Semiclassical Solution of the Quantum Hydrodynamic Equation for Trapped Bose-condensed Gas in the $l = 0$ Case

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I. INTRODUCTION

The main aim of this work is to study the applicability of the JWKB semiclassical quantization method for the solution of the quantum hydrodynamical equation describing the collective excitations of a trapped Bose-condensed gas. Although the analogous equation for homogeneous systems has been known for long, the equation for trapped gases has first been written down and solved for special confining potential only recently by S. Stringari [1] (see also the review articles [2, 3, 4]), after the first experimental successes of realizing Bose-Einstein condensation with trapped atomic vapors [5, 6, 7]. This quantum hydrodynamic equation, shortly called ‘Stringari equation’, has a quite similar structure to the ordinary three dimensional Schrödinger equation. The motive of the present investigation is provided by the fact that similarly to the ordinary Schrödinger equation, the solution of the Stringari equation is also only in very rare, special cases available in analytic form [1, 2, 3, 4, 8–10]. For this reason it is important to see how the semiclassical methods developed for the ordinary Schrödinger equation can be transported to the present problem, which has a turning point structure of unusual type. In this work we demonstrate it in the simplest, also analytically treatable case, where the confining potential is spherically symmetric and harmonic. A comparison to the exact eigenvalues shows that the accuracy of the semiclassical method is rather good, the deviation is only a small shift in the energy square.

In the first section the Stringari equation is introduced, rewritten in dimensionless form, separated in spherical polar coordinates, and the analytic results are revisited.

In the second section we summarize briefly all the necessary information we need to know about the JWKB method for the present purposes, and also derive how the form of a second order linear differential equation (containing no term of first order derivative) changes under a general transformation of the independent variable. This result will be used in the next section to eliminate an undesirable term from the radial Stringari equation.

In the third section, after applying the above mentioned transformation the semiclassical solution and the semiclassical quantization condition are derived rigorously.

Finally the semiclassical results are compared to the exact analytic ones, and some further possible applications of the JWKB method are proposed.

II. THE STRINGARI EQUATION — ANALYTIC RESULTS

Let us consider a dilute Bose gas trapped in a confining potential $V(r)$. If the interparticle interaction is dominated by weak s-wave scattering, i.e., the (magnitude) of the s-wave scattering length $a$ is much smaller than the average distance between the particles, and the temperature $T < T_c$ is much less than the condensation temperature $T_c$, i.e., the effect of the thermal cloud on the condensate is negligible then the ‘condensate wave function’ $\Psi(r, t) = |\langle \Psi(r, t) \rangle|$, which is the expectation value of the field operator $\hat{N}(r, t)$, satisfies a nonlinear partial differential equation, the so called Gross-Pitaevskii equation [1, 2, 3, 11, 12, 13] of the field operator $\hat{N}(r, t)$:

\[
\frac{1}{\hbar} \frac{\partial \Psi(r, t)}{\partial t} = \left( -\frac{\hbar^2}{2M} \Delta + V(r) + g|\Psi(r, t)|^2 \right) \Psi(r, t),
\]

where $M$ is the mass of the particles, and $g = \frac{4\pi\hbar^2}{M}$ is a constant depending on the scattering length $a$. The validity conditions of the equation are

\[
LN^{-1} \gg |a| \quad \text{and} \quad T < T_c,
\]

where $L$ is the linear size of the condensate, and $N$ is the number of condensed atoms. In this paper we restrict our attention to the $a > 0$ (i.e., $g > 0$) case, which means that the interaction is repulsive.

In the hydrodynamic formalism two real quantities are introduced instead of the complex wave function of the condensate $\Psi(r, t) = \sqrt{\rho(r, t)} e^{i\phi(r, t)}$, which are the...
condensate density \( \rho(r, t) = \Psi \Psi^* \) describing the magnitude of the complex wave function and the velocity field \( \mathbf{v}(r, t) = \frac{i}{\hbar} \nabla (\Psi \Psi^* - \Psi^* \Psi) \) related to the phase of the condensate wave function. (For the sake of simplicity the space and time arguments \( r, t \) of the functions are omitted, if there is no danger of confusion.) Taking the time derivative of the density \( \rho \) and using the time dependent Gross-Pitaevskii equation (1a) for substituting the time derivatives of \( \Psi \) and \( \Psi^* \) the following continuity equation is obtained:

\[
\frac{\partial \rho}{\partial t} + \nabla (\rho \mathbf{v}) = 0. \quad (2)
\]

Inserting the formula \( \Psi = \sqrt{\rho} e^{i \varphi} \) into the Gross-Pitaevskii equation (1a), and separating the real part of it, we get

\[
\hbar \frac{\partial \varphi}{\partial t} = \frac{\hbar^2}{2 M \sqrt{\rho}} \Delta \sqrt{\rho} - \frac{\hbar^2}{2 M} (\nabla \varphi)^2 - V - g \rho. \quad (3)
\]

Taking the gradient of this equation, and using the expression \( \mathbf{v} = \frac{\hbar}{M} \nabla \varphi \) for the velocity field, we obtain that the equation describing the time evolution of the velocity field \( \mathbf{v}(r, t) \) is

\[
M \frac{\partial \mathbf{v}}{\partial t} + \nabla \left( V + g \rho + \frac{M \mathbf{v}^2}{2} - \frac{\hbar^2}{2 M \sqrt{\rho}} \Delta \sqrt{\rho} \right) = 0. \quad (4)
\]

It is worth stressing that the quantum hydrodynamic equations (3) and (4) are equivalent with the time dependent Gross-Pitaevskii equation (1a), they do not involve any further approximation, and they are valid in the linear as well as in the nonlinear regimes.

