A sufficient condition for finiteness of Frobenius test exponents

Kyle Maddox
26 September, 2018

Abstract

The Frobenius test exponent \( Fte(R) \) of a local ring \((R, m)\) of prime characteristic \(p > 0\) is the smallest \(e_0 \in \mathbb{N}\) such that for every ideal \(q\) generated by a (full) system of parameters, the Frobenius closure \(q^p\) has \((q^p)^{[p^{e_0}]} = q^{[p^{e_0}]}\). We establish a sufficient condition for \(Fte(R) < \infty\) and use it to show that if \(R\) is such that the Frobenius closure of the zero submodule in the lower local cohomology modules has finite colength, i.e. \(H^j_m(R)/0^F\) has finite length for \(0 \leq j < d\), then \(Fte(R) < \infty\).

0 Introduction

Let \((R, m)\) be a commutative Noetherian local ring of characteristic \(p > 0\). The Frobenius closure of an ideal \(I \subset R\) is defined to be the ideal \(I^F = \{x \in R \mid x^{p^e} \in I^{[p^e]} \text{ for some } e \in \mathbb{N}\}\}. In general, computing the Frobenius closure of an ideal should be expected to be difficult, because we must check infinitely many equations for every element of the ring. However, Frobenius bracket powers are much simpler to compute, and since \(R\) is Noetherian we must have an \(e_0 \in \mathbb{N}\) such that \((I^F)^{[p^{e_0}]} = I^{[p^{e_0}]}\) for all \(e \geq e_0\), and so we can simply check one equation – \(x \in I^F\) if and only if \(x^{p^{e_0}} \in I^{[p^{e_0}]}\). However, computing the required \(e_0\) for each \(I\) might also be difficult, so it would be desirable to get bounds for each \(I\) depending only on the ring.

It turns out that one cannot expect nice behaviour like this even in nice rings – Brenner [Bre06] showed that in a two-dimensional domain standard graded over a field we can have a sequence of ideals where the required exponent tends to infinity. However, some finiteness results are known if we restrict to the class of parameter ideals – the Frobenius test exponent for (parameter ideals of) \(R\) is the smallest \(e_0\) such that for any \(q \subset R\) a parameter ideal, \((q^F)^{[p^{e_0}]} = q^{[p^{e_0}]}\).

It was shown by Katzman and Sharp [KS05] that \(Fte(R) < \infty\) if \(R\) is Cohen-Macaulay, and later Huneke, Katzman, Sharp, and Yao [HKSY06] showed \(Fte(R) < \infty\) if \(R\) is generalized Cohen-Macaulay using some very involved techniques. More recently, Quy [Quy18] vastly simplified the proof for generalized Cohen-Macaulay rings, and also showed for F-nilpotent rings that \(Fte(R) < \infty\) using the relative Frobenius action on local cohomology introduced by Quy and Polstra [PQ18]. Quy’s proofs lend themselves to a sufficient condition for finiteness of the Frobenius test exponent, the main theorem of this paper (Theorem 3.1), and we can extend his techniques to a new class of F-singularity (generalized F-nilpotent rings, see Definition 3.9 for a definition and Theorem 3.11 for the proof of the theorem). The author is interested in seeing how much further this sufficient condition can be pressed and if it is indeed a necessary condition for finite Frobenius test exponents.
Notation and conventions: Throughout, \((R, \mathfrak{m})\) will be a Noetherian local ring of dimension \(d\) and of prime characteristic \(p > 0\). By a parameter ideal, we will mean an ideal generated by a full system of \(d\) parameters. Write \(\text{Spec}^\circ(R) = \text{Spec}(R) \setminus \{\mathfrak{m}\}\) and for any subset \(X \subset \text{Spec}(R)\), write \(X^\circ = X \cap \text{Spec}^\circ(R)\). In particular, \(\text{Ass}_R^\circ(M) = \text{Ass}_R(M) \cap \text{Spec}^\circ(R)\).

If \(x_1, \cdots, x_t\) is an (ordered) sequence of elements of \(R\), write \(\underline{x} = x_1, \cdots, x_t\) for the list of elements. Given \(\underline{x} = x_1, \cdots, x_t\) and a sequence \(n_1, \cdots, n_t \in \mathbb{N}\), write \(\underline{x}^n = x_1^{n_1}, \cdots, x_t^{n_t}\) for the new sequence obtained by taking powers. In particular, if \(n \in \mathbb{N}\) then \(\underline{x}^n = x_1^n, \cdots, x_t^n\). If a sequence \(\underline{x} = x_1, \cdots, x_t\) is given and \(J = (\underline{x})\), then \(J_i = (x_1, \cdots, x_i)\) (this is where order may come into play). Set \(J_0 = 0\).

The set \(\mathbb{N}\) contains 0 (so that it may be treated as a commutative semiring) and \(\mathbb{Z}_+\) will be used for the set of positive integers.

Acknowledgments The author would like to thank his advisor Ian Aberbach for suggesting generalized \(F\)-nilpotent rings as a case study for the sufficient condition and countless grammar corrections. Also, the author would like to thank Thomas Polstra and Pham Hung Quy for looking over initial drafts of this paper and providing a multitude of helpful comments.

