THE RIGIDITY THEOREM OF
FANO–SEGRE–ISKOVSKIKH–MANIN–CORTI–PUKHLIKOV–
CHELTSOV–DE FERNEX–EIN–MUSTA_TA–ZHUANG

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Abstract. We prove that $n$-dimensional smooth hypersurfaces of degree $n+1$
are superrigid. Starting with the work of Fano in 1915, the proof of this
Theorem took 100 years and a dozen researchers to construct. Here I give
complete proofs, aiming to use only basic knowledge of algebraic geometry
and some Kodaira type vanishing theorems.

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The classification theory of algebraic varieties—developed by Enriques for sur-
faces and extended by Iitaka and then Mori to higher dimensions—says that every
variety can be built from 3 basic types:

- (General type) $K_X$ is ample,
- (Calabi-Yau) $K_X$ is trivial and
- (Fano) $-K_X$ is ample.

Moreover, in the Fano case the truly basic ones are those that have class number
equal to 1. That is, every divisor $D$ on $X$ is linearly equivalent to a (possibly
rational) multiple of $-K_X$.

If 2 varieties $X_1, X_2$ on the basic type list are birationally equivalent then they
have the same type. In the general type case they are even isomorphic and in
the Calabi-Yau case the possible birational maps are reasonably well understood,
especially for 3-folds, see [Kol89, Kol91].

By contrast, Fano varieties are sometimes birationally equivalent in quite unex-
pected ways and the Noether–Fano method aims to understand what happens.
Definition 1. I call a Fano variety $X$ with class number 1 \textit{weakly superrigid} if every birational map $\Phi : X \dasharrow Y$ to another Fano variety $Y$ with class number 1 is an isomorphism.

The adjective “weakly” is not standard. The definition of \textit{superrigid} is similar, but allows $Y$ to have terminal singularities and to be a Mori fiber space; see \cite{Puk95, Che05}. There are Fano varieties $X$, especially in dimensions 2 and 3, that are not birational to any other Fano variety but there are many birational self-maps $\Phi : X \dasharrow X$ that are not isomorphisms. Such Fano varieties are called \textit{rigid}. It seems to me that superrigidity is the more basic notion, though, in dimension 3, the theory of rigid Fano varieties is very rich.

The aim of these notes is to explain the proof of the following theorem. From now on we work over a field of characteristic 0. It is not important, but we may as well assume that it is algebraically closed.

Main Theorem 2. Every smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of dimension $n \geq 3$ and of degree $n+1$ is weakly superrigid.

Equivalently, every birational map $\Phi : X \dasharrow Y$ to another smooth Fano variety $Y$ with Picard number 1 is an isomorphism.

If $n = 2$ then $X$ is a cubic surface, hence it has class number 7. However, if the base field is not algebraically closed, it frequently happens that $X$ has class number 1, in which case it is weakly superrigid by \cite{Seg43}; see \cite{KSC04, Chap.2} for a modern treatment.

3 (The history of Theorem 2). The first similar result is Max Noether’s description of all birational maps $\mathbb{P}^2 \dasharrow \mathbb{P}^2$ \cite{Noe1870}, whose method formed the basis of all further developments.

Theorem 2 was first stated by Fano for 3-folds \cite{Fan1908, Fan1915}. His arguments contain many of the key ideas, but they also have gaps. I call this approach the \textit{Noether–Fano method}. The first complete proof for 3-folds, along the lines indicated by Fano, is in Iskovskikh-Manin \cite{IM71}. Iskovskikh and his school used this method to prove similar results for many other 3-folds, see \cite{Isk79, Sar80, IP99, Isk01}. This approach was gradually extended to higher dimensions by Pukhlikov \cite{Puk87, Puk98, Puk02} and Cheltsov \cite{Che00}. These results were complete up to dimension 8, but needed some additional general position assumptions in higher dimensions. A detailed survey of this direction is in \cite{Puk13}.

The theory of Fano varieties may be the oldest topic of higher dimensional birational geometry, but for a long time it grew almost independently of Mori’s Minimal Model Program. The Fano–Iskovskikh classification of Fano 3–folds using extremal rays and flops was first treated by Mori \cite{Mor83} and later improved by Takeuchi \cite{Tak89}.

The Noether–Fano method and the Minimal Model Program were brought together by Corti \cite{Cor95}. Corti’s technique has been very successful in many cases, especially for 3–folds; see \cite{CR00} for a detailed study and \cite{KSC04, Chap.5} for an introduction. However, usually one needs some special tricks to make the last steps work, and a good higher dimensional version proved elusive for a long time.

New methods involving multiplier ideals were introduced by de Fernex-Ein-Mustată \cite{dFEM03}; these led to a more streamlined proof that worked up to dimension 12. The proof of Theorem 2 was finally completed by de Fernex \cite{dF16}.
The recent paper of Zhuang [Zhu18] makes the final step of the Corti approach much easier in higher dimensions. The papers [SZ18, Zhu18, LZ18] contain more general results and applications.

The name Fano–Segre–Iskovskikh–Manin–Corti–Pukhlikov–Cheltsov–deFernex–Ein–Mustaţă–Zhuang theorem was chosen to give credit to all those with a substantial contribution to the proof, though this underemphasizes the major contributions of Fano, Iskovskikh and Pukhlikov.

The proofs in the theory are designed to prove superrigidity, and the optimal version of Theorem 2. says that a smooth hypersurface \( X \subset \mathbb{P}^{n+1} \) of dimension \( \geq 3 \) is superrigid if and only if \( \deg X = n + 1 \); see [Che05a, Puk13]. The proof of this version needs only some new definitions and very minor changes in Step 1.1.

The methods apply to many other Fano varieties for which \(-K_X\) is a generator of the class group; see [Puk13, Zhu18] for several examples. One of the big challenges is to understand what happens if \(-K_X\) is a multiple of the generator, see [Puk16].

Open problems about hypersurfaces

The following questions are stated in the strongest forms that are consistent with the known examples. I have no reasons to believe that the answer to either of them is positive and there may well be rather simple counter examples. As far as I know, there has been very little work on low degree hypersurfaces beyond cubics in dimension 4.

**Question 4.** Is every smooth hypersurface of degree \( \geq 4 \) non-rational?

**Question 5.** Is every smooth hypersurface of degree \( \geq 5 \) weakly superrigid?

Here \( \geq 5 \) is necessary since there are some smooth quartics with nontrivial birational maps.

**Example 6.** Let \( X \subset \mathbb{P}^{2n+1} \) be a quartic hypersurface that contains 2 disjoint linear subspaces \( L_1, L_2 \) of dimension \( n \). For every \( p \in \mathbb{P}^{2n+1} \setminus (L_1 \cup L_2) \) there is a unique line \( \ell_p \) through \( p \) that meets both \( L_1, L_2 \). This line meets \( X \) in 4 points, two of these are on \( L_1, L_2 \). If \( p \in X \) then this leaves a unique 4th intersection point, call it \( \Phi(p) \). Clearly \( \Phi \) is an involution which is not defined at \( p \) if either \( p \in L_1 \cup L_2 \) or if \( \ell_p \subset X \).

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1. RIGIDITY AND SUPERRIGIDITY, AN OVERVIEW

In the following outlines and in subsequent Sections I aim to put the pieces together and write down a simple proof of the superrigidity of smooth hypersurfaces \( V^n_{n+1} \subset \mathbb{P}^{n+1} \), where most steps are either easy or direct applications of some general principle of the Minimal Model Program.

The key notion we need is canonical and log canonical pairs involving linear systems.
Definition 7 (Log resolution). Assume that we have a variety $X$, a linear system $M$ on $X$ and a divisor $D$ on $X$. Somewhat sloppily, we say that a log resolution of these data is a proper birational morphism $\pi : X' \to X$ such that

1. $X'$ is smooth,
2. $\pi^*|M| = |M'| + B$ where $|M'|$ is base-point free and $B$ is the fixed part of $\pi^*|M|$ and
3. $B + \pi^{-1}(D) + \text{Ex}(\pi)$ is a simple normal crossing divisor.

(Here Ex($\pi$) denotes the exceptional set of $\pi$ and simple normal crossing means that the irreducible components are smooth and they intersect transversally. The adjective “log” loosely refers to condition (3).)

The existence of log resolutions was proved by Hironaka; see [Kol07 Chap.3] for a recent treatment.

Definition 8. Let $X$ be a smooth variety and $|M|$ a linear system on $X$. Let $\pi : X' \to X$ be a log resolution of $|M|$ as in Definition 7. Write $\pi^*|M| = |M'| + B$ where $|M'|$ is base-point free and $B$ is the fixed part of $\pi^*|M|$, and $K_{X'} \sim \pi^*K_X + E$ where $E$ is effective and $\pi$-exceptional. For any nonnegative rational number $c$ we can thus formally write

$$K_{X'} + c|M'| \sim_\mathbb{Q} \pi^*(K_X + c|M|) + (E - cB),$$

where $A_1 \sim_\mathbb{Q} A_2$ means that $N \cdot A_1$ is linearly equivalent to $N \cdot A_2$ for some $N > 0$.

A pair $(X, c|M|)$ is called canonical (resp. log canonical) if every divisor appears in $E - cB$ with coefficient $\geq 0$ (resp. $\geq -1$). This is independent of the log resolution [KM98 2.32].

If $|M|$ is base point free then $B = 0$, thus $(X, c|M|)$ is canonical for any $c$. In all other cases, $(X, c|M|)$ is canonical (resp. log canonical) for small values of $c$ but not for large values. (The transitional value of $c$ is called the canonical (resp. log canonical) threshold.) Roughly speaking, small thresholds correspond to very singular linear systems.

A divisor $E_i$ on $X'$ that appears in $E - cB$ with coefficient $< 0$ (resp. $< -1$) is a non-canonical divisor (resp. a non-log-canonical divisor) of $c|M|$. I call their images $\pi(E_i) \subset X$ non-canonical centers (resp. non-log-canonical centers) of $c|M|$ or of $(X, c|M|)$.

Using Remark 9 these formulas also define the above notions for pairs $(X, \Delta)$ where $\Delta$ is a divisor and pairs $(X, I^c)$ where $I$ is an ideal sheaf.

Remark 9 (Divisors, linear systems and ideal sheaves). Much of the Minimal Model Program literature works with pairs $(X, \Delta)$ where $\Delta$ is a divisor (with rational or real coefficients), see [KM98] [Kol13]. For rigidity questions, the natural object seems to be a pair $(X, c|M|)$ where $|M|$ is a linear system and $c$ is a rational or real coefficient. It is easy to see that if $c \in [0, 1)$ (which will always be the case for us) and $D \in |M|$ is a general divisor then the definitions and theorems for $(X, c|M|)$ and $(X, cD)$ are equivalent.

Let $X$ be an affine variety and $I \subset O_X$ an ideal sheaf. Many papers, for example [FJLM03], work with pairs $(X, F^c)$ where $c$ is viewed as a formal exponent. If $I$ is generated by global sections $g_1, \ldots, g_m$, we can consider the linear system $|M| := |\sum \lambda_i g_i = 0|$. Again we find that the definitions and theorems for $(X, c|M|)$ and $(X, F^c)$ are equivalent.

Here I follow the language of linear systems, since this seems best suited to our current aims. I will also always assume that $c$ is rational. This is always the case in
our applications and makes some statements simpler. However, it does not cause any essential difference at the end.

