Equation of state of strongly coupled Hamiltonian lattice QCD at finite density

Yasuo Umino

ECT*
Strada delle Tabarelle 286
I–38050 Villazzano (Trento), Italy
and
Instituto de Física Corpuscular – C.S.I.C.
Departamento de Física Teórica, Universitat de València
E–46100 Burjassot, València, Spain

January 24, 2019

Abstract

We calculate the equation of state of strongly coupled Hamiltonian lattice QCD at finite density by constructing a solution to the equation of motion corresponding to an effective Hamiltonian using Wilson fermions. We find that up to and beyond the chiral symmetry restoration density the pressure of the quark Fermi sea can be negative indicating its mechanical instability. This result is in qualitative agreement with continuum models and should be verifiable by future numerical simulations.

PACS numbers: 11.15.H, 12.38

Keywords: Lattice Field Theory, Strong Coupling QCD

Submitted to Physics Letters B
Simulating Quantum Chromodynamics (QCD) at finite density is one of the outstanding problems in lattice gauge theory [1]. In fact, because of the sign problem there are currently no reliable numerical simulations of finite density QCD with three colors even in the strong coupling limit [2]. This is a rather frustrating situation in view of the current intense interest in finite density QCD fueled by the phenomenology of heavy ion collisions, neutron stars, early universe and color superconductivity. Therefore even a qualitative description of finite density lattice QCD is welcome.

One method of studying finite density lattice QCD is to invoke the strong coupling approximation where analytical methods are applicable. Strongly coupled lattice QCD at finite quark chemical potential \( \mu \) and temperature \( T \) has previously been studied analytically both in the Euclidean [3, 4, 5] and in the Hamiltonian [6, 7, 8, 9] formulations. One of the main objectives of these studies was to investigate the nature of chiral symmetry restoration at finite \( T \) and/or \( \mu \). This has been accomplished by constructing some effective action or Hamiltonian for strongly coupled lattice QCD using Kogut–Susskind fermions. Except for [8] these effective descriptions involve composite meson and baryon fields which are treated in the mean field approximation.\(^1\) The consensus is that at zero or low \( T \), strongly coupled lattice QCD at finite \( \mu \) undergoes a first order chiral phase transition from the broken symmetry phase below a critical chemical potential \( \mu_C \) to a chirally symmetric phase above \( \mu_C \).

In this letter we present a calculation of the equation of state of strongly coupled lattice QCD at finite density in the Hamiltonian formulation using Wilson fermions. We find that up to and beyond the chiral symmetry restoration density the pressure of the many body system can be negative indicating its mechanical instability. This new result is in qualitative agreement with those obtained using continuum effective QCD models at finite density [10, 11] and should be verifiable by future numerical simulations.

As in previous studies on this subject we begin with an effective description of strongly coupled lattice QCD. We shall use Smit’s effective Hamiltonian [12] which involves only the quark field \( \Psi \) with a nearest neighbour interaction. This effective Hamiltonian has been studied in free space by Smit [12] and by Le Yaouanc et al. [13] who subsequently extended their analysis to finite \( T \) and \( \mu \) using Kogut–Susskind fermions [8]. A similar effective Hamiltonian has recently been derived by Gregory et al. [9] to study strongly coupled lattice QCD at finite \( \mu \), again using Kogut–Susskind fermions.

Henceforth we shall adopt the notation of Smit [12], set the lattice spacing

\(^1\)The work of [8] does not involve composite fields but the approach is equivalent to the mean field approximation.
to unity and work in momentum space. Then the charge conjugation symmetric form of Smit’s Hamiltonian using Wilson fermions may be written as

\[ H_{\text{eff}} = \frac{1}{2} M_0 (\gamma_0)_{\rho\nu} \sum_{p} \left[ (\Psi_{a\alpha}^i)_{\rho}(p) \right] \left( \Psi_{a\alpha}^\dagger \right)_{\nu}(-p) \]

\[- \frac{K}{8N_c} \sum_{\vec{p}_1, \ldots, \vec{p}_4} \sum_l \delta_{\vec{p}_1 + \ldots + \vec{p}_4, \vec{p}_l} \left[ e^{i((\vec{p}_1 + \vec{p}_2) \cdot \hat{n}_l)} + e^{i((\vec{p}_3 + \vec{p}_4) \cdot \hat{n}_l)} \right] \]

