Asynchronous Online Testing of Multiple Hypotheses

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Abstract

We consider the problem of asynchronous online testing, aimed at providing control of the false discovery rate (FDR) during a continual stream of data collection and testing, where each test may be a sequential test that can start and stop at arbitrary times. This setting increasingly characterizes real-world applications in science and industry, where teams of researchers across large organizations may conduct tests of hypotheses in a decentralized manner. The overlap in time and space also tends to induce dependencies among test statistics, a challenge for classical methodology, which either assumes (overly optimistically) independence or (overly pessimistically) arbitrary dependence between test statistics. We present a general framework that addresses both of these issues via a unified computational abstraction that we refer to as “conflict sets.” We show how this framework yields algorithms with formal FDR guarantees under a more intermediate, local notion of dependence. We illustrate these algorithms in simulation experiments, comparing to existing algorithms for online FDR control.

1 Introduction

Multiple testing has come of age, with steady progress in statistical theory and practice matched by a wide range of applications in science and technology. But the scale and scope of these applications has begun to outstrip the available theory and methodology. Statistical applications in domains such as medicine, commerce, finance and transportation are increasingly of planetary scale, with statistical analysis and decision-making tools being used to evaluate hundreds or thousands of related hypotheses in small windows of time [see, e.g., 28, 30]. This testing process is often sequential, conducted in the context of a continuing stream of data analysis. The sequentiality is at two levels—each individual test is often a sequential procedure, terminating at a random time when a stopping criterion is satisfied, and also the overall set of tests is carried out sequentially, with possible overlap in time. In this setting—which we refer to as online asynchronous testing—the goal is to control a criterion such as the false discovery rate (FDR) not merely at the end of a batch of tests, but at any moment in time, and to do so while recognizing that the decision for a given test must generally be made while other tests are ongoing.

The recent literature on “online FDR control” has responded to one aspect of this problem, namely the problem of providing FDR control during a sequence of tests, and not merely at the end, by adaptively setting the test levels for the tests\textsuperscript{[13] [17] [22] [24]}. These methods are synchronous, meaning that each test can only start when the previous test has finished. Our goal is to consider the more realistic setting in which each test is itself a sequential process and where tests can overlap in time. This is done in real applications to gain time efficiency, and because of the difficulties of coordination in a large-scale, decentralized setting. To illustrate this point, Figure\textsuperscript{[1]} compares the testing of five hypotheses within an asynchronous setting and a synchronous setting. In the asynchronous setting, the test level $\alpha_t$ used to test hypothesis $H_t$ is allowed to depend only on the outcomes of the previously completed tests—for example, $\alpha_3$ can depend on the outcome of $H_1$, however not on the outcome of $H_2$. In the synchronous setting, on the other hand, the test level $\alpha_t$ can depend on all previously started (hence also completed) tests. To account for the uncertainty about the tests in progress, the test levels assigned by asynchronous online procedures must be more conservative. Thus, there is a tradeoff—although asynchronous procedures take less time to perform
a given number of tests they are necessarily less powerful than their synchronous counterparts. The management of this tradeoff involves consideration of the overall power achieved per unit of real time, and consideration of the complexity of the coordination required in the synchronous setting.

Figure 1. Testing five hypotheses synchronously (top) and asynchronously (bottom). In both cases, the test levels $\alpha_t$ depend on the outcomes of previously completed tests, which in the synchronous case includes all previously started tests. At the start time of experiment $t$, $W_t$ is used to denote the remaining “wealth” for making false discoveries. At the end of experiment $t$, a $p$-value $P_t$ and its rejection indicator $R_t := 1 \{P_t \leq \alpha_t\}$ are known, which is used to adjust the available wealth at the start time of the next new test.

There is a vast literature on sequential testing [see, e.g., 29, 11, 10, 2, 21]. We do not aim to contribute to that literature per se; rather, our goal is to consider multiple testing through a more realistic lens as an outer sequential process, one that acknowledges the existence of inner sequential processes that are based on sequential testing.

Another limitation of existing work on online multiple testing is that the dependence assumptions on the tested $p$-value sequence, under which the formal false discovery rate guarantees hold, are usually at one of two extremes—they are either assumed to be independent, or arbitrarily dependent. From a practical perspective, independence seems overly optimistic as new tests may use previously collected data to formulate hypotheses, or to form a prior, or as evidence while testing. On the other hand, arbitrary dependence is likely too pessimistic, as older data and test outcomes with time become “stale,” and no longer have direct influence on newly created tests. We see that a reconsideration of dependence is natural in the setting of online FDR control, and is particularly natural in the asynchronous setting, given that tests that are being conducted concurrently are often likely to be dependent, since they may use the same or highly correlated data during their overlap.

We therefore define and study a notion of local dependence, and place it within the context of asynchronous multiple testing. Letting $P_t$ denote the $t$-th tested $p$-value, we say that a sequence of $p$-values $\{P_t\}$ satisfies local dependence if the following condition holds:

$$\text{for all } t > 0, \text{ there exists } L_t \in \mathbb{N} \text{ such that } P_t \perp P_{t-L_t-1}, P_{t-L_t-2}, \ldots, P_1,$$

(1)

where $\{L_t\}$ is a fixed sequence of parameters which we will refer to as “lags.” Clearly, when $L = 0$, we obtain the
independent setting, and when \( L = \infty \), we recover the arbitrarily dependent setting. If \( L_t = L \) for all \( t \), condition [1] captures a lagged dependence of order \( L \).

To further emphasize the natural connection between asynchrony and local dependence, consider the simple setting in Figure 2. This diagram captures the setting in which a research team is collecting data over time, and decides to run multiple tests in a relatively short time interval. For example, such a situation might arise when testing multiple treatments against a common control [25], or in large-scale A/B testing by internet companies [30]. Since there is overlap in the data these tests use to compute their test statistics, the corresponding \( p \)-values could be arbitrarily dependent. In general several tests might share data with the first test. Thus the \( p \)-values are locally dependent, with the lag parameter being equal to the number of consecutive tests that share data.

Figure 2. Example of \( p \)-values within a short interval computed on overlapping data. They exhibit local dependence; for example, \( P_3 \) and \( P_4 \) are independent of \( P_1 \).

In this paper, we reinforce this connection between asynchronous online testing and dependence by developing a general abstract framework in which, from an algorithmic point of view, these two issues are treated with a single formal structure. We do so by associating with each test a *conflict set*, which consists of other tests that have a potentially adversarial relationship with the test in question. Within this framework, we develop algorithms with provable guarantees on the FDR (technically, the modified FDR introduced in the next section). The core idea is to enforce a notion of pessimism with regard to the conflict set—when computing a new test level, the algorithm “hallucinates” the worst-case outcomes of the conflicting tests.

We derive procedures that handle conflict sets as strict generalizations of current state-of-the-art online FDR procedures; indeed, when there are no conflicts, for example when there is no asynchrony and when the \( p \)-values are independent, our solutions recover LORD [17], LOND [16] and SAFFRON [24], the latter of which recovers alpha-investing [13] as special cases for a particular choice of parameters. On the other hand, if the conflict sets are as large as possible—for example, if the parameter \( L_t \) or the number of tests run in parallel tend to infinity—our algorithms behave like alpha-spending [1], which was designed to control a more stringent criterion called the family-wise error rate (FWER), under any dependence structure. Independently, we also prove that the original LOND procedure controls the FDR even under positive dependence (PRDS), the first online procedure to provably have this guarantee under the PRDS condition that is popular in the offline FDR literature [6, 23].

The rest of this paper is organized as follows. After a presentation of technical preliminaries and related work, Section 2 presents the key notion of conflict sets. We present two general procedures based on conflict sets, deferring their formal FDR guarantees to Section 6. In Section 5 we provide further details of the way in which asynchronous testing can be couched in terms of conflict sets. In a similar fashion, in Section 4 we describe synchronous testing of locally dependent \( p \)-values using conflict sets, and present procedures having FDR guarantees within this environment. Section 5 then combines the ideas of local dependence and asynchronous testing into an overall framework designed for testing asynchronous batches of dependent \( p \)-values. Section 7 provides additional, stronger guarantees of the presented algorithms, which hold under more stringent assumptions on the \( p \)-value sequence. In Section 8 we present the numerical results of simulations designed to explore our methods, comparing them to existing procedures that handle dependent \( p \)-values. Finally, we conclude the paper with a short summary in Section 9.

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1Alpha-spending is a generalization of the Bonferroni correction in which the assigned test levels do not have to be equal. In other words, the Bonferroni correction suggests testing \( n \) hypotheses under level \( \frac{\alpha}{n} \), while alpha-spending merely requires \( \sum_{i=1}^{n} \alpha_i \leq \alpha \), where \( \alpha_i \) is the test level for the \( i \)-th hypothesis.
1.1 Technical preliminaries

We briefly overview the technical background upon which our work builds. Recall that the false discovery rate (FDR) is defined as follows:

\[ \text{FDR} \equiv \mathbb{E} \left[ \text{FDP} \right] = \mathbb{E} \left[ \frac{|H^0 \cap \mathcal{R}|}{|\mathcal{R}| \lor 1} \right], \]

where \( H^0 \) is the unknown set of true null hypotheses and \( \mathcal{R} \) is the set of hypotheses rejected by some procedure. Formally we have:

\[ H^0 = \{ i : H_i \text{ is true} \}, \quad \mathcal{R} = \{ i : H_i \text{ is rejected} \}. \]

The random ratio appearing inside the expectation is called the false discovery proportion (FDP). (The FDR is the expectation of the FDP.) It is also of theoretical and practical interest to study a related metric called the modified false discovery rate (mFDR):

\[ \text{mFDR} \equiv \frac{\mathbb{E} \left[ |H^0 \cap \mathcal{R}| \right]}{\mathbb{E} \left[ ||\mathcal{R}|| \lor 1 \right]} . \]

Foster and Stine have shown that in the long run the mFDR behaves similarly to the FDR, and that the mFDR is of particular interest in the analysis of online algorithms. We will mainly focus on the control of mFDR, as we can provide simple proofs under less restrictive assumptions. Importantly, in the Appendix we provide a side-by-side comparison of mFDR and FDR for all of the experiments in this paper; as we show there, the plots for mFDR and FDR are visually indistinguishable in every single case. To simplify our presentation, we will often suppress the distinction, referring to both of these metrics as “FDR.”

In online FDR control, the set of rejections possibly changes at each time step, implying changes in mFDR and FDR. Therefore, in online settings, we have to consider \( \mathcal{R}(t) \), which is the set of rejections up to time \( t \), and the naturally implied mFDR(\( t \)) and FDR(\( t \)). We will also use the symbol \( V(t) := \mathcal{R}(t) \cap H^0 \) to denote the set of false rejections made up to time \( t \). The main objective of online FDR algorithms is to ensure \( \text{mFDR}(t) \leq \alpha \) or \( \text{FDR}(t) \leq \alpha \), for a chosen level \( \alpha \) and for all times \( t \).

Many of the online FDR algorithms that have been proposed to date in the literature are special cases of the generalized alpha-investing (GAI) framework. These algorithms maintain an “alpha-wealth” (or “wealth” for short) \( W_t \), which changes dynamically with each new test. It is initialized at a value \( W_0 > 0 \). At time \( t \), a hypothesis \( H_t \) is tested, and the algorithm assigns a level \( \alpha_t \) to the current hypothesis test and computes the corresponding \( p \)-value \( P_t \). This causes the wealth to decrease by \( \phi_t \), and, if a rejection is made, meaning that \( P_t \leq \alpha_t \), the wealth increases by \( \psi_t \). This can be written as:

\[ W_t = W_{t-1} - \phi_t + \psi_t \mathbb{1} \{ P_t \leq \alpha_t \} . \]

The sequences \( \{ \alpha_t \} \), \( \{ \phi_t \} \), \( \{ \psi_t \} \) depend on the past tests and the target FDR level \( \alpha \). They also vary across different algorithms, with each algorithm incorporating some device to ensure that the wealth is always non-negative, otherwise no further discoveries can be made.

Ramdas et al. recently presented an alternative perspective on GAI algorithms. In this view, GAI algorithms are viewed as keeping track of an empirical estimate of the true false discovery proportion, denoted FDP(\( t \)), and they assign test levels \( \alpha_t \) in a way that ensures FDP(\( t \)) \leq \alpha for all time steps \( t \), where \( \alpha \) is the pre-specified FDR level. In the earlier paper, they show that such control on FDP estimates also yields FDR control. This perspective—which is equivalent to the wealth characterization of GAI algorithms—will provide the mathematical framework upon which we build in this paper.

Finally, we recap the typical assumptions made for null \( p \)-values in the FDR literature. If a hypothesis \( H_i \) is truly null, then the corresponding \( p \)-value \( P_i \) is stochastically larger than the uniform distribution (“super-uniformly distributed,” or “super-uniform” for short), meaning that:

\[ \text{If the null hypothesis } H_i \text{ is true, then } \Pr\{ P_i \leq u \} \leq u \text{ for all } u \in [0, 1]. \]
This assumption is sometimes generalized to the online FDR setting by incorporating a filtration $\mathcal{F}^{i-1}$:

If the null hypothesis $H_i$ is true, then $\Pr\{P_i \leq u \mid \mathcal{F}^{i-1}\} \leq u$ for all $u \in [0, 1]$,

(3)

As we discuss in later sections, however, this condition can be overly stringent when there are interactions between $p$-values, and we will accordingly introduce a weaker super-uniformity assumption.

1.2 Related work

There is a large and growing literature on false discoveries in multiple testing, aimed at solving a range of problems, often addressing issues of scientific reproducibility in research [15]. Here we focus on work whose methods or objectives have the most overlap with ours. In particular, we focus on literature on “online” methods in multiple testing, and compare and contrast those solutions to the ones we propose. The most salient difference is that we address the general problem of asynchrony; when there is no asynchrony, our approach recovers a slew of existing methods, including work by Foster and Stine [13], Aharoni and Rosset [1], Javanmard and Montanari [16, 17], Ramdas et al. [22, 24].

Most previous work also differs from ours in that it assumes that condition (3) holds. This condition is too strong for the notion of local dependence this paper considers; indeed, in Section 4 we present a simple toy example in which this assumption fails. An exception is the work of Javanmard and Montanari [16, 17], who discuss sufficient conditions for achieving FDR control under arbitrary dependence within the $p$-value sequence. However, these conditions essentially imply an alpha-spending-like correction for the test levels, making their proposed procedure overly conservative.

Robertson and Wason [25] have investigated the performance of several online FDR algorithms empirically, including all of those listed above, when the $p$-value sequence is positively dependent. They do not, however, provide any formal guarantees for those procedures that have thus far been shown to work only under independence. We make partial progress in this paper to justifying their empirical observations by proving that LOND provably controls FDR under positive dependence.

Recently, there has also been some work specifically motivated by controlling false discoveries in A/B testing in the tech industry [31]. However, their setup was again fully synchronous, and assume that the observations are independent across all experiments, which are the two assumptions this paper deems too strong and circumvents.

The vast literature on adaptive data analysis [12, 4, 9, 8] focuses on an online setting where a distribution is adaptively queried for a chosen functional, and at each step these queries are answered by making use of a single data set coming from that distribution. However, this line of work does not aim to provide FDR guarantees in online multiple testing.

Ordered hypothesis testing considers tests for which additional prior information is available, and allows sorting null hypotheses from least to most promising [19, 18, 20, 14]. In these papers, however, the word “sequential” or “ordered” does not refer to online testing; these methods are set in an offline environment, requiring access to all $p$-values at once. In our approach, we allow testing a possibly infinite number of hypotheses with no available knowledge of the future $p$-values.

2 Conflict sets: the unifying approach

In this section we describe a general, abstract formulation of multiple testing under asynchrony and dependence, which unifies the seemingly disparate solutions of this paper and provides the point of departure for deriving specific algorithms. We describe two such procedures, that we will refer to as LORD* and SAFFRON*, that control mFDR within this framework. The formal proof of mFDR control is provided in Section 6.

At time step $t \in \mathbb{N}$, the test of hypothesis $H_t$ begins, and the $p$-value resulting from this test is denoted $P_t$. In contradistinction to the standard online FDR paradigm, we do not require $P_t$ to be known at time $t$; indeed, this test is not fully performed at time $t$, but is only initiated at time $t$. The decision time for $H_t$ is denoted $E_t$; this is the time of possible rejection. Note that $E_t \neq t$ in the general case; moreover, $E_t$ might even be random. Throughout we assume that $P_t$ and $E_t$ are independent. It is worth pointing out that, unlike in the classical online FDR problem, the sets $R(t)$ and $V(t)$ now consider not all $\{P_i : i \leq t\}$, but only $\{P_i : E_i \leq t\}$. 
We let $\alpha_t$ denote the test level assigned to testing $H_t$. Unlike LORD*, SAFFRON* additionally lets the user choose $\lambda_t \geq \alpha_t$, which is the “candidacy threshold” at time $t$, meaning that, if $P_t \leq \lambda_t$, then $P_t$ is referred to as a candidate for rejection. Let $R_t := 1 \{ P_t \leq \alpha_t \}$ denote the indicator for rejection, and $C_t := 1 \{ P_t \leq \lambda_t \}$ denote the indicator for candidacy. This extension was introduced in the SAFFRON procedure of Ramdas et al. [24] and we will discuss it further below; for now, we simply note that it is an analog of the notion of “null-proportion adaptivity” in the offline multiple testing literature. Indeed, Ramdas et al. [24] argue that LORD can be seen as the online analog of the Benjamini-Hochberg (BH) procedure [5], while SAFFRON can be seen as the online analog of the adaptive Storey-BH procedure [26].

