On Extensions of a Theorem of Baxter

J. S. Geronimo
Georgia Institute of Technology, USA

A. Martínez-Finkelshtein*
Universidad de Almería, Spain

Dedicated to Barry Simon on the occasion of his 60th birthday.

Abstract

We combine the Riemann-Hilbert approach with the techniques of Banach algebras to obtain an extension of Baxter’s Theorem for polynomials orthogonal on the unit circle. This is accomplished by using the link between the negative Fourier coefficients of the scattering function and the coefficients in the recurrence formula satisfied by these polynomials.

1 Introduction

Recently there has been a upsurge in interest in the theory of orthogonal polynomials on the unit circle (OPUC). To a large extent this can be attributed to the two volumes by Barry Simon devoted to the theory of these polynomials. If \( \mu \) is a positive probability measure supported on the unit circle with an infinite number of points of increase then a sequence of polynomials \( \varphi_n(z) = \kappa_n z^n + \ldots \) of degree \( n, n = 0, 1, \ldots \) satisfying

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_n(e^{i\theta}) \overline{\varphi_m(e^{i\theta})} d\mu = \delta_{n,m},
\]

defines a set of orthonormal polynomials on the unit circle \( T \). We will assume that the leading coefficient \( \kappa_n \) in each polynomial \( \varphi_n \) is positive so the above equation uniquely specifies the polynomials. It is well know, [6], [12], [13] that these polynomials satisfy the following recurrence formula,

\[
\varphi_{n+1}(z) = A_n(z) \varphi_n(z),
\]

with

\[
\varphi_n(z) = \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix}, \quad \varphi_0(z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

*Corresponding author. E-mail andrei@ual.es
and
\[ A_n(z) = \rho_n^{-1} \begin{pmatrix} z & -\bar{\alpha}_n \\ -\alpha_n z & 1 \end{pmatrix}. \] (2)

In the formula above \( \varphi_n^*(z) = z^n \varphi_n(1/z) \) is called the reversed polynomial. The \( \alpha_n \) were known as the recurrence or Verblunsky coefficients [12], and have the property that \( |\alpha_n| < 1 \).

Also \( \rho_n = (1 - |\alpha_n|^2)^{1/2} = k_n/k_{n+1} \). It is well known that for any \( n \in \mathbb{N} \), \( \varphi_n \) has all its zeros strictly inside the unit circle.

Let us recall (see [12, Chapter 5]) that a Beurling weight is a two-sided sequence \( \nu = \{ \nu(n) \}_{n=\infty}^{\infty} \) with the properties
\[ \nu(0) = 1, \quad \nu(n) \geq 1, \] (3)
\[ \nu(n) = \nu(-n), \] (4)
\[ \nu(n+m) \leq \nu(n)\nu(m). \] (5)

These properties imply the existence of the limit
\[ \lim_{n \to +\infty} \nu(n)^{1/n} = R \geq 1. \] (6)

If \( R = 1 \), then \( \nu \) is a strong Beurling weight.

Each Beurling weight \( \nu \) has associated the Banach spaces \( \ell_\nu \) of two-sided sequences \( f = \{ f_n \}_{n=\infty}^{\infty} \) with
\[ \| f \|_\nu \overset{\text{def}}{=} \sum_{n=\infty}^{\infty} \nu(n)|f_n| < \infty; \]
this norm extends naturally to any one-sided sequence by completing the latter with zeros.

In what follows for a positive measure \( \mu \) on \( \mathbb{T} \) we will write
\[ d\mu = w(\theta) \, d\theta + d\mu_s, \] (7)
after the Lebesgue-Radon-Nikodym decomposition theorem. We will abuse notation using \( w(\theta) \) and \( w(e^{i\theta}) \) interchangeably, expecting that this will not cause confusion.

