QCD Analysis of Hadronic $\tau$ Decays Revisited

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Abstract

The calculation of perturbative corrections to the spectral moments observable in hadronic $\tau$ decays is reconsidered. The exact order-$\alpha_s^3$ results and the resummation procedure of Le Diberder and Pich are compared with a partial resummation of the perturbative series based on the analysis of so-called renormalon chains. The perturbative analysis is complemented by a model-independent description of power corrections. For the contributions of dimension four and six in the OPE, it is demonstrated how infrared renormalon ambiguities in the definition of perturbation theory can be absorbed by a redefinition of nonperturbative parameters. We find that previous determinations of QCD parameters from a measurement of spectral moments in $\tau$ decays have underestimated the theoretical uncertainties. Given the present understanding of the asymptotic behaviour of perturbation theory, the running coupling constant can be measured at best with a theoretical uncertainty $\delta \alpha_s(m_\tau^2) \simeq 0.05$, and the gluon condensate with an uncertainty of order its magnitude. Two weighted integrals of the hadronic spectral function are constructed, which can be used to test the absence of dimension-two operators and to measure directly the gluon condensate.

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1 Introduction

The $\tau$ lepton is the only known lepton heavy enough to decay into hadrons. Its decays provide a unique environment to study hadronic weak interactions at low energies. Because of its inclusive character, the total $\tau$ hadronic width is expected to be calculable in QCD using analyticity and the operator product expansion (OPE) [1]–[6]. Since nonperturbative contributions to the total width turn out to be strongly suppressed, a measurement of the $\tau$ width is considered an excellent way to extract the strong coupling constant $\alpha_s$ with high precision at low energies. It has also been pointed out that hadronic $\tau$ decays can more generally be used to test QCD and to extract some of its nonperturbative parameters [8]–[12]. The idea is to measure spectral moments built from weighted integrals of the invariant hadronic mass distribution. As the total width, such moments are calculable in QCD, but they are more sensitive to nonperturbative contributions. An analysis of the spectral moments can also be used to search for non-standard effects, such as power corrections of order $(\Lambda/m_\tau)^2$ [13, 14], which are absent in the standard OPE approach of Shifman, Vainshtein and Zakharov (SVZ) [15].

A QCD-based analysis of power-suppressed effects by measuring spectral moments in hadronic $\tau$ decays has been pursued by two experimental groups [16, 17]. There is a theoretical obstacle to any such analysis, however. The separation of perturbative and nonperturbative effects in QCD is intrinsically ambiguous. In order to define and extract power corrections in a meaningful way, one has to control the perturbative contributions to sufficiently high accuracy. But it is known that QCD perturbation theory provides an asymptotic expansion, which is factorially divergent at large orders. Related to this behaviour are the so-called renormalon singularities in the Borel transform of a perturbative series with respect to $1/\alpha_s$ [18]–[26]. The definition of the (resummed) perturbative expansion by itself is ambiguous; only the sum of perturbative and nonperturbative contributions in the OPE is well defined [24]. Recently, the study of renormalons has received renewed attention. Efficient techniques have been developed to resum so-called renormalon chains, i.e. terms of order $\beta_0^{n-1} \alpha_s^n$ (where $\beta_0$ denotes the first coefficient of the $\beta$-function), to all orders in perturbation theory [27]–[30]. These investigations are interesting, since they provide an estimate of the importance of higher-order terms in a series, and moreover they elucidate the structure of nonperturbative contributions, which have to be included in the OPE in order to cancel the ambiguities of resummed perturbation theory. They are also useful for estimating the uncertainty in finite-order perturbative calculations.

In the present paper, we reconsider the theoretical analysis of hadronic $\tau$ decays in the light of these developments. This extends some recent analyses of renormalon contributions to the $\tau$ hadronic width [28]–[30]. In Sect. 2 we define the spectral moments and discuss existing perturbative predictions for them. In

\footnote{The applicability of the OPE in $\tau$ decays has, however, been questioned recently [10].}
Sects. 3 and 4 we discuss the resummation of renormalon contributions for euclidean current correlators and for the spectral moments. Section 5 is devoted to a model-independent analysis of power corrections. In Sect. 6 we show in detail how the ambiguities of perturbation theory, which are due to the first two infrared (IR) renormalons, can be absorbed into a redefinition of some nonperturbative parameters. A discussion of our results and a study of the feasibility of extracting the running coupling constant and some nonperturbative parameters from data are presented in Sect. 7. There we construct two weighted integrals of the hadronic spectral function, which can be used to test the absence of dimension-two operators in QCD and to measure the gluon condensate. Technical details of our calculations are relegated to three appendices.

2 Spectral moments

The theoretical description of inclusive hadronic $\tau$ decays involves two-point correlation functions of flavour-changing vector and axial vector currents, $V^\mu_{ij} = \bar{q}_i \gamma^\mu q_j$ and $A^\mu_{ij} = \bar{q}_i \gamma^\mu \gamma^5 q_j$, where $i$ and $j$ are flavour labels ($i = u$, $j = d$ or $s$). These correlators admit the Lorentz decomposition ($\Gamma = V$ or $A$)

$$i \int d^4x e^{iq\cdot x} \langle 0 \mid T \{ \Gamma^\mu_{ij}(x), \Gamma^\nu_{ij}(0) \} \mid 0 \rangle = (q^\mu q^\nu - g^\mu^\nu q^2) \Pi_{ij}(q^2) + q^\mu q^\nu \Pi_{ij,\Gamma}(q^2).$$

The superscript $(J)$ denotes the angular momentum in the rest frame of the hadronic final state. The total hadronic $\tau$ decay rate, normalized to the electronic one, can be written as

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{\Gamma(\tau \to \nu_\tau e^+ e^-)} = \frac{m_\tau^2}{s} \frac{dR_\tau(s)}{ds},$$

where

$$\frac{dR_\tau(s)}{ds} = \frac{24\pi S_{\text{EW}}}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[ \left(1 + \frac{2s}{m_\tau^2}\right) \text{Im} \Pi^{(0+1)}(s - i\epsilon) - \frac{2s}{m_\tau^2} \text{Im} \Pi^{(0)}(s - i\epsilon) \right]$$

is the inclusive hadronic spectrum, and

$$\Pi^{(J)}(s) = \frac{1}{2} \left( |V_{ud}|^2 \Pi^{(J)}_{ud,V+A}(s) + |V_{us}|^2 \Pi^{(J)}_{us,V+A}(s) \right).$$

The factor $S_{\text{EW}} \simeq 1.0194$ accounts for electroweak radiative corrections [31].

Le Diberder and Pich have pointed out the usefulness of considering weighted integrals of the spectrum $dR_\tau/ds$ [12]. Such integrals are linear combinations of the spectral moments

$$\mathcal{M}^{(J)}_k = \frac{4\pi(k + 1)}{m_{\tau}^{2k+2}} \int_0^{m_\tau^2} ds \ s^k \text{Im} \Pi^{(J)}(s - i\epsilon); \ k \geq 0,$$
which contain the dynamical information that can be extracted from hadronic \( \tau \) decays.\footnote{Here we shall not pursue the idea to disentangle vector from axial vector, or strange from non-strange modes, although in principle this may be possible.}

The weighted integrals

\[
R_k = \frac{(k+1)(k+3)(k+4)}{36S_{\text{EW}}} \int_0^{m_{\tau}^2} \left( \frac{s}{m_{\tau}^2} \right)^k \frac{dR_{\tau}(s)}{ds},
\]

which are normalized such that their perturbative expansion is

\[
R_k = 1 + \frac{\alpha_s}{\pi} + \ldots,
\]

are given by

\[
R_k = \frac{(k+3)(k+4)}{6} M_k^{(0+1)} - \frac{(k+1)(k+4)}{2} M_{k+2}^{(0+1)} + \frac{(k+1)(k+3)}{3} M_{k+3}^{(0+1)}

- \frac{(k+1)(k+3)(k+4)}{3(k+2)} M_{k+1}^{(0)} + \frac{2(k+1)(k+4)}{3} M_{k+2}^{(0)} - \frac{(k+1)(k+3)}{3} M_{k+3}^{(0)}.
\]

In particular, we note that

\[
R_0 = \frac{R_{\tau}}{3S_{\text{EW}}} = 2M_0^{(0+1)} - 2M_2^{(0+1)} + M_3^{(0+1)} - 2M_1^{(0)} + \frac{8}{3} M_2^{(0)} - M_3^{(0)}.
\]

Using the analyticity properties of the correlators in (4), the spectral moments can be written as contour integrals in the complex \( s \)-plane. One finds

\[
M_k^{(J)} = \frac{1}{2\pi i} \oint_{|s|=m_{\tau}^2} \frac{ds}{s} \left[ 1 - \left( \frac{s}{m_{\tau}^2} \right)^{k+1} \right] D^{(J)}(-s),
\]

where

\[
D^{(J)}(-s) = 4\pi^2 s \frac{d}{ds} \Pi^{(J)}(s)
\]

are the logarithmic derivatives of the correlators, which are ultraviolet (UV) finite. These functions are analytic in the complex \( s \)-plane, with discontinuities on the positive real axis. The contour integrals have two attractive features. First, to perform the integration along a circle with radius \( |s| = m_{\tau}^2 \) requires knowledge of the correlation functions only for large (complex) momenta. Second, the integrand in (9) vanishes for \( s = m_{\tau}^2 \), where the contour touches the physical cut. Therefore, unlike the spectral functions themselves, their moments are expected to admit a theoretical description in QCD, which can be organized as an expansion in powers of \( (\Lambda/m_{\tau})^{2n} \)[4][12].

One of the purposes of this paper is to reconsider the calculation of the perturbative contribution in this expansion, i.e. the term with \( n = 0 \). This contribution can be calculated by setting the current quark masses of the light quarks \( u, d \), \( s, c, b \).
and $s$ to zero. In this limit the flavour-changing currents are conserved, so that the correlators with $J = 0$ in (4) vanish. Moreover, chiral invariance implies that the spectral functions in the vector and axial vector channels are the same. It follows that

$$D^{(0+1)}_{\text{pert}}(-s) = D(-s), \quad D^{(0)}_{\text{pert}}(-s) = 0. \quad (11)$$

The perturbative expansion of the function $D(-s)$,

$$D(-s) = 1 + \sum_{n=1}^{\infty} K_n \left( \frac{\alpha_s(-s)}{\pi} \right)^n, \quad (12)$$

is known to order $\alpha^3$. For $n \geq 3$, the coefficients $K_n$ depend on the renormalization scheme. In the $\overline{\text{MS}}$ scheme, one finds (with $n_f = 3$ quark flavours) \[32\]–\[36\]

$$K_1 = 1, \quad K_2 = \frac{299}{24} - 9\zeta(3) \simeq 1.63982, \quad K_{\overline{\text{MS}}} = \frac{58057}{288} - \frac{779}{4}\zeta(3) + \frac{75}{2}\zeta(5) \simeq 6.37101, \quad (13)$$

with $\zeta(3) \simeq 1.20206$ and $\zeta(5) \simeq 1.03693$. The next coefficient, $K_4$, has been estimated using the principle of minimal sensitivity \[37\] and the effective charge approach \[38\], with the result that \[39\]

$$K_{4,\text{est}} \simeq 27.5. \quad (14)$$

The perturbative contributions to the moments in (9) are

$$M_{k,\text{pert}}^{(0+1)} = M_k, \quad M_{k,\text{pert}}^{(0)} = 0, \quad (15)$$

with

$$M_k = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1 - x^{k+1}) D(-xm_\tau^2) = 1 + \sum_{n=1}^{\infty} d_n \left( \frac{\alpha_s(m_\tau^2)}{\pi} \right)^n. \quad (16)$$

The coefficients $d_n$ can be obtained using the renormalization-group equation (RGE)

$$\mu^2 \frac{d\alpha_s(\mu^2)}{d\mu^2} = -\alpha_s(\mu^2) \beta[\alpha_s(\mu^2)] = -\alpha_s(\mu^2) \sum_{n=0}^{\infty} \beta_n \left( \frac{\alpha_s(\mu^2)}{4\pi} \right)^{n+1} \quad (17)$$

We set $|V_{ud}|^2 + |V_{us}|^2 = 1$, which is an excellent approximation, since experimentally $|V_{ub}|^2 < 1.6 \times 10^{-5}$. \[3We set $|V_{ud}|^2 + |V_{us}|^2 = 1$, which is an excellent approximation, since experimentally $|V_{ub}|^2 < 1.6 \times 10^{-5}$.}
Table 1: Perturbative contributions to the moments at order $\alpha_s^3$.

| $\alpha_s(m_\tau^2)$ | $M_0$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ |
|----------------------|-------|-------|-------|-------|-------|-------|
| 0.26                 | 1.116 | 1.101 | 1.096 | 1.094 | 1.093 | 1.092 |
| 0.28                 | 1.128 | 1.110 | 1.104 | 1.102 | 1.100 | 1.099 |
| 0.30                 | 1.141 | 1.119 | 1.113 | 1.110 | 1.108 | 1.107 |
| 0.32                 | 1.154 | 1.128 | 1.121 | 1.118 | 1.116 | 1.114 |
| 0.34                 | 1.168 | 1.138 | 1.129 | 1.125 | 1.123 | 1.122 |
| 0.36                 | 1.183 | 1.148 | 1.138 | 1.133 | 1.131 | 1.129 |
| 0.38                 | 1.198 | 1.158 | 1.146 | 1.141 | 1.138 | 1.136 |

for the running coupling constant. For $n_f = 3$, the first three coefficients of the $\beta$-function are $\beta_0 = 9$, $\beta_1 = 64$ and $\beta_2^{\text{MS}} = 3863/6$ [40]. The result is [12]

$$d_1 = 1,$$

$$d_2 = K_2 + \frac{\beta_0}{4(k+1)} \simeq 1.63982 + \frac{2.25}{k+1},$$

$$d_3^{\text{MS}} = K_3^{\text{MS}} + \frac{1}{2(k+1)} \left( \frac{\beta_0 K_2 + \beta_1}{8} \right) + \frac{\beta_0^2}{8} \left( \frac{1}{(k+1)^2} - \frac{\pi^2}{6} \right)$$

$$\simeq -10.2839 + \frac{11.3792}{k+1} + \frac{10.125}{(k+1)^2},$$

$$d_4^{\text{MS}} = K_4^{\text{MS}} + \left( \frac{3}{8(k+1)^2} - \frac{\pi^2}{16} \right) \left( \beta_0^2 K_2 + \frac{5}{24} \beta_0 \beta_1 \right)$$

$$+ \frac{3}{4(k+1)} \left( \frac{\beta_0 K_3^{\text{MS}} + \beta_1}{6} K_2 + \frac{\beta_2^{\text{MS}}}{48} - \frac{\beta_0^3 \pi^2}{48} \right) + \frac{3\beta_0^3}{32(k+1)^3}$$

$$\simeq K_4^{\text{MS}} - 155.955 - \frac{46.238}{k+1} + \frac{94.810}{(k+1)^2} + \frac{68.344}{(k+1)^3}.$$  (18)

As mentioned above, the exact result for the coefficient $K_4^{\text{MS}}$ is unknown. Nevertheless, it follows that ratios and differences of moments are known exactly to order $\alpha_s^4$, since this unknown coefficient cancels out. In Table I we show the order-$\alpha_s^3$ predictions for the first six moments as a function of the value of $\alpha_s(m_\tau^2)$ in the $\overline{\text{MS}}$ scheme.