We discuss the canonical and log canonical property of linear systems in detail in Section 4. For now we mainly need to know that canonical means mild singularities and log canonical means somewhat worse singularities. In some sense the main question of the theory was how to describe these properties in terms of other, better understood, measures of singularities.

10 (Main steps of the proof). The proof can be organized into 6 fairly independent steps. Roughly speaking, Steps 1, 2 and 5 are essentially in the works of Fano, at least for 3-folds. Steps 3 and 4 are substantial reinterpretations of the classical ideas while Step 6 is a new way of finishing the proof.

For the rest of this section I write $Y$ for a smooth, projective variety, $X$ for a smooth, projective Fano variety with class number 1 and $V$ (or $V_{n+1}$ or $V_{n+1}^n$) for a smooth hypersurface $V_{n+1} \subset \mathbb{P}^{n+1}$ of degree $n + 1$ and of dimension $n \geq 3$. The base field has characteristic 0.

Step 10.1 (Noether-Fano criterion, Section 2). A smooth Fano variety $X$ of class number 1 is (weakly) superrigid if for every movable linear system $|M| \subset |-mK_X|$ the pair $(X, \frac{1}{m}|M|)$ is canonical.

Comments 10.1.1. Movable means that there are no fixed components, some authors use mobile instead. If dim $X = 2$ then $(X, \frac{1}{m}|M|)$ is not canonical iff $\text{mult}_x|M| > m$ for some point $x \in X$ by Lemma 27; this equivalence made Noether’s proof work well. If dim $X = 3$ then Fano tried to prove that if $(X, \frac{1}{m}|M|)$ is not canonical then either $\text{mult}_C|M| > m$ for some curve $C \subset X$ or $\text{mult}_x|M| > 2m$ for some point $x \in X$. Fano understood that the latter condition for points is not right, one needs instead only a consequence of it: The local intersection number at $x$ is $(M \cdot M \cdot H)_x > 4m^2$, where $H$ is a hyperplane through $x$. In higher dimensions it does not seem possible to define canonical in terms of just multiplicities and intersection numbers, this is one reason why the above form of Step 10.1 was established only in [Cor95]. We prove Step 10.1 in Theorem 14. Although historically the notion of “canonical” was defined starting from varieties of general type (see [Rei80, KM98]), the Noether-Fano criterion leads to the exact same definition.

If $X$ is not (weakly) superrigid then there is a movable linear system $|M| \subset |-mK_X|$ such that $(X, \frac{1}{m}|M|)$ is not canonical, thus it has some non-canonical divisors and centers as in Definition 8. (The “worst” non-canonical centers are called maximal centers by the Iskovskikh school.) From now on we focus entirely on understanding movable linear systems and their possible non-canonical centers on $X$. There are 2 persistent problems that we encounter.

- We can usually bound the multiplicities of $|M|$, but there is a gap—growing with the dimension—between multiplicity and the canonical property.
- We are better at understanding when a pair is log canonical, instead of canonical.

While we try to make statements about arbitrary Fano varieties, at some point we need to use special properties of the $V_{n+1}^n$. The following bounds, going back to Fano and Segre, were put into final form by Pukhlikov [Puk02, Prop.5] and later generalized by Cheltsov [Che05b, Lem.13] and Suzuki [Suz17, 2.1] to complete intersections.
Step 10.2 (Multiplicity bounds, Fano, Segre, Pukhlikov, Section 3) Let $Y \subset \mathbb{P}^{n+1}$ be a smooth hypersurface and $[H]$ the hyperplane class. Let $D \in [mH]$ be a divisor, $[M] \subset [mH]$ a movable linear system and $Z \subset Y$ an irreducible subvariety.

(a) If $\dim Z \geq 1$ then $\text{mult}_Z D \leq m$.
(b) If $\dim Z \geq 2$ then $\text{mult}_Z (M \cdot M) \leq m^2$,

where $M \cdot M$ denotes the intersection of 2 general members of $[M]$.

Comments 10.2.1. Note that (10.1) works with $[M] \subset |−mK_X|$ and (10.2) with $[M] \subset [mH]$. The two match up iff $−K_X \sim H$; the latter holds for $X = V_{n+1}^n$, the case that we are considering. In general, the method works best for those Fano varieties where every divisor is an integral multiple of $−K_X$ (up to linear equivalence).

As an easy corollary of (10.2.a) we get that $(Y, \frac{1}{m}D)$ is canonical, except at a finite point set $P \subset Y$. We already mentioned this in Comments 10.1.1; see Lemma [27] or [KSC04, 6.18] for proofs.

Relating (10.2.b) to the canonical or log canonical property was less obvious; it was done by Corti [Cor00, 3.1] (see also [KSC04, Sec.6.6]), then very much generalized by de Fernex-Ein-Mustaţă [dFEM03] and sharpened by Y. Liu [Liu16].

Step 10.3 (Non-canonical centers and multiplicity, Corti, Section 4) Let $[M]$ be a movable linear system on a smooth variety $Y$ and $Z \subset Y$ a non-canonical center of $(Y, \frac{1}{m}[M])$. Then

(a) $\text{mult}_Z [M] > m$ and
(b) $\text{mult}_Z (M \cdot M) > 4m^2$ if $\text{codim}_Y Z \geq 3$.

Comments 10.3.1. Using a by now standard method called inversion of adjunction, which we discuss in Section 5, both parts follow from claims about linear systems on algebraic surfaces:

Claim 10.3.2. Let $[M]$ be a movable linear system on a smooth surface $S$.

(a') If $s \in S$ is a non-canonical center of $(S, \frac{1}{m}[M])$ then $\text{mult}_s [M] > m$.
(b') If $s \in S$ is a non-log-canonical center of $(S, \frac{1}{m}[M])$ then $\text{mult}_s (M \cdot M) > 4m^2$.

There are 2 ways to use (10.3.2.b). The first gives a lower bound on some intersection numbers. Namely, we get that if $x \in X$ is a 0-dimensional non-canonical center on a 3-fold then $\text{mult}_x (M \cdot M) > 4m^2$. Thus if $H$ is a general hyperplane passing through $x$ then $(M \cdot M \cdot H) > 4m^2$. On the other hand, if $X$ is a quartic 3-fold then $M \sim mH$ and hence $(M \cdot M \cdot H) = 4m^2$, a contradiction. (One dimensional non-canonical centers are excluded by (10.2.a) and (10.3.a).) Thus smooth quartic 3-folds are superrigid.

It would be very nice to continue the claims (10.3.a–b) to stronger and stronger inequalities for higher codimension non-canonical centers. This was done in [dFEM03]. This is very useful if by chance the base locus of $[M]$ has codimension $> 2$. However, in almost all cases the base locus of $[M]$ has codimension 2 and it is not easy to apply the estimates of [dFEM03] directly.

Thus the method so far works well only if $−K_X \sim H$ and $(−K_X)^n \leq 4$. Among hypersurfaces in $\mathbb{P}^{n+1}$, this holds only for the quartic 3-folds. However there are smooth hypersurfaces in weighted projective spaces with these properties. For example, fix $m > 1$ and let $X$ be a smooth hypersurface of degree $4m + 2$ and dimension $2m$ in the weighted projective space $\mathbb{P}(1^{2m}, 2, 2m+1)$ (the notation means
that we have $2m$ coordinates of weight 1, see [KSC04, 3.48] for an introduction).
Then $-K_X \sim H$ and $(-K_X)^{2m} = 1$. With small changes the method proves that they are superrigid; see [KSC04, 5.22] for details.

We need a new way of exploiting the tension between the estimates (10.2.b) and (10.3.b). This is the other way of using (10.3.2.b’).

**Step 10.4** (Doubling the linear system, de Fernex, Sections 11.3) Let $Y \subset \mathbb{P}^{n+1}$ be a smooth hypersurface and $M \subset |mH|$ a movable linear system. Then

(a) $(Y, \frac{1}{m}|M|)$ is canonical outside a finite set of points $P \subset X$ and

(b) $(Y, \frac{1}{m}|2M|)$ is log canonical outside a finite set of curves $C \subset X$.

**Comments 10.4.1.** Note the key shift of having $|2M|$ instead of $|M|$ in part (b).

As an illustration, consider the linear system $|x, x - y'|$ in the plane. A general member of it is a smooth curve, and in $\frac{1}{m}|M|$ we take this curve with coefficient $\frac{1}{m}$. The $y'$ term seems to play no role. By contrast the linear system $|2M|$ is $|x^2, x(x - y'), (x - y')^2|$, its general member is (after a coordinate change) of the form $x^2 - y'^2$. Now we see both the original smoothness (since $x^2$ is there) and the order of tangency between two members of $|M|$ (shown by $y'^2$). It turns out that this is computationally not important since $(X, \frac{1}{m}|2M|)$ is log canonical iff $(X, \frac{1}{m}|M|)$ is log canonical. However, conceptually it seems clearer to me that information about intersections of 2 divisors in $|M|$ is now visible on individual divisors in $|2M|$.

We still have a problem with the lack of control along the curves $C$ in (10.4.4.b). We go around it by taking a hyperplane section.

**Step 10.5** (Cutting down to isolated centers, Section 5) Assume that $(Y, \frac{1}{m}|M|)$ satisfies the properties (10.4.a–b) and $(Y, \frac{1}{m}|M|)$ is not canonical at some point $p \in Y$. Let $W$ be a general member of a very ample linear system $|H|$ that passes through $p$. Then

(a) $(W, \frac{1}{m}|M|_W)$ is not log canonical at $p$ and

(b) $(W, \frac{1}{m}|2M|_W)$ is log canonical outside a finite set of points $p \in P \subset W$.

**Comments 10.5.1.** The multiplicity version of this goes back to Fano, but the above form may have been first made explicit in [Cor00].

One should think of (10.5) as saying that $|2M|_W$ is very singular at $p$ and at most half as singular everywhere else. This is quite important in applications. Usually we can get a lot of information if we have an isolated “very singular” point $p$ but much less if $p$ is a limit of other singular points that are “almost as singular.”

Cutting down has a very curious effect on the singularities. If $(X, c|M|)$ is a canonical (resp. log canonical) pair then its restriction to a general member of a base point free linear system is still canonical (resp. log canonical); see (3.4.1). Applying this to $X := Y \setminus \{p\}$ gives (b).

However, if $p \in X$ is a non-canonical center of $(X, c|M|)$ and $W$ passes through $p$ then $(W, c|M|_W)$ is not log canonical at $p$. That is, the singularity is made worse by restriction to a general hypersurface through a non-canonical center. This is in marked contrast with multiplicity, which is preserved by such restrictions. We discuss this in Section 5.

From now on we work with $(W, \frac{1}{m}|M|_W)$ and try to reach a contradiction.
Building on [dFCM04], [dF16] focuses on bounding multiplicities and valuations of exceptional divisors. The new observation of [Zhu18] is that focusing on the multiplier ideal of a multiplier ideal gives a very strong bound.

**Step 10.6** (Zhuang, Section 6) Let \( Y \) be a smooth projective variety of dimension \( d \) and \( L \) an ample divisor on \( Y \). Further let \( |M| \subset |mL| \) be a movable linear system and \( P \subset Y \) a finite (nonempty) subset of \( Y \). Assume that

- \( (Y, \frac{1}{m} |M|) \) is not log canonical at some \( p \in P \), but
- \( (Y, \frac{1}{m} |2M|) \) is log canonical outside \( P \).

