \]

provided \( \alpha \) sufficiently small, from which the assertion follows.

For the estimates of the Bergman kernel and the Bergman distance, we need some quantitative estimates of the sublevel set \( \{ g_{\Omega}(\cdot, w) < -c \} \). We start with the following known fact.

**Lemma 3.3** (cf. [18]). Let \( \Omega \) be a bounded domain in \( \mathbb{C} \) and \( U \) a relatively compact open set in \( \Omega \). For every \( w \in U \), we have

\[ \min_{\partial U} (-g_{\Omega}(\cdot, w)) \leq \frac{2\pi}{C_d(U, \Omega)} \leq \max_{\partial U} (-g_{\Omega}(\cdot, w)). \]  

(3.10)

**Proof.** For the sake of completeness, we include a proof here. As \( g_{\Omega}(\cdot, w) \) is harmonic on \( \Omega \setminus \overline{U} \) and vanishes n.e on \( \partial \Omega \), it follows from the maximum principle that

\[ \sup_{\Omega \setminus U} (-g_{\Omega}(\cdot, w)) = \max_{\partial U} (-g_{\Omega}(\cdot, w)) \quad \text{and} \quad \inf_{\overline{U}} (-g_{\Omega}(\cdot, w)) = \min_{\partial U} (-g_{\Omega}(\cdot, w)). \]

Then we have

\[ \left\{ -g_{\Omega}(\cdot, w) \geq \max_{\partial U} (-g_{\Omega}(\cdot, w)) \right\} \subset \overline{U} \subset \left\{ -g_{\Omega}(\cdot, w) \geq \min_{\partial U} (-g_{\Omega}(\cdot, w)) \right\}. \]  

(3.11)

Set \( F_c = \{ -g_{\Omega}(\cdot, w) \geq c \} \). It suffices to show

\[ C_d(F_c, \Omega) = 2\pi/c. \]

To see this, first note that \( \phi_c := -c^{-1}g_{\Omega}(\cdot, w) \) is the capacity potential of \( F_c \) relative to \( \Omega \). Then we have

\[ C_d(F_c, \Omega) = -\text{flux}_{\partial F_c} \phi_c = -\text{flux}_{\partial \Omega} \phi_c = c^{-1} \text{flux}_{\partial \Omega} g_{\Omega}(\cdot, w) = 2\pi/c, \]

where the second and last equalities follow from Green’s formula. \( \square \)
Proposition 3.4. Let $\Omega$ be a bounded domain in $\mathbb{C}$ with $0 \in \partial \Omega$. Let $\beta \geq 2\alpha > 0$. Suppose $C_i(K_r) \geq \varepsilon r$ for some $\varepsilon, r > 0$. There exists a positive number $c$ depending only on $\alpha, \beta, \varepsilon$ such that for every point $w$ with $|w| = \beta r$ and $D_{2\alpha r}(w) \subset \Omega$ we have

$$\{ g_{\Omega}(\cdot, w) \leq -c \} \subset D_{\alpha r}(w). \quad (3.12)$$

Proof. By (3.10), (3.11), and Harnack’s inequality, it suffices to show

$$C_d \left( D_{\alpha r}, \Omega \right) \geq c' \qquad (3.13)$$

for some positive constant $c'$ depending only on $\alpha, \beta, \varepsilon$. From the definition, we see that

$$C_d \left( D_{\alpha r}, \Omega \right) = C_d \left( \Omega^c, C_\infty - D_{\alpha r} \right) \geq C_d \left( K_r, C_\infty - D_{\alpha r} \right), \quad (3.14)$$

where $C_\infty$ denotes the Riemann sphere. Let us consider the conformal map

$$T : C_\infty - D_{\alpha r}(w) \to \mathbb{D}, \quad z \mapsto \frac{\alpha r}{z - w}.$$

As the Dirichlet energy is invariant under conformal maps, it follows that

$$C_d \left( K_r, C_\infty - D_{\alpha r}(w) \right) = C_d (T(K_r), \mathbb{D}). \quad (3.15)$$

As

$$K_r \subset D_{(1+\beta)r}(w) - D_{2\alpha r}(w),$$

we have

$$T(K_r) \subset \overline{D_{1/2}} - D_{\alpha/(1+\beta)},$$

so that

$$C_d (T(K_r), \mathbb{D}) = -\frac{2\pi}{\log C_g(T(K_r), \mathbb{D})} \geq \frac{2\pi}{\log 2 - \log C_i(T(K_r))}. \quad (3.16)$$