As noted by B. Simon [12], Baxter’s Theorem which relates the decay rate of the recurrence coefficients \( \alpha = \{ \alpha_n \}_{n=0}^{\infty} \) to the decay rate of the Fourier coefficients \( e = \{ c_n \}_{n=-\infty}^{\infty} \) of the log \( w \),
\[ c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \log w(\theta) d\theta, \quad n \in \mathbb{Z}, \] (8)
plays an important role in the theory:

**Theorem 1** (11) Suppose \( \nu \) is a strong Beurling weight (\( R = 1 \)). Then, \( \alpha \in \ell_\nu \) if and only if \( \mu_s = 0 \), \( w \in C[-\pi, \pi] \) is strictly positive, and \( e \in \ell_\nu \).
In the summer of 2003 Barry Simon asked one of us (JSG) whether it was possible to extend the above Theorem for the case of Beurling weights with \( R > 1 \) as had been done for polynomials orthogonal on the real line (see Geronimo [5]). What is needed is to remove poles from the weight function and to be able to control the decay of the coefficients of the new weight function. For polynomials orthogonal on the real line this is accomplished by the Christoffel-Uvarov formulas (see [13], [14] and [5]). However such formulas are not available at present in the case of the unit circle. The necessary control over the coefficients however can be established using Banach algebra techniques coupled with those coming from the Riemann-Hilbert approach (see [2], [8], or for an alternate approach [11]). We begin by collecting the results from scattering theory that will be needed. In particular we introduce the scattering function \( S \) and show that its projection \( P^-(S) \) governs the rate of decay \( \alpha \).

We then show that the product of two weights whose coefficients decay exponentially with the same rate give a weight whose coefficients decay with the same rate. We then apply this to Berstein–Szegö perturbations of weights. Finally we obtain a generalization of Baxter’s Theorem.

2 A generalization of Baxter’s theorem

As was noticed by Baxter [1], Banach algebras may be associated with the Beurling weights \( \nu \) in the following manner. Let \( a, b \in \ell_\nu \); then their convolution is given by

\[
(a \ast b)(n) = \sum_{k=-\infty}^{\infty} a(k)b(n-k),
\]

which is absolutely convergent by (3) and by (5),

\[
\|a \ast b\|_\nu = \sum_n \|f\|_\nu \sum_k |a(k)b(n-k)| \leq \|a\|_\nu \|b\|_\nu.
\] (9)

Thus if we consider the space of functions

\[
A_\nu \equiv \left\{ f(z) = \sum_{k \in \mathbb{Z}} f_k z^k : \|f\|_\nu = \sum_k \nu(k)|f_k| < \infty \right\},
\] (10)

equation (9) shows that it is closed under multiplication and so forms an algebra. For \( \nu \equiv 1 \) we obtain the Wiener algebra \( A_1 \), containing \( A_\nu \) for any Beurling weight \( \nu \).

On the set of all two-sided sequences \( \{d(n)\}_{n \in \mathbb{Z}} \) we define the projectors \( P_- \) and \( P_+ \):

\[
(P_- d)(n) = \begin{cases} d(n), & \text{if } n \leq 0, \\ 0, & \text{if } n > 0, \end{cases}
\]

and

\[
(P_+ d)(n) = \begin{cases} d(n), & \text{if } n \geq 0, \\ 0, & \text{if } n < 0. \end{cases}
\]
These projectors can be naturally extended to $A_\nu$, and give rise to two subalgebras associated with $A_\nu$: $A^+_\nu \overset{\text{def}}{=} \mathcal{P}_+(A_\nu)$ and $A^-_\nu \overset{\text{def}}{=} \mathcal{P}_-(A_\nu)$. In other words, $A^\pm_\nu$ are the set of functions $f \in A_\nu$ whose Fourier coefficients $f_n$ vanish for $n < 0$ or $n > 0$ respectively. It is easy to see because of (3) that if $f \in A_\nu$ then $f(z)$ is continuous for $1/R \leq |z| \leq R$ (which is the maximal ideal space associated to $A_\nu$) and analytic for $|z| < R$ (in which case the series in (10) is its Maclaurin expansion convergent at least in this disk), and if $f$ is in $A^-_\nu$ then $f$ is continuous for $|z| \geq 1/R$ and analytic for $|z| > 1/R$.

