POINTEWISE CONVERGENCE OF FOURIER SERIES ON THE RING OF INTEGERS OF LOCAL FIELDS WITH AN APPLICATION TO GABOR SYSTEMS

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Abstract. We construct a simple example of an integrable function on the ring of integers of the \( p \)-adic field \( \mathbb{Q}_p \) having an almost everywhere divergent Fourier series. On the other hand, we prove the pointwise convergence of the Fourier series of functions in \( L^p(\mathcal{D}, w) \), \( 1 < p < \infty \), where \( \mathcal{D} \) is the ring of integers of a local field \( K \) and \( w \) is a weight in the Muckenhoupt \( A_p \) class. This result includes, as special cases, when \( \mathcal{D} \) is the ring of integers of \( \mathbb{Q}_p \) or the field \( \mathbb{F}_q(\{X\}) \) of formal Laurent series over a finite field \( \mathbb{F}_q \), and in particular, when \( \mathcal{D} \) is the Walsh-Paley or dyadic group \( 2^\omega \). To achieve this, we establish a weighted estimate for the maximal operator corresponding to the Fourier partial sum operators for functions in \( L^p(\mathcal{D}, w) \). As an application, we characterize the Schauder basis property of the Gabor systems in a local field \( K \) of positive characteristic in terms of the \( A_2 \) weights on \( \mathcal{D} \times \mathcal{D} \) and the Zak transform \( Zg \) of the window function \( g \) that generates the Gabor system. Some examples are given to illustrate this result. In particular, we construct an example of a Gabor system which is complete and minimal, but fails to be a Schauder basis for \( L^2(K) \).

1. Introduction and Main Results

Let \( G \) be a compact abelian group. The dual group \( \hat{G} \), consisting of all continuous characters of \( G \), is a discrete group. Suppose the characters of \( G \) are arranged in a particular order \( \gamma_1, \gamma_2, \ldots \). We consider the Lebesgue spaces \( L^p(G) \), \( 1 \leq p \leq \infty \), with respect to the normalized Haar measure on \( G \). If \( f \in L^1(G) \), then the Fourier series of \( f \) is given by

\[
\hat{f}(x) \sim \sum_{n=1}^{\infty} \hat{f}(\gamma_n) \gamma_n(x),
\]

where \( \hat{f}(\gamma_n) = \int_G f(x) \gamma_n(x) \, dx \). We say that the Fourier series of \( f \) diverges at a point \( x \) if the sequence of partial sum \( S_n f(x) = \sum_{k=1}^{n} \hat{f}(\gamma_k) \gamma_k(x) \) is not convergent.

In the case of \( G = \mathbb{T} \), the problem of the divergence of trigonometric Fourier series has been a subject of investigation for a long time. Kolmogorov [19] constructed his famous example of a function \( f \in L^1(\mathbb{T}) \) whose trigonometric Fourier series is divergent almost everywhere on \( \mathbb{T} \). The analogous question for other compact abelian groups has been studied by, among others, Stein [26] on the Walsh-Paley group or the dyadic group \( 2^\omega \) and Kheladze [18] for the Vilenkin systems with certain boundedness criterion.

The characters of the compact abelian group \( \mathbb{Z}_p \), the ring of integers of the \( p \)-adic field \( \mathbb{Q}_p \), can be arranged in several natural ways and \( \lim_{k \to \infty} S_{p^k} f(x) \) exists almost everywhere for all \( f \in L^1(\mathbb{Z}_p) \). This is not true in the case of trigonometric Fourier series. Moreover, computing the Dirichlet kernels \( D_m \) in \( \mathbb{Z}_p \) for a general \( m \) is not easy unlike the trigonometric Fourier series. This makes the problem of divergence of Fourier series in \( \mathbb{Z}_p \) somewhat complicated and different from that of trigonometric Fourier series. Chaney [4] fixed an ordering of the characters and constructed a function in \( L^1(\mathbb{Z}_p) \) whose Fourier series diverges almost everywhere. In some respects, Chaney modelled his construction after Kolmogorov’s example, though Kolmogorov’s construction is quite involved. One of the objectives of this article is to construct a relatively simple example of an integrable function on \( \mathbb{Z}_p \) having divergent Fourier series for almost all \( x \in \mathbb{Z}_p \), keeping the ordering of the characters as in [4]. For this construction, we will use ideas associated with the construction of Kolmogorov’s example, a basic lemma.

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in [4] on Dirichlet kernels and an existence theorem of everywhere diverging Fourier series by Katznelson [17] on the circle \( \mathbb{T} \).

The group \( \mathbb{Z}_p \) is the ring of integers of the \( p \)-adic field \( \mathbb{Q}_p \). The field \( \mathbb{Q}_p \), of characteristic zero, is a canonical example of a local field. Other examples of local fields \( K \) are finite extensions of \( \mathbb{Q}_p \), if \( K \) has characteristic zero and the field \( \mathbb{F}_q((X)) \) of formal Laurent series over a finite field \( \mathbb{F}_q \), if \( K \) has positive characteristic. These are essentially all the local fields. The reader is referred to Chapter 1 of [28] and Chapter 4 of [24] for the details of the classification of nondiscrete local fields. Our second objective of this article is to present a positive result regarding pointwise convergence of Fourier series for a class of functions, defined on the ring of integers \( \mathcal{O} \) of a local field \( K \), which are subspaces of \( L^1(\mathcal{O}) \). Our result is applicable to all such fields without any restriction. In order to state our results, we briefly recall some basic facts of local fields. For more details, we refer the reader to Section 2.

Let \( K \) be a local field with a non-archimedean absolute value \( | \cdot | \). The closed unit ball \( \mathcal{D} = \{ x \in K : |x| \leq 1 \} \) is the unique maximal compact subring of \( K \). This is called the ring of integers of \( K \). Let \( \mathbb{N}_0 = \mathbb{N} \cup \{ 0 \} \) and \( \Lambda = \{ u(n) : n \in \mathbb{N}_0 \} \) be a complete set of distinct coset representatives of \( \mathcal{D} \) in \( K \). A function \( f \) on \( K \) is said to be \( \Lambda \)-periodic if \( f(x + u(n)) = f(x) \) for all \( n \geq 0 \) and for a.e. \( x \in K \). Let \( p \) be a fixed element of the set \( \mathfrak{P} = \{ x \in K : |x| < 1 \} \) of maximal absolute value. Such an element is called a prime element. The dual group \( \hat{K} \) of \( K^+ \) is isomorphic to \( K^+ \). We fix a character \( \chi \) of \( K^+ \) such that \( \chi \) is equal to 1 on \( \mathcal{D} \) but is non-constant on \( p^{-1} \mathcal{D} \). For \( \xi \in K \), let \( \chi_\xi(x) = \chi(\xi x) \). Then it follows that \( \{ \chi_{u(n)} : n \in \mathbb{N}_0 \} \) is a complete set of distinct characters of \( \mathcal{D} \). Now we list the characters as \( \chi_{u(0)}, \chi_{u(1)}, \chi_{u(2)}, \ldots \). Later, we shall see that for \( \mathcal{D} = \mathbb{Z}_p \), this ordering of the characters is different from the one given in [4]. With respect to the above ordering of the characters \( \{ \chi_{u(n)} : n \in \mathbb{N}_0 \} \), we have the following result.

**Theorem 1.1.** Let \( 1 < p < \infty \), \( w \) be a weight satisfying Muckenhoupt \( A_p \) condition and \( f \in L^p(\mathcal{D}, w) \). Then \( f \in L^1(\mathcal{D}) \) and the Fourier series of \( f \) converges almost everywhere with respect to the measure \( w(x) \, dx \).

A weight on \( K \) is a nonnegative locally integrable function. The weight \( w \) is said to satisfy Muckenhoupt \( A_p \) condition if there exists a constant \( C > 0 \) such that

\[
\left( \frac{1}{|B|} \int_B w(x) \, dx \right) \left( \frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} \leq C
\]

for every ball \( B \) in \( K \). In this case, we also say that \( w \) is an \( A_p \) weight or \( w \in A_p \). The infimum of all such constants \( C \) is called the \( A_p \) characteristic of the weight \( w \) and is denoted by \( \|w\|_{A_p} \).

The spaces \( L^p(w) \) and \( L^p(\mathcal{D}, w) \) are the usual spaces of \( p \)-integrable functions on \( K \) and \( \mathcal{D} \) respectively with respect to the weight \( w \). The space \( L^1_{\text{loc}}(K) \) denotes the locally integrable functions on \( K \).

The standard way to tackle this pointwise convergence problem is to consider the corresponding maximal operator \( Mf(x) = \sup_n |S_nf(x)| \). Hunt and Taibleson [15] showed that the operator \( M \) is bounded on \( L^p(\mathcal{D}) \) and obtained almost everywhere convergence of the Fourier series of functions in \( L^p(\mathcal{D}), 1 < p < \infty \). We shall deduce Theorem 1.1 from the following estimate.

**Theorem 1.2.** Let \( w \in A_p, 1 < p < \infty \). Then the operator \( M \) is bounded on \( L^p(\mathcal{D}, w) \). Hence, for every \( 1 < p < \infty \), there exists a constant \( C_p > 0 \) such that

\[
\int_{\mathcal{D}} |Mf(x)|^p w(x) \, dx \leq C_p \int_{\mathcal{D}} |f(x)|^p w(x) \, dx
\]

for all \( f \in L^p(\mathcal{D}, w) \).

Since \( |S_nf(x)| \) is dominated by \( Mf(x) \) for all \( n \), we obtain the uniform boundedness of the Fourier partial sum operators \( S_n, n \in \mathbb{N}_0 \), on the weighted spaces \( L^p(\mathcal{D}, w) \) for \( w \in A_p \). Moreover, it turns out that the \( A_p \) condition is also necessary. For a proof of this fact, we refer to [22]. Therefore, we have the following result.

**Theorem 1.3.** Let \( K \) be a local field, \( w \) a \( \Lambda \)-periodic weight on \( K \) and \( 1 < p < \infty \). Then the following statements are equivalent.
(a) There is a positive constant $C$ such that
\[
\left( \frac{1}{|B|} \int_B w(x) \, dx \right) \left( \frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} \leq C
\]
for every ball $B$ in $K$.

(b) There is a positive constant $C$, depending only on $w$ and $p$, such that for every $f \in L^p(D, w)$, we have $f \in L^1(D)$ and
\[
\int_D |S_n f(x)|^p w(x) \, dx \leq C \int_D |f(x)|^p w(x) \, dx
\]
for $n = 0, 1, 2, \ldots$.

We now turn to an application of Theorem 1.3. The special case $p = 2$ in Theorem 1.3 will be crucial for us to find a necessary and sufficient condition for a Gabor system to form a Schauder basis for $L^2(K)$. We first recall some definitions.

A Schauder basis in $L^2(K)$ is a sequence of functions $\{f_k\}$ such that for every $f \in L^2(K)$ there exists a unique sequence $\{\alpha_k\}$ of scalars with $f = \sum \alpha_k f_k$, where the series converges in the $L^2$-norm with respect to a fixed order.

The Gabor system generated by the time-frequency shifts of a window function $g \in L^2(K)$ is the collection of functions $\{M_{u(n)} T_{u(k)} g : n, k \in \mathbb{N}_0\}$, where $T_y f(x) = f(x - y)$ and $M_\xi f(x) = \chi_\xi(x) f(x)$, $y, \xi \in K$, are the usual translation and modulation operators, respectively.

In this article, as an application of Theorem 1.3, we characterize the Schauder basis property of the Gabor system $\{M_{u(n)} T_{u(k)} g : n, k \in \mathbb{N}_0\}$ in terms of the Zak transform of $g$ and $A_2$ weights on $D \times D$.

For $f \in L^2(\mathbb{R})$, the Zak transform $Zf$ of $f$ is the function on $\mathbb{R} \times \mathbb{R}$ defined by
\[
Zf(x, \xi) = \sum_{k \in \mathbb{Z}} f(x + k) e^{2\pi i x \xi}.
\]

The notion of Zak transform admits a natural generalization to local fields of positive characteristic as follows:
\[
Zf(x, \xi) = \sum_{k \in \mathbb{N}_0} T_{u(k)} f(x) \chi_k(\xi) = \sum_{k \in \mathbb{N}_0} f(x - u(k)) \chi(u(k)\xi)
\]
for $x, \xi \in K$.

The Zak transform was first introduced by Gelfand [4] and it was defined on an arbitrary locally compact abelian group by Weil [23]. It was later rediscovered by Zak [30] and Brezin [2]. For more details about the history of the Zak transform, we refer to [13] and [9].

