Generalization of Doob Decomposition Theorem and Risk Assessment in Incomplete Markets

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Abstract

In the paper, we introduce the notion of a local regular supermartingale relative to a convex set of equivalent measures and prove for it the necessary and sufficient conditions of optional Doob decomposition in the discrete case. This theorem is a generalization of the famous Doob decomposition onto the case of supermartingales relative to a convex set of equivalent measures. The description of all local regular supermartingales relative to a convex set of equivalent measures is presented. A notion of complete set of equivalent measures is introduced. We prove that every non-negative bounded supermartingale relative to a complete set of equivalent measures is local regular. A new definition of fair price of contingent claim in incomplete market is given and a formula for fair price of Standard option of European type is found.

Keywords: random process, convex set of equivalent measures, optional Doob decomposition, regular supermartingale, martingale, fair price of contingent claim

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1 Introduction.

In the paper, martingales and supermartingales relative to a convex set of equivalent measures are systematically studied. The notion of local regular supermartingale relative to a convex set of equivalent measures is introduced and the necessary and sufficient are found under that a supermartingale is local regular one. Complete description of local regular supermartingales is given. The notion of complete convex set of equivalent measures is introduced and established that every nonnegative supermartingale is local regular relative this set of measures. The notion of local regular supermartingale is used for definition of fair price of contingent claim relative to a convex set of equivalent measures. Sufficient conditions of the existence of fair price of contingent claim relative to a convex set of equivalent measures are presented. All these notions are used in the case as a convex set of equivalent measures is a set of equivalent martingale measures for evolution as risk and non-risk assets. Formulas for fair price of standard contract with option of European type in incomplete are found.

The notion of complete convex set of equivalent measures permits to give a new proof of optional decomposition for non negative supermartingale. This proof do not use no-arbitrage arguments and measurable choice [15], [7], [6], [8].

First, optional decomposition for supermartingale was opened by El Karoui N. and Quenez M. C. [5] for diffusion processes. After that, Kramkov D. O. and Follmer H. [15], [7] proved the optional decomposition for nonnegative bounded supermartingales. Follmer H. and Kabanov Yu. M. [6], [8] proved analogous result for an arbitrary supermartingale. Recently, Bouchard B. and Nutz M. [1] considered a class of discrete models and proved the necessary and sufficient conditions for validity of optional decomposition.

The optional decomposition for supermartingales plays fundamental role for risk assessment in incomplete markets [5], [15], [7], [6], [10], [11]. Considered in the paper problem is generalization of corresponding one that appeared in mathematical
finance about optional decomposition for supermartingale and which is related with
collection of superhedge strategy in incomplete financial markets.

Our statement of the problem unlike the above-mentioned one and it is more
general: a supermartingale relative to a convex set of equivalent measures is given
and it is necessary to find conditions on the supermartingale and the set of measures
under that optional decomposition exists.

Generality of our statement of the problem is that we do not require that the
considered set of measures was generated by random process that is a local martin-
gale as it is done in the papers [1,5,8,15] and that is important for the proof of the
optional decomposition in these papers.

2 Optional decomposition for supermartingales
relative to a convex set of equivalent measures.

We assume that on a measurable space \( \{\Omega, \mathcal{F}\} \) a filtration \( \mathcal{F}_m \subset \mathcal{F}_{m+1} \subset \mathcal{F}, \ m = 0, \infty \), and a family of measures \( M \) on \( \mathcal{F} \) are given. Further, we assume that \( \mathcal{F}_0 = \{\emptyset, \Omega\} \).

A random process \( \psi = \{\psi_m\}_{m=0}^{\infty} \) is said to be adapted one relative to the
filtration \( \{\mathcal{F}_m\}_{m=0}^{\infty} \) if \( \psi_m \) is \( \mathcal{F}_m \) measurable random value for all \( m = 0, \infty \).

**Definition 2.1** An adapted random process \( f = \{f_m\}_{m=0}^{\infty} \) is said to be a super-
martingale relative to the filtration \( \mathcal{F}_m, m = 0, \infty, \) and the family of measures \( M \) if
\[
E_P|f_m| < \infty, \ m = 1, \infty, \ P \in M, \tag{2.1}
\]
are valid.

We consider that the filtration \( \mathcal{F}_m, m = 0, \infty, \) is fixed. Further, for a supermartin-
gale \( f \) we use as denotation \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) and denotation \( \{f_m\}_{m=0}^{\infty} \).

Below, in a few theorems, we consider a convex set of equivalent measures \( M \) satisfying conditions: Radon – Nicodym derivative of any measure \( Q_1 \in M \) with
respect to any measure \( Q_2 \in M \) satisfies inequalities
\[
0 < l \leq \frac{dQ_1}{dQ_2} \leq L < \infty, \ Q_1, Q_2 \in M, \tag{2.2}
\]
where real numbers \( l, L \) do not depend on \( Q_1, Q_2 \in M \).

**Theorem 2.1** Let \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) be a supermartingale concerning a convex set of
equivalent measures \( M \) satisfying conditions \( \text{(2.2)} \). If for a certain measure \( P_1 \in M \) there exist a natural number \( 1 \leq m_0 < \infty, \) and \( \mathcal{F}_{m_0} \) measurable nonnegative random value \( \varphi_{m_0}, P_1(\varphi_{m_0} > 0) > 0, \) such that the inequality
\[
f_{m_0-1} - E^{P_1}\{f_{m_0}|\mathcal{F}_{m_0-1}\} \geq \varphi_{m_0},
\]
is valid, then
\[
f_{m_0-1} - E^Q\{f_{m_0}|\mathcal{F}_{m_0-1}\} \geq \frac{l}{1 + L}\varphi_{m_0}, \ Q \in M_{\varepsilon_0},
\]
where
\[
M_{\varepsilon_0} = \{Q \in M, \ Q = (1 - \alpha)P_1 + \alpha P_2, \ 0 \leq \alpha \leq \varepsilon_0, \ P_2 \in M\}, \ P_1 \in M,
\]
\[
\varepsilon_0 = \frac{L}{1 + L}.
\]
Proof. Let $B \in \mathcal{F}_{m_0-1}$ and $Q = (1 - \alpha)P_1 + \alpha P_2$, $P_2 \in M$, $0 < \alpha < 1$. Then

$$\int_B [f_{m_0-1} - E^Q\{f_{m_0}|\mathcal{F}_{m_0-1}\}]dQ =$$

$$\int_B E^Q\{[f_{m_0-1} - f_{m_0}]|\mathcal{F}_{m_0-1}\}dQ =$$

$$\int_B [f_{m_0-1} - f_{m_0}]dQ =$$

$$(1 - \alpha) \int_B [f_{m_0-1} - f_{m_0}]dP_1 +$$

$$\alpha \int_B [f_{m_0-1} - f_{m_0}]dP_2 =$$

$$(1 - \alpha) \int_B [f_{m_0-1} - E^{P_1}\{f_{m_0}|\mathcal{F}_{m_0-1}\}]dP_1 +$$

$$\alpha \int_B [f_{m_0-1} - E^{P_2}\{f_{m_0}|\mathcal{F}_{m_0-1}\}]dP_2 \geq$$

$$(1 - \alpha) \int_B [f_{m_0-1} - E^{P_1}\{f_{m_0}|\mathcal{F}_{m_0-1}\}]dP_1 =$$

$$(1 - \alpha) \int_B [f_{m_0-1} - E^{P_1}\{f_{m_0}|\mathcal{F}_{m_0-1}\}] \frac{dP_1}{dQ}dQ \geq$$

$$(1 - \alpha) \int_B \varphi_{m_0}dQ \geq (1 - \bar{\varepsilon}_0)l \int_B \varphi_{m_0}dQ = \frac{l}{1 + L} \int_B \varphi_{m_0}dQ.$$

Arbitrariness of $B \in \mathcal{F}_{m_0-1}$ proves the needed inequality.

Lemma 2.1 Any supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a family of measures $M$ for which there hold equalities $E^P f_m = f_0$, $m = 1, \infty$, $P \in M$, is a martingale with respect to this family of measures and the filtration $\mathcal{F}_m$, $m = 1, \infty$.

Proof. The proof of Lemma 2.1 see [16].
Remark 2.1 If the conditions of Lemma 2.1 are valid, then there hold equalities
\[ E^P\{f_m|\mathcal{F}_k\} = f_k, \quad 0 \leq k \leq m, \quad m = \overline{1, \infty}, \quad P \in M. \] (2.3)

Let \( f = \{f_m, \mathcal{F}_m\}_{m=0}^\infty \) be a supermartingale relative to a convex set of equivalent measures \( M \) and the filtration \( \mathcal{F}_m, \quad m = 0, \infty \). And let \( G \) be a set of adapted non-decreasing processes \( g = \{g_m\}_{m=0}^\infty \), such that \( f + g = \{f_m + g_m\}_{m=0}^\infty \) is a supermartingale concerning the family of measures \( M \) and the filtration \( \mathcal{F}_m, \quad m = 0, \infty \).

Introduce a partial ordering \( \preceq \) in the set of adapted non-decreasing processes \( G \).

Definition 2.2 We say that an adapted non-decreasing process \( g_1 = \{g_1^m\}_{m=0}^\infty \), \( g_1^1 = 0, \quad g_1 \in G \), does not exceed an adapted non-decreasing process \( g_2 = \{g_2^m\}_{m=0}^\infty \), \( g_2^0 = 0, \quad g_2 \in G \), if \( P(g_2^m - g_1^m \geq 0) = 1, \quad m = \overline{1, \infty} \). This partial ordering we denote by \( g_1 \preceq g_2 \).

For every nonnegative adapted non-decreasing process \( g = \{g_m\}_{m=0}^\infty \in G \) there exists \( \lim_{m \to \infty} g_m \) which we denote by \( g_\infty \).

Lemma 2.2 Let \( \tilde{G} \) be a maximal chain in \( G \) and for a certain \( Q \in M \) \( \sup_{g \in \tilde{G}} E^Q g = \alpha^Q < \infty \). Then there exists a sequence \( g^s = \{g^s_m\}_{m=0}^\infty \in \tilde{G}, \quad s = 1, 2, ... \), such that
\[ \sup_{g \in \tilde{G}} E^Q g = \sup_{s \geq 1} E^Q g^s, \]
where
\[ E^Q g = \sum_{m=0}^\infty E^Q g_m, \quad g \in G. \]

Proof. Let \( 0 < \varepsilon_s < \alpha^Q, \quad s = \overline{1, \infty}, \) be a sequence of real numbers satisfying conditions \( \varepsilon_s > \varepsilon_{s+1}, \quad \varepsilon_s \to 0, \quad s \to \infty \). Then there exists an element \( g^s \in \tilde{G} \) such that \( \alpha^Q - \varepsilon_s \leq E^Q g^s \leq \alpha^Q, \quad s = \overline{1, \infty} \). The sequence \( g^s \in \tilde{G}, \quad s = \overline{1, \infty} \), satisfies Lemma 2.2 conditions.

Lemma 2.3 If a supermartingale \( \{f_m, \mathcal{F}_m\}_{m=0}^\infty \) relative to a convex set of equivalent measures \( M \) is such that
\[ |f_m| \leq \varphi, \quad m = \overline{0, \infty}, \quad E^Q \varphi < T < \infty, \quad Q \in M, \] (2.4)
where a real number \( T \) does not depend on \( Q \in M \), then every maximal chain \( \tilde{G} \subseteq G \) contains a maximal element.

Proof. Let \( g = \{g_m\}_{m=0}^\infty \) belong to \( G \), then
\[ E^Q(f_m + \varphi + g_m) \leq f_0 + T, \quad m = \overline{1, \infty}, \quad Q \in M. \]
Then inequalities \( f_m + \varphi \geq 0, \quad m = \overline{1, \infty} \), yield
\[ E^Q g_m \leq f_0 + T, \quad m = \overline{1, \infty}, \quad \{g_m\}_{m=0}^\infty \in G. \]
Introduce for a certain $Q \in M$ an expectation for $g = \{g_m\}_{m=0}^\infty \in G$

$$E_1^Q g = \sum_{m=0}^\infty \frac{E^Q g_m}{2^m}, \quad g \in G.$$ 

Let $\mathcal{G} \subseteq G$ be a certain maximal chain. Therefore, we have inequality

$$\sup_{g \in \mathcal{G}} E_1^Q g = \alpha_0^Q \leq f_0 + T < \infty,$$

where $Q \in M$ and is fixed. Due to Lemma 2.2,

$$\sup_{g \in \mathcal{G}} E_1^Q g = \sup_{s \geq 1} E_1^Q g^s.$$ 

In consequence of the linear ordering of elements of $\mathcal{G}$,

$$\max_{1 \leq s \leq k} g^s = g^{s_0(k)}, \quad 1 \leq s_0(k) \leq k,$$

where $s_0(k)$ is one of elements of the set $\{1, 2, \ldots, k\}$ on which the considered maximum is reached, that is, $1 \leq s_0(k) \leq k$, and, moreover,

$$g^{s_0(k)} \preceq g^{s_0(k)+1}.$$

It is evident that

$$\max_{1 \leq s \leq k} E_1^Q g^s = E_1^Q g^{s_0(k)}.$$ 

So, we obtain

$$\sup_{s \geq 1} E_1^Q g^s = \lim_{k \to \infty} \max_{1 \leq s \leq k} E_1^Q g^s = \lim_{k \to \infty} E_1^Q g^{s_0(k)} = E_1^Q g^0,$$

where $g^0 = \lim_{k \to \infty} g^{s_0(k)}$, and that there exists, due to monotony of $g^{s_0(k)}$. Thus,

$$\sup_{s \geq 1} E_1^Q g^s = E_1^Q g^0 = \alpha_0^Q.$$ 

Show that $g^0 = \{g^0_m\}_{m=0}^\infty$ is a maximal element in $\mathcal{G}$. It is evident that $g^0$ belongs to $G$. For every element $g = \{g_m\}_{m=0}^\infty \in \mathcal{G}$ two cases are possible:

1) $\exists k$ such that $g \preceq g^{s_0(k)}$.

2) $\forall k \quad g^{s_0(k)} \prec g$.

In the first case $g \preceq g^0$. In the second one from 2) we have $g^0 \preceq g$. At the same time

$$E_1^Q g^{s_0(k)} \leq E_1^Q g.$$ 

By passing to the limit in (2.5), we obtain

$$E_1^Q g^0 \leq E_1^Q g.$$ 

The strict inequality in (2.6) is impossible, since $E_1^Q g^0 = \sup_{g \in \mathcal{G}} E_1^Q g$. Therefore,

$$E_1^Q g^0 = E_1^Q g.$$ 

The inequality $g^0 \preceq g$ and the equality (2.7) imply that $g = g^0$.

Let $\mathcal{M}$ be a convex set of equivalent probability measures on $\{\Omega, \mathcal{F}\}$. Introduce into $M$ a metric $|Q_1 - Q_2| = \sup_{A \in \mathcal{F}} |Q_1(A) - Q_2(A)|$, $Q_1, Q_2 \in \mathcal{M}$. 

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Lemma 2.4 Let \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) be a supermartingale relative to a compact convex set of equivalent measures \( M \) satisfying conditions (2.2). If for every set of measures \( \{P_1, P_2, \ldots, P_s\}, s < \infty, P_i \in M, i = \overline{1, s} \), there exist a natural number \( 1 \leq m_0 < \infty \), and depending on this set of measures \( \mathcal{F}_{m_0-1} \) measurable nonnegative random variable \( \Delta_{m_0} \), \( P_1(\Delta_{m_0} > 0) > 0 \), satisfying conditions
\[
f_{m_0-1} - E^{P_1}\{f_{m_0} | \mathcal{F}_{m_0-1}\} \geq \Delta_{m_0}^*, \quad i = \overline{1, s}, \tag{2.8}
\]
then the set \( G \) of adapted non-decreasing processes \( g = \{g_m\}_{m=0}^{\infty}, \ g_0 = 0, \) for which \( \{f_m + g_m\}_{m=0}^{\infty} \) is a supermartingale relative to the set of measures \( M \) contains nonzero element.

Proof. For any point \( P_0 \in M \) let us define a set of measures
\[
M^{P_0, \varepsilon_0} = \{Q \in M, \ Q = (1 - \alpha)P_0 + \alpha P, \ P \in M, \ 0 \leq \alpha \leq \varepsilon_0\}, \tag{2.9}
\]
\[
\varepsilon_0 = \frac{L}{1 + L}.
\]
Prove that the set of measures \( M^{P_0, \varepsilon_0} \) contains some ball of a positive radius, that is, there exists a real number \( \rho_0 > 0 \) such that \( M^{P_0, \varepsilon_0} \supseteq C(P_0, \rho_0) \), where \( C(P_0, \rho_0) = \{P \in M, \ |P_0 - P| < \rho_0\} \).

Let \( C(P_0, \bar{\rho}) = \{P \in M, \ |P_0 - P| < \bar{\rho}\} \) be an open ball in \( M \) with the center at the point \( P_0 \) of a radius \( 0 < \bar{\rho} < 1 \). Consider a map of the set \( M \) into itself given by the law: \( f(P) = (1 - \varepsilon_0)P_0 + \varepsilon_0 P, \ P \in M \).

The mapping \( f(P) \) maps an open ball \( C(P_2, \delta) = \{P \in M, \ |P_2 - P| < \delta\} \) with the center at the point \( P_2 \) of a radius \( \delta > 0 \) into an open ball with the center at the point \( (1 - \varepsilon_0)P_0 + \varepsilon_0 P_2 \) of the radius \( \varepsilon_0 \delta \), since \( |(1 - \varepsilon_0)P_0 + \varepsilon_0 P_2 - (1 - \varepsilon_0)P_0 - \varepsilon_0 P| = \varepsilon_0|P_2 - P| < \varepsilon_0 \delta \). Therefore, an image of an open set \( M_0 \subseteq M \) is an open set \( f(M_0) \subseteq M \), thus \( f(P) \) is an open mapping. Since \( f(P_0) = P_0 \), then the image of the ball \( C(P_0, \bar{\rho}) = \{P \in M, \ |P_0 - P| < \bar{\rho}\} \) is a ball \( C(P_0, \varepsilon_0 \bar{\rho}) = \{P \in M, \ |P_0 - P| < \varepsilon_0 \bar{\rho}\} \) and it is contained in \( f(M) \). Thus, inclusions \( M^{P_0, \varepsilon_0} \supseteq f(M) \supseteq C(P_0, \varepsilon_0 \bar{\rho}) \) are valid. Let us put \( \varepsilon_0 \bar{\rho} = \rho_0 \). Then we have \( M^{P_0, \varepsilon_0} \supseteq C(P_0, \rho_0) \), where \( C(P_0, \rho_0) = \{P \in M, \ |P_0 - P| < \rho_0\} \). Consider an open covering \( \bigcup_{P_0 \in M} C(P_0, \rho_0) \) of the compact set \( M \).

Due to compactness of \( M \), there exists a finite subcovering
\[
M = \bigcup_{i=1}^{v} C(P_0^i, \rho_0) \tag{2.10}
\]
with the center at the points \( P_0^i \in M, \ i = \overline{1, v} \), and a covering by sets \( M^{P_0^i, \varepsilon_0} \supseteq C(P_0^i, \rho_0), \ i = \overline{1, v} \),
\[
M = \bigcup_{i=1}^{v} M^{P_0^i, \varepsilon_0}. \tag{2.11}
\]

Consider the set of measures \( P_0^i \in M, \ i = \overline{1, v} \). From Lemma 2.4 conditions, there exist a natural number \( 1 \leq m_0 < \infty \), and depending on the set of measures \( P_0^i \in M, \ i = \overline{1, v}, \mathcal{F}_{m_0-1} \) measurable nonnegative random variable \( \Delta_{m_0}^v \), \( P_0^i(\Delta_{m_0}^v > 0) > 0 \), such that
\[
f_{m_0-1} - E^{P_0^i}\{f_{m_0} | \mathcal{F}_{m_0-1}\} \geq \Delta_{m_0}^v, \quad i = \overline{1, v}. \tag{2.12}
\]
Due to Theorem 2.1, we have

$$f_{m_0-1} - E^Q\{f_{m_0}|\mathcal{F}_{m_0-1}\} \geq \frac{l}{1+L} \Delta_{m_0} = \varphi_{m_0}, \quad Q \in M.$$  \hfill (2.13)

The last inequality imply

$$E^Q\{f_{m_0-1}|\mathcal{F}_s\} - E^Q\{f_{m_0}|\mathcal{F}_s\} \geq E^Q\{\varphi_{m_0}^v|\mathcal{F}_s\}, \quad Q \in M, \quad s < m_0.$$  \hfill (2.14)

But $E^Q\{f_{m_0-1}|\mathcal{F}_s\} \leq f_s, \quad s < m_0$. Therefore,

$$f_s - E^Q\{f_{m_0}|\mathcal{F}_s\} \geq E^Q\{\varphi_{m_0}^v|\mathcal{F}_s\}, \quad Q \in M, \quad s < m_0.$$  \hfill (2.15)

Since

$$f_{m_0} - E^Q\{f_m|\mathcal{F}_{m_0}\} \geq 0, \quad Q \in M, \quad m \geq m_0,$$
we have

$$E^Q\{f_{m_0}|\mathcal{F}_s\} - E^Q\{f_m|\mathcal{F}_s\} \geq 0, \quad Q \in M, \quad s < m_0, \quad m \geq m_0.$$  \hfill (2.16)

Adding (2.17) to (2.15), we obtain

$$f_s - E^Q\{f_{m_0}|\mathcal{F}_s\} \geq E^Q\{\varphi_{m_0}^v|\mathcal{F}_s\}, \quad Q \in M, \quad s < m_0, \quad m \geq m_0.$$  \hfill (2.18)

or

$$f_s - E^Q\{f_{m_0}|\mathcal{F}_s\} \geq E^Q\{\varphi_{m_0}^v|\mathcal{F}_s\} \chi_{[m_0,\infty)}(m) - \varphi_{m_0}^v \chi_{[m_0,\infty)}(s), \quad Q \in M, \quad s \leq m_0, \quad m \geq m_0.$$  \hfill (2.19)

Introduce an adapted non-decreasing process

$$g^{m_0} = \{g^{m_0}_m\}_{m=0}^\infty, \quad g^{m_0}_0 = 0,$$
where $\chi_{[m_0,\infty)}(m)$ is an indicator function of the set $[m_0,\infty)$. Then (2.19) implies that

$$E^Q\{f_{m} + g^{m_0}_m|\mathcal{F}_k\} \leq f_k + g^{m_0}_k, \quad 0 \leq k \leq m, \quad Q \in M.$$  \hfill (2.20)

In the Theorem 2.2 a convex set of equivalent measures

$$M = \{Q, \quad Q = \sum_{i=1}^n \alpha_i P_i, \quad \alpha_i \geq 0, \quad i = 1,n, \quad \sum_{i=1}^n \alpha_i = 1\}$$

satisfies conditions

$$0 < l \leq \frac{dP_i}{dP_j} \leq L < \infty, \quad i,j = 1,n.$$  \hfill (2.21)

where $l, \quad L$ are real numbers.

