A Note on Non-tangential Convergence for Schrödinger Operators

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Abstract

The goal of this note is to establish non-tangential convergence results for Schrödinger operators along restricted curves. We consider the relationship between the dimension of this kind of approach region and the regularity for the initial data which implies convergence. As a consequence, we obtain a upper bound for $p$ such that the Schrödinger maximal function is bounded from $H^s(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for any $s > \frac{n}{2(n+1)}$.

1 Introduction

The solution to the Schrödinger equation

$$iu_t - \Delta u = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+,$$

with initial datum $u(x,0) = f$, is formally written as

$$e^{it\Delta} f(x) := \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t|\xi|^2)} \hat{f}(\xi) d\xi.$$

The problem about finding optimal $s$ for which

$$\lim_{t \to 0^+} e^{it\Delta} f(x) = f(x) \text{ a.e.}$$

whenever $f \in H^s(\mathbb{R}^n)$, was first considered by Carleson [4], and extensively studied by Sjölin [20] and Vega [21], who proved independently the convergence for $s > 1/2$ in all dimensions. Dahlberg-Kenig [8] showed that the convergence does not hold for $s < 1/4$ in any dimension. In 2016, Bourgain [3] gave counterexample showing that convergence can fail if $s < \frac{n}{2(n+1)}$.

Very recently, Du-Guth-Li [9] and Du-Zhang [11] obtained the sharp results by the polynomial partitioning and decoupling method.

The natural generalization of the pointwise convergence problem is to ask a.e. convergence along a wider approach region instead of vertical lines. One of such problems is to consider non-tangential convergence to the initial data, it is natural to expect that more regularity on the

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initial data is necessary to guarantee a.e. existence of the non-tangential limit. It was shown by Sjölin-Sjögren [19] that non-tangential convergence fails for \( s \leq n/2 \), i.e., there exists a function \( f \in H^s(\mathbb{R}^n) \) such that

\[
\limsup_{(y,t) \to (x,0)} \frac{|e^{it\Delta}f(y)|}{|x-y|<\gamma(t), t>0} = \infty,
\]

for all \( x \in \mathbb{R}^n \), where \( \gamma \) is strictly increasing and \( \gamma(0) = 0 \). Cho-Lee-Vargas [6] raised a question about how the size or dimension of the approach region and the regularity which implies convergence are related.

In [6], this question is considered in the one dimensional case. More concretely, let \( \Gamma_x = \{ x + t\theta : t \in [-1,1] \text{ and } \theta \in \Theta \} \), where \( \Theta \) is a given compact set in \( \mathbb{R}^1 \). In [6], they proved that the corresponding non-tangential convergence result holds for \( s > \frac{\beta(\Theta)+1}{2} \), here \( \beta(\Theta) \) denotes the upper Minkowski dimension of \( \Theta \). This result in [6] was established by the \( TT^* \) method and a time localizing lemma. Recently, by getting around the key localizing lemma in [6], Shiraki [18] generalized this result to a wider class of equations which includes the fractional Schrödinger equation.

However, the above question remains open in higher dimensional case until recently. In this article, we consider the non-tangential convergence problem along the approach region in \( \mathbb{R}^n \) given by

\[
\Gamma_x = \{ \gamma(x,t,\theta) : t \in [0,1], \theta \in \Theta \},
\]

where \( \Theta \) is a given compact set in \( \mathbb{R}^n \). \( \gamma \) is a map from \( \mathbb{R}^n \times [0,1] \times \Theta \) to \( \mathbb{R}^n \), which satisfies \( \gamma(x,0,\theta) = x \) for all \( x \in \mathbb{R}^n \), \( \theta \in \Theta \), and the following (C1)-(C3) hold:

(C1) For fixed \( t \in [0,1] \), \( \theta \in \Theta \), \( \gamma \) has at least \( C^1 \) regularity in \( x \), and there exists a constant \( C_1 \geq 1 \) such that for each \( x,x' \in \mathbb{R}^n \), \( \theta \in \Theta \), \( t \in [0,1] \),

\[
C_1^{-1}|x-x'| \leq |\gamma(x,t,\theta) - \gamma(x',t,\theta)| \leq C_1|x-x'|; \quad (1.3)
\]

(C2) There exists a constant \( C_2 > 0 \) such that for each \( x \in \mathbb{R}^n \), \( \theta \in \Theta \), \( t,t' \in [0,1] \),

\[
|\gamma(x,t,\theta) - \gamma(x,t',\theta)| \leq C_2|t-t'|; \quad (1.4)
\]

(C3) There exists a constant \( C_3 > 0 \) such that for each \( x \in \mathbb{R}^n \), \( t \in [0,1] \), \( \theta,\theta' \in \Theta \),

\[
|\gamma(x,t,\theta) - \gamma(x,t,\theta')| \leq C_3|\theta - \theta'|. \quad (1.5)
\]

We consider the relationship between the dimension of \( \Theta \) and the optimal \( s \) for which

\[
\lim_{(y,t) \to (x,0)} \frac{e^{it\Delta}f(y)}{y \in \Gamma_x} = f(x) \text{ a.e.} \quad (1.6)
\]

whenever \( f \in H^s(\mathbb{R}^n) \).

