FINITE ELEMENT APPROXIMATIONS OF SYMMETRIC TENSORS IN ANY DIMENSION

JUN HU

ABSTRACT. For the first time, we construct, in a unified fashion, finite element subspaces of spaces of symmetric tensors with square-integrable divergence on a domain in any dimension. A set of degrees of freedom is defined and proved to be unisolvent. These spaces can be used to approximate symmetric matrix fields in first order systems where symmetric tensors are one variable, when standard discontinuous finite element spaces are used to approximate another variable. These finite element spaces are defined with respect to an arbitrary simplicial triangulation of the domain, and can be regarded as extensions to any dimension of those in two and three dimensions by Hu and Zhang.

Keywords. mixed finite element, symmetric finite element, first order system, conforming finite element, simplicial grids, inf-sup condition.

AMS subject classifications. 65N30, 73C02.

1. Introduction

We consider mixed finite element methods of first order systems with symmetric tensors: Find \((\sigma, u) \in \Sigma \times V := H(\text{div}, \Omega, S) \times L^2(\Omega, \mathbb{R}^n)\), such that

\[
\begin{aligned}
(A\sigma, \tau) + (\text{div}\tau, u) &= 0 \quad \text{for all } \tau \in \Sigma, \\
(\text{div}\sigma, v) &= (f, v) \quad \text{for all } v \in V.
\end{aligned}
\]

(1.1)

Here the symmetric tensor space for stress \(\Sigma\) is defined by

\[
H(\text{div}, \Omega, S) := \left\{ \tau = \begin{pmatrix} \tau_{11} & \cdots & \tau_{1n} \\ \vdots & \ddots & \vdots \\ \tau_{n1} & \cdots & \tau_{nn} \end{pmatrix} \in H(\text{div}, \Omega, \mathbb{R}^{n \times n}) \mid \tau^T = \tau \right\},
\]

(1.2)

and the space for vector displacement \(V\) is

\[
L^2(\Omega, \mathbb{R}^n) := \left\{ (u_1, \cdots, u_n)^T \mid u_i \in L^2(\Omega) \right\}.
\]

(1.3)

This paper denotes by \(H^k(T, X)\) the Sobolev space consisting of functions with domain \(T \subset \mathbb{R}^n\), taking values in the finite-dimensional vector space \(X\), and with all derivatives of order at most \(k\) square-integrable. For our purposes, the range space \(X\) will be either \(S\), \(\mathbb{R}^n\), or \(\mathbb{R}\). \(\| \cdot \|_{k,T}\) is the norm of \(H^k(T)\). \(S\) denotes the space of symmetric tensors, \(H(\text{div}, T, S)\) consists of square-integrable symmetric matrix fields with square-integrable divergence. The \(H(\text{div})\) norm is defined by

\[
\|\tau\|_{H(\text{div}, T)}^2 := \|\tau\|_{S,T}^2 + \|\text{div}\tau\|_{S,T}^2.
\]

The author was supported by the NSFC Projects 11271035, 11031006, 91430213 and 11421101.
\[ L^2(T, \mathbb{R}^n) \] is the space of vector-valued functions which are square-integrable. Here, the compliance tensor \( A = A(x) : \mathbb{S} \to \mathbb{S} \), characterizing the properties of the material, is bounded and symmetric positive definite uniformly for \( x \in \Omega \).

One celebrated example of (11) is the classical Hellinger-Reissner mixed formulation of the elasticity equations, the stress is sought in \( H(\text{div}, \Omega, \mathbb{S}) \) and the displacement in \( L^2(\Omega, \mathbb{R}^2) \) for two dimensions and in \( L^2(\Omega, \mathbb{R}^3) \) for three dimensions. Even for these systems in both two and three dimensions, the constructions of stable mixed finite elements using polynomial shape functions are a longstanding and challenging problem, see [6]. It was remarked in [38] that “It is, of course, possible to derive elements that exhibit complete continuity of the appropriate components along interfaces and indeed this was achieved by Raviart and Thomas in the case of the heat conduction problem discussed previously. Extension to the full stress problem is difficult and as yet such elements have not been successfully noted”. As a matter of fact, “four decades of searching for mixed finite elements for elasticity beginning in the 1960s did not yield any stable elements with polynomial shape functions”, see [4]. To overcome this difficulty, earliest works adopted composite element techniques or weakly symmetric methods, cf. [3, 7, 8, 29, 31, 33, 34, 35, 36]. In [10], Arnold and Winther designed the first family of mixed finite element methods in 2D, based on polynomial shape function spaces. From then on, various stable mixed elements have been constructed, see [2, 5, 6, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 26, 25].

