FREDHOLM CONDITIONS AND INDEX FOR RESTRICTIONS
OF INVARIANT PSEUDODIFFERENTIAL OPERATORS TO
ISOTYPICAL COMPONENTS

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Abstract. Let $\Gamma$ be a finite group acting on a smooth, compact manifold $M$, let $P \in \psi^m(M; E_0, E_1)$ be a $\Gamma$-invariant, classical pseudodifferential operator acting between sections of two equivariant vector bundles $E_i \to M$, $i = 0, 1$, and let $\alpha$ be an irreducible representation of the group $\Gamma$. Then $P$ induces a map $\pi_\alpha(P) : H^s(M; E_0) \to H^{s+m}(M; E_1)$ between the $\alpha$-isotypical components of the corresponding Sobolev spaces of sections. We prove that the map $\pi_\alpha(P)$ is Fredholm if, and only if, $P$ is $\alpha$-elliptic, an explicit condition that we define in terms of the principal symbol of $P$ and the action of $\Gamma$ on the vector bundles $E_i$. The result is not true for non-discrete groups. In the process, we also obtain several results on the structure of the algebra of invariant pseudodifferential operators on $E_0 \oplus E_1$, especially in relation to induced representations. We include applications to Hodge theory and to index theory of singular quotient spaces.

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Fredholm operators have been extensively studied and appear in many questions in Mathematical Physics, in Partial Differential Equations (linear and non-linear), in Algebraic and Differential Geometry, in Index Theory, and in other areas. On a compact manifold, a classical pseudodifferential operator is Fredholm between suitable Sobolev spaces if, and only if, it is elliptic. In this paper, we obtain an analogous result for the restrictions to isotypical components of a classical pseudodifferential operator $P$ invariant with respect to the action of a finite group $\Gamma$. Namely, the restriction of $P$ to the isotypical component corresponding to an irreducible representation $\alpha$ of $\Gamma$ is Fredholm if, and only if, the operator is $\alpha$-elliptic (Definition 1.1 and Theorem 1.2).

Let us now formulate and explain this result in more detail.

1.1. The setting and general notation. We shall work essentially in the same setting as the one considered in [11], but for a general finite group $\Gamma$. Thus, throughout this paper, $\Gamma$ will be a finite group acting by diffeomorphisms on a smooth Riemannian manifold $M$. As our main result is only valid for a compact manifold, we assume for this introduction that $M$ is compact. Certain intermediate results hold, however, also for open manifolds and non-discrete groups, see [11]. For the main result (Theorem 1.2), we do need $\Gamma$ to be discrete and finite, so our main result is optimal. Again, see [11]. There is no loss of generality to assume that $M$ is endowed with an invariant Riemannian metric, so we will assume that this is the case also.

As usual, $\hat{\Gamma}$ denotes the finite set of equivalence classes of irreducible $\Gamma$-modules (or representations). Let $T : V_0 \to V_1$ be a $\Gamma$-equivariant linear map of $\Gamma$-modules and $\alpha \in \hat{\Gamma}$. Then $T$ induces by restriction a $\Gamma$-equivariant linear map

$$\pi_\alpha(T) : V_{0\alpha} \to V_{1\alpha},$$

between the $\alpha$-isotypical components of the $\Gamma$-modules $V_i, i = 0, 1$.

We are mostly interested in this restriction morphism $\pi_\alpha$ in the following case. Let $P \in \psi^m(M; E_0, E_1)$ be a classical, $\Gamma$-invariant pseudodifferential operator acting between sections of two $\Gamma$-equivariant vector bundles $E_i \to M, i = 0, 1$. Then we obtain the operator

$$\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha,$$

which acts between the $\alpha$-isotypical components of the corresponding Sobolev spaces of sections. Our main result concerns this operator $\pi_\alpha(P)$. For simplicity, we will consider only classical pseudodifferential operators in this article [43, 44, 76, 80].
1.2. The \( \alpha \)-principal symbol and \( \alpha \)-ellipticity. To put our result into the right perspective, recall that a classical, order \( m \), pseudodifferential operator \( P \) is called elliptic if its principal symbol

\[
\sigma_m(P) \in C^\infty(T^*M \setminus \{0\}; \text{Hom}(E_0, E_1)),
\]

is invertible. Also, recall that a linear operator \( T : X_0 \to X_1 \) acting between Banach spaces is Fredholm if, and only if, the vector spaces

\[
\ker(T) := T^{-1}(0) \quad \text{and} \quad \text{coker}(T) := X_1/TX_0
\]

are (both) finite dimensional. Since \( M \) is compact, a very well known and widely used result states that \( P : H^s(M; E_0) \to H^{s-m}(M; E_1) \) is Fredholm if, and only if, \( P \) is elliptic \([38, 39, 57, 58, 72]\). Consequently, if \( P \) is elliptic, then \( \pi_\alpha(P) \) is also Fredholm. The converse is not true, however, in general.

To state our main result characterizing the Fredholm property of \( \pi_\alpha(P) \) in terms of the “\( \alpha \)-principal symbol” \( \sigma_m^\alpha(P) \) of \( P \), Theorem 1.2, we shall need to introduce \( \sigma_m^\Gamma(P) \), the “\( \Gamma \)-equivariant principal symbol” of \( P \), which is a refinement of the principal symbol \( \sigma_m(P) \) of \( P \) that takes into account the action of the group \( \Gamma \). The \( \alpha \)-principal symbol \( \sigma_m^\alpha(P) \) of \( P \) is a suitable restriction of the \( \Gamma \)-equivariant principal symbol \( \sigma_m^\Gamma(P) \). Let us formulate now the precise definition of these concepts.

The main question that we answer in this paper is to determine when the induced operator \( \pi_\alpha(P) \) of Equation (2) is Fredholm in terms of its \( \Gamma \)-equivariant principal symbol \( \sigma_m^\Gamma(P) \), see Theorem 1.2 below for the precise statement.

The \( \Gamma \)-invariance of \( P \) implies that its principal symbol is also \( \Gamma \) invariant:

\[
\sigma_m(P) \in C^\infty(T^*M \setminus \{0\}; \text{Hom}(E_0, E_1))^\Gamma.
\]

Let \( \Gamma_x := \{ \gamma \in \Gamma \mid \gamma \xi = \xi \} \) denote the isotropy of a \( \xi \in T^*_x M, x \in M \), as usual. The isotropy \( \Gamma_x \) of \( x \in M \) is defined similarly. Then \( \Gamma_x \subset \Gamma_x \) acts on \( E_0x \) and on \( E_1x \), the fibers of \( E_0, E_1 \to M \) at \( x \). If \( Q \in C^\infty(T^*M \setminus \{0\}; \text{Hom}(E_0, E_1))^\Gamma \), then \( Q(\xi) \in \text{Hom}(E_0x, E_1x)^\Gamma \). Let \( \rho \in \hat{\Gamma}_x \) be an irreducible representation of \( \Gamma_x \), then

\[
\tilde{Q}(\xi, \rho) := \pi_{\rho} [Q(\xi)] \in \text{Hom}(E_0x, E_1x)^{\Gamma_x}
\]

denotes the restriction of \( Q \) to the isotypical component corresponding to \( \rho \), with \( \pi_{\rho} \) defined in Equation (1). Let

\[
X_{M, \Gamma} := \{ (\xi, \rho) \mid \xi \in T^*M \setminus \{0\} \text{ and } \rho \in \hat{\Gamma}_x \}.
\]

Thus \( Q \) defines a function on \( X_{M, \Gamma} \). Applying this construction to \( \sigma_m(P) \in C^\infty(T^*M \setminus \{0\}; \text{Hom}(E_0, E_1))^\Gamma \) we obtain a function, the \( \Gamma \)-principal symbol

\[
\sigma_m^\Gamma(P) : X_{M, \Gamma} \to \bigcup_{(x, \rho) \in X_{M, \Gamma}} \text{Hom}(E_0x, E_1x)^{\Gamma_x},
\]

\[
\sigma_m^\Gamma(P)(\xi, \rho) := \pi_{\rho} [\sigma_m(P)(\xi)] \in \text{Hom}(E_0x, E_1x)^{\Gamma_x}, \quad \xi \in T^*_x M.
\]

That is \( \sigma_m^\Gamma(P) := \sigma_m(\alpha(P)) \).

The \( \alpha \)-principal symbol \( \sigma_m^\alpha(P) \) of \( P, \alpha \in \hat{\Gamma} \), is defined in terms of \( \sigma_m^\Gamma(P) \), but we need a crucial additional ingredient that takes \( \alpha \) into account.

Recall that \( \Gamma_{g\xi} = g\Gamma_{\xi}g^{-1} \) and that this defines an action of \( \Gamma \) on the set of stabilizer subgroups \( \text{Stab}_P(T^*M) := \{ \Gamma_x \mid \xi \in T^*M \} \) given by \( g \cdot \Gamma_x = \Gamma_{g\xi} \). For \( \rho \in \hat{\Gamma}_x \) define \( g \cdot \rho \in \hat{\Gamma}_x \) by \( (g \cdot \rho)(h) = \rho(g^{-1}h) \), for all \( h \in \Gamma_{g\xi} \). Let \( \Gamma_0 \subset \Gamma \) be a minimal element (for inclusion) among the isotropy groups \( \Gamma_x \) of elements \( x \in M \). Such a minimal element exists trivially, since \( \Gamma \) is finite. Moreover, if \( M/\Gamma \)
is connected, then \( \Gamma_0 \) is unique up to conjugacy (see Subsection 2.3.3). We assume for a moment that this is the case, that is, that \( M/\Gamma \) is connected. Then we let

\[
X^\alpha_{M,\Gamma} := \{ (\zeta, \rho) \in X_{M,\Gamma} \mid \exists g \in \Gamma, \text{ Hom}_{\Gamma_0}(g \cdot \rho, \alpha) \neq 0 \}.
\]

(Note that it is implicit in the definition of \( X^\alpha_{M,\Gamma} \) that \( \Gamma_0 \subset \Gamma^\alpha = g \cdot \Gamma \).) In general (if \( M/\Gamma \) is not connected), we define \( X^\alpha_{M,\Gamma} \) by taking the disjoint union of the corresponding spaces for each connected component of \( M/\Gamma \), see Subsection 4.1.

**Definition 1.1.** The \( \alpha \)-principal symbol \( \sigma^\alpha_m(P) \) of \( P \) is the restriction of the \( \Gamma \)-principal symbol \( \sigma^\Gamma_m(P) \) to \( X^\alpha_{M,\Gamma} \):

\[
\sigma^\alpha_m(P) := \sigma^\Gamma_m(P)|_{X^\alpha_{M,\Gamma}}.
\]

We shall say that \( P \in \psi^m(M; E_0, E_1)^\Gamma \) is \( \alpha \)-elliptic if its \( \alpha \)-principal symbol \( \sigma^\alpha_m(P) \) is invertible everywhere on its domain of definition. Note that when \( (\xi, \rho) \in X^\alpha_{M,\Gamma} \)

is such that \( E_{xp} = 0 \), then \( \sigma^\Gamma_m(P)(\xi, \rho) : 0 \to 0 \) is always invertible.

1.3. **Statement of the main result.** An alternative formulation of Definition 1.1 is that \( P \) is \( \alpha \)-elliptic if, and only if, \( \sigma^\Gamma_m \) is invertible on \( X^\alpha_{M,\Gamma} \) (this is, of course, a condition only for those \( \rho \) such that \( E_{ip} \neq 0 \), because, otherwise, we get an operator acting on the zero spaces, which we admit to be invertible). We then have the following result extending the classical result (i.e. \( \Gamma = \{1\} \)) and the one from [11] (i.e. \( \Gamma \) finite abelian) to a general finite group \( \Gamma \).

**Theorem 1.2.** Let \( \Gamma \) be a finite group acting on a smooth, compact manifold \( M \) and let \( P \in \psi^m(M; E_0, E_1)^\Gamma \) be a \( \Gamma \)-invariant classical pseudodifferential operator acting between sections of two \( \Gamma \)-equivariant bundles \( E_i \to M \), \( i = 0, 1 \), \( m \in \mathbb{R} \), and \( \alpha \in \hat{\Gamma} \). We have that

\[
\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha
\]

is Fredholm if, and only if it is \( \alpha \)-elliptic.

As in the abelian case, if \( \Gamma \) acts without fixed points on a dense open subset of \( M \), then \( X_{M,\Gamma} = X^\alpha_{M,\Gamma} \) for all \( \alpha \in \hat{\Gamma} \), by Corollary 4.10. Hence, in this case, \( P \) is \( \alpha \)-elliptic if, and only if, it is elliptic. The ellipticity of \( P \) can thus be checked in this case simply by looking at the action of \( P \) on a single isotypical component. We stress, however, that if \( \Gamma \) is not discrete, this statement, as well as the statement of the above theorem, are no longer true. However, many intermediate results remain valid for compact Lie groups.

A motivation for our result comes from index theory. Let us assume that \( P \) is \( \Gamma \)-invariant and elliptic. Atiyah and Singer have determined, for any \( \gamma \in \Gamma \), the value at \( \gamma \) of the character of \( \text{ind}_\Gamma(P) \in R(G) \). More precisely, they have computed \( \text{ind}_\Gamma(P)(\gamma) \in \mathbb{C} \) in terms of data at the fixed points of \( \gamma \) on \( M \) [6]. (Here \( R(G) := \mathbb{Z}^\hat{G} \) is the representation ring of \( G \) and is identified with a subalgebra of \( C^\infty(G)^G \), the ring of conjugacy invariant functions on \( G \) via the characters of representations.) By contrast, the multiplicity of \( \alpha \in \hat{\Gamma} \) in \( \text{ind}_\Gamma(P) \) was much less studied. It did appear, however, implicitly in the work of Brüning [17, 21], who initiated the program of studying the “isotypical heat trace” \( \text{tr}(p_\alpha e^{-t\Delta}) \) and its short time asymptotic expansion. See also the recent Preprint [20]. Its heat trace is nothing but the heat trace of \( \pi_\alpha(\Delta) \). This question was addressed also by Paradan and Vergne, who
obtained several important related results, see [66, 67] and the references therein. Brüning’s program would lead, in particular, to a heat equation determination of the \( \alpha \)-isotypical component of the \( \Gamma \)-equivariant index \( \text{ind}_\Gamma (D) \) for Dirac type operators \( D \). Carrying out this program is one of the motivations of this paper. Indeed, we obtain that the (Fredholm) index of \( \pi_\alpha (P) \) depends only on the homotopy class of its \( \alpha \)-principal symbol. An explicit determination of this index would require significant tools and input from Noncommutative Geometry [23, 28, 29, 40, 55, 56, 81], so we leave it for future work. Some partial results are contained, however, in Theorem 1.9 and in the remark following it. In particular, this yields results on the index theory of singular quotient spaces. We therefore expect our results to have applications to the Hodge theory of algebraic varieties [2, 15, 26, 40], see Remark 4.11. In the case of a non discrete compact Lie group, the computation of this index is related to the index class of \( G \)-transversally elliptic operators initiated in [5, 75]. Since then, this has been studied in \( K \)-theory [45, 47, 8, 10] and in equivariant cohomology [9, 13, 14, 65].

The formulation of our main result does not use \( C^* \)-algebras, but its proof does. \( C^* \)-algebras were used recently to obtain Fredholm conditions in [31, 33, 50, 62], for example. Some of the algebras involved were groupoid algebras [4, 3, 25, 34, 59, 70]. Fredholm conditions play an important role in the study of the essential spectrum of Quantum Hamiltonians [12, 37, 36, 42, 49]. The technique of “limit operators” [48, 53, 54, 69] is related to groupoids. Some of the most recent papers using related ideas include [7, 24, 25, 27, 60, 61, 82], to which we refer for further references. Besides \( C^* \)-algebras, pseudodifferential operators were also used to obtain Fredholm conditions, see [32, 51, 41] and the references therein.

1.4. Contents of the paper. We start in Section 2 with some preliminaries. We recall some facts about group actions, most notably the induction of representations and Frobenius reciprocity for finite groups. We also review some notions concerning the primitive spectrum of a \( C^* \)-algebra, as well as basic facts concerning (equivariant) pseudodifferential operators.

As in [11], we may assume \( E_0 = E_1 = E \) and \( P \) to be of order zero. Let \( A_M := \text{C}(S^* M; \text{End}(E)) \). The most substantial technical results are in Section 3. There, we identify the primitive spectrum of the \( C^* \)-algebra \( A^\Gamma_M \) of \( \Gamma \)-invariant symbols with the set \( X_{M, \Gamma}/\Gamma \) described above. Some care is taken to describe the corresponding topology on \( X_{M, \Gamma}/\Gamma \). We then consider the canonical map from \( A^\Gamma_M \) to the Calkin algebra of \( L^2 (M; E)_\alpha \) and show that the closed subset of \( \text{Prim}(A^\Gamma_M) \) associated to its kernel is \( X_{M, \Gamma}/\Gamma \).

These descriptions are used in Section 4 to prove the main result of the paper, Theorem 1.2. This section also addresses some particular cases of the Theorem and gives a few examples. We also explain the relation with previously known results, namely:

- the particular formulation in the abelian case, which was established in [11],
- Fredholm conditions for transversally elliptic operators when the group \( \Gamma \) is not discrete,
- Simonenko’s local principle for Fredholm operators.

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2. Preliminaries

This section is devoted to background material. For the most part, it will consist of a brief review of sections 2 and 3 of [11], where the reader will find more details, as well as definitions and results not repeated here. Note, however, that we need certain preliminary results for the case $\Gamma$ non-commutative that we are not needed in the abelian case. Nevertheless, the reader familiar with [11] can skip this section at a first reading.

For simplicity, let us from now on for the rest of this paper also assume that $M/\Gamma$ is connected, except in Subsection 4.7, where we explain how the disconnected case reduces to the connected case.