It is important to remark that we cannot simply ignore the asynchronous parallel aspects of the problem, and just run an online FDR algorithm on the p-values obtained whenever each test finishes (that is, whichever test is the $t$-th one to finish, test it at level $\alpha_t$). Firstly, this would completely ignore the possible dependence between tests, for example those that were running in parallel. Secondly, this scheme would only assign $\alpha_t$ to a test at the end of that test, which is unrealistic because sequential hypothesis tests (parametric ones such as Wald’s sequential probability ratio test (SPRT), and nonparametric ones as well [3]) usually require the type-1 target error level in advance because it is an important component of their stopping rule. For both these reasons, we need to specify $\alpha_t$ at the start of test $t$.

With the above remark in mind, we now define several filtrations. Let $I^t$ represent all the information known to the experimenter at time $t$, meaning that $I^t := \sigma(\{P_i; i : E_i \leq t\})$.

By $L^t$, we denote a filtration that captures all relevant information about the tests that started up to, and including, time $t$, for the LORD* procedure. Formally, $L^t := \sigma(\{R_t, E_1, \ldots, R_t, E_t\})$. For SAFFRON*, we also incorporate candidates in the filtration: $S^t := \sigma(\{R_t, C_1, E_1, \ldots, R_t, C_t, E_t\})$. Many of our arguments will apply to both algorithms; we accordingly use $F^t$ to indicate a generic filtration that can be either $L^t$ or $S^t$.

With each test and its corresponding hypothesis, we associate a conflict set. For the test starting at step $t$, we denote this set $X^t$; it consists of a (not necessarily strict) subset of $\{1, \ldots, t - 1\}$. For example, $X^t = \{3, 4\}$ could be $\{1, 5\}$. The reason why we refer to this set as conflicting for test $t$ is because it contains the indices of tests that interact with the $t$-th test in some unknown way. This could mean that, at time $t$, there is missing information about these tests, or that there potentially exists some arbitrary dependence between those tests and the upcoming one. A conflict set can also be random, however it needs to be measurable with respect to the filtration $F^{t-1}$, so that the test level $\alpha_t$ and candidacy threshold $\lambda_t$ can be computed accordingly.

We require the conflict sets to be monotone: each index $t$ has to be in a continuous “block” of conflict sets. More formally, if there exists $j$ such that $t \in X^j$, then $t \in X^i$, for all $i \in \{t + 1, \ldots, j\}$. This lets us define the last-conflict time of test $t$ as $\tau_t := \max\{j : t \in X^j\}$. If test $t$ never appears in a conflict set, we take $\tau_t = t$. Additionally, we make the natural assumption that $\tau_t \geq E_t$; if we proclaim a test as no longer conflicting, its decision has to be known.

Consider again the filtration $F^t$. A subtlety we initially ignored is that the superscript $t$ does not correspond to the physical quantity of time, meaning that $F^t \not\subset I^t$. In particular, different tests may run for different lengths of time and the decision time for each test may even be random; therefore, $R_t$ might be known before $R_{t-1}$. This motivates us to define a filtration as a counterpart of $F^t$ whose increase at each step corresponds to the real increase in knowledge with time. We introduce $F^{-X^t}$ as the non-conflicting filtration; the sigma-algebra $F^{-X^t}$ contains information about the tests that started before time $t$ which are not in the conflict set of test $t$. In particular, $L^{-X^t} := \sigma(\{R_i, E_i : i \leq t - 1, i \not\in X^t\})$ for LORD*, and $S^{-X^t} := \sigma(\{R_i, C_i, E_i : i \leq t - 1, i \not\in X^t\})$ for SAFFRON*. We have that $F^{-X^t} \subseteq F^{t-1}$. Notice that we promised to make this set a filtration; if $X^t$ was an arbitrary set of indices, this would not in general be satisfied. However, it is straightforward to verify that the monotonicity property of conflict sets ensures that $F^{-X^t}$ indeed forms a filtration.

We require $\alpha_t$ and $\lambda_t$ to be $F^{-X^t}$-measurable (as opposed to $I^t$-measurable). This is essentially the idea of pessimism mentioned earlier—among all tests that finished before the $t$-th one starts, $\alpha_t$ and $\lambda_t$ have to ignore the ones conflicting with test $t$ in order to guard against unknown interactions that the conflicting tests have with the upcoming one.

Further, we require the existence of a filtration $G^t$ such that the following holds:

\[
\text{If the null hypothesis } H_t \text{ is true, then } \Pr\{P_t \leq u \mid G^{E_t-1}\} \leq u, \text{ for all } u \in [0, 1],
\]
which can be rephrased as:

\[
\mathbb{E} \left[ \frac{1 \{ P_t > u \}}{1 - u} \left| G^{E_t - 1}_t \right. \right] \geq 1 \geq \mathbb{E} \left[ \frac{1 \{ P_t \leq u \}}{u} \left| G^{E_t - 1}_t \right. \right],
\]

where \( G^t \) additionally satisfies \( G^{E_t - 1}_t \supseteq F^{-X^t} \). This is just a condition that requires validity of null \( p \)-values: given the knowledge one has before making a decision, if a hypothesis is truly null, it has to be well-behaved. However, unlike in classical online FDR work, we do not require \( G^t = F^t \), as we will see in later sections. The additional assumption \( G^{E_t - 1}_t \supseteq F^{-X^t} \) can be interpreted in the following fashion: if \( F^{-X^t} \) was knowledge about non-conflicting tests before test \( t \) started, that knowledge should remain non-antagonistic also at decision time. We will choose \( G^t \) appropriately in what follows.

### 2.1 Oracle estimate under conflict sets

Following a recently proposed framework [22], we derive LORD\(^*\) and SAFFRON\(^*\) through an oracle estimate of the false discovery proportion. On a high level, this quantity serves as a good estimate of the true false discovery proportion, and controlling it under a pre-specified level guarantees that FDR is also controlled. Let the oracle estimate of the FDP be defined as:

\[
\text{FDP}^*(t) := \frac{\sum_{j \leq t, j \in H^0} \alpha_j}{(\sum_{E_t \leq t} R_j) \lor 1},
\]

where we recall that \( \alpha_j \) is required to be \( F^{-X^t} \)-measurable, across all \( j \). The following proposition gives formal justification for using \( \text{FDP}^*(t) \) as a proxy for the true FDP.

**Proposition 1.** Assume that we can define a filtration \( G^{E_t - 1}_t \supseteq F^{-X^t} \) such that the null \( p \)-values \( P_t \) are super-uniform conditional on \( G^{E_t - 1}_t \) as given by definition 4, and let \( \alpha_t \) be \( F^{-X^t} \)-measurable. Then, for all times \( t \in \mathbb{N} \), the condition \( \text{FDP}^*_{eq}(t) \leq \alpha \) implies that \( m\text{FDR}(t) \leq \alpha \).

**Proof.** Fix a time step \( t \in \mathbb{N} \). By this time, exactly \( t \) tests have started, and hence at most those \( t \) decisions are known. Therefore, by linearity of expectation:

\[
\mathbb{E} \left[ |V(t)| \right] = \mathbb{E} \left[ \sum_{E_t \leq t, j \in H^0} 1 \{ P_j \leq \alpha_j \} \right] \\
\leq \sum_{j \leq t, j \in H^0} \mathbb{E} \left[ 1 \{ P_j \leq \alpha_j \} \right].
\]

Applying the law of iterated expectations by conditioning on \( G^{E_j - 1}_j \) for each term, we obtain:

\[
\sum_{j \leq t, j \in H^0} \mathbb{E} \left[ 1 \{ P_j \leq \alpha_j \} \right] = \sum_{j \leq t, j \in H^0} \mathbb{E} \left[ \mathbb{E} \left[ 1 \{ P_j \leq \alpha_j \} \left| G^{E_j - 1}_j \right. \right. \right] \right] \\
\leq \sum_{j \leq t, j \in H^0} \mathbb{E} \left[ \alpha_j \right],
\]

which follows due to measurability of \( \alpha_j \) with respect to \( F^{-X^j} \subseteq G^{E_j - 1}_j \), and the super-uniformity property (4). If we assume \( \text{FDP}^*(t) := \frac{\sum_{j \leq t, j \in H^0} \alpha_j}{(\sum_{E_t \leq t} R_j) \lor 1} \leq \alpha \), then it follows that:

\[
\sum_{j \leq t, j \in H^0} \mathbb{E} \left[ \alpha_j \right] = \mathbb{E} \left[ \sum_{j \leq t, j \in H^0} \alpha_j \right] \\
\leq \alpha \mathbb{E} \left[ \left( \sum_{E_t \leq t} R_j \right) \lor 1 \right] \\
= \alpha \mathbb{E} \left[ |R(t)| \lor 1 \right],
\]
which follows by linearity of expectation and the assumption on $\text{FDP}^*(t)$. Rearranging yields the inequality

$$m\text{FDR}(t) := \mathbb{E}[|V(t)|] \leq \mathbb{E}[|R(t)|] \lor 1 \leq \alpha,$$

which completes the proof. 

The fact that $\alpha_t$ has to be measurable with respect to $\mathcal{F}^X_t$ should give us pause. Even though we are only required to guarantee $\text{FDP}^*(t) \leq \alpha$, we cannot rely on the rejection indicators that push down the value of $\text{FDP}^*(t)$, if they are in the current conflict set. As a consequence, $\alpha_t$ has to ensure $\text{FDP}^*(t) \leq \alpha$ for the worst-case configuration of conflicting rejections; that is, when $R_j = 0$ for all $j \in \mathcal{X}_t$. This motivates us to define the oracle estimate of the FDP under conflict sets:

$$\text{FDP}^*_{\text{conf}}(t) := \sum_{j \leq t, j \in \mathcal{H}_0} \alpha_j \left( \sum_{j \leq t, j \not\in \mathcal{X}_t} R_j \right) \lor 1.$$

Since this quantity is only more conservative than the oracle estimate, controlling it under $\alpha$ will preserve the guarantees given by Proposition 1. However, notice an unfortunate fact about both oracle estimates—they depend on the unobservable set $\mathcal{H}_0$. This implies that not even $\text{FDP}^*_{\text{conf}}(t)$ can be controlled exactly. Nevertheless, the take-away message of this discussion should be the following: if one can construct a procedure such that it assigns test levels implying, on average, that $\text{FDP}^*_{\text{conf}}(t) \leq \alpha$, then that procedure controls mFDR. In light of this result, our two proposed algorithms aim to construct empirical estimates of $\text{FDP}^*_{\text{conf}}(t)$ such that the properties given in Proposition 1 are retained.

### 2.2 The Lord* algorithm

A simple way to construct an empirical estimate that mimics the oracle would be to conservatively increase its numerator by summing over all test levels $\alpha_j$, regardless of the ground truth, obtaining thus a quantity almost surely greater than or equal to $\text{FDP}^*_{\text{conf}}(t)$ for all $t \in \mathbb{N}$. With this idea in mind, Lord* maintains control over the following estimate:

$$\text{FDP}^\text{LORD*}(t) := \frac{\sum_{j \leq t} \alpha_j \left( \sum_{j \leq t, j \not\in \mathcal{X}_t} R_j \right) \lor 1}{\left( \sum_{j \leq t, j \not\in \mathcal{X}_t} R_j \right) \lor 1}.$$

In other words, Lord* is defined as any update rule for $\alpha_t$ that ensures $\text{FDP}^\text{LORD*}(t) \leq \alpha$ for all $t \in \mathbb{N}$. Claiming mFDR control at fixed times now boils down to a simple observation: for any chosen $\alpha$, $\text{FDP}^\text{conf}(t) \leq \text{FDP}^\text{LORD*}(t) \leq \alpha$, hence by Proposition 1 mFDR is controlled.

A more sophisticated argument given in Section 6 will let us conclude that mFDR is also controlled at certain stopping times. This argument makes use of the wealth dynamics of Lord*; for this reason, Algorithm 1 and Algorithm 2 given below state the algorithm in terms of step-by-step wealth changes, using two different test level updates. Algorithm 1 generalizes the Lord++ procedure [17, 22], while Algorithm 2 generalizes its predecessor, the Lord procedure [16]. These are not the only ways of assigning $\alpha_j$ that are consistent with the assumptions and satisfy the definition of Lord* in their control of $\text{FDP}^\text{LORD*}$, but they are our focus in the remainder of the paper. Other rules can be found in the original Lord paper [17].

To state the algorithms in this paper, we will make use of the variable $r_k$, which refers to the first time that $k$ rejections are non-conflicting, meaning that there exist $k$ rejected hypotheses which are no longer in the conflict set at that time. That is, we define $r_k$ as:

$$r_k := \min\{i \in [t] : \sum_{j=1}^{i} R_j 1 \{ \tau_j \leq i \} \geq k \}.$$  

\[^2\]Here, as well as in the rest of this paper, we define the minimum of an empty set to be $-\infty$. 

Algorithm 1 The LORD++ algorithm under general conflict sets (a special case of LORD*)

**input:** FDR level \( \alpha \), non-negative non-increasing sequence \( \{\gamma_j\}_{j=1}^\infty \) such that \( \sum_j \gamma_j = 1 \), initial wealth \( W_0 \leq \alpha \)

Set \( \alpha_1 = \gamma_1 W_0 \)

for \( t = 1, 2, \ldots \) do

\( W_t := W_{t-1} - \alpha_t + \alpha \sum_{j=1}^t 1 \{ P_j \leq \alpha_j, r_j = t \} - W_0 1 \{ t = r_1 \} \)

\( \alpha_{t+1} = \gamma_{t+1} W_0 + \gamma_{t+1-r_1} (\alpha - W_0) + \left( \sum_{j=2}^t \gamma_{t+1-r_j} \right) \alpha \)

end

Algorithm 2 The LOND algorithm under general conflict sets (a special case of LORD*)

**input:** FDR level \( \alpha \), non-negative non-increasing sequence \( \{\gamma_j\}_{j=1}^\infty \) such that \( \sum_j \gamma_j = 1 \)

Set \( W_0 = \alpha, \alpha_1 = \gamma_1 W_0 \)

for \( t = 1, 2, \ldots \) do

\( W_t := W_{t-1} - \alpha_t + \alpha \sum_{j=1}^t 1 \{ P_j \leq \alpha_j, r_j = t \} - W_0 1 \{ t = r_1 \} \)

\( \alpha_{t+1} = \gamma_{t+1} \left( \sum_{j=1}^t 1 \{ P_j \leq \alpha_j, r_j \leq t \} \right) \lor 1 \)

end

In words, LORD* starts off with wealth \( W_0 \), whenever it starts a new test at time \( t \) it loses \( \alpha_t \) of wealth, and whenever a test \( t \) exits the conflict sets, if \( P_t \) was rejected, it earns back \( \alpha_t \), with the exception of earning \( \alpha - W_0 \) at the first such rejection. It is a simple algebraic exercise to verify that the two update rules given for \( \alpha_t \) indeed guarantee that \( \text{FDP}_{\text{LORD}}(t) \leq \alpha \) for all \( t \in \mathbb{N} \).

### 2.3 The SAFFRON* algorithm

Unlike LORD and LOND, the SAFFRON algorithm was originally derived through an FDP estimate, after observing that \( \sum_{j \leq t} \alpha_j \) might be overly conservative as an estimate of \( \sum_{j \leq t, j \in \mathcal{H}_0} \alpha_j \). Indeed, if the tested sequence contains a significant fraction of non-nulls, and if the non-nulls yield strong signals for rejection, these two quantities may be very far apart. Motivated by this observation, SAFFRON was developed as the adaptive counterpart of LORD which keeps track of an empirical estimate of how much alpha-wealth was spent on testing nulls, similar to the way in which Storey et al. [26, 27] improved upon the BH procedure [5]. We thus propose the SAFFRON* algorithm to maintain control over the following estimate:

\[
\text{FDP}_{\text{SAFFRON}^*}(t) := \frac{\sum_{j < t, j \notin X^t} \frac{\alpha_j}{1 - \lambda_j} 1 \{ P_j > \lambda_j \} + \sum_{j \in X^t \cup \{t\}} \frac{\alpha_j}{1 - \lambda_j}}{\sum_{j < t, j \notin X^t} R_j} \lor 1.
\]

Any update rule for \( \alpha_t \) and \( \lambda_t \) ensuring \( \text{FDP}_{\text{SAFFRON}^*}(t) \leq \alpha \) for all \( t \in \mathbb{N} \) satisfies the definition of SAFFRON*, and below we give a particular rule satisfying this inequality. This definition will be enough to prove mFDR guarantees at fixed times. Our stopping time proof, however, is easiest to spell out in the language of wealth; Algorithm 3 and Algorithm 4 describe two particularly useful instances of SAFFRON* in this fashion, obtained for specific choices of the sequence \( \{\lambda_j\} \). We present an algorithmic specification of SAFFRON* for the constant sequence \( \{\lambda_j\} = \lambda \) in Algorithm 3. A different case of SAFFRON* is presented in Algorithm 4, where we use the alpha-investing strategy \( \lambda_j = \alpha_j \).