As \( u \) is in \( L^2(\Omega, \mathbb{R}^n) \), a natural discretization is the piecewise \( P_{k-1} \) polynomial without interelement continuity. Even for two and three dimensional cases, it is a surprisingly hard problem if the stress tensor can be discretized by an appropriate \( P_k \) finite element subspace of \( H(\text{div}, \Omega, \mathbb{S}) \). In fact, in [10], Arnold and Winther designed the first family of mixed finite elements where the discrete stress space is the space of \( H(\text{div}, \Omega, \mathbb{S})-P_{k+1} \) tenors whose divergence is a \( P_{k-1} \) polynomial on each triangle. Such a two dimensional family was extended to a three dimensional family of mixed elements where the discrete stress space is the space of \( H(\text{div}, \Omega, \mathbb{S})-P_{k+2} \) tenors, while the lowest order element with \( k = 2 \) was first proposed in [2]. Mathematically speaking, these methods are one order (2D) or two-order (3D) sub-optimal. In very recent papers [27] and [28], Hu and Zhang attacked this open problem by constructing a suitable \( H(\text{div}, \Omega, \mathbb{S})-P_k \) instead of above \( P_{k+1} \) (2D, \( k \geq 3 \)) or \( P_{k+2} \) (3D \( k \geq 4 \)), finite element space for the stress discretization. The analysis there is based on a new idea for analyzing the discrete inf–sup condition. More precisely, they first decomposed the discontinuous displacement space into a subspace containing lower order polynomials and its orthogonal complement space. Second they found that the discrete stress space contains the full \( C^0-P_k \) space and some so-called \( H(\text{div}) \) bubble function space on each triangle (2D) or tetrahedron (3D). Third they proved that the full \( C^0-P_k \) space can control the subspace containing lower order polynomials while the \( H(\text{div}) \) bubble function space is able to deal with that orthogonal complement space. We refer interested readers to Hu [23] for similar mixed elements on rectangular and cubic meshes.

The purpose of this paper is two fold. First we generalize, in a unified fashion, the elements in [27] and [28] to any dimension. Second we define a set of degrees of freedom for the shape function spaces for the stress. The analysis here is based on three key ingredients. The first one is a new basis of the space \( \mathbb{S} \), which helps to define a \( H(\text{div}) \) bubble function space consisting of polynomials of degree \( \leq k \),
on each element and prove that it is indeed the full \(H(\text{div})\) bubble function space which is the second key ingredient. The second one is crucial for the definition of these degrees of freedom of the spaces (for any \(k\)) for the stress. The third one is that the divergence of the \(H(\text{div})\) bubble function space is equal to the orthogonal complement space of the rigid motion space with respect to the discrete displacement on each element. We stress that such a result holds for any \(k \geq 1\). The third one is indispensable for the proof of the discrete inf–sup condition and also for designing simple lower order stable elements.

2. Finite elements for symmetric tensors

Suppose that the domain \(\Omega\) is subdivided by a family of shape regular simplicial grids \(T_h\) (with the grid size \(h\)). We introduce the finite element space of order \(k \geq 1\) on \(T_h\).

\[
\Sigma_{k,h} := \{ \sigma \in H(\text{div}, \Omega, \mathbb{S}) \mid \sigma|_K \in P_k(K, \mathbb{S}) \forall K \in T_h \},
\]

where \(P_k(K, \mathbb{X})\) denotes the space of polynomials of degree \(\leq k\), taking value in the space \(\mathbb{X}\).

2.1. A new basis of the symmetric matrices.

To define the degrees of freedom for the shape function space \(P_k(K, \mathbb{S})\), let \(x_0, \ldots, x_n\) be the vertices of simplex \(K\). The referencing mapping is then

\[
x : = F_K(\hat{x}) = x_0 + (x_1 - x_0, \ldots, x_n - x_0) \hat{x},
\]

mapping the reference tetrahedron \(\hat{K} := \{ 0 \leq \hat{x}_1, \ldots, \hat{x}_n, 1 - \sum_{i=1}^n \hat{x}_i \leq 1 \}\) to \(K\).

Then the inverse mapping is

\[
\hat{x} = \begin{pmatrix} \nu_1^T \\ \vdots \\ \nu_n^T \end{pmatrix} (x - x_0),
\]

where

\[
\begin{pmatrix} \nu_1^T \\ \vdots \\ \nu_n^T \end{pmatrix} = (x_1 - x_0, \ldots, x_n - x_0)^{-1}.
\]

By (2.2), these normal vectors are coefficients of the barycentric variables:

\[
\lambda_1 = \nu_1 \cdot (x - x_0),
\]

\[
\vdots
\]

\[
\lambda_n = \nu_n \cdot (x - x_0),
\]

\[
\lambda_0 = 1 - \sum_{i=1}^n \lambda_i.
\]

For any edge \(x_ix_j\) of element \(K\), \(i \neq j\), let \(t_{i,j}\) denote associated tangent vectors, which allow for us to introduce the following symmetric matrices of rank one

\[
T_{i,j} := t_{i,j}t_{i,j}^T, 0 \leq i < j \leq n.
\]

For these matrices of rank one, we have the following important result.
Lemma 2.1. The \((n+1)n\) symmetric tensors \(T_{i,j}\) in (2.1) are linearly independent, and form a basis of \(S\).

