Abstract

The hydrogen atom theory in developed in de Sitter spaces of constant and negative curvature on the base of the Klein–Fock–Gordon wave equation in static coordinates. After separation of the variables in both models, the problem reduces to the Heun type second order differential equation with four singular points. Qualitative examination shows that the energy spectrum for the hydrogen atom in de Sitter space should be quasi-stationary, and the hydrogen atom becomes principally unstable. We have calculated approximate expression for energy levels within quasi-classical method; also the probability of decay of the hydrogen atom has been estimated. Similar analysis shows that in anti de Sitter model, the hydrogen atom should be stable in the sense of the quantum mechanics, approximate formulas for energy level have been found also within semiclassical approach. Extension to the case of a spin 1/2 particle in both de Sitter models is given, at these there arise complicated differential equations with 8 singular points.

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1 Introduction

To de Sitter geometrical models are given steady attention in the context of developing quantum theory in a curved space-time – for instance, see in [1, 2]. In particular, the problem of description of the particles with different spins on these curved backgrounds has a long history – see [3]–[41]. In the present paper we will examine influence of de Sitter’s geometries on quantum-mechanical description of the ”hydrogen atom” on the base of Klein–Fock–Gordon (KFG) equation – see preliminary analysis of this system in [42]. Both cases, de Sitter $dS$ and anti de Sitter are considered. Also briefly we consider the model of the hydrogen atom on the base of the Dirac equation in $dS$ model.

In $dS$ space, qualitative analysis of expression for classical squared radial momentum $p_r^2(r)$ is performed. Equation $p_r^2 = 0$ reduces to 4-th order polynomial, location of its roots says that the
situation of three positive and one negative roots is possible, which corresponds to a particle moving in a potential well, and there exist the one domains forbidden for classical motion and then one domain permitted for classical motion. In other word, de Sitter geometry acts effectively so that the ”hydrogen atom” turns to be an unstable system in a quantum-mechanical sense. Corresponding radial quantum-mechanical equation reduces to general Heun type, with four singular points. Some preliminary study of possible solutions of this equation is performed. Approximate expressions for the roots of 4-th order polynomial have been found, with the help of which we have calculated approximate expressions for energy levels within WKB-method; probability of decay of the hydrogen atom has been estimated.

Similar treatment is given to the case of anti de Sitter model. Equation \[ p_r^2 = 0 \] reduces to 4-th order polynomial as well. The character of location of its roots says that the case of two positive and to negative roots is possible, which corresponds to a particle in a pure potential well without any tunneling effect. In other words, this means that in the AdS-model, the hydrogen atom is stable quantum-mechanical system. Corresponding radial quantum-mechanical equation reduces to the general Heun type [43], with four singular points. Some preliminary study of possible solutions of the corresponding equation is performed. Approximate expressions for the roots of 4-th order polynomial have been found, with the help of which we have calculated approximate expressions for energy levels within WKB-method.

When treating the hydrogen atom with the help of the Dirac equation, final differential radial differential equations arising after separating the variables turn out to be much more complicated than in the scalar theory – the task reduces to a differential equation with 8 singular points.\footnote{Note that the problem of Dirac equation with Coulomb potential in hyperbolic Lobachevsky and spherical Riemann models reduces to differential equations with 5 singular points – see in [44].} However qualitatively the physical situations should be similar to these arising in scalar models. In particular, the hydrogen atom in dS space should not be a stable quantum system, whereas in AdS space it is stable.

### 2 Separation of the variables in dS-space

First, let us establish the form of the electric field created by a point charge in de Sitter space model (we employ the known static metrics; note that in these coordinates any nonrelativistic Schrödinger-like equation does not exist at all, so only relativistic description based on either scalar KFG-equation or Dirac equation are possible)

\[
dS^2 = (1 - \frac{r^2}{\rho^2})dt^2 - (1 - \frac{r^2}{\rho^2})^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{1}
\]

The origin of the coordinate system coincides with the point of the point charge location. Maxwell equations provide us with the following solution

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} F^{\alpha \beta} = -4\pi J^\alpha, \quad A^\alpha = (\frac{e}{r}, 0, 0, 0).
\]

Now, consider the scalar equation

\[
[ (ih\nabla_\alpha + \frac{e}{c} A_\alpha)(ih\nabla^\alpha + \frac{e}{c} A^\alpha) - M^2 c^2 ] \Phi = 0
\]
in $dS$-space (1) with 4-potential (2). After separation of the variables with the help of the substitution $\Phi = e^{-i\epsilon t/\hbar}Y_{lm}(\theta\phi)f(r)$:

$$\frac{d}{dr} \frac{d}{dr} r^2 g^{11} \frac{df}{dr} + \left[ \frac{g^{00}(\epsilon + eA_0)^2 - M^2 c^4}{c^2\hbar^2} - \frac{l(l+1)}{r^2} \right] f = 0,$$

we arrive at the radial equation

$$\frac{d^2}{dr^2} f + \frac{2(1 - 2r^2/\rho^2)}{r(1 - r^2/\rho^2)} \frac{df}{dr} f + \left[ \frac{(\epsilon + e^2/r)^2}{c^2\hbar^2} - \frac{1}{(1 - r^2/\rho^2)^2} - \frac{M^2 c^2}{r^2} + \frac{l(l+1)}{1 - r^2/\rho^2} \right] f = 0.$$  \hspace{1cm} (3)

Corresponding Hamilton–Jakobi equation reads

$$g^{00} \left( \frac{\partial S}{\partial x^0} - \frac{e}{c} A_0 \right)^2 + g^{rr} \left( \frac{\partial S}{\partial r} \right)^2 + g^{\theta\theta} \left( \frac{\partial S}{\partial \theta} \right)^2 + g^{\phi\phi} \left( \frac{\partial S}{\partial \phi} \right)^2 - M^2 c^2 = 0.$$  \hspace{1cm} (4)

Because the trajectory should be flat, one can fix the coordinate $\theta = \pi/2$, the the action function is to be searched in the form

$$S(t, r, \phi) = -\epsilon t + L\phi + \int p(r)dr,$$

so eq. (4) gives an expression for the squared radial momentum

$$p^2_r = \frac{(\epsilon + e^2/r)^2}{c^2} - \frac{1}{(1 - r^2/\rho^2)^2} - \frac{(M^2 c^2 + L^2/2)}{1 - r^2/\rho^2}.$$  \hspace{1cm} (5)

Thus, the classical problem of a particle in the Coulomb field reduces to the integral $\int p_r dr$, where $p_r$ is given by (5), whereas the quantum-mechanical problem reduces to the differential equation (3).

### 3 Qualitative treatment

Let us study the above expression for the squared radial momentum. It is convenient to consider both the flat Minkowski and the curved de Sitter models.

In Minkowski space, $p^2_r$ for a free particle vanishes at two points:

$$p^2_r = \left( \frac{\epsilon^2}{c^2} - M^2 c^2 - \frac{L^2}{r^2} \right) \Rightarrow r_0 = \pm \sqrt{\frac{L^2}{c^2 m^2 c^4}}.$$  \hspace{1cm} (6)

Classical motion is possible only in the domain where $p^2_r > 0$, and at $\epsilon > mc^2$ such a domain does exist.

Consider in the similar manner a free particle in $dS$-space:

$$p^2_r = \frac{\epsilon^2}{c^2(1 - r^2/\rho^2)^2} - \frac{(M^2 c^2 + L^2/2)}{1 - r^2/\rho^2};$$

at the origin ($r \sim 0$) and at the horizon ($r \sim \rho$), the momentum behaves respectively as follows

$$p^2_r(r \sim 0) \sim -L^2/r^2, \quad p^2_r(r \sim \rho) \sim \frac{\epsilon^2}{c^2(1 - r^2/\rho^2)^2} \rightarrow \infty.$$
Find the points of vanishing momentum:

\[ r^4 + r^2 \rho^2 \frac{\epsilon^2 - M^2 c^4}{c^2} + \frac{L^2}{\rho^2} - L^2 \rho^2 \frac{\epsilon^2 - M^2 c^4}{c^2} = 0, \]

since the roots of the 4-th order equation are

\[ r_0 = \pm \rho \sqrt{-A \pm \sqrt{A^2 + \frac{L^2}{M^2 c^2 \rho^2}}}, \quad A = \frac{\epsilon^2 - M^2 c^4}{M^2 c^4} + \frac{L^2}{M^2 c^2 \rho^2}. \quad (7) \]

Two roots are conjugate, and two are real-valued, one is negative, another is positive. We conclude that the character of the classical motion of a free particle in de Sitter space is much the same as in the flat Minkowski space.

Now, let us turn to the particle in the Coulomb potential on the background of the flat Minkowski space

\[ p_r^2 = \left( \frac{\epsilon + e^2}{c^2} \right)^2 - \left( M^2 c^2 + \frac{L^2}{r^2} \right); \quad (8) \]

behavior of the momentum in singular points is given by

\[ p_r^2(r \sim 0) \sim -\frac{L^2 - e^4/c^2}{r^2}, \quad p_r^2(r \sim \infty) \sim \frac{\epsilon^2 - M^2 c^4}{c^2}. \]

The turning points of \( p_r \) are given by

\[ r_0 = \frac{c^2}{\epsilon^2 - m^2 c^4} \left[ \frac{\epsilon^2}{c^2} \pm \sqrt{\frac{\epsilon^4}{c^4} + \frac{L^2 - e^4/c^2}{c^2}} \right]. \quad (9) \]

When \( L^2 - e^4/c^2 > 0 \) and \( \epsilon < Mc^2 \), with an additional requirement that the term under the square root be positive

\[ \epsilon > Mc^2 \sqrt{1 - \frac{e^4/c^2}{L^2}}, \quad (10) \]

according to (9) we get two positive roots and two negative (nonphysical) ones; The graph of \( p_r^2 \) schematically looks as shown below

![Fig. 1. Coulomb problem in Minkowski space](image-url)
Let us compare this with the case of de Sitter space:

\[ p_r^2 = \left(\frac{\epsilon + e^2/r}{c^2}\right)^2 \frac{1}{(1 - \frac{r^2}{\rho^2})^2} - \left(M^2 c^2 + \frac{L^2}{r^2}\right) \frac{1}{1 - \frac{r^2}{\rho^2}} = \]

\[ = \frac{1}{r^2(1 - \frac{r^2}{\rho^2})^2} \left[ \frac{M^2 c^2}{\rho^2} r^4 + \frac{e^2}{c^2} - M^2 c^2 + \frac{L^2}{\rho^2} r^2 + \frac{2e^2 \epsilon}{r^2} + \left(\frac{e^4}{c^2} - L^2\right) \right] = \]

\[ = \frac{1}{r^2(1 - \frac{r^2}{\rho^2})^2} \frac{M^2 c^2}{\rho^2} (r - r_1)(r - r_2)(r - r_3)(r - r_4). \quad (11) \]

At the horizon \((r \to \rho)\) and at the origin \((r \to 0)\) it behaves as

\[ p_r^2(r \to \rho) \sim \frac{1}{c^2(1 - \frac{r^2}{\rho^2})^2} \Rightarrow +\infty, \quad p_r^2(r \to 0) \sim -\frac{L^2 - e^4/c^2}{r^2}; \quad (12) \]

schematically the graph of \(p_r^2\) looks as (location of the rots \(r_1, \ldots, r_4\) will be studied later in Section 3)

\[ \text{Fig. 2. Coulomb problem in } dS \text{ space} \]

This means that we face here with a nonstable quantum-mechanical system. We should expect similar behavior when considering the same problem in \(dS\) model but on the base of the Dirac equation.

Let us discuss in more detail behavior of \(p_r\) at the horizon \(r \to \rho\). To this end, let us find the radial velocity measured in the proper time

\[ p_r = Mc \frac{dr}{dS}, \quad dS = (1 - \frac{r^2}{\rho^2})c d\tau, \quad V_r^2 = \left(\frac{dr}{d\tau}\right)^2 = \frac{p_r^2}{M^2 c^2} (1 - \frac{r^2}{\rho^2})^2; \quad (13) \]

that is

\[ V_r = c \sqrt{\frac{(\epsilon + e^2/r)^2}{M^2 c^4} - (1 + \frac{L^2}{M^2 c^2 r^2}) (1 - \frac{r^2}{\rho^2})}. \quad (3.5b) \]

Correspondingly, at the horizon we have

\[ V_r = c \sqrt{\frac{(\epsilon + e^2/\rho)^2}{M^2 c^2}} \Rightarrow | \epsilon + \frac{e^2}{\rho} | < M^2 c^2; \quad (14) \]
we assume that \( V_r \) cannot be greater the speed of the light \( c \).

