FURSTENBERG BOUNDARY OF MINIMAL ACTIONS

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Abstract. For a countable discrete group $\Gamma$ and a minimal $\Gamma$-space $X$, we study the notion of $(\Gamma, X)$-boundary, which is a natural generalization of the notion of topological $\Gamma$-boundary in the sense of Furstenberg. We give characterizations of the $(\Gamma, X)$-boundary in terms of essential or proximal extensions. The characterization is used to answer a problem of Hadwin and Paulsen in negative. As an application, we find necessary and sufficient condition for the corresponding reduced crossed product to be exact.

1. Introduction

The notion of (topological) $\Gamma$-boundaries of a group $\Gamma$ were introduced in the 60's by Furstenberg [10]. This notion was used to be considered as a tool to study rigidity problems in the context of semisimple Lie groups. The pioneering work of Kalantar and Kennedy [18], showed the key role of Furstenburg boundary in certain problems in operator algebras (see also, [5] and [22]).

There are several natural generalizations of the notion of Furstenberg boundary, including that of Bearden and Kalantar [3], Monod [23], Kennedy and Schafhauser [19], Amini and Behrouzi [1] and Borys [4]. In this paper, we introduce a dynamical version of the boundary for minimal actions on compact spaces. This is essential when one deals with the notion of minimality in dynamical setting [15], [14].

Throughout this paper $\Gamma$ is a countable discrete group, unless otherwise stated. Let $Y$ be a $\Gamma$-boundary, then by a result of Kalantar and Kennedy $C(Y)$ can be considered as a $\Gamma$-essential extension of $C$ [18]. This especially tells us that any $\Gamma$-boundary can be observed as a boundary of the trivial $\Gamma$-space. On the other hand, the notion of $\Gamma$-boundary, as a minimal strongly proximal $\Gamma$-space, can be extended to the notion of minimal strongly proximal extension of a $\Gamma$-space. The latter is introduced by Glasner in [12]. In this paper, we generalize the notion of $\Gamma$-boundary through the following characterization (c.f., Theorem 3.2).

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Theorem A. For a countable discrete group $\Gamma$, let $X$ be a minimal $\Gamma$-space and $(Y, \varphi)$ be an extension of $X$, inducing an extension $(C(Y), \tilde{\varphi})$ of $C(X)$. The following are equivalent:

1. $(C(Y), \tilde{\varphi})$ is a $\Gamma$-essential extension of $C(X)$.
2. $Y$ is minimal and, for every $\nu \in \text{Prob}(Y)$, if the restriction of Poisson map $P_\nu : C(Y) \to \ell^\infty(\Gamma)$ to $C(X)$ via $\tilde{\varphi}$ is isometric, then $P_\nu$ is isometric on $C(Y)$.
3. $Y$ is minimal and, for every $\nu \in \text{Prob}(Y)$, if the push forward of $\nu$ on $X$ via $\varphi$ is contractible, then $\nu$ is contractible.
4. $(Y, \varphi)$ is a minimal strongly proximal extension of $X$.

We say that $(Y, \varphi)$ is a $(\Gamma, X)$-boundary (or an $X$-boundary for short) if it satisfies any of the above equivalent conditions.

Though we focus on minimal $\Gamma$-spaces, the above theorem (excluding the last item) holds for arbitrary $\Gamma$-space as well (with slight modification of the proof). We also point out that our definition of a $(\Gamma, X)$-boundary is equivalent to the definition proposed by Kennedy and Schafhauser ([19, Remark 2.3, Corollary 2.7]).

To make the construction of $(\Gamma, X)$-boundaries somewhat clearer, we completely describe them when $X$ is minimal and finite. Indeed we show that for a minimal finite $\Gamma$-space $X$, any $(\Gamma, X)$-boundary can be characterized by induced action of some $\Lambda$-boundary, where $\Lambda$ is a subgroup of $\Gamma$ of finite index (cf. Theorem 3.5).

Theorem B. Let $\Gamma$ be a countable discrete group, and let $\Lambda$ be a finite index subgroup of $\Gamma$. If $Y$ is a $\Lambda$-boundary, the induced $\Gamma$-space $\tilde{Y}$ is a $(\Gamma, \Gamma/\Lambda)$-boundary. Conversely, for a minimal finite $\Gamma$-space $X$, every $(\Gamma, X)$-boundary is the induced $\Gamma$-space of a $\Lambda$-boundary, for some finite index subgroup $\Lambda \leq \Gamma$. In particular, when $X$ is finite, the universal $(\Gamma, X)$-boundary $\partial_F(\Gamma, X)$ is the induced $\Gamma$-space of the Furstenberg boundary $\partial_F\Lambda$, for some subgroup $\Lambda \leq \Gamma$ of finite index.

Hadwin and Paulsen in [14] asked the following question: Let $\Gamma$ be a countable discrete group and $X$ be a minimal $\Gamma$-space. For the universal minimal $\Gamma$-space $L$, is $C(L)$ the $\Gamma$-injective envelope of $C(X)$? Using the notion of $(\Gamma, X)$-boundary, we give a negative answer to this question as follows (c.f., Theorem 4.2).

Theorem C. If $L$ is the universal minimal $\Gamma$-space and $Y$ is an $X$-boundary for a minimal finite $\Gamma$-space $X$, then $I_\Gamma(C(Y)) \not\cong C(L)$.

Finally we use $(\Gamma, X)$-boundaries to study the problem of exactness for the corresponding reduced crossed product $C(X) \rtimes_r \Gamma$. A discrete group $\Gamma$ is exact if the reduced group $C^*$-algebra $C^*_r(\Gamma)$ is exact. We show that if $\Gamma$ is exact and $C(X)$ is $\Gamma$-injective, the action $\Gamma \curvearrowright X$ is amenable. In particular, for the universal $(\Gamma, X)$-boundary $\partial_F(\Gamma, X)$, we have the following (c.f., Theorem 5.2).
**Theorem D.** Let $\Gamma$ be a countable discrete group. The following are equivalent:

1. $\Gamma$ is exact,
2. For every minimal $\Gamma$-space $X$, the $\Gamma$-action on $\partial F(\Gamma, X)$ is amenable,
3. For every minimal $\Gamma$-space $X$, $C(\partial F(\Gamma, X)) \rtimes_r \Gamma$ is nuclear,
4. For every minimal $\Gamma$-space $X$, $C(X) \rtimes_r \Gamma$ is exact.