\[
\otimes \left[ (\Sigma_l)_{\rho\nu} (\Psi_{a\alpha}^i)^{\dagger}_{\rho}(\vec{p}_1)(\Psi_{b\beta}^\dagger)_{\nu}(\vec{p}_2) \right] - (\Sigma_l)_{\rho\nu}^{\dagger} \left( (\Psi_{a\alpha}^i)_{\rho}(\vec{p}_1)(\Psi_{b\beta}^\dagger)_{\nu}(\vec{p}_2) \right) \]

\[
\otimes \left[ (\Sigma_l)^{\dagger}_{\gamma\delta} (\Psi_{b\beta}^i)^{\dagger}_{\gamma}(\vec{p}_3)(\Psi_{a\alpha}^\dagger)_{\delta}(\vec{p}_4) \right] - (\Sigma_l)_{\gamma\delta} (\Psi_{b\beta}^i)(\vec{p}_3)(\Psi_{a\alpha}^\dagger)(\vec{p}_4) \right] \]

(1)

where \((\Sigma_l) = -i (\gamma_0 \gamma_l - ir \gamma_0)\) with the Wilson parameter \(r\) taking on values between 0 and 1. In the above Hamiltonian color, flavor and Dirac indices are denoted by \((a, b), (\alpha, \beta)\) and \((\rho, \nu, \gamma, \delta)\), respectively, and summation convention is implied. \(N_c\) is the number of colors.

The three parameters in \(H_{\text{eff}}\) are the Wilson parameter \(r\), the current quark mass \(M_0\) and the effective coupling constant \(K = 2N_c/(N_c^2 - 1) \frac{1}{g^2}\) where \(g\) is the QCD coupling constant. When \(r = M_0 = 0\) the Hamiltonian possesses a \(U(4N_f)\) symmetry with \(N_f\) being the number of flavors. This symmetry is spontaneously broken down to \(U(2N_f) \otimes U(2N_f)\) accompanied by the appearance of \(8N_f^2\) Goldstone bosons [12]. A finite current quark mass also breaks the original \(U(4N_f)\) symmetry, albeit explicitly, down to \((2N_f) \otimes (2N_f)\). Introduction of a finite Wilson parameter further breaks the latter symmetry explicitly down to \(U(N_f)\) thereby solving the fermion doubling problem.

The above Hamiltonian has been derived in the temporal gauge using second order degenerate perturbation theory, and provides an effective description of only the ground state of strongly coupled lattice QCD [12]. This ground state is the one in which no links are excited by the color electric flux. In the strong coupling limit the energy of one excited color electric flux link is

\[ E = \frac{1}{2N_c} \left( N_c^2 - 1 \right) g^2 = \frac{1}{K} \]

(2)

Therefore an extension of \(H_{\text{eff}}\) to finite \(T\) and/or \(\mu\) will be valid as long as \(T, \mu < 1/K\) [8] We shall see that this condition is satisfied in the present work.

**Note that in [8] E has been approximated by \(E \approx N_c g^2\).**
Our method for obtaining the equation of state of strongly coupled lattice QCD at finite density using $H_{\text{eff}}$ does not involve composite fields. Instead we explicitly construct a solution to the equation of motion corresponding to $H_{\text{eff}}$ for all densities and use it to calculate the equation of state. For free space such a solution has been found in [14]. This solution has the same structure as the free lattice Dirac field and exactly diagonalizes $H_{\text{eff}}$ to second order in field operators. It obeys the free lattice Dirac equation with a dynamical quark mass which is determined by solving a gap equation.

Temporarily dropping color and flavor indices this solution is given by

$$
\Psi_{\nu}(t, \vec{p}) = b(\vec{p})\xi_{\nu}(\vec{p})e^{-i\omega(\vec{p})t} + d^{\dagger}(-\vec{p})\eta_{\nu}(-\vec{p})e^{+i\omega(\vec{p})t}
$$

(3)

with $\nu$ denoting the Dirac index. The annihilation operators for particles $b$ and anti–particles $d$ annihilate an interacting vacuum state and obey the free fermion anti–commutation relations. The properties of the spinors $\xi$ and $\eta$ are given in [14]. The equation of motion for a free lattice Dirac field fixes the excitation energy $\omega(\vec{p})$ to be

$$
\omega(\vec{p}) = \left(\sum_l \sin^2(\vec{p} \cdot \hat{n}_l) + M^2(\vec{p})\right)^{1/2}
$$

(4)

where $M(\vec{p})$ is the dynamical quark mass.

