ZERO SETS OF FUNCTIONS IN THE NEVANLINNA CLASS AND THE
∂₀-EQUATION ON CONVEX DOMAINS OF GENERAL TYPE IN $\mathbb{C}^2$

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Abstract. The purpose of this paper is to characterize the zero sets of holomorphic functions in the Nevanlinna class on a class of convex domains of infinite type in $\mathbb{C}^2$. Moreover, we also obtain $L^p$ estimates, $1 \leq p \leq \infty$, for a particular solution of the tangential Cauchy-Riemann equation on the boundaries of these domains.

1. Introduction

1.1. Background. Let $\Omega$ be a $C^\infty$-smooth, bounded domain with smooth defining function $\rho$. In several complex variables, characterizing the zero sets of holomorphic functions in the Nevanlinna class is closely related to the Poincaré-Lelong equation

$$\text{i} \partial \bar{\partial} u = \alpha \quad \text{on } \Omega$$

for $d$-closed, smooth $(1,1)$-forms $\alpha$. Not surprisingly, solving the Poincaré-Lelong equation with estimates often amounts to studying the Cauchy-Riemann equation

$$\bar{\partial} u = f \quad \text{on } \Omega$$

where $f$ is a $\bar{\partial}$-closed $(0,1)$-form on $f$. For investigating the Nevanlinna class, the type of estimates that are useful are solutions $u$ with boundary values in $L^1(b\Omega)$. The technique that we use to solve the boundary value estimate in turn yields a solution to the tangential Cauchy-Riemann equation

$$\bar{\partial}_b u = f$$

in $L^p(b\Omega)$, $1 \leq p \leq \infty$ where $f$ is $\bar{\partial}_b$-closed $(0,1)$. In fact, our technique is produces a gain in $L^\infty$ and maps $L^\infty$ to an appropriate $f$-Hölder space. Therefore, our results concern the related problems of complex varieties that are zero sets of Nevanlinna functions, the Cauchy-Riemann, and tangential Cauchy-Riemann equations.

It is well-known that if $h \in N(\Omega)$ then the zero divisor $X_h$ of $h$ satisfies the Blaschke condition. Whether or not the converse is true, namely, “If $h \in H(\Omega)$ and $X_h$ satisfies the Blaschke condition, does there exists $f \in N(\Omega)$ so that $X_f = X_h$?” has been extensively studied over the past forty years. The $n=1$ case is a classic one-variable result in the complex plane. In contrast, for $n \geq 2$, the Blaschke condition for a divisor no longer suffices to be the zero set of a Nevanlinna function or even the zero set of an $H^p$ function [Var80]. There are cases, however, where the Blaschke condition is sufficient. Namely, the sufficiency is known when $\Omega$ is

- a strongly pseudoconvex domain [Gru75, Sko76];
- a pseudoconvex domain in $\mathbb{C}^2$ of finite type [CNS92, Sha89];
- a complex or real ellipsoid by [BCS82, Sha91].

2010 Mathematics Subject Classification. Primary 32F20; 32F10; 32T25; 32N15.

Key words and phrases. Cauchy-Riemann equation, infinite type, tangential Cauchy-Riemann equation, Nevanlinna class, Henkin solution, general Hölder space, $L^p$ regularity.

The first author was partially supported by Australian Research Council DE160100173. The second author was partially supported by NSF grant DMS-1405100. He also gratefully acknowledges the support provided by Wolfson College at the University of Cambridge during his Fellowship year there.
• a convex domain of strictly finite type and/or finite type \cite{BCD98,CDM14,Cum01,DM01}. Furthermore, the positive answer still holds on the following infinite type example:

\[
D_\alpha = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + \exp \left( 1 + \frac{2}{\alpha} - \frac{1}{|z_2|^\alpha} \right) < 1 \right\}
\]

with \(0 < \alpha < 1\) (see \cite{AC02}).

In this paper, we shall prove the converse is true for the class of convex domains in \(\mathbb{C}^2\) of general type by Khanh in \cite{Kha13,HKR14}. All known examples of convex domains in \(\mathbb{C}^2\) are covered by our class.

Establishing \(L^p\) and Hölder estimates for solutions of the \(\bar{\partial}\)-equation is a fundamental question in several complex variables. It has been extensively investigated on classes of domains of finite type such as strongly pseudoconvex domains \cite{FST4,Hen77a,Hen77b}, convex domains \cite{Ale05,Sha91}, domains with a diagonalizable Levi form \cite{FKM90}, domains where the Levi form has comparable eigenvalues \cite{Koe02}, decoupled domains \cite{NS06}, and pseudo convex domains in \(\mathbb{C}^2\) \cite{Chr88,FK88}. See also \cite{Wu98,LTS05}. Theorem 3 provides the first example of \(L^p, 1 \leq p \leq \infty\), and Hölder estimates on an infinite type domain.

1.2. The class of general type convex domains. Our setup is the following: \(\Omega \subset \mathbb{C}^2\) is a smooth, bounded domain. For each \(p \in \partial \Omega\), the curvature of \(b\Omega\) at \(p\) is captured by local coordinates \(z_p = T_p(z)\) where \(T_p\) is a \(C\)-linear transformation that sends \(p\) to the origin. Additionally, there exist a global defining function \(\rho\) and functions \(F_p\) and \(r_p\) satisfying

\[
\Omega_p = T_p(\Omega) = \{z_p = (z_{p,1}, z_{p,2}) \in \mathbb{C}^2 : \rho(T_p^{-1}(z_p)) = F_p(|z_{p,1}|^2) + r_p(z_p) < 0\}
\]

with \(\rho(0) = 0\) and \(\rho(\xi) > 0\) on \(\partial \Omega\) for some \(\xi \in \mathbb{C}^2\).

(i) \(F_p(0) = 0\);

(ii) \(F_p'(t), F_p''(t), m(t)\), and \(\left(\frac{F_p(t)}{t}\right)'\) are nonnegative on \([0, d_p^2)\) for some \(d_p > d_p\);

(iii) \(r_p(0) = 0\) and \(\frac{\partial r_p}{\partial z_2} \neq 0\) on \(b\Omega\) with \(|z_{1,p}| \leq \delta\);

(iv) \(r_p\) is convex,

where \(d_p\) is the diameter of \(\Omega_p\) and \(\delta\) is a small number independent of \(p\).

This class of domains includes several well-known examples. If \(\Omega\) is of finite type \(2m\), then \(F_p(t) = t^m\) at the points of type \(2m\). On the other hand, if \(F_p(t) = \exp(-1/t^\alpha)\), then \(\Omega\) is of infinite type at \(p\), and this is our main case of interest. Our hypotheses include the following three classes of infinite type domains: the complex ellipsoid

\[
\Omega = \left\{ z = (z_1, z_2) \in \mathbb{C}^2 : \sum_{j=1}^{2} \exp \left( - \frac{1}{|z_j|^\alpha_j} \right) \leq e^{-1} \right\};
\]

the real ellipsoid

\[
\Omega = \left\{ z = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2 : \sum_{j=1}^{2} \exp \left( - \frac{1}{|x_j|^\alpha_j} \right) + \exp \left( - \frac{1}{|y_j|^\beta_j} \right) \leq e^{-1} \right\};
\]
and the mixed case
\[ \Omega = \left\{ z = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2 : \exp \left( -\frac{1}{|x_1|^{\alpha_1}} \right) + \exp \left( -\frac{1}{|y_1|^{\beta_1}} \right) + \exp \left( -\frac{1}{|x_2|^{\alpha_2}} \right) \leq e^{-1} \right\} \tag{1.6} \]
where \( \alpha_j, \beta_j > 0 \). Moreover, our setting also includes a tube domain of infinite type at 0
\[ \Omega = \left\{ z = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2 : \exp \left( 1 - \frac{1}{|x_1|^{\alpha_1}} \right) + \chi(y_1) + |z_2|^2 \leq 1 \} \tag{1.7} \]
where \( \chi \) is a convex function and \( \chi(y_1) = 0 \) when \( |y_1| < \delta \) and \( \alpha_1 > 0 \) for \( j = 1, 2 \).