One subtlety should be pointed out here. If \( \lambda_j \) is constant and equal to \( \lambda \) for all \( j \in \mathbb{N} \), \( \text{FDP}_{\text{SAFFRON}^*}(t) \) could be interpreted in two ways, as inducing two different wealth updates: one, in which the rewards are \( (1 - \lambda)\alpha \) and the penalties are \( \alpha \), as in Algorithm 3, or another in which the rewards are \( \alpha \) and the penalties are \( \frac{\alpha}{1 - \lambda} \), similarly to alpha-investing. Although our preferred interpretation is the first one, both versions are valid instances of SAFFRON* and the proved guarantees simultaneously hold for both.
Recall the definition \( r_k := \min\{i \in [t] : \sum_{j=1}^{i} R_j 1\{\tau_j \leq i\} \geq k\} \). For SAFFRON*, if \( \lambda_j \) is not constant, the algorithm statement additionally requires the identifier of the hypothesis that corresponds to the \( k \)-th non-conflicting rejection. Moreover, if more than one hypothesis becomes non-conflicting at the same time, i.e., if \( r_k = r_{k+1} \) for some \( k \), then we also require \( s_k \neq s_{k+1} \). Formally, we can write this identifier as:

\[
s_k := \min\{i : \sum_{j \leq \tau_i} 1\{\tau_j \leq \tau_i\} R_j \geq k \text{ and } i \neq s_m \text{ for all } m < k\}.
\]

### Algorithm 3 The SAFFRON* algorithm for constant \( \lambda \) under general conflict sets

**Input:** FDR level \( \alpha \), non-negative non-increasing sequence \( \{\gamma_j\}_{j=1}^{\infty} \) such that \( \sum_j \gamma_j = 1 \), candidate threshold \( \lambda \in (0, 1) \), initial wealth \( W_0 \leq (1 - \lambda)\alpha \)

**Output:** \( \gamma_j \)

**Algorithm:**

\[\alpha_1 = \gamma_1 W_0\]

**for** \( t = 1, 2, \ldots \) **do**

- start \( t \)-th test with level \( \alpha_t \)
  
  \[
  W_t := W_{t-1} - \alpha_t + \sum_{j=1}^{t} 1\{P_j \leq \alpha_j, \tau_j = t\} ((1 - \lambda)\alpha_1 1\{P_j \leq \alpha_j\} + \alpha_j 1\{P_j \leq \lambda\}) - W_0 1\{t = r_1\}
  \]
  
  \[
  \alpha_{t+1} = \min\{\lambda, W_t \gamma_{t+1} - C_{t+1} + ((1 - \lambda)\alpha - W_0)\gamma_{t+1} r_1 - C_{t+1} + \sum_{j=1}^{t+1} (1 - \lambda)\alpha_j \gamma_{t+1} r_{j-1} - C_{j+1}\},
  \]
  
  where \( C_j := \sum_{i=r_j+1}^{t} C_i 1\{i \notin X^t\} \)

**end**

### Algorithm 4 The alpha-investing algorithm under general conflict sets as a special case of SAFFRON*

**Input:** FDR level \( \alpha \), non-negative non-increasing sequence \( \{\gamma_j\}_{j=1}^{\infty} \) such that \( \sum_j \gamma_j = 1 \), initial wealth \( W_0 \)

**Output:** \( \gamma_j \)

**Algorithm:**

\[\alpha_1 = \gamma_1 W_0\]

**for** \( t = 1, 2, \ldots \) **do**

- start \( t \)-th test with level \( \alpha_t \)
  
  \[
  W_t := W_{t-1} - \frac{\alpha_t}{1 - \alpha_t} + \sum_{j=1}^{t} 1\{P_j \leq \alpha_j, \tau_j = t\} \left( \alpha + \frac{\alpha_j}{1 - \alpha_t} \right) - W_0 1\{t = r_1\}
  \]
  
  \[
  \alpha_{t+1} = W_0 \gamma_{t+1} r_0 + ((1 - \alpha_t)\alpha - W_0)\gamma_{t+1} r_{t-1} - R_{t+1} + \sum_{j=1}^{t+1} (1 - \alpha_{j+1})\alpha_j \gamma_{t+1} r_{j-1} - R_{j+1},
  \]
  
  where \( R_j := \sum_{i=r_j+1}^{t+1} R_i 1\{i \notin X^t\} \)

**end**

In the original SAFFRON algorithm, if the \( p \)-value \( P_t \) is a candidate, the algorithm does not lose wealth at time \( t \). Accordingly, depending on the candidacy threshold, the algorithm would also earn less wealth by making a discovery. From the wealth dynamics given above, we can see that SAFFRON* is indeed a pessimistic version of SAFFRON: it preemptively decreases wealth for each new test, regardless of the candidacy outcome, and as long as a particular test is in the conflict set, it cannot reward the current wealth. Only later, after the last-conflict time of that test, the algorithm adjusts for this pessimism. More explicitly, SAFFRON* starts off with wealth \( W_0 \), at the start time of a test at time \( t \) it decreases its wealth by \( \frac{\alpha_t}{1 - \lambda} \), and at \( \tau_t \) when test \( t \) is no longer a “threat,” it earns back \( \alpha + \frac{\alpha_t}{1 - \lambda} \) if the test resulted in a rejection, or \( \frac{\alpha_t}{1 - \lambda} \), if the test resulted in a non-rejected candidate. Otherwise, it leaves the wealth unchanged if the test resulted in a non-candidate. Again we have the exception that the first observed rejection results in a smaller reward, namely \( \alpha - W_0 + \frac{\alpha_t}{1 - \lambda} \), given that \( s_1 = t \).

### 3 Example 1: Asynchronous online FDR control

In this section, we formally introduce the problem of FDR control in asynchronous hypothesis testing, and show how it fits into the framework of conflict sets. This immediately gives two procedures for asynchronous online testing as special cases of LORD* and SAFFRON*. From here forward we will refer to these methods as LORD\textsubscript{async} and SAFFRON\textsubscript{async}, respectively. In Section 6, we provide mFDR guarantees of these procedures in terms of the general conflict-set setting, while in Section 7 we also prove FDR guarantees, although under strict independence of the tested \( p \)-values.
An asynchronous testing process consists of tests that start and finish at random times. Without loss of generality, one can think of the start times as fixed, and the finish times as random, which is achieved by discretizing time based on the start time of each test: at each time \( t \in \mathbb{N} \), a new test starts. Naturally, the finish time of test \( T \) might occur between the start times of two different tests, say \( j \) and \( j+1 \). However, the result of \( T \) is for the first time relevant at time \( j+1 \), when a new test is about to start, so that the error budget one has at disposal is known. In the classical online FDR setup, the test level at time \( j+1 \) is allowed to use information available up to time \( j \), so to retain this convention we assign the finish time of test \( T \) to time \( j \), thereby yielding a full discretization of time in the asynchronous testing. Note that this discussion is simply a mathematical convenience; the actual finish time can still fall anywhere between \( j \) and \( j+1 \). This discretized finish time coincides with the decision time of the conflict-set framework; therefore, we denote the finish time of the test that starts at time \( t \) as \( E_t \). Fully synchronous testing is thus an instance of this setting in which \( E_t = t \), as assumed in classical online FDR work.

As before, we use \( P_t \) to refer to the \( p \)-value that results from the test that starts at time \( t \). In Section 2, we made no assumption on when \( P_t \) is actually computed, however in asynchronous testing it is important to notice that \( P_t \) is not known at time \( t \), but only at time \( E_t \) (unless they are identical). That is, at time \( t \) not all information about the tests that previously started is available. Therefore, a natural definition of the asynchronous conflict set at time \( t \) is:

\[
\mathcal{X}^t_{\text{async}} = \{ i \in [t-1] : E_i \geq t \},
\]

which is observable at time \( t - 1 \). It is straightforward to verify that this conflict set satisfies the monotonicity property; every index \( i \) is indeed in a block of conflict sets, starting from \( \mathcal{X}^{i+1}_{\text{async}} \) and ending at \( \mathcal{X}^{E_i}_{\text{async}} \). This implies that \( \tau_i = E_i \), which, as we will see in the next section, need not be the case.

Denote by \( \mathcal{R}_t \) the set of rejections at time \( t \), meaning:

\[
\mathcal{R}_t = \{ i \in [t] : E_i = t, P_i \leq \alpha_i \}.
\]

Analogously, let \( \mathcal{C}_t \) denote the set of candidates at time \( t \):

\[
\mathcal{C}_t = \{ i \in [t] : E_i = t, P_i \leq \lambda_i \}.
\]

With this, we can write the non-conflicting filtration \( \mathcal{L}^{-\mathcal{X}^t}_{\text{async}} \) compactly as:

\[
\mathcal{L}^{-\mathcal{X}^t}_{\text{async}} := \sigma(\mathcal{R}_1, \ldots, \mathcal{R}_{t-1}),
\]

and similarly:

\[
\mathcal{S}^{-\mathcal{X}^t}_{\text{async}} := \sigma(\mathcal{R}_1, \mathcal{C}_1, \ldots, \mathcal{R}_{t-1}, \mathcal{C}_{t-1}).
\]

Since the arguments for LORD\textsubscript{async} and SAFFRON\textsubscript{async} have significant overlap, for brevity we write \( \mathcal{F}^{-\mathcal{X}^t}_{\text{async}} \) to refer to both \( \mathcal{L}^{-\mathcal{X}^t}_{\text{async}} \) and \( \mathcal{S}^{-\mathcal{X}^t}_{\text{async}} \), where possible. Recall the condition from Section 2 that \( \alpha_t \) has to be measurable with respect to \( \mathcal{F}^{-\mathcal{X}^t}_{\text{async}} \); here this essentially means that it has to be computed as a function of outcomes known by time \( t \). For SAFFRON\textsubscript{async}, additionally \( \lambda_t \) is \( \mathcal{S}^{-\mathcal{X}^t}_{\text{async}} \)-measurable. More specifically, for LORD\textsubscript{async}, we choose \( \alpha_t = f_t(\mathcal{R}_1, \ldots, \mathcal{R}_{t-1}) \), for some deterministic function \( f_t \). The SAFFRON\textsubscript{async} procedure also keeps track of encountered candidates, hence we take \( \alpha_t = g_t(\mathcal{R}_1, \mathcal{C}_1, \ldots, \mathcal{R}_{t-1}, \mathcal{C}_{t-1}) \) and \( \lambda_t = h_t(\mathcal{R}_1, \mathcal{C}_1, \ldots, \mathcal{R}_{t-1}, \mathcal{C}_{t-1}) \), for deterministic functions \( g_t \) and \( h_t \). Moreover, we require that the functions \( f_t, h_t \) and \( g_t \) are monotone, meaning that they are non-decreasing functions when keeping all inputs fixed but one. Formally, this requirement is only necessary for FDR control, so if one’s objective is to have mFDR guarantees, one can neglect this assertion. The assertion is, however, a natural one for the online FDR setting—intuitively, it captures the notion that the more one has discovered in the past, the more one can discover in the future.

One subtlety regarding test level assignment should be emphasized. In certain applications, \( \alpha_t \) has to be determined before starting the testing procedure, while in others it suffices to obtain \( \alpha_t \) at the very end of the test, at decision time. However, from the algorithmic point of view, the latter situation is equivalent to starting and finishing the test at the same time, which corresponds to the actual decision time. Therefore, in such applications, one should interpret the term “start time” loosely; it is merely the time at which the test level should be assigned.

11
Finally, we need to determine a filtration \( G^t \) in the definition of the super-uniformity condition \( \text{(4)} \), such that it yields an appropriate requirement. Since this section does not consider any specific dependence structure between \( p \)-values, we expect there to be no adversarial dependencies. Therefore, it is reasonable to assume \( P_t \) is well-behaved, given all the knowledge one has before the decision time regarding \( H_t \). We state this assumption in formal terms as the asynchronous super-uniformity condition:

\[
\text{If the null hypothesis } H_t \text{ is true, then } \Pr \left\{ P_t \leq u \mid F_{\text{async}}^{X_t} \right\} \leq u, \text{ for all } u \in [0, 1]. \tag{7}
\]

### The LORD\textsubscript{async} and SAFFRON\textsubscript{async} algorithms

We turn to an analysis of how the abstract LORD* and SAFFRON* procedures translate into our asynchronous testing scenario, for the particular choice of conflict set \( \chi^t_{\text{async}} \). We will see that they utilize all available information; the conflict set—the tests whose outcomes the algorithms ignore—consists only of the tests about which we temporarily lack information.

Plugging in the definition of \( \chi^t_{\text{async}} \), we obtain the following empirical estimate of the false discovery proportion for LORD\textsubscript{async}:

\[
\text{FDP}_{\text{LORD}_{\text{async}}} (t) : = \frac{\sum_{j \leq t} \alpha_j}{(\sum_{j \leq t} 1 \{ P_j \leq \alpha_j, E_j < t \}) \lor 1},
\]

whereas SAFFRON\textsubscript{async} controls:

\[
\text{FDP}_{\text{SAFFRON}_{\text{async}}} (t) : = \frac{\sum_{j \leq t} \alpha_j (1 \{ P_j > \lambda_j, E_j < t \} + 1 \{ E_j \geq t \})}{(\sum_{j \leq t} 1 \{ P_j \leq \alpha_j, E_j < t \}) \lor 1}.
\]

Consider the wealth dynamics of these two algorithms, and how their pessimism comes into play. Whenever a test starts, they simply decrease wealth, expecting that the resulting \( p \)-value will have no positive contribution to their wealth. However, when the test in question ends, they earn back wealth if they see a positive outcome, namely a candidate and/or rejection. This shows that testing in parallel indeed has a cost—due to pessimistic expectations about the tests in progress, the algorithms remain conservative when assigning a new test level. For this reason, asynchronous testing should be used with caution, and the number of tests run in parallel should be monitored closely. Indeed, in the asymptotic limit where the number of parallel tests tends to infinity the algorithm behaves like alpha-spending; i.e., the sum of all assigned test levels converges to the error budget \( \alpha \).

Substituting \( \chi^t \) for \( \chi^t_{\text{async}} \) in Algorithms 1-4 yields procedures for asynchronous online FDR control. The explicit statements of these algorithms, which correspond to asynchronous versions of LORD++, LOND, SAFFRON and alpha-investing, are given in the Appendix.

### 4 Example 2: Online FDR control under local dependence

In this section, we present a mathematical representation of local dependence and present online FDR procedures that handle such dependencies. We begin with the fully synchronous environment studied in classical online FDR literature, and turn to the asynchronous setting in the next section. We will see that dependencies also imply the existence of conflict sets, thus unifying our treatment of dependence and asynchrony.

A standard assumption in existing work on online FDR has been independence of \( p \)-values, a requirement that is rarely justified in practice. Tests that cluster in time often use the same data, null hypotheses depend on the outcomes of recent tests, etc. On the other hand, arbitrary dependence between any two \( p \)-values in the sequence is also arguably unreasonable—very old data used for testing in the past is usually considered “stale,” and hypotheses tested a long time ago may bear little relevance to current hypotheses. In light of this, we consider a notion of local dependence:

\[
\text{for all } t > 0, \text{ there exists } L_t \in \mathbb{N} \text{ such that } P_t \perp P_{t-L_t-1}, P_{t-L_t-2}, \ldots, P_1, \tag{8}
\]

where \( \{ L_t \} \) is a fixed sequence of parameters which we refer to as lags. When \( L_t \equiv L \) is an invariant sequence, we refer to condition \( \text{(8)} \) as lagged dependence of order \( L \).
Since we allow \( P_t \) to have arbitrary dependence on the previous \( L_t \) \( p \)-values, some of these dependencies might be adversarial toward the statistician, and, with “peeking” into this adversarial set, the nulls might no longer behave super-uniformly. Let us give a toy example in which two consecutive tests exhibit this type of behavior. Suppose we observe a sample \( X \sim N(\mu, 1) \), and wish to test two hypotheses using this sample. Let the two hypotheses be \( H_1 : \mu < 0 \) and \( H_2 : \mu \geq 0 \). If, for instance, \( R_1 = 0 \), we know that \( P_2 \leq 1 - \alpha \) almost surely, which implies that \( P_2 \) is not super-uniform, given the information about past tests. On the other hand, if we were to ignore the outcome of the first test, \( P_2 \) would indeed be super-uniform.

This observation motivates us to define the conflict set for testing under local dependence as:

\[
\mathcal{X}_{\text{dep}}^t := \{ t - L_t, \ldots, t - 1 \}.
\]

The hope is that ignorance really is bliss, and thus ignoring the tests corresponding to the conflict set should make all \( p \)-values well-behaved.

Since the decision time of each test is deterministic in the synchronous setting, we omit \( E_t \) from further consideration in this section; in particular we omit it from our filtrations since it makes them no richer. The non-conflicting filtration \( \mathcal{L}_{\text{dep}}^{-X^t} \) for LORD_{\text{dep}} is given by:

\[
\mathcal{L}_{\text{dep}}^{-X^t} := \sigma(R_1, \ldots, R_{t-L_t-1}),
\]

and similarly for SAFFRON_{\text{dep}}:

\[
\mathcal{S}_{\text{dep}}^{-X^t} := \sigma(R_1, C_1, \ldots, R_{t-L_t-1}, C_{t-L_t-1}).
\]

Since most formal arguments in this section apply to both procedures, we use \( \mathcal{F}_{\text{dep}}^{-X^t} \) to indicate that the filtration in question could be both \( \mathcal{L}_{\text{dep}}^{-X^t} \) and \( \mathcal{S}_{\text{dep}}^{-X^t} \).

In contrast to asynchronous testing, the levels \( \alpha_t \) and \( \lambda_t \) under local dependence ignore some portion of available information, specifically the outcomes of the last \( L_t \) tests. Notice the difference between these two settings—in the asynchronous setting, pessimism guards against unknown outcomes, while here pessimism guards against known outcomes. Perhaps counterintuitively, this observation means that the pessimism of LORD_{\text{dep}} and SAFFRON_{\text{dep}} actually guards against possible disadvantageous direct impact of the last \( L_t \) \( p \)-values on the upcoming one. To define the test levels and candidacy thresholds more formally, let \( \alpha_t := f_t(R_1, \ldots, R_{t-L_t-1}) \) for LORD_{\text{dep}} and \( \alpha_t := g_t(R_1, C_1, \ldots, R_{t-L_t-1}, C_{t-L_t-1}) \) and \( \lambda_t := h_t(R_1, C_1, \ldots, R_{t-L_t-1}, C_{t-L_t-1}) \) for SAFFRON_{\text{dep}}. As before, \( \{ \alpha_t \} \) and \( \{ \lambda_t \} \) are assumed to be monotone; that is, \( f_t, g_t \) and \( h_t \) are coordinate-wise non-decreasing functions. With these definitions, let us revisit the idea of pessimism; one could equivalently think of \( f_t \) as a function of \( t - 1 \) arguments, in which the last \( L_t \) are identically set to zero, and \( g_t \) and \( h_t \) as functions of \( 2(t - 1) \) arguments, in which the last \( 2L_t \) are set to zero. In other words, the last \( L_t \) tests are hallucinated to have resulted in no rejections nor candidates, despite the fact that the truth about them is known at time \( t \).

