Structure preserving schemes for nonlinear Fokker-Planck
equations with anisotropic diffusion

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Abstract

In this work we propose novel numerical schemes for nonlinear Fokker-Planck-type equations with anisotropic diffusion matrix that preserve fundamental structural properties like non negativity of the solution, entropy dissipation and which guarantees an arbitrarily accurate approximation of the steady state of the problem. All the methods presented are at least second order accurate in the transient regimes and high order for large times. Applications of the schemes to models for collective phenomena and life sciences are considered, in these examples anomalous diffusion is often observed and must be taken into account in realistic models.

Keywords: Fokker-Planck equations, collective behaviour, structure preserving methods
Mathematics Subject Classification: 35Q70, 35Q84, 65N06

1 Introduction

We are interested in nonlinear Fokker-Planck equations describing the evolution of a multivariate distribution function \( f(w, t) \geq 0 \), with \( t \geq 0, w \in \Omega \subseteq \mathbb{R}^d \) of the following form

\[
\begin{aligned}
\partial_t f(w, t) &= \nabla_w \cdot \left[ B[f](w, t) f(w, t) + \nabla_w \cdot (\mathbb{D}(w) f(w, t)) \right] \\
 f(w, 0) &= f_0(w),
\end{aligned}
\]

(1)

where \( f_0(w) \geq 0 \) is a given initial distribution, \( B[f](\cdot, t) \) is a general nonlocal operator. Among the possible forms of the operator \( B[\cdot] \) with interest in collective phenomena we can consider

\[
B[f](\cdot, t) : \Omega \longrightarrow \mathbb{R}^d \\
w \longmapsto S(w) + \int_\Omega P(w, w_\star)(w - w_\star)f(w_\star, t)dw_\star,
\]

(2)

being \( S(\cdot) : \Omega \rightarrow \mathbb{R}^d \) and \( P(\cdot, \cdot) : \Omega \times \Omega \rightarrow \mathbb{R}^+ \). Furthermore, in (1) we indicated with \( \mathbb{D}(w) \in \text{Mat}_{d \times d}(\mathbb{R}) \) a nonconstant matrix characterizing the diffusion which is supposed to be positive definite in \( \Omega \subseteq \mathbb{R}^d \) and therefore invertible in \( \Omega \). We couple (1) with no-flux boundary conditions so that the total mass is conserved at each time \( t \geq 0 \), and \( f_0(w) \) is the initial distribution.

Kinetic-type equations with general diffusion often arise in the derivation of aggregate descriptions of many particles systems. Without intending to review the very huge literature on this topic we mention [9, 2, 5, 15, 15, 23] for applications to collective phenomena, [8, 20, 28, 29, 34, 48] for related models in self-organized biological aggregations, and [24, 45, 36, 46, 47] for their relation with Boltzmann-type modeling. These equations possess a strong physical interpretation.

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First, their solution are density functions which should be therefore nonnegative, under suitable assumptions a entropy functional is defined and the entropy is dissipated in time, and a unique equilibrium is reached for sufficiently regular initial distributions. The necessity to deal with a general diffusion matrix arise from various applications where heterogeneity appears in the evolution of the distribution function.

In this manuscript we concentrate on the construction of numerical schemes for the introduced problem that preserve structural properties like nonnegativity of the solution, entropy dissipation and that approximate with arbitrary accuracy the steady state of the problem. Furthermore, the methods here developed are second order accurate in the transient regime and do not require restrictions on the mesh size. The schemes here derived are based on recent works on this direction [14, 22, 38, 39] and follow pioneering works on linear Fokker-Planck equations [19, 32], see also [11, 12, 42]. We refer to [6, 7, 13, 18, 40] for related methods in the case of degenerate diffusion and to [25] for a recent survey on methods preserving steady states of balance laws.

Despite the apparently simple structure of the introduced class of problems, a purely analytical insight of equilibrium states of nonlinear Fokker-Planck equations with anisotropic diffusion matrix is generally unfeasible, see [41, 43] for an introduction. Hence, numerical methods preserving the mentioned structural properties is in this case essential. We remark how the schemes here derived can be applied to Vlasov-type PDEs with relaxation. The accurate description of steady states is therefore of paramount importance to find a solution compatible with the fluid regime through asymptotic preserving methods [21, 30, 31].

In more details the paper is organized as follows. In Section 2 we derive the SP scheme in dimension 2 by exploiting the large time properties of the introduced problems. We will compare the obtained scheme with recent results for isotropic problems. Hence, in Section 3 we prove nonnegativity of the numerical solution in the case of explicit and semi-implicit time integration. Sufficient conditions will be explicited in terms of bounds on the time step. The trend to equilibrium is then investigated in Section 4, here we prove that the SP scheme dissipates the numerical entropy. Finally in Section 5 we present several applications of the schemes in Fokker-Planck problems describing emerging patterns in interacting systems. Some conclusions are reported at the end of the manuscript.

2 Structure preserving schemes and anisotropic diffusion

In this section we focus on the design of a numerical scheme for nonlinear Fokker-Planck equations with general diffusion matrix of the form (1). For clarity of presentation we focus on the two-dimensional case, i.e. $d = 2$. The generalization of the scheme to the three dimensional case will be presented in Appendix A.

We rewrite equation (1) in divergence form as follows

$$
\partial_t f = \nabla_w \cdot \left[(B[f](w, t) + \nabla_w \cdot D(w)) f(w, t) + D(w) \cdot \nabla_w f(w, t)\right],
$$

where $t \geq 0, w = [w_x, w_y]^T \in \Omega \subset \mathbb{R}^2$, $B[f](\cdot, t)$ is a bounded operator defined as

$$
B[f](\cdot, t) : \Omega \rightarrow \mathbb{R}^2
$$

$$
w \mapsto B[f](w, t) = [B_x[f](w, t), B_y[f](w, t)]^T
$$

and $D$ is a positive definite nonconstant diffusion matrix of the form

$$
D(w) = \begin{bmatrix}
D^{1,1}(w) & D^{1,2}(w) \\
D^{2,1}(w) & D^{2,2}(w)
\end{bmatrix}.
$$

Equation (3) can be rewritten in flux form

$$
\partial_t f = \nabla_w \cdot F[f](w, t).
$$

2
where $\mathcal{F}[f]$ is the following flux
\[ \mathcal{F}[f](w, t) = (\mathcal{B}[f](w, t) + \nabla_w \cdot \mathcal{D}(w)) f(w, t) + \mathcal{D}(w) \cdot \nabla_w f(w, t), \] (5)
where $\nabla_w \cdot \mathcal{D} = [\partial_x \mathcal{D}^{1,1} + \partial_y \mathcal{D}^{2,1}, \partial_x \mathcal{D}^{1,2} + \partial_y \mathcal{D}^{2,2}]^T$, where we indicated with $\partial_x$, $\partial_y$ the partial derivatives in the directions defined by the components $w_x$ and $w_y$, respectively. In particular, let us introduce the following notation
\[ \mathcal{C}[f](w, t) = B[f](w, t) + \nabla_w \cdot \mathcal{D}(w) \]
so that $\mathcal{C}[f](w, t) = [C_x[f](w, t), C_y[f](w, t)]^T$ being
\[ C_x[f](w, t) = B_x[f](w, t) + \partial_x \mathcal{D}^{1,1}(w) + \partial_y \mathcal{D}^{1,2}(w) \]
and
\[ C_y[f](w, t) = B_y[f](w, t) + \partial_x \mathcal{D}^{1,2}(w) + \partial_y \mathcal{D}^{2,2}(w). \]
Therefore we may rewrite the flux defined in (5) as
\[ \mathcal{F}[f](w, t) = [\mathcal{F}_x[f](w, t), \mathcal{F}_y[f](w, t)]^T \]
whose components are given by
\[ \mathcal{F}_x[f](w, t) = C_x[f](w, t) f(w, t) + \mathcal{D}^{1,1}(w) \partial_x f(w, t) + \mathcal{D}^{1,2}(w) \partial_y f(w, t) \] (6)
\[ \mathcal{F}_y[f](w, t) = C_y[f](w, t) f(w, t) + \mathcal{D}^{2,1}(w) \partial_x f(w, t) + \mathcal{D}^{2,2}(w) \partial_y f(w, t). \] (7)

2.1 Derivation of the scheme

We are interested in constructing a scheme capturing the steady states of our problem. In the one-dimensional case, we observe that in correspondence of the steady state the flux is constant and it vanishes with suitable boundary conditions [38]. In the two-dimensional case, we will concentrate on the particular case in which the long time distribution leads to the two components of the analytical flux $\mathcal{F}_x[f]$ and $\mathcal{F}_y[f]$ annihilate. We will show in Section 5 that this constraint links the choice of the drift and diffusion terms.

Under the introduced hypothesis, we can define the following quasi-stationary system for the components of the flux
\[ \mathcal{D}^{1,1} \partial_x f = -f \left( C_x[f] - \mathcal{D}^{1,2} \partial_y f \right) \]
\[ \mathcal{D}^{2,2} \partial_y f = -f \left( C_y[f] - \mathcal{D}^{2,1} \partial_x f \right). \] (8)
Let us observe that, thanks to the introduction of the matrix characterizing anisotropic diffusions the equations (8) are not decoupled unless $\mathcal{D}$ is diagonal. By solving the introduced system first in terms of $\partial_x f$, and then in terms of $\partial_y f$, we find that (8) is equivalent to
\[ \left( \mathcal{D}^{1,1} - \frac{\mathcal{D}^{1,2} \mathcal{D}^{2,1}}{\mathcal{D}^{2,2}} \right) \partial_x f = - f \left( C_x[f] - \frac{\mathcal{D}^{1,2}}{\mathcal{D}^{2,2}} C_y[f] \right) \]
\[ \left( \mathcal{D}^{2,2} - \frac{\mathcal{D}^{2,1} \mathcal{D}^{1,2}}{\mathcal{D}^{1,1}} \right) \partial_y f = - f \left( C_y[f] - \frac{\mathcal{D}^{2,1}}{\mathcal{D}^{1,1}} C_x[f] \right). \] (9)
In the following we adopt the following notations
\[ \mathcal{D}^1(w) = \mathcal{D}^{1,1} - \frac{\mathcal{D}^{1,2} \mathcal{D}^{2,1}}{\mathcal{D}^{2,2}} > 0, \quad \mathcal{D}^2(w) = \mathcal{D}^{2,2} - \frac{\mathcal{D}^{2,1} \mathcal{D}^{1,2}}{\mathcal{D}^{1,1}} > 0. \]
It is worth stressing how in the case $\mathcal{D}^{1,2} = \mathcal{D}^{2,1} = 0$, the two equations in (9) can be decoupled and we basically recover the classical quasi-stationary formulation in each direction, we refer to
for more details on the concept of quasi-equilibrium distribution and to [35] for further applications. Furthermore, we remark how system (9) is in general not solvable, except in some special cases due to the nonlinearity on the right hand side and the intrinsically coupled nature of the system. We overcome this difficulty in the quasi steady-state approximation integrating the equations of system (9) over numerical grids.

