An approximation trichotomy for Boolean $\#\text{CSP}$

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Abstract

We give a trichotomy theorem for the complexity of approximately counting the number of satisfying assignments of a Boolean CSP instance. Such problems are parameterised by a constraint language specifying the relations that may be used in constraints. If every relation in the constraint language is affine then the number of satisfying assignments can be exactly counted in polynomial time. Otherwise, if every relation in the constraint language is in the co-clone $\text{IM}_2$ from Post’s lattice, then the problem of counting satisfying assignments is complete with respect to approximation-preserving reductions for the complexity class $\#\text{RH}_{\Pi_1}$. This means that the problem of approximately counting satisfying assignments of such a CSP instance is equivalent in complexity to several other known counting problems, including the problem of approximately counting the number of independent sets in a bipartite graph. For every other fixed constraint language, the problem is complete for $\#\text{P}$ with respect to approximation-preserving reductions, meaning that there is no fully polynomial randomised approximation scheme for counting satisfying assignments unless $\text{NP}=\text{RP}$.

1 Introduction

This paper gives a trichotomy theorem for the complexity of approximately counting the number of satisfying assignments of a Boolean CSP instance. Such problems are parameterised by a constraint language $\Gamma$ which specifies relations that may be used in constraints. In the Boolean case, the relations are on a domain which has two elements. Then $\#\text{CSP}(\Gamma)$ will denote the problem of

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determining the number of (distinct) satisfying assignments of a CSP instance with constraint language \( \Gamma \). Further details are given in Section 1.1 below.

Creignou and Hermann [6] have given a dichotomy theorem for the exact counting problem. They have shown that if every relation in \( \Gamma \) is affine, then \#CSP(\( \Gamma \)) is in \( \text{FP} \). Otherwise, it is \#P-complete. The complexity classes \( \text{FP} \) and \#P are the analogues of \( \text{P} \) and \( \text{NP} \) for counting problems. \#P is the class of functions computable in deterministic polynomial time. \#P is the class of integer functions that can be expressed as the number of accepting computations of a polynomial-time non-deterministic Turing machine.

In this paper we build on previous work on the complexity of approximate counting to identify a trichotomy in the complexity of approximate counting for Boolean \#CSP.

Together with Greenhill [9], we have previously studied approximation-preserving reductions (AP-reductions) between counting problems. We will give details of AP-reductions in Section 1.2. For now it suffices to note that if an AP-reduction exists from a counting problem \( f \) to a counting problem \( g \) and \( g \) has a Fully Polynomial Randomised Approximation Scheme (FPRAS) then \( f \) also has an FPRAS.

If an AP-reduction from \( f \) to \( g \) exists we write \( f \leq_{\text{AP}} g \), and say that \( f \) is AP-reducible to \( g \). If \( f \leq_{\text{AP}} g \) and \( g \leq_{\text{AP}} f \) then we say that \( f \) and \( g \) are AP-interreducible, and write \( f =_{\text{AP}} g \).

We previously identified [9] three natural classes of counting problems that are interreducible under AP-reductions. These are (i) those problems that have an FPRAS, (ii) those problems that are complete for \#P with respect to AP-reducibility, and a third class of intermediate complexity. Two counting problems played a special role in [9].

**Name.** \#SAT.

**Instance.** A Boolean formula \( \varphi \) in conjunctive normal form.

**Output.** The number of satisfying assignments of \( \varphi \).

**Name.** \#BIS.

**Instance.** A bipartite graph \( B \).

**Output.** The number of independent sets in \( B \).

All problems in \#P are AP-reducible to \#SAT (see [9], Section 3). Thus \#SAT is complete for \#P with respect to AP-reducibility. This means that \#SAT cannot have an FPRAS unless \( \text{NP} = \text{RP} \). The same is true of any problem in \#P to which \#SAT is AP-reducible.

We showed in [9] Sections 4, 5] that \#BIS is AP-interreducible with many other natural counting problems such as counting downsets in a partial order. Moreover, \#BIS is complete for \#RHΠ₁, a logically-defined subclass of \#P, with respect to AP-reductions.

The main theorem of our current paper (Theorem 3) shows that every problem \#CSP(\( \Gamma \)) falls neatly into one of the three classes from [9]: If every relation
in $\Gamma$ is affine, then trivially $\#\text{CSP}(\Gamma)$ has an FPRAS since it is in $\text{FP}$. Otherwise, if every relation in $\Gamma$ is in a certain set $\text{IM}_2$, then $\#\text{CSP}(\Gamma) =_{\text{AP}} \#\text{BIS}$. Otherwise $\#\text{CSP}(\Gamma) =_{\text{AP}} \#\text{SAT}$. A formal definition of $\text{IM}_2$ appears in Section 1.4 — it is the set of relations which can be expressed as conjunctions involving only binary implication and unary relations.

It is worth pointing out that, while every problem $\#\text{CSP}(\Gamma)$ falls into one of the three approximation classes from [9], the three classes may well not provide a partition of all approximate counting problems in $\#P$. For example, the problem of approximately counting 3-colourings of a bipartite graph is a problem that may well lie between $\#\text{BIS}$ and $\#\text{SAT}$ in approximability (see [9]).

1.1 Constraint satisfaction

*Constraint Satisfaction*, which originated in Artificial Intelligence, provides a general framework for modelling decision problems, and has many practical applications. (See, for example [18].) Decisions are modelled by *variables*, which are subject to *constraints*, modelling logical and resource restrictions. The paradigm is sufficiently broad that many interesting problems can be modelled, from satisfiability problems to scheduling problems and graph-theory problems. Understanding the complexity of constraint satisfaction problems has become a major and active area within computational complexity [7, 11].

A Constraint Satisfaction Problem (CSP) typically has a finite *domain*, which we denote by $\{0, \ldots, q - 1\}$ for a positive integer $q$. In this paper we are interested in the Boolean case $q = 2$. A constraint language $\Gamma$ with domain $\{0, \ldots, q - 1\}$ is a set of relations on $\{0, \ldots, q - 1\}$. For example, take $q = 2$. The relation $R = \{(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)\}$ is a 3-ary relation on the domain $\{0, 1\}$, with four tuples.

Once we have fixed a constraint language $\Gamma$, an *instance* of the CSP is a set of *variables* $V = \{v_1, \ldots, v_n\}$ and a set of *constraints*. Each constraint has a *scope*, which is a tuple of variables (for example, $(v_4, v_5, v_1)$) and a relation from $\Gamma$ of the same arity, which constrains the variables in the scope. An *assignment* $\sigma$ is a function from $V$ to $\{0, \ldots, q - 1\}$. The assignment $\sigma$ is *satisfying* if the scope of every constraint is mapped to a tuple that is in the corresponding relation. In our example above, an assignment $\sigma$ satisfies the constraint with scope $(v_4, v_5, v_1)$ and relation $R$, written $R(v_4, v_5, v_1)$, if and only if it maps an odd number of the variables in $\{v_1, v_4, v_5\}$ to the value 1. Given an instance $I$ of a CSP with constraint language $\Gamma$, the *decision problem* $\text{CSP}(\Gamma)$ asks us to determine whether any assignment satisfies $I$. The *counting problem* $\#\text{CSP}(\Gamma)$ asks us to determine the *number* of (distinct) satisfying assignments of $I$, which we will denote by $\#\text{csp}(I)$.