The extension of the method developed in [14] to finite $T$ and $\mu$ is accomplished in two steps. The first one is to make the following trivial replacement of the current quark mass term in $H_{\text{eff}}$ Eq. (1)

$$
M_0(\gamma_0)_{\rho\nu} \rightarrow M_0(\gamma_0)_{\rho\nu} - \mu_0\delta_{\rho\nu}
$$

(5)

where $\mu_0$ is the quark chemical potential. Note that $\mu_0$ should not be identified with the total chemical potential $\mu_{\text{tot}}$ of the interacting many body system. As we shall see below the interaction will induce a correction to $\mu_0$ which in general is momentum dependent. We shall therefore refer to $\mu_0$ as the ”bare” quark chemical potential and treat it as a parameter.

The second step is to observe that the annihilation operators $b$ and $d$ in Eq. (3) no longer annihilate the interacting vacuum state at finite $T$ and $\mu$ denoted as $|\mathcal{G}(T, \mu)\rangle$. In order to construct operators that annihilate $|\mathcal{G}(T, \mu)\rangle$ we apply a generalized thermal Bogoliubov transformation to the $b$ and $d$ operators following the formalism of thermal field dynamics [15]

$$
b(\vec{p}) = \alpha_p B(\vec{p}) - \beta_p \tilde{B}^\dagger(-\vec{p})
$$

(6)

$$
d(\vec{p}) = \gamma_p D(\vec{p}) - \delta_p \tilde{D}^\dagger(-\vec{p})
$$

(7)
The thermal field operators $B$ and $\tilde{B}^\dagger$ annihilate a quasi–particle and create a quasi–hole at finite $T$ and $\mu$, respectively, while $D$ and $\tilde{D}^\dagger$ are the annihilation operator for a quasi–anti–particle and creation operator for a quasi–anti–hole, respectively.

These thermal annihilation operators annihilate the interacting thermal vacuum state for each $T$ and $\mu$.

$$B(\vec{p})|G(T, \mu)\rangle = \tilde{B}(\vec{p})|G(T, \mu)\rangle = D(\vec{p})|G(T, \mu)\rangle = \tilde{D}(\vec{p})|G(T, \mu)\rangle = 0 \quad (8)$$

The thermal doubling of the Hilbert space accompanying the thermal Bogoliubov transformation is implicit in Eq. (8) where the vacuum state which is annihilated by thermal operators $B$, $\tilde{B}$, $D$ and $\tilde{D}$ is defined. Since we shall be working only in the space of quantum field operators it is not necessary to specify the structure of $|G(T, \mu)\rangle$.

The thermal operators also satisfy the fermion anti–commutation relations

$$\delta_{\vec{p}, \vec{q}} = \left[ B^\dagger(\vec{p}), B(\vec{q}) \right]_+ = \left[ \tilde{B}^\dagger(\vec{p}), \tilde{B}(\vec{q}) \right]_+$$
$$= \left[ D^\dagger(\vec{p}), D(\vec{q}) \right]_+ = \left[ \tilde{D}^\dagger(\vec{p}), \tilde{D}(\vec{q}) \right]_+ \quad (9)$$

with vanishing anti–commutators for the remaining combinations. The coefficients of the transformation are $\alpha_p = \sqrt{1 - n_p}$, $\beta_p = \sqrt{n_p}$, $\gamma_p = \sqrt{1 - n_p}$ and $\delta_p = \sqrt{n_p}$, where $n_p^\pm = [e^{(\omega_p^\pm + \mu)/(k_B T)} + 1]^{-1}$ are the Fermi distribution functions for particles and anti–particles. They are chosen so that the total particle number densities are given by

$$n_p^- = \langle G(T, \mu) | b^\dagger(\vec{p}) b(\vec{p}) | G(T, \mu) \rangle \quad (10)$$
$$n_p^+ = \langle G(T, \mu) | d^\dagger(\vec{p}) d(\vec{p}) | G(T, \mu) \rangle \quad (11)$$

Hence in this approach temperature and chemical potential are introduced simultaneously through the coefficients of the thermal Bogoliubov transformation and are treated on an equal footing. We stress that the chemical potential appearing in the Fermi distribution functions is the total chemical potential of the interacting many body system.

