The influence of the broadness of the degree distribution on network’s robustness: comparing localized attack and random attack

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The stability of networks is greatly influenced by their degree distributions and in particular by their broadness. Networks with broader degree distributions are usually more robust to random failures but less robust to localized attacks. To better understand the effect of the broadness of the degree distribution we study here two models where the broadness is controlled and compare their robustness against localized attacks (LA) and random attacks (RA). We study analytically and by numerical simulations the cases where the degrees in the networks follow a Bi-Poisson distribution

\[ P(k) = \alpha e^{\lambda_1 k} + (1 - \alpha) e^{\lambda_2 k}, \alpha \in [0,1], \] with a normalization constant \( A \) where \( k \geq 0 \). In the Bi-Poisson distribution the broadness is controlled by the values of \( \alpha, \lambda_1 \) and \( \lambda_2 \), while in the Gaussian distribution it is controlled by the standard deviation, \( \sigma \). We find that only for \( \alpha = 0 \) or \( \alpha = 1 \), namely degrees obeying a pure Poisson distribution, LA and RA are the same but for all other cases networks are more vulnerable under LA compared to RA. For Gaussian distribution, with an average degree \( \mu \) fixed, we find that when \( \sigma^2 \) is smaller than \( \mu \) the network is more vulnerable against random attack. However, when \( \sigma^2 \) is larger than \( \mu \) the network becomes more vulnerable against localized attack. Similar qualitative results are also shown for interdependent networks.

I. INTRODUCTION

Complex networks are widely used as models to understand many features of complex systems, such as structure, stability and function [1, 20]. Robustness of network suffering site or link attacks is of much interest, since it has been observed in many real-world networks. Approaches such as site percolation on a network where nodes suffer either random attack (RA) [24] or targeted attack (TA) based on nodes connectivity [2, 3] emerged to study these phenomena. A new type of attack, localized attack (LA), in which nodes surrounding a seed node are removed layer by layer, has been introduced recently [21, 22]. Moreover, interdependent networks have shown significantly more vulnerability under RA and TA compared to their single counterparts [23, 25]. LA on spatially embedded interdependent networks has been addressed, and a significant metastable regime where LA above a critical size propagates throughout the whole system has been found [22].

Until now, well-defined tools have been established to probe the robustness of networks against all the aforementioned attack scenarios and the broadness of the degree distributions is found to be influential on the stability of networks [5]. However, a systematic study on the effect of the broadness of the degree distribution on robustness is still missing. Here we study and compare LA and RA on two networks models where the broadness is controlled. One model is a Bi-Poisson with two distinct groups having different average degrees. The difference between the two average degrees characterizes the broadness of the degree distribution of the network. Usually, research focused on a network with a pure Poisson degree distribution, but in many cases there could be two or more degree distributions in a network [31, 32]. For example, a network of two groups of people, of whom one group has high degree (many friends) and the other has low degree (few friends), might be better captured as a Bi-Poisson distribution. Note that Bi-Poissonian networks were found to be optimally robust against TA [31].

The second model where the broadness can be controlled is a Gaussian degree distribution. Here, the standard deviation \( \sigma \) characterizes the broadness of the degree distribution. Such a distribution is realistic; e.g., the distribution of Web links has been found to resemble a Gaussian distribution [32].

Hence, we here analyze the robustness of networks with tunable broadness of degree distributions, such as Bi-Poisson and Gaussian degree distributions, under attack. Here, limiting ourselves to LA and RA only, we follow the frameworks developed in [4, 21] and extend them (i) to the study of single networks with a Bi-Poisson distribution and (ii) to the study of single networks with a Gaussian distribution and (iii) to the study of fully interdependent networks with the same Bi-Poisson distribution in each network and (iv) to the study of fully interdependent networks with the same Gaussian distribution in each network. By changing \( \alpha \) of the Bi-Poisson distribution

