Efficient Fréchet distance queries for segments

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Abstract
We study the problem of constructing a data structure that can store a two-dimensional polygonal curve P, such that for any query segment ab one can efficiently compute the Fréchet distance between P and ab. First we present a data structure of size O(n log n) that can compute the Fréchet distance between P and a horizontal query segment ab in O(log n) time, where n is the number of vertices of P. In comparison to prior work, this significantly reduces the required space. We extend the type of queries allowed, as we allow a query to be a horizontal segment ab together with two points s, t ∈ P (not necessarily vertices), and ask for the Fréchet distance between ab and the curve of P in between s and t. Using O(n log 2 n) storage, such queries take O(log 3 n) time, simplifying and significantly improving previous results. We then generalize our results to query segments of arbitrary orientation. We present an O(nk 3+ε + n 2) size data structure, where k ∈ [1..n] is a parameter the user can choose, and ε > 0 is an arbitrarily small constant, such that given any segment ab and two points s, t ∈ P we can compute the Fréchet distance between ab and the curve of P in between s and t in O((n/k) log 2 n + log 4 n) time. This is the first result that allows efficient exact Fréchet distance queries for arbitrarily oriented segments.

We also present two applications of our data structure: we show that we can compute a local δ-simplification (with respect to the Fréchet distance) of a polygonal curve in O(n 5/2+ε) time, and that we can efficiently find a translation of an arbitrary query segment ab that minimizes the Fréchet distance with respect to a subcurve of P.
1 Introduction

Comparing the shape of polygonal curves is an important task that arises in many contexts such as GIS applications [4, 13], protein classification [29], curve simplification [11], curve clustering [2] and even speech recognition [30]. Within computational geometry, there are two well studied distance measures for polygonal curves: the Hausdorff and the Fréchet distance. The Fréchet distance has proven particularly useful as it takes the course of the curves into account. However, the Fréchet distance between curves is costly to compute, as its computation requires roughly quadratic time [5, 12]. When a large number of Fréchet distance queries are required, we would like to have a data structure, a so-called distance oracle, to answer these queries more efficiently.

This leads to a fundamental data structuring problem: preprocess a polygonal curve such that, given a query polygonal curve, their Fréchet distance can be computed efficiently. (Here, the query curves are assumed to be of small size compared to the preprocessed one.) It turns out that this problem is extremely challenging. Firstly, polygonal curves are complex objects of non-constant complexity, where most operations have a non-negligible cost. Secondly, the high computational cost of the Fréchet distance complicates all attempts to design exact algorithms for the problem. Indeed, even though great efforts have been devoted to design data structures to answer Fréchet distance queries, there is still no distance oracle known that is able to answer exactly queries for a general query curve.

To make progress on this important problem, in this work we focus on a more restrictive but fundamental setting: when the query curve is a single segment. The reasons to study this variant of the problem are twofold. On the one hand, it is a necessary step to solve the general problem. On the other hand, it is a setting that has its own applications. For example, in trajectory simplification, or when trying to find subtrajectories that are geometrically close to a given query segment (e.g. when computing shortcut-variants of the Fréchet distance [18], or in trajectory analysis [16] on sports data). A similar strategy of tackling segment queries has also been successfully applied in nearest neighbor queries with the Fréchet distance [8].

More precisely, we study the problem of preprocessing a polygonal curve \( P \) to determine the exact continuous Fréchet distance between \( P \) and a query segment in sublinear time. Specifically, we study preprocessing a polygonal curve \( P \) of \( n \) vertices in the plane, such that given a query segment \( \overline{ab} \), traversed from \( a \) to \( b \), the Fréchet distance between \( P \) and \( \overline{ab} \) can be computed in sublinear time. Note that without preprocessing, this problem can be solved in \( O(n \log n) \) time.

Related work. Data structures that support (approximate) nearest neighbor queries with respect to the Fréchet distance have received considerable attention throughout the years, see for instance, these recent papers [8, 19, 22] and the references therein. In these problems, the goal is typically to store a set of polygonal curves such that given a query curve and a query threshold \( \Delta \) one can quickly report (or count) the curves that are within (discrete) Fréchet distance \((1 + \epsilon)\Delta\), for some \( \epsilon > 0 \), of the query curve. Some of these data structures even allow approximately counting the number of curves that have a subcurve within Fréchet distance \( \Delta \) [16]. Also highlighting its practical importance, the near neighbor problem using Fréchet distance was posed as ACM Sigspatial GIS Cup in 2017 [38].

Here we consider the problem in which we want to compute the Fréchet distance of (part of) a curve to a low complexity query curve. For the discrete Fréchet distance, efficient \((1 + \epsilon)\)-approximate distance oracles are known, even when \( P \) is given in an online fashion [21]. For the continuous Fréchet distance that we consider the results are more restrictive. Driemel and Har-Peled [18] present an \( O(n \epsilon^{-4} \log \epsilon^{-1}) \) size data structure that given a query segment \( \overline{ab} \) can compute a \((1 + \epsilon)\)-approximation of the Fréchet distance between \( P \) and \( \overline{ab} \) in \( O(\epsilon^{-2} \log n \log \log n) \) time. Their approach extends to higher dimensions and low complexity polygonal query curves.
Gudmundsson et al. [24] present an $O(n \log n)$ sized data structure that can decide if the Fréchet distance to $\overline{ab}$ is smaller than a given value $\Delta$ in $O((\log^2 n)$ time (so with some parametric search approach one could consider computing the Fréchet distance itself). However, their result holds only when the length of $\overline{ab}$ and all edges in $P$ is relatively large compared to $\Delta$. De Berg et al. [17] presented an $O(n^2)$ size data structure that does not have any restrictions on the length of the query segment or the edges of $P$. However, the orientation of the query segment is restricted to be horizontal. Queries are supported in $O(\log^2 n)$ time, and even queries asking for the Fréchet distance to a vertex-to-vertex subcurve are allowed (in that case, using $O(n^2 \log^2 n)$ space). Recently, Gudmundsson et al. [25] extended this result to allow the subcurve to start and end anywhere within $P$. Their data structure has size $O(n^2 \log^2 n)$ and queries take $O(\log^8 n)$ time. In their journal version, Gudmundsson et al. [26] directly apply the main result of a preliminary version of this paper [14] to immediately improve space usage of their data structure to $O(n^{3/2})$; their preprocessing time remains $O(n^2 \log^2 n)$. The current version of this paper significantly improves these results. Moreover, we present data structures that allow for arbitrarily oriented query segments.

**Problem statement & our results.** Let $P$ be a polygonal curve in $\mathbb{R}^2$ with $n$ vertices $p_1, \ldots, p_n$. For ease of exposition, we assume that the vertices of $P$ are in general position, i.e., all $x$- and $y$-coordinates are unique, no three points lie on a line, and no four points are cocircular. We consider $P$ as a function mapping any time $t \in [0, 1]$ to a point $P(t)$ in the plane. Our ultimate goal is to store $P$ such that we can quickly compute the Fréchet distance $D_F(P, Q)$ between $P$ and a query curve $Q$. The Fréchet distance is defined as

$$D_F(P, Q) = \inf_{\alpha, \beta} \max_{t \in [0, 1]} \| P(\alpha(t)) - Q(\beta(t)) \|,$$

where $\alpha, \beta : [0, 1] \to [0, 1]$ are nondecreasing surjections, also called reparameterizations of $P$ and $Q$, respectively, and $\| p - q \|$ denotes the Euclidean distance between $p$ and $q$.

In this work we focus on the case where $Q$ is a single line segment $\overline{ab}$ starting at $a$ and ending at $b$. Note that $P$ may self-intersect and $\overline{ab}$ may intersect $P$. Our first main result deals with the case where $\overline{ab}$ is horizontal:

**Theorem 1.** Let $P$ be a polygonal curve in $\mathbb{R}^2$ with $n$ vertices. There is an $O(n \log n)$ size data structure that can be built in $O(n \log^2 n)$ time such that given a horizontal query segment $\overline{ab}$ it can report $D_F(P, \overline{ab})$ in $O(\log n)$ time.

This significantly improves over the earlier result of de Berg et al. [17], as we reduce the required space and preprocessing time from quadratic to near linear. We simultaneously improve the query time from $O(\log^2 n)$ to $O(\log n)$. We further extend our results to allow queries against subcurves of $P$. Let $s, t$ be two points on $P$, we use $P[s, t]$ to denote the subcurve of $P$ from $s$ to $t$. For horizontal query segments we then get:

**Theorem 2.** Let $P$ be a polygonal curve in $\mathbb{R}^2$ with $n$ vertices. There is an $O(n \log^2 n)$ size data structure that can be built in $O(n \log^2 n)$ time such that given a horizontal query segment $\overline{ab}$ and two query points $s$ and $t$ on $P$ it can report $D_F(P[s, t], \overline{ab})$ in $O(\log^3 n)$ time.

De Berg et al. presented a data structure that could handle such queries in $O(\log^2 n)$ time (using $O(n^2 \log^2 n)$ space), provided that $s$ and $t$ were vertices of $P$. Compared to their data structure we thus again significantly improve the space usage, while allowing more general queries. The recently presented data structure of Gudmundsson et al. [25] does allow $s$ and $t$ to lie on the interior of edges of $P$ (and thus supports queries against arbitrary subcurves). Their original data structure uses $O(n^2 \log^2 n)$ space and allows for $O(\log^8 n)$ time queries. Compared to their result we use significantly less space, while also improving the query time.
There is an \( - \rightarrow D \) directed Hausdorff distance given two query points. We illustrate the main ideas of our approach, in particular for the case where the query segment is horizontal, with \( a \) left of \( b \). We can build a symmetric data structure in case \( a \) lies right of \( b \). We now first review some definitions based on those in [17].

Let \( P \subseteq \) be the set of ordered pairs of vertices \( (p, q) \in P \times P \) where \( p \) precedes or equals \( q \) along \( P \). An ordered pair \( (p, q) \in P \) forms a \textit{backward pair} if \( x_q \leq x_p \). Here, and throughout

**Theorem 3.** Let \( P \) be a polygonal curve in \( \mathbb{R}^2 \) with \( n \) vertices, and let \( k \in [1..n] \) be a parameter. There is an \( O(nk^{3+\varepsilon} + n^2) \) size data structure that can be built in \( O(nk^{3+\varepsilon} + n^2) \) time such that given an arbitrary query segment \( \overline{ab} \) it can report \( D_H(P, \overline{ab}) \) in \( O((n/k) \log^2 n) \) time. In addition, \textit{Applications.} In Section 6 we show how to efficiently solve two problems using our data structure. First, we show how to compute a local \( \delta \)-simplification of \( P \)—that is, a minimum complexity curve whose edges are within Fréchet distance \( \delta \) to the corresponding subcurve of \( P \)—in \( O(n^{5/2+\varepsilon}) \) time. This improves existing \( O(n^3) \) time algorithms [23]. Second, given a query segment \( \overline{ab} \) we show how to efficiently find a translation of \( \overline{ab} \) that minimizes the Fréchet distance to (a given subcurve of) \( P \). This extends the work of Gudmundsson et al. [26] to arbitrarily oriented segments. Furthermore, we answer one of their open problems by showing how to find a scaling of the segment that minimizes the Fréchet distance.

**Global approach**

We illustrate the main ideas of our approach, in particular for the case where the query segment \( \overline{ab} \) is horizontal, with \( a \) left of \( b \). We can build a symmetric data structure in case \( a \) lies right of \( b \). We now first review some definitions based on those in [17].

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the rest of the paper, \(x_p\) and \(y_p\) denote the \(x\)- and \(y\)-coordinates of point \(p\), respectively. The set of all backward pairs of \(P\) will be denoted \(\mathcal{B}(P)\). A backward pair \((p, q)\) is trivial if \(p = q\). See Fig. 1 for an example of backward pairs (omitting trivial pairs). For two points \(p, q \in P\), we then define \(\delta_{pq}(y) = \min_x \max \{||(x, y) - p||, ||(x, y) - q||\}\). That is, \(\delta_{pq}(y)\) is a function that for any \(y\)-coordinate gives the minimum possible distance between a point at height \(y\) and both \(p\) and \(q\). We will use the function \(\delta_{pq}\) only when \((p, q) \in \mathcal{B}(P)\) is a backward pair. We then define the function \(\mathcal{D}_B(y) = \max \{\delta_{pq}(y) \mid (p, q) \in \mathcal{B}(P)\}\), which we refer to as the backward pair distance of a horizontal segment at height \(y\) with respect to \(P\). Note that \(\mathcal{D}_B(y)\) is the upper envelope of the functions \(\delta_{pq}\) for all backward pairs \((p, q)\) of \(P\).

De Berg et al. [17] prove that the Fréchet distance is the maximum of four terms:

\[
\mathcal{D}_F(P, \overline{ab}) = \max \left\{ ||p_1 - a||, \ ||p_n - b||, \ \overline{D}_H(P, \overline{ab}), \ \mathcal{D}_B(y_a) \right\}.
\] (1)

The first two terms are trivial to compute in \(O(1)\) time. Like de Berg et al., we build separate data structures that allow us to efficiently compute the third and fourth terms.

A key insight is that we can compute \(\overline{D}_H(P, \overline{ab})\) by building the furthest segment Voronoi diagrams (FSVD) of two sets of horizontal halflines, and querying these diagrams with the endpoints \(a\) and \(b\). See Section 3.1. This allows for a linear space data structure that supports querying \(\overline{D}_H(P, \overline{ab})\) in \(O(n \log n)\) time, improving both the space and query time over [17].

However, in [17] the data structure that supports computing the backward pair distance dominates the required space and preprocessing time, as there may be \(\Omega(n^2)\) backward pairs, see Fig. 1. Via a divide and conquer argument we show that the number of backward pairs that show up on the upper envelope \(\mathcal{D}_B\) is only \(O(n \log n)\), see Section 3.2. The crucial ingredient is that there are only \(O(n)\) backward pairs \((p, q)\) contributing to \(\mathcal{D}_B\) in which \(p\) is a vertex among the first \(n/2\) vertices of \(P\), and \(q\) is a vertex in the remaining \(n/2\) vertices. Surprisingly, we can again argue this using furthest segment Voronoi diagrams of sets of horizontal halflines. This allows us to build \(\mathcal{D}_B\) in \(O(n \log^2 n)\) time in total. In Section 3.3 we extend these results to support queries against an arbitrary subcurve \(P[s, t]\) of \(P\).

For arbitrarily oriented query segments we similarly decompose \(\mathcal{D}_F(P, \overline{ab})\) into four terms, and build a data structure for each term separately, see Section 4. The directed Hausdorff term can still be queried efficiently using an \(O(n \log^2 n)\) size data structure. However, our initial data structure for the backward pair distance uses \(O(n^{1+\varepsilon})\) space. The main reason for this is that functions \(\delta_{pq}\) expressing the cost of a backward pair are now bivariate, depending on both the slope and intercept of the supporting line of \(\overline{ab}\). The upper envelope of a set of \(n\) such functions may have quadratic complexity. While our divide and conquer strategy does not help us to directly bound the complexity of the (appropriately generalized function) \(\mathcal{D}_B\) in this case, it does allow us to support queries against subcurves of \(P\). Moreover, we can use it to obtain a favourable query time vs. space trade off. In Section 6 we then apply our data structure to efficiently solve various Fréchet distance related problems.

