Dynamical Breaking to Special or Regular Subgroups in $SO(N)$ Nambu–Jona-Lasinio Model

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Abstract

It is recently shown that in 4D $SU(N)$ Nambu–Jona-Lasinio (NJL) type models, the $SU(N)$ symmetry breaking into its special subgroups is not special but much more common than that into the regular subgroups, where the fermions belong to complex representations of $SU(N)$. We perform the same analysis for $SO(N)$ NJL model for various $N$ with fermions belonging to an irreducible spinor representation of $SO(N)$. We find that the symmetry breaking into special or regular subgroups has some correlation with the type of fermion representations; i.e., complex, real, pseudo-real representations.

1 Introduction

Symmetries and their breaking [1–3] play a crucial role in constructing unified theories beyond the Standard Model (SM) as well as the SM. There have been already known several symmetry breaking mechanisms in quantum field theories; e.g., the Higgs mechanism [4–6], the dynamical symmetry breaking [1, 2, 7–20], the Hosotani mechanism [21–23], magnetic flux [24, 25] and orbifold breaking [26, 27].

Recently, in Ref. [20], present authors have examined dynamical symmetry breaking pattern in 4D $SU(N)$ Nambu–Jona-Lasinio (NJL) type models in which the fermion belongs to an irreducible representation of $SU(N)$ or $SU(n)$ or $SU(n-1)/2$. The potential analysis has shown that for almost all cases at the potential minimum, the $SU(N)$ group symmetry is broken to its special subgroups such as $SO(N)$ or $USp(N)$ when symmetry breaking occurs. (Note that special subgroups may be referred to as non-regular or irregular subgroups depending on literature. For more information about Lie subgroups, see, e.g., Refs. [28–32].) Also, in Ref. [19], one of the author (T.K.) and J. Sato have performed the same analysis in a 4D $E_6$ NJL type model in which the fermion belongs to an irreducible representation $27$ of $E_6$. The potential analysis showed that for all cases at the potential minimum the $E_6$ group symmetry is broken to its special subgroups such as $SU(3)$, $USp(8)$, $G_2$, $F_4$ when symmetry breaking occurs. These results clearly show that the symmetry breaking into special subgroups is not special at least in 4D NJL type models.

One might think that the above results may highly depend on a particular feature of the NJL type models, that is, fermion pair condensation in a specific type of effective theories, so the symmetry breaking into special subgroups may still be special, e.g., in Higgs mechanism [4–5]. In Ref. [33], however, L.-F. Li investigated the symmetry breaking patterns in gauge theories with Higgs scalars in several representations. The analysis showed that $SU(n)$ symmetry is broken to its regular subgroups for the cases of Higgs field in an $SU(n)$ fundamental and adjoint

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representation, while \(SU(n)\) symmetry is broken to its regular or special subgroups for the cases of Higgs field in an \(SU(n)\) rank-2 symmetric or anti-symmetric tensor representation. Therefore, the symmetry breaking into special subgroups is not so special even in the Higgs models.

One of the attractive ideas in constructing theories beyond the SM is the grand unification theories (GUTs) proposed in Ref. \[34\]. To realize the SM, which is a 4D chiral gauge theory, as a low energy effective theory, GUTs must be chiral gauge theories from the viewpoint of 4D effective theories. In other words, in 4D frameworks, 4D GUTs must contain at least a Weyl fermion in a complex representation of the GUT gauge group \(G_{\text{GUT}}\). Thus, the candidates of 4D GUT gauge group \(G_{\text{GUT}}\) may allow the existence of complex representations. (For the type of representations of Lie groups, see e.g., Refs. \[31,32,35\].) The condition suggests that \(G_{\text{GUT}}\) may be \(SU(N)(N \geq 5), SO(4n+2) (n \geq 2), \) or \(E_6\). There are many proposals for 4D GUTs \[34,36–42\] and 5D or higher dimensional GUTs \[41–57\]. Note that in higher dimensional theories on e.g., orbifold spaces unconventional GUT groups such as \(SO(11)\) \[50,52,55,56\] and \(SO(32)\) \[57\] are allowed because zero modes of some higher dimensional fields with appropriate boundary conditions can be 4D Weyl fermions in complex representations of their Lie subgroups.

From the usage in GUT model buildings in 4D framework, we are primarily motivated in studying 4D NJL type models in which the fermion matter belongs to complex representations of \(SU(N)(N \geq 5), SO(4n+2) (n \geq 2), \) and \(E_6\). We have already known some examples for \(SU(N)\) and \(E_6\) cases in Ref. \[19,20\], so if we investigate \(SO(4n+2) (n \geq 2)\) cases, our primary purpose will be completed at least for 4D cases. As is well-known in e.g., Refs. \[31,32,35\], the irreducible \(SO(N)\) spinor representations are classified into three types depending on \(N \mod 8\); i.e., complex representations for \(N = 8\ell \pm 2 (\ell \geq 1)\); real representations for \(N = 8\ell, 8\ell \pm 1(\ell \geq 1)\); pseudo-real representations for \(N = 8\ell \pm 3, 8\ell + 4(\ell \geq 1)\), which are summarized in Table 1. Some higher dimensional models, e.g., 5 and 6 dimensional \(SO(11)\) gauge-Higgs GUTs \[50,52,55,56\] contain the fermions in the spinor representation of \(SO(11)\) \[32\], which is a pseudo-real representation.

| \(N \mod 8\) | Spinor rep. | \(2^\frac{N}{2} - 1\) | Spinor type |
|-------------|-------------|----------------|-------------|
| 0           | \(r, r'\)   | \(2^\frac{N}{2} - 1\) | \(R\)       |
| 1           | \(r\)       | \(2^\frac{N}{2} - 1\) | \(R\)       |
| 2           | \(r, \bar{r}\) | \(2^\frac{N}{2} - 1\) | \(C\)       |
| 3           | \(r\)       | \(2^\frac{N}{2} - 1\) | \(PR\)      |
| 4           | \(r, r'\)   | \(2^\frac{N}{2} - 1\) | \(PR\)      |
| 5           | \(r\)       | \(2^\frac{N}{2} - 1\) | \(PR\)      |
| 6           | \(r, \bar{r}\) | \(2^\frac{N}{2} - 1\) | \(C\)       |
| 7           | \(r\)       | \(2^\frac{N}{2} - 1\) | \(R\)       |

Table 1: \(SO(N)\) spinor representations: \(R, PR, C\) stand for real, pseudo-real, and complex representations. For even \(N\), there are two irreducible spinors, denoted here as \(r\) and \(r'\)(or \(\bar{r}\)), with opposite 'chirality'.

In this paper, we analyze the \(SO(N)\) symmetry breaking patterns in 4D \(SO(N)\) NJL type models with fermions of \(SO(N)\) irreducible spinor representation which is complex for \(N = 4n+2\).
and self-conjugate otherwise. The main purpose of this study is to show that for \( N = 4n + 2 \) \((n \in \mathbb{Z}_{\geq 1})\), \(SO(N)\) symmetry is broken to its special subgroups such as \(SO(\frac{N}{2}) \times SO(\frac{N}{2})(S)\), \(G_2(S)\) for \( N = 14 \) and \( F_4(S)\) for \( N = 26\). For \( N \neq 4n + 2\), on the other hand, the prime number cases \( N \in \mathbb{P}_{\geq 3} \) and the other composite number cases \( N \in \mathbb{Z}_{\geq 3}(\mathbb{P})\) are distinguished. For prime numbers \( N \in \mathbb{P}_{\geq 3}\) such as \(3, 5, 7, 11, 13, \ldots\), \(SO(N)\) symmetry is always broken to its regular subgroups such as \(SO(\frac{(N+1)}{2}) \times SO(\frac{(N-1)}{2})(R)\). For composite numbers \( N \in \mathbb{Z}_{\geq 3}(\mathbb{P})\) (other than \( N = 4n + 2 \)) such as \(4, 8, 9, 12, 16, \ldots\), \(SO(N)\) symmetry is broken to its regular or special subgroups such as \(SO(\frac{(N+1)}{2}) \times SO(\frac{(N-1)}{2})(R)\). (Note that \((R)\) and \((S)\) here stand for regular and special subgroups, respectively.)

This paper is organized as follows. In Sec. 2 we quickly review the \(SO(N)\) NJL models and examine \(SO(N)\) spinor properties and maximal little groups of \(SO(N)\) for the irreducible representations of the fermion bilinear composite scalar fields. By using the knowledges in Sec. 2 we discuss addition-type, product-type, and embedding-type subgroups in Sec. 3 and 4 respectively. Section 5 is devoted to summary and discussion.

Note that we treat the same type of 4D NJL models and use the same notation as in Ref. [20], so we will not repeat them in detail in this paper. In the paper, we denote Dynkin labels such as \((0,0,\cdots,0,2)\) = \((0^{n-1},2), (0,0,\cdots,0,1,0,0,0)\) = \((0^{n-5},1,0^2), \cdots\) for rank-\(n\) groups, for simplicity.

2 NJL-type model and \(SO(N)\) spinor properties

We consider the NJL-type model where the fermion \(\psi_I\) \((I = 1, \cdots, d)\) belongs to the irreducible \(SO(N)\) spinor representation \(R\) of dimension \(d := 2^{\frac{(N-1)}{2}}\), denoted by using Dynkin label as

\[
R = \begin{cases} 
 r = (0^{n-1},1) \text{ or } r' = (0^{n-2},1,0) & \text{for even } N = 2n \\
 r = (0^{n-1},1) & \text{for odd } N = 2n + 1 
\end{cases} 
\tag{2.1}
\]

Since the fermion \(\psi_I\), for each \(I\), actually stands for 2-component Weyl spinor of Lorentz group, the fermion bilinear scalar \(\Phi_{IJ} \sim \psi_I\psi_J\) gives symmetric product \((R \times R)_S\) decomposed into the following \(\left[\frac{N}{2}\right]\) irreducible representations \(R_p\) and also \(\left[\frac{N+1}{2}\right]\) irreducible representations \(R'_q\) for odd \(N\):

\[
\left(2^{\frac{(N-1)}{2}} \otimes 2^{\frac{(N+1)}{2}}\right)_S = \left\{ \begin{array}{ll}
 R_0 \oplus \bigoplus_{p=1}^{\frac{N}{2}} R_p & \text{for even } N = 2n \\
 R_0 \oplus \bigoplus_{p=1}^{\frac{N}{2}} R_p \oplus \bigoplus_{q=1}^{\frac{N+1}{2}} R'_q & \text{for odd } N = 2n + 1 
\end{array} \right. 
\tag{2.2}
\]

where \(R_0, R_p(p = 1, 2, \cdots, \left[\frac{N}{2}\right]), R'_q(q = 1, 2, \cdots, \left[\frac{N+1}{2}\right])\) are the representations denoted by the following Dynkin labels and correspond to the denoted anti-symmetric (AS) tensors:

\[
R_0 := \begin{cases} 
 (0^{n-1},2) & \text{for } R = r \\
 (0^{n-2},2,0) & \text{for } R = r' \text{or } \bar{r} 
\end{cases} \sim \psi_I(CT_{a_1\cdots a_n})^{IJ}\psi_J \quad \text{rank-}n \text{ AS tensor (self-dual if } N = 2n),
\]

\[
R_p := \begin{cases} 
 (0^{n-1-4p},1,0^{4p}) & \text{for } 4p \leq n - 1 \\
 (0^n) = 1 & \text{for } 4p = n \sim \psi_I(CT_{a_1\cdots a_{n-4p}})^{IJ}\psi_J \quad \text{rank-} (n-4p) \text{ AS tensor,} \\
 \text{non-existing} & \text{for } 4p \geq n + 1 
\end{cases}
\]

\[
R'_q := \begin{cases} 
 (0^n) = 1 & \text{for } 4q = n + 1 \sim \psi_I(CT_{a_1\cdots a_{n-4q+1}})^{IJ}\psi_J \quad \text{rank-} (n-4q+1) \text{ AS tensor,} \\
 \text{non-existing} & \text{for } 4q \geq n + 2 
\end{cases}
\tag{2.3}
\]
where $C$ is $SO(N)$ charge conjugation matrix and $\Gamma_{a_1\cdots a_r}$ are rank-$r$ AS tensor gamma matrices defined by

$$
\Gamma_{a_1a_2\cdots a_r} := \Gamma_{[a_1} \Gamma_{a_2} \cdots \Gamma_{a_r]} = \begin{cases} 
\Gamma_{a_1} \Gamma_{a_2} \cdots \Gamma_{a_r} & \text{if } a_1, a_2, \cdots, a_r \text{ are all different} \\
0 & \text{otherwise}
\end{cases} \quad (2.4)
$$

in terms of the $SO(N)$ gamma matrices $\Gamma_a$ ($a = 1, 2, \cdots, N$) given explicitly later below. (For $SO(N)$ tensor products, see e.g., Ref. [32].) In the following, $R_k$ denotes $R_0$ and $R_p$ for even $N$; $R_0$, $R_p$ and $R'_p$ for odd $N$, respectively.

Now the Lagrangian of the NJL model we discuss is given by

$$
\mathcal{L} = \bar{\psi} \not\!D \psi + \sum_k \frac{1}{4} G_{R_k} (\psi_I \psi_J) R_k (\bar{\psi}^I \bar{\psi}^J) \mathcal{R}_k,
$$

where $(\psi_I \psi_J)_{R_k}$ is the projection of the fermion bilinear scalar $\psi_I \psi_J$ into the irreducible channel $R_k$, and $G_{R_k}$ is the 4-Fermi coupling constants in that channel. As in the previous papers [19, 20], we introduce a set of irreducible auxiliary scalar fields $\Phi_{R_k I J}$ standing for the composite operators $-(G_{R_k}/2)(\psi_I \psi_J)_{G_{R_k}}$. Then, we can equivalently rewrite this Lagrangian into the form

$$
\mathcal{L} = \bar{\psi} i \not\!\partial \psi - \frac{1}{2} \psi_I \psi_J \Phi^I \Phi^J - \frac{1}{2} \bar{\psi}^I \bar{\psi}^J \Phi_{I J} - \sum_k M^2_{R_k} \text{tr}(\Phi^I_{R_k} \Phi^J_{R_k}),
$$

and obtain the effective potential of this system in the leading order in $1/N$ as [20]

$$
V_{\text{leading}}(\Phi) = \sum_k M^2_{R_k} \text{tr}(\Phi^I_{R_k} \Phi^J_{R_k}) + V_{\text{1-loop}}(\Phi),
$$

$$
V_{\text{1-loop}}(\Phi) = - \frac{d}{d} f(\Phi^I) = - \frac{d}{d} f(\Phi^I) = - \sum_{I=1}^d f(m^2_I), \quad (2.7)
$$

where $M^2_{R_k} := 1/G_{R_k}$, and $\Phi_{R_k}$’s are constrained symmetric $d \times d$ matrices subject to non-trivial condition belonging to the irreducible representation $R_k$, so satisfying the orthogonality

$$
\text{tr}(\Phi^I_{R_k} \Phi^J_{R_{k'}}) = 0 \quad \text{for } k \neq k', \quad \text{while } \Phi := \sum_k \Phi_{R_k} \quad \text{without the irreducible suffix } R_k
$$

is the general (unconstrained) symmetric $d \times d$ matrix which appears in the Yukawa interaction terms in Eq. (2.6) and hence in the 1-loop part potential $V_{\text{1-loop}}$. Here $m^2_I$ are $d$ eigenvalues of the Hermitian matrix $\Phi^I \Phi$, which stand for $d$ mass-square eigenvalues of the fermion $\psi_I$, and the function $f(m^2)$ is given by

$$
f(m^2) := \int_{p^2 E^2 \Lambda^4} \frac{d^4 p_E}{(2\pi)^4} \ln \left( \frac{p^2 + m^2}{p_E^2} \right) = \frac{1}{32\pi^2} \left\{ \Lambda^4 \ln \left( 1 + \frac{m^2}{\Lambda^2} \right) - m^4 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) + m^2 \Lambda^2 \right\}, \quad (2.8)
$$

where $\Lambda$ is UV cutoff on the Euclideanized momentum. We also use $F(x; M^2)$ defined as

$$
F(x; M^2) := M^2 x - f(x). \quad (2.9)
$$

To discuss spontaneous symmetry breaking, we have to check little groups, where a little group $H_\phi$ of a vector $\phi$ in a representation $R$ of $G$ is defined by

$$
H_\phi := \left\{ g \mid g \phi = \phi, \; g \in G \right\}.
$$

This little group $H_\phi$ of $G$ depends not only on the representation $R$ of $\phi$ but also the vector (value) $\phi$ itself. The vector $\phi$ must be an $H_{\phi}$-singlet, so that a subgroup $H$ can be a little group of $G$ for some representation $R$ only when $R$ contains at least one $H$-singlet. Following
so-called Michel’s conjecture [58], we assume that the potential minimum of the scalar field in an irreducible representation \( R \) of \( G \) is located at one of the little group \( H \) of \( R \).

