Single time dynamical model for equations of motion of relativistic retarded systems

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Abstract

We present a procedure to build a single time model for the equations of motion of relativistic retarded systems composed of several particles; at any desired level of accuracy. We treat the especial case of a binary system.

We apply this model to the classical electromagnetic binary particle system.

We mentioned some differences with previous approaches and discussed the implications for linear gravitational models.

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1 Introduction

1.1 Content of this article

Models for retarded particle systems that use a single dynamical time are very convenient, since the numerical calculation simplifies considerably; instead to deal with several proper times and exact retarded effects. The way in which one approaches this procedure influences the precision and predictive power of the final dynamical equations. It has been customary to emphasize either an expansion in terms of interaction constants and/or in terms of the velocity of the particles. We here instead emphasize the precision in the calculation of the retarded quantities involved; which provides a new perspective into this topic.

The subject of this work is of interest to the dynamics of charged particles in Minkowski spacetime and to models for the motion of black holes, when treated as ‘particles’ in first order of the field equations. In this setting, then each particle feels the retarded fields generated by the other particle, with respect to a flat background metric. Although we will have in mind a binary system, the study can be extended to any number of particles.

The problem treated here has been studied in the past by several authors; but normally severe approximations have been made that had as a consequence that the retarded effects had a crude estimation. We instead, see the problem from a different perspective; so that for each dynamical model for a physical system, we take the forces as the source of information; and then we try to build an approximation that calculates the retarded effects with the desired accuracy. For this reason we will obtain equations of motions that differ from previous results.

When constructing a single time dynamical model for equations of motion of relativistic retarded systems one would like to reproduce as precisely as possible the global properties of the original retarded system. In our approach the aim is to avoid the introduction of cumulative effects on the relativistic dynamics, due to inaccurate retarded times, positions and velocities calculations. These calculations are independent of the nature of the theory, the field equations, or of the forces; they just take into account the Lorentzian relativistic nature of past null cones.

Since we are showing a new point of view to an old problem, we are going through a detailed presentation of the involved topics. We first, in section 2, concentrate on the calculations of the retarded times in Minkowski spacetime; and use them to approximate the retarded forces by a set that depends on a single dynamical time. We present a procedure that only takes into account the value of the positions and velocities of the particles and also use a single evaluation of the forces; which we call the order one calculation.

In section 3 we discuss the possibility to approximate the dynamics by a Lagrangian system.

The order two retarded forces are calculated in section 4, where we apply a Runge-Kutta like approach.

We present the forces for a charged binary system in section 5.

The final section is reserved for comments of our approach and its relation with previous works.

Although we have in mind a binary system, whose components we label with A and B; in order to simplify the reading, whenever possible, when there is no room for ambiguities, we neglect the corresponding subindices

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2 Retarded effects in a binary system

2.1 The retarded times issue

We assume the dynamics of a relativistic binary system is determined from the equations of motion of the form:

\[ m_A \frac{d\vec{r}_A}{dt} = \vec{F}_A(\vec{r}_A(t), \vec{v}_A(t); \vec{r}_B(t_{IB}), \vec{v}_B(t_{IB})), \]

and

\[ m_B \frac{d\vec{v}_B}{dt} = \int_B(\vec{r}_B(t), \vec{v}_B(t); \vec{r}_A(t_{IA}), \vec{v}_A(t_{IA})); \]

where \( t_{IA} \) and \( t_{IB} \) are the corresponding retarded times.

Then, the main objective is to obtain a new approximate version of (1) and (2) which will only include reference to the coordinate time \( t \); and any appearance of quantities evaluated at retarded times has been replaced by an appropriate expansion, in terms of a number of evaluations of the forces used to calculate the retarded times.

One can see that to determine the force on body \( A \) one has to know the retarded position, velocity and acceleration of body \( B \) at \( t_{IB} \); but we have the information \( \vec{r}_B(t), \vec{v}_B(t) \). In turn, to calculate the time derivative of the velocity for body \( B \) one needs to know the retarded position, velocity and acceleration of body \( A \); but we have the information \( \vec{r}_A(t), \vec{v}_A(t) \).

Using a four dimensional notation in which \( z_B^\mu(\tau_B) \) denotes the position of particle \( B \), in terms of a Cartesian coordinate system, at proper time \( \tau_B \), and \( y_A^\mu \) the field position of particle \( A \), one notes that since the point \( z_B^\mu(\tau_B) \) is in the past null cone of \( y_A^\mu \), one has

\[ y_A^\mu - z_B^\mu(\tau_B) = |y_A^\mu - z_B^\mu(\tau_B)|; \]

where \( |.| \) means the modulus of the spacelike relative position vector in the Cartesian frame one is using; that we could call \( \hat{r}(\tau_B) \).

We would like to refer to the binary system in terms of a common coordinate time, so that we will use:

\[ t - t_{IB} = |\vec{r}_A(t) - \vec{r}_B(t_{IB})|/c, \]

and

\[ t - t_{IA} = |\vec{r}_B(t) - \vec{r}_A(t_{IA})|/c. \]

These can also be expressed in terms of \( \Delta t_B = t - t_{IB} \) and \( \Delta t_A = t - t_{IA} \) as

\[ \Delta t_B = |\vec{r}_A(t) - \vec{r}_B(t - \Delta t_B)|/c, \]

and

\[ \Delta t_A = |\vec{r}_B(t) - \vec{r}_A(t - \Delta t_A)|/c. \]

Therefore, in order to be able to calculate accurately the retarded times, we need to know the trajectories to the past of the coordinate time \( t \) with the required precision. Different approaches to do this are mentioned below.

In reference [4] the author estimates the retarded times by approximating equations (6) and (7) by quadratic equations; where it was assumed an expansion in terms of forces, or accelerations. The approach used in the Landau-Lifshitz textbook [2] is based on the idea that if the coordinate velocities of system \( B \) are small, then its configuration will not change significantly during the time \( \hat{r}(\tau_B)/c \) (where we introduce explicitly the velocity of light \( c \)). Then it is natural to think in the coordinate retarded time \( z_B^0 \) as

\[ z_B^0 = y_B^0 - \hat{r}/c; \]

and expand any reference to \( z_B^0 \) in series of powers of \( \hat{r}/c \). Let us call \( t = y_B^0 \) de value of the time coordinate of one particle, then the fields of the other particle are evaluated at the retarded time \( t = z_B^0 = t - \hat{r}(t)/c \).

