On large sieve inequalities involving $p$th powers of trigonometric polynomials

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Abstract. In this paper, we extend the large sieve type estimates to sums involving $p$th powers of trigonometric polynomials. An approach to such estimates that does not rely on the usual $L^2$-technique is given. Our method is based on comparing the norm and the spectral radius of convolution operators on a normed space of trigonometric polynomials.

1 Introduction

We denote by $T_N$ the set of trigonometric polynomials of degree at most $N$

$$s(x) = \sum_{k=-N}^{N} c_k e^{ikx}$$

with a positive integer $N$ and $c_k \in \mathbb{C}$, $k = -N, \ldots, N$. Suppose that $x_1 < x_2 < \cdots < x_r$, $r > 1$, is any sequence in $\mathbb{T} = (-\pi, \pi]$ such that

$$\min\{x_{j+1} - x_j, j = 1, \ldots, r - 1; 2\pi - (x_r - x_1)\} = \delta > 0.$$  

The usual large sieve inequality states that

$$\sum_{j=1}^{r} |s(x_j)|^2 \leq \left(\frac{N}{2\pi} + \frac{1}{\delta}\right) \int_{\mathbb{T}} |s(x)|^2 \, dx.$$  

See Selberg [8] p. 221], but note the different notation. Gallagher [4] has given a simple derivation of the large sieve inequalities. It turned out that the method of [4] can be applied to $L^p(\mathbb{T})$-norm. More precisely, in [3] p. 96] was proved that if $s \in T_N$ and $2 \leq p < \infty$, then

$$\sum_{j=1}^{r} |s(x_j)|^p \leq \Delta_p p^{1/2} \max\left(\frac{4\pi}{\delta} \right) \left(\frac{1}{\delta}\right)^{p/q} \left(\sum_{k=-N}^{N} |c_k|^q\right)^{p/q},$$  

where $\Delta_p$ is an absolute constant and $1/p + 1/q = 1$. Next, in [6] p. 533] the inequality (3) was extended for all $0 < p < \infty$ as follows. Let $\Psi$ be a convex, nonnegative, and nondecreasing
function in \([0, \infty)\). Then for any \(s \in T_N\),

\[
(4) \quad \sum_{j=1}^{r} \Psi(|s(x_j)|^p) \leq \left(\frac{N}{\pi} + \frac{1}{\delta} \right) \int_{\mathbb{T}} \Psi\left(|s(x)|^p (p+1)e/2\right) dx,
\]

whenever \(-\pi < x_1 < x_2 < \cdots < x_r \leq \pi\) and \(\delta\) is defined by (1). If \(\Psi(t) = t, 0 \leq t < \infty\), then (4) implies

\[
(5) \quad \sum_{j=1}^{r} |s(x_j)|^p \leq \left(\frac{N}{\pi} + \frac{1}{\delta} \right) \frac{(p+1)e}{2} ||s||_{L^p(\mathbb{T})}^p.
\]

In [5, p. 164] the estimate (4) was extended to the case of generalized trigonometric polynomials. In particular, for usual trigonometric polynomials, the inequality (5) was improved as follows:

\[
(6) \quad \sum_{j=1}^{r} |s(x_j)|^p \leq \left(\frac{N+1}{2\pi} + \frac{1}{\delta} \right) \frac{(p+1)e}{2} ||s||_{L^p(\mathbb{T})}^p.
\]

Note that inequalities (2)-(6) are also called forward Marcinkiewicz-Zygmund inequalities (see, e.g., [2]).

In this paper, we will develop an approach to inequalities of the type (3) and (5)-(6) that not use the usual \(L^2\)-technique. Our approach is based on the spectral theory of convolution operators on \(T_N\).

The main result is given in the following theorem. Note that in the sequel, \([x]\) denotes the integer part of a positive number \(x\). Also we use the notation \(\Gamma(\cdot)\) for the standard gamma function.