We first give two examples for \( \Gamma_x \). It is not hard to check that all the conditions mentioned above can be satisfied if we take (E1): \( \gamma(x,t,\theta) = x + t\theta \), \( \Theta \) is a compact subset of the unit ball in \( \mathbb{R}^n \). When \( n = 1 \), this is just the problem considered in [6]. Another example is (E2):

\[
\gamma(x,t,\theta) = x + t\theta, \theta^i = (t^{\theta_1}, t^{\theta_2}, \ldots, t^{\theta_n}), \theta = (\theta_1, \theta_2, \ldots, \theta_n), \Theta \text{ is a compact subset in the first quadrant away from the axis of } \mathbb{R}^n. \text{ For this example, it is worth to mention that when } \theta \text{ is fixed, Lee-Rogers [14] have obtained that the convergence along the curve } (\gamma_0(x,t),t) \text{ is equivalent to the convergence along the vertical line.}
Figure 1. $\Theta = \{2, 5/2, \cdots, 3 - 1/k, \cdots, 3 : k = 1, 2, \cdots\}$.

Figure 2. $\Gamma_x$ is consist of all black points which lie on the line $y = x + t^\theta$, $\theta \in \Theta$, $t \in [0, 1/2]$. For every layer $t = t_0$, there are countable black points corresponding to $\Theta$. We try to seek the optimal $s$ for $e^{it\Delta}f(y) \to f(x)$ along different green-path whose points from $\Gamma_x$ whenever $f \in H^s$.

In order to characterize the size of $\Theta$, we introduce the so called logarithmic density or upper Minkowski dimension of $\Theta$,

$$\beta(\Theta) = \limsup_{\delta \to 0^+} \frac{\log N(\delta)}{-\log \delta},$$

where $N(\delta)$ is the minimum number of closed balls of diameter $\delta$ to cover $\Theta$. It is not hard to see that when $\Theta$ is a single point, $\beta(\Theta) = 0$; when $\Theta$ is a compact subset of $\mathbb{R}^n$ with positive Lebesgue measure, $\beta(\Theta) = n$. 


By standard arguments, in order to obtain the convergence result, it is sufficient to establish the bounded estimates for the maximal operator defined by

\[
\sup_{(t,\theta) \in (0,1) \times \Theta} |e^{it\Delta} f(\gamma(x, t, \theta))|.
\]

Our main results are as follows. Firstly, we show the maximal operator estimate in the two dimensional case.

**Theorem 1.1.** When \( n = 2 \), given \( B(x_0, R) \subset \mathbb{R}^2 \), \( R \lesssim 1 \), then for any \( s > \beta(\Theta) + \frac{1}{3} \),

\[
\left\| \sup_{(t,\theta) \in (0,1) \times \Theta} |e^{it\Delta} f(\gamma(x, t, \theta))| \right\|_{L^3(B(x_0, R))} \leq C \| f \|_{H^s(\mathbb{R}^2)}, \tag{1.7}
\]

whenever \( f \in H^s(\mathbb{R}^2) \), where the constant \( C \) depends on \( s, C_1, C_2, C_3 \), and the chosen of \( B(x_0, R) \), but does not depend on \( f \).

Then we obtain the following non-tangential convergence result.

**Theorem 1.2.** When \( n = 2 \), if \( s > \beta(\Theta) + \frac{1}{3} \), then

\[
\lim_{{(y, t) \to (x, 0)} \atop {y \in \Gamma_x}} e^{it\Delta} f(y) = f(x) \text{ a.e.} \tag{1.8}
\]

whenever \( f \in H^s(\mathbb{R}^2) \).

We notice that the convergence result obtained in Theorem 1.2 is sharp when \( \beta(\Theta) = 0 \) ([9] and [3]) or \( \beta(\Theta) = 2 \) ([19]). It is quite interesting to seek whether (1.8) is sharp when \( 0 < \beta(\Theta) < 2 \).

