FRACTIONAL CALCULUS AND APPLICATIONS OF FAMILY OF EXTENDED GENERALIZED GAUSS HYPERGEOMETRIC FUNCTIONS

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Abstract. The aim of the present paper is to establish certain new image formulae of family of some extended generalized Gauss hypergeometric functions by applying the operators of fractional derivative involving $\mathbf{2F}_1(.)$ due to Saigo. Furthermore, by employing some integral transforms on the resulting formulas, we obtained some more image formulas and also develop a new and further generalized form of the fractional kinetic equation involving the family of some extended generalized Gauss hypergeometric functions and the manifold generality of the family of functions is discussed in terms of the solution of the fractional kinetic equation. The results obtained here are quite general in nature.

1. Introduction and preliminaries. The Fractional Order Calculus (FOC) constitutes the branch of mathematics dealing with differentiation and integration under an arbitrary order of the operation, i.e. the order can be any real or even complex number, not only the integer one [38, 36, 43]. Although the FOC represents more than 300-year-old issue [12, 42], its great consequences in contemporary theoretical research and real world applications have been widely discussed relatively recently (see [1, 55, 2, 3, 13, 14, 4, 5, 57, 21, 6, 22, 47]). The idea of non-integer derivative was mentioned for the first time probably in a letter from Leibniz to L’Hospital in 1695. Later on, the pioneering works related to FOC have been elaborated by

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personalities such as Euler, Fourier, Abel, Liouville or Riemann. The interested reader can find the more detailed historical background of FOC in [38].

According to [12, 27], the reason why FOC remained practically unexplored for engineering applications and why only pure mathematics were privileged to deal with it for so long can be seen in multiple definitions of FOC, such as missing simple geometrical interpretation, absence of solution methods for fractional order differential equations and seeming adequateness of the Integer Order Calculus (IOC) for majority of problems. Nowadays, the situation is going better and the FOC provides efficient tool for many issues related to fractal dimension, “infinite memory”, chaotic behaviour, etc. Thus, the FOC has already come in useful in engineering areas such as bioengineering, viscoelasticity, electronics, robotics, control theory and signal processing [27]. Several control applications are available e.g. in [7, 28, 29].

A great number of additional results of fractional calculus was presented in the twentieth century, but at this point we only concentrate on one more, given by M. Caputo and first used extensively in [8]. Given a function \( f \) with an \( n - 1 \) absolute continuous derivative, Caputo defined a fractional derivative by

\[
D^\alpha_\ast f(x) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} \left( \frac{d}{ds} \right)^n f(s)ds
\]  

(1)

today usually named \textit{Caputo fractional derivative}. The fractional derivative given in equation 1 is strongly connected to the Riemann-Liouville fractional derivative (see [31]) and is today frequently used in applications. This is because using the Caputo derivative one can specify the initial conditions of fractional differential equations in classical form, i.e.

\[
y^{(k)}(0) = b_k, \quad k = 0, 1, ..., n - 1,
\]  

(2)
in contrast to differential equations containing the Riemann-Liouville differential operator (see [31]). While the operator \( D^\alpha_\ast \) is denoted today as Caputo operator, Y. N. Rabotnov had already introduced this differential operator into the Russian viscoelastic literature in [45], a year before Caputo’s paper was published.

By the second half of the twentieth century the field of fractional calculus had grown to such extent, that in 1974 the first conference concerned solely with the theory and applications of fractional calculus was held in New Haven [48]. In the same year the first book on fractional calculus by Oldham and Spanier [38] was published. A number of additional books have appeared since then, the most popular the ones by Miller and Ross [36], Samko et al. [50] and Podlubny [43]. In 1998 the first issue of the mathematical journal “Fractional calculus & applied analysis” was printed. This journal is solely concerned with topics on the theory of fractional calculus and its applications. Finally in 2004 the large conference “Fractional differentiation and its applications” was held in Bordeaux, where no less than 104 talks were given in the field of fractional calculus.

From its birth - a simple question from L’Hospital to Leibniz – to its today’s wide use in numerous scientific fields fractional calculus has come a long way. Even though it is nearly as old as classical calculus itself, it flourished mainly over the last decades because of its good applicability on models describing complex real life problems (see recent work [18, 19, 20, 23, 24]). And even though the term fractional calculus is a misnomer we will use it throughout this text, which will be concerned with theoretical and, more importantly, numerical aspects of problems arising in this field.
In our investigation, we need to recall the following pair of Saigo hypergeometric fractional integral operators. For \( x > 0, \mu, \nu, \eta \in \mathbb{C} \) and \( \Re(\alpha) > 0 \), we have

\[
(I^\lambda f(t)) (x) = \frac{x^{-\lambda-\vartheta}}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} t^{\vartheta-1} f(t) \, dt
\]

and

\[
(J_{x,\infty}^\lambda f(t)) (x) = \frac{1}{\Gamma(\lambda)} \int_x^{\infty} (t-x)^{\lambda-1} (t-x)^{\vartheta-1} f(t) \, dt
\]

where the \( _2F_1(.) \), a special case of the generalized hypergeometric function, is the Gauss hypergeometric function. The Saigo fractional integral operators given in equations 3 and 4 can be written in the following form:

\[
(I^\lambda f(x)) = \left( \frac{d}{dx} \right)^k I^\lambda f(x), \\
(\Re(\lambda) \leq 0; \ k = [\Re(-\lambda)] + 1)
\]

and

\[
(J_{x,\infty}^\lambda f(x)) = \left( -\frac{d}{dx} \right)^k J_{x,\infty}^\lambda f(x), \\
(\Re(\lambda) \leq 0; \ k = [\Re(-\lambda)] + 1)
\]

The Erdélyi-Kober type fractional integral operators are defined as follows (see Kober [33]):

\[
(E^\lambda f)(x) = \frac{x^{-\lambda-\vartheta}}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} t^{\vartheta-1} f(t) \, dt, \quad (\Re(\lambda) > 0)
\]

and

\[
(K_{x,\infty}^\lambda f)(x) = \frac{x^{\vartheta}}{\Gamma(\lambda)} \int_x^{\infty} (t-x)^{\lambda-1} (t-x)^{\vartheta-1} f(t) \, dt, \quad (\Re(\lambda) > 0)
\]

where the function \( f(t) \) is so constrained that the defining integrals in equations 7 and 8 converge.

The Riemann-Liouville fractional integral and the Weyl fractional integral operators defined as the follows (see, e.g., [38]):

\[
(R^\lambda f)(x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} f(t) \, dt, \quad (\Re(\lambda) > 0)
\]

and

\[
(W_{x,\infty}^\lambda f)(x) = \frac{1}{\Gamma(\lambda)} \int_x^{\infty} (t-x)^{\lambda-1} f(t) \, dt, \quad (\Re(\lambda) > 0)
\]

provided both the integrals converge.

The operator \( I^\lambda_{0,x} f(t) \) contains both the Riemann-Liouville and Erdélyi-Kober fractional integral operators by means of the following relationships (see Kilbas [31]):

\[
(R^\lambda f)(x) = (I^\lambda_{0,x} f)(x)
\]

and

\[
(E^\lambda f)(x) = (I^\lambda_{0,x} f)(x).
\]
While the operator $I_{x,\infty}^{\lambda,\sigma,\vartheta}(\cdot)$ unifies the Weyl and Erdélyi-Kober fractional integral operators as follows:

$$ (W_{x,\infty}^\lambda f)(x) = (J_{x,\infty}^{\lambda -\lambda,\vartheta} f)(x) $$

and

$$ (K_{x,\infty}^{\lambda,\vartheta} f)(x) = (J_{x,\infty}^{\lambda,0,\vartheta} f)(x). $$

The following lemmas obtained by Kilbas and Sebastian [32] are also required for our investigation.

**Lemma 1.1.** (Kilbas and Sebastian [32]) Let $\lambda, \sigma, \vartheta \in \mathbb{C}$ be such that $\Re(\lambda) > 0$. Then, we have the following relation

$$ I_{0,x}^{\lambda,\sigma,\vartheta}(t^{\rho-1}) (x) = \frac{\Gamma(\rho)\Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma)\Gamma(\rho + \lambda + \vartheta)} x^{\rho - \sigma - 1}, \quad (\Re(\rho) > \max\{0, \Re(\vartheta - \sigma)\}) $$

and

$$ J_{x,\infty}^{\lambda,\sigma,\vartheta}(t^{\rho-1}) (x) = \frac{\Gamma(\sigma - \rho + 1)\Gamma(\vartheta - \rho + 1)}{\Gamma(1 - \rho)\Gamma(\lambda + \sigma + \vartheta - \rho + 1)} x^{\rho - \vartheta - 1}, \quad \Re(\rho) < 1 + \min\{\Re(\sigma), \Re(\vartheta)\}. $$

The special cases of the formulae given in equation 15 and 16 when $\sigma = -\lambda$ yield, respectively, two formulas involving the Riemann-Liouville and Weyl type fractional operators as in the following lemma (see, [35]).