1.3. Notation. For an excellent discussion of the Nevanlinna class, complex varieties, (positive) currents, and (irreducible) divisors, we strongly encourage the reader to consult Range [Ran86] and Noguchi and Ochiai [NO90].

The Nevanlinna class for \( \Omega \), denoted by \( N(\Omega) \), is defined by
\[ N(\Omega) = \left\{ h \in H(\Omega) : \sup_{\varepsilon > 0} \int_{\Omega_{\varepsilon}} \log |h(z)| \, d\sigma_{\Omega_{\varepsilon}}(z) < +\infty \right\}, \]
where \( H(\Omega) \) is the space of holomorphic functions on \( \Omega \), \( b\Omega_{\varepsilon} = \left\{ z : \rho(z) = -\varepsilon \right\} \) for small \( \varepsilon > 0 \), and \( d\sigma_{\Omega_{\varepsilon}}(z) \) denotes the Euclidean surface measure on \( b\Omega_{\varepsilon} \). If \( X \subset \Omega \) is a complex variety with irreducible decomposition
\[ X = \bigcup_k X_k \]
and \( n_k \in \mathbb{N} \) are positive integers for each \( k \), the divisor \( X := \{ X_k, n_k \} \) is said to satisfy the Blaschke condition if
\[ \sum_k n_k \int_{X_k} |\rho(z)| \, d\mu_{X_k}(z) < \infty \]
where \( d\mu_{X_k} \) is the induced surface area measure on \( X_k \).

1.4. Main results. We have three main results.

**Theorem 1.** Let \( \Omega \) be a bounded domain in \( \mathbb{C}^2 \). Assume that for any \( p \in b\Omega \)

i. \( \Omega \) is defined by (1.2) and \( \int_0^{d\rho} |\log F_p(t^2)| \, dt < \infty \) for all \( p \in b\Omega \), or

ii. \( \Omega \) is defined by (1.3) and \( \int_0^{d\rho} |\log(t) \log F_p(t^2)| \, dt < \infty \) for all \( p \in b\Omega \).

Then for any divisors \( X \) in \( \Omega \) satisfying the Blaschke condition there is a function \( h \in N(\Omega) \) such that \( X \) is the zero divisor of \( h \).

To apply Theorem 1 to the domains \( \Omega \) defined by any of (1.2) - (1.7), we are forced to require that \( \alpha_j < 1 \), though any \( \beta_j > 0 \) is permissible. The crucial step to prove Theorem 1 is the existence of the \( \partial \)-equation that satisfies both of the following conditions: the solution is (i) smooth if data is smooth, and (ii) bounded in \( L^1(b\Omega) \).

**Theorem 2.** Let \( \Omega \) be a domain satisfying the hypotheses of Theorem 1. Then for any \( \partial \)-closed, smooth \((0,1)\)-form \( \phi \) on \( \Omega \) so that \( \|\phi\|_{L^1(b\Omega)} < \infty \), there exists a smooth function \( u \) such that
\[ \partial u = \phi \quad \text{on} \quad \Omega \tag{1.8} \]
and
\[ \|u\|_{L^1(b\Omega)} \leq c\|\phi\|_{L^1(b\Omega)} \tag{1.9} \]
where \( c > 0 \) is independent of \( \phi \).

**Remark 1.** The constant \( c \) in (1.9) depends on the geometric type and diameter of \( \Omega \). It is, however, uniformly bounded if \( \Omega \) satisfies the hypothesis of Theorem 1 and diameter of \( \Omega \) is bounded.
Let $u$ be the Henkin solution to \((1.8)\) given by \((2.18)\) below. Then the smoothness of $u$, given the smoothness of $\phi$, is a consequence of Theorem 3 in [Ran92]. Therefore we only need to prove that the inequality \((1.9)\) holds for this $u$. The proof will be given in Section 2. A small modification of the technique to prove \((1.3)\) yields $L^p$-estimates for the tangential Cauchy-Riemann equation, a significant and new result in its own right.

**Theorem 3.** Let $\Omega$ be a domain satisfying the hypotheses of Theorem 1 and $p \in [1, \infty]$. Let $\phi$ be a $(0,1)$-form in $L^p(b\Omega)$, satisfying the compatibility condition $\int_{b\Omega} \phi \wedge \alpha = 0$ for every continuous up the boundary, $\bar{\partial}$-closed $(2,0)$-form $\alpha$ on $\Omega$. Then there exists a function $u$ on $b\Omega$ such that

$$\bar{\partial}_b u = \phi \quad \text{on } b\Omega$$

and

$$\|u\|_{L^p(b\Omega)} \leq c\|\phi\|_{L^p(b\Omega)}$$

where $c > 0$ is independent of $\phi$.

Moreover, in the case $p = \infty$, we obtain a “gain” for the solution of $\bar{\partial}_b$ into the $f$-Hölder spaces, that is,

$$\|u\|_{\Lambda^f(b\Omega)} \leq c\|\phi\|_{L^\infty(b\Omega)}.$$  \hspace{1cm} \text{(1.12)}

The function $f$ is defined by $f(d^{-1}) := \inf_{p \in \Omega} \left( \int_0^d \frac{F^*(t)}{t} dt \right)^{-1}$ when $\Omega$ is defined by \((1.2)\) and by $f(d^{-1}) := \inf_{p \in \Omega} \left( \int_0^d \frac{F^*(t) \log F^*(t)}{t} dt \right)^{-1}$ when $\Omega$ is defined by \((1.3)\). Here the superscript $*$ denotes the inverse function and the $f$-Hölder space $\Lambda^f$ is defined by

$$\Lambda^f(b\Omega) = \left\{ u : \|u\|_{\Lambda^f(b\Omega)} := \|u\|_{L^\infty(b\Omega)} + \sup_{X(t) \in \mathcal{C}, 0 \leq t \leq 1} \{ f(|t|^{-1})|u(X(t)) - u(X(0))| \} < \infty \right\}$$

where $\mathcal{C} = \{ X(t) : t \in [0,1] \rightarrow X(t) \in b\Omega \text{ is } C^1 \text{ and } |X'(t)| \leq 1 \}$.

