Uniform bounds for solutions to elliptic problems on simply connected planar domains

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Abstract

We consider the following elliptic problems on simply connected planar domains

\[
\begin{cases}
-\Delta u = \lambda |x|^{2\alpha} K(x)e^u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

and

\[
\begin{cases}
-\Delta u = |x|^{2\alpha} K(x)u^p & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

with \( \alpha > -1, \lambda > 0, p > 1, 0 < K(x) \in C^1(\Omega) \).

We show that any solution to each problem must satisfy a uniform bound on the mass, which is given respectively by \( \lambda \int_{\Omega} |x|^{2\alpha} K(x)e^u \, dx \) and \( p \int_{\Omega} |x|^{2\alpha} K(x)u^{p+1} \, dx \). The same results applies to some systems and more general non-linearities.

The proofs are based on the Riemann mapping theorem and a Pohožaev-type identity.

1 Introduction

We are interested in the following PDE, known as Liouville equation:

\[
\begin{cases}
-\Delta u = \lambda |x|^{2\alpha} K(x)e^u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

with \( \Omega \ni 0 \) being a smooth bounded planar domain, \( \alpha > -1, \lambda > 0 \) and \( 0 < K(x) \in C^1(\Omega) \).

Equation (1.1) has been very deeply studied in the last decades due to its applications in geometry and physics. It may be considered as a critical elliptic problem on planar domain, as the exponential nonlinearity is a natural counterpart of the Sobolev critical exponent in dimension greater or equal than 3.

Solutions to (1.1) can be found either variationally ([23, 24, 2, 11]) or by computing the Leray-Schauder degree ([12, 13]), and blowing-up families have also been constructed ([25, 19]). In all of these cases the geometry and topology of the domain \( \Omega \) play a fundamental role.

In this paper we give a mass bound for solutions to (1.1) when \( \Omega \) is simply connected, namely we show that any solution must satisfy a uniform bound on the \( L^1 \) norm of the laplacian \( \rho := \lambda \int_{\Omega} |x|^{2\alpha} K(x)e^u \). Such a quantity plays an important role especially in the variational formulation of the problem and sometimes, to stress its importance, it is used as a parameter in place of \( \lambda \), with the equation in (1.1) rewritten as \( -\Delta u = \rho \frac{|x|^{2\alpha} K(x)e^u}{\int_{\Omega} |x|^{2\alpha} K(x)e^u \, dx} \).

The following results extends a previous one on the unit disk ([3], Proposition 5.7).

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Theorem 1.1.
Let \( \Omega \) be a simply connected planar domain and \( u \) be a solution to (1.1).
Then, there exists \( \rho_0 > 0 \), not depending on \( u \), such that \( \lambda \int_{\Omega} |x|^{2\alpha} K(x) e^u dx \leq \rho_0 \).

We are also considering the Hénon-Lane-Emden equation:
\[
\begin{cases}
-\Delta u = |x|^{2\alpha} K(x) u^p & \text{in } \Omega \\
\quad u > 0 & \text{in } \Omega \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
with \( p > 1 \) and \( \Omega, \alpha, K(x) \) as before. The power-type nonlinearity in (1.1) is subcritical on planar domains for any \( p \), hence positive solutions can be easily found. Nonetheless, it is interesting to investigate the asymptotic behavior of solutions as the exponent \( p \) goes to \(+\infty\).

Despite the different structure, the latter problem shares surprising similarities with blow-up analysis for equation (1.1) (see [16, 15]), and in particular the problem heavily depends on the shape of \( \Omega \). In the regular case \( \alpha = 0 \) solutions to (1.2) have been found on multiply connected domains with arbitrarily large values of the mass, which in this case is given by \( p \int_{\Omega} |x|^{2\alpha} K(x) u^{p+1} dx \) (see [26]). On the other hand, when \( \Omega \) is convex ([30, 15]) or strictly star-shaped ([31]) different bounds on the mass have been given, which are equivalent to giving an upper bound on the number of blow-up points.

Here we fill the gap between the two results by showing that, in the same spirit as Theorem 1.1, similar bounds hold true for solutions to (1.2) on simply connected domains.