The paper is organized as follows. In addition to this introduction, we have four other sections. In Section 2, we briefly review the background material. In Section 3, we discuss topological $\Gamma$-boundaries and introduce the notion of $(\Gamma, X)$-boundary for a $\Gamma$-space $X$. We show that the $(\Gamma, X)$-boundaries are the same as minimal strongly proximal extensions of $X$. This is employed to deal with the $(\Gamma, X)$-boundaries of finite $\Gamma$-space $X$, which in turn provides a negative answer to the Hadwin-Paulsen problem in Section 4. In section 5, we find conditions for the exactness of the reduced crossed product $C(X) \rtimes_r \Gamma$.

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2. **Preliminaries**

Let $\Gamma$ be a discrete group. A compact Hausdorff space $X$ is a $\Gamma$-space if there is a group homomorphism from $\Gamma$ into the group of homeomorphisms on $X$. In this case we write $\Gamma \curvearrowright X$. For $s \in \Gamma$ and $x \in X$ we denote the image of $x$ under $s$ by $sx$. The action $\Gamma \curvearrowright X$ induces an action on the algebra $C(X)$ of continuous functions on $X$ given by

$$(sf)(x) = f(s^{-1}x), \quad (s \in \Gamma, \ f \in C(X), \ x \in X).$$

Similarly, $\Gamma$ acts on the set $\text{Prob}(X)$ of probability measures on $X$ via

$$s\nu(Y) = \nu(s^{-1}Y), \quad (s \in \Gamma, \ \nu \in \text{Prob}(X), \ Y \in \mathcal{B}_X).$$

A map $\varphi : Y \rightarrow X$ between $\Gamma$-spaces is a $\Gamma$-map when $\varphi$ is continuous and $\varphi(sy) = s\varphi(y)$, for each $y \in Y$ and $s \in \Gamma$. If $\varphi : Y \rightarrow X$ is a surjective $\Gamma$-map, the pair $(Y, \varphi)$ is called an extension (or, in some texts, a cover) of $X$. We also refer to $Y$ or $\varphi$ as an extension of $X$.

A $\Gamma$-space $Y$ is minimal if for every $y \in Y$, the $\Gamma$-orbit $\Gamma y = \{tx \mid t \in \Gamma\}$ is dense in $Y$, and strongly proximal if for every probability measure $\nu \in \text{Prob}(Y)$, the weak* closure of the $\Gamma$-orbit $\Gamma \nu = \{s\nu : s \in \Gamma\}$ contains a point mass $\delta_y \in \text{Prob}(Y)$, for some $y \in Y$. A $\Gamma$-space $Y$ is said to be a $\Gamma$-boundary if $Y$ is both minimal and strongly proximal. Furstenberg in [11]
proved that every group \( \Gamma \) has a unique \( \Gamma \)-boundary \( \partial \Gamma \), which is universal, in the sense that every \( \Gamma \)-boundary is an image of \( \partial \Gamma \).

Consider the Stone-Čech compactification \( \beta \Gamma \) of \( \Gamma \). The action \( \Gamma \curvearrowright \beta \Gamma \) induces a semigroup structure on \( \beta \Gamma \). A subset \( I \) of \( \beta \Gamma \) is a left ideal if \( (\beta \Gamma)I \subseteq I \). By Zorn lemma, \( \beta \Gamma \) has a minimal left ideal which is unique up to homeomorphism [17, 2.9]. We here denote this \( \Gamma \)-space by \( L \). It is known that \( L \) is the universal minimal \( \Gamma \)-space [13, I.4], i.e., every minimal \( \Gamma \)-space is an image of \( L \) through a surjective \( \Gamma \)-map. In addition \( L \) is \( \Gamma \)-projective [14, 3.17], in the sense that, for any \( \Gamma \)-spaces \( X \) and \( Y \), any \( \Gamma \)-map \( \psi : L \to X \), and any surjective \( \Gamma \)-map \( \varphi : Y \to X \), there exists a \( \Gamma \)-map \( \theta : L \to Y \) such that \( \varphi \theta = \psi \).

An operator system \( V \) is a unital self-adjoint subspace of a unital \( C^\ast \)-algebra. We say that \( V \) is a \( \Gamma \)-operator system if there is a homomorphism from \( \Gamma \) into the group of order isomorphisms of \( V \). A linear map \( \phi : V \to W \) between \( \Gamma \)-operator systems is unital if \( \phi(1_V) = 1_W \), it is positive if it sends positive elements to positive elements, and completely positive (completely isometric) if the maps \( \text{id} \otimes \phi : M_n(\mathbb{C}) \otimes V \to M_n(\mathbb{C}) \otimes W \) are positive (isometric), for all \( n \in \mathbb{N} \). We call \( \phi : V \to W \) a \( \Gamma \)-map if it is unital completely positive and \( \Gamma \)-equivariant, that is \( \phi(sv) = s\phi(v) \), for each \( s \in \Gamma \) and \( v \in V \). If a \( \Gamma \)-map \( \phi : V \to W \) is completely isometric, the pair \((W, \phi)\) is called an extension of \( V \). In this case, we also refer to \( W \) or \( \phi \) as an extension of \( V \). An extension \((W, \phi)\) of \( V \) is \( \Gamma \)-essential if for every \( \Gamma \)-map \( \psi : W \to U \) such that \( \psi \phi \) is completely isometric on \( V \), \( \psi \) is completely isometric on \( W \). It is \( \Gamma \)-rigid if for every \( \Gamma \)-map \( \psi : W \to W \) such that \( \psi \phi = \phi \) on \( V \), \( \psi \) is the identity map on \( W \).

A \( \Gamma \)-operator system \( U \) is \( \Gamma \)-injective if for every \( \Gamma \)-map \( \phi : V \to U \) and every extension \( \iota : V \to W \), there is a \( \Gamma \)-map \( \psi : W \to U \) such that \( \psi \iota = \phi \). Given a \( \Gamma \)-operator system \( V \), we say that \((\mathcal{I}, \kappa)\) is the \( \Gamma \)-injective envelope of \( V \) provided that \( \mathcal{I} \) is \( \Gamma \)-injective and \((\mathcal{I}, \kappa)\) is an extension of \( V \) such that any other \( \Gamma \)-injective \( \Gamma \)-operator system \( \mathcal{I}_1 \) with \( \kappa(V) \subseteq \mathcal{I}_1 \subseteq \mathcal{I} \), we have \( \mathcal{I}_1 = \mathcal{I} \). In the other words, \((\mathcal{I}, \kappa)\) is the \( \Gamma \)-injective envelope of \( V \) if \((\mathcal{I}, \kappa)\) is a minimal \( \Gamma \)-injective extension of \( V \). Hamana showed that for a discrete group \( \Gamma \) and a \( \Gamma \)-operator system \( V \), the \( \Gamma \)-injective envelope always exists and is unique up to complete isometric \( \Gamma \)-equivariant isomorphism. In addition, the \( \Gamma \)-injective envelope is exactly the maximal \( \Gamma \)-essential extension, and it is \( \Gamma \)-rigid [15, 2.4, 2.6]. We denote the \( \Gamma \)-injective envelope of a \( \Gamma \)-operator system \( V \) by \( I_\Gamma(V) \). For a \( \Gamma \)-space \( X \), \( I_\Gamma(C(X)) \) is a commutative \( C^\ast \)-algebra under the Choi-Effros product [8].

