Research Article

Numerical Method to Modify the Fractional-Order Diffusion Equation

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Received 30 September 2021; Accepted 28 January 2022; Published 7 March 2022

Academic Editor: Soheil Salahshour

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Time or space or time-space fractional-order diffusion equations (FODEs) are widely used to describe anomalous diffusion processes in many physical and biological systems. In recent years, many authors have proposed different numerical methods to solve the modified fractional-order diffusion equations, and some achievements have been obtained. However, to our knowledge of the literature, up to date, all the proposed numerical methods to modify FODE have achieved at most a second-order time accuracy. In this study, we focus mainly on the numerical methods based on numerical integration in order to modify the fractional-order diffusion equation: 

\[ \frac{\partial p(x,t)}{\partial t} = \left( A \frac{\partial^{1-a}}{\partial t^{1-a}} + B \frac{\partial^{1-\beta}}{\partial t^{1-\beta}} \right) \frac{\partial^2 p(x,t)}{\partial x^2} \]  

(1)

Some scholars have constructed different numerical methods to solve the modified FODE, and accordingly, some useful achievements have been obtained. For one-dimensional positive FODE, Langlands [4] proposed an interpretation with an infinite series form of the Fox function over an infinite region. Liu et al. [5, 6] discussed numerical methods and analytical techniques to develop a finite element approximation with first-order time accuracy and \(m\)th order spatial accuracy, where \(m\) is the number of segmented polynomials. Mohebbi et al. [7] applied a fourth-order compact formula for the second-order spatial partial derivatives and the discretization of Riemann-Liouville fractional-order time derivatives to provide a higher-order and absolutely stable format with first-order time accuracy and fourth-order spatial accuracy. Bhrawy [8] studied the compact subdiffusion scheme including second-order time accuracy and fourth-order spatial accuracy. For the multi-repair positive FODE, Zhang et al. [9] developed a finite difference/finite element method with \((1+\min\{\alpha, \beta\})\) order time accuracy and \(m\)th order spatial accuracy, where \(m\) represents the number of segmented polynomials. Mohebbi et al. [10] proposed a fourth-order compact solution method with first-order time accuracy and fourth-order spatial accuracy. Abbaszadeh and Mohebbi [11] discussed a solution obtained by a radial basis function (RBF) meshless method with \(\min\{\alpha, \beta\}\) order time accuracy. Wang and Wang [12] analyzed the tight LOD method and its extrapolation method including \(2\min\{\alpha, \beta\}\) order time accuracy and fourth-order spatial accuracy. For two-dimensional variable-order modified FODE, Chen and Liu [13] examined a
1.1. Basic Concepts and Properties

**Definition 1** (Gr"unwald–Letnikov fractional stratification number). Let $\alpha$ be a positive real number, and $n - 1 \leq \alpha < n$, $n$ is a positive integer, and let the function $f(x)$ be defined on the interval $[a, b]$ as

$$aD_0^\alpha f(x) = \lim_{h \to 0} \sum_{j=0}^{[x-z]/h} (-1)^j \left( \begin{array}{c} \alpha \\ j \end{array} \right) f(x - jh).$$

(2)

It is the $\alpha$-order Gr"unwald–Letnikov (G-L) fractional-order derivative of function $f(x)$, $[z]$ is the largest integer that does not exceed $z$, and

$$\left( \begin{array}{c} \alpha \\ j \end{array} \right) = \frac{\alpha (\alpha - 1) \ldots (\alpha - j + 1)}{j!}.$$ (3)

**Definition 2** (Riemann-Liouville fractional stratification number). Let $\alpha$ be a positive real number, and $n - 1 \leq \alpha < n$, $n$ is a positive integer, and let the function $f(x)$ be defined on the interval $[a, b]$, respectively. Here,

$$aD_0^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x f(\tau)d\tau,$$

(4)

$$aD_\infty^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^\infty f(\tau)d\tau,$$

represent the left and right $\alpha$-order Riemann-Liouville fractional derivatives.

**Property 1.** Let $\alpha$ be a positive real number, and $n - 1 \leq \alpha < n$, $n$ is a positive integer defined in the interval of $[a, b]$, the function $f(x)$ has up to $n - 1$ continuous functions, and $f^{(n)}(x)$ is integrable in $[a, b]$. Then, the Riemann-Liouville fractional derivative is equivalent to the Gr"unwald–Letnikov (GL) fractional derivative.

1.2. Construction of Numerical Methods. In this research, the following numerical method is developed to modify the initial and boundary values of the fractional diffusion equation (MFDE):

$$\frac{\partial p(x,t)}{\partial t} = \left( \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} + B \frac{\partial^{1-\beta}}{\partial t^{1-\beta}} \right) \frac{\partial^2 p(x,t)}{\partial x^2} + f(x,t),$$

(5)

$$p(x,0) = \omega(x), \quad 0 \leq x \leq L,$$ (6)

$$p(0,t) = \varphi(t), \quad p(L,t) = \psi(t), \quad 0 \leq t \leq T,$$ (7)

where $0 < \alpha < \beta < 1$, $D_0^{1-\gamma}p(x,t)$ is the fractional large derivative of $1 - \gamma$,

$$aD_0^{1-\gamma}p(x,t) = \frac{1}{\Gamma(\gamma)} \frac{\partial}{\partial t} \int_0^t \frac{p(x,s)}{(t-s)^{1-\gamma}} ds.$$ (8)

Suppose $p(x,t) \in P(\Omega), p(x,t)$ is the exact solution of problems (1)–(6) and $(\partial^2 f(x,t)/\partial t^2) \in C(\Omega)$, where

$$\Omega = \{(x,t) | 0 \leq x \leq L, 0 \leq t \leq T\},$$

$$P(\Omega) = \left\{ p(x,t) \left| \frac{\partial^6 p(x,t)}{\partial x^6}, \frac{\partial^3 p(x,t)}{\partial x^2 \partial t^2} \in \Omega \right. \right\}.$$ (9)

Suppose that

$$x_j = jh, \quad j = 0, 1, \ldots, J,$$

$$t_k = kr, \quad k = 0, 1, \ldots, K,$$ (10)

where $h = L/J$ and $r = T/K$ are the space step and time step, respectively.