Then

\[
\h^0(Y, \mathcal{O}_Y(K_Y + 2L)) \geq \frac{1}{4}3^d.
\]

**Comments 10.6.1.** One should think of this as saying that if \( |M| \) is more singular at a finite set of points than elsewhere then the linear system \( |K_Y + 2L| \) is large. I stated the case where we compare \( \frac{1}{m} |M| \) and \( \frac{1}{m} |2M| \), the complete version in [Zhu18] also applies if we have \( \frac{1}{m} |M| \) and \( \frac{1}{mc} |M| \) for some \( c > 0 \).

It is quite remarkable that there is also a rather easy converse.

Let \( |L| \) be any linear system on \( Y \) and \( y \in Y \) a point. If \( \dim |2L| \geq \left( \frac{3d}{2d} \right) \) then there is a linear subsystem \( |N| \subset |2L| \) that has multiplicity \( \geq 2d \) at \( y \). In particular, \( (Y, \frac{1}{2} |N|) \) is not log canonical at \( y \). As \( d \to \infty \), \( \left( \frac{3d}{2d} \right) \) grows like \( 6.75^d \).

Thus if \( \dim |2L| \geq 6.75^d \) then we can find a linear system \( |N| \) that satisfies (10.6.a), and usually also (10.6.b). Informally we can restate (10.6) as

**Principle 10.6.2.** There are no accidental isolated singularities.

11 (Proof of Theorem 2 using (10.1–6)). Assuming the contrary, by Step 10.1 we get \( (V, \frac{1}{m} |M|) \) that is not canonical. Thus Steps 10.3–4 give a \( W = W_{n+1}^+ \subset \mathbb{P}^n \) and an \( |M|_W \) such that

- (1) \( (W, \frac{1}{m} |M|_W) \) is not log canonical at finitely many points \( P \subset W \), but
- (2) \( (W, \frac{1}{m} |M|_W) \) is log canonical outside \( P \).

By Step 10.6 this implies that

\[
\h^0(W, \mathcal{O}_W(K_W + 2H)) \geq \frac{1}{4}3^{n-1}.
\]

On the other hand \( \h^0(W, \mathcal{O}_W(K_W + 2H)) = \h^0(W, \mathcal{O}_W(2H)) = \h^0(\mathbb{P}^n, \mathcal{O}_p(2)) = \binom{n+2}{2} \), so

\[
\binom{n+2}{2} \geq \frac{1}{4}3^{n-1}.
\]

The left hand side is quadratic in \( n \), the right hand side is exponential, so for \( n \gg 1 \) this cannot hold. (In fact, we have a lot of room, leading to many other cases where the method applies in large dimensions; see [Zhu18].)

By direct computation, we get a contradiction for \( n \geq 5 \), hence we get the superrigidity of \( V_{n+1}^+ \subset \mathbb{P}^{n+1} \) for \( n \geq 5 \).

One can improve the lower bound in (10.6) to \( \frac{1}{4}3^d + \frac{3}{2} \), and then for \( n = 4 \) we get an equality \( \binom{6}{2} = 15 = \frac{1}{4}3^3 + \frac{3}{2} \). So there is no contradiction, but it is quite likely that a small change can make the proof work.

However, the \( n = 3 \) case does not seem to follow.

12 (Attribution of the Steps). In rereading many of the contributions to the proof I was really struck by how gradual the progress was and how difficult it is to attribute various ideas to a particular author or paper.

\[\square\]
Fano’s papers are quite hard to read, and some people who spent years on trying to learn from them came away with feeling that Fano got most parts of the proof wrong. Others who looked at Fano’s works feel that he had all the essential points right. In particular, the attribution of Step 2 has been controversial.

I think of Corti’s work as a major conceptual step forward, but some authors felt that it did not add anything new, at least initially. The idea of doubling the linear system is in retrospect already in [Cor00], in fact I had a hard time formulating Steps 3 and 4 in way that shows the difference between them meaningfully. In turn [dF16] contains most of the ingredients of Step 6. Nonetheless, at least in hindsight, each of the steps represents a major new idea, though this was not always immediately understood.

No doubt several people will feel that my presentation is flawed in many ways. Luckily the reader can consult the excellent survey [Che05a] and books [CR00, Puk13] for different viewpoints.

13 (What is missing?). My aim was to write down a proof of Theorem 2 that is short and focuses on the key ideas. My preference is for steps that follow from general results and techniques of the MMP. Thus several important developments have been left out.

After proving rigidity for quartic 3–folds, the Russian school went on to study other Fano 3–folds. They found that they are frequently rigid but not superrigid and the main question is how to find generators for Bir($X$). The contributions of Iskovskikh, Sarkisov, Pukhlikov and Cheltsov are especially significant. These results and their higher dimensional extensions are surveyed in [Che05a, Puk13].

The first major applications of the Corti method were also in dimension 3, see [CR00] for a survey and [HM13] for a higher dimensional extension.

In our proof we need to understand 0-dimensional log canonical centers, but the theory of arbitrary log canonical centers has been quite important in higher dimensional geometry. The first structure theorem was proved by Ambro [Amb03]; see [Kol13, Chaps.4–5] and [Fuj17] for recent treatments and generalizations.

2. THE NOETHER-FANO METHOD

We start the proof of Theorem 2 by establishing Step 10.1.

Theorem 14 (Noether-Fano inequality). Let $\Phi : X \dasharrow X'$ be a birational map between smooth Fano varieties of class number $1$. Then

1. either $\Phi$ is an isomorphism,
2. or there is a movable linear system $|M| \subset |−mK_X|$ for some $m > 0$ on $X$ such that $(X, \frac{1}{m}|M|)$ is not canonical.

Proof. Let $Z$ be the normalization of the closure of the graph of $\Phi$ with projections $p : Z \to X$ and $q : Z \to X'$. Pick any base-point-free linear system $|M'| \subset |−m'K_X'|$ and let $|M| := \Phi^{-1}|M'|$ denote its birational transform on $X$. Set $|M_Z| = q^*|M'|$. Since the class number of $X$ is 1, $|M| \sim_{\mathbb{Q}} −mK_X$ for some $m > 0$. (If $m$ is not an integer, we replace $|M'|$ by a suitable multiple. Thus we may as well assume that $|M| \subset |−mK_X|$.) We define a $q$-exceptional divisor $E_q$ and $p$-exceptional divisors $E_p, F_p$ by the formulas

$$K_Z = q^*K_{X'} + E_q, \quad |M_Z| = q^*|M'|$$

and

$$K_Z = p^*K_X + E_p, \quad |M_Z| = p^*|M| − F_p.$$
Since $X', \ X$ are smooth, $E_q, E_p$ are effective (cf. [Sha74, III.6.1]) and $F_p$ is effective since $p_*|M_Z| = |M|$. For any rational number $c$ we can rearrange (14.3) to get
\begin{align*}
K_Z + c|M_Z| &\sim_\mathbb{Q} q^*(K_{X'} + c|M'|) + E_q \quad \text{and} \\
K_Z + c|M_Z| &\sim_\mathbb{Q} p^*(K_X + c|M|) + E_p - cF_p. \quad (14.4)
\end{align*}
First we set $c = \frac{1}{m}$. Then $K_{X'} + \frac{1}{m}|M'| \sim_\mathbb{Q} 0$, hence
\[K_Z + \frac{1}{m}|M_Z| \sim_\mathbb{Q} q^*(K_{X'} + \frac{1}{m}|M'|) + E_q \sim_\mathbb{Q} E_q \geq 0.\]
Pushing this forward to $X$ we get that
\[K_X + \frac{1}{m}|M| = p_*\left(K_Z + \frac{1}{m}|M_Z|\right) \sim_\mathbb{Q} p_*(E_q) \geq 0.\]
Since
\[p_*(E_q) \sim_\mathbb{Q} K_X + \frac{1}{m}|M| \sim_\mathbb{Q} K_X - \frac{1}{m}mK_X = \frac{m-m'}{m}(\mathcal{K}_X), \quad (14.5)\]
we see that $m \geq m'$. Next set $c = \frac{1}{m}$. Then we get that
\[K_Z + \frac{1}{m}|M_Z| \sim_\mathbb{Q} p^*(K_X + \frac{1}{m}|M|) + E_p - \frac{1}{m}F_p \sim_\mathbb{Q} E_p - \frac{1}{m}F_p.\]
Pushing this forward to $X$ yields
\[K_{X'} + \frac{1}{m}|M'| = q_*(K_Z + \frac{1}{m}|M_Z|) \sim_\mathbb{Q} q_*(E_p) - \frac{1}{m}F_p.\]
As in (14.5) we obtain that
\[\frac{m-m'}{m}(\mathcal{K}_X') \sim_\mathbb{Q} K_{X'} + \frac{1}{m}|M'| \sim_\mathbb{Q} q_*(E_p) - \frac{1}{m}F_p. \quad (14.6)\]
**Basic alternative** (14.7).
- If $E_p - \frac{1}{m}F_p$ is not effective, then we declare the linear system $|M|$ to be "very singular." In our terminology, $(X, \frac{1}{m}|M|)$ is not canonical. This is case (2).
- If $E_p - \frac{1}{m}F_p$ is effective, then we declare the linear system $|M|$ to be "mildly singular." In our terminology, $(X, \frac{1}{m}|M|)$ is canonical. We need to prove that in this case $\Phi$ is an isomorphism.

Thus assume from now on that $E_p - \frac{1}{m}F_p$ is effective. Then (14.6) implies that $m' \geq m$. Combining it with (14.5) gives that $m' = m$ and then (14.5) shows that $p_*(E_q) = 0$. That is, $\text{Supp} \ E_q$ is $p$-exceptional. Since $X'$ is smooth, the support of $E_q$ is the whole $q$-exceptional divisor $\text{Ex}(q)$. Thus every $q$-exceptional divisor is also $p$-exceptional.

To see the converse, let $D \subset Z$ be an irreducible divisor that is not $q$-exceptional. Then $q_*(D) \sim_\mathbb{Q} r|M'|$ for some $r > 0$. Thus
\[r|M_Z| \sim_\mathbb{Q} q^*(r|M'|) \sim_\mathbb{Q} D + (q\text{-exceptional divisor}).\]
Pushing forward to $X$ now gives that $r|M| \sim_\mathbb{Q} p_*(D)$, since every $q$-exceptional divisor is also $p$-exceptional. Here $p_*(D) \neq 0$ since $r > 0$, so $D$ is not $p$-exceptional. This shows that $\text{Ex}(p) = \text{Ex}(q)$.

Finally set $Z := p(\text{Ex}(p)) \subset X$, $Z' := q(\text{Ex}(q)) \subset X'$ and apply the following result of Matsusaka and Mumford [MM64] to conclude that $\Phi$ is an isomorphism. $\Box$
Lemma 15. Let $\Psi : Y \rightarrow Y'$ be a birational map between smooth projective varieties. Let $Z \subset Y$ and $Z' \subset Y'$ be closed sets of codimension $\geq 2$ such that $\Psi$ restricts to an isomorphism $Y \setminus Z \cong Y' \setminus Z'$. Let $H$ be an ample divisor on $Y$ such that $H' := \Psi^* H$ is also ample. Then $\Psi$ is an isomorphism.

Proof. We may assume that $H$ and $H'$ are both very ample. Then $|H'| = |H_Y \setminus Z| = \Psi_* |H_Y \setminus Z| = \Psi_* |H|$. Thus $\Psi_* |H|$ is base point free, hence $\Psi^{-1}$ is everywhere defined. The same argument, with the roles of $Y, Y'$ reversed, shows that $\Psi$ is also everywhere defined. So $\Psi$ is an isomorphism. □

Remark 16. The proof of Theorem 14 also works if $X$ has canonical singularities, $X'$ has terminal singularities and they both have class number 1.

