Evolving networks with disadvantaged long-range connections

R. Xulvi-Brunet\textsuperscript{1} and I.M. Sokolov\textsuperscript{1,2}

\textsuperscript{1}Institut für Physik, Humboldt Universität zu Berlin, Invalidenstraße 110, D-10115 Berlin, Germany
\textsuperscript{2}Theoretische Polymerphysik, Universität Freiburg, Hermann Herder Str. 3, D-79104 Freiburg, Germany

Abstract. We consider a growing network, whose growth algorithm is based on the preferential attachment typical for scale-free constructions, but where the long-range bonds are disadvantaged. Thus, the probability to get connected to a site at distance $d$ is proportional to $d^{-\alpha}$, where $\alpha$ is a tunable parameter of the model. We show that the properties of the networks grown with $\alpha < 1$ are close to those of the genuine scale-free construction, while for $\alpha > 1$ the structure of the network is vastly different. Thus, in this regime, the node degree distribution is no more a power law, and it is well-represented by a stretched exponential. On the other hand, the small-world property of the growing networks is preserved at all values of $\alpha$.

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Complex weblike structures (the small-world or scale-free networks) have recently become an object of extensive investigation, and in the last years a great success in understanding the properties of these structures was achieved (see Ref. \textsuperscript{1} as a review). Apart from appealing mathematics, this recent interest is due to the fact that many natural and technological systems, like polymer networks,\textsuperscript{2} the science collaboration network,\textsuperscript{3} or networks of chemical reactions in a living cell\textsuperscript{4,7} seem to be organized according to some internal principles. Thus, the Internet,\textsuperscript{9} the network of human sexual contacts,\textsuperscript{4,9} or the WWW\textsuperscript{11} possess a similar structure, e.g. are they all based on the preferential attachment of the newly introduced nodes to the highly connected old ones. All these networks show the small-world property: the typical distance (in terms of the number of intermediate connections) between two nodes grows logarithmically with the web’s size.

One of the prominent examples of a mathematical model of such a growing network is the scale-free (SF) construction of Barabási and Albert\textsuperscript{1,12}; and one of its most interesting properties is the very specific form of the probability distribution of the degree of nodes (i.e. of the numbers of bonds connecting any given node $i$ with other ones in the network): $P(k) \propto k^{-3}$\textsuperscript{13,15}. Many models have been presented, based on the same two most important ingredients: growth and preferential attachment. Examples are models with an accelerated growth of the network\textsuperscript{16,17}, models with a non-linear preferential attachment\textsuperscript{19}, with nodes provided by a initial attractiveness\textsuperscript{13,19,20}, with growth constraints as aging and cost\textsuperscript{19,20}, models that have a competitive aspect of the nodes\textsuperscript{21}, or models of networks that incorporate local events as the addition, rewiring or removal of nodes or edges\textsuperscript{22}.

The SF-construction may be a reasonable approximation for such world-spanning networks like one of the Internet’s information transmission channels or one of the formal links of WWW. On the other hand, in many situations (like in a network of human sexual contacts) a connection means a physical contact, i.e. means that the contacting individuals, representing the nodes of the network, have to occur at the same site and at the same time, thus introducing a clear geographical aspect. In what follows we present a simple model taking into account this metrical ("geographical") aspect, where the probability to connect two nodes depends both on the number of connections that the nodes already have (as in the genuine SF-construction), and on the distance between them. That is, we treat an emerging network in a metric space. In this emerging network the probability that a newly introduced node $n$ is connected to a previously existing node $i$ is proportional to the number $k_i$ of the already existing connections of node $i$ (preferential attachment prescription), but on the other hand the too long bonds are disadvantaged, because this probability depends on the Euclidean distance $d_{in}$ between the nodes $n$ and $i$ as $d_{in}^{-\alpha}$, (clearly, a "scale-free" function), with $\alpha > 0$.