We classify little groups \( H \) of \( G = SO(N) \) into three types: “addition-type,” “product-type,” and “embedding-type” subgroups. The examples of these three types of subgroups are listed in Tables 2 and 3, Table 4 and Table 5, respectively, from which the meaning of the name for these three types will be clear. An addition-type subgroup is usually a regular subgroup except for the cases \( N = 4\ell + 2 \) for which \( n := N/2 \) is odd and the subgroup \( H = SO(n + 4p) \times SO(n - 4p) \) has lower rank than \( G = SO(N = 2n) \). Any embedding-type subgroup is a special subgroup.

In this paper, we follow the notation and convention of Ref. [32]. More explicitly, we normalize the Dynkin index of the \( SO(n) \) defining representation \( \mathbf{n} \) as \( T(\mathbf{n}) = 1 \), the Dynkin index of the \( SU(n) \) defining representation \( \mathbf{n} \) as \( T(\mathbf{n}) = 1/2 \), and the Dynkin index of USp(2n) \( 2\mathbf{n} \) representation as \( T(2\mathbf{n}) = 1/2 \). Note that \( T(3) = 1 \) for \( SO(3) \), while \( T(3) = 2 \) for \( SU(2) \).

### Table 2: “Addition-type” little groups \( H \) of \( SO(N = 2n) \) for irreducible representation \( R_p \) scalars possessing \( H \)-singlet are listed, where \( p = 0, 1, 2, \cdots, \left[ \frac{N}{8} \right] \). (Regular/Special) in front of each little group name shows that it is a regular or special subgroup of \( SO(N = 2n) \) depending on whether \( n \) is even or odd, respectively. Note that the rank of the subgroup is lowered than that of \( SO(n) \) when \( n \) is odd.

| Maximal little group \( H \) of \( SO(N = 2n + 1) \) | \( H \)-singlet in | \( n \) |
|---|---|---|
| \( p \) | (Regular) \( SO(n + 4p + 1) \times SO(n - 4p) \) case | \( R_p \) | (even/odd) |

### Table 3: “Addition-type” little groups \( H \) of \( SO(N = 2n + 1) \) for irreducible representation \( R_p \) and \( R'_q \) scalars possessing \( H \)-singlet are listed, where \( p = 0, 1, 2, \cdots, \left[ \frac{N}{8} \right] \) and \( q = 1, 2, \cdots, \left[ \frac{N+1}{8} \right] \). (Regular) in front of each little group name shows that they are all regular subgroups of \( SO(N = 2n + 1) \).

| Maximal little group \( H \) of \( SO(N) \) |
|---|
| \( N = mn(m, n \geq 3; m, n \neq 4) \): (Special) \( SO(m) \times SO(n) \) case |
| \( N = 4mn(m, n \geq 1 \text{ except } m = n = 1) \): (Special) \( USp(2m) \times USp(2n) \) case |
| \( N = 2n(n \geq 2) \): (Regular) \( SU(n) \times U(1) \) case |

Table 4: “Product-type” little groups \( H \) of \( SO(N) \) are listed. \( SO(N) \) irreducible representation \( R_0, R_p, \) and \( R'_q \) scalars possessing \( H \)-singlet up to \( N = 26 \) are listed in Table 4. (Note there the isomorphism \( USp(2) \simeq SO(3) \simeq SU(2), SO(5) \simeq USp(4) \) and \( U(1) \simeq SO(2) \).) (Special) and (Regular) in front of each little group name show that they are all special or regular subgroups of \( SO(N) \) with \( N \) indicated.

In the next section, we discuss the effective potential VEV of addition-type, product-type, and embedding-type subgroups one by one.

Before starting the discussion, let us here recapitulate the basic properties of spinor representation briefly.
Table 5: "Embedding-type" little groups \( H \) of \( SO(N) \) where the \( SO(N) \) defining representation \( N \) is identified with an irreducible representation of \( H \). Here such Embedding-type little groups \( H \) of \( SO(N) \) are listed up to \( N = 26 \) (except \( H = SU(2) \) cases), and which irreducible scalar among \( SO(N) \) \( R_0 \), \( R_p \), and \( R'_q \) contains the \( H \)-singlet is shown in Table 6. (Special) in front of each little group name indicates that they are all special subgroups of \( SO(N) \).

The \( SO(N = 2n) \) or \( SO(N = 2n+1) \) spinor representation can be described by \( 2^n \times 2^n \) gamma matrices \( \Gamma_a \) satisfying

\[
\{ \Gamma_a, \Gamma_b \} = 2\delta_{ab} 1_{2^n}, \quad \Gamma_a^T = \Gamma_a \quad (a = 1, 2, \cdots, 2n+1), \quad n := \lfloor N/2 \rfloor. \tag{2.11}
\]

When necessary in this paper, we will use the following expression for the \( SO(N) \) gamma matrices \( \Gamma_a \) and the corresponding charge conjugation matrix \( C \):

\[
\begin{align*}
\Gamma_{1,2,3} &= \sigma_{1,2,3} \otimes \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1, \\
\Gamma_{4,5} &= \sigma_0 \otimes \sigma_{2,3} \otimes \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1, \\
\Gamma_{6,7} &= \sigma_0 \otimes \sigma_0 \otimes \sigma_{2,3} \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1, \\
&\vdots \\
\Gamma_{2n-2,2n-1} &= \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \cdots \otimes \sigma_{2,3} \otimes \sigma_1, \\
\Gamma_{2n,2n+1} &= \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \cdots \otimes \sigma_0 \otimes \sigma_{2,3}, \tag{2.12}
\end{align*}
\]

\[
C = \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \cdots \otimes \begin{cases} \sigma_3 \otimes \sigma_2 & \text{for } n: \text{odd} \\ \sigma_2 \otimes \sigma_3 & \text{for } n: \text{even} \end{cases}, \tag{2.13}
\]

where \( \sigma_j (j = 1, 2, 3) \) are the Pauli matrices,

\[
\begin{align*}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.14}
\end{align*}
\]

and \( \Gamma_{2n+1} \) plays the role of ‘chirality’ operator for \( SO(2n) \) case. The charge conjugation matrix \( C \) \eqref{2.13} in this representation satisfies

\[
\begin{align*}
C \Gamma_a C^{-1} &= \eta \Gamma_a^T, \quad C^T = \varepsilon C, \\
C &= C^\dagger = C^{-1}, \quad \text{with } \eta = (-1)^n, \quad \varepsilon = (-1)^{[(n+1)/2]} \tag{2.15}.
\end{align*}
\]
As we will see below, the auxiliary scalar field $\Phi_{R_k}$ in any $SO(N)$ irreducible representation $R_k$ has an $H$-singlet of an addition-type little group listed in Tables 2 and 3. Also, in any breaking into an addition-type subgroup, squared masses of the $SO(N)$ spinor fermion will be found all degenerate. As has been discussed in Refs. [19, 20], if the fermion masses are all degenerate, the corresponding vacuum realizes the global minimum of the effective potential in the leading order in NJL model, and so giving a candidate for the true vacuum.

In the following sections, we will often use the “traceless condition” for the mass matrix of our $SO(N)$ spinor fermion $\psi$. The fermion mass term is given by the VEV of the auxiliary scalar field $\Phi = \sum_k \Phi_{R_k}$ as is seen from the Yukawa term

\[ - \frac{(1/2)}{2} \psi_I \psi_J \Phi^I J + h.c. \]  

in the Lagrangian (2.6). This VEV is developed in an $H$-singlet component of one of irreducible scalar fields $\Phi_{R_k}$, which, as already shown in Eq. (2.16), corresponds to a certain rank-\(r\) $AS$ tensor $\Gamma$-matrix $\Gamma_{\alpha_1\alpha_2\cdots\alpha_r}$, so that the fermion mass matrix takes the form

\[ \langle \Phi_{R_k} \rangle = C M_r, \quad M_r = \sum_c c^{a_1a_2\cdots a_r} \Gamma_{a_1a_2\cdots a_r}, \]  

with charge conjugation matrix $C$ and $H$-singlet linear combination $M_r$ of rank-$r$ tensor gamma matrices. Note that $C$ should be present here since the scalar $\Phi$ is symmetric. Note, however, that both $\langle \Phi_{R_k} \rangle = C M_r$ and $M_r$ realize the same quadratic mass matrix which actually determines the value of the effective potential:

\[ \langle \Phi_{R_k} \rangle \langle \Phi_{R_k} \rangle^\dagger = M_r^I C M_r^J = M_r^I M_r^J. \]  

Moreover, the mass matrix which becomes proportional to a unit matrix on each $H$-irreducible sector of the fermion $\psi$ is not $CM_r$, but $M_r$. This is because the $SO(N)$ invariance of the Yukawa term Eq. (2.17) implies that their transformation law under the $g \in SO(N)$ transformation $\psi \to \psi' = g \psi$ are given by

\[ \Phi^\dagger \to \Phi'^\dagger = g^{-1} \Phi^\dagger g \]  

so that

\[ CM_r \to CM_r' = g^{T-1}(CM_r)g^{-1} \quad M_r \to M_r' = g M_r g^{-1} \]  

where use has been made of $C^{-1}g^{T-1}C = g$ which follows from the fact that the $SO(N)$ spinor rotation $g$ is written as $g = \exp(\theta^{ab} \Gamma_{ab})$ with real angle $\theta^{ab}$ and rank-2 gamma matrix $\Gamma_{ab}$ and the property of the charge conjugation matrix $C^{-1} \Gamma^T_{ab} C = -\Gamma_{ab}$. The $H$-invariance of the VEV implies that Eq. (2.19) for $g = h \in H$ holds with $M_r' = M_r$. It is $M_r$ but not $CM_r$ that satisfies

\[ M_r h = h M_r \quad \text{for} \quad \forall h \in H \]  

taking the form for which Schur’s lemma can be applied and tells us that, when the components of the fermion $\psi_I$ are decomposed and ordered into $H$-irreducible blocks, then the matrix $M_r$ becomes block diagonal and is proportional to an identity matrix in each $H$-irreducible block (so being actually diagonal matrix on this basis).

Therefore we can call $M_r$ ‘fermion mass matrix’, since it gives a constant mass eigenvalue on each $H$-irreducible sector of $\psi$ and its square $M_r M_r^\dagger$ gives square mass eigenvalues $m_1^2$ appearing in the effective potential.

Now we can state our “traceless condition” for the fermion mass matrix $M_r$: Since the rank-$r$ $AS$ tensor gamma matrices are traceless,$^4$

\[ \text{tr} [\Gamma_{\alpha_1\alpha_2\cdots\alpha_r}] = 0, \]  

\[ \text{tr}[\Gamma_{\alpha_1\alpha_2\cdots\alpha_r} \Gamma_{b_1b_2\cdots b_s}] = \begin{cases} 0 & \text{for } r \neq s \\ \delta_{\alpha_1}^{[b_1} \delta_{\alpha_2}^{a_2} \cdots \delta_{\alpha_r]}^{a_r} & \text{for } r = s \end{cases} \]  

\[ (2.21) \]
the fermion mass matrix $M_r$ given as their linear combination \( \text{(2.17)} \) is also traceless and hence the sum of mass eigenvalues should vanish.

This conclusion is valid for $SO(N=2n+1)$ cases, but does not apply as it stands for $SO(N=2n)$ cases in which the irreducible spinor fermion $\psi$ is ‘Weyl fermion’ possessing ‘chirality’ $\Gamma_{2n+1} = +1$ or $-1$. When we use the representation of $\Gamma_{2n+1}$ given in Eq. \( \text{(2.12)} \), Weyl spinors with chirality $\Gamma_{2n+1} = \pm 1$ have only upper-half or lower-half non-vanishing components. We refer to $\Gamma$-matrices possessing non-vanishing matrix elements only on diagonal (off-diagonal) blocks as “$\gamma_5$-diagonal ($\gamma_5$-off-diagonal)”\(^2\). Then, assuming positive chirality for our spinor $\psi$, the fermion mass matrix in Eq. \( \text{(2.17)} \) is now replaced by the following chiral projected one:

$$M_r = \sum_{a_1, a_2, \ldots, a_r = 1}^{2n} c^{a_1 a_2 \ldots a_r} \Gamma_{a_1 a_2 \ldots a_r} \left( \frac{\Gamma_{2n+1} + 1}{2} \right). \quad \text{(2.23)}$$

In order for this matrix $M_r$ not to vanish, $\Gamma_{a_1 a_2 \ldots a_r}$ must be $\gamma_5$-diagonal and hence $r$ must be even. Moreover, in order for the mass term $\psi^T (C M_r) \psi$ not to vanish, the matrix $CM_r$ should also be $\gamma_5$-diagonal, and hence $C$ as well as $M_r$ must be $\gamma_5$-diagonal. From Eq. \( \text{(2.13)} \), we find that $C$ is $\gamma_5$-diagonal only for even $n$; that is, the cases where $SO(N = 2n)$ spinor representations are real for $n \equiv 0 \pmod{4}$, or pseudo-real for $n \equiv 2 \pmod{4}$. For those cases, the mass matrix $M_r$ in \( \text{(2.23)} \) is still traceless since both $\text{tr} [\Gamma_{a_1 a_2 \ldots a_r}] = 0$ and $\text{tr} [\Gamma_{a_1 a_2 \ldots a_r} \Gamma_{2n+1}] = 0$ hold, and can give non-trivial constraint on the fermion mass that the sum of mass eigenvalues should vanish. Note, therefore, that the traceless condition on the mass eigenvalues is valid except only for $N = 4n + 2$ cases where the $SO(N)$ spinor representation is complex.

