INSTANTONS ON THE QUANTUM 4-SPHERES $S_q^4$

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Abstract

We introduce noncommutative algebras $A_q$ of quantum 4-spheres $S_q^4$, with $q \in \mathbb{R}$, defined via a suspension of the quantum group $SU_q(2)$, and a quantum instanton bundle described by a selfadjoint idempotent $e \in \text{Mat}_4(A_q)$, $e^2 = e = e^*$. Contrary to what happens for the classical case or for the noncommutative instanton constructed in \cite{8}, the first Chern-Connes class $ch_1(e)$ does not vanish thus signaling a dimension drop. The second Chern-Connes class $ch_2(e)$ does not vanish as well and the couple $(ch_1(e), ch_2(e))$ defines a cycle in the $(b, B)$ bicomplex of cyclic homology.
1 Introduction

The goal of this paper is to provide more interesting examples of globally nontrivial four dimensional quantum manifolds and vector bundles (finite projective modules) over them. Most of the literature concentrated so far on noncommutative tori or Moyal deformations of $\mathbb{R}^4$ [17, 18, 14, 10, 11].

In [8] an instanton bundle over a family of noncommutative 4-spheres $S^4_\theta$ which are suspensions of a class of noncommutative 3-spheres, were introduced. These spaces $S^4_\theta$ fulfill all the axioms of Riemannian spin geometry as formulated in [5, 6]. Moreover, as it happens [7] for the instanton bundle over the ordinary 4-sphere $S^4$, the 0th and the 1st Chern-Connes classes of the instanton idempotent $e$ vanish, $\text{ch}_j(e) = 0$, $j = 0, 1$, while $\text{ch}_2(e)$ is a nontrivial Hochschild cycle.

In the present paper, we exhibit another family of quantum 4-spheres $S^4_q$, $q \in \mathbb{R}$, defined also via a suspension but now of the quantum 3-sphere $S^3_q$, which we take just as the underlying ‘space’ of the quantum group $SU_q(2)$. Though these two families are in a sense related by analytic continuation of the deformation parameter $[9]$, they present some different and worth mentioning properties. First of all, $S^4_q$ does not seem to obey (all of) the axioms of a noncommutative manifold as given in [3, 8], a feature which is shared with most of the quantum spaces defined in the framework of the so called $q$-deformations and quantum groups. Indeed, this fact may be an inspiration for weakening some of those axioms in order to embrace such a class of spaces. Furthermore, the quantum spheres $S^4_q$ come equipped with a natural idempotent $e$ as well which determines a vector bundle (i.e. a finite projective module of sections) over it. However, contrary to what happens for the classical case or for the instanton bundle constructed in [8], now the first Chern-Connes class $\text{ch}_1(e)$ does not vanish. Thus the idempotent $e$ does not provide a representation of the universal instanton algebras as defined in [8]. It turns out that the second Chern-Connes class $\text{ch}_2(e)$ does not vanish as well and the couple $(\text{ch}_1(e), \text{ch}_2(e))$ defines a cycle in the $(b, B)$ bicomplex of cyclic homology [3, 12].

2 The algebra of $S^4_q$

We define the quantum 4-sphere $S^4_q$ as the suspension of the quantum 3-sphere $S^3_q$ which we take as the underlying ‘space’ of the quantum group $SU_q(2)$. Thus, using the definition of the $C^*$-algebra of $S_\mu U(2)$ as given in [19, 20] (with the convenient replacements $\mu \mapsto q$, $\alpha \mapsto \alpha^*$ and $\gamma \mapsto \beta^*$), for a parameter $q \in [-1, 0) \cup (0, 1]$ we define $A_q$ as the $C^*$-algebra with unit $I$ generated by three elements $\alpha, \beta$ and $z$ satisfying the relations

$$\beta \alpha = q \alpha \beta, \quad \beta^* \alpha = q \alpha^* \beta, \quad \beta \beta^* = \beta^* \beta, \quad (1)$$

$$z = z^*, \quad z \alpha = a z, \quad z \beta = \beta z,$$

$$\alpha^* \alpha + q^2 \beta^* \beta + z^2 = I, \quad a \alpha^* + \beta \beta^* + z^2 = I.$$

In particular the ‘suspension’ generator $z$ is central and selfadjoint. It should be clear from the previous relations that, as it happens for the quantum group $SU_q(2)$, the generator $\alpha$ is not normal, $\alpha^* \alpha \neq \alpha \alpha^*$. 