Le Diberder and Pich have argued that one should improve the perturbative prediction for the spectral moments by performing a partial resummation of higher-order terms [5, 12]. Their observation was that the evolution of the running coupling constant along the integration contour in (16) generates higher-order corrections that are expected to be larger than the “genuine” higher-order corrections to the correlator $D(-s)$. These corrections can be resummed by writ-
Table 2: Perturbative contributions to the moments obtained from the re-
summation procedure of Le Diberder and Pich (LP) [12].

| $\alpha_s(m^2)$ | $M^0_{LP}$ | $M^1_{LP}$ | $M^2_{LP}$ | $M^3_{LP}$ | $M^4_{LP}$ | $M^5_{LP}$ |
|------------------|------------|------------|------------|------------|------------|------------|
| 0.26             | 1.110      | 1.090      | 1.087      | 1.086      | 1.085      | 1.084      |
| 0.28             | 1.120      | 1.095      | 1.093      | 1.091      | 1.090      | 1.089      |
| 0.30             | 1.130      | 1.100      | 1.098      | 1.096      | 1.095      | 1.094      |
| 0.32             | 1.140      | 1.105      | 1.103      | 1.101      | 1.100      | 1.099      |
| 0.34             | 1.150      | 1.109      | 1.107      | 1.105      | 1.104      | 1.103      |
| 0.36             | 1.160      | 1.112      | 1.112      | 1.109      | 1.108      | 1.107      |
| 0.38             | 1.169      | 1.114      | 1.116      | 1.113      | 1.111      | 1.111      |

The contour integrals over the running coupling constant can be performed numerically (see Appendix A). The effect of this resummation is significant. In Table 2, we show the results obtained using the two-loop RGE for the running coupling constant. The differences with respect to fixed-order perturbation theory are of order a few per cent. We shall comment on the usefulness of this resummation procedure at the end of Sect. 4.

In this paper we investigate another type of resummation, which deals with the so-called renormalon-chain contributions [18]–[30]. These are terms of order $\beta_0^{n-1}\alpha_s^n$ in the perturbative series for $D(-s)$ and for the moments $M_k$, where $\beta_0 = 11 - \frac{2}{3} n_f$ is the first coefficient of the $\beta$-function. A graphical representation of a renormalon chain is shown in Fig. 4. The motivation to resum these terms to all orders is the following: The coefficients in the perturbative series (12) and (16), which are polynomials in the number of quark flavours $n_f$, can be reorganized as polynomials in $\beta_0$, i.e.

$$M_k^{LP} = 1 + \sum_{n=1}^{3} K_n I_n^{k+1}(m_\tau),$$

$$I_n^{k+1}(m_\tau) = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} \frac{1-x^{k+1}}{x} \left(\frac{\alpha_s(-xm_\tau^2)}{\pi}\right)^n.$$

(19)

Since in QCD the value of $\beta_0$ is large, it is conceivable that a reasonable approximation to these coefficients is provided by the first term, i.e. the one with the highest power of $\beta_0$. Some formal arguments supporting this assertion can be
found in Ref. [30]. Below, we shall refer to this approximation as the “large-\(\beta_0\) limit”. In this limit, one finds

\[
\begin{align*}
\kappa_1 &= 1, \\
\kappa_2 \beta_0 &= \frac{297}{24} - 9\zeta(3) \simeq 1.55649, \\
\kappa_3 \beta_0^2 &= \frac{48924}{288} - \frac{513}{4} \zeta(3) \simeq 15.7112, \\
\kappa_4 \beta_0^3 &= \frac{165537}{64} - \frac{49329}{32} \zeta(3) - \frac{10935}{16} \zeta(5) \simeq 24.8320, \\
\end{align*}
\]

which can be compared with the exact values of \(K_1, K_2\) and \(K_3^{\text{MS}}\) given in (13), as well as with the estimated value of \(K_4^{\text{MS}}\) given in (14). With the exception of \(K_3^{\text{MS}}\), the large-\(\beta_0\) limit seems to work quite well. We also quote the corresponding results for the coefficients in the expansion of the moments. They are

\[
\begin{align*}
\delta_1 &= 1, \\
\delta_2 \beta_0 &= \left(\frac{11}{8} - \zeta(3) + \frac{1}{4(k+1)}\right) \beta_0 \simeq 1.55649 + \frac{2.25}{k+1}, \\
\delta_3 \beta_0^2 &= \left(\frac{151}{72} - \frac{19}{12} \zeta(3) - \frac{\pi^2}{48} + \left(\frac{11}{16} - \frac{1}{2} \zeta(3)\right) \frac{1}{k+1} + \frac{1}{8(k+1)^2}\right) \beta_0^2 \\
&\simeq -0.9442 + \frac{7.0041}{k+1} + \frac{10.125}{(k+1)^2}, \\
\end{align*}
\]

which can be compared with the exact values of \(d_1, d_2\) and \(d_3^{\text{MS}}\) given in (13).

\[\text{Figure 1: Renormalon-chain contribution to the correlator } D(-q^2).\text{ The shaded bubble represents a self-energy insertion on the gluon propagator. We note that bubble summation is not a gauge-invariant procedure in a non-abelian theory; the figure is meant as an illustration only.}\]

In Table 3, we show the results obtained in the large-\(\beta_0\) limit for the perturbative contributions to the moments at order \(\alpha_s^3\). For the difference between the “large-\(\beta_0\) results” and the exact ones, we find

\[
M_k - M_k^{\text{large-\(\beta_0\)}} = \frac{1}{12} \left(\frac{\alpha_s}{\pi}\right)^2 + \left(\frac{9133}{288} - \frac{133}{2} \zeta(3) + \frac{75}{2} \zeta(5) + \frac{35}{8(k+1)}\right) \left(\frac{\alpha_s}{\pi}\right)^3 + O(\alpha_s^4),
\]

(23)
Table 3: Large-\(\beta_0\) approximation to the perturbative contributions to the moments at order \(\alpha_s^3\).

| \(\alpha_s(m_s^2)\) | \(M_0\,_{\text{large-}}\beta_0\) | \(M_1\,_{\text{large-}}\beta_0\) | \(M_2\,_{\text{large-}}\beta_0\) | \(M_3\,_{\text{large-}}\beta_0\) | \(M_4\,_{\text{large-}}\beta_0\) | \(M_5\,_{\text{large-}}\beta_0\) |
|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 0.26                | 1.118               | 1.104               | 1.100               | 1.098               | 1.097               | 1.096               |
| 0.28                | 1.131               | 1.114               | 1.109               | 1.107               | 1.106               | 1.105               |
| 0.30                | 1.144               | 1.124               | 1.119               | 1.116               | 1.115               | 1.114               |
| 0.32                | 1.158               | 1.135               | 1.128               | 1.125               | 1.124               | 1.122               |
| 0.34                | 1.173               | 1.146               | 1.138               | 1.135               | 1.133               | 1.131               |
| 0.36                | 1.189               | 1.157               | 1.149               | 1.145               | 1.142               | 1.141               |
| 0.38                | 1.205               | 1.169               | 1.159               | 1.155               | 1.152               | 1.150               |

where \(\alpha_s = \alpha_s(m_s^2)\) in the \(\overline{\text{MS}}\) scheme. A comparison of Tables 1 and 3 shows that these differences are rather small, supporting to some extent the usefulness of the large-\(\beta_0\) limit.

In Refs. [27]–[30], efficient techniques have been developed to resum the terms of order \(\beta_0^{n-1}\alpha_s^n\) exactly to all orders in perturbation theory. Note that such a resummation includes the main part of the resummation of Le Diberder and Pich. In addition, a partial resummation of the higher-order coefficients \(K_n\) in the series (12) is achieved. This is important, since these coefficients are known to diverge factorially for large \(n\), reflecting the asymptotic behaviour of perturbative expansions in QCD. The perturbative series for \(D(-s)\) and for the moments \(M_k\) are not Borel summable, implying that the definition of these quantities in perturbation theory has an intrinsic ambiguity, which has to be cured by adding nonperturbative contributions [24]. The resummation of renormalon-chain contributions allows one to quantify (or at least estimate) this ambiguity. Moreover, unlike any truncated perturbative series, the resummed expressions for \(D(-s)\) and for the moments \(M_k\) are independent of the renormalization scheme. Nevertheless, one has to keep in mind that this method provides only a partial resummation of a perturbative series, and it is not obvious that it gives a better approximation to the full series than fixed-order perturbation theory. Therefore, to be conservative we shall take the differences between our procedure and others as an indication of the perturbative uncertainty.

### 3 Renormalon resummation for the \(D\) function

We start by considering the correlator \(D(-s)\) in the euclidean region, where \(-s = Q^2 > 0\). The terms of order \(\beta_0^{n-1}\alpha_s^n\) in the perturbative series (12) can be
resummed by evaluating the Borel integral

\[
D_{\text{Borel}}(Q^2) = 1 + \frac{4}{\beta_0} \int_0^\infty du \hat{S}_D(u) \exp \left( -\frac{4\pi u}{\beta_0 \alpha_s(Q^2)} \right)
\]

\[= 1 + \frac{4}{\beta_0} \int_0^\infty du \hat{S}_D(u) \left( \frac{\Lambda^2}{Q^2} \right)^u, \tag{24}\]

with the Borel transform \[25, 26\]

\[
\hat{S}_D(u) = \sum_{n=1}^\infty \frac{(4u)^{n-1}}{\Gamma(n)} \kappa_n = \frac{32 e^{-Cu}}{3(2-u)} \sum_{k=2}^\infty \frac{(-1)^k k}{[k^2 - (1-u)^2]^2}. \tag{25}\]

\(\Lambda\) is the scale parameter in the one-loop expression for the running coupling constant, and \(C\) is a scheme-dependent constant, with \(C = -5/3\) in the \(\overline{\text{MS}}\) scheme (\(C = 0\) in the V scheme, see below). The coefficients \(\kappa_n\) have been defined in (20). The function \(\hat{S}_D(u)\) contains pole singularities on the real \(u\) axis. The singularities on the negative axis are related to the UV behaviour of Feynman diagrams and are called UV renormalons. The singularities on the positive axis, which arise from the integration over low virtual momenta in Feynman diagrams, are called IR renormalons \[18\]. In the present case there are UV renormalon poles located at \(u = -1, -2, \ldots\) and IR renormalon poles at \(u = 2, 3, \ldots\). In general, a pole at \(u = u_i\) is associated with a factorial growth of the expansion coefficients \(\kappa_n\) of the form \(\Gamma(n)(4u_i)^{-n}\) (there is an extra factor \(n\) for a double pole), so that the renormalon singularity closest to the origin determines the asymptotic behaviour of the expansion coefficients. In the case of UV renormalons, these contributions have alternating sign and can be resummed by means of the Borel integral (24). However, for IR renormalons all terms have the same sign and the series is divergent. In fact, IR renormalon singularities fall on the integration contour in (24), making the value of the Borel integral ambiguous. A measure of the ambiguity is provided by the residue of the nearest IR renormalon pole. In the present case this is located at \(u = 2\) and leads to an ambiguity of order \(\Lambda^4/Q^4\).

The appearance of IR renormalons indicates that perturbation theory is incomplete; it must be supplemented by nonperturbative corrections. Only the sum of all perturbative and nonperturbative contributions is unambiguous. For euclidean correlation functions of currents, the OPE provides a consistent framework for a systematic incorporation of nonperturbative effects \[11\]. In the case of the function \(D(Q^2)\), such effects appear first at order \(1/Q^4\) and are parametrized by the gluon condensate \[15\], which has an ambiguity that compensates the ambiguity in the resummed perturbative series \[20 - 24\]. This will be discussed in more detail in Sect. 6.