The ground state solution of the hydrodynamic equations is characterized by the overall zero value of the velocity field \( \mathbf{v}(r, t) \), in which case the equation (4) reduces to the time independent Gross-Pitaevskii equation

\[
\mu \sqrt{\rho}(r) = \left( -\frac{\hbar^2}{2 M} \Delta + V(r) + g \rho(r) \right) \sqrt{\rho}(r), \quad (5)
\]

where \( \mu \) is the chemical potential of the system, and \( \Psi(r, t) = \sqrt{\rho(r)} \exp(-i \frac{\mu}{\hbar} t) \).

If the number \( N \) of the atoms in the condensate is sufficiently large, i.e., \( Na \gg L \) (where \( a > 0 \) is the scattering length of the repulsive interaction between the atoms, and \( L \) is the linear size of the condensate) than the kinetic energy term \( \frac{\hbar^2}{2 \sqrt{\rho}} \Delta \sqrt{\rho} \) is almost everywhere negligible with respect to the other terms in equation (5). (This approximation fails only in a narrow region at the boundary of the condensate where \( \rho \rightarrow 0 \).) The hydrodynamic equations have the following simpler form in this limit:

\[
\frac{\partial \rho}{\partial t} + \nabla (\rho \mathbf{v}) = 0, \quad (6a)
\]

\[
M \frac{\partial \mathbf{v}}{\partial t} + \nabla \left( V + g \rho + \frac{M \mathbf{v}^2}{2} \right) = 0. \quad (6b)
\]

The validity conditions of the above applied Thomas-Fermi approximation, together with the initially imposed conditions (1) are

\[
Na \gg L \gg N^{\frac{1}{2}} a > 0, \quad \text{and} \quad T \ll T_c. \quad (6c)
\]

It is worth noting that this approximation has a great relevance, since the validity conditions (6c) can be well satisfied in experiments.

Stringari [1] investigated the time-periodic collective excitational solutions of the Thomas-Fermi hydrodynamic equations (1) in the linear (low energy) regime. The Thomas-Fermi ground state \( \varrho_0(r) \) corresponds to the zero velocity field \( \mathbf{v}_0(r) = 0 \), in which case the solution

\[
\varrho_0(r) = \begin{cases} 
\varrho^{-1}(\mu - V(r)), & \text{if} \quad \mu - V(r) > 0 \\
0, & \text{otherwise}
\end{cases} \quad (7)
\]

is obtained from the equation (6b). Linearizing the equations (3) around the ground state \( \varrho_0 \) and \( \mathbf{v}_0 \), (substituting \( \varrho(r, t) = \varrho_0(r) + \delta \varrho(r, t) \) and \( \mathbf{v}(r, t) = \mathbf{v}_0(r) + \delta \mathbf{v}(r, t) \) and keeping the first order terms) the following equations are obtained for the density and velocity fluctuations \( \delta \varrho(r, t) \) and \( \delta \mathbf{v}(r, t) \)

\[
\frac{\partial \delta \varrho}{\partial t} + \nabla (\varrho \delta \mathbf{v}) = 0, \quad (8a)
\]

\[
M \frac{\partial \delta \mathbf{v}}{\partial t} + g \nabla \delta \varrho = 0. \quad (8b)
\]

Taking the time derivative of (8a) and using (8b) to eliminate (the time derivative of) the velocity fluctuation \( \delta \mathbf{v} \) we obtain the time dependent Stringari equation

\[
M \frac{\partial^2 \delta \varrho}{\partial t^2} = \nabla ((\mu - V) \nabla \delta \varrho), \quad (9)
\]

and using the Ansatz \( \delta \varrho(r, t) = e^{-i \omega t} \tilde{\xi}(r) \) for the time dependence of the density fluctuation \( \delta \varrho \) we arrive to the time independent Stringari equation

\[
-\nabla \left((\mu - V(r)) \nabla \tilde{\xi}(r)\right) = M \omega^2 \tilde{\xi}(r), \quad (10)
\]

where \( \omega \) is the angular frequency of the excitation, and \( \tilde{\xi} \) is defined only in the region \( \{r | \mu - V(r) > 0\} \). (The ‘\( \sim \)’ (tilde) sign above \( \tilde{\xi} \) just designates that the variable is not dimensionless.) This equation is the starting point of our semiclassical investigations.

Let us suppose that the external trap potential function is isotropic. In this case it is natural to rewrite equation (10) in spherical polar coordinates \( \{r, \vartheta, \varphi\} \) and separate the equation using the Ansatz \( \tilde{\xi}(r, \vartheta, \varphi) = \tilde{\chi}_l(m) Y_{l,m}(\vartheta, \varphi) \). (As usual, \( r = |r| \) and \( Y_{l,m}(\vartheta, \varphi) \) is the spherical harmonic function, \( l, m \in \mathbb{N} \). Because of the denominator \( r \) in the Ansatz, we have \( \int_{|r|} |\tilde{\xi}(r)|^2 d^3r = \int_{|r|} |\tilde{\chi}_l(m)|^2 dr \). After separation, one obtains the following second order ordinary differential equation for the radial component \( \tilde{\chi}(r) \):

\[
U(r) \tilde{\chi}''(r) + U'(r) \tilde{\chi}'(r) +
\left(M \omega^2 - \frac{U(r)}{r^2} (l(l+1) - \frac{U(r)}{r})\right) \tilde{\chi}(r) = 0, \quad (11)
\]
where $U(r)$ stands for $\mu - V(r)$.