Theorem 1.2.
Let \( \Omega \) be a simply connected planar domain, \( p_0 > 1 \) and \( u \) be a solution to (1.2) with \( p \geq p_0 \).
Then, there exists \( \rho_0 > 0 \), not depending on \( u \) nor on \( p \), such that \( p \int_{\Omega} |x|^{2\alpha} K(x) u^{p+1} dx \leq \rho_0 \).

We will also provide results similar to Theorems 1.1, 1.2 to some Liouville systems, namely systems of PDEs with the same features as (1.1). Such problems have been increasingly studied in the last years, especially in the case when the matrix of coefficients \( A \) is a Cartan matrix of some Lie algebra.

To get a mass bound for solutions on simply connected domains, we need the matrix \( A \) to be positive definite, which in the case of Cartan matrices holds true. Such a result had already been proven when \( \Omega \) is the unit disk, in [8] (Theorem 1.3) for the SU(3) Toda system and in the author’s PhD thesis [3] for general systems. A similar estimate was proved also in [1] for general systems on strictly star-shaped domains.

Theorem 1.3.
Let \( \Omega \) be a simply connected planar domain, \( A = \{a_{ij}\}_{i,j=1}^N \) be a positive definite matrix and \( u = (u_1, \ldots, u_N) \) be a solution to
\[
\begin{cases}
-\Delta u_i = \sum_{j=1}^{N} a_{ij} \lambda_j |x|^{2\alpha_j} K_j(x) e^{u_j} & \text{in } \Omega \\
u_i = 0 & \text{on } \partial \Omega
\end{cases}
\]
with \( \Omega \ni 0, \alpha_i > -1, \lambda_i > 0, 0 < K_i(x) \in C^1(\Omega) \).

Then, there exists \( \rho_0 > 0 \), not depending on \( u \), such that \( \lambda_i \int_{\Omega} |x|^{2\alpha_i} K_i(x) e^{u_i} dx \leq \rho_0 \) for all \( i \)’s.

Finally, similar estimates also hold true for some more general critical linearities.
Roughly speaking, we need a positive potential \( W(x) \) to be not too singular at 0 and the non-linearity \( F'(u) \) to grow at least as fast as its anti-derivative \( F(u) \). This case includes most exponential functions, including \( F'(u) = \lambda u^{p-1} e^{u^p} \) with \( 1 < p < 2 \), which was studied in [21, 20, 22].
Theorem 1.4.
Let $\Omega$ be a simply connected planar domain and $u$ be a solution to
\[
\begin{cases}
-\Delta u = W(x)F'(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
with $W(x), F(u)$ satisfying
\[
\begin{aligned}
W(x) &\in C^1_{\text{loc}}(\Omega \setminus \{0\}) \\
0 &\leq W(x) \leq C|x|^{2\alpha} \\
0 &\leq F'(u) \leq Ce^{Cu^2} \\
F(u) &\leq C(1 + F'(u)).
\end{aligned}
\]
Then, there exists $\rho_0 > 0$, not depending on $u$, such that $\int_{\Omega} W(x)F'(u)dx \leq \rho_0$.

It is interesting to compare all these results with previous works concerning existence and non-existence of solutions. This is the content of the following remark.

Remark 1.5.

- If the domain $\Omega$ is not simply connected, then each of the problem we are considering can have solutions with arbitrarily high values of the mass. This was done in [19, 23, 24, 2, 11, 10] for problem (1.1), in [26, 27] for (1.2), in [7, 4, 6] for some systems of the type (1.3) and in [21, 20, 22] for some nonlinearities of the type (1.4).

- In [25] the authors prove that for any $M > 0$ there exist a simply connected dumbbell-shaped domain $\Omega_M$ and a solution to (1.1) on $\Omega_M$ with $\lambda \int_{\Omega_M} |x|^{2\alpha}K(x)e^u dx \geq M$; the same argument also works for problems (1.2), (1.4) (see [27] and [21], respectively). The results presented here complement the latter, since Theorem 1.1 implies that there cannot exists any $\Omega_M$ such that the property shown in [25] holds for any $M$.

