Massless picture, massive picture, and symmetry in the Gaussian renormalization group

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Abstract
We consider renormalization groups of transformations composed of a Gaussian convolution and a field dilatation. As an example, we consider perturbations of a single component real Euclidean free field $\phi$ with covariance $(-\triangle)^{-1+\frac{2}{\epsilon}}$. We show that the renormalization group admits two equivalent formulations called massless picture and massive picture respectively. We then show in the massive picture that the renormalization group has a symmetry. The symmetry consists of global scale transformations composed with certain Gaussian convolutions. We translate the symmetry back to the massless picture. The relation between the symmetry and the notion of an anomalous dimension is briefly discussed.
1 Introduction

We consider renormalization groups \cite{1} of transformations, which are composed of a Gaussian convolution and a field dilatation. Transformations of this kind \cite{2} are tailor made to study perturbations of free fields. Specifically, we consider effective field theories given by measures

\[d\rho(\phi) = \frac{d\mu_{C_\infty}(\phi) Z(\phi)}{\int d\mu_{C_\infty}(\phi) Z(\phi)}\]  \hspace{1cm} (1)

on field space \(\{\phi : \mathbb{R}^D \rightarrow \mathbb{R}\}\), where \(d\mu_{C_\infty}(\phi)\) is the Gaussian measure with covariance \(C_\infty\), and where \(Z(\phi)\) is a perturbation, which is normalized such that \(Z(0) = 1\). We consider perturbations of a massless free field with \(C_\infty(x, y)\) of the order \(O(\left|x - y\right|^{2-D-\epsilon})\) at \(\left|x - y\right| \rightarrow \infty\). We use a unit ultraviolet cutoff, built in as a smooth momentum space regulator, whereby \(C_\infty(x, y)\) becomes a regular function of \(\left|x - y\right|\) at short distances.

The following renormalization group \cite{2} is custom built to study the properties of (1). Let \(D_L\) be the dilatation operator given by

\[D^T_L f(x) = L^{1-D+\epsilon} f\left(\frac{x}{L}\right)\].

We decompose \(C_\infty\) into \(C_\infty = D^T_L C_\infty D_L + C_L\) such that \(C_L(x, y)\) is of the order \(O\left(F\left(\frac{|x-y|}{L}\right)\right)\) at \(\left|x - y\right| \rightarrow \infty\) with a rapidly decreasing function \(F(t)\). (We will choose an exponential cutoff where \(F(t) = e^{-t^2}\).) For \(C_L(x, y)\) to be short ranged, the asymptotic of \(C_\infty(x, y)\) and its scaled version \((D^T_L C_\infty D_L)(x, y) = L^{2-D-\epsilon} C\left(\frac{x}{L}, \frac{y}{L}\right)\) have to agree. This condition determines the scaling dimension in the dilatation operator \(D_L\). We cannot decompose \(C_\infty\) with an anomalous dilatation, unless we give up that \(C_L\) be short ranged. At this instant, we have a pair of operators \((C_L, D_L)\), where \(C_L\) is a short ranged covariance, and where \(D_L\) is a dilatation. With this pair \((C_L, D_L)\) is associated a renormalization group transformation

\[R(C_L, D_L)(Z)(\Phi) = \frac{\int d\mu_{C_L}(\phi) Z(D^T_L(\Phi) + \phi)}{\int d\mu_{C_L}(\phi) Z(\phi)}\]  \hspace{1cm} (2)

of the perturbation \(Z(\phi)\). It defines a representation of the semi-group of scale transformations with scale factor \(L > 1\) on a suitable space of perturbations \(Z(\phi)\). (For \(L < 1\), \(C_L\) becomes negative.) Transformations of this kind have a wide range of applications. See \cite{3} for a recent review.

We will show that the transformation given by \((C_L, D_L)\) is equivalent to the transformation given by another pair \((c_L, d_L)\). The equivalence is of the form

\[Z(\phi) \xleftarrow{Q(K_L)} Z(\phi) \xrightarrow{Q(K_L)} z(\phi) \quad R(C_L, D_L) \quad \downarrow \quad R(c_L, d_L) \quad \downarrow \]

\[Z(\phi|L) \xleftarrow{Q(K_L)} z(\phi|L) \]  \hspace{1cm} (3)
where $Q(K_\star)(z)(\phi) = Z_\star(\phi) z(\phi)$ and $Z_\star(\phi) = e^{-\frac{1}{2}(\phi,K^{-1}\phi)}$ is a short ranged Gaussian fixed point of $R_L(C_L,D_L)$. The new pair $(c_L,d_L)$ defines a scale decomposition $c_\infty = d_L^T c_\infty d_L + c_L$ of a covariance $c_\infty$. The two formulations will be called the massless picture and the massive picture respectively. They are well known in the hierarchical (or ultra-local) approximation \[4\].

In the massive picture, both $c_L$ and $c_\infty$ are short ranged. Therefore, the scaling dimension of $d_L$ is variable. In the massive picture, we will show the transformation defined by yet another pair $(\gamma_L,\delta_L)$, which defines a global (space-time independent) scale decomposition of $c_\infty$, with $\delta_L$ the multiplication by a global factor $L^{-1}$, is a symmetry. Renormalization group and symmetry together spread a perturbation $z(\phi)$ to a surface

$$z(\phi) \xrightarrow{R(\gamma_L',\delta_L')} z(\phi|1',L') \xrightarrow{R(c_L,d_L)} z(\phi|L',L)$$

We will then use the equivalence to translate the symmetry back to the massless picture. The composition of a renormalization group transformation, which scales by a factor $L$, and a symmetry transformation, which scales by a factor $L^{\frac{\eta}{2}}$, defines an anomalous scale decomposition of $c_\infty$.