Due to the small velocity assumption one can expand any field as

\[ F(t, \vec{r}_B(t)) = F(t, \hat{r}_B(t)) - \frac{\hat{r}}{c} \frac{\partial F}{\partial t} + \frac{\hat{r}}{c} \nabla \hat{r}_B(t) \cdot \vec{v}_B + O(\hat{r}^2); \]

which is in fact a Taylor expansion around the fields evaluated at time \( t \) to the time \( t' = t - \Delta t \) with \( \Delta t \approx \hat{r}/c \); but of course, \( \Delta t \) need not be small in any sense. It is only assumed that the velocities are small. They apply this to a smooth energy-momentum tensor; but the procedure does not seem to have a regular behavior when one takes the limit to a Dirac delta distribution. Our approach differs from these works in that our primary objective is the precision for the calculation of the retarded quantities; which is gained at the expenses of evaluations of the force terms. This is in contrast to fix first an order for the evaluations of the forces and then calculate the retarded effects. In any case, since we are concerned with numerical evaluation of the dynamics, we next organize the presentation in terms of the number of evaluations of the forces.

2.2 Order one retarded approach

We would like to think of this problem by concentrating in a point like bodies approach. The guiding idea is to obtain an approximate expression, that uses all the available kinematical and dynamical information at a common time coordinate \( t \), instead of the proper times of each particles.

By order one retarded effects we mean those that can be calculated using one evaluation of the forces for each trajectory, using the kinematical data at our disposal at time \( t \); namely positions and velocities of the particles. More specifically we approximate the trajectory into the past by the corresponding parabolic motion determined by the acceleration at the present time \( t \).
Then, in turn, by order two retarded calculation of the retarded time we mean those that are calculated by making two evaluations of the forces in each trajectory. However it must be noted that when using this information to evaluate the force of one particle, it will include the evaluation of the force of the other particle; in order to estimate the retarded effects. Then, to avoid the inclusion of separate order keeping, we will also say that the order of the force calculated with an order n retarded time, will also be called of order n.

Let us think in an iterative procedure such that we start by evaluating a zero order retardation effect force by assuming a linear trajectory to the past; namely: Let us call \( t_{\text{ta0}} = t - \Delta t_{\text{a0}} \) the solution of

\[
\Delta t_{\text{a0}} = t - t_{\text{ta0}} = |\vec{r}_A(t) - \left( \vec{r}_B(t) - (t - t_{\text{ta0}})\vec{v}_B(t) \right)|/c
\]

or

\[
|\vec{r}_A(t) - \vec{r}_B(t) - \Delta t_{\text{a0}}\vec{v}_B(t)|/c
\]

(10)

which is the solution to the intersection of a linear motion with the past null cone of \((t, \vec{r}_A(t))\). We also define \( t_{\text{ra0}} = t - \Delta t_{\text{a0}} \) to be the solution of

\[
\Delta t_{\text{a0}} = t - t_{\text{ra0}} = |\vec{r}_B(t) - \left( \vec{r}_A(t) - (t - t_{\text{ra0}})\vec{v}_A(t) \right)|/c
\]

(11)

Expressing (11) as a quadratic equation, namely

\[
\Delta t_{\text{a0}}^2 = 2\Delta t_{\text{a0}}\vec{r}_A(t)\cdot\vec{v}_B(t)/c^2 - r_{AB}^2/c^2 = 0;
\]

one has the solution

\[
\Delta t_{\text{a0}} = \sqrt{(1 - v_B^2/c^2) r_{AB}^2 + \left( \frac{r_{AB}\vec{v}_B}{c} \right)^2 - \left( \frac{r_{AB}\vec{v}_B}{c} \right)^2} / (1 - v_B^2/c^2);
\]

(13)

and similarly for \( \Delta t_{\text{a0}} \). The notation we are using is: \( \vec{r}_{AB}(t) = \vec{r}_A(t) - \vec{r}_B(t) \), \( r_{AB} = |\vec{r}_{AB}(t)| \), for the scalar product \( r_{AB}\vec{v}_B = \vec{r}_A(t)\cdot\vec{v}_B(t) \) and \( \vec{v}_B = |\vec{v}_B| \).

Then, we define the zero order retarded forces as

\[
\vec{f}_{\text{a0}}(\vec{r}_A, \vec{v}_A; t) = \vec{f}_A \left( \vec{r}_A, \vec{v}_A; \vec{r}_B(t) - \Delta t_{\text{a0}}\vec{v}_B(t), \vec{v}_B(t) \right),
\]

(14)

and

\[
\vec{f}_{\text{b0}}(\vec{r}_B, \vec{v}_B; t) = \vec{f}_B \left( \vec{r}_B, \vec{v}_B; \vec{r}_A(t) - \Delta t_{\text{a0}}\vec{v}_A(t), \vec{v}_A(t) \right); \]

(15)

that is, in this first stage we assume the linear motion. With this we define \( t_{\text{ta1}} = t - \Delta t_{\text{a1}} \) the solution of

\[
t - t_{\text{ta1}} = |\vec{r}_A(t) - \left( \vec{r}_B(t) - (t - t_{\text{ta1}})\vec{v}_B(t) \right)|/c
\]

(16)

or

\[
\Delta t_{\text{a1}} = |\vec{r}_A(t) - \left( \vec{r}_B(t) - \Delta t_{\text{a1}}\vec{v}_B(t) \right)|/c
\]

(17)

which is the solution to the intersection of a quadratic motion with the past null cone of \((t, \vec{r}_A(t))\). If one expresses \( \Delta t_{\text{a1}} = \Delta t_{\text{a0}} + \delta t_{\text{a1}} \), one can see that \( \delta t_{\text{a1}} = \Theta(\Delta B_{\text{a0}}/m_B) \), so that in first order of \( \Theta(\Delta B_{\text{a0}}/m_B) \) one could instead solve for

\[
\Delta t_{\text{a1}} = |\vec{r}_A(t) - \left( \vec{r}_B(t) - \Delta t_{\text{a1}}\vec{v}_B(t) \right)|/c
\]

(18)

since \( \Delta t_{\text{a1}} - \Delta t_{\text{a0}} = \Theta(\Delta B_{\text{a0}}^2/m_B^2) \). In this way one again can deal with a quadratic equation which is simpler to handle. But conceptually we would like to deal with the exact solution for this retarded time, and since all this calculation will end up to be carried out numerically, we could resort to an iterative scheme of the form

\[
\Delta t_{\text{a(n+1)}} = |\vec{r}_A(t) - \left( \vec{r}_B(t) - \Delta t_{\text{a(n+1)}}\vec{v}_B(t) \right)|/c
\]

(19)

for \( n = 1, 2, 3, \ldots \) whose solutions are expressed in the form of (14), namely

\[
\Delta t_{\text{a(n+1)}} = \sqrt{(1 - v_B^2/c^2) r_{AB}^2 + \left( \frac{r_{AB}\vec{v}_B}{c} \right)^2 - \left( \frac{r_{AB}\vec{v}_B}{c} \right)^2} / (1 - v_B^2/c^2);
\]

(20)

where \( r_{AB} = \vec{r}_A(t) + \Delta t_{\text{a(n)}}\vec{v}_B(t) - \vec{r}_B(t) \). Of course one could also use the analytic expressions for the solutions of quartic polynomial equations. It is clear that the previous iterative scheme will produce the desired solution with a simplification of a numerical code.