If we denote by $\hat{\varphi}_n(z) \overset{\text{def}}{=} z^{-n} \varphi_n(z)$, then $\hat{\varphi}_n \in A^-_\nu$, and $\varphi^*_n \in A^+_\nu$, and we may recast (11) as

$$G_{n+1}(z) = B_n(z)G_n,$$

where

$$G_n(z) = \left( \begin{array}{c} \varphi_n(z) \\ \varphi^*_n(z) \end{array} \right), \quad \text{and} \quad B_n(z) = \rho_n^{-1} \left( \begin{array}{cc} 1 & -z^{-n-1} \tilde{\alpha}_n \\ -\alpha_n z^{n+1} & 1 \end{array} \right).$$

The next lemma is implicitly given by Baxter [1] and Simon [12, Ch. 5]. We present its proof for completeness of the reading.

**Lemma 1** Suppose $\nu$ is a Beurling weight, and $\{\alpha\}_0^\infty \in l_\nu$. Then

$$\lim_\nu \varphi^*_n = f_+ \in A^+_\nu;$$
$$\lim_\nu \hat{\varphi}_n = f_- \in A^-_\nu,$$

where convergence is understood in the $\| \cdot \|_\nu$ norm. For $|z| = 1$, $f_-(z) = \overline{f_+(z)} = \tilde{f}_+(1/z)$.

Furthermore, in the decomposition (7), $d\mu_* = 0$, and

$$w = \frac{1}{f_+ f_-}$$

on $\mathbb{T}$; hence, $w$ is a strictly positive continuous function on $\mathbb{T}$.

**Proof.** Let us denote by $\Phi_n(z) = \kappa_n^{-1} \varphi_n(z)$ the monic orthogonal polynomial, and $\tilde{\Phi}_n(z) = z^{-n}\Phi_n(z)$. Then from equation (11) we find

$$\Phi^*_{n+1}(z) = \Phi^*_n(z) - \alpha_n z^{n+1} \tilde{\Phi}_n(z),$$
$$\tilde{\Phi}_{n+1}(z) = \tilde{\Phi}_n(z) - \overline{\alpha_n} z^{-(n+1)} \Phi^*_n(z).$$

A consequence of (5) is that for $f \in A_\nu$ and $k \in \mathbb{Z}$, $\|z^k f(z)\|_\nu \leq \nu(k) \|f\|_\nu$. Thus,

$$\|\Phi^*_{n+1}\|_\nu \leq \|\Phi^*_n\|_\nu + \nu(n+1)|\alpha_n| \|	ilde{\Phi}_n\|_\nu.$$
A similar formula holds for \( \hat{\Phi}_{n+1} \), so that by induction it follows that

\[
\|g_{n+1}\|_\nu \leq \prod_{i=0}^{n} (1 + \nu(i + 1)|\alpha_i|),
\]

with \( g_n = \Phi^*_n \) or \( \hat{\Phi}_n \), so that \( \sup_n \|\Phi^*_n\|_\nu < \infty \), \( \sup_n \|\hat{\Phi}_n\|_\nu < \infty \).

By (15),

\[
\|\Phi^*_n - \Phi^*_m\|_\nu \leq \sum_{i=m}^{n-1} \nu(i + 1)|\alpha_i|\|\hat{\Phi}_i\|_\nu,
\]

which yields that \( \{\Phi^*_n\} \) is a Cauchy sequence in \( A^+_\nu \), and hence has a limit in \( A^+_\nu \). Since

\[
\kappa_n = \prod_{i=0}^{n-1} \left( 1 - |\alpha_i|^2 \right)^{-1/2}
\]

converge, (12) follows. Similar reasoning gives (13).

Since for \( |z| = 1 \), \( \hat{\varphi}_n = \overline{\varphi_n^*} \) we find for \( |z| = 1 \) that \( f_-(z) = \overline{f_+(z)} = \overline{\hat{f}_+(1/z)} \).