In addition to these changes, we demand that our ansatz satisfies the equation of motion corresponding to the free lattice Dirac Hamiltonian with a chemical potential term given by

$$H^0 = \frac{1}{2} \sum_{\vec{p}} \left[ - \sum_l \sin(\vec{p} \cdot \hat{n}_l)(\gamma_0 \gamma_l)_{\rho \nu} + M(\vec{p})(\gamma_0)_{\rho \nu} - \mu_{\text{tot}} \delta_{\rho \nu} \right] \bigotimes \left[ \Psi_{\rho}^\dagger(t, \vec{p}), \Psi_{\nu}(t, \vec{p}) \right] \quad (12)$$
As in [14] the mass $M(\vec{p})$ is identified with the dynamical quark mass. Thus our ansatz at finite $T$ and $\mu$ is

$$\Psi_\nu(t, \vec{p}) = \left[ \alpha_p B(\vec{p}) - \beta_p \tilde{B}^\dagger(-\vec{p}) \right] \xi_\nu(\vec{p}) e^{-i[\omega(\vec{p}) - \mu_{\text{tot}}]t} + \left[ \gamma_p D^\dagger(-\vec{p}) - \delta_p \tilde{D}(\vec{p}) \right] \eta_\nu(\vec{p}) e^{+i[\omega(\vec{p}) + \mu_{\text{tot}}]t}$$

(13)

The spinors $\xi$ and $\eta$ obey the same properties as in free space and the excitation energy $\omega(\vec{p})$ has the same form as in Eq. (4). The unknown quantities in Eq. (13) are the dynamical quark mass and the total chemical potential.

In this work we shall take the $T \to 0$ limit which amounts to setting $\gamma_p = 1$ and $\delta_p = 0$ in Eq. (7) thereby suppressing the excitation of anti–holes. In this limit $\beta_p^2$ becomes the Heaviside function $\beta_p^2 = \theta(\mu_{\text{tot}} - \omega(\vec{p}))$ defining the Fermi momentum $\vec{p}_F$ where

$$\mu_{\text{tot}} = \left( \sum l \sin^2(\vec{p}_F \cdot \hat{n}_l) + M^2(\vec{p}_F) \right)^{1/2}$$

(14)

One of the simplest quantities to calculate using the ansatz of Eq. (13) in the $T \to 0$ limit is the quark number density $n$ given by

$$n = \frac{1}{2V N_f N_c} \langle \bar{\Psi} \gamma_0 \Psi \rangle = \sum_{\vec{p}} \theta(\mu_{\text{tot}} - \omega(\vec{p}))$$

(15)

Therefore, above a sufficiently large value of $\mu_{\text{tot}}$ the quark number density becomes a constant which with the present normalization will equal unity. This saturation effect is purely a lattice artifact originating from the $\sin^2(\vec{p} \cdot \hat{n}_l)$ term in $\omega(\vec{p})$.

Another quantity that may be readily calculated using the $T \to 0$ ansatz is the chiral condensate. It is found to be proportional to the dynamical quark mass

$$\frac{1}{2V N_f N_c} \langle \bar{\Psi} \Psi \rangle = -\sum_{\vec{p}} \alpha_p^2 \frac{M(\vec{p})}{\omega(\vec{p})}$$

(16)

Below we shall derive a gap equation for $M(\vec{p})$ and show that for a given physically reasonable set of parameters there exists a critical chemical potential above which $M(\vec{p}) = 0$. Thus the chiral condensate may be identified as being the order parameter for the chiral phase transition at finite density.

However before deriving the gap equation we shall demonstrate that in the $T \to 0$ limit the ansatz shown in Eq. (13) exactly diagonalizes the effective Hamiltonian to second order in field operators for all densities. We make use of the fact that our ansatz satisfies the equation of motion corresponding to
the free lattice Dirac Hamiltonian $H^0$ given in Eq. (12). Therefore we have the relation

$$: \left[ (\Psi_{\alpha\alpha})_\mu(t, \vec{q}), H^0 \right] : = : \left[ (\Psi_{\alpha\alpha})_\mu(t, \vec{q}), H_{\text{eff}} \right] :$$

(17)

where the symbol $: :$ denotes normal ordering with respect to the vacuum at zero temperature $| G(T = 0, \mu) \rangle$. Evaluating both sides of Eq. (17) we obtain

$$\left[ \sum_l \sin(\vec{q} \cdot \hat{n}_l)(\gamma_0 \gamma_l)_{\rho\delta} + M(\vec{q})(\gamma_0)_{\rho\delta} - \mu_{\text{tot}} \delta_{\rho\delta} \right] (\Psi_{\alpha\alpha})_\delta(t, \vec{q}) =$$