We briefly sketch the proof of Theorem 1.1, and leave the details to Section 2. We decompose \( \Theta \) into small subsets \( \Theta = \bigcup_k \Theta_k \) with bounded overlap, for each \( \Theta_k \), the size is small enough such that our problem can be reduced to estimate the maximal function for Schrödinger operator along certain curves, i.e. the maximal operator defined by

\[
\sup_{t \in (0,1)} |e^{it\Delta} f(\gamma(x, t, \theta_k^0))| \tag{1.9}
\]

for some \( \theta_k^0 \in \Theta_k \). The number of \( \Theta_k \) is determined by \( \beta(\Theta) \). Finally, in order to get the bounded estimate for maximal function defined by (1.9), we still need the following theorem.

**Theorem 1.3.** ([9]) For any \( s > 1/3 \), the following bound holds: for any function \( f \in H^s(\mathbb{R}^2) \),

\[
\left\| \sup_{0 < t < 1} |e^{it\Delta} f(x)| \right\|_{L^3(B(0,1))} \leq C_s \| f \|_{H^s(\mathbb{R}^2)}. \]

The idea to establish the bounded estimate for maximal function define by (1.9) using Theorem 1.3 comes from the method adopted by Lee-Rogers [14] to show equivalence between convergence result for Schrödinger operators along smooth curves and vertical lines. However, we should be more careful since we need an estimate uniformly in \( k \). In our case, this can be realized sine \( \Theta \) is compact.

The above argument can also be applied to obtain the corresponding convergence problem in general dimensions. So we have the following result.
THEOREM 1.4. For general positive integer \( n \), if there exists \( p > 1 \) such that for any \( s > \frac{n}{2(n+1)} \),

\[
\left\| \sup_{0 < t < \lambda} |e^{it \Delta} f(x)| \right\|_{L^p(B(0,1))} \leq C_s \| f \|_{H^s(\mathbb{R}^n)}
\]  

(1.10)

whenever \( f \in H^s(\mathbb{R}^n) \), then given \( B(x_0, R) \subset \mathbb{R}^n, R \leq 1, \) for any \( s > \frac{\beta(\Theta)}{p} + \frac{n}{2(n+1)} \),

\[
\left\| \sup_{(t,\theta) \in (0,1) \times \Theta} |e^{it \Delta} f(\gamma(x, t, \theta))| \right\|_{L^p(B(x_0, R))} \leq C \| f \|_{H^s(\mathbb{R}^n)},
\]  

(1.11)

whenever \( f \in H^s(\mathbb{R}^n) \), where the constant \( C \) depends on \( s, C_1, C_2, C_3, \) and the chosen of \( B(x_0, R) \), but does not depend on \( f \).

Theorem 1.4 implies that if (1.10) holds for some \( p > 1 \) whenever \( f \in H^s(\mathbb{R}^n) \) for any \( s > \frac{n}{2(n+1)} \), then for any \( s > \frac{\beta(\Theta)}{p} + \frac{n}{2(n+1)} \),

\[
\lim_{(y,t) \to (x,0)} e^{it \Delta} f(y) = f(x) \text{ a.e.}
\]  

(1.12)

whenever \( f \in H^s(\mathbb{R}^n) \). Then it comes to the question about what is the optimal \( p \) for (1.10) to hold for any \( s > \frac{n}{2(n+1)} \). This question is still open to our best knowledge, but combining with the counterexample given by Sjölin-Sjögren [19], we get a upper bound for \( p \).

THEOREM 1.5. For general positive integer \( n \), if there exists \( p > 1 \) such that for any \( s > \frac{n}{2(n+1)} \),

(1.10) holds whenever \( f \in H^s(\mathbb{R}^n) \), then \( p \leq \frac{2(n+1)}{n} \).

The upper bound given by Theorem 1.5 is sharp for \( n = 1 \) ([4]) and \( n = 2 \) ([9]), but we do not know if it is also sharp for \( n \geq 3 \). By parabolic rescaling and time localizing lemma, inequality (1.10) is equivalent to

\[
\left\| \sup_{0 < t < \lambda} |e^{it \Delta} f(x)| \right\|_{L^p(B(0,\lambda))} \leq C_s \lambda^{\frac{n}{p} - \frac{n}{2(n+1)}} \| f \|_{L^2(\mathbb{R}^n)}
\]  

(1.13)

whenever \( \sup f \subset \{ \xi \in \mathbb{R}^n : |\xi| \sim 1 \} \), where \( \lambda \gg 1 \). The range of \( p \) has been discussed in Du-Kim-Wang-Zhang [10], but the optimal range of \( p \) is still unknown.

Conventions: Throughout this article, we shall use the well known notation \( A \gg B \), which means if there is a sufficiently large constant \( G \), which does not depend on the relevant parameters arising in the context in which the quantities \( A \) and \( B \) appear, such that \( A \geq GB \). We write \( A \sim B \), and mean that \( A \) and \( B \) are comparable. By \( A \lesssim B \) we mean that \( A \leq CB \) for some constant \( C \) independent of the parameters related to \( A \) and \( B \). Given \( \mathbb{R}^n \), we write \( B(0,1) \) instead of the unit ball \( B^n(0,1) \) in \( \mathbb{R}^n \) centered at the origin for short, and the same notation is valid for \( B(x_0, R) \).