On the other hand, by (12)–(13),

\[
\lim_n \|\varphi_n^*|^2 - f_- f_+\|_\nu = 0.
\]

Using the well know fact from the theory of OPUC [9, 12, 13] that \( \varphi_n^*(z) \neq 0, |z| \leq 1 \), and for any \( h \in C(\mathbb{T}) \), where \( C(\mathbb{T}) \) is the set of continuous functions on the unit circle,

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} h(\theta) |\varphi_n^*(e^{i\theta})|^2 \, d\theta \rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\theta) d\mu(\theta)
\]

the result follows.

**Remark 1** We may recast the last result in terms more familiar to the orthogonal polynomials community. The *Szegő function* of \( w \) (see e.g. [13, Ch. X, §10.2]) is defined by

\[
D(z) \overset{\text{def}}{=} \exp \left( \frac{1}{4\pi} \int_0^{2\pi} \log w(e^{i\theta}) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta \right).
\]

This function is piecewise analytic and non-vanishing, defined for \( |z| \neq 1 \), and we denote by \( D_i \) and \( D_e \) its values for \( |z| < 1 \) and \( |z| > 1 \), respectively. Then

\[
f_+(z) = D_i^{-1}(z), \quad f_-(z) = D_e(z).
\]

Also the last identity of Lemma 1 is consistent with the property

\[
D_i \left( \frac{1}{z} \right) = \frac{1}{D_e(z)}, \quad |z| > 1.
\]
Lemma 1 and equation (14) show that $1/w \in A_\nu$, which gives a natural meromorphic extension of $w$ such that $w \neq 0$ for $R^{-1} \leq |z| \leq R$, and this is the meaning we will be giving to $w$ in what follows.

From the previous works [2], [4], [8], [12] it is clear that the function

$$S = \frac{f^-}{f^+} = D_iD_e$$

plays a prominent role in the theory. By Lemma 1 we have that $|S(z)| = 1$ for $|z| = 1$. Following Faddeyev’s classic article on inverse scattering [3] and Newton’s book on Scattering theory [10] we call the unimodular function $S$ the scattering function.

Straightforward computation shows that

$$S(z) = \exp\left(\sum_{k=1}^{\infty} (c_k z^k - c_{-k} z^{-k})\right) = \exp\left(\sum_{k=1}^{\infty} (c_k z^k - c_{-k} z^{-k})\right),$$

where $c_n$’s are defined in [3]. Hence, if $\nu$ is a Beurling weight, then

$$\log w \in A_\nu \iff \log S \in A_\nu \iff \text{both } S \text{ and } S^{-1} \in A_\nu.$$  

In particular, if $\nu$ is a strong Beurling weight, then from Baxter’s Theorem it follows that

$$\log w \in A_\nu \iff w \in A_\nu \iff \log S \in A_\nu \iff \{\alpha\}_{0}^\infty \in l_\nu.$$  

Regarding the case of a Beurling weight with $R > 1$, the situation is different. It was observed by B. Simon [12] that the following three statements are equivalent:

(i) the Fourier coefficients of $\log w$ exhibit an exponential decay;

(ii) the Fourier coefficients of $w$ exhibit an exponential decay;

(iii) $\alpha_n$ exhibit an exponential decay.

As (17) shows, we may replace $w$ by $S$ in (i) and (ii).

However, the decay rates above are not necessary the same; typically the decay in (ii) and (iii) is faster than in (i). As examples in [12] Chapter 7 show, the decay rate in (ii) can be greater or less than in (iii).

Our goal is to investigate these rates more precisely, extending Baxter’s theorem to the case of $R > 1$. Our main results are gathered in Theorems 2, 3 and 4.

Let us start with the following technical result:

**Lemma 2** Let $h \in A_1$, and let $\nu$ be a Beurling weight which increases on $\mathbb{Z}_+$. Assume that $\mathcal{P}_-(h) \in A_\nu$. Then for a function $g$

$$g \in A_1^+ \text{ or } g \in A_\nu^- \Rightarrow \mathcal{P}_-(gh) \in A_\nu^-.$$
Proof. We begin by proving the first assertion. Set \( h = \sum_{k \in \mathbb{Z}} h_k z^k \) and \( g = \sum_{n \geq 0} g_n z^n \).

