Application of RC delay time for estimation of thermal properties

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Abstract. The analogy between the electrical and thermal system has been extensively used to solve different kinds of direct heat transfer problems. However, this analogy has not been explored much to obtain solutions of inverse heat transfer problems like estimation of thermal properties. This paper presents an approach of estimation of thermal properties using the correspondence between the thermal and electrical domains by exploiting the concept of RC delay time in the resistance-capacitance (RC) circuit. Simulations and experiments have been performed on stainless steel and glass samples to show the applicability of the proposed approach for materials belonging to different conductivity range.

1. Introduction
Estimation of material thermal properties is an inverse heat conduction problem. In order to obtain solution of such inverse problems, closed-form solutions are needed. However, such closed-form analytical solutions of heat conduction problem derived using the classical heat transfer approach exists only for simple cases like one dimension (1-D) geometry, infinite solid etc. [1]. For more practical cases like finite solids, finite composite media, the analytical solutions involve infinite series terms [1]. Obtaining inverse solutions from such non-closed form expression is tedious and complex. The transformation of a heat transfer problem into its electrical equivalent due to the correspondence between the thermal and electrical domain can aid in utilizing the closed-form solutions existing in electrical engineering in the field of thermal engineering.

The RC delay time i.e., the time corresponding to 63% of the applied voltage, is an important characteristic parameter on the voltage response of a resistance-capacitance (RC) circuit. Estimation of RC delay time for a circuit consisting of only one resistance (R) and capacitance (C) is well known and is referred as time constant. However, that for a circuit composed of more than one R and C element is difficult to obtain due to the presence of more than one time constant. For estimation of the overall delay time of such circuits, Elmore proposed a closed-form solution [2]. In this paper, this closed-form solution has been exploited to obtain the thermal properties of a material.

2. Methodology
One dimensional (1-D) heat conduction in a finite material of thickness ‘L’ subjected to a constant temperature boundary condition ( \( T = T_1 \) at \( x = 0 \) and \( -k \frac{\partial T}{\partial x} = 0 \) at \( x = L \) ) (figure 1(a)) can be
electrically modelled as an RC ladder circuit with a voltage source (figure 1(b)). The voltage at the opposite end of the RC ladder correspond to surface temperature rise at the rear end (i.e., $x=L$) of the finite thickness material as per the electro-thermal analogy [3].

![Figure 1](image1.png)

**Figure 1.** (a) Material subjected to constant temperature boundary condition; (b) Equivalent electrical model of the material

The propagation of signal from a voltage source to the opposite end of an RC ladder is not instantaneous but is associated with some delay, just like the lag in the thermal response during the heat conduction process. This lag in the thermal response is inversely related to the thermal diffusivity of the material. The electrical quantity analogous to the thermal diffusivity is reciprocal of RC delay time (thermal delay corresponds to RC delay time) [3]. The thermal delay of material undergoing 1-D heat conduction can be obtained by estimating the RC delay time of an equivalent RC ladder circuit using the Elmore delay formula [2] and can be expressed as

$$\tau_{TD} = \sum_{p=1}^{N} (c_{tl} \cdot \Delta l)_p \sum_{q=1}^{p} (r_{tl} \cdot \Delta l)_q$$

where $\tau_{TD}$ is the thermal delay, $N$ is the total number of elements in the equivalent RC ladder, $c_{tl}$ and $r_{tl}$ corresponds to capacitance and resistance of a single element calculated by the electro-thermal analogy approach [3], and $\Delta l$ is thickness of a single element. On expanding equation (1), the thermal delay becomes

$$\tau_{TD} = \Delta l^2 [c_{tl}r_{tl} + c_{tl}(2r_{tl}) \cdots + c_{tl}(Nr_{tl})] = \frac{\Delta l^2 c_{tl} r_{tl} N(N + 1)}{2}$$

For completely distributed circuit ($\Delta l \to 0, N \to \infty$)

$$\tau_{TD} = \frac{r_{tl} \cdot c_{tl} \cdot L^2}{2}$$

(3)

On converting equation (3) into thermal equivalents (by electro-thermal analogy [3]) and rearranging, an expression of thermal diffusivity ($\alpha$) is obtained

$$\alpha = \frac{L^2}{2\tau_{TD}}$$

(4)

