Non-covariance of the generalized holonomies: Examples

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Abstract

A key aspect of a recent proposal for a generalized loop representation of quantum Yang-Mills theory and gravity is considered. Such a representation of the quantum theory has been expected to arise via consideration of a particular algebra of observables – given by the traces of the holonomies of generalized loops. We notice, however, a technical subtlety, which prevents us from reaching the conclusion that the generalized holonomies are covariant with respect to small gauge transformations. Further analysis is given which shows that they are not covariant with respect to small gauge transformations; their traces are not observables of the gauge theory. This result indicates what may be a serious complication to the use of generalized loops in physics.

I. Introduction

There have recently been a variety of attempts to formulate gauge theories in terms of loops. One of the key technical developments which suggests such a formulation of gauge theory is Giles’ result that, for SU(N) theories, the information contained in the Wilson loops (i.e. the traces of all holonomies around closed curves) is sufficient for the reconstruction of the connection up to local gauge transformations. That is, the Wilson loops contain all of the gauge-invariant information about the connection. Since the Wilson loops separate points of the space of connections modulo gauge transformations, there is a sense in which any gauge-invariant function on the relevant space of connections (and hence any configuration observable of the gauge theory) may be expressed in terms of the Wilson loops. The (over-)completeness of these observables suggests that they be taken as the basic configuration observables in the quantization scheme.

This idea, along with Ashtekar’s connection-dynamic formulation of general relativity, provide the foundation of the Ashtekar-Rovelli-Smolin approach to quantum gravity. The duality between connections and loops suggests the possibility of representing states by functions of loops. The idea of the loop representation was first introduced to gravity by Rovelli and Smolin, and has resulted in a formalism with several attractive features. Most notable are the relationship between diffeomorphism invariance and knot theory, the combinatorial aspect of the formalism, and a sense in which discreteness emerges. For details of this approach

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to quantum gravity see [6].

Despite the merits of the loop-representation, it is generally believed that ordinary loops are not sufficient for the description of gravity. The problem of regularization of the Wilson loop operators suggests that one introduce a thickening, or framing, of the loops. A novel suggestion is presented by recent results of Di Bartolo, Gambini and Griego [7, 8]. The space of loops based at a fixed point forms a group [9], but not a Lie group. In an attempt to “coordinate” the group of based loops, the authors of [8] came upon a generalization of the notion of a loop. The result is an infinite-dimensional Lie group, which contains the group of (based) loops as a subgroup. The elements of this extended loop group\(^1\) are (sequences of) distributional quantities, ordinary loops being the “most distributional” elements. There are also elements, however, which are “less distributional” than loops, objects that we may think of as “smoothened loops”. With the above regularization issue in mind, the existence of the smoothened loops is of obvious interest. An anticipated benefit of the use of generalized loops is the ability to apply familiar functional methods to the study of the (generalized) loop representation. Further, we will see that the extended loop group has the global structure of a vector space. Hence, integration on the generalized loop space is a fairly straight-forward matter. Integration techniques may supply a form of the inner-product which is inherent to the loop space.

For these reasons, one is motivated to examine the role of generalized loops in gauge theory. Recall that the idea of the loop representation is based on the Wilson loops. Since, as we will see, the generalized loops are defined via inspection of the functional form of the holonomies of ordinary loops, the holonomy formally extends to the extended loop group. It is through this extension of the holonomy that one can imagine the construction of a generalized loop representation. For such a formulation of quantum Yang-Mills theory or gravity to make sense, the generalized holonomies must be gauge-covariant with respect to (small) gauge transformations. At the very least, the traced holonomies should be gauge-invariant. The main result presented here is the fact that the generalized holonomies are not covariant with respect to small gauge transformations. Despite the beauty of the extended loop group, its use in gauge theory may therefore be limited.

In Sec. II, we review the construction of the extended loop group and discuss some useful properties of its elements. Section III focuses on the generalized holonomies, with particular attention given to their transformation properties. Consideration of the Abelian case will suggest simple examples of non-covariance of the generalized SU(2) holonomies. Two such examples are presented in Sec. IV. Finally, in Sec. V, we conclude with generalizations of the results and remarks concerning their relevance in physics.

### II. Generalized loops

The purpose of this section is to recall the basic ideas regarding generalized loops (for details, see the original work [7]). After introducing the group of loops on an arbitrary connected manifold, consideration of the holonomies will suggest a generalization of the notion of loops. The set of these generalized loops – the extended loop group – forms an infinite dimensional Lie group.

Fix a point \(p\) on an arbitrary connected manifold \(\mathcal{M}\) and consider the space \(C_{\mathcal{M}}\) of closed

\(^1\)Note that the “extended loop group” and the “group of loops” discussed below are unrelated to what mathematicians call the “loop group”.

curves based at p. Elements of \( \mathcal{C}_p \) are piecewise-smooth maps \( C : I \to \mathcal{M} \) such that \( C(0) = p = C(1) \), where \( I \) is the unit interval. Now, our main motivation for such consideration is that the trace of the holonomy of a physical gauge field around any closed curve is gauge-invariant; i.e. an observable of the classical field theory. Since we are primarily concerned with the observables, we are not interested in the space \( \mathcal{C}_p \) itself, but in a space of equivalence classes of elements of \( \mathcal{C}_p \). For example, two closed curves which differ merely by reparametrization yield the same holonomies for an arbitrary smooth connection over \( \mathcal{M} \). Gambini and Trias [3] provide an appropriate identification of elements of \( \mathcal{C}_p \): Two closed curves \( C, C' \in \mathcal{C}_p \) are deemed equivalent if \( C \circ C' \) is contractible within itself to the trivial curve \( \iota(s) \equiv p \), where \( C' \) is the reversed path \( C'(s) = C'(1 - s) \). With this equivalence relation on \( \mathcal{C}_p \), it is easy to see that two closed curves which are equivalent give the same holonomies for any smooth connection over \( \mathcal{M} \). Denote by \( \mathcal{L}_p \) the space obtained by dividing \( \mathcal{C}_p \) by this equivalence relation. The obvious composition of paths induces a group operation on \( \mathcal{L}_p \).