Varying the constraint language $\Gamma$ defines the classes $\text{CSP}$ and $\#\text{CSP}$ of decision and counting problems. These contain problems of different computational complexities. For example, consider the binary relations defined by $\text{OR} = \{(0, 1), (1, 0), (1, 1)\}$, $\text{Implies} = \{(0, 0), (0, 1), (1, 1)\}$, and $\text{NAND} = \{(0, 0), (0, 1), (1, 0)\}$. If $\Gamma = \{\text{OR}, \text{Implies}, \text{NAND}\}$ then $\text{CSP}(\Gamma)$ is the classical 2-Satisfiability problem, which is in $\text{P}$. On the other hand, there is a similar constraint language $\Gamma'$ with four relations of arity 3 such that 3-Satisfiability (which
is NP-complete) can be represented in CSP(Γ’). It may happen, as here, that the counting problem is harder than the decision problem: #CSP(Γ) contains the problem of counting independent sets in graph, and is thus #P-complete.

Any decision problem CSP(Γ) is in NP, but not every problem in NP can be represented as a CSP. For example, the question “Is G Hamiltonian?” cannot be expressed as a CSP, because the property of being Hamiltonian cannot be captured by relations of bounded size. This limitation of the class CSP has an important advantage. If P ≠ NP, then there are problems which are neither in P nor NP-complete [15]. But, for well-behaved smaller classes of decision problems, the situation can be simpler. We may have a dichotomy theorem, partitioning all problems in the class into those which are in P and those which are NP-complete. There are no “leftover” problems of intermediate complexity. It has been conjectured that there is a dichotomy theorem for CSP. The conjecture is that CSP(Γ) is in P for some constraint languages Γ, and CSP(Γ) is NP-complete for all other constraint languages Γ. This conjecture appeared in a seminal paper of Feder and Vardi [13], but has not yet been proved. A similar dichotomy, between FP and #P-complete, is conjectured for #CSP [4]. Recently, Bulatov [3] has announced a positive resolution of this conjecture.

There have been many important results for subclasses of CSP and #CSP. We mention the most relevant to our paper here. The first decision dichotomy was that of Schaefer [19], for the Boolean domain {0, 1}. Schaefer’s result is as follows.

**Theorem 1** (Schaefer [19]). Let Γ be a constraint language with domain {0, 1}. The problem CSP(Γ) is in P if Γ satisfies one of the conditions below. Otherwise, CSP(Γ) is NP-complete.

(i) Γ is 0-valid or 1-valid.

(ii) Γ is weakly positive or weakly negative.

(iii) Γ is affine.

(iv) Γ is bijunctive.

We will not give detailed definitions of the conditions in Theorem 1 but the interested reader is referred to the paper [19] or to Theorem 6.2 of the textbook [7]. An interesting feature is that the conditions in [7, Theorem 6.2] are all checkable. That is, there is an algorithm to determine whether CSP(Γ) is in P or NP-complete, given a constraint language Γ with domain {0, 1}. We say in this case that the dichotomy is effective.

A Boolean relation R is said to be affine if the set of tuples x ∈ R is the set of solutions to a system of linear equations over GF(2). Creignou and Hermann [6] adapted Schaefer’s decision dichotomy to obtain a counting dichotomy for the Boolean domain. Their result is as follows.

**Theorem 2** (Creignou and Hermann [6]). Let Γ be a constraint language with domain {0, 1}. The problem #CSP(Γ) is in FP if every relation in Γ is affine. Otherwise, #CSP(Γ) is #P-complete.

Creignou and Hermann’s result is an important starting point for our work, and we will discuss it further below. Note that there is an algorithm for determining whether a relation is affine, so the dichotomy is effective.
We have recently [10] extended Creignou and Hermann’s dichotomy to the domain of weighted Boolean #CSP giving an effective dichotomy between \( \text{FP} \) and \( \text{FP}^{\#P} \) for the problem of computing the partition function of a weighted Boolean CSP instance.

1.2 The complexity of approximate counting

We now recall the necessary background from [9]. A randomised approximation scheme is an algorithm for approximately computing the value of a function \( f : \Sigma^* \to \mathbb{N} \). The approximation scheme has a parameter \( \varepsilon > 0 \) which specifies the error tolerance. A randomised approximation scheme for \( f \) is a randomised algorithm that takes as input an instance \( x \in \Sigma^* \) (e.g., an encoding of a CSP instance) and an error tolerance \( \varepsilon > 0 \), and outputs an integer \( z \) (a random variable on the “coin tosses” made by the algorithm) such that, for every instance \( x \),

\[
\Pr [e^{-\varepsilon} f(x) \leq z \leq e^{\varepsilon} f(x)] \geq \frac{3}{4}.
\]

The randomised approximation scheme is said to be a fully polynomial randomised approximation scheme, or \( \text{FPRAS} \), if it runs in time bounded by a polynomial in \( |x| \) and \( \varepsilon^{-1} \). (See Mitzenmacher and Upfal [16, Definition 10.2].) Note that the quantity \( 3/4 \) in Equation (1) could be changed to any value in the open interval \((\frac{1}{2}, 1)\) without changing the set of problems that have randomised approximation schemes [14, Lemma 6.1].

Suppose that \( f \) and \( g \) are functions from \( \Sigma^* \) to \( \mathbb{N} \). An “approximation-preserving reduction” (AP-reduction) from \( f \) to \( g \) gives a way to turn an FPRAS for \( g \) into an FPRAS for \( f \). An AP-reduction from \( f \) to \( g \) is a randomised algorithm \( A \) for computing \( f \) using an oracle for \( g \). The algorithm \( A \) takes as input a pair \((x, \varepsilon) \in \Sigma^* \times (0, 1)\), and satisfies the following three conditions:

(i) every oracle call made by \( A \) is of the form \((w, \delta)\), where \( w \in \Sigma^* \) is an instance of \( g \), and \( 0 < \delta < 1 \) is an error bound satisfying \( \delta^{-1} \leq \text{poly}(|x|, \varepsilon^{-1}) \); (ii) the algorithm \( A \) meets the specification for being a randomised approximation scheme for \( f \) (as described above) whenever the oracle meets the specification for being a randomised approximation scheme for \( g \); and (iii) the run-time of \( A \) is polynomial in \( |x| \) and \( \varepsilon^{-1} \). In formulating a definition of approximation-preserving reduction, a number of choices must be faced. The key requirement is that the class of functions computable by an FPRAS should be closed under AP-reducibility. Informally, we have gone for the most liberal notion of reduction meeting this requirement.

1.3 Notation for relations

Define the unary relations \( \delta_0 = \{(0)\} \) and \( \delta_1 = \{(1)\} \). Recall the binary relation \( \text{Implies} = \{(0, 0), (0, 1), (1, 1)\} \).