As

$$|T^{-1}(z_1) - T^{-1}(z_2)| = \frac{\alpha r}{|z_1 z_2|} \cdot |z_1 - z_2| \leq \frac{(1 + \beta)^2 r}{\alpha} \cdot |z_1 - z_2|$$

for all $z_1, z_2 \in T(K_r)$, we infer from Proposition 2.1/(2) that

$$C_i(T(K_r)) \geq \frac{\alpha}{(1 + \beta)^2 r} \cdot C_i(K_r) \geq \frac{\alpha \varepsilon}{(1 + \beta)^2 r}.$$

This combined with (3.14) $\sim$ (3.16) gives (3.13). \qed
Proposition 3.5. Suppose $D_\gamma(\varepsilon, \lambda) > 0$ for some $\varepsilon, \lambda$. There exists $c \gg 1$ such that for every $k \in \bigcap_{\alpha \in T} N_\gamma^k(\varepsilon, \lambda)$ and every $w$ with $\lambda^{k-1/3} \leq \delta_{\Omega}(w) \leq \lambda^{k-1/2}$,

$$\{ g_\Omega(\cdot, w) \leq -c \} \subset \{ \lambda^k < \delta_{\Omega} < \lambda^{k-1} \}. \quad (3.17)$$

Proof. Take $a(w) \in \partial \Omega$ such that $|w - a(w)| = \delta_{\Omega}(w)$. Note that

$$C_j(K_{\lambda^k}(a(w))) \geq \varepsilon \lambda^k.$$ 

Thus, Proposition 3.4 applies. \hfill \Box

Proposition 3.6. If $D_W(\varepsilon, \lambda, \gamma) > 0$ for some $\varepsilon, \lambda, \gamma$, then there exists $c \gg 1$ such that

$$\{ g_\Omega(\cdot, w) \leq -1 \} \subset \left\{ \frac{\phi_E(w)}{\lambda^{1+\beta}} < \phi_E < c \phi_E(w)^{1+\beta} \right\} \quad (3.18)$$

for all $w$ sufficiently close to $\partial \Omega$, where $\beta$ is given as (1.9).

Proof. We shall first adopt a trick from [6]. Consider two points $z, w$ with $|z - w|, \delta_{\Omega}(z)$ and $\delta_{\Omega}(w)$ are sufficiently small. We want to show

$$|\phi_E(z) - \phi_E(w)| \leq c_0(-\log |z - w|)^{-\beta} \quad (3.19)$$

for some constant $c_0 > 0$. Without loss of generality, we assume $\delta_{\Omega}(w) \geq \delta_{\Omega}(z)$. If $|z - w| \geq \delta_{\Omega}(w)/2$, this follows directly from (1.9). As $\phi_E$ is a positive harmonic function on $D_{\delta_{\Omega}(w)}(w)$, we see that if $|z - w| \leq \delta_{\Omega}(w)/2$ then

$$|\phi_E(z) - \phi_E(w)| \leq \sup_{D_{\delta_{\Omega}(w)/2}(w)} |\nabla \phi_E| |z - w| \leq c_1 \delta_{\Omega}(w)^{-1}(-\log \delta_{\Omega}(w))^{-\beta} |z - w| \quad (by \ (1.9))$$

$$\leq c_1 (2|z - w|)^{-1}(-\log(2|z - w|))^{-\beta} |z - w| \leq c_0(-\log |z - w|)^{-\beta}.$$

The remaining argument is standard. Let $R$ be the diameter of $\Omega$. By (3.19) we conclude that if $\phi_E(z) = \phi_E(w)/2$ then

$$\log \frac{|z - w|}{R} \geq -\left( \frac{2c_0}{\phi_E(w)} \right)^{1/\beta} - \log R \geq -c_2 \phi_E(w)^{-1/\beta} \quad (\text{by } \phi_E).$$

As $-\phi_E$ is subharmonic on $\Omega$, it follows that

$$\psi(z) := \begin{cases} \log \frac{|z - w|}{R}, & \text{if } \phi_E(z) \geq \phi_E(w)/2 \\ \max \left\{ \log \frac{|z - w|}{R}, -2c_2 \phi_E(w)^{-1-1/\beta} \phi_E(z) \right\}, & \text{otherwise} \end{cases}$$

\hfill \Box
is a well-defined negative subharmonic function on $\Omega$ with a logarithmic pole at $w$, and if $\phi_E(z) \leq \phi_E(w)/2$ then we have

$$g_{\Omega}(z, w) \geq \psi(z) \geq -2c_2\phi_E(w)^{-1-1/\beta} \phi_E(z),$$

so that

$$\{g_{\Omega}(\cdot, w) \leq -1\} \cap \{\phi_E \leq \phi_E(w)/2\} \subset \{\phi_E \geq (2c_2)^{-1}\phi_E(w)^{1+1/\beta}\}.$$