**Lemma 1.2.** Let $\lambda, \rho \in \mathbb{C}$. Then, we have the following relation

$$ R_{0,x}^{\lambda}(t^{\rho-1}) (x) = \frac{\Gamma(\rho)}{\Gamma(\rho + \lambda)} x^{\rho - \lambda - 1}, \quad (\Re(\lambda) > 0, \Re(\rho) > 0) $$

and

$$ W_{x,\infty}^{\lambda}(t^{\rho-1}) (x) = \frac{\Gamma(1 - \lambda - \rho)}{\Gamma(1 - \rho)} x^{\rho + \lambda - 1}, \quad \Re(\rho) > \Re(\lambda) > -1. $$

Setting $\sigma = 0$, the equations 15 and 16 yields, two formulas involving the Erdélyi-Kober fractional integral operators as in the following lemma (see, [35]).

**Lemma 1.3.** Let $\lambda, \sigma, \vartheta \in \mathbb{C}$ be such that $\Re(\lambda) > 0$. Then, we have the following relation

$$ E_{0,x}^{\lambda,\vartheta}(t^{\rho-1}) (x) = \frac{\Gamma(\rho + \vartheta)}{\Gamma(\rho + \lambda + \vartheta)} x^{\rho - 1}, \quad (\Re(\rho + \vartheta) > 0) $$

and

$$ K_{x,\infty}^{\lambda,\vartheta}(t^{\rho-1}) (x) = \frac{\Gamma(\vartheta - \rho + 1)}{\Gamma(\lambda + \vartheta - \rho + 1)} x^{\rho - 1}, \quad (\Re(\vartheta) > \Re(\rho) > -1). $$

2. Extensions of beta function and hypergeometric function. No doubt the classical beta function $B(x, y)$ is one of the most fundamental special functions, because of its precious role in several field of sciences such as mathematical, physical, engineering and statistical sciences. In many areas of applied mathematics, different types of special functions have become necessary tool for the scientists and engineers. During the recent decades or so, numerous interesting and useful extensions of the different special functions (such as the Gamma and Beta functions, the Gauss hypergeometric function, and so on) have been introduced by different authors. The
details of the extensions of beta function and Gauss hypergeometric function are presented here in this section.

The generalized Beta function \( B_p^{(\delta, \zeta, \kappa, \mu)}(x, y) \) is defined as (see, [59]):

\[
B_p^{(\delta, \zeta, \kappa, \mu)}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} F_1 \left( \delta; \zeta; -\frac{p}{t^\kappa(1-t)\mu} \right) dt,
\]

\[(\Re(p) \geq 0; \min\{\Re(x), \Re(y), \Re(\delta), \Re(\zeta)\} > 0; \max\{\Re(\kappa), \Re(\mu)\} > 0).\] \hspace{1cm} (21)

When \( \kappa = \mu \), the equation 21 reduces to the generalized extended beta function \( B_p^{(\delta, \zeta)}(x, y) \) defined as (see, [41, p. 37]):

\[
B_p^{(\delta, \zeta)}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} F_1 \left( \delta; \zeta; -\frac{p}{t^\mu(1-t)^\mu} \right) dt,
\]

\[(\Re(p) \geq 0; \min\{\Re(x), \Re(y), \Re(\delta), \Re(\zeta)\} > 0; \Re(\mu) > 0).\] \hspace{1cm} (22)

The special case of equation 22, when \( \mu = 1 \) reduces to the generalized Beta type function as follows (see [39, p.4602]):

\[
B_p^{(\delta, \zeta)}(x, y) = B_p^{(\delta, \zeta, 1)}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} F_1 \left( \delta; \zeta; -\frac{p}{t(1-t)} \right) dt,
\]

\[(\Re(p) \geq 0; \min\{\Re(x), \Re(y), \Re(\delta), \Re(\zeta)\} > 0).\] \hspace{1cm} (23)

The further special case of equation 23, when \( \delta = \zeta \) is given due to Choudhary et al. [9] by

\[
B_p(x, y) = B_p^{(\delta, \delta)}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp \left( -\frac{p}{t(1-t)} \right) dt, \quad (\Re(p) \geq 0).\] \hspace{1cm} (24)

If we choose \( p = 0 \), the extended beta function given in equation 24 reduces to the classical beta function \( B(x, y) \), which is defined as:

\[
B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad (\Re(x) > 0, \Re(y) > 0).\] \hspace{1cm} (25)

It is clear that there is following relationship between the classical Beta function \( B(x, y) \) and its extensions:

\[
B(x, y) = B_0(x, y) = B_p^{(\delta, \zeta)}(x, y) = B_0^{(\delta, \zeta, 1)}(x, y) = B_0^{(\delta, \zeta, 1, 1)}(x, y).\] \hspace{1cm} (26)

The generalized hypergeometric series \( pF_q(p, q \in \mathbb{N}) \) is defined as (see [46, p.73]) and [60, pp. 71-75]:

\[pF_q \left[ \begin{array}{c} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \\ z \end{array} \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}
\]

\[= pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z),\] \hspace{1cm} (27)

where \((\lambda)_n\) is the Pochhammer symbol defined (for \( \lambda \in \mathbb{C} \)) by (see [60, p.2 and p.5]):

\[(\lambda)_n := \begin{cases} 
1 & (n = 0) \\
\lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N})
\end{cases}\] \hspace{1cm} (28)

\[= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-),\] \hspace{1cm} (29)

and \(\mathbb{Z}_0^-\) denotes the set of Non-positive integers, where \(\Gamma(\lambda)\) is familiar Gamma function.
Chaudhry et al. [10, p. 591, Eqs. (2.1) and (2.2)] made use of the extended Beta function $B_p(x, y)$ in 24 to extend the Gauss hypergeometric function $2F_1$ as follow: The extended Gauss hypergeometric function $F_p(a, b; c; z)$ is defined as

$$F_p(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!},$$

(30) $|z| < 1; \Re(c) > \Re(b) > 0; \Re(p) \geq 0$.

Similarly, by appealing to the generalized Beta function $B_p(\delta, \xi, \mu)$ in equation 23 Özergin [40] and Özergin et al. [39] introduced and investigated a further potentially useful extensions of the generalized Gauss hypergeometric functions. The extended generalized Gauss hypergeometric function $F_p(\delta, \xi, \mu)$ is defined as:

$$F_p(\delta, \xi, \mu)(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p(\delta, \xi, \mu)(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!},$$

(31) $|z| < 1; \min\{\Re(\delta), \Re(\xi), \Re(\mu)\} > 0; \Re(c) > \Re(b) > 0; \Re(p) \geq 0$.

Based on the generalized Beta function in equation 22, Parmar [41] introduced and studied a family of the generalized Gauss hypergeometric functions $F_p(\delta, \xi, \mu)(a, b; c; z)$ defined by

$$F_p(\delta, \xi, \mu)(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p(\delta, \xi, \mu)(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!},$$

(32) $|z| < 1; \min\{\Re(\delta), \Re(\xi), \Re(\mu)\} > 0; \Re(c) > \Re(b) > 0; \Re(p) \geq 0$.

Recently, Srivastava et al. [61] used the generalized Beta function in equation 21 to introduce a family of some extended generalized Gauss hypergeometric functions defined by

$$F_p(\delta, \xi, \mu, \omega)(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p(\delta, \xi, \mu, \omega)(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!},$$

(33) $|z| < 1; \min\{\Re(\delta), \Re(\xi), \Re(\omega), \Re(\mu)\} > 0; \Re(c) > \Re(b) > 0; \Re(p) \geq 0$.

It is easy to see the following relationships:

$$F_p(\delta, \xi, \omega, \mu)(a, b; c; z) = F_p(\delta, \xi)(a, b; c; z)$$

(34)

$$F_p(\delta, \xi, \omega)(a, b; c; z) = F_p(\delta, \xi)(a, b; c; z)$$

(35)

$$F_p(\delta, \omega, \mu)(a, b; c; z) = F_p(a, b; c; z)$$

(36)

$$F_p(\delta, \gamma, \mu)(a, b; c; z) = 2F_1(a, b; c; z).$$

(37)

The above-mentioned detailed and systematic investigation by many authors (see, for example, [40], [39]) have largely motivated our present study. In this paper, our aim is to establish fractional integral and derivative formulas. Furthermore, by employing some integral transforms on the resulting formulas, we obtained also some more image formulas and we develop a new and further generalized form of the fractional kinetic equation involving the family of some extended generalized Gauss hypergeometric functions and the manifold generality of the family of functions to discussed in terms of the solution of the fractional kinetic equation.
3. Fractional calculus of the generalized Gauss hypergeometric functions.