In case of the real line $\mathbb{R}$, the Zak transform has proved to be a fundamental tool in analyzing Gabor frames. It is worth mentioning that the necessity of weights in various aspects of harmonic analysis and time-frequency analysis is quite well known. We refer to the survey article [10] for different types of weight functions which occur in various problems and contexts in time-frequency analysis. One such important class of weights is the Muckenhoupt $A_p$ weights which characterize the validity of certain weighted norm inequalities, e.g., for the Hardy-Littlewood maximal function, Hilbert transform, etc. An important application of Muckenhoupt weights in time-frequency analysis was given by Heil and Powell in [12]. They proved that Gabor Schauder bases admit a simple characterization in the Zak transform domain. Since Gabor theory rests mainly on the structure of translations and modulations, it is natural to extend this theory to other locally compact abelian groups. Such a development is quite useful as it includes all other examples which are important for applications and it emphasizes the basic features of time-frequency analysis in a comprehensive way.

Even though the generalization of many aspects of Gabor theory to locally compact abelian groups is routine and is based on standard harmonic analysis on such groups (see, for example, [8], [14] and [20]), extension of Schauder basis property of a Gabor system to a general locally compact abelian group is much harder. In this article, for any local field $K$ of positive characteristic, we shall define a family of permutations of $\mathbb{N}_0 \times \mathbb{N}_0$ and we will show that, with respect to each of these permutations, the Gabor system $\{M_{u(n)} T_{u(k)} g : n, k \in \mathbb{N}_0\}$ generated by the window function $g \in L^2(K)$ is a Schauder basis for $L^2(K)$ if and only if $|Zg|^2$ is an $A_2$ weight.
weight on $\mathcal{D} \times \mathcal{D}$. The proof of this characterization is based on Theorem 1.3 and some ideas from [12]. Consequently, we shall supply details only for those portions of the argument which are significantly different from the real line settings.

The article is organized as follows. In Section 2 we establish notation and introduce various concepts of local fields and state some results on Fourier analysis on such fields. We define the Hardy-Littlewood maximal function and some of its variants, and also describe the relevant weighted norm inequalities. In Section 3, we construct an example of an integrable function on $\mathbb{Z}_p$ whose Fourier series diverges almost everywhere. The pointwise convergence of Fourier series in Theorem 1.1 and the boundedness of the maximal operator corresponding to the Fourier partial sum operators in Theorem 1.2 will be proved in Section 4. The concepts of Gabor systems, Zak transform and the $A_p$ weights on $\mathcal{D} \times \mathcal{D}$ are introduced in Section 5. Further, we develop the machinery required to prove the characterization of Gabor Schauder bases which is done in Section 6.

2. Preliminaries on Fourier analysis on local fields

Here we present some of the basic facts about local fields $K$ and then recall some results on Fourier series of functions defined on the ring of integers $\mathcal{D}$ of $K$. Many of these facts are well-known and can be found, for example, in the books [28, 24]. Finally, we describe some tools from the theory of maximal functions [22, 23] that we need to prove our results.

2.1. Local fields. Let $K$ be a field and a topological space. Then $K$ is called a locally compact field or a local field if both $K^+$ and $K^*$ are locally compact abelian groups, where $K^+$ and $K^*$ denote the additive and multiplicative groups of $K$ respectively.

If $K$ is any field and is endowed with the discrete topology, then $K$ is a locally compact field. Further, if $K$ is connected, then $K$ is either $\mathbb{R}$ or $\mathbb{C}$. If $K$ is not connected, then it is totally disconnected. If the characteristic of $K$ is zero, then $K$ is either $\mathbb{Q}_p$ or a finite algebraic extension of $\mathbb{Q}_p$. If $K$ has finite characteristic, then it is a field of formal Laurent series over a finite field $GF(p^f)$. In this article, by a local field, we mean a field $K$ which is locally compact, non-discrete and totally disconnected. We use the notation of the book by Taibleson [28].

Let $K$ be a local field. Since $K^+$ is a locally compact abelian group, we choose a Haar measure $dx$ for $K^+$. If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. By uniqueness of the Haar measure, $d(\alpha x) = c \, dx$ for some $c > 0$. Let $c = |\alpha|$. We call $|\alpha|$ the absolute value or the valuation of $\alpha$. We also let $|0| = 0$. The map $x \rightarrow |x|$ satisfies the following properties:

(a) $|x| = 0$ if and only if $x = 0$;
(b) $|xy| = |x||y|$ for all $x, y \in K$;
(c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.

Property (c) is called the ultrametric inequality. It follows that

$$|x + y| = \max\{|x|, |y|\} \text{ if } |x| \neq |y|.$$

The set $\mathcal{D} = \{x \in K : |x| \leq 1\}$ is called the ring of integers in $K$. It is the unique maximal compact subring of $K$. Define $\mathfrak{p} = \{x \in K : |x| < 1\}$. The set $\mathfrak{p}$ is called the prime ideal in $K$. The prime ideal in $K$ is the unique maximal ideal in $\mathcal{D}$. It is principal and prime.

Since $K$ is totally disconnected, the set of values $|x|$, as $x$ varies over $K$, is a discrete set of the form $\{s^k : k \in \mathbb{Z}\} \cup \{0\}$ for some $s > 0$. Hence, there is an element of $\mathfrak{p}$ of maximal absolute value. Let $p$ be a fixed element of maximum absolute value in $\mathfrak{p}$. Such an element is called a prime element of $K$. It can be proved that $\mathcal{D}$ is compact and open. Hence, $\mathfrak{p}$ is compact and open. Therefore, the residue space $\mathcal{D}/\mathfrak{p}$ is isomorphic to a finite field $GF(q)$, where $q = p^f$ for some prime $p$ and $c \in \mathbb{N}$. For a proof of this fact, we refer to [28].

For a measurable subset $E$ of $K$, let $|E| = \int_K 1_E(x) \, dx$, where $1_E$ is the characteristic function of $E$ and $dx$ is the Haar measure of $K$ normalized so that $|\mathcal{D}| = 1$. Then, it is easy to see that $|\mathfrak{p}| = q^{-1}$ and $|p| = q^{-1}$ (see [28]). It follows that if $x \neq 0$, and $x \in K$, then $|x| = q^k$ for some $k \in \mathbb{Z}$.

Let $\mathcal{D}^* = \mathcal{D} \setminus \mathfrak{p} = \{x \in K : |x| = 1\}$. It is the group of units in $K^*$. If $x \neq 0$, we can write $x = p^k x'$, with $x' \in \mathcal{D}^*$. Let $\mathfrak{p}^k = p^k \mathcal{D} = \{x \in K : |x| \leq q^{-k}\}, k \in \mathbb{Z}$. These are called
fractional ideals. Each $\mathfrak{P}^k$ is compact and open and is a subgroup of $K^+$ (see [24]). It follows that $|\mathfrak{P}^k| = q^{-k}$ for $k \in \mathbb{Z}$.

Let $h \in K$ and $k \in \mathbb{Z}$. A set $B$ of the form $h + \mathfrak{P}^k$ will be called a ball with centre $h$ and radius $q^{-k}$. Using the ultrametric inequality, it is easy to verify the following facts. For a proof, we refer to [28].

**Proposition 2.1.** Let $K$ be a local field.

(a) Every point of a ball in $K$ is its centre.
(b) If two balls intersect, then one contains the other.
(c) The number of balls in $K$ is countable.
(d) Every ball of radius $\mathfrak{P}^k$ is a disjoint union of $q$ balls of radius $\mathfrak{P}^{k+1}$, $k \in \mathbb{Z}$.

We define the set $\mathcal{S}(K)$ to be the space of all finite linear combinations of functions of the form $1_{h+\mathfrak{P}^k}$, $h \in K$, $k \in \mathbb{Z}$. The following two theorems describe the properties of functions in $\mathcal{S}(K)$ and their Fourier transforms. For proofs of these statements, see Theorem 3.1 and Theorem 3.2, Chapter II in [28].

**Theorem 2.1.** The function $g \in \mathcal{S}(K)$ if and only if there exist integers $k$ and $l$ such that $g$ is constant on cosets of $\mathfrak{P}^k$ and is supported on $\mathfrak{P}^l$.

**Theorem 2.2.** If $g \in \mathcal{S}(K)$ is constant on cosets of $\mathfrak{P}^k$ and is supported on $\mathfrak{P}^l$, then $\hat{g} \in \mathcal{S}(K)$ is constant on cosets of $\mathfrak{P}^{-1}$ and is supported on $\mathfrak{P}^{-k}$.

2.2. Fourier series on the compact abelian group $\mathfrak{D}$. Let $\chi$ be a continuous character of $K^+$. It turns out that any other character of is of the form $\chi_y(x) = \chi(yx)$ for some $y \in K$. In fact, the map $y \mapsto \chi_y$ is a topological isomorphism of $K^+$ with its dual group $\hat{K}^+$ so that we can identify $K^+$ with $\hat{K}^+$ (see [28]).

Let $\chi_u$ be any character of $K^+$. Since $\mathfrak{D}$ is a subgroup of $K^+$, the restriction $\chi_u|\mathfrak{D}$ is a character of $\mathfrak{D}$. Also, as characters of $\mathfrak{D}$, $\chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. That is, $\chi_u = \chi_v$ if $u + \mathfrak{D} = v + \mathfrak{D}$ and $\chi_u \neq \chi_v$ if $(u + \mathfrak{D}) \cap (v + \mathfrak{D}) = \emptyset$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in $K^+$, then $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ is a list of distinct characters of $\mathfrak{D}$. It is proved in [28] that this list is complete.

**Proposition 2.2.** Let $\{u(n) : n \in \mathbb{N}_0\}$ be a complete list of coset representatives of $\mathfrak{D}$ in $K^+$. Then $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ is a complete list of characters of $\mathfrak{D}$. Moreover, it is an orthonormal basis for $L^2(\mathfrak{D})$.

Given such a list of characters $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$, we define the Fourier coefficients of $f \in L^1(\mathfrak{D})$ as

$$\hat{f}(u(n)) = \langle f, \chi_{u(n)} \rangle = \int_{\mathfrak{D}} f(x) \overline{\chi_{u(n)}(x)} \, dx, \quad n \in \mathbb{N}_0.$$  

For $N \in \mathbb{N}_0$, let $S_N f$ be the $N$-th partial sum of the Fourier series of $f$:

$$S_N f(x) = \sum_{n=0}^{N} \langle f, \chi_{u(n)} \rangle \chi_{u(n)}(x).$$

The series $\sum_{n=0}^{\infty} \langle f, \chi_{u(n)} \rangle \chi_{u(n)}(x)$ is called the Fourier series of $f$.

For brevity, we will now write $\chi_n = \chi_{u(n)}|\mathfrak{D}$ for $n \in \mathbb{N}_0$. With this notation, we have

$$S_N f(x) = \sum_{n=0}^{N} \langle f, \chi_n \rangle \chi_n(x).$$

From the standard $L^2$-theory for compact abelian groups, we conclude that the Fourier series of $f$ converges to $f$ in $L^2(\mathfrak{D})$ and Parseval’s identity holds:

$$\int_{\mathfrak{D}} |f(x)|^2 \, dx = \sum_{n \in \mathbb{N}_0} |\hat{f}(u(n))|^2.$$

Also, we have the uniqueness of the Fourier coefficients. That is, if $f \in L^1(\mathfrak{D})$ and $\hat{f}(u(n)) = 0$ for all $n \in \mathbb{N}_0$, then $f = 0$ a.e.
We now proceed to impose a natural order on the sequence \( \{u(n) : n \in \mathbb{N}_0\} \). Note that \( \mathcal{D}/\mathcal{P} \) is isomorphic to the finite field \( GF(q) \) and \( GF(q) \) is a \( c \)-dimensional vector space over the field \( GF(p) \). We choose a set \( \{1 = \epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{c-1}\} \subset \mathcal{D}^* \) such that \( \text{span}\{\epsilon_j\}_{j=0}^{c-1} \cong GF(q) \).

For \( n \in \mathbb{N}_0 \) such that \( 0 \leq n < q \), we have

\[
n = a_0 + a_1 p + \cdots + a_{c-1} p^{c-1}, \quad \text{with} \quad 0 \leq a_k < p, \quad k = 0, 1, \ldots, c - 1.
\]

Define

\[
(2.1) \quad u(n) = (a_0 + a_1 \epsilon_1 + \cdots + a_{c-1} \epsilon_{c-1}) p^{-1}
\]

Note that \( \{u(n) : n = 0, 1, \ldots, q-1\} \) is a complete set of coset representatives of \( \mathcal{D} \) in \( \mathcal{P}^{-1} \) so that we can write

\[
\mathcal{P}^{-1} = \bigcup_{l=0}^{q-1} (u(l) + \mathcal{D}).
\]

Now, for \( n \geq 0 \), write

\[
n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s, \quad 0 \leq b_k < q, \quad k = 0, 1, 2, \ldots, s,
\]

and define

\[
(2.2) \quad u(n) = u(b_0) + u(b_1) p^{-1} + \cdots + u(b_s) p^{-s}.
\]

This defines \( u(n) \) for all \( n \in \mathbb{N}_0 \). In general, it is not true that \( u(m + n) = u(m) + u(n) \). But it follows that

\[
u(rq^k + s) = u(r) p^{-k} + u(s) \quad \text{if} \quad r \geq 0, k \geq 0 \text{ and } 0 \leq s < q^k.
\]

In the following proposition we list some properties of \( \Lambda = \{u(n) : n \in \mathbb{N}_0\} \) which will be used later. We refer to [1] for a proof.