2 Proof of the main theorems for \( n = 2 \)

Proof of Theorem 1.1. In order to prove Theorem 1.1, using Littlewood-Paley decomposition, we only need to show that for \( f \) with \( \text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^2 : |\xi| \sim \lambda \} \), \( \lambda \gg 1 \),

\[
\left\| \sup_{(t,\theta) \in (0,1) \times \Theta} |e^{it \Delta} f(\gamma(x, t, \theta))| \right\|_{L^2(B(x_0, R))} \leq C \lambda^\alpha \| f \|_{L^2}, \ \forall \alpha > 0,
\]  

(2.1)
where \( s_0 = \frac{\beta(\Theta)+1}{3} \).

We decompose \( \Theta \) into subsets \( \Theta = \bigcup_k \Theta_k \) with bounded overlap, where each \( \Theta_k \) is contained in a closed ball with diameter \( \lambda^{-1} \). Then we have

\[
1 \leq k \leq \lambda^{\beta(\Theta)+\epsilon}.
\]

(2.2)

We claim that for each \( k \),

\[
\| \sup_{(t,\theta) \in (0,1) \times \Theta_k} |e^{it\Delta} f(\gamma(x,t,\theta))| \|_{L^3(B(x_0, R))} \leq C \lambda^{\frac{1}{3} + \frac{2}{3} \epsilon} \| f \|_{L^2}.
\]

(2.3)

Then inequality (2.1) follows from (2.2) and (2.3). More concretely, we have

\[
\| \sup_{(t,\theta) \in (0,1) \times \Theta} |e^{it\Delta} f(\gamma(x,t,\theta))| \|_{L^3(B(x_0, R))} \leq \left( \sum_k \| \sup_{(t,\theta) \in (0,1) \times \Theta_k} |e^{it\Delta} f(\gamma(x,t,\theta))| \|_{L^3(B(x_0, R))} \right)^{1/3}
\]

\[
\leq C \left( \sum_k \lambda^{1+2\epsilon} \| f \|_{L^3}^3 \right)^{1/3}
\]

\[
\leq C \lambda^{\frac{3\epsilon+1}{3}} \| f \|_{L^2},
\]

(2.4)

which implies inequality (2.1).

Now we are left to prove inequality (2.3). For this goal, we first show the following Lemma 2.1. The original idea comes from Lemma 2.2 in [14].

**Lemma 2.1.** Assume that \( g \) is a Schwartz function whose Fourier transform is supported in \( \{ \xi \in \mathbb{R}^n : |\xi| \sim \lambda \} \). If

\[
|\theta - \theta'| \leq \lambda^{-1},
\]

then for each \( x \in B(x_0, R) \) and \( t \in (0,1) \),

\[
|e^{it\Delta} g(\gamma(x,t,\theta))| \leq \sum_{l \in \mathbb{Z}^n} \frac{C}{(1+|l|)^{n+1}} \left| \int_{\mathbb{R}^n} e^{i\gamma(x,t,\theta') + \frac{1}{2} |\xi+it| |\xi|} \hat{g}(\xi) d\xi \right|, 
\]

(2.5)

where the constant \( C \) depends on \( n \) and \( C_3 \) in inequality (1.5).

**Proof.** We introduce a cut-off function \( \phi \) which is smooth and equal to 1 on \( B(0,2) \) and supported on \( (-\pi, \pi)^n \). After scaling we have

\[
e^{it\Delta} \hat{g}(\gamma(x,t,\theta)) = \lambda^n \int_{\mathbb{R}^n} e^{i\lambda \gamma(x,t,\theta) \cdot \eta + it|\lambda \eta|^2} \phi(\eta) \hat{g}(\lambda \eta) d\eta
\]

\[
= \lambda^n \int_{\mathbb{R}^n} e^{i\lambda \gamma(x,t,\theta) \cdot \eta - i\lambda \gamma(x,t,\theta') \cdot \eta + i\lambda \gamma(x,t,\theta') \cdot \eta + it|\lambda \eta|^2} \phi(\eta) \hat{g}(\lambda \eta) d\eta.
\]

(2.6)