Similarly we define \( t_{\text{ra1}} = t - \Delta t_{\text{a1}} \) to be the solution of

\[
t - t_{\text{ra1}} = |\vec{r}_B(t) - \left( \vec{r}_A(t) - (t - t_{\text{ra1}})\vec{v}_A(t) \right)|/c
\]

(21)

Then, we define the first order retarded forces by

\[
\vec{f}_{\text{a1}}(\vec{r}_A, \vec{v}_A; t) = \vec{f}_A \left( \vec{r}_A, \vec{v}_A; \vec{r}_B(t) - \Delta t_{\text{a1}}\vec{v}_B(t), \vec{v}_B(t) \right)
\]

(22)

where \( \vec{r}_{AB} = \vec{r}_A(t) + \Delta t_{\text{a1}}\vec{v}_B(t) - \vec{r}_B(t) \). Of course one could also use the analytic expressions for the solutions of quartic polynomial equations. It is clear that the previous iterative scheme will produce the desired solution with a simplification of a numerical code.
and
\[ \vec{f}_B(\vec{r}_B, \vec{v}_B; t) = \vec{f}_B(\vec{r}_A, \vec{v}_B); \]
\[ \vec{r}_A(t) - \Delta t_A \vec{v}_A(t) + \frac{\Delta t_A^2}{2m_A} \vec{f}_A, \]
\[ \vec{v}_A(t) - \frac{\Delta t_A}{m_A} \int_{t_0}^t \vec{f}_A, \]
\[ \frac{1}{m_A} \vec{f}_A. \]

This is an improvement to the calculations carried out in \([1]\), since in that reference he neglects higher order effects of the forces. That is, he also applies a Taylor expansion of the fields in the Lagrangian, around the time \(\Delta t\) that facilitates the writing of a numerical algorithm. Let

\[ \Delta t = t - t_A \]

return to this issue in section 4.

Suppose that one would like to calculate the dynamic evolution numerically. We have seen in this procedure that we are taking as initial data the positions and velocities; which at this stage are taken as exact quantities. With these we estimate two set of quantities: the retarded times and the forces taking into account retarded effects. When integrating the corresponding equations of motions one will deal with errors in the trajectories, that is in position and velocities, which where non-existent in the initial data. This indicates the important role played by the errors in the determination of retarded times and first order retarded forces; since the numerical calculation should have a precision according to the quality of the calculation of the retarded time and forces. We will return to this issue in section 4.

**Summary:** We can express this approach in a way that facilitates the writing of a numerical algorithm. Let \(\vec{r}_A, \vec{v}_A, \vec{r}_B, \vec{v}_B\) be the position and velocity vectors in the three dimensional Cartesian system at time \(t\). Then let us define the zero order retarded lapse of times

\[ \Delta t_{B0} = \sqrt{(1 - \frac{v^2}{c^2}) \frac{\Delta t_A^2}{c^2} + \frac{(\vec{v}_A \cdot \vec{v}_B)^2}{c^2} - \frac{(\vec{r}_A \cdot \vec{v}_B)^2}{c^2}}, \]

\[ \Delta t_{A0} = \sqrt{(1 - \frac{v^2}{c^2}) \frac{\Delta t_B^2}{c^2} + \frac{(\vec{v}_A \cdot \vec{v}_B)^2}{c^2} + \frac{(\vec{r}_A \cdot \vec{v}_B)^2}{c^2}}. \]

The zero order retarded forces are defined by

\[ \vec{f}_A(\vec{r}_A, \vec{v}_A; t) = \vec{f}_A(\vec{r}_A, \vec{v}_A; t_{B0} - \vec{v}_B(t) - \vec{v}_B(t), 0), \]

\[ \vec{f}_B(\vec{r}_B, \vec{v}_B; t) = \vec{f}_B(\vec{r}_B, \vec{v}_B; t_{A0} - \vec{v}_A(t) - \vec{v}_A(t), 0). \]

Then, we define \(\Delta t_{B1} = t - t_{B1}\) to be the appropriate iterative solution of

\[ \Delta t_{B1(n+1)} = \sqrt{(1 - \frac{v^2}{c^2}) \frac{\Delta t_{A0}}{c^2} + \frac{(\vec{r}_A \cdot \vec{v}_B)^2}{c^2} - \frac{(\vec{r}_A \cdot \vec{v}_B)^2}{c^2}}; \]

for \(n = 0, 1, 2, 3, \ldots\), where

\[ \vec{r}_{AfnB} = \vec{r}_A(t) + \frac{\Delta t_{B1(n)}}{2m_B} \vec{f}_{B0}(t, \vec{r}_B, \vec{v}_B) - \vec{r}_B(t); \]

with \(\Delta t_{B1(0)} = \Delta t_{B0}\). Similarly, we define \(\Delta t_{A1} = t - t_{A1}\) to be the appropriate iterative solution of

\[ \Delta t_{A1(n+1)} = \sqrt{(1 - \frac{v^2}{c^2}) \frac{\Delta t_{B1(n)}}{c^2} + \frac{(\vec{r}_A \cdot \vec{v}_B)^2}{c^2} - \frac{(\vec{r}_A \cdot \vec{v}_B)^2}{c^2}}; \]

where

\[ \vec{r}_{BfnA} = \vec{r}_B(t) + \frac{\Delta t_{A1(n)}}{2m_B} \vec{f}_{A0}(t, \vec{r}_A, \vec{v}_A) - \vec{r}_A(t); \]

with \(\Delta t_{A1(0)} = \Delta t_{A0}\). Then, the order one forces single time model of the retarded system \([1, 2]\) can be expressed as:

\[ \vec{f}_{A1}(\vec{r}_A, \vec{v}_A; t) = \vec{f}_A(\vec{r}_A, \vec{v}_A; t_{B1} - \vec{v}_B(t) - \vec{v}_B(t), 0), \]

\[ \vec{f}_{B1}(\vec{r}_B, \vec{v}_B; t) = \vec{f}_B(\vec{r}_B, \vec{v}_B; t_{A1} - \vec{v}_A(t) - \vec{v}_A(t), 0). \]

It should be emphasized that in order to minimally improve in the calculation of the forces, using the universal time \(t\), and taking into account first order retarded effects, one must make an evaluation of the force in the argument of the corrected positions and velocities. This is completely missing in the Darwin and Landau-Lifschitz approaches. In fact, it is not at all clear at this stage that there exists a normal Lagrangian formulation of this system.

