Literal Rippling of Spacetime

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ABSTRACT:
The metric perturbation tensor corresponding to a transverse oscillation of spacetime is composed of products of cosines. When averaged over many wavelengths, such a metric may look either Minkowskian or Euclidean at large scales, depending on the amplitude and wavelength of the oscillation.
A curved surface may be described in terms of its intrinsic geometry or in relation to a higher-dimensional space in which it is embedded. In the case of a four-dimensional hypersurface, an embedding space of ten dimensions is required. It is well known that solutions of Einstein’s equation can be embedded in a space of ten dimensions; see for example [Rosen 1965], [Dirac 1975] or [Kramer et al. 1980].

Consider a surface which, like that of the ocean, experiences oscillations transverse to itself into the embedding space. If the surface is four-dimensional, there may be as many as six normal directions. Let

\[ w(x^\mu) = \sum_{I=1}^{N} A_I \sin k_{I\mu} x^\mu \]

(The component oscillations may be superposed because they are not assumed to be physical gravitational waves but are merely Fourier components used to build up the surface.) Then the 4-space metric is

\[ g_{\mu\nu} = \gamma_{\mu\nu} + w_{,\mu} w_{,\nu} = \gamma_{\mu\nu} + \sum_{I,J} k_{I\mu} k_{J\nu} A_I A_J \cos k_{I\alpha} x^\alpha \cos k_{J\beta} x^\beta \]

where \( \gamma_{\mu\nu} \) is the metric of the (flat) hyperplane tangent to the surface. It may be verified that

\[ g_{\mu\nu} = (\gamma^{-1})^{\mu\nu} - H(\gamma^{-1})^{\mu\lambda}(\gamma^{-1})^{\nu\kappa} w_{,\lambda} w_{,\kappa} \]

where

\[ H = 1/(1 + (\gamma^{-1})^{\rho\sigma} w_{,\rho} w_{,\sigma}) \]

In all of these equations, expressions such as \( w_{,\mu} w_{,\nu} \) are to be interpreted as dot products in the normal part of hyperspace.

The salient point is that \( w_{,\mu} w_{,\nu} \) is always a sum of the products of cosines. Therefore, if one averages over distances much larger than the longest wavelength \( 1/k_{I\mu} \), one has

\[ \langle g_{\mu\nu} \rangle = \gamma_{\mu\nu} + (1/2) \sum_{I=1}^{N} A^2_I k_{I\mu} k_{I\nu} \]

where the factor of one-half comes from the average value of the squared cosine.

Thus the elaborately dimpled surface looks flat in the large: hardly a surprise. However, an interesting effect can be produced if \( \gamma_{\mu\nu} \) contains negative terms: the waves can make the corresponding terms in \( \langle g_{\mu\nu} \rangle \) positive if the amplitude-to-wavelength ratios are large enough. In particular, if \( \gamma_{\mu\nu} \) is the
Minkowski metric, the oscillating terms can be chosen to make \( g_{\mu\nu} \) Euclidean (or pseudo-Euclidean). If, on the other hand, \( \gamma_{\mu\nu} \) is pseudo-Euclidean, the oscillating terms can make \( g_{\mu\nu} \) the metric of a Minkowski space with trace \( \pm 2 \).

It is instructive to look at the case of four waves of the same amplitude, each with

\[
k_I\mu = k_I\delta_{I\mu}.
\]

Then,

\[
y = A(sin k_1x^1 + sin k_2x^2 + sin k_3x^3 + sin k_4x^4)
\]

and,

\[
< g_{\mu\nu} >= \gamma_{\mu\nu} + (1/2) \sum_{I=1}^{N} A^2(k_I)^2\delta_{I\mu}\delta_{I\nu}.
\]

If \( \gamma_{\mu\nu} \) is the Minkowski metric with trace +2, \( < g_{\mu\nu} > \) will look Euclidean at large scales if \((k_1)^2 = (k_2)^2 = (k_3)^3 \equiv k^2\) and

\[
A^2(k_4)^2 = 4 + A^2k^2
\]

In the same way, a pseudo-Euclidean metric will look Minkowskian at large scales if

\[
A^2(k_4)^2 = 4 - A^2k^2.
\]

It is natural to ask what sort of matter distribution would produce such an effect; actually, in the most trivial cases, no matter is required at all. For example, if \( k = 0 \) and \( A = 2/k_4 \), the metric is just

\[
g_{\mu\nu} = \gamma_{\mu\nu} + (4/k_4^2)\cos^2k_4x^4\delta_{\mu4}\delta_{\nu4}.
\]

This is simply a rescaling of the time coordinate if \( \gamma_{\mu\nu} \) is diagonal:

\[
(dx^4)^2 = (dx^4)^2[(1 + (4/k_4^2)\cos^2k_4x^4)/\gamma_{44}]
\]

The space has no curvature, much like a rippling flag.

A rather artificial example of a matter distribution in pseudo-Euclidean 4-space which results in a Minkowskian geometry when averaged over large distances is the case \( A^2k^2 = 4 \) and \( k_4 = 0 \). Then the time components of the metric have no effect, and:

\[
g_{ij} = -\delta_{ij} + 4\cos kx^r\cos kx^s\delta_{ir}\delta_{js};
\]

\[
g^{ij} = -\delta^{ij} + 4\cos kx^i\cos kx^j/(-1 + 4\cos kx^r\cos kx^s\delta_{rs}).
\]

The only non-vanishing Christoffel symbols are

\[
\Gamma^i_{jj} = -J\sin kx^i\cos kx^j
\]
where
\[ J = 4k/(-1 + 4(\cos^2 kx + \cos^2 ky + \cos^2 kz)). \]

The corresponding Ricci tensor has components:

\[ R_{xx} = J \sin kx[\sin ky + \sin kz - J(\sin ky \cos^2 ky + \sin kz \cos^2 kz)] \]

\[ R_{xy} = J^2 \sin kx \cos kx \sin ky \cos ky \]

plus three more found by cyclically permuting \( x, y, \) and \( z \) in these expressions.

It is hard to see how the resulting stress tensor could be established physically. More general oscillations producing curvature of the time coordinate would probably look even less physical, with the mass density also oscillating and taking on negative values.

Nevertheless, transverse oscillations of spacetime may have some physical interest. Four interfering plane-wave like oscillations would create a cellular structure in spacetime on the scale of the wavelengths; this is very reminiscent of the various discrete-space approaches to quantum gravity, reviewed in [Gibbs 1995]. In addition, an undulating pseudo-Euclidean metric which looks Minkowskian in the large and which features alternating regions of positive and negative mass sounds remarkably like a geometrization of Winterberg’s “Planck aether” Winterberg 1994 and 1995. Finally, if rather obviously, transverse oscillations could be used to Fourier-analyze embedded solutions of Einstein’s equation.

**BIBLIOGRAPHY**

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