2.1. Group representations. We follow the standard terminology and conventions. See, for instance, [11, 16, 73], where one can find further details. Most of the needed basic background material was recalled in greater detail in [11].

Throughout the paper, we denote by $\Gamma$ a finite group acting by isometries on a smooth, Riemannian manifold $M$ (without boundary). We use the standard notations, see [11, 16, 73], to which we refer for further details. If $x \in M$, then $\Gamma_x$ is the $\Gamma$ orbit of $x$ and

$$
\Gamma_x := \{ \gamma \in \Gamma \mid \gamma x = x \} \subset \Gamma
$$

the isotropy group of the action at $x$.

We shall write $H \sim H'$ if the subgroups $H$ and $H'$ are conjugated in $\Gamma$. If $H \subset \Gamma$ is a subgroup, then $M_H$ will denote the set of elements of $M$ whose isotropy $\Gamma_x$ is conjugated to $H$ (in $\Gamma$), that is, the set of elements $x \in M$ such that $\Gamma_x \sim H$.

Assuming that $\Gamma$ acts on a space $X$, we denote by $\Gamma \times_H X$ the space

$$
\Gamma \times_H X := (\Gamma \times X) / \sim,
$$

where $(\gamma h, x) \sim (\gamma, hx)$, $\forall \gamma \in \Gamma, h \in H$ and $x \in X$.

Let $V$ be a normed complex vector space and $L(V)$ be the set of bounded operators on $V$. A representation of $\Gamma$ on $V$ is a group morphism $\Gamma \to L(V)$; in that case we also call $V$ a $\Gamma$-module.

For any two $\Gamma$-modules $\mathcal{H}$ and $\mathcal{H}_1$, we shall denote by

$$
\text{Hom}_{\Gamma}(\mathcal{H}, \mathcal{H}_1) = \text{Hom}(\mathcal{H}, \mathcal{H}_1)^\Gamma = L(\mathcal{H}, \mathcal{H}_1)^\Gamma
$$

the set of continuous linear maps $T : \mathcal{H} \to \mathcal{H}_1$ that commute with the action of $\Gamma$, that is, $T(\gamma \xi) = \gamma T(\xi)$ for all $\xi \in \mathcal{H}$ and $\gamma \in \Gamma$.

Let $\mathcal{H}$ be a $\Gamma$-module and $\alpha$ an irreducible $\Gamma$-module. Then $p_\alpha$ will denote the $\Gamma$-invariant projection onto the $\alpha$-isotypical component $\mathcal{H}_\alpha$ of $\mathcal{H}$, defined as the largest (closed) $\Gamma$ submodule of $\mathcal{H}$ that is isomorphic to a multiple of $\alpha$. In other words, $\mathcal{H}_\alpha$ is the sum of all $\Gamma$-submodules of $\mathcal{H}$ that are isomorphic to $\alpha$. Notice that $\mathcal{H}_\alpha \simeq \alpha \otimes \text{Hom}_{\Gamma}(\alpha, \mathcal{H})$.

Since $\Gamma$ is finite, it is, in particular, compact, and hence we have

$$
\mathcal{H}_\alpha \neq 0 \Leftrightarrow \text{Hom}_{\Gamma}(\alpha, \mathcal{H}) \neq 0 \Leftrightarrow \text{Hom}_{\Gamma}(\mathcal{H}, \alpha) \neq 0.
$$

If $T \in L(\mathcal{H})^\Gamma$ (i.e. $T$ is $\Gamma$-equivariant), then $T(\mathcal{H}_\alpha) \subset \mathcal{H}_\alpha$ and we let

$$
\pi_\alpha : L(\mathcal{H})^\Gamma \to L(\mathcal{H}_\alpha), \quad \pi_\alpha(T) := T|_{\mathcal{H}_\alpha},
$$

be the associated morphism, as in Equation (1) of the Introduction. The morphism $\pi_\alpha$ will play an essential role in what follows.
2.2. Induction and Frobenius reciprocity. We now recall some definitions and results for induced representations mainly to set up notation and to obtain some intermediate results.

We let $\mathcal{V}(I) := \{ f : I \to V \}$ for $I$ finite. If $H \subseteq \Gamma$ is a subgroup (hence also finite) and $V$ is a $H$-module, we define, as usual,

$$\text{Ind}^{\Gamma}_H(V) := \mathbb{C}[\Gamma] \otimes_{\mathbb{C}[H]} V$$

(12)

$$\simeq \{ f : \Gamma \to V \mid f(gh^{-1}) = hf(g) \} \simeq V^{(\Gamma/H)}$$

to be the induced representation. The last isomorphism is obtained using a set of representatives of the right cosets $\Gamma/H$. The action of the group $\Gamma$ on $\text{Ind}^{\Gamma}_H(V)$ is obtained from the left multiplication on $\mathbb{C}[\Gamma]$ and the first isomorphism defines the $\Gamma$-module structure on $\text{Ind}^{\Gamma}_H(V)$. The induction is a functor, that is, the $\Gamma$-module $\text{Ind}^{\Gamma}_H(V)$ depends functorially on $V$.

Remark 2.1. Summarizing Remark 2.2 of [11], we have that

(1) if $V$ is a $H$-algebra, then $\text{Ind}^{\Gamma}_H(V)$ is an algebra for the pointwise product,

(2) if $V$ is a left $R$-module (with compatible actions of $\Gamma$), then $\text{Ind}^{\Gamma}_H(V)$ is a $\text{Ind}^{\Gamma}_H(R)$ module, again with the pointwise multiplication,

(3) the induction is compatible with morphisms of modules and algebras (change of scalars), again by the function representation of the induced representation.

See [11, Remark 2.2] for more details.

We shall use the Frobenius reciprocity in the form that states that we have an isomorphism

$$\Phi = \Phi^{\Gamma,H,V} : \text{Hom}_H(\mathcal{H}, V) \to \text{Hom}_\Gamma(\mathcal{H}, \text{Ind}^{\Gamma}_H(V)),$$

(13)

$$\Phi(f)(\xi) := \frac{1}{|H|} \sum_{g \in \Gamma} g \otimes_{\mathbb{C}[H]} f(g^{-1}\xi), \quad \xi \in \mathcal{H}, \ f \in \text{Hom}_H(\mathcal{H}, V).$$

The version of the Frobenius reciprocity used in this paper is valid only for finite groups [10, 73] (although it can be suitably be generalized to the compact case). We note that a more precise notation would be to write $\text{Hom}_H(\text{Res}^\Gamma_H(\mathcal{H}), V)$ instead of our simplified notation $\text{Hom}_H(\mathcal{H}, V)$.

Definition 2.2. Let $A$ and $B$ be finite groups and let $H$ a subgroup of both $A$ and $B$. Let $\alpha \in \hat{A}$ and $\beta \in \hat{B}$. We say that $\alpha$ and $\beta$ are $H$-disjoint if $\text{Hom}_H(\alpha, \beta) = 0$, otherwise we say that they are $H$-associated (to each other).

Let $\alpha \in \hat{\Gamma}$, let $H \subseteq \Gamma$ be a subgroup, and $\beta \in \hat{H}$. A useful consequence of the Frobenius reciprocity is that the multiplicity of $\alpha$ in $\text{ind}^{\Gamma}_H(\beta)$ is the same as the multiplicity of $\beta$ in the restriction of $\alpha$ to $H$. In particular, $\alpha$ is contained in $\text{ind}^{\Gamma}_H(\beta)$ if, and only if, $\beta$ is contained in the restriction of $\alpha$ to $H$, in which case recall that we say that $\alpha$ and $\beta$ are $H$-associated (Definition 2.2). On the other hand, recall that if $\beta$ is not contained in the restriction of $\alpha$ to $H$, we say that $\alpha$ and $\beta$ are $H$-disjoint.
Let $V$ be a $H$-module and $\mathcal{H}$ be the trivial $\Gamma$-module $\mathbb{C}$. Then we obtain, in particular, an isomorphism
\begin{equation}
\Phi : V^H = \text{Hom}_H(\mathbb{C}, V) \simeq \text{Hom}_\Gamma(\mathbb{C}, \text{Ind}^\Gamma_H(V)) = \text{Ind}^\Gamma_H(V)^\Gamma,
\end{equation}
(14)

\[ \Phi(\xi) := \frac{1}{|H|} \sum_{g \in \Gamma} g \otimes_{\mathbb{C}[H]} \xi = \sum_{x \in V^\Gamma / H} x \otimes \xi. \]

If $V$ is an algebra, then the map $\Phi$ is an isomorphism of algebras. In particular, we obtain the following consequences.

**Remark 2.3.** Let $H \subset \Gamma$ be a subgroup of $\Gamma$, $\beta_j$ be non-isomorphic simple $H$-modules, $j = 1, \ldots, N$, and
\begin{equation}
\beta := \oplus_{j=1}^N \beta_j.
\end{equation}
(15)

We then have that $\text{Ind}^\Gamma_H(\beta) \simeq \oplus_{j=1}^N \text{Ind}^\Gamma_H(\beta_j)$ and the Frobenius isomorphism gives
\begin{equation}
\text{Ind}^\Gamma_H(\text{End}(\beta))^\Gamma \simeq \text{End}(\beta)^H \simeq \oplus_{j=1}^N \text{End}(\beta_j)^H \simeq \oplus_{j=1}^N M_{k_j}(\mathbb{C}),
\end{equation}
(16)

which is a semi-simple algebra and where the first isomorphism is induced by $\Phi$ of Equation (14).

We shall need the following refinement of the above remark.

**Lemma 2.4.** Let $\beta := \oplus_{j=1}^N \beta_j$ be as in Equation (15), let $T = (T_j) \in \text{End}(\beta)^H \simeq \oplus_{j=1}^N \text{End}(\beta_j)^H$, with $T_j \in \text{End}(\beta_j)^H$, and let $\xi_j \in \text{Ind}^\Gamma_H(\beta_j)$. We let $\xi := (\xi_j) \in \oplus_{j=1}^N \text{Ind}^\Gamma_H(\beta_j)$. Then $\Phi(T)(\xi) = (\Phi(T_j)\xi_j)_{j=1, \ldots, N}$.

**Proof.** See for example [11, Lemma 2.4].

For the abelian case, the following elementary result was proved in [11] Proposition 2.5. That proof does not generalize to our case.

**Proposition 2.5.** Let $\beta := \oplus_{j=1}^N \beta_j$ be as in Equation (15). Let $J \subset \{1, 2, \ldots, N\}$ be the set of indices $j$ such that $\alpha$ and $\beta_j$ are $H$-disjoint (i.e. $\beta_j$ is not contained in the restriction of $\alpha$ to $H$). Then the morphism
\[ \pi_\alpha : \text{Ind}^\Gamma_H(\text{End}(\beta))^\Gamma \to \text{End}(p_\alpha \text{Ind}^\Gamma_H(\beta)) \]
is such that
\[ \ker(\pi_\alpha) = \bigoplus_{j \in J} \text{Ind}^\Gamma_H(\text{End}(\beta_j))^\Gamma \text{ and } \text{Im}(\pi_\alpha) \simeq \bigoplus_{j \notin J} \text{Ind}^\Gamma_H(\text{End}(\beta_j))^\Gamma. \]

**Proof.** By Lemma 2.4 we can assume that $N = 1$. Therefore the algebra $\text{End}(\beta)^H$ is simple (more precisely, isomorphic to a matrix algebra $M_q(\mathbb{C})$, $q = k_1$). We shall use the isomorphism of Equation (16). The action of $\text{Ind}^\Gamma_H(\text{End}(\beta))^\Gamma \simeq \text{End}(\beta)^H \simeq M_q(\mathbb{C})$ on $\text{Ind}^\Gamma_H(\beta)$ is unital (i.e. non-degenerate), so the morphism
\begin{equation}
M_q(\mathbb{C}) \simeq \text{Ind}^\Gamma_H(\text{End}(\beta))^\Gamma \to \text{End}(p_\alpha \text{Ind}^\Gamma_H(\beta))
\end{equation}
(17)
is injective if, and only if, $p_\alpha \text{Ind}^\Gamma_H(\beta) \neq 0$. Notice the following equivalences
\[ p_\alpha \text{Ind}^\Gamma_H(\beta) \neq 0 \Leftrightarrow \text{Hom}(\alpha, \text{Ind}^\Gamma_H(\beta))^\Gamma \neq 0 \Leftrightarrow \text{Hom}(\alpha, \beta)^H \neq 0. \]
2.3. The primitive ideal spectrum of a $C^\ast$-algebra. We shall need a few basic concepts and facts about $C^\ast$-algebras. A general reference is [35]. Recall that a two-sided ideal $I \subset A$ of a $C^\ast$-algebra $A$ is called primitive if it is the kernel of a non-zero, irreducible $\ast$-representation of $A$. Hence $A$ is not a primitive ideal of itself. By $\text{Prim}(A)$ we shall denote the set of primitive ideals of $A$, called the primitive ideal spectrum of $A$. If $X$ is a locally compact space, then $C_0(X)$ denotes the space of continuous functions $X \to \mathbb{C}$ that vanish at infinity. The concept of primitive ideal spectrum is important for us since we have a natural homeomorphism

$$\text{Prim}(C_0(X)) \simeq X.$$ 

This identification lies at the heart of non-commutative geometry [28, 29]. See also [23, 56].

If $A$ is a type I $C^\ast$-algebra, then $\text{Prim}(A)$ identifies with the set of isomorphism classes of irreducible representations of $A$. Any $C^\ast$-algebra with only finite dimensional irreducible representations is a type I algebra [35]. Most of the algebras considered in this paper (a notable exception are the algebras of compact operators), have this property.

Example 2.6. Let $H$ be a finite group and $\beta = \oplus_{j=1}^N \beta_j$ be as in Remark 2.3. Then, as explained in that remark, $L(\beta)^H \simeq \oplus_j M_{k_j}(\mathbb{C})$. The algebra $L(\beta)^H = \text{End}_H(\beta)$ is thus a $C^\ast$-algebra with only finite dimensional representations and we have natural homeomorphisms

$$\text{Prim}(\text{End}_H(\beta)) \leftrightarrow \{\beta_1, \beta_2, \ldots, \beta_N\} \leftrightarrow \{1, 2, \ldots, N\}.$$ 

The space $\text{Prim}(A)$ is a topological space for the Jacobson topology: we refer to [35] for more details. We will recall some facts about this topology when we need it, see Lemma 3.2 below.

We shall need the following “central character” map.

Remark 2.7. Let $Z$ be a commutative $C^\ast$-algebra and $\phi: Z \to M(A)$ be a $\ast$-morphism to the multiplier algebra $M(A)$ of $A$ [11, 22]. Assume that $\phi(Z)$ commutes with $A$ and $\phi(Z)A = A$. Then Schur’s lemma gives that every irreducible representation of $A$ restricts to (a multiple of) a character of $Z$ and hence there exists a natural continuous map

$$\phi^*: \text{Prim}(A) \to \text{Prim}(Z),$$

which we shall call also the central character map (associated to $\phi$).

We conclude our discussion with the following simple result.

Lemma 2.8. We freely use the notation of Example 2.6. The inclusion of the unit $\mathbb{C} \to \text{End}_H(\beta)$ induces a morphism $j: C_0(X) \to C_0(X; \text{End}_H(\beta)) \simeq C_0(X) \otimes \text{End}_H(\beta)$. The resulting central character map is the first projection

$$j^*: \text{Prim}(C_0(X; \text{End}_H(\beta))) \simeq X \times \{1, 2, \ldots, N\} \to X \simeq \text{Prim}(C_0(X)).$$

2.4. Group actions on manifolds. As before, we consider a finite group $\Gamma$ acting by isometries on a compact Riemannian manifold $M$. The result then follows from Equation (11). □
2.4.1. Slices and tubes. Given \( x \in M \), the isotropy group \( \Gamma_x \) acts linearly and isometrically on \( T_xM \). For \( r > 0 \), let \( U_x := (T_xM)_r \) denote the set of vectors of length \( < r \) in \( T_xM \). It is known then that, for \( r > 0 \) small enough, the exponential map gives a \( \Gamma \)-equivariant isometric diffeomorphism

\[
W_x = \exp(\Gamma \times \Gamma_x U_x) \simeq \Gamma \times \Gamma_x U_x
\]

where \( W_x \) is a \( \Gamma \)-invariant neighborhood of \( x \) in \( M \) and \( \Gamma \times \Gamma_x U_x \) is defined in equation (9). More precisely, \( W_x \) is the set of \( y \in M \) at distance \( < r \) to the orbit \( \Gamma x \), if \( r > 0 \) is small enough. The set \( W_x \) is called a tube around \( x \) (or \( \Gamma x \)) and the set \( U_x \) is called the slice at \( x \). When \( M \) is compact, the injectivity radius is bounded from below, so we may assume that the constant \( r \) does not depend on \( x \).

2.4.2. Equivariant vector bundles. Let us consider now a \( \Gamma \)-equivariant smooth vector bundle \( E \rightarrow M \). Let us fix \( x \in M \) and consider as above the tube \( W_x \simeq \Gamma \times \Gamma_x U_x \) around \( x \), see Equation (21). We use this diffeomorphism to identify \( U_x \) to \( \Gamma \times \Gamma_x U_x \). If, on the other hand, \( x \notin \Gamma \times \Gamma_x U_x \), then \( \Gamma \times \Gamma_x U_x \) is a \( \Gamma \)-invariant neighborhood of \( x \) in \( M \) to assume that \( \Gamma \) is a finite Lie group that acts smoothly and isometrically on \( M \). Recall that we have assumed that \( \Gamma \) is connected, making \( \Gamma \times \Gamma_x U_x \) dense open subset of \( M \). It is known then that there exists a \( \Gamma \)-equivariant isometric diffeomorphism \( W_x \rightarrow M \). Hence \( W_x \) is called the slice at \( x \). When \( M \) is compact, the injectivity radius is bounded from below, so we may assume that the constant \( r \) does not depend on \( x \).