Proof. Each matrix \(T_{i,j} = t_{i,j}t_{i,j}^T\) is a positive semi-definite matrix, on a simplex \(K\). We would show that the constants \(c_{i,j}\) are all equal to zero in

\[\tau = \sum_{0 \leq i < j \leq n} c_{i,j}T_{i,j} = 0.\]

By the definition of \(\nu_1\),

\[\nu_1^T \tau = \nu_1^T \sum_{0 \leq i < j \leq n} c_{i,j}t_{i,j}t_{i,j}^T = \nu_1^T (c_{0,1}t_{0,1}t_{0,1}^T + \sum_{j>1} c_{1,j}t_{1,j}t_{1,j}^T) = \tilde{c}_{0,1}t_{0,1}^T + \sum_{j>1} \tilde{c}_{1,j}t_{1,j}^T = 0,\]

where \(\tilde{c}_{0,1} = c_{0,1}\nu_1^T t_{0,1}\) and \(\tilde{c}_{1,j} = c_{1,j}\nu_1^T t_{1,j}\). This leads to

\[\tilde{c}_{0,1}t_{0,1}^T + \sum_{j>1} \tilde{c}_{1,j}t_{1,j}^T t_{1,l} = 0, l = 0, 2, \cdots, n.\]

Since \(t_{1,l}, l = 0, 2, \cdots, n\), are linearly independent, this yields

\[\tilde{c}_{0,1} = \tilde{c}_{1,j} = 0, j > 1.\]

Since \(\nu_1\) is perpendicular to the \(n - 1\) dimensional sub-simplex \(\triangle_{n-1} x_0 x_2 \cdots x_n\),

\[\nu_1^T t_{1,j} \neq 0\] for any \(j = 0, 2, \cdots, n.\)

This and (2.8) yield

\[c_{0,1} = c_{1,j} = 0, j > 1.\]

A similar argument by using \(\nu_i, i \neq 1\), proves the desired result.

2.2. The bubble functions space. With these symmetric matrices \(T_{i,j}\) of rank one, we define a \(H(\text{div}, K, S)\) bubble function space

\[\Sigma_{K,k,b} := \sum_{0 \leq i < j \leq n} \lambda_i \lambda_j P_{k-2}(K, \mathbb{R}) T_{i,j}\]

Define the full \(H(\text{div}, K, S)\) bubble function space consisting of polynomials of degree \(\leq k\)

\[\Sigma_{\partial K,k,0} := \{\tau \in H(\text{div}, K, S) \cap P_k(K, S), \tau \nu|_{\partial K} = 0\}.\]

Here \(\nu\) is the unit normal vector of \(\partial K\).

We have the following important result.

Lemma 2.2. It holds that

\[\Sigma_{K,k,b} = \Sigma_{\partial K,k,0}.\]
Proof: We start to consider function \( \tau \in \lambda_i \lambda_j, P_{k-2}(K, \mathbb{R})T_{i,j}, \) \( 0 \leq i < j \leq n. \) Note that \( \tau \) vanishes on the \( n - 1 \) dimensional simplices
\[
\triangle_{n-1} x_0 \cdots x_{i-1} x_{i+1} \cdots x_j \cdots x_n,
\]
and
\[
\triangle_{n-1} x_0 \cdots x_i \cdots x_{j-1} x_{j+1} \cdots x_n.
\]
For any \( n - 1 \) dimensional simplex which takes edge \( x_i x_j, \) its unit normal vector, say \( \nu, \) is perpendicular to the tangent vector \( t_{i, j} \) of edge \( x_i, x_j, \) which implies that \( \tau \nu = 0 \) on such a \( n - 1 \) dimensional simplex and consequently \( \tau \in \Sigma_{\partial K, k, 0}. \) Hence
\[
(2.14) \quad \Sigma_{K, k, b} \subset \Sigma_{\partial K, k, 0}.
\]
Next we show the converse of (2.14). Given \( \tau \in \Sigma_{\partial K, k, 0}, \) the boundary condition \( \tau \nu |_{\partial K} = 0 \) indicates that \( \tau \) vanishes at all the vertices of \( K. \) Let \( \mathbb{N}_b \) denote all the nodes except the vertices of \( K. \) \( \tau \) vanishes at all the vertices of \( K, \) let \( \varphi_{\ell} \in P_k(K, \mathbb{R}), \) denote the usual associated nodal Lagrange basis function, namely, \( \varphi_{\ell}(\mathbb{P}_\ell) = 1 \) and \( \varphi_{\ell} \) vanishes at all the other nodes for the space \( P_k(K, \mathbb{R}). \) It follows from Lemma 2.1 that
\[
(2.15) \quad \tau = \sum_{0 \leq i < j \leq n} T_{i,j} \left( \sum_{P_{\ell} \in \mathbb{N}_b} c_{\ell,i,j} \varphi_{\ell} \right).
\]
Note that \( \varphi_{\ell} \) has a homogeneous expression by \( \lambda_0, \ldots, \lambda_n. \) Therefore, we have
\[
(2.16) \quad \sum_{P_{\ell} \in \mathbb{N}_b} c_{\ell,i,j} \varphi_{\ell} = \sum_{m_0 + m_1 + \cdots + m_n = k} c_{(ij), m_0, m_1, \ldots, m_n} \lambda_0^{m_0} \cdots \lambda_n^{m_n}.
\]
We claim that \( \sum_{P_{\ell} \in \mathbb{N}_b} c_{\ell,i,j} \varphi_{\ell} \) has a factor \( \lambda_i \lambda_j, \) namely,
\[
(2.17) \quad \sum_{P_{\ell} \in \mathbb{N}_b} c_{\ell,i,j} \varphi_{\ell} = \lambda_i \lambda_j \sum_{m_0 + m_1 + \cdots + m_n = k-2} c_{(ij), m_0, m_1, \ldots, m_n} \lambda_0^{m_0} \cdots \lambda_n^{m_n}.
\]
Without loss of generality, we consider the case where \( i = 0 \) and \( j = 1. \) Suppose that there is a term \( f_1 T_{0,1} \) such that \( f_1 \) is a polynomial of degree \( \leq k \) and does not contain a factor \( \lambda_0. \) Next we shall show that \( f_1 = 0. \) In fact, all the terms of \( (2.16) \) which do not contain the factor \( \lambda_0 \) and whose normal components (namely \( T_{0,j} \nu_0 \neq 0, \nu_0 \) is the unit normal vector of \( \triangle_{n-1} x_1 \cdots x_n \)) do not vanish on the \( n - 1 \) dimensional simplex \( \triangle_{n-1} x_1 \cdots x_n, \) can be expressed as
\[
(2.18) \quad \sum_{j=1}^n f_j T_{0,j},
\]
where \( f_j, j = 1, \ldots, n, \) are polynomials of degree \( \leq k. \) Since \( f_j \) do not contain the factor \( \lambda_0, \) it is of the form
\[
(2.19) \quad f_j = \sum_{r_1 + \cdots + r_n = k} c_{r_1, \ldots, r_n} \lambda_1^{r_1} \cdots \lambda_n^{r_n}.
\]
Since \( \tau \nu_0 = 0 \) on the \( n - 1 \) dimensional simplex \( \triangle_{n-1} x_1 \cdots x_n, \)
\[
(2.20) \quad \sum_{j=1}^n (t_{0,j} \nu_0) f_j T_{0,j} \bigg|_{\triangle_{n-1} x_1 \cdots x_n} = 0.
\]
Since, for \( j = 1, \ldots, n, \) \( t_{0,j} \nu_0^T \neq 0, \) and \( t_{0,j} \) are linearly independent, this leads to
\[
(2.21) \quad f_j \big|_{\triangle_{n-1} x_1 \cdots x_n} = 0.
\]
Note that $\lambda_1^{r_1} \cdots \lambda_n^{r_n} |_{\triangle_{n-1} \mathbf{x}_1 \cdots \mathbf{x}_n}$, $\sum_{i=1}^n r_i = k$, form a basis of $P_k(\triangle_{n-1} \mathbf{x}_1 \cdots \mathbf{x}_n, \mathbb{R})$. This and the above equation show that
\[(2.22) \quad c_{j, r_1, \cdots, r_n} = 0.\]
This, in turn, implies that
\[(2.23) \quad f_j \equiv 0.\]
Therefore $f_1 = 0$ which implies that all the terms on the right hand side of (2.16) has a factor $\lambda_0$. A similar argument shows that all the terms on the right hand side of (2.16) has a factor $\lambda_1$. Hence
\[(2.24) \quad \tau \in \Sigma_{K,k,b}.\]
This completes the proof.