**THEOREM 1.** Let \(\{x_j\}_{j=1}^{r}\) be a sequence in \(\mathbb{T}\) that satisfies (1). If \(s \in T_N\) and \(1 \leq p < \infty\), then

\[
(7) \quad \sum_{j=1}^{r} |s(x_j)|^p \leq \frac{pN\sigma(\delta;N)}{2\sqrt{\pi}} \cdot \frac{\Gamma\left(p/2\right)}{\Gamma\left(p/2+1/2\right)} ||s||_{L^p(\mathbb{T})}^p
\]

with

\[
(8) \quad \sigma(\delta;N) = \begin{cases} \frac{\pi}{N\delta}, & \text{if } \frac{\pi}{N\delta} \in \mathbb{Z}, \\ 1 + \left[\frac{\pi}{N\delta}\right], & \text{otherwise}. \end{cases}
\]

If \(p\) is a positive integer, then the quantity

\[
\frac{\Gamma\left(p/2\right)}{\Gamma\left(p/2+1/2\right)}
\]

can be calculated directly by using the relations between \(\Gamma\left(p/2\right)\) and \(\Gamma\left(p/2+1/2\right)\).

**COROLLARY 2.** Let \(l\) be a positive integer. Then under the conditions of Theorem 1 it follows that:
(1) If $p = 2l$, then

\[
\sum_{j=1}^{r} |s(x_j)|^p \leq \frac{pN\sigma(\delta; N)}{2\sqrt{\pi}} \cdot \frac{\Gamma(p/2)}{\Gamma(p/2 + 1/2)} \|s\|^p_{L_p(\mathbb{T})} = \frac{pN\sigma(\delta; N) \cdot 2^{l-1}(l-1)!}{\pi(1 \cdot 3 \cdot 5 \cdots (2l-1))} \|s\|^p_{L_p(\mathbb{T})};
\]

(ii) if $p = 2l + 1$, then

\[
\sum_{j=1}^{r} |s(x_j)|^p \leq \frac{pN\sigma(\delta; N)}{2\sqrt{\pi}} \cdot \frac{\Gamma(p/2)}{\Gamma(p/2 + 1/2)} \|s\|^p_{L_p(\mathbb{T})} = \frac{pN\sigma(\delta; N) \cdot (1 \cdot 3 \cdot 5 \cdots (2l-1))}{2^{l+1} \cdot l!} \|s\|^p_{L_p(\mathbb{T})}.
\]

**COROLLARY 3.** Assume that $s \in T_N$, $\|s\|_{L_p(\mathbb{T})} = 1$, $p \geq 1$, and $\{x_j\}_{j=1}^{r}$ satisfies (1).

(i) If $\frac{\pi}{N\delta} \in \mathbb{Z}$, then

\[
\sum_{j=1}^{r} |s(x_j)|^p < \frac{p+1}{\delta}.
\]

(ii) If $\frac{\pi}{N\delta} \notin \mathbb{Z}$, then

\[
\sum_{j=1}^{r} |s(x_j)|^p < (p+1)\left(\frac{N}{\pi} + \frac{1}{\delta}\right).
\]

2 Proofs

Let $M(\mathbb{T})$ be the Banach algebra of finite complex-valued regular Borel measures on $\mathbb{T}$. The norm in $M(\mathbb{T})$ is given by the total variation $\|\mu\|$ of $\mu \in M(\mathbb{T})$. Therefore, the usual Banach space $L^1(\mathbb{T})$ can be identified with the closed ideal in $M(\mathbb{T})$ of all measures which are absolutely continuous with respect to the Lebesgue measure $dt$ on $\mathbb{T}$.