In this section, we will establish some fractional integral formulas for the generalized Gauss hypergeometric type functions \( F_p^{(\delta,\zeta,\kappa,\rho)}(a, b; c; z) \) by using certain general pair of fractional integral operators. To establish the image formulas, we require the following concept of the Hadamard products (see [44]).

**Definition 3.1.** Let \( f(z) := \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) := \sum_{n=0}^{\infty} b_n z^n \) be two power series whose radii of convergence are given by \( R_f \) and \( R_g \), respectively. Then their Hadamard product is power series defined by

\[
(f \ast g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n,
\]

whose radius of convergence \( R \) satisfies \( R_f R_g \leq R \).

In particular, if one of the power series defines an entire function and the radius of convergence of the other is greater than 0, then the Hadamard product series defines an entire function, too. We can use the Hadamard product to decompose a newly-emerged function into two known functions. For example, the function \( p \cdot F_q^{(\alpha, \beta; \rho; \lambda)}[z; b] \) can be decomposed as follows:

\[
p \cdot F_q^{(\alpha, \beta; \rho; \lambda)}[z; b] = 0 \cdot F_r \left[ \begin{array}{c} x_1, \ldots, x_p \\ y_1, \ldots, y_{q+r} \end{array} ; z \right] \ast p \cdot F_q^{(\alpha, \beta; \rho; \lambda)} \left[ \begin{array}{c} x_1, \ldots, x_p \\ y_1, \ldots, y_{q+r} \end{array} ; b \right],
\]

\[
\text{valid under the conditions of Theorem 3.2, we have}
\]

\[
\mathcal{F} = \sum_{n=0}^{\infty} \frac{B_p^{(\delta, \zeta, \kappa, \rho)}(b + n, c - b)}{B(b, c - b)} \frac{1}{n!} \left( \frac{\Gamma(\rho) \Gamma(\rho + \theta - \sigma)}{\Gamma(\rho + \lambda + \theta)} \right) x^n \tag{41}
\]

applying the result 15, the above equation 41 reduces to

\[
\mathcal{F} = x^{\rho - \sigma - 1} \sum_{n=0}^{\infty} \frac{B_p^{(\delta, \zeta, \kappa, \rho)}(b + n, c - b)}{B(b, c - b)} \frac{1}{n!} \left( \frac{\Gamma(\rho + n) \Gamma(\rho + \theta - \sigma + n)}{\Gamma(\rho - \sigma + n) \Gamma(\rho + \lambda + \theta + n)} \right) x^n \tag{42}
\]

The main results are given in the following theorems. The main results are given in the following theorems.

**Theorem 3.2.** Let \( \lambda, \sigma, \vartheta, \rho, \delta, \zeta, \kappa, \mu, a, b, c, p \in \mathbb{C} \) be such that \( \Re(\lambda) > 0 \), \( \Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\} \), \( \min\{\Re(\delta), \Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0 \), \( \Re(c) > \Re(b) > 0 \), \( \Re(p) \geq 0 \), then

\[
\left( I_{0, x}^{\lambda, \sigma, \vartheta} x^{\rho - 1} F_p^{(\delta, \zeta, \kappa, \rho)}(a, b; c; t) \right)(x) = x^{\rho - \sigma - 1} \frac{\Gamma(\rho) \Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} \times F_p^{(\delta, \zeta, \kappa, \mu)}(a, b; c; x) \ast F_2(\rho, \rho + \vartheta - \sigma; \rho - \sigma, \rho + \lambda + \vartheta; x).
\]

**Proof.** For convenience, we denote the left-hand side of the equation 40 by \( \mathcal{F} \). Using the formula 33, and then changing the order of integration and summation, which is valid under the conditions of Theorem 3.2, we have

\[
\mathcal{F} = \sum_{n=0}^{\infty} \frac{B_p^{(\delta, \zeta, \kappa, \rho)}(b + n, c - b)}{B(b, c - b)} \frac{1}{n!} \left( \frac{\Gamma(\rho) \Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} \right) x^n \tag{41}
\]
Let Theorem 3.3. after simplification, the above equation 42 reduces to
\[ J = x^{\rho-\sigma-1} \frac{\Gamma(\rho)\Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma)\Gamma(\rho + \lambda + \vartheta)} \sum_{n=0}^{\infty} \frac{(a)_n B_p^{(\delta,\zeta;\rho)}(b + n, c - b)}{B(b, c - b)} \times \frac{(\rho)_n(\rho + \vartheta - \sigma)_n x^n}{(\rho - \sigma)_n(\rho + \lambda + \vartheta)_n n!} \] (43)

Further interpret in the view of 33, the above equation 43 we have
\[ J = x^{\rho-\sigma-1} \frac{\Gamma(\rho)\Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma)\Gamma(\rho + \lambda + \vartheta)} \times 2 F_p^{(\delta,\zeta;\rho)}(a, b, \rho + \vartheta - \sigma; \rho - \sigma, \rho + \lambda + \vartheta, c, x), \] (44)
in the view of Hadamard case given in equation 39, from the above equation 44, we have the required result.

**Theorem 3.3.** Let \( \lambda, \sigma, \vartheta, \rho, \delta, \zeta, \kappa, \mu, a, b, c, p \in \mathbb{C} \) be such that \( \Re(\lambda) > 0, \Re(\rho) < 1 + \min\{|\Re(\sigma)|, |\Re(\vartheta)|\}; \min\{|\Re(\delta)|, |\Re(\zeta)|, |\Re(\kappa)|, |\Re(\mu)|\} > 0; \Re(c) > \Re(b) > 0; \Re(\rho) \geq 0, \) then
\[ \left( J_{\lambda, \sigma, \vartheta}^{\delta, \zeta, \kappa, \mu} F_p^{(\delta, \zeta; \rho)}(a, b; c; 1/t) \right)(x) = x^{\rho-\sigma-1} \frac{\Gamma(\sigma - \rho + 1)\Gamma(\vartheta - \rho + 1)}{\Gamma(1 - \rho)\Gamma(\lambda + \sigma + \vartheta - \rho + 1)} F_p^{(\delta, \zeta; \rho)}(a, b; c; 1/x) \] (45)
\[ \times 2 F_p^{(\delta, \zeta; \rho)}(a, b; \rho + \vartheta; \rho - \sigma, \rho + \lambda + \vartheta; c, x). \]

**Proof.** The proof of the above theorem would be parallel to those of the Theorem 3.2.

Setting \( \sigma = 0 \) in Theorems 3.2 & 3.3 and employing the relations 40 and 45 yield certain interesting results asserted by the following corollaries.

**Corollary 1.** Let \( \lambda, \vartheta, \rho, \delta, \zeta, \kappa, \mu, a, b, c, p \in \mathbb{C} \) be such that \( \Re(\lambda) > 0, \Re(\vartheta) > 0; \min\{|\Re(\delta)|, |\Re(\zeta)|, |\Re(\kappa)|, |\Re(\mu)|\} > 0; \Re(c) > \Re(b) > 0; \Re(\rho) \geq 0, \) then
\[ \left( J_{\lambda, \vartheta}^{\delta, \zeta, \kappa, \mu} F_p^{(\delta, \zeta; \rho)}(a, b; c; t) \right)(x) = x^{\rho-\sigma-1} \frac{\Gamma(\rho + \vartheta)}{\Gamma(\rho + \lambda + \vartheta)} \times F_p^{(\delta, \zeta; \rho)}(a, b; c; x) \times 1 F_1(\rho + \vartheta; \rho + \lambda + \vartheta; x). \] (46)

**Corollary 2.** Let \( \lambda, \vartheta, \rho, \delta, \zeta, \kappa, \mu, a, b, c, p \in \mathbb{C} \) be such that \( \Re(\lambda) > 0, \Re(\vartheta) > \Re(\rho) > -1; \min\{|\Re(\delta)|, |\Re(\zeta)|, |\Re(\kappa)|, |\Re(\mu)|\} > 0; \Re(c) > \Re(b) > 0; \Re(\rho) \geq 0, \) then
\[ \left( J_{\lambda, \vartheta}^{\delta, \zeta, \kappa, \mu} F_p^{(\delta, \zeta; \rho)}(a, b; c; 1/t) \right)(x) = x^{\rho-\sigma-1} \frac{\Gamma(\vartheta - \rho + 1)}{\Gamma(\lambda + \vartheta - \rho + 1)} \times F_p^{(\delta, \zeta; \rho)}(a, b; c; 1/x) \times 1 F_1(\vartheta - \rho + 1; \lambda + \vartheta - \rho + 1; 1/x). \] (47)