2.3. Degrees of freedom. Before we define the degrees of freedom, we need a classical result and its variant.

**Lemma 2.3.** It holds the following Chu-Vandermonde combinatorial identity and its variant
\[(2.25) \quad \sum_{\ell=0}^{n} C_{n+1}^{\ell+1} C_{k-1}^{\ell} = C_{n+k}^{n}, \]
and
\[(2.26) \quad \sum_{\ell=0}^{n} C_{n+1}^{\ell+1} C_{k-1}^{\ell} C_{\ell+1}^{2} = \frac{(n+1)n}{2} C_{n+k-2}^{n}, \]
where the combinatorial number $C_n = \frac{n\cdots(n-m+1)}{m\cdots 1}$.

**Theorem 2.1.** A matrix field $\tau \in P_k(K, \mathbb{S})$ can be uniquely determined by the following degrees of freedom:

1. For each $\ell$ dimensional simplex $\triangle_{\ell}$ of $K$, $0 \leq \ell \leq n-1$, with $\ell$ linearly independent tangential vectors $t_1, \cdots, t_\ell$, and $n-\ell$ linearly independent normal vectors $\nu_1, \cdots, \nu_{n-\ell}$, the mean moments of degree at most $k-\ell-1$ over $\triangle_{\ell}$, of $t^T \tau t_i, \nu^T \tau \nu_j$, $i = 1, \cdots, \ell$, $j = 1, \cdots, n-\ell$, $(C_{n+1}^{\ell+1} + (n-\ell)) C_{k-1}^{\ell}$ degrees of freedom for each $\triangle_{\ell}$;

2. the values $\int_K \tau : \theta d\mathbf{x}$ for any $\theta \in \Sigma_{K,k,b}$, $\frac{(n+1)n}{2} C_{n+k-2}^{n}$ degrees of freedom.

**Proof.** We assume that all degrees of freedom vanish and show that $\tau = 0$. Note that the mean moment become the value of $\tau$ for a 0 dimensional simplex $\triangle_0$, namely, a vertex, of $K$. The first set of degrees of freedom imply that $\tau \nu = 0$ on $\partial K$. Then the second set of degrees of freedom and Lemma 2.2 show $\tau = 0$. Next we shall prove that the sum of these degrees of freedom is equal to the dimension \(\frac{(n+1)n}{2} C_{n+k}^{n}\) of the space $P_k(K, \mathbb{S})$. In fact this sum is
\[(2.27) \quad \sum_{\ell=0}^{n-1} \frac{n-\ell}{2} (n+\ell+1) C_{k-1}^{\ell} + \frac{(n+1)n}{2} C_{n+k-2}^{n}.\]
Then the desired result follows from the Chu-Vandermonde combinatorial identity (2.25) and its variant (2.26).
Remark 2.1. It follows from Theorem [2.7] that, for any dimension, if $k = 1$, $\Sigma_{k,h}$ becomes a $H^1$ conforming approximation of $\Sigma := H(\text{div}, \Omega, S)$. For one dimensional case with $n = 1$, for any $k$, $\Sigma_{k,h}$ becomes the usual $H^1$ finite element space of degree $k$.