As $\{\phi_E \geq \phi_E(w)/2\} \subset \{\phi_E > c^{-1}\phi_E(w)^{1+1/\beta}\}$ if $c \gg 1$, we have

$$\{g_{\Omega}(\cdot, w) \leq -1\} \subset \{\phi_E > c^{-1}\phi_E(w)^{1+1/\beta}\}.$$

By the symmetry of $g_{\Omega}$, we immediately get

$$\{g_{\Omega}(\cdot, w) \leq -1\} \subset \{\phi_E < (c\phi_E(w))^{\beta/(1+\beta)}\}. \quad \Box$$

## 4 LOWER BOUNDS OF THE BERGMAN KERNEL

### 4.1 Proof of Theorem 1.1

Take $z^* \in L_z \cap B(z, \alpha \delta_{\Omega}(z)) \cap \partial \Omega$. As

$$B(z, \alpha \delta_{\Omega}(z)) \subset B(z^*, 2\alpha \delta_{\Omega}(z)),$$

we have

$$C_1\left(L_z \cap B(z^*, 2\alpha \delta_{\Omega}(z)) \cap \Omega^c\right) \geq \frac{\varepsilon}{2\alpha} \cdot (2\alpha \delta_{\Omega}(z)).$$

By Proposition 3.4, there exists a constant $c = c(\alpha, \varepsilon) > 0$ such that

$$\{g_{L_z \cap \Omega}(\cdot, z) \leq -c\} \subset L_z \cap B(z, \delta_{\Omega}(z)/2).$$

Hence, it follows from (1.14) that

$$K_{L_z \cap \Omega}(z) \gtrsim |\{g_{L_z \cap \Omega}(\cdot, z) \leq -c\}|^{-1} \gtrsim \delta_{\Omega}(z)^{-2},$$

where the implicit constants depend only on $\alpha, \varepsilon$. This together with the Ohsawa–Takegoshi extension theorem yield (1.4).

### 4.2 Proof of Theorem 1.2

**Proof of (b).** Take $z^* \in \partial \Omega$ such that $|z - z^*| = \delta_{\Omega}(z)$. As

$$B(z^*, \delta_{\Omega}(z)) \subset B(z, 2\delta_{\Omega}(z)),$$
it follows from (1.5) that

$$\left| B(z, 2\delta_\Omega(z)) \cap \Omega^c \right| \geq C_n \varepsilon \delta_\Omega(z)^{2n},$$

where $C_n$ stands for a constant depending only on $n$. We claim that there exists a real line $l_z \ni z$, such that

$$\left| l_z \cap B(z, 2\delta_\Omega(z)) \cap \Omega^c \right|_1 \geq \varepsilon' \delta_\Omega(z), \tag{4.1}$$

where $\varepsilon' = \varepsilon'(n, \varepsilon) > 0$, $| \cdot |_1$ stands for the 1-dimensional Lebesgue measure. Indeed, if we denote $E := B(z, 2\delta_\Omega(z)) \cap \Omega^c$ and $E_\xi := E \cap l_{z, \xi}$, where $\xi \in \mathbb{C}^n$, $l_{z, \xi} = \{z + t\xi : t \in \mathbb{R}\}$, then

$$|E| = \int_{\xi \in S^{2n-1}} \int_{r>0 : z+r\xi \in E} r^{2n-1} drd\sigma(\xi) \tag{4.2}$$

$$\leq 2^{2n-1}\delta_\Omega(z)^{2n-1} \int_{\xi \in S^{2n-1}} |E_\xi|_1 d\sigma(\xi),$$

where $S^{2n-1}$ denotes the unit sphere in $\mathbb{C}^n$ and $d\sigma$ denotes the surface element. Thus,

$$\int_{\xi \in S^{2n-1}} |E_\xi|_1 d\sigma(\xi) \geq C_n 2^{1-2n} \varepsilon \delta_\Omega(z),$$

so that there exists at least one point $\xi_0 \in S^{2n-1}$, such that

$$|E_{\xi_0}|_1 \geq \frac{C_n \varepsilon}{2^{2n}|S^{2n-1}|} \cdot \delta_\Omega(z).$$

Take $l_z = l_{z, \xi_0}$ and $\varepsilon' = \frac{C_n \varepsilon}{2^{2n}|S^{2n-1}|}$, we get (4.1).

