Embedding Graphs into Embedded Graphs

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Abstract
A (possibly degenerate) drawing of a graph \( G \) in the plane is approximable by an embedding if it can be turned into an embedding by an arbitrarily small perturbation. We show that testing whether a piece-wise linear drawing of a planar graph \( G \) in the plane is approximable by an embedding can be carried out in polynomial time, if a desired embedding of \( G \) belongs to a fixed isotopy class. In other words, we show that c-planarity with embedded pipes is tractable for graphs with prescribed combinatorial embedding. To the best of our knowledge, an analogous result was previously known essentially only when \( G \) is a cycle.

Keywords Graph embedding · Clustered planarity · Weakly simple polygons · Euler circuit

1 Introduction
In the theory of graph visualization a drawing of a graph \( G = (V, E) \) in the plane is usually assumed to be free of degeneracies, that is, edge overlaps and edges passing through a vertex. However, in practice degenerate drawings often arise and need to be dealt with. The present work concerns the algorithmic problem of detecting denegerate drawings that can be perturbed into embeddings. (See Sect. 2 for the definitions of a generic and degenerate drawing, and an embedding.)

Recent papers [1, 9] address this problem for simple polygons, which can be thought of as straight-line (rectilinear) embeddings of graph cycles. Chang et al. [9] gave an \( O(n^2 \log n) \)-time algorithm to detect if a given polygon with \( n \) vertices can be turned into a simple (non self-intersecting) one by small perturbations of its vertices, or in other words if the polygon is weakly simple. We mention that there exists an earlier closely related definition of weakly simple polygons by Toussaint [8, 31].

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however, as pointed out in [9], this notion is not well-defined for general polygons with so-called spurs. A spur can be thought of as a U-turn, or more formally, a connected portion of the polygon consisting of two identical overlapping parts, each of which is free of self-intersections, that immediately follow one another during polygon’s traversal; see [9] for an overview of attempts at combinatorial definitions of a polygon not intersecting itself.

Akitaya et al. [1] improved the running time of the algorithm by Chang et al. to $O(n \log n)$. The combinatorial formulation of this problem corresponds to the setting of c-planarity with embedded pipes introduced by Cortese et al. [12] well before the two aforementioned papers. Therein only an $O(n^3)$-time algorithm for the problem was given. Nevertheless, the algorithms in [1, 9] were built upon the ideas from [12]. Moreover, to the best of our knowledge, the complexity status of the c-planarity with embedded pipes is essentially known only for cycles. Recently the problem was studied for general planar graphs by Angelini and Da Lozzo [3], but they gave only an FPT algorithm. The introduction of this problem was motivated by a more general and well known problem, called c-planarity by Feng et al. [16, 17], whose tractability status was open for 25 years. The author jointly with Cs. Tóth recently announced [21] that c-planarity can be solved in $O(n^8)$ time. Prior to that work the tractability of c-planarity was unknown in much more restricted cases than the one that we consider. Biedl [6] gave a polynomial-time algorithm for c-planarity with two clusters. Beyond two clusters a polynomial-time algorithm for c-planarity was obtained in the past only in special cases, e.g., [11, 22, 23, 25, 26], and most recently in [7, 10, 18].

There is, however, another tightly related line of research on approximability or realizations of maps pioneered by Sieklucki [29], Minc [27] and Skopenkov [30] that is completely independent from the aforementioned developments, and that is also a major source of inspiration for our work. It can be easily seen that the result [30, Theorem 1.5] implies that c-planarity is tractable for flat instances with three clusters or cyclic clustered graphs [20, Section 6] with a fixed isotopy class of a desired embedding, where the isotopy class of an embedding of a graph is given by a choice of the outer face, a set of rotations at its vertices, and a containment relation of its connected components as described in Sect. 2. An algorithm with a better running time was given by the author in [18].

The aim of the present work is to show that c-planarity with embedded pipes is tractable for planar graphs with a fixed isotopy class of embeddings, which extends results of [3, 4, 18]. Our work also implies the tractability of deciding whether a drawing is approximable by an embedding in a fixed isotopy class, which extends results of [1, 9]. This also answers in the affirmative a question posed in [9, Section 8.2] if the isotopy class of an embedding of $G$ is fixed.

Roughly, we are to decide if in the given isotopy class of $G$ an embedding approximating a given (possibly degenerate) drawing of $G$ exists. The degenerate drawing of $G$ is viewed as a plane graph $H$ and the degeneracies (if any) are captured by a simplicial map between $G$ and $H$. Let $G$ and $H$ be a pair of graphs such that $H$ contains neither loops nor multiple edges, that is, $H$ is simple. A map $\gamma : V(G) \to V(H)$ is simplicial if for every edge $uv \in E(G)$ either $\gamma(u) = \gamma(v)$ or $\gamma(u)\gamma(v)$ is an edge of $H$. We partition $V(G)$ into clusters $V_v$ so that $\gamma(v) = v$ if and only if $v \in V_v$. If it leads
to no confusion, we do not distinguish between a vertex or an edge and its representation in the drawing, and we use the words “vertex” and “edge” in both contexts. We are in the position to state our problem formally.

We are given an ordered triple \((G, H, \gamma)\), where \(G\) is a planar graph (possibly with loops and multiple edges) together with an isotopy class of an embedding of \(G\) in the plane, \(H\) is a plane simple graph, and \(\gamma : V(G) \rightarrow V(H)\) is a simplicial map. We assume that the drawing given by \(H\) is piece-wise linear. The feature size of \(H\) is the minimum of the set consisting of the Euclidean non-zero distances between the endpoints of the line segments in the drawing of \(H\), and the Euclidean distances between the line segments defining the drawing of \(H\). By treating a graph as a 1-dimensional topological space we extend the definition of \(\gamma\) linearly to the edges of \(G\). We want to decide if the given isotopy class of \(G\) contains an embedding \(\mathcal{E}\) such that \(\|\mathcal{E}(x) - \gamma(x)\|_2 \ll \epsilon\), for all \(x \in G\), where \(\epsilon := \epsilon(H) > 0\) is smaller than half of the feature size of \(H\). Thus, seeing \(H\) as subset of the plan, a desired embedding of \(G\) lies in a small neighborhood of \(H\) preserving the facial structure of the embedding of \(H\).

However, to view the problem from a perspective that is more combinatorial, we put further restrictions on a desired embedding of \(G\), which lead to the equivalent problem of \(c\)-planarity with embedded pipes, see Fig. 1. To this end, we need to introduce a couple of notions. Let \(\text{dist}(\mathbf{p}, \mathbf{q})\) denote the Euclidean distance between \(\mathbf{p}, \mathbf{q} \in \mathbb{R}^2\). Let \(\text{dist}(\mathbf{p}, S) = \min_{\mathbf{q} \in S} \text{dist}(\mathbf{p}, \mathbf{q})\), where \(S \subset \mathbb{R}^2\). Let \(N_{\epsilon}(S)\) for \(S \subset \mathbb{R}^2\) denote the \(\epsilon\)-neighborhood of \(S\), that is, \(N_{\epsilon}(S) = \{\mathbf{p} \in \mathbb{R}^2 | \text{dist}(\mathbf{p}, S) \leq \epsilon\}\). Let \(\epsilon' > 0\) be a small value as described later. The thickening \(\mathcal{H}\) of \(H\) is the union of \(N_{\epsilon'}(\mathbf{v})\), for all \(\mathbf{v} \in V(H)\) and \(N_{\epsilon'}(\rho)\), for all \(\rho \in E(H)\). (Throughout the paper we denote vertices and edges of \(H\) by Greek letters.) Let the pipe of \(\rho \in E(H)\) be the closure of \(N_{\epsilon'}(\rho) \setminus (N_{\epsilon'}(\mathbf{v}) \cup N_{\epsilon'}(\mu))\), where \(\rho = \mathbf{v}\mu\). Let the valve of \(\rho\) at \(\mathbf{v}\) be the curve obtained as the intersection of \(N_{\epsilon'}(\mathbf{v})\) and the pipe of \(\rho\). We put \(\epsilon' < \epsilon = \epsilon(H)\), where \(\epsilon(H)\) is the same as in the previous paragraph, so that the valves are pairwise disjoint in \(\mathcal{H}\).

In the combinatorial formulation of the problem, we are to decide if the given isotopy class of \(G\) contains an embedding contained in \(\mathcal{H}\), where the vertices in \(V_{\mathbf{v}}\), for every \(\mathbf{v}\), are drawn in the interior of \(N_{\epsilon'}(\mathbf{v})\) and every edge crosses

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1 In other words, a (planar) graph drawn in the plane without edge crossings.
the boundary of \( N_\ell (v) \), for every \( v \in V(H) \), at most once. This does not change the problem as observed in [9]. Such an embedding of \( G \) is \( H \)-compatible. Let \( E_{\mu\nu} = \{ uv \in E(G) \mid v \in V_\nu, u \in V_\mu \} \). An \( H \)-compatible embedding of \( G \) is encoded by \( G, H \), and a set of total orders \( (E_{\nu\mu}, <_\omega) \), for every \( \nu \mu \in E(H) \) and a valve \( \omega \) of \( \nu \mu \), where \( (E_{\nu\mu}, <_\omega) \) encodes the order of crossings of \( \omega \) with edges along \( \omega \). Since we are interested only in combinatorial aspects of the problem, \( H \) is also given by the isotopy class of its embedding. Throughout the paper we assume that \( G \) and \( H \) are given as above.

**Theorem 1.1** There exists an \( O(n^2) \)-time algorithm that decides if the given isotopy class of \( G \) contains an \( H \)-compatible embedding, where \( n = |V(G)| \). An \( H \)-compatible embedding of \( G \) can be also constructed in \( O(n^2) \) time if it exists. In other words, \( c \)-planarity with embedded pipes is tractable when an isotopy class of a desired embedding of \( G \) is fixed.