Further, if we replace \( \sigma \) with \( -\lambda \) in Theorems 3.2 & 3.3 reduced to the following form

**Corollary 3.** Let \( \lambda, \sigma, \vartheta, \rho, \delta, \zeta, \kappa, \mu, a, b, c, p \in \mathbb{C} \) be such that \( \Re(\lambda) > 0, \Re(\rho) > 0; \min\{|\Re(\delta)|, |\Re(\zeta)|, |\Re(\kappa)|, |\Re(\mu)|\} > 0; \Re(c) > \Re(b) > 0; \Re(\rho) \geq 0, \) then
\[ \left( J_{\lambda, \sigma, \vartheta}^{\delta, \zeta, \kappa, \mu} F_p^{(\delta, \zeta; \rho)}(a, b; c; t) \right)(x) = x^{\rho+\sigma-1} \frac{\Gamma(\rho)}{\Gamma(\rho + \lambda)} \times F_p^{(\delta, \zeta; \rho)}(a, b; c; x) \times 1 F_1(\rho; \rho + \lambda; x). \] (48)
Corollary 4. Let $\lambda, \sigma, \vartheta, \rho, \delta, \zeta, \kappa, \mu, a, b, c, p \in \mathbb{C}$ be such that $\Re(\lambda) > 0, \Re(\rho) > \Re(\lambda) > -1$; 
\[ \min\{\Re(\delta), \Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0; \Re(\sigma) > \Re(b) > 0; \Re(\rho) \geq 0, \] 
then
\[ \left( W_{2,\infty}^{\lambda} \mu^{\rho-1} F_p^{(\delta, \zeta, \kappa, \mu)} (a, b; c; 1/t) \right) (x) = x^{\rho+\sigma-1} \frac{\Gamma(1-\rho-\lambda)}{\Gamma(1-\rho)} \] 
\[ \times F_p^{(\delta, \zeta, \kappa, \mu)} (a, b; c; 1/x) *_1 F_1(1-\rho-\lambda; 1-\rho; 1/x). \] 

If we choose $\kappa = \mu$ then the formulae in equations 40 and 45 reduced to the following form:

Corollary 5. Let $\lambda, \sigma, \vartheta, \rho, \delta, \zeta, \kappa, a, b, c, p \in \mathbb{C}$ be such that $\Re(\lambda) > 0, \Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}; \min\{\Re(\delta), \Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0; \Re(\sigma) > \Re(b) > 0; \Re(\rho) \geq 0, \] 
then
\[ \left( I_{0,x}^{\lambda, \sigma, \vartheta, \rho-1} F_p^{(\delta, \zeta, \kappa)} (a, b; c; t) \right) (x) = x^{\rho+\sigma-1} \frac{\Gamma(\rho + \sigma - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \sigma)} \] 
\[ \times F_p^{(\delta, \zeta, \kappa)} (a, b; c; x) *_2 F_2(\rho, \rho + \sigma - \rho; \rho - \sigma, \rho + \lambda + \sigma; x). \]

Corollary 6. Let $\lambda, \sigma, \vartheta, \rho, \delta, \zeta, \kappa, a, b, c, p \in \mathbb{C}$ be such that $\Re(\lambda) > 0, \Re(\rho) < 1 + \min\{\Re(\sigma), \Re(\rho)\}; \min\{\Re(\delta), \Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0; \Re(\sigma) > \Re(b) > 0; \Re(\rho) \geq 0, \] 
then
\[ \left( I_{0,x}^{\lambda, \sigma, \vartheta, \rho-1} F_p^{(\delta, \zeta, \kappa)} (a, b; c; 1/t) \right) (x) = x^{\rho+\sigma-1} \frac{\Gamma(\rho + \sigma - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \sigma)} \] 
\[ \times F_p^{(\delta, \zeta, \kappa)} (a, b; c; 1/x) *_2 F_2(\rho - \rho + 1, \sigma - \rho + 1; 1 - \rho, \lambda + \sigma + \sigma - \rho + 1; 1/x). \]

If we choose $\kappa = \mu = 1$ then the formulae in equations 40 and 45 reduced to the following form:

Corollary 7. Let $\lambda, \sigma, \vartheta, \rho, \delta, \zeta, \kappa, a, b, c, p \in \mathbb{C}$ be such that $\Re(\lambda) > 0, \Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}; \min\{\Re(\delta), \Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0; \Re(\sigma) > \Re(b) > 0; \Re(\rho) \geq 0, \] 
then
\[ \left( I_{0,x}^{\lambda, \sigma, \vartheta, \rho-1} F_p^{(\delta, \zeta, \kappa)} (a, b; c; t) \right) (x) = x^{\rho+\sigma-1} \frac{\Gamma(\rho + \sigma - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \sigma)} \] 
\[ \times F_p^{(\delta, \zeta, \kappa)} (a, b; c; x) *_2 F_2(\rho + \sigma - \rho + 1, \sigma - \rho + 1; 1 - \rho, \lambda + \sigma + \sigma - \rho + 1; 1/x). \]

If we choose $\kappa = \mu = 1$ and $\delta = \zeta$ then the formulae in equations 40 and 45 reduced to the following form:

Corollary 8. Let $a, b, c, p \in \mathbb{C}$ be such that $\Re(\lambda) > 0, \Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}; \min\{\Re(\delta), \Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0; \Re(\sigma) > \Re(b) > 0; \Re(\rho) \geq 0, \] 
then
\[ \left( I_{0,x}^{\lambda, \sigma, \vartheta, \rho-1} F_p^{(\delta, \zeta)} (a, b; c; 1/t) \right) (x) = x^{\rho+\sigma-1} \frac{\Gamma(\rho + \sigma - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \sigma)} \] 
\[ \times F_p^{(\delta, \zeta)} (a, b; c; 1/x) *_2 F_2(\sigma - \rho + 1, \vartheta - \rho + 1; 1 - \rho, \lambda + \sigma + \sigma - \rho + 1; 1/x). \]

If we choose $\kappa = \mu = 1$ and $\delta = \zeta$ then the formulae in equations 40 and 45 reduced to the following form:

Corollary 9. Let $a, b, c, p \in \mathbb{C}$ be such that $\Re(\lambda) > 0, \Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}; \min\{\Re(\delta), \Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0; \Re(\sigma) > \Re(b) > 0; \Re(\rho) \geq 0, \] 
then
\[ \left( I_{0,x}^{\lambda, \sigma, \vartheta, \rho-1} F_p^{(\delta, \zeta)} (a, b; c; t) \right) (x) = x^{\rho+\sigma-1} \frac{\Gamma(\rho + \sigma - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \sigma)} \] 
\[ \times F_p^{(\delta, \zeta)} (a, b; c; x) *_2 F_2(\rho + \sigma - \rho; \rho - \sigma, \rho + \lambda + \sigma; x). \]
Corollary 10. Let \( \lambda, \sigma, \vartheta, \rho, \delta, \zeta, a, b, c, p \in \mathbb{C} \) be such that \( \Re(\lambda) > 0, \Re(\rho) < 1 + \min\{\Re(\sigma), \Re(\vartheta)\} \); \( \min\{\Re(\delta), \Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0; \Re(c) > \Re(b) > 0; \Re(\rho) \geq 0 \), then
\[
(I_{x,\infty}^{\lambda, \sigma, \vartheta, \rho, \delta, \zeta}(a, b; c; 1/t)) (x) = x^{\rho-1} \frac{\Gamma(\sigma - \rho + 1) \Gamma(\vartheta - \rho + 1)}{\Gamma(1 - \rho) \Gamma(\lambda + \sigma + \vartheta - \rho + 1)}
\]
\[
\times F_p(a; b; c; 1/x) \ast_{n-1} F_3(\sigma - \rho + 1, \vartheta - \rho + 1, l + m; x).
\]

4. Image formulas associated with integral transform. Certain theorems involving the results obtained in previous section associated with the integral transforms like, Beta transform, Laplace transform and Whittaker transform obtained in this section.