### 3 Addition-type subgroups

We can write down the form of the auxiliary scalar VEV which breaks $SO(N)$ into an addition-type subgroup $H$:

$$\langle \Phi^\dagger \rangle = C \cdot V_{SO(N) \rightarrow H}, \quad \text{(3.1)}$$

where we have factored out the charge conjugation matrix $C$ from the VEV $\langle \Phi^\dagger \rangle$ in conformity with Eq. \( \text{(2.17)} \), so $V_{SO(N) \rightarrow H}$ corresponds to the fermion mass matrix $M_r$ to which the “traceless condition” can be applied. $SO(N)$ spinor representations has $N$ modulo 8 structure. $H$ represents one of the little group of $SO(N)$. From Tables 2 and 3 and by using the expression of the gamma matrices in Eq. \( \text{(2.12)} \), we find the following VEVs in general; for $\langle \Phi^\dagger \rangle$,

$$V_{SO(4\ell) \rightarrow SO(2\ell) \times SO(2\ell)} \propto \Gamma_{1 \ldots 2\ell} \propto \sigma^{0}_{\ell-1} \sigma^{0}_{\ell} \sigma^{0}_{\ell+1},$$

$$V_{SO(4\ell+1) \rightarrow SO(2\ell+1) \times SO(2\ell+1)} \propto \Gamma_{1 \ldots 2\ell+1} \propto \sigma^{0}_{\ell-1} \sigma^{0}_{\ell} \sigma^{0}_{\ell+1},$$

$$V_{SO(4\ell+2) \rightarrow SO(2\ell+1) \times SO(2\ell+1)} \propto \Gamma_{1 \ldots 2\ell+1} \propto \sigma^{0}_{\ell} \sigma^{0}_{\ell+1},$$

$$V_{SO(4\ell+3) \rightarrow SO(2\ell+1) \times SO(2\ell+2)} \propto \Gamma_{1 \ldots 2\ell+1} \propto \sigma^{0}_{\ell} \sigma^{0}_{\ell+1},$$

for $\langle \Phi^\dagger \rangle \ (q = 1, 2, \ldots, \left[\frac{N+1}{8}\right])$,

$$V_{SO(4\ell+1) \rightarrow SO(2\ell+1) \times SO(2\ell+1) \times SO(2\ell+1) \times SO(2\ell+1)} \propto \Gamma_{1 \ldots 2\ell+1} \propto \sigma^{0}_{\ell} \sigma^{0}_{\ell+1},$$

$$V_{SO(4\ell+1) \rightarrow SO(2\ell+1) \times SO(2\ell+1) \times SO(2\ell+1) \times SO(2\ell+1)} \propto \Gamma_{1 \ldots 2\ell+1} \propto \sigma^{0}_{\ell} \sigma^{0}_{\ell+1},$$

for $\langle \Phi^\dagger \rangle \ (p = 1, 2, \ldots, \left[\frac{N}{8}\right])$.

\(^2\) Since $\Gamma_a (a = 1, 2, \ldots, 2n)$ flip the chirality $\Gamma_{2n+1}$, the rank-$r$ anti-symmetric tensor $\Gamma$-matrices $\Gamma_{a_1 a_2 \ldots a_r}$ are $\gamma_5$-diagonal for even $r$ and $\gamma_5$-off-diagonal for odd $r$. 

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It should be understood that the chiral projection matrix
\[ \mathcal{P} := \frac{\Gamma_{2n+1} \pm 1}{2} \] (3.5)
is multiplied to these expressions for \( V_{SO(N) \rightarrow H} \) when \( N \) is even \( N = 2n \). Any VEV of these is proportional to \( \Gamma_{1-r} \) of a certain rank \( r \), so the quadratic mass matrix of the \( SO(N) \) spinor fermion is found to be
\[ \langle \Phi_{R_k} \rangle \langle \Phi_{R_k}^\dagger \rangle \propto \mathcal{P} \Gamma_1^1 \cdots \Gamma_1^1 \mathcal{C}^\dagger \mathcal{T}_{1-r} \mathcal{P} \propto 1 \left[ \frac{1}{2} \frac{N}{2} \right] \mathcal{P} \] (3.6)
with understanding that \( \mathcal{P} = 1 \) for \( SO(N) \) with odd \( N = 2n + 1 \) case. That is, all the \( d = 2 \left[ \frac{N-1}{2} \right] \) components of \( \psi \) obtain a degenerate mass square \( m^2 \) realizing the minimum of \( F(x, M^2_{R_k}) \) for all cases. Therefore the global minimum of the potential is realized by the VEV of \( \Phi_{R_k} \) for which the coupling constant \( G_{R_k} \) is the strongest. So the symmetry breaking pattern is determined depending on which coupling constant \( G_{R_k} \) is the strongest: We find the following breaking pattern. For even \( N = 2n \),
\[
SO(N = 2n) \rightarrow \begin{cases} 
SO(n) \times SO(n) & \text{for } G_{R_0} \text{: strongest} \\
SO(n+4) \times SO(n-4) & \text{for } G_{R_1} \text{: strongest} \\
SO(n+8) \times SO(n-8) & \text{for } G_{R_2} \text{: strongest} \\
SO(n+12) \times SO(n-12) & \text{for } G_{R_3} \text{: strongest} \\
& \vdots 
\end{cases}
\] (3.7)
For odd \( N = 2n + 1 \),
\[
SO(N = 2n + 1) \rightarrow \begin{cases} 
SO(n+1) \times SO(n) & \text{for } G_{R_0} \text{: strongest} \\
SO(n+4) \times SO(n-3) & \text{for } G_{R_1} \text{: strongest} \\
SO(n+5) \times SO(n-4) & \text{for } G_{R_2} \text{: strongest} \\
SO(n+8) \times SO(n-7) & \text{for } G_{R_3} \text{: strongest} \\
SO(n+9) \times SO(n-8) & \text{for } G_{R_4} \text{: strongest} \\
& \vdots 
\end{cases}
\] (3.8)

### 4 Product-type subgroups

Next, we discuss “product-type” little group cases. From Table 1 we see that this “product-type” subgroups are further classified into three sub-types; i.e., “orthogonal-type,” “symplectic-type,” and “unitary-type,” subgroups. More explicitly, here we consider \( SO(mn) \rightarrow SO(m) \times SO(n)(S) (m, n \in \mathbb{Z}_{\geq 3}) \), and \( SO(4mn) \rightarrow USp(2m) \times USp(2n)(S) (m, n \in \mathbb{Z}_{\geq 1}) \), and \( SO(2n) \rightarrow SU(n) \times U(1)(R) (n \in \mathbb{Z}_{\geq 2}) \). Note that \( USp(2) \simeq SU(2) \) and \( SO(4) \simeq SU(2) \times SU(2) \) lead to \( SO(4n) \supset USp(2) \times USp(2n) \simeq SU(2) \times USp(2n) \supset SU(2) \times SU(2) \times SO(n) \simeq SO(4) \times SO(n) \). Thus, \( SO(4) \times SO(n) \) is not a maximal subgroup of \( SO(4n) \).

For the branching rules for the cases \( SO(mn) \supset SO(m) \times SO(n) (m, n \in \mathbb{Z}_{\geq 3}) \) and \( SO(4mn) \supset USp(2m) \times USp(2n) (m, n \in \mathbb{Z}_{\geq 1}) \), many examples are listed in the tables in Ref. 42.

As we will see below, we find that only \( SO(9) \rightarrow SO(3) \times SO(3)(S) \) and \( SO(16) \rightarrow USp(4) \times USp(4)(S) \), \( SO(4) \rightarrow SU(2) \times U(1)(R) \) breaking cases can satisfy the global minimum condition in appropriate NJL coupling constant region because all the fermion masses are degenerate for those breaking.
4.1 Orthogonal-type

For the breaking $SO(mn) \supset SO(m)_A \times SO(n)_B$, the $SO(mn)$ vector index $a (1 \leq a \leq mn)$ is identified with a pair $\{iI\}$ of $SO(m)_A$ vector index $i$ ($1 \leq i \leq m$) and $SO(n)_B$ vector index $I$ ($1 \leq I \leq n$); more explicitly, the $SO(mn)$ Gamma matrices $\Gamma_a$ are denoted in both way as

$$\Gamma_{\{iI\}} \leftrightarrow \Gamma_a \left(\text{with } a = n(i-1) + I\right). \quad (4.1)$$

Then the $SO(m)_A$ and $SO(n)_B$ generators $J^A_{ij}$ and $J^B_{IJ}$ are given by the following linear combinations of the $SO(mn)$ generators $\propto \Gamma_{\{iI\}} = \Gamma_a \Gamma_b = -\Gamma_b \Gamma_a \left(\text{with } a \neq b\right)$:

$$J^A_{ij} = \frac{1}{2\sqrt{2i}} \sum_{I,J=1}^{m} \delta^{IJ} \Gamma_{\{iI\}\{jJ\}}, \quad J^B_{IJ} = \frac{1}{2\sqrt{2i}} \sum_{i,j=1}^{n} \delta^{ij} \Gamma_{\{iI\}\{jJ\}}. \quad (4.2)$$

The quadratic Casimir operator of $SO(m)_A$ is calculated as

$$\sum_{j>i=1}^{m} (J^A_{ij})^2 = -\frac{1}{(2\sqrt{2i})^2} \sum_{j>i=1}^{m} \sum_{I,J=1}^{n} \Gamma_{\{iI\}\{jJ\}} \Gamma_{\{iI\}\{jJ\}} = -\frac{1}{8} \left( -1 \times n \times \frac{m(m-1)}{2} + 2M_4 \right), \quad (4.3)$$

where the first terms $\propto 1$ come from $I = J$ terms and the second term $2M_4$ from $I \neq J$, which is a (linear combination of) rank-4 anti-symmetric tensor $\Gamma_{\{iI\}\{jJ\}\{jJ\}}$ since $i \neq j, I \neq J$. Note that since $SO(m)_A$ generators $J^A_{ij}$ are $SO(n)_B$-invariant by construction and so the Casimir operator $(J^A_{ij})^2$ is both invariant under $SO(m)_A$ and $SO(n)_B$. But the first term $\propto 1$ is $G = SO(mn)$-invariant, so that the rest rank-4 tensor part $M_4$ gives an $H = SO(m)_A \times SO(n)_B$ singlet component in the rank-4 $SO(mn)$ tensor $\Gamma_{abcd}$. Note also that the quadratic Casimir of the other factor group $SO(n)_B$ reproduce the same rank-4 $H$-singlet $M_4$ with minus sign as

$$\sum_{j>i=1}^{m} (J^B_{ij})^2 = -\frac{1}{(2\sqrt{2i})^2} \sum_{j>i=1}^{m} \sum_{I,J=1}^{n} \Gamma_{\{iI\}\{jJ\}} \Gamma_{\{iI\}\{jJ\}} = -\frac{1}{8} \left( -1 \times m \times \frac{n(n-1)}{2} - 2M_4 \right). \quad (4.4)$$

This should be so because the rank-4 $H$-singlet is unique in the $SO(mn)$ rank-4 tensor $\Gamma_{abcd}$, or more technically, because

$$\Gamma_{\{iI\}\{jJ\}} \Gamma_{\{iI\}\{jJ\}} = \Gamma_{\{iI\}} \Gamma_{\{jJ\}} \Gamma_{\{iI\}} \Gamma_{\{jJ\}} = -\Gamma_{\{iI\}} \Gamma_{\{iI\}} \Gamma_{\{jJ\}} \Gamma_{\{jJ\}} = -\Gamma_{\{iI\}\{iI\}\{jJ\}\{jJ\}}.$$

First, we show that the $SO(9) \rightarrow SO(3)_A \times SO(3)_B$ breaking VEV of the $SO(9)$ 126 (rank-4 AS tensor) scalar field $\Phi_{R_o}$ generates a common mass for the $SO(9)$ spinor fermion 16. The $SO(9)$ spinor 16 is decomposed into two representations under $SO(9) \supset SO(3)_A \times SO(3)_B$ as

$$16 = (4, 2) \oplus (2, 4). \quad (4.5)$$

(See e.g., Ref. [32].) With $SO(3)$ generator denoted in 3-vector notation $\vec{J} = (J_{23}, J_{31}, J_{12})$ as usual, the quadratic Casimir operator $(\vec{J}^A)^2$ of $SO(3)_A$ subgroup possesses the following eigenvalues on the $SO(9)$ spinor 16:

$$\langle \vec{J}^A \rangle^2 \left( \begin{array}{c|c} (4, 2) & \langle (4, 2) \rangle \\ \hline (2, 4) & \langle (2, 4) \rangle \end{array} \right) = \left( \begin{array}{c} C_2(4)|4, 2\rangle \\ C_2(2)|2, 4\rangle \end{array} \right) = \left( \begin{array}{c} \frac{15}{8} |(4, 2)\rangle \\ \frac{9}{8} |(2, 4)\rangle \end{array} \right), \quad (4.6)$$

where the $SO(3)$ Casimir eigenvalues $C_2(4) = 15/8, C_2(2) = 3/8$ here are halves of the $SU(2)$ ones $j(j+1)$ for spin-$j$ representation listed in Ref. [32] because of our convention $T(3) = 1$ for $SO(3)$. The quadratic Casimir operators (4.3) and (4.4) in this $m = n = 3$ case are given by

$$\langle \vec{J}^A \rangle^2 = -\frac{1}{8} \langle 16 \times 3 \times 3 + 2M_4 \rangle, \quad \langle \vec{J}^B \rangle^2 = -\frac{1}{8} \langle 16 \times 3 \times 3 - 2M_4 \rangle, \quad (4.7)$$

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with the rank-4 tensor part $M_4$ which more explicitly reads by using $SO(9)$ vector index $a = 2i + I$ in place of \{ij\} notation ($a = 1, 2, \cdots, 9$; $i, I = 1, 2, 3$),

$$-M_4 := \Gamma_{1245} + \Gamma_{1346} + \Gamma_{2356} + \Gamma_{1278} + \Gamma_{1379} + \Gamma_{2389} + \Gamma_{4578} + \Gamma_{4679} + \Gamma_{5689}. \tag{4.8}$$

Since the $M_4$ is an $SO(3)_A \times SO(3)_B$-singlet in the rank-4 tensor $\Gamma_{abcd}$ as noted above, the fermion mass matrix is proportional to $M_4$ and acts on the $SO(9)$ spinor $16$ as

$$-vM_4 \begin{pmatrix} \langle(4, 2)\rangle \\ \langle(2, 4)\rangle \end{pmatrix} = \frac{1}{2}v \begin{pmatrix} (8\langle J^A \rangle)^2 - 9 \times 1_{16} \end{pmatrix} \begin{pmatrix} \langle(4, 2)\rangle \\ \langle(2, 4)\rangle \end{pmatrix} + \begin{pmatrix} 3v\langle(4, 2)\rangle \\ -3v\langle(2, 4)\rangle \end{pmatrix}. \tag{4.9}$$

This leads to the fermion mass square matrix proportional to $(M_4^2M_4) \propto 1_{16}$. Thus, we find that all the fermion states in $(4, 2)$ and $(2, 4)$ of $SO(3)_A \times SO(3)_B \subset SO(9)$ get the same mass from the condensation of $SO(3)_A \times SO(3)_B$-singlet combination (4.3) of rank-4 $\Gamma_{abcd}$. Note, however, that we could have obtained this result (4.9) more quickly from the “traceless condition” of the fermion mass matrix $M_4$ since it can be applied for $SO(9)$ with odd 9 and the $H$-irreducible sectors $(4, 2)$ and $(2, 4)$ have the same dimensions 8.