As a corollary of our result we obtain that we can test in polynomial time if a piece-wise linear drawing of a graph in the plane is approximable by an embedding and construct such an embedding if it exists. We defer the definition of the approximability by an embedding to Sect. 2.1, where also the proof of the corollary can be found. As we previously discussed, this extends results in [1, 9] and also [30].

**Corollary 1.2** There exists an \( O(n^4) \)-time algorithm that decides if a piece-wise linear degenerate drawing of a graph in the plane is approximable by an embedding, and constructs such an embedding if it exists, where \( n \) is the size of the representation of the drawing.

**Extensions of our results** By [28, Theorem 3.1] and Fáry–Wagner theorem [15], Theorem 1.1 holds also in the setting of rectilinear, that is, straight-line, drawings of graphs. An analogous extension of Corollary 1.2 should hold as well, but showing this is just a little bit technical.

In a recent paper [2], see also [19], both of which can be regarded as an extension of this work, an almost linear-time algorithm for our problem was given, if we lift the restriction on the isotopy class \( G \). On the one hand, these results do not directly imply that the problem with the restriction on the isotopy class \( G \) is tractable except when \( G \) is connected. If \( G \) is disconnected, the algorithms differ significantly, and we do not see how to augment \( G \) in order to make it connected, while preserving the equivalence with the original input. On the other hand, a restriction on the isotopy class makes the problem arguably easier.

As noted by Chang et al. [9], the technique of Cortese et al. [12] extends directly from the plane to any closed two-dimensional surface. The same holds for our method, but since considering general two-dimensional surfaces does not bring anything substantially new to our treatment of the problem, for the sake of simplicity we consider only the planar case.

**Strategy of the proof of Theorem 1.1** Recall that the input of our algorithm is a triple \((G, H, \gamma)\), where the partition of the vertex set of \( G \) corresponds to the
simplicial map $\gamma$ from the set of vertices of $G$ to the set of vertices of $H$. Hence, for $v \in V_\gamma$, where $v \in V(H)$, we have $\gamma(v) = v$. The input $(G, H, \gamma)$ is positive if there exists an $H$-compatible embedding of $G$ in the given isotopy class of $G$, and negative otherwise.

Main troubles in constructing a polynomial-time algorithm for our problem even when $G$ is a path are caused by spurs such as the red subgraph of $G$ in Fig. 1 (left). In general graphs, a spur is a connected subgraph $G'$ of $G$, such that $G'$ is an induced connected subgraph of $G[\bigcup_{v \in V(P)} V'_v]$, where $P$ is a path (possibly consisting of a single vertex) of $H$, and such that all the edges connecting $G'$ with a vertex in $V(G) \setminus V(G')$ are mapped by $\gamma$ to the same edge of $H$. The length of spur $G'$ is $|V(P)|$. For example, according to our more general definition in Fig. 1 on the left, the red vertex forms a spur of length 1. Due to the presence of spurs it is hard to see that our problem is tractable even in the case, when $G$ is a path.

The centerpiece of our method is an extension of the definition of the derivative of maps of intervals/loops (corresponding to the case when $G$ is a path/cycle in our terminology) in the plane introduced by Minc [27]. We adapt this notion to the setting of $c$-planarity with embedded pipes. The derivative is an operator that takes $(G, H, \gamma)$, and either detects that there exists no $H$-compatible embedding of $G$ in the given isotopy class of $G$, or outputs $(G', H', \gamma')$, which is also a valid input for our algorithm, such that $(G, H, \gamma)$ is positive if and only if $(G', H', \gamma')$ is positive (Lemma 3.4). The proof of this equivalence relies on the planar case of Belyi’s theorem [5], which is a simple consequence of the fact that the planar dual of an embedded bipartite graph is Eulerian, that is, it admits a closed walk traversing every edge exactly once.

Intuitively, $H'$ is reminiscent of the line graph of $H$ and the subgraphs of $G$, that are mapped by $\gamma$ to the edges of $H$, are turned into subgraphs of $G'$ mapped by $\gamma'$ into vertices of $H'$. One of the important features of the derivative, which we do not prove formally, is that if $(G, H, \gamma)$ contains a spur then the length of the longest spur in $(G', H', \gamma')$ is less than in $(G, H, \gamma)$, or $(G', H', \gamma')$ does not have any spur. For example, in the case when $H$ is a path and $G$ is connected, the graph $H'$ is the line graph of $H$ and therefore its length as well as the length of all the spurs decreases after a single application of the derivative.

We show that by iterating the derivative $|E(G)|$ times we either detect that there exists no $H$-compatible embedding of $G$ in the given isotopy class of $G$, or we arrive at an input without problematic spurs (Lemma 3.5). Since it is fairly easy to solve the problem for the latter inputs; the derivative at every iteration can be computed in linear time in $|V(G)|$; and by derivating the size of the input is increased only by a little, the tractability follows.

The operation of node expansion and base contraction introduced by Cortese et al. [12] resemble the derivative. The main difference is that these two operations affect only a single cluster or a pair of clusters in $(G, H, \gamma)$, and therefore they are local, whereas the derivative changes the whole input. We are very positive that essential ideas underpinning our method are applicable to other graph drawing problems related to $c$-planarity whose tractability is open, which is documented by a growing body of follow-up work [2, 19].
The derivative is applied to an input \((G, H, \gamma)\), in which every cluster \(V_v\) induces in \(G\) an independent set. Such an input is in the normal form. The detailed description of the algorithm proving Theorem 1.1 is in Sect. 3. We show in Sect. 3.1 that an input can be assumed to be in the normal form. The definition of the derivative is given in Sect. 3.2, and sufficiently simplified inputs are dealt with in Sect. 3.3.

2 Preliminaries

Throughout the paper we tacitly use Jordan-Schönflies theorem for polygons to conclude that a simple closed curve bounds a topological disc.

Let \(G = (V, E)\) denote a planar graph possibly with multiple edges and loops. For \(V' \subseteq V\) we denote by \(G[V']\) the sub-graph of \(G\) induced by \(V'\). The star \(St(v)\) of a vertex \(v\) in a graph \(G\) is the subgraph of \(G\) consisting of all the edges incident to \(v\). Throughout the paper we use standard graph notions such as path, cycle, walk, vertex degree \(deg(v)\), etc., see [14].

A generic drawing \(D(G)\), just drawing for short, is a representation of \(G\) in the plane, where every vertex in \(V\) is represented by a point and every edge \(e = uv\) in \(E\) is represented by a simple piece-wise linear curve joining the points that represent \(u\) and \(v\). Thus, a drawing can be thought of as a map from \(G\) understood as a topological space into the plane. In a drawing, we additionally require every pair of distinct curves representing edges to meet only in finitely many points each of which is a proper crossing or a common endpoint. Drawings that are not generic are degenerate. Therefore in a degenerate drawing, we allow a pair of distinct vertices to be represented by the same point, an edge to pass through a vertex, and a pair of edges to overlap. In a generic drawing, multiple edges are mapped to distinct arcs meeting at their endpoints. In the paper we consider generic drawings, except for Corollary 1.2. An embedding is a generic drawing with no edge crossings. A graph given by an embedding in the plane is a plane graph.

The following lemma is well known.

Lemma 2.1 Let \(G\) be a plane graph with \(n\) vertices such that \(G\) does not contain a pair of multiple edges joining the same pair of vertices that form a face of size two, that is, a lens, except for the outer face. The graph \(G\) has \(O(n)\) edges.

The rotation at a vertex in an embedding of \(G\) is the counterclockwise cyclic order of the edges that are incident to the vertex, which is defined by the order of their end pieces at the vertex in the embedding. The rotation at a vertex is stored as a doubly-linked list of edges. Furthermore, we assume that for every edge of \(G\) we store a pointer to its preceding and succeeding edge in the rotation at both of its end vertices. The interior (resp., exterior) of a cycle in an embedded graph is the bounded (resp., unbounded) connected component of its complement in the plane. Similarly, the interior of an inner face (resp., outer face) in an embedded connected graph is the bounded (resp., unbounded) connected component of the complement of its facial walk in the plane. An embedding of a connected
graph $G$ is up to an isotopy described by the rotations at its vertices and the choice of its outer (unbounded) face. If $G$ is not connected the isotopy class of its embedding is described by isotopy classes of its connected components $G_1, \ldots, G_l$ and the containment relation $G_i \subset f$, for every $G_i$, where $f$ is a face of $G_j$, $j \neq i$, such that $G_i$ is embedded in the interior of $f$.

2.1 Approximation of Maps by Embeddings

The aim of this section is to derive Corollary 1.2 from Theorem 1.1. By treating a graph $G$ as a 1-dimensional topological space, a drawing $D$ of $G$ is understood as a piece-wise linear continuous map $D$ mapping every $x \in G$ to $\mathbb{R}^2$. Such a drawing is given by the finite set of pairs of real values representing the end points of line segments of polylines corresponding in the drawing to edges of $G$.

Let $G = (V, E)$ denote a planar graph. Let $D$ be a degenerate drawing of $G$. In $D$, we subdivide edges at vertices drawn in their interiors. This will result in at most a quadratic blow-up of the input size and allows us to pass to the setting of $H$-compatible embeddings. The claimed running time then follows from Theorem 1.1. Thus, we assume that in $D$ no vertex is drawn in a relative interior of an edge.

An $\epsilon$-approximation of a drawing $D$ of a graph $G$ is a drawing $D'$ of $G$ such that $\text{dist}(D(x), D'(x)) < \epsilon$, for all $x \in G$. A drawing $D$ is approximable by an embedding if there exists $\epsilon(D) > 0$ such that for every $\epsilon$, $0 < \epsilon < \epsilon(D)$, there exists an $\epsilon$-approximation $D$ that is an embedding. It is clear that a drawing with edge crossings is not approximable by an embedding, and thus, in the sequel we consider only drawings without edge crossings.

