THE AUTOMORPHISM GROUP OF
A RATIONAL PROJECTIVE $K^*$-SURFACE

JÜRGEN HAUSEN, TIMO HUMMEL

Abstract. We consider non-toric possibly singular rational projective $K^*$-surfaces and provide an explicit description of the unit component of the automorphism group in terms of isotropy group orders and intersection numbers of suitable invariant curves.

1. Introduction

We work over an algebraically closed field $K$ of characteristic zero. By a $K^*$-surface we mean a normal irreducible surface $X$ endowed with an effective morphical action $K^* \times X \to X$ of the multiplicative group $K^*$. The geometry of $K^*$-surfaces has been intensely studied by many authors; see for instance [10–12, 23–26]. We consider the automorphism group $\text{Aut}(X)$ of a rational projective $K^*$-surface $X$. This is an affine algebraic group and our aim is to give a detailed explicit description of the unit component of $\text{Aut}(X)$ in terms of basic data of the action.

In order to formulate our main result, let us recall the basic geometric features of $K^*$-surfaces. One calls a fixed point elliptic (hyperbolic, parabolic) if it lies in the closure of infinitely many (precisely two, precisely one) non-trivial $K^*$-orbit(s). Elliptic and hyperbolic fixed points are isolated, whereas the parabolic fixed points form a closed smooth curve with at most two connected components. Every projective normal $K^*$-surface $X$ has a source and a sink, that means irreducible components $F^+, F^- \subseteq X$ of the fixed point set admitting open $K^*$-invariant neighborhoods $U^+, U^- \subseteq X$ such that

$$\lim_{t \to 0} t \cdot x \in F^+ \text{ for all } x \in U^+,$$

$$\lim_{t \to \infty} t \cdot x \in F^- \text{ for all } x \in U^-,$$

where these limits are the respective values at the points 0 and $\infty$ of the unique morphism $\mathbb{P}_1 \to X$ extending the orbit map $t \mapsto t \cdot x$. The source, and as well the sink, consists either of a single elliptic fixed point or it is a smooth irreducible curve of parabolic fixed points; we write $x^+$ and $x^-$ in the elliptic case and $D^+$ and $D^-$ in the parabolic case. Apart from the source and the sink, we find at most hyperbolic fixed points. The raw geometric picture of a rational projective $K^*$-surface $X$ is as follows:

![Diagram of a $K^*$-surface with sources and sinks]

The general orbit $K^* \cdot x \subseteq X$ has trivial isotropy group $K^*_x$ and its closure connects the source and the sink in the sense that it contains one fixed point from $F^+$ and $F^-$. 

2010 Mathematics Subject Classification. 14L30,14J26.
one from $F^-$. Besides the general orbits, we have the special non-trivial ones. Their closures are rational curves $D_{ij} \subseteq X$ forming the arms $A_i = D_{i1} \cup \ldots \cup D_{in}$ of $X$, where $i = 0, \ldots, r$, the intersections $F^+ \cap D_{i1}$ and $D_{in} \cap F^-$ consist each of a fixed point and any two subsequent $D_{ij}, D_{ij+1}$ intersect in a hyperbolic fixed point.

To every such rational curve $D_{ij}$ we associate an integer, namely the order of the isotropy group of the general point of $D_{ij}$.

Every $\mathbb{K}^*$-surface $X$ admits a minimal equivariant resolution $\pi : \tilde{X} \rightarrow X$ of singularities. If there is a parabolic fixed point curve $D^+ \subseteq X$, then we consider the points $x_i \in X$ lying in $D^+ \cap D_{i1}$. If $x_i$ is singular, then the fibre $\pi^{-1}(x_i)$ of the minimal resolution is a connected part $E_{i1} \cup \ldots \cup E_{iq}$ of an arm of $\tilde{X}$, where the curve $E_{i1}$ intersects the proper transform of $D^+$. We define

$$c_i(D^+) := \left( -E_{i1}^2 - \frac{1}{-E_{i2}^2 - \frac{1}{\ldots - E_{iq}^2}} \right)^{-1}$$

if $x_i$ is singular and $c_i(D^+) = 0$ else. We call an elliptic fixed point $x \in X$ simple, if $\pi^{-1}(x)$ is contained in an arm of $\tilde{X}$. If $X$ admits a simple elliptic fixed point, then we may assume that this is $x^-$. The fibre $\pi^{-1}(x^-) = E_1 \cup \ldots \cup E_q$ is a connected part of an arm of $\tilde{X}$ and $E_q$ contains a smooth elliptic fixed point of $\tilde{X}$. In this situation, we define

$$c(x^-) := \left( -E_q^2 - \frac{1}{-E_{q-1}^2 - \frac{1}{\ldots - E_1^2}} \right)^{-1}$$

if $x^-$ is singular and $c(x^-) := 0$ else. Finally, a point $x \in X$ is called quasismooth if it is a toric surface singularity; see Definition 6.7 and Corollary 6.12 for more background. In case of a quasismooth simple elliptic fixed point $x^- \in X$, we can always assume the numbering of the arms $A_0, \ldots, A_r$ to be normalized in the sense that $l_0, \ldots, l_{n_1}$, and $l_{n_2} = l_{n_3}$ implies $D_{i1}^2 \leq D_{i2}^2$, whenever $i < j$ and $n_i, n_j \geq 2$. In this situation, the exceptional curves $E_1, \ldots, E_q \subseteq \tilde{X}$ belong to the arm $A_0 \subseteq \tilde{X}$ mapping onto the arm $A_0 \subseteq X$; see Proposition 7.12. We denote by $l_{0n}$ the order of the isotropy group of the general point of $E_q = D_{0n} \subseteq A_0$.

We are ready to state the main result of this article.

**Theorem 1.1.** Let $X$ be a non-toric rational projective $\mathbb{K}^*$-surface. Then the unit component of the automorphism group $\text{Aut}(X)$ of $X$ is given as a semidirect product

$$\text{Aut}(X)^0 = (\mathbb{K}^* \rtimes \mathbb{K}^*) \rtimes_{\psi} \mathbb{K}^*, \quad \rho \in \mathbb{Z}_{\geq 0}, \quad \zeta \in \{0, 1\}.$$ 

If $X$ has neither a non-negative fixed point curve nor a quasismooth simple elliptic fixed point, then $\rho = \zeta = 0$ holds. Otherwise, precisely one of the following holds.

(i) There is a non-negative fixed point curve. Then we can assume that this curve is $D^+ \subseteq X$. In this situation, we have $\zeta = 0$ and

$$\rho = \max(0, (D^+)^2 + 1 - \sum_{i=0}^{r} c_i(D^+)).$$

The group homomorphism $\psi : \mathbb{K}^* \rightarrow \text{Aut}((\mathbb{K}^*)^0)$ fixing the semidirect product structure is given by $t \mapsto t^{-1}E_q$.

(ii) There is exactly one quasismooth simple elliptic fixed point. We can assume that this is $x^-$ and the numbering of the arms is normalized. Then

$$\rho = \max\left(0, \left[ l_{1n_1} \min_{i \neq 0} (l_{in}, D_{in}^2) + (l_{in} - l_{1n_1})D_{in}D_{1n_1} - c(x^-) \right] + 1 \right)$$
holds. Moreover, we have $\zeta = 1$ if and only if for all $i \neq 1$ the following inequalities are satisfied

$$D_{n_i}^2 \geq (l_{on_i} - l_{n_i})D_{on_i}D_{n_i}.$$  

The semidirect product structure on $\text{Aut}(X)$ is determined by the following. For $\zeta = 1$, the homomorphism $\varphi: \mathbb{K} \to \text{Aut}(\mathbb{K}^P)$ is given by

$$s \mapsto A = (a_{\mu \alpha}), \text{ where } a_{\mu \alpha} = \begin{cases} \left(\frac{\mu - 1}{\alpha - \mu}\right)s^\alpha - \mu, & \alpha \geq \mu, \\ 0, & \alpha < \mu. \end{cases}$$

Moreover, the group homomorphism $\psi: \mathbb{K}^* \to \text{Aut}(\mathbb{K}^P \rtimes_\varphi \mathbb{K}^\zeta)$ is given by

$$t \mapsto \begin{cases} \text{diag}(t_{l_{on_1}}, \ldots, t_{l_{on_1} - (\rho - 1)i_{1n_1}}), & \zeta = 0, \\ \text{diag}(t_{l_{on_1}}, \ldots, t_{l_{on_1} - (\rho - 1)i_{1n_1}}, t_{i_{1n_1}}), & \zeta = 1. \end{cases}$$

Automorphism groups of rational surfaces have also been considered by several other authors. For instance, Sakamaki [28] studied the case of cubic surfaces without parameters. More generally, Cheltsov and Prokhorov [3] and, independently, also Mast and Stadlmayr [22] determined the Gorenstein log del Pezzo surfaces with infinite automorphism groups; see also [8] for earlier work in that direction. It turns out that 50 out of the 53 listed surfaces of [3, 22] are in fact rational surfaces with infinite automorphism groups; see also [8] for further work in that direction. Moreover, the group homomorphism $\psi: \mathbb{K}^* \to \text{Aut}(\mathbb{K}^P \rtimes_\varphi \mathbb{K}^\zeta)$ is given by

$$t \mapsto \begin{cases} \text{diag}(t_{l_{on_1}}, \ldots, t_{l_{on_1} - (\rho - 1)i_{1n_1}}), & \zeta = 0, \\ \text{diag}(t_{l_{on_1}}, \ldots, t_{l_{on_1} - (\rho - 1)i_{1n_1}}, t_{i_{1n_1}}), & \zeta = 1. \end{cases}$$

Let us give an outline of the article, showing its basic ingredients and some main ideas. Our working environment is the Cox ring based approach of [15, 18] to rational $T$-varieties $X$ of complexity one, that means that $X$ is normal, rational and comes with an effective torus action $T \times X \to X$ such that the general $T$-orbit is of codimension one in $X$. One of the basic features of this approach is that it provides a natural $T$-equivariant closed embedding $X \subseteq Z$ into a toric variety $Z$. In Section 2, we present a brief general reminder.

We will also make use of the description of the automorphism group of a toric variety via the Demazure roots of its defining fan; see [4, 7] and Section 3 for a quick summary. First applications are the tools provided in Section 4 and the explicit description of the automorphism groups of toric surfaces given there. The understanding of $\text{Aut}(X)$ for a complete rational $T$-variety $X$ of complexity one is not yet as developed as in the toric case. However, the main results of [2] show that $\text{Aut}(X)^0$ is generated by $T$ and the additive one-parameter groups arising from Demazure $P$-roots; see also Section 5. In Theorem 6.4 we provide a presentation of the automorphisms arising from Demazure $P$-roots as restrictions of automorphisms of the ambient toric variety $Z$ which are explicitly given in Cox coordinates. The explicit nature of the result is crucial for our purposes. The general question to which extent a variety inherits its automorphisms from a suitable ambient variety is interesting as well. For Mori dream spaces $X$, a positive result concerning $\text{Aut}(X)^0$ is given in [17, Thm. 4.4]; see also [24] for further results in the case of quasismooth Fano weighted complete intersections.

From Section 6 on, we focus on rational projective $\mathbb{K}^*$-surfaces. We first recall basics on their geometry and relate defining data to self intersection numbers, see Sections 6 and 7. In Theorem 8.4 we figure out geometric implications of the existence of a quasismooth simple elliptic fixed point: a non-toric rational projective $\mathbb{K}^*$-surface $X$ can have at most one such fixed point and if there is one, then any
parabolic fixed point curve is contractible or its intersection with any arm of \( X \) is a singularity of \( X \). In Section 9, we introduce horizontal and vertical \( P \)-roots, which basically means adapting the more involved notion of a Demazure \( P \)-root to the surface case. Together with \( \mathbb{K}^* \), the additive one-parameter groups arising from the \( P \)-roots generate \( \text{Aut}(X)^0 \). We link existence of \( P \)-roots to the geometry of \( X \). From [2] we infer that \( \text{Aut}(X) \) acts with an open orbit if and only if there is a horizontal \( P \)-root. Proposition 9.6 shows that the presence of a horizontal \( P \)-root forces a quasismooth simple elliptic fixed point. By Proposition 9.17 existence of vertical \( P \)-roots exclude quasismooth simple elliptic fixed points. Consequently, \( \text{Aut}(X) \) does not act almost transitively if we have vertical \( P \)-roots. Each vertical root is uniquely associated with a parabolic fixed point curve in the sense that the corresponding additive one-parameter group moves that curve. Proposition 9.18 shows that if there are vertical roots, then they are all associated with the same fixed point curve.

Starting with Section 10, we study the structure of the unit component of \( \text{Aut}(X) \). The first task is to figure out relations among the additive one-parameter groups arising from the \( P \)-roots. A sufficiently detailed study allows us to figure out minimal generating systems of \( P \)-roots. Proposition 10.2 does this for the case that \( \text{Aut}(X) \) acts with an open orbit and Proposition 10.3 settles the remaining case. In Section 11, we show in terms of the combinatorics of defining data that for the minimal resolution of singularities \( \tilde{X} \to X \) of a rational projective \( \mathbb{K}^* \)-surface, the groups \( \text{Aut}(\tilde{X})^0 \) and \( \text{Aut}(X)^0 \) coincide. Whereas the latter can as well be deduced from the general existence of a functorial resolution in characteristic zero, our investigation is more specific and allows us to relate the root groups of \( X \) with those of \( \tilde{X} \) in an explicit manner. Finally, in Section 12, we prove Theorem 1.1. The basic idea is to extract the structural information on \( \text{Aut}(X)^0 \) from \( X \to \tilde{X} \to X' \), where \( X \to \tilde{X} \) is the minimal resolution and \( \tilde{X} \to X' \) a suitable birational contraction to a certain toric surface that allows to keep track on the relevant root groups.

**Contents**

1. Introduction 4
2. \( T \)-varieties of low complexity 4
3. Demazure roots and automorphisms 8
4. Automorphisms of complete toric surfaces 12
5. Representation via toric ambient automorphisms 18
6. Rational projective \( \mathbb{K}^* \)-surfaces 21
7. Self intersection numbers and continued fractions 25
8. Quasismooth simple elliptic fixed points 28
9. Horizontal and vertical \( P \)-roots 34
10. Generating root groups 41
11. Root groups and resolution of singularities 47
12. Structure of the automorphism group 52
References 55

2. \( T \)-varieties of low complexity

Here we provide the necessary background on toric varieties and rational varieties with torus action of complexity one. Throughout the whole article, the ground field \( \mathbb{K} \) is algebraically closed and of characteristic zero. We simply write \( \mathbb{K} \) for the additive group of the ground field, \( \mathbb{K}^* \) for the multiplicative one and \( T^n \) for the \( n \)-fold direct product \( (\mathbb{K}^*)^n \).
By a torus we mean an algebraic group $T$ isomorphic to some $T^n$. A quasitorus is a direct product of a torus and a finite abelian group. By a $T$-variety $X$ we mean a normal, irreducible variety $X$ with an effective action of a torus $T$ given by a morphism $T \times X \to X$. The complexity of a $T$-variety $X$ is the difference $\dim(X) - \dim(T)$.

We turn to toric varieties, which by definition are the $T$-varieties of complexity zero. The basic feature of toric varieties is that they are completely described via lattice fans. We assume the reader to be familiar with the foundations of this theory as explained for instance in [5, 6, 14].

We will intensely use the Cox ring and Cox’s quotient construction for toric varieties [4]. Recall that for any normal variety $X$ with only constant global invertible functions and finitely generated divisor class group $\text{Cl}(X)$, one associates a Cox sheaf $\mathcal{R} = \bigoplus_{D \in \text{Cl}(X)} \mathcal{O}_X(D)$, see [1, Chap. 1] for details. The Cox ring $\mathcal{R}(X)$ is the $\text{Cl}(X)$-graded algebra of global sections of the Cox sheaf. If the Cox ring is finitely generated, we can establish the following picture

$$
\begin{array}{cccc}
\text{Spec}_X \mathcal{R} & \longrightarrow & \hat{X} & \subseteq & \text{Spec}_X \mathcal{R}(X) \\
\downarrow H & & & & \downarrow H \\
X & & & & X
\end{array}
$$

where $\hat{X}$ is the total coordinate space coming with an action of the characteristic quasitorus $H = \text{Spec} \mathbb{K}[\text{Cl}(X)]$ and the characteristic space $\hat{X}$ which is an open $H$-invariant subset of $\hat{X}$ and has $X$ as a good quotient for the induced $H$-action. In the case of toric varieties, this picture can be established in terms of defining lattice fans as follows.

**Construction 2.1.** Let $Z$ be the toric variety defined by a fan $\Sigma$ in a lattice $N$ such that the primitive generators $v_1, \ldots, v_r$ (of the rays) of $\Sigma$ span the rational vector space $N_\mathbb{Q} = N \otimes \mathbb{Z} \mathbb{Q}$. We have mutually dual exact sequences

$$
\begin{array}{cccc}
0 & \longrightarrow & L & \longrightarrow & \mathbb{Z}^r \\
0 & \longrightarrow & K & \longrightarrow & \mathbb{Z}^r \overset{P}{\longrightarrow} M \longrightarrow 0
\end{array}
$$

where $P: \mathbb{Z}^r \to N$ sends the $i$-th canonical basis vector $e_i \in \mathbb{Z}^r$ to the $i$-th primitive generator $v_i \in N$; we also speak of the generator matrix $P = [v_1, \ldots, v_r]$ of $\Sigma$. The lower sequence gives rise to an exact sequence

$$
\begin{array}{cccc}
1 & \longrightarrow & H & \longrightarrow & \mathbb{T}^r \\
& & \overset{P}{\longrightarrow} & T_Z & \longrightarrow 1
\end{array}
$$

involving the quasitorus $H = \text{Spec} \mathbb{K}[K]$ and the acting torus $T_Z = \text{Spec} \mathbb{K}[M]$ of $Z$. Moreover, the divisor class group and the Cox ring of $Z$ are given as

$$
\text{Cl}(Z) = K, \quad \mathcal{R}(Z) = \mathbb{K}[T_1, \ldots, T_s],
$$

where the $\text{Cl}(Z)$-grading of $\mathcal{R}(Z)$ is given by $\text{deg}(T_i) = Q(e_i)$. Finally, we obtain a fan $\hat{\Sigma}$ in $\mathbb{Z}^r$ consisting of certain faces of the positive orthant, namely

$$
\hat{\Sigma} := \{ \delta_0 \geq Q_{\geq 0}: P(\delta_0) \subseteq \sigma \text{ for some } \sigma \in \Sigma \}.
$$

The toric variety $\hat{Z}$ associated with $\hat{\Sigma}$ is the characteristic space of $Z$, sitting as an open toric subset in the total coordinate space $\hat{Z} := \mathbb{K}^r$. As $P$ is a map of the
Example 2.3. Consider the surface $X$ in the weighted projective space $\mathbb{P}_{2,7,1,13}$ given as the zero set of a weighted homogeneous trinomial equation:

$$X = V(T_{o1}^7 + T_{12}^2 + T_{21}T_{22}) \subseteq \mathbb{P}_{2,7,1,13},$$

where each of the variables appears in exactly one monomial as indicated by the double-indexing $T_{ij}$. Then $X$ comes with a $K^*$-action, given by

$$t \cdot [z] = [z_{o1}, z_{11}, t^{-1}z_{21}, tz_{22}].$$

The ambient space $\mathbb{P}_{2,7,1,13}$ is a toric variety. Its defining fan $\Sigma$ lives in $\mathbb{Z}^3$ and its rays are generated by the columns $v_{o1}, v_{11}, v_{21}$ and $v_{22}$ of the matrix

$$P = \begin{bmatrix}
-7 & 2 & 0 & 0 \\
-7 & 0 & 1 & 1 \\
-4 & 1 & 1 & 0
\end{bmatrix}.$$ 

This setting reflects the key features of the $K^*$-action on our surface $X$. For instance, setting $D_{ij} := X \cap V(T_{ij})$, we obtain the arms of $X$ as

$$A_0 = D_{o1}, \quad A_1 = D_{11}, \quad A_2 = D_{21} \cup D_{22}.$$ 

Moreover, the order $l_{ij}$ of the isotropy group of the general point in $D_{ij}$ shows up in the upper two rows of the matrix $P$, as we have

$$l_{o1} = 7, \quad l_{11} = 2, \quad l_{21} = 1, \quad l_{22} = 1.$$ 

Finally, $X$ inherits many geometric properties from its ambient space $Z := \mathbb{P}_{2,7,1,13}$. Most significantly, the Cox ring of $X$ is the factor algebra

$$\mathcal{R}(X) = \mathcal{R}(Z)/(g) = \mathbb{K}[T_{o1}, T_{11}, T_{21}, T_{22}]/(T_{o1}^7 + T_{12}^2 + T_{21}T_{22}),$$

where the grading of the Cox rings $\mathcal{R}(X)$ and $\mathcal{R}(Z)$ by the divisor class group $\text{Cl}(X) = \text{Cl}(Z) = \mathbb{Z}$ are given by

$$\deg(T_{o1}) = 2, \quad \deg(T_{o2}) = 7, \quad \deg(T_{11}) = 1, \quad \deg(T_{21}) = 13.$$ 

This picture extends as follows. The arbitrary rational projective $K^*$-surface $X$ comes embedded into a certain toric variety, is given by specific trinomial equations as above and the key features of the $K^*$-action as well as the geometry of $X$ can be extracted from the defining data. Here comes the construction provided in \cite{15, 18}; see also \cite{1} Sec. 3.4.

Construction 2.4. Fix $r \in \mathbb{Z}_{\geq 1}$, a sequence $n_0, \ldots, n_r \in \mathbb{Z}_{\geq 1}$, set $n := n_0 + \ldots + n_r$, and fix integers $m \in \mathbb{Z}_{\geq 0}$ and $0 < s < n + m - r$. The input data are matrices

$$A = [a_0, \ldots, a_r] \in \text{Mat}(2, r + 1; \mathbb{K}), \quad P = \begin{bmatrix}
L & 0 \\
d & d'
\end{bmatrix} \in \text{Mat}(r + s, n + m; \mathbb{Z}),$$

fans $\Sigma$ and $\Sigma$, it defines a toric morphism $p: \hat{Z} \to Z$, the good quotient for the action of the quasitorus $H = \ker(p) \subseteq T^r$ on $\hat{Z}$. 

Remark 2.2. Construction 2.1 allows to put hands on the points of a toric variety: every $x \in Z$ can be written as $x = p(z)$, where $z \in \hat{Z}$ is a point with closed $H$-orbit in $\hat{Z}$. Such a presentation is unique up to multiplication by elements of $H$ and we call $z = (z_1, \ldots, z_r)$ Cox coordinates for the point $x \in Z$.

We will use the Cox ring based approach to torus actions as developed for the case of rational $T$-varieties of complexity one in \cite{15, 18}, and, more recently, in widest possible generality in \cite{16}. Let us first have a look at an example, showing some of the main ideas.
where $A$ has pairwise linearly independent columns and $P$ is built from an $(s \times n)$-block $d$, an $(s \times m)$-block $d'$ and an $(r \times n)$-block $L$ of the form

$$L = \begin{bmatrix} -l_0 & l_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \cdots & l_r \end{bmatrix},$$

such that the columns $v_{ij}$, $v_k$ of $P$ are pairwise different primitive vectors generating $\mathbb{Q}^{r+s}$ as a cone. Consider the polynomial algebra

$$K[T_{ij}, S_k] := K[T_{ij}, S_k; 0 \leq i \leq r, 1 \leq j \leq n_i, 1 \leq k \leq m].$$

Denote by $\mathfrak{I}$ the set of all triples $I = (i_1, i_2, i_3)$ with $0 \leq i_1 < i_2 < i_3 \leq r$ and define for any $I \in \mathfrak{I}$ a trinomial

$$g_I := g_{i_1,i_2,i_3} := \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_i & a_i & a_i \end{bmatrix},$$

$$T_i^i := T_{i_1}^{l_{i_1}} \cdots T_{i_m}^{l_{i_m}}.$$ Consider the factor group $K := \mathbb{Z}^{n+m}/\text{im}(P^*)$ and the projection $Q: \mathbb{Z}^{n+m} \to K$. We define a $K$-grading on $K[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) := w_{ij} := Q(e_{i,j}), \quad \deg(S_k) := w_k := Q(e_k).$$

Then the trinomials $g_I$ just introduced are $K$-homogeneous, all of the same degree. In particular, we obtain a $K$-graded factor algebra

$$R(A, P) := K[T_{ij}, S_k] / \langle g_I; I \in \mathfrak{I} \rangle.$$

The ring $R(A, P)$ just constructed is a normal complete intersection ring and its ideal of relations is, for example, generated by $g_{i,i+1,i+2}$, where $i = 0, \ldots, r - 2$. The varieties $X$ with torus action of complexity one are constructed as quotients of $\text{Spec} R(A,P)$ by the quasitorus $H = \text{Spec} K[K]$. Each of them comes embedded into a toric variety.

**Construction 2.5.** Situation in Construction[2.4] Consider the common zero set of the defining relations of $R(A, P)$:

$$\bar{X} := V(g_I; I \in \mathfrak{I}) \subseteq \bar{Z} := K^{n+m}$$

Let $\Sigma$ be any fan in the lattice $N = \mathbb{Z}^{r+s}$ having the columns of $P$ as the primitive generators of its rays. Construction[2.4] leads to a commutative diagram

$$\begin{array}{ccc}
\bar{X} & \subseteq & \bar{Z} \\
\cup & & \cup \\
\bar{X} & \xrightarrow{\mathcal{J} H} & \bar{Z} \\
\uparrow & \uparrow & \uparrow \\
X & \xrightarrow{p} & Z
\end{array}$$

with a variety $X = X(A, P, \Sigma)$ embedded into the toric variety $Z$ associated with $\Sigma$. Dimension, divisor class group and Cox ring of $X$ are given by

$$\dim(X) = s + 1, \quad \text{Cl}(X) \cong K, \quad \mathcal{R}(X) \cong R(A, P).$$

The subtorus $T \subseteq \mathbb{T}^{r+s}$ of the acting torus of $Z$ associated with the sublattice $\mathbb{Z}^s \subseteq \mathbb{Z}^{r+s}$ leaves $X$ invariant and the induced $T$-action on $X$ is of complexity one.

**Remark 2.6.** In Construction[2.5] the group $H \cong \text{Spec} K[\text{Cl}(X)]$ is the characteristic quasitorus and $\bar{X} \cong \text{Spec} \mathcal{R}(X)$ is the total coordinate space of $X$. Moreover, $p: \bar{X} \to X$ is the characteristic space over $X$. 
Remark 2.7. As in the toric case, Construction 2.5 yields Cox coordinates for the points of \( X = X(A, P, \Sigma) \). Every \( x \in X \subseteq Z \) can be written as \( x = p(z) \), where \( z \in X \subseteq Z \) is a point with closed \( H \)-orbit in \( X \) and this presentation is unique up to multiplication by elements of \( H \).

Remark 2.8. We say that the matrix \( P \) from Construction 2.5 is irredundant if we have \( l_i q_i v_i \geq 2 \) for \( i = 0, \ldots, r \). In Construction 2.5, we may assume without loss of generality that \( P \) is irredundant. An \( X(A, P, \Sigma) \) with irredundant \( P \) is a toric variety if and only if \( r = 1 \) holds.

The results of [1, 15, 18] tell us in particular the following; see also [16] for a generalization to higher complexity.

Theorem 2.9. Every normal rational projective variety with a torus action of complexity one is equivariantly isomorphic to some \( X(A, P, \Sigma) \).

3. DEMAZURE ROOTS AND AUTOMORPHISMS

Here we present the necessary general background and facts on automorphisms of toric varieties and rational varieties with a torus action of complexity one.

The approach to automorphisms via Demazure roots involves locally nilpotent derivations. Let us briefly recall some basics from [13]. A derivation \( \delta: R \to R \) satisfying the Leibniz rule

\[
\delta(fg) = \delta(f)g + f\delta(g).
\]

A derivation \( \delta: R \to R \) is locally nilpotent if every \( f \in R \) admits an \( n \in \mathbb{N} \) with \( \delta^n(f) = 0 \). Any locally nilpotent derivation \( \delta: R \to R \) defines a representation

\[
\bar{\lambda}_\delta: K \to \text{Aut}(R), \quad \bar{\lambda}_\delta(s)(f) := \exp(s\delta)(f) := \sum_{k=0}^\infty \frac{s^k}{k!}\delta^k(f).
\]

In fact this yields a bijection between the locally nilpotent derivations of \( R \) and the rational representations of \( K \) by automorphisms of \( R \). Consequently,

\[
\bar{\lambda}_\delta: K \to \text{Aut}(\text{Spec}(R)), \quad s \mapsto \text{Spec}(\bar{\lambda}_\delta(s))
\]

is a group homomorphism and, by construction, each of the automorphisms \( \bar{\lambda}_\delta(s) \) of \( \text{Spec}(R) \) has \( \bar{\lambda}_\delta(s)^* = \lambda_\delta^* \) as its comorphism.

As for any complete rational variety, the automorphism group of a toric variety is an affine algebraic group. Its structure has been studied by Demazure [7] and Cox [4]. The following is a key concept.

Definition 3.1. Notation as in [21] A Demazure root at the primitive generator \( v_i \in N \) of \( \Sigma \) is an integral linear form \( u \in M \) satisfying the conditions

\[
\langle u, v_i \rangle = -1, \quad \langle u, v_j \rangle \geq 0 \text{ for all } j \neq i.
\]

Construction 3.2. Notation as in [21]. Let \( u \in M \) be a Demazure root at the primitive generator \( v_i \in N \) of \( \Sigma \). The associated locally nilpotent derivation \( \delta_u \) on \( K[T_1, \ldots, T_r] \) is defined by its values on the variables:

\[
\delta_u(T_j) := \begin{cases} T_i T^{P^*(u)}, & j = i, \\ 0, & j \neq i, \end{cases} \text{ where } T^{P^*(u)} = T_1^{(u,v_1)} \cdots T_r^{(u,v_r)}.
\]

Observe that \( \delta_u^2 = 0 \) holds. Moreover, we have \( Q(P^*(u)) = 0 \) and thus \( \delta_u \) preserves the \( K \)-grading of \( K[T_1, \ldots, T_r] \). The corresponding rational representation of \( K \) on \( K[T_1, \ldots, T_n] \) is given by

\[
\bar{\lambda}_u^*(s)(T_j) := \begin{cases} T_i + s T^{P^*(u)}, & j = i, \\ T_j, & j \neq i, \end{cases}
\]
Example 3.5. Consider the defining matrix in Example 2.3, that means $P_{Demazure}$.