There exist another argumentation, more appropriate in the context of the quantum-mechanics. Indeed, because the coordinate \( r \), entering the wave equation, has no direct metrical sense, one can use instead of it any other. To find such a more convenient variable, let us require that the radial momentum in that new variable be finite at \( r \to \rho \). However, \( p_r = \frac{\partial S}{\partial r} \); therefore one should perform the following change of the radial variable

\[
p_r^2 \sim \frac{(\epsilon + e^2/\rho)^2}{c^2} \frac{1}{(1 - \frac{r^2}{\rho^2})^2}, \quad r \implies r^* \quad \frac{d}{dr} = \frac{1}{1 - \frac{r^2}{\rho^2}} \frac{d}{dr^*}.
\]  

This results in

\[
p_{r^*} = \frac{d}{dr^*}S = (1 - \frac{r^2}{\rho^2}) \frac{dS}{dr} = (1 - \frac{r^2}{\rho^2})p_r \implies p_{r^*} \to \frac{(\epsilon + e^2/\rho)^2}{c^2}.
\]

Relationship between \( r \) and \( r^* \) is given by

\[
\frac{dr^*}{1 - \frac{r^2}{\rho^2}} \implies r^* = \frac{\rho}{2} \ln \frac{1 + r/\rho}{1 - r/\rho}
\]  

Correspondingly, the graph of \( p_{r^*}^2 \) looks as

![Graph of p_{r^*}^2](image)

**Fig. 3. Coulomb problem in dS space, the coordinate \( r^* \)**

This picture is much more understandable from physical point of view as representing a non-stable quantum-mechanical hydrogen atom model in de Sitter space.

## 4 The turning points of the radial momentum

To work with the squared radial momentum

\[
p_r^2 = \frac{(\epsilon + e^2/r)^2}{c^2} \frac{1}{(1 - r^2/\rho^2)^2} - (M^2c^2 + \frac{L^2}{r^2}) \frac{1}{1 - r^2/\rho^2}.
\]  

it will be convenient to use the following dimensionless quantities

\[
\frac{p_r^2}{M^2c^2} \implies p^2, \quad \frac{r}{\rho} \implies r, \quad \frac{L^2}{M^2c^2\rho^2} \implies L^2, \quad \frac{\epsilon^2}{\rho M^2c^2} = q;
\]
so \( p^2(r) \) reads

\[
p^2(r) = (\epsilon + \frac{q}{r})^2 \frac{1}{(1 - r^2)^2} - (1 + \frac{L^2}{r^2}) \left( \frac{1}{1 - r^2} \right).
\]

(18)

Physical domain for the variable \( r \) is the interval \( r \in [0, +1) \). On the boundaries, \( p^2(r) \) behaves as

\[
r \rightarrow 0, \quad p^2(r) \sim -\frac{L^2 - q^2}{r^2}, \quad (L^2 - q^2) > 0;
\]

\[
r \rightarrow +1 - 0, \quad p^2(r) = +\frac{\epsilon + q}{(1 - r^2)^2}.
\]

(19)

Possible turning point coincide with the roots of the equation \( p^2 = 0 \), which are the roots of 4-th order polynomial

\[
r^4 + Ar^2 + Br + C = 0,
\]

\[
A = \epsilon^2 + L^2 - 1, \quad B = +2\epsilon q, \quad C = -(L^2 - q^2).
\]

(20)

A cubic resolvent of this 4-th order equation is

\[
z^3 + 2Az^2 + (A^2 - 4C)z - B^2 = 0.
\]

(4.6)

In new variable \( z = y - \frac{2A}{3} \), it take on a reduced form

\[
y^3 + Py + Q = 0,
\]

\[
P = -\left( \frac{A^2}{3} + 4C \right), \quad Q = -\frac{2}{27}A^3 + \frac{8}{3}AC - B^2.
\]

(21)

Solutions \( y_i \) of the cubic equation ar given by Cardano formulas:

\[
y = \alpha + \beta,
\]

where

\[
\alpha = \sqrt[3]{\frac{Q}{2} + \sqrt{\left( \frac{Q}{2} \right)^2 + \left( \frac{P}{3} \right)^3}}, \quad \beta = \sqrt[3]{-\frac{Q}{2} - \sqrt{\left( \frac{Q}{2} \right)^2 + \left( \frac{P}{3} \right)^3}};
\]

(22)

the cubic roots \( \alpha \) and \( \beta \) are to correlate, so that

\[
\alpha \beta = -\frac{P}{3}
\]

Instead of (22), one can employ slightly different representation for these solutions

\[
y_1 = \alpha + \beta, \quad y_{2,3} = -\frac{\alpha + \beta}{2} \pm i\frac{\alpha - \beta}{2} \sqrt{3}.
\]

(23)

From the roots \( y_i \) we can obtains the root \( z_i \) of eq. (21):

\[
z_1 = y_1 - \frac{2A}{3}, \quad z_2 = y_2 - \frac{2A}{3}, \quad z_3 = y_3 - \frac{2A}{3}.
\]

(24)
Four roots of eq. (20) can be found by Euler formulas:

\[
\begin{align*}
    r_1 &= \frac{1}{2}(\sqrt{z_0} + \sqrt{z_1} + \sqrt{z_2}) , \\
    r_2 &= \frac{1}{2}(\sqrt{z_0} - \sqrt{z_1} - \sqrt{z_2}) , \\
    r_3 &= \frac{1}{2}(-\sqrt{z_0} + \sqrt{z_1} - \sqrt{z_2}) , \\
    r_4 &= \frac{1}{2}(-\sqrt{z_0} - \sqrt{z_1} + \sqrt{z_2}) ,
\end{align*}
\]  

(25)

where the roots \(\sqrt{z_0}, \sqrt{z_1}, \sqrt{z_2}\) are to satisfy the following constrain

\[
\sqrt{z_0} \sqrt{z_1} \sqrt{z_2} = -Q .
\]

(26)

The explicit form of the roots (25) is hardly convenient in practical use. By this reason, let us examine qualitatively the possible location of these roots. Starting with the identity

\[
r^4 + Ar^2 + Br + C = \left[ (r - r_1)(r - r_2) \right] \left[ (r - r_3)(r - r_4) \right] =
\]

\[
= r^4 - (r_1 + r_2 + r_3 + r_4)r^3 + (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4)r^2 -
\]

\[
-(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)r + r_1r_2r_3r_4
\]

we derive

\[
\begin{align*}
    r_1 + r_2 + r_3 + r_4 &= 0 , \\
    A &= r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 , \\
    B &= -(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4) , \\
    C &= r_1r_2r_3r_4 ;
\end{align*}
\]

(27)

recall that

\[
A = \epsilon^2 + L^2 - 1 , \quad B = +2\epsilon q > 0 , \quad C = -(L^2 - q^2) < 0 .
\]

At ones, we conclude that three next variants,

\[
\begin{align*}
    (r_1, r_2, r_3, r_4) &\sim (-, -, +, +) , \\
    (r_1, r_2, r_3, r_4) &\sim (-, -, -, -) , \\
    (r_1, r_2, r_3, r_4) &\sim (+, +, +, +) ,
\end{align*}
\]

cannot be realized in virtue of (ref4.10b). Assuming two real-valued roots, and two conjugate ones, we get the variant

\[
(r_1, r_2, r_3, r_4) \sim (+, -, z, z^*) ,
\]

it has no physical sense. Relationship

\[
r_1r_2r_3r_4 = C < 0
\]

(28)

says that assuming all four roots be real, then only two variant are possible:

\[
(r_1, r_2, r_3, r_4) \sim (-, -, +, +) , \quad (r_1, r_2, r_3, r_4) \sim (+, +, +, -).
\]

(29)

The first possibility in (29) is not interesting physically. To the second one there corresponds the graph in Fig. 3.
5 Reducing the radial equation to the Heun type

In dimensionless quantities

\[ x = \frac{r}{\rho}, \quad \frac{\varepsilon \rho}{c \hbar} = E, \quad \frac{c^2}{c \hbar} = \alpha, \quad \frac{M^2 c^2 \rho^2}{\hbar^2} \Rightarrow M^2, \quad (30) \]

the radial equation takes the form

\[
\left[ \frac{d^2}{dx^2} + \frac{2(1 - 2x^2)}{x(1 - x^2)} \frac{d}{dx} + (E + \frac{\alpha}{x})^2 - \frac{1}{(1 - x^2)} (M^2 + \frac{l(l + 1)}{x^2}) \frac{1}{1 - x^2} \right] f = 0, \quad (31)
\]

or differently

\[
\frac{d^2 f}{dx^2} + \left[ \frac{2}{x} + \frac{1}{x - 1} + \frac{1}{x + 1} \right] \frac{df}{dx} + \left[ \frac{\alpha^2 - l(l + 1)}{x^2} + \frac{1}{4} \frac{(E + \alpha)^2}{(x - 1)^2} + \frac{1}{4} \frac{(E - \alpha)^2}{(x + 1)^2} + \right.
\]

\[
+ \frac{2E \alpha}{x} - \frac{1}{4} \frac{(E + 2\alpha)^2 - \alpha^2 - 2[M^2 + l(l + 1)]}{x - 1}
\]

\[
+ \left. \frac{1}{4} \frac{(E - 2\alpha)^2 - \alpha^2 - 2[M^2 + l(l + 1)]}{x + 1} \right] f = 0. \quad (32)
\]

Near singular points, solutions of eq. (32) behave in accordance with the relations below:

\[ x = 0, \quad \frac{d^2 f}{dx^2} + \frac{2}{x} \frac{df}{dx} + \frac{\alpha^2 - l(l + 1)}{x^2} f = 0, \]

\[ f = x^A, \quad A = -\frac{1}{2} \pm \sqrt{(l + 1/2)^2 - \alpha^2}; \]

\[ x = 1, \quad \frac{d^2 f}{dx^2} + \frac{1}{x - 1} \frac{df}{dx} + \frac{1}{4} \frac{(E + \alpha)^2}{(x - 1)^2} f = 0, \]

\[ f = (x - 1)^B, \quad B = \pm \frac{i}{2} (E + \alpha); \]

\[ x = -1, \quad \frac{d^2 f}{dx^2} + \frac{1}{x + 1} \frac{df}{dx} + \frac{1}{4} \frac{(E - \alpha)^2}{(x + 1)^2} f = 0, \]

\[ f = (x + 1)^C, \quad C = \pm \frac{i}{2} (E - \alpha). \quad (33)\]

Making the substitution \( f = x^A (1 - x)^B (1 + x)^C F \), from (32) we get

\[
\frac{d^2 f}{dx^2} + \left[ \frac{2A + 2}{x} + \frac{2B + 1}{x - 1} + \frac{2C + 1}{x + 1} \right] \frac{df}{dx} +
\]

\[
+ \left[ \frac{A(A + 1) + \alpha^2 - l(l + 1)}{x^2} + \frac{4B^2 + (E + \alpha)^2}{4(x - 1)^2} + \frac{4C^2 + (E - \alpha)^2}{4(x + 1)^2} + \right.
\]

\[
+ \frac{2A + 3B + 3C + 2(AB + AC + BC)}{(x - 1)(x + 1)} \frac{M^2 - E^2/2 - 3\alpha^2/2 + l(l + 1)}{x - 1} \frac{E^2/2 - 3\alpha^2/2 + l(l + 1)}{(x - 1)(x + 1)}, \]

\[ (x - 1)(x + 1) \]

\[ (x - 1)(x + 1) \]
\[- \frac{2(E \alpha - B + C - AB + AC)}{x(x - 1)(x + 1)} \] \( f = 0 \).  