Given a drawing $D$ of a graph $G$ in the plane, in order to decide if $D$ is approximable by an embedding in a fixed isotopy class of $G$, we construct an input $(G, H, \gamma)$ for c-planarity with embedded pipes. The graph $H$ is the embedded graph given by the image of $D$. We put $\gamma(v) := D(v)$, for $v \in V(G)$.

The input $(G, H, \gamma)$ is positive if and only if $D$ is approximable by an embedding. The “only if” direction is easy. If $(G, H, \gamma)$ is positive, let $\epsilon(D') := \epsilon'$, where $\epsilon'$ is the same as in the definition of the thickening of $H$, witnessing that $D'$ is approximable by an embedding. Let $\epsilon'$ be as in the definition of the thickening of $H$. To prove the “if” direction, it is enough to show that an $\epsilon'$-approximation of $D$, that is an embedding, can be constructed such that for every $v \mu \in E(H)$, $v \in V_v$ and $u \in V_\mu$ we have $|\omega \cap D(uv)| \leq 1$, where $\omega$ is a valve of $v \mu \in E(H)$. This can be achieved by an appropriate local deformation of the $\epsilon'$-approximation as we show next.

Suppose that a valve $\omega$ at, say $v$, crosses an edge at least twice in the $\epsilon'$-approximation. We consider a pair of consecutive crossings with $\omega$ along an edge $e$ such that the piece of $e$ between the crossings in the pair is contained in the pipe. We choose the pair so that the distance between the crossings is minimal, and eliminate the crossings as illustrated in Fig. 2a. By repeating this procedure, we eventually obtain an $H$-compatible embedding of $G$. 
3 Proof of Theorem 1.1

Let \((G, H, \gamma)\) be the input of our algorithm. We naturally extend \(\gamma\) to the edges of \(G\) by mapping edges and subgraphs of \(G\) to subgraphs of \(H\): \(\gamma(e) = \rho = \nu \mu\), for \(\nu \in V_{\nu}\) and \(\mu \in V_{\mu}\), and to subgraphs \(G_1\) of \(G\): \(\gamma(G_1) = H_1 = (V(H_1), E(H_1))\) such that \(V(H_1) = \{\nu \in V(H) | \gamma(\nu) = \nu, \nu \in V(G_1)\}\) and \(E(H_1) = \{\rho \in E(H) | \gamma(e) = \rho, e \in E(G_1)\}\).

A vertex \(\nu \in V(H)\) of degree 2 is redundant if \(V_{\nu} \subseteq V(G)\) is an independent set of \(G\), each vertex \(\nu \in V_{\nu}\) has degree 2, and \(\gamma(\nu \mu) \neq \gamma(\nu \nu)\), where \(\mu\) and \(\nu\) are the neighbors of \(\nu\). We assume that every edge of \(H\) is used by at least one edge of \(G\), that is, for every \(\rho \in E(H)\) there exists \(e \in E(G)\) such that \(\gamma(e) = \rho\).

3.1 The Normal Form

A connected component of a graph is trivial if it consists of a single vertex. Similarly as in [18], the input \((G, H, \gamma)\) is in the normal form if

1. \(G\) does not contain a trivial connected component;
2. every cluster \(V_{\nu} \subseteq V(G)\), for \(\nu \in V(H)\), is an independent set of \(G\); and
3. \(H\) does not contain a pair of redundant vertices joined by an edge.

We remark that (3) is required only due to the running time analysis. We do not forbid redundant vertices completely, since we do not allow \(H\) to contain multiple edges. Indeed, suppressing all vertices of degree two in a graph can lead to multiple edges. In what follows we show how to either detect that no \(H\)-compatible embedding in the given isotopy class of \(G\) exists just by considering the subgraph of \(G\) induced by a single cluster \(V_{\nu}\), or construct an input \((G^N, H^N, \gamma^N)\), see Fig. 2b, in the normal form, which is positive if and only if the input \((G, H, \gamma)\) is positive. Clearly, (3) can be assumed without loss of generality. Before establishing the other conditions we introduce a couple of definitions.

A contraction of an edge \(e = uv\) in an embedding of a graph is an operation that turns \(e\) into a vertex by moving \(v\) along \(e\) towards \(u\) while dragging all the other edges incident to \(v\) along \(e\). By a contraction we can introduce multiple edges or loops at the vertices. We will also use the following operation, which can be
A vertex split, see Fig. 3a, in an embedding of a graph \( G \) is an operation that replaces a vertex \( v \) by two vertices \( u \) and \( w \) joined by a crossing-free edge so that the neighbors of \( v \) are partitioned into two parts according to whether they are joined with \( u \) or \( w \) in the resulting drawing. The rotations at \( u \) and \( w \) are inherited from the rotation at \( v \). When applied to \( G \), the operations are meant to return a graph given by an isotopy class of its embedding; the same applies to vertex multisplit defined later. Note that a contraction can be carried out in \( O(1) \) time, since it amounts to merging a pair of doubly-linked lists, and redirecting at most four pointers. The same applies to the vertex split.

In order to satisfy (2), by a series of successive edge contractions we contract each connected component of \( G[V_{u}] \), for all \( u \in V(H) \), to a vertex. Since rotations are stored as doubly-linked lists, contracting all such connected components can be carried out in linear time. If a loop at a vertex from \( V_{u} \) contains a vertex from a different cluster \( V_{v} \), \( u \neq v \), in its interior we know that the input is negative, since for every \( u \) all the vertices in \( V_{u} \) must be contained in the outer face of \( G[V_{v}] \) if the input is positive. All this can be easily checked in time linear in \( |V(G)| \) by the breadth-first or depth-first search algorithm. If a loop at a vertex from \( V_{v} \) does not contain a vertex from a different cluster in its interior, deleting the loop preserves the existence of an \( H \)-compatible embedding in the given isotopy class of \( G \). Indeed, deleted empty loops can be reintroduced in an \( H \)-compatible embedding of the resulting graph, and contracted edges recovered via vertex splits. Hence, we delete such loops. It remains to deal with trivial connected components of \( G \) in order to establish (1).

For a non-trivial connected component \( C \) of \( G \) and a trivial connected component consisting of a single vertex \( v \), let \( f_{C}(v) \) denote the face of \( C \) containing the vertex \( v \) in its interior in the given isotopy class of \( G \). By the next lemma, we are done unless there exists an isolated vertex \( v \) of \( G \), and a non-trivial connected component \( C \) of \( G \), such that \( \gamma(v) = v \), and such that \( f_{C}(v) \) satisfies the following. We have \( \gamma(u) \neq v \), for every vertex \( u \) incident to \( f_{C}(v) \). Note that \( f_{C}(v) \) can be the outer face of \( C \).

**Lemma 3.1** Let \( v \in V(G) \) form a trivial connected component of \( G \). If for every non-trivial connected component \( C \) of \( G \) there exists a vertex \( u \) incident to \( f_{C}(v) \), such that \( \gamma(u) = \gamma(v) \), then deleting \( v \) does not change if the input is positive or negative.
Proof Since in an $H$-compatible embedding of $G$, the bounding walk of $f_C(v)$ passes through $N_v(\gamma(v))$, for every non-trivial connected component $C$ of $G$, vertex $v$ can be reintroduced into an $H$-compatible embedding of $G \setminus v$ while maintaining $H$-compatibility.

If Lemma 3.1 does not apply, we would like to either detect that $v$ is a reason that the instance is negative, or delete $v$ and argue that this does not change if the input is positive or negative. To this end, however, it is not sufficient to take only $G[V_v]$ into account, since we need to detect if there exists an $H$-compatible embedding of $G$, such that the vertex $v$ is contained in the interior of $f_C(v)$, for every non-trivial connected component $C$ of $G$. The most problematic faces are those without incident vertices mapped to $v$ by $\gamma$. In the following, we show that for $f_C(v)$, none of whose incident vertices is mapped to $v$ by $\gamma$, it does not depend on the choice of $H$-compatible embedding of $G \setminus v$, whether $v$ is contained in the interior of $f_C(v)$

In order to simplify the notation we put $f := f_C(v)$, for some $C$ and $v$ as above. Let $H_f$ denote the subgraph of $H$ such that $H_f$ is the image of $\gamma(E_f)$, where $E_f \subseteq E(G)$ is the set of edges incident to $f$. An escaping arc $A$ of $v$ from $f$ is a Jordan arc joining $v$ with a point in the outer face of $H$ such that $A$ does not pass through vertices of $H$ and crosses every edge of $H$ in a proper crossing. Let $E_A$ denote the set of edges $e$ of $H$ such that $e$ is crossed by $A$ an odd number of times and such that $|\gamma^{-1}[e]|$ is odd, that is, the number of edges of $G$ mapped to $e$ by $\gamma$ is odd. If $f$ is an inner (resp., outer) face, the vertex $v$ is virtually contained in the interior of $f$ if $|E_A|$ is odd (resp., even).

Lemma 3.2 If $v$ is not virtually contained in $f$ then the input $(G, H, \gamma)$ is negative.

Proof We prove the counterpositive. Fix an $H$-compatible embedding of $G$ and consider a Jordan arc $A$ closely following $A$, that is, $A$ (i) starts in a point not contained in $\mathcal{H}$, but in a close neighborhood of $v$; (ii) intersects $\mathcal{H}$ in finitely many line segments each of which is contained in a pipe; and (iii) ends in the outer face of $H$ in a point not contained in $\mathcal{H}$. Note that the parity of the total number of crossings of $A$ with the edges of $E_A$ is equal to $|E_A| \mod 2$. This parity is odd (resp., even) if and only if $v$ is contained in the interior of $f$ given that $f$ is inner (resp., outer) face, or in other words, if and only if $v$ is virtually contained in $f$, which concludes the proof.

