Averages of Ramanujan sums: Note on two papers by E. Alkan

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Abstract

We give a simple proof and a multivariable generalization of an identity due to E. Alkan concerning a weighted average of the Ramanujan sums. We deduce identities for other weighted averages of the Ramanujan sums with weights concerning logarithms, values of arithmetic functions for gcd’s, the Gamma function, the Bernoulli polynomials and binomial coefficients.

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1 Introduction

Let \( c_k(j) \) denote the Ramanujan sums defined for \( k \in \mathbb{N} := \{1, 2, \ldots \} \) and \( j \in \mathbb{Z} \) by

\[
c_k(j) := \sum_{\gcd(m, k) = 1}^{k} \exp(2\pi imj/k).
\]

Other notations used throughout this note are the following: \( \lfloor x \rfloor \) is the integer part of \( x \), \( B_m \ (m \in \mathbb{N} \cup \{0\}) \) are the Bernoulli numbers, \( \varphi \) is Euler’s totient function, \( \tau(n) \) and \( \sigma(n) \) stand for the number and the sum of the divisors of \( n \), respectively, \( \mu \) is the Möbius function, \( \Lambda \) is the von Mangoldt function, \( * \) denotes the Dirichlet convolution of arithmetical functions. Other notations will be fixed inside the note.

E. Alkan [2] considered for \( r \in \mathbb{N} \) the weighted average

\[
S_r(k) := \frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r c_k(j),
\]

being motivated by the use of (1) in proving exact formulas for certain mean square averages of special values of \( L \)-functions. See [1]. He proved an asymptotic formula for \( \sum_{k \leq x} S_r(k) \) ([2, Th. 1]), based on the following identity.
Proposition 1. ([2, Eq. 2.19]) For every $k, r \in \mathbb{N}$,

$$S_r(k) = \frac{\varphi(k)}{2^k} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} \prod_{p | k} \left( 1 - \frac{1}{p^{2m}} \right).$$

(2)

Note that $\prod_{p | k} (1 - p^{-2m}) = J_{2m}(k) k^{-2m}$, where $J_{2m}$ is the Jordan function of order $2m$. For the proof of (2) E. Alkan used Hölder’s evaluation of the Ramanujan sums given by

$$c_k(j) = \frac{\varphi(k) \mu(k \gcd(k,j))}{\varphi(k \gcd(k,j))} \quad (k \in \mathbb{N}, j \in \mathbb{Z}),$$

applied the formula

$$n \sum_{j=1 \atop \gcd(j,n)=1}^n j^r = \frac{n^{r+1}}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} \prod_{p | n} \left( 1 - \frac{1}{p^{2m}} \right) \quad (n, r \in \mathbb{N}, n > 1)$$

(4)

(see [10, Cor. 4]), and then considered the cases $r$ even and $r$ odd, respectively. The same identity (2) and the same proof were presented by E. Alkan also in [3, Proof of Th. 1].

In this note we offer a more simple proof of (2). Furthermore, we establish identities for other weighted averages of the Ramanujan sums with weights concerning logarithms, values of arithmetic functions for gcd’s, the Gamma function, the Bernoulli polynomials and binomial coefficients. It is possible to derive similar formulas for the corresponding weighted averages of $|c_k(j)|$ and $(c_k(j))^2$, but we will not go into details. We remark that properties of the polynomials $\sum_{j=0}^{k-1} c_k(j)x^j$ were investigated by the author in [11]. We also present a multivariable generalization of the formula (2) connected to the “orbicyclic” arithmetic function, discussed by V. A. Liskovets [9] and the author [12].