Denote by $G$ the set of all adapted non-decreasing processes $g = \{g_m\}_{m=0}^\infty, \quad g_0 = 0,$
such that $\{f_{m} + g_m\}_{m=0}^\infty$ is a supermartingale relative to all measures from $M$.  

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Theorem 2.2 Let a supermartingale \( \{f_m, \mathcal{F}_m\}_{m=0}^\infty \) relative to the set of measures (2.20) satisfy the conditions [2.4], and let there exist a natural number \( 1 \leq m_0 < \infty \), and \( \mathcal{F}_{m_0-1} \) measurable nonnegative random value \( \varphi_{m_0}^n, P_1(\varphi_{m_0}^n > 0) > 0 \), such that

\[
f_{m_0-1} - E^{P_i}\{f_{m_0}|\mathcal{F}_{m_0-1}\} \geq \varphi_{m_0}^n, \quad i = 1, n. \tag{2.22}
\]

If for the maximal element \( g^0 = \{g^0_m\}_{m=0}^\infty \) in a certain maximal chain \( \tilde{G} \subseteq G \) the equalities

\[
E^{P_i}(f_\infty + g^0_\infty) = f_0, \quad P_i \in M, \quad i = 1, n, \tag{2.23}
\]

are valid, where \( f_\infty = \lim_{m \to \infty} f_m, g^0_\infty = \lim_{m \to \infty} g^0_m \), then there hold equalities

\[
E^P\{f_m + g^0_m|\mathcal{F}_k\} = f_k + g^0_k, \quad 0 \leq k \leq m, \quad m = 1, \infty, \quad P \in M. \tag{2.24}
\]

Proof. The set \( M \) is compact one in the introduced metric topology. From the inequalities (2.22) and the formula

\[
E^Q\{f_{m_0}|\mathcal{F}_{m_0-1}\} = \frac{\sum_{i=1}^{n} \alpha_i E^{P_i}\{\varphi_i|\mathcal{F}_{m_0-1}\} E^{P_i}\{f_{m_0}|\mathcal{F}_{m_0-1}\}}{\sum_{i=1}^{n} \alpha_i E^{P_i}\{\varphi_i|\mathcal{F}_{m_0-1}\}}, \quad Q \in M, \tag{2.25}
\]

where \( \varphi_i = \frac{dP_i}{dP} \), we obtain

\[
f_{m_0-1} - E^Q\{f_{m_0}|\mathcal{F}_{m_0-1}\} \geq \varphi_{m_0}^n, \quad Q \in M. \tag{2.26}
\]

The inequalities (2.21) lead to inequalities

\[
\frac{1}{nL} \leq \frac{dQ}{dP} \leq nL, \quad P, Q \in M. \tag{2.27}
\]

Inequalities (2.26) and (2.27) imply that conditions of Lemma 2.4 are satisfied for any set of measures \( Q_1, \ldots, Q_s \in M \). Hence, it follows that the set \( G \) contains nonzero element. Let \( \tilde{G} \subseteq G \) be a maximal chain in \( G \) satisfying condition of Theorem 2.2 Denote by \( g^0 = \{g^0_m\}_{m=0}^\infty \), \( g^0_0 = 0 \), a maximal element in \( \tilde{G} \subseteq G \). Theorem 2.2 and Lemma 2.3 yield that as \( \{f_m\}_{m=0}^\infty \) and \( \{g^0_m\}_{m=0}^\infty \) are uniformly integrable relative to each measure from \( M \). There exist therefore limits

\[
\lim_{m \to \infty} f_m = f_\infty, \quad \lim_{m \to \infty} g^0_m = g^0_\infty
\]

with probability 1. Due to Theorem 2.2 condition, in this maximal chain

\[
E^{P_i}(f_\infty + g^0_\infty) = f_0, \quad P_i \in M, \quad i = 1, n.
\]

Since \( \{f_m + g^0_m\}_{m=0}^\infty \) is a supermartingale concerning all measures from \( M \), we have

\[
E^{P_i}(f_m + g^0_m) \leq E^{P_i}(f_k + g^0_k) \leq f_0, \quad k < m, \quad m = 1, \infty, \quad P_i \in M. \tag{2.28}
\]

By passing to the limit in (2.28), as \( m \to \infty \), we obtain

\[
f_0 = E^{P_i}(f_\infty + g^0_\infty) \leq E^{P_i}(f_k + g^0_k) \leq f_0, \quad k = 1, \infty, \quad P_i \in M. \tag{2.29}
\]
So, $E^P(f_k + g^0_k) = f_0, \ k = 1, \infty, \ P_i \in M, \ i = 1, n$. Taking into account Remark 2.1 we have

$$E^P \{ f_m + g^0_m | \mathcal{F}_k \} = f_k + g^0_k, \ 0 \leq k \leq m, \ m = 1, \infty, \ P_i \in M, \ i = 1, n.$$ \hfill (2.30)

Hence,

$$E^P \{ f_m + g^0_m | \mathcal{F}_k \} =$$

$$\sum_{i=1}^{n} \alpha_i E^P \{ \varphi_i | \mathcal{F}_k \} E^P \{ f_m + g^0_m | \mathcal{F}_k \} = f_k + g^0_k, \ 0 \leq k \leq m, \ P \in M,$$

where $\varphi_i = \frac{dP_i}{dP}, \ i = 1, n$. Theorem 2.2 is proved.

Let $M$ be a convex set of equivalent measures. Below, $G_s$ is a set of adapted non-decreasing processes $\{g_m\}_{m=0}^\infty, g_0 = 0$, for which $\{f_m + g_m\}_{m=0}^\infty$ is a supermartingale relative to all measures from

$$\hat{M}_s = \{Q, Q = \sum_{i=1}^{s} \gamma_i \hat{P}_i, \gamma_i \geq 0, \ i = 1, s, \sum_{i=1}^{s} \gamma_i = 1 \}, \hfill (2.32)$$

where $\hat{P}_1, \ldots, \hat{P}_s \in M$ and satisfy conditions

$$0 < l \leq \frac{d\hat{P}_i}{d\hat{P}_j} \leq L < \infty, \quad i, j = 1, s, \hfill (2.33)$$

$l, L$ are real numbers depending on the set of measures $\hat{P}_1, \ldots, \hat{P}_s \in M$.

**Definition 2.3** Let a supermartingale $\{f_m, \mathcal{F}_m\}_{m=0}^\infty$ relative to a convex set of equivalent measures $M$ satisfy conditions (2.32). We call it regular one if for every set of measures $\{\hat{P}_i\}_{i=1}^{s}, \mathcal{F}_m$, satisfying conditions (2.33) there exist a natural number $1 \leq m_0 < \infty$, and $\mathcal{F}_{m_0-1}$ measurable nonnegative random value $\varphi_{m_0}^s, \hat{P}_1(\varphi_{m_0}^s > 0) > 0$, such that the inequalities

$$f_{m_0-1} - E^{\hat{P}_1} \{ f_{m_0} | \mathcal{F}_{m_0-1} \} \geq \varphi_{m_0}^s, \quad i = 1, s,$$

hold and for the maximal element $g^s = \{g^s_m\}_{m=0}^\infty$ in a certain maximal chain $\tilde{G}_s \subseteq G_s$ the equalities

$$E^{\hat{P}_1} \{ f_m + g^s_m | \mathcal{F}_k \} = f_k + g^s_k, \ 0 \leq k \leq m, \quad i = 1, s, \quad m = 1, \infty, \hfill (2.34)$$

are valid. Moreover, there exists an adapted nonnegative process $\tilde{g}_0 = \{\tilde{g}_m^0\}_{m=0}^\infty, \tilde{g}_0^0 = 0$, $E^{\hat{P}_1} \tilde{g}_m^0 < \infty, m = 1, \infty, P \in M$, not depending on the set of measures $\hat{P}_1, \ldots, \hat{P}_s$ such that

$$E^{\hat{P}_1} \{ g_m^s - g_{m-1}^s | \mathcal{F}_{m-1} \} = E^{\hat{P}_1} \{ \tilde{g}_m^0 | \mathcal{F}_{m-1} \}, \quad m = 1, \infty, \quad i = 1, s. \hfill (2.35)$$
The next Theorem describes regular supermartingales.

**Theorem 2.3** Let \( \{f_m, \mathcal{F}_m\}_{m=0}^\infty \) be a regular supermartingale relative to a convex set of equivalent measures \( M \). Then for the maximal element \( g^0 = \{g^0_m\}_{m=0}^\infty \) in a certain maximal chain \( \hat{G} \subseteq G \) the equalities

\[
E^{P_0}(f_m + g^0_m) = f_0, \quad m = 1, \infty, \quad P_0 \in M,
\]

are valid. There exists a martingale \( \{\hat{M}_m, \mathcal{F}_m\}_{m=0}^\infty \) relative to the family of measures \( M \) such that

\[
f_m = \hat{M}_m - g^0_m, \quad m = 1, \infty.
\]

Moreover, for the martingale \( \{\hat{M}_m, \mathcal{F}_m\}_{m=0}^\infty \) the representation

\[
\hat{M}_m = E^{P_0}(f_\infty + g_\infty | \mathcal{F}_m), \quad m = 1, \infty, \quad P_0 \in M,
\]

holds, where \( f_\infty + g_\infty = \lim_{m \to \infty} (f_m + g_m) \).

**Proof.** For any finite set of measures \( P_1, \ldots, P_n, P_i \in M, i = 1, n \), let us introduce into consideration two sets of measures

\[
M_n = \{P, P = \sum_{i=1}^n \alpha_i P_i, \alpha_i \geq 0, i = 1, n, \sum_{i=1}^n \alpha_i = 1\},
\]

\[
\tilde{M}_n = \{P, P = \sum_{i=1}^n \alpha_i P_i, \alpha_i > 0, i = 1, n, \sum_{i=1}^n \alpha_i = 1\}.
\]

Let \( \hat{P}_i, \ldots, \hat{P}_s \) be a certain subset of measures from \( \tilde{M}_n \). For every measure \( \hat{P}_i \in \tilde{M}_n \) the representation \( \hat{P}_i = \sum_{k=1}^n \alpha^i_k P_k \) is valid, where \( \alpha^i_k > 0, i = 1, s, k = 1, n \). The representation for \( \hat{P}_i, i = 1, s \), imply the validity of inequalities

\[
0 < l = \min_{i,j} \frac{\min_k \alpha^i_k}{\max_k \alpha^j_k} \leq \frac{d\hat{P}_i}{d\hat{P}_j} \leq \max_{i,j} \frac{\max_k \alpha^i_k}{\min_k \alpha^j_k} = L < \infty, \quad i, j = 1, s.
\]

Denote by \( G_s \) a set of adapted non-decreasing processes \( \{g_m\}_{m=0}^\infty \), \( g_0 = 0 \), for which \( \{f_m + g_m\}_{m=0}^\infty \) is a supermartingale relative to all measures from

\[
\hat{M}_s = \{Q, Q = \sum_{i=1}^s \gamma_i \hat{P}_i, \gamma_i \geq 0, i = 1, s, \sum_{i=1}^s \gamma_i = 1\}.
\]

In accordance with the definition of a regular supermartingale, there exist a natural number \( 1 \leq m_0 < \infty \), and \( \mathcal{F}_{m_0-1} \) measurable nonnegative random value \( \varphi^s_{m_0} \), \( \hat{P}_1(\varphi^s_{m_0} > 0) > 0 \), such that the inequalities there hold

\[
f_{m_0-1} - E^{\hat{P}_i}(f_{m_0} | \mathcal{F}_{m_0-1}) \geq \varphi^s_{m_0}, \quad i = 1, s.
\]
and for a maximal element $g_s^s = \{g_m^s\}_{m=0}^\infty$ in a certain maximal chain $\hat{G}_s \subseteq G_s$ there hold equalities (2.34), (2.35). Equalities (2.35) yield the equalities

$$E^Q \{g_m^s - g_{m-1}^s | F_{m-1} \} =$$

$$\sum_{i=1}^s \gamma_i E^{\hat{P}_i} \{\hat{\varphi}_i | F_{m-1} \} E^{\hat{P}_i} \{g_m^s - g_{m-1}^s | F_{m-1} \} = \sum_{i=1}^s \gamma_i E^{\hat{P}_i} \{\hat{\varphi}_i | F_{m-1} \}, \quad (2.36)$$

$$m = \overline{1, \infty}, \quad Q \in \hat{M}_s,$$

where $\hat{\varphi}_i = \frac{d\hat{P}_i}{dP_i}$, $i = \overline{1, n}$. Taking into account the equalities (2.34), we obtain

$$E^Q \{f_m + g_m^s | F_{m-1} \} =$$

$$\sum_{i=1}^s \gamma_i E^{\hat{P}_i} \{\hat{\varphi}_i | F_{m-1} \} E^{\hat{P}_i} \{f_m + g_m^s | F_{m-1} \} = \sum_{i=1}^s \gamma_i E^{\hat{P}_i} \{\hat{\varphi}_i | F_{m-1} \}$$

$$f_{m-1} + g_{m-1}^s, \quad m = \overline{1, \infty}, \quad Q \in \hat{M}_s. \quad (2.37)$$

Thus, we have

$$E^Q \{g_m^s - g_{m-1}^s | F_{m-1} \} = E^Q \{g_m^0 | F_{m-1} \}, \quad m = \overline{1, \infty}, \quad Q \in \hat{M}_s. \quad (2.38)$$

$$E^Q \{f_m + g_m^s | F_{m-1} \} = f_{m-1} + g_{m-1}^s, \quad m = \overline{1, \infty}, \quad Q \in \hat{M}_s. \quad (2.39)$$

Let us introduce into consideration a random process $\{N_m, F_m\}_{m=0}^\infty$, where

$$N_0 = f_0, \quad N_m = f_m + \sum_{i=1}^m g_m^i, \quad m = \overline{1, \infty}.$$

It is evident that $E^Q |N_m| < \infty, \ m = \overline{1, \infty}, \ Q \in \hat{M}_s$. The definition of $\{N_m, F_m\}_{m=0}^\infty$ and the formulae (2.38), (2.39) yield

$$E^Q \{N_{m-1} - N_m | F_{m-1} \} = \sum_{i=1}^m \gamma_i E^{\hat{P}_i} \{\hat{\varphi}_i | F_{m-1} \} E^{\hat{P}_i} \{g_m^s - g_{m-1}^s - g_m^0 | F_{m-1} \} = 0, \quad m = \overline{1, \infty}, \quad Q \in \hat{M}_s.
The last equalities imply
\[ E^Q \{ N_m | \mathcal{F}_{m-1} \} = N_{m-1}, \quad m = 1, \infty, \quad Q \in \tilde{M}_s. \]

Due to arbitrariness of the set of measures \( \tilde{P}_1, \ldots, \tilde{P}_s, \tilde{P}_i \in \tilde{M}_n \), we have
\[ E^P \{ N_m | \mathcal{F}_{m-1} \} = N_{m-1}, \quad P \in \tilde{M}_n, \quad m = 1, \infty. \quad (2.40) \]

So, the set \( G_0 \) of adapted non-decreasing processes \( \{ g_m \}_{m=0}^\infty, \; g_0 = 0 \), for which \( \{ f_m + g_m \}_{m=0}^\infty \) is a supermartingale relative to all measures from \( \tilde{M}_n \) contains nonzero element \( \tilde{g}^0 = \{ \tilde{g}^0_m \}_{m=0}^\infty, \; \tilde{g}^0_0 = 0, \; \tilde{g}^0_m = \sum_{i=1}^m \tilde{g}^0_i, \; m = 1, \infty \), which is a maximal element in a maximal chain \( \tilde{G}_0 \) containing this element. Really, if \( g^0 = \{ g^0_m \}_{m=0}^\infty, \; g^0_0 = 0 \), is a maximal element in the maximal chain \( \tilde{G}_0 \subseteq G_0 \), then there hold inequalities
\[ E^{P_0} \{ f_m + g^0_m | \mathcal{F}_k \} \leq f_k + g^0_k, \quad m = 1, \infty, \quad 1 \leq k \leq m, \quad P_0 \in \tilde{M}_n, \quad (2.41) \]
\[ E^{P_0} (f_m + g^0_m) \leq f_0, \quad m = 1, \infty, \quad P_0 \in \tilde{M}_n. \quad (2.42) \]

and inequality \( \tilde{g}^0 \leq g^0 \) meaning that \( \tilde{g}^0_m \leq g^0_m, \; m = 0, \infty \). Equalities (2.40) yield
\[ E^{P_0} (f_m + \tilde{g}^0_m) = f_0, \quad m = 1, \infty, \quad P_0 \in \tilde{M}_n. \quad (2.43) \]

Inequalities (2.42) and equalities (2.43) imply
\[ f_0 \geq E^{P_0} (f_m + g^0_m) \geq E^{P_0} (f_m + \tilde{g}^0_m) = f_0, \quad m = 1, \infty, \quad P_0 \in \tilde{M}_n. \quad (2.44) \]

The last inequalities lead to equalities
\[ E^{P_0} (g^0_m - \tilde{g}^0_m) = 0, \quad m = 1, \infty, \quad P_0 \in \tilde{M}_n. \quad (2.45) \]

But
\[ g^0_m - \tilde{g}^0_m \geq 0, \quad m = 0, \infty. \quad (2.46) \]

The equalities (2.45) and inequalities (2.46) yield \( g^0_m = \tilde{g}^0_m, \; m = 0, \infty \), or \( g^0 = \tilde{g}^0 \).

Prove that \( G_n = G_0 \), where \( G_n \) is a set of non-decreasing processes \( g = \{ g_m \}_{m=0}^\infty \) such that \( \{ f_m + g_m \}_{m=0}^\infty \) is a supermartingale relative to all measures from \( \tilde{M}_n \). Really, if \( g = \{ g_m \}_{m=0}^\infty \) is a non-decreasing process from \( G_n \), then it belongs to \( G_0 \), owing to that \( M_n \supset \tilde{M}_n \) and \( G_n \subseteq G_0 \). Suppose that \( g = \{ g_m \}_{m=0}^\infty, \; g_0 = 0 \), is a non-decreasing process from \( G_0 \). It means that
\[ E^Q \{ f_m + g_m | \mathcal{F}_k \} \leq f_k + g_k, \quad m = 1, \infty, \quad 0 \leq k \leq m, \quad Q \in \tilde{M}_n. \quad (2.47) \]

The last inequalities can be written in the form
\[ \sum_{i=1}^n \alpha_i \int_A (f_m + g_m) dP_i \leq \sum_{i=1}^n \alpha_i \int_A (f_k + g_k) dP_i, \quad m = 1, \infty, \quad 0 \leq k \leq m, \quad A \in \mathcal{F}_k, \quad \alpha_i > 0, \quad i = 1, n. \]
By passing to the limit, as $\alpha_j \to 0$, $\alpha_j > 0$, $j \neq i$, $\alpha_i \to 1$, we obtain
\[
\int_A (f_m + g_m) dP_i \leq \int_A (f_k + g_k) dP_i, \quad i = 1, n, \quad A \in F_k, \quad m = 1, \infty, \quad 0 \leq k \leq m.
\]

The last inequalities yield inequalities
\[
\sum_{i=1}^n \alpha_i \int_A (f_m + g_m) dP_i \leq \sum_{i=1}^n \alpha_i \int_A (f_k + g_k) dP_i, \quad m = 1, \infty, \quad 0 \leq k \leq m,
\]
or
\[
E^Q \{f_m + g_m | F_k\} \leq f_k + g_k, \quad m = 1, \infty, \quad 0 \leq k \leq m, \quad Q \in M_n.
\]
It means that $g = \{g_m\}_{m=0}^\infty$ belongs to $G_n$. On the basis of the above proved, for the maximal element $\tilde{g}^0 = \{\tilde{g}_m\}_{m=0}^\infty$ in the maximal chain $\tilde{G}_0 \subseteq G_0$ the equalities
\[
E^Q \{f_m + \tilde{g}_m | F_k\} = f_k + \tilde{g}_k, \quad m = 1, \infty, \quad 1 \leq k \leq m, \quad Q \in \tilde{M}_n, \quad (2.48)
\]
\[
E^Q (f_m + \tilde{g}_m) = f_0, \quad m = 1, \infty, \quad Q \in \tilde{M}_n, \quad (2.49)
\]
are valid. From proved equality $G_n = G_0$, it follows that $\tilde{G}_0$ is a maximal chain in $G_n$.