Let us introduce the grid \( w_{i,j} = (w_{x,i}, w_{y,j}) \in \Omega \) with \( N \times N \) points, such that \( w_{x,i+1} - w_{x,i} = \Delta w \) and \( w_{y,j+1} - w_{y,j} = \Delta w \) and we denote by \( w_{i,\pm 1/2, j, \pm 1/2} = w_{i,j} \pm (\Delta y, \Delta x) \). Without lack of generality and to avoid unnecessary complications we considered a uniform grid, anyway what presented in the following can be easily generalized to the non uniform case. In the numerical cells \([w_{i,j}, w_{i+1,j}]\) and \([w_{i,j}, w_{i,j+1}]\) from (9) we have

\[
\begin{align*}
\int_{w_{i,j}}^{w_{i+1,j}} \frac{\partial_x f(w,t)}{f(w,t)} \, dw_x &= - \int_{w_{i,j}}^{w_{i+1,j}} \frac{1}{\mathcal{D}^1(w)} \left( C_x[f](w,t) - \frac{\mathcal{D}^{1,2}(w)}{\mathcal{D}^{2,2}(w)} C_y[f](w,t) \right) \, dw_x, \\
\int_{w_{i,j}}^{w_{i+1,j}} \frac{\partial_y f(w,t)}{f(w,t)} \, dw_y &= - \int_{w_{i,j}}^{w_{i+1,j}} \frac{1}{\mathcal{D}^2(w)} \left( C_y[f](w,t) - \frac{\mathcal{D}^{2,1}(w)}{\mathcal{D}^{1,1}(w)} C_x[f](w,t) \right) \, dw_y,
\end{align*}
\]

leading respectively to

\[
\begin{align*}
\frac{f(w_{i+1,j},t)}{f(w_{i,j},t)} &= \exp \left\{ - \int_{w_{i,j}}^{w_{i+1,j}} \frac{1}{\mathcal{D}^1(w)} \left( C_x[f](w,t) - \frac{\mathcal{D}^{1,2}(w)}{\mathcal{D}^{2,2}(w)} C_y[f](w,t) \right) \, dw_x \right\}, \\
\frac{f(w_{i,j+1},t)}{f(w_{i,j},t)} &= \exp \left\{ - \int_{w_{i,j}}^{w_{i+1,j}} \frac{1}{\mathcal{D}^2(w)} \left( C_y[f](w,t) - \frac{\mathcal{D}^{2,1}(w)}{\mathcal{D}^{1,1}(w)} C_x[f](w,t) \right) \, dw_y \right\}.
\end{align*}
\]

Let us consider a conservative discretization of (4)

\[
\frac{d}{dt} f_{i,j}(t) = \frac{\mathcal{F}_x[f]_{i+1/2,j}(t) - \mathcal{F}_x[f]_{i-1/2,j}(t)}{\Delta w} + \frac{\mathcal{F}_y[f]_{i,j+1/2}(t) - \mathcal{F}_y[f]_{i,j-1/2}(t)}{\Delta w},
\]

being \( f_{i,j}(t) \) a numerical approximation of \( f(w_{i,j},t) \) for each \( t \geq 0 \) and \( \mathcal{F}_x[f]_{i+1/2,j}, \mathcal{F}_y[f]_{i,j+1/2} \) the numerical flux functions relative to the introduced numerical discretization. A general second order definition for the numerical fluxes is given by

\[
\begin{align*}
\mathcal{F}_x[f]_{i+1/2,j} &= \tilde{C}_x[f]_{i+1/2,j} \hat{f}_{i+1/2,j} + \mathcal{D}^{1,1}_{i+1/2,j} \frac{\hat{f}_{i+1,j} - \hat{f}_{i,j}}{\Delta w} + \mathcal{D}^{1,2} \partial_y f(w,t), \\
\mathcal{F}_y[f]_{i,j+1/2} &= \tilde{C}_y[f]_{i,j+1/2} \hat{f}_{i,j+1/2} + \mathcal{D}^{2,2}_{i,j+1/2} \frac{\hat{f}_{i,j+1} - \hat{f}_{i,j}}{\Delta w} + \mathcal{D}^{2,1} \partial_x f(w,t),
\end{align*}
\]

where

\[
\begin{align*}
\hat{f}_{i+1/2,j} &= (1 - \delta_{i+1/2,j}) \hat{f}_{i+1,j} + \delta_{i+1/2,j} \hat{f}_{i,j}, \\
\hat{f}_{i,j+1/2} &= (1 - \delta_{i,j+1/2}) \hat{f}_{i,j+1} + \delta_{i,j+1/2} \hat{f}_{i,j},
\end{align*}
\]

We need to define suitable weight functions \( \delta_{i+1/2,j}, \delta_{i,j+1/2} \) and numerical drifts \( \tilde{C}_x[f], \tilde{C}_y[f] \) so that the method preserves the steady state of the problem with arbitrary accuracy and so that its numerical solution defines nonnegative solutions without additional restrictions on the grid \( \Delta w \).

To produce an effective scheme we still need to define a proper discretization for the partial derivative in (13) in the complementary direction of the flux. Hence, in addition to \( \mathcal{F}_x[f]_{i+1/2,j} = 0 \) and \( \mathcal{F}_y[f]_{i,j+1/2} = 0 \) we consider the conditions for \( \partial_x f \) and \( \partial_y f \) coming from (6)-(7) at the steady state in the vanishing flux case and corresponding to the steady state in the present setting. We
obtain the following explicit expressions of the numerical fluxes, which corresponds to consider simultaneously $F_x[f]_{i+1/2,j} = 0$ and $F_y[f]_{i+1/2,j} = 0$

$$
\begin{align}
\partial_x f(w, t) &= -\frac{1}{\Delta w} \left[ \tilde{C}_x[f]_{i+1/2,j} \tilde{f}_{i+1/2,j} + \tilde{D}^1_{i+1/2,j} \frac{f_{i+1,j} - f_{i,j}}{\Delta w} \right], \\
\partial_y f(w, t) &= -\frac{1}{\Delta w} \left[ \tilde{C}_y[f]_{i+1/2,j} \tilde{f}_{i+1/2,j} + \tilde{D}^2_{i+1/2,j} \frac{f_{i+1,j} - f_{i,j}}{\Delta w} \right],
\end{align}
$$

and

$$
\begin{align}
\partial_x f(w, t) &= -\frac{1}{\Delta w} \left[ \tilde{C}_x[f]_{i,j+1/2} \tilde{f}_{i,j+1/2} + \tilde{D}^1_{i,j+1/2} \frac{f_{i,j+1} - f_{i,j}}{\Delta w} \right], \\
\partial_y f(w, t) &= -\frac{1}{\Delta w} \left[ \tilde{C}_y[f]_{i,j+1/2} \tilde{f}_{i,j+1/2} + \tilde{D}^2_{i,j+1/2} \frac{f_{i,j+1} - f_{i,j}}{\Delta w} \right].
\end{align}
$$

Then, from the derived systems (15)-(16) and by expressing $\tilde{f}_{i+1/2,j}$, $\tilde{f}_{i,j+1/2}$ as in (14) we have

$$
f_{i+1,j} = \frac{\delta_{i+1/2,j}}{\tilde{f}_{i,j}} \tilde{G}_x[f]_{i+1/2,j} + \frac{\tilde{D}^1_{i+1/2,j}}{\Delta w} (1 - \delta_{i+1/2,j}) \tilde{G}_x[f]_{i+1/2,j} + \frac{\tilde{D}^2_{i+1/2,j}}{\Delta w} \tilde{C}_x[f]_{i+1/2,j},
$$

and

$$
f_{i,j+1} = \frac{\delta_{i,j+1/2}}{\tilde{f}_{i,j}} \tilde{G}_y[f]_{i,j+1/2} + \frac{\tilde{D}^2_{i,j+1/2}}{\Delta w} (1 - \delta_{i,j+1/2}) \tilde{G}_y[f]_{i,j+1/2} + \frac{\tilde{D}^1_{i,j+1/2}}{\Delta w} \tilde{C}_y[f]_{i,j+1/2},
$$

where $\tilde{D}^\alpha_{i,j} = \tilde{D}^\alpha(w_{i,j})$ for $\alpha = 1, 2$. In (17)-(18) we have also introduced the following notations

$$
\tilde{G}_x[f]_{i+1/2,j} = \tilde{C}_x[f]_{i+1/2,j} - \frac{\tilde{D}^1_{i+1/2,j}}{\tilde{D}^1_{i+1/2,j}} \tilde{C}_y[f]_{i+1/2,j},
\tilde{G}_y[f]_{i,j+1/2} = \tilde{C}_y[f]_{i,j+1/2} - \frac{\tilde{D}^2_{i,j+1/2}}{\tilde{D}^2_{i,j+1/2}} \tilde{C}_x[f]_{i,j+1/2}.
$$

Finally, by equating analytical and the numerical form of the flux, i.e. $f(w_{i+1,j}, t)/f(w_{i,j}, t)$ in (10) with $f_{i+1,j}/f_{i,j}$ in (17), and $f(w_{i,j+1}, t)/f(w_{i,j}, t)$ in (11) with $f_{i,j+1}/f_{i,j}$ in (18), and setting

$$
\tilde{C}_x[f]_{i+1/2,j} = \frac{\tilde{D}^1_{i+1/2,j}}{\Delta w} \int_{w_{i,j}}^{w_{i+1,j}} \frac{C_x[f](w, t)}{D^1(w)} dw_w
$$

and

$$
\tilde{C}_y[f]_{i+1/2,j} = \frac{\tilde{D}^2_{i+1/2,j}}{\Delta w} \int_{w_{i,j}}^{w_{i+1,j}} \frac{C_y[f](w, t)}{D^2(w)} dw_w,
$$

and

$$
\tilde{C}_x[f]_{i,j+1/2} = \frac{\tilde{D}^2_{i,j+1/2}}{\Delta w} \int_{w_{i,j}}^{w_{i,j+1}} \frac{C_x[f](w, t)}{D^2(w)} dw_w
$$

and

$$
\tilde{C}_y[f]_{i,j+1/2} = \frac{\tilde{D}^1_{i,j+1/2}}{\Delta w} \int_{w_{i,j}}^{w_{i,j+1}} \frac{C_y[f](w, t)}{D^1(w)} dw_w,
$$

we finally get

$$
\delta_{i+1/2,j} = \frac{1}{\lambda_{i+1/2,j}} + \frac{1}{1 - \exp(\lambda_{i+1/2,j})}, \quad \delta_{i,j+1/2} = \frac{1}{\lambda_{i,j+1/2}} + \frac{1}{1 - \exp(\lambda_{i,j+1/2})},
$$

(20)
where
\[ \lambda_{i+1/2,j} = \int_{w_{i,j}}^{w_{i+1,j}} \frac{1}{D_1(w)} \left( C_x[f](w,t) - \frac{D_1^{1,2}}{D_2^{1,2}} C_y[f](w,t) \right) \, dw_x = \frac{\Delta w}{D_1^{1+1/2,j}} \tilde{G}_x[f]_{i+1/2,j}, \]
\[ \lambda_{i,j+1/2} = \int_{w_{i,j}}^{w_{i,j+1}} \frac{1}{D_2(w)} \left( C_y[f](w,t) - \frac{D_2^{2,1}}{D_1^{2,1}} C_x[f](w,t) \right) \, dw_y = \frac{\Delta w}{D_2^{i+1/2,j}} \tilde{G}_y[f]_{i,j+1/2}. \]  

(21)

We have proven the following result

**Theorem 1.** The numerical flux defined from the solution of (15)-(16) is given by

\[ F_x[f]_{i+1/2,j} = \tilde{G}_x[f]_{i+1/2,j} f_{i+1/2,j} + D_1^{1} \frac{f_{i+1,j} - f_{i,j}}{\Delta w}, \]
\[ F_y[f]_{i,j+1/2} = \tilde{G}_y[f]_{i,j+1/2} f_{i,j+1/2} + D_2^{2} \frac{f_{i,j+1} - f_{i,j}}{\Delta w}, \]  

(22)

with \( \tilde{G}_x[f]_{i+1/2,j} \) and \( \tilde{G}_y[f]_{i,j+1/2} \) defined in (19) and with \( \delta_{i+1/2,j} \) defined in (20), vanishes when the flux (6)-(7) annihilates over the cell \([w_{i,j},w_{i+1,j}] \times [w_{i,j},w_{i,j+1}]\). The nonlinear weights \( \delta_{i+1/2,j} \) defined in (20)-(21) are such that \( \delta_{i+1/2,j} \in (0,1) \), \( \delta_{i,j+1/2} \in (0,1) \).

**Proof.** The form of the flux comes from the computations present in this section. The solution of (15)-(16) guarantees that the exact flux vanishes in the derived numerical approximation in the case where the components of the analytical flux vanish in the presence of a steady state. Finally, the latter result follows from the inequality \( \exp(x) \geq 1 + x \).

**Remark 2.** The derived scheme may be seen as a generalization of the classical second-order Chang-Cooper scheme \([19, 32]\) to anisotropic nonlinear Fokker-Planck equations. In their original formulation, these works focussed on linear and isotropic Fokker-Planck equations, a recent generalization to the nonlinear case has been proposed in [38]. We highlight how our scheme is coherent to the original one by approximating the functions (21) through a midpoint quadrature rule as follows

\[ \lambda_{i+1/2,j}^{\text{mid}} = \int_{w_{i,j}}^{w_{i+1,j}} \frac{1}{D_1^{1}(w)} \left( C_x[f](w,t) - \frac{D_1^{1,2}}{D_2^{1,2}} C_y[f](w,t) \right) \, dw_x = \frac{\Delta w}{D_1^{1+1/2,j}} \tilde{G}_x[f]_{i+1/2,j}, \]
\[ \lambda_{i,j+1/2}^{\text{mid}} = \int_{w_{i,j}}^{w_{i,j+1}} \frac{1}{D_2^{2}(w)} \left( C_y[f](w,t) - \frac{D_2^{2,1}}{D_1^{2,1}} C_x[f](w,t) \right) \, dw_y = \frac{\Delta w}{D_2^{i+1/2,j}} \tilde{G}_y[f]_{i,j+1/2}. \]  

leading to the following weights

\[ \delta_{i+1/2,j}^{\text{mid}} = \frac{D_1^{1}}{\Delta w} \left( C_x[f]_{i+1/2,j} - \frac{D_1^{1,2}}{D_2^{1,2}} C_y[f]_{i+1/2,j} \right) + \frac{1}{1 - \exp(\lambda_{i+1/2,j}^{\text{mid}})} \]
\[ \delta_{i,j+1/2}^{\text{mid}} = \frac{D_2^{2}}{\Delta w} \left( C_y[f]_{i,j+1/2} - \frac{D_2^{2,1}}{D_1^{2,1}} C_x[f]_{i,j+1/2} \right) + \frac{1}{1 - \exp(\lambda_{i,j+1/2}^{\text{mid}})}. \]

Hence, in the case \( D_1^{1,2} = D_2^{2,1} = 0 \) we recover the classical formulation. Furthermore, if \( B[f](w,t) = B(w) \) has components which are first order polynomials, the midpoint rule gives an exact evaluation of the integrals in (21).
Remark 3. For linear problems in the form $\mathcal{B}[f](w, t) = B(w)$, the exact stationary profile making the flux vanish may be directly computed by solving (9) under suitable assumptions and together with the boundary conditions. A possible form is $B(w) = -\nabla_w \cdot \mathbb{D}(w) - \mathbb{D}(w) \cdot \nabla_w \phi(w)$ with $\phi(\cdot) : \mathbb{R}^d \to \mathbb{R}$.