1 Background

1.1 Filter regular sequences

This section is characteristic-independent.

Definition 1.1. An element \(x \in R\) is filter regular or \(\mathfrak{m}\) filter regular if \(x \in \mathfrak{m}\) and \(x \not\in p\) for any \(p \in \text{Ass}_R^\circ(R)\). A sequence \(\underline{x} = x_1, \cdots, x_t\) is a filter regular sequence if \(x_1\) is filter regular, \(x_2 + x_1R\) is filter regular in \(R/x_1R\), and so on – equivalently that \(x_i \not\in p\) for all \(p \in \text{Ass}_R^\circ(R/(x_1, \cdots, x_{i-1}))\).

Remark: The sequence \(\underline{x} = x_1, \cdots, x_t\) is filter regular if and only if \(\underline{x}^n = x_1^{n_1}, \cdots, x_t^{n_t}\) is a filter regular sequence for any \(n \in (\mathbb{Z}_+)^t\).

Proposition 1.2. Let \(q \subset R\) be a parameter ideal. Then, there is a filter regular system of parameters \(\underline{x} = x_1, \cdots, x_d\) such that \(q = (\underline{x})\).

Proof. If \(d = 0\), there is nothing to prove. Otherwise, pick the first parameter:

\[
x_1 \in q \setminus \left( mq \cup \bigcup_{p \in \text{Ass}_R^\circ(R)} p \right),
\]

which is a nonempty set by prime avoidance. Then we repeat in \(R/x_1R\). After we have selected \(d\) such elements, we have \(d\) minimal generators \(\underline{x} = x_1, \cdots, x_d\) of \(q\), a parameter ideal, so \((\underline{x}) = q\).

Proposition 1.3. Let \(\underline{x} = x_1, \cdots, x_t\) be a filter regular sequence in \(R\) and let \(J = (\underline{x})\). (Recall \(J_i = (x_1, \cdots, x_i)\).) Then \((J_{i-1} : R x_i)/J_{i-1}\) is finite length as an \(R\)-module.

Proof. Since \(x_i + J_{i-1} \not\in p/J_{i-1}\) for any \(p \in \text{Ass}_R^\circ(R/J_{i-1})\), this forces \(\text{Ass}_R((J_{i-1} : R x_i)/J_{i-1}) \subset \{\mathfrak{m}\}\). Then since \((J_{i-1} : x_i)/J_{i-1}\) is finitely generated, it must be finite length.
Proposition 1.4. Let \( \underline{x} = x_1, \cdots, x_t \) be a filter regular sequence in \( R \) and let \( J = (\underline{x}) \). Then for any \( j > 0 \) and any \( I \subset R \) an ideal, we have \( H^{j}_I(R/J) \simeq H^{j}_I(R/(J_{i-1} : R x_i)) \). Consequently, for \( j > 0 \) the map \( x_i : R/J_{i-1} \rightarrow R/J_i \) induces the long exact sequence:

\[
\cdots \rightarrow H^{j}_I(R/J_{i-1}) \xrightarrow{x_i} H^{j}_I(R/J_i) \xrightarrow{\alpha} H^{j}_I(R/J_i) \xrightarrow{\beta} H^{j+1}_I(R/J_{i-1}) \xrightarrow{x_i} \cdots
\]

where \( j > 0 \) and \( \beta \) is the connecting morphism.

We will refer to this as the filter regular long exact sequence in local cohomology.

Proof. By the previous proposition, \( (J_{i-1} : R x_i)/J_{i-1} \) is finite length and hence dimension 0, and thus \( H^{j}_I((J_{i-1} : R x_i)/J_{i-1}) = 0 \) if \( j > 0 \) as local cohomology vanishes above the dimension. Hence the canonical ideal short exact sequence:

\[
0 \longrightarrow (J_{i-1} : R x_i)/J_{i-1} \longrightarrow R/J_{i-1} \longrightarrow R/(J_{i-1} : R x_i) \longrightarrow 0
\]

gives the desired isomorphism after applying \( H^{j}_I(\bullet) \). Then we factor the map \( x_i : R/J_{i-1} \rightarrow R/J_i \) into the short exact sequence:

\[
0 \longrightarrow R/(J_{i-1} : R x_i) \xrightarrow{x_i} R/J_{i-1} \xrightarrow{\pi} R/J_i \longrightarrow 0
\]

to which we also apply \( H^{j}_I(\bullet) \) and use the previous result. \( \square \)

1.2 Frobenius closure of an ideal and Frobenius test exponents

Now we return to the case that \( R \) is of prime characteristic \( p > 0 \).

Definition 1.5. Let \( I \subset R \) be an ideal. The Frobenius closure of \( I \) is the ideal:

\[
I^F = \left\{ x \in R \mid x^{p^e} \in I^{[p^e]} \text{ for all } e \gg 0 \right\},
\]

which contains \( I \). Furthermore, \( \bullet^F \) is a closure operation on ideals, i.e. if \( I \subset J \), then \( I^F \subset J^F \) and \( I^F = (I^F)^F \).