One can notice that, without any constraint on the sequence \( \{ L_t \} \), the non-conflicting “filtration” need not be a filtration by definition, because the conflict sets may not be monotone. Therefore, we translate the condition of monotonicity of conflict sets into a constraint on the sequence \( \{ L_t \} \) as:

\[
L_{t+1} \leq L_t + 1.
\]

Informally, this is just a requirement that the observable information does not decrease with time. Consequently, this would ensure that the test level \( \alpha_t \) and candidacy threshold \( \lambda_t \) have at least as much knowledge about prior tests as \( \alpha_{t-1} \) and \( \lambda_{t-1} \). This requirement is indeed a natural one, and usual testing practices satisfy it; for example, this condition holds if dependent \( p \)-values come in disjoint blocks (like in Section 5).

Consider some \( P_t \) which is from a null hypothesis. As previously emphasized, we cannot trust \( P_t \) to behave like a true null, given that we already know its last \( L_t \) predecessors that have a direct impact on it. The appropriate super-uniformity assumption then ignores these last \( L_t \) \( p \)-values and is of the following form:

\[
\text{If the null hypothesis } H_t \text{ is true, then } \Pr\left\{ P_t \leq u \ \bigg| \ \mathcal{F}_{\text{dep}}^{-X^t} \right\} \leq u, \text{ for all } u \in [0, 1]. \tag{9}
\]
The LORD\textsubscript{dep} and SAFFRON\textsubscript{dep} algorithms

As in Section 3, we analyze the particular instances of LORD\textsuperscript{*} and SAFFRON\textsuperscript{*} that are obtained by taking the conflict set of Section 2 to be $X'_{\text{dep}} = \{ t - L_t, \ldots , t - 1 \}$. Since this conflict set is deterministic, unlike $X'_{\text{async}}$, the estimate of the false discovery proportion that LORD\textsubscript{dep} and SAFFRON\textsubscript{dep} keep track of is completely determined $L_t$ steps ahead, that is at time $t - L_t - 1$.

By definition of the general estimate and the conflict set in consideration, LORD\textsubscript{dep} controls the following quantity:

$$\hat{\text{FDP}}_{\text{LORD}_{\text{dep}}}(t) : = \sum_{j \leq t} \alpha_j \frac{1}{\sum_{j \leq t, j \notin \{ t - L_t, \ldots , t - 1 \}} R_j} \lor 1,$$

while SAFFRON\textsubscript{dep} controls a more adaptive ratio:

$$\hat{\text{FDP}}_{\text{SAFFRON}_{\text{dep}}}(t) : = \sum_{j < t - L_t} \frac{\alpha_j}{\sum_{j \leq t, j \notin \{ t - L_t, \ldots , t - 1 \}} R_j} \lor 1 + \sum_{t \leq j < t - L_t, j \notin \{ t - L_t, \ldots , t - 1 \}} \frac{\alpha_j}{\sum_{j \leq t, j \notin \{ t - L_t, \ldots , t - 1 \}} R_j} \lor 1.$$

The wealth changes of these two algorithms are perhaps somewhat surprising. In the case of running asynchronous tests, the algorithms were constructed as pessimistic; however, they had access to as much information as the statistician performing the tests. Here, that is not the case—LORD\textsubscript{dep} and SAFFRON\textsubscript{dep} decrease wealth at time $t$, and choose to ignore the outcome of this test as long as it is in the conflict set of subsequent tests. Only after the last-conflict time $\tau_i$, positive outcomes are rewarded by earning some wealth. On the other hand, the statistician’s perspective is different—as soon as round $t$ is over, the statistician knows the outcome of the $t$-th test. Just like testing in parallel, testing locally dependent $p$-values comes at a cost—if the lags are large, the algorithm keeps subtracting ever smaller fractions of wealth, assigning ever smaller test levels, waiting for rewards from tests performed a very long time ago. In the extreme case of $L_t$ trending to infinity, the test levels steadily decrease so that their sum converges to $\alpha$, regardless of the fact that discoveries have possibly been made.

Explicit setting-specific algorithms can be obtained by substituting $X'$ for $X'_{\text{dep}}$ in Algorithms 1-4, resulting in LORD++, LOND, SAFFRON and alpha-investing under local dependence. Their detailed specifications are given in the Appendix.

5 Example 3: Controlling FDR in asynchronous mini-batch testing

Here we merge the ideas of the previous two sections, bringing together asynchronous testing and local dependence of $p$-values. Although there are various ways one could think of in which these two concepts intertwine, here we discuss a particularly simple and natural one.

Let a mini-batch represent a grouping of an arbitrary number of tests that are run asynchronously, which result in dependent $p$-values; for instance, these tests could be run on the same data. After a mini-batch of tests is fully executed, a new one can start, testing new hypotheses, independent of the previous batch, and doing so on fresh data. From the point of view of asynchrony, such a process could be thought of as a compromise between synchronous and asynchronous testing—batches are internally asynchronous, however they are globally sequential and synchronous. If all batches are of size one, one recovers classical online testing; if the batch-size tends to infinity, the usual notion of asynchronous testing is obtained. Figure 3 depicts an example of a mini-batch testing process with three mini-batches.
values have dependence on $P$ necessarily expect the $h$-th one in the $b$-th batch, testing hypothesis $H_{b,t}$. We allow any two $p$-values in the same batch to have arbitrary dependence; however, we require any two $p$-values in different batches to be independent. This can be written compactly as:

$$\alpha_{b,t} \perp P_{b,j}, \text{ for any } b_1, b_2, i, j, \text{ such that } b_1 \neq b_2.$$  

We will denote the size of the $b$-th batch as $n_b$. Thus, the first batch results in $p$-values $P_{1,1}, \ldots, P_{1,n_1}$, the second one in $P_{2,1}, \ldots, P_{2,n_2}$, etc. Analogously, the test levels and candidacy thresholds will also be doubly-indexed; $\alpha_{b,t}$ and $\lambda_{b,t}$ are used for testing $P_{b,t}$. Further, we define $R_{b,t} := 1 \{ P_{b,t} \leq \alpha_{b,t} \}$, and $C_{b,t} := 1 \{ P_{b,t} \leq \lambda_{b,t} \}$ as the rejection and candidacy indicators, respectively. By $R_b$ we will denote the set of rejections in the $b$-th batch, and by $C_b$ the set of candidates in the $b$-th batch.

Recall the key ideas of the previous two sections—tests running in parallel, or those resulting in dependent $p$-values, are seen as conflicting. We again pursue this approach, and let the conflict set of $P_{b,t}$-values, are seen as conflicting. We again pursue this approach, and let the conflict set of $P_{b,t}$-values in the same batch. More formally, the mini-batch conflict set can be defined as:

$$\lambda_{b,t}^{i} = \{ (b, i) : i < t \}.$$  

Notice that in Section 3 the conflicts arise solely due to missing information, in Section 4 solely due to dependence, while here they are due to both. It is straightforward to verify that these conflict sets are monotone—i.e., the test indexed by $(b, t)$ is in all conflict sets $\lambda_{b,t}^{i}$, where $i \in \{ t + 1, \ldots, n_b \}$.

The instances of LORD* and SAFFRON* used to test mini-batches will be referred to as LORD$_{mini}$ and SAFFRON$_{mini}$. As before, we will define the past-describing filtrations for both of these algorithms, however first we need to discuss a technical remark. Due to local dependence, as in Section 4 the wealth of an algorithm cannot be rewarded while a batch is running, due to the whole batch being mutually conflicted. Only at the finish time of that batch are the discoveries taken into account. For this reason, from the perspective of any batch, all rejections in any prior batch happened at one time step. Consequently, there is no need to consider the actual finish time of any test from previous batches. This implies that, for LORD$_{mini}$, the non-conflicting filtration will be of the form:

$$L_{\lambda_{b,t}}^{-} = \sigma(R_1, \ldots, R_{b-1}),$$  

while for SAFFRON$_{mini}$, this filtration is:

$$S_{\lambda_{b,t}}^{-} = \sigma(R_1, C_1, \ldots, R_{b-1}, C_{b-1}).$$  

As before, we use $F_{\lambda_{b,t}}^{-}$ to refer to both of these two filtrations simultaneously. The test levels $\{\alpha_{b,t}\}$ and candidacy thresholds $\{\lambda_{b,t}\}$ are therefore computed as functions of the outcomes of the tests in previous batches, i.e., we can write $\alpha_{b,t} = f_t(R_1, \ldots, R_{b-1})$ for LORD$_{mini}$, and similarly, $\alpha_{b,t} = g_t(R_1, C_1, \ldots, R_{b-1}, C_{b-1})$ and $\alpha_{b,t} = h_t(R_1, C_1, \ldots, R_{b-1}, C_{b-1})$ for SAFFRON$_{mini}$. We keep the monotonicity assumption, meaning that $f_t$, $g_t$, and $h_t$ are non-decreasing functions when keeping all inputs fixed but one.

By analogy with the last section, our super-uniformity assumption expresses an admission that we do not necessarily expect the $p$-value $P_{b,t}$ to be well-behaved, given that we have seen the outcomes of tests whose $p$-values have dependence on $P_{b,t}$. Therefore, the mini-batch algorithms assume the following:

$$\text{If the null hypothesis } H_{b,t} \text{ is true, then } \Pr \{ P_{b,t} \leq u \mid F_{\lambda_{b,t}}^{-}\} \leq u, \text{ for all } u \in [0, 1]. \quad (10)$$
The LORD* and SAFFRON* algorithms

Here we give explicit statements of LORD* and SAFFRON*, both in terms of the empirical estimate of the false discovery proportion, as well as their wealth updates. By definition of the mini-batch conflict set and the general estimate or LORD*, LORD* is obtained as any rule for assigning \( \alpha_{b,t} \) such that the following quantity is controlled:

\[
\widehat{\text{FDP}}_{\text{LORD}*}(b, t) := \frac{\sum_{i<b} \sum_{j \leq n_i} \alpha_{i,j} + \sum_{j \leq t} \alpha_{b,j}}{(\sum_{i<b} \sum_{j \leq n_i} R_{i,j}) \vee 1},
\]

for all \( b, t \in \mathbb{N} \). Using the same conflict set, SAFFRON* controls:

\[
\text{FDP}_{\text{SAFFRON}*}(b, t) := \frac{\sum_{i<b} \sum_{j \leq n_i} \alpha_{i,j} \sum_{P_{i,j} \geq \lambda_{i,j}} 1 \{ P_{i,j} > \lambda_{i,j} \} + \sum_{j \leq t} \alpha_{b,j}}{(\sum_{i<b} \sum_{j \leq n_i} R_{i,j}) \vee 1}.
\]

Since the set of rejections corresponding to tests that are not in the current conflict set is invariant throughout the testing of any whole batch, the wealth gradually decreases while a batch is being tested. Only when the batch has finished testing in its entirety does the algorithm earn back wealth for every rejection it made in that batch. This implies that the batch size should be carefully chosen, as the achieved power decreases with batch size. This is numerically verified in Section 8.

The LORD++, LOND, SAFFRON and alpha-investing procedures for mini-batch testing are explicitly stated in the Appendix, obtained by substituting \( \lambda_{\text{mini}} \) into Algorithms 1-4.

6 Controlling mFDR at fixed and stopping times

The previous three sections have shown that the abstract framework of conflict sets is a useful representational tool for expressing interactions across different tests, yielding three natural specific testing protocols. In this section, we return to the abstract unified framework in order to prove mFDR guarantees of LORD* and SAFFRON*, which implies mFDR control of all of the setting-specific algorithms.

We begin by focusing on fixed-time mFDR control. As mentioned earlier, the claim for LORD* follows trivially from Proposition 1, so Theorem 1 focuses on providing guarantees for SAFFRON*.

**Theorem 1.** Let \( P_t \) denote the p-value that results from the test that starts at time \( t \), and let \( \mathcal{X}^t \) denote its conflict set. Further, let the null p-values be super-uniform conditional on \( G^{E_t-1} \supseteq \mathcal{F} - \mathcal{X}^t \) (4). Then, LORD* and SAFFRON* both guarantee that mFDR \((t) \leq \alpha \) for all \( t \in \mathbb{N} \).

**Proof.** As stated before, the guarantees for LORD* follow directly from Proposition 1 after observing that \( \text{FDP}_{\text{cond}}(t) \leq \text{FDP}_{\text{LORD}^*}(t) \leq \alpha \) holds almost surely for all \( t \in \mathbb{N} \). Therefore, in the rest of this proof, we focus on SAFFRON*.

Fix a time \( t \). Then, we have:

\[
\mathbb{E}[|V(t)|] = \mathbb{E} \left[ \sum_{E_t \leq t, j \in \mathcal{H}^0} 1 \{ P_j \leq \alpha_j \} \right] \\
\leq \sum_{j \leq t, j \in \mathcal{H}^0} \mathbb{E}[1 \{ P_j \leq \alpha_j \}],
\]

where the inequality follows because the set of rejections made by time \( t \) could be at most the set \([t]\). Note that \( \alpha_j \) and \( \lambda_j \) are measurable with respect to \( G^{E_t-1} \), since \( \mathcal{S} - \mathcal{X}^t \subseteq G^{E_t-1} \). Therefore, applying iterated expectations by conditioning on \( G^{E_t-1} \) gives:

\[
\sum_{j \leq t, j \in \mathcal{H}^0} \mathbb{E}[1 \{ P_j \leq \alpha_j \}] \leq \sum_{j \leq t, j \in \mathcal{H}^0} \mathbb{E}[\alpha_j] \\
\leq \sum_{j \leq t, j \in \mathcal{H}^0} \mathbb{E} \left[ \alpha_j \frac{1 \{ P_j > \lambda_j \}}{1 - \lambda_j} \right],
\]
where we apply the super-uniformity property \( \text{(4)} \). If we assume that

\[
\widehat{\text{FDP}}_{\text{SAFFRON}}^*(t) := \frac{\sum_{j<t, j \notin X^t} \frac{\alpha_j}{1 - \lambda_j} \mathbb{1}\{P_j > \lambda_j\}}{\sum_{j<t, j \notin X^t} R_j} \leq \alpha,
\]

then it follows that:

\[
\sum_{j \leq t, j \in \mathcal{H}^0} E \left[ \alpha_j \frac{1}{1 - \lambda_j} \mathbb{1}\{P_j > \lambda_j\} \right] \leq \sum_{j \leq t} E \left[ \frac{\alpha_j}{1 - \lambda_j} \mathbb{1}\{P_j > \lambda_j\} \right]
\]

\[
\leq E \left[ \sum_{j<t, j \notin X^t} \frac{\alpha_j}{1 - \lambda_j} \mathbb{1}\{P_j > \lambda_j\} + \sum_{j \in X^t \cup \{t\}} \frac{\alpha_j}{1 - \lambda_j} \right]
\]

\[
\leq \alpha E \left[ \sum_{j<t, j \notin X^t} R_j \right]
\]

\[
\leq \alpha E [R(t)],
\]

where the first inequality drops the condition \( j \in \mathcal{H}^0 \), the second one ignores the condition \( \mathbb{1}\{P_j > \lambda_j\} \) for some terms, the third inequality applies the assumption on \( \widehat{\text{FDP}}_{\text{SAFFRON}}^*(t) \) and the last inequality uses the fact that \( R(t) \) contains all past rejections that are no longer conflicting. Rearranging the terms in the previous derivation, we reach the conclusion that \( \text{mFDR}(t) \leq \alpha \), which concludes the proof of the theorem.

The result of Theorem 1 actually holds much more generally; in particular, in the next two theorems we show that mFDR is also controlled at certain stopping times. Our approach is based on constructing a process which behaves similarly to a submartingale, which allows us to derive a result mimicking optional stopping. This process, however, is not a submartingale in the general case. For example, it is not a submartingale in the synchronous setting under local dependence, described in Section 4.

More specifically, we show that LORD* and SAFFRON* control mFDR at any stopping time \( T \) which satisfies the following conditions:

(C1) \( T \) is defined with respect to the filtration \( \mathcal{G}^t \) used in the super-uniformity condition \( \text{(4)} \), \( \{T = t\} \in \mathcal{G}^t \).

(C2) \( T \) has finite expectation, \( E[T] < \infty \).

(C3) Let \( J(t) := \{i \in \mathbb{N} : R_i \in \mathcal{G}^t\} \). We require that, on the event \( \{T = t\} \), it almost surely holds that \( \sum_{i \in J(t)} R_i \geq 1 \). We then say that \( T \) is non-trivial.

Condition (C2) is a mild one, as in practice we primarily care about stopping times with finite expectation. For instance, one would not wait infinitely long to observe the first rejection; if \( T_{\text{r}_1} \) denotes the time of the first rejection, a natural stopping time would be \( T := T_{\text{r}_1} \wedge t_{\text{max}} \), where \( t_{\text{max}} \) is the fixed longest time one is willing to wait for a rejection. Note also that Condition (C3) is not necessary if \( E_i = \tau_i \); in particular, it is not necessary in the asynchronous setting described in Section 3.

Here we present a proof for LORD*. We defer the full proof for SAFFRON* to the Appendix, as it utilizes similar ideas. We begin with a lemma.

**Lemma 1.** If \( T \) is a random variable supported on \( \mathbb{N} \) with finite expectation, then the random variable

\[
Y_1 := \sum_{j \leq T, j \in \mathcal{H}^0} \left( \mathbb{1}\{P_j \leq \alpha_j\} + \alpha_j \right)
\]

also has finite expectation.