Indeed the quasi-stationary formulation becomes stationary. Once we know the stationary state $f^\infty(w)$, we can compute the weights $\delta_{i+1/2,j}$, $\delta_{i,j+1/2}$ exactly. In fact, we have

$$
\frac{f_{i+1,j}^\infty}{f_{i,j}^\infty} = \exp \left\{ - \int_{w_{i,j}}^{w_{i+1,j}} \frac{1}{D^1(w)} \left( C_x[f](w, t) - \frac{\partial w_{i,j}}{2} C_y[f](w, t) \right) dw_x \right\} = \exp \left\{ -\lambda_{i+1/2,j}^\infty \right\},
$$

$$
\frac{f_{i,j+1}^\infty}{f_{i,j}^\infty} = \exp \left\{ - \int_{w_{i,j}}^{w_{i,j+1}} \frac{1}{D^2(w)} \left( C_y[f](w, t) - \frac{\partial w_{i,j}}{2} C_x[f](w, t) \right) dw_y \right\} = \exp \left\{ -\lambda_{i,j+1/2}^\infty \right\},
$$

that define the following weights

$$
\delta_{i+1/2,j}^\infty = \frac{1}{\log f_{i,j}^\infty - \log f_{i+1,j}^\infty} + \frac{f_{i+1,j}^\infty}{f_{i,j}^\infty - f_{i+1,j}^\infty},
$$

$$
\delta_{i,j+1/2}^\infty = \frac{1}{\log f_{i,j}^\infty - \log f_{i,j+1}^\infty} + \frac{f_{i,j+1}^\infty}{f_{i,j}^\infty - f_{i,j+1}^\infty}.
$$

Remark 4. If we consider the limit case in which the diffusion tensor tends to be singular and the elements of $\nabla \cdot \mathbb{D}$ tend to vanish, we obtain

$$
\delta_{i+1/2,j} = \begin{cases} 0, & B_{i+1/2,j} > 0 \\ 1, & B_{i+1/2,j} < 0 \end{cases},
$$

$$
\delta_{i,j+1/2} = \begin{cases} 0, & B_{i,j+1/2} > 0 \\ 1, & B_{i,j+1/2} < 0. \end{cases}
$$

Therefore the scheme reduces to a first order upwind scheme.

### 3 Main properties

In this section we show the properties of the derived numerical schemes. In particular, we will prove how the present method enforces conservations, non negativity of the numerical solution and correctly dissipates the entropy.

Lemma 5. Let us consider problem (1) complemented with no-flux boundary conditions, i.e.

$$
F_x[f]_{i+1/2,j} = F_x[f]_{i-1/2,j} = 0, \quad \text{and} \quad F_y[f]_{i,N+1/2} = F_y[f]_{i,-1/2} = 0,
$$

for all $i, j = 0, \ldots, N$. Then we have

$$
\frac{d}{dt} \sum_{i=0}^{N} \sum_{j=0}^{N} f_{i,j}(t) = 0.
$$

Proof. From (12) we have

$$
\sum_{i=0}^{N} \sum_{j=0}^{N} \frac{d}{dt} f_{i,j} = \frac{1}{\Delta w} \sum_{j=0}^{N} (F_x[f]_{i+1/2,j} - F_x[f]_{i-1/2,j}) + \frac{1}{\Delta w} \sum_{i=0}^{N} (F_y[f]_{i,N+1/2,j} - F_y[f]_{i,-1/2,j}),
$$

from which we conclude. \qed
3.1 Positivity

In this section we will provide results for non-negativity of the scheme with explicit time integration. To this end, we introduce a time discretization \( t_n = n \Delta t \) with \( \Delta t > 0 \) and \( n = 0, ..., T \) and consider the simple forward Euler method

\[
f^{n+1}_{i,j} = f^n_{i,j} + \frac{\Delta t}{\Delta w} \left( \tilde{G}_x [f^n_{i+1/2,j}] (1 - \delta^n_{i+1/2,j}) + \frac{\partial^1_{i+1/2,j}}{\Delta w} \right) f^n_{i+1,j}
\]

we can prove the following result

**Theorem 6.** Under the time step restriction

\[
\Delta t \leq \frac{\Delta w^2}{2[(G_x + G_y)\Delta w + (D^1 + D^2)]}
\]

where

\[
G_x = \max_{i,j} |\tilde{G}_x[f^n_{i+1/2,j}]|, \quad G_y = \max_{i,j} |\tilde{G}_y[f^n_{i,j+1/2}]|
\]

and

\[
D^1 = \max_{i,j} D^1_{i+1/2,j}, \quad D^2 = \max_{i,j} D^2_{i,j+1/2},
\]

the explicit scheme preserves nonnegativity, i.e. \( f^{n+1}_{i,j} \geq 0 \) if \( f^n_{i,j} \geq 0 \).

**Proof.** We will adopt the structure of the scheme introduced in Theorem 1. In details, the scheme reads

\[
f^{n+1}_{i,j} = f^n_{i,j} + \frac{\Delta t}{\Delta w} \left( \tilde{G}_x [f^n_{i+1/2,j}] (1 - \delta^n_{i+1/2,j}) + \frac{\partial^1_{i+1/2,j}}{\Delta w} \right) f^n_{i+1,j}
\]

This is a sum of convex combinations of \( f^n_{i+1,j}, f^n_{i-1,j} \) and \( f^n_{i,j+1}, f^n_{i,j-1} \) if the following conditions are satisfied

\[
\tilde{G}_x [f^n_{i+1/2,j}] (1 - \delta^n_{i+1/2,j}) + \frac{\partial^1_{i+1/2,j}}{\Delta w} \geq 0, \quad -\tilde{G}_x [f^n_{i-1/2,j}] \delta^n_{i-1/2,j} + \frac{\partial^1_{i-1/2,j}}{\Delta w} \geq 0,
\]

\[
\tilde{G}_y [f^n_{i,j+1/2}] (1 - \delta^n_{i,j+1/2}) + \frac{\partial^2_{i,j+1/2}}{\Delta w} \geq 0, \quad -\tilde{G}_y [f^n_{i,j-1/2}] \delta^n_{i,j-1/2} + \frac{\partial^2_{i,j-1/2}}{\Delta w} \geq 0,
\]

that is equivalent to

\[
\lambda^n_{i+1/2,j} \left( 1 - \frac{1}{1 - \exp(\lambda^n_{i+1/2,j})} \right) \geq 0, \quad \frac{\lambda^n_{i-1/2,j}}{\exp(\lambda^n_{i-1/2,j}) - 1} \geq 0,
\]

\[
\lambda^n_{i,j+1/2} \left( 1 - \frac{1}{1 - \exp(\lambda^n_{i,j+1/2})} \right) \geq 0, \quad \frac{\lambda^n_{i,j-1/2}}{\exp(\lambda^n_{i,j-1/2}) - 1} \geq 0,
\]

\[8\]
which hold true thanks to the basic properties of the exponential function. In order to ensure positivity for $f_{i,j}^{n+1}$ if $f_{i,j}^n \geq 0$ we must have for all $i,j$

$$
\left(1 - (\nu_x + \nu_y) \frac{\Delta t}{\Delta w}\right) f_{i,j}^n \geq 0
$$

where

$$
\nu_x = \max_{i,j} \left\{ -\tilde{G}_x[f_i^n] \delta_{i+1/2,j}^n + \tilde{G}_x[f_{i-1/2,j}^n] (1 - \delta_{i-1/2,j}^n) + \frac{D_{i+1/2,j}^1 + D_{i-1/2,j}^1}{\Delta w} \right\},
$$

$$
\nu_y = \max_{i,j} \left\{ -\tilde{G}_y[f_{i,j+1/2}^n] \delta_{i,j+1/2}^n + \tilde{G}_y[f_{i,j-1/2}^n] (1 - \delta_{i,j-1/2}^n) + \frac{D_{i,j+1/2}^2 + D_{i,j-1/2}^2}{\Delta w} \right\},
$$

from which we can conclude being $0 \leq \delta_{i\pm1/2,j} \leq 1$, $0 \leq \delta_{i,j\pm1/2} \leq 1$. \(\square\)

We highlight how the restriction on $\Delta t$ in (24) ensures positivity of the numerical solution of the problem without additional bounds on the spatial grids as happens for central schemes, see [38] for additional details. This remarkable property holds also for higher order strong stability preserving (SSP) methods like SSP Runge-Kutta and SSP multistep methods [26] since these are convex combinations of the forward Euler integration. Hence, the proved non-negativity of the scheme is automatically extended to each general SSP type time integration.

Even if in (24) we obtained an effective the time step bound for the positivity of the explicit scheme, for practical purposes this parabolic restriction is very heavy especially in genuine nonlinear type problems. Usually the strategy to overcome this problem relies in the adoption of IMEX schemes [21]. Nevertheless, this is not always possible if the due to the strong nonlinearities embedded in problem (1) coming from the nonlocal drift term. Further, the defined weights $\delta_{i+1/2,j}, \delta_{i,j+1/2}$ depend in general on $f$ introducing additional difficulties. An efficient way to overcome this problem relies in the semi-implicit integration technique, see [10].

To apply semi-implicit schemes we integrate (12) as follows

$$
f_{i,j}^{n+1} = f_{i,j}^n + \Delta t \frac{\hat{x}[f_{i+1/2,j}]^{n+1} - \hat{x}[f_{i-1/2,j}]^{n+1}}{\Delta w} + \Delta t \frac{\hat{y}[f_{i,j+1/2}]^{n+1} - \hat{y}[f_{i,j-1/2}]^{n+1}}{\Delta w}
$$

where now the discretized flux terms $\hat{x}[f_{i+1/2,j}]$, $\hat{y}[f_{i,j+1/2}]$ are defined as

$$
\hat{x}[f_{i+1/2,j}] = \tilde{G}_x[f_{i+1/2,j}^n] \left[ (1 - \delta_{i+1/2,j}^n) f_{i+1,j}^{n+1} + \delta_{i+1/2,j}^n f_{i,j}^{n+1} \right] + \frac{D_{i+1/2,j}^1 - f_{i,j}^{n+1}}{\Delta w},
$$

$$
\hat{y}[f_{i,j+1/2}] = \tilde{G}_y[f_{i,j+1/2}^n] \left[ (1 - \delta_{i,j+1/2}^n) f_{i,j+1}^{n+1} + \delta_{i,j+1/2}^n f_{i,j}^{n+1} \right] + \frac{D_{i,j+1/2}^2 - f_{i,j}^{n+1}}{\Delta w}.
$$

**Theorem 7.** Under the time step restriction

$$
\Delta t \leq \frac{\Delta w}{2(G_x + G_y)}, \quad G_x = \max_{i,j} \|\tilde{G}_x[f_{i+1/2,j}]\|, \quad G_y = \max_{i,j} \|\tilde{G}_y[f_{i,j+1/2}]\|
$$

the semi-implicit scheme (25) preserves nonnegativity, i.e., $f_{i,j}^{n+1} \geq 0$ if $f_{i,j}^n \geq 0$. 