Since it follows by inequality (1.5),

\[
\lambda |\gamma(x,t,\theta) - \gamma(x,t,\theta')| \leq C_3,
\]

then by Fourier expansion,

\[
\phi(\eta) e^{i\lambda |\gamma(x,t,\theta) - \gamma(x,t,\theta')|^2} = \sum_{l \in \mathbb{Z}^n} c_l(x, t, \theta, \theta') e^{il \eta},
\]
where
\[ |c_l(x, t, \theta, \theta')| \leq \frac{C}{(1 + |l|)^{n+1}} \]
uniformly for each \( l \in \mathbb{Z}^n \), \( x \in B(x_0, R) \) and \( t \in (0, 1) \). Then we have
\[ |e^{it\Delta}g(\gamma(x, t, \theta))| \leq \sum_{l \in \mathbb{Z}^n} \frac{C \gamma^n}{(1 + |l|)^{n+1}} \left| \int_{\mathbb{R}^n} e^{it\gamma(x, t, \theta') \cdot \gamma + i|l\gamma|^2} \hat{g}(\gamma) d\gamma \right| \]
\[ = \sum_{l \in \mathbb{Z}^n} \frac{C}{(1 + |l|)^{n+1}} \left| \int_{\mathbb{R}^n} e^{i\frac{1}{2}\xi + i\gamma(x, t, \theta') \cdot \xi + i|\xi|^2} \hat{g}(\xi) d\xi \right|, \]
then we arrive at (2.5).

By the similar argument, we can prove the following lemma.

**Lemma 2.2.** Assume that \( g \) is a Schwartz function whose Fourier transform is supported in \( \{ \xi \in \mathbb{R}^n : |\xi| \sim \lambda \} \). If
\[ |t - t'| \leq \lambda^{-1}, \]
then for each \( x \in B(x_0, R) \) and \( \theta \in \Theta \),
\[ |e^{it\Delta}g(\gamma(x, t, \theta))| \leq \sum_{l \in \mathbb{Z}^n} \frac{C}{(1 + |l|)^{n+1}} \left| \int_{\mathbb{R}^n} e^{i|\gamma(x, t', \theta') + \frac{1}{2}|\xi|^2} \hat{g}(\xi) d\xi \right|, \quad (2.7) \]
where the constant \( C \) depends on \( n \) and \( C_2 \) in inequality (1.4).

We now prove inequality (2.1). For fixed \( k \), by the construction of \( \Theta_k \), there is a \( \theta^0_k \in \Theta_k \) such that
\[ |\theta - \theta^0_k| \leq \lambda^{-1} \]
holds for each \( \theta \in \Theta_k \). Then according to Lemma 2.1, for each \( x \in B(x_0, R) \), \( t \in (0, 1) \) and \( \theta \in \Theta_k \), we have
\[ |e^{it\Delta}f(\gamma(x, t, \theta))| \leq \sum_{l \in \mathbb{Z}^2} \frac{C}{(1 + |l|)^2} \left| \int_{\mathbb{R}^2} e^{i\gamma(x, t, \theta^0_k) \cdot \xi + i|\xi|^2} e^{i\frac{1}{2}\xi \cdot \xi} \hat{f}(\xi) d\xi \right| \]
\[ = \sum_{l \in \mathbb{Z}^2} \frac{C}{(1 + |l|)^2} \left| \int_{\mathbb{R}^2} e^{i\gamma(x, t, \theta^0_k) \cdot \xi + i|\xi|^2} \hat{f}_\lambda^1(\xi) d\xi \right| \]
\[ = \sum_{l \in \mathbb{Z}^2} \frac{C}{(1 + |l|)^2} \left| e^{it\Delta} \hat{f}_\lambda^1(\gamma(x, t, \theta^0_k)) \right|, \quad (2.8) \]
where
\[ \hat{f}_\lambda^1(\xi) = e^{i\frac{1}{2}\xi \cdot \xi} \hat{f}(\xi). \]
It follows that
\[ \left\| \sup_{(t, \theta) \in (0, 1) \times \Theta_k} |e^{it\Delta}f(\gamma(x, t, \theta))| \right\|_{L^3(B(x_0, R))} \leq \sum_{l \in \mathbb{Z}^2} \frac{C}{(1 + |l|)^2} \left\| \sup_{(t, \theta) \in (0, 1) \times \Theta_k} |e^{it\Delta} \hat{f}_\lambda^1(\gamma(x, t, \theta^0_k))| \right\|_{L^3(B(x_0, R))} \]
provided that we have proved the following lemma.

**Lemma 2.3.** Assume that $g$ is a Schwartz function whose Fourier transform is supported in the annulus $\{\xi \in \mathbb{R}^2 : |\xi| \sim \lambda\}$. Then for each $k$,

$$
\left\| \sum \frac{1}{(1 + |k|)^\beta} \sup_{t \in (0, 1)} |e^{it\Delta} f_k^i(\gamma(x, t, \theta_k^0)(\gamma(x, t, \theta_k^0))| \right\|_{L^3(B(x_0, R))} \\
\leq \sum \frac{1}{(1 + |k|)^\beta} \lambda^{1 + \frac{2\beta}{3}} \|f_k\|_{L^2}
$$

where the constant $C$ is independent of $k$.

