Abstract differential equations and Caputo fractional derivative

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Abstract

In this work we consider the abstract Cauchy problem with Caputo fractional time derivative of order $\alpha \in (0, 1]$, and discuss the continuity of the respective solutions regarding the parameter $\alpha$. We also present a study about the continuity of the Mittag-Leffler families of operators (for $\alpha \in (0, 1]$), when they are induced by sectorial operators.

Keywords Fractional differential equations · Mittag-Leffler operators · Sectorial operators · Semilinear equations

1 Contextualization of the main problem

We begin by presenting the abstract Cauchy problem

$$\begin{cases}
  cD_t^\alpha u(t) = Au(t) + f(t, u(t)), & t \geq 0, \\
  u(0) = u_0 \in X,
\end{cases} \quad (P_\alpha)$$

where $X$ is a Banach space over $\mathbb{C}$, $\alpha \in (0, 1]$, $A : D(A) \subset X \to X$ is a sectorial operator, $f : [0, \infty) \times X \to X$ is a continuous function and $cD_t^\alpha$ denotes Caputo fractional derivative of order $\alpha$, when $\alpha \in (0, 1]$, and denotes the standard derivative when $\alpha = 1$. (see Sect. 2 for more details).

In a previous work [2] (see also [4]), which was done with some collaborators, we have presented several results about the existence, uniqueness and continuation of mild solutions to problem $(P_\alpha)$, when $\alpha \in (0, 1)$. In fact, the results obtained in [2] are similar to the classical results that are obtained when $\alpha = 1$. Therefore, in order
to clarify the differences between these two situations, let us present the following notions and results.

**Definition 1** Consider $\alpha \in (0, 1]$ and $T$ a positive real value.

(a) A function $\phi_\alpha : [0, T] \to X$ is called a local mild solution of $(P_\alpha)$ in $[0, T]$ if $\phi_\alpha \in C([0, T]; X)$ and

$$\phi_\alpha(t) = \begin{cases} E_\alpha(At^\alpha)u_0 + \int_0^t (t-s)^{\alpha-1}E_{\alpha}(A(t-s)^\alpha)f(s, \phi_\alpha(s))ds, & \text{if } \alpha \in (0, 1), \\ e^{At}u_0 + \int_0^t e^{A(t-s)}f(s, \phi_\alpha(s))ds, & \text{if } \alpha = 1, \end{cases}$$

for every $t \in [0, T]$.

(b) A function $\phi : [0, T) \to X$ is called a local mild solution of $(P_\alpha)$ in $[0, T)$, if for any $T^* \in (0, T)$, $\phi(t)$ is a local mild solution of $(P_\alpha)$ in $[0, T^*]$.

**Definition 2** Let $\alpha \in (0, 1]$ be given. A function $\phi : [0, \infty) \to X$ is called a global mild solution of $(P_\alpha)$, if for any $T^* > 0, \phi(t)$ is a local mild solution of $(P_\alpha)$ in $[0, T^*]$.

It is worth to recall that the family of operators $\{e^{At} : t \geq 0\} \subset \mathcal{L}(X)$ is the standard analytic semigroup generated by the sectorial operator $A$, while the families $\{E_\alpha(At^\alpha) : t \geq 0\} \subset \mathcal{L}(X)$ and $\{E_{\alpha,a}(At^\alpha) : t \geq 0\} \subset \mathcal{L}(X)$, which are called Mittag-Leffler operators, are standard from the literature of fractional differential equations (see for instance [2], and references therein). However, for the completeness of this paper we also present more details about these operators in Section 2.

By introducing the notions of continuation of mild solution and maximal local mild solution, it is possible to prove an interesting result, which was called in [2] “Blow up Alternative”. The “Blow up Alternative” was what motivated the discussion proposed in this work.

**Definition 3** Let $\alpha \in (0, 1]$ and let $\phi_\alpha : [0, T) \to X$ be a local mild solution in $[0, T)$ of problem $(P_\alpha)$. If $T^* \geq T$ and $\phi_\alpha^* : [0, T^*) \to X$ is a local mild solution to $(P_\alpha)$ in $[0, T^*)$, with $\phi_\alpha^*(t) = \phi_\alpha(t)$, for every $t \in [0, T)$, then we say that $\phi_\alpha^*(t)$ is a continuation of $\phi_\alpha(t)$ over $[0, T^*)$.

**Definition 4** Consider $\alpha \in (0, 1]$ and $\phi_\alpha : [0, T) \to X$. If $\phi(t)$ is a local mild solution of $(P_\alpha)$ in $[0, T)$ that does not have a continuation in $[0, T)$, then we call it a maximal local mild solution of $(P_\alpha)$ in $[0, T)$.

**Theorem 1** (Blow up Alternative) Let $\alpha \in (0, 1]$ and $f : [0, \infty) \times X \to X$ be a continuous function, locally Lipschitz in the second variable, uniformly with respect to the first variable, and bounded (i.e. it maps bounded sets onto bounded sets). Then problem $(P_\alpha)$ has a global mild solution $\phi_\alpha(t)$ in $[0, \infty)$ or there exists $\omega_\alpha \in (0, \infty)$ such that $\phi_\alpha : [0, \omega_\alpha) \to X$ is a maximal local mild solution in $[0, \omega_\alpha)$, and in such a case, $\limsup_{t \to \omega_\alpha^-} \|\phi_\alpha(t)\| = \infty$. 

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Remark 1 Although in [2] they have not presented a proof for Theorem 1 in the case $\alpha = 1$, it is simple to adapt such result. Therefore, since this proof does not bring any gain to the results discussed in this paper, we prefer to omit it from this paper.

Theorem 1 is our main concern from now on. Observe that for each $\alpha \in (0, 1]$ there exists $\omega_\alpha > 0$ (it may occur $\omega_\alpha = \infty$ for some values of $\alpha \in (0, 1]$) and a respective maximal local mild solution (or global mild solution) $\phi_\alpha : [0, \omega_\alpha) \to X$ to problem $(P_\alpha)$.

It is very natural that several questions can be raised from the facts emphasized above. In this paper, the main questions we discuss are the following:

(Q1) Does $E_\alpha(At^\alpha)$ converge to the semigroup $e^{At}$, when $\alpha \to 1^-$, in some sense?
(Q2) Does $E_{\alpha, \alpha}(At^\alpha)$ converge to the semigroup $e^{At}$, when $\alpha \to 1^-$, in some sense?
(Q3) Is the intersection of every $[0, \omega_\alpha)$ non trivial, i.e.,

$$[0] \subset \bigcap_{\alpha \in (0, 1]} [0, \omega_\alpha)?$$

(Q4) Does $\phi_\alpha(t)$ converge to $\phi_1(t)$, when $\alpha \to 1^-$, in some sense?

The set of questions made above are called here the “Limit Problems”. The main objective of this work is to answer these questions as fully as possible.

In order to describe the subjects dealt with in this work, we present a short summary of each of the following sections.

Section 2 makes a small survey about the notions used in this paper. To be more precise, in this section we recall the notions of sectorial operators, analytical semigroups, Mittag-Leffler operators and Caputo fractional derivative of order $\alpha \in (0, 1)$.

Finally, in Sect. 3 we present the results that answer the four questions called the Limit Problems, while in Sect. 4 we present a short last discussion about the results obtained in Sect. 3.

2 A small survey on sectorial operators and fractional calculus

Let us begin by recalling the notions of sectorial operators and analytical semigroups. It is worth to emphasize that for a complete discussion of these notions we may refer to [9–11, 14] and the references therein.

Definition 5 A semigroup of linear operators on $X$, or simply semigroup, is a family $\{T(t) : t \geq 0\} \subset \mathcal{L}(X)$ such that:

i) $T(0) = I_X$, where $I = I_X$ is the identity operator in $X$.
ii) It holds that $T(t)T(s) = T(t+s)$, for any $t, s \geq 0$.

Definition 6 Assume that $\{T(t) : t \geq 0\}$ is a semigroup. If it holds that

$$\lim_{t \to 0^+} T(t)x = x,$$

for every $x \in X$, we say that it is a $C_0$-semigroup.
**Definition 7** Let \( \{ T(t) : t \geq 0 \} \) be a \( C_0 \)-semigroup. Its infinitesimal generator is the linear operator \( A : D(A) \subset X \to X \), where

\[
D(A) := \left\{ x \in X : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}
\]

and

\[
Ax := \lim_{t \to 0^+} \frac{T(t)x - x}{t}.
\]

From the definitions above, we may present the classical result that relates semigroups and the linear part of \((P_\alpha)\), when \( \alpha = 1 \).

