Free Vibration Analysis of Multilayered Arches using a Layerwise Theory

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Abstract. This work presents a layerwise formulation for the dynamic analysis of multilayered arches. For modelling the laminate mechanics and layerwise inhomogeneity across the thickness accurately in a laminated arch, an efficient layerwise theory has been developed. This theory has only three independent variables like the most equivalent single layer theories such as the third order theory and the first order shear deformation theory. Hamilton’s principle has been adopted to derive equations of motion and consistent boundary conditions. Analytical solution has been presented for the arches having simply supported ends using n terms of the Fourier series. Results of the present theory have been compared with the results available in literature and 2D finite element results obtained using ABAQUS. Natural frequencies for a variety of lamination schemes, radii of curvature and span to thickness ratios have been presented.

1. Introduction
Laminated composite arches have been extensively used as structural components in varied industrial domains ranging from aerospace industries as helicopter blades and aeroplane wings, in energy sector as turbine blades, in construction industry as arches and many more. This is due to the facts that composite material itself gives leverage over other conventional material owing to its flexible properties and high strength to weight ratio. The possibility of altering the mechanical properties of the material for the purpose of making it utile for particular application can be easily provided by composite materials. Structures made up of composites can be made safer by modifying the layup orientation of the various plies so as to curtail the response of the structure to the vibratory actuating forces. Owing the profound use of laminated curved structures in the various fields of engineering, a deeper understanding of vibration characteristics of the composite arches is obligatory.

Some of the earlier reviews of the work done in the field of arches and rings have been compiled by Markus and Nanasi [1], Laura and Maurizi [2] and Chidamparam and Leissa [3]. These literatures are phenomenal in terms of the knowledge they address, containing numerous references on the vibration of curved beam; nevertheless most of it is pertaining to isotropic material. However vibration analysis of laminated arches made of orthotropic material has been carried out by Bhimaraddi et al. [4], Qatu [5], Khdeir and Reddy [6], Matsumaga [7] and many more. These studies are based on equivalent single layer (ESL) theories. The accuracy of these theories are dependent on the order of the polynomial considered to define the displacement field. Two dimensional(2D) elasticity and finite element solutions display higher accuracy in the results with larger number of variables than the one dimensional ESL theories.
In the present work, we develop an efficient layerwise theory (ZIGT) for the free vibration analysis of laminated arches. In this theory, layerwise cubic variation of the displacement field is assumed. Equations of motion and boundary conditions are derived using Hamilton’s principle. Analytical solution based on Fourier series expansion is obtained for laminated arches having simply supported ends.

2. Problem Formulation

Consider a multilayered arch with circumferential length \( l \) width \( b \) thickness \( h \) and radius of curvature \( R \) (figure 1). The coordinate axis has been shown in figure 1. The bottom surface of the arch lies at \( z = z_0 = -h/2 \) and its top surface lies at \( z = z_L = h/2 \). The \( z \)-coordinate of the bottom surface of the \( k^{th} \) layer mentioned from the bottom is denoted by \( z_{k-1} \). The reference plane \( z = 0 \) either passes through or is the bottom surface of the \( k^{th} \) layer. The arch is loaded with transverse pressure \( p \) applied on the top surface, with no variation along the width \( b \).

Considering small width, assuming plane stress condition (\( \sigma_y = \tau_yz = \tau_{xy} = 0 \)) and considering negligible transverse normal stress (\( \sigma_z \simeq 0 \)), the general 3D constitutive equations reduce to

\[
\sigma_x = \bar{Q}_{11}\varepsilon_x, \quad \tau_{zx} = \bar{Q}_{55}\gamma_{zx}
\]

where \( \bar{Q}_{11} = 1/\bar{s}_{11} \) and \( \bar{Q}_{55} = 1/\bar{s}_{55} \) are reduced stiffness coefficients and

\[
\bar{s}_{11} = s_{11}\cos^4\theta - 2s_{16}\cos^3\theta\sin\theta + (2s_{12} + s_{66})\cos^2\theta\sin^2\theta - 2s_{26}\cos\theta\sin^3\theta + s_{22}\sin^4\theta
\]

\[
\bar{s}_{55} = s_{55}\cos^2\theta + s_{44}\sin^2\theta - 2s_{45}\cos\theta\sin\theta
\]

are the compliance coefficients transformed from material coordinate system (1,2,3) to the laminate coordinate system (\( x,y,z \)).