- In Theorem 1.3 it is essential to assume the matrix $A$ to be positive definite. Otherwise, in [29, 33, 34, 9] the authors build solutions to (1.3) whose masses can be arbitrarily large also on simply connected domains.

- If one allows more than one singularity, namely replaces the singular term $|x|^{2\alpha}$ with $|x|^{2\alpha}\prod_{i=1}^N|x-x_i|^{2\alpha_i}$ for some $x_i \in \Omega \setminus \{0\}, \alpha_i > -1$, then uniform mass bounds do not seem to be true anymore. In fact, in this case Theorem 1.4 in [13] shows that the Leray-Schauder degree of (1.1) does not vanish for arbitrarily high values of the mass.

- Finally, assuming $K(x)$ to be positive is essential. In fact, in [32, 17, 18] the authors show existence of solutions to the Liouville equation (1.1) with sign-changing potential even in the case of simply connected domain; here, a crucial role seems to be played not by the topology of $\Omega$ but rather of the set $\{x \in \Omega : K(x) > 0\}$.

The main tools to prove Theorems 1.1, 1.2, 1.3, 1.4 will be the Riemann Mapping Theorem and a Pohožaev-type identity. We will recall these very well-known results in Section 2, as well as some other preliminary. Then, in Section 3 we will prove the main results of this paper.
2 Preliminaries

Let us recall some facts which will be used in the proof of the results of this paper. We start with a very classical and powerful tool, the Riemann Mapping Theorem. Such a result will allow to conformally deform the simply connected domain $\Omega$ into the unit disk; in such a way, the PDE defined on $\Omega$ is transformed into a new equation on the disk, different from the original but with similar features. We actually need a refined version of the theorem by Carathéodory, which ensures that the conformal factor appearing in the new PDE is not singular on the boundary of $\Omega$.

**Theorem 2.1** (Riemann Mapping Theorem, Carathéodory’s Theorem).

Let $\Omega \ni 0$ be a smooth simply connected planar domain and $D \subset \mathbb{R}^2$ be the unit disk. Then, there exists a conformal diffeomorphism $\Phi : \Omega \rightarrow D$, smooth up to $\partial \Omega$, such that $\Phi(0) = 0$.

Moreover, if $u$ solves
\[
\begin{aligned}
-\Delta u &= f(x, u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega ,
\end{aligned}
\]
then $v := u \circ \Phi^{-1}$ solves
\[
\begin{aligned}
-\Delta v &= \frac{1}{\det(D\Phi(y))} f(\Phi^{-1}(y), v) \quad \text{in } D \\
v &= 0 \quad \text{on } \partial D .
\end{aligned}
\]

Our proofs will also use the Pohožaev identity, an often-used instrument to show non-existence of solutions to elliptic PDEs. Such a result is usually stated for solutions having at least a $W^{2,2}$ regularity, which in general does not hold true for solutions to (1.1), (1.2), (1.3), (1.4) if $\alpha$ is negative. Anyway, in the proof of the theorems we will verify that we are still in position to apply the following result.

**Theorem 2.2** (Pohožaev Identity).

Let $u$ be a sufficiently regular solution to
\[
\begin{aligned}
-\Delta v &= \partial_v G(y, v) \quad \text{in } D \\
v &= 0 \quad \text{on } \partial D .
\end{aligned}
\]

Then, it satisfies
\[
\frac{1}{2} \int_{\partial D} (\nabla v \cdot \nu(y))^2 \, d\sigma(y) = 2 \int_D G(y, v) \, dy + \int_D (\nabla_y G(y, v) \cdot y) \, dy - \int_{\partial D} G(y, v) \, d\sigma(y).
\]