**Theorem 2** Suppose that \( \{ T(t) : t \geq 0 \} \subset \mathcal{L}(X) \) is a \( C_0 \)-semigroup on \( X \). If \( A : D(A) \subset X \to X \) is the infinitesimal generator of \( \{ T(t) : t \geq 0 \} \), then \( A \) is a closed and densely defined linear operator. Also, for any \( x \in D(A) \) the application \([0, \infty) \ni t \mapsto T(t)x \in X\) is continuously differentiable and

\[
\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax, \quad \forall t \geq 0.
\]

**Proof** See Theorems 10.3.1 and 10.3.3 in [10]. \( \square \)

**Remark 2** From now on, to avoid confusion in this paper, whenever \( A \) is the infinitesimal generator of \( \{ T(t) : t \geq 0 \} \), we shall write that \( e^{At} \) instead of \( T(t) \).

Now we describe an important curve in the complex plane, that is used throughout this paper.

**Definition 8** The symbol \( Ha \) denotes Hankel’s path, if there exist \( \epsilon > 0 \) and \( \theta \in (\pi/2, \pi) \), where \( Ha = Ha_1 + Ha_2 - Ha_3 \), and the paths \( Ha_i \) are given by

\[
Ha = \begin{cases} 
Ha_1 := \{ te^{i\theta} : t \in [\epsilon, \infty) \}, \\
Ha_2 := \{ e^{it} : t \in [-\theta, \theta) \}, \\
Ha_3 := \{ te^{-i\theta} : t \in [\epsilon, \infty) \}.
\end{cases}
\]

The following discussion provides a brief description of the basic results of the theory of sectorial operators and analytical semigroups.

**Definition 9** Let \( A : D(A) \subset X \to X \) be a closed and densely defined operator. The operator \( A \) is said to be a sectorial operator if there exist \( \theta \in (\pi/2, \pi) \) and \( M_\theta > 0 \) such that

\[
S_\theta := \{ \lambda \in \mathbb{C} : |\arg(\lambda)| \leq \theta, \lambda \neq 0 \} \subset \rho(A)
\]

and

\[
\| (\lambda - A)^{-1} \|_{\mathcal{L}(X)} \leq \frac{M_\theta}{|\lambda|}, \quad \forall \lambda \in S_\theta.
\]
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**Theorem 3** If $A : D(A) \subset X \to X$ is a sectorial operator, then $A$ generates a $C_0$-semigroup $\{e^{At} : t \geq 0\}$, which is given by

$$e^{At} = \frac{1}{2\pi i} \int_{Ha} e^{\lambda t} (\lambda - A)^{-1} d\lambda, \quad \forall t \geq 0,$$

where $Ha \subset \rho(A)$. Moreover, there exists $M_1 > 0$ such that

$$\|e^{At}\|_{\mathcal{L}(X)} \leq M_1, \quad \|Ae^{At}\|_{\mathcal{L}(X)} \leq (M_1/t),$$

for every $t > 0$. Finally, for any $x \in X$ the function $[0, \infty) \ni t \mapsto e^{At}x \in X$ is analytic and

$$\frac{d}{dt} e^{At}x = Ae^{At}x,$$

for every $t > 0$.

**Proof** See Theorem 1.3.4 in [9] or Theorem 6.13 in [14, Chapter 2].

Now, let us draw the attention to the fractional calculus, the Mittag-Leffler functions and their relations with the sectorial operators.

**Definition 10** For $\alpha \in (0, 1)$ and $f : [0, T] \to X$, the Riemann-Liouville fractional integral of order $\alpha > 0$ of $f(t)$ is defined by

$$J^\alpha_I t f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) \, ds,$$

for every $t \in [0, T]$ such that integral (1) exists. Above $\Gamma$ denotes the classical Euler gamma function.

**Definition 11** For $\alpha \in (0, 1)$ and $f : [0, T] \to X$, the Caputo fractional derivative of order $\alpha \in (0, 1)$ of $f(t)$ is defined by

$$cD^\alpha_{0+, t} f(t) := \frac{d}{dt} \left\{ J^{1-\alpha}_{t} \left[ f(t) - f(0) \right] \right\},$$

for every $t \in [0, T]$ such that (2) exists.

**Remark 3** We could have defined Caputo fractional derivative for any $\alpha > 0$, however this is not the focus of this work. It also worths to emphasize that $cD^1_0$ is equal to $(d/dt)$.

We end this section presenting the Mittag-Leffler operators which are related with the solution of the linear part of $(P_\alpha)$, when $\alpha \in (0, 1)$ (confront the following result with Theorem 3).
Theorem 4  Let $\alpha \in (0, 1)$ and suppose that $A : D(A) \subset X \to X$ is a sectorial operator. Then, the operators

$$E_\alpha(A^\alpha) := \frac{1}{2\pi i} \int_{H_A} e^{\lambda t} \lambda^\alpha \lambda^{-1}(\lambda^\alpha - A)^{-1} d\lambda, \quad t \geq 0,$$

and

$$E_{\alpha,\alpha}(A^\alpha) := \frac{t^{1-\alpha}}{2\pi i} \int_{H_A} e^{\lambda t} (\lambda^\alpha - A)^{-1} d\lambda, \quad t \geq 0,$$

where $H_A \subset \rho(A)$, are well defined. Furthermore, it holds that:

(i) $E_\alpha(A^\alpha)x|_{t=0} = E_{\alpha,\alpha}(A^\alpha)x|_{t=0} = x$, for any $x \in X$;

(ii) $E_\alpha(A^\alpha)$ and $E_{\alpha,\alpha}(A^\alpha)$ are strongly continuous in $0$, i.e., for each $x \in X$

$$\lim_{t \to 0^+} \|E_\alpha(A^\alpha)x - x\| = 0 \quad \text{and} \quad \lim_{t \to 0^+} \|E_{\alpha,\alpha}(A^\alpha)x - x\| = 0;$$

(iii) There exists a constant $M_2 > 0$ (uniform on $\alpha$) such that

$$\sup_{t \geq 0} \|E_\alpha(A^\alpha)\|_{\mathcal{L}(X)} \leq M_2 \quad \text{and} \quad \sup_{t \geq 0} \|E_{\alpha,\alpha}(A^\alpha)\|_{\mathcal{L}(X)} \leq M_2;$$

(iv) For each $x \in X$, the function $[0, \infty) \ni t \mapsto E_\alpha(A^\alpha)x$ is analytic and it is the unique solution of

$$cD_1^{\alpha} E_\alpha(A^\alpha)x = AE_\alpha(A^\alpha)x, \quad t > 0.$$

Proof  See Theorem 2.1 in [2] and Theorem 2.44 in [4].  

Remark 4  For a more complete survey on the special functions named Mittag-Leffler, we suggest the very interesting book [7].

3 The limit problems

Like it was pointed out in Section 1, here we focus our attention in the Limit Problems. At first, we emphasize that the study proposed here was already discussed in the literature, however with other objectives. We can cite as examples two papers, namely [1, 12].

To be more specific, Almeida and Ferreira in [1] studied this kind of limit questions involving the solutions of the semilinear fractional partial differential equation,

$$\begin{cases}
    cD_1^{1+\alpha} u(t, x) = \Delta_x u(t, x) + |u(t, x)|^\rho u(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}^n,
    \\
    u(0, x) = u_0 \quad \text{and} \quad \partial_t u(0, x) = 0,
\end{cases}$$

for $\alpha \in (0, 1)$, which interpolates the semilinear heat and wave equations. They studied the existence, uniqueness and regularity of solution to this problem in Morrey
spaces and then showed that when \( \alpha \to 1^- \), the solution of the problem loses its native regularity. On the other hand, in [12] the authors discuss these kind of limits to \( \alpha \)-times resolvent families, which are not our main objective here.

### 3.1 Questions (Q1) and (Q2)

In order to give a complete answer, let us begin by proving the following limit theorems.