3 Horizontal queries

3.1 The Hausdorff term

In this section we show that there is a linear-size data structure to query the Hausdorff term in \(O(\log n)\) time, which can be built in \(O(n \log n)\) time.

For a point \(p \in \mathbb{R}^2\), define \(\overrightarrow{p}\) to be the “leftward” horizontal halfline starting at \(p\) and containing all points directly to the left of \(p\). Analogously, we define \(\overleftarrow{p}\) as the “rightward” horizontal halfline
starting at \( p \), so that \( p = \overrightarrow{p} \cap \overrightarrow{p} \). We extend this notation to any set of points \( S \), that is, \( \overrightarrow{S} = \{ \overrightarrow{s} \mid s \in S \} \) denotes the set of “leftward” halflines starting at the points in \( S \subseteq \mathbb{R}^2 \). We define \( \overrightarrow{S} \) analogously. Let \( S \) and \( T \) be two (possibly overlapping) point sets in the plane. We define the following distance functions (see also Fig. 2.):

\[
\begin{align*}
\overrightarrow{h}_S(q) &= \overrightarrow{D}_H(\{q\}, \overrightarrow{p}) = \min \left\{ \|p' - q\| \mid p' \in \overrightarrow{p} \right\}, \\
\overrightarrow{h}_T(q) &= \overrightarrow{D}_H(\{q\}, \overrightarrow{p}) = \min \left\{ \|p' - q\| \mid p' \in \overrightarrow{p} \right\}, \\
\overleftarrow{h}_S(q) &= \max \{ h_S(q) \mid p \in S \}, \\
\overleftarrow{h}_T(q) &= \max \{ h_T(q) \mid p \in S \}.
\end{align*}
\]

Note that \( h_S^{-1} \) (resp., \( h_T^{-1} \)) is the upper envelope of the distance functions to the halflines in \( \overrightarrow{S} \) (resp., \( \overrightarrow{T} \)). Since \( h_S^{-1} \) and \( h_T^{-1} \) map each point in the plane to a distance, the envelopes live in \( \mathbb{R}^3 \).

We begin by providing some observations on the computation of the Hausdorff term.

**Observation 4.** Let \( S \) be a set of points in the plane. The (graph of the) function \( h_S^{-1} \) is an upper envelope whose orthogonal projection is the furthest segment Voronoi diagram of the set of halflines \( \overrightarrow{S} \). A symmetrical property holds for \( h_T^{-1} \).

For points \( p = (x_p, y_p) \) and \( q = (x_q, y_q) \in \mathbb{R}^2 \), we have

\[
h_S(q) = \begin{cases} 
\|p - q\| & \text{if } x_p \leq x_q \\
|y_p - y_q| & \text{if } x_p \geq x_q,
\end{cases} \quad \text{and} \quad h_T(q) = \begin{cases} 
\|p - q\| & \text{if } x_p \geq x_q \\
|y_p - y_q| & \text{if } x_p \leq x_q.
\end{cases}
\]

**Observation 5.** For any fixed \( y \) and \( p \in S \), the function \( x \mapsto h_S^{-1}(x, y) \) for a point \( p \) is monotonically increasing, and \( x \mapsto h_T^{-1}(x, y) \) is monotonically decreasing. Consequently, also for any point set \( S \), the function \( x \mapsto h_S^{-1}(x, y) \) is monotonically decreasing, and \( x \mapsto h_T^{-1}(x, y) \) is monotonically increasing.

**Lemma 6.** For any horizontal segment \( \overrightarrow{ab} \) and any point set \( S \subseteq \mathbb{R}^2 \), we have \( \overrightarrow{D}_H(S, \overrightarrow{ab}) = \max \{ h_S^{-1}(a), h_T^{-1}(b) \} \).

**Proof.** Assume w.l.o.g. that \( x_a \leq x_b \), and let \( y = y_a = y_b \). Partition \( S \) into three disjoint subsets \( L \), \( R \), and \( M \), where \( L \subseteq S \) contains all points in \( S \) strictly left of \( a \), \( R \) all points strictly right of \( b \), and \( M \) all (remaining) points that lie in the vertical slab defined by \( x_a \) and \( x_b \). We then have:

\[
\overrightarrow{D}_H(S, \overrightarrow{ab}) = \max \min \left\{ s \in S, r \in \overrightarrow{ab} \right\} = \max \left\{ \sup_{s \in L} \|s - a\|, \max_{s \in R} \|s - b\|, \max_{s \in M} |y_s - y| \right\} = \max \left\{ h_S^{-1}(a), h_T^{-1}(b), \max_{s \in M} |y_s - y| \right\}.
\]
For all points \( s \) not strictly left of \( a \), so in particular those in \( M \), we have that \( h^-_S(a) = |y_s - y_a| = |y_s - y| \), and thus \( \sup_{s \in M} |y_s - y| = h^-_M(a) \). For the points \( s \in R \) (these are the points right of \( a \), that even lie right of \( b \)) we have \( h^-_S(a) = |y_s - y_b| = |y_s - y_b| \leq \| s - b \| = h^-_R(b) \). It therefore follows that \( h^-_R(a) \leq h^-_R(b) \). Consequently we observe that by definition:

\[
\overrightarrow{D}_H(S, \overline{ab}) = \max \left\{ h^-_L(a), h^-_R(b), \sup_{s \in M} |y_s - y| \right\}.
\]

Finally we conclude:

\[
\overrightarrow{D}_H(S, \overline{ab}) = \max \left\{ h^-_S(a), h^-_R(b) \right\}.
\]

Symmetrically, we obtain that \( \sup_{s \in M} |y_s - y| = h^-_M(b) \), and that \( h^-_L(b) \leq h^-_L(a) \). Therefore

\[
\overrightarrow{D}_H(S, \overline{ab}) = \max \left\{ h^-_L(a), h^-_L(b) \right\}.
\]

The lemma now follows since \( L, R \subseteq S \).

\[\square\]

**Corollary 7.** For any point \( p \) and point set \( S \subseteq \mathbb{R}^2 \), \( \overrightarrow{D}_H(S, p) = \max \left\{ h^-_S(p), h^-_R(p) \right\} \).

By Observation 4, \( h^-_S \) corresponds to the furthest segment Voronoi diagram (FSVD) of \( S \). For \( d = 2 \), this diagram has size \( O(n) \) and can be computed in \( O(n \log n) \) time [32]. Thus, by preprocessing the FSVD for planar point location queries [33] we obtain a linear space data structure that allows us to evaluate \( h^-_S(q) \) for any query point \( q \in \mathbb{R}^2 \) in \( O(n \log n) \) time. Note that Sarnak and Tarjan [33] define their point location data structure (a sweep with a partially persistent red-black tree) for planar subdivisions whose edges are line segments. However, their result directly applies to subdivisions with \( x \)-monotone curved segments of low algebraic degree. Since the edges in the FSVD are line segments, rays, or parabolic arcs [32] we can easily split each such edge into \( O(1) \) \( x \)-monotone curved segments, and thus use their result as well. Analogously, we build a linear space data structure for querying \( h^-_S \), and obtain the following result through Lemma 6.

**Theorem 8.** Let \( S \) be a set of \( n \) points in \( \mathbb{R}^2 \). In \( O(n \log n) \) time we can build a data structure of linear size so that given a horizontal query segment \( \overline{ab} \), \( \overrightarrow{D}_H(S, \overline{ab}) \) can be computed in \( O(n \log n) \) time.

Note that the directed Hausdorff distance from a polygonal curve \( P \) to a (horizontal) line segment is attained at a vertex of \( P \) [17], thus, we can use Theorem 8 to compute it.

### 3.2 The backward pairs term

In this section we show that the function \( \mathcal{D}_R \), representing the backward pair distance, has complexity \( O(n \log n) \), can be computed in \( O(n \log^2 n) \) time, and can be evaluated for some query value \( y \) in \( O(\log n) \) time. This leads to an efficient data structure for querying \( P \) for the Fréchet distance to a horizontal query segment \( \overline{ab} \), proving Theorem 1.

Recall that \( \mathcal{D}_B(y) \) is the maximum over all function values \( \delta_{pq}(y) \) for all backward pairs \( (p, q) \in \mathcal{B}(P) \). To avoid computing \( \mathcal{B}(P) \), we define a new function \( \delta'_{pq}(y) \) that applies to any ordered pair of points \( (p, q) \in P^\leq \). We show that for all backward pairs \( (p, q) \in \mathcal{B}(P) \), we have \( \delta'_{pq}(y) = \delta_{pq}(y) \). For any pair \( (p, q) \in P^\leq \) that is not a backward pair, we show that there exists a backward pair \( (p', q') \in \mathcal{B}(P) \) such that \( \delta'_{pq}(y) \leq \delta'_{p'q'}(y) = \delta_{p'q'}(y) \). Consequently, we can compute the value \( \mathcal{D}_B(y) \) by computing the maximum value of \( \delta'_{pq}(y) \) over all pairs in \( P^\leq \). We will show how to do this in an efficient manner.

#### 3.2.1 Decomposing the backward pair distance

Recall that for any two points \( (p, q) \), we denote by \( \overleftarrow{q} \) the leftward horizontal ray originating from \( q \) and by \( \overrightarrow{p} \) the rightward horizontal ray originating from \( p \). For each pair of points \( (p, q) \in P^\leq \), we
Fig. 3: An illustration of the argument of Lemma 9.

define the pair distance between a query \( \overrightarrow{ab} \) at height \( y \) and \( (p, q) \) as the Hausdorff distance from a horizontal line of height \( y \) to \( (\overrightarrow{p} \cup \overrightarrow{q}) \), or more formally:

\[
\delta'_{pq}(y) = \min_x \max \{h_q^-(x, y), h_p^-(x, y)\}.
\]

First we analyze the case where \((x, y)\) is a backward pair.

**Lemma 9.** Let \((p, q) \in P^e\) be a pair of points with \(x_p \geq x_q\). Then for all \(y\), \(\delta_{pq}(y) = \delta'_{pq}(y)\).

**Proof.** For all points \(p\), the function \(x \mapsto \|(x, y) - p\|\) is convex, and minimized at \(x = x_p\). The function \(\delta_{pq}(y)\) is defined as \(\delta_{pq}(y) = \min_x \max \{\|(x, y) - p\|, \|(x, y) - q\|\}\). Since \(x_q \leq x_p\), it follows from the convexity that the function value \(\max \{\|(x, y) - p\|, \|(x, y) - q\|\}\) is minimized for an \(x\)-coordinate in \([x_q, x_p]\) (and thus \(\delta_{pq}(y)\) is realized by an \(x\)-value in \([x_q, x_p]\)). For all points \(p\), \(\|(x, y) - p\| = \max \{h_p^-(x, y), h_q^-(x, y)\}\), thus we observe that:

\[
\delta_{pq}(y) = \min_{x \in [x_q, x_p]} \max \{\|(x, y) - p\|, \|(x, y) - q\|\} = \min_{x \in [x_q, x_p]} \max \{h_q^-(x, y), h_p^-((x, y))\} = \min_{x \in [x_q, x_p]} \max \{h_q^-(x, y), h_p^-((x, y))\} = \delta'_{pq}(y).
\]

The consequence of the above lemma is that for each \((p, q) \in B(P)\), \(\delta_{pq}(y) = \delta'_{pq}(y)\).

Next we make an observation about pairs of points that are not a backward pair:

**Lemma 10.** Let \((p, q) \in P^e\) be a pair of points with \(x_p < x_q\), then \(\delta'_{pq}(y) = \max \{\delta_{qy}(y), \delta_{py}(y)\}\).

**Proof.** By definition, \(\delta'_{pq}(y) = \min_x \max \{h_q^-((x, y)), h_p^-((x, y))\}\). For any \(y\), the function \(x \mapsto h_q^-((x, y))\) is minimal and constant for all \(x \leq x_q\). Similarly, the function \(x \mapsto h_p^-((x, y))\) is minimal and constant for all \(x_p \leq x\). Since \((-\infty, x_q] \cap [x_p, \infty) = [x_p, x_q]\), it follows that:

\[
\delta'_{pq}(y) = \min_x \max \{h_q^-((x, y)), h_p^-((x, y))\} = \min_{x \in [x_p, x_q]} \max \{h_q^-((x, y)), h_p^-((x, y))\} = \max \{|y_q - y|, |y_p - y|\} = \max \{\delta_{qy}(y), \delta_{py}(y)\}.
\]

Where the last equality follows from the observation in the proof of Lemma 9 that for any point \(p\), the function \(x \mapsto \|(x, y) - p\|\) is convex, and minimized at \(x = x_p\).
For all pairs of points \((p, q) \in P^\le\) either \((p, q)\) is a backward pair or \(x_p < x_q\), and thus we obtain
the following lemma.

**Lemma 11.** For any polygonal curve \(P\) and any \(y\),
\[
\mathcal{D}_B(y) = \max \{\delta_{pq}(y) \mid (p, q) \in \mathcal{B}(P)\} = \max \{\delta'_{pq}(y) \mid (p, q) \in P^\le\}.
\]

### 3.2.2 Relating \(\mathcal{D}_B(y)\) to furthest segment Voronoi diagrams

In this section we devise a divide and conquer algorithm that computes \(\mathcal{D}_B(y)\) by computing it for
subsets of vertices of \(P\). Lemma 11 allows us to express \(\mathcal{D}_B(y)\) in terms of \(P^\le\) instead of \(\mathcal{B}(P)\).
Next we refine the definition of \(\mathcal{D}_B(y)\) to make it decomposable. To that end, we define \(\mathcal{D}_B(y)\) on
pairs of subsets of \(P\). Let \(S, T\) be any two subsets of vertices of \(P\), we define:
\[
\mathcal{D}^{S \times T}_B(y) = \max \{\delta'_{pq}(y) \mid (p, q) \in (S \times T) \cap P^\le\}.
\]

We show that we can compute \(\mathcal{D}^{S \times T}_B(y)\) efficiently using the \(\delta'\) functions. To this end, we fix a
value of \(y\) and show that computing \(\mathcal{D}^{S \times T}_B(y)\) is equivalent to computing an intersection between two
curves that consist of a linear number of pieces, each of constant complexity. We then argue that as \(y\)
changes, the intersection point moves along a linear complexity curve that can be computed in
\(O(n \log n)\) time. This allows us to query \(\mathcal{D}_B(y) = D^{P \times P}_B(y)\) in \(O(\log n)\) time, for any query height \(y\).

