Gravitational lensing by traversable Lorentzian wormholes is a new possibility which is analyzed here in the strong field limit. Wormhole solutions are considered in the Einstein minimally coupled theory and in the brane world model. The observables in both the theories show significant differences from those arising in the Schwarzschild black hole lensing. As a corollary, it follows that wormholes with zero Keplerian mass exhibit lensing properties which are qualitatively (though not quantitatively) the same as those of a Schwarzschild black hole. Some special features of the considered solutions are pointed out.

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I. Introduction

Gravitational lensing is an important and effective window to look for signatures of peculiar astrophysical objects such as black holes (BH). This field of activity has lately attracted a lot of interest among the physics community. Early works focussed on the lensing phenomenon in the weak field (for a review, see [1]), but weak field results can not distinguish between various different solutions that are asymptotically flat. What one needs for this purpose is a method of calculation in the strong field regime. Progress in this direction have been initiated by Fritelli, Kling and Newman [2], and by Virbhadra and Ellis [3]. However, the difficulty is that, in the strong field, light deflection diverges at the photon sphere. (The conditions for the existence of photon surfaces have been rigorously analyzed by Claudel, Virbhadra and Ellis [4]). By an analytic approximation method, Bozza et al [5] have shown that the nature of divergence of the deflection angle becomes logarithmic as the light rays approach the photon sphere of a Schwarzschild BH. This method has been successfully applied also in the Reissner-Nordström BH [6]. Virbhadra and Ellis [7] have further extended the method of strong field lensing to cover the cases of Weak Naked Singularity (WNS) and Strong Naked Singularity (SNS). Bozza [8] has subsequently extended his analytic theory to show that the logarithmic divergence near the photon sphere is a generic feature for static, spherically symmetric spacetimes. This work is a remarkable step forward in the arena of gravitational lensing. Bhadra [9] has applied the procedure to BHs in string theory. The extension of the strong field limit to Kerr BH has also been worked out recently [10,11]. All these investigations have indeed thrown up a richesse of information about the signatures of BH via lensing mechanism.
There is however another exciting possibility that has not received enough attention to date: It is lensing by stellar size traversable wormholes (WH) which are just as interesting objects as BHs are. WHs have “handles” (throats) that connect two asymptotically flat regions of spacetime and many interesting effects including light propagation, especially in the Morris-Thorne-Yurtsever (MTY) WH spacetime, have been extensively investigated in the literature [12]. WHs require exotic matter (that is, matter violating at least some of the known energy conditions) for their construction. The idea of this kind of matter has received further justification in the notion of “phantom field” or “dark matter” invoked to interpret the observed galactic flat rotation curves or the present acceleration of the Universe. Some works on lensing on a cosmological scale involving dark matter do exist [13,14] but they have nothing to do with Lorentzian WHs on a stellar scale. Nonetheless, it might be noted that recent works by Onemli [15] show that the gravitational lensing by the dual cusps of the caustic rings at cosmological distances may provide the tantalizing opportunity to detect Cold Dark Matter (CDM) indirectly, and discriminate between axions and weakly interacting massive particles (WIMPs). It is also to be noted that local, static WH solutions threaded by phantom matter have also been worked out recently [16].

Work in the direction of WH lensing has been initiated by Cramer et al [17] not very long ago and recently Safonova et al [18] have investigated the problem of lensing by negative mass WHs. A most recent work by Tejeiro and Larra˜naga [19] shows that Morris-Thorne type WHs generally act like convergent lenses. Unfortunately, work on WH lensing, let alone the strong field analysis, is still relatively scarce though observables in WH lensing have the potential to serve a dual purpose: They would establish not only the WH itself but also throw light on the existence of classical exotic matter. This fact provides the basic motivation for the present theoretical investigation.

We shall investigate the strong field lensing phenomenon in the WH solutions belonging to the Einstein minimally coupled scalar field theory (EMS) as well as the brane world model. (It is to be noted that novel effects of the scalar field on gravitational lensing have been analyzed in Ref.[20] way back in 1998 in the context of point like naked singularity lens.) Importance of the EMS theory need not be repeated here. Suffice it to say that it is the simplest scalar field theory. It can be connected to the vacuum Brans-Dicke theory via the so-called Dicke transformations and to the vacuum heterotic string theory. The static WH solutions in all these theories have been well investigated [21-30]. On the other hand, brane theory is a completely different proposition of great interest. The brane world paradigm envisages that only gravity propagates in the 5-D bulk while all other fields are confined to the 4-D brane. This idea leads to newer models of local self-gravitating objects. It would therefore be interesting to calculate the lensing effect in these models, especially in the strong field limit.