If $v = (v_1, \ldots, v_N)$ solves
\[
\begin{aligned}
-\Delta v_i &= \sum_{j=1}^N a_{ij} \partial_{v_j} G_j(y, v_j) \quad \text{in } D \\
v_i &= 0 \quad \text{on } \partial D 
\end{aligned}
\]
with $A = \{a_{ij}\}_{i,j=1,...,N}$ being a non-singular matrix, then $u$ satisfies
\[
\frac{1}{2} \sum_{i,j=1}^N a_{ij} \int_{\partial D} (\nabla v_i \cdot \nu(y)) (\nabla v_j \cdot \nu(y)) \, d\sigma(y) = \sum_{i=1}^N \left(2 \int_D G_i(y, v_i) \, dy + \int_D (\nabla_y G_i(y, v_i) \cdot y) \, dy - \int_{\partial D} G_i(y, v_i) \, d\sigma(y)\right).
\]

We finally need some *a priori* estimates for solutions to (1.2), which are essential to adapt the argument for (1.1).

The following result was originally stated in [31] for the case $K \equiv 1, \alpha = 0$ but the same argument, based on estimates from [14] and the celebrated moving plane technique from [28], seems to be working more generally.

**Theorem 2.3.**

Let $u$ be a solution to (1.2) with $p \geq p_0$.

Then, there exists $C_0 > 0$, not depending on $p$ nor $u$, such that $\sup_{\Omega} u = \|u\|_{L^\infty(\Omega)} \leq C_0$.  

4
3 Proofs

We are now in position to prove the results stated in the introduction. Since all the proofs are rather similar to each other, we will give more details for Theorem 1.1 but we will skip some for the other theorems.

Proof of Theorem 1.1.
Let \( u \) be a solution to (1.1) on \( \Omega \) and \( \Phi : \Omega \to \mathbb{D} \) be the Riemann mapping described in Theorem 2.1. Then, \( v := u \circ \Phi^{-1} \) will solve \[ \begin{cases} -\Delta v = \lambda \tilde{K}(y)|y|^{2\alpha}e^v & \text{in } \mathbb{D} \\ u = 0 & \text{on } \partial \mathbb{D} \end{cases} \] for some \( 0 < \tilde{K}(y) \in C^1(\mathbb{D}) \).

We want to apply to \( v \) Theorem 2.2 with \( G(y,v) = \lambda|y|^{2\alpha}\tilde{K}(y)e^v \). If \( \alpha > 0 \) this is immediate since standard regularity gives \( v \in C^{2,2\alpha}(\mathbb{D}) \), but in case \( \alpha < 0 \) we only have \( v \in W^{2,q}(\mathbb{D}) \) with \( q < \frac{1}{-\alpha} \), therefore we need an ad hoc argument. Pohožaev identity is based on applying the divergence theorem to \( (\nabla v \cdot y) \nabla v - \frac{|\nabla v|^2}{2} y \) and \( \lambda \tilde{K}(y)|y|^{2\alpha}e^v y \), so we need to check that both vector fields are in \( W^{1,1}(\mathbb{D}) \).

Concerning the former field, we have
\[
D \left( (\nabla v \cdot y) \nabla v - \frac{|\nabla v|^2}{2} y \right) = (\nabla v \cdot y)D^2v + \nabla v \otimes \nabla v + (D^2v, y) \otimes \nabla v - \frac{|\nabla v|^2}{2} y - (D^2v, \nabla v) \otimes y;
\]
since we already know that \( |\nabla v|^2 \in L^1(\mathbb{D}) \), we suffice to check that \( |D^2v||\nabla v|y| \in L^1(\mathbb{D}) \). We have \( D^2v \in L^1(\mathbb{D}) \) and moreover, since \( |\Delta v| \leq C|y|^{2\alpha} \), by the Green’s representation formula we deduce
\[
|\nabla v| \leq C \int_{\mathbb{D}} \frac{|\eta|^{2\alpha}}{|y - \eta|} d\sigma(\eta) \leq \frac{C}{|y|};
\]
hence \( |\nabla v|y| \in L^\infty(\mathbb{D}) \) and we are done.

The other vector field verifies
\[
D \left( \lambda \tilde{K}(y)|y|^{2\alpha}e^v y \right) = \lambda \tilde{K}(y)|y|^{2\alpha}e^v y + \lambda |y|^{2\alpha}e^v \left( \nabla \tilde{K}(y) \otimes y \right) + 2\alpha \lambda |y|^{2\alpha-2} \tilde{K}(y)e^v (y \otimes y),
\]
which is in \( L^1(\mathbb{D}) \) because each term can be estimated by constant times \( |y|^{2\alpha} \).

