Eleven-dimensional supergravity as a gauge theory for the M-algebra

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Abstract: An eleven-dimensional supergravity constructed as a gauge theory for the M-algebra is presented. The gravitational sector is described by the dimensional continuation of the Euler density in ten dimensions. The theory admits a class of vacuum solutions of the form $S^{10-d} \times X_{d+1}$, where $X_{d+1}$ is a warped product of $\mathbb{R}$ with a $d$-dimensional spacetime. It is shown that a nontrivial propagator for the graviton exists only for $d = 4$ and positive cosmological constant. Perturbations of the metric around this solution reproduce linearized General Relativity around four-dimensional de Sitter spacetime.

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Introduction- A consensus has emerged in the high energy community that a consistent unified theory of all interactions and matter should be formulated in some dimension higher than four. Strong theoretical evidence, both in supergravity and in string theory, leads to conjecture the existence of an underlying fundamental theory in eleven dimensions [1], [2, 3]. This is nowadays called M-Theory (see, e.g., [4]). The standard procedure to link the higher dimensional theory with four-dimensional physics has been either to compactify the extra dimensions by the Kaluza-Klein reduction (see, e.g., [5]), or through some more recent alternatives [6].

In these frameworks, however, the physical spacetime dimension is an input rather than a prediction of the theory. In fact, in standard theories, there is no obstruction to perform dimensional reductions to spacetimes of dimensions \( d \neq 4 \). An ideal situation, instead, would be that the eleven-dimensional theory dynamically predicted a low energy regime which could only be a four-dimensional effective theory. In this scenario, a background solution with an effective spacetime dimension \( d > 4 \) should be expected to be a false vacuum where the propagators for the dynamical fields are ill-defined, lest a low energy effective theory could exist in dimensions higher than four.

In this paper, a new eleven-dimensional supergravity theory consistent with this new scenario is constructed. Indeed, for this theory, eleven-dimensional Minkowski spacetime is a maximally supersymmetric solution that would be a natural candidate for the vacuum. However, propagators around this background are ill-defined and hence it is a false vacuum. On the other hand, the theory admits vacuum geometries of the form \( S^{10-d} \times X_{d+1} \), where \( X_{d+1} \) is a domain wall whose worldsheet is a \( d \)-dimensional constant curvature spacetime \( M_d \). These solutions exist only if \( M_d \) has a non-negative cosmological constant, and the graviton can only propagate provided \( M_d \) is a four-dimensional de-Sitter space. Moreover, the gravitational perturbations reproduce linearized General Relativity in four dimensions.

The action we propose is constructed as a gauge theory for the M-algebra which is the maximal extension of the \( \mathcal{N}=1 \) super-Poincaré algebra in eleven dimensions. This algebra is spanned by the set \( G_A = \{ J_{ab}, P_a, Q_\alpha, Z_{ab}, Z_{abcde} \} \), where \( J_{ab} \) and \( P_a \) are the generators of the Poincaré group and \( Q_\alpha \) is a Majorana spinor supercharge whose anticommutator is

\[
\{ Q_\alpha, Q_\beta \} = (CT^a)_{\alpha\beta} P_a + (CT^{ab})_{\alpha\beta} Z_{ab} + (CT^{abcde})_{\alpha\beta} Z_{abcde}.
\]  

The charge conjugation matrix \( C \) is antisymmetric, and the “central charges” \( Z_{ab} \) and \( Z_{abcde} \) are tensors under Lorentz rotations, and correspond to the “electric” and “magnetic” charges of the \( M_2 \) and \( M_5 \) branes, respectively. As shown below, the algebra fixes the field content to include, apart from the graviton \( g_\mu^a \), the spin connection \( \omega_\mu^{ab} \) and the gravitino \( \psi_\mu \), two one-form fields \( b_\mu^{ab}, b_\mu^{abcde} \), which are rank two and five antisymmetric tensors under the Lorentz group, respectively. The local supersymmetry transformations close off-shell without requiring auxiliary fields.