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Proof. Equation (25) corresponds to

\[
\begin{align*}
 f^{n+1}_{i+1,j} &= 
 - \frac{\Delta t}{\Delta w} \left[ \hat{g}_x \left[ f^n_{i+1/2,j} \right] (1 - \delta_{i+1/2,j}) + \frac{D^1_{i+1/2,j}}{\Delta w} \right] \\
 &+ f^{n+1}_{i,j} \left[ 1 - \frac{\Delta t}{\Delta w} \left[ \hat{g}_x \left[ f^n_{i+1/2,j} \right] \delta_{i+1/2,j} - \hat{g}_x \left[ f^n_{i-1/2,j} \right] (1 - \delta_{i-1/2,j}) - \frac{D^1_{i-1/2,j}}{\Delta w} \right] \right] \\
 &+ f^{n+1}_{i-1,j} \left[ - \frac{\Delta t}{\Delta w} \left[ \hat{g}_x \left[ f^n_{i-1/2,j} \right] \delta_{i-1/2,j} + \frac{D^1_{i-1/2,j}}{\Delta w} \right] \right] \\
 &+ f^{n+1}_{i,j+1} \left[ - \frac{\Delta t}{\Delta w} \hat{g}_y \left[ f^n_{i,j+1/2} \right] (1 - \delta_{i,j+1/2}) + \frac{D^2_{i,j+1/2}}{\Delta w} \right] \\
 &+ f^{n+1}_{i,j-1} \left[ - \frac{\Delta t}{\Delta w} \hat{g}_y \left[ f^n_{i,j-1/2} \right] \delta_{i,j-1/2} + \frac{D^2_{i,j-1/2}}{\Delta w} \right] = f^n_{i,j}.
\end{align*}
\]

Using the identities in (21), we have that

\[
\begin{align*}
 f^{n+1}_{i+1,j} &= 
 - \frac{\Delta t}{\Delta w^2} D^1_{i+1/2,j} \frac{\lambda^n_{i+1/2,j}}{\exp(\lambda^n_{i+1/2,j}) - 1} \exp(\lambda^n_{i+1/2,j}) \\
 &+ f^{n+1}_{i,j} \left[ 1 + \frac{\Delta t}{\Delta w^2} \left[ D^1_{i+1/2,j} \frac{\lambda^n_{i+1/2,j}}{\exp(\lambda^n_{i+1/2,j}) - 1} + \frac{\lambda^n_{i+1/2,j}}{\exp(\lambda^n_{i+1/2,j}) - 1} \right] \right] \\
 &+ f^{n+1}_{i-1,j} \left[ - \frac{\Delta t}{\Delta w^2} D^1_{i-1/2,j} \frac{\lambda^n_{i-1/2,j}}{\exp(\lambda^n_{i-1/2,j}) - 1} \right] \\
 &+ f^{n+1}_{i,j+1} \left[ - \frac{\Delta t}{\Delta w^2} D^2_{i,j+1/2} \frac{\lambda^n_{i,j+1/2}}{\exp(\lambda^n_{i,j+1/2}) - 1} \right] \\
 &+ f^{n+1}_{i,j-1} \left[ - \frac{\Delta t}{\Delta w^2} D^2_{i,j-1/2} \frac{\lambda^n_{i,j-1/2}}{\exp(\lambda^n_{i,j-1/2}) - 1} \right] = f^n_{i,j}.
\end{align*}
\]

Then by introducing the quantities

\[
\alpha^n_{i+1/2,j} = \frac{\lambda^n_{i+1/2,j}}{\exp(\lambda^n_{i+1/2,j}) - 1} \geq 0, \quad \text{and} \quad \alpha^n_{i,j+1/2} = \frac{\lambda^n_{i,j+1/2}}{\exp(\lambda^n_{i,j+1/2}) - 1} \geq 0.
\]
and setting

\[ R_x(j)^n = 1 + \frac{\Delta t}{\Delta x^2} \left[ D_{i+1/2,j}^1 \alpha_{i+1/2,j}^n - D_{i-1/2,j}^1 \alpha_{i-1/2,j}^n \exp(\lambda_{i-1/2,j}^n) \right] \]

\[ Q_x(j)_i^n = -\frac{\Delta t}{\Delta x^2} D_{i+1/2,j}^1 \alpha_{i+1/2,j}^n \exp(\lambda_{i+1/2,j}^n) \]

\[ P_x(j)_i^n = -\frac{\Delta t}{\Delta x^2} D_{i-1/2,j}^1 \alpha_{i-1/2,j}^n \]

\[ R_y(i)_j^n = 1 + \frac{\Delta t}{\Delta y^2} \left[ D_{i,j+1/2}^2 \alpha_{i,j+1/2}^n - D_{i,j-1/2}^2 \alpha_{i,j-1/2}^n \exp(\lambda_{i,j-1/2}^n) \right] \]

\[ Q_y(i)_j^n = -\frac{\Delta t}{\Delta y^2} D_{i,j+1/2}^2 \alpha_{i,j+1/2}^n \exp(\lambda_{i,j+1/2}^n) \]

\[ P_y(i)_j^n = -\frac{\Delta t}{\Delta y^2} D_{i,j-1/2}^2 \alpha_{i,j-1/2}^n \]

the latter equation reduces to

\[ R_x(j)^n f_{i,j}^{n+1} - Q_x(j)^n f_{i+1,j}^{n+1} - P_x(j)^n f_{i-1,j}^{n+1} + R_y(i)_j^n f_{i,j+1}^{n+1} - Q_y(i)_j^n f_{i,j-1}^{n+1} - P_y(i)_j^n f_{i,j-1}^{n+1} = f_{i,j}^n. \]

Now, by denoting \( F^n = \{ f_{i,j}^n \}_{i=1,...,N} \) we can define the matrices

\[ A_x[F^n]_{ik} = \begin{cases} R_x(j)^n & k = i \\ -Q_x(j)^n & k = i + 1, \ 0 \leq i \leq N - 1 \\ -P_x(j)^n & k = i - 1, \ 1 \leq i \leq N. \end{cases} \]

\[ A_y[F^n]_{jk} = \begin{cases} R_y(i)_j^n & k = j \\ -Q_y(i)_j^n & k = j + 1, \ 0 \leq j \leq N - 1 \\ -P_y(i)_j^n & k = j - 1, \ 1 \leq j \leq N. \end{cases} \]

we reduce to study

\[ (A_x[F^n] + A_y[F^n]) f^{n+1} = F^n. \]

If \( F^n \geq 0 \), in order to prove that \( F^{n+1} \geq 0 \) it is sufficient to prove that \((A_x[F^n] + A_y[F^n])^{-1} \) is non-negative. Let us observe that since \((A_x[F^n] + A_y[F^n])\) is tridiagonal we only need to prove that it is a diagonally dominant matrix. In particular, this is true if for each \( i, j = 1, \ldots, N \) the following inequality is verified

\[ |R_x(j)^n + R_y(i)_j^n| > |Q_x(j)^n + Q_y(i)_j^n| + |P_x(j)^n + P_y(i)_j^n|, \]

which is true provided

\[ 1 > \frac{\Delta t}{\Delta x^2} \left[ D_{i+1/2,j}^1 \alpha_{i+1/2,j}^n (\exp(\lambda_{i+1/2,j}^n) - 1) - D_{i-1/2,j}^1 \alpha_{i-1/2,j}^n \exp(\lambda_{i-1/2,j}^n) - 1) \right] \]

\[ \frac{\Delta t}{\Delta x^2} \left[ D_{i,j+1/2}^2 \alpha_{i,j+1/2}^n (\exp(\lambda_{i,j+1/2}^n) - 1) - D_{i,j-1/2}^2 \alpha_{i,j-1/2}^n \exp(\lambda_{i,j-1/2}^n) - 1) \right] \]

\[ \frac{\Delta t}{\Delta x^2} \left[ D_{i+1/2,j}^1 \alpha_{i+1/2,j}^n \exp(\lambda_{i+1/2,j}^n) - 1) - D_{i-1/2,j}^1 \alpha_{i-1/2,j}^n \exp(\lambda_{i-1/2,j}^n) - 1) \right] \]

\[ \frac{\Delta t}{\Delta x^2} \left[ D_{i,j+1/2}^2 \alpha_{i,j+1/2}^n \exp(\lambda_{i,j+1/2}^n) - 1) - D_{i,j-1/2}^2 \alpha_{i,j-1/2}^n \exp(\lambda_{i,j-1/2}^n) - 1) \right] \]
4 Trends to equilibrium

A classical question in kinetic theory pertains the determination of the rate of exponential convergence to equilibrium. To this end the consolidated approach relies on entropy production arguments for which lower bounds are explicitly computable thanks to log-Sobolev inequalities, see [44, 46]. In particular, the convergence to the stationary state of the standard Fokker-Planck equation can be achieved by looking at the monotonicity in time of various Lyapunov functionals like the relative entropy. In the nonconstant diffusion case additional difficulties arise since standard log-Sobolev inequality are not available [33].

In order to study the entropy properties, we consider

\[ B[f](w,t) = B[w] \]

leading to the following prototype equation

\[ \partial_t f = \nabla_w \cdot [B(w)f(w,t) + \nabla_w \cdot (Df)] , \quad w \in \Omega \]  \hspace{1cm} (26)

with \( D \) positive definite and also symmetric and no-flux boundary conditions

\[ B(w)f(w,t) + \nabla_w \cdot (Df) = 0, \quad w \in \partial\Omega. \]

A possible choice considered in the literature is

\[ B(w) = w - U, \quad U \in \Omega, \]

which results from the nonlocal operator (2) with \( S \equiv 0 \) and \( P \equiv 1 \), see [24].

We start to observe that if the stationary state \( f^\infty \) of (26) exists, it satisfies

\[ B(w)f^\infty(w,t) + \nabla_w \cdot (Df^\infty) = 0, \quad w \in \Omega. \]

Then

\[ B(w) = -f^\infty \nabla_w \cdot \frac{\nabla f^\infty}{f^\infty} = -\nabla_w \cdot \frac{\nabla f^\infty}{f^\infty} \] \hspace{1cm} (27)

Therefore, equation (26) may be written for \( f = f(w,t), \ w \in \Omega, \) in the form

\[ \partial_t f = \nabla_w \cdot \left( f^\infty D \nabla_w \frac{f}{f^\infty} \right), \] \hspace{1cm} (28)

since

\[ \nabla_w \cdot [B(w)f + \nabla_w \cdot (Df)] = \nabla_w \cdot \left[ -D\nabla_w \cdot \frac{\nabla f^\infty}{f^\infty} + \nabla_w \cdot (Df) \right] \]

\[ = \nabla_w \cdot \left[ D\nabla_w f^\infty - \nabla f^\infty \right] \]

\[ = \nabla_w \cdot \left[ fD \nabla_w f^\infty - \nabla f^\infty \right] \]

\[ = \nabla_w \cdot \left[ fD \nabla_w f^\infty \right] \]

considering, as usual, the boundary conditions

\[ f^\infty \nabla_w \frac{f}{f^\infty} = 0, \quad w \in \partial\Omega. \]

Therefore, from the Landau’s formulation (28), we get the equation satisfied by \( F = f/f^\infty \) that is

\[
\partial_t F = \frac{\partial_t f}{f^\infty} = \nabla_w \cdot \left[ f^\infty D \nabla_w \frac{f}{f^\infty} \right]
\]

\[ = \nabla_w \cdot \left( D \nabla_w F + D \nabla_w F \cdot \frac{\nabla f^\infty}{f^\infty} \right) \]

\[ = \nabla_w \cdot \left( D \nabla_w F - B(w) \cdot \nabla_w F - \nabla f^\infty \cdot \nabla f^\infty \right) \cdot \nabla f^\infty, \]
where the last equality holds true since \( D \) is a symmetric matrix and thanks to the relation (27). Now, since
\[
\nabla_w \cdot (D \nabla_w F) = (\nabla_w \cdot D) \cdot \nabla_w F + D : \nabla_w(\nabla_w F),
\]
where \( \nabla_w(\nabla_w F) \) is the covariant derivative of the vector \( \nabla_w F \), i.e. \( \nabla_w(\nabla_w F) = (\partial_{w_i} \nabla_w F) \), and it is the Hessian matrix of \( F \), which we will denote \( H_w[F] \). is the inner tensorial product that is for definition
\[
D : H_w[F] = \text{tr} \left[ (H_w[F])^T D \right].
\]
In conclusion
\[
\partial_t F = D : H_w[F] - B(w) \cdot \nabla_w F.
\]

4.1 Lyapunov functionals

We will focus on the study of relative Shannon entropy for the problem (1) with nonconstant diffusion. We will extend the results proved in [24] to the two-dimensional case where the diffusion is a nonconstant positive definite tensor of the second order and the drift term is general in the form \( B(w) \).

Let \( f, g : \Omega \to \mathbb{R}^+ \) denote two probability densities. Then, the relative Shannon entropy of \( f \) and \( g \) is defined by
\[
H(f|g) = \int_{\Omega} f \log \frac{f}{g} dw.
\]
It is a Lyapunov functional since the following result can be established.