We are therefore in position to use Theorem 2.2, which gives:
\[
\frac{1}{2} \int_{\partial \mathbb{D}} (\nabla v \cdot \nu(y))^2 d\sigma(y) = 2\lambda \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y)e^v dy + \lambda \int_{\mathbb{D}} \left( 2\alpha |y|^{2\alpha} \tilde{K}(y) + |y|^{2\alpha} \left( \nabla \tilde{K}(y) \cdot y \right) \right) e^v dy - \lambda \int_{\partial \mathbb{D}} \tilde{K}(y)e^v d\sigma(y).
\]

On the left-hand side we can use Hölder’s inequality and integrate by parts:
\[
\frac{1}{2} \int_{\partial \mathbb{D}} (\nabla v \cdot \nu(y))^2 d\sigma(y) \geq 1 - \frac{1}{4\pi} \left( \int_{\partial \mathbb{D}} \nabla v \cdot \nu(y) d\sigma(y) \right)^2 = \frac{1}{4\pi} \left( \int_{\mathbb{D}} \Delta v dy \right)^2 = \frac{1}{4\pi} \lambda \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y)e^v dy ;
\]
on the right-hand side we exploit the positivity of \( \tilde{K}(y) \) and the boundedness of \( \nabla \tilde{K}(y) \):
\[
2\lambda \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y)e^v dy + \lambda \int_{\mathbb{D}} \left( 2\alpha |y|^{2\alpha} \tilde{K}(y) + |y|^{2\alpha} \left( \nabla \tilde{K}(y) \cdot y \right) \right) e^v dy - \lambda \int_{\partial \mathbb{D}} \tilde{K}(y)e^v d\sigma(y) \\
\leq 2(1 + \alpha) \lambda \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y)e^v dy + \lambda \int_{\mathbb{D}} |y|^{2\alpha} \left( \nabla \tilde{K}(y) \cdot y \right) e^v dy \\
\leq \left( 2(1 + \alpha) + \sup_{y \in \mathbb{D}} \frac{|\nabla \tilde{K}(y)|}{\inf_{y \in \mathbb{D}} \tilde{K}(y)} \right) \left( \lambda \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y)e^v dy \right).
\]
Putting the two estimates together we get
\[
\frac{1}{4\pi} \left( \lambda \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y)e^v dy \right)^2 \leq \frac{2\alpha}{4\pi} \left( \lambda \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y)e^v dy \right),
\]
hence we conclude
\[
\lambda \int_\Omega |x|^{2\alpha} K(x) e^{u} dx = \lambda \int_\mathbb{D} |y|^{2\alpha} \tilde{K}(y) e^{v} dy \leq \rho_0.
\]

\[\Box\]

**Proof of Theorem 1.2.**

As in the proof of Theorem 1.1, we take a solution to (1.2) and transform it, via the Riemann mapping, into a solution to \(-\Delta v = |y|^{2\alpha} \tilde{K}(y) y^p \) \(v > 0\) in \(\mathbb{D}\) and \(v = 0\) on \(\partial \mathbb{D}\), with some \(0 < \tilde{K}(y) \in C^1(\mathbb{D})\).

By the same argument as in Theorem 1.1, we are allowed to apply Theorem 2.2, this time with \(G(y, v) = |y|^{2\alpha} \tilde{K}(y) v^{p+1}\), which reads as

\[
\frac{1}{2} \int_{\partial \mathbb{D}} (\nabla v \cdot \nu(y))^2 d\sigma(y) = \frac{1}{2} \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y) v^{p+1} dy + \int_{\mathbb{D}} \left(2\alpha |y|^{2\alpha} \tilde{K}(y) + |y|^{2\alpha} \left(\nabla \tilde{K}(y) \cdot y\right)\right) \frac{v^{p+1}}{p+1} dy.
\]