**From distance to intersections.** For a fixed value \(y'\), computing \(\mathcal{D}^{S \times T}_B(y')\) is equivalent to computing
an intersection point between two curves:

**Lemma 12.** Let \(y' \in \mathbb{R}\) be a fixed height, let \(p\) be a point in \(P\), and let \(T\) be a subset of the vertices
of \(P[p, p_n]\). The graphs of the functions \(x \mapsto h_p^\le((x, y'))\) and \(x \mapsto h_T^\le((x, y'))\) intersect at a single
point \((x^*, y')\). Moreover, \(\mathcal{D}^{(p) \times T}_B(y') = h_T^\le((x^*, y'))\)

**Proof.** Recall that \(P[p, p_n]\) is the subcurve of \(P\) from \(p\) to \(p_n\). In Observation 5 we noted that for
all fixed \(y'\), the function \(x \mapsto h_p^\le((x, y'))\) is monotonically increasing. Similarly, for any
point \(p\), the function \(x \mapsto h_T^\le((x, y'))\) is monotonically decreasing. Meaning that the value
\(\min_x \max \{h_p^\le((x, y'), h_T^\le((x, y'))\}\) is realized at \(x^*\). Notice that \((x^*, y')\) is a unique point as
we assume general position, i.e., no two points have the same \(y\)-coordinate. Next, we apply the
definition of \(\mathcal{D}^{(p) \times T}_B(y')\):
\[
\mathcal{D}^{(p) \times T}_B(y') = \max \{\delta'_{pq}(y')\} = \max \{\min \max \{h_p^\le((x, y'), h_T^\le((x, y'))\}\} =
\min \max \{h_p^\le((x, y'), h_T^\le((x, y'))\} = \min \max \{h_T^\le((x, y'), h_T^\le((x, y'))\} \Rightarrow
\mathcal{D}^{(p) \times T}_B(y') = h_T^\le((x^*, y')) = h_p^\le((x^*, y')).
\]

Lemma 13 now follows easily from the previous.

**Lemma 13.** Let \(y' \in \mathbb{R}\) be a fixed height, and let \(S, T\) be subsets of vertices of \(P\) such that the vertices
in \(S\) precede all vertices of \(T\). The graphs of the functions \(x \mapsto h_S^\le((x, y'))\) and \(x \mapsto h_T^\le((x, y'))\)
intersect at a single point \((x^*, y')\). Moreover, \(\mathcal{D}^{S \times T}_B(y') = h_S^\le((x^*, y')) = h_T^\le((x^*, y')).\)
Fig. 4: (a) A set \( \hat{T} \) of rays arising from a set \( T \) of points, with their FSVD. (b) \( h_T((x, y)) \) is the distance from \((x, y)\) to the ray corresponding to the Voronoi cell at \((x, y)\). (c) For a fixed \( y' \), \( x \mapsto h_T((x, y')) \) is monotonically increasing. (d) The value \( x \) for which \( h_T((x, y')) = h_{\hat{T}}((x, y')) \) corresponds to \( f(\beta, \gamma)((y')) \), and to \( D_B^{S \times T}(y') \).

**Proof.** If all points in \( S \) precede all points in \( T \), then all elements in \( S \times T \) are in \( P \leq \) and we note: 
\[
D_B^{S \times T}(y') = \max_{p \in S} \left\{ D_B^{p \times T}(y') \right\}.
\]
The equality then follows from Lemma 12. \( \square \)

Given such a pair \( S, T \), for a fixed value \( y' \), we can compute a linear-size representation of \( x \mapsto h_T((x, y')) \) in \( O(n \log n) \) time as follows (see Fig. 4). We compute the FSVD of \( \hat{T} \) in \( O(n \log n) \) time. Then, we compute the Voronoi cells intersected by a line of height \( y' \) (denoted by \( \ell_{y'} \)) in left-to-right order in \( O(n \log n) \) time. Suppose that a segment of \( \ell_{y'} \) intersects only the Voronoi cell belonging to a halfline \( \hat{q} \in \hat{T} \), then on this domain the function \( h_T((x, y')) = h_{\hat{T}}((x, y')) \), and thus it has constant complexity. A horizontal line can intersect at most a linear number of Voronoi cells, hence the function has total linear complexity. Analogous arguments apply to \( x \mapsto h_{\hat{T}}((x, y')) \).

**Varying the y-coordinate.** Let \( f(\beta, \gamma): y \mapsto x^* \) be the function that for each \( y \) gives the intersection point \( x^* \) such that \( h_{\hat{T}}((x^*, y)) = h_T((x^*, y)) \). The intersection point \( (x^*, y') \) lies on a Voronoi edge of the FSVD of \( (\hat{T} \cup \hat{S}) \). More precisely, it lies on the bichromatic bisector of the FSVD of \( \hat{T} \) and the one of \( \hat{S} \) (see Fig. 5). When we vary the \( y \)-coordinate, the intersection point traces this bisector. This implies that, given the FSVD of \( \hat{S} \) and the FSVD of \( \hat{T} \), the graph of \( f(\beta, \gamma) \) can be computed in \( O(n) \) time.

**Lemma 14.** Let \( S, T \) be subsets of vertices of \( P \) such that all vertices in \( S \) precede all vertices in \( T \). The function \( D_B^{S \times T} \) has complexity \( O(n) \) and can be computed in \( O(n \log n) \) time. Evaluating \( D_B^{S \times T}(y) \), for some query value \( y \in \mathbb{R} \), takes \( O(\log n) \) time.

**Proof.** We consider the function \( y \mapsto (f(\beta, \gamma)(y), D_B^{S \times T}(y)) \). The graph of this curve is \( y \)-monotone. Each \( y \) has a unique value \( f(\beta, \gamma)(y) \), and we need to compute \( D_B^{S \times T}(y) = h_{\hat{T}}(f(\beta, \gamma)(y), y) = h_T(f(\beta, \gamma)(y), y) \). Hence the function \( y \mapsto (f(\beta, \gamma)(y), D_B^{S \times T}(y)) \) is a well-defined curve in \( \mathbb{R}^3 \) parameterized only by \( y \). Thus, we consider the graph of the function \( y \mapsto (f(\beta, \gamma)(y), D_B^{S \times T}(y)) \) projected into the \((y, z)\)-plane. It has linear complexity and can be computed in \( O(n \log n) \) time. By storing the breakpoints of this function in a balanced binary search tree we can then evaluate \( D_B^{S \times T}(y) \), for any \( y \), in \( O(\log n) \) time. \( \square \)
We begin by analyzing the complexity of the function $D_B(y)$. Consider a partition of $P$ into subcurves $S$ and $T$ with at most $\lceil n/2 \rceil$ vertices each, and with $S$ occurring before $T$ along $P$. Our approach relies on the following fact.

**Observation 15.** Let $P$ be partitioned into two subcurves $S$ and $T$ with all vertices in $S$ occurring on $P$ before the vertices of $T$. We have that $D_B(y) = D_B^{P \times P}(y) = \max \{ D_B^{S \times S}(y), D_B^{S \times T}(y), D_B^{T \times T}(y) \}$.

Note that we can omit $D_B^{T \times S}$ because $(T \times S) \cap P^\leq = \emptyset$.

**Lemma 16.** Let $P$ be a polygonal curve with $n$ vertices. Function $D_B$ has complexity $O(n \log n)$ and can be computed in $O(n \log^2 n)$ time. Evaluating $D_B(y)$, for a given $y \in \mathbb{R}$, takes $O(\log n)$ time.

**Proof.** By Observation 15, $D_B(y) = D_B^{P \times P}(y) = \max \{ D_B^{S \times S}(y), D_B^{S \times T}(y), D_B^{T \times T}(y) \}$, and by Lemma 14, the complexity of $D_B^{S \times T}$ is $O(n)$. Hence, there are $O(n)$ backward pairs from $S \times T$ that could contribute to $D_B^{P \times P}$. Let $C(n)$ denote the number of backward pairs contributing to $D_B^{P \times P}$. It follows that $C(n) = 2C(\lfloor n/2 \rfloor) + O(n)$, which solves to $O(n \log n)$. Since the complexity of $D_B = D_B^{P \times P}$ is linear in the number of contributing backward pairs [17], $D_B$ has complexity $O(n \log n)$.

To compute $D_B$ we apply the same divide and conquer strategy. We recursively partition $P$ into roughly equal size subcurves $S$ and $T$. At each step, we compute the (graph of the) function $D_B^{S \times T}$, and merge it with the recursively computed functions $D_B^{S \times S}$ and $D_B^{T \times T}$. By Lemma 14, computing $D_B^{S \times T}$ takes $O(n \log n)$ time. Computing the upper envelope of $D_B^{S \times T}$, $D_B^{S \times S}$, and $D_B^{T \times T}$, takes time linear in the complexity of the functions involved. The function $D_B^{S \times T}$ has complexity $O(n)$. However, $D_B^{S \times S}$, $D_B^{T \times T}$, and the output $D_B^{P \times P}$, have complexity $O(n \log n)$. Hence, we spend $O(n \log n)$ time to compute $D_B^{P \times P}$. The total running time obeys the recurrence $R(n) = 2R(n/2) + O(n \log n)$, that resolves to $O(n \log^2 n)$ time.

We can easily store (the breakpoints of) $D_B^{P \times P}$ in a balanced binary search tree so that we can evaluate $D_B(y)$ for some query value $y$ in $O(\log(\log n)) = O(\log n)$ time. \hfill \qed
Eq. 1 together with Theorem 8 and Lemma 16 thus imply that we can store $P$ in an $O(n \log n)$ size data structure so that we can compute $D_F(P, \overline{ab})$ for some horizontal query segment $\overline{ab}$ in $O(\log n)$ time. That is, we have established Theorem 1.

3.3 Querying for subcurves

In this section we extend our data structure to support Fréchet distance queries to subcurves of $P$, establishing Theorem 2. A query now consists of two points $s$ and $t$ on $P$ and the horizontal query segment $\overline{ab}$, and we wish to efficiently report the Fréchet distance $D_F(P[s, t], \overline{ab})$ between the subcurve $P[s, t]$, from $s$ to $t$, and $\overline{ab}$. We show that we can support such queries in $O(\log^3 n)$ time using $O(n \log^2 n)$ space.

We assume that given $s$ and $t$ we can determine the edges of $P$ containing $s$ and $t$ respectively, in constant time. Note that this is the case, for instance, when $s$ is given as a pointer to its containing edge together with a location. If $s$ and $t$ are given only as points in the plane, and $P$ is not self-intersecting, we can find these edges in $O(\log n)$ time using a linear-size data structure for vertical ray-shooting [33] on $P$. If $P$ does contain self-intersections this requires more space and preprocessing time [1].

By Eq. 1, $D_F(P[s, t], \overline{ab})$ can again be decomposed into four terms, the first two of which can be trivially computed in constant time. We build two separate data structures for the remaining two terms: the Hausdorff distance and the backward pair distance terms.

3.3.1 Hausdorff distance for subcurves

We build a data structure on $P$ such that given points $s, t$ on $P$ and $\overline{ab}$ we can report $D_H(P[s, t], \overline{ab})$ efficiently. In particular, we use a two-level data structure of size $O(n \log n)$ that supports queries in $O(\log^2 n)$ time, after $O(n \log n)$ preprocessing time, based on two observations:

1. The Hausdorff distance is decomposable in its first argument. That is:
   \[ D_H(P[s, t], \overline{ab}) = \max \left\{ D_H(P[s, m], \overline{ab}), D_H(P[m, t], \overline{ab}) \right\}, \]

2. The Hausdorff distance $D_H(P[s, t], \overline{ab})$ is realized by $s$, $t$, or a vertex $p$ of $P[s, t]$, that is:
   \[ D_H(P[s, t], \overline{ab}) = \max \{ D_H(p, \overline{ab}), D_H(s, \overline{ab}), D_H(t, \overline{ab}) \}. \]

The data structure is a balanced binary search tree (essentially, a 1D-range tree) in which the leaves store the vertices of $P$, in the order along $P$. Each internal node $\nu$ represents a canonical subcurve $P_\nu$ and stores the vertices of $P_\nu$ in the data structure of Theorem 8. Since these associated data structures use linear space, the total space used is $O(n \log n)$. Building the associated data structures from scratch would take $O(n \log^2 n)$ time. However, the following lemma immediately implies that this time can be reduced to $O(n \log n)$:

**Lemma 17.** Two instances of the data structure of Theorem 8, consisting of a furthest segment Voronoi diagram preprocessed for point location, can be merged in linear time.

**Proof.** Since two furthest segment Voronoi diagrams can be merged in linear time [32], it remains to show that the associated point location data structure can also be constructed in linear time. This is a well-known result if the regions are $y$-monotone [20].
We prove that the furthest segment Voronoi diagram of $\vec{S}$ has $y$-monotone cells (the case of $\vec{S}$ is symmetric).

Suppose for the sake of contradiction that there is some horizontal line $\ell_y$ at height $y$, and a ray $\vec{p}_i \in \vec{S}$, such that the furthest Voronoi region of $\vec{p}_i$, intersected by $\ell_y$, has at least two maximal disjoint intervals $A$ and $B$ where $A$ is left of $B$ (these intervals contain the boundary of the associated Voronoi region). Consider the rightmost point $x$ in $A$. Since $x$ coincides with the right boundary of $A$, there is a ray $\vec{p}_j \in \vec{S}$ such that $d((x,y), \vec{p}_j) = d((x,y), \vec{p}_i)$, and for an arbitrary small $\varepsilon > 0$, $d((x+\varepsilon,y), \vec{p}_i) < d((x+\varepsilon,y), \vec{p}_j)$. We make a distinction based on whether the distance from $(x,y)$ to $\vec{p}_i$ is realized by the distance to the point $p_i$ or by the vertical distance to the line supporting $\vec{p}_i$ (similarly for the distance from $(x,y)$ to $\vec{p}_j$). Refer to Fig. 6. The first two cases show that the interval $B$ is empty and the last two show that the interval $A$ does not end at $x$.

**Case 1:** $d((x,y), \vec{p}_i) = d((x,y), p_i)$ and $d((x,y), \vec{p}_j) = d((x,y), p_j)$. In this case, $x$ lies on the bisector between $p_i$ and $p_j$ and the interval $A$ lies left of this bisector. Hence $p_i$ must lie right of $p_j$. It follows that for all $x' > x$, $d((x',y), \vec{p}_i) < d((x',y), \vec{p}_j)$. Indeed if for such $x'$, $d((x',y), \vec{p}_j) = d((x',y), p_j)$ then it must be that $d((x',y), \vec{p}_i) = d((x',y), p_i)$. Since $x'$ lies right of the bisector between $p_i$ and $p_j$ it follows that $d((x',y), \vec{p}_i) = d((x',y), p_i) < d((x',y), \vec{p}_j) = d((x',y), p_j)$. For all $x' > x$ where $d((x',y), \vec{p}_j) \neq d((x',y), p_j)$, the value $d((x',y), \vec{p}_j)$ remains constant whilst the value $d((x',y), \vec{p}_i)$ decreases or stays constant. This contradicts the assumption that the farthest Voronoi cell of $\vec{p}_i$ intersects $\ell_y$ right of $x$.

