CENSORED STABLE SUBORDINATORS AND FRACTIONAL DERIVATIVES

QIANG DU, LORENZO TONIAZZI, AND ZIRUI XU

Abstract. Based on the popular Caputo fractional derivative of order $\beta$ in $(0,1)$, we define the censored fractional derivative on the positive half-line $\mathbb{R}_+$. This derivative proves to be the Feller generator of the censored (or resurrected) decreasing $\beta$-stable process in $\mathbb{R}_+$. We provide a series representation for the inverse of this censored fractional derivative, which we use to study general censored initial value problems. We are then able to prove that this censored process hits the boundary in a finite time $\tau_\infty$, whose expectation is proportional to that of the first passage time of the $\beta$-stable subordinator. We also show that the censored relaxation equation is solved by the Laplace transform of $\tau_\infty$. This relaxation solution proves to be a completely monotone series, with algebraic decay one order faster than its Caputo counterpart, leading, surprisingly, to a new regime of fractional relaxation models. Lastly, we discuss how this work identifies a new sub-diffusion model.

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1. Introduction

Fractional derivatives, a special class of nonlocal integral and pseudo-differential operators [15, 21, 46, 27], have been successfully employed to model heterogeneities and nonlocal interactions in many applications (see, e.g., [38, 41, 43, 11]). They also enjoy an interesting mathematical theory with deep connections to Lévy processes (see, e.g., [40, 7, 32, 33, 34]). For example, the Caputo derivative [14] of order \( \beta \in (0, 1) \) on the positive half-line \( \mathbb{R}^+ \), plays important roles in modelling non-exponential relaxation [14, 41] and non-Markovian sub-diffusive dynamics [39, 1, 22]. For a smooth function \( u \) vanishing outside \( \mathbb{R}^+ \), the Caputo derivative equals the Riemann–Liouville (R–L) derivative \( D_0^\beta \) [14] given by

\[
D_0^\beta u(x) = \int_0^x (u(x) - u(x - r)) \frac{r^{-1-\beta}}{\Gamma(-\beta)} \, dr + \frac{u(x)}{\Gamma(1-\beta)}, \quad x > 0.
\]

(1.1)

Probabilistically, \(-D_0^\beta\) generates a killed Lévy process, which is the decreasing \( \beta \)-stable process \( S_1 = \{S_1^s\}_{s \geq 0} \) killed at time \( \tau_1 \), the first exit time from \( \mathbb{R}^+ \) [3, 28]. Intuitively, the first summand in (1.1) describes the decreasing \( \beta \)-stable jumps landing inside \( \mathbb{R}^+ \), while \( x^{-\beta}/\Gamma(1-\beta) = \int_x^\infty r^{-1-\beta}/\Gamma(-\beta) \, dr \) is the killing coefficient for the jumps landing outside \( \mathbb{R}^+ \). In this work, we introduce what we call the censored fractional derivative \( \partial_0^\beta \), allowing the representation

\[
\partial_0^\beta u(x) = \int_0^x (u(x) - u(x - r)) \frac{r^{-1-\beta}}{\Gamma(-\beta)} \, dr, \quad x > 0.
\]

(1.2)

It is intuitively clear that \(-\partial_0^\beta\) only allows the decreasing \( \beta \)-stable jumps to land inside \( \mathbb{R}^+ \), and suppresses those landing outside \( \mathbb{R}^+ \). Indeed we prove that it is the (Feller) generator of \( S_1^0 = \{S_1^0\}_{s \geq 0} \), the censored decreasing \( \beta \)-stable process in \( \mathbb{R}^+ \). We will construct \( S_1^0 \) by repeatedly resurrecting in situ the killed decreasing \( \beta \)-stable process, following the canonical
Ikeda–Nagasawa–Watanabe (INW) piecing together procedure [25]. (Cf. [36, Remark 3.3] for two other notions of “censoring” a process.)

We initiate the study of the censored fractional derivative, and then apply its theory to derive several new and non-trivial results about the censored stable subordinator, as we now explain. We first prove the well-posedness of the basic initial value problem (IVP)

\[
\begin{aligned}
\partial_0^\beta u(x) &= g(x), \quad x \in (0, T], \\
u(x) &= u_0, \quad x = 0,
\end{aligned}
\]

for any \( T > 0, \ u_0 \in \mathbb{R} \) and certain \( g \in C(0, T] \). Our proof is based on constructing the candidate solution \( u = u_0 + J_0^\beta g \), where \( J_0^\beta \) allows a probabilistic series representation and the expected potential representation, namely

\[
I_0^\beta g(x) = J_0^\beta g(x) + \sum_{j=1}^\infty \mathbb{E}_x \left[ J_0^\beta g(X_j) \right] = \mathbb{E}_x \left[ \int_0^{\tau_{\infty}} g(S_c^x) \, ds \right].
\]

Here, \( J_0^\beta \) is the R–L integral, i.e. the inverse of \( D_0^\beta \), given by

\[
J_0^\beta g(x) = \int_0^x g(y) \frac{(x - y)^{\beta - 1}}{\Gamma(\beta)} \, dy = \mathbb{E}_x \left[ \int_0^{\tau_{i+n}} g(S_i^x) \, ds \right],
\]

where the second identity is the known potential representation for \( J_0^\beta \); the discrete-time process \( X \mid X_0 = x \) is defined as \( X_j := x \prod_{i=1}^j B_i \), where \( \{B_i\}_{i \in \mathbb{N}} \) is an i.i.d. collection of beta-distributed random variables with parameters \( 1 - \beta \) and \( \beta \); and \( \tau_{\infty} \) is the lifetime of \( S^c \). The equivalence of (1.4) and (1.5) is due to the equality in law between \( X_j \) and \( S^c \) at its \( j \)-th resurrection time, combined with the second identity in (1.6) (see Remark 4.6 for more details). The way we solve (1.3) is to regard it as a linear R–L IVP \( D_0^\beta u = ku + g, \ u(0) = 0 \) with the coefficient \( k(x) = x^{-\beta}/\Gamma(1-\beta) \). It turns out that the formula given in [14, Theorem 7.10] for bounded \( k \) still converges for this specific unbounded \( k \), allowing us to construct the solution. (As for more general \( k \) that may diverge as \( O(x^{-\beta}) \), [37, Example 3.4] gave a non-constructive proof of the existence result.) This explicit solution then allows us to establish the (global) well-posedness of general IVPs \( \partial_0^\beta u = f(x, u), \ u(0) = u_0 \), for certain Lipschitz data \( f \).

Using the results above, we are able to solve the linear IVP \( \partial_0^\beta u = \lambda u, \ u(0) = u_0 \), for any \( \lambda \in \mathbb{R} \). We obtain the Mittag-Leffler-type representation for its solution

\[
u(x) = u_0 \sum_{N=0}^\infty \lambda^N x^{\beta N} \prod_{n=1}^N \left( \frac{\Gamma(1+n\beta)}{\Gamma(n\beta+1-\beta)} - \frac{1}{\Gamma(1-\beta)} \right)^{-1},
\]

where an empty product equals 1 by convention (also, \( u(x) = u_0 \) if \( \lambda = 0 \)) and each factor of the indexed product is positive by (2.1). Surprisingly, for \( \lambda < 0 \), this solution decays at the fast algebraic rate \( x^{-1-\beta} \) (Theorem 3.17), which we believe is a new regime for
fractional relaxation models. Indeed the Caputo fractional relaxation solution \( u_0 E_\beta(\lambda x^\beta) \) decays at the rate \( x^{-\beta} \) [14, Theorem 7.3], where \( E_\beta(x) = \sum_{n=0}^{\infty} x^n / \Gamma(n\beta + 1) \) is the Mittag-Leffler function. Moreover, the lagging and leading coupled fractional relaxation equations in [2, 49] model the decay rate \( x^{-\gamma} \) for some \( \gamma \in (0, 1) \). Our proof (inspired by [17, Theorem 3.2]) is based on maximum principle and turns out to be versatile, albeit elementary. Indeed the same argument proves the decay rate \( x^{-1-\alpha} \) of the solution to \( \partial_0^\beta u = \lambda x^{\alpha-\beta} u \) (\( \lambda < 0, \alpha > 0 \)) (see Proposition 3.24), which is again one order faster than its Caputo counterpart (expressed by the Kilbas–Saigo function [42]). Moreover, we will show how to adapt this argument to the Caputo setting to give new and simple proofs of the two-sided uniform bounds of \( E_\beta \) and more generally, a class of Kilbas–Saigo functions, which are the recent results in [45, Theorem 4] and [10, Proposition 4.12], respectively. This very argument may have even broader applications, e.g., in general Caputo-type relaxation problems (corresponding to general killed subordinators), as we discuss in Remark 3.18-(iii).

As a special case of (1.5), we have the identity

\[
E_x[\tau_\infty] = E_x[\tau_1] \frac{\beta \pi}{\beta \pi - \sin(\beta \pi)}, \quad \text{where} \quad E_x[\tau_1] = \frac{x^\beta}{\Gamma(\beta + 1)},
\]

which implies that \( S^c \) hits 0 in finite time, a fact that we believe has not been shown before. This is fundamental and not obvious, especially in view of [6, Theorem 1.1-(1)], which proves that the censored symmetric \( \beta \)-stable Lévy process never hits the boundary, whether censored in an interval or \( \mathbb{R}_+ \). (Also, censored decreasing compound Poisson processes do not hit the barrier in finite time, and our numerical simulations suggest neither do censored gamma subordinators.) We are then able to show several more connections between the analytic and probabilistic aspects of \( \partial_0^\beta \). That is, we will prove that \( S^c \) is indeed a Feller process generated by \( -\partial_0^\beta \), and that the exit problem for \( \tau_\infty \) is solved by (1.7), i.e.

\[
u_0 E_x[\exp\{\lambda \tau_\infty\}] \text{ equals the series } (1.7), \text{ for all } x > 0 \text{ and } \lambda \in \mathbb{R}.
\]

As a consequence of (1.9), we can obtain all the moments of \( \tau_\infty \) and confirm the complete monotonicity of (1.7). We emphasise that (1.9) is significantly harder to prove than Caputo’s counterpart \( E_x[\exp\{\lambda \tau_1\}] = E_\beta(\lambda x^\beta) \). This is mainly due to the inapplicability of Laplace transforms to \( S^c \) and the complexity of the coefficients in (1.7) (see Remark 4.14 for more detail). Nonetheless, we obtain a proof by combining our series solution to the resolvent equation \( \partial_0^\beta u = \lambda u + g \) with a simple semigroup theory argument, following [23, Corollary 5.1]. We could alternatively try combining our IVP theory with standard potential theory (see, e.g., [13, Chapter 3]) applied to the Feynman–Kac semigroup of \( -D_0^\beta + (\lambda + k) \), but it would be more involved. (We also remark that (1.9) serves as an efficient alternative to numerically compute (1.7) for \( \lambda < 0 \).)

Lastly, we discuss how this work sets the foundations for the study of a new time-fractional diffusion equation \( \partial_0^\beta u = \Delta u / 2 \), solved by the process \{\( B_{\tau_\infty(t)} \)\}\( t \geq 0 \). (Here \( \partial_0^\beta \) acts on the time variable, \( B \) is a Brownian motion independent of \( \tau_\infty(t) := \tau_\infty | S_0^c = t \).) This
is the censored analogue of the Caputo time-fractional diffusion equation $D^\beta_0 [u - u(0)] = \Delta u / 2$, which is solved by the fractional kinetic process $\{B_{\tau_1(t)} \}_{t \geq 0}$ (with $B$ independent of the inverse stable subordinator $\tau_1(t) := \inf \{ s : t < -S^1_s \}$), a non-Markovian sub-diffusion process arising from several central limit theorems [39, 1, 22]. As we discuss in Remark 4.15-(i), although both $B_{\tau_1}$ and $B_{\tau_\infty}$ are sub-diffusion processes (due to (1.8)), their respective characteristic functions, $E_{\beta}(\lambda t^\beta)$ and (1.7) (for some $\lambda < 0$), display strikingly different decay rates (due to our results on relaxation solutions).

This work is organized as follows: Section 2 introduces notation, recalls basic results on fractional calculus, defines the censored fractional derivative, and studies the solution kernels; Section 3 focuses on the well-posedness and series representation of the solution to (1.3), then addresses linear and non-linear censored IVPs; in Section 4 we construct the censored decreasing $\beta$-stable process and apply our IVP theory to its study.

2. Preliminary notation and definitions

Throughout this article, we denote by $\beta$ ($0 < \beta < 1$) the order of fractional derivatives, and by $[0, T]$ ($0 < T < \infty$) the interval of interest. We denote by $\mathbb{N}, \mathbb{R}$ and $\mathbb{R}_+$ the sets of positive integers, real numbers and positive numbers, respectively. For any interval $\Omega \subseteq \mathbb{R}$ we denote by $C(\Omega), C^1(\Omega), C^{0,\beta}(\Omega)$ and $L^1(\Omega)$ the real functions on $\Omega$ that are continuous, continuously differentiable, $\beta$-Hölder continuous and Lebesgue integrable, respectively. We abbreviate $C(\Omega) \cap L^1(\Omega)$ to $C \cap L^1(\Omega)$. For compact $\Omega$ we denote by $\| \cdot \|_{C(\Omega)}$ the sup norm.

We denote by $\Gamma$ the gamma function and frequently use without mention the standard identities $\Gamma(2 - \alpha) = (1 - \alpha)\Gamma(1 - \alpha)$ for all $\alpha \in \mathbb{R} \setminus \mathbb{N}$, $\Gamma(\beta + 1)\Gamma(1 - \beta) = \beta \pi / \sin(\beta \pi)$ and

$$\int_0^x (x - r)^{\gamma - 1} r^{\alpha - 1} \, dr = x^{\gamma + \alpha - 1} \frac{\Gamma(\gamma)\Gamma(\alpha)}{\Gamma(\gamma + \alpha)} \text{ for all } \alpha, \gamma, x > 0.$$ 

We also rely crucially on the inequality (which we prove in Lemma 2.14)

$$\Gamma(\alpha + 1 - \beta) < \Gamma(1 + \alpha)\Gamma(1 - \beta) \text{ for all } \alpha > 0.$$ \hspace{1cm} (2.1)

2.1. R–L calculus and fractional function spaces. We present some basic results about the R–L fractional derivative, and proofs are given to make our presentation self-contained. We refer to [14] for a general study of Caputo/R–L derivatives.

**Definition 2.1.** For $\beta \in (0, 1)$, $u \in C \cap L^1(0, T]$, we define the R–L integral

$$J_0^\beta u(x) = \int_0^x \frac{(x - r)^{\beta - 1}}{\Gamma(\beta)} u(r) \, dr, \quad x \in (0, T],$$

we define the function spaces

$$C_{\beta}(0, T] = \left\{ u \in C \cap L^1(0, T] : J_0^{1-\beta} u \in C^1(0, T] \right\},$$

$$C_{\beta}[0, T] = C[0, T] \cap C_{\beta}(0, T].$$
and for $u \in C_\beta(0,T]$, $x \in (0,T]$, we define the $R-L$ derivative

$$D_0^\beta u(x) = \frac{d}{dx} J_0^{1-\beta} u(x) = \frac{d}{dx} \int_0^x \frac{(x-r)^{-\beta}}{\Gamma(1-\beta)} u(r) \, dr.$$

**Remark 2.2.**

(i) Note that $C_\beta(0,T]$ is chosen so that the image of $D_0^\beta$ is contained in $C(0,T]$. Moreover, $C_\beta[0,T]$ is chosen to be the solution space, as we will explain in Remark 2.8.

(ii) Note that $C_\beta[0,T]$ is not a subspace of $C^{0,\beta}[0,T]$. Indeed $x^\alpha$ is in $C_\beta[0,T]$ for all $\alpha \geq 0$, but in $C^{0,\beta}[0,T]$ only if $\alpha \geq \beta$.

**Lemma 2.3.** The following relations between $D_0^\beta$ and $J_0^\beta$ hold (proved in Appendix A.1).

(i) If $u \in C \cap L^1(0,T]$, then $J_0^\beta u \in C \cap L^1(0,T]$.

(ii) If $u \in C \cap L^1(0,T]$, then $J_0^\beta u \in C_\beta(0,T]$ and $D_0^\beta J_0^\beta u = u$.

(iii) Assume $g \in C \cap L^1(0,T]$. Then

$$u = J_0^\beta g \quad \text{if and only if} \quad \begin{cases} u \in C_\beta(0,T], \\ D_0^\beta u = g, \\ \lim_{x \to 0} J_0^{1-\beta} u(x) = 0. \end{cases}$$

(iv) If $u \in C_\beta[0,T]$ satisfies $D_0^\beta u = 0$, then $u = 0$.

**Remark 2.4.** Note that in Lemma 2.3-(iv), the condition $u \in C_\beta[0,T]$ cannot be weakened to $u \in C_\beta(0,T]$, since $D_0^\beta x^{\beta-1}$ is also 0.

**Remark 2.5.** For $g \in L^1(0,T]$ satisfying $|g(x)| \leq M x^{\alpha-\beta}$ for some $\alpha, M \geq 0$ and all $x \in (0,T]$, the Hölder regularity of $J_0^\beta g$ is summarized as follows. See Lemmata A.2 and A.4 for the proofs.

(i) For any $T_1 \in (0,T)$, $J_0^\beta g \in C^{0,\beta}[T_1,T]$ with a Hölder constant $2M \max\{T_1^{\alpha-\beta}, T^{\alpha-\beta}\}/\Gamma(1+\beta)$.

(ii) If $0 < \alpha < \beta$, then $J_0^\beta g \in C^{0,\alpha}[0,T]$ with a Hölder constant $2M\Gamma(\alpha+1-\beta)/\Gamma(1+\alpha)$.

2.2. **Censored fractional derivative.** We now define our censored fractional derivative.

**Definition 2.6.** Given $\beta \in (0,1)$, we define the censored fractional derivative of any $u \in C_\beta(0,T]$ as

$$\partial_0^\beta u(x) = D_0^\beta u(x) - \frac{x^{-\beta}}{\Gamma(1-\beta)} u(x), \quad \text{for all } x \in (0,T].$$

**Remark 2.7.**

(i) Like the Caputo derivative, the censored fractional derivative maps constants to 0, and satisfies the scaling property

$$\partial_0^\beta v(x) = c^{-\beta} \partial_0^\beta u(x/c),$$

where $u \in C_\beta(0,T]$, $c$ is a positive constant and $v(x) := u(x/c) \in C_\beta(0,cT]$. 
(ii) For functions of the form \( x^\alpha (\alpha > 0) \), the censored fractional derivative equals the R–L derivative up to a constant multiple: 
\[
\partial_0^\alpha x^\alpha = c_{\alpha,\beta} D_0^\beta x^\alpha, \quad \text{where}
\]
\[
c_{\alpha,\beta} = 1 - \frac{\Gamma(\alpha + 1 - \beta)}{\Gamma(\alpha + 1)\Gamma(1 - \beta)}, \quad D_0^\beta x^\alpha = x^{\alpha - \beta} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - \beta)}.
\]
By (2.1), \( c_{\alpha,\beta} \) is in \((0, 1)\). In particular, for \( \alpha = \beta \), we have 
\[
\partial_0^\beta x^\alpha = \Gamma(\beta + 1)(\beta\pi - \sin(\beta\pi))/(\beta\pi).
\]
While we can talk about the semigroup property for \( D_0^\beta \) and the Caputo derivative [14, Theorem 2.13 and Lemma 3.13], we cannot for \( \partial_0^\beta \). For instance,
\[
\partial_0^\beta \partial_0^\beta x^\alpha = c_{\alpha - \gamma,\beta} c_{\alpha,\gamma} D_0^{\beta + \gamma} x^\alpha,
\]
\[
\partial_0^\beta \partial_0^\alpha x^\alpha = c_{\alpha - \beta,\gamma} c_{\alpha,\beta} D_0^{\beta + \gamma} x^\alpha,
\]
however \( c_{\alpha - \gamma,\beta} c_{\alpha,\gamma} \neq c_{\alpha - \beta,\gamma} c_{\alpha,\beta} \) unless \( \beta = \gamma \).

(iii) If \( u \in C^1[0, T] \cap L^1(0, T) \), then on \((0, T)\), \( \partial_0^\beta u \) allows the representation (1.2), from which it is clear that \(-\partial_0^\beta\) satisfies the positive maximum principle [9], and hence it is dissipative in the sense that 
\[
\|\lambda u + \partial_0^\beta u\|_C[0,T] \geq \lambda\|u\|_C[0,T]
\]
for any \( \lambda > 0 \) and \( u \in C^1[0, T] \).

(iv) The Laplace transform of the censored fractional derivative can be written as
\[
\mathcal{L} [\partial_0^\beta u](k) = k^\beta \left( \mathcal{L}[u](k) - k^{-1} \mathcal{L} \left[ \frac{u(x/k)x^{-\beta}}{\Gamma(1 - \beta)} \right](1) \right), \quad k > 0,
\]
which differs from \( k^\beta (\mathcal{L}[u](k) - k^{-1} u(0)) \), the Laplace transform of the Caputo derivative [40, Chapter 2.4]. One can notice that even in Laplace space, it is unclear if the initial conditions can be imposed on the problem \( \partial_0^\beta u = g \).

