Pauli-type coupling between spinors and curved spacetime

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On the basis of the regularized Dirac Lagrangian, we derive the Pauli interaction term of the subsequent field equation from the minimal coupling of the spinor $\psi$ to an external electromagnetic field $A_\mu$. An analogous coupling term emerges from the spinor’s coupling to the spinor connection $\omega_\mu(x)$ in a curved spacetime. In the ensuing field equation for the spinor, one thereby encounters an additional effective mass term, which is associated with a particular coupling constant $M$. This modifies the description of the dynamics of spinors in a gravitational field. For neutrinos, this could explain measurements indicating that all flavors of neutrinos apparently exhibit small but non-zero rest masses.

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I. Introduction. The regularized Dirac Lagrangian $\mathcal{L}$\cite{1,2,3}, which can be converted into an equivalent covariant Dirac Hamiltonian $\mathcal{H}$ by means of a regular Legendre transformation, differs from the conventional Dirac Hamiltonian\cite{1–3}, which can be converted into an equivalent covariant Hamiltonian description\cite{2,3} necessarily embraces the quadratic term which there appears as a term quadratic in the canonical momenta $\pi^\alpha$ and $\pi^\beta$:

$$\mathcal{H}_0 = \left(\pi^\alpha - \frac{i}{2} \bar{\psi} \gamma^\alpha \psi\right) \frac{3M\tau_{\alpha\beta}}{i} \left(\pi^\beta + \frac{i}{2} \bar{\gamma}^\beta \psi\right) + m\bar{\psi}\psi.$$

(2)

$\tau_{\alpha\beta}$ is the inverse of the matrix $\sigma^{\alpha\beta}$,

$$\tau_{\alpha\beta} = \frac{i}{6} \left(\gamma_\alpha \gamma_\beta + 3\gamma_\beta \gamma_\alpha\right), \quad \tau_{\mu\alpha}\sigma^{\mu\nu} = \delta^\nu_\nu 1,$$

which entails the identities

$$\gamma^\alpha \tau^\alpha_\beta = \frac{1}{3i} \gamma^\beta , \quad \tau^\alpha_\beta \gamma^\alpha = \frac{1}{3i} \gamma^\beta ,$$

$$\gamma^\alpha \sigma^{\alpha\beta} = 3\gamma^\beta , \quad \sigma^{\alpha\beta} \gamma^\beta = 3\gamma^\alpha.$$

Similar to the Dirac Hamiltonian, the regularized Dirac Lagrangian\cite{1} can equivalently be written in the symmetric form

$$\mathcal{L}_0 = \left(\bar{\psi} \left(\frac{\partial}{\partial x^\alpha} - \frac{iM}{2} \bar{\gamma}^\alpha \psi\right) \right) \frac{\sigma^{\alpha\beta}}{3M} \left(\bar{\psi} \left(\frac{\partial}{\partial x^\beta} + \frac{iM}{2} \gamma^\beta \psi\right)\right) - (m - M) \bar{\psi}\psi$$

(3)

which suggests a minimum coupling of the spinor to the $\gamma$-matrices with coupling constant $M/2$. Setting up the covariant canonical equations

$$\frac{\partial \psi}{\partial x^\mu} = \frac{\partial \mathcal{H}}{\partial \pi^\mu}, \quad \frac{\partial \bar{\psi}}{\partial \pi^\mu} = - \frac{\partial \mathcal{H}}{\partial \psi}, \quad \frac{\partial \pi^\alpha}{\partial \bar{\psi}} = - \frac{\partial \mathcal{H}}{\partial \pi^\alpha}$$

(4)

for the Hamiltonian\cite{2} or the Euler-Lagrange equations

$$\frac{\partial}{\partial x^3} \left(\frac{\partial \mathcal{L}}{\partial \bar{\psi}}\right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0, \quad \frac{\partial}{\partial x^3} \left(\frac{\partial \mathcal{L}}{\partial \psi}\right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0$$

(5)

for the Lagrangian\cite{1} or\cite{3}, one encounters the common Dirac equations in a static spacetime as all terms depending on $M$ cancel

$$i \frac{\partial \psi}{\partial x^\alpha} \gamma^\alpha + m\bar{\psi} = 0, \quad i\gamma^\beta \frac{\partial \bar{\psi}}{\partial x^\beta} - m\psi = 0.$$

(6)

We note that in the case of the Hamiltonian description the term quadratic in the canonical momenta is mandatory as otherwise no correlation is obtained between canonical momenta and “velocities”, i.e., the partial derivatives of the spinors.

