Symmetries at and Near Critical Points of Quantum Phase Transitions in Nuclei

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Abstract. We examine several types of symmetries which are relevant to quantum phase transitions in nuclei. These include: critical-point, quasidynamical, and partial dynamical symmetries.

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Symmetry plays a profound role in thermal and quantum phase transitions (QPT). The latter occur at zero temperature as a function of a coupling constant in the Hamiltonian. Such ground-state energy phase transitions are a pervasive phenomenon observed in many branches of physics, and are realized empirically in nuclei as transitions between different shapes. QPT occur as a result of a competition between terms in the Hamiltonian with different symmetry character, which lead to considerable mixing in the eigenfunctions, especially at the critical-point where the structure changes most rapidly. In the present contribution we address the question whether there are any symmetries (or traces of) still present at and near critical points of QPT. As shown below, unexpectedly, several types of intermediate-symmetries can survive in spite of the strong mixing.

The first indication that symmetries can persist at critical points of QPT came from the recent recognition that the dynamics at these critical-points is amenable to analytic descriptions [1, 2]. For nuclei, such analytic benchmarks of criticality, called “critical-point symmetries”, were obtained in the geometric framework of a Bohr Hamiltonian for macroscopic quadrupole shapes. The E(5) benchmark [1] is applicable to a second-order shape-phase transition between spherical and deformed $\gamma$-soft nuclei. The X(5) benchmark [2] is applicable to a low-barrier first-order phase transition between spherical and axially-deformed nuclei. Both benchmarks employ an infinite square-well potential, which is $\gamma$-independent for E(5) and axially-deformed, with an assumed $\beta \gamma$ decoupling, for X(5). The eigenvalues are proportional to squared zeroes of Bessel functions, and display spectral features which are in-between the indicated limiting phases. The importance of these analytic benchmarks lies in the fact that they provide a classification of states (quantum numbers) and analytic expressions (parameter-free except for scale) for observables in regions of rapid structural changes. Empirical evidence for these “critical-point symmetries” has been presented in nuclei [3, 4]. An example of X(5)-like structure found in $^{152}$Sm and $^{150}$Nd is shown in Fig. 1.

A convenient framework to study symmetry-aspects of QPT is the interacting boson model (IBM) [5], which describes quadrupole collective states in nuclei in terms of a
system of $N$ monopole ($s$) and quadrupole ($d$) bosons, representing valence nucleon pairs. The model is based on a $U(6)$ spectrum generating algebra. Dynamical symmetries occur when the Hamiltonian is written in terms of the Casimir operators of a chain of nested algebras of $U(6)$. They provide analytically solvable limits and quantum numbers, which are the labels of the irreducible representations (irreps) of the algebras in the chain. The three dynamical symmetry limits of the IBM and associated bases are

$$\begin{align*}
U(6) \supset U(5) \supset O(5) \supset O(3) & \quad |N, n_d, \tau, \bar{\nu}, L \rangle \quad \text{spherical vibrator} \\
U(6) \supset SU(3) \supset O(3) & \quad |N, (\lambda, \mu), K, L \rangle \quad \text{axially deformed rotor} \\
U(6) \supset O(6) \supset O(5) \supset O(3) & \quad |N, \sigma, \tau, \bar{\nu}, L \rangle \quad \gamma{-}\text{soft deformed rotor}
\end{align*}$$

The corresponding analytic solutions resemble spectral features of known geometric paradigms, indicated above. This identification is consistent with the geometric visualization of the model in terms of a potential surface defined by the expectation value of the Hamiltonian in a coherent (intrinsic) state $[6, 7]$. For one- and two-body interactions the surface has the form $E(\beta, \gamma) = E_0 + N(N-1)f(\beta, \gamma)$, with $f(\beta, \gamma) = (1 + \beta^2)^{-2} \beta^2 [a - b\beta \cos 3\gamma + c\beta^2]$. The quadrupole shape parameters $(\beta, \gamma)$ at the global minimum of $E(\beta, \gamma)$ define the equilibrium shape for a given Hamiltonian. Each dynamical symmetry corresponds to a possible phase of the system. Phase transitions can be studied by IBM Hamiltonians of the form $[7]$,

$$H(\alpha) = (1 - \alpha) H_1 + \alpha H_2 ,$$

involving terms from different dynamical symmetry chains. The nature of the phase transition is dictated by the topology of the underlying surface. The energy surfaces at the critical-points of first- and second-order transitions have the form

$$\begin{align*}
1^{\text{st}} \text{order} : \quad f(\beta, \gamma = 0) = c(1 + \beta^2)^{-2}\beta^2 (\beta - \beta_0)^2 , \\
2^{\text{nd}} \text{order} : \quad f(\beta, \gamma) = c(1 + \beta^2)^{-2}\beta^4.
\end{align*}$$

**FIGURE 1.** Energy spectrum of $^{152}\text{Sm}$ and $^{150}\text{Nd}$ compared to that of the $X(5)$ “critical-point symmetry”. From $[4]$. 
As shown in Fig. 2, the first-order critical-surface has degenerate spherical and deformed minima at \( \beta = 0 \) and \( (\beta = \beta_0 > 0, \gamma = 0) \). The position \((\beta_+\)) and height \((h)\) of the barrier are indicated in the caption. The second-order critical-surface is independent of \(\gamma\) and is flat bottomed \((\sim \beta^4)\) for small \(\beta\). By requiring the Hamiltonian \(H(\alpha)\) of Eq. (2) to have a critical energy-surface, one pins down the value of the control parameter at the critical-point \((\alpha = \alpha_c)\). The critical-point Hamiltonians, obtained in this manner, for the \(U(5) - SU(3)\) and \(U(5) - O(6)\) phase transitions are given by

\[
\begin{align*}
U(5) - SU(3) : & \quad H(\alpha) = (1 - \alpha) \hat{n}_d - \alpha \frac{1}{4N} Q \cdot Q \quad \alpha_c = \frac{16N}{(34N - 27)}, \\
U(5) - O(6) : & \quad H(\alpha) = (1 - \alpha) \hat{n}_d + \alpha \frac{1}{N} \hat{P}_6 \quad \alpha_c = \frac{N}{(2N - 1)}.
\end{align*}
\]