(34)

At restrictions

\[ A = -\frac{1}{2} \pm \sqrt{(l + 1/2)^2 - \alpha^2}, \quad B = \pm \frac{i}{2} (E + \alpha), \quad C = \pm \frac{i}{2} (E - \alpha) \]

eq. (34) becomes simpler

\[
\frac{d^2 f}{dx^2} + \left[ \frac{2A + 2}{x} + \frac{2B + 1}{x - 1} + \frac{2C + 1}{x + 1} \right] \frac{df}{dx} + \\
\left[ \frac{2A + 3B + 3C + 2(AB + AC + BC) + M^2 - E^2/2 - 3\alpha^2/2 + l(l + 1)}{x(x - 1)(x + 1)} \right] f = 0.
\]

(35)

This equation can be identified with the general Heun equation for \( G(-1; q, \lambda, \beta, \gamma, \delta; z) \)

\[
\frac{d^2 G}{dx^2} + \left[ \frac{\gamma}{x} + \frac{\delta}{x - 1} + \frac{\epsilon}{x + 1} \right] \frac{dG}{dx} + \frac{\lambda \beta x - q}{x(x - 1)(x + 1)} G = 0;
\]

(36)

relations determining its parameters are

\[ \epsilon = \lambda + \beta - \gamma - \delta + 1, \quad \gamma = 2(A + 1), \quad \delta = 2B + 1, \]

\[ q = 2(E \alpha - B + C - AB + AC), \]

\[ \lambda + \beta = 3 + 2(A + B + C), \]

\[ \lambda \beta = 2A + 3B + 3C + 2(AB + AC + BC) + M^2 - \frac{E^2}{2} - \frac{3\alpha^2}{2} + l(l + 1). \]

From two last relations we get

\[ \lambda = \frac{3}{2} + A + B + C + \sqrt{\frac{9}{4} + A^2 + B^2 + C^2 + A + \frac{3\alpha^2}{2} + \frac{E^2}{2} - M^2 - l(l + 1)}, \]

\[ \beta = \frac{3}{2} + A + B + C - \sqrt{\frac{9}{4} + A^2 + B^2 + C^2 + A + \frac{3\alpha^2}{2} + \frac{E^2}{2} - M^2 - l(l + 1)}; \]

the Heun function is symmetrical in \( \alpha, \beta \). Next, allowing for the identity

\[ A^2 + A + \alpha^2 - l(l + 1) = 0, \]

one can simplify expressions for \( \alpha \) and \( \beta \):

\[ \lambda = \frac{3}{2} + A + B + C + \sqrt{B^2 + C^2 + \frac{E^2 + \alpha^2}{2} - M^2 + \frac{9}{4}}, \]

\[ \beta = \frac{3}{2} + A + B + C - \sqrt{B^2 + C^2 + \frac{E^2 + \alpha^2}{2} - M^2 + \frac{9}{4}}. \]

(37)
To have solutions vanishing in the origin \( r = 0 \), one takes

\[
A = -\frac{1}{2} + \sqrt{(l + 1/2)^2 - \alpha^2}.
\] (38)

The singular point \( x = +1 \) is physical, because it represents the de Sitter horizon. The point \( x = -1 \) does not belong to physical domain – the interval \( x \in [0, +1) \).

Depending on the choice signs in expressions for \( B, C \) we have four possibilities:

\[
\begin{align*}
(+) & \quad B = +\frac{i}{2} (E + \alpha), \quad C = +\frac{i}{2} (E - \alpha), \quad B + C = +iE, \quad B^2 + C^2 = -\frac{E^2 + \alpha^2}{2}; \\
(-, -) & \quad B = -\frac{i}{2} (E + \alpha), \quad C = -\frac{i}{2} (E - \alpha), \quad B + C = -iE, \quad B^2 + C^2 = -\frac{E^2 + \alpha^2}{2}; \\
(+) & \quad B = +\frac{i}{2} (E + \alpha), \quad C = -\frac{i}{2} (E - \alpha), \quad B + C = +i\alpha, \quad B^2 + C^2 = -\frac{E^2 + \alpha^2}{2}; \\
(-, +) & \quad B = -\frac{i}{2} (E + \alpha), \quad C = +\frac{i}{2} (E - \alpha), \quad B + C = -i\alpha, \quad B^2 + C^2 = -\frac{E^2 + \alpha^2}{2}.
\end{align*}
\] (39)

Note that for all four cases we have one the same identity

\[
B^2 + C^2 + \frac{E^2 + \alpha^2}{2} = -\frac{E^2 + \alpha^2}{2} + \frac{E^2 + \alpha^2}{2} = 0;
\]

therefore expressions (37) for \( \alpha, \beta \) becomes yet simpler

\[
\lambda = \frac{3}{2} + A + B + C + \sqrt{-M^2 + \frac{9}{4}}, \quad \beta = \frac{3}{2} + A + B + C - \sqrt{-M^2 + \frac{9}{4}}.
\] (40)

Taking into account identities

\[
\begin{align*}
(+) & \quad C - B = -i\alpha, \quad (-, -) \quad C - B = +i\alpha, \\
(+) & \quad C - B = -iE, \quad (-, +) \quad C - B = +iE,
\end{align*}
\]

we readily find expressions for parameters \( q, \alpha, \beta \) for each type of solutions:

\[
\begin{align*}
(+) & \quad q = 2\alpha \left[ E - i(A + 1) \right], \quad (-, -) \quad q = 2\alpha \left[ E + i(A + 1) \right], \\
(+) & \quad q = 2E \left[ \alpha - i(A + 1) \right], \quad (-, +) \quad q = 2E \left[ \alpha + i(A + 1) \right],
\end{align*}
\] (41)

\[
\begin{align*}
(+) & \quad \lambda, \beta = \frac{1}{2} + A + 1 + iE \pm i \sqrt{M^2 - \frac{9}{4}}, \\
(-, -) & \quad \lambda, \beta = \frac{1}{2} + A + 1 - iE \pm i \sqrt{M^2 - \frac{9}{4}}, \\
(+) & \quad \lambda, \beta = \frac{1}{2} + A + 1 + i\alpha \pm i \sqrt{M^2 - \frac{9}{4}}, \\
(-, +) & \quad \lambda, \beta = \frac{1}{2} + A + 1 - i\alpha \pm i \sqrt{M^2 - \frac{9}{4}}.
\end{align*}
\] (42)
Note that by physical reasons we can assume inequality $M^2 \to (M^2 \rho^2 c^2 / \hbar^2) \gg 1$.

In conclusion of this section let us discuss one speculative possibility to get complex values for energy. Indeed, let impose the following quantization rule: $\lambda = -n$, which results complex values for energy:

$$(+, +) \quad E = +i \left( \frac{3}{2} n + A \right) - \sqrt{M^2 - \frac{9}{4}};$$

$$(-, -) \quad E = -i \left( \frac{3}{2} n + A \right) + \sqrt{M^2 - \frac{9}{4}}. \quad (43)$$

In turn, further these formulas lead to

$$(+, +) \quad e^{-ict/\hbar} = e^{i(ct/\rho) + (3/2 + n + A) + i\sqrt{M^2 - 9/4}}$$

$$(-, -) \quad e^{-ict/\hbar} = e^{i(ct/\rho) - (3/2 + n + A) - i\sqrt{M^2 - 9/4}}. \quad (45)$$

It seems that more physical is the variant $(-, -)$. The question on physical meaning of the produced formulas related to complex values for energy remains open.

### 6 Approximate roots $r_1, \ldots, r_4$ of equation $p_t^2 = 0$

Let us find approximate expressions for the roots $r_1, \ldots, r_4$ of equation

$$\frac{M^2 c^2 \rho^2}{\rho^2} r^4 + \left(\frac{\epsilon^2}{c^2} - M^2 c^2 + \frac{L^2}{\rho^2}\right) r^2 + \frac{2 \epsilon^2 c^2}{c^2} r + \left(\frac{\epsilon^4}{c^2} - L^2\right) = 0. \quad (47)$$

We will use expansions in small parameter $\rho^{-1}$:

$$r_{1,2} = a_0 + \frac{a_1}{\rho} + \frac{a_2}{\rho^2} + \ldots, \quad r_{3,4} = \rho \left( b_0 + \frac{b_1}{\rho} + \frac{b_2}{\rho^2} + \ldots \right); \quad (48)$$

First, we study $r_{1,2} \sim (a_0 + a_1 \rho^{-1})$, then eq. (47) gives

$$\frac{M^2 c^2}{\rho^2} \left[ a_0^4 + 4a_0^3 \frac{a_1}{\rho} + 6a_0^2 \frac{a_1^2}{\rho^2} + 4a_0 \frac{a_1^3}{\rho^3} + \frac{a_1^4}{\rho^4} \right] +$$

$$+ \left(\frac{\epsilon^2}{c^2} - M^2 c^2 + \frac{L^2}{\rho^2}\right) a_0^4 + 2a_0 a_1 \frac{a_1}{\rho^2} + \frac{a_1^2}{\rho^4} a_2^2 + 2 \frac{\epsilon^2 c^2}{c^2} a_0 + \frac{a_1}{\rho} + \left(\frac{\epsilon^4}{c^2} - L^2\right) = 0.$$

We are interested in first two terms of the order $\rho^0$ and $\rho^{-1}$:

$$\rho^0 : \quad \left(\frac{\epsilon^2}{c^2} - M^2 c^2\right) a_0^2 + \frac{2 \epsilon^2 c^2}{c^2} a_0 + \left(\frac{\epsilon^4}{c^2} - L^2\right) = 0,$$

$$\rho^{-1} : \quad 2a_0 a_1 \left(\frac{\epsilon^2}{c^2} - M^2 c^2\right) + \frac{2 \epsilon^2 c^2}{c^2} a_1 = 0 \implies a_1 = 0. \quad (49)$$

As $a_1 = 0$, let us find next term $a_2$. With notation

$$r_i = r_{0i} + \frac{\Delta_i}{\rho^2}, \quad i = 1, 2,$$
we get

\[ \Delta_i = -r_{0i}^2 (L^2 + M^2 e^2 r_{0i}^2) \left[ \frac{2\epsilon^2 c^2}{c^2} + 2r_{0i} \left( \frac{\epsilon^2}{c^2} - M^2 c^2 \right) \right]^{-1} ; \]  

(50)

let us write down expressions for \( r_{0i} \) as well:

\[ r_{01} = \frac{c^2}{m^2 c^4 - e^2} \left[ \frac{e^2 \epsilon}{c^2} + m \sqrt{\frac{e^4 c^2}{c^4} - \left( \frac{L^2}{c^2} - \frac{e^2}{c^2} \right) M^2 c^2 - \epsilon^2} \right] > 0 , \]

\[ r_{02} = \frac{c^2}{m^2 c^4 - e^2} \left[ \frac{e^2 \epsilon}{c^2} - \sqrt{\frac{e^4 c^2}{c^4} - \left( \frac{L^2}{c^2} - \frac{e^2}{c^2} \right) M^2 c^2 - \epsilon^2} \right] > 0 . \]  

(51)

Starting from expansions

\[ r_{3,4} = \rho \left( b_0 + \frac{b_1}{\rho} \right) , \]

we get

\[ \rho^2 M^2 c^2 \left( b_0^4 + 4b_0^3 b_1 \frac{b_1}{\rho} + 6b_0^2 b_1^2 \frac{b_1}{\rho^2} + 4b_0 b_1^3 \frac{b_1}{\rho^3} + \frac{b_1^4}{\rho^4} + \right) + \]

\[ + \rho^2 \left( \frac{e^2 c^2}{c^2} - M^2 c^2 + \frac{L^2}{\rho^2} \right) \left( b_0^4 + 2b_0 \frac{b_1}{\rho} + \frac{b_1^2}{\rho^2} \right) + \frac{2e^2 \epsilon}{c^2} \rho \left( b_0 + \frac{b_1}{\rho} \right) + \left( \frac{e^4}{c^2} - L^2 \right) = 0 ; \]

from this it follows

\[ M^2 c^4 b_0^4 + (e^2 - M^2 c^4) b_0^2 = 0 , \]

\[ 4b_0^3 b_1 M^2 c^4 + (e^2 - M^2 c^4) 2b_0 b_1 + 2 e^2 \epsilon b_1 = 0 ; \]  

(52)

that is

\[ r_3 = \rho (b_{03} + \frac{b_1}{\rho}) , \quad r_4 = \rho (b_{04} + \frac{b_1}{\rho}) , \]

\[ b_{03} = + \sqrt{1 - \frac{\epsilon^2}{m^2 c^4}} > 0 , \quad b_{04} = - \sqrt{1 - \frac{\epsilon^2}{m^2 c^4}} < 0 , \]

\[ b_1 = - \frac{\epsilon^2 \epsilon}{M^2 c^4 - \epsilon^2} \]  

(53)

It should be noted that (at the large curvature radius \( \rho \)) the domain between \( r_1 \) and \( r_2 \) lays far from \( r_3 > 0 \) and \( r_4 < 0 \).

7 WKB-analysis

Let us study the problem within WKB-method, starting from

\[ p_r^2 = \frac{1}{r^2(1 - r^2)^2} \frac{M^2 c^2}{\rho^2} (r - r_1)(r - r_2)(r - r_3)(r - r_4) . \]
Allowing for (53) we obtain approximate form for \( p_r \) in the domain \( r << \rho \) where the potential well is located:

\[
P_r \sim mc \sqrt{(r-r_1)(r-r_2)} \left( \frac{\epsilon^2}{m^2c^4} - 1 \right)^{1/2} \left( 1 + \frac{A \rho^2}{\rho^2} \right),
\]

\( A = 1 - \frac{1}{2(1 - \epsilon^2/m^2c^4)} \).