Then

\[
P_- (gh) = \sum_{i \geq 0} d_i z^{-i}, \quad \text{with} \quad d_i = \sum_{n \geq 0} g_n h_{-n-i}.
\]

Hence,

\[
\nu(i) |d_i| \leq \nu(i) \sum_{n \geq 0} |g_n| |h_{-n-i}| \leq \sum_{n \geq 0} \nu(n + i) |g_n| |h_{-n-i}|.
\]

Therefore, using Fubini’s theorem we get that

\[
\|P_- (gh)\|_{\nu} \leq \|P_- (h)\|_{\nu} \|g\|_1.
\]

For the second assertion, write \( h_- = P_- (h) \) and \( h_+ = h - h_- \in A_1^+ \). Then \( P_- (gh_-) \in A_\nu^- \) by assumption, and \( P_- (gh_+) \in A_\nu^- \) by the previous argument. Since \( P_- (gh) = P_- (gh_-) + P_- (gh_+) \), the statement follows.

**Corollary 1** Let \( g(z) = \sum_{k \in \mathbb{Z}} d_k z^k \in A_1 \), and let \( \nu \) be a Beurling weight \( \nu \) which increases on \( \mathbb{Z}_+ \). For a rational function \( r \) denote \( r(z)g(z) = \sum_{k \in \mathbb{Z}} \hat{d}_k z^k \). If \( r \) has no poles in \( 1/R \leq r \leq 1 \) then

\[
\sum_{k < 0} d_k z^k \in A_\nu^+ \Rightarrow \sum_{k < 0} \hat{d}_k z^k \in A_\nu^-.
\]

Let us recall some facts that we will need further (see [8] for details). If

\[
\rho^{-1} \overset{\text{def}}{=} \lim_{n \to \infty} |\alpha_n|^{1/n} < 1,
\]

then (see [9]) also

\[
\rho^{-1} = \inf \{0 < r < 1 : f_- (z) \text{ is holomorphic in } |z| > r\}.
\]

For the scattering function \( S \) defined in (16), let

\[
S(z) = \sum_{k \in \mathbb{Z}} d_k z^k
\]

be its Laurent expansion. An important fact, established independently in [2], [11] and [8] (cf. Corollary 2 therein) is that for \( n \in \mathbb{N} \),

\[
\overline{\alpha_n} = -d_{-n-1} + e_n, \quad \text{with} \quad |e_n| \leq C r^{3n}, \quad (18)
\]

for an arbitrary \( r \) such that \( 1/\rho < r \leq 1 \), and where \( C > 0 \) does not depend on \( r \) or \( n \).

**Remark 2** Since \( S(z) = 1 \) for \( |z| = 1 \), we may state this result equivalently in terms of the coefficients of \( P_+ (1/S) \), see [7].
The above formula indicate the close connection between the rate of decay of \( \alpha \)'s and the decay of the negative Fourier coefficients of the scattering function \( S \), which is made explicit in:

**Theorem 2** Let \( d\mu = w(\theta)\,d\theta \) with \( w \in A_1 \) and \( \nu \) be a Beurling weight. Then

\[
\alpha \in \ell_\nu \iff \mathcal{P}_-(S) \in A^{-\nu}_\nu.
\]

**Proof.** Since for any Beurling weight, \( A_\nu \subset A_1 \), Baxter’s theorem (Theorem 1) shows that \( f_\pm \) exist, \( f_\pm \in A_1^\pm \), and therefore \( S \in A_1 \). Furthermore, for any strong Beurling weight the results follow from Theorem 1 and Lemma 2, so we consider only the case when \( R > 1 \).

Assume that \( \alpha \in \ell_\nu \), then \( 1 < R \leq \rho \). For an sufficiently small \( \varepsilon > 0 \) let us take \( r = (1 + \varepsilon)/R \in (1/\rho, 1) \). Since we may take \( r \) arbitrary close to \( 1/\rho \), assumption \( \alpha \in \ell_\nu \) and formula (18) imply that \( \{d_k\}_{k<0} \in \ell_\nu \), so that \( \mathcal{P}_-(S) \in A^{-\nu}_\nu \).