II. Coupling to EM. We know from standard U(1) gauge theory that a spinor couples minimally to an electromagnetic field $A_\mu$ with coupling constant $q$

$$\frac{\partial \bar{\psi}}{\partial x^\alpha} \gamma^\alpha \rightarrow \frac{\partial \bar{\psi}}{\partial x^\alpha} + iq A_\alpha, \quad \frac{\partial \psi}{\partial x^\beta} \gamma_\beta \rightarrow \frac{\partial \psi}{\partial x^\beta} - iq A_\beta \psi,$$

(7)

whereas $-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$ introduced by hand to describe the dynamics of the “free” external electromagnetic field, hence its dynamics in the absence of any spinor $\psi$

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}.$$

This system is described by the generalized QED Lagrangian

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\[ L_1 = \left( \frac{\partial \bar{\psi}}{\partial x^\alpha} + ig \bar{\psi} A_\alpha - \frac{iM}{2} \bar{\psi} \gamma_\alpha \right) \left( \frac{i\sigma^{\alpha\beta}}{3M} \left( \frac{\partial \psi}{\partial x^\beta} - ig A_\beta \psi + \frac{iM}{2} \gamma_\beta \psi \right) \right) - (m - M) \bar{\psi} \psi - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta}, \tag{8} \]

which writes in expanded form

\[ L_1 = \frac{i}{2} \bar{\psi} \gamma^\alpha \left( \frac{\partial \psi}{\partial x^\alpha} - ig A_\alpha \psi \right) - \frac{i}{2} \left( \frac{\partial \bar{\psi}}{\partial x^\alpha} + ig \bar{\psi} A_\alpha \right) \gamma^\alpha \psi - m \bar{\psi} \psi - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} - \frac{ig}{3M} \bar{\psi} \gamma^\alpha \gamma^\beta \psi F_{\alpha\beta} \]

\[ + \frac{\partial \bar{\psi}}{\partial x^\alpha} \frac{i\sigma^{\alpha\beta}}{3M} \frac{\partial \psi}{\partial x^\beta} + \frac{\partial G^\alpha}{\partial x^\alpha}, \text{ where } G^\alpha = \frac{q}{3M} \bar{\psi} \sigma^{\alpha\beta} \psi A_\beta. \tag{9} \]

The last two terms do not contribute to the subsequent Euler-Lagrange field equations. The equivalent covariant Hamiltonian is then

\[ H_1 = \left( \bar{\pi}^\alpha - \frac{i}{2} \bar{\psi} \gamma^\alpha \right) \frac{3M \tau^{\alpha\beta}}{1} \left( \pi^\beta + \frac{i}{2} \gamma^\beta \psi \right) + m \bar{\psi} \psi - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + i q \left( \bar{\pi}^\alpha A_\alpha - \bar{\psi} A_\alpha \pi^\alpha \right), \tag{10} \]

which writes in terms of the Dirac Hamiltonian \( H_0 \) proper from Eq. \( \text{(2)} \)

\[ H_1 = H_0 - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + H_g, \]

wherein the “gauge Hamiltonian” \( H_g \) describes the coupling of the Dirac spinor to the electromagnetic field \( A_\alpha \),

\[ H_g = i q \left( \bar{\pi}^\alpha A_\alpha - \bar{\psi} A_\alpha \pi^\alpha \right). \]

The term proportional to \( \bar{\psi} \gamma^\alpha \gamma^\beta \psi F_{\alpha\beta} \) in Eq. \( \text{(8)} \) is exactly the interaction term proposed by Pauli [1], which is apparently a non-minimal coupling term. As the Lagrangians \( \text{(8)} \) and \( \text{(9)} \), and the Hamiltonian \( \text{(10)} \) are classically equivalent, one finds that the Pauli term can be cast into the minimal coupling form of the Lagrangian \( \text{(9)} \) by adding two terms, namely the “Gasiorowicz term” and the divergence \( \partial G^\alpha / \partial x^\alpha \), both of which do not contribute to the ensuing Euler-Lagrange field equations [5]. The Pauli interaction term in Eq. \( \text{(9)} \) satisfies the requirement to be Hermitian [5]. Also note that the Hamiltonian description \( \text{(10)} \) necessarily includes the Pauli coupling effect.