We call $f$ $v$-bad if $\gamma(v)$ is not virtually contained in $f$. On the one hand, Lemma 3.2 shows that if $f$ is a $v$-bad then the input is negative. On the other hand, if $f_C(v)$ is not $v$-bad, for every $C$ as above, then we can safely delete $v$, since $v$ can be reintroduced in an $H$-compatible embedding of $G \setminus v$ in the given isotopy class by Lemma 3.1.

In light of the previous paragraph, we delete every isolated vertex $v$ of $G$ such that $f_C(v)$ is not $v$-bad face, for every non-trivial $C$. Let $(G^N, H^N, \gamma^N)$ denote the resulting input, which is clearly in the normal form. Lemmas 3.1 and 3.2 give the following.
Lemma 3.3 If in the given isotopy class of embeddings $G$ contains a v-bad face $f_C(v)$ for some isolated vertex $v \in V(G)$, and a non-trivial connected component $C$ of $G$, then the input $(G, H, \gamma)$ is negative. Otherwise, the input $(G, H, \gamma)$ is positive if and only if $(G^N, H^N, \gamma^N)$ is positive.

3.2 Derivative

We present the operation of derivative that simplifies the input in the normal form. By iterating the derivative we either detect that the input is negative, or we eventually end up with an input that is easy to deal with. We call such inputs locally injective (cf. Sect. 3.3). From the perspective of the running time analysis, the crucial property of the derivative is that its application to an input, which is not locally injective, decreases the value of the following expression $|E(G)| - |E(H)|$, which is always non-negative. Before we describe the derivative formally, we give a couple of necessary definitions.

In the following, we define the operation of vertex multisplit that, roughly speaking, turns a vertex $v$ of $G$ into a star, and groups together the edges incident to $v$ that are mapped by $\gamma$ to the same edge of $H$. A vertex multisplit, see Fig. 3a, in an embedding of a graph $G$ is an operation producing an embedding of a graph obtained from $G$ by replacing a vertex $v$ and its adjacent edges with a star $\{(v, v_1, \ldots, v_l), \{v v_1, \ldots, v v_{deg(v)}\}\}$, where $l \leq deg(v)$, so that the resulting underlying graph has vertex set $V(G) \cup \{v_1, \ldots, v_l\}$ and edge set $(E(G) \setminus \{vu_1, \ldots, vu_{deg(v)}\}) \cup \{v_i u_j \mid j = 1, \ldots, deg(v)\} \cup \{v v_1, \ldots, v v_l\}$, where $u_1, \ldots, u_{deg(v)}$ are neighbors of $v$ in $G$ and $1 \leq i_j \leq l$, for all $j$. The rotations at $v_1, \ldots, v_l$ are inherited from the rotation at $v$ so that by contracting all the edges of $St(v)$ in the resulting graph we obtain the original embedding of $G$. Note that a vertex multisplit can be carried out in $O(deg(v))$ time.

Let $(vv_1, \ldots, vv_{deg(v)})$ be the rotation at $v \in V_\gamma$ in an embedding of $G$ in the given isotopy class. The rotation at $v \in V(H)$ is consistent with the rotation at $v$ if in the rotation at $v$ in the embedding of $H$ the edges $\gamma(vv_1), \ldots, \gamma(vv_{deg(v)})$ appear in this order in the embedding of $H$ (possibly after removing duplicates).

The derivative of $(G, H, \gamma)$ is the input $(G', H', \gamma')$ obtained as follows, see Fig. 3b. First, we construct the graph $G'$ from $G$ by applying the following procedure to every vertex $v \in V(G)$ such that the star $\gamma(St(v))$ has at least two edges, and thus, $v$ is not a spur. In fact, we construct an auxiliary input $(G', H, \gamma)$, where by slightly abusing the notation we will extend $\gamma$ to take values on the vertices of $G'$. (In the second step we use $(G', H, \gamma)$ to construct $(G', H', \gamma')$.) The input $(G, H, \gamma)$ is clearly negative, if there exists a vertex $v$ in $G$ with four incident edges $vv_1, \ldots, vv_4 \in E(G)$ such that $vv_1, vv_2, vv_3$ and $vv_4$ appear in the rotation at $v$ in the given order and $\gamma(vv_1) = \gamma(vv_3) \neq \gamma(vv_2), \gamma(vv_4)$. Otherwise, the following operations of vertex split and multisplit are applicable to $G$. Let the valency of a vertex $v \in V(G)$ be $val(v) := |E(\gamma(St(v)))|$. Thus, the valency counts the size of the set of edges of $H$ that the edges incident to $v$ are mapped to.
If $\text{val}(v) = 2$, we apply the operation of vertex split to $v$ thereby turning it into an edge $uw$ as follows. Let $E(\gamma(\text{St}(v))) = \{\rho_1, \rho_2\}$. Let $v_1, \ldots, v_{\text{deg}(v)}$ be the neighbors of $v$. Let $\{v_1 \ldots v_l\} \cup \{v_{l+1} \ldots v_{\text{deg}(v)}\}$ be the partition of the neighbors of $v$ such that $\gamma(v_1) = \ldots = \gamma(v_l) = \rho_1$ and $\gamma(v_{l+1}) = \ldots = \gamma(v_{\text{deg}(v)}) = \rho_2$. We put $\gamma(u), \gamma(w) := \gamma(v)$, and join $u$ by an edge with the vertices in $\{v_1 \ldots v_l\}$ and $w$ with the vertices in $\{v_{l+1} \ldots v_{\text{deg}(v)}\}$.

If $\text{val}(v) \geq 3$, we analogously apply the operation of vertex multisplit to $v$ so that we replace $v$ with a star $(\{v, v_1, \ldots, v_l\}, \{v_{l+1}, \ldots, v_{\text{deg}(v)}\})$ with $l := \text{val}(v)$ edges, in which the set of incident edges of every leaf vertex $v_i$ is $\{v_{l+i}\} \cup \{v_iu\}$, where $E(\gamma(\text{St}(v))) = \{\rho_1, \ldots, \rho_l\}$, and for every such leaf $\gamma(v_i) := \gamma(v)$.

Let $V_{\geq 3} \subseteq V(G)$ denote the set of vertices in $G$ consisting of the vertices $v \in V(G)$ with $\text{val}(v) \geq 3$. Note that $V_{\geq 3}$ can be treated also as a subset of $V(G')$. Now, every vertex of $G'$ has valency 0 or 1 in $(G', H, \gamma)$, and the vertices in $V_{\geq 3}$ have valency 0. Let $E_2 \subseteq E(G')$ denote the set of edges in $G'$ consisting of every edge $uv$ obtained by splitting $v \in V(G)$ such that $\text{val}(v) = 2$. Let $C$ denote the set of connected components of $G' \setminus E_2 \setminus V_{\geq 3}$. Note that every connected component of $C$ is mapped to an edge of $H$ by $\gamma$.

Second, we construct $H'$ and $\gamma'$. As mentioned in the introduction the graph $H'$ is reminiscent of the line graph of $H$. More precisely, an induced subgraph, say $L$, of $H'$ is a subgraph of this line graph. Therefore the edges of $H$ will play the role of vertices in $H'$. In order to distinguish an edge $\rho$ of $H$ and its corresponding vertex in $H'$, we add the superscript $*$ to $\rho$. The remaining vertices of $H'$, that is, the vertices not belonging to $L$, are in a one-to-one correspondence with the vertices in $V_{\geq 3}$. We add an edge to $H'$ joining a pair of vertices that correspond to the edges $\rho_1$ and $\rho_2$ of $H$ if a connected component in $C$ is incident to an edge mapped by $\gamma$ to $\rho_1$ and an edge mapped by $\gamma$ to $\rho_2$. We add an edge to $H'$ joining a pair of vertices $\rho^*$ and $v_v$, such that $v_v$ corresponds to $v \in V_{\geq 3}$, if an edge of $G$ incident to $v$ is mapped by $\gamma$ to $\rho$. Formally, we put $V(H') := \{\rho^* | \rho \in E(H)\} \cup \{v_v | v \in V_{\geq 3}\}$, and $E(H') := \{\rho^* \rho | \rho \in E(\gamma(\text{St}(v)))\} \cup \{\gamma(C) \gamma(D) | C, D \in C \text{ s.t. there exists } e \in E_2 \text{ joining } C \text{ with } D\}$. We put $\gamma'(v) := \gamma(C)^*$, for $v \in V(C)$ where $C \in C$; and $\gamma(v) := v_v$, for $v \in V_{\geq 3}$.

Refer to Fig. 4a. Finally, the embedding of $H'$, if it exists, is constructed as follows. For $v \in V(H)$, let $C_v$ be the cycle with the vertex set $\{\rho^* | \rho = v \mu \in E(H)\}$ that captures the rotation at $v$, that is, a pair of vertices $\rho^*_0$ and $\rho^*_1$ is joined by an
edge in \( C_v \) if \( \rho_0 \) and \( \rho_1 \) are consecutive in the rotation at \( v \). Hence, \( C_v \) could be possibly a pair of multiple edges. Let \( H'_v \), for \( v \in V(H) \), be the local subgraph of \( H' \) induced by \( \{ \rho^* \mid \rho = v \mu \in E(H) \} \cup \{ v \gamma(v) = v \} \). Let \( \tilde{H}'_v \) be obtained from \( H'_v \) by adding to \( H'_v \) (1) the missing edges of the cycle \( C_v \); and (2) a new vertex joined by the edges exactly with all the vertices of \( C_v \). Note that \( \tilde{H}'_v \) is vertex three-connected, and hence, if \( \tilde{H}'_v \) is planar, then the rotations at vertices in its embedding are determined up to the choice of orientation.