One may wonder whether for other cases such as $SO(15) \rightarrow SO(3)_A \times SO(5)_B$ the VEV of the scalar field can also realize the global minimum or not. For example, for $SO(15) \rightarrow SO(3)_A \times SO(5)_B$, the branching rules of the $SO(15)$ spinor representation is given by

$$128 = (6, 4) \oplus (4, 16) \oplus (2, 20). \tag{4.10}$$

The eigenvalues of $SO(3)_A$ and $SO(5)_B$ Casimir operators $(\langle J^A \rangle)^2$ and $(\langle J^B \rangle)^2$ of the $SO(15)$ spinor $128$ are given by

$$(\langle J^A \rangle)^2 \begin{pmatrix} \langle(6, 4)\rangle \\ \langle(4, 16)\rangle \\ \langle(2, 20)\rangle \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 35 \langle(6, 4)\rangle \\ 15 \langle(4, 16)\rangle \\ 3 \langle(2, 20)\rangle \end{pmatrix}, \tag{4.11}$$

$$(\langle J^B \rangle)^2 \begin{pmatrix} \langle(6, 4)\rangle \\ \langle(4, 16)\rangle \\ \langle(2, 20)\rangle \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 10 \langle(6, 4)\rangle \\ 30 \langle(4, 16)\rangle \\ 42 \langle(2, 20)\rangle \end{pmatrix}. \tag{4.12}$$

where use has been made of, for $SO(3)$, $C_2(6) = 35/8, C_2(4) = 15/8, C_2(2) = 3/8$; for $SO(5)$, $C_2(20) = 21/4, C_2(16) = 15/4, C_2(4) = 5/4$. (See e.g., Ref. [32].) From Eq. (4.3) with $m = 3, n = 5$, we find the quadratic Casimir operators for the subgroups $SO(3)_A$ and $SO(5)_B$:

$$(\langle J^A \rangle)^2 = -\frac{1}{8} (-1_{128} \times 5 \times 3 + 2M_4), \quad (\langle J^B \rangle)^2 = -\frac{1}{8} (-1_{128} \times 3 \times 10 - 2M_4), \tag{4.13}$$

with $M_4 = \sum_{j=1}^{3} \sum_{I=1}^{5} \Gamma_{\{ji\}\{jI\}\{ij\}\{jI\}}$. The fermion mass is proportional to the $SO(3)_A \times SO(5)_B$-singlet $M_4$ which acts on the $SO(15)$-spinor $128$ as either $-8(\langle J^A \rangle)^2 - 15 1_{128}/2$ or $(\langle J^B \rangle)^2 - 30 1_{128}/2$. In either way, we find

$$-vM_4 \begin{pmatrix} \langle(6, 4)\rangle \\ \langle(4, 16)\rangle \\ \langle(2, 20)\rangle \end{pmatrix} \propto \begin{pmatrix} 10v \langle(6, 4)\rangle \\ 0 \langle(4, 16)\rangle \\ -6v \langle(2, 20)\rangle \end{pmatrix}. \tag{4.14}$$

This leads to $(M_4^2M_4) \neq 1$. Thus, we find that the fermion states in $\langle(6, 4)\rangle$, $\langle(4, 16)\rangle$, and $\langle(2, 20)\rangle$ of $SO(3)_A \times SO(5)_B \subset SO(15)$ obtain different masses from the rank-4 tensor $\Gamma_{abcd}$ condensation. Note here also that the fermion mass matrix $-vM_4$ satisfies the traceless property, $\text{tr} M_4 = 0$; indeed

$$\text{tr}(-vM_4) = 10v \times (6 \times 4) + 0 \times (4 \times 16) - 6v \times (2 \times 20) = 0. \tag{4.15}$$
This again confirms the validity of our traceless condition of the fermion mass matrix. In this case, however, the original spinor is decomposed into three $H$-irreducible components so that the traceless condition alone cannot determine the three unknown mass eigenvalues even aside from the overall scale.

In general, when the breaking $SO(mn) \rightarrow SO(m) \times SO(n)(m,n \geq 3)$ occurs, the $SO(mn)$ spinor representation is decomposed into at least two different $H$-irreducible representations. (Several examples can be found in Ref. [22].) The $SO(9) \rightarrow SO(3) \times SO(3)$ case is rather exceptional where only two $H$-irreducible components appear and they have the same dimensions. Then, by the traceless condition, the mass matrix $M_4$ turns to have two mass eigenvalues $\pm v$ with same magnitude and opposite sign, so that the fermion quadratic mass matrix $(M_4^\dagger M_4)$ was proportional to an identity. For general $m \neq n$ cases, however, the dimensions of the $H$-irreducible components are mutually different and so, even for the case where only two $H$-irreducible components appear, the traceless arguments for the mass matrix $M_r$ gives two different eigenvalues for those $H$-irreducible components, so $(M_4^\dagger M_4) \neq 1$. Even for $m = n$ cases, the $SO(mn)$ spinor splits into more than or equal to four $H$-irreducible representations for $m = n \geq 5$, and one easily see by the argument of quadratic Casimir and tracelessness that the mass matrix $M_4$ has eigenvalues of at least two different absolute values, so that $(M_4^\dagger M_4) \neq 1$. Therefore, the $SO(mn) \rightarrow SO(m) \times SO(n)(m,n \geq 3)$ breaking VEV cannot be the global minimum of the potential in the $SO(mn)$ NJL-type model except for the $m = n = 3$ case.

4.2 Symplectic-type

Here we discuss the $SO(4mn) \rightarrow USp(2m) \times USp(2n)$ breaking case. A close parallelism holds between the previous breaking $SO(mn) \rightarrow SO(m) \times SO(n)$ in Sec. 4.1 and the present one $SO(4mn) \rightarrow USp(2m) \times USp(2n)$.

From the tables in Ref. [22] and additional searches, we find that under the subgroup $H = USp(2m) \times USp(2n) \subset SO(4mn)$ except the $m = 1, n = 2$ and $m = n = 2$ cases, the $SO(4mn)$ spinor representation is decomposed into two essentially different $H$-irreducible representations. So the quadratic mass matrix $M_r M_r^\dagger$ of the fermion cannot be proportional to identity. Thus, they cannot realize the global minimum of the effective potential.

Consider the first exceptional case of $m = 1, n = 2$. For $USp(2) \times USp(4) \subset SO(8)$ the branching rules of the $SO(8)$ spinor representations $r = 8_s$ and $r' = 8_c$ (both are real) are given by

$$8_s = (2, 4), \quad 8_c = (2, 4);$$

(4.16)

that is, they are also $H$-irreducible. In this case, however, the symmetric tensor product of the $SO(8)$ spinor gives $(8 \times 8)_S = 35(R_0) + 1(R_1)$ (for both cases $8_s$ and $8_c$), but $\Phi_{R_0}$ does not have a singlet of $H = USp(2) \times USp(4) \subset SO(8)$ and $\Phi_{R_1}$ is $SO(8)$-singlet. So the breaking $SO(8) \rightarrow USp(2) \times USp(4)$ actually does not occur in this case.

Next consider the second exceptional case $m = n = 2$. This case, the two $SO(16)$ irreducible spinor representations $r = 128$ and $r' = 128'$ have different branching rules under $USp(4) \times USp(4) \subset SO(16)$ as

$$128 = (14, 1) \oplus (10, 5) \oplus (5, 10) \oplus (1, 14), \quad 128' = (16, 4) \oplus (4, 16).$$

(4.17)

The pair condensation of the $128$ spinor fermion cannot realize the global minimum of the potential since the fermion mass eigenvalues are not degenerate in this case. This can be shown as follows.

The traceless condition holds in this case $SO(16)$ possessing real spinor. For the case of $128$ spinor which decomposes into more than two $H$-irreducible components, the traceless condition alone is not enough to calculate the fermion mass eigenvalues. Since the $USp(4) \times USp(4)$-singlet exists in the channel $\Phi_{R_1}$ (rank-4 tensor), the eigenvalue of the mass matrix $M_4$ in the rank-4 tensor can be calculated by computing the $USp(4)$ quadratic Casimir $C_2$ which must be a
linear combination of $G$-singlet matrix $1 \in \Phi_{R_2}$ and the $H$-singlet $M_4 \in \Phi_{R_1}$. Hence we have $M_4 \propto C_2 - x1$:

$$M_4 \left( \begin{array}{c} \langle 14,1 \rangle \\ \langle 10,5 \rangle \\ \langle 5,10 \rangle \\ \langle 1,14 \rangle \end{array} \right) \propto \left( \begin{array}{c} (5-x) \langle 14,1 \rangle \\ (3-x) \langle 10,5 \rangle \\ (2-x) \langle 5,10 \rangle \\ (0-x) \langle 1,14 \rangle \end{array} \right) = \left( \begin{array}{c} \frac{1}{2} \langle 14,1 \rangle \\ \frac{1}{2} \langle 10,5 \rangle \\ -\frac{1}{2} \langle 5,10 \rangle \\ -\frac{1}{2} \langle 1,14 \rangle \end{array} \right), \quad (4.18)$$

where the $USp(4)$ Casimir eigenvalues $C_2(14) = 5$, $C_2(10) = 3$, $C_2(5) = 2$, $C_2(1) = 0$ have been used and, in going to the last expression, the $G$-singlet coefficient $x$ was found to be $x = 5/2$ by using the traceless condition:

$$5 \times 14 + 3 \times (10 \times 5) + 2 \times (5 \times 10) + 0 \times 14 - 128x = 0 \quad \Rightarrow \quad x = \frac{320}{128} = \frac{5}{2}. \quad (4.19)$$

Note that we were able to skip the laborious explicit identification of the $USp(4)$ generators in terms of the original $SO(16)$ generators $\Gamma_{ab}$.

For the case of the spinor $r' = 128'$, on the other hand, the traceless condition is enough to determine the (ratio of) the mass eigenvalues since $r' = 128'$ has only two $H$-irreducible components under $H = USp(4) \times USp(4) \subset SO(16)$. Again the $H$-singlet is in the $\Phi_{R_1}$ (rank-4 tensor) so the mass matrix $M_4$ must be traceless:

$$M_4 \left( \begin{array}{c} \langle 16,4 \rangle \\ \langle 4,16 \rangle \end{array} \right) \propto \left( \begin{array}{c} +1 \langle 16,4 \rangle \\ -1 \langle 4,16 \rangle \end{array} \right). \quad (4.20)$$

Thus, their quadratic mass matrix is proportional to identity and can realize the global minimum.

For $USp(2) \times USp(8) \subset SO(16)$, the branching rules of the $SO(16)$ spinor representations are given by

$$128 = (5,1) \oplus (3,27) \oplus (1,42), \quad 128' = (4,8) \oplus (2,48). \quad (4.21)$$

Again the $USp(2) \times USp(8)$-singlet is in the $\Phi_{R_1}$ (rank-4 tensor) both for $128$ and $128'$ spinors. The traceless condition and quadratic Casimir lead to

$$M_4 \left( \begin{array}{c} \langle 5,1 \rangle \\ \langle 3,27 \rangle \\ \langle 1,42 \rangle \end{array} \right) \propto \left( \begin{array}{c} (6-x) \langle 5,1 \rangle \\ (2-x) \langle 3,27 \rangle \\ (0-x) \langle 1,42 \rangle \end{array} \right) = \left( \begin{array}{c} \frac{6}{7} \langle 5,1 \rangle \\ \frac{2}{7} \langle 3,27 \rangle \\ -\frac{1}{7} \langle 1,42 \rangle \end{array} \right), \quad (4.22)$$

$$M_4 \left( \begin{array}{c} \langle 4,8 \rangle \\ \langle 2,48 \rangle \end{array} \right) \propto \left( \begin{array}{c} \left( \frac{15}{7} - \frac{2}{7} \right) \langle 4,8 \rangle \\ \left( \frac{3}{7} - \frac{3}{2} \right) \langle 2,48 \rangle \end{array} \right) = \left( \begin{array}{c} \frac{13}{7} \langle 4,8 \rangle \\ -\frac{3}{7} \langle 2,48 \rangle \end{array} \right), \quad (4.23)$$

where the $USp(2) \simeq SU(2)$ Casimir eigenvalues $C_2(5) = 6$, $C_2(3) = 2$, $C_2(1) = 0$, $C_2(4) = 15/4$, $C_2(2) = 3/4$ have been used. The pair condensation of the $128$ and $128'$ spinor fermion, therefore, cannot realize the global minimum of the potential.

### 4.3 Unitary-type

Here we consider the $SO(2n) \rightarrow SU(n) \times U(1)$ or $SU(n)$ ($n \geq 2$) breaking cases for even and odd $n$, respectively. For $SO(4k) \rightarrow SU(2k) \times U(1)$ (even $n = 2k$), the spinor representations are real or pseudo-real representations; for $SO(4k + 2) \rightarrow SU(2k + 1) \times U(1)$ (odd $n = 2k + 1$), the spinor representations are complex representations. For $SO(4k)$, the fermion mass matrix must satisfy the traceless condition. From the following discussion, we find that for only $n = 2$, i.e., $SO(4) \simeq SU(2) \times SU(2) \rightarrow SU(2) \times U(1)$, the pair condensation of the spinor fermion realizes the global minimum of the potential.
To calculate the $SU(n)$ decomposition of the $SO(2n)$ spinor fermion, we here follow the well-known trick to rewrite the $SO(2n)$ gamma matrices $\Gamma_a$ into the creation and annihilation operators of $SU(n)$ fermion, $a_k^\dagger, a^k (k = 1, 2, \ldots, n)$ [59]:

\[
\begin{align*}
  a_k^\dagger & := \frac{1}{2} (\Gamma_{2k-1} + i \Gamma_{2k}) \\
  a^k & := \frac{1}{2} (\Gamma_{2k-1} - i \Gamma_{2k}) \\
\end{align*}
\]

Then the $SO(2n)$ (reducible) $2^n$-dimensional spinor is represented in the form

\[
|\psi\rangle = |0\rangle \psi_0 + \frac{1}{2} a_k^\dagger |0\rangle \psi^j + \frac{1}{2} a_k^\dagger a_j^\dagger |0\rangle \psi^{ij} + \cdots \\
+ \varepsilon^{k_1 k_2 \cdots k_{n-1}} a_{k_1}^\dagger a_{k_2}^\dagger \cdots a_{k_{n-1}}^\dagger |0\rangle \tilde{\psi}_j + a_1^\dagger a_2^\dagger \cdots a_n^\dagger |0\rangle \tilde{\psi}_0,
\]

giving a decomposition into irreducible representations of $SU(n)$. The $SO(2n)$ chirality matrix ($\gamma_5$-analogue) is represented by

\[
\Gamma_\chi := i^n \Gamma_1 \Gamma_2 \cdots \Gamma_n = \prod_{k=1}^{n} \left(1 - 2 a_k^\dagger a^k\right) = \prod_{k=1}^{n} (-1)^{a_k^\dagger a^k} = (-1)^{\tilde{N}},
\]

with the total fermion number operator $\tilde{N} := \sum_k a_k^\dagger a^k$. So $SO(2n)$ irreducible chiral spinors $|\psi_{\pm}\rangle$ with $\Gamma_\chi = \pm 1$ are given by half sums of states in the RHS of Eq. (4.25) possessing even and odd number of fermion excitations, respectively.

Since the gamma matrices $\Gamma_{2k-1}$ are symmetric and $\Gamma_{2k}$ are anti-symmetric in the spinor basis of Eq. (4.25), the charge conjugation matrix $C$ satisfying

\[
C^{-1} \Gamma_a^T C = - \Gamma_a
\]

can be chosen to be the operator

\[
C = i^n \prod_{k=1}^{n} \Gamma_{2k} = \prod_{k=1}^{n} (a_k^\dagger - a^k).
\]

First, begin with the $n = 2$ case, i.e., $SO(4) \simeq SU(2) \times SU(2) \rightarrow SU(2)(\times U(1))$. The branching rules of the $SO(4)$ spinor representations are given by

\[
(2, 1) = (2)(0), \quad (1, 2) = (1)(+1) \oplus (1)(-1).
\]

The symmetric tensor product of $SO(4)$ spinor gives $((2, 1) \otimes (2, 1))_S = (3, 1)(R_0)$ and $((1, 2) \otimes (1, 2))_S = (1, 3)(R_0)$, which are respectively decomposed under under $SU(2) \subset SO(4)$ as $SO(4)$ has a singlet so the the pair condensation of the $(1, 2)$ spinor fermion can cause this breaking.