**Case 2:** $d((x,y), \vec{p}_i) = d((x,y), p_i)$ and $d((x,y), \vec{p}_j) \neq d((x,y), p_j)$. In this case, for all $x' > x$, $d((x',y), \vec{p}_j) = d((x,y), \vec{p}_j)$ and $d((x',y), \vec{p}_i) < d((x,y), \vec{p}_i)$. This contradicts the assumption that the farthest Voronoi cell of $\vec{p}_i$ intersects $\ell_y$ right of $x$.

**Case 3:** $d((x,y), \vec{p}_i) \neq d((x,y), p_i)$ and $d((x,y), \vec{p}_j) = d((x,y), p_j)$ In this case, for all $x' > x$, $d((x',y), \vec{p}_j) < d((x,y), \vec{p}_j)$ and $d((x',y), \vec{p}_i) = d((x,y), \vec{p}_i)$ which contradicts the assumption that for an arbitrary small $\varepsilon > 0$, $d((x+\varepsilon,y), \vec{p}_i) < d((x+\varepsilon,y), \vec{p}_j)$.

**Case 4:** $d((x,y), \vec{p}_i) \neq d((x,y), p_i)$ and $d((x,y), \vec{p}_j) \neq d((x,y), p_j)$. In this case, for all $x' > x$, $d((x',y), \vec{p}_j) = d((x,y), \vec{p}_j)$ and $d((x',y), \vec{p}_i) = d((x,y), \vec{p}_i)$ which contradicts the assumption that for an arbitrary small $\varepsilon > 0$, $d((x+\varepsilon,y), \vec{p}_i) < d((x+\varepsilon,y), \vec{p}_j)$.

Let $s,t$ be two points on $P$, and let $s'$ and $t'$ be the first vertex succeeding $s$ and preceding $t$, respectively. The terms $\vec{D}_H(P[s,s'], \vec{a}b)$ and $\vec{D}_H(P[t',t], \vec{a}b)$ can be computed in $O(1)$ time. There are $O(\log n)$ internal nodes in our range tree, whose canonical subcurves together form $P[s',t']$ (see e.g. [15, Chapter 5]). For each such node $\nu$, corresponding to a subcurve $P_\nu$, we query its associated data structures in time logarithmic in the number of vertices in $P_\nu$ to compute $\vec{D}_H(P_\nu, \vec{a}b)$. We report the maximum distance found and conclude:
Lemma 18. Let $P$ be a polygonal curve in $\mathbb{R}^2$ with $n$ vertices. In $O(n \log n)$ time we can construct a data structure of size $O(n \log n)$ so that given a horizontal query segment $\overline{ab}$, and two points $s, t$ on $P$, $\overline{D_H}(P[s, t], \overline{ab})$ can be computed in $O(\log^2 n)$ time.

Note that since we are given pointers to $s$ and $t$ we can make the query time sensitive to the complexity $|P[s, t]|$ of the query subcurve. Let $\nu_1, \ldots, \nu_k$ be the nodes whose canonical subcurves (ordered along $P$) make up $P[s', t']$. It is well known that a prefix of $\nu_1, \ldots, \nu_k$ are (a subset of the) right children on the path connecting the leaf representing $s'$ to the lowest common ancestor (LCA) $\mu$ of this leaf and the leaf representing $t'$, whereas the remaining nodes $\nu_{k+1}, \ldots, \nu_k$ are left children along the path from $\mu$ to the leaf representing $t'$. By preprocessing our tree for $O(1)$ time LCA queries [9] and storing a pointer from every internal node to its last leaf, we can find these $k$ nodes in $O(k)$ time: given $\nu_i$, jump to the last leaf in its subtree. This leaf represents some vertex $p_j$. We can then compute the LCA of $p_{j+1}$ and $\nu_i$. The next node $\nu_{i+1}$ is the right child of the node found. Moreover, since the sizes of successive nodes in $\nu_1, \ldots, \nu_k$ (at least) double there are at most $\ell = O(\log |P[s, t]|)$ such nodes. Symmetrically $k - \ell = O(\log |P[s, t]|)$, hence $k = O(\log |P[s, t]|)$. Since each of the canonical subcurves has size at most $|P[s, t]|$, querying their associated data structures also takes at most $O(\log |P[s, t]|)$ time. Hence, our queries take only $O(\log^2 |P[s, t]|)$ time in total.

3.3.2 Backward pair distance for subcurves

In this section we describe how to store $P$ so that given points $s$ and $t$ on $P$ and the horizontal query segment $\overline{ab}$ we can compute the backward pair distance $D_B^{P[s, t] \times P[s, t]}(y_a)$ efficiently. The main idea to support subcurve queries is to store all intermediate results of the divide and conquer algorithm from Section 3.2.3. Hence, our main data structure is a 1D-range tree whose leaves store the vertices of $P$, ordered along $P$. Each internal node $\nu$ corresponds to some subcurve $P_{\nu}$ of $P$, and will store the function $D_B^{P_{\nu} \times P_{\nu}}(y)$ (Lemma 16) as well as the functions $h_{P_{\nu}}(x, y)$ and $h_{P_{\nu}}^x(y, x)$ represented by two new data structures that we will denote by $\Delta_{P_{\nu}}(y)$ and $\Delta_{P_{\nu}}^x(y)$.

We first sketch our query approach, in the following subsections we show how to construct the specific data structure and queries. We are given $y'$, and $s, t \in P$. We then identify the vertex $s'$ succeeding $s$ and the vertex $t'$ preceding $t$, and define $P_0 = P[s, s']$ and $P_{k+1} = P[t', t]$. There are $k = O(\log n)$ internal nodes whose canonical subcurves $P_1, P_2, \ldots$ together form $P[s', t']$. We prove that $D_B^{P_i \times P_j}(y')$ is the maximum over two terms:

$$D_B^{P_i \times P_j}(y'), \text{ for all } i, \text{ and } D_B^{P_i \times P_j}(y') \text{ for } i < j.$$

The $O(\log n)$ values of the first term can be computed in $O(\log^2 n)$ total time, using the data structures of Lemma 16 stored in each node. The second term contains $O(\log^2 n)$ values, and we show how to compute each value in $O(\log n)$ time, using the new data structures, for a total query time of $O(\log^3 n)$. This query time then dominates the time it takes to compute the maximum of all these terms.

3.3.3 Using furthest segment Voronoi diagrams to compute $D_B^{S \times T}(y')$

Let $S, T$ be two contiguous subcurves of $P$, with $S$ occurring strictly before $T$ on $P$. We first study how to compute $D_B^{S \times T}(y')$ efficiently for any given $y'$. For ease of exposition, we first show how to construct a linear-size data structure on $T$ such that given a query point $p$ that precedes $T$ along $P$ and a value $y'$, we can compute $D_B^{P \times T}(y')$ in $O(\log |T|)$ time. By Lemma 12 this amounts to computing the point of intersection between the functions $x \mapsto h_T^p((x, y'))$ and $x \mapsto h_T^T((x, y'))$. 

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Our global approach is illustrated in Fig. 4. Suppose for the ease of exposition, that \( y' \) is fixed. Given the query \( p \), we can compute the convex function \( x \mapsto h_{\overrightarrow{T}}((x, y')) \) in constant time. For the set \( \overrightarrow{T} \), we note that Observation 4 shows that the (graph of the) function \( (x, y) \mapsto h_{\overrightarrow{T}}((x, y)) \) is an upper envelope whose maximization diagram is the furthest segment Voronoi diagram of the set of halflines \( \overrightarrow{T} \). For any fixed \( y' \) and \( \overrightarrow{T} \), we can construct a data structure \( \Delta_{\overrightarrow{T}}(y') \) that stores the Voronoi cell edges that are intersected by a horizontal line of height \( y' \) in their left-to-right order. Specifically, we store a red-black tree where each node stores a consecutive pair of edges (in this way, a node in the tree corresponds to a unique Voronoi cell in the diagram). Since a furthest segment Voronoi diagram has a linear number of edges, \( \Delta_{\overrightarrow{T}}(y') \) has \( O(|\overrightarrow{T}|) \) size.

**Lemma 19.** Let \( p \in P \) be a query point, let \( y' \) be some given value, and let \( T \) be a sequence of points of \( P \) that succeed \( p \). Given \( \Delta_{\overrightarrow{T}}(y') \) we can compute the point \((x^*, y')\) where \( h_{\overrightarrow{T}}((x^*, y')) = h_{\overrightarrow{T}}((x, y')) \) in \( O(\log |\overrightarrow{T}|) \) time.

**Proof.** Consider the two edges stored at the root of \( \Delta_{\overrightarrow{T}}(y') \). These two edges partially bound a Voronoi cell corresponding to a halfline \( \overrightarrow{T} \in \overrightarrow{T} \). In constant time, we can compute the point of intersection \((x^*, y')\) between \( x \mapsto h_{\overrightarrow{T}}((x, y')) \) and \( x \mapsto h_{\overrightarrow{T}}((x, y')) \).

Using the two edges, we can compute the \( x \)-interval of the Voronoi cell of \( \overrightarrow{T} \) that intersects the horizontal line of height \( y' \) in constant time. If the value \( x' \) lies in this interval, then we have found the unique point of intersection between \( h_{\overrightarrow{T}} \) and \( h_{\overrightarrow{T}} \) (in other words, \( (x^*, y') = (x^*, y') \)). If \( x' \) lies left of this \( x \)-interval, we can disregard all nodes in the right subtree of the root. This is because for all \( x \geq x' \), \( h_{\overrightarrow{T}}((x, y')) \geq h_{\overrightarrow{T}}((x, y')) > h_{\overrightarrow{T}}((x^*, y')). \) A symmetrical property holds when \( x' \) lies right of this \( x \)-interval. Hence at every step, we can discard at least half of the remaining candidate halflines in \( \overrightarrow{T} \) and thus we can compute the halfline in \( \overrightarrow{T} \) that forms the intersection with \( x \mapsto h_{\overrightarrow{T}}((x, y')) \) in logarithmic time. \( \square \)

Now, we extend our approach from products of points and sets, to products between sets. Recall that for any \( \overrightarrow{S} \), the function \( x \mapsto h_{\overrightarrow{S}}((x, y')) \) is piecewise monotone, where each piece is a constant-complexity segment coinciding with \( x \mapsto h_{\overrightarrow{S}}((x, y')) \) for some \( \overrightarrow{s} \in \overrightarrow{S} \).

**Lemma 20.** Let \( S, T \subset P \) such that all points in \( S \) precede all points in \( T \) and let \( y' \) be some given value. Let \( \sigma_1, \sigma_2 \) be two points on \( x \mapsto h_{\overrightarrow{S}}((x, y')) \) that bound a curve segment with \( \sigma_1 \) left of \( \sigma_2 \). Moreover, let \( \tau_1, \tau_2 \) be defined analogously. If the segments bounded by \( (\sigma_1, \sigma_2) \) and \( (\tau_1, \tau_2) \) do not intersect then the intersection between \( h_{\overrightarrow{S}}((x, y')) \) and \( h_{\overrightarrow{T}}((x, y')) \) lies either: left of \( \sigma_1 \), right of \( \sigma_2 \), left of \( \tau_1 \), or right of \( \tau_2 \), and we can identify the case in \( O(1) \) time.

**Proof.** Recall that \( x \mapsto h_{\overrightarrow{S}}((x, y')) \) and \( x \mapsto h_{\overrightarrow{T}}((x, y')) \) are monotonically decreasing and increasing functions, respectively. The proof is a case distinction, illustrated by Fig. 7.
Case 1: there is no vertical or horizontal line that intersects both segments. In this case, there is a separation of the \((x,z)\)-plane into four quadrants, such that one segment lies in a top quadrant and the other segment in the opposite bottom quadrant. Let the segment of \(h_2\) lie in the top left quadrant, then all points on \(h_2\) left of this segment, are higher than \(\sigma_1\). All points left of \(h_2\) must be lower than \(\tau_1\). Hence all segments of \(h_2\) left \(\sigma_1\) cannot form the point of intersection and can be discarded. Given \((\sigma_1,\sigma_2,\tau_1,\tau_2)\) we can identify this case in constant time. The three remaining (sub-)cases are symmetrical.

Case 2: At least one horizontal line intersects both segments. If the segments between \((\sigma_1,\sigma_2)\) and \((\tau_1,\tau_2)\) do not intersect, all horizontal lines intersect these two segments in the same order. Via the same argument as above we note that if the segment of \(h_2\) is intersected first, then all segments on \(h_2\) left of \(\sigma_1\) can be discarded. Similarly then all segments of \(h_2\) right of \(\tau_2\) can be discarded. The argument for when the order is reversed is symmetrical and we can compute the intersections with a horizontal line in \(O(1)\) time.

Case 3: At least one vertical line intersects both segments. In this case, each vertical line that intersects both segments must intersect them in the same order. We note that if the segment between \((\sigma_1,\sigma_2)\) is intersected first, then all segments on \(h_2\) right of \(\sigma_2\) can be discarded, along with all segments right of \(\tau_2\). The argument for when the order is reversed is symmetrical and we can compute the intersections with a vertical line in \(O(1)\) time.

Lemma 21. Let \(S, T \subset P\), all points in \(S\) precede all points in \(T\) and let \(y'\) be some given value. Given \(\Delta_S(y)\) and \(\Delta_T(y')\), we can compute \(\mathcal{D}_B^{S \times T}(y)\) in \(O(|S| + \log |T|)\) time.

Proof. Consider the Voronoi edges stored at the root of \(\Delta_S(y')\), these are intersected by a horizontal line of height \(y'\) in the \(x\)-coordinates \(x_1\) and \(x_2\) respectively. The edges of the Voronoi cell point to the cell \(\vec{s} \in \tilde{S}\) where we know that for all \(x'\) with \(x_1 < x' < x_2\), \(h_2(x',y') = h_2(x_1,y')\). Given the pointer to \(\vec{s} \in \tilde{S}\) we compute the points \(\sigma_1 = h_2(x_1,y') = h_2(x_1,y')\) and \(\sigma_2 = h_2(x_1,y') = h_2(x_1,y')\) in constant time and the segment of \(h_2(x,y')\) connecting them. Similarly, we can compute a point \(\tau_1\) and point \(\tau_2\) for the edges stored at the root of \(\Delta_T(y')\), and the segment of \(h_2(x,y')\) connecting \((\tau_1,\tau_2)\).

Then we can apply Lemma 20 to the pair \((\sigma_1,\sigma_2), (\tau_1,\tau_2)\), to discard at least all segments left/right of \((\sigma_1,\sigma_2)\) or all segments left/right of \((\tau_1,\tau_2)\). Repeating this recursively on the remaining subtrees of \(\Delta_S(y')\) and \(\Delta_T(y')\), we can find the two segments defining the point of intersection in \(O(|S| + \log |T|)\) time.

All that remains is to show that we can apply Lemma 21 not only to a given \(y'\), but to any \(y\).

To this end we sweep the plane with a horizontal line at height \(y\) while maintaining \(\Delta_T(y)\) as a partially persistent red black tree [33]. We do the same for \(\Delta_S(y)\).

