Non-Gaussian generalization of the Kazantsev-Kraichnan model for turbulent dynamo

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ABSTRACT

We consider a natural generalization of the Kazantsev-Kraichnan model for small-scale turbulent dynamo. This generalization takes account of statistical time asymmetry of a turbulent flow, and, thus, allows to describe velocity fields with energy cascade. For three-dimensional velocity field, generalized Kazantsev equation is derived, and evolution of the second order magnetic field correlator is investigated for large but finite magnetic Prandtl numbers. It is shown that as \( Pr_m \to \infty \), the growth increment tends to the limit known from the T-exponential (Lagrangian deformation) method. Magnetic field generation is shown to be weaker than that in the Gaussian velocity field for any direction of the energy cascade, and depends essentially on the Prandtl number.

Keywords: dynamo — magnetohydrodynamics — turbulence — methods: analytical — ISM: magnetic fields

1. INTRODUCTION

Magnetic field generation in turbulent plasma is one of the most probable mechanisms responsible for stellar, interstellar and intergalactic magnetism (see, e.g., Zeldovich et al. 1984a; Moffatt 1978; Parker 1979; Brandenburg & Subramanian 2005; Schober et al. 2018). Small-scale turbulent dynamo has been the object of interest of many researchers (see, e.g., Falkovich et al. 2001; Brandenburg et al. 2012; Alexakis & Biferale 2018) as it can provide intensive increase of magnetic field. In these problems, characteristic scale of magnetic field fluctuations is much smaller than the scale at which turbulence is generated: this corresponds to inertial and viscous scale ranges of turbulence.

The conception of small (seed) initial magnetic field fluctuations implies that there exists an important stage of kinematic dynamo: magnetic field is small enough to cause no feedback on the velocity distribution, so it is passively advected by the turbulent flow. Magnetic Prandtl number, i.e. ratio of the kinematic viscosity \( \nu \) to the magnetic diffusivity \( \kappa \), is the most important characteristic of this advection process. In the paper we consider large Prandtl numbers:

\[
Pr_m = \frac{\nu}{\kappa} \gg 1.
\]

Such situation is observed, e.g., in interstellar medium (Brandenburg & Subramanian 2005; Rincon 2019). This means that the magnetic diffusive scale length \( r_d \) is much smaller than the Kolmogorov viscous scale \( r_\nu \). We assume the characteristic scale length \( l \) of initial magnetic field fluctuation to lie between these two scales,

\[
r_d \ll l \ll r_\nu.
\]

Evolution of magnetic field is described by a stochastic partial differential equation with random velocity field acting as multiplicative noise. The velocity statistics is assumed to be stationary and known. The problem is to find the statistics of the magnetic field, in particular, its correlations.

Kazantsev-Kraichnan model (Kazantsev 1968; Kraichnan & Nagarajan 1967) is the simplest and natural approximation for the velocity statistics: the velocity field is assumed to be Gaussian and \( \delta \)-correlated in time.
In this model, all magnetic field correlators are governed by the only two-point velocity correlator.

This model is an essential simplification. Actually, unlike the additive random processes, in stochastic equations with multiplicative noise the cumulants of all orders give comparable contributions to any statistical moment. So, the Central limit theorem 'does not work' for these processes, and, to calculate even the second-order correlators of magnetic field, one should use the Large deviation principle and take all velocity correlators into account. So, the replacement of arbitrary random velocity field by Gaussian process can change the result crucially.

Besides, in Gaussian approximation for velocity field and hence, in Kazantsev-Kraichnan model, there is no energy cascade. The energy of magnetic field excitation comes from the energy of the turbulent flow, which is generated at large scales; thus, the cascade may be important for dynamo. The non-zero third-order velocity correlator is responsible for energy cascade and for time asymmetry in general (Kolmogorov 1941; Frisch 1995): indeed, the inversion of time would result in the change of sign of all velocities, and time symmetry implies that the statistics would not change; hence, the third order correlator is zero for time symmetric flows. Its presence indicates time asymmetry. So, the account of non-Gaussianity is highly desirable.

There are two different theoretical approaches to investigate the magnetic field statistics. One of them is based on the Lagrangian deformations statistics (see Zel’dovich et al. 1984b; Chertkov et al. 1999; Il’yn et al. 2018): it implies direct solution of the magnetic field evolution equation by means of the T-exponential formalism. The physical meaning of this method can be formulated in terms of independent magnetic blobs, each of them undergoing its evolution in the turbulent flow (Moffat & Saffman 1964; Kolokolov 2017; Il’yn et al. 2019, 2021). This approach allows to calculate the magnetic field correlators of all orders, and to consider inhomogenous, in particular, localized initial magnetic field distributions. In this frame, it is possible to deal with arbitrary (not necessarily Gaussian) velocity statistics. So, this approach allows to consider velocity statistics wider than the Kazantsev-Kraichnan model.

However, this approach is restricted to so-called Batchelor regime (Batchelor 1959): the characteristic scale of the magnetic field must lie deep inside the viscous range of turbulence, so that the velocity field can be approximated by a linear function. This means that the solutions found by this method are definitely applicable for some finite time range \( t \propto \ln r_0/l \). Later on, the characteristic scale of the magnetic field continues to increase and reaches the inertial range of turbulence. The 'Lagrangian deformation' approach may fail to predict the behavior of correlators at this stage. The details of applicability of the method to the inertial stage are considered by Il’yn et al. (2021).

The other approach is based on statistical properties of pair correlators and allows to derive a closed differential equation for the pair correlator of magnetic field (Kazantsev equation). It was used and developed in many papers (see, e.g., Kazantsev 1968; Kraichnan 1968; Vainshtein & Kichatinov 1986; Kolokolov 2017; Seshasayanan & Alexakis 2016; Schekochihin et al. 2002a; Malyshev & Boldyrev 2007; Istomin & Kiselev 2013); hereafter we will refer to it as to Kazantsev approach. The advantage of the method is its applicability to any stage of the magnetic field evolution. However, it is restricted to statistically homogenous magnetic field configurations, and it allows to calculate the second-order two-point correlator only. There is one more vice of this approach: it requires Gaussian and \( \delta \)-correlated in time velocity statistics, so it is restricted to the Kazantsev-Kraichnan model of velocity field. Only for several models with some special additional conditions its applicability has been enlarged (Schekochihin & Kulsrud 2001; Bhat & Subramanian 2014; Kleerorin et al. 2002).

The two approaches produce concordant results wherever their domains of applicability overlap (Chertkov et al. 1999; Il’yn et al. 2021), they are also verified by numerical simulations (Mason et al. 2011; Schekochihin et al. 2004; Seta et al. 2020). However, there remains the domain where neither of them can be applied: processes with non-Gaussian and/or not \( \delta \)-correlated velocity statistics cannot be analyzed at late (inertial) stage of their evolution neither by the Lagrangian deformations approach nor by the classical Kazantsev method. The finite correlation time was taken into account in Bhat & Subramanian (2014); Mason et al. (2011); Kleerorin et al. (2002); Schekochihin & Kulsrud (2001) for some special types of flows. The non-Gaussian velocity statistics in combination with the inertial stage has not been considered yet.

To fill this gap, in this paper we consider the simplest non-Gaussian generalization of the Kazantsev-Kraichnan model introduced in Il’yn et al. (2016, 2019). It implies a non-zero third-order velocity correlator, and thus, takes into account the time asymmetry of the flow. This model allows to investigate the long-time evolution of statistics for advection (and, more generally, multiplicative) equations for arbitrary velocity field with small but non-zero third order correlator. We gen-
eralize the Kazantsev method to apply it to this \( V^3 \) model, and find the two-point pair magnetic field correlator. We show that inside the Batchelor regime, the results obtained for the \( V^3 \)-model by means of the Lagrangian deformations approach and by means of the generalized Kazantsev method coincide, which verifies the new generalized method. Now, for later (inertial) stage, we show that the \( V^3 \)-model is stable relative to the limit of 'zero non-Gaussianity' as it turns into the Kazantsev-Kraichnan model. We calculate the magnetic field growth increment for finite time asymmetry, and evaluate the correction produced by finite Prandtl number.

It appears that small time asymmetry decreases the magnetic field generation, independently of the direction of the energy cascade. The range of Prandtl numbers that produce effective generation is also narrower as non-Gaussian time-asymmetry increases. The \( V^3 \) model is shown to be a useful and effective instrument to investigate magnetic field advection in finite-Prandtl model is stable relative to the applicability of \( \text{Pr} \) numbers, validation of the method and check of the increment of the pair magnetic field correlator. The appears to be possible to solve the equation and to find the equation in order to apply the Kazantsev approach to this equation III). In Section IV we derive the modified Kazantsev equation in order to apply the Kazantsev model to apply it to this \( V^3 \) model.

