FINITE GROUPS WITH LARGE CHEBOTAREV INVARIANT

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Abstract. A subset \( \{g_1, \ldots, g_d\} \) of a finite group \( G \) is said to invariably generate \( G \) if the set \( \{g_1^{x_1}, \ldots, g_d^{x_d}\} \) generates \( G \) for every choice of \( x_i \in G \). The Chebotarev invariant \( C(G) \) of \( G \) is the expected value of the random variable \( n \) that is minimal subject to the requirement that \( n \) randomly chosen elements of \( G \) invariably generate \( G \). The authors recently showed that for each \( \epsilon > 0 \), there exists a constant \( c_\epsilon \) such that \( C(G) \leq (1 + \epsilon)\sqrt{|G|} + c_\epsilon \). This bound is asymptotically best possible. In this paper we prove a partial converse: namely, for each \( \alpha > 0 \) there exists an absolute constant \( \delta_\alpha \) such that if \( G \) is a finite group and \( C(G) > \alpha \sqrt{|G|} \), then \( G \) has a section \( X/Y \) such that \( |X/Y| \geq \delta_\alpha \sqrt{|G|} \), and \( X/Y \cong \mathbb{F}_q \times H \) for some prime power \( q \), with \( H \leq \mathbb{F}_q^* \).

1. Introduction

Following [10] and [5], we say that a subset \( \{g_1, g_2, \ldots, g_d\} \) of a group \( G \) invariably generates \( G \) if \( \{g_1^{x_1}, g_2^{x_2}, \ldots, g_d^{x_d}\} \) generates \( G \) for each \( d \)-tuple \( (x_1, x_2, \ldots, x_d) \in G^d \). The Chebotarev invariant \( C(G) \) of \( G \) is the expected value of the random variable \( n \) which is minimal subject to the requirement that \( n \) randomly chosen elements of \( G \) invariably generate \( G \).

Motivated by the problem of finding field extensions \( K/F \) such that a fixed finite group \( G \) occurs as the Galois group of \( K/F \), E. Kowalski and D. Zywina carried out a detailed investigation of the invariant \( C(G) \) in [12]. Amongst many interesting results, they show that \( C(G) \) can be quite large in comparison to \( |G| \). More precisely, it is shown that if \( G \cong G_q := \mathbb{F}_q \times \mathbb{F}_q^* \), then

\[
C(G) = q - \sum_{1 \neq d | q - 1} \frac{\mu(d)}{q(1 - d^{-1})(1 - d^{-1} + q^{-1})}.
\]

In particular, \( C(G_q) \sim \sqrt{|G_q|} \) as \( q \to \infty \). It was also conjectured in [12] that these are the “worst” cases: that is, that \( C(G) = O(\sqrt{|G|}) \) as \( |G| \to \infty \). The conjecture was proved by the first author in [15], and was later improved in [17] where it is shown that for each \( \epsilon > 0 \), there exists a constant \( c_\epsilon \) such that \( C(G) \leq (1 + \epsilon)\sqrt{|G|} + c_\epsilon \). Furthermore, one has \( C(G) \leq \frac{5}{4} \sqrt{|G|} \) when \( G \) is soluble.

In this paper, we prove a partial converse. Informally, we prove that the only examples where \( C(G) \) is a constant times \( \sqrt{|G|} \) are those groups with a “large” section isomorphic to a subgroup of \( G_q \), for some prime power \( q \). Our main result reads as follows.

Theorem 1. Fix a constant \( \alpha > 0 \). There exists absolute constants \( \beta_\alpha, \gamma_\alpha, \delta_\alpha \) and \( k_\alpha \), depending only on \( \alpha \), such that whenever \( G \) is a finite group with the property that \( C(G) > \alpha \sqrt{|G|} \), then \( G \) has a factor group \( \overline{G} \) such that

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(i) \( G \cong V \rtimes H \), with \( V \cong \mathbb{F}_q^k \), and \( H \leq \Gamma L_1(q) \wr \text{Sym}(k) \), with \( q \) a prime power and \( k \leq k_0 \); 
(ii) \(|G| \geq \delta_n \sqrt{|G|} \); and 
(iii) \( \beta_a|V| \leq |H| \leq \gamma_a|V| \).

Our approach utilises the theory of crowns in finite groups, which we describe in Section 2. We also require a characterisation of those irreducible linear groups \( H \leq GL(V) \) such that the set \( H^*(V) := \{ h \in H : v^h = v \text{ for some } v \in V \setminus \{0\} \} \) is bounded above by an absolute constant, and this is the content of Section 3.