Suppose that every \( \tilde{H}'_v \), for \( v \in V(H) \), is a planar graph. Let us fix for every \( v \in V(H) \) an embedding of \( H'_v \), in which the cycle \( C_v \) bounds the outer face and its orientation corresponds to the rotation at \( v \). Such an embedding is obtained as a restriction of an embedding of \( \tilde{H}'_v \). Note that for every \( v \) the graph \( H'_v \) does not have multiple edges. Since \( H \) also does not have multiple edges, \( H'_v \) and \( H''_v \), for \( v \neq \mu \), are either disjoint (if \( v \mu \notin E(H) \)) or intersect in a single vertex \( (v \mu)^* \) (if \( v \mu \in E(H) \)). It follows that \( H' \) does not have multiple edges. The desired embedding of \( H' \) is obtained by combining embeddings of \( H'_v \), for \( v \in V(H) \), in the same isotopy class as the embeddings of \( H'_v \), that we fixed above, by identifying the corresponding vertices so that the restriction of the obtained embedding of \( H' \) to every \( H'_v \) has the rest of \( H' \) in the interior of the outer face (of this restriction).

Note that the construction of \((G', H', \gamma')\) can be carried out in \( O(\sum_{v \in V(G)} \deg(v)) = O(|V(G)|) \) time thanks to the doubly-linked lists that we use to store the rotations of the vertices of \( G \) and \( H \).

**Lemma 3.4** The input \((G, H, \gamma)\) is negative if one of the following three conditions is satisfied. (1) There exists a vertex \( v \) in \( G \) with four incident edges \( vv_1, \ldots, vv_4 \in E(G) \) such that \( vv_1, vv_2, vv_3 \) and \( vv_4 \) appear in the rotation at \( v \) in the given order and \( \gamma(vv_1) = \gamma(vv_3) \neq \gamma(vv_2), \gamma(vv_4) \). (2) The graph \( \tilde{H}'_v \), for some \( v \in V(H) \), is not planar. (3) The rotation at a vertex \( v \in V(H'_v) \), for some \( v \in V(H) \) and \( v \in V(G') \), in the obtained embedding of \( H'_v \) is not consistent with the rotation at \( v \) in \( G' \).

Otherwise, the input \((G, H, \gamma)\) is positive if and only if the input \((G', H', \gamma')\) is positive.

**Proof** The first part of the claim is obvious. For the second part, we start with the “only if” direction, which is easier. Recall that \( C \) denotes the set of connected components of \( G' \backslash E_{\geq 3} \backslash V_{\geq 3} \).

To this end, given an \( H \)-compatible embedding of \( G \), we first easily construct an \( H \)-compatible embedding of \( G' \) with respect to the input \((G', H, \gamma)\). In the second step, for every \( \rho \in E(H) \), we construct a disc \( D_\rho \), containing the restriction to \( G'_\rho = \bigcup_{C \in \mathcal{C}, \gamma(C) = \rho} C \) of the \( H \)-compatible embedding of \( G \) in its interior as follows. Let \((G'_\rho)_{E} \) be a plane graph obtained from the embedding of \( G'_\rho \) by turning the crossings of edges of \( G'_\rho \) with both valves of \( \rho \) into vertices; and parts of the valves joining closest pairs of crossing (which were turned into vertices) into edges. The disc \( D_\rho \), see Fig. 4b, is a small neighborhood of the union of the inner faces in the embedding of \((G'_\rho)_{E} \). Finally, we apply a homeomorphism of the plane that maps the union of the discs \( D_\rho \)'s with \( G'_\rho \) into the thickening of \( H' \), so that every \( D_\rho \) is mapped onto
Fig. 5 The curve $K_\rho$ splitting the cluster $\rho^*$ in the derivative on the left. The construction of the curve $(K_\rho)_0$ and its deformation into $K_\rho$ on the right.

Fig. 6 Construction of an $H$-compatible embedding of $G'$ from the $H'$-compatible embedding of $G'$ in Fig. 3b.

$N_\epsilon(\rho^*)$ and a small neighborhood of every $v \in V_{\geq 3}$ onto $N_\epsilon(v)$. This concludes the proof of the “only if” direction.

It remains to prove the “if” direction. We show by using Belyi’s theorem [5] that given an $H'$-embedding of $G'$ in the given isotopy class, every $N_\epsilon(\rho^*)$, for $\rho^* \in V(H')$, $\rho = v\mu$, can be split by a simple continuous curve $K_\rho$ into two parts, see Fig. 5 left, so that the curve is disjoint from every edge mapped by $\gamma'$ to an edge of $H'$. Furthermore, vertices $v \in \gamma^{-1}[\rho]$, for which $\gamma(v) = v$, are in one part of $N_\epsilon(\rho^*)$ and vertices $v \in \gamma^{-1}[\rho]$, for which $\gamma(v) = \mu$, are in the other part $N_\epsilon(\rho^*)$.

For a while suppose that $K_\rho$’s exist. Then it follows that an $H$-compatible embedding of $G$ in the given isotopy class exists. Analogously to the previous paragraph, for every $v \in V(H)$, we construct a disc $D_v$, see Fig. 6, containing the subgraph of $G'$ induced by the vertex set $\gamma^{-1}[v]$. We construct $D_v$ so that (1) the intersection of the boundary of $D_v$ with the thickening of $H'$ is $\bigcup_{\rho=\mu, \epsilon \in E(H)} K_\rho$, which is intersected by the boundary in the order given by the rotation at $v$; (2) $N_\epsilon(v) \cap D_v = \emptyset$, for $v \notin V(H')$; and (3) every pair of discs $D_v$ and $D_\mu$, for $v \neq \mu$, is internally disjoint. We perturb discs $D_v$’s a little bit in order to make them pairwise disjoint. Then we apply a homeomorphism of the plane that maps the union of the discs $D_v$’s with $G'$ into the thickening of $H$, so that every $D_v$ is mapped onto $N_\epsilon(v)$. Finally, we contract the edges incident to the vertices in $V_{\geq 3}$ and contract edges in $E_2$ in order to obtain a desired $H$-compatible embedding of $G$. It remains to show that $K_\rho$’s exist, which is rather simple, but a detailed argument requires some work. The claim essentially follows due to the fact that given an embedded bipartite graph in the plane there exists a simple closed curve that crosses every edge of the graph exactly once.

Let $G'_\rho$ be as above. We construct an auxiliary graph $(G'_\rho)_0$ in five steps, see Fig. 7. The graph $G'_\rho$ has the bipartition $V_1 \cup V_2 = V(G'_\rho)$ such that $\gamma(V_1) = \nu$ and $\gamma(V_2) = \mu$, where $\rho = v\mu$. Let $(G'_\rho)_0$ be the union of $G'_\rho$ with its incident edges in $G'_\rho$. Let $(G'_\rho)'_1$ be a plane graph obtained from the embedding of $(G'_\rho)_0$ by turning the crossings of edges of $G'_\rho$ with valves into vertices; and parts of the boundary of $N_\epsilon(\rho^*)$ joining
consecutive pairs of crossings into edges as follows. A consecutive pair of vertices both of which are joined by an edge with a vertex of \( V_1 \) (or \( V_2 \)), is joined by an edge contained in the boundary of \( N_\varepsilon'(\rho^*) \) so that the edge is disjoint from every valve of an edge in \( H'_\mu \) (or \( H'_\nu \)). Let \( (G'_{\rho})_2 \) be the subgraph of \( (G'_{\rho})_1 \) contained in \( N_\varepsilon'(\rho^*) \). Let \( (G'_{\rho})_3 \) be the plane graph obtained from \( (G'_{\rho})_2 \) by contracting all the edges that do not join a vertex of \( V_1 \) with a vertex of \( V_2 \). Note that in the previous step all the edges, that we contracted, were contracted into at most two vertices. Let us denote them by \( v_1 \) and \( v_2 \). Let \( V'_1 \) and \( V'_2 \) be the partition of the vertices of \( (G'_{\rho})_3 \) inherited from \( V_1 \) and \( V_2 \). We assume that \( v_1 \in V'_1 \) and \( v_2 \in V'_2 \), if they exist. If none of \( v_1 \) and \( v_2 \) exists we simply have \( (G'_{\rho})_3 = G'_{\rho} \). If one of them exists suppose without loss of generality that it is \( v_2 \). Finally, let \( (G'_{\rho})_4 \) be the plane graph obtained from \( (G'_{\rho})_3 \) by applying the vertex split to \( v_2 \) so that the newly created edge is incident to a vertex of degree 1.

By a slight abuse of notation we denote this vertex by \( v_2 \). We can assume that \( (G'_{\rho})_4 \) is drawn in \( N_\varepsilon'(\rho^*) \) such that \( v_1 \) is contained in the valve of an edge of \( H'_\nu \) and \( v_2 \) in the valve of an edge of \( H'_\mu \). We obtain the bipartition \( V'_1 \) and \( V'_2 \) of \( (G'_{\rho})_4 \) from the bipartition \( V'_1 \) and \( V'_2 \) of \( (G'_{\rho})_4 \) by putting \( V'_1 := V'_1 \cup \{ v_2 \} \).

By taking the bipartition \( V'_1 \) and \( V'_2 \) of \( (G'_{\rho})_4 \), it follows by Belyi’s theorem that there exists a simple closed curve \( (K'_{\rho})_0 \subset N_\varepsilon'(\rho^*) \) intersecting every edge of \( (G'_{\rho})_4 \) exactly once. Finally, we construct \( K'_{\rho} \) by cutting and deforming \( (K'_{\rho})_0 \) as follows, see Fig. 5 right. We distinguish two cases depending on whether \( v_2 \) exists.

First, suppose that \( v_2 \) exists. The desired curve \( K'_{\rho} \) is obtained by cutting \( (K'_{\rho})_0 \) at its crossing point with the edge incident to \( v_2 \), and applying a homeomorphism of \( N_\varepsilon'(\rho^*) \) that takes the severed end points very close to a pair of the boundary points of \( N_\varepsilon'(\rho^*) \) that split the boundary into two parts, one of which contains the valves of the edges in \( H'_\mu \) and the other the valves of the edges in \( H'_\nu \). Second, if \( v_2 \) does not exist, we cut \( (K'_{\rho})_0 \) at an arbitrary point in the outer face of \( (G'_{\rho})_4 \), and apply a similar homeomorphism of \( N_\varepsilon'(\rho^*) \).