Let us rewrite the radial equation into dimensionless form! As length unit it is reasonable to choose the Thomas-Fermi length $r_0$ which is defined by the equation $V(r_0) = \mu$ i.e. $U(r_0) = 0$, thus the new independent radial variable is $\rho = \frac{r}{r_0} \in [0,1]$. For the dimensionless energy (frequency) and potential we introduce the variable $\varepsilon = \sqrt{\frac{2}{\mu} r_0^2}$ and the function $u(\rho) = \frac{U(\rho)}{\varepsilon^2}$, respectively. Using these new notations, the equation (12) for the dimensionless radial function $\chi(\rho) = \chi(r) = \chi(r_0 \rho)$ takes the form

\[
u(\rho)\nu'(\rho) + u'(\rho)\chi'(\rho) + 
\left(2\varepsilon^2 - \frac{2\varepsilon^2}{\rho^2} + \frac{l(l+1)}{\rho^2} - \frac{u'(\rho)}{\rho}\right) \chi(\rho) = 0. \tag{12}\]

Further we are interested in the case when the external trapping potential is harmonic oscillator potential. This assumption makes the problem not only analytically solvable [16] but also has experimental relevance [14]. The Thomas-Fermi length is $r_0 = \sqrt{\frac{2\mu}{\varepsilon^2}}$, for the dimensionless energy $\varepsilon = \frac{\varepsilon}{\varepsilon_0}$ holds and the dimensionless potential function becomes $u(\rho) = 1 - \rho^2$. In this case the equation (12) takes the form

\[
\nu''(\rho) - \frac{2\rho}{1-\rho^2}\nu'(\rho) + 
\left(2\varepsilon^2 + 1 - \frac{l(l+1)}{\rho^2} - \frac{2}{\rho^2}\right) \chi(\rho) = 0, \quad \rho \in [0,1] \tag{13a}
\]

with the normalization and continuity condition

\[
\int_{\rho=0}^{1} |\chi(\rho)|^2 d\rho < \infty, \quad \lim_{\rho \rightarrow 0} \left(\frac{\nu(\rho)}{\rho}\right)' = 0, \text{ if } l = 0, 
\lim_{\rho \rightarrow 0} \nu(\rho) = 0, \text{ if } l \neq 0. \tag{13b}
\]

This equation is an ordinary second order linear homogeneous differential equation with three regular singular points at 0, 1 and $\infty$. It is easy to see that in the Frobenius series expansion $\chi(\rho) = \rho^2 \sum_{n \in N} \chi_n \rho^n$ of the solution $\chi(\rho)$ only the coefficients $\chi_n$ of even indices $n$ are nonzero, thus the equation (13) can be further simplified by the substitution $t = \rho^2$, $\kappa(t) = \chi(\rho)$ of the independent variable:

\[
\kappa''(t) + \left(\frac{1}{2t} - \frac{1}{1-t}\right) \kappa'(t) + 
\left(\frac{\varepsilon^2 + 1}{2t(1-t)} - \frac{l(l+1)}{4t^2}\right) \kappa(t) = 0, \quad t \in [0,1] \tag{14a}
\]

\[
\int_{t=0}^{1} |\kappa(t)|^2 dt < \infty, \quad \lim_{t \rightarrow 0} 2\nu'(t) - \kappa(t) = 0, \text{ if } l = 0, 
\lim_{t \rightarrow 0} \frac{\kappa(t)}{t} = 0, \text{ if } l \neq 0. \tag{14b}
\]

This equation has again three regular singularities at $t^{(0)} = 0$, $t^{(1)} = 1$ and $t^{(\infty)} = \infty$ with indices $\alpha_1^{(1)} = \frac{1}{2}$, $\alpha_2^{(1)} = -\frac{1}{2}$; $\alpha_1^{(2)} = \alpha_2^{(2)} = 0$; and $\alpha_1^{(\infty)} = \frac{1}{2} + \sqrt{2} \varepsilon^2 + 1 + (l + 1/2)^2$, respectively, what means that the basic solutions of the equation (14a) have power-law behavior $\kappa(t) \propto t^{\alpha_1^{(0)}}$ at the origin $t = 0$, and one of them is locally linear (i.e., $\kappa(1) < \infty$, $\kappa'(1) < \infty$), and the other possesses logarithmic singularity at $t = 1$. Because of the conditions (14b) the physically meaningful solutions must have the index $\alpha_1^{(0)} = \frac{1}{2}$ at the origin and they should not have logarithmic singularity at $t = 1$.