Let $L_z$ be the complex line determined by $l_z$. Then from Proposition 2.1/(1),(3), we obtain

$$C_l \left( L_z \cap \overline{B(z, 2\delta_\Omega(z))} \cap \Omega^c \right) \geq \frac{|E_{\xi_0}|_1}{4} \geq \frac{\varepsilon'}{4} \cdot \delta_\Omega(z).$$

Thus, Theorem 1.1 applies. \(\square\)

**Proof of (d).** By Theorem 1.1, it suffices to verify that there exists a complex line $L_z \ni z$, such that

$$C_l \left( L_z \cap \overline{B(z, 2\delta_\Omega(z))} \cap \Omega^c \right) \geq \delta_\Omega(z)/4. \tag{4.3}$$

This follows from an argument of Pflug–Zwonek [29]: Take $c_0 > 0$, so that $\{\rho < c_0\} \subset U$, where $\rho$ is defined on an open set $U \supset \overline{\Omega}$. Take $z^* \in \partial\Omega$ with $|z - z^*| = \delta_\Omega(z)$ and let $L_z$ be the complex line determined by $z$ and $z^*$. We claim that

$$L_z \cap \partial B(z, r) \cap \Omega^c \neq \emptyset, \tag{4.4}$$

for any $r$ satisfying $\delta_\Omega(z) < r < d_0 - \delta_\Omega(z)$, where $d_0 = d(\overline{\Omega}, \{\rho = c_0\})$. Indeed, if $L_z \cap \partial B(z, r) \cap \Omega^c = \emptyset$, that is,

$$L_z \cap \partial B(z, r) \subset \Omega,$$
then the maximum principle and continuity of \( \rho \) yield
\[
\rho(z^*) \leq \max \{ \rho(w) : w \in L_z \cap \partial B(z, r) \} < 0 = \rho(z^*),
\]
this is a contradiction!

As the circular projection \( T : w \mapsto |w| \) satisfies \( |T(w) - T(w')| \leq |w - w'| \), it follows from Proposition 2.1/(2),(3) that (4.4) gives (4.3).

\( \square \)

**Proof of (c).** Given \( 0 < \varepsilon \ll 1 \), define

\[
\Omega_\varepsilon := \left\{ z \in U : d(z, \overline{\Omega}) < \varepsilon \right\}.
\]

As \( \overline{\Omega} = \Omega \), it is not difficult to show that \( \delta_{\Omega_\varepsilon}(z) \to \delta_{\Omega}(z) \) uniformly on \( \overline{\Omega} \) as \( \varepsilon \to 0 \). Fix a domain \( V \) with \( \overline{\Omega} \subset V \subset \subset U \). As \( \overline{\Omega} \) is \( O(U) \)-convex, there exist for each \( \varepsilon \ll 1 \) a finite number of holomorphic functions \( f_1, \ldots, f_m \) on \( U \), where \( m = m_\varepsilon \), such that

\[
\sup_{\Omega} \max_{1 \leq j \leq m} |f_j| < 1 \quad \text{and} \quad \inf_{V \setminus \Omega_\varepsilon} \max_{1 \leq j \leq m} |f_j| > 1.
\]

Clearly, \( \rho := \max_{1 \leq j \leq m} |f_j| - 1 \) is a continuous psh function on \( U \), and we have

\[
\overline{\Omega} \subset \{ z \in V : \rho(z) < 0 \} =: \Omega' \subset \Omega_\varepsilon.
\]

Note that \( \Omega \) is pseudoconvex because \( \overline{\Omega} \) is the limit of a decreasing family of analytic polyhedrons.

Let \( z \in \Omega \) be a fixed point that is sufficiently close to \( \partial \Omega \). Take \( \varepsilon \ll 1 \) such that \( \delta_{\Omega_\varepsilon}(z) \leq 3 \delta_{\Omega}(z)/2 \). Then we have

\[
\delta_{\Omega'}(z) \leq \delta_{\Omega_\varepsilon}(z) \leq 3 \delta_{\Omega}(z)/2.
\]

Take \( z^* \in \partial \Omega' \cap \partial B(z, \delta_{\Omega'}(z)) \) and let \( L_z \) be the complex line determined by \( z \) and \( z^* \). From the proof of (d), we know that

\[
L_z \cap \partial B(z, r) \cap (\mathbb{C}^n \setminus \Omega') \neq \emptyset, \quad \delta_{\Omega'}(z) < r < r_0,
\]

where \( r_0 > 0 \) is suitable constant depending only on \( \Omega \) and \( V \). As \( \delta_{\Omega'}(z) \leq \frac{3}{2} \delta_{\Omega}(z) \), we conclude that

\[
C_1 \left( \frac{L_z \cap \overline{B(z, 2\delta_{\Omega}(z))} \cap \Omega^c}{\Omega^c} \right) \geq C_1 \left( \frac{L_z \cap \overline{B(z, 2\delta_{\Omega}(z))} \cap (\mathbb{C}^n \setminus \Omega')} {\Omega'} \right) \geq \frac{1}{4} \cdot \frac{\delta_{\Omega}(z)}{2}
\]

in view of the circular projection. Thus, Theorem 1.1 applies.