Lemma 22. Let \(S\) and \(T\) be two sets of vertices of \(P\). We can store \(S\) in a data structure of size \(O(|S|)\), and \(T\) in a data structure of size \(O(|T|)\) such that, if all vertices in \(S\) precede all vertices in \(T\), we can compute \(\mathcal{D}_B^{S \times T}(y')\) for any fixed \(y' \in \mathbb{R}\) in \(O(|S| + \log |T|)\) time. Building these data structures takes \(O(|S| \log |S|)\) and \(O(|T| \log |T|)\) time, respectively.

Proof. Consider a horizontal sweepline at height \(y'\), and consider the continuously changing data structure \(\Delta_T(y')\). For any set \(\vec{T}\), the FSVD has linear complexity [32]. Hence, for any \(y'\), \(\Delta_T(y')\) contains at most a \(O(|T|)\) edges. Moreover, there are only \(O(|T|)\) \(y\)-coordinates at which the combinatorial structure of \(\Delta_{LLT}(y)\) changes. At each such an event we make a constant number of updates to \(\Delta_T\). Since \(\Delta_T\) is a (partially persistent) red black tree, these changes take \(O(\log |T|)\)
time, and \( O(1) \) space \([33]\). We use the same preprocessing for \( \tilde{S} \). Finally, we observe that we can obtain \( \Delta_{\tilde{T}}(y') \) and \( \Delta_{\tilde{S}}(y') \) for a given \( y' \in \mathbb{R} \) in \( O(\log |T| + \log |S|) \) time.

### 3.3.4 Divide and conquer for subcurves

We denote by \( \Delta_{\tilde{T}}(y) \) the partially persistent red-black tree on the furthest-segment Voronoi diagram of \( \tilde{T} \). We now show how for any two vertices \( s, t \in P \) and height \( y' \), we can compute \( D_B^{P[s,t] \times P[s,t]}(y') \) efficiently through repeated application of Observation 23 and Lemma 22.

**Observation 23.** Let \( S \) and \( T \) be two sets of vertices of \( P \) such that all points in \( S \) precede all points in \( T \) and \( S = S' \cup S'' \) and \( T = T' \cup T'' \). Then for all \( y \in \mathbb{R} \):

\[
D_B^{S \times T}(y) = \max \left\{ D_B^{S' \times T}(y), D_B^{S'' \times T}(y) \right\} \quad \text{and} \quad D_B^{S \times T}(y) = \max \left\{ D_B^{S \times T'}(y), D_B^{S \times T''}(y) \right\}.
\]

**Proof.** Recall that \( D_B^{S \times T}(y) = \max \left\{ \delta_{pq}(y) \mid (p, q) \in (S \times T) \cap P \leq \right\} \). Using that \( S = S' \cup S'' \) we then get \( S \times T = (S' \times T) \cup (S'' \times T) \), and thus also \( (S \times T) \cap P \leq = ((S' \times T) \cap P \leq) \cup ((S'' \times T) \cap P \leq) \). Since computing a maximum is decomposable we therefore get \( D_B^{S \times T}(y) = \max \left\{ D_B^{S' \times T}(y), D_B^{S'' \times T}(y) \right\} \).

Analogously, \( D_B^{S \times T}(y) = \max \left\{ D_B^{S \times T'}(y), D_B^{S \times T''}(y) \right\} \).

**Lemma 24.** Let \( P \) be a polygonal curve in \( \mathbb{R}^2 \) with \( n \) vertices. We can build an \( O(n \log^2 n) \) size data structure in \( O(n \log^2 n) \) time, such that given any query \( (s, t, y) \), for \( s, t \in P \) (not necessarily vertices) we can compute \( D_B^{P[s,t] \times P[s,t]}(y) \) in \( O(\log^3 n) \) time.

**Proof.** Our main data structure is a range tree whose leaves store the vertices of \( P \), ordered along \( P \). Each internal node \( \nu \) corresponds to some subcurve \( P_{\nu} = P[p, q] \) for two vertices \( p, q \in P \). We assume that \( P_{\nu} \) consists of \( m \) vertices. Each node will store the representation of function \( D_B^{P_{\nu}, P_{\nu}}(y) \) as an upper envelope. By Lemma 16, this envelope has complexity \( O(m \log m) \). The algorithm to compute this envelope is a divide and conquer algorithm whose recursion tree matches our range tree. Hence, when we compute \( D_B^{P_{\nu}, P_{\nu}}(y) \) for the root of our tree in \( O(n \log^2 n) \) time we actually also construct all the envelopes associated with the other nodes. In addition, each node will store \( \Delta_{\tilde{T}_{\nu}}(y) \) and \( \Delta_{\tilde{T}_{\nu}}(y) \) which require \( O(m) \) space and can be constructed by a divide and conquer approach in \( O(m \log m) \) time. Hence, the resulting data structure requires \( O(n \log^2 n) \) space and can be constructed in \( O(n \log^2 n) \) total time.

Let \( s, t \) be two points on \( P \), and \( s' \) and \( t' \) be the first vertex succeeding \( s \) and preceding \( t \), resp. There are \( O(\log n) \) subtrees in our range search tree, whose subcurves \( P[p, q] \) together form \( P[s', t'] \). We can identify the nodes bounding these subtrees in \( O(\log n) \) time. We denote these nodes by \( P_1, .., P_k \), where the index matches the order along \( P \). Furthermore, we define \( P_0 = P[s, s'] \) and \( P_{k+1} = P[t', t] \) and observe:

\[
D_B^{P[s,t] \times P[s,t]}(y) = \max_{0 \leq i, j \leq k+1} D_B^{P_{i} \times P_{j}}(y)
\]

(2)

Indeed, by Observation 23, \( D_B^{P[s,t] \times P[s,t]}(y) \) is decomposable. By repeated application of this observation we have that \( D_B^{P[s,t] \times P[s,t]}(y) = \max_{i, j} D_B^{P_{i} \times P_{j}}(y) \). Moreover, we only need to consider pairs with \( i \leq j \), since if \( j < i \), we have \( P_{i} \times P_{j} \subseteq P \leq \). Thus, we can compute \( D_B^{P[s,t] \times P[s,t]}(y') \) by only computing the values \( D_B^{P_{i} \times P_{j}}(y') \) with \( i \leq j \).

By construction, for each of the \( O(\log^2 n) \) pairs \( S, T \) in our decomposition (Eq. 2) it holds that all points in \( S \) precede all points in \( T \). Thus we can compute \( D_B^{S \times T}(y') \) in \( O(\log(|S| + |T|)) = O(\log n) \)
time by Lemma 22. Computing this value for each pair takes $O(\log^3 n)$ total time. For each subcurve $P_i$, we compute $D_B^{P_i \times P_i}(y)$ in $O(\log n)$ time using the data structure for $D_B^{P_i \times P_i}(y)$ of Lemma 16 stored in the node $\nu$ corresponding to $P_i$ (or in $O(1)$ time if $i \in \{0, k + 1\}$). The lemma follows by taking the maximum of these $O(\log^2 n)$ values.

As with the Hausdorff term we can make the query time sensitive to the complexity of $P[s, t]$, and obtain a query time of $O(\log^3 |P[s, t]|)$. Theorem 2 now follows.

4 Arbitrary orientation queries

In this section we extend our results to arbitrarily oriented query segments, proving Theorem 3. Let $\alpha$ be the slope of the line containing the query segment $\overline{ab}$, and let $\beta$ be its intercept (note that vertical query segments can be handled by a rotated version of our data structure for horizontal queries). We again consider the case where $a$ is left of $b$; the other case is symmetric. Following Eq. 1, we can write $D(x, \overline{ab})$ as the maximum of four terms: $\|p_1 - a\|$, $\|p_n - b\|$, $\overrightarrow{D_H}(P, \overline{ab})$, and the backward pair distance $D_B(\alpha, \beta)$ with respect to $\alpha$. The backward pair distance is now defined as

$$D_B(\alpha, \beta) = \max_{x} \{\delta_{pq}(\alpha, \beta) \mid (p, q) \in B(P)\},$$

where

$$\delta_{pq}(\alpha, \beta) = \min_x \max \{\|(x, \alpha x + \beta) - p\|, \|(x, \alpha x + \beta) - q\|\}.$$

In Section 4.1 we present an $O(n \log n)$ size data structure that supports querying $\overrightarrow{D_H}(P, \overline{ab})$ in $O(\log^2 n)$ time. The key insight is that we can use furthest point Voronoi diagrams instead of furthest segment Voronoi diagrams. In Section 4.2 we present a data structure that efficiently supports querying $D_B(\alpha, \beta)$. In Section 5 we extend our results to support queries against subcurves of $P$ as well. This combines our insights from the horizontal queries with our results from Sections 4.1 and 4.2. Finally, in Section 5.1 we then show how this also leads to a space-time trade off.

4.1 The Hausdorff distance term

For any point $p$ and slope $\alpha$ we denote by $\overset{\leftarrow}{p}^{\alpha}$ the ray with slope $\alpha$ that points in the leftward direction. Similarly, for any point set $T$, we define $\overset{\leftarrow}{T}$ as $\{\overset{\leftarrow}{p}^{\alpha} \mid p \in T\}$. Furthermore, we define $h_{\overset{\leftarrow}{T}^{\alpha}}(x, y)$ to be the directed Hausdorff distance from $(x, y)$ to the ray $\overset{\leftarrow}{p}^{\alpha}$, and $h_{\overset{\leftarrow}{T}^{\alpha}}(x, y) = \max\{h_{\overset{\leftarrow}{p}^{\alpha}}(x, y) \mid p \in T\}$. Let $CH(T)$ be the convex hull of $T$. For a given slope $\alpha$ and a point $r$, we denote by $\ell_r$ the line that is perpendicular to a line with slope $\alpha$ that goes through $r$. Observe that $\ell_r$ is the halfplane to the left of $\ell_r$.

Lemma 25. Let $\overset{\leftarrow}{T}$ be a collection of halflines containing $\overset{\leftarrow}{p}^{\alpha}$ as a topmost and $\overset{\leftarrow}{q}^{\alpha}$ as a bottommost halfline, respectively (always with respect to $\alpha$). We have that

$$h_{\overset{\leftarrow}{T}^{\alpha}}(x, y) = \max\{h_{\overset{\leftarrow}{p}^{\alpha}}(x, y), h_{\overset{\leftarrow}{q}^{\alpha}}(x, y), \max\{||s - (x, y)|| \mid s \in CH(T) \cap \overset{\leftarrow}{\ell_{(x, y)}}\}\}$$

Proof. Let $t \in T$ be the point realizing $h_{\overset{\leftarrow}{T}^{\alpha}}(x, y) = \max\{h_{\overset{\leftarrow}{s}^{\alpha}}(x, y) \mid s \in T\}$, for some query point $(x, y)$. Without loss of generality, we can assume $\alpha = 0$, and hence the line $\ell_{(x, y)}$ is vertical. We distinguish two cases depending on the position of $t$.

Case $t$ right of $\ell_{(x, y)}$. In this case, $h_{\overset{\leftarrow}{t}^{\alpha}}(x, y)$ must be given by the vertical distance $|y_t - y|$, therefore we must have that $|y_t - y| = \max\{|y_p - y|, |y_q - y|\}$. Thus $\overset{\leftarrow}{t}^{\alpha}$ must be a topmost or bottommost halfline.
Case $t$ left of $\ell_{(x,y)}$. Observe that for all points $u \in T$ left of $\ell_{(x,y)}$ we have that $h_{-\alpha}(x,y) = \|u - (x,y)\|$. Now assume by contradiction that $t \notin \{p,q\} \cup (CH(T) \cap \ell_{(x,y)})$, and let $\overline{cd}$ be the (leftmost) edge of $CH(T) \cap \ell_{(x,y)}$ hit by $\overline{t}$. Since $t$ realizes $h_{-\alpha}(x,y)$ we claim that $c$ lies inside the disk $D$ centered at $(x,y)$ that has $t$ on its boundary. If $c \in T$ and outside $D$ this would immediately contradict that $t$ realizes $h_{-\alpha}(x,y)$. If $c$ is the intersection point of an edge of $CH(T)$ with $\ell_{(x,y)}$ and lies outside $D$ the endpoint $c'$ of this edge that lies right of $\ell_{(x,y)}$ would have $h_{-\alpha}(c',x,y) > \|c - (x,y)\| > \|t - (x,y)\| = h_{T}^{-}(x,y)$, contradicting that $t$ realizes $h_{-\alpha}(x,y)$. Hence, $c$ lies inside $D$.

Via the same argument as above $d$ lies inside $D$. So, by convexity, the line segment $\overline{cd}$ is completely contained in this disk $D$. However, since (by definition of $h_{-\alpha}(x,y)$) $t$ is the point on $\overline{t}$ closest to $(x,y)$, the remainder of this halfline is outside $D$. Hence, $\overline{t}$ does not intersect $\overline{cd}$. Contradiction.

Our data structure will store $CH(T)$ in a 1D-range tree whose internal nodes store FPVDs. This allows us to evaluate $h_{-\alpha}(x,y)$ and $h_{-\alpha}(x,y)$ for a query point $(x,y)$ and slope $\alpha$.

**Lemma 26.** Let $T$ be a set of $n$ points in $\mathbb{R}^2$. In $O(n \log n)$ time we can construct a data structure of size $O(n \log n)$ so that given a query point $(x,y)$ and query slope $\alpha$ we can compute $h_{-\alpha}(x,y)$ in $O((\log n)^2)$ time.

**Proof.** Consider a clockwise traversal of the convex hull of $T$ that visits every vertex twice. Let $t_1, \ldots, t_{2k}$ denote the vertices in this order (so $t_{i+k} = t_i$). We store these vertices $t_1, \ldots, t_{2k}$ in the leaves of a range tree. Each internal node $\nu$ corresponds to some contiguous subsequence $T_\nu = t_i, \ldots, t_j$ of these vertices, and stores the furthest point Voronoi diagram (FPVD) of $T_\nu$. Since the FPVD has linear size and the points are in convex position, it can be computed in linear time [3]. Thus our data structure has size $O(n \log n)$ and can be computed in $O(n \log n)$ time.

To answer a query, we first find the bottom- and topmost points of $CH(T)$ (and thus of $T$) with respect to slope $\alpha$. Let $q$ and $p$ be these points, respectively. We now find a contiguous subsequence $t_i, \ldots, t_j$ of the vertices of $CH(T)$ in the halfplane $\ell_{\alpha}(x,y)$. Note that since $t_1, \ldots, t_{2k}$ traverses $CH(T)$ twice such a contiguous sequence exists. More specifically, we only compute the first vertex $t_i$ and the last vertex $t_j$. We then query the data structure to obtain $O(\log n)$ internal nodes $\nu$ whose associated sets $T_\nu$ together represent $t_i, \ldots, t_j$, and query their furthest point Voronoi diagrams to find the point $s$ in $t_i, \ldots, t_j$ furthest from $(x,y)$. We report the maximum of $h_{-\alpha}(p,x,y), h_{-\alpha}(q,x,y)$, and $\|s - (x,y)\|$. By Lemma 25 this is $h_{-\alpha}(x,y)$.