Now let’s turn to prove Lemma 2.3. The following theorem is required.

**Theorem 2.4.** ([14]) Let $\rho : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $q, r \in [2, +\infty]$, $\lambda \geq 1$, $supp \nu \subset [-2, 2]$, $\lambda \geq \|1\|_{L^2_{\nu}L^2_{\nu}}$, and suppose that

$$
\sup_{x \in supp(\nu), t \in supp(\nu)} |\rho(x, t)| \leq M,
$$

where $M > 1$. Suppose that for a collection of boundedly overlapping intervals $I$ of length $\lambda^{-1}$, there exists a $C_0 > 1$ such that

$$
\|e^{it\Delta} f(\rho(x, t))\|_{L^6_{\rho}L^6_{\rho}(I)} \leq C_0 \|f\|_{L^2(\mathbb{R}^n)},
$$

whenever $f$ is supported in $\{\xi \in \mathbb{R}^n : |\xi| \sim \lambda\}$. Then there is a constant $C_n > 1$ such that

$$
\|e^{it\Delta} f(\rho(x, t))\|_{L^6_{\rho}L^6_{\rho}(\cup I)} \leq C_n M^{1/2}C_0 \|f\|_{L^2(\mathbb{R}^n)}
$$

whenever $\hat{f}$ is supported in $\{\xi \in \mathbb{R}^n : |\xi| \sim \lambda\}$.

Notice that in our case, for each $k$, we have

$$
\sup_{(x,t) \in B(x_0,R) \times (0, 1)} |\gamma(x, t, \theta_k^0)| \leq \sup_{(x,t,\theta) \in B(x_0,R) \times (0, 1) \times \Theta} |\gamma(x, t, \theta)|.
$$

By inequality (1.4), for each $(x, t, \theta) \in B(x_0, R) \times (0, 1) \times \Theta$,

$$
|\gamma(x, t, \theta) - \gamma(x, 0, \theta)| \leq C_2,
$$

then $|\gamma(x, t, \theta)|$ is uniformly bounded for $(x, t, \theta) \in B(x_0, R) \times (0, 1) \times \Theta$, and the upper bound is determined by $C_2$ and the chosen of $B(x_0, R)$, but independent of $k$.

Therefore, according to Theorem 2.4, in order to prove Lemma 2.3, we only need to show that for each interval $I \subset (0, 1)$ of length $\lambda^{-1}$, and any function $g$ such that $\hat{g}$ is supported in $\{\xi \in \mathbb{R}^2 : |\xi| \sim \lambda\}$, we have

$$
\left\| \sup_{t \in I} |e^{it\Delta} g(x, t, \theta_k^0)(x, t, \theta_k^0)| \right\|_{L^3(B(x_0, R))} \leq C\lambda^{1 + \frac{2\beta}{3}} \|g\|_{L^2}.
$$

(2.11)
Since \( I \) is an interval of length \( \lambda^{-1} \), there exists \( t^0_I \in I \) such that for each \( t \in I \),
\[
|t - t^0_I| \leq \lambda^{-1}.
\]
Then by Lemma 2.2, for each \( x \in B(x_0, R) \), \( t \in I \), we have
\[
|e^{it\Delta} g(\gamma(x, t, \theta^0_k))| \leq \sum_{l \in \mathbb{Z}^2} \frac{C}{(1 + |l|)^3} \left| \int_{\mathbb{R}^2} e^{i\gamma(x, t^0_I, \theta^0_k) \cdot \xi + u|\xi|^2} e^{i\frac{\xi}{2} \cdot \xi} \hat{g}(\xi) d\xi \right|
\]
\[
= \sum_{l \in \mathbb{Z}^2} \frac{C}{(1 + |l|)^3} \left| e^{it\lambda \hat{g}_\lambda} (\gamma(x, t^0_I, \theta^0_k)) \right|,
\]
where
\[
\hat{g}_\lambda(\xi) = e^{i\frac{\xi}{2} \cdot \xi} \hat{g}(\xi).
\]
It follows that
\[
\left\| \sup_{t \in I} |e^{it\Delta} g(\gamma(x, t, \theta^0_k))| \right\|_{L^3(B(x_0, R))} \leq \sum_{l \in \mathbb{Z}^2} \frac{C}{(1 + |l|)^3} \left\| \sup_{t \in I} |e^{it\lambda \hat{g}_\lambda} (\gamma(x, t^0_I, \theta^0_k))| \right\|_{L^3(B(x_0, R))}.
\]
(2.13)
For each \( t^0_I, \theta^0_k, \gamma^0_I, \theta^0_k \) is at least \( C^1 \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \). By inequality (1.3), for each \( x \in \mathbb{R}^2 \),
\[
C^{-1}_1 \leq |\nabla_x \gamma(x, t^0_I, \theta^0_k)| \leq C_1.
\]
By the same reason, for each \( x \in B(x_0, R) \),
\[
|\gamma(x, t^0_I, \theta^0_k) - \gamma(x_0, t^0_I, \theta^0_k)| \leq C_1 R,
\]
which implies \( \gamma^0_I, \theta^0_k(B(x_0, R)) \subset B(\gamma(x_0, t^0_I, \theta^0_k), C_1 R) \). Therefore, changes of variables and Theorem 1.3 imply that
\[
\left\| \sup_{t \in I} |e^{it\Delta} g(\gamma(x, t, \theta^0_k))| \right\|_{L^3(B(x_0, R))} \leq C \lambda^{\frac{1}{2} + \frac{2}{3}} \|g\|_{L^2}.
\]
(2.14)
Combining inequality (2.13) with inequality (2.14), we have
\[
\left\| \sup_{t \in I} |e^{it\Delta} g(\gamma(x, t, \theta^0_k))| \right\|_{L^3(B(x_0, R))} \leq \sum_{l \in \mathbb{Z}^2} \frac{C}{(1 + |l|)^3} \lambda^{\frac{1}{2} + \frac{2}{3}} \|\hat{g}_\lambda\|_{L^2}
\]
\[
\leq C \lambda^{\frac{1}{2} + \frac{2}{3}} \|g\|_{L^2}.
\]
(2.15)
This completes the proof of Lemma 2.3.