Next, consider a connection \( \mathbf{A} \) on a principle bundle \( P(\mathcal{M}, G) \), where \( G \) is a compact, connected Lie group. For the sake of simplicity, we shall assume that \( P \) is trivial and view connections as Lie algebra-valued one-forms on \( \mathcal{M} \). (Below, we will restrict attention to the case \( G = SU(2) \), for which every bundle is trivial.) To an element \( \gamma \in \mathcal{L}_p \) we may associate the holonomy, \( U_A[\gamma] \), around any path \( C \) in the equivalence class defining \( \gamma \). Expressed in terms of the fundamental representations of the gauge group \( G \) and its Lie algebra, the holonomy takes the form of the path-ordered exponential, which may be written explicitly as

\[
U_A[\gamma] = P \exp \oint_C \mathbf{A} = \sum_{n=0}^{\infty} \int \cdots \int X_{\gamma}^{a_1 \cdots a_n}(x_1, \ldots, x_n) \mathbf{A}_{a_1}(x_1) \cdots \mathbf{A}_{a_n}(x_n), \tag{2.1}
\]

where

\[
X_{\gamma}^{a_1 \cdots a_n}(x_1, \ldots, x_n) := \oint dy_1 \int_{y_1}^{1} dy_2 \cdots \int_{y_{n-1}}^{1} dy_n \delta(x_1, y_1) \cdots \delta(x_n, y_n), \tag{2.2}
\]

and the zeroth term in Eq. (2.1) is taken to be the identity. In Eq. (2.2) the index \( a_k \) is “attached” to the point \( x_k \), and for each \( n \), \( X_{\gamma}^{a_1 \cdots a_n}(x_1, \ldots, x_n) \) is an \( n \)-point distributional vector density of weight one in each argument \( x_k \). As suggested by the subscript on the \( X \)’s, these \( n \)-point distributions are independent of the particular path \( C \) chosen from the equivalence class determined by \( \gamma \in \mathcal{L}_p \). It will be convenient to employ the notation

\[
X_{\gamma}^{\mu_1 \cdots \mu_n} := X_{\gamma}^{a_1 \cdots a_n}(x_1, \ldots, x_n), \tag{2.3}
\]

where the index \( \mu_k \) now represents the pair \( (a_k, x_k) \), and contraction of greek indices represents both contraction of the latin index and the integration over \( \mathcal{M} \).

Thus, to every element \( \gamma \in \mathcal{L}_p \) is associated a string

\[
X_{\gamma} := (1, X_{\gamma}^{\mu_1}, \ldots, X_{\gamma}^{\mu_1 \cdots \mu_n}, \ldots)
\]

of multi-vector densities. As is observed in [7], if \( \gamma, \eta \in \mathcal{L}_p \), the multi-densities corresponding to their product may be expressed as

\[
X_{\gamma \eta}^{\mu_1 \cdots \mu_n} = \sum_{k=0}^{\eta} X_{\gamma}^{\mu_1 \cdots \mu_k} X_{\eta}^{\mu_{k+1} \cdots \mu_n}, \tag{2.4}
\]
with the convention that
\[ X^{\mu_1 \cdots \mu_0} := 1. \]

These strings of multi-densities satisfy two useful identities. Denote by \( X \) the string corresponding to an arbitrary loop in \( \mathcal{L}_p \). The first identity reflects the ordering of points on the image of the loop;
\[ X^{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_n} = X^{\mu_1 \cdots \mu_k} X^{\mu_{k+1} \cdots \mu_n}, \]
where the left side is obtained by summing over all permutations of the \( \mu_i \) which preserve the relative ordering of the first \( k \) indices and also the relative ordering of remaining \( n - k \). For example,
\[ X^{\mu_1 \mu_2 \mu_3 \mu_4} = X^{\mu_1 \mu_2 \mu_3 \mu_4} + X^{\mu_1 \mu_3 \mu_2 \mu_4} + X^{\mu_1 \mu_3 \mu_4 \mu_2} + X^{\mu_2 \mu_3 \mu_1 \mu_4} + X^{\mu_2 \mu_3 \mu_4 \mu_1} + X^{\mu_3 \mu_1 \mu_4 \mu_2} + X^{\mu_3 \mu_4 \mu_1 \mu_2}. \]

Next, since taking the divergence of a vector density requires no additional structure (e.g. a metric or derivative operator) on \( \mathcal{M} \), it is natural to ask whether the divergence of an entry of \( X \) satisfies any useful property. The answer is in the affirmative;
\[ \frac{\partial}{\partial x_{ak}} X^{\mu_1 \cdots \mu_n} = [\delta(x_k, x_{k-1}) - \delta(x_k, x_{k+1})] X^{\mu_1 \cdots \mu_k \cdots \mu_n}, \]
where the caret over the \( \mu_k \) is intended to indicate its absence, and the mixed notation on the left side should be transparent. By definition, \( x_0 \) and \( x_{n+1} \) are taken to be the base point, \( p \). Thus the divergence of the rank-\( n \) entry of \( X \) is directly related to the rank-(\( n-1 \)) entry.