Definition 1.6. Since \( R \) is Noetherian, it is clear that there is an \( e_0 \in \mathbb{N} \) such that \( (I^F)^{[p^e]} = I^{[p^e]} \) for all \( e \geq e_0 \). Call the smallest such exponent the Frobenius test exponent for \( I \), and denote it \( \text{Fte}(I) \).

Given that there is an \( e_0 \) for each \( I \), it is natural to try and make the statement uniform – does there exist an \( e_0 \in \mathbb{N} \) such that for all ideals \( I \subset R \), \( (I^F)^{[p^e]} = I^{[p^e]} \)? However, even in relatively nice rings we can run into trouble – Brenner \[Bre06\] showed that there is a sequence of ideals in a two-dimensional normal domain, standard graded over a field, such that some sequence of ideals has required exponents tending to infinity.

However, if we restrict the class of ideals and rings we consider, then we can show some uniformity. In particular, Katzmann and Sharp \[KS06\] showed that if \( R \) is Cohen-Macaulay and \( I \) is a parameter ideal, then there is a uniform bound depending only on the ring.

Definition 1.7. We define the Frobenius test exponent (for parameter ideals) of \( R \) to be:

\[
\text{Fte}(R) = \inf_{\epsilon \in \mathbb{N}} \left\{ (q^F)^{[p^\epsilon]} = q^{[p^\epsilon]} \text{ for all parameter ideals } q \subset R \right\} \in \mathbb{N} \cup \{\infty\}.
\]
This number is a coarse measure of singularity in characteristic \( p \) — that is, if \( R \) is regular than \( \text{Fte}(R) = 0 \) since all ideals of \( R \) must be tightly closed (and hence Frobenius closed). However, there are non-regular rings with \( \text{Fte}(R) = 0 \), for instance F-injective Cohen-Macaulay rings or F-pure rings. The authors of [QS16] define parameter F-closed rings to be rings for which \( \text{Fte}(R) = 0 \), so that \( \text{Fte}(R) \) is a measure of how close \( R \) is to being parameter F-closed. We are interested in cases where \( \text{Fte}(R) < \infty \).

2 Frobenius actions on modules

2.1 Basics of Frobenius actions

**Definition 2.1.** Let \( M \) and \( N \) be \( R \)-modules and let \( \alpha : M \to N \) be an abelian group homomorphism. Say \( \alpha \) is \( p^e \text{-linear} \) for some \( e \in \mathbb{N} \) if \( \alpha(xm) = xp^e\alpha(m) \) for any \( x \in R \) and \( m \in M \). A \( p \)-linear endomorphism \( f_M \) on \( M \) is called a Frobenius action on \( M \). We will ruthlessly suppress the subscript and write \( f = f_M \). If \( M = R \), then there is a standard choice of \( f \) — the Frobenius endomorphism \( F(r) = r^p \). \( R \) will always be considered as an \( R \)-module with a Frobenius action via the Frobenius endomorphism.

**Definition 2.2.** Let \( M \) be an \( R \)-module with a Frobenius action \( f \). A submodule \( M' \subset M \) is an \( f \)-submodule or is said to be \( f \)-stable if \( f(M') \subset M' \). If \( M' \) is an \( f \)-submodule, we can define the Frobenius orbit closure of \( M' \) to be \( (M')_{f} = \{ m \in M | f^e(m) \in M' \text{ for some } e \in \mathbb{N} \} \). Clearly if \( M' \subset M \) is an \( f \)-submodule, then \( f|_{M'} : M' \to M' \) is a Frobenius action on \( M' \).

**Remark:** This is sometimes called the Frobenius closure of \( M' \), but in general does not necessarily agree with other notions of Frobenius closure. For instance, considering the case of the submodule \( I \) in \( R \), we do not get the original definition of Frobenius closure established earlier.

**Definition 2.3.** Let \( M \) and \( N \) be \( R \)-modules with Frobenius actions \( f_M \) and \( f_N \) respectively. An \( R \)-linear map \( \alpha : M \to N \) commutes with Frobenius if \( f_N \circ \alpha = \alpha \circ f_M \).

**Example 2.4.** Let \( S \) be another characteristic \( p \) ring and \( \varphi : R \to S \) be a ring homomorphism. Then \( \varphi \) commutes with Frobenius, as it is multiplicative — i.e., \( \varphi(F(x)) = \varphi(x^p) = \varphi(x)^p = F(\varphi(x)) \).

**Proposition 2.5.** Let \( M, M', N, \text{ and } N' \) be \( R \)-modules with Frobenius actions, and let \( \alpha : M \to N \) and \( \beta : M' \to N' \) be maps which commute with Frobenius. Furthermore, let \( N'' \subset N \) be an \( f \)-submodule. Then:

- a) \( \text{im}(\alpha) \) and \( \text{ker}(\alpha) \) are \( f \)-submodules.
- b) \( \alpha(0^f_M) \subset 0^f_N \), and moreover \( \alpha^-1((N'')^f_M) = (\alpha^-1(N''))^f_M \).
- c) \( N/N'' \) has an induced Frobenius action given by \( f(n + N'') = f(n) + N'' \), and the projection map \( \pi : N \to N/N'' \) commutes with this action.
- d) \( M \oplus M' \) and \( N \oplus N' \) have induced Frobenius actions given by \( f(a, b) = (f(a), f(b)) \) and \( \alpha \oplus \beta \) commutes with the actions given. Furthermore, \( M \leftrightarrow M \oplus M' \) commutes with the actions, i.e. \( M \oplus 0 \) is an \( f \)-submodule of \( M \oplus M' \).