**Proposition 2.3.** For \( n \in \mathbb{N}_0 \), let \( u(n) \) be defined as in (2.1) and (2.2). Then

(a) \( u(n) = 0 \) if and only if \( n = 0 \). If \( k \geq 1 \), then \( |u(n)| = q^k \) if and only if \( q^{k-1} \leq n < q^k \).

Moreover, if \( K \) is a local field of positive characteristic, then

(b) \( \{u(k) : k \in \mathbb{N}_0\} \subseteq \{u(k) : k \in \mathbb{N}_0\} \);

(c) \( \{u(l) + u(k) : k \in \mathbb{N}_0\} \subseteq \{u(k) : k \in \mathbb{N}_0\} \) for any fixed \( l \in \mathbb{N}_0 \).

In particular, \( \{u(n) : n \in \mathbb{N}_0\} \) is a subgroup of \( K^+ \) if \( K \) is of positive characteristic.

The Dirichlet kernels are the functions

\[
(2.3) \quad D_0 \equiv 0 \quad \text{and} \quad D_n(x) = \sum_{k=0}^{n-1} \chi_k(x), n \geq 1.
\]

We will need the following property of the Dirichlet kernels for some specific values of \( n \).

**Lemma 2.1.** For \( k \in \mathbb{N}_0 \), we have \( D_{q^k} = q^k 1_{\mathcal{P}^k} \).

**Proof.** See Page 86 in [28].

Let \( f \) be a locally integrable function in \( K \). We say that \( x \in K \) is a regular point of \( f \) if

\[
q^k \int_{\{y : |x-y| \leq q^{-k}\}} f(y) \, dy = \frac{1}{|x + \mathcal{P}^k|} \int_{x + \mathcal{P}^k} f(y) \, dy \to f(x) \quad \text{as} \quad k \to \infty.
\]

The following result, called Lebesgue differentiation theorem, asserts that almost every point of a locally integrable function is a regular point. We refer to [28] for a proof of this theorem.

**Theorem 2.3.** If \( f \) is a locally integrable function in \( K \), then almost every \( x \in K \) is a regular point of \( f \).
2.3. Maximal functions and their weighted norm estimates. For \( f \in L^1_{\text{loc}}(K) \), the Hardy-Littlewood maximal function \( Mf \) is defined by

\[
Mf(x) = \sup_{k \in \mathbb{Z}} \frac{1}{|P^k|} \int_{x+P^k} |f(y)| \, dy, \quad x \in K.
\]

Recall that a ball \( B \) with centre at \( h \) of radius \( q^{-k} \) is of the form \( h + P^k \). Also, observe that the balls in a local field are dyadic in nature, in the sense that given any two balls, either they are disjoint or one ball contains the other (see Proposition 2.1). So, the maximal function \( M \) is more or less like the dyadic maximal function in \( \mathbb{R}^n \), but in contrast to \( \mathbb{R}^n \), in a local field \( Mf \) can never be zero at any point unless \( f = 0 \).

For \( 1 < p < \infty \), the weighted \( L^p \)-norm of any function \( f \in L^p(w) \) and the corresponding maximal function \( Mf \) are comparable. For a proof of the following theorem, see [22] or [23].

**Theorem 2.4.** Let \( M \) be the Hardy-Littlewood maximal function on \( K \) and \( w \in A_p \), \( 1 < p < \infty \). Then there exists a constant \( C_p > 0 \) such that

\[
\int_K Mf(x)^p w(x) \, dx \leq C_p \int_K |f(x)|^p w(x) \, dx.
\]

We define a variant of the maximal operator by putting

\[
M_s f(x) = (M|f|^s(x))^{1/s}, \quad 1 < s < \infty.
\]

Finally, for \( f \in L^1_{\text{loc}}(K) \), we introduce the Fefferman-Stein maximal function \( f^\sharp \) analogously as in \( \mathbb{R}^n \) by setting

\[
f^\sharp(x) = \sup_{k \in \mathbb{Z}} \frac{1}{|P^k|} \int_{x+P^k} |f(y) - f_{x+P^k}| \, dy, \quad x \in K,
\]

where \( f_{x+P^k} = \frac{1}{|P^k|} \int_{x+P^k} f(x) \, dx \) is the average of \( f \) over the ball \( x+P^k \). We will also need the following result, proved in [22], which relates the weighted \( L^p \)-norm of the Hardy-Littlewood maximal function \( Mf \) and the Fefferman-Stein maximal function \( f^\sharp \).

**Lemma 2.2.** Let \( 1 < p < \infty \) and \( w \in A_p \). Then there exists a constant \( C > 0 \), depending only on \( p \) and \( w \), such that

\[
\int_K Mf(x)^p w(x) \, dx \leq C \int_K f^\sharp(x)^p w(x) \, dx
\]

for every \( f \in L^p(w) \).

3. Example of a Fourier series diverging almost everywhere

3.1. Some background on \( \mathbb{Z}_p \). We start with a quick review of the \( p \)-adic numbers. Let \( p \) be a prime. Any nonzero rational number \( s \in \mathbb{Q} \) can be uniquely written as \( s = p^r \frac{a}{b} \) where \( r, a, b \in \mathbb{Z}, b \neq 0 \) and \( (p,a) = (p,b) = 1 \). Here \( (x,y) \) denotes the greatest common divisor of the integers \( x \) and \( y \). We define \( |s|_p = p^{-r} \) for \( s \neq 0 \) and \( |0|_p = 0 \). Then \( | \cdot |_p \) is a non-archimedean absolute value. The field \( \mathbb{Q}_p \) of \( p \)-adic numbers is the completion of \( \mathbb{Q} \) with respect to \( | \cdot |_p \).

The ring \( \mathbb{Z}_p \) of \( p \)-adic integers is the set of \( p \)-adic numbers \( x \) with \( |x|_p \leq 1 \). The family \( \{ x + P^k : x \in \mathbb{Z}_p, k \in \mathbb{N} \} \) is a base for a topology on \( \mathbb{Z}_p \), and with this topology \( \mathbb{Z}_p \) is a compact, Hausdorff, 0-dimensional, metric topological group. The set \( \mathbb{Z}_p \) can also be identified with the direct product \( \prod_{j \in \mathbb{N}_0} \Delta_j \), where each \( \Delta_j \) is the cyclic group of order \( p \), addition being done modulo \( p \) carries from left to right. A typical element \( x \) of \( \mathbb{Z}_p \) is of the form

\[
x = \sum_{j=0}^{\infty} x_j p^j, \quad x_j \in \{0, 1, 2, \ldots, p - 1\}.
\]
3.2. The character group of $\mathbb{Z}_p$. The dual group or the character group $\Gamma_p$ of $\mathbb{Z}_p$ can be described as follows. Consider the set of all complex numbers $\exp(2\pi ik/p^n)$, where $k$ runs through all integers and $n$ through all nonnegative integers. Denote this set by $Z(p^\infty)$. Note that $Z(p^\infty)$ is a group under multiplication of complex numbers. Let us define a map $\Pi$ from $Z(p^\infty)$ onto the character group of $\mathbb{Z}_p$ as follows:

$\Pi(1)$ is the constant character 1. That is, $\Pi(1)(x) = 1$ for all $x \in \mathbb{Z}_p$. For $\lambda \in Z(p^\infty)$ and $\lambda \neq 1$, there are unique $n, k \in \mathbb{N}$ such that $(k, p) = 1$, $\lambda = \exp(2\pi ik/p^n)$, and $k < p^n$. For $x = \sum_{i=0}^{\infty} x_ip^i$, we define

$$\Pi(\lambda)(x) = \lambda^{x_0+x_1p+\cdots+x_{n-1}p^{n-1}} = \exp \frac{2\pi ik(x_0 + x_1p + \cdots + x_{n-1}p^{n-1})}{p^n}.$$ 

Lemma 3.1 (Sec 25.2, [14]). The map $\Pi$ is a canonical isomorphism from $Z(p^\infty)$ onto $\Gamma_p$.

Now, we enumerate the characters of $\mathbb{Z}_p$ into a sequence $\{\gamma_1, \gamma_2, \ldots\}$ as done in Chaney [4] in the following manner. The constant character 1 precedes all other characters. If $\gamma_1$ and $\gamma_2$ are distinct elements in $Z(p^\infty)$ and if $\gamma_1 \neq 1, \gamma_2 \neq 1$, there exist unique positive integers $n_j, k_j, j = 1, 2$, such that

$$\lambda_j = \exp \frac{2\pi ik_j}{p^{n_j}}, \quad 0 < k_j < p^{n_j}, \quad (k_j, p) = 1.$$ 

We say that the character $\Pi(\lambda_2)$ follows the character $\Pi(\lambda_1)$ in case $n_2 > n_1$ or in case $n_1 = n_2$ and $k_2 > k_1$. The characters of $\mathbb{Z}_p$ are well ordered by this relation. We enumerate the characters of $\mathbb{Z}_p$ as $\gamma_1 = 1, \gamma_2, \gamma_3, \ldots$, with the understanding that, given $n_1, n_2 \in \mathbb{N}$, $\gamma_{n_2}$ follows $\gamma_{n_1}$ if and only if $n_1 < n_2$.

It should be noted that this enumeration $\{\gamma_k\}_{k \in \mathbb{N}}$ of the characters of $\mathbb{Z}_p$ is not identical with that of $\{\chi_{u(k)}\}_{k \in \mathbb{N}_0}$ given in the previous section for general local fields. For example, by an easy calculation one can see that the characters $\gamma_6$ and $\chi_{u(5)}$ disagree on a set of positive measure in $\mathbb{Z}_p$. However, applying Lemma 2.3 and Theorem 2.3, it is easy to see that $\lim_{k \to \infty} S_{p^k}f(x)$ exists for almost every $x \in \mathbb{Z}_p$ and for all $f \in L^1(\mathbb{Z}_p)$ with respect to the ordering $\{\chi_{u(k)}\}_{k \in \mathbb{N}_0}$.

For the ordering $\{\gamma_k\}_{k \in \mathbb{N}}$ of the countable character group of $\mathbb{Z}_p$, we define Fourier series as in the introduction. For $m \in \mathbb{N}$, by $S_mf(x)$ we shall denote the $m$-th partial sum of the Fourier series expansion of the function $f \in L^1(\mathbb{Z}_p)$ at $x \in \mathbb{Z}_p$. More generally, $S_m\mu(x)$ will denote the $m$-th partial sum of the Fourier series of a Borel measure $\mu$.

For each $m \in \mathbb{N}$, we define the Dirichlet kernel $D_m$ on $\mathbb{Z}_p$ by

$$D_m(y) = \gamma_1(y) + \gamma_2(y) + \cdots + \gamma_m(y)$$ 

and the Fejér kernel $K_m$ by setting

$$K_m(y) = \frac{1}{m} \sum_{k=1}^{m} D_k(y) = \frac{1}{m} \sum_{k=1}^{m} \left( \frac{m+k-1}{m} \right) \gamma_k.$$ 

Therefore, for every $x, m$ and $f$, we have

$$(3.1) \quad S_mf(x) = \int_{\mathbb{Z}_p} D_m(x - y)f(y)\,dy = D_m * f(x).$$

A $p$-adic polynomial of degree $n$ is a function of the form

$$P(x) = a_1\gamma_1(x) + a_2\gamma_2(x) + \cdots + a_n\gamma_n(x),$$ 

where $a_i$’s are scalars.

In general, $D_m(x)$ is not easy to compute for every $m \in \mathbb{N}$. However, if $m = p^r$, $r \in \mathbb{N}$, then the task is easier. In fact, we have (see [3])

$$D_{p^r}(x) = p^r1_{p^r}(x), \quad x \in \mathbb{Z}_p.$$ 

From (3.1), for every $x \in \mathbb{Z}_p$ and $r \geq 0$, we have

$$S_{p^r}f(x) = \frac{1}{|x + p^r|} \int_{x + p^r} f(y)\,dy.$$
By Lebesgue differentiation theorem (see Theorem 2.3), $S_p f(x) \to f(x)$ for almost every $x \in \mathbb{Z}_p$. As we mentioned before, this is not true for Fourier series in $\mathbb{T}$.

3.3. **Construction of the example.** We now come to our construction of a function $\Psi$ on $\mathbb{Z}_p$ whose Fourier series diverges almost everywhere. The basic idea is to first construct a sequence $\{\psi_k\}_{k \in \mathbb{N}}$ of pathological functions in $L^1(\mathbb{Z}_p)$ close to having divergent Fourier series. More precisely, on a subset $\Omega_k$ of $\mathbb{Z}_p$ of large measure, in the sense that the measure of $\Omega_k$ approaches 1, the function $\psi_k$ must have a maximal Fourier partial sum that is large as a function of $k$. Then a suitable infinite linear combination of them will give the desired function. For this, we also need some control over the $L^1$-norm of $\psi_k$’s.

For the construction of the functions $\psi_k$, we use the following crucial property of the Dirichlet kernels. This is proved in Lemma 2.4 in [4].