An efficient technique for evaluating the Borel integral (24) has been developed.
in Ref. [28], where it was shown that

\[ D_{\text{Borel}}(Q^2) = 1 + \frac{1}{\beta_0} \int_0^\infty \frac{d\tau}{\tau} w_D(\tau) a(\tau Q^2), \]  

(26)

where

\[ a(\tau Q^2) = \frac{1}{\ln \tau + \ln(Q^2/\Lambda_V^2)} = \frac{\beta_0}{4\pi} \alpha_s^{(V)}(\tau Q^2). \]

(27)

Here \( \alpha_s^{(V)}(\tau Q^2) \) is the one-loop running coupling constant in the so-called V scheme, in which the coupling constant is defined in terms of the heavy-quark potential [42]. The relation between the V scheme and the \( \overline{\text{MS}} \) scheme is such that

\[ \alpha_s(V)(\mu^2) = \alpha_s(\mu^2) = \frac{5}{3}. \]

Thus, the parameter \( \Lambda_V \) can be obtained from the value of \( \alpha_s(m_T^2) \) in the MS scheme by

\[ \Lambda_V = m_T \exp \left( \frac{5}{6} - \frac{2\pi}{9\alpha_s(m_T^2)} \right) = e^{5/6} \Lambda_{\overline{\text{MS}}}. \]

(28)

The function \( w_D(\tau) \) in (26) describes the distribution of the virtuality of the gluon in the diagrams associated with the order-\( \alpha_s \) correction to \( D(Q^2) \). It is given by [28]

\begin{align*}
   w_D(\tau) &= \frac{32}{3} \left\{ \left( \frac{7}{4} - \ln \tau \right) \tau^2 + \tau (1 + \tau) \left[ L_2(-\tau) + \ln \ln(1 + \tau) \right] \right\}; \quad \tau < 1, \\
   w_D(\tau) &= \frac{32}{3} \left\{ \frac{3}{4} + \frac{1}{2} \ln \tau + (1 + \ln \tau) \tau \\
   &\quad \quad + \tau (1 + \tau) \left[ L_2(-\tau^{-1}) - \ln \ln(1 + \tau^{-1}) \right] \right\}; \quad \tau > 1.
\end{align*}

(29)

\( L_2(x) = -\int_0^x \frac{dt}{t} \ln(1 - t) \) is the dilogarithm. In Fig. [4], we show \( w_D(\tau) \) as a function of \( \ln \tau \), which is the natural integration variable in (26). The expansion coefficients \( \kappa_n \) are related to the moments of this distribution by

\[ \kappa_n = \int_0^\infty \frac{d\tau}{\tau} w_D(\tau) (-\ln \tau - C)^{n-1}. \]

(30)

The value of the integral in (26) is independent of the renormalization scheme; changing the scheme simply amounts to rescaling the integration variable \( \tau \).\footnote{In Ref. [28], the function \( w_D(\tau) \) was denoted \( \tau \tilde{w}_D(\tau) \).}

However, as written above the integral is ambiguous because the integration contour runs over the Landau pole in the running coupling constant \( a(\tau Q^2) \), which is located at \( \tau = \tau_L = \Lambda_V^2/Q^2 \). This is nothing than the IR renormalon ambiguity mentioned above. Different regularization prescriptions for the Landau pole lead

\footnote{This is true for regular (MS-like) schemes only.}
to different results. A measure of the ambiguity is provided by the residue of the pole, i.e.

$$\Delta D_{\text{ren}}(Q^2) = \frac{1}{\beta_0} w_D(\tau_L) = \frac{8}{\beta_0} \frac{\Lambda_V^4}{Q^4} - \frac{16}{3\beta_0} \left( \ln \frac{Q^2}{\Lambda_V^2} + \frac{3}{2} \right) \frac{\Lambda_V^6}{Q^6} + O(Q^{-8}). \quad (31)$$

The quantity $\Delta D_{\text{ren}}(Q^2)$ is equal to the sum of the residues of the IR renormalon poles in the integrand of the Borel integral in (24) [28]. Below we shall use the principle value prescription to regulate the Landau pole in (26), which is equivalent to calculating the principle value of the Borel integral.

To illustrate the importance of the all-order resummation, we quote numerical results for the case $Q^2 = m^2_\tau$ and $\alpha_s(m^2_\tau) = 0.32$ in the $\overline{\text{MS}}$ scheme. For these parameters, the order-$\alpha_s^3$ result in the large-$\beta_0$ limit is $D(m^2_\tau) = 1.135$ (the exact result is 1.126). The resummation of all terms of order $\beta_0^{n-1} \alpha_s^n$ leads to a value that is significantly larger, $D_{\text{Borel}}(m^2_\tau) = 1.151 \pm 0.003$. The central value corresponds to the principle value of the Borel integral, whereas the error is given by the renormalon ambiguity $\Delta D_{\text{ren}}$ in (31). It is instructive to investigate the behaviour of the perturbative series in the large-$\beta_0$ limit. The expansion coefficients $\kappa_n$ can be obtained by expanding the Borel transform $\tilde{S}_D(u)$ in (23) in powers of $u$. For the $\overline{\text{MS}}$ and the V scheme, the results are given in Table 4. In the V scheme, the coefficients have alternating sign, and the nearest UV renormalon at $u = -1$ is dominant already at low orders [30]. In the $\overline{\text{MS}}$ scheme, UV renormalons are not yet dominant at low orders due to an extra factor $e^{5u/3}$ in the Borel transform, which suppresses the region of negative $u$. The first few coefficients have the same sign, and their growth is determined by the interplay of the UV renormalon at $u = -1$ and the IR renormalon at $u = 2$.

In Table 5, we show the partial sums $D^{(N)} = 1 + \sum_{n=1}^N \kappa_n \beta_0^{n-1} (\alpha_s/\pi)^n$ in the two schemes. Note that the asymptotic value, which is given by the integral

![Figure 2: Distribution function $w_D(\tau)$ versus $\ln \tau$.](image)
Table 4: Perturbative coefficients for the $D$ function in the large-$\beta_0$ limit.

| $n$ | $\kappa_{n}^{\text{MS}}\beta_0^{n-1}$ | $\kappa_{n}^{\text{V}}\beta_0^{n-1}$ |
|-----|---------------------------------|---------------------------------|
| 1   | 1.55649                         | -2.19351                        |
| 2   | 15.7112                         | 18.1000                         |
| 3   | 24.8320                         | -138.989                        |
| 4   | 787.827                         | 1610.41                         |

in (26), is scheme independent. In the $\overline{\text{MS}}$ scheme, the minimal term in the series is reached at $n = 4$ and gives a contribution of 0.3%. If one truncates the series at this point, the difference to the resummed result is 1.4%, which is about four times the renormalon ambiguity. In the $\text{V}$ scheme, the minimal term is reached already at $n = 2$, and its contribution is 6.0%. If the series is truncated at this point, the difference to the resummed result is 4.6%. Since in the $\text{V}$ scheme the leading UV renormalon is dominant, the error due to the truncation of the series is necessarily of order $(\Lambda_\text{V}/m_\tau)^2 \simeq 6.7\%$. The fact that this error is much larger than the IR renormalon ambiguity, and moreover has a different power dependence on $m_\tau$, has been emphasized in Ref. [43]. For comparison, we note that the leading nonperturbative contribution to the $D$ function, which is proportional to the gluon condensate [15], is given by (see Appendix C)

$$D_{\text{power}}(m_\tau^2) = \frac{2\pi^2}{3m_\tau^4} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \simeq (1.2 \pm 0.6)\%.$$  \hspace{1cm} (32)

Clearly, to give meaning to the value of nonperturbative parameters such as the vacuum condensates one has to control the higher-order behaviour of perturbation theory; in particular, it is necessary to resum the contribution of the first UV renormalon. If one believes that the large-$\beta_0$ limit provides a reasonable description of the nature of this singularity, the resummation is achieved by performing the Borel integral, and then the residual ambiguity is due to the nearest IR renormalon and thus of order $(\Lambda_\text{V}/m_\tau)^4$. If, however, one wants to be more conservative and truncate the series at the last term that is exactly known, one is left with an uncertainty that is larger, namely of order the last term itself.

## 4 Renormalon resummation for the moments

Let us now turn to the resummation of renormalon-chain contributions for the spectral moments. In the Minkowski region, a simple representation of the Borel integral of the form (26) does not exist. [28]. Beneke et al. have shown that the
Table 5: Partial sums of the perturbative series for the $D$ function in the large-$\beta_0$ limit. In the second column we show for comparison the known exact results in the $\overline{\text{MS}}$ scheme.

| $N$ | $D^{(N)}_{\overline{\text{MS}}}(m_z^2)$ | exact result (\overline{\text{MS}}) | $D^{(N)}_{\overline{\text{V}}}(m_z^2)$ |
|-----|---------------------------------|---------------------------------|---------------------------------|
| 1   | 1.1019                          | 1.1019                          | 1.1648                          |
| 2   | 1.1180                          | 1.1189                          | 1.1052                          |
| 3   | 1.1346                          | 1.1256                          | 1.1863                          |
| 4   | 1.1373                          | 1.0837                          |                                 |
| 5   | 1.1459                          |                                 | 1.2795                          |
| $\infty$ | 1.151 ± 0.003                | 1.151 ± 0.003                |                                 |

The principle value of the Borel integral can be obtained from [27, 29]

$$M_k^{\text{Borel}} = 1 + \frac{1}{\pi \beta_0} \int_0^\infty \frac{d\tau}{\tau} W_k(\tau) \arctan[\pi a(\tau m_z^2)] + \frac{1}{\beta_0} \Re \int_{-\tau_L}^{\tau_L} \frac{d\tau}{\tau} W_k(\tau - i\epsilon),$$

(33)

where $\tau_L = \Lambda^2_{\overline{\text{V}}}/m_z^2$ is the position of the Landau pole in the coupling constant $a(\tau m_z^2)$ defined in (27). The functions $W_k(\tau)$ are related to weighted integrals of the distribution function $w_D(\tau)$ [28]:

$$W_k(\tau) = (k + 1) \int_0^1 dx x^k w_D(\tau/x).$$

(34)

It is possible to perform these integrals explicitly. For $k = 0$, we find that

$$W_0(\tau) = \frac{32\tau}{3} \left\{ 4 - 3\zeta(3) - \frac{15}{4} \tau - 2L_3(-\tau) + \left[ 2\tau + L_2(-\tau) \right] \ln \tau \right. \right.$$

$$\left. - (1 + \tau) \left[ L_2(-\tau) + \ln \tau \ln(1 + \tau) \right] \right\}; \quad \tau < 1,$$

(35)

$$W_0(\tau) = \frac{32\tau}{3} \left\{ -1 + \frac{5}{4\tau} + 2L_3(-\tau^{-1}) - \left( 1 - \frac{1}{2\tau} - L_2(-\tau^{-1}) \right) \ln \tau \right.$$

$$\left. - (1 + \tau) \left[ L_2(-\tau^{-1}) - \ln \tau \ln(1 + \tau^{-1}) \right] \right\}; \quad \tau > 1.$$
\[ W_1(\tau) = \frac{64\tau}{3} \left\{ 3 + \frac{1}{2\tau} + 2\tau L_3(-\tau^{-1}) + \left( 2 + \frac{1}{4\tau} + \tau L_2(-\tau^{-1}) \right) \ln \tau + (1 + \tau) \left[ L_2(-\tau^{-1}) - \ln \tau \ln(1 + \tau^{-1}) \right] \right\}; \quad \tau > 1. \]  

For the remaining cases, \( k \geq 2 \), we find

\[ W_k(\tau) = \frac{32\tau}{3} (k + 1) \left\{ \left( \frac{k^2 + k + 1}{k^3} + \frac{3k + 5}{4(k + 1)^2} - \frac{7k^2 - 18k + 15}{4(k - 1)^3} - \frac{2(-1)^k}{k(k - 1)} S_2^{(k)}(-1) \right) \tau^k + \frac{7k^2 - 18k + 15}{4(k - 1)^3} \tau - \frac{k - 2}{(k - 1)^2} \tau \ln \tau + \left( \frac{1}{k} + \frac{\tau}{k - 1} + \frac{(-\tau)^k}{k(k - 1)} \right) \left[ L_2(-\tau) + \ln \tau \ln(1 + \tau) \right] + \frac{(-\tau)^k}{k(k - 1)} \left( S_2^{(k)}(-\tau) + S_1^{(k)}(-\tau) \ln \tau - \frac{1}{2} \ln^2 \tau \right) \right\}; \quad \tau < 1, \]  

\[ W_k(\tau) = \frac{32\tau}{3} (k + 1) \left\{ \left( \frac{k^2 + k + 1}{k^3} + \frac{3k + 5}{4(k + 1)^2} + \frac{k + 1}{k^2} + \frac{1}{2(k + 1)} \frac{1}{\tau} \right) \ln \tau + \left( \frac{1}{k} + \frac{\tau}{k - 1} + \frac{(-\tau)^k}{k(k - 1)} \right) \left[ L_2(-\tau^{-1}) - \ln \tau \ln(1 + \tau^{-1}) \right] - \frac{(-\tau)^k}{k(k - 1)} \left( S_2^{(k)}(-\tau) + S_1^{(k)}(-\tau) \ln \tau \right) \right\}; \quad \tau > 1, \]  

with

\[ S_n^{(k)}(x) = \sum_{l=1}^{k-1} \frac{x^{-l}}{l^n}. \]  

These functions obey the normalization

\[ \int_0^\infty \frac{d\tau}{\tau} W_k(\tau) = 4, \]  

which yields the unit coefficient in front of \( \alpha_s/\pi \) in the perturbative series for \( M_k \). Moreover, we note that \( W_\infty(\tau) = w_D(\tau) \). In Fig. 3 we show \( W_k(\tau) \) as a function of \( \ln \tau \) for a few values of \( k \). For \( k \to \infty \), these functions converge towards the function \( w_D(\tau) \) depicted in Fig. 2.