Inspired by the construction of a non-trivial $\phi^4_4,\epsilon$ fixed point in the planar approximation \[3\] and beyond it \[1\], we consider the renormalization group given by $C_\infty = F(-\Delta,\epsilon) (-\Delta)^{-1+\frac{\eta}{2}}$, where $\epsilon$ is a small parameter, and where $F(-\Delta,\epsilon)$ is a certain cutoff function. We will argue that the two cases $\epsilon = 0$ and $\epsilon \neq 0$ are different. We will briefly discuss the relation between the anomalous dimension $\eta$ and the symmetry applying the concepts of \[7\] to this case. The anomalous dimension of the $\phi^4_4,\epsilon$ fixed point is by definition zero.

## 2 Renormalization group – massless picture

We consider renormalization groups of transformations $R(C_L,D_L)$, defined by a pair of operators $(C_L,D_L)$, where $C_L$ is a covariance, $D_L$ is a dilatation, and $L$ is a scale, with $L > 1$. Our starting point and setup is a particular pair $(C_L,D_L)$, which defines the renormalization group in the massless picture.
2.1 Covariance $C_L$

Let $C_L$ be the covariance operator $C_L(f)(x) = \int d^D y \ C_L(x,y) \ f(y)$ given by

$$C_L(x,y) = \int \frac{d^D p}{(2\pi)^D} \ e^{ip(x-y)} \ \hat{C}_L(p),$$  \hspace{1cm} (5)

$$\hat{C}_L(p) = \frac{\hat{\chi}(p) - \hat{\chi}(Lp)}{(p^2)^{1+\frac{D}{2}}},$$ \hspace{1cm} (6)

where $\hat{\chi}(p)$ is a cutoff function, and $\epsilon$ is a small parameter (possibly zero). Let $\hat{\chi}(p) = F(p^2)$, where $F(s)$ is a monotonously decreasing function, which takes values between $F(0) = 1$ and $\lim_{s \to \infty} F(s) = 0$. We choose the incomplete $\Gamma$-function

$$\hat{\chi}(p) = \frac{\int p^2 \frac{da}{a} \alpha^{1-\frac{D}{2}} e^{-\alpha}}{\Gamma (1-\frac{D}{2})} = \frac{\Gamma (1 - \frac{D}{2}, p^2)}{\Gamma (1-\frac{D}{2})}.$$ \hspace{1cm} (7)

For $\epsilon = 0$, it becomes $\hat{\chi}(p) = e^{-p^2}$. The advantages of this particular cutoff function are apparent from the integral representations

$$\hat{C}_L(p) = \frac{\int p^2 \frac{da}{a} \alpha^{1-\frac{D}{2}} e^{-\alpha p^2}}{\Gamma (1-\frac{D}{2})},$$ \hspace{1cm} (8)

$$C_L(x,y) = \frac{\int_{L-2}^{L} \frac{da}{a} \alpha^{D/2-1} e^{-\alpha (\frac{x-y}{2})^2}}{(4\pi)^{\frac{D}{2}} \Gamma (1-\frac{D}{2})}.$$ \hspace{1cm} (9)

The covariance $C_L$ is a two sided regularization of $(-\triangle)^{-1+\frac{D}{2}}$, with unit ultraviolet cutoff and infrared cutoff $L^{-1}$ (in units of mass). We included a parameter $\epsilon$ for the sake of generality, and to distinguish it from another parameter, the anomalous dimension $\eta$ to appear below. We will assume that $D + \epsilon > 2$.

2.2 Dilatation $D_L$

When the infrared cutoff is removed, we obtain a massless covariance $C_\infty = \lim_{L \to \infty} C_L$ with unit ultraviolet cutoff. In this limit, (8) and (9) become

$$\hat{C}_\infty(p) = \frac{\Gamma (1 - \frac{D}{2}, p^2)}{\Gamma (1-\frac{D}{2})} \left( \frac{D+\epsilon}{2} - 1 \right),$$ \hspace{1cm} (10)

$$C_\infty(x,y) = \frac{\gamma \left( \frac{D+\epsilon}{2} - 1, \left( \frac{x-y}{2} \right)^2 \right)}{(4\pi)^{\frac{D}{2}} \Gamma (1-\frac{D}{2}) \left( \left( \frac{x-y}{2} \right)^2 \right)^{\frac{D+\epsilon}{2}}}.$$ \hspace{1cm} (11)
In other words, $C_\infty = \chi/(-\triangle)^{1-\frac{D}{2}}$. Conversely, $C_L$ is the difference of $C_\infty$ and a scaled version of $C_\infty$. We have that

$$C_L(p) = C_\infty(p) - L^{2-\epsilon} C_\infty(Lp), \quad (12)$$

$$C_L(x,y) = C_\infty(x,y) - L^{2-D-\epsilon} C_\infty \left( \frac{x}{L}, \frac{y}{L} \right). \quad (13)$$

Eqs. (12) and (13) read $C_L = C_\infty - D^T_L C_\infty D_L$ in operator notation, where $D_L$ is the dilatation operator, and where $D^T_L$ is the transposed of $D_L$, given by

$$D^T_L(f)(x) = L^{1-\frac{D-\epsilon}{2}} f \left( \frac{x}{L} \right), \quad (14)$$

$$D_L(f)(x) = L^{1+\frac{D-\epsilon}{2}} f(Lx) \quad (15)$$

respectively. Thus $D_L$ represents a dilatation by the scale factor $L$ on function space $\{f : \mathbb{R}^D \rightarrow \mathbb{R}\}$, with scaling dimension $\sigma = 1 + \frac{D-\epsilon}{2}$. The dilatations by scale factors $L > 1$ form a semi-group.

To summarize, we have a pair of operators $(C_L, D_L)$, where $C_L$ is a short ranged covariance, $D_L$ is a dilatation, and $L$ is a scale such that $D_1 = 1$, $D_L D_L' = D_{LL'}$, and $C_L = C_\infty - D^T_L C_\infty D_L$. We say that $(C_L, D_L)$ defines a scale decomposition of $C_\infty$ with respect to $D_L$.