WKB quantization rule is

\[
\int L_P \, dr = 2\pi \hbar (n + 1/2).
\]

From this, taking \( P_r \) according to (54) and choosing the path of integration surrounding the turning points \( r_1 \) and \( r_2 \), we reduce the problem to calculating residues in two points

\[
\sum_{0, \infty} \text{res} \ P_r = i\hbar (n + 1/2).
\]

Taking into account such residues in 0 and \( r = \infty \):

\[
\text{res}_{r=0} \ P_r = \sqrt{\frac{\epsilon^2}{c^2} - m^2c^2} r_1 r_2.
\]

\[
\text{res}_{r=\infty} = (y = \frac{1}{r}) = -\sqrt{\frac{\epsilon^2}{c^2} - m^2c^2} \text{res}_{y=0} \frac{1}{2} \sqrt{(1 - yr_1)(1 - yr_2)} \left( 1 + \frac{A}{\rho^2} \right) y = \frac{1}{2} \sqrt{\frac{\epsilon^2}{c^2} - m^2c^2} \left[ (r_1 + r_2) - \frac{A}{\rho^2} r_1 r_2 (r_1 + r_2) \right].
\]

we arrive at the formula

\[
\sqrt{\frac{\epsilon^2}{c^2} - m^2c^2} \left[ (\sqrt{r_1 r_2} + \frac{r_1 + r_2}{2}) - \frac{A}{\rho^2} r_1 r_2 \frac{r_1 + r_2}{2} \right] = i\hbar (n + 1/2).
\]

Allowing for relations

\[
r_1 = r_{01} + \frac{\Delta_1}{\rho^2}, \quad r_2 = r_{02} + \frac{\Delta_2}{\rho^2},
\]

\[
r_{01} + r_{02} = \frac{2\epsilon^2}{m^2c^4 - \epsilon^2}, \quad r_{01} r_{02} = (L^2 - \frac{\epsilon^4}{c^2}) \frac{\epsilon^2}{m^2c^4 - \epsilon^2}, \quad \Delta_1 = -\frac{r_{01}^2 (L^2 + m^2c^2 r_{01}^2)}{2\epsilon^2 c^2 + 2r_{01} (\epsilon^2 c^2 - m^2c^2)}, \quad \Delta_2 = -\frac{r_{02}^2 (L^2 + m^2c^2 r_{02}^2)}{2\epsilon^2 c^2 + 2r_{02} (\epsilon^2 c^2 - m^2c^2)},
\]

we produce

\[
\sqrt{r_1 r_2} + \frac{r_1 r_2}{2} = \sqrt{r_{01} r_{02}} + \frac{r_{01} r_{02}}{2} + \frac{1}{\rho^2} \left( \frac{\Delta_1 + \Delta_2}{2} + \frac{1}{2} \Delta_1 \sqrt{\frac{r_{01}}{r_{01}}} + \frac{1}{2} \Delta_2 \sqrt{\frac{r_{02}}{r_{02}}} \right).
\]
Correspondingly, eq. (57) takes the form
\[
\sqrt{\frac{e^2}{c^2} - m^2c^2} \left[ \sqrt{\frac{r_{01}r_{02}}{r_{01}}} - \frac{r_{01}r_{02}}{2ρ^2} + \frac{1}{2ρ^2} \left( \frac{Δ_1 + Δ_2}{2} + r_{01}r_{02}(r_{01} + r_{02}) \right) \right] = \hbar(n + 1/2) .
\] (58)

Resolving energy \( ǫ \) in a series of \( ρ^{-2} \):
\[
Δ = \left( \frac{m^2c^4 - ϵ_0^2}{4e^2m^2c^4} \right)^2 \left[ Δ_1 + Δ_2 + Δ_1 \sqrt{\frac{r_{02}}{r_{01}}} + Δ_2 \sqrt{\frac{r_{01}}{r_{02}}} - Ar_{01}r_{02}(r_{01} + r_{02}) \right] .
\] (59)

From this, it follows an approximate formula for energy levels in de Sitter space
\[
ε = ε_0 + \frac{Δ}{ρ^2} , \quad ε_0 = mc^2 \left[ 1 + \frac{e^4/c^2}{[h(n + 1/2) + √h^2(l + 1/2)^2 - e^4/c^2])^2} \right]^{-1/2} ,
\] (60)

where
\[
Δ = \left( \frac{m^2c^4 - ϵ_0^2}{4e^2m^2c^4} \right)^2 \left[ Δ_1 + Δ_2 + Δ_1 \sqrt{\frac{r_{02}}{r_{01}}} + Δ_2 \sqrt{\frac{r_{01}}{r_{02}}} - Ar_{01}r_{02}(r_{01} + r_{02}) \right] .
\]

In accordance with behavior of the function \( P^2 \), the particle can go from the domain between \( r_1 \) \( r_2 \) to the domain near the horizon \( r \sim ρ \) due to the quantum-mechanical tunneling effect. Probability of this process is given by the formula
\[
W = \exp \left( -\frac{2}{h} ∫_{r_2}^{r_3} √{-P^2_r} \, dr \right) ,
\] (61)

from whence we can obtain a rough estimation \( W \sim e^{-2ρ/λ} \), where \( λ \) stands for the Compton wave length.

8 The hydrogen atom in AdS space and the Heun functions

Now, let us consider the Coulomb problem in AdS space
\[
dS^2 = (1 + r^2/ρ^2)dt^2 - (1 + r^2/ρ^2)^{-1}dr^2 - r^2(dθ^2 + sin^2θdφ^2)
\] (62)
In scalar wave equation with the potential \( A_0 = (\vec{r}, 0, 0, 0) \), the variables are separated by substitution \( \Phi = e^{-i\epsilon t/\hbar} Y_{lm}(\theta \phi) f(r) \), and the radial equation takes the form

\[
\frac{d^2 f}{dr^2} + \frac{2(1 + 2r^2/\rho^2)}{r(1 + r^2/\rho^2)} \frac{df}{dr} + \left[ \frac{(\epsilon + e^2/r)^2}{c^2 \hbar^2} - \frac{1}{(1 + r^2/\rho^2)^2} - \frac{M^2 c^2}{\hbar^2} - \frac{l(l+1)}{r^2} \right] \frac{1}{1 + r^2/\rho^2} f = 0. \tag{63}
\]

In dimensionless form

\[
x = \frac{ir}{\rho}, \quad \frac{\epsilon \rho}{c \hbar} = E, \quad \frac{e^2}{c \hbar} = \alpha, \quad \frac{M^2 c^2 \rho^2}{\hbar^2} \implies M^2,
\]

eq. (63) will read

\[
\left[ \frac{d^2 f}{dx^2} + \frac{2(1 - 2x^2)}{x(1 - x^2)} \frac{df}{dx} - \left( E + \frac{i\alpha}{x} \right)^2 \frac{1}{(1 - x^2)^2} + \frac{M^2 - l(l+1)}{x^2} \frac{1}{1 - x^2} \right] f = 0, \tag{64}
\]

or

\[
\frac{d^2 f}{dx^2} + \left[ \frac{2}{x} + \frac{1}{x - 1} + \frac{1}{x + 1} \right] \frac{df}{dx} + \left[ \frac{\alpha^2 - l(l+1)}{x^2} - \frac{1}{4} \frac{(E + i\alpha)^2}{(x - 1)^2} - \frac{1}{4} \frac{(E - i\alpha)^2}{(x + 1)^2} - \frac{2Ei\alpha}{x} \right.
\]

\[
\left. + \frac{1}{4} \frac{(E - 2i\alpha)^2 - \alpha^2 + 2[M^2 - l(l+1)]}{x + 1} \right] f = 0. \tag{65}
\]

Near singular points, solutions behave as follows:

\[
x = 0, \quad \frac{d^2 f}{dx^2} + \frac{2}{x} \frac{df}{dx} + \frac{\alpha^2 - l(l+1)}{x^2} f = 0,
\]

\( f = x^A, \quad A = -\frac{1}{2} \pm \sqrt{(l + 1/2)^2 - \alpha^2}; \)

\[
x = 1, \quad \frac{d^2 f}{dx^2} + \frac{1}{x - 1} \frac{df}{dx} \frac{1}{4} \frac{(E + i\alpha)^2}{(x - 1)^2} f = 0,
\]

\( f = (x - 1)^B, \quad B = \pm \frac{1}{2} (E + i\alpha); \)

\[
x = -1, \quad \frac{d^2 f}{dx^2} + \frac{1}{x + 1} \frac{df}{dx} \frac{1}{4} \frac{(E - i\alpha)^2}{(x + 1)^2} f = 0,
\]

\( f = (x + 1)^C, \quad C = \pm \frac{1}{2} (E - i\alpha). \)
Correspondingly, we should use the substitution \( f = x^A (x - 1)^B (x + 1)^C \); eq. (65) gives

\[
\frac{d^2 f}{dx^2} + \left[ \frac{2A + 2}{x} + \frac{2B + 1}{x - 1} + \frac{2C + 1}{x + 1} \right] \frac{df}{dx} + \\
\frac{A(A + 1) + \alpha^2 - l(l + 1)}{x^2} + \frac{1}{4} \frac{4B^2 + (E + \alpha)^2}{(x - 1)^2} + \frac{1}{4} \frac{4C^2 + (E - \alpha)^2}{(x + 1)^2} + \\
2A + 3B + 3C + 2(AB + AC + BC) - M^2 + E^2/2 - 3\alpha^2/2 + l(l + 1) + \\
\frac{2(iE\alpha + B - C + AB - AC)}{x(x - 1)(x + 1)} \right] f = 0.
\] (66)

At restrictions on \( A, B, C \) as shown

\[ A = \frac{1}{2} \pm \sqrt{(l + 1/2)^2 - \alpha^2}, \quad B = \frac{1}{2}(E + i\alpha), \quad C = \frac{1}{2}(E - i\alpha) \]

eq. (66) becomes simpler

\[
\frac{d^2 f}{dx^2} + \left[ \frac{2A + 2}{x} + \frac{2B + 1}{x - 1} + \frac{2C + 1}{x + 1} \right] \frac{df}{dx} + \\
\frac{2A + 3B + 3C + 2(AB + AC + BC) - M^2 + E^2/2 - 3\alpha^2/2 + l(l + 1)}{x(x - 1)(x + 1)} + \\
\frac{2(iE\alpha + B - C + AB - AC)}{x(x - 1)(x + 1)} \right] f = 0.
\] (67)

This can be identified with the generalized Heun equation for \( G(-1; q, \lambda, \beta, \gamma, \delta, x) \)

\[
\frac{d^2 G}{dx^2} + \left[ \frac{\gamma}{x} + \frac{\delta}{x - 1} + \frac{\epsilon}{x + 1} \right] \frac{dG}{dx} + \frac{\lambda\beta x - q}{x(x - 1)(x + 1)} G = 0;
\] (68)

its parameters are to be determined from relations

\[
\epsilon = \lambda + \beta - \gamma - \delta + 1, \quad \gamma = 2A + 2, \quad \delta = 2B + 1, \\
q = -2(iE\alpha + B - C + AB - AC), \\
\lambda + \beta = 3 + 2(A + B + C), \\
\lambda\beta = 2A + 3B + 3C + 2(AB + AC + BC) - M^2 + \frac{E^2}{2} - \frac{3\alpha^2}{2} + l(l + 1).
\]

From two last relations we get \( \lambda \) and \( \beta \):

\[
\lambda = \frac{3}{2} + A + B + C + \sqrt{\frac{9}{4} + A^2 + B^2 + C^2 + A + \frac{3\alpha^2}{2} - \frac{E^2}{2} + M^2 - l(l + 1)}, \\
\beta = \frac{3}{2} + A + B + C - \sqrt{\frac{9}{4} + A^2 + B^2 + C^2 + A + \frac{3\alpha^2}{2} - \frac{E^2}{2} + M^2 - l(l + 1)}.
\]
Allowing for the identity $A^2 + A + \alpha^2 - l(l + 1) = 0$, they can be read simpler

$$\lambda = \frac{3}{2} + A + B + C + \sqrt{B^2 + C^2 + \frac{\alpha^2 - E^2}{2} + M^2 + \frac{9}{4}},$$

$$\beta = \frac{3}{2} + A + B + C - \sqrt{B^2 + C^2 + \frac{\alpha^2 - E^2}{2} + M^2 + \frac{9}{4}}. \quad (69)$$

To have solutions finite at the origin, we should use the positive value of $A$:

$$A = -\frac{1}{2} + \sqrt{(l + 1/2)^2 - \alpha^2} > 0. \quad (70)$$

recall that physical domain for radial variable here is the interval $x \in (0, +i\infty)$. Depending on the signs in $B$ and $C$, we have four different possibilities:

$$(+,-), \quad B = +\frac{1}{2} (E + i\alpha), \quad C = +\frac{1}{2} (E - i\alpha), \quad B + C = +E, \quad B^2 + C^2 = \frac{E^2 - \alpha^2}{2};$$

$$(-,-), \quad B = -\frac{1}{2} (E + i\alpha), \quad C = -\frac{1}{2} (E - i\alpha), \quad B + C = -E, \quad B^2 + C^2 = \frac{E^2 - \alpha^2}{2};$$

$$(+,-), \quad B = +\frac{1}{2} (E + i\alpha), \quad C = -\frac{1}{2} (E - i\alpha), \quad B + C = +i\alpha, \quad B^2 + C^2 = \frac{E^2 - \alpha^2}{2};$$

$$(-,+), \quad B = -\frac{1}{2} (E + i\alpha), \quad C = +\frac{1}{2} (E - i\alpha), \quad B + C = -i\alpha, \quad B^2 + C^2 = \frac{E^2 - \alpha^2}{2}. \quad (71)$$