Given $\mu \in M(\mathbb{T})$ and $f \in L^1(\mathbb{T})$, we define the Fourier transform of $\mu$ and $f$ by

\[
\hat{\mu}(x) = \int_{\mathbb{T}} e^{-ixt} \mu(t) \quad \text{and} \quad \hat{f}(x) = \int_{\mathbb{T}} e^{-ixt} f(t) \ dt,
\]

respectively. For each $u \in L^r(\mathbb{T})$, $1 \leq r \leq \infty$, on $T_N$ is well defined the convolution operator

\[
A_{u}(s)(x) = s * u(x) = \int_{\mathbb{T}} s(x-t) u(t) \ dt = \int_{\mathbb{T}} \left( \sum_{k=-N}^{N} c_k e^{ik(\pi x)} \right) u(t) \ dt
\]

\[
= \sum_{k=-N}^{N} c_k \hat{u}(k) e^{ikx},
\]

for each $x \in \mathbb{T}$, where $s \in T_N$. Note that in this definition and also below we assume that $s$ is a periodic function on the real line with the period equal to $2\pi$. Below, the notation $T^p_N$, $1 \leq p \leq \infty$, means that $T_N$ is equipped with the usual $L^p(\mathbb{T})$ norm.
Suppose that $S_1$ and $S_2$ are two measurable subsets of $\mathbb{T}$, $\mu_1$ and $\mu_2$ are two non-negative finite measures on $\mathbb{T}$, and $F : \mathbb{T}^2 \to \mathbb{R}$ is a measurable function. For $1 \leq p < \infty$, Minkowski’s integral inequality \[9, p. 37\] states that
\[
\left[ \int_{S_2} \left| \int_{S_1} F(x,y) \, d\mu_1(x) \right| \right]^p \, d\mu_2(y) \right]^{1/p} 
\leq \int_{S_1} \left( \int_{S_2} |F(x,y)|^p \, d\mu_2(y) \right)^{1/p} \, d\mu_1(x).
\] (14)

**PROPOSITION 4.** Let $u \in L^q(\mathbb{T})$, $1 \leq q \leq \infty$, $\|u\|_{L^q(\mathbb{T})} \neq 0$. Assume that $u$ is continuous, non-negative and even on $\mathbb{T}$. If
\[
(15) \supp u \subset \left[ -\frac{\pi}{2N}, \frac{\pi}{2N} \right],
\] then there exists a trigonometric sum
\[
(16) p_u(x) = \sum_{m=-N+1}^{N} \tau_m e^{-i\pi mx/N}
\] such that
\[
(17) (-1)^m \tau_m > 0,
\] $m = -N + 1, \ldots, N$, and
\[
(18) p_u(n) = \frac{1}{\hat{u}(n)}
\] for all $n = -N, \ldots, N$.

**PROOF.** We start by examining in more details the Fourier transform of $u$. Under the assumptions on $u$, we see that
\[
(19) \hat{u}(x) = \int_\mathbb{T} u(t) e^{-i xt} \, dt = 2 \int_0^{\pi/2N} u(t) \cos xt \, dt > 0
\] for all $x \in [-N, N]$. From this it follows that
\[
(20) (\hat{u})'(x) = -2 \int_0^{\pi/2N} tu(t) \sin xt \, dt < 0
\] and
\[
(21) (\hat{u})''(x) = -2 \int_0^{\pi/2N} t^2 u(t) \cos xt \, dt < 0
\] for all $x \in [0, N]$.
Let $v = 1/\hat{u}$. By (19), the function $v$ is well defined and positive on $[-N, N]$. Moreover, we conclude from (19)-(21) that $v$ is an even function and
\[
(22) v'(x) > 0 \quad \text{and} \quad v''(x) > 0
\]
for all $x \in [0,N]$. Therefore, $v$ is increasing and convex on $[0,N]$, in particular $v$ is of bounded variation on $[0,N]$. In particular, this means that $v$ is a function of bounded variation on $\mathbb{T}$. The following is well known: If $f$ is an $2\pi$-periodic continuously differentiable even function on $\mathbb{T}$ such that $f$ is of bounded variation on $\mathbb{T}$, then the Fourier series of $f$ converges absolutely (see, eg. [10, p. 241]). Thus,