If \( X \) is a \( \Gamma \)-space, for every \( \nu \in \text{Prob}(X) \), the \( \Gamma \)-map \( \mathcal{P}_\nu : C(X) \to \ell^\infty(\Gamma) \), called the Poisson map, is defined as follows:

\[
\mathcal{P}_\nu(f)(s) = \langle f, s\nu \rangle = \int_X f(sx) d\nu(x), \quad (s \in \Gamma, \ f \in C(X)).
\]
Every $\Gamma$-map $\varphi : C(X) \to \ell^\infty(\Gamma)$ is a Poisson map for some probability measure on $X$: for $\nu = \hat{\varphi}(\delta_e)$, where $\hat{\varphi}$ is the adjoint of $\varphi$ and $\delta_e$ is the Dirac measure at the identity element of $\Gamma$, we have $\varphi = \mathcal{P}_\nu$.

3. Boundary Extensions

Kalantar and Kennedy proved in [18] that $I_\Gamma(\mathbb{C})$, as the maximal $\Gamma$-essential extension of $\mathbb{C}$, can be identified with $C(\partial_F \Gamma)$, for the universal $\Gamma$-boundary $\partial_F \Gamma$. In particular, a $\Gamma$-space $Y$ is a $\Gamma$-boundary precisely when $C(Y)$ is a $\Gamma$-essential extension of $\mathbb{C}$. We wish to replace $\mathbb{C}$ by $C(X)$, for a minimal $\Gamma$-space $X$. For this, let us first review the construction of Kalantar and Kennedy.

For a $\Gamma$-space $Y$, $C(Y)$ is a $\Gamma$-essential extension of $\mathbb{C}$, if for any $\Gamma$-operator system $V$, every $\Gamma$-map $\theta : C(Y) \to V$ is isometric. Let us note that one need to verify this only for the $\Gamma$-operator system $\ell^\infty(\Gamma)$: if $\tau$ is in the state space of $V$, there is a $\Gamma$-map $\mathcal{P}_\tau : V \to \ell^\infty(\Gamma)$, given by $\mathcal{P}_\tau(v)(t) = \langle v, t\tau \rangle$, and $\mathcal{P}_\tau \theta$ is isometric. On the other hand, every $\Gamma$-map from $C(Y)$ into $\ell^\infty(\Gamma)$ is a Poisson map for some probability measure on $Y$. Hence $C(Y)$ is a $\Gamma$-essential extension of $\mathbb{C}$ exactly when for every $\nu \in \text{Prob}(Y)$, every Poisson map $\mathcal{P}_\nu : C(Y) \to \ell^\infty(\Gamma)$ is isometric. In [2], Azencott showed that for a measure $\nu \in \text{Prob}(Y)$, the Poisson map $\mathcal{P}_\nu : C(Y) \to \ell^\infty(\Gamma)$ is isometric if and only if $\{\delta_y : y \in Y\} \subseteq \Gamma \nu^{-1}$. If a measure has this property, we say that it is contractible. Note that if $Y$ is minimal, all measures in $\text{Prob}(Y)$ are contractible precisely when $Y$ is strongly proximal. This is to say that $C(Y)$ is a $\Gamma$-essential extension of $\mathbb{C}$ if and only if $Y$ is a $\Gamma$-boundary. In particular, by considering the contravariant functor between $\Gamma$-spaces and commutative $\Gamma$-$C^*$-algebras, the maximal $\Gamma$-essential extension of $\mathbb{C}$ is $C(\partial_F \Gamma)$.

Next definition is due to Glasner [12].

**Definition 3.1.** Let $X$ be a $\Gamma$-space and $\varphi : Y \to X$ be an extension of $X$.

(i) $(Y, \varphi)$ is called a **minimal** extension if $Y$ is minimal.

(ii) $(Y, \varphi)$ is called a **strongly proximal** extension if for every $\nu \in \text{Prob}(Y)$ with $\text{supp}(\nu) \subseteq \varphi^{-1}(x)$, for some $x \in X$, $\delta_y \in \overline{\Gamma \nu^{-1}}$, for some $y \in Y$.

When $X$ is singleton with trivial action, the minimal strongly proximal extensions of $X$ are exactly topological boundaries in the sense of Furstenberg. Following the above observations, we are lead to introduce a generalization of the notion of topological $\Gamma$-boundaries.

**Theorem 3.2.** For a countable discrete group $\Gamma$, let $X$ be a minimal $\Gamma$-space and $(Y, \varphi)$ be an extension of $X$, inducing an extension $(C(Y), \hat{\varphi})$ of $C(X)$. The following are equivalent:

1. $(C(Y), \hat{\varphi})$ is a $\Gamma$-essential extension of $C(X)$.
2. $Y$ is minimal and, for every $\nu \in \text{Prob}(Y)$, if the restriction of Poisson map $\mathcal{P}_\nu : C(Y) \to \ell^\infty(\Gamma)$ to $C(X)$ via $\hat{\varphi}$ is isometric, then $\mathcal{P}_\nu$ is isometric on $C(Y)$.
(3) \(Y\) is minimal and for every \(\nu \in \text{Prob}(Y)\), if the push forward of \(\nu\) on \(X\) via \(\varphi\) is contractible, then \(\nu\) is contractible.

(4) \((Y, \varphi)\) is a minimal strongly proximal extension of \(X\).

**Proof.** (1) \(\Rightarrow\) (2). First we show that \(Y\) is minimal. Let \(L\) be the universal minimal \(\Gamma\)-space. Since \(X\) is minimal, there is an extension \(\psi : L \to X\), inducing a \(\Gamma\)-equivariant \(*\)-monomorphism \(\tilde{\psi} : C(X) \to C(L)\). The \(\Gamma\)-space \(L\) is \(\Gamma\)-projective, and so there is a \(\Gamma\)-map \(\pi : L \to Y\) with \(\varphi \pi = \psi\).

