Polarization modes for strong-field gravitational waves

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Abstract. Strong-field gravitational plane waves are often represented in either the Rosen or Brinkmann forms. These forms are related by a coordinate transformation, so they should describe essentially the same physics, but the two forms treat polarization states quite differently. Both deal well with linear polarizations, but there is a qualitative difference in the way they deal with circular, elliptic, and more general polarization states. In this article we will describe a general algorithm for constructing arbitrary polarization states in the Rosen form.

1. Brinkmann form
Consider the general pp spacetime geometry [1, 2, 3, 4, 5]
\[ ds^2 = -2\, du\, dv + H(u, x, y)\, du^2 + dx^2 + dy^2. \] (1)
It is then a standard result that the only nonzero component of the Ricci tensor is
\[ R_{uu} = -\frac{1}{2} \left\{ \partial_x^2 H(u, x, y) + \partial_y^2 H(u, x, y) \right\}. \] (2)
Restricting attention to vacuum plane waves [6] gives us the form:
\[ ds^2 = -2\, du\, dv + \left\{ [x^2 - y^2] H_+(u) + 2xy H_\times(u) \right\} du^2 + dx^2 + dy^2. \] (3)
In this form of the metric the two polarization modes are explicitly seen to decouple. By choosing \( H_+(u) \) and \( H_\times(u) \) appropriately we can construct any general polarization state.

2. Rosen form
The “most general” form of the Rosen metric is [1, 7]
\[ ds^2 = -2\, du\, dv + g_{AB}(u)\, dx^A\, dx^B, \] (4)
where \( x^A = \{x, y\} \). The only non-zero component of the Ricci tensor is [1]:
\[ R_{uu} = -\left\{ \frac{1}{2} g^{AB} g_{AB}'' - \frac{1}{4} g^{AB} g_{BC} g^{CD} g_{DA} \right\}. \] (5)
Though relatively compact, because of the implicit matrix inversions this is a grossly nonlinear function of the metric components. In particular, in this form of the metric the + and \( \times \) linear polarizations do not decouple in any obvious way [8].
3. Linear Polarization
Consider the strong-field gravity wave metric in the + linear polarization. That is, set $g_{xy} = 0$. It is found most useful to put the resulting metric in the form [8]

$$ds^2 = -2 du dv + S^2(u) \left\{e^{+X(u)} dx^2 + e^{-X(u)} dy^2\right\}. \tag{6}$$

Then

$$R_{uu} = -\frac{1}{2} \left(4 \frac{S''}{S} + (X')^2\right). \tag{7}$$

In vacuum we have the general vacuum wave for + polarization in the form

$$ds^2 = -2 du dv + S^2(u) \left\{\exp\left(2 \int^u \sqrt{-S''/S} \, du\right) dx^2 + \exp\left(-2 \int^u \sqrt{-S''/S} \, du\right) dy^2\right\}. \tag{8}$$

From the + polarization, by rotating the $x$--$y$ plane through a fixed but arbitrary angle $\Theta_0$, we can easily deal with linear polarization modes along any desired axis.

4. Arbitrary Polarization
Now take an arbitrary, possibly $u$ dependent, polarization and consider the following metric ansatz [8]:

$$ds^2 = -2 du dv + S^2(u) \left\{[\cosh(X(u)) + \cos(\theta(u)) \sinh(X(u))] dx^2 + 2 \sin(\theta(u)) \sinh(X(u)) dx dy + \left[\cosh(X(u)) - \cos(\theta(u)) \sinh(X(u))\right] dy^2\right\}. \tag{9}$$

Note setting $\theta(u) = \Theta_0$ corresponds to linear polarization. The vacuum field equations imply

$$4 \frac{S''}{S} + (X')^2 + \sinh^2(X(u)) (\theta')^2 = 0. \tag{10}$$

Let us introduce a dummy function $L(u)$ and split this into the two equations

$$4 \frac{S''}{S} + (L')^2 = 0, \quad (L')^2 = (X')^2 + \sinh^2(X(u)) (\theta')^2. \tag{11}$$

The first of these equations is just the equation you would have to solve for a pure + (or in fact any linear) polarization. The second of these equations can be rewritten as

$$dL^2 = dX^2 + \sinh^2(X) \, d\theta^2, \tag{12}$$

and is the statement that $L$ can be interpreted as distance in the hyperbolic plane $H_2$.

Compare this to Maxwell electromagnetism, where polarizations can be specified by

$$\vec{E}(u) = E_x(u) \, \hat{x} + E_y(u) \, \hat{y}, \tag{13}$$

with no additional constraints. Thus an electromagnetic wavepacket of arbitrary polarization can be viewed as an arbitrary “walk” in the $(E_x, E_y)$ plane. We could also go to a magnitude-phase representation $(E, \theta)$ where

$$\vec{E}(u) = E(u) \cos \theta(u) \, \hat{x} + E(u) \sin \theta(u) \, \hat{y}. \tag{14}$$

So an electromagnetic wavepacket of arbitrary polarization can also be viewed as an arbitrary “walk” in the $(E, \theta)$ plane, where the $(E, \theta)$ plane is provided with the natural Euclidean metric

$$dL^2 = dE^2 + E^2 \, d\theta^2. \tag{15}$$

In contrast for gravitational waves in the Rosen form we are now dealing with an arbitrary “walk” in the hyperbolic plane, $H_2$. Furthermore, because of the nonlinearity of general relativity, there is still one remaining differential equation to solve.
5. Circular Polarization