Because in all four cases we have one the same identity $B^2 + C^2 + \frac{\alpha^2 - E^2}{2} = 0$, expressions $(69)$ for $\lambda$ and $\beta$ simplify

$$\lambda = \frac{3}{2} + A + B + C + \sqrt{M^2 + \frac{9}{4}}, \quad \beta = \frac{3}{2} + A + B + C - \sqrt{M^2 + \frac{9}{4}}. \quad (72)$$

Therefore, we easily derive

$$(+,+) \quad \lambda, \beta = \frac{3}{2} + A + E \pm \sqrt{M^2 + \frac{9}{4}};$$

$$(-,-) \quad \lambda, \beta = \frac{3}{2} + A - E \pm \sqrt{M^2 + \frac{9}{4}}; \quad (73)$$

$$(+,-) \quad \lambda, \beta = \frac{3}{2} + A + i\alpha \pm \sqrt{M^2 + \frac{9}{4}};$$

$$(-,+) \quad \lambda, \beta = \frac{3}{2} + A - i\alpha \pm \sqrt{M^2 + \frac{9}{4}}. \quad (74)$$

Similarly, simpler can be written the parameter $q = -2 [i\alpha E + (B - C)(A + 1) ]$:

$$(+,+) \quad q = -2i\alpha [ E + (A + 1) ], \quad (-,-) \quad q = -2i\alpha [ E - (A + 1) ],$$

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\( q = -2E \left[ i\alpha + (A + 1) \right], \quad (-,+) \quad q = -2E \left[ i\alpha - (A + 1) \right]. \) 

(75)

The quantization rule for energy levels, \( \lambda = -n \), gives

\( (-,-) \quad E_n = (n + A + \frac{3}{2}) + \sqrt{M^2 + 9/4}, \) 

\( (1,+) \quad E_n = (n + A + \frac{3}{2}) - \sqrt{M^2 + 9/4}, \) 

(76)

(77)

In absence of the Coulomb potential (\( \alpha = 0 \)) the task can be solved in hypergeometric functions. In this case, we have more simple radial equation

\[
\frac{d^2 f}{dx^2} + \left[ \frac{2}{x} + \frac{1}{x-1} + \frac{1}{x+1} \right] \frac{df}{dx} + \left[ -\frac{l(l+1)}{x^2} - \frac{1}{4} \frac{E^2}{(x-1)^2} - \frac{1}{4} \frac{E^2}{(x+1)^2} \right]
\]

\[
+ \frac{1}{4} \frac{E^2 - 2(M^2 - l(l+1))}{x-1} + \frac{1}{4} \frac{-E^2 + 2(M^2 - l(l+1))}{x+1} \right] f = 0. \] 

(78)

In the new variable \( y = x^2 \) we will obtain

\[
\frac{d^2 f}{dy^2} + \left[ \frac{3/2}{y} + \frac{1}{y-1} \right] \frac{df}{dy} + \left[ -\frac{1}{4} \frac{l(l+1)}{y^2} - \frac{1}{4} \frac{E^2}{(y-1)^2} \right]
\]

\[
+ \frac{1}{4} \frac{E^2 - M^2 + l(l+1)}{y-1} + \frac{1}{4} \frac{-E^2 + M^2 - l(l+1)}{y} \right] f = 0. \] 

(79)

After using the substitution \( f = y^A(y-1)^B F(y) \) we arrive at

\[
\frac{d^2 f}{dy^2} + \left[ \frac{2A + 3/2}{y} + \frac{2B + 1}{y-1} \right] \frac{df}{dy} + \left[ \frac{1}{4} \frac{2A(2A+1) - l(l+1)}{y^2} + \frac{1}{4} \frac{4B^2 - E^2}{(y-1)^2} \right]
\]

\[
+ \frac{1}{4} \frac{4A + 6B + 8AB + E^2 - M^2 + l(l+1)}{(y-1)y} \right] f = 0. \] 

(80)

At restrictions on \( A, B \) as shown

\( A = +l/2, -(l+1)/2, \quad B = +E/2, -E/2 \)

eq. \( (78) \) will simplify

\[
\frac{d^2 f}{dy^2} + \left[ \frac{2A + 3/2}{y} + \frac{2B + 1}{y-1} \right] \frac{df}{dy} + \frac{1}{4} \frac{4A + 6B + 8AB + E^2 - M^2 + l(l+1)}{(y-1)y} f = 0 \] 

(81)

which is the hypergeometric equation with parameters

\[ \alpha = \frac{3}{4} + A + B - \frac{1}{4} \sqrt{16(A^2 + B^2) + 8A + 9 - 4E^2 + 4M^2 - 4l(l+1)}, \]
\[ \beta = \frac{3}{4} + A + B + \frac{1}{4} \sqrt{16(A^2 + B^2) + 8A + 9 - 4E^2 + 4M^2 - 4l(l+1)}, \]
\[ \gamma = 2A + \frac{3}{2}. \]

Let \( A = +l/2, \ B = -E/2, \) this can provide us with solutions finite at the origin and at the infinity, \( y = 0, +\infty; \) then
\[ \alpha = \frac{1}{2} \left( \frac{3}{2} + l - E - \sqrt{9/4 + M^2} \right), \quad \beta = \frac{1}{2} \left( \frac{3}{2} + l - E + \sqrt{9/4 + M^2} \right). \] (82)

There are possible two possibilities to get polynomial solutions:

(a) \( \alpha = -n \quad \Rightarrow \quad E = 2n + l + 3/2 - \sqrt{9/4 + M^2} \)

(b) \( \beta = -n \quad \Rightarrow \quad E = 2n + l + 3/2 + \sqrt{9/4 + M^2} \) (83)

The complete radial function will vanish at the infinity if
\[ y \to \infty, \quad f = y^A(y-1)^BF(y) \to 0, \quad \text{when} \quad A + B + n < 0; \]
this takes correspondingly the form

(a) \( \frac{l}{2} - \frac{E}{2} + n < 0 \quad \Rightarrow \quad +\sqrt{9/4 + M^2} < \frac{3}{2}; \)

(b) \( \frac{l}{2} - \frac{E}{2} + n < 0 \quad \Rightarrow \quad -\sqrt{9/4 + M^2} < \frac{3}{2}. \) (84)

Evidently, that only the case (b) is realizable. Thus, in AdS space the energy spectrum is discrete and is given by
\[ E = 2n + l + 3/2 + \sqrt{9/4 + M^2}, \quad n = 0, 1, 2, ... \] (85)

Evidently, the last spectrum correlates with (77).

9 Qualitative study of the problem in AdS

To work with the momentum
\[ p_r^2 = \frac{(\epsilon + e^2/r)^2}{c^2} \frac{1}{(1 + r^2/p^2)^2} - (M^2c^2 + \frac{L^2}{r^2}) \frac{1}{1 + r^2/p^2}. \]

it is convenient to use dimensionless quantities
\[ \frac{p_r^2}{M^2c^2} \Rightarrow p_r^2, \quad \frac{r}{\rho} \Rightarrow r, \quad \frac{L^2}{M^2c^2\rho^2} \Rightarrow L^2, \quad \frac{e^2}{\rho \, Mc^2} = q; \]
so that
\[ p_r^2(r) = (\epsilon + \frac{q}{r})^2 \frac{1}{(1 + r^2)^2} - (1 + \frac{L^2}{r^2}) \frac{1}{1 + r^2}. \] (86)
In the origin and the infinity behavior is as shown
\[ r \to 0, \quad p^2(r) \sim -\frac{L^2 - q^2}{r^2} < 0, \]
\[ r \to +\infty, \quad p^2(r) = -\frac{1}{r^2} \to 0; \]
therefore both these domain are forbidden for classical motion.

Let us examine the location of the turning points.\(^/\)
\[ p^2 = 0 \quad \implies \quad (\epsilon r + q)^2 = (r^2 + L^2)(1 + r^2), \]
that is
\[ r^4 - r^2(\epsilon^2 - L^2 - 1) - 2\epsilon q r + (L^2 - q^2) = 0. \quad (87) \]

Taking into account the evident identity
\[ r^4 + Ar^2 + Br + C = [(r - r_1)(r - r_2)](r - r_3)(r - r_4) = \]
\[ = r^4 - (r_1 + r_2 + r_3 + r_4)r^3 + (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4)r^2 - \]
\[ -(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)r + r_1r_2r_3r_4; \]
we derive identities
\[ r_1 + r_2 + r_3 + r_4 = 0, \quad A = r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4, \]
\[ B = -(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4), \quad C = r_1r_2r_3r_4 > 0, \quad (88) \]
where
\[ A = -\left(\epsilon^2 - L^2 - 1\right), \quad B = -2\epsilon q, \quad C = (L^2 - q^2). \quad (9.10c) \]
The relation \( r_1r_2r_3r_4 = C > 0 \) says that assuming all four roots real-valued, then there are forbidden variants below:
\[ (r_1, r_2, r_3, r_4) \sim (-, -, -, +), \quad (r_1, r_2, r_3, r_4) \sim (+, +, +, -). \]
In turn, variants
\[ (r_1, r_2, r_3, r_4) \sim (-, -, -, -), \quad (r_1, r_2, r_3, r_4) \sim (+, +, +, +), \]
are not forbidden as well because of the the identity \( r_1 + r_2 + r_3 + r_4 = 0 \) – see \[SS\]. So we conclude that there exists only one possibility of the real roots location
\[ (r_1, r_2, r_3, r_4) \sim (-, -, +, +), \quad r_4 > r_3 > 0; \quad (89) \]
where \( r_4, r_3 \) stand for physical turning points.

Note that eqs. \[SS\] can be presented in the form
\[ r_1 + r_2 = -(r_3 + r_4), \quad r_1r_2 = \frac{C}{r_3r_4} > 0 \]
\[ A - r_3r_4 = r_1r_2 + (r_1 + r_2)(r_3 + r_4), \]
\[ r_1 + r_2 = -\frac{B + r_1r_2(r_3 + r_4)}{r_3r_4}. \quad (90) \]
Two first equation in (90) can be solved under \( r_1, r_2 \) (we assume that \( r_1 < r_2 < 0 \)):

\[
\begin{align*}
  r_1 &= -\frac{r_3 + r_4}{2} - \sqrt{\frac{1}{4} (r_3 + r_4)^2 - \frac{C}{r_3 r_4}}, \\
  r_2 &= -\frac{r_3 + r_4}{2} + \sqrt{\frac{1}{4} (r_3 + r_4)^2 - \frac{C}{r_3 r_4}}.
\end{align*}
\]

(91)

When the expression under the square root in (91) is positive,

\[
\frac{1}{4} (r_3 + r_4)^2 - \frac{C}{r_3 r_4} \geq 0,
\]

then the roots \( r_1, r_2 \) are real and negative.

When

\[
\frac{1}{4} (r_3 + r_4)^2 - \frac{C}{r_3 r_4} \leq 0,
\]

then the roots \( r_1, r_2 \) are complex and conjugate. To the last possibility there correspond the following roots

\[
(r_1, r_2, r_3, r_4) \sim (z, z^*, +, +), \quad \text{Re} \ z < 0.
\]

Thus, qualitative study shows that in \( \text{AdS} \) space we can expect for the Coulomb problem the situation of a complicated potential well with normal (real) energy spectrum, so the hydrogen atom seems to be a stable system in the sense of the quantum-mechanical sense.

10 WKB-treatment in \( \text{AdS} \) space

Let us make use of formal method of analytical continuation

\[
dS = \Rightarrow \text{AdS} \quad \rho = i\rho;
\]

(92)

this gives possibility to get replies without repeating much the same calculation.