Remark 2.8. We will spend the next few pages establishing the well-posedness of \( \partial_0^\beta u = g \) with \( u(0) = u_0 \), for certain \( g \). With the initial condition imposed, \( C_\beta[0, T] \), which equals \( \{ u \in C[0, T] : J_0^\beta u \in C^1[0, T] \} \), now becomes a natural function space for solutions. A large part of the Caputo literature (e.g., [14]), however, chose \( J_0^\beta [C \cap L^1(0, T)] \), i.e., the image of \( J_0^\beta \) over \( C \cap L^1(0, T) \), as the solution space. This difference seems not to matter, at least to our studies. Indeed, the set \( U \) consisting of the solutions to (1.3) (for those \( g \) of interest) is contained in the intersection of those two spaces, as shown in the diagram below (see Appendix A.3 for the proof of the diagram)

\[
\begin{array}{ccc}
C_\beta(0, T) & C_\beta[0, T] & U \\
\end{array}
\]

where we define \( U = \{ u \in C_\beta[0, T] : x^{\beta - \alpha} \partial_0^\beta u \in C[0, T] \text{ for some } \alpha > 0 \} \). Lastly, let us mention that \( J_0^\beta C[0, T] = \{ u \in C[0, T] : J_0^\beta u \in C^1[0, T] \} \) [47, Proposition 4.1].

2.3. An integral operator and related kernels. As we can see from (1.4), the solution to the IVP (1.3) may be seen as a variation of the R–L integral. In this subsection we introduce an integral operator and related kernels for the convergence study of (1.4). This
leads to Lemma 2.14, which is a crucial bound in this work. The probabilistic interpretation of the kernels under consideration will be presented in Section 4.

**Definition 2.9.** For $0 < r < x$, we define the following kernels recursively

$$k_j(x, r) = \begin{cases} (x-r)^{\beta-1}r^{-\beta} \frac{\Gamma(\beta)}{\Gamma(1-\beta)}, & j = 1, \\ \int_r^x k_1(x, s)k_{j-1}(s, r) \, ds, & j \geq 2. \end{cases} \quad (2.2)$$

**Remark 2.10.** Note that for each $x > 0$, $k_1(x, \cdot)$ is a beta distribution on $(0, x)$ with parameters $(1 - \beta, \beta)$, and straightforward induction arguments can be used to prove that

$$\int_0^x k_j(x, r) \, dr = 1 \quad (j \geq 1, \ x > 0)$$

and

$$k_j(x, r) = \int_r^x k_{j-1}(x, s)k_1(s, r) \, ds \quad (j \geq 2, \ x > r > 0).$$

**Definition 2.11.** For $\psi \in C[0, T]$, we define

$$K\psi(x) = \begin{cases} \int_0^x k_1(x, r)\psi(r) \, dr, & x > 0, \\ \psi(0), & x = 0, \end{cases}$$

where the explicit dependence of $K$ on $\beta$ is suppressed to ease notation.

**Remark 2.12.** It is easy to see that $K\psi(x) = J_0^{\beta}[x^{-\beta}\psi(x)/\Gamma(1-\beta)]$ for $\psi \in C[0, T]$ and $x \in (0, T)$, and that $K$ is a linear operator preserving positivity ($K\psi \geq 0$ if $\psi \geq 0$).

**Lemma 2.13.** For any $\alpha \geq 0$, we have

$$Kx^{\alpha} = x^{\alpha}\Gamma(\alpha + 1 - \beta)/\Gamma(\alpha + 1 - \beta).$$

If $\psi \in C[0, T]$ satisfies $|\psi(x)| \leq Mx^{\alpha}$ for some $M > 0$ and all $x \in (0, T]$, then $K\psi \in C[0, T]$, and $|K\psi(x)| \leq MKx^{\alpha}$ for all $x \in (0, T]$.

**Proof.** The first claim is immediate from the definition of $K$, and by the assumption on $\psi$, we have $|K\psi(x)| \leq K|\psi|(x) \leq MKx^{\alpha}$. We now prove that $K\psi$ is continuous on $(0, T]$. For $\varepsilon \in (0, x/2)$, define

$$K_{\varepsilon}\psi(x) = \int_{x-\varepsilon}^{x-}\!\! k_1(x, r)\psi(r) \, dr.$$

Given $T_1 \in (0, T]$, for every $x \in [T_1, T]$ and $\varepsilon \in (0, T_1/2)$, we have

$$|K_{\varepsilon}\psi(x) - K\psi(x)| \leq \int_0^\varepsilon k_1(x, r)|\psi(r)| \, dr + \int_{x-\varepsilon}^x k_1(x, r)|\psi(r)| \, dr \leq \frac{\beta(x/\varepsilon - 1)^{\beta-1} + (1 - \beta)(x/\varepsilon - 1)^{-\beta}}{\beta(1 - \beta)\Gamma(\beta)\Gamma(1-\beta)}\|\psi\|_{C[0, T]} \leq \frac{\beta(T_1/\varepsilon - 1)^{\beta-1} + (1 - \beta)(T_1/\varepsilon - 1)^{-\beta}}{\beta(1 - \beta)\Gamma(\beta)\Gamma(1-\beta)}\|\psi\|_{C[0, T]},$$
therefore, as \( \varepsilon \to 0 \), \( K_{x} \psi \to K \psi \) uniformly on \([T_{1}, T]\). Because \( K_{x} \psi \) is continuous on \([T_{1}, T]\), \( K \psi \) must be continuous on \([T_{1}, T]\), and thus on \((0, T]\). In addition, by the continuity of \( \psi \) at \( x = 0 \), \( K \psi(x) \to \psi(0) \) as \( x \to 0 \), and therefore \( K \psi \in C[0, T] \).

We can now obtain the crucial bound that will help us adapt \cite[Theorem 7.10]{14} to the censored IVP (1.3) in order to express the solution as a series.

**Lemma 2.14.** For any \( \alpha > 0 \), we have

\[
\sum_{j=1}^{\infty} K^{j} x^{\alpha} = x^{\alpha} \left( \frac{\Gamma(1 + \alpha) \Gamma(1 - \beta)}{\Gamma(\alpha + 1 - \beta)} - 1 \right)^{-1}. \tag{2.3}
\]

If \( \psi \in C[0, T] \) satisfies \( |\psi(x)| \leq M x^{\alpha} \) for some \( M > 0 \) and all \( x \in (0, T] \), then

\[
\sum_{j=1}^{\infty} K^{j} \psi \in C[0, T], \text{ and } \left| \sum_{j=1}^{\infty} K^{j} \psi(x) \right| \leq M \sum_{j=1}^{\infty} K^{j} x^{\alpha} \text{ for all } x \in (0, T].
\]

In addition, \( K^{j} \psi(x) = \int_{0}^{x} k_{j}(x, r) \psi(r) \, dr \) for all \( j \in \mathbb{N}, x \in (0, T] \).

**Proof.** We first confirm (2.1) using the fact that \( t^{\alpha} \) and \((1-t)^{-\beta}\) strictly increase, so that

\[
\frac{1}{\alpha - \beta + 1} = \int_{0}^{1} (1-t)^{\alpha} (1-t)^{-\beta} \, dt < \int_{0}^{1} t^{\alpha} (1-t)^{-\beta} \, dt = \frac{\Gamma(1 + \alpha) \Gamma(1 - \beta)}{\Gamma(1 + \alpha + 1 - \beta)}.
\]

Applying Lemma 2.13 for \( j \) times, we get \( K^{j} x^{\alpha} = x^{\alpha} \left( \frac{\Gamma(1 + \alpha) \Gamma(1 - \beta)}{\Gamma(\alpha + 1 - \beta)} \right)^{-j} \). Then, by summing over \( j \), we obtain (2.3) from (2.1). Meanwhile, we have \( |K^{j} \psi(x)| \leq M K^{j} x^{\alpha} \) and \( K^{j} \psi \in C[0, T] \), so \( \sum_{j=1}^{\infty} K^{j} \psi \) converges uniformly to a limit in \( C[0, T] \), whose absolute value is pointwise bounded by \( M \sum_{j=1}^{\infty} K^{j} x^{\alpha} \). Finally, by induction,

\[
K^{j} \psi(x) = K K^{j-1} \psi(x) = \int_{0}^{x} k_{1}(x, r) \int_{0}^{r} k_{j-1}(r, s) \psi(s) \, ds \, dr
= \int_{0}^{x} \int_{s}^{x} k_{1}(x, r) k_{j-1}(r, s) \psi(s) \, ds \, dr
= \int_{0}^{x} k_{j}(x, s) \psi(s) \, ds.
\]

**Remark 2.15.** In Lemma 2.14, we require \( \alpha > 0 \) (though the last statement there holds for all \( \alpha \geq 0 \)), in fact, if \( \alpha = 0 \), let \( \psi = 1 \), then

\[
\sum_{j=1}^{\infty} K^{j} \psi(x) = \sum_{j=1}^{\infty} \int_{0}^{x} k_{j}(x, r) \, dr = \infty.
\]
3. Well-posedness of the censored IVPs

3.1. Inverse of $D_0^\beta$. We begin with the basic censored IVP (1.3) with $g \in C(0,T]$ and $u_0 \in \mathbb{R}$. Our strategy is to consider the equivalent Caputo/R–L problem for $\bar{u} = u - u_0$ with the unbounded coefficient $x^{-\beta}/\Gamma(1-\beta)$,

$$D_0^\beta \bar{u}(x) = \frac{x^{-\beta}}{\Gamma(1-\beta)} \bar{u}(x) + g(x), \quad x > 0, \quad \bar{u}(0) = 0, \quad (3.1)$$

and then show that for certain forcing terms $g$, the formula [14, Theorem 7.10] for bounded coefficients still yields a solution to (3.1), and thus to (1.3).

Remark 3.1.

(i) We can solve (3.1) using Picard iteration, i.e., $\bar{u}_{m+1}(x) = J_0^\beta \left[ x^{-\beta} \bar{u}_m(x)/\Gamma(1-\beta) + g(x) \right]$ $(m = 1, 2, \cdots)$ with $\bar{u}_1 = 0$. By Remark 2.12, the limit equals $I_0^\beta g$ defined in (3.2) if the iteration converges.

(ii) For $g \in C[0,T]$ and $u_0 = 0$, [37, Example 3.4] guarantees (after change of variables) that there is a unique solution in $J_0^\beta C[0,T]$ to (3.1) and thus to (1.3) (and also to (3.22), a linear IVP we will study later). However, [37] does not cover nonlinear IVPs (3.10). Moreover, it provides neither explicit expressions nor stochastic interpretations for the solutions, and seemingly cannot obtain the continuous dependence. Lastly, no singularity of $g$ at $x = 0$ is allowed there either. Let us also mention that, as stated beneath [37, Equation (5)], for singular fractional differential equations one should not expect to impose initial conditions without losing regularity. But (3.1) proves to be an exception. In fact, constant functions solve its homogeneous version because of the specific coefficient $x^{-\beta}/\Gamma(1-\beta)$.

(iii) If one replaces the coefficient in the R–L problem (3.1) by $C x^{-\beta}/\Gamma(1-\beta)$, then the series representation for the solution would be $\bar{u} = \sum_{j=0}^{\infty} C^j \mathcal{K}^j J_0^\beta g$, which does not converge for important data (like $g = 1$) if $|C| \geq \Gamma(1+\beta)\Gamma(1-\beta)$. In other words, the Picard iteration will not converge for such $C$. The above threshold can be obtained from the proof of Lemma 2.14, and is consistent with the condition “$b(0) < \Gamma(\alpha+1)$” in [37, Example 3.4].

We now present a key result concerning IVP (1.3), which serves as the fundamental theorem of calculus for $D_0^\beta$. Or simply put, $I_0^\beta$ is to $D_0^\beta$ as $J_0^\beta$ is to $D_0^\beta$.

Theorem 3.2. Let $u_0 \in \mathbb{R}$ and $g \in C(0,T]$ such that $|g(x)| \leq M x^{\alpha-\beta}$ for some $M, \alpha > 0$ and all $x \in (0,T]$. Then there exists a unique function $u \in C_0^\beta[0,T]$ satisfying (1.3), and it has the series representation

$$u(x) - u_0 = I_0^\beta g(x) := \sum_{j=0}^{\infty} \mathcal{K}^j J_0^\beta g(x), \quad (3.2)$$

where $\mathcal{K}^0$ is the identity operator by convention. Moreover, $u$ depends on $u_0$ and $g$ continuously in the sense of Remark 3.4.
Theorem 3.2 is an immediate consequence of Lemmata 3.5 and 3.6.

**Remark 3.3.** For $g$ satisfying the conditions in Theorem 3.2, $I_0^\beta g$ can be equivalently represented as

$$I_0^\beta g(x) = J_0^\beta g(x) + \sum_{j=1}^{\infty} \int_0^x k_j(x,r)J_0^\beta g(r) \, dr,$$

(3.3)

$$= \sum_{j=1}^{\infty} K^j \left[ (1-\beta)x^\beta g(x) \right],$$

(3.4)

$$= \sum_{j=0}^{\infty} J_0^\beta \left[ K^j \left[ x^\beta g(x) \right] \right],$$

(3.5)

where (3.3) is due to Lemma 2.14, while (3.4) and (3.5) are due to Remark 2.12. From any representation, we can see that $I_0^\beta$ is a linear operator preserving positivity ($I_0^\beta g \geq 0$ if $g \geq 0$).

**Remark 3.4.** For $g \in C[0,T]$, we can prove the continuous dependence by showing that $\|u - u_0\|_{C[0,T]} \leq C\|g\|_{C[0,T]}$ for some $C$ dependent only on $\beta$ and $T$. For a more general $g$ which may diverge at $x = 0$, $C$ will depend also on $\alpha$, and $\|g\|_{C[0,T]}$ needs to be replaced by $\|g\|_{C^{\alpha-\beta}[0,T]}$, where we define for any $\gamma \in \mathbb{R}$ a Banach space $G^\gamma(0,T) = \{ h \in C(0,T) : \|h\|_{G^\gamma(0,T)} < \infty \}$, with the norm $\|h\|_{G^\gamma(0,T)} := \sup \{ |x^{-\gamma}h(x)| : x \in (0,T) \}$. In particular, if $g \in C[0,T]$ and $\alpha = \beta$, then $\|g\|_{C^{\alpha-\beta}[0,T]} = \|g\|_{C[0,T]}$. (Note that $G^\gamma$ is the same as $\bar{B}$ defined in [14, Proof of Lemma 5.3].)

**Lemma 3.5.** Solutions to problem (1.3) are unique in $C_\beta[0,T]$.

**Proof.** Let $u_1, u_2 \in C_\beta[0,T]$ be two solutions to problem (1.3). By linearity of $\partial_0^\beta$, $u := u_1 - u_2 \in C_\beta[0,T]$ satisfies $\partial_0^\beta u = 0$ on $(0,T)$. Therefore for every $x \in (0,T)$, $D_0^\beta u(x) = \Gamma(1-\beta)^{-1} x^{-\beta} u(x)$, where the right-hand side is in $C \cap L^1(0,T)$. Using Lemma 2.3-(ii) as well as Remark 2.12, we obtain

$$D_0^\beta u(x) = \frac{x^{-\beta}}{\Gamma(1-\beta)} u(x) = D_0^\beta J_0^{\beta} \left[ \frac{x^{-\beta}}{\Gamma(1-\beta)} u(x) \right] = D_0^\beta K u(x),$$

where $K u \in C_\beta[0,T]$. By Lemma 2.13, $K u$ is in $C[0,T]$, and so is $u - K u$. Consequently, $u - K u \in C_\beta[0,T]$. By the linearity of $D_0^\beta$, we know $D_0^\beta [u - K u] = 0$. According to Lemma 2.3-(iv), we obtain $u = K u$.

Let $\xi \in \arg \max_{r \in (0,T)} |u(r)|$. If $\xi = 0$, then $u = 0$ on $[0,T]$ because $u(0) = 0$. If $\xi > 0$, using the fact that $u(\xi) = K u(\xi)$, we have $\int_0^\xi k_1(\xi, r) (u(\xi) - u(r)) \, dr = 0$, where $u(\xi) - u(r)$ never changes sign for all $r \in [0, \xi]$, according to the definition of $\xi$. So $u(\xi) = u(r)$ for all $r \in [0, \xi]$, therefore $u(\xi) = u(0) = 0$, and we still obtain $u = 0$ on $[0,T]$. This proves $u_1 = u_2$, and we are done. \qed
Lemma 3.6. For \( g \) satisfying the conditions in Theorem 3.2, \( I_0^\beta g \) is in \( C_\beta[0,T] \) with \( I_0^\beta g(0) = 0 \) and \( \partial_0^\beta I_0^\beta g = g \). In addition, \( I_0^\beta g \) depends on \( g \) continuously in the sense of Remark 3.4.

Proof. Using representation (3.4), we can see \( I_0^\beta g(0) = 0 \) from the assumptions on \( g \) and Definition 2.11, then from Lemma 2.14 we obtain for all \( x \in (0,T) \),

\[
|I_0^\beta g(x)| \leq I_0^\beta |g|(x) \leq M x^\alpha \left( \frac{\Gamma(1+\alpha)}{\Gamma(\alpha+1-\beta)} - \frac{1}{\Gamma(1-\beta)} \right)^{-1}, \text{ and } I_0^\beta g \in C[0,T]. \tag{3.6}
\]

Note that in (3.5), the summation commutes with \( J_0^\beta \), by Fubini’s Theorem and the above bound. So

\[
I_0^\beta g(x) = J_0^\beta \sum_{j=0}^\infty \frac{K_j [x^\beta g(x)]}{x^\beta} = J_0^\beta \left[ g(x) + x^{-\beta} \sum_{j=1}^\infty K_j [x^\beta g(x)] \right] \tag{3.7}
\]

with the last equality due to (3.4). Therefore, \( I_0^\beta g = J_0^\beta \psi \) for a \( \psi \in C(0,T) \) satisfying

\[
|\psi(x)| \leq M x^{\alpha-\beta} \left( 1 - \frac{\Gamma(\alpha+1-\beta)}{\Gamma(1+\alpha)\Gamma(1-\beta)} \right)^{-1}, \text{ for all } x \in (0,T), \tag{3.8}
\]

so Lemma 2.3-(ii) proves that \( I_0^\beta g \) is in \( C_\beta(0,T) \) and thus \( C_\beta[0,T] \). Lemma 2.3-(ii) also proves that

\[
D_0^\beta I_0^\beta g(x) = \psi(x) = g(x) + \frac{x^{-\beta} I_0^\beta g(x)}{\Gamma(1-\beta)}, \text{ for all } x \in (0,T),
\]

which rewrites as \( \partial_0^\beta I_0^\beta g = g \) by Definition 2.6.

To see the continuity of \( I_0^\beta \), let the \( M \) in (3.6) be \( \|g\|_{G^{\alpha-\beta}(0,T)} \) (\( G^\gamma(0,T) \) is defined in Remark 3.4), then we obtain

\[
\|I_0^\beta g\|_{G^{\alpha}(0,T)} \leq \left( \frac{\Gamma(1+\alpha)}{\Gamma(\alpha+1-\beta)} - \frac{1}{\Gamma(1-\beta)} \right)^{-1} \|g\|_{G^{\alpha-\beta}(0,T)}.
\]

Since \( \alpha > 0 \), we have \( \|I_0^\beta g\|_{C[0,T]} \leq T^{\alpha} \|I_0^\beta g\|_{G^{\alpha}(0,T)} \leq C \|g\|_{G^{\alpha-\beta}(0,T)} \) for some \( C \) dependent only on \( \alpha, \beta \) and \( T \).

Example 3.7. Recall that for the Caputo IVP \( D_0^\beta [u - u(0)] = x^\alpha \) \((\alpha > -1)\) with \( u(0) = u_0 \), the solution is \( u_0 + J_0^\beta x^\alpha \) [14]. By (3.4) and Lemma 2.14, the solution to (1.3) for \( g(x) = x^\alpha \) \((\alpha > -\beta)\) is

\[
u(x) - u_0 = J_0^\beta x^\alpha = \left( \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} - \frac{1}{\Gamma(1-\beta)} \right)^{-1} x^{\alpha+\beta} = c_{\alpha+\beta,\beta}^1 J_0^\beta x^\alpha, \tag{3.9}
\]
where $c_{\alpha+\beta, \beta}$ is defined in Remark 2.7-(ii). In particular, when $\alpha = 0$, $c_{\alpha+\beta, \beta}^{-1} = \beta \pi / (\beta \pi - \sin(\beta \pi))$. If $\alpha \in (-1, -\beta]$, we may not be able to impose the initial condition in (1.3), since the solution may explode at 0. For example, when $\alpha = -\beta$, one can verify that a particular solution is $-\Gamma(1 - \beta) \ln(x) / H_{-\beta}$, where $H_{-\beta}$ is the Harmonic number.