2.4. The divergence space of the bubble function space. Before ending this section, we prove an important result concerning the divergence space of the bubble function space. To this end, we introduce the following rigid motion space on each element $K$.

$$R(K) := \{ v \in H^1(K, \mathbb{R}^n), (\nabla v + \nabla v^T)/2 = 0 \}.$$  

(2.28)

It follows from the definition that $R(K)$ is a subspace of $P_1(K, \mathbb{R}^n)$. For $n = 1$, $R(K)$ is the constant function space over $K$. The dimension of $R(K)$ is $n(n+1)/2$.

This allows for defining the orthogonal complement space of $R(K)$ with respect to $P_{k-1}(K, \mathbb{R}^n)$ by

$$R^\perp(K) := \{ v \in P_{k-1}(K, \mathbb{R}^n), (v, w)_K = 0 \text{ for any } w \in R(K) \},$$  

(2.29)

where the inner product $(v, w)_K$ over $K$ reads $(v, w)_K = \int_K v \cdot w \, dx$.  

Theorem 2.2. For any $K \in T_h$, it holds that

$$\text{div} \Sigma_{K,k,b} = R^\perp(K).$$  

(2.30)

Proof. For any $\tau \in \Sigma_{K,k,b}$, an integration by parts yields

$$\int_K \text{div} \tau \cdot w \, dx = 0 \text{ for any } w \in R(K).$$

This implies that

$$\text{div} \Sigma_{K,k,b} \subset R^\perp(K).$$  

(2.31)

Next we show the converse. In fact, if $\text{div} \Sigma_{K,k,b} \neq R^\perp(K)$, there is a nonzero $v \in R^\perp(K)$ such that

$$\int_K \text{div} \tau \cdot v \, dx = 0 \quad \forall \tau \in \Sigma_{K,k,b}.$$

By integration by parts, for $\tau \in \Sigma_{K,k,b}$, we have

$$\int_K \text{div} \tau \cdot v \, dx = \int_K \tau : \epsilon(v) \, dx = 0,$$

(2.32)

where $\epsilon(v)$ is the symmetric gradient, $(\nabla v + \nabla^T v)/2$.

By Lemma [2.1], $T_{i,j}$, $0 \leq i < j \leq n$ defined in [2.4], form a basis of the space of symmetric matrices in $\mathbb{R}^{n \times n}$. Then there exists an associated dual basis, say $M_{i,j}$, $0 \leq i < j \leq n$, such that

$$T_{i,j} : M_{k,l} = \delta_{i,k} \delta_{j,l}, 0 \leq i < j \leq n, 0 \leq k < l \leq n.$$  

(2.33)

Here the inner product of two matrices $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$ is defined as

$$A : B = \sum_{i=1}^n \sum_{j=1}^n a_{ij}b_{ij}.$$
As \( \epsilon(v) \in P_{k-2}(K, \mathbb{S}) \), it follows that there exist \( q_{i,j} \in P_{k-2}(K, \mathbb{R}) \), \( 0 \leq i < j \leq n \), such that

\[
\epsilon(v) = \sum_{0 \leq i < j \leq n} q_{i,j} M_{i,j}.
\]

(2.34)

Selecting \( \tau = \sum_{0 \leq i < j \leq n} \lambda_i \lambda_j q_{i,j} T_{i,j} \in \Sigma_{K,k,b} \), we have,

\[
0 = \int_K \tau : \epsilon(v) \, dx = \sum_{0 \leq i < j \leq n} \int_K \lambda_i \lambda_j q_{i,j}^2(\mathbf{x}) \, dx.
\]

As \( \lambda_i \lambda_j > 0 \) on \( K \), we conclude that \( q_{i,j} \equiv 0 \), which implies that \( v \) is a rigid motion. This contradicts with \( v \in \mathbb{R}^+(K) \). Hence \( \mathbb{R}^+(K) \subset \text{div} \Sigma_{K,k,b} \), which completes the proof.

3. Mixed methods of first order systems with symmetric tensors

3.1. Mixed methods. We propose to use the space \( \Sigma_{k,h} \), with \( k \geq n + 1 \), defined in (2.1) to approximate \( \Sigma \). In order get a stable pair of spaces, we take the discrete displacement space as the full \( C^{-1}P_{k-1} \) space

\[
V_{k,h} := \{ v \in L^2(\Omega, \mathbb{R}^n) \mid v|_K \in P_{k-1}(K, \mathbb{R}^n) \text{ for all } K \in T_h \}.
\]

(3.1)

It follows from the definition of \( V_{k,h} \) (\( P_{k-1} \) polynomials) and \( \Sigma_{k,h} \) (\( P_k \) polynomials) that

\[
\text{div} \Sigma_{k,h} \subset V_{k,h}.
\]