(23)  
$$v(x) = \sum_{k \in \mathbb{Z}} a_k e^{ik\pi x/N} = \sum_{k \in \mathbb{Z}} a_k \cos\left(\frac{\pi k x}{N}\right)$$

with

(24)  
$$\sum_{k \in \mathbb{Z}} |a_k| < \infty,$$

where

(25)  
$$a_k = \int_{-N}^{N} v(t) e^{-ik\pi/N} dt = 2 \int_{0}^{N} v(t) \cos\left(\frac{\pi k t}{N}\right) dt.$$

We claim that

(26)  
$$(-1)^k a_k > 0$$

for all $k \in \mathbb{Z}$. Combining (19) with (25), we see that $a_0 > 0$. Let $k \geq 1$. Then using integration by parts, we conclude from (15) that

(27)  
$$a_k = -\frac{2N}{\pi k} \int_{0}^{N} v'(t) \sin\left(\frac{\pi k t}{N}\right) dt = -\frac{2N}{\pi k} \sum_{j=0}^{k-1} I_j,$$

where

(28)  
$$I_j = \int_{E_j} v'(t) \sin\left(\frac{\pi k t}{N}\right) dt$$

and $E_j = [Nj/k,N(j + 1)/k]$. Note that the length of $E_j$ is exactly half length of period for $\sin\left(\pi k x/N\right)$. Combining this with (22), we see that

(29)  
$$(-1)^j I_j > 0 \quad \text{and} \quad |I_j| < |I_{j+1}|$$

for all $j = 0, \ldots, k - 1$. From this, it is easily seen that

$$(-1)^{k-1} \sum_{j=0}^{k-1} I_j > 0.$$

In light of (27) this proving the claim (26).

For $m = -N + 1, \ldots, N$, let $\tau_m$ be defined by

(30)  
$$\tau_m = \sum_{j \in \mathbb{Z}} a_{m+2jN}.$$
From (24), we see that the series in (30) converges absolutely. Next, from (26) it follows that, for each \( m \), all terms of the sequence \( \{a_{m+2jN}\}_{j \in \mathbb{Z}} \) have the same sign. In particular, (26) shows that \((-1)^m a_{m+2jN} > 0\). Therefore, the trigonometric sum (16) is well defined and satisfies (17).

Finally, for any \( n \in \{-N, \ldots, N\} \), combining (23) with (30), we get

\[
\frac{1}{\hat{u}(n)} = v(n) = \sum_{k \in \mathbb{Z}} a_k e^{ik \pi n/N} = \sum_{m=-N+1}^{N} \left( \sum_{j \in \mathbb{Z}} a_{m+2jN} e^{i(m+2jN) \pi n/N} \right)
\]

\[
= \sum_{m=-N+1}^{N} \left( \sum_{j \in \mathbb{Z}} a_{m+2jN} e^{im \pi n/N} \right) = \sum_{m=-N+1}^{N} \left( \sum_{j \in \mathbb{Z}} a_{m+2jN} e^{im \pi n/N} \right)
\]

\[
= \sum_{m=-N+1}^{N} \tau_m e^{im \pi n/N} = p_u(-n).
\]

As \( u \) and \( \hat{u} \) are even functions we have \( p_u(-n) = p(n) \) for all \( n \). Proposition 4 is proved.

We will denote by \( \delta_a \) the usual Dirac measure supported on \( a \in \mathbb{T} \).