**Remark 3.8.** Assume $g$ satisfies the conditions in Theorem 3.2. By (3.8) and Remark 2.5, $I_0^\beta g$ is Hölder continuous. More specifically,

(i) for all $T_1 \in (0, T)$, $I_0^\beta g \in C^{0, \beta}[T_1, T]$ with a Hölder constant being

$$2M \max\{T_1^{\alpha-\beta}, T^{\alpha-\beta}\} \Gamma(1 + \alpha) \Gamma(1 - \beta) \over \Gamma(1 + \alpha) \Gamma(1 - \beta) - \Gamma(\alpha + 1 - \beta) \Gamma(1 + \beta),$$

(ii) additionally assume $\alpha < \beta$, then $I_0^\beta g \in C^{0, \alpha}[0, T]$ with a Hölder constant being

$$2M \Gamma(1 - \beta) \Gamma(\alpha + 1 - \beta) \over \Gamma(1 + \alpha) \Gamma(1 - \beta) - \Gamma(\alpha + 1 - \beta).$$

**3.2. General censored IVPs.** We now study the censored IVP with more general Lipschitz data $f$:

$$\begin{cases} 
\partial_0^\beta u(x) = f(x, u(x)), & x \in (0, T], \\
 u(x) = u_0, & x = 0.
\end{cases} \tag{3.10}$$

Analogously to Caputo IVPs [14, Chapter 6] and classical ODEs, the censored IVP (3.10) can be solved by Picard iteration, with Theorem 3.2 acting as the fundamental theorem of calculus.

**Proposition 3.9.** For $f : (0, T] \times [u_0 - Y, u_0 + Y] \to \mathbb{R}$, where $u_0 \in \mathbb{R}$, $Y > 0$, assume that there exist $L, \alpha, M > 0$ such that for all $x \in (0, T]$ and all $y, \tilde{y} \in [u_0 - Y, u_0 + Y]$,

(i) $f(\cdot, y) \in C(0, T],$

(ii) $|f(x, y)| \leq Mx^{\alpha-\beta},$

(iii) $|f(x, y) - f(x, \tilde{y})| \leq Lx^{\alpha-\beta}|y - \tilde{y}|.$

Then there exists a unique $u \in C_\beta[0, \bar{T}]$ solving (3.10) on $[0, \bar{T}]$, where either $\bar{T} = T$, or $\bar{T} \in (0, T)$ with $u(\bar{T}) \in \{u_0 - Y, u_0 + Y\}$. Furthermore, $u$ depends on $u_0$ and $f$ continuously, in the sense of Lemma 3.13.

We leave the proof of Proposition 3.9 at the end of this subsection. Now we are going to prove the following lemmata. Lemma 3.10 gives us a bound which is essential to the convergence of our Picard iteration. Lemmata 3.12, 3.13 and 3.14 guarantee the uniqueness, continuous dependence and the local existence of the solutions, respectively. Finally we will prove the global existence by extending the local solution.

**Lemma 3.10.** For $x, \alpha > 0$ and $N \in \mathbb{N}$, we have

$$(I_0^\beta x^{\alpha-\beta} \cdot )^N 1(x) \leq C \frac{2^N x^{N\alpha}}{(N! \alpha N)^\beta},$$
where \(1(x)\) is the constant function 1, \(C\) is a positive constant dependent only on \(\alpha\) and \(\beta\), and we denote
\[
(I_0^\beta[x^{\alpha-\beta}] )^N g(x) = I_0^\beta [x^{\alpha-\beta} \cdots I_0^\beta [x^{\alpha-\beta} g(x)] \cdots] .
\]

**Proof.** From Example 3.7 we know that
\[
(I_0^\beta[x^{\alpha-\beta}] )^N 1(x) = \prod_{n=1}^N \left( \frac{\Gamma(1+n\alpha)}{\Gamma(n\alpha+1-\beta)} - \frac{1}{\Gamma(1-\beta)} \right)^{-1} x^{n\alpha} ,
\]
where each factor is positive. Using Stirling’s formula for the gamma function, i.e.
\[
\Gamma(z) = \sqrt{2\pi} \left( \frac{z}{e} \right)^z \left( 1 + O\left( \frac{1}{z} \right) \right),
\]
we have the following approximation
\[
\frac{\Gamma(1+n\alpha)}{\Gamma(n\alpha+1-\beta)} = (n\alpha)^\beta \left( 1 + O\left( \frac{1}{n} \right) \right),
\]
which indicates that there exists \(\tilde{n} \in \mathbb{N}\) such that for all \(n > \tilde{n}\),
\[
\frac{\Gamma(1+n\alpha)}{\Gamma(n\alpha+1-\beta)} - \frac{1}{\Gamma(1-\beta)} \geq \frac{(n\alpha)^\beta}{2},
\]
so there exists \(C > 0\) such that Lemma 3.10 holds for all \(N \in \mathbb{N}\). \(\square\)

**Remark 3.11.** In Proposition 3.9, if \(\alpha = \beta\), then we only need \((I_0^\beta)^N 1(x)\) for its proof. The multiplier \(x^{\alpha-\beta}\) is to accommodate more general \(f\) that may diverge at \(x = 0\).

**Lemma 3.12.** If \(f\) satisfies the condition (iii) in Proposition 3.9, and both \(u_1, u_2 \in C_\beta[0,T]\) solve IVP (3.10), then \(u_1 = u_2\).

**Proof.** By the linearity of \(\partial_0^\beta\), the difference \(u := u_1 - u_2 \in C_\beta[0,T]\) satisfies
\[
\begin{cases}
\partial_0^\beta u(x) = f(x, u_1(x)) - f(x, u_2(x)), & x \in (0,T], \\
u(x) = 0, & x = 0.
\end{cases}
\]
Since \(u \in C_\beta[0,T]\), we know \(\partial_0^\beta u(x) \in C(0,T)\), so \(f(x, u_1(x)) - f(x, u_2(x)) \in C(0,T)\).

By assumption, for all \(x \in (0,T]\),
\[
\left| f(x, u_1(x)) - f(x, u_2(x)) \right| \leq Lx^{\alpha-\beta}|u_1(x) - u_2(x)| = Lx^{\alpha-\beta}|u(x)|.
\]

Then by Theorem 3.2 and the positivity preserving property of \(I_0^\beta\), for all \(x \in (0,T]\),
\[
|u(x)| \leq I_0^\beta \left| f(x, u_1(x)) - f(x, u_2(x)) \right| \leq LI_0^\beta \left[ x^{\alpha-\beta}|u(x)| \right].
\]

Iterating the above inequality, we obtain for all \(N \in \mathbb{N}\), \(x \in (0,T]\),
\[
|u(x)| \leq L^N (I_0^\beta [x^{\alpha-\beta}] )^N |u||x(x) \leq \|u\|_{C[0,T]} L^N (I_0^\beta [x^{\alpha-\beta}] )^N 1(x).
\]
By Lemma 3.10, as $N \to \infty$, we obtain $u(x) = 0$ for all $x \in (0, T]$. □

We prove below the continuous dependence of $u$ on $f$. Then the continuous dependence on $u_0$ is a simple corollary if we take $\tilde{f}(x, y) = f(x, y + \delta)$, where $\delta$ is the tiny change in $u_0$.

**Lemma 3.13.** If $f$ and $\tilde{f}$ satisfy conditions (ii) and (iii) in Proposition 3.9, and $u, \tilde{u} \in C_\beta[0,T]$ satisfy $\partial_0^\beta u = f(x, u)$ and $\partial_0^\beta \tilde{u} = \tilde{f}(x, \tilde{u})$ with $u(0) = \tilde{u}(0) = u_0$, then $\|u - \tilde{u}\|_{C[0,T]} \leq C\varepsilon$, where $C$ depends only on $\alpha, \beta, L, T$, and $\varepsilon = \sup \{x^{\beta-\alpha}|f(x, y) - \tilde{f}(x, y)| : x \in (0,T), y \in [u_0 - Y, u_0 + Y]\}$ (the definition of $\varepsilon$ is analogous to $\|g\|_{C^{\alpha-\beta}(0,T)}$ in Remark 3.4).

**Proof.** By assumption, $\partial_0^\beta u, \partial_0^\beta \tilde{u} \in C(0,T]$, so $f(x, u(x)), \tilde{f}(x, \tilde{u}(x))$ are in $C(0,T]$ and bounded by $Mx^{\alpha-\beta}$. According to Theorem 3.2 and the positivity preserving property of $I_0^\beta$, for all $x \in (0,T]$, we have

$$|u(x) - \tilde{u}(x)| = \left|I_0^\beta f(x, u(x)) - I_0^\beta \tilde{f}(x, \tilde{u}(x))\right|$$

$$\leq \left|I_0^\beta f(x, u(x)) - I_0^\beta f(x, \tilde{u}(x))\right| + \left|I_0^\beta f(x, \tilde{u}(x)) - I_0^\beta \tilde{f}(x, \tilde{u}(x))\right|$$

$$\leq I_0^\beta \left|f(x, u(x)) - f(x, \tilde{u}(x))\right| + I_0^\beta \left|f(x, \tilde{u}(x)) - \tilde{f}(x, \tilde{u}(x))\right|$$

$$\leq LI_0^\beta x^{\alpha-\beta}|u(x) - \tilde{u}(x)| + \varepsilon I_0^\beta x^{\alpha-\beta}.$$ 

Like the proof of Lemma 3.12, by iterating the above inequality, we obtain for all $N \in \mathbb{N}$, $x \in (0,T]$,

$$|u(x) - \tilde{u}(x)| \leq \|u - \tilde{u}\|_{C[0,T]} L^N (I_0^\beta [x^{\alpha-\beta}])^N 1(x) + \varepsilon \sum_{n=0}^{N-1} L^n (I_0^\beta [x^{\alpha-\beta}])^{n+1} 1(x).$$

By Lemma 3.10, as $N \to \infty$, the first summand goes to 0 uniformly in $x$, and the second summand can be bounded by $\varepsilon C$ for some finite $C$ dependent only on $\alpha, \beta, L, T$. □

**Lemma 3.14.** If $f$ satisfies conditions (i), (ii) and (iii) in Proposition 3.9, then there exists $u \in C_\beta[0,h]$ solving the IVP (3.10) on $[0,h]$, as long as $h \in (0,T]$ satisfies

$$h^\alpha \leq \frac{Y}{M} \left( \frac{\Gamma(1 + \alpha)}{\Gamma(\alpha + 1 - \beta)} - \frac{1}{\Gamma(1 - \beta)} \right).$$

**Proof.** Define the function space $U = \{ \varphi \in C[0,h] : \|\varphi - u_0\|_{C[0,h]} \leq Y \}$. For $\varphi \in U$, we know that $f(x, \varphi(x)) \in C(0,h)$ and $|f(x, \varphi(x))| \leq Mx^{\alpha-\beta}$, so by Theorem 3.2 we can define the following Picard iteration operator

$$\mathcal{P}\varphi = u_0 + I_0^\beta f(x, \varphi(x)).$$

From (3.6) we know that $\mathcal{P}\varphi \in C[0,h]$, and

$$\|\mathcal{P}\varphi - u_0\|_{C[0,h]} = \left\|I_0^\beta f(x, \varphi(x))\right\|_{C[0,h]} \leq Mh^\alpha \left( \frac{\Gamma(1 + \alpha)}{\Gamma(\alpha + 1 - \beta)} - \frac{1}{\Gamma(1 - \beta)} \right)^{-1} \leq Y.$$
Therefore, \( \mathcal{P}[U] \subseteq U \). In addition, for any \( \varphi, \psi \in U \) and any \( x \in (0, h] \),

\[
|\mathcal{P}\varphi(x) - \mathcal{P}\psi(x)| \leq I_0^\beta \left[ f(x, \varphi(x)) - f(x, \psi(x)) \right] \leq I_0^\beta \left[ L x^{\alpha-\beta} |\varphi(x) - \psi(x)| \right].
\]

Like the proof of Lemma 3.12, by iterating the above inequality, we obtain for all \( N \in \mathbb{N} \), \( x \in (0, h] \),

\[
|\mathcal{P}^N \varphi(x) - \mathcal{P}^N \psi(x)| \leq L^N \left( I_0^\beta \left[ x^{\alpha-\beta} \right] \right)^N |\varphi - \psi|(x) \leq \|\varphi - \psi\|_{C[0,h]} L^N \left( I_0^\beta \left[ x^{\alpha-\beta} \right] \right)^N 1(x).
\]

By Lemma 3.10, there exists \( N \) large enough, such that \( \mathcal{P}^N \) is a contraction on \( U \), which is a complete metric space under the metric induced by \( \| \cdot \|_{C[0,h]} \). By a corollary of the Banach fixed point theorem, \( \mathcal{P} \) has a unique fixed point \( u_* \in U \). Then, by Theorem 3.2, \( u_* \) is a solution to (3.10) on \([0, h]\).

With the preceding lemmata, we are ready to establish the global well-posedness of IVP (3.10).

**Proof.** [of Proposition 3.9] The uniqueness and continuous dependence of the solution are already shown in Lemmata 3.12 and 3.13, respectively.

By Lemma 3.14, for some \( h \in (0, T] \), there exists \( u_* \in C_\beta[0, h] \) solving (3.10) on \([0, h]\). If \( h < T \) and \( |u_*(h) - u_0| < Y \), we are going to extend \( u_* \) beyond \( h \), in a manner similar to the proof of [14, Theorem 6.8]. Let us choose \( \tilde{h} \in (h, T] \) such that

\[
|h - \tilde{h}| \leq \frac{Y - |u_*(h) - u_0|}{2M \max\{h^{\alpha-\beta}, T^{\alpha-\beta}\}} \left( 1 - \frac{\Gamma(\alpha + 1 - \beta)}{\Gamma(1 + \alpha)\Gamma(1 - \beta)} \right)^{-1} \Gamma(1 + \beta).
\]

Define the function space

\[
V = \left\{ \varphi \in C[0, \tilde{h}] : \|\varphi - u_*\|_{C[0, h]} = 0 \text{ and } \|\varphi - u_* (h)\|_{C[\tilde{h}, \tilde{h}]} \leq Y - |u_*(h) - u_0| \right\},
\]

which is a complete metric space under the metric induced by \( \| \cdot \|_{C[0, \tilde{h}]} \).

For \( \varphi \in V \), we know that \( f(x, \varphi(x)) \in C(0, \tilde{h}) \) and \( |f(x, \varphi(x))| \leq M x^{\alpha-\beta} \), so \( \mathcal{P}\varphi \in C(0, \tilde{h}) \), where \( \mathcal{P} \) is introduced in the proof of Lemma 3.14. By Remark 3.8-(i), \( \mathcal{P}\varphi \in C^{\alpha,\beta}[h, \tilde{h}] \) with a Hölder constant

\[
C = \frac{2M \max\{h^{\alpha-\beta}, T^{\alpha-\beta}\}}{\Gamma(1 + \beta)} \left( 1 - \frac{\Gamma(\alpha + 1 - \beta)}{\Gamma(1 + \alpha)\Gamma(1 - \beta)} \right)^{-1}.
\]

For all \( x \in [h, \tilde{h}] \), \( |\mathcal{P}\varphi(x) - \mathcal{P}\varphi(h)| \leq C |x - h|^{\beta} \leq C |\tilde{h} - h|^{\beta} \leq Y - |u_*(h) - u_0| \). Recall that \( u_* \) is already a fixed point of \( \mathcal{P} \) on \([0, h] \), we have \( \mathcal{P}\varphi(h) = \mathcal{P}u_*(h) = u_*(h) \). Therefore \( \mathcal{P}\varphi \in V \) and \( \mathcal{P}[V] \subseteq V \). Similar to the proof of Lemma 3.14, we can show that \( \mathcal{P}^N \) is a contraction on \( V \) for some large \( N \), so \( \mathcal{P} \) has a unique fixed point \( v_* \in V \), which is a solution to (3.10) on \([0, \tilde{h}] \). Choose \( \tilde{h} \) as large as possible and repeat the procedure, then we can extend the solution all the way to the boundary of the domain. That is, the solution must exist on the entire interval \([0, T]\), or attains \( u_0 + Y \) or \( u_0 - Y \).
3.3. A linear censored IVP. We now consider the IVP $\partial_t^\beta u = \lambda u$ for a constant $\lambda$. Like its counterpart in classical ODEs, such IVP can play important roles in more general equations.

**Lemma 3.15.** For any $\lambda, u_0 \in \mathbb{R}$, the linear IVP

\[
\begin{aligned}
\begin{cases}
\partial_t^\beta u(x) = \lambda u(x), & x \in (0, T], \\
u(x) = u_0, & x = 0,
\end{cases}
\end{aligned}
\tag{3.12}
\]

has a unique solution in $C_\beta[0, T]$ given by the series $u(x) = u_0 \sum_{N=0}^\infty \lambda^N (I_0^\beta)^N 1(x)$, which is equivalent to (1.7) by letting $\alpha = \beta$ in (3.11).

**Proof.** By Lemma 3.10, we know that $u = u_0 \sum_{N=0}^\infty \lambda^N (I_0^\beta)^N 1$ converges uniformly on $[0, T]$, thus $u \in C[0, T]$ with $u(0) = u_0$. By the continuous dependence in Theorem 3.2,

\[
I_0^\beta u = u_0 \left[ \sum_{N=0}^\infty \lambda^N (I_0^\beta)^N 1 \right] = u_0 \sum_{N=0}^\infty I_0^\beta \left[ \lambda^N (I_0^\beta)^N 1 \right] = u_0 \sum_{N=0}^\infty \lambda^N (I_0^\beta)^{N+1} 1,
\]

therefore $u_0 + \lambda I_0^\beta u = u$. By Theorem 3.2, $u \in C_\beta[0, T]$ and solves (3.12). Uniqueness is a consequence of Proposition 3.9.

**Remark 3.16.** We can obtain the solution (1.7) by Picard iteration, i.e. recursively solving the IVPs: $\partial_t^\beta u_{m+1} = \lambda u_m$ with $u_m(0) = u_0$ ($m = 1, 2, \ldots$), where $u_1(x) = u_0$ for all $x$.

Although the series (1.7) looks cumbersome, it surprisingly decays at the simple algebraic rate $x^{-1-\beta}$ for $\lambda < 0$.

**Theorem 3.17.** For $\lambda < 0$ and $u_0 > 0$, the solution $u$ to (3.12) is completely monotone (i.e., $(-1)^n u^{(n)} \geq 0$ on $\mathbb{R}_+$ for $n = 0, 1, 2, \ldots$) and there exists a constant $C > 1$ such that

\[
\frac{C^{-1}}{x^{1+\beta}} \leq u(x) \leq \frac{C}{x^{1+\beta}}, \quad \text{for all } x \geq 1.
\]

**Proof.** The complete monotonicity will be proved in Corollary 4.13, using a probabilistic argument. The upper and lower bounds are proved in Lemmata 3.20 and 3.21 below, using a maximum principle argument.

**Remark 3.18.**

(i) For Caputo’s counterpart of IVP (3.12), i.e., $D_0^\beta [u - u(0)] = \lambda u$ with $u(0) = u_0$, the solution can be expressed in terms of the Mittag-Leffler function

\[
u(x) = u_0 \sum_{N=0}^\infty \frac{(\lambda x^\beta)^N}{\Gamma(N\beta + 1)}.
\tag{3.13}
\]

For $\lambda < 0$ and $u_0 > 0$, it is completely monotone and decays at the rate $x^{-\beta}$ [14, Theorem 7.3]. By contrast, the censored relaxation equation (3.12) models a new decay regime $x^{-1-\beta}$. (See also [49, Page 1623] for related fractional relaxation equations, where the decay rate is $x^{-\gamma}$ for some $\gamma \in (0, 1)$.)
As a side note, for $\lambda, u_0 > 0$, obviously both (1.7) and (3.13) increase in $x$ faster than any polynomial. Indeed, for $\lambda = 1$, the latter grows at the rate $e^x$ [20, Proposition 3.5], and our numerical results suggest $\exp\{x + cx^{1-\beta}\}$ for the former, where $c$ is positive and depends only on $\beta$.

(ii) For (3.13) with $\lambda = -1$, $u_0 = 1$, [45, Theorem 4] gave the uniform estimates with optimal constants: $$(1 + \Gamma(1-\beta)x^\beta)^{-1} \leq u(x) \leq (1 + \Gamma(1+\beta)^{-1}x^\beta)^{-1}.$$ In Proposition B.1 we give what we believe to be a new and simple proof of those bounds, using the same strategy used for the uniform bounds of (1.7). Recently [10, Proposition 4.12] gave another new proof by showing the generalized results for a class of Kilbas–Saigo functions. Our simple proof can also be applied with few modifications to prove those generalized results (see Proposition B.4). In Section 3.4 we will use it again, to prove the uniform bounds of (3.20), the solution to $\partial_0^\beta u = \lambda x^{\alpha-\beta} u$.

(iii) The reason why our proof is both simple and versatile is that it involves only maximum principle (mentioned in Remark 2.7-(iii)) and some suitable candidate bounds (e.g. $(1 + cx)^{-1-\beta}$), but no specific representation of the solution (e.g. (1.7) or (1.9)). In fact, we expect this strategy to have broader applications. As an example, consider a general Caputo-type derivative $D_{\alpha_0}^\psi u$ (so $-D_{\alpha_0}^\psi$ generates a non-increasing pure jump Lévy process killed upon leaving $\mathbb{R}^+$ [33]) for a Lévy measure $\psi$ with $\int_0^\infty \min\{r, 1\} \psi(dr) < \infty$,

$$D_{\alpha_0}^\psi u(x) = \int_0^x (u(x) - u(x-r)) \psi(dr) + (u(x) - u(0)) \psi((x, \infty)),$$

and its relaxation equation $D_{\alpha_0}^\psi u = \lambda u$ ($\lambda < 0$). The solution is given as an expectation or a series under mild assumptions [30, Lemma 3.4]. It is possible for our strategy to prove two-sided bounds of this solution without those representations of it. Indeed, this has already been done for certain absolutely continuous $\psi$ (so $\psi(dr) = \psi(r) dr$).