Finally, Section 4 is reserved for the proof of Theorem 1.

2. Crowns in finite groups

Before defining the notion of a crown in a finite group, we require some terminology. First, let \( L \) be a monolithic primitive group. That is, \( L \) is a finite group with a unique minimal normal subgroup \( V \not\leq \text{Frattini}(L) \). For each positive integer \( k \), write \( L^k \) for the \( k \)-fold direct product of \( L \). The crown-based power of \( L \) of size \( k \), denoted \( L_k \), is the subgroup \( L_k \) of \( L^k \) defined by
\[
L_k = \{(l_1, \ldots, l_k) \in L^k \mid l_1 \equiv \cdots \equiv l_k \equiv v \mod |V| \}.
\]
Equivalently, \( L_k = V^k \text{Diag}(L) \).

Next, let \( G \) be a finite group. We say that a group \( V \) is a \( G \)-group if \( G \) acts on \( V \) via automorphisms. Following [9], we say that two irreducible \( G \)-groups \( V_1 \) and \( V_2 \) are \( G \)-equivalent and we put \( V_1 \sim_G V_2 \), if there are isomorphisms \( \phi : V_1 \to V_2 \) and \( \Phi : V_1 \rtimes G \to V_2 \rtimes G \) such that the following diagram commutes:
\[
\begin{array}{ccccccccc}
1 & \longrightarrow & V_1 & \longrightarrow & V_1 \rtimes G & \longrightarrow & G & \longrightarrow & 1 \\
\phi & & \downarrow \phi & & \downarrow \Phi & & & & \\
1 & \longrightarrow & V_2 & \longrightarrow & V_2 \rtimes G & \longrightarrow & G & \longrightarrow & 1.
\end{array}
\]

Note that two \( G \)-isomorphic \( G \)-groups are \( G \)-equivalent. In the abelian case, the converse is true: if \( V_1 \) and \( V_2 \) are abelian and \( G \)-equivalent, then \( V_1 \) and \( V_2 \) are also \( G \)-isomorphic. It is proved (see for example [9, Proposition 1.4]) that two chief factors \( V_1 \) and \( V_2 \) of \( G \) are \( G \)-equivalent if and only if either they are \( G \)-isomorphic, or there exists a maximal subgroup \( M \) of \( G \) such that \( G/\text{Core}_G(M) \) has two minimal normal subgroups \( N_1 \) and \( N_2 \) \( G \)-isomorphic to \( V_1 \) and \( V_2 \) respectively. For example, the minimal normal subgroups of a crown-based power \( L_k \) are all \( L_k \)-equivalent.

Let \( V = X/Y \) be a chief factor of \( G \). A complement \( U \) to \( V \) in \( G \) is a subgroup \( U \) of \( G \) such that \( UV = G \) and \( U \cap X = Y \). We say that \( V = X/Y \) is a Frattini chief factor if \( X/Y \) is contained in the Frattini subgroup of \( G/Y \); this is equivalent to saying that \( V \) is abelian and there is no complement to \( V \) in \( G \). The number of non-Frattini chief factors \( G \)-equivalent to \( V \) in any chief series of \( G \) does not depend on the series, and so this number is well-defined: we will write it as \( \delta_V(G) \).

We now define \( L_V \), the monolithic primitive group associated to \( V \), by
\[
L_V := \begin{cases} 
V \rtimes (G/C_G(V)) & \text{if } V \text{ is abelian}, \\
G/C_G(V) & \text{otherwise}.
\end{cases}
\]

If \( V \) is a non-Frattini chief factor of \( G \), then \( L_V \) is a homomorphic image of \( G \). More precisely, there exists a normal subgroup \( N \) of \( G \) such that \( G/N \cong L_V \).
and $\text{soc}(G/N) \sim_G V$. Consider now all the normal subgroups $N$ of $G$ with the property that $G/N \cong L_V$ and $\text{soc}(G/N) \sim_G V$: the intersection $R_G(V)$ of all these subgroups has the property that $G/R_G(V)$ is isomorphic to the crown-based group $(L_V)_{\delta_V(G)}$. The socle $I_G(V)/R_G(V)$ of $G/R_G(V)$ is called the $V$-crown of $G$ and it is a direct product of $\delta_V(G)$ minimal normal subgroups $G$-equivalent to $V$.

We now record a lemma and two propositions which will be crucial in our proof of Theorem 5. The lemma reads as follows.

**Lemma 2.** [17, Lemma 1.3.6] Let $G$ be a finite group with trivial Frattini subgroup. There exists a chief factor $V$ of $G$ and a non trivial normal subgroup $U$ of $G$ such that $I_G(V) = R_G(V) \times U$.