\(^1\)In fact, if \( F \) is the Frobenius endomorphism on \( R \), \( I^e_F = \sqrt{I} \) instead of the (usually strictly smaller) ideal \( I^e \).
Remark: One can view a Frobenius action $f$ on an $R$-module $M$ as a left module structure over a noncommutative $R$ algebra $R[F]$. This perspective is not necessary for the rest of this paper but it makes proving the previous proposition just observing a $f$-submodule is a $R[F]$-submodule, a map which commutes with Frobenius is simply $R[F]$-linear, etc. We can also consider complexes of $R$-modules which are also left $R[F]$-modules, and consequently the homology of these complexes will also have Frobenius actions.

Example 2.6. Recall the Čech cocomplex on the elements $\underline{x} = x_1, \cdots, x_t$ of $R$, defined below.

$$\check{C}^j := \check{C}^j(\underline{x}; R) = \bigoplus_{1 \leq i_1 < \cdots < i_j \leq t} R_{x_{i_1} \cdots x_{i_j}}$$

Each summand of $\check{C}^j$ is a characteristic $p$ ring and so has a Frobenius endomorphism. By Proposition 2.3, $\check{C}^j$ has the action on the direct sum. The maps in the Čech cocomplex are simply given by taking signed linear combinations of the elements in each place, and hence the Čech cohomology – the local cohomology of the ideal $(\underline{x})$ on $R$– has a Frobenius action as well. This induced action on the quotient will be referred to as the standard Frobenius action on local cohomology.

This action is the primary one of interest throughout the rest of this paper.

Remark: Notice that parts b) and c) of Proposition 2.5 show that to study $(N^\prime \prime)^f_N$, we need only study $0^f M / N^\prime \prime$ since $\pi^{-1}(0^f M / N^\prime \prime) = (N^\prime \prime)^f_N$. This will then be the primary case of interest.

2.2 Hartshorne-Speiser-Lyubeznik numbers

Similar to the desire for a finite test exponent for $R$, it is natural to seek a uniform “test exponent” $e$ such that $f^e : 0^f M \to 0^f M$ is the zero map.

Definition 2.7. Let $M$ be an $R$-module with a Frobenius action. Define the Hartshorne-Speiser-Lyubeznik number of $M$ to be:

$$\text{HSL}(M) = \inf_{e \in \mathbb{N}} \{f^e(0^f M) = 0\} \in \mathbb{N} \cup \{\infty\}.$$  

If $M$ is finitely generated, it is clear that $\text{HSL}(M) < \infty$ – since for any generating set $m_1, \cdots, m_r$ of $0^f M$ we have for each $i$ an $e_i \in \mathbb{N}$ such that $f^{e_i}(m_i) = 0$, so we have $\text{HSL}(M) = \max e_i$. Another important case is also known.

Theorem 2.8 ([HS77], [Lyu97], [Sha07]). Let $A$ be an Artinian $R$-module with a Frobenius action. Then $\text{HSL}(A) < \infty$. In particular, (re)-define $\text{HSL}(R) := \max\{\text{HSL}(H^j_{\mathfrak{m}}(R))\}$, and we have $\text{HSL}(R) < \infty$.

2.3 The relative Frobenius action on local cohomology

This section summarizes material from [PQ18] and [Quy18].

Definition 2.9. Let $I, J \subset R$ be ideals. The Frobenius endomorphism $F : R/J \to R/J$ can be factored as follows:
where \( f_R(x + J) = x^p + J^{[p]} \). Call the map \( f_R \) the **relative Frobenius map** on \( R/J \). Taking the Čech complex on generators of an ideal \( I \) on the modules \( R/J \) and \( R/J^{[p]} \), the \( p \)-linear map \( f_R^e : R/J \to R/J^{[p]} \) induces the **relative Frobenius action on local cohomology**, \( f_R^e : H^j_I(R/J) \to H^j_I(R/J^{[p]}) \).

There does not seem to be another definition of “relative Frobenius closure” so we omit the descriptor “orbit” used earlier. Now we will define many of the same earlier Frobenius action ideas with respect to this relative map – however keep in mind that it is not a Frobenius action since it is not an endomorphism.

**Definition 2.10.** Let \( I, J \subset R \) be ideals. The **relative Frobenius closure of zero in** \( H^j_I(R/J) \) is the submodule:

\[
0^{f_R^e}_{H^j_I(R/J)} = \left\{ \xi \in H^j_I(R/J) \mid f_R^e(\xi) = 0 \in H^j_I(R/J^{[p]}) \text{ for some } e \in \mathbb{N} \right\}.
\]

Notice that if \( f_R^e(\xi) \in 0^{f_R^e}_{H^j_I(R/J^{[p]})} \) then \( \xi \in 0^{f_R^e}_{H^j_I(R/J)} \).