For convenience, according to context, we view a \( k \)-ary relation \( R \) either as a set of \( k \)-tuples or as a \( k \)-ary predicate. Thus the notations \( R(x_1, \ldots, x_k) = 1 \) (or

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1The reader who is not familiar with oracle Turing machines can just think of this as an imaginary (unwritten) subroutine for computing \( g \).
just \( R(x_1,\ldots,x_k) \) and \((x_1,\ldots,x_k) \in R\) are equivalent. For example, \( \delta_0(x) = \overline{x} \), \( \delta_1(x) = x \) and \( \text{Implies}(x,y) = \overline{x} \lor y \).

1.4 The set of relations IM

An \( n \)-ary relation \( R \) is in \( IM_2 \) if and only if \( R(x_1,\ldots,x_n) \) is logically equivalent to a conjunction of predicates of the form \( \delta_0(x_i) \), \( \delta_1(x_i) \) and \( \text{Implies}(x_i,x_j) \).

As we will discuss below, Creignou, Kolaitis, and Zanuttini \[8\] have shown that \( IM_2 \) is a co-clone in Post’s lattice (see \[2\]).

1.5 Our result

We can now state our main theorem.

**Theorem 3.** Let \( \Gamma \) be a constraint language with domain \{0,1\}. If every relation in \( \Gamma \) is affine then \( \#CSP(\Gamma) \) is in \( FP \). Otherwise if every relation in \( \Gamma \) is in \( IM_2 \) then \( \#CSP(\Gamma) =_{AP} \#BIS \). Otherwise \( \#CSP(\Gamma) =_{AP} \#SAT \).

The main ingredients in the proof are: (1) the AP-reduction technology of [9], which allows us to effectively “pin” certain CSP variables in hardness proofs (see Section 2.3); (2) the “implementations” of Creignou, Khanna and Sudan \[7\], which show how to construct the key relations OR, Implies, and NAND from a non-affine relation and \( \delta_0 \) or \( \delta_1 \) (see Section 2.5); (3) the complexity class \#RHΠ₁ from \[9\], consisting of those problems which are AP-interreducible with \#BIS; and (4) the co-clone \( IM_2 \) in Post’s lattice (see Section 2.8), since the complexity of \( \#CSP(\Gamma) \) for \( \Gamma \subseteq IM_2 \) turns out to be closely connected to the complexity of \#BIS.

2 The pieces of the proof

2.1 Types of relations

A relation \( R \) is \( 0 \)-valid if the all-zero tuple is in \( R \). Similarly, \( R \) is \( 1 \)-valid if the all-ones tuple is in \( R \). Following \[7\], we say that a \( k \)-ary relation \( R \) is complement-closed (C-closed in \[7\]) if

\[
(x_1,\ldots,x_k) \in R \iff (x_1 \oplus 1,\ldots,x_k \oplus 1) \in R,
\]

where \( \oplus \) is the exclusive or operator.

We say that \( \Gamma \) is \( 0 \)-valid if every \( R \in \Gamma \) is \( 0 \)-valid and we define what it means for \( \Gamma \) to be \( 1 \)-valid or complement-closed similarly.

2.2 Some preliminary complexity results

We start by observing that every problem \( \#CSP(\Gamma) \) is AP-reducible to \( \#SAT \).

**Observation 4.** Let \( \Gamma \) be a constraint language with domain \{0,1\}. Then \( \#CSP(\Gamma) \leq_{AP} \#SAT \).
Observation 4 follows from the fact that all problems in \#P are AP-reducible to \#SAT \[9\]. Another, very simple, but useful, observation is the following.

**Observation 5.** Let \(\Gamma\) be a constraint language with domain \(\{0, 1\}\). Suppose \(\Gamma' \subseteq \Gamma\). Then \(#\text{CSP}(\Gamma') \leq_{\text{AP}} #\text{CSP}(\Gamma)\).

Observation 5 is true for the simple reason that every instance of \(#\text{CSP}(\Gamma')\) is an instance of \(#\text{CSP}(\Gamma)\).

Recall the relations \(\text{OR} = \{(0, 1), (1, 0), (1, 1)\}\) and \(\text{NAND} = \{(0, 0), (0, 1), (1, 0)\}\). These relations are particularly fundamental for us, and we start with complexity results about these.

**Lemma 6.** \(#\text{SAT} \leq_{\text{AP}} #\text{CSP}(\{\text{NAND}\})\).

**Proof.** It was shown in \[9\] that the following problem is AP-interreducible with \#SAT.

**Name.** \#IS.

**Instance.** A graph \(G\).

**Output.** The number of independent sets in \(G\).

We show that \(#\text{IS} \leq_{\text{AP}} #\text{CSP}(\{\text{NAND}\})\). Let \(G = (V, E)\) be an instance of \#IS. Construct an instance \(I\) of \#CSP(\{NAND\}) with variable set \(V\). For every edge \((u, v) \in E\), add constraint \(\text{NAND}(u, v)\). There is now a bijection between independent sets of \(G\) and satisfying assignments \(\sigma\) of \(I\): variables \(v\) with \(\sigma(v) = 1\) correspond to vertices in the independent set.

**Lemma 7.** \(#\text{SAT} \leq_{\text{AP}} #\text{CSP}(\{\text{OR}\})\).

**Proof.** The proof that \(#\text{IS} \leq_{\text{AP}} #\text{CSP}(\{\text{OR}\})\) is similar (just associate variables \(v\) with \(\sigma(v) = 1\) with vertices that are out of the independent set).

Finally, we will need a couple of complexity results involving \#BIS.

**Lemma 8.** \(#\text{BIS} \leq_{\text{AP}} #\text{CSP}(\{\text{Implies}\})\).

**Proof.** Let \(G\) be an instance of \#BIS with vertex sets \(U\) and \(V\) and edge set \(E\). Construct an instance \(I\) of \#CSP(\{Implies\}) with variable set \(U \cup V\). For every edge \((u, v) \in E\) with \(u \in U\) add constraint \(\text{Implies}(u, v)\). There is now a bijection between independent sets of \(G\) and satisfying assignments \(\sigma\) of \(I\): a variable \(u \in U\) with \(\sigma(u) = 1\) is in the independent set and a variable \(v \in V\) with \(\sigma(v) = 0\) is in the independent set.

**Lemma 9.** Suppose \(\Gamma \subseteq IM_2\). Then \(#\text{CSP}(\Gamma) \leq_{\text{AP}} #\text{BIS}\).
Proof. It is straightforward to show that \( \#\text{CSP}(\Gamma) \) is in the complexity class \( \#\text{RH\Pi}_1 \) which has \( \#\text{BIS} \) as a complete problem \([9]\).

However, to avoid giving a definition of \( \#\text{RH\Pi}_1 \), which requires some notation, we will instead show \( \#\text{CSP}(\Gamma) \leq_{\text{AP}} \#\text{Downsets} \), where \( \#\text{Downsets} \) is the following counting problem which was shown in \([9]\) to be \( \text{AP} \)-interreducible with \( \#\text{BIS} \).