Gravitational sector.- A supergravity action containing (1) as a local symmetry must be,
in particular, invariant under local translations,
\[ \delta e^a = D\lambda^a = d\lambda^a + \omega^a_b \lambda^b, \delta \omega^{ab} = 0. \]  

(2)

The only gravitational action in eleven dimensions constructed out of the vielbein \(e^a\) and the spin connection \(\omega^{ab}\), leading to second order field equations for the metric, invariant under diffeomorphisms and local Poincaré transformations is given by \[8, 9\]

\[
I_G[e, \omega] = \int_{\mathcal{M}_{11}} \epsilon_{a_1 \ldots a_{11}} R^{a_1 a_2} \cdots R^{a_9 a_{10}} e_{a_{11}}.
\]

(3)

Here \(R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb}\) is the curvature two-form, and wedge product between forms is understood \[14\]. For the reason given above, we take \(I_G\) as the gravitational sector of our theory rather than the Einstein-Hilbert action which, is not invariant under (2) \[11\]. The Lagrangian in (3) is the dimensional continuation of the Euler density from ten to eleven dimensions and contains the degrees of freedom of eleven dimensional gravity \[12\].

A local Poincaré transformation acting on the dynamical fields is a gauge transformation \(\delta \lambda = d\lambda + [A, \lambda]\), with parameter \(\lambda = \lambda^a P_a + \frac{1}{2} \lambda^{ab} J_{ab}\); provided \(e^a\) and \(\omega^{ab}\) are the components of a single connection for the Poincaré group, \(A = e^a P_a + \frac{1}{2} \omega^{ab} J_{ab}\). This observation will be the guiding principle for the construction of the locally supersymmetric extension of \(I_G\).

**Supersymmetric extension.**- A natural way to construct the locally supersymmetric extension of (3) without breaking local Poincaré invariance is that the extra fields required by supersymmetry enter on a similar footing with the original fields. In other words, all dynamical fields will be assumed to belong to a connection for a supersymmetric extension of the Poincaré group. The simplest option would be to consider the \(\mathcal{N} = 1\) super Poincaré algebra without central extensions. However, this algebra is not rich enough to ensure the off-shell supersymmetry of the action. Indeed, in this case, the connection would be extended by the addition of a gravitino as \(A \rightarrow A + \psi Q/\sqrt{2}\), and the gauge generator would change as \(\lambda \rightarrow \lambda + \epsilon Q/\sqrt{2}\), where \(\epsilon\) is a zero-form Majorana spinor. This fixes the supersymmetric transformations to be \(\delta e^a = \bar{\epsilon} \Gamma^a \psi/2\), \(\delta \psi = D\epsilon\) and \(\delta \omega^{ab} = 0\). Then, in order to cancel the variation of (3) under supersymmetry, a kinetic term for the gravitino of the form

\[
I_\psi = -\frac{1}{6} \int_{\mathcal{M}_{11}} R_{abc} \bar{\psi} \Gamma^{abc} D\psi,
\]

(4)

is required, where \(R_{abc} := e_{abca_1 \ldots a_8} R^{a_1 a_2} \cdots R^{a_7 a_8}\). However, the variation of \(I_\psi\) produces, in turn, an extra piece which cannot be cancelled by a local Lagrangian for \(e^a, \omega^{ab}\), and \(\psi\), and hence the super Poincaré algebra is not sufficient to achieve supersymmetry. Nevertheless, following the Noether procedure, it can be seen that supersymmetry can be ensured introducing additional bosonic fields. These fields can be either a second-rank or a fifth-rank tensor one-forms \(b^{ab}\), and \(b^{abcdef}\), that transform like \(\bar{\epsilon} \Gamma^{ab} \psi\) and \(\bar{\epsilon} \Gamma^{abcdef} \psi\), respectively. The M-algebra \[1\] naturally brings in these extra bosonic fields and also prescribes their supersymmetry transformations in the expected form, provided the field content is given by the
components of a single fundamental field, the M-algebra connection,
\[ A = \frac{1}{2} \omega^{ab} J_{ab} + e^a P_a + \frac{1}{\sqrt{2}} \psi^a Q_a + b^{ab} Z_{ab} + b^{abcde} Z_{abcde}. \]  