**2. Proof of Theorem 2 and Theorem 3**

We first assume that the origin is in $b\Omega$, the functions $F = F_0$ and $r = r_0$ satisfy conditions (i)-(iv) from Section 1 and

$$\Omega = \{ z = (z_1, z_2) \in \mathbb{C}^2 : \rho(z) = F(|z_1|^2) + r(z) < 0 \}$$

or

$$\Omega = \{ z = (z_1, z_2) \in \mathbb{C}^2 : \rho(z) = F(|x_1|^2) + r(z) < 0 \}$$

where $z_j = x_j + iy_j, x_j, y_j \in \mathbb{R}, j = 1, 2$. Here we only need to consider $F'(t^2) \neq 0$, for otherwise $\Omega$ is strictly convex, and the proof of Theorem 2 and 3 is known. Let the support function for $\Omega$ be defined by

$$\Phi(\zeta, z) = \sum_{j=1}^2 \frac{\partial \rho(\zeta)}{\partial \zeta_j} (\zeta - z_j).$$

We are going to estimate $\text{Re}\{\Phi(\zeta, z)\}$ for $\zeta, z$ in a neighborhood of $b\Omega$.

**Setting 1: $\Omega$ is defined by (2.1).** The convexity of $r$ yields that

$$2 \text{Re}\{\Phi(\zeta, z)\} \geq \rho(\zeta) - \rho(z) + F'(|\zeta_1|^2)|z_1 - \zeta_1|^2 + \left[ F(|z_1|^2) - F(|\zeta_1|^2) - F'(|\zeta_1|^2)(|z_1|^2 - |\zeta_1|^2) \right].$$  \hspace{1cm} \text{(2.3)}
for any $\zeta, z$ in a neighborhood of $b\Omega$.

**Lemma 4.** Let $\Omega$ be defined by (2.1) and $F$ satisfy both conditions (i)-(iv) from Section 7 and $F'(0) = 0$. Let $\zeta, z$ be in a neighborhood of $b\Omega$ and satisfy $\rho(\zeta) - \rho(z) \geq 0$.

- If $|\zeta| \geq |z_1 - \zeta|$ then
  \[
  |\Phi(\zeta, z)|^k|z - \zeta| \geq |\mathrm{Im}\, \Phi(\zeta, z)| + \rho(\zeta) - \rho(z) + F(|z_1 - \zeta|)k|z_1 - \zeta|.
  \]

- Otherwise, if $|\zeta| \leq |z_1 - \zeta|$, then
  \[
  |\Phi(\zeta, z)|^k|z - \zeta| \geq |\mathrm{Im}\, \Phi(\zeta, z)| + \rho(\zeta) - \rho(z) + F(1/2|\zeta|)k|\zeta_1| + (|\mathrm{Im}\, \Phi(\zeta, z)| + \rho(\zeta) - \rho(z) + F(1/2|x_1|)k|x_1| + |y_1 - \eta_1|) \tag{2.5}
  \]

for $k = 1, 2$.

**Proof.** The term in $[\cdots]$ in (2.3) is nonnegative for any $\zeta, z$ near $b\Omega$, so the hypothesis on the sizes of $|\zeta_1|$ and $|z_1 - \zeta_1|$ allow us to obtain

\[
|\Phi(\zeta, z)|^k|z - \zeta| \geq \begin{cases}
  |\mathrm{Im}\, \Phi(\zeta, z)| + \rho(\zeta) - \rho(z) + F(|z_1 - \zeta|)k|z_1 - \zeta| & \text{if } |\zeta| \geq |z_1 - \zeta|, \\
  |\mathrm{Im}\, \Phi(\zeta, z)| + \rho(\zeta) - \rho(z) + F(|\zeta|)k|\zeta_1| & \text{if } |\zeta| \leq |z_1 - \zeta|.
\end{cases}
\]

The remaining estimate to show is

\[
|\Phi(\zeta, z)|^k|z - \zeta| \geq (|\mathrm{Im}\, \Phi(\zeta, z)| + \rho(\zeta) - \rho(z) + F(1/2|z_1|)k|z_1| \tag{2.5} \tag{2.5}
\]

in the case $|\zeta| \leq |z_1 - \zeta|$. It can be obtained using the argument of Lemma 3.2 in [HKR14]. For the reader’s convenience, we outline the proof here. Start by comparing the relative sizes of $|\zeta_1|$ and $1/2|z_1|$. If $|\zeta_1| \geq 1/2|z_1|$, then the argument follows from the second line of (2.5). Otherwise, $|\zeta_1| \leq 1/2|z_1|$, and this inequality implies both $|z_1| \geq |\zeta_1|$ and $|z_1 - \zeta_1| \geq (1 - 1/\sqrt{2})|z_1|$. Then estimate the $[\cdots]$ in (2.3) by

\[
[\cdots] := F(|z_1|) - F(|\zeta|) \geq F(|z_1|) \geq F(1/2|z_1|), \tag{2.7}
\]

where the inequality uses the facts that $F'(0) = 0$ and $F''$ is nondecreasing (see [FLZ11] Lemma 4) or [HKR14] Lemma 3.1 for details). This completes the proof. \(\square\)

**Setting 2:** $\Omega$ is defined by (2.2). An argument analogous to that of Lemma 4 produces the following lemma.

**Lemma 5.** Let $\Omega$ be defined by (2.2) and $F$ satisfy both conditions (i)-(iv) from Section 7 and $F'(0) = 0$. Suppose $\zeta, z$ are in a neighborhood of $b\Omega$ and $\rho(\zeta) - \rho(z) \geq 0$.

- If $|\xi| \geq |x_1 - \xi_1|$, then
  \[
  |\Phi(\zeta, z)|^k|z - \zeta| \geq |\mathrm{Im}\, \Phi(\zeta, z)| + \rho(\zeta) - \rho(z) + F(|x_1 - \zeta_1|)k(|x_1 - \zeta_1| + |y_1 - \eta_1|).
  \]

- Otherwise, if $|\xi| \leq |x_1 - \xi_1|$, then
  \[
  |\Phi(\zeta, z)|^k|z - \zeta| \geq (|\mathrm{Im}\, \Phi(\zeta, z)| + \rho(\zeta) - \rho(z) + F(1/2|x_1|)k|x_1| + |y_1 - \eta_1|) \tag{2.8}
  \]

for $k = 1, 2$, where $z_1 = x_1 + iy_1$ and $\zeta_1 = \xi_1 + i\eta_1$. \(\square\)
Both the proofs of Theorem 2 and 3 use the supporting function estimates of Lemma 4 and 5. We first give the full proof of Theorem 3 since it is more delicate and indicate the changes necessary to prove Theorem 2.

2.1. **Proof of Theorem 3**

**Proof of Theorem 3.** Let \( \phi \) be a \((0,1)\)-form satisfying the compatibility condition, that is, \( \int_{\partial \Omega} \phi \wedge \alpha = 0 \) for every continuous up to the boundary \( \partial \)-closed \((2,0)\)-form \( \alpha \) on \( \Omega \). M.C. Shaw [Sha91, Sha89] showed that

\[
u(\zeta, z) := \frac{\partial \Phi(\zeta, z)}{\partial \zeta_1} (\tilde{\zeta}_2 - \tilde{z}_2) - \frac{\partial \Phi(\zeta, z)}{\partial \zeta_2} (\tilde{\zeta}_1 - \tilde{z}_1). \tag{2.10}\]

In order to prove (1.11), we will prove that \( \|u\|_{L_p(\Omega)} \lesssim \|\phi\|_{L_p(\Omega)} \) only for \( p = 1 \) and \( p = \infty \); then using Riesz-Thorin Interpolation Theorem we obtain \( L_p \) estimates for all \( p \in [1, \infty] \).