Based on extensive numerical simulations of a one-dimensional situation, we show that even if the length penalties are mild, the model exhibits properties which differ strongly from those of the usual scale-free networks. Thus, the corresponding degree distribution function $P(k)$ depends strongly on $\alpha$. We show, in particular, that for $\alpha < 1$ the behavior of $P(k)$ is similar to the behavior of the SF model without penalties, so that asymptotically $P(k) \propto k^{-3}$, (which distribution possesses a mean, but no dispersion, and corresponds to strong, universal fluctuations). On the other hand, for $\alpha > 1$ the behavior of $P(k)$ is well-described by a stretched-exponential $P(k) \propto \exp(-bk^{\gamma})$, with the power $\gamma$ depending on $\alpha$, so that the fluctuations in $k$ are rather weak. We discuss the reasons for such a dramatic change, being rooted in the probability of connection between the nodes as function of the distance, and the overall structure of the emerging network, preserving its small-world nature even at large (probably at all) $\alpha$-values.
We start from a one-dimensional lattice of $L$ sites, spaced by a unit distance and apply cyclic boundary conditions. On this structure we will let our network grow, so that each lattice site will be a possible location of a network’s node. We denote by $n_i$ the position in the lattice of a node $i$. The distance $d_{ij}$ between any two nodes $i$ and $j$ is defined as:

\[ d_{ij} = \min\{|n_i - n_j|, (L - |n_i - n_j|)\}. \quad (1) \]

Let us now construct the network. First, we choose randomly an even number $m_0$ of sites from the lattice and we bind them in pairs with one bond each. This will be our initial condition. That is, at $t = 0$, our network will consist from $m_0$ nodes connected in pairs. As in the SF model we will add at every time step a new node to our network (linear growth). We proceed according to the following rule: at every time step we choose at random a free site of our lattice, and pose the new node there. This new node is then connected through a free site of our lattice, and is connected to an old one $i$ (instead of $n_i$). This will be our initial condition. That is, at $t = 0$, our network will consist from $m_0$ nodes connected in pairs. As in the SF model we will add at every time step a new node to our network (linear growth). We proceed according to the following rule: at every time step we choose at random a free site of our lattice, and pose the new node there. This new node is then connected through a free site of our lattice, and is connected to an old one $i$ (instead of $n_i$).

Our initial condition is slightly different from one of Barabási and Albert, where the initial $m_0$ nodes are not connected: in our case all nodes introduced at $t = 0$ have exactly one edge, which allows to use Eq. (2) from the very beginning. This simplifies the algorithm, since we do not have to distinguish between the initial and the further steps. The only difference with the genuine SF construction is that at time $t$ one has $mt + m_0/2$ edges present; hence, the asymptotic behavior of both models for $t \to \infty$ is the same.

Three examples of the evolving networks of such a kind are given in Fig. 1. Here is $m = 3$, $L = 10^6$, $N = 102$, and $m = 3$. All simulations are done for several values of $\alpha$ and $m$ are present; hence, the asymptotic behavior of both models for $t \to \infty$ is the same.

In our further simulations we use a lattice of $L = 2 \cdot 10^7$ sites; the maximal number of the introduced nodes is $N = 2 \cdot 10^5$. All simulation results are based on the average of 10 realizations of this structure. The error bars on Figs. 3-5 correspond just to this ensemble average. The simulations are done for several values of $\alpha$ and for two

We contrast with the SF model, the probability $\Pi$ for the new network. The network (a) is a genuine SF construction while (c) strongly resembles the Watts and Strogatz’s small-world network. We denote by $\alpha$ and (c)

![Networks generated using the simulation prescription, Eq.(2), with different values of $\alpha$: (a) $\alpha = 0$, (b) $\alpha = 1.5$ and (c) $\alpha = 15$. All three examples have 300 edges, $L = 10^6$, $N = 102$, and $m = 3$. Note the change in the appearance of the networks. The network (a) is a genuine SF construction while (c) strongly resembles the Watts and Strogatz’s small-world network.](image-url)
values of $m$, the number of the outgoing bonds: $m = 1$ and $m = 3$; $m_0 = 2m$.

One of the prominent features of the scale free-model is that the distribution of the degrees of the nodes decays as a power law, i.e. like $P(k) \sim k^{-\gamma}$, with $\gamma = 3$. This corresponds to the fact that the mean number of connections per site exists, but its dispersion diverges. Let us now discuss, how this distribution changes if the long-range connections are penalized. In Fig. 2 we plot the probability distribution of $k$ for different values of $\alpha$ on double logarithmic scales. One readily infers that for all $0 < \alpha < 1$ no important differences with the scale free model ($\alpha = 0$) can be detected: in any case the asymptotic behavior of $P(k)$ is well-described by $P(k) \sim k^{-3}$.

The distributions seem to be almost identical; however, small but statistically significant deviations can be detected for small $k$-values. At $\alpha \approx 1$ the degree distribution shows a pronounced change in its behavior and ceases being a power law; now the behavior of the model with distance penalties is quite different.