For instance, for compactly supported $\psi$, the solution is given an upper bound of the decay rate $x^{-1}$ [17, Remark 3.5]. A special case is the truncated fractional kernel $\psi(r) = 1_{\{r \in [0, \delta]\}} r^{-1-\beta}$ with $\delta > 0$ (see [17, Theorem 3.2], which inspired our proof). Even if $\psi$ is not compactly supported, as long as $\int_0^\infty r \psi(r) dr < \infty$, the same argument applies. Another instance is when $r^{1+\beta} \psi(r)$ is continuous on $\mathbb{R}^+$ and bounded within $[C^{-1}, C]$ for some $C > 1$, our strategy (in Proposition B.1) can still prove the two-sided bounds, both of $x^{-\beta}$ decay.

**Lemma 3.19.** If $\lambda < 0$ and $v \in C^1([0, T]) \cap C[0, T]$ satisfies $\partial_0^\beta v \geq \lambda v$, then $v$ is nonnegative if $v(0) \geq 0$, and positive if $v(0) > 0$.

**Proof.** If $v(0) \geq 0$ but $v$ is not nonnegative, let $x_0$ be a minimum point of $v$ on $[0, T]$, then $x_0 > 0$ and $v(x_0) < 0$. So we have $0 < \lambda v(x_0) \leq \partial_0^\beta v(x_0)$. However, since $v \in C^1([0, T])$, by Remark 2.7-(iii) we know that

$$\partial_0^\beta v(x_0) = \int_0^{x_0} (v(x_0) - v(x_0 - r)) \frac{r^{-1-\beta}}{\Gamma(-\beta)} dr \leq 0,$$
which is a contradiction. Similarly, we can prove that $v$ is positive if $v(0) > 0$.

With the above lemma, we can get the desired bounds for the solution to (3.12).

**Lemma 3.20.** For $\lambda < 0$ and $u_0 = 1$, the solution $u$ to (3.12) is positive and can be bounded from above by $v(x) = (1 + c|\lambda|^{1/\beta} x)^{-\beta}$, where

$$c = \frac{|\Gamma(-\beta)|^{1/\beta}}{2} \left( \frac{2^{1+\beta} - 1}{1-\beta} + \frac{2}{\beta} \right)^{-1/\beta}.$$  \hspace{1cm} (3.14)

**Proof.** We know from Lemma 3.15 that $u \in C[0, T]$. We also know that $u \in C^1(0, T)$ from the uniform convergence of the series representation of its derivative on any closed interval not containing 0. Thus $u$ remains positive by Lemma 3.19.

Let $v(x) = (1 + x/c)^{-1-\beta}$ and first assume that there is a constant $c > 0$ such that $v$ satisfies the condition in Lemma 3.19. Under this assumption, we get $\partial_0^\beta (v-u) - \lambda (v-u) \geq 0$ with $v(0) - u(0) = 0$. By Lemma 3.19, we have $v \geq u$ on $[0, T]$.

Now, given $\beta \in (0, 1)$ and $\lambda < 0$, up to a constant multiple, it remains to find a constant $c > 0$ such that $v(x) = (x + c)^{1-\beta}$ satisfies $\partial_0^\beta v \geq \lambda v$, i.e., for all $x > 0$,

$$\int_0^x (v(x) - v(x-r)) \frac{r^{-1-\beta}}{|\Gamma(-\beta)|} dr \geq \lambda v(x),$$

or equivalently, for all $x > 0$,

$$|\lambda |^{-\beta} \Gamma(-\beta) \geq \int_0^x \left( \frac{x+c}{x+c-r} \right)^{1+\beta} - 1 \frac{dr}{r^{1+\beta}}.$$  \hspace{1cm} (3.15)

Let $y = x + c$, then the right-hand side of (3.15) equals

$$\int_0^{y-c} \left( \frac{y^{1+\beta}}{(y-r)^{1+\beta}} - 1 \right) \frac{dr}{r^{1+\beta}} = y^{-\beta} \int_0^{1-c/y} \left( \frac{1}{(1-s)^{1+\beta}} - 1 \right) \frac{ds}{s^{1+\beta}}.$$  \hspace{1cm} (3.16)

If $y \leq 2c$, then the right-hand side of (3.16) can be bounded from above by

$$y^{-\beta} \int_0^{1/2} \left( \frac{1}{(1-s)^{1+\beta}} - 1 \right) \frac{ds}{s^{1+\beta}} \leq 2^{1+\beta} \int_0^{1/2} \frac{2s(2^{1+\beta} - 1)}{y^{2+\beta}}.$$

If $y > 2c$, we split the interval $[0, 1 - c/y]$ into two parts $[0, 1/2]$ and $[1/2, 1 - c/y]$. For the second subinterval,

$$\int_{1/2}^{1-c/y} \left( \frac{1}{(1-s)^{1+\beta}} - 1 \right) \frac{ds}{s^{1+\beta}} \leq 2^{1+\beta} \int_{1/2}^{1-c/y} \frac{ds}{(1-s)^{1+\beta}} \leq \frac{2^{1+\beta} (1/2)}{\beta (y/c)^{\beta}}.$$

Therefore the right-hand side of (3.15) can be bounded from above by

$$\frac{2^{1+\beta} - 1}{y^{1-\beta}} + \frac{2^{1+\beta}}{y^{1+\beta}} \left( \frac{y}{c} \right)^{\beta} \leq \frac{2^{1+\beta} - 1}{c^{1+\beta}} + \frac{2^{1+\beta}}{c\beta}.$$

Let

$$c \geq \frac{2}{|\lambda |^{1/\beta}} \left( \frac{2^{1+\beta} - 1}{1-\beta} + \frac{2}{\beta} \right)^{1/\beta},$$
then (3.15) will be satisfied and we are done.

Lemma 3.21. If \( \lambda < 0 \) and \( u_0 = 1 \), then the solution to (3.12) is bounded from below by
\[
w(x) = (1 + d|\lambda|^{1/\beta}x)^{-1}(1 + d^2|\lambda|x^\beta)^{-1},
\]
where \( d = \left| \Gamma(-\beta) \right|^{1/\beta} \max \left\{ 4, (1-\beta)(1+2\beta)/\beta \right\}^{1/\beta}. \)

The proof is similar to that of Lemma 3.20, so we put it in Appendix B.2.

3.4. Other linear censored IVPs. We conclude this section by generalizing IVP (3.12) to the inhomogeneous version with a variable coefficient.

Proposition 3.22. Let \( \lambda, u_0 \in \mathbb{R} \) and \( g \in C(0, T) \) such that \( |g(x)| \leq M x^{\alpha-\beta} \) for some \( M, \alpha > 0 \) and all \( x \in (0, T] \). Then the inhomogeneous linear IVP
\[
\begin{cases}
\partial_0^\beta u(x) = \lambda u(x) + g(x), & x \in (0, T], \\
u(x) = u_0, & x = 0,
\end{cases}
\]  
(3.17)

has a unique solution in \( C_\beta[0, T] \) given by the following series
\[
u(x) = u_0 \sum_{N=0}^{\infty} \lambda^N (I_0^\beta)^N 1(x) + \sum_{N=0}^{\infty} \lambda^N (I_0^\beta)^{N+1} g(x), \quad (3.18)
\]
and from Proposition 3.9, \( u \) inherits the continuous dependence on \( u_0 \) and \( g \).

Proof. From (3.6) we know \( I_0^\beta g \in C[0, T] \) and thus \( (I_0^\beta)^{N+1} g \in C[0, T] \) for \( N = 0, 1, 2, \ldots \).

Then, by the positivity preserving property of \( I_0^\beta \), we have
\[
| (I_0^\beta)^{N+1} g | \leq (I_0^\beta)^N |I_0^\beta g| \leq (I_0^\beta)^N \cdot \|I_0^\beta g\|_{C[0,T]}.
\]

By Lemma 3.10, the series \( \sum_{N=0}^{\infty} \lambda^N (I_0^\beta)^N 1 \) converges uniformly on \([0, T]\), and so does \( \sum_{N=0}^{\infty} \lambda^N (I_0^\beta)^{N+1} g \). So the function \( u \) given by (3.18) is in \( C[0, T] \) and \( I_0^\beta u \) is well-defined. Therefore,
\[
\lambda I_0^\beta u + I_0^\beta g = \lambda u_0 \sum_{N=0}^{\infty} \lambda^N (I_0^\beta)^N 1 + \sum_{N=0}^{\infty} \lambda^N (I_0^\beta)^{N+1} g + I_0^\beta g
\]
\[
= \lambda u_0 \sum_{N=0}^{\infty} \lambda^N (I_0^\beta)^N 1 + \sum_{N=0}^{\infty} \lambda^N (I_0^\beta)^{N+2} g + I_0^\beta g
\]
\[
= u - u_0,
\]
where the second equality is due to the continuous dependence in Theorem 3.2. Using Theorem 3.2 again, we know that \( u \) solves (3.17). By Proposition 3.9, \( u \) is actually the unique solution in \( C_\beta[0, T] \), and depends on \( (u_0, g) \) continuously. \( \square \)

Lemma 3.23. For \( \lambda, u_0 \in \mathbb{R} \) and \( \alpha > 0 \), the linear IVP
\[
\begin{cases}
\partial_0^\beta u(x) = \lambda x^{\alpha-\beta} u(x), & x \in (0, T], \\
u(x) = u_0, & x = 0.
\end{cases}
\]
(3.19)
has a unique solution in $C\beta[0,T]$ given by the following series

$$
\begin{align*}
u(x) &= u_0 \sum_{N=0}^{\infty} \lambda^N \left(I_0^\beta [x^{\alpha-\beta} \cdot] \right)^N 1(x) \\
&= u_0 \sum_{N=0}^{\infty} \left(\lambda x^\alpha \right)^N \prod_{n=1}^{N} \left( \frac{\Gamma(1+n\alpha)}{\Gamma(n\alpha+1-\beta)} - \frac{1}{\Gamma(1-\beta)} \right)^{-1}.
\end{align*} 
(3.20)
$$

We omit the proof since it is a special case of Proposition 3.28. Surprisingly, the solution (3.20) has a decay property analogous to what we see in Section 3.3.

**Proposition 3.24.** For $\lambda < 0$ and $u_0 > 0$, there exists a constant $C > 1$ such that the solution $\nu$ to (3.19) satisfies

$$
C^{-1} x^{-\alpha} \leq \nu(x) \leq C x^{-\alpha}, \quad \text{for all } x \geq 1.
$$

**Proof.** See Lemmas 3.26 and 3.27, which can be shown by maximum principle arguments.

**Remark 3.25.**

(i) For Caputo’s counterpart of IVP (3.19), the solution can be expressed in terms of the Kilbas–Saigo function

$$
\begin{align*}
u(x) &= u_0 \sum_{N=0}^{\infty} \left(\lambda x^\alpha \right)^N \prod_{n=1}^{N} \left( \frac{\Gamma(1+n\alpha)}{\Gamma(n\alpha+1-\beta)} - \frac{1}{\Gamma(1-\beta)} \right)^{-1}.
\end{align*} 
(3.21)
$$

For $\lambda < 0$ and $u_0 > 0$, the solution (3.21) decays at the rate $x^{-\alpha}$ [10, Remark 4.6 (c)] (and is completely monotone [10, Remark 3.1 (d)] if $\alpha \leq 1$). On the other hand, the censored IVP (3.19) once again models a new decay regime $x^{-1-\alpha}$.

As a side note, for $\lambda, u_0 > 0$, both (3.20) and (3.21) increase in $x$ faster than any polynomial. Indeed, for $\lambda = 1$, the latter can be bounded by $\exp \left\{ (\frac{\beta}{\alpha} + \varepsilon) x^{\alpha/\beta} \right\}$ for any $\varepsilon$ positive and $x$ large enough [20, Theorem 5.9], and our numerical results suggest the same for the former.

(ii) For (3.21) with $\lambda = -1$, $u_0 = 1$, [10, Proposition 4.12] proved the uniform bounds

$$
(1 + \Gamma(1-\beta)x^\alpha)^{-1} \leq \nu(x) \leq (1 + \Gamma(1-\beta))\Gamma(1+\alpha)^{-1} x^\alpha.
$$

As mentioned in Remark 3.18-(ii), our maximum principle argument can give a new and simple proof of those bounds.

(iii) For $\alpha = 1$, (3.19) can be seen as a linear equation $\sigma \partial_0^\beta u = \lambda u$, where we let $\sigma(x) = x^{\beta-1}$ so that the rescaled fractional derivative $\sigma \partial_0^\beta$ acts like the classical first order derivative on linear functions. This kind of rescaling naturally extends to more general nonlocal derivatives, and we refer to [15] for a discussion of nonlocal calculus and rescaling.
Lemma 3.26. For \( \lambda < 0 \) and \( u_0 = 1 \), the solution to (3.19) is positive and can be bounded from above by \( v(x) = (1 + (c|\lambda|^{1/\alpha}x)^{1+\alpha})^{-1} \), where
\[
c = \frac{\Gamma(-\beta)^{1/\alpha}}{2^{\beta/\alpha}} \left( \frac{2^{1+\alpha} + \alpha^{\alpha/(1+\alpha)}}{1-\beta} \right)^{-1/\alpha}.
\]

The proof is similar to that of Lemma 3.20, so we put it in Appendix B.2.

Lemma 3.27. If \( \lambda < 0 \) and \( u_0 = 1 \), then the solution to (3.19) is bounded from below by \( w(x) = (1+d|\lambda|^{1/\alpha}x)^{-1} (1+d^{\alpha}|\lambda|x^{\alpha})^{-1} \), where \( d = |\Gamma(-\beta)|^{1/\alpha} \max \{ 4, (1+2^\alpha)(1-\beta)/\alpha \}^{1/\alpha} \).

The proof is parallel to that of Lemma 3.21, so we omit it.

Proposition 3.28. Let \( \lambda, u_0 \in \mathbb{R} \), \( \alpha > 0 \) and \( g \in C(0, T) \) such that \( |g(x)| \leq Mx^{\gamma-\beta} \) for some \( M, \gamma > 0 \) and all \( x \in (0, T) \). Then the inhomogeneous linear IVP
\[
\begin{cases}
\partial_0^\beta u(x) = \lambda x^{\alpha-\beta} u(x) + g(x), & x \in (0, T), \\
u(x) = u_0, & x = 0,
\end{cases}
\] (3.22)
has a unique solution in \( C_\beta[0, T] \) given by the following series
\[
u(x) = u_0 \sum_{N=0}^{\infty} \lambda^N \left( I^\beta_0 \left[ x^{\alpha-\beta} \cdot \right] \right)^N 1(x) + \sum_{N=0}^{\infty} \lambda^N \left( I^\beta_0 \left[ x^{\alpha-\beta} \cdot \right] \right)^N I^\beta_0 g(x).
\] (3.23)

From Proposition 3.9, \( u \) inherits the continuous dependence on \( u_0 \) and \( g \).

The proof is parallel to that of Proposition 3.22, so we omit it. A quick check can be done by Picard iteration.

4. Censored decreasing \( \beta \)-stable process

In this section we first prove that the hitting time of 0 (or lifetime) for the censored decreasing \( \beta \)-stable process is finite and that \( f_0^{\beta} \) has probabilistic representations (1.4) and (1.5). We then use these results to prove that our censored process is Feller with generator \( -\partial_0^\beta \), which in turn leads us to show that the Laplace transform of the lifetime equals the series (1.7), and thus they are completely monotone. We denote by \( 1_A \) the indicator function of a set \( A \). All our stochastic processes are real-valued right-continuous with left limits (càdlàg), hence we always assume the canonical underlying filtered probability space as in [4, Chapter O]. For a stochastic process \( Y = \{ Y_s \}_{s \geq 0} \) and a real-valued integrable function \( f \) on the probability space of \( Y \), we use the notation \( \mathbb{E}_y[f(Y)] = \mathbb{E}[f(Y) | Y_0 = y], \mathbb{E}[f(Y)] = \mathbb{E}_0[f(Y)] \), and correspondingly \( \mathbb{P}_y[A], \mathbb{P}[A] \) when \( f = 1_A \). We write \( Y_{t-} = \lim_{s \uparrow t} Y_s \). By a \( \beta \)-stable subordinator (\( \beta \in (0, 1) \)) we mean the Lévy process \( -S^k = \{-S^k_s\}_{s \geq 0} \) characterised by the Laplace transform \( \mathbb{E}[\exp\{kS^k_s\}] = \exp\{-sk^\beta\}, k, s > 0 \) [4, Chapter III]. We denote by \( B[0, T] \) the set of real-valued bounded Borel measurable functions on \( [0, T] \) and define \( C^\infty_\beta(0, T) = \{ u \in C[0, T] : u(0) = 0 \} \), both understood as Banach spaces with the sup norm. We extend the domain of any \( f \in B[0, T] \) to a cemetery state \( \partial \) imposing \( f(\partial) = 0 \). As discussed in Section 1, we treat the censored decreasing stable process in \( \mathbb{R}^+ \).
because it is generated by $-\partial_0^\beta$, where $\partial_0^\beta$ is the “left” censored derivative (at 0). However, it should be clear that all the results in this section translate immediately to the censored stable subordinator in $(-\infty, b)$ when paired with the “right” censored derivative at $b \in \mathbb{R}$.

4.1. **Construction and finite lifetime.** The starting point of the censored process is always assumed to be fixed to some $x > 0$. We define the censored decreasing $\beta$-stable process $S^c$ by the INW piecing together construction, then [25, Theorem 1.1 and Section 5.i] guarantees us a càdlàg strong (sub-)Markov process. The construction is: run $x + S^1_t$ until $\tau_1$, the time when it first exits $(0, T)$, where $-S^1$ is a $\beta$-stable subordinator (started at 0); then kill the process if $x + S^1_t \leq 0$; otherwise piece together an independent copy of $S^1$ started at $x + S^1_{\tau_1-}$ and repeat the same procedure for at most countably many times.

With Lemma 4.1 we prove that we can directly define the censored decreasing $\beta$-stable process $S^c | S^c_0 = x$ as

$$S^c_t := \begin{cases} \overline{S}^j_t, & \tau_{j-1} \leq t < \tau_j, \ j \in \mathbb{N}, \\ \partial, & t \geq \tau_\infty, \end{cases}$$

with

$$\overline{S}^j_t := \begin{cases} x + S^j_t, & j = 1, \\ \overline{S}^{j-1}_{\tau_{j-1}-} + S^j_t, & j \geq 2, \end{cases}$$

and $\tau_j := \begin{cases} 0, & j = 0, \\ \inf \{ s > \tau_{j-1} : \overline{S}^j_s \leq 0 \}, & j \in \mathbb{N}, \\ \lim_{j \to \infty} \tau_j, & j = \infty, \end{cases}$

where $\{ -S^j \}_{j \in \mathbb{N}}$ is an $i.i.d.$ collection of $\beta$-stable subordinators. Recall [4, Chapter III] the expectation of the inverse stable subordinator

$$E[E_1(y)] = y^\beta / \Gamma(\beta + 1), \text{ where } E_j(y) := \inf \{ s > 0 : y < -S^j_s \}, \ j \in \mathbb{N}, y > 0. \quad (4.2)$$

**Lemma 4.1.** For any $x > 0$ and $j \in \mathbb{N}$, assuming $S^c_0 = x$, we have

(i) $E_x[\tau_j] < \infty$, $P_x[S^c_{\tau_j} \in (0, x)] = 1$ and $S^c_{\tau_j}$ has the density $k_j(x, \cdot)$, as defined in (2.2);

(ii) $S^c_{\tau_j} > 0$, and (4.1) equals the INW construction of the censored decreasing $\beta$-stable process;

(iii) $E_x[\tau_{j+1} - \tau_j] = E_x[E_{j+1}(S^c_{\tau_j})] = \int_0^x \frac{y^\beta}{\Gamma(\beta + 1)} k_j(x, y) \, dy; \quad (4.3)$

(iv) $P_x[\tau_\infty < \infty] = 1$ and $P_x[S^c_{\tau_\infty-} = 0] = 1$.

**Proof.** The statement (ii) follows immediately from (i). We now prove (i) by induction. For $j = 1$, $E_x[\tau_1] = E[E_1(x)] = x^\beta / \Gamma(\beta + 1) < \infty$, and it is known that $S^c_{\tau_1} = x + S^1_{\tau_1-}$ is beta-distributed on $(0, x)$ with density $k_1(x, \cdot)$ [4, Chapter III, Proposition 2]. Then we perform induction for each $j \geq 1$: since $\tau_j < \infty$, $S^c_{\tau_j} > 0$ and $S^j_{\tau_j}$ is independent of $(S^c_{\tau_j}, \tau_j)$, we have

$$\tau_{j+1} - \tau_j = \inf \{ s > \tau_j : S^c_{\tau_j} < -S^j_{\tau_j} \} - \tau_j = \inf \{ r > 0 : S^c_{\tau_j} < -S^j_{\tau_j} \} = E_{j+1}(S^c_{\tau_j}). \quad (4.4)$$
Combining (4.4) with $S_{\tau_j}^c < x$ and (4.2), we obtain
\[ E_x[\tau_{j+1}] = E_x[E_{j+1}(S_{\tau_j}^c)] + E_x[\tau_j] \leq E[E_{j+1}(x)] + E_x[\tau_j] < \infty. \]
By definition and (4.4), we have
\[ S_{\tau_{j+1}}^c = S_{\tau_j}^c + S_{\tau_{j+1} - \tau_j}^c = S_{\tau_j}^c + S_{E_{j+1}(S_{\tau_j}^c)}^c \in (0, S_{\tau_j}^c) \subseteq (0, x). \]
Therefore for any bounded measurable $f$, we have
\[
\begin{align*}
E_x[f(S_{\tau_{j+1}}^c)] &= E_x[f(S_{\tau_j}^c + S_{E_{j+1}(S_{\tau_j}^c)}^c)] \\
&= \int_0^x \left( \int_0^y f(z) k_j(y, z) \, dz \right) k_j(x, y) \, dy \\
&= \int_0^x f(z) \left( \int_z^x k_j(x, y) k_1(y, z) \, dy \right) \, dz,
\end{align*}
\]
where the second equality holds because $S_{\tau_j}^c$ is independent of $S_{E_{j+1}(S_{\tau_j}^c)}^c$ and has the density $k_j(x, \cdot)$; the last equality is due to Fubini’s theorem. By Remark 2.10 we know that $S_{\tau_{j+1}}^c$ has the density $k_{j+1}(x, \cdot)$. The induction step is now complete.