As an important example, we consider strong-field circular polarization. Circular polarization corresponds to a fixed distortion \( X_0 \) with a linear advancement of \( \theta(u) \):

\[
\theta(u) = \Omega_0 u; \quad X(u) = X_0.
\]  

(16)

Then \[8\]

\[
ds^2 = -2 \, du \, dv + S^2(u) \left\{ [\cosh(X_0) + \cos(\Omega_0 u) \sinh(X_0)] dx^2 + \sin(\Omega_0 u) \sinh(X_0) dy \right\} \]  

(17)

So the only nontrivial component of the Ricci tensor is

\[
R_{uu} = -\frac{1}{2} \left\{ \frac{4}{S} S'' + \sinh^2(X_0) \Omega_0^2 \right\}.
\]  

(18)

Solving the vacuum equations gives

\[
S(u) = S_0 \cos \left( \frac{\sinh(X_0) \Omega_0 (u - u_0)}{2} \right).
\]  

(19)

This now describes a spacetime that has good reason to be called a strong-field circularly polarized gravity wave. Note the weak-field limit corresponds to \( X_0 \ll 1 \) so for an arbitrarily long interval in \( u \) we have \( S \approx S_0 \), and without loss of generality we can set \( S \approx 1 \). Then, as expected, we obtain \[8\]

\[
ds^2 \approx -2 \, du \, dv + dx^2 + dy^2 + X_0 \left\{ \cos(\Omega_0 u)[dx^2 - dy^2] + 2 \sin(\Omega_0 u) \, dx \, dy \right\}.
\]  

(20)

6. Decoupling the general form

Let us return to considering the metric in general Rosen form

\[
ds^2 = -2 \, du \, dv + g_{AB}(u) \, dx^A \, dx^B,
\]  

(21)

where \( x^A, x^B \) represent any arbitrary number of dimensions \( (d_\perp \geq 2) \) transverse to the \( (u, v) \) plane. It is easy to check that the only non-zero component of the Ricci tensor is still

\[
R_{uu} = -\left\{ \frac{1}{2} g^{AB} g''_{AB} - \frac{1}{4} \, g^{AB} \, g_{BC} \, g^{CD} \, g_{DA} \right\}.
\]  

(22)

Let us now decompose the \( d_\perp \times d_\perp \) matrix \( g_{AB} \) into an “envelope” \( S(u) \) and a unit determinant related to the “direction of oscillation” \[8\]. That is, let us take

\[
g_{AB}(u) = S^2(u) \, \hat{g}_{AB}(u),
\]  

(23)

where \( \text{det}(\hat{g}) \equiv 1 \). (A related discussion can be found in \[9\].) Calculating the various terms of the Ricci tensor using the relations

\[
[\hat{g}^{AB} \hat{g}_{AB}] = 0, \quad [\hat{g}^{AB} \hat{g}''_{AB}] - [\hat{g}^{AB} \hat{g}_{BC} \hat{g}^{CD} \hat{g}_{DA}] = 0,
\]  

(24)

it is found that \[8\]

\[
R_{uu} = -d_\perp \frac{S''}{S} - \frac{1}{2} [\hat{g}^{AB} \hat{g}_{BC} \hat{g}^{CD} \hat{g}_{DA}].
\]  

(25)
Note that we have now succeeded in decoupling the determinant \( \det(g) = S^2 d_{\perp} \); effectively the “envelope” \( S(u) \) of the gravitational wave) from the unit-determinant matrix \( \hat{g}(u) \).

Now consider the set \( SS(\mathcal{R}, d_{\perp}) \) of all unit determinant real symmetric matrices, and on that set consider the Riemannian metric

\[
\mathrm{d}L^2 = \mathrm{tr} \left\{ [\hat{g}]^{-1} \, \mathrm{d}[\hat{g}] \, [\hat{g}]^{-1} \, \mathrm{d}[\hat{g}] \right\}.
\]

(26)

Then

\[
R_{\alpha\alpha} = -\frac{1}{2} \left\{ 2d_{\perp} \frac{S''(u)}{S(u)} + \left( \frac{\mathrm{d}L}{\mathrm{d}u} \right)^2 \right\}.
\]

(27)

This means an arbitrary polarization vacuum Rosen wave is an arbitrary walk in \( SS(\mathcal{R}, d_{\perp}) \), with distance along the walk \( L(u) \) related to the envelope function \( S(u) \) as in the discussion above.

7. Discussion

Arbitrary polarizations, while trivial in the Brinkmann form, are difficult to implement in the Rosen form. To address this puzzle we have re-analyzed the Rosen strong-field gravity wave in terms of an “envelope” function and two freely specifiable functions. The vacuum field equations can be interpreted in terms of a single differential equation governing the “envelope”, coupled with an arbitrary walk in polarization space. In particular we have indicated how to construct a circularly polarized Rosen form gravity wave, and how to generalize this central idea beyond (3+1) dimensions. Further detailed calculations and discussions of these Rosen form polarizations can be found in [8].

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