The proof of Lemma 1 is deferred to the Appendix.
Proof. For all \(t \in \mathbb{N}\), define the process \(A(t)\) as:

\[
A(t) := - \sum_{i \leq t, i \in \mathcal{H}} 1\{E_i \leq t\} \{1\{P_i \leq \alpha_i\} - \alpha_i\}
\]

\[
= A(t - 1) - \sum_{i \leq t, i \in \mathcal{H}} 1\{E_i = t\} \{1\{P_i \leq \alpha_i\} - \alpha_i\},
\]

where we take \(A(0) = 0\). Let \(H(t) := 1\{T \geq t\}\). Since \(T\) is a stopping time, it holds that \(\{T \geq t + 1\} = \{T \leq t\}^c \in \mathcal{G}\), therefore \(H(t + 1)\) is predictable, that is it is measurable with respect to \(\mathcal{G}\). Define the transform \((H \cdot A)\) of \(H\) by \(A\) as follows:

\[
(H \cdot A)(t) := \sum_{m=1}^{t} H(m)(A(m) - A(m - 1))
\]

\[
= \sum_{m=1}^{t} H(m)(\sum_{i \leq m, i \in \mathcal{H}} 1\{E_i = m\} \{1\{P_i \leq \alpha_i\} - \alpha_i\}).
\]

By taking conditional expectations, we can obtain:

\[
\mathbb{E}[(H \cdot A)(t + 1) \mid \mathcal{G}^t] = \mathbb{E}[(H \cdot A)(t) \mid \mathcal{G}^t] + \mathbb{E}[H(t + 1)(A(t + 1) - A(t)) \mid \mathcal{G}^t]
\]

\[
= \mathbb{E}[(H \cdot A)(t) \mid \mathcal{G}^t] + H(t + 1)\mathbb{E}\left[- \sum_{i \leq t + 1, i \in \mathcal{H}} 1\{E_i = t + 1\} \{1\{P_i \leq \alpha_i\} - \alpha_i\} \mid \mathcal{G}^t\right]
\]

\[
= \mathbb{E}[(H \cdot A)(t) \mid \mathcal{G}^t] + H(t + 1) \sum_{i \leq t + 1, i \in \mathcal{H}} \mathbb{E}[-1\{E_i = t + 1\} \{1\{P_i \leq \alpha_i\} - \alpha_i\} \mid \mathcal{G}^t],
\]

where the first and last equality follow by linearity of expectation, and the second one uses the predictability of \(H(t + 1)\). The term \(-1\{E_i = t + 1\} \{1\{P_i \leq \alpha_i\} - \alpha_i\}\) is clearly non-negative when \(E_i \neq t + 1\). If \(E_i = t + 1\), we can invoke the super-uniformity condition \((4)\), since we are summing over null indices:

\[
\mathbb{E}[-1\{P_i \leq \alpha_i\} + \alpha_i \mid \mathcal{G}^t] \geq -\alpha_i + \alpha_i = 0.
\]

Therefore, additionally applying the law of iterated expectations, it follows that:

\[
\mathbb{E}[(H \cdot A)(t + 1)] \geq \mathbb{E}[(H \cdot A)(t)].
\]

Iteratively applying the same argument, we reach the conclusion that, for all \(t \in \mathbb{N}\):

\[
\mathbb{E}[(H \cdot A)(t)] \geq 0. \tag{11}
\]

So far we have only used the predictability of \(H(t)\); observe that, by its definition, and the definition of \(A(t)\):

\[
(H \cdot A)(t) = A(T \wedge t) - A(0) = A(T \wedge t),
\]

and hence by equation \((11)\), we obtain:

\[
\mathbb{E}[(H \cdot A)(t)] = \mathbb{E}[A(T \wedge t)] \geq 0.
\]

Define \(Y_1 := \sum_{j \leq T, j \in \mathcal{H}} (1\{P_j \leq \alpha_j\} + \alpha_j)\), and observe that \(Y_1 \geq |A(T \wedge t)|\) almost surely. Since \(A(T \wedge t) \to A(T)\) almost surely as \(t \to \infty\), by Lemma \((4)\) and dominated convergence we can conclude that \(\mathbb{E}[A(T \wedge t)] \to \mathbb{E}[A(T)]\) as \(t \to \infty\). With this we obtain a useful intermediate result:

\[
\mathbb{E}[A(T)] \geq 0. \tag{12}
\]
Recall that \( R(t) \) denotes the set of all rejections made by time \( t \), and \( V(t) \) denotes the set of false rejections made by time \( t \). Consider the following process:

\[
B(t) := \alpha |R(t)| - |V(t)| - W_t \\
= \alpha |R(t)| - |V(t)| + \sum_{j \leq t} \alpha_j - \sum_{j < t, j \notin \mathcal{X}^t} R_j \alpha + W_0 \{ t \geq r_1 \} - W_0 \\
\geq -|V(t)| + \sum_{j \leq t} \alpha_j + W_0 \{ t \geq r_1 \} - W_0 \\
\geq -|V(t)| + \sum_{j \leq t} \alpha_j \\
\geq A(t),
\]

where the second equality follows by the wealth dynamics of LORD*, the second inequality uses the fact that \( T \) is non-trivial, and the third inequality applies the definition of \( A(t) \) together with the fact that \( \sum_{j \leq t} \alpha_j \geq \sum_{j \leq t} 1 \{ E_j \leq t \} \alpha_j \).

Now take a stopping time \( T \) such that the conditions of the theorem are satisfied, then:

\[
E[|R(T)| - |V(T)|] = E[B(T) + W_T] \\
\geq E[B(T)] \\
\geq E[A(T)] \\
\geq 0,
\]

where the first inequality follows by the non-negativity of wealth, the second one by the relationship already established between \( A(T) \) and \( B(T) \), and the third inequality applies the intermediate result (12). Rearranging the terms we have that \( mFDR(T) \leq \alpha \), as desired.

The guarantees for SAFFRON* follow in a similar fashion. A minor technical condition one needs to ensure is that the sequence \( \{ \lambda_j \} \) is uniformly bounded away from zero. This condition is easily satisfied; for example, \( \lambda \) can be chosen as a fixed constant. We first introduce Lemma 2 which is SAFFRON*'s analog of Lemma 1 after which we state the second part of our main result regarding stopping-time mFDR control.

**Lemma 2.** If \( \min_{j \in \mathbb{N}} \lambda_j \geq \epsilon \) for some \( \epsilon > 0 \) and \( T \) is a random variable supported on \( \mathbb{N} \) with finite expectation, then the random variable

\[
Y_2 := \sum_{j \leq T, j \in \mathcal{X}^t} \left( 1 \{ P_j \leq \alpha_j \} + 1 \{ P_j > \lambda_j \} \frac{\alpha_j}{\lambda_j} \right)
\]

also has finite expectation.

The proof of Lemma 2 can be found in the Appendix.

**Theorem 3.** Let \( \alpha \) be the target FDR level of SAFFRON*, and let the null \( p \)-values be super-uniform conditional on \( \mathcal{G}^{E_t-1} \). Consider any stopping time \( T \) such that it satisfies conditions (C1-C3). Let also \( \min_{j \in \mathbb{N}} \lambda_j \geq \epsilon \) for some \( \epsilon > 0 \). Then, the SAFFRON* algorithm controls mFDR at \( T \): \( mFDR(T) \leq \alpha \).

Due to its similarity to the proof of Theorem 2 the proof of Theorem 3 is deferred to the Appendix.

### 7 Additional results on strict FDR control

#### 7.1 FDR control of LORD\(_{async} \) and SAFFRON\(_{async} \)

Even though the main objective of the paper is to provide mFDR guarantees for LORD\(_{async} \) and SAFFRON\(_{async} \), one can also obtain FDR control, provided that the \( p \)-values in the sequence are independent. This is in line
with earlier work where (synchronous) online FDR control has only been proved under independence assumptions [17][22][24]. While our arguments below generalize the earlier ones, we stress that the independence assumption may not be reasonable in asynchronous settings, which is why we focused on the mFDR for most of the paper and we only present the argument below for completeness. We briefly present the argument here.

First we state a technical lemma that is the key ingredient in proving FDR control of our asynchronous procedures. To introduce the lemma, note that by definition of a null p-value $P$, we have for any $x, y \in (0, 1)$, we have $\Pr\{P \leq x\} \leq x$ and $\Pr\{P > y\} \geq 1 - y$, and hence

$$\mathbb{E}\left[\frac{x1\{P > y\}}{(1 - y)}\right] \geq x \geq \mathbb{E}\left[1\{P \leq x\}\right].$$

The following lemma is a generalization of the above fact, and also a generalization of similar lemmas that have appeared before [17][22][24].

**Lemma 3.** Assume that the p-values $P_1, P_2, \ldots$ are mutually independent and each $P_t$ is conditionally independent of its respective decision time: $P_t \perp E_t|\mathcal{F}_{\text{async}}^{E_{t-1}}$. Moreover, let $g : \{\mathbb{N} \cup \{0\}\}^M \rightarrow \mathbb{R}$ be any coordinate-wise non-decreasing function. Then, for any index $t \leq M$ such that $t \in \mathcal{H}^0$, we have:

$$\mathbb{E}\left[\frac{\alpha_t1\{P_t > \lambda_t\}}{(1 - \lambda_t)g(|R|_{1:M})}\right] \mathcal{F}_{\text{async}}^{E_{t-1}} \geq \mathbb{E}\left[\frac{\alpha_t1\{P_t \leq \lambda_t\}}{g(|R|_{1:M})}\right] \mathcal{F}_{\text{async}}^{E_{t-1}},$$

where $|R|_{1:M} = (|R_1|, \ldots, |R_M|)$.

With this lemma, we directly obtain FDR guarantees of LORD$_{\text{async}}$ and SAFFRON$_{\text{async}}$ under independence, as stated in the following theorem.

**Theorem 4.** If the null p-values are independent of each other and of the non-nulls, and all p-values are conditionally independent of their decision time, then LORD$_{\text{async}}$ and SAFFRON$_{\text{async}}$ both achieve FDR control. In mathematical terms, FDR$(t) \leq \alpha$ for all $t \in \mathbb{N}$ is implied by either:

1. $\text{FDP}_{\text{LORD}_{\text{async}}} (t) \leq \alpha$ for all $t \in \mathbb{N}$,

2. $\text{FDP}_{\text{SAFFRON}_{\text{async}}} (t) \leq \alpha$ for all $t \in \mathbb{N}$.

The proofs of Lemma 3 and Theorem 4 are given in the Appendix.

### 7.2 FDR control of LOND under positive dependence (PRDS)

We can also prove that the original LOND algorithm [16] controls FDR for an arbitrary sequence of p-values that satisfy positive regression dependency on a subset (PRDS) [6], without any correction. In other words, under the PRDS assumption, it suffices to take all conflict sets in the sequence to be empty. For convenience, we state the formal definition of PRDS in the Appendix.

Recall the setup of the LOND algorithm. Given a non-negative sequence $\{\gamma_j\}_{j=1}^{\infty}$ such that $\sum_{j=1}^{\infty} \gamma_j = 1$, the test levels are set as $\alpha_t = \alpha \gamma_t(|R|(t - 1) \lor 1)$, where $|R|(t - 1)$ denotes the number of rejections at time $t - 1$. Note that this rule is monotone, in the sense that $\alpha_t$ is coordinate-wise non-decreasing in the vector of rejection indicators $(R_1, \ldots, R_{t-1})$. Below, we prove that LOND controls the FDR at any time $t \in \mathbb{N}$ under PRDS.

Recalling the definition of reshaping [23][24], we will also prove that if $\{\beta_t\}$ is a sequence of reshaping functions, then using the test levels $\bar{\alpha}_t := \alpha \gamma_t \beta_t(|R|(t - 1) \lor 1)$ controls FDR under arbitrary dependence. We call this the reshaped LOND algorithm. As one example, using the Benjamini-Yekutieli reshaping yields $\bar{\alpha}_t := \alpha \gamma_t \frac{|R|(t - 1) \lor 1}{\sum_{i=1}^{t-1} \frac{1}{\gamma_i}}$.

**Theorem 5.** (a) The LOND algorithm satisfies $\text{FDR}(t) \leq \alpha$ for all $t \in \mathbb{N}$ under positive dependence (PRDS).

(b) The reshaped LOND algorithm satisfies $\text{FDR}(t) \leq \alpha$ for all $t \in \mathbb{N}$ under arbitrary dependence.

The proof of Theorem 5 can be found in the Appendix.
8 Numerical experiments

Here we present the results of several numerical simulations, which show the gradual change in performance of LORD* and SAFFRON* with the increase of asynchrony and the lags of local dependence. We also compare these solutions to existing procedures with formal FDR guarantees under dependence. The plots in this section compare the achieved power and FDR of LORD async, SAFFRON async, LORD dep, SAFFRON dep, LORD mini and SAFFRON mini for different problem parameters, in settings with p-values computed from Gaussian observations.

The justification for using synthetic Gaussian data is three-fold. First, there is no standardized real data set for testing online FDR procedures. The quintessential applications of these methods involve testing with sensitive data, which are not publicly available due to privacy concerns. Second, even if these data were obtainable, it is unclear how one would evaluate the ground truth. Third, due to the central limit theorem, averages of many samples behave like Gaussian random variables. Practitioners exploit this property and often use variants of a t-test in applications such as clinical trials or A/B testing.

In all presented simulations we control the FDR under $\alpha = 0.05$, and estimate the FDR and power by averaging the results of 200 independent trials. The SAFFRON-type algorithms use the constant candidacy threshold sequence $\lambda = 1/2$, across all tests. The LORD-type algorithms use the LORD++ update for test levels. Each figure additionally plots the performance of uncorrected testing, in which the constant test level $\alpha_t = \alpha = 0.05$ is used across all $t \in \mathbb{N}$, and alpha-spending, whose test levels decay according to the $\{\gamma_t\}_{t=1}^{\infty}$ sequence of LORD* and SAFFRON*.

The experiments test for the means of $M = 1000$ Gaussian observations, and each null hypothesis takes the form $H_i : \mu_i = 0$, where $\mu_i$ is the mean of the Gaussian sample. We generate samples $\{Z_i\}_{i=1}^{M}$, where $Z_i \sim \mathcal{N}(\mu_i, 1)$ and the parameter $\mu_i$ is chosen according to the following model:

$$\mu_i = \begin{cases} 0 & \text{with probability } 1 - \pi_1, \\ F_1 & \text{with probability } \pi_1, \end{cases}$$

for a fixed proportion of non-nulls in the sequence $\pi_1$, and some random variable $F_1$. We consider two distributions for $F_1$—a degenerate distribution with a point mass at $\mu_c$, where $\mu_c$ is a fixed constant for the whole sequence, or $\mathcal{N}(0, 2 \log(M))$. The motivation for the latter is that $\sqrt{2 \log(M)}$ is the minimax amplitude for estimation under the sparse Gaussian sequence model. In the case of the mean coming from a degenerate distribution, we form one-sided $p$-values as $P_i = \Phi(-Z_i)$, where $\Phi$ is the standard Gaussian CDF. If the mean has a Gaussian distribution, we form two-sided $p$-values, i.e. $P_i = 2\Phi(-|Z_i|)$.

8.1 Varying asynchrony

First we show the results of simulated asynchronous tests, in which the $p$-values are independent. At each time step, the test duration is sampled randomly from a geometric distribution with parameter $p$: $E_j \sim j - 1 + \text{Geom}(p)$ for all $j$. Notice that this implies that $p = 1$ yields the fully synchronous setting, as $E_j \equiv j$ in this case. As $p$ gets smaller, the expectation of the test duration grows larger, hence the procedure gets more asynchronous, and consequently less powerful. Figure 4 and Figure 5 show numerically how changing $p$ affects the achieved power of LORD async and SAFFRON async, respectively, across different non-null proportions $\pi_1$, when the mean of the alternative has a degenerate distribution at $\mu_c = 3$. Figure 6 and Figure 7 also plot power and FDR of LORD async and SAFFRON async against $\pi_1$, however for normally distributed means, showing a more gradual change in performance with the increase of asynchrony.
Figure 4. Power and FDR of LORD\textsubscript{async} with varying the parameter of asynchrony $p$ of the tests. In all five runs LORD\textsubscript{async} has the same parameters ($\{\gamma_j\}_{j=1}^\infty$, $W_0$). The mean of observations under the alternative is a point mass at $\mu_c = 3$.

Figure 5. Power and FDR of SAFFRON\textsubscript{async} with varying the parameter of asynchrony $p$ of the tests. In all five runs SAFFRON\textsubscript{async} has the same parameters ($\{\gamma_j\}_{j=1}^\infty$, $W_0$). The mean of observations under the alternative is a point mass at $\mu_c = 3$.

Figure 6. Power and FDR of LORD\textsubscript{async} with varying the parameter of asynchrony $p$ of the tests. In all five runs LORD\textsubscript{async} has the same parameters ($\{\gamma_j\}_{j=1}^\infty$, $W_0$). The mean of observations under the alternative is $N(0, 2 \log(M))$. 

\[ N(0, 2 \log(M)) \]
8.2 Varying the lag of dependence

The second set of simulations considers synchronous testing of locally dependent p-values. We take $L_t$ to be invariant and equal to $L$, which reduces to lagged dependence between p-values. In particular, we generate an $M$-dimensional vector of Gaussian observations $(Z_1, \ldots, Z_M)$, with the following Toeplitz covariance matrix:

$$
\Sigma(M, L, \rho) = \\
\begin{bmatrix}
1 & \rho & \rho^2 & \ldots & \rho^L & 0 & \ldots & 0 & 0 \\
\rho & 1 & \rho & \ldots & \rho^{L-1} & \rho & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 0 & 0 & \ldots & \rho & 1
\end{bmatrix},
$$

where we set $\rho = 0.5$. Marginally, the observations are distributed according to the Gaussian model described at the beginning of the section. Figure 8 and Figure 9 compare the power and FDR of LORD$_{dep}$ and SAFFRON$_{dep}$ under local dependence, when the mean of the observations under the alternative is $\mu_c = 3$ with probability 1. Figure 10 and Figure 11 give the same comparison when the mean of non-null samples is normally distributed, which yields a slower decrease in performance with increasing the lag.