For part (iii), by (4.4) we have $E_x[\tau_{j+1} - \tau_j] = E_x[E_{j+1}(S_{\tau_j}^c)]$, meanwhile, since $S_{E_{j+1}(S_{\tau_j}^c)}^c$ is independent of $S_{\tau_j}^c$, by (4.2) we have
\[
E_x[E_{j+1}(S_{\tau_j}^c)] = \int_0^x E[E_{j+1}(y)] k_j(x, y) \, dy = \int_0^x y^{\beta} \Gamma(\beta + 1) k_j(x, y) \, dy.
\]
We now prove part (iv). The results obtained so far are enough to derive Theorem 4.2 below, which immediately implies that $P_x[\tau_\infty < \infty] = 1$. To prove $P_x[S_{\tau_\infty}^c > 0] = 0$, first, observe that
\[ P_x[S_{\tau_\infty}^c > 0] \leq \sum_{n=1}^{\infty} P_x[S_{\tau_\infty}^c \geq n^{-1}], \]
and for each $n \in \mathbb{N}$
\[ P_x[S_{\tau_\infty}^c \geq n^{-1}] = P_x \left[ \bigcap_{j=1}^\infty \left\{ S_{\tau_j}^c \geq n^{-1} \right\} \right] = \lim_{j \to \infty} P_x [S_{\tau_j}^c \geq n^{-1}], \]
where we used $\{ S_{\tau_j}^c \geq n^{-1} \} \supseteq \{ S_{\tau_{j+1}}^c \geq n^{-1} \}$ for each $j \in \mathbb{N}$ and convergence from above of finite measures. Then, Chebyshev’s inequality and the above results guarantee that
\[
\frac{1}{n} P_x [S_{\tau_j}^c \geq n^{-1}] \leq E_x [S_{\tau_j}^c] = \int_0^x k_j(x, y) y \, dy,
\]
and the right-hand side goes to 0 as $j \to \infty$ by Lemma 2.14.

We can now prove our main result of this subsection, which gives (1.8).

**Theorem 4.2.** The hitting time of 0 of the censored $\beta$-stable Lévy process (4.1) is finite in expectation, with $E_x[\tau_\infty] = E_x[\tau_1](1 - \sin(\beta \pi)/(\beta \pi))^{-1}, x > 0$. \[ \square \]
Remark 4.3. Our key ingredient for proving Theorem 4.2 is the following closed formula for (4.3) (obtained in the proof of Lemma 2.14)

\[ \int_{0}^{x} y^\beta k_j(x, y) \, dy = x^\beta \left( \Gamma(\beta + 1) \Gamma(1 - \beta) \right)^{-j}, \quad \text{for all } j \in \mathbb{N} \text{ and } x > 0. \]

**Proof.** [of Theorem 4.2] On the one hand, by Monotone Convergence Theorem, \( E_x[\tau_\infty] = \lim_{j \to \infty} E_x[\tau_{j+1}] \). On the other hand, by (4.2), (4.3) and Remark 4.3, for each \( j \in \mathbb{N} \),

\[
E_x[\tau_{j+1}] = E_x[\tau_1] + \sum_{i=1}^{j} E_x[\tau_{i+1} - \tau_i] = \frac{x^\beta}{\Gamma(\beta + 1)} \sum_{i=0}^{j} \left( \Gamma(\beta + 1) \Gamma(1 - \beta) \right)^{-i},
\]

and as \( \Gamma(\beta + 1) \Gamma(1 - \beta) = \beta \pi / \sin(\beta \pi) > 1 \), the result follows letting \( j \to \infty \). \( \square \)

Remark 4.4.

(i) Theorem 4.2 is not obvious. For instance, the censored symmetric \( \beta \)-stable process for \( \beta \in (0, 1) \) never hits the boundary, whether the censoring is performed in an interval or \( \mathbb{R}^+ \) [6, Theorem 1.1-(1)].

(ii) Any compound Poisson process in \( \mathbb{R}^d \) censored upon exiting an open set must have infinite lifetime, and so does a non-increasing compound Poisson process censored in \( (0, T] \). This is because the lifetime can be bounded from below by \( \sum_{n=1}^{\infty} e_n = \infty \), where \( \{e_n\}_{n \in \mathbb{N}} \) is an infinite subset of the i.i.d. exponential waiting times of the process.

(iii) The censored gamma subordinator with Lévy measure \( \psi(r) = e^{-r}/r \) [7, Example 5.10] seems to have infinite lifetime, because our numerical simulations indicate pathwise that \( \tau_j \approx 2 \sqrt{j/3} \) and \( S_{\tau_j}^c \approx \exp\{-\sqrt{3j}\} \) for \( x = 1 \) and \( j \gg 1 \). We do not know whether other censored (driftless) subordinators hit the barrier in finite time. If they do, it is not clear if our proof strategy can be extended to such cases, as it relies on the closed formula for the potential kernel, which is only available for the stable case.

4.2. **Probabilistic representations of** \( I_0^\beta \). Firstly, we prove that \( I_0^\beta \) is equal to the potential of the semigroup of the censored process \( S^c \). Secondly, we give a representation of \( I_0^\beta \) in terms of products of i.i.d. beta-distributed random variables.

**Proposition 4.5.** If \( g \) satisfies the assumption in Theorem 3.2 or if \( g \in B[0, T] \), it holds that \( I_0^\beta g \in C_\infty(0, T] \), and for all \( x \in (0, T] \) we have the identity

\[
I_0^\beta g(x) = E_x \left[ \int_{0}^{\tau_\infty} g(S^c_s) \, ds \right].
\]

**Proof.** For \( g \geq 0 \) we justify the following equalities

\[
E_x \left[ \int_{0}^{\tau_\infty} g(S^c_s) \, ds \right] = \sum_{j=0}^{\infty} E_x \left[ \int_{0}^{\tau_{j+1} - \tau_j} g(S^c_{\tau_j + s}) \, ds \right] = \sum_{j=0}^{\infty} E_x \left[ \int_{0}^{E_{j+1}S^c_{\tau_j}} g(S^c_{\tau_j} + S^c_{j+1}) \, ds \right].
\]
\[
\begin{align*}
&= \sum_{j=0}^{\infty} \mathbb{E}_x \left[ \mathbb{E} \left[ \int_0^{E_{j+1}(S_{r_j}^c)} g(S_{r_j}^c + S_{s_{j+1}}) \, ds \mid S_{r_j}^c \right] \right] \\
&= \sum_{j=0}^{\infty} \mathbb{E}_x \left[ J_0^\beta g(S_{r_j}^c) \right] = \sum_{j=0}^{\infty} J_0^\beta J_0^\beta g(x) = I_0^\beta g(x). \quad (4.6)
\end{align*}
\]

The first equality is an application of Tonelli's Theorem and a simple change of variables; the second follows from (4.4); the third is due to the law of total expectation; the fourth is due to the independence of \( S_{r_j}^{j+1} \) and \( S_{r_j}^c \) along with the known identity (1.6) (which is a straightforward consequence of \([7, \text{Eq. (1.38)})\)); the fifth follows from Lemmata 4.1-(i) and 2.14; the last follows the definition of \( I_0^\beta \). If \( g \in B[0, T] \), recalling that \( J_0^\beta |g|(x) \leq \sup \{|g(y)| : y \in [0, x]\} x^\beta/\Gamma(\beta + 1) \) and \( J_0^\beta g \in C[0, T] \), by Lemma 2.14 we know that \( \sum_{j=0}^{\infty} J_0^\beta J_0^\beta g \in C_{\infty}(0, T) \), and that we can apply Fubini's Theorem to the above equalities. If \( g \) satisfies the condition in Theorem 3.2, then Theorem 3.2 proves \( I_0^\beta g \in C_{\infty}(0, T) \) and justifies the application of Fubini's Theorem. \( \square \)

**Remark 4.6.**

(i) The above proof provides the following intuition for how \( \partial_0^\beta \) extends the memory effect of \( D_0^\beta \). Rewrite the right-hand side of (4.5) as
\[
\mathbb{E} \left[ \int_{0}^{E_1(x)} g(x + S_{s_{1}}^1) \, ds \right] + \mathbb{E}_x \left[ J_0^\beta g(S_{r_1}^c) \right]. \quad (4.7)
\]

Then the first term in (4.7) weights the past values of \( g \) on the interval \((x + S_{E_1(x)}^1, x]\), just like (1.6) in the Caputo case (note that (1.6) takes a slightly different form, just because there we assume \( S_{r}^1 \) starts from \( x \) instead of 0). Meanwhile, the second term proceeds on the interval \((0, x + S_{E_1(x)}^1]\) according to the censored process. This second term can be simplified further using the distribution of \( S_{r_j}^c \) and written in terms of products of i.i.d. beta-distributed random variables, as we will see in Proposition 4.8.

(ii) Proposition 4.5 proves that \( \mathbb{E}_x \left[ J_0^{r_{\infty}} (S_{s_j}^c)^{\alpha} \, ds \right] \) equals the right-hand side of (3.9). If \( \alpha > -\beta \), it is finite and yields Theorem 4.2 (by letting \( \alpha = 0 \)). If \( \alpha \leq -\beta \), then it is infinite by Remark 2.15. In contrast, \( \mathbb{E} \left[ \int_0^{E_1(x)} (x + S_{s_1}^1)^{\alpha} \, ds \right] < \infty \) for all \( \alpha > -1 \).

**Definition 4.7.** For any \( x > 0 \), we define the \((0, x]\)-valued discrete time Markov process \( X_j = x \prod_{i=1}^{j} B_i, \ j \in \mathbb{N} \), with \( X_0 = x \), and \{\( B_i \}_{i \in \mathbb{N}} \) being an i.i.d. collection of beta-distributed random variables on \((0, 1]\) with parameters \((1 - \beta, \beta)\).

**Proposition 4.8.** Under the assumption of Proposition 4.5, \( S_{r_j}^c \) equals \( X_j \) in law for each \( j \in \mathbb{N} \), and \( I_0^\beta \) allows the probabilistic series representation
\[
I_0^\beta g(x) = \sum_{j=0}^{\infty} \mathbb{E}_x \left[ J_0^\beta g(X_j) \right], \quad x \in (0, T]. \quad (4.8)
\]
Proof. We use induction to prove that $k_j(x, \cdot)$ is the density of $X_j \mid X_0 = x$, for each $j \in \mathbb{N}$. The case when $j = 1$ is clear. By the independence of $B_{j+1}$ and $X_j$, and the induction hypothesis

$$P_x[X_{j+1} \leq r] = P_x[X_j B_{j+1} \leq r] = P_x \left[ B_{j+1} \leq \frac{r}{X_j} \right] = \int_0^r k_j(x, s) P \left[ B_{j+1} \leq \frac{r}{s} \right] ds.$$ 

Then, recalling that $k_1(1, \cdot)$ is the density of $B_{j+1}$ and that it is supported on $(0, 1)$,

$$\frac{d}{dr} P_x[X_{j+1} \leq r] = \int_0^r k_j(x, s) \frac{d}{dr} P \left[ B_{j+1} \leq \frac{r}{s} \right] ds$$

$$= \int_r^x k_j(x, s) \frac{1}{s} k_1(1, \frac{r}{s}) ds$$

$$= \int_r^x k_j(x, s) k_1(s, r) ds.$$ 

Now apply Remark 2.10 and we know that $k_{j+1}(x, \cdot)$ is the density of $X_{j+1} \mid X_0 = x$. The induction step is now complete. By Lemma 4.1, we know that $S^c_j$ equals $X_j$ in law for each $j \in \mathbb{N}$, therefore the left-hand side of (4.6) equals the right-hand side of (4.8).

**Remark 4.9.** Clearly Proposition 4.8 can be strengthened into that \{S^c_j\}_{j \in \mathbb{N}} equals \{X_j\}_{j \in \mathbb{N}} in law. It is also clear that the series in (4.8) equals $\Gamma(1-\beta) \sum_{j=1}^{\infty} \mathbb{E}_x [(X_j)^{\beta} g(X_j)]$, since we know $J_0^{\beta} g(x) = \Gamma(1-\beta) \mathbb{E}_x [(X_1)^{\beta} g(X_1)]$ from Remark 2.12.

4.3. **Laplace transform of $\tau_\infty$.** We recall some definitions adapted to our setting that relate to Feller semigroups [9]. A collection of operators $P = \{P_t\}_{t \geq 0}$ is said to be a *semigroup* on a Banach space $X$ if $P_t : X \to X$ is bounded and linear for any $s > 0$, $P_s P_t = P_{s+t}$ for all $t, s > 0$, and $P_0$ is the identity operator. We say that $P$ is *strongly continuous* on $\mathcal{L} \subseteq X$ if for any $f \in \mathcal{L}$, $P_t f \to f$ in $X$ as $s \to 0$, and that $P$ is *strongly continuous* if $P$ is strongly continuous on $X$. We define the *generator* of $P$ to be the pair $(\mathcal{G}, \mathcal{D})$, where $\mathcal{D} := \{f \in X : \mathcal{G} f \text{ converges in } X \}$ with $\mathcal{G} f := \lim_{s \to 0}(P_s f - f)/s$, and we call $\mathcal{D}$ the *domain of the generator* of $P$. Moreover, $P$ is said to be a *positivity preserving contraction* on $X \subseteq B[0, T]$ if $0 \leq P_t f \leq 1$ for any $s > 0$ and $f \in X$ such that $0 \leq f \leq 1$.

Finally, a semigroup $P$ on $X = C_\infty(0, T]$ is said to be a *Feller semigroup* if it is a strongly continuous positivity preserving contraction on $X$. We recall [9, Page 15] that there exists a one-to-one correspondence between Feller semigroups and Markov processes $\{Y_s\}_{s \geq 0}$ such that $f(\cdot) \mapsto P_s f(\cdot) := \mathbb{E}[f(Y_s) \mid Y_0 = \cdot], s \geq 0$, is a Feller semigroup [9, Page 15].

**Proposition 4.10.** For any $T > 0$, the censored decreasing $\beta$-stable process $S^c$ induces a Feller semigroup on $C_\infty(0, T]$, whose generator is

$$\left( -\partial_0^{\beta} I_0^{\beta} C_\infty(0, T) \right).$$

**Proof.** Let $P^c_t f(x) = \mathbb{E}_x [f(S^c_t)]$ for $t \geq 0$ and $f \in B[0, T]$ (defining $S^c_t = \emptyset$ for all $t > 0$ if $S^c_0 = 0$). Then $P^c = \{P^c_t\}_{t \geq 0}$ is a positivity preserving contraction semigroup on $B[0, T]$, due to $S^c$ being a Markov process. We denote by $\mathcal{L}^c$ the largest subset of $B[0, T]$ on which
$P_c$ is strongly continuous and by $D^c$ the domain of the generator of $P_c$. First we prove $L^c \supseteq C_\infty(0, T)$. For $f \in C_\infty(0, T)$, let $\tilde{f}(x) := f(\max\{x, 0\})$ for any $x \in (-\infty, T]$, and compute

$$
|P_c^\beta f(x) - f(x)| \leq \left| \mathbb{E}_x \left[ 1_{\{t < \tau_1\}}(f(S_t^\beta) - f(x)) \right] \right| + \mathbb{E}_x \left[ 1_{\{t \geq \tau_1\}}|f(S_t^\beta) - f(x)| \right] \\
\leq \left| \mathbb{E} \left[ 1_{\{t < E_1(x)\}}(\tilde{f}(x + S_t^\beta) - \tilde{f}(x)) \right] \right| + 2\|f\|_{C[0, x]}\mathbb{P}[t \geq E_1(x)] \\
\leq \left| \mathbb{E} \left[ \tilde{f}(x + S_t^\beta) - \tilde{f}(x) \right] \right| + 3\|f\|_{C[0, x]}\mathbb{P}[t \geq E_1(x)],
$$

where the first summand vanishes uniformly in $x$ as $t \to 0$ because $S^1$ is a Feller process on $\{g \in C(-\infty, T) : \lim_{x \to -\infty} g(x) = 0\}$ [9]. Meanwhile for the second summand, for any $\varepsilon > 0$ we can choose $\delta > 0$ so that $\|\cdot\|_{C[0, x]} \leq \varepsilon$ for all $x \in (0, \delta]$ and then we choose $\tilde{t}$ small so that

$$
\mathbb{P}[t \geq E_1(x)] = \mathbb{P}[x + S_t^\beta \leq 0] \leq \mathbb{P}[\delta \leq -S_t^\beta] \leq \varepsilon,
$$

for all $x \geq \delta$ and $t \leq \tilde{t}$, so for all $t \leq \tilde{t}$

$$
3\|f\|_{C[0, x]}\mathbb{P}[t \geq E_1(x)] \leq \left\{ \begin{array}{ll}
3\varepsilon, & 0 \leq x \leq \delta, \\
3\varepsilon\|f\|_{C[0, x]}, & \delta < x \leq T.
\end{array} \right.
$$

Therefore we have proved the strong continuity of $P_c$ on $C_\infty(0, T)$ and thus $C_\infty(0, T) \subseteq L^c$.

We now prove that $C_\infty(0, T)$ is invariant under $P_c$. The key ingredients are Theorem 4.2 and Proposition 4.5, which prove that $I^\beta_0$ equals the potential $\int_0^\infty P_c^s ds$ and is a bounded operator from $B[0, T]$ to $C_\infty(0, T)$. Then [18, Theorem 1.1'] implies $D^c = I^\beta_0 L^c$, and we have

$$
I^\beta_0 C_\infty(0, T) \subseteq I^\beta_0 L^c \subseteq I^\beta_0 B[0, T] \subseteq C_\infty(0, T).
$$

Because Stone–Weierstrass Theorem and Example 3.7 prove that $I^\beta_0 C_\infty(0, T)$ is dense in $C_\infty(0, T)$, by [18, Property 1.3.B] and the above inclusions we obtain $L^c = C_\infty(0, T)$. Because $P_c L^c \subseteq L^c$ [18, Property 1.3.A], we have proved that $P_c$ is a Feller semigroup on $C_\infty(0, T)$. Since its potential is $I^\beta_0$ and a bounded potential determines the generator [18, Theorem 1.1'], Theorem 3.2 implies that the generator of $P_c$ is $(-\partial^\beta_0, I^\beta_0 C_\infty(0, T))$. \hfill \Box

**Remark 4.11.**

(i) As a corollary of Proposition 4.10, for any $b > 0$ and $f \in I^\beta_0 C_\infty(0, b]$, $u(t, x) = \mathbb{E}_x \left[ f(S^\beta_t) \right]$ is the unique solution satisfying the conditions [18, a, b and c of Theorem 1.3] to the evolution equation (with $\partial^\beta_0$ acting on the spatial variable)

$$
\frac{du}{dt} = -\partial^\beta_0 u, \ (t, x) \in \mathbb{R}_+ \times (0, b]; \ u(t, 0) = 0, t \in \mathbb{R}_+; \ u(0, x) = f(x), x \in (0, b].
$$

(ii) Note that Proposition 4.10 relies crucially on Theorem 3.2. Also, it is not a special case of [26, Theorem 3.3], primarily because the latter needs the assumption that $S^c$ is a Feller process, which is not clear from the INW construction.
We are now ready to prove a Mittag-Leffler-type representation for \( E_x[e^{\lambda \tau}] \), whose analogue in the Caputo setting is the probabilistic identity

\[
E_x[e^{\lambda \tau}] = \sum_{j=0}^{\infty} \frac{\lambda^j x^{\beta j}}{\Gamma(j\beta + 1)},
\]

(4.10)

first proved in [5]. Our proof follows the approach of [23, Corollary 5.1] to (4.10). This approach allows one to solve exit problems by computing the Laplace transform of the lifetime of a killed Markov process when one knows the analytical solution to the resolvent equation \(-G u = \lambda u + g\) (\(G\) being the generator of the process). In our case, the analytical solution is given by Proposition 3.22.

**Theorem 4.12.** For every \( \lambda < 0, T > 0 \) and \( g \in C[0,T] \),

\[
E_x \left[ \int_0^{\tau} e^{\lambda s} g(S_s^c) \, ds \right] = \sum_{j=0}^{\infty} \lambda^j (I_0^\beta)^j g(x), \quad x \in (0,T].
\]

(4.11)

Moreover, the Mittag-Leffler-type series (1.7) \((u_0 = 1)\) equals \( E_x[e^{\lambda \tau}] \) for all \( \lambda \in \mathbb{R} \) and \( x > 0 \).