The paper is organized as follows. In the next section, we briefly review the basic ideas and equations of the two approaches by the example of the Kazantsev-Kraichnan model. Then we recall the formulation and restrictions of the \( V^3 \) model and the results obtained for this model by means of the Lagrangian deformations method (Section III). In Section IV we derive the modified Kazantsev equation in order to apply the Kazantsev approach to the \( V^3 \) model. In the limit of infinite Prandtl number, it appears to be possible to solve the equation and to find the increment of the pair magnetic field correlator. The results of numerical solution of the equation for finite Prandtl numbers, validation of the method and check of its stability is performed in Section V. In Discussion we analyze the obtained results, and pay special attention to the applicability of \( \delta \)-correlated in time velocity distribution and compare the with the models with finite correlation time.

2. GAUSSIAN VELOCITY FIELD: RECALL OF CLASSIC RESULTS

To introduce the notations and equations needed, we start from the classical problem statement. Kinematic transport of magnetic field \( \mathbf{B}(t, \mathbf{r}) \) advected by random statistically homogenous and isotropic nondivergent velocity field \( \mathbf{v}(t, \mathbf{r}) \), \( \nabla \cdot \mathbf{v} = 0 \), is described by the evolution equation

\[
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \nabla) \mathbf{B} - (\mathbf{B} \nabla) \mathbf{v} = \kappa \Delta \mathbf{B},
\]

where \( \kappa \) is the diffusivity. The random process \( \mathbf{v}(t, \mathbf{r}) \) is assumed to be stationary, and to have given statistical properties. The initial conditions for magnetic field are also stochastically isotropic and homogenous. The aim is to find statistical characteristics of the process \( \mathbf{B} \), in particular, its pair correlator.

From statistical homogeneity and isotropy, and non-divergency of \( \mathbf{B} \) it follows that its simultaneous pair correlator has the form

\[
\langle B_i(\mathbf{r}, t)B_j(\mathbf{r} + \mathbf{r}, t) \rangle = G(r, t)\delta_{ij} + \frac{1}{2} r G'(r, t)(\delta_{ij} - r_i r_j/r^2). \tag{2}
\]

The average here is taken over the initial conditions \( \mathbf{B}(\mathbf{r}, 0) \) and over the possible realizations of the velocity field \( \mathbf{v}(\mathbf{r}, t) \).

So, we are interested in time dependence of \( G \): if it increases exponentially,

\[
G \sim e^{\gamma t} G(r), \tag{3}
\]

one calls the process 'turbulent dynamo', and \( \gamma \) is called the magnetic field increment.

The Kazantsev-Kraichnan model implies that the velocity field is Gaussian, \( \delta \)-correlated in time; its statistics is completely determined by the second-order correlator

\[
\langle v_i(\mathbf{r}, t)v_j(\mathbf{r} + \mathbf{r}, t + \tau) \rangle = D_{ij}(\mathbf{r})\delta(\tau). \tag{4}
\]

The correlator does not depend on \( \mathbf{R} \) and \( t \) because of homogeneity and stationarity. To make contact with finite correlation time real flows, one can define \( D_{ij} \) by

\[
D_{ij}(\mathbf{r}) = \int \langle v_i(\mathbf{r}, t)v_j(\mathbf{r} + \mathbf{r}, t + \tau) \rangle d\tau. \tag{5}
\]

Just as in (2), non-divergency and isotropy oblige the tensor \( D_{ij} \) to be determined by only one scalar function of a scalar argument. For the purposes of next subsection, it is convenient to consider the (time-integrated) longitudinal structure function

\[
\sigma(r) = \frac{1}{2} \int d\tau \left\langle \left( (\mathbf{v}(\mathbf{r}, \tau) - \mathbf{v}(0, \tau)) \cdot \mathbf{r} \right)/(r^2) \right\rangle - \left( (\mathbf{v}(\mathbf{r}, 0) - \mathbf{v}(0, 0)) \cdot \mathbf{r} \right)/(r^2). \tag{6}
\]

Then

\[
\sigma(r) = \frac{1}{2} \left( D_{ij}(0)\frac{r_i r_j}{r^2} - D_{ij}(r)\frac{r_i r_j}{r^2} \right). \tag{7}
\]

If one presents \( D_{ij} \) in the form analogous to (2),

\[
D_{ij} = P(r)\delta_{ij} + \frac{1}{2} r P'(r)(\delta_{ij} - r_i r_j/r^2),
\]

then

\[
\sigma(r) = \frac{1}{2} (P(0) - P(r)).
\]

In presence of viscosity, the velocity field is smooth at the smallest scales, and

\[
P(r) = r \to 0 P(0) - \frac{2}{3} D r^2 + O(r^3), \quad D = -\frac{3}{4} P''(0), \tag{8}
\]
2.1. Kazantsev Equation

The equation to describe the evolution of the second-order correlator (2) can be found from Eq.(1) by means of multiplying and subsequent averaging. The cross-correlations of magnetic field and velocity can be split by means of the Furutsu-Novikov theorem due to Gaussianity and delta-correlation (Furutsu 1963; Novikov 1965):

$$\langle v_i(r, t)g[v]\rangle = \frac{1}{2} \int dr' D_{ij}(r - r') \left\langle \frac{\delta g[v]}{\delta v_j(r', t)} \right\rangle,$$  \hspace{1cm} (9)

where $g[v]$ is an arbitrary analytic functional of $v(r, t)$ and $\delta/\delta v$ is a functional derivative. Thus, for $G(r, t)$ one gets the equation

$$\frac{\partial G(r, t)}{\partial t} = L_{Gauss}G,$$  \hspace{1cm} (10)

$$L_{Gauss} = 2\sigma(r) \frac{\partial^2}{\partial r^2} + 2(\sigma' + \frac{4}{r}\sigma) \frac{\partial}{\partial r} + 2(\sigma'' + \frac{4}{r}\sigma'') + 2\kappa \left( \frac{\partial^2}{\partial r^2} + \frac{4}{r} \frac{\partial}{\partial r} \right).$$

In a turbulent hydrodynamic flow, $\sigma$ has the following asymptotics:

$$\sigma(r) =\begin{cases} \frac{2}{3} r^2, & r \ll r_v, \\ \text{const} \cdot r^2, & r_v \ll r \ll L, \\ \frac{2}{9} P(0), & r \gg L, \end{cases}$$  \hspace{1cm} (11)

where $r_v$ is the viscous dissipation scale, and $L$ is the integral scale of turbulence. The time scale

$$D^{-1} = \frac{2r_v}{v_v}$$

is of the order of the eddy turnover time at the viscous scale (see Brandenburg & Subramanian 2005). The first asymptote in (11) corresponds to the viscous range of scales, the second presents the inertial range, and the last string is for the integral range of turbulence (see Landau & Lifshitz 1987).

The well-known result (Kleeorin et al. 2002; Schekochihin et al. 2002b) is that for large Prandtl numbers, independently of the parameter $\xi$ (characterizing the inertial range), the equation (10) has a growing mode with the increment

$$\gamma = \frac{5}{2} D - O(\ln^{-2} \frac{r_d}{r_v}),$$

where

$$r_d = \sqrt{\kappa/\Omega} \propto r_v/\sqrt{Pr_m}.$$  \hspace{1cm}

Thus, turbulent dynamo exists at large Prandtl numbers, and the increment is determined by the viscous range of turbulence.