As far as, $G_0$ coincides with $G_n$ we proved that the maximal element $\tilde{g}^0$ in a certain maximal chain in $G_n$ satisfies equalities
\[
E^P \{f_m + \tilde{g}_m | F_k\} = f_k + \tilde{g}_k, \quad m = 1, \infty, \quad 1 \leq k \leq m, \quad P_0 \in M_n, \quad (2.50)
\]
\[
E^P (f_m + \tilde{g}_m) = f_0, \quad m = 1, \infty, \quad P_0 \in M_n. \quad (2.51)
\]
Due to arbitrariness of the set of measure $P_1, \ldots, P_n, P_1 \in M$, the set $G$ contains nonzero element $\tilde{g}^0$ and in the maximal chain $\tilde{G} \subseteq G$ containing element $\tilde{g}^0$ the maximal element $g^0 = \{g_m^0\}_{m=0}^\infty, g_0^0 = 0$, coincides with $\tilde{g}^0$. The last statement can be proved as in the case of maximal chain $\tilde{G}_0$. So,
\[
E^P \{f_m + g_m^0 | F_k\} = f_k + g_k^0, \quad m = 1, \infty, \quad 1 \leq k \leq m, \quad P_0 \in M, \quad (2.52)
\]
\[
E^P (f_m + g_m^0) = f_0, \quad m = 1, \infty, \quad P_0 \in M. \quad (2.53)
\]
Denote by $\{\tilde{M}_m, F_m\}_{m=0}^\infty$ a martingale relative to the set of measures $M$, where $\tilde{M}_m = f_m + g_m^0, m = 1, \infty$. Due to Theorem 233 conditions, the supermartingale $\{f_m, F_m\}_{m=0}^\infty$ and non-decreasing process $g^0 = \{g_m^0\}_{m=0}^\infty$ are uniformly integrable relative to any measure from $M$, since for the non-decreasing process $g^0 = \{g_m^0\}_{m=0}^\infty$ there hold bounds $E^P g_m^0 < T + f_0, m = 1, \infty, \quad P \in M$. Therefore, the martingale $\{\tilde{M}_m, F_m\}_{m=0}^\infty$ is uniformly integrable relative to any measure from $M$. So, with probability 1 relative to every measure from $M$ there exist limits
\[
\lim_{m \to \infty} \tilde{M}_m = M_\infty = f_\infty + g_\infty^0, \quad \lim_{m \to \infty} f_m = f_\infty, \quad \lim_{m \to \infty} g_m^0 = g_\infty^0.
\]
Moreover, the representation
\[ M_m = E^P \{(f_{\infty} + g^0_{\infty})|F_m\}, \quad m = 1, \infty, \quad P \in M, \]
holds, where \( M = \{M_m\}_{m=0}^\infty \) does not depend on \( P \in M \).

In the next theorem we give the necessary and sufficient conditions of regularity of supermartingales.

**Theorem 2.4** Let a supermartingale \( \{f_m, F_m\}_{m=0}^\infty \) relative to a convex set of equivalent measures \( M \) satisfy conditions \([2.4]\). The necessary and sufficient conditions for it to be a regular one is the existence of adapted nonnegative random process \( g^0 = \{g^0_m\}_{m=0}^\infty, \quad g^0_0 = 0, \quad E^P g^0_m < \infty, \quad m = 1, \infty, \quad P \in M, \) such that equalities \( E^P \{f_{m-1} - f_m|F_{m-1}\} = E^P \{g^0_m|F_{m-1}\}, \quad m = 1, \infty, \quad P \in M, \) (2.55)
are valid.

**Proof.** **Necessity.** If \( \{f_m, F_m\}_{m=0}^\infty \) is a regular supermartingale, then there exist a martingale \( \{\bar{M}_m, F_m\}_{m=0}^\infty \) and a non-decreasing nonnegative random process \( \{g_m, F_m\}_{m=0}^\infty, \quad g_0 = 0, \) such that
\[ f_m = \bar{M}_m - g_m, \quad m = 1, \infty. \] (2.56)
As before, equalities (2.56) yield inequalities \( E^P g_m \leq f_0 + T, \quad m = 1, \infty, \) and equalities
\[ E^P \{f_{m-1} - f_m|F_{m-1}\} = E^P \{g^0_m|F_{m-1}\}, \quad m = 1, \infty, \quad P \in M, \] (2.57)
where we introduced the denotation \( g^0_m = g_m - g_{m-1} \geq 0. \) It is evident that \( E^P g^0_m \leq 2(f_0 + T) \).

**Sufficiency.** If there exists an adapted nonnegative random process \( g^0 = \{g^0_m\}_{m=0}^\infty, \quad g^0_0 = 0, \quad E^P g^0_m < \infty, \quad m = 1, \infty, \) such that the equalities (2.55) are valid, then let us consider a random process \( \{\bar{M}_m, F_m\}_{m=0}^\infty, \) where
\[ \bar{M}_0 = f_0, \quad \bar{M}_m = f_m + \sum_{i=1}^m g^0_i, \quad m = 1, \infty. \] (2.58)
It is evident that \( E^P |\bar{M}_m| < \infty \) and
\[ E^P \{\bar{M}_{m-1} - \bar{M}_m|F_{m-1}\} = E^P \{f_{m-1} - f_m - g^0_m|F_{m-1}\} = 0. \]
Theorem 2.4 is proven.

In the next Theorem we describe the structure of non-decreasing process for a regular supermartingale.

**Theorem 2.5** Let a supermartingale \( \{f_m, F_m\}_{m=0}^\infty \) relative to a convex set of equivalent measures \( M \) satisfy conditions \([2.4]\). The necessary and sufficient conditions for it to be regular one is the existence of non-decreasing adapted process \( g = \{g_m\}_{m=0}^\infty, \quad g_0 = 0, \) and adapted processes \( \bar{\Psi}^j = \{\bar{\Psi}^j_m\}_{m=0}^\infty, \quad \bar{\Psi}^j_0 = 0, \quad j = 1, n, \) such that between elements \( g_m, \quad m = 1, \infty, \) of non-decreasing process \( g = \{g_m\}_{m=0}^\infty \) the relations
\[ g_m - g_{m-1} = f_{m-1} - E^P \{f_m|F_{m-1}\} + \bar{\Psi}^j_m, \quad m = 1, \infty, \quad j = 1, n, \] (2.59)
are valid for each set of measures \( P_1, \ldots, P_n \in M, \) where \( E^P |\bar{\Psi}^j_m| < \infty, \quad E^P \{\bar{\Psi}^j_m|F_{m-1}\} = 0, \quad j = 1, n, \quad m = 1, \infty. \)
Proof. The necessity. Let \( \{f_m, F_m\}_{m=0}^{\infty} \) be a regular supermartingale. Then for it the representation
\[
f_m + g_m = M_m, \quad m = 1, \infty, \quad j = 1, n,
\]
is valid, where \( \{g_m\}_{m=0}^{\infty}, g_0 = 0 \), is a non-decreasing adapted process, \( \{M_m, F_m\}_{m=0}^{\infty} \) is a martingale relative to the set of measures \( M \). For any finite set of measures \( P_1, \ldots, P_n \in M \), we have
\[
E^P\{f_m + g_m \mid F_{m-1}\} = f_{m-1} + g_{m-1}, \quad m = 1, \infty, \quad j = 1, n.
\]
Hence, we have
\[
E^P\{g_m - g_{m-1} \mid F_{m-1}\} =
\]
\[
f_{m-1} - E^P\{f_m \mid F_{m-1}\}, \quad m = 1, \infty, \quad j = 1, n.
\]
Let us put
\[
\Psi_m^j = g_m - g_{m-1} - E^P\{g_m - g_{m-1} \mid F_{m-1}\}.
\]
The assumptions of Theorem \([2.5]\) and Lemma \([2.3]\) the representation \((2.63)\) imply
\[
E^P|\Psi_m^j| < 4(f_0 + T), \quad E^P\{\Psi_m^j \mid F_{m-1}\} = 0, \quad j = 1, n, \quad m = 1, \infty. \quad \text{This proves the necessity.}
\]

The sufficiency. For any set of measures \( P_1, \ldots, P_n \in M \) the representation \((2.59)\) for a non-decreasing adapted process \( g = \{g_m\}_{m=0}^{\infty}, g_0 = 0 \), is valid. Hence, we obtain \((2.62)\) and \((2.61)\). The equalities \((2.62), (2.61)\) and the formula
\[
E^P\{f_m + g_m \mid F_{m-1}\} = \frac{\sum_{i=1}^{n} \alpha_i E^P\{\varphi_i \mid F_{m-1}\} E^P\{f_m + g_m \mid F_{m-1}\}}{\sum_{i=1}^{n} \alpha_i E^P\{\varphi_i \mid F_{m-1}\}}, \quad P \in M_n,
\]
\[
\varphi_i = \frac{dP_i}{dP_1}, \quad i = 1, n,
\]
imply
\[
E^P\{f_m + g_m \mid F_{m-1}\} = f_{m-1} + g_{m-1}, \quad m = 1, \infty, \quad P \in M_n.
\]
Arbitrariness of the set of measures \( P_1, \ldots, P_n \in M \) and fulfillment of the condition \((2.4)\) for the supermartingale \( \{f_m, F_m\}_{m=0}^{\infty} \) imply its regularity.

Further, we consider a class of supermartingales \( F \) satisfying conditions
\[
\sup_{P \in M} E^P|f_m| < \infty, \quad m = 0, \infty.
\]

Definition 2.4 A supermartingale \( f = \{f_m, F_m\}_{m=0}^{\infty} \in F \) is said to be local regular one if there exists an increasing sequence of nonrandom stopping times \( \tau_s = k_s, k_s < \infty, s = 1, \infty, \lim k_s = \infty \), such that the stopped process \( f^{\tau_s} = \{f_{m \wedge \tau_s}, F_m\}_{m=0}^{\infty} \) is a regular supermartingale for every \( \tau_{k_s} = k_s, k_s < \infty, s = 1, \infty \).
\textbf{Theorem 2.6} Let \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) be a supermartingale relative to a convex set of equivalent measures \( M \), belonging to the class \( F \), for which the representation
\[ f_m = M_m - g_m^0, \quad m = 0, \infty, \] (2.64)

is valid, where \( \{M_m\}_{m=0}^{\infty} \) is a martingale relative to a convex set of equivalent measures \( M \) such that
\[ E^P|M_m| < \infty, \quad m = 0, \infty, \quad P \in M, \]
g\(^0 = \{g_m^0\}_{m=0}^{\infty}, \quad g_0^0 = 0, \]
is a non-decreasing adapted process. Then \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) is a local regular supermartingale.

\textbf{Proof.} The representation (2.64) and assumptions of Theorem 2.6 imply inequalities
\[ E^P g_m^0 < \infty, \quad m = 1, \infty, \quad P \in M. \]
For any measure \( P \in M \), therefore we have
\[ E^P \{f_m + g_m^0|\mathcal{F}_{m-1}\} = M_{m-1} = f_{m-1} + g_{m-1}^0, \quad m = 1, \infty. \] (2.65)
Consider a sequence of stopping times \( \tau_s = s, \quad s = 1, \infty \). Equalities (2.65) yield
\[ E^P \{f_{m \wedge \tau_s} + g_{m \wedge \tau_s}^0|\mathcal{F}_{m-1}\} = M_{(m-1) \wedge \tau_s} = f_{(m-1) \wedge \tau_s} + g_{(m-1) \wedge \tau_s}^0, \] (2.66)
\[ m = 1, \infty, \quad P \in M. \]
For the stopped supermartingale \( \{f_{m \wedge \tau_s}, \mathcal{F}_m\}_{m=0}^{\infty} \), the set \( G \) of adapted non-decreasing processes \( g = \{g_m\}_{m=0}^{\infty}, \quad g_0 = 0, \) such that \( \{f_{m \wedge \tau_s} + g_m, \mathcal{F}_m\}_{m=0}^{\infty} \) is a supermartingale relative to a convex set of equivalent measures \( M \) contains nonzero element \( g^{0, \tau_s} = \{g_m^{0, \tau_s}\}_{m=0}^{\infty}, \quad g_0^0 = 0 \). Consider a maximal chain \( \bar{G} \subseteq G \) containing this element and let \( g = \{g_m\}_{m=0}^{\infty}, \quad g_0 = 0, \) be a maximal element in \( \bar{G} \) which exists, since the stopped supermartingale \( \{f_{m \wedge \tau_s}, \mathcal{F}_m\}_{m=0}^{\infty} \) is such that
\[ |f_{m \wedge \tau_s}| \leq \sum_{i=0}^{s} |f_i| = \varphi, \quad m = 0, \infty, \quad E^P \varphi \leq \sum_{i=0}^{s} \sup_{P \in M} E^P |f_i| = T < \infty. \]
Then
\[ E^P \{f_{m \wedge \tau_s} + g_m|\mathcal{F}_{m-1}\} \leq f_{(m-1) \wedge \tau_s} + g_{m-1}, \quad m = 1, \infty. \] (2.67)
Equalities (2.66) and inequality \( g^{0, \tau_s} \leq g \) imply
\[ f_0 = E^P \{f_{m \wedge \tau_s} + g_m^{0, \tau_s}\} \leq E^P \{f_{m \wedge \tau_s} + g_m\} \leq f_0, \quad m = 1, \infty, \quad P \in M. \] (2.68)
The last inequalities yield
\[ E^P \{f_{m \wedge \tau_s} + g_m\} = f_0, \quad m = 1, \infty, \quad P \in M. \] (2.69)
The equalities (2.69), inequality \( g^{0, \tau_s} \leq g \), and equalities
\[ E^P \{f_{m \wedge \tau_s} + g_{m \wedge \tau_s}^0\} = M_0 = f_0, \quad m = 1, \infty, \quad P \in M, \] (2.70)
imply that \( g^{0, \tau_s} = g \).
So, we proved that the stopped supermartingale \( \{f_{m \wedge \tau_s}, \mathcal{F}_m\}_{m=0}^{\infty} \) is regular one for every stopping time \( \tau_s, \quad s = 1, \infty \), converging to the infinity, as \( s \to \infty \). This proves Theorem 2.6.
Theorem 2.7  On a measurable space \( \{\Omega, \mathcal{F}\} \), let a supermartingale \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) relative to a convex set of equivalent measures \( M \) belongs to a class \( F \) and there exists a nonnegative adapted random process \( \{\bar{g}_0^m\}_{m=1}^{\infty}, E^P \bar{g}_0^m < \infty, m = 1, \infty, P \in M \), such that

\[
f_{m-1} - E^P \{f_m|F_{m-1}\} = E^P \{\bar{g}_0^m|F_{m-1}\}, \quad m = 1, \infty, \ P \in M,
\]

(2.71) then \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) is a local regular supermartingale.

**Proof.** To prove Theorem 2.7 let us consider a random process

\[
\bar{M}_m = f_m + \sum_{i=1}^{m} \bar{g}_i^0, \quad m = 1, \infty, \ P \in M, \ f_0 = \bar{M}_0.
\]

It is evident that \( E^P |\bar{M}_m| < \infty, m = 1, \infty, \ P \in M \), and \( E^P \{\bar{M}_m|\mathcal{F}_{m-1}\} = \bar{M}_{m-1}, m = 1, \infty, \ P \in M \). Therefore, for \( f_m \) the representation

\[
f_m = \bar{M}_m - g_m, \quad m = 0, \infty,
\]

(2.72)
is valid, where \( g_m = \sum_{i=1}^{m} \bar{g}_i^0 \). Supermartingale (2.72) satisfies conditions of the Theorem 6. The Theorem 2.7 is proved.

3 Description of local regular supermartingales.

Below, we describe local regular supermartingales. For this we need some auxiliary statements.

Let \( P_1, \ldots, P_k \) be a family of equivalent measures on a measurable space \( \{\Omega, \mathcal{F}\} \) and let us introduce denotation \( M \) for a convex set of equivalent measures

\[
M = \left\{ Q, Q = \sum_{i=1}^{k} \alpha_i P_i, \ \alpha_i \geq 0, \ i = 1, k, \ \sum_{i=1}^{k} \alpha_i = 1 \right\}.
\]

**Lemma 3.1** If \( \xi \) is an integrable random value relative to the set of equivalent measures \( P_1, \ldots, P_k \), then the formula

\[
\text{ess sup}_{Q \in M} E^Q \{\xi|\mathcal{F}_n\} = \max_{1 \leq i \leq k} E^{P_i} \{\xi|\mathcal{F}_n\}
\]

(3.1)
is valid almost everywhere relative to the measure \( P_1 \).

**Proof.** The definition for ess sup of non countable family of random variable see [2]. Using the formula

\[
E^Q \{\xi|\mathcal{F}_n\} = \frac{\sum_{i=1}^{k} \alpha_i E^{P_i} \{\varphi_i|\mathcal{F}_n\} E^{P_i} \{\xi|\mathcal{F}_n\}}{\sum_{i=1}^{k} \alpha_i E^{P_i} \{\varphi_i|\mathcal{F}_n\}}, \quad Q \in M,
\]

(3.2)
where \( \varphi_i = \frac{dP_i}{dP_1} \), we obtain the inequality

\[
E^Q \{\xi|\mathcal{F}_n\} \leq \max_{1 \leq i \leq k} E^{P_i} \{\xi|\mathcal{F}_n\}, \quad Q \in M,
\]

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or,

\[
\text{ess sup}_{Q \in M} E^Q \{ \xi | \mathcal{F}_n \} \leq \max_{1 \leq i \leq k} E^{P_i} \{ \xi | \mathcal{F}_n \}.
\]

On the other side \[2\],

\[
E^{P_i} \{ \xi | \mathcal{F}_n \} \leq \text{ess sup}_{Q \in M} E^Q \{ \xi | \mathcal{F}_n \}, \quad i = 1, k.
\]

Therefore,

\[
\max_{1 \leq i \leq k} E^{P_i} \{ \xi | \mathcal{F}_n \} \leq \text{ess sup}_{Q \in M} E^Q \{ \xi | \mathcal{F}_n \}.
\]

The Lemma 3.1 is proved.

**Lemma 3.2** Let \( G \) be a sub-\( \sigma \)-algebra of \( \sigma \)-algebra \( \mathcal{F} \) and \( f_s, s \in S \), be a nonnegative bounded family of random values relative to every measure from \( M \). Then

\[
E^P \{ \text{ess sup}_{s \in S} f_s | G \} \geq \text{ess sup}_{s \in S} E^P \{ f_s | G \}, \quad P \in M.
\] (3.3)

**Proof.** From the definition of \( \text{ess sup} \) \[2\], we have the inequalities

\[
\text{ess sup}_{s \in S} f_s \geq f_t, \quad t \in S.
\] (3.4)

Therefore,

\[
E^P \{ \text{ess sup}_{s \in S} f_s | G \} \geq E^P \{ f_t | G \}, \quad t \in S.
\] (3.5)

The last implies

\[
E^P \{ \text{ess sup}_{s \in S} f_s | G \} \geq \text{ess sup}_{s \in S} E^P \{ f_s | G \}.
\] (3.6)

In the next Lemma, we present formula for calculation of conditional expectation relative to another measure from \( M \).