Generically, the brane world BHs are far richer in structure than ordinary BHs as they embody a synthesis of wormhole and black hole features. That is why we refer to those objects here as WH/BHs. For instance, the effective stress energy tensor could violate some of the energy conditions, though it need not always be the case. This feature is not unexpected as the stress tensor contains imprints of the nonlocal free gravitational field existing in the 5-D bulk which contributes negative energy [31]. Several observable effects of the extra dimension on quasar luminosity in the rotating models have been recently reported [32]. In the context of spherical symmetry, the extra-dimensional bulk contribution essentially implies a correction to the Schwarzschild solution but its horizon structure remains undisturbed. The brane theory we have in mind is described by the RS2 framework, that is, a single brane in a $Z_2$-symmetric 5-D asymptotically anti-de Sitter bulk in which only gravity propagates while all other fields are confined to the brane [33]. Strong field lensing in one of the brane world BHs [34] have been carried out in Ref.[35] (and the weak field lensing is calculated in Ref.[36]). Lensing in another class of brane world BH (see below, Sec.IVA) has been investigated by Whisker [37]. Authors in Refs.[35] and [37] have shown that such BHs could produce observables that are significantly different from the Schwarzschild BH.

In this paper, we shall apply the strong field limit procedure, due to Bozza [8], in the standard lensing (distinct from retrolensing) phenomenon by static spherically symmetric WH solutions in
the EMS theory and by the WH/BH solutions in the brane theory. (Lensing in the weak field regime has been investigated in Refs. [38].) We show that more spectacular differences can appear in the observables in the strong field limit. This is our key result.

The paper is organized as follows. In Sec.IIA, we outline the procedure of the strong field limit including the expressions for observables in Sec.IIB. In Sec.IIA, we deal with the lensing by a massive WH and in Sec.IIB, with the zero mass WH. In Sec.IVA, we point out that the brane world BH, considered recently in Ref. [37] can also be interpreted as a self dual WH harboring a globally strong naked singularity. Sec.IVB reveals characteristic features of the strong field lensing by a different brane world WH/BH solution. In Sec.V, we point out certain important aspects of the considered WH solutions that should be useful in understanding the lensing behavior. Finally, Sec.VI summarizes the results.

II. Strong Field Limit

A. Deflection angle

We assume that the asymptotically flat spacetime describing a BH or WH is centered at $L$ which serves as the lens. The observer $O$ and the source $S$, which is to be lensed, are positioned in the flat region on either side of $L$, but not necessarily along the same line. This is a plane configuration of ordinary lensing, as distinct from retrolensing where both $O$ and $S$ are positioned only on one side of $L$. Let $I$ be the location of the image of $S$ as observed by $O$ and that the extended IS segment meet the extended OL segment at $X$. Defining the angles as $(OL,OS) = \beta$, $(OL,OI) = \theta$, the lens equation follows from the plane geometry [3]:

$$\tan \beta = \tan \theta - \frac{D_{OX}}{D_{OL}} [\tan \theta + \tan (\alpha - \theta)]$$

(1)

where $D_{OX} = D_{OL} + D_{LX}$ and $D_{PQ}$ is the Euclidean distance between $P$ and $Q$, and $\alpha$ is the deflection angle.

The generic spherically symmetric static metric for our purposes is (we take $8\pi G = c = 1$):

$$ds^2 = A(x)dt^2 - B(x)dx^2 - C(x)d\Omega^2$$

(2)

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the metric on a unit sphere. The photon sphere $x = x_{ps}$ is a solution of the equation

$$\frac{C'(x)}{C(x)} = \frac{A'(x)}{A(x)}$$

(3)

in which the primes represent derivatives with respect to $x$. The impact parameter $u$ is defined in terms of the closest approach distance $x = x_0$ as

$$u = \sqrt{\frac{C(x_0)}{A(x_0)}}$$

(4)

The minimum impact parameter is given by

$$u_{ps} = \sqrt{\frac{C(x_{ps})}{A(x_{ps})}}$$

(5)

From the equation of photon trajectory, it is easy to derive the deflection angle

$$\alpha(x_0) = -\pi + I(x_0)$$

(6)

$$I(x_0) = \int_{x_0}^{\infty} \frac{2\sqrt{B(x)}dx}{\sqrt{C(x)} \sqrt{\frac{C(x)}{C(x_0)}} \frac{A(x_0)}{A(x)}} - 1$$

(7)
Bozza’s procedure [8] for the strong field limit is based on the following conditions: (a) The photon sphere $x = x_{ps}$ must exist, (b) The functions $A, B, C, A', C'$ must be positive for $x > x_{ps}$, (c) There should exist a static limit, or horizon where $A(x_0) = 0$. The last condition is sufficient but not necessary. Then, define $y = A(x), y_0 = A(x_0)$ and

$$z = \frac{y - y_0}{1 - y_0} \quad (8)$$

and rewrite the integral $I(x_0)$ as

$$I(x_0) = \int_0^1 R(z, x_0) f(z, x_0) dz \quad (9)$$

$$R(z, x_0) = \frac{2\sqrt{By}}{CA'} (1 - y_0) \sqrt{C_0} \quad (10)$$

$$f(z, x_0) = \frac{1}{\sqrt{y_0 - [(1 - y_0)z + y_0]y_0}} \quad (11)$$

where all functions without the subscript 0 are evaluated at $x = A^{-1}[1 - y_0]z + y_0$. The function $R(z, x_0)$ is regular for all values of its arguments, but the function $f(z, x_0)$ diverges as $z \to 0$ and it expands to second order like