Assume \( E \) is endowed with a \( \Gamma \)-invariant hermitian metric. We then have isomorphisms of \( \Gamma \)-modules:

\[
\langle W_x; E|W_x \rangle \simeq \text{Ind}_{\Gamma_x}^{\Gamma}(L^2(U_x; \beta)) \quad \text{and} \quad \langle C_0(W_x; E|W_x) \simeq \text{Ind}_{\Gamma_x}^{\Gamma}(C_0(U_x; \beta)).
\]

In view of the previous isomorphism, we will often identify \( W_x \) and \( \Gamma \times \Gamma_x U_x \), making no distinction between them to simplify notations.

2.4.3. The principal orbit bundle. Recall that \( M/H \) denotes the set of points of \( M \) whose stabilizer is conjugated in \( \Gamma \) to \( H \). Recall that we have assumed that \( M/\Gamma \) is connected. It is known then that there exists a minimal isotropy subgroup \( \Gamma_0 \subset \Gamma \), in the sense that \( M/\Gamma_0 \) is a dense open subset of \( M \), with measure zero complement in \( M \).

In particular, the fact that \( M/\Gamma \) is connected gives that there exist minimal elements for the set of isotropy groups of points in \( M \) (with respect to inclusion) and all minimal isotropy groups are conjugated to a fixed subgroup \( \Gamma_0 \subset \Gamma \). By the definition, the set \( M_{(\Gamma_0)} \) consists of the points whose stabilizer is conjugated to that minimal subgroup. The set \( M_{(\Gamma_0)} \) is called the principal orbit bundle of \( M \). We will denote \( M_{(\Gamma_0)} \) by \( M_0 \) in the sequel.

The principal orbit bundle \( M_0 := M_{(\Gamma_0)} \) has the following useful property. If \( x \in M_0 \), then \( \Gamma_x \) acts trivially on the slice \( U_x \) at \( x \), by the minimality of \( \Gamma_0 \). Hence \( \Gamma_0 \) acts trivially on \( T_x^*M \) as well, which implies that \( \Gamma_0 \subset \Gamma_0 \) for any \( \xi \in T_x^*M \).

If, on the other hand, \( x \in M \) is arbitrary (not necessarily in the principal orbit bundle), then the isotropy of \( \Gamma_x \) will contain a subgroup conjugated to \( \Gamma_0 \).

2.5. Pseudodifferential operators. We continue to follow [11]. We also continue to assume that \( \Gamma \) is a finite Lie group that acts smoothly and isometrically on a smooth Riemannian manifold \( M \). Let \( \psi^m(M; E) \) denote the space of order \( m \),
classical pseudodifferential operators on $M$ with compactly supported distribution kernel.

Let $\psi^0(M; E)$ and $\widetilde{\psi^{-1}}(M; E)$ denote the respective norm closures of $\psi^0(M; E)$ and $\psi^{-1}(M; E)$. The action of $\Gamma$ then extends to an action on $\psi^m(M; E), \psi^0(M; E),$ and $\psi^{-1}(M; E)$. We shall denote by $\mathcal{K}(\mathcal{H})$ the algebra of compact operators acting on a Hilbert space $\mathcal{H}$. Of course, we have $\psi^{-1}(M; E) = \mathcal{K}(L^2(M; E))$, since we have considered only pseudodifferential operators with compactly supported distribution kernels.

Let $S^*M$ denote the unit cosphere bundle of a smooth manifold $M$, that is, the set of unit vectors in $T^*M$, as usual. We shall denote, as usual, by $\mathcal{C}_0(S^*M; \text{End}(E))$ the set of continuous sections of the lift of the vector bundle $\text{End}(E)$ to $S^*M$.

**Corollary 2.29.** We have an exact sequence

$$0 \to \mathcal{K}(L^2(M; E))^\Gamma \to \overline{\psi^0(M; E)}^\Gamma \xrightarrow{\sigma_\alpha} \mathcal{C}_0(S^*M; \text{End}(E))^\Gamma \to 0.$$  

**Proof.** See, for instance, [11, Corollary 2.7].

2.5.1. The structure of regularizing operators. From now on, all our vector bundles will be $\Gamma$-equivariant vector bundles. We want to understand the structure of the algebra $\pi_\alpha(\overline{\psi^0(M; E)}^\Gamma)$, for any fixed $\alpha \in \widehat{\Gamma}$ (see Equations (1) and (11) for the definition of the restriction morphism $\pi_\alpha$ and of the projectors $p_\alpha \in C^*(\Gamma)$).

We shall need the following standard result about negative order operators. Recall that, for $\alpha \in \widehat{\Gamma}$, we let $\pi_\alpha$ be the representation of $\overline{\psi^0(M; E)}^\Gamma$ on $L^2(M; E)_\alpha$ defined by restriction as before, Equations (1) and (11).

**Proposition 2.10.** We have the identifications

$$p_\alpha \psi^{-1}(M; E)^\Gamma \simeq \pi_\alpha(\psi^{-1}(M; E)^\Gamma) = \pi_\alpha(\mathcal{K}(L^2(M; E))^\Gamma) = \mathcal{K}(L^2(M; E)_\alpha)^\Gamma,$$

where the first isomorphism map is simply $\pi_\alpha$ and

$$\mathcal{K}(L^2(M; E))^\Gamma = \psi^{-1}(M; E)^\Gamma \simeq \bigoplus_{\alpha \in \widehat{\Gamma}} \mathcal{K}(L^2(M; E)_\alpha)^\Gamma.$$  

**Proof.** See, for example, [11, Section 3] for a proof.

3. The principal symbol

From now on we assume that $M$ is compact and that $M/\Gamma$ is connected. Let us fix an irreducible representation $\alpha$ of $\Gamma$ and consider the restriction morphism $\pi_\alpha$ to the $\alpha$-isotypical component of $L^2(M; E)$. Recall that this morphism was first introduced in Equation (11) and discussed in detail in Section 2.1. As in (11), we now turn to the identification of the quotient

$$\pi_\alpha(\psi^0(M; E)^\Gamma)/\pi_\alpha(\psi^{-1}(M; E)^\Gamma).$$

The methods used in this paper diverge, however, drastically from those of [11].

Since $\pi_\alpha(\psi^{-1}(M; E)^\Gamma)$ was identified in the previous section, the promised identification of the quotient $\pi_\alpha(\psi^0(M; E)^\Gamma)/\pi_\alpha(\psi^{-1}(M; E)^\Gamma)$ will give further insight into the structure of the algebra $\pi_\alpha(\psi^0(M; E)^\Gamma)$ and will provide us, eventually, with Fredholm conditions. Recall that, in this paper, we are assuming $\Gamma$ to be finite. Nevertheless, a several intermediate results hold also in the case $\Gamma$ compact.
3.1. The primitive ideal spectrum of $A_M^\Gamma$. As before, $S^*M$ denotes the unit cosphere bundle of $M$. For the simplicity of the notation, we shall write

$$A_M := C(S^*M; \text{End}(E)),$$

as in the Introduction. Recall from Corollary 2.9 that we have an algebra isomorphism

$$\psi^0(M; E)^{\Gamma}/\psi^{-1}(M; E)^{\Gamma} \simeq A_M^\Gamma.$$

In our case, the inclusion $j : C(S^*M/\Gamma) \subset A_M^\Gamma$ as a central subalgebra induces, as in Equation (19), a central character map

$$j^* : \text{Prim}(A_M^\Gamma) \to S^*M/\Gamma,$$

that underscores the local nature of the structure of the primitive ideal spectrum of $A_M^\Gamma$. We introduce the representation $\pi_{\xi, \rho}$ defined for any $f \in A_M^\Gamma$ by

$$\pi_{\xi, \rho}(f) = \pi_{\rho}(f(\xi)),$$

that is $\pi_{\xi, \rho}(f)$ is the restriction of $f(\xi) \in \text{End}(E_x)$ to the $\rho$-isotypical component of $E_x$. The central character map $j^*$ was used in [11], Corollary 4.2, to obtain the following identification of $\text{Prim}(A_M^\Gamma)$.

**Proposition 3.1** ([11]). Let $\Omega_M$ be the set of pairs $(\xi, \rho)$, where $\xi \in S_x^*M$, $x \in M$, and $\rho \in \hat{\Gamma}_\xi$ appears in $E_x$ (i.e. $\text{Hom}_{\Gamma_\xi}(\rho, E_x) \neq 0$).

1. The map $\Omega_M/\Gamma \ni (\xi, \rho) \mapsto \ker(\pi_{\xi, \rho}) \in \text{Prim}(A_M^\Gamma)$ is bijective.
2. The central character map $\Omega_M/\Gamma \simeq \text{Prim}(A_M^\Gamma) \to S^*M/\Gamma$ maps $\Gamma(\xi, \rho) \in \Omega_M/\Gamma$ to $\Gamma\xi$ and is continuous and finite-to-one.

The space $\text{Prim}(A_M^\Gamma)$ is endowed with the Jacobson topology, which was recalled in Subsection 2.3; thus Proposition 3.1 allows us to use the central character map $j^*$ to obtain a topology on $\Omega_M/\Gamma$ that will play a crucial role in what follows. We thus now turn to the study of this topology on $\Omega_M/\Gamma$. We begin with the following standard lemma.

**Lemma 3.2.** Let $A$ be a $C^*$-algebra. The family $(V_a)_{a \in A}$ defined by

$$V_a = \{J \in \text{Prim}(A) \mid a \notin J\},$$

for any $a \in A$, is a basis of open sets for $\text{Prim}(A)$.

**Proof.** Following [35], we know that the open, non-empty subsets of $\text{Prim}(A)$ are exactly the sets

$$\{J \in \text{Prim}(A) \mid I \not\subset J\} \simeq \text{Prim}(I)$$

where $I$ ranges through the closed, non-zero, two-sided ideals of $A$. If $a \in A$, let us denote by $I_a := \overline{aAa}$ the closed, two-sided ideal generated by $a$. Then $a \notin J \iff I_a \not\subset J$, and hence $V_a = \text{Prim}(I_a)$. This shows that $V_a$ is open.

Next, let $V \subset \text{Prim}(A)$ be a non-empty open subset and $J_0 \in V$. We know then that there exists a closed, two-sided ideal $I$, $0 \neq I \subset A$, such that $V = \text{Prim}(I)$. We have $I \not\subset J_0$, and hence we can choose $a \in I \cap J_0$. If $J \subset A$ is a primitive ideal such that $a \notin J$, then a fortiori $I \not\subset J$. Therefore $V_a \subset \text{Prim}(I)$. This shows that $J_0 \in V_a \subset V$. Therefore the family $(V_a)_{a \in A}$ is a basis for the topology on $\text{Prim}(A)$. \hfill \square

We shall use the bijection of Proposition 3.1 to conclude the following.
Corollary 3.3. A basis for the induced topology on $\Omega_M/\Gamma \simeq \text{Prim}(A^*_M)$ is given by the sets

$$V_f := \{ \Gamma(\xi, \rho) \in \Omega_M/\Gamma \mid \pi_{\xi, \rho}(f) \neq 0 \},$$

where $f$ ranges through the non-zero elements of $A^*_M$.

3.2. The restriction morphisms. Let $O \subset M$ be an open subset. Then $S^*O$ is the restriction of $S^*M$ to $O$. We shall need the algebras

$$A_O := C_0(S^*O; \text{End}(E)) \quad \text{and} \quad B_O := \psi^R(O; E).$$

Assume that $O \subset M$ is $\Gamma$-invariant. The group $\Gamma$ does not act, in general, as multipliers on the $C^*$-algebra $B_O := \psi^R(O; E)$ (it does however act by conjugation), so the method used in [11] to compute $\overline{\psi^{-1}(O; E)}^\Gamma \simeq K(L^2(O; E))$ does not extend to compute $B^R_O$. We shall thus consider the natural, surjective map

$$\mathcal{R}_O : A^*_O := C_0(S^*O; \text{End}(E))^\Gamma \simeq B^R_O/\overline{\psi^{-1}(O; E)}^\Gamma$$

$$\to \pi_\alpha(B^R_O)/\pi_\alpha(\overline{\psi^{-1}(O; E)}^\Gamma).$$

Recall from Corollary 2.10 that $\pi_\alpha(\overline{\psi^{-1}(M; E)}^\Gamma) = K(L^2(M; E))$. Therefore, for a given $P \in \psi^R(M; E)$, we have that $\pi_\alpha(P)$ is Fredholm if, and only if, the principal symbol of $P$ is invertible in $A^*_M/\ker(\mathcal{R}_M)$. This will be discussed in more detail in the next section.

We shall approach the computation of $\ker(\mathcal{R}_M) \subset A^*_M$ by determining the closed subset

$$\Xi := \text{Prim}(A^*_M/\ker(\mathcal{R}_M)) \subset \text{Prim}(A^*_M)$$

of the primitive ideal spectrum of $A^*_M$ corresponding to $\ker(\mathcal{R}_M)$. Once we have determined $\Xi$, we will also have determined $\ker(\mathcal{R}_M)$, in view of the definitions recalled in Subsection 2.3 that put in bijection the closed, two-sided ideals of a $C^*$-algebra with the closed subsets of its primitive ideal spectrum.

Since $\mathcal{C}(M/\Gamma) \subset B_M$, it follows from the definition of $\mathcal{R}_M$ that it is a $\mathcal{C}(M/\Gamma)$-module morphism, and hence that $\ker(\mathcal{R}_M)$ is a $\mathcal{C}(M/\Gamma)$-module. Let us also recall that

$$\mathcal{C}(M/\Gamma) = \mathcal{C}(M)^\Gamma \subset Z_M := \mathcal{C}(S^*M)^\Gamma \subset Z(A^*_M) \subset A^*_M \subset A_M.$$

The local nature of $\ker(\mathcal{R}_M)$ and of the space $\Xi$ is explained in the following remark.

Remark 3.4. Let $M/\Gamma = \bigcup V_k$ be an open cover and

$$\ker(\mathcal{R}_M)|_{V_k} := C_0(V_k) \ker(\mathcal{R}_M) = \ker(\mathcal{R}_{V_k}).$$

If we determine each $\ker(\mathcal{R}_M)|_{V_k}$, then we determine $\ker(\mathcal{R}_M)$ using a partition of unity through:

$$\ker(\mathcal{R}_M) = \sum' \phi_k \ker(\mathcal{R}_{V_k}),$$

where $\sum'$ refers to sums with only finitely many non-zero terms and $(\phi_k)$ is a partition of unity of $M/\Gamma$ with continuous functions subordinated to the covering $(V_k)$ (thus, in particular, $\text{supp}(\phi_k) \subset V_k$). Since $M$ is compact, we can assume the covering to be finite. (If $M$ was non-compact, then we would need to take the closure of the right hand side in Equation (28).) To determine $\mathcal{R}_M$, we can therefore replace $M$ by any of the open sets $V_k$ in the covering and study $\ker(\mathcal{R}_{V_k})$. 
We shall do that for the covering of $M/\Gamma$ with the tubes $W_x \simeq \Gamma \times_{\Gamma_x} U_x$ considered in \cite[see Equation (21)]{2.4.1}.

3.3. Local calculations. In view of Remark 3.4, we shall concentrate now on the local structure of $\ker(\mathcal{R}_M)$, that is, on the structure of $\ker(\mathcal{R}_O)$ for suitable ("small") open, $\Gamma$-invariant subsets $O \subset M$. Let us fix then $x \in M$ and let $W_x \simeq \Gamma \times_{\Gamma_x} U_x$ be the tube around $x$, Equation (21). For simplicity, we shall write

$$\psi := C_0(S^*U_x;\text{End}(E))$$

and let $Z_x := Z(A_{\Gamma_x}^{\Gamma})$.

For these algebras, the role of $\Gamma$ will be played by $\Gamma_x$. For the statement of the following lemma, recall the definitions in Subsection 2.4, especially Equation (21).

Lemma 3.5. Let $W_x \simeq \Gamma \times_{\Gamma_x} U_x$. Then $S^*W_x \simeq \Gamma \times_{\Gamma_x} S^*U_x$ and we have $\Gamma$-equivariant algebra isomorphisms

$$A_{W_x} := C_0(S^*W_x;\text{End}(E)) \simeq \text{Ind}_{\Gamma_x}^\Gamma (C_0(S^*U_x;\text{End}(E))) =: \text{Ind}_{\Gamma_x}^\Gamma (A_x).$$

Consequently, the Frobenius isomorphism $\Phi$ of Equation (14) induces an isomorphism

$$\Phi^{-1} : A_{W_x}^\Gamma \rightarrow A_{x}^\Gamma.$$

Proof. We have that $E|_{W_x} \simeq \Gamma \times_{\Gamma_x} (E|_{U_x})$, hence $\text{End}(E)|_{W_x} \simeq \Gamma \times_{\Gamma_x} (\text{End}(E)|_{U_x})$. Equation (23) then gives that $C_0(W_x,\text{End}(E)) \simeq \text{Ind}_{\Gamma_x}^\Gamma (C_0(U_x,\text{End}(E)))$. The rest follows right away from the Frobenius reciprocity (more precisely, from Equation (14)) and from Equation (23), with $\beta$ replaced with $\text{End}(E)$. \hfill $\Box$

Remark 3.6. In view of Equation (14), the isomorphism $\Phi$ of Lemma 3.5 can be written explicitly as follows. Let $f \in A_{\Gamma_x}^{\Gamma_x}$. Then, for any equivalence class $[\gamma,\xi] := \Gamma_x(\gamma,\xi) \in \Gamma \times_{\Gamma_x} S^*U_x \simeq S^*W_x$ we have

$$\Phi(f)([\gamma,\xi]) = [\gamma, f(\xi)],$$

where $[\gamma, f(\xi)] \in \Gamma \times_{\Gamma_x} (U_x \times \text{End}(E_x))^{\Gamma_x} \simeq \Gamma \times_{\Gamma_x} \text{End}(E|_{U_x})^{\Gamma_x} \simeq \text{End}(E|_{W_x})^{\Gamma}$. This defines $\Phi(f) \in C_0(S^*W_x;\text{End}(E|_{W_x}))^{\Gamma} = A_{W_x}^\Gamma$.