4.1. Beta transform. The Beta transform of \( f(z) \) is defined as [56]:
\[
B\{f(z); a, b\} = \int_0^1 z^{a-1}(1 - z)^{b-1}f(z)dz
\]

Theorem 4.1. Let \( \lambda, \sigma, \vartheta, \rho, \delta, \zeta, \kappa, \mu, a, b, c, p \in \mathbb{C} \) be such that \( \Re(\lambda) > 0, \Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}; \min\{\Re(\delta), \Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0; \Re(c) > \Re(b) > 0; \Re(\rho) \geq 0 \), then
\[
B \left\{ \left( I_{0,\infty}^{\lambda, \sigma, \vartheta, \rho, \delta, \zeta}(a, b; c; t) \right) (x) : \lambda, \sigma, \vartheta \right\} = x^{\rho-1} B(l, m) \frac{\Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} F_p^{(\delta, \zeta, \kappa, \mu)}(a, b; c; x)
\]
\[
\ast_{n-1} F_3(\rho, \vartheta, \rho - \vartheta + 1, l; \rho - \vartheta, \rho + \lambda + \vartheta, l + m; x).
\]

Proof. For convenience, we denote the left-hand side of the equation 57 by \( \mathcal{B} \), then using the definition of beta transform, we have:
\[
\mathcal{B} = \int_0^1 z^{l-1}(1 - z)^{m-1} \left( I_{0,\infty}^{\lambda, \sigma, \vartheta, \rho, \delta, \zeta}(a, b; c; t) \right) (x)dz,
\]

further using the formula given in equation 33 and then changing the order of integration and summation, which is valid under the conditions of Theorem 4.1, then
\[
\mathcal{B} = \sum_{n=0}^{\infty} (a)_n B_p^{(\delta, \zeta, \kappa, \mu)}(b + n, c - b) \frac{1}{n!} \frac{1}{B(b, c - b)} \int_0^1 z^{l+n-1}(1 - z)^{m-1}dz
\]
\[
\times \left( I_{0,\infty}^{\lambda, \sigma, \vartheta, \rho, \delta, \zeta}(a, b; c; t) \right) (x) \int_0^1 z^{l+n-1}(1 - z)^{m-1}dz,
\]

applying the formulae given in equation 15, after simplification the above equation 59 gives
\[
\mathcal{B} = x^{\rho-1} \sum_{n=0}^{\infty} (a)_n B_p^{(\delta, \zeta, \kappa, \mu)}(b + n, c - b) \frac{x^n}{n!} \frac{1}{B(b, c - b)} \frac{\Gamma(\rho + n) \Gamma(\rho + \vartheta - \sigma + n)}{\Gamma(\rho - \sigma + n) \Gamma(\rho + \lambda + \vartheta + n)} \int_0^1 z^{l+n-1}(1 - z)^{m-1}dz,
\]
\[
\times \frac{\Gamma(\rho + n) \Gamma(\rho + \vartheta - \sigma + n)}{\Gamma(\rho - \sigma + n) \Gamma(\rho + \lambda + \vartheta + n)} \frac{\Gamma(l + n) \Gamma(m)}{\Gamma(l + m + n)}.
\]

Applying the definition of beta transform, the above equation 60 leads to
\[
\mathcal{B} = x^{\rho-1} \sum_{n=0}^{\infty} (a)_n B_p^{(\delta, \zeta, \kappa, \mu)}(b + n, c - b) \frac{x^n}{n!} \frac{1}{B(b, c - b)} \frac{\Gamma(\rho + n) \Gamma(\rho + \vartheta - \sigma + n)}{\Gamma(\rho - \sigma + n) \Gamma(\rho + \lambda + \vartheta + n)} \frac{\Gamma(l + n) \Gamma(m)}{\Gamma(l + m + n)}
\]
after simplification the above equation 61 reduces to

\[ B = x^{\rho-\sigma-1} B(l, m) \frac{\Gamma(\rho) \Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} \sum_{n=0}^{\infty} (a)_{n} B_{p}^{(\delta, \zeta; \kappa, \mu)} (b + n, c - b) (l)_{n} x^{n} \frac{(\rho)_{n}(\rho + \vartheta - \sigma)}{(\rho - \sigma)_{n}(\rho + \lambda + \vartheta)} (l + m)_{n} n! \]

interpret the above equation with the help of equation 33, we arrived at

\[ B = x^{\rho-\sigma-1} B(l, m) \frac{\Gamma(\rho) \Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} \times F_{3}^{(\delta, \zeta; \kappa, \mu)} (a, b, \rho, \rho + \vartheta - \sigma, l; 1 + m, \rho - \sigma, \rho + \lambda + \vartheta, c; x) \]

in the view of the equations 39, and 62 we have the required result.

**Theorem 4.2.** Let \( \lambda, \sigma, \vartheta, \rho, \delta, \zeta, \kappa, \mu, a, b, c, p \in \mathbb{C} \) be such that \( \Re(\lambda) > 0, \Re(\rho) < 1 + \min\{\Re(\sigma), \Re(\vartheta)\}; \min\{\Re(\delta), \Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0; \Re(c) > \Re(b) > 0; \Re(p) \geq 0, \)

\[ B \left\{ J_{x, \infty}^{\lambda, \sigma, \vartheta} F_{p}^{(\delta, \zeta; \kappa, \mu)} (a, b; c; z/t)(x : l, m) \right\} = x^{\rho-\sigma-1} B(l, m) \frac{\Gamma(\sigma - \rho + 1) \Gamma(\vartheta - \rho + 1)}{\Gamma(1 - \rho) \Gamma(\lambda + \sigma + \vartheta - \rho + 1)} F_{3}^{(\delta, \zeta; \kappa, \mu)} (a, b; c; 1/x) \]

\[ \times F_{3}^{(\delta, \zeta; \kappa, \mu)} (\sigma - \rho + 1, \vartheta - \rho + 1, l; 1 - \rho, \lambda + \sigma + \vartheta - \rho + 1, l + m; 1/x). \]

**Proof.** The proof of Theorem 4.2 is the same as that of Theorem 4.1.

**4.2. Laplace transform.** The Laplace transform of \( f(z) \) is defined as [56]:

\[ L\{f(z)\} = \int_{0}^{\infty} e^{-sz} f(z)dz. \]

**Theorem 4.3.** Let \( \lambda, \sigma, \vartheta, \rho, \delta, \zeta, \kappa, \mu, a, b, c, p \in \mathbb{C} \) be such that \( \Re(\lambda) > 0, \Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}; \min\{\Re(\delta), \Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0; \Re(c) > \Re(b) > 0; \Re(p) \geq 0, \)

\[ L \left\{ z^{l-1} J_{0, x}^{\lambda, \sigma, \vartheta} F_{p}^{(\delta, \zeta; \kappa, \mu)} (a, b; c; tz)(x) \right\} = x^{\rho-\sigma-1} \frac{\Gamma(l) \Gamma(\rho) \Gamma(\vartheta - \rho - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\lambda + \vartheta)} F_{p}^{(\delta, \zeta; \kappa, \mu)} (a, b; c; x) \]

\[ \times F_{3}^{(\delta, \zeta; \kappa, \mu)} (\rho, \rho + \vartheta - \sigma, l; \rho - \sigma, \rho + \lambda + \vartheta; x). \]

**Proof.** For convenience, we denote the left-hand side of the equation 66 by \( \mathcal{L} \). Then applying the Laplace, we have:

\[ \mathcal{L} = \int_{0}^{\infty} e^{-sz} z^{l-1} \left( J_{0, x}^{\lambda, \sigma, \vartheta} F_{p}^{(\delta, \zeta; \kappa, \mu)} (a, b; c; tz)(x) \right) dz, \]

further using the definition of function, defined in equation 33 and then changing the order of integration and summation, which is valid under the conditions of
Theorem 4.3, then
\[
\mathcal{L} = x^{\rho - \sigma - 1} \frac{\Gamma(\rho) \Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta, \zeta; \kappa, \mu)}(b + n, c - b)}{B(b, c - b)} \times \left( \frac{\rho_n}{\rho - (\rho + \lambda + \vartheta)} \right)_n \frac{x^n}{n!} \int_0^\infty e^{-s z} z^{n+l-1} dz
\]
\[
= x^{\rho - \sigma - 1} \frac{\Gamma(l)}{s!} \frac{\Gamma(\rho) \Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta, \zeta; \kappa, \mu)}(b + n, c - b)}{B(b, c - b)} \times \left( \frac{\rho_n}{\rho - (\rho + \lambda + \vartheta)} \right)_n \frac{x^n}{s! n!}
\]

further interpret in the view of equations 33 and 68 we have
\[
\mathcal{L} = x^{\rho - \sigma - 1} \frac{\Gamma(l)}{s!} \frac{\Gamma(\rho) \Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} \times 3 F_{p,2}^{(\delta, \zeta; \kappa, \mu)}(a, b, \rho, \rho + \vartheta - \sigma, l; \rho - \sigma, \rho + \lambda + \vartheta, c; x/s),
\]
in the view of Hadamard type 39, the equation 69 yields the required result.