As for Frobenius actions, we define the **relative Hartshorne-Speiser-Lyubeznik number** of \( H^j_I(R/J) \) or the **Hartshorne-Speiser-Lyubeznik number** of \( H^j_I(R/J) \) with respect to \( R \) to be:

\[
\text{HSL}_R(H^j_I(R/J)) = \inf_{e \in \mathbb{N}} \left\{ f_R^e \left( 0^{f_R^e}_{H^j_I(R/J)} \right) = 0 \subset H^j_I(R/J^{[p]}) \right\} \in \mathbb{N} \cup \{ \infty \}.
\]

**Proposition 2.11.** The maps given in Proposition 1.4 commute with \( f_R \). That is, given any ideal \( I \subset R \) and any filter regular sequence \( x = x_1, \ldots, x_t \) with \( J = (x) \), we have a commutative diagram with exact rows for any \( e, e' \in \mathbb{N} \), any \( j > 0 \) and any \( 1 \leq i \leq t \):

\[
\cdots \xrightarrow{x_i} H^j_I(R/J_{i-1}) \xrightarrow{\alpha_i} H^j_I(R/J_i) \xrightarrow{\beta_i} H^{j+1}_I(R/J_{i-1}) \xrightarrow{} \cdots
\]

\[
\downarrow f_R^e \quad \quad \downarrow f_R^e \quad \quad \downarrow f_R^e
\]

\[
\cdots \xrightarrow{x_i^{e'}} H^j_I(R/J_{i-1}^{[p]}) \xrightarrow{\alpha_i} H^j_I(R/J_i^{[p]}) \xrightarrow{\beta_i} H^{j+1}_I(R/J_{i-1}^{[p]}) \xrightarrow{} \cdots
\]

**Proof.** Consider the following commutative diagram of short exact sequences for any \( e \) and \( e' \) in \( \mathbb{N} \):

\[
0 \to R/(J_{i-1} :_R x_i) \xrightarrow{x_i} R/J_{i-1} \xrightarrow{\pi} R/J_i \xrightarrow{} 0
\]

\[
\downarrow (f_R^e)' \quad \quad \downarrow f_R^e \quad \quad \downarrow f_R^e
\]

\[
0 \to R/(J_{i-1}^{[p]} :_R x_i^{e'}) \xrightarrow{x_i^{e'}} R/J_{i-1}^{[p]} \xrightarrow{\pi e} R/J_i^{[p]} \xrightarrow{} 0
\]

where \( (f_R^e)' : R/(J_{i-1} :_R x_i) \to R/(J_{i-1}^{[p]} :_R x_i^{e'}) \) by \( x + (J_{i-1} :_R x_i) \mapsto x^{e'} + (J_{i-1}^{[p]} :_R x_i^{e'}) \). We then apply the functor \( H^j_I(\bullet) \) and one can check that the map \( (f_R^e)' \) composed with the isomorphism given in Proposition 1.4 gives \( f_R \). \( \square \)
Proposition 2.12. Let \( \mathfrak{p} = x_1, \cdots, x_d \) be a filter regular system of parameters and \( q = (\mathfrak{p}) \). Then \( \text{HSL}_R(H^0_m(R/q)) = \text{Fte}(q) \).

Proof. Since \( q \) is \( m \)-primary, \( R/q \) is \( m \)-torsion, and hence \( H^0_m(R/q) = R/q \). Note the relative Frobenius action \( f^e_R : R/q \to R/q^{[p^e]} \) has \( 0^{f^e_R}_{R/q} = q^F/q \) and \( f^e_R(q^F/q) = (q^F)^{[p^e]}/q^{[p^e]} \). Hence the smallest \( e_0 \) such that \( (q^F)^{[p^e_0]} = q^{[p^e_0]} \), i.e. \( \text{Fte}(q) \), is the same as the smallest \( e \) such that \( f^e_R(0^{f^e_R}_{R/q}) = 0 \), i.e. \( \text{HSL}_R(R/q) \).

This connection and the commutativity of the diagram in Proposition 2.11 gives us the ability to use \( \text{HSL}(H^i_m(R)) \) to control \( \text{Fte}(q) \), as long as we know that uniformly in \( e \) the maps \( \alpha_e \) eventually do not map too many elements into the Frobenius closure of zero. Note that commutative of the same diagram shows that

\[
\alpha_e \left( 0^{f^e_R}_{H^i_m(R/q^{[p^e]})} \right) \subset 0^{f^e_R}_{H^i_m(R/q^{[p^e]})}
\]

just as when \( M \) and \( N \) are modules with a Frobenius action and \( \alpha : M \to N \) is \( F \) equivariant, then \( \alpha(0^F_M) \subset 0^F_N \).

3 Finite Frobenius test exponents

3.1 The sufficient condition

In [Quy18], Quy essentially uses this condition in his proofs of Corollary 3.5 and Corollary 3.8.