Inserting the Frobenius series expansion

\[
\kappa(t) = t^{\frac{1}{2}} \sum_{n=0}^{\infty} \kappa_n t^n, \quad \kappa_0 \neq 0 \tag{15a}
\]

into the differential equation (14a), the recursion

\[
n + l + \frac{3}{2} \kappa_n+1 = \left(n^2 + \frac{3}{2} n + nl + \frac{l}{2} - \varepsilon^2 \frac{l}{2}\right) \kappa_n \tag{15b}
\]

is obtained for the coefficients $\kappa_n$, with the initial condition $\kappa_0 \neq 0$. The regularity condition of $\kappa(t)$ at $t = 1$ can be satisfied by requiring the expansion (15a) to be finite, i.e., that $\kappa_k = 0$ for all $k \geq n_0$, what results in the discrete energy spectrum [16]

\[
\varepsilon(n,l) = \sqrt{2n^2 + 2nl + 3n + l}, \quad n,l \in N. \tag{16}
\]

The figures 1 and 2 show the exact radial wave function $\chi(\rho)$ for a few quantum numbers.

It is worth noting that by changing the dependent variable according to the formula $\kappa(t) = t^{\frac{l}{2}} w(t)$ the equation can be transformed to the hypergeometric form [14]

\[
t(1-t) \frac{d^2w}{dt^2} + (c - (a + b + 1)t) \frac{dw}{dt} - abw = 0 \tag{17a}
\]

with parameter values

\[
ab = \frac{-\varepsilon^2 - l}{2}, \quad a + b = c = l + \frac{3}{2},
\]

\[
a, b = \frac{l + \frac{3}{2} \pm \sqrt{l^2 + l + \frac{9}{4} + 2\varepsilon^2}}{2}, \tag{17b}
\]

and additional physical constrains

\[
\int_{t=0}^{1} t^{l+1/2}|w(t)|^2 dt < \infty, \quad \lim_{t \rightarrow 0} (w(t) + 2tw'(t)) t^{\frac{l}{2}} = 0, \text{ if } l = 0, 
\lim_{t \rightarrow 0} w(t) t^{\frac{l-1}{2}} = 0, \text{ if } l \neq 0. \tag{17c}
\]
The physically meaningful solutions [satisfying (17c)] fall into the so called degenerate case of the hypergeometric equation, in which case the solution has the form $w(t) = t^\alpha (1-t)^\beta p_n(t)$ where $p_n(t)$ is a finite polynomial of degree $n$. The condition for this case is that either $a = c - b$ or $b = c - a$ is an integer, and it reproduces the exact quantization condition (16).

III. PRELIMINARY STEPS

In this section first the necessary formulae of the JWKB method are summarized, and then, for later use, we briefly derive how the equation subjected to JWKB approximation transforms under a general change of the independent variable.

A. The JWKB semiclassical method

The JWKB method, originally developed by H. Jeffreys [17], G. Wentzel [18], H. A. Kramers [19] and L. Brillouin [20, 21] has still remained one of the most powerful tools for the approximative solution of wave equations since its birth in the very early days of quantum mechanics [22, 23, 24, 25]. Basically it can be used for obtaining a global approximation to the solution of a linear differential equation whose highest derivative is multiplied by a small parameter, and the method gives an asymptotic approximation of the real solution on a certain interval in terms of increasing powers of this parameter.

In this paper we consider only second order homogeneous linear differential equations of the following type:

$$\delta^2 y''(x) + Q(x)y(x) = 0,$$

where $\delta^2$ is the small real parameter. In this case the
JWKB Ansatz has the form

\[ y(x) = \exp \left( \frac{1}{\delta} S_\delta(x) \right), \quad S_\delta(x) = \sum_{n=0}^{\infty} \delta^n S_n(x). \]  

(19)

Substituting this Ansatz into the differential equation [13] and requiring the equivalence in any orders of \( \delta \) the following equations are obtained for the multiplier functions \( S_n \) in the exponent:

\[ S_0(x) = \pm i \int_{t_0}^{x} \sqrt{Q(t)} \, dt, \quad (20a) \]

\[ S_1(x) = -\frac{1}{4} \ln Q(x), \quad (20b) \]

\[ S_2(x) = \pm i \int_{t_0}^{x} \left( \frac{5Q'^2(t)}{32Q^{5/2}(t)} - \frac{Q''(t)}{8Q^{3/2}(t)} \right) \, dt, \quad \text{etc} \quad (20c) \]

\[ y_1(x) = C_1 \exp \left( \frac{S_0(x)}{\delta} + S_1(x) \right) + C_2 \exp \left( -\frac{S_0(x)}{\delta} + S_1(x) \right) = C_1 Q^{-1/4} \exp \left( \frac{i}{\delta} \int_{t_0}^{x} \sqrt{Q(t)} \, dt \right) + C_2 Q^{-1/4} \exp \left( -\frac{i}{\delta} \int_{t_0}^{x} \sqrt{Q(t)} \, dt \right), \quad C_1, C_2 \in \mathbb{R}. \]  

(21)

The JWKB solution, truncated at the \( N \)-th order \( y_N(x) = \exp \left( \frac{1}{\delta} \sum_{n=0}^{N} \delta^n S_n(x) \right) \) is a uniformly valid approximation on the interval \( I \subset \mathbb{R} \) in the \( \delta \to 0 \) limit, if the following relations [24] hold for the succeeding terms in the exponent:

\[ \frac{1}{\delta} S_0(x) \gg S_1(x) \gg \delta S_2(x) \gg \cdots \gg \delta^{N-1} S_N(x), \quad (22a) \]

\[ 1 \gg \delta^N S_{N+1}(x), \quad \text{for} \quad \delta \to 0, \quad x \in I. \quad (22b) \]