We define

$$\delta^2_{x,j,j} = \frac{1}{\Delta x^2}, \quad \delta^2_{t,k,k} = \frac{1}{\Delta t^2},$$

and integrating the two sides of equation (5) with respect to $t$ on the interval $[t_{k-1}, t_k]$, we get

$$P(x_j,t_k) - P(x_j,t_{k-1}) = \frac{1}{\Gamma(\alpha)} \int_0^{t_k} \frac{\partial^2 p(x_j,s)}{\partial x^2} \frac{ds}{(t_k - s)^{1-\alpha}} + \int_0^{t_k} \frac{\partial^2 p(x_j,s)}{\partial x^2} \frac{ds}{(t_k - s)^{1-\beta}}.$$ (12)

Hence, we get
\[
\left( 1 + \frac{1}{12} \delta_{x}^2 \right) p(x_j, t_k) - \left( 1 + \frac{1}{12} \delta_{x}^2 \right) p(x_j, t_{k+1}) \\
= \left( 1 + \frac{1}{12} \delta_{x}^2 \right) \frac{1}{\Gamma(\alpha)} \left[ \int_{0}^{t_k} \frac{\partial^2 p(x_j, s)}{\partial x^2} \frac{ds}{(t_k - s)^{1-\alpha}} - \int_{0}^{t_{k+1}} \frac{\partial^2 p(x_j, s)}{\partial x^2} \frac{ds}{(t_{k+1} - s)^{1-\alpha}} \right] \\
+ \left( 1 + \frac{1}{12} \delta_{x}^2 \right) \frac{1}{\Gamma(\beta)} \left[ \int_{0}^{t_k} \frac{\partial^2 p(x_j, s)}{\partial x^2} \frac{ds}{(t_k - s)^{1-\beta}} - \int_{0}^{t_{k+1}} \frac{\partial^2 p(x_j, s)}{\partial x^2} \frac{ds}{(t_{k+1} - s)^{1-\beta}} \right] \\
+ \left( 1 + \frac{1}{12} \delta_{x}^2 \right) \int_{t_{k+1}}^{t_k} f(x_j, s) ds.
\]

Since \( p(x, t) \in P(\Omega) \), then
\[
\left( 1 + \frac{1}{12} \delta_{x}^2 \right) \frac{\partial^2 p(x_j, t_i)}{\partial x^2} = \frac{\delta_{x}^2 p(x_j, t_i)}{h^2} + O(h^4),
\]
\[
\frac{\partial^2 p(x_j, s)}{\partial x^2} = \frac{t_1 - s}{\tau} \frac{\partial^2 p(x_j, t_{i-1})}{\partial x^2} + \frac{s - t_{i-1}}{\tau} \frac{\partial^2 p(x_j, t_i)}{\partial x^2} + O(\tau^2),
\]

hence,
\[
\left( 1 + \frac{1}{12} \delta_{x}^2 \right) \frac{1}{\Gamma(\alpha)} \left[ \int_{0}^{t_k} \frac{\partial^2 p(x_j, s)}{\partial x^2} \frac{ds}{(t_k - s)^{1-\alpha}} - \int_{0}^{t_{k+1}} \frac{\partial^2 p(x_j, s)}{\partial x^2} \frac{ds}{(t_{k+1} - s)^{1-\alpha}} \right] \\
= \left( 1 + \frac{1}{12} \delta_{x}^2 \right) \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \sum_{j=1}^{l} \left[ \frac{t_i - s}{\tau} \frac{\partial^2 p(x_j, t_{i-1})}{\partial x^2} + \frac{s - t_{i-1}}{\tau} \frac{\partial^2 p(x_j, t_i)}{\partial x^2} + O(\tau^2) \right] \frac{ds}{(t_k - s)^{1-\alpha}} \\
- \left( 1 + \frac{1}{12} \delta_{x}^2 \right) \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left[ \frac{t_i - s}{\tau} \left( 1 + \frac{1}{12} \delta_{x}^2 \right) \frac{\partial^2 p(x_j, t_{i-1})}{\partial x^2} + \frac{s - t_{i-1}}{\tau} \left( 1 + \frac{1}{12} \delta_{x}^2 \right) \frac{\partial^2 p(x_j, t_i)}{\partial x^2} + O(\tau^2) \right] \frac{ds}{(t_{k+1} - s)^{1-\alpha}} \\
- \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k-1} \sum_{j=1}^{l} \left[ \frac{t_i - s}{\tau} \left( \delta_x^2 p(x_j, t_{i-1}) + O(h^4) \right) + O(\tau^2) \right] \frac{ds}{(t_{k+1} - s)^{1-\alpha}} \\
- \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left[ \frac{t_i - s}{\tau} \left( \frac{\delta_x^2 p(x_j, t_{i-1})}{h^2} + O(h^4) \right) + \frac{s - t_{i-1}}{\tau} \left( \frac{\delta_x^2 p(x_j, t_i)}{h^2} + O(h^4) \right) + O(\tau^2) \right] \frac{ds}{(t_{k+1} - s)^{1-\alpha}} \\
= \mu_n \sum_{i=1}^{k} \lambda_{n}^{(i)} \delta_{x}^2 p(x_j, t_{k+1}) + R_{i}^{k},
\]
where