**Theorem 5** Let \( A : D(A) \subset X \to X \) be a sectorial operator and \( \{ e^{At} : t \geq 0 \} \) the \( C_0 \)-semigroup generated by \( A \). Now assume that \( \alpha \in (0, 1) \) and consider the Mittag-Leffler family of operators \( \{ E_\alpha(At^\alpha) : t \geq 0 \} \). Then, for each \( t \geq 0 \) we have that

\[
\lim_{\alpha \to 1^-} \| E_\alpha(At^\alpha) - e^{At} \|_{\mathcal{L}(X)} = 0.
\]

Moreover, the convergence is uniform on every compact \( S \subset (0, \infty) \), i.e.,

\[
\lim_{\alpha \to 1^-} \left\{ \max_{s \in S} \| E_\alpha(As^\alpha) - e^{As} \|_{\mathcal{L}(X)} \right\} = 0.
\]

**Proof.** Let us assume that \( x \in X \) and \( t > 0 \). Consider \( \theta \in (\pi/2, \pi) \), \( S_\theta \) the sector associated with the sectorial operator \( A \) and \( M_\theta > 0 \) such that

\[
\| (\lambda - A)^{-1} \|_{\mathcal{L}(X)} \leq \frac{M_\theta}{|\lambda|}, \quad \forall \lambda \in S_\theta.
\]

Choose \( \epsilon > 1 \) and consider Hankel’s path \( H_a = H_{a_1} + H_{a_2} - H_{a_3} \), where \( H_{a_1} := \{ te^{i\theta} : t \in [\epsilon, \infty) \} \), \( H_{a_2} := \{ e^{it} : t \in [-\theta, \theta] \} \), \( H_{a_3} := \{ te^{-i\theta} : t \in [\epsilon, \infty) \} \).

The choices made above ensure that \( H_a \subset \rho(A) \), therefore observe by Theorem 3 and Theorem 4 that

\[
E_\alpha(At^\alpha)x - e^{At}x = \frac{1}{2\pi i} \int_{H_a} e^{\lambda t} \left[ \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x - \lambda^{\alpha-1}(\lambda - A)^{-1}x \right] d\lambda
\]

\[
+ \frac{1}{2\pi i} \int_{H_a} e^{\lambda t} \left[ \lambda^{\alpha-1}(\lambda - A)^{-1}x - (\lambda - A)^{-1}x \right] d\lambda,
\]

which is equivalent to

\[
E_\alpha(At^\alpha)x - e^{At}x = \sum_{j=1}^{3} \left\{ \frac{1}{2\pi i} \int_{H_{a_j}} e^{\lambda t} \left[ \lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x - \lambda^{\alpha-1}(\lambda - A)^{-1}x \right] d\lambda
\]

\[
+ \frac{1}{2\pi i} \int_{H_{a_j}} e^{\lambda t} \left[ \lambda^{\alpha-1}(\lambda - A)^{-1}x - (\lambda - A)^{-1}x \right] d\lambda \right\}.
\]
Using the first resolvent identity (see [6, Lemma 6]) in the first term in the right side of the above equality, we obtain the identity

\[
E_\alpha(At^\alpha)x - e^{At}x
= \frac{1}{2\pi i} \left\{ \sum_{j=1}^{3} \left\{ \int_{H_{\alpha j}} e^{\lambda t} \left[ \lambda^{\alpha - 1} (\lambda - \lambda^\alpha) A^{-1} (\lambda - A)^{-1} x \right] d\lambda \right. \right.
\]
\[
+ \left. \left. \int_{H_{\alpha j}} e^{\lambda t} [\lambda^{\alpha - 1} - 1] (\lambda - A)^{-1} x d\lambda \right\} \right\}.
\]

Now assume that \( S \subset (0, \infty) \) is a compact set and define the real values

\[
M_1 = \min\{ s : s \in S \} \quad \text{and} \quad M_2 = \max\{ \epsilon \cos(s) t : s \in [-\theta, \theta] \text{ and } t \in S \}.
\]

If \( t \in S \), we have that:

(i) Over \( H_{\alpha 1} \)

\[
\int_{H_{\alpha 1}} e^{\lambda t} \left\{ \lambda^{\alpha - 1} [\lambda - \lambda^\alpha] (\lambda^\alpha - A)^{-1} (\lambda - A)^{-1} x \right\} d\lambda
\]
\[
= \int_{\epsilon}^{\infty} e^{(\tau e^{i\theta}) t} \left\{ (\tau e^{i\theta})^{\alpha - 1} - (\tau e^{i\theta})^{\alpha} \right\}
\]
\[
\times \left( (\tau e^{i\theta})^{\alpha} - A \right)^{-1} (\tau e^{i\theta} - A)^{-1} x \right\} e^{i\theta} d\tau,
\]

and

\[
\int_{H_{\alpha 1}} e^{\lambda t} [\lambda^{\alpha - 1} - 1] (\lambda - A)^{-1} x d\lambda
\]
\[
= \int_{\epsilon}^{\infty} e^{(\tau e^{i\theta}) t} \left\{ (\tau e^{i\theta})^{\alpha - 1} - 1 \right\} (\tau e^{i\theta} - A)^{-1} x \} e^{i\theta} d\tau,
\]

from which, by considering the sectorial operator estimate (3), we deduce

\[
\left\| \int_{H_{\alpha 1}} e^{\lambda t} \left\{ \lambda^{\alpha - 1} [\lambda - \lambda^\alpha] (\lambda^\alpha - A)^{-1} (\lambda - A)^{-1} x \right\} d\lambda \right\|
\]
\[
\leq (M_{\theta})^2 \| x \| \left[ \int_{\epsilon}^{\infty} e^{\tau \cos(\theta)|M_1|} \frac{\left| (\tau e^{i\theta})^{\alpha} - (\tau e^{i\theta})^{\alpha} \right|}{\tau^2} d\tau \right] =: \mathcal{J}_1(\alpha) \| x \|.
\]
for every $t \in S$, and
\[
\left\| \int_{H_{a_1}} e^{\lambda t} [\lambda^{\alpha-1} - 1](\lambda - A)^{-1} x \ d\lambda \right\| \leq M_{\theta} \|x\| \left[ \int_{e}^{\tau \cos(\theta)} \frac{M_{1} \alpha^{-1} - 1}{\tau} \ d\tau \right] =: \mathcal{J}_2(\alpha) \|x\|
\]
for every $t \in S$.

(ii) Over $H_{a_2}$
\[
\int_{H_{a_2}} e^{\lambda t} \left\{ [\lambda^{\alpha-1} - 1](\lambda - A)^{-1} \right\} d\lambda \\
= \int_{-\theta}^{\theta} e^{(ee^{i\tau})t} \left\{ (ee^{i\tau})^{\alpha-1} - (ee^{i\tau})^{\alpha} \right\} \left( (ee^{i\tau} - A)^{-1} \right) x \ d\tau,
\]
and
\[
\int_{H_{a_2}} e^{\lambda t} [\lambda^{\alpha-1} - 1](\lambda - A)^{-1} x \ d\lambda \\
= \int_{-\theta}^{\theta} e^{(ee^{i\tau})t} \left\{ (ee^{i\tau})^{\alpha-1} - (ee^{i\tau})^{\alpha} \right\} \left( (ee^{i\tau} - A)^{-1} \right) x \ d\tau,
\]
by considering the sectorial operator estimate (3), we deduce
\[
\left\| \int_{H_{a_2}} e^{\lambda t} \left\{ [\lambda^{\alpha-1} - 1](\lambda - A)^{-1} \right\} d\lambda \right\| \leq \frac{(M_{\theta})^2 \|x\|}{\epsilon^2} \left[ \int_{-\theta}^{\theta} e^{(ee^{i\tau})t} \left\{ (ee^{i\tau})^{\alpha-1} - (ee^{i\tau})^{\alpha} \right\} d\tau \right] \leq \frac{(M_{\theta})^2 \|x\|}{\epsilon^2} \left[ \int_{-\theta}^{\theta} e^{M_2} \left( (ee^{i\tau})^{\alpha-1} - (ee^{i\tau})^{\alpha} \right) d\tau \right] =: \mathcal{J}_3(\alpha) \|x\|
\]
for every $t \in S$, and
\[
\left\| \int_{H_{a_2}} e^{\lambda t} [\lambda^{\alpha-1} - 1](\lambda - A)^{-1} x \ d\lambda \right\| \leq \frac{M_{\theta} \|x\|}{\epsilon} \left[ \int_{-\theta}^{\theta} e^{M_2} \left( (ee^{i\tau})^{\alpha-1} - 1 \right) d\tau \right] =: \mathcal{J}_4(\alpha) \|x\|
\]
for every $t \in S$.  