Finding $p, q, t_i$, and $t_j$ takes $O(\log n)$ time. This is dominated by the $O((\log n)^2)$ time to query all FPVDs. Thus, we can compute $h_{-\alpha}(x,y)$ in $O((\log n)^2)$ time. Symmetrically, we can query $h_{-\alpha}(x,y)$ in $O((\log n)^2)$ time.

By using similar ideas to those of range minimum queries [7, 9] we can achieve $O(\log n)$ query time using $O(n^2)$ space. Then following Corollary 7 we can also compute $\overline{D_H}(P, \overline{ab})$ by querying either of these data structures with points $a$ and $b$. Hence, we obtain:

**Lemma 27.** Let $T$ be a set of $n$ points in $\mathbb{R}^2$. In $O(n^2)$ time we can construct a data structure of size $O(n^2)$ so that given a query point $(x,y)$ and query slope $\alpha$ we can compute $h_{-\alpha}(x,y)$ in $O(\log n)$ time.
Proof. For every vertex \( t_i \) on \( CH(T) \), and every \( \ell \in 1, \ldots, \log n \) we store the FPVD of \( t_i, \ldots, t_{i+2^\ell} \). This takes a total of \( \sum_{\ell=1}^{\log n} O(2^\ell) = O(2^{\log n}) = O(n) \) space per vertex, and thus \( O(n^2) \) space in total. Building the FPVDs takes linear time, so the total construction time is \( O(n^2) \) as well.

To evaluate \( h_T(x, y) \) for some query point \( (x, y) \) and query slope \( \alpha \) we again have to find the furthest point among some interval \( t_i, \ldots, t_j \) on \( CH(T) \). We compute the largest value \( \ell \) such that \( 2^\ell \leq j - i \). This allows us to decompose the “query range” \( t_i, \ldots, t_j \) into two overlapping intervals \( t_i, \ldots, t_{i+2^\ell} \) and \( t_{j-2^\ell}, \ldots, t_j \) for which we have pre-stored the FPVD. We can thus query both these FPVDs, in \( O(\log n) \) time, and report the furthest point found. Computing \( \ell \) and finding the points \( t_{i+2^\ell} \) and \( t_{j-2^\ell} \) takes \( O(\log n) \) time as well, by a binary search on \( t_i, \ldots, t_j \). Hence the total query time is \( O(\log n) \).

\[\text{Theorem 28.} \text{ Let } P \text{ be a polygonal curve in } \mathbb{R}^2 \text{ with } n \text{ vertices.} \]

- In \( O(n \log n) \) time we can construct a data structure of size \( O(n \log n) \) so that given a query segment \( \overline{ab}, D_H(P, \overline{ab}) \) can be computed in \( O(\log^2 n) \) time.
- In \( O(n^2) \) time we can construct a data structure of size \( O(n^2) \) so that given a query segment \( \overline{ab}, D_H(P, \overline{ab}) \) can be computed in \( O(\log n) \) time.

4.2 The backward pair distance term

Let \( (p_i, p_j) \in P^\leq \) be an ordered pair. We restrict \( \delta_{pq}(\alpha, \beta) \) to the interval of \( \alpha \) values for which \( (p_i, p_j) \) is a backward pair with respect to orientation \( \alpha \). Hence, each \( \delta_{pq} \) is a partially defined, constant algebraic degree, constant complexity, bivariate function. The backward pair distance \( D_B \) is the upper envelope of \( O(n^2) \) such functions. This envelope has complexity \( O(n^{4+\varepsilon}) \), for some arbitrarily small \( \varepsilon > 0 \), and can be computed in \( O(n^{4+\varepsilon}) \) time [34]. Evaluating \( D_B(\alpha, \beta) \) for some given \( \alpha, \beta \) takes \( O(\log n) \) time. The following lemma, together with Theorem 28 then gives an \( O(n^{4+\varepsilon}) \) size data structure that supports \( O(\log n) \) time Fréchet distance queries.

\[\text{Lemma 29.} \text{ Let } P \text{ be an } n \text{-vertex polygonal curve in } \mathbb{R}^2. \text{ In } O(n^{4+\varepsilon}) \text{ time we can construct a size } O(n^{4+\varepsilon}) \text{ data structure so that given a query segment } \overline{ab}, D_B(\overline{ab}) \text{ can be computed in } O(\log n) \text{ time.}\]

5 Arbitrary Orientation Subcurve queries

Next, we show how to support querying against subcurves \( P[s, t] \) of \( P \) in \( O(\log^4 n) \) time as well. We use the same approach as for the horizontal query segment case: we store the vertices of \( P \) into the leaves of a range tree where each internal node \( \nu \) corresponds to some canonical subcurve \( P_{\nu} \), so that any subcurve \( P[s, t] \) can be represented by \( O(\log n) \) nodes.

The Hausdorff distance term. Since computing the directed Hausdorff distance is decomposable, using this approach with the data structure of Theorem 28 immediately gives us a data structure that allows us to compute \( \overline{D}_H(P[s, t], \overline{ab}) \) in \( O(\log^2 n) \) time. Since the space usage satisfies the recurrence \( S(n) = 2S(n/2) + O(n^2) \), this uses \( O(n^2) \) space in total.

The backward pair distance term. By storing the data structure of Lemma 29 at every node of the tree, we can efficiently compute the contribution of the backward pairs inside each of the \( O(\log n) \) canonical subcurves that make up \( P[s, t] \). However, as in Section 3.3, we are still missing the contribution of the backward pairs from different canonical subcurves. We again store additional data structures at every node of the tree that allow us to efficiently compute this contribution at query time.

Let \( S \) and \( T \) be (the vertices of) two such canonical subcurves, with all vertices of \( S \) occurring before \( T \) along \( P \). Analogous to Section 3.3 we will argue that for some given \( \alpha \) and \( \beta \) the
functions \( x \mapsto h_{\rightarrow \alpha}(x, \alpha x + \beta) \) and \( x \mapsto h_{\leftarrow \alpha}(x, \alpha x + \beta) \) are monotonically increasing and decreasing, respectively, and that the intersection point of (the graphs of) these functions corresponds to the contribution of the backward pairs in \( S \times T \). So, our goal is to build data structures storing \( S \) and \( T \) that given a query \( \alpha, \beta \) allow us to efficiently compute the intersection point of these functions. As we will argue next, we can use the data structure of Lemma 27 to support such queries in \( O(\log^2 n) \) time.

We generalize some of our earlier geometric observations to arbitrary orientations. Let \( p, q \) be vertices of \( P \), and let \( S \) and \( T \) be subsets of vertices of \( P \). We define

\[
\delta_{pq}(\alpha, \beta) = \min_x \max \{ h_{\rightarrow \alpha}((x, \alpha x + \beta)), h_{\leftarrow \alpha}((x, \alpha x + \beta)) \}
\]

and

\[
\mathcal{D}^{S \times T}_B(\alpha, \beta) = \max \{ \delta_{pq}(\alpha, \beta) \mid (p, q) \in (S \times T) \cap P^\leq \}.
\]

and prove the following lemmas (by essentially rotating the plane so that the query segment becomes horizontal, and applying the appropriate lemmas from earlier sections):

**Lemma 30.** Let \( P \) be partitioned into two subcurves \( S \) and \( T \) with all vertices in \( S \) occurring on \( P \) before the vertices of \( T \). We have that

\[
\mathcal{D}_B(\alpha, \beta) = \mathcal{D}^{P \times P}_B(\alpha, \beta) = \max \{ \mathcal{D}^{S \times S}_B(\alpha, \beta), \mathcal{D}^{T \times T}_B(\alpha, \beta), \mathcal{D}^{S \times T}_B(\alpha, \beta) \}.
\]

**Proof.** Consider the rotation that transforms the line \( y = \alpha x + \beta \) into a horizontal line, keeping \( a \) to the left of \( b \). Applying this rotation to \( P \) and \( \overline{ab} \) produces an instance of the horizontal query problem. Since the same rotation is applied to all vertices of \( P \), the set of backward pairs and the relative distances from the query segment to them remain unchanged. Therefore, we can apply the results for the horizontal case, in particular Lemma 11, to obtain that \( \mathcal{D}_B(\alpha, \beta) = \mathcal{D}^{P \times P}_B(\alpha, \beta) \).

For the second equality we use some basic set theory together with the facts that (i) \( S \) and \( T \) are a partition of \( P \) and thus \( P \times P = (S \times S) \cup (T \times T) \cup (S \times T) \cup (T \times S) \) and (ii) that all vertices of \( S \) occur before \( T \) and thus \((T \times S) \cap P^\leq = \emptyset \).

**Lemma 31.** Let \( S \) and \( T \) be subsets of vertices of \( P \), with \( S \) occurring before \( T \) along \( P \) and let \( \alpha, \beta \) denote some query parameters. The function \( x \mapsto h_{\rightarrow \alpha}(x, \alpha x + \beta) \) is monotonically increasing, whereas \( x \mapsto h_{\leftarrow \alpha}(x, \alpha x + \beta) \) is monotonically decreasing. These functions intersect at a point \((x^*, \alpha x^* + \beta)\), for which \( \mathcal{D}^{S \times T}_B(\alpha, \beta) = h_{\rightarrow \alpha}(x^*, \alpha x^* + \beta) = h_{\leftarrow \alpha}(x^*, \alpha x^* + \beta) \).

**Proof.** Consider the rotation that transforms the line \( y = \alpha x + \beta \) into a horizontal line, keeping \( a \) to the left of \( b \). Note again that this preserves distances. It follows then from Observation 5 that \( x \mapsto h_{\rightarrow \alpha}(x, \alpha x + \beta) \) and \( x \mapsto h_{\leftarrow \alpha}(x, \alpha x + \beta) \) are monotonically increasing and decreasing, respectively. Furthermore, by Lemma 13 they intersect in a single point \((x^*, y) = (x^*, \alpha x^* + \beta)\), at which \( \mathcal{D}^{S \times T}_B(\alpha, \beta) = h_{\rightarrow \alpha}(x^*, y) = h_{\leftarrow \alpha}(x^*, y) \).

**Querying** \( \mathcal{D}^{S \times T}_B(\alpha, \beta) \). Consider the predicate \( Q(x) = h_{\rightarrow \alpha}(x, \alpha x + \beta) < h_{\leftarrow \alpha}(x, \alpha x + \beta) \). It follows from Lemma 31 that there is a single value \( x^* \) so that \( Q(x) = \text{FALSE} \) for all \( x < x^* \) and \( Q(x) = \text{TRUE} \) for all \( x \geq x^* \). Moreover, \( x^* \) realizes \( \mathcal{D}^{S \times T}_B(\alpha, \beta) \). By storing \( S \) and \( T \), each in a separate copy of the data structure of Lemma 27, we can evaluate \( Q(x) \), for any value \( x \), in \( O(\log n) \) time. We then use parametric search [31] to find \( x^* \) in \( O(\log^2 n) \) time.

**Lemma 32.** Let \( S, T \) be subsets of vertices of \( P \) such that all vertices in \( S \) precede all vertices in \( T \), stored in the data structure of Lemma 27. For any query \( \alpha, \beta \) we can compute \( \mathcal{D}^{S \times T}_B(\alpha, \beta) \) in \( O(\log^2 n) \) time.
Proof. We treat \( x^* \) as a variable, and evaluate \( Q \) on the (unknown) value \( x^* \). While doing so, we maintain an interval that is known to contain \( x^* \). Initially this interval is \( \mathbb{R} \) itself. When the algorithm to evaluate \( Q(x^*) \) reaches a comparison involving \( x^* \), we obtain a low degree polynomial in \( x^* \) (as all comparisons in the query algorithm of Lemma 27 test if the query point lies left or right of some line, or compare the Euclidean distance between two pairs of points). We compute the (constantly many) roots of this polynomial, and evaluate \( Q \) again at each root. This shrinks the interval known to contain \( x^* \), and allows the evaluation of \( Q(x^*) \) to proceed. When the evaluation of \( Q(x^*) \) finishes, the interval known to contain \( x^* \) has shrunk to a single point, \( x^* \), or \( x^* \) can be computed from it by solving one more equation in constant time. Evaluating \( Q \) takes \( O(\log n) \) time, and thus encounters at most \( O(\log n) \) comparisons. For each such comparison we again evaluate \( Q \) at a constant number of roots, taking \( O(\log n) \) time each. Hence the total time required to compute \( x^* \) is \( O(\log^2 n) \). Given \( x^* \) we can obtain \( D_B^{\times T}(\alpha, \beta) = h_{\nu}(x^*, \alpha x + \beta) \) in \( O(\log n) \) time. □

Note that the parametric search approach we used here is an \( O(\log n) \) factor slower compared to the approach we used for horizontal queries only (Lemma 22).

For every node \( \nu \) of the recursion tree on \( P \) we store: (i) the data structure of Lemma 29 built on its canonical subcurves \( P_\nu \), and (ii) the data structure of Lemma 27 built on the vertices of \( P_\nu \). The total space usage of the data structure follows the recurrence \( S(n) = 2S(n/2) + O(n^{4+\varepsilon}) \), which solves to \( O(n^{4+\varepsilon}) \). To query the data structure with some subcurve \( P[s, t] \) from some vertex \( s \) to a vertex \( t \) we again find the \( O(\log n) \) nodes whose canonical subcurves together define \( P[s, t] \), query the Lemma 29 data structure for each of them, and run the algorithm from Lemma 32 for each pair. The running time is dominated by this last step, as this requires \( O(\log^2 n) \) time for each pair, and we have \( O(\log^2 n) \) pairs to consider. Hence, the total running time is \( O(\log^4 n) \). As before, the procedure can be easily extended to the case where \( s \) and \( t \) lie on the interior of an edge. We conclude:

**Lemma 33.** Let \( P \) be a polygonal curve in \( \mathbb{R}^2 \) with \( n \) vertices. There is an \( O(n^{4+\varepsilon}) \) size data structure that can be built in \( O(n^{4+\varepsilon}) \) time such that given an arbitrary query segment \( \overline{ab} \) and two query points \( s \) and \( t \) on \( P \) it can report \( D_B^{P[s, t]}(\alpha, \beta) \) in \( O(\log^4 n) \) time. Here, \( \varepsilon > 0 \) is an arbitrarily small constant.

Since we can compute all four terms \( ||s - a||, \|t - b\|, D_B(P[s, t], \overline{ab}), \) and \( D_B^{P[s, t]}(\alpha, \beta) \) in \( O(\log^4 n) \) time, it follows that we can efficiently answer Fréchet distance queries against subcurves.