Name. \#\text{Downsets}.

Instance. A partially ordered set \((X, \preceq)\).

Output. The number of downsets\(^2\) in \((X, \preceq)\).

Consider an instance \(I\) of \( \#\text{CSP}(\Gamma) \) with variables \(v_1, \ldots, v_n\). The set of constraints can be viewed as an equivalent set of constraints of the form \( \delta_0(v_i) \), \( \delta_1(v_i) \) or \( \text{Implies}(v_i, v_j) \). Denote by \( \text{Implies}^*(v_i, v_j) \) if there is a sequence of variables, starting with \( v_i \) and ending with \( v_j \), such that every adjacent pair in the sequence is constrained by \( \text{Implies} \).

Let \( N_0(I) \) be the set of variables \( v_i \) for which either (i) a constraint \( \delta_0(v_i) \) occurs in \( I \), or (ii) there exists a variable \( v_j \) such that \( \text{Implies}^*(v_i, v_j) \) and a constraint \( \delta_0(v_j) \) occurs in \( I \). These are the variables that are forced to be 0 in any satisfying assignment of \( I \). Define \( N_1(I) \) analogously to be the set of variables that are forced to be 1 in any satisfying assignment. We can assume without loss of generality that \( N_0(I) \) and \( N_1(I) \) are disjoint. Otherwise the instance \( I \) has no satisfying assignments, and we can determine this without even using the downsets oracle.

Now remove all the variables in \( N_0(I) \) and \( N_1(I) \) from the instance \( I \): this does not affect the number of satisfying assignments, since these variables do not constrain any of the others. Also identify all pairs of variables \( v_i, v_j \) such that \( \text{Implies}^*(v_i, v_j) \) and \( \text{Implies}^*(v_j, v_i) \): again, this does not affect the number of satisfying assignments.

The remaining variables and relations define a partial order \((X, \preceq)\) since our construction forces antisymmetry. The satisfying assignments of \( I \) correspond 1–1 with the downsets of \((X, \preceq)\). \( \square \)

2.3 A useful tool: pinning

Pinning is the ability to tie certain CSP variables to specific values in hardness proofs. This idea was used by Creignou and Hermann in their dichotomy theorem \([6]\). Similar ideas have been used in many other hardness proofs and dichotomy theorems \([4, 5, 10, 12]\). As we show in this section, AP-reductions facilitate a particularly useful form of pinning.

Lemma 10. Let \( \Gamma \) be a constraint language with domain \( \{0, 1\} \). Suppose there is a relation \( R \in \Gamma \) for which, for some position \( j \), \( R \) has more tuples \( t \) with \( t_j = 0 \) than with \( t_j = 1 \). Then \( \#\text{CSP}(\Gamma \cup \{\delta_0\}) \leq_{\text{AP}} \#\text{CSP}(\Gamma) \). Similarly, if

\[^{2}\text{A downset in } (X, \preceq) \text{ is a subset } D \subseteq X \text{ that is closed under } \preceq; \text{ i.e., } x \preceq y \text{ and } y \in D \implies x \in D.\]
there is a relation \( R \in \Gamma \) for which, for some position \( j \), \( R \) has more tuples \( t \) with \( t_j = 1 \) than with \( t_j = 0 \) then \( \#CSP(\Gamma \cup \{\delta_1\}) \leq_{AP} \#CSP(\Gamma) \).

**Proof.** Consider an instance \( I \) of \( \#CSP(\Gamma \cup \{\delta_0\}) \) with \( n \) variables. Suppose there is an arity-\( k \) relation \( R \in \Gamma \) for which, for position \( j \), \( R \) has \( w \) tuples \( t \) with \( t_j = 0 \) and \( w' < w \) tuples \( t \) with \( t_j = 1 \).

As in the proof of Lemma \( 9 \) let \( N_0(I) \) be the set of variables \( x \) to which one or more constraints \( \delta_0(x) \) occurs in \( I \) and let \( N_1(I) \) be the set of variables \( y \) to which one or more constraints \( \delta_1(y) \) occurs. Let \( n_0 = |N_0(I)| \). Let \( m = \lceil (n + 2)/\lg(w/w') \rceil \). Construct an instance \( I' \) of \( \#CSP(\Gamma) \). Include all constraints in \( I \) other than those involving \( \delta_0 \). For each variable \( x \in N_0(I) \), and every \( a \in \{1, \ldots, m\} \), introduce \( k - 1 \) new variables \( x'_{a,b} \) for \( b \in \{1, \ldots, k\} - \{j\} \). Introduce a new constraint in \( I' \) with relation \( R \) and variable \( x \) in the \( j \)th position, and \( x'_{a,b} \) in the \( b \)th position, for all \( b \).

Now a satisfying assignment for \( I \) can be extended in \( w^{mn_0} \) ways to satisfying assignments of \( I' \). An assignment for \( I \) that violates one of the \( \delta_0(x) \) constraints can be extended in at most \( w^{m(n_0 + 1)}w'^m \) ways to satisfying assignments of \( I' \). Thus,

\[
\#csp(I)w^{mn_0} \leq \#csp(I') \leq \#csp(I)w^{mn_0} + 2^n w^{m(n_0 - 1)}w'^m,
\]

i.e.,

\[
\#csp(I) \leq \frac{\#csp(I')}{w^{mn_0}} \leq \#csp(I) + 2^n (w'/w)^m.
\]

So, by definition of \( m \),

\[
\#csp(I) \leq \frac{\#csp(I')}{w^{mn_0}} \leq \#csp(I) + \frac{1}{4}.
\]

Thus we have constructed a reduction from \( \#CSP(\Gamma \cup \{\delta_0\}) \) to \( \#CSP(\Gamma) \): Given an instance \( I \) of \( \#CSP(\Gamma \cup \{\delta_0\}) \), use an oracle for \( \#CSP(\Gamma) \) to approximate \( \#csp(I') \), divide by \( w^{mn_0} \), and round to the nearest integer (always down). Note that the reduction makes only one oracle call (and uses no randomisation).

To show that the reduction is indeed an AP-reduction, we add some technical details concerning the choice of the accuracy parameter \( \delta \) in the oracle call (see the definition of AP-reduction in Section \( 1.2 \)). These details are here to make the proof complete, but they are not essential for understanding the rest of the paper.

If we had

\[
\#csp(I) = \frac{\#csp(I')}{w^{mn_0}},
\]

we could simply set \( \delta = \varepsilon \), since division by a constant preserves relative error. Instead we have

\[
\#csp(I) = \left\lfloor \frac{\#csp(I')}{w^{mn_0}} \right\rfloor.
\]

The discontinuous floor function could spoil the approximation when its argument is small.