**Theorem 9.** Let \( F(w,t) \) be the solution to Eq. (30) in \( \Omega \). Then, if \( \Psi(w) \) is a smooth function such that
\[
|\Psi| \leq c \leq \infty \text{ on } \partial \Omega
\]
the following relation holds
\[
\int_{\Omega} f^{\infty}(w,t) \Psi(w) \partial_t F(w,t) dw = \int_{\Omega} f^{\infty}(w,t) \nabla_w \Psi \cdot (D \nabla_w F(w,t)) dw.
\]
**Proof.** From (30) it follows that
\[
\int_{\Omega} f^{\infty}(w) \Psi(w) \partial_t F dw = \int_{\Omega} f^{\infty}(w) \Psi(w) (D : H_w[F] - B(w) \cdot \nabla_w F) dw
\]
and from (31) the latter term is equal to
\[ \int_{\Omega} f^\infty(w) \Psi(w) \left( \nabla_w (D \nabla_w F) - \nabla_w \cdot D \nabla_w F \right) dw - \int_{\Omega} f^\infty(w) \Phi(w) B(w) \cdot \nabla_w F dw \]
\[ = - \int_{\Omega} \nabla_w (f^\infty(w) \Psi(w)) \cdot (D \nabla_w F) dw + \int_{\partial \Omega} f^\infty(w) \Psi(w) (D \nabla_w F) d\sigma(w) \]
\[ - \int_{\Omega} \left[ B(w) f^\infty(w) + \nabla_w \cdot D f^\infty(w) \right] \cdot \nabla_w F \Psi(w) dw \]
\[ = - \int_{\Omega} \nabla_w \cdot (f^\infty(w) \Psi(w)) \cdot (D \nabla_w F) dw \]
\[ - \int_{\Omega} \left[ B(w) f^\infty(w) + \nabla_w \cdot D f^\infty(w) \right] \cdot \nabla_w F \Psi(w) dw \]
\[ = - \int_{\Omega} f^\infty(w) \nabla_w \Psi(w) \cdot (D \nabla_w F) dw - \int_{\Omega} \left[ B(w) f^\infty(w) + \nabla_w \cdot (D f^\infty(w)) \right] \cdot \nabla_w F \Psi(w) dw \]
\[ = - \int_{\Omega} f^\infty(w) \nabla_w \Psi(w) \cdot (D \nabla_w F) dw, \]
as the border terms vanish because of the boundary conditions and we used (29) and the divergence theorem.

**Theorem 10.** Let the smooth function $\Phi(x), x \in \mathbb{R}^+$ be convex. Then, if $F(t, w)$ is the solution to Eq. (30) in $\Omega$, and $c \leq F(t, w) \leq C$ for some positive constants $c < C$, the functional
\[ \Theta(F(t)) = \int_{\Omega} f^\infty(w) \Phi(F(w, t)) dw \]
is monotonically decreasing in time, and the following equality holds
\[ \frac{d}{dt} \Theta(F(t)) = -I_\Theta(F(t)) \]
where $I_\Theta$ denotes the quantity
\[ I_\Theta = \int_{\Omega} f^\infty(w) \Phi''(F(t, w)) \nabla_w F D(w) \nabla_w F dw \]
(32)
that is non-negative because $\Phi$ is convex and $D(w)$ is positive definite.

**Proof.** The relation (32) follows from Theorem 9 by choosing $\Psi(w) = \Phi'(F(w, t))$ for a fixed $t > 0$.

The Shannon entropy of $f$ relative to $f^\infty$, defined by (31) with $g = f^\infty$, is obtained by choosing $\Phi(x) = x \log x$. In this case
\[ I_\Theta = \int_{\Omega} f \frac{\nabla_w F}{F} D(w) \frac{\nabla_w F}{F} dw \]
that may be re-written as
\[ I_\Theta = \int_{\Omega} f \left( \frac{\nabla_w f}{f} - \frac{\nabla_w f^\infty}{f^\infty} \right) D(w) \left( \frac{\nabla_w f}{f} - \frac{\nabla_w f^\infty}{f^\infty} \right) dw \]
that is the Fisher information of $f$ relative to $f^\infty$. We might also consider the weighted $L^2$ distance that is obtained by considering $\Phi(x) = (x - 1)^2$. In this case
\[ \Theta(F(t)) = L^2(f, f^\infty) = \int_{\Omega} \frac{(f - f^\infty)^2}{f^\infty} dw \]
and

\[ I(\Theta) = 2 \int_{\Omega} \nabla_u F \nabla(w) \nabla_u F dw. \]

### 4.1.1 Dissipation of the numerical entropy

In the following results we show how the derived schemes dissipate in the introduced setting a Shannon-type numerical entropy functional.

**Theorem 11.** In the case \( B[f](w, t) = B(w) \) the numerical flux function (22) with \( \delta_{i+1/2,j}, \delta_{i,j+1/2} \) given by (20) can be written in the form (28) and reads

\[
\begin{align*}
F_x^n[f]_{i+1/2,j} &= \frac{D_1^{i+1/2,j}}{\Delta w} \left( \lambda_{i+1/2,j} \hat{f}_{i+1/2,j}^\infty + \left( f_{i+1,j}^n - f_{i,j}^n \right) \right) \\
F_y^n[f]_{i,j+1/2} &= \frac{D_2^{i,j+1/2}}{\Delta w} \left( \lambda_{i,j+1/2} \hat{f}_{i,j+1/2}^\infty + \left( f_{i,j+1}^n - f_{i,j}^n \right) \right)
\end{align*}
\]

where

\[
\hat{f}_{i+1/2,j}^\infty = \frac{f_{i+1,j}^\infty - f_{i,j}^\infty}{\Delta w} \log \left( \frac{f_{i+1,j}^\infty}{f_{i,j}^\infty} \right) \quad \hat{f}_{i,j+1/2}^\infty = \frac{f_{i,j+1}^\infty - f_{i,j}^\infty}{\Delta w} \log \left( \frac{f_{i,j+1}^\infty}{f_{i,j}^\infty} \right)
\]

**Proof.** If \( B[f] = B(w) \), we have that the definitions of \( \lambda_{i+1/2,j} \) and \( \lambda_{i,j+1/2} \) do not depend on time. Hence, we may denote \( \lambda_{i+1/2,j} = \lambda_{i+1/2,j}^\infty \) and \( \lambda_{i,j+1/2} = \lambda_{i,j+1/2}^\infty \) and we have

\[
\begin{align*}
\log f_{i+1,j}^\infty - \log f_{i,j}^\infty &= \lambda_{i+1/2,j} \\
\log f_{i,j+1}^\infty - \log f_{i,j}^\infty &= \lambda_{i,j+1/2}
\end{align*}
\]

and \( \delta_{i+1/2,j}, \delta_{i,j+1/2} \) are of the form (23). Therefore, under these assumptions the flux function writes

\[
\begin{align*}
F_x^n[f]_{i+1/2,j} &= \frac{D_1^{i+1/2,j}}{\Delta w} \left( \lambda_{i+1/2,j} \hat{f}_{i+1/2,j}^\infty + \left( f_{i+1,j}^n - f_{i,j}^n \right) \right) \\
F_y^n[f]_{i,j+1/2} &= \frac{D_2^{i,j+1/2}}{\Delta w} \left( \lambda_{i,j+1/2} \hat{f}_{i,j+1/2}^\infty + \left( f_{i,j+1}^n - f_{i,j}^n \right) \right)
\end{align*}
\]

and

\[
\begin{align*}
F_x^n[f]_{i+1/2,j} &= \frac{D_1^{i+1/2,j}}{\Delta w} \left( \lambda_{i+1/2,j} \hat{f}_{i+1/2,j}^\infty + \left( f_{i+1,j}^n - f_{i,j}^n \right) \right) \\
F_y^n[f]_{i,j+1/2} &= \frac{D_2^{i,j+1/2}}{\Delta w} \left( \lambda_{i,j+1/2} \hat{f}_{i,j+1/2}^\infty + \left( f_{i,j+1}^n - f_{i,j}^n \right) \right)
\end{align*}
\]

By substituting (23) in (33)-(34) we obtain the thesis. \( \square \)

**Theorem 12.** Let us consider \( B[f](w, t) = B(w) \) as in equation (26). The numerical flux satisfies the discrete entropy dissipation

\[
\frac{d}{dt} \mathcal{H}_\Delta(f, f^\infty) = -I_\Delta(f, f^\infty)
\]

where

\[
\mathcal{H}_\Delta(f, f^\infty) = \Delta w^2 \sum_{j=0}^{N} \sum_{i=0}^{N} f_{i,j} \log \frac{f_{i,j}}{f_{i,j}^\infty}
\]
and $\mathcal{I}_{\Delta}$ is the positive discrete dissipation function

$$
\mathcal{I}_{\Delta} = \Delta w \sum_{j=0}^{N} \sum_{i=0}^{N} \left[ \log \left( \frac{f_{i,j+1}}{f_{\infty,i,j}} \right) - \log \left( \frac{f_{i,j}}{f_{\infty,i,j}} \right) \right] \left( \frac{f_{i,j+1} - f_{i,j}}{f_{\infty,i,j}} \right) f_{i+1/2,j}^1 \mathcal{D}_{i+1/2,j}^1
$$

$$
+ \sum_{i=0}^{N} \sum_{j=0}^{N} f_{i,j+1} \left[ \log \left( \frac{f_{i,j+1}}{f_{\infty,i,j+1}} \right) - \log \left( \frac{f_{i,j}}{f_{\infty,i,j}} \right) \right] \left( \frac{f_{i,j+1} - f_{i,j}}{f_{\infty,i,j}} \right) f_{i,j+1/2}^2 \mathcal{D}_{i,j+1/2}^2.
$$

(35)

Proof. If we compute the time derivative of the discrte relative entropy we have that

$$
\frac{d}{dt} \mathcal{H}(f,f^{\infty}) = \Delta w^2 \sum_{j=0}^{N} \sum_{i=0}^{N} \frac{df_{i,j}}{dt} \left( 1 + \log \left( \frac{f_{i,j}}{f_{\infty,i,j}} \right) \right)
$$

$$
= \Delta w \sum_{j=0}^{N} \sum_{i=0}^{N} \left( 1 + \log \left( \frac{f_{i,j}}{f_{\infty,i,j}} \right) \right)
$$

$$
\times \left( \mathcal{F}^{1}[f]_{i,j+1/2}^n(t) - \mathcal{F}^{1}[f]_{i,j-1/2}^n(t) + \mathcal{F}^{2}[f]_{i,j+1}^n(t) - \mathcal{F}^{2}[f]_{i,j-1/2}^n(t) \right).
$$

After telescopic summation and thanks to the identity of Proposition 11 we obtain (35), which is positive because $\mathcal{D}^\alpha > 0$, $\alpha = 1,2$ and $(x - y) \log(\frac{x}{y})$ is positive for all $x, y \geq 0$.

\[ \square \]

Remark 13. We highlight that in the case in which $\mathcal{D}_{1,2} = \mathcal{D}_{2,1} = 0$ and $\mathcal{D}$ is isotropic, if we define an energy in the form

$$
\xi(w,t) = (U_p \ast f)(w,t) + \frac{\text{tr}(\mathcal{D})}{2} \log(f)
$$

which in our case corresponds to

$$
\mathcal{B}[f](w,t) = \nabla_w (U_p \ast f)(w,t),
$$

with $U_p = U_p(|w|)$ an interaction potential, then we have that

$$
\nabla_w \xi(w,t) = \mathcal{B}[f](w,t) + \mathcal{D} \nabla_w \log(f).
$$

Therefore, Eq. (3) may be written in the form

$$
\partial_t f(w,t) = \nabla_w \cdot [f(w,t) \nabla_w \xi(w,t)], \quad w \in \Omega,
$$

and therefore in a gradient flow structure for which entropic averaged schemes may be used [38].

5 Applications

In this section we present some numerical examples of Fokker-Planck equations with anisotropic diffusion matrix solved through structure-preserving schemes that have been introduced in the previous sections. As we have shown, the key point for an accurate approximation of the long time behavior of (1) is reduced to a high order numerical approximation of the nonlinear weights (21)-(20). In the following numerical examples we consider open Newton-Cotes methods up to order 6 and a Gauss-Legendre quadrature. For the Gaussian quadrature we considered 8 points in each numerical cell. In the sequel, we will adopt the notation $SP_k$, with $k = 2, 4, 6, G$, to denote the SP schemes with (21) that is evaluated with second, fourth, sixth order Newton-Cotes quadrature or Gaussian quadrature, respectively. We highlight how possible singularities at the boundaries are avoided using open quadrature rules.
5.1 Test 1. Validation

In this subsection we consider the evolution of a distribution function \( f(w, t) \), \( w \in [-1, 1] \times [-1, 1] \), whose evolution is given by (1) in which, given the diffusion tensor \( D \), we chose the drift operator in such a way that the flux vanishes. In particular, we consider a linear drift term of the following form

\[
B[f](w, t) = B(w) := -\nabla_w \cdot D(w) - D(w) \cdot \nabla_w \phi(w)
\]

being \( \phi(w) \) a given function of the state variable and \( D(w) \) a \( 2 \times 2 \) matrix of the form

\[
D = \begin{bmatrix}
\frac{\sigma^2}{2}(1-w_x^2) & \rho \frac{\sigma_1 \sigma_2}{4}(1-w_x^2)(1-w_y^2) \\
\rho \frac{\sigma_1 \sigma_2}{4}(1-w_x^2)(1-w_y^2) & \frac{\sigma^2}{2}(1-w_y^2)^2
\end{bmatrix}, \quad w_x, w_y \in [-1, 1].
\] (36)

For the above choice the exact stationary state is explicitly computable and is given by

\[
f_\infty(w) = C \exp\{-\phi(w)\}
\] (37)

being \( C > 0 \) a normalization constant. As initial condition we consider

\[
f_0(w) = \beta \left[ \exp(-c(w_x + \mu)^2) \exp(-c(w_y + \mu)^2) + \exp(-c(w_x - \mu)^2) \exp(-c(w_y - \mu)^2) \right]
\] (38)

with \( \mu = \frac{1}{2} \), \( c = 30 \) and where \( \beta > 0 \) is a normalization constant.