Weighted integrals of the functions \( W_k(\tau) \) with powers of \( \ln \tau \) determine the coefficients in the perturbative expansion of the moments \( M_k \) in the large-\( \beta_0 \) limit.
Expanding (33) to order $\alpha_s^3$, one finds \cite{28}

$$
M_k^\text{large-\beta_0} = 1 + \frac{\alpha_s}{\pi} \left( \frac{\beta_0}{4} \right) \langle \ln \tau \rangle + C \left( \frac{\alpha_s}{\pi} \right)^2 + \frac{\beta_0^2}{16} \left( \langle \ln^2 \tau \rangle + 2 \langle \ln \tau \rangle + C^2 - \frac{\pi^2}{3} \right) \left( \frac{\alpha_s}{\pi} \right)^3 + O(\alpha_s^4),
$$

where $\alpha_s = \alpha_s(m_r^2)$, $C = -5/3$ for the $\overline{\text{MS}}$ scheme ($C = 0$ for the V scheme), and

$$
\langle \ln \tau \rangle = \frac{1}{4} \int_0^\infty \frac{d\tau}{\tau} \ln \tau W_k(\tau) = 4\zeta(3) - \frac{23}{6} - \frac{1}{k + 1},
$$

$$
\langle \ln^2 \tau \rangle = \frac{1}{4} \int_0^\infty \frac{d\tau}{\tau} \ln^2 \tau W_k(\tau) = 18 - 12\zeta(3) + \left( \frac{23}{3} - 8\zeta(3) \right) \frac{1}{k + 1} + \frac{2}{(k + 1)^2}.
$$

This does indeed reproduce the result given in (22).

In Table 6, we show the resummed results for the first six moments as a function of the value of the coupling constant $\alpha_s(m_r^2)$ in the $\overline{\text{MS}}$ scheme, which according to (28) defines the scale parameter $\Lambda_V$. In order to perform the numerical integrations in (33), it is convenient to use the asymptotic behaviour of the functions $W_k(\tau)$ for large and small values of $\tau$, which is given in Appendix B. When expanded in powers of the coupling constant, our resummation reproduces the large-$\beta_0$ limit for the perturbative coefficients given in (22). We can correct for the missing pieces, which are known up to order $\alpha_s^3$, using (23). This leads to the numbers given in the lower portion of the table.\footnote{We note, however, that this correction depends on the renormalization scheme. The results presented here refer to the $\overline{\text{MS}}$ scheme.}
Table 6: Predictions for the resummed perturbative contributions to the moments obtained from the principle value of the Borel integral. The lower portion of the table contains the results corrected for the exact coefficients up to order $\alpha_s^3$.

| $\alpha_s(m_T^2)$ | $M_0^{\text{Borel}}$ | $M_1^{\text{Borel}}$ | $M_2^{\text{Borel}}$ | $M_3^{\text{Borel}}$ | $M_4^{\text{Borel}}$ | $M_5^{\text{Borel}}$ |
|-------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 0.26              | 1.122               | 1.097               | 1.094               | 1.092               | 1.092               | 1.091               |
| 0.28              | 1.136               | 1.103               | 1.100               | 1.099               | 1.098               | 1.097               |
| 0.30              | 1.151               | 1.108               | 1.105               | 1.104               | 1.103               | 1.103               |
| 0.32              | 1.168               | 1.113               | 1.108               | 1.108               | 1.108               | 1.107               |
| 0.34              | 1.186               | 1.117               | 1.109               | 1.110               | 1.111               | 1.111               |
| 0.36              | 1.205               | 1.121               | 1.109               | 1.111               | 1.112               | 1.112               |
| 0.38              | 1.226               | 1.126               | 1.105               | 1.109               | 1.111               | 1.112               |
| 0.26              | 1.119               | 1.093               | 1.090               | 1.088               | 1.087               | 1.087               |
| 0.28              | 1.133               | 1.098               | 1.095               | 1.093               | 1.092               | 1.092               |
| 0.30              | 1.148               | 1.103               | 1.098               | 1.097               | 1.097               | 1.096               |
| 0.32              | 1.163               | 1.106               | 1.101               | 1.100               | 1.100               | 1.099               |
| 0.34              | 1.180               | 1.109               | 1.100               | 1.101               | 1.101               | 1.101               |
| 0.36              | 1.199               | 1.112               | 1.098               | 1.100               | 1.100               | 1.101               |
| 0.38              | 1.218               | 1.114               | 1.093               | 1.095               | 1.097               | 1.098               |

The renormalon ambiguity in the perturbative contribution to the moments $M_k$ can be obtained by inspecting the corresponding Borel transforms

$$S_k(u) = \frac{k + 1}{k + 1 - u} \sin \frac{\pi u}{u} S_D(u),$$

or by performing the integrals [27]

$$\Delta M_k^{\text{ren}} = \frac{1}{\pi \beta_0} \text{Im} \int_{-\gamma_L}^{0} \frac{d\tau}{\tau} W_k(\tau - i\epsilon).$$

The structure of IR renormalon poles is as follows: For $k = 0$ there are single poles at $u = 3, 4, \ldots$; for $k = 1$ there are single poles at $u = 2, 3, \ldots$; for $k = 2$ there is a double pole at $u = 3$ and singles poles at $u = 4, 5, \ldots$; for $k \geq 2$ there is a double pole at $u = k + 1$ and single poles at $u \geq 3$ ($u \neq k + 1$). For the corresponding renormalon ambiguities, we obtain

$$\Delta M_k^{\text{ren}} = -\frac{16}{9 \beta_0} \frac{k + 1}{k - 2} \frac{\Lambda^6_{\text{V}}}{m_T^6} m_T^6 + O(m_T^{-8}); \quad k \neq 1, 2,$$

$$\Delta M_1^{\text{ren}} = -\frac{8}{3 \beta_0} \frac{\Lambda^4_{\text{V}}}{m_T^4} + \frac{32 \Lambda^6_{\text{V}}}{9 \beta_0} \frac{m_T^6}{m_T^6} + O(m_T^{-8}).$$
\[
\Delta M_{2}^{\text{ren}} = \frac{-16}{3\beta_0} \left( \ln \frac{m_\tau^2}{\Lambda_{V}^2} + \frac{11}{6} \right) \frac{\Lambda_{V}^6}{m_\tau^6} + O(m_\tau^{-8}). \tag{44}
\]

Table 7: Perturbative coefficients for the first three moments in the large-\(\beta_0\) limit.

| \(k\) | \(n\) | \(\delta_{n}^{\text{MS}}\beta_0^{n-1}\) | \(\delta_{n}^{V}\beta_0^{n-1}\) |
|-------|-------|-----------------|-----------------|
| 0     | 1     | 3.80649         | 0.05649         |
|       | 2     | 16.1854         | 1.69928         |
|       | 3     | 56.3140         | -17.9203        |
|       | 4     | 223.935         | 139.694         |
|       | 5     |                 |                 |
| 1     | 1     | 2.68149         | -1.06851        |
|       | 2     | 5.08959         | -0.95907        |
|       | 3     | -35.7603        | -32.6273        |
|       | 4     | -443.813        | 154.154         |
| 2     | 1     | 2.30649         | -1.44351        |
|       | 2     | 2.51598         | -0.72018        |
|       | 3     | -47.2767        | -31.0109        |
|       | 4     | -424.721        | 207.944         |

Let us again study the behaviour of the perturbative series in the large-\(\beta_0\) limit. The asymptotic growth of the expansion coefficients \(\delta_{n}\) in (20), which can be obtained by expanding the Borel transform \(\hat{S}_{k}(u)\) in powers of \(u\), is determined by the UV renormalon at \(u = -1\). However, the behaviour in low orders is governed by a complicated interplay of this UV renormalon with the nearest IR renormalon, which is located at \(u = 2\) for \(k = 1\) and \(u = 3\) for \(k \neq 1\). For the first three moments, we show the coefficients \(\delta_{n}\) in Table 7. In Table 8, we show the partial sums \(M_{k}^{(N)} = 1 + \sum_{n=1}^{N} \delta_{n} \beta_0^{n-1} (\alpha_s/\pi)^n\) and compare them to the asymptotic value obtained from the resummation of renormalon chains. As before, the minimal term in the series is reached later in the \(\overline{\text{MS}}\) scheme than in the V scheme. If one truncates the series at the minimal term, in both schemes the differences to the resummed results are typically of order \(10^{-3}\) for \(k = 0\), and of order 1–2\% for \(k = 1, 2\). In all cases these differences are much larger than the renormalon ambiguities.

An interesting question that can be addressed using the large-\(\beta_0\) limit is whether the resummation procedure of Le Diberder and Pich [5] improves the convergence of the perturbative series. To answer this question, we investigate the partial sums

\[
M_{k}^{\text{LP},(N)} = 1 + \sum_{n=1}^{N} \kappa_{n} \beta_0^{n-1} I_{n}^{k+1}(m_\tau), \tag{45}
\]

where the integrals \(I_{n}^{k+1}(m_\tau)\) are evaluate using the one-loop \(\beta\)-function for the running coupling constant (see Appendix A). This ensures that for \(N \to \infty\) one recovers the large-\(\beta_0\) limit of the series. The partial sums for the first three moments are shown in Table 9. The results are interesting, as they indicate
Table 8: Partial sums of the perturbative series for the first three moments in the large-$\beta_0$ limit.

| $N$ | $M_{0,\text{MS}}^{(N)}$ | $M_{0,V}^{(N)}$ | $M_{1,\text{MS}}^{(N)}$ | $M_{1,V}^{(N)}$ | $M_{2,\text{MS}}^{(N)}$ | $M_{2,V}^{(N)}$ |
|-----|------------------|----------------|------------------|----------------|------------------|----------------|
| 1   | 1.1019           | 1.1648         | 1.1019           | 1.1648         | 1.1019           | 1.1648         |
| 2   | 1.1414           | 1.1663         | 1.1297           | 1.1358         | 1.1258           | 1.1256         |
| 3   | 1.1585           | 1.1740         | 1.1351           | 1.1315         | 1.1284           | 1.1224         |
| 4   | 1.1645           | 1.1607         | 1.1312           | 1.1074         | 1.1234           | 1.0995         |
| 5   | 1.1670           | 1.1777         | 1.1263           | 1.1262         | 1.1187           | 1.1248         |
| $\infty$ | 1.1678 ± .00003 | 1.1131 ± 0.0039 | 1.1080 ± 0.0008 |

that the convergence is not improved in a significant way. In some cases (such as $k = 0$), fixed-order perturbation theory even converges better towards the asymptotic result. We believe that this observation is not specific for the large-$\beta_0$ limit, since the large $\pi^2$-terms resummed in the approach of Le Diberder and Pich are retained in this limit.

Table 9: Partial sums of the perturbative series for the first three moments obtained using the large-$\beta_0$ limit of the resummation procedure of Le Diberder and Pich.

| $N$ | $M_{0,\text{MS}}^{\text{LP},(N)}$ | $M_{0,V}^{\text{LP},(N)}$ | $M_{1,\text{MS}}^{\text{LP},(N)}$ | $M_{1,V}^{\text{LP},(N)}$ | $M_{2,\text{MS}}^{\text{LP},(N)}$ | $M_{2,V}^{\text{LP},(N)}$ |
|-----|------------------|----------------|------------------|----------------|------------------|----------------|
| 1   | 1.1155           | 1.1852         | 1.0973           | 1.1350         | 1.0940           | 1.1316         |
| 2   | 1.1343           | 1.1235         | 1.1098           | 1.1094         | 1.1059           | 1.1031         |
| 3   | 1.1526           | 1.1903         | 1.1180           | 1.1104         | 1.1141           | 1.1208         |
| 4   | 1.1552           | 1.1309         | 1.1185           | 1.1415         | 1.1148           | 1.1103         |
| 5   | 1.1622           | 1.2018         | 1.1175           | 1.0503         | 1.1155           | 1.1341         |
| $\infty$ | 1.1678 ± .00003 | 1.1131 ± 0.0039 | 1.1080 ± 0.0008 |

To summarize, in the first part of this paper we have investigated three schemes to calculate the perturbative contributions to the spectral moments $M_k$: fixed-order perturbation theory, the resummation procedure of Le Diberder and Pich, and resummed perturbation theory in the large-$\beta_0$ limit. The results are summarized in Tables 1, 2 and 6. As there is no strong argument to prefer one of these schemes over the others, the differences between the numerical results must be considered as theoretical uncertainties in the perturbative calculation of the moments. These differences are of order a few per cent. We note that taking ratios of moments (as proposed in Ref. [12]) does not improve the accuracy, although such ratios are known exactly to order $\alpha_s^4$. For instance, the predictions
for the ratio $M_1/M_0$ differ by a larger amount than the predictions for the individual moments. In the second part of the paper, we shall investigate what this uncertainty implies for the sensitivity to power corrections.

5 Power corrections

One of the goals of analysing moments of the $\tau$ hadronic spectrum is to test QCD at the level of nonperturbative effects, which manifest themselves in the form of power corrections [12]. In the standard approach of SVZ, these corrections are parametrized in terms of vacuum expectation values of local, gauge-invariant operators, the so-called condensates [15]. There have been attempts to extract the condensates of dimension four, six and even eight from the analysis of hadronic $\tau$ decays [9]–[11], [16, 17]. However, the feasibility of such determinations is limited by the fact that, in order to be sensitive to power-suppressed effects, one has to control perturbation theory to a sufficient level of accuracy.