The basic idea of Di Bartolo, Gambini and Griego \cite{7} is to consider the space of all objects satisfying these relations. To be precise, let \( \mathcal{E} \) be the space of all sequences \((E^0, E^{\mu_1}, \ldots, E^{\mu_1 \cdots \mu_n}, \ldots)\), where \( E^0 \) is a real number and \( E^{\mu_1 \cdots \mu_n} \) is a distributional vector-density in each index. \( \mathcal{E} \) becomes an associative algebra when equipped with the product motivated by Eq. (2.4); given \( E_1, E_2 \in \mathcal{E} \), we define
\[ (E_1 \times E_2)^{\mu_1 \cdots \mu_n} := \sum_{k=0}^{n} E_1^{\mu_1 \cdots \mu_k} E_2^{\mu_{k+1} \cdots \mu_n}, \]
where \( E^{\mu_1 \cdots \mu_0} := E^0 \in \mathbb{R} \). The extended loop group (based at \( p \in \mathcal{M} \)) is defined to consist of those elements of \( \mathcal{E} \) which satisfy the algebraic relation (2.3) and the differential relation (2.6) and for which the rank-zero entry is unity. This set, denoted by \( \mathcal{X}_p \), is closed under the product defined above and every element is seen to have an inverse with respect to the identity element, \( I := (1, 0, 0 \ldots) \). \( \mathcal{X}_p \) is then a group which contains \( \mathcal{L}_p \) as a subgroup.

Next, \( \mathcal{X}_p \) is an infinite-dimensional Lie group in the following sense. For any element \( X \in \mathcal{X}_p \), the logarithm
\[ \ln(X) := \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}(X - I)^m \]
is a well-defined element of \( \mathcal{E} \) (with vanishing rank-zero entry) which satisfies the differential relation given by Eq. (2.6) and the homogeneous algebraic relation
\[ \ln(X)^{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_n} = 0 \quad \forall \quad 0 < k < n. \]

Let \( \mathcal{F}_p \) consist of all elements of \( \mathcal{E} \) which satisfy these two conditions and with vanishing rank-zero entry. One can show that if \( F \in \mathcal{F}_p \) then \( \exp(F) := \sum_{k=0}^{\infty} \frac{1}{k!} F^k \) is a well-defined
Further, for any element $X \in \mathcal{X}_p$ the logarithm $F = \ln X$ is the unique element of $F_p$ for which $X = \exp(F)$. $F_p$ is closed under the Lie bracket given by the commutator with respect to the associative product (2.7), defined on $\mathcal{E}$; this is the Lie bracket relevant to the group operation on $\mathcal{X}_p$. Thus, $F_p$ is simply the Lie algebra corresponding to $\mathcal{X}_p$. Note also that $\mathcal{X}_p$ has the global structure of an infinite-dimensional vector space since there is a one-one correspondence between its elements and elements of its Lie algebra. In particular, we may unambiguously take the real power of any element in $\mathcal{X}_p$: $X^t := \exp(t \ln X)$.

Given a $G$-connection, one may consider the formal expression for the holonomy around an arbitrary extended loop,

$$U_A[X] := \sum_{n=0}^{\infty} X^{\mu_1 \cdots \mu_n} A_{\mu_1} \cdots A_{\mu_n}.$$  (2.9)

There is no claim that the extended holonomies take values in the gauge group, or even that they converge. However, in [7] it is formally shown that $U_A[X_1 \times X_2] = U_A[X_1]U_A[X_2]$, where the right side is given by matrix multiplication in the fundamental representation. At least at the formal level, the holonomy extends to a homomorphism on $\mathcal{X}_p$. It is worth noticing one particular situation in which the extended holonomies converge to elements of the gauge group. Suppose that $A$ is an Abelian connection; i.e. $[A(x), A(y)] \equiv 0$. Then, using the algebraic relation (2.5), it is a simple matter to show that for any $X \in \mathcal{X}_p$

$$U_A^{(n)}[X] := X^{\mu_1 \cdots \mu_n} A_{\mu_1} \cdots A_{\mu_n} = \frac{1}{n!} (X^\mu A_\mu)^n.$$  (2.10)

So the extended holonomy corresponding to any Abelian connection is just given by the exponential of $X^\mu A_\mu$. (The result (2.10) depends only on the fact that the restriction of $A$ to the support of $X$ is Abelian.) If, for example, the support of $A$ is also of compact closure, the holonomy is convergent and group-valued on all of $\mathcal{X}_p$. This result will be used extensively in what follows.

### III. The generalized holonomies

The construction of the extended loop group is elegant and of considerable interest from a purely mathematical point of view. However, the intention extends to physics as well. The idea is simply to generalize the formalism used in the ordinary loop representations of gauge theories, i.e. to consider the traces of the extended holonomies as observables for Yang-Mills theory and, perhaps more importantly, general relativity. In particular, an extended loop representation for quantum general relativity may be an especially useful setting for consideration of an inverse loop transform and the framing problem of knot invariants [8, 10].