The coupling constant of the Pauli interaction has dimension of inverse mass. As is discussed by Weinberg [6], the corresponding Pauli interaction formally represents a non-renormalizable interaction. However, the actual contribution of a Pauli-type interaction to physical processes is suppressed by factors of \( k / M \), if \( k \) characterizes the momentum scale and \( k \ll M \). In order to be effective, such a suppression requires that divergencies at high energy scales were removed in the procedure of renormalization. As an example illustrating the successful calculation of a term formally equivalent to the Pauli term, one can refer to Schwinger’s result [7] for the one-loop contribution to the anomalous magnetic moment of a point-like fermion. Without fixing the mass scale \( M \), Weinberg estimates that the contribution of a Pauli term to the anomalous magnetic moment could be of order \( 4qM / M^2 \) if the Pauli term in the Lagrangian \( \text{(1)} \) is to preserve the chiral symmetry \( \psi \rightarrow \gamma_5 \psi, \ m \rightarrow -m \). In this context, it is interesting to note that the experimental value of the anomalous magnetic moment \( \delta \mu \) of the muon deviates from the theoretical prediction of the Standard Model at a level of 3.5 \( \sigma \) [8]. An additional contribution to \( \delta \mu \) from the Pauli term might also lead to the observed deviation.

Setting up the canonical equations \( \text{(4)} \) for the Hamiltonian \( \text{(10)} \), or the Euler-Lagrange equations \( \text{(5)} \) for the Lagrangian \( \text{(8)} \) or \( \text{(9)} \), one encounters the field equations

\[ i \gamma^\alpha \frac{\partial \psi}{\partial x^\alpha} + q A_\alpha \gamma^\alpha \psi - m \psi + q \frac{F_{\alpha\beta}}{6M} \sigma^{\alpha\beta} \psi = 0 \tag{11} \]

\[ \frac{1}{2} \frac{\partial \bar{\psi}}{\partial x^\alpha} \gamma^\alpha - q \bar{\psi} \gamma^\alpha A_\alpha + m \bar{\psi} - \frac{q}{6M} \bar{\psi} \sigma^{\alpha\beta} F_{\alpha\beta} = 0. \]

With \( M = 2m/3 \) and \( \mu = q/2m \) the spinor’s magneton, the last term on the left-hand side of Eq. \( \text{(11)} \) writes, explicitly,

\[ \frac{\mu}{2} F_{\alpha\beta} \sigma^{\alpha\beta} \psi. \tag{12} \]

This term describes an additional coupling of the electromagnetic field with the anomalous magnetic moment of the fermion \( \psi \). We conclude that the “Gasiorowicz term”, i.e., the last term of Eq. \( \text{(1)} \)—which does not modify the subsequent field equation of the pure fermion system—actually modifies the field equation of the minimally coupled system \( \text{(8)} \).

III. Coupling to Spacetime. As was shown by e.g. Frankel [9], the transition of the Dirac equations \( \text{(6)} \) in a fixed flat spacetime background to the Dirac equations in a dynamic curved spacetime can be formulated applying the minimal coupling rules

\[ \frac{\partial \psi}{\partial x^\beta} \rightarrow \frac{\partial \psi}{\partial x^\beta} - \omega_\beta \psi, \quad \frac{\partial \bar{\psi}}{\partial x^\alpha} \rightarrow \frac{\partial \bar{\psi}}{\partial x^\alpha} + \bar{\psi} \omega_\alpha, \tag{13} \]