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More precisely the algebra $A_q$ can be defined in the following way. Consider the free (noncommutative) $*$-algebra with unity $\mathbb{C}[\alpha, \beta, z, \alpha^*, \beta^*, z^*]$ generated by three elements $\alpha, \beta$ and $z$. A $*$-representation $\pi$ of $\mathbb{C}[\alpha, \beta, z, \alpha^*, \beta^*, z^*]$ in terms of bounded operators on a Hilbert space $H$ is said to be admissible if the operators $\pi(\alpha), \pi(\beta), \pi(z)$ satisfy the relations (I). Then for arbitrary $a \in \mathbb{C}[\alpha, \beta, z, \alpha^*, \beta^*, z^*]$ we set $\|a\|$ to be the supremum, over all admissible representations $\pi$ of $\mathbb{C}[\alpha, \beta, z, \alpha^*, \beta^*, z^*]$, of the operator norms $\|\pi(a)\|$. It can be seen that $\|a\| < \infty$ and that $\| \cdot \|$ is a $C^*$-semi-norm. As a consequence the set $\mathcal{J}$ of all those $a$ in $\mathbb{C}[\alpha, \beta, z, \alpha^*, \beta^*, z^*]$ with $\|a\| = 0$ is a two-sided ideal in $\mathbb{C}[\alpha, \beta, z, \alpha^*, \beta^*, z^*]$. Then one obtains a $C^*$-norm on the quotient algebra $\mathbb{C}[\alpha, \beta, z, \alpha^*, \beta^*, z^*]/\mathcal{J}$, the completion of which is the $C^*$-algebra $A_q$ in question.

In the sequel, to simplify the notation we shall denote the $\mathcal{J}$-equivalence class $a + \mathcal{J}$ simply by $a$. With this convention, one sees that the $*$-subalgebra generated by $\alpha, \beta$ and $z$ is dense in $A_q$. Moreover, for any triple $\hat{\alpha}, \hat{\beta}, \hat{z}$ of bounded operators on $H$ satisfying the relations (I) there exists exactly one representation $\pi : A_q \to B(H)$ such that $\pi(\alpha) = \hat{\alpha}$, $\pi(\beta) = \hat{\beta}$ and $\pi(z) = \hat{z}$. It can be also verified that the ideal $\mathcal{J}$ actually coincides with the ideal in $\mathbb{C}[\alpha, \beta, z, \alpha^*, \beta^*, z^*]$ generated by the following elements:

$$
\begin{align*}
\beta\alpha - q\alpha\beta, & \quad \beta^*\alpha - q\alpha\beta^*, & \quad \beta^*\beta - \beta\beta^*, \\
\alpha^*\beta^* - q\beta^*\alpha^*, & \quad \alpha^*\beta - q\beta\alpha^*, \\
z - z^*, & \quad z\alpha - \alpha z, & \quad z\beta - \beta z, \\
\alpha^*\alpha + q^2\beta^*\beta + z^2 - 1, & \quad \alpha\alpha^* + \beta^*\beta + z^2 - 1.
\end{align*}
$$

Using the relations (I) a linear basis for $A_q$ can be taken as $a_{kmn\ell}$, with $k \in \mathbb{Z}$ and $m, n, \ell$ non negative integers, of the form

$$
a_{kmn\ell} = \begin{cases} 
\alpha^k\beta^m\beta^nz^\ell & \text{for } k = 0, 1, 2 \ldots \\
\alpha^{-k}\beta^m\beta^nz^\ell & \text{for } k = -1, -2 \ldots .
\end{cases}
$$

Notice that the quantum sphere $S_q^4$ may be defined also for $q \in \mathbb{R}$, $|q| > 1$, but with the transformation $q \mapsto 1/q$, $\alpha \mapsto \alpha^*$, $\beta \mapsto q\beta$ and $z \mapsto z$, we get a sphere which is $C^*$-isomorphic to one for $|q| < 1$.

It is clear that the quotient of the $C^*$-algebra $A_q$ by the ideal generated by $z$ can be identified with the $C^*$-algebra of the compact quantum group $SU_q(2)$. However, in this paper we shall not make any use of additional structures (like coproduct, counit, and antipode) coming from $SU_q(2)$. In [3] it was shown that for $q \in (-1, 0) \cup (0, 1)$ the spaces $SU_q(2)$ are all homeomorphic in the sense that the corresponding $C^*$-algebras are isomorphic. Then, for $q \in (-1, 0) \cup (0, 1)$, all our $C^*$-algebras $A_q$ are isomorphic as well and all corresponding spheres are homeomorphic.