With the intended application of the generalized loops in mind, it is natural to examine the behavior of the extended holonomies under gauge transformations. One often distinguishes between two types of gauge freedom. The gauge which is generated by the (first-class) constraints is physical gauge freedom, while that which is not is symmetry. The physical gauge freedom then corresponds to that generated by the infinitesimal gauge transformations. Thus, in order for the traces of the extended holonomies to give observables, it is necessary that they be invariant under small gauge transformations. This issue was considered in [7], but as
is usual in pioneering work, a detailed analysis was sacrificed for the sake of progress in other directions.

Such an analysis is the purpose of this section. We will find that there is a technical subtlety which prevents us from accepting the naive conclusion that the extended holonomies are (formally) gauge covariant with respect to infinitesimal gauge transformations. In order to gain some insight about the transformations properties of the holonomies, we will then consider the Abelian case. In all that follows, we will restrict attention to the manifold $\mathcal{M} \approx \mathbb{R}^3$.

Recall that an infinitesimal gauge transformation is given by a map $\Lambda : \mathcal{M} \to L_{\mathbb{G}}$, where $L_{\mathbb{G}}$ is the Lie algebra of $\mathbb{G}$. To first order in $\Lambda$, the gauge-transformed connection is given by $A^{\Lambda} = A + d\Lambda + [A, \Lambda]$.

Set $U^{(n)}_A[X] := X^{\mu_1 \cdots \mu_n} A_{\mu_1} \cdots A_{\mu_n}$ as in Eq. (2.10), so that

$$U_A[X] = \sum_{n=0}^{\infty} U^{(n)}_A[X].$$

Using the differential relation (2.6) satisfied by the generalized loops, one may obtain the holonomy corresponding to the gauge-transformed connection;

$$U^{(n)}_{A^{\Lambda}}[X] = U^{(n)}_A[X] + \left[ U^{(n-1)}_A[X], \Lambda(p) \right] + f^{(n)}_{(A,\Lambda)}[X] - f^{(n-1)}_{(A,\Lambda)}[X],$$

(3.1)

where

$$f^{(n)}_{(A,\Lambda)}[X] := \sum_{k=1}^{n} X^{\mu_1 \cdots \mu_k} A_{\mu_1} \cdots A_{\mu_{k-1}} [A, A]_{\mu_k} A_{\mu_{k+1}} \cdots A_{\mu_n}.$$  

(3.2)

One is tempted to conclude from Eq. (3.1) that, since the $f^{(n)}$ cancel upon summation of the series for the transformed holonomy, all extended holonomies are formally gauge-covariant. However, let us proceed more carefully. Consider the partial sum,

$$\sum_{n=0}^{N} U^{(n)}_{A^{\Lambda}}[X] = \sum_{n=0}^{N} U^{(n)}_A[X] + \left[ \sum_{n=0}^{N} U^{(n)}_A[X], \Lambda(p) \right] - \left[ U^{(N)}_A[X], \Lambda(p) \right] + f^{(N)}_{(A,\Lambda)}[X].$$

(3.3)

If we suppose that $U_A[X]$ converges, then $U^{(N)}_{A^{\Lambda}}[X] \to 0$ as $N \to \infty$ and the extended holonomy is covariant under infinitesimal gauge-transformations only if

$$f^{(N)}_{(A,\Lambda)}[X] \to 0 \quad \text{as} \quad N \to \infty.$$  

(3.4)

Thus, gauge-covariance of the extended holonomies does not follow trivially from Eq. (3.1).

We are now faced with the problem of whether the notion of holonomy generalizes to the extended loop group. Although, by inspection of Eq. (3.2), the Abelian case is trivial, an example will lead the way to an understanding of the non-Abelian case. Therefore, let us consider $G = U(1)$. Let $\gamma$ be the loop determined by the curve

$$C(s) = (\cos(2\pi s), \sin(2\pi s), 0).$$

This loop determines an element $X_\gamma \in \mathcal{X}_p$, where the base point has been fixed as $p = (1,0,0)$. We will focus on the generalized holonomies of an arbitrary real power, $X_\gamma^t$, of this particular loop.
The generic $U(1)$-connection is of the form

$$\mathbf{A}(x) = -i\omega(x),$$

where $\omega$ is a real one-form on $\mathcal{M}$. To compute the holonomy of $\mathbf{A}$ around $X_\gamma^t$, we need only know the rank-1 entry of $X_\gamma^t$,

$$(X_\gamma^t)_\mu = t X_\gamma^\mu,$$

and that, with respect to the cylindrical coordinates $z, r, \theta$,

$$X_\gamma^\mu = \frac{1}{r} \delta^1(r, 1) \delta^1(z, 0) \left( \frac{\partial}{\partial \theta} \right)^\mu.$$

By Eq. (2.10), the holonomy is then given by

$$U_A[X_\gamma^t] = \exp(tX^\mu A_\mu) = \exp(-it \oint_\Gamma \omega), \quad (3.5)$$

where $\Gamma$ is the unit circle in the $x$-$y$ plane (the image of $C$). The holonomy of the gauge-transformed connection $\mathbf{A}^g = \mathbf{A} + g^{-1}dg$ is

$$U_{A^g}[X_\gamma^t] = U_A[X_\gamma^t] \cdot \exp(-2\pi it w_\gamma[g]), \quad (3.6)$$

where

$$w_\gamma[g] := \frac{i}{2\pi} \oint_\Gamma g^{-1}dg \in \mathbb{Z}. \quad (3.7)$$

