Asymptotic interaction between solitons in the Nielsen - Olesen model

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Abstract. The objective of this paper was to find the interaction between two vortex solitons in the Nielsen – Olensen model, these solitons are the asymptotic solutions in said model, in order to calculate this interaction it was assumed that these solitons are quasi-static and that they are very far between if and with the help of the momentum energy tensor, this interaction could be found in the z direction in a plane that symmetrically divides these two solitons, which presents two cases since there are two possible asymptotic solutions for the field function according to the coefficient $\beta$, the possible cases are for $\beta < 4$ and $\beta > 4$, in each of the cases interaction is found that depends on exponential functions, the distance between these solitons and the parameters found at the time of calculating said asymptotic solution that are constants $\{1, -1\}$. Furthermore, this force is similar to the force experienced by two turns with current that are very far from each other.

1. Introduction

One of the most important contributions in the second half of the twentieth century was the theory of solitons. Solitons that are nonlinear waves born as solutions of nonlinear dynamics have the following properties: they are nonlinear waves that can travel very large distances without changing their shape, the amplitude of the wave depends directly on the speed [1]. Because solitons have served as a particle model for particle physics, therefore it is necessary to study the interaction between solitons, papers referring to the calculation of the interaction between two solitons in non-linear models, they showed that such interaction is similar to the potential of Yukawa [2], to calculate the interaction between two quasi-static solitons in said papers, the asymptotic solution of each model used was used and with the help of the energy-momentum tensor [4] said interaction was calculated in the Z direction in an equidistant plane of the two solitons taking into account that the separation between them is very large, therefore the same procedure will be applied to find the interaction between two solitons in the Nielsen - Olensen model which is a nonlinear model that presents a vortex type solution [3].

2. Calculus

2.1. Nielsen - Olesen model

The Lagrangian density of the nonlinear model of Nielsen - Olensen is written as:

$$
\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} |D_\mu \Phi|^2 - \frac{\lambda}{4} (|\Phi|^2 - \eta^2)^2
$$

(1)
where Φ it is a complex scalar field, \( D_\mu = \partial_\mu - ieA_\mu \) and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \).

Replacing in the equation of Euler - Lagrange we obtain the equations of motion of the model.

\[
- \lambda \Phi(|\Phi|^2 - \eta^2) = D_\mu D^\mu \Phi \tag{2}
\]

\[
\partial^\nu F_{\mu\nu} = \frac{ie}{2} [(D_\mu \phi)\phi^* - (D_\mu \phi^*)\phi] = j_\mu \tag{3}
\]

Using Nielsen - Olensen ansatz [5]:

\[
\Phi = \eta g(r)e^{i\theta} \tag{4}
\]

\[
A_\mu = \hat{e}_\theta a(r) \frac{e^r}{r} \tag{5}
\]

and replacing in equations (2) and (3) we obtain the differential equations based on the dimensionless functions \( g(r) \) and \( a(r) \)

\[
g''(r) + \frac{1}{r} g'(r) - \frac{(1-a(r))^2}{r^2}g(r) - \beta(g(r)^2 - 1)g(r) = 0 \tag{6}
\]

\[
a''(r) - \frac{1}{r} a'(r) + 2(1-a(r))g(r)^2 = 0 \tag{7}
\]

where \( \beta = \lambda \eta^2 \).

2.2. Asymptotic solution

According to the paper “On the asymptotics of Nielsen - Olensen Vortices” [3] the asymptotic value of the functions of \( g(r) \) and \( a(r) \) when \( r \to \infty \) can be found using the following ansatz

\[
g \to 1 + \delta g \tag{8}
\]

\[
a \to 1 + \delta a \tag{9}
\]

replacing this ansatz in equations (6) and (7) and considering only the first order terms of \( \delta g \), you get the equations:

\[
\delta g'' + \frac{1}{r} \delta g' - \frac{(\delta a)^2}{r^2} - 2\beta \delta g = 0 \tag{10}
\]

\[
\delta a'' - \frac{1}{r} \delta a' - 2\delta a = 0 \tag{11}
\]