\[ \mu_\alpha = \frac{\tau^\alpha}{\Gamma(2 + \alpha)h^\alpha}, \]

\[ \mathcal{R}_j^k = (O(\tau^3) + O(h^4)) \left[ \sum_{l=1}^{k-1} \int_{t_{l-1}}^{t_l} \frac{ds}{(t_k - s)^{1+\alpha}} - \sum_{l=1}^{k-1} \int_{t_{l-1}}^{t_l} \frac{ds}{(t_k - s)^{1+\alpha}} \right]. \]

(16)

Similarly, then, we have

\[ \left( 1 + \frac{1}{12} \frac{\partial^2}{\partial x^2} \right) \left( \frac{1}{\Gamma(\beta)} \int_0^{\tau^\beta} \frac{ds}{(t_k - s)^{1+\beta}} \right) \]

\[ = \mu_\beta \sum_{l=0}^k \lambda_\beta^{(l)} \delta_x^l \mathcal{P}(x_j, t_k) + \mathcal{R}_j^k, \]

(17)

where

\[ \mu_\beta = \frac{\tau^\beta}{\Gamma(2 + \beta)h^\beta}, \]

\[ \mathcal{R}_j^k = (O(\tau^3) + O(h^4)) \left[ \sum_{l=1}^{k-1} \int_{t_{l-1}}^{t_l} \frac{ds}{(t_k - s)^{1+\beta}} - \sum_{l=1}^{k-1} \int_{t_{l-1}}^{t_l} \frac{ds}{(t_k - s)^{1+\beta}} \right]. \]

(18)

Because \((\partial^2 f(x,t) / \partial t^2) \in C(\Omega)\), the following trapezoidal formula holds:

\[ \int_{t_{l-1}}^{t_l} f(x_j, s) ds = \frac{\tau}{2} \left[ f(x_j, t_{k-1}) + f(x_j, t_k) \right] + O(\tau^3), \]

(19)

and according to the above analysis, we have

\[ \left( 1 + \frac{1}{12} \delta_x^2 \right) \mathcal{P}(x_j, t_k) - \left( 1 + \frac{1}{12} \delta_x^2 \right) \mathcal{P}(x_j, t_{k-1}) \]

\[ = \mu_\beta \sum_{l=0}^k \lambda_\beta^{(l)} \delta_x^l \mathcal{P}(x_j, t_k) + \mu_\beta \sum_{l=0}^k \lambda_\beta^{(l)} \delta_x^l \mathcal{P}(x_j, t_{k-1}) \]

\[ + \frac{\tau}{2} \left( 1 + \frac{1}{12} \delta_x^2 \right) \left[ f(x_j, t_{k-1}) + f(x_j, t_k) \right] + \mathcal{R}_j^k, \]

(20)

where

\[ \mathcal{R}_j^k = \mathcal{R}_j^k + \mathcal{R}_j^k + O(\tau^3). \]

(21)

Please note that

\[ \sum_{l=1}^{k-1} \int_{t_{l-1}}^{t_l} \frac{ds}{(t_k - s)^{1+\alpha}} = \int_0^{t_k} \frac{ds}{(t_k - s)^{1+\alpha}} = \frac{(k\tau)\alpha}{\alpha}. \]

\[ \sum_{l=1}^{k-1} \int_{t_{l-1}}^{t_l} \frac{ds}{(t_k - s)^{1+\beta}} = \int_0^{t_k} \frac{ds}{(t_k - s)^{1+\beta}} = \frac{(k\tau)\beta}{\beta}. \]

(22)
with $kr \leq T$; then,
\begin{equation}
R^j_k = O\left(r^2 + h^4\right).
\end{equation}

Now, the numerical methods should be taken into account for solving problems (5)–(7):
\begin{equation}
\begin{aligned}
\left(1 + \frac{1}{12} \delta^2_x\right) p^k_j - \left(1 + \frac{1}{12} \delta^2_x\right) p^{k-1}_j &= \mu_a \sum_{l=0}^k \lambda_{\alpha}^{(l)} \delta^2_x p^{k-1}_j + \mu_p \sum_{l=0}^k \lambda_{\beta}^{(l)} \delta^2_x p^{k-1}_j \\
&+ \frac{r}{2} \left(1 + \frac{1}{12} \delta^2_x\right) \left(f^{k-1}_j + f^k_j\right),
\end{aligned}
\end{equation}
\begin{equation}
k = 1, 2, \ldots, K; j = 1, 2, \ldots, j - 1,
\end{equation}
\begin{equation}
p^0_j = \varphi(t_j), p^k_j = \psi(t_j), \quad k = 0, 1, \ldots, K,
\end{equation}
where $f^j_j = f(x_j, t_j)$, $\omega_j = \omega(x_j)$, and $p^k_j$ is the approximation of the exact solution $p(x_j, t_j)$.