3. Subvarieties of hypersurfaces

Our aim is to prove that a subvariety of a smooth hypersurface can not be unexpectedly singular along a large dimensional subset. The claim and the method go back to Fano and Segre; the first complete statement and proof is in [Puk02, Prop.5].

Theorem 17. Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface and $Z \subset W \subset X$ irreducible subvarieties such that $\dim Z + \dim W \geq \dim X$. Then

$$\text{mult}_Z W \leq \frac{\deg W}{\deg X}. \tag{17.1}$$

We define the multiplicity $\text{mult}_Z W$ in Paragraph 19. See [Che05b, Lem.13] and [Suz17, 2.1] for generalizations of the theorem to complete intersections.

Remark 18. The simplest special case of the theorem is when $W$ is an intersection of $X$ with a hyperplane. Then $\deg W = \deg X$ hence we claim that $W$ has only finitely many singular points. Equivalently, a given hyperplane can be tangent to a smooth hypersurface only at finitely many points. I encourage the reader to prove this, there are very easy proofs but also messy ones. Note that this is truly a projective statement. For example, $(z - y^2 x = 0)$ is a smooth surface in $\mathbb{A}^3$ and the plane $z = 0$ is tangent to it everywhere along the $x$-axis.

Consider next the case when $W$ is an intersection of $X$ with a hypersurface of degree $d$. Then (17.1) says that $W$ has multiplicity $\leq d$ at all but finitely many of its points $p$. The easy geometric way to prove this would be to find a line $\ell$ in $X$ that passes through $p$ but not contained in $W$. This sounds like a reasonable plan if $\deg X \leq n$, since in these cases there is a line through every point of $X$, see [Kol96, V.4.3]. However, if $\deg X \geq 2n$ then a general $X$ does not contain any lines [Kol96, V.4.3].

In Proposition 21 as replacements of lines, we construct certain auxiliary subvarieties $Z^*$ that have surprisingly many intersections with $W$.

The assumption $Z \neq W$ is necessary. Indeed, there are smooth hypersurfaces $X \subset \mathbb{P}^{2n+1}$ that contain a linear space $L$ of dimension $n$. Setting $Z = W = L$ we get that $\text{mult}_Z W = 1$ but $\frac{\deg W}{\deg X} = \frac{1}{\deg X}$. 
19 (Multiplicity). The simplest measure of a singularity is its multiplicity. Let $X = (h = 0) \subset A^n$ be an affine hypersurface and $p = (p_1, \ldots, p_n)$ a point on $X$. We can write the equation as

$$h = \sum a_{i_1, \ldots, i_n} (x_1 - p_1)^{i_1} \cdots (x_n - p_n)^{i_n}.$$  

The multiplicity of $X$ at $p$, denoted by $\text{mult}_p X$, is defined as

$$\text{mult}_p X := \min \{i_1 + \cdots + i_n : a_{i_1, \ldots, i_n} \neq 0\}. \quad (19.1)$$

The definition of multiplicity for other varieties is, unfortunately, more complicated. Here are two old-style definitions that capture the essence but are not easy to work with rigorously.

Let $Y \subset A^n$ be a variety of dimension $m$ and $p = (p_1, \ldots, p_n)$ a point on $Y$. The following give the correct definition of the multiplicity $\text{mult}_p Y$, see [Mum76, Chap.5] for details.

1. Let $\pi : A^n \rightarrow A^{m+1}$ be a general projection. Then $\pi(Y)$ is a hypersurface and $\text{mult}_p Y = \text{mult}_{\pi(p)} \pi(Y)$.
2. If we are over $\mathbb{C}$, this also equals the number of intersection points of $Y$ with a general small translate of general linear subspace of dimension $n - m$ through $p$, that are contained in a small Euclidean neighborhood of $p$.

Finally we set

$$\text{mult}_Z Y := \min \{\text{mult}_p Y : p \in Z\}, \quad (19.5)$$

and note that the minimum is achieved on a dense open subset.

We will also need the following.

**Theorem 19.6.** Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface and $Z, W \subset X$ irreducible subvarieties such that $Z \cap W$ is finite and $\dim Z + \dim W = \dim X$. Assume furthermore that neither of them has dimension $\frac{2}{3}$. Then

$$\sum_p \text{mult}_p Z \cdot \text{mult}_p W \leq \frac{\deg Z \deg W}{\deg X}. \quad (19.7)$$

**Comments on the proof.** There are several theorems rolled into one here.

Intersection theory says that if $X$ is any smooth projective variety and $Z, W \subset X$ irreducible subvarieties such that $\dim Z + \dim W = \dim X$ then they have a natural intersection number, denoted by $(Z \cdot W)$. Intersection theory can be developed completely algebraically, but working over $\mathbb{C}$ there is a shortcut. Both $Z, W$ have a homology class $[Z] \in H_2 \dim Z (X(\mathbb{C}), \mathbb{Z})$ and $[W] \in H_2 \dim W (X(\mathbb{C}), \mathbb{Z})$ and then

$$(Z \cdot W) = [Z] \cap [W] \in H_0 (X(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}. \quad (19.8)$$

Furthermore, if $Z \cap W$ is finite then their intersection number $(Z \cdot W)$ is the sum of local terms, denoted by $(Z \cdot W)_p$, computed at each $p \in Z \cap W$. Next we need that

$$(Z \cdot W)_p \geq \text{mult}_p Z \cdot \text{mult}_p W. \quad (19.9)$$

This very useful inequality used to be well known (see, for instance, [Sam55, p.95]) but it does not seem to be included in more recent books. It is, however, easy to derive it from [Mum76, Cor.A.14].

Assume next that $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of degree $d$ and $W$ is obtained as the intersection of $X$ by $n - r$ hypersurfaces of degrees $m_{r+1}, \ldots, m_n$. Then, by Bézout’s theorem,

$$\deg W = d \cdot m_{r+1} \cdots m_n \quad \text{and} \quad (Z \cdot W) = \deg Z \cdot m_{r+1} \cdots m_n.$$
Thus we obtain that
\[ (Z \cdot W) = \frac{\deg Z \deg W}{\deg X}. \tag{19.9} \]
It is not at all true that every \( W \) can be obtained this way, but by the Lefschetz hyperplane theorem (see [Lef50] or [GH78, p.156]), every subvariety is homologically equivalent to a rational multiple of a power of the hyperplane class, except in the middle dimension \( \frac{n}{2} \). Thus the above computation applies to every \( W \) as in Theorem 17.

20 (Proof of Theorem 17). If \( \dim Z + \dim W > \dim X \) and the claim holds for all subvarieties \( Z' \subset Z \) of codimension 1 then it also holds for \( Z \). Thus we may assume from now on that \( \dim Z + \dim W = \dim X \).

In Proposition 21 we construct a subvariety \( Z^* \subset X \) such that \( \dim Z^* = \dim Z \), \( \deg Z^* = (d-1)^r \deg Z \), \( Z \cap Z^* \) consists of at least \( (d-1)^r \deg Z \) distinct points and \( W \cap Z^* \) is finite.

By assumption \( \dim Z < \frac{1}{2} \dim X < \dim W \) thus the intersection number \( (W \cdot Z^*) \) is well defined and \( (W \cdot Z^*) = \frac{\deg W \deg Z^*}{\deg X} \) by (19.9). At each point of \( Z \cap Z^* \) the intersection multiplicity of \( Z^* \) and \( W \) is at least \( \mult_Z W \) by (19.8). Therefore
\[ ((d-1)^r \deg Z) \cdot \mult_Z W \leq (W \cdot Z^*) \]
\[ \leq \frac{\deg W \deg Z^*}{\deg X} = \frac{\deg W (d-1)^r \deg Z}{\deg X}. \tag{20.1} \]

Canceling \((d-1)^r \deg Z\) gives (17.1).

Next we construct the subvariety \( Z^* \) used in the above proof.

Proposition 21. Let \( X \subset \mathbb{P}^{n+1} \) be a smooth hypersurface of degree \( d \). Let \( Z \subset X \) be a subvariety of dimension \( r \leq \frac{d}{2} \) and \( W_i \subset X \) a finite set of subvarieties. Then there is a subvariety \( Z^* \) of dimension \( r \) such that

1. \( \deg Z^* = (d-1)^r \deg Z \),
2. \( Z \cap Z^* \) consists of at least \( (d-1)^r \deg Z \) distinct points, and
3. \( \dim (Z^* \cap (W_i \setminus Z)) \leq \dim Z + \dim W_i - \dim X \) for every \( i \).

The proof relies on the studying certain residual intersections.

22 (Residual intersection with cones). Let \( X \subset \mathbb{P}^{n+1} \) be a hypersurface of degree \( d \) and \( Z \subset X \) a subvariety. Pick a point \( v \in \mathbb{P}^{n+1} \) and let \( \langle v, Z \rangle \) denote the cone over \( Z \) with vertex \( v \), that is, the union of all lines \( \langle v, z \rangle : z \in Z \).

If \( \dim Z \leq n - 1 \) and \( v \) is general then \( \langle v, Z \rangle \) has the same degree as \( Z \) but 1 larger dimension. If \( \langle v, Z \rangle \) is not contained in \( X \) then \( X \cap \langle v, Z \rangle \) is a subscheme of \( X \) of degree \( = d \cdot \deg Z \). This subscheme contains \( Z \), thus we can write
\[ X \cap \langle v, Z \rangle = Z \cup Z^*_v, \tag{22.1} \]
where \( Z^*_v \) is called the residual intersection of the cone with \( X \). Note that
\[ \deg Z^*_v = (d-1) \cdot \deg Z. \tag{22.2} \]

We are a little sloppy here; if \( X \) is singular along \( Z \) then \( Z^*_v \) is well defined as a cycle but not well defined as a subscheme. We will always consider the case when \( X \) is smooth at general points \( z \in Z \) and \( v \) is not contained in the tangent plane of \( X \) at \( z \). If these hold then \( \langle v, Z \rangle \) is also smooth at \( z \) and hence \( Z \not\subset Z^*_v \). Our aim is to understand their intersection, \( Z \cap Z^*_v \).

Note that the intersection \( Z \cap Z^*_v \) can be quite degenerate. For example, let \( X \) be the cone \( (x^n + y^n - z^n) \subset \mathbb{P}^3 \) with vertex at \((0:0:0:1)\) and \( Z \) the line \((x-z=y=0)\).
Then \( \langle v, Z \rangle \) is a plane that contains \( Z \), hence it contains the vertex of the cone. Thus \( X \cap \langle v, Z \rangle \) is a union of \( n \) lines through \((0:0:0:1)\). Thus \( Z^\text{res}_v \) is a union of \( n - 1 \) lines and \( Z \cap Z^\text{res}_v = (0:0:0:1) \), a single point.

We see below that similar bad behavior does not happen for smooth hypersurfaces.