2.2. ‘Lagrangian Deformations’ Approach

This alternative way can only be applied for scales deep inside the viscous range, which corresponds to the early stage of evolution of initially small-scale ($l \ll r_v$) magnetic fluctuation (Chertkov et al. 1999). For scales much smaller than $r_v$, the velocity field is smooth. So, one chooses a co-moving quasi-Lagrangian reference frame (Belinicher & Lvov 1987) associated to some fluid particle $r_0(t)$, and in this frame expands the (relative) velocity into a series up to the first order:

$$\delta v_i(r, t) = A_{ij}(t)r_j.$$  \hspace{1cm}

This is called Batchelor approximation (Batchelor 1959), and $A_{ij}$ is the velocity gradient tensor:

$$A_{ij} = \partial_j v_i(r_0(t), t).$$

The transport equation (1) now takes the form:

$$\partial_t B_i + A_{jk} r_k \partial_j B_i - A_{ij} B_j = \kappa \Delta B_i$$  \hspace{1cm} (12)

In Kazantsev-Kraichnan model, the statistics of $A$ is determined by its second-order correlator:

$$\langle A_{ij}(t) A_{kp}(t') \rangle = D_{ijkp} \delta(t - t'),$$

where, in accordance with (4),

$$D_{ijkp} = -\partial_j \partial_p D_{ik}(0).$$

The equation (12) can be solved explicitly by means of the Fourier transform (Zel’dovich et al. 1984b); for brevity, here we restrict ourselves to one spatial point and hence, to one-point correlator:

$$B_m(t) = Q_{mn}^{-1} \int B_n(p, 0)e^{-i\mathbf{p}\cdot \mathbf{r}} f(QQ^T)_{ij}(t')dt'dp.$$  \hspace{1cm} (13)

Here $Q(t)$ is the evolution matrix defined by the equation

$$\frac{dQ}{dt} = -QA, \quad Q(0) = 1.$$  \hspace{1cm} (14)

It is convenient to use the polar decomposition for the evolution matrix:

$$Q = sdR, \quad s, R \in SO(3), \quad d = \text{diag} \{e^{-\zeta_i t}\},$$

From incompressibility it follows $\det Q = 1$, hence, $\zeta_1 + \zeta_2 + \zeta_3 = 0$.

It is well known (Furstenberg 1963; Letchikov 1996) that the long-time asymptotic behavior of these three components is quite different: as $Q$ obeys Eq. (14), $s(t)$ stabilizes at some random value that depends on the
realization of the process; $\zeta(t)$ are asymptotically stationary random processes and tend (with unitary probability) to the limits $\lambda_i$,

$$\lambda_1 \geq \lambda_2 \geq \lambda_3,$$

the set of $\lambda_i$ is called the Lyapunov spectrum (Oseledets 1968); and $R(t)$ rotates randomly. We note that since $QQ^T = sd^2s^T$ and $(Q^{-1})^TQ^{-1} = sd^{-2}s^T$, the matrix $R$ vanishes in the expression for $B^2(t)$. This simplifies the calculation of the statistical moments. The Kazantsev-Kraichnan model provides one more significant simplification: in particular, it corresponds to time-reversible flow, which means $\lambda_3 = 0$.

Without loss of generality, the initial conditions for $B(p, 0)$ can be chosen Gaussian, with pair correlator

$$\langle B_n(0, p)B_m(0, p') \rangle = \delta(p + p')e^{-p^2t^2} \langle \gamma^2 \delta_{mn} - p_mp_n \rangle,$$

(15)

Raising (13) to the square and taking average over the initial conditions, one can obtain (Chertkov et al. 1999; Zel’dovich et al. 1984b)

$$\langle B^2 \rangle_{i.c.} \sim \left\{ \frac{d_2}{d_1}, \begin{array}{l} \ln d_2 > 0, \\
\ln d_2 < 0, \end{array} \right.$$

To average this expression over all possible realizations of $A$, one considers the probability density of $\zeta$:

$$P(y, t) = \prod_{j=1,3} \delta(\zeta_j(t) - y_j),$$

(16)

$$\langle B^2 \rangle_{i.c.,v} = \int P(\zeta, t)\langle B^2 \rangle_{i.c.}d\zeta_1 \ldots d\zeta_3.$$n

The incompressibility condition leaves only two independent variables (e.g., $\zeta_1, \zeta_2$). The probability density of $\zeta_j$ for any (not necessarily Gaussian) $A(t)$ can be expressed in terms of statistics of the process $A(t)$ (Il’yn et al. 2016, 2019).

Eventually, for the Kazantsev-Kraichnan model one gets (Chertkov et al. 1999)

$$\lim_{t \to \infty} \frac{1}{t} \ln \langle B^2 \rangle = \frac{5}{2}D,$$

(17)

$$\lim_{t \to \infty} \frac{1}{t} \ln \langle B^{2n} \rangle = \left( 2n + \frac{n^2}{2} \right) D.$$n

We see that (17) coincides with the increment obtained in the Kazantsev approach. So, it appears that the asymptote found in Batchelor approximation remains to be valid not only during the initial stage, $t < \frac{1}{\gamma} \ln(r_p/l)$, but also at later stages of evolution. This fact is non-trivial, since, e.g., in two-dimensional flows the Kazantsev equations shows no growing modes, and the exponential increase of magnetic field at the initial Batchelor stage changes to decrease at larger time (Schekochihin et al. 2002a; Kolokolov 2017).

3. V$^3$ MODEL IN THE METHOD OF LAGRANGIAN DEFORMATIONS

To generalize the Kazantsev-Kraichnan model, one has to add higher-order connected correlators; in particular, to take time asymmetric processes into account, one has to deal with third-order correlators. The isotropy and incompressibility conditions reduce the degrees of freedom of the whole tensor $\langle A_{ij}A_{km}A_{nl} \rangle$ to one arbitrary multiplier $F$. The general expression for all the components is given by Pumir (2017); here we restrict our consideration to the correlators of the diagonal elements $A_{jj}$ (no summation), since these components are the only ones needed for calculation of $\langle B^2 \rangle$ (Il’yn et al. 2016, 2019):

$$\langle A_{pp}(t)A_{qq}(t')A_{rr}(t'') \rangle = F f_{ppr}\delta(t - t')\delta(t - t''),$$

(19)

$$f_{111} = f_{222} = f_{333} = f_{123} = -\frac{4}{3},$$

$$f_{112} = f_{113} = f_{221} = \ldots = \frac{2}{3}.$$n

We note that here $f_{ppr}$ is not a tensor. The right-hand side of (19) is written in the form corresponding to an effective $\delta$-process (Il’yn et al. 2016). The validity of this approximation is verified by the possibility to reduce any finite-correlation time non-Gaussian process to some delta-correlated process, see Appendix A.

In the frame of the $V^3$ model, we set all the higher order connected correlators zero. A vice of this simplification is that the probability density is negative in some range of its argument (Monin & Yaglom 1987; Rytov et al. 1978) as only the second and the third connected correlators are unequal to zero. This artefact can be fixed in the case of small $F$ by addition negligibly small but non-zero higher-order correlators. These higher-order corrections would not affect the magnetic field increment.

So, the time asymmetry of the velocity field in Batchelor regime is governed by only one parameter $F$. The coefficient $F$ is the index of asymmetry of the flow. The direction of cascade observed in real three-dimensional hydrodynamic flows corresponds to $F > 0$ (Il’yn & Zybin 2015). The numerical simulation (Girimaji & Pope 1990) and the experiment (Luthi et al. 2005) give an estimate of the relation between the Lyapunov exponents $\lambda_2/\lambda_1 \approx 0.14$. The Lyapunov exponents are related to the parameters $F$ and $D$ (Il’yn et al. 2016; Balkovsky & Fouxon 1999) by $\lambda_1/\lambda_2 = (2D - F)/(2F)$. So, this result for the Lyapunov spectrum corresponds to

$$F/D \approx 0.13$$

(20)
We see that the average coincides with that for Kazantsev-Kraichnan model if $F = 0$. Analogous calculations for the higher order increments lead to

$$\lim_{t \to \infty} \frac{1}{t} \ln \langle B^2(t) \rangle \simeq \left( 2n + \frac{n^2}{2} \right) D - \frac{3}{32} (2 + n) D^2$$

The dependence $\gamma(F)$ for the exact and approximated equations (21) and (22) is presented in Figure 1. One can see that the approximation (22) works well for $F/D \lesssim 0.1$.

The expression (22) generalizes (17) for time asymmetric flows, but it is still derived for linear velocity field, and thus, is only valid at Batchelor stage of magnetic field evolution. To investigate the dynamo generation at longer time, one should apply the Kazantsev approach to the $V^3$ model.

\section*{4. V$^3$ Model in the Generalized Kazantsev Theory}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{$\gamma$ as a function of $F$, all values normalized by $D$. The solid line corresponds to (21), and the dashed line represents (22).}
\end{figure}

In the Kazantsev-Kraichnan model there is only one nontrivial velocity correlator:

$$\langle v_i(r, t)v_j(r', t') \rangle = \delta_{ij}(r - r') \delta(t - t'). \quad (23)$$

In the frame of the $V^3$ model ideology, we add the third order correlator: 2

$$\langle v_i(r, t)v_j(r', t')v_k(r'', t'') \rangle = F_{ijk}(r - r', r - r'') \delta(t - t') \delta(t - t''). \quad (24)$$

The correspondence with velocity gradients statistics (8),(19) requires

\begin{align}
D &= -\frac{3}{4} \frac{d^2}{dt^2} \left( \frac{1}{r^2} r^i D_{ij} \right) \bigg|_{r = 0}, \quad (25) \\
F &= -\frac{3}{4} \frac{\partial^3}{\partial r_1 \partial r_1' \partial r_1''} F_{111}(0, 0). \quad (26)
\end{align}

Again, just as in the case of velocity gradients, the requirement of isotropy and incompressibility of the flow leaves only one free parameter for the tensor $F_{ijk}$: it is the multiplier $F$ that plays the role.