### 2.3 Transformation $R(C_L, D_L)$

With the pair $(C_L, D_L)$ is associated a renormalization group transformation $R(C_L, D_L)$, defined by

$$R(C_L, D_L)(Z)(\Phi) = \int \frac{d\mu_{C_L}(\phi) Z(D^T_L(\Phi) + \phi)}{d\mu_{C_L}(\phi) Z(\phi)}, \quad (16)$$

where $d\mu_{C_L}(\phi)$ is the Gaussian measure with mean zero and covariance $C_L$, and $Z(\phi)$ is a perturbation, which is normalized such that $Z(0) = 1$. Recall that

$$\int d\mu_{C_L}(\phi) e^{(\phi,f)} = e^{\frac{1}{2}(f,C_L f)}. \quad (17)$$

### 2.3.1 Averaged perturbation $Z(\Phi)$

Admit a brief detour around the physical meaning of (16). We consider effective quantum field theories defined by the covariance $C_\infty$ and an interaction $V(\phi) = -\log Z(\phi)$. Their physical content is coded in the generating function

$$G(f) = \int \frac{d\mu_{C_\infty}(\phi) Z(\phi) e^{(\phi,f)}}{d\mu_{C_\infty}(\phi) Z(\phi)} = e^{-\frac{1}{2}(f,C_\infty f)} Z(C_\infty(f)), \quad (18)$$
where $Z(\Phi)$ is the convolution of $Z(\phi)$ with $d\mu_{C_\infty}(\phi)$, divided by a normalization constant. Its explicit form is

$$Z(\Phi) = \frac{\int d\mu_{C_\infty}(\phi) \ Z(\Phi + \phi) \ Z}{\int d\mu_{C_\infty}(\phi) \ Z(\phi)}.$$  \hspace{1cm} (19)

The transformation $R(C_L, D_L)$ performs a dilatation of $Z(\Phi)$. We have that

$$Z(D_T L(\Phi)) = \frac{\int d\mu_{C_\infty}(\phi) \ R(C_L, D_L)(Z)(\Phi + \phi) \ Z}{\int d\mu_{C_\infty}(\phi) \ R(C_L, D_L)(Z)(\phi)}.$$  \hspace{1cm} (20)

The transformation $R(C_L, D_L)$ thus enables a study of the behavior of $Z(\Phi)$ under $D_T L$, without having to perform the convolution with respect to $d\mu_{C_\infty}(\phi)$, which is difficult because $C_\infty$ is long range.

If $R(C_L, D_L)(Z_*)(\Phi) = Z_*(\Phi)$ then $Z(D_T L(\Phi)) = Z(\Phi)$. In other words, if we have a fixed point of the renormalization group, then we have dilatation invariant correlators.

### 2.3.2 Semi-group property of $R(C_L, D_L)$

We are led to consider the following structure. We have a pair of operators $(A, B)$ (with certain properties). With this pair, we associate a functional transformation

$$R(A, B)(Z)(\Phi) = \frac{\int d\mu_A(\phi) \ Z(B^T(\Phi) + \phi)}{\int d\mu_A(\phi) \ Z(\phi)}.$$  \hspace{1cm} (21)

The particular pair $(A, B) = (0, 1)$ gives the identity transformation. Since $(C_1, D_1) = (0, 1)$, we have that $R(C_1, D_1) = 1$. The composition of two transformations (21) is another transformation of this kind. Let $(A, B)$ and $(A', B')$ be two pairs of operators. An standard computation of Gaussian integrals yields

$$R(A, B) R(A', B') = R(B'^T A B + A', B B').$$  \hspace{1cm} (22)

Eq. (22) can be thought of as a product of $(A, B)$ and $(A', B')$, given $(A, B)(A', B') = (B'^T A B + A', B B')$. From the two properties $D_L D_{L'} = D_{L L'}$ and $C_L = C_\infty - D_{L}^T C_\infty D_L$, it follows that $(C_L, D_L)$ $(C_{L'}, D_{L'}) = (C_{L L'}, D_{L L'})$. Therefore, the transformation $R(C_L, D_L)$ satisfies the semi-group property

$$R(C_L, D_L) R(C_{L'}, D_{L'}) = R(C_{L L'}, D_{L L'}).$$  \hspace{1cm} (23)

### 2.4 Generator $\dot{R}(\dot{C}, \dot{D})$

The transformation $R(C_L, D_L)$ associates with a given perturbation $Z(\phi)$ a renormalization group trajectory

$$Z(\phi|L) = R(C_L, D_L)(Z)(\phi).$$  \hspace{1cm} (24)
Due to the semi-group property, $Z(\phi|L)$ is the solution of the discrete flow equation

$$R(C_L, D_L)(Z(\phi|L')) = Z(\phi|L')$$

(25)

to the initial condition $Z(\phi|1) = Z(\phi)$. The discrete flow equation implies a continuous flow equation for $Z(\phi|L)$, the renormalization group differential equation

$$L \frac{\partial}{\partial L} Z(\phi|L) = \hat{R}(\hat{C}, \hat{D})(Z(\phi|L)),$$

(26)

where $\hat{R}(\hat{C}, \hat{D}) = \frac{\partial}{\partial L} |_{L=1} R(C_L, D_L)$ is the generator of the renormalization group. Its explicit form is

$$L \frac{\partial}{\partial L} Z(\phi|L) = \left[ \frac{1}{2} \left( \frac{\delta}{\delta \phi}, \hat{C} \frac{\delta}{\delta \phi} \right) + \left( \hat{D}^T \phi, \frac{\delta}{\delta \phi} \right) \right] Z(\phi|L)$$

$$- Z(\phi|L) \left[ \frac{1}{2} \left( \frac{\delta}{\delta \phi}, \hat{C} \frac{\delta}{\delta \phi} \right) + \left( \hat{D}^T \phi, \frac{\delta}{\delta \phi} \right) \right] \bigg|_{\phi=0} Z(\phi|L),$$