**Lemma 3.2.** Let $y$ be an element in $\mathbb{Z}_p$ for which $y_{s+1} = y_{s+2} = \cdots = y_{s+n} = 0$, where $s$ is a nonnegative integer and $n$ is an integer exceeding 2. Then there exists an integer $m$ with $p^{s+n} < m < p^{s+n+1}$ such that $|D_m(y)| \geq \left(\frac{n}{p^n} - 1\right)^{1/2p}$.

Let $X_0$ be the set of all $y \in \mathbb{Z}_p$ with the following property: For every $n \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that $y_{s+1} = y_{s+2} = \cdots = y_{s+n} = 0$. Then $X_0$ is almost all of $\mathbb{Z}_p$, that is, $|X_0| = 1$ (see Lemma 3.1 in [4]). Let $\delta_0$ be the Dirac mass at the origin. Since $S_n\delta_0(x) = D_n(x)$, in view of Lemma 3.2 we observe that $\mathfrak{M}\delta_0$ is unbounded almost everywhere on $\mathbb{Z}_p$, i.e.,

$$\mathfrak{M}\delta_0(x) = \sup_n |S_n\delta_0(x)| = \sup_n |D_n(x)| > p^k$$

for almost every $x \in \mathbb{Z}_p$ and for each $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}$, there exists an integer $N_k$ and a set $\Omega_k \subset \mathbb{Z}_p$ with $|\Omega_k| > 1 - \frac{1}{p^k}$ such that

$$\sup_{n \leq N_k} |S_n\delta_0(x)| > p^k \quad \text{for} \quad x \in \Omega_k. \quad (3.2)$$

Note that the sequence of integers $\{N_k\}_{k \in \mathbb{N}}$ is non-decreasing. Now, we set

$$\psi_k = \frac{1}{k^2} (\delta_0 * L_{N_k}),$$

where $L_n = 2K_{2n} - K_n$ and $K_n$ is the $n$-th Fejér kernel, $n \in \mathbb{N}$.

Observe that $\psi_k$ is a $p$-adic polynomial of degree $2N_k$ and $\|\psi_k\|_{L^1(\mathbb{Z}_p)} \leq 3/k^2$. For our convenience, we write

$$\psi_k(x) = \sum_{i=1}^{N_k} a_i^k \gamma_i(x), \quad k \in \mathbb{N}.$$ 

These $\psi_k$’s will be the required pathological functions for our purpose.

Now, we define a sequence of characters $\{\gamma_k\}_{k \in \mathbb{N}}$ to form the linear combination mentioned earlier. Put $\gamma_1 = \gamma_1$. Observe that there exists a character $\gamma$ and integers $r_2 < r_2'$ such that the set $\{\gamma \cdot \gamma_1, \gamma \cdot \gamma_2, \ldots, \gamma \cdot \gamma_{2N_2}\}$ is a subset of $\{\gamma_{r_2}, \gamma_{r_2+1}, \ldots, \gamma_{r_2'}\}$ and $2N_1 < r_2$. Otherwise, for each $\gamma$ in $\Gamma_p$, there would be an integer $t$ (depending on $\gamma$) in $\{1, 2, \ldots, 2N_2\}$ such that $\gamma \cdot \gamma_t$ is in $\{\gamma_1, \gamma_2, \ldots, \gamma_{2N_1}\}$. Since $\Gamma_p$ is infinite, this is not possible. We choose any such character and call it $\gamma_k$. By the same argument, there exists a character $\gamma$ and integers $r_3 < r_3'$ such that the set $\{\gamma \cdot \gamma_{r_3}, \gamma \cdot \gamma_{r_3+1}, \ldots, \gamma_{r_3'}\}$ is a subset of $\{\gamma_{r_3}, \gamma_{r_3+1}, \ldots, \gamma_{r_3'}\}$ and $r_2' < r_3$. Call any such character $\gamma_l$ and continue the process.

We define a function on $\mathbb{Z}_p$ by $\Psi = \sum_{k=1}^{\infty} \gamma_k \psi_k$. Then

$$\Psi = \sum_{k=1}^{\infty} \sum_{i=1}^{2N_k} a_i^k (\gamma_i \cdot \gamma_k).$$

Note that $\Psi$ is an integrable function since

$$\|\Psi\|_{L^1(\mathbb{Z}_p)} \leq \sum_{k=1}^{\infty} \|\psi_k\|_{L^1(\mathbb{Z}_p)} \leq 3 \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$
We now prove that the Fourier series of $\Psi$ diverges almost everywhere on $\mathbb{Z}_p$. Let us first fix an integer $n$. By the construction of the sequence $\{\gamma_k\}_{k \in \mathbb{N}}$, we have
$$\{\gamma_i \cdot \gamma_{n-1} : i = 1, 2, \ldots, 2N_{n-1}\} \cap \{\gamma_i \cdot \gamma_n : i = 1, 2, \ldots, 2N_n\} = \emptyset.$$ Therefore,
$$S_{r'} \Psi(x) - S_{r_{n-1}} \Psi(x) = \sum_{i=1}^{2N_n} a^n \gamma_i(x) \gamma_n(x) = \gamma_n(x) \psi_n(x).$$
Further, on account of $S_k \psi_n(x) = \psi_n(x)$ for any $k \geq 2N_n$, we get the following estimate:
$$|S_k \psi_n(x)| \leq 2 \sup_j |S_j \Psi(x)| \text{ for all } k \geq 2N_n.$$
Hence,
$$\mathcal{M} \Psi(x) \geq \frac{1}{2} \mathcal{M} \psi_n(x), \quad n \in \mathbb{N}.$$ But $\sup_k \mathcal{M} \psi_k(x) = \infty$ for almost every $x \in \mathbb{Z}_p$. This may be seen as follows. First, we set $\Omega = \cap_m \cup_{m < p^k} \Omega_k$ so that $|\Omega| = 1$. Next, note that $\hat{L}_n(j) = 1$ for $j \leq n$. Using this in (3.2), we obtain
$$\mathcal{M} \psi_k(x) \geq \sup_{n \leq N_k} |S_n \psi_k(x)| = \frac{1}{k^2} \sup_{n \leq N_k} |S_n \delta_0(x)| > \frac{p^k}{k^2} \text{ on } \Omega_k.$$ This gives $\sup_k \mathcal{M} \psi_k(x) = \infty$ on $\Omega$ and hence the Fourier series of $\Psi$ diverges almost everywhere on $\mathbb{Z}_p$.

4. Pointwise convergence of Fourier series

For the moment let us take Theorem 1.2 for granted and proceed to prove the almost everywhere pointwise convergence result. Consider the space $\mathcal{S} = \mathcal{S}(\mathfrak{D})$ of test functions on $\mathfrak{D}$ consisting of locally constant functions. A function $\phi$ on $\mathfrak{D}$ is said to be \textit{locally constant} if there exists $k \in \mathbb{N}_0$ such that $\phi(x + u) = \phi(x)$ for all $x \in \mathfrak{D}$ and for all $u \in \mathfrak{D}^k$. In other words, $\phi$ is constant on cosets of $\mathfrak{D}^k$. By Theorem 2.1, such a function is a finite linear combination of the characteristic functions of the form $\chi_{x + p^k \mathfrak{D}}$, $x \in \mathfrak{D}$ and $k \in \mathbb{N}_0$.

Let $\phi \in \mathcal{S}$ so that $\phi$ is constant on cosets of $\mathfrak{D}^k$ for some $k \in \mathbb{N}_0$. Consider $\phi$ as a function on $K$ and supported on $\mathfrak{D} = \mathfrak{D}^k$. Then by Theorem 2.2, $\hat{\phi} \in \mathcal{S}$ is constant on cosets of $\mathfrak{D}$ and is supported on $\mathfrak{D}^{1-k}$. Hence, $\hat{\phi}(u(n)) = 0$ if $|u(n)| > q^k$, that is, if $n \geq q^k$ (see Proposition 2.3(a)). Hence,
$$\phi(x) = \sum_{|u(n)| \leq q^k} \hat{\phi}(u(n)) \chi_n(x) = \sum_{n=0}^{q^k-1} \hat{\phi}(u(n)) \chi_n(x) = S_{q^k} \phi(x).$$ Therefore, $\phi(x) = S_m \phi(x)$ for all $m \geq q^k$.

The space $\mathcal{S}$ is dense in $L^p(\mathfrak{D}, w)$ for $w \in A_p$, $1 < p < \infty$. Indeed, by Stone-Weierstrass theorem, $\mathcal{S}$ is dense in $C(\mathfrak{D})$, the space of continuous functions on $\mathfrak{D}$. Also, the space $C(\mathfrak{D})$ is dense in $L^p(\mathfrak{D}, w)$. For a proof of the density of $C(\mathfrak{D})$ in $L^p(\mathfrak{D}, w)$, see the proof of Theorem 1.1 in [22]. Then by a standard argument, it follows that $\mathcal{S}$ is dense in $L^p(\mathfrak{D}, w)$. We observed that if $\phi \in \mathcal{S}$, then $S_n \phi = \phi$ for large $n$. Using these two facts, the proof of a.e. pointwise convergence result for functions in $L^p(\mathfrak{D}, w)$ is essentially the same as the proof of Carleson-Hunt theorem using the boundedness of $\mathcal{M}$ on $L^p(\mathfrak{D})$ and can be omitted (see Corollary 1.2, Chapter VIII in [28]).

We now prove Theorem 1.2.

Proof of Theorem 1.2. Throughout the proof, a function on $\mathfrak{D}$ will be treated as a function defined on $K$ but supported on $\mathfrak{D}$. Let $1 < p < \infty$, $w \in A_p$ and $f \in L^p(\mathfrak{D}, w)$. Observe that the $A_p$ condition of the weight $w$ together with Hölder’s inequality immediately implies that $f \in L^1(\mathfrak{D})$ so that $S_n f(x)$ makes sense for all $n \in \mathbb{N}$.

There are several stages in the proof of inequality (1.1). First we define the modified Dirichlet kernels and the corresponding modified partial sum operators as follows:
$$D_n^* = \mathcal{M} D_n \quad \text{and} \quad S_n^* f = D_n^* \ast f, \quad n \in \mathbb{N},$$
where the integration that defines the convolution is over all of $K$. Let $\Phi_0 = 1_D$. Therefore, by our convention, $D_n^* = \Phi_0 D_n^*$ is the kernel of the operator $S_n^*$. We denote it by $k_n$. Then we observe that $S_n^* f$ is supported on $\mathcal{D}$ for $n \in \mathbb{N}$. Next, by an easy calculation, we see that the usual partial sum operators $S_n$ and the modified partial sum operators $S_n^*$ are related by the equation

$$
S_n f = \chi_n (S_n^* (\chi_n f)), \quad n \in \mathbb{N}.
$$

Define the operator $\mathcal{M}$ by $\mathcal{M} f(x) = \sup_n |S_n^* f(x)|$. Therefore, in order to prove Theorem 1.2 it suffices to show that there exists $C_p > 0$ such that

$$
(4.1) \quad \int_D |\mathcal{M} f(x)|^p w(x) \, dx \leq C_p \int_D |f(x)|^p w(x) \, dx
$$

for all $f \in L^p(D, w)$.

It is proved in [15] (see also [28]) that $\mathcal{M}$ is bounded on $L^p(D)$. Using this and the reverse Hölder inequality for $A_p$ weights ([22] and [23]), it is easy to see that $\mathcal{M} f \in L^p(D, w)$. Since $S_n^* f$ are supported on $\mathcal{D}$, so is $\mathcal{M} f$ and hence $\mathcal{M} f$ also lies in $L^p(w)$.

We apply Theorem 2.4 and Lemma 2.2 to $\mathcal{M} f$ and obtain

$$
\|M(\mathcal{M} f)\|_{L^p(w)} \leq C \|\mathcal{M} f\|_{L^p(w)}
$$

and

$$
\|M(\mathcal{M} f)\|_{L^p(w)} \leq C \|\mathcal{M} f\|^2_{L^p(w)}.
$$

These two inequalities, along with the fact that $\mathcal{M} f(x) \leq M(\mathcal{M} f)(x)$ for almost every $x$, yield

$$
\int_D |\mathcal{M} f(x)|^p w(x) \, dx \leq C \int_D (\mathcal{M} f(x))^{p-2} (\mathcal{M} f(x))^2 w(x) \, dx.
$$

Finally, in order to estimate the last integral, we dominate $(\mathcal{M} f)^2$ pointwise a.e. by the function $M_r f$ as follows.

**Proposition 4.1.** Let $1 < r < \infty$. There is a constant $C > 0$, depending only on $r$, such that for any $f \in L^r(D, w)$

$$
(\mathcal{M} f)^2(x) \leq CM_r f(x) \quad \text{for a.e.} \ x.
$$

We postpone the proof of this proposition and continue with the proof of inequality (4.1). We recall an important property of the $A_p$ weights, see [22] and [23].