Figure 7. Power and FDR of SAFFRON$_{async}$ with varying the parameter of asynchrony $p$ of the tests. In all five runs SAFFRON$_{async}$ has the same parameters $(\{\gamma_j\}_{j=1}^\infty, W_0)$. The mean of observations under the alternative is $N(0, 2 \log(M))$.

Figure 8. Power and FDR of LORD$_{dep}$ with varying the dependence lag $L$ in the $p$-value sequence. In all five runs LORD$_{dep}$ has the same parameters $(\{\gamma_j\}_{j=1}^\infty, W_0)$. The mean of observations under the alternative is a point mass at $\mu_c = 3$. 

Figure 9. Power and FDR of LORD$_{dep}$ with varying the dependence lag $L$ in the $p$-value sequence. In all five runs LORD$_{dep}$ has the same parameters $(\{\gamma_j\}_{j=1}^\infty, W_0)$. The mean of observations under the alternative is a point mass at $\mu_c = 3$. 

Figure 10. Power and FDR of LORD$_{dep}$ with varying the dependence lag $L$ in the $p$-value sequence. In all five runs LORD$_{dep}$ has the same parameters $(\{\gamma_j\}_{j=1}^\infty, W_0)$. The mean of observations under the alternative is a point mass at $\mu_c = 3$. 

Figure 11. Power and FDR of LORD$_{dep}$ with varying the dependence lag $L$ in the $p$-value sequence. In all five runs LORD$_{dep}$ has the same parameters $(\{\gamma_j\}_{j=1}^\infty, W_0)$. The mean of observations under the alternative is a point mass at $\mu_c = 3$. 

23
Figure 9. Power and FDR of SAFFRON$_{dep}$ with varying the dependence lag $L$ in the $p$-value sequence. In all five runs SAFFRON$_{dep}$ has the same parameters ($\{\gamma_j\}_{j=1}^\infty$, $W_0$). The mean of observations under the alternative is a point mass at $\mu_c = 3$.

Figure 10. Power and FDR of LORD$_{dep}$ with varying the dependence lag $L$ in the $p$-value sequence. In all five runs LORD$_{dep}$ has the same parameters ($\{\gamma_j\}_{j=1}^\infty$, $W_0$). The mean of observations under the alternative is $N(0, 2 \log(M))$.

Figure 11. Power and FDR of SAFFRON$_{dep}$ with varying the Markov lag $L$ in the $p$-value sequence. In all five runs SAFFRON$_{dep}$ has the same parameters ($\{\gamma_j\}_{j=1}^\infty$, $W_0$). The mean of observations under the alternative is $N(0, 2 \log(M))$. 
8.3 Varying mini-batch sizes

Here we analyze the change in performance of LORD_{\text{mini}} and SAFFRON_{\text{mini}} when the size of mini-batches varies. We fix the batch size $n_b \equiv n$ for all batches $b$. Within each batch tests are performed asynchronously, and all p-values within the same batch are dependent. In particular, they follow a multivariate normal distribution, where the marginal distributions are as described at the beginning of this section, and the covariance matrix is a Toeplitz matrix of the form:

$$
\Sigma_{\text{mini}}(n, \rho) = \begin{bmatrix}
1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\
\rho & 1 & \rho & \cdots & \rho^{n-2} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1
\end{bmatrix},
$$

(14)

where $\rho$ is a fixed hyperparameter. Dependent p-values come in “blocks” of size $n$, implying that any two p-values belonging to two different batches are independent. Figure 12 and Figure 13 compare the power and FDR of LORD_{\text{mini}} and SAFFRON_{\text{mini}} for different batch sizes when the mean of the non-null $Z_i$ is a point mass at $\mu_c = 3$, and Figure 14 and Figure 15 plot the same comparison when the mean of the non-null observations is normally distributed. In all plots, we fix $\rho = 0.5$.
8.4 Comparison with LORD under dependence

The final set of experiments contrasts LORD_{dep} and SAFFRON_{dep} to the original LORD algorithm under dependence. The latter controls FDR under arbitrary dependence at the price of being overly conservative, as shown by the following plots.

As mentioned earlier, LORD under dependence entails a similar update to alpha-investing; more precisely, the test levels \( \alpha_j^{\text{indep}} \) of LORD under independence have to be discounted by a convergent sequence \( \{\xi_j\}_{j=1}^{\infty} \), resulting in new test levels \( \alpha_j = \xi_j \alpha_j^{\text{indep}} \), which essentially diminishes the effect of \( \alpha_j^{\text{indep}} \) earning wealth through discoveries.

We generate the \( p \)-value sequence using the same scheme as in Subsection 7.2; they are computed from Gaussian observations with covariance matrix \( \Sigma(M, L, \rho) \) \([3]\), where we fix \( \rho = 0.5 \) and \( L = 150 \). By construction, this sequence is only locally dependent, which implies that the application of our algorithms comes with provable guarantees. Figure 16 and Figure 17 compare the power and FDR of SAFFRON_{dep}, LORD_{dep}, LORD under dependence and alpha-spending when the mean of the non-null \( Z_i \) is a point mass at \( \mu_c = 3 \), and Figure 17 shows the same comparison in the setting with a normally distributed mean under the alternative.
Figure 16. Power and FDR of SAFFRON<sub>dep</sub>, LORD<sub>dep</sub>, LORD under dependence and alpha-spending. The decay of test levels in alpha-spending and discount sequence $\{\xi_j\}_{j=1}^{\infty}$ act according to the sequence $\{\gamma_j\}_{j=1}^{\infty}$ used for SAFFRON<sub>dep</sub> and LORD<sub>dep</sub>. The mean of observations under the alternative is a point mass at $\mu_c = 3$, and we fix parameters $\rho = 0.5$ and $L = 150$.

Figure 17. Power and FDR of SAFFRON<sub>dep</sub>, LORD<sub>dep</sub>, LORD under dependence and alpha-spending. The decay of test levels in alpha-spending and discount sequence $\{\xi_j\}_{j=1}^{\infty}$ act according to the sequence $\{\gamma_j\}_{j=1}^{\infty}$ used for SAFFRON<sub>dep</sub> and LORD<sub>dep</sub>. The mean of observations under the alternative is $N(0, 2 \log(M))$, and we fix parameters $\rho = 0.5$ and $L = 150$.

9 Summary

We have presented a unified framework for the design and analysis of online FDR procedures for asynchronous testing, as well as testing locally dependent $p$-values. Our framework reposes on the concept of “conflict sets,” and we show the value of this concept for the study of both asynchronous testing and local dependence and for their combination. We derive two specific procedures that make use of conflict sets to yield algorithms that provide online mFDR and FDR control. Further, we give formal guarantees on mFDR at fixed times and certain stopping times. Finally, we present simulation experiments that demonstrate how varying parameters of asynchrony and local dependence in the $p$-value sequence affect the performance of the algorithms.

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10 Deferred proofs

10.1 Proof of Lemma 1

We can reformulate $Y_1$ as:

$$Y_1 = \sum_{j=1}^{\infty} \left( 1 \{ P_j \leq \alpha_j, P_j \in \mathcal{H}_0^0 \} + \alpha_j \right) 1 \{ j \leq T \}.$$
Since $1 \{ P_j \leq \alpha_j \} + \alpha_j \leq 2$, we can bound the expectation of $Y_1$ as:

$$
\mathbb{E}[Y_1] = \mathbb{E} \left[ \sum_{j \in \mathbb{N}, j \in H^0} (1 \{ P_j \leq \alpha_j \} + \alpha_j) 1 \{ j \leq T \} \right]
$$

$$
\leq \sum_{j=1}^{\infty} 2 \mathbb{Pr}(T \geq j)
$$

$$
= 2 \mathbb{E}[T]
$$

$$
< \infty,
$$

where the last step uses the condition that $\mathbb{E}[T] < \infty$.

### 10.2 Proof of Lemma 2

We can reformulate $Y_2$ as:

$$
Y_2 = \sum_{j=1}^{\infty} \left( 1 \{ P_j \leq \alpha_j, P_j \in H^0 \} + 1 \{ P_j > \lambda_j \} \frac{\alpha_j}{1-\lambda_j} \right) 1 \{ j \leq T \}.
$$

Since $1 \{ P_j \leq \alpha_j \} + 1 \{ P_j > \lambda_j \} \frac{\alpha_j}{1-\lambda_j} \leq 1 + \frac{1}{1-\epsilon} = c$ for all $j$, we can bound the expectation of $Y_2$ as:

$$
\mathbb{E}[Y_2] = \mathbb{E} \left[ \sum_{j \in \mathbb{N}, j \in H^0} \left( 1 \{ P_j \leq \alpha_j \} + 1 \{ P_j > \lambda_j \} \frac{\alpha_j}{1-\lambda_j} \right) 1 \{ j \leq T \} \right]
$$

$$
\leq c \sum_{j=1}^{\infty} \mathbb{Pr}(T \geq j)
$$

$$
= c \mathbb{E}[T]
$$

$$
< \infty,
$$

where the last step uses the condition that $\mathbb{E}[T] < \infty$.

### 10.3 Proof of Theorem 3

For all $t \in \mathbb{N}$, define the process $A(t)$ as:

$$
A(t) := \sum_{i \leq t, i \in H^0} 1 \{ E_i \leq t \} \left( 1 \{ P_i \leq \alpha_i \} + 1 \{ P_i > \lambda_i \} \frac{\alpha_i}{1-\lambda_i} \right)
$$

$$
= A(t-1) - \sum_{i \leq t, i \in H^0} 1 \{ E_i = t \} \left( 1 \{ P_i \leq \alpha_i \} + 1 \{ P_i > \lambda_i \} \frac{\alpha_i}{1-\lambda_i} \right),
$$

where we take $A(0) = 0$. Let $H(t) := 1 \{ T \geq t \}$. Since $T$ is a stopping time, it holds that $\{ T \geq t + 1 \} = \{ T \leq t \} \cap G^t$, therefore $H(t+1)$ is measurable with respect to $G^t$. Define the following transform of $H$ by $A$:

$$
(H \cdot A)(t) := \sum_{m=1}^{t} H(m)(A(m) - A(m-1))
$$

$$
= \sum_{m=1}^{t} H(m) \left( - \sum_{i \leq m, i \in H^0} 1 \{ E_i = m \} \left( 1 \{ P_i \leq \alpha_i \} + 1 \{ P_i > \lambda_i \} \frac{\alpha_i}{1-\lambda_i} \right) \right).
$$
By taking conditional expectations, we can obtain:

\[
\mathbb{E} \left[ (H \cdot A)(t + 1) \mid \mathcal{G}^t \right] = \mathbb{E} \left[ (H \cdot A)(t) \mid \mathcal{G}^t \right] + \mathbb{E} \left[ H(t + 1)(A(t + 1) - A(t)) \mid \mathcal{G}^t \right] \\
= \mathbb{E} \left[ (H \cdot A)(t) \mid \mathcal{G}^t \right] + H(t + 1) \mathbb{E} \left[ - \sum_{i \leq t+1, j \in \mathcal{H}^0} 1 \{ E_i = t + 1 \} \left( 1 \{ P_i \leq \alpha_i \} + 1 \{ P_i > \lambda_i \} \frac{\alpha_i}{\lambda_i} \right) \mid \mathcal{G}^t \right] \\
= \mathbb{E} \left[ (H \cdot A)(t) \mid \mathcal{G}^t \right] + H(t + 1) \sum_{i \leq t+1, j \in \mathcal{H}^0} \mathbb{E} \left[ -1 \{ E_i = t + 1 \} \left( 1 \{ P_i \leq \alpha_i \} + 1 \{ P_i > \lambda_i \} \frac{\alpha_i}{\lambda_i} \right) \mid \mathcal{G}^t \right],
\]

where the first equality follows by linearity of expectation and the definition of the transform and the second one uses measurability of \( H(t + 1) \). The term \(-1 \{ E_i = t + 1 \} (1 \{ P_i \leq \alpha_i \} + 1 \{ P_i > \lambda_i \} \frac{\alpha_i}{\lambda_i})\) is clearly non-negative when \( E_i \neq t + 1 \). If \( E_i = t + 1 \) however, we can invoke the super-uniformity condition \([4]\), since we are summing over null indices:

\[
\mathbb{E} \left[ -\left( 1 \{ P_i \leq \alpha_i \} + 1 \{ P_i > \lambda_i \} \frac{\alpha_i}{\lambda_i} \right) \mid \mathcal{G}^{t+1} \right] \geq -\alpha_i + (1 - \lambda_i) \frac{\alpha_i}{\lambda_i} = 0.
\]

Therefore, additionally applying the law of iterated expectations, it follows that:

\[
\mathbb{E} \left[ (H \cdot A)(t + 1) \right] \geq \mathbb{E} \left[ (H \cdot A)(t) \right].
\]

Iteratively applying the same argument, we reach the conclusion that, for all \( t \in \mathbb{N} \):

\[
\mathbb{E} \left[ (H \cdot A)(t) \right] \geq 0. \tag{15}
\]

So far we have only used the predictability of \( H(t) \); observe that, by its definition:

\[
(H \cdot A)(t) = A(T \wedge t) - A(0) = A(T \wedge t),
\]

and hence by equation \((15)\), we obtain:

\[
\mathbb{E} \left[ (H \cdot A)(t) \right] = \mathbb{E} \left[ A(T \wedge t) \right] \geq 0.
\]

Define \( Y_2 := \sum_{j \leq T, j \notin \mathcal{X}^0} (1 \{ P_j \leq \alpha_j \} + 1 \{ P_j > \lambda_j \} \frac{\alpha_j}{\lambda_j}) \), and observe that \( Y_2 \geq \left| A(T \wedge t) \right| \) almost surely. Since \( A(T \wedge t) \rightarrow A(T) \) almost surely as \( t \rightarrow \infty \), by Lemma \([2]\) and dominated convergence we can conclude that \( \mathbb{E} \left[ A(T \wedge t) \right] \rightarrow \mathbb{E} \left[ A(T) \right] \) as \( t \rightarrow \infty \). As in Theorem \([2]\) we reach the result that states:

\[
\mathbb{E} \left[ A(T) \right] \geq 0. \tag{16}
\]

Recall \( \mathcal{R}(t) \), the set of all rejections made by time \( t \), and \( \mathcal{V}(t) \), the set of false rejections made by time \( t \). Consider the following process:

\[
B(t) : = \alpha |\mathcal{R}(t)| |\mathcal{V}(t)| - W_t \\
= \alpha |\mathcal{R}(t)| |\mathcal{V}(t)| + \sum_{j \leq t, j \notin \mathcal{X}^t} 1 \{ P_j > \lambda_j \} \frac{\alpha_j}{\lambda_j} + \sum_{j \in \mathcal{X}^t \cup \{ t \}} \frac{\alpha_j}{\lambda_j} - \sum_{j \leq t, j \notin \mathcal{X}^t} R_j \alpha + W_0 1 \{ t \geq r_1 \} - W_0 \\
\geq -|\mathcal{V}(t)| + \sum_{j \leq t} 1 \{ P_j > \lambda_j \} \frac{\alpha_j}{\lambda_j} + W_0 1 \{ t \geq r_1 \} - W_0 \\
\geq -|\mathcal{V}(t)| + \sum_{j \leq t} 1 \{ P_j > \lambda_j \} \frac{\alpha_j}{\lambda_j} \\
\geq A(t),
\]
where the first inequality follows because \( 1 \{ P_j > \lambda_j \} \) is almost surely bounded by 1, the second inequality uses the fact that \( T \) is non-trivial, and the last equality applies the definition of \( A(t) \) together with the fact that \( \sum_{j \leq t} 1 \{ P_j > \lambda_j \} \). 

Now take a stopping time \( T \) with respect to \( G^t \) such that it satisfies the conditions of the theorem, then:

\[
\mathbb{E} \left[ \alpha | R(t) | - | V(t) | \right] = \mathbb{E} [B(T) + W_T] \\
\geq \mathbb{E} [B(T)] \\
\geq \mathbb{E} [A(T)] \\
= 0,
\]

where the first inequality follows because wealth is non-negative at all times by construction, the second inequality uses the proved relationship between \( A(t) \) and \( B(t) \), and the third inequality applies equation (16). Rearranging the terms we have that \( mFDR(T) \leq \alpha \), as desired.

### 10.4 Proof of Lemma 3

We begin by focusing on the first inequality. Letting \( P_{1:M} = (P_1, \ldots, P_M) \) be the original vector of \( p \)-values, we define a “hallucinated” vector of \( p \)-values \( \tilde{P}_{1:M}^{-1} := (\tilde{P}_1, \ldots, \tilde{P}_M) \) that equals \( P_{1:M} \), except that the \( t \)-th component is set to one:

\[
\tilde{P}_i = \begin{cases} 
1 & \text{if } i = t, \\
\hat{P}_i & \text{if } i \neq t.
\end{cases}
\]

Further, denote by \( \tilde{E}_j \) the finish times of the tests that yield \( \tilde{P}_j \), and let \( \tilde{E}_j \) be equal to \( E_j \) for all \( 1 \leq j \leq M \). Denote the number of candidates and rejections in the hallucinated sequence at time \( t \) by \( \tilde{C}_t \) and \( \tilde{R}_t \), respectively, and let \( \alpha_t \) be the test level for \( \tilde{P}_t \). Also, let \( R_{1:M} = (R_1, \ldots, R_M) \) and \( \tilde{R}_{1:M}^{-1} = (\tilde{R}_1, \ldots, \tilde{R}_M) \) denote the vectors of the numbers of rejections using \( P_{1:M} \) and \( \tilde{P}_{1:M}^{-1} \), respectively. Similarly, let \( C_{1:M} = (C_1, \ldots, C_M) \) and \( \tilde{C}_{1:M}^{-1} = (\tilde{C}_1, \ldots, \tilde{C}_M) \) denote the vectors of the numbers of candidates using \( P_{1:M} \) and \( \tilde{P}_{1:M}^{-1} \), respectively.