2 Simple proof of Proposition 1

To derive (2) use the familiar formula (see, e.g., [6, Prop. 10.1.6], [8, Th. 271]),

$$c_k(j) = \sum_{d | \gcd(k,j)} d \mu(k/d) \quad (k \in \mathbb{N}, j \in \mathbb{Z}).$$

(5)

We obtain

$$S_r(k) = \frac{1}{kr+1} \sum_{j=1}^k j^r \sum_{d | \gcd(k,j)} d \mu(k/d) = \frac{1}{kr+1} \sum_{d | k} d^{r+1} \mu(k/d) \sum_{m=1}^{k/d} m^r$$

$$= \sum_{d | k} \mu(d) \frac{1}{d^{r+1}} \sum_{m=1}^d m^r.$$
It is well known that for every \( n, r \in \mathbb{N} \) (see, e.g., [6, Prop. 9.2.12], [7, Sect. 3.9]),

\[
\sum_{j=1}^{n} j^r = \frac{1}{r+1} \sum_{m=0}^{r} (-1)^m \binom{r+1}{m} B_m n^{r+1-m}
\]

\[
= \frac{n^r}{2} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} n^{r+1-2m}.
\]

We deduce

\[
S_r(k) = \sum_{d | k} \mu(d) \left( \frac{d^r}{2} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} d^{r+1-2m} \right)
\]

\[
= \frac{1}{2} \sum_{d | k} \mu(d) \frac{d^r}{d^{r+1}} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} \sum_{d | k} \frac{\mu(d)}{d^{2m}},
\]

giving (2) by using the elementary convolutional identities on \( \varphi(k) \) and \( J_{2m}(k) \).

Note that the original proof presented in [2] and [3], based on the application of (3) and (4) can be shortened. Using that

\[
\sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} = \frac{r+1}{2},
\]

valid for every \( r \in \mathbb{N} \cup \{0\} \) it is not necessary to split the proof into the cases \( r \) even and odd.

3 Other weighted averages

Proposition 2. For every \( k \in \mathbb{N} \),

\[
\frac{1}{k} \sum_{j=1}^{k} (\log j)c_k(j) = \Lambda(k) + \sum_{d | k} \mu(d) \frac{d}{d!} \log(d!)
\]

Proof. We obtain, using formula (5),

\[
\frac{1}{k} \sum_{j=1}^{k} (\log j)c_k(j) = \frac{1}{k} \sum_{j=1}^{k} (\log j) \sum_{d | \gcd(k,j)} d \mu(k/d)
\]

\[
= \sum_{d | k} (d/k) \mu(k/d) \sum_{m=1}^{k/d} d \log(dm) = \sum_{d | k} \frac{\mu(d)}{d} \sum_{m=1}^{d} \log(mk/d)
\]

\[
= \sum_{d | k} \mu(d) \log(k/d) + \sum_{d | k} \frac{\mu(d)}{d} \log(d!),
\]

where the first sum is \( \mu * \log = \Lambda \), in terms of the Dirichlet convolution, and the proof is complete. \( \square \)
Proposition 3. Let $f$ be an arbitrary arithmetic function. Then for every $k \in \mathbb{N}$,

$$
\sum_{j=1}^{k} f(\gcd(j, k)) c_k(j) = \varphi(k)(\mu \ast f)(k).
$$

Proof. Here it is convenient to use Hölder’s formula (3), although the proof works out also applying (5) instead. We have

$$
T_f(k) := \sum_{j=1}^{k} f(\gcd(j, k)) c_k(j) = \sum_{j=1}^{k} f(\gcd(j, k)) \frac{\varphi(k) \mu(k / \gcd(k, j))}{\varphi(k / \gcd(k, j))},
$$

and grouping the terms according to the values of $d = \gcd(k, j)$ we deduce

$$
T_f(k) = \varphi(k) \sum_{d|k} f(d) \frac{\mu(k/d)}{\varphi(k/d)} \sum_{m=1}^{k/d} \frac{1}{\gcd(m, k/d)} = \varphi(k)(\mu \ast f)(k).
$$

\[ \square \]