**Lemma 3.3** On a measurable space \( \{ \Omega, \mathcal{F} \} \) with filtration \( \mathcal{F}_n \) on it, let \( M \) be a convex set of equivalent measures and let \( \xi \) be a bounded random value. Then the following formulas

\[
E^{P_{n+1}} \{ \xi | \mathcal{F}_{n+1} \} = E^{P_n} \{ \xi \varphi_n^{P_{n+1}} | \mathcal{F}_n \}, \quad n = 1, \infty,
\] (3.7)

are valid, where

\[
\varphi_n^{P_{n+1}} = \frac{dP_1}{dP_2} \left[ E^{P_1} \left\{ \frac{dP_1}{dP_2} \{ \xi | \mathcal{F}_n \} \right\} \right]^{-1}, \quad P_1, P_2 \in M.
\]

**Proof.** The proof of the Lemma 3.3 is evident.

**Lemma 3.4** On a measurable space \( \{ \Omega, \mathcal{F} \} \) with filtration \( \mathcal{F}_n \) on it, let \( \xi \) be a nonnegative bounded random value. Then the formulas

\[
E^Q \{ \text{ess sup}_{P \in M} E^P \{ \xi | \mathcal{F}_n \} | \mathcal{F}_m \} = \text{ess sup}_{P \in M} E^Q \{ \xi \varphi_n^P | \mathcal{F}_m \}, \quad Q \in M, \quad n > m,
\] (3.8)

are valid, where

\[
\varphi_n^P = \frac{dP}{dQ} \left[ E^Q \left\{ \frac{dP}{dQ} \{ \xi | \mathcal{F}_n \} \right\} \right]^{-1}, \quad P \in M.
\]
Proof. From the Lemma [3.3], we obtain
\[\text{ess sup}_{P \in M} E^P \{\xi | F_n\} = \text{ess sup}_{P \in M} E^Q \{\xi \varphi^n_P | F_n\}, \quad Q \in M.\]

Due to Lemma [3.2], we obtain the inequality
\[E^Q \{\text{ess sup}_{P \in M} E^P \{\xi | F_n\} | F_m\} = E^Q \{\text{ess sup}_{P \in M} E^Q \{\xi \varphi^n_P | F_n\} | F_m\} \geq \text{ess sup}_{P \in M} E^Q \{\xi \varphi^n_P | F_m\}.\]

Let us prove reciprocal inequality
\[E^Q \{\text{ess sup}_{P \in M} E^P \{\xi | F_n\} | F_m\} \leq \text{ess sup}_{P \in M} E^Q \{\xi \varphi^n_P | F_m\}.\]

From the definition of \(\text{ess sup}_{P \in M} E^Q \{\xi \varphi^n_P | F_n\}\), there exists a countable set \(D = \{\bar{P}_i \in M, i = 1, \infty\} [2]\) such that
\[\text{ess sup}_{P \in M} E^Q \{\xi \varphi^n_P | F_n\} = \sup_{P \in D} E^Q \{\xi \varphi^n_P | F_n\}.\]

The sequence \(\varphi_k = \sup_{P \in D} E^Q \{\xi \varphi^n_P | F_n\} - \max_{1 \leq i \leq k} E^Q \{\xi \varphi^n_{\bar{P}_i} | F_n\}, k = 1, \infty,\) converges to zero with probability one, as \(k\) tends to infinity. It is evident that
\[\max_{1 \leq i \leq k} E^Q \{\xi \varphi^n_{\bar{P}_i} | F_n\} = E^Q \{\xi \varphi^n_{\bar{P}_k} | F_n\},\]
where
\[\tau_i = \begin{cases} \tau_{i-1}, & E^Q \{\xi \varphi^n_{\bar{P}_{i-1}} | F_n\} > E^Q \{\xi \varphi^n_{\bar{P}_i} | F_n\}, \\ i, & E^Q \{\xi \varphi^n_{\bar{P}_i} | F_n\} \geq E^Q \{\xi \varphi^n_{\bar{P}_{i-1}} | F_n\}, \end{cases} \quad i = 2, \infty.\]

Therefore,
\[E^Q \{\text{ess sup}_{P \in M} E^P \{\xi | F_n\} | F_m\} = E^Q \{\text{ess sup}_{P \in M} E^Q \{\xi \varphi^n_P | F_n\} | F_m\} =\]
\[= E^Q \{\sup_{P \in D} E^Q \{\xi \varphi^n_P | F_n\} | F_m\} = E^Q \{\lim_{k \to \infty} \max_{1 \leq i \leq k} E^Q \{\xi \varphi^n_{\bar{P}_i} | F_n\} | F_m\} =\]
\[\lim_{k \to \infty} E^Q \{E^Q \{\xi \max_{1 \leq i \leq k} \varphi^n_{\bar{P}_i} | F_n\} | F_m\} = \lim_{k \to \infty} E^Q \{\xi \varphi^n_{\bar{P}_k} | F_m\} \leq \text{ess sup}_{P \in M} E^Q \{\xi \varphi^n_P | F_m\}.\]

In equalities above, we can change the limits under conditional expectation sign, since with probability one the inequalities
\[\max_{1 \leq i \leq k} \varphi^n_{\bar{P}_i} \leq \max_{1 \leq i \leq k+1} \varphi^n_{\bar{P}_i}, \quad k = 1, \infty,\]
are valid. Lemma [3.4] is proved.

The next Lemma is proved, as Lemma [3.4].
Lemma 3.5 On a measurable space \( \{ \Omega, \mathcal{F} \} \) with filtration \( \mathcal{F}_n \) on it, let \( \xi \) be a non-negative bounded random value. Then the equalities

\[
E^Q \{ \xi \text{ess sup}_{P \in M} \phi^P_n | \mathcal{F}_n \} = \text{ess sup}_{P \in M} E^Q \{ \xi \phi^P_n | \mathcal{F}_n \}, \quad Q \in M, \quad n = 0, \infty,
\]

are valid, where

\[
\phi^P_n = \frac{dP}{dQ} \left[ E^Q \left( \frac{dP}{dQ} | \mathcal{F}_n \right) \right]^{-1}.
\]

Lemma 3.6 On a measurable space \( \{ \Omega, \mathcal{F} \} \) with filtration \( \mathcal{F}_n \) on it, for every non-negative bounded random value \( \xi \) the inequalities

\[
E^Q \{ \text{ess sup}_{P \in M} E^P \{ \xi | \mathcal{F}_n \} | \mathcal{F}_m \} \leq \text{ess sup}_{P \in M} E^P \{ \xi | \mathcal{F}_m \}, \quad n > m,
\]

are valid.

Proof. From the Lemma 3.4, we have

\[
E^Q \{ \sup_{P \in D} E^P \{ \xi | \mathcal{F}_n \} | \mathcal{F}_m \} = E^Q \{ \sup_{P \in D} E^Q \{ \xi \phi^P_n | \mathcal{F}_n \} | \mathcal{F}_m \} =
\]

\[
\sup_{P \in D} E^Q \{ \xi \phi^P_n | \mathcal{F}_m \}, \quad n > m,
\]

where \( D \) is a countable subset of the set \( M \). Without loss of generality, we assume that the set of measures \( \{ P_1, \ldots, P_k \} \) belongs to the countable set \( D = \{ \bar{P}_i \in M, i = 1, \infty \} \). First, we assume that \( Q \in D \). Then, it is evident that the following equalities

\[
\bigcup_{i=1}^\infty \left\{ \omega, \ E^Q \left( \frac{d\bar{P}_i}{d\bar{Q}} | \mathcal{F}_n \right) \geq E^Q \left( \frac{d\bar{P}_i}{d\bar{Q}} | \mathcal{F}_m \right) \right\} = \Omega, \quad n > m,
\]

are valid. Due to (3.12), for every \( \omega \in \Omega \) there exist \( 1 \leq i < \infty \) such that

\[
\frac{\xi \frac{d\bar{P}_i}{d\bar{Q}}}{E^Q \{ \frac{d\bar{P}_i}{d\bar{Q}} | \mathcal{F}_n \}} \leq \frac{\xi \frac{d\bar{P}_i}{d\bar{Q}}}{E^Q \{ \frac{d\bar{P}_i}{d\bar{Q}} | \mathcal{F}_m \}}.
\]

Therefore,

\[
\sup_{P_i \in D} \frac{\xi \frac{d\bar{P}_i}{d\bar{Q}}}{E^Q \{ \frac{d\bar{P}_i}{d\bar{Q}} | \mathcal{F}_n \}} \leq \sup_{P_i \in D} \frac{\xi \frac{d\bar{P}_i}{d\bar{Q}}}{E^Q \{ \frac{d\bar{P}_i}{d\bar{Q}} | \mathcal{F}_m \}}.
\]

From (3.14), we obtain the inequality

\[
E^Q \{ \sup_{P_i \in D} \frac{\xi \frac{d\bar{P}_i}{d\bar{Q}}}{E^Q \{ \frac{d\bar{P}_i}{d\bar{Q}} | \mathcal{F}_n \}} | \mathcal{F}_m \} \leq E^Q \{ \sup_{P_i \in D} \frac{\xi \frac{d\bar{P}_i}{d\bar{Q}}}{E^Q \{ \frac{d\bar{P}_i}{d\bar{Q}} | \mathcal{F}_m \}} | \mathcal{F}_m \}.
\]

Or,

\[
E^Q \{ E^Q \{ \sup_{P_i \in D} \frac{\xi \frac{d\bar{P}_i}{d\bar{Q}}}{E^Q \{ \frac{d\bar{P}_i}{d\bar{Q}} | \mathcal{F}_n \}} | \mathcal{F}_n \} | \mathcal{F}_m \} \leq E^Q \{ \sup_{P_i \in D} \frac{\xi \frac{d\bar{P}_i}{d\bar{Q}}}{E^Q \{ \frac{d\bar{P}_i}{d\bar{Q}} | \mathcal{F}_m \}} | \mathcal{F}_m \}.
\]
The Lemmas 3.4, 3.5 and inequality (3.16) prove Lemma 3.6 as $Q \in D$. Let $Q \in M$. Since the set of measures $\{P_1, \ldots, P_k\}$ belongs to $D$ we complete the proof of the Lemma 3.6 using the formula

$$E^Q\{\Phi|\mathcal{F}_m\} = \frac{\sum_{i=1}^{k} \alpha_i E^{P_i}\{\varphi_i|\mathcal{F}_m\} E^{P_i}\{\Phi|\mathcal{F}_m\}}{\sum_{i=1}^{k} \alpha_i E^{P_i}\{\varphi_i|\mathcal{F}_m\}}, \quad Q \in M, \quad (3.17)$$

and proved above inequalities, as $Q \in D$, where $\Phi = \sup_{P_i \in D} E^{P_i}\{\xi|\mathcal{F}_m\}, \varphi_i = \frac{dP_i}{dP}, \quad i = 1, k$. The Lemma 3.6 is proved.

**Lemma 3.7** On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration $\mathcal{F}_n$ on it, let $\xi$ be an integrable relative to the set of equivalent measures $P_1, \ldots, P_k$ random value. Then the inequalities

$$E^Q\{\max_{1 \leq i \leq k} E^{P_i}\{\xi|\mathcal{F}_n\}|\mathcal{F}_m\} \leq \max_{1 \leq i \leq k} E^{P_i}\{\xi|\mathcal{F}_m\}, \quad n > m, \quad Q \in M, \quad (3.18)$$

are valid.

**Proof.** Using the Lemma 3.1 and the Lemma 3.6 for a bounded $\xi$, we prove the Lemma 3.7 inequalities (3.18). Let us consider the case, as $\max_{1 \leq i \leq k} E^{P_i}\xi < \infty$. Let $\xi_s, s = 1, \infty$, be a sequence of bounded random values converging to $\xi$ monotomusly. Then

$$E^Q\{\max_{1 \leq i \leq k} E^{P_i}\{\xi_s|\mathcal{F}_n\}|\mathcal{F}_m\} \leq \max_{1 \leq i \leq k} E^Q\{\xi_s|\mathcal{F}_m\}, \quad l = 1, \infty. \quad (3.19)$$

Due to monotony convergence of $\xi_s$ to $\xi$, as $s \to \infty$, we can pass to the limit under conditional expectations on the left and on the right in inequalities (3.19) that proves the Lemma 3.7.

**Lemma 3.8** On a measurable space $\{\Omega, \mathcal{F}\}$ with filtration $\mathcal{F}_m$ on it, let $\xi$ be a nonnegative integrable random value with respect to a set of equivalent measures $\{P_1, \ldots, P_k\}$ and such that

$$E^{P_i}\xi = M_0, \quad i = 1, k, \quad (3.20)$$

then the random process $\{M_m = \text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is a martingale relative to a convex set of equivalent measures $M$.

**Proof.** Due to Lemma 3.7 a random process $\{M_m = \text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^\infty$ is a supermartingale, that is,

$$E^P\{M_m|\mathcal{F}_{m-1}\} \leq M_{m-1}, \quad m = 1, \infty, \quad P \in M. $$

Or, $E^P M_m \leq M_0$. From the other side,

$$E^{P_i}\{\max_{1 \leq i \leq k} E^{P_i}\{\xi|\mathcal{F}_m\}\} \geq \max_{1 \leq i \leq k} E^{P_i}\{\xi|\mathcal{F}_m\} \geq M_0, \quad s = 1, k. $$

The above inequalities imply $E^{P_i}M_m = M_0, \quad m = 1, \infty, \quad s = 1, k$. The last equalities lead to equalities $E^P M_m = M_0, \quad m = 1, \infty, \quad P \in M$. The fact that $M_m$ is a supermartingale relative to the set of measures $M$ and the above equalities prove the Lemma 3.8 since the Lemma 2.1 conditions are valid.
Theorem 3.1 On a measurable space \( \{ \Omega, \mathcal{F} \} \) with filtration \( \mathcal{F}_m \) on it, let \( \xi \) be a \( \mathcal{F}_N \)-measurable nonnegative integrable relative to a set of equivalent measures \( \{ P_1, \ldots, P_k \} \) random value, \( N < \infty \). Then a supermartingale \( \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \), where

\[
f_m = \text{ess sup}_{P \in M} E^P \{ \xi | \mathcal{F}_m \}, \quad m = 1, \infty, \quad \max_{1 \leq i \leq k} E^P \xi < \infty, \tag{3.21}
\]

is local regular one if and only if

\[
E^P \xi = f_0, \quad i = 1, k. \tag{3.22}
\]

Proof. The necessity. Let \( \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \) be a local regular supermartingale. Then there exists a sequence of nonrandom stopping times \( \tau_s = n_s, s = 1, \infty \), such that for every \( n_s \) there exists \( \varphi = \sum_{m=1}^{n_s} \sum_{i=1}^{k} E^{P_i} \{ \xi | \mathcal{F}_m \} \) satisfying inequalities

\[
\max_{1 \leq j \leq k} E^{P_j} \varphi \leq \sum_{m=1}^{n_s} \sum_{i=1}^{k} \max_{1 \leq i \leq k} E^{P_i} E^{P_i} \{ \xi | \mathcal{F}_m \} \leq
\]

\[
\sum_{m=1}^{n_s} \sum_{i=1}^{k} \max_{1 \leq j \leq k} E^{P_j} \max_{1 \leq i \leq k} E^{P_i} \{ \xi | \mathcal{F}_m \} \leq
\]

\[
\sum_{m=1}^{n_s} \sum_{i=1}^{k} \max_{1 \leq j \leq k} E^{P_j} \max_{1 \leq i \leq k} E^{P_i} \xi = n_s k \max_{1 \leq i \leq k} E^{P_i} \xi,
\]

and nonnegative adapted random process \( \{ \bar{g}^0_m \}_{m=0}^{\infty}, \bar{g}^0_0 = 0, E^P \bar{g}^0_m < \infty, 0 \leq m \leq n_s \) such that

\[
f_m + \sum_{i=1}^{m} \bar{g}^0_i = \bar{M}_m, \quad E^P \bar{M}_m = f_0, \quad 0 \leq m \leq n_s, \quad P \in M.
\]

If \( n_s > N \), then

\[
E^{P_i} (\xi + \sum_{i=1}^{N} \bar{g}^0_i) = E^{P_i} \xi + E^{P_i} \sum_{i=1}^{N} \bar{g}^0_i = f_0.
\]

But there exists \( 1 \leq i_1 \leq k \) such that \( E^{P_i_1} \xi = f_0 \). Therefore, \( E^{P_i_1} \sum_{i=1}^{N} \bar{g}^0_i = 0 \). Due to equivalence of measures \( P_i, i = 1, k \), we obtain

\[
E^{P_i} \xi = f_0, \quad i = 1, k, \tag{3.23}
\]

where \( f_0 = \sup_{P \in M} E^P \xi \).

Sufficiency. If conditions (3.23) are satisfied, then \( \{ \bar{M}_m, \mathcal{F}_m \}_{m=0}^{\infty} \) is a martingale, where \( \bar{M}_m = \sup_{P \in M} E^P \{ \xi | \mathcal{F}_m \} \). The last implies local regularity of \( \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \). The Theorem 3.1 is proved.
Below we consider an arbitrary convex set of equivalent measures $M$ on a measurable space $\{\Omega, F\}$ and a filtration $F_n$ on it. Introduce into consideration a set $A_0$ of all integrable nonnegative random values $\xi$ relative to a convex set of equivalent measures $M$ satisfying conditions

$$E^P\xi = 1, \quad P \in M.$$ \hfill (3.24)

It is evident that the set $A_0$ is not empty, since contains random value $\xi = 1$. More interesting case is as $A_0$ contains more than one element.

**Lemma 3.9** On measurable space $\{\Omega, F\}$ and a filtration $F_n$ on it, let $M$ be an arbitrary convex set of equivalent measures. If nonnegative random value $\xi$ is such that $\sup_{P \in M} E^P\xi < \infty$, then $\{f_m = \operatorname{ess sup}_{P \in M} E^P\{\xi|F_m\}, F_m\}_{m=0}^{\infty}$ is a supermartingale relative to the convex set of equivalent measures $M$.

**Proof.** From definition of ess sup [2], for every $\operatorname{ess sup}_{P \in M} E^P\{\xi|F_m\}$ there exists a countable set $D_m$ such that

$$\operatorname{ess sup}_{P \in M} E^P\{\xi|F_m\} = \sup_{P \in D_m} E^P\{\xi|F_m\}, \quad m = 0, \infty.$$ \hfill (3.25)

The set $D = \bigcup_{m=0}^{\infty} D_m$ is also countable and

$$\operatorname{ess sup}_{P \in M} E^P\{\xi|F_m\} = \sup_{P \in D} E^P\{\xi|F_m\}. \quad (3.26)$$

Really, since

$$\sup_{P \in D} E^P\{\xi|F_m\} \geq \sup_{P \in D_m} E^P\{\xi|F_m\} = \operatorname{ess sup}_{P \in M} E^P\{\xi|F_m\}. \quad (3.27)$$

From the other side,

$$\operatorname{ess sup}_{P \in M} E^P\{\xi|F_m\} \geq E^Q\{\xi|F_m\}, \quad Q \in M.$$ \hfill (3.28)

The last gives

$$\operatorname{ess sup}_{P \in M} E^P\{\xi|F_m\} \geq \sup_{P \in D} E^P\{\xi|F_m\}. \quad (3.29)$$

The inequalities (3.26), (3.28) prove the needed. So, for all $m$ we can choose the common set $D$. Let $D = \{P_1, \ldots, P_n, \ldots\}$. Due to Lemma 3.7 for every $Q \in \bar{M}_k$, where

$$\bar{M}_k = \{P \in M, P = \sum_{i=1}^{k} \alpha_i \bar{P}_i, \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1\}, \quad (3.29)$$

we have

$$E^Q\{\max_{1 \leq i \leq k} E^P_i\{\xi|F_n\}|F_m\} \leq \max_{1 \leq i \leq k} E^P_i\{\xi|F_m\}, \quad n > m, \quad Q \in \bar{M}_k. \quad (3.30)$$ \hfill 23
It is evident that \( \max_{1 \leq i \leq k} E^P \{ \xi | \mathcal{F}_n \} \) tends to \( \sup_{P \in D} E^P \{ \xi | \mathcal{F}_n \} \) monotonously increasing, as \( k \to \infty \). Fixing \( Q \in \bar{M}_k \subset \bar{M}_{k+1} \) and tending \( k \) to the infinity in inequalities (3.30) we obtain
\[
E^Q \{ \sup_{P \in D} E^P \{ \xi | \mathcal{F}_m \} | \mathcal{F}_m \} \leq \sup_{P \in D} E^P \{ \xi | \mathcal{F}_m \}, \quad n > m, \quad Q \in \bar{M}_k, \quad (3.31)
\]
The last inequalities implies that for every measure \( Q \), belonging to the convex span, constructed on the set \( D \), \( \{ f_m = \text{ess sup}_{P \in M} E^P \{ \xi | \mathcal{F}_m \}, \mathcal{F}_m \}_{m=0}^\infty \) is a supermartingale relative to the convex set of equivalent measures, generated by set \( D \). Now, if a measure \( Q_0 \) does not belong to the convex span, constructed on the set \( D \), then we can add it to the set \( D \) and repeat the proof made above. As a result, we proved that \( \{ f_m = \text{ess sup}_{P \in M} E^P \{ \xi | \mathcal{F}_m \}, \mathcal{F}_m \}_{m=0}^\infty \) is also a supermartingale relative to the measure \( Q_0 \). The Zorn Lemma \[17\] complete the proof of the Lemma 3.9.

**Theorem 3.2** On measurable space \( \{ \Omega, \mathcal{F} \} \) and a filtration \( \mathcal{F}_n \) on it, let \( M \) be an arbitrary convex set of equivalent measures. For a random value \( \xi \in A_0 \) the random process \( \{ E^P \{ \xi | \mathcal{F}_m \}, \mathcal{F}_m \}_{m=0}^\infty, \quad P \in M, \) is a local regular martingale relative to a convex set of equivalent measures \( M \).