So, approximate roots \( r_1, r_2 \) of the equation \( P^2(r) = 0 \) will read

\[
 r_i = r_{0i} - \frac{\Delta_i}{\rho^2}, \quad i = 1, 2,
\]

(93)

where

\[
 r_{01} = \frac{c^2}{m^2c^4 - \epsilon^2} \left[ \frac{e^2 \epsilon}{c^2} + m \sqrt{\frac{e^4 \epsilon^2}{c^4} - (L^2 - \frac{e^4}{c^2}) \frac{M^2 c^2 - \epsilon^2}{c^2}} \right] > 0,
\]

\[
 r_{02} = \frac{c^2}{m^2c^4 - \epsilon^2} \left[ \frac{e^2 \epsilon}{c^2} - \sqrt{\frac{e^4 \epsilon^2}{c^4} - (L^2 - \frac{e^4}{c^2}) \frac{M^2 c^2 - \epsilon^2}{c^2}} \right] > 0,
\]

\[
 \Delta_i = -r_{0i}^2 (L^2 + M^2 c^2 r_{0i}^2) \left[ \frac{2e^2 \epsilon}{c^2} + 2r_{0i}(\frac{e^2}{c^2} - M^2 c^2) \right]^{-1}.
\]

Approximate roots \( r_3, r_4 \) will be

\[
r_3 = i\rho(b_{03} + \frac{b_1}{i\rho}), \quad r_4 = i\rho(b_{04} + \frac{b_1}{i\rho}),
\]

(94)
where
\[ b_{03} = +\sqrt{1 - \frac{\epsilon^2}{m^2c^4}} > 0, \quad b_{04} = -\sqrt{1 - \frac{\epsilon^2}{m^2c^4}} < 0, \]
\[ b_1 = -\frac{e^2\epsilon}{M^2c^4 - \epsilon^2}. \]

Approximate form of \( P_r \) in the region \( r \ll \rho \) (where the potential well is located) is
\[ P_r \sim mc\sqrt{(r - r_1)((r - r_2)} - 1)^{1/2} \left( 1 - A \frac{r^2}{\rho^2} \right), \tag{95} \]

where
\[ A = 1 - \frac{1}{2(1 - \epsilon^2/m^2c^4)}. \]

Quantization condition has the form
\[ \int_L P_r dr = 2\pi\hbar(n + 1/2). \]

Choosing the contour of integration \( L \) to surround the points \( r_1 \) and \( r_2 \), we reduce the problem to finding residues
\[ \sum_{0,\infty} \text{res} \ P_r = i\hbar(n + 1/2). \]

Allowing for these two residues
\[ \text{res}_{r=0} P_r = \sqrt{\left( \frac{\epsilon^2}{c^2} - m^2c^2 \right)r_1r_2}, \]
\[ \text{res}_{r=\infty} P_r = \frac{1}{2}\sqrt{\frac{\epsilon^2}{c^2} - m^2c^2} \left[ \frac{(r_1 + r_2) + A}{\rho^2} r_1r_2(r_1 + r_2) \right], \tag{96} \]

we arrive at
\[ \sqrt{\frac{\epsilon^2}{c^2} - m^2c^2} \left[ \sqrt{r_{12}^2 + \frac{r_1 + r_2}{2}} + A \frac{r_1r_2}{\rho^2} \frac{r_1 + r_2}{2} \right] = i\hbar(n + 1/2). \]

From whence it follows an approximate form of the energy spectrum in AdS space:
\[ \epsilon = \epsilon_0 - \frac{\Delta}{\rho^2}, \quad \epsilon_0 = mc^2 \left[ 1 + \frac{e^4/c^2}{[\hbar(n + 1/2) + \sqrt{\hbar^2(l + 1/2)^2 - e^4/c^2]}} \right]^{-1/2}, \tag{97} \]
\[ \Delta = \frac{(m^2e^4 - \epsilon^2)^2}{4e^2m^2c^4} \left[ \Delta_1 + \Delta_2 + \Delta_1 \sqrt{\frac{r_{02}}{r_{01}}} + \Delta_2 \sqrt{\frac{r_{01}}{r_{02}}} + Ar_{01}r_{02}(r_{01} + r_{02}) \right]. \]
11 Spin 1/2 particle in the Coulomb field in dS space

Let start with the Dirac free wave equation (the notation according [47] is used)

\[
[i \gamma^c (e^a_c) \partial_\alpha + \frac{1}{2} \sigma^{ab} \gamma_{abc} - m] \Psi = 0 ; \tag{98}
\]

in static coordinates and tetrad of the Sitter space it takes the form

\[
[ i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \gamma^3 \partial_r + \frac{\gamma^1 \sigma^{31} + \gamma^2 j^{32}}{r} + \frac{\Phi'}{2 \Phi} \gamma^0 \sigma^{03} ] + \frac{1}{r} \Sigma_{\theta,\phi} - M ] \Psi(x) = 0 , \quad \phi = 1 - r^2 ,
\]

\[
\Sigma_{\theta,\phi} = i \gamma^1 \partial_{\theta} + \gamma^2 \frac{i \partial + i \sigma^{12} \cos \theta}{\sin \theta} \quad . \tag{99}
\]

Below the spinor basis will be used

\[
\gamma^0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} , \quad \gamma^j = \begin{bmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{bmatrix} , \quad i \sigma^{12} = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} .
\]

Allowing for

\[ \gamma^1 \sigma^{31} + \gamma^2 j^{32} = \gamma^3 , \quad \gamma^0 \sigma^{03} = \gamma^3 / 2 , \]

eq. (99) reads

\[
[ i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \gamma^3 ( \partial_r + \frac{1}{r} \frac{\Phi'}{\Phi} ) + \frac{1}{r} \Sigma_{\theta,\phi} - M ] \Psi(x) = 0 . \tag{100}
\]

One can simplify the problem with the help of substitution \( \Psi(x) = r^{-1} \Phi^{-1/4} F(x) \), then

\[
[ i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \gamma^3 \partial_r + \frac{1}{r} \Sigma_{\theta,\phi} - M ] F(x) = 0 . \tag{101}
\]

Spherical waves are constructed through the substitution (see in [47])

\[
\Psi_{\epsilon jm}(x) = e^{-i \epsilon t} \frac{r}{f_1(r)} D_{-1/2} \begin{vmatrix} f_1(r) & D_{-1/2} \\ f_2(r) & D_{+1/2} \\ f_3(r) & D_{-1/2} \\ f_4(r) & D_{+1/2} \end{vmatrix} . \tag{102}
\]

With the use of the recursive relations [45]

\[
\partial_\theta D_{+1/2} = a D_{-1/2} - b D_{+3/2} ,
\]

\[
- \frac{m - 1/2 \cos \theta}{\sin \theta} D_{+1/2} = -a D_{-1/2} - b D_{+3/2} ,
\]

\[
\partial_\theta D_{-1/2} = b D_{-3/2} - a D_{+1/2} ,
\]

\[
- \frac{m + 1/2 \cos \theta}{\sin \theta} D_{-1/2} = -b D_{-3/2} - a D_{+1/2} ,
\]

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where $a = (j + 1)/2$, $b = (1/2) \sqrt{(j - 1/2)(j + 3/2)}$, we get

$$\sum_{\theta, \phi} \Psi_{\epsilon jm}(x) = i \nu \frac{e^{-i \epsilon t}}{r} \begin{vmatrix} -f_4(r) & D_{-1/2} \\ + f_3(r) & D_{+1/2} \\ + f_2(r) & D_{-1/2} \\ - f_1(r) & D_{+1/2} \end{vmatrix}, \quad \nu = (j + 1/2). \quad (103)$$

Then we arrive at the radial system

$$\begin{align*}
\frac{\epsilon}{\sqrt{\Phi}} f_3 - i \sqrt{\Phi} \frac{d}{dr} f_3 - \frac{\nu}{r} f_4 - M f_1 &= 0, \\
\frac{\epsilon}{\sqrt{\Phi}} f_4 + i \sqrt{\Phi} \frac{d}{dr} f_4 + i \frac{\nu}{r} f_3 - M f_2 &= 0, \\
\frac{\epsilon}{\sqrt{\Phi}} f_1 + i \sqrt{\Phi} \frac{d}{dr} f_1 + i \frac{\nu}{r} f_2 - M f_3 &= 0, \\
\frac{\epsilon}{\sqrt{\Phi}} f_2 - i \sqrt{\Phi} \frac{d}{dr} f_2 - i \frac{\nu}{r} f_1 - M f_4 &= 0. \quad (104)
\end{align*}$$

To simplify the system, let us diagonalize $P$-operator [47]. In Cartesian basis, $\Pi_C = i \gamma^0 \otimes \hat{P}$, after transition to spherical tetrad gives

$$\Pi_{sph.} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} \otimes \hat{P}. \quad (105)$$

From the equation $\Pi_{sph.} \Psi_{jm} = \Pi \Psi_{jm}$ it follows that $\Pi = \delta (-1)^j + 1$, $\delta = \pm 1$ and

$$f_4 = \delta f_1, \quad f_3 = \delta f_2, \quad \Psi(x)_{\epsilon jm} = \epsilon^{-i \epsilon t} \begin{vmatrix} f_1(r) & D_{-1/2} \\ f_2(r) & D_{+1/2} \\
\delta f_2(r) & D_{-1/2} \\ - \delta f_1(r) & D_{+1/2} \end{vmatrix}. \quad (106)$$

Allowing for (106), we simplify the system (104) as

$$\begin{align*}
(\sqrt{\Phi} \frac{d}{dr} + \frac{\nu}{r}) f + (\frac{\epsilon}{\sqrt{\Phi}} + \delta M) g &= 0, \\
(\sqrt{\Phi} \frac{d}{dr} - \frac{\nu}{r}) g - (\frac{\epsilon}{\sqrt{\Phi}} - \delta M) f &= 0; \quad (107)
\end{align*}$$

where new functions $f = (f_1 + f_2)/\sqrt{2}$, $g = (f_1 - f_2)/i \sqrt{2}$ are used instead of $f_1$ and $f_2$. For definiteness, let us consider eqs. (107) at $\delta = +1$ (formally the second case $\delta = -1$ corresponds to the change $M \Rightarrow -M$):

$$\begin{align*}
(\sqrt{\Phi} \frac{d}{dr} + \frac{\nu}{r}) f + (\frac{\epsilon}{\sqrt{\Phi}} + M) g &= 0, \\
(\sqrt{\Phi} \frac{d}{dr} - \frac{\nu}{r}) g - (\frac{\epsilon}{\sqrt{\Phi}} - M) f &= 0. \quad (108)
\end{align*}$$

To take into account the presence of external Coulomb field it suffices to make one formal change

$$\epsilon \quad \Rightarrow \quad \epsilon + \frac{\epsilon^2}{r};$$

---

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in this way we get the system

\[
\begin{align*}
(\sqrt{1-r^2} r') + \frac{\nu}{r} f + \frac{1}{\sqrt{1-r^2}} (\epsilon + \frac{e^2}{r}) + M g &= 0 , \\
(\sqrt{1-r^2} r') - \frac{\nu}{r} g - \frac{1}{\sqrt{1-r^2}} (\epsilon + \frac{e^2}{r}) - M f &= 0 .
\end{align*}
\] (109)

In the variable \( r = \sin \rho \), these equation look shorter

\[
\begin{align*}
\left( \frac{d}{d\rho} + \frac{\nu}{\sin \rho} \right) f + \frac{1}{\cos \rho} (\epsilon + \frac{e^2}{\sin \rho}) g &= 0 , \\
\left( \frac{d}{d\rho} - \frac{\nu}{\sin \rho} \right) g - \frac{1}{\cos \rho} (\epsilon + \frac{e^2}{\sin \rho}) f &= 0 .
\end{align*}
\] (110)

Summing and subtracting two last equations, we get

\[
\begin{align*}
\frac{d}{d\rho} (f + g) + \frac{\nu}{\sin \rho} (f - g) - \frac{1}{\cos \rho} (\epsilon + \frac{e^2}{\sin \rho}) (f - g) + M (f + g) &= 0 , \\
\frac{d}{d\rho} (f - g) + \frac{\nu}{\sin \rho} (f + g) + \frac{1}{\cos \rho} (\epsilon + \frac{e^2}{\sin \rho}) (f + g) - M (f - g) &= 0 .
\end{align*}
\] (111)