Since \(\tilde{\pi} \tilde{\varphi} = \tilde{\psi}\) is a \(*\-\)monomorphism and \(C(Y)\) is \(\Gamma\)-essential, \(\tilde{\pi}\) is a \(*\)-monomorphism. This means that \(\pi\) is surjective. Thus \(Y\) is a minimal \(\Gamma\)-space. Now (2) follows, because every \(\Gamma\)-map from \(C(Y)\) to \(\ell^\infty(\Gamma)\) is isometric, and a Poisson map is isometric if and only if the corresponding measure is contractible. By the construction of \(Y\), \(\mathcal{P}_\tau \varphi |_{C(X)}\) is a Poisson map with measure \(\delta_x\). Since \(\delta_x\) is contractible, \(\mathcal{P}_\tau \varphi |_{C(X)}\) is an isometry. By (2), \(\mathcal{P}_\tau \varphi\) is also isometric. Therefore, for \(f \in C(Y)\),

\[
\|f\| = \|\mathcal{P}_\tau \varphi(f)\| \leq \|\varphi(f)\| \leq \|f\|,
\]

which means that \(\theta\) is isometric.

(2) \(\iff\) (3). It is straightforward to see that the restrictions of \(\mathcal{P}_\nu\) to \(C(X)\) is a Poisson map, with the push forward of \(\nu\) as its measure. Now apply the result of Azencott.

(3) \(\Rightarrow\) (4). Let \(\varphi_* \nu\) be the push forward of \(\nu\). Let us observe that \(\varphi_* \nu(x) = \delta_x\), for some \(x \in X\), if and only if, \(\text{supp}(\nu) \subseteq \varphi^{-1}(x)\). For any Borel set \(E \subseteq B_X\), if \(x \in E\) then \(\varphi^{-1}(x) \subseteq \varphi^{-1}(E)\). Thus \(\text{supp}(\nu) \subseteq \varphi^{-1}(x)\) implies \(\varphi_* \nu = \delta_x\). Conversely, let \(\varphi_* \nu = \delta_x\), for some \(x \in X\), and \(y \notin \varphi^{-1}(x)\). Since \(\varphi(y) \neq x\), there are open neighborhoods \(V_{\varphi(y)}\) and \(V_x\) such that \(V_{\varphi(y)} \cap V_x = \emptyset\). Put \(U_y = \varphi^{-1}(V_{\varphi(y)})\). Then \(U_y\) is an open neighborhood of \(y\) such that \(\nu(U_y) = \nu(\varphi^{-1}(V_{\varphi(y)})) = 0\). Hence \(y \notin \text{supp}(\nu)\). Now if \(\text{supp}(\nu) \subseteq \varphi^{-1}(x)\), for some \(x\), (4) follows because \(\varphi_* \nu = \delta_x\) is contractible.

(4) \(\Rightarrow\) (3). Suppose \(\nu \in \text{Prob}(Y)\) such that \(\varphi_* \nu\) is contractible. Then \(\delta_x \in \overline{\varphi_* \nu}^{\ast}\), for some \(x \in X\). Since \(\varphi_*\) is isometric, this is equivalent to the existence of \(\nu' \in \text{Prob}(Y)\) with \(\varphi_* \nu' = \delta_x\), and \(\nu' \in \overline{\nu}^{\ast}\). This is to say that there exists \(\nu' \in \text{Prob}(Y)\) such that \(\text{supp}(\nu') \subseteq \varphi^{-1}(x)\) and \(\nu' \in \overline{\nu}^{\ast}\).

By (3), \(\delta_y \in \overline{\delta_y}^{\ast}\), for some \(y \in Y\). Since \(\overline{\delta_y}^{\ast} \subseteq \overline{\nu}^{\ast}\), we get \(\delta_y \in \overline{\nu}^{\ast}\). This plus minimality of \(Y\) finishes the proof.

**Definition 3.3.** We say that \((Y, \varphi)\) is a \((\Gamma, X)\)-boundary (or simply a \(X\)-boundary), if \((Y, \varphi)\) satisfies any of the above equivalent conditions.
When $X$ is singleton with trivial action, the $X$-boundaries are exactly the topological boundaries in the sense of Furstenberg.

Let $X$ be a $\Gamma$-space. the commutative $C^*$-algebra $I_F(C(X))$ is the maximal $\Gamma$-essential extension of $C(X)$. By Definition 3.3, the spectrum of $I_F(C(X))$ is a $(\Gamma, X)$-boundary. In fact, by contravariance, it is universal among all $(\Gamma, X)$-boundaries. We denote this universal $(\Gamma, X)$-boundary, which is unique up to homeomorphism, by $\partial_F(\Gamma, X)$, and write $I_F(C(X)) = C(\partial_F(\Gamma, X))$. In particular, the universal strongly proximal extension of a minimal $\Gamma$-space $X$ always exists. Note that if $X$ is singleton, $\partial_F(\Gamma, X)$ is nothing but the Furstenberg universal $\Gamma$-boundary $\partial_F \Gamma$.

**Corollary 3.4.** For countable discrete group $\Gamma$ and $\Gamma$-space $X$, $C(\partial_F(\Gamma, X))$ is $\Gamma$-essential and $\Gamma$-rigid, and the only unital positive $\Gamma$-equivariant map from $C(\partial_F(\Gamma, X))$ to itself is the identity map.

Next we investigate the structure of $(\Gamma, X)$-boundaries when $X$ is a finite minimal $\Gamma$-space.

Let $\Gamma \acts X$ is an action of $\Gamma$ on $X$ and $\Lambda$ be another discrete group. A cocycle of the action in $\Lambda$ is a map $\alpha : \Gamma \times X \to \Lambda$ such that

$$\alpha(\gamma_1\gamma_2, x) = \alpha(\gamma_1, \gamma_2 x)\alpha(\gamma_2, x),$$

for all $\gamma_1, \gamma_2 \in \Gamma$ and $x \in X$.

We need the notion of induced $\Gamma$-spaces [27, 4.2.21] (c.f., [9, 2.2.4]). Let $\Lambda$ be a finite index subgroup of a countable discrete group $\Gamma$, $Y$ be a $\Lambda$-space, and $\tilde{Y} = \Gamma/\Lambda \times Y$. Take a transversal $T = \{t_1, \ldots, t_n\}$ for $\Gamma/\Lambda$ such that $t_1 = e$. Define the cocycle $\alpha : \Gamma \times \Gamma/\Lambda \to \Lambda$ by $\alpha(\gamma, t_i\Lambda) = \lambda$, such that $\gamma t_i \lambda \in T$, and observe that such a $\lambda$ is unique. Now $\Gamma$ acts on $\Gamma/\Lambda \times Y$ by

$$\gamma.(t_i\Lambda, y) = (\gamma t_i\alpha(\gamma, t_i\Lambda)\Lambda, \alpha(\gamma, t_i\Lambda)^{-1}y), \quad (\gamma \in \Gamma, y \in Y).$$

The $\Gamma$-space $\tilde{Y}$ is called the induced $\Gamma$-space of the $\Lambda$-space $Y$.