**Proof.** Let \( P_1, \ldots, P_n \) be a certain subset of measures from \( M \). Denote by \( M_n \) a convex set of equivalent measures
\[
M_n = \{ P \in M, \quad P = \sum_{i=1}^n \alpha_i P_i, \quad \alpha_i \geq 0, \quad i = 1, n, \quad \sum_{i=1}^n \alpha_i = 1 \}. \quad (3.32)
\]
Due to Lemma 3.8 \( \{ \bar{M}_m, \mathcal{F}_m \}_{m=0}^\infty \) is a martingale relative to the set of measures \( M_n \), where \( \bar{M}_m = \text{ess sup}_{P \in M_n} E^P \{ \xi | \mathcal{F}_m \}, \xi \in A_0 \). Let us consider an arbitrary measure \( P_0 \in M \) and let
\[
M_{n_{P_0}}^m = \{ P \in M, \quad P = \sum_{i=0}^n \alpha_i P_i, \quad \alpha_i \geq 0, \quad i = 0, n, \quad \sum_{i=0}^n \alpha_i = 1 \}. \quad (3.33)
\]
Then \( \{ \bar{M}_{n_{P_0}}^m, \mathcal{F}_m \}_{m=0}^\infty \), where \( \bar{M}_{n_{P_0}}^m = \text{ess sup}_{P \in M_{n_{P_0}}} E^P \{ \xi | \mathcal{F}_m \} \), is a martingale relative to the set of measures \( M_{n_{P_0}}^m \). It is evident that
\[
\bar{M}_m \leq \bar{M}_{n_{P_0}}^m, \quad m = 0, \infty. \quad (3.34)
\]
Since \( E^P \bar{M}_m = E^P \bar{M}_{n_{P_0}}^m = 1, \quad m = 0, \infty, \quad P \in M_n \), the inequalities (3.34) give \( \bar{M}_m = \bar{M}_{n_{P_0}}^m \). Analogously, \( E^{P_0} \{ \xi | \mathcal{F}_m \} \leq \bar{M}_{n_{P_0}}^m \). From equalities \( E^{P_0} E^{P_0} \{ \xi | \mathcal{F}_m \} = E^{P_0} \bar{M}_{n_{P_0}}^m = 1 \) we obtain \( E^{P_0} \{ \xi | \mathcal{F}_m \} = \bar{M}_{n_{P_0}}^m = \bar{M}_m \). Since the measure \( P_0 \) is arbitrary it implies that \( \{ E^P \{ \xi | \mathcal{F}_m \}, \mathcal{F}_m \}_{m=0}^\infty \) is a martingale relative to all measures from \( M \). Due to Theorem 2.7 it is a local regular supermartingale with random process \( g_m^0 = 0, \quad m = 0, \infty \). The Theorem 3.2 is proved.

**Theorem 3.3** On measurable space \( \{ \Omega, \mathcal{F} \} \) and a filtration \( \mathcal{F}_n \) on it, let \( M \) be an arbitrary convex set of equivalent measures. If \( \{ f_m, \mathcal{F}_m \}_{m=0}^\infty \) is an adapted random process satisfying conditions
\[
f_m \leq f_{m-1}, \quad E^P \xi | f_m | < \infty, \quad P \in M, \quad m = 1, \infty, \quad \xi \in A_0, \quad (3.35)
\]

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then the random process
\[ \{ f_m E^P \{ \xi | F_m \}, F_m \}_{m=0}^\infty, \quad P \in M, \]  
(3.36)
is a local regular supermartingale relative to a convex set of equivalent measures \( M \).

**Proof.** Due to Theorem 3.2, the random process \( \{ E^P \{ \xi | F_m \}, F_m \}_{m=0}^\infty \) is a martingale relative to the convex set of equivalent measures \( M \). Therefore,
\[ f_{m-1} E^P \{ \xi | F_{m-1} \} - E^P \{ f_m E^P \{ \xi | F_m \} | F_{m-1} \} = \]
\[ E^P \{ (f_{m-1} - f_m) E^P \{ \xi | F_m \} | F_{m-1} \}, \quad m = 1, \infty. \]  
(3.37)
So, if to put \( \bar{g}_m^0 = (f_{m-1} - f_m) E^P \{ \xi | F_m \}, m = 1, \infty, \) then \( \bar{g}_m^0 \geq 0 \), it is \( F_m \)-measurable and \( E^P \bar{g}_m^0 \leq E^P \xi (|f_{m-1}| + |f_m|) < \infty \). It proves the needed statement.

**Corollary 3.1** If \( f_m = \alpha, \ m = 1, \infty, \alpha \in R^1, \xi \in A_0, \) then \( \{ \alpha E^P \{ \xi | F_m \}, F_m \}_{m=0}^\infty \) is a local regular martingale. Assume that \( \xi = 1 \), then \( \{ f_m, F_m \}_{m=0}^\infty \) is a local regular supermartingale relative to a convex set of equivalent measures \( M \).

Denote by \( F_0 \) the set of adapted processes
\[ F_0 = \{ f = \{ f_m \}_{m=0}^\infty, \ P(\{ f_m \} < \infty) = 1, \ P \in M, \ f_m \leq f_{m-1}, \ m = 1, \infty \}. \]
For every \( \xi \in A_0 \) let us introduce the set of adapted processes
\[ L_\xi = \]
\[ \{ \tilde{f} = \{ f_m E^P \{ \xi | F_m \} \}_{m=0}^\infty, \ f_m E^P \{ \xi | F_m \} \in F_0, \ E^P \xi |f_m| < \infty, \ P \in M, \ m = 1, \infty \}, \]
and
\[ V = \bigcup_{\xi \in A_0} L_\xi. \]

**Corollary 3.2** Every random process from the set \( K \), where
\[ K = \left\{ \sum_{i=1}^m C_i \tilde{f}_i, \ \tilde{f}_i \in V, \ C_i \geq 0, \ i = 1, m, \ m = 1, \infty \right\}, \]  
(3.38)
is a local regular supermartingale relative to the convex set of equivalent measures \( M \) on a measurable space \( \{ \Omega, F \} \) with filtration \( F_m \) on it.

**Proof.** The proof is evident.

**Theorem 3.4** On measurable space \( \{ \Omega, F \} \) and a filtration \( F_n \) on it, let \( M \) be an arbitrary convex set of equivalent measures. Suppose that \( \{ f_m, F_m \}_{m=0}^\infty \) is a non-negative uniformly integrable supermartingale relative to a convex set of equivalent measures \( M \), then the necessary and sufficient conditions for it to be a local regular one is belonging it to the set \( K \).
Theorem 3.4 is proved.

**Proof.** Necessity. It is evident that if \( \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \) belongs to \( K \), then it is a local regular supermartingale.

Sufficiency. Suppose that \( \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \) is a local regular supermartingale. Then there exists nonnegative adapted process \( \{ \bar{g}_m \}_{m=1}^{\infty} \), \( E^P \bar{g}_m^0 < \infty \), \( m = 1, \infty \), and a martingale \( \{ M_m \}_{m=0}^{\infty} \), such that

\[
f_m = M_m - \sum_{i=1}^{m} \bar{g}_i^0, \quad m = 0, \infty.
\]

Then \( M_m \geq 0 \), \( m = 0, \infty \), \( E^P M_m < \infty \), \( P \in M \). Since \( 0 < E^P M_m = f_0 < \infty \) we have \( E^P \sum_{i=1}^{m} \bar{g}_i^0 < f_0 \). Let us put \( g_\infty = \lim_{m \to \infty} \sum_{i=1}^{m} \bar{g}_i^0 \). Using uniform integrability of \( f_m \), we can pass to the limit in the equality

\[
E^P(f_m + \sum_{i=1}^{m} \bar{g}_i^0) = f_0, \quad P \in M,
\]

as \( m \to \infty \). Passing to the limit in the last equality, as \( m \to \infty \), we obtain

\[
E^P(f_\infty + g_\infty) = f_0, \quad P \in M.
\]

Introduce into consideration a random value \( \xi = \frac{f_\infty + g_\infty}{f_0} \). Then \( E^P \xi = 1 \), \( P \in M \). From here we obtain that \( \xi \in A_0 \) and

\[
M_m = f_0 E^P(\xi | \mathcal{F}_m), \quad m = 0, \infty.
\]

Let us put \( \bar{f}_m^2 = - \sum_{i=1}^{m} \bar{g}_i^0 \). It is easy to see that an adapted random process \( \bar{f}_2 = \{ \bar{f}_m^2, \mathcal{F}_m \}_{m=0}^{\infty} \) belongs to \( F_0 \). Therefore, for the supermartingale \( f = \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \) the representation

\[
f = \bar{f}_1 + \bar{f}_2,
\]

is valid, where \( \bar{f}_1 = \{ f_0 E^P(\xi | \mathcal{F}_m), \mathcal{F}_m \}_{m=0}^{\infty} \) belongs to \( L_\xi \) with \( \xi = \frac{f_\infty + g_\infty}{f_0} \) and \( f_1^0 = f_0 \), \( m = 0, \infty \). The same is valid for \( \bar{f}_2 \) with \( \xi = 1 \). This implies that \( f \) belongs to the set \( K \). The Theorem 3.4 is proved.

**Corollary 3.3** Let \( f_N, N < \infty \), be a \( \mathcal{F}_N \)-measurable integrable random value, \( \sup_{P \in M} E^P |f_N| < \infty \), and let there exist \( \alpha_0 \in R^1 \) such that

\[
-\alpha_0 M_N + f_N \leq 0, \quad \omega \in \Omega,
\]

where \( \{ M_m, \mathcal{F}_m \}_{m=0}^{\infty} = \{ E^P(\xi | \mathcal{F}_m), \mathcal{F}_m \}_{m=0}^{\infty}, \xi \in A_0 \). Then a supermartingale \( \{ f^0_m + \bar{f}_m \}_{m=0}^{\infty} \) is local regular one relative to a convex set of equivalent measures \( M \), where

\[
f^0_m = \alpha_0 M_m,
\]

\[
\bar{f}_m = \begin{cases} 0, & m < N, \\
\bar{f}_N - \alpha_0 M_N, & m \geq N.
\end{cases}
\]

**Proof.** It is evident that \( \bar{f}_{m-1} - \bar{f}_m \geq 0 \), \( m = 0, \infty \). Therefore, the supermartingale

\[
f^0_m + \bar{f}_m = \begin{cases} \alpha_0 M_m, & m < N, \\
\bar{f}_N, & m = N, \\
f_N - \alpha_0 M_N + \alpha_0 M_m, & m > N,
\end{cases}
\]

is local regular one relative to a convex set of equivalent measures \( M \). The Corollary 3.3 is proved.
4 Optional decomposition for non negative supermartingales.

In this section we introduce the notion of complete set of equivalent measures and prove that non negative supermartingales are local regular with respect to this set of measures. For this purpose we are needed the next auxiliary statement.

**Theorem 4.1** The necessary and sufficient condition of local regularity of non negative supermartingale \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) relative to a convex set of equivalent measures \( M \) is the existence of \( \mathcal{F}_m \)-measurable random value \( \xi_0^m \in A_0 \) such that

\[
\frac{f_m}{f_{m-1}} \leq \xi_0^m, \quad E^P\{\xi_0^m|\mathcal{F}_{m-1}\} = 1, \quad P \in M, \quad m = 1, \infty. \tag{4.1}
\]

**Proof. The necessity.** Let \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) be a local regular supermartingale. Then there exists non negative adapted random process \( \{g_m\}_{m=0}^{\infty}, \quad g_0 = 0, \) such that \( \sup_{P \in M} E^P g_m < \infty, \)

\[
f_{m-1} - E^P\{f_m|\mathcal{F}_{m-1}\} = E^P\{g_m|\mathcal{F}_{m-1}\}, \quad P \in M, \quad m = 1, \infty. \tag{4.2}
\]

Let us put \( \xi_0^m = \frac{f_m + g_m}{f_{m-1}} \), \( m = 1, \infty \). Then from (4.2) \( E^P\{\xi_0^m|\mathcal{F}_{m-1}\} = 1, \) \( P \in M, \) \( m = 1, \infty \). It is evident that inequalities (4.1) are valid.

**The sufficiency.** Suppose that conditions of the Theorem 4.1 are valid. Then \( f_m \leq f_{m-1} + f_m - \xi_0^m \). Introduce denotation \( g_m = -f_m + f_{m-1}\xi_0^m \). Then \( g_m \geq 0, \)

\[
\sup_{P \in M} E^P g_m \leq \sup_{P \in M} E^P f_m + \sup_{P \in M} E^P f_{m-1} < \infty, \quad m = 1, \infty. \]

The last inequalities and equality give

\[
f_m = f_0 + \sum_{i=1}^{m} f_{i-1}(\xi_0^i - 1) - \sum_{i=1}^{m} g_i, \quad m = 1, \infty. \tag{4.3}
\]

Let us consider \( \{M_m, \mathcal{F}_m\}_{m=0}^{\infty} \), where \( M_m = f_0 + \sum_{i=1}^{m} f_{i-1}(\xi_0^i - 1) \). Then \( E^P\{M_m|\mathcal{F}_{m-1}\} = M_{m-1}, \) \( P \in M, \) \( m = 1, \infty \). The Theorem 4.1 is proved.

### 4.1 Space of finite set of elementary events.

In this subsection we assume that a space of elementary events \( \Omega \) is finite, that is, \( N_0 = |\Omega| < \infty \), and we give new proof of optional decomposition for non negative supermartingale relative to some convex set of equivalent measures.

Let \( \mathcal{F} \) be some algebra of subsets of \( \Omega \) and let \( \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F} \) be an increasing set of algebras, where \( \mathcal{F}_0 = \{\emptyset, \Omega\}, \) \( \mathcal{F}_N = \mathcal{F} \). Denote by \( M \) a set of equivalent measures on a measurable space \( \{\Omega, \mathcal{F}\} \). Further, we assume that a set \( A_0 \) contains an element \( \xi_0 \neq 1 \). It is evident that every algebra \( \mathcal{F}_n \) is generated by sets \( A^n_i, \) \( i = 1, N, A^n_i \cap A^n_j = \emptyset, \) \( i \neq j, \) \( N_0 < \infty, \sum_{i=1}^{N_0} A^n_i = \Omega, \) \( n = 1, N. \) Between the sets \( A^n_i \) and \( A^n_j \) the relations \( A^n_j^{-1} = \bigcup_{s \in I_j} A^n_s \) are valid, where \( I_j \subseteq T_n, \) \( T_n = \{1, 2, \ldots, N\}, \) \( I_k \cap I_j = \emptyset, \) \( s \neq k, \) \( \bigcup_{j=1}^{N_n} I_j = T_n. \) Let \( m_n = E^P\{\xi_0|\mathcal{F}_n\}, \) \( P \in M, \) \( n = 1, N. \) Then for \( m_n \)
the representation

\[ m_n = \sum_{i=1}^{N_n} m_i^n \chi_{A_i^n}(\omega), \quad n = 1, N, \quad (4.4) \]

is valid. Consider the difference \( m_n - m_{n-1} \). Then

\[ m_n - m_{n-1} = \sum_{s=1}^{N_{n-1}} \sum_{j \in I_s} \left( m_j^n - m_s^{n-1} \right) \chi_{A_j^n}(\omega) = \sum_{s=1}^{N_{n-1}} \sum_{j \in I_s} \chi_{I_s}(j)(m_j^n - m_s^{n-1}) \chi_{A_j^n} = \sum_{j=1}^{N_n} \left[ m_j^n - \sum_{s=1}^{N_{n-1}} \chi_{I_s}(j)m_s^{n-1} \right] \chi_{A_j^n}. \quad (4.5) \]

Introduce the set of numbers \( a_{js}^n = m_j^n - m_s^{n-1}, j \in I_s, s = 1, N_{n-1}, \) and sets

\[ I_s^- = \{ j \in I_s, a_{js}^n \leq 0 \}, \quad I_s^+ = \{ j \in I_s, a_{js}^n > 0 \}, \quad I^- = \bigcup_{s=1}^{N_{n-1}} I_s^-, \quad I^+ = \bigcup_{s=1}^{N_{n-1}} I_s^+. \]

Then

\[ m_n - m_{n-1} = \sum_{j \in I^-} d_j^n \chi_{A_j^n}(\omega) + \sum_{j \in I^+} d_j^n \chi_{A_j^n}(\omega), \quad (4.6) \]

\[ \sum_{j \in I^-} \chi_{A_j^n}(\omega) + \sum_{j \in I^+} \chi_{A_j^n}(\omega) = 1, \quad (4.7) \]

where \( d_j^n = a_{js}^n, \) as \( j \in I_s^-, \) or \( j \in I_s^+. \) From equalities (4.6), (4.7) we obtain

\[ \sum_{j \in I^-} d_j^n P(A_j^n) + \sum_{j \in I^+} d_j^n P(A_j^n) = 0, \quad P \in M, \quad (4.8) \]

\[ \sum_{j \in I^-} P(A_j^n) + \sum_{j \in I^+} P(A_j^n) = 1, \quad \in M. \quad (4.9) \]

Denote by \( M_n \) the contraction of the set of measures \( M \) on the algebra \( \mathcal{F}_n \). Introduce into the set \( M_n \) metrics

\[ \rho_n(P_1, P_2) = \sum_{j \in I^-} |P_1(A_j^n) - P_2(A_j^n)| + \sum_{j \in I^+} |P_1(A_j^n) - P_2(A_j^n)|, \quad n = 1, N. \quad (4.10) \]

**Definition 4.1** On a measurable space \( \{ \Omega, \mathcal{F} \} \), a set of measure \( M \) we call complete if for every \( 1 \leq n \leq N \) the closure of the set of measures \( M_n \) in metrics (4.10) contains measures

\[ P_{ij}^n(A) = \begin{cases} 0, & A \neq A_i^n, A_j^n, \\ \frac{d_i^n}{d_i^n + d_j^n}, & A = A_i^n, \\ -\frac{d_i^n}{d_i^n + d_j^n}, & A = A_j^n, \end{cases} \quad (4.11) \]

for every \( i \in I^- \) and \( j \in I^+ \).

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Lemma 4.1 Let a family of measures $M$ be complete and the set $A_0$ contains an element $\xi_0 \neq 1$. Then for every non negative $\mathcal{F}_n$-measurable random value $\xi_n = \sum_{i=1}^{N_n} C_i^n \chi_{A_i^n}$ there exists a real number $\alpha_n$ such that

$$\sup_{P \in M_n} \sum_{i=1}^{N_n} C_i^n P(A_i^n) \leq 1 + \alpha_n (m_n - m_{n-1}), \quad n = 1, N. \quad (4.12)$$

Proof. On the set $\bar{M}_n$, a functional $\varphi(P) = \sum_{i=1}^{N_n} C_i^n P(A_i^n)$ is continuous one, where $\bar{M}_n$ is the closure of the set $M_n$ in the metrics $\rho_n(P_1, P_2)$. From this it follows that the equality

$$\sup_{P \in M_n} \sum_{i=1}^{N_n} C_i^n P(A_i^n) = \sup_{P \in \bar{M}_n} \sum_{i=1}^{N_n} C_i^n P(A_i^n) \quad (4.13)$$

is valid. Denote by $f^n_i = \frac{C^n_i}{\sup_{P \in \bar{M}_n} \sum_{i=1}^{N_n} C_i^n P(A_i^n)}$, $i = 1, N_n$. Then

$$\sum_{i=1}^{N_n} f^n_i P(A_i^n) \leq 1, \quad P \in \bar{M}_n. \quad (4.14)$$

In every set $I^-$ there are strictly negative elements and in the every set $I^+$ there are strictly positive elements. For those $i \in I^-$ for which $d^n_i < 0$ and those $j \in I^+$ for which $d^n_j > 0$ the inequality (4.14) is as follows

$$f^n_i \frac{d^n_i}{-d^n_i + d^n_j} + \frac{-d^n_i}{-d^n_i + d^n_j} f^n_j \leq 1, \quad (4.15)$$

$$d^n_i < 0, \quad i \in I^-, \quad d^n_j > 0, \quad j \in I^+. \quad (4.16)$$

From (4.15) we obtain inequalities

$$f^n_j \leq 1 + \frac{1 - f^n_i}{-d^n_i} d^n_j, \quad d^n_i < 0, \quad i \in I^-, \quad d^n_j > 0, \quad j \in I^+. \quad (4.16)$$

Since the inequalities (4.16) are valid for every $\frac{1 - f^n_i}{-d^n_i}$, as $d^n_i < 0$, and since the set of such elements is finite, then if to denote

$$\alpha_n = \min_{\{i, d^n_i < 0\}} \frac{1 - f^n_i}{-d^n_i},$$

then we have

$$f^n_j \leq 1 + \alpha_n d^n_j, \quad d^n_j > 0, \quad j \in I^+. \quad (4.17)$$
From the definition of $\alpha_n$ we obtain inequalities
\[ f^n_i \leq 1 + \alpha_n d^n_i, \quad d^n_i < 0, \quad i \in I^- . \]
Now if $d^n_i = 0$ for some $i \in I^-$, then in this case $f^n_i \leq 1$. All these inequalities give
\[ f^n_i \leq 1 + \alpha_n d^n_i, \quad i \in I^- \cup I^+. \]  \hfill (4.18)
Multiplying on $\chi_{A^n_i}$ the inequalities (4.18) and summing over all $i \in I^- \cup I^+$ we obtain the needed inequality. The Lemma 4.1 is proved.

**Theorem 4.2** Suppose that conditions of the Lemma 4.1 are valid. Then for every non negative supermartingale $\{f_m, F_m\}_{m=0}^N$ optional decomposition is valid.

**Proof.** Consider random value $\xi_n = \frac{f_n}{f_{n-1}}$. Due to Lemma 4.1
\[ \sup_{P \in M} E^P \xi_n \leq 1 + \alpha_n (m_n - m_{n-1}) = \xi_n^0, \quad n = 1, N. \]
It is evident that $E^P \{\xi_n^0 | F_{n-1}\} = 1, \quad P \in M, \quad n = 1, N$. Since $\sup_{P \in M} E^P \xi_n \leq 1$, then
\[ \frac{f_n}{f_{n-1}} \leq \xi_n^0, \quad n = 1, N. \]  \hfill (4.19)
The Theorem 4.1 and inequalities (4.19) prove the Lemma 4.2.