**Part I: Proof of** \( \|u\|_{L^1(\Omega)} \lesssim \|\phi\|_{L^1(\Omega)} \). **Denote by**

\[
u(\zeta, z) := \frac{1}{4\pi^2} \int_{\partial \Omega} H(\zeta, z - \epsilon \nu(z)) \phi(\zeta) \wedge \omega(\zeta) - \frac{1}{4\pi^2} \int_{\partial \Omega} H(z + \epsilon \nu(z), \zeta) \phi(\zeta) \wedge \omega(\zeta), \quad z \in b\Omega.
\]

It follows that \( \lim_{\epsilon \to 0^+} \nu_\epsilon = u \) a.e. For \( \phi \in L^1(b\Omega) \) and we need to prove that \( \|u_\epsilon\|_{L^1(\Omega)} \leq \|\phi\|_{L^1(\Omega)} \) uniformly for small \( \epsilon > 0 \). It then follows that \( u_\epsilon \to u \) in \( L^1(b\Omega) \) by the Dominated Convergence Theorem. We observe

\[
\|u_\epsilon\|_{L^1(\Omega)} = \frac{1}{4\pi^2} \int_{\partial \Omega} \int_{\zeta \in \Omega} |H(\zeta, z - \epsilon \nu(z) - H(z + \epsilon \nu(z), \zeta) \phi(\zeta) \wedge \omega(\zeta)| dS(z) \\
\lesssim \int_{\zeta \in \Omega} \int_{\zeta \in \Omega} |H(\zeta, z - \epsilon \nu(z)) \phi(\zeta) dS(z) dS(z) + \int_{\zeta \in \Omega} \|H(z - \epsilon \nu(z), \zeta) \phi(\zeta) dS(\zeta) dS(z) \\
\lesssim \int_{\zeta \in \Omega} \int_{\zeta \in \Omega} |H(\zeta, z) \phi(\zeta) dS(\zeta) dS(z) + \int_{\zeta \in \Omega} |H(\zeta, z) \phi(\zeta) dS(z) dS(z)
\]

where \( \Omega_\epsilon = \{ z \in \Omega : \rho(z) < -\epsilon \} \) and \( \Omega^\epsilon = \{ z \in \mathbb{C}^2 \setminus \tilde{\Omega} : \rho(z) > \epsilon \} \). As a consequence of Tonelli’s Theorem, it suffices to prove that

\[
\int_{(\zeta, z) \in \Omega \times \Omega^\epsilon} |H(\zeta, z) \phi(\zeta)| dS(\zeta, z) + \int_{(z, \zeta) \in \Omega^\epsilon \times \Omega^\epsilon} |H(z, \zeta) \phi(\zeta)| dS(z, \zeta) \leq \|\phi\|_{L^1(\Omega)} < \infty. \tag{2.11}
\]

Since \( b\Omega \) is compact, for any \( \delta > 0 \), there exist points \( p_1, \ldots, p_N \in b\Omega \) so that \( b\Omega \) is covered by \( \{B(p_j, \delta)\}_{j=1}^N \). After changing coordinates with the linear transformation \( T_{p_j} \) as in Section 1
(keeping in mind $T_{p_j}(p_j) = 0$), we may assume the goal is to prove
\[
\int_{\Omega} \left| H(T_{p_j}^{-1}(\zeta), T_{p_j}^{-1}(z)) \right| dS(\zeta, z) \leq \|\phi(T_{p_j}^{-1}(\cdot))\|_{L^1(\partial \Omega)} + \int_{\Omega} \left| H(T_{p_j}^{-1}(\zeta), T_{p_j}^{-1}(z)) \right| dS(\zeta, z) \tag{2.12}
\]

\[
\leq \|\phi(T_{p_j}^{-1}(\cdot))\|_{L^1(\partial \Omega)} \approx \|\phi\|_{L^1(\partial \Omega)},
\]

where
\[
\Omega_{p_j} = \{\rho_{p_j}(z_{p_j}) := \rho(T_{p_j}^{-1}(z_{p_j})) = P_{p_j}(z_{p_j,1}) + r_{p_j}(z_{p_j}) < 0\}
\]
and $P_{p_j}(z_{p_j,1}) = F_{p_j}(|z_{p_j,1}|^2)$ or $P(z_{p,j}) = F(x_{p,j,1})$ as in Section II. Although the integrals in (2.12) do not cover the full boundaries, the estimate on the complement is trivial because $H$ and its derivatives are uniformly bounded. Next, note that
\[
\Phi \left( T_{p_j}^{-1}(\zeta), T_{p_j}^{-1}(z) \right) = \Phi_{p_j}(\zeta, z),
\]
where $\Phi_{p_j}$ is the support function of $\Omega_{p_j}$, and we can therefore estimate
\[
\left| H(T_{p_j}^{-1}(\zeta), T_{p_j}^{-1}(z)) \right| \lesssim \frac{1}{|\Phi_{p_j}(\zeta, z)||\zeta - z|} \tag{2.13}
\]
for any $\zeta_{p_j}, z_{p_j} \in b\Omega$. Nearby each point $p_j$, we will consider domains $\Omega_{p_j}$ defined by either (1.2) or (1.3). Here and in what follows, we abuse notation slightly and omit the subscript $p_j$ as well as writing $\phi(\cdot)$ for $\phi(T_{p_j}^{-1}(\cdot))$.

**Setting 1: $\Omega$ is defined by (1.2).** We start our estimate of (I) from (2.12) by decomposing the domain of integration and applying Lemma 2 with $k = 1$ to obtain
\[
I = \int_I \cdots = \int_{(\zeta, z) \in \partial (b\Omega \cap B(0, \delta)) \times b\Omega} \cdots + \int_{(\zeta, z) \in \partial (b\Omega \cap B(0, \delta)) \times b\Omega_{\epsilon,1} \mid \zeta - z \leq \epsilon} \cdots + \int_{(\zeta, z) \in \partial (b\Omega \cap B(0, \delta)) \times b\Omega_{\epsilon,1} \mid \zeta - z \leq \epsilon} \cdots \leq (A) + (B) + (C), \tag{2.14}
\]
where
\[
(A) := \int_{(\zeta, z) \in (B(0, 2\delta) \cap \partial \Omega) \times (B(0, 2\delta) \cap \partial \Omega)} \left| \phi(\zeta) \right| dS(\zeta, z) \left| \epsilon + \text{Im} \Phi(\zeta, z) \right| + F(|\zeta - \frac{1}{2}|^2) |\zeta - \zeta_1| \tag{2.15}
\]
\[
(B) := \int_{(\zeta, z) \in (B(0, 2\delta) \cap \partial \Omega) \times (B(0, 2\delta) \cap \partial \Omega)} \left| \phi(\zeta) \right| dS(\zeta, z) \left| \epsilon + \text{Im} \Phi(\zeta, z) \right| + F \left( \frac{1}{2} |\zeta - \frac{1}{2}|^2 \right) |\zeta - \zeta_1| \tag{2.15}
\]
\[
(C) := \int_{(\zeta, z) \in (B(0, 2\delta) \cap \partial \Omega) \times \partial \Omega_{\epsilon,1} \mid \zeta - z \leq \epsilon} \cdots \leq (A) + (B) + (C), \tag{2.14}
\]
where
\[
(A) := \int_{(\zeta, z) \in (B(0, 2\delta) \cap \partial \Omega) \times (B(0, 2\delta) \cap \partial \Omega)} \left| \phi(\zeta) \right| dS(\zeta, z) \left| \epsilon + \text{Im} \Phi(\zeta, z) \right| + F(|\zeta - \frac{1}{2}|^2) |\zeta - \zeta_1| \tag{2.15}
\]
\[
(B) := \int_{(\zeta, z) \in (B(0, 2\delta) \cap \partial \Omega) \times (B(0, 2\delta) \cap \partial \Omega)} \left| \phi(\zeta) \right| dS(\zeta, z) \left| \epsilon + \text{Im} \Phi(\zeta, z) \right| + F \left( \frac{1}{2} |\zeta - \frac{1}{2}|^2 \right) |\zeta - \zeta_1| \tag{2.15}
\]
\[
(C) := \int_{(\zeta, z) \in (B(0, 2\delta) \cap \partial \Omega) \times \partial \Omega_{\epsilon,1} \mid \zeta - z \leq \epsilon} \cdots \leq (A) + (B) + (C), \tag{2.14}
\]
$F$ is increasing, so it easily follows that $(C) \lesssim (F(2\delta^2)\delta)^{-1}\|\phi\|_{L^1(\Omega)}$. For (A), we make the change of variables $(\alpha, w) = (\alpha_1, \alpha_2, w_1, w_2) = (\zeta_1, \zeta_2, z_1 - \zeta_1, \rho(z) + i\Im \Phi(\zeta, z))$. A direct calculation then establishes that if $\delta$ is chosen sufficiently small then the Jacobian of this transform does not vanish on the domain of integration. Since $\Phi$ is smooth, we can assume that there exists $\delta' > 0$ that depends on $\Omega$, $\delta$, and $\rho$ so that if integrate $w_1$ in polar coordinates,