To apply the Kazantsev method to the non-Gaussian velocity field, one has to use the non-Gaussian version of the Furutsu-Novikov relation (Klyatskin 2005; Rytov et al. 1978) to take the nonzero third order correlator into account:

$$\langle v_i(r, t)g[v] \rangle = \frac{1}{4} \int d r' D_{ij}(r - r') \langle \frac{\delta}{\delta v_j(r', t')} g[v] \rangle + \frac{1}{8} \int d r' \int d r'' F_{ijk}(r - r', r - r'') \langle \frac{\delta}{\delta v_j(r', t')} \frac{\delta}{\delta v_k(r'', t'')} g[v] \rangle. \quad (27)$$

In this equation the time integral is already calculated; see the details in Appendix B.

Taking average of the square of Eq.(1), we then arrive to the modification of the Kazantsev equation (10)

$$\frac{\partial G(r, t)}{\partial t} = (L_{\text{Gauss}} + \delta L) G(r, t). \quad (28)$$

The expression for $\delta L$ is very cumbersome. Shorter expressions for important particular choices of $F_{ijk}(r)$ will be presented in the next Subsections.

We are interested in the long-time asymptotics of the magnetic field correlator, so we seek for the solutions

$$G(r, t) = e^{\gamma t} G(r),$$

1 In fact, this expression is not accurate: to calculate the averages to get the generalized Kazantsev equation, one needs to ‘regularize’ the $\delta$-functions and take the limit of zero correlation time in the end of the calculation. So, in more accurate writing, the arguments of the $\delta$-functions must be symmetricized. See Appendix B for more details.

2 There is a misprint in the last formula of Section 10 in Il’yn et al. (2019); here we give the corrected expression.
which transform (28) into the ordinary differential equation

$$\gamma G(r) = (L_{\text{Gauss}} + \delta L)G(r).$$  \hspace{1cm} (29)

The important feature is that Eqs.(28),(29) are third order differential equations with a small multiplier at the elder derivative. This results in appearance of a non-physical solution that does not coincide with the solutions of (10) as $F \to 0$; instead, it goes to infinity. This solution must not be taken into consideration (see also more details in Appendix C). Technically, this non-physical solution is a consequence of the truncated sequence of correlators; if one adds higher order correlators of $A$ into consideration, the number of solutions of the Kazantsev equation would increase in accordance with the order of the highest correlator. The non-physical solutions would grow unrestrictedly as the magnitudes of the higher order correlators tend to zero. These solutions have to be excluded by accurate choice of the boundary conditions.

4.1. Batchelor Regime

In the Batchelor approximation, velocity gradients are assumed to be constant in space (although dependent on time). Accordingly, the second derivatives of $D_{ij}$ and the third derivatives of $F_{ijk}$ are assumed to be constant all over the liquid volume. Then, in accordance with (7),(8),

$$\sigma(r) = \frac{D}{3}r^2.$$  \hspace{1cm} (30)

The exact expression for $F_{ijk}(r-r', r-r'')$ in the case is presented in Appendix B. Substituting this in (27), we get

$$L_{\text{Gauss}}^B = \frac{2}{3}D \left( r^2 \frac{\partial^2}{\partial r^2} + 6r \frac{\partial}{\partial r} + 10 \right) + 2\kappa \left( \frac{\partial^2}{\partial r^2} + \frac{4}{r} \frac{\partial}{\partial r} \right),$$  \hspace{1cm} (31)

and the additional term in (28),(29) (see details of the derivation in Appendix B):

$$\delta L^B = \frac{1}{9}F \left( 2r^3 \frac{\partial^3}{\partial r^3} + 21r^2 \frac{\partial^2}{\partial r^2} + 14r \frac{\partial}{\partial r} + 70 \right).$$  \hspace{1cm} (32)

The analytic analysis of Eq.(28),(30),(32) is performed in Appendix C. We show that the fastest-growing mode corresponds to

$$\gamma = \frac{5}{2}D \left( 1 - \frac{243}{80} \left( \frac{F}{D} \right)^2 + o \left( \frac{F}{D} \right)^2 \right).$$  \hspace{1cm} (33)

This coincides with the exponents (22) found in the previous Section. The important consequence from this expression is that, independently of the sign of $F$, the resulting $\gamma$ is smaller than that found in the Kazantsev-Kraichnan model. This means that the magnetic field generation is weaker in time-asymmetric flows than in the flow with Gaussian velocity gradients, independently of the direction of the energy cascade. Also, $\gamma$ is monotonic function of $F^2$; the time asymmetry of the flow decreases the generation.

The coincidence of the results obtained by means of the Lagrangian deformation method and of the modified Kazantsev approach is not trivial and not evident, and we will consider it in more details in Discussion. Anyway, it proves that both methods work well as long as Batchelor approximation is valid. However, the Kazantsev approach allows to investigate later stages of magnetic field evolution, when characteristic scales of magnetic lines lengths exceed the viscous scale, and spatial inhomogeneity of velocity gradients cannot be neglected.

4.2. Nonlinear Velocity Field

To take this inhomogeneity into account, one should consider scales comparable or larger than the viscous scale: outside this scale the correlators change essentially. In (11) this is expressed by means of three different ranges. We also have to cut off the third-order correlator. Since it is known (Novikov et al. 1983; Kulsrud & Anderson 1992) that (in the case of large $Pr_m$) the details of the outer ranges do not affect the result significantly, we simplify (11) to

$$\sigma(r) = \begin{cases} \frac{D}{3}r^2, & r \leq r_\nu, \\ \frac{D}{3r_\nu^2}, & r > r_\nu. \end{cases}$$  \hspace{1cm} (34)

In the $V^3$ model, we cut $F_{ijk}$ at the same boundary $r = r_\nu$. The first term in (29) then takes the form

$$L_{\text{Gauss}} = \begin{cases} L_{\text{Gauss}}^B, & r \leq r_\nu, \\ \left( \frac{2}{3}D \frac{\partial^2}{\partial r^2} + 2\kappa \right) \left( \frac{D}{3}r^2 + \frac{4}{r} \frac{\partial}{\partial r} \right), & r > r_\nu, \end{cases}$$  \hspace{1cm} (35)

and the second term is

$$\delta L = \begin{cases} \delta L^B, & r \leq r_\nu, \\ 0, & r > r_\nu. \end{cases}$$  \hspace{1cm} (36)

Since we consider large magnetic Prandtl numbers, viscosity is large as compared to magnetic diffusivity, and

$$r_\nu^2 = \frac{\kappa}{D} \ll r_m^2.$$  \hspace{1cm} (37)

So, in addition to the small parameter $F/D$ in the Batchelor problem statement, here we have one more small parameter

$$1/\mu = r_\nu/r_m.$$  \hspace{1cm} (38)

The equation (29) with (34), (36) can be solved in special functions. However, we make an analytic estimate
for the contribution of large but finite \( \mu \) to the magnetic increment:

\[
\gamma_{\max} \gtrsim D \left( \frac{5}{2} - \frac{243 F^2}{32 D^2} - \frac{2 \pi^2}{3 \ln^2 \mu} (1 + O(F^2/D^2)) \right)
\]

(38)

(See Appendix C for the derivation). This estimate shows that, in the first approximation, the non-Gaussianity and the finiteness of the magnetic Prandtl number (\( \mu < \infty \)) act independently.

In the next Section we will consider the numerical solutions to the equation (29) for finite \( \mu \), investigate the dependence of magnetic field generation on the parameters \( F \) and \( \mu \), and check the reliability of the model.

5. NUMERICAL SOLUTION OF THE GENERALIZED KAZANTSEV EQUATION

The generalized Kazantsev equation (28) is a third-order differential equation with small multiplier \( F/D \) at the elder derivative. It is not evident if its solutions are stable and converge to the solutions of (10) in the limit \( F/D \to 0 \). We also test our theoretical conclusion (38) and show that the behavior of the increment does not depend on the details of the cutoff at large \( r \).