(27)

where $\hat{C} = \frac{\partial}{\partial L} |_{L=1} C_L$ and $\hat{D}^T = \frac{\partial}{\partial L} |_{L=1} D_L^T$. See [8, 1]. For the interaction $V(\phi|L) = -\log Z(\phi|L)$, eq. (27) implies that

$$L \frac{\partial}{\partial L} V(\phi|L) = \left[ \frac{1}{2} \left( \frac{\delta}{\delta \phi}, \hat{C} \frac{\delta}{\delta \phi} \right) + \left( \hat{D}^T \phi, \frac{\delta}{\delta \phi} \right) \right] V(\phi|L)$$

$$- \left( \frac{\delta}{\delta \phi} V(\phi|L), \hat{C} \frac{\delta}{\delta \phi} V(\phi|L) \right)$$

$$- E(\hat{C})(V(\cdot|L)),$$

(28)

where $E(\hat{C})(V(\cdot|L))$ denotes the (field independent) normalization constant

$$E(\hat{C})(V(\cdot|L)) = \frac{1}{2} \left[ \left( \frac{\delta}{\delta \phi}, \hat{C} \frac{\delta}{\delta \phi} \right) V(\phi|L) \right.$$}

$$- \left. \left( \frac{\delta}{\delta \phi} V(\phi|L), \hat{C} \frac{\delta}{\delta \phi} V(\phi|L) \right) \right]_{\phi=0}.$$

(29)

Indispensable intermediate volume cutoffs are here treated casually. For the instant pair $(C_L, D_L)$, the differential version $(\hat{C}, \hat{D})$ is explicitly given by

$$\hat{C}(p) = \frac{2 e^{-p^2}}{\Gamma \left( 1 - \frac{\epsilon}{2} \right)},$$

(30)

$$\hat{C}(x, y) = \frac{2 e^{-\left( \frac{x+y}{2} \right)^2}}{(4\pi)^{\frac{d}{2}} \Gamma \left( 1 - \frac{\epsilon}{2} \right)},$$

(31)
together with
\[
\hat{D}^T(f)(x) = \left[ 1 - \frac{D + \epsilon}{2} - x \frac{\partial}{\partial x} \right] f(x), \quad (32)
\]
\[
\hat{D}^T(f)(p) = \left[ 1 + \frac{D - \epsilon}{2} + p \frac{\partial}{\partial p} \right] \hat{f}(p). \quad (33)
\]
The differential pair \((\hat{C}, \hat{D})\) in the flow equations (27) and (28) is independent of \(L\). This is a characteristic feature of the renormalization group with rescaling.

To summarize, the renormalization group associated with the pair of operators \((C_L, D_L)\) comes in two equivalent formulations. In its discrete version, one deals with difference equations. In its continuous version, one deals with differential equations.

### 3 Renormalization group – massive picture

In this section, we show that the renormalization group defined by the pair \((C_L, D_L)\) is equivalent to a renormalization group defined by another pair \((c_L, d_L)\).

#### 3.1 Gaussian fixed point

The transformation \(R(C_L, D_L)\) has a fixed point \(Z_\star(\phi) = e^{-\frac{1}{2}(\phi, K_\star^{-1} \phi)}\), with \(R(C_L, D_L)(Z_\star)(\phi) = Z_\star(\phi)\) for all \(L > 1\), given by \(K_\star = (1 - \chi)/(\triangle)^{1 - \frac{\epsilon}{2}}\).

With the cutoff function \(\tilde{K}\), \(K_\star\) has the integral representations
\[
\tilde{K}_\star(p) = \int_0^1 \frac{d\alpha}{\alpha} \alpha^{1 - \frac{\epsilon}{2}} e^{-\alpha p^2} \Gamma \left(1 - \frac{\epsilon}{2}\right), \quad (34)
\]
\[
K_\star(x, y) = \int \frac{d\alpha}{\alpha} \alpha^{\frac{\delta p + 1}{2}} e^{-\alpha \frac{(x-y)^2}{2}} \Gamma \left(1 - \frac{\epsilon}{2}\right). \quad (35)
\]
\(K_\star\) is short ranged, and \(\tilde{K}_\star(p) = 1/\Gamma \left(2 - \frac{\epsilon}{2}\right) + O(p^2)\). More generally, we have a line of fixed points given by \(K_\star + \text{const.}/(\triangle)^{1 - \frac{\epsilon}{2}}\). All of them are long ranged, except for the special point with \(\text{const.} = 0\).

#### 3.2 Transformation \(Q(C)\)

Let \(C\) be an operator. Let \(Q(C)\) be the functional operator given by
\[
Q(C)(Z)(\phi) = e^{-\frac{1}{2}(\phi, C^{-1} \phi)} Z(\phi). \quad (36)
\]
Thus \(Q(C)\) multiplies a perturbation with a Gauss function. The two transformations \(R(A, B)\) and \(Q(C)\) conspire to the identity
\[
R(A, B) Q(C) = Q(B^{-1T} \mathcal{L}^{-1} C B^{-1}) R(A, B \mathcal{L}), \quad (37)
\]
where $\mathcal{L}^{-1} = 1 + AC^{-1}$, assuming that $A = A^T$, $C = C^T$, invertibility of $A$, $B$, and $C$, and existence of the Gaussian convolutions.

### 3.3 Substitution $Z(\phi) = Z_\ast(\phi) z(\phi)$

We write the perturbation $Z(\phi)$ as a product of the fixed point $Z_\ast(\phi)$ and a new perturbation $z(\phi)$. Substitute $Z(\phi) = Z_\ast(\phi) z(\phi)$ to obtain

$$R(C_L, D_L)(Z_\ast z)(\phi) = Z_\ast(\phi) R(c_L, d_L)(z)(\phi),$$

with a new pair $(c_L, d_L)$ of operators, defining the massive picture of the renormalization group. They are given by

$$c_L = \mathcal{L}_L C_L, \quad d_L = D_L \mathcal{L}_L,$$

where $\mathcal{L}_L$ is the operator $\mathcal{L}_L = K_\ast/(K_\ast + C_L)$. Notice that $C_L$ and $K_\ast$ commute. $\mathcal{L}_L$ is explicitly given by

$$\hat{\mathcal{L}}_L(p) = \frac{1 - \hat{\chi}(p)}{1 - \hat{\chi}(Lp)} = L^{-2 + \epsilon} \frac{\hat{K}_\ast(p)}{\hat{K}_\ast(Lp)}.$$

Think of $\hat{\mathcal{L}}_L(p)$ as a $p$-dependent correction to the scale factor in the field dilatation. At $p^2 = 0$ it yields an extra factor $L^{-2 + \epsilon}$ to the dilataton, while it becomes one as $p^2 \to \infty$.