1.3. Convergence of Numerical Methods. In this section, we discuss the convergence of numerical equations (24)–(26). Subtract (24) from (20) and then obtain the following error equation:
\begin{equation}
\begin{aligned}
\left(1 + \frac{1}{12} \delta^2_x\right) E^k_j - \left(1 + \frac{1}{12} \delta^2_x\right) E^{k-1}_j &= \mu_a \sum_{l=0}^k \lambda_{\alpha}^{(l)} \delta^2_x E^{k-1}_j + \mu_p \sum_{l=0}^k \lambda_{\beta}^{(l)} \delta^2_x E^{k-1}_j + R^k_j,
\end{aligned}
\end{equation}
\begin{equation}
k = 1, 2, \ldots, K; j = 1, 2, \ldots, j - 1,
\end{equation}
where
\begin{equation}
E^k_j = p(x_j, t_k) - p^k_j.
\end{equation}

For $k = 0, 1, \ldots, K$, the following grid functions are defined, respectively:
\begin{equation}
E^k(x) = \begin{cases}
E^k_j, & x \in \left(x_{j-(1/2)}, x_{j+(1/2)}\right], j = 1, 2, \ldots, j - 1, \\
0, & x \in \left[0, \frac{h}{2}\right] \cup \left(L - \frac{H}{2}, L\right],
\end{cases}
\end{equation}
\begin{equation}
R^k(x) = \begin{cases}
R^k_j, & x \in \left(x_{j-(1/2)}, x_{j+(1/2)}\right], j = 1, 2, \ldots, j - 1, \\
0, & x \in \left[0, \frac{h}{2}\right] \cup \left(L - \frac{H}{2}, L\right],
\end{cases}
\end{equation}
then, $E^k(x)$ and $R^k(x)$ can be expanded with the following Fourier series, respectively:
\begin{equation}
E^k(x) = \sum_{l=0}^{\infty} \xi_k(l)e^{(i2\pi lx)/L}, \quad k = 0, 1, \ldots, K,
\end{equation}
\begin{equation}
R^k(x) = \sum_{l=0}^{\infty} \eta_k(l)e^{(i2\pi lx)/L}, \quad k = 0, 1, \ldots, K,
\end{equation}
where
\begin{equation}
\xi_k(l) = \int_0^L E^k(x)e^{-(i2\pi lx)/L}dx,
\end{equation}
\begin{equation}
\eta_k(l) = \int_0^L R^k(x)e^{-(i2\pi lx)/L}dx.
\end{equation}
Let
\begin{equation}
E^k(x) = [E^k_1, E^k_2, \ldots, E^k_{j-1}]^T,
\end{equation}
\begin{equation}
R^k(x) = [R^k_1, R^k_2, \ldots, R^k_{j-1}]^T.
\end{equation}
The following Parseval equation can be derived:
\begin{equation}
\int_0^L \left|E^k(x)\right|^2 dx = \int_0^L \left|E^k(x)\right| \cdot \overline{E^k(x)} dx
\end{equation}
\begin{equation}
= \int_0^L E^k(x) \cdot \sum_{l=0}^{\infty} \xi_k(l)e^{(i2\pi lx)/L} dx,
\end{equation}
\begin{equation}
= \int_0^L E^k(x) \cdot \sum_{l=0}^{\infty} \overline{\xi_k(l)}e^{-(i2\pi lx)/L} dx,
\end{equation}
\begin{equation}
= \sum_{l=0}^{\infty} \overline{\xi_k(l)} \int_0^L E^k(x)e^{-(i2\pi lx)/L} dx,
\end{equation}
\begin{equation}
= L \sum_{l=0}^{\infty} \overline{\xi_k(l)} \xi_k(l)
\end{equation}
\begin{equation}
= L \sum_{l=0}^{\infty} \left|\xi_k(l)\right|^2.
\end{equation}
Similarly, the following Parseval equation can also be derived:
\begin{equation}
\int_0^L \left|R^k(x)\right|^2 dx = L \sum_{l=0}^{\infty} \left|\eta_k(l)\right|^2.
\end{equation}
Notice that
\begin{equation}
\int_0^L \left|E^k(x)\right|^2 dx = \left(\sum_{l=1}^{j-1} h^2 |E^k_l|^2\right)^{(1/2)},
\end{equation}
\begin{equation}
\int_0^L \left|R^k(x)\right|^2 dx = \left(\sum_{l=1}^{j-1} h^2 |R^k_l|^2\right)^{(1/2)}.
\end{equation}
So far, we obtain
Lemma 1. If \( t \) Advances in Mathematical Physics satisfies \( \tau \) Lemma 2. where \( \tau \), we get
\[
\left( \frac{1}{3} - \sin^2 \frac{\sigma h}{2} \right) \xi_k - \left( \frac{1}{3} - \sin^2 \frac{\sigma h}{2} \right) \xi_{k-1}
\]
where \( \sigma = (2\pi/L) \). Substituting the above expression into (27), we get
\[
\frac{1}{2} \sin \frac{\sigma h}{2} \sum_{l=0}^{h} \lambda_{1}^{(0)} \xi_{k-l} - 4 \mu_{1} \sin^2 \frac{\sigma h}{2} \sum_{l=0}^{h} \lambda_{1}^{(0)} \xi_{k-l} + \eta_k,
\]
k = 1, 2, . . . , K.
After rearrangement,
\[
\frac{1}{m} \Psi \xi_{k-l} - 4 \mu_{1} \sin \frac{\sigma h}{2} \sum_{l=0}^{h} \lambda_{1}^{(0)} \xi_{k-l} + \eta_k,
\]
k = 1, 2, . . . , K,
where
\[
\Phi = 1 - \frac{1}{3} \sin^2 \frac{\sigma h}{2} + 4 \left( \mu_{1} + \mu_{2} \right) \sin \frac{\sigma h}{2},
\]
\[
\psi = 1 - \frac{1}{3} \sin^2 \frac{\sigma h}{2} - 4 \left( \mu_{1}^{(0)} + \mu_{1}^{(0)} \right) \sin^2 \frac{\sigma h}{2}.
\]
Lemma 3. If \( 0 < \alpha < \beta < 1 \), \( \mu_{1} \lambda_{1}^{(0)} + \mu_{2} \lambda_{2}^{(0)} \leq 0 \), or \( \mu_{1} \lambda_{1}^{(0)} + \mu_{2} \lambda_{2}^{(0)} \leq (1/6) \), then
\[
\Phi \geq 0.
\]
From (23), therefore, constant \( C \), exists and makes
\[
\left| \xi_k \right| \leq C_1 \tau^2 + h^4, \quad k = 1, 2, \ldots , K.
\]
Again, paying attention to the first equation in (37), we have
\[
\left| \eta_k \right| \equiv \left| \eta_k (l) \right| \leq C_2 \tau \left| \eta_1 \right| \equiv C_2 \tau \left| \eta_1 \right|, \quad k = 1, 2, \ldots , K.
\]
Theorem 1. Assuming that \( p(x, t) \in P (\Omega) \) is the exact solution of problems (5)–(7), \( \frac{\partial^2 f(x, t)/\partial t^2}{} \in C (\Omega) \), and time and space steps satisfy \( \tau^2 = h^4 \), if \( \mu_{1} \lambda_{1}^{(0)} + \mu_{2} \lambda_{2}^{(0)} \leq 0 \), or \( \mu_{1} \lambda_{1}^{(0)} + \mu_{2} \lambda_{2}^{(0)} \leq (1/6) \), then the numerical equations (24)–(26) converge, and the convergence order is \( O \tau^2 + h^4 \).
Proof. First, we prove the following conclusion, where there is a positive constant \( C \), such that
\[
\left| \xi_k \right| \leq C \left| \eta_1 \right| , \quad k = 1, 2, \ldots , K,
\]
where \( \xi_k (k = 1, 2, \ldots , K) \) is the solution of (40).
For \( k = 1 \), since \( \tau^2 = 0 \), get \( \xi_0 = 0 \), and further applying Lemma 2 and (47), we can obtain from equation (40),
\[
\left| \xi_1 \right| \leq \frac{1}{\Phi} \left| \eta_1 \right| \leq \left| \eta_1 \right| \leq C_2 \tau \left| \eta_1 \right|.
\]
Assume
\[
\left| \xi_k \right| \leq C_2 \tau \left| \eta_1 \right| , \quad k = 1, 2, \ldots , K.
\]
According to Lemmas 1–3, we have through (40),
\[
\| \xi_1 \| = \frac{1}{\Phi} \left| \psi^{(1)}_{k,-1} - 4\mu_a \sin^2 \frac{\alpha h}{2} \sum_{l=2}^{k} \lambda_{\alpha}^{(l)} \xi_{k-l} - 4\mu_\beta \sin^2 \frac{\alpha h}{2} \sum_{l=2}^{k} \lambda_{\beta}^{(l)} \xi_{k-l} + \eta_k \right|
\]