The situation here is that the true answer \( N = \#csp(I) \) is obtained by rounding the fraction \( Q = \frac{\#csp(I')}{w^{mn_0}} \) where we have \(|Q - N| \leq 1/4\).
Suppose that the oracle provides an approximation \( \hat{Q} \) to \( Q \) satisfying \( Q e^{-\delta} \leq \hat{Q} \leq Q e^\delta \) (as it is required to do with probability at least \( 3/4 \)). Set \( \delta = \varepsilon/21 \), where \( \varepsilon \) is the accuracy parameter governing the final result. There are two cases. If \( N \leq 2/\varepsilon \), then a short calculation yields \( |\hat{Q} - Q| < 1/4 \) implying that the result returned by the algorithm is exact. If \( N > 2/\varepsilon \), then the result returned is in the range \( [(N - 1/4)e^{-\delta} - 1/2, (N + 1/4)e^\delta + 1/2] \) which, for the chosen \( \delta \), is contained in \([Ne^{-\varepsilon}, Ne^\varepsilon]\).

Thus, we have an AP-reduction from \( \#\text{CSP}(\Gamma \cup \{\delta_0\}) \) to \( \#\text{CSP}(\Gamma) \). The reduction showing \( \#\text{CSP}(\Gamma \cup \{\delta_1\}) \leq_{\text{AP}} \#\text{CSP}(\Gamma) \) is similar. \( \square \)

### 2.4 Affine relations

We use the following well-known facts about affine relations.

**Lemma 11.**

(i) A \( k \)-ary Boolean relation \( R \) is affine if and only if \( a, b, c \in R \) implies \( d = a \oplus b \oplus c \in R \), where the \( \oplus \) operator is applied componentwise.

(ii) If \( R \) is not affine, then for any fixed \( a \in R \) there are \( b, c \in R \) such that \( a \oplus b \oplus c \not\in R \).

(iii) If \( R \) is not affine, then there are \( a, b \in R \) such that \( a \oplus b \not\in R \).

**Proof.** For Part (i) see, for example, Lemma 4.10 of [7]). Part (ii) is proved in the same place, but since it is a little less well-known, we provide the proof: Suppose the contrary that \( R \) is not affine, but for all \( b, c \in R, a \oplus b \oplus c \in R \). Choose \( s_0, s_1, s_2 \in R \) such that \( s_0 \oplus s_1 \oplus s_2 \not\in R \). From \( b = s_0, c = s_1, d = a \oplus s_0 \oplus s_1 \) we have \( d \in R \). From \( b = s_2, c = d \) we have \( a \oplus s_2 \oplus d = s_0 \oplus s_1 \oplus s_2 \not\in R \), a contradiction.

To see Part (iii), note that the condition \( \forall a, b : a, b \in R \) implies \( a \oplus b \in R \) implies that \( R \) is affine, so, if \( R \) is not affine then the condition is false. \( \square \)

### 2.5 Implementation

Let \( \Gamma \) be a constraint language with domain \( \{0, 1\} \). \( \Gamma \) is said to implement a \( k \)-ary relation \( R \) if, for some \( k' \geq k \) there is a CSP instance \( I \) with variables \( x_1, \ldots, x_{k'} \) and constraints in \( \Gamma \) such that, for every tuple \( (s_1, \ldots, s_k) \in R \), there is exactly one satisfying assignment \( \sigma \) of \( I \) with \( \sigma(x_1) = s_1, \ldots, \sigma(x_k) = s_k \) and for every tuple \( (s_1, \ldots, s_k) \not\in R \), there are no satisfying assignments \( \sigma \) of \( I \) with \( \sigma(x_1) = s_1, \ldots, \sigma(x_k) = s_k \). Note the following straightforward observation, which is essentially a parsimonious reduction [17] p.441).

**Observation 12.** If \( \Gamma \) implements \( R \) then \( \#\text{CSP}(\Gamma \cup \{R\}) \leq_{\text{AP}} \#\text{CSP}(\Gamma) \).

We will use several implementations of Creignou, Khanna and Sudan. Proofs are provided in the appendix in order to make the paper self-contained.

**Lemma 13.** (Creignou, Khanna and Sudan, [2] Lemmas 5.24 and 5.25) Let \( \Gamma \) be a constraint language with domain \( \{0, 1\} \).

\(^3\)There are many variants of “implement” defined in the literature. See [7] Chapter 5, where the kind of implementation we define here is called “faithful” and “perfect”.

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(i) If $\Gamma$ contains a relation $R$ that is 0-valid, 1-valid and not complement-closed then $\Gamma$ implements the relation $R' = \{(0,0),(1,1),(1,0)\}$.

(ii) If $\Gamma$ contains a relation $R$ that is not 0-valid, not 1-valid and not complement-closed then $\Gamma$ implements $\delta_0$ and $\delta_1$.

(iii) If $\Gamma$ contains a relation $R$ that is 0-valid and not 1-valid then $\Gamma$ implements $\delta_0$.

(iv) If $\Gamma$ contains a relation $R$ that is 1-valid and not 0-valid then $\Gamma$ implements $\delta_1$.

Lemma 14. (Creignou, Khanna and Sudan, [7, Claim 5.31]) Let $R$ be a ternary relation containing $(0,0,0)$, $(0,1,1)$ and $(1,0,1)$ but not $(1,1,0)$. Then $\{R, \delta_0\}$ implements one of $\text{Implies}$ and $\text{NAND}$.

Lemma 15. (Creignou, Khanna and Sudan, [7, Lemma 5.30]) If $R$ is a relation over $\{0,1\}$ that is not affine then $\{R, \delta_0\}$ implements one of $\text{OR}$, $\text{Implies}$, and $\text{NAND}$ and so does $\{R, \delta_1\}$.

2.6 Pinning revisited

Combining the useful pinning that we get from AP-reductions (Lemma 10) with the implementations of $\text{OR}$, $\text{Implies}$ and $\text{NAND}$ in Section 2.5, we obtain a useful lemma which says that we can always do some pinning.

Lemma 16. Let $\Gamma$ be a constraint language with domain $\{0,1\}$. Then either $\#\text{CSP}(\Gamma \cup \{\delta_0\}) \leq_{\text{AP}} \#\text{CSP}(\Gamma)$ or $\#\text{CSP}(\Gamma \cup \{\delta_1\}) \leq_{\text{AP}} \#\text{CSP}(\Gamma)$ (or both).

Proof. First, suppose that $\Gamma$ is not complement-closed. If $\Gamma$ contains a relation $R$ that is not 0-valid, not 1-valid and not complement-closed then we finish by Observation 12 and Part (ii) of Lemma 13. If $\Gamma$ contains a relation $R$ that is 0-valid, 1-valid and not complement-closed then it implements the relation $R'$ from Part (i) of Lemma 13 so by Observation 12 $\#\text{CSP}(\Gamma \cup \{R'\}) \leq_{\text{AP}} \#\text{CSP}(\Gamma)$. But Lemma 10 shows both $\#\text{CSP}(\Gamma \cup \{R', \delta_0\}) \leq_{\text{AP}} \#\text{CSP}(\Gamma \cup \{R'\})$ and $\#\text{CSP}(\Gamma \cup \{R', \delta_1\}) \leq_{\text{AP}} \#\text{CSP}(\Gamma \cup \{R'\})$. Otherwise $\Gamma$ contains a relation $R$ that is 0-valid and not 1-valid (or vice-versa) and we finish by Part (iii) (or Part (iv)) of Lemma 13 and Observation 12.