Corollary 1. For every $k \in \mathbb{N}$,

$$
\sum_{j=1}^{k} \gcd(j, k) c_k(j) = (\varphi(k))^2.
$$

$$
\sum_{j=1}^{k} \tau(\gcd(j, k)) c_k(j) = \varphi(k).
$$

$$
\sum_{j=1}^{k} \sigma(\gcd(j, k)) c_k(j) = k \varphi(k).
$$

Let $\Gamma$ be the Gamma function defined for $x > 0$ by

$$
\Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt.
$$

Proposition 4. For every $k \in \mathbb{N}$, $k > 1$,

$$
\frac{1}{\varphi(k)} \sum_{j=1}^{k} (\log \Gamma(j/k)) c_k(j) = \frac{1}{2} \sum_{p|k} \log p \frac{\log p}{p - 1} - \frac{\log 2\pi}{2}.
$$
Proof. It is well known that for every $n \in \mathbb{N}$,
\[
\prod_{k=1}^{n} \Gamma(k/n) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}},
\] (6)
which is a consequence of Gauss’ multiplication formula, cf., e.g., [4, Eq. (3.10)], [6, Prop. 9.6.33]. We obtain from (5) and (6) that
\[
\sum_{j=1}^{k} (\log \Gamma(j/k))c_k(j) = \sum_{j=1}^{k} (\log \Gamma(j/k)) \sum_{d \mid \gcd(k,j)} d\mu(k/d)
\]
\[
= \sum_{d \mid k} d\mu(k/d) \sum_{m=1}^{k/d} \log \Gamma(md/k) = \sum_{d \mid k} (k/d)\mu(d) \sum_{m=1}^{d} \log \Gamma(m/d)
\]
\[
= \sum_{d \mid k} (k/d)\mu(d) \log \frac{(2\pi)^{(d-1)/2}}{\sqrt{d}}
\]
\[
= \log(2\pi) \sum_{d \mid k} \frac{k}{d}\mu(d) \frac{d-1}{2} - \frac{1}{2} \sum_{d \mid k} \frac{k}{d}\mu(d) \log d
\]
\[
= \log(2\pi) \left( \frac{k}{2} \sum_{d \mid k} \mu(d) - \frac{1}{2} \sum_{d \mid k} \frac{k}{d}\mu(d) \right) - \frac{k}{2} \sum_{d \mid k} \frac{\mu(d)}{d} \log d,
\]
where the first sum is zero for $k > 1$ and the second sum is $\varphi(k)$. Now the use of the identity
\[
\sum_{d \mid k} \frac{\mu(d)}{d} \log d = -\frac{\varphi(k)}{k} \sum_{p \mid k} \log \frac{p}{p-1},
\]
(see, e.g., [5], [6, Ex. 10.8.45]) completes the proof. 

Proposition 5. For every $k \in \mathbb{N}$,
\[
\frac{1}{2k} \sum_{j=0}^{k} {k \choose j}c_k(j) = \sum_{d \mid k} \mu(k/d) \sum_{\ell=1}^{d} (-1)^{\ell k/d} \cos^k(\ell \pi/d).
\] (7)

Note the symmetry property ${k \choose j}c_k(j) = {k \choose k-j}c_k(k-j)$, valid for every $0 \leq j \leq k$.

Proof. By using (5) and the formula
\[
\sum_{k=0}^{[n/r]} {n \choose kr} = \frac{2^n}{r} \sum_{j=1}^{r} \cos^n(j \pi/r) \cos(nj \pi/r) \quad (n, r \in \mathbb{N}),
\]
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\[ \sum_{j=0}^{k} \binom{k}{j} c_k(j) = \sum_{d|k} \frac{d}{\gcd(k,j)} \sum_{d} \sum_{m=0}^{k/d} \binom{k}{d} \sum_{\ell=1}^{d} \cos^k(\ell\pi/d) \cos(k\ell\pi/d), \]

where the last factor is \((-1)^{\ell k/d}\). This gives (7). \(\square\)

Let \(B_m(x) (m \in \mathbb{N} \cup \{0\})\) be the Bernoulli polynomials defined by the expansion

\[ \frac{te^{xt}}{e^t-1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}. \]

It is well known (see, e.g., [6, Prop. 9.1.3]) that for every \(k, m \in \mathbb{N}\),

\[ \sum_{j=0}^{k-1} B_m(j/k) = \frac{B_m}{k^{m-1}}. \]

We obtain by (5), similarly as in the proofs of above the next formula.