By construction, we have the following properties:

1. \( \tilde{E}_j = E_j \) implies \( \alpha_t = \alpha_i \) for all \( i \leq E_t \).
2. \( \tilde{R}_i = R_i \) and \( \tilde{C}_i = C_i \) for all \( i < E_t \), since the finished tests and the respective test levels are the same in the original and hallucinated setting.
3. \( |\tilde{R}_{E_t}| \leq |R_{E_t}| \) and \( |\tilde{C}_{E_t}| \leq |C_{E_t}| \), and hence \( |\tilde{R}_i| \leq |R_i| \) also for all \( i > E_t \), due to monotonicity of the test levels \( \alpha_i \).

Therefore, on the event \( \{ P_t > \lambda_t \} \), we have \( R_{E_t} = \tilde{R}_{E_t} \) and \( C_{E_t} = \tilde{C}_{E_t} \), and hence also \( R_{1:M} = \tilde{R}_{1:M}^{-1} \) and \( C_{1:M} = \tilde{C}_{1:M}^{-1} \). This allows us to conclude that:

\[
\frac{\alpha_t 1 \{ P_t > \lambda_t \}}{(1 - \lambda_t)g(R_{1:M})} = \frac{\tilde{\alpha}_t 1 \{ P_t > \lambda_t \}}{(1 - \lambda_t)g(R_{1:M}^{-1})}.
\]

Since the \( p \)-values \( P_{1:M} \) are mutually independent, and \( E_t \) is by assumption independent of \( P_t \) given \( F_{E_t}^{-1} \), we conclude that \( \tilde{R}_{1:M}^{-1} \) is independent of \( P_t \) conditioned on \( F_{E_t}^{-1} \). With this, we can obtain:

\[
\mathbb{E} \left[ \frac{\alpha_t 1 \{ P_t > \lambda_t \}}{(1 - \lambda_t)g(R_{1:M}^{-1})} \right] = \mathbb{E} \left[ \frac{\tilde{\alpha}_t 1 \{ P_t > \lambda_t \}}{(1 - \lambda_t)g(R_{1:M}^{-1})} \right] \\
\geq \mathbb{E} \left[ \frac{\alpha_t 1 \{ P_t > \lambda_t \}}{g(|\tilde{R}_{1:M}^{-1}|)} \right] \cdot \mathbb{E} \left[ F_{E_t}^{-1} \right] \\
\geq \mathbb{E} \left[ \frac{\alpha_t 1 \{ P_t > \lambda_t \}}{g(|R_{1:M}^{-1}|)} \right] \cdot \mathbb{E} \left[ F_{E_t}^{-1} \right],
\]

32
where the first inequality follows by taking an expectation only with respect to $P_t$ by invoking the asynchronous super-uniformity property \(\mathcal{H} \), and the second inequality follows because $g(|\mathcal{R}_{1:M}|) \geq g(|\mathcal{R}_{i+1}^{t-1}|)$ since $|\mathcal{R}_{i}| \geq |\mathcal{R}_{i+1}|$ for all $i$ by monotonicity of the test levels. This concludes the proof of the first inequality.

The second inequality uses a similar idea of hallucinating tests with identical finish times, only now the $p$-values that these tests result in are:

\[
\tilde{P}_i = \begin{cases} 0 & \text{if } i = t, \\ P_i & \text{if } i \neq t, \end{cases}
\]

where $P_i$ are the $p$-values in the original sequence. In a similar fashion, the following observations hold:

1. $\tilde{E}_j = E_j$ implies $\alpha_i = \tilde{\alpha}_i$ for all $i \leq E_t$.

2. $\tilde{\mathcal{R}}_i = \mathcal{R}_i$ and $\tilde{C}_i = C_i$ for all $i < E_t$, since the finished tests and the respective test levels are the same in the original and hallucinated setting.

3. $|\tilde{\mathcal{R}}_{E_t}| \geq |\mathcal{R}_{E_t}|$ and $|\tilde{C}_{E_t}| \geq |C_{E_t}|$, and hence $|\tilde{\mathcal{R}}_i| \geq |\mathcal{R}_i|$ also for all $i > E_t$, due to monotonicity of the test levels $\alpha_i$.

Then, on the event $\{P_t \leq \alpha_t\}$, we have $\mathcal{R}_{E_t} = \tilde{\mathcal{R}}_{E_t}$ and $C_{E_t} = \tilde{C}_{E_t}$, and hence also $\mathcal{R}_{1:M} = \tilde{\mathcal{R}}_{1:M}^{t-1}$ and $C_{1:M} = \tilde{C}_{1:M}^{t-1}$. From this we conclude that:

\[
\frac{1 \{P_t \leq \alpha_t\}}{g(|\mathcal{R}_{1:M}|)} = \frac{1 \{P_t \leq \alpha_t\}}{g(|\tilde{\mathcal{R}}_{1:M}^{t-1}|)}.
\]

As in the first part of the proof, we use the fact that the $p$-values $P_{1:M}$ are mutually independent, and $E_t$ is by assumption independent of $P_t$ given $\mathcal{F}_{\text{async}}^{E_{t-1}}$, which allows us to conclude that $\mathcal{R}_{1:M}^{t-1}$ is independent of $P_t$ conditioned on $\mathcal{F}_{\text{async}}^{E_{t-1}}$. This observation results in the following:

\[
\begin{align*}
\mathbb{E}\left[ \frac{1 \{P_t \leq \alpha_t\}}{g(|\mathcal{R}_{1:M}|)} \mid \mathcal{F}_{\text{async}}^{E_{t-1}} \right] &= \mathbb{E}\left[ \frac{1 \{P_t \leq \alpha_t\}}{g(|\mathcal{R}_{1:M}^{t-1}|)} \mid \mathcal{F}_{\text{async}}^{E_{t-1}} \right] \\
&\leq \mathbb{E}\left[ \frac{\alpha_t}{g(|\mathcal{R}_{1:M}^{t-1}|)} \mid \mathcal{F}_{\text{async}}^{E_{t-1}} \right] \\
&\leq \mathbb{E}\left[ \frac{\alpha_t}{g(|\mathcal{R}_{1:M}|)} \mid \mathcal{F}_{\text{async}}^{E_{t-1}} \right],
\end{align*}
\]

where the first inequality follows by taking an expectation only with respect to $P_t$ by invoking the asynchronous super-uniformity property \(\mathcal{H} \), and the second inequality follows because $g(|\mathcal{R}_{1:M}|) \leq g(|\mathcal{R}_{1:M}^{t-1}|)$ since $|\mathcal{R}_i| \leq |\tilde{\mathcal{R}}_i|$ for all $i$ by monotonicity of the test levels. This concludes the proof of the lemma.

### 10.5 Proof of Theorem 4

Fix a time step $t$. First we show the claim for $\text{LORD}_{\text{async}}$, so suppose that $\widehat{\text{FDP}}_{\text{LORD}_{\text{async}}} (t) := \frac{\sum_{j \leq t} \alpha_j}{\sum_{j \leq t} 1 \{P_j \leq \alpha_j, E_j \leq t\}} \leq \alpha$. Then:

\[
\text{FDR}(t) := \mathbb{E}\left[ \frac{|\mathcal{Y}(t)|}{|\mathcal{R}(t)|} \right] = \mathbb{E}\left[ \frac{\sum_{j \leq t, j \in \mathcal{H}^o} 1 \{P_j \leq \alpha_j, E_j \leq t\}}{\sum_{j \leq t} 1 \{P_j \leq \alpha_j, E_j \leq t\}} \right] \\
\leq \sum_{i \leq t, i \in \mathcal{H}^o} \mathbb{E}\left[ \frac{1 \{P_i \leq \alpha_i\}}{\sum_{j \leq t} 1 \{P_j \leq \alpha_j, E_j \leq t\}} \right],
\]

33
where the second equality follows by definition of $\mathcal{V}(t)$ and $\mathbb{R}(t)$, and the inequality drops the condition $E_i \leq t$ from the numerator and applies linearity of expectation. Now we can apply Lemma 3 with $g(|\mathcal{R}_{1:t}|) = \sum_{i=1}^{t} |\mathcal{R}_i|$, together with iterated expectations, to obtain:

$$
\sum_{i \leq t, i \in \mathcal{H}^0} \mathbb{E} \left[ \sum_{j \leq t} \frac{1 \{ P_i \leq \alpha_i \}}{\sum_{j \leq t} 1 \{ P_j \leq \alpha_j, E_j \leq t \}} \right] \leq \sum_{i \leq t, i \in \mathcal{H}^0} \mathbb{E} \left[ \sum_{j \leq t} \frac{\alpha_i}{\sum_{j \leq t} 1 \{ P_j \leq \alpha_j, E_j \leq t \}} \right] \leq \mathbb{E} \left[ \frac{\text{FDP}_{\text{LOND}}(t)}{\text{FDP}_{\text{SAFFRON}}(t)} \right] \leq \alpha,
$$

where the second inequality follows by dropping the condition $i \in \mathcal{H}^0$, and ignoring the rejections at time $t$ in the denominator. This completes the first claim of the theorem.

Now we move on to SAFFRON$_{\text{async}}$. Using the same steps as above, for any fixed time $t$, we can conclude the following inequality:

$$
\text{FDR}(t) \leq \sum_{i \leq t, i \in \mathcal{H}^0} \mathbb{E} \left[ \sum_{j \leq t} \frac{\alpha_i}{\sum_{j \leq t} 1 \{ P_j \leq \alpha_j, E_j \leq t \}} \right].
$$

Here we additionally apply the other inequality of Lemma 3 with the same choice $g(|\mathcal{R}_{1:t}|) = \sum_{i=1}^{t} |\mathcal{R}_i|$, again with iterated expectations:

$$
\sum_{i \leq t, i \in \mathcal{H}^0} \mathbb{E} \left[ \sum_{j \leq t} \frac{\alpha_i}{\sum_{j \leq t} 1 \{ P_j \leq \alpha_j, E_j \leq t \}} \right] \leq \sum_{i \leq t, i \in \mathcal{H}^0} \mathbb{E} \left[ \frac{\alpha_i 1 \{ P_i > \lambda_i \}}{(1 - \lambda_i) \sum_{j \leq t} 1 \{ P_j \leq \alpha_j, E_j \leq t \}} \right].
$$

Assuming that the inequality $\frac{\text{FDP}_{\text{SAFFRON}}(t)}{\text{FDP}_{\text{SAFFRON}}(t)} \leq \alpha$ holds, it follows that:

$$
\sum_{j \leq t, j \in \mathcal{H}^0} \mathbb{E} \left[ \frac{\alpha_j 1 \{ P_j > \lambda_j \}}{(1 - \lambda_j) \sum_{j \leq t} 1 \{ P_j \leq \alpha_j, E_j \leq t \}} \right] \leq \mathbb{E} \left[ \frac{\sum_{j \leq t} \frac{\alpha_j}{1 - \lambda_j} (1 \{ P_j > \lambda_j, E_j \leq t \} + 1 \{ E_j > t \})}{\sum_{j \leq t} 1 \{ P_j \leq \alpha_j, E_j \leq t \}} \right] \leq \mathbb{E} \left[ \frac{\text{FDP}_{\text{SAFFRON}}(t)}{\text{FDP}_{\text{SAFFRON}}(t)} \right] \leq \alpha,
$$

where the first inequality follows by dropping the conditions $j \in \mathcal{H}^0$ and $P_j > \lambda_j$ for some rounds, and the second inequality ignores the rejections at time $t$ in the denominator. The second inequality follows by assumption, hence proving the theorem.

### 10.6 Proof of Theorem 5

For statement (a), we begin by noting that for any $t \in \mathbb{N}$:

$$
\text{FDR}(t) = \mathbb{E} \left[ \frac{\sum_{i \leq t, i \in \mathcal{H}^0} 1 \{ P_i \leq \alpha_i \}}{|\mathcal{R}(t)| \lor 1} \right] \leq \sum_{i \leq t, i \in \mathcal{H}^0} \mathbb{E} \left[ \frac{1 \{ P_i \leq \alpha_i \}}{|\mathcal{R}(i - 1)| \lor 1} \right] = \sum_{i \leq t, i \in \mathcal{H}^0} \gamma_i \alpha \mathbb{E} \left[ \frac{1 \{ P_i \leq \alpha_i \}}{\alpha_i} \right],
$$

where the first equality follows by definition of FDR, the sole inequality follows because the number of rejections can only increase with time, and the second equality follows by definition of the LOND rule for $\alpha_i$. Lemma 1 from Ramdas et al. [23] now asserts that the term in the expectation is bounded by one under PRDS. Hence, by also noting that $\sum_{i \leq t} \gamma_i \leq 1$ we immediately deduce statement (a).
For statement (b), we follow almost the same sequence of steps to note that:

\[
FDR(t) = E \left[ \sum_{i \leq t, i \in H_0} \frac{1 \{ P_i \leq \alpha_i \}}{|R(t)| \lor 1} \right] \leq \sum_{i \leq t, i \in H_0} E \left[ \frac{1 \{ P_i \leq \alpha_i \}}{|R(i-1)| \lor 1} \right]
\]

\[
= \sum_{i \leq t, i \in H_0} \gamma_i \alpha E \left[ \frac{1 \{ P_i \leq \gamma_i \alpha_i \beta_i(|R(i-1)| \lor 1) \}}{\gamma_i \alpha(|R(i-1)| \lor 1)} \right].
\]

We now apply Lemma 1 from Ramdas et al. [23] with \( c = \gamma_i \alpha \) and \( f(P) = |R(i-1)| \lor 1 \) to again assert that the term in the expectation is bounded by one under arbitrary dependence, hence establishing statement (b).

### 11 Different instantiations of LORD* and SAFFRON*

Here we give explicit statements of different instances of LORD* and SAFFRON* described in Section 3, Section 4, and Section 5. All of the following algorithms are special instances of Algorithms 1-4, given in Section 2.

Here we give explicit statements of different instances of LORD* and SAFFRON* described in Section 3, Section 4, and Section 5. All of the following algorithms are special instances of Algorithms 1-4, given in Section 2.

First, we state LORDasync and SAFFRONasync explicitly, by taking \( X^t = \lambda_{async} \) in the statement of LORD* and SAFFRON*. Algorithm 5 and Algorithm 6 state the LORD++ and LOND versions of LORDasync. Algorithm 7 states SAFFRONasync for constant candidacy thresholds, i.e. \( \{\lambda_j\} = \lambda \), and Algorithm 8 states asynchronous alpha-investing, i.e. SAFFRONasync when \( \lambda_j = \alpha_j \). Recall the definitions of \( r_k \) and \( s_k \), which in this setting adopt the form:

\[
\begin{align*}
    r_k &= \min\{i \in [t]: \sum_{j=1}^{i} R_j 1 \{ E_j \leq i \} \geq k\}, \\
    s_k &= \min\{i: \sum_{j \leq t} 1 \{ E_j \leq E_i \} R_j \geq k \text{ and } i \neq s_m \text{ for all } m < k\}.
\end{align*}
\]

**Algorithm 5** The asynchronous LORD++ algorithm as a version of LORDasync

**Input:** FDR level \( \alpha \), non-negative non-increasing sequence \( \{\gamma_j\}_{j=1}^{\infty} \) such that \( \sum_j \gamma_j = 1 \), initial wealth \( W_0 \leq \alpha \)

\( \alpha_1 = \gamma_1 W_0 \)

**For** \( t = 1, 2, \ldots \) **do**

\[
\begin{align*}
    &\text{start } t\text{-th test with level } \alpha_t \\
    &W_t := W_{t-1} - \alpha_t + \alpha \sum_{j=1}^{t} 1 \{ P_j \leq \alpha_j, E_j = t \} - W_0 1 \{ t = r_1 \} \\
    &\alpha_{t+1} = \gamma_{t+1} W_0 + \gamma_{t+1-r_1} (\alpha - W_0) + \left( \sum_{j \geq 2} \gamma_{t+1-r_j} \right) \alpha
\end{align*}
\]

end

**Algorithm 6** The asynchronous LOND algorithm as a version of LORDasync

**Input:** FDR level \( \alpha \), non-negative non-increasing sequence \( \{\gamma_j\}_{j=1}^{\infty} \) such that \( \sum_j \gamma_j = 1 \)

\( W_0 = \alpha \)

\( \alpha_1 = \gamma_1 W_0 \)

**For** \( t = 1, 2, \ldots \) **do**

\[
\begin{align*}
    &\text{start } t\text{-th test with level } \alpha_t \\
    &W_t := W_{t-1} - \alpha_t + \alpha \sum_{j=1}^{t} 1 \{ P_j \leq \alpha_j, E_j = t \} - W_0 1 \{ t = r_1 \} \\
    &\alpha_{t+1} = \alpha \gamma_{t+1} \left( \sum_{j=1}^{t} 1 \{ P_j \leq \alpha_j, E_j \leq t \} \lor 1 \right)
\end{align*}
\]

end
Below we give explicit statements of LORD* and SAFFRON*. Algorithm 9 and Algorithm 10 state LORD++ and LOND under local dependence, both as instances of LORD*.
Algorithm 11 states SAFFRON* for the constant sequence \( \{\lambda_j\} \equiv \lambda \), and Algorithm 12 states alpha-investing under local dependence, which is a particular instance of SAFFRON* obtained by taking \( \lambda_j = \alpha_j \). The definitions of \( r_k \) and \( s_k \) under local dependence simplify to:

\[
r_k = \min\{i \in [t] : \sum_{j=1}^{i-L_{i+1}} R_j \geq k\}, \quad s_k = \min\{i : \sum_{j \leq i} R_j \geq k \text{ and } i \neq s_m \text{ for all } m < k\}.
\]