### 3 Possibility of a Lagrangian treatment of the retarded effects

Let us discuss the situation in which, if system \(B\) where given, then there exists a Lagrangian \(L_A\) for particle \(A\) of the form

\[ L_A = L_A(\vec{r}_A(t), \vec{v}_A(t), \vec{r}_B(t - \Delta t_B), \vec{v}_B(t - \Delta t_B)); \]
where \( t_rB = t - \Delta t_B \) is the retarded time of particle \( B \) as seen from particle \( A \) at \( \tilde{r}_A(t) \), as discussed above.

Then, using the procedure described in the previous section, we will obtain approximate expressions

\[
\tilde{r}_B(t - \Delta t_B) = \tilde{r}_B(t) - \Delta t_B \tilde{v}_B(t) + \frac{\Delta t_B^2}{2m_B} \tilde{f}_B(\tilde{r}_B, \tilde{v}_B; t) + \mathcal{O}(r^2),
\]

(29)

and

\[
\tilde{v}_B(t - \Delta t_B) = \tilde{v}_B(t) - \Delta t_B \tilde{f}_B(\tilde{r}_B, \tilde{v}_B; t) + \mathcal{O}(r^2);
\]

(30)

where \( \mathcal{O}(r^2) \) means order two in retarded effects and where \( \Delta t_B \) and \( f_B(\tilde{r}_B, \tilde{v}_B; t) \) are normally given in terms of approximations.

Note that we have used explicitly the appearance of force terms, although the starting equations are \( \frac{d\tilde{v}_B(t)}{dt} = \frac{1}{m_B} \tilde{f}_B \). Therefore, in looking for a Lagrangian that approximates the system (1) and (2) we observe two attitudes: one in which one uses in the above expressions \( \frac{d\tilde{v}_B(t)}{dt} \), and the other in which one uses \( \frac{1}{m_B} \tilde{f}_B(\tilde{r}_B, \tilde{v}_B; t) \). When using the first approach, one is concerned with the apparition, in the Lagrangian, of terms of the form

\[
\frac{d\tilde{v}_B}{dt}, \tilde{\mathcal{v}}_A,
\]

(31)

and

\[
\frac{d\tilde{v}_B}{dt}, \tilde{\mathcal{v}}_A.
\]

(32)

The first type probably might be dealt with by adding to the Lagrangian a new term involving the time derivative of \( \tilde{v}_B - \tilde{\mathcal{v}}_A \). But the second type, seems much more difficult to handle. Terms of this type seem to pose an obstruction for building the desired Lagrangian with this approach.

Instead when using the second approach, the introduction of expressions involving \( \frac{1}{m_B} \tilde{f}_B(\tilde{r}_B, \tilde{v}_B; t) \) will not introduce any problem form the Lagrangian program point of view, since all the expressions still depend on positions and velocities only. In a sense one is introducing more precision at the cost of introducing higher order expressions in the forces; since these expressions appear in the interactions terms of the Lagrangian.

The first approach was used in reference [1]; and we have seen that this Lagrangian model simplifies severely the dynamics, in several ways. So, we recommend the second approach based on the approximation given by [25], [28] and [30], or their higher order versions; since, although it introduces higher order force terms in the Lagrangian, it is based on a more precise approximation to the original dynamics of equations [1] and [2].

4 Building an order two retarded approach

Although we have improved over reference [1] by taking into account the higher order retardation effects on the accelerations, we still share the shortcoming associated to the fact that we are using a second order Taylor expansion for the position around time \( t \). It is not clear at this stage what cumulative effects this will produce in the dynamics of the solutions to the approximate equations of motion, based on this Taylor expansion. The question is: suppose that instead to calculate the retarded times based on a second order Taylor expansions of the positions, we use an approximation of the trajectory based on a higher order of accuracy Runge-Kutta (R-K) time step calculation; then, how does the dynamic changes in the equation of motion based in this new approximation? Is it possible to improve on the previous estimate, by considering more evaluations of the force? A Runge-Kutta like method are techniques developed for ‘first order’ ordinary differential equations. This in principle would introduce much complication in the calculation since one has to consider in the analysis the retarded effects at each step. But, since the problem we are concerned with is actually a ‘second order’ differential equation, it is worthwhile to review the R-K logic, to see if one can improve in the calculation of the retarded times \( \Delta t \), and in the evaluation of the forces depending on a single dynamical time.

The R-K integration methods deal with ‘first order’ ordinary differential equations; namely

\[
\frac{dx}{dt} = d(t, x);
\]

(33)

where \( x \) is defined in terms of position and velocities of the point like objects [3].

Let us review here the second order R-K method[4] given by:

\[
x(t + h) = x(t) + w_1 k_1 + w_2 k_2;
\]

(34)

with

\[
k_1 = \alpha h d(t, x),
\]

(35)

\[
k_2 = \beta h d(t + \alpha h, x + \beta k_1);
\]

(36)

where we could take [3][4]

\[
\alpha = \beta = \frac{2}{3},
\]

(37)

\[
w_1 = \frac{1}{4},
\]

(38)

\[
w_2 = \frac{3}{4}.
\]

(39)

In the case of a ‘second order’ ordinary differential equation one has to solve

\[
\frac{d^2 y}{dt^2} = f(t, y, \frac{dy}{dt});
\]

(40)

\footnote{We will refer to the “order” of the R-K method in italics to differentiate it from the order of the retarded effects; and we refer to ‘first order’ or ‘second order’ of the ordinary differential equation between quotes, also to differentiate from the other uses of the word “order”.
}
so that normally one transforms to the ‘first order’ version by defining
\[ x = \left( \frac{\partial y}{\partial t} \right); \]  
(41)
so that
\[ d(t, x) = \left( \frac{\partial f}{\partial y} \right). \]  
(42)
Then, using the R-K procedure one has
\[
\begin{align*}
\left( \frac{\partial y}{\partial t} (t+h) \right) & = \left( \frac{\partial y}{\partial t} (t) \right) + w_1 h \left( f(t, y, \frac{\partial y}{\partial t}) \right) \\
& \quad + w_2 h \left( f(t + \alpha h, y + \beta h \frac{\partial y}{\partial t}, \frac{\partial y}{\partial t} + \beta h f(t, y, \frac{\partial y}{\partial t})) \right);
\end{align*}
\]  
(43)
that is, for the position one has
\[
y(t+h) = y(t) + w_1 h \frac{\partial y}{\partial t} + w_2 h \left( \frac{\partial y}{\partial t} + \beta h f(t, y, \frac{\partial y}{\partial t}) \right); \]  
(44)
from which, in order to agree with the second order Taylor expansion, we need
\[ w_1 + w_2 = 1, \]  
(45)
and
\[ w_2 \beta = \frac{1}{2}; \]  
(46)
which are part of the R-K conditions, but without requirements on \( \alpha \).