On the left-hand side, we integrate by parts as before and then we apply Theorem 2.3:

\[
\frac{1}{2} \int_{\partial \mathbb{D}} (\nabla v \cdot \nu(y))^2 d\sigma(y) \geq \frac{1}{4\pi} \left(\int_{\Delta} \Delta v dy\right)^2 = \frac{1}{4\pi} \left(\int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y) v^p dy\right)^2 \geq \frac{1}{4\pi C_0^2 p^2} \left(p \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y) v^{p+1} dy\right)^2.
\]

On the right-hand side, we argue as before exploiting the properties of \(\tilde{K}(y)\):

\[
2 \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y) \frac{v^{p+1}}{p+1} dy + \int_{\mathbb{D}} \left(2\alpha |y|^{2\alpha} \tilde{K}(y) + |y|^{2\alpha} \left(\nabla \tilde{K}(y) \cdot y\right)\right) \frac{v^{p+1}}{p+1} dy \leq \left(2(1 + \alpha) + \sup_{y \in \mathbb{D}} \frac{\nabla \tilde{K}(y)}{\inf_{y \in \mathbb{D}} \tilde{K}(y)}\right) \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y) \frac{v^{p+1}}{p+1} dy.
\]

Therefore we get

\[
\frac{1}{4\pi C_0^2 p^2} \left(p \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y) v^{p+1} dy\right)^2 \leq \frac{\rho_0}{4\pi C_0^2 p^2} \left(p \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y) v^{p+1} dy\right),
\]

namely

\[
p \int_{\Omega} |x|^{2\alpha} K(x) u^{p+1} dx = p \int_{\mathbb{D}} |y|^{2\alpha} \tilde{K}(y) v^{p+1} dy \leq \rho_0.
\]

\[\Box\]

**Proof of Theorem 1.3.**

We apply the Riemann mapping to a solution \(u = (u_1, \ldots, u_N)\) to (1.3) and we get a solution \(v = (v_1, \ldots, v_N)\) to

\[
-\Delta v_i = \sum_{j=1}^N a_{ij} \lambda_j |y|^{2\alpha} \tilde{K}_j(y) e^{v_j} \quad \text{in } \mathbb{D}, \quad u_i = 0 \quad \text{on } \partial \mathbb{D}
\]

apply to the latter Pohožaev identity, which gets

\[
\frac{1}{2} \sum_{i,j=1}^N a_{ij} \int_{\partial \mathbb{D}} (\nabla v_i \cdot \nu(y))(\nabla v_j \cdot \nu(y)) d\sigma(y)
\]
\[ = \sum_{i=1}^{N} \left( 2\lambda_i \int_{\mathbb{D}} |y|^{2\alpha_i} \tilde{K}_i(y) e^{\nu_i} dy + \lambda_i \int_{\mathbb{D}} (2\alpha_i |y|^{2\alpha_i} \tilde{K}_i(y) + |y|^{2\alpha_i} \left( \nabla \tilde{K}_i(y) \cdot y \right)) e^{\nu_i} dy - \lambda \int_{\partial \mathbb{D}} \tilde{K}_i(y) e^{\nu_i} d\sigma(y) \right). \]

On the left-hand side we use the positivity of \( A^{-1} \), as well as previous arguments, and we get \[
\frac{1}{2} \sum_{i,j=1}^{N} a^{ij} \int_{\partial \mathbb{D}} (\nabla v_i \cdot \nu(y)) (\nabla v_j \cdot \nu(y)) d\sigma(y) \geq \frac{1}{4\pi} \sum_{i,j=1}^{N} a^{ij} \left( \int_{\partial \mathbb{D}} |\nabla v_i| \, d\sigma(y) \right) \left( \int_{\partial \mathbb{D}} |\nabla v_j| \, d\sigma(y) \right) = \frac{1}{4\pi} \sum_{i,j=1}^{N} a^{ij} \left( \lambda_i \int_{\mathbb{D}} |y|^{2\alpha_i} \tilde{K}_i(y) e^{\nu_i} dy \right) \left( \lambda_j \int_{\mathbb{D}} |y|^{2\alpha_j} \tilde{K}_j(y) e^{\nu_j} dy \right); \]