There is also a conceptual need for an analysis of power corrections. As we have seen, the perturbative definition of the moments is ambiguous because of the presence of IR renormalons. In order to deal with these ambiguities in a consistent way, one is forced to add other nonperturbative contributions. Hence, we shall write

$$M^{(J)}_k = M^{(J)}_{k,\text{pert}} + M^{(J)}_{k,\text{power}} = M_k \delta_{J=1} + M^{(J)}_{k,\text{power}}. \quad (46)$$

Only the sum of the perturbative and nonperturbative contributions can be expected to be well defined and unambiguous. We shall now present a model-independent analysis of power corrections, which does not rely on assumptions about their origin. Our goal is to relate the power corrections to the moments to those of the euclidean correlators $D^{(J)}(Q^2)$ defined in (10), for which we write

$$D^{(J)}(Q^2) = D^{(J)}_{\text{pert}}(Q^2) + D^{(J)}_{\text{power}}(Q^2) = D(Q^2) \delta_{J=1} + \sum_{n=1}^{\infty} \frac{\langle O_{2n}^{(J)}(Q^2) \rangle}{Q^{2n}}. \quad (47)$$

The nonperturbative quantities $\langle O_{2n}^{(J)}(Q^2) \rangle$ scale like $\Lambda^{2n}$. The only assumption we shall make is that these quantities have a weak, logarithmic dependence on $Q^2$. Writing then

$$\langle O_{2n}^{(J)}(Q^2) \rangle = \exp \left( \ln \frac{Q^2}{m_\tau^2} \frac{d}{d \ln m_\tau^2} \right) \langle O_{2n}^{(J)}(m_\tau^2) \rangle \equiv \left( \frac{Q^2}{m_\tau^2} \right)^\nabla \langle O_{2n}(m_\tau^2) \rangle, \quad (48)$$

it is justified to treat $\nabla = d/d \ln m_\tau^2$ as a small parameter. In fact, in QCD the nonperturbative quantities $\langle O_{2n}^{(J)}(m_\tau^2) \rangle$ depend on $m_\tau$ through the running
coupling constant and through the running quark masses,\footnote{This becomes explicit if these quantities are expressed in terms of scale-invariant condensates, see Appendix \ref{app:scaleinvariance}.} so that

\[
\nabla = \gamma_m [\alpha_s(m^2_T)] \frac{m_q^2 \partial}{\partial m_q^2} - \beta [\alpha_s(m^2_T)] \alpha_s \frac{\partial}{\partial \alpha_s} \\
\simeq \frac{-\alpha_s(m^2_T)}{\pi} \left(2m_q^2 \frac{\partial}{\partial m_q^2} + \frac{9}{4} \alpha_s \frac{\partial}{\partial \alpha_s}\right).
\]

(49)

Here \(\gamma_m = -2\alpha_s/\pi + \ldots\) is the anomalous dimension of the running quark mass. Inserting expression (48) into the contour integral (9), we find

\[
\mathcal{M}_{k,\text{power}}(J) = \sum_{n=1}^{\infty} \frac{1}{m^2_T} I_{k,n}(\nabla) \langle O^{(J)}_{2n}(m^2_T) \rangle ,
\]

(50)

with

\[
I_{k,n}(\nabla) = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1 - x^{k+1}) (-x)^{n-1} \\
= (-1)^{n+1} \frac{k + 1}{(k + 1 - n + \nabla)(n - \nabla)} \sin(\pi \nabla) \frac{\sin(\pi \nabla)}{\pi} .
\]

(51)

To second order in \(\nabla\), we obtain

\[
I_{k,k+1}(\nabla) = (-1)^k \left\{ 1 + \frac{\nabla}{k + 1} + \left(\frac{1}{(k + 1)^2} - \frac{\pi^2}{6}\right) \nabla^2 + \mathcal{O}(\nabla^3) \right\}
\]

(52)

for \(n = k + 1\), and

\[
I_{k,n}(\nabla) = (-1)^{n+1} \frac{k + 1}{(k + 1 - n)n} \left\{ \nabla + \frac{k + 1 - 2n}{(k + 1 - n)n} \nabla^2 + \mathcal{O}(\nabla^3) \right\}
\]

(53)

for \(n \neq k + 1\). Hence, if the logarithmic dependence of \(\langle O_{2n}(m^2_T) \rangle\) on \(m_T\) is neglected, the \(k\)-th moment projects out the power corrections of dimension \(d = 2k + 2\):

\[
\mathcal{M}_{k,\text{power}}^{(J)} = (-1)^k \frac{\langle O_{2k+2}^{(J)}(m^2_T) \rangle}{m_T^{2k+2}} + \mathcal{O}(\nabla).
\]

(54)

The most important effect of the scale dependence is to induce contributions from lower-dimensional operators.

Using the above results we can derive explicit expressions for the power corrections to the moments, treating both \(\nabla\) and \(1/m^2_T\) as small parameters. Neglecting terms of order \(\nabla^3 \langle O_2 \rangle/m^2_T\), \(\nabla^2 \langle O_4 \rangle/m^4_T\), \(\nabla \langle O_6 \rangle/m^6_T\) and operators of dimension
larger or equal to eight, we find

\[ M_{0,\text{power}}^{(0+1)} = \left[ 1 + \nabla + \left( 1 - \frac{\pi^2}{6} \right) \nabla^2 \right] \frac{\langle O_2^{(0+1)} \rangle}{m_\tau^2} + \frac{\nabla}{2} \frac{\langle O_4^{(0+1)} \rangle}{m_\tau^4} + \ldots, \]

\[ M_{1,\text{power}}^{(j)} = 2 \nabla \frac{\langle O_2^{(j)} \rangle}{m_\tau^2} - \left( 1 + \frac{\nabla}{2} \right) \frac{\langle O_4^{(j)} \rangle}{m_\tau^4} + \ldots, \]

\[ M_{2,\text{power}}^{(j)} = \left( \frac{3}{2} \nabla + \frac{3}{4} \nabla^2 \right) \frac{\langle O_2^{(j)} \rangle}{m_\tau^2} - \frac{3}{2} \nabla \frac{\langle O_4^{(j)} \rangle}{m_\tau^4} + \frac{\langle O_6^{(j)} \rangle}{m_\tau^6} + \ldots, \]

\[ M_{k,\text{power}}^{(j)} = \left( k + \frac{1}{k} \nabla + \frac{k^2 - 1}{k^2} \nabla^2 \right) \frac{\langle O_2^{(j)} \rangle}{m_\tau^2} - \frac{k + 1}{2(k - 1)} \nabla \frac{\langle O_4^{(j)} \rangle}{m_\tau^4} + \ldots; \quad k \geq 56. \]

Thus, the power corrections are parametrized by a set of fundamental nonperturbative parameters \( \nabla^k \langle O_n^{(j)} \rangle \).

We now apply this general formalism to calculate the leading power corrections in the SVZ approach \[15\]. The corresponding corrections \( \langle O_n^{(j)} \rangle \) to the current correlators \( D^{(j)} \) have been calculated, at next-to-leading order in the coupling constant, by several authors. They have been summarized and rewritten in terms of scale-invariant condensates in Ref. \[4\]. For the corrections of dimension two and four, the results are \[14\]–\[17\]

\[ \langle O_2^{(0+1)} \rangle = -3 \left( 1 + \frac{13}{3} \frac{\alpha_s}{\pi} \right) |V_{ud}|^2 \left( m_u^2 + m_d^2 \right), \]

\[ \langle O_4^{(0+1)} \rangle = - \frac{2\pi^2}{3} \left( 1 - \frac{11}{18} \frac{\alpha_s}{\pi} \right) \left( \frac{\alpha_s}{\pi} G^2 \right) \]

\[ + \frac{8\pi^2}{3} \left( 1 - \frac{\alpha_s}{\pi} \right) |V_{uj}|^2 \left( m_u \bar{\psi}_u \psi_u + m_j \bar{\psi}_j \psi_j \right) + \frac{32\pi^2}{27} \frac{\alpha_s}{\pi} \sum_k \langle m_k \bar{\psi}_k \psi_k \rangle \]

\[ + 12 |V_{uj}|^2 m_u^2 m_j^2 - \frac{2}{7} \sum_k m_k^4 - \left( \frac{24}{7} \frac{\pi}{\alpha_s} + 1 \right) |V_{uj}|^2 \left( m_u^4 + m_j^4 \right), \]

\[ \langle O_2^{(0)} \rangle = 6 \left( \frac{\pi}{\alpha_s} - \frac{11}{4} \right) |V_{ud}|^2 \left( m_u^2 + m_d^2 \right) + \text{const.}, \]

\[ \langle O_4^{(0)} \rangle = 8\pi^2 |V_{ud}|^2 \left( m_u \bar{\psi}_u \psi_u + m_j \bar{\psi}_j \psi_j \right) + 12 |V_{uj}|^2 m_u^2 m_j^2 \]

\[ - \left( \frac{24}{7} \frac{\pi}{\alpha_s} \right) |V_{uj}|^2 \left( m_u^4 + m_j^4 \right). \]

Here \( \alpha_s = \alpha_s(m_\tau^2) \) and \( m_q = m_q(m_\tau^2) \). A summation over \( j = d, s \) is understood. The terms proportional to \( 1/\alpha_s \) arises from IR logarithms, which can be resummed into the running coupling constant by means of the RGE \[16\]. Note that \( \langle O_2^{(0)} \rangle \) is defined only up to a constant (with respect to \( m_\tau^2 \)), which depends on the renormalization scheme. However, only derivatives of \( \langle O_2^{(0)} \rangle \) appear in physical quantities.

The most important power corrections of dimension six come from four-quark operators; the coefficient of the operator \( G^3 \) vanishes to leading order in \( \alpha_s \).
Operators containing powers of the light quark masses are strongly suppressed and can safely be neglected. The coefficient functions for the four-quark operators were calculated to next-to-leading order in Ref. [50]. In the chiral limit there are no contributions to the $J = 0$ correlator, and the contributions to the $J = 1$ correlator are

$$\langle O_{6}^{(0+1)} \rangle = - \frac{64\pi^4}{3} \left[ 1 + \left( \frac{9}{2} - 2L \right) \frac{\alpha_s(\mu^2)}{\pi} \right] \frac{\alpha_s(\mu^2)}{\pi} \left( O_{ij}^{V1} + O_{ij}^{A1} \right)$$

$$+ 16\pi^4 \left[ 1 + \left( \frac{87}{16} + \frac{1}{4} L \right) \frac{\alpha_s(\mu^2)}{\pi} \right] \frac{\alpha_s(\mu^2)}{\pi} \left( O_{ij}^{V8} + O_{ij}^{A8} \right)$$

$$- \frac{32\pi^4}{3} \left[ 1 + \left( \frac{1153}{432} - \frac{95}{72} L \right) \frac{\alpha_s(\mu^2)}{\pi} \right] \frac{\alpha_s(\mu^2)}{\pi} \sum_k \left( O_{ik}^{V8} + O_{jk}^{V8} \right)$$

$$- \frac{16\pi^4}{27} \left( 1 - \frac{2}{3} L \right) \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^2 \sum_{k,l} O_{kl}^{V8}.$$ 

(57)

The dependence on $m_r$ resides in $L = \ln m_r^2 / \mu^2$, and $\mu$ is an arbitrary renormalization scale. Since we work in the chiral limit, it is not necessary to specify the flavour labels; however, it is important that $i \neq j$. The four-quark operators, which are renormalized in the $\overline{\text{MS}}$ scheme, are defined as

$$O_{ij}^{V1} = \langle \bar{\psi}_i \gamma_\mu \psi_i \bar{\psi}_j \gamma_\mu \psi_j(\mu^2) \rangle,$$

$$O_{ij}^{A1} = \langle \bar{\psi}_i \gamma_\mu \gamma_5 \psi_i \bar{\psi}_j \gamma_\mu \gamma_5 \psi_j(\mu^2) \rangle,$$

$$O_{ij}^{V8} = \langle \bar{\psi}_i \gamma_\mu T_a \psi_i \bar{\psi}_j \gamma_\mu T_a \psi_j(\mu^2) \rangle,$$

$$O_{ij}^{A8} = \langle \bar{\psi}_i \gamma_\mu T_a \psi_i \bar{\psi}_j \gamma_\mu \gamma_5 T_a \psi_j(\mu^2) \rangle.$$ 

(58)

Since there exist no reliable estimates for these condensates, the traditional approach is to apply the vacuum saturation (or factorization) approximation [13] to relate them to the quark condensate. Unfortunately, this approximation is inconsistent with renormalization-group (RG) invariance [51, 52]. Moreover, phenomenological analyses indicate that the factorization approximation underestimates the contribution of four-quark operators. One introduces a fudge factor $\rho$ to compensate for this. At leading order in $\alpha_s$, one then obtains

$$\langle O_{6}^{(0+1)} \rangle_{\text{fact}} = \frac{256\pi^3}{27} \rho \alpha_s \langle \bar{\psi} \psi \rangle^2.$$ 

(59)

The combination $\rho \alpha_s \langle \bar{\psi} \psi \rangle^2$ is treated as a scale-invariant phenomenological parameter, whose value has been determined to be of order $2 \times 10^{-4}$ GeV$^6$ [9]–[11].

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We have rewritten the result given in Ref. [54] using Fierz identities. Note that, in the notation of Ref. [52], one has $\langle O_{6}^{(0+1)} \rangle = 6\pi^2 O(6)$. 

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corresponding to \( \rho \sim 3 \text{--} 8 \) at \( \mu = 1 \text{ GeV} \). The fact that phenomenology requires \( \rho \gg 1 \) is unsatisfactory and indicates large violations of the factorization hypothesis.