Now in the $SU(n)$ fermion representation, the $SO(4)$ chiral spinors with $\Gamma_\chi = \pm 1$ are given by

\[
\begin{align*}
  |\psi_+\rangle &= |0\rangle \psi_0 + a_1^\dagger a_2^\dagger |0\rangle \tilde{\psi}_0 \\
  |\psi_-\rangle &= a_1^\dagger |0\rangle \psi^j
\end{align*}
\]

\[\text{In this spinor basis, the creation and annihilation operators } a^\dagger \text{ and } a \text{ are represented by } 2 \times 2 \text{ Pauli matrices as } \sigma^+ = (\sigma_1 + i \sigma_2)/2 \text{ and } \sigma^- = (\sigma_1 - i \sigma_2)/2, \text{ respectively, in each } k \text{ sector. So, for the present case with } n \text{ species of fermions, the gamma matrices are represented as}
\]

\[
\Gamma_{2k-1} = \sigma_3^{\otimes k-1} \otimes \sigma_1 \otimes \sigma_0^{\otimes n-k}, \quad \Gamma_{2k} = \sigma_3^{\otimes k-1} \otimes \sigma_2 \otimes \sigma_0^{\otimes n-k}.
\]

This representation of the $\Gamma_a$ matrices are different from that in Eq. (2.12).
where $|\psi_+\rangle$ and $|\psi_-\rangle$ correspond to $(1, 2)$ and $(2, 1)$ in Eq. (4.30), respectively. The bispinor representation $R_0$ is now a rank-2 tensor $\Gamma_{a_1a_2}$ and so can be described by the operators 
\{a_1^\dagger a_j^\dagger, a_i a^\dagger a^\dagger j\}. The SU(2)-singlet in $R_0$ is clearly given by a linear combination \{a_1^\dagger a_2^\dagger, a^\dagger a^\dagger j\}: $c_1 a_1^\dagger a_2^\dagger + c_2 a^\dagger a^\dagger j$ with arbitrary coefficients $c_j (j = 1, 2)$. The SU(2)-singlet mass matrix $M_2$ should also satisfy $M_2^\dagger M_2 = M_2 M_2^\dagger$ since it is diagonalizable as we have shown before, which further leads to a condition $|c_1|^2 = |c_2|^2$. We choose the phase convention of $a^\dagger$ and $a^\dagger j$ such that $c_1 = c_2 =: m$ is satisfied with real $m$ and take $M_2$ as a Hermitian operator:

$$M_2 = m \left( a_1^\dagger a_2^\dagger + a^\dagger a^\dagger j \right).$$

(4.32)

Acting this on the positive chiral spinor $|\psi_+\rangle$,

$$M_2 |\psi_+\rangle = m \left( a_1^\dagger a_2^\dagger + a^\dagger a^\dagger j \right) \left( |0\rangle \psi_0 + a_1^\dagger a_2^\dagger |0\rangle \tilde{\psi}_0 \right)$$

$$= m \left( a_1^\dagger a_2^\dagger |0\rangle \psi_0 + |0\rangle \tilde{\psi}_0 \right) = \begin{pmatrix} 0 & m & 0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \tilde{\psi}_0 \end{pmatrix}. \tag{4.33}$$

We can diagonalize the mass matrix $M_2$ as

$$M_2 \begin{pmatrix} |(1)(+1)\rangle \\ |(1)(-1)\rangle \end{pmatrix} = \begin{pmatrix} m |(1)(+1)\rangle \\ -m |(1)(-1)\rangle \end{pmatrix}. \tag{4.34}$$

The pair condensation of the $(1, 2)$ spinor fermion thus yields a degenerate mass square eigenvalue for the spinor $(1, 2)$ and hence realizes the global minimum of the potential.

Next, for $n = 3$, i.e., $SO(6) \simeq SU(4)$, the branching rules of the irreducible spinor representations $r = 4$ and $\bar{r} = \bar{4}$ are given by

$$4 = (3)(1) \oplus (1)(-3), \quad \bar{4} = (3)(-1) \oplus (1)(3). \tag{4.35}$$

The symmetric tensor product of $SO(6)$ spinor gives $(4 \otimes 4)_S = 10(R_0)$ ($(\bar{4} \otimes 4)_S = \bar{10}(R_0)$).

$10(R_0)$ has a singlet under $SU(3) \times U(1)$:

$$10 = (\bar{6})(2) \oplus (3)(-2) \oplus (1)(-6), \quad \bar{10} = (6)(-2) \oplus (\bar{3})(2) \oplus (1)(6). \tag{4.36}$$

The $SO(6)$ chiral spinors are given in the $SU(n)$ fermion representation as

$$\begin{cases} |4\rangle = |\psi_-\rangle = a_1^\dagger |0\rangle \psi^i + a_2^\dagger a_3^\dagger |0\rangle \tilde{\psi}_0 \\ |\bar{4}\rangle = |\psi_+\rangle = |0\rangle \psi_0 + \frac{1}{2} \epsilon_{ijk} a_1^\dagger a_2^\dagger a_3^\dagger |0\rangle \tilde{\psi}_k \end{cases}. \tag{4.37}$$

The mass matrix $M_3(10)$ ($M_3(\bar{10})$) in the rank-3 tensor bispinor $10(R_0)$ ($\bar{10}(R_0)$) corresponds to an $SU(3)$ singlet operator:

$$M_3(10) = m a_1^\dagger a_2^\dagger a_3^\dagger, \quad M_3(\bar{10}) = m a^\dagger a^\dagger j a^\dagger a^\dagger j. \tag{4.38}$$

For odd $n$, the mass matrix $M_n$ and the charge conjugation matrix $C$ are both $\Gamma_\chi$-off diagonal, while $CM_n$ is $\Gamma_\chi$-diagonal. So we directly have to calculate the fermion bilinear mass term $\psi^\dagger_i \Gamma_j CM_n \psi_{\pm}$. Noting $\psi^\dagger \Gamma_j$ is represented by the state

$$\begin{cases} \langle 4^* \rangle = \langle \psi^+_\pm \rangle = \psi^i(0) a^i + \tilde{\psi}_0(0) a^\dagger a^\dagger j \\ \langle \bar{4}^* \rangle = \langle \psi^-_\pm \rangle = \psi_0(0) + \frac{1}{2} \psi^{iji}(0) a^\dagger a^\dagger j \end{cases}, \tag{4.39}$$
and using the expression (4.39) of the charge conjugation operator $C$ for $n = 3$, the mass terms $\psi^\dagger \hat{M}_3 \psi_{\pm}$ can be calculated as follows:

$$
\psi^\dagger \hat{M}_3 \psi_{\pm} = (\mathbf{4}^* | C \ M_3 (\mathbf{10}) | \mathbf{4}) = \langle \psi_{\pm}^{\dagger} | (a_1^1 - a^1) \ (a_2^2 - a^2) \ (a_3^3 - a^3) \ (ma^3 a^2 a^1) | \psi_{\pm} \rangle \\
= \langle \psi_{\pm}^{\dagger} | (a_1^1 - a^1) \ (a_2^2 - a^2) \ (a_3^3 - a^3) \ m | 0 \rangle \ \tilde{\psi}_0 = m \langle \psi_{\pm}^{\dagger} | a_1^1 a_2^2 a_3^3 | 0 \rangle \ \tilde{\psi}_0 = m \tilde{\psi}_0 \tilde{\psi}_0,
$$

$$
\psi^\dagger \hat{M}_3 \psi_{\pm} = (\mathbf{4}^* | C \ M_3 (\mathbf{10}) | \mathbf{4}) = \langle \psi_{\pm}^{\dagger} | (a_1^1 - a^1) \ (a_2^2 - a^2) \ (a_3^3 - a^3) \ (ma_1^1 a_2^2 a_3^3) | \psi_{\pm} \rangle \\
= m \langle \psi_{\pm}^{\dagger} | (-a^1 a^2 a^3) a_1^1 a_2^2 a_3^3 | 0 \rangle \ \tilde{\psi}_0 = m \langle \psi_{\pm}^{\dagger} | 0 \rangle \ \tilde{\psi}_0 = m \psi_{\pm} \psi_{\pm}.
$$

That is, only the singlet $\tilde{\psi}_0$ in $\mathbf{4}$ ($\psi_0$ in $\mathbf{4}$) gets the mass $m$ while the rest triplet $\mathbf{3} (\psi^i) \in \mathbf{4} \ (\bar{\mathbf{3}} (\bar{\psi}_i) \in \bar{\mathbf{4}})$ remains massless. Thus, the pair condensation of the $\mathbf{4}$ ($\bar{\mathbf{4}}$) spinor fermion cannot realize the global minimum of the potential.

This argument actually holds for all cases of odd $n = 2\ell + 1$, for which the chiral spinor is complex

$$
| \psi_+ \rangle = | 0 \rangle \ \psi_0 + \frac{1}{2} a_1^1 a_2^2 | 0 \rangle \ \psi^{ij} + \ldots + \frac{1}{(n-1)!} e^{i\epsilon_1 \epsilon_2 \cdots \epsilon_{n-1} - j} a_{1i} a_{1j} \cdots a_{i_{n-1}} | 0 \rangle \ \psi_i,
$$

$$
| \psi_- \rangle = a_1^1 | 0 \rangle \ \psi^i + \frac{1}{2} a_1^1 a_2^2 a_3^3 | 0 \rangle \ \psi^{ij} + \ldots + a_1^1 a_2^2 a_3^3 | 0 \rangle \ \psi_0,
$$

and the symmetric product of $SO(2n) = SO(4\ell + 2)$ spinor contains the $SU(n)$ singlet only in the $R_0$ (rank $n$ tensor). The mass operator $M_n$ is thus given by

$$
M_n = ma_1^1 a_2^2 \cdots a_n^a \text{ or } ma^n a^{n-1} \cdots a^2 a^1.
$$

Then the fermion mass term $\psi^\dagger \hat{M}_n \psi_{\pm}$ is calculated in quite the same way as above; for instance, we have for positive chiral spinor $\psi_+$:

$$
\psi^\dagger \hat{M}_n \psi_+ = (\psi_+^{\dagger} | C \ (ma^n \cdots a^2 a^1) | \psi_+ \rangle = (\psi_+^{\dagger} | \prod_{k=1}^n (a_k^a - a^k) | 0 \rangle \ \tilde{\psi}_0 \\
= \langle \psi_+^{\dagger} | a_1 a_2 \cdots a_n | 0 \rangle \ \tilde{\psi}_0 = m \tilde{\psi}_0 \tilde{\psi}_0,
$$

$$
\psi^\dagger \hat{M}_n \psi_+ = (\psi_+^{\dagger} | C \ (ma_1^1 a_2^2 \cdots a_n^a) | \psi_+ \rangle = (\psi_+^{\dagger} | \prod_{k=1}^n (a_k^a - a^k) | 0 \rangle \ \tilde{\psi}_0 \\
= m \langle \psi_+^{\dagger} | (-1)^n a^1 a^2 \cdots a^n a_1^1 a_2^2 \cdots a_n^a | 0 \rangle \ \tilde{\psi}_0 = (-1)^{n+1} m \langle \psi_+^{\dagger} | 0 \rangle \ \tilde{\psi}_0 = (-1)^{\ell+1} m \psi_+ \psi_+.
$$

In either case of $\pm$ chiral spinors, only the singlet component $\tilde{\psi}_0$ or $\psi_0$ obtains the mass while all the other $SU(n)$ non-singlet components remain massless, so that the pair condensation of the $SO(2n = 4\ell + 2)$ chiral spinor fermion $\psi_{\pm}$ cannot realize the global minimum of the potential for $\ell \geq 1$.

Also for the cases of $SO(2n)$ with even $n = 2\ell$, it is easy to see that the $SU(n)$-invariant mass operator $M_n$ (4.42) in the channel $R_0$ can give a mass only to the $SU(n)$ singlet components and leaves the other non-singlet components (existing for $\ell \geq 2$) massless. For this even $n = 2\ell$ cases, however, the symmetric tensor product of $SO(2n)$ chiral spinors also contains other channels $R_p$ ($p \geq 1$) that contains $SU(n)$-singlets. Since the only invariant tensors of $SU(n)$ is $\delta_1^i$ other than $\epsilon_{1i2\cdots in}^i$, the only possible $SU(n)$-invariant mass operators of lower rank than $a^1 a^2 \cdots a^n = \epsilon_{1i2\cdots in} a_i^a a_j^b a_k^c a_l^d / n!$ must be constructed with the total number operator $\hat{N} = \delta_i^i a_i^i a_j^j$ (rank 2). So the $SU(n)$-invariant mass operator contained in the channel $R_p$ (rank $n - 4p$ tensor) must be a polynomial $m(N^q + c_1 \hat{N}^{q-1} + \cdots + c_q)$ of degree $q = (n - 4p)/2.$
(See Appendix A for the explicit form of this SU(n)-singlet operators.) In any case, those mass operators have eigenvalues depending on the SU(n) fermion number so the SU(n) multiplets have non-degenerate masses so that the spinor pair condensation in those channels \( R_p \) (\( p \geq 1 \)) cannot realize the global minimum of the potential.

Although this general discussion is enough, let us demonstrate this general feature in an explicit example of \( n = 10 \) case in which two bispinor channels \( R_1 \) and \( R_2 \) containing SU(n)-singlets exist other than the usual \( R_0 \) of rank \( n \).

For \( n = 10 \), i.e., \( SO(20) \rightarrow SU(10) \times U(1) \), the \( SO(20) \) irreducible spinors are represented as follows in terms of \( SU(10) \) fermion operators giving the branching rule:

\[
|\psi_+\rangle = |0\rangle \psi_0 + \frac{1}{2} a^\dagger_j a^\dagger_k |0\rangle \psi_{ij} + (4 a^\dagger \text{term}) + (6 a^\dagger \text{term}) + (8 a^\dagger \text{term}) + a^\dagger_j a^\dagger_k \cdots a^\dagger_9 a^\dagger_{10} |0\rangle \tilde{\psi}_0
\]

\[
512' = (1)(-5) \oplus (45)(-3) \oplus (210)(-1) \oplus (210)(1) \oplus (45)(3) \oplus (1)(5),
\]

\[
|\psi_-\rangle = a^\dagger_j |0\rangle \psi^j + \frac{1}{3!} a^\dagger_j a^\dagger_k a^\dagger_l |0\rangle \psi^{ijk} + (5 a^\dagger \text{term}) + (7 a^\dagger \text{term}) + \frac{1}{9!} 2^{1+2+\cdots} n_j a^\dagger_1 a^\dagger_2 \cdots a^\dagger_9 |0\rangle \tilde{\psi}_j
\]

\[
512 = (10)(-4) \oplus (120)(-2) \oplus 252(0) \oplus (\overline{210})(2) \oplus (\overline{10})(4).
\]

The symmetric tensor product of these \( SO(20) \) spinors gives

\[
\begin{align*}
(512' \otimes 512)_{S} &= 92378'(R_0) \oplus 38760(R_1) \oplus 190(R_2), \\
(512 \otimes 512)_{S} &= 92378'(R_0) \oplus 38760(R_1) \oplus 190(R_2).
\end{align*}
\]  