5.1. Technical Details

We consider the Eq.(29) with \( \sigma \) and \( \delta L \) determined by (34) and (36). In dimensionless notations

\[
x = r/r_d, \quad \Gamma = \gamma/D, \quad f = F/D, \quad \mu = r_\nu/r_d
\]

we get the equation for \( G(x) \):

\[
\Gamma G = \frac{2}{3} \left( x^2 \theta(\mu - x) + \mu^2 \theta(x - \mu) + 3 \right) G'' + \frac{4}{3} \left( 3 x \theta(\mu - x) + 2 \mu^2 \theta(x - \mu) + \frac{6}{x} \right) G' + \frac{2\mu}{x} \theta(\mu - x) G + \frac{4}{3} f \left( 2 x^3 G'' + 21 x^2 G'' + 14 x G' - 70 \right) \theta(\mu - x), \quad (39)
\]

where \( \theta(y) \) denotes the Heaviside function.

The solution \( G(x) \) depends on two parameters \( f \) and \( \mu \). The asymptotes of the solutions can be found analytically.

In the limit \( x \to 0 \), (39) has three modes:

\[
G_{(1)}(x) = 1 - \frac{x^2}{2}, \quad Y = \frac{1}{3} - \frac{7 f}{18} - \frac{\Gamma}{20}, \quad (40)
\]

\[
G_{(2)}(x) \sim x^{-3}, \quad G_{(3)}(x) \sim x^6 \exp(-\frac{9}{2 f x^2}). \quad (41)
\]

The first two of them are close to the corresponding solutions for the Kazantsev-Kraichnan model; the third one is ‘produced’ by the third-order term. The last two modes diverge as \( x \to 0 \), so, they must be excluded from the physical solution.

The other asymptote is \( x \gg \mu \). The equation is significantly simplified in this limit because \( \sigma(r) \) becomes a constant:

\[
p^2 G = G'' + \frac{4G'}{x}, \quad p^2 = \frac{\Gamma}{2(1 + \frac{3}{16} \mu^2)}. \quad (42)
\]

The exact solution to this equation is

\[
G(x) = Y_1 \frac{e^{-p x}}{x} (1 + \frac{1}{p x}) + Y_2 \frac{e^{p x}}{x} (1 - \frac{1}{p x}). \quad (43)
\]

Again, the divergence condition \( G(x) \to x \to \infty 0 \) requires \( Y_2 = 0 \) and leaves only one of the two modes.

To solve equation (39) numerically, we fix some \( \Gamma \) and the initial point \( x_1 \ll 1 \). The initial conditions \( G(x_1), G'(x_1), G''(x_1) \) are determined by the asymptote (40). Then we get the numerical solution \( G(x) \) up to \( x = \mu \), and match it with the asymptote (43); the condition \( Y_2 = 0 \) singles out a discrete spectrum of possible values \( \Gamma \). We are looking for the maximal value \( \Gamma = \Gamma_{\max} \). We also check the stability of the solution relative to the choice of \( x_1 \).

5.2. Results

As it follows from (38), the theoretical prediction for \( \gamma(f, \mu) \) is

\[
\Gamma = \frac{\gamma}{D} \simeq \left( \frac{5}{2} - c_2 f^2 - \frac{c_3 + O(f^2)}{\ln^2 \mu} \right), \quad (44)
\]

\[
c_2 = \frac{243}{32} \simeq 7.59, \quad c_3 \lesssim \frac{2 \pi^2}{3} \simeq 6.6
\]

First, we analyze the dependence \( \Gamma(f) \) for some fixed \( \mu \). The results for three values of \( \mu \) are presented in Figure 2. We see that, in accordance with (44), \( \Gamma(f) \) has parabolic shape at small \( f \) for all considered values of \( \mu \). To observe the dependence of the second-order term on \( \mu \), we fit the results of the simulation.
by parabolic functions in the range $0 \leq f \leq 0.1$; at larger $f$, the higher-order terms in (44) may come into play. The results are presented in Table 1. One can see that the absolute term is smaller than 2.5 and increases as a function $\mu$. The magnitude of the declination agrees with (44). The absolute value of the coefficient at $f^2$ decreases as a function of $\mu$, approaching the theoretical prediction $c_2$ for $\mu = \infty$.

Table 1. Fit of the numerical data (Fig.2) for $0 < f < 0.1$: $\Gamma(f) = C_1 - C_2 f^2$

| $\mu$  | $C_1$             | $C_2$             |
|--------|-------------------|-------------------|
| 30     | 1.884 ± 0.003     | 11.5 ± 0.5        |
| 100    | 2.159 ± 0.001     | 9.5 ± 0.2         |
| 300    | 2.277 ± 0.001     | 8.7 ± 0.3         |
| $\infty$ (theory) | 2.5         | 7.59            |

Second, we calculate the function $\Gamma(\mu)$ at some given $f$. We take $f = 0.13$ because it corresponds to the asymmetry of a real flow (20) observed in the numerical calculation (Girimaji & Pope 1990) and experiment (Luthi et al. 2005). We also calculate $\Gamma(\mu)$ for a symmetric flow, $f = 0$: in this case, the equation (39) becomes a second order equation. The results are presented in Figure 3. We fit the graphs by the ansatz

$$\Gamma(\mu) = C_1 - \frac{C_3}{\ln^2 \mu}. \quad (45)$$

The correspondence is good enough; for $f = 0$, we get $C_1 = 2.50 \pm 0.02$ and $C_3 = 6.64 \pm 0.33$, which coincides with the theoretical prediction $c_1$ and $c_3$ in (44).

The choice of the $\sigma(r)$ profile at $r \approx r_v$ is rather conditional; the details of the velocity structure function at this range of scales are believed not to effect the result significantly. This has been checked numerically (Novikov et al. 1983) for the Kazantsev-Kraichnan model, but it has to be also proved for $F \neq 0$. So, apart from (34), we consider a smooth function

$$\tilde{\sigma}(r) = \frac{D}{3} \times \frac{(r/r_\mu)^2}{1 + (r/r_\nu)^2}. \quad (46)$$

At $r \to \infty$ and $r \to 0$ it has the same asymptotes as $\sigma(r)$. We perform the same calculations with this function. In Figure 4, the dependence $\Gamma(\mu)$ for $f = 0.13$ is presented for both choices of $\sigma$. We see that the details of the saturation do not affect the increment behavior crucially.

Finally, from (44) it follows that $\gamma(f, \mu)$ becomes negative outside some region in the $f, \mu$ plane. So, the range of the parameters at which the generation of magnetic field is possible is restricted by some $\mu > \mu_{crit}(f)$. The numerical calculations confirm this prediction. We also find the dependence $\mu_{crit}(f)$: it is presented in Figure 5. One can see that the presence of non-Gaussian term decreases the range of $\mu$ that permits the generation.

6. DISCUSSION

Thus, in this paper we consider the magnetic field generation in a turbulent hydrodynamic flow. We derive the generalization of the Kazantsev equation for the case of time asymmetric flows with small but finite third order velocity correlator, which is, probably, the general case of real hydrodynamic turbulent flows. The non-zero third order correlator corresponds to the time asymmetry of the flow and is responsible for the energy cascade.

We use the $V^3$ model: slightly non-Gaussian stochastic velocity field is replaced by an effective $\delta$-process, and the velocity connected correlators of the order higher than three are assumed to be zero. The validity of this model is argued in Section III and Appendix A. We also
use one more (unessential) simplifying assumption on the shape of velocity structure function, which is considered piecewise.

The main results are:

- We show that the magnetic field generation weakens in presence of time asymmetry, independently of the direction of the energy cascade (38): the increment of average magnetic energy density decreases proportionally to the square of the magnitude $F$ of the third order correlator.

- The range of magnetic Prandtl numbers (presented in the considered model by the parameter $\mu$) that allow the magnetic field generation also becomes narrower for finite $F$: the critical Prandtl number increases as a function of $F$ (Figure 5).

- We show that, despite the presence of a small higher-order derivative, the numerical solution of the generalized Kazantsev equation converges in the limit $F \to 0$ to the solution of the Kazantsev equation. This proves that the generalized Kazantsev equation allows to investigate numerically the magnetic field evolution in time-asymmetric flows with intermediate magnetic Prandtl numbers.

Now we proceed to the discussion of some interesting particular consequences.