IBM Hamiltonians of this kind have been studied extensively \[8\], concluding that the \(U(5) - SU(3)\) transition is of first order, with an extremely low barrier, (corresponding to \(\beta_0 = \sqrt{2}/4\) and \(h \approx 10^{-3}\) in Fig. 2). The \(U(5) - O(6)\) transition is found to be of second order. The corresponding critical-point Hamiltonians, Eq. (4), exhibit \(X(5)\)- and \(E(5)\)-like spectrum, respectively, albeit finite-N modifications \[8, 9, 10\].

From the point of view of symmetry, \(H(\alpha),\) Eq. (2), involves competing incompatible (non-commuting) symmetries. For \(\alpha = 0\) or \(\alpha = 1\), one recovers the limiting symmetries. For \(0 < \alpha < 1\), both symmetries are broken and the mixing is particularly strong at the critical-point \((\alpha_c \approx 1/2)\). A detailed study of the symmetry content of the IBM Hamiltonians, Eq. (4), upon variation of the control parameter \(\alpha\), has found that for most values of \(\alpha\), except for a narrow region near the critical-point \((\alpha = \alpha_c)\), selected low-lying states continue to exhibit characteristic properties (energy and \(B(E2)\) ratios) of the closest dynamical symmetry limit. Such an “apparent” persistence of symmetry in the face of strong symmetry-breaking interactions, was called “quasidynamical sym-
FIGURE 3. Left panel: energy spectrum for the $U(5)-SU(3)$ Hamiltonian, Eq. (4), for $N = 32$, as a function of the control parameter $\alpha$. Right panel: Squared amplitudes for angular momenta $L = 0,2,4$ yrast states in the $SU(3)$ basis and $\alpha = 0.6$. The critical-point value is $\alpha_c = 0.482$. From [13].

The indicated persistence is clearly evident in the spectrum shown in Fig. 3, for the $U(5)-SU(3)$ transition. This “apparent” symmetry is due to the coherent nature of the mixing. As seen on the right hand side of Fig. 3, the mixing of $SU(3)$ irreps is large, but is approximately independent of the angular momentum of the yrast states.

The IBM can also accommodate spherical to prolate-deformed first-order phase transitions, with a high barrier. The relevant critical-point Hamiltonian can be transcribed in the form [14]

$$H(\beta_0) = h_2 P^\dagger_{2\mu}(\beta_0) \cdot \tilde{P}_{2\mu}(\beta_0).$$

Here $P^\dagger_{2\mu}(\beta_0) = \beta_0 s^\dagger d^\dagger_{\mu} + \sqrt{7/2} (d^\dagger d^\dagger_{\mu})^{(2)}$, $\tilde{P}_{2\mu}(\beta_0) = (-1)^\mu P_{-\mu}(\beta_0)$ and $h_2, \beta_0 > 0$. The energy surface of $H(\beta_0)$ coincides with the first-order critical surface given in Eq. (3) and shown in Fig. (2a). For $\beta_0 = \sqrt{2}$, $H(\beta_0 = \sqrt{2})$ has a subset of solvable states with good $SU(3)$ symmetry $(\lambda, \mu) = (2N - 4k, 2k)$ [15], where $k = 0, 1, 2, \ldots$

$$|N, (2N-4k, 2k) \rangle \quad L = K, K+1, K+2, \ldots, (2N-2k) \quad E = 3h_2(2N+1-2k)k, \quad k > 0.$$  

For $k = 0$, the solvable states are members of a prolate-deformed ground band. For $k > 0$, the solvable states are members of the $\gamma^k$ bands, with $K = 2k$. In addition, $H(\beta_0 = \sqrt{2})$ has solvable spherical eigenstates with good $U(5)$ symmetry,

$$|N, n_d = \tau = L = 0 \rangle \quad E = 0,$$

$$|N, n_d = \tau = L = 3 \rangle \quad E = 3h_2(2N-1).$$

The remaining levels in the spectrum, shown in Fig. 4, are either predominantly spherical or deformed states arranged in several excited bands. Their wave functions are spread
over many $U(5)$ and $SU(3)$ irreps. This situation, for which only a subset of states obey an exact dynamical symmetry, while other states are mixed, is referred to as a partial dynamical symmetry (PDS) \cite{15}. For the first-order critical-point Hamiltonian considered here, some states are solvable with good $U(5)$ symmetry, some are solvable with good $SU(3)$ symmetry and all other states are mixed with respect to both $U(5)$ and $SU(3)$. This behavior defines a $U(5)$ PDS coexisting with a $SU(3)$ PDS.

In summary, the study of quantum phase transitions in nuclei provides a fertile test-ground for the development of novel concepts of symmetry. The latter include “critical-point symmetries” and partial dynamical symmetries at the critical-point and quasidynamical symmetry away from the critical point.

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