Second (and finally), suppose that $\Gamma$ is complement-closed. Here is a simple AP-reduction from $\#\text{CSP}(\Gamma \cup \{\delta_0\})$ to $\#\text{CSP}(\Gamma)$. Let $I$ be an instance of $\#\text{CSP}(\Gamma \cup \{\delta_0\})$. Construct an instance $I'$ of $\#\text{CSP}(\Gamma)$ by adding a new variable $z_0$. For all $x \in N_0(I)$ (all variables $x$ to which one or more constraints $\delta_0(x)$ in $I$ apply), replace all occurrences of variable $x$ with $z_0$ in $I'$. Now note that $2\#\text{csp}(I) = \#\text{csp}(I')$ since there is a one-to-two map from satisfying assignments of $I$ and satisfying assignments of $I'$. In particular, if $s$ is an assignment to all variables of $I$ other than those in $N_0(I)$ and $s$ is satisfying, provided the rest of the variables are assigned value 0, then $s$ is mapped to $s; z_0 = 0$ and $\overline{s}; z_0 = 1$, where $\overline{s}$ is the tuple obtained from $s$ by complementing the assignment of every variable. Both satisfy $I'$ since $\Gamma$ is complement-closed. It is clear that all satisfying assignments of $I'$ arise in this way. 

\hfill \qed
2.7 Notation for Boolean functions

The following definitions are from [1, 2]. An $m$-ary Boolean function $f$ is monotonic if and only if $(a_1, \ldots, a_m) \leq (b_1, \ldots, b_m)$ componentwise implies $f(a_1, \ldots, a_m) \leq f(b_1, \ldots, b_m)$. Let $M_2$ be the set of all monotone Boolean functions $f$ satisfying $f(0, \ldots, 0) = 0$ and $f(1, \ldots, 1) = 1$. Given a set $B$ of Boolean functions, the closure $[B]$ consists of all functions that can be defined by propositional formulas with connectives from $B$ (see [1]).

An $m$-ary Boolean function $f$ is said to be a polymorphism of an $n$-ary relation $R(x_1, \ldots, x_n)$ if applying $f$ componentwise to $m$ tuples in $R$ results in a tuple that is also in $R$.

2.8 Polymorphisms and $IM_2$

In the terminology of universal algebra, Creignou, Kolaitis, and Zanuttini [8] have shown that $IM_2$ is precisely the co-clone corresponding to $M_2$, which is a clone in Post’s lattice (see [2]). The direction of this result that we will use is the following.

Lemma 17. (Creignou, Kolaitis, Zanuttini, [8]) If the relation $R$ is not in $IM_2$ then there is an $f \in M_2$ that is not a polymorphism of $R$.

Corollary 18. If the $n$-ary relation $R$ is not in $IM_2$ then there are Boolean tuples $(a_1, \ldots, a_n) \in R$ and $(b_1, \ldots, b_n) \in R$ such that either $(a_1 \land b_1, \ldots, a_n \land b_n) \notin R$ or $(a_1 \lor b_1, \ldots, a_n \lor b_n) \notin R$ (or both).

Proof. We will use the fact (see [1]) that $M_2 = [\{\lor, \land\}]$ where $x \lor y$ is the OR of the Boolean values $x$ and $y$ and $x \land y$ is the AND of $x$ and $y$. Thus, every function $f \in M_2$ can be defined by a propositional formula using the 2-ary connectives $\lor$ and $\land$.

The proof is by induction on the number of connectives used in the propositional formula used to represent the function $f$ from Lemma 17.

The case $f(x) = x$ (in which $f$ has no connectives) cannot arise since the identity function is a polymorphism of every relation. The cases $f(x,y) = x \lor y$ and $f(x,y) = x \land y$ (in which $f$ has one connective) immediately give the corollary.

For the inductive step, we assume either $f(x_1, \ldots, x_m) = f'(x_1, \ldots, x_m) \lor f''(x_1, \ldots, x_m)$ or $f(x_1, \ldots, x_m) = f'(x_1, \ldots, x_m) \land f''(x_1, \ldots, x_m)$ where $f'$ and $f''$ have fewer connectives than $f$. Note that $f'$ and $f''$ may not actually use all of the variables in $x_1, \ldots, x_m$.

These two cases are similar, so suppose we are in the first of them. That is, suppose

$$f(x_1, \ldots, x_m) = f'(x_1, \ldots, x_m) \lor f''(x_1, \ldots, x_m).$$

Suppose also that $f'$ and $f''$ are polymorphisms of $R$ (otherwise we will apply the inductive hypothesis to one of these functions which has fewer connectives). Let $t^1, \ldots, t^m$ be $m$ $n$-tuples in $R$, such that the tuple obtained by applying $f$ componentwise to $t^1, \ldots, t^m$ is not in $R$. Let $t'$ be the $n$-tuple obtained by applying $f'$ componentwise to $t^1, \ldots, t^m$ and let $t''$ be the $n$-tuple obtained by
applying \( f'' \) componentwise to \( t^1, \ldots, t^m \). Since \( f' \) and \( f'' \) are polymorphisms of \( R \), we know that \( t' \) and \( t'' \) are in \( R \). However, since \( f \) is not a polymorphism of \( R \), the tuple \( t' \lor t'' \) is not in \( R \), proving the corollary.

\( \square \)

3 Putting it all together: the proof of Theorem 3

We start with a lemma establishing a reduction from \#SAT.

**Lemma 19.** Let \( R_1 \) and \( R_2 \) be relations on \{0, 1\}. If \( R_1 \) is not affine and \( R_2 \) is not in \( IM_2 \) then \#SAT \( \leq_{AP} \#CSP(\{R_1, R_2\}) \).

**Proof.** Apply Lemma 16 with \( \Gamma = \{R_1, R_2\} \). Then either \#CSP(\{\( R_1, R_2, \delta_0 \) \}) \( \leq_{AP} \#CSP(\{R_1, R_2\}) \) or \#CSP(\{\( R_1, R_2, \delta_1 \)\}) \( \leq_{AP} \#CSP(\{R_1, R_2\}) \). Assume the former (the latter case is symmetric).

Now use Lemma 15 together with Observation 12. Since \( R_1 \) is not affine this shows one of the following.

- \#CSP(\{\( R_1, R_2, \delta_0, OR \)\}) \( \leq_{AP} \#CSP(\{R_1, R_2, \delta_0\}) \), or
- \#CSP(\{\( R_1, R_2, \delta_0, NAND \)\}) \( \leq_{AP} \#CSP(\{R_1, R_2, \delta_0\}) \), or
- \#CSP(\{\( R_1, R_2, \delta_0, Implies \)\}) \( \leq_{AP} \#CSP(\{R_1, R_2, \delta_0\}) \).

In the first two of these cases, we are finished by Observation 5 and Lemmas 6 and 7 so assume the final case. Using Lemma 10 with the second position of Implies, we get \#CSP(\{\( R_1, R_2, \delta_0, Implies, \delta_1 \)\}) \( \leq_{AP} \#CSP(\{R_1, R_2, \delta_0\}) \).