**PROPOSITION 5.** Let positive numbers \( p \) and \( q \) satisfy \( 1/p + 1/q = 1 \). Under the conditions of Proposition 4 on \( u \in L^q(\mathbb{T}) \) it follows that the operator (13) possesses on \( T_N^p \) a bounded inverse of the type

\[
A_u^{-1}(s)(x) = \int_{\mathbb{T}} s(x-t) d\mu(t),
\]

where

\[
\mu = \sum_{m=-N+1}^{N} \tau_m \delta_{\pi m/N}
\]

and \( \{\tau_m\}_{-N+1}^{N} \) are defined by (30), using (23) for \( v = 1/\hat{u} \). Furthermore,

\[
\|A_u^{-1}\|_{T_N^p} = \|\mu\|_{M(\mathbb{T})} = \sum_{m=-N+1}^{N} |\tau_m| = \frac{1}{\hat{u}(N)}.
\]

**PROOF.** Let \( s(x) = \sum_{k=-N}^{N} c_k e^{ikx} \in T_N \). From (16) it follows that

\[
s * \mu(x) = \sum_{k=-N}^{N} c_k \left( \int_{\mathbb{T}} e^{ik(x-t)} d\mu(t) \right) = \sum_{k=-N}^{N} c_k \tilde{\mu}(k) e^{ikx},
\]

where

\[
\tilde{\mu}(k) = \int_{\mathbb{T}} e^{-ikt} d\mu(t) = \sum_{m=-N+1}^{N} \tau_m e^{-i\pi km/N} = p_u(k),
\]

for all \( k = -N, \ldots, N \). Now, taking into account (18), we conclude from from (13) that (31) defines the inverse of the operator \( A_u \).

Now, according to (34), we see that the set \( \{\tilde{\mu}(k) = p_u(k) : k = -N, \ldots, N\} \) coincides with the spectrum of \( A_u^{-1} \). Therefore, if \( |A_u^{-1}|_{T_N^p} \) denotes the spectral radius of \( A_u^{-1} \), then

\[
|A_u^{-1}|_{T_N^p} = \max\{|\tilde{\mu}(k)| = |p_u(k)| : k = -N, \ldots, N\}.
\]
Let us recall that $|H|_X \leq \|H\|_X$, i.e. the spectral radius is not greater than the operator norm, for any bounded linear operator $H$ on a normed space $X$. Combining this with (18), we get

$$\|A_u^{-1}\|_{T^p_N} \geq \max_{-N \leq k \leq N} |\hat{u}(k)| = \max_{-N \leq k \leq N} |p_u(k)| = \max_{-N \leq k \leq N} \left\{ \frac{1}{|\hat{u}(k)|} \right\}$$

$$= \frac{1}{\min_{-N \leq k \leq N} |\hat{u}(k)|} \geq \frac{1}{\hat{u}(N)} = p_u(N) = \sum_{M=-N+1}^{N} \tau_m e^{imN/N} = \sum_{M=-N+1}^{N} (-1)^m \tau_m = \sum_{M=-N+1}^{N} |\tau_m| = \|\mu\|.$$  

(35)

Other hands, from Minkowski’s inequality (14) it is easily to see that

$$\|A_u^{-1}(s)\|_{T^p_N} = \left( \int_{|s|_{T^p_N}} \int_{|s|} \left| s(x-t) d\mu(t) \right|^p dx \right)^{1/p} \leq \left( \int_{|s|} \int_{|s|} |s(x-t)| d|\mu|(t) dx \right)^{1/p} \leq \int_{|s|} \left( \int_{|s|} |s(x-t)|^p dx \right)^{1/p} d|\mu|(t) = \|s\|_{T^p_N} |\mu|,$$

(36)

where $|\mu|$ denote the variation of $\mu$. Thus, (35) with (36) show (33), and the Proposition 5 is proved.