\[ P(k) = \alpha e^{\lambda_1 k} + (1 - \alpha) e^{\lambda_2 k}, \alpha \in [0,1] \] with fixed \( \lambda_1 \) and \( \lambda_2 \), and \( \sigma^2 \) of the Gaussian distribution

\[ P(k) = A \cdot \exp(-\frac{(k - \mu)^2}{2\sigma^2}), k \geq 0 \] with \( \mu \) fixed, we investigate how the distribution broadness influences the percolation properties. These include the critical threshold \( p_c \) at which the giant component \( P_\infty \) first collapses, and the size of the giant components...
as a function \( p \), the fraction of unremoved nodes. In all cases, extensive simulations and analytical calculations are performed, showing good agreement with each other. Qualitative characteristics about robustness of both single and interdependent networks under LA and RA are presented.

II. RA AND LA ON A SINGLE NETWORK

A. Theory

As in Ref. [33], we introduce the generating function of the degree distribution \( P(k) \) of a certain network as

\[
G_0(x) = \sum_k P(k)x^k.
\]

(Similarly, for the generating function of the underlying branching processes, we have

\[
G_1(x) = \sum_k \frac{P(k)k}{k}x^{k-1} = \frac{G_0(x)}{G_0(1)}.
\]

Likely, the size distribution of the clusters that can be reached from a randomly chosen link is generated in a self-consistent equation

\[
H_1(x) = xG_1(H_1(x)).
\]

Then the size distribution of the clusters that can be traversed by randomly following a starting vertex is generated by

\[
H_0(x) = xG_0(H_1(x)).
\]

Next we distinguish between random attack and localized attack.

(I) Random Attack: An initial attack with the random removal of a fraction \( 1 - p \) of nodes from the network will result in changing the cluster size distribution of the remaining network, so that the generating functions of the surviving clusters’ size distribution are

\[
H_1(x) = 1 - p + pxG_1(H_1(x)),
\]

and analogously,

\[
H_0(x) = 1 - p + pxG_0(H_1(x)).
\]

Now \( p_c \), the critical value at which the giant component collapses, is determined by the following equation

\[
p_c = \frac{1}{G_1(1)},
\]

namely,

\[
p_c = \frac{1}{G_1(1)} = \frac{G_0'(1)}{G_0(1)}.
\]

which is equivalent to the expression \( p_c = \langle k \rangle / \langle k(k-1) \rangle \) found in [3].

Thus, for a Bi-Poisson distribution, since we have \( G_0(x) = \alpha x^{\lambda_1} + (1 - \alpha) x^{\lambda_2} \), \( p_c \) can be obtained as

\[
p_c = \frac{\alpha \lambda_1 + (1 - \alpha) \lambda_2}{\alpha \lambda_1^2 + (1 - \alpha) \lambda_2^2}.
\]

For a Gaussian distribution we have

\[
p_c = \frac{\sum_{k=1}^\infty ke^{-(k-\mu)^2/2\sigma^2}}{\sum_{k=1}^\infty k(k-1)e^{-(k-\mu)^2/2\sigma^2}}.
\]

And the size of the resultant giant component is

\[
P_\infty(p) = 1 - H_0(1) = p[1 - G_0(H_1(1))],
\]

which can be numerically determined by solving \( H_1(1) \) from its self-consistent equation

\[
H_1(1) = 1 - p + pG_1(H_1(1)).
\]

(II) Localized Attack: We next consider the initial removal of a fraction \( 1 - p \) of nodes locally, starting with a randomly chosen seed node. In this case we remove this seed node and its nearest neighbors, next nearest neighbors etc. until a fraction \( 1 - p \) of nodes are removed from the network. This kind of attack may be realistic in cases such as earthquakes or in cases of weapons of mass destruction. As in [21], the localized attack can be separated into two stages: (i) at the first stage nodes belonging to the attacked area (the seed node and the layers surrounding it) are removed but the links connecting them to the remaining nodes of the network are kept; (ii) at the second stage, these links are also removed. Following the method introduced in [21, 33], we get the generating function of the degree distribution of the remaining network as

\[
G_{p0}(x) = \frac{1}{G_0(f)} G_0[f + \frac{G_0(f)}{G_0(1)} (x - 1)],
\]

where \( f \equiv G_0^{-1}(p) \). Therefore, the generating function of the underlying branching processes is

\[
G_{p1}(x) = \frac{G_{p0}'(x)}{G_{p0}'(1)}.
\]