with

\[ \omega_\alpha (x) = \frac{i}{4} \sigma^{kj} \omega_{k\alpha}(x) \tag{14} \]
the spinor connection and $\omega_{k\alpha}(x)$ the spin connection. As a matter of fact, the Hamiltonian formulation of gauge theory of gravity for spin-$\frac{1}{2}$ particles adds up in the Lagrangian formulation to replace the partial derivatives of the spinors in the regularized Lagrangian according to the prescriptions, in conjunction with their respective derivatives. The Euler-Lagrange equation for the spinor follows from the prescriptions, in conjunction with their respective derivatives. The Euler-Lagrange equation for the spinor follows from

$$\mathcal{L}_2 = \left( \frac{\partial \bar{\psi}}{\partial x^\alpha} + \bar{\psi} \omega_\alpha - \frac{iM}{2} \bar{\psi} \gamma_\alpha \right) \frac{i\sigma^{\alpha\beta}}{3M} \left( \frac{\partial \psi}{\partial x^\beta} - \omega_\beta \psi + \frac{iM}{2} \bar{\psi} \gamma_\beta \right) - (m - M) \bar{\psi} \psi \sqrt{-g} + \mathcal{L}_{\text{Dyn}}. \quad (15)$$

Herein $\mathcal{L}_{\text{Dyn}}(\partial \omega, \omega, \partial \gamma, \gamma)$ denotes the scalar density Lagrangian for the dynamics of the “free” gravitational field, which is expressed here in terms of the spinor connection and the spacetime-dependent $\gamma^\mu(x)$-matrices, in conjunction with their respective derivatives. The Euler-Lagrange equation for the spinor follows from (15) as

$$i\gamma^\alpha \left( \frac{\partial \psi}{\partial x^\alpha} - \omega_\alpha \psi \right) - m \psi + \frac{i}{2} \left( \frac{\partial \bar{\psi}}{\partial x^\alpha} + \bar{\psi} \omega_\alpha - \omega_\alpha \gamma^\alpha + \gamma^\alpha \Gamma^\xi_{\xi\alpha} \right) \psi + \frac{i}{3M} \left[ \sigma^{\alpha\beta} \left( \frac{\partial \omega_\beta}{\partial x^\alpha} - \omega_\alpha \omega_\beta \right) \psi - \left( \frac{\partial \sigma^{\alpha\beta}}{\partial x^\alpha} + \sigma^{\alpha\beta} \omega_\alpha - \omega_\alpha \sigma^{\alpha\beta} + \sigma^{\alpha\beta} \Gamma^\xi_{\xi\alpha} \right) \left( \frac{\partial \psi}{\partial x^\beta} - \omega_\beta \psi \right) \right] = 0, \quad (16)$$

where $\Gamma^\xi_{\xi\alpha}$ stands for the affine connection, which emerges from the derivative of $\sqrt{-g}$

$$\frac{\partial \sqrt{-g}}{\partial x^\alpha} = \Gamma^\xi_{\xi\alpha} \sqrt{-g}.$$  

Equation (16) describes the spinor dynamics in a curved spacetime with spacetime-dependent $\gamma^\mu$-matrices by its coupling to the gauge field $\omega_\alpha$. According to Refs. [11][12], metric compatibility, i.e., a covariantly conserved metric $\partial_{\langle\mu\nu\rangle} \equiv 0$, is warranted provided that $\mathcal{L}_{\text{Dyn}}$ of Eq. (15) is postulated not to depend on the derivatives $\partial_\mu \gamma^\mu$ of the dynamic Dirac matrices $\gamma_\mu(x)$. Then, the field equation for the spacetime-dependent $\gamma$ matrices follows as

$$\gamma_{\mu\nu} = \frac{\partial \gamma^\mu}{\partial x^\nu} - \gamma^\xi \Gamma^\mu_{\xi\nu} = \omega_\nu \gamma^\mu - \gamma^\mu \omega_\nu,$$

hence

$$\gamma^\beta_{\alpha \nu} = \frac{\partial \gamma^\beta}{\partial x^\alpha} + \gamma^\gamma \Gamma^\beta_{\gamma\alpha} = \omega_\nu \gamma^\beta - \gamma^\beta \omega_\nu$$

and

$$\sigma^{\alpha\beta}_{\alpha \nu} = \frac{\partial \sigma^{\alpha\beta}}{\partial x^\alpha} + \sigma^{\alpha\mu} \Gamma^\beta_{\mu\alpha} = \omega_\nu \sigma^{\alpha\beta} - \sigma^{\alpha\beta} \omega_\nu.$$