An interesting and, to our knowledge, new alternative to the vacuum saturation approximation is to consider the large-\( \beta_0 \) limit, which we have employed already in the calculation of the perturbative corrections. This limit respects exact RG invariance, yet reducing the dimension-six corrections to a single phenomenological parameter. To obtain it, we analyse expression (57) in the limit \( n_f \to \infty \). Note that there is \( n_f \) dependence in the running coupling constant as well as in the flavour sums. We find

\[
\langle O_6^{(0+1)} \rangle = -\frac{64\pi^4}{3} \left( \frac{n_f \alpha_s(\mu^2)}{\pi} \right)^2 O_{ij}^{V8}(\mu^2) \left[ \frac{\pi}{n_f \alpha_s(\mu^2)} + \frac{1}{36} (1 - 6L) \right] + O(1/n_f),
\]

(60)

where the product \( n_f \alpha_s(\mu^2) \) is formally of order \( n_f^0 \). The two terms shown in parenthesis correspond to the diagrams depicted in Fig. 4. In the next step, we replace \( n_f \) by \( -\frac{2}{3} \beta_0 \) and use the one-loop expression for the running coupling constant to find that

\[
\langle O_6^{(0+1)} \rangle_{\text{large-} \beta_0} = 128\pi^4 \left( \ln \frac{m_u^2}{\Lambda_V^2} + \frac{3}{2} \right) a^2(\mu^2) O_{ij}^{V8}(\mu^2),
\]

(61)

where \( \Lambda_V = e^{5/6} \Lambda_{\overline{\text{MS}}} \), and \( a(\mu^2) \) has been defined in (27). The \( \mu \) dependence of \( a(\mu^2) \) is exactly cancelled by the \( \mu \) dependence of the four-quark condensate \( [52] \). Note that the contribution (61) vanishes in the factorization approximation. In this sense the above two approximations are orthogonal to each other. If we define

\[
\varepsilon = \frac{a^2(\mu^2) O_{ij}^{V8}(\mu^2)}{\langle \overline{\psi} \psi \rangle^2},
\]

(62)

where \( \langle \overline{\psi} \psi \rangle^2 \) is evaluated at \( \mu = 1 \text{ GeV} \), we find that the results (61) and (59) are of similar magnitude if \( \varepsilon \sim 1 \text{--} 2\% \), corresponding to only a small deviation from vacuum saturation, which is certainly not excluded. Hence, if one accepts that the large-\( \beta_0 \) limit gives a reasonable approximation to QCD, it provides a natural explanation of the empirical fact that the values of four-quark condensates are much larger than predicted in the vacuum saturation approximation.

Starting from (56) and using the \( \beta \)-function and the anomalous dimension of the running quark masses from Appendix [C], we find

\[
\nabla \langle O_2^{(0+1)} \rangle = 6 \frac{\alpha_s}{\pi} |V_{uj}|^2 (m_u^2 + m_j^2),
\]

\[
\nabla \langle O_4^{(0+1)} \rangle = \alpha_s^2 \left( \frac{11}{12} \left( \frac{\alpha_s}{\pi} G^2 \right) + 18 |V_{uj}|^2 \right) \langle \overline{\psi} u \psi u + \overline{\psi} j \psi j \rangle - \frac{8}{3} \sum_k \langle m_k \overline{\psi} k \psi k \rangle
\]

\[
+ 6 |V_{uj}|^2 (m_u^4 + m_j^4),
\]
Figure 4: Examples of leading contributions of four-quark condensates in the large-$n_f$ limit. The first diagram is of order $n_f \alpha_s$, the second of order $(n_f \alpha_s)^2$.

\[
\begin{align*}
\nabla \langle O_2^{(0)} \rangle &= \frac{3}{2} \left( 1 + \frac{23}{3} \frac{\alpha_s}{\pi} \right) |V_{uj}|^2 (m_u^2 + m_j^2), \\
\nabla^2 \langle O_2^{(0)} \rangle &= -3 \frac{\alpha_s}{\pi} |V_{uj}|^2 (m_u^2 + m_j^2), \\
\nabla \langle O_4^{(0)} \rangle &= 6 |V_{uj}|^2 (m_u^4 + m_j^4).
\end{align*}
\]

The quantity $\nabla^2 \langle O_2^{(0+1)} \rangle$ appearing in (55) is of order $\alpha_s^2$ and will thus be neglected. For completeness, we also quote the results for the derivatives of the dimension-six operator. They are

\[
\begin{align*}
\nabla \langle O_6^{(0+1)} \rangle_{\text{fact}} &= -\frac{2176\pi^3}{243} \frac{\alpha_s}{\pi} \rho \alpha_s \langle \bar{\psi} \psi \rangle^2, \\
\nabla \langle O_6^{(0+1)} \rangle_{\text{large-} \beta_0} &= 128\pi^4 \varepsilon \langle \bar{\psi} \psi \rangle^2.
\end{align*}
\]

To estimate the size of the various power corrections we use the standard values of the QCD parameters, which are collected in Appendix C. We find the following results for the nonperturbative parameters corresponding to $J = 0 + 1$:

\[
\begin{align*}
\frac{\langle O_2^{(0+1)} \rangle}{m_r^2} &\simeq -(1.42 \pm 0.27) \times 10^{-3}, \\
\nabla \frac{\langle O_2^{(0+1)} \rangle}{m_r^2} &\simeq (0.20 \pm 0.04) \times 10^{-3}, \\
\frac{\langle O_4^{(0+1)} \rangle}{m_r^4} &\simeq (9.2 \pm 5.6) \times 10^{-3}, \\
\nabla \frac{\langle O_4^{(0+1)} \rangle}{m_r^4} &\simeq (0.18 \pm 0.08) \times 10^{-3}, \\
\frac{\langle O_6^{(0+1)} \rangle}{m_r^6} &\simeq (3.3 \pm 1.9) \times 10^{-3}.
\end{align*}
\]

The quoted errors reflect the uncertainty in the values of the quark masses and vacuum condensates. We note that the values of $\langle O_4^{(0+1)} \rangle$ and $\nabla \langle O_4^{(0+1)} \rangle$ are

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dominated by the gluon condensate. For \( J = 0 \), we obtain:

\[
\begin{align*}
\nabla \left\langle O^0_2 \right\rangle/m^2_F & \simeq (0.88 \pm 0.17) \times 10^{-3}, \\
\nabla^2 \left\langle O^0_2 \right\rangle/m^2_F & \simeq -(0.10 \pm 0.02) \times 10^{-3}, \\
\left\langle O^0_4 \right\rangle/m^4_F & \simeq -(1.92 \pm 0.72) \times 10^{-3}, \\
\nabla \left\langle O^0_4 \right\rangle/m^4_F & \simeq 0.01 \times 10^{-3}.
\end{align*}
\]

Most of the contributions are strongly suppressed by powers of the small quark masses and can safely be neglected. Contributions to the moments of order \( 10^{-3} \) are certainly not detectable given the uncertainties in the perturbative calculation, and also given that any extraction of the moments will be affected by experimental uncertainties. Therefore, it is a safe approximation to consider the power corrections in the chiral limit, in which case only contributions with \( J = 1 \) remain. The most striking prediction of the SVZ approach is that \( \left\langle O^{(1)}_2 \right\rangle = 0 \) in the chiral limit, since there is no gauge-invariant operator of dimension two in QCD. As a consequence, the leading nonperturbative corrections are induced by the gluon condensate and are of order \( \left\langle O^{(1)}_4 \right\rangle/m^4_F \sim 1\% \). We expect that corrections of dimension six are suppressed, relative to this, by about a factor 3, which is beyond the precision reachable in a realistic analysis. Therefore, we believe that the primary goals must be to measure \( \alpha_s(m^2_F) \) with a minimum contamination from power corrections, to test the absence of dimension-two operators in QCD, and to extract a value for the gluon condensate. We shall discuss some strategies how to pursue these goals in Sect. 7. Before, however, we will demonstrate how the inclusion of power corrections cures the IR renormalon problem.

### 6 Cancellation of renormalon ambiguities

The structure of power corrections simplifies greatly in the combined large-\( \beta_0 \) and chiral limit. The nonperturbative parameters remaining in this limit are the ones needed to absorb the renormalon ambiguities in the perturbative calculations of Sects. 3 and 4. We find

\[
\begin{align*}
\left\langle O^{(1)}_4 \right\rangle_{\text{large-}\beta_0} & = \frac{2\pi^2}{3} \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle, \\
\left\langle O^{(1)}_6 \right\rangle_{\text{large-}\beta_0} & = 128\pi^4 \left( \ln \frac{m^2_F}{\Lambda_V} + \frac{3}{2} \right) \varepsilon \langle \bar{\psi}\psi \rangle^2, \\
\nabla \left\langle O^{(1)}_6 \right\rangle_{\text{large-}\beta_0} & = 128\pi^4 \varepsilon \langle \bar{\psi}\psi \rangle^2.
\end{align*}
\]
All other contributions of dimension up to six vanish in this limit; in particular all contributions for $J = 0$. It is satisfying to observe that the potentially large power corrections survive in the large-$\beta_0$ limit. From the numerical discussion of the preceding section, it follows that all other terms are of order $10^{-3}$ or less (in the appropriate units of $m^\tau$).

Let us now demonstrate how the renormalon ambiguities of perturbation theory can be cured by adding nonperturbative corrections. To start with, consider the euclidean correlator $D^{(1)}(Q^2)$. In the combined large-$\beta_0$ and chiral limit, we find from (26) and (47)

$$D^{(1)}(Q^2) = 1 + \frac{1}{\beta_0} \int_0^\infty \frac{d\tau}{\tau} w_D(\tau) a(\tau Q^2) + \frac{\langle O_4^{(1)} \rangle}{Q^4} + \frac{\langle O_6^{(1)}(Q^2) \rangle}{Q^6} + O(Q^{-8}).$$

(68)

The perturbative contribution is ambiguous because of the Landau pole in the running coupling constant $a(\tau Q^2)$. The corresponding renormalon ambiguity has been given in (31). To obtain a meaningful result, we have to require that, order by order in $1/Q^2$, these ambiguities are cancelled by corresponding ambiguities in the definition of the nonperturbative parameters $\langle O_{2n}^{(1)} \rangle$. The philosophy is the following: The values of the nonperturbative parameters become meaningful only after one specifies a resummation prescription for the perturbative series, i.e. a prescription to regulate the Landau pole in the integral over $\tau$ in (68). Changing this prescription changes the values of the nonperturbative parameters in such a way that the total answer remains the same [24]. Hence, to order $1/Q^6$ we have to require that

$$\frac{8}{\beta_0} \frac{\Lambda_V^4}{Q^4} - \frac{16}{3\beta_0} \left( \ln \frac{Q^2}{\Lambda_V^2} + \frac{3}{2} \right) \frac{\Lambda_V^6}{Q^6} + \frac{\Delta \langle O_4^{(1)} \rangle_{\text{ren}}}{Q^4} + \frac{\Delta \langle O_6^{(1)}(Q^2) \rangle_{\text{ren}}}{Q^6} = 0.$$

(69)

Note that the momentum dependence of $\langle O_6^{(1)}(Q^2) \rangle$, which is obtained from (61) by the replacement $m_\tau^2 \to Q^2$, is precisely of the form required to fulfill this condition. It follows that

$$\Delta \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle_{\text{ren}} = -\frac{12 \Lambda_V^4}{\pi^2 \beta_0} \approx -0.006 \text{ GeV}^4,$n

$$\Delta \left[ \varepsilon \langle \bar{\psi} \psi \rangle^2 \right]_{\text{ren}} = \frac{\Lambda_V^6}{24 \pi^4 \beta_0} \approx 5 \times 10^{-7} \text{ GeV}^6,$n

(70)

where we have used $\Lambda_V = 0.461$ GeV corresponding to $\alpha_s(m_\tau^2) = 0.32$. Numerically, the ambiguity of the gluon condensate is about one third of its value; the ambiguity of the dimension-six condensate is about $3 \times 10^{-3}/\varepsilon$ times its value, which amounts to 30% if $\varepsilon = 1\%$.

Once the ambiguities cancel for the euclidean correlation function, they also cancel for the moments. We find that to order $1/m_\tau^6$ the conditions for this
cancellation read

\[ \Delta M^\text{ren}_k + \frac{1}{3} k + \frac{1}{2} \frac{\Delta \langle O_6^{(1)} \rangle}{m_\tau^6} = 0 ; \quad k \neq 1, 2, \]
\[ \Delta M^\text{ren}_1 - \frac{\Delta \langle O_4^{(1)} \rangle}{m_\tau^2} - \frac{2}{3} \frac{\Delta \langle O_6^{(1)} \rangle}{m_\tau^6} = 0 , \]
\[ \Delta M^\text{ren}_2 + \frac{\Delta \langle O_6^{(1)} (m_\tau^2) \rangle}{m_\tau^6} + \frac{1}{3} \frac{\Delta \langle O_6^{(1)} \rangle}{m_\tau^6} = 0 , \]

where the perturbative ambiguities \( \Delta M^\text{ren}_k \) have been given in (44). These conditions are indeed satisfied with (70).

### 7 Discussion of the results

We have presented a new analysis of the spectral moments measurable in hadronic \( \tau \) decays, with the main focus on the uncertainties inherent in the perturbative calculation of the leading terms in the OPE. Comparing fixed-order perturbation theory (which is known to order \( \alpha_s^3 \) in this case) with a partial resummation of the perturbative series based on the analysis of so-called renormalon chains, we conclude that the uncertainties in the calculation of the moments are of order a few per cent. We have also considered the resummation prescription of Le Diberder and Pich [5] and find that it does not reduce this uncertainty in a significant way. These observations imply important limitations for tests of QCD at the level of power corrections. We have presented a model-independent description of such corrections and argued that they have to be included, for reasons of consistency, in order to cure certain ambiguities of perturbation theory, which are related to IR renormalons. For the power corrections of dimension four and six we have demonstrated in detail how these ambiguities can be absorbed into a redefinition of some nonperturbative parameters (vacuum condensates). Based on our results, we conclude that previous analyses of spectral moments in hadronic \( \tau \) decays [12, 16, 17], which were aiming at an extraction of power corrections up to dimension six and even eight, have underestimated the theoretical uncertainties. In the remainder of this section we shall reconsider the feasibility of extracting fundamental QCD parameters, such as the running coupling constant and some of the vacuum condensates, in the light of our results.