In concrete cases these asymptotic inequalities define the interval \( I \) on which the JWKB approximation can be applied. On the regions where the approximation fails (in most cases at the classical turning points, where \( Q(x) = 0 \)), usually the multiplier function \( Q(x) \) itself should be approximated by a simpler function, for which the equation [13] is exactly solvable, and then the semiclassical solution (of the exact equation) and the exact solution (of the approximate equation) should be patched together. This procedure will be carried out in detail for the radial Stringari equation, (which is an equation with \( \delta \) variable)

thus in the first order the semiclassical solution has the general form

\[ \bar{y}(t) = \exp \left( \frac{1}{2} \int_{z=\delta}^{t} \bar{P}(z) dz \right) \bar{y}(t) = \frac{\bar{y}(t)}{\sqrt{\lambda'(t)}} \]  

(25)

B. The effect of the change of the independent variable

It is well known that the applicability of the JWKB method highly depends on the proper choice (transformation) of the independent variable of the differential equation investigated. For this reason the formulae describing the transformation of the differential equation induced by a general change of the independent variable are stated here.

The starting point is the equation [18]. Let us transform the independent variable \( x \) in the equation to the new variable \( t \) by the invertible transformation

\[ \mathcal{X} : \mathbb{R} \rightarrow \mathbb{R}, \quad t \mapsto \mathcal{X}(t), \quad (23a) \]

\[ \mathcal{T} = \mathcal{X}^{-1} : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \mathcal{T}(x), \quad (23b) \]

which is defined on a suitable interval of the real line. Denoting the new dependent variable by \( \tilde{y}(t) = y(\mathcal{X}(t)) \), the original equation [18] has the transformed form

\[ \delta^2 \tilde{y}''(t) + \delta^2 \bar{P}(t) \tilde{y}'(t) + \bar{Q}(t) \tilde{y}(t) = 0, \quad (24a) \]

where the multiplier functions are

\[ \bar{P} = -\frac{\mathcal{X}''}{\mathcal{X}'} = -\left( \ln(\mathcal{X}') \right)' = \frac{\mathcal{T}''}{\mathcal{T}'^2} \circ \mathcal{X} = \left( \ln(\mathcal{T}') \circ \mathcal{X} \right)', \]

\[ \bar{Q} = (\mathcal{Q} \circ \mathcal{X}) \mathcal{X}'^2 = \frac{\mathcal{Q}}{\mathcal{T}'^2} \circ \mathcal{X}. \quad (24b) \]

The term containing the first order derivative \( \tilde{y}'(t) \) can be eliminated by applying the

\[ \tilde{y}(t) = \exp \left( \frac{1}{2} \int_{z=\delta}^{t} \bar{P}(z) dz \right) \tilde{y}(t), \quad \tilde{y}(t) = \frac{\tilde{y}(t)}{\sqrt{\lambda'(t)}} \]  

(25)
transformation of the dependent variable. After the transformation the equation
\[ \delta^2 \tilde{y}'(t) + \tilde{Q}(t) \tilde{y}(t) = 0 \] (26a)
is obtained, where the new coefficient function is
\[ \tilde{Q} = (Q \circ \mathcal{X}) \mathcal{X}' + \delta^2 2 \chi'' \mathcal{X}' - 3 \chi'^2 = \left( \frac{1}{T^2} \right) \left( Q + \delta^2 \left( \frac{1}{2} \Pi' + \frac{1}{4} \Pi^2 \right) \right) \circ \mathcal{X} \] (26b)
with
\[ \Pi = \frac{T''}{T'} = (\ln T')', \] (26c)
and the new unknown function \( \tilde{y}(t) \) is related to the original unknown function \( y(x) \) via the formula
\[ \tilde{y} = \frac{1}{\sqrt{\mathcal{X}'}} (y \circ \mathcal{X}) = \left( \sqrt{T'} y \right) \circ \mathcal{X}. \] (27)

Thus applying two consecutive transformations to the differential equation (13), first an arbitrary invertible change of the independent variable (23) and then an appropriate transformation (25) of the dependent variable, one arrives to the equation (26) of the same form, but with different coefficient function (26b). This freedom of the choice of the independent variable enables us to transform the investigated equation to a more appropriate form before applying the JWKB method.

IV. THE SEMICLASSICAL SOLUTION OF THE RADIAL STRINGARI EQUATION (IN CASE OF \( l = 0 \))

In this section the JWKB method, summarized in the previous section is applied to the (dimensionless radial) Stringari equation (13) derived in Section II.

As we did in the previous subsection [formula (23)], we can get rid of the term containing the first order derivative \( \mathcal{X}'(\rho) \) in equation (13a) by applying the transformation \( y(\rho) := e^{- \int_{x=0}^x \sqrt{2 \chi}(\rho)} = \sqrt{1 - \rho^2 \chi(\rho)}. \) As result, the equation
\[ \delta^2 y''(x) + (Q(x) + \delta^2 W_l(x)) y(x) = 0, \quad x \in [0, 1] \] (28a)
\[ Q(x) = \frac{1}{1 - x^2}, \quad W_l(x) = - \frac{L}{x^2} + \frac{1}{(1 - x^2)^2} \] (28b)
is obtained with the normalization and continuity conditions
\[ \int_0^1 \frac{y^2(x)}{1 - x^2} dx < \infty, \quad \lim_{x \to 0} \left( \frac{y(x)}{x} \right)' = 0, \quad \text{if } l = 0, \quad \lim_{x \to \infty} \frac{y(x)}{x} = 0, \quad \text{if } l \neq 0, \] (28c)
where the abbreviations
\[ \delta^2 = \frac{1}{2 e^2 + 2}, \quad L = l(l + 1) \] (28d)
were used. (For convenience we denoted here the independent variable with \( x \) instead of \( \rho \).) This equation is the starting point of the semiclassical quantization, and \( \delta^2 \) is used as small parameter.