The required local supersymmetry transformations can be read from a gauge transformation of the M-connection (5) with parameter \( \lambda = 1/\sqrt{2} e^a Q_a \),
\[ \delta_\varepsilon e^a = \frac{1}{2} \varepsilon \Gamma^a \psi, \quad \delta_\varepsilon \psi = D\varepsilon, \quad \delta_\varepsilon \omega^{ab} = 0, \]
\[ \delta_\varepsilon b^{ab} = \frac{1}{2} \varepsilon \Gamma^{ab} \psi, \quad \delta_\varepsilon b^{abcde} = \frac{1}{2} \varepsilon \Gamma^{abcde} \psi. \]  

Thus, the supersymmetric extension of (3), invariant under (8), is found to be
\[ I_\alpha = I_G + I_\psi - \frac{\alpha}{6} \int_{M_{11}} R_{abc} R_{de} b^{abcde} \]
\[ + 8(1 - \alpha) \int_{M_{11}} [R^2 R_{ab} - 6 (R^3)_{ab}] R_{cd} \left( \bar{\psi} \Gamma^{abcde} D \psi - 12 R^{[ab} b^{cd]} \right), \]
where \( \alpha \) is a dimensionless constant whose meaning will be discussed below. Apart from diffeomorphism invariance, that is guaranteed by the use of forms, this action is also invariant under (2) as well as under the set of local Abelian transformations
\[ \delta b^{ab} = D\theta^{ab}, \quad \delta b^{abcde} = D\theta^{abcde}. \]  

It is simple to see that the local invariances of the action, including Poincaré transformations, supersymmetry (3) together with (8), are a gauge transformation for the M-connection (5) with parameter \( \lambda = \lambda^a P_a + \frac{1}{2} \lambda^{ab} J_{ab} + \frac{1}{2} \lambda^{abcde} Z_{abcde} + 1/\sqrt{2} e^a Q_a \). As a consequence, the off-shell closure of the supersymmetry algebra is ensured by construction without requiring auxiliary fields.

**Manifest M-Covariance.** The action (7) describes a gauge theory for the M-algebra with fiber bundle structure, which can be seen explicitly by writing the Lagrangian as a Chern-Simons form (13) for the M-connection (5). Indeed, the Lagrangian satisfies \( dL = \langle F^6 \rangle \), where the curvature \( F = dA + A^2 \) is given by \( F = \frac{1}{2} R_{ab} J_{ab} + \bar{T}^a P_a + 1/\sqrt{2} D\psi^a Q_a + \bar{F}^{(2)} Z_{[2]} + \bar{F}^{(5)} Z_{[5]} \); with \( \bar{T}^a = De^a - (1/4) \bar{\psi} \Gamma^a \psi \) and \( \bar{F}^{(k)} = D b^{[k]} - (1/4) \bar{\psi} \Gamma^k \psi \) for \( k = 2, 5 \). The bracket \( \langle ... \rangle \) stands for a multilinear form of the M-algebra generators \( G_A \) whose only nonvanishing components are given by
\[ \langle J_{a_1 a_2 \ldots} J_{a_9 a_{10}}, P_{a_{11}} \rangle = \frac{16}{3} \epsilon_{a_1 \ldots a_{11}}, \]
\[ \langle J_{a_1 a_2 \ldots} J_{a_9 a_{10}}, Z_{abcde} \rangle = -\alpha \frac{1}{4} \epsilon_{a_1 \ldots a_9 b c d e} \eta_{[a_9 a_{10}] [d e]}, \]
\[ \langle J_{a_1 a_2 \ldots} J_{a_3 a_4}, J_{a_5 a_6}, J_{a_7 a_8}, J_{a_9 a_{10}}, Z^{(ab)} \rangle = (1 - \alpha) \frac{16}{3} \left[ \delta_{a_1 \ldots a_{10}, a_6} - \delta_{a_1 \ldots a_{4}, a_9 a_{10}, a_{5 6}} \right], \]
\[ \langle Q, J_{a_1 a_2 \ldots} J_{a_3 a_4}, J_{a_5 a_6}, J_{a_7 a_8}, Q \rangle = \frac{1}{48} \left[ C T_{a_1 a_2} a_{3 \ldots a_{8}} + (1 - \alpha) \left( 3 \delta_{a_1 a_2 a_6} C T \gamma_{a_{7 8} a_{9} a_{10} a_{5 6}} + 2 C T^{a_3 a_7 a_8 a_9} \delta_{a_1 a_{2 6}} \right) \right], \]
where (anti-) symmetrization under permutations of each pair of generators is understood when all the indices are lowered. The existence of this bracket allows writing the field equations in a manifestly covariant form as
\[ \langle F^5 G_A \rangle = 0. \]
In addition, if the spacetime is the boundary of a twelve-dimensional manifold, \( \partial \Omega = M_{11} \), the action (7) can also be written as \( I = \int_{\Omega_{12}} \langle F^6 \rangle \), which describes a topological theory in twelve dimensions.