on the right-hand side we just use the same estimates as in Theorem 1.1 to each term and get \[
\sum_{i=1}^{N} \left( 2\lambda_i \int_{\mathbb{D}} |y|^{2\alpha_i} \tilde{K}_i(y) e^{\nu_i} dy + \lambda_i \int_{\mathbb{D}} (2\alpha_i |y|^{2\alpha_i} \tilde{K}_i(y) + |y|^{2\alpha_i} \left( \nabla \tilde{K}_i(y) \cdot y \right)) e^{\nu_i} dy - \lambda_i \int_{\partial \mathbb{D}} \tilde{K}_i(y) e^{\nu_i} d\sigma(y) \right) \leq C \sum_{i=1}^{N} \lambda_i \int_{\mathbb{D}} |y|^{2\alpha_i} \tilde{K}_i(y) e^{\nu_i} dy. \]

Finally, since \( A^{-1} \) is positive definite, the relation we just found \[
\frac{1}{4\pi} \sum_{i,j=1}^{N} a^{ij} \left( \lambda_i \int_{\mathbb{D}} |y|^{2\alpha_i} \tilde{K}_i(y) e^{\nu_i} dy \right) \left( \lambda_j \int_{\mathbb{D}} |y|^{2\alpha_j} \tilde{K}_j(y) e^{\nu_j} dy \right) \leq C \sum_{i=1}^{N} \lambda_i \int_{\mathbb{D}} |y|^{2\alpha_i} \tilde{K}_i(y) e^{\nu_i} dy \]
can only be satisfied if every integral belongs to a bounded region, namely \[
\lambda_i \int_{\Omega} |x|^{2\alpha_i} K_i(x) e^{\nu_i} dx = \lambda_i \int_{\mathbb{D}} |y|^{2\alpha_i} \tilde{K}_i(y) e^{\nu_i} dy \leq \rho_0 \quad i = 1, \ldots, N. \]

Proof of Theorem 1.4.
As before, we take a solution \( u \) to (1.4) and apply the Riemann mapping theorem, thus getting a solution to \( -\Delta v = \tilde{W}(y) F'(v) \) in \( \mathbb{D} \) on \( \partial \mathbb{D} \), with \( \tilde{W}(y) \) satisfying the same conditions as (1.5).

Because of the properties of \( \tilde{W}(y) \) and \( F(v) \), we have \( v \in W^{2,q}(\mathbb{D}) \) for some \( q > 1 \), \( |\Delta v| \leq C |y|^{2\alpha} \) and \( \left| \nabla \tilde{W}(y) \right| |y| \leq C |y|^{2\alpha} \), therefore, as we argued in the proof of Theorem 1.1, we can apply Theorem 2.2 to get:
\[ \frac{1}{2} \int_{\partial \mathbb{D}} (\nabla v \cdot \nu(y))^2 d\sigma(y) = 2 \int_{\mathbb{D}} \tilde{W}(y) F'(v) dy + \int_{\mathbb{D}} \left( \nabla \tilde{W}(y) \cdot y \right) F(v) dy - \int_{\partial \mathbb{D}} \tilde{W}(y) F'(v) d\sigma(y). \]

Arguing as before we obtain
\[ \frac{1}{4\pi} \left( \int_{\mathbb{D}} \tilde{W}(y) F'(v) dy \right)^2 \leq \frac{1}{2} \int_{\partial \mathbb{D}} (\nabla v \cdot \nu(y))^2 d\sigma(y). \]
\[
2 \int_D \tilde{W}(y) F(v) \, dy + \int_D \left( \nabla_y \tilde{W}(y) \cdot y \right) F(v) \, dy - \int_{\partial D} \tilde{W}(y) F(v) \, d\sigma(y) \\
\leq (2 + C) \int_D \tilde{W}(y) F(v) \, dy \\
\leq C(2 + C) \int_D \tilde{W}(y) F'(v) \, dy + C(2 + C) \int_D \tilde{W}(y) \, dy,
\]
which means the mass must be uniformly bounded.

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