**Proposition 6.** For every \(k, m \in \mathbb{N}\),

\[ \sum_{j=0}^{k-1} B_m(j/k) c_k(j) = \frac{B_m}{k^{m-1}} J_m(k). \]

**Remark 1.** The Ramanujan sum \(j \mapsto c_k(j)\) is the discrete Fourier transform of the function \(n \mapsto g_k(n)\) given below. By the formula of the inverse discrete Fourier transform we immediately obtain that

\[ \frac{1}{k} \sum_{j=1}^{k} \exp(2\pi ij n/k) c_k(j) = \begin{cases} 1, & \gcd(k, n) = 1 \\ 0, & \gcd(k, n) > 1 \end{cases} =: g_k(n), \]

valid for every \(k, n \in \mathbb{N}\). We refer to [13] for details.

**4 A multivariable generalization**

Let \(k_1, \ldots, k_n \in \mathbb{N} (n \in \mathbb{N})\) and let \(k := \text{lcm}(k_1, \ldots, k_n)\). The function of \(n\) variables

\[ E(k_1, \ldots, k_n) := \frac{1}{k} \sum_{j=1}^{k} c_{k_1}(j) \cdots c_{k_n}(j) \]
has combinatorial and topological applications, and was investigated in the papers of V. A Liskovets [9] and of the author [12]. Note that all values of $E(k_1, \ldots, k_n)$ are nonnegative integers. Furthermore, the function $E$ is multiplicative as a function of several variables (see [12] for this concept). Furthermore, it has the following representation ([12, Prop. 3]):

$$E(k_1, \ldots, k_n) = \sum_{d_1 | k_1, \ldots, d_n | k_n} \frac{d_1 \mu(k_1/d_1) \cdots d_n \mu(k_n/d_n)}{\text{lcm}(d_1, \ldots, d_n)}.$$

Consider now the average

$$S_r(k_1, \ldots, k_n) := \frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r c_{k_1}(j) \cdots c_{k_n}(j).$$

**Proposition 7.** Let $k_1, \ldots, k_n \in \mathbb{N}$, $k = \text{lcm}(k_1, \ldots, k_n)$ and let $r \in \mathbb{N}$. Then

$$S_r(k_1, \ldots, k_n) = \frac{\varphi(k_1) \cdots \varphi(k_n)}{2k} + \frac{1}{r+1} \sum_{m=0}^{[r/2]} \binom{r+1}{2m} B_{2m} k^{2m} g_m(k_1, \ldots, k_n),$$

where

$$g_m(k_1, \ldots, k_n) = \sum_{d_1 | k_1, \ldots, d_n | k_n} \frac{d_1 \mu(k_1/d_1) \cdots d_n \mu(k_n/d_n)}{(\text{lcm}(d_1, \ldots, d_n))^{1-2m}}$$

is a multiplicative function in $n$ variables.

**Proof.** Similar to the proof of Proposition 1 by using formula (5) for each of $c_{k_1}(j), \ldots, c_{k_n}(j)$.

For $n = 1$ Proposition 7 reduces to Proposition 1.

**Corollary 2.** ($r = 1$) For every $k_1, \ldots, k_n \in \mathbb{N}$, with $k = \text{lcm}(k_1, \ldots, k_n)$,

$$S_1(k_1, \ldots, k_n) = \frac{\varphi(k_1) \cdots \varphi(k_n)}{2k} + \frac{E(k_1, \ldots, k_n)}{2}.$$
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