\( \square \)

**Proof of (a).** Let \( z_0 \in \partial \Omega \). After a complex affine transformation, we may assume that \( z_0 = 0 \) and there exist a ball in \( \mathbb{C}^{n-1} \times \mathbb{R} \) given by

\[
B'_{r_0} := \{ (z', \Re z_n) \in \mathbb{C}^{n-1} \times \mathbb{R} : |z'|^2 + (\Re z_n)^2 < r_0^2 \},
\]

and a continuous real-valued function \( g \) on \( B'_{r_0} \) such that

\[
\Omega \cap \left\{ B'_{r_0} \times (-r_0, r_0) \right\} = \left\{ z \in B'_{r_0} \times (-r_0, r_0) : \Im z_n < g(z', \Re z_n) \right\}.
\]
Given $0 \leq t \leq \varepsilon_0 \ll r_0$, define

$$
\Omega_t := \left\{ z \in B'_{\frac{r_0}{2} + t} \times (-r_0/2 - t, r_0) : \text{Im } z_n < g(z', \text{Re } z_n) + t \right\}.
$$

Note that $\Omega_0 = \Omega \cap \{B'_{\frac{r_0}{2}} \times (-r_0/2, r_0)\}$ and $\{\Omega_t\}_{t \in [0, \varepsilon_0]}$ forms an increasing 1-parameter family (in the sense of Docquier–Grauert (cf. [14]; see also [32, section 5])) of bounded pseudoconvex fat domains.

By (c) and the well-known localization principle of the Bergman kernel, it suffices to show that the closure $\overline{\Omega}_0$ of $\Omega_0$ is $\mathcal{O}(\Omega_{\varepsilon_0})$-convex. Given $w \in \Omega_0 \setminus \overline{\Omega}_0$, take $0 < t_1 < \varepsilon_0$ such that $w \in \partial \Omega_{t_1}$. Take $\gamma > 0$ and $t_1 < t_2 < \varepsilon_0$ such that $\min_{\overline{\Omega}_0} \delta_{\Omega_{t_1}} > \gamma$ and $\delta_{\Omega_{t_2}}(w) = \gamma/2$. We are going to construct a holomorphic function $f$ on $\Omega_{t_2}$ satisfying $|f(w)| > 1$ and $|f| < 1$ on $\overline{\Omega}_0$. The argument is essentially standard, but we include the detail here for the sake of completeness. Define $\psi := -\log \delta_{\Omega_{t_2}} + \log \gamma$. Clearly, $\psi$ is a locally Lipschitz continuous psh exhaustion function on $\Omega_{t_2}$ satisfying

$$
\Omega_0 \subset \{\psi < 0\} \quad \text{and} \quad \psi(w) = \log 2.
$$

Let $\kappa$ be a convex nondecreasing function on $\mathbb{R}$ with $\kappa = 0$ on $(-\infty, 0]$ and $\kappa', \kappa'' > 0$ on $(0, \infty)$. Take a smooth cut-off function $\chi$ on $\mathbb{R}$ such that $\chi = 0$ on $\left[\frac{\varepsilon_0}{4} \log 2, \frac{\varepsilon_0}{3} \log 2\right] \cup [3 \log 2, \infty)$ and $\chi = 1$ on $(-\infty, 0] \cup \left[\frac{1}{2} \log 2, 2 \log 2\right]$. Define $h = \frac{1}{2}$ on $\{\psi \leq \frac{1}{4} \log 2\}$ and $h = 2$ on $\left\{\frac{1}{3} \log 2 \leq \psi \leq 3 \log 2\right\}$. Then $\chi(\psi)h$ defines a Lipschitz continuous function with compact support in $\Omega_{t_2}$. By Hörmander’s $L^2$-estimates (cf. [21]), there exists a solution of $\bar{\partial} u = \bar{\partial}(\chi(\psi)h)$ such that

$$
\int_{\Omega_{t_2}} |u|^2 e^{-\kappa(\psi) - 2n \log |z-w|} \lesssim \int_{\Omega_{t_2}} |h|^2 \left|\chi'(\psi)\right|^2 \left|\bar{\partial} \chi\right|^2 e^{-\kappa(\psi) - 2n \log |z-w|} e^{\kappa(\psi)} \left|\bar{\partial} \psi\right|^2,
$$

where the implicit constants are independent of the choice of $\kappa$, which can be arbitrarily small provided that $\kappa$ increases on $(0, \infty)$ sufficiently rapidly. Thus $f := \chi(\psi)h - u$ is a holomorphic function on $\Omega_{t_2}$ that satisfies $f(w) = h(w) = 2$ and as $f - h = -u$ is holomorphic on $\{\psi < 0\}$, it follows the mean-value inequality that