**Theorem 3.5.** Let $\Gamma$ be a countable discrete group, and let $\Lambda$ be a finite index subgroup of $\Gamma$. If $Y$ is a $\Lambda$-boundary, the induced $\Gamma$-space $\tilde{Y}$ is a $(\Gamma, \Gamma/\Lambda)$-boundary. Conversely, for a minimal finite $\Gamma$-space $X$, every $(\Gamma, X)$-boundary is the induced $\Gamma$-space of a $\Lambda$-boundary, for some finite index subgroup $\Lambda \leq \Gamma$. In particular, when $X$ is finite, the universal $(\Gamma, X)$-boundary $\partial_F(\Gamma, X)$ is the induced $\Gamma$-space of the Furstenberg boundary $\partial_F \Lambda$, for some subgroup $\Lambda \leq \Gamma$ of finite index.

**Proof.** With the above notations, we show that $\tilde{Y}$ is a $(\Gamma, \Gamma/\Lambda)$-boundary. To see that $\tilde{Y}$ is $\Gamma$-minimal, let $t \in T$ and $y, y' \in Y$. Since $Y$ is $\Lambda$-minimal, there exists $\{\lambda_t\} \subseteq \Lambda$ such that $\lambda_t y \to y'$. Fix $\lambda_t \in \Lambda$, and observe that $\alpha(t\lambda_t, \Lambda) = \lambda_t^{-1}$. Thus

$$t\lambda_t(\Lambda, y) = (t\lambda_t\alpha(t\lambda_t, \Lambda)\Lambda, \alpha(t\lambda_t, \Lambda)^{-1}y) = (t\Lambda, \lambda_t y).$$

When $\ell$ tends to infinity, $t\lambda_{t}(\Lambda, y) \to (t\Lambda, y')$. Therefore, $\overline{\Gamma(\Lambda, y)} = \tilde{Y}$. Similarly, $\lambda_{t^{-1}}(t\Lambda, y) \to (\Lambda, y')$, which implies that $(\Lambda, y') \in \overline{\Gamma(t\Lambda, y)}$, for
t \in T$. Hence $\overline{\Gamma(\Lambda, \gamma')} \subseteq \overline{\Gamma(t\Lambda, y)}$. We have shown that

$$\overline{\Gamma(t\Lambda, y)} = \overline{\Gamma(\Lambda, \gamma')} \subseteq \overline{\Gamma(t\Lambda, y)} \subseteq \overline{Y}.$$  

Therefore, $\overline{\Gamma(t\Lambda, y)} = \overline{Y}$, for $t \in T$ and $y \in Y$.

Next let us observe that $\overline{Y}$ is a strongly proximal extension of $\Gamma/\Lambda$. Consider the continuous surjective map $\varphi : \overline{Y} \to \Gamma/\Lambda$ given by $\varphi(t\Lambda, y) = t\Lambda$, for $y \in Y$. We show that $\varphi$ is $\Gamma$-equivariant. Put $\Lambda_j = t_j\Lambda t_j^{-1}$. Then $\Lambda_j$ is a subgroup of $\Gamma$, and $\{t_j \Lambda t_j^{-1} : 1 \leq j \leq n\}$ is a transversal for $\Gamma/\Lambda_j$. Therefore, $\Gamma = \bigsqcup_{j=1}^nt_j \Lambda t_j^{-1}$. If $\gamma = t_j \Lambda t_j^{-1} \lambda_j \in t_j \Lambda t_j^{-1} \Lambda_j$, since $t_j^{-1} \lambda_j t_j \in \Lambda$, we have

$$\varphi(t_j \Lambda, y) = t_j \Lambda.$$  

On the other hand, $\alpha(\gamma, t_j \Lambda) = e$, and so

$$\varphi(\gamma(t_j \Lambda, y)) = \varphi(\gamma(t_j \Lambda) \alpha(\gamma, t_j \Lambda)) \alpha(\gamma, t_j \Lambda)^{-1}.$$  

Thus, $\varphi(t_j \Lambda, y) = \varphi(\gamma(t_j \Lambda, y))$. We have shown that $\varphi : \overline{Y} \to \Gamma/\Lambda$ is an extension of $\Gamma/\Lambda$. To show that the extension is strongly proximal, let $\nu \in \text{Prob}(\overline{Y})$ such that $\text{supp}(\nu) \subseteq \varphi^{-1}(t\Lambda) = \{t\Lambda \times Y\}$, for some $t \in T$. Then $\nu = \delta_{t\Lambda} \times \mu$, for some $\mu \in \text{Prob}(Y)$. Since $Y$ is $\Lambda$-strongly proximal, there exists $\{\lambda\} \subseteq \Lambda$ such that $\lambda \mu \xrightarrow{\text{weak}^*} \delta_{y_0}$, for some $y_0 \in Y$. We claim that if $\{\gamma_t\}_{t} = \{t_i \lambda t_i^{-1}\}_{t} \subseteq \Gamma$, then $\gamma_t \nu \xrightarrow{\text{weak}^*} \delta_{(t_i \Lambda, y_0)}$. Let $h \in C(\Gamma/\Lambda \times Y)$ and fix $\lambda_t \in \Lambda$. For $t_i \in T$, $\alpha(t_i \lambda t_i^{-1}, t_i \Lambda) = \lambda_t^{-1}$,

$$\int h((t_i \lambda t_i^{-1})(t\Lambda, y))d\nu(t\Lambda, y) = \int h((t_i \lambda t_i^{-1})(t\Lambda, y))d(\delta_{t\Lambda} \times \mu)(t\Lambda, y)$$

$$= \int h((t_i \lambda t_i^{-1})(t\Lambda, y))d(\delta_{t\Lambda} \times \mu)(t\Lambda, y)$$

$$= \int h(t_i \lambda t_i^{-1} \alpha(t_i \lambda t_i^{-1}, t_i \Lambda) \alpha(t_i \lambda t_i^{-1}, t_i \Lambda)^{-1} y)d(\delta_{t\Lambda} \times \mu)(t\Lambda, y)$$

$$= \int h((t\Lambda, \lambda t y))d(\delta_{t\Lambda} \times \mu)(t\Lambda, y).$$

When $\lambda_t$ tends to infinity,

$$\int h((t_i \lambda t_i^{-1})(t\Lambda, y))d\nu(t\Lambda, y) \to h(t_i \Lambda, y_0).$$

Therefore, $\gamma_t \nu \xrightarrow{\text{weak}^*} \delta_{(t_i \Lambda, y_0)}$. This means that $\overline{Y}$ is a $\Gamma$-strongly proximal extension of $X$, and so is a $(\Gamma, \Gamma/\Lambda)$-boundary.