A comparison of the equation (28a) to the general form (13) shows that (28a) does not have the desired form since it contains a small term \( \delta^2 W_l(x) y(x) \).

First, to eliminate the \( W_0(x) = \frac{1}{1 - x^2} \) part of this term, (which does not depend on the angular momentum number \( l \)), we apply a transformation (23–25), discussed in Subsection IIIA, to the equation (28a), and choose the transforming function \( T(x) \) to be a solution of the Ricatti-equation
\[ \frac{1}{2} \Pi'(x) - \frac{1}{4} \Pi^2(x) = W_0(x) = \frac{1}{(1 - x^2)^2}, \] (29)
where \( \Pi = \frac{T''}{T'} = (\ln T')' \). It is easy to see that
\[ \Pi(x) = \frac{2x}{1 - x^2}, \quad T(x) = \text{artanh}(x), \quad \chi(t) = \text{tanh}(t) \] (30)
is a simple solution of the nonlinear differential equation (25), and for this choice the transformed equation (26) has the form
\[ \delta^2 y''(t) + (\tilde{Q}(t) + \delta^2 \tilde{W}_l(t)) \tilde{y}(t) = 0, \quad t \in [0, \infty) \] (31a)
with
\[ \tilde{Q}(t) = \frac{1}{\cosh^2(t)}, \quad \tilde{W}_l(t) = \frac{-L}{\sinh^2(t)} \] (31b)
for the transformed variable \( \tilde{y}(t) = \cosh(t) y(t) \text{tanh}(t) \). Furthermore, the physically relevant solutions must obey the supplementary conditions
\[ \int_{t=0}^\infty \frac{y^2(t)}{\cosh^2(t)} dt < \infty, \quad \lim_{t \to 0} \left( \frac{\tilde{y}(t)}{t} \right)' = 0, \quad \text{if } l = 0, \quad \lim_{t \to \infty} \frac{\tilde{y}(t)}{t} = 0, \quad \text{if } l \neq 0. \] (31c)

We note that the correspondence
\[ \chi(t \text{tanh}(t)) = \tilde{y}(t) \] (32)
holds between the original variable \( \chi(\rho) \) of the equation (13a) and the transformed variable \( \tilde{y}(t) \) of (31a).

To get rid of the undesirable term \( \delta^2 \tilde{W}_l \) in the followings we restrict our attention to the \( l = L = 0 \) case, where this problem does not occur, since \( \tilde{W}_0(t) = 0 \).

In this case the differential equation (31a) has the simpler form
\[ \delta^2 \tilde{y}''(t) + \tilde{Q}(t) \tilde{y}(t) = 0 \] with (33a)
\[
\hat{Q}(t) = \frac{1}{\cosh^2(t)} \quad \delta = \frac{1}{\sqrt{2\pi^2 + 2}} \quad (33b)
\]

where the dimensionless potential function \(\hat{Q}(t)\), depicted in figure 3, has a quite unusual form. There are no real turning points, so the function \(\hat{Q}(t)\) builds a parabolic potential well around the origin (for \(0 \leq t < 1\)), and it approaches exponentially to zero from below as \(t \to \infty\).

According to the formulae (21) in Subsection III A, the asymptotic behavior in the semiclassical solution of equation (33) are

\[
S_0(t) = \pm 2i \left(\arctan(e^t) - \frac{\pi}{2}\right) \sim \pm 2i e^{-t} \quad (t \gg 1), \quad (34a)
\]

\[
S_1(t) = \frac{1}{2} \ln (\cosh(t)) - \frac{i\pi}{4} \sim \frac{t}{2} \quad (t \gg 1), \quad (34b)
\]

\[
S_2(t) \sim \pm \frac{i}{16} e^t \quad (t \gg 1), \quad (34c)
\]

Applying the formula (21) and the validity criterion (22), the following first order semiclassical solution is obtained for the equation (33):

\[
\hat{y}_1(t) = A_1 \sqrt{cosh(t)} \sin \left(\frac{2}{\delta} \left(\arctan(e^t) - \frac{\pi}{4}\right)\right) \quad \text{if } 0 < t < -\ln(\delta), \quad (35a)
\]

\[
\sim \frac{A_1}{\sqrt{2}} e^{\frac{1}{2t}} \sin \left(\frac{2}{\delta} \left(\frac{\pi}{4} - e^{-t}\right)\right) \quad \text{if } 1 < t < -\ln(\delta), \quad (35b)
\]

where \(A_1 \in \mathbb{R}\). (In the \(t \gg 1\) approximation the estimation \(\frac{\pi}{4} - \arctan x = \int_{0}^{\infty} \frac{2}{1+x^2} \approx \int_{0}^{\infty} \frac{2}{1+e^{-2t}} = \frac{1}{4}\) was used inside the argument of the sine function, which is valid in the \(x = e^t \gg 1\) limit.)