In the end, we extend \( K'_{\rho} \) a little bit so that both of its end points are contained in the boundary of \( N_\varepsilon'(\rho^*) \) and split the contracted vertices in \( (G'_{\rho})_4 \) thereby recovering \( G'_{\rho} \). 

\[ \square \]

### 3.3 Locally Injective Inputs

Let the potential \( p(G, H, \gamma) = |E(G)| - |E(H)| \). Obviously, \( p(G, H, \gamma) \geq 0 \), and \( p(G, H, \gamma) = 0 \) if \( G \) is isomorphic to \( H \). The input in the normal form \( (G, H, \gamma) \) is locally injective, see Fig. 8 (right) for an illustration, if
Fig. 8 Constructing the normal form and derivating one more time the derivative from Fig. 3b on the left, we obtain an input that is strongly locally injective in the normal form on the right

(i) the restriction of \( \gamma \) to \( V(S_t(v)) \) is injective, for all \( v \in V(G) \); and
(ii) for every degree-1 vertex \( v \) in \( G \) its unique incident edge \( e \) satisfies the following. If \( \gamma(e) = \gamma(f) \) then \( e = f \) for all \( f \in E(G) \).

Given an input \((G, H, \gamma)\), the vertex \( v \in V(G) \) is fixed if the condition of property (i) holds for \( v \), and \( v \) is alone in its cluster, that is, \( \gamma(u) = \gamma(v) \) implies \( u = v \). If \( v \) is fixed then we call \( \gamma(v) = v \in V(H) \) also fixed. Note that the edges incident to fixed vertices do not contribute to the potential. Indeed, for every edge \( v\mu \in E(H) \) incident to \( \gamma(v) = v \) such that \( v \) is fixed, there exists exactly one edge \( e \in E(G) \), for which \( \gamma(e) = v\mu \).

For a non-locally injective \((G, H, \gamma)\) in the normal form, by Lemma 3.4 we either easily detect that there exists no \( H \)-compatible embedding of \( G \) in the given isotopy class, or we construct the input \((G', H', \gamma')\) having a smaller potential after being brought to the normal form, such that \((G', H', \gamma')\) is positive if and only if \((G, H, \gamma)\) is positive. The following lemma implies that by iterating the derivative at most \(|E(G)| = O(|V(G)|)\) times we obtain an input that is locally injective.

**Lemma 3.5** If \((G, H, \gamma)\) is in the normal form then \( p((G')^N, (H')^N, (\gamma')^N) \leq p(G, H, \gamma) \). If additionally \((G, H, \gamma)\) is not locally injective then the inequality is strict.

Moreover, \( p((G')^N, (H')^N, (\gamma')^N) \leq p(G, H, \gamma) - \frac{1}{2} t \), where \( t \) is the number of vertices in \( G \) that do not satisfy the condition in property (i) or (ii) of locally injective inputs.

**Proof** Note that edges incident to fixed vertices in \((G', H', \gamma')\) do not contribute to the potential \( p((G')^N, (H')^N, (\gamma')^N) \), and thus, we will deal only with the remaining edges. Recall the definition of \( V_{\geq 3} \) and \( E_2 \) from Sect. 3.2. Since suppressing the vertices of degree two in \( G' \) and \( H' \) violating property (3) of the normal form in order to make the property satisfied does not increase the value of the potential, for the purpose of the proof of the lemma by somewhat abusing the notation, we assume that such vertices are still present in \((G')^N\) and \((H')^N\).

Let \( H'_0 \) be the subgraph of \((H')^N = H'\) induced by its vertex subset \( \{\rho^+ | \rho \in E(H)\} \). Every connected graph on \( n \) vertices has at least \( n - 1 \) edges. It follows that the number of edges in \( H'_0 \) is at least \( |V(H'_0)| - c_r = |E(H)| - c_r \), where \( c_r \) is the number of connected components of \( H'_0 \) that are trees. Note
that \(|E(G')| - |E(H')| = |E_2| - |E(H_0')|\). We claim that to establish the first part of the lemma, it is enough to prove that \(|E_2| \leq |E(G)| - c_r\), where the inequality is strict for non-locally injective inputs. Indeed, summing up the inequalities \(|E(H) - c_r \leq |E(H_0')|\) and \(|E_2| \leq |E(G)| - c_r\) gives \(|E(G')| - |E(H')| = |E_2| - |E(H_0')| \leq |E(G)| - |E(H)|\). In the following we prove the inequality.

To make the argument easier to follow, we first consider the case when \(H_0'\) is connected. The set of edges \(E_2\) in \(G'\) forms a matching. Note that none of the end vertices of edges in \(E_2\) is of degree 1 in \(G'\). Let \(I_2 = \{(v, e_2) | v \in e_2 \in E_2\}\). Note that \(\text{deg}(v) > 1\) if \((v, e_2) \in I_2\). Let \(\psi\) denote the injective map from \(I_2\) taking \((v, e_2)\), \(v \in e_2 \in E_2\) to a pair \((v, e)\), \(e \in E(G')\), such that \(e \cap e_2 = \{v\}\). By using the natural one-to-one correspondence between the edges of \(G\) \(\backslash E_2 \backslash V_{\geq 3}\) and the edges of \(G\), it follows that the size of \(E_2\) is upper bounded by the size of \(E(G)\), since \(|E_2| = |I_2| \leq \{|(v, e)| e \cap e_2 = \{v\}, e_2 \in E_2, e \in E(G')\| \leq 2|E(G)|\). Hence, we have

\[
|E_2| \leq |E(G)|
\] (1)

We show that \(|E_2| = |E(G)|\), only if \((G, H, \gamma)\) is locally injective, and \(H_0'\) contains a cycle. Indeed, if \(H_0'\) does not contain a cycle, it is either a trivial graph consisting of a single vertex, or it contains a pair of vertices \((\rho_0)^*\) and \((\rho_1)^*\) of degree 1 such that \(\gamma'(v_0) = (\rho_0)^*\) and \(\gamma'(v_1) = (\rho_1)^*\), where \(v_0 \in e_0 \in E_2\), \(v_0 \in f_0 = v_0u_0 \notin E_2\), and \(v_1 \in e_1 \in E_2\), \(v_1 \in f_1 = v_1u_1 \notin E_2\). It holds that \(|E_2| < |E(G)|\), because \((u_0^0, f_0)\) and \((u_1^1, f_1)\) are not in the image of the map \(\psi\). Note that there exists at least two such pairs also if \(H_0'\) is trivial (which is a fact that we will need later). Namely, \((u, uv)\) and \((v, uv)\), for some \(\gamma'(uv) = \rho^* \in V(H_0')\). By the same token, we have that \(|E_2| < |E(G)|\), if \((G, H, \gamma)\) is not locally injective. In fact, if \(|E_2| = |E(G)|\) then every connected component of \(G'\) must be a cycle.

Suppose that \(H_0'\) consists of more than one connected component. In the following we extend the previous considerations to prove that \(|E_2| \leq |E(G)| - c_r\), and that the inequality is strict if \((G, H, \gamma)\) is not locally injective. It cannot happen that \(|E_2| > |E(G)| - c_r\) by the analysis in the case when \(H_0'\) is connected. If \(|E_2| = |E(G)| - c_r\), then there exist exactly \(2c_r\) pairs \((v, e)\), \(v \in e \in E(G') \backslash E_2 \backslash V_{\geq 3}\), that are not in the image of the map \(\psi\). However, we showed in the previous paragraph that there are at least \(2c_r\) such pairs \((u, f)\), where both \(u\) and \(f\) are mapped by \(\gamma'\) to a vertex of degree at most one in \(H_0'\). Hence, if \(|E_2| = |E(G)| - c_r\) then all the pairs that are not contained in the image of \(\psi\) are accounted for by such \((u, f)\)'s, in which case \(V_{\geq 3}\) is exactly the subset of \(V(G')\) of vertices of degree at least three. This establishes property (i) of locally injective inputs if \(|E_2| = |E(G)| - c_r\). We establish also (ii) in this case as follows.

We consider the natural correspondence between the vertices of degree 1 in \(G\) and the vertices of degree 1 in \(G'\). Note that none of the pairs \((u, f), u \in f \in E(G')\), where \(u\) is of degree 1, is in the image of \(\psi\). Thus, if \(|E_2| = |E(G)| - c_r\), then every leaf \(u \in V(G')\) is mapped by \(\gamma'\) to some \(\rho^*\), \(\rho \in E(H)\), which is an isolated vertex or a leaf of \(H_0'\). We need to show that there is no other edge besides \(f \supset u\) mapped by \(\gamma'\) to \(\rho^*\). If \(\rho^*\) is an isolated vertex of \(H_0'\) this is immediate, since otherwise
\(|E_2| < |E(G)| - c_r\). If \(\rho^*\) is a leaf of \(H'_1\), every other edge \(g \neq f\) such that \(\gamma'(g) = \rho^*\) must share both end vertices with an edge of \(E_2\), but then \(\rho^*\) has degree at least two in \(H'_1\) (contradiction).

The “moreover” part follows immediately due to the fact that every vertex of \(G\) not satisfying (i) or (ii) causes the slack of \(\frac{1}{2}\) in (1) as shown by the previous analysis.

\[\square\]

Given an input \((G, H, \gamma)\) in the normal form. As in Sect. 3.2, let \(V_{\geq 3} \subseteq V(G)\) denote the set of vertices in \(G\) consisting of the vertices \(v \in V(G)\) with \(\text{val}(v) \geq 3\). The input is strongly locally injective if it is locally injective and

(iii) every vertex in \(V_{\geq 3}\) is fixed.