This, in turn, leads to a strong divergence-free space:

\[
Z_h := \{ \tau_h \in \Sigma_{k,h} \mid (\text{div} \tau_h, v) = 0 \text{ for all } v \in V_{k,h} \}
\]

(3.2)

\[
= \{ \tau_h \in \Sigma_{k,h} \mid \text{div} \tau_h = 0 \text{ pointwise } \}.
\]

The mixed finite element approximation of Problem (1.1) reads: Find \( (\sigma_h, u_h) \in \Sigma_{k,h} \times V_{k,h} \) such that

\[
\begin{cases}
(A\sigma_h, \tau) + (\text{div} \tau, u_h) = 0 & \text{for all } \tau \in \Sigma_{k,h}, \\
(\text{div} \sigma_h, v) = (f, v) & \text{for all } v \in V_{k,h}.
\end{cases}
\]

(3.3)

3.2. Stability analysis and error estimates. The convergence of the finite element solutions follows the stability and the standard approximation property. So we consider first the well-posedness of the discrete problem (3.3). By the standard theory, we only need to prove the following two conditions, based on their counterpart at the continuous level.

(1) K-ellipticity. There exists a constant \( C > 0 \), independent of the meshsize \( h \) such that

\[
(A\tau, \tau) \geq C \| \tau \|_{H(\text{div})}^2 \text{ for all } \tau \in Z_h,
\]

where \( Z_h \) is the divergence-free space defined in (3.2).

(2) Discrete B-B condition. There exists a positive constant \( C > 0 \) independent of the meshsize \( h \), such that

\[
\inf_{0 \neq \tau \in V_{k,h}} \sup_{0 \neq v \in \Sigma_{k,h}} \frac{(\text{div} \tau, v)}{\| \tau \|_{H(\text{div})} \| v \|_0} \geq C.
\]

(3.5)
It follows from \( \text{div} \Sigma_{k,h} \subset V_{k,h} \) that \( \text{div} \tau = 0 \) for any \( \tau \in Z_h \). This implies the above K-ellipticity condition (3.4). It remains to show the discrete B-B condition (3.5), in the following two lemmas.

For the analysis, we need a subspace \( \tilde{\Sigma}_{k,h} := \Sigma_{k,h} \cap H^1(\Omega, S) \) of \( \Sigma_{k,h} \). \( \tau \in \tilde{\Sigma}_{k,h} \), the degrees of freedom on any element \( K \) are: for each \( \ell \) dimensional simplex \( \Delta_\ell \) of \( K \), \( 0 \leq \ell \leq n \), the mean moments of degree at most \( k - \ell - 1 \) over \( \Delta_\ell \), of \( \tau \). A standard argument is able to prove that these degrees of freedom are unisolvent.

**Lemma 3.1.** For any \( v_h \in V_{k,h} \), there is a \( \tau_h \in \tilde{\Sigma}_{k,h} \) such that, for all polynomial \( p \in R(K) \), \( K \in T_h \),

\[
(3.6) \quad \int_K (\text{div} \tau - v_h) \cdot p \, dx = 0 \quad \text{and} \quad \| \tau_h \|_{H(\text{div})} \leq C \| v_h \|_0.
\]

**Proof.** Let \( v_h \in V_{k,h} \). By the stability of the continuous formulation, cf. [10] for two dimensional case, there is a \( \tau \in H^1(\Omega, S) \) such that,

\[
\text{div} \tau = v_h \quad \text{and} \quad \| \tau \|_1 \leq C \| v_h \|_0.
\]

First let \( I_h \) be a Scott-Zhang [32] interpolation operator such that

\[
(3.7) \quad \| \tau - I_h \tau \|_0 + h \| \nabla I_h \tau \|_0 \leq C \| \nabla \tau \|_0.
\]

Since \( k \geq n + 1, k - (n - 1) - 1 \geq 1 \), for each \( n - 1 \) dimensional simplex \( \Delta_{n-1} \) of \( K \), there are at least \( n \) bubble functions which vanish on the boundary \( \partial \Delta_{n-1} \) of \( \Delta_{n-1} \) for each component of \( \tau \). These bubble functions allow for defining a correction \( \delta_h \in \Sigma_{k,h} \) such that

\[
(3.8) \quad \int_{\Delta_{n-1}} \delta_h \nu \cdot p ds = \int_{\Delta_{n-1}} (\tau - I_h \tau) \nu \cdot p ds \quad \text{for any} \quad p \in R(K)|_{\Delta_{n-1}}.
\]

Finally we take

\[
(3.9) \quad \tau_h = I_h \tau + \delta_h.
\]

We get a partial-divergence matching property of \( \tau_h \): for any \( p \in R(K) \), as the symmetric gradient \( \epsilon(p) = 0 \),

\[
\int_K (\text{div} \tau - v_h) \cdot p \, dx = \int_K (\text{div} \tau - \text{div} \tau) \cdot p \, dx = \int_{\partial K} (\tau_h - \tau) \nu \cdot p \, ds = 0.
\]

The stability estimate follows from (3.7) and the definition of the correction \( \delta_h \). \( \blacksquare \)

We are in the position to show the well-posedness of the discrete problem.

**Theorem 3.1.** For the discrete problem (3.3), the K-ellipticity (3.4) and the discrete B-B condition (3.5) hold uniformly. Consequently, the discrete mixed problem (3.3) has a unique solution \( (\sigma_h, u_h) \in \Sigma_{k,h} \times V_{k,h} \).