Theorem 3.1. Suppose there is an \( e_0 \in \mathbb{N} \) such that, for any \( e \geq e_0 \), any parameter ideal \( q \) generated by a filter regular system of parameters \( \mathfrak{p} = x_1, \cdots, x_d \), and all \( 0 \leq i + j < d \) we have the map \( \alpha_e : H^i_m \left( R/q^{[p^e]} \right) \to H^i_m \left( R/q^{[p^e]} \right) \) induced by the map \( \pi_e : R/q^{[p^e]} \to R/q^{[p^e]} \) has the property that

\[
\alpha_e^{-1} \left( 0^{f^e_R}_{H^i_m(R/q^{[p^e]})} \right) = 0^{f^e_R}_{H^i_m(R/q^{[p^e]})}
\]

for all \( i + j < d \). Then, \( \text{Fte}(R) \leq e_0 + \sum_{k=0}^{d} \binom{d}{k} \text{HSL}(H^k_m(R)) \).

Proof. Replace \( q \) by \( q^{[p^e]} \) and note \( \text{Fte}(q) = \text{Fte} (q^{[p^e]} ) + e_0 \), so it suffices to assume each \( \alpha_e \) has the property. For notational convenience, we set \( S_{i,e} = R/q^{[p^e]} \) and \( S_i = S_{i,0} \).

We claim now that:

\[
\text{HSL}_R(H^i_m(S_i)) \leq \sum_{k=j}^{i} \binom{i}{k-j} \text{HSL}(H^k_m(R))
\]

for all \( i + j \leq d \). We will show this by induction on \( i \). If \( i = 0 \), then \( \text{HSL}_R(H^0_m(R)) = \binom{0}{0} \text{HSL}(H^0_m(R)) \) (as when \( i = 0 \), \( q_i = 0 \) so \( f_R = F \)), so there’s nothing to show.

Now suppose for any \( 0 \leq j \leq d - i + 1 \) the result holds. Consider the commutative diagram of long exact sequences in local cohomology from Proposition 2.11. Let \( e = \text{HSL}_R(H^{i+1}_m(S_{i-1,e})) \) and \( e' = \text{HSL}_R(H^k_m(S_{i-1,e})) \). By induction and manipulation of the binomial coefficients, \( e + e' \leq \sum_{k=j}^{i+j} \binom{i}{k-j} \text{HSL}(H^k_m(R)) \) and so if we can show that \( \text{HSL}_R(H^i_m(S_i)) \leq e + e' \), we are finished.
To that end, take $\xi \in H^i_m(S_i)$ which is in the relative Frobenius closure of zero. Then $\beta_0(\xi)$ is in the relative Frobenius closure of zero in $H^i_{m+1}(S_{i-1})$, so $0 = f^e_R(\beta_0(\xi)) = \beta_0(f^e_R(\xi))$ by choice of $e$. But by exactness, we then have an $\xi' \in H^i_m(S_{i-1})$ such that $\alpha_\xi(\xi') = f^e_R(\xi)$, and since $f^e_R(\xi)$ is in the relative Frobenius closure of zero in $H^i_m(S_{i-1})$, by hypothesis on $\alpha_\xi$ we have $\xi'$ is in the relative Frobenius closure of zero in $H^i_m(S_{i-1})$. But then by choice of $e'$, we have:

$$f^{e+e'}_R(\xi) = f^{e'}_R(\alpha_e(\xi')) = \alpha_{e+e'}(f^{e'}_R(\xi')) = \alpha_{e+e'}(0) = 0.$$  

Since $\xi$ was arbitrary, $HSL_R(H_m^j(S_i)) \leq e + e'$, as required.

3.2 Some cases

We now consider some cases where we may apply the condition.

Definition 3.2. $R$ is generalized Cohen-Macaulay if for all $0 \leq j < d$, $H^j_m(R)$ is finite length.

Definition 3.3. Let $(R, m)$ be a local ring of dimension $d$ and let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters. Let $q = (x_1, \ldots, x_d)$ and $q_i = (x_1, \ldots, x_i)$. Then say $\underline{x}$ is standard if $q \cdot H^j_m(R/q_i) = 0$ for all $i + j < d$.

Lemma 3.4. Suppose $R$ is generalized Cohen-Macaulay. Then there is an $n_0 \in \mathbb{N}$ such that for any filter regular system of parameters $\underline{x} = x_1, \ldots, x_d$, we have $\underline{x}^n = x_1^n, \ldots, x_d^n$ is standard for all $n \geq n_0$.

Proof. This follows from a slight modification of the proof of Lemma 3.10 replacing the given ideals $b_j$ with $a_j = \text{Ann}_R(H^j_m(R))$ and $a = a_1 \cdots a_{d-1}$, and then ignoring the Frobenius parts of the proof.

To be precise, for any $q$ a parameter ideal of $R$ generated by a filter regular system of parameters $\underline{x} = x_1, \ldots, x_d$, we have:

$$a^{n_0} \subset \text{Ann}_R(H^j_m(R/q_i)),$$

using Proposition 1.4 and a similar argument as in Lemma 3.10. But $a$ is $m$-primary and consequently $m^N \subset a$, so we can take $n_0 = 2^dN$ to be the required exponent.