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(iii) Over $H\alpha_3$ it is analogous to item (i), therefore
\[
\left\| \int_{H\alpha_1} e^{\lambda t} \left\{ \lambda^{-\alpha} - 1 \right\} \left( \lambda^{-\alpha} - A \right)^{-1} \left( \lambda - A \right)^{-1} x \right\| d\lambda \leq (M_0)^2 \|x\| \left[ \int_{\epsilon}^{\infty} \frac{e^{\tau \cos(\theta)}}{\tau^2} \left| \left( \tau e^{-i\theta} \right)^{\alpha-1} - 1 \right| d\tau \right] =: J_5(\alpha) \|x\|,
\]
and
\[
\left\| \int_{H\alpha_1} e^{\lambda t} \left( \lambda^{-\alpha} - 1 \right) \left( \lambda - A \right)^{-1} x \right\| d\lambda \leq M_0 \|x\| \left[ \int_{\epsilon}^{\infty} \frac{e^{\tau \cos(\theta)}}{\tau^2} \left| \left( \tau e^{-i\theta} \right)^{\alpha-1} - 1 \right| d\tau \right] =: J_6(\alpha) \|x\|.
\]
Hence, we can estimate (4) by
\[
\|E_\alpha(At^\alpha)-e^{At}\|_{L^2(X)} \leq \sum_{i=1}^{6} J_i(\alpha),
\]
for every $t \in S$.

A consequence of the dominated convergence theorem ensures that
\[
\lim_{\alpha \to 1^-} J_i(\alpha) = 0,
\]
for every $i \in \{1, \ldots, 6\}$, therefore we deduce that
\[
\lim_{\alpha \to 1^-} \max_{s \in S} \left[ \|E_\alpha(As^\alpha) - e^{As}\|_{L^2(X)} \right] = 0.
\]
This is the second part of the proof. The first part, for the case $t > 0$, follows from this second part, while the first part is trivial when $t = 0$. \hfill \Box

**Theorem 6** Consider $A : D(A) \subset X \to X$ a sectorial operator; $\{e^{At} : t \geq 0\}$ the $C_0$-semigroup generated by $A$ and $\{E_\alpha(At^\alpha) : t \geq 0\}$ the Mittag-Leffler family of operators. Now assume that:

(i) $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1]$ and $\alpha \in (0, 1]$ is such that
\[
\lim_{n \to \infty} \alpha_n = \alpha;
\]

(ii) $\{\sigma_n\}_{n=1}^{\infty} \subset [0, \infty)$ such that
\[
\lim_{n \to \infty} \sigma_n = 0.$
Then, it holds that

$$\lim_{n \to \infty} \| E_{\alpha_n} (A(\sigma_n)^{\alpha_n}) - E_{\alpha} (A(\sigma)^{\alpha}) \|_{\mathcal{L}(X)} = 0.$$ 

**Proof** Following the same process as presented in the proof of Theorem 5, we deduce that

$$E_{\alpha_n} (A(\sigma_n)^{\alpha_n}) x - E_{\alpha} (A(\sigma)^{\alpha}) x = \frac{1}{2\pi i} \left\{ \sum_{j=1}^{3} \int_{H_{\lambda_j}} e^{\lambda \sigma_n} \left[ \lambda \alpha_n^{-1} (\lambda \alpha_n - A) (\lambda - A)^{-1} x \right] d\lambda + \int_{H_{\lambda_j}} e^{\lambda \sigma_n} \left[ \lambda \alpha_n^{-1} - \lambda^{-1} (\lambda - A)^{-1} x \right] d\lambda \right\}.$$ 

Last identity allows us to obtain the inequality

$$\| E_{\alpha} (-At^{\alpha}) - e^{At} \|_{\mathcal{L}(X)} \leq \sum_{i=1}^{6} \mathcal{J}_{i}(\alpha, \alpha_n),$$

for every $t \in S$, where

$$\mathcal{J}_{1}(\alpha, \alpha_n) = (M_{\theta})^2 \left[ \int_{\epsilon}^{\infty} e^{[\tau \cos(\theta)] \sigma_n} \left( (\tau e^{i\theta})^{\alpha_n} - (\tau e^{i\theta})^{\alpha_n} \right) \frac{d\tau}{\tau^2} \right],$$

$$\mathcal{J}_{2}(\alpha, \alpha_n) = M_{\theta} \left[ \int_{\epsilon}^{\infty} e^{[\tau \cos(\theta)] \sigma_n} \left( (\tau e^{i\theta})^{\alpha_n-1} - (\tau e^{i\theta})^{\alpha_n-1} \right) \frac{d\tau}{\tau^2} \right],$$

$$\mathcal{J}_{3}(\alpha, \alpha_n) = \epsilon \left[ \int_{-\theta}^{\theta} e^{M_2 \tau} \left( (\epsilon e^{i\tau})^{\alpha_n} - (\epsilon e^{i\tau})^{\alpha_n} \right) \frac{d\tau}{\tau^2} \right],$$

$$\mathcal{J}_{4}(\alpha, \alpha_n) = \epsilon \left[ \int_{-\theta}^{\theta} e^{M_2 \tau} \left( (\epsilon e^{i\tau})^{\alpha_n-1} - (\epsilon e^{i\tau})^{\alpha_n-1} \right) \frac{d\tau}{\tau^2} \right],$$

$$\mathcal{J}_{5}(\alpha, \alpha_n) = (M_{\theta})^2 \left[ \int_{\epsilon}^{\infty} e^{[\tau \cos(\theta)] \sigma_n} \left( (\tau e^{-i\theta})^{\alpha} - (\tau e^{-i\theta})^{\alpha_n} \right) \frac{d\tau}{\tau^2} \right],$$

$$\mathcal{J}_{6}(\alpha, \alpha_n) = M_{\theta} \left[ \int_{\epsilon}^{\infty} e^{[\tau \cos(\theta)] \sigma_n} \left( (\tau e^{-i\theta})^{\alpha_n-1} - (\tau e^{-i\theta})^{\alpha_n-1} \right) \frac{d\tau}{\tau^2} \right].$$

Again, the dominated convergence theorem ensures that

$$\lim_{n \to \infty} \mathcal{J}_{i}(\alpha, \alpha_n) = 0,$$
for every $i \in \{1, \ldots, 6\}$, therefore we obtain
\[
\lim_{n \to \infty} \| E_{\alpha_n}(A(\sigma_n)^\alpha_n) - E_{\alpha}(A(\sigma_n)^\alpha) \|_{L(X)} = 0.
\]

Theorem 7 Let $A : D(A) \subset X \to X$ be a sectorial operator and \{e^{At} : t \geq 0\} the $C_0$-semigroup generated by $A$. Now assume that $\alpha \in (0, 1)$ and consider the Mittag-Leffler family of operators \{E_{\alpha,\alpha}(At^\alpha) : t \geq 0\}. Then, for each $t \geq 0$ we have that
\[
\lim_{\alpha \to 1^-} \| E_{\alpha,\alpha}(At^\alpha) - e^{At} \|_{L(X)} = 0.
\]
Moreover, the convergence is uniform on every compact $S \subset (0, \infty)$, i.e.,
\[
\lim_{\alpha \to 1^-} \max_{s \in S} \| E_{\alpha,\alpha}(As^\alpha) - e^{As} \|_{L(X)} = 0.
\]

Proof Like we did in the proof of Theorem 5, let us assume that $x \in X$ and $t > 0$. Consider $\theta \in (\pi/2, \pi)$, $S_\theta$ the sector associated with the sectorial operator $A$ and $M_\theta > 0$ such that
\[
\| (\lambda - A)^{-1} \|_{L(X)} \leq \frac{M_\theta}{|\lambda|}, \quad \forall \lambda \in S_\theta.
\]
(5)
Choose $\epsilon > 1$ and consider Hankel’s path $Ha = Ha_1 + Ha_2 - Ha_3$, where
\[
Ha_1 := \{ te^{i\theta} : t \in [\epsilon, \infty) \}, \quad Ha_2 := \{ \epsilon e^{it} : t \in [-\theta, \theta] \},
\]
\[
Ha_3 := \{ te^{-i\theta} : t \in [\epsilon, \infty) \}.
\]

The choices made above ensures that $Ha \subset \rho(A)$, therefore observe by Theorem 3 and Theorem 4 that
\[
t^{\alpha - 1} E_{\alpha,\alpha}(At^\alpha)x - e^{At}x = \frac{1}{2\pi i} \int_{Ha_i} e^{\lambda t} \left[ (\lambda^\alpha - A)^{-1}x - (\lambda - A)^{-1}x \right] d\lambda,
\]
what, thanks to the first resolvent identity (see [6, Lemma 6]), is equivalent to
\[
t^{\alpha - 1} E_{\alpha,\alpha}(At^\alpha)x - e^{At}x
\]
\[
= \frac{1}{2\pi i} \sum_{j=1}^{3} \left\{ \int_{Ha_i} e^{\lambda t} [\lambda - \lambda^\alpha] (\lambda^\alpha - A)^{-1}(\lambda - A)^{-1}x d\lambda \right\}.
\]