Note that the pull-back of $g$ to $\Gamma$ is a map from the circle into $U(1)$ and that $w_\gamma[g]$ is simply the winding number of this map, i.e. the number of times $g$ wraps $U(1)$ around $\Gamma$. Suppose $g$ is a small gauge transformation. Then, by definition, there exists a homotopy $g_\lambda$ (a smooth one-parameter family of gauge transformations, $\lambda \in [0, 1]$) connecting $g = g_1$ to the trivial map $g_0 \equiv 1$. By pulling the homotopy back to $\Gamma$, one then obtains a one-parameter family of maps from the circle into $U(1)$. Since the winding number is integral, it must be the same for each $g_\lambda$. Hence, $w_\gamma[g] = w_\gamma[g_1] = w_\gamma[g_0] = 0$. We then see that the holonomy (3.5) is covariant with respect to small gauge transformations. Note that this was not a general proof of covariance of the $U(1)$-holonomies (the general proof is much simpler that what we have done above!). The above reasoning applies only to the arbitrary real power of the particular loop $\gamma$. The utility of our result, however, lies not in the conclusion, but in the methodology. The above ansatz, when applied to the case $G = SU(2)$, will suggest simple examples which show that the non-Abelian holonomies are not covariant with respect to small gauge.

**IV. Non-covariance of the generalized holonomies**

We can study the non-Abelian case, by “embedding” the above result into $SU(2)$. The idea in mind is to replace $U(1)$ by an Abelian subgroup of $SU(2)$. After making this idea more precise, natural examples of non-covariance will be presented. The first is, perhaps, the most natural; it involves the holonomy of the real power of an ordinary loop. For the second example, we will consider the holonomies of generalized loops which are “least distributional”, in a sense to be explained below.
Let us consider an arbitrary Abelian subgroup of $SU(2)$. This subgroup is generated by an element, $T$, of the Lie algebra, $\mathcal{L}SU(2)$. We may assume, without loss of generality, that $T$ is normalized as $\text{tr}(T^2) = -2$, so that, for example,

$$\exp[rT] = \exp[(r + 2\pi n)T] \quad \forall \ n \in \mathbb{Z}, \ r \in \mathbb{R}. \quad (4.1)$$

Now suppose $A$ is an $SU(2)$-connection which is proportional to $T$; i.e.

$$A = \omega T \quad (4.2)$$

for some one-form $\omega$. Suppose further that $g : \mathbb{R}^3 \to SU(2)$ is an $SU(2)$ gauge transformation whose restriction to $\Gamma$ is contained in the $U(1)$-subgroup generated by $T$. We can now mimic the discussion leading to Eq. (3.6) by making the replacement $-i \mapsto T$. We obtain

$$U_A[X^\gamma] = U_A[X^\gamma] \cdot \exp(2\pi t v_\gamma[g]T), \quad (4.3)$$

where $v_\gamma$ is defined as

$$v_\gamma[g]T := \frac{1}{2\pi} \oint_{\Gamma} g^{-1} dg. \quad (4.4)$$

The meaning of $v_\gamma$ is analogous to that of $w_\gamma$; it is simply the number of times $g$ winds the $U(1)$-subgroup generated by $T$ around the circle $\Gamma$. Of course, $v_\gamma$ is only defined for such an Abelian gauge transformation.

For integral $t$, $X^\gamma$ corresponds to an ordinary loop and the holonomy is covariant under all gauge transformations. But for the above holonomy to transform covariantly for all real $t$, $v_\gamma[g]$ must be trivial. Recall that in Sec. III it was the non-simple connectivity of $U(1)$ that prevented us from finding small gauge transformations with non-trivial winding number around $\Gamma$. $SU(2)$ is, of course, simply connected; hence, there is no immediate suspicion that there do not exist small gauge transformations with nontrivial $v_\gamma$. In fact, there do exist such gauge transformations. The task at hand is to produce an explicit expression for a small gauge transformation whose restriction to $\Gamma$ lies in an Abelian subgroup of $SU(2)$, and which winds $\Gamma$ non-trivially around this Abelian subgroup. Since the exponential factor in Eq. (4.4) is independent of the connection, we will then have shown that the generalized holonomies are not gauge covariant.

To this end, let us first focus on a convenient description of the manifold structure of $SU(2)$. The Lie algebra, $\mathcal{L}SU(2)$, of $SU(2)$ is a real, three-dimensional vector space with a natural (Killing-Cartan) inner-product which, in the fundamental representation, takes the form

$$(T_1, T_2) := -2 \text{tr}(T_1 T_2). \quad (4.5)$$

Fix a basis $\{\tau_1, \tau_2, \tau_3\}$ for the Lie algebra, which is ortho-normal with respect to this inner-product. For any element $L \in \mathcal{L}SU(2)$, $\exp L$ may be uniquely written as

$$\exp(L) = a_0 \mathbf{1} + 2[a_1 \tau_1 + a_2 \tau_2 + a_3 \tau_3], \quad (4.6)$$