Theorem 4.4. Let \( \lambda, \sigma, \vartheta, \rho, \delta, \zeta, \kappa, \mu, a, b, c, p, \in \mathbb{C} \) be such that \( \Re(\lambda) > 0, \Re(\rho) < 1 + \min\{\Re(\sigma), \Re(\vartheta)\}; \min\{\Re(\delta), \Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0; \Re(\sigma) > \Re(b) > 0; \Re(\rho) \geq 0, \)
then
\[
\mathcal{L} \left\{ z^{l-1} \left( J_{\lambda, \vartheta, \rho}^{(\delta, \zeta; \kappa, \mu)} \left( a, b, z/t \right) \right) (x) \right\}
\]
\[
= x^{\rho - \sigma - 1} \frac{\Gamma(l)}{s!} \frac{\Gamma(\sigma - \rho + 1)}{\Gamma(1 - \rho) \Gamma(\rho + 2 + \sigma - \vartheta + \rho)} \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta, \zeta; \kappa, \mu)}(b + n, c - b)}{B(b, c - b)} \times \left( \frac{\rho_n}{\rho - (\rho + \lambda + \vartheta)} \right)_n \frac{x^n}{s! n!} \int_0^\infty e^{-s z} z^{n+l-1} dz
\]

(70)

Proof. The proof is parallel to Theorem 4.3.

4.3. Whittaker transform.

Theorem 4.5. Let \( \lambda, \sigma, \vartheta, \rho, \delta, \zeta, \kappa, \mu, a, b, c, p, \in \mathbb{C} \) be such that \( \Re(\lambda) > 0, \Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}; \min\{\Re(\delta), \Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0; \Re(\sigma) > \Re(b) > 0; \Re(\rho) \geq 0; \)
\( \Re(\xi + \omega) > -1/2 \), then
\[
\int_0^\infty z^{\xi-1} e^{-s z/2} W_{\tau, \omega}(\eta z) \left\{ \left( J_{\lambda, \vartheta, \rho}^{(\delta, \zeta; \kappa, \mu)} \left( a, b; z/t \right) \right) \right\} dz
\]
\[
= x^{\rho - \sigma - 1} \frac{\Gamma(\rho) \Gamma(\rho + \vartheta - \sigma)}{\eta^{\xi-1} \Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta, \zeta; \kappa, \mu)}(b + n, c - b)}{B(b, c - b)} \times \left( \frac{\rho_n}{\rho - (\rho + \lambda + \vartheta)} \right)_n \frac{x^n}{s! n!} \int_0^\infty e^{-s z} z^{n+l-1} dz
\]

(71)

Proof. For convenience, we denote the left-hand side of the equation 71 by \( \mathcal{W} \). Using the result from equation 43, after changing the order of integration and summation, we get:

\[
\mathcal{W} = x^{\rho - \sigma - 1} \frac{\Gamma(\rho) \Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta, \zeta; \kappa, \mu)}(b + n, c - b)}{B(b, c - b)} \times \left( \frac{\rho_n}{\rho - (\rho + \lambda + \vartheta)} \right)_n \frac{x^n}{s! n!} \int_0^\infty e^{-s z} z^{n+l-1} dz
\]

(72)
by substituting $\eta z = \zeta$, equation 72 becomes:

$$
\mathcal{W} = x^{\rho-\sigma-1} \frac{\Gamma(\rho) \Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta,\xi,\kappa,\mu)}(b + n, c - b)}{n!} \frac{(\rho)_n}{\Gamma(\rho + \lambda + \vartheta)} \frac{1}{n!} \frac{x^n}{\eta^{n+\xi-1}} \int_{0}^{\infty} \zeta^{n+\xi-1} e^{-\zeta/2} W_{\tau,\omega}(t) d\zeta.
$$

(73)

Now we use the following integral formula involving Whittaker function,

$$
\int_{0}^{\infty} t^{\nu-1} e^{-t/2} W_{\tau,\omega}(t) dt = \frac{\Gamma(1/2 + \omega + \nu) \Gamma(1/2 - \omega + \nu)}{\Gamma(1/2 - \tau + \nu)}, \quad \left(\Re(\nu) > -\frac{1}{2}\right).
$$

(74)

After simplification, we have

$$
\mathcal{W} = x^{\rho-\sigma-1} \frac{\Gamma(\rho) \Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} \frac{\Gamma(1/2 + \omega + \xi) \Gamma(1/2 - \omega + \xi)}{\Gamma(1/2 - \tau + \xi)} \times \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta,\xi,\kappa,\mu)}(b + n, c - b)}{n!} \frac{(\rho)_n}{\Gamma(\rho + \lambda + \vartheta)} \frac{1}{n!} \frac{x^n}{\eta^{n+\xi-1}} \left(\frac{x}{\eta}\right)^n,
$$

(75)

further interpret in the view of function given in 33, equation 75 can be written as

$$
\mathcal{W} = x^{\rho-\sigma-1} \frac{\Gamma(\rho) \Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} \frac{\Gamma(1/2 + \omega + \xi) \Gamma(1/2 - \omega + \xi)}{\Gamma(1/2 - \tau + \xi)} \times {}_4F_p^{(\delta,\xi,\kappa,\mu)}(a, b, \rho, \rho + \vartheta - \sigma, 1/2 + \omega + \xi, 1/2 - \omega + \xi; \rho - \sigma, \rho + \lambda + \vartheta, 1/2 - \tau + \xi, c; x/\eta),
$$

(76)

Applying the definition of Hadamard given in 39, we get the desired result.

\[\square\]

**Theorem 4.6.** Let $\lambda, \sigma, \vartheta, \rho, \delta, \xi, \kappa, \mu, a, b, c, p \in \mathbb{C}$ be such that $\Re(\lambda) > 0$, $\Re(\rho) < 1 + \min\{\Re(\sigma), \Re(\vartheta)\}$; $\min\{\Re(\delta), \Re(\xi), \Re(\kappa), \Re(\mu)\} > 0$; $\Re(c) > \Re(b) > 0$; $\Re(\rho) \geq 0$; $\Re(\xi, \zeta, \omega) > -\frac{1}{2}$, then

$$
\int_{0}^{\infty} z^{\xi-1} e^{-z/2} W_{\tau,\omega}(\eta z) \left\{ \left( J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} F_p^{(\delta,\xi,\kappa,\mu)}(a, b; c; z/t) \right) (x) \right\} dz
$$

$$
= x^{\rho-\sigma-1} \frac{\Gamma(\rho) \Gamma(\rho + \vartheta - \sigma)}{\Gamma(\rho - \sigma) \Gamma(\rho + \lambda + \vartheta)} \frac{\Gamma(1/2 + \omega + \xi) \Gamma(1/2 - \omega + \xi)}{\Gamma(1/2 - \tau + \xi)} \times {}_3F_p^{(\delta,\xi,\kappa,\mu)}(a, b; c; 1/\eta x) \times {}_3F_5(\sigma - \rho + 1, \vartheta - \rho + 1, 1/2 + \omega + \xi, 1/2 - \omega + \xi; 1 - \rho, \lambda + \sigma + \vartheta - \rho + 1, 1/2 - \tau + \xi; 1/\eta x).
$$

(77)

\[\square\]

**5. Fractional differential equations.** The importance of fractional differential equations in the field of applied sciences gained more attention not only in mathematics but also in physics, dynamical systems, control systems and engineering, to create the mathematical model of physical phenomena. Especially, the kinetic equations describe the continuity of motion of substance. The extension and generalization of fractional kinetic equations involving many fractional operators were found in [11, 15, 16, 25, 26, 30, 34, 51, 52, 53, 54, 49, 63].
In view of the effectiveness and a great importance of the kinetic equation in certain astrophysical problems the authors develop a further generalized form of the fractional kinetic equation involving family of extended generalized Gauss hypergeometric functions.

The fractional differential equation between rate of change of the reaction, the destruction rate and the production rate was established by Haubold and Mathai [30] given as follows:

\[
\frac{dN}{dt} = -d(N_t) + p(N_t),
\]  

(78)

where \( N = N(t) \) the rate of reaction, \( d = d(N) \) the rate of destruction, \( p = p(N) \) the rate of production and \( N_t \) denotes the function defined by \( N_t(t^*) = N(t-t^*), t^* > 0 \).

Special case of equation 78 for spatial fluctuations and inhomogeneities in \( N(t) \) the quantities are neglected , that is the equation

\[
\frac{dN}{dt} = -c_i N_i(t),
\]  

(79)

with the initial condition that \( N_i(t = 0) = N_0 \) is the number density of the species \( i \) at time \( t = 0 \) and \( c_i > 0 \). If we remove the index \( i \) and integrate the standard kinetic equation 79, we have

\[
N(t) - N_0 = -c_0 D_t^{-1} N(t)
\]  

(80)

where \( D_t^{-1} \) is the special case of the Riemann-Liouville integral operator \( D_t^{-\nu} \) defined as

\[
D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s) ds, \quad (t > 0, R(\nu) > 0)
\]  

(81)

The fractional generalization of the standard kinetic equation 80 is given by Haubold and Mathai (see for more detail, [30]) as follows:

\[
N(t) - N_0 = -c^\nu_0 D_t^{-\nu} N(t)
\]  

(82)

and obtained the solution of 82 as follows:

\[
N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-c^\nu)^k}{\Gamma(\nu k + 1)} (ct)^{\nu k}
\]  

(83)

Further, Saxena and Kalla [54] considered the the following fractional kinetic equation:

\[
N(t) - N_0 f(t) = -c^\nu_0 D_t^{-\nu} N(t), \quad (R(\nu) > 0),
\]  

(84)

where \( N(t) \) denotes the number density of a given species at time \( t \), \( N_0 = N(0) \) is the number density of that species at time \( t = 0 \), \( c \) is a constant and \( f \in \mathcal{L}(0, \infty) \).