Now consider $S \subset (0, \infty)$ a compact set and
\[
M_1 = \min\{s : s \in S\} \quad \text{and} \quad M_2 = \max\{\epsilon \cos(s)t : s \in [-\theta, \theta] \text{ and } t \in S\}.
\]

If $t \in S$, we have that:
(i) Over $Ha_1$

$$\int_{Ha_1} e^{\lambda t} \left\{ (\lambda - \lambda^\alpha)(\lambda^\alpha - A)^{-1}(\lambda - A)^{-1}x \right\} d\lambda$$

$$= \int e^{(\tau e^{i\theta})t} \left\{ \left( \tau e^{i\theta} \right)^\alpha - \left( \tau e^{i\theta} \right)^\alpha \right\} \left( \left( \tau e^{i\theta} \right)^\alpha - A \right)^{-1} \left( \left( \tau e^{i\theta} \right)^\alpha - A \right)^{-1}x \right\} e^{i\theta} d\tau,$$

from which, by considering the sectorial operator estimate (5), we deduce

$$\left\| \int_{Ha_1} e^{\lambda t} \left\{ (\lambda - \lambda^\alpha)(\lambda^\alpha - A)^{-1}(\lambda - A)^{-1}x \right\} d\lambda \right\| \leq (M_\theta)^2 \|x\| \left[ \int e^{\tau \cos(\theta)M_1} \left\| \left( \tau e^{i\theta} \right)^\alpha - \left( \tau e^{i\theta} \right)^\alpha \right\| d\tau \right] =: J_1(\alpha) \|x\|,$$

for every $t \in S$.

(ii) Over $Ha_2$

$$\int_{Ha_2} e^{\lambda t} \left\{ (\lambda - \lambda^\alpha)(\lambda^\alpha - A)^{-1}(\lambda - A)^{-1}x \right\} d\lambda$$

$$= \int_{-\theta}^{\theta} e^{(\epsilon e^{i\tau})t} \left\{ (\epsilon e^{i\tau})^\alpha - (\epsilon e^{i\tau})^\alpha \right\} \left( \left( \epsilon e^{i\tau} \right)^\alpha - A \right)^{-1} \left( \left( \epsilon e^{i\tau} \right)^\alpha - A \right)^{-1}x \right\} i \epsilon e^{i\tau} d\tau,$$

and therefore, if we consider the sectorial operator estimate (5), we obtain

$$\left\| \int_{Ha_2} e^{\lambda t} \left\{ (\lambda - \lambda^\alpha)(\lambda^\alpha - A)^{-1}(\lambda - A)^{-1}x \right\} d\lambda \right\| \leq (M_\theta)^2 \epsilon^{-\alpha} \|x\| \left[ \int_{-\theta}^{\theta} \left\| e^{M_2} \right\| \left( (\epsilon e^{i\tau})^\alpha - (\epsilon e^{i\tau})^\alpha \right\| d\tau \right] =: J_2(\alpha) \|x\|,$$

for every $t \in S$.

(iii) Over $Ha_3$ it is analogous to item (i), therefore

$$\left\| \int_{Ha_1} e^{\lambda t} \left\{ (\lambda - \lambda^\alpha)(\lambda^\alpha - A)^{-1}(\lambda - A)^{-1}x \right\} d\lambda \right\|$$

$$\leq (M_\theta)^2 \|x\| \left[ \int_{\epsilon}^{\infty} e^{\tau \cos(\theta)M_1} \left\| \left( \tau e^{i\theta} \right)^\alpha - \left( \tau e^{i\theta} \right)^\alpha \right\| d\tau \right] =: J_3(\alpha) \|x\|.$$

Thus, we achieve the estimate

$$\|t^{\alpha-1} E_\alpha(At^\alpha) - e^{At}\|_{\mathcal{L}(X)} \leq J_1(\alpha) + J_2(\alpha) + J_3(\alpha),$$

for every $t \in S$. 
A consequence of the dominated convergence theorem ensures
\[
\lim_{\alpha \to 1^-} \mathcal{J}_1(\alpha) = 0, \quad \lim_{\alpha \to 1^-} \mathcal{J}_2(\alpha) = 0, \quad \lim_{\alpha \to 1^-} \mathcal{J}_3(\alpha) = 0.
\]
Now, since
\[
\| E_{\alpha,\alpha}(\alpha) - e^{\alpha} \|_{L^p(X)} \leq |1 - t^{\alpha-1}| \| E_{\alpha,\alpha}(\alpha) \|_{L^p(X)} + |t^{\alpha-1} E_{\alpha,\alpha}(\alpha) - e^{\alpha} \|_{L^p(X)},
\]
for every \( t \in S \), we deduce that
\[
\lim_{\alpha \to 1^-} \left\{ \max_{s \in S} \left[ \| E_{\alpha,\alpha}(\alpha) - e^{\alpha} \|_{L^p(X)} \right] \right\} = 0.
\]
This is the second part of the proof. The first part, for the case \( t > 0 \), follows from this second part, while the first part is trivial when \( t = 0 \). \( \square \)

A natural question that arises at this point of this paper is: Is it possible to obtain other modes of convergence to these operators? To answer this question, we begin by presenting the following notion.

**Definition 12** Let \( S \subset \mathbb{R} \) and \( p \geq 1 \). The symbol \( L^p(S; L^p(X)) \) is used to represent the set of all Bochner measurable functions \( f : S \to L^p(X) \) for which \( \| f \|_{L^p(X)} \in L^p(S; \mathbb{R}) \), where \( L^p(S; \mathbb{R}) \) stands for the classical Lebesgue space. Moreover, \( L^p(S; L^p(X)) \) is a Banach space when considered with the norm
\[
\| f \|_{L^p(S; L^p(X))} := \left[ \int_S \| f(s) \|_{L^p(X)}^p ds \right]^{1/p}.
\]
The vectorial spaces \( L^p(S; L^p(X)) \) are called Bochner–Lebesgue spaces.

For more details on the Bochner measurable functions and Bochner–Lebesgue integrable functions, we may refer to [3, 13] and references therein.

Now we present the last theorem of this subsection.

**Theorem 8** Let \( A : D(A) \subset X \to X \) be a sectorial operator and \( \{ e^{At} : t \geq 0 \} \) the \( C_0 \)-semigroup generated by \( A \). Now assume that \( \alpha \in (0, 1) \) and consider the Mittag-Leffler families \( \{ E_\alpha(At^\alpha) : t \geq 0 \} \) and \( \{ E_{\alpha,\alpha}(At^\alpha) : t \geq 0 \} \). If \( p \geq 1 \) and \( S \subset \mathbb{R}^+ \) is compact, then
\[
e^{At}, \quad E_\alpha(At^\alpha) \quad \text{and} \quad E_{\alpha,\alpha}(At^\alpha) \in L^p(S; L^p(X)).
\]
Moreover, it holds that
\[
\lim_{\alpha \to 1^-} \left[ \int_S \| E_\alpha(At^\alpha) - e^{At} \|_{L^p(X)}^p ds \right]^{1/p} = 0,
\]
\[
\lim_{\alpha \to 1^-} \left[ \int_S \| E_{\alpha,\alpha}(At^\alpha) - e^{At} \|_{L^p(X)}^p ds \right]^{1/p} = 0.
\]
Proof Since Theorems 3 and 4 ensure that

\[ \sup_{t \geq 0} \| e^{At} \|_{L(X)} \leq M_1, \quad \sup_{t \geq 0} \| E_\alpha(At^\alpha) \|_{L(X)} \leq M_2, \quad \sup_{t \geq 0} \| E_{\alpha,\alpha}(At^\alpha) \|_{L(X)} \leq M_2, \]

we trivially deduce that \( e^{At}, E_\alpha(At^\alpha), E_{\alpha,\alpha}(At^\alpha) \in L^p(S; L(X)). \)