### 4.2 Countable set of elementary events.

In this subsection we generalize the results of the previous subsection onto the countable space of elementary events.

Let $\mathcal{F}$ be some $\sigma$-algebra of subsets of the countable set of elementary events $\Omega$ and let $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ be a certain increasing set of $\sigma$-algebras, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Denote by $M$ a set of equivalent measures on a measurable space $\{\Omega, \mathcal{F}\}$. Further, we assume that the set $A_0$ contains an element $\xi_0 \neq 1$. Suppose that $\sigma$-algebra $\mathcal{F}_n$ is generated by sets $A^n_i, \quad i = 1, \infty, \quad A^n_i \cap A^n_j = \emptyset, \quad i \neq j, \quad \bigcup_{i=1}^{\infty} A^n_i = \Omega, \quad n = 1, \infty$. We also assume that between the sets $A^n_i$ and $A^{n-1}_j$ the relations $A^{n-1}_j = \bigcup_{s \in I_j} A^n_s$ are valid, where $I_j \subseteq N_0 = \{1, 2, \ldots, n, \ldots\}, \quad I_s \cap I_k = \emptyset, \quad s \neq k, \quad \bigcup_{j=1}^{\infty} I_j = N_0$. Introduce into consideration a martingale $m_n = E^P \{\xi_0 | F_n\}, \quad P \in M, \quad n = 1, \infty$. Then for $m_n$ the representation
\[ m_n = \sum_{i=1}^{\infty} m^n_i \chi_{A^n_i}(\omega), \quad n = 1, \infty, \]  \hfill (4.20)
is valid. Consider the difference $m_n - m_{n-1}$. Then
\[ m_n - m_{n-1} = \sum_{s=1}^{\infty} \sum_{j \in I_s} (m^n_j - m^{n-1}_s) \chi_{A^n_s}(\omega) = \]
\[
\sum_{s=1}^{\infty} \sum_{j=1}^{\infty} \chi_{I_s}(j)(m^n_j - m^{n-1}_s)\chi_{A^n_j} = \sum_{j=1}^{\infty} [m^n_j - \sum_{s=1}^{\infty} \chi_{I_s}(j)m^{n-1}_s] \chi_{A^n_j}. \tag{4.21}
\]

Introduce the set of numbers \(a^n_{js} = m^n_j - m^{n-1}_s, \ j \in I_s, \ s = 1, \infty, \) and sets \(I^-_s = \{j \in I_s, \ a^n_{js} \leq 0\}, \ I^+_s = \{j \in I_s, \ a^n_{js} > 0\}, \ I^- = \bigcup_{s=1}^{\infty} I^-_s, \ I^+ = \bigcup_{s=1}^{\infty} I^+_s. \) Then

\[
m_n - m_{n-1} = \sum_{j \in I^-} d^n_j \chi_{A^n_j}(\omega) + \sum_{j \in I^+} d^n_j \chi_{A^n_j}(\omega), \tag{4.22}
\]

\[
\sum_{j \in I^-} \chi_{A^n_j}(\omega) + \sum_{j \in I^+} \chi_{A^n_j}(\omega) = 1, \tag{4.23}
\]

where \(d^n_j = a^n_{js}, \) as \(j \in I^-_s, \) or \(j \in I^+_s. \) From equalities (4.22), (4.23) we obtain

\[
\sum_{j \in I^-} d^n_j P(A^n_j) + \sum_{j \in I^+} d^n_j P(A^n_j) = 0, \ P \in M, \tag{4.24}
\]

\[
\sum_{j \in I^-} P(A^n_j) + \sum_{j \in I^+} P(A^n_j) = 1, \ P \in M. \tag{4.25}
\]

Denote by \(M_n\) the contraction of the set of measures \(M\) on the \(\sigma\)-algebra \(\mathcal{F}_n.\) Introduce into the set \(M_n\) metrics

\[
\rho_n(P_1, P_2) = \sum_{j \in I^-} |P_1(A^n_j) - P_2(A^n_j)| + \sum_{j \in I^+} |P_1(A^n_j) - P_2(A^n_j)|, \ n = 1, \infty. \tag{4.26}
\]

**Definition 4.2** On a measurable space \(\{\Omega, \mathcal{F}\},\) a set of measure \(M\) we call complete if for every \(1 \leq n < \infty\) the closure of the set of measures \(M_n\) in metrics (4.26) contains measures

\[
P^n_{ij}(A) = \begin{cases} 
0, & A \neq A^n_i, A^n_j, \\
\frac{d^n_i}{d^n_i + d^n_j}, & A = A^n_i, \\
\frac{-d^n_j}{d^n_i + d^n_j}, & A = A^n_j,
\end{cases} \tag{4.27}
\]

for every \(i \in I^-\) and \(j \in I^+.\)

**Lemma 4.2** Let a family of measures \(M\) be complete and the set \(A_0\) contains an element \(\xi_0 \neq 1.\) Then for every non negative bounded \(\mathcal{F}_n\)-measurable random value \(\xi_n = \sum_{i=1}^{\infty} C^n_i \chi_{A^n_i}\) there exists real number \(\alpha_n\) such that

\[
\sup_{P \in M_n} \sum_{i=1}^{\infty} C^n_i P(A^n_i) \leq 1 + \alpha_n(m_n - m_{n-1}), \quad n = 1, \infty. \tag{4.28}
\]
**Proof.** On the set $\bar{M}_n$, a functional $\varphi(P) = \sum_{i=1}^{\infty} C_i^n P(A_i^n)$ is continuous one, where $\bar{M}_n$ is the closure of the set $M_n$ in metrics $\rho_n(P_1, P_2)$. From this it follows that the equality

$$\sup_{P \in \bar{M}_n} \sum_{i=1}^{\infty} C_i^n P(A_i^n) = \sup_{P \in M_n} \sum_{i=1}^{\infty} C_i^n P(A_i^n)$$

(4.29)

is valid. Denote by $f_i^n = \frac{C_i^n}{\sup_{P \in M_n} \sum_{i=1}^{\infty} C_i^n P(A_i^n)}$, $i = 1, \infty$. Then

$$\sum_{i=1}^{\infty} f_i^n P(A_i^n) \leq 1, \quad P \in \bar{M}_n.$$  

The last inequalities can be written in the form

$$\sum_{i \in I^-} f_i^n P(A_i^n) + \sum_{i \in I^+} f_i^n P(A_i^n) \leq 1, \quad P \in \bar{M}_n.$$  

(4.30)

In every set $I_s^-$ there are strictly negative elements and in the every set $I_s^+$ there are strictly positive elements. For those $i \in I^-$ for which $d_i^n < 0$ and those $j \in I^+$ for which $d_j^n > 0$ the inequality (4.30) is as follows

$$f_i^n d_j^n + \frac{-d_i^n}{-d_i^n + d_j^n} f_j^n \leq 1,$$

(4.31)

$$d_i^n < 0, \quad d_j^n > 0, \quad i \in I^-, \quad j \in I^+.$$  

From (4.31) we obtain inequalities

$$f_j^n \leq 1 + \frac{1 - f_i^n}{-d_i^n} d_j^n, \quad d_i^n < 0, \quad d_j^n > 0, \quad i \in I^-, \quad j \in I^+.$$  

(4.32)

Two cases are possible: a) for all $i \in I^-$, $f_i^n \leq 1$; b) there exists $i \in I^-$ such that $f_i^n > 1$. First, let us consider the case a).

Since inequalities (4.32) are valid for every $\frac{1 - f_i^n}{-d_i^n}$, as $d_i^n < 0$, and $f_i^n \leq 1$, $i \in I^-$, then if to denote

$$\alpha_n = \inf_{\{i, d_i^n < 0\}} \frac{1 - f_i^n}{-d_i^n},$$

we have $0 \leq \alpha_n < \infty$ and

$$f_j^n \leq 1 + \alpha_n d_j^n, \quad d_j^n > 0, \quad j \in I^+.$$  

(4.33)

From the definition of $\alpha_n$ we obtain inequalities

$$f_i^n \leq 1 + \alpha_n d_i^n, \quad d_i^n < 0, \quad i \in I^-.$$  

Now, if $d_i^n = 0$ for some $i \in I^-$, then in this case $f_i^n \leq 1$. All these inequalities give

$$f_i^n \leq 1 + \alpha_n d_i^n, \quad i \in I^- \cup I^+.$$  

(4.34)
Consider the case b). From the inequality (4.32) we obtain
\[ f_j^n \leq 1 - \frac{1 - f_i^n}{d_i^n} d_j^n, \quad d_i^n < 0, \quad d_j^n > 0, \quad i \in I^-, \quad j \in I^+. \] (4.35)

The last inequalities give
\[ \frac{1 - f_i^n}{d_i^n} \leq \min_{\{j, d_j^n > 0\}} \frac{1}{d_j^n} < \infty, \quad d_i^n < 0, \quad i \in I^- . \] (4.36)

Let us define \( \alpha_n = \sup \{i, d_i^n < 0\} \frac{1 - f_i^n}{d_i^n} < \infty . \) Then from (4.35) we obtain
\[ f_j^n \leq 1 - \alpha_n d_j^n, \quad d_j^n > 0, \quad j \in I^+. \] (4.37)

From the definition of \( \alpha_n \) we have
\[ f_i^n \leq 1 - \alpha_n d_i^n, \quad d_i^n < 0, \quad i \in I^- . \] (4.38)

The inequalities (4.37), (4.38) give
\[ f_j^n \leq 1 - \alpha_n d_j^n, \quad j \in I^- \cup I^+. \] (4.39)

Multiplying on \( \chi_{A_n} \) the inequalities (4.39) and summing over all \( j \in I^- \cup I^+ \) we obtain the needed inequality. The Lemma 4.2 is proved.

**Theorem 4.3** Suppose that conditions of the Lemma 4.2 are valid. Then for every non negative supermartingale \( \{f_m, \mathcal{F}_m\}_{m=0}^\infty \), satisfying conditions
\[ \sup_{P \in M} E^P f_m < \infty, \quad \frac{f_m}{f_{m-1}} \leq C_m < \infty, \quad m = 1, \infty, \] (4.40)

optional decomposition is valid.

**Proof.** Consider random value \( \xi_n = \frac{f_n}{f_{n-1}} \). Due to Lemma 4.2
\[ \sup_{P \in M} E^P \xi_n \leq 1 + \alpha_n (m_n - m_{n-1}) = \xi_0^n. \]

It is evident that \( E^P \{\xi_0^n | \mathcal{F}_{n-1}\} = 1, \quad P \in M, \quad n = 1, \infty. \) Since \( \sup_{P \in M} E^P \xi_n \leq 1 \), then
\[ \frac{f_n}{f_{n-1}} \leq \xi_0^n, \quad n = 1, N. \] (4.41)

The Theorem 4.1 and inequalities (4.41) prove the Lemma 4.3.
4.3 An arbitrary space of elementary events.

In this subsection we consider an arbitrary space of elementary events and prove optional decomposition for non negative supermartingales.

Let $\mathcal{F}$ be some $\sigma$-algebra of subsets of the set of elementary events $\Omega$ and let $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ be an increasing set of $\sigma$-algebras, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Denote by $M$ a set of equivalent measures on a measurable space $\{\Omega, \mathcal{F}\}$. We assume that $\sigma$-algebras $\mathcal{F}_n$, $n = \overline{1, \infty}$, and $\mathcal{F}$ are complete relative to all measure $P \in M$. Further, we suppose that a set $A_0$ contains an element $\xi_0 \neq 1$. Let\( m_n = E^P \{\xi_0 | \mathcal{F}_n\} \), $P \in M$, $n = \overline{1, \infty}$. Then for $m_n$ the representation

\[
m_n = \sum_{i=1}^{\infty} m_i^n\chi_{A_i^n}(\omega), \quad n = \overline{1, \infty},
\]

is valid for some $A_i^n \in \mathcal{F}_n$, $A_i^n$, $i = \overline{1, \infty}$, $A_i^n \cap A_j^n = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{\infty} A_i^n = \Omega$, $n = \overline{1, \infty}$.

Really, let us consider a sequence of random values \( m_n = E^P \{\xi_0 | \mathcal{F}_n\} \), $P \in M$, $n = \overline{1, \infty}$. It is evident that $E^P \{m_n | \mathcal{F}_{n-1}\} = m_{n-1}$. For every random value $m_n$ there exists not more then a countable set of non negative real number $m_s^n \geq 0$ such that $P(A_s^n) > 0$, where $A_s^n = \{\omega \in \Omega, m_n = m_s^n\}$. It is evident that $A_i^n \cap A_j^n = \emptyset$, $i \neq j$. Since $m_n$ is defined on all $\Omega$, then $P(\bigcup_{i=1}^{\infty} A_i^n) = 1$. From this it follows that $P(\Omega \setminus \bigcup_{i=1}^{\infty} A_i^n) = 0$. The set $\Omega \setminus \bigcup_{i=1}^{\infty} A_i^n$ we can join, for example, to $A_i^n$ and to put $m_n = m_i^n$, $\omega \in A_i^n \cup (\Omega \setminus \bigcup_{i=1}^{\infty} A_i^n)$. If to change denotation we come to the above statement. Further, let us prove that we can choose the sets $A_i^n$ such that the relations $A_{j-1}^{n-1} = \bigcup_{s \in I_j} A_s^n$ are valid, where $I_j \subseteq N_0 = \{1, 2, \ldots, n, \ldots\}$, $I_s \cap I_k = \emptyset$, $s \neq k$, $\bigcup_{j=1}^{\infty} I_j = N_0$. Really, if it is not so then we can choose countable set of subsets $B_{ij}^n = A_i^n \cap A_{j-1}^{n-1}$, $i, j = \overline{1, \infty}$. It is evident that $\bigcup_{j=1}^{\infty} B_{ij}^n = A_i^n$, $\bigcup_{i=1}^{\infty} B_{ij}^n = A_{j-1}^{n-1}$.

For fixed $j$ denote by $I_j$ those indexes $i$ for which $P(B_{ij}^n) > 0$. Then $\bigcup_{i \in I_j} B_{ij}^n = A_{j-1}^{n-1}$.

Let us define on $B_{ij}^n$ random value putting $m_{ij}^n = m_i^n$, $\omega \in B_{ij}^n$, $i \in I_j$, $j = \overline{1, \infty}$. Then

\[
\sum_{j=1}^{\infty} \sum_{i \in I_j} m_{ij}^n\chi_{B_{ij}^n}(\omega) = \sum_{j=1}^{\infty} \sum_{i \in I_j} m_i^n\chi_{B_{ij}^n}(\omega) = m_n, \quad n = \overline{1, \infty}.
\]

Taking into account these facts, further without loss of generality we suppose that between the sets $A_i^n$ and $A_{j-1}^{n-1}$ the relations $A_{j-1}^{n-1} = \bigcup_{s \in I_j} A_s^n$ are valid, where $I_j \subseteq N_0 = \{1, 2, \ldots, n, \ldots\}$, $I_s \cap I_k = \emptyset$, $s \neq k$, $\bigcup_{j=1}^{\infty} I_j = N_0$.

Consider the difference $m_n - m_{n-1}$. Then

\[
m_n - m_{n-1} = \sum_{s=1}^{\infty} \sum_{j \in I_s} (m_j^n - m_{s}^{n-1})\chi_{A_s^n}(\omega) =
\]
\[
\sum_{s=1}^{\infty} \sum_{j=1}^{\infty} \chi_{I_s}(j)(m_j^n - m_s^{n-1}) \chi_{A_j^n} = \sum_{j=1}^{\infty} [m_j^n - \sum_{s=1}^{\infty} \chi_{I_s}(j)m_s^{n-1}] \chi_{A_j^n}. \tag{4.44}
\]

Introduce the set of numbers \(a_{js}^n = m_j^n - m_s^{n-1}, j \in I_s, s = 1, \infty, \) and sets \(I_s^- = \{j \in I_s, a_{js}^n \leq 0\}, I_s^+ = \{j \in I_s, a_{js}^n > 0\}, I^- = \bigcup_{s=1}^{\infty} I_s^-, I^+ = \bigcup_{s=1}^{\infty} I_s^+. \) Then

\[
m_n - m_{n-1} = \sum_{j \in I^-} d_j^n \chi_{A_j^n}(\omega) + \sum_{j \in I^+} d_j^n \chi_{A_j^n}(\omega), \tag{4.45}
\]

\[
\sum_{j \in I^-} \chi_{A_j^n}(\omega) + \sum_{j \in I^+} \chi_{A_j^n}(\omega) = 1, \tag{4.46}
\]

where \(d_j^n = a_{js}^n, j \in I_s^-, \) or \(j \in I_s^+. \)

Let a countable set of subsets \(D_j^n \in F_n, j = 1, \infty, \) be such that \(D_i^n \cap D_j^n = \emptyset, i \neq j, \bigcup_{i=1}^{\infty} D_i^n = \Omega, n = 1, \infty. \) Denote by \(\hat{F}_n^D \subseteq F_n \) a sub \(\sigma\)-algebra of the \(\sigma\)-algebra \(F_n, \)
generated by the countable set of subsets \(D_j^n \in F_n, j = 1, \infty. \)

Let \(M_n^D \) be the contraction of the set of measures \(M \) on the sub \(\sigma\)-algebra \(\hat{F}_n^D \subseteq F_n. \) Introduce into the set \(M_n^D \) metrics

\[
\rho_n^D(P_1, P_2) = \sum_{j \in I^-} |P_1(D_j^n) - P_2(D_j^n)| + \sum_{j \in I^+} |P_1(D_j^n) - P_2(D_j^n)|, \tag{4.47}
\]

\[
n = 1, \infty.
\]

**Definition 4.3** On a measurable space \(\{\Omega, F\}, \) a set of measure \(M \) we call complete if for every \(1 \leq n < \infty \) and every countable set of subsets \(D_j^n \in F_n, j = 1, \infty, \)

\(D_i^n \cap D_j^n = \emptyset, i \neq j, \bigcup_{i=1}^{\infty} D_i^n = \Omega, n = 1, \infty, \) the closure in metrics \((4.47)\) of the set of measures \(M_n^D \) contains measures

\[
P_{ij}^n(B) = \begin{cases} 
0, & B \neq D_i^n, D_j^n, \\
\frac{d_i^n}{d_i^n + d_j^n}, & B = D_i^n, \\
\frac{d_j^n}{d_i^n + d_j^n}, & B = D_j^n,
\end{cases} \tag{4.48}
\]

for every \(i \in I^- \) and \(j \in I^+. \)

**Lemma 4.3** Let a family of measures \(M \) be complete and the set \(A_0 \) contains an element \(\xi_0 \neq 1. \) Then for every non negative bounded \(F_n\)-measurable random value \(\xi_n \) there exists a real number \(\alpha_n \) such that

\[
\sup_{P \in M} E_P^\xi_n \leq 1 + \alpha_n(m_n - m_{n-1}), \quad n = 1, \infty. \tag{4.49}
\]

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Proof. For the random value $\xi_n$, the representation $\sum_{j=1}^{\infty} \xi_j^n \chi_{V_j^n}$ is valid, where $V_j^n \in \mathcal{F}_n$, $j = 1, \infty$, $V_i^n \cap V_j^n = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{\infty} V_i^n = \Omega$, $n = 1, \infty$. Introduce into consideration the countable set of subsets $U_{ij}^n = A_i^n \cap V_j^n$, $i, j = 1, \infty$. It is evident that $\bigcup_{i,j=1}^{\infty} U_{ij}^n = \Omega$, $U_{ij}^n \cap U_{rs}^n = \emptyset$, $\{ij\} \neq \{rs\}$.

Then
\[
m_n - m_{n-1} = \sum_{s=1}^{\infty} \sum_{i \in I^-} d_i^n \chi_{A_i^n \cap V_s^n}(\omega) + \sum_{t=1}^{\infty} \sum_{j \in I^+} d_j^n \chi_{A_j^n \cap V_t^n}(\omega), \tag{4.50}
\]
\[
\sum_{s=1}^{\infty} \sum_{i \in I^-} \chi_{A_i^n \cap V_s^n}(\omega) + \sum_{t=1}^{\infty} \sum_{j \in I^+} \chi_{A_j^n \cap V_t^n}(\omega) = 1, \tag{4.51}
\]
where $d_j^n = a_{jn}^n$, as $j \in I_-$, or $j \in I_+$. From equalities (4.50), (4.51) we obtain
\[
\sum_{s=1}^{\infty} \sum_{i \in I^-} d_i^n P(A_i^n \cap V_s^n) + \sum_{t=1}^{\infty} \sum_{j \in I^+} d_j^n P(A_j^n \cap V_t^n) = 0, \quad P \in M, \tag{4.52}
\]
\[
\sum_{s=1}^{\infty} \sum_{i \in I^-} P(A_i^n \cap V_s^n) + \sum_{t=1}^{\infty} \sum_{j \in I^+} P(A_j^n \cap V_t^n) = 1, \quad P \in M. \tag{4.53}
\]
The random value $\xi_n$ can be written in the form
\[
\xi_n = \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \xi_j^n \chi_{A_j^n \cap V_s^n}. \tag{4.54}
\]
Let $M_n^U$ be the contraction of the set of measures $M$ on the sub $\sigma$-algebra $\tilde{\mathcal{F}}_n^U$, generated by the countable set of subsets $U_{ij}^n$, $i, j = 1, \infty$. On the set $\tilde{M}_n^U$, a functional $\varphi(P) = \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \xi_j^n P(A_j^n \cap V_s^n)$, $P \in \tilde{M}_n^U$, is continuous one in the metrics $\rho_n^U(P_1, P_2)$, where $\tilde{M}_n^U$ is the closure of the set $M_n^U$ in the metrics. From this it follows that the equality
\[
\sup_{P \in M_n^U} \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \xi_j^n P(A_j^n \cap V_s^n) = \sup_{P \in \tilde{M}_n^U} \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \xi_j^n P(A_j^n \cap V_s^n) \tag{4.55}
\]
is valid.