$$\begin{cases} M_0 (\gamma_0)_{\rho\delta} - \mu_0 \delta_{\rho\delta} \\ + \frac{1}{N_c} K \sum_p \sum_l \alpha_p^2 \Lambda_{\nu\gamma}^+(\vec{p}) \\
\otimes \left[ \cos (\vec{p} - \vec{q}) \cdot \hat{n}_l \left( (\Sigma_l)_{\gamma\nu} (\Sigma_l)_{\rho\delta}^\dagger + (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta} \right) \\
+ \cos (\vec{p} + \vec{q}) \cdot \hat{n}_l \left( (\Sigma_l)_{\gamma\nu}^\dagger (\Sigma_l)_{\rho\delta}^\dagger + (\Sigma_l)_{\mu\nu} (\Sigma_l)_{\gamma\delta} \right) \right] \\
- \frac{1}{N_c} K \sum_{\vec{p},\vec{q}} \sum_l \left[ 2 \alpha_p^2 \Lambda_{\nu\gamma}^+(\vec{p}) - \delta_{\nu\gamma} \right] \\
\otimes \left[ N_c \left( (\Sigma_l)_{\rho\nu} (\Sigma_l)_{\gamma\delta}^\dagger + (\Sigma_l)_{\rho\nu} (\Sigma_l)_{\gamma\delta} \right) \\
+ \cos (\vec{p} + \vec{q}) \cdot \hat{n}_l \left( (\Sigma_l)_{\rho\nu}^\dagger (\Sigma_l)_{\gamma\delta}^\dagger + (\Sigma_l)_{\rho\nu} (\Sigma_l)_{\gamma\delta} \right) \right] \right) (\Psi_{\alpha\alpha})_\delta(t, \vec{q})
\end{cases}$$

(18)

with $\Lambda^+(\vec{p}) \equiv \xi(\vec{p}) \otimes \xi^\dagger(\vec{p})$ being the positive energy projection operator defined in [14].

To second order in field operators the off–diagonal Hamiltonian is given by

$$H_{\text{off}} | G(0, \mu) \rangle = \sum_{\vec{q}} \left\{ \alpha_q \xi_{\rho\nu}^\dagger(\vec{q}) \left[ M_0 (\gamma_0)_{\rho\delta} - \mu_0 \delta_{\rho\delta} \right] \\
+ \frac{1}{N_c} K \sum_{\vec{p},\vec{q}} \sum_l \alpha_p^2 \alpha_q \Lambda_{\nu\rho\nu\rho}^+(\vec{p}) \\
\otimes \xi_{\rho\nu}^\dagger(\vec{q}) \left[ \cos (\vec{p} - \vec{q}) \cdot \hat{n}_l \left( (\Sigma_l)_{\rho\nu} (\Sigma_l)_{\gamma\delta}^\dagger + (\Sigma_l)_{\rho\nu} (\Sigma_l)_{\gamma\delta} \right) \\
+ \cos (\vec{p} + \vec{q}) \cdot \hat{n}_l \left( (\Sigma_l)_{\rho\nu}^\dagger (\Sigma_l)_{\gamma\delta}^\dagger + (\Sigma_l)_{\rho\nu} (\Sigma_l)_{\gamma\delta} \right) \right] \right\}$$
\[-\frac{1}{N_c} K \sum_{\vec{p}, \vec{q}} \sum_l \alpha_q \left[ 2\alpha'_q \Lambda^+_{\nu\gamma}(\vec{p}) - \delta_{\nu\gamma} \right] \]
\[\otimes \xi^\dagger_{\nu}(\vec{q}) \left[ N_c \left( (\Sigma_l)_{\rho\nu}(\Sigma_l)^{\dagger}_{\gamma\delta} + (\Sigma_l)^{\dagger}_{\rho\nu}(\Sigma_l)_{\gamma\delta} \right) \right. \]
\[+ \cos(\vec{p} + \vec{q}) \cdot \hat{n}_l \left( (\Sigma_l)_{\rho\nu}(\Sigma_l)^{\dagger}_{\gamma\delta} + (\Sigma_l)^{\dagger}_{\rho\nu}(\Sigma_l)_{\gamma\delta} \right) \left\{ \eta_{\delta}(\vec{q}) \right. \]
\[\otimes B^\dagger_{\alpha,a}(\vec{q}) D^\dagger_{\alpha,a}(\vec{q}) | G(0, \mu) \rangle \] (19)