\( 92378'(R_0) \) (rank-10 tensor) contains the usual \( SU(10) \) singlet which gives a mass operator

\[
m_0 \left( a^\dagger_j a^\dagger_k \cdots a^\dagger_9 + a^{10} a^9 \cdots a^1 \right)
\]

for the spinor \( \psi_+(512') \), while \( 92378'(R_0) \) does not. The other channels \( R_1 \) (rank-6 tensor) and \( R_2 \) (rank-2 tensor) both contain \( SU(10) \) singlets of the form \([\overline{10}]^5 \) and \([\overline{4}]^5 \), respectively, which give the following mass operators for the both spinors \( \psi_+(512') \) and \( \psi_-(512) \):

\[
m_1 \left( (\tilde{N} - 5)^3 - 7(\tilde{N} - 5) \right) + m_2 \left( \tilde{N} - 5 \right),
\]

with \( \tilde{N} = \sum_{j=1}^{5} a^\dagger_j a^j \). Thus the mass matrix \( M_+ \) of the positive chiral fermion \( \psi_+ \) is given by

\[
M_+ |\psi_+\rangle = \begin{pmatrix} m_0 \left( a^\dagger_1 a^\dagger_2 \cdots a^\dagger_9 + a^{10} a^9 \cdots a^1 \right) + m_1 \left( (\tilde{N} - 5)^3 - 7(\tilde{N} - 5) \right) + m_2 \left( \tilde{N} - 5 \right) \end{pmatrix} |\psi_+\rangle
\]

\[
= \begin{pmatrix} -90m_1-5m_2 & -6m_1-3m_2 & 6m_1-m_2 & -6m_1+m_2 & 6m_1+3m_2 & 90m_1+5m_2 \\
-6m_1-3m_2 & 6m_1-m_2 & -6m_1+m_2 & 6m_1+3m_2 & 90m_1+5m_2 & m_0 \end{pmatrix}
\]

\[
\begin{pmatrix} \psi_0(1) \\
\psi_{ij}(45) \\
\psi^{ijk}(210) \\
\psi^{i(6)}(210) \\
\psi^{i(8)}(45) \\
\tilde{\psi}_0(1) \end{pmatrix}
\]

\( \psi_0(1) \)

\( \psi_{ij}(45) \)

\( \psi_{ijk}(210) \)

\( \psi^{i(6)}(210) \)

\( \psi^{i(8)}(45) \)

\( \tilde{\psi}_0(1) \)

\( \psi_+ \)

\( \psi_- \)

\( m_+ \)

\( m_- \)

\( m_0 \)

\( m_1 \)

\( m_2 \)

\( \psi_0 \)

\( \psi_{ij} \)

\( \psi_{ijk} \)

\( \psi^{i(6)} \)

\( \psi^{i(8)} \)

\( \tilde{\psi}_0 \)

The mass matrix \( M_- \) of the negative chiral fermion \( \psi_- \) is given by

\[
M_- |\psi_-\rangle = \begin{pmatrix} m_1 \left( (\tilde{N} - 5)^3 - 7(\tilde{N} - 5) \right) + m_2 \left( \tilde{N} - 5 \right) \end{pmatrix} |\psi_-\rangle
\]

\[
= \begin{pmatrix} -36m_1-4m_2 & -6m_1-2m_2 & 6m_1-m_2 & -6m_1+2m_2 & 36m_1+4m_2 \\
-6m_1-2m_2 & 6m_1-m_2 & -6m_1+2m_2 & 36m_1+4m_2 & \end{pmatrix}
\]

\[
\begin{pmatrix} \psi^{i(10)} \\
\psi^{ijk}(120) \\
\psi^{i(5)}(252) \\
\psi^{i(7)}(120) \\
\tilde{\psi}_i(10) \end{pmatrix}
\]

\( \psi_0 \)

\( \psi_{ij} \)

\( \psi_{ijk} \)

\( \psi^{i(6)} \)

\( \psi^{i(8)} \)

\( \tilde{\psi}_0 \)

\( \psi_+ \)

\( \psi_- \)

\( m_+ \)

\( m_- \)

\( m_0 \)

\( m_1 \)

\( m_2 \)
These masses proportional to \(m_0\), \(m_1\) and \(m_2\) are generated by the fermion pair condensation in the bispinor channels \(R_0\), \(R_1\) and \(R_2\), respectively. Consider the case of pair condensation into a single channel. For the \(\psi_+\) case,

\[
\langle R_0 \rangle \neq 0 \rightarrow \text{Only singlets } \psi_0(1) \text{ and } \tilde{\psi}_0(1) \text{ get non-zero mass square } m_0^2
\]
and other components \(45, 210, 210, 45\) remain massless;

\[
\langle R_1 \rangle \neq 0 \rightarrow \text{Singlets } \psi_0(1) \text{ and } \tilde{\psi}_0(1) \text{ get a mass square } (90m_1)^2
\]
and other components \(45, 210, 210, 45\) get another common mass square \((6m_1)^2\);

\[
\langle R_2 \rangle \neq 0 \rightarrow \text{All the } \psi_0(1), \tilde{\psi}_0(1), 45, 45 \text{ and } 210, 210
\]
get different mass squares \((5m_2)^2, (3m_2)^2, (m_2)^2\), respectively. \(4.51\)

It is interesting that all the non-singlet components \(45, 45, 210, 210\) have a degenerate mass for the \(R_1\) condensation. The situation is also similar for the \(\psi_-\) case. In any case, however, the fermion mass spectrum does not realize the total degeneracy, so the pair condensation in any channels \(R_0, R_1\) and \(R_2\) cannot realize the global minimum of the potential.

5 Embedding-type subgroups

Here we discuss “embedding-type” little group cases. Since there are infinite “embedding-type” little groups, we restrict this type up to \(N = 26\) for practicability. Even for up to \(N = 26\), in view of Table 5, we classify this “embedding-type” subgroups into seven sub-types; i.e., “orthogonal adjoint-type (rank-2 anti-symmetric tensor-type),” “orthogonal rank-2 symmetric tensor-type,” “symplectic rank-2 anti-symmetric tensor-type,” “symplectic adjoint-type (rank-2 symmetric tensor-type),” “unitary adjoint-type,” “spinor-type,” and “exceptional-type” subgroups.

In the following, we summarize the branching rules of these sub-type subgroups. Several examples can be found in Ref. [52].

5.1 Orthogonal adjoint-type (rank-2 anti-symmetric tensor-type)

We consider \(SO(N = \frac{n(n-1)}{2}) \supset SO(n)\) \((n \in \mathbb{Z}_{\geq 5})\) for \(N \leq 26\), in which the rank-2 anti-symmetric tensor representation \(\frac{n(n-1)}{2}\) of the subgroup \(SO(n)\) is identified with the defining representation \(N\) of \(SO(N)\). So the branching rule for the \(SO(N)\) vector \(N\) under this subgroup \(SO(n)\) is

\[
N = \frac{n(n-1)}{2}. \quad (5.1)
\]

First, for the \(n = 5, N = 10\) case, the breaking \(SO(10) \rightarrow SO(5)\) does not occur via the pair condensation of the \(SO(10)\) spinor fermion. This is because the branching rule of the \(SO(10)\) (complex) spinor representation \(16\) or \(\overline{16}\) under \(SO(5)(\simeq USp(4)) \subset SO(10)\) is

\[
16 = (16), \quad \overline{16} = (16). \quad (5.2)
\]

So the fermion pair \((16 \otimes 16)_S\) of \(SO(10)\) is viewed as a tensor product \((16 \otimes 16)_S\) of \(SO(5)\), but it has no \(SO(5)\)-singlet since \(SO(5) 16\) is a pseudo-real representation.

Second, consider the \(n = 6, N = 15\) case. This time, the \(SO(15) \rightarrow SO(6)(\simeq SU(4))\) breaking can occur via the pair condensation of the \(SO(15)\) spinor fermion \(128\) (real representation, now). This is because the branching rule of \(SO(15) \supset SO(6)(\simeq SU(4))\) for the spinor representation is given by

\[
128 = 2(64). \quad (5.3)
\]

\(SO(15)\) spinor \(128\) is real and so a tensor product \((128 \otimes 128)_S\) has a singlet of \(SO(15)\). \(SO(6)\) \(64\) is also a real representation, so a tensor product \([64 \oplus 64] \otimes (64 \oplus 64)]_S\) of \(SO(6)\) has three
independent singlets of $SO(6) \subset SO(15)$. One of them is an $SO(15)$ singlet. From Table 6, the other $SO(6)$ singlets are in $R_0 = 64_{35}$ and $R_1 = 455$ representations, which correspond to rank-7 and rank-3 anti-symmetric tensor representations, respectively. The fermion mass matrix from the rank-3 tensor representations $M_3$ is traceless as noted before, so we have

$$M_3 \propto \begin{pmatrix} (|64\rangle)_1 & +1(|64\rangle)_1 \\ (|64\rangle)_2 & -1(|64\rangle)_2 \end{pmatrix},$$

which yields the quadratic mass matrix $M_3^T M_3$ proportional to identity matrix. This can also be confirmed in another way as follows. Since $M_3$ is given by a linear combination of rank-3 gamma matrices $\Gamma_{a_1a_2a_3}$, the quadratic mass matrix $(M_3^T M_3)$ takes the form

$$M_3^T M_3 = (\text{rank-6 part}) + (\text{rank-4 part}) + (\text{rank-2 part}) + 1(\# \text{ of terms}) \quad (5.5)$$

but the rank 6, 4, 2 parts must vanish since there are no such $H$-singlets now, thus leaving only the part proportional to the identity matrix. Also, the mass matrix from the rank-7 tensor representation $M_7$ is traceless, so the quadratic mass $(M_7^T M_7)$ is proportional to the identity matrix. Therefore, the VEVs of the auxiliary field $\Phi_{R_0} = 64_{35}$ and $\Phi_{R_1} = 455$ lead to the identity quadratic mass matrix $(M_j^T M_j) \propto 1 (j = 3, 7)$.

Third, consider the $n = 7, N = 21$ case. There is an $SO(21) \rightarrow SO(7)$ breaking via the pair condensation of the $SO(21)$ spinor fermion. This is because the branching rule of $SO(21) \supset SO(7)$ for the spinor representation is given by

$$1024 = 2(512). \quad (5.6)$$

$SO(21)$ spinor 1024 is pseudo-real, so a tensor product $(128 \otimes 128)_S$ has no singlet of $SO(21)$. $SO(7)$ 512 is a real representation, so a tensor product $[(512 \oplus 512) \otimes (512 \oplus 512)]_S$ of $SO(6)$ has three independent singlets of $SO(7) \subset SO(21)$. From Table 6, the three singlets of $SO(7)$ exist in $R_0 = 3527_{16}, R_1' = 1162_{80}$, and $R_2' = 1330$ representations. The VEVs of the scalar fields $\Phi_{R_0}, \Phi_{R_1'},$ and $\Phi_{R_2'}$ lead to the quadratic mass matrix proportional to an identity, $(M_j^T M_j) \propto 1 (j = 10, 7, 3)$, because of the traceless condition of $M_j$, again.

### 5.2 Orthogonal rank-2 symmetric tensor-type

We consider $SO(N) = \{(n-1)(n+2)\}$ for $N \leq 26$, where the rank-2 symmetric tensor representation $\frac{n(n+1)}{2}$ of the subgroup $SO(N)$ is identified with the defining representation $N$ of $SO(N)$. Then the branching rule for the $SO(N)$ vector $N$ under this subgroup $SO(n)$ is

$$N = \frac{(n-1)(n+2)}{2}. \quad (5.7)$$

First, for the $n = 5, N = 14$ case, there is no $SO(14) \rightarrow SO(5)(\simeq USp(4))$ breaking via the pair condensation of the $SO(14)$ spinor fermion. This is because the branching rule of the $SO(14)$ (complex) spinor representation 64 or $\overline{64}$ under $SO(5)(\simeq USp(4)) \subset SO(14)$ is

$$64 = (64), \quad \overline{64} = (64). \quad (5.8)$$

But the $SO(5)$ 64 is a pseudo-real representation, so a tensor product $(64 \otimes 64)_S$ of $SO(5)$ has no singlet of $SO(5) \subset SO(14)$.

Second, for the $n = 6, N = 20$ case, the breaking $SO(20) \rightarrow SO(6)(\simeq SU(4))$ can occur via the pair condensation of the $SO(20)$ spinor fermion. This is because the branching rule of $SO(20)$ (pseudo-real) spinor 512 or $512'$ under $SO(6) \subset SO(20)$ is given by

$$512 = (256) \oplus (\overline{256}), \quad 512' = (256) \oplus (\overline{256}). \quad (5.9)$$
But $SO(6)$ 256 and 256 are complex representations, and the tensor product $[(256 \oplus 256) \otimes (256 \oplus 256)]_S$ of $SO(6)$ has one independent $SO(6)$ singlet, which is unique invariant 256. From Table 1, the singlet of $SO(6)$ exists in $R_1 = 38760$ (rank-6 tensor) representation. The VEV of the auxiliary field $\Phi_{R_1}$ must lead to an identity-proportional quadratic mass matrix $(M_0^a M_0^a) \propto 1$ since it gives the unique $SO(6)$ invariant mass 256 $\cdot$ 256.

5.3 Symplectic rank-2 anti-symmetric tensor-type

We consider $SO(N = (n-1)(2n+1)) \supset USp(2n)$ $(n \in \mathbb{Z}_{\geq 3})$ for $N \leq 26$, in which the rank-2 anti-symmetric $\Omega$-traceless tensor representation of the subgroup $USp(2n)$, $(n-1)(2n+1) = \frac{2n(2n-1)}{2} - 1$, is identified with the defining representation $N$ of $SO(N)$. So the branching rule for the $SO(N)$ vector $N$ under this subgroup $USp(n)$ is

$$N = (n-1)(2n+1). \quad (5.10)$$

For the $n = 3, N = 14$ case, there is no $SO(14) \to USp(6)$ breaking via the pair condensation of the $SO(14)$ spinor fermion. This is because the branching rule of $SO(14)$ (complex) spinor representation 64 or 64 under $USp(6) \subset SO(14)$ is given by

$$64 = (64), \quad \overline{64} = (64). \quad (5.11)$$

But $USp(6)$ 64 is a pseudo-real representation, so a tensor product $(64 \otimes 64)_S$ of $USp(6)$ has no $USp(6)$ singlet.

5.4 Symplectic adjoint-type (rank-2 symmetric tensor-type)

We consider $SO(N = n(2n+1)) \supset USp(2n)$ for $N \leq 26$, in which the adjoint (rank-2 symmetric tensor) representation of the subgroup $USp(2n)$, $n(2n+1)$, is identified with the defining representation $N$ of $SO(N)$. So the branching rule for the $SO(N)$ vector $N$ under this subgroup $USp(n)$ is

$$N = n(2n+1). \quad (5.12)$$

First, for the $n = 2, N = 10$ case, there is no $SO(10) \to USp(4)$ breaking via the pair condensation of the $SO(10)$ spinor fermion. This is because the branching rule of $SO(10)$ (complex) spinor 16 or 16 under $USp(4) \subset SO(10)$ is given by

$$16 = (16), \quad \overline{16} = (16). \quad (5.13)$$

But $USp(4)$ 16 is a pseudo-real representation, so a tensor product $(16 \otimes 16)_S$ of $USp(4)$ has no $USp(4)$ singlet.

Second, for the $n = 3, N = 21$ case, there is an $SO(21) \to USp(6)$ breaking via the pair condensation of the $SO(21)$ spinor fermion. This is because the branching rule of $SO(21) \supset USp(6)$ for the spinor representation is given by

$$1024 = 2(512). \quad (5.14)$$

$SO(21)$ 1024 is a pseudo-real representation, so a tensor product $(1024 \otimes 1024)_S$ of $SO(21)$ has no singlet of $SO(21)$. $USp(6)$ 512 is a real representation, so a tensor product $[(512 \oplus 512) \otimes (512 \oplus 512)]_S$ of $USp(6)$ has three independent singlets of $USp(6) \subset SO(21)$. From Table 6 the three singles of $USp(6)$ exist in $R_0 = 352716$, $R_1^0 = 116280$, and $R_2^0 = 1330$ representations. The VEVs of the auxiliary fields $\Phi_{R_0}$, $\Phi_{R_1^0}$, and $\Phi_{R_2^0}$ lead to the identity-proportional quadratic mass matrix $(M_0^a M_0^a) \propto 1$ ($j = 10, 7, 3$) because of the tracelessness condition for each $M_j$. 