Now, let us prove the convergence. Like it was done in the proof of Theorem 5, we deduce the existence of \( \tilde{M} > 0 \) such that

\[
\| E_\alpha(At^\alpha)x - e^{At}x \| \leq \tilde{M} \| x \| \left[ \int_{\epsilon}^{\infty} e^{[r \cos(\theta)]t} \left| \frac{(\tau e^{i\theta})^{\alpha} - (\tau e^{i\theta})^{\alpha}}{\tau^2} \right| d\tau \right. \\
+ \int_{-\theta}^{\theta} e^{M_2} \left| (\epsilon e^{i\tau})^{\alpha-1} - 1 \right| d\tau \left. + \int_{\epsilon}^{\infty} e^{[r \cos(\theta)]t} \left| \left( \frac{e^{-i\theta}}{\tau^2} - \frac{e^{-i\theta}}{\tau^2} \right)^{\alpha} \right| d\tau \right],
\]

for every \( t \in S. \)

At first, observe that the triangular inequality of \( L^p(S; X) \) ensures that

\[
\left( \int_S \| E_\alpha(At^\alpha) - e^{At} \|^p_{L(X)} dt \right)^{1/p} \leq \tilde{M} \left[ \mathcal{J}_\alpha(t) + \mathcal{J}_\alpha + \mathcal{K}_\alpha(t) \right],
\]

where

\[
\mathcal{J}_\alpha(t) = \left( \int_S \left[ \int_{\epsilon}^{\infty} e^{[r \cos(\theta)]t} \left| \frac{(\tau e^{i\theta})^{\alpha} - (\tau e^{i\theta})^{\alpha}}{\tau^2} \right| d\tau \right] dt \right)^{1/p},
\]

\[
\mathcal{J}_\alpha = |S|^{1/p} e^{M_2} \left[ \int_{-\theta}^{\theta} \left| (\epsilon e^{i\tau})^{\alpha-1} - 1 \right| d\tau \right],
\]

and

\[
\mathcal{K}_\alpha(t) = \left( \int_S \left[ \int_{\epsilon}^{\infty} e^{[r \cos(\theta)]t} \left| \left( \frac{e^{-i\theta}}{\tau^2} - \frac{e^{-i\theta}}{\tau^2} \right)^{\alpha} \right| d\tau \right] dt \right)^{1/p}.
\]

Like above, by applying Minkowski’s integral inequality (see Theorem 202 in [8]) we deduce that

\[
\mathcal{J}_\alpha(t) \leq \int_{\epsilon}^{\infty} \left[ \left( \int_{M_1} e^{[p \cos(\theta)]t} dt \right)^{1/p} \left| \frac{(\tau e^{i\theta})^{\alpha} - (\tau e^{i\theta})^{\alpha}}{\tau^2} \right| \right] d\tau.
\]
what implies in the estimate
\[ I_\alpha(t) \leq \int_\epsilon^\infty e^{[\tau \cos(\theta)]M_1^2} \left| \left( \tau e^{i\theta} \right) - \left( \tau e^{-i\theta} \right)^\alpha \right| \frac{1}{p \tau^{2p+1}} \left( - \cos(\theta) \right) \frac{1}{p} d\tau =: I_\alpha. \]

Similarly, we obtain
\[ K_\alpha(t) \leq \int_\epsilon^\infty e^{[\tau \cos(\theta)]M_1^2} \left| \left( \tau e^{-i\theta} \right)^\alpha - \left( \tau e^{-i\theta} \right)^\alpha \right| \frac{1}{p \tau^{2p+1}} \left( - \cos(\theta) \right) \frac{1}{p} d\tau =: K_\alpha. \]

Again, as a consequence of the dominated convergence theorem we deduce that
\[ \lim_{\alpha \to 1^-} I_\alpha = 0, \quad \lim_{\alpha \to 1^-} J_\alpha = 0 \quad \text{and} \quad \lim_{\alpha \to 1^-} K_\alpha = 0, \]

therefore
\[ \lim_{\alpha \to 1^-} \left( \int_S \| E_\alpha(At^\alpha) - e^{At^\alpha} \|_{L^2(X)}^p \right)^{1/p} = 0. \]

By analogous arguments we also conclude that
\[ \lim_{\alpha \to 1^-} \left( \int_S \| E_{\alpha,\alpha}(At^\alpha) - e^{At^\alpha} \|_{L^2(X)}^p \right)^{1/p} = 0. \]

\[ \square \]

**Remark 5** Observe that for \( S \subset (0, \infty) \) the convergence presented in the above result becomes a consequence of Theorems 5 and 7. However, since we are considering \( S \subset \mathbb{R}^+ \), the above result is in fact more general.

### 3.2 Questions (Q3) and (Q4)

In this subsection we begin to deal with the nonlinearity \( f(t, x) \), introducing new computations to the results obtained so far. Recall that if \( f(t, x) \) is a continuous function, locally Lipschitz in the second variable, uniformly with respect to the first variable, and bounded (i.e. it maps bounded sets onto bounded sets), Theorem 1 and Remark 1 ensure that for any \( \alpha \in (0, 1) \), there exist \( w_\alpha > 0 \) (which can be \( \infty \)) and a unique maximal local mild solution (or global mild solution) \( \phi_\alpha : [0, \omega_\alpha) \to X \) of the Cauchy problem \((P_\alpha)\).

However there is no simple way to know if
\[ \bigcap_{\alpha \in (0, 1)} [0, \omega_\alpha) = \{0\}, \tag{6} \]
which, if this is the case, would lead us to face the possibility of calculate the fractional limit of the family of functions \( \{ \phi_\alpha(t) : \alpha \in (0, 1) \} \), when \( \alpha \to 1^- \), just to the constant value \( \phi_\alpha(0) = u_0 \). This is exactly what question (Q3) intends to discuss.

Let us begin by presenting the following result.

**Theorem 9** Assume that \( f : [0, \infty) \times X \to X \) is a continuous function, locally Lipschitz in the second variable, uniformly with respect to the first variable, and bounded (i.e., it maps bounded sets onto bounded sets), and consider problem \((P_\alpha)\), for \( \alpha \in (0, 1) \). If \( \phi_\alpha(t) \) is the maximal local mild solution of \((P_\alpha)\) defined over the interval \([0, \omega_\alpha)\) (which can be \((0, \infty)\) for some values of \( \alpha \)), then for each \( \alpha_0 \in (0, 1) \), there exists \( t_{\alpha_0} > 0 \) such that

\[
[0, t_{\alpha_0}] \subset \bigcap_{\alpha \in [\alpha_0, 1]} [0, \omega_\alpha).
\]

**Proof** Observe that for each \( \alpha \in (0, 1) \), Theorem 1 and Remark 1 ensure the existence and uniqueness of a maximal local mild solution (or global mild solution) of \((P_\alpha)\), such that the maximal interval of existence for the solution \( \phi_\alpha(t) \) is \([0, \infty)\) or a finite interval \([0, \omega_\alpha)\). If \( f(t, x) \) is locally Lipschitz in the second variable, there exist \( R_0, L_0 > 0 \) such that

\[
\|f(t, x) - f(t, y)\| \leq L_0\|x - y\|
\]

for every \((t, x), (t, y) \in B((0, u_0), R_0)\).

Let \( \alpha_0 \in (0, 1) \) and suppose that

\[
\bigcap_{\alpha \in [\alpha_0, 1]} [0, \omega_\alpha) = \{0\}.
\]

Then, there should exist a sequence \( \{\alpha_n\}_{n=1}^\infty \subset [\alpha_0, 1) \) and a related decreasing sequence of positive real numbers \( \{\omega_{\alpha_n}\}_{n=1}^\infty \), such that \( \lim_{n \to \infty} \omega_{\alpha_n} = 0 \). By choosing a subsequence of \( \{\omega_{\alpha_n}\}_{n=1}^\infty \), if necessary, we can suppose that there exists \( \alpha \in [\alpha_0, 1) \) such that \( \alpha_n \) converges to \( \alpha \) and a sequence \( \{\sigma_n\}_{n=1}^\infty \) with the following properties:

(i) \( 0 < \omega_{\alpha_n+1} < \sigma_n < \omega_{\alpha_n} \), for every \( n \in \mathbb{N} \);

(ii) \( \|\phi_{\alpha_n}(t) - u_0\| < R_0/2 \), for \( t \in [0, \sigma_n) \), and \( \|\phi_{\alpha_n}(\sigma_n) - u_0\| = R_0/2 \), for every \( n \in \mathbb{N} \). This assertion follows mainly from (7).