Note that $\overline{Y} = \bigsqcup_{i=1}^n t_i \Lambda \times Y$. It is not hard to see that any $t_i \Lambda \times Y$ is a $\Lambda_i$-boundary when $\Lambda_i = t_i \Lambda t_i^{-1}$. Conversely, Let $X = \{x_1, \ldots, x_n\}$ be a minimal $\Gamma$-space. Let $\varphi : Y \to X$ be a surjective $\Gamma$-map making $(Y, \varphi)$ a $(\Gamma, X)$-boundary. We write $Y = \bigsqcup_{i=1}^n Y_i$, for $Y_i = \varphi^{-1}(x_i)$, $1 \leq i \leq n$. Fix $x_i \in X$, and consider the stabilizer subgroup $\Lambda_i = \{\gamma \in \Gamma : \gamma x_i = x_i\}$. We claim that $Y_i$ is a $\Lambda_i$-boundary. First let us show that $Y_i$ is $\Lambda_i$-minimal. Given $y, y' \in Y_i$,
since \( Y \) is \( \Gamma \)-minimal, there exists \( \{ \gamma_\ell \} \subseteq \Gamma \) such that \( \gamma_\ell y \to y' \). Since \( y', y \in Y_\ell = \varphi^{-1}(x_i) \) and \( \varphi \) is a \( \Gamma \)-map, \( \gamma_\ell x_i \to x_i \). Therefore, for sufficiently large \( \ell \), \( \gamma_\ell \in \Lambda_i \). Hence \( Y_\ell \) is \( \Lambda_i \)-minimal. To show that \( Y_\ell \) is \( \Lambda_i \)-strongly proximal, take \( \nu \in \text{Prob}(Y_\ell) \), then since \( (Y, \varphi) \) is a \((\Gamma, X)\)-boundary and \( \text{supp}(\nu) \subseteq Y_\ell = \varphi^{-1}(x_i) \), there exists \( y \in \varphi^{-1}(x_i) \) and \( \{ \gamma_\ell \} \subseteq \Gamma \) such that \( \gamma_\ell \nu \xrightarrow{\text{weak*}} \delta_y \). Thus \( \gamma_\ell \varphi_* \nu = \varphi_*(\gamma_\ell \nu) \xrightarrow{\text{weak*}} \delta_{x_i} \). Also, \( \text{supp}(\nu) \subseteq \varphi^{-1}(x_i) \), exactly when \( \varphi_* \nu = \delta_{x_i} \). This implies that \( \delta_{\gamma_\ell x_i} = \gamma_\ell \delta_{x_i} \xrightarrow{\text{weak*}} \delta_{x_i} \). Thus, for sufficiently large \( \ell \), \( \delta_{\gamma_\ell x_i} = \delta_{x_i} \). Therefore, for sufficiently large \( \ell \), \( \gamma_\ell \in \Lambda_i \).

Without loss of generality, we suppose \( i = 1 \). Let \( y \in Y_1 = \varphi^{-1}(x_1) \), and let \( T = \{ t_1, \ldots, t_n \} \) be a transversal for \( \Gamma/\Lambda_1 \). For every \( 1 \leq i \leq n \), \( t_i x_1 = x_i \), which implies that \( t_i y \in Y_1 \). Thus, \( t_i Y_1 \subseteq Y_1 \). Similarly, \( Y_1 \subseteq t_i Y_1 \). Therefore, \( Y \) is a disjoint union of \( n \)-copies of \( Y_1 \). By considering the homeomorphism \( \bigsqcup_{t_i \in T} t_i Y_1 \to \Gamma/\Lambda_1 \times Y_1 \), given by \( t_i y \mapsto (t_i \Lambda_1, y) \), and inducing the action of \( \bigsqcup_{t_i \in T} t_i Y_1 \) to \( \Gamma/\Lambda_1 \times Y_1 \), the \((\Gamma, X)\)-boundary \( Y \) is in the form of an induced \( \Gamma \)-space of \( \Lambda_1 \)-space \( Y_1 \). Note that \( \Gamma/\Lambda_1 \cong X \).

To prove the last part, since \( \partial_F(\Gamma, X) \) is a \((\Gamma, X)\)-boundary, \( \partial_F(\Gamma, X) \) is in the form of an induced \( \Gamma \)-space \( X \times Y \), where \( Y \) is a \( \Lambda \)-boundary, for some finite index subgroup \( \Lambda \). On the other hand, the induced \( \Gamma \)-space \( X \times \partial_F \Lambda \) is a \((\Gamma, X)\)-boundary. There exists a surjective \( \Lambda \)-map \( \theta : \partial_F \Lambda \to Y \) which induces a surjective \( \Gamma \)-map \( \Theta : X \times \partial_F \Lambda \to X \times Y \), given by \( \Theta(x, f) = (x, \theta(f)) \). Also \( \partial_F(\Gamma, X) \cong X \times Y \) is universal, so there exists a surjective \( \Gamma \)-map \( \Psi : X \times Y \to X \times \partial_F \Lambda \). Now by Corollary 3.4, \( \Theta \Psi \) is the identity and \( X \times Y \cong X \times \partial_F \Lambda \).

By the above theorem, if \( \Gamma \) is amenable and \( (Y, \varphi) \) is a \((\Gamma, X)\)-boundary for minimal finite \( \Gamma \)-space \( X = \{ x_1, \ldots, x_n \} \), then \( Y \) has exactly \( n \) elements. This is because \( Y = \bigsqcup_{i=1}^{n} \varphi^{-1}(x_i) \), where for every \( i \), \( \varphi^{-1}(x_i) \) is a \( \Lambda_i \)-boundary for \( \Lambda_i = \{ \gamma \in \Gamma : \gamma x_i = x_i \} \). Now every \( \Lambda_i \) is amenable, which implies that every \( \varphi^{-1}(x_i) \) is singleton.

4. On a problem of Hadwin and Paulsen

There is a contravariant functor between the category of compact Hausdorff spaces with continuous maps and the category of unital commutative \( C^* \)-algebras with \(*\)-homomorphisms, sending projective objects to injective objects. Hadwin and Paulsen [14] showed that for every compact Hausdorff space \( X \), there is a unique projective cover \( P \), which is minimal among all projective covers of \( X \). As a result, the injective envelope \( I(C(X)) \) of \( C(X) \) is \(*\)-isomorphic to \( C(P) \). They also extended this to the case when a countable discrete group \( \Gamma \) acts on a compact space \( X \), using the functor between compact \( \Gamma \)-spaces and unital commutative \( \Gamma \)-\( C^* \)-algebras. In this case, unlike the previous case where rigidity and essentiality of projective covers are equivalent [14, Proposition 2.11], there is no \( \Gamma \)-projective, \( \Gamma \)-rigid cover, even when \( X \) is a singleton [14, Proposition 3.1]. However, one still could work with \( \Gamma \)-essential covers.
Hadwin and Paulsen showed that if $\Gamma$ is a countable discrete group and $X$ is a minimal $\Gamma$-space, a minimal left ideal of the Stone-Cech compactification of $\Gamma$, is the minimal $\Gamma$-projective cover of $X$. This leads naturally to the question that for a minimal left ideal $L$ in $\beta\Gamma$, is $C(L)$ $\ast$-isomorphic to $I_\Gamma(C(X))$? Recall that any two minimal left ideals in $\beta\Gamma$ are homeomorphic, and the minimal left ideal in $\beta\Gamma$ is nothing but the universal minimal $\Gamma$-space.