Let us concentrate on the case $\alpha > 1$ and try to describe the shape of the degree distribution under such conditions. The analysis of the simulations suggests that the corresponding mathematical expression could be a stretched-exponential function of the form:

$$P(k) = a \exp(-b k^\gamma),$$

where the parameters $a$, $b$, and $\gamma$ depend on $\alpha$ and $m$. To obtain the values of these parameter and to analyze the goodness of this fitting function we have fitted the data to Eq.(3) using the nonlinear least-squares Levenberg-Marquardt algorithm [23], taking into consideration the error bars as coming out of 10 realizations of each situation. The data is replotted together with the outcomes of the fits in Fig. 4 on the scales in which the fitting function, Eq.(3), is represented by a straight line. One namely takes $k^\gamma$ as the abscissa and $\ln P(k)$ as the ordinate of the graph. Fig. 4 shows that such a fit (straight line) is surprisingly good!

The values of the exponent $\gamma$ are shown as a function of $\alpha$ ($\alpha > 1$) in Fig. 3, for the two different situations corresponding to $m = 1$ and $m = 3$. We see that $\gamma$ monotonously grows with $\alpha$, and that the dependences $m = 1$ and $m = 3$ differ, i.e. that the $\gamma(\alpha)$ dependence is nonuniversal.

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FIG. 4. Shown is $\ln P(k)$ as a function of $k^\gamma$, where $\gamma$ is the output of the fit, Eq.(3). (See text for details). The parameters are: (a): $m = 1$, $\alpha = 1.5$, $\gamma = 0.37$. (b): $m = 3$, $\alpha = 1.5$, $\gamma = 0.33$. (c): $m = 1$, $\alpha = 3$, $\gamma = 0.69$. (d): $m = 3$, $\alpha = 3$, $\gamma = 0.64$. (e): $m = 1$, $\alpha = 10$, $\gamma = 1.07$. (f): $m = 3$, $\alpha = 10$, $\gamma = 0.96$.

\[
P(k) = ak^\gamma \exp(-bk).
\] (4)

We tested also this fit function and found out that it gives a good fit for larger $\alpha$-values, but is definitely inferior to our fit, Eq.(3), for $1 < \alpha < 3$.

A growing network with disadvantaged long bonds is a very interesting hierarchical construction. Thus, for large $\alpha$, the strong correlation between the age of the connection and its length exists. The old connections, made when the nodes were sparse, are typically long, while the younger connections get shorter and shorter, since more sites in the immediate vicinity of a newly introduced site
can be found. The simulations show that for $\alpha$ large, the nodes are almost surely connected to their nearest neighbors. On the other hand, the old, long-range connections are of great importance for the overall topology of the lattice, since they guarantee that for any $\alpha$ the network is a small-world one.

In Fig. 5 we plot the mean number of connections between each two nodes of the network for two different values of $\alpha$ ($\alpha = 1.5$ and $\alpha = 5$) and for the two values $m = 1$ and $m = 3$ as a function of the network size $N$. The algorithm here is trivial: starting from a node (labeled 0) we pass to all nodes connected to it (nodes of the first generation, labeled 1), then to nodes of the second generation (labeled 2), etc; until all nodes are labeled. The mean distance between this node (labeled 0) and any other given node of the network is then the sum of all values of these labels divided by $N - 1$. This procedure is repeated for each node, and the overall mean value, the so-called path diameter of the network ($l$), is evaluated. The error bars of the figure correspond to the average of the mean diameters over 10 realizations of the network.

Fig. 5 shows that the mean diameter of the network grows linearly in $\ln N$, i.e. it shows the typical small-world behavior. This behavior is preserved for all tested values of $\alpha$; the largest value tested was $\alpha = 45$, which, for $m = 1$, corresponds to a practically sure connection of a newly introduced node to its nearest neighbor. The high-$\alpha$ networks resemble closely the simple small-world constructions [23].

Let us summarize our findings. We considered a growing network, whose growth algorithm is based, as in the SF construction, on a preferential attachment of the newly introduced nodes to the highly connected old ones. However, here the too long connections are disadvantaged by introducing penalties. Thus, the probability to connect two nodes separated by a distance $d$ is proportional to $d^{-\alpha}$, where $\alpha$ is a variable parameter. We found out that for $\alpha < 1$ the degree distribution $P(k)$ decays, as in the SF model, like $P(k) \sim k^3$, whereas for $\alpha > 1$ a stretched exponential form $P(k) = a \exp(-bk^\gamma)$ gives an extremely good description of this distribution. On the other hand, the small-world property is preserved at all checked values of $\alpha$.

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