Applying the same boundary conditions to a composite media (figure 2), the thermal delay in a composite media undergoing 1-D heat conduction can also be obtained by estimating the RC delay time of an equivalent RC ladder circuit, which can be written as
\[
\tau_{TD} = \Delta l^2 \left\{ \left( r_{ati} \cdot c_{ati} + 2 r_{ati} \cdot c_{ati} \cdots + N_1 \cdot r_{ati} \cdot c_{ati} \right) + \left( N_1 \cdot r_{ati} + r_{btl} \right) c_{btl} + \left( N_1 \cdot r_{ati} + 2 r_{btl} \right) c_{btl} + \cdots + \left( N_1 \cdot r_{ati} + N_2 \cdot r_{btl} \right) c_{btl} \right\}
\]

where \( N_1 \) and \( N_2 \) are the number of elements of the first and second material of the composite media respectively, \( \Delta l \) is the thickness of single element of composite media, \( c_{ati} \) and \( c_{btl} \) are the capacitance of single element of first and second material (obtained by electro-thermal analogy [3]) whereas \( r_{ati} \) and \( r_{btl} \) are the resistance of first and second material respectively (obtained by electro-thermal analogy [3]). When equation (5) is expanded and simplified, an expression of thermal delay in terms of material dimension and thermal properties is obtained

\[
\tau_{TD} = \frac{l_a^2}{2 \alpha_a} + \frac{l_b^2}{2 \alpha_b} + \frac{l_a l_b k_b}{\alpha_b \alpha_a} \quad (6)
\]

where \((l_a, l_b, \alpha_a, \alpha_b)\) and \((k_a, k_b)\) are the thickness, thermal diffusivity and thermal conductivity of first and second material of composite media respectively. Rearranging equation (6), expression of thermal conductivities \((k_a, k_b)\) can be obtained

\[
k_a = \frac{l_a l_b k_b}{\left( \tau_{TD} - \frac{l_a^2}{2 \alpha_a} - \frac{l_b^2}{2 \alpha_b} \right) \alpha_b} \quad (7)
\]

\[
k_b = \frac{\left( \tau_{TD} - \frac{l_a^2}{2 \alpha_a} - \frac{l_b^2}{2 \alpha_b} \right) \alpha_b k_a}{l_a l_b} \quad (8)
\]

Simulations and experiments have been done on stainless steel and glass sample for verifying the applicability of the approach. The 1-D heat conduction is simulated using PSpice, which is a widely used circuit simulator. A finite element approach was used to model the circuit. The thermal properties given in table 1 has been used for simulation. For experimental verification, the sample was placed on the heat source which was maintained at constant temperature and the thermal response on the other side (i.e., the non-heating side) was measured using a thermocouple. The schematic of the experimental setup is shown in figure 3.

![Figure 2. Composite media subjected to constant temperature boundary condition](image)

![Figure 3. Schematic of the experimental setup](image)

| Table 1. Thermal properties of materials (used for simulation) [4, 5] |
| Material property | Stainless steel | Glass |
|-------------------|----------------|------|
| Thermal conductivity, \( k \) (W/m-K) | 16.2 | 1.38 |
| Density, \( \rho \) (kg/m\(^3\)) | 8000 | 2200 |
| Specific heat, \( c_p \) (J/kg-K) | 500 | 740 |
| Thermal diffusivity, \( \alpha \) (\( \times 10^{-6} \) m\(^2\)/s) | 4 | 0.8 |

3. Results and Discussions

The simulated and experimental findings have been discussed in this section. Simulation and experiments have been performed on stainless steel and glass samples for corroborating the proposed approach. Figure 4 shows the thermal response of the two samples obtained both by simulation and experiment. The thermal response initially evolves exponentially and saturates after some time. The time taken by the thermal response to achieve saturation depends on the thermal properties and dimension of the sample under investigation. It is also observed that the difference between the experimental and simulated thermal response is less initially but increases with time. The thermal
diffusivity estimated from the simulated and experimental thermal response are shown in Figure 5. The thermal diffusivity obtained from the simulated thermal response show good accordance with the actual values. However, the thermal diffusivity obtained from experimental thermal response is underestimated which is due to the overestimation of the thermal delay. The overestimation of the thermal delay is mainly due to the effect of undesired losses (like convective and radiative heat loss, conductivity loss to thermocouple and ineffective thermal insulation etc.) on the sample thermal response [3]. For the estimation of thermal conductivity, simulation and experiment was performed on a glass-stainless steel composite media. The thermal response trend followed by the composite media was similar to that discussed in case thermal diffusivity estimation. In case of composite media, the saturation time depends on the thermal properties and dimension of both the constituent materials of the composite media. Figure 6 shows the thermal response obtained both by simulation and experiment. The experimental thermal response differs from the simulated response for the same reasons as discussed previously. Additionally, the resistance at the interface of the constituent materials of the composite media also contributes to the difference. The estimated thermal conductivities of the constituent material of composite media are shown in figure 7. The simulated thermal conductivities were in agreement with the actual values. But that estimated from the experiment laid close to the actual values.
4. Conclusion

This paper demonstrated the application of the RC delay time in the thermal domain for the estimation of thermal properties. The proposed approach was verified through experiments and simulations on stainless steel and glass sample, which showed the pertinence of the approach for materials belonging to diverse conductivity range. It was also found that the effect of undesired loss on the thermal response of materials (or composite media) was inevitable and increases with time. Hence the approach would yield better results for materials (or composite media) exhibiting an overall thermally thin behaviour. There is further scope of improving the approach by including the effect convection in the equivalent electrical model.

References

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