To state the propositions, we need some additional notation. For a finite group $G$, and an abelian chief factor $V$ of $G$, set $H_V = H_V(G) := G/C_G(V)$, $m = m_V = \dim_{\text{End}_G(V)} H_V^1(H_V, V)$, and write $H^* = H^*(V) = H^*_G(V)$ for the set of elements $h$ of $H_V$ which fix a non-zero vector in $V$. Also, let $\delta_V = \delta_V(G)$, and set $\theta_V = \theta_V(G) = 0$ if $\delta_V = 1$, and $\theta_V = 1$ otherwise. Finally, let $q_V = q_V(G) := |\text{End}_G(V)|$ and $n_V = n_V(G) := \dim_{\text{End}_G(V)} V$. Note that $\text{End}_G(V)$ is a finite field, since $V$ is finite and irreducible.

**Proposition 3.** [17, Proposition 8 and the Proof of Theorem 1] Let $G$ be a finite group with trivial Frattini subgroup, and let $U$, $V$ and $R = R_G(V)$ be as in Lemma 2. If $U$ is non-abelian, then there exists absolute constants $b_1$, $b_2$ and $b_3$ such that

$$C(G) \leq C(G/U) + [b_3(\log |G|)^2] + \frac{b_1}{b_2} \sqrt{|G|^2 \log |G|((1 - b_2/\log |G|)^{b_3(\log |G|)^2}}.$$

**Proposition 4.** [17, Proposition 8 and the Proof of Theorem 1] Let $G$ be a finite group with trivial Frattini subgroup, and let $U$, $V$ and $R = R_G(V)$ be as in Lemma 2. Suppose that $V$ is abelian, and write $q = q_V$, $n = n_V$ and $H = H_V$, $H^* = H^*(V)$ and $m = m_V$. Also, set $\delta = \delta_V$ and $\theta = \theta_V$. Set

$$\alpha_U := \begin{cases} \sum_{0 \leq i \leq \delta - 1} \frac{(q - q^i)}{q - q^i} \leq \delta + \frac{q}{(q - 1)^2}, & \text{if } H = 1, \\ \min \left\{ \left( \delta \cdot \theta + m + \frac{q^n}{q - q^m} \right) \frac{|H|}{m}, \left( \frac{\delta \cdot \theta}{m} + \frac{q^n}{q - q^m} \right) |H| \right\} & \text{otherwise.} \end{cases}$$

Then

$$C(G) \leq C(G/U) + \alpha_U.$$
Proposition 6. Let $V$ be a vector space of dimension $n$ over a field $F$, and fix a constant $c > 0$. Suppose that $H$ is an irreducible subgroup of $GL(V)$ with the property that $|H^*| \leq c$. Then there exists positive integers $m$ and $k$ such that $n = mk$, and $H \leq R \wr \text{Sym}(k)$, where either $|R|$ has order bounded above by a function of $|H^*|$, or $R \cong \Gamma_1(F_m)$ for some extension field $F_m$ of $F$ of degree $m$.

Proposition 6 will follow almost immediately from our next result. Recall that if $F$ is a field, then an irreducible subgroup $H$ of a linear group $GL_n(F)$ is called weakly quasiprimitive if every characteristic subgroup of $G$ is homogeneous.

Proposition 7. There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if $F$ is a field, $n$ is a positive integer, and $H \leq GL_n(F)$ is finite and weakly quasiprimitive, then either $|H| \leq f(|H^*|)$, or $H$ is a subgroup of $\Gamma L_1(F_n)$, for some extension field $F_n$ of $F$ of degree $n$.

Proof. If $n = 1$, then $\Gamma L_n(F) = GL_n(F)$. Thus, we may assume that $n > 1$. Fix a subgroup $H$ of $GL_n(F)$. We want to prove that if $H$ is not a subgroup of $\Gamma L_1(F_n)$ for some extension field $F_n$ of $F$ of degree $n$, then $|H|$ is bounded in terms of $|H^*|$. Suppose first that every characteristic abelian subgroup of $H$ is contained in $Z(GL_n(F))$. Let $L$ be the generalised Fitting subgroup of $H$. Our aim is to prove that $|L|$ is bounded above in terms of $|H^*|$. Since $L$ is self-centralising, this will show that $|H|$ is bounded above in terms of $|H^*|$, which will give us what we need.