For converse, if \( \mathcal{P}_-(S) \in A^{-\nu}_\nu \), then Lemma 2 implies that \( f_- \in A^{-\nu}_\nu \), showing that again \( \rho > 1 \). Now an application of formula (18) gives the assertion.

A consequence of the previous theorem is the following result which shows that the product of two measures whose coefficients are \( A_\nu \) is a new measure whose coefficients are also \( A_\nu \).

**Theorem 3** For \( d\mu_i = w_i(\theta)\,d\theta \), \( i = 1, 2, 3 \), with \( w_i \in A_1 \) denote by \( \alpha^{(i)} \) the corresponding sequences of the recurrence coefficients. Assume that \( w_3 = w_1w_2 \), and let \( \nu \) be a Beurling weight. Then

\[
\alpha^{(1)}, \alpha^{(2)} \in \ell_\nu \Rightarrow \alpha^{(3)} \in \ell_\nu.
\]

**Proof.** Let us denote by \( S_i \) the scattering function corresponding to \( \mu_i \). By Baxter’s theorem, \( S_1, S_2 \in A_1 \), and by Theorem 2 for \( i = 1, 2 \),

\[
\alpha^{(i)} \in \ell_\nu \Rightarrow S_i^- \overset{\text{def}}{=} \mathcal{P}_-(S_i) \in A^{-\nu}_\nu, \quad S_i^+ \overset{\text{def}}{=} S_i - S_i^- \in A_1^+.
\]

Hence,

\[
\mathcal{P}_-(S_3) = \mathcal{P}_-(S_1S_2) = \mathcal{P}_-(S_1^-S_2^- + S_1^+S_2^- + S_1^-S_2^+) \in A^{-\nu}_\nu,
\]

where we have used Lemma 2. The result now follows from Theorem 2.

Now we are in the position of discussing several corollaries of the previous results.

**Corollary 2** Let \( \nu \) be a Beurling weight, and for the decomposition (7), \( d\mu_s = 0 \) and \( w \in A_\nu \). If \( w(z) \neq 0 \) in \( R^{-1} \leq |z| \leq R \), where \( R \) is defined in (6), then \( \{\alpha\}_{0}^{\infty} \in \ell_\nu \).

This result for complex weights with zero winding numbers has been proved in [2].
Proof. According to (17) and Theorem 2, under assumptions of the corollary,

\[
\log w \in A_\nu \implies \log S \in A_\nu \implies S \in A_\nu \implies P_- (S) \in A_\nu \implies \alpha \in \ell_\nu.
\]

Another straightforward consequence of Theorem 3 is that the multiplication of the original weight by a Bernstein-Szegő weight does not affect the rate of decay of the recurrence coefficients:

**Corollary 3** Let \( d\mu = w(\theta) \, d\theta \), with \( w \in A_1 \), and let \( p \) be a polynomial with no zeros on \( \mathbb{T} \). Denote by \( \alpha \) and \( \hat{\alpha} \) the sequences of the recurrence coefficients corresponding to measures \( d\mu = w(\theta) \, d\theta \) and \( d\hat{\mu} = |p|^{-2} w(\theta) \, d\theta \), respectively. Then for any Beurling weight \( \nu \),

\[
\alpha \in \ell_\nu \implies \hat{\alpha} \in \ell_\nu.
\]

The proof is an immediate consequence of Theorem 3 by observing that for the weight \( |p|^{-2} \) the corresponding recurrence coefficients vanish for a sufficiently large \( n \), and hence belong to \( \ell_\nu \) for any Beurling weight \( \nu \).

Finally we formulate an extension of Baxter’s theorem (Theorem 1) for the case \( R > 1 \).