**Proof.** The K-ellipticity immediately follows from the fact that \( \text{div} \Sigma_{k,h} \subset V_{k,h} \). To prove the discrete B-B condition (3.5), for any \( v_h \in V_{k,h} \), it follows from Lemma 3.1 that there exists a \( \tau_1 \in \Sigma_{k,h} \) such that, for any polynomial \( p \in R(K) \),

\[
(3.10) \quad \int_K (\text{div} \tau - v_h) \cdot p \, dx = 0 \quad \text{and} \quad \| \tau_1 \|_{H(\text{div})} \leq C \| v_h \|_0.
\]
Then it follows from Lemma 2.2 that there is a \( \tau_2 \in \Sigma_{k,h} \) such that

\[
\text{div } \tau_2 = v_h - \text{div } \tau_1.
\]

(3.11)

In addition, a scaling argument proves

\[
\| \tau_2 \|_{H(\text{div})} \leq C \| \text{div } \tau_1 - v_h \|_0.
\]

(3.12)

Let \( \tau = \tau_1 + \tau_2 \). This implies that

\[
\text{div } \tau = v_h \text{ and } \| \tau \|_{H(\text{div})} \leq C \| v_h \|_0,
\]

(3.13) this proves the discrete B-B condition (3.5).

**Theorem 3.2.** Let \((\sigma, u) \in \Sigma \times V\) be the exact solution of problem (1.1) and \((\tau_h, u_h) \in \Sigma_{k,h} \times V_{k,h}\) the finite element solution of (3.3). Then, for \(k \geq n + 1\),

\[
\| \sigma - \sigma_h \|_{H(\text{div})} + \| u - u_h \|_0 \leq C h^k (\| \sigma \|_{k+1} + \| u \|_k).
\]

(3.14)

**Proof.** The stability of the elements and the standard theory of mixed finite element methods [14, 15] give the following quasioptimal error estimate immediately

\[
\| \sigma - \sigma_h \|_{H(\text{div})} + \| u - u_h \|_0 \leq C \inf_{\tau_h \in \Sigma_{k,h}, v_h \in V_{k,h}} (\| \sigma - \tau_h \|_{H(\text{div})} + \| u - v_h \|_0).
\]

(3.15)

Let \( P_h \) denote the local \( L^2 \) projection operator, or element-wise interpolation operator, from \( V \) to \( V_{k,h} \), satisfying the error estimate

\[
\| v - P_h v \|_0 \leq C h^k \| v \|_k \text{ for any } v \in H^k(\Omega, \mathbb{R}^n).
\]

(3.16)

Choosing \( \tau_h = I_h \sigma \in \Sigma_{k,h} \) where \( I_h \) is defined in (3.7) as \( I_h \) preserves symmetric \( P_k \) functions locally,

\[
\| \sigma - \tau_h \|_0 + h \| \sigma - \tau_h \|_{H(\text{div})} \leq C h^{k+1} \| \sigma \|_{k+1}.
\]

(3.17)

Let \( v_h = P_h v \) and \( \tau_h = I_h \sigma \) in (3.15), by (3.16) and (3.17), we obtain (3.14).

4. Conclusions

In this paper we propose a family of mixed elements of symmetric tensors in any dimension. For stability, we require in Section 3 that the polynomial degree for the stress be greater than \( n \). Note that one key result, namely, Theorem 3.2 holds for an arbitrary \( k \). Therefore, to design lower order methods, we only need to add some \( n - 1 \) dimensional simplex bubbles such that Lemma 3.1 holds. In fact, it has been done for two and three dimensions in [27] and [28], respectively. We shall work on the general case in a forthcoming paper.

Acknowledgement

The author would like to thank Professor Jinchao Xu for suggesting him to write such a paper.
References