For convenience, we would like to work with strongly locally injective inputs, see Fig. 8. The following lemma shows that if the input \((G, H, \gamma)\) is locally injective, but not strongly, we just derivate it one more time in order to arrive at a strongly locally injective input.

**Lemma 3.6** Suppose that \((G, H, \gamma)\) in the normal form is locally injective.

Then in \(((G')^N, (H')^N, (\gamma')^N)\), every vertex \(v \in V((G')^N)\), such that \(\text{val}(v) \geq 3\), is fixed. Moreover, \(((G')^N, (H')^N, (\gamma')^N)\) is still locally injective.

**Proof** The lemma follows directly from the definition of the derivative. \[\square\]

Deciding in, roughly, quadratic time in \(p(G, H, \gamma)\), which is sufficient for us, whether the strongly locally injective input \((G, H, \gamma)\) is positive, is quite straightforward. The reason is that in this case the order of crossings of a valve with edges, that are incident to the same vertex \(v\) of \(G\), along the valve in an \(H\)-compatible embedding of \(G\) is determined by the rotation at \(v\). In order to decide if a desired \(H\)-compatible embedding of \(G\) exists, we just detect if for every valve \(\omega\) such an order of all the edges crossing \(\omega\) exists, such that together the orders are compatible. To this end, we consider relations between unordered pairs of edges of \(G\) such that the edges in a pair are mapped by \(\gamma\) to the same edge of \(H\), and two pairs are related if they intersect in a pair of vertices. In the following we assume that \((G, H, \gamma)\) is strongly locally injective.

Refer to Fig. 9a. Let \(\Xi = \{\{e, f\}| e, f \in E(G) \text{ s.t. } e \neq f \text{ and } \gamma(e) = \gamma(f)\}\). Two elements \(\{e_1, f_1\} \in \Xi\) and \(\{e_2, f_2\} \in \Xi\) are neighboring if \(|e_1 \cap e_2| = 1, |f_1 \cap f_2| = 1\) and \(\gamma(e_1 \cap e_2) = \gamma(f_1 \cap f_2)\); we write \(\{e_1, f_1\} \sim \{e_2, f_2\}\). Note that no pair of adjacent edges is an element of \(\Xi\) by the fact that \((G, H, \gamma)\) is strongly locally injective. An element \(\{e_1, f_1\} \in \Xi\) is a boundary pair if there exists at most one \(\{e_2, f_2\} \in \Xi\) such that \(\{e_1, f_1\} \) and \(\{e_2, f_2\}\) are neighboring. Let \(\Xi_1, \ldots, \Xi_l\) be equivalence classes of the transitive closure of the relation \(\sim\). A boundary pair \(\{e_1, f_1\} \in \Xi_i\) is determined if there exists \(\{e_2, f_2\}\) that is neighboring with \(\{e_1, f_1\}\) such that \(\gamma(e_2) \neq \gamma(f_2)\). By properties (i) and (iii) of strong local injectivity, the subgraph \(G_\Xi\) of \(G\) induced by \(\bigcup_{\{e, f\} \in \Xi} \{e, f\}\) has maximum degree two. First, we consider
the case when a connected component of $G_\Xi$ does not contain a vertex of degree 1, see Fig. 9b (left) for an illustration.

**Lemma 3.7** If there exists an equivalence class $\Xi_c$, for some $c$ in $\{1, \ldots, l\}$, such that the subgraph $G_\Xi$ of $G$ induced by $\bigcup_{(e,f) \in \Xi_c} \{e,f\}$ is a cycle, then $(G,H,\gamma)$ is a negative input.

**Proof** Let $\Xi_c = \{(e_0,f_0), \ldots, (e_{m-1},f_{m-1})\}$, where $\{e_p,f_p\} \sim \{e_{p+1}\mod m,f_{p+1}\mod m\}$. Since $\bigcup_{(e,f) \in \Xi_c} \{e,f\}$ induces a cycle $C$ of $G$, there exists the minimum value $p_0 < m$ such that $e_{p_0} = f_0$ or $f_{p_0} = e_0$. Note that $p_0$ divides $m$, since $\gamma(e_g) = \gamma(e_{g+p_0}\mod m)$, for every $0 \leq g,a < m$, and that $C = e_0, \ldots, e_{m-1}$. Due to the fact that the plane is orientable, it follows that the cycle $C$ does not admit an $H$-compatible embedding, since in an $H$-compatible embedding $C$ must wind around a point in the plane more than once.

Note that Lemma 3.7 does not cover the case when $G_\Xi$ is a union of two cycles, see Fig. 9b (right) for an illustration. By (ii), it must be that if $\Xi_c$ contains a boundary pair, then it, in fact, contains exactly two boundary pairs, both of which are determined. Hence, in the following we assume that every $\Xi_c$ either gives rise to a pair of cycles, or contains exactly two determined boundary pairs. We construct for every valve $\omega$ of $\rho \in E(H)$ the relation $(E_\rho,\omega)$, where $E_\rho = \{e \in E(H) | \gamma(e) = \rho\}$. We define relations $(E_\rho,\omega)$ by propagating relations enforced by the determined boundary pairs, for every determined pair contained in $\Xi$. We assume that $(E_\rho,\omega)$ encodes the increasing order of the crossing points of edges with $\omega$ as encountered when traversing $\omega \subset N_\epsilon(\nu)$ in the direction inherited from the counterclockwise orientation of the boundary of $N_\epsilon(\nu)$.

Let $\{e_1,f_1\} \in \Xi_c \subseteq \Xi$ be determined. Let $\Xi_c = \{\{e_1,f_1\}, \ldots, \{e_{m},f_{m}\}\}$ such that $\{e_p,f_p\} \sim \{e_{p+1},f_{p+1}\}$. Let $\gamma(e_1) = \gamma(f_1) = \nu \mu$, $\gamma(e_0) = \nu \mu'$, $\gamma(f_0) = \nu \mu''$, where $\mu' \neq \mu''$ and $|e_0 \cap f_1| = 1$ and $|f_0 \cap f_1| = 1$. W.l.o.g. we suppose that $\nu \mu, \nu \mu'$ and $\nu \mu''$ appear in the rotation at $\nu$ in this order counterclockwise. Let $\omega_1$ be the value of $\nu \mu$ at $\nu$. Let $\omega_2$ be the value of $\nu \mu$ at $\mu$. We put the relation $f_1 <_{\omega_2} e_1$ into $(E_{\nu \mu}, <_{\omega_2})$ and $e_1 <_{\omega_1} f_1$ into $(E_{\nu \mu}, <_{\omega_1})$. Recursively, we put $f_{p+1} <_{\omega_{2p+1}} e_{p+1}$ into $(E_{\nu \mu}, <_{\omega_{2p+1}})$ and $e_{p+1} <_{\omega_{2p+1}} f_{p+1}$ into $(E_{\nu \mu}, <_{\omega_{2p+1}})$, if $f_p <_{\omega_{2p+1}} e_p$ and $e_p <_{\omega_{2p}} f_p$, and vice-versa, where $\omega_{2p}$ and $\omega_{2p+1}$ are valves contained in the boundary of the same disc.

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**Fig. 9** a A neighboring pair $\{e_1,f_1\}$ and $\{e_2,f_2\}$ (left), and a determined boundary pair $\{e,f\}$ (right). b A negative input detected by Lemma 3.7 (left), in which $G$ consists of a cycle $C$. A pair of cycles $C_1$ and $C_2$ of $G$ whose edges form an equivalence class $\Xi_c$ (right).
If \( G_{\Xi_c} \) is a union of two disjoint cycles we add \( f_p <_{\omega_{2p}} e_p \) and \( e_p <_{\omega_{2p-1}} f_p \), or \( f_p >_{\omega_{2p}} e_p \) and \( e_p >_{\omega_{2p-1}} f_p \) for every \( p \), in correspondence with the isotopy class of \( G \).

**Lemma 3.8** Suppose that every equivalence class \( \Xi_c \) contains exactly two determined boundary pairs, or \( G_{\Xi_c} \) is a union of two disjoint cycles. We can test in \( O((p(G,H,\gamma))^2 + |V(G)|) \) time if \((G,H,\gamma)\) is positive or negative.

**Proof** The relations \((E_p, <_\omega)\) can be clearly constructed in total \( O((p(G,H,\gamma))^2) \) time, since only edges not incident to fixed vertices are contained in pairs of \( \Xi \). If the constructed \((E_p, <_\omega)\) is a total order for all \( \rho \in E(H) \) and its valve \( \omega \), the isotopy class of every \( H \)-compatible embedding of \( G \) is determined by an embedding constructed as follows. We first draw the crossings of valves with edges of \( G \) according to the orders \((E_p, <_\omega)\); join every pair of consecutive crossing on the same edge of \( G \) by a piece-wise linear segment contained in a pipe of an edge of \( H \); and finish by drawing the piece-wise linear segments joining vertices of \( G \) with the already drawn parts of edges contained in pipes. It is enough to check if the obtained embedding is in the desired isotopy class of \( G \), which can be easily done in \( O(|V(G)|) \) time by traversing orders \((E_p, <_\omega)\). Note that the only thing that can make the input negative is the containment of connected components of \( G \) in the interiors of its faces.

If \((E_p, <_\omega)\), for some \( \rho \in E(H) \), contains a cyclic chain of inequalities, the input is clearly negative. \( \square \)

### 3.4 Algorithm

We give a description of the decision algorithm proving the first part of the theorem. The running time analysis using Lemmas 3.5, 3.6 and 3.8 follows afterwards.

**Decision Algorithm** Let \((G,H,\gamma) = (G_0, H_0, \gamma_0)\) be the input. We work with inputs in which \( G \) contains multiple edges and loops. However, w.l.o.g. we assume that \( G \) does not contain a pair of multiple edges joining the same pair of vertices that form a face of size two, that is, a lens, except for the outer face. Moreover, we assume that whenever a lens is created during the execution of the algorithm, the lens is eliminated by deleting one of its edges.