\footnote{The usual factor of $i$ has been absorbed into the definition of the Lie algebra elements. With this convention, $\mathcal{L}SU(2)$ is represented by traceless anti-Hermitian matrices. This eliminates annoying powers of $i$ which otherwise would have appeared in Eq. (2.3).}

\footnote{The common example is obtained by choosing $\tau_i = (-i/2)\sigma_i$, where $\sigma_i$ are the Pauli matrices.}
where \( a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1 \). While \( a_1, \ldots, a_4 \) may be written in terms of \( L \), we will not find their explicit form useful. Having chosen a basis \( \{ \tau_1, \tau_2, \tau_3 \} \) for the Lie algebra, we then obtain an isomorphism of \( SU(2) \) with the unit 3-sphere in \( \mathbb{R}^4 \). Of course, the identity \( \mathbf{1} \) is represented by the point \((1,0,0,0)\). We will abuse notation and write \( \exp(\mathbf{L}) = (a_0, a_1, a_2, a_3) = a_i \vec{e}_i \), where the \( a_i \) are those appearing in Eq. (4.6), and \( \vec{e}_i \) form the obvious orthonormal basis of \( \mathbb{R}^4 \). The algebra \( LSU(2) \) may be viewed as the tangent space to \( SU(2) \) at the identity, and with the Lie algebra element, \( \mathbf{L} = L^1 \tau_1 + L^2 \tau_2 + L^3 \tau_3 \), we may identify the vector \((0, L^1, L^2, L^3) \in \mathbb{R}^4 \). The \( U(1) \)-subgroup generated by \( \mathbf{L} \) is now simply represented by the great circle (through \( \mathbf{1} \)) whose tangent at the identity is proportional to \( \mathbf{L} \). For example, \( \exp(\alpha \tau_1) = (\cos(\alpha/2), \sin(\alpha/2), 0, 0) \).

We may now view a gauge transformation as a smooth map \( g : \mathbb{R}^3 \to \mathbb{R}^4 \) whose image is contained in the unit 3-sphere. Choose the basis \( \{ \tau_1, \tau_2, \tau_3 \} \) so that the (arbitrary) algebra element considered above is given by \( \mathbf{T} = 2 \tau_1 \) and let us look for a small gauge transformation \( g \) for which \( g(\cos \theta, \sin \theta, 0) = \exp(\theta \mathbf{T}) = (\cos \theta, \sin \theta, 0, 0) \). We will then have \( v, g = 1 \), and our goal will have been accomplished.

At this point, a brief digression will be quite instructive. Let us display a particular homotopy connecting the curve \( h(\theta) = \exp(\theta \mathbf{T}) \) to the trivial curve \( \iota(\theta) \equiv \mathbf{1} \). This can be done geometrically, as follows. Consider the intersection of \( SU(2) \) (the 3-sphere in \( \mathbb{R}^4 \)) and the hyperplane \( P^3 = \{ \vec{a} \in \mathbb{R}^4 | a_3 = 0 \} \). This is a 2-sphere in \( \mathbb{R}^4 \), which we will denote as \( S \). Let \( P^2(\alpha) \) be the 2-plane in \( P^3 \) consisting of points of the form \( \vec{e}_0 + \vec{v} \) such that \( \vec{v} \cdot \vec{n}(\alpha) = 0 \), where \( \vec{n}(\alpha) = \vec{e}_2 \cos \alpha + \vec{e}_0 \sin \alpha \). The intersection of \( S \) with \( P^2(\alpha) \) is a circle of radius \( r(\alpha) = \sin \alpha \), which may be parameterized as

\[
h_\alpha(\theta) = \begin{pmatrix} 1 - (1 - \cos \theta) \sin^2 \alpha \\ \sin \theta \sin \alpha \\ (1 - \cos \theta) \sin \alpha \cos \alpha \\ 0 \end{pmatrix},
\]

for \( \theta \in [0, 2\pi] \). As \( \alpha \) varies from 0 to \( \pi \), these circles “foliate” the sphere \( S \). Notice, in particular, that \( h_0(\theta) \equiv \mathbf{1} \) and \( h_{\pi/2}(\theta) = (\cos \theta, \sin \theta, 0, 0) = \exp(\theta \mathbf{T}) \). Therefore, \( h_\alpha \) provides the desired homotopy (see Fig. 1.) This homotopy will play a very important role in the examples that follow.
IV.1. Example 1

We can now suggest the form of a gauge transformation \( g : \mathbb{R}^3 \to SU(2) \) which is demonstrably small, whose restriction to the unit circle in the \( x-y \) plane lies in the Abelian subgroup generated by \( T \), and which winds this subgroup non-trivially around the unit circle. In order to satisfy generic boundary conditions at infinity, we will also demand that \( g \) be trivial outside of a compact region.

The desired gauge transformation may be obtained from a smooth assignment of the angle \( \alpha \) to each pair of cylindrical coordinates \( r, z \); i.e. we try

\[
g(r \cos \theta, r \sin \theta, z) := h_{\alpha(r,z)}(\theta),
\]

such that \( \alpha(1, 0) = \pi/2 \). Any such gauge transformation is obviously small since the one-parameter family of gauge transformations

\[
g_\lambda(r \cos \theta, r \sin \theta, z) := h_{\lambda \alpha(r,z)}(\theta), \quad \lambda \in [0, 1]
\]

provides a homotopy connecting \( g \) to the trivial map.