\[
\leq \frac{1}{\Phi} \left( \psi^{(1)}_{k,-1} - 4\mu_a \sin^2 \frac{\alpha h}{2} \sum_{l=2}^{k} \lambda_{\alpha}^{(l)} \xi_{k-l} - 4\mu_\beta \sin^2 \frac{\alpha h}{2} \sum_{l=2}^{k} \lambda_{\beta}^{(l)} \xi_{k-l} + |\eta_k| \right)
\]

\[
\leq \frac{1}{\Phi} \left( \psi^{(1)}_{k,-1} - 4\mu_a \sin^2 \frac{\alpha h}{2} \sum_{l=2}^{k} \lambda_{\alpha}^{(l)} \xi_{k-l} - 4\mu_\beta \sin^2 \frac{\alpha h}{2} \sum_{l=2}^{k} \lambda_{\beta}^{(l)} \xi_{k-l} + |\eta_k| \right)
\]

\[
\leq \frac{1}{\Phi} \left( |\psi^{(1)}_{k,-1}| - 4\mu_a \sin^2 \frac{\alpha h}{2} \sum_{l=2}^{k} \lambda_{\alpha}^{(l)} \xi_{k-l} - 4\mu_\beta \sin^2 \frac{\alpha h}{2} \sum_{l=2}^{k} \lambda_{\beta}^{(l)} \xi_{k-l} + |\eta_k| \right)
\]

\[
\leq \frac{1}{\Phi} \left( |\psi^{(1)}_{k,-1}| - 4\mu_a \sin^2 \frac{\alpha h}{2} \sum_{l=2}^{k} \lambda_{\alpha}^{(l)} \xi_{k-l} - 4\mu_\beta \sin^2 \frac{\alpha h}{2} \sum_{l=2}^{k} \lambda_{\beta}^{(l)} \xi_{k-l} + |\eta_k| \right)
\]

\[
\leq \frac{1}{\Phi} \left( |\psi^{(1)}_{k,-1}| - 4\mu_a \sin^2 \frac{\alpha h}{2} \sum_{l=2}^{k} \lambda_{\alpha}^{(l)} \xi_{k-l} - 4\mu_\beta \sin^2 \frac{\alpha h}{2} \sum_{l=2}^{k} \lambda_{\beta}^{(l)} \xi_{k-l} + |\eta_k| \right)
\]

\[
= \left( K - 1 + \frac{1}{\Phi} \right) C_2 \tau |\eta_k| \]

\[
\leq C_2 k \tau |\eta_k|.
\]