**Proof of Theorem 1.2.** The proof of Theorem 1.2 is quite standard. We write the details for completeness. In fact, for any \( s > \frac{\beta(\Theta) + 1}{3} \), \( f \in H^s(\mathbb{R}^2) \), fix \( \lambda > 0 \), choose \( g \in C_c^\infty(\mathbb{R}^2) \) such that
\[
\|f - g\|_{H^s(\mathbb{R}^2)} \leq \frac{\lambda^{1/3}}{2C},
\]
where the constant $C$ is the constant in inequality (1.7), which follows

$$\left\{ x \in B(x_0, R) : \sup_{(t, \theta) \in (0,1) \times \Theta} |e^{it\Delta}(f - g)(\gamma(x, t, \theta))| > \frac{\lambda}{2} \right\}$$

$$\leq \frac{2^3}{\lambda^3} \left\| \sup_{(t, \theta) \in (0,1) \times \Theta} |e^{it\Delta}(f - g)(\gamma(x, t, \theta))| \right\|_{L^3(B(x_0, R))}^3$$

$$\leq \frac{2^3 C^3}{\lambda^3} \|f - g\|_{L^3(\mathbb{R}^2)}^3$$

Moreover,

$$\leq \epsilon.$$  

(2.16)

Moreover,

$$\lim_{y \to x} e^{it\Delta}g(y) = g(x)$$  

(2.17)

uniformly for $x \in B(x_0, R)$. Indeed, for each $x \in B(x_0, R)$,

$$\limsup_{y \to x} |e^{it\Delta}g(y) - g(x)| \leq \limsup_{y \to x} |e^{it\Delta}g(y) - e^{it\Delta}g(x)| + \limsup_{y \to x} |e^{it\Delta}g(x) - g(x)|$$

$$= \limsup_{y \to x} |e^{it\Delta}g(y) - e^{it\Delta}g(x)| + \limsup_{t \to 0^+} |e^{it\Delta}g(x) - g(x)|.$$  

(2.18)

By mean value theorem and inequality (1.4), we have

$$|e^{it\Delta}(\gamma(x, t, \theta)) - e^{it\Delta}g(x)| \leq t \int_{\mathbb{R}^2} |\xi| |\hat{g}(\xi)| d\xi,$$  

(2.19)

and

$$|e^{it\Delta}g(x) - g(x)| \leq t \int_{\mathbb{R}^2} |\xi|^2 |\hat{g}(\xi)| d\xi.$$  

(2.20)

Inequalities (2.18) - (2.20) imply (2.17).