To this end, extend the field $F$ so that $F$ is a splitting field for all subgroups of $L$. Then $L$ may no longer be homogeneous, but its irreducible constituents are algebraic conjugates of each other, so $L$ acts faithfully on them. Let $W$ be such a constituent, and let $r_i, m_i, s_i, t_i, S_i$ and $T_i$ be as in [Lemma 2.14]. By Lemmas 2.15, 2.16 and 2.17, $W$ decomposes as a tensor product

$$W = W_Z \otimes W_{r_1} \otimes \ldots \otimes W_{r_n} \otimes W_{s_1} \otimes \ldots \otimes W_{s_k},$$

where $W_Z$ is a 1-dimensional module for $Z$; $W_{r_i}$ is an irreducible module for $O_{r_i}(G)$ of dimension $r_i^{m_i}$; and $W_{s_i}$ is an irreducible module for $T_i$ of dimension $s_i^{t_i}$. In particular, $[O_{r_i}(H), W_{r_i}] = [T_i, W_{s_j}] = 1$ for $i \neq j$, and $[O_{r_i}(H), W_{s_j}] = [T_i, W_{r_i}] = 1$, for all $i, j$. Hence, if $a + b > 1$, then $|L|$ is bounded above in terms of $|H^*|$, as needed. So we may assume that either $L = Z(G) \circ O_{r}(H)$, for some prime $r$, or $L = Z(G) \circ T$ is a central product of $t$ copies of a quasisimple group $S$. If $Z(G) \not\subseteq O_{r}(H)$ in the first case, or $Z(G) \not\subseteq T$ in the second case, then the same argument as above gives that $|L|$ is bounded above in terms of $|H^*|$. So we may assume that either $L = O_{r}(H)$, for some prime $r$, or $L = T$ is a central product of $t$ copies of a quasisimple group $S$. Hence, $W$ is a tensor product of $m$ [respectively $t$] copies of an irreducible module for an extraspecial group of order $r^3$ [resp. quasisimple group]. Thus, by arguing as in the paragraph above, we can immediately reduce to the case $m = 1$ [resp. $t = 1$].

Suppose first that $L = O_{r}(H) = M \times \langle x \rangle$ is extraspecial of order $r^3$, for a prime $r$, where $M$ is cyclic of order $r^2$ if $L$ has exponent $r^2$, and $M$ is elementary abelian of order $r^2$ otherwise. Then, being an absolutely irreducible module for $L$ of dimension $r$, $W$ is isomorphic to $U \uparrow_{M} \Gamma_1$, where $U$ is a one dimensional module for $M$ in which $Z(L)$ acts non-trivially. Hence, we may write $W = \bigoplus_{j=0}^{r-1} U \otimes x^j$. It follows that for each non-zero vector $u \in U$, $x^j$ fixes the non-zero vector $u \otimes 1 + u \otimes x + \ldots + u \otimes x^{r-1}$. Thus, $r \leq |H^*|$, from which it follows that $|L| = r^3$ is bounded above in terms of $|H^*|$, as needed.
Finally, assume that $L$ is quasisimple. Since $L$ acts on $L^*$ by conjugation, we may assume that $L^* \leq Z$ (otherwise $L \leq \text{Sym}(L^*)$, which would imply that $|L|$ is bounded above in terms of $|H^*|$). However, since $Z = Z(H) \leq Z(GL_n(F))$, $Z$ acts on $V$ by scalar multiplication. Hence, $Z \cap H^* = 1$. It follows that $L^* = 1$, and hence that $L$ is a Frobenius complement in the group $V \rtimes L$. Since $L$ is perfect, it now follows from Zassenhaus' Theorem that $L \cong SL_2(5)$. Whence, $|L|$ is bounded, and this prove our claim.

Finally, assume that $H$ has a characteristic abelian subgroup not contained in $Z(GL_n(F))$, and let $M \leq H$ be maximal with this property. Then by \cite{16} Lemma 1.10], $M$ is contained in $Z(GL_m(F))$ for some $m$ dividing $n$, and some extension field $F_m$ of $F$ of degree $m$. Hence, $H_1 := C_H(M)$ is a subgroup of $GL_m(F_m)$ with the property that every characteristic abelian subgroup of $H_1$ is contained in $Z(GL_m(F_m))$. Furthermore, $H_1$ is weakly quasiprimitive, since it is characteristic in $H$. Also, the group $H/H_1$ is naturally embedded in $\text{Gal}(F_m/F)$, its action induced by a vector space isomorphism $F_m \rightarrow F^n$. Since $H_1^1(F_m) = H_1^1(F^n)$, it follows from the arguments above that either $|H_1|$ is bounded in terms of $|H^*|$; or $n = 1$. If $|H_1|$ is bounded in terms of $|H^*|$, then so is $|H|$, since $H_1$ is self-centralising and normal in $H$. If $n = 1$, then $H_1 \leq GL_1(F_n)$, so $H \leq \Gamma L_1(F_n)$, since $H/H_1$ acts on $M = Z(H_1)$ via the Galois group, as described above. This completes the proof.