In Figure 1 we compute the relative \( L^1 \) error of the numerical solution with respect to the exact stationary state (37) using \( N = 81 \) gridpoints for the \( SP_k \) scheme with various quadrature rules. The different integration methods capture the steady state with different accuracy. In particular low order quadrature rules achieve their numerical steady state faster due to a saturation effect, whereas high order quadratures essentially reach machine precision in finite time. We considered in this plot semi-implicit time integration. In the same figure we illustrate how \( SP_k \) scheme dissipates the relative entropy (32) in the case of two coarse grids with \( N = 10 \) and \( N = 20 \) points.

In Table 1 we estimate the order of convergence of the schemes for first order time integration and a fourth order Runge-Kutta integration. The time step is chosen such that the CFL condition for the positivity of the scheme is satisfied, \( i.e., \Delta t = O(\Delta w^2) \). We may observe that in the transient regime the second order is maintained, whilst we reach higher orders for larger times.
Table 1: **Test 1.** Estimation of the order of convergence for $SP_k$ scheme with explicit Euler (left) and RK4 (right). Rates have been computed using $N = 21, 41, 81$ gridpoints in each component of the computational cell. We considered $\sigma_1^2 = \sigma_2^2 = 1$, $\rho = 0.1$, $\Delta t = \Delta w^2/(10\sigma_1^2\Delta w + 10)$.

Table 2: **Test 1.** Estimation of the order of convergence for $SP_k$ scheme with first (left) and second order (right) semi-implicit methods. Rates have been computed using $N = 21, 41, 81$ gridpoints, $\sigma_1^2 = \sigma_2^2 = 1$, $\Delta t = \Delta w/(20\sigma_1^2)$, and two correlation coefficients $\rho = 0.1$ (top) and $\rho = 0.9$ (bottom).

5.2 **Test 2. Alignment dynamics in bounded domains**

Let us consider the evolution of a distribution function as in (1) with $w \in [-1,1] \times [-1,1]$, anisotropic diffusion introduced in (36), and

$$B[f](w,t) = \int_{[-1,1] \times [-1,1]} P(w,w_*) (w-w_*) f(w_*,t) dw_*$$

(39)

with $P \equiv 1$, and we considered as initial distribution (38). We note that in this case we have no guarantee that the flux vanishes for large times.

In Table 3 we estimate the order of convergence of the $SP_k$ scheme with explicit time integration methods. In details, we computed the relative $L^1$ error for $N = 21, 41, 81$ gridpoints by considering as before as reference solution the one of the successive refinement of the computational grid. We present the case of first order forward Euler method and fourth order Runge-Kutta with suitable time step to guarantee positivity of the scheme, i.e. $\Delta t = O(\Delta w^2)$. In Table (4) we estimate the order of convergence of the method in the case of semi-implicit time integration taking into account first and second order semi-implicit methods with $\Delta t = O(\Delta w)$. We may observe that in this case only the second order is globally conserved. The scheme increases its order but is not capable to assume the order of the quadrature.
Table 3: Test 2. Estimation of the order of convergence for SPCC scheme with explicit Euler (left) and RK4 (right). Rates have been computed using \( N = 21, 41, 81 \) gridpoints in each component of the computational cell. We considered \( \sigma_1^2 = \sigma_2^2 = 1, \rho = 0.1, \Delta t = \Delta w^2/(10\sigma_1^2\Delta w + 10) \).

| Time | \( SP_k \) | \( SP_k \) |
|------|------------|------------|
| 1    | 2.0830     | 2.1320     |
| 10   | 2.0914     | 2.1102     |
| 20   | 2.0914     | 2.1102     |

Table 4: Test 2. Estimation of the order of convergence for SP scheme with first (left) and second order (right) semi-implicit integration. Rates have been computed using \( N = 21, 41, 81, \sigma_1^2 = \sigma_2^2 = 1, \rho = 0.1, \Delta t = \Delta w/(20\sigma_1^2) \).

| Time | \( SP_k \) | \( SP_k \) |
|------|------------|------------|
| 1    | 1.9585     | 1.9612     |
| 10   | 2.0694     | 2.0685     |
| 20   | 2.0695     | 2.0686     |

Figure 2: Test 2. Evolution of the nonlinear FP equation with drift (39), \( P \equiv 1 \), and anisotropic diffusion matrix (36) with \( \sigma_1^2 = 0.1, \sigma_2^2 = 0.5 \) and correlation coefficient \( \rho = 0.1 \) (top row) and \( \rho = 0.9 \) (bottom row). The numerical solution has been computed with \( N = 101 \) gridpoints in both components and semi-implicit time integration with \( \Delta t = \Delta w/(20\max\{\sigma_1^2, \sigma_2^2\}) \).
In Figure 2 we present the evolution of the 2D Fokker-Planck equation with drift term of the form (39) with $P \equiv \chi(||w-w_\star|| \leq \Delta)$ and anisotropic diffusion (36) for several choices of $\sigma_1, \sigma_2$ and correlation coefficient $\rho \in (0,1)$. We consider as initial distribution the one introduced in (38). In Figure 3 we present the evolution of the 2D Fokker-Planck equation with drift term of bounded confidence type (39) with $P = \chi(||w-w_\star|| \leq \Delta)$, being $|| \cdot ||$ the standard Euclidean distance, $\chi(\cdot)$ the indicator function, and $0 \leq \Delta \leq 2$ a given constant measuring the maximum distance for which interaction is activated. The resulting model has been introduced in [27] in the microscopic setting and has been deeply investigated in the kinetic community in the isotropic case, see for example [1, 37, 47]. We remark how the present setting corresponds to a multidimensional opinion formation process where consensus may be reached also in the anisotropic case. In particular, if the correlation $\rho$ is not zero, there is an anisotropic consensus for sufficiently big parameter $\Delta$, in particular we considered the case $\Delta = 0.8$. On the other hand, for smaller values of the parameter $\Delta$ consensus is not achieved and clustered distributions typically appear for long time. We present the case $\Delta = 0.4$ for which we have anisotropic clustering. In all the presented examples it is easily observed how the anisotropy strongly modifies the observed large time behavior of the system. Here the integral $\mathcal{B}[f](w; t)$ has been evaluated through a trapezoidal rule.

5.3 Test 3. Anisotropy in swarming modelling

Let us consider a self-propelled swarming model of Cucker-Smale type with anisotropic diffusion. This model has been proposed in [3] in the case of constant diffusion. In the original model a density of individuals $f(x, w, t)$ is considered, representing the density of individuals in position $x \in \mathbb{R}^{d_x}$ having velocity $w \in \mathbb{R}^{d_w}$, $d_x \geq 1$, $d_w \geq 1$, at time $t > 0$, which is solution of the following

\begin{align*}
\frac{\partial f}{\partial t} + v \cdot \nabla_x f &= \nabla_x \cdot (f(v \cdot \nabla_w f) - (\sigma_x(v) \chi(||w-w_\star|| \leq \Delta) - \sigma_w(v) \chi(||x-x_\star|| \leq \Delta)) f), \\
\frac{\partial v}{\partial t} &= f(v) - f_\star(v).
\end{align*}
We may observe that for large values of the diffusion coefficients we have a symmetric steady state, whilst for small diffusion coefficients the steady state is not symmetric. This behavior suggests that there is a phase transition like the one stated by the result proved in [3], in which they have used, with a Gauss-Legendre quadrature method. The second order semi-implicit numerical scheme has been used, with a Gauss-Legendre quadrature method. Each method reach spectral accuracy in the case \(\alpha = 0\) since all the quadrature methods become exact since we need to integrate a first order polynomial to find the weights (20). We note that since the diffusion matrix does not depend on \(w\) we are again in the case where the steady state distribution of the problem corresponds to a vanishing flux. The main feature of this model is to enclose a phase transition between the ordered states and a chaotic state characterized by a null asymptotic velocity of the system of agents, [3, 4]. Several examples at the PDE level has been given in [38], see also [16, 17]. In the following we investigate the performance of the derived SP scheme for the introduced model in the case of anisotropic diffusion.

The space homogeneous version of the introduced model can be formulated in terms of the inhomogeneous equation

\[
\frac{\partial f}{\partial t}(x, w, t) + w \cdot \nabla_x f(x, w, t) = \nabla_w \cdot [\alpha w(|w|^2 - 1)f(x, w, t) + \rho_f(w - w_f)f(x, w, t) + \rho \nabla_w f(x, w, t)],
\]

where \(\alpha \geq 0\), \(\nabla = D \cdot I\) with \(D > 0\) and \(I\) the identity matrix, are respectively self-propulsion strength and intensity of the diffusion operator, and where \(w_f\) is the mean velocity of the system which is not conserved due to the presence of the self-propelling term

\[
\rho_f(x, t) = \int_{\mathbb{R}^2} f(x, w, t) dw, \quad \rho_f(x, t)u_f(x, t) = \int_{\mathbb{R}^2} w f(x, w, t) dw.
\]

In Table 5 we estimate the order of convergence of the SP scheme in the case of semi-implicit time integration and two different self-propulsion strengths \(\alpha = 0, \alpha = 1\), in the case of different quadrature methods. Each method reach spectral accuracy in the case \(\alpha = 0\) since all the quadrature methods become exact since we need to integrate a first order polynomial to find the weights (20). We note that since the diffusion matrix does not depend on \(w \in \mathbb{R}\) we are again in the case where the steady state distribution of the problem corresponds to a vanishing flux.

Finally, in Figure 4 we present the stationary state for the resulting 2D model for several values of the diffusion tensor and self-propulsion coefficient \(\alpha \geq 0\). We consider as initial distribution a bivariate normal distribution of the form

\[
f_0(w) = \frac{1}{2\pi\sigma^2} \exp \left\{ -\frac{1}{2} \left[ \frac{(w_x - \mu_x)^2}{2\sigma^2} + \frac{(w_y - \mu_y)^2}{2\sigma^2} \right] \right\}
\]

where \(\mu_x = 0.5, \mu_y = -0.5\) and \(\sigma = 0.01\). The second order semi-implicit numerical scheme has been used, with a Gauss-Legendre quadrature method. We may observe that for large values of the diffusion coefficients we have a symmetric steady state, whilst for small diffusion coefficients the steady state is not symmetric. This behavior suggests that there is a phase transition like the one stated by the result proved in [3], in which they consider an isotropic diffusion.

| Time | \(SP_k, \alpha = 0\) | \(SP_k, \alpha = 1\) |
|------|----------------|----------------|
| 1    | 2.1105 2.1105 2.1105 2.1105 | 1.9016 1.9716 1.9716 1.9716 |
| 5    | 8.2885 8.2885 8.2885 8.2885 | 1.9615 8.2913 8.2913 8.2913 |
| 10   | 23.1521 23.1521 23.1521 23.1521 | 1.9621 11.2461 11.2461 11.2461 |

Table 5: Test 3. Estimation of the order of convergence for \(SP_k\) scheme with second order semi-implicit methods for \(\alpha = 0\) (left) and \(\alpha = 1\) (right). Rates have been computed using \(N = 21, 41, 81\), \(\sigma_1^2 = \sigma_2^2 = 0.4\), \(\rho = 0.1\) in the computational domain \([-L, L] = [-6, 6]^2\), \(\Delta t = \Delta \omega/(20L1^2)\).
Figure 4: Test 3. Large time distributions at time $T = 20$ for the two-dimensional swarming model (40) in the homogeneous case with diffusion matrix (41) with correlation coefficient $\rho = 0.1$ and two choices of the diffusion coefficients $\sigma_1^2 \neq \sigma_2^2$. We considered as initial distribution (42). The left column corresponds to the case $\alpha = 5$ and the right column to the case $\alpha = 10$. The numerical solution has been computed through a $SP_G$ scheme with $N = 101$ gridpoints in both directions of the domain $[-3, 3] \times [-3, 3]$ and over the time interval $[0, T]$, $T = 20$ with $\Delta t = \Delta v/9$. 
5.4 Test 4. 3D numerical test

In the present section we present extension to the 3D case for the introduced scheme. We report the nonlinear weights in the Appendix A. In order to show the effectiveness of the approach we extend to the three dimensional case the latter test describing the self-propelled swarming model. In particular, we consider a density of individuals \( f(w, t) \) such that \( \int_{\mathbb{R}^3} f(w, t) = 1 \) having velocity \( w \in \mathbb{R}^3 \) at time \( t > 0 \), and solution of the following homogeneous equation

\[
\partial_t f(w, t) = \nabla_w \cdot [\alpha w(|w|^2 - 1)f(w, t) + (w - u_f)f(w, t) + D \nabla_w f(w, t)],
\]

where \( \alpha \geq 0 \), is the self-propulsion strength and intensity of the diffusion operator, and where \( u_f = \int_{\mathbb{R}^3} w f(w, t) dw \) is the mean velocity of the system which is not conserved due to the presence of the self-propulsion term.