Lemma 3.5 together with the following remark will allow us to reduce the study of the algebra $A_M^\Gamma$ to that of its analogues defined for slices.

Remark 3.7. Let $U$ be an open set of some euclidean space and $W = U \times \{1,2,\ldots,N\}$, where the space on the second factor is endowed with the discrete topology. For simplicity, we identify $L^2(W)$ with $L^2(U)^N$ using the map $f \mapsto (f(i))_{i=1\ldots N}$. Then

$$\psi^{-1}(U) = M_N(\psi^{-1}(U)) \simeq \psi^{-1}(U) \otimes M_N(\mathbb{C})$$

and hence

$$\overline{\psi^{-1}}(W) = M_N(\overline{\psi^{-1}}(U)) \simeq \overline{\psi^{-1}}(U) \otimes M_N(\mathbb{C}).$$

On the other hand, if $A^N$ denotes the direct sum of $N$-copies of the algebra $A$, then we have the following inclusions of algebras

$$\psi^0(U)^N \subset \psi^0(W) \subset M_N(\psi^0(U)) \simeq \psi^0(U) \otimes M_N(\mathbb{C}) \text{, and hence}$$

$$\overline{\psi^0}(U)^N \subset \overline{\psi^0}(W) \subset M_N(\overline{\psi^0}(U)) \simeq \overline{\psi^0}(U) \otimes M_N(\mathbb{C}).$$

The following lemma makes explicit the group actions in the isomorphisms of the last remark. Thus, in analogy with the definitions of the algebras $A_{W_x} = \text{C}_0(S^*W_x;\text{End}(E))$ and $A_x = \text{C}_0(S^*U_x;\text{End}(E))$, we consider the algebras

$B_{W_x} := \overline{\psi^0}(W_x;E)$ \quad and \quad $B_x := \overline{\psi^0}(U_x;E)$. 
We shall also use the standard notation $V^{(f)} := \{ f : I \to V \}$ for $I$ finite, as before.

**Lemma 3.8.** We keep the notation of Lemma 3.6 and of Equation (32) above. Then we have $\Gamma$-equivariant algebra isomorphisms

$$B_{W_x} \simeq \text{Ind}_{\Gamma_x}^{\Gamma} (B_x) + \overline{\psi^{-1}}(W_x; E).$$

Consequently, $B_{W_x}^\Gamma \simeq \Phi(B_{W_x}^{\Gamma_x}) + \overline{\psi^{-1}}(W_x; E)^\Gamma$. 

**Proof.** Since $B_y = B_{U_y} \subset B_{W_x}$ for all $y \in \Gamma x$ and since $U_x$ and $U_y$ are diffeomorphic through any $\gamma \in \Gamma$ such that $\gamma x = y$ we obtain the inclusion $B_x^{(\Gamma/\Gamma_x)} \subset B_{W_x}$, as in Remark 3.7. Similarly, since $B_x \to A_x$ is surjective, we obtain the equality $B_{W_x} = B_x^{(\Gamma/\Gamma_x)} + \overline{\psi^{-1}}(W_x; E)$ as in the same remark. From Equation (24) and Lemma 3.5 we know that $B_{W_x} / \overline{\psi^{-1}}(W_x; E) \simeq A_{W_x} \simeq A_x^{(\Gamma/\Gamma_x)} = \text{Ind}_{\Gamma_x}^{\Gamma} (A_x)$, and hence we obtain $B_{W_x} \simeq \text{Ind}_{\Gamma_x}^{\Gamma} (B_x) + \overline{\psi^{-1}}(W_x; E)$. The last isomorphism follows from the Frobenius reciprocity (more precisely, from Equation (23), with $\beta$ replaced with $B_x$ and from the exactness of the functor $V \to V^\Gamma$. 

To be able to make further progress, it will be convenient to look first at the case when $x \in M$ has minimal isotropy $\Gamma_x \simeq \Gamma_0$, that is, when $x$ belongs to the principal orbit bundle $M_0 := M(\Gamma_0)$. The notation $\Gamma_0$ will remain fixed from now on.

### 3.4. Calculations for the principal orbit bundle

We assume as before that $M/\Gamma$ is connected. Let $\Gamma_0$ be a minimal isotropy group (which, we recall, is unique up to conjugation). Let $x \in M$ be our fixed point and $\Gamma_x$ its isotropy, as before. The case when $\Gamma_x$ is conjugated to $\Gamma_0$ is simpler since, as noticed already, then $\Gamma_x$ acts trivially on $U_x$.

Let us fix $x \in M$ with isotropy group $\Gamma_x = \Gamma_0$. As before, we let

$$W_x \simeq \Gamma \times_{\Gamma_0} U_x \text{ and } E_{W_x} \simeq \Gamma \times_{\Gamma_0} (U_x \times \beta),$$

where $\beta$ is some $\Gamma_0$-module, as in Equations (21) and (22). We decompose $\beta$ into a direct sum of representations of the form $\beta_j^{k_j}$ for some non-isomorphic irreducible module (or representation) $\beta_j$ of $\Gamma_0$, again as before:

$$E_x = \beta \simeq \oplus \beta_j^{k_j}. $$

**Remark 3.9.** We have noticed earlier that $\Gamma_0$ acts trivially on $U_x$, hence on $T^*_x M$. In particular $S^* M$ also has $\Gamma_0$ as minimal isotropy subgroup, and $S^* M_0$ is a dense subset of the principal bundle of $S^* M$.

**Corollary 3.10.** Let $x \in M$ be such that $\Gamma_x = \Gamma_0$ and $\beta = \oplus_{j=1}^N \beta_j^{k_j}$, for some non-isomorphic, irreducible $\Gamma_0$-modules $\beta_j$. Then

$$A_{W_x}^\Gamma \simeq A_x^{*\Gamma} \simeq C_0(S^* U_x) \otimes \text{End}_{\Gamma_0} (\beta) \simeq \oplus_{j=1}^N M_{k_j} (C_0(S^* U_x)).$$

In particular, the canonical central character map

$$\text{Prim}(A_{W_x}^{\Gamma_0}) \to S^* U_x \simeq \text{Prim}(C_0(S^* U_x)^{\Gamma_0})$$

of Proposition 3.7 corresponds to the trivial finite covering $S^* U_x \times \text{Prim} (\text{End}_{\Gamma_0} (\beta)) \to S^* U_x$.

**Proof.** The first isomorphism is repeated from Lemma 3.5. The second one is obtained from the following:
(i) from the definition of $A_x = A_{U_x}$, 
(ii) from the assumption that $\Gamma_x = \Gamma_0$, 
(iii) from the fact that $\Gamma_0$ acts trivially on $U_x$, and 
(iv) from the identifications 

$$A_x^0 := C_0(S^*U_x; \text{End}(E))^{\Gamma_0} \simeq C_0(S^*U_x) \otimes \text{End}(\beta)^{\Gamma_0}.$$ 

The last isomorphism follows from Example 2.6 and the isomorphism $M_n(C) \otimes A \simeq M_n(A)$, valid for any algebra $A$. The rest follows from Lemma 2.8.

Indeed, since both $C_0(S^*U_x)$ and $\text{End}(\beta)^{\Gamma_0}$ have only finite dimensional irreducible representations, we obtain $\text{Prim}(A_x^0) = S^*U_x \times \text{Prim}(\text{End}_{\Gamma_0}(\beta)) \simeq S^*U_x \times \{1, 2, \ldots, N\}$, where we use the identification $\text{Prim}(C_0(S^*U_x)) \simeq S^*U_x$ and where the set $\{1, 2, \ldots, N\}$ is in natural bijection with the primitive ideal spectrum of the algebra $\text{End}_{\Gamma_0}(\beta) \simeq \bigoplus_{\beta_j=1}^{N} M_{\beta_j}(C)$. The inclusion $C_0(S^*U_x) = C_0(S^*U_x)^{\Gamma_0} \to A_x^0$ is given by the unital inclusion $C \to \bigoplus_{\beta_j=1}^{N} M_{\beta_j}(C)$. Hence the map $\text{Prim}(A_x^0) \to S^*U_x$ identifies with the first projection in $S^*U_x \times \{1, 2, \ldots, N\} \to S^*U_x$. That is, it is a trivial covering, as claimed. $\square$

The fibers of $\text{Prim}(A_{M_x}^0) \to M_0/\Gamma$ are thus the simple factors of $\text{End}(E_x)^{\Gamma_0}$, whose structure was determined in Example 2.6. We shall need the following remark similar to Remark 3.7, but simpler.

**Remark 3.11.** Let $U$ be an open subset of a euclidean space, let $V$ be a finite dimensional vector space and let $V$ denote, by abuse of notation, also the trivial, vector bundle with fiber $V$. Then we have natural isomorphisms

$$\psi^{-1}(U; V) \simeq \psi^{-1}(U) \otimes \text{End}(V)$$

and

$$\psi^0(U; V) \simeq \psi^0(U) \otimes \text{End}(V).$$

Consequently, we also have the analogous isomorphisms for the completions

$$\overline{\psi^{-1}}(U; V) \simeq \overline{\psi^{-1}}(U) \otimes \text{End}(V)$$

and

$$\overline{\psi^0}(U; V) \simeq \overline{\psi^0}(U) \otimes \text{End}(V).$$

We are in position now to determine the kernel of $R_{W_x}$, when $x$ is in the principal orbit bundle. We will use the notation of Subsection 2.4 that was recalled at the beginning of this subsection as well as the notation of Subsection 2.2. In particular, recall that $\beta_j \in \widehat{\Gamma}_0$ and $\alpha \in \widehat{\Gamma}$ are said to be $\Gamma_0$-disjoint if $\beta_j$ is not contained in the restriction of $\alpha$ to $\Gamma_0$. Also, $\Phi$ is the Frobenius isomorphism, Equations (13) and (14) and Corollary 3.10.

**Proposition 3.12.** Let $\Gamma_x = \Gamma_0$, let $E_x = \beta = \bigoplus_{j=1}^{N} \beta_j^{k_j}$, and $\Phi : C_0(S^*U_x) \otimes \text{End}_{\Gamma_0}(\beta) \simeq A_x^0 \to A_{W_x}^0$ be the Frobenius isomorphism of Corollary 3.10. Then

1. $C_0(S^*U_x) \otimes \text{End}_{\Gamma_0}(\beta_j^{k_j}) \subset \Phi^{-1}(\ker(R_{W_x}))$ if $\beta_j$ and $\alpha$ are $\Gamma_0$-disjoint, and
2. $C_0(S^*U_x) \otimes \text{End}_{\Gamma_0}(\beta_j^{k_j}) \cap \Phi^{-1}(\ker(R_{W_x})) = 0$ if $\beta_j$ and $\alpha$ are $\Gamma_0$-associated.

In particular, let $J \subset \{1, 2, \ldots, N\}$ be the set of indices $j$ such that $\beta_j$ and $\alpha$ are $\Gamma_0$-disjoint, then

$$\ker(R_{W_x}) = \Phi(\bigoplus_{j \in J} C_0(S^*U_x) \otimes \text{End}_{\Gamma_0}(\beta_j^{k_j}))$$

and

$$\pi_\alpha(B_{M_0}/\pi_\alpha(M; E)^{\Gamma}) \simeq \Phi(\bigoplus_{j \notin J} C_0(S^*U_x) \otimes \text{End}_{\Gamma_0}(\beta_j^{k_j})).$$
Proof: The proof is essentially a consequence of Proposition 2.5 by including $U_x$ as a parameter, using also Lemma 3.8. To see how this is done, we will use the notation of that lemma, in particular, $W_x \simeq \Gamma \times \Gamma_0 \times U_x \simeq (\Gamma/\Gamma_0) \times U_x$ and $E \simeq \Gamma \times \Gamma_0 (U_x \times \beta)$.

We identify $W_x$ with $\Gamma \times \Gamma_0 \times U_x$, i.e. we work with $W_x = \Gamma \times \Gamma_0 \times U_x$.

Let $\pi_\alpha$ the fundamental morphism of restriction to the $\alpha$-isotypical component, see Equations (11) and (11). Recall that $B_{\alpha} := \overline{\psi}(U_x; E)$. Since $\Gamma_x$ acts trivially on $U_x$, Remark 3.11 yields the $\Gamma$-equivariant isomorphisms
\begin{equation}
\text{Ind}_{\Gamma_0}^\Gamma(B_{\alpha}) \simeq \overline{\psi}(U_x) \otimes \text{Ind}_{\Gamma_0}^\Gamma(\text{End}(\beta)) \subset B_{W_x},
\end{equation}
where the last inclusion is modulo the trivial identification given by $P \otimes f(s)(\gamma, x) = P(f(s))((\gamma)(x))$, $P \in \overline{\psi}(U_x)$, $f \in \text{Ind}_{\Gamma_0}^\Gamma(\text{End}(\beta))$ and $s \in C_c(W_x, \text{End}(E))$. Combining further Remark 3.11 with Remark 3.7 we further obtain the isomorphism
\begin{equation}
\overline{\psi^{-1}}(W_x; E) \simeq \overline{\psi^{-1}}(U_x) \otimes \text{End}(\text{Ind}_{\Gamma_0}^\Gamma(\beta)).
\end{equation}

Lemma 3.8 and the exactness of the functor $V \rightarrow V^\Gamma$ give $\pi_\alpha(B_{W_x}) = \pi_\alpha \circ \Phi(B_{W_x}^\Gamma) / \pi_\alpha \circ \Phi(B_{W_x}^\Gamma) \cap \pi_\alpha(\overline{\psi^{-1}}(W_x)^\Gamma)$.

Let $\mathfrak{A}$ and $\mathfrak{J}$ be the image and, respectively, the kernel of $\pi_\alpha : \text{Ind}_{\Gamma_0}^\Gamma(\text{End}(\beta))^\Gamma \rightarrow \text{End}(\rho_0 \text{Ind}_{\Gamma_0}^\Gamma(\beta))$, which have been identified in Proposition 2.5 in terms of the set $J$. Recall next from Equation (23) that $L^2(W_x; E) = L^2(U_x) \otimes \text{Ind}_{\Gamma_0}^\Gamma(\beta)$, again $\Gamma$-equivariantly. Each time, the action is on the second component, since $\Gamma_0 = \Gamma_x$ acts trivially on $\overline{\psi}(U_x)$. The action of $\text{Ind}_{\Gamma_0}^\Gamma(B_{\alpha}) \subset B_{W_x}$ on $L^2(U_x; E) = L^2(U_x) \otimes \text{Ind}_{\Gamma_0}^\Gamma(\beta)$ is compatible with the tensor product decomposition of Equation (33), in the sense that $\overline{\psi}(U_x)$ acts on $L^2(U_x)$ and $\text{Ind}_{\Gamma_0}^\Gamma(\beta)$ acts on $\text{Ind}_{\Gamma_0}^\Gamma(\beta)$. Also, $\text{Ind}_{\Gamma_0}^\Gamma(B_{\alpha})^\Gamma \simeq \overline{\psi}(U_x) \otimes \text{Ind}_{\Gamma_0}^\Gamma(\beta)^\Gamma$, (we use this isomorphism to identify them).

We obtain that
\begin{equation}
\pi_\alpha \circ \Phi(B_{W_x}^\Gamma) = \pi_\alpha \text{Ind}_{\Gamma_0}^\Gamma(B_{\alpha})^\Gamma = \overline{\psi}(U_x) \otimes \mathfrak{A}.
\end{equation}

On the other hand, Corollary 2.10 then gives that $\pi_\alpha(\overline{\psi^{-1}}(W_x; E))^\Gamma$ is the algebra of $\Gamma$-invariant compact operators acting on the space $p_0(L^2(W_x, \text{End}(E)))$. Therefore, $\overline{\psi^{-1}}(U_x) \otimes \mathfrak{A} \subset \pi_\alpha(\overline{\psi^{-1}}(W_x; E))^\Gamma$, since $\overline{\psi^{-1}}(U_x) \otimes \mathfrak{A}$ consists of compact, $\Gamma$-invariant operators acting on $p_0(L^2(W_x, E))$. Consequently,
\begin{equation}
\pi_\alpha(\text{Ind}_{\Gamma_0}^\Gamma(B_{\alpha})^\Gamma) \subset \overline{\psi}(U_x) \otimes \mathfrak{A} \cap \mathcal{K}(p_0 L^2(W_x; E))^\Gamma \subset \overline{\psi^{-1}}(U_x) \otimes \mathfrak{A},
\end{equation}
and hence we have equalities everywhere.

Recall from Corollary 2.10 that $A_{W_x}^\Gamma \simeq A_{W_x}^\Gamma$. We obtain that the map
\begin{equation}
\mathcal{R}_{W_x} : A_{W_x}^\Gamma \simeq B_{W_x}^\Gamma / \overline{\psi^{-1}}(W_x; E)^\Gamma \rightarrow \pi_\alpha(B_{W_x}^\Gamma) / \pi_\alpha(\overline{\psi^{-1}}(W_x)^\Gamma)
\end{equation}
becomes, up to the canonical isomorphisms above, the map
\begin{equation}
A_{W_x}^\Gamma \simeq C_0(S^*U_x) \otimes \text{End}_{\Gamma_0}^\Gamma(\beta) \rightarrow \pi_\alpha(B_{W_x}^\Gamma) / \pi_\alpha(\overline{\psi^{-1}}(W_x)^\Gamma)
\end{equation}
\begin{equation}
\simeq \overline{\psi}(U_x) \otimes \mathfrak{A} / \overline{\psi^{-1}}(U_x) \otimes \mathfrak{A} \simeq C_0(S^*U_x) \otimes \mathfrak{A},
\end{equation}
with all maps being surjective and preserving the tensor product decompositions. This identifies the kernel of $\mathcal{R}_{W_x}$ with $C_0(S^*U_x) \otimes \mathfrak{J}$ and the image of $\mathcal{R}_{W_x}$ with $C_0(S^*U_x)$.
\( C_0(S^*U_x) \otimes A \). The rest of the statement follows from the identification of \( J \) and \( A \) in Proposition 3.13.