Introducing two new functions

\[
f + g = e^{-i\rho/2} (F + G) , \quad f - g = e^{+i\rho/2} (F - G) ,
\] (112)

one transforms (111) into

\[
\begin{align*}
\frac{d}{d\rho} e^{-i\rho/2} (F + G) + \frac{\nu}{\sin \rho} e^{+i\rho/2} (F - G) \\
- \frac{1}{\cos \rho} (\epsilon + \frac{e^2}{\sin \rho}) e^{+i\rho/2} (F - G) + M e^{-i\rho/2} (F + G) &= 0 , \\
\frac{d}{d\rho} e^{+i\rho/2} (F - G) + \frac{\nu}{\sin \rho} e^{-i\rho/2} (F + G) \\
+ \frac{1}{\cos \rho} (\epsilon + \frac{e^2}{\sin \rho}) e^{-i\rho/2} (F + G) - M e^{+i\rho/2} (F - G) &= 0 ,
\end{align*}
\]
or

\[
\begin{align*}
\frac{d}{d\rho} (F + G) - \frac{i}{2} (F + G) + \frac{\nu}{\sin \rho} (\cos \rho + i \sin \rho) (F - G) \\
- \frac{1}{\cos \rho} (\epsilon + \frac{e^2}{\sin \rho}) (\cos \rho + i \sin \rho) (F - G) + M (F + G) &= 0 , \\
\frac{d}{d\rho} (F - G) + \frac{i}{2} (F - G) + \frac{\nu}{\sin \rho} (\cos \rho - i \sin \rho) (F + G) \\
+ \frac{1}{\cos \rho} (\epsilon + \frac{e^2}{\sin \rho}) (\cos \rho - i \sin \rho) (F + G) - M (F - G) &= 0 .
\end{align*}
\]
Again, summing and subtracting two last equations, we arrive at
\[
\left( \frac{d}{d\rho} + \nu \frac{\cos \rho}{\sin \rho} - i \frac{\sin \rho}{\cos \rho} (\epsilon + \frac{e^2}{\sin \rho}) \right) F + \left( \epsilon + \frac{e^2}{\sin \rho} + M - i\nu - \frac{i}{2} \right) G = 0 ,
\]
\[
\left( \frac{d}{d\rho} - \nu \frac{\cos \rho}{\sin \rho} + i \frac{\sin \rho}{\cos \rho} (\epsilon + \frac{e^2}{\sin \rho}) \right) G + \left( -\epsilon - \frac{e^2}{\sin \rho} + M + i\nu - \frac{i}{2} \right) F = 0 .
\]  
(113)

To have coefficients as rational functions, one should use a new variable:
\[
y = \tan \frac{\rho}{2}, \quad r = \sin \rho = \frac{2y}{1 + y^2}, \quad yY = 1 ,
\]
\[
y = 1 + \sqrt{1 - r^2}, \quad Y = 1 - \sqrt{1 - r^2}, \quad y \in (1, +\infty), \quad Y \in (0, 1) ,
\]
the above system takes the form
\[
\left[ (1 + y^2) \frac{d}{dy} - \nu y + \frac{\nu}{y} - i(2\epsilon + 2e^2) \right] F + \left( e^2y + \frac{e^2}{y} + 2\epsilon + 2M - 2i\nu - i \right) G = 0 ,
\]
\[
\left[ (1 + y^2) \frac{d}{dy} + \nu y - \frac{\nu}{y} + i(2\epsilon + 2e^2) \right] G - \left( e^2y + \frac{e^2}{y} + 2\epsilon - 2M - 2i\nu + i \right) F = 0 .
\]  
(114)

Introducing the notations
\[
\left( e^2y + \frac{e^2}{y} + 2\epsilon + 2M - 2i\nu - i \right) = \frac{e^2}{y} (y - y_1)(y - y_2) ,
\]
\[
y_1 = - \frac{(\epsilon + M - i\nu - i/2) + \sqrt{(\epsilon + M - i\nu - i/2)^2 + e^4}}{e^2},
\]
\[
y_2 = - \frac{(\epsilon + M - i\nu - i/2) - \sqrt{(\epsilon + M - i\nu - i/2)^2 + e^4}}{e^2},
\]  
(115)

and
\[
\left( e^2y + \frac{e^2}{y} + 2\epsilon - 2M - 2i\nu + i \right) = \frac{e^2}{y} (y - Y_1)(y - Y_2) ,
\]
\[
y_1 = - \frac{(\epsilon - M - i\nu + i/2) + \sqrt{(\epsilon - M - i\nu + i/2)^2 + e^4}}{e^2},
\]
\[
Y_2 = - \frac{(\epsilon - M - i\nu + i/2) - \sqrt{(\epsilon - M - i\nu + i/2)^2 + e^4}}{e^2},
\]  
(116)

we rewrite the system (114) shorter
\[
\left[ (1 + y^2) \frac{d}{dy} - \nu y + \frac{\nu}{1 - y} + \frac{a}{1 + y} + \frac{b}{2} \right] F + \frac{e^2}{y} (y - y_1)(y - y_2) G = 0 ,
\]
\[
\left[ (1 + y^2) \frac{d}{dy} + \nu y - \frac{\nu}{1 - y} - \frac{a}{1 + y} - \frac{b}{2} \right] G - \frac{e^2}{y} (y - Y_1)(y - Y_2) F = 0 ,
\]  
(117)
where

\[ a = i(2\epsilon + 2e^2), \quad b = i(2\epsilon - 2e^2). \]

Second order differential equation \( F(y) \) will contain singular points \( 0, \infty, \pm 1, \pm i, y_1, y_2 \).
Spin 1/2 particle in the Coulomb field in $AdS$ space

Let start with the Dirac free wave equation

$$[ i \gamma^c (e^a_c) \partial_\alpha + \frac{1}{2} \sigma^{ab} \gamma_{abc} ) - m ] \Psi = 0 ; \quad (118) $$

in static coordinates and tetrad of the anti Sitter space it takes the form

$$[ i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \gamma^3 ( \partial_r + \frac{1}{r} \Phi' ) + \frac{1}{r} \Sigma_{\theta,\phi} - M ] \Psi(x) = 0 , \quad \Phi = 1 + r^2 , \quad \Sigma_{\theta,\phi} = i \gamma^1 \partial_\theta + \gamma^2 \frac{i\partial + i\sigma^{12} \cos \theta}{\sin \theta} . \quad (119)$$

Below the spinor basis will be used, eq. (119) reads

$$[ i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \gamma^3 ( \partial_r + \frac{1}{r} \Phi' ) + \frac{1}{r} \Sigma_{\theta,\phi} - M ] F(x) = 0 . \quad (120)$$

One can simplify the problem with the help of substitution $\Psi(x) = r^{-1} \Phi^{-1/4} F(x)$, then

$$[ i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \gamma^3 ( \partial_r + \frac{1}{r} \Sigma_{\theta,\phi} - M ) F(x) = 0 . \quad (121)$$

Spherical waves are constructed through the substitution (see in [47])

$$\Psi_{\epsilon jm}(x) = \frac{e^{-i\epsilon t}}{r} \left| \begin{array}{c} f_1(r) D_{-1/2} \\ f_2(r) D_{+1/2} \\ f_3(r) D_{-1/2} \\ f_4(r) D_{+1/2} \end{array} \right| . \quad (122)$$

Then we arrive at the radial system

$$\frac{\epsilon}{\sqrt{\Phi}} f_3 - i \sqrt{\Phi} \frac{d}{dr} f_3 - \frac{\nu}{r} f_4 - M f_4 = 0 , \quad \frac{\epsilon}{\sqrt{\Phi}} f_4 + i \sqrt{\Phi} \frac{d}{dr} f_4 + \frac{\nu}{r} f_3 - M f_3 = 0 , \quad \frac{\epsilon}{\sqrt{\Phi}} f_1 + i \sqrt{\Phi} \frac{d}{dr} f_1 + \frac{\nu}{r} f_2 - M f_2 = 0 , \quad \frac{\epsilon}{\sqrt{\Phi}} f_2 - i \sqrt{\Phi} \frac{d}{dr} f_2 - \frac{\nu}{r} f_1 - M f_1 = 0 . \quad (123)$$

To simplify the system, let us diagonalize $P$-operator [47], then $\delta = \pm 1$ and

$$f_4 = \delta f_1 , \quad f_3 = \delta f_2 , \quad \Psi(x)_{\epsilon jm\delta} = \frac{e^{-i\epsilon t}}{r} \left| \begin{array}{c} f_1(r) D_{-1/2} \\ f_2(r) D_{+1/2} \\ \delta f_2(r) D_{-1/2} \\ \delta f_1(r) D_{+1/2} \end{array} \right| . \quad (124)$$

Allowing for (124), we simplify the system (123) as

$$(\sqrt{\Phi} \frac{d}{dr} \frac{\nu}{r} ) f + (\frac{\epsilon}{\sqrt{\Phi}} + \delta M ) g = 0 ,$$

$$(\sqrt{\Phi} \frac{d}{dr} - \frac{\nu}{r} ) g - (\frac{\epsilon}{\sqrt{\Phi}} - \delta M ) f = 0 ; \quad (125)$$
where new functions \( f = (f_1 + f_2)/\sqrt{2} \), \( g = (f_1 - f_2)/i\sqrt{2} \) are used instead of \( f_1 \) and \( f_2 \). For definiteness, let us consider eqs. (125) at \( \delta = +1 \) (formally the second case \( \delta = -1 \) corresponds to the change \( M \Rightarrow -M \)):

\[
\begin{align*}
(\sqrt{\Phi} \frac{d}{dr} + \frac{\nu}{r}) f + (\frac{e}{\sqrt{\Phi}} + M) g &= 0, \\
(\sqrt{\Phi} \frac{d}{dr} - \frac{\nu}{r}) g - (\frac{e}{\sqrt{\Phi}} - M) f &= 0.
\end{align*}
\]

(126)

To take into account the presence of external Coulomb field it suffices to make one formal change \( \epsilon \Rightarrow \epsilon + \frac{e^2}{r} \); in this way we get the system

\[
\begin{align*}
(\sqrt{1 + \frac{r^2}{d^2}} \frac{d}{dr} + \frac{\nu}{r}) f + (\frac{1}{\sqrt{1 + r^2}}(\epsilon + \frac{e^2}{r}) + M) g &= 0, \\
(\sqrt{1 + \frac{r^2}{d^2}} \frac{d}{dr} - \frac{\nu}{r}) g - (\frac{1}{\sqrt{1 + r^2}}(\epsilon + \frac{e^2}{r}) - M) f &= 0.
\end{align*}
\]

(127)

In the variable \( r = \sinh \rho \), these equation look shorter

\[
\begin{align*}
(\frac{d}{d\rho} + \frac{\nu}{\sinh \rho}) f + \left(\frac{1}{\cosh \rho}(\epsilon + \frac{e^2}{\sinh \rho}) + M\right) g &= 0, \\
(\frac{d}{d\rho} - \frac{\nu}{\sinh \rho}) g - \left(\frac{1}{\cosh \rho}(\epsilon + \frac{e^2}{\sinh \rho}) - M\right) f &= 0.
\end{align*}
\]

(128)

Summing and subtracting two last equations, we get

\[
\begin{align*}
\frac{d}{d\rho}(f + g) + \frac{\nu}{\sinh \rho}(f - g) - \frac{1}{\cosh \rho}(\epsilon + \frac{e^2}{\sinh \rho})(f - g) + M(f + g) &= 0, \\
\frac{d}{d\rho}(f - g) + \frac{\nu}{\sinh \rho}(f + g) + \frac{1}{\cosh \rho}(\epsilon + \frac{e^2}{\sinh \rho})(f + g) - M(f - g) &= 0.
\end{align*}
\]

(129)

Introducing two new functions

\[
\begin{align*}
   f + g = e^{-\rho/2}(F + G), & \quad f - g = e^{+\rho/2}(F - G),
\end{align*}
\]

(130)

one transforms (129) into

\[
\begin{align*}
\frac{d}{d\rho}e^{-\rho/2}(F + G) + \frac{\nu}{\sinh \rho}e^{+\rho/2}(F - G) \\
- \frac{1}{\cosh \rho}(\epsilon + \frac{e^2}{\sinh \rho})e^{+\rho/2}(F - G) + Me^{-\rho/2}(F + G) &= 0, \\
\frac{d}{d\rho}e^{+\rho/2}(F - G) + \frac{\nu}{\sinh \rho}e^{-\rho/2}(F + G) \\
+ \frac{1}{\cosh \rho}(\epsilon + \frac{e^2}{\sinh \rho})e^{-\rho/2}(F + G) - Me^{+\rho/2}(F - G) &= 0,
\end{align*}
\]
\[
\frac{d}{d\rho}(F + G) - \frac{1}{2}(F + G) + \frac{\nu}{\sinh \rho} (\cosh \rho + \sinh \rho)(F - G) - \frac{1}{\cosh \rho}(\epsilon + \frac{e^2}{\sinh \rho})(\cosh \rho + \sinh \rho)(F - G) + M(F + G) = 0 ,
\]
\[
\frac{d}{d\rho}(F - G) + \frac{1}{2}(F - G) + \frac{\nu}{\sinh \rho} (\cosh \rho - \sinh \rho)(F + G) + \frac{1}{\cosh \rho}(\epsilon + \frac{e^2}{\sinh \rho})(\cosh \rho - \sinh \rho)(F + G) - M(F - G) = 0 .
\]

Again, summing and subtracting two last equations, we arrive at
\[
\left(\frac{d}{d\rho} + \nu \frac{\cosh \rho}{\sinh \rho} - \sinh \rho \left(\epsilon + \frac{e^2}{\sinh \rho}\right)\right) F + \left(\epsilon + \frac{e^2}{\sinh \rho} + M - \nu - \frac{1}{2}\right) G = 0 ,
\]
\[
\left(\frac{d}{d\rho} - \nu \frac{\cosh \rho}{\sinh \rho} + \sinh \rho \left(\epsilon + \frac{e^2}{\sinh \rho}\right)\right) G + \left(-\epsilon - \frac{e^2}{\sinh \rho} + M + \nu - \frac{1}{2}\right) F = 0 .
\] (131)