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5.5 Spinor-type

We consider $SO(N = 2^{\left\lfloor \frac{n-1}{2} \right\rfloor}) \supset SO(n)$, $(n = 8\ell, 8\ell \pm 1, \ell \in \mathbb{Z}_{\geq 1}; n \geq 9)$, where the real spinor representation $2^{\left\lfloor \frac{n-1}{2} \right\rfloor}$ of the subgroup $SO(n)$, existing for $n = 8\ell, 8\ell \pm 1$, is identified with the defining representation $N$ of $SO(N)$. So the branching rule for the $SO(N)$ vector $N$ under this subgroup $SO(n)$ is

$$N = 2^{\left\lfloor \frac{n-1}{2} \right\rfloor}. \quad (5.15)$$

For $SO(16) \supset SO(9)$, there is an $SO(16) \to SO(9)$ breaking via the pair condensation of the $SO(16)$ spinor fermion. This is because the branching rule of $SO(16)$ real spinor, $128$ or $128'$, under $SO(9) \subset SO(16)$ is given by

$$128 = (44) \oplus (84), \quad 128' = (128). \quad (5.16)$$

Since $SO(16)$ $128$ is real, a tensor product $(128 \otimes 128)_S$ of $SO(16)$ has one $SO(16)$ singlet, and contains two $SO(9)$ singlets which come from $44 \otimes 44$ and $84 \otimes 84$. The $SO(9)$ singlets are in the rank-0 and rank-8 tensor representations. The rank-0 tensor representation, proportional to identity, is an $SO(16)$ singlet. The quadratic mass matrix from the rank-8 tensor representation $(M_8^1 M_8)$ is not proportional to identity and gives two different eigenvalues for $44$ and $84$ because of their dimension difference and the traceless condition for $M_8$.

$SO(16)$ $128'$ is a real representation, so a tensor product $(128' \otimes 128')_S$ of $SO(16)$ has a singlet of $SO(16)$. $SO(9)$ $128$ ($\gamma$-traceless vector-spinor), on the other hand, is also a real representation, so the fermion pair $(128' \otimes 128')_S$ of $SO(16)$ spinor contains an $SO(9)$-singlet, but it is also a singlet of $SO(16)$, so does not cause the breaking $SO(16) \to SO(9)$.

5.6 Unitary adjoint-type

We consider $SO\left(N = n^2 - 1 \right) \supset SU(n)$ for $N \leq 26$, where the adjoint representation of the $SU(n)$ subgroup is identified with the defining representation $N$ of $SO(N)$. So the branching rule for the $SO(N)$ vector $N$ under this subgroup $SU(n)$ is

$$N = n^2 - 1. \quad (5.17)$$

First, for the $n = 3, N = 8$ case, there is no $SO(8) \to SU(3)$ breaking via the pair condensation of the $SO(8)$ spinor fermion. This is because the branching rule of $SO(8)$ real spinor, $8_s$ or $8_c$, under $SU(3) \subset SO(8)$ is given by

$$8_s = (8), \quad 8_c = (8). \quad (5.18)$$

Since $SO(8)$ spinors $8_s$ and $8_c$ are real, a tensor product $(8_s \otimes 8_s)_S$ or $(8_c \otimes 8_c)_S$ is a singlet of $SO(8)$. $SU(3)$ adjoint 8 is also a real representation, so the tensor product $(8 \otimes 8)_S$ has an $SU(3)$ singlet, but it is also the $SO(8)$ singlet, so does not cause the breaking $SO(8) \to SU(3)$.

Second, consider the $n = 4, N = 15$ case. There is an $SO(15) \to SU(4)$ breaking via the pair condensation of the $SO(15)$ spinor fermion. This is because the branching rule of $SO(15)$ real spinor representation $128$ under $SU(4) \subset SO(15)$ is given by

$$128 = 2(64). \quad (5.19)$$

$SO(15)$ spinor $128$ is real, so a tensor product $(128 \otimes 128)_S$ has an $SO(15)$ singlet. $SO(6)$ $64$ is also a real representation, so a tensor product $[(64 \oplus 64) \otimes (64 \oplus 64)]_S$ contains three independent $SU(4)$ singlets. One of them is the $SO(15)$ singlet. From Table 6, the remaining singlets of $SU(4) \simeq SO(6)$ exist in $R_0 = 6435$ and $R_1 = 455$ representations. The VEVs of the scalar fields $\Phi_R$ and $\Phi^{\dagger}_R$ lead to the quadratic mass matrix $(M_j^1 M_j) \propto 1$ $(j = 7, 3)$ proportional to identity because of the traceless condition for $M_j$. 

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Third, for the \( n = 5, N = 24 \) case also, we quite similarly see that there is an \( SO(24) \to SU(5) \) breaking via the pair condensation of the \( SU(24) \) spinor fermion. This is because the branching rule of \( SO(24) \) real spinor representation, \( 2048 \) or \( 2048' \), under \( SU(5) \subset SO(24) \) is given by

\[
2048 = 2(1024), \quad 2048' = 2(1024).
\]  

\( SO(24) \) spinor \( 2048 \) is real, so a tensor product \( (2048 \otimes 2048)_S \) has an \( SO(24) \) singlet. \( SU(5) \) \( 1024 \) is also a real representation, so a tensor product \( [(1024 \oplus 1024) \otimes (1024 \oplus 1024)]_S \) has three independent \( SU(5) \) singlets, one of which is the \( SO(24) \) singlet. From Table 6, the remaining two \( SU(5) \) singlets exist in \( R_0 = 1352078(14) \) and \( R_1 = 735471(5.20) \) representations. The VEVs of the scalar fields \( \Phi_{R_0} \), and \( \Phi_{R_1} \), lead to the quadratic mass matrix \( (M_j^R)^\dagger M_j^R \propto 1 \) (\( j = 12, 7 \)) proportional to identity because of the traceless property of \( M_j \). Thus, their quadratic mass matrix is proportional to identity and can realize the global minimum.

### 5.7 Exceptional-type

We consider exceptional-type maximal subgroups up to \( N = 26 \). Only three cases exist: \( SO(7) \subset G_2 \), \( SO(14) \subset G_2 \), and \( SO(26) \subset F_4 \).

First, consider \( SO(7) \subset G_2 \) case. There is an \( SO(7) \to G_2 \) breaking via the pair condensation of the \( SO(7) \) spinor fermion. This is because the branching rule of \( SO(7) \) real spinor representation \( 8 \) under \( G_2 \subset SO(7) \) is given by

\[
8 = (7) \oplus (1).
\]  

\( G_2 \) singlet exists in \( 35 = R_0 \) (rank-3 tensor) in the fermion pair \( (8 \otimes 8)_S \) of \( SO(7) \) spinor.

We can apply the traceless condition to the mass matrix from the rank-3 tensor representation \( M_3 \) also here and obtain

\[
M_3 \left( \begin{array}{c} \langle 7 \rangle \\ \langle 1 \rangle \end{array} \right) \propto \left( \begin{array}{c} +1 \langle 7 \rangle \\ -7 \langle 1 \rangle \end{array} \right).
\]  

Thus, the quadratic mass matrix \( (M_j^R)^\dagger M_j^R \) is not proportional to identity, so for this case, the pair condensation of the \( SO(7) \) spinor cannot realize the global minimum of the potential.

Second, consider the case \( SO(14) \subset G_2 \). There is an \( SO(14) \to G_2 \) breaking via the pair condensation of the \( SO(14) \) spinor fermion. This is because the branching rule of \( SO(14) \) complex spinor representation, \( 64 \) or \( \bar{64} \), under \( G_2 \subset SO(14) \) is given by

\[
64 = (64), \quad \bar{64} = (64).
\]  

\( SO(14) \) spinor \( 64 \) (or \( \bar{64} \)) is complex, so a tensor product \( (64 \otimes 64)_S \) has no \( SO(14) \) singlet, while \( G_2 \) \( 64 \) is a real representation, so a tensor product \( (64 \otimes 64)_S \) of \( G_2 \) has one \( G_2 \) singlet. From Table 6, this \( G_2 \) singlet exists in \( R_1 = 364 \) (rank-3 tensor) representation. The VEVs of the scalar field \( \Phi_{R_1} \) must give the quadratic mass matrix \( (M_j^R)^\dagger M_j^R \propto 1 \) proportional to identity since \( 64 \) is also irreducible under \( G_2 \). (Recall that the traceless condition does not apply to the chiral projected mass matrix \( M_j \) for the \( SO(N = 8 \ell \pm 2) \) cases with complex spinor.)

Third, for \( SO(26) \subset F_4 \) case, the situation is almost the same as for the previous \( SO(14) \subset G_2 \) case. There is an \( SO(26) \to F_4 \) breaking via the pair condensation of the \( SO(26) \) spinor fermion. This is because the branching rule of \( SO(26) \subset F_4 \) for the spinor representation is given by

\[
4096 = (4096), \quad \bar{4096} = (4096).
\]  

\( SO(26) \) spinor \( 4096 \) (or \( \bar{4096} \)) is complex, so a tensor product \( (4096 \otimes 4096)_S \) of \( SO(26) \) contains no \( SO(26) \) singlet, while \( F_4 \) \( 4096 \) is a real representation, so a tensor product \( (4096 \otimes 4096)_S \) of \( F_4 \) has one \( F_4 \) singlet. From Table 6, the \( F_4 \) singlet exists in \( R_1 = 3124550 \) (rank-9 tensor) representation. The VEV of the scalar field \( \Phi_{R_1} \) must give the quadratic mass matrix \( (M_j^R)^\dagger M_j^R \propto 1 \) proportional to identity since the \( SO(26) \) spinor \( 4096 \) is also irreducible under \( F_4 \).
6 Summary and discussions

We summarize the symmetry breaking patterns in the $SO(N)$ NJL models whose fermions belong to an irreducible spinor representation of $SO(N)$. The summary Table 7 shows which symmetry vacuum realizes (or vacua realize) the lowest potential energy when which 4-Fermi coupling constant $G_{R_k}$ beyond critical is the strongest. For all of them, the spinor fermions get a totally degenerate mass so realizing the possible lowest potential energy. All the vacua are written there if they realize such a lowest potential energy in the $1/N$ leading order. The perturbation will determine the true vacuum among those degenerate vacua in the leading order. From Table 7, we find that when $SO(N)$ spinor representations are complex, the $SO(N)$ symmetry is always broken into special subgroups. When $SO(N)$ spinor representations are pseudo-real, the $SO(N)$ symmetry is broken into regular subgroups in most cases and into special subgroups in some cases. When $SO(N)$ spinor representations are real, the $SO(N)$ symmetry is unbroken or broken into regular subgroups in most cases and into special subgroups in some cases. That is, our analysis have shown that the symmetry breaking into special or regular subgroups is strongly correlated with the type of representations; i.e., complex, or real, or pseudo-real representations.

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Table 6: Maximal little groups $H$ or itself of $SO(N)$ spinor tensor product representations in the maximal subgroups of $SO(N)$ ($N \leq 26$)

| $N$ | R/PR/C | Maximal little groups $H$ of $SO(N)$ | A/P/E | $H$-singlet in |
|-----|--------|------------------------------------|-------|----------------|
| 3   | PR     | $SO(2)(R)$                         | A     | $R_0$          |
| 4   | PR     | $SU(2) \times U(1)(R)$             | A     | $R_0$          |
| 5   | PR     | $SO(3) \times SO(2)(R)$           | A     | $R_0$          |
| 6   | C      | $SU(3)(R)$                         | A     | $R_0$          |
|     |        | $SO(3) \times SO(3)(S)$           | P     | $R_0$          |
| 7   | R      | $SO(4) \times SO(3)(R)$           | A     | $R_0$          |
|     |        | $SO(7)(No)$                        | A     | $R_0$          |
|     |        | $G_2(S)$                           | E     | $R_0$          |
| 8   | R      | $SU(4) \times U(1)(R)$            | P     | $R_0^{(*)}$    |
|     |        | $SO(4) \times SO(4)(R)$           | A     | $R_0$          |
|     |        | $SO(8)(No)$                        | A     | $R_1$          |
| 9   | R      | $SO(5) \times SO(4)(R)$           | A     | $R_0$          |
|     |        | $SO(8)(R)$                         | A     | $R_1$          |
|     |        | $SO(9)(No)$                        | A     | $R_1$          |
|     |        | $SO(3) \times SO(3)(S)$           | P     | $R_0$          |
|     |        | $SU(2)(S)$                         | E     | $R_0$          |
| 10  | C      | $SU(5)(R)$                         | P     | $R_0$          |
|     |        | $SO(5) \times SO(5)(S)$           | A     | $R_0$          |
|     |        | $SO(9)(S)$                         | A     | $R_1$          |
| 11  | PR     | $SO(6) \times SO(5)(R)$           | A     | $R_0$          |
|     |        | $SO(9) \times SO(2)(R)$           | A     | $R_1^{(*)}$    |
|     |        | $SO(10)(R)$                        | A     | $R_1$          |
| 12  | PR     | $SU(6) \times U(1)(R)$            | P     | $R_0^{(*)}$, $R_1$ |
|     |        | $SO(6) \times SO(6)(R)$           | A     | $R_0$          |
|     |        | $SO(10) \times SO(2)(R)$          | A     | $R_1$          |
| 13  | PR     | $SO(7) \times SO(6)(R)$           | A     | $R_0$          |
|     |        | $SO(10) \times SO(3)(R)$          | A     | $R_1^{(*)}$    |
|     |        | $SO(11) \times SO(2)(R)$          | A     | $R_1$          |
| N  | R/PR/C | Maximal little groups $H$ of $SO(N)$ | A/P/E | $H$-singlet in |
|----|--------|------------------------------------|-------|---------------|
| 14 | C      | $SU(7)(R)$                         | P     | $R_0$         |
|    |        | $SO(7) \times SO(7)(S)$            | A     | $R_0$         |
|    |        | $SO(11) \times SO(3)(S)$           | A     | $R_1$         |
|    |        | $G_2(S)$                           | E     | $R_1$         |
| 15 | R      | $SO(8) \times SO(7)(R)$            | A     | $R_0$         |
|    |        | $SO(11) \times SO(4)(R)$           | A     | $R_1$         |
|    |        | $SO(12) \times SO(3)(R)$           | A     | $R_1$         |
|    |        | $SO(15)(No)$                       | A     | $R_2$         |
|    |        | $SO(5) \times SO(3)(S)$            | P     | $R_0, R_1$    |
|    |        | $SU(6)(S)$                         | E     | $R_0, R_1$    |
|    |        | $SU(2)(S)$                         | E     | $R_0, R_1, R_1$ |
| 16 | R      | $SU(8) \times U(1)(R)$             | P     | $R_0(*)$, $R_1$ |
|    |        | $SO(8) \times SO(8)(R)$            | A     | $R_0$         |
|    |        | $SO(12) \times SO(4)(R)$           | A     | $R_1$         |
|    |        | $SO(16)(No)$                       | A     | $R_2$         |
|    |        | $SU(2) \times USp(8)(S)$           | P     | $R_0(*)$, $R_1$ |
|    |        | $USp(4) \times USp(4)(S)$          | P     | $R_0(*)$, $R_1$ |
|    |        | $SO(9)(S)$                         | E     | $R_0(*)$      |
| 17 | R      | $SO(9) \times SO(8)(R)$            | A     | $R_0$         |
|    |        | $SO(12) \times SO(5)(R)$           | A     | $R_1$         |
|    |        | $SO(13) \times SO(4)(R)$           | A     | $R_1$         |
|    |        | $SO(16)(R)$                        | A     | $R_2$         |
|    |        | $SO(17)(No)$                       | A     | $R_2$         |
|    |        | $SU(2)(S)$                         | E     | $R_0, R_1, R_1$ |
| 18 | C      | $SU(9)(R)$                         | P     | $R_0$         |
|    |        | $SO(9) \times SO(9)(S)$            | A     | $R_0$         |
|    |        | $SO(13) \times SO(5)(S)$           | A     | $R_1$         |
|    |        | $SO(17)(S)$                        | A     | $R_2$         |
| 19 | PR     | $SO(10) \times SO(9)(R)$           | A     | $R_0$         |
|    |        | $SO(13) \times SO(6)(R)$           | A     | $R_1$         |
|    |        | $SO(14) \times SO(5)(R)$           | A     | $R_1$         |
|    |        | $SO(17) \times SO(2)(R)$           | A     | $R_2$         |
|    |        | $SO(18)(R)$                        | A     | $R_2$         |
|    |        | $SU(2)(S)$                         | E     | $R_0$         |
| 20 | PR     | $SU(10) \times U(1)(R)$            | P     | $R_0(*)$, $R_1, R_2$ |
|    |        | $SO(10) \times SO(10)(R)$          | A     | $R_0$         |
|    |        | $SO(14) \times SO(6)(R)$           | A     | $R_1$         |
|    |        | $SO(18) \times SO(2)(R)$           | A     | $R_2$         |
|    |        | $SO(6)(S)$                         | E     | $R_1$         |
| 21 | PR     | $SO(11) \times SO(10)(R)$          | A     | $R_0$         |
|    |        | $SO(14) \times SO(7)(R)$           | A     | $R_1$         |
|    |        | $SO(15) \times SO(6)(R)$           | A     | $R_1$         |
|    |        | $SO(18) \times SO(3)(R)$           | A     | $R_2$         |
|    |        | $SO(19) \times SO(2)(R)$           | A     | $R_2$         |
|    |        | $SO(7)(S)$                         | E     | $R_0, R_1, R_2$ |
|    |        | $USp(6)(S)$                        | E     | $R_0, R_1, R_2$ |
|    |        | $SU(2)(S)$                         | E     | $R_0, R_1, R_1$ |
| 22 | C      | $SU(11)(R)$                        | P     | $R_0$         |
|    |        | $SO(11) \times SO(11)(S)$          | A     | $R_0$         |
|    |        | $SO(15) \times SO(7)(S)$           | A     | $R_1$         |
|    |        | $SO(19) \times SO(3)(S)$           | A     | $R_2$         |
| 23 | R      | $SO(12) \times SO(11)(R)$          | A     | $R_0$         |
|    |        | $SO(15) \times SO(8)(R)$           | A     | $R_1$         |
|    |        | $SO(16) \times SO(7)(R)$           | A     | $R_1$         |
|    |        | $SO(19) \times SO(4)(R)$           | A     | $R_2$         |
|    |        | $SO(20) \times SO(3)(R)$           | A     | $R_2$         |
|    |        | $SO(23)(No)$                       | A     | $R_3$         |
Table 6 (continued)