**Proof.** For the first claim, if \( g \in C_x(0,T), \) recalling [18, Theorem 1.1], the equality (4.11) holds by Propositions 4.10 and 3.22, as both sides of it are the unique solution in \( C^\beta[0,T] \) to the resolvent equation

\[
\partial_0^\beta u = \lambda u + g, \quad u(0) = 0, \quad g \in C_x(0,T),
\]

where we used \( I_0^\beta C_x(0,T) \subseteq C^\beta[0,T] \) given by Lemma 3.6. Now, for any \( g \in C[0,T], \) take \( g_n \in C_x(0,T) \) so that \( g_n \to g \) uniformly on every compact subset of \((0,T]\) and \( \sup_n \|g_n\|_{C[0,T]} < \infty \). Fix \( x \in (0,T], \) then for any \( s > 0, \)

\[
E_x \left[ g_n(S_s^c) \right] \to E_x \left[ g(S_s^c) \right], \quad \text{as} \quad n \to \infty,
\]

by Dominated Convergence Theorem. Then another application of Dominated Convergence Theorem, with the dominating function \( \sup_n \|g_n\|_{C[0,T]} e^{\lambda s} \), yields

\[
\int_0^{\infty} e^{\lambda s} E_x \left[ g_n(S_s^c) \right] \, ds \to \int_0^{\infty} e^{\lambda s} E_x \left[ g(S_s^c) \right] \, ds, \quad \text{as} \quad n \to \infty.
\]

On the other hand, by the continuous dependence in Proposition 3.22 (let the \( \alpha \) there be e.g. \( \beta/2 \)),

\[
\sum_{j=0}^{\infty} \lambda^j (I_0^\beta)^j g_n(x) \to \sum_{j=0}^{\infty} \lambda^j (I_0^\beta)^j g(x), \quad \text{as} \quad n \to \infty.
\]

Therefore we have proved (4.11) for all \( g \in C[0,T]. \)

To prove the second claim for \( \lambda < 0 \), in (4.11) let \( g = \lambda \), so that on the left-hand side

\[
E_x \left[ \int_0^{\tau} e^{\lambda s} \lambda \, ds \right] = \frac{\lambda E_x[e^{\lambda \tau}] - 1}{\lambda} = E_x[e^{\lambda \tau}] - 1,
\]
and on the right-hand side
\[\sum_{j=0}^{\infty} \lambda^{j+1}(I_0^\beta)^{j+1}(x) = \sum_{j=0}^{\infty} \lambda^{j}(I_0^\beta)^{j}(x) - 1.\]

Hence by (3.11) (with \(\alpha = \beta\)) we have proved the second claim for \(\lambda \leq 0\) (with \(\lambda = 0\) being a trivial case), which combined with Lemma 3.10 allows us to compute the moments \(E_x[(\tau_\infty)^j] (j \in \mathbb{N})\) by differentiating \(E_x[e^{\lambda \tau_\infty}]\) in \(\lambda (\lambda \leq 0)\) for \(j\) times. Those moments are displayed in (4.12), and in turn they allow us to prove the second claim also for \(\lambda > 0\), since we have
\[E_x[e^{\lambda \tau_\infty}] = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} E_x[(\tau_\infty)^j], \quad \lambda, x > 0,\]
where the series in the right-hand side converges to (1.7) by Lemma 3.10. \(\square\)

**Corollary 4.13.** For any \(\lambda < 0\), the Mittag-Leffler-type series (1.7) is completely monotone. More generally, for any Bernstein function \(f\) the series (1.7) composed with \(f^{1/\beta}\) is completely monotone.

**Proof.** Denote by \(\mu_1\) the law of \(\tau_\infty\) for \(S_0^c = 1\) and by \(M_\lambda(x)\) the series (1.7) \((u_0 = 1)\). Then
\[M_\lambda(x) = M_{x^\beta}(1) = E_1[e^{(x^\beta)\tau_\infty}] = \int_{[0,\infty)} e^{x^\beta y} \mu_1(dy),\]
where the second equality is due to Theorem 4.12. The second claim now follows from [44, Theorem 3.7] because \(M_\lambda\) composed with \(f^{1/\beta}\) equals
\[x \mapsto \int_{[0,\infty)} e^{\lambda f(x)y} \mu_1(dy),\]
the composition of \(x \mapsto \int_{[0,\infty)} e^{\lambda x^\beta y} \mu_1(dy)\) (which is completely monotone [44, Theorem 1.4]) with the Bernstein function \(f\). The first claim corresponds to the Bernstein function \(f(x) = x^\beta\). \(\square\)

**Remark 4.14.** The proof of (1.9) in Theorem 4.12 suits well our IVP theory, but is rather indirect, especially when compared to the standard proofs of (4.10). Below we discuss the issues encountered with adapting those standard proofs to our censored setting, suggesting that our strategy is quite efficient.

(i) A simple proof of (4.10) follows the known direct evaluation of the moments \(E_x[(\tau_1)^n] = x^{\beta n} n!/\Gamma(\beta n + 1)\), as then one can write \(E_x[e^{\lambda \tau_1}] = \sum_{n=0}^{\infty} \lambda^n E_x[(\tau_1)^n]/n!\) for \(\lambda > 0\) and check the convergence [8]. In Theorem 4.12 after proving that (1.9) holds for all \(\lambda \leq 0\), we obtain the closed form for the moments \(E_x[(\tau_\infty)^n] = x^{\beta n} n! C_n\) where
\[C_n = \prod_{j=1}^{n} \left( \frac{\Gamma(1+j\beta)}{\Gamma(j\beta + 1 - \beta)} - \frac{1}{\Gamma(1-\beta)} \right)^{-1}, \quad n \in \mathbb{N},\]
and those moments in turn allow us to prove the case when \(\lambda > 0\). But obtaining those moments by a direct evaluation is not easy. Indeed, although one can directly
compute \( \mathbb{E}_x[(\tau_\infty)^n] = \lim_{j \to \infty} \mathbb{E}_x[(\tau_j)^n] = x^{n\beta} n! \lim_{j \to \infty} C_{n,j} \) (see Appendix C for details), where

\[
C_{n,j} = \sum_{n_1 + \ldots + n_j = n} \frac{\Gamma(1 - \beta)^{1-j}}{\Gamma(\beta n_j + 1)} \prod_{i=1}^{j-1} \frac{\Gamma(1 - \beta + \beta \sum_{i=0}^{j-2} n_{j-i})}{\Gamma(1 + \beta \sum_{i=0}^{j-2} n_{j-i})}, \quad n \in \mathbb{N},
\]
a direct proof of \( C_{n,j} \to C_n \) as \( j \to \infty \) appears hard (nonetheless, Theorem 4.12 can serve as an indirect proof).

(ii) There exist several proofs of (4.10) whose key steps rely on the Laplace transform of \(-S_1^t\) or infinitely divisible random variables (see [5, Proposition 1.a], [19, XIII.8 Example (b), p. 453], [50, Theorem 2.10.2], [30, Lemma 3.4] and [35, 6.6 (ii)]), and therefore these proofs do not apply to \( S^c \). Also, the recent approach in [10, Section 4.1], which characterises complete monotonicity of Kilbas–Saigo functions, poses many challenges to being adapted to our censored setting (in particular obtaining an appropriate analogue of [10, Lemma 4.1]).

(iii) We could not apply the strategy used by the proof of Proposition 4.5 to prove (1.9), as one would need a closed form of the complicated expectations \( \mathbb{E}_x[(E_\beta(\lambda(S^c_{\tau_j})^\beta) - 1) e^{\lambda \tau_j}] \) for \( j \in \mathbb{N} \), where \( E_\beta(x) = \sum_{n=0}^{\infty} x^n / \Gamma(\beta n + 1) \) (not to be confused with the inverse stable subordinator \( E_j(y) \) defined in (4.2)).

(iv) The simple observation that the relaxation equation (3.12) rewrites as the Dirichlet problem \((-D_0^\beta + q)u = 0, u(0) = 1\) with the unbounded potential \( q = \lambda + x^{-\beta} / \Gamma(1 - \beta) \), suggests combining our IVP theory with potential theoretic techniques to prove (1.9).

Namely, show that the gauge function \( u(x) = \mathbb{E}\left[ \exp\left\{ \int_0^{E_1(x)} q(x + S_t^1) \, dt \right\} \right] \) solves (3.12). Although this approach appears feasible (as Proposition 4.5 should prove gaugeability by [7, Theorem 2.9(ii)]), we expect it to be more involved than our strategy. The main reason is that, to show that the gauge function equals \( \mathbb{E}_x\left[ \exp\{\lambda \tau_\infty\} \right] \), one would need to derive a relationship between the INW construction of \( S^c \) and the coefficient \( x^{-\beta} / \Gamma(1 - \beta) \). Such relationship seems difficult to derive without the knowledge of the generator of \( S^c \) (cf. [6, Theorem 2.1]), on the other hand this knowledge is already enough for our strategy to work. Another reason is that one would have to employ general results from the potential theory of Feynman–Kac semigroups (cf. [13, Chapter 3]), only making the proof more technical.

Remark 4.15.

(i) Let \( \tau_1(t), \tau_\infty(t) \) and \( B \) denote \( E_1(t) \) (i.e. the inverse stable subordinator), \( \tau_\infty \mid S_0^c = t \) and an independent Brownian motion, respectively. It is known (e.g. [39]) that the Caputo time-fractional diffusion equation \( D_0^\beta [u - u(0)] = \Delta u / 2 \) is solved by the fractional kinetic process \( \{B_{\tau_1(t)}\}_{t \geq 0} \). This process is well-known as sub-diffusion since (4.2) implies \( \mathbb{E}[|B_{\tau_1(t)}|^2] = \mathbb{E}[\tau_1(t)] = t^\beta / \Gamma(\beta + 1) \), which is slower than normal diffusion \( \mathbb{E}[|B_t|^2] = t \). Our work suggests that the censored counterpart \( \partial_0^\beta u = \Delta u / 2 \) is solved by a new sub-diffusion process \( \{B_{\tau_\infty(t)}\}_{t \geq 0} \). Indeed, Theorem 4.2 shows that \( \mathbb{E}[|B_{\tau_\infty(t)}|^2] = ct^\beta \) (\( c > 0 \)), and we expect the time-fractional evolution equation
\[ \partial_0^\beta u = Gu + g, \ u(0) = \phi \text{ to have a unique (generalised) solution} \]
\[
\begin{align*}
u(t, x) &= \mathbb{E}\left[ \phi(X_{\tau_\infty}) + \int_0^{\tau_\infty} g(S_{s}^c, X_s) \, ds \mid (S_0^c, X_0) = (t, x) \right], \quad (t, x) \in (0, T) \times \mathbb{R}^d,
\end{align*}
\]
where \( \phi \in \text{Dom}(G) \), \( g \in C([0, T] \times \mathbb{R}^d) \), and \( (G, \text{Dom}(G)) \) is the generator of any Feller process \( X \) on \( \mathbb{R}^d \) independent of \( S^c \). (We think the last claim can be proved using the techniques from \([16, 24]\), in the light of Proposition 4.10.) Let us also mention that to find strong solutions to \( \partial_0^\beta u = \Delta u/2 \), Theorem 4.12 opens up the possibility of applying the spectral decomposition method of \([12]\).

(ii) Although both \( B_{\tau_1} \) and \( B_{\tau_\infty} \) spread like \( t^\beta \), their respective Fourier modes model entirely different relaxation regimes. Namely, for any \( \lambda \in \mathbb{R}^d \) we have,
\[
\begin{align*}
\mathbb{E}\left[ \exp\left\{ i\lambda \cdot B_{\tau_1}(t) \right\} \right] &= \mathbb{E}\left[ \exp\left\{ -|\lambda|^2 \tau_1(t)/2 \right\} \right] \asymp t^{-\beta} ; \\
\mathbb{E}\left[ \exp\left\{ i\lambda \cdot B_{\tau_\infty}(t) \right\} \right] &= \mathbb{E}\left[ \exp\left\{ -|\lambda|^2 \tau_\infty(t)/2 \right\} \right] \asymp t^{-1-\beta},
\end{align*}
\]
by (4.10), Theorem 4.12 and Remark 3.18-(i). Here \( f \asymp g \) means \( C^{-1}g \leq f \leq Cg \) for some constant \( C > 1 \).

(iii) There are several interesting questions revolving around \( B_{\tau_\infty} \), a new example of anomalous diffusion. For instance, it is natural to ask if there is a continuous-time-random-walk-type framework which scales to \( B_{\tau_\infty} \), as is the case for \( B_{\tau_1} \) \([1, 39]\) and several other anomalous diffusion processes \([2, 49]\) related to Caputo derivatives. Moreover, it is challenging and interesting to study the difference in path regularity between \( B_{\tau_\infty} \) and \( B_{\tau_1} \), in particular because the latter can be “trapped” \([39]\).

(iv) We mention that sub-diffusion and fractional relaxation equations are widely used to model anomalous (non-Debye) relaxation in dielectrics, see \([29, 48, 31, 49]\) and references therein. Their role is to provide a probabilistic theoretic explanation of the empirical (Havriliak–Negami) formula \( \chi(\omega) = (1 + (i\omega)^\alpha)^{-\gamma} \). (Here \( \omega \) is the electric field’s frequency and \( \chi \) is the electric susceptibility. This formula fits well a majority of experimental data.) A typical example (Cole–Cole) is \( \alpha = \beta \in (0, 1) \) and \( \gamma = 1 \), which is modelled by the sub-diffusion \( B_{\tau_1} \) \([48, \text{ Page 3}]\). On the other hand, we expect \( B_{\tau_\infty} \) to model a new regime with \( \alpha = 1 + \beta \in (1, 2) \) and \( \gamma = \beta/(1 + \beta) \) (by \([48, \text{ Eq. (1) and (5)}]\)), although in the literature (e.g. \([31, 48]\)) we have not seen the parameter range \( \alpha > 1 \).

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**Appendix A. Proofs of elementary properties of \( J_0^\beta \) and \( D_0^\beta \)**

A.1. **Proof of Lemma 2.3.** The proof consists of four parts.
(i) Since both \( u \) and \( x^{\beta-1} \) are in \( C \cap L^1(0, T) \), for any \( x \in (0, T) \), \( J_0^\beta u(x) \) is well-defined and finite. For \( \varepsilon \in (0, x) \), define
\[
J_\varepsilon^\beta u(x) = \int_0^{x-\varepsilon} \frac{(x-r)^{\beta-1}}{\Gamma(\beta)} u(r) \, dr.
\]
Given \( T_1 \in (0, T) \), for all \( x \in [T_1, T] \) and \( \varepsilon \in (0, T_1) \), we have
\[
\left| J_\varepsilon^\beta u(x) - J_0^\beta u(x) \right| \leq \int_{x-\varepsilon}^x \left| \frac{(x-r)^{\beta-1}}{\Gamma(\beta)} u(r) \right| \, dr \leq \frac{\varepsilon\beta}{\beta\Gamma(\beta)} \| u \|_{C[T_1-\varepsilon, T]} \\
\]
therefore, as \( \varepsilon \to 0 \), \( J_\varepsilon^\beta u \to J_0^\beta u \) uniformly on \( [T_1, T] \). By Dominated Convergence Theorem, \( J_\varepsilon^\beta u \) is continuous on \( [T_1, T] \). So \( J_0^\beta u \) is also continuous on \( [T_1, T] \), and thus on \( (0, T) \). Integrability of \( J_0^\beta u \) follows by
\[
\int_0^T |J_0^\beta u(x)| \, dx \leq \int_0^T \int_0^x \frac{(x-r)^{\beta-1}}{\Gamma(\beta)} |u(r)| \, dr \, dx \\
= \int_0^T \left| u(r) \right| \int_0^T (x-r)^{\beta-1} \, dx \, dr \\
\leq \frac{T^\beta}{\Gamma(1+\beta)} \int_0^T |u(r)| \, dr < \infty.
\]
(ii) By Lemma 2.3-(i), \( J_0^\beta u \in C \cap L^1(0, T) \), and \( J_0^{1-\beta} J_0^\beta u \) is well-defined on \( (0, T) \). For \( x \in (0, T) \),
\[
J_0^{1-\beta} J_0^\beta u(x) = \int_0^x \frac{(x-r)^{1-\beta}}{\Gamma(1-\beta)} \int_0^r \frac{(r-s)^{\beta-1}}{\Gamma(\beta)} u(s) \, ds \, dr \\
= \int_0^x u(s) \int_s^x \frac{(x-r)^{1-\beta}}{\Gamma(1-\beta)} \frac{(r-s)^{\beta-1}}{\Gamma(\beta)} \, dr \, ds \\
= \int_0^x u(s) \, ds,
\]
where the second identity is due to Fubini’s Theorem. Therefore \( J_0^{1-\beta} J_0^\beta u \in C^1(0, T) \) and \( D_0^\beta J_0^\beta u = u \).

(iii) The “only if” is due to Lemma 2.3-(ii) and (A.1) as well as the assumption that \( g \in L^1(0, T) \). For the “if”, we use Lemma 2.3-(ii) to get \( D_0^\beta u = g = D_0^\beta J_0^\beta g \), so \( D_0^\beta [u - J_0^\beta g] = 0 \). By the definition of \( D_0^\beta \), we know that \( J_0^{1-\beta} [u - J_0^\beta g] \) is constant. By (A.1) we know that \( \lim_{x \to 0} J_0^{1-\beta} J_0^\beta g(x) = 0 \), and by assumption \( \lim_{x \to 0} J_0^{1-\beta} J_0^\beta u(x) = 0 \).

Therefore \( J_0^{1-\beta} [u - J_0^\beta g] \) must be 0. Conclude with Lemma 2.3-(ii) which proves \( u - J_0^\beta g = D_0^{1-\beta} J_0^\beta [u - J_0^\beta g] = D_0^{1-\beta} 0 = 0 \).

(iv) For \( u \in C([0, T]) \), we have \( |J_0^{1-\beta} u(x)| \leq \| u \|_{C([0, T])} x^{1-\beta} / \Gamma(2-\beta) \to 0 \) as \( x \to 0 \). Then the “if” of Lemma 2.3-(iii) applies, giving \( u = J_0^\beta 0 = 0 \). \( \square \)
A.2. Proof of Hölder regularity of $J_0^x g$. For any interval $\Omega \subseteq \mathbb{R}$, we denote by $L^\infty(\Omega)$ the essentially bounded functions in $L^1(\Omega)$.

**Lemma A.1.** If $g \in L^\infty(0, T]$, then $J_0^x g \in C^{0, \beta}[0, T]$ with $J_0^x g(0)$ being 0 and a Hölder constant being $2||g||_{L^\infty(0, T]}/\Gamma(1 + \beta)$.

**Proof.** For $0 < x < x + h \leq T$, the difference $J_0^x g(x + h) - J_0^x g(x)$ equals

$$
\frac{\Gamma(\beta)}{\Gamma(\beta + 1)} - \frac{\Gamma(\beta)}{\Gamma(\beta + 1)} g(r) dr.
$$

The absolute value of the first summand can be bounded from above by

$$
\|g\|_{L^\infty(0, T]} (x + h)^\beta - h^\beta - x^\beta \leq \frac{h^\beta \|g\|_{L^\infty(0, T]}}{\Gamma(1 + \beta)},
$$

where the inequality is obtained by observing that $(x + h)^\beta - h^\beta - x^\beta$ is decreasing from 0 to $-h^\beta$ (unattainable) with respect to $x \in [0, \infty)$. The absolute value of the second summand can be bounded from above by $h^\beta \|g\|_{L^\infty(0, T]}/\Gamma(1 + \beta)$. Therefore

$$
|J_0^x g(x + h) - J_0^x g(x)| \leq 2h^\beta \|g\|_{L^\infty(0, T]}/\Gamma(1 + \beta).
$$

Note that for all $x \in (0, T]$,

$$
|J_0^x g(x) - J_0^x g(0)| \leq x^\beta \|g\|_{L^\infty(0, T]}/\Gamma(1 + \beta),
$$

we know $J_0^x g \in C^{0, \beta}[0, T]$ with $2||g||_{L^\infty(0, T]}/\Gamma(1 + \beta)$ as a Hölder constant.

**Lemma A.2.** If $g \in L^1(0, T]$ satisfies $|g(x)| \leq Mx^{\alpha-\beta}$ for some $\alpha, M \geq 0$ and all $x \in (0, T]$, then for all $T_1 \in (0, T)$, $J_0^x g \in C^0, 0, [T_1, T]$ with a Hölder constant being $2M \max\{T_1^{\alpha-\beta}, T^{\alpha-\beta}\}/\Gamma(1 + \beta)$.