Since $L$ is $\Gamma$-projective, $C(L)$ is $\Gamma$-injective in the category of $\Gamma$-$C^*$-algebras, and so $I_\Gamma(C(L)) = C(L)$. However, as we see soon, the problem of Hadwin and Paulsen has negative answer in general. For this let us first observe that for an arbitrary countable infinite group $\Gamma$, there is an infinite compact minimal $\Gamma$-space which has an invariant measure.

A probability-measure-preserving (p.m.p.) action of a group $\Gamma$ on a probability measure space $(X, \mu)$ is a homomorphism of $\Gamma$ into the group of measure-preserving transformations on $X$, parameterized by $\Gamma$. In this context, the action $\Gamma$ on $(X, \mu)$ is said to be free if there is a $\Gamma$-invariant set $X_0 \subseteq X$ with $\mu(X_0) = 1$, such that if $sx = x$, for some $x \in X_0$ and $s \in \Gamma$, then $s = e$.

Let $\Gamma$ be a countable infinite group. The Bernoulli shift action of $\Gamma$ on the space $\{0, 1\}^\Gamma$, with any invariant probability measure (for example, the product of equiprobability measure on $\{0, 1\}$) is a free p.m.p. action. Thus, every countable infinite group admits at least one non-trivial infinite free p.m.p. action. Benjamin Weiss in [26, 6.1] has shown that if $\Gamma$ is any countable infinite group and $\Gamma \acts (X, \mu)$ is any free p.m.p. action, there is a minimal continuous action as a subshift of $([0, 1] \times \mathbb{N})^\Gamma$, which admits an invariant measure and is a model for $(X, \mu)$ (that is, an isomorphic copy of the action which is also continuous). In particular, there exists a non-trivial minimal $\Gamma$-space with an invariant measure. We note that this $\Gamma$-space is infinite.

**Lemma 4.1.** Suppose that $L$ is the universal minimal $\Gamma$-space, $X$ is a minimal $\Gamma$-space, and $Z$ is an infinite minimal $\Gamma$-space with an invariant measure $\mu$. Consider the surjective $\Gamma$-maps $\varphi : L \to X$ and $\theta : L \to Z$. If the set of all pull backs of $\mu$ under $\theta$ contains a measure $\nu$ such that $\nu(\varphi^{-1}(x)) \neq 0$, for some $x \in X$, then $I_\Gamma(C(X)) \ncong C(L)$.

**Proof.** Suppose that $I_\Gamma(C(X)) \cong C(L)$. Fix a $\Gamma$-map $\varphi : L \to X$. By definition of the $\Gamma$-injective envelope, $C(L)$ is an $\Gamma$-essential $\Gamma$-extension of $C(X)$, and $L$ is a $(\Gamma, X)$-boundary. Let $\nu$ be a pull back of $\mu$ under $\theta$ such that $\nu(\varphi^{-1}(x)) \neq 0$. Define $\nu_x \in \text{Prob}(L)$ by $\nu_x(E) = \frac{\nu(E \cap \varphi^{-1}(x))}{\nu(\varphi^{-1}(x))}$. It is easy to see that $\text{supp}(\nu_x) \subseteq \varphi^{-1}(x)$ and $\nu_x \ll \nu$. Since $\text{supp}(\nu_x) \subseteq \varphi^{-1}(x)$ and $L$ is a $(\Gamma, X)$-boundary, there exists $\{t_\alpha\} \subseteq \Gamma$ and $\ell \in L$ such that $t_\alpha \nu_x \xrightarrow{\text{weak}} \delta_\ell$. Let $\varphi_x \nu_x = \mu_x$, then $t_\alpha \mu_x \xrightarrow{\text{weak}} \delta_{\theta(\ell)}$. Moreover, $\mu_x = \theta_x \nu_x \ll \theta_x \nu = \mu$. Thus $\mu$ is invariant, and $\delta_{\theta(\ell)} = \lim_\alpha t_\alpha \mu_x \ll \lim_\alpha t_\alpha \mu = \mu$. Therefore, $\mu(\{\theta(\ell)\}) > 0$. 

Consider the orbit $\Gamma\theta(\ell)$ of $\theta(\ell)$. For $x \in \Gamma\theta(\ell)$, $\mu(\{x\}) = \mu(\{\theta(\ell)\})$. In particular, $\Gamma\theta(\ell)$ must be finite, since otherwise, $\mu(Z) > \mu(\Gamma\theta(\ell)) > 1$. On the other hand, since $Z$ is minimal, $\Gamma\theta(\ell) = \overline{\theta(\ell)} = Z$. This implies that $Z$ is finite, which is a contradiction. \hfill $\square$

**Theorem 4.2.** If $L$ is the universal minimal $\Gamma$-space and $Y$ is an $X$-boundary for a minimal finite $\Gamma$-space $X$, then $I_\Gamma(C(Y)) \not\cong C(L)$.

**Proof.** Let us first observe that $I_\Gamma(C(X)) \not\cong C(L)$. For this, in the above notations let $\nu$ be any member of the set of all pull backs of $\mu$. Let $X = \{x_1, \ldots, x_n\}$. Since $L = \bigcup_{i=1}^n \varphi^{-1}(x_i)$, $0 \neq \nu(\ell) = \sum_{i=1}^n \nu(\varphi^{-1}(x_i))$. Hence there exists $x \in X$ such that $\nu(\varphi^{-1}(x)) \neq 0$.

On the other hand, $I_\Gamma(C(X)) = C(\partial_F(\Gamma, X))$, for the universal $X$-boundary $\partial_F(\Gamma, X)$, thus $C(\partial_F(\Gamma, X))$ is $\Gamma$-injective. By definition of $\Gamma$-injective envelope, $I_\Gamma(C(\partial_F(\Gamma, X))) = C(\partial_F(\Gamma, X))$, that is, $I_\Gamma(C(\partial_F(\Gamma, X))) \neq C(L)$.