Proposition 3.12 above and its proof give the following corollary.

**Corollary 3.13.** We use the notation of Proposition 3.12 and we identify the space \( \text{Prim}(\text{End}(\beta)) \) with \( \{1, 2, \ldots, N\} \) as in Remark 2.7. Then the homeomorphism \( \text{Prim}(A^\Gamma_{W_x}) \simeq S^*U_x \times \{1, 2, \ldots, N\} \) maps the set \( \Xi \cap \text{Prim}(A^\Gamma_{W_x}) \) to \( S^*U_x \times J \). In particular, the restriction \( \Xi \cap \text{Prim}(A^\Gamma_{W_x}) \to S^*U_x \) of the central character is a covering as well.

**Proof.** Using the notations of the proof of Proposition 3.12, we have that \( \ker(\mathcal{R}_{W_x}) \) has primitive ideal spectrum \( S^*U_x \times \text{Prim}(J) \). We have \( \Xi \cap \text{Prim}(A^\Gamma_{W_x}) = S^*U_x \times \text{Prim}(\mathcal{A}) \).

The same methods yield the following result (recall that \( M_0 = M(\Gamma_0) \) is the principal orbit bundle).

**Corollary 3.14.** Let \( M_0 := M(\Gamma_0) \), the principal orbit bundle. The central character map \( \text{Prim}(A^\Gamma_{M_0}) \to S^*M_0/\Gamma \) defined by the inclusion \( C_0(S^*M_0/\Gamma) \subset Z(A^\Gamma_{M_0}) \) is a covering with typical fiber \( \text{Prim}(\text{End}(E_x)^{\Gamma_0}) \) such that \( \Xi \cap \text{Prim}(A^\Gamma_{M_0}) \to S^*M_0/\Gamma \) is a subcovering, see (27) for the definition of \( \Xi \). In particular, \( \Xi \cap \text{Prim}(A^\Gamma_{M_0}) \) is open and closed in \( \text{Prim}(A^\Gamma_{M_0}) \).

**Proof.** The first statement is true locally, by Corollary 3.14 and hence it is true globally. Indeed, let \( x \in M_0 \), let \( \xi \in S^*M_0 \), and let \( \rho \in \widehat{\Gamma}_x \) that appears in \( E_x \) (so \( (\xi, \rho) \in \Omega_\Gamma \)). We let \( W_x \subset M_0 \subset M \) be the typical tube with minimal isotropy \( \Gamma_x = \Gamma_0 \), as before. Let \( Z_x := C_0(S^*W_x)^{\Gamma} \subset Z_M = C(S^*M)^{\Gamma} \). Then \( \text{Prim}(Z_x A^\Gamma_{M_0}) \) is an open neighborhood in \( \text{Prim}(A^\Gamma_{M_0}) \) of the primitive ideal \( \ker(\pi_{\xi, \rho}) \), see Proposition 3.11 for notation and details. We have that \( Z_x A^\Gamma_{M_0} = A^\Gamma_{W_x} \Gamma_0 \) and hence, on \( \text{Prim}(Z_x A^\Gamma_{M_0}) \), the central character is a covering, by Corollary 3.10. Similarly, its restriction to \( \Xi \cap \text{Prim}(Z_x A^\Gamma_{M_0}) \) is a covering by Corollary 3.13.

Putting Corollary 3.14 and Proposition 3.12 together we obtain the following results.

**Corollary 3.15.** Let \( M_0 \) be the principal orbit type of \( M \). The ideal \( \ker(\mathcal{R}_{M_0}) = A^\Gamma_{M_0} \cap \ker(\mathcal{R}_M) \) is defined by the closed subset \( \Xi_0 := \Xi \cap \text{Prim}(A^\Gamma_{M_0}) \) of \( \text{Prim}(A^\Gamma_{M_0}) \) consisting of the sheets of \( \text{Prim}(A^\Gamma_{M_0}) \to S^*M_0/\Gamma \) that correspond to the simple factors \( \text{End}(E_{\xi, \rho})^{\Gamma_0} \) of \( \text{End}(E_x)^{\Gamma_0} \) with \( \rho \) and \( \alpha \) \( \Gamma_0 \)-associated.

If \( \Gamma \) is abelian, then \( \rho \) and \( \alpha \) are characters and saying that they are \( \Gamma_0 \)-associated means, simply, that their restrictions to \( \Gamma_0 \) coincide: \( \rho|_{\Gamma_0} = \alpha|_{\Gamma_0} \). This is consistent with the definition given in (11).

### 3.5. The non-principal orbit case

As in the rest of the paper, we assume \( M/\Gamma \) to be connected. We will show in Theorem 3.17 that \( \Xi \) is the closure of \( \Xi_0 \) in \( \text{Prim}(A^\Gamma_{M_0}) \). To that end, we first construct a suitable basis of neighborhoods of \( \text{Prim}(A^\Gamma_{M_0}) \) using Lemma 3.12.

**Remark 3.16.** Let \( \Gamma(\xi, \rho) \in \text{Prim}(A^\Gamma_{M_0}) \), where we have used the description of \( \text{Prim}(A^\Gamma_{M_0}) \) provided in Proposition 3.11 as orbits of pairs \( \xi \in S^*M \) and suitable \( \rho \in \widehat{\Gamma}_\xi \). We construct a basis of neighborhoods \( (V_{\xi, \rho, n})_{n \in \mathbb{N}} \) of \( \Gamma(\xi, \rho) \) in \( \text{Prim}(A^\Gamma_{M_0}) \).
as follows. Let $\xi \in S^r_x M$ (that is, $\xi$ sits above $x \in M$) and we use the notation $U_x$ and $W_x$ of Equation (21), as always.

First, by choosing a different point $\xi$ in its orbit, if necessary, we may assume that $\Gamma_0 \subset \Gamma_\xi$. Now let $(O_n)_{n \in \mathbb{N}}$ be a family of $\Gamma_\xi$-invariant neighborhoods of $\xi$ in $S^* U_x$ such that:

- for all $n$ and $\gamma \in \Gamma \setminus \Gamma_\xi$, we have $\gamma O_n \cap O_n = \emptyset$,
- $O_{n+1} \subset O_n$ and $\bigcap_{n \in \mathbb{N}} O_n = \{\xi\}$.

For any $n \in \mathbb{N}$, we choose a function $\varphi_n \in \mathcal{C}_c(O_n)^{\Gamma_\xi}$ such that $\varphi_n \equiv 1$ on $O_{n+1}$. Let $p_\rho \in \text{End}(E_x)^{\Gamma_\xi}$ be the projection onto $E_x\mathcal{P}$. We can assume the bundle $E$ to be trivial on $U_x$ and, using that, we first extend $p_\rho$ constantly on $O_n$ and then as an element $q_n \in \mathcal{C}_c(S^* U_x; \text{End}(E_x))^{\Gamma_\xi}$ defined as

$$q_n := \begin{cases} \Phi_{\Gamma_\xi, \Gamma_x}(\varphi_n p_\rho) & \text{on } \Gamma_x O_n, \\ 0 & \text{on } S^* U_x \setminus \Gamma_x O_n, \end{cases}$$

with $\Phi_{\Gamma_\xi, \Gamma_x}$ the Frobenius isomorphism of Equation (14). Let us set $\tilde{q}_n := \Phi_{\Gamma_x, \Gamma}(q_n) \in A^F_n$, where $\Phi_{\Gamma_x, \Gamma}$ is the Frobenius isomorphism of Equation (14). Finally, we associate to $\tilde{q}_n$ the open set

$$V_{\xi, \rho, n} := \{ J \in \text{Prim}(A^F_n) \mid \tilde{q}_n \notin J \}.$$ 

Recall from 3.2 that $V_{\xi, \rho, n}$ is an open subset of $\text{Prim}(A^F_n)$. Moreover, it follows from our definition that $V_{\xi, \rho, n+1} \subset V_{\xi, \rho, n}$ and that $\bigcap_{n \in \mathbb{N}} V_{\xi, \rho, n} = \{ \Gamma(\xi, \rho) \}$.

Recall that we are assuming that $M/\Gamma$ is connected.

**Theorem 3.17.** Let $\Xi := \text{Prim}(A^F_M / \ker(\mathcal{R}_M)) \subset \text{Prim}(A^F_n)$ be the closed subset defined by the ideal $\ker(\mathcal{R}_M)$. Then $\Xi$ is the closure in $\text{Prim}(A^F_M)$ of the set $\Xi_0 := \Xi \cap \text{Prim}(A^F_M)$, where $M_0$ is the principal orbit bundle of $M$.

**Proof.** We have that $\Xi_0 \subset \Xi$ since $\Xi_0 \subset \Xi$ and the latter is a closed set. Conversely, let $\Psi \in \text{Prim}(A^F_M) \setminus \Xi_0$. We will show that $\Psi \notin \Xi$. Let $\Psi$ correspond to $(\xi, \rho) \in \Omega_M$, as in Proposition 3.1. We may assume that $\Gamma_0 \subset \Gamma_\xi$. Let $x$ be projection of $\xi$ onto $M$. Since the problem is local, we may also assume that $U_x \subset T_x M$, that $M = W_x := \Gamma \times_{\Gamma_x} U_x$, and that $E := \Gamma \times_{\Gamma_x} (U_x \times \beta)$ for some $\Gamma_x$-module $\beta$.

Using the notations of Remark 3.16, there exists $n > 0$ such that $V_{\xi, \rho, n} \cap \Xi_0 = \emptyset$. Let $\tilde{q}_n = \Phi_{\Gamma_x, \Gamma}(q_n)$ be the symbol defined in Remark 3.16. The description of $\Xi_0$ provided in Corollary 3.15.12 the definition of $V_{\xi, \rho, n}$, and the definition of $\tilde{q}_n$ imply that $\pi_{\xi, \rho'}(\tilde{q}_n) = 0$ for any $\zeta \in S^* M_0$ and $\rho' \in \Gamma_0$ such that $\Gamma(\zeta, \rho') \in \Xi_0$, that is, such that $\rho'$ and $\alpha$ are $\Gamma_0$-associated.

We next “quantize” $\tilde{q}_n$ in an appropriate way, that is, we construct an operator $\tilde{Q}_n \in B(T^*_x M)$ with symbol $\tilde{q}_n$ and with other convenient properties as follows. First, let $\chi \in \mathcal{C}^\infty_c(U_x)^{\Gamma_\xi}$ be such that $\chi \varphi_n = \varphi_n$, which is possible since $\varphi_n$ has compact support. Then let $\psi \in \mathcal{C}^\infty(T^*_x M)^{\Gamma_\xi}$ be such that $\psi(0) = 0$ if $|\eta| < 1/2$ and $\psi(\eta) = 1$ whenever $|\eta| \geq 1$. Recall that in this proof $U_x \subset T_x M$ is identified with its image in $M = \Gamma \times_{\Gamma_x} U_x$ through the exponential map. Let for any symbol $a$

$$\text{Op}(a)(y) := \int_{T^*_x M} \int_{U_x} e^{i(y - z) \eta} a(y, z, \eta) f(z) dz d\eta.$$
We shall use this for \( a_n(y, z, \eta) := \chi(y)\psi(\eta)\tilde{q}_n(\frac{z}{\eta})\chi(z) \), then set
\[
Q_n := Op(a_n),
\]
that is
\[
Q_nf(y) := \int_{T^*M} \int_{U_x} e^{i(y-z)\cdot \eta} \chi(y)\psi(\eta)\tilde{q}_n(\frac{z}{\eta})\chi(z)f(z)dzd\eta
\]
to be the standard pseudodifferential operator on \( U_x \), associated to the symbol \( a_n(y, z, \eta) := \chi(y)\psi(\eta)\tilde{q}_n(\frac{z}{\eta})\chi(z) \). The operator \( Q_n \) is \( \Gamma_x \)-invariant by construction. Using the Frobenius isomorphism of Equation (14), we extend \( Q_n \) to the operator \( \tilde{Q}_n := \Phi(Q_n) \), which acts on \( M = W_x = \Gamma \times \Gamma_x U_x \) (see also Equation (23) with regards to this isomorphism). Then \( \tilde{Q}_n \in \Psi^0(M; E)^\Gamma \), that is, it is \( \Gamma \)-invariant, by construction, and has principal symbol \( \sigma_0(\tilde{Q}_n) = \tilde{q}_n \).

Now let \( x_0 \in M_0 \cap U_x \), where, we recall, \( M_0 := M_{(\Gamma_0)} \) denotes the principal orbit bundle. We have
\[
L^2(W_{x_0}; E) = \text{Ind}_{\Gamma_0}^\Gamma\left( L^2(U_{x_0}; \beta) \right) = L^2(U_{x_0}; \text{Ind}_{\Gamma_0}^\Gamma(\beta)),
\]
where \( \beta = E_{x_0} = E_x \) by the assumption that \( E := \Gamma \times \Gamma_x (U_x \times \beta) \).

Let \( \beta_j \in \tilde{\Gamma}_0 \) be the isomorphism classes of the \( \Gamma_\xi \)-submodules of \( \beta \) and \( k_j \geq 0 \) is the dimension of the corresponding \( \beta_j \)-isotypical component in \( \beta \), so that \( \beta \cong \bigoplus_{j=1}^N \beta_j^{k_j} \), as \( \Gamma_0 \)-modules, as before. Thus
\[
L^2(W_{x_0}; E) \simeq \bigoplus_{j=1}^N L^2(U_{x_0}; \text{Ind}_{\Gamma_0}^\Gamma(\beta_j^{k_j})).
\]

Recall that the \( \alpha \)-isotypical component of \( \text{Ind}_{\Gamma_0}^\Gamma(\beta_j^{k_j}) \) is \( \alpha \otimes \text{Hom}_\Gamma(\alpha, \text{Ind}_{\Gamma_0}^\Gamma(\beta_j^{k_j})) \), which is non-zero if, and only if, \( \alpha \) and \( \beta_j \) are \( \Gamma_0 \)-associated, by the Frobenius isomorphism. Hence, passing to the \( \alpha \)-isotypical components, we have
\[
(38) \quad L^2(W_{x_0}; E)_\alpha := p_\alpha L^2(W_{x_0}; E) = \bigoplus_{j \in J^c} L^2(U_{x_0}; \text{Ind}_{\Gamma_0}^\Gamma(\beta_j^{k_j}))_\alpha,
\]
where \( J \subset \{1, \ldots, N\} \) is the set of indices such that \( \beta_j \in \tilde{\Gamma}_0 \) and \( \alpha \) are \( \Gamma_0 \)-disjoint; \( J^c \) is its complement (i.e. \( \beta_j \in \tilde{\Gamma}_0 \) and \( \alpha \) are \( \Gamma_0 \)-associated).

Let \( p_J \in \text{End}(\beta)^{J^c} \) be the projector onto \( \bigoplus_{j \in J^c} \beta_j^{k_j} \). Recall that \( \pi_{\xi, \beta_j}^\Gamma(\tilde{q}_n) = 0 \) for any \( (\xi, \beta_j) \in S^*M_0 \times \tilde{\Gamma}_0 \) with \( j \notin J \). Therefore \( \tilde{q}_n(\xi)p_J = 0 \), for all \( \xi \in S^*M_0 \).

Since \( S^*M_0 \) is dense in \( S^*M_0 \), this implies that \( \tilde{q}_n(\xi)p_J = 0 \). Thus
\[
\tilde{Q}_np_J = Op(\chi \tilde{q}_n \chi)p_J = Op(\chi \tilde{q}_n \chi p_J) = 0.
\]
Hence for any \( f \in L^2(W_{x_0}; E)_\alpha \), we have that \( \tilde{Q}_nf = 0 \). This is true for any \( x_0 \in M_0 \), so we conclude that \( \tilde{Q}_n \) is zero on \( L^2(M_0; E)_\alpha \). Since \( M_0 \) has measure zero complement in \( M \), we have \( L^2(M_0; E)_\alpha = L^2(M; E)_\alpha \); therefore \( \pi_\alpha(\tilde{Q}_n) = 0 \). This implies that \( R_M(\tilde{q}_n) = 0 \), while \( \pi_{\xi, \rho}(\tilde{q}_n) = 1 \). Thus \( \Gamma(\xi, \rho) \notin \Xi \), which concludes the proof.

Our question now is to decide whether some given \( \Gamma(\xi, \rho) \) is in \( \Xi \) or not. Recall that \( \rho \) and \( \alpha \) are said to be \( \Gamma_0 \)-associated if \( \text{Hom}_{\Gamma_0}(\rho, \alpha) \neq 0 \). The set \( X_{\alpha, \Gamma}^\ast \) was defined in the introduction as the set of pairs \( (\xi, \rho) \in T^*M \setminus \{0\} \times \tilde{\Gamma}_\xi \) for which there is an element \( g \in \Gamma \) such that \( g \cdot \rho \) and \( \alpha \) are \( \Gamma_0 \)-associated.
Remark 3.18. Let us highlight the following interesting fact, implied by the proof of Theorem \ref{t:3.17}. We have that $E_{x\rho} = 0$ for any $(\xi, \rho) \in X_{M,0}^\alpha$ (with $x$ the projection of $\xi$ on $M_0$) if, and only if, $L^2(M; E)^\alpha = 0$.