Denote by $f_s^n = \frac{\sup_{P \in M_n} \sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \xi_j^n P(A_j^n \cap V_s^n)}{\sum_{j=1}^{\infty} \sum_{s=1}^{\infty} \xi_j^n P(A_j^n \cap V_s^n)}$, $s = 1, \infty$. Then
\[
\sum_{j=1}^{\infty} \sum_{s=1}^{\infty} f_s^n P(A_j^n \cap V_s^n) \leq 1, \quad P \in \tilde{M}_n^U. \tag{4.56}
\]
The last inequalities can be written in the form

\[ \sum_{s=1}^{\infty} \sum_{i \in I^-} f_{is}^n P(A_i^n \cap V_s^n) + \sum_{t=1}^{\infty} \sum_{j \in I^+} f_{jt}^n P(A_j^n \cap V_t^n) \leq 1, \quad P \in \tilde{M}_n^U. \]  

Let us write equalities (4.50), (4.51) in more general form

\[ m_n - m_{n-1} = \sum_{s=1}^{\infty} \sum_{i \in I^-} d_{is}^n \chi_{A_i^n \cap V_s^n}(\omega) + \sum_{t=1}^{\infty} \sum_{j \in I^+} d_{jt}^n \chi_{A_j^n \cap V_t^n}(\omega), \]  

where \( d_{is}^n = d_i^n, \ s = 1, \infty, \ d_{jt}^n = d_j^n, \ t = 1, \infty. \)

From equalities (4.58), (4.59) we obtain

\[ \sum_{s=1}^{\infty} \sum_{i \in I^-} d_{is}^n P(A_i^n \cap V_s^n) + \sum_{t=1}^{\infty} \sum_{j \in I^+} d_{jt}^n P(A_j^n \cap V_t^n) = 0, \quad P \in M, \]  

\[ \sum_{s=1}^{\infty} \sum_{i \in I^-} P(A_i^n \cap V_s^n) + \sum_{t=1}^{\infty} \sum_{j \in I^+} P(A_j^n \cap V_t^n) = 1, \quad P \in M. \]  

Let us write inequality (4.57) in the form

\[ \sum_{s=1}^{\infty} \sum_{i \in I^-} f_{is}^n P(A_i^n \cap V_s^n) + \sum_{t=1}^{\infty} \sum_{j \in I^+} f_{jt}^n P(A_j^n \cap V_t^n) \leq 1, \quad P \in \tilde{M}_n^U, \]  

where \( f_{is}^n = f_i^n, \ i = 1, \infty, \ f_{jt}^n = f_j^n, \ j = 1, \infty. \)

Due to completeness of the set of measures \( M, \) for those \( i \in I^- \) for which \( d_{is}^n < 0, \ s = 1, \infty, \) and those \( j \in I^+ \) for which \( d_{jt}^m > 0, \ t = 1, \infty, \) the inequality (4.62) is as follows

\[ f_{is}^n \frac{d_{jt}^m}{-d_{is}^m + d_{jt}^m} + \frac{-d_{is}^m}{-d_{is}^m + d_{jt}^m} f_{jt}^m \leq 1, \quad d_{is}^m < 0, \ s = 1, \infty, \]  

\[ d_{jt}^m > 0, \ t = 1, \infty. \]  

From inequalities (4.63) we obtain

\[ f_{jt}^m \leq 1 + \frac{1 - f_{is}^n}{-d_{is}^m} d_{jt}^m, \quad d_{is}^m < 0, \ s = 1, \infty, \ d_{jt}^m > 0, \ t = 1, \infty. \]  

Two cases are possible: a) \( f_{is}^n \leq 1, \ i \in I^-, \ s = 1, \infty; \) b) there exists \( i \in I^- \) such that \( f_{is}^n > 1, \ s = 1, \infty. \) Consider the case a).
Since inequalities (4.64) are valid for every $\frac{1 - f^n_{is}}{d^n_{is}}$, as $d^n_{is} < 0$, $s = 1, \infty$, and $f^n_{is} \leq 1$, $i \in I^-$, $s = 1, \infty$, then if to denote
\[
\alpha_n = \inf_{\{is, d^n_{is} < 0\}} \frac{1 - f^n_{is}}{-d^n_{is}},
\]
we have $0 \leq \alpha_n < \infty$, and
\[
f^n_{jt} \leq 1 + \alpha_n d^n_{jt}, \quad d^n_{jt} > 0, \quad j \in I^+, \quad t = 1, \infty.
\]
(4.65)
From the definition of $\alpha_n$ we obtain inequalities
\[
f^n_{is} \leq 1 + \alpha_n d^n_{is}, \quad d^n_{is} < 0, \quad i \in I^-, \quad s = 1, \infty.
\]
Now if $d^n_{is} = 0$ for some $i \in I^-$, $s = 1, \infty$, then in this case $f^n_{is} \leq 1$. All these inequalities give
\[
f^n_{is} \leq 1 + \alpha_n d^n_{is}, \quad i \in I^- \cup I^+, \quad s = 1, \infty.
\]
(4.66)
Consider the case b). From the inequality (4.64) we obtain
\[
f^n_{jt} \leq 1 - \frac{1 - f^n_{is}}{d^n_{is}} d^n_{jt}, \quad d^n_{is} < 0, \quad i \in I^-, \quad s = 1, \infty,
\]
(4.67)
\[
d^n_{jt} > 0, \quad j \in I^+, \quad t = 1, \infty.
\]
The last inequality gives
\[
\frac{1 - f^n_{is}}{d^n_{is}} \leq \min_{\{jt, d^n_{jt} > 0\}} \frac{1}{d^n_{jt}} < \infty, \quad d^n_{is} < 0, \quad i \in I^-, \quad s = 1, \infty.
\]
(4.68)
Let us define $\alpha_n = \sup_{\{is, d^n_{is} < 0\}} \frac{1 - f^n_{is}}{d^n_{is}} < \infty$. Then from (4.67) we obtain
\[
f^n_{jt} \leq 1 - \alpha_n d^n_{jt}, \quad d^n_{jt} > 0, \quad j \in I^+, \quad t = 1, \infty.
\]
(4.69)
From the definition of $\alpha_n$ we obtain
\[
f^n_{is} \leq 1 - \alpha_n d^n_{is}, \quad d^n_{is} < 0, \quad i \in I^-, \quad s = 1, \infty.
\]
(4.70)
The inequalities (4.69), (4.70) give
\[
f^n_{is} \leq 1 - \alpha_n d^n_{is}, \quad i \in I^- \cup I^+, \quad s = 1, \infty.
\]
(4.71)
Multiplying on $\chi_{A^n \cap V^n}$ the inequalities (4.71) and summing over all $i \in I^- \cup I^+$ and $s = 1, \infty$ we obtain
\[
\sum_{s=1}^{\infty} \sum_{i=1}^{\infty} f^n_{is} \chi_{A^n \cap V^n} = \sum_{s=1}^{\infty} \sum_{i=1}^{\infty} f^n_{is} \chi_{A^n \cap V^n} = \frac{\xi_n}{\sup_{P \in M} E^n P \xi_n} \leq
\]
\[
1 - \alpha_n \sum_{s=1}^{\infty} \sum_{i=1}^{\infty} d^n_{is} \chi_{A^n \cap V^n} = 1 - \alpha_n (m_n - m_{n-1}).
\]
(4.72)
The Lemma 4.3 is proved.
Theorem 4.4 Suppose that conditions of the Lemma 4.3 are valid. Then for every non negative supermartingale \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \), satisfying conditions

\[
\sup_{P \in \mathcal{M}} E^P f_m < \infty, \quad \frac{f_m}{f_{m-1}} \leq C_m < \infty, \quad m = 1, \infty,
\]  

optional decomposition is valid.

Proof. Consider random value \( \xi_n = \frac{f_n}{f_{n-1}} \). Due to Lemma 4.3

\[
\sup_{P \in \mathcal{M}} E^P \xi_n \leq 1 + \alpha_n (m_n - m_{n-1}) = \xi^0_n.
\]

It is evident that \( E^P \{\xi^0_n|\mathcal{F}_{n-1}\} = 1, \ P \in \mathcal{M}, \ n = 1, \infty \). Since \( \sup_{P \in \mathcal{M}} E^P \xi_n \leq 1 \), then

\[
\frac{f_n}{f_{n-1}} \leq \xi^0_{n}, \quad n = 1, N.
\]  

(4.74)

The Theorem 4.1 and inequalities (4.74) prove the Lemma 4.4.

5 Application to Mathematical Finance.

Due to Corollary 3.3, we can give the following definition of fair price of contingent claim \( f_N \) relative to a convex set of equivalent measures \( \mathcal{M} \).

Definition 5.1 Let \( f_N, \ N < \infty, \) be a \( \mathcal{F}_N \)-measurable integrable relative to a convex set of equivalent measures \( \mathcal{M} \) random value such that for some \( 0 \leq \alpha_0 < \infty \) and \( \xi_0 \in A_0 \)

\[
P(f_N - \alpha_0 E^P \{\xi_0|\mathcal{F}_N\} \leq 0) = 1.
\]  

(5.1)

Denote \( G_{\alpha_0} = \{\alpha \in [0, \alpha_0], \ \exists \xi_\alpha \in A_0, \ P(f_N - \alpha E^P \{\xi_\alpha|\mathcal{F}_N\} \leq 0) = 1\} \). We call

\[
f_0 = \inf_{\alpha \in G_{\alpha_0}} \alpha
\]  

(5.2)

a fair price of contingent claim \( f_N \) relative to a convex set of equivalent measures \( \mathcal{M} \), if there exists \( \zeta_0 \in A_0 \) and a sequences \( \alpha_n \in [0, \alpha_0], \ \xi_{\alpha_n} \in A_0 \), satisfying conditions \( \alpha_n \to f_0, \ \xi_{\alpha_n} \to \zeta_0 \) by probability, as \( n \to \infty \), and such that

\[
P(f_N - \alpha_n E^P \{\xi_{\alpha_n}|\mathcal{F}_N\} \leq 0) = 1, \quad n = 1, \infty.
\]  

(5.3)

Theorem 5.1 Let the set \( A_0 \) be uniformly integrable one relative to every measure \( P \in \mathcal{M} \). Suppose that for a nonnegative \( \mathcal{F}_N \)-measurable integrable relative to every measure \( P \in \mathcal{M} \) contingent claim \( f_N, \ N < \infty \), there exist \( \alpha_0 < \infty \) and \( \zeta_0 \in A_0 \) such that

\[
P(f_N - \alpha_0 E^P \{\xi_0|\mathcal{F}_N\} \leq 0) = 1,
\]  

(5.4)

then a fair price \( f_0 \) of contingent claim \( f_N \) exists. For \( f_0 \) the inequality

\[
\sup_{P \in \mathcal{M}} E^P f_N \leq f_0
\]  

(5.5)

is valid. If a supermartingale \( \{f_m = \text{ess sup}_{P \in \mathcal{M}} E^P \{f_N|\mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^{\infty} \) is local regular one, then \( f_0 = \sup_{P \in \mathcal{M}} E^P f_N \).
Proof. If \( f_0 = \alpha_0 \), then Theorem 5.1 is proved. Suppose that \( f_0 < \alpha_0 \). Then there exists a sequence \( \alpha_n \to f_0 \), and \( \xi_{\alpha_n} \in A_0 \), \( n \to \infty \), such that

\[
P(f_N - \alpha_n E^P \{ \xi_{\alpha_n} | F_N \} \leq 0) = 1, \quad P \in M.
\] (5.6)

Due to uniform integrability \( A_0 \) we obtain

\[
1 = \lim_{n \to \infty} \int_{\Omega} \xi_{\alpha_n} dP = \int_{\Omega} \xi_0 dP, \quad P \in M.
\] (5.7)

Using again uniform integrability and going to the limit in (5.6) we obtain

\[
P(f_N - f_0 E^P \{ \xi_0 | F_N \} \leq 0) = 1, \quad P \in M.
\] (5.8)

From the inequality \( f_N - f_0 E^P \{ \xi_0 | F_N \} \leq 0 \) it follows inequality (5.5). If \( f_m = \text{ess sup}_{P \in M} E^P \{ f_N | F_m \}, \ m = 0, N \), is a local regular supermartingale, then

\[
f_m = M_m - g_m, \quad m = 0, N, \quad g_0 = 0,
\] (5.9)

where a martingale \( M_m, \ m = 0, N \), is a nonnegative and \( E^P M_m = \sup_{P \in M} E^P f_N \).

Introduce into consideration a random value \( \xi_0 = \frac{M_N}{f_0}, \hat{f}_0 = \sup_{P \in M} E^P f_N \). Then \( \xi_0 \) belongs to the set \( A_0 \) and

\[
P(f_N - \hat{f}_0 E^P \{ \xi_0 | F_N \} \leq 0) = 1.
\] (5.10)

From this it follows that \( f_0 = \sup_{P \in M} E^P f_N \).

Let us prove that \( f_0 = \text{sup} \ P \in M \ f_N \).

Suppose that evolution of risk asset is given by the law \( S_m = f_0 E^P \{ \xi_0 | F_m \}, \ m = 0, N \), and evolution of non risk asset is given by the formula \( B_m = 1, \ m = 0, N \).

As proved above, for \( f_0 = \inf_{\alpha \in G_{n_0}} \alpha \) there exists \( \xi_0 \in A_0 \) such that the inequality

\[
f_N - f_0 E^P \{ \xi_0 | F_N \} \leq 0
\]
is valid. Let us put

\[
f_m^0 = f_0 E^P \{ \xi_0 | F_m \}, \quad P \in M,
\]

\[
\bar{f}_m = \begin{cases} 0, & m < N, \\ f_N - f_0 E^P \{ \xi_0 | F_m \}, & m = N. \end{cases}
\]

It is evident that \( \bar{f}_{m-1} - \bar{f}_m \geq 0, \ m = 0, N \). Therefore, the supermartingale

\[
f_m^0 + \bar{f}_m = \begin{cases} f_0 E^P \{ \xi_0 | F_m \}, & m < N, \\ f_N, & m = N, \end{cases}
\]
is a local regular one. It is evident that

\[
f_m^0 + \bar{f}_m = M_m - g_m, \quad m = 0, N,
\]

where

\[
M_m = f_0 E^P \{ \xi_0 | F_m \}, \quad m = 0, N.
\]
\[ g_m = 0, \quad m = 0, N - 1, \]
\[ g_N = f_0 E^P \{ \zeta_0 | \mathcal{F}_N \} - f_N. \]

For martingale \( \{ M_m \}_{m=0}^N \) the representation
\[ M_m = f_0 + \sum_{i=1}^m H_i \Delta S_i, \quad m = 0, N, \]
is valid, where \( H_i = 1, \ i = 1, N \). Let us consider a trading strategy \( \pi = \{ \bar{H}_m, \bar{H}_{m+1} \}_{m=0}^N \), where
\[ \bar{H}_0^0 = f_0, \quad \bar{H}_m^0 = M_m - H_m S_m, \quad m = 1, N, \quad \bar{H}_0 = 0, \quad \bar{H}_m = H_m, \quad m = 1, N. \]

It is evident that \( \bar{H}_m^0, \bar{H}_m \) are \( \mathcal{F}_{m-1} \) measurable and the trading strategy \( \pi \) satisfy self-financed condition
\[ \Delta \bar{H}_m^0 + \Delta \bar{H}_m S_{m-1} = 0. \]
Moreover, a capital corresponding to the self-financed trading strategy \( \pi \) is given by the formula
\[ X^\pi_m = \bar{H}_m^0 + \bar{H}_m S_m = M_m. \]

Herefrom, \( X_0^\pi = f_0 \). Further,
\[ X_N^\pi = f_N + g_N \geq f_N. \]

The last proves the Theorem 5.1. From (5.8) and Corollary 3.3 the Theorem 5.2 follows.

**Theorem 5.2** Suppose that the set \( A_0 \) contains only \( 1 \leq k < \infty \) linear independent elements \( \xi_1, \ldots, \xi_k \). If there exist \( \xi_0 \in T \) and \( \alpha_0 \geq 0 \) such that
\[ P(f_N - \alpha_0 E^P \{ \xi_0 | \mathcal{F}_N \} \leq 0) = 1, \quad P \in M, \] where
\[ T = \{ \xi \geq 0, \ \xi = \sum_{i=1}^k \alpha_i \xi_i, \ \alpha_i \geq 0, \ i = 1, k, \ \sum_{i=1}^k \alpha_i = 1 \}, \]
then a fair price \( f_0 \) of contingent claim \( f_N \geq 0 \) exists, where \( f_N \) is \( \mathcal{F}_N \) measurable and integrable relative to every measure \( P \in M, \ N < \infty \).

**Proof.** The proof is evident, as the set \( T \) is uniformly integrable relative to every measure from \( M \).

**Corollary 5.1** On a measurable space \( \{ \Omega, \mathcal{F} \} \) with filtration \( \mathcal{F}_m \) on it, let
\( \{ f_m, \mathcal{F}_m \}_{m=0}^N \) be a non negative local regular supermartingale relative to a convex set of equivalent measures \( M \). If the set \( A_0 \) is uniformly integrable relative to every measure \( P \in M, \) then the fair price of contingent claim \( f_N \) exists.

**Proof.** From optional decomposition we have \( f_m = M_m - g_m, \ m = 0, N \). Therefore, \( P(f_N - \alpha_0 \zeta_0 \leq 0) = 1 \), where \( \alpha_0 = E^P M_N, \ P \in M, \zeta_0 = \frac{M_N}{E^P M_N} \). From the last it follows that conditions of Theorem 5.2 are satisfied. The Corollary 5.1 is proved.

On a probability space \( \{ \Omega, \mathcal{F}, P \} \) let us consider an evolution of one risk asset given by the law \( \{ S_m \}_{m=0}^N \), where \( S_m \) is a random value taking values in \( R_+^1 \). Suppose
that $\mathcal{F}_m$ is a filtration on $\{\Omega, \mathcal{F}, P\}$. We assume that non risk asset evolve by the law $B_0^m = 1$, $m = 1, N$. Denote by $M^e(S)$ the set of all martingale measures being equivalent to the measure $P$. We assume that the set $M^e(S)$ of such martingale measures is not empty and effective market is non complete, see, for example, [3], [13], [4], [12]. So, we have that

$$E^Q\{S_m|\mathcal{F}_{m-1}\} = S_{m-1}, \quad m = 1, \ldots, N, \quad Q \in M^e(S). \quad (5.13)$$

The next Theorem justify the Definition 5.1.

**Theorem 5.3** Let a contingent claim $f_N$ be a $\mathcal{F}_N$-measurable integrable random value with respect to every measure from $M^e(S)$ and conditions of the Theorem 5.2 are satisfied with $\xi_i = \frac{S_i}{S_0}$, $i = 0, \ldots, N$. Then there exists self-financed trade strategy $\pi$ the capital evolution $\{X^\pi_m\}_{m=0}^N$ of which is a martingale relative to every measure from $M^e(S)$ satisfying conditions $X_0^\pi = f_0$, $X_N^\pi \ge f_N$, where $f_0$ is a fair price of contingent claim $f_N$.

**Proof.** Due to Theorems 5.1, 5.2 for $f_0 = \inf_{\alpha \in G_{a_0}} \alpha$ there exists $\zeta_0 \in A_0$ such that the inequality

$$f_N - f_0 E^P\{\zeta_0|\mathcal{F}_N\} \le 0 \quad (5.14)$$

is valid. Let us put

$$f_m^0 = f_0 E^P\{\zeta_0|\mathcal{F}_m\}, \quad P \in M^e(S),$$

$$f_m = \begin{cases} 0, & m < N, \\ f_N - f_0 E^P\{\zeta_0|\mathcal{F}_m\}, & m = N. \end{cases}$$

It is evident that $f_{m-1} - f_m \ge 0$, $m = 0, \ldots, N$. Therefore, the supermartingale

$$f_m^0 + f_m = \begin{cases} f_0 E^P\{\zeta_0|\mathcal{F}_m\}, & m < N, \\ f_N, & m = N, \end{cases}$$

is a local regular one. It is evident that

$$f_m^0 + f_m = M_m - g_m, \quad m = 0, \ldots, N,$$

where

$$M_m = f_0 E^P\{\zeta_0|\mathcal{F}_m\}, \quad m = 0, \ldots, N,$$

$$g_m = 0, \quad m = 0, \ldots, N - 1,$$

$$g_N = f_0 E^P\{\zeta_0|\mathcal{F}_N\} - f_N.$$

Due to Theorem 6.2 for martingale $\{M_m\}_{m=0}^N$ the representation

$$M_m = f_0 + \sum_{i=1}^m H_i \Delta S_i, \quad m = 0, \ldots, N,$$

is valid. Let us consider a trading strategy $\pi = \{H^0_m, H_m\}_{m=0}^N$, where

$$\bar{H}_0 = f_0, \quad \bar{H}_m = M_m - H_mS_m, \quad m = 1, \ldots, N, \quad \bar{H}_0 = 0, \quad \bar{H}_m = H_m, \quad m = 1, \ldots, N.$$
It is evident that $\tilde{H}_0^m, \tilde{H}_m$ are $\mathcal{F}_{m-1}$ measurable and the trading strategy $\pi$ satisfy self-financed condition
\[ \Delta \tilde{H}_0^m + \Delta \tilde{H}_m S_{m-1} = 0. \]
Moreover, a capital corresponding to the self-financed trading strategy $\pi$ is given by the formula
\[ X_\pi^m = \tilde{H}_0^m + \tilde{H}_m S_m = M_m. \]
Herefrom, $X_\pi^0 = f_0$. Further,
\[ X_\pi^N = f_N + g_N. \]
Therefore $X_\pi^N \geq f_N$. Theorem 5.3 is proved.