From Eq. (19) we see that the elementary excitations of the effective Hamiltonian are color singlet (quasi) quark–anti–quark pairs coupled to zero total three momentum. With the use of Eq. (18), the equation of motion for the \( \eta \) spinor and the orthonormality condition \( \xi^\dagger_{\nu}(\vec{p}) \eta_{\nu}(\vec{q}) = 0 \) [14] we can show that

\[H_{\text{off}} | G(0, \mu) \rangle = \sum_{\vec{q}} \left\{ \alpha_q \xi^\dagger_{\nu}(\vec{q}) \left[ - \sum_l \sin(\vec{q} \cdot \hat{n}_l)(\gamma_0\gamma_l)_{\nu\delta} \right. \right. \]
\[-M(\vec{q})(\gamma_0)_{\nu\delta} + \mu_{\text{tot}} \delta_{\nu\delta} \left] \eta_{\delta}(\vec{q}) \right\} \]
\[\otimes B^\dagger_{\alpha,a}(\vec{q}) D^\dagger_{\alpha,a}(\vec{q}) | G(0, \mu) \rangle \]
\[= \sum_{\vec{q}} \left\{ \alpha_q \xi^\dagger_{\nu}(\vec{q}) \left[ \omega(\vec{q}) + \mu_{\text{tot}} \right] \eta_{\nu}(\vec{q}) \right\} \]
\[\otimes B^\dagger_{\alpha,a}(\vec{q}) D^\dagger_{\alpha,a}(\vec{q}) | G(0, \mu) \rangle \]
\[= 0 \] (20)

Therefore our ansatz exactly diagonalizes the effective Hamiltonian to second order in field operators for all densities.

We now derive the equations for the dynamical quark mass and the total chemical potential and solve them to determine our solution Eq. (13) for each density. To accomplish this we explicitly evaluate the right hand side of Eq. (18) to reveal its Dirac structure. The result may be cast in the following compact form

\[\left( \sum_l \sin(\vec{q} \cdot \hat{n}_l)(\gamma_0\gamma_l)_{\nu\delta} + M(\vec{q})(\gamma_0)_{\nu\delta} - \mu_{\text{tot}} \delta_{\nu\delta} \right)(\Psi_{aa})_{\delta}(t, \vec{q}) = \]
\[A(\vec{q})(\gamma_0\gamma_l)_{\nu\delta} + B(\vec{q})(\gamma_0)_{\nu\delta} + C(\vec{q})\delta_{\nu\delta} \right)(\Psi_{aa})_{\delta}(t, \vec{q}) \] (21)

The equations for \( M(\vec{p}) \) and \( \mu_{\text{tot}} \) are obtained by equating the coefficients of the \( \gamma_0 \) operator and the Kronecker delta function, respectively.
The gap equation determining $M(\vec{p})$ is given by the coefficient $B(\vec{q})$

\[
M(\vec{q}) = B(\vec{q}) = M_0 + \frac{3}{2}K(1 - r^2) \sum_{\vec{p}} \left(1 - \beta_p^2\right) \frac{M(\vec{p})}{\omega(\vec{p})} \\
+ \frac{K}{N_c} \sum_{\vec{p},l} \left(1 - \beta_p^2\right) \frac{M(\vec{p})}{\omega(\vec{p})} \left\{ 8r^2 \cos(\vec{p} \cdot \hat{n}_l) \cos(\vec{q} \cdot \hat{n}_l) \\
- \frac{1}{2}(1 + r^2) \cos(\vec{p} + \vec{q}) \cdot \hat{n}_l \right\}
\]  

(22)

The structure of this gap equation is very similar to the one in free space ($\beta_p^2 = 0$) found in [14]. The dynamical quark mass is a constant to lowest order in $N_c$ but becomes momentum dependent once $1/N_c$ correction is taken into account.

Similarly, the total chemical potential is given by the coefficient $C(\vec{q})$

\[
\mu_{\text{tot}} = -C(\vec{q}) = \mu_0 + \frac{1}{4}K \sum_{l} \beta_p^2 \left[ 2N_c \left(1 + r^2 \right) - 2 \left(1 - r^2 \right) \cos(\vec{p} + \vec{q}) \right]
\]  

(23)

Thus $\mu_{\text{tot}}$ is a sum of the bare chemical potential $\mu_0$ and an interaction induced chemical potential which is proportional to the effective coupling constant $K$. Furthermore, the latter contribution to $\mu_{\text{tot}}$ is momentum dependent and this dependence is a $1/N_c$ correction just as in the case of the gap equation. It should be noted that the above shifting of the bare chemical potential by the interaction is not a new effect. For example, in the well–known and well–studied Nambu–Jona–Lasinio model [14] at finite $T$ and $\mu$ the interaction induces a contribution to the total chemical potential which is proportional to the number density [17, 18].

The two equations (22) and (23) are coupled and therefore solutions for $M$ and $\mu_{\text{tot}}$ must be found self–consistently for each value of the input parameter $\mu_0$. In Figure 1 we present $M$ as a function of $\mu_{\text{tot}}$ for two values of $K$ determined by solving Eqs. (22) and (23) to $O(N_c^0)$, which is the same order in the $1/N_c$ expansion used to obtain results in all previous studies of strongly coupled lattice QCD. At this order in $N_c$ both the dynamical mass and the total chemical potential are momentum independent. The values of input parameters are $M_0 = 0$, $r = 0.25$ and $N_c = 3$.