**Proof.** For $0 < T_1 \leq x < x + h \leq T$, we have

$$
J_0^x g(x + h) - J_0^x g(x) = \int_0^x \frac{(x + h - r)^{\beta-1} - (x - r)^{\beta-1}}{\Gamma(\beta)} g(r) dr + \int_x^{x+h} \frac{(x + h - r)^{\beta-1}}{\Gamma(\beta)} g(r) dr.
$$
The absolute value of the first summand can be bounded from above by

\[
\int_0^x \frac{(x-r)^{\beta-1} - (x+h-r)^{\beta-1}}{\Gamma(\beta)} M r^{\alpha-\beta} \, dr
\]

\[
= \frac{M}{\Gamma(\beta)} \left( x^\alpha B(1; \alpha+1-\beta, \beta) - (x+h)^\alpha B\left(\frac{x}{x+h}; \alpha+1-\beta, \beta\right) \right)
\]

\[
\leq \frac{M x^\alpha}{\Gamma(\beta)} \left( B(1; \alpha+1-\beta, \beta) - B\left(\frac{x}{x+h}; \alpha+1-\beta, \beta\right) \right)
\]

\[
= \frac{M x^\alpha}{\Gamma(\beta)} \int_{x/(x+h)}^1 r^{\alpha-\beta} (1-r)^{\beta-1} \, dr
\]

\[
\leq \frac{M x^\alpha}{\Gamma(\beta)} \max \left\{ 1, \left(\frac{x}{x+h}\right)^{\alpha-\beta}\right\} \frac{h^\beta}{\beta}
\]

\[
\leq \frac{M x^\alpha - h^\beta \Gamma(1+\beta)}{\Gamma(1+\beta)}
\]

where B is the incomplete beta function. The absolute value of the second summand can be bounded from above by

\[
\int_{x}^{x+h} \frac{(x+h-r)^{\beta-1}}{\Gamma(\beta)} M r^{\alpha-\beta} \, dr
\]

\[
= \frac{M}{\Gamma(\beta)} \int_0^h (h-r)^{\beta-1} (x+r)^{\alpha-\beta} \, dr
\]

\[
\leq \frac{M}{\Gamma(\beta)} \max \{ x^{\alpha-\beta}, (x+h)^{\alpha-\beta} \} \frac{h^\beta}{\beta}
\]

\[
\leq \frac{M h^\beta \max \{T_1^{\alpha-\beta}, T^{\alpha-\beta}\}}{\Gamma(1+\beta)}
\]

Therefore \( |J_0^\beta g(x+h) - J_0^\beta g(x)| \leq 2M h^\beta \max \{T_1^{\alpha-\beta}, T^{\alpha-\beta}\}/\Gamma(1+\beta) \), we know \( J_0^\beta g \in C^{0,\beta}[T_1, T] \) with \( 2M \max \{T_1^{\alpha-\beta}, T^{\alpha-\beta}\}/\Gamma(1+\beta) \) as a Hölder constant. \( \square \)

**Remark A.3.** Lemma A.2 is a natural generalization of Lemma A.1, since \( \|g\|_{L^\infty[T_1, T]} \leq M \max \{T_1^{\alpha-\beta}, T^{\alpha-\beta}\} \). For \( \alpha < \beta \), the \( \beta \)-Hölder continuity of \( J_0^\beta g \) is only local (away from 0), since \( T_1^{\alpha-\beta} \) explodes as \( T_1 \to 0 \). Indeed, in such cases, \( |J_0^\beta g(x) - J_0^\beta g(0)| \leq M x^\alpha \Gamma(\alpha+1-\beta)/\Gamma(1+\alpha) \) for \( x \in (0, T] \), with \( J_0^\beta g(0) \) defined to be 0. So the uniform Hölder continuity of \( J_0^\beta g \) on \( [0, T] \) is probably only of exponent \( \alpha \). Sure enough, we have the following result.

**Lemma A.4.** For the same \( g \) in Lemma A.2, additionally assume \( \alpha \in (0, \beta) \), then \( J_0^\beta g \) \( \in C^{0,\alpha}[0, T] \) with a Hölder constant being \( 2M \Gamma(\alpha+1-\beta)/\Gamma(1+\alpha) \).

**Proof.** By assumptions on \( g \), we have

\[
|J_0^{\beta-\alpha} g(x)| \leq J_0^{\beta-\alpha} |g|(x) \leq M J_0^{\beta-\alpha} [x^{\alpha-\beta}] = M \Gamma(\alpha+1-\beta),
\]
then by Lemma A.1, \( J_0^\beta \phi \in C^{0,\alpha}[0,T] \) with a Hölder constant being \( 2M\Gamma(\alpha + 1 - \beta)/\Gamma(1+\alpha) \), and consequently \( J_0^\beta \phi \in C[0,T] \). By Lemma A.2, \( J_0^\beta \phi \in C(0,T) \). Recalling [14, Theorem 2.2], we know \( J_0^\beta \phi \) is defined everywhere, with \( J_0^\beta \phi(0) \) defined to be 0. Therefore, \( J_0^\beta \phi \) is also in \( C^{0,\alpha}[0,T] \) with the same Hölder constant as \( J_0^\alpha J_0^\beta \phi \).

### A.3. Proof of the diagram in Remark 2.8

The proof can be decomposed into several statements.

(i) \( J_0^\beta [C \cap L^1(0,T)] \subseteq C_\beta(0,T) \): follows Lemma 2.3-(ii).

(ii) \( J_0^\beta [C \cap L^1(0,T)] \not\subseteq C_\beta(0,T) \): this is because \( x^\gamma \) is in \( C_\beta(0,T) \) but not in \( J_0^\beta [C \cap L^1(0,T)] \). The latter fact can be proven by Lemma 2.3-(ii), noticing \( D_0^\beta x^\gamma \notin L^1(0,T) \).

(iii) \( J_0^\beta [C \cap L^1(0,T)] \not\subseteq C_\beta(0,T) \): for example, \( J_0^\beta (x^{-\frac{1+\alpha}{2}}) \) is not in \( C[0,T] \).

(iv) \( J_0^\beta [C \cap L^1(0,T)] \not\subseteq C_\beta(0,T) \): see Lemma A.5.

(v) \( J_0^\beta [C \cap L^1(0,T)] \supseteq U \): follows Lemma 2.3-(iii).

(vi) \( J_0^\beta [C \cap L^1(0,T)] \cap C_\beta(0,T) \not\subseteq U \): to see this, in the proof of Lemma A.5, we may take \( \alpha - \beta - \gamma \beta \in (-1, -\beta) \) instead of \( \alpha - \beta - \gamma \beta < -1 \), then \( (J_0^\beta \phi)' = O(x^{\alpha - \beta - \gamma \beta}) + O(x^{\alpha - \beta}) \) is absolutely integrable. We know such \( u \) is in \( C_\beta(0,\bar{T}) \), and by Lemma 2.3-(iii), \( u \notin J_0^\beta [C \cap L^1(0,T)] \). Yet \( u \notin U \), because \( \partial_0^\beta u = (J_0^\beta \phi)' - x^{-\beta}u/\Gamma(1-\beta) = O(x^{\alpha - \beta - \gamma \beta}) + O(x^{\alpha - \beta}) \), which cannot be controlled by \( x^{\alpha - \beta} \) for any \( \bar{\alpha} > 0 \).

### Lemma A.5

We have \( J_0^\beta [C \cap L^1(0,T)] \not\subseteq C_\beta(0,T) \).

**Proof.** We only need to find \( u \in C[0,T] \), such that \( J_0^{1-\beta} u \in C^1(0,T) \) but \( (J_0^{1-\beta} u)' \notin L^1(0,T) \). Once succeed, we know such \( u \in C_\beta(0,T) \) by definition. If \( u \notin J_0^\beta [C \cap L^1(0,T)] \), then \( \exists v \in C \cap L^1(0,T) \) such that \( u = J_0^\beta v \). By Lemma 2.3-(iii), \( v \notin J_0^\beta [C \cap L^1(0,T)] \). This contradiction implies \( u \notin J_0^\beta [C \cap L^1(0,T)] \) and thus \( C_\beta(0,T) \not\subseteq J_0^\beta [C \cap L^1(0,T)] \).

In order for \( J_0^{1-\beta} u \) to have continuous but not absolutely integrable derivative on \((0,T]\), and thus have unbounded variation, \( u \) had better oscillate quickly near 0. Let \( u(x) = x^\alpha \sin(x^{-\gamma}) \), where \( \alpha, \gamma > 0 \). Then

\[
J_0^{1-\beta} u(x) = \int_0^x \frac{r^\alpha \sin(r^{-\gamma})}{\Gamma(1-\beta)(x-r)^\beta} \, dr = \frac{x^{1-\beta+\alpha}}{\Gamma(1-\beta)} \int_0^1 s^{\alpha} \left( \frac{(xs)^{-\gamma}}{(1-s)^\beta} \right) \, ds = \frac{x^{1-\beta+\alpha}}{\Gamma(1-\beta)} \phi(x),
\]

where \( \phi \) is defined in Lemma A.6. So \( J_0^{1-\beta} u \in C^1(0,T) \) and

\[
(J_0^{1-\beta} \phi)'(x) = \frac{1 - \beta + \alpha}{\Gamma(1-\beta)x^{\beta-\alpha}} \phi(x) + \frac{x^{1-\beta+\alpha}}{\Gamma(1-\beta)} \phi'(x)
\]

Note that \( \|\phi\|_{L^\infty(0,T]} \leq \Gamma(1+\alpha)/\Gamma(1-\beta)/\Gamma(2-\beta+\alpha) \), we know the first summand is absolutely integrable, so \( (J_0^{1-\beta} u)' \) has the same absolute integrability with the second summand. By
Lemma A.6, we know
\[
\frac{x^{1-\beta+\alpha}}{\Gamma(1-\beta)} \phi'(x) = \gamma^\beta x^{\alpha-\beta-\gamma} \sin \left( \frac{1}{x^\gamma} - \frac{\pi \beta}{2} \right) + O(x^{\alpha-\beta}).
\]

Choose \( \gamma \) large enough so that \( \alpha - \beta - \gamma \beta < -1 \), then \( (J_0^{1-\beta} u)' \notin L^1(0,T) \) and we are done. Note although the above estimate is enough for us, it may be tightened into \( (J_0^{1-\beta} u)'(x) = \gamma^\beta x^{\alpha-\beta-\gamma} \left[ \sin(x^{-\gamma} - \pi \beta/2) + O(x^\gamma) \right] \).

\( \square \)

**Lemma A.6.** For \( \alpha, \gamma > 0 \), define
\[
\phi(x) = \int_0^1 s^\alpha \sin \left( (xs)^{-\gamma} \right) \frac{ds}{(1-s)^\beta}, \quad x \in (0,T],
\]
then \( \phi \in C^1(0,T] \), and \( \phi'(x) \) is given by an improper integral which converges absolutely or conditionally,
\[
\phi'(x) = \frac{-\gamma}{x^{1+\gamma}} \int_0^1 \frac{\cos \left( (xs)^{-\gamma} \right)}{s^{\gamma-\alpha}(1-s)^\beta} \, ds = \int_1^\infty \frac{-x^{1-\gamma} \cos(x^{-\gamma} r)}{r^{(1+\alpha)/\gamma}(1-r^{-1/\gamma})^\beta} \, dr, \quad x \in (0,T],
\]
and we have the following estimate for \( \phi' \),
\[
\phi'(x) = \gamma^\beta \frac{\Gamma(1-\beta)}{x^{1+\gamma} x^{\alpha-\beta-\gamma}} \sin \left( \frac{1}{x^\gamma} - \frac{\pi \beta}{2} \right) + O\left( \frac{1}{x} \right),
\]
which means \( \phi' \) explodes like \( x^{-1-\gamma \beta} \) at 0.

**Proof.** For \( 0 < \epsilon < 1/2 \), define
\[
\phi_\epsilon(x) = \int_\epsilon^{1-\epsilon} s^\alpha \sin \left( (xs)^{-\gamma} \right) \frac{ds}{(1-s)^\beta}, \quad x \in (0,T],
\]
then \( \phi_\epsilon \in C^1(0,T] \), and
\[
\phi_\epsilon'(x) = \frac{-\gamma}{x^{1+\gamma}} \int_\epsilon^{1-\epsilon} \frac{\cos \left( (xs)^{-\gamma} \right)}{s^{\gamma-\alpha}(1-s)^\beta} \, ds = \int_{(1-\epsilon)^{-\gamma}}^{\epsilon^{-\gamma}} \frac{-x^{1-\gamma} \cos(x^{-\gamma} r)}{r^{(1+\alpha)/\gamma}(1-r^{-1/\gamma})^\beta} \, dr.
\]
As \( \epsilon \to 0 \), obviously \( \phi_\epsilon \) converges to \( \phi \) uniformly on any closed subinterval of \( (0,T] \). Our next step is to verify \( \phi_\epsilon' \) also converges uniformly on the closed subinterval. After that, we can obtain \( \phi' \) by exchanging limit and differentiation. To this end, we will prove that \( \phi_\epsilon' \) satisfies the Cauchy criterion for uniform convergence.

For \( 0 < \epsilon < \delta < 1/2 \) and \( x \in (0,T] \),
\[
\phi_\epsilon'(x) - \phi_\delta'(x) = \frac{-1}{x^{1+\gamma}} \left( \int_{(1-\epsilon)^{-\gamma}}^{(1-\delta)^{-\gamma}} + \int_{\delta^{-\gamma}}^{\epsilon^{-\gamma}} \right) \frac{\cos(x^{\gamma} r)}{r^{(1+\alpha)/\gamma}(1-r^{-1/\gamma})^\beta} dr.
\]
We can bound the above integral over the first subinterval as follows,
\[
\left| \int_{(1-\varepsilon)\gamma}^{1-\gamma} \frac{\cos(x^{-\gamma} r) dr}{r^{(1+\alpha)/\gamma}(1-r^{-1/\gamma})^\beta} \right| \leq \int_{(1-\varepsilon)\gamma}^{1-\gamma} \frac{r^{-1}(1+\alpha)/\gamma (1-r^{-1/\gamma})^{-\beta} dr}{r^{1+1/\gamma}} \\
= \int_{(1-\varepsilon)\gamma}^{1-\gamma} \frac{r^{1-\alpha/\gamma} (1-r^{-1/\gamma})^{-\beta} dr}{r^{1+1/\gamma}} \\
\leq \int_{(1-\varepsilon)\gamma}^{(1-\delta)-\gamma} \frac{(1-\delta)\gamma}{r^{1+1/\gamma}} (1-r^{-1/\gamma})^{-\beta} dr \\
= \frac{\gamma}{1-\beta} \frac{\delta^{1-\beta} - \varepsilon^{1-\beta}}{(1-\delta)^\gamma}.
\]

For the second subinterval, we have
\[
\left| \int_{\delta\gamma}^{\varepsilon\gamma} \frac{\cos(x^{-\gamma} r) dr}{r^{(1+\alpha)/\gamma}(1-r^{-1/\gamma})^\beta} \right| = \left| \int_{\delta\gamma}^{\varepsilon\gamma} \frac{x^{\gamma} \sin(x^{-\gamma} r) dr}{r^{(1+\alpha)/\gamma}(1-r^{-1/\gamma})^\beta} \right| \\
\leq \left| \frac{x^{\gamma} \sin(x^{-\gamma} r) dr}{r^{(1+\alpha)/\gamma}(1-r^{-1/\gamma})^\beta} \right|_{\delta\gamma}^{\varepsilon\gamma} \\
+ \left| x^{\gamma} \int_{\delta\gamma}^{\varepsilon\gamma} \sin(x^{-\gamma} r) dr \right| \left| (r^{-1}(1+\alpha)/\gamma (1-r^{-1/\gamma})^{-\beta}) \right|,
\]

note that \(r^{-1}(1+\alpha)/\gamma (1-r^{-1/\gamma})^{-\beta}\) is decreasing on \((1, \infty)\), so the second summand can be bounded from above by \(-x^{\gamma} \int_{\delta\gamma}^{\varepsilon\gamma} \sin(x^{-\gamma} r) dr\left[ r^{-1}(1+\alpha)/\gamma (1-r^{-1/\gamma})^{-\beta} \right] = x^{\gamma} \left[ \delta^{1+\alpha}(1-\delta) - \varepsilon^{1+\alpha}(1-\varepsilon) \right]. \)

To sum up, for \(0 < \varepsilon < \delta < 1/2\) and \(x \in (0, T]\), we have
\[
\left| \phi'(x) - \phi'(\delta) \right| \leq \frac{1}{x^{1+\gamma}} \left( \frac{\gamma}{1-\beta} \frac{\delta^{1-\beta} - \varepsilon^{1-\beta}}{(1-\delta)^\gamma} + 2x^{\gamma} \frac{\delta^{1+\alpha}}{1-\beta} \frac{\delta^{1-\beta}}{(1-\delta)^\gamma} \right) \leq \frac{2\gamma}{1-\beta} \frac{\delta^{1-\beta}}{x^{1+\gamma}} + 2^{1+\beta} \frac{\delta^{1+\alpha}}{x},
\]

and thus Cauchy criterion is satisfied on any closed subinterval of \((0, T]\).

So far we have proved \(\phi \in C^1(0, T]\) and
\[
\phi'(x) = \int_{1}^{\infty} \frac{-x^{-1}\gamma \cos(x^{-\gamma} r) dr}{r^{(1+\alpha)/\gamma}(1-r^{-1/\gamma})^\beta}, \quad x \in (0, T]. \tag{A.2}
\]

The only job left is to estimate \(\phi'\) around \(x = 0\). The essential observation is that the denominator in (A.2) behaves like \(\gamma^{-\beta}(r-1)^\beta\) as \(r \to 1\), and that this singularity at \(r = 1\) has the dominating effect on the whole integral, so that we need not worry such behavior is only asymptotic at \(r = 1\). Our strategy is to replace the denominator with \(\gamma^{-\beta}(r-1)^\beta\), which will make our life much easier, and then we only need to show the above replacement will only cause higher order effect. To this end, we need the boundedness of the total variation of \(\eta\) in Lemma A.7. Let us rewrite (A.2) in terms of \(\eta\) and \((r-1)^{-\beta}\),
\[
\phi'(x) = \frac{-1}{x^{1+\gamma}} \int_{1}^{\infty} \left( \eta(r) \frac{\gamma^{\beta}}{(r-1)^{\beta}} \right) \cos(x^{-\gamma} r) dr.
\]
The effect of $\eta$ can be bounded as follows,
\[
\left| \int_1^\infty \eta(r) \cos(x^{-\gamma}r) \, dr \right| = x^\gamma \left| \int_1^\infty \eta(r) \, d\sin(x^{-\gamma}r) \right|
= x^\gamma \left| \eta(r) \sin(x^{-\gamma}r) \right|_1^\infty - \int_1^\infty \sin(x^{-\gamma}r) \, d\eta(r)
= x^\gamma \left| \int_1^\infty \sin(x^{-\gamma}r) \, d\eta(r) \right|
\leq x^\gamma V_1^\infty(\eta),
\]
where the third identity is due to $\eta \in C^\infty$. For $(r - 1)^{-\beta}$, we can evaluate the integral exactly,
\[
\int_1^\infty \frac{\cos(x^{-\gamma}r)}{(r - 1)^\beta} \, dr = \int_0^\infty \frac{\cos(x^{-\gamma}r + x^{-\gamma})}{r^\beta} \, dr
= \cos(x^{-\gamma}) \int_0^\infty \frac{\cos(x^{-\gamma}r)}{r^\beta} \, dr - \sin(x^{-\gamma}) \int_0^\infty \frac{\sin(x^{-\gamma}r)}{r^\beta} \, dr
= x^{\gamma-\beta} \cos(x^{-\gamma}) \int_0^\infty \frac{\cos(r)}{r^\beta} \, dr - x^{\gamma-\beta} \sin(x^{-\gamma}) \int_0^\infty \frac{\sin(r)}{r^\beta} \, dr
= x^{\gamma-\beta} \Gamma(1 - \beta) \left( \cos(x^{-\gamma}) \sin(\frac{\pi \beta}{2}) - \sin(x^{-\gamma}) \cos(\frac{\pi \beta}{2}) \right)
= x^{\gamma-\beta} \Gamma(1 - \beta) \sin(\frac{\pi \beta}{2} - x^{-\gamma}).
\]
In conclusion
\[
\phi'(x) = \gamma \frac{\Gamma(1-\beta)}{x^{1+\gamma \beta}} \sin \left( \frac{1}{x^{\gamma}} - \frac{\pi \beta}{2} \right) + O \left( \frac{1}{x} \right),
\]
where $|O(1/x)| \leq V_1^\infty(\eta)/x$. \hfill \Box

**Lemma A.7.** For $\alpha, \gamma > 0$, denote $\eta(r) = r^{-(1+\alpha)/\gamma} (1 - r^{-1/\gamma})^{-\beta} - \gamma^\beta (r - 1)^{-\beta}$, then $\eta \in C^1(1, \infty)$, $\lim_{r \to 1} \eta(r) = \lim_{r \to \infty} \eta(r) = 0$, and $V_1^\infty(\eta) = \int_1^\infty |\eta'(r)| \, dr < \infty$.