Now observe that

\[
\|E_{\alpha_n}(A(\sigma_n)^{\alpha_n})u_0 - u_0\| \leq \|E_{\alpha_n}(A(\sigma_n)^{\alpha_n})u_0 - E_{\alpha}(A(\sigma_n)^{\alpha})u_0\| + \|E_{\alpha}(A(\sigma_n)^{\alpha})u_0 - u_0\|,
\]
therefore, Theorems 4 and 6 ensure that
\[
\lim_{n \to \infty} \| E_{\alpha_n} (A(\sigma_n)^{\alpha_n}) u_0 - u_0 \| = 0.
\]

Thus, choose \( N_0 \in \mathbb{N} \) such that
\[
\sigma_n < R_0/2 \quad \text{and} \quad \| E_{\alpha_n} (A(\sigma_n)^{\alpha_n}) u_0 - u_0 \| \leq R_0/4,
\]
for any \( n \geq N_0 \). Then, if \( n \geq N_0 \) we have
\[
\| \phi_{\alpha_n}(\sigma_n) - u_0 \| \leq \| E_{\alpha_n} (A(\sigma_n)^{\alpha_n}) u_0 - u_0 \|
+ \left\| \int_0^{\sigma_n} (\sigma_n - s)^{\alpha_n-1} E_{\alpha_n,\alpha_n} (A(\sigma_n - s)^{\alpha_n}) f(s, \phi_{\alpha_n}(s)) \, ds \right\|
+ \left\| \int_0^{\sigma_n} (\sigma_n - s)^{\alpha_n-1} \left[ f(s, \phi_{\alpha_n}(s)) - f(s, u_0) \right] \, ds \right\|
\]
and therefore
\[
\| \phi_{\alpha_n}(\sigma_n) - u_0 \| \leq (R_0/4)
+ M_2 \int_0^{\sigma_n} (\sigma_n - s)^{\alpha_n-1} \left[ f(s, \phi_{\alpha_n}(s)) - f(s, u_0) \right] \, ds
+ \left\| \int_0^{\sigma_n} (\sigma_n - s)^{\alpha_n-1} f(s, u_0) \, ds \right\|
\]
where \( M_2 > 0 \) is given uniformly on \( \alpha \in [\alpha_0, 1] \) by Theorem 4.

Now, since \( (\sigma_n, \phi_{\alpha_n}(\sigma_n)), (\sigma_n, u_0) \in B((0, u_0), R_0) \), we have that
\[
\| \phi_{\alpha_n}(\sigma_n) - u_0 \| \leq (R_0/4) + M_2 M_3 \sigma_n + M_2 L_0 \int_0^{\sigma_n} (\sigma_n - s)^{\alpha_n-1} \| \phi_{\alpha_n}(s) - u_0 \| \, ds,
\]
where \( M_3 = \max_{s \in [0, \omega_1]} \| f(s, 0) \| \).

Finally, from [9, Lemma 7.1.1] we have that
\[
\| \phi_{\alpha_n}(\sigma_n) - u_0 \| \leq \theta_n + \eta_n \int_0^{\sigma_n} (\sigma_n - s)^{\alpha_n-1} E_{\alpha_n,\alpha_n} (\eta_n(\sigma_n - s)) \theta_n \, ds,
\]
where \( \eta_n = [M_2 L_0 \Gamma(\alpha_n)]^{1/\alpha_n} \) and \( \theta_n = (R_0/4) + M_2 M_3 \sigma_n \). Thus, using the dominated convergence theorem and Theorem 4, we deduce that
\[
R_0/2 = \lim_{n \to \infty} \| \phi_{\alpha_n}(\sigma_n) - u_0 \|
\leq \lim_{n \to \infty} \theta_n + \left( \lim_{n \to \infty} \eta_n \right) \left\{ \lim_{n \to \infty} \int_0^{\sigma_n} (\sigma_n - s)^{\alpha_n-1} E_{\alpha_n,\alpha_n} (\eta_n(\sigma_n - s)) \theta_n \, ds \right\}
= (R_0/4),
\]
what is a contradiction. This concludes the proof of this theorem. \( \square \)
Let us give an interesting example of the behavior described by Theorem 9. We may stress that it is very difficult to obtain analytical solutions to fractional differential equations (partial or ordinary), therefore we rely on numerical solutions to give this example.

**Example 1** In [5, Example 3] the authors consider the following real fractional differential equation (with Caputo fractional derivative of order $\alpha \in (0, 1)$):

\[
\begin{cases}
c D^\alpha_t u(t) = 1 - u^4(t), & t \geq 0, \\
u(0) = 0 \in \mathbb{R}.
\end{cases}
\]

It is worth to emphasize that when $\alpha = 1/2$ this problem governs the radiation of heat from a semi-infinite solid having a constant heat source; see [15] for details.

The technique used in this example is the Adomian decomposition method (or simply ADM). The authors prove that the solution $\phi_\alpha(t)$ of (8) is given by

$$\phi_\alpha(t) = \sum_{j=0}^{\infty} a_j, t \in [0, \omega_{\alpha})$$

and estimate the radius of convergence of this series. The estimate to the radius of convergence, in our situation, give an estimate to the values $\omega_\alpha$ (cf. Theorems 1 and 9). For instance, when $\alpha = 1$ the radius of convergence is $\omega_1 = 1.55$ approximately, when $\alpha = 0.75$ the radius of convergence is $\omega_{0.75} = 1.08$ approximately, when $\alpha = 0.5$ the radius of convergence is $\omega_{0.5} = 0.59$ approximately and when $\alpha = 0.25$ the radius of convergence is $\omega_{0.25} = 0.13$ approximately.

The behavior of $\omega_\alpha$ elucidates the conclusions obtained by Theorem 9, i.e., that if we fix $\alpha_0 \in (0, 1)$ then $[0, \omega_\alpha) \subset \bigcap_{\alpha \in [\alpha_0, 1]} [0, \omega_{\alpha})$.

In order to give another characterization to the biggest interval contained in the set $\bigcap_{\alpha \in [\alpha_0, 1]} [0, \omega_{\alpha})$, for each $\alpha_0 \in (0, 1)$, we present the following result.

**Theorem 10** Consider same hypotheses of Theorem 9 and assume that for any $\alpha_0$ in $(0, 1)$, there exists $\alpha \in [\alpha_0, 1]$ such that $\omega_\alpha < \infty$. Define

$$\Omega_{\alpha_0} := \sup \{ t > 0 : [0, t] \subset \bigcap_{\alpha \in [\alpha_0, 1]} [0, \omega_{\alpha}) \}.$$ 

Then, we have that:

(i) for each $\alpha_0 \in (0, 1)$ it holds that $\inf \{ \omega_\alpha : \alpha \in [\alpha_0, 1] \text{ with } \omega_\alpha < \infty \} = \Omega_{\alpha_0}$;

(ii) for each $\alpha_0 \in (0, 1)$ we have that $[0, \Omega_{\alpha_0}) \subset [0, \omega_{\alpha})$, for every $\alpha \in [\alpha_0, 1]$;

(iii) if $\alpha_1, \alpha_2 \in (0, 1)$, with $\alpha_1 < \alpha_2$, then $\Omega_{\alpha_1} \leq \Omega_{\alpha_2}$.

**Proof** (i) Since by definition it holds that

$$\Omega_{\alpha_0} \leq \inf \{ \omega_\alpha : \alpha \in [\alpha_0, 1] \text{ with } \omega_\alpha < \infty \},$$

let us assume for an instant that $\Omega_{\alpha_0} < \inf \{ \omega_\alpha : \alpha \in [\alpha_0, 1] \text{ with } \omega_\alpha < \infty \}$. If such is the case, by setting

$$r := \left[ \Omega_{\alpha_0} + \inf \{ \omega_\alpha : \alpha \in [\alpha_0, 1] \text{ with } \omega_\alpha < \infty \} \right]/2,$$
we could verify that \( \Omega \alpha_0 + r \in \{ t > 0 : [0, t] \subset \cap_{\alpha \in [\alpha_0, 1]}[0, \omega_\alpha) \} \), what would lead us to the contradiction \( \Omega \alpha_0 + r \leq \Omega \alpha_0 \). Therefore

\[
\Omega \alpha_0 = \inf \{ \omega_\alpha : \alpha \in [\alpha_0, 1] \text{ with } \omega_\alpha < \infty \},
\]

as we wanted.