### 3.4 Covariance $c_L$

Like $C_L$, the covariance $c_L$ is short ranged. The explicit form of $c_L$ is

$$\tilde{c}_L(p) = \hat{\mathcal{L}}_L(p) \hat{C}_L(p) = \frac{1 - \hat{\chi}(p)}{1 - \hat{\chi}(Lp)} \frac{\hat{\chi}(p) - \hat{\chi}(Lp)}{(p^2)^{1 - \frac{d}{2}}}.$$

Analogous to $C_1 = 0$, it satisfies $c_1 = 0$. In the limit $L \to \infty$, (41) turns into a covariance

$$c_\infty(p) = \frac{(1 - \hat{\chi}(p)) \hat{\chi}(p)}{(p^2)^{1 - \frac{d}{2}}} = (1 - \hat{\chi}(p)) \hat{C}_\infty(p).$$

Recall that $C_\infty$ is a massless covariance with unit ultraviolet cutoff. In contrast, $c_\infty$ is a massive (short ranged) covariance with two-sided unit cutoffs.

### 3.5 Dilatation $d_L$

From (40), it follows that $L_1 = 1$ and consequently $d_1 = 1$, in analogy to $D_1 = 1$. Furthermore, $d_L$ satisfies the composition law $d_L d_{L'} = d_{L + L'}$. The pair $(c_L, d_L)$ can be shown to be a scale decomposition of the covariance

$$5.$$
with respect to \(d_L\). We have that \(c_L = c_\infty - d_L^T c_\infty d_L\). Therefrom, we conclude that \(R(c_L, d_L)\) satisfies the semi-group properties \(R(c_1, d_1) = 1\) and
\[
R(c_L, d_L) R(c_L', d_L') = R(c_L L', d_L L'). \tag{43}
\]
The reasoning is completely analogous to the massless case, since the algebraic properties of the pair \((c_L, d_L)\) are identical to those of \((C_L, D_L)\).

### 3.6 Generator \(\hat{R}(\dot{c}, \dot{d})\)

To compute the renormalization group differential equation in the massive picture, all we have to do, is to compute the differential pair \((\dot{c}, \dot{d})\) and to replace \((\dot{C}, \dot{D})\) by it in (27) and (28). The differential covariance and dilatation pick up an extra term \(\hat{\dot{L}}\) as compared to the massless picture. The extra term is
\[
\hat{\dot{L}}(p) = \frac{-2 (p^2)^{1-\frac{\epsilon}{2}} e^{-p^2}}{\gamma (1 - \frac{\epsilon}{2} p^2)} = \frac{-2}{\int_0^1 d\alpha (1 - \alpha) \frac{\gamma}{1 - \frac{\epsilon}{2} p^2}} e^{\alpha p^2}. \tag{45}
\]
In the special case when \(\epsilon = 0\), it takes the form of the generating function of the Bernoulli numbers, namely \(\hat{\dot{L}}(p) = -2 p^2/(e^{p^2} - 1)\). In the massive picture, we thus have a differential pair of operators \((\dot{c}, \dot{d})\), given by
\[
\hat{\dot{c}}(p) = \hat{\dot{L}}(p) + \hat{\dot{C}}(p) = 2 e^{-p^2} \left( \frac{1}{\Gamma (1 - \frac{\epsilon}{2})} - \frac{(p^2)^{1-\frac{\epsilon}{2}}}{\gamma (1 - \frac{\epsilon}{2} p^2)} \right) \tag{46}
\]

Together with
\[
\hat{\dot{d}}(f)(p) = \hat{\dot{L}}(f)(p) + \hat{\dot{D}}(f)(p) = \left[ \hat{\dot{L}}(p) + 1 + \frac{D - \epsilon}{2} + p \frac{\partial}{\partial p} \right] \hat{f}(p). \tag{47}
\]

The change from the massless picture to the massive picture is thus particularly simple in the differential formulation of the renormalization group. Finally, we remark that \(d_L = D_L \mathcal{L}_L D^{-1}_L\) with
\[
(D_L \mathcal{L}_L D^{-1}_L)(p) = \hat{\mathcal{L}}_L(L^{-1} p) = \frac{1 - \hat{\chi}(L^{-1} p)}{1 - \hat{\chi}(p)}. \tag{48}
\]

Which of the two pictures is better suited, depends on the kind of perturbation one is considering, whether it is functionally closer to the trivial fixed point or to the Gaussian fixed point. In the hierarchical approximation for instance, the massive picture has proved to be better suited for an investigation of the non-trivial scalar fixed point in three dimensions [4].
4 Symmetry – massive picture

In this section, we show that the renormalization group defined by the pair \((c_L, d_L)\) has a symmetry, the transformation defined by another pair \((\gamma_L, \delta_L)\).