### 5.1 Space vs Query time tradeoff

We can use our approach for subcurve queries from Section 5 to obtain a space vs query time tradeoff for queries against the entire curve. Let \( k \in [1..n] \) be a parameter. We trim the recursion tree on \( P \) at a node \( \nu \) of size \( O(k) \). Let \( T \) denote the resulting tree (i.e. the top \( \log(n/k) \) levels of the full recursion tree), and let \( L(T) \) denote the set of leaves of \( T \), each of which thus corresponds to a subcurve of length \( O(k) \). Let \( \ell(\nu) \) and \( r(\nu) \) be the left and right child of \( \nu \), respectively. By repeated application of the second equality in Lemma 30 we have that

\[
D_B^{P \times P}(\alpha, \beta) = \max_{\nu \in T} \left\{ \max_{\nu \in L(T)} D_B^{P_{\ell(\nu)} \times P_{r(\nu)}}(\alpha, \beta), \max_{\nu \in L(T)} D_B^{P_{r(\nu)} \times P_{\ell(\nu)}}(\alpha, \beta) \right\}
\]

At every leaf of \( T \) we now store the data structure of Lemma 29, and at every internal node the data structure of Lemma 27. The space required by all Lemma 29 data structures is \( O((n/k)k^{4+\varepsilon}) = O(nk^{3+\varepsilon}) \). The total size for all Lemma 27 data structures follows the recurrence

\[
B(n) = \max_{\nu \in L(T)} B(\ell(\nu)) + B(r(\nu)) + O(nk^{3+\varepsilon})
\]
Fig. 8: A query against a subcurve \( P_{[s,t]} \) selects \( O(\log n) \) nodes. When (the subcurve \( P_{\nu} \) of) such a node \( \nu \) is small enough (the blue nodes) we can directly query their Lemma 29 data structures. When the subtree is too large (e.g. \( \nu' \)) we visit its top part (in orange) computing the their contribution to \( \mathcal{D}_{P_{[s,t]} \times P_{[s,t]}}(\alpha, \beta) \) using the Lemma 27 data structures until we reach the (blue) subtrees of size \( O(k) \). The total number of nodes visited is \( O(n/k) \).

\[ S(n) = 2S(n/2) + O(n^2) \] which solves to \( O(n^2) \). Hence, the total space used is \( O(nk^{3+\epsilon} + n^2) \). The preprocessing time is \( O(nk^{3+\epsilon} + n^2) \) as well.

To answer a query \((\alpha, \beta)\) we now query the Lemma 29 data structures at the leaves of \( \mathcal{T} \) in \( O(\log k) \) time each. For every internal node \( \nu \) we use Lemma 32 to compute the contribution of \( \mathcal{D}_{B}^{P_{[\nu,\nu]} \times P_{[\nu,\nu]}}(\alpha, \beta) \) in \( O(\log^2 n) \) time. Hence, the total query time is \( O((n/k) \log k + (n/k) \log^2 n) = O((n/k) \log^2 n) \). So, e.g., choosing \( k = n^{1/3} \) yields an \( O(n^{2+\epsilon}) \) size data structure supporting \( O(n^{2/3} \log^2 n) \) time queries. We can extend this idea to support subcurve queries in \( O((n/k) \log^2 n + \log^4 n) \) time as well, giving us the following result:

**Lemma 34.** Let \( P \) be a polygonal curve in \( \mathbb{R}^2 \) with \( n \) vertices, and let \( k \in [1..n] \) be a parameter. In \( O(nk^{3+\epsilon} + n^2) \) time we can construct a data structure of size \( O(nk^{3+\epsilon} + n^2) \) so that given a query segment \( \overline{ab} \), \( \mathcal{D}_{B}(\overline{ab}) \) can be computed in \( O((n/k) \log^2 n) \) time. If, in addition we are also given two points \( s \) and \( t \) on \( P \), \( \mathcal{D}_{B}^{P_{[s,t]} \times P_{[s,t]}}(\overline{ab}) \) can be computed in \( O((n/k) \log^2 n + \log^4 n) \) time.

**Proof.** We can also apply the time space trade off in case of subcurve queries. We use essentially the structure as above: for all nodes in the full recursion tree that represent subcurves of length at most \( k \) we store both the Lemma 29 and the Lemma 27 data structures, whereas for the topmost nodes in the tree (with a subcurve of size \( > k \)) we store only the Lemma 27 data structure. The space usage remains \( O(nk^{3+\epsilon} + n^2) \).

We can again find a set \( Q \) of \( O(\log n) \) nodes \( \nu \) whose subcurves together represent the query subcurve \( P_{[ps,t]} \). We then have

\[
\mathcal{D}_{B}^{P_{[s,t]} \times P_{[s,t]}}(\alpha, \beta) = \max \left\{ \max_{\nu \in Q} \mathcal{D}_{B}^{P_{\nu} \times P_{\nu}}(\alpha, \beta), \max_{\mu, \nu \in Q, \mu \neq \nu} \mathcal{D}_{B}^{P_{\mu} \times P_{\nu}}(\alpha, \beta) \right\}.
\] (3)

Observe that for each of distinct nodes \( \mu, \nu \in Q \) we can compute \( \mathcal{D}_{B}^{P_{\nu} \times P_{\nu}}(\alpha, \beta) \) in \( O(\log^2 n) \) time using the Lemma 27 data structures stored at nodes \( \mu \) and \( \nu \) (Lemma 32). Hence, computing the contribution of the second term in Equation 3 takes \( O(\log^4 n) \) time. To compute \( \mathcal{D}_{B}^{P_{\mu} \times P_{\nu}}(\alpha, \beta) \) for some \( \nu \in Q \) there are two cases. If \( P_{\nu} \) has length at most \( k \) and thus we can compute the term directly by querying the Lemma 29 data structure of node \( \nu \) in \( O(\log k) \) time. If \( P_{\nu} \) has size more than \( k \), we essentially run the query algorithm from the beginning of this section on the subtree rooted at \( \nu \) (See Fig. 8): i.e. we traverse the tree starting from the root to find a minimal set of
nodes $Q_\nu$ that store a Lemma 29 data structure, and whose associated subcurves make up $P_\nu$. We query their associated data structures, as well as the Lemma 27 data structures of their subtree ancestors. Observe that the subcurve of each node of $Q_\nu$ has size $\Theta(k)$ (otherwise we would have picked its parent instead). Therefore, the total size of all sets $Q_\nu$ over all visited nodes $\nu$ is $O(n/k)$. The total number of nodes whose Lemma 27 data structure we query is thus also $O(n/k)$, and hence the cost of querying these nodes is $O((n/k) \log^2 n)$.

Since computing $D_H(P[s,t], a\nu b)$ can be done in $O(\log^2 n)$ time using only $O(n^2)$ space, we thus established Theorem 3. Once again, it is possible to make the query time proportional to the complexity of $P[s,t]$ rather than to $n$.

6 Applications

In this section we discuss how to apply our result to improve the results of some prior publications.

Curve simplification. Our tools can be used for local curve simplification under the Fréchet distance. In curve simplification, the input is a polygonal curve $P = (p_1, p_2, \ldots, p_n)$ with $n$ vertices and some parameter $\delta$ and the goal is to compute a polygonal curve $S$, whose vertices are vertices of $P$, that is within distance $\delta$ of $P$ for some distance metric and has a minimum number of vertices. The simplification is a local simplification if for every edge $e_{ij}$ in $S$ from $p_i$ to $p_j$, the distance between $e_{ij}$ and $P[i,j]$ is at most $\delta$. Curve simplification is well-studied within computational geometry. Recent examples that use the Fréchet distance include algorithms for global curve simplification by Kerkhof et al. [36], a polynomial-time algorithm for computing an optimal local simplification for the Fréchet distance by van Kreveld et al. [37], and an $O(n^3)$ lower bound on computing a local simplification in higher dimensions using $L_p$ ($p \neq 2$) norms by Bringmann and Chaudhury [10].

The state-of-the-art approach to compute a local $\delta$-simplification using Fréchet distance and the $L_2$ norm in 2D is the Imai-Iri line simplification algorithm [28] (see also Godau [23]). This algorithm considers all $O(n^2)$ edges between vertices of $P$ and computes for every edge $e_{ij}$ from $p_i$ to $p_j$ ($i < j$), the Fréchet distance between $e_{ij}$ and $P[p_i,p_j]$ in total $O(n^3)$ time. They assign the corresponding Fréchet distance as a weight to $e_{ij}$. This results in $O(n^2)$ weighted edges (links). Computing a minimum link path from $p_1$ to $p_n$ in this graph takes $O(n^2)$ time, and gives the desired simplification. Thus, a local $\delta$-simplification can be computed in $O(n^3)$ time and $O(n^2)$ space.

We can improve this state-of-the-art algorithm by applying Theorem 3. Specifically, if we choose our parameter $k = n^{1/2}$, we can construct the corresponding data structure in $O(n^{2.5+\varepsilon})$ time and space (where $\varepsilon$ is an arbitrarily small positive constant). For each edge $e_{ij}$, we can compute the Fréchet distance between $e_{ij}$ and the subcurve $P[p_i,p_j]$ in $O(\sqrt{n} \log^2 n)$ time, and we conclude:

**Theorem 35.** Let $P$ be a polygonal curve in $\mathbb{R}^2$ with $n$ vertices, and let $\delta > 0$. We can compute a local $\delta$-simplification of $P$ with respect to the Fréchet distance using $O(n^{5/2+\varepsilon})$ time and space.

Fréchet distance queries under translation. There are many geometric pattern matching application where one would like to perform a transformation (e.g., translation or rotation) that minimizes the Fréchet distance between two input curves. One typical example is handwriting recognition [35]. Our results can be used to compute a data structure for Fréchet distance queries under translation. Given are two polygonal curves $P$ and $Q$ in the plane, the goal is to find an optimal translation of $Q$ such that the Fréchet distance between the translated version of $P$ and $Q$ is minimized. This problem can be solved in $O((nm)^3(n + m)^2 \log(n + m))$ time [6] where $n$ and $m$ are the number of vertices of $P$ and $Q$, respectively. Recently, Gudmundsson et al. [25] (full version [27]) studied the query version of this problem, where the goal is to preprocess $P$, such that given a query curve $Q$
and two points $s$ and $t$ on $P$, one can find the translation of $Q$ that minimizes the Fréchet distance between $P[s, t]$ and $Q$ efficiently. They study this query version in a restricted setting, where $Q$ is a horizontal segment. Their data structure uses $O(n^2 \log^2 n)$ space and allows for $O(\log^{32} n)$ time queries. By applying our data structure, we obtain the following result:

**Theorem 36.** Let $P$ be a polygonal curve in $\mathbb{R}^2$ with $n$ vertices. There is an $O(n \log^2 n)$ size data structure that can be built in $O(n \log^2 n)$ time such that given any two points $s, t \in P$ and a horizontal query segment $\overline{ab}$, one can report the translation of $\overline{ab}$ that minimizes its Fréchet distance to $P[s, t]$ in $O(\log^{12} n)$ time.

**Proof.** The approach by Gudmundsson et al. [25] essentially uses four levels of parametric search [31] to turn a Fréchet distance data structure into a data structure that can find the translation of $\overline{ab}$ that minimizes the distance to $P[s, t]$.

Specifically, suppose that you have access to a data structure that for a horizontal query segment $\overline{ab}$ can decide if the Fréchet distance between $P[s', t']$ and $\overline{ab}$ is at most $R$ in $Q(n)$ time (where $s'$ and $t'$ are vertices of $P$).

The authors then use the data structure for four levels of parametric search. The first two levels are used to, given two points $s, t$ on $P$ that are not necessarily vertices, compute the Fréchet distance between $\overline{ab}$ and the subcurve $P[s, t]$ (as opposed to the Fréchet distance between $\overline{ab}$ and $P$). The third level decides (given a fixed $x$-coordinate for the point $a$ of $\overline{ab}$) the vertical translation of $\overline{ab}$ that minimizes the Fréchet distance to $P[s, t]$. The fourth level decides the arbitrary translation of $\overline{ab}$ that minimizes the Fréchet distance to $P[s, t]$, by deciding, for a given $x$-coordinate, whether the endpoint $a$ of $\overline{ab}$ lies left or right of this $x$-coordinate.

Each parametric search squares the running time of the decision algorithm that it has access to, and their final solution runs in $O(Q(n)^{16})$ time. By replacing the data structure of de Berg et al. [17] by Theorem 2 we note that we can skip the first two levels of parametric search (since our data structure already supports arbitrary subcurves $P[s, t]$), getting the total query time down to $O(Q(n)^4)$. With our data structure, we can decide if the Fréchet distance between $P[s, t]$ and $\overline{ab}$ is at most $R$ in $O(\log^3 n)$ time. Hence our total running time is $O((\log^3 n)^4) = O(\log^{12} n)$. 

Since the Theorem 2 data structure is essentially used as a black-box, replacing it with Theorem 3 then yields a data structure supporting arbitrarily oriented query segments.

**Theorem 37.** Let $P$ be a polygonal curve in $\mathbb{R}^2$ with $n$ vertices, let $k \in [1..n]$, and let $\varepsilon > 0$ be an arbitrarily small constant. There is an $O(n k^{3+\varepsilon} + n^2)$ size data structure that can be built in $O(n k^{3+\varepsilon} + n^2)$ time such that given a query segment $\overline{ab}$ and two points $s$ and $t$ on $P$, it can report the translation of $\overline{ab}$ that minimizes its Fréchet distance to $P[s, t]$ in $O((n/k)^4 \log^8 n + \log^{16} n)$ time.

**Fréchet distance queries under translation and scaling.** We answer an open question by Gudmundsson et al. by showing how to find a horizontal segment that minimizes its Fréchet distance to a given polygonal curve $P$. We start with the case that the vertical position of the segment is given and fixed. That is, given a height $y'$, we want to determine a horizontal segment $\overline{ab}$ with $y_a = y'$ such that the Fréchet distance between $P$ and $\overline{ab}$ is minimized. Observe that $x_a \leq x_{p_1}$ and $x_b \geq x_{p_n}$. Recall that the Fréchet distance $\mathcal{D}_F(P, \overline{ab})$ is the maximum of the four terms $\|p_1 - a\|$, $\|p_n - b\|$, $\overset{\rightarrow}{D}_H(P, \overline{ab})$, and $\mathcal{D}_B(y_a)$. Note that $\mathcal{D}_B(y_a)$ is independent of the length of $\overline{ab}$.

We divide the set of vertices of $P$ into three subsets $P^<$, $P^>$, and $P'$ where $P^<$ contains all vertices that have an $x$-coordinate smaller than $x_{p_1}$, $P^>$ contains all vertices that have an $x$-coordinate greater than $x_{p_n}$, and $P'$ contains all vertices that have an $x$-coordinate in $[x_{p_1}, x_{p_n}]$. Note that for all $p_i \in P^<$, the pair $(p_1, p_i)$ is a backward pair and for all $p_j \in P^>$, the pair $(p_j, p_n)$ is a backward pair.
Lemma 38. Let $\overline{ab}$ be a horizontal segment with height $y'$ that minimizes the Fréchet distance to a curve $P$. Then, $D_F(P, \overline{ab}) = \max \{ D_H(P, \ell), D_B(y') \}$ where $\ell$ is a horizontal line at height $y'$.