The present case can be framed in the general setting introduced in (1) with drift term

\[
B[f](w, t) = \alpha w(|w|^2 - 1) + \int_{\mathbb{R}^3} P(w, w_*) (w - w_*) f(w_*, t) dw_*
\]

with \( P \equiv 1 \), and we will consider \( D \) a matrix with constant components

\[
D = \begin{bmatrix}
\sigma_1^2 & \rho \sigma_1 \sigma_2 & \rho \sigma_1 \sigma_3 \\
\rho \sigma_1 \sigma_2 & \sigma_2^2 & \rho \sigma_2 \sigma_3 \\
\rho \sigma_1 \sigma_3 & \rho \sigma_2 \sigma_3 & \sigma_3^2
\end{bmatrix}, \quad \rho \in (0, 1), \sigma_k > 0 \quad k = 1, 2, 3. \tag{44}
\]

In Fig. 5 we present two numerical tests in analogy with the tests that we considered in the previous section. In particular, we considered for a given self-propulsion coefficient \( \alpha = 2 \) first the regime of large diffusion coefficients \( \sigma_1 \neq \sigma_2 \neq \sigma_3 \). Hence, we investigated the small diffusion coefficient case. As initial distribution we considered a multivariate normal distribution which has the following form

\[
f_0(w) = \frac{1}{2\pi \sigma^2} \exp \left\{ -\frac{1}{2} \left( \frac{(w_x - \mu_x)^2}{2\sigma^2} + \frac{(w_y - \mu_y)^2}{2\sigma^2} + \frac{(w_z - \mu_z)^2}{2\sigma^2} \right) \right\}
\]

where we fixed \( \mu_x = \mu_y = \mu_z = 0.3 \) and \( \sigma = 0.01 \). In Figure 5 we report the distribution at time \( T = 20 \) over the domain \([-3, 3]^3\) discretized with \( N = 61 \) gridpoints and obtained through a \( SPG \) scheme with semi-implicit time integration, \( \Delta t = O(\Delta w) \). In particular, we can observe that the emerging distribution has isotropic 1D marginals. This behavior is coherent with the case discussed in the 2D case in Section 5.3. In Figure 6 we present the related case with vanishing diffusion coefficients. The numerical parameters have been chosen in the same way of Figure 5. We can easily observe that the model for small diffusion parameters looses isotropy by components characterizing the large diffusion case even in case of anistropic diffusion. Hence, the behavior of the solution in the two numerical tests suggests the existence of a phase transition also in the 3D case.

Conclusion

We studied the construction of structure preserving methods for Fokker-Planck equations with anistropic nonconstant diffusion matrix. Under suitable assumptions we have been able to derive schemes that approximate with arbitrary accuracy the steady state of those problems that are in general analytically unknown. All the methods are second order accurate in the transient regimes even for problems whose flux does not vanish at equilibrium. Furthermore, the methods here developed are positivity preserving without any restriction on the discretization of the state variable both in the case of SSP and of semi-implicit time integration methods, the latter in particular lead to more mild restrictions on the time step that are very useful in the high-dimensional case. Trends to equilibrium have been studied in relation to the dissipation of the numerical entropy and in particular we proved that the introduced schemes dissipates the numerical entropy. We presented
Figure 5: **Test 4.** Large time distribution for the 3D model of swarming (43) with anisotropic diffusion $D$ of the form (44) with $\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 5$, constant self-propulsion coefficient $\alpha = 2$ and correlation coefficient $\rho = 0.1$. The numerical domain is $[-3, 3]^3$ discretized with $N = 61$ gridpoints. We evolution over the time interval $[0, 20]$ has been computed through $SP_G$ scheme with second-order semi-implicit time integration and $\Delta t = \Delta w/9$. In the top row there are two perspectives of the three-dimensional distribution, in the bottom row there are the one dimensional marginal density functions.
Figure 6: **Test 4.** Large time distribution for the 3D model of swarming (43) with anisotropic diffusion $D$ of the form (44) with $\sigma_1 = 1 \cdot 10^{-2}, \sigma_2 = 2 \cdot 10^{-2}, \sigma_3 = 5 \cdot 10^{-2}$, constant self-propulsion coefficient $\alpha = 2$ and correlation coefficient $\rho = 0.1$. The numerical domain is $[-3,3]^3$ discretized with $N = 61$ gridpoints. We evolution over the time interval $[0,20]$ has been computed through $SPG_2$ scheme with second-order semi-implicit time integration and $\Delta t = \Delta w/9$. In the top row there are two perspectives of the three-dimensional distribution, in the bottom row there are the one dimensional marginal density functions.
several application in the context of collective phenomena in the 2D case. Extension of the present set-up to the 3D case have been applied for a swarming model that exhibit phase transition in the isotropic case. Fully nonlinear diffusion problems together with the case of vanishing diffusion are currently under study and will be presented elsewhere.

Acknowledgements

This research was partially supported by the Italian Ministry of Education, University and Research (MIUR) through the “Dipartimenti di Eccellenza” Programme (2018-2022) – Department of Mathematical Sciences “G. L. Lagrange”, Politecnico di Torino (CUP: E11G18000350001).

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NL would like to thank Compagnia San Paolo for financing her PhD scholarship.

A The three-dimensional case

Let us now consider the three-dimensional case, i.e. to (1) with \( d = 3 \). In this case the three components of the flux \( \mathcal{F}[f] \) read

\[
\begin{align*}
\mathcal{F}_x[f] &= C_x[f]f + \Delta x^1 \partial_x f + \Delta x^2 \partial_y f + \Delta x^3 \partial_z f, \\
\mathcal{F}_y[f] &= C_y[f]f + \Delta y^1 \partial_y f + \Delta y^2 \partial_y f + \Delta y^3 \partial_z f, \\
\mathcal{F}_z[f] &= C_z[f]f + \Delta z^1 \partial_x f + \Delta z^2 \partial_y f + \Delta z^3 \partial_z f,
\end{align*}
\]

where

\[
\begin{align*}
C_x[f] &= B_x[f] + \partial_x \Delta x^1 + \partial_y \Delta x^2 + \partial_z \Delta x^3, \\
C_y[f] &= B_y[f] + \partial_x \Delta y^1 + \partial_y \Delta y^2 + \partial_z \Delta y^3, \\
C_z[f] &= B_z[f] + \partial_x \Delta z^1 + \partial_y \Delta z^2 + \partial_z \Delta z^3.
\end{align*}
\]

The method may be easily generalized to the three dimensional case, by following the same procedure as illustrated in subsection 2.1. We introduce a uniform mesh \((w_{x,i}, w_{y,j}, w_{z,k})\) with \( \Delta w = w_{x,i+1} - w_{x,i} = w_{y,j+1} - w_{y,j} = w_{z,k+1} - w_{z,k} \). We shall denote \( w_{x,i \pm 1/2} = w_{x,i} \pm \Delta w/2, w_{y,j \pm 1/2} = w_{y,j} \pm \Delta w/2, w_{z,k \pm 1/2} = w_{z,k} \pm \Delta w/2 \). Let \( f_{i,j,k}(t) \) be an approximation of the solution \( f(w_{x,i}, w_{y,j}, w_{z,k}, t) \) and consider the following conservative discretization

\[
\frac{d}{dt} f_{i,j,k}(t) = \frac{\mathcal{F}_x[f]_{i \pm 1/2,j,k}(t) - \mathcal{F}_x[f]_{i-1/2,j,k}(t)}{\Delta w}
+ \frac{\mathcal{F}_y[f]_{i,j \pm 1/2,k}(t) - \mathcal{F}_y[f]_{i,j-1/2,k}(t)}{\Delta w}
+ \frac{\mathcal{F}_z[f]_{i,j,k \pm 1/2}(t) - \mathcal{F}_z[f]_{i,j,k-1/2}(t)}{\Delta w},
\]

being \( \mathcal{F}_x[f]_{i \pm 1/2,j,k}, \mathcal{F}_y[f]_{i,j \pm 1/2,k}, \mathcal{F}_z[f]_{i,j,k \pm 1/2} \) flux functions characterizing the numerical discretization. In order to find the quasi-stationary approximations over the cell \([w_{x,i}, w_{x,i+1}] \times [w_{y,j}, w_{y,j+1}] \times [w_{z,k}, w_{z,k+1}]\) and to discretize each component of the flux function in its direction, we need to annihilate all the other flux functions discretized in the complementary directions. Coherently with the 2D case the quasi-stationary approximations over the cell \([w_{x,i}, w_{x,i+1}] \times [w_{y,j}, w_{y,j+1}] \times [w_{z,k}, w_{z,k+1}]\) of the 3D problem reads

\[
\int_{w_{i,j,k}}^{w_{i+1,j,k}} \frac{\partial_x f(w,t)}{f(w,t)} \, dw_x = - \int_{w_{i,j,k}}^{w_{i+1,j,k}} \frac{1}{\mathcal{D}} \left[ C_x[f](\Delta x^2 \Delta y^2 - \Delta x^2 \Delta z^2) \right]
+ C_y[f](\Delta y^2 \Delta z^2 - \Delta y^2 \Delta z^2) + C_z[f](\Delta z^2 \Delta z^2 - \Delta z^2 \Delta z^2) \right] \, dw_x
\]

(46a)
\[ \int_{w_i,j,k}^{w_i,j+1,k} \partial_y f(w,t) \, dw_y = - \int_{w_i,j,k}^{w_i,j+1,k} \frac{1}{\mathcal{D}} \left[ \mathcal{C}_x[f](\mathcal{D}^{3.1}\mathcal{D}^{2,3} - \mathcal{D}^{2.1}\mathcal{D}^{3,3}) + \mathcal{C}_y[f](\mathcal{D}^{1,1}\mathcal{D}^{1,3} - \mathcal{D}^{1,3}\mathcal{D}^{1,1}) + \mathcal{C}_z[f](\mathcal{D}^{2,1}\mathcal{D}^{1,3} - \mathcal{D}^{1,1}\mathcal{D}^{2,3}) \right] \, dw_y \]  

(46b)

\[ \int_{w_i,j,k}^{w_i,j,k+1} \partial_z f(w,t) \, dw_z = - \int_{w_i,j,k}^{w_i,j,k+1} \frac{1}{\mathcal{D}} \left[ \mathcal{C}_x[f](\mathcal{D}^{2,1}\mathcal{D}^{3,2} - \mathcal{D}^{3,1}\mathcal{D}^{2,2}) + \mathcal{C}_y[f](\mathcal{D}^{1,1}\mathcal{D}^{1,2} - \mathcal{D}^{1,2}\mathcal{D}^{1,1}) \right] \, dw_z \]  