$$
\sup_{\overline{\Omega}_0} |f - h|^2 \lesssim \int_{\{\psi < 0\}} |u|^2 e^{-2n \log |z-w|}
$$

$$
\lesssim \int_{\Omega_{t_2}} |u|^2 e^{-\kappa(\psi) - 2n \log |z-w|},
$$

which can be arbitrarily small provided that $\kappa$ increases on $(0, \infty)$ sufficiently rapidly. Thus $|f| < 1$ on $\overline{\Omega}_0$.

Finally, as $\{\Omega_t\}_{t \in [0, \varepsilon_0]}$ forms an increasing 1-parameter family of bounded pseudoconvex domains, it follows from a classical result of Docquier–Grauert (cf. [14]; see also [32, section 5]) that $(\Omega_{t_2}, \Omega_{\varepsilon_0})$ forms a Runge pair, that is, every holomorphic function on $\Omega_{t_2}$ can be approximated locally uniformly on $\Omega_{t_2}$ by holomorphic functions on $\Omega_{\varepsilon_0}$. From this, we obtain a holomorphic function $\tilde{f}$ on $\Omega_{\varepsilon_0}$ satisfying $|\tilde{f}(w)| > 1$ and $|\tilde{f}| < 1$ on $\overline{\Omega}_0$. Thus, $\overline{\Omega}_0$ is $\mathcal{O}(\Omega_{\varepsilon_0})$-convex. □
4.3 | Proof of Proposition 1.3

We first give an elementary lemma as follows.

**Lemma 4.1.** If $D \subset \mathbb{C}$ is simply-connected, then

$$K_D(z) \geq \frac{1}{16\pi} \cdot \delta_D(z)^{-2}, \quad z \in D.$$ 

*Proof.* Given $z \in D$, consider the Riemann mapping $F : D \to \mathbb{D}$ satisfying

$$F(z) = 0, \quad F'(z) > 0.$$ 

It is a standard consequence of the Koebe one-quarter theorem that $\delta_D(z) \geq \frac{1}{4|F'(z)|}$ (cf. [6, Theorem 1.4]). Thus,

$$K_D(z) = K_\mathbb{D}(0)|F'(z)|^2 \geq \frac{1}{16\pi} \cdot \delta_D(z)^{-2}. \quad \square$$

*Proof of Proposition 1.3.* Note that the Runge domain $\Omega$ enjoys the following geometric property: for every complex line $\mathcal{L}$, all components of $\Omega \cap \mathcal{L}$ are simply-connected. To see this, let $D$ be a component of $\Omega \cap \mathcal{L}$. Given $f \in \mathcal{O}(D)$, the holomorphic function on $\Omega \cap \mathcal{L}$ that equals $f$ on $D$ and 0 on other components can be extended to some $\tilde{f} \in \mathcal{O}(\Omega)$, for $\Omega$ is pseudoconvex. As $\Omega$ is Runge, there exists a sequence of complex polynomials $\{P_j\}$ converging locally uniformly to $\tilde{f}$ on $\Omega$. Thus, the polynomials $\{P_j|_{\mathcal{L}}\}$ converge locally uniformly on $D$ to $f$, so that $D$ has to be simply connected.

Given $z \in \Omega$, take $z^* \in \partial \Omega$ so that $|z - z^*| = \delta_\Omega(z)$. Let $\mathcal{L}_z$ be the complex line determined by $z, z^*$. Let $D_z$ be the component of $\Omega \cap \mathcal{L}_z$ that contains $z$. Clearly, we have

$$\delta_{D_z}(z) = \delta_\Omega(z). \quad (4.5)$$

Set $f = K_{D_z}(\cdot, z)/\sqrt{K_{D_z}(z)}$ on $D_z$ and $f = 0$ on other components of $\Omega \cap \mathcal{L}_z$. By the Ohsawa–Takegoshi extension theorem, there is a holomorphic function $\tilde{f}$ on $\Omega$ such that $\tilde{f}|_{\Omega \cap \mathcal{L}_z} = f$ and

$$\int_{\Omega} |\tilde{f}|^2 \lesssim \int_{\Omega \cap \mathcal{L}_z} |f|^2 \lesssim 1,$$

where the implicit constants depend only on the diameter of $\Omega$. Thus,

$$K_\Omega(z) \geq \frac{|\tilde{f}(z)|^2}{\int_{\Omega} |\tilde{f}|^2} \geq K_{D_z}(z) \geq \delta_{D_z}(z)^{-2},$$

in view of Lemma 4.1. This together with (4.5) conclude the proof. \square

4.4 | Proofs of Proposition 1.4

The following lemma is essentially implicit in [36].
Lemma 4.2. Let $\Omega$ be a bounded domain in $\mathbb{C}$ and $\alpha > 1$. Then for any $z \in \Omega$ with $\alpha \delta_\Omega(z) \leq 1/2$,

$$K_\Omega(z) \geq (\alpha \delta_\Omega(z))^{-2} \left(- \log C_l(B(z, \alpha \delta_\Omega(z)) \cap \Omega^c) \right)^{-1},$$

where the implicit constant depends only on $\text{diam} \, \Omega$.