![Figure 1. Geometry of the laminated curved beam.](image-url)
The transverse deflection is taken to be independent of \( z \) and the circumferential displacement \( u \) is taken as the combination of a global third-order polynomial variation in thickness coordinate \( z \) and layerwise linear variation
\[
w(x, z, t) = w_0(x, t) \\
u(x, z, t) = u(x, t) - zw_{0,z} + z\psi_k(x, t) + z^2\xi(x, t) + z^3\eta(x, t).
\]
where \( u_k \) and \( \psi_k \) are the translation and rotation of the \( k^{th} \) layer, and \( \xi \) and \( \eta \) are the coefficients of the quadratic and cubic terms in the cubic polynomial. These \( (2L + 2) \) variables are defined in terms of displacement and rotation of reference surface i.e., \( u_0(x, t) \) and \( \psi_0(x, t) \) using the continuity of \( u \) and \( \tau_{zx} \) at \( (L - 1) \) interfaces and 2 traction free conditions \( (\tau_{zx} = 0) \) at the bottom and top surfaces to express \( u \) in terms of only 3 independent variables i.e. \( u_0, w_0 \) and \( \psi_0 \)
\[
u(x, z, t) = u_0(x, t) - zw_{0,z}(x, t) + R^k(z)\psi_0(x, t).
\]
where
\[
R^k(z) = R_1^k + zR_2^k + z^2R_3^k + z^3R_4^k
\]
is a cubic layerwise function of \( z \). The coefficients \( R_1^k, R_2^k, R_3 \) and \( R_4 \) are functions of material properties of the layers and the laminate scheme.
The expression of \( u \) in third order theory can be shown as
\[
u(x, z, t) = u_0(x, t) - zw_{0,z}(x, t) + R(z)\psi_0(x, t)
\]
\[
R(z) = z + z^2R_1' + z^3R_2'
\]
\[
R_1' = \frac{3(z_L^2 - z_0^2)}{6(z_0^2 z_L - z_L^2 z_0)}, \quad R_2' = -\frac{2(z_L - z_0)}{6(z_0^2 z_L - z_L^2 z_0)}.
\]
The dynamic equations, variationally consistent boundary conditions, curved beam stiffnesses and inertia parameters for the present theory have been derived via the Hamilton’s principle.
Using the notation \( \langle \ldots \rangle = \sum_{k=1}^L \int_0^{z_k} \ldots dz \), the Hamilton’s principle for the laminated arch can be expressed as
\[
\int_x \left[ \langle \left( \rho^k\ddot{w}u + \rho^k\ddot{w}\delta w + \sigma_x\delta\varepsilon_x + \tau_{zx}\delta\gamma_{zx} \right) \rangle - \frac{1 + \frac{h}{2R}}{p_z} \delta w(x, z_L, t) \right] dx - \langle \sigma_x\delta u + \tau_{zx}\delta w \rangle |_x = 0
\]
\( \forall \delta u_0, \delta w_0 \) and \( \delta \psi_0 \), where over-dot represents the time derivative, \( \rho^k \) is the material mass density of the \( k^{th} \) layer and \( p_z \) is the applied normal load intensity on the top surface of the arch. Substituting the displacements and strains and integrating Eq. (8) by parts along circumferential axis yields the following equations of motion and consistent boundary conditions:

**Equations of motion**
\[
I_{11}\ddot{u}_0 - I_{12}\dddot{w}_0 - I_{13}\dddot{\psi}_0 - N_{x,x} = 0 \\
-I_{21}\dddot{u}_0 + I_{22}\dddot{w}_0 - I_{23}\dddot{\psi}_0 - \dddot{w}_0 - \frac{N_x}{R} + M_{x,x} + \dddot{P}_2 = 0 \\
I_{31}\dddot{u}_0 - I_{32}\dddot{w}_0 + I_{33}\dddot{\psi}_0 - P_{x,x} + Q_x = 0
\]

**Boundary conditions**
\[
u_0 = \ddot{u}_0 \quad \text{or} \quad N_x = \dddot{N}_x \\
w_0 = \ddot{w}_0 \quad \text{or} \quad -I_{21}\dddot{u}_0 + I_{22}\dddot{w}_0 - I_{23}\dddot{\psi}_0 + M_{x,x} = \dddot{V}_x \\
w_{0,z} = \dddot{w}_{0,z} \quad \text{or} \quad M_x = \dddot{M}_x \\
\psi_0 = \dddot{\psi}_0 \quad \text{or} \quad P_x = \dddot{P}_x
\]
where \( I_{ij} \) and \( \hat{I} \) are the inertia terms which are the functions of material density, radius of curvature and layerwise function \( R^k(z) \). \( N_x, M_x, P_x \) and \( Q_x \) are the stress resultants for the arch and \( \hat{P}_2 \) is the load vector. Over-bar on the variable denotes its prescribed value.

We obtain exact analytical solution for simply supported arch considering \( n \) terms of the Fourier series. For this purpose different primary and secondary varibles are expanded as sine and cosine functions to satisfy the simply supported boundary conditions,

\[
\begin{align*}
& w_0 = \sum_{n=1}^{\infty} (u_0)_n \sin n\bar{x}, \quad u_0 = \sum_{n=1}^{\infty} (u_0)_n \cos n\bar{x}, \quad \psi_0 = \sum_{n=1}^{\infty} (\psi_0)_n \cos n\bar{x}, \\
& N_x = \sum_{n=1}^{\infty} (N_x)_n \sin n\bar{x}, \quad M_x = \sum_{n=1}^{\infty} (M_x)_n \sin n\bar{x}, \quad P_x = \sum_{n=1}^{\infty} (P_x)_n \sin n\bar{x}, \\
& Q_x = \sum_{n=1}^{\infty} (Q_x)_n \cos n\bar{x}, \quad \hat{P}_2 = \sum_{n=1}^{\infty} (\hat{P}_2)_n \sin n\bar{x}.
\end{align*}
\]

with \( \bar{n} = n\pi/l \). These solution have been substituted in the equations of motion Eq. (9), yielding

\[
M\ddot{U}^n + KU^n = \dot{P}^n
\]  

where \( \dot{U}^n = [u_0 \ w_0 \ \psi_0]^T \) is the displacement vector for the \( n \)th Fourier term, \( \dot{P}^n = [P_1 - P_2 \ P_3]^T \) is the corresponding load vector. \( M \) and \( K \) are the mass and stiffness matrices. For performing the free vibration analysis, \( \dot{P}^n \) is set to zero to transform Eq. (12) to a generalised eigenvalue problem.

3. Results and Discussions

For validating the present efficient layerwise theory (ZIGT), we consider a 4 layer straight beam \((R/l = \infty)\) having equal ply-thickness and symmeytric lamination scheme [0°/90°/90°/0°]. This has been previously studied by Kapuria et al. [8]. The beam is simply supported on both ends. The material properties are taken as \( E_1=181\text{GPa}, \ E_2=10.3\text{GPa}, \ E_3=10.3\text{GPa}, \ G_{23}=2.87\text{GPa}, \ G_{31}=7.17\text{GPa}, \ G_{12}=7.17\text{GPa}, \nu_{23}=0.33\text{GPa}, \nu_{31}=0.25\text{GPa}, \nu_{12}=0.25\text{GPa}, \) and \( \rho=1578\text{kg/m}^3 \). First three natural frequencies are obtained using the present solution and their non dimensionalised values \((\bar{\omega}_n = \omega_n l S \sqrt{\rho/E_2})\) have been tabulated in table 1 and compared with 2D elasticity solution presented in [8]. It can be very well stated that present ZIGT produce better results in comparison to TOT. Also error tends to reduce with the increase in span to thickness ratio, \( S \).