For the generic situation when $-1 < q < 0$ or $0 < q < 1$ any character $\chi$ of $A_q$ has to satisfy the equations

$$
\begin{align*}
\chi(\alpha^*) = \overline{\chi(\alpha)}, & \quad \chi(\beta^*) = \overline{\chi(\beta)}, & \quad \chi(z^*) = \chi(z), \\
\chi(\beta) = 0 & \text{ and } |\chi(\alpha)|^2 + (\chi(z))^2 = 1.
\end{align*}
$$

To show that the space of all characters is homeomorphic to the two dimensional sphere $S^2$, we take a generic $\alpha' \in \mathbb{C}$ and $z' \in \mathbb{R}$ such that $|\alpha'|^2 + z'^2 = 1$. Then, from the
general considerations presented above, there is a 1-dimensional representation (that is a character) \( \chi \) of \( A_q \) such that \( \chi(\alpha) = \alpha' \), \( \chi(\beta) = 0 \) and \( \chi(z) = z' \) and this proves the homeomorphism in question. Hence, for \(-1 < q < 0 \) or \( 0 < q < 1 \) the space of (nonzero) characters of \( A_q \), which can be thought of as the space of ‘classical points’ of \( S^4_q \), is homeomorphic to the classical \( S^4 \).

For the particular case \( q = 1 \) the algebra of the sphere \( S^4_q \) is commutative. The associated space of characters is homeomorphic to the 4-dimensional sphere \( S^4 \). Indeed any character \( \chi \) of \( A_{q=1} \) satisfies the equations

\[
\chi(\alpha^*) = \overline{\chi(\alpha)} , \quad \chi(\beta^*) = \overline{\chi(\beta)} , \quad \chi(z^*) = \chi(z) , \quad (5)
\]

and

\[
|\chi(\alpha)|^2 + |\chi(\beta)|^2 + (\chi(z))^2 = 1 .
\]

To show that any element of \( S^4 \) arises in this way, similarly to what we did before we take generic \( \alpha', \beta' \in \mathbb{C} \) and \( z' \in \mathbb{R} \) such that \( |\alpha'|^2 + |\beta'|^2 + z'^2 = 1 \). Thus they satisfy relations (3) (or relations (2) for \( q = 1 \)) and there is a 1-dimensional representation \( \chi \) of \( A_q (q = 1) \) such that \( \chi(\alpha) = \alpha', \chi(\beta) = \beta' \) and \( \chi(z) = z' \). This proves the homeomorphism in question and shows that the algebra \( A_q \) for \( q = 1 \) can be identified with the algebra of all continuous functions on the 4-dimensional sphere \( S^4 \). It is in this sense that \( S^4_q \) provides a deformation of the classical \( S^4 \).

Next, we describe irreducible representations of the algebra \( A_q \) (for \(-1 < q < 0 \) or \( 0 < q < 1 \)) as bounded operators on an infinite dimensional Hilbert space \( H \) with an orthonormal basis \( \{ \psi_n, n = 0, 1, 2, \cdots \} \). With \( \lambda \in \mathbb{C} \), \( |\lambda| \leq 1 \), we get two families of representations \( \pi_{\lambda, \pm} : A_q \to B(H) \) given by

\[
\pi_{\lambda, \pm}(z) \psi_n = \pi_{\lambda, \pm}(z^*) \psi_n = \pm \sqrt{1 - |\lambda|^2} \psi_n ,
\]
\[
\pi_{\lambda, \pm}(\alpha) \psi_n = \lambda \sqrt{1 - q^{2(n+1)}} \psi_{n+1} , \quad \pi_{\lambda, \pm}(\beta) \psi_n = \lambda q^n \psi_n ,
\]
\[
\pi_{\lambda, \pm}(\alpha^*) \psi_n = \overline{\lambda} \sqrt{1 - q^{2n}} \psi_{n-1} , \quad \pi_{\lambda, \pm}(\beta^*) \psi_n = \overline{\lambda} q^n \psi_n .
\]

To be precise, for \( \lambda \) such that \( |\lambda| = 1 \), the two representations \( \pi_{\lambda, +} \) and \( \pi_{\lambda, -} \) are identical so that, in fact, we have a family of representations parametrized by points on a classical sphere \( S^2 \), similarly to what happens for one dimensional representations (characters) as described before.