It remains only to produce the assignment \( \alpha(r, z) \). This may be accomplished by use of the smearing function

\[
\sigma_\Delta(x) := \begin{cases} \ e \cdot \exp \left( \frac{-\Delta^2}{x^2 + \Delta^2} \right) & : |x| \leq \Delta \\ 0 & : |x| \geq \Delta. \end{cases}
\]

(4.10)

\( \sigma_\Delta \) is symmetric about \( x = 0 \), at which it attains its maximum value \( \sigma_\Delta(0) = 1 \). Most importantly, \( \sigma_\Delta \) is an infinitely differentiable function, the support of which is compact. Putting

\[
\alpha(r, z) := \frac{\pi}{2} \sigma_\frac{1}{2}(z) \sigma_\frac{1}{2}(r-1),
\]

we obtain

\[
g(r \cos \theta, r \sin \theta, z) := h_{\frac{\pi}{2} \sigma_\frac{1}{2}(z) \sigma_\frac{1}{2}(r-1)}(\theta)
\]

\[
= \begin{pmatrix}
1 & -\left(1 - \cos \theta\right) \sin^2 \left[\frac{\pi}{2} \sigma_\frac{1}{2}(z) \sigma_\frac{1}{2}(r-1)\right] \\
\sin \theta \sin \left[\frac{\pi}{2} \sigma_\frac{1}{2}(z) \sigma_\frac{1}{2}(r-1)\right] & \sin \theta \sin \left[\frac{\pi}{2} \sigma_\frac{1}{2}(z) \sigma_\frac{1}{2}(r-1)\right] \\
\left(1 - \cos \theta\right) \cos \left[\frac{\pi}{2} \sigma_\frac{1}{2}(z) \sigma_\frac{1}{2}(r-1)\right] & 0
\end{pmatrix}.
\]

(4.11)

Note that \( g \) is infinitely differentiable and is trivial outside a compact region. Further, since \( \alpha_\frac{1}{2}(r = 1, z = 0) = \pi/2 \), its restriction to the circle \( \Gamma \) is given by \( g(\cos \theta, \sin \theta, 0) = \exp(\theta T) \), as desired.

The purpose of this sub-section was the construction of this gauge-transformation. For any connection of the form written in Eq. (4.2), the holonomy “around” \( X_\gamma \) is non-covariant, according to Eq. (4.3). For example, choose

\[
A(r \cos \theta, r \sin \theta, z) := A \sigma_\frac{1}{2}(z) \sigma_\frac{1}{2}(r-1) T d\theta, \quad A \in \mathbb{R}
\]

(4.12)

(which also vanishes outside of a compact region). We then have

\[
U_A[X_\gamma] = \exp[2\pi A t T],
\]

(4.13)
and
\[ U_A^\theta [X^\mu_\gamma] = \exp[2\pi A t(T)] \cdot \exp[2\pi t(T)]. \] (4.14)

This completes the first example.

IV.2. Example 2

The above example involved what may be, from the Lie algebraic point of view, the most notable element of the extended loop group – the real power of an ordinary loop. The extended loop group also contains elements which seem radically different than ordinary loops, i.e. those for which all entries are smooth; they must be genuinely distributional. However, it is a trivial application of the results of [7] to show that given an arbitrary multi-vector density \( Y^\mu_1 \cdots^\mu_m \) which is divergence-free in each index and satisfies the homogeneous algebraic relation (2.8), there exists an element \( X \in \mathcal{X}_p \) such that
\[ X^\mu_1 \cdots^\mu_k = 0 \quad \forall \quad k < m \quad \text{and} \quad X^\mu_1 \cdots^\mu_m = Y^\mu_1 \cdots^\mu_m. \]

In particular, we may choose \( Y^\mu_1 \cdots^\mu_m \) to be smooth. The set of all extended loops whose first non-vanishing entry is smooth is a sub-Lie group of \( \mathcal{X}_p \). One might think of these elements as “smoothened loops”.

As was mentioned in the introduction, the existence of the smoothened loops is a nice feature of \( \mathcal{X}_p \). If \( X \) is as described above, then due to Eq. (2.6), \( X^\mu_1 \cdots^\mu_n \) must be a genuine distribution for each \( n > m \). Therefore, the hope of obtaining a gauge-invariant smearing of the connection by smooth functions is not borne out. Nonetheless, one might hope that some light may be shed on the problem of regularization of the Wilson loop variables. Could it be that, by some mathematical miracle, the holonomies of smoothened loops do not suffer from the problem of non-covariance illustrated above? By “smearing out” the previous example, we will see that the answer to this question is, unfortunately, negative.

Recall that for Abelian connections, it is only the rank-one entry of \( X \) on which the holonomy \( U_A[X] \) depends. Choose an element \( X \in \mathcal{X}_p \) for which
\[ X^\mu = \left( \frac{\partial}{\partial \theta} \right)^\mu \sigma_1(z) \sigma_\frac{1}{2}(r-1). \] (4.15)

\( X^\mu \) is a smooth vector density of compact support, which may be viewed as a smoothened version of the \( X^\mu_\gamma \) considered above. Let \( A \) be as in the definition (4.12). A short calculation yields the holonomy
\[ U_A[X] = \exp(X^\mu A_\mu) = \exp(2\pi A e^6 T). \] (4.16)