Indeed, for any $x \in M_0$ with $\Gamma_x = \Gamma_0$, we have noted in Equation \eqref{eq:3.58} that

$$L^2(W_x; E)^\alpha = \bigoplus_{\rho} L^2(U_x; \text{Ind}_{\Gamma_0}^\Gamma(E_{x\rho}))^\alpha,$$

where the direct sum is indexed by the representations $\rho \in \tilde{\Gamma}_0$ that are $\Gamma_0$-associated to $\alpha$. If $E_{x\rho} = 0$ for any such representation, then $L^2(W_x; E)^\alpha = 0$. Such open sets $W_x$ cover $M_0$, so $L^2(M_0; E)^\alpha = 0$. Since $M_0$ has measure zero complement, we conclude that $L^2(M; E)^\alpha = 0$.

Proposition 3.19. We use the notation in the last two paragraphs. We have $\Gamma(\xi, \rho) \in \Xi$ if, and only if, there is a $g \in \Gamma$ such that $g \cdot \rho$ and $\alpha$ are $\Gamma_0$-associated.

Proof. Let $\Gamma(\xi, \rho) \in \text{Prim}(A_M^\alpha)$, with $x \in M$ the base point of $\xi$. We can assume (by choosing a different element in the orbit if needed) that $\Gamma_0 \subseteq \Gamma_x$. Let $\tilde{q}_n \in A_M^\alpha$ be the element defined in Remark \ref{r:8.16} and $V_{\xi,\rho,n}$ the corresponding neighbourhood of $\Gamma(\xi, \rho)$ in $\text{Prim}(A_M^\alpha)$.

There is a $\Gamma_x$-equivariant isomorphism $E|_{U_x} \cong U_x \times \beta$, where $\beta = E_x$ is a $\Gamma_x$-module. Since $\Gamma_0 \subseteq \Gamma_x$, we may decompose $\beta$ into $\Gamma_0$-isotypical components, i.e. $\beta = \bigoplus_{j=1}^N \beta^j$, with the usual notation. If $\eta \in \mathcal{O}_n$, then $\pi_{\eta,\beta_j}(\tilde{q}_n) = \varphi_n(\eta)\pi_{\beta_j}(p_{\rho})$. Therefore, for any $\eta \in S^*M$, we have

$$\pi_{\eta,\beta_j}(\tilde{q}_n) = 0 \iff \text{Hom}_{\Gamma_0}(\beta_j, \rho) = 0 \text{ or } \tilde{q}_n(\eta) = 0.$$

This implies that

$$V_{\xi,\rho,n} \cap \Xi_0 = \{ \Gamma(\eta, \beta) \in \Xi_0 \mid \tilde{q}_n(\eta) \neq 0 \text{ and } \text{Hom}_{\Gamma_0}(\beta, \rho) \neq 0 \}.$$

It follows from the determination of $\Xi_0$ in Corollary \ref{c:3.15} that $V_{\xi,\rho,n} \cap \Xi_0 \neq \emptyset$ if, and only if, we have $\text{Hom}_{\Gamma_0}(\rho, \alpha) \neq 0$. Now $\Xi = \Xi_0$ by Theorem \ref{t:3.17}. Since the open sets $(V_{\xi,\rho,n})_{n \in \mathbb{N}}$ form a basis of neighborhoods of $\Gamma(\xi, \rho)$, we conclude that $\Gamma(\xi, \rho) \in \Xi$ if, and only if, we have $\text{Hom}_{\Gamma_0}(\rho, \alpha) \neq 0$. \hfill $\square$

Remark 3.20. Our definition of $\alpha$-ellipticity for an operator $P \in \overline{\mathcal{P}}(M; E)^\Gamma$ was stated in terms of the set $X_{M,\Gamma}^\alpha$, defined in Equation \eqref{eq:3.19}. Proposition \ref{p:3.19} establishes that $\Xi \cong \tilde{X}_{M,\Gamma}^\alpha / \Gamma$, where $\tilde{X}_{M,\Gamma}^\alpha$ is the (possibly smaller) subset of pairs $(\xi, \rho) \in X_{M,\Gamma}^\alpha$ such that $E_{x\rho} \neq 0$ (with $x$ the projection of $\xi$ on $M$). Keeping in mind the fact that the null operator on a trivial vector space is invertible, we have that $\sigma_{\alpha}^k(P)(\xi, \rho)$ is invertible for any $(\xi, \rho) \in X_{M,\Gamma}^\alpha$, if, and only if, it is invertible for any $(\xi, \rho) \in \tilde{X}_{M,\Gamma}^\alpha$. The pathological case $\Xi = \emptyset$, for which $E_{x\rho} = 0$ for any $(\xi, \rho) \in X_{M,\Gamma}^\alpha$, causes no problem: indeed, as noticed in Remark \ref{r:3.18}, we then have $L^2(M; E)^\alpha = 0$. In that case $\pi_{\alpha}(P)$ is Fredholm for any $P \in \mathcal{P}(M; E)^\Gamma$, which is consistent with the invertibility of $\sigma_{\alpha}^k(P)(\xi, \rho) : 0 \to 0$ for any $(\xi, \rho) \in X_{M,\Gamma}^\alpha$.

We summarize part of the above discussions in the following proposition.

Proposition 3.21. Let $\tilde{X}_{M,\Gamma}^\alpha$ be as in Remark \ref{r:3.20}. The primitive ideal spectrum $\Xi = \text{Prim}(A_M^\Gamma / \ker(\mathcal{R}_M))$ is canonically homeomorphic to $\tilde{X}_{M,\Gamma}^\alpha / \Gamma$ via the restriction map from $A_M^\Gamma := C(S^*M; \text{End}(E))^\Gamma$ to sections over $X_{M,\Gamma}^\alpha$. 
4. Applications and Extensions

We now prove the main result of the paper, Theorem 1.2, on the characterization of Fredholm operators, and discuss some extensions of our results. We first explain how to reduce the proof to the case $M/\Gamma$ connected and we discuss in more detail the $\Gamma$-principal symbol and $\alpha$-ellipticity (this discussion can be skipped at a first lecture).

4.1. Reduction to the connected case and $\alpha$-ellipticity. In this subsection, unlike most of the rest of the paper, we do not assume that $M/\Gamma$ is connected in order to explain how to reduce the general case to the connected one. We do assume however, as always, that $M$ is compact. We also provide some other reductions of our proof.

Let $\pi_{M,\Gamma} : M \to M/\Gamma$ be the quotient map and let us write then $M/\Gamma = \bigcup_{i=1}^{N} C_i$ as the disjoint union of its connected components. We let $M_i := \pi_{M,\Gamma}^{-1}(C_i)$ be the preimages of these connected components. Note that, in general, the submanifolds $M_i$ are not connected, but, for each $i$, $M_i/\Gamma = C_i$ is connected. In particular, a similar discussion applies to yield the definition of the space

$$X_{M,\Gamma}^\alpha := \bigcup_{i=1}^{N} X_{M_i,\Gamma}^\alpha$$

as a disjoint union of the spaces $X_{M_i,\Gamma}^\alpha$, which makes sense since each of the spaces $M_i$ is invariant for $\Gamma$ and $M_i/\Gamma$ is connected. (See Equation (4) of the Introduction for the definition of the spaces $X_{M_i,\Gamma}^\alpha$.)

We shall decorate with the index $i$ the restrictions of objects on $M$ to $M_i$. Thus, $E_i := E|_{M_i}$, and so on and so forth. This almost works for an operator $P \in \overline{\psi^0}(M; E)^\Gamma$. Indeed, we first notice that

$$L^2(M; E) \simeq \bigoplus_{i=1}^{N} L^2(M_i; E_i) \quad \text{and} \quad \bigoplus_{i=1}^{N} \overline{\psi^0}(M_i; E_i) \subset \overline{\psi^0}(M; E).$$

Recall that $\mathcal{K}(V)$ denotes the algebra of compact operators on a Hilbert space $V$. The following proposition provides the desired reduction to the connected case.

**Proposition 4.1.** Let $p_i : L^2(M; E) \to L^2(M_i; E_i)$ be the canonical orthogonal projection. For $P \in \overline{\psi^0}(M; E)$, we let $P_i := p_i P p_i \in \overline{\psi^0}(M_i; E_i)$. Then $P - \sum_{i=1}^{N} P_i \in \mathcal{K}(L^2(M; E))$. If we regard $\sum_{i=1}^{N} P_i = \bigoplus_{i=1}^{N} P_i$ as an element of $\bigoplus_{i=1}^{N} \overline{\psi^0}(M_i; E_i)$, then we see that

$$\overline{\psi^0}(M; E) = \bigoplus_{i=1}^{N} \overline{\psi^0}(M_i; E_i) + \mathcal{K}(L^2(M; E)).$$

Moreover, $\pi_\alpha(P) - \bigoplus_{i=1}^{N} \pi_\alpha(P_i)$ is compact and hence $\pi_\alpha(P)$ is Fredholm if, and only if, each $\pi_\alpha(P_i)$ is Fredholm, for $i = 1, \ldots, N$.

**Proof.** If $i \neq j$, $p_i P p_j$ has zero principal symbol, and hence it is compact. Therefore $P - \sum_{i=1}^{N} P_i = \sum_{i\neq j} p_i P p_j$ is compact. The rest follows from Equation (10), its corollary $L^2(M; E)_\alpha \simeq \bigoplus_{i=1}^{N} L^2(M_i; E_i)_\alpha$, and the fact that $\pi_\alpha$ respects these direct sum decompositions. \qed

Recall the algebras $A_{C_1}$ of symbols of the previous section, see Equation (20), where $\mathcal{O}$ is an open subset of $M$.

**Remark 4.2.** The $\Gamma$-principal symbol $\sigma_m^\Gamma(P)$ was defined in (19), and we stress that the definition of the space $X_{M,\Gamma}$ did not require that $M/\Gamma$ be connected. The disjoint union definition of the space $X_{M,\Gamma} = \bigcup_{i=1}^{N} X_{M_i,\Gamma}$ means that

$$\sigma_m^\Gamma(P)|_{X_{M,\Gamma}} = \sigma_m^\Gamma(P_1)$$
for each $i = 1, \ldots, N$. The analogous disjoint union decomposition of $X^\alpha_{M, \Gamma} := \sqcup_{i=1}^N X^\alpha_{M_i, \Gamma}$ gives that $P$ is $\alpha$-elliptic if, and only if, for each $i$, $P_i$ is $\alpha$-elliptic.

This allows us to reduce the proof of our main theorem, Theorem 4.2 to the connected case since, assuming that the connected case has been proved, we have

$$\pi_\alpha(P) \text{ is Fredholm} \Leftrightarrow \forall i, \pi_\alpha(P_i) \text{ is Fredholm}$$

$$\Leftrightarrow \forall i, P_i \text{ is } \alpha\text{-elliptic}$$

$$\Leftrightarrow P \text{ is } \alpha\text{-elliptic},$$

where the first equivalence is by Proposition 4.1, the second equivalence is by the assumption that our main theorem has been proved in the connected case, and the last equivalence is by the first part of this remark.

We now resume our assumption that $M/\Gamma$ is connected, for convenience. In particular, $\Gamma_0$ will be a minimal isotropy group, which is unique up to conjugacy (since we are again assuming that $M/\Gamma$ is connected). We shall take a closer look next at the $\Gamma$- and $\alpha$-principal symbols, so the following simple discussion will be useful. Recall that if $K \subset \Gamma$, $\rho \in \hat{\Gamma}$, and $g \in \Gamma$, then $g \cdot K := gKg^{-1}$ and $(g \cdot \rho)(\gamma) := \rho(g^{-1}\gamma g)$, so that $g \cdot \rho$ is an irreducible representation of $g \cdot K$ (i.e. $g \cdot \rho \in \hat{g \cdot K}$).

**Remark 4.3.** Let $\xi \in T^*M \setminus \{0\}$ and $\rho \in \hat{\Gamma}$ (that is, $(\xi, \rho) \in X_{M, \Gamma}$). Then the following three statements are equivalent:

(i) the pair $(\xi, \rho) \in X^\alpha_{M, \Gamma}$;

(ii) there is $g \in \Gamma$ such that $\Gamma_0 \subset g \cdot \Gamma_\xi = \Gamma_{g\xi}$ and such that $g \cdot \rho$ and $\alpha$ are $\Gamma_0$-associated;

(iii) There is $\gamma \in \Gamma$ such that $\gamma \cdot \Gamma_0 \subset \Gamma_\xi$ and $\text{Hom}_{\gamma \Gamma_0}(\rho, \alpha) \neq 0$.

Indeed, if (i) is satisfied, then the definition of $X^\alpha_{M, \Gamma}$ (see Equation (7) and Definition 2.2) is equivalent to the existence of $g$, i.e. (i) $\Leftrightarrow$ (ii). Recalling that $g \cdot \rho \in \Gamma_{g\xi}$, we stress then that we need $\Gamma_0 \subset g \cdot \Gamma_\xi = \Gamma_{g\xi}$ for $\alpha$ and $g \cdot \rho$ to be associated.

To prove (ii) $\Leftrightarrow$ (iii), let $g = g^{-1}$. We have $\Gamma_0 \subset g \cdot \Gamma_\xi$ and $\text{Hom}_{\gamma \Gamma_0}(g \cdot \rho, \alpha) \neq 0$ if, and only if, $\gamma \cdot \Gamma_0 \subset \Gamma_\xi$ and $\text{Hom}_{\gamma \Gamma_0}(\rho, \gamma \cdot \alpha) \neq 0$. The result follows from the fact that $\alpha$ and $\gamma \cdot \alpha$ are equivalent (since $\gamma \in \Gamma$ and $\alpha$ is a representation of $\Gamma$).

We include next below, in Proposition 4.5, a reformulation of our $\alpha$-ellipticity condition in terms of the fixed point manifold $S^*M^{\Gamma_0}$, with $\Gamma_0$ a minimal isotropy subgroup as before. This result was suggested by some discussions with P.-E. Paradan, whom we thank for his useful input.

In the following, $\text{Stab}_\Gamma(M)$ will denote the set of stabilizer subgroups $K$ of $\Gamma$, that is, the set of subgroups $K \subset \Gamma$ such that there is $m \in M$ with $K = \Gamma_m$. It is a finite set, since $\Gamma$ is finite. Similarly, we let

$$\text{Stab}^{\Gamma_0}_\Gamma(M) := \{K \in \text{Stab}_\Gamma(M) \mid \Gamma_0 \subset K\}.$$ 

Note that $\text{Stab}_\Gamma(T^*M) = \text{Stab}_\Gamma(M)$. Recall also that $(T^*M)^K = T^*(M^K)$, where $M^K$ is the submanifold of fixed points of $M$ by $K$, as usual. For a $\Gamma$-space $X$ and $K \subset \Gamma$ a subgroup, we shall let $X_K := \{x \in X \mid \Gamma_x = K\} \subset X^K$ denote the set of points of $X$ with isotropy $K$. Note that, in general, $T^*(M_K) \neq (T^*M)_{\langle K \rangle}$.

**Lemma 4.4.** The set $M_K := \{m \in M \mid \Gamma_m = K\}$ is a submanifold.
Proof. Let \( x \in M_K \), that is, \( \Gamma_x = K \). The problem is local, so, using, \([79\text{ Proposition 5.13}]\), we see that it suffices to consider the case \( M = \Gamma \times_K V \), where \( V \) is a \( K \)-representation. Then, if \( z = (\gamma, y) \in \Gamma \times_K V \), we have \( \Gamma_z = \gamma K_y \gamma^{-1} \) and hence, if \( \Gamma_z = K \), we obtain \( K = \gamma K_y \gamma^{-1} \), which, in turn, gives \( K_y = K \) and \( \gamma \in N(K) := \{ g \in \Gamma \mid g K g^{-1} = K \} \). We thus obtain that
\[
M_K = \{ (\gamma, y) \in \Gamma \times_K M \mid K_y = \gamma^{-1} K \gamma \} = N(K) \times_K V^K,
\]
which is a submanifold of \( M \). \( \square \)

Let \( K \subset \Gamma \) be a subgroup and \( \rho \in \hat{K} \). Then \( E_\rho := \bigcup_{x \in M_K} E_{x\rho} \) is a smooth vector bundle over \( M^K \), the set of fixed points of \( M \) with respect to \( K \). Similarly, \((E \otimes \rho)^K \to M^K \) is a smooth vector bundle (over \( M^K \)). Moreover, we have an isomorphism
\[
\text{End}(E_\rho)^K \cong \text{End}((E \otimes \rho)^K \otimes \rho)^K \cong \text{End}((E \otimes \rho)^K),
\]
of vector bundles over \( M^K \), where the last isomorphism comes from the fact that \( \text{End}(\rho)^K = \mathbb{C} \). In view of this discussion, we choose to state the following result in terms of the vector bundle \((E \otimes \rho)^K \) over \( M^K \) rather than in terms of \( E_\rho \). This discussion shows also that it is enough in our proofs to assume that \( \alpha \) is the trivial (one-dimensional) representation.

**Proposition 4.5.** Let \( \alpha \in \hat{\Gamma} \) and \( P \in \psi^m (M; E) \), for some \( m \in \mathbb{R} \). Recall the vector bundle \((M \otimes \rho)^K \to M^K \supset M_K \). The following are equivalent:

1. \( P \) is \( \alpha \)-elliptic (Definition \([77]\)).
2. For all \( K \in \text{Stab}_1^\Gamma_0(M) \) and all \( \rho \in \hat{K} \) that are \( \Gamma_0 \)-associated with \( \alpha \), we have that \( (\sigma_m(P) \otimes \text{id}_\rho)|_{(E \otimes \rho)^K} \) defines by restriction an invertible element of
\[
\mathcal{C}^\infty \left( (T^*M \setminus \{ 0 \})_K, \text{End}((E \otimes \rho)^K) \right).
\]
3. The principal symbol \( (\sigma_m(P) \otimes \text{id}_\alpha)|_{(E \otimes \alpha)^\Gamma_0} \) defines by restriction an invertible element in
\[
\mathcal{C}^\infty \left( (T^*M)^{\Gamma_0} \setminus \{ 0 \}; \text{End}((E \otimes \alpha)^{\Gamma_0}) \right).
\]

Recall that for representations \( \alpha \) and \( \beta \) to be \( H \)-associated, they have to be defined, after restriction, on \( H \). See Definition \([22]\).