As mentioned already, the quotient of the \( C^* \)-algebra \( A_q \) by the ideal generated by \( z \) is the \( C^* \)-algebra of the compact quantum group \( SU_q(2) \). Then, with \( |\lambda| = 1 \), the representations \( \pi_{\lambda, +} = \pi_{\lambda, -} =: \pi_{\lambda} \) yield representations of \( SU_q(2) \) which are unitary equivalent to the ones constructed by Woronowicz (see for instance ([21])).

### 3 The instanton and its classes

Consider now the following element \( e \) in the algebra \( \text{Mat}_4(A_q) \simeq \text{Mat}_4(\mathbb{C}) \otimes A_q \)

\[
e = \frac{1}{2} \begin{pmatrix}
1 + z, & 0, & \alpha, & \beta \\
0, & 1 + z, & -q\beta^*, & \alpha^* \\
\alpha^*, & -q\beta, & 1 - z, & 0 \\
\beta^*, & \alpha, & 0, & 1 - z
\end{pmatrix} . \quad (7)
\]
Using the relations (1) it can be verified that $e$ is a selfadjoint idempotent (projection)

$$e^2 = e = e^*.$$ 

It operates on the right $A_q$-module $A^4_q = A_q \otimes \mathbb{C}^4$ and its range may be thought of as sections of a vector bundle over $S^4_q$. It is easy to see that $eA^4_q$ is a deformation of the classical instanton bundle over $S^4$ in the sense that for $q = 1$, the module $eA^4_q$ is the module of sections of the complex rank two instanton bundle over $S^4$ [1].

Next, we compute the Chern-Connes Character of the idempotent $e$ given in (7). If $\langle \rangle$ is the projection on the commutant of $4 \times 4$ matrices, up to normalization the component of the (reduced) Chern-Connes Character are given by

$$ch_n(e) = \langle e - \frac{1}{2} \otimes \underbrace{e \otimes \cdots \otimes e}_{2n} \rangle, \quad n = 0, 1, 2, \ldots ,$$

and they are elements of

$$A_q \otimes \underbrace{\mathbb{A}_q \otimes \cdots \otimes \mathbb{A}_q}_{2n},$$

where $\mathbb{A}_q = A_q/\mathbb{C}\mathbb{I}$ is the quotient of the algebra $A_q$ by the scalar multiples of the unit $\mathbb{I}$. The crucial property of the components $ch_n(e)$ is that they define a cycle in the $(b, B)$ bicomplex of cyclic homology [3, 12], that is,

$$B ch_n(e) = b ch_{n+1}(e).$$

The operator $b$ is defined by

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_m) = \sum_{j=0}^{m-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_m + (-1)^m a_m a_0 \otimes a_1 \otimes \cdots \otimes a_{m-1}$$

while the operator $B$ is written as

$$B = AB_0,$$

where

$$B_0(a_0 \otimes a_1 \otimes \cdots \otimes a_m) = \mathbb{I} \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_m$$

and

$$A(a_0 \otimes a_1 \otimes \cdots \otimes a_m) = \frac{1}{m} \sum_{j=0}^{m} (-1)^{mj} a_j \otimes a_{j+1} \otimes \cdots \otimes a_{j-1},$$

with the obvious cyclic identification $m + 1 = 0$. To be precise, in formulæ (11), (13) and (14), all elements in the tensor products but the first one should be taken modulo complex multiples of the unit $\mathbb{I}$, that is one has to project onto $\mathbb{A}_q = A_q/\mathbb{C}\mathbb{I}$.

For the 0th component of the Chern-Connes Character of the idempotent (4) on the spheres $S^4_q$ we find,

$$ch_0(e) = \langle e - \frac{1}{2} \rangle = 0.$$
This could be interpreted as saying that the idempotent and the corresponding module (the ‘vector bundle’) has complex rank equal to 2.

Next for the 1st component we have,

\[
ch_1(e) = \left\langle \left( e - \frac{1}{2} \right) \otimes e \otimes e \right\rangle
= \frac{1}{8}(1 - q^2) \left\{ z \otimes (\beta \otimes \beta^* - \beta^* \otimes \beta) + \beta^* \otimes (z \otimes \beta - \beta \otimes z) + \beta \otimes (\beta^* \otimes z - z \otimes \beta^*) \right\}.
\]

It is straightforward to check that

\[
bch_1(e) = 0 = Bch_0(e)
\]