The generating function of the cluster size distribution following a random starting node in the remaining network is

\[
H_{p0}(x) = xG_{p0}(H_{p1}(x)),
\]

where \( H_{p1}(x) \), standing for the generating function of the cluster size distribution by randomly traversing a link, satisfies the self-consistent condition

\[
H_{p1}(x) = xG_{p1}(H_{p1}(x)).
\]
Eq. (21) reduces to
\[ \alpha y + (1 - \alpha) y^2 = \alpha \lambda_1 + (1 - \alpha) \lambda_2, \]
which could be viewed as a function of \( \alpha \) under LA and RA with \( \lambda_1 = 4 \) and \( \lambda_2 = 12 \). Here solid lines are theoretical predictions, from Eq. (19) for RA (red line) and Eq. (20) for LA (green line), and symbols are simulation results with network size \( N = 10^4 \), where averages are taken over 10 realizations, under LA (○) and RA (□).

The condition for the network to start having a giant component is \( G'_{p_1}(1) = 1 \) [21], which yields \( p_c \) as the solution of
\[ G''_0(G^{-1}_0(p_c)) = G'_0(1). \]

The size of the giant component \( P_\infty(p) \) as a fraction of the remaining network thus satisfies [21],
\[ P_\infty(p) = p \left[ 1 - G_{p_0}(H_{p_1}(1)) \right], \]
which can be numerically determined by first solving \( H_{p_1}(1) \) from Eq. (18), i.e. \( H_{p_1}(1) = G_{p_1}(H_{p_1}(1)) \).

In order to get \( p_c \) explicitly, we first need to get \( f_c \) from equation \( f_c = G^{-1}_0(p_c) \), namely \( f_c \) from \( G_0(f_c) = p_c \). And then from Eq. (19), \( f_c \) must also satisfy \( G_0(f_c) = G_0(1) \). In the general case, \( p_c \) and \( P_\infty \) can be obtained only by solving numerically Eqs. (19) and (20). In certain limiting cases, however, one can derive explicit analytical expressions for \( p_c \) from which more physical insight can be obtained. An example of such a specific case is given in the next subsection.

1. **Analytic solution of \( p_c \) for Bi-Poisson distribution with \( \lambda_2 = 2 \lambda_1 \)**

For a Bi-Poisson distribution, using its generating function and \( G_0(f_c) = p_c \), \( f_c \) and \( p_c \) satisfy the relation
\[ G_0(f_c) = \alpha [e^{(f_c-1)\lambda_1} + (1 - \alpha) e^{(f_c-1)\lambda_2}] = p_c. \]

Assuming \( \lambda_2 = 2 \lambda_1 \), we denote \( e^{\lambda_1(f_c-1)} = y \) such that Eq. (21) reduces to \( \alpha y + (1 - \alpha) y^2 = p_c \), which, for \( \alpha \neq 1 \), is a quadratic equation of \( y \) and its positive solution is
\[ y = \frac{\sqrt{\alpha^2 + 4 p_c (1 - \alpha)} - \alpha}{2(1 - \alpha)}. \]

Plugging \( f_c \) into Eq. (19) we get another quadratic equation of \( y \)
\[ \alpha \lambda_1^2 y + (1 - \alpha) \lambda_2^2 y^2 = \alpha \lambda_1 + (1 - \alpha) \lambda_2, \]
for which the physical solution of \( y \) is
\[ y = \frac{\sqrt{\alpha^2 \lambda_1^2 + 4(1 - \alpha) \lambda_2^2 \alpha (\lambda_1 - \lambda_2) + \lambda_2^2} - \alpha \lambda_1^2}{2(1 - \alpha) \lambda_2^2}. \]