(46c)

where \( \mathcal{D} \) is the determinant of the matrix \( \mathcal{D} \) and, for brevity of notation, we dropped the dependence in the integrands of (46a), (46b) and (46c). Let us set \( \lambda_{i+1/2,j,k}, \lambda_{i,j+1/2,k}, \lambda_{i,j,k+1/2} \) equal to the right hand sides of (46a), (46b) and (46c) respectively. In the three dimensional case, the numerical flux is obtained by imposing that the discretization of the three flux components in every direction vanish. The three flux components can then be defined in the same spirit of the 2D case as follows

\[ F_x[f]_{i+1/2,j,k} = G_x[f]_{i+1/2,j,k} f_{i+1/2,j,k} + \frac{\mathcal{D}_{i+1/2,j,k}}{(\mathcal{D}^{2,2}\mathcal{D}^{3,3} - \mathcal{D}^{2,3}\mathcal{D}^{3,2})} \frac{f_{i+1,j,k} - f_{i,j,k}}{\Delta w} \]

\[ F_y[f]_{i,j+1/2,k} = G_y[f]_{i,j+1/2,k} f_{i,j+1/2,k} + \frac{\mathcal{D}_{i,j+1/2,k}}{(\mathcal{D}^{2,1}\mathcal{D}^{3,3} - \mathcal{D}^{3,1}\mathcal{D}^{2,3})} \frac{f_{i,j+1,k} - f_{i,j,k}}{\Delta w} \]

\[ F_z[f]_{i,j,k+1/2} = G_z[f]_{i,j,k+1/2} f_{i,j,k+1/2} + \frac{\mathcal{D}_{i,j,k+1/2}}{(\mathcal{D}^{2,1}\mathcal{D}^{3,2} - \mathcal{D}^{3,1}\mathcal{D}^{2,2})} \frac{f_{i,j,k+1} - f_{i,j,k}}{\Delta w} \]

where

\[ f_{i+1/2,j,k} = (1 - \delta_{i+1/2,j,k}) f_{i+1,j,k} + \delta_{i+1/2,j,k} f_{i,j,k}, \]

\[ f_{i,j+1/2,k} = (1 - \delta_{i,j+1/2,k}) f_{i,j+1,k} + \delta_{i,j+1/2,k} f_{i,j,k}, \]

\[ f_{i,j,k+1/2} = (1 - \delta_{i,j,k+1/2}) f_{i,j,k+1} + \delta_{i,j,k+1/2} f_{i,j,k}, \]

and the weight functions \( \delta_{i+1/2,j,k}, \delta_{i,j+1/2,k}, \delta_{i,j,k+1/2} \) are defined as

\[ \delta_{i+1/2,j,k} = \frac{1}{\lambda_{i+1/2,j,k}} + \frac{1}{1 - \exp(\lambda_{i+1/2,j,k})}, \]

\[ \delta_{i,j+1/2,k} = \frac{1}{\lambda_{i,j+1/2,k}} + \frac{1}{1 - \exp(\lambda_{i,j+1/2,k})}, \]

\[ \delta_{i,j,k+1/2} = \frac{1}{\lambda_{i,j,k+1/2}} + \frac{1}{1 - \exp(\lambda_{i,j,k+1/2})} \]

and

\[ G_x[f]_{i+1/2,j,k} = \frac{\mathcal{D}_{i+1/2,j,k}}{(\mathcal{D}^{2,2}\mathcal{D}^{3,3} - \mathcal{D}^{2,3}\mathcal{D}^{3,2})} \frac{\lambda_{i+1/2,j,k}}{\Delta w} \]

\[ G_y[f]_{i,j+1/2,k} = \frac{\mathcal{D}_{i,j+1/2,k}}{(\mathcal{D}^{2,1}\mathcal{D}^{3,3} - \mathcal{D}^{3,1}\mathcal{D}^{2,3})} \frac{\lambda_{i,j+1/2,k}}{\Delta w} \]

\[ G_z[f]_{i,j,k+1/2} = \frac{\mathcal{D}_{i,j,k+1/2}}{(\mathcal{D}^{2,1}\mathcal{D}^{3,2} - \mathcal{D}^{3,1}\mathcal{D}^{2,2})} \frac{\lambda_{i,j,k+1/2}}{\Delta w} \]

References

[1] G. Albi, L. Pareschi, G. Toscani, and M. Zanella. Recent advances in opinion modeling: control and social influence. In N. Bellomo, P. Degond, and E. Tadmor, editors, Active Particles Volume 1, Theory, Methods, and Applications, Modeling and Simulation in Science, Engineering and Technology. Birkhäuser, 2017.
[2] R. Bailo, J. A Carrillo, and J. Hu. Fully discrete positivity-preserving and energy-decaying schemes for aggregation-diffusion equations with a gradient flow structure. Preprint arxiv:1811.11502, 2018.

[3] A. B. Barbaro, J. A. Cañizo, J. A. Carrillo, and P. Degond. Phase transitions in a kinetic model of Cucker-Smale type, *Multiscale Model. Simul.*, 14(3): 1063–1088, 2016.

[4] A. B. Barbaro, and P. Degond. Phase transition and diffusion among socially interacting self-propelled agents, *Discrete Contin. Dyn. Syst. Ser. B*, 19: 1249–1278, 2014.

[5] J. Barré, P. Degond, and E. Zatorska. Kinetic theory of particle interactions mediated by dynamical networks, *Multiscale Model. Simul.*, 15(3): 1294–1323, 2017.

[6] M. Bessemoulin-Chatard, and F. Filbet. A finite volume scheme for nonlinear degenerate parabolic equations, *SIAM J. Sci. Comput.*, 34: 559–582, 2012.

[7] M. Bessemoulin-Chatard, M. Herda, and T. Rey. Hypocoercivity and diffusion limit of a finite volume scheme for linear kinetic equations. Preprint arXiv:1812.05967, 2018.

[8] A. Blanchet, J. A. Carrillo, and N. Masmoudi. Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$, *Commun. Pure Appl. Math.*, 61(10): 1449–1481, 2008.

[9] F. Bolley, J. A. Cañizo, and J. A. Carrillo. Stochastic mean-field limit: non-Lipschitz forces & swarming, *Math. Mod. Meth. Appl. Sci.*, 21: 2179–2210, 2011.

[10] S. Boscarino, F. Filbet, and G. Russo. High order semi-implicit schemes for time dependent partial differential equations, *J. Sci. Comput.*, 68: 975–1001, 2016.

[11] C. Buet, S. Cordier, and V. Dos Santos. A conservative and entropy scheme for a simplified model of granular media. *Transp. Theory Stat. Phys.*, 33(2): 125–155, 2004.

[12] C. Buet, and S. Dellacherie. On the Chang and Cooper numerical scheme applied to a linear Fokker-Planck equations, *Commun. Math. Sci.*, 8(4): 1079–1090, 2010.

[13] J. A. Carrillo, A. Chertock, Y. Huang. A finite-volume method for nonlinear nonlocal equations with a gradient flow structure, *Commun. Comput. Phys.*, 17(1): 233–258, 2015.

[14] J. A. Carrillo, Y.-P. Choi, and L. Pareschi. Structure preserving schemes for the continuum Kuramoto model: phase transitions, *J. Comput. Phys.*, 376: 365–389, 2019.

[15] J. A. Carrillo, M. Fornasier, G. Toscani, and F. Vecil. Particle, kinetic, and hydrodynamic models of swarming. In G. Naldi, L. Pareschi, G. Toscani, editors, *Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences*, Modeling and Simulation in Science, Engineering and Technology, Birkhäuser Boston, pp. 297–336, 2010.

[16] J. A. Carrillo, L. Pareschi, and M. Zanella. Particle based gPC methods for mean-field models of swarming with uncertainty, *Commun. Comput. Phys.*, 25(2): 508–531, 2019.

[17] J. A. Carrillo, and M. Zanella. Monte Carlo gPC methods for diffusive kinetic flocking models with uncertainties. Preprint 2019.

[18] C. Chainais-Hillairet, A. Jüngel, S. Schnechmigg. Entropy-dissipative discretization of nonliner diffusion equations and discrete Beckner inequalities, *ESAIM Math. Model. Numer. Anal.*, 50(1): 135–162, 2016.

[19] J. S. Chang, and G. Cooper. A practical difference scheme for Fokker-Planck equations, *J. Comput. Phys.*, 6(1): 1–16, 1970.

[20] A. Chauviere, T. Hillen and L. Preziosi. Modeling cell movement in anisotropic and heterogeneous network tissues, *Networks and Heterogeneous Media* 2 (2), 333–357, 2007.
[21] G. Dimarco, and L. Pareschi. Numerical methods for kinetic equations, *Acta Numerica*, **23**: 369–520, 2014.

[22] G. Dimarco, L. Pareschi, and M. Zanella. Uncertainty quantification for kinetic models in socio-economic and life sciences. In S. Jin, L. Pareschi, editors, *Uncertainty Quantification for Hyperbolic and Kinetic Equations*, SEMA SIMAI Springer Series, vol. 14, pp. 151–191, 2017.

[23] R. Duan, M. Fornasier, and G. Toscani. A kinetic flocking model with diffusion, *Commun. Math. Phys.*, **300**: 95-145, 2010.

[24] G. Furioli, A. Pulvirenti, E. Terraneo, and G. Toscani. Fokker–Planck equations in the modeling of socio-economic phenomena, *Math. Mod. Meth. Appl. Sci.*, **27**(1): 115–158, 2017.

[25] L. Gosse. *Computing Qualitatively Correct Approximations of Balance Laws.*, SEMA SIMAI Springer Series, Springer, Berlin, 2013.

[26] S. Gottlieb, C.-W. Shu, and E. Tadmor. Strong stability-preserving high-order time discretization methods, *SIAM Rev.*, **43**(1): 89–112, 2001.

[27] R. Hegselmann, and U. Krause. Opinion dynamics and bounded confidence: Models, analysis, and simulation, *J. Artif. Soc. Soc. Simulat.*, **5**(3):1–33, 2002.

[28] T. Hillen. M5 mesoscopic and macroscopic models for mesenchymal motion. *J. Math. Biol.*, **53**(4): 585–616, 2005.

[29] T. Hillen, and K. J. Painter. Transport and anisotropic diffusion models for movement in oriented habitats. In *Dispersal, Individual Movement and Spatial Ecology*, eds. M. A. Lewis, P. K. Maini, S. V. Petrovskii, Lecture Notes in Mathematics, pp. 177–222, 2013.

[30] J. Hu, R. Shu, and X. Zhang. Asymptotic-preserving and positivity-preserving implicit-explicit schemes for the stiff BGK equations. *SIAM J. Numer. Anal.*, **56**: 942–973, 2018.

[31] S. Jin. Asymptotic preserving (AP) schemes for multiscale kinetic and hyperbolic equations: a review. In Lecture Notes for Summer School on Methods and Models of Kinetic Theory, (M&MKT), Porto Ercole (Grosseto, Italy), *Riv. Mat. Univ. Parma* **3**(2), 177216, 2012.

[32] E. W. Larsen, C. D. Levermore, G. C. Pomraning, and J. G. Sanderson. Discretization methods for one-dimensional Fokker-Planck operators, *J. Comput. Phys.*, **61**(3): 359–390, 1985.

[33] D. Matthes, A. Jüngel, and G. Toscani. Convex Sobolev inequalities derived from entropy dissipation, *Arch. Rat. Mech. Anal.*, **199**(2): 563–596, 2011.

[34] A. Okubo, and S.A. Levin. Diffusion and Ecological Problems: Modern Perspectives, *Springer, New York*, 2002.

[35] L. Pareschi, and T. Rey. Residual equilibrium schemes for time dependent partial differential equations, *Comput. Fluids*, **156**: 329–342, 2017.

[36] L. Pareschi, G. Toscani. *Interacting Multiagent Systems: Kinetic equations and Monte Carlo methods*, Oxford University Press, 2013.

[37] L. Pareschi, G. Toscani, A. Tosin, and M. Zanella. Hydrodynamic models of preference formation in multi-agent societies. Preprint arXiv: 1901.00486, 2019.

[38] L. Pareschi, and M. Zanella. Structure preserving schemes for nonlinear Fokker-Planck equations and applications, *J. Sci. Comput.*, **74**(3):1575-1600, 2018.
[39] L. Pareschi, and M. Zanella. Structure preserving schemes for mean-field equations of collective behavior. In M. Westdickenberg, C. Klingenberg, editors, Theory, Numerics and Applications of Hyperbolic Problems II. HYP2016, vol. 237 of Springer Proceedings in Mathematics & Statistics, pp. 405–421, Springer, Cham, 2018

[40] Y. Qian, Z. Wang, and S. Zhou. A conservative, free energy dissipating, and positivity preserving finite difference scheme for multi-dimensional nonlocal Fokker-Planck equation, J. Comput. Phys., 386: 22–36, 2019.

[41] H. Risken. The Fokker-Planck Equation, Methods of solution and Applications, Springer-Verlag, Berlin, 1996.

[42] D. L. Scharfetter, and H. K. Gummel. Large-signal analysis of a silicon Read diode oscillator, IEEE Trans. Electron Devices, 16(1): 64–77, 1969.

[43] P. C. da Silva, L. R. da Silva, E. K. Lenzi, R. S. Mendes, and L. C. Malacarne. Anomalous diffusion and anisotropic nonlinear Fokker-Planck equation, Physica A, 342: 16–21, 2004.

[44] G. Toscani. Entropy production and the rate of convergence to equilibrium for the Fokker-Planck equation, Quart. Appl. Math., 57: 521–541, 1999.

[45] G. Toscani. Kinetic models of opinion formation, Commun. Math. Sci., 4(3): 481–496, 2006.

[46] G. Toscani, and C. Villani. Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation, Commun. Math. Phys., 203(3): 667–706, 1999.

[47] A. Tosin, and M. Zanella. Boltzmann-type models with uncertain binary interactions, Commun. Math. Sci., 16(4): 962-984, 2018.

[48] C. Yates, R. Erban, C. Escudero, L. Couzin, J. Buhl, L. Kevrekidis, P. Maini and D. Sumpter. Inherent noise can facilitate coherence in collective swarm motion Proc. Nat. Acad. Sci., 106(14): 54645469, 2009.