Now, if $\lambda_u : \mathbb{K} \to \text{Aut}(\mathbb{Z})$ leaves $\mathbb{Z}$ invariant, for instance, if $\mathbb{Z}$ is complete, then $\lambda_u$ descends to a homomorphism $\lambda_u : \mathbb{K} \to \text{Aut}(\mathbb{Z})$, the root group associated with the Demazure root $u$. In Cox coordinates, we have

$$\lambda_u(s)(z) = z + sz\pi^r(u)e_i.$$ 

**Theorem 3.3.** See [2, Cor. 4.7]. Let $Z$ be a complete toric variety arising from a fan $\Sigma$ in a lattice $N$. Then $\text{Aut}(Z)^0$ is generated as group by the acting torus $T_Z$ of $Z$ and the images $\lambda_u(\mathbb{K})$, where $u$ runs through the Demazure roots of $\Sigma$.

The concept of Demazure roots was extended in [2] to the case of normal rational varieties $X$ with an effective torus action $T \times X \to X$ of complexity one. Let us recall the basic notions and facts.

**Definition 3.4.** See [2, Def. 5.2]. Let $P$ be a matrix as in Construction 2.4. Consider the columns $v_{ij}, v_k \in N = \mathbb{Z}^{r+s}$ of $P$ and the dual lattice $M$ of $N$.

(i) A vertical Demazure $P$-root is a tuple $(u, k_0)$ with a linear form $u \in M$ and an index $1 \leq k_0 \leq m$ satisfying

$$\langle u, v_{ij} \rangle \geq 0 \quad \text{for all } i, j,$$

$$\langle u, v_k \rangle \geq 0 \quad \text{for all } k \neq k_0,$$

$$\langle u, v_{k_0} \rangle = -1.$$

(ii) A horizontal Demazure $P$-root is a tuple $(u, i_0, i_1, C)$, where $u \in M$ is a linear form, $i_0 \neq i_1$ are indices with $0 \leq i_0, i_1 \leq r$, and $C = (c_0, \ldots, c_r)$ is a sequence with $1 \leq c_i \leq n_i$ such that

$$l_{ic_i} = 1 \quad \text{for all } i \neq i_0, i_1,$$

$$\langle u, v_{ic_i} \rangle = \begin{cases} 0, & i \neq i_0, i_1, \\ -1, & i = i_1, \end{cases}$$

$$\langle u, v_{ij} \rangle \geq \begin{cases} l_{ij}, & i \neq i_0, i_1, j \neq c_i, \\ 0, & i = i_0, i_1, j \neq c_i, \\ 0, & i = i_0, j = c_i, \end{cases}$$

$$\langle u, v_k \rangle \geq 0 \quad \text{for all } k.$$

**Example 3.5.** Consider the defining matrix $P$ of the $K^*$-surface discussed before in Example 2.3 that means

$$P = \begin{bmatrix} -7 & 2 & 0 & 0 \\ -7 & 0 & 1 & 1 \\ -4 & 1 & 1 & 0 \end{bmatrix}.$$ 

As $m = 0$, there are no vertical Demazure $P$-roots, but we have a horizontal Demazure $P$-root $(u, i_0, i_1, C)$ given by

$$u = (-1, 0, 1), \quad i_0 = 0, \quad i_1 = 1, \quad C = (1, 1, 2).$$

**Construction 3.6.** See [2, Constr. 3.4 and 5.7]. Let $A$ and $P$ be as in Construction 2.4. Given $i_0 \neq i_1$ with $0 \leq i_0, i_1 \leq r$ and $C = (c_0, \ldots, c_r)$ with $1 \leq c_i \leq n_i$ we define $\zeta = \zeta(i_0, i_1, C) = (\zeta_{ij}, \zeta_k) \in \mathbb{Z}^{n+m}$ by

$$\zeta_{ij} := \begin{cases} l_{ij}, & i \neq i_0, i_1, j \neq c_i, \\ -1, & i = i_1, j \neq c_i, \\ 0, & i = i_1, j = c_i, \end{cases} \quad \zeta_k := 0, k = 1, \ldots, m.$$
Moreover, to \( u \in M \) and the lattice vectors \( \zeta \in \mathbb{Z}^{n+m} \) just introduced we assign the following monomials

\[
h^u = \prod_{i,j} T_{ij}^{(u_{ij})} \prod_k S_k^{(u_{jk})}, \quad h^\zeta := \prod_{i,j} T_{ij}^{(\zeta_{ij})} \prod_k S_k^{(\zeta_k)}.
\]

Every Demazure \( P \)-root \( \kappa \) defines a locally nilpotent derivation \( \delta_\kappa \) on \( \mathbb{K}[T_{ij}, S_k] \). If \( \kappa = (u, k_0) \) is vertical, then one sets

\[
\delta_\kappa(T_{ij}) := 0 \quad \text{for all } i,j, \quad \delta_\kappa(S_k) := \begin{cases} h^u S_{k_0}, & k = k_0, \\ 0, & k \neq k_0. \end{cases}
\]

If \( \kappa = (i_0, i_1, C) \) is horizontal, then there is a unique vector \( \beta = \beta(A, i_0, i_1) \) in the row space of \( A \) with \( \beta_{i_0} = 0, \beta_{i_1} = 1 \) and one sets

\[
\delta_\kappa(T_{ij}) := \begin{cases} \beta \frac{\partial^u}{\partial T_{ij}} \prod_{k \neq i_0} \frac{\partial T_{ik}}{\partial T_{j1}}, & j = c_i, \\ 0, & j \neq c_i, \end{cases} \quad \delta_\kappa(S_k) := 0, \quad k = 1, \ldots, m.
\]

In both cases, the derivation \( \delta_\kappa \) respects the \( K \)-grading. Moreover, \( \delta_\kappa \) leaves the ideal of defining relations of \( R(A, P) \) invariant and thus induces a locally nilpotent derivation on \( R(A, P) \). This gives us

\[
\lambda_\kappa := \tilde{\lambda}_\kappa : \mathbb{K} \rightarrow \text{Aut}(\bar{X}),
\]

where \( \tilde{\lambda}_\kappa \) is the additive one-parameter group associated with the locally nilpotent derivation \( \delta_\kappa \) of \( R(A, P) \). If \( \bar{X} \) is invariant under \( \lambda_\kappa \), for example, if \( X \) is complete, then the root group associated with \( \kappa \) is

\[
\lambda_\kappa := \lambda_{\tilde{\lambda}_\kappa} : \mathbb{K} \rightarrow \text{Aut}(X).
\]

**Example 3.7.** We continue the discussion started in [23] and [20]. Recall that the \( \mathbb{K}^* \)-surface \( X \) comes embedded into a weighted projective space \( Z \) via

\[
X = V(T_{01}^7 + T_{11}^2 + T_{21}T_{22}) \subseteq \mathbb{P}_{2, 7, 1, 13} = Z.
\]

We determine the root group \( \lambda_\kappa : \mathbb{K} \rightarrow \text{Aut}(X) \) arising from the horizontal Demazure-\( P \)-root \( \kappa = (u, i_0, i_1, C) \) given by

\[
u = (-1, 0, 1), \quad i_0 = 0, \quad i_1 = 1, \quad C = (1, 1, 2).
\]

First we have to write down the monomials \( h^u \) and \( h^\zeta \) and the vector \( \beta \) from Construction 3.6. These are

\[
h^u = T_{01}^3 T_{11}^{-1} T_{21}, \quad h^\zeta = T_{11}^{-1} T_{21}, \quad \frac{h^u}{h^\zeta} = T_{01}^3, \quad \beta = (0, 1, -1).
\]

Next we describe the derivation \( \delta_\kappa : R(A, P) \rightarrow R(A, P) \). It is determined by its values on the variables \( T_{ij} \), which in turn are given as

\[
\delta_\kappa(T_{01}) = 0, \quad \delta_\kappa(T_{11}) = T_{01} \frac{\partial T_{11}}{\partial T_{22}} = T_{01}^3 T_{21},
\]

\[
\delta_\kappa(T_{21}) = 0, \quad \delta_\kappa(T_{22}) = -T_{01} T_{01} \frac{\partial T_{21}}{\partial T_{11}} = -2 T_{01}^3 T_{11}.
\]

For computing the exponential map, we have to evaluate the powers of \( \delta_\kappa \) on the variables. For \( T_{11} \) and \( T_{22} \) this needs further computation:

\[
\begin{align*}
\delta_\kappa^2(T_{11}) &= \delta_\kappa(T_{01}^3 T_{21}) = T_{01}^3 \delta_\kappa(T_{21}) = 0, \\
\delta_\kappa^2(T_{22}) &= \delta_\kappa(-T_{01}^3 T_{11}) = -2 T_{01}^3 \delta_\kappa(T_{11}) = -2 T_{01}^6 T_{21}, \\
\delta_\kappa^3(T_{22}) &= \delta_\kappa(-2 T_{01}^3 T_{21}) = -2 T_{01}^6 \delta_\kappa(T_{21}) = 0.
\end{align*}
\]
Using this, we directly obtain the comorphism \( \hat{\lambda}_u(s) = \exp(s\delta_u) \). On the variables \( T_{ij} \) it is given by

\[
\begin{align*}
T_{01} &\mapsto T_{01}, \\
T_{11} &\mapsto T_{11} + sT_{01}^3 T_{21}, \\
T_{21} &\mapsto T_{21}, \\
T_{22} &\mapsto T_{22} - 2sT_{01}^3 T_{11} - s^2 T_{01}^6 T_{21}.
\end{align*}
\]

Consequently, we can represent each automorphism \( \lambda_u(s) : X \to X \), where \( s \in \mathbb{K} \), explicitly in Cox coordinates as

\[
[z_{01}, z_{11}, z_{21}, z_{22}] \mapsto [z_{01}, z_{11} + s z_{01}^3 z_{21}, z_{21}, z_{22} - 2sz_{01}^3 z_{11} - s^2 z_{01}^6 z_{21}].
\]

One of the central ingredients of our article is the following result on automorphisms of varieties with a torus action of complexity one.

**Theorem 3.8.** See [2] Thm. 5.5 and Cor. 5.11. Let \( X = X(A, P, \Sigma) \) be a complete variety arising from Construction 2.5. Then \( \text{Aut}(X)^0 \) is generated as a group by the acting torus \( TX \) of \( X \) and the root groups associated with the Demazure-\( P \)-roots.

Every Demazure \( P \)-root in the sense of Definition 3.3 also hosts a Demazure root in the sense of Definition 3.4. We spend a few words on the relations among the associated automorphisms.

**Remark 3.9.** Consider a complete variety \( X = X(A, P, \Sigma) \) and its ambient toric variety \( Z \) as in Construction 2.5.

(i) Let \( \kappa = (u, k_0) \) be a vertical Demazure \( P \)-root. Then \( u \) is a Demazure root at \( v_{k_0} \), each \( \lambda_u(s) \in \text{Aut}(Z) \) leaves \( X \subseteq Z \) invariant and the restriction of \( \lambda_u(s) \) to \( X \) equals \( \lambda_u(s) \in \text{Aut}(X) \).

(ii) Let \( \kappa = (u, i_0, i_1, C) \) be a horizontal Demazure \( P \)-root. Then \( u \) is a Demazure root at \( v_{i_0} C \). In general, the automorphism \( \lambda_u(s) \in \text{Aut}(Z) \) does not leave \( X \subseteq Z \) invariant.

**Example 3.10.** Consider once more the \( \mathbb{K}^* \)-surface \( X \subseteq Z = \mathbb{P}^2 \cdot 1 \cdot 13 \) from 2.3. As seen in 3.3 we have the horizontal Demazure-\( P \)-root \( \kappa = (u, i_0, i_1, C) \), where

\[
u = (-1, 0, 1), \quad i_0 = 0, \quad i_1 = 1, \quad C = (1, 1, 2).
\]

In Example 3.4 we computed the associated root group. In Cox coordinates the automorphisms \( \lambda_u(s) \) are given by

\[
[z_{01}, z_{11}, z_{21}, z_{22}] \mapsto [z_{01}, z_{11} + s z_{01}^3 z_{21}, z_{21}, z_{22} - 2sz_{01}^3 z_{11} - s^2 z_{01}^6 z_{21}].
\]

The linear form \( u \) also defines a Demazure root at \( v_{i_1} \) for \( Z \) in the sense of Definition 3.4. By Construction 3.2 the corresponding automorphisms are

\[
\lambda_u(s) : Z \to Z, \quad [z_{01}, z_{11}, z_{21}, z_{22}] \mapsto [z_{01}, z_{11} + s T_{01}^3 T_{21}, z_{21}, z_{22}].
\]

Observe that these ambient automorphisms do not leave the surface \( X \subseteq Z \) invariant. For instance, we have

\[
x = [1, 0, -1, 1] \in X, \quad \lambda_u(1)(x) = [1, 1, -1, 1] \notin X.
\]

The toric variety \( Z = \mathbb{P}^2 \cdot 1 \cdot 13 \) admits two further Demazure roots, each at the primitive ray generator \( v_{22} \), namely

\[
u' := (0, -1, 1), \quad \nu'' := (-1, -1, 2).
\]

The corresponding derivations vanish at all variables \( T_{ij} \) except for \( T_{22} \), and on \( T_{22} \) the evaluations are given as

\[
\delta_{\nu'}(T_{22}) = T_{01}^3 T_{11}, \quad \delta_{\nu''}(T_{22}) = T_{01}^6 T_{21}.
\]
With the associated automorphisms of $Z$, we can represent $\lambda_u(s)$ on $X$ as a composition of ambient automorphisms

$$\lambda_u(s) = \lambda_u(s) \circ \lambda_{u^*}(-2s) \circ \lambda_{u^*}(-s^2)|_X$$

4. Automorphisms of complete toric surfaces

We consider the toric variety arising from a complete fan and investigate groups of automorphisms generated by the acting torus and root groups arising from Demazure roots at one or two generators. The results are given in Propositions 4.2 and 4.3. As an independent application we present in Proposition 4.3 the automorphism groups of the projective toric surfaces.

**Reminder 4.1.** Let $G,H$ be groups and $\varphi : H \rightarrow \text{Aut}(G)$ a homomorphism. The semidirect product is the set $G \times H$ together with the group law

$$(g,h) \cdot (g',h') := (g\varphi(h)(g'),hh').$$

The notation for the semidirect product is $G \rtimes H$ and we call $\varphi$ the twisting homomorphism. Observe that $G = G \times \{e_H\}$ is a normal subgroup in $G \rtimes H$.

**Proposition 4.2.** Let $\Sigma$ be a complete fan in $\mathbb{Z}^n$, denote by $P = [v_1, \ldots, v_r]$ the generator matrix and let $Z$ be the associated toric variety. Moreover, fix $0 \leq i_0 \leq r$ and let $u_1, \ldots, u_r$ be pairwise distinct Demazure roots at $v_{i_0}$.

(i) Let $1 \leq j \leq k \leq \rho$. Then $\delta_{u_j} \delta_{u_k} = \delta_{u_k} \delta_{u_j} = 0$ holds. In particular, for any two $s_j, s_k \in \mathbb{K}$, we have

$$\tilde{\lambda}_{u_j}(s_j)^* \tilde{\lambda}_{u_k}(s_k)^* = \tilde{\lambda}_{u_k}(s_k)^* \tilde{\lambda}_{u_j}(s_j)^*$$

(ii) Let $U \subseteq \text{Aut}(Z)$ be the subgroup generated by the root groups $\lambda_{u_1}, \ldots, \lambda_{u_r}$. Then we have an isomorphism of algebraic groups

$$\Theta : \mathbb{K}^\rho \rightarrow U, \quad (s_1, \ldots, s_\rho) \mapsto \lambda_{u_1}(s_1) \cdots \lambda_{u_\rho}(s_\rho).$$

(iii) The acting torus $\mathbb{T}^n \subseteq \text{Aut}(Z)$ normalizes $U \subseteq \text{Aut}(Z)$ and we have an isomorphism of algebraic groups

$$\Psi : \mathbb{K}^\rho \rtimes \mathbb{T}^n \rightarrow U \mathbb{T}^n, \quad (s,t) \mapsto \Theta(s)t,$$

where the twisting homomorphism $\psi : \mathbb{T}^n \rightarrow \text{Aut}(\mathbb{K}^\rho)$ sends $t \in \mathbb{T}^n$ to the diagonal matrix $\text{diag}(\chi^{u_1}(t), \ldots, \chi^{u_r}(t))$.

In the proof and also later, we will make use of the following fact; see for instance [3] Lemma 1.

**Lemma 4.3.** Consider a complete fan $\Sigma$ in $\mathbb{Z}^n$, the associated toric variety $Z$ and a Demazure root $u$ of $\Sigma$. Then, for all $t \in \mathbb{T}^n$ and $s \in \mathbb{K}$, the root group $\lambda_u$ associated with $u$ satisfies $t^{-1}\lambda_u(s)t = \lambda_u(\chi^u(t)s)$.

**Proof of Proposition 4.2.** We prove (i). Due to the definition of $\delta_{u_j}$ and $\delta_{u_k}$ given in Construction 3.2, we have $\delta_{u_k}(T_i) = \delta_{u_i}(T_i) = 0$ for all $i \neq i_0$. We conclude

$$\delta_{u_j} \delta_{u_k}(T_i) = 0 = \delta_{u_k} \delta_{u_j}(T_i),$$

whenever $i \neq i_0$. Moreover, using the Leibniz rule, we compare the evaluations at $T_{i_0}$ and obtain

$$\delta_{u_j} \delta_{u_k}(T_{i_0}) = \delta_{u_j} \prod_{i \neq i_0} T_i^{(u_k,v_i)} = 0 = \delta_{u_k} \prod_{i \neq i_0} T_i^{(u_j,v_i)} = \delta_{u_k} \delta_{u_j}(T_{i_0}).$$

As $\delta_j$ and $\delta_k$ commute, we can apply the homomorphism property of the exponential map which yields

$$\tilde{\lambda}_{u_j}(s_j)^* \tilde{\lambda}_{u_k}(s_k)^* = \exp(s_j \delta_{u_j} + s_k \delta_{u_k}) = \exp(s_k \delta_{u_k} + s_j \delta_{u_j}) = \tilde{\lambda}_{u_k}(s_k)^* \tilde{\lambda}_{u_j}(s_j)^*.$$
This proves (i). As a consequence, \( \lambda_u \) and \( \lambda_u \) commute. Thus, the map \( \Theta \) from (ii) is a homomorphism. Let us see why \( \Theta \) is injective. Assume
\[
\Theta(s_1, \ldots, s_p) = \lambda_u(s_1) \cdots \lambda_u(s_p) = \text{id}_Z.
\]
The task is to show \( s_1 = \ldots = s_p = 0 \). In Cox coordinates, the automorphism \( \vartheta := \Theta(s_1, \ldots, s_p) \) of \( Z \) is given by
\[
\vartheta: z \mapsto \hat{z}, \quad \hat{z}_i = \begin{cases} z_{i_0} + s_1 z^{P^*} u_1 z_{i_0} + \ldots + s_p z^{P^*} u_p z_{i_0}, & i = i_0; \\ z_i, & i \neq i_0. \end{cases}
\]

Consider the set \( \hat{Z}_0 \subseteq \hat{Z} \) obtained by removing all \( V(T_1, T_3) \) from \( \hat{Z} = \mathbb{K}^r \), where \( i \neq j \). Then \( \vartheta = \text{id}_Z \) implies that there is a morphism \( h: \hat{Z}_0 \to H = \ker(p) \) with
\[
\vartheta(z) = h(z) \cdot z \quad \text{for all } z \in \hat{Z}_0.
\]
As \( H \) is a quasitorus, \( h \) must be constant. Since \( H \) acts freely on \( V(T_1, \hat{Z}_0) \), we obtain \( h(z) = \epsilon_H \) for all \( z \in \hat{Z}_0 \). Consequently,
\[
s_1 z^{P^*} u_1 + \ldots + s_p z^{P^*} u_p = 0 \quad \text{for all } z \in \mathbb{T}^r.
\]
Since \( v_1, \ldots, v_r \) generate \( \mathbb{Q}^n \) as a vector space, the dual map \( P^* \) is injective. Thus, as \( u_1, \ldots, u_p \) are pairwise distinct, we can conclude
\[
s_1 = \ldots = s_p = 0.
\]
We verify (iii). Using (ii) and Lemma 14.5 we see that \( \mathbb{T}^n \) normalizes \( U \). In particular, \( U \cap \mathbb{T}^n \subseteq \text{Aut}(Z) \) is a closed subgroup. By the definition of \( \Theta \) and applying again Lemma 14.5 we obtain
\[
\Theta(s)\Theta(\varphi(t)(s')) = \Theta(s)\Theta(\chi^{u_1}(t)s_1' \cdot \ldots \cdot \chi^{u_\alpha}(t)s_\alpha') = \Theta(s)\varphi(t)\Theta(s)\varphi(t).
\]
We conclude that \( \Psi \) is a group homomorphism. Since \( U \) is unipotent and every \( t \in \mathbb{T} \) is semisimple, we have \( U \cap \mathbb{T}^n = \{\text{id}_Z\} \). Thus, \( \Psi \) injective. Using (ii) again, we see that \( \Psi \) is surjective. \( \square \)

**Proposition 4.4.** Let \( \Sigma \) be a complete fan with primitive generators \( v_1, \ldots, v_r \) and \( Z \) the associated toric variety. For \( i_0 \neq i_1 \) and \( \varepsilon \geq 0 \), let \( u_\varepsilon \) be Demazure roots at \( v_{i_0}, v_{i_1} \) respectively, such that \( \langle u, v_{i_0} \rangle = 0 \) and \( \langle u_\varepsilon, v_{i_0} \rangle = \varepsilon \).

(i) For \( \mu = 0, \ldots, \varepsilon \), set \( u_\mu \varepsilon := u_\varepsilon + (\varepsilon - \mu)u \). Then each \( u_\mu \varepsilon \) is a Demazure root at \( v_{i_1} \) with \( \langle u_\mu, v_{i_1} \rangle = \mu \) and for every \( 0 \leq \alpha \leq \varepsilon \) we have
\[
\lambda_u(\lambda_\varepsilon(\mu)q) = \lambda_u(q) \prod_{\mu=0}^{\alpha} \lambda_\mu(u_\alpha(q) q^{\alpha-\mu}).
\]

(ii) Let \( U, V \subseteq \text{Aut}(Z) \) be the subgroups generated by \( \lambda_u \), \( \ldots, \lambda_u \), and by \( \lambda_u \), respectively. Then \( V \) normalizes \( U \) and
\[
\Phi: \mathbb{K}^{\varepsilon+1} \times \mathbb{K} \to UV, \quad (s_0, \ldots, s_\varepsilon, q) \mapsto \lambda_u(s_0) \cdots \lambda_u(s_\varepsilon) \lambda_u(q)
\]
is an isomorphism of algebraic groups, where the twisting homomorphism \( \varphi: \mathbb{K} \to \text{Aut}(\mathbb{K}^{\varepsilon+1}) \) is given by the matrix valued map
\[
q \mapsto A(q) = (a_{\mu\alpha}(q)), \quad \text{where } a_{\mu\alpha}(q) = \begin{cases} \binom{\alpha}{\mu} q^{\alpha-\mu}, & \alpha \geq \mu; \\ 0, & \alpha < \mu. \end{cases}
\]

(iii) The acting torus \( \mathbb{T}^n \subseteq \text{Aut}(Z) \) normalizes \( UV \subseteq \text{Aut}(Z) \) and we have an isomorphism of algebraic groups
\[
\Psi: (\mathbb{K}^{\varepsilon+1} \times \mathbb{K}) \times \mathbb{T}^n \to UV\mathbb{T}^n, \quad (s, q, t) \mapsto \Phi(s, t)\varphi(t),
\]
where the twisting homomorphism \( \varphi: \mathbb{T}^n \to \text{Aut}(\mathbb{K}^{\varepsilon+1} \times \mathbb{K}) \) sends \( t \in \mathbb{T}^n \) to the diagonal matrix \( \text{diag}(\chi^{u_\varepsilon}(t), \ldots, \chi^{u_\varepsilon}(t)) \).
Proof. We show (i). The fact that each $u_{i}$ is a Demazure root as claimed is directly verified. Now, write $P = [v_{1}, \ldots, v_{r}]$. Then we compute in Cox coordinates:

\[
\lambda_{u}(q)(z)^{P^{*}(u)} = \left( z + qz_{i_{0}}z^{P^{*}(u)}e_{i_{0}} \right)^{P^{*}(u)} = (z_{i_{0}} \left( \prod_{i=0}^{r} z_{i}^{\langle u_{i}, v_{i} \rangle} \right) \prod_{i \neq i_{0}} z_{i}^{\langle u_{i}, v_{i} \rangle}) = \left( \sum_{\mu=0}^{r} \left( \frac{\alpha}{\mu} \right) q^{\alpha - \mu} \prod_{i=0}^{r} z_{i}^{\langle \mu - \alpha \rangle(u_{i}, v_{i})} \right) \prod_{i \neq i_{0}} z_{i}^{\langle u_{i}, v_{i} \rangle} = \sum_{\mu=0}^{r} \left( \frac{\alpha}{\mu} \right) q^{\alpha - \mu} z^{P^{*}(u_{i})}.
\]

Next, using $\langle u, v_{i} \rangle = 0$, we see that the monomial $z^{P^{*}(u)}$ does not depend on $z_{i}$, and thus, for any $a \in K$, obtain

\[
\lambda_{u}(q)(z + ae_{i_{1}}) = z + ae_{i_{1}} + q(z + ae_{i_{1}})^{P^{*}(u)} = z + qz^{P^{*}(u)} + ae_{i_{1}} = \lambda_{u}(q)(z) + ae_{i_{1}}.
\]

Set for short $t_{\mu} := s_{\mu}(a)q^{\alpha - \mu}$. Then, applying the two computations just performed, we can verify the displayed formula as follows:

\[
\lambda_{u_{\alpha}}(s)\lambda_{u_{\alpha}}(q)(z) = \lambda_{u_{\alpha}}(q)(z) + s_{z_{i_{1}}} \lambda_{u_{\alpha}}(q)(z)^{P^{*}(u_{0})}e_{i_{1}} = \lambda_{u_{\alpha}}(q)(z) + \sum_{\mu=0}^{r} t_{\mu}z_{i_{1}}z^{P^{*}(u_{\mu})}e_{i_{1}} = \lambda_{u_{\alpha}}(q)(\lambda_{u_{\alpha}}(to_{\alpha}))(z) + \sum_{\mu=1}^{r} t_{\mu}z_{i_{1}}z^{P^{*}(u_{\mu})}e_{i_{1}}
\]

where we used that all the $u_{\mu}$ are Demazure roots at $v_{i_{1}}$ and hence for every $\mu = 0, \ldots, \alpha - 1$ we have

\[
z_{i_{1}}z^{P^{*}(u_{\alpha+1})} = \lambda(t_{\mu})(z)_{i_{1}}\lambda(t_{\mu})(z)^{P^{*}(u_{\alpha})}.
\]

We turn to (ii). First note that $V$ normalizes $U$, since by the identity of (i) it normalizes each $\lambda_{u_{\alpha}}(K)$, where $\alpha = 0, \ldots, \varepsilon$. In particular, $UV$ is a closed subgroup of $\text{Aut}(Z)$. Now, for $s \in K^{\varepsilon+1}$ and $q \in K$, set

\[
\Psi(s) := \Psi(s, 0) \quad \Psi(r) := \Psi(0, q).
\]

By the nature of $\Psi$, we then have $\Psi(s, q) = \Psi(s)\Psi(q)$. Moreover, showing that $\Psi$ respects the multiplication of $(s, q)$ and $(s', q')$ means to verify

\[
\Psi(q)\Psi(s') = \Psi(\varphi(q)(s')) = \Psi(q)(s') = \Psi(q)(s')\Psi(0, q).
\]

For this, write $s' = s'_{0}e_{0} + \ldots + s'_{\varepsilon}e_{\varepsilon}$ with the canonical basis vectors $e_{i} \in K^{\varepsilon+1}$. Then for each $s'_{\varepsilon}e_{i_{1}}$, the above equality is a direct consequence of the definition of $\varphi$ and the identity provided by (i). Thus, $\Psi$ is a homomorphism.

Proposition 4.2 tells us that $\Psi$ maps $K^{\varepsilon+1}$ isomorphically onto $U$ and $K$ isomorphically onto $V$. In particular, $\Psi$ is surjective. Now consider an element
(s, q) ∈ ker(Ψ) with s ≠ 0 and q ≠ 0. As in the proof of Proposition 4.2, we work in Cox coordinates. The element (s, q) restricted to the identity on V(T_{i_0}, T_{i_1}) ⊆ Z by our assumptions, v_{i_0} and v_{i_1} form part of a lattice basis of Z^n and thus H acts freely on V(T_{i_0}, T_{i_1}); see [H Prop. 2.1.4.2]. We conclude s = 0 and q = 0.

Finally we note that the verification of Assertion (iii) runs exactly as for the corresponding statement of Proposition 4.2.

\[\square\]

We enter the surface case. The first step towards our description of the automorphism groups is combinatorial: we specify the fans admitting Demazure roots at two or more primitive generators.