**Proof.** Recall that \( P \) is \( \alpha \)-elliptic if the restriction of \( \sigma_m^\Gamma_0(P) \) to \( X^\alpha_{M, \Gamma} \) is invertible (see Remark \([43]\)) for a detailed definition and discussion of the space \( X^\alpha_{M, \Gamma} \), appearing in the definition of \( \alpha \)-ellipticity).

Let \( K \in \text{Stab}_1^\Gamma_0(M) \) (so \( \Gamma_0 \subset K \)), \( \rho \in \hat{K} \), and \( \xi \in T^*_x M \setminus \{ 0 \} \) with \( \Gamma_\xi = K \). We have that \( (\sigma_m(P) \otimes \text{id}_\rho)|_{(E \otimes \rho)^K} \) is invertible at \( \xi \in (T^*M)_K \) if, and only if, the restriction of \( \sigma_m(P)(\xi) \) to \( E_{x\rho} \) is invertible, since they correspond to each other under the isomorphism of Equation \([41]\). The relation (2) thus means that the restriction of the principal symbol \( \sigma_m(P) \) is invertible on a subset of \( X^\alpha_{M, \Gamma} \), so (1) implies (2) right away.

Let us show next that (2) implies (1), let \( \xi \in T^*M \setminus \{ 0 \} \) and let \( K' := \Gamma_\xi \). By definition, \( \xi \) belongs to \((T^*M)_{K'} \). Assume that \( (\xi, \rho) \in X^\alpha_{M, \Gamma} \). This means that there exists \( g \in \Gamma \) such that \( \rho' := g \cdot \rho \) and \( \alpha \) are \( \Gamma_0 \) associated (see Equation \([11]\) and Definition \([22]\) alternatively, this is also recalled in Remark \([13]\)). For this to make sense, it is implicit that
\[
\Gamma_0 \subset \Gamma_{g \xi} = g \cdot \Gamma_\xi = g \cdot K' =: K
\]
(again, see Remark 4.3). Then \( g : (T^*M)_{K'} \to (T^*M)_K \) is a diffeomorphism. Condition (2) for the group \( K \) gives that \( \pi_{g\xi,\rho'}(\sigma_m(P)) \) is invertible, since the irreducible representation \( \rho' \) of \( \Gamma_{g\xi} \) is \( \Gamma_0 \)-associated to \( \alpha \) (we have used here again the isomorphism (11)). Furthermore, \( g : E_{\xi,\rho} \to E_{g\xi,\rho'} \) is an isomorphism. Now, by the \( \Gamma \)-invariance of \( \sigma := \sigma_m(P) \), we have \((g^{-1}\sigma)(\xi) = g^{-1}(\sigma(g\xi))g = \sigma(\xi)\) therefore \( \pi_{g\xi,\rho'}(\sigma) \) is invertible if, and only if, \( \pi_{\xi,\rho}(\sigma) \) is.

For the equivalence of (1) and (3), we can assume that \( m = 0 \). Recall first that the density of \( \Xi_0 \) in \( \Xi \) established in Theorem 3.17 gives that the family of representations

\[
\mathcal{F}_0 := \{ \pi_{\xi,\rho} \mid (\xi, \rho) \in X^\alpha_{M,\Gamma}, \Gamma_{\xi} = \Gamma_0 \}
\]

is faithful for the \( C^* \)-algebra \( A^\Gamma_M / \ker(\mathcal{R}_M) \) (see e.g. [71, Theorem 5.1]). In other words, the restriction morphism

\[
A^\Gamma_M / \ker(\mathcal{R}_M) \to \bigoplus_{\rho \in \Gamma_0} C((T^*M \setminus \{0\})_{\Gamma_0}, \text{End}(E_\rho)_{\Gamma_0})
\]

is injective. Since \((T^*M)_{\Gamma_0} \) is dense in \( T^*M_{\Gamma_0} \), it follows that the restriction morphism

\[
R_M : A^\Gamma_M / \ker(\mathcal{R}_M) \to \bigoplus_{\rho \in \Gamma_0} C(T^*M_{\Gamma_0} \setminus \{0\}, \text{End}(E_\rho)_{\Gamma_0})
\]

is also injective.

Let us write \( \alpha|_{\Gamma_0} = \bigoplus_{\rho \in \Gamma_0} m_\rho \rho \), with multiplicities \( m_\rho \geq 0 \). By considering the representations \( \rho \) with \( m_\rho > 0 \), we see that there is an injective vector bundle morphism over the manifold \( M_{\Gamma_0} \) defined by

\[
\Psi : \bigoplus_{\rho \in \Gamma_0} \text{End}(E_\rho)_{\Gamma_0} \simeq \bigoplus_{\rho \in \Gamma_0} \text{End}((E \otimes \rho)_{\Gamma_0}^{\Gamma_0}) \hookrightarrow \text{End}((E \otimes \alpha)_{\Gamma_0}^{\Gamma_0})
\]

where the last morphism maps any element \( T \in \text{End}((E \otimes \rho)_{\Gamma_0}^{\Gamma_0}) \) to a direct sum of copies of \( T \) acting on the direct summand \( [(E \otimes \rho)_{\Gamma_0}^{\Gamma_0}]^{m_\rho} \subset (E \otimes \alpha)_{\Gamma_0}^{\Gamma_0} \).

Condition (3) amounts to the fact that

\[
\Psi(R_M(\sigma^\Gamma_0(P))) \in C^\infty(T^*M_{\Gamma_0} \setminus \{0\}; \text{End}((E \otimes \alpha)_{\Gamma_0}^{\Gamma_0})
\]

is invertible. To establish that (1) \( \iff \) (3), we thus need to prove that \( P \) is \( \alpha \)-elliptic if, and only if, \( \Psi(R_M(\sigma^\Gamma_0(P))) \) is invertible.

Recall the definition of the symbol algebras \( A_M \) from Equation (25). We have that \( P \in \mathfrak{so}(M; E) \) is \( \alpha \)-elliptic if, and only if, the image of \( \sigma^\Gamma_0(P) \) in the quotient algebra \( A^\Gamma_M / \ker(\mathcal{R}_M) \) is invertible (by the determination of \( \ker(\mathcal{R}_M) \) in Remark 3.20 or Proposition 3.21). But since both \( \Psi \) and \( R_M \) are injective, \( \Psi \circ R_M \) is injective on \( A^\Gamma_M / \ker(\mathcal{R}_M) \). Thus \( \sigma^\Gamma_0(P) \) is invertible in the quotient algebra \( A^\Gamma_M / \ker(\mathcal{R}_M) \) if, and only if, \( \Psi(R_M(\sigma^\Gamma_0(P))) \) is invertible. As we have seen above, this amounts to (1) \( \iff \) (3).

\[
\square
\]

4.2. Fredholm conditions and Hodge and index theory. We continue to assume that \( M \) is a compact smooth manifold. We have the following \( \Gamma \)-equivariant version of Atkinson’s theorem. (Recall that \( \mathcal{K}(V) \) denotes the algebra of compact operators acting on the Hilbert space \( V \).)
Proposition 4.6. Let \( V \) be a Hilbert space with a unitary action of \( \Gamma \) and \( P \in \mathcal{L}(V)^{\Gamma} \) be a \( \Gamma \)-equivariant bounded operator on \( V \). We have that \( P \) is Fredholm if, and only if, it is invertible modulo \( \mathcal{K}(V)^{\Gamma} \), in which case, we can choose the parametrix (i.e. the inverse modulo the compacts) to also be \( \Gamma \)-invariant.

**Proof.** See for example [11 Proposition 5.1]. \( \square \)

Since \( \pi_\alpha(\mathcal{K}(L^2(M; E))^\Gamma) = \mathcal{K}(L^2(M; E))^{\alpha\Gamma} \), we obtain the following corollary.

**Corollary 4.7.** Let \( P \in \mathcal{L}(M; E)^\Gamma \) and \( \alpha \in \widehat{\Gamma} \). We have that \( \pi_\alpha(P) \) is Fredholm on \( L^2(M; E)_{\alpha\Gamma} \) if, and only if, \( \pi_\alpha(P) \) is invertible modulo \( \pi_\alpha(\mathcal{K}(L^2(M; E))^\Gamma) \) in \( \pi_\alpha(\mathcal{L}(M; E)^\Gamma) \).

We are now in a position to prove the main result of this paper, Theorem 1.2.

**Proof of Theorem 1.2.** As in [11 Section 2.6], we may assume that \( P \in \mathcal{L}(M; E)^\Gamma \). Corollary 4.7 then states that \( \pi_\alpha(P) \) is Fredholm if, and only if, the image of its symbol \( \sigma(P) \) is invertible in the quotient algebra

\[
\mathcal{R}_M(A^\Gamma_{M,G}) = \pi_\alpha(\mathcal{L}(M; E)^\Gamma)/\pi_\alpha(\mathcal{K}(L^2(M; E))^\Gamma).
\]

According to Proposition 3.19 and Remark 3.20 following it, the primitive spectrum \( \Xi \) of \( \mathcal{R}_M(A^\Gamma_{M,G}) \) identifies with \( \Sigma^\alpha_{M,G} \). Therefore \( \mathcal{R}_M(\sigma(P)) \) is invertible if, and only if, the endomorphism \( \pi_{\xi,\rho}(\sigma(P)) \) is invertible for all \( (\xi, \rho) \in \Sigma^\alpha_{M,G} \); i.e. if, and only if, \( P \) is \( \alpha \)-elliptic (see Definition 1.1). \( \square \)

**Remark 4.8.** Let \( P : H^s(M; E) \to H^{s-m}(M; E) \) be an order \( m \), classical pseudodifferential operator. Since the index of Fredholm operators is invariant under small perturbations and under compact perturbations, we obtain that the index of \( \pi_\alpha(P) \) depends only on the homotopy class of its \( \alpha \)-principal symbol \( \sigma^\alpha_m(P) \).

An alternative approach to the Fredholm property (Theorem 1.2) can be obtained from the following theorem. Recall that \( X^\alpha_{M,G} \) was defined in (7). Below, by \( \partial \) we shall denote the connecting morphism in the six-term \( K \)-theory exact sequence associated to a short exact sequence of \( C^* \)-algebras. Recall that \( \sigma^\alpha_0 \) is the \( \alpha \)-principal symbol map, see Definition 1.1

**Theorem 4.9.** Let us denote by \( \mathcal{C}(X^\alpha_{M,G}) \) the algebra of restrictions of \( A_M := \mathcal{C}(S^*M; \text{End}(E))^\Gamma \) to \( X^\alpha_{M,G} \). Using the notation of Corollary 4.7, we have an exact sequence

\[
0 \to \mathcal{K} \to \pi_\alpha(\mathcal{L}(M; E)^\Gamma) \xrightarrow{\sigma^\alpha_0} \mathcal{C}(X^\alpha_{M,G}) \to 0.
\]

Let \( \partial : K_1(\mathcal{C}(X^\alpha_{M,G})) \to \mathbb{Z} \simeq K_0(\mathcal{K}) \) be the associated connecting morphism and let \( P \in \mathcal{L}(M; E)^\Gamma \) such that \( \pi_\alpha(P) \) is Fredholm. Then \( \text{ind}(\pi_\alpha(P)) = \dim(\alpha)\partial[\sigma^\alpha_0(P)] \).

**Proof.** The exactness of the sequence follows from the proof of Corollary 4.7 and the fact that \( \mathcal{K}(L^2(M; E))_{\alpha\Gamma} \simeq \mathcal{K} \), the algebra of compact operators on a model separable Hilbert space \( \mathcal{H} \). Under this isomorphism, the resulting representation of \( K \) on \( L^2(M; E)_{\alpha\Gamma} \) is isomorphic to \( \dim(\alpha) \) times the standard representation of \( K \) on \( \mathcal{H} \). This justifies the factor \( \dim(\alpha) \). The rest follows from the fact that the index is the connecting morphism in \( K \)-theory for the Calkin exact sequence. See [63] for more details. \( \square \)
Remark 4.10. As in [63], it follows that the index of $\pi_\alpha(P)$ with $P \in \psi^0(M; E)^\Gamma$ is the pairing between a cyclic cocycle $\phi$ on $\mathcal{C}_c^\infty(X_{M, \Gamma})$ (the algebra of principal symbols of operators in $\psi^0(M; E)^\Gamma$) and the $K$-theory class of the $\alpha$-principal symbol of $P$ [28]. See also [23, 29, 46, 55, 66, 81]. Lemma 3.8 gives that the restriction of this cyclic cocycle to the principal orbit bundle is the usual Atiyah-Singer cocycle (i.e. the cocycle that yields the Atiyah-Singer index theorem in cyclic homology [30, 52, 63, 68], which thus corresponds, after suitable rescaling, to the Todd class). The full determination of the class of the index cyclic cocycle $\phi$ require, however, a non-trivial use of cyclic homology, since the quotient algebra $\mathcal{C}_c^\infty(X_{M, \Gamma})$ is non-commutative, in general.

Remark 4.11. As for the case of compact complex varieties [40, 83], we can consider complexes of operators [18] and the corresponding notion of $\alpha$-ellipticity. In particular, we obtain the finiteness of the corresponding cohomology groups if the complex is $\alpha$-elliptic. This is related to the Hodge theory of singular spaces [2, 15, 19, 26, 77, 78] Moreover, since a general $P$ may act between different bundles $E_0$ and $E_1$, it would be convenient to extend our framework to Connes’ tangent groupoid [29, 3]. However, this goes beyond the scope of this paper.

4.3. Special cases. We now specialize our main result to some particular cases.

4.3.1. The abelian group case [11]. Many statements and definitions become easier in the case of abelian groups. In particular, if $\Gamma_i$, $i = 1, 2$, are both abelian, then the irreducible representations $\alpha_i \in \hat{\Gamma}_i$ are characters, that is, morphisms $\alpha_i : \Gamma_i \to \mathbb{C}^*$, and we have that they are $H$-associated for some subgroup $H$ if, and only if, $\alpha_1|_H = \alpha_2|_H$.

Let $\alpha$ be an irreducible representation of $\Gamma$. When $\Gamma$ is abelian, the conjugacy class of isotropy subgroups corresponding to the principal orbit type of the action has only one element, namely $\Gamma_0$. In that case, the set $X_{M, \Gamma}^\alpha$ defined in Equation 17 of the introduction has the simpler expression:

$$X_{M, \Gamma}^\alpha = \{(x, \rho) \mid x \in T^* M \setminus \{0\}, \rho \in \hat{\Gamma}_x, \rho|_{\Gamma_0} = \alpha|_{\Gamma_0}\}.$$  

As a consequence, it is easier to check the $\alpha$-ellipticity for an operator $P$ in the abelian case. Let $E, F$ be $\Gamma$-equivariant vector bundles over $M$ and set $\alpha_0 := \alpha|_{\Gamma_0}$. Recall that, for any $x \in M$, we denote by $E_{x\alpha_0}$ the $\alpha_0$-isotypical component of $E_x$, seen as a $\Gamma_0$-representation. We then recover the main result of [11]. Indeed, Theorem 1.2 can then be stated as follows:

Theorem 4.12. [11] Theorem 1.2] Let $\Gamma$ be a finite, abelian group acting on a smooth, compact manifold $M$ and let $P \in \psi^m(M; E, F)^\Gamma$. Then, for any $s \in \mathbb{R}$, the following are equivalent:

1) the operator $\pi_\alpha(P) : H^s(M; E)^\alpha \to H^{s-m}(M; F)^\alpha$ is Fredholm,
2) for all $(x, \xi) \in T^*M \setminus \{0\}$, the restriction of $\sigma(P)(x, \xi)$ defines an isomorphism $\pi_{x_0}(\sigma(P)(x, \xi)) : E_{x_0} \to F_{x_0}.$

4.3.2. Scalar operators. Our main theorem becomes quite explicit when we are dealing with scalar operators, i.e. when the vector bundles $E_i = M \times \mathbb{C}$, where $\mathbb{C}$ denotes the trivial representation of $\Gamma$. 

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Proposition 4.13. Let $P : H^s(M) \to H^{s-m}(M)$ be a $\Gamma$-invariant pseudodifferential operator. Let $\alpha \in \hat{\Gamma}$. Then $P$ is $\alpha$-elliptic if, and only if, $\sigma(P)(\xi)$ is invertible for all $\xi \in T^*M \setminus \{0\}$ such that $\alpha$ is $\Gamma_0$-associated to the trivial (constant 1) representation of $\Gamma_\xi$.

Proof. Let $\hat{\Gamma}_\xi$ denote the trivial representation of $\Gamma_\xi$ and let $(\xi, \rho) \in X^\alpha_{M, \Gamma}$. If $\rho \neq \hat{\Gamma}_\xi$ then $C_\rho = 0$ and then $\pi_\rho(\sigma(P)(\xi)) : 0 \to 0$ is invertible. Now if $\rho = \hat{\Gamma}_\xi$ then $(\xi, \rho) \in X^\alpha_{M, \Gamma}$ if, and only if, $\alpha$ is $\Gamma_0$-associated to $\hat{\Gamma}_\xi$. \hfill $\square$

4.3.3. Trivial actions. Assume that $\Gamma$ acts trivially on $M$ (in particular, $M$ is then also connected). Our assumption implies that $\Gamma_0 = \Gamma_\xi = \Gamma$, for all $\xi \in T^*M \setminus \{0\}$. It follows that $\rho \in \hat{\Gamma}_\xi$ is $\Gamma_0$-associated to $\alpha \in \hat{\Gamma}$ if, and only if, $\alpha = \rho$.

Let $E \to M$ be a $\Gamma$-equivariant vector bundle. For any $x \in M$, recall that we denote $E_x$ the $\alpha$-isotypical component of $E_x$. Assuming $M$ to be connected, we have that $E_\alpha = \bigcup_{x \in M} E_x$ is a $\Gamma$-equivariant sub-vector bundle of $E$. Our main result then becomes the following statement.