**Lemma 4.1.** Suppose $w \in A_p$, $1 < p < \infty$. Then there exists $s$ with $1 < s < p$ such that $w \in A_s$.

Taking the $s$ obtained from this lemma, we choose $r = \frac{p}{s}$. Then $\frac{p}{s} = s > 1$ and $w \in A_{\frac{p}{s}}$. Hence, by Proposition 4.1 we have

$$
\int_D |\mathcal{M} f(x)|^p w(x) \, dx \leq C \int_K ((\mathcal{M} f)^2(x))^{p-2} (\mathcal{M} f(x))^2 w(x) \, dx
$$

$$
\leq C_{r,p} \int_K M_r f(x)^p w(x) \, dx
$$

$$
= C_{r,p} \int_K (M|f|^r(x))^\frac{p}{r} w(x) \, dx.
$$

Since $|f|^r \in L^\frac{p}{s}(w)$ and $w \in A_{\frac{p}{s}}$, we apply Theorem 2.4 to obtain

$$
\int_D |\mathcal{M} f(x)|^p w(x) \, dx \leq C \int_D |f(x)|^p w(x) \, dx.
$$

Therefore, to complete the proof of (4.1), all that remains is to prove Proposition 4.1. We first observe that

$$
(4.2) \quad \frac{1}{2} \|f^2\|_{\infty} \leq \sup_B \inf_{\alpha \in \mathbb{C}} \frac{1}{|B|} \int_B |f(x) - \alpha| \, dx.
$$

Indeed, first we note that, for all $\alpha \in \mathbb{C}$,

$$
\int_B |f(x) - f_B| \, dx \leq \int_B |f(x) - \alpha| \, dx + \int_B |\alpha - f_B| \, dx \leq 2 \int_B |f(x) - \alpha| \, dx.
$$
Dividing both sides by $|B|$, taking the infimum over $\alpha \in \mathbb{C}$ and then supremum over all balls $B$, we get (4.2).

Now, fix $r > 1$ and assume $f \in L^r(\mathcal{D}, w)$. Let $x \in K$ and $B$ be any ball containing $x$. In view of (4.2), it is enough to show that there exists a constant $\alpha$ depending on the ball $B$ such that

$$\frac{1}{|B|} \int_B |\mathfrak{M}^* f(y) - \alpha| \, dy \leq CM_r f(x). \tag{4.3}$$

By Proposition 2.1(a), $B = \{y \in K : |y - x| < q\}$ for some $l \in \mathbb{Z}$. We write $f_1 = f \cdot 1_B$ and $f_2 = f - f_1$ and set $\alpha = \mathfrak{M}^* f_2(x)$. Then for any $y \in B$,

$$|\mathfrak{M}^* f(y) - \mathfrak{M}^* f_2(x)| = \left| \sup_n |S_n^* f(y)| - \sup_n |S_n^* f_2(x)| \right| \leq \sup_n |S_n^* f(y) - S_n^* f_2(x)| \leq \sup_n |S_n^* f_1(y)| + \sup_n |S_n^* f_2(y) - S_n^* f_2(x)| \tag{4.4}$$

For the first term of (4.4), we use the boundedness of $\mathfrak{M}^*$ on $L^r(\mathcal{D})$ and Hölder’s inequality to get

$$\frac{1}{|B|} \int_B |\mathfrak{M}^* f_1(y)| \, dy \leq \frac{1}{q^{1/r}} \left( \int_{|y-x|<q} |\mathfrak{M}^* f_1(y)|^r \, dy \right)^{\frac{1}{r}} \cdot (q^{1-1/r} \frac{1}{2})^{\frac{1}{r}} \leq \frac{C}{q^{1/r}} \left( \int_{|y-x|<q} |f(y)|^r \, dy \right)^{\frac{1}{r}} \cdot (q^{1-1/r} \frac{1}{2})^{\frac{1}{r}} \leq CM_r f(x).$$

Now, integrating the second term of (4.4) over the ball $B$, we get

$$\frac{1}{|B|} \int_B \sup_n |S_n^* f_2(y) - S_n^* f_2(x)| \, dy$$

$$= \frac{1}{q^{1/r}} \int_{|y-x|<q} \sup_n \int_K |k_n(y - z) - k_n(x - z)| f_2(z) \, dz \, dy$$

$$\leq \frac{1}{q^{1/r}} \int_{|y-x|<q} \sup_n \int_{|z-x|\geq q} |k_n(y - z) - k_n(x - z)| \, dz \, dy.$$ 

We now show that the above integral is zero. This will follow once we show that

$$k_n(x-y) = k_n(x) \quad \text{for all } x, y \in K, |y| < |x|.$$ 

To see this, observe that since the kernels $k_n$ are supported on $\mathcal{D}$ and the norm is ultrametric, we only need to consider $x, y \in \mathcal{D}$ such that $|y| < |x|$. In the proof of Theorem 4 in [27], it was shown that $\hat{k_n}(x+y) = \hat{k_n}(x)$ whenever $|y| < |x|$. For $m \in L^\infty$, set $m_k = m1_{\bar{\mathbb{Q}}^k}$, $k \in \mathbb{Z}$. We will now use the following lemma proved in [28] (see Lemma 2.1, Chapter VI).

**Lemma 4.2.** If $m(x+y) = m(x)$ whenever $|y| < q^{-s}|x|$, $s$ a non-negative integer, then for all $k \in \mathbb{Z}$, $\hat{m_k}(x+y) = \hat{m_k}(x)$, whenever $|y| < q^{-s}|x|$. 

Applying this lemma to $\Phi_0 \hat{k_n}$ for $s = 0$, we obtain $k_n(x+y) = k_n(x)$ whenever $|y| < |x|$ and $x, y \in \mathcal{D}$. Therefore, (4.3) holds with $\alpha = \mathfrak{M}^* f_2(x)$. This completes the proof Proposition 4.1.

**Remark 4.1.** In the case of the circle $\mathbb{T}$, Hunt and Young [10] proved the boundedness of $\mathfrak{M}$ on $L^p(\mathbb{T}, w)$ by verifying Burkholder-Gundy [2] type weighted distribution function inequality which relates a modified version of $\mathfrak{M}$ and the standard Hardy-Littlewood maximal function. Following their method, Gosselin [2] obtained a similar result for Vilenkin-Fourier series. A Vilenkin group $G$ is a direct product of cyclic groups of order $p_i$, where each $p_i$ is an integer greater than or equal to 2. In particular, if we take each $p_i$ to be equal to a fixed prime $p$, then $G$ becomes the ring of integers of the field $\mathbb{F}_p(X)$, the $p$-series field. Our approach is
different from that of \[10\] and \[7\] and hence it also gives another proof of the boundedness of \(M\) for the case of Vilenkin groups.

5. **Gabor systems, Zak transform and \(A_p\) weights on \(\mathfrak{D} \times \mathfrak{D}\)**

In this section we develop the results required to prove the characterization of Gabor Schauder bases. We first present some background information on Schauder basis, Gabor system, and Zak transform. Then we define the \(A_p\) weights on the product space \(\mathfrak{D} \times \mathfrak{D}\) in two different ways. We show the equivalence of these definitions by showing that each of them is equivalent to the boundedness of a maximal operator on \(\mathfrak{D} \times \mathfrak{D}\). Finally, we characterize the uniform boundedness of a sequence of partial sum operators on \(\mathfrak{D} \times \mathfrak{D}\).

5.1. **Schauder bases.** Let us recall some standard facts about bases in a Banach space \(\mathbb{B}\). A sequence \(\{x_k : k \in \mathbb{N}_0\}\) of elements of \(\mathbb{B}\) is called a Schauder basis for \(\mathbb{B}\) if for every \(x \in \mathbb{B}\) there exists a unique sequence \(\{\alpha_k : k \in \mathbb{N}_0\}\) of scalars such that

\[
x = \sum_{k \in \mathbb{N}_0} \alpha_k x_k,
\]

where the partial sums of the series converge in the norm of \(\mathbb{B}\), that is,

\[
\lim_{N \to \infty} \left\| x - \sum_{k=0}^{N} \alpha_k x_k \right\| = 0.
\]

The sequence \(\{x_k : k \in \mathbb{N}_0\}\) is said to be complete if \(\text{span}\{x_k : k \in \mathbb{N}_0, k \neq n\} = \mathbb{B}\). It is said to be minimal if \(x_n \not\in \text{span}\{x_k : k \in \mathbb{N}_0, k \neq n\}\) for all \(n \in \mathbb{N}_0\).

Let \(\{x_n : n \in \mathbb{N}_0\}\) be a sequence in a Hilbert space \(\mathbb{H}\). A sequence \(\{\bar{x}_n : n \in \mathbb{N}_0\}\) in \(\mathbb{H}\) is said to be biorthogonal to \(\{x_n : n \in \mathbb{N}_0\}\) if \(\langle x_k, \bar{x}_l \rangle = \delta_{k,l}\) for all \(k, l \in \mathbb{N}_0\). It is easy to verify that if \(\{x_n : n \in \mathbb{N}_0\}\) is complete in \(\mathbb{H}\), then there is a unique sequence \(\{\bar{x}_n : n \in \mathbb{N}_0\} \subset \mathbb{H}\) which is biorthogonal to \(\{x_n : n \in \mathbb{N}_0\}\). Such a sequence is called the biorthogonal dual of \(\{x_n : n \in \mathbb{N}_0\}\) and vice versa. A dual sequence is not necessarily uniquely defined. In fact, it is unique if and only if the original sequence is complete in \(\mathbb{H}\). Therefore, every Schauder basis has a unique biorthogonal dual.

In general, a complete sequence need not be a Schauder basis for a Hilbert space. The following theorem determines when this is the case. For a proof, we refer to \[25\] (see Theorem 4.1, Chapter 1).

**Theorem 5.1.** A complete sequence \(\{x_n : n \in \mathbb{N}_0\}\) with biorthogonal dual \(\{\bar{x}_n : n \in \mathbb{N}_0\}\) is a Schauder basis for \(\mathbb{H}\) if and only if the partial sum operators \(s_n, n \geq 0\), defined by

\[
s_n(x) = \sum_{k=0}^{n} \langle x, \bar{x}_k \rangle x_k,
\]

are uniformly bounded in \(\mathbb{H}\).

5.2. **Gabor systems and Zak transform on local fields.** We defined the translation operators (time shifts) \(T_y\) and the modulation operators (frequency shifts) \(M_\xi\) in Section 1. Their compositions \(T_y M_\xi\) and \(M_\xi T_y\) are called time-frequency shift operators.

Let \(K\) be a local field of positive characteristic. Fix \(a, b \in K\) and \(g \in L^2(K)\). The Gabor system \(\mathcal{G}(g, a, b)\) is the collection of functions

\[
\mathcal{G}(g, a, b) = \{ M_{bu(n)} T_{au(k)} g : n, k \in \mathbb{N}_0\}.
\]

The function \(g\) is called the window function or atom. In this article, we are concerned about the Schauder basis property of the system

\[
\mathcal{G}(g) = \mathcal{G}(g, 1, 1) = \{ M_{u(n)} T_{u(k)} g : n, k \in \mathbb{N}_0\}.
\]

As mentioned earlier, Weil \[20\] introduced the concept of Zak transform on locally compact abelian groups and formulated its basic properties. For the definition of Zak transform and derivation of its properties on certain locally compact nonabelian groups, we refer to \[20\]. In order to define Zak transform on a locally compact abelian group \(G\), we need a lattice. Recall that a discrete subgroup \(D\) of \(G\) is called a lattice if the quotient \(G/D\) is a compact group.
Since the characteristic of $K$ is positive, it follows that $\Lambda = \{ u(k) : k \in \mathbb{N}_0 \}$ is a subgroup of $K^+$ (see Proposition 2.3), and hence is a lattice in $K^+$. With respect this lattice, we make the following definition.

**Definition 5.1.** Let $K$ be a local field of positive characteristic. The Zak transform of a function $f \in L^2(K)$ is the function of two variables defined by

$$ Zf(x, \xi) = \sum_{k \in \mathbb{N}_0} T_{u(k)} f(x) \chi_k(\xi) = \sum_{k \in \mathbb{N}_0} f(x - u(k)) \chi(u(k)\xi), \quad x, \xi \in K. $$

Using the $\Lambda$-periodicity of the characters $\chi_k$, we can show that $|Zf|$ is $\Lambda$-periodic in both the variables. It turns out that $Z$ maps $L^2(K)$ isometrically onto $L^2(\mathfrak{D} \times \mathfrak{D})$. The proof of this fact can be obtained from the corresponding result on Euclidean spaces with necessary modifications, see e.g. [13].

For $n, k \in \mathbb{N}_0$, define

$$ E_{n,k}(x, \xi) = \chi_n(x)\overline{\chi_k(\xi)} = \chi(u(n)x - u(k)\xi). $$

The following theorem shows that the Zak transform diagonalizes the time-frequency shifts.