| N | R/PR/C | Maximal little groups \( H \) of \( SO(N) \) | A/P/E | \( H \)-singlet in |
|---|---|---|---|---|
| 24 | R | \( SU(2)(S) \) | E | \( R_0, R_1, R_2, R_2' \) |
|  | | \( SU(12) \times U(1)(R) \) | P | \( R_0(*) \) |
|  | | \( SO(12) \times SO(12)(R) \) | A | \( R_0 \) |
|  | | \( SO(16) \times SO(8)(R) \) | A | \( R_1 \) |
|  | | \( SO(20) \times SO(4)(R) \) | A | \( R_2 \) |
|  | | \( SO(24)(No) \) | A | \( R_3 \) |
|  | | \( USp(6) \times USp(4)(S) \) | P | \( R_0(*) \) |
|  | | \( SO(3) \times SO(8)(S) \) | P | \( R_0, R_1, R_2 \) |
|  | | \( SU(2) \times USp(12)(S) \) | P | \( R_0, R_1, R_2 \) |
|  | | \( SU(5)(S) \) | E | \( R_0, R_1 \) |
| 25 | R | \( SO(13) \times SO(12)(R) \) | A | \( R_0 \) |
|  | | \( SO(16) \times SO(9)(R) \) | A | \( R_1' \) |
|  | | \( SO(17) \times SO(8)(R) \) | A | \( R_2' \) |
|  | | \( SO(20) \times SO(5)(R) \) | A | \( R_3' \) |
|  | | \( SO(24)(R) \) | A | \( R_3 \) |
|  | | \( SO(25)(No) \) | A | \( R_3 \) |
|  | | \( SO(5) \times SO(5)(S) \) | P | \( R_0, R_1, R_1, R_2 \) |
|  | | \( SU(2)(S) \) | E | \( R_0, R_1, R_1, R_2, R_2 \) |
| 26 | C | \( SU(13)(R) \) | P | \( R_0 \) |
|  | | \( SO(13) \times SO(13)(S) \) | A | \( R_0 \) |
|  | | \( SO(17) \times SO(9)(S) \) | A | \( R_1 \) |
|  | | \( SO(21) \times SO(5)(S) \) | A | \( R_2 \) |
|  | | \( SO(25)(S) \) | A | \( R_3 \) |
|  | | \( F_4(S) \) | E | \( R_1 \) |

Note that R, PR and C in the R/PR/C column indicate that the \( SO(N) \) spinor representation is real, pseudo-real and complex, respectively. For (*) attached subgroup \( R_0 \) for \( N = 4k \) cases, which has two real (or pseudo-real) irreducible spinor \( r \) and \( r' \), only one \( R_0 \) either in \( r \times r \) or in \( r' \times r' \) contains the \( H \)-singlet. A, P and E in the A/P/E column indicate addition-type, product-type, and embedding-type subgroup, respectively. \((R), (S) \) and (No) indicate that the subgroup \( H \) is “regular”, “special” and “No breaking” (i.e., \( G = H \)), respectively. This list omitted the possible maximal little groups which are not maximal subgroups, since those vacua in any case cannot realize the global minimum of the potential.
Table 7: Symmetry breaking pattern in 4D $SO(N)$ NJL model

| $N$ | $R_0$          | $R_1^{(i)}$ | $R_2^{(i)}$ | $R_3^{(i)}$ |
|-----|---------------|-------------|-------------|-------------|
| 3   | $SO(2)(R)$    |             |             |             |
| 4   | $SU(2) \times U(1)(R)$ |             |             |             |
| 5   | $SO(3) \times SO(2)(R)$ |             |             |             |
| 6   | $SO(3) \times SO(3)(S)$ |             |             |             |
| 7   | $SO(4) \times SO(3)(R)$ | $R'_1 : SO(7)(\text{No})$ |             |             |
| 8   | $SO(4) \times SO(4)(R)$ |             | $R'_1 : SO(8)(\text{No})$ |             |
| 9   | $SO(5) \times SO(4)(R)$ |             | $R'_1 : SO(8)(\text{No})$ |             |
|     | $SO(3) \times SO(3)(S)$ |             | $R_1 : SO(9)(\text{No})$ |             |
| 10  | $SO(5) \times SO(5)(S)$ |             | $R'_1 : SO(9)(\text{No})$ |             |
| 11  | $SO(6) \times SO(5)(R)$ |             | $R'_1 : SO(10)(R)$ |             |
|     | $SO(6)(S)$ | $R_1 : SO(10) \times SO(2)(R)$ |             |             |
| 12  | $SO(6) \times SO(6)(R)$ | $SO(10) \times SO(2)(R)$ |             |             |
| 13  | $SO(7) \times SO(6)(R)$ | $R'_1 : SO(10) \times SO(3)(R)$ |             |             |
|     | $SO(7)(S)$ | $R_1 : SO(11) \times SO(2)(R)$ |             |             |
| 14  | $SO(7) \times SO(7)(S)$ | $G_2(S)$ |             |             |
|     | $SO(8)(S)$ | $R'_1 : SO(11) \times SO(4)(\text{No})$ | $R'_2 : SO(15)(\text{No})$ |             |
| 15  | $SO(8) \times SO(7)(R)$ |             |             |             |
|     | $SO(6)(S)$ | $R_1 : SO(12) \times SO(3)(R)$ |             |             |
| 16  | $SO(8) \times SO(8)(R)$ | $SO(12) \times SO(4)(\text{No})$ | $SO(16)(\text{No})$ |             |
|     | $USp(4) \times USp(4)(S)$ |             |             |             |
| 17  | $SO(9) \times SO(8)(R)$ | $R'_1 : SO(12) \times SO(5)(R)$ | $R'_2 : SO(16)(R)$ |             |
|     | $SO(9)(S)$ | $R_1 : SO(13) \times SO(4)(\text{No})$ | $R_2 : SO(17)(\text{No})$ |             |
| 18  | $SO(9) \times SO(9)(S)$ | $SO(13) \times SO(5)(S)$ | $SO(17)(S)$ |             |
| 19  | $SO(10) \times SO(9)(R)$ | $R'_1 : SO(13) \times SO(6)(R)$ | $R'_2 : SO(17) \times SO(2)(R)$ |             |
|     | $SO(10)(S)$ | $R_1 : SO(14) \times SO(5)(R)$ | $R_2 : SO(18)(R)$ |             |
| 20  | $SO(10) \times SO(10)(R)$ | $SO(14) \times SO(6)(R)$ | $SO(18) \times SO(2)(R)$ |             |
|     | $SO(6)(S)$ |             |             |             |
| 21  | $SO(11) \times SO(10)(R)$ | $R'_1 : SO(14) \times SO(7)(R)$ | $R'_2 : SO(18) \times SO(3)(R)$ | $R'_3 : SO(21)(\text{No})$ |
|     | $SO(11)(S)$ | $SO(7)(S)$ | $SO(7)(S)$ |             |
|     | $USp(6)(S)$ | $USp(6)(S)$ | $USp(6)(S)$ |             |
| 22  | $SO(11) \times SO(11)(S)$ | $SO(15) \times SO(7)(S)$ | $SO(19) \times SO(3)(S)$ |             |
|     | $SO(12)(S)$ | $SO(16) \times SO(8)(R)$ | $SO(19) \times SO(4)(\text{No})$ | $R'_3 : SO(23)(\text{No})$ |
|     | $SO(12)(S)$ | $SO(16) \times SO(9)(R)$ | $SO(20) \times SO(3)(R)$ |             |
| 23  | $SO(12) \times SO(12)(R)$ | $SO(16) \times SO(8)(R)$ | $SO(20) \times SO(4)(\text{No})$ | $SO(24)(\text{No})$ |
|     | $SU(5)(S)$ | $SO(17) \times SO(8)(R)$ |             |             |
|     | $SU(5)(S)$ | $SO(17) \times SO(9)(R)$ |             |             |
| 24  | $SO(13) \times SO(12)(R)$ | $SO(16) \times SO(5)(S)$ | $SO(21) \times SO(4)(R)$ | $SO(24)(\text{No})$ |
|     | $SO(13)(S)$ | $SO(17) \times SO(9)(S)$ | $SO(21) \times SO(5)(S)$ | $SO(25)(\text{No})$ |

In the row of indicated value $N$, the possible subgroups $H \subset G$ to which the symmetry $G = SO(N)$ is broken are shown in the column $R_k^{(i)}$ for the cases where the coupling constant $G_{R_k}^{(i)}$ is beyond critical and the strongest. $(R), (S)$ and (No) indicate that the subgroup $H$ is “regular”, “special” and “No breaking” (i.e., $G = H$), respectively. $R_k^{(i)}$ denotes both channels $R_k$ and $R_k'$, the latter channel $R_k'$ with prime exists only for odd $N$. In the odd $N$ row, therefore, $R_k$: and $R_k'$: are put in front of the $H$ names to denote which channel of the two is the strongest.
SU\((n)\)-singlet operator in rank-2\(k\) tensor \(\Gamma_{a_1a_2\cdots a_{2k}}\) of \(SO(2n)\)

SU\((n)\)-singlet operator in rank-2\(k\) tensor \(\Gamma_{a_1a_2\cdots a_{2k}}\) can be found as follows. First, we see from \(\Gamma_{2j-1}\Gamma_{2j} = i(2a_j^\dagger a^j - 1)\) that the rank-2 \(SU(n)\)-singlet operator is clearly given by

\[
\Gamma^{(2)} := \sum_{j=1}^{n} \Gamma^{(2)}_j = (2i)(\hat{N} - \frac{n}{2}), \quad \left( \Gamma^{(2)}_j := \Gamma_{2j-1}\Gamma_{2j}, \quad \hat{N} := \sum_{j=1}^{n} a_j^\dagger a^j \right). \tag{A.1}
\]

Then, it is also clear that the rank-2\(k\) \(SU(n)\)-singlet operator for \(k < n\) is given by

\[
\Gamma^{(2k)} := \sum_{1\leq j_1 < j_2 < \cdots < j_k \leq n} \Gamma^{(2)}_{j_1}\Gamma^{(2)}_{j_2} \cdots \Gamma^{(2)}_{j_k} = \frac{1}{k!} \sum_{j_1,j_2,\ldots,j_k=1}^{n} \Gamma_{2j_1-1,2j_2-1,2j_3-1,\ldots,2j_k-1,2j_k}. \tag{A.2}
\]

Using the \(\Gamma_a\) matrix algebra (or the fusion rule), we can derive the identity

\[
\Gamma^{(2)}_a \Gamma^{(2k)} = (k+1)\Gamma^{(2k+2)} - (n-k+1)\Gamma^{(2k-2)} \tag{A.3}
\]

which recursively gives \(\Gamma^{(2k)}\) as a polynomial of \(\Gamma^{(2)}\) and hence as a polynomial of \((\hat{N} - n/2)\); e.g.,

\[
\Gamma^{(4)} = \frac{(2i)^2}{2!} \left( (\hat{N} - \frac{n}{2})^2 - \frac{n}{4} \right), \tag{A.4}
\]

\[
\Gamma^{(6)} = \frac{(2i)^3}{3!} \left( (\hat{N} - \frac{n}{2})^3 - \frac{3n - 2}{4}(\hat{N} - \frac{n}{2}) \right), \tag{A.5}
\]

\[
\Gamma^{(8)} = \frac{(2i)^4}{4!} \left( (\hat{N} - \frac{n}{2})^4 - \frac{3n - 4}{2}(\hat{N} - \frac{n}{2})^2 + \frac{3n(n - 2)}{16} \right). \tag{A.6}
\]

These expressions of \(\Gamma^{(2k)}\) in terms of \(\Gamma^{(2)}\) can also be obtained directly from the relation

\[
\sum_{k=0}^{n} t^k \Gamma^{(2k)} = (1 + t^2)^\frac{n}{2} e^{2t^2}, \tag{A.7}
\]

by expanding the RHS in powers of \(t\), where \(t = \tan x\) so that

\[
x = \frac{1}{2i} \ln \frac{1 + it}{1 - it} = t \sum_{\ell=0}^{\infty} \frac{(-t^2)^\ell}{2\ell + 1} = t - \frac{t^3}{3} + \frac{t^5}{5} + \cdots, \quad \cos^2 x = \frac{dx}{dt} = \frac{1}{1 + t^2}. \tag{A.8}
\]

Noting that \(\Gamma^{(2)}_j\) for each \(j\) satisfies \((\Gamma^{(2)}_j)^2 = -1\), we can obtain Eq. (A.7) as follows:

\[
e^{x^2} = \prod_{j=1}^{n} e^{-\Gamma^{(2)}_j} = \prod_{j=1}^{n} (\cos x + \Gamma^{(2)}_j \sin x) = (\cos x)^n \prod_{j=1}^{n} (1 + \Gamma^{(2)}_j \tan x)
= (\cos x)^n \sum_{k=0}^{n} (\tan x)^k \sum_{j_1 < j_2 < \cdots < j_k} \Gamma^{(2)}_{j_1}\Gamma^{(2)}_{j_2} \cdots \Gamma^{(2)}_{j_k} = (1 + t^2)^{-\frac{n}{2}} \sum_{k=0}^{n} t^k \Gamma^{(2k)}. \tag{A.9}
\]

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