In other words, if we had carried out the calculation of the retarded times \( \Delta t \) in terms of a second order R-K calculation, we would have arrived at the same results as above.

But the order two R-K scheme also introduces an improvement on the calculation of the retarded velocity, which enters as argument in the force at time \( t \). However the new evaluation of the force at time \( t + \alpha h = t + \frac{2}{3} \Delta t \), forces another evaluation of the force of the other particle, at an earlier retarded time. We could use here the order one forces, for this task.

That is, we now evaluate
\[
\begin{align*}
\vec{v}_B(t - \Delta t_B) &= \vec{v}_B(t) - \frac{1}{3} \Delta t_B \frac{1}{m_B} \vec{f}_B1(\vec{r}_B, \vec{v}_B; t) \\
& \quad - \frac{3}{4} \Delta t_B \frac{1}{m_B} \vec{f}_B1 \left( \vec{r}_B - \frac{2}{3} \Delta t_B \vec{v}_B, \vec{v}_B \right) \\
& \quad - t \frac{2}{3} \Delta t_B \vec{v}_B \\
& \quad - \frac{2}{3} \Delta t_B \vec{f}_B1(t, \vec{r}_B, \vec{v}_B); \\
& \quad - \frac{2}{3} \Delta t_B \vec{f}_B1(t, \vec{r}_B, \vec{v}_B);
\end{align*}
\]  
(47)
In the evaluation of \( \vec{f}_B1(., t - \frac{2}{3} \Delta t_B) \) it is required the knowledge of the value of the kinematical variables of particle \( A \) at retarded time \( t - \Delta t_A \) given by
\[
(t - \frac{2}{3} \Delta t_B) - (t - \Delta t_A) = \\
\Delta t_A - \frac{2}{3} \Delta t_B = |\vec{r}_B(t - \frac{2}{3} \Delta t_B) - \vec{r}_A(t - \Delta t_A)|/c; \\
\]  
(48)
Figure 1: Sketch of two arbitrary world lines and corresponding retarded times. The evaluation of the force, for particle \( B \), at retarded time \( t - \frac{2}{3} \Delta t \) requires the estimation of the position of particle \( A \) at a previous retarded time \( t - \Delta t_A \), not drawn in the figure.

where we are taking
\[
\vec{r}_A(t - \Delta t_A) = \vec{r}_A(t) - \Delta t_A \vec{v}_A(t) + \frac{\Delta t_A^2}{2 m_A} \vec{f}_A1. \]  
(49)
In the evaluation of \( \Delta t_A \) we can either use the iterative approach presented previously, or we can solve exactly the quartic equation.

With this then we define the order two retarded force by
\[
\begin{align*}
\vec{f}_{A2}(\vec{r}_A, \vec{v}_A; t) &= \vec{f}_A \left( \vec{r}_A, \vec{v}_A; \right) \\
& \quad \vec{r}_B(t - \Delta t_B1), \\
& \quad \vec{v}_B(t - \Delta t_B1), \\
& \quad \vec{a}_B(t - \Delta t_B1), \\
& \quad \vec{a}_B(t - \Delta t_B1),
\end{align*}
\]  
(50)
where we estimate
\[
\begin{align*}
\vec{r}_B(t - \Delta t_B1) &= \vec{r}_B(t) - \Delta t_B1 \vec{v}_B(t) + \frac{\Delta t_B1^2}{2 m_B} \vec{f}_B1, \\
\vec{v}_B(t - \Delta t_B1) &= \vec{v}_B(t) - \frac{1}{4} \Delta t_B1 \vec{f}_B1(\vec{r}_B, \vec{v}_B; t) \\
& \quad - \frac{3}{4} \Delta t_B1 \vec{f}_B1 \left( \vec{r}_B - \frac{3}{4} \Delta t_B1 \vec{v}_B, \vec{v}_B \right) \\
& \quad - \frac{2}{3} \Delta t_B1 \vec{v}_B, \\
& \quad \vec{r}_A(t - \Delta t_A), \\
& \quad \vec{v}_A(t - \Delta t_A), \\
& \quad \vec{a}_A(t - \Delta t_A),
\end{align*}
\]  
(51)
and
\[
\vec{a}_B(t - \Delta t_B1) = \frac{1}{m_B} \vec{f}_B1(t, \vec{r}_B, \vec{v}_B); \\
\]  
(53)
with \( r_A(t - \Delta t_A) \) given by (49), and

\[
\vec{v}_A(t - \Delta t_A) = \vec{v}_A(t) - \frac{\Delta t_A}{m_A} \vec{f}_{A1}, 
\]

and

\[
\vec{a}_A(t - \Delta t_A) = \frac{1}{m_A} \vec{f}_{A1},
\]

Similarly, for particle \( B \), one has the corresponding order two force given by

\[
\vec{f}_{B2}(\vec{r}_B, \vec{v}_B; t) = \vec{f}_B(\vec{r}_B, \vec{v}_B);
\]

\[
\vec{r}_{A2}(t - \Delta t_{A1}) = \vec{r}_A(t) - \Delta t_{A1} \vec{v}_A(t) + \frac{\Delta t_{A1}^2}{2 m_A} \vec{f}_{A1},
\]

\[
\vec{v}_{A2}(t - \Delta t_{A1}) = \vec{v}_A(t) - \frac{\Delta t_{A1}}{4 m_A} \vec{f}_{A1}(t, \vec{r}_A, \vec{v}_A)
\]

\[
- \frac{3 \Delta t_{A1}}{4 m_A} \vec{f}_{A1}(\vec{r}_A - \frac{2}{3} \Delta t_{A1} \vec{v}_A)
\]

\[
\vec{v}_A - \frac{2 \Delta t_{A1}}{3 m_A} \vec{f}_{A1}(t, \vec{r}_A, \vec{v}_A),
\]

\[
\vec{r}_B(t - \Delta t_{B1}),
\]

\[
\vec{v}_B(t - \Delta t_{B1}),
\]

\[
\vec{a}_B(t - \Delta t_{B1}),
\]

and

\[
\vec{a}_{A2}(t - \Delta t_{A1}) = \frac{1}{m_A} \vec{f}_{A1}(\vec{r}_A, \vec{v}_A; t);
\]

with

\[
\vec{r}_B(t - \Delta t_{B1}) = \vec{r}_B(t) - \Delta t_{B1} \vec{v}_B(t) + \frac{\Delta t_{B1}^2}{2 m_B} \vec{f}_{B1},
\]

\[
\vec{v}_B(t - \Delta t_{B1}) = \vec{v}_B(t) - \frac{\Delta t_{B1}}{4 m_B} \vec{f}_{B1},
\]

and

\[
\vec{a}_B(t - \Delta t_{B1}) = \frac{1}{m_B} \vec{f}_{B1};
\]

where \( \Delta t_{B1} \) is the solution to

\[
(t - \frac{2}{3} \Delta t_A) - (t - \Delta t_{B1}) = \\
\Delta t_{B1} - \frac{2}{3} \Delta t_A = |\vec{r}_A(t - \frac{2}{3} \Delta t_A) - \vec{r}_B(t - \Delta t_{B1})|/c.
\]