Even in the ideal case in which the spectral moments \( \mathcal{M}_k^{(J)} \) can be measured with arbitrary precision, and in which the perturbative contributions to these moments can be calculated to very high order, there is an unavoidable ambiguity in the definition of power corrections, which results from the presence of renormalons in perturbative QCD. The renormalon ambiguities in the values of the gluon condensate and the dimension-six condensate have been given in (70).
They amount to about 30% of the expected values of the condensates. However, to achieve such a level of precision would not only require zero experimental errors, but also a control of the asymptotic behaviour of perturbation theory, which is lacking at present. One cannot trust the large-$\beta_0$ limit to provide a very accurate description of this behaviour. To be conservative, we shall consider the difference between third-order truncated perturbation theory and the resummed results in the large-$\beta_0$ limit as an estimate of the perturbative uncertainty. We will now discuss what this implies for measurements of $\alpha_s(m_T^2)$ and some of the vacuum condensates.

The most accurate way to determine the running coupling constant is to measure the total inclusive ratio $R_\tau$ defined in (2). We separate the perturbative contribution to this quantity from nonperturbative corrections by writing

$$R_\tau = 3S_{\text{EW}} (1 + \delta_{\text{pert}} + \delta_{\text{power}}). \quad (72)$$

The perturbative contribution, $\delta_{\text{pert}} = 2M_0 - 2M_2 + M_3 - 1$, depends strongly on the value of the running coupling constant. In Fig. 5, we compare the exact order-$\alpha^3_s$ prediction for this quantity (Table 1) to the result obtained from the resummation of renormalon chains (lower portion of Table 6). For comparison, we also show the result obtained using the resummation method of Le Diberder and Pich [5] (Table 2), which is routinely used in the determinations of $\alpha_s$ from $R_\tau$. The differences between these perturbative approximations are of order few to several per cent, increasing as the value of $\alpha_s$ increases. The quantity

$$\delta_{\text{power}} = \left(2 + \frac{\nabla}{3}\right) \frac{\langle O^{(0+1)}_2 \rangle}{m_T^2} + 3 \frac{\nabla \langle O^{(0+1)}_4 \rangle}{m_T^4} - 2 \frac{\langle O^{(0+1)}_6 \rangle}{m_T^6}$$

$$- \left(\frac{4}{3} \nabla - \frac{10}{9} \nabla^2\right) \frac{\langle O^{(0)}_2 \rangle}{m_T^2} + 2(1 - \nabla) \frac{\langle O^{(0)}_4 \rangle}{m_T^2} + \ldots \simeq -(1.4 \pm 0.5\%) \quad (73)$$

contains the power corrections. In the SVZ approach, the largest contribution (about 50%) comes from dimension-six operators; the contribution of the gluon condensate is suppressed by two powers of $\alpha_s$. The remaining corrections come mainly from quark-mass effects and from the quark condensate, each of which contribute about 25%. The fact that the power corrections to $R_\tau$ are very small has been noted in Ref. [4]. It is for this reason that measurements of $R_\tau$ are believed to provide a reliable determination of the running coupling constant. Note that the uncertainty in the value of the power corrections is much less than the uncertainty in the perturbative calculation of $\delta_{\text{pert}}$. Thus, it is the perturbative uncertainty which limits the accuracy in the extraction of $\alpha_s(m_T^2)$. Even if one could determine $\delta_{\text{pert}}$ with a very small error (in practice this error cannot be much smaller than 1% because of the theoretical uncertainty in $\delta_{\text{power}}$), there is a rather large uncertainty in the corresponding value of the running coupling constant, depending on which perturbative approximation one uses.
For $\delta_{\text{pert}} \simeq 0.2$, which is the value preferred by experiments [16, 17], one obtains $\alpha_s(m_\tau^2) \simeq 0.303$ from the resummation of renormalon chains, $\alpha_s(m_\tau^2) \simeq 0.337$ from the exact order-$\alpha_s^3$ calculation, and $\alpha_s(m_\tau^2) \simeq 0.353$ from the resummation of Le Diberder and Pich. Hence, we conclude that

$$\delta \alpha_s(m_\tau^2) \simeq 0.05$$

(74)

is a reasonable estimate of the theoretical uncertainty. This is a factor 3 larger than the total error estimated in Ref. [5, 12] and quoted in the experimental analyses [11, 17].

Next consider the extraction of the leading nonperturbative parameter contributing to the moments, which is the gluon condensate or, more generally, the quantity $\langle O_4^{(0+1)} \rangle$. From (55) it follows that the moment $M_1^{(0+1)}$ is most sensitive to this parameter. Let us assume an ideal experiment, in which this moment can be extracted with high accuracy.$^9$ Moreover, assume that $\alpha_s(m_\tau^2)$ has been determined independently from a measurement of $R_\tau$. As previously, we write

$$M_1^{(0+1)} = 1 + \eta_{\text{pert}} + \eta_{\text{power}}.$$  

(75)

The theoretical predictions for the perturbative contribution, $\eta_{\text{pert}} = M_1 - 1$, can be obtained directly from the numbers given in Tables 1, 2 and 6. For values of

$^9$ In practice, this would require to subtract the contributions with $J = 0$. Moreover, what is accessible in experiments are not the spectral moments $M_k^{(J)}$ themselves, but rather the moments $R_k$ of the $\tau$ hadronic spectrum, which according to (4) are combinations of the spectral moments.
\( \alpha_s(m_\tau^2) \) in the range from 0.30 to 0.34, the results differ by 1–3\%. For the leading power corrections we obtain, neglecting terms of order \( 10^{-4} \),

\[
\eta_{\text{power}} = -\frac{\langle O_4^{(0+1)} \rangle}{m_T^4} + \ldots \simeq 0.2\% - \frac{2\pi^2}{3m_T^4} \left( 1 - \frac{11}{18} \frac{\alpha_s}{\pi} \right) \left( \frac{\alpha_s}{\pi} G^2 \right).
\] (76)

For \( \langle \frac{\alpha_s}{\pi} G^2 \rangle \simeq 0.018 \text{ GeV}^2 \), the contribution of the gluon condensate is \(-1.1\%\), thus providing the dominant correction. The precision in the determination of the gluon condensate is limited by the theoretical uncertainty in the value of \( \eta_{\text{pert}} \). An uncertainty of 1\% in the value of \( \eta_{\text{pert}} \) implies an uncertainty of 0.02 GeV\(^4\) in the value of the gluon condensate, which is of the same magnitude as the expected value of the condensate itself. We conclude that, unless one manages to suppress the perturbative uncertainty, the gluon condensate cannot be extracted with a precision of much better than 100\%.

Consider finally the extraction of power corrections of dimension six. An ideal observable to determine these would be the combination

\[
3M_3^{(0+1)} - 2M_2^{(0+1)} = 1 + \rho_{\text{pert}} + \rho_{\text{power}},
\] (77)

since it does not receive power corrections from \( \langle O_4^{(0+1)} \rangle \) and \( \nabla \langle O_4^{(0+1)} \rangle \). Neglecting a contribution of order \( 10^{-4} \) from \( \nabla \langle O_2^{(0+1)} \rangle \), we find

\[
\rho_{\text{power}} = -2 \frac{\langle O_6^{(0+1)} \rangle}{m_T^6} + \ldots
\] (78)

In this case, the perturbative uncertainty in the value of \( \rho_{\text{pert}} = 3M_3 - 2M_2 - 1 \) is of order 1–2\%, leading to \( \delta \langle O_6^{(0+1)} \rangle \simeq 0.2–0.3 \text{ GeV}^6 \), which is larger than expected value of the condensate. Given the present level of control over the perturbative contributions to the moments, it is thus not possible to extract power corrections of dimension six or higher from hadronic \( \tau \) decays, even in an ideal experiment.

To summarize, we believe that beyond extracting the running coupling constant \( \alpha_s(m_\tau^2) \) the goals of a realistic analysis of spectral moments in hadronic \( \tau \) decays should be (i) to test QCD, or more specifically the SVZ approach, by checking the absence of large dimension-two operators; (ii) to extract a value of the gluon condensate. To pursue these goals, we propose to measure differences of the moments \( R_k \) defined in (7). The perturbative contributions to these differences start at order \( \alpha_s^2 \) and are known exactly to order \( \alpha_s^4 \). Since these contributions are very small, the quantities we shall construct provide a more direct measurement of power corrections, which is very little affected by uncertainties in the value of \( \alpha_s(m_\tau^2) \).

A quantity that is sensitive to the presence of non-standard power corrections of dimension two is

\[
D_2 = \frac{1}{2} (R_0 - R_1) = \frac{1}{6S_{\text{EW}}} \int_0^{m_T^2} ds \left( 1 - \frac{10}{3} \frac{s}{m_T^2} \right) \frac{dR_\tau(s)}{ds}
\]
\[ = M_0^{(0+1)} - \frac{5}{3} M_1^{(0+1)} - M_2^{(0+1)} + 3 M_3^{(0+1)} - \frac{4}{3} M_4^{(0+1)} \\
- M_1^{(0)} + \frac{32}{9} M_2^{(0)} - \frac{23}{6} M_3^{(0)} + \frac{4}{3} M_4^{(0)}. \] (79)

Using (18), we find that in the \( \overline{\text{MS}} \) scheme the perturbative contribution to this difference is

\[
D_{\text{pert}}^2 = \frac{57}{80} \left( \frac{\alpha_s}{\pi} \right)^2 + c_1 \left( \frac{\alpha_s}{\pi} \right)^3 + c_2 \left( \frac{\alpha_s}{\pi} \right)^4 + \ldots \simeq (2.8 \pm 1.0)\%,
\] (80)

where

\[
c_1 = \frac{241531}{9600} - \frac{513}{40} \zeta(3) \simeq 9.7431, \\
c_2 = \frac{596310293}{768000} - \frac{4617}{1280} \pi^2 - \frac{193519}{3200} \zeta(3) + \frac{2565}{32} \zeta(5) \simeq 96.898, \]

and we have used \( \alpha_s = \alpha_s(m_\tau^2) = 0.32 \). Numerically, we find that every term in this series is of the same magnitude, and we have used the last term to estimate the uncertainty. If we perform a partial resummation of the series in the large-\( \beta_0 \) limit and correct for the known pieces to order \( \alpha_s^4 \) (this is a small corrections in this case), we find the larger value \( D_{\text{Borel}}^2 = (5.5 \pm 0.7)\% \), where the error is given by the ambiguity due to the nearest IR renormalon. The leading power corrections of dimension less or equal to six are

\[
D_{\text{power}}^2 = \left( 1 - \frac{3}{2} \nabla \right) \frac{\langle O_2^{(0+1)} \rangle}{m_\tau^2} + \left( \frac{5}{3} + \frac{17}{18} \nabla \right) \frac{\langle O_4^{(0+1)} \rangle}{m_\tau^4} - \frac{\langle O_6^{(0+1)} \rangle}{m_\tau^6} \\
- \left( \nabla \frac{55}{9} \right) \frac{\langle O_2^{(0)} \rangle}{m_\tau^2} + \left( 1 - \frac{19}{9} \nabla \right) \frac{\langle O_4^{(0)} \rangle}{m_\tau^4} + \ldots. \] (82)

In the SVZ approach the largest contribution comes from the gluon condensate, and we find \( D_{\text{power}}^2 \simeq (0.8 \pm 1.0)\% \). We conclude that

\[
D_2 \simeq \begin{cases} 
(3.6 \pm 1.3)\%; & \text{fixed-order perturbation theory}, \\
(6.3 \pm 1.2)\%; & \text{resummed perturbation theory}.
\end{cases} \] (83)

A non-standard dimension-two contribution of size \( (\Lambda/m_\tau)^2 \sim 3-8\% \) (for \( \Lambda = 300-500 \text{ MeV} \)) and possibly negative sign could be large enough to change this value in a significant way. Such a contribution is forbidden in the standard SVZ approach, since there is no local, gauge-invariant operator of dimension two in QCD. We note, however, that the truncation of the perturbative series for \( D_2 \) can fake such a term, because this series contains an UV renormalon at \( u = -1 \) \[13\]. It would be interesting to see whether the resummation of renormalon chains, which resums this contribution in the large-\( \beta_0 \) limit, improves the agreement with experiment.
Another interesting quantity, which can be used to measure the gluon condensate, is

\[ D_4 = \frac{3}{10} (R_2 - R_1) = \frac{1}{3S_{\text{EW}}} \int_0^{m_T^2} ds \frac{s}{m_T^2} \left( \frac{9}{4} \frac{s}{m_T^2} - 1 \right) \frac{dR_T(s)}{ds} \]

\[ = -M_1^{(0+1)} + \frac{3}{2} M_2^{(0+1)} + \frac{3}{2} M_3^{(0+1)} - \frac{7}{2} M_4^{(0+1)} + \frac{3}{2} M_5^{(0+1)} \]

\[ + \frac{4}{3} M_2^{(0)} - \frac{17}{4} M_3^{(0)} + \frac{22}{5} M_4^{(0)} - \frac{3}{2} M_5^{(0)}. \]  

(84)

The perturbative contribution to this difference is

\[ D_4^{\text{pert}} = -\frac{27}{160} \left( \frac{\alpha_s}{\pi} \right)^2 - d_1 \left( \frac{\alpha_s}{\pi} \right)^3 - d_2 \left( \frac{\alpha_s}{\pi} \right)^4 + \ldots \simeq -(0.46 \pm 0.10)\%, \]  

(85)

with

\[ d_1 = \frac{34527}{6400} - \frac{243}{80} \zeta(3) \simeq 1.7436, \]

\[ d_2 = \frac{77871081}{512000} - \frac{2187}{2560} - \frac{819369}{6400} \zeta(3) + \frac{1215}{64} \zeta(5) \simeq 9.4508. \]  

(86)

The resummation of renormalon chains gives a very similar result, \( D_4^{\text{Borel}} = -(0.55 \pm 0.40)\% \). In both cases the perturbative contribution is very small, and the accuracy seems to be limited by the renormalon ambiguity. We also note that the coefficient of the UV renormalon at \( u = -1 \) is very small in this case, so that one does not expect large uncertainties due to the truncation of the series. The leading power corrections to the quantity \( D_4 \) are

\[ D_4^{\text{power}} = -\frac{13}{40} \nabla \langle O_2^{(0+1)} \rangle + \left( 1 - \frac{35}{24} \nabla \right) \langle O_4^{(0+1)} \rangle \frac{m_T^4}{m_T^4} + 3 \langle O_6^{(0+1)} \rangle \frac{m_T^6}{m_T^6} \]

\[ + \left( \nabla^2 - \frac{167}{1800} \nabla^2 \right) \langle O_2^{(0)} \rangle \frac{m_T^2}{m_T^2} - \frac{7}{24} \nabla \langle O_4^{(0)} \rangle \frac{m_T^4}{m_T^4} + \ldots \]

\[ \simeq 0.3\% + \frac{2\pi^2}{3m_T^2} \left( 1 - \frac{11}{18} \frac{\alpha_s}{\pi} \right) \frac{\alpha_s}{\pi} G^2 \].  