Proof. Let $z = \max \{ D_H(P, \ell), D_B(y') \}$. Let $a'$ be the leftmost intersection point of $\ell$ with a circle with radius $z$ and center $p_1$ and let $b'$ be the rightmost intersection point of $\ell$ with a circle with radius $z$ and center $p_n$. We show that $D_F(P, \overline{ab'}) = z$ which proves the lemma. Recall that $D_F(P, \overline{ab'}) = \max \{ ||p_1 - a'||, ||p_n - b'||, D_H(P, \overline{ab'}), D_B(y') \}$. Clearly, $||p_1 - a'|| = ||p_n - b'|| = z$ and $D_B(y') \leq z$.

We need to show that $D_H(P, \overline{ab'}) \leq z$. As $x_a' \leq x_{p_1}$ and $x_b' \geq x_{p_n}$, $D_H(P', \overline{ab'}) = D_H(P', \ell)$ (for points in $P'$, the Hausdorff distance is the vertical distance to $\ell$).

We show that $D_H(P^<, \overline{ab'}) \leq z$ by contradiction. Suppose instead that $D_H(P^<, \overline{ab'}) > z$. Because $D_H(P, \ell) \leq z$, the vertical distance from $P^<$ to $\ell$ is at most $z$. Because all points of $P^<$ lie to the left of $p_1$ and hence $b'$, points $p_j \in P^<$ with distance greater than $z$ to $\overline{ab'}$ must lie left of $a'$, and their distance to $\overline{ab'}$ is $||p_j - a'||$. Because $D_H(P^<, \overline{ab'}) > z$, there exists some point $p_j \in P^<$ that lies left of $a'$ such that $||p_j - a'|| > z$. Because $p_j \in P^<$, $(p_1, p_j)$ is a backward pair. However, $||p_1 - a'|| = z$ and $x_{p_1} \leq x_{a'} \leq x_{p_1}$, so the backward pair distance of $(p_1, p_j)$ is $||p_1 - q||$ for some $q = (x^*, y')$ with $x^* < x_{a'}$ (recall that the backward pair distance is the minimum possible distance between a point at height $y'$ and both $p_1$ and $p_j$, and that the vertical distance from $p_1$ and $p_j$ to $\ell$ is at most $z$). Thus, the backward pair distance is strictly greater than $z$, which contradicts $D_B(y') \leq z$.

We can prove that $D_H(P^>, \overline{ab'}) \leq z$ using a symmetric argument. 

Note that given the value $z = \max \{ D_H(P, \ell), D_B(y') \}$, we can find a segment $\overline{ab'}$ on $\ell$ with Fréchet distance $z$ to $P$ in constant time.

Next, we want to determine the height $y^*$ such that the Fréchet distance between the curve $P$ and a horizontal segment $\overline{ab}$ with $y_a = y^*$ is minimized. We show that this height can be computed in $O(n \log^2 n)$ time, moreover, we present a data structure with $O(n \log^2 n)$ size (that can be built in the same time) that reports a horizontal segment that minimizes the Fréchet distance to a subcurve of $P$ in polylogarithmic time.

First, we describe a decision algorithm that decides whether the height of a current candidate segment is larger or smaller than the height $y^*$ of an optimal segment. Let the height of the current candidate segment be $y'$ and let $\ell$ be a line with height $y'$. Recall that by Lemma 38 the Fréchet distance between $P$ and this segment is either determined by $D_H(P, \ell)$ or $D_B(y')$. We have the following cases:

- the Fréchet distance is determined by $D_H(P, \ell)$: If the point that has the largest vertical distance to $\ell$ lies below $\ell$, then the optimal height has to be smaller than $y'$. If this point lies above $\ell$, the optimal height has to be larger than $y'$. If this point lies on $\ell$, we stop and the Fréchet distance is 0.

- the Fréchet distance is determined by $D_B(y')$: If the midpoint of a segment connecting the two points of the backward pair determining $D_B(y')$ lies below $\ell$, then the optimal height has to be smaller than $y'$. If this midpoint lies above $\ell$, the optimal height has to be larger than $y'$. If this midpoint lies on $\ell$, we found the optimal height.

It can be the case that more than one term determines the current Fréchet distance. Then we decide for each of these terms whether the next candidate height has to be larger or smaller than the current one. If the decisions are the same for all these terms, we move the height in the
corresponding direction. Otherwise, we stop, as moving the height in any direction will increase the Fréchet distance. (Note that this decision algorithm is quite similar to the one of Gudmundsson et al. [26].)

The Fréchet distance between a polygonal curve and a horizontal segment at optimal height as a function of \( y' \) is convex, and has complexity \( O(n \log n) \). By Lemma 38 it is the maximum of the backward pair distance, and the Hausdorff distance from \( P \) to the line at height \( y' \). De Berg et al. [17] already showed that the backward pair distance is convex, and by Lemma 16 it has complexity \( O(n \log n) \). The Hausdorff distance is determined only by the top and bottommost point in \( P \), and is also easily seen to be convex and of constant complexity. Therefore, the maximum of these two functions is also convex and has \( O(n \log n) \) breakpoints; at most a constant number per piece of \( D_B(y') \).

It then follows that given \( P \) we can compute a horizontal segment that minimizes the Fréchet distance to \( P \) in \( O(n \log^2 n) \) time; we use Lemma 16 to compute \( D_B \), and construct the function representing \( y \mapsto \max \{ D_B(y), \overrightarrow{D}_H(P, \ell) \} \) in \( O(n \log n) \) time, and use the above binary search procedure to find a height \( y^* \) where this function is minimized, and an optimal segment at height \( y^* \) that realizes this Fréchet distance. (Note that we can achieve the same running time without performing a binary search. As there are only \( O(n \log n) \) break points, we could compute for each of them explicitly the corresponding height and Fréchet distance, go through all of them and report the minimum one.)

We can also support queries where we find a horizontal segment minimizing the Fréchet distance to a query subcurve \( P[s, t] \) of \( P \). One option is to simply use parametric search with the above algorithm as decision algorithm. We show that we can do slightly better by explicitly binary searching over the critical values.

**Theorem 39.** Let \( P \) be a polygonal curve in \( \mathbb{R}^2 \) with \( n \) vertices. There is an \( O(n \log^2 n) \) size data structure that can be built in \( O(n \log^2 n) \) time such that given any two points \( s, t \in P \), one can report a horizontal segment that minimizes its Fréchet distance to \( P[s, t] \) in \( O(\log n) \) time.

**Proof.** In this case we store the data structure from Lemma 24, and a data structure that can report the minimum and maximum \( y \)-coordinate of a query curve \( P[s, t] \).

To answer a query we now wish to binary search (using the above decision procedure) on the breakpoints of the function \( y \mapsto \max \{ D_B^{P[s, t] \times P[s, t]}(y), \overrightarrow{D}_H(P[s, t], \ell) \} \). We will simply binary search on the breakpoints \( y_1, \ldots, y_m \) of \( D_B^{P[s, t] \times P[s, t]} \); for each candidate breakpoint \( y_i \) we find its successor \( y_{i+1} \), and explicitly compute the \( O(1) \) additional breakpoints in \( [y_i, y_{i+1}] \) contributed by the Hausdorff term (by intersecting \( D_B^{P[s, t] \times P[s, t]} \) with the function describing the Hausdorff distance).

As in Lemma 24, the function \( D_B^{P[s, t] \times P[s, t]} \) is not stored explicitly, but represented by \( O(\log^2 n) \) functions \( D_B^{P[s, t] \times P[s, t]} \). Therefore, we cannot access the breakpoints \( y_1, \ldots, y_m \) directly; some of them are not even represented explicitly. Instead, we use a two phase approach in which we maintain an interval \( I^{\mu, \nu} \) for each function \( D_B^{P[s, t] \times P[s, t]} \) that is known to contain the value \( y^* \) that we are searching for. We keep shrinking these intervals until each function has no more breakpoints in its interval. In the second phase we can then explicitly compute \( D_B^{P[s, t] \times P[s, t]} \) inside \( \bigcap I^{\mu, \nu} \) by constructing the upper envelope of the functions \( D_B^{P[s, t] \times P[s, t]} \) (restricted to their intervals). Since the total complexity of all functions (restricted to their intervals) is only \( O(\log^2 n) \), this takes \( O(\log^2 n \log \log n) \) time. We can now find \( y^* \) (which is guaranteed to lie in \( \bigcap I^{\mu, \nu} \)) by explicitly traversing (the breakpoints of) \( D_B^{P[s, t] \times P[s, t]} \).

In the first phase, we will shrink the intervals by simultaneously binary searching over all \( O(\log^2 n) \) functions. Let \( n^{\nu, \mu} \) be the complexity of \( D_B^{P[s, t] \times P[s, t]} \) restricted to the interval \( I^{\nu, \mu} \), and let
\[ N = \sum n^{\mu,\nu} \] be the total remaining complexity. Assume that for every function \( B^{P_{\mu} \times P_{\nu}} \) we can find a halving point \( y^{\mu,\nu} \in I^{\mu,\nu} \) for which the complexity of \( B^{P_{\mu} \times P_{\nu}} \) in \( I^{\mu,\nu} \) before and after \( y^{\mu,\nu} \) is roughly halved. That is, let \( c \in (1, 2] \) be a constant such that the complexity of \( B^{P_{\mu} \times P_{\nu}} \) in both intervals is at most \( n^{\mu,\nu}/c \leq N/c \).

We can then binary search as follows: for each function (that still has breakpoints inside its interval), compute its halving point \( y^{\mu,\nu} \), and its complexity \( n^{\mu,\nu} \). Consider the halving points in increasing order, and for \( y^{\mu,\nu} \) compute the sum \( M^{\mu,\nu} \) of the complexities before \( y^{\mu,\nu} \), i.e.,

\[ M^{\mu,\nu} = \sum_{y^{\mu',\nu'} \leq y^{\mu,\nu}} n^{\mu',\nu'}. \]

Now there are two cases, either there is a halving value \( y^{\mu,\nu} \) for which \( M^{\mu,\nu} \) lies in the range \( [N(\frac{1}{2} - \frac{1}{2c}), N(\frac{1}{2} + \frac{1}{2c})] \), or there is a single function \( B^{P_{\mu} \times P_{\nu}} \) that has at least complexity \( N(1/c) \). In either case, we pick \( y^{\mu,\nu} \) to reduce the size of the intervals. That is, we evaluate \( D^{P_{\mu} \times P_{\nu}}_{[s,t]} \) at \( y^{\mu,\nu} \) and use the decision procedure to test if \( y^{\nu} \) occurs before or after \( y^{\mu,\nu} \) (this means we may have to construct the piece of \( D^{P_{\mu} \times P_{\nu}}_{[s,t]} \) at \( y^{\mu,\nu} \) and compute its intersection with the function representing the Hausdorff distance term). We now argue that in both cases we can discard at least a constant fraction of the total complexity.

In the former case, we pick a halving point for which \( M^{\mu,\nu} \) lies in the range \( [N(\frac{1}{2} - \frac{1}{2c}), N(\frac{1}{2} + \frac{1}{2c})] \). If we discard the first half of the intervals, i.e., up to \( y^{\mu,\nu} \), we can discard the first halves for all functions for which \( y^{\mu',\nu'} < y^{\mu,\nu} \). Since these functions have total complexity \( M^{\mu,\nu} \geq N(\frac{1}{2} - \frac{1}{2c}) \) and we discard (at least) a \( (1 - \frac{1}{c}) \) fraction of each function, we reduce the complexity by at least a fraction \( (\frac{1}{2} - \frac{1}{2c})(1 - \frac{1}{c}) = (\frac{1}{2} - \frac{1}{c} + \frac{1}{2c^2}) \). Note that for any \( c \in (1, 2] \) this value is strictly positive, i.e., for \( c = 2 \) we would discard a fraction of 1/8, for something like \( c = 3/2 \) we would discard a 1/18 fraction. Similarly, if we discard the second half of the intervals, we can discard the second halves for all functions for which \( y^{\mu',\nu'} \geq y^{\mu,\nu} \). These functions have total complexity at least \( N - N(\frac{1}{2} + \frac{1}{2c}) = N(\frac{1}{2} - \frac{1}{2c}) \). Since again we discard a \( (1 - \frac{1}{c}) \) fraction from each function, we reduce the complexity by at least a \( (\frac{1}{2} - \frac{1}{c} + \frac{1}{2c^2}) \) fraction.

In the later case, we discard at least a \( (1 - \frac{1}{c}) \) fraction of \( B^{P_{\mu} \times P_{\nu}} \). Since this function has complexity at least \( N/c \), we reduce the total complexity by at least \( N\frac{1}{c}(1 - \frac{1}{c}) = N\left(\frac{1}{c} - \frac{1}{c^2}\right) \).

In either case we thus reduce the complexity by a constant fraction. Hence, after a total of \( O(\log N) \) rounds, there are no more interior breakpoints, and we can start phase two. Apart from finding the halving points and complexities, each such round takes \( O(\log^{3} n) \) time, since we have to evaluate \( D^{P_{\mu} \times P_{\nu}}_{[s,t]} \) at \( y^{\mu,\nu} \) using Lemma 24.

All that remains is to argue that we can also find the halving points and complexities (in at most \( O(\log n) \) time per round) for each function. The functions \( B^{P_{\mu} \times P_{\nu}} \) are directly represented using binary search trees, and hence finding a halving point is trivial. The functions \( B^{P_{\mu} \times P_{\nu}} \), with \( \nu \neq \mu \), are represented using Lemma 22 data structures. These structures are essentially also just binary search trees, hence the same applies here.

Since the total complexity \( N \) of all \( B^{P_{\mu} \times P_{\nu}} \) functions is at most \( O(n \log^{2} n) \), our binary search finishes in \( O(\log n) \) rounds. The theorem now follows.

By using parametric search instead of the binary search and replacing our data structure for horizontal query segments with the one for arbitrarily oriented query segment, we immediately get the following result.

**Theorem 40.** Let \( P \) be a polygonal curve in \( \mathbb{R}^2 \) with \( n \) vertices, let \( k \in [1..n] \), and let \( \varepsilon > 0 \) be an arbitrarily small constant. There is an \( O(nk^{3+\varepsilon} + n^2) \) size data structure that can be built in
$O(nk^{3+\varepsilon} + n^2)$ time such that given a query slope $\alpha$ and two points $s$ and $t$ on $P$, one can report a segment with slope $\alpha$ that minimizes its Fréchet distance to $P[s, t]$ in $O((n/k)^2 \log^3 n + \log^8 n)$ time.

7 Concluding Remarks

We presented data structures for efficiently computing the Fréchet distance of (part of) a curve to a query segment. Our results improve over previous work for horizontal segments and are the first for arbitrarily oriented segments. However, we are left with the challenge of reducing the space used for arbitrary orientations. There are two main issues. The first issue is that even for a small interval of query orientations (e.g., one of the $O(n^2)$ angular intervals defined by lines through a pair of points) it is difficult to limit the number of relevant backward pairs to $o(n^2)$. The second issue is how to combine the backward pair distance values contributed by various subcurves. For (low algebraic degree) univariate functions, the upper envelope has near linear complexity, whereas for bivariate functions the complexity is near quadratic. The combination of these issues makes it hard to improve over the somewhat straightforward $O(n^{4+\varepsilon})$ space bound we build upon.

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