Proposition 4.5. Let N and M be mutually dual two-dimensional lattices. Consider distinct primitive vectors v, v', w, w' ∈ N and u, u' ∈ M such that
\[ \langle u, v \rangle = -1, \quad \xi := \langle u, v' \rangle \geq 0, \quad \langle u, w \rangle \geq 0, \quad \langle u, w' \rangle \geq 0, \]
\[ \langle u', v' \rangle = -1, \quad \xi' := \langle u', v' \rangle \geq 0, \quad \langle u', w \rangle \geq 0, \quad \langle u', w' \rangle \geq 0. \]

Assume ξ = 0 or ξ' = 0. Then we have w ∈ cone(−v, −v') and, choosing suitable Z-linear coordinates on N, we achieve
\[ v = (1, 0), \quad v' = (0, 1), \quad u = (-1, \xi), \quad u' = (\xi', -1). \]

Assume ξ, ξ' > 0, w \not\in cone(v, v') and w' \neq w. Then each of u, u' annihilates w and w'. Choosing suitable Z-linear coordinates on N and b ∈ Z_{≥ 1}, we have
\[ v = (1, 0), \quad u = (-1, 1), \quad w = (-1, -1), \]
\[ v' = (b - 1, b), \quad u' = (1, -1), \quad w' = (1, 1). \]

Proof. By assumption v and v' generate N_Q as a vector space and w is not a multiple of one of v, v'. Thus, we can write w = −ηv − η'v' with η, η' ∈ Q. Evaluating the linear forms u and u' yields
\[ \eta - \xi' \eta' = \langle u, w \rangle \geq 0, \quad \eta' - \xi \eta = \langle u', w \rangle \geq 0. \]

Assume ξ\xi' = 0. Then η, η' ≥ 0, hence w ∈ cone(−v, −v'). Moreover, (v, v') is a basis of N as it is sent via (u, u') to a basis of Z^2. Clearly, (v, v') provides the desired coordinates. Now assume ξ, ξ' > 0 and w \not\in cone(v_1, v_2). Then
\[ \eta \geq \xi' \eta' \geq \xi \xi' \eta > 0, \quad \eta' \geq \xi' \eta \geq \xi \xi' \eta' > 0. \]

We conclude ξ = ξ' = 1 and η = η'. This implies u' = −u. Consequently, each of the linear forms u and u' annihilates w and w'. With respect to suitable Z-linear coordinates on N, we have
\[ v = (1, 0), \quad v' = (a, b), \text{ where } 0 \leq a < b. \]

Then \langle u, v \rangle = -1 implies u = (-1, x) with x ∈ Z. Moreover, \langle u, v' \rangle = 1 gives us bx = a + 1. Because of b > a, we must have x = 1. Consequently, b = a + 1 ≥ 1 holds. We conclude
\[ u = (-1, 1), \quad u' = (1, -1), \quad w = (-1, -1), \quad w' = (1, 1). \]

\[ \square\]

Corollary 4.6. Let Σ be a complete fan in Z^2 and P = [v_1, ..., v_r] the generator matrix of Σ. Let Demazure roots m_1 at v_1 and m_2 at v_2 be given.

(i) Assume that \{m_1, v_2\} = 0 or \{m_2, v_1\} = 0 holds. Then, with respect to suitable Z-linear coordinates, we have
\[ v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_i \in cone(-v_1, -v_2), \quad i = 3, ..., r. \]
and, denoting by $\xi(\Sigma)$ the minimum of the slopes of the lines $Qv_3, \ldots, Qv_r$, the Demazure roots of $\Sigma$ at $v_1$ and $v_2$ are given as

$$u = (-1,0), \quad u_\xi = (\xi,-1), \quad 0 \leq \xi \leq \xi(\Sigma).$$

Assume in addition that some $v_i$ with $i \geq 3$ admits a Demazure root $u'$.

Then, according to the value of $\xi(\Sigma)$ and suitably renumbering, we have

$$\xi(\Sigma) = 0 : \quad v_3 = (-1,0), \quad u' = (1,0), \quad v_4 = (0,-1), \quad u'' = (0,1),$$

$$\xi(\Sigma) > 0 : \quad v_3 = (-1,-\xi(\Sigma)), \quad u' = (1,0), \quad v_4 = (0,-1)$$

as the lists of the remaining primitive generators and the remaining Demazure roots of the fan $\Sigma$, where $v_4$ is optional in the second case.

Remark 4.7. The toric surfaces behind the fans of Corollary 4.6 provided we have straightforward calculation. □

Proposition 4.8. Situation as in Corollary 4.6 (ii). Then for $r = 3$ and $r = 4$ the following holds. For $0 \leq \xi \leq b$, the root subgroups given by $u$, $u'$ and $u_\xi$ satisfy

$$\lambda_{u_\xi}(r)\lambda_u(s)\lambda_{u'}(s') = \lambda_u(s)\lambda_{u'}(s') \prod_{\mu=0}^{b} \lambda_{u_{\mu}} \left( r \sum_{\mu=0}^{\min(\nu,\xi)} \left( \begin{array}{c} \xi \\ \mu \end{array} \right) \left( \begin{array}{c} b-\mu \\ \nu-\mu \end{array} \right) s^{\xi-\mu}(s')^{\nu-\mu} \right).$$

Let $U \subseteq \text{Aut}(X)$ be the subgroup generated by $\lambda_{u_0}, \ldots, \lambda_{u_{b-1}}, \lambda_u$ and $\lambda_{u'}$. Then we have an isomorphism of algebraic groups

$$\Psi_2: \mathbb{K}^{b+1} \times \mathbb{K}^2, \quad U, \quad (r_0, \ldots, r_b, s, s') \mapsto \lambda_u(s) \circ \lambda_{u'}(s') \lambda_{u_0}(r_0) \cdots \lambda_{u_b}(r_b),$$
Here, the twisting homomorphism \( \varphi_2 : \mathbb{K}^2 \to \text{Aut}(\mathbb{K}^{b+1}) \) is given by the matrix valued map \( (s, s') \mapsto B(s, s') = (b_{ij}(s, s')) \), where \( b_{ij}(s, s') = 0 \) for \( i < j \)

\[
b_{ji}(s, s') = \sum_{\mu=0}^{\min(j-1, i-1)} \left( \frac{i-1}{\mu} \right) \left( \frac{b-\mu}{b-j+1} \right) s^{i-1-\mu} (s')^{j-1-\mu} \quad \text{for } i \geq j.
\]

Proof. First recall that both \( u \) and \( u' \) evaluate to zero on \( e_3 \). Furthermore, one directly computes

\[
\langle u_\xi, v_1 \rangle = \xi, \quad u_\xi + (\xi - \mu)u = u_\nu, \quad \langle u_\mu, v_2 \rangle = b - \mu, \quad u_\mu + (b - \mu - \nu)u' = u_{b-\nu}.
\]

Thus, applying Proposition 4.4 to the pairs roots \( u, u_\xi \) and \( u', u_\xi \), we can verify the first assertion:

\[
\lambda_{u_\xi}(r) \lambda_u(s) \lambda_{u'}(s') = \lambda_u(s) \prod_{\mu=0}^{\xi} \lambda_{u_\mu} \left( r \left( \frac{\xi}{\mu} \right) s^{\xi-\mu} \right) \lambda_{u'}(s')
\]

\[
= \lambda_u(s) \lambda_{u'}(s') \prod_{\mu=0}^{b-\mu} \lambda_{b-\mu} \left( r \left( \frac{\xi}{\mu} \right) s^{\xi-\mu} \right) (s')^{b-\mu-\nu}
\]

\[
= \lambda_u(s) \lambda_{u'}(s') \prod_{\nu=0}^{\min(\nu, \xi)} \lambda_{\nu} \left( r \left( \frac{\xi}{\mu} \right) s^{\xi-\mu} \right) (s')^{\nu-\mu}
\]

\[
= \lambda_u(s) \lambda_{u'}(s') \prod_{\mu=0}^{\min(\nu, \xi)} \lambda_{\nu} \left( r \sum_{\mu=0}^{\xi} \left( \frac{\xi}{\mu} \right) (b-\mu) s^{\xi-\mu} (s')^{\nu-\mu} \right)
\]

For second assertion, we proceed similarly as in the proof of Proposition 4.4. The property that \( \Psi_2 \) is a group homomorphism reduces to the following, which is a consequence of the first assertion.

\[
\Psi_2(r) \Psi_2(s, s') = \Psi_2(\varphi_2(s, s')) \Psi_2(r)
\]

By definition, \( \Psi_2 \) is surjective. To see injectivity, take \( (s, s', r) \in \ker(\Psi_2) \). Then, working Cox coordinates, we find an \( h \in H \) with \( \Psi_2(s, s', r)(z) = h \cdot z \). We conclude \( s = 0, s' = 0 \) and \( r = 0 \), using the fact that the \( H \)-action is explicitly known in our situation.

establish.

\[\]

Proposition 4.9. Let \( \Sigma \) be a complete fan in \( \mathbb{Z}^2 \), denote by \( \ell(\Sigma) \) the number of primitive ray generators admitting a Demazure root and by \( \rho(\Sigma) \) the total number of Demazure roots of \( \Sigma \). Then the unit component of the automorphism group of the toric surface \( Z \) defined by \( \Sigma \) is given as follows:

| \( Z \) | \( \ell(\Sigma) \) | \( \text{Aut}^0(Z) \) | \( \rho(\Sigma) \) |
|---|---|---|---|
| \( - \) | \( 0 \) | \( \mathbb{K}^2 \) | \( 0 \) |
| \( - \) | \( 1 \) | \( \mathbb{K}^\rho \times_{\psi_1} \mathbb{T}^2 \) | \( \rho \) |
| \( - \) | \( 2 \) | \( (\mathbb{K}^\rho \times_{\psi_2} \mathbb{T}^2) \times_{\psi_3} \mathbb{T}^2 \) | \( \rho + 1 \) |
| \( \mathbb{P}_{1,1,b}, b \geq 2 \) | \( 3 \) | \( (\mathbb{K}^{\rho+1} \times_{\psi_1} \mathbb{T}^2) \times_{\psi_2} \mathbb{T}^2 \) | \( b + 3 \) |
| \( \mathbb{Z}_b, b \geq 1 \) | \( 3 \) | \( (\mathbb{K}^{\rho+1} \times_{\psi_1} \mathbb{T}^2) \times_{\psi_2} \mathbb{T}^2 \) | \( b + 3 \) |
| \( \mathbb{P}_2 \) | \( 3 \) | \( \text{PGL}_3(\mathbb{K}) \) | \( 6 \) |
| \( \mathbb{P}_1 \times \mathbb{P}_1 \) | \( 4 \) | \( \text{PGL}_2(\mathbb{K}) \times \text{PGL}_2(\mathbb{K}) \) | \( 4 \) |

The twisting homomorphisms \( \varphi \) and \( \varphi' \) are given by upper triangular matrices \( \varphi(s) = A = (a_{ij}) \) and \( \varphi'(s, s') = B = (b_{ij}) \), where for \( i \geq j \) we have

\[
a_{ij} = \left( \frac{i-1}{j-1} \right) s^{i-j}, \quad b_{ji} = \sum_{\mu=0}^{\min(i-1,j-1)} \left( \frac{i-1}{\mu} \right) \left( \frac{b-\mu}{b-j+1} \right) s^{i-1-\mu} (s')^{j-1-\mu}.
\]
Furthermore the twisting homomorphism for the toric factors are given by the following diagonal matrices:

\[
\begin{align*}
\psi_1(t_1, t_2) &= \text{diag}(t_1^{-1}, \ldots, t_1^{-1}) \\
\psi_2(t_1, t_2) &= \text{diag}(t_2^{-1}, t_1^{-1}t_2^{-1}, \ldots, t_1^{-1}t_2^{-1}, t_1^{-1}) \\
\psi_3(t_1, t_2) &= \text{diag}(t_2, t_1, t_1t_2, \ldots, t_1^{\rho}t_2t_1^{-1}, t_1t_2^{-1}, t_1^{-1}t_2).
\end{align*}
\]

Proof. The cases of \(\mathbb{P}_2\) and \(\mathbb{P}_1 \times \mathbb{P}_1\) are well known. So, let \(Z\) arise from a complete fan admitting Demazure roots and denote by \(U \subseteq Z\) the subgroup generated by all the root groups. If only one primitive generator of \(\Sigma\) allows Demazure roots, then Proposition 4.2 yields \(U \cong K^\rho\). If there are roots at exactly two primitive ray generators, Proposition 4.3 shows that \(U \cong K^\rho \times \mathbb{K}\) is as claimed. In the cases \(Z = \mathbb{P}_{1,1,6}\) or \(Z = \mathbb{Z}_6\), Proposition 4.8 tells us that \(U \cong K^\rho \times \mathbb{K}^2\) is as in the assertion. Finally, using Lemma 3.3 we see that also the twisting homomorphisms \(\psi_1, \psi_2, \psi_3\) are the right ones.

5. Representation via toric ambient automorphisms

The main result of this section, Theorem 5.4, is an important ingredient for the explicit handling of automorphisms in the subsequent sections; it represents automorphisms of a variety with torus action of complexity one as restrictions of explicitly accessible automorphisms of its ambient toric variety.

Construction 5.1. Situation as in Construction 2.1. Let \(k = (u, i_0, i_1, C)\) be a horizontal Demazure \(P\)-root. Define linear forms

\[
u u, v_j := \nu u + e'_i - e'_i \in M, \quad \nu = 1, \ldots, l_{1, c_i}, \quad i \neq i_0, i_1,
\]

where \(e'_0 = 0\) and \(e'_1, \ldots, e'_r \in M\) are the first \(r\) canonical basis vectors. So, the values of \(u_{\nu, i}\) on the columns \(v_j\) and \(v_k\) of \(P\) are given by

\[
\langle u_{\nu, i}, v_j \rangle = \begin{cases} 
\nu(u, v_j) + l_{ij}, & i = i_1, \\
\nu(u, v_j) - l_{ij}, & i = i, \\
\nu(u, v_j), & i \neq i_1, i.
\end{cases}
\]

Lemma 5.2. Let \(\delta\) and \(\delta'\) be derivations on the polynomial ring \(K[T_1, \ldots, T_n]\).

(i) We have \(\delta(a) = 0\) for every \(a \in K\).
(ii) If \(\delta(T_i) = \delta(0)\) holds for \(i = 1, \ldots, n\), then \(\delta = 0\) holds.
(iii) If \(\delta'(T_i) = \delta(T_i)\) holds for \(i = 1, \ldots, n\), then \(\delta' = \delta\) holds.
(iv) If \(\delta\delta(T_i) = 0\) holds for \(i = 1, \ldots, n\), then \(\delta'\delta = 0\) holds.
(v) If \(\delta\delta(T_i) = \delta\delta'(T_i)\) holds for \(i = 1, \ldots, n\), then \(\delta'\delta = \delta'\delta\) holds.

Proof. The statements are directly verified by using the defining properties of derivations and fact that \(K[T_1, \ldots, T_n]\) is generated as a \(K\)-algebra by the variables \(T_1, \ldots, T_n\).

Lemma 5.3. Consider linear forms \(u_{\nu, i}\) and \(u'_{\nu, i'}\) as in Construction 5.1 and the associated derivations \(\delta_{u_{\nu, i}}\) and \(\delta_{u'_{\nu, i'}}\) on \(K[T_{ij}, S_k]\) as provided by Construction 5.2.

(i) The linear form \(u_{\nu, i} \in M\) is a Demazure root at \(v_{i, c_i} \in N\) in the sense of Definition 5.1.
(ii) The derivation \(\delta_{u_{\nu, i}}\) annihilates all variables \(T_{ij}\) and \(S_k\) except \(T_{i, c_i}\), where we have

\[
\delta_{u_{\nu, i}}(T_{c_i}) = f_{u_{\nu, i}} T_{i, c_i}^{\nu} - v
\]

with a monomial \(f_{u_{\nu, i}}\) in the variables \(T_{ij}, S_k\) but not depending on any \(T_{i, c_i}\) with \(i \neq i_0\).
More explicitly, the comorphism satisfies
\[ \bar{\lambda}_{u,v} = \delta_u \delta_v \bar{\lambda}_{u,v} \] in particular, the derivations $\delta_{u,v}$ and $\bar{\lambda}_{u,v}$ commute.

For any two $s, s' \in \mathbb{K}$, the automorphisms $\bar{\lambda}_{u,v}(s)$ and $\bar{\lambda}_{u,v}^*(s')$ of $\mathbb{K}[T_1, \ldots, T_n]$ satisfy
\[ \bar{\lambda}_{u,v}^*(s') \circ \bar{\lambda}_{u,v}(s) = \exp(s' \delta_{u,v} + s \delta_{u,v}). \]
In particular, $\bar{\lambda}_{u,v}(s)$ and $\bar{\lambda}_{u,v}^*(s')$ as well as the associated automorphisms $\bar{\lambda}_{u,v}(s)$ and $\bar{\lambda}_{u,v}^*(s')$ of $\bar{Z}$ commute.

**Proof.** For (i), we need that $u_{i,c_1}^t$ evaluates to $-1$ on $v_{i,c_1}$ and is non-negative on all other columns of $P$. By Definition 3.3, the latter is clear for all $v_k$ and $v_{ij}$ with $(i, j) \neq (i_1, c_1)$ or $i \neq i_1$. We compute
\[
\langle u_{i,c_1}^t, v_{i,c_1} \rangle = \nu' \langle u, v_{i,c_1} \rangle + l_{i_1c_1} = l_{i_1c_1} - \nu,
\]
\[
\langle u_{i,c_1}^t, v_{i,c_1} \rangle = \nu' \langle u, v_{i,c_1} \rangle - l_{i_1c_1} \begin{cases} \geq (\nu - 1)l_{i_1c_1}, & j \neq c_1, \\ = -1, & j = c_1. \end{cases}
\]

We turn to (ii). We infer directly from Construction 3.2 that $\delta_{u,v}^t$ annihilates all variables $T_{i_1}$, $S_k$ except $T_{i_1}$, and satisfies
\[ \delta_{u,v}^t(T_{i_1}) = T_{i_1} \prod T_{i_1} \prod S_k^{(u_{i_1c_1},v_{i_1c_1})} = f_{u,v_{i_1}}(u_{i_1c_1}) - \nu, \]
where by the computation proving (i) the monomial $f_{u,v_{i_1}}$ depends neither on $T_{i_1}, c_1$, nor on $T_{i_1c_1}$ and the $T_{i_1}$ with $i \neq i_0, i_1, t$ have by Definition 3.3 the exponent
\[ \langle u_{i_1c_1}, v_{i_1c_1} \rangle = \nu' \langle u, v_{i_1c_1} \rangle = 0. \]

We show (iii). From (ii) we infer that $\delta_{u,v}^t, \delta_{u,v}^t$, as well as $\delta_{u,v}^t, \delta_{u,v}^t$, annihilate all variables $T_{i_1}$ and $S_k$; use Lemma 5.2 (ii) and (iii). Thus, Lemma 5.2 (v) gives
\[ \delta_{u,v}^t, \delta_{u,v}^t = \delta_{u,v}^t, \delta_{u,v}^t = 0. \]

To obtain (iv), first note that due to (iii), the linear endomorphisms $\delta_{u,v}^t$ and $s' \delta_{u,v}^t$ of $\mathbb{K}[T_1, \ldots, T_n]$ commute. Thus, the assertion follows from the definition $\bar{\lambda}_{s}(s)^* := \exp(s \delta)$ and the homomorphism property of the exponential series. \qed

**Theorem 5.4.** Let $X = X(A, P, \Sigma)$ be as in Construction 2.4 with $\Sigma$ complete and $\kappa = (u, i_0, i_1, C)$ a horizontal Demazure $P$-root. For $s \in \mathbb{K}$ set
\[ \alpha(s, \nu, \tau) := \beta_1 \left( l_{i_1c_1} \right)^{s'}, \quad \nu = 0, \ldots, 1, \quad \nu \neq i_0, i_1, \quad \nu = 1, \ldots, i_1c_1, \]
where $\beta$ is the unique vector in the row space of $A$ with $\beta_{i_0} = 0$ and $\beta_{i_1} = 1$. Consider the linear forms $w_{u,v}$ from Construction 2.7 and the automorphisms
\[ \varphi_u(s) := \prod_{i \neq i_0, i_1} \prod_{\nu = 1}^{l_{i_1c_1}} \lambda_{u,v}(\alpha(s, \nu, \tau)) \in \text{Aut}(Z). \]

Then the automorphism $\lambda_{s}(s)$ of $X$ can be presented as the restriction of an automorphism of $Z$ as follows:
\[ \lambda_{s}(s) = \lambda_{s}(s) \circ \varphi_u(s)|_X. \]
More explicitly, the comorphism satisfies $\bar{\lambda}_{s}(s)^*(S_k) = S_k$ and $\bar{\lambda}_{s}(s)^*(T_{i_1}) = T_{i_1}$, whenever $i = i_0$ or $j \neq i_1$ and in the remaining cases
\[ \bar{\lambda}_{s}(s)^*(T_{i_1c_1}) = T_{i_1c_1} + s \delta_u(T_{i_1c_1}), \]
\[ \bar{\lambda}_{s}(s)^*(T_{i_1c_1}) = T_{i_1c_1} + \sum_{\nu=1}^{l_{i_1c_1}} \alpha(s, \nu, \tau) \delta_{u,v}(T_{i_1c_1}). \]
Proof. By definition, the comorphism of $\lambda_\kappa(s)$ equals $\exp(s\delta_\kappa)$. In a first step, we compute the powers $\delta_\kappa^\nu$ occurring in the exponential series. We will make repeated use of the fact

$$\delta_{nu,v}(T_{ic_i}) = T_{ic_i}h^{nu}\frac{\partial l_{ij}}{T_i} = f_{nu,v}T_{ic_i}^{l_{ij} - v},$$

where $h^{nu}$ is as in Construction 3.9 and $f_{nu,v}$ is a monomial in the variables $T_{ij}$ and $S_k$ but not depending on any $T_{ic_i}$ with $i \neq i_0$; see Lemma 5.3 (ii). Now, recall from Construction 3.9 that, apart from the $T_{ic_i}$, all variables $T_{ij}$ and $S_k$ are annihilated by $\delta_\kappa$. Moreover, we have

$$\delta_\kappa(T_{ic_0}) = \beta_{i_0} = 0, \quad \delta_\kappa(T_{ic_1}) = \beta_{i_1}h^{nu}\frac{\partial l_{ij}}{T_i} = \beta_{i_1}T_{ic_1}h^{nu} = \delta_u(T_{ic_1}).$$

Since $\delta_u(T_{ic_1})$ does not depend on any $T_{ic_i}$ with $i \neq i_0$, we conclude $\delta_\kappa^0(T_{ic_1}) = 0$. Finally, for $i \neq i_0, i_1$, Construction 3.9 and the above formula for $\nu = 1$ give us

$$\delta_\kappa(T_{ic_i}) = \beta_i h^{nu}\frac{\partial l_{ij}}{T_i} = \beta_i l_{ic_i}h^{nu}\frac{\partial l_{ij}}{T_i} = \beta_i l_{ic_i} \delta_u(T_{ic_i}).$$

To evaluate higher powers of $\delta_\kappa$, we use the representation $\delta_{nu,v}(T_{ic_i}) = f_{nu,v}T_{ic_i}^{l_{ij} - v}$ given above. Applying the Leibniz rule yields

$$\delta_\kappa(\delta_{nu,v}(T_{ic_i})) = (l_{ic_i} - v)\delta_{nu+1,v}(T_{ic_i}).$$

Putting things together, we arrive at

$$\delta_\kappa^\nu(T_{ic_i}) = \beta_{l_{ic_i}} \frac{l_{ic_i}}{(l_{ic_i} - v)!} \delta_{nu,v}(T_{ic_i}).$$

In the next step, we compute the values of $\lambda_\kappa(s)^* = \exp(s\delta_\kappa)$ on the generators $T_{ij}$ and $S_k$. From above, we infer $\lambda_\kappa(s)^*(T_{ij}) = T_{ij}$, whenever $i = i_0$ or $j \neq c_i$. Moreover, we have

$$\lambda_\kappa^*(s)(T_{ic_1}) = T_{ic_1} + s\delta_u(T_{ic_1}).$$

Finally, for $i \neq i_0, i_1$, plugging the above representations of the $\delta_\kappa^\nu(T_{ic_i})$ into the exponential series gives the remaining statements on comorphisms:

$$\lambda_\kappa^*(s)(T_{ic_i}) = T_{ic_i} + \sum_{\nu=1}^{l_{ic_i}} \beta_{l_{ic_i}} \frac{(l_{ic_i} - \nu)}{\nu} \delta_{nu,v}(T_{ic_i}) = T_{ic_i} + \sum_{\nu=1}^{l_{ic_i}} \alpha(s,\nu,i)\delta_{nu,v}(T_{ic_i}).$$

We turn to $\lambda_u(s) = \lambda_u(s) \circ \varphi_u(s)|_x$. We verify the corresponding identity on $\lambda_\kappa$ and $\lambda_\kappa \circ \varphi_u$ by comparing the comorphisms. First recall from Construction 5.2 that we have

$$\lambda_\kappa^*(s) = \text{id} + s\delta_u,$$

where $\delta_u$ annihilates all variables $T_{ij}$ and $S_k$ except $T_{ic_1}$. Next note that $\varphi_u(s)$ doesn’t depend on the order of composition due to Lemma 5.2 (v). Moreover, Lemma 5.2 (iv) allows to compute

$$\varphi_u(s)^* = \prod_{i \neq i_0, i_1} \lambda_u(\alpha(s,\nu,i))^* = \text{id} + \sum_{i \neq i_0, i_1, \nu=1}^{l_{ic_1}} \alpha(s,\nu,i)\delta_{nu,v}.$$
Using $\delta_u(T_{i_1c_{i_1}}) = h^uT_{i_1c_{i_1}}$, where the latter monomial doesn’t depend on any $T_{i c_i}$ with $i \neq i_0$, we compute
\[ \varphi_u(s)^* \circ \lambda_u(s)^*(T_{i_1c_{i_1}}) = \varphi_u(s)^*(h^uT_{i_1c_{i_1}}) + \varphi_u(s)^*(h^uT_{i_1c_{i_1}}) = T_{i_1c_{i_1}} + \delta_u(T_{i_1c_{i_1}}). \]
Finally, for any $i \neq i_0, i_1$, we obtain
\[ \varphi_u(s)^* \circ \lambda_u(s)^*(T_{c_i}) = \varphi_u(s)^*(T_{c_i}) = T_{c_i} + \sum_{\nu=1}^{l_{i_1c_{i_1}}} \alpha(s, \nu, i)\delta_{\nu,i}(T_{c_i}). \]
Thus, comparing with the previously obtained values of $\bar{\lambda}_u(s)^*$ on the generators, we arrive at the identity $\hat{\lambda}_u(s)^* = \varphi_u(s)^* \circ \lambda_u(s)^*$ of isomorphisms, which in turn induces the desired equation $\lambda_u(s) = \lambda_u(s) \circ \varphi_u(s)$ on $Z$ and hence $X$. \hfill \square

6. Rational projective $K^*$-surfaces

We will use the approach provided by Constructions 2.4 and 2.5 producing all rational projective varieties with torus action of complexity one as $X = X(A, P, \Sigma)$. Recall that the defining $(r + s) \times (n + m)$ block matrix $P$ is of the form
\[ P = \begin{bmatrix} L & 0 \\ d & d' \end{bmatrix} = [v_{01}, \ldots, v_{0n}, \ldots, v_{r1}, \ldots, v_{rn}, v_1, \ldots, v_m], \]
where the columns $v_{ij}$ and $v_k$ are pairwise distinct primitive integral vectors generating $\mathbb{Q}^{r+s}$ as a vector space. In the case of a $K^*$-surface $X$, several aspects simplify. First, we have $s = 1$. Thus, the lower part $[d, d']$ of $P$ is just one row and $m \leq 2$ holds. Observe
\[ v_{0j} = (-l_{0j}, \ldots, -l_{0j}, d_{0j}), \quad v_{ij} = (0, \ldots, 0, l_{ij}, 0 \ldots, 0, d_{ij}), \quad i = 1, \ldots, r, \]
where $l_{ij}$ sits at the $i$-th place for $i = 1, \ldots, r$ and we always have $\gcd(l_{ij}, d_{ij}) = 1$. Moreover, we arrange $P$ to be slope ordered, that means that for each $0 \leq i \leq r$, we order the block $v_{i1}, \ldots, v_{im}$ of columns in such a way that
\[ m_{i1} > \ldots > m_{in}, \quad \text{where } m_{ij} := \frac{d_{ij}}{l_{ij}}. \]
Finally, in the surface case the defining fan $\Sigma$ of the ambient toric variety $Z$ is basically unique and needs no extra specification. More precisely, the rays of $\Sigma$ are the columns over the columns of $P$ and we always have the maximal cones
\[ \tau_{ij} := \text{cone}(v_{ij}, v_{ij+1}) \in \Sigma, \quad i = 0, \ldots, r, \quad j = 1, \ldots, n_i - 1. \]
Writing $v^+ := v_1 = (0, \ldots, 0, 1)$ and $v^- := v_2 = (0, \ldots, 0, -1)$ for the columns of $P$ that arise for $m = 1, 2$, the collection of maximal cones of $\Sigma$ is complemented depending on the value of $m$ as follows
\[ m = 2 : \quad \text{(p-p)} \quad \tau_i^+ := \text{cone}(v^+, v_{i1}) \]
\[ m = 1 : \quad \text{(p-e)} \quad \tau_i^+ := \text{cone}(v^+, v_{i1}) \]
\[ m = 0 : \quad \text{(e-e)} \quad \sigma_i^+ := \text{cone}(v_0, \ldots, v_{r1}) \]
\[ m = 0 : \quad \text{(e-e)} \quad \sigma^- := \text{cone}(v_0, \ldots, v_{r1}) \]

\[ \tau_i^- := \text{cone}(v_{in}, v^-) \]
\[ \sigma^- := \text{cone}(v_{in}, v^-) \]
In particular, the $\mathbb{K}^*$-surfaces delivered by Construction 2.5 only depend on the matrices $A$ and $P$, which allows us to denote them as $X = X(A, P)$. The $\mathbb{K}^*$-action on $X$ is given on the torus $\mathbb{T}^{r+1} \subseteq \mathbb{Z}$ by $t \cdot z = (z_1, \ldots, z_r, tz_{r+1})$.

**Remark 6.1.** Let $X = X(A, P)$ be a $\mathbb{K}^*$-surface as above. Then the fan $\Sigma$ of the ambient toric variety $Z$ of $X$ reflects the geometry of the $\mathbb{K}^*$-action on $X$, as outlined in the introduction, in the following way.

(i) If $P$ has a column $v^+$ or $v^-$, then the toric prime divisor on $Z$ corresponding to $\tau^+_i = \text{cone}(v^+_i)$, or $\tau^-_i = \text{cone}(v^-_i)$, cuts out a parabolic fixed point curve forming source or sink:

$$D^+ = (B^+ \cap X) \cup \{x^+_0\} \cup \ldots \cup \{x^+_r\},$$

$$D^- = (B^- \cap X) \cup \{x^-_0\} \cup \ldots \cup \{x^-_r\}.$$  

Here $B^+, B^- \subseteq Z$ denote the toric orbits corresponding to $\tau^+_i, \tau^-_i \in \Sigma$ and $x^+_i \in B^+_i \cap X, x^-_i \in B^-_i \cap X$ are the unique points in the intersections with the toric orbits $B^+_i, B^-_i \subseteq Z$ corresponding to $\tau^+_i, \tau^-_i \in \Sigma$.

(ii) If we have a cone $\sigma^+, \sigma^-$, resp. $\sigma^-, \sigma^+$, in $\Sigma$, then the associated toric fixed point $x^+, \text{resp. } x^-$, of $Z$ is an elliptic fixed point of the $\mathbb{K}^*$-action on $X$ forming the source, resp. the sink.