PROOF OF THEOREM 1. Assume that $u \in L^q(\mathbb{T})$, $\|u\|_{L^q(\mathbb{T})} = 1$ and $u$ satisfies the conditions of Proposition 4. If $s \in T_N$, then by Hölder’s inequality, we get

$$|s(x)|^p = \left| \int_{|s|} A_u^{-1}(s)(x-t)u(t) dt \right|^p = \left| \int_{-\pi/2N}^{\pi/2N} A_u^{-1}(s)(x-t)u(t) dt \right|^p \leq \left( \int_{-\pi/2N}^{\pi/2N} \left| A_u^{-1}(s)(x-t) \right|^p dt \right)^{p/q} \left( \int_{-\pi/2N}^{\pi/2N} |u(t)|^q dt \right)^{p/q} \leq \left( \int_{-\pi/2N}^{\pi/2N} \left| A_u^{-1}(s)(x-t) \right|^p dt \right)^{p/q}$$

for each $x \in \mathbb{T}$. Therefore,

$$\sum_{j=1}^{r} |s(x_j)|^p \leq \sum_{j=1}^{r} \int_{-\pi/2N}^{\pi/2N} \left| A_u^{-1}(s)(x_j-t) \right|^p dt = \sum_{j=1}^{r} \int_{x_j-\pi/2N}^{x_j+\pi/2N} \left| A_u^{-1}(s)(y) \right|^p dy.$$

(37)

We will denote by $E_j$ the set

$$E_j = \left( x_j - \frac{\pi}{2N}, x_j + \frac{\pi}{2N} \right],$$

$j = 1, \ldots, r$. Using the fact that $A_u^{-1}(s)$ is a trigonometric polynomial, i.e., a continuous and periodical function on $\mathbb{R}$, we conclude that

$$\int_{x_j-\pi/2N}^{x_j+\pi/2N} \left| A_u^{-1}(s)(y) \right|^p dy = \int_{E_j} \left| A_u^{-1}(s)(y) \right|^p dy.$$

(38)
Let \( x \in \mathbb{R} \) and assume that \( x \in E_j \), for \( j = i, i+1, \ldots, i+k \) with some non-negative integer \( k \).
Then we claim that:

(i) if \( \pi/N\delta \in \mathbb{Z} \), then

\[
k + 1 \leq \frac{\pi}{N\delta};
\]

(ii) if \( \pi/N\delta \not\in \mathbb{Z} \), then

\[
k \leq \left\lfloor \frac{\pi}{N\delta} \right\rfloor,
\]

where \( \lfloor \cdot \rfloor \) is the usual integer part of a real number.
Indeed, since \( x \in E_i \cap E_{i+k} \), it follows that \( x_i + \pi/2N > x_{i+k} - \pi/2N \). Then

\[
x_{i+k} - x_i < \frac{\pi}{N}.
\]

Other hands, by (1) we get

\[
x_{i+k} - x_i \geq k\delta.
\]

Combining (42) with (41), we see that

\[
k < \frac{\pi}{N\delta}.
\]

Now note that \( k \) is an integer. Hence if \( \pi/N\delta \in \mathbb{Z} \), then (43) implies (39). For \( \pi/N\delta \not\in \mathbb{Z} \), we get (40), which yields our claim.
Thus, each \( x \in \mathbb{T} \) can belong to at most \( \sigma_p \) intervals \( E_j, j = 1, \ldots, p \), where \( \sigma_p \) was defined by (8).

Now combining (35), (36) with (37), (38), (31) and (8), we get

\[
\sum_{j=1}^{r} |s(x_j)|^p \leq \sum_{j=1}^{r} \int_{x_j - \pi/2N}^{x_j + \pi/2N} |A_u^{-1}(s)(y)|^p \, dy \leq \sigma(\delta; N) \int_{\mathbb{T}} |A_u^{-1}(s)(y)|^p \, dy = \sigma(\delta; N)\|A_u^{-1}(s)\|_{L^p(\mathbb{T})}^p \leq \sigma(\delta; N)\|A_u^{-1}\|_{L^p(\mathbb{T})}^p \|s\|_{L^p(\mathbb{T})}^p\|A_u^{-1}\|_{L^p(\mathbb{T})}^p \cdot \sigma_p \left(\hat{u}(N)\right)^p \|s\|_{L^p(\mathbb{T})}^p,
\]