$$(A) \lesssim \|\phi\|_{L^1(\Omega)} \int_0^{\delta'} \int_0^{\delta'} \frac{r}{(|\Im w_2| + F(r^2))r} dr d\Im w_2 \lesssim \|\phi\|_{L^1(\Omega)} \int_0^{\delta'} \log F(r^2) dr < \infty.$$  

That the integral is finite follows by the hypotheses on $\phi$ and $F$.

Repeating this argument with the change of variables $(\alpha, w) = (\alpha_1, \alpha_2, w_1, w_2) = (\zeta_1, \zeta_2, 1/\sqrt{2}z_1, \rho(z) + i\Im \Phi(\zeta, z))$ for the integral (B), we can obtain the same conclusion.

To estimate (II) in (2.12), we use Lemma 4 with $k = 1$ and to show the interchanging of $\zeta$ and $z$ is benign. In then follows by the same argument as for (I), we obtain (II) $\lesssim \|\phi\|_{L^1(\Omega)}$. Therefore, the estimate in Setting 1 is complete.

**Setting 2: $\Omega$ is defined by (1.3).** We omit the proof because it is analogous to Setting 1 with Lemma 5 replacing Lemma 4. For details, see Section 3.2 in [HKR13].

**Part II: Proof of $\|u\|_{L^\infty(\Omega)} \lesssim \|\phi\|_{L^\infty(\Omega)}$.** The proof of this part is similar to, but simpler than, the argument for Part III, so we omit it.

**Part III: Proof of $\|u\|_{\Lambda^1(\Omega)} \lesssim \|\phi\|_{L^\infty(\Omega)}$.** We need a general Hardy-Littlewood type lemma to prove $f$-Hölder estimates on the boundary.

**Lemma 6.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $\frac{G(t)}{t}$ is decreasing and $\int_0^s \frac{G(t)}{t} dt < \infty$ for $s > 0$ small enough. If $v \in C^1(\mathbb{C}^n \setminus b\Omega)$ such that

$$|\nabla v(x \pm sv(x))| \lesssim \frac{G(s)}{s} \quad \text{for any } x \in b\Omega. \quad (2.16)$$

Then $v \in \Lambda^1(b\Omega)$ where $f(s^{-1}) = \left(\int_0^s \frac{G(t)}{t} dt\right)^{-1}$.

The proof is basically identical to the corresponding result for domains. See [Kha13, Theorem 5.1]) for details. Consequently, the focus is now to control the gradient of $H^+$ and $H^-$.  

**Lemma 7.** For $z \in b\Omega$, we have

1. $\sup_{z \in \Omega_\delta} |\nabla H^+ \phi(z)| \lesssim \frac{G(s)}{s} \|\phi\|_{L^\infty(\Omega)}$, and
2. $\sup_{z \in \Omega_\delta} |\nabla H^- \phi(z)| \lesssim \frac{G(s)}{s} \|\phi\|_{L^\infty(\Omega)}$

where $G(s) = \sup_{p \in \Omega_1} \{F_p^*(s)\}$ if $\Omega$ is defined by (1.3) and $G(s) = \sup_{p \in \Omega_1} \{\sqrt{F_p^*(s)} \log \sqrt{F_p^*(s)}\}$ if $\Omega$ is defined by (1.3).
where the last inequality follows by Lemma 3.2 in [Kha13]. For the proof of (2), direct calculations show
\[ |\nabla H^{-}\phi(z)| \lesssim \|\phi\|_{L^\infty(\partial\Omega)} \int_{M^2} \left( \frac{1}{\Phi(z,\zeta)||z-\zeta|^2} + \frac{1}{\Phi(z,\zeta)|z-\zeta|} \right) dS(\zeta) \]
\[ \lesssim \|\phi\|_{L^\infty(\partial\Omega)} \int_{M^2} \frac{dS(\zeta)}{|\Phi(z,\zeta)|^2|z-\zeta|} \]
for \( z \in \mathbb{C}^2 \setminus \bar{\Omega} \) near \( \bar{\Omega} \). We choose a covering \( \{ B(p_j, \delta) \}_{j=1}^N \) of \( \partial\Omega \) and change coordinates to set \( p_j \) to 0 as in the proof of Theorem 2; thus our proof reduces to showing
\[ L(z) := \int_{\zeta \in \partial\Omega \cap B(0,\delta)} \frac{dS}{|\Phi(z,\zeta)|^2|z-\zeta|} \lesssim \frac{G(\rho(z))}{\rho(z)}, \quad z \in \mathbb{C}^n \setminus \bar{\Omega}. \]

For Setting 1, we use Lemma 3 with \( k = 2 \) to interchange the roles of \( \zeta \) and \( z \). We then estimate
\[ L(z) \lesssim \int_{\zeta \in \partial\Omega \cap B(0,\delta)} \frac{dS}{|\Phi(z,\zeta)|^2|z-\zeta|} \]
\[ \lesssim \int_{\zeta \in \partial\Omega \cap B(0,\delta)} \frac{dS}{|\Phi(z,\zeta)|^2|z-\zeta|} \]
where
\[ (D) = \int_{\zeta \in \partial\Omega \cap B(0,\delta)} \frac{dS}{|\Phi(z,\zeta)|^2|z-\zeta|}, \]
\[ (E) = \int_{\zeta \in \partial\Omega \cap B(0,\delta)} \frac{dS}{|\Phi(z,\zeta)|^2|z-\zeta|}. \]

For the integral \( (D) \), if \( |z_1 - \zeta_1| \geq \delta \) then \( (D) \lesssim (F^2(\delta^2)\delta)^{-1} \); otherwise we make the change of variable \( (w,t) = (z_1 - \zeta_1, \text{Im} \Phi(z,\zeta)) \). We can check that the Jacobian of this transformation is nonzero on the domain of integration \( \delta \) is chosen sufficiently small. Thus,
\[ (D) \lesssim (F^2(\delta^2)\delta)^{-1} + \int_{|w| \leq 2\delta} \frac{dw}{|\rho(z)| + F(|w|^2)|w|} \leq C_{\delta} \sqrt{F^*(|\rho(z)|)} \]

where the last inequality follows by Lemma 3.2 in [Kha13].