In the next theorem we assume that evolutions of risk and non-risk assets generate incomplete market [3], [14], [4], [12], that is, the set of martingale measures contains more than one element.

**Theorem 5.4** Let an evolution $\{S_m\}_{m=0}^N$ of risk asset satisfy conditions $P(D_1^1 \leq S_m \leq D_2^1) = 1$, $D_{m-1}^1 \geq D_m^1 > 0$, $D_{m-1}^2 \leq D_m^2 < \infty$, $m = 1, \ldots, N$, and let non-risk asset evolution be deterministic one given by the law $\{B_m\}_{m=0}^N$, $B_m = 1$, $m = 0, \ldots, N$. The fair price of standard European call option with payment function $f_N = (S_N - K)^+$ is given by the formula
\[
 f_0 = \begin{cases} 
 S_0(1 - \frac{K}{D_N^1}), & K \leq D_N^2, \\
 0, & K > D_N^2. 
\end{cases} 
\]

The fair price of standard European put option with payment function $f_N = (K - S_N)^+$ is given by the formula
\[
 f_0 = \begin{cases} 
 K - D_N^1, & K \geq D_N^1, \\
 0, & K < D_N^1. 
\end{cases} 
\]

**Proof.** In the Theorem 5.4 conditions the set of equations $E^P \zeta = 1, \zeta \geq 0$, has solutions $\zeta_i = \frac{S_i}{S_0}, i = 0, \ldots, N$. It is evident that $\alpha_0 = S_0$ and $\zeta_N = \frac{S_N}{S_0}$, since
\[
 \frac{(S_N - K)^+}{B_N} - \alpha_0 \frac{S_N}{S_0} \leq 0, \quad \omega \in \Omega.
\]
Let us prove the needed formula. Consider the inequality
\[
 (S_N - K) - \alpha \sum_{i=0}^N \gamma_i \frac{S_i}{S_0} \leq 0, \quad \gamma \in V_0,
\]
where $V_0 = \{\gamma = \{\gamma_i\}_{i=0}^N, \gamma_i \geq 0, \sum_{i=0}^N \gamma_i = 1\}$. Or,
\[
 S_N \left(1 - \frac{\alpha \gamma_N}{S_0}\right) - K - \alpha \sum_{i=0}^{N-1} \gamma_i \frac{S_i}{S_0} \leq 0. \tag{5.18}
\]
Suppose that $\alpha$ satisfies inequality
\[
 1 - \frac{\alpha}{S_0} > 0. \tag{5.19}
\]
If $\alpha$ satisfies additionally the equality
\[ D_N^2 \left( 1 - \frac{\alpha \gamma}{S_0} \right) - K - \alpha \sum_{i=0}^{N-1} \gamma_i \frac{D^1_i}{S_0} = 0, \]  
then for all $\omega \in \Omega$ (5.18) is valid. From (5.20) we obtain for $\alpha$
\[ \alpha = \frac{S_0(D^2_N - K)}{(D^2_N \gamma + \sum_{i=0}^{N-1} \gamma_i D^1_i)}. \]  
If $D^2_N - K > 0$, then
\[ \inf_{\gamma \in V_0} \frac{S_0(D^2_N - K)}{(D^2_N \gamma + \sum_{i=0}^{N-1} \gamma_i D^1_i)} = \frac{S_0(D^2_N - K)}{D^2_N}, \]  
since $D^2_N \geq D^1_i$. From here we obtain
\[ f_0 = S_0 \left( 1 - \frac{K}{D^2_N} \right). \]  
It is evident that $\alpha = f_0$ satisfies inequality (5.19).

If $D^2_N - K \leq 0$, then $S_N - K \leq 0$ and from (5.17) we can put $\alpha = 0$. Then, the formula (5.18) is valid for all $\omega \in \Omega$.

Let us prove the formula (5.16) for standard European put option. If $S_N \leq K$ it is evident that $\alpha_0 = K$, and $\zeta_0 = 1$, since
\[ (K - S_N) - \alpha_0 \leq 0, \quad \omega \in \Omega. \]  
Let us prove the needed formula. Consider the inequality
\[ (K - S_N)^+ - \alpha \sum_{i=0}^{N} \gamma_i \frac{S_i}{S_0} \leq 0, \quad \gamma \in V_0. \]  
Or, for $S_N \leq K$
\[ - S_N \left( 1 + \frac{\alpha \gamma}{S_0} \right) + K - \alpha \sum_{i=0}^{N-1} \gamma_i \frac{S_i}{S_0} \leq 0. \]  
If $\alpha$ is a solution of the equality
\[ - D_N^1 \left( 1 + \frac{\alpha \gamma}{S_0} \right) + K - \alpha \sum_{i=0}^{N-1} \gamma_i \frac{D^1_i}{S_0} = 0, \]  
then for all $\omega \in \Omega$ (5.23) is valid. From (5.20) we obtain for $\alpha$
\[ \alpha = \frac{S_0(K - D_N^1)}{\sum_{i=0}^{N} \gamma_i D^1_i}. \]
Therefore,
\[ \inf_{\gamma \in \mathcal{V}_0} \sum_{i=0}^{N} \gamma_i D_i^1 = K - D_N^1, \quad (5.28) \]

since \( D_i^1 \leq S_0, \ i = 1, N, \ D_0^1 = S_0 \). From here we obtain
\[ f_0 = K - D_N^1. \quad (5.29) \]

If \( D_N^1 - K > 0 \), then \( S_N - K > 0 \) and from (5.24) we can put \( \alpha = 0 \). Then, (5.25) is valid for all \( \omega \in \Omega \). The Theorem 5.4 is proved.

6 Some auxiliary results.

On a measurable space \( \{ \Omega, \mathcal{F} \} \) with filtration \( \mathcal{F}_n \) on it, let us consider a convex set of equivalent measures \( M \). Suppose that \( \xi_1, \ldots, \xi_d \) is a set of random values belonging to the set \( A_0 \). Introduce \( d \) martingales relative to a set of measures \( \mathcal{M} \{S_n, \mathcal{F}_n\}_{n=0}^{\infty} \), \( i = 1, \ldots, d \), \( P \in M \). Denote by \( M^e(S) \) a set of all equivalent to a measure \( P \in M \) martingale measures, that is, \( Q \in M^e(S) \) if
\[ E_Q \{S_n|\mathcal{F}_{n-1}\} = S_{n-1}, \quad E_Q|S_n| < \infty, \quad Q \in M^e(S), \quad n = 1, \infty. \]

It is evident that \( M \subseteq M^e(S) \) and \( M^e(S) \) is a convex set. Denote by \( P_0 \) a certain fixed measure from \( M^e(S) \) and let \( L_0^d(\mathbb{R}^d) \) be a set of finite valued random values on a probability space \( \{ \Omega, \mathcal{F}, P_0 \} \), taking values in \( R^d \).

Let \( H_0^d \) be a set of finite valued predictable processes \( H = \{ H_n \}_{n=1}^{N} \), where \( H_n = \{ H_n^s \}_{s=1}^{d} \) takes values in \( R^d \) and \( H_n \) is \( \mathcal{F}_{n-1} \)-measurable. Introduce into consideration a set of random values
\[ K_N^1 = \{ \xi \in L_0^d(\mathbb{R}^1), \ \xi = \sum_{k=1}^{N} \langle H_k, \Delta S_k \rangle, \ H \in H_0^d \}, \quad N < \infty, \quad (6.1) \]

\[ \Delta S_k = S_k - S_{k-1}, \quad \langle H_k, \Delta S_k \rangle = \sum_{s=1}^{d} H_k^s (S_k^s - S_{k-1}^s). \]

Lemma 6.1 The set of random values \( K_N^1 \) is a closed subset in the set of finite valued random values \( L_0^d(\mathbb{R}^1) \) relative to convergence by measure \( P \in M \).

The proof of the Lemma 6.1 see, for example, [3]. Introduce into consideration a subset
\[ V_0 = \{ H \in H_0^d, \ ||H_n|| < \infty, \ n = 1, N \} \]

of the set \( H_0^d \), where \( ||H_n|| = \sup_{\omega \in \Omega} \sum_{i=1}^{d} |H^i_n| \). Let \( K_N \) be a subset of the set \( K_N^1 \)
\[ K_N = \{ \xi \in L_0^d(\mathbb{R}^1), \ \xi = \sum_{k=1}^{N} \langle H_k, \Delta S_k \rangle, \ H \in V_0 \}. \]
Denote also a set
\[ C = \{ k - f, \, k \in K_N, \, f \in L^\infty(\Omega, \mathcal{F}, P_0) \} , \]
where \( L^\infty(\Omega, \mathcal{F}, P_0) \) is a set of bounded nonnegative random values. Let \( \bar{C} \) be a closure of \( C \) in \( L^1(\Omega, \mathcal{F}, P_0) \) metrics.

**Lemma 6.2** If \( \zeta \in \bar{C} \) and such that \( E^{P_0} \zeta = 0 \), then for \( \zeta \) the representation
\[
\zeta = \sum_{k=1}^{N} \langle H_k, \Delta S_k \rangle
\]
is valid for a certain finite valued predictable process \( H = \{ H_n \}_{n=1}^{N} \).

**Proof.** If \( \zeta \in K_N \), then Lemma 6.2 is proved. Suppose that \( \zeta \in \bar{C} \), then there exists a sequence \( k_n - f_n, \, k_n \in K_N, \, f_n \in L^\infty(\Omega, \mathcal{F}, P_0) \) such that \( \| k_n - f_n - \zeta \|_{P_0} \to 0, \, n \to \infty \), where \( \| g \|_{P_0} = E^{P_0} |g| \). Since \( |E^{P_0}(k_n - f_n - \zeta)| \leq \| k_n - f_n - \zeta \|_{P_0} \), we have \( E^{P_0} f_n \leq \| k_n - f_n - \zeta \|_{P_0} \). From here we obtain \( \| k_n - \zeta \|_{P_0} \leq 2 \| k_n - f_n - \zeta \|_{P_0} \). Therefore, \( k_n \to \zeta \) by measure \( P_0 \). On the basis of the Lemma 6.1 a set
\[ K_N^1 = \{ \xi \in L^0(\mathbb{R}^1), \, \xi = \sum_{k=1}^{N} \langle H_k, \Delta S_k \rangle, \, H \in H^0 \}, \]

is a closed subset of \( L^0(\mathbb{R}^1) \) relative to convergence by measure \( P_0 \). From this fact, we obtain the proof of Lemma 6.2 since there exists finite valued predictable process \( H \in H^0 \) such that for \( \zeta \) the representation
\[
\zeta = \sum_{k=1}^{N} \langle H_k, \Delta S_k \rangle
\]
is valid.

**Theorem 6.1** Let \( E^Q |\zeta| < \infty, \, Q \in M^c(S) \). If for every \( Q \in M^c(S), \, E^Q \zeta = 0 \), then there exists finite valued predictable process \( H \) such that for \( \zeta \) the representation
\[
\zeta = \sum_{k=1}^{N} \langle H_k, \Delta S_k \rangle \tag{6.2}
\]
is valid.

**Proof.** If \( \zeta \in C \), then 6.2 follows from Lemma 6.2. So, let \( \zeta \) does not belong to \( \bar{C} \). As in Lemma 6.2 \( \bar{C} \) is a closure of \( C \) in \( L^1(\Omega, \mathcal{F}, P_0) \) metrics for the fixed measure \( P_0 \). The set \( \bar{C} \) is a closed convex set in \( L^1(\Omega, \mathcal{F}, P_0) \). Consider the other convex closed set that consists from one element \( \zeta \). Due to Han–Banach Theorem, there exists a linear continuous functional \( l_1 \), which belongs to \( L^\infty(\Omega, \mathcal{F}, P_0) \), and real numbers \( \alpha > \beta \) such that
\[
l_1(\xi) = \int_\Omega \xi(\omega) g(\omega) dP_0, \quad q(\omega) \in L^\infty(\Omega, \mathcal{F}, P_0), \tag{6.3}
\]
and inequalities $l_1(\xi) > \alpha$, $l_1(\xi) \leq \beta$, $\xi \in \bar{C}$, are valid. Since $\bar{C}$ is a convex cone we can put $\beta = 0$. From condition $l_1(\xi) \leq 0$, $\xi \in \bar{C}$ we have $l_1(\xi) = 0$, $\xi \in K^1_N \cap L^1(\Omega, \mathcal{F}, P_0)$. From (6.3) and inclusions $\bar{C} \supset C \supset -L^\infty(\Omega, \mathcal{F}, P_0)$ we have $q(\omega) \geq 0$. Introduce a measure

$$Q^*(A) = \int_A q(\omega) dP_0 \left(\int_\Omega q(\omega) dP_0 \right)^{-1}.$$

Then, we have

$$\int_\Omega \xi(\omega) dQ^* = 0, \quad \xi \in K^1_N \cap L^1(\Omega, \mathcal{F}, P_0).$$

(6.4)

Let us choose $\xi = \chi_A(\omega)(S^j_i - S^j_{i-1})$, $A \in \mathcal{F}_{i-1}$, where $\chi_A(\omega)$ is an indicator of a set $A$. We obtain

$$\int_A (S^j_i - S^j_{i-1}) dQ^* = 0, \quad A \in \mathcal{F}_{i-1}.$$

So, $Q^*$ is a martingale measure that belongs to the set $M^a(S)$, which is a set of absolutely continuous martingale measures. Let us choose $Q \in M^e(S)$ and consider a measure $Q_1 = (1 - \gamma)Q + \gamma Q^*$, $0 < \gamma < 1$. A measure $Q_1 \in M^e(S)$ and, moreover, $E^{Q_1}\zeta = \gamma E^{Q^*}\zeta > 0$. We come to the contradiction with conditions of Theorem 6.1 since for $Q \in M^e(S)$, $E^Q\zeta = 0$. So, $\zeta \in \bar{C}$, and in accordance with the Lemma 6.2 for $\zeta$ the declared representation in Theorem 6.1 is valid.

**Theorem 6.2** For every martingale $\{M_n, \mathcal{F}_n\}_{n=0}^\infty$ relative to the set of measures $M^e(S)$, there exists a predictable random process $H$ such that for $M_n$, $n = 0, \infty$, the representation

$$M_n = M_0 + \sum_{i=1}^n \langle H_i, \Delta S_i \rangle, \quad n = 1, \infty,$$

(6.5)

is valid.

**Proof.** For fixed natural $N \geq 1$, let us consider random value $M_N - M_0 = \zeta$. Since

$$E^Q|\zeta| < \infty, \quad E^Q\zeta = 0, \quad Q \in M^e(S),$$

then $\zeta$ satisfies conditions of Theorem 6.1 and, therefore, belongs to $\bar{C}$, so, there exists a sequence $k_n = \sum_{i=1}^N \langle H_i^n, \Delta S_i \rangle \in K_N$ such that

$$\int_\Omega |k_n - \zeta| dP_0 \to 0, \quad n \to \infty.$$

From here, we obtain

$$\int_\Omega |E^{P_0}\{(k_n - \zeta)|\mathcal{F}_n\}| dP_0 \leq \int_\Omega |k_n - \zeta| dP_0 \to 0, \quad n \to \infty.$$
But $E^{P_0}\{k_n|\mathcal{F}_m\} = \sum_{i=1}^{m} \langle H^0_i, \Delta S_i \rangle$. Hence, we obtain that as $\sum_{i=1}^{m} \langle H^0_i, \Delta S_i \rangle$ and $\sum_{i=1}^{N} \langle H^0_i, \Delta S_i \rangle$ converges by measure $P_0$ to $E^{P_0}\{\zeta|\mathcal{F}_m\}$ and $\zeta$, correspondingly. There exists a subsequence $n_k$ such that $H^{n_k}$ converges everywhere to predictable process $H$. From here, we have $\zeta = \sum_{i=1}^{N} \langle H^0_i, \Delta S_i \rangle$ and $E^{P_0}\{\zeta|\mathcal{F}_m\} = \sum_{i=1}^{m} \langle H^0_i, \Delta S_i \rangle$. It proves that for all $m < N$

$$M_m = M_0 + \sum_{i=1}^{m} \langle H^0_i, \Delta S_i \rangle.$$ 

Theorem 6.2 is proved.

7 Conclusions.

In the paper, we generalize Doob decomposition for supermartingales relative to one measure onto the case of supermartingales relative to a convex set of equivalent measures. For supermartingales relative to one measure for continuous time Doob’s result was generalized in papers [18] [19].

Section 2 contains the auxiliary statements giving sufficient conditions of the existence of maximal element in a maximal chain, of the existence of nonzero non-decreasing process such that the sum of a supermartingale and this process is again a supermartingale relative to a convex set of equivalent measures needed for the main Theorems. In Theorem 2.2 we give sufficient conditions of the existence of the optional Doob decomposition for the special case as the set of measures is generated by finite set of equivalent measures with bounded as below and above the Radon - Nicodym derivatives. After that, we introduce the notion of a regular supermartingale. Theorem 2.3 describes regular supermartingales. In Theorem 2.4 we give the necessary and sufficient conditions of regularity of supermartingales. After that we introduce a notion of local regular supermartingale. At last, we prove Theorem 2.6 asserting that if the optional decomposition for a supermartingale is valid, then it is local regular one. Essentially, Theorem 2.6 and 2.7 give the necessary and sufficient conditions of local regularity of supermartingale.

In section 3 we prove auxiliary statements needed for the description of local regular supermartingales. The notion of a local regular supermartingale relative to a convex set of equivalent measures is equivalent to the existence of non negative adapted process such that the equalities (2.71) are valid. Since the existence of optional decomposition for supermartingale and existence of adapted non negative process entering (2.71) are equivalent ones, then it would seem to obtain new information from the set of equation (2.71) is impossible. As it was found, this new formulation are proved to be fruitful, since it turned out to describe the structure of all local regular supermartingales relative to a convex set of equivalent measures. For this purpose we investigate the structure of supermartingales of special types relative to a convex set of equivalent measures, generated by a certain finite set of equivalent measures. The main result of this investigation is the Lemma 3.7 which allowed us to prove Lemma 3.8 stating sufficient conditions of existence of a martingale on a measurable space with respect to a convex set of equivalent measures generated by finite set of equivalent measures. The existence of non trivial random value satisfying conditions (3.20) is sufficient condition for the existence of non trivial martingale with respect to a convex set of equivalent measures, generated by finite set of equivalent measures. Theorem 3.1 describes all local regular non negative supermartingales of special type (3.21) relative to constructed above set of equivalent measures.
In the Theorem 3.2 we give sufficient conditions of the existence of local regular martingale relative to arbitrary set of equivalent measures and arbitrary filtration. If time interval is finite these conditions are also necessary. After that, we present in Theorem 3.3 important construction of local regular supermartingales which we sum up in Corollary 3.2. Theorem 3.4 proves that every non negative uniformly integrable supermartingale belongs to described class (3.38) of local regular supermartingales.

Section 4 contains the Theorem 4.1 giving a variant of the necessary and sufficient conditions of local regularity of non negative supermartingale relative to a convex set of equivalent measures. In subsection 1 the Definition 4.1 determine a class of complete set of equivalent measures. The Lemma 4.1 guarantee a bound (4.12) for all non negative random values allowing us to prove the Theorem 4.2 stating that for every non negative supermartingale optional decomposition is valid. In subsection 2 we extend the results of subsection 1 onto the case as a space of elementary events is countable. At last, subsection 3 contains the generalization of the result obtained in subsection 2 onto the case of arbitrary space of elementary events. We prove that for every non negative supermartingale optional decomposition is valid.

Corollary 3.3 of the Section 5 contains important construction of the local regular supermartingales playing important role in definition of fair price of contingent claim relative to a convex set of equivalent measures. The Definition 5.1 is fundamental for evaluation of risk in incomplete markets. Theorem 5.1 gives sufficient conditions of the existence of fair price of contingent claim relative to a convex set of equivalent measures. It also gives sufficient conditions when defined fair price coincides with classical value. In the Theorem 5.2 simple conditions of the existence of fair price of contingent claim are given. In Theorem 5.3 we prove the existence of self-financed trading strategy confirming a Definition 5.1 of fair price as parity between long and short positions in contracts. As application of the result obtained we prove Theorem 5.4 where the formulas for standard European call and put options in incomplete market we present. Section 6 contains auxiliary results needed for previous sections.

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