By applying the Laplace transform to 84 (see [34]),

\[
L \{ N(t); p \} = N_0 \frac{F(p)}{1 + c^\nu p^{-\nu}} = N_0 \left( \sum_{n=0}^{\infty} (-c^\nu)^n p^{-\nu n} \right) F(p),
\]  

(85)

\[
\left( n \in N_0, \left| \frac{c}{p} \right| < 1 \right)
\]

where the Laplace transform [58] is given by

\[
F(p) = L \{ N(t); p \} = \int_0^\infty e^{-pt} f(t) dt, \quad (R(p) > 0).
\]  

(86)
5.1. Solution of generalized fractional kinetic equations. In this section, the solution of the generalized fractional kinetic equations by considering family of extended generalized Gauss hypergeometric functions are obtained.

Remark 1. Solutions of the fractional kinetic equations in this section are obtained in terms of the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ (Mittag-Leffler [37]), which is defined as:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}.$$  \hfill (87)

Theorem 5.1. If $d > 0, \nu > 0; \delta, \xi, \kappa, \mu, a, b, c, p \in \mathbb{C}$ be such that $\Re(c) > \Re(b) > 0$; $\min\{\Re(\delta), \Re(\xi), \Re(\kappa), \Re(\mu)\} > 0$ and $\Re(p) \geq 0$, then the solution of the following fractional kinetic equation

$$N(t) - N_0 F_p^{(\delta,\xi,\kappa,\mu)}(a, b; c; d^\nu t^\nu) = -\delta^\nu_0 D^{-\nu}_t N(t)$$  \hfill (88)

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} \left( \sum_{n=0}^{\infty} \frac{B_p^{(\delta,\xi,\kappa,\mu)}(b + n, c - b)}{B(b, c - b)} \frac{\Gamma(\nu n + 1)(d^\nu t^\nu)^n}{n!} \right) E_{\nu,\nu n+1}(-\delta^\nu t^\nu).$$  \hfill (89)

Proof. Laplace transform of Riemann-Liouville fractional integral operator is given by (Erdelyi et al. [17], Srivastava and Saxena [62]):

$$L \{ D^{-\nu}_t f(t); p \} = p^{-\nu} F(p)$$  \hfill (90)

where $F(p)$ is defined in 86. Now applying the Laplace transform to the both sides of 88, we have:

$$L \{ N(t); p \} = N_0 L \left\{ F_p^{(\delta,\xi,\kappa,\mu)}(a, b; c; d^\nu t^\nu); p \right\} - \delta^\nu L \left\{ D^{-\nu}_t N(t); p \right\}$$  \hfill (91)

$$N(p) = N_0 \left( \int_0^{\infty} e^{-pt} \sum_{n=0}^{\infty} \frac{B_p^{(\delta,\xi,\kappa,\mu)}(b + n, c - b)}{B(b, c - b)} \frac{(d^\nu t^\nu)^n}{n!} dt \right) - \delta^\nu p^{-\nu} N(p)$$  \hfill (92)

$$N(p) + \delta^\nu p^{-\nu} N(p) = N_0 \sum_{n=0}^{\infty} \frac{B_p^{(\delta,\xi,\kappa,\mu)}(b + n, c - b)}{B(b, c - b)} \frac{(d^\nu t^\nu)^n}{n!} \int_0^{\infty} e^{-pt}(t^\nu)^n dt$$  \hfill (93)

$$= N_0 \sum_{n=0}^{\infty} \frac{B_p^{(\delta,\xi,\kappa,\mu)}(b + n, c - b)}{B(b, c - b)} \frac{(d^\nu t^\nu)^n}{n!} \Gamma(\nu n + 1) p^{\nu n+1}$$  \hfill (94)

$$N(p) = N_0 \sum_{n=0}^{\infty} \frac{B_p^{(\delta,\xi,\kappa,\mu)}(b + n, c - b)}{B(b, c - b)} \frac{(d^\nu t^\nu)^n}{n!} \Gamma(\nu n + 1) p^{\nu n+1} \times \sum_{r=0}^{\infty} \left\{ -\left( \frac{p}{\nu} \right)^{-\nu} \right\}^r$$  \hfill (95)

The inverse Laplace transform of 95 is given by

$$L^{-1} \{ p^{-\nu}; t \} = \frac{t^{\nu-1}}{\Gamma(\nu)} (R(\nu) > 0)$$  \hfill (96)
we have

\[ L^{-1} \{ N(p) \} = N_0 \sum_{n=0}^{\infty} \frac{B_p^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)} \frac{\Gamma(\nu n + 1)(\nu'^n)}{n!} \]

\[ \times L^{-1} \left\{ \sum_{r=0}^{\infty} (-1)^r \delta^{\nu r} p^{-(\nu(n+r)+1)} \right\} \tag{97} \]

\[ N(t) = N_0 \sum_{n=0}^{\infty} \frac{B_p^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)} \frac{\Gamma(\nu n + 1)(\nu'^n)}{n!} \]

\[ \times \left\{ \sum_{r=0}^{\infty} (-1)^r \delta^{\nu r} \frac{t^{\nu(n+r)}}{\Gamma(\nu(n+r)+1)} \right\} \tag{98} \]

\[ = N_0 \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)} \frac{\Gamma(\nu n + 1)(\nu'^n)}{n!} E_{\nu,n+1}^{-\delta^{\nu n}}. \tag{99} \]

This is the required solution of the equation \(88\).

\[ \square \]

**Theorem 5.2.** If \( d > 0, \nu > 0; \delta, \zeta, \kappa, \mu, a, b, c, p \in \mathbb{C} \) be such that \( \Re(\delta) > \Re(b) > 0; \min\{\Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0 \) and \( \Re(p) \geq 0 \), then the solution of the following fractional kinetic equation

\[ N(t) = N_0 F_p^{(\delta,\zeta;\kappa,\mu)}(a,b,c;\nu,t^{\nu}) = -d^{\nu} a D_1^{-\nu} N(t) \tag{100} \]

is given by

\[ N(t) = N_0 \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)} \frac{\Gamma(\nu n + 1)(\nu'^n)}{n!} E_{\nu,n+1}^{-\delta^{\nu n}}. \tag{101} \]

**Proof.** If we choose \( \delta = d \) in equation \(88\), we can easily obtained the required result.

\[ \square \]

**Theorem 5.3.** If \( d > 0, \nu > 0; \delta, \zeta, \kappa, \mu, a, b, c, p \in \mathbb{C} \) be such that \( \Re(\delta) > \Re(b) > 0; \min\{\Re(\zeta), \Re(\kappa), \Re(\mu)\} > 0 \) and \( \Re(p) \geq 0 \), then the solution of the following fractional kinetic equation

\[ N(t) = N_0 F_p^{(\delta,\zeta;\kappa,\mu)}(a,b,c;t) = -d^{\nu} a D_1^{-\nu} N(t) \tag{102} \]

is given by

\[ N(t) = N_0 \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta,\zeta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)} t^{\nu} E_{\nu,n+1}^{-\delta^{\nu n}}. \tag{103} \]

**Proof.** If we choose \( d = \nu = 1 \) in the middle term of \(101\), then we have the required result.

\[ \square \]
5.2. Special cases. If we choose $\kappa = \mu$ the Theorems 5.1, 5.2 and 5.3 reduced to the following form:

**Corollary 11.** If $d > 0, \nu > 0; \delta, \zeta, \kappa, a, b, c, p \in \mathbb{C}$ be such that $\Re(c) > \Re(b) > 0; \min\{\Re(\delta), \Re(\zeta), \Re(\kappa)\} > 0$ and $\Re(p) \geq 0$, then the solution of the following fractional kinetic equation

$$N(t) - N_0 F_p^{(\delta, \zeta, \kappa)}(a, b; c; d^\nu t^\nu) = -\delta^\nu a D_t^{-\nu} N(t)$$  (105)