We solve the generalized Kazantsev equation analytically in the limit of the Batchelor regime, $r_\nu = \infty$. In this limit, the resulting increment (33) coincides with the result that follows from the Lagrangian deformation method (22). This coincidence not only proves the reliability of both methods. It also reflects a non-trivial physical fact: the time-averaged magnetic energy density measured at some fixed point coincides with that measured along a liquid particle trajectory. Not only the methods of calculation differ in the two approaches but also the averages $\langle B^2 \rangle$ are taken over different ensembles. So, the coincidence not only verifies the results but also proves the equivalence of these two averages. This coincidence is not a consequence of ergodicity, since the trajectory of a particle as well as magnetic energy are functionals of the velocity field, and, thus, are not independent.

The equivalence of the two average was found for passive scalar (Balkovsky & Fouxon 1999) and vector (Chertkov et al. 1999) advection in the case of Kazanatsev-Kraichnan velocity field. Now we see that it also holds for vector advection in time-asymmetric velocity fields.

Another important coincidence establishes the relation between the magnetic increments calculated for the Batchelor limit and for the case of finite Prandtl number. Namely, the theoretical analysis and numerical simulation confirm that, as $\mu \to \infty$, the increment $\gamma(\mu, F)$ converges to the value $\gamma(F)$ found for the Batchelor regime. This is also not trivial: for instance, this equality does not hold for two-dimensional flows (Kolokolov 2017; Schekochihin et al. 2002a) or for higher-order correlators in three-dimensional flows (Zybin et al. 2020). The Kazantsev equation corresponds to the infinite-time limit; the convergence of the solutions in the limit $Pr_m \to \infty$ (and its coincidence with the value obtained for the Batchelor case) means that the limits $Pr_m \to \infty$ and $t \to \infty$ commute. This also has been known for Gaussian flows (Kulsrud & Anderson 1992; Kazantsev 1968; Novikov et al. 1983), now this is also checked for the time asymmetric case.

Eventually, the deformation of the magnetic energy spectrum and its evolution is also an interesting and important question (Kazantsev 1968; Kulsrud & Anderson 1992; Schekochihin et al. 2002a; Bhat & Subramanian 2014; Aiyer et al. 2017). In the frame of $V^3$ model it is also possible to find the spectrum evolution, however this problem is rather complicated and deserves separate investigation. We will explore it in the next paper.

Summarizing, we stress that the asymmetry of the hydrodynamic flow statistics is an essential feature that may affect the process of magnetic field generation. The $V^3$ model allows to investigate the asymmetric flows, and thus opens the prospective for investigation of transport problems in the flows with energy cascade.

![Figure 5. The boundary of the magnetic field generation range $\gamma > 0$, $\mu_{\text{crit}}$ as a function of $f$. The triangle corresponds to the point (20).](image-url)
The authors are grateful to Professor A.V. Gurevich for his permanent attention to their work. A.M.K. thanks Professor Ya.N. Istomin for considerable contribution to the work on its early stage. The work of A.V.K. and A.M.K. was supported by the RSF Grant No. 20-12-00047.

APPENDIX

A. GROUNDS FOR THE EFFECTIVE $\delta$-PROCESS INTRODUCTION

The equation (12) is a stochastic differential equation, the random velocity gradient tensor $A$ acts as a multiplicative noise (Batchelor 1959).

One can show that for any non-Gaussian process with finite correlation time, there exists some corresponding effective $\delta$-process (Il’yn et al. 2016). This means that

$$\lim_{T \to \infty} \frac{1}{T} \log \langle B^2(T) \rangle = \lim_{T \to \infty} \frac{1}{T} \log \langle B^2(T) \rangle_{\text{eff}},$$

where the average in the right-hand side is calculated for velocity statistics defined by the effective $\delta$-process. The reason to replace an arbitrary finite-correlation time process by the corresponding $\delta$-process is that in the equation with multiplicative noise, the higher order connected correlators of the noise contribute to the long-time statistical properties of the solutions only via their integrals. This allows to substitute singular correlation functions for the real correlators.

We demonstrate this fact by a simple example. Consider a one-dimensional stochastic equation with multiplicative noise $\xi(t)$:

$$\frac{\partial x(t)}{\partial t} = \xi(t) x(t), \quad x(0) = 0,$$

where $\xi(t)$ is a continuous stationary random process with finite correlation time. Let it have regular fast decaying connected correlation functions (cumulants):

$$\langle \xi(t_1) \ldots \xi(t_n) \rangle_c = W^{(n)}(t_1 - t_2, \ldots, t_1 - t_n).$$

The cumulant generating functional is defined by

$$\langle e^{\int \xi(t) \eta(t) dt} \rangle = e^{W[\eta(t)]},$$

then

$$W[\eta(t)] = \sum_n \frac{1}{n!} \int W^{(n)}(t_1 - t_2, \ldots, t_1 - t_n) \eta(t_1) \ldots \eta(t_n) dt_1 \ldots dt_n$$

The solution of the Eq. (A2) for each continuous realization of $\xi(t)$ can be written as

$$x(T) = e^{\int_0^T \xi(t) dt}$$

We are interested in the statistical moments

$$\langle x^m(T) \rangle = \langle e^{\int_0^T m \xi(t) dt} \rangle$$

From (A3) it then follows that

$$\langle x^m(T) \rangle = e^{W[m \theta(T-t)]},$$

where $\theta$ is the Heaviside step-like function. According to (A4), we then have

$$\lim_{T \to \infty} \frac{1}{T} \log \langle x^m(T) \rangle = w(m),$$
\[ w(m) = \sum_n \frac{m^n}{n!} w^{(n)}. \]
\[ w^{(n)} = \int W^{(n)}(\tau_2, \ldots, \tau_n)d\tau_2 \ldots d\tau_n, \]

We see that as \( T \to \infty \) the statistical moments of \( x \) depend only on the integrals \( w^{(n)} \) and do not depend on the detailed shape of the correlators. So, for any random process \( \xi(t) \) (i.e., for any given \( W^{(n)} \)), we can consider a series of random processes \( \xi_\epsilon(t) \) with
\[ W_\epsilon^{(n)} = \frac{1}{\epsilon^{n-1}} W^{(n)}(\tau_2/\epsilon, \ldots, \tau_n/\epsilon) \]
and all these processes would produce the same long-time asymptotes for the moments of \( x(t) \). The effective process \( \xi_{eff}(t) \) is defined as the formal limit of this consequence as \( \epsilon \to 0 \). Its connected correlation functions are
\[ \langle \xi_{eff}(t_1) \ldots \xi_{eff}(t_n) \rangle_c = w^{(n)} \delta(t_1 - t_2) \ldots \delta(t_1 - t_n) \]

In the case of multidimensional stochastic processes and stochastic fields the expression for the exact solution looks much more complicated than (A5), since the integral transforms into a continuous matrix product (T-exponent). However the long-time asymptotes of the solutions’ moments still depend only on the integrals of the connected correlators of the noise. This is the reason and the justification for substitution of the effective \( \delta \)-process for any random process. This tool is convenient in turbulence and turbulent transport problems, since it allows to get closed equations for statistical moments.

B. DERIVATION OF THE GENERALIZED KAZANTSEV EQUATION

We start with Eq. (1); to get the equation for the pair correlator, we multiply it by \( B \) and take the average. For brevity, we denote
\[ B_\alpha = B_\alpha(r, t), \quad B'_\alpha = B_\alpha(r', t), \]
\[ v_\alpha = v_\alpha(r, t), \quad v'_\alpha = v_\alpha(r', t), \]
\[ \partial_\alpha = \frac{\partial}{\partial r_\alpha}, \quad \partial'_\alpha = \frac{\partial}{\partial r'_\alpha}, \quad \partial_\rho = \frac{\partial}{\partial (r-r_\rho)}. \]

We also take into account that homogeneity of the flow, which results in \( \partial_m \langle \ldots \rangle = -\partial'_m \langle \ldots \rangle = -\partial''_m \langle \ldots \rangle \) and
\[ \langle v'_p B'_q B_\alpha \rangle = -\langle v_p B_q B'_\alpha \rangle. \]

Then the equation for the pair correlator takes the form
\[ \frac{\partial}{\partial t} B_\alpha B'_\beta = -\varepsilon_{amn} \varepsilon_{npq} \partial'_m \langle v_p B_q B'_\beta \rangle - \varepsilon_{bmn} \varepsilon_{npo} \partial'_m \langle v_p B_q B'_\alpha \rangle + 2\varepsilon \partial'_m \partial''_m \langle B_\alpha B'_\beta \rangle. \]