(iii) The toric prime divisor of $Z$ corresponding to the ray $\varrho_{ij} = \text{cone}(v_{ij})$ of $\Sigma$ cuts out the closure $D_{ij} \subseteq X$ of an orbit $\mathbb{K}^* \cdot x_{ij} \subseteq X$. If $P$ is irredundant, then the arms of $X$ are precisely

$$A_i = D_{ij} \cup \ldots \cup D_{in_i}, \quad i = 0, \ldots, r.$$  

The order of the isotropy group $K^*_x$ equals $l_{ij}$. The hyperbolic fixed point forming $D_{ij} \cap D_{ij+1}$ is the point cut out from $X$ by the toric orbit of $Z$ corresponding to the cone $\tau_{ij} \in \Sigma$.

**Definition 6.2.** For any rational projective $\mathbb{K}^*$-surface $X = X(A, P)$, we define the numbers

$$l^+ := l_0 l_1 \ldots l_r, \quad m^+ := m_0 + \ldots + m_r,$$

$$l^- := l_{0n} l_{1n} \ldots l_{rn}, \quad m^- := m_{0n} + \ldots + m_{rn}.$$  

**Remark 6.3.** Let $X = X(A, P)$. We have $l^+ m^+ \in \mathbb{Z}$ and if there is an elliptic fixed point $x^+ \in X$, then

$$\det(\sigma^+) := \det(v_{01}, \ldots, v_{r1}) = l^+ m^+ > 0.$$  

Similarly, $l^- m^- \in \mathbb{Z}$ holds and if there is an elliptic fixed point $x^- \in X$, then we obtain

$$\det(\sigma^-) := \det(v_{0n}, \ldots, v_{rn}) = l^- m^- < 0.$$  

Rational projective $\mathbb{K}^*$-surfaces turn out to be always $\mathbb{Q}$-factorial. That means in particular that intersection numbers are well defined. Let us recall [3, Cor. 5.4.4.2].
Remark 6.4. For $X = X(A, P)$, the self intersection numbers of the orbit closures $D_{ij} \subseteq X$ and possible parabolic fixed point curves $D^{+}, D^{-} \subseteq X$ are given by

$$D^2_{ij} = \begin{cases} \frac{1}{l_{ij}} \left( \frac{1}{m_i} - \frac{1}{m_j} \right), & (e-e), \\ 0, & (p-p), \\ \frac{1}{l_{ij}} m_j, & (e-p), \\ \frac{1}{l_{ij}} m_i, & (p-e), \\ \frac{1}{l_{ij}} \left( \frac{1}{m_i} - \frac{1}{m_j} \right), & (p-e), \\ \frac{1}{l_{ij}} \left( \frac{1}{m_i} - \frac{1}{m_j} \right), & (e-e), \end{cases}$$

for $n_i = 1$, $j = 1$ and $(e-e)$ or $(e-p),$ $j = 1$ and $(p-p)$ or $(e-p),$ $\frac{1}{l_{ij}} \left( \frac{1}{m_i} - \frac{1}{m_j} \right), 1 < j < n_i,$ for $n_i > 1$, $\frac{1}{l_{ij}} \left( \frac{1}{m_i} - \frac{1}{m_j} \right), j = n_i$ and $(p-p)$ or $(e-p),$ $\frac{1}{l_{ij}} \left( \frac{1}{m_i} - \frac{1}{m_j} \right), j = n_i$ and $(e-e)$ or $(e-p),$

$$(D^+)^2 = -m^+,$$

$$(D^-)^2 = m^-.$$

Recall that an irreducible curve $D$ on a normal projective surface $X$ is called contractible if there is a morphism $\pi: X \to X'$ mapping $D$ to a point $x' \in X'$ and inducing an isomorphism from $X \setminus D$ onto $X' \setminus \{x'\}$.

Remark 6.5. Consider $X = X(A, P)$. Then, provided that they are present, the vectors $v^+$ and $v^-$ satisfy the identities

$$l_{01}^{-1} v_{01} + \ldots + l_{r1}^{-1} v_{r1} = m^+ v^+, \quad l_{0n}^{-1} v_{0n} + \ldots + l_{rn}^{-1} v_{rn} = -m^- v^-.$$

Combining this with Remark 6.4, we rediscover that $D^+$ and $D^-$ are contractible if and only if they are of negative self intersection.

More generally, contractibility of invariant curves on rational normal projective $K^*$-surfaces is characterized as follows.

Remark 6.6. Consider $X = X(A, P)$. Given a column $v$ of $P$, let $D \subseteq X$ be the corresponding curve and $P'$ the matrix obtained from $P$ by removing $v$. Then the following statements are equivalent.

(i) The curve $D \subseteq X$ is contractible.

(ii) The matrices $A$ and $P'$ define a $K^*$-surface $X' = X(A, P')$.

(iii) We have $D^2 < 0$ for the self intersection number.

If one of these statements holds, then $D$ is contracted by the $K^*$-equivariant morphism $X \to X'$ induced by the map of fans $\Sigma \to \Sigma'$ and there is a unique cone $\sigma' \in \Sigma'$ containing $v$ in its relative interior.

We turn to singularities of $K^*$-surfaces $X = X(A, P)$. Note that due to normality of the surfaces, every singularity is a fixed point.

Definition 6.7. Let $X$ be a rational projective $K^*$-surface and $p: \hat{X} \to X$ its characteristic space. A point $x \in X$ is quasismooth if $x = p(z)$ holds for a smooth point $z \in \hat{X}$.

We characterize quasismoothness and smoothness of parabolic, hyperbolic and elliptic fixed points in terms of the defining matrix $P$ of $X = X(A, P)$. All the statements are direct consequences of the general (quasi-)smoothness criterion [16 Cor. 7.16].
Proposition 6.8. Consider $X = X(A,P)$. Then we have the following statements on (quasi-)smoothness of possible parabolic fixed points.

(i) All points of $B^+ \subseteq D^+$ are smooth and all points $x_i^+ \in D^+$ are quasi-smooth. Moreover, $x_i^+ \in D^+$ is smooth if and only if $l_{i1} = 1$ holds.

(ii) All points of $B^- \subseteq D^-$ are smooth and all points $x_i^- \in D^-$ are quasi-smooth. Moreover, $x_i^- \in D^-$ is smooth if and only if $l_{in_i} = 1$ holds.

Proposition 6.9. Consider $X = X(A,P)$. Then every hyperbolic fixed point of $X$ is quasismooth. Moreover, the hyperbolic fixed point corresponding to $\tau_{ij} \in \Sigma$ is smooth if and only if $l_{ij+1}d_{ij} - l_{ij}d_{ij+1} = 1$ holds.

Proposition 6.10. Assume that the $K^*$-surface $X = X(A,P)$ has an elliptic fixed point $x \in X$.

(i) If $x = x^+$, then $x$ is quasismooth if and only if there are $0 \leq \epsilon_0, \epsilon_1 \leq r$ with $l_{i1} = 1$ for every $i \neq \epsilon_0, \epsilon_1$.

(ii) If $x = x^+$, then $x$ is smooth if and only if there are $0 \leq \epsilon_0, \epsilon_1 \leq r$ with $l_{i1} = 1$ for every $i \neq \epsilon_0, \epsilon_1$ and

$$\det(\sigma^+) = l^+m^+ = l_{\epsilon_01}d_{\epsilon_11} + l_{\epsilon_11}d_{\epsilon_01} = 1.$$ 

(iii) If $x = x^-$, then $x$ is quasismooth if and only if there are $0 \leq \epsilon_0, \epsilon_1 \leq r$ with $l_{n_i} = 1$ for every $i \neq \epsilon_0, \epsilon_1$.

(iv) If $x = x^-$, then $x$ is smooth if and only if there are $0 \leq \epsilon_0, \epsilon_1 \leq r$ with $l_{n_i} = 1$ for every $i \neq \epsilon_0, \epsilon_1$ and

$$\det(\sigma^-) = l^-m^- = l_{n_{i0}}d_{n_1} + l_{i,n_1}d_{n_{i0}} = -1.$$ 

Definition 6.11. Given an elliptic fixed point $x \in X = X(A,P)$, we call the numbers $0 \leq \epsilon_0, \epsilon_1 \leq r$ from Proposition 6.10 leading indices for $x$.

As a consequence of Proposition 6.10, we obtain the following characterization of quasismoothness of $K^*$-surface $X$. We say that a singularity $x \in X$ is a toric surface singularity if there is a $K^*$-invariant open neighbourhood $x \in U \subseteq X$ such that $U$ is a toric surface. Recall from [5] that toric surface singularities are quotients of $K^2$ by finite cyclic groups.

Corollary 6.12. A rational projective $K^*$-surface is quasismooth if and only if it has at most toric surface singularities.

Proof. We may assume that our $K^*$-surface is given as $X = X(A,P)$. By normality, any singular point of $X$ is a $K^*$-fixed point. The parabolic and hyperbolic fixed point are toric surfaces singularities due to [19] Prop. 3.4.4.6.

Thus, we are left with discussing quasismooth elliptic fixed points. It suffices to consider $x^{-} \in X$. Let $0 \leq \epsilon_0, \epsilon_1 \leq r$ be leading indices for $x^{-}$. The cone $\sigma^- \in \Sigma$ defines affine open subsets

$$Z^- \subseteq Z, \quad X^- := X \cap Z^- \subseteq X.$$ 

Recall that $x^{-}$ is the toric fixed point of $Z^-$. Then $X^- \in X$ is the affine $K^*$-surface given by the data $A$ and $P^- = [v_{0m_0}, \ldots, v_{rn_r}]$ in the sense of [19] Constr. 1.6 and Cor. 1.9. Consider the defining relations

$$g_{i_1,i_2,i_3} := \det \begin{bmatrix} T_{i_1}^{l_{i11}} & T_{i_1}^{l_{i12}} & T_{i_1}^{l_{i13}} \\ a_{i1} & a_{i2} & a_{i3} \end{bmatrix}$$

of the Cox ring $R(X^-)$ of $X^-$. By Proposition 6.10 the point $x^{-}$ is quasismooth if and only if $l_{in_i} = 1$ for all $i \neq \epsilon_0, \epsilon_1$. The latter is equivalent to the fact that $R(X^-)$ is a polynomial ring. This in turn holds if and only if $X^-$ is an affine toric surface. □
Remark 6.13. Consider \( X = X(A, P) \). The canonical resolution of singularities \( X'' \rightarrow X \) from [11 Constr. 5.4.3.2] is obtained by the following two step procedure.

(i) Enlarge \( P \) to a matrix \( P' \) by adding \( e_{r+1} \) and \( -e_{r+1} \), if not already present. Then the surface \( X' := X(A, P') \) is quasismooth and there is a canonical morphism \( X' \rightarrow X \).

(ii) Let \( P'' \) be the slope ordered matrix having the primitive generators of the regular subdivision of \( \Sigma(P') \) as its columns. Then \( X'' := X(A, P'') \) is smooth and there is a canonical morphism \( X'' \rightarrow X' \).

Contracting all \((-1)\)-curves inside the smooth locus that lie over singularities of \( X \) gives \( X'' \rightarrow \tilde{X} \rightarrow X \), where \( \tilde{X} = X(A, \tilde{P}) \) is the minimal resolution of \( X \).

7. Self intersection numbers and continued fractions

This section presents some variations on [25, Thm. 2.5] (iii) and (iv) on continued fractions over the numbers \(-D_1^2, \ldots, -D_n^2\) given by an arm of a smooth \( \mathbb{K}^* \)-surface with two parabolic fixed point curves. Proposition 7.5 shows how to express the numbers of \( X \) and the entries \( l_{ij} \) of the defining matrix \( D \).

Remark 6.4. Consider the given in Remark 6.4.

□

Definition 7.1. Consider the defining matrix \( P \) of a smooth rational projective \( \mathbb{K}^* \)-surface \( X(A, P) \). By our assumptions, \( P \) is irredundant and slope ordered.

(i) We call \( P \) adapted to the source if it satisfies

(a) \(-l_{i1} < d_{i1} \leq 0 \) for \( i = 1, \ldots, r \),
(b) \( l_{01}, l_{11}, \geq l_{21}, \geq \ldots, \geq l_{r1} \).

(ii) We call \( P \) adapted to the sink if it satisfies

(a) \( 0 \leq d_{in_i} < l_{in_i} \) for \( i = 1, \ldots, r \),
(b) \( l_{01}, l_{11}, \geq l_{21}, \geq \ldots, \geq l_{rn_r} \).

Definition 7.2. Consider a smooth rational projective \( \mathbb{K}^* \)-surface \( X = X(A, P) \) and the entries \( l_{ij} \) of the defining matrix \( P \).

(i) Assume that \( X \) has a parabolic fixed point curve \( D^+ \) and let \( P \) be adapted to the source. Set

\[
\begin{align*}
\text{for } i = 0, \ldots, r : & \quad l_{i0} := 0, \quad d_{i0} := 1, \quad D_{i0}^2 := (D^+)^2, \\
\text{for } i = 0 : & \quad l_{i(-1)} := -l_{i1}, \quad d_{i(-1)} := d_{i1}, \\
\text{for } i = 1, \ldots, r : & \quad l_{i(-1)} := -l_{01}, \quad d_{i(-1)} := d_{01}.
\end{align*}
\]

(ii) Assume that \( X \) has an elliptic fixed point \( x^- \) and let \( P \) be adapted to the sink. Set

\[
\begin{align*}
\text{for } i = 0 : & \quad l_{0n_{i+1}} := -l_{1n_i}, \quad d_{0n_{i+1}} := d_{1n_i}, \\
\text{for } i = 1 : & \quad l_{1n_{i+1}} := -l_{0n_i}, \quad d_{1n_{i+1}} := d_{0n_i}, \\
\text{for } i = 2, \ldots, r : & \quad l_{in_{i+1}} := -l_{0n_i}, l_{1n_i}, \quad d_{in_{i+1}} := -1.
\end{align*}
\]

Lemma 7.3. Consider a smooth rational projective \( \mathbb{K}^* \)-surface \( X = X(A, P) \) and the curves \( D_{ij} \) in the arms of \( X \).

(i) Assume that \( X \) has a parabolic fixed point curve \( D^+ \) and let \( P \) be adapted to the source. Then, for all \( i = 0, \ldots, r \) and \( j = 0, \ldots, n_i - 1 \), we have

\[
-l_{ij}D_{ij}^2 = l_{ij-1} + l_{ij+1}, \quad -d_{ij}D_{ij}^2 = d_{ij-1} + d_{ij+1}.
\]

(ii) Assume that \( X \) has an elliptic fixed point curve \( x^- \) and let \( P \) be adapted to the sink. Then, for all \( i = 0, \ldots, r \) and \( j = 2, \ldots, n_i \), we have

\[
-l_{ij}D_{ij}^2 = l_{ij-1} + l_{ij+1}, \quad -d_{ij}D_{ij}^2 = d_{ij-1} + d_{ij+1}.
\]

Proof. The statements follow directly from the computation of self intersection numbers of \( X = X(A, P) \) in terms of the entries of \( P \) given in Remark 6.4. □
Reminder 7.4. Given any finite sequence \(a_1, \ldots, a_k\) of rational numbers, consider the process

\[
\text{CF}_1(a_1) = a_1, \quad \text{CF}_2(a_1, a_2) = a_1 - \frac{1}{a_2}, \quad \text{CF}_3(a_1, a_2, a_3) = a_1 - \frac{1}{a_2 - \frac{1}{a_3}} \quad \ldots
\]

Provided there is no division by zero, these numbers are called continued fractions.

The formal definition runs inductively:

\[
\text{CF}_1(a_1) := a_1, \quad \text{CF}_k(a_1, \ldots, a_k) := a_1 - \frac{1}{\text{CF}_{k-1}(a_2, \ldots, a_k)}.
\]

Proposition 7.5. Consider a smooth rational projective \(K^*\)-surface \(X = X(A, P)\) and the curves \(D_{ij}\) in the arms of \(X\).

(i) Assume that \(X\) has a parabolic fixed point curve \(D^+\) and \(P\) is adapted to the source. Fix \(0 \leq i \leq r\) and \(1 \leq j \leq n_i\), and for \(k = 1, \ldots, j - 1\) set

\[
f_{ijk} := \text{CF}_k(-D^2_{ij-k}, \ldots, -D^2_{ij-1}).
\]

Then the entries \(l_{ij}\) and \(d_{ij}\) of the matrix \(P\) can be expressed in terms of the above continued fractions \(f_{ijk}\) as

\[
l_{ij} = \prod_{k=1}^{j-1} f_{ijk}, \quad d_{ij} = \begin{cases} -((D^+)^2 f_{0j-1} + 1) \prod_{k=1}^{j-2} f_{ijk} & i = 0, \\
- \prod_{k=1}^{j-2} f_{ijk}, & i \neq 0. \end{cases}
\]

(ii) Assume that \(X\) has an elliptic fixed point \(x^-\) and \(P\) is adapted to the sink. Fix \(0 \leq i \leq r\) and \(1 \leq j \leq n_i\), and for \(k = 1, \ldots, j - 1\) set

\[
h_{ijk} := \text{CF}_k(-D^2_{m_i-j+k}, \ldots, -D^2_{m_i}).
\]

Then the entries \(l_{m_i-j}\) and \(d_{m_i-j}\) of the matrix \(P\) can be expressed in terms of the above continued fractions as

\[
l_{m_i-j} = \left( \prod_{k=1}^j h_{ijk} \right) l_{m_i} - \left( \prod_{k=1}^{j-1} h_{ijk} \right) l_{m_i+1},
\]

\[
d_{m_i-j} = \left( \prod_{k=1}^j h_{ijk} \right) d_{m_i} - \left( \prod_{k=1}^{j-1} h_{ijk} \right) d_{m_i+1}.
\]

Proof. For both assertions, the proof relies on partially solving tridigoal systems of linear equations of the following form:

\[
\begin{bmatrix}
  a_1 & -1 & 0 \\
  -1 & a_2 & -1 \\
  \ddots & \ddots & \ddots \\
  0 & -1 & a_{n-1} & -1 \\
  & & & -1 & a_n
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_{n-1} \\
  x_n
\end{bmatrix}
= \begin{bmatrix}
  b_1 \\
  0 \\
  \vdots \\
  0 \\
  b_n
\end{bmatrix}
\]

In [20] Thm. 15, the solutions are explicitly computed via continued fractions in the entries. In particular, with \(f_k := \text{CF}_k(a_k, \ldots, a_1)\), it gives us

\[
b_1 = \left( \prod_{k=1}^n f_k \right) x_n - \left( \prod_{k=1}^{n-1} f_k \right) b_n.
\]
We verify (i). Due to smoothness, Proposition 6.5 (i) yields \( l_{i1} = 1 \). Now, the relations among the \( l_{ij} \) provided by Lemma 7.3 (i) can be written as follows:

\[
\begin{pmatrix}
-D_{ij}^2 & -1 & 0 \\
-1 & -D_{i,j-2}^2 & -1 \\
\vdots & \ddots & \ddots \\
0 & -1 & -D_{i,j-1}^2 \\
& -1 & -D_{i1}^2 \\
\end{pmatrix}
\begin{pmatrix}
l_{ij-1} \\
l_{ij-2} \\
\vdots \\
l_{i2} \\
l_{i1} \\
\end{pmatrix}
= \begin{pmatrix}
l_{ij} \\
l_{i-1} \\
\vdots \\
l_{i0} \\
\end{pmatrix}
\]

Thus, the above formula for \( b_i \) gives the desired presentation of \( l_{ij} \). With the \( d_{ij} \), we proceed analogously. In order to verify (ii), look at

\[
\begin{pmatrix}
-D_{i1}^2 & -1 & 0 \\
-1 & -D_{i,j+1}^2 & -1 \\
\vdots & \ddots & \ddots \\
0 & -1 & -D_{in}^2 \\
& -1 & -D_{i1}^2 \\
\end{pmatrix}
\begin{pmatrix}
l_{i,j+1} \\
l_{i,j+2} \\
\vdots \\
l_{in-1} \\
l_{in} \\
\end{pmatrix}
= \begin{pmatrix}
l_{ij} \\
l_{i-1} \\
\vdots \\
l_{i1} \\
\end{pmatrix}
\]

encoding the relations among the \( l_{ij} \) from Lemma 7.3 (ii) and apply the above presentation of \( b_1 \). Again the \( d_{ij} \) are settled analogously. \( \square \)

**Corollary 7.6.** Consider a smooth rational projective \( \mathbb{K}^* \)-surface \( X = X(A, P) \).

(i) Assume that there is a fixed point curve \( D^+ \subseteq X \) and that \( P \) is adapted to the source. Fix any choice of indices \( 1 \leq j_i \leq n_i \), where \( i = 0, \ldots, r \). Then we have

\[
(D^+)^2 = -\sum_{i=0}^{r} m_{ij_i} - \sum_{i=0}^{r} CF_{j_i-1}(-D_{i1}^2, \ldots, -D_{i,j_i-1}^2)^{-1}.
\]

(ii) Assume that there is a fixed point \( x^- \in X \) and that \( P \) is adapted to the sink. Fix \( 0 \leq j \leq n_0 - 1 \) and set \( \sigma^- := \text{cone}(v_{0n_0-1}, v_{1n_1}, \ldots, v_{rn_r}) \). Then we have:

\[
\frac{l_{0n_0-j}}{\det(\sigma^-)} = \text{CF}_j(-D_{in_1}, \ldots, -D_{in_n-j+1}^2)^{-1}l_{in_1}.
\]

*Proof.* We prove (i). Proposition 7.3 (i) allows us to express the slopes \( m_{ij_i} \) in the following way:

\[
m_{0j_0} = -(D^+)^2 - f_{j_0, j_0-1}^{-1}, \quad m_{ij_i} = -f_{j_i, j_i-1}^{-1}, \quad i = 1, \ldots, r.
\]

By the definition of the \( f_{ijk} \), this directly leads to the desired representation of the self intersection number:

\[
(D^+)^2 = -m_{0j_0} - f_{j_0, j_0-1}^{-1} = -\sum_{i=0}^{r} m_{ij_i} - \sum_{i=0}^{r} f_{ij_i, j_i-1}^{-1}.
\]

We turn to (i). Since \( X \) is smooth, Proposition 6.10 (iv) tells us \( \det(\sigma^-) = -1 \). Thus, setting \( h_0 := h_{0j_1} \cdots h_{0j_r} \), we have

\[
-1 = \det(\sigma^-) = l_{0n_0}d_{1n_1} + l_{1n_1}d_{0n_0} = (h_0^{-1}d_{0n_0-j} + h_{0j}^{-1}d_{1n_1})l_{1n_1} + d_{1n_1}(h_0^{-1}l_{0n_0-j} - h_{0j}^{-1}l_{1n_1}) = h_0^{-1}(d_{0n_0-j}l_{1n_1} + d_{1n_1}l_{0n_0-j}) = h_0^{-1}\det(\sigma^-),
\]

\]
as is seen by a direct computation. Using the representation of $l_{0\eta_0}$ provided by Proposition 7.5 (ii), we obtain

$$\frac{l_{0\eta_0-j}}{\det(\bar{\sigma})} - l_{0\eta_0} = -h_0^{-1}l_{0\eta_0-j} - l_{0\eta_0} = h_0^{-1}l_{1n_1}.$$

\[\Box\]

8. Quasismooth simple elliptic fixed points

The aim of this section is to establish Theorem 8.4 which specifies obstructions to the existence of quasismooth simple elliptic fixed points. This is a first step towards the proof of Theorem 1.1, but also has some general applications to the geometry of rational projective $\mathbb{K}^*$-surfaces, see Corollaries 8.6 to 8.8. Let us recall our definition of a simple elliptic fixed point.

**Definition 8.1.** We say that an elliptic fixed point $x$ of a rational projective $\mathbb{K}^*$-surface $X$ is **simple** if for the minimal resolution $\pi: \tilde{X} \to X$ of singularities, the fiber $\pi^{-1}(x)$ is contained in an arm of $\tilde{X}$.

Note that a simple elliptic fixed point has in particular no parabolic fixed point curve in its fiber of the minimal resolution of singularities. We discuss two examples of simple elliptic fixed points, making use of the resolution of singularities provided by Remark 6.13 and the formulae for the self intersection numbers given in Remark 6.14.

**Example 8.2.** The following matrices $P$ and $\tilde{P}$ define the minimal resolutions $\tilde{X} \to X$ of non-toric $\mathbb{K}^*$-surfaces $X$, each of them with a simple elliptic fixed point $x^- \in X$.

(i) Here $x^-$ is quasismooth but singular and the fiber over $x^-$ equals the curve $\tilde{D}_{04} \subset \tilde{X}$ in the 0-th arm:

$$P = \begin{bmatrix} -1 & -2 & -3 & 1 & 1 & 0 & 0 & 0 \\ -1 & -3 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\tilde{P} = \begin{bmatrix} -1 & -2 & -3 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & -2 & -3 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$  

The curve $\tilde{D}_{04}$ is isomorphic to a projective line and has self intersection number equal to $-2$. In other words, $x^- \in X$ is an $A_2$-singularity.

(ii) Here $x^-$ is not quasismooth and the fiber over $x^-$ equals the curve $\tilde{D}_{08} \subset \tilde{X}$ in the 0-th arm:

$$P = \begin{bmatrix} -1 & -2 & -3 & -4 & -5 & -6 & -7 & 1 & 2 & 3 & 0 & 0 & 0 \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & -3 & -4 & -5 & -6 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\tilde{P} = \begin{bmatrix} -1 & -2 & -3 & -4 & -5 & -6 & -7 & -1 & 1 & 2 & 3 & 0 & 0 & 0 \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & -1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & -3 & -4 & -5 & -6 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$  

The curve $\tilde{D}_{08}$ is of intersection number $-1$ and has a cusp singularity. The singularity $x^- \in X$ is isomorphic to the Brieskorn-Pham singularity

$$0 \in V(T_1^7 + T_2^7 + T_3^7) \subset \mathbb{K}^3,$$

where we gain this presentation by looking at $X \cap Z_{\sigma^-}$ for the (smooth) affine toric chart $Z_{\sigma^-} \subset Z$; compare also [21, No. 2.5 on p. 72].
Definition 8.3. We say that a parabolic fixed point curve $D \subseteq X$ is of a rational projective $\mathbb{K}^*$-surface is gentle if there is an arm $A_i = D_{1i} \cup \ldots \cup D_{mi}$, such that the (unique) point $x \in D \cap A_i$ is a smooth point of $X$.

Theorem 8.4. Let $X$ be a non-toric rational projective $\mathbb{K}^*$-surface $X$ with a quasismooth simple elliptic fixed point $x \in X$.

(i) There is no gentle non-negative parabolic fixed point curve in $X$.
(ii) There is no other quasismooth simple elliptic fixed point in $X$.

Remark 8.5. The assumption that $X$ is non-toric is essential in Theorem 8.4. A cheap smooth toric counterexample is given by the projective plane $\mathbb{P}_2$: Consider the two $\mathbb{K}^*$-actions given by

$$t \cdot [z] = [z_0, z_1, tz_2], \quad t \cdot [z] = [z_0, tz_1, t^2z_2].$$

For the first one, $[0, 0, 1]$ is an elliptic fixed point and $V(T_2)$ a parabolic fixed point curve of self intersection one. The second one has $[1, 0, 0]$ and $[0, 0, 1]$ as elliptic fixed points.

Corollary 8.6. Every rational projective $\mathbb{K}^*$-surface with two quasismooth simple elliptic fixed points is a toric surface.

Corollary 8.7. Every quasismooth non-toric rational projective $\mathbb{K}^*$-surface with a simple elliptic fixed point has a fixed point curve.

The latter says that, when considering quasismooth non-toric rational projective $\mathbb{K}^*$-surfaces $X$, we always may assume the we have a curve $D^+ \subseteq X$. For smooth $X = X(A, P)$, this allows us to complement [23 Thm. 2.5] by showing that the defining matrix $P$ is basically determined by the self intersection numbers of invariant curves. More precisely, we obtain the following.

Corollary 8.8. Let $X = X(A, P)$ be smooth, non-toric with $P$ adapted to $D^+ \subseteq X$. Then all entries $l_{ij}$ and $d_{ij}$ of $P$ can be expressed via self intersection numbers according to Corollary 7.7.

Definition 8.9. Let $X = X(A, P)$ have a simple elliptic fixed point $x \in X$. We call $0 \leq i \leq r$ an exceptional index of $x$ if $\pi^{-1}(x)$ is contained in the $i$-th arm of $\tilde{X} = X(A, \tilde{P})$, where $\pi: \tilde{X} \to X$ is the minimal resolution of singularities.

Note that for any singular simple elliptic fixed point the exceptional index is unique. The following characterization of simple quasismooth elliptic fixed points is an important ingredient for the proof of Theorem 8.4.

Proposition 8.10. Let $x \in X = X(A, P)$ be a quasismooth elliptic fixed point with leading indices $t_0, t_1$.

(i) Assume $x = x^+$. Then $x$ is simple with exceptional index $t_0$ if and only if there exists a vector $u \in \mathbb{Z}^r \times \mathbb{Z}_{<0}$ such that

$$\langle u, v_{i1} \rangle = -1, \quad \langle u, v_{i0} \rangle = 0, \quad i \neq t_0, t_1, \quad \langle u, v_{ij} \rangle \geq 0, \quad i \neq t_1.$$

We have $l_{i1} = 1$ whenever $i \neq t_0, t_1$. Moreover, if $u \in \mathbb{Z}^r \times \mathbb{Z}_{<0}$ is a vector as above, then the following holds:

$$0 < m^+ \leq -u_{r+1}m^+ \leq \frac{1}{l_{i1}}.$$

(ii) Assume $x = x^-$. Then $x$ is simple with exceptional index $t_0$ if and only if there exists a vector $u \in \mathbb{Z}^r \times \mathbb{Z}_{<0}$ such that

$$\langle u, v_{i1n_1} \rangle = -1, \quad \langle u, v_{m0} \rangle = 0, \quad i \neq t_0, t_1, \quad \langle u, v_{ij} \rangle \geq 0, \quad i \neq t_1.$$
We have \( l_{in_i} = 1 \) whenever \( i \neq t_0, t_1 \). Moreover, if \( u \in \mathbb{Z}^* \times \mathbb{Z}_{>0} \) is a vector as above, then the following holds:
\[
0 > m^- \geq u_{r+1}m^- \geq -\frac{1}{l_{1n_1}}.
\]

Lemma 8.11. Consider a sequence of vectors \( v_0, \ldots, v_k \in \mathbb{Q}^2 \) such that there are \( c_1, \ldots, c_{k-1} \in \mathbb{Z}_{\geq 2} \) with
\[
v_{j+1} = c_jv_j - v_{j-1}, \quad j = 1, \ldots, k-1.
\]
Then, for \( k \geq 2 \), the difference \( v_k - v_{k-1} \) lies in \( \tau := \text{cone}(v_1, v_1 - v_0) \) and the vector \( v_k \) lies in the shifted cone \( \tau + v_1 \).