23 (Ramification linear system). Let \( X = X(G) \subset \mathbb{P}^{n+1} \) be a hypersurface. The tangent plane \( T_pX \) at a smooth point \((p_0: \cdots :p_{n+1})\) is given by the equation

\[
\sum_i x_i \cdot \frac{\partial G}{\partial x_i}(p) = 0.
\] (221)

Let \( v := (v_0: \cdots :v_{n+1}) \in \mathbb{P}^{n+1} \) be a point and \( \pi_v : \mathbb{P}^{n+1} \rightarrow \mathbb{P}^n \) the projection from \( v \). The \textit{ramification divisor} \( R_v \) of \( \pi_v|_X \) is the set of points whose tangent plane passes through \( v \). Thus

\[
R_v = (\sum_i v_i \cdot \frac{\partial G}{\partial x_i} = 0) \cap X.
\] (222)

Thus the \(|R_v|\) form a linear system, called the \textit{ramification linear system}, which is the restriction of the linear system of all first derivatives of \( G \). We denote it by \(|R_X|\). The base locus of \(|R_X|\) is exactly the singular locus \( \text{Sing} \, X \).

Note that \(|R_X| \subset (\deg X - 1)H|_X \), where \( H \) is the hyperplane class.

Lemma 24. Let \( X \subset \mathbb{P}^{n+1} \) be a smooth hypersurface of degree \( d \). Let \( Z \subset X \) be a subvariety of dimension \( r \) and \( W_i \subset X \) a finite set of subvarieties. Then, for general \( v \in \mathbb{P}^{n+1} \),

1. \( Z^\text{res}_v \cap Z = R_v \cap Z \) (set theoretically) and
2. \( Z^\text{res}_v \cap (W_i \setminus Z) \) has dimension \( \leq \dim Z + \dim W_i - n \).

Proof. Set \( \tau := \pi_v|_X \). If \( \tau \) is unramified at \( x \in X \) then it is a local isomorphism near \( x \), thus \( \langle v, Z \rangle \cap X = \tau^{-1}(\tau(Z)) \) equals \( Z \) near \( X \). Thus \( Z \cap Z^\text{res}_v \subset \tau \cap R_v \). To see the converse, it is enough to prove that \( Z \cap Z^\text{res}_v \) contains a dense open subset of \( Z \cap R_v \). Thus choose a point \( x \in Z \) that is smooth both on \( X, Z \) and such that \( \tau \) ramifies at \( x \) but \( \tau|_Z \) does not. Then the vector pointing from \( x \) to \( v \) is also a tangent vector of \( \langle v, Z \rangle \cap X \), hence \( x \) is a singular point of \( \langle v, Z \rangle \cap X \). So \( x \in Z^\text{res}_v \), proving (1).

Note that \( p \in Z^\text{res}_v \cap (W_i \setminus Z) \) iff the secant lines connecting \( p \) with some point of \( Z \) passes through \( v \). The union of all secant lines connecting a point of \( Z \) with a different point of \( W_i \) has dimension \( \dim Z + \dim W_i + 1 \). Thus only a \( \dim Z + \dim W_i + 1 - (n + 1) \) dimensional family of secant lines passes through a general point of \( \mathbb{P}^{n+1} \), proving (2). \( \square \)

25 (Proof of Proposition[21]). Set \( r = \dim Z \) and \( Z_0 := Z \). We inductively define

\[
Z_{i+1} := (Z_i)^{\text{res}}_{v_i} \quad \text{for general} \quad v_i \in \mathbb{P}^{n+1}.
\] (223)

We claim that \( Z^* := Z_r \) has the right properties. First note that (221) follows from (222).

Using (241) \( r \) times we see that \( Z \cap Z^* \) consists of the intersection points

\[
Z \cap R_{v_1} \cap \cdots \cap R_{v_r}.
\] (224)

for general \( v_i \). (If \( r = \frac{d}{2} \), we may also get finitely many other points \( Z_{i+1} \cap (Z \setminus Z_i) \); these we can ignore.) Since \( X \) is smooth, \(|R_X|\) is base point free, thus (222) consists of \((d - 1)^r \deg Z\) points in general position. (We used characteristic 0 at the last step.) \( \square \)
4. Multiplicity and canonical singularities

One can usually compute or at least estimate the multiplicity of a divisor or a linear system at a point quite easily, thus it would be useful to be able decide using multiplicities whether a pair \( (X, c|M|) \) is canonical or log canonical. This turns out to be possible for surfaces, less so for 3–folds but the notions diverge more and more as the dimension grows.

If a pair \( (X, c|M|) \) is not canonical, then there is a non-canonical exceptional divisor. We start with an example where this divisor is obtained by just one blow-up. Note that every exceptional divisor can be obtained by repeatedly blowing up subvarieties, but the more blow-ups we need, the harder it is to connect the multiplicity with being canonical.

**Example 26.** Let \( X \) be a smooth variety, \( Z \subset X \) a smooth subvariety of codimension \( r \) and \( |M| \) a linear system. Let \( \pi : X' \to X \) denote the blow-up of \( Z \) with exceptional divisor \( E \). Then

\[
K_{X'} = \pi^* K_X + (r-1)E \quad \text{and} \quad \pi^* |M| = |M'| + \text{mult}_Z |M| \cdot E.
\]

Thus

\[
K_{X'} + c|M'| \sim_\mathbb{Q} \pi^* (K_X + c|M|) + (r-1 - c \cdot \text{mult}_Z |M|)E. \tag{26.1}
\]

Note that we can apply this to any subvariety, after we replace \( X \) by \( X \setminus \text{Sing} Z \). We have thus proved the following.

**Claim 26.2.** Let \( X \) be a smooth variety, \( |M| \) a linear system and \( Z \subset X \) a subvariety. Then the following hold.

(a) If \( (X, c|M|) \) is canonical then \( c \cdot \text{mult}_Z |M| \leq \text{codim}_X Z - 1 \).

(b) If \( (X, c|M|) \) is log canonical then \( c \cdot \text{mult}_Z |M| \leq \text{codim}_X Z \). \( \square \)

The problem we have is that the converse holds only for \( n = 2 \) and only for part (a). Thus here our aim is to get some weaker converse statements in dimensions 2 and 3. In order to do this, we need a good series of examples.

**Claim 26.3.** \( (\mathbb{A}^n, c|\sum \lambda_i x_i^{m_i}|) \) is log canonical iff

\[
c \leq \frac{1}{m_1} + \cdots + \frac{1}{m_n}.
\]

A very useful way to think about this is the following. If we assign weights to the variables \( w(x_i) = \frac{1}{m_i} \) then the linear system becomes weighted homogeneous of weight 1. Thus, our condition says that

\[
c \cdot w(\sum \lambda_i x_i^{m_i}) \leq w(x_1 \cdots x_n). \tag{26.1}
\]

The claim is quite easy to prove if all the \( m_i \) are the same or if you know how to use weighted blow-ups, but can be very messy otherwise. The case \( n = 2 \) and \( m_1 = 2 \) is quite instructive and worth trying.

See [KSC04, Sec.6.5] for details in general (using weighted blow-ups).

The following lemma, which is a partial converse to (26.2.a) is used in the proof of Step 10.3.a.

**Lemma 27.** Let \( X \) be a smooth variety and \( |M| \) a linear system. Assume that \( c \cdot \text{mult}_p |M| \leq 1 \) for every point \( p \in X \). Then \( (X, c|M|) \) is canonical.
Proof. For one blow-up $\pi : X' \to X$ as in (26.1) we have the formula
$$K_{X'} + c|M'| \sim_{Q} \pi^* (K_X + c|M|) + (r - 1 - c \cdot \mult_Z |M|)E.$$ Since $r \geq 2$, our assumption $c \cdot \mult_Z |M| \leq 1$ implies that $r - 1 - c \cdot \mult_Z |M| \geq 0$.

If $\tau : X'' \to X'$ is any birational morphism and
$$K_{X''} + c|M''| \sim_{Q} \tau^* (K_{X'} + c|M'|) + E'',$n then we get that
$$K_{X''} + c|M''| \sim_{Q} (\tau \circ \pi)^* (K_X + c|M|) + E'' + (r - 1 - c \cdot \mult_Z |M|)\tau^* E.$$ If $(X', c|M'|)$ is canonical then $E''$ is effective and so is
$$E'' + (r - 1 - c \cdot \mult_Z |M|)\tau^* E.$$ Thus $(X, c|M|)$ is also canonical. If $p' \in X'$ is any point and $p = \pi(p')$ then
$$\mult_{p'} |M'| \leq \mult_p |M|,$$ thus $c \cdot \mult_{p'} |M'| \leq 1$ and we can use induction.

The problem is that this seems to be an infinite induction, since we can keep blowing up forever. There are 2 ways of fixing this.

The easiest is to use a log resolution as in Definition 7 and stop when the birational transform of $|M|$ becomes base point free, hence canonical.

Theoretically it is better to focus on one divisor at a time and use a lemma of Zariski and Abhyankar, which is a very weak form of resolution; see [KM98, 2.45] or [KSC04, 4.26].

The following partial converse to (26.2.b) is a reformulation of [Var76], see also [KSC04, 6.40] for a proof.

**Theorem 28.** Let $S$ be a smooth surface and $|M|$ a linear system such that $p \in S$ is a non-log-canonical center of $(S, c|M|)$. Then one can choose local coordinates $(x, y)$ at $p$ and weights $w(x) = a$ and $w(y) = b$ such that
$$|M| \subset |x^i y^j : w(x^i y^j) > \frac{1}{c}w(xy) = \frac{1}{c}(a + b)|.$$ (28.1)

**Example 29.** It can be quite hard to find the right coordinate system that works, they are usually given by complicated power series. For example, [Yos79] writes down a degree 6 polynomial $g(x, y)$ that, in suitable local coordinates becomes $x^2 + y^{20}$. (I do not doubt the claim but I have been unable to find a clear, non-computational explanation.) Taking $a = 10$ and $b = 1$ shows that $(k^2, c(g = 0))$ is log canonical for $c \leq \frac{1}{20}$. Related bounds and examples are given in [JK11].

The following consequence proves Step 10.3.2.b'.

**Corollary 30.** [Cor00] Let $S$ be a smooth surface and $|M|$ a linear system such that $p \in S$ is a non-log-canonical center of $(S, c|M|)$. Then $(M \cdot M)_p > \frac{1}{4}$. 

**Remark.** Unlike for Lemma 27 a direct induction does not seem to work, but [Cor00] sets up a more complicated inductive assumption and proves it blow up at a time. The following argument, relying on Theorem 28, easily generalizes to all dimensions. (Unfortunately, this is less useful since Theorem 28 does not generalize to higher dimensions.)

Proof. Assume first that in (28.1) we have $a = b$. Then every member of $|M|$ is a curve that has multiplicity $\geq \frac{1}{2}$ at $p$ and the intersection multiplicity is at least the product of the multiplicities. (This is a special case of [1998], but it is much simpler; see [Sha74, IV.3.2].) Hence the intersection multiplicity is $> \frac{1}{\sqrt{2}}$. 

In general we get that members of $|M|$ have multiplicity $>\frac{1}{r}(1+\min\{\frac{a}{r}, \frac{b}{r}\})$ at $p$ and this only gives that $(M\cdot M)_p > \frac{1}{r}$. Thus we need to equalize $a$ and $b$. The best way to do this is by a weighted blow-up, see \cite[Sec.6.5]{KSC04}, but here the following trick works.

