Fault tolerant quantum computation with high threshold in two dimensions

Kovid Goyal
with Robert Raussendorf and Jim Harrington

Institute for Quantum Information
Caltech

December 20
Motivation
Operational requirements

We need experimentally viable methods for fault tolerance

- High threshold
- Threshold should be robust against variations in the error model
- Moderate overhead
- Simple architecture (e.g. no long range interaction)
Motivation
Operational requirements

- Preparation of 2D cluster state by translation invariant nearest-neighbor Ising type interactions
- Hadamard gate
- Single qubit measurements in the $X + Y$, $X$, $Y$ and $Z$ bases
- Classical post-processing of measurement results
Motivation
Operational requirements

2D, short range translation invariant interactions
Outline

1. Introduction

2. The fault tolerant $QC_C$

3. Threshold and overhead
The one way quantum computer measurement of $Z (\bigcirc)$, $X (\uparrow)$, $\cos \alpha X + \sin \alpha Y (\nearrow)$

- Universal computational resource: 2D cluster state.
- Information is written onto the cluster, processed and read out by single qubit measurements only.
The threshold theorem

Theorem

If the noise per elementary operation is below a constant non-zero threshold then an arbitrarily long quantum computation can be performed with arbitrary accuracy and small operational overhead.\(^a\)

\(^a\)Ahronov & Ben-Or (1996), Kitaev (1997), Knill, Laflamme & Zurek (1998), Aliferis, Gottesman & Preskill (2005)

- What is the threshold value?
- What is the overhead?
- What are the requirements on interaction?
### Known thresholds

| No constraint | Geometric constraint |
|---------------|----------------------|
| [1] 0.03, est. | [5] $7.5 \cdot 10^{-3}$, est. |
| [2] $10^{-3}$, est. | |
| [3] $10^{-4}$, est. | [6] $2 \cdot 10^{-5}$, est. |
| [4] $10^{-5}$, bd. | |
| [7] $10^{-8}$, bd. | |

[1] Knill, (2005); [2] Zalka (1999); [3] Dawson & Nielsen (2005); [4] Aliferis, Gottesman & Preskill (2005); [5] Raussendorf, Harrington & Goyal quant-ph/0703143; [6] Svore, DiVincenzo & Terhal, quant-ph/0604090; [7] Aharonov & Ben-Or (1999)
Main idea

Replace 2D cluster state with 3D cluster state

- The 3D cluster state is a fault tolerant substrate
- Topological quantum logic via lattice defects
- Mapping to 2D physical lattice
- Threshold: $7.5 \times 10^{-3}$
Macroscopic view

- Example CNOT

\[
\exp\left(i \frac{\pi}{8} X\right) \text{ gate}
\]

- Three cluster regions
  - \( V \) (Vacuum), \( D \) (Defect) and \( S \) (Singular)
    - \( V \): local \( X \) measurements
    - \( D \): local \( Z \) measurements
    - \( S \): local \( \frac{X+Y}{2} \), \( Y \) measurements

- Defect region \( D \) is string like. The quantum circuit is encoded in the topology of \( D \).
Microscopic view

Cluster edges

Elementary cell of the \( L \)attice

qubit location : (even, odd, odd) - face of \( L \)
qubit location : (odd, even, even) - edge of \( L \)
syndrome location : (odd, odd, odd) - cube of \( L \)
syndrome location : (even, even, even) - site of \( L \)
Key to the scheme

3D cluster state

code plane for surface code

simulated time

$t$
Surface codes

Surface codes\textsuperscript{\dag} are CSS codes associated with planar lattices

Harmful errors stretch across the entire lattice (rare events)

\textsuperscript{\dag} A. Kitaev, quant-ph/9707021 (1997)
$QCC$: topological error correction in $V$

Fault tolerant quantum memory with planar code $\Leftrightarrow$ Random plaquette $Z_2$ gauge model (RPGM)$^\dagger$.

Same error correction applies to the 3D cluster state

$^\dagger$Dennis et al., quant-ph/0110143 (2001).
Phase diagram of the RPGM

Have an error budget of 3%

- Error correction [1]
- Minimum weight chain matching [2]

[1] T. Ohno et al., quant-ph/0401101 (2004).
[2] E. Dennis et al., quant-ph/0110143 (2001); J. Edmonds, Canadian J. Math. 17, 449 (1965).
Fault tolerant quantum logic

Encoding capacity of the code depends on the topology of the code surface

code plane for surface code

3D cluster state

Plane segment
  1 Qubit

Torus
  2 Qubits

Plane with two holes
  1 Qubit
Surface code on a plane with holes

- There are two types of hole: primal and dual
- A pair of same-type holes form an encoded qubit
Defects are the extension of holes in the code plane to the third dimension.
Quantum logic via defect topology

C-NOT gate

Topological quantum gates are encoded in the way primal and dual defects are wound around each other.
Quantum gates

Clifford gates

control

target

CNOT

Out

Z-prep.

Out

Z-meas.

In

X-prep.

Out

X-meas.

In
Quantum gates

Non-Clifford gates

- Need one non-Clifford element:
  
  fault tolerant preparation of \( |A\rangle := \frac{X + Y}{\sqrt{2}} |A\rangle \)

- FT prep. of \( |A\rangle \) is achieved via concatenated magic state distillation\(^\dagger\) of logical qubits

\(^\dagger\) S. Bravyi and A. Kitaev, Phys. Rev. A 71, 022316 (2005).
Mapping to 2D

- Turn simulated time into real time
- Require a single 2D layer

![Diagram showing code plane for surface code and 3D cluster state](image)
Fault tolerance threshold

Error sources after mapping to 2D

1. **|+\rangle preparation**: Perfect preparation followed by single qubit partially depolarizing noise with probability $p_P$.

2. **$\Lambda(Z)$ gates** (space like edges of $\mathcal{L}$): Perfect gates followed by two qubit partially depolarizing noise with probability $p_2$.

3. **Hadamard gates** (time like edges of $\mathcal{L}$): Perfect gates followed by single qubit partially depolarizing noise with probability $p_1$.

4. **Measurement**: Perfect measurement preceded by single qubit partially depolarizing noise with probability $p_M$.

No qubit is idle between preparation and measurement – no memory error.
Fault tolerance threshold

Threshold estimate \( (p := p_1 = p_2 = p_P = p_M) \)

- Topological threshold in cluster region \( V: \)
  \[
  p_c = 7.5 \times 10^{-3}
  \]

- Threshold for magic state distillation:
  \[
  p_c = 2.8 \times 10^{-2}
  \]

- The threshold is robust against variations in the error model such as higher weight elementary errors or decaying long distance errors.

*Topological EC sets the overall threshold*
Operational overhead

(At \( p = \frac{1}{3} p_c \))

\[
\Lambda(X) = \exp \left( i \frac{\pi}{4} X \right)
\]

\[
\exp \left( i \frac{\pi}{8} Z \right)
\]

\[\overline{\Omega} \sim \log(\Omega)^3\]
Summary

Scenario
- Local and nearest neighbor gates on a 2D lattice

Performance
- Threshold: $7.5 \times 10^{-3}$
- Overhead: $\Omega \sim \log(\Omega)^3$

Method
- 3D cluster states provide intrinsic topological error correction and topologically protected quantum gates

Suitable systems for implementation
- Cold atoms in optical lattices, segmented ion traps, superconducting qubits, quantum dots...
Cold atoms in an optical lattice

(a) Shift

(b) Translation invariant 2D C-PHASE†

†Greiner et. al., Nature (2002)

Individual atom readout