Proof. Clearly, \( v_1 \in \tau + v_1 \) and \( v_1 - v_0 \in \tau \). We proceed inductively. For \( j \geq 1 \), assume \( v_j \in \tau + v_1 \) and \( v_j - v_{j-1} \in \tau \). Write \( v_j = v_j' + v_1 \) with \( v_j' \in \tau \). Then, using \( c_j \geq 2 \) we see
\[
v_{j+1} = c_jv_j - v_{j-1} = (c_j - 1)v_j' + (c_j - 2)v_1 + (v_j - v_{j-1}) + v_1 \in \tau + v_1.
\]

Lemma 8.12. Consider \( H := \{(x, y) \in \mathbb{Q}^2; \ x - y \geq 1 \} \). Given \( v_0 = (a, b) \) and \( v_1 = (c, d) \) in \( H \) with \( a < 0 \) and \( c > 0 \), there is no \( u \in \mathbb{Z}^* \times \mathbb{Z}_{>0} \) satisfying
(i) \( \langle u, v_0 \rangle = u_1a + u_2b \geq 0 \), \quad (ii) \( \langle u, v_1 \rangle = u_1c + u_2d = -1 \).

Proof. Since \( b < a < 0 \) and \( u_2 > 0 \) hold, we infer \( u_1 \leq -2 \) from (i). Then (ii) tells us \( d > 0 \). Now, plugging \( u_1 = -(u_2d + 1)/c \) into (i) leads to a contradiction:
\[
u_2 \leq \frac{a}{bc - ad} \leq \frac{b+1}{bc - ad} \leq \frac{b+1}{bc} \leq 1 + \frac{1}{bc} < 1.
\]

Lemma 8.13. Consider four vectors \( \xi_1, \xi_2 \) and \( \eta_1, \eta_2 \) in \( \mathbb{Q}^2 \) satisfying the following conditions:
\[
\det(\xi_2, \xi_1) = \det(\xi_1, \eta_1) = \det(\eta_1, \eta_2) = 1, \quad \det(\xi_2, \eta_2) \geq 1.
\]
Then \( \xi_1 = a\eta_1 - \eta_2 \) and \( \xi_2 = b\eta_1 - c\eta_2 \), where \( a, b, c > 0 \) and \( c = ab - 1 \). In particular,
\[
a = 1 \Rightarrow \det(\xi_2, \eta_2) = 1, \quad a \geq 2 \Rightarrow c - b \geq 1.
\]

Proof. In suitable linear coordinates, we have \( \eta_1 = (0, -1) \) and \( \eta_2 = (1, 0) \) and moreover \( \xi_1, \xi_2 \in \mathbb{Z}_{\geq 0}^2 \). In this situation, the assertion can be directly verified.

Proof of Proposition 6.10. First observe that multiplying the last row of \( P \) by \(-1\) interchanges source and sink and thus it suffices to prove Assertion (ii). For this we may assume that \( P \) is adapted to the sink. Consider the minimal resolution \( \pi: \tilde{X} \rightarrow X \), where \( \tilde{X} = X(A, \tilde{P}) \), as provided by Remark 6.13. Then, for every \( i = 0, \ldots, r \), the columns \( v_{1i}, \ldots, v_{ni} \) of \( P \) occur among the columns \( \tilde{v}_{1i}, \ldots, \tilde{v}_{ni} \) of \( \tilde{P} \). Moreover, Proposition 6.10 yields
\[
\tilde{l}_{ni} = l_{ni} = 1, \quad \tilde{d}_{bi} = d_{ni} = 0 \quad \text{for} \ i = 2, \ldots, r.
\]

First suppose that \( x^- \in X \) is simple. We may assume that the exceptional index of \( x^- \) is 0 that means that the divisors inside \( \pi^-(x) \) are located in the 0-th arm of \( \tilde{X} \). Then \( \tilde{l}_{ni} = l_{ni} \) and \( \tilde{d}_{bi} = d_{ni} \) hold. Define \( u \in \mathbb{Z}^{r+1} \) by \( u_1 := \tilde{d}_{b0} \) and \( u_i := 0 \) for \( i = 1, \ldots, r \) and \( u_{r+1} := \tilde{l}_{b0} \). Then Proposition 6.10 yields
\[
\langle u, v_{1ni} \rangle = \tilde{d}_{b0}l_{ni} + \tilde{l}_{b0}d_{ni} = \tilde{l}_{b0}d_{ni} + \tilde{l}_{ni}d_{b0} = -1.
\]
According to the definition of $u$, we have $\langle u, v_{i\eta_i} \rangle = 0$ for $i = 2, \ldots, r$. Moreover, for $j = 0, \ldots, \tilde{n}_0$, we use slope orderedness of $\tilde{P}$ to see
\[ \langle u, v_{ij} \rangle = -\tilde{d}_{0\eta_0} \tilde{l}_{ij} + \tilde{l}_{0\eta_0} \tilde{d}_{ij} = \tilde{l}_{0\eta_0} \tilde{l}_{ij} \left( \frac{\tilde{d}_{ij}}{\tilde{l}_{ij}} - \frac{\tilde{d}_{0\eta_0}}{\tilde{l}_{0\eta_0}} \right) \geq 0. \]
This means in particular $\langle u, v_{ij} \rangle \geq 0$ for $j = 1, \ldots, n_0$. Moreover, since $P$ is slope ordered and adapted to the sink, we have $d_{ij} \geq 0$ for all $i \geq 1$ and thus
\[ \langle u, v_{ij} \rangle = \tilde{l}_{0\eta_0} d_{ij} \geq 0, \quad i = 2, \ldots, r, \quad j = 1, \ldots, \eta_i. \]
Let us care about the estimate for the slope sum $m^-$. Evaluating $u$ at the vectors $v_{0n_0}$ and $v_{1n_1}$ gives us
\[-u_1 l_{0n_0} + u_{r+1} d_{0n_0} \geq 0, \quad u_1 l_{1n_1} + u_{r+1} d_{1n_1} = -1.\]
Solving the second condition for $u_1$ and plugging the result into the first one, gives us the estimate
\[ u_{r+1} (l_{0n_1} d_{1n_1} + l_{1n_1} d_{0n_0}) \geq -l_{0n_0}. \]
The expression $l_{0n_1} d_{1n_1} + l_{1n_1} d_{0n_0}$ equals $l_{0n_1} l_{1n_1} m^-$ and is negative due to Proposition 6.10. We conclude
\[ 1 \leq u_{r+1} \leq -\frac{1}{l_{1n_1} m^-}. \]
This directly yields the desired lower bound for $u_{r+1} m^-$. The upper bound $0 > m^-$ is guaranteed by Remark 6.3.

Now suppose that there is a vector $u \in \mathbb{Z} \times \mathbb{Z}_{>0}$ as in the proposition. Suitably arranging $P$ and adapting $u$, we achieve $t_0 = 0$ and $t_1 = 1$. Note that for each $i = 2, \ldots, r$, we have $u_i = 0$ due to $d_{in_i} = 0$. We assume that $x^-$ is not simple and show that this leads to a contradiction. For this, it suffices to verify
\[ (-l_{0n_o}, d_{0n_0}), (l_{1n_1}, d_{1n_1}) \in H := \{(x, y) \in \mathbb{Q}^2; \ x - y \geq 1\}, \]
because then Lemma 8.12 implies that $u$ cannot evaluate non-negatively on $v_{0n_0}$ and to $-1$ on $v_{1n_1}$. Since $P$ is slope ordered, $(l_{1n_1}, d_{1n_1})$ lies in $H$. In order to see that also $(-l_{0n_o}, d_{0n_0})$ belongs to $H$, we use the assumption that $x^-$ is not simple and thus we are in one of the following two cases.

Case 1: The fiber $\pi^-(x) \subseteq \tilde{X}$ contains a parabolic fixed point curve $\tilde{D}^-$. Consider the sequence of divisors $\tilde{D}^-, \tilde{D}_{\tilde{0}n_0}, \ldots, \tilde{D}_{\tilde{0}n_0-k}$ connecting $\tilde{D}^-$ with the proper transform $\tilde{D}_{\tilde{0}n_0-k}$ of $D_{0n_0} \subseteq X$. This gives us a sequence of pairs
\[ (0, -1), (-1, \tilde{d}_{\tilde{0}n_0}), \ldots, (-\tilde{l}_{\tilde{0}n_0-k}, \tilde{d}_{\tilde{0}n_0-k}) = (-l_{0n_0}, d_{0n_0}), \]
where $\tilde{l}_{\tilde{0}n_0} = 1$ due to Proposition 6.8. Since $\tilde{X} \rightarrow X$ is the minimal resolution, we have $(\tilde{D}^-)^2 \leq -2$ and $\tilde{D}_{\tilde{0}j} \leq -2$ whenever $j > \tilde{n}_0 - k$. Thus, Remark 6.3 shows $\tilde{d}_{\tilde{0}n_0} \leq -2$. According to Lemma 7.3, the above sequence of pairs satisfies the assumptions of Lemma 8.11. Applying the latter yields $(-l_{0n_0}, d_{0n_0}) \in H$.

Case 2: The fiber $\pi^-(x) \subseteq \tilde{X}$ contains an elliptic fixed point $\tilde{x}^-$ and curves from the arms 0 and 1 of $\tilde{X}$. The curves in $\pi^-(x)$ are $\tilde{D}_{\tilde{1}n_1}, \ldots, \tilde{D}_{\tilde{1}n_1-k}$, where $i = 0, 1$ and $\tilde{D}_{\tilde{1}n_1-k-1}$ is the proper transform of $D_{0n_0} \subseteq X$. Consider the associated sequences of pairs
\[ (\tilde{l}_{\tilde{0}n_0}, \tilde{d}_{\tilde{0}n_0}), \ldots, (-\tilde{l}_{\tilde{0}n_0-k_0}, \tilde{d}_{\tilde{0}n_0-k_0}) = (-l_{0n_0}, d_{0n_0}), \]
\[ (\tilde{l}_{\tilde{1}n_1}, \tilde{d}_{\tilde{1}n_1}), \ldots, (\tilde{l}_{\tilde{1}n_1-k_1}, \tilde{d}_{\tilde{1}n_1-k_1}) = (l_{1n_1}, d_{1n_1}). \]
By slope orderedness of $\tilde{P}$, all members of the second sequence lie in $H$. Now, write $\xi_1, \xi_2$ for the first two pairs of the first sequence and $\eta_1, \eta_2$ for the first two pairs...
of the second one. Then Lemma 8.13 provides us with integers \(a, b, c > 1\) such that

\[
\xi_1 = a\eta_1 - \eta_2, \quad \xi_2 = b\eta_1 - c\eta_2.
\]

Here \(a \geq 2\) holds as otherwise Proposition 6.10 shows that \(D_{0\eta_0}\) and \(D_{1\eta_1}\) contract smoothly which contradicts to minimality of the resolution. Thus, \(\xi_1, \xi_2 \in H\). Again by minimality of the resolution, all \(D_{ij} \subseteq \pi^{-1}(x)\) are of self intersection at most \(-2\). Using Lemmas 7.3 and 8.11 we arrive at \((l_{i1n_i}, d_{i1n_i}) \in H\). \(\Box\)

**Lemma 8.14.** Consider the defining matrix \(P\) of a rational projective \(K^*\)-surface \(X(A, P)\).

(i) If \(m_{ij} = 0\) holds for \(0 \leq i \leq r\) and \(1 \leq j \leq n_i\), then \(d_{ij} = 0\) and \(l_{ij} = 1\).

(ii) If \(P\) is adapted to the sink, then \(0 \leq m_{ii} < m_{ii} + l_{i1}^{-1} \leq 1\) for \(i = 1, \ldots, r\).

(iii) If \(P\) is irredundant and adapted to the sink, then \(m_{11} > 0\) for \(i = 1, \ldots, r\).

**Proof.** We verify (i). If \(m_{ij} = 0\) holds, then we must have \(d_{ij} = 0\) and thus primitivity of the column \(v_{ij}\) yields \(l_{ij} = 1\). We turn to (ii). As \(P\) is adapted to the sink, we have \(0 \leq d_{in_i} < l_{ni}\) whenever \(i \geq 1\) and the desired estimate follows. We prove (iii). By slope orderedness of \(P\) and (ii), we have \(m_{11} \geq m_{in_i} \geq 0\). We exclude \(m_{11} = 0\). Otherwise, \(m_{ii} = 0\) holds. Thus, \(d_{i1} = d_{in_i} = 0\) and (i) yields \(l_{i1} = l_{in_i} = 1\) and \(n_i = 1\). This is a contradiction to irredundance of \(P\). \(\Box\)

**Proof of Theorem 8.4.** We may assume \(X = X(A, P)\) and that the quasismooth simple elliptic fixed point is \(x^- \in X\). Moreover, we may assume that \(P\) is irredundant, adapted to the sink and that the two leading indices from Proposition 8.10 are 0 and 1. Then we have

\[
m_{01} \geq m_{0\eta_0}, \quad m_{11} \geq m_{1\eta_1} \geq 0, \quad m_{11} > m_{in_i} = 0, \quad i = 2, \ldots, r,
\]

by slope orderedness, Proposition 6.10 and Lemma 8.14. In particular, \(m^-\) equals \(m_{0\eta_0} + m_{1\eta_1}\). Using the estimate on \(m^-\) from Proposition 8.10 and Lemma 8.14(ii), we see

\[
0 \geq -m_{1n_1} > m_{0\eta_0} \geq -m_{1n_1} - \frac{1}{l_{1n_1}} \geq -1.
\]

We prove (i). Let \(D^+ \subseteq X\) be a non-negative parabolic fixed point curve. We have to show that \(D^+\) is not gentle. According to Proposition 6.3 this means to verify \(l_{11} \geq 2\) for \(i = 0, \ldots, r\). Remark 6.4 yields

\[
0 \geq m^+ = m_{01} + m_{11} + m_{21} + \ldots + m_r,
\]

where \(m_{11}, \ldots, m_r > 0\) and \(r \geq 2\), hence \(0 > m_{01}\). Moreover, \(m_{01} \geq m_{0\eta_0} \geq 0\) yields \(0 < m_{1i} \leq 1\) for \(i = 1, \ldots, r\). This implies \(l_{1i} \geq 2\). We show \(l_{01} \geq 2\). Otherwise, \(l_{01} = 1\) and thus \(m_{01} = 1\). Then \(m_{0\eta_0} = 1\). Consequently \(n_0 = 1\) and \(l_{01} = -d_{01} = 1\) by primitivity of \(v_{01}\). This is a contradiction to irredundance of \(P\).

We prove (ii). Suppose that there is also a quasismooth simple elliptic fixed point \(x^+ \in X\). Let \(0 \leq l_{01}, l_{11} \leq r\) be the leading indices as in Proposition 8.10(i). Then \(l_{i1} = 1\) holds whenever \(i \neq l_{01}, l_{11}\). This allows us to assume \(l_{01}, l_{11} \leq 3\). The estimates from Proposition 8.10 yield

\[
m^+ - m^- = m_{01} - m_{0\eta_0} + m_{11} - m_{1\eta_1} + m_{21} + m_{31} + \sum_{i=4}^{r} d_{i1} \leq \frac{1}{l_{1n_1}} + \frac{1}{l_{11}} \leq 2.
\]

**Case** \(l_{01} \leq 1\) and \(l_{11} \leq 1\). Then we have \(1 \leq d_{21} = m_{21}\). From the above estimate, we infer

\[
0 \leq m^+ - m^- - 1 \leq \frac{1}{l_{1n_1}} + \frac{1}{l_{11}} - 1 = \frac{l_{1n_1} + l_{11} - l_{1n_1}l_{11}}{l_{1n_1}l_{11}}.
\]
This leaves us with the following possibilities: first $l_{1n_1} = l_{i,1} = 2$, second $l_{1n_1} = 1$ and third $l_{i,1} = 1$. We go through these cases.

Let $l_{1n_1} = l_{i,1} = 2$. Then $m_{01} = m_{0n_0}$ and $m_{11} = m_{1n_1}$ as well as $d_{21} = 1$ hold. Thus, $n_0 = n_1 = 1$. Moreover, $l_{11} = 2$ implies $d_{11} = 1$. Proposition 8.10 tells us

$$-2 \left( m_{01} + \frac{1}{2} \right) = -l_{1n_1} m^{-} \leq 1, \quad 2 \left( m_{01} + \frac{1}{2} + 1 \right) \leq l_{11} m^{+} \leq 1.$$

Thus $m_{01} \geq -1$ and $m_{01} \leq -1$. As $v_{01}$ is primitive, we arrive at $l_{01} = -d_{01} = 1$, which is a contradiction to the irredundance of $P$.

Let $l_{1n_1} = 1$. As $P$ is adapted to the sink, $d_{1n_1} = 0$ holds. Irredundance yields $n_1 \geq 2$ and $d_{11} > 0$. As noted before, $m_{01} \geq m_{0n_0} \geq -1$. If $t_1 = 1$, then

$$0 \leq d_{11} + l_{11}(m_{01} + 1) = l_{11}(m_{01} + m_{11} + 1) \leq l_{11} m^{+} \leq 1$$

due to Proposition 8.10. This implies $d_{11} = 1$ and $m_{01} = m_{0n_0} = -1$. Thus, $n_0 = 1$ and $-d_{01} = l_{01} = 1$ holds; a contradiction. If $t_1 = 0$, then we have

$$0 \leq d_{01} + l_{01} < l_{01}(m_{01} + m_{11} + 1) \leq l_{01} m^{+} \leq 1.$$ We conclude $-d_{01} = l_{01} = 1$, using primitivity of $v_{01}$. Then $m_{01} \geq m_{0n_0} \geq -1$ implies $v_{01} = 1$. A contradiction to irredundance of $P$.

Let $l_{i,1} = 1$. This case transforms into the preceding one by switching source and sink via multiplying the last row of $P$ by $-1$ and adapting to the new sink.

Case $\iota_0 \leq 1$ and $t_1 \geq 2$. Then we may assume $t_1 = 2$. If $\iota_0 = 0$, then we have $l_{11} = 1$. Thus, $n_1 \geq 2$ and $d_{11} > 0$. Proposition 8.10 and Lemma 8.14(ii) show

$$d_{11} + \frac{d_{21}}{l_{21}} = m_{11} + m_{21} \leq m_{1n_1} + m^{+} - m^{-} \leq m_{1n_1} + \frac{1}{l_{1n_1}} + \frac{1}{l_{21}} \leq 2.$$ Now, $d_{11} = 2$ yields $m_{21} = 0 = m_{2n_2}$, which is impossible by irredundance of $P$. Thus, $d_{11} = 1$. The above inequality gives $d_{21} = 1$ and $d_{1n_1} = l_{1n_1} - 1$. Hence,

$$m_{01} \leq m^{+} - m_{11} - m_{21} \leq \frac{1}{l_{21}} - 1 - \frac{1}{l_{21}} = -1.$$ So, $m_{01} \geq m_{0n_0} \geq -1$ yields $m_{0n_0} = m_{01} = -1$. Thus, $n_0 = 1$ and $l_{01} = 1$; a contradiction. If $\iota_0 = 1$, then $l_{01} = 1$. Hence $m_{01} \in \mathbb{Z}$ and $n_0 \geq 2$. Now

$$-1 \leq m_{0n_0} < m_{01} \leq m^{+} - m_{21} - m_{11} \leq \frac{1}{l_{21}} - \frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}}$$

and $d_{21} > 0$ imply $d_{11} = 0$. Then $0 = m_{11} \geq m_{1n_1} \geq 0$ yields $n_1 = 1$ and $l_{11} = 1$. A contradiction to irredundance of $P$.

Case $\iota_0 \geq 2$ and $t_1 \leq 1$. We may assume $\iota_0 = 2$. If $t_1 = 1$, then $l_{01} = 1$. Hence $m_{01} \in \mathbb{Z}$ and $n_0 \geq 2$. We derive $m_{1n_1} = m_{11} = d_{11} = 0$ and thus $l_{11} = 1$ from

$$-1 \leq m_{0n_0} < m_{01} \leq m^{+} - m_{11} - m_{21} \leq \frac{1}{l_{11}} - \frac{d_{11}}{l_{11}} - \frac{d_{21}}{l_{21}}.$$ A contradiction to irredundance of $P$. The case $t_1 = 0$ transforms to the case $\iota_0 = 1$ and $t_2 = 1$ settled before by switching sink and source.

Case $\iota_0 \geq 2$ and $t_1 \geq 2$. We may assume $\iota_0 = 2$ and $t_1 = 3$. Then $l_{01} = l_{11} = 1$ and $n_0, n_1 \geq 2$ hold. In particular, $d_{01} = m_{01} > m_{0n_0} \geq -1$. Moreover, $d_{11}, d_{21}$ and $d_{31}$ are all strictly positive. This contradicts to the estimate

$$d_{01} + d_{11} + \frac{d_{21}}{l_{21}} + \frac{d_{31}}{l_{31}} \leq m^{+} \leq \frac{1}{l_{31}}.$$ \qed
Example 8.15. We present $\mathbb{K}$-surfaces $X = X(A, P)$ with a smooth elliptic fixed point $x^-$ and a positive parabolic fixed point curve $D^+$. Consider
\[
P = \begin{bmatrix}
-l_{01} & l_{11} & 0 & 0 \\
-l_{01} & 0 & l_{22} & 1 \\
d_{01} & d_{11} & 1 & 0 & 1
\end{bmatrix}.
\]
Now choose the entries $l_{ij}$ and $d_{ij}$ in such a way that $P$ is adapted to the sink and we have
\[
l_{01}d_{11} + l_{11}d_{01} = -1, \quad l_{21} > l_{01}l_{11}
\]
Then $x^-$ is smooth and $D^+$ has self intersection number $l_{21} - l_{01}l_{11}$. For example we can take
\[
P = \begin{bmatrix}
-2 & 3 & 0 & 0 \\
-2 & 0 & 7 & 1 \\
-1 & 1 & 1 & 0 & 1
\end{bmatrix}.
\]
Then $x^-$ is smooth and we have $m^+ = -1/42$. Consequently, $D^+$ has self intersection number $-m^+ = 1/42$. Observe that $D^+$ is not gentle.

9. Horizontal and Vertical $P$-roots

In section, we introduce horizontal and vertical $P$-roots as adapted versions of the general Demazure $P$-roots to the special case of rational projective $\mathbb{K}$-surfaces $X = X(A, P)$. This allows a less technical treatment. The main results of this section are Propositions 9.6, 9.11 and 9.18 showing geometric constraints to the existence of $P$-roots and Propositions 9.10, 9.15 which identify the $P$-roots in terms of the defining matrix $P$.

Definition 9.1. Consider a rational projective $\mathbb{K}$-surface $X = X(A, P)$ and assume that $P$ is irredundant.

(i) Let $x^+ \in X$ be an elliptic fixed point and $0 \leq i_0, i_1 \leq r$. A horizontal $P$-root at $(x^+, i_0, i_1)$ is a vector $u \in \mathbb{Z}^r \times \mathbb{Z}_{\leq 0}$ such that
\[
\langle u, v_{i_1} \rangle = -1, \quad \langle u, v_{i_0} \rangle = 0, \ i \neq i_0, i_1, \quad l_{i_1} = 1, \ i \neq i_0, i_1.
\]
(ii) Let $x^- \in X$ be an elliptic fixed point and $0 \leq i_0, i_1 \leq r$. A horizontal $P$-root at $(x^-, i_0, i_1)$ is a vector $u \in \mathbb{Z}^r \times \mathbb{Z}_{\geq 0}$ such that
\[
\langle u, v_{i_1} \rangle = -1, \quad \langle u, v_{i_0} \rangle = 0, \ i \neq i_0, i_1, \quad l_{i_0} = 1, \ i \neq i_0, i_1,
\]
\[
\langle u, v_{i_0n_{i_0}} \rangle \geq 0, \ i \neq i_0, i_1, \quad l_{i_0} = 1, \ i \neq i_0, i_1,
\]
\[
\langle u, v_{i_1n_{i_1}} \rangle \geq 0, \ i \neq i_0, i_1, \quad l_{i_1} = 1, \ i \neq i_0, i_1,
\]
\[
\langle u, v_{i_0n_{i_0}} \rangle \geq 0, \ i \neq i_0, i_1, \quad l_{i_0} = 1, \ i \neq i_0, i_1.
\]

We say that an elliptic fixed point $x \in X$ admits a horizontal $P$-root if there is a vector $u$ as in (i) if $x = x^+$, respectively a vector $u$ as in (ii) if $x = x^-$. 

Remark 9.2. Given $u = (u_1, \ldots, u_{r+1}) \in \mathbb{Q}^{r+1}$, set $u_0 := -u_1 - \ldots - u_r$. For $i = 0, \ldots, r$, the linear form $u$ evaluates at the columns $v_{ij}$ of a defining matrix $P$ as
\[
\langle u, v_{ij} \rangle = u_i l_{ij} + u_{r+1} d_{ij}.
\]
This allows a unified treatment of the cases $i = 0$ and $i \neq 0$ and will be used frequently in the sequel.

The subsequent Propositions 9.3 and 9.4 together with Remark 9.5 give the precise relations between the horizontal $P$-roots just defined and the horizontal Demazure $P$-roots recalled in Definition 5.3.

Proposition 9.3. Let $X = X(A, P)$ be non-toric, $P$ irredundant and $(u, i_0, i_1, C)$ a horizontal Demazure $P$-root. Then precisely one of the following statements holds:

(i) We have $u_{r+1} < 0$, there is an elliptic fixed point $x^+ \in X$, the vector $u$ is a horizontal $P$-root at $(x^+, i_0, i_1)$ and $c_i = 1$ holds for all $i \neq i_0$.

(ii) We have \( u_{r+1} > 0 \), there is an elliptic fixed point \( x^− \in X \), the vector \( u \) is a horizontal \( P \)-root at \( (x^−, i_0, i_1) \) and \( c_i = n_i \) holds for all \( i \neq i_0 \).

**Proof.** We exemplarily prove (i). Recall that here we have \( u \) impossible due to inequalities from Definition 9.1 (i) and slope orderedness of \( \mathcal{X} \). Clearly, \( u \) Definition 3.4 and \( x \leq 1 \) holds, there is a \( 1 \leq j \leq n_i \) different from \( c_i \) and we have \( \langle u, v_{ij} \rangle \geq l_{ij} > 0 \). This is impossible due to \( u = u_{r+1} = 0 \).

Next we claim \( u_{r+1} m_{ic_i} \leq u_{r+1} m_{ij} \) holds for every \( 0 \leq i \leq r \) with \( i \neq i_0 \) and every \( 1 \leq j \leq n_i \). Indeed, we infer

\[
\begin{align*}
  u_i &= \begin{cases} 
    -u_{r+1} m_{ic_i}, & i \neq i_1, \\
    -\frac{1}{u_{r+1}} - u_{r+1} m_{ic_i}, & i = i_1,
  \end{cases} \\
  u_i d_{ic_i} + u_{r+1} d_{ic_i} &\leq u_i d_{ij} + u_{r+1} d_{ij}
\end{align*}
\]

from the conditions on \( \langle u, v_{ij} \rangle \) for \( i \neq i_0 \) stated in Definition 3.4. Eliminating \( u_i \) in the above inequalities then directly yields the claim.

We show that for \( u_{r+1} < 0 \), we arrive at (i). First, \( P \) has no column \( v^+ \) by Definition 3.4 and \( u_{r+1} < 0 \). Thus, there is an elliptic fixed point \( x^+ \in X \). We have \( m_{ic_i} \geq m_{ij} \) for \( i \neq i_0 \). Hence, slope orderedness of \( P \) forces \( c_i = 1 \) for all \( i \neq i_0 \). Clearly, \( u \) fulfills the conditions of a horizontal \( P \)-root at \( (x^+, i_0, i_1) \). Similarly, we see that \( u_{r+1} > 0 \) leads to (ii).

**Proposition 9.4.** Consider \( X = X(A, P) \) and assume that \( P \) is irredudant. Define \((r+1)\)-tuples \( C^+ := (1, \ldots, 1) \) and \( C^- := (n_0, \ldots, n_r) \).

(i) Let \( x^+ \in X \) be an elliptic fixed point and \( u \) a horizontal \( P \)-root at \( (x^+, i_0, i_1) \). Then \((u, i_0, i_1, C^+) \) is a Demazure \( P \)-root.

(ii) Let \( x^- \in X \) be an elliptic fixed point and \( u \) a horizontal \( P \)-root at \( (x^-, i_0, i_1) \). Then \((u, i_0, i_1, C^-) \) is a Demazure \( P \)-root.

**Proof.** We exemplarily prove (i). Recall that here we have \( u_{r+1} < 0 \). Using the inequalities from Definition 9.3 (i) and slope orderedness of \( P \), we obtain

\[
\begin{align*}
  \frac{u_{i_0}}{u_{r+1}} &\geq m_{i_0} \geq m_{i_0 j}, \quad j = 1, \ldots, n_{i_0}, \\
  -\frac{u_{i_1}}{u_{r+1}} &\geq m_{i_1} \geq m_{i_1 j}, \quad j = 2, \ldots, n_{i_1}, \\
  -\frac{u_{1}}{u_{r+1}} &\geq m_{i_2} \geq m_{i_2 j}, \quad j = 1, \ldots, n_{i_2}, \quad i \neq i_0, i_1, \\
  -\frac{u_{r+1}}{u_{r+1}} &\geq m_{ij} \geq m_{ij} \quad \text{for all } i, j.
\end{align*}
\]

Together with the two equations of Definition 4.4 (i), this directly leads to the conditions of a Demazure \( P \)-root for \((u, i_0, i_1, C^+)\).

**Remark 9.5.** Let \( X = X(A, P) \) be non-toric and \( P \) irredudant. By Propositions 9.3 and 9.4, the horizontal Demazure \( P \)-roots map surjectively to the horizontal \( P \)-roots. Here \( \kappa \) and \( \kappa' \) have the same image if and only if \( \kappa = (u, i_0, i_1, C) \) and \( \kappa' = (u, i_0, i_1, C') \), where \( C \) and \( C' \) differ at most in the \( i_0 \)-th entry. In this case, the locally nilpotent derivations \( \delta_\kappa \) and \( \delta_{\kappa'} \) on \( \mathcal{R}(X) = R(A, P) \) coincide.