To get a good approximative solution of the equation (33) for large \(t \gg 1\) values it is straightforward to approximate the coefficient function by \(\hat{Q}_0(t) = \frac{1}{\cosh^2(t)} \sim 4e^{-2t}\) if \(1 < t\). With this approximation, however, the obtained differential equation \(\delta^2 \hat{y}''_1(t) + 4e^{-2t} \hat{y}_1(t) = 0\) is exactly solvable; the two basic solutions \(\hat{y}_1, \hat{y}^\dagger_1\) are

\[
\hat{y}_1(t) = A_1 J_0 \left(\frac{2}{\delta} e^{-t}\right) \quad \text{and} \quad \hat{y}^\dagger_1(t) = A_1 Y_0 \left(\frac{2}{\delta} e^{-t}\right), \quad (36)
\]

where \(J_0\) and \(Y_0\) are the zeroth order Bessel functions of the first and second kind, respectively (14), and \(A_1, A_1^\dagger \in \mathbb{R}\). The function \(\hat{y}^\dagger_1(t)\) is linearly diverging for \(t \to \infty\) values what, according to the correspondence (22), causes the divergence of \(\chi(\rho)\) at \(\rho \to 1\), so the physically meaningful solution of our problem is only \(\hat{y}_1(t)\).

Thus we have obtained two approximative solutions for the equation (33), which are valid in two different, but especially for small \(\delta \to 0\) values exceedingily overlapping intervals, as it is shown in figure 3. To get a semiclassical quantization condition the functions \(\hat{y}_1(t)\) [equation (33)] and \(\hat{y}_1(t)\) [equation (33)] should be matched in the overlapping region \(1 < t < -\ln(\delta)\). At first sight the two functions bear no readily visible resemblance of form, however, using the asymptotic expansion

\[
\hat{y}_1(t) \sim \sqrt{\frac{\delta}{\pi}} A_{11} e^{\frac{1}{2}t} \cos \left(\frac{2}{\delta} e^{-t} - \frac{\pi}{4}\right), \quad (36)
\]

if \(1 < t < -\ln(\delta)\) is obtained, which can well be compared to the formula (35). The matching condition between the two asymptotic formulae (35a) and (37) is clearly the requirement \(A_1 = \pm \sqrt{\frac{\delta}{2\pi}} A_1\) and \(\sin \left(\frac{2}{3} \left(\frac{\pi}{4} - e^{-t}\right)\right) = \pm \cos \left(\frac{2}{3} \left(\frac{\pi}{4} - e^{-t}\right)\right)\) for every \(t\) value in the interval \(1 < t < -\ln(\delta)\). This means that the sum of the arguments of the trigonometric functions must be \(\pi n + \pi/2\), where \(n \in \mathbb{N}\) is a nonnegative integer. Using the definition (33) of \(\delta\) the following semiclassical energy spectrum is obtained

\[
\varepsilon_{sc}(n) = \sqrt{2n^2 + 3n + \frac{1}{8}}, \quad l = 0, \quad n \in \mathbb{N}, \quad (38)
\]

which differs from the exact spectrum (14) only by a constant shift of \(\frac{1}{8}\) under the square root sign.

With the help of the formula (23) the semiclassical radial wave functions \(\hat{y}_1(t)\) [equation (35a)] and \(\hat{y}_1(t)\) [equation (33)] can be transformed back to approximate the original radial function \(\chi(\rho)\) [in equation (13)]:

\[
\chi_1(\rho) = \hat{y}_1 \left(\arctan(\rho)\right) = \delta (1 - \rho^2)^{-\frac{1}{4}} x \times \sin \left(\frac{2}{\delta} \left(\arctan \sqrt{\frac{1 - \rho}{1 - \rho^2} - \frac{\pi}{4}}\right)\right), \quad (39a)
\]

\[
\chi_{11}(\rho) = \hat{y}_{11} \left(\arctan(\rho)\right) = (-1)^n \sqrt{\frac{\pi \delta}{2}} J_n \left(\frac{2}{\delta} \sqrt{\frac{1 - \rho}{1 - \rho^2}}\right). \quad (39b)
\]

(Here the ‘normalization’ condition \(\kappa_0 = \hat{y}'(0) = \chi'(0) = 1\) was used, which implies that the multiplier constants in the semiclassical formulae (33) and (34) are \(A_1 = \delta\) and \(A_{11} = (-1)^n \sqrt{\frac{2\pi}{\delta}}\).)

To get an insight into the accuracy of the semiclassical approximation, in figures 4 and 5 we plotted the exact \(\chi(\rho) = \kappa(\rho^2)\), equation (13)] and the semiclassical wave functions \(\chi_1(\rho), \chi_{11}(\rho)\) equations (33) as well as the relative errors \(\frac{\chi_1(\rho) - \chi(\rho)}{\chi(\rho)}\) and \(\frac{\chi_{11}(\rho) - \chi(\rho)}{\chi(\rho)}\) for the lowest two radial quantum numbers \(n = 0, 1\). The exact radial function was calculated with the exact energy eigenvalues (14) \([l = 0]\) and with the ‘normalization’ condition \(\kappa_0 = \hat{y}'(0) = 1\), while the semiclassical functions \(\chi_1(\rho)\) and \(\chi_{11}(\rho)\) were calculated with the semiclassical
FIG. 3: The dimensionless potential function $-\tilde{Q}(t) = \frac{-1}{\cosh^2(t)}$.