To have coefficients as rational functions, one should use a new variable:
\[
y = \tanh \frac{\rho}{2} , \quad r = \sinh \rho = \frac{2y}{1 - y^2} , \quad yY = -1 ,
\]
\[
y = \frac{-1 + \sqrt{1 + r^2}}{r} , \quad Y = \frac{-1 - \sqrt{1 + r^2}}{r} ,
\]

the above system takes the form
\[
\left[(1 - y^2) \frac{d}{dy} + \nu y + \frac{\nu}{y} - \frac{4(\epsilon y + e^2)}{1 + y^2} + 2e^2\right] F +
\]
\[
\left[\left(-e^2 y + \frac{e^2}{y} + 2\epsilon + 2M - 2\nu - 1\right) G = 0 ,
\right.
\]
\[
\left[\left((1 - y^2) \frac{d}{dy} - \nu y - \frac{\nu}{y} + \frac{4(\epsilon y + e^2)}{1 + y^2} - 2e^2\right] G +
\right.
\]
\[
\left[\left(e^2 y - \frac{e^2}{y} - 2\epsilon + 2M + 2\nu - 1\right) F = 0 .
\] (132)

Introducing the notations
\[
\left(-e^2 y + \frac{e^2}{y} + 2\epsilon + 2M - 2\nu - 1\right) = \frac{e^2}{y}(y - y_1)(y - y_2) ,
\]
\[
y_1 = \frac{(\epsilon + M - \nu - 1/2) + \sqrt{(\epsilon + M - \nu - 1/2)^2 + e^4}}{e^2}
\]
\[
y_2 = \frac{(\epsilon + M - \nu - 1/2) - \sqrt{(\epsilon + M - \nu - 1/2)^2 + e^4}}{e^2}
\] (133)
and

\[
\left( e^2 y - \frac{e^2}{y} - 2\epsilon + 2M + 2\nu - 1 \right) = \frac{e^2}{y} (y - Y_1)(y - Y_2),
\]

\[
Y_1 = \frac{(\epsilon - M - \nu + 1/2) + \sqrt{(\epsilon - M - \nu + 1/2)^2 + e^4}}{e^2}
\]

\[
Y_2 = \frac{(\epsilon - M - \nu + 1/2) - \sqrt{(\epsilon - M - \nu + 1/2)^2 + e^4}}{e^2},
\]

we rewrite the system (132) shorter

\[
\begin{align*}
\left[ (1 - y^2) \frac{d}{dy} + \nu y + \frac{\nu}{y} - \frac{4(\epsilon y + e^2)}{1 + y^2} + 2e^2 \right] F + \frac{e^2}{y} (y - y_1)(y - y_2) & G = 0, \\
\left[ (1 - y^2) \frac{d}{dy} - \nu y - \frac{\nu}{y} + \frac{4(\epsilon y + e^2)}{1 + y^2} - 2e^2 \right] G + \frac{e^2}{y} (y - Y_1)(y - Y_2) & F = 0.
\end{align*}
\]

Second order differential equation \( F(y) \) will contain singular points \( 0, \infty, \pm 1, \pm i, y_1, y_2 \).

13 Conclusion

The hydrogen atom theory in developed in de Sitter spaces of constant and negative curvature on the base of the Klein–Fock–Gordon wave equation in static coordinates. After separation of the variables in both models, the problem reduces to the Heun type second order differential equation with four singular points. Qualitative examination shows that the energy spectrum for the hydrogen atom in de Sitter space should be quasi-stationary, and the hydrogen atom becomes principally unstable. We have calculated approximate expression for energy levels within semiclassical method; also the probability of decay of the hydrogen atom has been estimated. Similar analysis shows that in anti de Sitter model, the hydrogen atom should be stable in the sense of the quantum mechanics, approximate formulas for energy level have been found also within semi-classical approach. Extension to the case of a spin 1/2 particle is given.

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References

[1] Hawking, S.W.; Ellis, G.F.R.; The large scale structure of space-time. Cambridge University Press, 1973

[2] Gibbons, G.W.: Anti-de-Sitter spacetime and its uses, in Mathematical and quantum aspects of relativity and cosmology (Pythagoreon, 1998), Lecture Notes in Phys., 537, Springer-Verlag, 102-142, 2000.
[3] Dirac, P.A.M.: The electron wave equation in the de Sitter space. Ann. Math. 36, 657–669 (1935)

[4] Dirac, P.A.M.: Wave equations in conformal space. Ann. of Math. 37, 429–442 (1936)

[5] Schrödinger, E.: The proper vibrations of the expanding universe. Physica. 6, 899–912 (1939)

[6] Schrödinger, E.: General theory of relativity and wave mechanics. Wiss. en Natuurkund. 10, 2–9 (1940)

[7] Goto, K.: Wave equations in de Sitter space. Progr. Theor. Phys. 6, 1013–1014 (1951)

[8] Nachtmann, O.: Quantum theory in de-Sitter space. Commun. Math. Phys. 6, 1–16 (1967)

[9] Chernikov, N.A.; Tagirov, E.A.: Quantum theory of scalar field in de Sitter space-time. Ann. Inst. Henri Poincare. IX, 109–141 (1968)

[10] Börner, G.; Dürr, H.P.: Classical and quantum theory in de Sitter space. Nuovo Cim. A. 64, 669–713 (1969)

[11] Fushchych, W.L.; Krivsky, I.Yu.: On representations of the inhomogeneous de Sitter group and equations in five-dimensional Minkowski space. Nucl. Phys. B. 14, 573–585 (1969)

[12] Börner, G.; Dürr, H.P.: Classical and Quantum Fields in de Sitter space. Nuovo Cim. LXIV, 669 (1969)

[13] Castagnino, M.: Champs spinoriels en Relativité générale; le cas particulier de l’espace-temps de De Sitter et les équations d’ond pour les sps élèves. Ann. Inst. Henri Poincaré. A. 16, 293–341 (1972)

[14] Tagirov, E.A.: Consequences of field quantization in de Sitter type cosmological models. Ann. Phys. 76, 561–579 (1973)

[15] Riordan, F.: Solutions of the Dirac equation in finite de Sitter space. Nuovo Cim. B. 20, 309–325 (1974)

[16] Candelas, P.; Raine, D.J.: General-relativistic quantum field theory: an exactly soluble model. Phys. Rev. D. 12, 965–974 (1975)

[17] Schomblond, Ch.; Spindel P.: Propagateurs des champs spinoriels et vectoriels dans l’univers de de Sitter. Bull. Cl. Sci., V. Ser., Acad. R. Belg. LXII, 124 (1976)

[18] Hawking, S.W.; Gibbons, G.W.: Cosmological event horizons, thermodynamics, and particle creation. Phys. Rev. D. 15, 2738–2751 (1977)

[19] Avis, S.J.; Isham, C.J.; Storey, D.: Quantum Field Theory In Anti-de Sitter Space-Time. Phys. Rev. D. 18, 3565 (1978)

[20] Avis, S.J.; Isham, C.J.; Storey, D.: Quantum field theory in anti-de Sitter space-time. Phys. Rev. D. 18, 3565–3576 (1978)

[21] Lohiya, D.; Panchapakesan, N.: Massless scalar field in a de Sitter universe and its thermal flux. J. Phys. A. 11, 1963–1968 (1978)
[22] Lohiya, D.; Panchapakesan, N.: Particle emission in the de Sitter universe for massless fields with spin. J. Phys. A. 12, 533–539 (1979)
[23] Hawking, S., Page, D.: Thermodynamics Of Black Holes In Anti-de Sitter Space. Commun. Math. Phys. 87, 577–588 (1983)
[24] Otchik, V.S.: On the Hawking radiation of spin 1/2 particles in the de Sitter space-time. Class. Quantum Crav. 2, 539–543 (1985)
[25] Motolla, F.: Particle creation in de Sitter space. Phys. Rev. D. 31, 754–766 (1985)
[26] Takashi Mishima; Akihiro Nakayama: Particle production in de Sitter spacetime. Progr. Theor. Phys. 77, 218–222 (1987)
[27] Polarski, D.: The scalar wave equation on static de Sitter and anti-de Sitter spaces. Class. Quantum Grav. 6, 893–900 (1989)
[28] Bros, J.; Gazeau, J.P.; Moschella, U.: Quantum Field Theory in the de Sitter Universe. Phys. Rev. Lett. 73, 1746 (1994)
[29] Suzuki, H.; Takasugi, E.: Absorption Probability of De Sitter Horizon for Massless Fields with Spin. Mod. Phys. Lett. A. 11, 431–436 (1996)
[30] Pol'shin, S.A.: Group Theoretical Examination of the Relativistic Wave Equations on Curved Spaces. I. Basic Principles. [http://arxiv.org/abs/gr-qc/9803091] II. De Sitter and Anti-de Sitter Spaces. [http://arxiv.org/abs/gr-qc/9803092] III. Real reducible spaces. [http://arxiv.org/abs/gr-qc/9809011]
[31] Cotaescu, I.I.: Normalized energy eigenspinors of the Dirac field on anti-de Sitter spacetime. Phys. Rev. D(3) 60, 124006, 4pp (1999)
[32] Garidi, T., Huguet, E., Renaud, J.: De Sitter Waves and the Zero Curvature Limit Comments. Phys. Rev. D. 67, 124028 (2003)
[33] Moradi, S., Rouhani, S., Takook, M.V.: Discrete Symmetries for Spinor Field in de Sitter Space. Phys. Lett. B. 613, 74–82 (2005)
[34] Bachelot, A.: The Dirac equation on the Anti-de-Sitter Universe. L’équation de Dirac sur l’univers Anti-de Sitter Comptes Rendus Mathematique. 345, Issue 8, 435–440 (2007)
[35] Ovsiyuk, E.M.; Red’kov, V.M.: Spherical waves of spin 1 particle in anti de Sitter space-time. Acta Physica Polonica. B. 41, no 6. 1247–1276 (2010)
[36] Red’kov, V.M.; Ovsiyuk, E.M.: On exact solutions for quantum particles with spin S= 0,1/2, 1 and de Sitter event horizon. Ricerche di matematica. 60, no 1. 57–88 (2011)
[37] Red’kov, V.M.; Ovsiyuk, E.M.; Krylov, G.G.: Hawking Radiation in de Sitter Space: Calculation of the Reflection Coefficient for Quantum Particles. Vestnik RUDN. ser. Matemika–Informatika-Fizika. no 4. 153–169 (2012).
[38] Ovsiyuk,E.M.; Kisel, V.V.; Red’kov, V.M.: Maxwell Electrodynamics and Boson Fields in Spaces of Constant Curvature. Nova Science Publishers Inc., New York, 2014. 486 pages.
[39] Ovsiyuk, E.M.l Veko, O.V.: Spin 1/2 particle in presence ob thre Abelian monopole on the background of anti de Sitter space-time. The case $j = j_{min}$. Doklady NAN of Belarus. 55. no 6. 49–55 (2011).

[40] Ovsiyuk, E.M.l Veko, O.V.: Spin 1/2 particle in presence ob thre Abelian monopole on the background of anti de Sitter space-time. The case $j > j_{min}$. Doklady NAN of Belarus. 56. no 1. 43–49 (2011).

[41] Red’kov, V.; Ovsiyuk, E.; Veko, O.: Spin 1/2 particle in the field of Dirac string on the background of de Sitter space–time. Uzhhorod University Scientific Herald. Series Physics. no 32. 141–150 (2012)

[42] V.S. Otchik, V.M. Red’kov. Quantum-mechanival Kepler problem in spaces of constant curvature. Preprint no 298, Institut of Physics. AN BSSR, 1986. 49 pages (in Russian).

[43] S.Ju. Slavyanov, W. Lay. Special functions. A unified theory based on singularities. Oxford: Oxford Univ.Press, 2000.

[44] V.M. Red’kov, E.M. Ovsiyuk. Quantum mechanics in spaces of constant curvature. Nova Science Publishers. Inc., New York, 2012.

[45] Varshalovich, D.A., Moskalev, A.N., Hersonskiy, V.K.: Quantum theory of angular moment. Nauka, Leningrad, 1975 (in Russian)

[46] Red’kov, V.M.: Fields in Riemannian space and the Lorentz group. Publishing House ”Belarusian Science”, Minsk, 496 pages, 2009 (in Russian)

[47] Red’kov, V.M.: Tetrad formalism, spherical symmetry and Schrödinger basis. Publishing House ”Belarusian Science”, Minsk, 339 pages, 2011 (in Russian)