**Theorem 4** Suppose that \( \nu \) is a Beurling weight with \( R > 1 \).

If there exists a polynomial \( p(z) \neq 0 \) for \( |z| = 1 \), such that for

\[
\hat{w}(z) \overset{\text{def}}{=} |p(z)|^2 w(z), \quad |z| = 1, \quad (19)
\]

we have \( \log(\hat{w}) \in A_\nu \), then \( \{\alpha\}_0^\infty \in \ell_\nu \).

Conversely, if \( \{\alpha\}_0^\infty \in \ell_\nu \) and \( 1/w \neq 0 \) for \( |z| = R \) then there exists a polynomial \( p \) such that for \( \hat{w}(z) \) defined by (19) we have \( \log(\hat{w}) \in A_\nu \).

Proof. Let us prove the first implication. Assume that \( \log(\hat{w}) \in A_\nu \). By Corollary 2 if \( \hat{\alpha} \) are the recurrence coefficients corresponding to \( \hat{w} \), then \( \{\hat{\alpha}\}_0^\infty \in \ell_\nu \). But \( w = |p|^{-2} \hat{w} \), and it remains to apply Corollary 3.

Conversely, let \( \{\alpha\}_0^\infty \in \ell_\nu \). Since by Lemma 1 \( 1/w = f_+ f_- \in A_\nu \), and \( 1/w(z) \neq 0 \) for \( |z| = R \), then \( f_+ \) has a finite number of zeros, \( \zeta_1, \ldots, \zeta_m \), in \( 1 < |z| < R \) (denoted with account of their multiplicity). Let

\[
p(z) = \prod_{k=1}^m (z - \zeta_k).
\]

Then \( \hat{w} \) satisfies the conditions of Corollary 2 which concludes the proof.
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References

[1] Glen Baxter. A convergence equivalence related to polynomials orthogonal on the unit circle. Trans. Amer. Math. Soc., 99:471–487, 1961.

[2] P. Deift and J. Östensson. A Riemann-Hilbert approach to some theorems on Toeplitz operators and orthogonal polynomials. Arxiv:math.FA/0504284.

[3] L. D. Faddeyev. The inverse problem in the quantum theory of scattering. J. Math Phys., 4:72–104, 1963.

[4] J. S. Geronimo and K. M. Case. Scattering theory and polynomials orthogonal on the unit circle. J. Math. Phys., 20(2):299–310, 1979.

[5] Jeffrey S. Geronimo. Scattering theory, orthogonal polynomials, and $q$-series. SIAM J. Math. Anal., 25(2):392–419, 1994.

[6] Ya. L. Geronimus. Orthogonal Polynomials: Estimates, Asymptotic Formulas, and Series of Polynomials Orthogonal on the Unit Circle and on an Interval. Consultants Bureau, New York, 1961.

[7] A. Martínez-Finkelshtein. Szegő polynomials: A view from the Riemann-Hilbert window. arXiv:math.CA/0508117

[8] A. Martínez-Finkelshtein, K. T.-R. McLaughlim, and E. B. Saff. Szegő orthogonal polynomials with respect to an analytic weight: canonical representation and strong asymptotics. arXiv:math.CA/0502300

[9] Paul Nevai and Vilmos Totik. Orthogonal polynomials and their zeros. Acta Sci. Math. (Szeged), 53(1-2):99–104, 1989.
[10] R. G. Newton. *Scattering Theory of Waves and Particles*. Springer Verlag, N. Y., 1982.

[11] B. Simon. Meromorphic Szegő functions and asymptotic series for Verblunsky coefficients. Preprint, SP/0502489, 2005.

[12] B. Simon. *Orthogonal Polynomials on the Unit Circle I and II*, volume 54 of *AMS Colloquium Publications*. American Mathematical Society, Providence, RI, 2005.

[13] G. Szegő. *Orthogonal Polynomials*, volume 23 of *Amer. Math. Soc. Colloq. Publ*. Amer. Math. Soc., Providence, RI, fourth edition, 1975.

[14] V. B. Uvarov. Relation between polynomials orthogonal with different weights. *Dokl. Akad. Nauk SSSR*, 126:33–36, 1959.