We have just presented what constitutes the adaptation of the second order R-K method to the case of a retarded system, for the purpose of providing with an approximate single time set of equations of motion with second order numerical accuracy for time steps of retarded times magnitude. At each step, we have made full use of the previous calculated quantities. One can notice that the evaluation of the couple order two forces requires the previous calculation of \( \vec{f}_{A0}(\vec{r}_A, \vec{v}_A; t); \vec{f}_{B0}(\vec{r}_B, \vec{v}_B; t); \vec{f}_{A1}(\vec{r}_A, \vec{v}_A; t); \vec{f}_{B1}(\vec{r}_B, \vec{v}_B; t); \vec{f}_{A1}(.; t - \frac{2}{3} \Delta t_A) \) and \( \vec{f}_{B1}(.; t - \frac{2}{3} \Delta t_B) \); and since the last two involve another couple of force evaluation, they form a total of eight evaluations of the force functions, instead of the four evaluations one would have in the non-retarded case. The extra evaluations are needed to maintain a second order numerical accuracy of the retarded effects.

All this suggests the following remark. If the retardation effects were small, then one would be tempted to make a numerical evaluation of the dynamics in which the time steps are of the same order of magnitude as the retarded times. But the second order retarded forces respect the second order R-K scheme; which means that from the physical point of view it is enough to carryout the numerical calculation with a second order numerical scheme. Then, from this point of view, it is clear that the standard methods of references [1, 2] do not have a second order numerical precision. So, it seems that it would be a waste of computational resources to make a numerical integration of equations of motions obtained from standard method [1, 2] with a, let us say, fourth order integration scheme; since the dynamical equations where calculated with less numerical precision in the retardation effects. This is irrespective from the fact that given a first order ordinary differential equation, the fourth order integration scheme will show better numerical properties than lower order ones, in general.

5 Applying the model to the binary electromagnetic case

5.1 The Lorentz force case

The binary system of electromagnetic charges provides us with the opportunity to study a couple of interesting physical systems. The first one is provided by the system of interacting particle with retarded fields through the Lorentz force. This is the simplest relativistic binary system one can study, which it can be applied to classical systems of particles with small charges.

In this case, each particle with charge \( q \) generates the electromagnetic field given by

\[
F_{ab} = 2q \left( \frac{1}{rV} l^l_a V_b + \frac{1}{rV^2} (1 - \frac{rV}{V}) l^l_a V_b \right)
\]

\[
= 2q \left( \frac{1}{r} l^l_a V_b - \frac{V l^l_a V_b}{V^2} + \frac{1}{r} l^l_a V_b \right);
\]

where we are using now a four dimensional notation, \( a, b, \ldots \) are abstract indices, a dot means covariant derivative in the direction of the four velocity \( V^l \), \( l \) and \( \dot{l} \) are proportional null vectors pointing from the retarded position to the field point, \( r \) is a retarded distance,
\[ V = \mathring{p} \nu_\mu \] and we have chosen Gaussian units. The notation is expanded in the Appendix.

Then, the forces for this system are

\[ m_A \mathring{v}_A^a = q_A F(B)^a_{\ b} \mathring{v}_A^b; \quad (65) \]

where \( F^a_{\ b}(B) \) is the electromagnetic field tensor generated by particle \( B \), in which all quantities are evaluated at the corresponding retarded time. Similarly one has, for particle \( B \)

\[ m_B \mathring{v}_B^a = q_B F(A)^a_{\ b} \mathring{v}_B^b. \quad (66) \]

These are the four dimensional version of equations [1] and [2]. In the appendix we recall the notation to write these equations of motion with a Galilean language.

We observe in this case that the force is linear in the single coupling constant determined from the product of both charges.

This case is a paradigmatic example, since it has a couple of important properties. Firstly, the field equations are linear in the intervening fields; and it is clear then that the issue of precision in the calculation of the retarded effects is completely unrelated to the subject of the nature of the field equation; since for instance, in this case there is no room for calculations of the field equations to higher orders. Secondly, although the forces seem to involve only linearly the coupling constant; the fact that involves the retarded accelerations means that they are already hiding multiple apparitions of the coupling constant, when the retarded effects are calculated exactly.

5.2 The back reaction force case

If one wants to improve on the physical precision of the binary particle electromagnetic system, one must also consider the effects of back reaction in the equations of motion due to the fact that accelerated charges emit electromagnetic radiation. In reference [5] we have presented the most general form of the equation of motion for charged particles which balance the electromagnetic radiation generated by the motion. We have argued there that the back reaction terms must be understood in terms of orders in the evaluation of the Lorentz force. In particular we gave the explicit form of the forces up to third order. Let us recall here these forces.

We first define the four force vector

\[ f^a_A = q_A F(B)^a_{\ b} \mathring{v}_A^b; \quad (67) \]

and

\[ f^a_A = -f_A f^A_a. \quad (68) \]

Then, we define the second order time derivative four vector \( f_{A(2)}^a \) from

\[ f_{A(2)}^a = q_A (F(B)^a_{\ b} \mathring{v}_B^b + q_A F(B)^a_{\ b} \mathring{v}_A^b) \cdot \frac{1}{m_A} f^a_A. \quad (69) \]

The third order equation of motion for particle \( A \) is given by

\[ m_A \mathring{v}_A^a = f^a_A + \frac{2}{3} q_A^2 (\mathring{v}_A A(2)) - a_{A(2)}^2 \mathring{v}_A^a, \quad (70) \]

where

\[ \mathring{v}_A A(2) = \frac{1}{m_A} f^A_A + \frac{2}{3} q_A^2 \mathring{v}_A A(2) - \frac{2}{3} q_A^2 \mathring{v}_A^a \mathring{v}_A^a, \quad (71) \]

and

\[ a_{A(2)}^2 = -\mathring{v}_A A(2) \mathring{v}_A A(2)\mathring{v}_A^a; \quad (72) \]

and it is understood that in the evaluation of the right hand side of \( f_{A(2)}^a \) it is only required to maintain third order terms in \( f_A \). The corresponding equation of motion for particle \( B \) is obtained from the above by interchanging the indices \( A \leftrightarrow B \) appropriately.