Gravitons and four dimensional spacetime. - We now turn to the problem of identifying the true vacuum of the theory. Obviously, a configuration with a locally flat connection, \( F = 0 \), solves the field equations and would be a natural candidate for vacuum in a standard field theory. However, no local degrees of freedom can propagate on such background because all perturbations around it are zero modes. Note that eleven-dimensional Minkowski spacetime is maximally supersymmetric by virtue of (6), but as it obeys \( F = 0 \), propagators on it are ill-defined, and hence it is a false vacuum.

In a matter-free configuration, Eq. (9) is a set of quintic polynomials for the Riemann two-form \( R^{ab} \). A necessary condition to have well-defined perturbations is that the background solution be a simple zero of at least one of the polynomials. In particular, this requires the curvature to be nonvanishing on a submanifold of a large enough dimension.

Let us consider a torsionless spacetime with a product geometry of the form \( X_{d+1} \times S^{10-d} \), where \( X_{d+1} \) is a domain wall whose worldsheet is a \( d \)-dimensional constant curvature spacetime \( M_d \). The line element is given by

\[
 ds^2 = \exp \left( -2a|z| \right) \left( dz^2 + \tilde{g}^{(d)}_{\mu\nu}(x) dx^\mu dx^\nu \right) + \gamma_{mn}^{(10-d)}(y) dy^m dy^n, \tag{10}
\]

where \( \tilde{g}^{(d)}_{\mu\nu} \) stands for the worldsheet metric with \( \mu, \nu = 0, \ldots, d - 1 \); \( \gamma_{mn}^{(10-d)} \) is the metric of \( S^{10-d} \) of radius \( r_0 \) and \( a \) is a constant. This Ansatz solves the vacuum field equations provided the projection of the Riemann tensor along the worldsheet, \( \tilde{R}^{ij} = \tilde{R}^{ij} - a^2 \tilde{e}^i \wedge \tilde{e}^j \), vanishes (here \( \tilde{e}^i \) and \( \tilde{R}^{ij} \) stand for the vielbein and the Riemann curvature of the worldsheet, respectively). This means that \( M_d \) is either locally de Sitter spacetime of radius \( a^{-1} \), or locally Minkowski for \( a = 0 \). The requirement that the curvature of (10) be a simple zero, implies, after a long but straightforward computation, that \( d \) cannot be greater than four. Then, the condition of having well-defined propagators singles out the dimension of the worldsheet to be \( d = 4 \), and \( a^2 > 0 \). For \( d = 4 \), the only relevant equation for the perturbations is the one that arises from the variation with respect to \( \tilde{e}^i \),

\[
 a \delta(z) \epsilon_{ijkl} \delta(\tilde{R}^{ik} - a^2 \tilde{e}^i \tilde{e}^k) \tilde{e}^l = 0. \tag{11}
\]

Since for \( a = 0 \) this equation becomes empty, Minkowski spacetime is ruled out. Thus, the existence of the propagator requires the four-dimensional cosmological constant to be strictly positive and given by \( \Lambda_4 = 3a^2 \).