For the integral \( (E) \), if \( |z_1| \geq \delta \) then \( (E) \lesssim (F^2(\frac{1}{4}\delta^2)\delta)^{-1} \); otherwise
\[ (E) \lesssim \int_{\zeta \in \partial\Omega \cap B(0,\delta)} \frac{dS}{|\Phi(z,\zeta)|^2|z-\zeta|}. \]

In this case, we make the change variable \( (w,t) = (\zeta_1, \text{Im} \Phi(z,\zeta)) \). The Jacobian of this transformation is also different zero on the domain of integration if \( \delta \) is small. We thus obtain the desired estimate for \( (E) \).

The proof for the real case follows by the same argument using Lemma 5 and Lemma 4.1 in [Kha13]. This is complete the proof of Lemma 7.

Lemma 7 allows us to apply Lemma 6 to \( H^+ \phi \) and \( H^- \phi \) and establish that \( H^+ \phi, H^- \phi \in \Lambda^J(b\Omega) \). We may now conclude that \( u \in \Lambda^J(b\Omega) \).

2.2. Proof of Theorem 2

Proof of Theorem 2 Let \( \phi = \sum_{j=1}^{2} \phi_j \partial z_j \) be a bounded, \( C^1 \), \( \partial \)-closed \( (0,1) \)-form on \( \Omega \). The solution \( u \) of the \( \partial \)-equation, \( \partial u = \phi \), provided by the Henkin kernel is given by
\[ u = T\phi(z) = H\phi(z) + K\phi(z). \]
where \( H \phi = \int_{\zeta \in \Omega} H(\zeta, z) \phi(\zeta) \wedge \omega(\zeta) \) and
\[
K \phi(z) = \frac{1}{4\pi^2} \int_{\Omega} \frac{\phi_1(\zeta)(\bar{\zeta}_1 - \bar{z}_1) - \phi_2(\zeta)(\bar{\zeta}_2 - \bar{z}_2)}{|\zeta - z|^4} \omega(\zeta) \wedge \omega(\zeta)
\] (2.19)

As mentioned in Section 1, the smoothness of \( u \) is a consequence of Theorem 3 in [Ran92]. In particular, Range proved
\[
\|T \phi\|_{\Lambda^s(\Omega)} \lesssim \|\phi\|_{\Lambda^s(\Omega)}
\]
for all \( \phi \) with \( \bar{\partial} \phi = 0 \) and all \( s > 0 \)
holds on any bounded convex domain \( \Omega \) in \( \mathbb{C}^2 \) with smooth boundary. Here \( \Lambda^s(\Omega) \) is the Hölder space of order \( s \). Thus the proof of Theorem 2 will be complete if we prove
\[
\|T \phi\|_{L^1(\Omega)} \lesssim \|\phi\|_{L^1(\Omega)}
\]
(2.20)
on our setting of \( \Omega \). For \( z \in \partial \Omega \) and \( \bar{\partial} \phi = 0 \) in \( \Omega \), Shaw [Sha89, pages 412-414] showed that
\[
K \phi(z) = -\int_{\zeta \in \partial \Omega} H(z, \zeta) \phi(\zeta) \wedge \omega(\zeta).
\]
Although Shaw uses the signed distance to the boundary defining function, her argument is essentially formal and holds for any \( C^1 \) defining function. Thus we have
\[
\forall z \in \partial \Omega, \quad u(z) = \int_{\zeta \in \partial \Omega} (H(\zeta, z) - H(z, \zeta)) \phi(\zeta) \wedge \omega(\zeta).
\]
By the same argument to Part I in Section 2, (2.20) is obtained.

\[\square\]

3. PROOF OF THEOREM 1

The next two lemmas are modified versions of Lemmas 4.3 and 4.8 in [Sha91].

**Lemma 8.** Suppose that \( \Omega \) is convex and contains the origin. Let \( \alpha \) be a positive, \( d \)-closed, smooth \((1,1)\)-form on \( \bar{\Omega} \) supported on \( \bar{\Omega} \setminus B(0, r) \) for some \( r > 0 \). This means
\[
\alpha = \sum_{j,k=1}^{2} \alpha_{j\bar{k}} dz_j \wedge d\bar{z}_k
\]
where \( \alpha_{j\bar{k}} \in C^\infty(\bar{\Omega}) \) and \( \alpha_{j\bar{k}} \equiv 0 \) on \( B(0, r) \). Then there exists a \((0,1)\)-form \( f \) on \( \bar{\Omega} \) so that
\begin{enumerate}
\item \( \bar{\partial} f = 0 \);
\item \( \bar{\partial} f = \bar{\partial} \bar{f} = \alpha \);
\item There exists \( c = c(\Omega, r) \) such that
\[
\|f\|_{L^1(\bar{\Omega})} + \|f\|_{L^1(\Omega)} \leq c\|\alpha\|_{L^1(\Omega)}.
\]
\end{enumerate}

**Proof.** Following Rudin [Rud80, Theorem 17.2.7], we let \( f(z) = \sum_{k=1}^{n} f_k(z) d\bar{z}_k \) where
\[
f_k(z) = \sum_{j=1}^{n} \bar{z}_j \int_{0}^{1} t \alpha_{j\bar{k}}(tz) dt.
\]
With this choice of \( f \), it follows that both \( \partial f - \bar{\partial} \bar{f} = \alpha \) and \( \bar{\partial} f = 0 \).
Since \( \| \alpha \|_{L^1(\Omega)} = \sum_{j,k=1}^2 \| \alpha_{j\bar{k}} \|_{L^1(\Omega)} \), it follows easily that with induced surface area measure \( d\sigma \),
\[
\int_{b\Omega} |f(z)|d\sigma(z) \leq \sum_{j,k=1}^2 \int_{b\Omega} |z_j| \int_0^1 |t \alpha_{j\bar{k}}(zt)| dt \, d\sigma(z) \leq \frac{c}{r^2} \sum_{j,k=1}^2 \int_{b\Omega} \int_0^1 t^2 |\alpha_{j\bar{k}}(zt)| dt \, d\sigma(z)
\]
\[
\leq \frac{c}{r^2} \sum_{j,k=1}^2 \int_{b\Omega} |\alpha_{j\bar{k}}(z)| \, dV(z) = \frac{c}{r^2} \| \alpha \|_{L^1(\Omega)},
\]
where \( dV \) is Lebesgue measure on \( \mathbb{C}^2 \) and \( c \) may change from line to line (and also depends on \( \text{dist}(b\Omega,0) \)). Additionally, a similar argument also shows \( \| f \|_{L^1(\Omega)} \leq c \| \alpha \|_{L^1(\Omega)} \). In particular, with the change of variables \( s = t\tau \),
\[
\int_{\Omega} |f(z)|dV(z) = \int_0^1 \int_{b\Omega} |f(\tau z)|d\sigma(z) \tau^3 d\tau \leq \sum_{j,k=1}^2 \int_0^1 \int_{b\Omega} \int_0^\tau |sz_j \alpha_{j\bar{k}}(sz)| \, ds \, d\sigma(z) \tau^2 d\tau
\]
\[
\leq \sum_{j,k=1}^2 \int_{b\Omega} \int_0^1 |sz_j \alpha_{j\bar{k}}(sz)| \, ds \, d\sigma(z) \leq \frac{c}{r^2} \| \alpha \|_{L^1(\Omega)}
\]
\( \square \)