By (2.16) and (2.17) we have

$$\left\{ x \in B(x_0, R) : \limsup_{(y, t) \to (x, 0)} \sup_{y \in \Gamma_x} |e^{it\Delta}(f - g)(y)| > \frac{\lambda}{2} \right\}$$

$$\leq \left\{ x \in B(x_0, R) : \limsup_{(y, t) \to (x, 0)} \sup_{y \in \Gamma_x} |e^{it\Delta}(f - g)(y)| > \frac{\lambda}{2} \right\}$$

$$+ \left\{ x \in B(x_0, R) : |f(x) - g(x)| > \frac{\lambda}{2} \right\}$$

$$\leq \left\{ x \in B(x_0, R) : \sup_{(t, \theta) \in (0,1) \times \Theta} |e^{it\Delta}(f - g)(\gamma(x, t, \theta))| > \frac{\lambda}{2} \right\}$$

$$+ \left\{ x \in B(x_0, R) : |f(x) - g(x)| > \frac{\lambda}{2} \right\}$$
\[ \lesssim \epsilon + \frac{2^2}{\lambda^2} \| f - g \|_{H^s(\mathbb{R}^2)}^2 \]
\[ \leq \epsilon + \frac{\epsilon^2}{C^2} \]
\[ \leq \epsilon + \epsilon^2, \quad (2.21) \]
since we can always assume that \( C \geq 1 \), which implies convergence for \( f \in H^s(\mathbb{R}^2) \) and almost every \( x \in B(x_0, R) \). By the arbitrariness of \( B(x_0, R) \), in fact we can get convergence for almost every \( x \in \mathbb{R}^2 \). This completes the proof of Theorem 1.2.

3 Proof of the main theorems for \( n \geq 3 \)

**Proof of Theorem 1.4.** We briefly explain the proof of Theorem 1.4 since most of the details are similar to the proof of Theorem 1.1. As in the proof of Theorem 1.1, we only need to prove that for \( f \), \( \text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^n : |\xi| \sim \lambda \} \), \( \lambda \gg 1 \),

\[ \left\| \sup_{(t, \theta) \in (0, 1) \times \Theta} |e^{it\Delta} f(\gamma(x, t, \theta))| \right\|_{L^p(B(x_0, R))} \leq C\lambda^{s_0 + \epsilon} \| f \|_{L^2}, \quad \forall \epsilon > 0, \quad (3.1) \]

where \( s_0 = \frac{\beta(\Theta)}{p} + \frac{n}{2(n+1)} \).

We decompose \( \Theta \) into subsets \( \Theta = \bigcup_k \Theta_k \) with bounded overlap, where each \( \Theta_k \) is contained in a closed ball with diameter \( \lambda^{-1} \). Then we have

\[ 1 \leq k \leq \lambda^{\beta(\Theta) + \epsilon}. \]

As in the proof of Theorem 1.1, we can show that for each \( k \),

\[ \left\| \sup_{(t, \theta) \in (0, 1) \times \Theta_k} |e^{it\Delta} f(\gamma(x, t, \theta))| \right\|_{L^p(B(x_0, R))} \leq C\lambda^{\frac{n}{2(n+1)} + \frac{p}{p-1}\epsilon} \| f \|_{L^2}. \]

Then we have

\[ \left\| \sup_{(t, \theta) \in (0, 1) \times \Theta} |e^{it\Delta} f(\gamma(x, t, \theta))| \right\|_{L^p(B(x_0, R))} \leq \left( \sum_k \left\| \sup_{(t, \theta) \in (0, 1) \times \Theta_k} |e^{it\Delta} f(\gamma(x, t, \theta))| \right\|_{L^p(B(x_0, R))} \right)^{1/p} \]
\[ \leq C \left( \sum_k \lambda^{\frac{n}{2(n+1)} + \frac{p}{p-1}\epsilon} \| f \|_{L^2}^p \right)^{1/p} \]
\[ \leq C \lambda^{\frac{\beta(\Theta)}{p} + \frac{n}{2(n+1)} + \epsilon} \| f \|_{L^2}, \]

which implies inequality (3.1).

**Proof of Theorem 1.5.** Taking \( \gamma(x, t, \theta) = x + t \theta \), \( \Theta \) is the interior of the unit ball in \( \mathbb{R}^n \). Then we have

\[ \beta(\Theta) = n, \quad (3.2) \]
and the approach region

\[ \Gamma_x = \{ \gamma(x, t, \theta) : t \in [0, 1], \theta \in \Theta \} = \{ y : |y - x| < t, t \in [0, 1] \}. \quad (3.3) \]
Assuming that (1.10) holds true, then it follows from Theorem 1.4 and inequality (3.3) that for any \( s > \beta(\Theta) + \frac{n}{2(n+1)} \),

\[
\lim_{(y,t) \to (x,0)} \lim_{|x-y|<t,t>0} e^{it\Delta} f(y) = f(x) \text{ a.e.} \quad (3.4)
\]

whenever \( f \in H^s(\mathbb{R}^2) \). But according to Theorem 3 in [19] by Sjölin-Sjögren, this result fails for any \( s \leq \frac{n}{2} \). Therefore, we get

\[
\beta(\Theta) + \frac{n}{2(n+1)} \geq \frac{n}{2} \quad (3.5)
\]

Then inequality (3.5) and inequality (3.2) imply Theorem 1.5.

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