Corollary 3.5 ([HKSY06], [Quy18]). Suppose $R$ is a generalized Cohen-Macaulay ring. Then $F\text{te}(R) < \infty$.

Proof. If $q$ is standard, then each $\alpha_e$ as in Theorem 3.1 is injective which implies the condition is satisfied trivially. If $q$ is not, replace $q$ by $q^{[p^{e_0}]}$ where $e_0$ is minimal with $p^{e_0} \geq n_0$ from Lemma 3.4.

Remark: This shows that Brenner’s example in [Bre06] has a finite Frobenius test exponent for parameter ideals – any domain is equidimensional, and any equidimensional ring $R$ with $\text{dim}(R) = 2$ is generalized Cohen-Macaulay.

Definition 3.6. Say $R$ is weakly F-nilpotent if the standard Frobenius actions $F$ on $H^j_m(R)$ have $F^e(H^j_m(R)) = 0$ for each $0 \leq j < d$ and some $e$ – which can be taken to be $HSL(R)$.
Lemma 3.7 ([PQ18], Theorems 4.2 and 4.4). Suppose \( R \) is weakly F-nilpotent. Then for any filter regular sequence \( \underline{x} = x_1, \ldots, x_t \) we have:

\[
H^j_m(R/\langle \underline{x} \rangle) = 0^{f_R}_{H^j_m(R/\langle \underline{x} \rangle)}
\]

for \( j < d - t \).

\[\text{Proof.}\] Again, this is a specialization of Lemma 3.10. As under the hypotheses, the ideal \( b \) in Lemma 3.10 is simply all of \( R \).

Corollary 3.8 ([Quy18]). Suppose \( R \) is a weakly F-nilpotent ring. Then \( \text{Fte}(R) < \infty \).

\[\text{Proof.}\] The condition on the maps \( \alpha_e \) in Theorem 3.1 are trivial in this case, as both modules are nilpotent with respect to \( f_R \) by Lemma 3.7.

Motivated by the similarity of the two proofs offered for Corollary 3.5 and Corollary 3.8 in [Quy18], we define a new class of F-singularities which we can also show have finite Frobenius test exponents.

3.3 A new case

Definition 3.9. Say \( R \) is generalized weakly F-nilpotent if \( H^j_m(R)/0^F_{H^j_m(R)} \) is finite length for all \( 0 \leq j < d \).

We shall see (Theorem 3.11) that such a ring has a finite Frobenius test exponent. As of now, however, the author does not have an example of a generalized weakly F-nilpotent ring which is not either generalized Cohen-Macaulay or F-nilpotent. We combine the ideas of Lemma 3.7 and Lemma 3.4 for generalized weakly F-nilpotent rings.

Lemma 3.10. Suppose \( R \) is generalized weakly F-nilpotent. Then there is an \( e_0 \in \mathbb{N} \) such that for all \( e \geq e_0 \) and for any parameter ideal \( q = (\underline{x}) \) generated by a filter regular system of parameters \( \underline{x} = x_1, \ldots, x_d \), any \( 0 \leq i \leq d - 1 \) and any \( 0 \leq j < d - i \), we have:

\[
q^{[p^j]} \cdot H^j_m(R/q^{[p^j]}) \subset 0^{f_R}_{H^j_m(R/q^{[p^j]})}.
\]

\[\text{Proof.}\] By hypothesis, the ideals:

\[
b_j = \text{Ann}_R \left( H^j_m(R)/0^F_{H^j_m(R)} \right)
\]

for \( 0 \leq j < d \) are \( m \)-primary or all of \( R \), and hence \( b = b_0 \cdots b_{d-1} \) is either \( m \)-primary or all of \( R \).

For any parameter ideal \( q \) generated by a filter regular system of parameters \( \underline{x} = x_1, \ldots, x_d \), recall \( q_i = (x_1, \ldots, x_i) \), and, as in the proof of Theorem 3.1, we set \( S_i,e = R/q^{[p^j]}_i \) and \( S_i = S_i,0 \).

We now claim that for any such \( q \), any \( 0 \leq i \leq d \), and any \( 0 \leq j < d - i \) we have:

\[
b^{p^j} \subset \text{Ann}_R \left( H^j_m(S_i)/0^F_{H^j_m(S_i)} \right).
\]

We induce on \( i \). If \( i = 0 \), then the statement is the simple fact that \( b \subset b_j \) for any \( 0 \leq j < d \).