After multiplying with the common denominator, we may assume that $a, b$ are integers. Set $x = s^a$ and $y = t^b$. These define a degree $ab$ morphism $\tau : \mathbb{A}^2_{st} \rightarrow \mathbb{A}^2_{xy}$. The inclusion

$$|M| \subset |x^iy^j : ai + bj > \frac{1}{2}(a + b)|$$

of (28.1) is now transformed into

$$\tau^*|M| \subset |s^{ai}t^{bj} : ai + bj > \frac{1}{2}(a + b)| \subset |s^mt^n : m + n > \frac{1}{2}(a + b)|.$$ 

That is, $\tau^*|M|$ has multiplicity $>\frac{1}{r}(a + b)$, hence $(\tau^* M \cdot \tau^* M)_p > \frac{1}{r^2}(a + b)^2$. Intersection multiplicities get multiplied by the degree of the map under pull-back, thus we conclude that $(M \cdot M)_p > \frac{1}{r} : \frac{(a + b)^2}{ab} \geq \frac{1}{r}$.

A 3-dimensional analog of Theorem 28 was conjectured in \cite{Cor00}. The method of \cite{Cor95} shows that it is a consequence of a result of Kawakita \cite{Kaw01}. See also \cite[Chap.5]{KSC04} for more details.

**Theorem 31.** Let $X$ be a smooth threefold and $|M|$ a linear system such that $p \in X$ is a non-canonical center of $(X, c|M|)$. Then one can choose local coordinates $(x, y, z)$ at $p$ and weights $w(x) = a$, $w(y) = b$ and $w(z) = 1$ such that

$$|M| \subset |x^iy^jz^k : w(x^iy^jz^k) > \frac{1}{r}w(xy) = \frac{1}{r}(a + b)|.$$ 

**32 (Summary).** Let $|M| := |\sum \lambda g_i|$ be a linear system on $\mathbb{A}^n$.

- If $n = 2$ then we can decide whether $(\mathbb{A}^2, c|M|)$ is canonical at the origin just by looking at the degrees of the monomials that occur in the $g_i$.
- If $n = 3$ then we can decide whether $(\mathbb{A}^3, c|M|)$ is log canonical at the origin by looking at the monomials that occur in the $g_i$, provided we use the right coordinate system.
- If $n = 3$ then we can decide whether $(\mathbb{A}^3, c|M|)$ is canonical at the origin by looking at the monomials that occur in the $g_i$, provided we use the right coordinate system.
- If $n \geq 4$ then the situation is more complicated, see Example 33. However, as we discuss in Section 3 there is the following partial replacement.

We can frequently show that $(\mathbb{A}^n, c|M|)$ is not canonical at the origin by looking at the monomials that occur in a Gröbner basis of the ideal $(g_i)$.

**Example 33.** \cite[6.45]{KSC04} For $r \geq 5$ consider the linear system

$$|M_r| := |(x^2 + y^2 + z^2)^2, x', y', z'|.$$ 

Show that $(\mathbb{C}^3, c|M_r|)$ is log canonical iff $c \leq \frac{1}{2} + \frac{1}{r}$. However, if $(x', y', z')$ is any coordinate system with weights $w$ then $w(|M_r|) \leq \frac{4}{r}w(x'y'z')$.

5. **Hyperplane sections and canonical singularities**

We start with the proof of Step 11.5.6.b.

34 (Bertini type theorems). The classical Bertini theorem—for differentiable maps also known as Sard’s theorem—says that a general member of a base point free linear system on a smooth variety is also smooth. This has numerous analogs, all
saying that if a variety has certain types of singularities then a general member of a base point free linear system also has only the same type of singularities. Thus it is not surprising that the same holds for canonical and log canonical singularities. The log canonical case of the following proves Step 10.5.b.

**Proposition 35.1.** Let \( H \subseteq X \) be a general member of a base point free linear system \(|H|\). If \((X, c|M|)\) is canonical (resp. log canonical) then so is \((H, c|M|_H)\).

**Proof.** Choose a log resolution \( \pi : X' \rightarrow X \) as in Definition 7 and write
\[
K_{X'} = \pi^*K_X + \sum e_iE_i \quad \text{and} \quad \pi^*|M| = |M'| + \sum a_iE_i, \tag{34.2}
\]
where \(|M'|\) is base point free and \(\sum E_i\) has simple normal crossing singularities only. Thus
\[
K_{X'} + c|M'| \sim_{\mathbb{Q}} \pi^*(K_X + c|M|) + \sum (e_i - ca_i)E_i, \tag{34.3}
\]
Note that \(|H|\) gives us base point free linear systems \(|H'|\) on \(X'\) and \(|H'|_{E_i}\) on each \(E_i\). The adjunction formula (stated only for curves but proved in general in [Sho74 VI.1.4]) says that \(K_H = (K_X + H)|_H\) and \(K_{H'} = (K_{X'} + H')|_{H'}\). Adding \(H' = \pi^*H\) to (34.3) and restricting to \(H\) and \(H'\) we get that
\[
K_{H'} + c|M'|_{H'} \sim_{\mathbb{Q}} \pi^*(K_H + c|M|_H) + \sum (e_i - ca_i)(E_i \cap H'), \tag{34.4}
\]
where \(H'\) is smooth and \(\sum (E_i \cap H')\) has simple normal crossing singularities only. If \((X, c|M|)\) is canonical (resp. log canonical) then \(e_i - ca_i \geq 0\) (resp. \(\geq -1\)) for every \(i\). The same \(e_i - ca_i\) are involved in (34.4), except that some of the \(E_j \cap H'\) may be empty, in which case \(e_j - ca_j\) does not matter for \((H, c|M|_H)\). In any case, \((H, c|M|_H)\) is also canonical (resp. log canonical).

Let us next see what happens if we try to use the same method to prove Step 10.5.a.

**35.** Here we have a non-canonical center \(p \in X\) and we take an \(H \subseteq |H|\) that passes through the point \(p\). If there is an exceptional divisor \(E_j \subseteq X'\) such that \(\pi(E_j) = \{p\}\), then \(\pi^*H \supset E_j\). Hence \(\pi^*H\) is not smooth, it is not even irreducible. In this case we write
\[
\pi^*H = H' + \sum m_iE_i. \tag{35.1}
\]
Adding \(H' = \pi^*H - \sum m_iE_i\) to (34.2) we get
\[
K_{X'} + H' + c|M'| \sim_{\mathbb{Q}} \pi^*(K_X + H + c|M|) + \sum (e_i - m_i - ca_i)E_i. \tag{34.2}
\]
Thus restricting (35.2) to \(H'\) and \(H\) we get that
\[
K_{H'} + c|M'_{H'} \sim_{\mathbb{Q}} \pi^*(K_H + c|M|_H) + \sum (e_i - m_i - ca_i)(E_i \cap H'). \tag{35.3}
\]
At first sight we are done. If \(p\) is a non-canonical center of \((X, c|M|)\) then there is an \(E_j\) such that \(\pi(E_j) = \{p\}\) and \(e_j - ca_j < 0\). Since \(H\) passes through \(p\), \(m_j \geq 1\) also holds, so \(e_j - m_j - ca_j < -1\). Thus \(E_j \cap H'\) shows that \(p\) is a non-log-canonical center of \((H, c|M|_H)\).

However, all this falls apart if \(E_j \cap H' = \emptyset\). This can easily happen for some \(E_j\), but it is enough to show that it can not happen for every \(E_j\) for which \(e_j - m_j - ca_j < -1\). This is what we discuss next.

The following result has many names. In [KM98] and [Kol13] it is called inversion of adjunction, while [Puk13] uses Shokurov-Kollár connectedness principle. A closely related result in complex analysis is the Ohsawa-Takegoshi extension theorem, proved in [OT87]. The theorem was conjectured in [Sho92] and proved in
The sharpest form was established in [Kaw07], see also [Kol13, Sec.17] for other generalizations.

For simplicity I state it only for smooth varieties, though the singular case is needed for most applications. The proof is actually a quite short application of Theorem 46.3; see [KM98, Sec.5.4] or [KSC04, Chap.6] for detailed treatments.

**Theorem 36.** Let $X$ be a smooth variety and $\Delta$ an effective $\mathbb{Q}$-divisor on $X$. Let $\pi : X' \to X$ be a proper, birational morphism and write

$$K_{X'} \sim_{\mathbb{Q}} \pi^*(K_X + \Delta) + \sum b_i B_i,$$

where the $B_i$ are either $\pi$-exceptional or lie over $\text{Supp} \Delta$. Then every fiber of

$$\pi : \text{Supp}(\sum b_i \leq 1 B_i) \to X$$

is connected. \qed

The following consequence is especially important. The first part of it directly implies Step 10.5.a, the second part is used in Section 8.

**Theorem 37.** Let $X$ be a smooth variety and $|M|$ a linear system on $X$. Let $H \subset X$ be a smooth divisor. Assume that

1. either $H$ contains a non-canonical center $Z^c$ of $(X, c|M|)$,
2. or $H$ has nonempty intersection with a non-log-canonical center $Z^c$ of $(X, c|M|)$.

Then $(H, c|M|_H)$ is not log-canonical.

Proof. Choose a log resolution $\pi : X' \to X$ and write $\pi^*H = H' + \sum m_i E_i$. Choosing a general member $M \in |M|$ gives $M' \in |M'|$. We can rearrange (37.2) as

$$K_{X'} \sim_{\mathbb{Q}} \pi^*(K_X + H + cM) - H' - cM' + \sum (e_i - m_i - ca_i) E_i.$$  \hspace{1cm} (37.3)

Pick a point $p \in Z^c$ (resp. $p \in H \cap Z^c$). If $c < 1$ (which is always the case for), then Theorem 38 say that

$$F_p := \pi^{-1}(p) \cap (H' \cup \sum e_i - m_i - ca_i \leq -1 E_i)$$

is connected.

If (1) holds then there is an $E_j$ such that $p \in \pi(E_j)$, $e_j - ca_j < 0$ and $m_j \geq 1$. If (2) holds then there is an $E_j$ such that $p \in \pi(E_j)$ and $e_j - ca_j < 1$. Thus, in both cases, $e_j - m_j - ca_j < -1$ and $\pi^{-1}(p) \cap \sum (e_i - m_i - ca_i \leq -1 E_i)$ is not empty. Since $F_p$ is connected, we obtain that

$$\pi^{-1}(p) \cap H' \cap \sum e_i - m_i - ca_i \leq -1 E_i \neq \emptyset.$$

Thus there is at least one divisor $E_{j_0}$ such that

$$e_{j_0} - m_{j_0} - ca_{j_0} \leq -1 \quad \text{and} \quad p \in \pi(E_{j_0} \cap H').$$

Hence $E_{j_0} \cap H'$ gives the non-empty divisor that we needed in (38.3). A small problem is that we would like a strict inequality $e_{j_0} - m_{j_0} - ca_{j_0} < -1$. To achieve this, run the same argument with some $c' < c$. Then we get a $j_0$ such that

$$e_{j_0} - m_{j_0} - ca_{j_0} < e_{j_0} - m_{j_0} - c'a_{j_0} \leq -1 \quad \text{and} \quad p \in \pi(E_{j_0} \cap H').$$

Thus $(H, c|M|_H)$ is not log-canonical. This completes the proof of Step 10.5.a. \qed

(Proof of Step 10.4). Let $Y \subset \mathbb{P}^{n+1}$ be a smooth hypersurface, $|M| \subset |mH|$ a movable linear system and $M \in |M|$ any member. Let $B_m \subset M$ be the set of points where $\text{mult}_p M > m$. This is a closed subset, thus it is either finite or it contains a curve $C$. The latter is impossible by Theorem 17. Thus $B_m$ is finite and $(Y, \frac{1}{m}|M|)$ is canonical outside $B_m$ by Lemma 22. This proves Step 10.4.a.
The argument for Step 4.b is similar. We use Theorem 17 to obtain that there is a 1-dimensional closed subset $C_m$ such that $\text{mult}_p(M \cdot M) \leq m^2$ for every $p \in Y \setminus C_m$. Then we claim that $(Y, \frac{1}{m} |2M|)$ is log canonical on $Y \setminus C_m$. So pick any $p \in Y \setminus C_m$. The multiplicity is preserved by general hyperplane cuts, thus eventually we get a surface $S = H_1 \cap \cdots \cap H_{n-2}$ containing $p$ such that $\text{mult}_p(|M|_S \cdot |M|_S) \leq m^2$. Thus $(S, \frac{1}{m} |2M|_S)$ is log canonical at $p$ by Corollary 30. Finally we use Theorem 37 to conclude that $(Y, \frac{1}{m} |2M|)$ is also log canonical at $p$. 