4.1 Symmetry pair \((\gamma_L, \delta_L)\)

Let \(L\) be a scale, with \(L > 1\). Let \(\gamma_L\) and \(\delta_L\) be the operators given by
\[
\gamma_L = \left(1 - L^{-2}\right) c_\infty, \quad \delta_L = L^{-1}.
\]
(49)
The operator \(\delta_L\) performs a global (space-time independent) multiplication by \(L^{-1}\). Obviously, \((\gamma_L, \delta_L)\) is a (global) scale decomposition of \(c_\infty\) with respect to \(\delta_L\), namely \(\gamma_L = c_\infty - \delta_L^T c_\infty \delta_L\). It follows that the transformation \(R(\gamma_L, \delta_L)\) satisfies the semi-group properties \(R(\gamma_1, \delta_1) = 1\) and
\[
R(\gamma_L, \delta_L) R(\gamma_{L'}, \delta_{L'}) = R(\gamma_{L L'}, \delta_{L L'}).
\]
(50)

4.2 Commutator of \(R(c_L, d_L)\) and \(R(\gamma_{L'}, \delta_{L'})\)

The two one parameter families of transformations \(R(c_L, d_L)\) and \(R(\gamma_{L'}, \delta_{L'})\) commute: For all \(L > 1\) and \(L' > 1\), we have that
\[
R(c_L, d_L) R(\gamma_{L'}, \delta_{L'}) = R(\gamma_{L'}, \delta_{L'}) R(c_L, d_L).
\]
(51)
This property defines a symmetry of \(R(c_L, d_L)\). An analogous construction can be made in the massless picture. The point here is that the symmetry covariance is short ranged in the massive picture. To prove (51), we compute
\[
(c_L, d_L) (\gamma_{L'}, \delta_{L'}) = (\delta_L^T c_L \delta_{L'} + \gamma_{L'}, d_L \delta_{L'})
\]
(52)
and compare the result with
\[
(\gamma_{L'}, \delta_{L'}) (c_L, d_L) = (d_L^T \gamma_{L'} d_L + c_L, \delta_{L'} d_L).
\]
(53)
The two dilatations commute, \(d_L \delta_{L'} = \delta_{L'} d_L\). The combined covariances are
\[
\delta_L^T c_L \delta_{L'} + \gamma_{L'} = (L')^{-2} \left( c_\infty - d_L^T c_\infty d_L \right) + \left( 1 - (L')^{-2} \right) c_\infty
\]
\[
= c_\infty - (L')^{-2} d_L^T c_\infty d_L
\]
(54)
and
\[
d_L^T \gamma_{L'} d_L + c_L = \left( 1 - (L')^{-2} \right) d_L^T c_\infty d_L + c_\infty - d_L^T c_\infty d_L
\]
\[
= c_\infty - (L')^{-2} d_L^T c_\infty d_L.
\]
(55)
These two covariances are the same. It follows that \(R(c_L, d_L)\) and \(R(\gamma_{L'}, \delta_{L'})\) indeed commute.
4.3 Composed transformation

To obtain a better understanding of the symmetry transformation, we can study the composition of a renormalization group transformation and a symmetry transformation. This composition turns out to be given by yet another pair. Let \( L' = L^{\frac{\eta}{2}} \), where \( \eta \) parametrizes the symmetry transformation. (A special value of \( \eta \) will be identified with the anomalous dimension.) Then we have that

\[
R(c_L, d_L) R(\gamma L^{\frac{\eta}{2}}, \delta L^{\frac{\eta}{2}}) = R(c_L, d_L). \tag{56}
\]

where

\[
d_L,\eta = d_L \delta L^{\frac{\eta}{2}} = L^{-\frac{\eta}{2}} d_L, \tag{57}
\]

and

\[
c_L,\eta = c_\infty - L^{-\eta} d_L^T c_\infty d_L = c_\infty - d_L^T c_L, \tag{58}
\]

In other words, the composition of the renormalization group and the symmetry is a renormalization group built from an anomalous scale decomposition \((c_L, d_L)\) of \(c_\infty\).

4.4 Differential pair \((\dot{\gamma}, \dot{\delta})\)

The generator of the symmetry is particularly simple. It is given by the differential pair of operators \((\dot{\gamma}, \dot{\delta})\) with \(\dot{\gamma} = 2 c_\infty\) and \(\dot{\delta} = -1\). The generators of the renormalization group and the symmetry are

\[
r = \frac{1}{2} \left( \frac{\delta}{\delta \phi}, \dot{c} \frac{\delta}{\delta \phi} \right) + \left( \dot{d}^T \phi, \frac{\delta}{\delta \phi} \right), \tag{59}
\]

\[
u = \frac{1}{2} \left( \frac{\delta}{\delta \phi}, \dot{\gamma} \frac{\delta}{\delta \phi} \right) + \left( \dot{\delta}^T \phi, \frac{\delta}{\delta \phi} \right). \tag{60}
\]

They satisfy \( [r, \nu] = 0 \), as can be checked directly. Renormalization group and symmetry associate with a perturbation \(z(\phi)\) a two parametric orbit

\[
z(\phi|L, L') = R(c_L, d_L) R(\gamma L, \delta L')(z(\phi)), \tag{61}
\]

whose dependence on \(L\) is governed by (59), and whose dependence on \(L'\) is governed by (60). Indeed, (51) satisfies the differential equations

\[
L \frac{\partial}{\partial L} z(\phi|L, L') = r(z)(\phi|L, L') - z(\phi|L, L') r(z)(0|L, L'), \tag{62}
\]

\[
L' \frac{\partial}{\partial L'} z(\phi|L, L') = \nu(z)(\phi|L, L') - z(\phi|L, L') \nu(z)(0|L, L'). \tag{63}
\]

The interaction \(v(\phi|L, L') = -\log z(\phi|L, L')\) obey an exponentiated version (32) and (33) of (62) and (63).
5 Symmetry – massless picture

In this section, we transform the symmetry \( R(\gamma_L, \delta_L) \) to the massless picture. Recall that the change of pictures is the transformation \( Z(\phi) = Q(K_*)(z)(\phi) \) and that

\[
R(C_L, D_L) \ Q(K_*) = Q(K_*) \ R(c_L, d_L). \tag{64}
\]

To translate \( R(\gamma_L, \delta_L) \) to the massless picture, we have to conjugate with \( Q(K_*) \). From (56) and

\[
Q(C) \ Q(D) = Q(C_1 C + D D) \tag{65}
\]

it follows that

\[
R(A, B) \ Q(C) = Q(C_1) \ Q\left(-C C' - C\right) \ R(A', B'), \tag{66}
\]

with \( A' = \mathcal{L} A, \ B' = B \ \mathcal{L}, \ C' = (B^{-1})^T (C + A) \ B^{-1} \), where \( \mathcal{L} = 1/(1 + A C^{-1}) \). In the massless picture, the symmetry transformation therefore consists of two parts. One part is the transformation given by a pair of operators. The second part is the multiplication with a certain Gauss function. This term is essentially a kinetic term as we will see.