The solutions of (10) and (11) are [3]:

\[
\delta a \approx c_a e^{-\sqrt{2}\beta r^{1/2}} \tag{12}
\]

\[
\delta g \approx c_g \frac{e^{-\sqrt{2}\beta r}}{\sqrt{r}} \frac{(c_a)^2 e^{-2\sqrt{2}\beta r}}{2(\beta - 4)r} \tag{13}
\]

where \( c_a, c_g \) are constants \( \{1, -1\} \), for \( \delta g \) there are two possible cases

\[
\delta g \to c_a e^{-\sqrt{2}\beta r} \frac{e^{-\sqrt{2}\beta r}}{\sqrt{r}} \quad \beta \lesssim 4 \tag{14}
\]

\[
\delta g \to -\frac{(c_a)^2 e^{-2\sqrt{2}\beta r}}{2(\beta - 4)r} \quad \beta > 4 \tag{15}
\]
2.3. Interaction Calculation
Let’s define the tensor energy-moment [4]:

\[ T^\mu_\nu = \frac{\partial L}{\partial (\partial^\mu \phi^I)} \partial^\nu \phi^I - \delta^\mu_\nu L \] (16)

Therefore the quadrivector. It can be defined as [4]:

\[ \mathcal{P}^\mu = \int T^{0\mu} d^3x \] (17)

From the equation of (17) we can find the quadrivector of the following equation.

\[ \mathcal{F}^\mu = \frac{d\mathcal{P}^\mu}{d\tau} \] (18)

Where \( d\tau \) is the proper time: \( d\tau = (1 - v^2)^{1/2} dt \), therefore the quadrivector is:

\[ \mathcal{F}^\mu = (1 - v^2)^{-1/2} \int \partial_0 T^{0\mu} \] (19)

By the law of conservation of linear momentum:

\[ \partial_0 T^{0i} + \partial_k T^{ki} = 0 \] (20)

Therefore the force will be expressed as follows:

\[ \mathcal{F}^i = (1 - v^2)^{-1/2} \int \partial_0 T^{0i} d^3x = -(1 - v^2)^{-1/2} \int \partial_k T^{ki} d^3x = -(1 - v^2)^{-1/2} \oint \mathcal{S}_k T^{ki} dS_k \] (21)

We consider that two vortex form a bound state of large radius \( R \gg R_0 \) (where \( R_0 \) is the size of the soliton) as shown in figure 1, such that the center of the solitons are on the axis \( Z \) and they are also separated symmetrically by the plane \( z = 0 \) where \( R_1 = R_2 = R/2 \), in the quasi-stationary approach it is placed as [2]

\[ \Phi = \Phi_1(r) + \Phi_2(r) \] (22)

\[ \begin{array}{c}
\text{Figure 1. Two vortexes that form a bound state of large radius } R \gg R_0 \text{ symmetrically separated by the plane } z = 0. 
\end{array} \]
Our goal is to find the interaction between these two vortexes in the direction $Z$ in the plane $(z = 0)$ which symmetrically divides these vortex. Therefore the force of interaction considering that they are quasi-stationary solutions ($v \ll 1$) it will be according to (21) as:

$$\mathcal{F}^3 = - \int T^{33} \, dx \, dy$$

The development of $T^{\mu \nu}$ for the Nielsen - Ollesen model is:

$$T^{\mu \nu} = \frac{1}{2} \left( \partial^\mu \Phi^* + ie A^\mu \Phi^* \right) \partial^\nu \Phi + \frac{1}{2} \left( \partial^\mu \Phi - ie A^\mu \Phi \right) \partial^\nu \Phi^* - F^{\mu \beta} \partial^\nu A_\beta - g^{\mu \nu} \mathcal{L}$$

$$T^{33} = \frac{1}{2} \partial^3 \Phi^* + ie A^3 \Phi^* \partial^3 \Phi + \frac{1}{2} \left( \partial^3 \Phi - ie A^3 \Phi \right) \partial^3 \Phi^* - F^{33} \partial^3 A_\beta + \mathcal{L}$$

But as it will only be integrated into the plane $z = 0$ so $\partial^3 \Phi_{|z=0}$ and $\partial^3 \Phi^*_{|z=0}$, also $A_\mu$ (according to eq. (5)) it has no component $z$ so $\partial^3 A_\beta = 0$, thus

$$T^{33} = \mathcal{L}$$

If the vortexes are the same $\Phi_1 = \Phi_2$, then the field of the linked system (eq. (22)) is $\Phi = 2\Phi_1$, therefore according to ansatz (eq. (4)) it will have to

$$\Phi = 2\eta g(r)e^{i\theta}$$

Then for this field solution your Lagrangian will be of the form:

$$\mathcal{L} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} + \frac{1}{2} \left( \partial^\mu \Phi \right)^2 - \frac{\lambda}{4} \left( |\Phi|^2 - 4\eta^2 \right)^2$$

The justification of factor $4\eta^2$ in the term of the Lagrangian potential will be explained later, expressing $\mathcal{L}$ in function of $g(r)$ and $a(r)$ the force will be defined by the following integral.

$$\mathcal{F}^3 = \int \left[ \frac{1}{2} \left( \frac{a'(r)}{r} - \frac{a(r)}{r^2} \right)^2 + 2\eta^2 \left( g^2(r) + \frac{g^2(r)}{r^2} \right) + 4\lambda \eta^2 \left( g^2(r) - 1 \right)^2 \right] \, dx \, dy$$

There will be two cases to find the interaction force, this is due to the factor $\beta$ because for $\beta \leq 4$ y $\beta > 4$ the function $g(r)$ take two possible asymptotic values.