In order to validate the present formulation for laminated arches, different lamination schemes, orthotropy ratios and span to thickness ratios have been considered. Results have been enlisted in table 2 and compared with the results available literature [9] for other higher order theories such as the trigonometric shear deformation theory (TSDBT). The material properties are \( E_2=6.98\text{GPa}, \ G_{12}=G_{13}=0.6\text{E}_2\text{GPa}, \ G_{23}=0.5\text{E}_2\text{GPa}, \nu_{12}=0.25\text{GPa}, \) \( h=0.254\text{m}, \) \( b=0.254\text{m} \) and \( \rho=1550\text{kg/m}^3 \). Four different cases of laminated curved beam are

\[
\begin{align*}
\text{Case 1} : & [0/90], \ E_1/E_2 = 40, l/h = 5, R/l = 5, \\
\text{Case 2} : & [0/90], \ E_1/E_2 = 40, l/h = 10, R/l = 5, \\
\text{Case 3} : & [0/90/0], \ E_1/E_2 = 40, l/h = 5, R/l = 5, \\
\text{Case 4} : & [0/90/0/90], \ E_1/E_2 = 40, l/h = 10, R/l = 5,
\end{align*}
\]

First five natural frequencies have been obtained using the present theory and their values in Hz have been reported in table 2. It is observed that the present results are in good agreement with the results given in [9].
Table 1. Nondimensionalised frequencies for symmetric cross-ply straight beam.

|   | $\bar{\omega}_1$ | $\bar{\omega}_2$ | $\bar{\omega}_3$ |
|---|-----------------|-----------------|-----------------|
| S | ZIGT | TOT | Exact 2D[8] | ZIGT | TOT | Exact 2D[8] | ZIGT | TOT | Exact 2D[8] |
| 5 | 6.815 | 6.978 | 6.806 | 16.729 | 16.877 | 16.515 | 27.873 | 27.140 | 26.688 |
| 10 | 9.344 | 9.471 | 9.343 | 27.259 | 27.911 | 27.224 | 46.656 | 47.615 | 46.419 |
| 20 | 10.640 | 10.689 | 10.640 | 37.376 | 37.884 | 37.374 | 71.775 | 73.261 | 71.744 |
| 100 | 11.192 | 11.195 | 11.193 | 44.473 | 44.509 | 44.477 | 98.985 | 99.161 | 98.988 |

Table 2. Natural frequencies (Hz) of cross-ply composite curved beams.

|   | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
|---|-------------|-------------|-------------|-------------|-------------|
| Case 1 | TSDBT[9] | 326.8 | 978.8 | 1719.9 | 2425.4 | 2522.4 |
|  | TOSDBT[9] | 325.5 | 968.2 | 1689.4 | 2435.3 | 2460.4 |
|  | ZIGT | 329.2 | 993.8 | 1749.1 | 2557.1 | 3422.5 |
| Case 2 | TSDBT[9] | 92.1 | 326.2 | 634.4 | 977.7 | 1340.7 |
|  | TOSDBT[9] | 92.0 | 325.0 | 629.8 | 967.1 | 1321.6 |
|  | ZIGT | 92.3 | 328.6 | 642.0 | 992.6 | 1363.4 |
| Case 3 | TSDBT[9] | 478.9 | 1119.3 | 1816.0 | 2608.2 | 2989.9 |
|  | TOSDBT[9] | 486.0 | 1132.3 | 1848.3 | 2608.2 | 3059.8 |
|  | ZIGT | 487.2 | 1148.8 | 1905.3 | 2804.3 | 3872.2 |
| Case 4 | TSDBT[9] | 134.9 | 411.5 | 715.7 | 1026.0 | 1343.4 |
|  | TOSDBT[9] | 135.0 | 411.9 | 715.7 | 1024.3 | 1338.1 |
|  | ZIGT | 134.2 | 406.8 | 704.7 | 1007.6 | 1316.7 |