Proof. Given $z \in \Omega$, set $E := B(z, \alpha \delta_\Omega(z)) \cap \Omega^c$. Without loss of generality, we assume $C_l(E) > 0$. Dividing $\mathbb{C}$ into three trisection angles with vertices at $z$, one may divide $E$ into three parts $E_1, E_2, E_3$. We may assume $C_l(E_1) \geq C_l(E_2) \geq C_l(E_3)$. Then

$$\frac{1}{- \log C_l(E)} \leq \frac{1}{- \log C_l(E_1)} + \frac{1}{- \log C_l(E_2)} + \frac{1}{- \log C_l(E_3)} \leq \frac{3}{- \log C_l(E_1)} \quad (4.7)$$

in view of Proposition 2.1/(4). Let $\mu_{E_1}$ denote the equilibrium measure of $E_1$. Consider the following function

$$f_{E_1}(\zeta) := \int_{E_1} \frac{d\mu_{E_1}(w)}{\zeta - w}, \quad \zeta \in \mathbb{C} \setminus E_1.$$ 

Then $f_{E_1}$ is holomorphic on $\mathbb{C} \setminus E_1 \supset \Omega$ and satisfies

$$\int_{\Omega} |f_{E_1}|^2 \lesssim - \log C_l(E_1) \lesssim - \log C_l(E),$$

where the implicit constant depends only on $\text{diam} \, \Omega$ (cf. [36, Lemma 3]). On the other hand, as $|z - w| \leq \alpha \delta_\Omega(z)$ for all $w \in E_1$ and $E_1$ is contained in an angle with vertex $z$ and of the apex angle $2\pi/3$, we conclude that there exists a numerical constant $C_0 > 0$ such that

$$|f_{E_1}(z)| \geq C_0 \int_{E_1} \frac{d\mu_{E_1}(w)}{|z - w|} \geq \frac{C_0}{\alpha \delta_\Omega(z)},$$

for $d\mu_{E_1}$ is a probability measure. Thus,

$$K_\Omega(z) \geq \frac{|f_{E_1}(z)|^2}{\|f\|^2_{L^2(\Omega)}} \geq (\alpha \delta_\Omega(z))^{-2} (- \log C_l(E))^{-1}. \quad \square$$

Proof of Proposition 1.4. The conclusion follows directly from the Ohsawa–Takegoshi extension theorem and Lemma 4.2. \quad \square

Corollary 4.3. Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose there are constants $\varepsilon, r_0 > 0$ and $\beta > 1$, such that

$$|B(\zeta, r) \cap \Omega^c| \geq \varepsilon |B(\zeta, r)|^\beta, \quad \zeta \in \partial \Omega, \quad 0 < r \leq r_0,$$

then (1.7) holds.
Proof. The argument is a mimic of the proof of Theorem 1.2/(b). Given \( z \in \Omega \), take \( z^* \in \partial \Omega \) such that \( |z - z^*| = \delta_\Omega(z) \). As \( B(z^*, \delta_\Omega(z)) \subset B(z, 2\delta_\Omega(z)) \), it follows from (4.8) that

\[
|B(z, 2\delta_\Omega(z)) \cap \Omega^c| \geq \varepsilon(C_n \delta_\Omega(z))^{2n\beta}.
\]

(4.9)

We claim that there exists a real line \( l_z \ni z \), such that

\[
|l_z \cap B(z, 2\delta_\Omega(z)) \cap \Omega^c| \geq \varepsilon' \delta_\Omega(z)^{2n\beta-2n+1}
\]

(4.10)

for suitable \( \varepsilon' = \varepsilon'(n, \varepsilon) > 0 \). Indeed, if we denote \( E := B(z, 2\delta_\Omega(z)) \cap \Omega^c \) and \( E_z := E \cap l_z \), then (4.2) together with (4.9) yield

\[
\int_{\zeta \in S^{2n-1}} |E_z| \, d\sigma(\zeta) \geq C_n^{2n\beta} 2^{1-2n} \varepsilon \delta_\Omega(z)^{2n\beta-2n+1},
\]

so that there exists at least one point \( \zeta_0 \in S^{2n-1} \), such that

\[
|E_{\zeta_0}|_1 \geq \frac{C_n^{2n\beta}\varepsilon}{2^{2n}|S|^{2n-1}} \cdot \delta_\Omega(z)^{2n\beta-2n+1}.
\]

Take \( l_z = l_{z, \zeta_0} \) and \( \varepsilon' = \frac{C_n^{2n\beta}\varepsilon}{2^{2n}|S|^{2n-1}} \), we get (4.10).