**Theorem 5.2.** Let $g \in L^2(K)$. Then

$$ Z(M_{u(n)}T_{u(k)}g)(x, \xi) = (E_{n,k} \cdot Zg)(x, \xi) = E_{n,k}(x, \xi) \cdot Zg(x, \xi). $$

**Proof.** This is a straightforward verification. \( \square \)

If $\mathcal{G}(g)$ is a Schauder basis for $L^2(K)$, then using Theorem 5.2, we can show that the biorthogonal system is of the form $\mathcal{G}(\tilde{g})$, where the dual window $\tilde{g} \in L^2(K)$ is defined by the condition $Z\tilde{g} = 1/\sqrt{g}$. Indeed, since $Z$ is an isometry, we have

\[
\langle M_{u(n)}T_{u(k)}g, M_{u(m)}T_{u(l)}\tilde{g} \rangle = \langle Z(M_{u(n)}T_{u(k)}g), Z(M_{u(m)}T_{u(l)}\tilde{g}) \rangle = \langle E_{n,k} \cdot Zg, E_{m,l} \cdot Z\tilde{g} \rangle = \int_{\mathfrak{D}} \int_{\mathfrak{D}} \chi_n(x)\overline{\chi_k(\xi)}Zg(x, \xi)\chi_m(x)\overline{\chi_l(\xi)}Z\tilde{g}(x, \xi) \, dx \, d\xi
\]

\[
= \langle \chi_n, \chi_m \rangle \langle \chi_k, \chi_l \rangle = \delta_{n,m}\delta_{k,l}.
\]

Therefore, $\mathcal{G}(\tilde{g})$ is biorthogonal to $\mathcal{G}(g)$. Since a Schauder basis has a unique biorthogonal dual, it follows that $\mathcal{G}(\tilde{g})$ is the biorthogonal dual of $\mathcal{G}(g)$.

**5.3. $A_p$ weights on the product space $\mathfrak{D} \times \mathfrak{D}$.** Let $w$ be a non-negative function on $\mathfrak{D} \times \mathfrak{D}$. Following Fefferman and Stein [5], we say that $w \in A_p(\mathfrak{D} \times \mathfrak{D})$ if for a.e. $y \in \mathfrak{D}$, the function $x \mapsto w(x, y)$ is an $A_p(\mathfrak{D})$ weight and the $A_p$ characteristic $[w(\cdot, y)]_{A_p}$ is independent of $y$, and a similar condition holds for the function $y \mapsto w(x, y)$ for a.e. $x \in \mathfrak{D}$.

That is, $w \in A_p(\mathfrak{D} \times \mathfrak{D})$ if there exists a constant $C > 0$ such that for a.e. $y \in \mathfrak{D}$ and all balls $B \subset \mathfrak{D}$

\[
(\frac{1}{|B|} \int_B w(x, y) \, dx) \left( \frac{1}{|B|} \int_S w(x, y)^{-\frac{1}{p-1}} \, dx \right)^{p-1} \leq C,
\]

and for a.e. $x \in \mathfrak{D}$ and all balls $B \subset \mathfrak{D}$

\[
(\frac{1}{|B|} \int_B w(x, y) \, dy) \left( \frac{1}{|B|} \int_S w(x, y)^{-\frac{1}{p-1}} \, dy \right)^{p-1} \leq C.
\]

We will show that the above definition is equivalent to the existence of a constant $C > 0$ such that for all balls $B_1, B_2 \subset \mathfrak{D}$, we have

\[
\left( \frac{1}{|B_1| |B_2|} \int_{B_1} \int_{B_2} w(x, y) \, dx \, dy \right)
\]

\[
\times \left( \frac{1}{|B_1| |B_2|} \int_{B_1} \int_{B_2} w(x, y)^{-\frac{1}{p-1}} \, dx \, dy \right)^{p-1} \leq C.
\]

Let us temporarily write $w \in A_{p,\ast}(\mathfrak{D} \times \mathfrak{D})$ if the weight $w$ satisfies (5.5) for all balls $B_1, B_2 \subset \mathfrak{D}$. Also, let us denote the supremum of the left hand side in (5.5) taken over all balls $B_1, B_2 \subset \mathfrak{D}$.
\( \mathcal{D} \) by \([w]_{A_p,*}\). We will now show that both these definitions of weights on \( \mathcal{D} \times \mathcal{D} \) are equivalent to the boundedness on \( L^p(\mathcal{D} \times \mathcal{D}, w) \) of the Hardy-Littlewood maximal operator \( M_w \) adapted to this definition.

For a function \( f \) on \( \mathcal{D} \), we define the maximal function \( M_w f \) as follows:

\[
M_w f(x,y) = \sup_{x \in B_1, y \in B_2} \frac{1}{|B_1| |B_2|} \int_{B_1} \int_{B_2} |f(u,v)| \, du \, dv,
\]

where the supremum is taken over all balls \( B_1 \) in \( \mathcal{D} \) containing \( x \) and \( B_2 \) in \( \mathcal{D} \) containing \( y \).

**Theorem 5.3.** Let \( w \) be a weight on \( \mathcal{D} \times \mathcal{D} \) and \( 1 < p < \infty \). Then the following are equivalent.

(a) \( w \in A_{p,*}(\mathcal{D} \times \mathcal{D}) \).

(b) \( w \in A_p(\mathcal{D} \times \mathcal{D}) \).

(c) \( M_w \) is a bounded operator on \( L^p(\mathcal{D} \times \mathcal{D}, w) \).

**Proof.** Suppose (a) holds. Then for all balls \( B_1, B_2 \subset \mathcal{D} \), we have

\[
\left( \frac{1}{|B_1| |B_2|} \int_{B_1} \int_{B_2} w(x,y) \, dx \, dy \right)^{\frac{1}{p}} \leq [w]_{A_{p,*}}.
\]

(5.6)

Fix a ball \( B_1 \subset \mathcal{D} \). By Lebesgue differentiation theorem (Theorem 2.3), for a.e. \( y \), we have

\[
\frac{1}{|x + \mathcal{P}^k|} \int_{x + \mathcal{P}^k} \left( \frac{1}{|B_1|} \int_{B_1} w(x,z) \, dx \right) \, dz \to \frac{1}{|B_1|} \int_{B_1} w(x,y) \, dx
\]
as \( k \to \infty \). Similarly,

\[
\left( \frac{1}{|x + \mathcal{P}^k|} \int_{x + \mathcal{P}^k} \left( \frac{1}{|B_1|} \int_{B_1} w(x,z)^{-\frac{1}{p-1}} \, dx \right) \, dz \right)^{p-1}
\]

\[
\to \left( \frac{1}{|B_1|} \int_{B_1} w(x,y)^{-\frac{1}{p-1}} \, dx \right)^{p-1}
\]
as \( k \to \infty \). Therefore, the product of the left sides in the last two equations converge to the product of the right sides. By (5.6) the product of the left sides is at most [\( w \)]_{A_{p,*}}. Hence,

\[
\left( \frac{1}{|B_1|} \int_{B_1} w(x,y) \, dx \right) \left( \frac{1}{|B_1|} \int_{B_1} w(x,y)^{-\frac{1}{p-1}} \, dx \right)^{p-1} \leq [w]_{A_{p,*}}.
\]

This is true for every ball \( B_1 \in \mathcal{D} \) and a.e. \( y \in \mathcal{D} \). Hence, \( w(\cdot,y) \in A_p(\mathcal{D}) \) and the \( A_p \) characteristic \([w(\cdot,y)]_{A_p}\) is independent of \( y \). Similarly, \( w(x,\cdot) \in A_p(\mathcal{D}) \) and \([w(x,\cdot)]_{A_p}\) is independent of \( x \). Therefore, \( w \in A_p(\mathcal{D} \times \mathcal{D}) \). So (b) is proved.

Assume (b). Define the Hardy-Littlewood maximal operators \( M_1 \) and \( M_2 \) corresponding to the first and second variables:

\[
M_1 f(x,y) = \sup_{B} \frac{1}{|B|} \int_{B} |f(s,y)| \, ds
\]

and

\[
M_2 f(x,y) = \sup_{B} \frac{1}{|B|} \int_{B} |f(x,t)| \, dt.
\]

Since \( w(x,\cdot) \) and \( (\cdot,y) \) are in \( A_p(\mathcal{D}) \), the operators \( M_1 \) and \( M_2 \) are bounded on \( L^p(\mathcal{D}, w(\cdot,y)) \) and \( L^p(\mathcal{D}, w(x,\cdot)) \) respectively, by Theorem 2.4. Clearly, \( M_w f(x,y) \leq M_1 \circ M_2 f(x,y) \). Since the \( A_p \) characteristic of \( w(x,\cdot) \) and \( w(\cdot,y) \) have uniform bound, by applying Fubini’s theorem, we get

\[
\int_{\mathcal{D}} \int_{\mathcal{D}} |M_w f(x,y)|^p w(x,y) \, dx \, dy \leq \int_{\mathcal{D}} \int_{\mathcal{D}} |M_1 \circ M_2 f(x,y)|^p w(x,y) \, dx \, dy
\]

\[
\leq C \int_{\mathcal{D}} \int_{\mathcal{D}} |M_2 f(x,y)|^p w(x,y) \, dx \, dy
\]

\[
\leq C^2 \int_{\mathcal{D}} \int_{\mathcal{D}} |f(x,y)|^p w(x,y) \, dx \, dy.
\]

This proves (c).
Now, assume that (c) is true. Let $B_1$ and $B_2$ be balls in $\mathcal{D}$. Then for each $(x, y) \in B_1 \times B_2$, we have

$$M_s(f \cdot 1_{B_1 \times B_2})(x, y) \geq \frac{1}{|B_1|} \frac{1}{|B_2|} \int_{B_1} \int_{B_2} |f(u, v)| \, du \, dv.$$ 

Hence, for any $f$

$$\left( \frac{1}{|B_1|} \frac{1}{|B_2|} \int_{B_1} \int_{B_2} |f(u, v)| \, du \, dv \right)^p \int_{B_1} \int_{B_2} w(x, y) \, dx \, dy \leq \int_{B_1} \int_{B_2} \|M_s(f \cdot 1_{B_1 \times B_2})(x, y)\|^p w(x, y) \, dx \, dy \leq C \int_{B_1} \int_{B_2} |f(x, y)|^p w(x, y) \, dx \, dy.$$

Taking $f = w^{-\frac{1}{p-1}}$, we see that (5.5) is satisfied so that we obtain (a). This completes the proof of the theorem. \hfill \Box

**Remark 5.1.**
(a) In view of Theorem 5.3 we say that $w \in A_p(\mathcal{D} \times \mathcal{D})$ if $w$ satisfies either (5.3) and (5.4) or (5.5).

(b) In [21], several characterizations are provided for the $A_p$ weights on $\mathbb{T}^n$ in terms of strong maximal functions, rectangular conjugate functions, and rectangular partial sums.

For $M, N \in \mathbb{N}_0$, define

$$S_{M,N}F = \sum_{m=0}^{M} \sum_{n=0}^{N} \langle F, E_{m,n} \rangle E_{m,n},$$

where $E_{n,k}(x, \xi) = \chi_n(x)\chi_k(\xi)$ and the inner product is in $L^2(\mathcal{D} \times \mathcal{D})$.

Let $T : L^p(\mathcal{D} \times \mathcal{D}, w) \to L^p(\mathcal{D} \times \mathcal{D}, w)$ be a bounded operator. The norm of this operator is denoted by $\|T\|_{p,w}$. Also, we denote $\|T\|_p$ to be the norm of the bounded operator $T : L^p(\mathcal{D} \times \mathcal{D}) \to L^p(\mathcal{D} \times \mathcal{D})$.

The following theorem characterizes the uniform boundedness of these operators on the weighted space $L^2(\mathcal{D} \times \mathcal{D}, w)$.

**Theorem 5.4.** Let $w$ be a non-negative function in $L^2(\mathcal{D} \times \mathcal{D})$. Then $\sup_{M,N} \|S_{M,N}\|_{2,w} < \infty$ if and only if $w \in A_2(\mathcal{D} \times \mathcal{D})$.

**Proof.** Suppose $C = \sup_{M,N} \|S_{M,N}\|_{2,w} < \infty$. In order to show that $w \in A_2(\mathcal{D} \times \mathcal{D})$, we will prove that

$$\left( \frac{1}{|B_1||B_2|} \int_{B_1} \int_{B_2} w \right) \left( \frac{1}{|B_1||B_2|} \int_{B_1} \int_{B_2} \frac{1}{w} \right) \leq C$$

for all balls $B_1, B_2 \subset \mathcal{D}$. Suppose $|B_1| = q^{-r}$ and $|B_2| = q^{-s}$, where $r, s \geq 0$. Choose $F$ to be non-negative on $B_1 \times B_2$ and 0 on $(\mathcal{D} \times \mathcal{D}) \setminus (B_1 \times B_2)$. Then extend $F$ $\Lambda \times \Lambda$-periodically.