Conclusion (48) is proven by induction. From conclusions (36), (37), (45), and (48), we obtain the following one:

\[
\| E^2 \| \leq C_2 \tau^3 \| R^2 \| \leq C \tau^3 + h^2,
\]

where \( C = C_1 C_2 \sqrt{LT} \).

We know that because there are many high-precision numerical integration formulas (45) and (46), it is not difficult to continue to choose a high time precision numerical method for solving the modified fractional diffusion equation (5). However, the problem is that the qualitative analysis of convergence and stability can be very difficult.

2. Stability of the Numerical Methods

The stability of numerical format was discussed in (24)–(26); consider the following difference equation:

\[
\left( 1 + \frac{1}{12} \delta^2 \right) \rho_j^{k-1} - \left( 1 + \frac{1}{12} \delta^2 \right) \rho_j^k = \mu_a \sum_{l=0}^{k} \lambda_{\alpha}^{(l)} \delta^2 \rho_j^{k-l} + \mu_\beta \sum_{l=0}^{k} \lambda_{\beta}^{(l)} \delta^2 \rho_j^{k-l} + R^k
\]

\[
k = 1, 2, \ldots, K; j = 1, 2, \ldots, j - 1,
\]

where \( \rho_j^k = \rho_j^k - \bar{\rho}_j^k, \) \( \bar{\rho}_j^k \) is the approximation of \( \rho_j^k. \)

For \( k = 0, 1, \ldots, K, \) respectively, define the following grid function:

\[
E_j^k = \zeta_k e^{i \phi_h},
\]

\[
\rho^k(x) = \left\{ \begin{array}{ll}
\rho_j^k & x \in [x_{j-1/2}, x_{j+1/2}], j = 1, 2, \ldots, j - 1,
0 & x \in \left[ 0, \frac{h}{2} \right] \cup \left[ L - \frac{h}{2}, L \right].
\end{array} \right.
\]

Then, \( \rho^k(x) \) can be expanded by Fourier series:

\[
\rho^k(x) = \sum_{k=0}^{\infty} \zeta_k(l) e^{(i2\pi x)/L}, \quad k = 0, 1, \ldots, K,
\]

where

\[
\zeta_k(l) = \frac{1}{L} \int_0^L \rho^k(l) e^{-i2\pi x/l} dx.
\]

Let

\[
\rho^k(x) = [\rho_1^k, \rho_2^k, \ldots, \rho_{j-1}^k]^T.
\]

Similar to the derivation of (36), the following Parseval equation can also be obtained:

\[
\| \rho^k \|_2 = \left( \sum_{j=1}^{j-1} |H^k_{j,j}|^2 \right)^{1/2}
\]

\[
= L \left( \sum_{l=-\infty}^{\infty} |\zeta_k(l)|^2 \right)^{1/2}, \quad k = 0, 1, \ldots, K.
\]

Assume

\[
E_j^k = \zeta_k e^{i \phi_h},
\]
where $\sigma = (2\pi/L)$.

Substituting the above expression into (53), we have

$$
\left(1 - \frac{1}{3} \sin \frac{\sigma h}{2}\right) \zeta_k - \left(1 - \frac{1}{3} \sin \frac{\sigma h}{2}\right) \zeta_{k-1}
$$

$$
= -4\mu_a \sin \frac{\sigma h}{2} \sum_{l=0}^{k} \lambda_a^{(l)} \zeta_{k-l} - 4\mu_\beta \sin \frac{\sigma h}{2} \sum_{l=0}^{k} \lambda_\beta^{(l)} \zeta_{k-l},
$$

(60)

$k = 1, 2, \ldots, K$.

After rearranging, we get

$$
\zeta_k = \frac{1}{\Phi} \left[ \Psi \zeta_{k-1} - 4\mu_a \sin \frac{\sigma h}{2} \sum_{l=0}^{k} \lambda_a^{(l)} \zeta_{k-l} - 4\mu_\beta \sin \frac{\sigma h}{2} \sum_{l=0}^{k} \lambda_\beta^{(l)} \zeta_{k-l} \right],
$$

(61)

$k = 1, 2, \ldots, K$.

**Theorem 2.** Assume that the time and space steps satisfy $\tau^2 = h^4$, if $\mu_a \lambda_a^{(l)} + \mu_\beta \lambda_\beta^{(l)} \leq 0$, or $0 < \mu_a \lambda_a^{(l)} + \mu_\beta \lambda_\beta^{(l)} \leq (1/6)$, then numerical equations (24)–(26) are stable.

**Proof.** First, the following conclusion is proved:

$$
|\zeta_k| \leq |\zeta_0|, \quad k = 1, 2, \ldots, K,
$$

(62)

where $\zeta_k (k = 1, 2, \ldots, K)$ is the solution of (61).