Finally item \((ii)\) follows from the definition of \( \Omega \alpha_0 \), while item \((iii)\) is a directly consequence of the infimum properties. \(\square\)

**Remark 6** It is worth to emphasize the importance of the hypotheses: “for any \( \alpha_0 \) in \((0, 1)\), there exists \( \alpha \in [\alpha_0, 1] \) such that \( \omega_\alpha < \infty \).” In fact, if this is not the case, there should exist \( \tilde{\alpha} \in (0, 1) \) such that for every \( \alpha \in [\tilde{\alpha}, 1] \) we would have

\[
[0, \omega_\alpha) = [0, \infty).
\]

Now, let us change the subject a little and begin to address question \((Q4)\).

**Theorem 11** Consider the Cauchy problem \((P_\alpha)\) with \( \alpha \in (0, 1) \) and assume that function \( f : [0, \infty) \times X \to X \) is continuous, locally Lipschitz in the second variable, uniformly with respect to the first variable, and bounded (i.e. it maps bounded sets onto bounded sets). Then, if \( \phi_\alpha(t) \) is the maximum local mild solution (or global mild solution) of \((P_\alpha)\) defined over \([0, \omega_\alpha)\) (or \([0, \infty)\)), there exists \( t^* > 0 \) such that

\[
\lim_{\alpha \to 1^-} \| \phi_\alpha(t) - \phi_1(t) \| = 0,
\]

for every \( t \in [0, t^*] \).

**Proof** Observe that Theorem 1 and Remark 1 guarantee the existence of a maximal mild solution (or global mild solution) \( \phi_\alpha(t) \) defined over a maximal interval of existence \([0, \omega_\alpha)\) (or \([0, \infty)\)), such that for \( \alpha \in (0, 1) \) satisfies

\[
\phi_\alpha(t) = \begin{cases} 
E_\alpha(At^\alpha)u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) f(s, \phi_\alpha(s)) ds, & \text{if } \alpha \in (0, 1), \\
e^{At}u_0 + \int_0^t e^{A(t-s)} f(s, \phi_\alpha(s)) ds, & \text{if } \alpha = 1.
\end{cases}
\]

Now we need to make two considerations.

(i) Note that Theorem 10 and Remark 6 ensure that

\[
[0, \Omega_{1/2}/2] \subset \cap_{\alpha \in [1/2, 1]}[0, \omega_\alpha) \quad \text{or} \quad [0, \infty) \subset \cap_{\alpha \in [1/2, 1]}[0, \omega_\alpha).
\]

In any case, it is not difficult to choose \( t_0 > 0 \) such that \([0, t_0] \subset [0, \omega_\alpha)\), for every \( \alpha \in [1/2, 1] \).
(ii) Recall that \( f(r, x) \) is locally Lipschitz in the second variable, i.e., there exist \( R_0, L_0 > 0 \) such that

\[
\| f(r, x) - f(r, y) \| \leq L_0 \| x - y \|
\]

for every \((r, x), (r, y) \in B((0, u_0), R_0)\). In that way, we can infer that there exists \( t_\alpha \in [0, t_0] \) such that

\[
\bigcup_{\alpha \in [1/2, 1]} \{ (t, \phi_\alpha(t)) : t \in [0, t_\alpha] \} \subset B((0, u_0), R_0/2),
\]

since otherwise, it would exist \( \alpha_\ast \in [1/2, 1] \) and two sequences of real numbers \( \{\alpha_n\}_{n=1}^\infty \subset [1/2, 1] \) and \( \{\tau_n\}_{n=1}^\infty \subset [0, t_0] \), with \( \alpha_n \to \alpha_\ast \) and \( \tau_n \to 0 \), when \( n \to \infty \), that also satisfies \( \|\phi_{\alpha_n}(t) - u_0\| < R_0/2 \), for all \( t \in [0, \tau_n] \), and \( \|\phi_{\alpha_\ast}(\tau_n) - u_0\| = R_0/2 \). However this construction, like the one given in the proof of Theorem 9, leads to a contradiction.

(iii) Define \( N := \sup\{|\| f(s, \phi_1(s)) \| : s \in [0, t_\alpha] \} \), which is finite since \( f(s, \phi_1(s)) \) is continuous and \([0, t_\alpha] \) is a compact interval.

Taking into account considerations (i), (ii) and (iii), we deduce that

\[
\| \phi_\alpha(t) - \phi_1(t) \| \leq \| E_\alpha(A t^\alpha) u_0 - e^{A t} u_0 \|
\]

\[
+ \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(A (t-s)^\alpha) \left[ f(s, \phi_\alpha(s)) - f(s, \phi_1(s)) \right] ds \right\|
\]

\[
+ \left\| \int_0^t (t-s)^{\alpha-1} \left[ E_{\alpha, \alpha}(A (t-s)^\alpha) - e^{A(t-s)} \right] f(s, \phi_1(s)) ds \right\|
\]

\[
+ \left\| \int_0^t [(t-s)^{\alpha-1} - 1] e^{A(t-s)} f(s, \phi_1(s)) ds \right\|
\]

for every \( \alpha \in [1/2, 1] \) and \( t \in [0, t_\alpha] \), therefore, by considering also Theorems 3 and 4, we obtain

\[
\| \phi_\alpha(t) - \phi_1(t) \| \leq \| E_\alpha(A t^\alpha) - e^{A t} \| \| \phi_1(t) \| \| u_0 \|
\]

\[
+ L_0 M_2 \int_0^t (t-s)^{\alpha-1} \| \phi_\alpha(s) - \phi_1(s) \| ds
\]

\[
+ N \int_0^t s^{\alpha-1} \left\| E_{\alpha, \alpha}(A s^\alpha) - e^{A s} \right\| ds
\]

\[
+ N M_1 \int_0^t [s^{\alpha-1} - 1] ds.
\]

for every \( \alpha \in [1/2, 1] \) and \( t \in [0, t_\alpha] \). Note that we may rewrite the above inequality as

\[
\| \phi_\alpha(t) - \phi_1(t) \| \leq g_\alpha(t) + L_0 M_2 \int_0^t (t-s)^{\alpha-1} \| \phi_\alpha(s) - \phi_1(s) \| ds, \tag{9}
\]
for every $\alpha \in [1/2, 1]$ and $t \in [0, t_*]$, where
\[
g_\alpha(t) = \| E_\alpha(At^\alpha) - e^{At} \|_{L(X)} \| u_0 \| + N \int_0^t s^{\alpha-1} \left\| E_{\alpha,\alpha}(As^\alpha) - e^{As} \right\|_{L(X)} ds \\
+N M_1 \int_0^t [s^{\alpha-1} - 1] ds.
\]

Finally, by considering [9, Lemma 7.1.1], we have that (9), the dominated convergence theorem, Theorems 5, 7 and 8 allow us to deduce that
\[
\lim_{\alpha \to 1^-} \| \phi_\alpha(t) - \phi_1(t) \| = 0,
\]
for each $t \in [0, t_*]$. This completes the proof of the theorem.

\section{4 Closing remarks}

All the computations done in this paper could have being more general. We could have considered $\alpha_0 \in (0, 1]$ and studied the limit
\[
\lim_{\alpha \to \alpha_0} u_\alpha(t) = u_{\alpha_0}(t),
\]
or, we could have considered the Cauchy problem with initial condition $u(t_0) = u_0$. However, these two more general problems would only bring technical difficulties, since the computations would be analogous. That is why we avoided these situations here.

As a final commentary, recall that we can consider “well-behaved” functions $f(t, x)$ in order to obtain better results.

\textbf{Proposition 1} If $f : [0, \infty) \times X \to X$ is globally Lipschitz on the second variable, uniformly with respect to the first variable, that is, there exists $L > 0$ such that
\[
\| f(t, x) - f(t, y) \| \leq L \| x - y \|
\]
for every $x, y \in X$ and every $t \in [0, \infty)$, then $f$ maps bounded sets into bounded sets.

\textbf{Proof} Just observe that for any $t \geq 0$ and $x \in X$ we have the inequality
\[
\| f(t, x) \| \leq \| f(t, x) - f(t, 0) \| + \| f(t, 0) \| \leq L \| x \| + \| f(t, 0) \|.
\]

Proposition 1 allows us to conclude that every solution of the Cauchy problem $(P_\alpha)$, when $f(t, x)$ is globally Lipschitz on the second variable, uniformly with respect to the first variable, is defined in $[0, \infty)$ (this is a consequence of estimates we have
already done in this paper, Theorem 1, Remark 1 and [9, Lemma 7.1.1]). Therefore, we would be able to compute the limit done in Theorem 11 for every $t \in [0, \infty)$, in order to obtain a more general result.

The discussion about functions $f(t, x)$ that are locally Lipschitz on the second variable, uniformly with respect to the first variable, deserves a more profound study, which is being developed.

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