FIG. 4: The validity regions of the two approximative solutions $\tilde{\gamma}_1(t)$ [equation (35)] and $\tilde{\gamma}_1(t)$ [equation (34)] on the $t - \delta$ plane.

FIG. 5: a) The exact and the semiclassical (dimensionless) radial wave functions [$\chi(\rho)$, $\chi_1(\rho)$ and $\chi_II(\rho)$]; b) The relative error $\frac{\Delta\chi(\rho)}{\chi(\rho)}$ of the semiclassical wave functions (where $\Delta\chi$ is $\chi_1 - \chi$ or $\chi_II - \chi$) for radial quantum number $n = 0$. 
energy spectrum \([35]\) (and with the same ‘normalization’ condition).

The figures 6, 7 demonstrate that even for the lowest energy levels the semiclassical wave function \(\chi_1(\rho)\) already approximates the exact function \(\chi(\rho)\) fairly well on almost the whole interval \([0, 1]\). The only failure of \(\chi_1(\rho)\) is that it diverges at \(\rho = 1\), while the exact function \(\chi(\rho)\) has finite value with finite slope at this point. This ‘illness’ is cured by the second approximative solution \(\chi_{11}(\rho)\) which is quite different from the exact function on almost the whole interval \([0, 1]\), but in the immediate vicinity of \(\rho = 1\), where \(\chi_1(\rho)\) deviates from \(\chi(\rho)\), gives a good approximation.

We remark that the divergence of the relative error at the roots \(\rho_\ast\) of the exact function \(\chi(\rho_\ast) = 0\) are due to an unavoidable, small shift between the roots of the exact and semiclassical functions.

In the end we find it worth making a final remark upon the numerical stability of the calculation of the exact and semiclassical wave function. Figure 6 shows the semiclassical and the exact wave function, \(\chi_{11}(\rho)\) and \(\chi(\rho)\) for a somewhat larger but still relatively small radial quantum number \(n = 13\), calculated with the help of standard \(C\) routines with the usual floating point accuracy. It is clearly visible that above a certain point \(\rho_0 \approx 0.7\) the exact polynomial totally loses its numerical stability, while the semiclassical function can still be easily calculated and serves as a very good approximation almost on the whole interval (apart from the II\(^{nd}\) region \([0.997, 1]\), where \(\chi_{11}(\rho)\) becomes the more appropriate approximation). The table I shows the (exact values of the coefficients \(k_i\) of the polynomial \([15a]\) obtained from the recursion \([15b]\) in the \(l = 0, n = 13\) case. The unexpectedly large values of these coefficients cause the numerical instability in the calculation of the exact function \(\chi(\rho)\). Therefore, in certain cases it may well be reasonable to use semiclassical wave functions instead of the exact ones in numerical calculations.

**V. DISCUSSION**

In the previous sections the JWKB semiclassical approximation has been rigorously carried out for a very special case of the Stringari equation \([10]\) (with spherically symmetric harmonic external potential \(V(r)\) and \(l = 0\) zero angular momentum quantum number). Because of the unusual type of the turning point, the vicinity of this point needed special treatment and careful analysis. A comparison of the semiclassical results to the known exact solutions and eigenvalues shows that the accuracy of the semiclassical approximation is quite satisfying; there is only a small \((\pm 1\%\) in dimensionless units) shift in the energy square, and the relative error of the semiclassical radial wave function is already for the lowest energy levels under a few percent. For larger quantum numbers the semiclassical wave function is numerically much more stable than the exact polynomial, what may make the use of the semiclassical functions preferable than the

**TABLE I:** The coefficients \(\kappa_i\) [equation \([13]\)] for \(l = 0\) and \(n = 13\)

| \(l\) | \(\kappa_i\) |
|---|---|
| 0 | 1 |
| 1 | -125.6667 |
| 2 | 4674.7998 |
| 3 | -80807.25 |
| 4 | 785626.0625 |
| 5 | -4756608.85 |
| 6 | 19926434. |
| 7 | -52005588. |
| 8 | 98657565. |
| 9 | -129812704. |
| 10 | 116213280. |
| 11 | -67523128. |
| 12 | 22957864. |
| 13 | -3466572. |
exact ones in certain numerical calculations.

We remark that a simpler, more practical way of achieving the same first order semiclassical solution (35a) of the radial Stringari equation (33) in the $l = 0$ case would have been just to insert the Ansatz (19) right into the equation (28a), without bothering us about the $\delta^2W_0(x)$ term. This term is so small that its presence disturbs only the second order term $S_2(x)$ of the JWKB approximation, which does not appear in the first order semiclassical solution (21), but it does turn up in the condition (22b) that defines the validity region $I$ of the semiclassical approximation. So it is safer anyway to get rid of the term $\delta^2W_0(x)$ before applying the JWKB method.

Finally we note that in the works [26, 27, 28] the same physical problem (collective excitations in Bose-condensed gases in spherically symmetric harmonic traps) has been studied with basically different semiclassical methods. The authors of these works could handle also the $l \neq 0$ case. They applied the Langer modification [29] $l(l+1) \rightarrow (l+\frac{1}{2})^2$, and in their formula the shift of the energy square is 1 in dimensionless units, as compared to the exact result.

As a next step, it would be useful to extend the present semiclassical method also for the $l \neq 0$ case, and then apply the same techniques for the solution of the Stringari equation in non-harmonic potentials, which are analytically not treatable, but which have more and more experimental relevance with the growth of the extent of the condensate.

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