Finally, we prove Proposition \cite{10}.

**Proof of Proposition \cite{10}** If $H$ is primitive, then the result follows immediately from Proposition \cite{11}. Thus, we may assume that $H$ is not primitive. Then $V$ may be decomposed into a system $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$ of imprimitivity for $H$. Let $\Gamma := \{W_1, \ldots, W_k\}$, let $S := H^\Gamma$ denote the induced (transitive) action of $H$ on $\Gamma$, and let $R := \text{Stab}_H(W_1)^W_1$ denote the induced action of $\text{Stab}_H(W_1)$ on $W_1$. Then $H$ is isomorphic to a subgroup of the wreath product $R \wr S$.

Finally, since $\text{Stab}_H(W)$ induces $R$ on $W$, we have $|R^*(W_1)| \leq |H^*(V)|$. Hence, Proposition \cite{10} implies that either $R \leq \Gamma L_1(F_m)$, for some extension $F_m$ of $F$ of degree $m$, or $|R|$ is bounded above by a function of $|H^*|$. This completes the proof.

\section{The proof of Theorem \cite{11}}

We begin our preparations towards the proof of Theorem \cite{11} with a lemma concerning the cohomology of an irreducible linear group which has a bounded number of elements fixing a non-zero vector.

**Lemma 8.** There exists an absolute constant $c$ such that if $V$ is a vector space of dimension $n$ over a field $F$ of characteristic $p > 0$, and $H$ is an irreducible subgroup of $GL(V)$ with the property that $|H| > \sqrt{|V|}$, then $2^m \leq c|H^*|^4$, where $m := \dim_F H^1(H, V)$ and $F := \text{End}_H V$.

**Proof.** Clearly we may assume that $m > 0$. Then, it is proven in \cite{15} Lemma 9] that
\begin{enumerate}
  \item $H$ has a unique minimal normal subgroup $N$, which is non-abelian.
  \item If $S$ is a component of $H$, then $C_H(S) \leq H^*$.
  \item If $W$ is an irreducible $N$-submodule of $V$ not centralised by $S$, then $m \leq \dim_F H^1(S, W)$.
\end{enumerate}
Write \( N = S_1 \times \ldots \times S_t \cong S^t \), and view \( H \) as a subgroup in the wreath product \( \text{Aut}(N) = \text{Aut}(S) \wr K \), where \( K \) denote the induced action of \( H \) on the components in \( N \). Suppose first that \( t > 1 \). Then (2) implies that \( S_i \subseteq H^* \) for all \( i \). Hence, \( |H^*| \geq 1 + t(|S| - 1) \). Also, \( |H^*| \geq C_H(S_t) \geq |H \cap B| |\text{Stab}_K(1)| = |H \cap B||K| \), where \( B := \text{Aut}(S_2) \times \ldots \times \text{Aut}(S_t) \). Note that \( |H| \leq |H \cap B| |\text{Aut}(S)||K| \). It follows that \( |H| \leq |H^*|t |\text{Aut}(S)| \leq |H^*|t(|S| - 1)^2 \leq |H^*|^3 \).

Next, it is shown by Guralnick and Hoffman in [7, Theorem 1] that \( m \leq \frac{n}{2} \).

Since we also have \( |H| > \sqrt{|V|} \), it follows that
\[
m \leq \frac{n}{2} \leq \log \sqrt{|V|} < \log |H| \leq \log |H^*|^3.
\]

Thus, we may assume that \( H \leq \text{Aut}(S) \) is almost simple. Before distinguishing cases, we make some remarks. First, \( p = \text{char } F \) divides \( |H| \), since \( H^1(H, V) \neq 0 \). Furthermore, \( |H^*| \geq |H|_p \), since every element of a Sylow \( p \)-subgroup of \( H \) fixes a non-zero vector in \( V \). Finally, note that we may assume that \( S \) is not sporadic, since there are a bounded number of such groups having an irreducible module with non-zero cohomology.

Thus, we have two cases.