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta, \zeta, \kappa)}(b+n, c-b) \Gamma(\nu n+1)(d^\nu t^\nu)^n}{B(b, c-b) n!} E_{\nu, \nu+n+1}(-\delta^\nu t^\nu).$$  (106)

**Corollary 12.** If $d > 0, \nu > 0; \delta, \zeta, \kappa, a, b, c, p \in \mathbb{C}$ be such that $\Re(c) > \Re(b) > 0; \min\{\Re(\delta), \Re(\zeta), \Re(\kappa)\} > 0$ and $\Re(p) \geq 0$, then the solution of the following fractional kinetic equation

$$N(t) - N_0 F_p^{(\delta, \zeta, \kappa)}(a, b; c; d^\nu t^\nu) = -d^\nu a D_t^{-\nu} N(t)$$  (107)

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta, \zeta, \kappa)}(b+n, c-b) \Gamma(\nu n+1)(d^\nu t^\nu)^n}{B(b, c-b) n!} E_{\nu, \nu+n+1}(-d^\nu t^\nu).$$  (108)

**Corollary 13.** If $d > 0, \nu > 0; \delta, \zeta, \kappa, a, b, c, p \in \mathbb{C}$ be such that $\Re(c) > \Re(b) > 0; \min\{\Re(\delta), \Re(\zeta), \Re(\kappa)\} > 0$ and $\Re(p) \geq 0$, then the solution of the following fractional kinetic equation

$$N(t) - N_0 F_p^{(\delta, \zeta, \kappa)}(a, b; c; d^\nu t^\nu) = -d^\nu a D_t^{-\nu} N(t)$$  (109)

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta, \zeta, \kappa)}(b+n, c-b) \Gamma(\nu n+1)(d^\nu t^\nu)^n}{B(b, c-b) n!} E_{\nu, \nu+n+1}(-d^\nu t^\nu).$$  (110)

If we choose $\kappa = \mu = 1$ the Theorems 5.1, 5.2 and 5.3 reduces to the following form:

**Corollary 14.** If $d > 0, \nu > 0; \delta, \zeta, \kappa, a, b, c, p \in \mathbb{C}$ be such that $\Re(c) > \Re(b) > 0; \min\{\Re(\delta), \Re(\zeta)\} > 0$ and $\Re(p) \geq 0$, then the solution of the following fractional kinetic equation

$$N(t) - N_0 F_p^{(\delta, \zeta)}(a, b; c; d^\nu t^\nu) = -\delta^\nu a D_t^{-\nu} N(t)$$  (111)

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta, \zeta)}(b+n, c-b) \Gamma(\nu n+1)(d^\nu t^\nu)^n}{B(b, c-b) n!} E_{\nu, \nu+n+1}(-\delta^\nu t^\nu).$$  (112)

**Corollary 15.** If $d > 0, \nu > 0; \delta, \zeta, \kappa, a, b, c, p \in \mathbb{C}$ be such that $\Re(c) > \Re(b) > 0; \min\{\Re(\delta), \Re(\zeta)\} > 0$ and $\Re(p) \geq 0$, then the solution of the following fractional kinetic equation

$$N(t) - N_0 F_p^{(\delta, \zeta)}(a, b; c; d^\nu t^\nu) = -d^\nu a D_t^{-\nu} N(t)$$  (113)

is given by

$$N(t) = N_0 \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\delta, \zeta)}(b+n, c-b) \Gamma(\nu n+1)(d^\nu t^\nu)^n}{B(b, c-b) n!} E_{\nu, \nu+n+1}(-d^\nu t^\nu).$$  (114)
Corollary 16. If $d > 0$, $\nu > 0; \delta, \zeta, a, b, c, p \in \mathbb{C}$ be such that $\Re(c) > \Re(b) > 0$; $\min\{\Re(\delta), \Re(\zeta)\} > 0$ and $\Re(p) \geq 0$, then the solution of the following fractional kinetic equation
\[ N(t) - N_0 F_p^{(\delta, \zeta)}(a, b; c; t) = -d_\nu^\nu D_t^{-\nu} N(t) \] (115)
is given by
\[ N(t) = N_0 \sum_{n=0}^{\infty} \frac{(a)_n B_p(b + n, c - b)}{B(b, c - b)} \frac{(\nu n + 1)(d_\nu^\nu)^n}{n!} t^n E_{\nu, n+1}(-d_\nu^\nu). \] (116)

For $\delta = \zeta; \kappa = \mu = 1$ the Theorems 5.1, 5.2 and 5.3 reduces to the following form:

Corollary 17. If $d > 0$, $\nu > 0; a, b, c, p \in \mathbb{C}$ be such that $\Re(c) > \Re(b) > 0$ and $\Re(p) \geq 0$, then the solution of the following fractional kinetic equation
\[ N(t) - N_0 F_p(a, b; c; d_\nu^\nu) = -d_\nu^\nu D_t^{-\nu} N(t) \] (117)
is given by
\[ N(t) = N_0 \sum_{n=0}^{\infty} \frac{B_p(b + n, c - b)}{B(b, c - b)} \frac{\Gamma(\nu n + 1)(d_\nu^\nu)^n}{n!} t^n E_{\nu, n+1}(-d_\nu^\nu). \] (118)

Corollary 18. If $d > 0$, $\nu > 0; a, b, c, p \in \mathbb{C}$ be such that $\Re(c) > \Re(b) > 0$ and $\Re(p) \geq 0$, then the solution of the following fractional kinetic equation
\[ N(t) - N_0 F_p(a, b; c; d_\nu^\nu) = -d_\nu^\nu D_t^{-\nu} N(t) \] (119)
is given by
\[ N(t) = N_0 \sum_{n=0}^{\infty} \frac{B_p(b + n, c - b)}{B(b, c - b)} \frac{\Gamma(\nu n + 1)(d_\nu^\nu)^n}{n!} t^n E_{\nu, n+1}(-d_\nu^\nu). \] (120)

Corollary 19. If $d > 0$, $\nu > 0; a, b, c, p \in \mathbb{C}$ be such that $\Re(c) > \Re(b) > 0$ and $\Re(p) \geq 0$, then the solution of the following fractional kinetic equation
\[ N(t) - N_0 F_p(a, b; c; t) = -d_\nu^\nu D_t^{-\nu} N(t) \] (121)
is given by
\[ N(t) = N_0 \sum_{n=0}^{\infty} \frac{(a)_n B_p(b + n, c - b)}{B(b, c - b)} \frac{\Gamma(\nu n + 1)(d_\nu^\nu)^n}{n!} t^n E_{\nu, n+1}(-d_\nu^\nu). \] (122)

Taking $p = 0; \kappa = \mu = 1$ the Theorems 5.1, 5.2 and 5.3 reduces to the following form:

Corollary 20. If $d > 0$, $\nu > 0; a, b, c \in \mathbb{C}$ be such that $\Re(c) > \Re(b) > 0$, then the solution of the following fractional kinetic equation
\[ N(t) - N_0 {}_2 F_1(a, b; c; d_\nu^\nu) = -d_\nu^\nu D_t^{-\nu} N(t) \] (123)
is given by
\[ N(t) = N_0 \sum_{n=0}^{\infty} \frac{(a)_n (b)_n \Gamma(\nu n + 1)(d_\nu^\nu)^n}{(c)_n n!} t^n E_{\nu, n+1}(-d_\nu^\nu). \] (124)
Corollary 21. If $d > 0, \nu > 0; a, b, c \in \mathbb{C}$ be such that $\Re(c) > \Re(b) > 0$, then the solution of the following fractional kinetic equation

$$N(t) - N_0 \; 2F_1(a, b; c; d^\nu t^\nu) = -d^\nu 0D^{-\nu}_t N(t) \tag{125}$$

is given by

$$N(t) = N_0 \; \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{\Gamma(\nu n + 1)(d^\nu t^\nu)^n}{n!} E_{\nu, \nu n + 1}(-d^\nu t^\nu). \tag{126}$$

Corollary 22. If $d > 0, \nu > 0; a, b, c \in \mathbb{C}$ be such that $\Re(c) > \Re(b) > 0$, then the solution of the following fractional kinetic equation

$$N(t) - N_0 \; 2F_1(a, b; c; t) = -d^\nu 0D^{-\nu}_t N(t) \tag{127}$$

is given by

$$N(t) = N_0 \; \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} n^n E_{\nu, \nu n + 1}(-d^\nu t^\nu). \tag{128}$$

6. Conclusion. We may also emphasize that results derived in this paper are of general character and can specialize to give further interesting and potentially useful formulas involving integral transform and fractional calculus. Also we give a new fractional generalization of the standard kinetic equation and derived solution for the same. From the close relationship of family of extended generalized Gauss hypergeometric functions with many special functions, we can easily construct various known and new fractional kinetic equations.

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