(87)

To good approximation, they are given by the gluon condensate. Combining them with the perturbative contribution, we obtain

\[ D_4 \simeq 1.1\% \times \frac{\langle \frac{\alpha_s}{\pi} G^2 \rangle}{0.018 \text{ GeV}^4} - (0.2 \pm 0.4)\%. \]  

(88)

Hence, a measurement of \( D_4 \) comes close to a “null measurement”. If a value \( D_4 \neq 0 \) can be established, it provides direct evidence for the existence of the gluon condensate. We would consider such a measurement more convincing than the existing fits [16, 17] of the gluon condensate from a moment analysis following the lines of Ref. [12].
Appendices

A Calculation of the integrals $I_{n}^{k+1}(m_{\tau})$

Le Diberder and Pich have introduced the integrals \cite{12}

$$I_{n}^{k+1}(m_{\tau}) = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1 - x^{k+1}) \left(\frac{\alpha_{s}(-x m_{\tau}^{2})}{\pi}\right)^{n}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \left[ 1 + (-1)^{k} e^{i(k+1)\phi} \right] \left(\frac{\alpha_{s}(e^{i\phi} m_{\tau}^{2})}{\pi}\right)^{n}.$$  \hspace{0.5cm} (89)

The running coupling constant along the contour in the complex plane can be obtained from the solution of the RGE \cite{17}, which reads

$$i\phi \frac{\beta_{0}}{4\pi} = \frac{1}{\alpha_{s}(\varphi)} - \frac{1}{\alpha_{s}(0)} + \frac{\beta_{1}}{4\pi \beta_{0}} \ln \frac{\alpha_{s}(\varphi)}{\alpha_{s}(0)} + \frac{\beta_{0} \beta_{2} - \beta_{1}^{2}}{16\pi^{2} \beta_{0}^{2}} \left[ \alpha_{s}(\varphi) - \alpha_{s}(0) \right] + \ldots, \hspace{0.5cm} (90)$$

with $\alpha_{s}(\varphi) \equiv \alpha_{s}(e^{i\phi} m_{\tau}^{2})$.

In the approximation where one uses the one-loop $\beta$-function (setting $\beta_{1} = \beta_{2} = \ldots = 0$), it is possible to perform the contour integrals explicitly. The result for $n = 1$ is

$$I_{1}^{k+1}(m_{\tau}) = \frac{4}{\beta_{0}} \left\{ \frac{(-1)^{k}}{\pi} e^{-(k+1)/a} \text{Im} \text{Ei} \left[ (k+1)(1/a + i\pi) \right] + \frac{1}{\pi} \text{arctan}(\pi a) \right\}, \hspace{0.5cm} (91)$$

where

$$a = \frac{\beta_{0}}{4\pi} \alpha_{s}(m_{\tau}^{2}) = \frac{1}{\ln(m_{\tau}^{2}/\Lambda^{2})}, \hspace{0.5cm} (92)$$

and $\text{Ei}(x) = \int_{-\infty}^{x} \frac{dt}{t} e^{t}$ is the exponential integral. Note that the value of $\text{Im} \text{Ei}(x + iy)$ is not affected by the pole at $t = 0$. The integrals with $n > 1$ can be obtained from the recursion relation

$$I_{n}^{k+1}(m_{\tau}) = \left(\frac{4}{\beta_{0}}\right)^{n-1} \frac{1}{(n-1)!} \left( - \frac{d}{da^{-1}} \right)^{n-1} I_{1}^{k+1}(m_{\tau}). \hspace{0.5cm} (93)$$

Beyond this approximation, the integrals have to be performed numerically. However, in the case of the two-loop $\beta$-function one can still obtain an approximate analytic expression by writing

$$i\phi \frac{\beta_{0}}{4\pi} = \frac{1}{\alpha_{s}(\varphi)} - \frac{1}{\alpha_{s}(0)} + \frac{\beta_{1}}{4\pi \beta_{0}} \ln \frac{\alpha_{s}(\varphi)}{\alpha_{s}(0)}$$

$$\approx \frac{1}{\alpha_{s}(\varphi)} - \frac{1}{\alpha_{s}(0)} - i\phi \frac{\beta_{1}}{16\pi^{2}} \alpha_{s}(0). \hspace{0.5cm} (94)$$
Table 10: Integrals $I_n^{k+1}$ evaluated using the one-loop (upper portion), two-loop (middle portion), and three-loop (lower portion) $\beta$-function. In the two-loop case, the approximate results obtained from (95) are given below the exact values.

| $k$ | $k = 0$ | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ | $k = 5$ |
|-----|---------|---------|---------|---------|---------|---------|
| $I_1^{k+1} (\times 10^{-2})$ | 11.55 | 9.73 | 9.40 | 9.24 | 9.16 | 9.10 |
| $I_2^{k+1} (\times 10^{-3})$ | 12.08 | 8.01 | 7.62 | 7.40 | 7.28 | 7.20 |
| $I_3^{k+1} (\times 10^{-4})$ | 11.67 | 5.25 | 5.22 | 5.05 | 4.95 | 4.89 |

This leads to the simple relation

$$ I_n^{k+1} \bigg|_{2-\text{loop}} \simeq \left(1 + \frac{\beta_1}{\beta_0}^*\right)^{-n} I_n^{k+1} \bigg|_{1-\text{loop}} (a^*), $$

where in the one-loop integrals one uses

$$ a^* = \left(1 + \frac{\beta_1}{\beta_0} a\right) a $$

instead of $a$. This provides a very good approximation to the exact numerical results.

In Table 10, we compare the various approximations for some of these integrals in the case of $\alpha_s(m_\tau^2) = 0.32$ (in the MS scheme).

**B  Asymptotic behaviour of the functions $W_k(\tau)$**

In order to perform numerical integrations with the distribution functions $W_k(\tau)$ defined in (35)–(37), it is convenient to use their asymptotic behaviour for large and small values of $\tau$. For $\tau \gg 1$, we find

$$ W_k(\tau) = \frac{32}{3} (k + 1) \left\{ \left( \frac{5k + 16}{36(k + 2)^2} + \frac{\ln \tau}{6(k + 2)} \right) \frac{1}{\tau} \right\} $$
\[- \left( \frac{7k + 33}{144(k+3)^2} + \frac{\ln \tau}{12(k+3)} \right) \frac{1}{\tau^2} + O(\tau^{-3}) \right\}, \quad (97)\]

which is valid for all \( k \geq 0 \). For \( \tau \ll 1 \), on the other hand, we obtain

\[
W_0(\tau) = \frac{32}{3} \left\{ \left( 4 - 3\zeta(3) \right) \tau - \frac{3}{4} \tau^2 + \left( \frac{1}{2} - \frac{1}{4} \ln \tau \right) \tau^3 + O(\tau^4) \right\},
\]

\[
W_1(\tau) = \frac{32}{3} \left\{ \left( 5 - 6\zeta(3) - \frac{3}{2} \ln \tau \right) \tau^2 + \left( \frac{5}{2} - \ln \tau \right) \tau^3 + O(\tau^4) \right\},
\]

\[
W_2(\tau) = \frac{32}{3} \left\{ \frac{9}{4} \tau^2 - \left( \frac{4}{3} - \frac{9}{4} \ln \tau + \frac{3}{4} \ln^2 \tau \right) \tau^3 + O(\tau^4) \right\}, \quad (98)
\]
as well as

\[
W_k(\tau) = \frac{32}{3} (k + 1) \left\{ \frac{3}{4(k-1)} \tau^2 - \left( \frac{3k - 8}{4(k-2)} - \frac{\ln \tau}{2(k-2)} \right) \tau^3 + O(\tau^4) \right\} \quad (99)
\]

for \( k \geq 3 \).

The asymptotic behaviour of the distribution function \( w_D(\tau) \) in (23) can be derived from the above results by taking the limit \( k \to \infty \). For \( \tau \gg 1 \), we find

\[
w_D(\tau) = \frac{16}{9} \left( \ln \tau + \frac{5}{6} \right) \frac{1}{\tau} - \frac{8}{9} \left( \ln \tau + \frac{7}{12} \right) \frac{1}{\tau^2} + O(\tau^{-3}), \quad (100)
\]

whereas for \( \tau \ll 1 \)

\[
w_D(\tau) = 8\tau^2 + 8 \left( \frac{2}{3} \ln \tau - 1 \right) \tau^3 + O(\tau^4). \quad (101)
\]

### C  QCD parameters

For the running quark masses at the scale \( \mu_0 = 1 \text{ GeV} \) (in the \( \overline{\text{MS}} \) scheme), we use

\[
m_u(\mu_0^2) = 5 \pm 1 \text{ MeV}, \]

\[
m_d(\mu_0^2) = 9 \pm 1 \text{ MeV}, \]

\[
m_s(\mu_0^2) = 178 \pm 18 \text{ MeV}. \quad (102)
\]

The value for \( m_s \) has been obtained in a recent QCD sum rule analysis [54]. The scale dependence of the running masses is governed by the RGE

\[
\frac{\text{d}m_q^2(\mu^2)}{\text{d} \ln \mu^2} = \gamma_m[\alpha_s(\mu^2)] m_q^2(\mu^2). \quad (103)
\]

At two-loop order,

\[
\gamma_m(\alpha_s) = \gamma_0 \frac{\alpha_s}{4\pi} + \gamma_1 \left( \frac{\alpha_s}{4\pi} \right)^2 + \ldots, \quad (104)
\]
with coefficients \( \gamma_0 = -8 \) and \( \gamma_1 = -364/3 \) (for \( n_f = 3 \)) \cite{55}. At this order, the solution of the RGE is

\[
m_q^2(\mu^2) = m_q^2(\mu_0^2) \left( \frac{\alpha_s(\mu_0^2)}{\alpha_s(\mu^2)} \right)^{\gamma_0/\beta_0} \left\{ 1 + \frac{\beta_0 \gamma_1 - \beta_1 \gamma_0}{4 \beta_0^2} \frac{\alpha_s(\mu_0^2) - \alpha_s(\mu^2)}{\pi} + \ldots \right\}. \quad (105)
\]

This can be used to calculate \( m_q(m_F^2) \). In the numerical analysis we use \( \alpha_s(m_F^2) = 0.32 \), unless otherwise specified.

In the calculation of the power corrections in Sect. 5 we also need the elements \( V_{uj} \) of the Cabibbo-Kobayashi-Maskawa matrix. We neglect \( V_{ub} \), so that \( |V_{ud}|^2 + |V_{us}|^2 = 1 \), and use \( |V_{ud}| = 0.9753 \), \( |V_{us}| = 0.2209 \). \quad (106)

Finally, we give the definition of the scale-invariant condensates of dimension four, which are relevant to our analysis. In the MS scheme, and to next-to-leading order in the coupling constant, they are defined as \cite{16, 53}

\[
\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = \left( 1 + \frac{16}{9} \frac{\alpha_s(\mu^2)}{\pi} \right) \frac{\alpha_s(\mu^2)}{\pi} \left\langle G_{\mu\nu} G^{\mu\nu}(\mu^2) \right\rangle \\
- \frac{16}{9} \frac{\alpha_s(\mu^2)}{\pi} \left( 1 + \frac{19}{24} \frac{\alpha_s(\mu^2)}{\pi} \right) \sum_k m_k(\mu^2) \left\langle \bar{\psi}_k \psi_k(\mu^2) \right\rangle \\
- \frac{1}{3 \pi^2} \left( 1 + \frac{4}{3} \frac{\alpha_s(\mu^2)}{\pi} \right) \sum_k m_k^4(\mu^2), \quad \langle m_i \bar{\psi}_i \psi_i \rangle = m_i(\mu^2) \left\langle \bar{\psi}_i \psi_i(\mu^2) \right\rangle + \frac{3}{7 \pi^2} \left( \frac{\pi}{\alpha_s(\mu^2)} - \frac{53}{24} \right) m_i^4(\mu^2). \quad (107)
\]

For the scale-invariant quark condensates we write \( \langle m_i \bar{\psi}_i \psi_i \rangle = -m_i(\mu_0^2) \kappa_i \) and take the standard values \cite{15}

\[
\kappa_u = \kappa_d = (230 \pm 30 \text{ MeV})^3, \quad \kappa_s = 0.65 \kappa_u. \quad (108)
\]

For the scale-invariant gluon condensate and for the four-quark condensate we use an average over some recent determinations \cite{3}–\cite{11}, however with conservative errors:

\[
\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle = (0.018 \pm 0.009) \text{ GeV}^4, \quad \rho\alpha_s\langle \bar{\psi}\psi \rangle^2 = (3.5 \pm 2.0) \text{ GeV}^6. \quad (109)
\]
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