Since \( f_c = \ln(y)/\lambda_1 + 1 \) and to obtain \( p_c \) we need to equate Eqs. (22) and (24); thus, we obtain
\[ p_c = \frac{1}{64 (1 - \alpha)} [\beta + 6 \alpha \sqrt{\alpha^2 + \beta} - 6 \alpha^2], \]
where \( \beta = 16(1 - \alpha)(2 - \alpha) \) and the relation of \( \lambda_2 = 2 \lambda_1 \) has been used for simplification. Plugging \( \alpha = 0 \) into Eq. (25), we get \( p_c = 1/\lambda_2 \) as found in [21]. Also, for \( \alpha \to 1 \), employing L’Hôpital’s rule we get \( \lim_{\alpha \to 1} p_c = 1/\lambda_1 \), as found in a pure Poisson distribution above.

Deriving \( p_c \) explicitly for a Gaussian distribution is not possible. Even for a Bi-Poisson distribution, other than special cases like the one discussed above, deriving \( p_c \) is also not possible as it requires solving first \( f_c = G^{-1}_0(p_c) \), namely \( f_c \) from Eq. (21), which could be viewed as \( \alpha \lambda_1 + (1 - \alpha) y^2 = p_c \), a polynomial equation of \( y = e^{(f_c-1)} \). Since in this paper, we consider also the cases of \( \lambda_2 > \lambda_1 \geq 4 \) and according to Abel-Ruffini theorem, there is no general algebraic solution to the above equation except some special cases. Hence Newton’s Method is employed to solve \( p_c \) and \( P_\infty \) numerically.
that the network is always more vulnerable under LA than under RA if degrees follow a Bi-Poisson distribution. Notice also that $p_c(LA)$ peaks at $\alpha = 0.79$.

For the special case of $\lambda_2 = 2\lambda_1$, we compare the analytical values of $p_c$ from Eqs. (11) and (26) using $\lambda_1 = 4$ and $\lambda_2 = 8$ with results obtained from Newton’s Method (see Fig. 3). For this combination of average degrees, $p_c(LA)$ peaks at $\alpha = 0.91$. Notice that they agree well with each other showing that Newton’s Method can deliver satisfactory results and therefore in the general case where $\lambda_2 \neq 2\lambda_1$ and in the cases of Gaussian distribution, it is used to get $p_c(LA)$.

2. Single Gaussian Networks

Fig. 4 shows the giant component $P_\infty(p)$ as a function of the occupation probability $p$ under LA and RA respectively for a single network with Gaussian degree distribution. Note that simulation results agree well with theoretical results obtained from Eqs. (15) and (20) and a second-order phase transition behavior is present for both attack scenarios. Notice here that $\mu = 4$ and $\sigma^2 = 2$, and we have $p_c(LA) < p_c(RA)$ which indicates that the network is more robust under LA than RA for this particular distribution.