6. Global sections from isolated singularities

The observation that one can use divisors with an isolated log canonical singularity to obtain global sections is an important ingredient of the Kawamata–Reid–Shokurov approach to the cone theorem (cf. [KM98 Chap.3]) and is central in the works around Fujita’s conjecture (cf. [Ko97 Secs.5–6]). In all these applications the aim is to get at least 1 section that does not vanish at a point. Although it is known that the process can be used to get several sections, this has not been important in past applications.

One of the key observations of [Zhu18] is that a suitable modification of this method leads to many sections, in fact their number grows exponentially with the dimensions. The proof of Step 10.6 is a combination of 4 lemmas, which are either quite easy to prove (Lemmas 39–40) or have been well known (Lemmas 41–42). Nonetheless, the power of their combination was not realized before [Zhu18].

We define the upper multiplier ideal $\mathcal{J}^+(c|N|)$ of a linear system or of an ideal $I$ in Definition 48. We use the following of its properties.

Lemma 39. Let $Y$ be a smooth variety and $|N|$ a linear system. Then the support of $\mathcal{O}_Y / \mathcal{J}^+(c|N|)$ is the union of all non-log-canonical centers of $(Y, c|N|)$.

Lemma 40. Let $Y$ be a smooth variety and $|N|$ a linear system. Assume that $(Y, \frac{1}{2} |N|)$ is not log canonical. Then

$$\mathcal{J}^+(\mathcal{J}^+(c|N|)) \neq \mathcal{O}_Y.$$

Lemma 41. Let $Y$ be a smooth variety and $I \subset \mathcal{O}_Y$ an ideal sheaf that vanishes only at finitely many points. Assume that $\mathcal{J}^+(I) \neq \mathcal{O}_Y$. Then

$$\dim(\mathcal{O}_Y / I) \geq \frac{1}{2^3 \dim Y}.$$

Lemma 42. Let $Y$ be a smooth, projective variety, $H$ an ample divisor on $Y$ and $|N|$ a linear system such that $H \sim c |N|$. Assume that $(Y, c|N|)$ is log canonical outside finitely many points. Then

$$H^0(Y, \mathcal{O}_Y(K_Y + H)) \geq \dim(\mathcal{O}_Y / \mathcal{J}^+(c|N|)).$$

43 (Proof of Step 10.6 using Lemmas 40–42). We apply Lemma 10 to $H := 2L$, $|N| := |2M|$ and $c = \frac{1}{m}$. We get the ideal sheaf $I := \mathcal{J}^+(\frac{1}{m} |2M|)$ such that $\mathcal{J}^+(I) \neq \mathcal{O}_Y$. By Lemma 39 $I$ vanishes only at finitely many points. Thus, by Lemma 41

$$\dim(\mathcal{O}_Y / \mathcal{J}^+(\frac{1}{m} |2M|)) \geq \frac{1}{2^3 \dim Y}.$$

Finally Lemma 42 says that

$$H^0(Y, \mathcal{O}_Y(K_Y + 2L)) \geq \dim(\mathcal{O}_Y / \mathcal{J}^+(\frac{1}{m} |2M|)) \geq \frac{1}{2^3 \dim Y}. \quad \Box$$
We prove Lemma 49 in Paragraph 59 and Lemma 51 is local at the points where $I$ vanishes, in fact, it is a quite general algebra statement about ideals. We discuss it in detail in Section 8. Lemma 52 is a restatement of Corollary 51 and we explain its proof in Section 7.

44 (Proof of Lemma 40). Take a log resolution $\pi : Y' \to Y$ as in Definition 7 and write

$$K_{Y'} \sim \pi^*K_Y + \sum e_i E_i \quad \text{and} \quad \pi^*|N| = |N'| + \sum a_i E_i,$$

where $|N'|$ is base point free. By definition

$$\mathcal{J}^+(c|N|) = \pi_* \mathcal{O}_{Y'}(\sum \lfloor e_i - c'a_i \rfloor E_i),$$

where $0 < c - c' \ll 1$. Thus if

$$\pi^* \mathcal{J}^+(c|N|) = \mathcal{O}_{Y'}(-\sum b_i E_i),$$

then $-b_i \leq \lfloor e_i - c'a_i \rfloor$. (We have to be a little careful here. We need to use a $\pi : Y' \to Y$ that is a log resolution for both $|N|$ and $\mathcal{J}^+(c|N|)$.) Therefore

$$\mathcal{J}^+(\mathcal{J}^+(c|N|)) = \pi_* \mathcal{O}_{Y'}(\sum \lfloor e_i - (1 - \epsilon)b_i \rfloor E_i), \quad (44.1)$$

for $0 < \epsilon \ll 1$. If $b_i = 0$ then $\lfloor e_i - (1 - \epsilon)b_i \rfloor = e_i - b_i$ and if $b_i > 0$ then

$$\lfloor e_i - (1 - \epsilon)b_i \rfloor = e_i - b_i + \lfloor \epsilon b_i \rfloor = e_i - b_i + 1,$$

since $b_i$ is an integer. Thus, in both cases

$$\lfloor e_i - (1 - \epsilon)b_i \rfloor \leq e_i - b_i + 1 \leq 2e_i + \lfloor -c'a_i \rfloor + 1 \leq 2e_i - ca_i + 2.$$

Since $(Y, \mathcal{O}_Y|N|)$ is not log canonical, there is an index $j$ such that $e_j - \frac{c}{2}a_j < -1$. Then

$$2e_j - ca_j + 2 = 2(e_j - \frac{c}{2}a_j) + 2 < -2 + 2 = 0.$$

Thus $\mathcal{J}^+(\mathcal{J}^+(c|N|))$ vanishes along $\pi(E_j)$.

7. Review of vanishing theorems

Here we prove Lemma 42. For this we need to use the cohomology of coherent sheaves. We use that the groups $H^i(Y, F)$ exist and that a short exact sequence of sheaves leads to a long exact sequence of the cohomology groups. For the uninitiated, [Rei97, Chap.B] is a very good introduction.

We also need a vanishing theorem which says that under certain assumptions the cohomology group $H^1(Y, F)$ is 0. The reader who is willing to believe Theorem 50 need not get into any further details. However, at first sight, the definition of the multiplier ideal may appear rather strange, so I include an explanation of where these definitions and results come from.

45 (Vanishing and global sections). Let $F$ be a coherent sheaf on a projective variety $Y$. One way to estimate the dimension of $H^0(Y, F)$ from below is to identify a subsheaf $\mathcal{S}(F) \hookrightarrow F$ and the corresponding quotient $F \to \mathcal{Q}(F)$, write down the short exact sequence

$$0 \to \mathcal{S}(F) \to F \to \mathcal{Q}(F) \to 0,$$

and the beginning of its long exact sequence

$$0 \to H^0(Y, \mathcal{S}(F)) \to H^0(Y, F) \to H^0(Y, \mathcal{Q}(F)) \to H^1(Y, \mathcal{S}(F)).$$

If the last term vanishes then

$$\dim H^0(Y, F) \geq \dim H^0(Y, \mathcal{Q}(F)).$$
In our case we have a divisor \( L \) and a linear system \( |M| \) such that \( c|M| \sim L \) for some \( c \). We will use these data to construct the subsheaf 
\[
S(O_Y(K_Y + L)) \subset O_Y(K_Y + L),
\]
such that a generalization of Kodaira’s vanishing theorem applies to \( S(O_Y(K_Y + L)) \). These vanishing theorems form a powerful machine which gives us a vanishing involving \( O_Y(aK_Y + bL) \) for our purposes in Section 6. For our purposes in Section 6, a vanishing involving pretty much any other \( O_Y(aK_Y + bL) \) would be good enough, as long as \( a, b \) are much smaller than \( \dim Y \).

46 (Generalizations of Kodaira’s vanishing theorem). Kodaira’s classical vanishing theorem says that if \( Y \) is a smooth, projective variety over \( \mathbb{C} \) and \( L \) an ample divisor on \( X \) then 
\[
H^i(Y, O_Y(K_Y + L)) = 0 \text{ for } i > 0.
\]
It has various generalizations when \( L \) is only close to being ample.

46.1 (Close-to-ample divisors). It turns out that Kodaira’s vanishing theorem also works for a divisor \( L \) if one can write it as \( L \sim cA + \Delta \) where
(a) \( c > 0 \) and \( A \) is nef and big (that is, \( (A \cdot C) \geq 0 \) for every curve \( C \subset Y \) and \( (A^{\dim Y}) > 0 \)), and
(b) \( \Delta := \sum d_i D_i \), where \( d_i \in [0, 1) \) and \( D_i \) has simple normal crossing singularities only.

In practice the condition that \( \sum D_i \) be a simple normal crossing divisor is very rarely satisfied, but log resolution (as in Definition 7) allows us to reduce almost everything to this case. The basic vanishing theorem is the following, see [KM98, Sec.2.5] or [Laz04, 9.1.18] for proofs.

Theorem 46.2 (Kawamata-Viehweg version). Let \( X \) be a smooth, projective variety and \( L \) a divisor as in (46.1). Then
\[
H^i(X, O_X(K_X + L)) = 0 \text{ for } i > 0. \tag{46.2}
\]

The following versions are easy to derive from (46.1); see [KM98, 2.68].

Theorem 46.3 (Grauert-Riemenschneider version). Let \( X \) be a smooth, projective variety, \( \pi : X \to Y \) a birational morphism and \( L \) a divisor as in (46.1). Then
\[
R^i \pi_* O_X(K_X + L) = 0 \text{ for } i > 0. \tag{46.3}
\]

Corollary 46.4. Let \( X \) be a smooth, projective variety, \( \pi : X \to Y \) a birational morphism and \( L \) a divisor as in (46.1). Then
\[
H^i(Y, \pi_* O_X(K_X + L)) = 0 \text{ for } i > 0. \tag{46.4}
\]

Next we show how we get vanishing theorems starting with a linear system.

47. Let \( Y \) be a smooth, projective variety over \( \mathbb{C} \) and \( |M| \) a linear system on \( Y \). Following (46.1) we would like to get a nef and big divisor plus a divisor with simple normal crossing support.