The generalized Dirac equation thus simplifies for metric compatibility to

$$\left[ i\gamma^\alpha - \frac{i}{3M} \left( \sigma^{\beta\xi} s^{\alpha}_{\beta \xi} - 2 \sigma^{\alpha\beta} \xi_{\beta \xi} \right) \right] \left( \frac{\partial \psi}{\partial x^\alpha} - \omega_\alpha \psi \right) - \left( m \mathbf{1} - i\gamma^\alpha s^{\beta}_{\beta \alpha} - \frac{i}{3M} \sigma^{\alpha\beta} \left( \frac{\partial \omega_\beta}{\partial x^\alpha} - \omega_\alpha \omega_\beta \right) \right) \psi = 0, \quad (17)$$

wherein $s^{\beta}_{\alpha \xi} = \Gamma^\beta_{\beta \alpha \xi}$ defines the Cartan torsion tensor. One thus encounters three terms describing a direct coupling of the spin with a torsion of spacetime. The spinor connection $\omega_\alpha$ in Eq. (21) defines the curvature spinor $k_{\alpha \beta}$ according to

$$k_{\alpha \beta} = \frac{\partial \omega_\beta}{\partial x^\alpha} + \omega_\beta \omega_\alpha - \omega_\alpha \omega_\beta, \quad (18)$$

The curvature spinor $k_{\alpha \beta}$ is related to the Riemann curvature tensor via

$$k_{\alpha \beta} = \frac{1}{4} R_{\xi\eta\alpha\beta} \xi^\xi \eta^\eta = -\frac{1}{4} R_{\xi\eta\alpha\beta} \xi^\xi \eta^\eta. \quad (19)$$

Equation (17) can thus be written equivalently in terms of the Riemann curvature tensor as

$$\left[ i\gamma^\alpha - \frac{i}{3M} \left( \sigma^{\beta\xi} s^{\alpha}_{\beta \xi} - 2 \sigma^{\alpha\beta} \xi_{\beta \xi} \right) \right] \left( \frac{\partial \psi}{\partial x^\alpha} - \omega_\alpha \psi \right) - \left( m \mathbf{1} - i\gamma^\alpha s^{\beta}_{\beta \alpha} - \frac{1}{24M} R_{\xi\eta\beta\alpha} \gamma^\alpha \gamma^\beta \gamma^\xi \gamma^\eta \right) \psi = 0. \quad (20)$$

The Pauli-type coupling terms disappear in the limit $M \to \infty$, in which case Eq. (20) reduces to

$$i\gamma^\alpha \left( \frac{\partial \psi}{\partial x^\alpha} - \omega_\alpha \psi \right) - m \psi + i\gamma^\alpha \psi s^{\beta}_{\beta \alpha} = 0. \quad (21)$$
With the last term on the left-hand side, Eq. (21) contains an additional torsion term as compared to the conventional result [9].

On the other hand, neglecting torsion means that Eq. (20) simplifies to

\[ i\gamma^\alpha \left( \frac{\partial \psi}{\partial x^\alpha} - \omega_\alpha \psi \right) - m\psi + \frac{1}{24M} R_{\xi\eta\alpha\beta} \gamma^\xi \gamma^\eta \gamma^\alpha \gamma^\beta \psi = 0. \]

Compared to the Dirac equation in flat space from Eq. (6), one encounters here two additional terms: the well-known coupling of the spinor \( \psi \) to the spinor connection \( \omega_\nu \) [9], and a new additional coupling to the Riemann tensor \( R_{\xi\eta\alpha\beta} \). For the particular case of metric compatibility and zero torsion, the "gravitational Pauli coupling term" can be expressed simply in terms of the Ricci scalar \( R = R_{\xi\eta\alpha\beta} g^{\xi\alpha} g^{\eta\beta} = R_{\eta\beta} g^{\eta\beta} \), making use of the symmetries of the Riemann tensor:

\[ R_{\xi\eta\alpha\beta} \gamma^\xi \gamma^\eta \gamma^\alpha \gamma^\beta = -2R \mathbf{1}. \] (22)

The "gravitational Pauli coupling effect" thus generally vanishes in Ricci-flat regions of spacetime \( (R = 0) \), hence in particular for the vacuum solutions of the Einstein equation with cosmological constant \( \Lambda = 0 \).