5.1 \( A', B', C', and C'' \) for the symmetry

Let \( A, B, \) and \( C \) be given by \( A = (1 - L^{-2}) c_\infty, \ B = L^{-1}, \) and \( C = -K_*. \) It follows that \( \mathcal{L} = 1/(1 - (1 - L^{-2}) \chi) \) and thus

\[
A' = \frac{(1 - L^{-2}) c_\infty}{1 - (1 - L^{-2}) \chi}, \tag{67}
\]

\[
B' = \frac{L^{-1}}{1 - (1 - L^{-2}) \chi}, \tag{68}
\]

\[
C' = \left( L^2 - 1 - \frac{L^2}{\chi} \right) c_\infty, \tag{69}
\]

and

\[
\frac{1}{C''} = \frac{C - C'}{C C'} = \frac{(1 - L^{-2}) (-\triangle)^{1-\frac{d}{2}}}{1 - (1 - L^{-2}) \chi}. \tag{70}
\]

The symmetry transformation \( R(\gamma_L, \delta_L) \) thus has the following form in the massless picture. It maps \( Z(\phi) \) to a symmetry orbit

\[
Z(\Phi|1, L) = e^{-\frac{1}{2}(\Phi, C''(\Phi) + C''(\Phi))} \frac{d\mu_{A'}(\phi) \ Z((B')^T(\Phi) + \phi)}{d\mu_{A'}(\phi) \ Z(\phi)}. \tag{71}
\]
Remarkably, the function $\hat{C}''(p)^{-1} = (1 - L^{-2}) (p^2)^{1-\frac{d}{2}}/(1 - (1 - L^{-2}) \hat{\chi}(p))$ is an analytic function of $p^2$ at the origin only if $\epsilon = 0$. A straightforward calculation confirms directly that the transformation (71) satisfies the properties of a semi-group and that it commutes with the renormalization group.

5.2 $A'$, $B'$, $C'$, and $C''$ for the composed transformation

The composition of a symmetry transformation, which scales by a factor $L'$, and a renormalization group transformation, which scales by a factor $L$, is computed analogously. Let $A = c_\infty - (L')^{-2} d_L^T c_\infty d_L$, $B = (L')^{-1} d_L$, and $C = -K_*$. Then

$$L = \frac{1 - \chi_L}{1 - \chi_L} \frac{1}{1 - (1 - (L')^{-2} \chi_L)},$$

(72)

where $\hat{\chi}_L(p) = \hat{\chi}(Lp)$. It follows that

$$A' = \frac{1}{(-\triangle)^{1-\frac{d}{2}}} \frac{\chi(1 - \chi_L) - (L')^{-2}(1 - \chi) \chi_L}{1 - (1 - (L')^{-2}) \chi_L},$$

(73)

$$B' = D_L \frac{(L')^{-1}}{1 - (1 - (L')^{-2}) \chi_L},$$

(74)

$$C' = \frac{1}{(-\triangle)^{1-\frac{d}{2}}} (1 - \chi)(L')^2 \left(-1 + (1 - (L')^{-2}) \chi \right),$$

(75)

and

$$\frac{1}{C''} = \frac{1 - (L')^{-2}}{1 - (1 - (L')^{-2}) \chi} (-\triangle)^{1-\frac{d}{2}}.$$ 

(76)

For $L = 1$, one reproduces the formulas (67), (68), and (69) for the pure symmetry transformation. Notice that (74) is independent of $L$, as it should. Eq. (71) generalizes to

$$Z(\Phi|L, L') = e^{-\frac{1}{2}(\Phi, \frac{1}{C''} \Phi)} \int d\mu_{A'}(\phi) Z((B')^T(\Phi) + \phi) \int d\mu_{A'}(\phi) Z(\phi).$$

(77)

5.3 Generators in the massless picture

We compute the generators of the double flow (77) for the interaction $V(\phi|L, L') = -\log Z(\phi|L, L')$ from

$$V(\Phi|L, L') = \frac{1}{2} \left( \Phi, \frac{1}{C''} \Phi \right)$$

$$- \log \int d\mu_{A'}(\phi) e^{-V((B')^T(\Phi) + \phi)}$$

$$+ \log \int d\mu_{A'}(\phi) e^{-V(\phi)},$$

(78)
where \( A', B', \) and \( \frac{1}{\lambda} \) are given by (73), (74), and (76) respectively. We find two flow equations. The first flow equation is identical with (28) and (29). It describes the flow in the renormalization group direction. The second flow equation is

\[
L' \frac{\partial}{\partial L'} V(\phi|L, L') = \left( \phi, (-\Delta)^{1-\frac{\gamma}{2}} \phi \right) + \left[ \frac{1}{2} \left( \frac{\delta}{\delta \phi}, 2c_{\infty} \frac{\delta}{\delta \phi} \right) + \left( -1 + 2\chi \right) \phi, \frac{\delta}{\delta \phi} \right] V(\phi|L, L') - \frac{1}{2} \left( \frac{\delta}{\delta \phi} V(\phi|L, L'), 2c_{\infty} \frac{\delta}{\delta \phi} V(\phi|L, L') \right)
\]

(79)

where

\[
E(V(\cdot|L, L')) = \frac{1}{2} \left[ \left( \frac{\delta}{\delta \phi}, 2c_{\infty} \frac{\delta}{\delta \phi} V(\phi|L, L') \right) - \left( \frac{\delta}{\delta \phi} V(\phi|L, L'), 2c_{\infty} \frac{\delta}{\delta \phi} V(\phi|L, L') \right) \right]_{\phi=0}
\]

(80)

It describes the flow in the symmetry direction. Both together, plus an initial condition, are equivalent to the integral (78). The novelty as compared to the massive picture is the appearance of a kinetic term.

6 Anomalous dimension \( \eta \)

This section contains a brief discussion of fixed points with anomalous dimension based on the following hypothesis, adapted from [7] and [9]. See also the recent discussion in [10, 11] and references therein.