Next we study the effects of radius of curvature on the natural frequencies of laminated arches having different lamination schemes. For this purpose, the variation of the fundamental natural frequency parameter ($\bar{\omega}_n = \omega_n l S \sqrt{\rho/E_2}$) for $R/l = 2$ and $R/l = 5$ has been shown in figure 2. Three different lamination schemes, symmetric (s), anti-symmetric (as) and three layer symmetric, are opted having material properties $E_1 = 40, G_{12} = G_{13} = 0.6 E_2, G_{23} = 0.5 E_2, \nu_{12} = 0.25$. Exact 2D elasticity results available in the literature [10] for $R/l = 5$ are also included in figure 2. It is quite evident from figure that the results are in good agreement with the exact 2D elasticity results.

Next, we consider a five layer antisymmetric sandwich arch having very soft core $[0^o/90^o/core/0^o/90^o]$. This lamination scheme has widely different material properties across the arch thickness. Material properties for the outer 2 layers of the face $E_1=224.25\text{GPa}$,
Figure 2. Variation of fundamental frequencies with span to thickness ratio ($S$) for various lamination scheme.

Figure 3. Variation of second and third mode frequencies with span to thickness ratio ($S$) for various lamination scheme.

$E_2=6.9\text{GPa},\ E_3=6.9\text{GPa},\ G_{23}=1.38\text{GPa},\ G_{31}=56.58\text{GPa},\ G_{12}=56.58\text{GPa},\ \nu_{23}=0.25\text{GPa},\ \nu_{13}=0.25\text{GPa},\ \nu_{12}=0.25\text{GPa}$ and $\rho=1578\text{kg/m}^3$ and the material properties for middle soft core are $E_1=0.2208\text{MPa},\ E_2=0.2001\text{MPa},\ E_3=2760\text{MPa},\ G_{23}=455.4\text{MPa},\ G_{31}=545.1\text{MPa},\ G_{12}=16.65\text{MPa},\ \nu_{23}=0.00003\text{GPa},\ \nu_{13}=0.00003\text{GPa},\ \nu_{12}=0.99\text{GPa}$ and $\rho=70\text{kg/m}^3$. The results for the first three natural frequencies have been obtained for different span to thickness ratio and radius to length ratio using both ZIGT and TOT presented above and compared their nondimensional values ($\tilde{\omega}_n = \omega_n l S \sqrt{\rho_0/E_0}$), $\rho_0 = 1578\text{kg/m}^3$ and $E_0 = 6.9\text{GPa}$ in table 3. To test the accuracy of the two theories, these arches have been modeled using finite element software ABAQUS adopting CPS8R element. Converged results have been produced and presented in the table 3. It has been observed that the results obtained using ZIGT are much closer to 2D FE results and the TOT performs badly for this configuration of the arch.


Table 3. Nondimensional natural frequencies of 5-layer antisymmetric arches.

| R/l | S | $\bar{\omega}_1$ | $\bar{\omega}_2$ | $\bar{\omega}_3$ |
|-----|---|-----------------|-----------------|-----------------|
|     | 5 | 7.956           | 17.611          | 27.210          |
|     | 10| 12.255          | 31.862          | 51.285          |
|     | 20| 15.164          | 49.083          | 88.042          |
|     | 5 | 7.882           | 17.585          | 27.210          |
|     | 10| 12.134          | 31.796          | 51.255          |
|     | 20| 15.010          | 48.969          | 87.965          |

4. Conclusions

Present efficient layerwise theory (ZIGT) yield accurate values of natural frequencies for arches of all lay-ups considered in the present study. These results are comparable with the 2D elasticity solution which has been found as the most accurate method for the analysis of 1D structures. For the case of sandwich beam, the TOT fails to give accurate response for all the thickness, radius of curvature considered in the present study.

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