**Proposition 9.6.** Consider a rational projective \( \mathbb{K}^* \)-surface \( X = X(A, P) \), assume \( P \) to be irredudant and let \( 0 \leq i_0, i_1 \leq r \).

(i) Let \( X \) have an elliptic fixed point \( x^+ \). If there is a horizontal \( P \)-root \( u \) at \( (x^+, i_0, i_1) \), then \( x^+ \) is simple, quasismooth with leading indices \( i_0, i_1 \) and

\[
0 < m^+ \leq -u_{r+1} m^+ \leq \frac{1}{l_{i_1 1}}.
\]

Additionally, the presence of a horizontal \( P \)-root \( u \) at \( (x^+, i_0, i_1) \), forces \( D_{i_1}^2 \geq 0 \) for all \( i = 0, \ldots, r \) with \( i \neq i_0 \).
Let \( X \) have an elliptic fixed point \( x^- \). If there is a horizontal \( P \)-root at \( (x^-, i_0, i_1) \), then \( x^- \) is simple, quasismooth with leading indices \( i_0, i_1 \) and
\[
0 > m^- \geq u_{r+1}m^- \geq \frac{1}{l_{i_1 n_{i_1}}}.
\]

Additionally, the presence of a horizontal \( P \)-root \( u \) at \( (x^-, i_0, i_1) \) forces \( D_{i_{n_i}}^2 \geq 0 \) for all \( i = 0, \ldots, r \) with \( i \neq i_0 \).

Moreover, there exists at most one elliptic fixed point in \( X \) admitting a horizontal \( P \)-root.

**Proof.** The estimates can be treated at once, writing \( x = x^+, x^- \). Proposition 6.10 tells us that \( x \) is quasismooth with leading indices \( i_0, i_1 \). Moreover, according to Proposition 9.7, the horizontal \( P \)-root \( u \) satisfies the assumptions of Proposition 8.10. Thus, \( x \) is simple and we obtain the desired estimates. For the self complement is a consequence of Theorem 8.4.

Our next step is to identify the horizontal \( P \)-roots as certain integers contained in intervals \( \Delta(\iota, \kappa) \subseteq \mathbb{Q}_{\geq 0} \), which in turn are extracted in the following way from the defining matrix \( P \).

**Construction 9.7.** Consider the defining matrix \( P \) of \( X(A, P) \). For \( 0 \leq i, k \leq r \), define rational numbers
\[
\eta_k := \frac{1}{l_{kn_k}m}, \quad \xi_i := \begin{cases} 0, & n_i = 1, \\
\frac{1}{l_{in_i}(m_{in_i-1} - m_{in_i})}, & n_i \geq 2.
\end{cases}
\]

Then all \( \xi_i \) and \( \eta_k \) are non-negative. Moreover, for \( 0 \leq i, k \leq r \) with \( i \neq k \), consider the sets
\[
[\xi_i, \eta_k] = \{ t \in \mathbb{Q} : \xi_i \leq t \leq \eta_k \} \subseteq \mathbb{Q}_{\geq 0}.
\]

Note that \( [\xi_i, \eta_k] \) may be empty. Finally, for any two \( 0 \leq \iota, \kappa \leq r \), we have the intersections
\[
\Delta(\iota, \kappa) = \bigcap_{i \neq \iota} [\xi_i, \eta_k] \subseteq \mathbb{Q}_{\geq 0}.
\]

**Remark 9.8.** Using Remark 8.4, we can express the length of the intervals \([\xi_i, \eta_k]\) from Construction 9.7 via intersection numbers:
\[
\eta_k - \xi_i = l_{in_i}D_{in_i}^2 + (l_{in_i} - l_{kn_k})D_{kn_k}D_{kn_k}.
\]

Moreover, for any two \( 0 \leq \iota, \kappa \leq r \), the (possibly empty) set \( \Delta(\iota, \kappa) \) is explicitly given as
\[
\Delta(\iota, \kappa) = \left[ \max \left( 0, \frac{1}{l_{in_i}(m_{in_i-1} - m_{in_i})} \right), \frac{1}{l_{kn_k}m} \right].
\]

**Definition 9.9.** Consider the defining matrix \( P \) of \( X(A, P) \) and let \( 0 \leq i_0, i_1 \leq r \). Set \( e'_0 := 0 \in \mathbb{Z}^{r+1} \) and \( e'_i := e_i \in \mathbb{Z}^{r+1} \) for \( i = 1, \ldots, r + 1 \). Given \( \gamma \in \mathbb{Q} \), define
\[
u(i_0, i_1, \gamma) := \gamma e'_{r+1} - \frac{1}{l_{i_1 n_{i_1}}} (e'_{i_1} - e'_0) - \gamma \sum_{i \neq i_0, r+1} m_{in_i} (e'_i - e'_0) \in \mathbb{Q}^{r+1}.
\]
Lemma 9.10. Let $u := u(i_0, i_1, \gamma)$ be as in Definition 9.5. Then the evaluation of $u$ at a column $v_{ij}$ of the matrix $P$ is given by

$$\langle u, v_{ij} \rangle = l_{ij} \left( \gamma m_{ij} + \frac{1}{l_{i_1 n_1}} + \gamma \sum_{i \neq i_0} m_{in} \right),$$

$$\langle u, v_{i1} \rangle = l_{i1} \left( \gamma m_{i1} - \frac{1}{l_{i_1 n_1}} - \gamma m_{i1 n_1} \right),$$

$$\langle u, v_{ij} \rangle = l_{ij} \left( \gamma m_{ij} - \gamma m_{i1 n_1} \right), \quad i \neq i_0, i_1.$$

Proof. Set $e_0 := -e_1 - \ldots - e_r \in \mathbb{Z}^{r+1}$. Then the evaluations of $e'_0 = 0 \in \mathbb{Z}^{r+1}$ and $e'_i = e_i \in \mathbb{Z}^{r+1}$, where $i = 0, \ldots, r+1$, at $e_0, e_1, \ldots, e_{r+1}$ are

$$\langle e'_i, e_k \rangle = \begin{cases} 0, & i = 0, \\ 1, & 1 \leq i \leq r, k = 0, \\ -1, & 1 \leq i \leq r, k = i, \\ 0, & 1 \leq i \leq r, k \neq i. \end{cases} \quad \langle e'_{r+1}, e_k \rangle = \delta_{r+1,k}.$$

Here, as usual, we define $\delta_{ik} := 1$ if $i = k$ and $\delta_{ik} := 0$ if $i \neq k$. Consequently, for all $0 \leq i, k \leq r$ with $i \neq i_0$ and $0 \leq i, k \leq r+1$ we obtain

$$\langle e'_i - e'_i, e_k \rangle = \begin{cases} \delta_{ik}, & k \neq i_0, \\ -1, & k = i_0. \end{cases}$$

This enables us to verify the assertion by explicitly evaluating $u = u(i_0, i_1, \gamma)$ at the vectors $v_{ij} = l_{ij} e_i + d_{ij} e_{r+1}$.

Proposition 9.11. Assume that $X = X(A, P)$ has an elliptic fixed point $x^- \in X$ and let $0 \leq i_0, i_1 \leq r$. Then we have mutually inverse bijections:

$$\begin{cases} \text{horizontal } P\text{-roots} \\ \text{u at } (x^-, i_0, i_1) \end{cases} \quad \longleftrightarrow \quad \begin{cases} \text{integers } \gamma \in \Delta(i_0, i_1) \text{ such} \\ \text{that } \gamma d_{i_1 n_1} \equiv -1 \mod l_{i_1 n_1} \end{cases} \quad \begin{cases} u(i_0, i_1, \gamma) \\ \gamma \end{cases}$$

Proof. Given a horizontal $P$-root $u$ at $(x^-, i_0, i_1)$, we use Lemma 9.10 to see $u = u(i_0, i_1, \gamma)$ for $\gamma := u_{r+1}$ by comparing the values of $u$ and $u(i_0, i_1, \gamma)$ at the vectors $v_{in}$ for $i \neq i_0$. Now consider $0 \leq i_0, i_1 \leq r$ and any vector $u \in \mathbb{Z}^r \times \mathbb{Z}_{>0}$. Then we have

$$\langle u, v_{i_0 n_0} \rangle \geq 0 \iff u_{r+1} \leq \eta_{i_0}.$$ 

Moreover, if $n_{i_1} > 1$, then

$$\langle u, v_{i_1 n_1-1} \rangle \geq 0 \iff u_{r+1} \geq \xi_{i_1}.$$ 

Finally, if $i \neq i_0, i_1$, then

$$\langle u, v_{i_1 n_1-1} \rangle \geq l_{i_1 n_1-1} \iff u_{r+1} \geq \xi_{i_1}.$$ 

So, the inequalities of Definition 9.11(ii) are satisfied if and only if $u_{r+1} \in \Delta(i_0, i_1)$ holds. Thus, if $u$ is a horizontal $P$-root $u$ at $(x^-, i_0, i_1)$, then $u_{r+1} \in \Delta(i_0, i_1)$ and $u_{r+1} d_{i_1 n_1} \equiv -1 \mod l_{i_1 n_1}$ holds due to

$$-1 = \langle u, v_{i_1 n_1} \rangle = u_{i_1} l_{i_1 n_1} + u_{r+1} d_{i_1 n_1}.$$ 

Conversely, given any $\gamma \in \Delta(i_0, i_1)$ with $\gamma d_{i_1 n_1} \equiv -1 \mod l_{i_1 n_1}$, we directly verify that the vector $u(i_0, i_1, \gamma)$ is a horizontal $P$-root at $(x^-, i_0, i_1)$ having $\gamma$ as its last coordinate. \qed
Proposition 9.12. Let \( X = X(A, P) \) have an elliptic fixed point \( x^- \in X \) and let \( u \) be a horizontal \( P \)-root at \( (x^-; i_0; i_1) \). Then \( x^- \in X \) is quasismooth simple with leading indices \( i_0; i_1 \).

(i) If \( l_{i_0 n_{i_0}} \leq l_{i_1 n_{i_1}} \) holds, then \( x^- \) is smooth, we have \( \langle u, v_{i_0 n_{i_0}} \rangle = 0 \) and \( u \) is the only horizontal \( P \)-root at \( (x^-; i_0; i_1) \).

(ii) If the point \( x^- \in X \) is singular, then \( l_{i_0 n_{i_0}} > l_{i_1 n_{i_1}} \) holds and \( i_0 \) is the exceptional index of \( x^- \in X \).

Proof. Proposition 9.6 tells us that \( x^- \in X \) is quasismooth simple with leading indices \( i_0; i_1 \). Moreover, the second assertion is an immediate consequence of the first one and Proposition 8.10. Thus, we only have to prove the first assertion.

Suppose that there are two distinct horizontal \( P \)-roots at \( (x^-; i_0; i_1) \). Then, by Proposition 9.11 they are given as \( u(i_0, i_1, \gamma) \) and \( u(i_0, i_1, \gamma') \) with positive integers \( \gamma, \gamma' \in \Delta(i_0, i_1) \) differing by a non-zero integral multiple of \( l_{i_1 n_{i_1}} \). We conclude

\[
\eta_{i_1} = -\frac{1}{l_{i_1 n_{i_1}} m} > l_{i_1 n_{i_1}} \geq l_{i_0 n_{i_0}}.
\]

This implies \( -m^- = l_{i_1 n_{i_1}} l_{i_0 n_{i_0}} m^- > -1 \) which contradicts to Remark 6.3. Thus, there exists only one horizontal \( P \)-root \( u = u(i_0, i_1, \gamma) \) at \( (x^-; i_0; i_1) \). We show \( \langle u, v_{i_0 n_{i_0}} \rangle = 0 \). Otherwise Lemma 9.10 yields

\[
\langle u, v_{i_0 n_{i_0}} \rangle = l_{i_0 n_{i_0}} \left( \gamma m^- + \frac{1}{l_{i_1 n_{i_1}}} \right) \geq 1.
\]

This implies

\[
\gamma m^- \geq \frac{1}{l_{i_0 n_{i_0}}} - \frac{1}{l_{i_1 n_{i_1}}} \geq 0.
\]

Again we arrive at a contradiction to Remark 6.3, telling us \( m^- < 0 \). Thus \( \langle u, v_{i_0 n_{i_0}} \rangle = 0 \) holds. According to Lemma 9.10 this forces

\[
\gamma = -\frac{1}{l_{i_1 n_{i_1}} m^-} = -\frac{l_{i_0 n_{i_0}}}{l^{-m^-}}.
\]

In particular, as \( \gamma \) is an integer, \( l^{-m^-} \) divides \( l_{i_0 n_{i_0}} \). Moreover, making use of \( \gamma d_{i_1 n_{i_1}} \equiv -1 \mod l_{i_1 n_{i_1}} \), we obtain an integer

\[
a := \frac{1}{l_{i_1 n_{i_1}}} + \gamma m_{i_1 n_{i_1}} = \frac{m^- - m_{i_1 n_{i_1}}}{l_{i_1 n_{i_1}} m} = \frac{d_{i_0 n_{i_0}}}{l^{-m^-}} + \frac{l_{i_0 n_{i_0}}}{l^{-m^-}} \left( \sum_{i \notin \{i_0; i_1\}} d_{i n_i} \right).
\]

Thus, \( l^{-m^-} \) also divides \( d_{i_0 n_{i_0}} \). Since \( l_{i_0 n_{i_0}} \) and \( d_{i_0 n_{i_0}} \) are coprime, we arrive at \( l^{-m^-} = -1 \). Proposition 6.10 yields that \( x^- \) is smooth. \( \square \)

Corollary 9.13. Consider \( X = X(A, P) \) with a smooth elliptic fixed point \( x^- \in X \) and fix \( 0 \leq i_0, i_1 \leq r \) such that \( l_{m_i} = 1 \) for all \( i \neq i_0, i_1 \). Let \( \varepsilon \in \mathbb{Z} \) be maximal with \( l_{i_0 n_{i_0}} - \varepsilon l_{i_1 n_{i_1}} \geq 0 \).

For every integer \( \mu \) with \( 0 \leq \mu \leq \varepsilon \), set \( u_\mu := u(i_0, i_1, l_{i_0 n_{i_0}} - \mu l_{i_1 n_{i_1}}) \) according to Definition 7.7. Then the following holds.

(i) For every \( 0 \leq \mu \leq \varepsilon \) the linear form \( u_\mu \) is a horizontal \( P \)-root at \( (x^-; i_0, i_1) \) and we have \( \langle u_\mu, v_{i_0} \rangle = \mu \).

(ii) There exist horizontal \( P \)-roots at \( (x^-; i_0, i_1) \) if and only if \( \varepsilon \geq 0 \) holds.

Moreover, \( u_0, \ldots, u_\varepsilon \) are the only horizontal \( P \)-roots at \( (x^-; i_0, i_1) \).

(iii) If \( u \) is a horizontal \( P \)-root at \( (x^-; i_0, i_1) \) then, for any two \( 0 \leq \mu \leq \alpha \leq \varepsilon \), we have \( u_\mu = u_\alpha - (\alpha - \mu)u \).
Proposition 9.15. Consider a rational projective $K^*$-surface $X = X(A, P)$ where $P$ is irredundant.

(i) Assume that there is a parabolic fixed point curve $D^+ \subseteq X$. A vertical $P$-root at $D^+$ is a vector $u \in Z^r \times Z_{<0}$ such that
\[ (u, v^+) = -1, \quad (u, v_i) \geq 0, \quad i = 0, \ldots, r. \]

(ii) Assume that there is a parabolic fixed point curve $D^- \subseteq X$. A vertical $P$-root at $D^-$ is a vector $u \in Z^r \times Z_{>0}$ such that
\[ (u, v^-) = -1, \quad (u, v_i) \geq 0, \quad i = 0, \ldots, r. \]

Proposition 9.15. Consider a rational projective $K^*$-surface $X(A, P)$, assume $P$ to be irredundant and let $u \in Z^{r+1}$.

(i) If there is a curve $D^+ \subseteq X$, then the following statements are equivalent:
(a) $u$ is a vertical $P$-root at $D^+$,
(b) $u_{r+1} = 1$ and $u_i \geq m_i$ for all $i = 0, \ldots, r$,
(c) $u_{r+1} = 1$ and $u_i \geq m_{ij}$ for all $i = 0, \ldots, r$, $j = 1, \ldots, n_i$.

If one of the statements (a), (b) or (c) holds, then we have $(D^+)^2 \geq 0$.

(ii) If there is a curve $D^- \subseteq X$, then the following statements are equivalent:
(a) $u$ is a vertical $P$-root at $D^-$,
(b) $u_{r+1} = 1$ and $u_i \leq m_i$ for all $i = 0, \ldots, r$,
(c) $u_{r+1} = 1$ and $u_i \leq m_{ij}$ for all $i = 0, \ldots, r$, $j = 1, \ldots, n_i$.

If one of the statements (a), (b) or (c) holds, then we have $(D^-)^2 \geq 0$.
In particular, \((u,k) \mapsto u\) defines a one-to-one correspondence between the vertical Demazure \(P\)-roots and the vertical \(P\)-roots.

**Proof.** In each of the items, the equivalence of (a) and (b) is clear by Remark 9.2 and the equivalence of (b) and (c) holds due to slope orderliness. The assertions on the self intersection numbers are clear by the definition of vertical \(P\)-roots and Remarks 9.4 and 9.5. □

**Corollary 9.16.** Let \(X(A,P)\) be a \(\mathbb{K}\)-surface, assume \(P\) to be irredundant, let \(u \in \mathbb{Z}^{r+1}\) and fix \(0 \leq i_0 \leq r\).

(i) Assume that there is a curve \(D^+ \subseteq X\). Then \(u\) is a vertical \(P\)-root at \(D^+\) if and only if

\[
u_i \geq m_{i_1} \text{ for all } i \neq i_0, r + 1, \sum_{i \neq i_0, r+1} u_i \leq -m_{i_1}.
\]

(ii) Assume that there is a curve \(D^- \subseteq X\). Then \(u\) is a vertical \(P\)-root at \(D^-\) if and only if

\[
u_i \geq -m_{i_{n_i}} \text{ for all } i \neq i_0, r + 1, \sum_{i \neq i_0, r+1} u_i \leq m_{i_{n_i}}.
\]

**Proposition 9.17.** Let \(X = X(A,P)\) be non-toric and \(P\) irredundant. If there is a quasismooth simple elliptic fixed point \(x \in X\), then there are no vertical \(P\)-roots.

**Proof.** We may assume \(x = x^-\) having leading indices 0, 1, exceptional index 0 and that \(P\) is adapted to the sink. Suppose that \(D^+ \subseteq X\) admits a vertical \(P\)-root \(u \in \mathbb{Z}^{r+1}\). Then Proposition 9.16 yields

\[
u_i \geq m_{i_1} \text{ for } 1 \leq i \leq r, \quad -u_0 = u_1 + \ldots + u_r \leq -m_{i_1}.
\]

For \(i = 1, \ldots, r\), we have \(l_{i_{n_i}} = 1\) and thus \(m_{i_{n_i}} = 0\), as \(P\) is adapted to the sink. Irredundance of \(P\) implies \(m_{i_1} > 0\) and hence \(u_i \geq 1\) for \(i = 2, \ldots, r\). Using Proposition 8.10 (ii) for the inequality, we obtain

\[
m_{i_1} + (r-1) \leq u_1 + \ldots + u_r \leq -m_{i_1} \leq -m_{0n_0} \leq m_{i_{n_1}} + \frac{1}{l_{i_{n_1}}}.
\]

We claim \(r \leq 1\). For \(l_{i_{n_1}} \geq 2\) this follows from \(m_{i_1} \geq m_{i_{n_1}}\). If \(l_{i_{n_1}} = 1\), then \(m_{i_{n_1}} = 0\), hence \(m_{i_1} > 0\) by irredundance and the claim follows. Now, \(r \leq 1\) means that \(X\) is toric, which contradicts to our assumptions. □

**Proposition 9.18.** Let \(X = X(A,P)\) with \(P\) irredundant and assume that \(X\) has fixed point curves \(D^+\) and \(D^-\). If \(D^+\) admits a vertical \(P\)-root, then there is no vertical \(P\)-root at \(D^-\).

**Proof.** We may assume that \(P\) is adapted to the source. Let \(u^+ \in \mathbb{Z}^{r+1}\) be a vertical root at \(D^+\). Proposition 9.15 yields

\[
u_i^+ \geq m_{i_1} > -1 \text{ for } 1 \leq i \leq r, \quad u_0^+ = -u_1^+ - \ldots - u_r^+ \geq m_{i_1}.
\]

We conclude \(u_i^+ \geq 0\) for \(i = 1, \ldots, r\) and hence \(m_{i_1} \leq 0\). Now suppose that there is a vertical \(P\)-root \(u^-\) at \(D^-\). Then

\[
u_i^- \geq -m_{i_{n_i}} \geq -m_{i_1} \geq 0 \text{ for } 1 \leq i \leq r, \quad 0 \leq -u_0^- \leq m_{0n_0} \leq m_{i_1} \leq 0.
\]

Consequently \(m_{i_1} = m_{0n_0} = 0\), which in turn implies \(n_0 = 1\) and \(l_{i_1} = 1\). This contradicts to the assumption that \(P\) is irredundant. □
10. Generating root groups

In this section, we provide suitable generators for the unipotent part of the automorphism group of a non-toric rational projective $K^*$-surface.

**Definition 10.1.** Consider $X = X(A,P)$. We denote by $U(X) \subseteq \text{Aut}(X)$ the subgroup generated by all root groups of $X$.

Note that $U(X) \subseteq \text{Aut}(X)^0$ holds. We have two cases. The first one is that $U(X)$ is generated by the root groups stemming from horizontal P-roots. In this situation, we prove the following.

**Proposition 10.2.** Let $X = X(A,P)$ be non-toric with horizontal P-roots. Then there are a quasismooth simple elliptic fixed point $x \in X$ and $0 \leq i_0, i_1 \leq r$ such that $U(X)$ is generated by the root groups arising from horizontal P-roots at $(x, i_0, i_1)$ or $(x, i_1, i_0)$.

According to Proposition 9.17, the remaining case is that $U(X)$ is generated by the root groups given by the vertical roots. Here we obtain the following.

**Proposition 10.3.** Let $X = X(A,P)$ be non-toric with vertical P-roots at $D^+$ and let $0 \leq i_0, i_1 \leq r$. Then $U(X)$ is generated by the root groups arising from vertical P-roots $u$ at $D^+$ with $0 \leq \langle u, v_{i_1} \rangle < l_{i_1}$ for all $0 \leq i \leq r$ different from $i_0, i_1$.

We begin with discussing the horizontal case. First, we summarize the necessary background. By Proposition 9.11 all horizontal P-roots at $x^-$ are of the form $u(i_0, i_1, \gamma)$. According to Proposition 9.4, each such $u(i_0, i_1, \gamma)$ defines a Demazure P-root in the sense of Definition 3.6. Moreover, the following will be frequently used.

**Construction 3.6** associates with $\tau(i_0, i_1, \gamma)$ a locally nilpotent derivation on $R(A,P)$ which in turn gives rise to a root group

$$\lambda_{\tau(i_0, i_1, \gamma)} : K \to \text{Aut}(X).$$

Our statement involves the unique vectors $\beta = \beta(A, i_0, i_1)$ in the row space of the defining matrix $A$ having $i_0$-th coordinate zero and $i_1$-th coordinate one as introduced in Construction 3.6. Moreover, the following will be frequently used.

**Definition 10.4.** For the defining matrix $P$ of $X(A,P)$, we denote by $I(P) \subseteq \{0, \ldots, r\}$ the set of all indices $i$ with $l_{i_0} = 1$.

**Proposition 10.5.** Let $X = X(A,P)$ be non-toric with an elliptic fixed point $x^- \in X$. Then we obtain the following relations among the root subgroups associated with horizontal P-roots at $x^-$.

(i) Let $i_1, t_1 \in I(P)$ and $0 \leq i_0 \leq r$. If there are horizontal P-roots $u(i_0, i_1, \gamma)$ and $u(i_0, t_1, \gamma)$, then, for every $s \in K$, we have

$$\lambda_{\tau(i_0, i_1, \gamma)}(s) = \lambda_{\tau(i_0, t_1, \gamma)}(\beta(A, i_0, t_1)i_{i_1}s).$$

(ii) Let $i_0, t_0 \in I(P)$ and $0 \leq i_1 \leq r$. If there are horizontal P-roots $u(i_0, i_1, 1)$, $u(i_0, i_1, 1)$ and $u(i_1, t_0, \nu)$, $\nu = 1, \ldots, l_{i_1}n_1$, then, for every $s \in K$, we have

$$\lambda_{\tau(i_0, i_1, 1)}(s) = \lambda_{\tau(i_0, i_1, 1)}(s) \prod_{\nu=1}^{l_{i_1}n_1} \lambda_{\tau(i_1, t_0, \nu)}(\beta(A, i_0,i_1)\lambda_{\nu}(i_{i_1}n_1))s_{\nu}. $$

**Lemma 10.6.** Consider the defining matrix $A$ of $X = X(A,P)$. Then the vectors $\beta \in K^{r+1}$ introduced in Construction 3.6 satisfy

$$\beta(A, i_0, i_1) = (\beta(A, i_0, i_1))_{ii_1}^{-1}\beta(A, i_0, t_1),$$

$$\beta(A, i_1, t_0) = (\beta(A, i_0, i_1))_{ii_1} \beta(A, i_0, t_1) - \beta(A, t_0, i_1).$$
Consequently, we obtain Construction 3.6 satisfy Lemma 10.8. We prove the second identity. Due to the assumptions, we have

$$i$$

with

$$u$$

By Lemma 10.7, the linear form

$$\text{Proof of Proposition 10.5.}$$

Let

$$\nu = 1, \ldots , l_{i_1 n_1}$$

as we did in Construction 3.6 in the case of Demazure P-roots. Then, for any two indices $$i_1, t_1 \in I(P)$$, we have

$$u(i_0, t_1, i_1) = u(i_0, t_1, i_1).$$

Moreover, if $$l^{-1}m^{-1} = 1$$ and there is an $$i_1$$ with $$i \in I(P)$$ for all $$i \neq i_1$$, then, for any two $$i_0, t_1 \in I(P)$$, we have

$$u(i_0, i_1, 1)_P, v, l.$$ 

Proof. For the first identity, observe that we have $$\nu = 1$$. Now, using the definition of $$u(i_0, i_1, \gamma)$$ and $$l_{i_1 n_1} = l_{i_1 n_1} = 1$$, we compute

$$u(i_0, i_1, \gamma)1, t_1 = \gamma e_{r+1} + (e_{r+1} - e_{i_0}) - \sum_{i \neq i_0} \gamma m_{in_i} (e_{i_1} - e_{i_0}) + e_{i_1} - e_{i_1}$$

$$= \gamma e_{r+1} - (e_{i_1} - e_{i_0}) - \sum_{i \neq i_0} \gamma m_{in_i} (e_{i_1} - e_{i_0})$$

$$= \gamma e_{r+1} - (e_{i_1} - e_{i_0}) - \sum_{i \neq i_1} \gamma m_{in_i} (e_{i_1} - e_{i_0})$$

We prove the second identity. Due to the assumptions, we have $$l_{i_1 n_1}^{-1} = -m_{i_1 n_1}$$. Consequently, we obtain

$$u(i_0, t_1, \gamma) = \nu (e_{r+1} + m_{i_1 n_1} (e_{i_1} - e_{i_0}) - \sum_{i \neq i_0} \gamma m_{in_i} (e_{i_1} - e_{i_0})) + e_{i_1} - e_{i_1}$$

$$= \nu e_{r+1} + (e_{r+1} - e_{i_0}) - \sum_{i \neq i_1} \gamma m_{in_i} (e_{i_1} - e_{i_0})$$

$$= u(i_0, i_1, \nu).$$

□

Lemma 10.8. Consider the defining matrix P of X(A, P) and the linear forms $$u(i_0, i_1, \gamma)$$ from Definition 3.3. Define

$$u(i_0, i_1, \gamma)_P, v, l.$$ 

Proof. By Lemma 10.7 the linear form $$u(i_0, i_1, \gamma)$$ equals $$u(i_0, t_1, i_1)1, t_1$$. From Construction 3.6 we infer how the latter evaluates and conclude

$$\frac{h^u(i_0, t_1, \gamma)}{h^l(i_0, i_1, \gamma)} = \frac{h^u(i_0, t_1, \gamma)}{h^l(i_0, i_1, \gamma)}.$$ 

□

Proof of Proposition 10.5. We prove (i). It suffices to verify the corresponding relation for the locally nilpotent derivations associated with $$\tau(i_0, i_1, \gamma)$$ and $$\tau(i_0, i_1, \gamma)$$; see Construction 3.6. Lemmas 10.6 and 10.8 yield

$$\beta(A, i_0, t_1)_{i_1} \delta_{\tau(i_0, i_1, \gamma)} = \delta_{\tau(i_0, i_1, \gamma)}.$$ 

We turn to (ii). First consider the map \( \varphi_{u(i_0,i_1,1)}(s) \) as given in Theorem 5.4. For the \( \alpha(s, \nu, \iota) \) defined there, we write

\[
\alpha_{i_0,i_1,1} := \alpha_{i_0,i_1,1}(s) := \beta(A, i_0, i_1) s_{i_1,1} \left( l_{i_1,1}^{n_{i_1}} \right) s_{\nu},
\]

which allows to specify in the case of varying \( i_0 \) and \( i_1 \). Now, using Lemma 10.6 and \( \iota, i_1 \in I(P) \), we obtain

\[
\varphi_{u(i_0,i_1,1)}(s) = \prod_{\nu \neq i_0, i_1} \lambda_{u(i_0,i_1,1),\nu}(\alpha_{i_0,i_1,1})
\]

\[
= \prod_{\nu = 1} \lambda_{u(i_1,1,\nu)}(\alpha_{i_0,i_1,1},-\alpha_{i_0,i_1,1}),
\]

where for the last equation, we used \( \varphi_{u(1,i_0,i_1)}(s) = \varphi_{u(1,i_0,i_1)}(-s) \). Next we observe

\[
\alpha_{i_0,i_1,1} = 0, \quad \alpha_{i_0,i_1,1} - \alpha_{i_0,i_1,1} = \beta(A, i_1, i_0) \beta(A, i_0, i_1) s_{i_0,1} \left( l_{i_0,1}^{n_{i_0}} \right) s_{\nu},
\]

where the second identity follows from Lemma 10.6. With the aid of these considerations, we compute

\[
\psi := \varphi_{u(i_0,i_1,1)}(s)\varphi_{u(i_0,i_1,1)}(s)
\]

\[
= \prod_{\nu = 1} \lambda_{u(i_1,1,\nu)}(\alpha_{i_0,i_1,1})\lambda_{u(i_1,1,\nu)}(-\alpha_{i_0,i_1,1})
\]

\[
= \prod_{\nu = 1} \lambda_{u(i_1,1,\nu)}(\alpha_{i_0,i_1,1})\lambda_{u(i_1,1,\nu)}(-\alpha_{i_0,i_1,1})
\]

\[
= \prod_{\nu = 1} \lambda_{u(i_1,1,\nu)}(\beta(A, i_0, i_1) s_{i_1,1} \left( l_{i_1,1}^{n_{i_1}} \right) s_{\nu})
\]

\[
= \prod_{\nu = 1} \lambda_{u(i_1,1,\nu)}(\beta(A, i_1, i_0) s_{i_0,1} \left( l_{i_0,1}^{n_{i_0}} \right) s_{\nu}),
\]

\[
= \prod_{\nu = 1} \lambda_{u(i_1,1,\nu)}(\beta(A, i_0, i_1) s_{i_0,1} \left( l_{i_0,1}^{n_{i_0}} \right) s_{\nu}),
\]

\[
= \prod_{\nu = 1} \lambda_{u(i_1,1,\nu)}(\beta(A, i_0, i_1) s_{i_0,1} \left( l_{i_0,1}^{n_{i_0}} \right) s_{\nu}),
\]

\[
= \prod_{\nu = 1} \lambda_{u(i_1,1,\nu)}(\beta(A, i_0, i_1) s_{i_0,1} \left( l_{i_0,1}^{n_{i_0}} \right) s_{\nu}).
\]
Now, let \( \lambda \in \) and \( i \). Moreover, there is a \( \kappa \) one, use the equality is due to (ii). The second one holds by definition. For the third equality, use Theorem 5.4 we obtain

\[
\lambda_{\tau(io,i1,1)}(s) = \lambda\varphi_{\tau(io,i1,1)}(s) = \lambda\varphi_{\tau(io,i1,1)}(s)\psi = \lambda_{\tau(io,i1,1)}(s)\psi.
\]

\( \Box \)

**Definition 10.9.** Provided that the defining matrix \( P \) of \( X = X(A,P) \) is adapted to the sink in the sense of Definition (7.4) (ii), we say that \( P \) is normalized if \( l_{0n} \geq \ldots \geq l_{rn} \), and for all \( i < j \) with \( l_{in} = l_{jn} \) and \( n_i, n_j \geq 2 \), we have

\[
m_{in} - m_{in} \leq m_{jn} - m_{jn}.
\]

**Remark 10.10.** The above definition of normalized coincides with the one given in the introduction as Remark 5.4 ensures

\[
m_{in} - m_{in} \leq m_{jn} - m_{jn} \iff D_{in}^2 \leq D_{jn}^2.
\]

**Lemma 10.11.** Let the defining matrix \( P \) of \( X = X(A,P) \) be adapted to the sink and normalized. Consider the intervals \( [\xi, \eta] \) and \( \Delta(i, \kappa) \) from Construction 9.7.