Proposition 4.14. Assume that $\Gamma$ acts trivially on $M$ and let $\alpha \in \hat{\Gamma}$. Let $E, F$ be two $\Gamma$-equivariant vector bundles over $M$ and let $P \in \psi^m(M; E, F)^\Gamma$. Then for any $s \in \mathbb{R}$, the following are equivalent

1. $\pi_\alpha(P) : H^s(M; E_\alpha) \to H^{s-m}(M; F_\alpha)$ is Fredholm,
2. for all $(x, \xi) \in T^*M \setminus \{0\}$, the morphism

$$\pi_\alpha(\sigma(P)(x, \xi)) : E_{x\alpha} \to F_{x\alpha}$$

is invertible,
3. for all $(x, \xi) \in T^*M \setminus \{0\}$, the morphism

$$\sigma_m(P) \otimes \text{id}_{\alpha^*}(x, \xi) : \text{Hom}_\Gamma(\alpha, E_x) \to \text{Hom}_\Gamma(\alpha, F_x)$$

is invertible.

Of course, the above result is nothing but the classical condition that the elliptic operator $p_\alpha P p_\alpha$ be Fredholm.

Proof. The equivalence between (1) and (2) is a direct consequence of Theorem 2. Let us check the equivalence of (1) and (3). First note that

$$(H^s(M, E) \otimes \alpha)^\Gamma = H^s(M, (E \otimes \alpha)^\Gamma),$$

since the action of $\Gamma$ on $M$ is trivial. The operator $\pi_\alpha(P)$ is Fredholm if, and only if, the pseudodifferential operator $P_\alpha : H^s(M, \text{Hom}(\alpha, E)^\Gamma) \to H^{s-m}(M, \text{Hom}(\alpha, F)^\Gamma)$ defined for any $\nu^* \in \alpha^*$ and $s \in C^\infty(M, E)$ by $P_\alpha(\nu^* s) = \nu^* P s$ is Fredholm. Furthermore, the operator $P_\alpha$ is Fredholm if, and only if, it is elliptic, that is if, and only if, $\sigma_m(P) \otimes \text{id}_{\alpha^*}(x, \xi) : \text{Hom}_\Gamma(\alpha, E_x) \to \text{Hom}_\Gamma(\alpha, F_x)$ is invertible for any $(x, \xi) \in T^*M \setminus \{0\}$. Note that the invertibility of $\pi_\alpha(\sigma_m(P)(x, \xi))$ is equivalent to the invertibility of $\pi_\alpha(\sigma_m(P)(x, \xi))$ by definition, which is consistent with (2). \hfill $\square$

4.3.4. Free action on a dense subset. As in the previous sections, the group $\Gamma$ is finite and acts continuously on the manifold $M$. We consider vector bundles $E, F \to M$.

We have following corollary of the last few results in Section 5

Corollary 4.15. Let us assume that $\Gamma$ acts freely on a dense open subset of $M$. Then $\Xi = \text{Prim}(A^1_M)$.
Proof. The assumption on the action implies that $\Gamma_0 = \{1\}$. If $\xi \in T^*M \setminus \{0\}$ and $\rho \in \tilde{\Gamma}_I$, then $\rho$ and $\alpha$ are always $\{1\}$-associated. The Corollary then follows from Proposition 3.19.

Similarly, we have the following result.

**Proposition 4.16.** Assume that $\Gamma$ acts freely on a dense subset in $M$, and let $P \in \psi^m(M; E, F)^\Gamma$. For any $\alpha \in \tilde{\Gamma}$, we have that $P$ is $\alpha$-elliptic if, and only if, $P$ is elliptic.

**Proof.** It follows from Corollary 4.15 that $X_{M, I} = X_{M, \Gamma}$. Thus the operator $P_\alpha$ is $\alpha$-elliptic if, and only if, the sum $\bigoplus_{\rho \in \tilde{\Gamma}_I} \rho(\sigma_m(P)(\xi)) = \sigma_m(P)(\xi)$ is invertible for all $\xi \in T^*M \setminus \{0\}$, that is, if, and only if, $P$ is elliptic.

4.4. Simoenko’s localization principle. In this section, we obtain an equivariant version of Simoenko’s principle [74]. In this subsection and the rest of the paper, we consider a compact Lie group $G$ instead of $\Gamma$.

4.4.1. Simoenko’s general principle. Let $A$ be a unital $C^*$-algebra and $Z \simeq C(\Omega_\Gamma)$ a unital sub-$C^*$-algebra in $A$, i.e. $1_Z = 1_A$. An element $a \in A$ is said to have the strong Simoenko local property with respect to $Z$ if, for every $\phi, \psi \in Z$ with compact disjoint supports, $\phi \psi = 0$.

**Lemma 4.17.** The set $B \subset A$ of elements $a$ satisfying the strong Simoenko local property is the set of elements of $A$ commuting with $Z$.

**Proof.** We are going to show that the set of elements $a \in A$ with the strong Simoenko local type property is a $C^*$-algebra $B$ containing $Z$ and that every irreducible representation of $B$ restricts to a scalar valued representation on $Z$, and hence that $Z$ commutes with $B$.

Let us show first that $B$ is a sub-$C^*$-algebra of $A$. Note that $B$ is not empty since $Z \subset B$. To show that $B$ is a sub-$C^*$-algebra, the only fact that is non-trivial to prove is that $ab \in B$, for all $a, b \in B$. Let $\phi$ and $\psi \in Z$ with disjoint compact supports and let $\theta$ be a function equal to 1 on $\text{supp}(\psi)$ and 0 on $\text{supp}(\phi)$, which exists by Urysohn’s lemma. Then we have

\begin{equation}
\phi ab\psi = \phi a(\theta + 1 - \theta)b\psi = \phi a\theta b\psi + \phi a(1 - \theta)b\psi = 0,
\end{equation}

since $\phi a\theta = 0$ and $(1 - \theta)b\psi = 0$.

Let $\pi : B \to \mathcal{L}(H)$ be an irreducible representation of $B$. First, let us show that for any $\phi, \psi \in Z$ with disjoint support, we either have $\pi(\phi) = 0$ or $\pi(\psi) = 0$. Indeed we have $\pi(\phi) \pi(a) \pi(\psi) = 0$ since $\phi a\psi = 0$, for any $a \in B$. Assume that $\pi(\psi) \neq 0$ then there is $\eta \in H$ such that $\pi(\psi)\eta \neq 0$. Now, $\pi$ is irreducible so we get that the set $\{\pi(a)\pi(\psi)\eta, \ a \in B\}$ is dense in $H$. Thus $\pi(\phi) = 0$ on a dense subspace of $H$ and so on $H$.

Assume now that $\pi(Z) \neq C1_H$. Then there exist two distinct characters $\chi_0, \chi_1 \in \text{Sp}(\pi(Z))$. Denote by $h_\pi : \text{Sp}(\pi(Z)) \to \text{Sp}(Z)$ the injective map adjoint to $\pi$, and choose $\phi, \psi \in \mathcal{C}(\text{Sp}(Z))$ with disjoint supports such that $\phi(h_\pi(\chi_0)) = 1$ and $\psi(h_\pi(\chi_1)) = 1$. Then $\pi(\phi)(\chi_0) = 1$ and $\pi(\psi)(\chi_1) = 1$, which contradicts the fact that either $\pi(\phi) = 0$ or $\pi(\psi) = 0$.

Recall that a family $(\varphi_i)_{i \in I}$ of morphisms of a $C^*$-algebra $A$ is said to be exhaustive if any primitive ideal contains some $\ker(\varphi_i)$, for a suitable $i \in I$ [64]. Then

\[ \ker(\varphi_i) = \{ x \in A : \varphi_i(x) = 0 \} \]
Remark 2.7 gives that the family of morphisms
\[(44) \quad \chi_\omega : A \to A/\omega A,\]
for \(\omega \in \Omega_Z\), is exhaustive for \(A\).

**Definition 4.18.** Denote by \(\mathcal{H} = L^2(M)\). An operator \(P \in \mathcal{L}(\mathcal{H})\) is said to be **locally invertible** at \(x \in M\) if there are a neighbourhood \(V_x\) of \(x\) and operators \(Q_1^x\) and \(Q_2^x \in \mathcal{L}(\mathcal{H})\) such that
\[(45) \quad Q_1^x P \phi = \phi \quad \text{and} \quad \phi P Q_2^x = \phi, \quad \text{for any } \phi \in C_c(V_x).
\]
The operator \(P\) is said to be **locally invertible** if it is locally invertible at any \(x \in M\).

Let \(\Psi_M \subset \mathcal{L}(\mathcal{H})\) be the \(C^*\)-algebra of all \(P \in \mathcal{L}(\mathcal{H})\) such that \(\phi P \psi \in \mathcal{K}(\mathcal{H})\), for all \(\phi, \psi \in \mathcal{C}(M)\) with disjoint support. We denote by \(\mathcal{B}_M\) the image of \(\Psi_M\) in the Calkin algebra \(\mathcal{Q}(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})\). We know by Lemma 4.17 that
\[(46) \quad \mathcal{B}_M = \{ P \in \mathcal{Q}(H) \mid \phi P = P \phi \text{ for all } \phi \in \mathcal{C}(M) \}.
\]
Simonenko’s principle is then [74]:

**Proposition 4.19 (Simonenko’s principle).** If \(P \in \Psi_M\), then \(P\) is locally invertible if, and only if, it is Fredholm.

We shall prove, in fact, a stronger version of this result, see Proposition 4.21.

4.5. **Compact (non-finite) groups.** We now allow for compact groups and try to see to what extent our results remain valid. To that end, we turn to an analog of Simonenko’s principle. Let then \(G\) be a compact Lie group acting smoothly on \(M\). We continue to study Fredholm conditions for \(\pi_\alpha(P)\), \(\alpha \in \hat{G}\). Denote by \(\mathcal{H} := L^2(M, E)\) and by \(\mathcal{H}_\alpha\) the \(\alpha\)-isotypical component associated to \(\alpha \in \hat{G}\).

**Definition 4.20.** We shall say that \(P \in \mathcal{L}(\mathcal{H})\) is locally \(\alpha\)-invertible at \(x \in M\) if there are a \(G\)-invariant neighbourhood \(V_x\) of \(\Gamma x\) and operators \(Q_1^x\) and \(Q_2^x \in \mathcal{L}(\mathcal{H}_\alpha)\) such that
\[(46) \quad Q_1^x \pi_\alpha(P) \phi = \phi \quad \text{and} \quad \phi \pi_\alpha(P) Q_2^x = \phi,
\]
as operators on \(\mathcal{H}_\alpha\), for any \(\phi \in \mathcal{C}(M)^G\) supported in \(V_x\).

We denote by \(\Psi_M^G\) the \(G\)-invariant elements in the \(C^*\)-algebra \(\Psi_M\), which was defined in the previous subsection.

**Proposition 4.21 (Simonenko’s equivariant principle).** Let \(P \in \Psi_M^G\). Then \(P\) is locally \(\alpha\)-invertible if, and only if, \(\pi_\alpha(P)\) is Fredholm.

**Proof.** We now use the results of the last section for \(Z = C(M)^G = C(M/G)\). Let \(\mathcal{B}_M^G\) be the image of \(\Psi_M^G\) in the Calkin algebra \(\mathcal{Q}(\mathcal{H}_\alpha)\). We know from Lemma 4.17 that
\[(46) \quad \mathcal{B}_M^G = \{ P \in \mathcal{Q}(\mathcal{H}_\alpha) \mid \phi P = P \phi, \quad \forall \phi \in \mathcal{C}(M)^G \}.
\]
Assume that \(P\) is locally \(\alpha\)-invertible, i.e. \(\forall x \in M\), there are a neighborhood \(V_x\) of \(Gx\) and operators \(Q_1^x, Q_2^x \in \mathcal{L}(\mathcal{H}_\alpha)\) such that \(Q_1^x \pi_\alpha(P) \phi = \phi\) and \(\phi \pi_\alpha(P) Q_2^x = \phi\), for any \(\phi \in \mathcal{C}(M)^G\) supported in \(V_x\). Let \(\chi_x\), be the family of representations of \(\mathcal{B}_M^G\) introduced in Equation 14. We use the same notation for \(\pi_\alpha(P)\) and its image in \(\mathcal{Q}(\mathcal{H}_\alpha)\). We have that
\[
\chi_x(Q_1^x \pi_\alpha(P) \phi) = \chi_x(Q_1^x) \chi_x(\pi_\alpha(P)) \chi_x(\phi) = \chi_x(\phi).
\]
Since $\chi_x(\phi) = 1$, we get:

$$\chi_x(Q_1^x)\chi_x(\pi_\alpha(P)) = 1.$$  

And similarly,

$$\chi_x(\pi_\alpha(P))\chi_x(Q_2^x) = 1.$$  

Therefore, $\chi_x(\pi_\alpha(P))$ is invertible for all $x$. Since the family $\chi_x$ is exhaustive, it follows that that $\pi_\alpha(P)$ is invertible in $B_{G}^0$ and so it is Fredholm.

Now assume that $\pi_\alpha(P)$ is Fredholm and let $Q$ be an inverse modulo $\mathcal{K}(\mathcal{H}_\alpha)$ for $\pi_\alpha(P)$, i.e. $\pi_\alpha(P)Q = id + K$ and $Q\pi_\alpha(P) = id + K'$, with $K, K' \in \mathcal{K}(\mathcal{H}_\alpha)$. Using Proposition 4.23 and Lemma 2.10, we can assume that $K = \pi_\alpha(k)$ and $K' = \pi_\alpha(k') \in \mathcal{K}(\mathcal{H}_\alpha) = \pi_\alpha(\mathcal{K}(\mathcal{H})^G)$, with $k, k' \in \mathcal{K}(\mathcal{H})^G$. Let $\chi \in C(M)^G$ be equal to 1 on a $G$-invariant neighbourhood $V_x$ of $Gx$ and let $\phi \in C(M)^G$ be supported in $V_x$ then

$$\phi \chi \pi_\alpha(P)Q\chi = \phi \chi^2 + \phi \chi K\chi \quad \text{ and } \quad \chi \pi_\alpha(P)Q\chi \phi = \chi^2 \phi + \chi K'\chi \phi.$$  

Since $\phi$ is supported in $V_x$, we have $\phi \chi = \phi$ and so

$$\phi \pi_\alpha(P)Q\chi = \phi (1 + \chi K\chi) \quad \text{ and } \quad \pi_\alpha(P)Q\chi \phi = (1 + \chi K'\chi)\phi.$$  

As $V_x$ becomes smaller and smaller, we have that $\chi$ converges strongly to 0. Since $K$ is compact, we obtain that $\|\chi K\chi\| \to 0$, Thus, by choosing $V_x$ small enough, we may assume that $\|\chi K\chi\| < 1$ and $\|\chi K'\chi\| < 1$.

It follows that $(1 + \chi K\chi)$ and $(1 + \chi K'\chi)$ are invertible and this implies

$$\phi \pi_\alpha(P)(Q\chi(1 + \chi K\chi)^{-1}) = \phi \quad \text{ and } \quad ((1 + \chi K'\chi)^{-1}Q)\pi_\alpha(P)\phi = \phi,$$

i.e. $P$ is locally $\alpha$-invertible. \hfill $\square$

**Corollary 4.22.** Assume that $M$ is compact, $\Gamma$ is a finite group and $M/\Gamma$ connected. Let $P \in \psi(M; E, F)^G$ and $\alpha \in \hat{\Gamma}$. Then the following are equivalent:

1. $\pi_\alpha(P) : H^s(M; E)_\alpha \to H^{s-m}(M; F)_\alpha$ is Fredholm for any $s \in \mathbb{R}$.
2. $P$ is $\alpha$-elliptic.
3. $P$ is locally $\alpha$-invertible.

**Proof.** The first equivalence is given by Theorem 1.2. Now since a finite group is compact Proposition 4.21 implies that (1) is equivalent to (3). \hfill $\square$

**4.5.1. Transversally elliptic operators.** Assume that $M$ is a compact smooth manifold and that $G$ is a compact Lie group acting on $M$. Denote by $\mathfrak{g}$ the Lie algebra of $G$. Then any $X \in \mathfrak{g}$ defines as usual the vector field $X_M$ given by $X_M(m) = \frac{d}{dt}_{t=0} e^{tX}m$. Denote by $\pi : T^*M \to M$ the canonical projection and let us introduce as in [5] the $G$-transversal space

$$T_G^*M := \{ \alpha \in T^*M \mid \alpha(X_M(\pi(\alpha))) = 0, \forall X \in \mathfrak{g} \}.$$  

Recall that a $G$-invariant classical pseudodifferential operator $P$ of order $m$ is said $G$-transversally elliptic if its principal symbol is invertible on $T_G^*M \setminus \{0\}$ [5, 65].

We may now state the classical result of Atiyah and Singer [5, Corollary 2.5].

**Theorem 4.23 (Atiyah-Singer [5]).** Assume $P$ is $G$-transversally elliptic. Then, for every irreducible representation $\alpha \in \hat{G}$, $\pi_\alpha(P) : H^s(M; E_0)_\alpha \to H^{s-m}(M; E_1)_\alpha$, is Fredholm.
Note that this implies that Theorem 1.2 is not true anymore for if $G$ is non-discrete. In particular, we obtain the following consequence of the localization principle.

**Corollary 4.24.** Assume that $M$ is compact and that $G$ is a compact Lie group and let $P \in \psi^m(M; E)^G$ be a $G$-transversally elliptic operator. Then $P$ is locally $\alpha$-invertible for any $\alpha \in \hat{G}$, as in Definition 4.20.

**Proof.** Using Theorem 4.23 we obtain that $\pi_\alpha(P)$ is Fredholm. Therefore by Proposition 4.21 $P$ is $\alpha$-invertible. \qed

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