Algorithm 7 The SAFFRON asynchronous algorithm for constant \( \lambda \)

**input:** FDR level \( \alpha \), non-negative non-increasing sequence \( \{\gamma_j\}_{j=1}^{\infty} \) such that \( \sum_j \gamma_j = 1 \), candidate threshold \( \lambda \in (0,1) \), initial wealth \( W_0 \leq (1-\lambda)\alpha \)

\[
\alpha_{t+1} = \gamma_t W_0
\]

**for** \( t = 1, 2, \ldots \) **do**

**start** \( t \)-th test with level \( \alpha_t \)

\[
W_t := W_{t-1} - \alpha_t + \sum_{j=1}^{t} 1 \{E_j = t, P_j \leq \alpha_j\} \{1 - (1-\lambda) \alpha P_j \leq \lambda\} - W_0 1 \{t = r_1\}
\]

\[
\alpha_{t+1} = \min\{\lambda, W_0 \gamma_{t+1-C_{j+}} + ((1-\lambda) \alpha - W_0) \gamma_{t+1-r_1-C_{j+}} + \sum_{j \geq 2} (1-\lambda) \alpha \gamma_{t+1-r_1-C_{j+}}\}
\]

where \( C_{j+} = \sum_{i=r_j+1}^{t} C_{i+} \)

end

Algorithm 8 The asynchronous alpha-investing algorithm as a special case of SAFFRON asynchronous

**input:** FDR level \( \alpha \), non-negative non-increasing sequence \( \{\gamma_j\}_{j=1}^{\infty} \) such that \( \sum_j \gamma_j = 1 \), initial wealth \( W_0 \)

\[
\alpha_{t+1} = \gamma_t W_0
\]

**for** \( t = 1, 2, \ldots \) **do**

**start** \( t \)-th test with level \( \alpha_t \)

\[
W_t := W_{t-1} - \frac{\alpha_t}{1-\alpha_t} + \sum_{j=1}^{t} 1 \{E_j = t, P_j \leq \alpha_j\} \{1 - (1-\lambda) \alpha P_j \leq \lambda\} - W_0 1 \{t = r_1\}
\]

\[
\alpha_{t+1} = W_0 \gamma_{t+1-R_{j+}} + ((1-\alpha_s) \alpha - W_0) \gamma_{t+1-r_1-R_{j+}} + \sum_{j \geq 2} (1-\alpha_s) \alpha \gamma_{t+1-r_1-R_{j+}}
\]

where \( R_{j+} = \sum_{i=r_j+1}^{t} R_{i+} \)

end

Algorithm 9 The LORD++ algorithm under local dependence as a version of LORD*
Algorithm 10: The LOND algorithm under local dependence as a version of LORD_{dep}

\textbf{input:} FDR level \( \alpha \), non-negative non-increasing sequence \( \{\gamma_j\}_{j=1}^{\infty} \) such that \( \sum_j \gamma_j = 1 \)

\( W_0 = \alpha \)
\( \alpha_1 = \gamma_1 W_0 \)

\[ \text{for} \ t = 1, 2, \ldots \ \text{do} \]
\[ \text{run} \ t\text{-th test with level} \ \alpha_t \]
\[ W_t := W_{t-1} - \alpha_t + \alpha \sum_{i=t-L_t}^{t-L_t+1} R_i - W_0 1 \{ t = r_1 \} \]
\[ \alpha_{t+1} = \alpha \gamma_{t+1} \left( \sum_{i=1}^{L_{t+1}} R_i \right) \vee 1 \]
\[ \text{end} \]

Algorithm 11: The SAFFRON_{dep} algorithm for constant \( \lambda \)

\textbf{input:} FDR level \( \alpha \), non-negative non-increasing sequence \( \{\gamma_j\}_{j=1}^{\infty} \) such that \( \sum_j \gamma_j = 1 \), candidate threshold \( \lambda \in (0, 1) \), initial wealth \( W_0 < (1 - \lambda)\alpha \)
\( \alpha_1 = \gamma_1 W_0 \)

\[ \text{for} \ t = 1, 2, \ldots \ \text{do} \]
\[ \text{run} \ t\text{-th test with level} \ \alpha_t \]
\[ W_t := W_{t-1} - \alpha_t + (1 - \lambda) \alpha \sum_{i=t-L_t}^{t-L_t+1} R_i + \sum_{i=t-L_t}^{t-L_t+1} \alpha_t C_i - W_0 1 \{ t = r_1 \} \]
\[ \alpha_{t+1} = \min \{ \lambda, W_0 \gamma_{t+1} - C_0 + \gamma_{t+1} - C_{t+1} + (1 - \lambda) \alpha \left( \sum_{j=2}^{\infty} \gamma_{t+1-r_j} - C_{j+} \right) \} \]
\[ \text{where} \ C_{j+} = \sum_{i=r_j+1}^{t-L_t+1} C_i \]
\[ \text{end} \]

Algorithm 12: The alpha-investing algorithm under local dependence as a special case of SAFFRON_{dep}

\textbf{input:} FDR level \( \alpha \), non-negative non-increasing sequence \( \{\gamma_j\}_{j=1}^{\infty} \) such that \( \sum_j \gamma_j = 1 \), initial wealth \( W_0 \)
\( \alpha_1 = \gamma_1 W_0 \)

\[ \text{for} \ t = 1, 2, \ldots \ \text{do} \]
\[ \text{run} \ t\text{-th test with level} \ \alpha_t \]
\[ W_t := W_{t-1} - \alpha_t + \alpha \sum_{i=t-L_t}^{t-L_t+1} R_i \left( \alpha + \frac{\alpha}{1 - \alpha_t} \right) - W_0 1 \{ t = r_1 \} \]
\[ \alpha_{t+1} = W_0 \gamma_{t+1} - C_0 + (1 - \alpha_{s_t}) \alpha - W_0 \gamma_{t+1-r_1} - C_{t+} + \sum_{j=2}^{\infty} (1 - \alpha_{s_j}) \alpha \gamma_{t+1-r_j} - R_{j+} \]
\[ \text{where} \ R_{j+} = \sum_{i=r_j+1}^{t-L_t+1} R_i \]
\[ \text{end} \]

Algorithm 13 describes the mini-batch version of LORD++, and Algorithm 14 states the mini-batch version of LOND, both as cases of LORD_{mini}. Algorithm 15 is a variant of SAFFRON_{mini} with \( \lambda_j \) chosen constant and equal to some \( \lambda \in (0, 1) \), and Algorithm 16 is the alpha-investing version of SAFFRON_{mini}, in which \( \lambda_j = \alpha_j \). In this setting, the definitions of \( r_k \) and \( s_k \) are slightly tweaked in order to satisfy the convention of double indexing; \( r_k \) refers to the \textit{batch} in which the \( k\)-th non-conflicting rejection occurs, while \( s_k \) is the \textit{second index} of the test that corresponds to the \( k\)-th non-conflicting rejection, which is necessarily in batch \( r_k \):

\[ r_k := \min \{ i \in [b - 1] : \sum_{j=1}^{i} \left| R_j \right| \geq k \} , \ s_k = \min \{ i : \sum_{s=1}^{r_k-1} \left| R_{s} \right| + \sum_{j \leq i} R_{r_k,j} \geq k \}. \]
Algorithm 13 The mini-batch LORD++ algorithm as a version of LORD\textsubscript{mini}

\textbf{input:} FDR level $\alpha$, non-negative non-increasing sequence $\{\gamma_j\}_{j=1}^\infty$ such that $\sum_j \gamma_j = 1$, initial wealth $W_{1,0}$, constant $\lambda$

\begin{align*}
\alpha_{1,1} &= \gamma_1 W_{1,0} \\
\text{for } b = 1, 2, \ldots &\text{ do} \\
\quad \text{if } b > 1 &\text{ then} \\
\quad &\quad W_{b,0} = W_{b-1,n} + \alpha |R_{b-1}| - W_{1,0} \mathbf{1} \{r_1 = b - 1\} \\
\quad \text{end} \\
\quad \text{for } t = 1, 2, \ldots, n_b &\text{ do} \\
\quad &\quad \text{start } t\text{-th test in the } b\text{-th batch with level } \alpha_{b,t} \\
\quad &\quad W_{b,t} := W_{b,t-1} - \alpha_{b,t} \\
\quad &\quad \alpha_{b,t+1} = \gamma_{\sum_{i=1}^{b-1} n_i + t + 1} W_{1,0} + \gamma_{\sum_{i=1}^{b-1} n_i + t + 1 - \sum_{i=1}^{r_1} n_i} (\alpha - W_{1,0}) + \left(\sum_{j=2}^{\infty} \gamma_{\sum_{i=1}^{b-1} n_i + t + 1 - \sum_{i=1}^{r_j} n_i} \right) \alpha \\
\quad \text{end} \\
\text{end} \\
\end{align*}

Algorithm 14 The mini-batch LORD algorithm as a version of LORD\textsubscript{mini}

\textbf{input:} FDR level $\alpha$, non-negative non-increasing sequence $\{\gamma_j\}_{j=1}^\infty$ such that $\sum_j \gamma_j = 1$, initial wealth $W_{1,0}$, constant $\lambda$

\begin{align*}
\alpha_{1,1} &= \gamma_1 W_{1,0} \\
\text{for } b = 1, 2, \ldots &\text{ do} \\
\quad \text{if } b > 1 &\text{ then} \\
\quad &\quad W_{b,0} = W_{b-1,n} + \alpha |R_{b-1}| - W_{1,0} \mathbf{1} \{r_1 = b - 1\} \\
\quad \text{end} \\
\quad \text{for } t = 1, 2, \ldots, n_b &\text{ do} \\
\quad &\quad \text{start } t\text{-th test in the } b\text{-th batch with level } \alpha_{b,t} \\
\quad &\quad W_{b,t} := W_{b,t-1} - \alpha_{b,t} \\
\quad &\quad \alpha_{b,t+1} = \alpha_{\sum_{i=1}^{b-1} n_i + t + 1} \left(\frac{\sum_{j=1}^{\infty} |R_j| \vee 1}{\sum_{j=1}^{b-1} n_i + t + 1} \right) \\
\quad \text{end} \\
\text{end} \\
\end{align*}

Algorithm 15 The SAFFRON\textsubscript{mini} algorithm for constant $\lambda$

\textbf{input:} FDR level $\alpha$, non-negative non-increasing sequence $\{\gamma_j\}_{j=1}^\infty$ such that $\sum_j \gamma_j = 1$, initial wealth $W_{1,0}$, constant $\lambda$

\begin{align*}
\alpha_{1,1} &= \gamma_1 W_{1,0} \\
\text{for } b = 1, 2, \ldots &\text{ do} \\
\quad \text{if } b > 1 &\text{ then} \\
\quad &\quad W_{b,0} = W_{b-1,n} + (1 - \lambda)\alpha |R_{b-1}| + \sum_{j \leq n_b-1} \alpha_{b-1,j} C_{b-1,j} - W_{1,0} \mathbf{1} \{r_1 = b - 1\} \\
\quad \text{end} \\
\quad \text{for } t = 1, 2, \ldots, n &\text{ do} \\
\quad &\quad \text{start } t\text{-th test in the } b\text{-th batch with level } \alpha_{b,t} \\
\quad &\quad W_{b,t} := W_{b,t-1} - \alpha_{b,t} \\
\quad &\quad \alpha_{b,t+1} = \gamma_{\sum_{i=1}^{b-1} n_i - |C_j^+| + t + 1} W_{1,0} + \gamma_{\sum_{i=1}^{b-1} n_i - |C_j^+| + t + 1 - \sum_{i=1}^{r_j} n_i} ((1 - \lambda)\alpha - W_{1,0}) + \\
\quad &\quad \left(\sum_{j=2}^{\infty} \gamma_{\sum_{i=1}^{b-1} n_i - |C_j^+| + t + 1 - \sum_{i=1}^{r_j} n_i} \right) (1 - \lambda)\alpha, \\
\quad &\quad \text{where } |C_j^+| = \sum_{j=r_j+1}^{b-1} |C_j| \\
\quad \text{end} \\
\text{end} \\
\end{align*}
Algorithm 16 The mini-batch alpha-investing as a special case of SAFFRON_\text{mini}

\begin{verbatim}
input: FDR level $\alpha$, non-negative non-increasing sequence $\{\gamma_j\}_{j=1}^\infty$ such that $\sum_j^\infty \gamma_j = 1$, initial wealth $W_{1,0}$
\begin{align*}
\alpha_{1,1} &= \gamma_1 W_{1,0} \\
\text{for } b = 1, 2, \ldots \text{ do} \\
\quad &\text{if } b > 1 \text{ then} \\
\quad &\quad W_{b,0} = W_{b-1,n} + \alpha |R_{b-1}| + \sum_{j \leq n_{b-1}} \alpha_{b-1,j} C_{b-1,j} - W_{1,0} \mathbf{1} \{r_1 = b - 1\}
\quad \text{end} \\
\quad \text{for } t = 1, 2, \ldots, n \text{ do} \\
\quad &\quad W_{b,t} := W_{b,t-1} - \frac{\alpha_{b,t}}{1-\alpha_{b,t}} \\
\quad &\quad \alpha_{b,t+1} = \gamma \sum_{j=1}^t n_j - |C_t^+|+t+1 W_{1,0} + \gamma \sum_{j=1}^t n_j - |C_t^+|+t+1-\sum_{j=1}^r n_j ((1 - \alpha_{r_1,s})\alpha - W_{1,0}) + \\
\quad &\quad \left(\sum_{j=2}^\gamma \gamma \sum_{j=1}^t n_j - |C_t^+|+t+1-\sum_{j=1}^r n_j \right) (1 - \alpha_{r_1,s})\alpha,
\quad \text{where } |C_t^+| = \sum_{j=r_1+1}^{t-1} |C_j|
\quad \text{end}
\text{end}
\end{verbatim}

12 Positive regression dependency on a subset (PRDS)

For convenience, here we briefly review the definition of positive regression dependency on a subset (PRDS).

**Definition 1.** Let $D \subseteq [0, 1]^n$ be any non-decreasing set, meaning that $x \in D$ implies $y \in D$, for all $y$ such that $y_i \geq x_i$ for all $i \in [n]$. We say that a vector of $p$-values $P = (P_1, \ldots, P_n)$ satisfies positive dependence (PRDS) if for any null index $i \in \mathcal{H}^0$ and arbitrary non-decreasing $D \subseteq [0, 1]^n$, the function $t \rightarrow \Pr\{P \in D \mid P_i \leq t\}$ is non-decreasing over $t \in (0, 1]$.

Clearly, independent $p$-values satisfy PRDS. Another important example is given for Gaussian observations. Suppose $Z = (Z_1, \ldots, Z_n)$ is a multivariate Gaussian with covariance matrix $\Sigma$, and let $P = (\Phi(Z_1), \ldots, \Phi(Z_n))$ be a vector of $p$-values, where $\Phi$ is the standard Gaussian CDF. Then, $P$ satisfies PRDS if and only if, for all $i \in \mathcal{H}^0$ and $j \in [n]$, $\Sigma_{ij} \geq 0$.

13 Experiments under model misspecification

We additionally test the robustness of our algorithms by testing LORD_{dep} and SAFFRON_{dep} on Markov dependent $p$-values, where the order of dependence is a fixed constant $L$. In particular, each $p$-value subsequence of length $L + 1$ comes from observations that follow a multivariate normal distribution, whose covariance matrix has the Toeplitz structure $\Sigma_{\text{min}}(L+1, \rho)$ [14]. Marginally, the observations are distributed according to the Gaussian model described at the beginning of this section. Figure 18 and Figure 19 compare the power and FDR of LORD_{dep} and SAFFRON_{dep} under local dependence, when the mean of the observations under the alternative is $\mu_c = 3$ with probability 1. Figure 20 and Figure 21 give the same comparison when the mean of non-null samples is normally distributed, which yields a slower decrease in performance with increasing the lag. In all four plots we fix $\rho = 0.5$. 

39
Figure 18. Power and FDR of LORD$_{dep}$ with varying the Markov lag $L$ in the $p$-value sequence. In all five runs LORD$_{dep}$ has the same parameters $(\{\gamma_j\}_1^\infty, W_0)$. The mean of observations under the alternative is a point mass at $\mu_c = 3$, and $\rho = 0.5$.

Figure 19. Power and FDR of SAFFRON$_{dep}$ with varying the Markov lag $L$ in the $p$-value sequence. In all five runs SAFFRON$_{dep}$ has the same parameters $(\{\gamma_j\}_1^\infty, W_0)$. The mean of observations under the alternative is a point mass at $\mu_c = 3$, and $\rho = 0.5$.

Figure 20. Power and FDR of LORD$_{dep}$ with varying the Markov lag $L$ in the $p$-value sequence. In all five runs LORD$_{dep}$ has the same parameters $(\{\gamma_j\}_1^\infty, W_0)$. The mean of observations under the alternative is $N(0, 2 \log(M))$, and $\rho = 0.5$.  

40
14 Examining the difference between mFDR and FDR

In Section 8, we plotted strict FDR estimates, obtained by averaging the false discovery proportion over 200 independent trials; on the other hand, the main guarantees of this paper apply to mFDR control. For this reason, here we provide the plot of both mFDR and FDR estimates, for all experiments in Section 8. We estimate mFDR by computing the ratio of the average number of false discoveries and the average total number of discoveries.
Figure 22. The left plots reproduce FDR from Figures 4, 5, 6, 7, while the right plots show mFDR for the same experiments.
Figure 23. The left plots reproduce FDR from Figures 8, 9, 10, 11, while the right plots show mFDR for the same experiments.
Figure 24. The left plots reproduce FDR from Figures 12, 13, 14, 15, while the right plots show mFDR for the same experiments.
Figure 25. The left plots reproduce FDR from Figures 16 and 17, while the right plots show mFDR for the same experiments.