### 3. Results and discussions

By integrating equation (29) as a function of polar coordinates on the plane that symmetrically divides the two vortexes, considering only the terms up to the order $R^{-1}$ and neglecting the terms of major order because $R \gg 1$, we get the interaction for both cases.

For $\beta \leq 4$

$$\mathcal{F}^3 = \left( \frac{\pi c^2 \sqrt{2}}{2} + \pi \eta^2 c_a^2 \sqrt{2} \right) e^{-\sqrt{2}R} + \frac{\pi c^2 e^{-\sqrt{2}R}}{R} + \left( 2\pi \eta^2 c_a^2 \sqrt{2} \beta \right) e^{-\sqrt{2}R} + \frac{8\pi \eta^4 \lambda \sqrt{2} \beta}{R} e^{-\sqrt{2}R} +$$

$$+ 4\pi \beta \eta^2 c_g^2 \left( \frac{e^{-\sqrt{2}R} \sqrt{2} + 2}{R} \right) + \frac{8\pi \beta \eta^2 c_g^2 \sqrt{2} e^{-\sqrt{2}R}}{R} +$$

$$+ \frac{32\lambda \beta \sqrt{2} \beta}{\beta} e^{-2\sqrt{2}R} + 2\pi \lambda \sqrt{2} \beta \frac{e^{-2\sqrt{2}R}}{R}$$
Figure 2. Graph of $F$ VS $R$ for $\beta \lesssim 4$.

Figure 2 shows the force between two solitons in the Nielsen-Olensen model, it can be seen according to the figure that it is a force of attraction when $\beta \lesssim 4$, it is more according to our ansatz (5) the vector $\vec{A}$ presents a Coulomb gauge ($\nabla \cdot \vec{A} = 0$) remembering the Ampere’s law considering that the field $\vec{E} = 0$, then you have to

$$\nabla^2 \vec{A} = -\mu \vec{J},$$

from (32) we can interpret that around the two solitons there is a current that circulates in the direction $\hat{e}_\theta$ in each one, similar to two parallel turns with current in the same direction. The force between two parallel turns in which you circulate a current in the same direction for both when they are sufficiently far apart can approximate the force between two magnetic dipoles whose graph is similar to the graph (2) according to the electromagnetic theory, by therefore it can be inferred that these vortexes are similar to two very small turns through which a current passes

For $\beta > 4$

$$\mathcal{F} = \left( \frac{\sqrt{2}\pi c_a^2}{2} - \sqrt{2}\pi \eta c_a^3 \right) e^{-\sqrt{2}R} + \pi c_a^2 e^{-\sqrt{2}R}$$

$$+ \left( \frac{\sqrt{2}\pi \eta c_a^4}{(\beta - 4)} - \frac{2\sqrt{2}\pi \eta c_a^4}{(\beta - 4)^2} - \frac{2\sqrt{2}\pi \eta \lambda c_a^4}{(\beta - 4)^2} \right) \frac{e^{-2\sqrt{2}R}}{R}$$

(33)
Figure 3. Graph of $F$ VS $R$ for $\beta > 4$.

Figure 3 shows a force of repulsion of the two solitons when $\beta > 4$, similar to the previous case, this graph is similar to the repulsive force of two parallel turns far apart in which a current flows one way in the opposite direction with respect to the other.

The factor $4\eta^2$ in the term of the Lagrangian potential was placed in such a way that at the time of developing the integral (eq. (30)) to calculate the interaction there are no divergences.

4. Conclusions
The interaction between two vortexes in the $Z$ direction in the Nielsen - Olesen model whose centers were placed on the $Z$ axis is similar to the electromagnetic interaction of two parallel turns far apart through which an electric current flows.

According to the coefficient $\beta$ two cases were obtained, for $\beta \lesssim 4$ the force between the vortexes is of attraction and for $\beta > 4$ the force is repulsive.

It was necessary to change the term $\eta$ of the vacuum in the term of the potential by the term $2\eta$ to avoid the divergences that may exist at the time of calculating the force between the vortexes.

5. References
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