[1] R. A. Adams, Sobolev Spaces, New York: Academic Press, 1975.
[2] S. Adams and B. Cockburn, A mixed finite element method for elasticity in three dimensions, J. Sci. Comput. 25 (2005), no. 3, 515–521.
[3] M. Amara and J. M. Thomas, Equilibrium finite elements for the linear elastic problem, Numer. Math. 33 (1979), 367–383.
[4] D. N. Arnold, Proceedings of the International Congress of Mathematicians, Vol. I: Plenary Lectures and Ceremonies (2002), 137–157.
[5] D. N. Arnold and G. Awanou, Rectangular mixed finite elements for elasticity, Math. Models Methods Appl. Sci. 15 (2005), 1417–1429.
[6] D. Arnold, G. Awanou and R. Winther, Finite elements for symmetric tensors in three dimensions, Math. Comp. 77 (2008), no. 263, 1229–1251.
[7] D. N. Arnold, F. Brezzi and J. Douglas Jr., PEERS: A new mixed finite element for plane elasticity, Jpn. J. Appl. Math. 1 (1984), 347–367.
[8] D. N. Arnold, J. Douglas Jr., and C. P. Gupta, A family of higher order mixed finite element methods for plane elasticity, Numer. Math. 45 (1984), 1–22.
[9] D. N. Arnold, R. Falk and R. Winther, Mixed finite element methods for linear elasticity with weakly imposed symmetry, Math. Comp. 76 (2007), no. 260, 1699–1723.
[10] D. N. Arnold and R. Winther, Mixed finite element for elasticity, Numer. Math. 92 (2002), 401–419.
[11] D. N. Arnold and R. Winther, Nonconforming mixed elements for elasticity, Math. Models Methods Appl. Sci. 13 (2003), 295–307.
[12] G. Awanou, Two remarks on rectangular mixed finite elements for elasticity, J. Sci. Comput. 50 (2012), 91–102.
[13] D. Boffi, F. Brezzi and M. Fortin, Reduced symmetry elements in linear elasticity, Commun. Pure Appl. Anal. 8 (2009), no. 1, 95–121.
[14] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, Rev. Francaise Automat. Informat. Recherche Operationnelle Ser. Rouge, 8(R-2) (1974), 129–151.
[15] F. Brezzi and M. Fortin, Mixed and hybrid finite element methods, Springer, 1991.
[16] C. Carstensen, M. Eigel, J. Gedicke, Computational competition of symmetric mixed FEM in linear elasticity, Comput. Methods Appl. Mech. Engrg. 200 (2011), 2903–2915.
[17] C. Carstensen, D. Günther, J. Rehbringhaus, J. Thiele, The Arnold–Winther mixed FEM in linear elasticity. Part I: Implementation and numerical verification, Comput. Methods Appl. Mech. Engrg. 197 (2008), 3014–3023.
[18] S. C. Chen and Y. N. Wang, Conforming rectangular mixed finite elements for elasticity, J. Sci. Comput. 47 (2011), no. 1, 93–108.
[19] B. Cockburn, J. Gopalakrishnan and J. Guzmán, A new elasticity element made for enforcing weak stress symmetry, Math. Comp. 79 (2010), no. 271, 1331–1349.
[20] J. Gopalakrishnan and J. Guzmán, Symmetric nonconforming mixed finite elements for linear elasticity, SIAM J. Numer. Anal. 49 (2011), no. 4, 1504–1520.
[21] J. Gopalakrishnan and J. Guzmán, A second elasticity element using the matrix bubble, IMA J. Numer. Anal. 32 (2012), no. 1, 352–372.
[22] J. Guzmán, A unified analysis of several mixed methods for elasticity with weak stress symmetry, J. Sci. Comput. 44 (2010), no. 2, 156–169.
[23] J. Hu, A new family of efficient rectangular conforming mixed elements for linear elasticity in the symmetric formulation, arXiv:1311.4715v2 [math.NA], 17 Dec 2013.
[24] J. Hu and Z. C. Shi, Lower order rectangular nonconforming mixed elements for plane elasticity, SIAM J. Numer. Anal. 46 (2007), 88–102.
[25] J. Hu, H. Y. Man and S. Zhang, The minimal mixed finite element method for the symmetric stress field on rectangular grids in any space dimension, arXiv:1304.5428 [math.NA] (2013).
[26] J. Hu, H. Y. Man and S. Zhang, A simple conforming mixed finite element for linear elasticity on rectangular grids in any space dimension, J. Sci. Comput. 58(2014), 367–379.
[27] J. Hu and S. Zhang, A family of conforming mixed finite elements for linear elasticity on tetrahedral grids, arXiv:1406.7457 [math.NA].
[28] J. Hu and S. Zhang, A family of conforming mixed finite elements for linear elasticity on triangle grids, arXiv:1407.4190 [math.NA].
[29] C. Johnson and B. Mercier, Some equilibrium finite element methods for two-dimensional elasticity problems, Numer. Math. 30 (1978), 103–116.
[30] H.-Y. Man, J. Hu and Z.-C. Shi, Lower order rectangular nonconforming mixed finite element for the three-dimensional elasticity problem, Math. Models Methods Appl. Sci. 19 (2009), no. 1, 51–65.
[31] M. Morley, A family of mixed finite elements for linear elasticity, Numer. Math. 55 (1989), no. 6, 633–666.
[32] L. R. Scott and S. Zhang, Finite-element interpolation of non-smooth functions satisfying boundary conditions, Math. Comp. 54 (1990), 483–493.
[33] R. Stenberg, On the construction of optimal mixed finite element methods for the linear elasticity problem, Numer. Math. 48 (1986), 447–462.
[34] R. Stenberg, Two low-order mixed methods for the elasticity problem, In: J. R. Whiteman (ed.): The Mathematics of Finite Elements and Applications, VI. London: Academic Press, 1988, 271–280.
[35] R. Stenberg, A family of mixed finite elements for the elasticity problem, Numer. Math. 53 (1988), no. 5, 513–538.
[36] S. Y. Yi, Nonconforming mixed finite element methods for linear elasticity using rectangular elements in two and three dimensions, CALCOLO 42 (2005), 115–133.
[37] S. Y. Yi, A New nonconforming mixed finite element method for linear elasticity, Math. Models Methods Appl. Sci. 16 (2006), 979–999.
[38] O. C. Zienkiewicz, R. L. Taylor, and J. Z. Zhu, The Finite Element Method: Its Basis and Fundamentals, 6th ed., vol. 1, Amsterdam–Boston–Heidelberg–London–New York–Oxford–Paris–San Diego–San Francisco–Singapore–Sydney–Tokyo, 2005.

LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China. hujun@math.pku.edu.cn