Next, the estimate (44) can be improved to

\[
\sum_{j=1}^{r} |s(x_j)|^p \leq \frac{\sigma(\delta; N)}{\sup_u (\hat{u}(N))} \|s\|_{L^p(\mathbb{T})}^p,
\]

where the supremum extends over all admissible \( u \) as described in the statement of Proposition 4.
We claim that for such an \( u \) we have

\[
\sup_u (\hat{u}(N))^p = \|\cos Nx\|_{L^p[\pi/2, \pi/2]}^p = \frac{\|\cos t\|_{L^p[-\pi/2, \pi/2]}^p}{N}.
\]
Indeed, Hölder’s inequality implies that
\[ |\hat{u}(N)|^p = \left| \int_T u(x) \cos Nx \, dx \right|^p = \left| \int_{-\pi/2N}^{\pi/2N} u(x) \cos Nx \, dx \right|^p \]
\[ \leq \int_{-\pi/2N}^{\pi/2N} |\cos Nx|^p \, dx \cdot \|u\|_{L^p([-\pi/2N, \pi/2N])}^p = \int_{-\pi/2N}^{\pi/2N} |\cos Nx|^p \, dx \]
\[ = \frac{1}{N} \int_{-\pi/2}^{\pi/2} |\cos t|^p \, dt = \frac{\|\cos t\|_{L^p([-\pi/2, \pi/2])}^p}{N}. \]
(47)

Moreover, since we used Hölder’s inequality, it follows that the estimate (47) is exact and the equality is attained if
\[ u(x) = \theta \cos^{p-1} Nx \cdot \chi_{[-\pi/2N, \pi/2N]}(x), \]
where \( \chi_{[-\pi/2N, \pi/2N]} \) is the indicator function of the interval \([-\pi/2N, \pi/2N]\) and \( \theta \in \mathbb{C} \) is such that \( \|u\|_{L^p([-\pi/2N, \pi/2N])} = 1 \). Therefore, our claim (46) is proved.

Combining (45) with (46), we get
\[ \sum_{j=1}^r |s(x_j)|^p \leq \frac{\sigma(\delta; N) \cdot N}{\|\cos t\|_{L^p([-\pi/2, \pi/2])}^p} \|s\|_{L^p(T)}^p. \]
(48)

Next,
\[ \|\cos t\|_{L^p([-\pi/2, \pi/2])}^p = \int_{-\pi/2}^{\pi/2} \cos^p t \, dt = B\left(\frac{1}{2}; \frac{p+1}{2}\right), \]
(49)

(see, e.g., [7, p. 142]), where \( B \) is Euler’s beta function defined by
\[ B(a; b) = 2 \int_0^{\pi/2} \sin^{2a-1} \theta \cos^{2b-1} \theta \, d\theta \]
for \( \Re a, \Re b > 0 \). Applying the following connection between the beta and the usual gamma function ([7] p. 142)
\[ B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}, \]
we conclude from (49) that
\[ \|\cos t\|_{L^p([-\pi/2, \pi/2])}^p = \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)} \]
(50)

Since \( \Gamma\left(\frac{p}{2}+1\right) = \Gamma\left(\frac{p}{2}\right) \cdot \frac{p}{2} \) and \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \) (see, e.g., [7] p.p. 137-138), it follows from (49) and (50) that
\[ \|\cos t\|_{L^p([-\pi/2, \pi/2])}^p = \frac{2 \sqrt{\pi} \cdot \Gamma\left(\frac{p}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)}. \]

Substituting this into (48), we obtain (7). Theorem 1 is proved.
POOF OF COROLLARY 2. It is known that for a nonnegative integer \( n \),
\[
\Gamma(n+1) = n! \quad \text{and} \quad \Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi} \cdot \frac{1 \cdot 3 \cdot 5 \cdots (2 - 1)}{2^n}
\]
(see [7, p. 139]). Using this, the representations (9) and (10) can be verified by straightforward calculation.

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