**Proof.** Obviously $\eta \in C^1(1, \infty)$, and $\eta(r) \to 0$ as $r \to \infty$. By Taylor expansion at $r = 1$, we obtain
\[
\eta(r) = \left( \frac{1 + \gamma}{2} \beta - 1 - \alpha \right) \gamma^{-1}(r - 1)^{1-\beta} + O((r - 1)^{-2}).
\]
So we know $\eta$ vanishes at both 1 and $\infty$. Now let us substitute $r$ with $s^{-\gamma}$, i.e., for $s \in (0, 1)$ define $\tilde{\eta}(s) = \eta(s^{-\gamma}) = s^{1+\alpha}(1 - s^{-\gamma})^{-\beta} - \gamma^\beta (s^{-\gamma} - 1)^{-\beta}$. Then $\tilde{\eta} \in C^1(0, 1)$ and
\[
\tilde{\eta}'(s) = s^{\alpha} \frac{1 + \alpha}{(1 - s)^\beta} + \beta \frac{s^{1+\alpha}}{(1 - s)^{1+\beta}} - \beta \frac{s^{1+\beta} s^{-1-\gamma}}{(s^{-\gamma} - 1)^{1+\beta}}.
\]
On $(0, 1/2)$, we can bound $|\tilde{\eta}'|$ by an integrable function,
\[
|\tilde{\eta}'(s)| \leq \frac{1 + \alpha + \beta}{2^{\alpha-\beta}} + \beta \frac{s^{\gamma+\beta-1}}{(1 - s^{-\gamma})^{1+\beta}}, \quad s \in (0, \frac{1}{2}).
\]
By Taylor expansion at $s = 1$, we obtain
\[
\eta'(s) = \frac{2(\alpha + 1) - \beta(\gamma + 1)}{2}(1 - \beta)(1 - s)^{-\beta} \\
+ \frac{\beta(\gamma + 1)(3\beta(\gamma + 1) - 5) - 12\alpha(\alpha + 1)}{24}(2 - \beta)(1 - s)^{1 - \beta} + O((1 - s)^{2 - \beta}).
\]
Note that $\eta' \in C(0, 1)$, we know $\eta'$ is absolutely integrable on $(0, 1)$ according to the above estimates. Therefore $V^1_0(\eta) = \int_0^1 |\eta'(s)|\, ds$ is a finite number which only depends on $\alpha, \beta, \gamma$. Since the change of variables from $r$ to $s$ is monotone, $\eta$ and $\tilde{\eta}$ have the same total variation which is bounded. \hfill \Box

APPENDIX B. UNIFORM BOUNDS FROM MAXIMUM PRINCIPLE ARGUMENTS

B.1. Bounds of Caputo relaxation solutions (Mittag-Leffler and Kilbas–Saigo functions). We define the Caputo derivative as $D^\beta_0 u := D^\beta_0 [u - u(0)]$.

**Proposition B.1.** For $\lambda < 0$ and $u_0 = 1$, the series (3.13), which solves $D^\beta_0 u = \lambda u$, has the following bounds for all $x \geq 0$,
\[
\frac{1}{1 + |\lambda|\Gamma(1 - \beta)x^\beta} \leq u(x) \leq \frac{1}{1 + |\lambda|\Gamma(1 + \beta)^{-1}x^\beta}.
\]

**Proof.** Obviously the $u$ defined by (3.13) is in $C^1(0, \infty) \cap C[0, \infty)$, so the Caputo derivative of $u$ allows the representation
\[
D^\beta_0 u(x) = \int_0^x (u(x) - u(x - r)) \frac{r^{-1 - \beta}}{\Gamma(-\beta)} \, dr + \frac{u(x) - u(0)}{x^\beta\Gamma(-\beta)}, \quad x > 0. \tag{B.1}
\]

Then Proposition B.1 is immediate from Lemma B.2 and Lemma B.3 below. \hfill \Box

**Lemma B.2.** The solution $u$ in Proposition B.1 stays positive and is bounded from above by $u(x) = (1 + c|\lambda|x^\beta)^{-1}$, where $c = \Gamma(1 + \beta)^{-1}$.

**Proof.** The positivity of $u$ can be proven similarly to that in Lemma 3.20. As for its upper bound, analogously we let $v(x) = (x^\beta + c)^{-1}$. Now, given $\beta \in (0, 1)$ and $\lambda < 0$, it remains to find a $c > 0$, such that $D^\beta_0 v \geq \lambda v$, i.e., for all $x > 0$,
\[
\int_0^x (v(x) - v(x - r)) \frac{r^{-1 - \beta}}{\Gamma(-\beta)} \, dr + \frac{v(x) - v(0)}{x^\beta\Gamma(-\beta)} \geq \lambda v(x),
\]
or equivalently, for all $x > 0$,
\[
|\lambda\Gamma(-\beta)| \geq \int_0^x \left( \frac{x^\beta + c}{(x - r)^\beta + c} - 1 \right) \frac{dr}{r^{1+\beta}} + \beta^{-1}x^{-\beta}\left( \frac{x^\beta + c}{c} - 1 \right), \tag{B.2}
\]
where the first summand can be bounded from above by
\[
c^{-1}\int_0^x (x^\beta - (x - r)^\beta) \frac{dr}{r^{1+\beta}} = c^{-1}|\lambda\Gamma(-\beta)|\Gamma(1 + \beta) - \Gamma(1 - \beta)^{-1}.
\]
In order to fulfil (B.2), we only need to require $c$ to satisfy $|\lambda\Gamma(-\beta)| \geq c^{-1}|\lambda\Gamma(-\beta)|\Gamma(1 + \beta)$, so we can choose $c = \Gamma(1 + \beta)/|\lambda|$. \hfill \Box
Lemma B.3. The solution \( u \) in Proposition B.1 is bounded from below by \( w(x) = (1 + d|x^\beta|)^{-1} \), where \( d = \Gamma(1 - \beta) \).

Proof. Similarly to the proof of Lemma B.2, given \( \beta \in (0,1) \) and \( \lambda < 0 \), it remains to find a \( d > 0 \) such that \( D_{x_0}^\beta w \leq \lambda w \), where \( w(x) = (x^\beta + d)^{-1} \), i.e., for all \( x > 0 \),

\[
\int_0^x (w(x) - w(x-r)) \frac{r^{-1-\beta}}{|\Gamma(-\beta)|} \, dr + \frac{w(x) - w(0)}{x^\beta |\Gamma(-\beta)|} \leq \lambda w(x),
\]

or equivalently, for all \( x > 0 \),

\[
|\lambda \Gamma(-\beta)| \leq \int_0^x \left( \frac{w(x-r)}{w(x)} - 1 \right) \frac{dr}{r^{1+\beta}} + \beta^{-1} x^{-\beta} \left( \frac{w(0)}{w(x)} - 1 \right).
\]

Note that the first summand is nonnegative, and the second summand equals \( \beta^{-1} x^{-\beta} ((x^\beta + d)/d - 1) \) and thus \( (d\beta)^{-1} \), so we only need choose such \( d \) that \( |\lambda \Gamma(-\beta)| \leq (d\beta)^{-1} \), i.e.,

\[
d \leq |\lambda \beta \Gamma(-\beta)|^{-1} = |\lambda \Gamma(1-\beta)|^{-1}.
\]

Proposition B.4. For \( \lambda < 0 \), \( u_0 = 1 \) and \( \alpha > 0 \), the series (3.21), which solves \( D_{x_0}^\beta u = \lambda x^{\alpha-\beta} u \), has the following bounds for all \( x \geq 0 \),

\[
\frac{1}{1 + |\lambda\Gamma(1-\beta)x^{\alpha}|} \leq u(x) \leq \frac{1}{1 + |\lambda\Gamma(1+\alpha-\beta)\Gamma(1+\alpha)^{-1}x^{\alpha}|}.
\]

Note that these two bounds are identical to those in [10, Proposition 4.12] (To see this, rewrite \( u(x) \) as \( E_{\beta,\alpha|\beta,\alpha|}(\lambda x^{\alpha}) \) following their notation \( E_{\cdots}(\cdots) \) which is the Kilbas–Saigo function). The proof is parallel to that of Proposition B.1, so we omit it. The only difference is that we now need the criterion \( D_{x_0}^\beta v \geq \lambda x^{\alpha-\beta} v \) to find \( v \) of the form \((1 + cx^{\alpha})^{-1}\) as the upper bound (the lower bound is analogous).

B.2. Bounds of censored relaxation solutions.

Proof. [of Lemma 3.21] According to Remark 2.7-(i), we only need to consider a particular \( \lambda \). Take \( \lambda = -d^{-\beta} \), and our job is to prove that the solution is bounded from below by \( w(x) = (1 + x)^{-1}(1 + x^\beta)^{-1} \), which is convex on \([0,\infty)\). Similar to the proof of Lemma 3.20, it remains to prove \( \partial_x^\beta w \leq \lambda w \), i.e., for all \( x > 0 \),

\[
\int_0^x (w(x) - w(x-r)) \frac{r^{-1-\beta}}{|\Gamma(-\beta)|} \, dr \leq \lambda w(x),
\]

or equivalently, for all \( x > 0 \),

\[
|\lambda \Gamma(-\beta)| \leq \int_0^x \frac{dr}{r^{1+\beta}} \left( \frac{w(x-r)}{w(x)} - 1 \right).
\]

For \( x > 2 \), the right-hand side of (B.3) can be bounded from below by

\[
\int_{x-1}^x \frac{dr}{r^{1+\beta}} \left( \frac{w(x-r)}{w(x)} - 1 \right) \geq \frac{1}{x^{1+\beta}} \left( \frac{w(1)}{w(x)} - 1 \right) \geq \frac{1}{x^{1+\beta}} \left( \frac{1 + x^\beta + x + x^{1+\beta}}{4} - 1 \right) \geq \frac{1}{4}.
\]
For $0 < x < 2$, since $w$ is convex, $w(x - r)$ is bounded from below by $w(x) - w'(x)r$, so
\[ \frac{w(x - r)}{w(x)} - 1 \geq r \left( \frac{1}{1 + x} + \frac{\beta x^{\beta - 1}}{1 + x^{\beta}} \right), \]
therefore the right-hand side of (B.3) can be bounded from below by
\[ \int_0^x \frac{dr}{r^\beta} \left( \frac{1}{1 + x} + \frac{\beta x^{\beta - 1}}{1 + x^{\beta}} \right) \geq \frac{x^{1-\beta}}{1 - \beta} \frac{\beta x^{\beta - 1}}{1 + x^{\beta}} \geq \frac{\beta}{(1 - \beta)(1 + 2^\beta)}. \]

Now we only need verify $\lambda = -d^{-\beta}$ satisfies $|\lambda \Gamma(\beta)| \leq \min \{1/4, (1 - \beta)^{-1}/(1 + 2^\beta)\}$, which is obvious by the definition of $d$.

**Proof.** [of Lemma 3.26] The positivity of $u$ can be proven similarly to that in Lemma 3.20. As for its upper bound, analogously we let $u(x) = 1/(x^{1+\alpha} + c)$. Now, given $\beta \in (0, 1), \alpha > 0$ and $\lambda < 0$, it remains to find a $c > 0$, such that $\partial_x^\beta u \geq \lambda x^{\alpha-\beta} u$, i.e., for all $x > 0$,
\[ \int_0^x (v(x) - v(x - r)) \frac{r^{-1-\beta}}{\Gamma(-\beta)} \frac{dr}{r^{1+\beta}} \geq \lambda x^{\alpha-\beta} v(x), \]
or equivalently, for all $x > 0$,
\[
|\lambda \Gamma(-\beta)| \geq x^{-\alpha} \int_0^x \frac{x^{1+\alpha} - (x-r)^{1+\alpha}}{c + (x-r)^{1+\alpha}} \cdot \frac{dr}{r^{1+\beta}} = x^{-\alpha} \int_0^1 \frac{1 - s^{1+\alpha}}{c/x^{1+\alpha} + s^{1+\alpha}} \cdot \frac{ds}{(1 - s)^{1+\beta}}, \tag{B.4}
\]
where $s = 1 - r/x$. To obtain an upper bound of the right-hand side of (B.4), we split the interval $[0, 1]$ into two halves. For the first half interval, we have
\[
\int_0^{1/2} \frac{1 - s^{1+\alpha}}{c/x^{1+\alpha} + s^{1+\alpha}} \cdot \frac{ds}{(1 - s)^{1+\beta}} \leq 2^{1+\beta} \int_0^{1/2} \frac{ds}{c/x^{1+\alpha} + s^{1+\alpha}} \leq 2^{1+\beta+\alpha} \frac{\alpha^{(1+\alpha)/(1+\alpha)}}{\alpha^{(1+\alpha)/(1+\alpha)}} \int_0^{1/2} \frac{ds}{(c^{1+\alpha})^{-1}/x^{1+\alpha}} \leq 2^{1+\beta+\alpha} \frac{x^{\alpha}}{\alpha^{(1+\alpha)/(1+\alpha)}},
\]
where the second inequality is due to Jensen’s inequality $t^{1+\alpha} + s^{1+\alpha} \geq 2^{-\alpha}(t + s)^{1+\alpha}$ for $t, s, \alpha > 0$. For the second half interval, we have
\[
\int_{1/2}^1 \frac{1 - s^{1+\alpha}}{c/x^{1+\alpha} + s^{1+\alpha}} \cdot \frac{ds}{(1 - s)^{1+\beta}} \leq \frac{\alpha^{(1+\alpha)/(1+\alpha)}}{1 + \alpha} \int_{1/2}^1 \frac{2x^{\alpha}}{\alpha^{(1+\alpha)}} \cdot \frac{ds}{(1 - s)^{1+\beta}} = \frac{\alpha^{(1+\alpha)/(1+\alpha)}}{1 + \alpha} \cdot \frac{2^\beta x^{\alpha}}{1 - \beta} \cdot \frac{1}{\alpha^{(1+\alpha)/(1+\alpha)}}.
\]
where the first inequality is due to Young’s inequality for products: $a^\gamma b^{1-\gamma} \leq \gamma a + (1-\gamma)b$ with $\gamma = \alpha/(1+\alpha)$, $a = c(1+\alpha^{-1})/x^{1+\alpha}$ and $b = (1+\alpha)s^{1+\alpha}$; the second inequality is due to the concavity of $1 - s^{1+\alpha}$ whose tangent at $s = 1$ is $(1+\alpha)(1-s)$.

Therefore the right-hand side of (B.4) can be bounded from above by

$$
\frac{2^{\frac{\beta}{\alpha}}}{c^{\alpha/(1+\alpha)}} \left( \frac{\alpha/(1+\alpha)}{\alpha} \right) \left( \frac{\alpha/(1+\alpha)}{1-\beta} \right) .
$$

Let

$$
c \geq \left( \frac{2^{\frac{\beta}{\alpha}}}{\alpha^{\alpha/(1+\alpha)}} \left( \frac{\alpha/(1+\alpha)}{1-\beta} \right) \right)^{1+1/\alpha},
$$

then (B.4) will be satisfied and we are done.

\[\square\]

**Appendix C. Derivation of $E_x[(\tau_j)^n]$ in Remark 4.14-(1)**

By (4.4) and the multinomial formula we have

$$
E_x[(\tau_j)^n] = E_x \left[ \left( \sum_{i=1}^{j} E_i(S_{\tau_{i-1}}^c) \right)^n \right] = \sum_{n_1 + \ldots + n_j = n} \frac{n!}{n_1! \ldots n_j!} E_x \left[ \prod_{i=1}^{j} E_i(S_{\tau_{i-1}}^c)^{n_i} \right], \quad n \in \mathbb{N},
$$

where the right-hand side can be calculated using Lemma C.1.

**Lemma C.1.** For $\{n_i\}_{i=1}^{j} \subseteq \mathbb{N} \cup \{0\}$ and $n = n_1 + n_2 + \ldots + n_j (j \in \mathbb{N})$, it holds that

$$
E_x \left[ \prod_{i=1}^{j} E_i(S_{\tau_{i-1}}^c)^{n_i} \right] = \frac{x^{\beta n} \prod_{i=1}^{j} \Gamma(n_i + 1)}{\Gamma(\beta n_j + 1) \Gamma(1-\beta)^{j-1}} \prod_{i=1}^{j-1} \frac{\Gamma(1-\beta + \beta \sum_{i=0}^{j-1} n_j-i)}{\Gamma(1 + \beta \sum_{i=0}^{j-1} n_j-i)}. \tag{C.1}
$$

**Proof.** By Lemma C.2 (with $m = 0$), the left-hand side of (C.1) equals

$$
E_x \left[ E_j(S_{\tau_{j-1}}^c)^{n_j} \prod_{i=1}^{j-1} E_i(S_{\tau_{i-1}}^c)^{n_i} \right] = E_x \left[ E_j(S_{\tau_{j-1}}^c)^{\beta n_j} \prod_{i=1}^{j-1} E_i(S_{\tau_{i-1}}^c)^{n_i} \right] \Gamma(n_j + 1) \Gamma(\beta n_j + 1). \tag{C.2}
$$

We now prove that the right-hand sides of (C.1) and (C.2) are equal, by induction on $j$. When $j = 1$, it is obvious; when $j \geq 2$, suppose it is true for $j - 1$, then we have

The right-hand side of (C.2)

\[
\begin{align*}
&= E_x \left[ \left( S_{\tau_{j-2}}^c + S_{E_{j-1}(S_{\tau_{j-2}}^c)}^{j-1} \right)^{\beta n_j} E_{j-1}(S_{\tau_{j-2}}^c)^{n_{j-1}} \prod_{i=1}^{j-2} E_i(S_{\tau_{i-1}}^c)^{n_i} \right] \Gamma(n_j + 1) \Gamma(\beta n_j + 1) \\
&= E_x \left[ \left( S_{\tau_{j-2}}^c + S_{E_{j-1}(S_{\tau_{j-2}}^c)}^{j-1} \right)^{\beta n_j} E_{j-1}(S_{\tau_{j-2}}^c)^{n_{j-1}} \prod_{i=1}^{j-2} E_i(S_{\tau_{i-1}}^c)^{n_i} \right] \Gamma(\beta n_j + 1) \\
&= E_x \left[ \Gamma(n_{j-1} + 1) \Gamma(\beta n_j - \beta + 1) (S_{\tau_{j-1}}^c)^{\beta n_{j-1} + \beta n_j} \prod_{i=1}^{j-2} E_i(S_{\tau_{i-1}}^c)^{n_i} \right] \Gamma(n_j + 1) \Gamma(\beta n_j + 1)
\end{align*}
\]
\[= \frac{\Gamma(n_j-1+1) \Gamma(\beta n - \beta + 1) \Gamma(n_j + 1)}{\Gamma(1 - \beta) \Gamma(n_j-1+n_j+1) \Gamma(\beta n_j+1)} \mathbb{E}_x \left[ \left( S_{\hat{t}_{j-2}} \right)^{\beta n_j-\beta n_j} \prod_{i=1}^{j-2} \mathbb{E}_i \left( \left( S_{\hat{t}_{i-1}} \right)^{\beta n} \right) \right] \frac{\Gamma(n_j-1+n_j+1)}{\Gamma(\beta n_j+1+\beta n_j+1)} \]

The right-hand side of (C.1),

where the third equality is due to Lemma C.2, the equality in law between \(S^1\) and \(S^{\beta^j} - 1\), as well as the independence of \(S^{\beta^j-1}\) and \(\{S^i_t\}_{t \in [0, \tau_{j-2}]}\); the last equality is due to the induction hypothesis.

\[\square\]

**Lemma C.2.** For all \(n \in \mathbb{N} \cup \{0\}\) and \(\alpha \geq 0\), we have

\[\mathbb{E} \left[ E_1(x)^n \left( x + S^1_{E_1(x)-} \right)^\alpha \right] = \frac{\Gamma(n+1) \Gamma(\alpha - \beta + 1)}{\Gamma(1 - \beta) \Gamma(\alpha + n \beta + 1)} x^{\beta n + \alpha}. \tag{C.3}\]

**Proof.** [suggested by Andreas Kyprianou and Jean Bertoin] Let \(p_s\) be the density of the \(\beta\)-stable subordinator \(-S^1\) at time \(s > 0\), and recall its Mellin transform for each \(n \in \mathbb{N}\),

\[\int_0^\infty s^{n-1} \frac{p_s(x)}{\Gamma(n)} \, ds = \frac{x^{n\beta-1}}{\Gamma(\beta n)}, \quad x > 0. \tag{C.4}\]

Applying the compensation formula as in the proof of [4, Proposition III.2-(i)] with \(f(x) = x^n\) and \(g(x) = x^\alpha\), we obtain

\[\mathbb{E} \left[ f(E_1(x)) g(x + S^1_{E_1(x)-}) \right] = \mathbb{E} \left[ \sum_{l \geq 0} f(t) g(x + S^1_{-l}) I_{\{ -S^1_{l} < x \}} I_{\{ -\Delta S_{l} > x + S^1_{l} \}} \right] \]

\[= \int_0^\infty \mathbb{E} \left[ f(t) g(x + S^1_{-l}) I_{\{ -S^1_{l} < x \}} \int_{x+S^1_{l}}^{\infty} \frac{z^{-1-\beta}}{|\Gamma(-\beta)|} \, dz \right] \, dt \]

\[= \int_0^x \int_0^x f(t) g(x-y) p_t(y) \int_{x-y}^{\infty} \frac{z^{-1-\beta}}{|\Gamma(-\beta)|} \, dz \, dy \, dt \]

\[= \int_0^x (x-y)^{\alpha} \frac{(x-y)^{-\beta}}{|\Gamma(1 - \beta)|} \int_0^\infty t^n \, p_t(y) \, dy \, dt \]

\[= \int_0^x (x-y)^{\alpha-\beta} \frac{\Gamma(n+1)}{\Gamma(1 - \beta)} \frac{y^{n\beta+\beta-1}}{\Gamma(n\beta + \beta)} \, dy \]

\[= x^{\beta n + \alpha} \frac{\Gamma(\alpha - \beta + 1)}{\Gamma(\alpha + n \beta + 1)} \frac{\Gamma(n+1)}{\Gamma(1 - \beta)}. \]

\[\square\]

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