Remark 2. In [Sha91], Shaw requires that \( \alpha \) is positive, i.e., \( (\alpha_{j\bar{k}}) \) is a positive definite matrix. In this case, basic linear algebra shows that \( 2|\alpha_{j\bar{k}}| \leq \alpha_{jj} + \alpha_{k\bar{k}} \). She then estimates the integral only on the diagonal of \( \alpha \). Implicit in her computation is Lelong’s computation that positive \((1,1)\) currents \( \alpha \) must satisfy \( \alpha = \bar{\alpha} \) and \( \alpha_{j\bar{k}} = -\alpha_{k\bar{j}} \). In contrast, we solve the Poincare-Lelong equation for general data with no assumption of positivity. However, our application to the Nevanlinna class argument only involves positive data.

Lemma 9. Suppose that \( \Omega \) is convex and contains the origin. Let \( \alpha \) be a \( d \)-closed, smooth \((1,1)\)-form on \( \Omega \) supported on \( \Omega \setminus B(0,r) \) for some \( r > 0 \). Then there exists a real-valued function \( u \in C^\infty(\Omega) \) so that
\[
(1) \quad i\partial\bar{\partial}u = \alpha;
(2) \quad \| u \|_{L^1(\Omega)} \leq c \| \alpha \|_{L^1(\Omega)} \text{ for some constant } c = c(r,\Omega) > 0 \text{ that is independent of } \alpha \text{ and } u.
\]

Proof. We use Lemma 8 to establish the existence of a \( \tilde{\partial} \)-closed \((0,1)\)-form \( f \) that satisfies \( \partial f - \tilde{\partial} \bar{f} = \alpha \) and (3.1). Since \( f \) is \( \tilde{\partial} \)-closed and in \( L^1(b\Omega) \), we can use Theorem 2 to establish a function \( v \) so that \( i\partial v = f \) and satisfies (1.9). Note then that
\[
\alpha = i\partial\bar{\partial}v - i\partial\bar{\partial}(v + \bar{v}).
\]
It now follows that \( u = v + \bar{v} \) is the desired function. \( \square \)

We are now ready to prove Theorem 1.

Proof of Theorem 1. The second Cousin problem can be solved on convex domains, so there exists \( h \in H(\Omega) \) with zero set \( \tilde{X} \). Extend \( h \) to \( \mathbb{C}^2 \) by setting \( h(z) \equiv 1 \) for \( z \in \mathbb{C}^2 \setminus \Omega \). Let \( \alpha = \alpha_X \) be the positive \((1,1)\)-current on \( \mathbb{C}^2 \) defined by \( \alpha = i\partial\bar{\partial} \log |h| \). Observe that \( \alpha \equiv 0 \) off of \( \tilde{\Omega} \). Let \( \varphi \in C_c^\infty(\mathbb{R}) \) be an approximation of the identity, in particular, let \( \varphi \in C_c^\infty(\mathbb{R}) \), \( \int_{\mathbb{R}} \varphi \, dx = 1 \), and \( \varphi_\epsilon(x) = \epsilon^{-1} \varphi(x/\epsilon) \). Let \( \Omega_\epsilon = \{ z \in \Omega : \rho(z) < -\epsilon \} \).
Define \( v_\epsilon \in C^\infty(\mathbb{C}^2) \) by
\[
v_\epsilon(z) = \int_{\mathbb{C}^2} \log |h(w)| \varphi_\epsilon(|z - w|) \, dV(w).
\]

Then \( v_\epsilon(z) \to \log |h(z)| \) for almost all \( z \in \Omega \). By the Poincaré-Lelong formula [NO90, Theorem 5.1.13], \( \alpha = 0 \) on \( \{ z : h(z) \neq 0 \} \), an open set. Therefore, there exists \( p \in \Omega \) and \( r > 0 \) for which \( \alpha|_{\partial B(p,r)} \equiv 0 \).

Set \( \alpha_\epsilon = i \partial \bar{\partial} v_\epsilon \). Then \( \alpha_\epsilon \in C^\infty_{1,1}(\Omega) \). Since \( d\partial \bar{\partial} = 0 \), \( \alpha_\epsilon \) is \( d \)-closed. Note that if \( \epsilon > 0 \) is small enough, \( \varphi_\epsilon|_{B(p,r)} \equiv 0 \). Therefore, by translating \( p \to 0 \), we can apply Lemma 9 to \( \alpha_\epsilon \) (which we shall do without any further comment regarding the support of \( \alpha \) of \( \alpha_\epsilon \)). Also, \( \alpha \) is positive, so \( \alpha_\epsilon \) is as well (on \( \Omega_\epsilon \)) [NO90, Lemma 3.2.13], and we write
\[
\alpha_\epsilon = \sum_{j,k=1}^2 \alpha^\epsilon_{jk} d z_j \wedge d \bar{z}_k.
\]

Recall that convolution of a distribution with a test function behaves as follows: \( \langle T * \varphi, \psi \rangle = \langle T, \varphi \ast \psi \rangle \) where \( \varphi(x) = \varphi(-x) \). This means
\[
\| \alpha_\epsilon \|_{L^1(\Omega)} = \sup_{g, \|g\|_{L^\infty(\Omega)} \leq 1} \int \alpha_\epsilon \wedge g \, dV.
\]

Each integral in the wedge product is an integral of the \( \langle \beta_\epsilon, \psi \rangle \) where \( \beta_\epsilon = \beta \ast \varphi \) where \( \beta \) is a (positive on \( \Omega_\epsilon \)) Radon measure built from the components of \( \alpha \). All of this means
\[
\| \beta_\epsilon \|_{L^1(\Omega)} = \sup_{\psi, \|\psi\|_{L^\infty(\Omega)} \leq 1} \langle \beta \ast \varphi_\epsilon, \psi \rangle = \sup_{\psi, \|\psi\|_{L^\infty(\Omega)} \leq 1} \langle \beta, \varphi_\epsilon \ast \psi \rangle
\]
\[
= \sup_{\psi, \|\psi\|_{L^\infty(\Omega)} \leq 1} \int \varphi_\epsilon \ast \psi \, d\beta \leq \sup_{\psi, \|\psi\|_{L^\infty(\Omega)} \leq 1} \int \int \varphi_\epsilon(x - y) \psi(y) \, dy \, d\beta(x)
\]
\[
\leq \sup_{\psi, \|\psi\|_{L^\infty(\Omega)} \leq 1} \beta(\Omega) \|\psi\|_{L^\infty(\Omega)} = \beta(\Omega).
\]

The upshot of this calculation is that because \( h \) exactly has \( \tilde{X} \) as its zero divisor, the finite area of \( \tilde{X} \) guarantees the existence of a constant \( A > 0 \) so that \( \| \alpha_\epsilon \|_{L^1(\Omega)} \leq A \) where the constant \( A \) is independent of \( \epsilon \).