Now, for $(x, \xi) \in B_1 \times B_2$, we have

$$S_{M,N}F(x, \xi) = \sum_{m=0}^{M} \sum_{n=0}^{N} \langle F, E_{m,n} \rangle E_{m,n}(x, \xi)$$

$$= \sum_{m=0}^{M} \sum_{n=0}^{N} \left[ \int_{B_1} \int_{B_2} F(y, \eta) \chi_m(y) \chi_n(\eta) \, dy \, dn \right] \chi_m(x) \chi_n(\xi)$$

$$= \int_{B_1} \int_{B_2} F(y, \eta) \left[ \sum_{m=0}^{M} \sum_{n=0}^{N} \chi_m(x-y) \chi_n(\eta-\xi) \right] \, dy \, dn$$

$$= \int_{B_1} \int_{B_2} F(y, \eta) D_{M+1}(x-y) D_{N+1}(\eta-\xi) \, dy \, dn,$$

where $D_m$ is the Dirichlet kernel defined in (2.23). Now, $B_1$ is a ball in $\mathcal{D}$ of measure $q^{-r}$. Since $\mathcal{D} = \bigcup_{i=0}^{q^{-r}-1} (a_i + \mathcal{Y}^r)$, we have $B_1 = a_i + \mathcal{Y}^r$ for some $a_i$, by Proposition 2.1. Since $x \in B_1$, we have $B_1 = x + \mathcal{Y}^r$ since every point of a ball is its centre (see Proposition 2.1). Also, we have
chosen $F$ to be supported on $B_1 \times B_2$ so that $y \in B_1$ in the above integral, that is, $y \in x + \mathfrak{P}^r$. Hence, $x - y \in \mathfrak{P}^r$. Similarly, $\eta - \xi \in \mathfrak{P}^r$. Now, we choose $M + 1 = q^r$ and $N + 1 = q^s$. By Lemma \[2.1\] $D_q = q^k 1_{\mathfrak{P}^k}$. Hence, we obtain

$$S_{q^r,q^s} F(x,\xi) = \int_{B_1} \int_{B_2} F(y,\eta) q^r q^s \, dy \, d\eta = \frac{1}{|B_1||B_2|} \int_{B_1} \int_{B_2} F(y,\eta) \, dy \, d\eta.$$ 

Therefore,

$$\frac{1}{|B_1|^2|B_2|^2} \left( \int_{B_1} \int_{B_2} F(y,\eta) \, dy \, d\eta \right)^2 \left( \int_{B_1} \int_{B_2} w(x,\xi) \, dx \, d\xi \right)$$

$$= \int_{B_1} \int_{B_2} \left| S_{q^r,q^s} F(x,\xi) \right|^2 w(x,\xi) \, dx \, d\xi$$

$$= \| S_{q^r,q^s} F \|_{L^2,w}^2 \leq C^2 \| F \|_{L^2,w}^2.$$ 

(5.7)

In particular, if $F$ is the $\Lambda \times \Lambda$-periodic extension of $\frac{1}{w} 1_{B_1 \times B_2}$, then

$$\| F \|_{L^2,w}^2 = \int_{B_1} \int_{B_2} \frac{1}{w^2} \, dx = \int_{B_1} \int_{B_2} \frac{1}{w}.$$ 

Hence,

$$\left( \frac{1}{|B_1|^2|B_2|^2} \int_{B_1} \int_{B_2} \frac{1}{w} \right) \left( \frac{1}{|B_1|^2|B_2|^2} \int_{B_1} \int_{B_2} \frac{1}{w} \right) \leq C^2 \int_{B_1} \int_{B_2} \frac{1}{w}.$$ 

From this it follows that if $\int_{B_1} \int_{B_2} \frac{1}{w} < \infty$, then

$$\left( \frac{1}{|B_1|^2|B_2|^2} \int_{B_1} \int_{B_2} \frac{1}{w} \right) \left( \frac{1}{|B_1|^2|B_2|^2} \int_{B_1} \int_{B_2} \frac{1}{w} \right) \leq C^2.$$ 

(5.8)

If $\int_{B_1} \int_{B_2} \frac{1}{w} = 0$, then (5.8) holds trivially. Finally, if $\int_{B_1} \int_{B_2} \frac{1}{w} = \infty$, then there exists $G \in L^2(B_1 \times B_2)$ such that $\frac{G}{w^{1/2}} \notin L^1(B_1 \times B_2)$. Let $F = \frac{|G|}{w^{1/2}}$. Then $\int_{B_1} \int_{B_2} F = \infty$ but $\| F \|_{L^2,w} = \| G \|_{L^2} < \infty$. So, from (5.7), we get $\int_{B_1} \int_{B_2} w = 0$. Hence, (5.8) holds in this case also. Therefore, $w \in A_2(\mathfrak{D} \times \mathfrak{D})$.

We will now prove the converse. Suppose $w \in A_2(\mathfrak{D} \times \mathfrak{D})$. Let $w_x = w(x,\cdot)$ and $w_\xi = w(\cdot,\xi)$. Since the $A_2$ characteristics of $w_x$ and $w_\xi$ are uniformly bounded, by Theorem \[1.3\] there exists $C > 0$ such that for a.e. $x, \xi$, we have

$$\int_{\mathfrak{D}} |S_N f(x)|^2 w_\xi(x) \, dx \leq C \int_{\mathfrak{D}} |f(x)|^2 w_\xi(x) \, dx, \quad f \in L^2(\mathfrak{D}, w_\xi)$$

and

$$\int_{\mathfrak{D}} |S_N f(\xi)|^2 w_x(\xi) \, d\xi \leq C \int_{\mathfrak{D}} |f(\xi)|^2 w_x(\xi) \, d\xi, \quad f \in L^2(\mathfrak{D}, w_x).$$

Now, let $F \in L^2(\mathfrak{D} \times \mathfrak{D}, w)$. By Fubini’s theorem, $F_x = F(x,\cdot) \in L^2(\mathfrak{D}, w_x)$ for a.e. $x$ and $F_\xi = F(\cdot,\xi) \in L^2(\mathfrak{D}, w_\xi)$ for a.e. $\xi$. Let

$$S_{N}^1 F(x,\xi) = S_N F_\xi(x) = \sum_{n=0}^{N} \langle F_\xi, \chi_n \rangle \chi_n(x)$$

and

$$S_{M}^2 F(x,\xi) = S_M F_x(\xi) = \sum_{m=0}^{M} \langle F_x, \chi_m \rangle \chi_m(\xi).$$

Then $S_N^1 S_M^2 F = S_{M,N} F$. Hence,

$$\| S_{M,N} F \|_{L^2,w}^2 = \int_{\mathfrak{D}} \int_{\mathfrak{D}} |S_N^1 S_M^2 F(x,\xi)|^2 w(x,\xi) \, dx \, d\xi$$

$$\leq C \int_{\mathfrak{D}} \int_{\mathfrak{D}} |S_M^2 F(x,\xi)|^2 w(x,\xi) \, dx \, d\xi$$

$$\leq C^2 \int_{\mathfrak{D}} \int_{\mathfrak{D}} |F(x,\xi)|^2 w(x,\xi) \, dx \, d\xi.$$
This completes the proof of the theorem. \hfill \square

We rewrite the above result in terms of the partial sum operators involving the windowed system $E_{m,n} \cdot (1/W)$.

**Corollary 5.1.** Let $W$ be a non-negative function in $L^2(\mathcal{D} \times \mathcal{D})$ and let $T_{M,N} : L^2(\mathcal{D} \times \mathcal{D}) \to L^2(\mathcal{D} \times \mathcal{D})$ be the operator

$$T_{M,N}F = \sum_{m=0}^{M} \sum_{n=0}^{N} \langle F, E_{m,n} \cdot \frac{1}{W} \rangle E_{m,n} \cdot W.$$ 

Then $\sup_{M,N} \|T_{M,N}\|_2 < \infty$ if and only if $|W|^2 \in A_2(\mathcal{D} \times \mathcal{D})$.

**Proof.** This is a reformulation of Theorem 5.4. We omit the proof since it is similar to that of Corollary 5.1. \hfill \square

### 6. An Application: Characterization of Gabor Schauder Bases

We consider the problem of determining whether the Gabor system $\mathcal{G}(g) = \{M_u(n)T_u(k)g : n, k \in \mathbb{N}_0\}$ is a Schauder basis for $L^2(K)$ or not. Since Schauder basis expansions may converge conditionally, the order of summation is important. The Gabor system $\mathcal{G}(g)$ involves two indices each in $\mathbb{N}_0$ so that we have to consider permutations of $\mathbb{N}_0 \times \mathbb{N}_0$. We define a family of permutations of $\mathbb{N}_0 \times \mathbb{N}_0$ which are compatible with the partial sum operators considered in the earlier sections. Following Heil and Powell \cite{12}, we define the following enumerations.

**Definition 6.1.** Let $\Gamma$ be the set of all enumerations $\{(k_j, n_j)\}_{j=1}^{\infty}$ of $\mathbb{N}_0 \times \mathbb{N}_0$ defined in the following recursive manner.

(a) The initial terms $(k_1, n_1), (k_2, n_2), \ldots, (k_{J_1}, n_{J_1})$ are either 

$$(0,0), (0,1), \ldots, (0,M_1) \quad \text{or} \quad (0,0), (1,0), \ldots, (N_1,0)$$

for some positive integers $M_1$ and $N_1$.

(b) If $\{(k_j, n_j)\}_{j=1}^{J_k}$ has been constructed to be of the form 

$$\{0,1,\ldots,M_k\} \times \{0,1,\ldots,N_k\}$$

for some positive integers $M_k$ and $N_k$, then terms are added to the top or the right side to obtain either the rectangle 

$$\{0,1,\ldots,M_k\} \times \{0,1,\ldots,N_k+1\}$$

or 

$$\{0,1,\ldots,M_k+1\} \times \{0,1,\ldots,N_k\}.$$ 

With respect to the above enumerations of $\mathbb{N}_0 \times \mathbb{N}_0$, we have the following result which characterizes the Schauder basis property of the Gabor system $\mathcal{G}(g)$ in terms of the Zak transform.

**Theorem 6.1.** Let $K$ be a local field of positive characteristic and $g \in L^2(K)$. A necessary and sufficient condition for the Gabor system $\mathcal{G}(g)$ to be a Schauder basis for $L^2(K)$ with respect to every enumeration $\sigma \in \Gamma$ is that $|Zg|^2 \in A_2(\mathcal{D} \times \mathcal{D})$.

**Proof.** We first prove the necessity of the condition. Let $\sigma = \{(k_j, n_j)\}_{j=1}^{\infty} \in \Gamma$ be an enumeration of $\mathbb{N}_0 \times \mathbb{N}_0$ and $\mathcal{G}(g)$ be a Schauder basis of $L^2(K)$ with respect to $\sigma$. Consider the partial sum operators $T_{N}^{\sigma} : L^2(\mathcal{D} \times \mathcal{D}) \to L^2(\mathcal{D} \times \mathcal{D})$ corresponding to $\sigma$, defined by 

$$T_{N}^{\sigma}F = \sum_{j=1}^{N} \langle F, E_{n_j,k_j} \cdot \frac{1}{Zg} \rangle E_{n_j,k_j} \cdot Zg,$$

where $E_{n,k}(x, \xi) = \chi_{n}(x)\chi_{k}(\xi)$ (see (5.2)). Then by Theorem 5.1 and Theorem 5.2 $|Zg|^2 \in L^2(\mathcal{D} \times \mathcal{D})$ and the operators $T_{N}^{\sigma}$ are uniformly bounded. Hence, by Corollary 5.1, $|Zg|^2 \in A_2(\mathcal{D} \times \mathcal{D})$.

We will now show the sufficiency. Let $|Zg|^2 \in A_2(\mathcal{D} \times \mathcal{D})$ and $\sigma \in \Gamma$ be an enumeration of $\mathbb{N}_0 \times \mathbb{N}_0$. Note that by the definition of the $A_2$ condition, it follows that $\frac{1}{Zg} \in L^2(\mathcal{D} \times \mathcal{D})$. 