In the spirit of the previous example, we construct a gauge transformation \( \tilde{g} \) which commutes with \( A \). The restriction of \( \tilde{g} \) to the support of \( A \) must then take values in the Abelian subgroup generated by \( T \). The idea is simply to replace the smearing function \( \sigma \) in Eq. (4.11). The function
\[ t_\Delta(x) := \frac{2}{\Delta \sigma} \int_{-\infty}^x dx' \sigma_\Delta(x') \] (4.17)
is a smoothened step function. It vanishes for all \( x \leq -\Delta \) and is unity for all \( x \geq \Delta \). Of course, \( t_\Delta \) is infinitely differentiable everywhere. Define
\[ s_\Delta(x) := \begin{cases} 
  t_\Delta(x + 3\Delta) & : \quad x \leq 0 \\
  1 - t_\Delta(x - 3\Delta) & : \quad x \geq 0.
\end{cases} \] (4.18)

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This function is non-vanishing only for $|x| \leq 4\Delta$ and is constant on the interval $|x| \leq 2\Delta$, on which it assumes the value one. Using $s_{\frac{1}{4}}$ in place of $\sigma_{\frac{1}{2}}$ in Eq. (4.11), one obtains the desired gauge transformation; put
\[ \tilde{g}(r \cos \theta, r \sin \theta, z) := h_{\frac{\pi}{2} s_{\frac{1}{4}} (z) s_{\frac{1}{4}} (r-1)}(\theta) \]
\[ = \left( \begin{array}{cccc}
1 - (1 - \cos \theta) \sin^2 \left[ \frac{\pi}{2} s_{\frac{1}{4}} (z) s_{\frac{1}{4}} (r-1) \right] \\
\sin \theta \sin \left[ \frac{\pi}{2} s_{\frac{1}{4}} (z) s_{\frac{1}{4}} (r-1) \right] \\
\sin \left[ \frac{\pi}{2} s_{\frac{1}{4}} (z) s_{\frac{1}{4}} (r-1) \right] (1 - \cos \theta) \cos \left[ \frac{\pi}{2} s_{\frac{1}{4}} (z) s_{\frac{1}{4}} (r-1) \right] \\
0 \end{array} \right). \] (4.19)

On the support of $X^\mu$, $\tilde{g}$ takes a very simple form; for $|z| \leq \frac{1}{\Delta}$ and $|r-1| \leq \frac{1}{\Delta}$,
\[ \tilde{g}(r \cos \theta, r \sin \theta, z) = (\cos \theta, \sin \theta, 0, 0) = \exp(\theta T). \] One may then obtain
\[ \int X^a \tilde{g}^{-1}(d\tilde{g})_a = \frac{\pi e^3}{8} T. \] (4.20)

Finally,
\[ U_{A^\theta}[X] = U_A[X] \cdot \exp \left( \frac{\pi e^3}{8} T \right). \] (4.21)

The extended “holonomies” of the smoothened loops are not covariant with respect to small gauge transformations.

V. Generalizations and conclusions

The extended loop group is a well-defined mathematical object. It is an infinite-dimensional group which encompasses the group of based loops on an arbitrary connected manifold, $\mathcal{M}$. (Note that we have used the term “Lie group” fairly loosely. For the sake of rigor, it should be shown that $\mathcal{X}_p$ admits a manifold structure with respect to which the group operations are continuous.) For applications to physics, however, one would also like to extend the concept of holonomy. In fact, the construction of the extended loop group was based on the functional form of the holonomy of ordinary loops. There is then the obvious candidate for a generalized holonomy. We have found, however, that for the case $\mathcal{M} \approx \mathbb{R}^3$ this generalized holonomy is not covariant with respect to small gauge transformations. Its trace does not provide gauge-invariant functionals on the space of connections for an $SU(2)$ gauge theory on Minkowski space, for example.

In fact, the result is of a very general validity. Since any simple Lie group contains an $SU(2)$ subgroup, it extends to the non-Abelian case with such gauge groups – those which are typically relevant in physics. The result also applies to the case of gravity in terms of the Ashtekar variables. Further, although we restricted our attention to $\mathcal{M} \approx \mathbb{R}^3$, all of the mappings used in the first example are of compact support. We may then extend the result to an arbitrary manifold. (Note, however, that since the topology of $\mathbb{R}^3$ was used in a critical way in defining the smoothened loop group, the second example does not extend to manifolds of arbitrary topology; i.e. it is not clear that one can even define the smoothened loop group for the arbitrary case.)
From our point of view, the potential power of the extended loop group involves the use of the traced holonomies as a large class of observables for gauge theories. Our results then suggest a re-evaluation of the extended loop group as an arena for quantum gravity and Yang-Mills theory.

There are three alternatives worth consideration. First, the following question arises: What characterizes those generalized loops for which the holonomies are covariant? Perhaps consideration of this question would shed some light on the appropriate extension of the loop representation. Note however, that by Example 1, one will not have the continuous structure of a Lie group at one’s disposal. Thus, techniques involving functional differentiation are not likely to be straight-forward in such a formulation. A second alternative, suggested by Gambini and Pullin [11], is to design a different extension of the holonomy which is covariant. Alternatively, since observables of the theory are our primary concern, it may be most productive to focus on an extension of the concept of the Wilson loops. It may be the case, for example, that such an extension exists which does not manifest itself as the trace of a holonomy. Lastly, while it seems that some generalization of the ordinary loop representation is needed, it may turn out that the appropriate generalization is altogether different than that suggested by the existence of the extended loop group.

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