The coupling is associated with the length parameter \( \ell = 1/(24M) \). It is analogous to the Pauli interaction term \( (12) \) of a spinor in an external electromagnetic gauge field, hence an additional interaction of the magnetic moment of the spinor with the magnetic field. From this analogy, we conclude that the new coupling term of spinors to a curved spacetime—described by a non-vanishing Riemann tensor—is as physical as the Pauli interaction effect. In the actual case, the additional term can be interpreted as an interaction of the intrinsic spin of the fermionic field with the curved spacetime, which yields an effectively shifted mass value according to

\[ i\gamma^\alpha \left( \frac{\partial \psi}{\partial x^\alpha} - \omega_\alpha \psi \right) - \left( m + \frac{R}{12M} \right) \psi = 0. \] (23)

Note that the new coupling term emerges independently of an eventual torsion of spacetime. For a hypothetical fermion with zero rest mass, i.e., for \( m = 0 \), the gravitational Pauli term \( R/(24M) \) thus constitutes the only mass-like term in regions with \( R \neq 0 \). The respective particle then behaves similar to a photon, which propagates at a reduced speed in an optically dense medium and returns to the speed of light in classical vacuum.

IV. Consequences for the neutrino mass question.

Neutrino masses are strongly constrained from cosmological observations as summarized in the review [14]. More explicitly, the sum of masses for neutrinos with electroweak interactions is constrained below 0.11 eV. On the other hand, measurements of neutrino oscillations correspond to a mass scale of at least 0.05 eV. The smallness of neutrino masses can be explained e.g. in the seesaw mechanism [15–17], where a very large mass scale for non-interacting right-handed neutrinos effectively leads to three light neutrinos.

An effective mass term for neutrinos could also arise from from Pauli-type coupling of Dirac neutrinos to curved spacetime via Eq. (23). Even if the Dirac-type mass term \( m_\nu \) for neutrinos vanishes, the corresponding term \( R/(12M) \) would effectively lead to neutrino masses.

In order to illustrate an extreme case of minimal curvature one can consider \( R_{\Lambda CDM} \approx 10^{-66} \text{ eV}^2 \), as explained in [18]. Then Eq. (23) leads to a minimal scale of \( M \approx 8 \times 10^{-67} \text{ eV} \) in order to reproduce neutrino mass scales around 0.1 eV.

For comparison, in the atmosphere of the earth the curvature \( R_{\text{atm}} \approx 10^{-50} \text{ eV}^2 \) [18] would correspond to a larger scale of \( M \approx 10^{-51} \text{ eV} \). The latter case would have a significant impact on the interpretation of measurements of atmospheric neutrino oscillations.

It should also be mentioned that this type of coupling will be strongly enhanced for neutrinos traversing more compact objects such as supernovae. Conversely, the effective neutrino masses in the vicinity of a supernova would be larger than in the interstellar medium. Such a new relation would also have an effect for the interpretation of cosmic neutrino results from supernovae such as investigated in [19].

Ultimately, any new contribution to neutrino masses from the Pauli-type coupling will have to be confronted with precision neutrino measurements. In this context, measurements of velocities of neutrinos within media of different curvatures would be particularly valuable. Measurements from OPERA and MINOS [20, 21] can also constrain such a new contribution. Also the constraints coming from neutrino magnetic dipole moments will constrain contributions from this Pauli-type coupling and might favor large values of the unknown mass scale \( M \) for neutrinos.

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The spacetime dependent $\gamma^\mu(x)$-matrices are built from the tetrads $e^a_\mu(x)$ and any representation of the static Dirac matrices $\gamma^a$ according to $\gamma^\mu(x) = e^a_\mu(x)\gamma^a$, where $\mu = 0, \ldots, 3$ is the coordinate index in curved spacetime and $a = 0, \ldots, 3$ the coordinate index in Minkowski space.