Now we have to split the mixed correlations by means of the generalized Furutsu-Novikov equation (Furutsu 1963; Klyatskin 2005):
\[ \langle v_p B_q B'_r \rangle = \int d\mathbf{r}_1 dt_1 \langle v_p(r, t)v_{i_1}(\mathbf{r}_1, t_1) \rangle \left( \frac{\delta(B_q(r, t)B_r(r', t))}{\delta v_{i_1}(\mathbf{r}_1, t_1)} \right) \]
\[ + \frac{1}{2} \int d\mathbf{r}_1 dt_1 d\mathbf{r}_2 dt_2 \langle v_p(r, t)v_{i_1}(\mathbf{r}_1, t_1)v_{i_2}(\mathbf{r}_2, t_2) \rangle \left( \frac{\delta^2(B_q(r, t)B_r(r', t))}{\delta v_{i_1}(\mathbf{r}_1, t_1) \delta v_{i_2}(\mathbf{r}_2, t_2)} \right). \]

Since the variational derivatives contain delta-functions, to avoid the products of delta-functions with coinciding arguments, we have to deal more accurately with the time coincidence in correlators. Namely, we have to introduce a ‘regularized \( \delta \)-function’: a bell-shaped function \( \delta_\epsilon(\tau) \) satisfying the normalization condition \( \int \delta_\epsilon(\tau)d\tau = 1 \) and with the width of the order of the correlation time. Then (23) and (24) can be written more precisely:
\[ \langle v_i(r, t)v_j(r, t_1) \rangle = D_{ij}(r - r_1) \delta_\epsilon(t - t_1), \]
and
\[ \langle v_i(r, t)v_j(r_1, t_1)v_k(r_2, t_2) \rangle = \frac{1}{3} F_{ijk}(r - r_1, r - r_2) \left( \delta_\epsilon(t - t_1) \delta_\epsilon(t - t_2) + \delta_\epsilon(t - t_1) \delta_\epsilon(t_1 - t_2) + \delta_\epsilon(t - t_2) \delta_\epsilon(t_1 - t_2) \right). \]
The variational derivative of this expression is:

\[ \delta \frac{\partial F_{ijk}}{\partial a_i} = \delta F_{ijk} \]

Here we omit the terms that become zero being multiplied by the regularized \( \delta \)-functions here are 'real', not regularized. The lower limits of the integrals in the last two summands are changed, and the non-divergency condition results in the requirement \( \partial F_{ijk}/\partial a_i = \partial F_{ijk}/\partial b_j = (\partial/\partial a_k + \partial/\partial b_k)F_{ijk} = 0 \); also, the existence of viscosity means that \( F_{ijk} \) is proportional to \( r^3 \), i.e., a third-order polynom. These conditions determine the tensor \( F_{ijk} \) to a constant multiplier:

\[
F_{ijk}(a, b) = -\frac{1}{18} F \left( (2a^2 - 15b^2 - 2a \cdot b)a_i \delta_{ij} + (2b^2 - 15a^2 - 2a \cdot b)b_j \delta_{ik} \right) + (20b^2 - 5a^2 - 16a \cdot b)a_i \delta_{jk} + (20a^2 - 5b^2 - 16a \cdot b)b_j \delta_{ik} - (5b^2 + a^2 - 26a \cdot b)b_j \delta_{ik} + 4a_i a_j a_k + 4b_i b_j b_k + 12 a_i a_j b_k + 12 b_i a_j b_k - 2 a_i b_j a_k - 2 b_i b_j a_k - 16 a_i b_j b_k - 16 b_i a_j a_k \right). 
\] (B10)

Analogous calculations for the inertial range were performed by Kopyev & Zybin (2018).

B.1. Taking the Variational Derivative

To take the variational derivative, we make a formal functional consequence from (1):

\[
B_q(r, t) = \varepsilon_{qij} \varepsilon_{jkl} \int_{t}^{t} \partial_{l1} (v_{l1}(r, \tau) B_{k1}(r, \tau)) d\tau + \kappa \partial_{l1} \int_{-\infty}^{t} B_q(r, \tau) d\tau. 
\] (B11)

The variational derivative of this expression is:

\[
\delta B_q(r, t) \over \delta v_j(r_1, t_1) = \varepsilon_{qij} \varepsilon_{jkl} \int_{-\infty}^{t} \partial_{l1} (\delta_{ji} \delta(r - r_1) \delta(\tau - t_1) B_{k1}(r, \tau)) d\tau 
+ \varepsilon_{qij} \varepsilon_{jkl} \int_{t}^{t} \partial_{l1} \left( v_{l1}(r, \tau) \frac{\delta B_{k1}(r, \tau)}{\delta v_j(r_1, t_1)} \right) d\tau + \kappa \partial_{l1} \int_{t_1}^{t} \frac{\delta B_q(r, \tau)}{\delta v_j(r_1, t_1)} d\tau. 
\] (B12)

The \( \delta \)-functions here are 'real', not regularized. The lower limits of the integrals in the last two summands are changed, in accordance with the causality principle; after multiplying by \( \delta \) in the velocity correlators in (B7), these lower limits will make no difference. Taking the second derivative, we get:

\[
\frac{\delta^2 B_q(r, t)}{\delta v_j(r_1, t_1) \delta v_k(r_2, t_2)} \overset{t_1, t_2 \to -0}{=} \varepsilon_{qij} \varepsilon_{jkl} \int_{-\infty}^{t} d\tau_1 \partial_{l1} \left( \delta(r - r_1) \delta(\tau - t_1) \frac{\delta B_{k1}(r, t)}{\delta v_k(r_2, t_2)} \right) \]

\[= \varepsilon_{qij} \varepsilon_{jkl} \varepsilon_{k1} \varepsilon_{k2} \varepsilon_{j2} \varepsilon_{j3} \varepsilon_{k1} \varepsilon_{k2} \varepsilon_{j3} \varepsilon_{j4} \theta(t - t_1) \theta(t - t_2) \times \partial_{l1} \left( \delta(r - r_1) \partial_{l2} \left( \delta(r - r_2) B_{k2} \right) \right) \] (B13)

Here we omit the terms that become zero being multiplied by the regularized \( \delta \)-function. Now we substitute these expressions into (B7). For the regularized delta functions we have

\[
\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\tau \delta_\tau(t - t_1) \delta(\tau - t_1) = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\tau \delta_\tau(t - t_1) \delta(t - t_1) = \int_{-\infty}^{+\infty} dt \delta_\tau(t - t_1) = \frac{1}{2}, \] (B14)

\[
\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\tau \delta_\tau(t - t_1) \delta(t - t_1) = \frac{1}{4} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\tau \delta_\tau(t - t_1) \delta(t - t_1) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \delta(x) \delta(y) = \frac{3}{8}. \] (B15)
These relations allow to take the time integrals in (B7). Thus, we arrive to (27). Taking the space integrals, we get the closed equation for the pair correlator:

\[
\frac{\partial}{\partial t}(B_n B'_n) = 2\alpha \partial^2_{\rho m} \partial^2_{\rho n} (B_n B'_n) + \frac{1}{2} (\varepsilon_{\alpha mn} \varepsilon_{\rho np} \delta_{\beta r} + \varepsilon_{\beta mn} \varepsilon_{\rho np} \delta_{\alpha r}) \varepsilon_{ij, k\ell} \partial^\rho_{i\ell} \partial^\rho_{j\ell} \left( \varepsilon_{\rho j i, jk} D_{jp}(0) \langle B_k B'_k \rangle - \varepsilon_{\rho 1 j, jk} D_{jp}(0) \langle B_k B'_k \rangle \right) + \frac{1}{6} (\varepsilon_{\alpha mn} \varepsilon_{\rho np} \delta_{\beta r} + \varepsilon_{\beta mn} \varepsilon_{\rho np} \delta_{\alpha r}) \varepsilon_{ij, k\ell} \varepsilon_{j k, k\ell} \partial^\rho_{i\ell} \left( 2\varepsilon_{\rho j i, jk} D_{jp}(0) \langle B_k B'_k \rangle + F_{kpj}(\rho, \rho) \langle B_k B'_k \rangle + F_{pkj}(\rho, \rho) \partial^\rho_{i\ell} \langle B_k B'_k \rangle \right),
\]

where \( \rho = r' - r \) and \( \partial^2_{(2)} \) denotes the derivative over the second argument.

Making use of (7), we express all the coefficients in the (B16) by means of the function \( P(r) \) and the longitudinal velocity structure function. After cumbersome symbol math-assisted calculations, we arrive to the generalized Kazantsev equation: (28), (31), (32) for the Batchelor regime and (28), (35), (36) for the nonlinear velocity field.