(i) If \( i, \ell \in I(P) \) satisfy \( i \leq \ell \), then we have

\[
[\xi, \eta] \subseteq [\xi, \eta], \quad \Delta(i,k) \subseteq \Delta(i,k).
\]

(ii) For any two \( k, \ell \in I(P) \) and every \( i = 0, \ldots, r \), we have

\[
[\xi, \eta] = [\xi, \eta], \quad \Delta(i,k) = \Delta(i,k).
\]

(iii) Assume \( l^m = -1 \). Let \( i \) with \( \ell \neq \ell \) and \( k \geq 2 \). Then

\[
1 \in \Delta(k,i) \Rightarrow \Delta(i,k) \cap \mathbb{Z} = [1, l_{in}] \cap \mathbb{Z}.
\]

**Proof.** Irredundance of \( P \) implies \( n_i \geq 2 \) for all \( i \in I(P) \). Consider \( i, \ell, k \in I(P) \) with \( i \leq \ell \). Then Construction 9.7 and the fact that \( P \) is normalized yield

\[
\xi_i = \frac{1}{m_{in} - m_{in}}, \quad \eta_i = \frac{1}{m_{in} - m_{in}}, \quad \xi_{\ell}, \quad \eta_{\ell} = -\frac{m_{rn}}{m_{rn}}.
\]

This gives the first assertion. The second one is obvious. For the third one, observe

\[
l_{in} = \ldots = l_{rn} = 1.
\]

Thus \( l^m = l_{0n} \) and \( l_{0n} \neq -1 \). We conclude

\[
\eta_k = -\frac{1}{m_{rn}} = \frac{l_{0n}}{l_{rn}}, \quad \kappa \geq 1, \quad \kappa = 0.
\]

Now, let \( 1 \in \Delta(k,i) \). Then, by the definition of \( \Delta(k,i) \), we have \( \xi_i \leq 1 \) for all \( i \neq k \). Moreover, there is a \( \kappa \in \{0,1\} \cap I(P) \) with \( k \neq i \). We claim

\[
\Delta(i,k) \cap \mathbb{Z} = \Delta(i,k) \cap \mathbb{Z} = \bigcap_{1 \neq \ell} [\xi_i, \eta_i] \cap \mathbb{Z} = [1, \eta_i] \cap \mathbb{Z} = [1, l_{in}] \cap \mathbb{Z}.
\]

The first equality is due to (ii). The second one holds by definition. For the third one, use \( k \geq 2 \) to see \( \xi_k \leq 1 \). For the last equality, let \( l_{in} = 1 \) for \( \kappa = 0 \) and \( \kappa = 1, i = 0 \Rightarrow l_{in} = l_{0n}, \quad \kappa = 1, i \geq 2 \Rightarrow l_{in} = 1 = l_{0n} \).

**Proof of Proposition 10.2.** Proposition 9.6 tells us that there is a unique elliptic fixed point \( x \in X \) admitting horizontal \( P \)-roots. We may assume that \( x = x^- \) holds and that \( P \) is adapted to the sink and normalized. We claim that then \( i_0 = 0 \) and \( i_1 = 1 \) are as wanted. So, given any horizontal \( P \)-root \( u(i_0, i_1, \gamma) \) at \( (x, i_0, i_1) \), the task is to show that the associated root group maps into the subgroup generated by all the root subgroups arising from horizontal \( P \)-roots at \( (x, 0, 1) \) and \( (x, 1, 0) \).
Lemma 9.10 shows \( \gamma \). Hence, there is a horizontal \( P \)-root \( \beta \). Let \( \iota \). Let \( \iota = 0 \) and \( \iota \neq 1 \). Then we have \( l_{1n_1} = l_{1n_1} = 1 \). Moreover, using Lemma 10.11 (i), we see \( \gamma \in \Delta(0, 1) = (0, 1) \). Thus, Proposition 10.5 (i) applies and we obtain

\[ \lambda_{\tau(0, 1, \gamma)}(K) = \lambda_{\tau(0, 1, \gamma)}(K). \]

Let \( \iota = 0 \) and \( \iota \neq 1 \). Then we have \( l_{1n_1} = 1 \). Proposition 0.12 (i) says that \( x^- \) is smooth and that \( \langle u, v_{10n_1} \rangle = 0 \) holds. According to Lemma 0.10 the latter means

\[ 0 = \gamma m^- + \frac{1}{l_{1n_1}} = \gamma m^- + \frac{1}{l_{0n_1}} = -\gamma l_{0n_1} + \frac{1}{l_{0n_1}}. \]

where the last equality is due to Proposition 6.10 showing \( l_{0n_1}m^- = l^-m^- = -1 \). We conclude \( \gamma = 1 \). Proposition 9.11 gives \( 1 \in \Delta(0, 0) \). Now, Lemma 10.11 (i) shows that \( 1 \in \Delta(0, 0) = \Delta(1, 0) \), hence there is a horizontal \( P \)-root \( u(0, 1, 1) \). Furthermore, Lemma 10.11 (i) shows

\[ \Delta(0, 1) \cap Z = [1, l_{0n_1}] \cap Z. \]

Consequently, there is a horizontal \( P \)-root \( u(0, 1, \nu) \) for every \( 1 \leq \nu \leq l_{0n_1} \). Now Proposition 10.5 tells us

\[ \lambda_{\tau(0, 0, 1)}(K) \subseteq \lambda_{\tau(0, 1, \nu)}(K) \prod_{\nu=1}^{l_{0n_1}} \lambda_{\tau(0, 1, \nu)}(K). \]

We enter the vertical case. According to Proposition 9.15 every vertical \( P \)-root \( u \) corresponds to a vertical Demazure \( P \)-root \( \kappa = (u, i) \) and via the associated locally nilpotent derivation of \( R(A, P) \) we obtain the root group

\[ \lambda_{\kappa} = \lambda_{\kappa} : K \to \text{Aut}(X). \]

Lemma 10.12. Let \( A \in \operatorname{Mat}(2, r + 1; K) \) and \( g_{i1, i2, i3} \) be as in Construction 2.4 and \( \beta(i1, i2, A), \beta(i2, i1, A) \) as in Construction 2.6. Then there is a \( b_{i1, i2, i3} \in K^* \) with

\[ b_{i1, i2, i3} g_{i1, i2, i3} = T_{i3}^{i13} - \beta(i1, i2, A)_{i3} T_{i1}^{i1} - \beta(i2, i1, A)_{i3} T_{i2}^{i1}. \]

Proof. Consider \( A' = [a_{i1}, a_{i2}, a_{i3}] \in \operatorname{Mat}(2, 3, K) \). As a direct computation shows, we have

\[ B \cdot A' = \begin{bmatrix} 1 & 0 & \beta(A, i2, i1)_{i3} \\ 0 & 1 & \beta(A, i1, i2)_{i3} \end{bmatrix} \]

with a unique matrix \( B \in \text{GL}(2, K) \). Setting \( b_{i1, i2, i3} := \det(B)^{-1} \), we infer the assertion from

\[ g_{i1, i2, i3} = b_{i1, i2, i3} \det \begin{bmatrix} T_{i11}^{i1} & T_{i12}^{i1} & T_{i13}^{i1} \\ 1 & 0 & \beta(A, i2, i1)_{i3} \\ 0 & 1 & \beta(A, i1, i2)_{i3} \end{bmatrix}. \]

\[ \square \]
Lemma 10.13. Consider an $X = X(A,P)$ with a curve $D^+ \subseteq X$. Fix $u \in \mathbb{Z}^{r+1}$ and $0 \leq i \leq r$. For any $0 \leq i_0 \leq r$ with $i_0 \neq i$ set
\[ u_{i,i_0} := u + e_i - e_i', \quad e_i' := 0, \quad e_i := e_i, \quad i = 0, \ldots, r. \]
Now assume that $u$ is a vertical $P$-root at $D^+$ with $(u,v_{ij}) \geq l_{ij}$ and let $0 \leq i_0,i_1 \leq r$ with $i_0 \neq i_1$. Then $u_{i,i_0}$ and $u_{i,i_1}$ are vertical $P$-roots at $D^+$ and we have
\[ \lambda_u(s) = \lambda_{u_{i,i_0}}(\beta(A,i_1,i_0),s)\lambda_{u_{i,i_1}}(\beta(A,i_0,i_1),s). \]

Proof. First let us see how $u_{i,i_0}$ evaluates on the vectors $v_{ij}$, $v^+$ and $v^-$, if present. We have
\[ \langle u_{i,i_0}, v_{ij} \rangle = \begin{cases} \langle u, v_{ij} \rangle, & i \neq i_0, \\ \langle u, v_{ij} \rangle - l_{ij}, & i = i_0, \\ \langle u_{i,i_0} v_j \rangle + l_{i0j}, & i = i_0, \end{cases} \]
In particular, we see that $u_{i,i_0}$ is a vertical $P$-root at $D^+$: using Remark 9.2 and slope-orderness of $P$, we infer $(u_{i,i_0}, v_{ij}) \geq 0$ for any $j = 1, \ldots, n$, from
\[ (u,v_{ij}) \geq l_{ij} \Rightarrow u_i \geq 1 + m_{i1} \Rightarrow u_i \geq 1 + m_{ij} \Rightarrow (u,v_{ij}) \geq l_{ij}. \]
Moreover, we observe that the locally nilpotent derivation $\delta_u$ provided by Construction 3.9 gives us a polynomial
\[ f := \delta_u(S_1)T_{i}^{-1} = S_1T_{i}^{-1}\prod_{i,j} T_{ij}^{l_{ij}} \in \mathbb{K}[T_{ij}, S_1]. \]
Next we claim that the locally nilpotent derivation $\delta_u$ on $R(A,P)$ coincides with $\beta(A,i_1,i_0)\delta_{u_{i,i_0}} + \beta(A,i_0,i_1)\delta_{u_{i,i_1}}$. Indeed, we compute
\[ \delta_u(S_1) - f g_{i_0,i_1} g_{i_1,i_0} = \delta_u(S_1) - f \left( T_{i} - \beta(A,i_1,i_0)T_{i}^{l_{i0}} - \beta(A,i_0,i_1)T_{i}^{l_{i1}} \right) = \beta(A,i_1,i_0)fT_{i}^{l_{i0}} + \beta(A,i_0,i_1)fT_{i}^{l_{i1}} = \beta(A,i_0,i_1)f_{i_0,i_1}(S_1) + \beta(A,i_1,i_0)f_{i_1,i_0}(S_1), \]
using Lemma 10.12. Computing the associated root groups according to Proposition 10.3 (i) gives the assertion. \hfill \Box

Proof of Proposition 10.3 According to Propositions 9.17 and 9.18 the group $U(X)$ is generated by the root groups arising from vertical $P$-roots at $D^+$. Given a vertical $P$-root $u$ with $(u,v_{ij}) \geq l_{ij}$, take any two distinct $0 \leq i_0,i_1 \leq r$ differing from $i$. Then Lemma 10.13 tells us
\[ \lambda_u(\mathbb{K}) \subseteq \lambda_{u_{i,i_0}}(\mathbb{K})\lambda_{u_{i,i_1}}(\mathbb{K}). \]
Recall that the evaluations of the linear forms $u_{i,i_0}$ and $u_{i,i_1}$ at the vectors $v_{ij}$ are given by
\[ \langle u_{i,i_0}, v_{ij} \rangle = \begin{cases} \langle u, v_{ij} \rangle, & i \neq i_0, \\ \langle u, v_{ij} \rangle - l_{ij}, & i = i_0, \end{cases} \quad \langle u_{i,i_1}, v_{ij} \rangle = \begin{cases} \langle u, v_{ij} \rangle, & i \neq i_1, \\ \langle u, v_{ij} \rangle - l_{ij}, & i = i_1. \end{cases} \]
Thus, the automorphism $\lambda_u(s)$ can be expressed as a composition of automorphisms stemming from vertical $P$-roots evaluating strictly smaller at $v_{ij}$ and equal to $u$ at all other $v_{ij}$ with $i \neq i_0,i_1$. Suitably iterating this process, we arrive at the assertion. \hfill \Box

Definition 10.14. Consider the defining matrix $P$ of $X(A,P)$ and let $0 \leq i_0,i_1 \leq r$. Define an interval
\[ \Gamma(i_0,i_1) := \left[ m_{i11}, -m_{i01} - \sum_{i \neq i_0,i_1} m_{i1} \right] \subseteq \mathbb{Q}. \]
Moreover, denote $e'_0 := 0 \in \mathbb{Z}^{r+1}$ and $e'_i := e_i \in \mathbb{Z}^{r+1}$ for $i = 1, \ldots, r + 1$. Given $\alpha \in \mathbb{Q}$, define
\[ u(i_0, i_1, \alpha) := -e'_{r+1} + \alpha(e'_{i_1} - e'_{i_0}) + \sum_{i \neq i_0, i_1, r+1} [m_{i1}](e'_i - e'_{i_0}) \in \mathbb{Q}^{r+1}. \]

**Proposition 10.15.** Assume that $X = X(A, P)$ has a parabolic fixed point curve $D^+ \subseteq X$. Then we have mutually inverse bijections
\[ \left\{ \begin{array}{l} \text{vertical } P \text{-roots } u \text{ at } D^+ \text{ such that } \langle u, v_{i1} \rangle < l_{i_1} \text{ for all } i \neq i_0, i_1 \end{array} \right\} \longmapsto \Gamma(i_0, i_1) \cap \mathbb{Z}
\]
\[ u \mapsto u_{i_1} \]
\[ (i_0, i_1, \alpha) \mapsto \alpha \]

**Proof.** First we consider any vertical $P$-root $u$ at $D^+$. Let $u_0 = -u_1 - \ldots - u_r$ as in Remark 9.2 and set $\varepsilon_i := u_i - m_i$. Using Proposition 9.15 we obtain
\[ m_{i1} \leq \langle u, v_{i1} \rangle = u_il_{i_1} - d_{i_1} = \varepsilon_il_{i_1}. \]
Now let $u$ stem from the left hand side set above. Then we must have $0 \leq \varepsilon_i < 1$ and hence $u_i = [m_{i1}]$ for all $i \neq i_0, i_1$. Corollary 11.6 yields
\[ m_{i11} \leq u_{i_1} \leq m_{i1} - \sum_{i \neq i_0, i_1} u_i = -m_{i01} - \sum_{i \neq i_0, i_1} [m_{i1}]. \]
One directly checks that any $\alpha \in \Gamma(i_0, i_1) \cap \mathbb{Z}$ delivers an $(i_0, i_1, \alpha)$ in the left hand side set and the assignments are inverse to each other. \hfill \Box

11. **Root groups and resolution of singularities**

In this section, we show how to lift the root groups arising from the horizontal or vertical $P$-roots of $X = X(A, P)$ with respect to the minimal resolution of singularities $\pi: \tilde{X} \to X$. The following theorem gathers the essential results; observe that items (iii) and (iv) are as well direct consequences of the general existence of a functorial resolution in characteristic zero, whereas (i) and (ii), used later, are more specific.

**Theorem 11.1.** Consider $X = X(A, P)$ and its minimal resolution $\pi: \tilde{X} \to X$, where $\tilde{X} = X(A, \tilde{P})$.

(i) There is a natural bijection $\lambda \mapsto \tilde{\lambda}$ between the root groups of $X$ and those of $\tilde{X}$, made concrete in terms of defining data in Propositions 11.5 and 11.7.

(ii) For every root group $\lambda: \mathbb{K} \to \text{Aut}(X)$ and every $s \in \mathbb{K}$ we have a commutative diagram
\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\lambda(s)} & \tilde{X} \\
\pi \downarrow & & \downarrow \pi \\
X & \xrightarrow{\lambda(s)} & X
\end{array}
\]

(iii) The isomorphism $\pi: \pi^{-1}(X_{\text{reg}}) \to X_{\text{reg}}$ gives rise to a canonical isomorphism of groups $\text{Aut}(X) \cong \text{Aut}(\tilde{X})$.

(iv) Every action $G \times X \to X$ of a connected algebraic group $G$ lifts to an action $G \times \tilde{X} \to \tilde{X}$.

The proof of Theorem 11.1 essentially relies on the preceding results showing that we either have only horizontal roots at a common simple quasismooth elliptic fixed point or there are only vertical roots at a common parabolic fixed point curve. This allows us to relate the resolution of singularities closely to resolving toric surface...
singularities. We begin the preparing discussion with a brief reminder on Hilbert bases of two-dimensional cones and then enter the horizontal case.

**Remark 11.2.** Consider two primitive vectors \( v_0 \) and \( v_1 \) in \( \mathbb{Z}^2 \). Assume \( \det(v_0, v_1) \) to be positive. Set \( v'_0 := v_0 \) and let \( v'_1 \in \mathbb{Z}^2 \) be the unique vector with
\[
v'_1 \in \cone(v'_0, v_1), \quad \det(v'_0, v'_1) = 1, \quad 0 \leq \det(v'_1, v_1) < \det(v'_0, v_1).
\]
Iterating gives us a finite sequence \( v_0 = v'_0, v'_1, \ldots, v'_q, v'_{q+1} = v_2 \), the Hilbert basis \( \mathcal{H}(\sigma) \) of \( \sigma = \cone(v_0, v_1) \in \mathbb{Z}^2 \). We have
\[
\det(v'_t, v'_{t+1}) = 1, \quad v'_{t-1} + v'_{t+1} = c_t v'_t
\]
with unique integers \( c_1, \ldots, c_q \in \mathbb{Z}_{\geq 2} \). Subdividing \( \sigma \) along the Hilbert basis gives us the fan of the minimal resolution of the affine toric surface \( Z_\sigma \).

**Construction 11.3.** Assume that \( X = X(A, P) \) has a quasismooth elliptic fixed point \( x^- \in X \) with leading indices \( i_0, i_1 \). Consider
\[
v_0 := (-l_{0n_0}, d_{0n_0}), \quad v_1 := (l_{1n_1}, d_{1n_1}), \quad \sigma := \cone(v_0, v_1).
\]
As earlier, let \( e_0 = -e_1 - \ldots - e_r \), where \( e_i \in \mathbb{Z}^{r+1} \) are the canonical basis vectors. Then, with every \( v' = (l', d') \in \mathcal{H}(\sigma) \), we associate \( \bar{v} \in \mathbb{Z}^{r+1} \) by
\[
\bar{v} := \begin{cases}
-l' e_{0n_0} + d' e_{r+1}, & l' < 0, \\
l' e_{1n_1} + d' e_{r+1}, & l' > 0, \\
-e_{r+1}, & l' = 0.
\end{cases}
\]
Inserting the columns \( \bar{v} \), where \( v_0, v_1 \neq v' \in \mathcal{H}(\sigma) \), at suitable places of \( P \) produces a slope ordered defining matrix \( P' \).

**Proposition 11.4.** Let \( A, P \) and \( P' \) be as in [11.3]. Consider \( X' := X(A, P') \) and the natural morphism \( \pi': X' \to X \). Then \( \pi' \) is an isomorphism over \( X \setminus \{x^-\} \), each \( x' \in X' \) over \( x^- \in X \) is smooth and the minimal resolution \( \bar{X} \to X \) factors as
\[
\begin{array}{ccc}
\bar{X} & \xrightarrow{\pi} & X' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\pi} & x'
\end{array}
\]

**Proof.** We may assume that \( P' \) is adapted to the sink. Remark 6.3 ensures \( \det(v_0, v_1) > 0 \). Let \( v'_0, \ldots, v'_{q+1} \) be the members of \( \mathcal{H}(\sigma) \), constructed as in Remark 11.2. Write \( v'_t = (l'_t, d'_t) \). There are unique integers \( 1 \leq k^- < k^+ \leq q \) with
\[
l'_i < 0 \quad \text{for} \quad i = 0, \ldots, k^-, \quad l'_i > 0 \quad \text{for} \quad i = k^+, \ldots, q + 1,
\]
where \( k^+ = k^- + 1 \) if all \( l'_i \) differ from zero and otherwise we have \( k^+ = k^- + 2 \) and \( l'_{k^0} = 0 \) for \( k^0 = k^- + 1 \). The curve of \( X' \) corresponding to a column \( \bar{v}_i \) of \( P' \) lies in \( A_{\bar{v}} \) if \( l'_i < 0 \), in \( A_i \) if \( l'_i > 0 \) and equals \( D^i \) if \( i = k^0 \).

We verify smoothness of the points \( x' \in X' \) lying over \( x^- \in X \). First consider the case that \( x' \in X' \) is a parabolic or a hyperbolic fixed point. According to Remark 11.2, we have
\[
\det(v'_t, v'_{t+1}) = 1.
\]
This gives us precisely the smoothness conditions from Propositions 6.8 (ii) and 6.9. If \( x' \in X' \) is an elliptic fixed point, then we can apply Proposition 6.10 to obtain smoothness of \( x' \) in terms of \( P' \):
\[
l^- \bar{m}^- = \sum l_{0n_0} d_{1n_1} \bar{a}_{1n_1} + \sum l_{1n_1} d_{0n_0} \bar{a}_{0n_0} = -l'_{k^-} d'_{k^+} + l'_{k^+} d'_{k^-} = -\det(v_{k^-}, v_{k^+}) = -1.
\]

We show minimality of \( X' \to X \). The conditions \( v'_{t-1} + v'_{t+1} = c_t v'_t \) from Remark 11.2 translate to the equations Lemma 7.3 for the exceptional curves \( E_i \) of \( X' \to X \) corresponding to \( v'_t \). We conclude \( E^2_t \leq -2 \).
Proposition 11.5. Consider the minimal resolution $\pi: \tilde{X} \to X$ of $K^*$-surfaces defined by $(A,P)$ and $(A,\tilde{P})$. Assume that there is a quasismooth simple elliptic fixed point $x^- \in X$ and let $\tilde{x}^- \in \pi^{-1}(x^-)$ be the corresponding elliptic fixed point. Given $0 \leq i_0, i_1 \leq r$ and $u \in \mathbb{Z}^{r+1}$, the following statements are equivalent.

(i) The linear form $u \in \mathbb{Z}^{r+1}$ is a horizontal $P$-root at $(x^-, i_0, i_1)$.

(ii) The linear form $u \in \mathbb{Z}^{r+1}$ is a horizontal $\tilde{P}$-root at $(\tilde{x}^-, i_0, i_1)$.

Proof. As $x^- \in X$ is a simple elliptic fixed point with exceptional index $i_0$, we have $\tilde{v}_{i_0} = v_{i_0}$ for all $i \neq i_0$. The fiber of $\pi: \tilde{X} \to X$ over $x^- \in X$ is of the form

$$\pi^{-1}(x^-) = \tilde{D}_{i_0} \cap \ldots \cap \tilde{D}_{i_0}.$$

By Proposition 11.4, the corresponding columns $\tilde{v}_{i_0} \in \mathbb{Z}^{r+1}$ of $\tilde{P}$ are obtained by running Construction 11.3 with the initial data

$$v_0 := (-l_{i_0} \cdot d_{i_0}), \quad v_1 := (l_{i_0} \cdot d_{i_0}).$$

In the notation of Remark 11.2, the vector $\tilde{v}_{i_0} \in \mathbb{Z}^{r+1}$ stems from the penultimate Hilbert basis member $v'_i \in \mathcal{H}(\sigma)$ of $\sigma = \text{cone}(v_0, v_1)$ which is determined by the conditions

$$\det(v'_i, v_1) = 1, \quad 0 \leq \det(v_0, v'_i) < \det(v_0, v_1).$$

In terms of $\tilde{P}$, we have $v'_i = -(\tilde{l}_{i_0} \cdot d_{i_0})$. Together with the definitions of $v_0$ and $v_1$, this gives us

$$\tilde{l}_{i_0} \cdot d_{i_0} \equiv -1 \mod l_{i_1}, \quad -1 \leq \tilde{l}_{i_0} \cdot d_{i_0} \leq -\frac{1}{l_{i_0}} \cdot d_{i_0},$$

where the estimate is obtained by resolving the first characterizing condition of $v'_i$ for $\tilde{d}_{i_0}$ and plugging the result into the second one. Next look at

$$v_0 := (l_{i_0} \cdot d_{i_1}), \quad v_1 := (l_{i_0} \cdot d_{i_1})$$

in case $n_i \geq 1$. Then, according to Remark 11.2 the Hilbert basis member $v'_i \in \mathcal{H}(\sigma)$ of $\sigma = \text{cone}(v_0, v_1)$ is characterized by the conditions

$$\det(v_0, v'_i) = 1, \quad 0 \leq \det(v'_i, v_1) < \det(v_0, v_1).$$

Similarly as before, we have $v'_1 = (\tilde{l}_{i_0} \cdot d_{i_1} \cdot d_{i_1})$ and making the above conditions explicit, we arrive at

$$\tilde{l}_{i_0} \cdot d_{i_1} \equiv -1 \mod l_{i_1}, \quad \frac{1}{l_{i_1} \cdot (m_{n_i} - m_{n_i})} \leq \tilde{l}_{i_0} \cdot d_{i_1} \leq \frac{1}{l_{i_0} \cdot (m_{n_i} - m_{n_i})} + l_{i_1},$$

Now consider a horizontal $P$-root $u \in \mathbb{Z}^{r+1}$ at $(i_0, i_1, x^-)$. Proposition 11.11 yields $u = u(i_0, i_1, \gamma)$ with a non-negative integer $\gamma$ satisfying $\gamma d_{i_1} \equiv -1 \mod l_{i_1}$ and

$$\frac{1}{l_{i_1} \cdot (m_{n_i} - m_{n_i})} \leq \gamma \leq \frac{1}{l_{i_0} \cdot m_{n_i}},$$

for all $i = 0, \ldots, r$ with $i \neq i_0$ and $n_i > 1$. We compare $\gamma$ with $\tilde{l}_{i_0} \cdot d_{i_1}$ and $\tilde{l}_{i_1} \cdot d_{i_1}$. First, using the modular identities, we observe

$$(\tilde{l}_{i_0} \cdot d_{i_1} \cdot d_{i_1} \cdot d_{i_1}) \in l_{i_1}, \quad (\gamma - \tilde{l}_{i_1} \cdot d_{i_1} \cdot d_{i_1}) \in l_{i_1}, \quad (\gamma - \tilde{l}_{i_1} \cdot d_{i_1} \cdot d_{i_1}) \in l_{i_1} \cdot \mathbb{Z}.$$

As $l_{i_1} \cdot d_{i_1}$ and $d_{i_1} \cdot d_{i_1}$ are coprime, $\tilde{l}_{i_0} \cdot d_{i_1} \cdot d_{i_1}$ as well as $\gamma - \tilde{l}_{i_1} \cdot d_{i_1} \cdot d_{i_1}$ are multiples of $l_{i_1} \cdot d_{i_1}$. Thus, the previous estimates and $l_{i_1} = 1$ for $i \neq i_0$ give us

$$\tilde{l}_{i_1} \leq \gamma \leq \tilde{l}_{i_0} \cdot d_{i_1}, \quad i = 0, \ldots, r, \quad i \neq i_0, \quad n_i > 1.$$
Now we can directly check the defining conditions of a horizontal $\tilde{P}$-root at $(\tilde{x}, i_0, i_1)$ for $u = u(i_0, i_1, \gamma)$: Lemma 9.10 together with Propositions 6.9 and 6.10 yields

$$ \langle u, \tilde{v}_{i_0 n_{i_0}} \rangle = \frac{i_0 n_{i_0} - \gamma}{l_{i_1 n_{i_1}}} \geq 0, $$

$$ \langle u, \tilde{v}_{i_1 n_{i_1} - 1} \rangle = \frac{\gamma - i_1 n_{i_1} - 1}{l_{i_1 n_{i_1} - 1}} \geq 0, $$

$$ \langle u, v_{i_1 n_{i_1}} \rangle = \gamma \geq l_{i_1 n_{i_1} - 1}, $$

where $i \neq i_0, i_1$ with $n_i > 1$ in the last case. This verifies "(i)⇒(ii)". The reverse implication is a direct consequence of Proposition 9.4.