In this case we use a more sophisticated form of the equation of motion with the objective to appropriately describe the effects of back reaction in the motion due to the emission of electromagnetic radiation of the accelerated charges. These techniques involve several evaluations of the Lorentz force and time derivatives of it; but it should be clear that these further evaluations are needed to cover a specific effect, which is completely unrelated to the need of evaluations of the forces to obtain a precise single time dynamical model of a relativistic retarded system.

It has been customary in the literature to assume that the number of evaluations of the forces should be universal for the construction of a model; but we are here presenting arguments against this assumption.

If we do not take into account the retardation effects with high enough precision, in the construction of a single time dynamical model of a relativistic retarded system, then the model will not be able to accurately describe the dynamics of the first order version of the force; and therefore it would be meaningless to correct this dynamics by taking higher order effects, as the back reaction effects discussed in the case just presented.

5.3 The single time dynamical model for binary charged particles

We have just presented two theoretical models for a physical system consisting of two charged particles; which in principle are exactly described in terms of the two proper times of the particles. To each of these theoretical models we can apply the techniques described previously, for the construction of a single time dynamical model. For a summary presentation of how to pass from the four dimensional description to the Newtonian language, we refer to the Appendix.

It is our freedom to choose the degree of accuracy that we would like to require to the single time model; and this choice normally depends only on the nature of initial conditions, and/or on the evolution of the system, but not on the nature of the theoretical model one is using.

6 Final comments

We have presented above the order one retarded forces, and have shown that they contain more dynamical information that the forces obtained by the standard methods, as those of reference [1].
We have also introduced the second order retarded forces; which has even more dynamical information than the first order ones.

The discussions, in section 3, on the possibility to construct a Lagrangian from the equations of motion can be extended to higher order forces, as those introduced in section 4, that we plan to carry out elsewhere.

From the discussion presented above, concerning the degree of accuracy of the single time equations of motion, it suggests that we should revise the standard view that is applied to different kind of approximations for the dynamics of compact objects; since it is customary to base the studies in the choice of a universal degree of approximation, normally based in a power of interactive constants, and/or powers of the velocities. This is normally suggested from the point of view of the nature of field equations, as is done in the gravitational case; and/or from the nature of the equations of motion, as is done in the electromagnetic case. That is, it is customary to first choose a universal power to be applied to any equation in the study, and then determine the dynamics from it. What we are suggesting with our previous discussion is that one should choose the degree of accuracy one desires for the evaluation of the retarded effects, independently from anything else; that is, nature of the field equations or any other independent physical consideration. Since this choice for precision will determine the degree in which the approximated forces will represent the real global physical implications of the retardation effects.

Our first example for the electromagnetic case, of two particles interacting with retarded fields through the Lorentz force, is a clear situation in which: the exact field equations are linear, and also the forces are linear in the fields. However, we can ask for any desired precision in the calculation of the retarded effects, what will be related with several evaluations of the force function along the trajectories, with the corresponding apparition of non-linear interactive terms in the final expressions for the forces. If one had used in this case the customary view point, one would be forced to only use ‘zero order forces’ (in our notation) which will severely restrict the precision of the final dynamical system.

The other electromagnetic case considered in section 5 reinforces the view that the degree needed for the appropriate calculation of different effects must be considered separately. This of course is in contrast to the customary attitude which considers a universal choice of order first, and then proceed with the calculation.

The consideration of binary gravitating systems also involve the issue of the retarded effects; however the retarded effects are rarely considered separately. For example in the seminal article [6] that derive the first order (PN), and then calculate all the dynamics in terms of this choice, these frameworks do not incorporate the retarded effects we have presented here. Naturally, when considering gravitating systems, the problem of the retarded effects complicates considerably when higher order geometries are taken into account; since the past null cones are calculated in the corresponding curved spacetime.

Although we have presented the order one and order two set of equations; it is clear that the procedure presented here can be extended to any order of accuracy one desires. This can be done, for example, by taking a standard nth order Runge-Kutta method, and at each stage use the previous elements to calculate the needed retarded times; in analogy to what we have presented here. As we have seen, to obtain a nth R-K order of accuracy we need about $2^n$ number of force evaluations for each particle.

It is worthwhile to remark, that our work gives an answer to the unsolved problem stated in [8]; namely we give a constructive way for the initial value problem of a set of relativistic particles with mechanical initial data, i.e. position and velocities, with arbitrary desired precision.

We plan to apply this discussion to numerical calculation of binary charged particle systems and to the problem of equations of motion for gravitating systems in future works.

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A Relations among coordinates and null vectors

A.1 The inertial system

Let us denote with $y^\mu$ the standard Cartesian coordinates and with $(\hat{x}^0 = \tilde{\alpha}, \hat{x}^1 = \tilde{\rho}, \hat{x}^2 = \hat{\sigma}, \hat{x}^3 = \hat{\phi})$, where $(\hat{x}^2, \hat{x}^3) = (\theta, \phi)$ or $\zeta = \sqrt{\hat{x}^2 + \hat{x}^3}$, the corresponding null polar coordinates; then, the relation between them is given by

$$y^\mu = \tilde{\alpha} \delta^\mu_0 + \hat{\rho} \tilde{\rho}^\mu (\zeta, \bar{\zeta});$$

with

$$\tilde{\rho}^\mu (\zeta, \bar{\zeta}) = \tilde{\rho}^\mu (\zeta, \bar{\zeta});$$

(74)
defined by
\[ \hat{P}_0(x^2, x^3) = \left(1, \sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta)\right) \]
\[ = \left(1, \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}}, \frac{\zeta - \bar{\zeta}}{1 + \zeta \bar{\zeta}}\right) \]
\[ = \left(\sqrt{2\pi} Y_{0,0}, -\frac{2\pi}{3} Y_{1,1} - Y_{1,-1}, \frac{4\pi}{3} Y_{0,1}\right); \quad (75) \]
where \(\mu, \nu, \cdots\) are \(0, 1, 2, 3\), and we are using either the standard sphere angular coordinates \((\theta, \phi)\) or the complex angular coordinates \(\zeta = \frac{r^2 + i z^2}{2}\), where \((\zeta, \bar{\zeta})\) are complex stereographic coordinates of the sphere; which are related to the standard coordinates by \(\zeta = e^{i\phi} \cos(\frac{\theta}{2})\).