Finally, the 2nd component

\[
ch_2(e) = \left\langle \left( e - \frac{1}{2} \right) \otimes e \otimes e \otimes e \right\rangle
\]

can be written as a sum of five terms

\[
ch_2(e) = \frac{1}{32} \left( z \otimes c_z + \alpha \otimes c_\alpha + \alpha^* \otimes c_{\alpha^*} + \beta \otimes c_\beta + \beta^* \otimes c_{\beta^*} \right),
\]

with

\[
c_z = (1 - q^4)(\beta \otimes \beta^* \otimes \beta \otimes \beta - \beta^* \otimes \beta \otimes \beta^* \otimes \beta)
+ (1 - q^2) \left\{ z \otimes z \otimes (\beta \otimes \beta^* - \beta^* \otimes \beta) + (\beta \otimes z \otimes z \otimes \beta^* - \beta^* \otimes z \otimes z \otimes \beta)
+ (\beta \otimes \beta^* - \beta^* \otimes \beta) \otimes z \otimes z + z \otimes (\beta \otimes \beta^* - \beta^* \otimes \beta) \otimes z
- z \otimes (\beta \otimes z \otimes \beta^* - \beta^* \otimes z \otimes \beta) - (\beta \otimes z \otimes \beta^* - \beta^* \otimes z \otimes \beta) \otimes z \right\}
+ (\alpha \otimes \alpha^* - q^2 \alpha^* \otimes \alpha) \otimes (\beta \otimes \beta^* - \beta^* \otimes \beta)
+ (\beta \otimes \beta^* - \beta^* \otimes \beta) \otimes (\alpha \otimes \alpha^* - q^2 \alpha^* \otimes \alpha)
+ (\beta \otimes \alpha - q \alpha \otimes \beta) \otimes (\alpha^* \otimes \beta^* - q \beta^* \otimes \alpha^*)
+ (\alpha^* \otimes \beta^* - q \beta^* \otimes \alpha^*) \otimes (\beta \otimes \alpha - q \alpha \otimes \beta)
+ (\alpha^* \otimes \beta - q \beta \otimes \alpha^*) \otimes (q \alpha \otimes \beta^* - \beta^* \otimes \alpha)
+ (q \alpha \otimes \beta^* - \beta^* \otimes \alpha) \otimes (\alpha^* \otimes \beta - q \beta \otimes \alpha^*);
\]

\[
c_\alpha = (z \otimes \alpha^* - \alpha^* \otimes z) \otimes (\beta^* \otimes \beta - \beta \otimes \beta^*)
+ q^2(\beta^* \otimes \beta - \beta \otimes \beta^*) \otimes (z \otimes \alpha^* - \alpha^* \otimes z)
+ q(z \otimes \beta - \beta \otimes z) \otimes (\alpha^* \otimes \beta^* - q \beta^* \otimes \alpha^*)
+ (\alpha^* \otimes \beta^* - q \beta^* \otimes \alpha^*) \otimes (z \otimes \beta - \beta \otimes z)
+ q(\beta^* \otimes z - z \otimes \beta^*) \otimes (\alpha^* \otimes \beta - q \beta \otimes \alpha^*)
+ (\alpha^* \otimes \beta - q \beta \otimes \alpha^*) \otimes (\beta^* \otimes z - z \otimes \beta^*);
\]
\[ c_{\alpha^*} = q^2 (z \otimes \alpha - \alpha \otimes z) \otimes (\beta \otimes \beta^* - \beta^* \otimes \beta) \]
\[ + (\beta \otimes \beta^* - \beta^* \otimes \beta) \otimes (z \otimes \alpha - \alpha \otimes z) \]
\[ + (\beta^* \otimes z - z \otimes \beta^*) \otimes (\beta \otimes \alpha - q \alpha \otimes \beta) \]
\[ + q (\beta \otimes \alpha - q \alpha \otimes \beta) \otimes (\beta^* \otimes z - z \otimes \beta^*) \]
\[ + (z \otimes \beta - \beta \otimes z) \otimes (\beta^* \otimes \alpha - q \alpha \otimes \beta^*) \]
\[ + q (\beta^* \otimes \alpha - q \alpha \otimes \beta^*) \otimes (z \otimes \beta - \beta \otimes z); \tag{22} \]