(a) \( S \cong \text{Alt}(k) \). In this case, we have \( \frac{n}{2} \leq \log \sqrt{|V|} \leq \log |H| \leq k \log k \), as long as \( k > 6 \). Hence, by [15 Proposition 10], we have \( m \leq 4 \log k \) and \( |H|_p > \frac{k}{2} \), if \( k \) is large enough. Hence \( 2^m \leq k^4 \leq 16|H^*|^4 \) in this case. If \( k \) is also bounded, then \( m \) is also bounded, since \( m \leq \frac{n}{2} \leq \log |H| \). Hence, the result also follows in this case.

(b) \( S \cong X_k(r) \) is a group of Lie type. Write \( R_F(S) \) for the smallest degree of a non-trivial irreducible representation of \( S \) over the field \( F \). If \( char \ F \) is different to the defining characteristic for \( S \), then we have \( \frac{p|R_F(S)|}{|S|} > |\text{Aut}(S)| \) for \( |S| \) large enough (see [13, 15, 20]). Since \( \sqrt{|V|} \leq |H| \), we conclude that either \( |S| \) is bounded, or \( char \ F \) coincides with the defining characteristic of \( S \). In the latter case, we have \( |H|_p > |S|^{1/2} \) by [11 Proposition 3.5]. Also, \( |S| \geq |\text{Aut}(S)|^{1/2} \) by [13 Proposition 4.4]. Hence,
\[
|H^*| > |S|^{1/2} \geq |\text{Aut}(S)|^{1/2} > |H|^{1/2} \geq 2^{1/4}.
\]

Thus, either \( |S| \) is bounded, or \( 2^m \leq |H^*|^4 \). This gives us what we need.

Next, we prove a reduction lemma.

**Lemma 9.** Fix a constant \( \alpha > 0 \). There exists absolute constants \( b = b(\alpha) \), \( c = c(\alpha) \) and \( c_i = c_i(\alpha) \), \( 1 \leq i \leq 4 \), depending only on \( \alpha \), such that: If \( G \) is a finite group with trivial Frattini subgroup with the property \( C(G) > \alpha \sqrt{|G|} \), and \( U \) is as in Lemma 2 then one of the following holds.

(i) \( U \) is non-abelian and \( |G| \leq b \).

(ii) \( U \) is abelian and \( |U| \leq c \).

(iii) \( U \) is abelian and \( G \) has a factor group \( \overline{G} \) such that

(a) \( \overline{G} \cong V \rtimes H \), with \( V \cong U \) an abelian chief factor of \( G \), and \( H \leq GL(V) \);

(b) \( |H^*(V)| \leq c_1 \);

(c) \( \dim_{\text{End}_H V} H^1(H, V) \leq c_2 \); and

(d) \( c_3 |V| \leq |H| \leq c_4 |V| \).

□
Proof. Adopt in its entirety the notation of Proposition \cite{4} so that $U$, $V$ and $R = R_G(V)$ are as in Lemma \cite{2} We first consider the case where $V$ is non-abelian. Then by Proposition \cite{3} we have
\[ \alpha \sqrt{|G|} < C(G/U) + \left[ b_3 \log |G| \right]^2 + \frac{b_1}{b_2} \sqrt{|G|}^3 \log |G| \left(1 - b_2/\log |G| \right) \left[ b_3 \log |G| \right]^2, \]
where $b_1$, $b_2$ and $b_3$ are the absolute constants from Proposition \cite{3}. Since $C(G/U) \leq C \sqrt{|G/U|}$, it follows that $\sqrt{|G|} \leq \alpha' \left[ b_3 \log |G| \right]^2 + \frac{b_1}{b_2} \sqrt{|G|}^3 \log |G| \left(1 - b_2/\log |G| \right) \left[ b_3 \log |G| \right]^2$ for some constant $\alpha'$ depending only on $\alpha$. Hence, since the square root of $|G|$ divided by the right hand side of the above equation tends to $\infty$ as $|G|$ tends to infinity, we must have that $|G|$ is bounded above by a constant $b = b(\alpha)$ depending only on $\alpha$.