Note that Eq. (11) has support only on the \( z = 0 \) plane. In this case, for perturbations along the worldsheet, \( \delta \tilde{g}_{\mu\nu} = h_{\mu\nu}(x) \), this equation reproduces the linearized Einstein equations in four-dimensional de Sitter spacetime. The modes that depend on the coordinates transverse to the worldsheet fall into two classes. Those of the form \( \delta \tilde{g}_{\mu\nu} = h_{\mu\nu}(x, y) \) are massive Kaluza-Klein modes with a discrete spectrum, while \( \delta \tilde{g}_{\mu\nu} = h_{\mu\nu}(x, z) \) correspond to Randall-Sundrum-like massive modes whose spectrum is continuous and has a mass gap. The
perturbations of the remaining metric components are zero modes which is related to the fact that the equations are not deterministic for the compact space. A detailed analysis of this, as well as of the perturbations of matter fields will be presented elsewhere [14].

Discussion.- We have presented a framework in which the spacetime dimension is dynamically selected to be four. The mechanism is based on a new eleven-dimensional action of the Chern-Simons type, which is a gauge theory for the M-algebra. The possibility of dynamical dimensional reduction arises because the theory has radically different dynamical behaviors around backgrounds of different effective spacetime dimensions. Thus, in a family of product spaces of the form $X_{d+1} \times S^{10-d}$, the only option that yields a well defined low energy propagator for the graviton is $d = 4$ and $\Lambda_4 > 0$. It should be stressed that for all gravity theories of the type discussed here, possessing local Poincaré invariance in dimensions $D = 2n + 1 \geq 5$, four-dimensional de Sitter spacetime is also uniquely selected by the same mechanism as the background for the low energy effective theory.

The action discussed in this paper has a free parameter $\alpha$, which reflects the fact that the theory contains two natural limits which correspond to different subalgebras of (1). For $\alpha = 0$, the action $I_0$ in Eq. (7) does not depend on $b^{[5]}$ and corresponds to a gauge theory for the supermembrane algebra, while for $\alpha = 1$, the bosonic field $b^{[2]}$ decouples, and $I_1$ is a gauge theory for the super five-brane algebra as discussed in [13]. It is interesting to note that the linear combination of both limits, $I_{\alpha} = I_0 + \alpha(I_1 - I_0)$, is not only invariant under the intersection of both algebras, but under the entire M-algebra. As the term $I_1 - I_0$ does not couple to the vielbein and is invariant under supersymmetry by itself, $\alpha$ is an independent coupling constant. A similar situation occurs in nine dimensions, where in one limit the theory corresponds to the super five-brane algebra while for the other it is a gauge theory for the super-Poincaré algebra with a central extension [16].

In the presence of negative cosmological constant, eleven-dimensional AdS supergravity [7] can be written as a Chern-Simons theory for $osp(32|1)$, which is the supersymmetric extension of AdS$_{11}$. It is natural to ask whether there is a link between that theory in the vanishing cosmological constant limit, and the one discussed here. Since the M-algebra has 55 bosonic generators more than $osp(32|1)$, these theories cannot be related through a Inönü-Wigner contraction for a generic value of $\alpha$. However, it has been recently pointed out in [3], generalizing the procedure of [19], that it is possible to obtain the M-algebra from an expansion of $osp(32|1)$. In this light, applying this procedure to the eleven-dimensional AdS supergravity theory, it should expected that the action presented here will be recovered up to some additional terms decoupled from the vielbein, that are supersymmetric by themselves.

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