Let \( \mathcal{L}_z \) be the complex line determined by \( l_z \). Then

\[
\mathcal{C}(\mathcal{L}_z \cap B(z, 2\delta_\Omega(z)) \cap \Omega^c) \geq \frac{|E_{\zeta_0}|_1}{4} \geq \frac{\varepsilon'}{4} \cdot \delta_\Omega(z)^{2n\beta-2n+1}.
\]

Thus, Proposition 1.4 applies.

\[\square\]

5 \quad LOWER BOUNDS OF THE BERGMAN DISTANCE

Proof of Theorem 1.7.

(1) Let \( c \) be as Proposition 3.5. Let \( z \) be sufficiently close to \( \partial \Omega \). Take \( n \in \mathbb{Z}^+ \) such that \( \lambda_n \leq \delta_\Omega(z) \leq \lambda_{n-1} \). Write

\[
\bigcap_{a \in \partial \Omega} \mathcal{N}_a^n(\varepsilon, \lambda) = \{k_1 < k_2 < \ldots < k_{m_n}\}.
\]

We may choose a Bergman geodesic jointing \( z_0 \) to \( z \), and a finite number of points on this geodesic with the following order

\[
z_0 \to z_{k_1} \to z_{k_2} \to \ldots \to z,
\]

such that

\[
\lambda_{k_j - 1/3} \leq \delta_\Omega(z_{k_j}) \leq \lambda_{k_j - 1/2}.
\]

By Proposition 3.5, we have

\[
\{g_\Omega(\cdot, z_{k_j}) \leq -c\} \cap \{g_\Omega(\cdot, z_{k_{j+1}}) \leq -c\} = \emptyset,
\]

where \( c = c(n, \varepsilon) > 0 \).
so that $d_B(z_{k_j}, z_{k_{j+1}}) \geq c_1 > 0$ for all $j$, in view of (1.15). As $m_n \geq n$, we have

$$d_B(z_0, z) \geq \sum_j d_B(z_{k_j}, z_{k_{j+1}}) \geq n \geq |\log \delta \Omega(z)|.$$

(2) The assertion follows directly from Theorem 1.5 and [11, Corollary 1.8].

(3) Let $c$ be as Proposition 3.6. Let $z$ be sufficiently close to $\partial \Omega$. We may choose a Bergman geodesic jointing $z_0$ to $z$, and a finite number of points $\{z_k\}_{k=1}^m$ on this geodesic with the following order

$$z_0 \rightarrow z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_m \rightarrow z,$$

where

$$c \phi_E(z_{k+1})^{1+\beta} = c^{-1} \phi_E(z_k)^{1+\beta}$$

and

$$c^{-1} \phi_E(z_m)^{1+\beta} \leq \phi_E(z) \leq c \phi_E(z_m)^{1+\beta}.$$

By Proposition 3.6, we have

$$\{g_\Omega(\cdot, z_k) \leq -1\} \cap \{g_\Omega(\cdot, z_{k+1}) \leq -1\} = \emptyset,$$

so that $d_B(z_k, z_{k+1}) \geq c_1 > 0$ for all $k$.

Note that

$$\log \phi_E(z_0) = \left(\frac{\beta}{1+\beta}\right)^2 \log \phi_E(z_1) + \frac{\beta}{1+\beta} \log c^2 = \cdots$$

$$= \left(\frac{\beta}{1+\beta}\right)^{2m} \log \phi_E(z_m) + \frac{\beta}{1+\beta} \frac{1 - \left(\frac{\beta}{1+\beta}\right)^2}{1 - \left(\frac{\beta}{1+\beta}\right)^2} \log c^2.$$

Thus, we have

$$m \simeq \log |\log \phi_E(z_m)| \simeq \log |\log \phi_E(z)| \gtrsim \log \log |\log \delta \Omega(z)|$$

in view of Theorem 1.6, so that

$$d_B(z_0, z) \gtrsim \sum_{k=1}^{m-1} d_B(z_k, z_{k+1}) \gtrsim c_1 (m - 1) \gtrsim \log \log |\log \delta \Omega(z)|.$$

□

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