Thus let \( \pi : Y' \to Y \) be a log resolution as in Definition 7. Write
\[
K_{Y'} = \pi^* K_Y + \sum c_i E_i \quad \text{and} \quad \pi^* |M| = |M' + \sum a_i E_i|. \tag{47.1}
\]
Thus the $E_i$ are either $\pi$-exceptional or belong to the base locus of $\pi^*|M|$, and we allow $e_i$ or $a_i$ to be 0. Let $L$ be an ample divisor such that $L \sim_Q c|M|$. Then we can write the pull-back of $K_Y + L$ as

$$\pi^*(K_Y + L) \sim_Q K_{Y'} + c|M'| + \sum (-e_i + ca_i)E_i. \quad (17.2)$$

The right hand side starts to look like we could apply (16.2) to it, but there are two problems. The coefficient $(-e_i + ca_i)$ need not lie in the interval $[0, 1)$ and, although $|M'|$ is nef, it need not be big. The latter can be arranged by keeping a little bit of $L$ unchanged. That is, pick $0 < c' < c$ and write the pull-back of $K_Y + L$ as

$$\pi^*(K_Y + L) \sim_Q K_{Y'} + (c - c')\pi^*L + c'|M'| + \sum (-e_i + c'a_i)E_i. \quad (17.3)$$

Now $\pi^*L + c'|M'|$ is nef and big, but the first problem remains. Here we use that any number $a$ can be uniquely written as $a = [a] + \{a\}$ where $[a]$ is an integer and $\{a\} \in [0, 1)$. Furthermore, let $[a] := -\lfloor -a \rfloor$ denote the rounding up of $a$. Applying this and rearranging we get that

$$\pi^*(K_Y + L) + \sum [e_i - c'a_i]E_i \sim_Q K_{Y'} + (c - c')\pi^*L + c'|M'| + \sum (-e_i + c'a_i)E_i. \quad (17.4)$$

Now the vanishing (16.4) applies to the right hand side of (17.4). Note also that $\pi_*\mathcal{O}_{Y'}(\pi^*(K_X + L) + \sum [e_i - c'a_i]E_i) = \mathcal{O}_Y(K_Y + L) \otimes \pi_*\mathcal{O}_{Y'}(\sum [e_i - c'a_i]E_i).$ This suggests that the basic object is $\pi_*\mathcal{O}_{Y'}(\sum [e_i - c'a_i]E_i).$ Note that this does not depend on $c'$ as long as $c - c'$ is small enough. Indeed, then

$$[e_i - c'a_i] = \begin{cases} 
[e_i - ca_i] & \text{if } e_i - ca_i \notin \mathbb{Z} \\
[e_i - ca_i] + 1 & \text{if } e_i - ca_i \in \mathbb{Z}.
\end{cases}$$

**Definition 48** (Multiplier ideal). Let $X$ be a smooth, projective variety over $\mathbb{C}$ and $|M|$ a linear system. The upper multiplier ideal of $c|M|$ is

$$\mathcal{J}^+(c|M|) := \pi_*\mathcal{O}_{Y'}(\sum [e_i - c'a_i]E_i)$$

for any $c'$ satisfying $0 < c - c' \ll 1$. It is not hard to see that this does not depend on the choice of $\pi : Y' \to Y$. Note that [Laz04: Sec.9.2] calls

$$\mathcal{J}(c|M|) := \pi_*\mathcal{O}_{Y'}(\sum e_i - ca_i)E_i$$

the multiplier ideal. Clearly $\mathcal{J}^+(c|M|) = \mathcal{J}(c'|M'|)$ for $0 < c - c' \ll 1$.

**49** (Proof of Lemma 39). Set $W := \text{Supp} \mathcal{O}_Y / \mathcal{J}^+(c|M|)$. Then $W$ is exactly the $\pi$-image of the support of the negative part of $\sum [e_i - c'a_i]E_i$. If $e_i - ca_i < -1$ then $e_i - c'a_i < -1$ so $[e_i - c'a_i] \leq -1$. If $e_i - ca_i \geq -1$ and $a_i > 0$ then $e_i - c'a_i > -1$ so $[e_i - c'a_i] \geq 0$. If $a_i = 0$ then $e_i - ca_i = e_i \geq 0$.

Now we can apply (16.4) and get the following, see [Laz04: Sec.9.4].

**Theorem 50** (Nadel vanishing). Let $Y$ be a smooth, projective variety, $L$ an ample divisor and $|M|$ a linear system such that $c|M| \sim_Q L$. Then

$$H^i(Y, \mathcal{O}_Y(K_Y + L) \otimes \mathcal{J}^+(c|M|)) = 0 \quad \text{for } i > 0. \quad \Box$$

As we discussed in Paragraph 15 this immediately implies the following.

**Corollary 51.** Let $Y$ be a smooth, projective variety, $L$ an ample divisor on $Y$ and $|M|$ a linear system such that $L \sim_Q c|M|$. Assume that $(Y, c|M|)$ is log canonical outside finitely many points. Then

$$H^0(Y, \mathcal{O}_Y(K_Y + L)) \geq \text{dim}(\mathcal{O}_Y / \mathcal{J}^+(c|M|)). \quad \Box$$
8. Review of monomial ideals

In this section we prove Lemma 41. Its claim is local at the points where \( I \) vanishes, we can thus work using local coordinates at a point. Though not important, it is notationally simpler to pretend that we work at the origin of \( \mathbb{A}^n \). (This is in fact completely correct, one needs to argue that they have the same completion, [Sha74, Sec.II.2.2].)

As a general rule, an ideal is log canonical iff it contains low multiplicity polynomials. In this section we give a precise version of this claim. Key special cases of the following are proved by Reid [Rei80] and Corti [Cor00]. More general versions are in [dFEM03, How01]. An excellent detailed treatment of this topic is given in [Laz04, Chap.9], so I concentrate on the definitions and explanations, leaving the details to [Laz04].

**Theorem 52.** Let \( I \subset R := k[x_1, \ldots, x_n] \) be an ideal vanishing only at the origin. Assume that \( I \) is not log canonical. Then

\[
\dim(R/I) \geq \min_{a_1, \ldots, a_n \geq 0} \# \left\{ \mathbb{N}^n \cap \left( \sum a_i r_i \leq \sum a_i \right) \right\}.
\]

The proof is given in 2 steps. We first reduce to the case of monomial ideals in (54.4) and then to counting lattice points in a simplex (55.2). Following the proof shows that the lower bound is sharp, but I do not know a closed formula for it. However, a simple argument, given in Paragraph 56, gives the following.

**Corollary 53.** Let \( I \subset k[x_1, \ldots, x_n] \) be a non-log-canonical ideal that vanishes only at the origin. Then

\[
\dim(R/I) \geq \frac{1}{3}3^n.
\]

54 (Deformation to monomial ideals). (See [CLO92, Chap.2] for details.) Let \( I \subset R := k[x_1, \ldots, x_n] \) be an ideal. Write every \( g \in R \) as \( g = \text{in}(g) + \text{rem}(g) \) where \( \text{in}(g) := a_g \prod x_i^{r_i} \) is the lexicographically lowest monomial that appears in \( g \) with nonzero coefficient. Define the \emph{initial ideal} of \( I \) (with respect to the lexicographic ordering) as

\[
\text{in}(I) := (\text{in}(g) : g \in I).
\]

Thus \( \text{in}(I) \) is generated by monomials and it is not hard to see that

\[
\dim(R/I) = \dim(R/\text{in}(I)).
\]

A key property is the following.

**Proposition 54.3.** If \( \text{in}(I) \) is log canonical then so is \( I \).

Comments on the proof. Choose integers \( 1 \leq w_1 \ll \cdots \ll w_n \). For \( g \in R \) let \( w(g) \) denote the largest \( t \) power that divides \( g(t^{w_1}x_1, \ldots, t^{w_n}x_n) \). Then

\[
t^{-w(g)}(t^{w_1}x_1, \ldots, t^{w_n}x_n) = \text{in}(g)(x_1, \ldots, x_n) + t(\text{other terms}).
\]

Any finite collection of these defines a linear system \( |M| \) on \( Y := \mathbb{A}^{n+1} \) with coordinates \( (x_1, \ldots, x_n, t) \).

If we choose \( w_i \) that work for a Gröbner basis \( g_i \in I \), then we get \( |M| \) whose restriction to \( (t = 0) \) gives \( I_0 = \text{in}(I) \) and to \( (t = \lambda) \) gives \( I_\lambda \cong I \) for \( \lambda \neq 0 \).

If \( I_0 \) is log canonical then \( (Y, |M|) \) is also log canonical by Theorem 57.2, and so is \( I_\lambda \cong I \) by (34.1). \( \square \)

Combining (54.2) and (54.3) gives the first reduction step of the proof of Theorem 52.
Corollary 54.4. If Theorem 52 holds for monomial ideals then it also holds for all ideals.

55 (Monomial ideals). Let $I \subset k[x_1, \ldots, x_n]$ be a monomial ideal, that is, an ideal generated by monomials. A very good description of $I$ is given by its Newton polytope.

For $\prod x_i^{r_i} \in I$ we mark the point $(r_1, \ldots, r_n)$ with a big dot for elements of $I\setminus(x_1, \ldots, x_n)I$ and with an invisible dot for elements of $(x_1, \ldots, x_n)I$. The Newton polytope is the boundary of the convex hull of the marked points, as in the next example.

The Newton polygon of $(y^7, y^5x, y^3x^2, yx^4, x^6)$.

A face of the Newton polytope is called central if it contains a point all of whose coordinates are equal.

The Newton polygon of $(y^7, y^5x, y^3x^2, yx^4, x^6)$, with central face extended.

The next generalization of (26.3) follows from [Rei80]; see [How01] for various generalizations and [Laz04, Sec.9.3.C] for proof.

Proposition 55.1. A monomial ideal $I$ is log canonical iff its Newton polytope contains the point $(1, \ldots, 1)$.

Thus $I$ is not log canonical iff a central face of its Newton polytope contains a point $(d, \ldots, d)$ with $d > 1$. The equation of this face can then be written as

$$\sum a_ir_i = d\sum a_i \quad \text{for some} \quad a_i \geq 0.$$  

In particular, a monomial $\prod x_i^{r_i}$ is not contained in $I$ if $\sum a_ir_i \leq \sum a_i$. We have thus proved the following.

Corollary 55.2. A monomial ideal $I$ is not log canonical iff there is a simplex

$$\Delta(a) := (0 \leq r_i, \sum a_ir_i \leq \sum a_i)$$

that is disjoint from the Newton polytope of $I$. If this holds then

$$\dim(R/I) \geq \left( \text{number of lattice points in the simplex } \Delta(a) \right).$$
56 (Lattice points in simplices). We thus need to estimate from below the number of lattice points in the $n$-simplex $\left(0 \leq r_i, \sum a_i r_i \leq \sum a_i\right)$, independent of the $a_i$. I could not find the optimal values.

The lower bound $\frac{1}{2}3^n$ comes from the observation that if $r_i \in \{0, 1, 2\}$ then either $(r_1, \ldots, r_n)$ or $(2 - r_1, \ldots, 2 - r_n)$ satisfies $\sum a_i r_i \leq \sum a_i$. We can do a little better by adding the points with coordinates 3, . . . , $n$ on at least 1 of the coordinate axes.

Another lower bound is $\frac{1}{n} \binom{2n}{n}$, which is asymptotically $4^n/(n\sqrt{n})$. This comes from the observation that if $\sum r_i \leq n$ then at least one of the cyclic permutations of $(r_1, \ldots, r_n)$ satisfies $\sum a_i r_i \leq \sum a_i$.

The first bound is better for $n \leq 5$, the second for $n \geq 6$.

We can also combine the 2 bounds to get

$$\frac{1}{2} \left(\binom{2n}{n} - \frac{1}{2}3^n\right) + \frac{1}{2}3^n = \frac{1}{2} \binom{2n}{n} + \frac{n-1}{2}3^n.$$

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