An execution of the algorithm is divided into steps. During the \( s \)-th step we process \((G_s, H_s, \gamma_s)\) and output \((G_{s+1}, H_{s+1}, \gamma_{s+1})\) as follows.

First, by following the procedure described in Sect. 3.1 we either construct an instance \(((G_s)^N, (H_s)^N, (\gamma_s)^N)\) in the normal form that is positive if and only if \((G_s, H_s, \gamma_s)\) is positive, or output that \((G,H,\gamma)\) is negative, if the hypothesis of the first part of Lemma 3.3 is satisfied.

Second, if \(((G_s)^N, (H_s)^N, (\gamma_s)^N)\) is not strongly locally injective we proceed as follows. If \((G_s, H_s, \gamma_s)\) satisfies the hypothesis of the first part of Lemma 3.4 with \(((G_s)^N, (H_s)^N, (\gamma_s)^N)\) playing the role of \((G,H,\gamma)\) we output that \((G,H,\gamma)\) is negative; otherwise we construct the derivative \(((G_s')^N, (H_s')^N, (\gamma_s')^N) = (G_{s+1}, H_{s+1}, \gamma_{s+1})\) defined in Sect. 3.2 and proceed to the \((s + 1)\)-st step. Otherwise, \(((G_s)^N, (H_s)^N, (\gamma_s)^N)\)
is strongly locally injective and we construct equivalence classes \( \Xi_1, \ldots, \Xi_t \) from Sect. 3.3 defined by \((G_s)^N, (H_s)^N, (\gamma_s)^N\) and proceed as follows.

We check if there exists a class \( \Xi_c \) satisfying the hypothesis of Lemma 3.7. If this is the case, then we output that \((G, H, \gamma)\) is negative. Otherwise, we construct relations \((E_p, <_o)\), for every \(p \in (H_s)^N\) and its value \(o\). If there exists \((E_p, <_o)\) that is not a total order we output that \((G, H, \gamma)\) is negative; otherwise we check if the isotopy class of an \(H\)-compatible embedding of \(G_s\) enforced by relations \((E_p, <_o)\) is the same as the given one and output that \((G, H, \gamma)\) is positive if and only if this is the case.

The correctness of the algorithm follows directly from Lemmas 3.3, 3.4, 3.7, and 3.8.

### 3.5 Running Time Analysis

We start with estimating the running time of the normalization in Sect. 3.1. The hypothesis of Lemma 3.1 can be checked in \(O(|V(G)|)\) for every trivial connected component \(v\) of \(G\). Similarly, checking the hypothesis of Lemma 3.2 can be carried out in \(O(|V(G)|)\) time for \(v\) by a simple graph search in a planar dual of a subgraph of \(H\), which can be constructed in \(O(|V(H)|)\) time. Note that we are not creating new connected components of \(G\) during the execution of the algorithm. By Lemma 2.1, 3.5, and 3.6, the number of steps of our algorithm is \(O(|V(G)|)\). Hence, we are done once we establish that \(|V((G_s)^N)| = O(|V(G)|)\), for every \(s\).

By Lemma 3.5, \(p((G_{s+1})^N, (H_{s+1})^N, (\gamma_{s+1})^N) \leq p((G_s)^N, (H_s)^N, (\gamma_s)^N) - \frac{1}{2}t\), where \(t\) is the number of vertices in \((G_s)^N\) that do not satisfy the condition in property (i) or (ii) of locally injective inputs. The number of newly created vertices in \((G_{s+1})^N\) of degree at least three satisfying the condition in property (i) during the \(s\)-th step of the algorithm is at most \(t = 2(p((G_s)^N, (H_s)^N, (\gamma_s)^N) - p((G_{s+1})^N, (H_{s+1})^N, (\gamma_{s+1})^N))\). Note that a vertex of degree \(d \geq 3\) in \((G_s)^N, (H_s)^N, (\gamma_s)^N\) satisfying the condition of property (i) becomes a fixed vertex of degree \(d\) in \((G_{s+1})^N, (H_{s+1})^N, (\gamma_{s+1})^N\).

Let \(V_{\geq 3}\) denote the set of fixed vertices of degree at least three in \((G_s)^N\). By the previous paragraph, \(|V_{\geq 3}| \leq |V(G)| + \sum_{s \geq 2} 2(p((G_s)^N, (H_s)^N, (\gamma_s)^N) - p((G_{s+1})^N, (H_{s+1})^N, (\gamma_{s+1})^N)) \leq |V(G)| + 2p((G_0)^N, (H_0)^N, (\gamma_0)^N) = O(|E(G)|) = O(|V(G)|)\) for every \(s\), due to the definition of the potential. The fact \(|V((G_s)^N)| = O(|V(G)|)\), for every \(s\), then follows by (3) in the definition of the normal form. Indeed, the number of vertices of degree two in \((G_s)^N\) mapped to redundant vertices in \((H_s)^N\) is linear in the number of remaining vertices in \((G_s)^N\) due to Lemma 2.1. Hence, the number of vertices in \((G_s)^N\) that are not fixed vertices of degree at least three is linear in \(p(G, H, \gamma) + |V((H_s)^N)| = p(G, H, \gamma) + |V(H_0')| + |V_{\geq 3}|\) due to Lemma 2.1, where \(H_0'\) is defined as in the proof of Lemma 3.5 with \((G_{s-1})^N, (H_{s-1})^N, (\gamma_{s-1})^N\) playing the role of \((G, H, \gamma)\). It follows that the number of vertices in \(|V((G_s)^N)|\), for every \(s\), is linear in \(p(G, H, \gamma) + |V(G)|\), since the size of the subset of \(V(H_0')\) of non-redundant vertices is upper bounded by \(|E(G)| = O(|V(G)|)\).

This more-or-less follows inductively from (1) in the proof of Lemma 3.5, except that in every step we consider the subgraph of \((G_s')^N, s' < s\), induced by the set \(E_2\), where \(E_2\) is defined with \((G_s')^N, (H_s)^N, (\gamma_s)^N\) playing the role of \((G, H, \gamma)\). Indeed, the rest of the edges in \(G_{s+1}\) are incident to fixed vertices or mapped by
\(y'_{s+1}\) to a redundant vertex of \(H'_{s+1}\). Formally, we show by induction on \(s'\) that
\[|E((G'_{s'})^N \setminus V_{s'})| \leq |E(G)|, \]
where \(V_{s'}\) is the set of fixed vertices of degree at least two in \(V((G'_{s'})^N)\) for all \(s'\). In the base case we have \(|E((G_1)^N \setminus V_1)| \leq |E_2| \leq |E(G)|\) by (1). For \(s' > 0\), we have \(|E((G'_{s'+1})^N \setminus V_{s'+1})| \leq |E((G_{s'})^N \setminus V_{s'})|\) by the argument that we used to prove (1).

After having shown that \(O(|V((G_{s-1})^N)|) = O(|V(G)|)\), it is easy to see that the \(s\)-th step of the algorithm can be easily carried out in \(O(|V(G)|)\) time, since the planarity testing and embedding construction of all \(\hat{H}'\)'s needed in the construction of the derivative can be done in linear time in \(O(|V(G_{s'})|) = O(|V((G_{s-1})^N)|)\) [24], and the construction of the instance in the normal form from the given one takes the same running time. The last step of the algorithm in which we construct orders \((E_\rho, \prec_\omega)\) can be easily done in \(O(|V(G)|^2)\), since the number of pairs in \(\Xi\) is \(O(|V(G)|^2)\) due to \(|V(G_s)| = O(|V(G)|)\).

**Algorithm constructing an embedding** The construction of an \(H\)-compatible embedding of \(G\) for strongly locally injective inputs is given by the set of total orders \((E_\rho, \prec_\omega)\), for every \(\rho \in H\) and a valve \(\omega\) of \(\rho\). Therefore in order to construct a desired \(H\)-compatible embedding of \(G\) we need to reverse the order of steps in the decision algorithm. To this end, we make the proof of the second part of Lemma 3.4 algorithmic. In order words, we need to construct the order in which a curve \(K_\rho\) intersects edges of \(G'\). Since we can construct a desired order in linear time by the following lemma, the overall quadratic running time follows. The proof of the lemma is reminiscent of Theorem 1 in [13].

**Lemma 3.9** Given a plane bipartite graph \(G\), we can construct in \(O(|V(G)|)\) time a cyclic order \(\mathcal{O}\) of edges of \(G\) such that there exists a simple closed curve in the plane properly crossing every edge of \(G\) exactly once, but otherwise disjoint from \(G\), in the order given by \(\mathcal{O}\).

**Proof** Let \(V_1 \sqcup V_2 = V(G)\) be the bipartition of \(V(G)\). We construct a plane graph \(G \cup T\) such that \(V(T) = V_1\), \(E(T) \cap E(G) = \emptyset\), and \(T\) is a spanning tree of \((G \cup T)[V_1] = T\). The tree \(T\) is constructed in linear time as follows. We subdivide every face of \(G\) by as many edges as possible in \(\binom{V_1}{2}\), while keeping the resulting graph \(G_0\) plane and its subgraph \(G_0[V_1]\) without multiple edges, e.g., we perform the subdivision so that every subgraph of \(G_0[V_1]\) subdividing a face of \(G\) is a star containing vertices incident to the face. Since rotations at vertices are stored in doubly-linked lists, \(G_0\) can be constructed in \(O(|V(G)|)\) time. Note that \(G_0[V_1]\) is connected, since every face of \(G\) is incident to a vertex in \(V_1\). The tree \(T\) is obtained as a spanning tree of \(G_0[V_1]\).

We contract all the edges of \(T\) in \(G \cup T\). Let \(\mathcal{O}'\) be the rotation at the vertex, that \(T\) was contracted into, in the resulting graph. The desired order \(\mathcal{O}\) is obtained by substituting in \(\mathcal{O}'\) for every edge its corresponding edge in \(G\).

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