When \( i > 0 \), we can consider the commutative diagram from Proposition 2.11.
Let $\xi \in H^j_m(S_i)$ and suppose $y \in b^{2^j-1}$. Then in $H^{j+1}_m(S_{i-1})$ we have for some $e \in \mathbb{N}$:

$$0 = f_R^e(y\xi_0(\xi)) = f_R^e(\beta_0(y\xi)) = \beta_e(f_R^e(y\xi))$$

and by exactness, there is an $\xi' \in H^j_m(S_{i-1,e})$ such that $\alpha_e(\xi') = f_R^e(\beta(y\xi))$. But $y^{p^e} \in b^{2^j-1}$ so $y^{p^e} \xi'$ is in the relative Frobenius closure of zero in $H^j_m(S_{i-1,e})$ and hence $\alpha_e(y^{p^e} \xi') = y^{p^e} f_R^e(\xi) = f_R^e(y^{2^j} \xi)$ is in the relative Frobenius closure of zero in $H^j_m(S_{i,e})$. But then $y^{2^j} \xi$ is in the relative Frobenius closure of zero in $H^j_m(S_i)$, so we have:

$$y^{2^j} \xi \in \operatorname{Ann}_R \left( H^j_m(S_i)/0_{H^j_m(S_i)} \right)$$

showing the claim.

Now, pick $N$ minimal so $m^N \subseteq b$. Then for any $e \in \mathbb{N}$ with $p^e \geq N 2^{d-1}$,

$$q^{[p^e]} H^j_m(S_{i,e}) \subseteq 0_{H^j_m(S_{i,e})}$$

for any parameter ideal $q$ and any $0 \leq i + j < d$, and so taking $e_0$ minimal among all such $e$, we have the result.

**Theorem 3.10.** Suppose $R$ is a generalized weakly F-nilpotent ring. Then $\operatorname{Fte}(R) < \infty$.

**Proof.** Adopt the notation in the proofs of Theorem 3.1 and Lemma 3.10. Let $q \subseteq R$ be any parameter ideal, and by replacing $q$ with $q^{[p^{e_0}]}$ as in the lemma, we may assume:

$$q H^j_m(S_i) \subseteq 0_{H^j_m(S_i)}$$

for any $0 \leq i + j < d$.

Fix $0 \leq i \leq d-1$ and pick $e \in \mathbb{N}$ and $0 < j < d-i$. Then suppose $\alpha_e(\xi)$ is in the relative Frobenius closure of zero in $H^j_m(S_{i,e})$. For some $e' \in \mathbb{N}$, we have:

$$f_R^{e'}(\xi) \in \ker(\alpha_{e+e'}) = \operatorname{im} \left( x_i^{p^e+e'} \right).$$

By hypothesis $q^{[p^e+e']}$ multiplies $H^j_m(S_{i,e+e'})$ into its relative Frobenius closure of zero. But then $f_R^{e'}(\xi)$ must be in the relative Frobenius closure of zero in $H^j_m(S_{i+1,e+e'})$ and hence $\xi$ is in the relative Frobenius closure of zero in $H^j_m(S_{i+1,e})$.

When $j = 0$ we can exploit that $H^0_m(R/J)$ is an ideal in $R/J$ for any ideal $J \subseteq R$. Now note that the map $x_i : H^0_m(R/(q_{i-1} : x_i)) \to H^0_m(S_{i-1})$ sends a class $r + (q_{i-1} : x_i)$ to $x_i r + q_{i-1} = x_i (r + q_{i-1})$, so that if:

$$x_i H^0_m(S_i) \subseteq 0_{H^0_m(S_i)}$$

then

$$x_i H^0_m(R/(q_{i-1} : x_i)) \subseteq 0_{H^0_m(S_i)}.$$

Thus we have shown the condition on $\alpha_e$ is satisfied for $0 \leq j < d-i$ and we may apply the sufficient condition. □
References

[Bre06] Holger Brenner. Bounds for test exponents. *Compositio Mathematica*, 142(02):451–463, Mar 2006.

[HKSY06] Craig Huneke, Mordechai Katzman, Rodney Y. Sharp, and Yongwei Yao. Frobenius test exponents for parameter ideals in generalized Cohen–Macaulay local rings. *Journal of Algebra*, 305(1):516–539, Nov 2006.

[HS77] Robin Hartshorne and Robert Speiser. Local Cohomological Dimension in Characteristic p. *The Annals of Mathematics*, 105(1):45, Jan 1977.

[KS06] Mordechai Katzman and Rodney Y. Sharp. Uniform behaviour of the Frobenius closures of ideals generated by regular sequences. *Journal of Algebra*, 295(1):231–246, Jan 2006.

[Lyu97] Gennady Lyubeznik. F-modules: applications to local cohomology and D-modules in characteristic p>0. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 1997(491), 1997.

[PQ18] Thomas Polstra and Pham Hung Quy. Nilpotence of Frobenius actions on local cohomology and Frobenius closure of ideals, 2018. Preprint, arXiv:1803.04081.

[QS16] Pham Hung Quy and Kazuma Shimomoto. F-injectivity and Frobenius closure of ideals in Noetherian rings of characteristic p > 0, 2016.

[Quy18] Pham Hung Quy. On the uniform bound of Frobenius test exponents, 2018. Preprint, arXiv:1804.01012.

[Sha07] Rodney Y. Sharp. On the Hartshorne-Speiser-Lyubeznik theorem about Artinian modules with a Frobenius action. *Proceedings of the American Mathematical Society*, 135(03):665–671, Mar 2007.