From the figure we see that the chiral phase transition can be either first or second order depending on the value of the effective coupling constant. When $K = 0.9$ we find a second order phase transition with a critical chemical
Figure 1: Dynamical quark mass $M$ as a function of total chemical potential $\mu_{\text{tot}}$ for two values of effective coupling constant $K$. These results were obtained by solving Eqs. (22) and (23) self-consistently to lowest order in $N_c$ using $M_0 = 0$, $r = 0.25$ and $N_c = 3$. There is a second order chiral phase transition when the effective coupling constant $K$ is 0.9. The order of the phase transition changes to first order when $K$ is increased to 1.0.

Potential of $(\mu_{\text{tot}})_C \approx 0.716$, while if the coupling constant is increased to $K = 1.0$ the phase transition becomes first order with a larger critical chemical potential of $(\mu_{\text{tot}})_C \approx 0.871$. Furthermore, we find that when $K = 0.9$ lattice saturation sets in around $\mu_{\text{tot}} \approx 0.898$ while this effect takes place immediately above $(\mu_{\text{tot}})_C$ for $K = 1.0$. These values of chemical potentials are smaller than the energy $E = 1/K$ required to excite one color electric flux link as given in Eq. (2). Therefore with a reasonable set of parameters it is possible to extend Smit’s effective Hamiltonian to finite density as was first pointed out in [8].

Having solved for the dynamical quark mass and the total chemical potential we have constructed a solution to the equation of motion for $H_{\text{eff}}$ in the $T \to 0$ limit to lowest order in $N_c$. In addition we have shown that this solution exactly diagonalizes the effective Hamiltonian to second order in field operators for all densities. Therefore we may use it to evaluate the vacuum energy density which to lowest order in $N_c$ is given by

$$\frac{1}{V} \langle \mathcal{G}(0, \mu) | H_{\text{eff}} | \mathcal{G}(0, \mu) \rangle =$$

$$-2N_c \sum_{\vec{p}} \left\{ \alpha^2_p \left[ \frac{3}{2} K(1 + r^2) + \omega(\vec{p}) + \frac{M}{\omega(\vec{p})} M_0 - \mu_{\text{tot}} \right] + (1 + \beta^2_p) \mu_0 \right\} \quad (24)$$
Numerically we find that the difference of the vacuum energy densities in the symmetry restored ($M = 0$) and broken ($M \neq 0$) phases of the theory is positive

$$\frac{1}{V} \langle G(0, \mu) | H_{\text{eff}} | G(0, \mu) \rangle |_{M=0} - \frac{1}{V} \langle G(0, \mu) | H_{\text{eff}} | G(0, \mu) \rangle |_{M \neq 0} > 0$$

(25)

Therefore the phase with broken chiral symmetry is the energetically favored phase.

The equation of state is obtained by numerically evaluating the thermodynamic potential density using Eq. (24). In Figure 2 we plot the pressure as a function of $\mu_{\text{tot}}$ for $K = 0.9$ and $1.0$ with $M_0 = 0$, $r = 0.25$ and $N_c = 3$. In both cases we find that the pressure of the quark Fermi sea is negative and monotonically decreasing in the broken symmetry phase. For $K = 0.9$ the pressure remains negative but increasing in the symmetry restored phase, at least until the lattice saturation point, and has a cusp where the two phases meet. Unfortunately, we cannot make a definite quantitative statement on the behaviour of the pressure in the symmetry restored phase for $K = 1.0$ due to lattice saturation, except to mention that there is a discontinuity when going from one phase to another. However, we may conclude that up to and beyond the chiral symmetry restoration point the quark Fermi sea can have negative pressure and therefore can be mechanically unstable with an imaginary speed of sound.

Our conclusion regarding the (strongly coupled) quark matter stability at finite density is consistent with similar studies using the Nambu–Jona–Lasinio model \cite{10} and the effective instanton induced 't Hooft interaction model \cite{11}. These mean field calculations show that cold and dense quark matter may be unstable in the phase with spontaneously broken chiral symmetry, but can become stable in the symmetry restored phase at high enough density. In particular, the result for the pressure of cold and dense quark matter obtained in \cite{11} is qualitatively the same as the one shown in Figure 2. The possibility of unstable quark matter lead the authors of \cite{10} and \cite{11} to speculate the formation of nucleon droplets, reminiscent of the MIT bag model, in the broken symmetry phase. We shall not dwell on such a speculation here since we are working in an artificial strong coupling regime. Nevertheless, it would be interesting to compare our result concerning the negative pressure with future lattice simulations of finite density QCD at strong coupling.