Next, each \( v_\epsilon \) is \( d \)-closed on \( \Omega \), so we may invoke Lemma 9 to establish the existence of a real-valued \( u_\epsilon \in C^\infty(\Omega) \) that satisfies \( i \partial \bar{\partial} u_\epsilon = \alpha_\epsilon \) on \( \Omega \) and
\[
\| u_\epsilon \|_{L^1(\Omega)} \leq c \|\alpha_\epsilon\|_{L^1(\Omega)} \leq c A.
\]

Set \( g_\epsilon = u_\epsilon - v_\epsilon \). Then \( g_\epsilon \) is a smooth function on \( \bar{\Omega} \) and pluriharmonic on \( \Omega \) since \( i \partial \bar{\partial} u_\epsilon = \alpha_\epsilon = i \partial \bar{\partial} v_\epsilon \). Moreover, for small \( \epsilon > 0 \), Lemma 10 proves that \( \{ g_\epsilon \} \) is a normal family of pluriharmonic functions on \( \Omega \) and therefore there exists a subsequence \( \epsilon_k \to 0 \) and a pluriharmonic function \( g \) so that \( g_\epsilon \to g \) uniformly on compact subsets of \( \Omega \).

Since \( g \) is pluriharmonic on \( \Omega \), there exists \( H \in H(\Omega) \) so that \( g = \text{Re} H \). By construction (and the uniform convergence on compacta), \( i \partial \bar{\partial} g = 0 \). Define
\[
U(z) = \log |h(z)| + g(z) = \log |e^{H(z)} h(z)|.
\]

The proof is complete once we show that
\[
\int_{\mathcal{M}_s} |U(z)| \, d\sigma_{\mathcal{M}_s} \leq C
\]
for some \( C > 0 \) and all \( s > 0 \) but this follows by the argument leading to [Gru75, (6)].
Lemma 10. For $\epsilon > 0$ small, the set of pluriharmonic functions $\{g_\epsilon\}$ from the proof of Theorem 1 comprises a normal family. Specifically, there exists $C > 0$ so that if $U \subset \Omega$, then there exists $C = C(U)$ that does not depend on $\epsilon$ so that $|g_{\epsilon,s}(z)| \leq C$.

Proof. Plurisubharmonic functions are in $L^1_{\text{loc}}(\Omega)$ so $v_\epsilon = \log |h|^e \phi_\epsilon$ satisfies the following inequality: for $K \subset \Omega$ compact, there exists $C_K > 0$ so that for every $\epsilon > 0$

$$\|v_\epsilon\|_{L^1(K)} \leq C_K.$$  

Following Gruman [Gru75], we let $U \subset \Omega$ have compact closure in $\Omega$. Let $\eta \in C_c^\infty(\Omega)$ so that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on a neighborhood of $\overline{U}$. Then for $z \in U$,

$$v_\epsilon(z) = \frac{2}{(2\pi)^2} \int_{\Omega} \frac{1}{|z-w|^2} \Delta(\eta v_\epsilon(w)) \, dV(w)$$

$$= \frac{2}{2\pi} \int_{\Omega} \frac{\eta(w)}{|z-w|^2} \Delta v_\epsilon(w) + \frac{v_\epsilon(w)}{|z-w|^2} \Delta \eta(w) + \frac{1}{|z-w|^2} \left( \nabla_w \eta \cdot \nabla_w v_\epsilon + \nabla_w \eta \cdot \nabla_w v_\epsilon \right) dV(w)$$

$$= \frac{2}{2\pi} \int_{\Omega} \frac{\eta(w)}{|z-w|^2} \Delta v_\epsilon(w) \, dV(w) + \frac{2}{2\pi} \int_{\Omega} v_\epsilon(w) \left( \frac{\Delta \eta(w)}{|z-w|^2} - \nabla_w \cdot \left[ \frac{\nabla_w \eta}{|z-w|^2} \right] + \nabla_w \left[ \frac{\nabla_w \eta}{|z-w|^2} \right] \right) dV(w)$$

The second integral is bounded by $C_\eta \|v_\epsilon\|_{L^1(\text{supp } \eta)}$ since $|w-z|$ is bounded away from 0 since $\text{supp } \nabla \eta$ is a positive distance away from $\overline{U}$. For the first integral, if $\epsilon$ is small enough, then $K = \{\xi : \xi \in \text{supp } \phi_\epsilon(w - \cdot) \text{ for any } w \in \overline{U}\}$ is a compact set in $\Omega$. This means

$$\left| \int_{\Omega} \frac{\eta(w)}{|z-w|^2} \Delta v_\epsilon(w) \, dV(w) \right| = \int_{\Omega} \int_{\Omega} \frac{\eta(w)}{|z-w|^2} \phi_\epsilon(w-x) \, d\alpha(x) \, dV(w)$$

$$\leq \int_{K} |\log |h|\xi|| \int_{\Omega} \phi_\epsilon(w-x) \frac{1}{|z-w|^2} \, dV(w) \, d\alpha(x)$$

$$\leq C_\alpha(\Omega) < \infty$$

since $\hat{X}$ has finite area. We therefore obtain the bound

$$\|v_\epsilon\|_{L^\infty(U)} \leq C_\eta$$

where $C_\eta$ does not depend on $\epsilon > 0$ (assuming that $\epsilon > 0$ is sufficiently small). The functions $g_\epsilon$ are pluriharmonic, so by the Poisson Integral Formula,

$$g_\epsilon(z) = \int_{\partial \Omega} P(z,w) g_\epsilon(w) \, d\sigma(w) = \int_{\partial \Omega} P(z,w) \left( u_\epsilon(w) + v_\epsilon(w) \right) \, d\sigma(w).$$

Since $u_\epsilon \in L^1(b\Omega)$ and $z \in U$ so that $|z-w|$ is bounded away from 0,

$$\left| \int_{\partial \Omega} P(z,w) u_\epsilon(w) \, d\sigma(w) \right| \leq C_U \|u_\epsilon\|_{L^1(b\Omega)}.$$

Also, recall that for each fixed $z \in \Omega$, $P(z,w) = -\frac{\partial}{\partial w} G(z,w)$ where $G(z,w)$ is the Green’s function for $\Omega$, and $G(z,w) = 0$ for all $y \in b\Omega$. By Green’s formula

$$\int_{\partial \Omega} P(z,w) v_\epsilon(w) \, d\sigma(w) = - \int_{\partial \Omega} \frac{\partial G(z,w)}{\partial w} v_\epsilon(w) \, d\sigma(w)$$

$$= \int_{\Omega} v_\epsilon(w) \Delta G(z,w) \, dV(w) - \int_{\Omega} G(z,w) \Delta v_\epsilon(w) \, dV(w).$$

Recall that $G(z,w)$ is integrable on $\Omega$ in $z$ and in $w$. Indeed, $G(z,w)$ blows up like the Newtonian potential (i.e., integrably) and is symmetric in its arguments. Consequently, by Folland [Fol99, Theorem 6.18], for $z \in U$, there exists $C = C(U) > 0$ so that

$$\left| \int_{\partial \Omega} P(z,w) v_\epsilon(w) \, d\sigma(w) \right| \leq C \left( \|v_\epsilon\|_{L^\infty(U)} + \|\alpha_\epsilon\|_{L^1(\Omega)} \right).$$
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