Thus, we may assume that $U$ is abelian. Then by Proposition \cite{4} and Theorem \cite{5} there exists an absolute constant $C$ such that
\[ \alpha \sqrt{|G|} \leq C(G) \leq C(G/U) + \alpha_U \leq c \sqrt{|G/U|} + \alpha_U. \]
In particular, using the definition of $\alpha_U$ from Proposition \cite{4} we conclude that
\begin{align}
\alpha \leq & \frac{c}{\sqrt{|U|}} + (\delta \cdot \theta + m + 2) \frac{\sqrt{|H|}}{\sqrt{|V|}}|H^*|, \\
\alpha \leq & \frac{c}{\sqrt{|U|}} + \left( \frac{\delta \cdot \theta}{n} + 2 \right) \frac{\sqrt{|H|}}{\sqrt{|V|}}|H^*|.
\end{align}
We claim first that $\delta = 1$. Indeed, assume otherwise, and note that $\frac{|H|}{|H^*|} \leq |H|/|H_v| \leq |V|$, for any non-zero $v \in V$. Hence, since $m \leq \frac{n}{\delta}$, we conclude from (4.1) that
\[ |V|^{\frac{\alpha}{n+\delta}} \leq C_1(n + \delta), \]
where $C_1 = C_1(\alpha)$ depending only on $\alpha$. Now, since $|U| = |V|^{\delta} = q^n$, we conclude that there exists a constant $c = c(\alpha)$ such that if $|U| > c$ and $\delta > 1$ then $|V|^{\frac{\alpha}{n+\delta}} > C_1(n + \delta)$.

Hence, we may assume that $\delta = 1$. We will first prove that the properties (b) and (c) of Part (iii) of the statement of the lemma hold in the factor group $G := G/R_G(V)$. If $|H| \leq \frac{|V|}{q^m}$ then (4.1) respectively (4.2) implies that $|H^*|$ [resp. $n$] is bounded above by a constant depending only on $\alpha$. Properties (b) and (c) then follow immediately.

So we may assume that $|H| > \frac{|V|}{q^m}$. We then use (4.1) and the fact that $|H|/|H_v| \leq |V|$ to deduce that $|H^*| \leq C_2(1 + m^2)$, where $C_2 = C_2(\alpha)$ is a constant depending only on $\alpha$. Since $|H| > \sqrt{|V|}$, if follows from Lemma \cite{3} that $|H^*| \leq C_3(1 + \log |H^*|^2)$, where $C_3 = C_3(\alpha)$ is a constant depending only on $\alpha$. It follows that $|H^*|$, and hence $m$, are bounded above by constants depending only on $\alpha$. This proves that Properties (b) and (c) hold.

Finally, the existence of $c_3$ follows immediately from (4.2), while the existence of $c_4$ follows from (4.1) and the bound $|H|/|H^*| \leq |V|$. This proves that Property (d) holds, and completes the proof. $\square$

We are now ready to prove Theorem \cite{1}.
Proof of Theorem 4. Let $C$ be the constant from Theorem 3 let $f$ be the function from Proposition 2 let $b_1$, $b_2$ and $b_3$ be the constants from Proposition 3 and let $b = b(\alpha)$ and $c = c(\alpha)$ be the constants from Lemma 3. Also, let $c_i, 1 \leq i \leq 4$, be the functions of $\alpha$ from Lemma 3. Note that we may assume that $f$, $c_1$, $c_2$ and $c_4$ are increasing functions, while $c_3$ is decreasing. Hence, we may also assume that $g$ satisfies $g(\alpha_1 \alpha_2) \geq g(\alpha_1) \alpha_2$, for $g \in \{f, c_1\}$. For ease of notation, we will sometimes write $c_i$ in place of $c_i(\alpha)$.

Set $b_4 := \max\{b, [b_3 (\log b)^2] + \frac{b_2}{2b_3} \log b (1 - b_2 / \log b) [b_3 (\log b)^2]\};$ $\alpha' := \max\{\alpha, C\};$ $c_5 := \max\{c, \frac{1}{\alpha(\alpha')} f(\alpha')\}$; $\alpha' = \max\{\alpha, C\};$ and $c_6 := (2 + c_2) c_5$. Then define

$$\delta(\alpha) := \min\{f(\alpha') : 0 < \beta \leq \alpha'\} \text{ and }$$

$$k(\alpha) := \frac{c_1(\alpha')}{c_3(\alpha')}.$$  

Finally, set $\beta := c_3$ and $\gamma := c_4$. Note that by construction $k$ is an increasing function of $\alpha$, and that

(4.4) \hspace{1cm} \delta(\beta \sqrt{\alpha}) \geq \delta(\beta) \sqrt{\alpha} \geq \delta(\alpha) \sqrt{\alpha},$$

whenever $\beta \leq \alpha$.

We will now prove by induction on $|G|$ that $G$ has a factor group $\overline{G}$ such that

(i) $\overline{G} \cong V \times H$, with $V \cong \mathbb{F}_q^k$, and $H \leq \Gamma L_1(q) \wr \text{Sym}(k)$, with $q$ a prime power and $k \leq k(\alpha)$;

(ii) $|\overline{G}| \geq \delta(\alpha) \sqrt{|G|}$; and

(iii) $\beta(\alpha)|V| \leq |H| \leq \gamma(\alpha)|V|$.