In this work we studied the equation of state of strongly coupled lattice QCD in the Hamiltonian formulation using Wilson fermions. This was accomplished by constructing a solution of the equation of motion corresponding

\cite{3}Compare Figure 1 of \cite{11} with Figure 2 of this letter.
to an effective Hamiltonian which exactly diagonalizes the Hamiltonian to second order in field operators for all densities. We found that: the dynamical quark mass is in general momentum dependent; the interaction induces a momentum dependent contribution to the total chemical potential making it necessary to solve for the dynamical quark mass self-consistently with $\mu_{\text{tot}}$; the elementary excitations of the theory consist of color singlet quark–anti–quark pairs coupled to zero total three momentum; and the broken symmetry phase is the energetically favored phase.

To leading order in $N_c$ we find that the chiral phase transition can be either first or second order depending on the value of the effective coupling constant. In addition, the pressure of the strongly interacting many body system is found to be negative up to and beyond the chiral phase transition density. A similar behaviour for the pressure has been obtained with $r = 0$ which corresponds to using Kogut–Susskind fermions. Therefore our result concerning the negative pressure seems to be robust, at least to leading order in $N_c$, and should be verifiable by future numerical simulations of strongly coupled lattice QCD at finite density.

Figure 2: Pressure $P$ as a function total chemical potential $\mu_{\text{tot}}$ for two values of effective coupling constant $K$. The results were obtained using the same parameters as in Figure 1. For $K = 0.9$, the critical chemical potential is $(\mu_{\text{tot}})_C \approx 0.716$ and lattice saturation sets in around $\mu_{\text{tot}} \approx 0.898$, while this effect takes place right above $(\mu_{\text{tot}})_C \approx 0.871$ for $K = 1.0$. 
Acknowledgements

I am indebted to M.–P. Lombardo for comments and suggestions which lead to an improvement of this manuscript, as well as to O.W. Greenberg for reading the first draft. Part of this work was completed while I was at ECT* as a Junior Visiting Scientist. I thank the Center for its hospitality and generous support.

References

[1] M. Creutz, Nucl. Phys. Proc. Suppl. 94 (2001) 219.
[2] R. Aloisio, V. Azcoiti, G. Di Carlo, A. Galante and A.F. Grillo, Nucl. Phys. B 564 (2000) 489.
[3] P.H. Damgaard, D. Hochberg, N. Kawamoto, Phys. Letts. B 158 (1985) 239.
[4] E.–M. Ilgenfritz and J. Kripfganz, Z. Phys. C 29 (1985) 79.
[5] N. Bilić, K. Demeterfi and B. Petersson, Nucl. Phys. B 377 (1992) 651.
[6] A. Patel, Phys. Letts. B 141 (1984) 244.
[7] C.P. Van Den Doel, Phys. Letts. B 143 (1984) 210.
[8] A. Le Yaouanc, L. Oliver, O. Pêne and J.–C. Raynal, M. Jarfi and O. Lazrak, Phys Rev D 37 (1986) 3691; 3702.
[9] E.B. Gregory, S.–H. Guo, H. Kröger and X.–Q. Luo, Phys. Rev. D 62 (2000) 054508.
[10] M. Buballa, Nucl. Phys. A 611 (1996) 393.
[11] M. Alford, K. Rajagopal and F. Wilczek, Phys. Letts. B 422 (1998) 247.
[12] J. Smit, Nucl. Phys. B 175 (1980) 307.
[13] A. Le Yaouanc, L. Oliver, O. Pêne and J.–C. Raynal, Phys. Rev D 33 (1986) 3098.
[14] Y. Umino, Phys. Letts. B 492 (2000) 385; The corrected version of Figure 1b is available at Los Alamos Archives arXiv:hep-lat/0007356 v.2.

[15] H. Umezawa, H. Matsumoto and M. Tachiki, Thermo Field Dynamics and Condensed States, (North–Holland, Amsterdam, 1982); H. Umezawa, Advanced Field Theory, (AIP, New York, 1992).

[16] Y. Nambu and G. Jona–Lasinio, Phys. Rev. 122 (1961) 345; 124 (1961) 246.

[17] M. Asakawa and K. Yazaki, Nucl. Phys. A 504 (1989) 668.

[18] S.P. Klevansky, Rev. Mod. Phys. 64 (1992) 649.