\[ c_{\beta} = (1 - q^4) \left[ (\beta^* \otimes z - z \otimes \beta^*) \otimes \beta \otimes \beta^* + \beta^* \otimes \beta \otimes (\beta^* \otimes z - z \otimes \beta^*) \right] \tag{23} \]
\[ + (1 - q^2) \left\{ (\beta^* \otimes z \otimes z \otimes z - z \otimes \beta^* \otimes z \otimes z \right. \]
\[ \left. + z \otimes z \otimes \beta^* \otimes z - z \otimes z \otimes z \otimes \beta^* \right\} \]
\[ + (\beta^* \otimes z - z \otimes \beta^*) \otimes (\alpha \otimes \alpha^* - q^2 \alpha^* \otimes \alpha) \]
\[ + (\alpha \otimes \alpha^* - q^2 \alpha^* \otimes \alpha) \otimes (\beta^* \otimes z - z \otimes \beta^*) \]
\[ + (\alpha \otimes z - z \otimes \alpha) \otimes (\alpha^* \otimes \beta^* - q \beta^* \otimes \alpha^*) \]
\[ + q (\alpha^* \otimes \beta^* - q \beta^* \otimes \alpha^*) \otimes (\alpha \otimes z - z \otimes \alpha) \]
\[ + (\beta^* \otimes \alpha - q \alpha \otimes \beta^*) \otimes (\alpha^* \otimes z - z \otimes \alpha^*) \]
\[ + q (\alpha^* \otimes z - z \otimes \alpha^*) \otimes (\beta^* \otimes \alpha - q \alpha \otimes \beta^*); \tag{24} \]

\[ c_{\beta^*} = (1 - q^4) \left[ (z \otimes \beta - \beta \otimes z) \otimes \beta^* \otimes \beta + \beta \otimes \beta^* \otimes (z \otimes \beta - \beta \otimes z) \right] \tag{24} \]
\[ + (1 - q^2) \left\{ - \beta \otimes z \otimes z \otimes z + z \otimes \beta \otimes z \otimes z \right. \]
\[ \left. - z \otimes z \otimes \beta \otimes z + z \otimes z \otimes z \otimes \beta \right\} \]
\[ + (z \otimes \beta - \beta \otimes z) \otimes (\alpha \otimes \alpha^* - q^2 \alpha^* \otimes \alpha) \]
\[ + (\alpha \otimes \alpha^* - q^2 \alpha^* \otimes \alpha) \otimes (z \otimes \beta - \beta \otimes z) \]
\[ + q (z \otimes \alpha^* - \alpha^* \otimes z) \otimes (\beta \otimes \alpha - q \alpha \otimes \beta) \]
\[ + (\beta \otimes \alpha - q \alpha \otimes \beta) \otimes (z \otimes \alpha^* - \alpha^* \otimes z) \]
\[ + q (\alpha^* \otimes \beta - q \beta \otimes \alpha^*) \otimes (z \otimes \alpha - \alpha \otimes z) \]
\[ + (z \otimes \alpha - \alpha \otimes z) \otimes (\alpha^* \otimes \beta - q \beta \otimes \alpha^*). \]

By using the relations (21) for our algebra, and remembering that we need to project on \( \hat{A}_q \) in all terms of the tensor product but the first one, a long (one needs to compute 750 terms) but straightforward computation gives

\[ bch_2(c) = \frac{1}{16} (1 - q^2) \left\{ \mathbb{I} \otimes z \otimes (\beta \otimes \beta^* - \beta^* \otimes \beta) \right. \]
\[ \left. + \mathbb{I} \otimes \beta \otimes (\beta^* \otimes z - z \otimes \beta^*) + \mathbb{I} \otimes \beta^* \otimes (z \otimes \beta - \beta \otimes z) \right\} \tag{25} \]

and this is exactly equal to \( Bch_1(c) \).
4 Final remarks

There are several directions in which one can proceed and we just mention some of them. It would be clearly very interesting to study differential calculi on our quantum 4-sphere and develop Yang-Mills theory.

Another natural question is to which extent the sphere $S^4_q$ could be endowed with a structure of a metric noncommutative manifold which fulfills (some of) the related axioms [5, 6]. In particular one should construct an appropriate Dirac operator. This will probably be possible along the lines of [8] where it was suggested that the true Dirac operator $D$ for the quantum $SU_q(2)$ (and also for the quantum Podleś 2-sphere $S^2_q$ [16]) should satisfy an equation of the form

$$\frac{q^{2D} - q^{-2D}}{q^2 - q^{-2}} = Q.$$  \tag{26}

where $Q$ is some $q$-analogue of the Dirac operator like the ones found in [2, 13]. Once found the operator $D$, one would easily ‘suspend’ it to the 4-sphere $S^4_q$.

Finally, we mention that it will be interesting to study if there is any relation with the sheaf-theoretic construction of a $q$-deformed instanton in [15].

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