**Remark 11.6.** From the proof of Proposition 11.5 we infer that $\gamma = \tilde{i}_{i_0 n_{i_0}}$ is the maximal integer such that $u(i_0, i_1, \gamma)$ is a horizontal $P$-root $\tilde{x}$.

**Proposition 11.7.** Consider the minimal resolution $\pi: \tilde{X} \to X$ of $K$-surfaces defined by $(A, P)$ and $(A, \tilde{P})$. Assume that there is a parabolic fixed point curve $D^+ \subseteq X$ and let $\tilde{D}^+ \subseteq \tilde{X}$ be the proper transform. Given $u \in Z^{r+1}$, the following statements are equivalent.

(i) The linear form $u \in Z^{r+1}$ is a vertical $P$-root at $D^+$.

(ii) The linear form $u \in Z^{r+1}$ is a vertical $\tilde{P}$-root at $\tilde{D}^+$.

**Proof.** The implication "(ii)⇒(i)" is clear due to Proposition 9.15. We care about "(i)⇒(ii)". By Remark 6.13, the columns $\tilde{v}_{i_1}, \ldots, \tilde{v}_{i_r} = v_{i_1}$ of $\tilde{P}$, where $i = 0, \ldots, r$, arise from subdividing cone$(v_{i_1}, v^+)$ along the Hilbert basis. Consider

$$ v_0 := (0, -1), \quad v_1 := (l_1, -d_{i_1}), $$

where $i = 0, \ldots, r$. In the setting of Remark 11.2, we have $v_i' = (\tilde{l}_{i_1}, -\tilde{d}_{i_1})$. Moreover, the conditions on the determinants lead to

$$ 1 = \det(v_0, v_1') = \tilde{l}_{i_1}, \quad l_1 \tilde{d}_{i_1} - d_{i_1} = \det(v', v_1) < \det(v_0, v_1) = l_{i_1}. $$

Using also slope orderedness of $\tilde{P}$, we see $m_{i_1} \leq \tilde{d}_{i_1} < m_{i_1} + 1$. Now, let $u \in Z^{r+1}$ be a vertical $P$-root at $D^+$. Proposition 9.15 ensures $u_i \geq m_{i_1}$. This implies $u_i \geq \tilde{d}_{i_1}$. Using Proposition 6.13 again, we obtain that $u$ is a vertical $\tilde{P}$-root at $\tilde{D}^+$. □

**Proposition 11.8.** Consider $\tilde{X} = X(A, \tilde{P})$ and $X = X(A, P)$, where each column of $P$ also occurs as a column of $\tilde{P}$.

(i) There is a proper birational morphism $\pi: \tilde{X} \to X$ contracting precisely the curves $\tilde{D}_{ij}$ and $\tilde{D}^+$, where $\tilde{v}_{ij}$ and $\tilde{v}^+$ is not a column of $P$.

(ii) If there is an elliptic fixed point $\tilde{x}^- \in \tilde{X}$ and $u$ is a horizontal $P$-root at $(\tilde{x}^-, i_0, i_1)$. Then $x^- = \pi(\tilde{x}^-) \in X$ is an elliptic fixed point forming the sink and $u$ is a horizontal $P$-root at $(x^-, i_0, i_1)$.

(iii) If we have a parabolic source $\tilde{D}^+ \subseteq X$ and $u$ is a vertical $\tilde{P}$-root at $\tilde{D}^+$, then $D^+ = \pi(\tilde{D}^+) \subseteq X$ is a curve forming the source and $u$ is a vertical $P$-root at $D^+$.

Moreover, the root groups $\lambda: K \to \text{Aut}(\tilde{X})$ and $\lambda: K \to \text{Aut}(X)$ arising from a common root $u$ fit for every $s \in K$ into the commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\lambda(s)} & \tilde{X} \\
\downarrow{\pi} & \quad & \downarrow{\pi} \\
X & \xrightarrow{\lambda(s)} & X
\end{array}
$$
Proof. For (i) observe that each cone of the fan $\tilde{\Sigma}$ of the ambient toric variety $\tilde{Z}$ of $\tilde{X}$ is contained in a cone of the fan $\Sigma$ of the ambient toric variety $Z$ of $X$. The corresponding toric morphism $\tilde{Z} \to Z$ restricts to the morphism $\pi : \tilde{X} \to X$. Assertion (ii) is clear by Proposition 9.4 and (iii) follows from Proposition 9.15.

We prove the supplement. Consider the Cox rings $R(A, P)$ of $X$ and $R(A, \tilde{P})$ of $\tilde{X}$. Recall that these are given as factor algebras

$$R(A, P) = \mathbb{K}[T_{ij}, S^\pm]/\langle g_I; I \in \mathfrak{I} \rangle, \quad R(A, \tilde{P}) = \mathbb{K}[\tilde{T}_{ij}, \tilde{S}^\pm]/\langle \tilde{g}_I; I \in \tilde{\mathfrak{I}} \rangle,$$

where $R(A, P)$ is graded by $K = \text{Cl}(X)$ and $R(A, \tilde{P})$ by $\tilde{K} = \text{Cl}(\tilde{X})$ see Construction 2.3 for the details. Define a homomorphism of the graded polynomial algebras

$$\Psi : \mathbb{K}[\tilde{T}_{ij}, \tilde{S}^\pm] \to \mathbb{K}[T_{ij}, S^\pm]$$

by sending $\tilde{T}_{ij}$ and $\tilde{S}^\pm$ to the variables of $\mathbb{K}[T_{ij}, S^\pm]$ corresponding to $\pi(\tilde{D}_{ij})$ and $\pi(\tilde{D}^\pm)$ in case these divisors are not exceptional and to $1 \in \mathbb{K}[T_{ij}, S^\pm]$ otherwise. Then $\Psi$ descends to a homomorphism

$$\psi : R(A, \tilde{P}) \to R(A, P).$$

Now note that any Demazure root $\delta_a$ on the fan $\tilde{\Sigma}$ having $\tilde{P}$ as generator matrix is as well a Demazure root on the fan $\Sigma$ having $P$ as generator matrix. Moreover, we have a commutative diagram

$$\begin{array}{ccc}
\mathbb{K}[T_{ij}, S^\pm] & \xrightarrow{\delta_a} & \mathbb{K}[T_{ij}, S^\pm] \\
\phi \downarrow & & \downarrow \phi \\
\mathbb{K}[T_{ij}, S^\pm] & \xrightarrow{\delta_a} & \mathbb{K}[T_{ij}, S^\pm]
\end{array}$$

Next, given a horizontal or vertical $P$-root and its corresponding $\tilde{P}$-root, we look at the associated Demazure $P$-root $\kappa$ and Demazure $\tilde{P}$-root $\tilde{\kappa}$. Presenting $\lambda_a(s)^*$ and $\lambda_{\tilde{\kappa}}(s)^*$ as in Theorem 5.4 and using commutativity of the previous diagram we see that the following diagram commutes as well:

$$\begin{array}{ccc}
R(A, \tilde{P}) & \xrightarrow{\lambda_{\tilde{\kappa}}(s)} & R(A, P) \\
\downarrow \psi & & \downarrow \psi \\
R(A, P) & \xrightarrow{\lambda_\kappa(s)} & R(A, P)
\end{array}$$

Cover $\pi^{-1}(X_{\text{reg}})$ by affine open subsets of the form $\tilde{X}_{[\tilde{D}], f}$, where $\tilde{D}$ is a Weil divisor of $\tilde{X}$ having $\tilde{f} \in R(A, \tilde{P})$ as a section and $\tilde{X}_{[\tilde{D}], f}$ is obtained by removing the support of $\tilde{D} + \text{div} (\tilde{f})$ from $\tilde{X}$. Set $D = \pi_*\tilde{D}$ and $f = \psi(\tilde{f})$. Then we have commutative diagrams

$$\begin{array}{ccc}
R(A, \tilde{P})_f & \xrightarrow{\psi} & R(A, P)_f \\
\downarrow & & \downarrow \\
\Gamma(\tilde{X}_{[\tilde{D}], f}, O_{\tilde{X}}) & \xrightarrow{\psi_0 \pi_*} & \Gamma(X_{[D], f}, O_X)
\end{array}$$

where the lower row represents the degree zero part of the upper one. The homomorphisms $\psi_0$ and $\pi^*$ in the lower row are directly seen to be inverse to each other; see [1] Prop. 1.5.2.4]. Passing to the spectra and gluing gives us a commutative
diagram

\[
p^{-1}(X_{\text{reg}}) \xrightarrow{\varphi} \tilde{p}^{-1}(\pi^{-1}(X_{\text{reg}}))
\]

\[
\xrightarrow{\varphi_0 \circ \pi} \pi^{-1}(X_{\text{reg}})
\]

where \( p \) and \( \tilde{p} \) denote the quotients of characteristic spaces of \( X \) and \( \tilde{X} \) by the respective characteristic quasitori; use again [11 Prop. 1.5.2.4]. By construction, the morphisms \( \varphi \) arising from \( \psi \) and \( \varphi_0 \) arising from \( \psi_0 \) satisfy

\[
\varphi \circ \tilde{\lambda}_a(s) = \tilde{\lambda}_a(s) \circ \varphi, \quad \varphi_0 \circ \lambda_a(s) = \lambda_a(s) \circ \varphi_0.
\]

\[\square\]

**Proof of Theorem 11.1** Propositions [11.3] and [11.7] provide us with a bijection between the \( P \)-roots and the \( P \)-roots. Applying Proposition [11.8], we obtain proves the second assertion of the Theorem. Assertions (iii) and (iv) are then direct consequences. \[\square\]

12. Structure of the automorphism group

Here we prove Theorem 11.1. In a first step, we express the number of necessary \( P \)-roots to generate the unipotent part of the automorphism group of a \( K^* \)-surface \( X = X(A, P) \) in terms of intersection numbers of invariant curves of \( X \). We make use of the numbers defined in the introduction:

\[
c_i(D^+) = \text{CF}_{\bar{q}}(-E_{i_1}^2, \ldots, -E_{i_1}^2, -1)^{-1}, \quad c(x^{-}) = \text{CF}_{\bar{q}}(-E_{q}^2, \ldots, -E_{1}^2, -1)^{-1},
\]

where \( E_{ij} \subseteq \tilde{X} \) are the exceptional curves lying over \( D^+ \subseteq X \) and the \( E_i \subseteq \tilde{X} \) over \( x^{-} \in X \) with respect to the minimal resolution of singularities \( \tilde{X} \rightarrow X \).

**Definition 12.1.** Let \( X = X(A, P) \) be non-toric with a fixed point curve \( D^+ \subseteq X \). Given \( 0 \leq i_0, i_1 \leq r \), we call a vertical \( P \)-root \( u \) at \( D^+ \) essential with respect to \( i_0, i_1 \), if \( 0 \leq (u, v_{i_1}) < l_{i_1} \) for all \( i \neq i_0, i_1 \).

**Proposition 12.2.** Consider a non-toric \( K^* \)-surface \( X = X(A, P) \).

(i) Assume that there is a curve \( D^+ \subseteq X \) and let \( 0 \leq i_0, i_1 \leq r \). Then the number of vertical \( P \)-roots at \( D^+ \) essential to \( i_0, i_1 \) is given by

\[
\max(0, (D^+)^2 + 1 - \sum_{i=0}^{r} c_i(D^+)).
\]

(ii) Assume that there is a quasismooth simple \( x^{-} \in X \) and that \( P \) is normalized. Then the number of horizontal \( P \)-roots at \( x^{-}, 0, 1 \) is given by

\[
\max\left(0, \left[l_{1n_1}^{-1} \min_{i \neq 0} \left( l_{in_1} D_{in_1}^2 + (l_{in_1} - l_{1n_1}) D_{in_1} D_{0n_1} - c(x^{-}) \right) + 1 \right] \right).
\]

(iii) Assume that there is a quasismooth simple \( x^{-} \in X \) and that \( P \) is normalized. Then there is a horizontal \( P \)-root at \( (x^{-}, 1, 0) \) if and only if

\[
D_{1n_1}^2 \geq (l_{0n_0} - l_{in_1}) D_{in_1} D_{0n_0}, \quad \text{for all } i \neq 1.
\]

Moreover, if these conditions hold, then \( x^{-} \in X \) is smooth and there exists precisely one horizontal \( P \)-root at \( (x^{-}, 1, 0) \).

**Proof.** Let \( \tilde{X} = X(A, \tilde{P}) \) be the minimal resolution. We verify (i). By Proposition [11.7], the numbers \( \rho \) of vertical \( P \)-roots and \( \tilde{\rho} \) of vertical \( \tilde{P} \)-roots essential to \( i_0, i_1 \) coincide. By Proposition [10.13] the number \( \tilde{\rho} \) equals the number of integers in the interval

\[
\tilde{\Gamma}(i_0, i_1) = [\bar{m}_{i_1}, -\bar{m}_{i_0}, 1 - \sum_{i \neq i_0, i_1} [\bar{m}]]
\]
Since $\tilde{X}$ is smooth, the slopes $\tilde{m}_{i0}, \ldots, \tilde{m}_{ir}$ are all integral numbers; see Proposition 6.8. Thus, we see that $\rho = \tilde{\rho}$ equals the maximum of zero and the number 
$$-	ilde{m}_{i0} - \tilde{m}_{i1} - \sum_{i \not= i_{0,1}} \tilde{m}_{i1} + 1 = -\tilde{m} + 1 = (\tilde{D}^+)^2 + 1,$$
where the last equality holds by Remark 6.4. We have $\tilde{D}_{ij} = E_i$ for $j = 1, \ldots, q_i$. Moreover, $\tilde{D}_{qi+1} = D_{i1}$ and $\tilde{m}_{i_{qi+1}} = m_{i1}$. Thus, applying Corollary 7.6 (i) with $j_i = q_i + 1$, we see that $\rho = \tilde{\rho}$ equals the maximum of zero and
$$(\tilde{D}^+)^2 + 1 = (D^+)^2 - \sum_{i=0}^r c_i(D^+) + 1.$$ We verify (ii). According to Proposition 9.11, the number $\rho$ of horizontal $P$-roots at $(x^-,0,1)$ equals the number of integers $\gamma$ satisfying
$$\gamma \in \Delta(0,1) = \bigcap_{i \not= 0} [\xi_i,\eta_i], \quad \gamma d_{i1n_1} \equiv -1 \mod l_{i1n_1},$$
see Construction 9.7 for the notation. By Remark 11.6, the maximal integer $\gamma$ satisfying these conditions is $\tilde{l}_{\eta_{0\eta}}$. Thus, we can replace $[\xi_i,\eta_i]$ with $[\xi_i,\tilde{l}_{\eta_{0\eta}}]$. So, the number of integers $\gamma$ in $[\xi_i,\tilde{l}_{\eta_{0\eta}}]$ with $\gamma d_{i1n_1} \equiv -1 \mod l_{i1n_1}$ is the maximum of zero and the round down $\vartheta(i) \in \mathbb{Z}$ of
$$\frac{\tilde{l}_{\eta_{0\eta}} - \xi_i}{l_{i1n_1}} + 1 = \frac{(\eta_i - \tilde{\xi}_i) - (\eta_i - \tilde{l}_{\eta_{0\eta}})}{l_{i1n_1}} + 1 = l_{i1n_1}^{-1}(l_{i1n_1}D_{i1n_1}^2 + (l_{i1n_1} - l_{i1n_2})D_{i1n_1} - c(x^-)) + 1.$$ Here, the second equality needs explanation. First, we express $\eta_i - \tilde{\xi}_i$ in terms of intersection numbers according to Remark 9.8. Moreover, the definition of $\eta_i$, Remark 6.3 quasismoothness of $x^-$ and Proposition 6.10 yield
$$\eta_i = \frac{1}{l_{i1n_1}m^-} - 1 - \det(\sigma^-), \quad l^- = l_{0n_0}l_{1n_1}.$$ Proposition 9.12 (ii) says that 0 is the exceptional index of $x_i \in X$ and thus $E_j = \tilde{D}_{0\eta_{0\eta}} - q_j$ holds for $j = 1, \ldots, q$. Using Corollary 7.6 (ii), we obtain
$$\eta_i - \tilde{l}_{\eta_{0\eta}} = \frac{l_{0\eta}}{\det(\sigma^-)} - \tilde{l}_{\eta_{0\eta}} = c(x^-).$$ Since $\Delta(0,1)$ is the intersection over the intervals $[\xi_i,\eta_i]$, where $i \not= i_0$, we see that the number of all the wanted $\gamma$ we have to take the minimum of the above round downs $\vartheta(i)$ as an upper bound.

We care about (iii). By Proposition 9.11, there exists a horizontal $P$-root at $(x^-,1,0)$ if and only if $\Delta(1,0)$ is non-empty. The latter precisely means $\eta_0 - \xi_i \geq 0$ for all $i \not= 1$. This in turn is equivalent to
$$l_{i1n_1}D_{i1n_1}^2 \geq (l_{0\eta} - l_{i1n_1})D_{i1n_1}D_{0\eta_{0\eta}}, \quad \text{for all } i \not= 1,$$ see Remark 9.8. Now, if there is a horizontal $P$-root at $(x^-,1,0)$, then Proposition 9.12 (i) yields that there is no further one and that $x^- \in X$ is smooth. \hfill $\Box$

The final puzzle piece for the proof of Theorem 1.1 is the following controled contraction $\tilde{X} \to X'$ of the minimal resolution $\tilde{X}$ of $X$ onto a toric surface that allows to keep track of the relevant roots.

**Proposition 12.3.** Let $\tilde{X} = X(A,\tilde{P})$ be a smooth non-toric $\mathbb{R}^*$-surface with a fixed point curve $\tilde{D}^+ \subseteq \tilde{X}$ and assume that $\tilde{P}$ is adapted to the sink.
Consider the data These, together with $v$ and $9.15$. Thus, $\text{Aut}(X)$,
the latter ones are given by horizontal and vertical one-parameter groups associated with the Demazure
$P$-roots at $(\tilde{x}^-,0,1)$ map injectively to the horizontal $P'$-roots at $(\pi(\tilde{x}^-),0,1)$ via
$\mathbb{Z}^{r+1} \ni u(0,1,\gamma) \mapsto u(0,1,\gamma) \in \mathbb{Z}^2$.
Similarly, the horizontal $\tilde{P}$-roots at $(\tilde{x}^-,1,0)$ map injectively to the horizontal $P'$-roots at $(\pi(\tilde{x}^-),1,0)$ via
$\mathbb{Z}^{r+1} \ni u(1,0,\gamma) \mapsto u(1,0,\gamma) \in \mathbb{Z}^2$.

(ii) Assume that $\tilde{D}^+$ admits vertical $\tilde{P}$-roots. Then there is a morphism $\tilde{X} \to X'$ onto the smooth toric $K^*$-surface $X'$ defined by the matrix
\[
P' = \begin{bmatrix} -l_{01} & \ldots & -l_{0n_0} & l_{11} & \ldots & l_{1n_1} & 0 \\
d_{01} & \ldots & d_{0n_0} & d_{11} & \ldots & d_{1n_1} & 1 \end{bmatrix}
\]

The image $\pi(\tilde{D}^+) \subseteq X'$ is a curve forming the source and the vertical $\tilde{P}$-roots at $\tilde{D}^+$ essential with respect to $0,1$ map injectively to the vertical $P'$-roots at $\pi(\tilde{D}^+)$ essential with respect to $0,1$ given by
$\mathbb{Z}^{r+1} \ni u(0,1,\alpha) \mapsto u(0,1,\alpha) \in \mathbb{Z}^2$.

Moreover, the root groups $\tilde{\lambda}: K \to \text{Aut}(\tilde{X})$ and $\lambda': K \to \text{Aut}(X')$ arising from a common root $u$ fit for every $s \in K$ into the commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\lambda}(s)} & \tilde{X} \\
\pi \downarrow & & \downarrow \pi \\
X' & \xrightarrow{\lambda'(s)} & X' \\
\end{array}
\]

Proof. First observe that we have $l_{in_i} = 1$ for all $i \geq 2$. In the setting of (i), this holds by Proposition 6.10. In the situation of (ii), Proposition 9.11 tells us that there must be a curve $D^- \subseteq \tilde{X}$. Thus, Proposition 6.8 ensures $l_{in_i} = 1$ even for all $i$. Since $P$ is adapted to the source, we have $d_{in_i} = 0$ for $i \geq 2$ in both cases. Consider the data

\[
n_{0i}^u = a_0, \quad v_{0j}^u = v_{0j}, \quad n_{1i}^u = a_1, \quad v_{1j}^u = v_{1j}, \quad n_{i}^u = 1, \quad v_{ji}^u = v_{ni}, \quad i \geq 2.
\]

These, together with $v^+$ in the setting of (i) and $v^+, v^-$ in the setting of (ii) are the columns of a matrix $P''$. It defines a $K^*$-surface $X'' = X(A,P'')$ which is smooth due to Propositions 6.8 and 6.10. Proposition 11.8 gives us a morphism $\tilde{X} \to X''$ having the desired properties concerning the roots and the associated root groups.

Now, the matrix $P''$ is highly redundant. Removing all these redundancies, that means erasing the column $v_{ni}$, and the $i$-th rows for $i = 2, \ldots, r$ turns $P''$ into $P'$. The $K^*$-surface $X''$ is isomorphic to the toric $K^*$-surface $X' = X(A',P')$, where $A'$ is the $2 \times 2$ unit matrix, and the $P''$-roots turn into $P'$-roots as claimed. \hfill \Box

Proof of Theorem 11.7 Let $X := X(A,P)$ be a non-toric $K^*$-surface. Theorem 5.8 on the automorphism group of a rational projective variety with torus action of complexity one says that $\text{Aut}(X)^0$ is generated by the acting torus and the additive one-parameter groups associated with the Demazure $P$-roots. In the surface case, the latter ones are given by horizontal and vertical $P$-roots; see Propositions 9.11 and 9.13. Thus, $\text{Aut}(X)^0 = K^*$ if and only if there are neither horizontal nor vertical
Horizontal $P$-roots only exist if $X$ admits a quasismooth simple elliptic fixed point, and in this case there is no other such fixed point; see Proposition 9.6 and Theorem 11.1. This setting is Case (ii) of Theorem 11.1. Moreover, existence of vertical $P$-roots requires a non-negative parabolic fixed point curve and excludes quasismooth simple elliptic fixed points; see Propositions 9.15 and 9.17. This setting restitutes Case (i) of Theorem 11.1. Recalling that $U(X) \subseteq \text{Aut}(X)^0$ denotes the subgroup generated by all root subgroups.

We determine Aut$(X)^0$ in Case (i) of Theorem 11.1. Thus, we have to deal with a non-negative parabolic fixed point curve hosting vertical $P$-roots, if present, and which we may assume to be $D^+ \subseteq X$. Moreover, we may assume that $P$ is adapted to the sink. Proposition 12.2 yields that $D^+$ is the only fixed point curve admitting vertical roots. Fix any two distinct $0 \leq i_0, i_1 \leq r$. Then Proposition 10.3 says that $U(X)$ is generated by all root groups arising from vertical $P$-roots being essential at $i_0, i_1$. Proposition 12.2 (i) shows that $\rho \in \text{Aut}(X)^0$ denotes the number of vertical $P$-roots essential to $i_0, i_1$. Theorems 11.1 and Proposition 12.3 realize $U(X)$ as the subgroup generated by Demazure roots at a common primitive ray generator of the automorphism group of a suitable toric surface $X'$. Moreover, the original $K^*$-action of $X$ is given on $X'$ by the one parameter group $K^* \to \mathbb{T}^2$ sending $t$ to $(1, t)$. Applying Proposition 12.2 yields the desired isomorphism $K^* \ltimes_{\psi} K^* \cong \text{Aut}(X)^0$.

We enter Case (ii) of Theorem 11.1. The pattern of arguments is similar to that of the preceding case. Now we have a unique quasismooth simple elliptic fixed point which we can assume to be $x^- \in X$. Moreover, we can assume $P$ to be normalized. By Proposition 12.2 the group $U(X)$ is generated by the root groups stemming from the horizontal $P$-roots at $(i_0, i_1, x^-)$ and $(i_1, i_0, x^-)$, where we may assume $i_0 = 0$ and $i_1 = 1$. Proposition 12.2 (ii) says that $i_0 = 0$ is the exceptional index of $x^- \in X$. Proposition 12.2 (ii) shows that $\rho$ and $\zeta$ from Theorem 11.1 (ii) equal the numbers of horizontal $P$-roots at $(0, 1, x^-)$ and $(1, 0, x^-)$, respectively. Theorems 11.1 and Proposition 12.3 realize $U(X)$ as the subgroup generated by Demazure roots at two common primitive ray generators of the automorphism group of a suitable toric surface $X'$. Here, using Proposition 10.12 (i) and Corollary 10.13 we see that the Demazure roots of $X'$ corresponding to horizontal $P$-roots at $(0, 1, x^-)$ and $(1, 0, x^-)$ are as in the setting of Proposition 14.4. Moreover, $K^*$ acts $X'$ via the one parameter group $K^* \to \mathbb{T}^2$ sending $t$ to $(1, t)$. Thus, Proposition 14.4 yields the desired isomorphism $(K^* \ltimes_{\psi} K^*) \ltimes_{\psi} K^* \cong \text{Aut}(X)^0$.

 References

[1] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, Cox rings, Cambridge Studies in Advanced Mathematics, vol. 144, Cambridge University Press, Cambridge, 2015. 33, 5, 6, 8, 15, 22, 24, 25, 51, 52

[2] Ivan Arzhantsev, Jürgen Hausen, Elaine Herppich, and Alvaro Liendo, The automorphism group of a variety with torus action of complexity one, Mosc. Math. J. 14 (2014), no. 3, 429–471, 641. 3, 4, 9, 11

[3] Ivan Cheltsov and Yuri Prokhorov, Del Pezzo surfaces with infinite automorphism groups, to appear in Algebraic Geometry, available at arxiv:2007.14202

[4] David A. Cox, The homogeneous coordinate ring of a toric variety, J. Algebraic Geom. 4 (1995), no. 1, 17–50. 3, 5, 8, 9

[5] David A. Cox, John B. Little, and Henry K. Schenck, Toric varieties, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. 5, 24

[6] V. I. Danilov, The geometry of toric varieties, Uspekhi Mat. Nauk 33 (1978), no. 2(200), 85–134, 247 (Russian). 5

[7] Michel Demazure, Sous-groupes algébriques de rang maximum du groupe de Cremona, Ann. Sci. École Norm. Sup. (4) 3 (1970), 507–588 (French). 3, 8

[8] Ulrich Derenthal and Daniel Loughran, Equivariant compactifications of two-dimensional algebraic groups, Proc. Edinb. Math. Soc. (2) 58 (2015), no. 1, 149–168. 3
56

JÜRGEN HAUSEN, TIMO HUMMEL

[9] Sergey Dzhunusov, Additive actions on complete toric surfaces (2019), available at 1908.03563
[10] Karl-Heinz Fieseler and Ludger Kaup, Fixed points, exceptional orbits, and homology of affine C∗-surfaces, Compositio Math. 78 (1991), no. 1, 79–115. ↑
[11] Sergey Dzhunusov, Additive actions on complete toric surfaces (2019), available at 1908.03563. ↑
[12] Karl-Heinz Fieseler and Ludger Kaup, Fixed points, exceptional orbits, and homology of affine C∗-surfaces, Compositio Math. 78 (1991), no. 1, 79–115. ↑
[13] Hubert Flenner and Mikhail Zaidenberg, Normal affine surfaces with C∗-actions, Osaka J. Math. 40 (2003), no. 4, 981–1009. ↑
[14] Gene Freudenburg, Algebraic theory of locally nilpotent derivations, 2nd ed., Encyclopedia of Mathematical Sciences, vol. 136, Springer-Verlag, Berlin, 2017. Invariant Theory and Algebraic Transformation Groups, VII. ↑
[15] William Fulton, Introduction to toric varieties, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry. ↑
[16] Jürgen Hausen and Elaine Herppich, Factorially graded rings of complexity one, Torsors, étale homotopy and applications to rational points, London Math. Soc. Lecture Note Ser., vol. 405, Cambridge Univ. Press, Cambridge, 2013, pp. 414–428. ↑
[17] Jürgen Hausen and Elaine Herppich, Factorially graded rings of complexity one, Torsors, étale homotopy and applications to rational points, London Math. Soc. Lecture Note Ser., vol. 405, Cambridge Univ. Press, Cambridge, 2013, pp. 414–428. ↑
[18] Jürgen Hausen and Hendrik Süß, The Cox ring of an algebraic variety with torus action, Adv. Math. 225 (2010), no. 2, 977–1012. ↑
[19] Jürgen Hausen and Milena Wrobel, On torus actions of higher complexity, Forum Math. Sigma 7 (2019), e38. ↑
[20] Jürgen Hausen and Hendrik Süß, The Cox ring of an algebraic variety with torus action, Adv. Math. 225 (2010), no. 2, 977–1012. ↑
[21] Jürgen Hausen and Hendrik Süß, The Cox ring of an algebraic variety with torus action, Adv. Math. 225 (2010), no. 2, 977–1012. ↑
[22] Gebhard Martin and Claudia Stadlmayr, Weak del Pezzo surfaces with global vector fields, available at arXiv:2007.03665. ↑
[23] Peter Orlik and Philip Wagreich, Isolated singularities of algebraic surfaces with C∗-action, Ann. of Math. (2) 93 (1971), 205–228. ↑
[24] Peter Orlik and Philip Wagreich, Isolated singularities of algebraic surfaces with C∗-action, Ann. of Math. (2) 93 (1971), 205–228. ↑
[25] Peter Orlik and Philip Wagreich, Isolated singularities of algebraic surfaces with C∗-action, Ann. of Math. (2) 93 (1971), 205–228. ↑
[26] H. Pinkham, Normal surface singularities with C∗-action, Math. Ann. 227 (1977), no. 2, 183–193. ↑
[27] Victor Przyjalkowski and Constantin Shramov, On automorphisms of quasi-smooth weighted complete intersections (2020), available at 2006.01213. ↑
[28] Yoshikyo Sakamaki, Automorphism groups on normal singular cubic surfaces with no parameters, Trans. Amer. Math. Soc. 362 (2010), no. 5, 2641–2666. ↑

Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany
Email address: juergen.hausen@uni-tuebingen.de

Email address: hummel@math.uni-tuebingen.de