Suppose first that Frat($G$) = 1, and let $U$, $V$ and $R = R_1(V)$ be as in Lemma 3. We would like to reduce to the case where $|G| > b$ if $V$ is non-abelian, and $|U| > c_5$ if $V$ is abelian. We first deal with the non-abelian case. So assume that $V$ is non-abelian and that $|G| \leq b$. In this case, we have

$$\alpha \sqrt{|G|} < C(G/U) + b_4 \leq (1 + b_4) C(G/U),$$

by Proposition 3. In particular, it follows that $C(G/U) > \alpha \sqrt{|G/U|}$, where

$$\alpha := \frac{\alpha \sqrt{|U|}}{1 + b_4}.$$  

Note that $\gamma(\alpha_1) \leq \gamma(\alpha)$, since $\alpha_1 \leq \alpha$, and $\gamma$ is an increasing function. Similarly, $k(\alpha_1) \leq k(\alpha)$ and $\beta(\alpha) \leq \beta(\alpha_1)$. Furthermore, $\delta(\alpha_1) \geq \delta(\alpha) \sqrt{|U|}$ by (1.3). The inductive hypothesis now implies that $G$, and hence $G/U$, has a factor group $\overline{G}$ with the desired properties.

Next, assume that $V$ is abelian, and that $|U| \leq c$. Then since $\alpha U \leq c_6$, Proposition 2 yields $C(G/U) > \alpha_2 \sqrt{|G/U|}$, where

$$\alpha_2 := \frac{\alpha \sqrt{|U|}}{1 + c_6}.$$  

As above, it now follows from the inductive hypothesis and the definitions of $\delta(\alpha)$ and $k(\alpha)$ that $G$ has a factor group $\overline{G}$ with the desired properties.

Thus, we may assume that $|G| > b$ if $U$ is non-abelian, and $|U| > c_5 \geq c$ otherwise. However, Lemma 3 then implies that $U$ must be abelian, and that $G$ has a factor group $\overline{G}$ such that
(a) $\overline{G} \cong V \times H$, with $V \cong U$ an abelian chief factor of $G$, and $H \leq GL(V)$;
(b) $|H^*(V)| \leq c_1(\alpha)$;
(c) $\dim_{\text{End}_H} V^1(H, V) \leq c_2(\alpha)$; and
(d) $c_3(\alpha)|V| \leq |H| \leq c_4(\alpha)|V|$.

Furthermore, Lemma 6 guarantees the existence of positive integers $m$ and $k$, and a transitive permutation group $S$ of degree $k$, such that $n = mk$ and $H \leq R \wr S$, with either $|R| \leq f(c_1)$, or $R \leq \Gamma L_1(p^m)$. Hence, we just need to prove that $k \leq k(\alpha)$. Indeed, if this is true then we must have $R \leq \Gamma L_1(p^m)$, since otherwise $|V| \leq \frac{1}{c_3(\alpha)}|H| \leq \frac{1}{c_3(\alpha)}f(c_1(\alpha)) \frac{c_1(\alpha)}{c_3(\alpha)}!, \text{ contradicting } |U| > c_5$.

Now, note that (b) and (d) above imply that the number of orbits of $H$ in its action on $V$ is bounded above by $1 + \frac{|c_4|}{c_3}$. Hence, the number of orbits of $X := GL_m(p) \wr \text{Sym}(k)$ is bounded above by $1 + \frac{|c_4|}{c_3}$. Then since $GL_m(p)$ has 2 orbits in its action on the natural module $(\mathbb{F}_p)^m$, it follows that the number of orbits of $X$ on $V$ is precisely the number of orbits of $\text{Sym}(k)$ in its action on the $k$-fold cartesian power $\{0, 1\}^k$ by permutation of coordinates. This number is precisely $k + 1$. Hence, we have $k + 1 \leq 1 + \frac{|c_4|}{c_3}$, and this completes the proof in the case $\text{Frat}(G) = 1$.

Finally, assume that $\text{Frat}(G) > 1$. Then $C(G/\text{Frat}(G)) = C(G) > \beta \sqrt{|G/\text{Frat}(G)|}$, where $\beta := \alpha \sqrt{|\text{Frat}(G)|}$. Now, since $\alpha \sqrt{|C|} < C(G/\text{Frat}(G)) \leq C(G/\text{Frat}(G))$, we have $|\text{Frat}(G)| \leq \left(\frac{2}{\alpha}\right)^2$. Hence, $\beta \leq C$. The result now follows from the inductive hypothesis and the definitions of $\delta(\alpha)$ and $k(\alpha)$. \qed

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