Simplifying the chain of reductions and using Observation 5 to drop \( R_1 \) from the left-hand side, we get \#CSP(\{\( Implies, R_2, \delta_0, \delta_1 \)\}) \( \leq_{AP} \#CSP(\{R_1, R_2\}) \).

We will now finish by showing \#SAT \( \leq_{AP} \#CSP(\{Implies, R_2, \delta_0, \delta_1\}) \).

**Case 1.** Using Corollary 18 suppose that \( t \) and \( t' \) are tuples in \( R_2 \) but the tuple \( t \land t' \) (in which the operator \( \land \) is applied componentwise) is not in \( R_2 \). We will show that \{\( Implies, R_2, \delta_0, \delta_1 \)\} implements one of OR and XOR = \{0, 1\}, (1, 0). Let \( k \) be the arity of \( R_2 \). As in the implementations of Creignou et al. [7], define \( r_i \) to be the actual value of \( t_i \) if \( t_i = t'_i = 0 \) or \( x \) if \( t_i = 0, t'_i = 1 \) or \( y \) if \( t_i = 1, t'_i = 0 \), or \( v \) if \( t_i = t'_i = 1 \). Let \( R' \) be the relation implemented by \( R'(x, y) = R_2(r_1, \ldots, r_k) \land \delta_0(x) \land \delta_1(v) \). Note that both \( x \) and \( y \) appear as arguments of \( R' \) since \( t \neq t \land t' \) and \( t' \neq t \land t' \). If \( t \lor t' \) is in \( R_2 \) then \( R'(x, y) \) implements OR(x, y), so we are finished. Otherwise \( R' = XOR \) (which we now assume).

Using Observation 12 and 5, we have

\[ \#CSP(\{Implies, XOR\}) \leq_{AP} \#CSP(\{R_1, R_2\}) \]

We will finish by showing that \{\( Implies, XOR \)\} implements NAND. (The result then follows by Lemma 6 and Observation 12.)

The implementation is given by \( NAND(x, z) = Implies(x, y) \land XOR(y, z) \).

**Case 2.** Otherwise, by Corollary 18 there are \( t \) and \( t' \) in \( R_2 \) such that \( t \lor t' \) is not in \( R_2 \). This case is dual to Case 1. \( \square \)
We can now prove the main theorem.

**Theorem 3.** Let $\Gamma$ be a constraint language with domain $\{0, 1\}$. If every relation in $\Gamma$ is affine then $\#\text{CSP}(\Gamma)$ is in FP. Otherwise if every relation in $\Gamma$ is in $IM_2$ then $\#\text{CSP}(\Gamma) = \text{AP} \#BIS$. Otherwise $\#\text{CSP}(\Gamma) = \text{AP} \#\text{SAT}$.

**Proof.** First, suppose that every relation in $\Gamma$ is affine. In this case, the number of satisfying assignments of an instance $I$ of $\#\text{CSP}(\Gamma)$ is the number of solutions to a system of linear equations over $GF(2)$. This can be computed exactly, by Gaussian elimination, in polynomial time, as Creignou and Hermann have noted [6].

Next, suppose that $\Gamma$ contains a relation $R$ that is not affine, but every relation in $\Gamma$ is in $IM_2$. By Lemma 9, $\#\text{CSP}(\Gamma) \leq \text{AP} \#\text{BIS}$.

To see that $\#\text{BIS} \leq \text{AP} \#\text{CSP}(\Gamma)$, apply Lemma 16. Then we know that either $\#\text{CSP}(\Gamma \cup \{\delta_0\}) \leq \text{AP} \#\text{CSP}(\Gamma)$ or $\#\text{CSP}(\Gamma \cup \{\delta_1\}) \leq \text{AP} \#\text{CSP}(\Gamma)$ (or both). We will show

\[ \#\text{BIS} \leq \text{AP} \#\text{CSP}(\Gamma \cup \{\delta_0\}) \] (2)

and

\[ \#\text{BIS} \leq \text{AP} \#\text{CSP}(\Gamma \cup \{\delta_1\}) \] (3)

and then we will be able to conclude $\#\text{BIS} \leq \text{AP} \#\text{CSP}(\Gamma)$. The proofs of Equations (2) and (3) are similar, so we just prove (2). By Lemma 15 $\Gamma \cup \{\delta_0\}$ implements one of OR, IMplies, and NAND. So by Observation 12 we have (at least) one of the following.

(i) $\#\text{CSP}(\Gamma \cup \{\delta_0, \text{OR}\}) \leq \text{AP} \#\text{CSP}(\Gamma \cup \{\delta_0\})$

(ii) $\#\text{CSP}(\Gamma \cup \{\delta_0, \text{Implies}\}) \leq \text{AP} \#\text{CSP}(\Gamma \cup \{\delta_0\})$

(iii) $\#\text{CSP}(\Gamma \cup \{\delta_0, \text{NAND}\}) \leq \text{AP} \#\text{CSP}(\Gamma \cup \{\delta_0\})$

Equation (2) follows from the combination of Lemma 8 and (ii) using Observation 5. Also, since $\#\text{BIS} \leq \text{AP} \#\text{SAT}$ (see [9]), Equation (2) follows from the combination of Lemma 7 and (i) using Observation 5. Similarly, it follows from the combination of Lemma 6 and (iii) using Observation 5.

Finally, suppose that $\Gamma$ contains a relation $R_1$ that is not affine and a relation $R_2$ that is not in $IM_2$. ($R_1$ and $R_2$ might possibly be the same relation.) The fact that $\#\text{CSP}(\Gamma) \leq \text{AP} \#\text{SAT}$ follows from Observation 4 and the fact that $\#\text{SAT} \leq \text{AP} \#\text{CSP}(\Gamma)$ follows from Lemma 19 and Observation 5. □

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Appendix: The implementations of Creignou, Khanna and Sudan

In order to make our paper self-contained, we give the details of the implementations of Creignou, Khanna and Sudan that we use. In particular, we provide the proofs for Lemmas 13, 14 and 15. (These proofs can be found in [7].)

We start with the construction for Lemma 13. Suppose \( R \in \Gamma \) is not complement-closed. Choose \((s_1, \ldots, s_k)\) in \( R \) such that \((s_1 \oplus 1, \ldots, s_k \oplus 1)\) is not in \( R \). Now consider the relation \( R' \) implemented by \( R'(x, y) = R(r_1, \ldots, r_k) \) where \( r_i = x \) if \( s_i = 1 \) and \( r_i = y \) otherwise. In the first case, \( R' \) is the relation \( \{(0, 0), (1, 1), (1, 0)\} \). In the second case, \( R' = \{(1, 0)\} \) so \( R' \) gives an implementation of both \( \delta_1 \) and \( \delta_0 \). The construction for the third and fourth cases are the trivial implementations \( \delta_0(x) = R(x, \ldots, x) \) and \( \delta_1(x) = R(x, \ldots, x) \).