When $k = 1$, from Lemmas 2 and 3, we get from equation (61)

$$
|\zeta_1| = \frac{1}{\Phi} |\psi| |\zeta_0| \leq |\zeta_0|,
$$

(63)

and assume

$$
|\zeta_n| \leq |\zeta_0|, \quad n = 1, 2, \ldots, K - 1.
$$

(64)

According to Lemmas 1–3, the following equation is obtained from (61):

$$
|\zeta_1| = \frac{1}{\Phi} |\psi \zeta_{k-1} - 4\mu_a \sin \frac{\sigma h}{2} \sum_{l=0}^{k} \lambda_a^{(l)} \zeta_{k-l} - 4\mu_\beta \sin \frac{\sigma h}{2} \sum_{l=0}^{k} \lambda_\beta^{(l)} \zeta_{k-l}| \leq \frac{1}{\Phi} \left( |\psi \zeta_{k-1}| + 4\mu_a \sin \frac{\sigma h}{2} \sum_{l=0}^{k} |\lambda_a^{(l)}| |\zeta_{k-l}| + 4\mu_\beta \sin \frac{\sigma h}{2} \sum_{l=0}^{k} |\lambda_\beta^{(l)}| |\zeta_{k-l}| \right)
$$

$$
\leq \frac{1}{\Phi} \left( |\psi| |\zeta_{k-1}| + 4\mu_a \sin \frac{\sigma h}{2} \sum_{l=0}^{k} |\lambda_a^{(l)}| |\zeta_{k-l}| + 4\mu_\beta \sin \frac{\sigma h}{2} \sum_{l=0}^{k} |\lambda_\beta^{(l)}| |\zeta_{k-l}| \right) |\zeta_0|
$$

$$
= \frac{1}{\Phi} \left( |\psi| + 4\mu_a (1 + \lambda_a^{(l)}) \sin \frac{\sigma h}{2} + 4\mu_\beta (1 + \lambda_\beta^{(l)}) \sin \frac{\sigma h}{2} \right) |\zeta_0| \leq |\zeta_0|.
$$

Conclusion (62) can be achieved by induction. From conclusion (58) and (62), we obtain the following one:

$$
\|p^k\|_2 \leq \|p^0\|_2, \quad k = 1, 2, \ldots, K.
$$

(66)

**3. Solvability of Numerical Methods**

This section discusses the solvability of numerical equations (24)–(26). The corresponding chi-square linear algebraic equations with numerical equations (24)–(26) are

$$
\left(1 + \frac{1}{12} \delta_x^2 \rho_j^0\right) p_j^k - \left(1 + \frac{1}{12} \delta_x^2 \rho_j^0\right) p_j^{k-1},
$$

(67)

$$
= \mu_a \sum_{l=0}^{k} \lambda_a^{(l)} \delta_x p_j^{k-l} + \mu_\beta \sum_{l=0}^{k} \lambda_\beta^{(l)} \delta_x p_j^{k-l},
$$

(68)

$k = 1, 2, \ldots, K; j = 1, 2, \ldots, j - 1$.

$$
p_j^0 = 0, \quad j = 0, 1, \ldots, J,
$$

$$
p_0^k = 0, \quad k = 0, 1, \ldots, K.
$$

(69)

(70)

Similar to the proof of Theorem 2, it can also be shown that the solution $p^k$ of the chi-square linear equations (67)–(71) satisfies

$$
\|p^k\|_2 \leq \|p^0\|_2, \quad k = 1, 2, \ldots, K,
$$

(71)

from $p^0 = 0$, we have

$$
p_0^k = 0, \quad k = 0, 1, \ldots, K.
$$

(72)

This means that the numerical equations (24)–(26) have only zero solutions for the corresponding linear algebraic
Table 1: Maximum error $E_{\infty}$, time accuracy, and spatial accuracy $b$ of numerical equations (24)–(26) for solving problems (73)–(75).

| $\alpha$ | $\beta$ | $t^2 = h^2 = (1/16)$ | $t^2 = h^4 = (1/81)$ | $a$ | $b$ | $t^2 = h^4 = (1/256)$ | $a$ | $b$ |
|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.7 | $3.5759 \times 10^{-5}$ | $7.4432 \times 10^{-6}$ | 1.94 | 3.87 | $2.4953 \times 10^{-6}$ | 1.90 | 3.80 |
| 0.2 | 0.8 | $3.5591 \times 10^{-5}$ | $7.4432 \times 10^{-6}$ | 1.93 | 3.86 | $2.4953 \times 10^{-6}$ | 1.90 | 3.80 |
| 0.3 | 0.9 | $3.5422 \times 10^{-5}$ | $7.3461 \times 10^{-6}$ | 1.94 | 3.88 | $2.2508 \times 10^{-6}$ | 2.06 | 4.11 |
| 0.4 | 0.5 | $3.5675 \times 10^{-5}$ | $7.4432 \times 10^{-6}$ | 1.93 | 3.87 | $2.4272 \times 10^{-6}$ | 1.95 | 3.90 |
| 0.5 | 0.6 | $3.5505 \times 10^{-5}$ | $7.3461 \times 10^{-6}$ | 1.94 | 3.89 | $2.4272 \times 10^{-6}$ | 1.92 | 3.85 |
| 0.6 | 0.7 | $3.5253 \times 10^{-5}$ | $7.2489 \times 10^{-6}$ | 1.95 | 3.90 | $2.4272 \times 10^{-6}$ | 1.90 | 3.80 |
| 0.7 | 0.71 | $3.5169 \times 10^{-5}$ | $7.2489 \times 10^{-6}$ | 1.95 | 3.90 | $2.2508 \times 10^{-6}$ | 2.03 | 4.07 |
| 0.8 | 0.82 | $3.4916 \times 10^{-5}$ | $7.3461 \times 10^{-6}$ | 1.93 | 3.84 | $2.4272 \times 10^{-6}$ | 1.92 | 3.85 |
| 0.9 | 0.93 | $3.4748 \times 10^{-5}$ | $7.1571 \times 10^{-6}$ | 1.95 | 3.90 | $2.0066 \times 10^{-6}$ | 2.21 | 4.42 |