C. SOLUTIONS OF THE GENERALIZED KAZANTSEV EQUATION

C.1. Kazantsev-Kraichnan Model, Linear Velocity Field

Consider first the simplest Kazantsev equation for the Kazantsev-Kraichnan model (10) in the Batchelor regime, i.e., with \( \sigma = (D/3)r^2 \) (which corresponds to \( r_v \to \infty \)). The substitution of the ansatz

\[
G(t, r) = e^{\gamma t} G(r)
\]

reduces this equation to the ordinary differential equation

\[
\gamma G = \frac{2}{3} D \left( r^2 G'' + 6r G' + 10G \right) + 2\alpha \left( G'' + \frac{4}{r} G' \right).
\]

We proceed to the dimensionless variables \( x = r/r_d \); with account of \( \alpha = Dr^2_d \), we get

\[
(x^2 + 3) G'' + \left( 6x + \frac{12}{x} \right) G' + \left( 10 - \frac{3}{2} \Gamma \right) G = 0,
\]

where

\[
\Gamma = \gamma / D.
\]

The two exact solutions of this equation can be written by means of hypergeometric functions. Here we restrict our consideration to the analysis of their asymptotic behavior, to get the experience necessary to generalize the solutions to the cases of nonzero \( F \) and/or finite \( r_v \).

One of two independent solutions diverges as \( x \to 0 \), so we are interested in the other one. It satisfies the boundary condition \( G'(0) = 0 \).

As \( x \to \infty \), the equation (C17) is simplified to a homogenous differential equation; its characteristic equation is

\[
\alpha (\alpha - 1) + 6\alpha + 10 - \frac{3}{2} \Gamma = 0,
\]

and the solution is

\[
G^{(1)}(x) \sim_{x \to \infty} A_+ x^{\alpha_+} + A_- x^{\alpha_-},
\]

where

\[
\alpha_{\pm} = -\frac{5}{2} \pm \frac{3}{2} \sqrt{2\Gamma - 5}.
\]

The real and imaginary parts of \( \alpha_+(\Gamma) \) are illustrated in Figure 6. One can see that for all \( \Gamma < 5/2 \), \( G^{(1)}(x) \) decreases as \( x \to \infty \) with the same rate and oscillates, while for \( \Gamma > 5/2 \) it decreases slower (and for \( \Gamma > 20/3 \) even grows), without oscillations.
Now, let us return to the evolution equation (10). The equation (C17) can be reduced to the Sturm-Liouville equation (by the change of variables \( y = \arcsinh(x/\sqrt{3}) \), \( q(y) = (3 + x^2)^{1/4} x^2 G(x) \)). So, an arbitrary initial perturbation \( G_0(x) \) can be decomposed into a sum (integral) of the eigenfunctions \( G^{(\Gamma)}(x) \). The time dependence of each summand is exponential with its own rate, \( G^{(\Gamma)}(t, x) \propto e^{\gamma t} \), so the term with the biggest \( \gamma \) would survive after large time. Now, if the initial function is localized (more precisely, decreases not faster than \( x^{-5/2} \) as \( x \to \infty \)), then all the terms in the decomposition must oscillate or decrease at \( x \to \infty \) faster than \( G_0(x) \). Since \( \text{Im} \alpha_++(\Gamma \geq 5/2) = 0 \) and \( \text{Re} \alpha_++(\Gamma \geq 5/2) > -5/2 \), we get the upper boundary of the spectrum:

\[ \Gamma_{\text{max}} = \frac{5}{2}, \quad G(x, t) \propto e^{\Gamma_{\text{max}} D t}. \]

**C.2. Non-Zero Time Asymmetry**

Now we take account of \( F \neq 0 \), and apply the same ideas to analyze the equation (28), (32). The equation for the eigenvalues is

\[ \gamma G = \frac{2}{3} D \left( r^2 G'' + 6 r G' + 10 G \right) + 2 \pi \left( G'' + \frac{4}{r} G' \right) + \frac{1}{9} F \left( 2 r^3 G'''' + 21 r^2 G'' + 14 r G' - 70 G \right). \]  

(C18)

In the dimensionless variables, this is equivalent to

\[ (x^2 + 3) G'' + \left( 6 x + \frac{12}{x} \right) G' + \left( 10 - \frac{3}{2} \right) G + \frac{1}{6} f \left( 2 x^3 G'''' + 21 x^2 G'' + 14 x G' - 70 G \right) = 0, \]  

(C19)

where \( f = F/D \) is the new small parameter. Again, we consider the asymptote \( x \to \infty \) and get the homogenous differential equation with the characteristic polynomial:

\[ \frac{f}{3} \beta (\beta - 1)(\beta - 2) + \left( 1 + \frac{7}{2} f \right) \beta (\beta - 1) + \left( 6 + \frac{7}{3} f \right) \beta + \left( 10 - \frac{35}{3} f - \frac{3}{2} \Gamma \right) = 0. \]  

(C20)

This equation has three solutions, two of them are close to \( \alpha_{\pm} \), and the third solution is real (negative) and very big:

\[ G^{(\Gamma)} \simeq B_+ \xi^{\beta_+} + B_- \xi^{\beta_-} + B_f \xi^{\beta_f}, \quad \xi \to \infty, \]  

(C21)

\[ \beta_f \propto -1/f, \]  

(C22)

\[ \beta_{\pm} - \alpha_{\pm} \propto f. \]  

(C23)

The third summand is artefact of the model; it corresponds to the non-physical solution and must be excluded by setting \( B_f = 0 \).

The internal limit \( x \to 0 \) gives also three solutions, only one of them is convergent. We assume that there exist modes that converge at both limits \( x \to 0 \) and \( x \to \infty \). Unlike the Gaussian (Kazantsev-Kraichnan) case, this is not guaranteed. This supposition is to be proved by numerical simulations. This is done in Section V. Also, the completeness of this set of functions is not evident; however, we suppose that the asymptotic time dependence of arbitrary solution is exponential, as it was in the case of the second order equation, so this arbitrary solution can be presented as a sum of eigenfunctions; we are again interested in the eigenfunction with the fastest increment.
But as soon as we suppose the existence of the spectrum, we can derive the upper boundary of the eigenvalue in the decomposition of an arbitrary initial distribution $G_0(x)$ based on the asymptotic behavior of the eigenfunctions. The eigenfunctions presented in the decomposition can either decrease faster than $G_0(x)$ of oscillate as $x \to \infty$. From the characteristic polynomial (C20) we find that the condition of oscillations Im$\beta \neq 0$ holds for

$$\Gamma < \Gamma_{\text{max}} = \frac{8 + 126 f^2 - (4 + 27 f^2)^{3/2}}{18 f^2}. \quad \text{(C24)}$$

This coincides with (21). In the limit $f \ll 1$, we arrive to the Equation (33).

**C.3. Finite Prandtl Numbers**

To evaluate the effect of the nonlinearity of velocity field, we consider the function $\sigma(r)$ truncated in accordance with (34). For this model, the Kazantsev equation is step-wise: it coincides with (C18) for $r < r_\nu$, while for $r > r_\nu$ it becomes

$$\gamma G = \frac{2}{3} D(G'' + \frac{4}{r} G') (r_\nu^2 + 2 r^2), \quad r > r_\nu. \quad \text{(C25)}$$

So, one has to match the solution of (C18) with the descending solution of this equation. The coincidence of $G$ and $G'$ determines the spectrum of $\gamma$. But the biggest $\gamma_{\text{max}}^{(\mu)}$ is still restricted by the condition Im$\beta \neq 0$.

To estimate the correction to $\gamma$ produced by large but finite $r_\nu$, we note that the solution of (C18) oscillates as a function of $r$, and the phase of the oscillations is determined by the imaginary part of $\beta_+$:

$$G(r) \propto r^\beta_+ \propto e^{i\text{Im} \beta_+ \ln \mu}. \quad \text{(C26)}$$

Matching the amplitudes of both branches of the solution at $r = r_\nu$ is provided by a multiplier; to match the derivatives, one has to choose the phase. However, half a period of oscillations is enough to ensure any phase that is needed. So, the correction to $\gamma_{\text{max}}$ produced by finite $r_{\text{nu}}$ at any rate leaves it within the range of $\gamma$ in which the phase of $G(r_\nu)$ changes by $\pi$:

$$\text{Im} \beta_+ (\Gamma_{\text{max}}^{(\mu)}) \ln \mu = \pi. \quad \text{(C27)}$$

The bigger $\mu$, the closer is $\Gamma_{\text{max}}^{(\mu)}$ to the 'Batchelor' value $\Gamma_{\text{max}}$. 
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