We now give the construction for Lemma 14. If \( R \) excludes exactly one of \((0, 1, 0)\) and \((1, 1, 1)\) then \( R(x, y, x) \) implements \( \text{implies}(y, x) \) or \( \text{NAND}(x, y) \) (depending on which is excluded). Similarly, if \( R \) excludes exactly one of \((1, 0, 0)\) and \((1, 1, 1)\) then \( R(x, y, y) \) implements \( \text{implies}(x, y) \) or \( \text{NAND}(x, y) \). If both \((0, 1, 0)\) and \((1, 0, 0)\) are in \( R \) then \( f_R(x, y, z) \land \delta_0(z) \) implements \( f_{\text{NAND}}(x, y) \).

If \((0, 1, 0)\), \((1, 1, 1)\) and \((1, 0, 0)\) are excluded from \( R \) and so is \((0, 0, 1)\) then \( R(x, y, z) \) implements \( \text{NAND}(x, y) \). Finally, if \((0, 1, 0)\), \((1, 1, 1)\) and \((1, 0, 0)\) are excluded but \((0, 0, 1)\) is in \( R \) then \( R(x, y, z) \land \delta_0(x) \) implements \( \text{implies}(y, z) \).

Finally, we give the construction for Lemma 15. We will show that \( \{R, \delta_1\} \) implements one of the named relations. A similar argument shows that \( \{R, \delta_0\} \) does. Let \( k \) be the arity of \( R \).

First, suppose that \( R \) is 0-valid. Using part (iii) of Lemma 11, let \( s \) and \( s' \) be tuples in \( R \) such that \( s \oplus s' \) is not in \( R \). Let \( r_i = w \) if \( s_i = s_i' = 0 \). Let \( r_i = x \) if \( s_i = 0, s_i' = 1 \). Let \( r_i = y \) if \( s_i = 1, s_i' = 0 \). Let \( r_i = z \) if \( s_i = s_i' = 1 \). Now we know that at least one of \( x \) and \( y \) occurs as an \( r_i \), since \( s \neq s' \). Let \( R' \) be the relation implemented by \( R(r_1, \ldots, r_k) \land \delta_0(w) \). There are a few cases to consider. If \( x \) occurs as an argument to \( R \) but \( y \) does not then \( z \) occurs since \( s \neq 0 \). Thus, the relation \( R'(x, z) \) is \( \text{implies} \). (Technically, this is a ternary relation in variables \( x, y \) and \( z \), but it can be viewed as a binary relation since \( y \) does not appear.) The situation is similar if \( y \) occurs as an argument to \( R \) but \( x \) does not. If both \( x \) and \( y \) occur as arguments but \( z \) does not then the relation \( R'(x, y) \) is \( \text{NAND} \). Otherwise, \( x, y \) and \( z \) all occur as arguments. Furthermore, since \( R \) is 0-valid, lemma 14 applies to the relation given by \( R'(x, y, z) \).

Second (and finally), suppose that \( R \) is not 0-valid. Note that \( \{R, \delta_0\} \) can implement \( \delta_1 \). To see this, let \( s \) be a tuple in \( R \). Let \( r_i = x \) if \( s_i = 1 \) and let \( r_i = y \) otherwise. Then \( \delta_1(x) \) is implemented by \( R(r_1, \ldots, r_k) \land \delta_0(y) \). Now consider two sub-cases.

For the first sub-case, suppose that for any two tuples, \( t \) and \( t' \), in \( R \), the tuple \( t \land t' \), where \( \land \) is applied componentwise, is also in \( R \). Let \( s \) be the intersection of all tuples in \( R \). Then \( s \in R \). By Part (ii) of Lemma 11, there are two tuples \( s' \) and \( s'' \) in \( R \) such that \( s \oplus s' \oplus s'' \) is not in \( R \). Let \( r_i = u \) if \( s_i = s_i' = s_i'' = 0 \). Let \( r_i = x \) if \( s_i = 0, s_i' = 0, s_i'' = 1 \). Let \( r_i = y \) if \( s_i = 0, s_i' = 1, s_i'' = 1 \). Let \( r_i = v \) if \( s_i = s_i' = s_i'' = 1 \).
Let \( R' \) be the relation implemented by \( R(r_1, \ldots, r_k) \land \delta_0(u) \land \delta_1(v) \). If \( y \) does not occur as an argument of \( R' \) then \( R'(x, z) \) implements \texttt{Implies}. Similarly, if \( x \) does not occur as an argument of \( R' \) then \( R'(y, z) \) implements \texttt{Implies}. If \( z \) does not occur as an argument of \( R' \) then \( R'(x, y) \) implements \texttt{NAND}. So we assume that \( x, y \) and \( z \) occur as arguments. Then apply Lemma 14 to \( R'(x, y, z) \).

For the final subcase, suppose that there are tuples \( t \) and \( t' \) in \( R \) such that \( t \land t' \) is not in \( R \). Define \( r_i \) to be \( u \) if \( t_i = t'_i = 0 \) or \( x \) if \( t_i = 0, t'_i = 1 \) or \( y \) if \( t_i = 1, t'_i = 0 \), or \( v \) if \( t_i = t'_i = 1 \). Let \( R' \) be the relation implemented by \( R'(x, y) = R(r_1, \ldots, r_k) \land \delta_0(u) \land \delta_1(v) \). If \( t \lor t' \) is in \( R \) then \( R'(x, y) \) implements \texttt{OR}(\( x, y \)), so we are finished. Otherwise \( R' = \{ (0,1), (1,0) \} \) (which we now assume).

Now using Part (i) of Lemma 11 let \( s, s' \) and \( s'' \) be tuples in \( R \) so that \( s \oplus s' \oplus s'' \) is not in \( R \). Define \( r_i \) as follows.

| \( s_i \) | \( s'_i \) | \( s''_i \) | \( r_i \) |
|---|---|---|---|
| 0  | 0  | 0  | \( u \) |
| 0  | 0  | 1  | \( x \) |
| 0  | 1  | 0  | \( y \) |
| 0  | 1  | 1  | \( z \) |
| 1  | 0  | 0  | \( z' \) |
| 1  | 0  | 1  | \( y' \) |
| 1  | 1  | 0  | \( x' \) |
| 1  | 1  | 1  | \( u' \) |

Let \( R'' \) be the relation implemented by

\[
R(r_1, \ldots, r_k) \land \delta_0(u) \land R'(u, u') \land R'(x, x') \land R'(y, y') \land R'(z, z').
\]

By writing \( x' = \overline{x}, y' = \overline{y} \) and \( z' = \overline{z}, \) we can think of \( R'' \) as a function of \( x, y \) and \( z \). If \( x \) does not occur as an argument then \( R''(y, z) \) implements \texttt{Implies}(\( y, z \)). Similarly, we can assume that \( y \) and \( z \) occur as arguments. Now consider the relation \( R''(x, y, z) \). We know that \( (0,0,0), (0,1,1), (1,0,1) \in R'' \), since \( s, s', s'' \in R \). Also \( (1,1,0) \notin R'' \) since \( s \oplus s' \oplus s'' \notin R \). Then apply Lemma 14 to \( R'' \).