Now if $Y$ is an $X$-boundary, we have $C(X) \hookrightarrow C(Y) \hookrightarrow C(\partial_F(\Gamma, X))$, with $I_\Gamma(C(X)) = I_\Gamma(C(\partial_F(\Gamma, X)))$. Thus $I_\Gamma(C(Y)) = C(\partial_F(\Gamma, X))$, which means $I_\Gamma(C(Y)) \not\cong C(L)$. \hfill $\square$

5. Applications to reduced crossed products

In this section we apply our results to find conditions for exactness of the reduced crossed product of the (minimal) action of a countable group on a compact space. For the general theory of discrete exact groups and amenable actions, we refer the reader to [6].

Recall that a group $\Gamma$ is exact if $C^*_r(\Gamma)$ is exact as a $C^*$-algebra. This is introduced by Kirchberg and Wasserman in [21] and is known to be equivalent to the amenability of $\Gamma$ actions on arbitrary compact spaces. Ozawa observed that one needs only the amenability of canonical action on $\beta\Gamma$ [24]. Exactness of $\Gamma$ is also known to be equivalent to the amenability of the $\Gamma$-action on the Furstenberg boundary $\partial_F\Gamma$ [18, 4.5]. In the latter case, the key point is that $C(\partial_F\Gamma)$ is $\Gamma$-injective. In this section we show that the same idea could be adapted to show that $\Gamma$ is exact if and only if the $\Gamma$-action on $\partial_F(\Gamma, X)$ is amenable, for every minimal $\Gamma$-space $X$.

**Lemma 5.1.** Suppose $\Gamma$ is exact and $X$ is a $\Gamma$-space. If $C(X)$ is $\Gamma$-injective, then the action $\Gamma \curvearrowright X$ is amenable.

**Proof.** Since $C(X)$ is $\Gamma$-injective, there is a $\Gamma$-map $\psi : \ell^\infty(\Gamma) \rightarrow C(X)$. Identifying $\ell^\infty(\Gamma)$ with $C(\beta\Gamma)$, restriction of the adjoint map $\psi^* : \mathcal{M}(X) \rightarrow \mathcal{M}(\beta\Gamma)$ to the space of point masses on $X$ gives a continuous $\Gamma$-equivariant map $\theta : X \rightarrow \text{Prob}(\beta\Gamma)$. Since $\Gamma$ is exact, the $\Gamma$-action on $\beta\Gamma$ is amenable and so is the $\Gamma$-action on $\text{Prob}(\beta\Gamma)$ [16, 3.6]. Thanks to the existence of the $\Gamma$-equivariant map $\theta$, the $\Gamma$-action on $X$ is also amenable. \hfill $\square$

Next we show that if $X$ is a minimal $\Gamma$-space, the exactness passes from $\Gamma$ to the reduced crossed product $C(X) \rtimes_\gamma \Gamma$. Recall that a $C^*$-algebra is exact if and only if it can be embedded into a nuclear $C^*$-algebra [20], [25].
Theorem 5.2. Let \( \Gamma \) be a countable discrete group. The following are equivalent:

1. \( \Gamma \) is exact,
2. For every minimal \( \Gamma \)-space \( X \), the \( \Gamma \)-action on \( \partial_F(\Gamma, X) \) is amenable,
3. For every minimal \( \Gamma \)-space \( X \), \( C(\partial_F(\Gamma, X)) \rtimes_r \Gamma \) is nuclear,
4. For every minimal \( \Gamma \)-space \( X \), \( C(X) \rtimes_r \Gamma \) is exact.

Proof. (1) \( \Rightarrow \) (2). Since \( \mathcal{N}(C(\partial_F(\Gamma, X))) = \mathcal{N}(I_\Gamma(C(X))) \), \( C(\partial_F(\Gamma, X)) \) is \( \Gamma \)-injective. Now the result follows from Lemma 5.1.

(2) \( \Rightarrow \) (3). This is well known (see, [6, 4.3.4, 4.3.7]).

(3) \( \Rightarrow \) (4). We have,
\[
C(X) \rtimes_r \Gamma \subseteq \mathcal{N}(C(X)) \rtimes_r \Gamma = C(\partial_F(\Gamma, X)) \rtimes_r \Gamma,
\]
and by assumption, \( C(\partial_F(\Gamma, X)) \rtimes_r \Gamma \) is nuclear. Thus \( C(X) \rtimes_r \Gamma \) is exact, as it is embedded into a nuclear \( C^* \)-algebra [25, Proposition 7].

(4) \( \Rightarrow \) (1). Just let \( X \) be a singleton. \( \square \)

Finding a tangible nuclear \( C^* \)-algebra containing a given exact \( C^* \)-algebra is the next natural thing to ask for. Ozawa has conjectured that for an exact \( C^* \)-algebra \( A \), there is a nuclear \( C^* \)-algebra \( \mathcal{N}(A) \) such that \( A \subset \mathcal{N}(A) \subset I(A) \), where \( I(A) \) is the injective envelope of \( A \). Kalantar and Kennedy proved that if \( \Gamma \) is a discrete exact group, for the reduced group \( C^* \)-algebra \( C^*_\Gamma(\Gamma) \), there is a canonical unital nuclear \( C^* \)-algebra \( \mathcal{N}(C^*_\Gamma(\Gamma)) \) such that \( C^*_\Gamma(\Gamma) \subset \mathcal{N}(C^*_\Gamma(\Gamma)) \subset I(C^*_\Gamma(\Gamma)) \) [18, 4.6]. Indeed, they observed that \( C(\partial_F(\Gamma)) \rtimes_r \Gamma \), which is nuclear when \( \Gamma \) is exact, could play the role of \( \mathcal{N}(C^*_\Gamma(\Gamma)) \). The above result shows that the same could be done in the more general case of minimal \( \Gamma \)-spaces.

Corollary 5.3. Let \( \Gamma \) be an exact group and let \( X \) be a minimal \( \Gamma \)-space. There is a canonical unital nuclear \( C^* \)-algebra \( \mathcal{N}(C(X) \rtimes_r \Gamma) \) such that
\[
C(X) \rtimes_r \Gamma \subset \mathcal{N}(C(X) \rtimes_r \Gamma) \subset I(C(X) \rtimes_r \Gamma),
\]
where \( I(C(X) \rtimes_r \Gamma) \) is the injective envelope of \( C(X) \rtimes_r \Gamma \).

Proof. Take \( \mathcal{N}(C(X) \rtimes_r \Gamma) = C(\partial_F(\Gamma, X)) \rtimes_r \Gamma \). Since \( \Gamma \) is exact, by Theorem 5.2, \( C(\partial_F(\Gamma, X)) \rtimes_r \Gamma \) is nuclear and \( C(X) \rtimes_r \Gamma \) is exact. The second inclusion follows from [15, 3.4]. \( \square \)

The results in this section hold also for arbitrary \( \Gamma \)-spaces. Thus (a slight modification of) the above corollary proves a recent result of Buss, Echterhoff and Willett [7, Corollary 8.4].

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