Further, with $\mu$ fixed, we find that when the Gaussian distribution gets broader, i.e., $\sigma$ increases, $p_c(RA)$ decreases whereas $p_c(LA)$ increases with $\sigma$ (see Fig. 5). Notice that, if $\sigma^2 < \mu$, then $p_c(LA) < p_c(RA)$; while the opposite when $\sigma^2 > \mu$. We also see that if $\sigma^2 \approx \mu$, then we have a crossing point with $p_c(RA) \approx p_c(LA)$, which is analogous to a Poisson ER networks with the same mean and variance, thus rendering the same robustness of the network under both LA and RA as reported in [21].
Indeed in Fig. 6 we plot \( \sigma^2 \) as a function of \( \mu \) when this particular intersection point occurs, namely \( p_c(LA) = p_c(RA) \). Notice that except for some minor deviations at small \( \mu \) values because the Gaussian distribution is deformed as we require \( k \geq 0 \), and the region above the extrapolation curve corresponds to \( p_c(LA) > p_c(RA) \) whereas the region below this curve corresponds to \( p_c(LA) < p_c(RA) \).
and described above, we can simply find an equivalent random network E with generating function \( G_{E0}(x) \) such that \( p_e(p) = \frac{1}{G_{E0}(p)} + 1 \). From Eq. (30), we will end up with \( G_{E0}(x) = G_A(x) \). Thus, for pure Poisson distributions, we will have exactly the same percolation properties for fully interdependent networks under LA with that under RA, as found in [23].

However, due to the complexities of the equations above, it is very difficult to obtain explicit expressions for \( p_c \) and \( P_\infty(p) \) except for quite simple degree distributions, and thus we resort to numerical calculations in general.

### B. Results

#### 1. Fully interdependent networks with Bi-Poisson degree distribution

For two fully interdependent networks where the degrees in each network follow the same Bi-Poisson distribution, we carry out RA on one of the networks to initiate the cascading failure process until equilibrium. With the same set-up, we also carry out LA on one of the networks to launch the cascading failure process until equilibrium.

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**FIG. 8.** (Color online) Percolation thresholds \( p_c \) of the fully interdependent Bi-Poisson networks with \( \lambda_1 = 4 \), \( \lambda_2 = 12 \) as a function of \( \alpha \) under LA and RA. Here solid lines are theoretical predictions, from Eq. (29) for RA (blue line) and similarly for LA (green line) and symbols (□ for RA and ○ for LA) with error bars are simulation results with network size \( N = 10^4 \) nodes, where averages and standard deviations are taken from 20 realizations. When \( \alpha \) is not 1 or 0, \( p_c(LA) \) is always larger than \( p_c(RA) \).

**FIG. 9.** (Color online) Sizes of mutual giant component of the fully interdependent Gaussian networks as a function of \( \mu = 4 \) and \( \sigma^2 = 2 \). Here solid lines are theoretical predictions, from Eq. (26) for RA (red line) and similarly for LA (black line), and symbols are simulation results with network size \( N = 10^4 \), where averages are taken over 10 realizations, under LA (○) and RA (□).
is reached. Fig. 7 shows the sizes of the giant component \( P_\infty(p) \) of the system as a function of the occupation probability \( p \) under LA and RA. Note that simulation results agree well with theoretical results obtained from Eq. 20. The results demonstrate a first-order phase transition behavior for both attack scenarios; however, the system is much more vulnerable compared to single networks under attack due to interdependency. Here with \( \mu = 4 \) and \( \sigma^2 = 2 \), the system represents more fragility under LA compared to RA with \( p_c(LA) < p_c(RA) \) and the giant components behave differently.

Further, with \( \mu \) fixed, we find that when the Gaussian distribution gets broader, i.e., as \( \sigma \) increases, analogous to what we find in a single Gaussian network, the system shows different behaviors of the critical \( p_c \) against LA and RA. More specifically, Fig. 10 shows the effect of \( \sigma \) on \( p_c \) of the fully interdependent Gaussian networks where if \( \sigma^2 < \mu \), then \( p_c(LA) < p_c(RA) \); the opposite occurs if \( \sigma^2 > \mu \). Again, the intersection point in this figure happens near the point \( \sigma^2 \approx \mu \) similar to the Poisson distribution networks and thus the system behaves the same under LA as that under RA, as found in the previous subsection.

IV. CONCLUSIONS

In summary, we show that LA on interdependent networks can be mapped to a RA problem by transforming the network under initial attack. We also show how the broadness of the degree distribution affects the robustness of networks against RA and LA respectively. We show here that generally the broader the degree distribution is the more risky the networks are under LA than under RA.

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