**An Application: Characterization of Gabor Schauder Bases**
We now estimate the second term. We have

Let $M$ be the largest integer $M < N$ such that $T^\sigma_M F = T_{J,K} F$ for some integers $J, K$. Observe that

$$\|T^\sigma_M F\|_2 = \|T^\sigma_M F + T^\sigma_M F - T^\sigma_M F\|_2 \leq \|T_{J,K} F\|_2 + \|(T^\sigma_M - T^\sigma_M) F\|_2.$$  

We now estimate the second term. We have

$$(T^\sigma_M - T^\sigma_M) F = \sum_{j=M+1}^{N} \left< F, E_{n,j,k} \cdot \frac{1}{Zg} \right> E_{n,j,k} \cdot Zg.$$  

According to the specific nature of the enumerations $\sigma$ and the definition of $M_N$, it follows that the terms in the above sum correspond to terms that have been added to a rectangle on top or on right. That is, the sum is equal to either of the following two sums:

1. $$\sum_{n=0}^{L} \left< F, E_{n,K+1} \cdot \frac{1}{Zg} \right> E_{n,K+1} \cdot Zg, \quad L \leq J,$$

2. $$\sum_{k=0}^{R} \left< F, E_{J+1,k} \cdot \frac{1}{Zg} \right> E_{J+1,k} \cdot Zg, \quad R \leq K.$$  

Note that by Proposition 2.3(b), $-u(K+1) = u(K')$ for some $K' \in \mathbb{N}$. Hence, $\chi_{-u(K+1)}(\xi) = \chi_{u(K')}(\xi)$ so that

$$E_{0,K'}(x, \xi) = \chi_0(x) \chi_{u(K')}(\xi) = \chi_{-u(K')}(\xi) = \chi_{u(K+1)}(\xi).$$

Also, $E_{n,0}(x, \xi) = \chi_{u(n)}(x)$. Therefore, we have

$$\left< F, E_{n,K+1} \cdot \frac{1}{Zg} \right> = \int_D \int_D F(x, \xi) \chi_{u(n)}(x) \chi_{u(K+1)}(\xi) \frac{1}{Zg(x, \xi)} \, dx \, d\xi$$

$$= \int_D \int_D F(x, \xi) \chi_{u(K+1)}(\xi) \chi_{u(n)}(x) \frac{1}{Zg(x, \xi)} \, dx \, d\xi$$

$$= \int_D \int_D (F \cdot E_{0,K'})(x, \xi) \chi_{u(n)}(x) \frac{1}{Zg(x, \xi)} \, dx \, d\xi$$

$$= \left< F \cdot E_{0,K'}, E_{n,0} \cdot \frac{1}{Zg} \right>.$$  

Hence, the first sum (6.2) is bounded by

$$\left\| \sum_{n=0}^{L} \left< F, E_{n,K+1} \cdot \frac{1}{Zg} \right> E_{n,K+1} \cdot Zg \right\|_2$$

$$= \left\| \sum_{n=0}^{L} \left< F \cdot E_{0,K'}, E_{n,0} \cdot \frac{1}{Zg} \right> E_{n,K+1} \cdot \frac{1}{Zg} \right\|_2$$

$$= \left\| \sum_{n=0}^{L} \left< F \cdot E_{0,K'}, E_{n,0} \frac{1}{Zg} \right> E_{n,K+1} \cdot \frac{1}{Zg} \right\|_2$$

$$= \left\| T_{L,0}(F \cdot E_{0,K'}) \right\|_2$$

$$\leq C \left\| F \cdot E_{0,K'} \right\|_2$$

$$= C \|F\|_2.$$
For the second sum we observe that
\[
\left\langle F, E_{J+1,k} \cdot \frac{1}{Zg} \right\rangle = \int_{\mathcal{D}} \int_{\mathcal{D}} F(x, \xi) \overline{\chi_{u(J+1)}(x) \chi_{u(k)}(\xi)} \frac{1}{Zg(x, \xi)} \, dx \, d\xi
\]
\[
\quad = \int_{\mathcal{D}} \int_{\mathcal{D}} F(x, \xi) E_{J+1,0}(x, \xi) E_{0,k}(x, \xi) \frac{1}{Zg(x, \xi)} \, dx \, d\xi
\]
\[
\quad = \left\langle F, E_{J+1,0} \cdot E_{0,k} \cdot \frac{1}{Zg} \right\rangle.
\]
Hence, the second sum (6.3) is bounded by
\[
\left\| \sum_{k=0}^{R} \left\langle F, E_{J+1,k} \cdot \frac{1}{Zg} \right\rangle E_{J+1,k} \cdot Zg \right\|_2
\]
\[
\quad = \left\| \sum_{k=0}^{R} \left\langle F, E_{J+1,0,0} \cdot E_{0,k} \cdot \frac{1}{Zg} \right\rangle E_{J+1,k} \cdot Zg \right\|_2
\]
\[
\quad = \left\| \sum_{k=0}^{R} \left\langle F, E_{J+1,0,0} \cdot E_{0,k} \cdot \frac{1}{Zg} \right\rangle E_{0,k} \cdot Zg \right\|_2
\]
\[
\quad = \left\| T_{0,R} \langle F, E_{J+1,0} \rangle \right\|_2
\]
\[
\quad \leq C \left\| F \cdot E_{J+1,0} \right\|_2
\]
\[
\quad = C \left\| F \right\|_2.
\]
Substituting these estimates in (6.1), we obtain \( \left\| T_{N}^\sigma F \right\|_2 \leq 2C \left\| F \right\|_2 \) for all enumeration \( \sigma \in \Gamma \) and all \( N \in \mathbb{N} \). Therefore, \( \sup_{N,\sigma} \left\| T_{N}^\sigma \right\|_2 \leq 2C < \infty \). Hence, the claim is proved and this completes the proof of the theorem. \( \square \)

For applications, it is important to know whether a Gabor system is complete, minimal, a frame, a Riesz basis or an orthonormal basis. We mention some results in Gabor theory on the characterizations of such systems in terms of the Zak transform. We refer to [20] for a proof of the following theorem. For an introduction to frame theory, and definitions of frames and Riesz bases, see [11].

**Theorem 6.2.** Let \( K \) be a local field of positive characteristic, \( g \in L^2(K) \) and \( \mathcal{G}(g) \) the Gabor system generated by \( g \). Then

(a) \( \mathcal{G}(g) \) is complete in \( L^2(K) \) if and only if \( Zg \neq 0 \) a.e.

(b) \( \mathcal{G}(g) \) is minimal and complete in \( L^2(K) \) if and only if \( \frac{1}{Zg} \in L^2(\mathcal{D} \times \mathcal{D}) \).

(c) \( \mathcal{G}(g) \) is a frame for \( L^2(K) \) with bounds \( A \) and \( B \) if and only if \( A \leq |Zg|^2 \leq B \) a.e. In this case \( \mathcal{G}(g) \) is a Riesz basis for \( L^2(K) \) with bounds \( A \) and \( B \).

(d) \( \mathcal{G}(g) \) is an orthonormal basis for \( L^2(K) \) if and only if \( |Zg| = 1 \) a.e.

Now, to illustrate Theorem 6.1, we present some examples of Gabor systems which form Schauder bases for \( L^2(K) \).

**Example 6.1.** Let \( -\frac{1}{2} < \alpha < \frac{1}{2} \). Consider the function \( g \) which is supported on \( \mathcal{D} \) and \( g(x) = |x|^{\alpha}, \ x \in \mathcal{D} \). Observe that \( g \in L^2(K) \) since \( \alpha > -\frac{1}{2} \). Since \( g \) is supported on \( \mathcal{D} \), the only term which contributes to the sum in the definition of \( Zg \) (see (1.2)) corresponds to \( k = 0 \). Hence, \( Zg(x, \xi) = g(x) \). It follows from the definition that for a function of the form \( w(x, y) = v(x) \), \( w \in A_p(\mathcal{D} \times \mathcal{D}) \) if \( v \in A_p(\mathcal{D}) \). The function \( g(x) = |x|^{\alpha} \) is an \( A_p(\mathcal{D}) \) weight if and only if \( -1 < \alpha < p-1 \) (see [22]). Hence, \( |g|^2 \) is an \( A_2(\mathcal{D}) \) weight if and only if \( -\frac{1}{2} < \alpha < \frac{1}{2} \). This shows that \( |Zg|^2 \in A_2(\mathcal{D} \times \mathcal{D}) \). Therefore, by Theorem 6.1, it follows that the Gabor system \( \mathcal{G}(g) \) is a Schauder basis for \( L^2(K) \) with respect to every enumeration \( \sigma \in \Gamma \).

Also, observe that \( |Zg| \) is not bounded away from zero. Therefore, by Theorem 6.2, it follows that \( \mathcal{G}(g) \) is not a Riesz basis for \( L^2(K) \).

**Example 6.2.** Let \( g \) be a function constructed in Example 6.1 and \( h \) be any function supported in \( \mathcal{D} \) such that \( h \in A_2(\mathcal{D}) \). For example, we can take \( h(\xi) = |\xi|^{\alpha} \) with \( -\frac{1}{2} < \alpha < \frac{1}{2} \). Let...
Let \( G(x, \xi) = g(x)h(\xi) \). Then \( |G|^2 \in A_2(\mathcal{D} \times \mathcal{D}) \) so that \( f = Z^{-1}G \in L^2(K) \). By Theorem 6.1 \( \mathcal{G}(f) \) is a Schauder basis for \( L^2(K) \) with respect to every enumeration \( \sigma \in \Gamma \).

Note that every Schauder basis is complete and minimal, but the converse need not be true in general. We now construct an example of a Gabor system which is complete and minimal, and using Theorem 6.1 we will show that this system cannot be a Schauder basis for some permutation of \( N_0 \times N_0 \).

**Example 6.3.** Let \( A_1 = \mathcal{P} \) and

\[
A_n = pu(1) + p^2 u(1) + \cdots + p^{n-1} u(1) + \mathcal{P}^n, \quad n \geq 2.
\]

We first show that the balls \( A_n, n \geq 1 \), are pairwise disjoint. Note that \( |A_n| = q^{-n} \). Let \( k, l \in \mathbb{N} \) with \( k < l \). Suppose \( A_k \) and \( A_l \) are not disjoint. By Proposition 2.1(b), \( A_l \subset A_k \). Let \( y = pu(1)+p^2 u(1)+\cdots+p^{k-1} u(1) \). Then \( A_k = y+\mathcal{P}^k \) and \( A_l = y+p^k u(1)+\cdots+p^{l-1} u(1)+\mathcal{P}^l \). Now, if \( A_l \subset A_k \), then \( (A_l-y) \subset (A_k-y) \). But, this is not possible since \( p^k u(1)+\cdots+p^{l-1} u(1) \in A_l-y \) and \( |p^k u(1)+\cdots+p^{l-1} u(1)| = q^{-k+1} \) whereas \( A_k-y = \mathcal{P}^k \) is the ball of radius \( q^{-k} \) centred at 0. Hence, \( A_n, n \geq 1 \), are disjoint.

By Proposition 2.1(d), each \( A_n \) is a union of \( q \) balls of radius \( q^{-n-1} \). Choose any two such balls and call them \( E_n \) and \( F_n \). Then \( |E_n| = |F_n| = q^{-n-1} \). Let \( S = \bigcup_{n=1}^{\infty} (E_n \cup F_n) \) and \( \alpha \) be a real number such that \( 1 < \alpha < q \). Define the function \( g(x) \) which is supported on \( \mathcal{D} \) and

\[
g(x) = \begin{cases} 
\alpha^{n/2} & x \in E_n, n \geq 1, \\
\alpha^{-n/2} & x \in F_n, n \geq 1, \\
1 & x \in \mathcal{D} \setminus S.
\end{cases}
\]

We have

\[
\int_{\mathcal{D}} |g(x)|^2 \, dx = \sum_{n=1}^{\infty} \left( \alpha^n |E_n| + \alpha^{-n} |F_n| \right) + |\mathcal{D} \setminus S| \\ 
\leq \sum_{n=1}^{\infty} (\alpha^n + \alpha^{-n})q^{-n-1} + 1 \\ 
= \frac{1}{q} \sum_{n=1}^{\infty} \left[ \left( \frac{\alpha}{q} \right)^n + \left( \frac{1}{\alpha q} \right)^n \right] + 1 < \infty,
\]

since \( \frac{\alpha}{q}, \frac{1}{\alpha q} < 1 \). Similarly,

\[
\int_{\mathcal{D}} \frac{1}{|g(x)|^2} \, dx = \sum_{n=1}^{\infty} \left( \alpha^{-n} |E_n| + \alpha^n |F_n| \right) + |\mathcal{D} \setminus S| < \infty.
\]

Since \( Zg(x, \xi) = g(x) \), it follows that \( Zg, \frac{1}{Zg} \in L^2(\mathcal{D} \times \mathcal{D}) \). Hence, by Theorem 6.2 \( \mathcal{G}(g) \) is minimal and complete in \( L^2(K) \).

We now compute the average of \( |Zg(x, \cdot)|^2 \) over the ball \( A_n \). Note that \( |A_n \setminus (E_n \cup F_n)| = \frac{q^{-2}}{q^{n+1}} \). Hence,

\[
\frac{1}{|A_n|} \int_{A_n} |Zg(x, \xi)|^2 \, dx = \frac{1}{|A_n|} \int_{A_n} |g(x)|^2 \, dx \\ 
= q^n \left[ (\alpha^n + \alpha^{-n})q^{-n-1} + \frac{q-2}{q^{n+1}} \right] \\ 
= \frac{1}{q} (\alpha^n + \alpha^{-n} + q - 2) \longrightarrow \infty \quad \text{as} \quad n \to \infty.
\]

Similarly,

\[
\frac{1}{|A_n|} \int_{A_n} \frac{1}{|Zg(x, \xi)|^2} \, dx \longrightarrow \infty \quad \text{as} \quad n \to \infty.
\]

Therefore, (6.3) does not hold for \( w = |Zg|^2 \) and \( p = 2 \). Hence, \( |Zg|^2 \not\in A_2(\mathcal{D} \times \mathcal{D}) \). By Theorem 6.1 there exists an enumeration \( \sigma \in \Gamma \) such that \( \mathcal{G}(g) \) is not a Schauder basis with respect to \( \sigma \).
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