Figure 1: When $\alpha = 0.5, \beta = 0.6$, and $t^2 = h^4 = (1/256)$, numerical methods (24)–(26) are utilized to compare the numerical solutions (N.S.) of problems (73)–(75) with the exact solutions (E.S.) of problems (73)–(75) in $t = (1/4), (2/4), (3/4), 1$.

Theorem 3. Suppose that the time and space steps satisfy $t^2 = h^4$, if $\mu_{\alpha} \lambda_{\alpha}^{(1)} + \mu_{\beta} \lambda_{\beta}^{(1)} \leq 0$, or $0 < \mu_{\alpha} \lambda_{\alpha}^{(1)} + \mu_{\beta} \lambda_{\beta}^{(1)} \leq (1/6)$, then the numerical equations (24)–(26) are uniquely solvable.

4. Numerical Tests

The numerical equations (24)–(26) are used to solve the following modified fractional-order diffusion equation for the initial value problem:

$$\frac{\partial p(x, t)}{\partial t} = \left(\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} + B \frac{\partial^{1-\beta}}{\partial t^{1-\beta}}\right) \frac{\partial^2 p(x, t)}{\partial x^2} + f(x, t), \quad 0 < t \leq 1, 0 < x < 1,$$

$$p(x, 0) = 0, \quad 0 \leq x \leq 1,$$

$$p(x, t) = t^2, \quad 0 \leq t \leq 1,$$

$$\int_{0}^{\infty} f(x, t) \, dx = 0, \quad 0 \leq t \leq 1,$$  \hspace{2cm} (73)

$$f(x, t) = 2e^x \left(t - \frac{t^{1+\alpha}}{\Gamma 2 + \alpha} - \frac{t^{1+\beta}}{\Gamma 2 + \beta}\right), \quad 0 \leq t \leq 1,$$  \hspace{2cm} (74)

where $f(x, t) = 2e^x (t - (t^{1+\alpha}/(\Gamma 2 + \alpha)) - (t^{1+\beta}/(\Gamma 2 + \beta)))$, and the exact solution of problems (73)–(75) is

$$p(x, t) = e^x t^2.$$  \hspace{2cm} (75)

Let the maximum error of the numerical solution be

$$E_{\infty} = \max_{0 \leq x \leq 1} \left\{ \| p(x, t) \|_{L_2} \right\} = O(t^2 + h^6).$$  \hspace{2cm} (76)

Assume that the maximum errors of spatial step, time step, and numerical solution are $h_f$, $\tau_f$, and $E_{\infty}^{(f)}$ in the previous experiment and the maximum error of the spatial step, time step, and numerical solution are $h_s, \tau_s$, and $E_{\infty}^{(s)}$ in the later experiment. Then, the time accuracy $a$ and spatial accuracy $b$ are calculated by the following equation:

$$a = \log_{(\tau_s/\tau_f)} \frac{E_{\infty}^{(s)}}{E_{\infty}^{(f)}}, \quad b = \log_{(h_s/h_f)} \frac{E_{\infty}^{(s)}}{E_{\infty}^{(f)}}.$$  \hspace{2cm} (77)
Table 1 provides that the numerical equations (24)–(26) for problems (73)–(75) are obtained for different $\alpha$, $\beta$, and $\tau^2 = h^2$ with maximum error $E_{\text{max}}$, time accuracy $\alpha$, and spatial accuracy $\beta$, respectively.

5. Conclusion

As given in Table 1, our theoretical analysis results are strongly supported by the numerical experiments.

Figure 1 shows that the numerical methods (24)–(26) are used to compare the numerical solutions (N.S.) of problems (73)–(75) with the exact solutions (E.S.) of problems (73)–(75) in $t = (1/4), (2/4), (3/4), 1$, when $\alpha = 0.5, \beta = 0.6, \tau^2 = h^2 = (1/256)$.

Figure 2 shows that the numerical methods (24)–(26) are used to compare the numerical solutions (N.S.) of problems (73)–(75) with the exact solutions (E.S.) of problems (73)–(75) in $t = (1/4), (2/4), (3/4), 1$, when $\alpha = 0.5, \beta = 0.6, \tau^2 = h^2 = (1/256)$.

As shown in Figures 1 and 2, the numerical solutions of problems (73)–(75) and the exact solutions of problems (73)–(75) obtained by numerical methods (24)–(26) have a good approximation effect. This finding revealed that our theoretical analysis results are reliable.

In conclusion, one-dimensional fractional diffusion equations were studied in the present study, with the purpose that the construction techniques of numerical methods for solving one-dimensional fractional diffusion equations and the corresponding numerical analysis can be extended to multidimensional fractional diffusion equations.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

This work was supported by Guizhou Science and Technology fund major support project, qiankehe (2020) (1z002).

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