6.1 Fixed point \((z_*(\phi), \eta)\)

We define a fixed point \( z(\phi) \) (in the massive picture) with anomalous dimension \( \eta \) as a fixed point of \( R(c_L, d_L) R(\gamma_L, \delta_L) \), the composition of renormalization group transformation \( R(c_L, d_L) \), which scales by a factor \( L \), and a symmetry transformation \( R(\gamma_L, \delta_L) \), which scales by a factor \( L^{\frac{\eta}{2}} \).

6.2 Discussion of \((z_*(\phi), \eta)\)

\( z_*(\phi) \) is a fixed point of the renormalization group transformation \( R(c_L, d_L) \) only if \( \eta = 0 \). In this case, there are two possibilities. Either \( z_*(\phi) \) is also a fixed point of \( R(\gamma_L, \delta_L) \), so to speak a double fixed point, or one has a line of fixed points \( z_*(\phi|L') \) generated as a symmetry orbit from an arbitrary representative \( z_*(\phi) \). This is the case for the trivial fixed point.
The case with anomalous dimension $\eta$ is a generalization hereof. There one has an invariant line of the renormalization group rather than a line of fixed points, where the renormalization group acts on the invariant line by an inverse symmetry transformation. A fixed point $z_*(\phi)$ with non-zero anomalous dimension $\eta$ is not a stationary renormalization group flow. (There remains the possibility to declare a composition of the renormalization group and the symmetry to be a new renormalization group.) The appropriate framework to analyze such a situation is to divide the space of perturbations into symmetry orbits and to consider the renormalization group on orbit space. A natural possibility is to consider one representative on each orbit by imposing a renormalization condition, which breaks the symmetry. Such a condition could be to demand the prefactor of the kinetic term in the interaction to be a given number. In the presence of a symmetry, the spectrum of a fixed point should include a marginal operator, the direction of the symmetry.

A principal problem is the following. To make the fixed point problem tractable, one has to truncate the space of interactions. Ideally, the truncation would be such that the symmetry leaves invariant the space of truncated interactions. Because then, the orbit construction can be done on the truncated space. If however the truncation breaks the symmetry, the topic of anomalous dimension is buried in the no-truncation limit, which is usually very difficult to analyze. The idea forwarded in \cite{7} for this situation is to truncate in such a way that a marginal operator survives truncation. In other words that the truncations is such that at least differentially the symmetry persists. The local approximation to the renormalization group unfortunately breaks the symmetry, wherefore the concept of anomalous dimension loses its meaning in the local approximation, and in particular in the hierarchical renormalization group.

### 6.3 $\epsilon$-model

Consider the fixed point constructions \cite{5} and \cite{6} for the $\epsilon$-model in this light. Their constructions work directly in the massless picture. When $\epsilon$ is non-zero, the situation is the following. If one restricts the renormalization group to vertices, which are $C^\infty$-functions of the momenta, then the symmetry can be ruled out because it generates a non-$C^\infty$ kinetic term. In this situation, the authors consider an invariant subspace of the renormalization group, which is not invariant under the symmetry. Remarkably, the authors succeed to construct a non-trivial fixed point in the $C^\infty$-subspace. (For the planar approximation, this is perhaps not so surprising because the planar approximation itself also breaks the symmetry.) Such a non-trivial fixed point is by definition a fixed point with anomalous dimension zero.

If on the other hand one wants to investigate flows modulo the symmetry (as one has to in the case of non-zero anomalous dimension) then one cannot
restrict the flow to $C^\infty$-vertices in this model. Whether the $\epsilon$-model has another fixed point in this bigger space, I do not know. If this is the case, it would be very interesting to determine $\eta(\epsilon)$. In the most interesting case, the three dimensional theory with $\epsilon = 0$, the situation is different. There the symmetry preserves the property of analyticity in the momenta. Therefore it cannot be ruled out.

6.4 Differential equation for $v_\star(\phi)$

The fixed point problem (81) in the massive picture takes the form of the following renormalization group differential equation for the fixed point interaction $v_\star(\phi) = -\log z_\star(\phi)$,

$$
\begin{align*}
&\frac{1}{2} \left( \frac{\delta}{\delta \phi}, [\dot{c} - \eta c_\infty] \frac{\delta}{\delta \phi} \right) + \left( \left[ d^T - \frac{\eta}{2} \phi, \frac{\delta}{\delta \phi} \right] \right) v_\star(\phi) \\
&- \frac{1}{2} \left( \frac{\delta}{\delta \phi} v_\star(\phi), [\dot{c} - \eta c_\infty] \frac{\delta}{\delta \phi} v_\star(\phi) \right) = E(\dot{c} - \eta c_\infty)(v_\star), \quad (81)
\end{align*}
$$

where

$$
E(\dot{c} - \eta c_\infty)(v_\star) = \frac{1}{2} \left[ \left( \frac{\delta}{\delta \phi}, [\dot{c} - \eta c_\infty] \frac{\delta}{\delta \phi} \right) v(\phi) \\
- \left( \frac{\delta}{\delta \phi} v_\star(\phi), [\dot{c} - \eta c_\infty] \frac{\delta}{\delta \phi} v_\star(\phi) \right) \right]_{\phi=0} \quad (82)
$$

A natural problem is to find good truncation schemes for (81) and (82) and to determine $\eta$ by the condition that the spectrum of $v_\star(\phi)$ have a marginal operator. An investigation of various schemes will be presented elsewhere.

Recall the generators (59) and (60) of the renormalization group and the symmetry respectively. If we neglect the trivial normalization constants, the fixed point problem becomes a linear eigenvalue problem

$$
r(z_\star)(\phi) = -\frac{\eta}{2} u(z_\star)(\phi) \quad (83)
$$

for $r u^{-1}$. The meaning of the anomalous dimension is thus to be an eigenvalue. Eq. (81) is just an exponentiated and thus non-linear form of (83). The analogous equations in the massless picture follow immediately from (78) and (79).

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