The \(Y_{l,m}\) are the usual spherical harmonic on the sphere.

A.2 The intrinsic non-inertial system

Let \(z^\mu(\tau_0)\) be the evolution of the particle, in terms of a Cartesian coordinate system, with proper time \(\tau_0\). We define a null function \(u^\mu\) as the future null cones emanating from \(z^\mu(\tau_0)\), such that \(u^\mu = \tau_0\) at the world line of the particle.

If \((x^0, x^1, x^2, x^3)\), where \((x^2, x^3) = (\theta', \phi')\) or \(\zeta' = \frac{x^2 + i x^3}{2}\), are the null polar coordinates adapted to an arbitrary timelike curve, determined by \(z(u^\mu)^\mu\), then one has
\[ y^\mu = z^\mu(u^\mu) + r P^\mu(u^\mu, \zeta', \bar{\zeta}); \quad (76) \]

Note that
\[ (y^\mu - z^\mu(u^\mu))|_\mu = r P^\mu|_\mu = 0, \quad (77) \]
and that
\[ (y^\mu - z^\mu(u^\mu)) v_\mu = r P^\mu v_\mu = r; \quad (78) \]
so that
\[ P^\mu = \frac{y^\mu - z^\mu(u^\mu)}{(y^\mu - z^\mu(u^\mu)) v_\mu}. \quad (79) \]

Given a fixed point \(y^\mu\) one has to take different spacelike directions, and therefore different angular coordinates, for the two null vectors to reach the fixed point. But, if given a particular future null cone determined by the apex \(z(u^\mu)\), one also chooses an inertial frame with origin at this apex, then, from equations (73) and (76) one deduces that at this cone the two null vectors \(P^\mu(u^\mu, \zeta')\) and \(\hat{P}_0^\mu(\zeta', \bar{\zeta}')\) must be proportional; so that
\[ P^\mu(u^\mu, \zeta', \bar{\zeta}') = \alpha(u^\mu, \zeta', \bar{\zeta}') \hat{P}_0^\mu(\zeta', \bar{\zeta}'); \quad (80) \]
with \(\alpha > 0\). But then we have
\[ 1 = \alpha P^\mu v_\mu = \alpha P_0^\mu v_\mu; \quad (81) \]
which implies that
\[ \frac{1}{\alpha} = V; \quad (82) \]

with
\[ V \equiv \hat{P}_0^\mu v_\mu, \quad (83) \]
and also
\[ \nu^\mu(u^\mu, \zeta', \bar{\zeta}') = \frac{1}{V(u^\mu, \zeta', \bar{\zeta}') \hat{P}_0^\mu(\zeta', \bar{\zeta}').} \quad (84) \]

A.3 Basic relations for coordinate velocities and accelerations

We use for the four velocity the notation
\[ \nu^\mu = \frac{dx^\mu}{d\tau}; \quad (85) \]
where \(\tau_0\) is the proper time with respect to the flat metric, and
\[ \bar{v} = (v^i) = \left(\frac{dx^i}{dt}\right), \quad (86) \]
for \(i = 1, 2, 3\) and \(t = x^0\). Then, one has
\[ \nu^\mu = \frac{1}{\sqrt{1 - v^2}}(1, v^i) = \Gamma(1, \bar{v}); \quad (87) \]
where we use the notation \(\Gamma = \frac{d\tau}{d\tau_0} = \frac{1}{\sqrt{1 - v^2}}\) and \(v^2 = \bar{v} \cdot \bar{v}\). With this notation one can also express the four acceleration as
\[ a^\mu = \frac{dv^\mu}{d\tau} = \Gamma^3 v^\mu = \Gamma^2(0, \bar{a}); \quad (88) \]
where \(a^i = (a^i) = \left(\frac{dv^i}{dt}\right) = \bar{d} \bar{v} \bar{d} \bar{v} \bar{d} \bar{v} \bar{d} \bar{v}; \quad (89) \]
and
\[ v^2 \frac{dv}{dt} = \frac{1}{2} d \bar{v} \bar{v} = \bar{a} \bar{v}; \quad (91) \]
so that one has
\[ a^\mu = \Gamma^4(0, \bar{v}) + \Gamma^2(0, \bar{a}) \]
\[ = \Gamma^4(\bar{a} \bar{v}), \Gamma^4(\bar{a} \bar{v}) \bar{v} + \Gamma^2(\bar{a}); \quad (92) \]
We also use the notation
\[ a^\mu a_\mu = -a^2; \quad (93) \]
so that \(a^2\) is a positive quantity.

From the point of view of equations of motion, the physical important quantity is the momentum of the particle defined by
\[ p^\mu = \nu v^\mu; \quad (94) \]
and the equation of motion is written in the form
\[ \frac{dp^\mu}{d\tau} = f^\mu. \quad (95) \]
In the case of the Lorentz force one can write
\[ \frac{dp^\mu}{d\tau} = \Gamma \frac{dp^\mu}{dt} = q F^\mu_\nu v^\nu; \] (96)
therefore for the spacelike components we can write the equation of motion in the form
\[ \frac{dp^i}{dt} = q F^i_\nu \frac{1}{\Gamma} v^\nu; \] (97)
where we have seen that \( \frac{1}{\Gamma} v^\nu = (1, \vec{v}) \); so that
\[ \frac{dp^i}{dt} = q \left( E^i + (\vec{v} \times \vec{B})^i \right); \] (98)
which is the standard way to write the Lorentz force in terms of the three dimensional variable. And using that \( p^i = \Gamma m v^i \), one can also write
\[ \frac{dp^i}{dt} = \Gamma m \vec{a} + \frac{d\Gamma}{dt} m \vec{v} = \Gamma m \vec{a} + \Gamma^3 (\vec{a} \cdot \vec{v}) m \vec{v}; \] (99)
so that in terms of the standard acceleration Newtonian form one can express the equation of motion as
\[ m \vec{a} = q \frac{1}{\Gamma} \left( \vec{E} + \vec{v} \times \vec{B} \right) - \Gamma^2 (\vec{a} \cdot \vec{v}) m \vec{v}; \] (100)
from which we can see that
\[ m \vec{a} \cdot \vec{v} (1 + \Gamma^2 v^2) = m \vec{a} \cdot \vec{v} \Gamma^2 = q \frac{1}{\Gamma} \vec{E} \cdot \vec{v}; \] (101)
so that the final equation of motion in Newtonian notation is
\[ m \vec{a} = q \frac{1}{\Gamma} \left( \vec{E} + \vec{v} \times \vec{B} \right) - q \frac{1}{\Gamma} (\vec{E} \cdot \vec{v}) \vec{v}; \] (102)
which is seldom shown explicitly in textbooks[9].

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