$$f(z, x_0) \sim f_0(z, x_0) = \frac{1}{\sqrt{\alpha_1 z + \beta_1 z^2}} \quad (12)$$

where the parameters $\alpha_1, \beta_1$ depend on the closest approach $x_0$ as

$$\alpha_1 = \frac{1 - y_0}{C_0 A_0'} [C_0' y_0 - C_0 A_0'] \quad (13)$$

$$\beta_1 = \frac{(1 - y_0)^2}{2C_0^2 A_0^3} [2C_0 C_0' A_0'^2 + (C_0 C_0'' - 2C_0'^2) y_0 A_0' - C_0 C_0 y_0 A_0''] \quad (14)$$

Then the integral $I(x_0)$ is resolved into a regular and a divergent part and the latter gives the deflection angle to order $O(x_0 - x_{ps})$ as

$$\alpha(\theta) = -\pi \ln \left( \frac{\theta D_{OL}}{u_{ps}} - 1 \right) + \bar{\theta} \quad (15)$$

where

$$\pi = \frac{R(0, x_{ps})}{2\sqrt{\beta_{ps}}} \quad (16)$$

$$\bar{\theta} = -\pi + b_R + \pi \ln \frac{2\beta_{ps}}{y_{ps}} \quad (17)$$

$$b_R = \int_0^1 g(z, x_{ps}) dz + O(x_0 - x_{ps}) \quad (18)$$

$$g(z, x_{ps}) = R(z, x_{ps}) f(z, x_{ps}) - R(0, x_{ps}) f_0(z, x_{ps}) \quad (19)$$

$$\beta_{ps} = \beta_1 \big|_{x_0 = x_{ps}, y_{ps} = A(x_{ps})} \quad (20)$$

The function $g(z, x_{ps})$ is regular at $z = 0$ [8].
B. Observables

The relativistic images of the source are greatly demagnified in comparison to weak field images because the photon trajectories wind several times around the photon sphere before emerging outside. Yet, best results are obtained when the source \( S \), lens \( L \) and the observer \( O \) are highly aligned. In this case we can assume that the angles \( \theta \) and \( \beta \) are small, but \( \alpha = 2n\pi + \Delta\alpha_n \), \( n \in \mathbb{Z} \) where \( \Delta\alpha_n \) is the residual angle after the trajectories wind the photon sphere \( n \) times. Under these conditions, the lens Eq.(1) reduces to

\[
\theta = \beta + \frac{D_{LX}}{D_{OX}} \Delta\alpha_n
\]

(21)

Defining \( \alpha(\theta^0_n) = 2n\pi \), and using Eq.(15), we can write

\[
\theta^0_n = \frac{u_{ps}}{D_{OL}} (1 + e_n)
\]

(22)

where

\[
e_n = e^{(\pi - 2n\pi)/\pi}
\]

(23)

The position \( \theta^0_n \) and the magnification \( \mu_n \) of the \( n \)th relativistic image are:

\[
\theta_n = \theta^0_n + \frac{e_n u_{ps} (\beta - \theta^0_n) D_{OX}}{\pi D_{LX} D_{OL}}
\]

(24)

\[
\mu_n = \frac{1}{(\beta/\theta) \partial \beta / \partial \theta} \bigg|_{\theta^0_n} \approx \frac{e_n u_{ps}^2 (1 + e_n) D_{OX}}{\pi \beta^2 D_{OL}^2 D_{LX}}
\]

(25)

Now we bunch all the images together at \( \theta_\infty = u_{ps}/D_{OL} \), so that the outermost single image appears at \( \theta_1 \). Then define the observables

\[
s = \theta_1 - \theta_\infty
\]

(26)

\[
r = \frac{\mu_1}{\sum_{n=2} \mu_n}
\]

(27)

which, respectively, are the separation and flux ratio between the bunch and the outermost image. Using the relevant expressions, they simplify to

\[
s = \theta_\infty e^{(\pi - 2n\pi)/\pi}
\]

(28)

\[
r = e^{2\pi/\pi}
\]

(29)

We shall calculate the strong field coefficients \( \bar{\pi}, \bar{\beta} \) and the observables \( s, r \) for some physically interesting WH solutions in the EMS and the brane world model.

III. EMS theory

The field equations of the EMS theory are

\[
R_{\mu\nu} = \kappa \Phi_{,\mu} \Phi_{,\nu}
\]

(30)

\[
\Phi_{,\mu} = 0
\]

(31)

where \( \Phi \) is the minimally coupled scalar field and \( \kappa \) is a constant free parameter. Note that the above equations are just the conformally rescaled vacuum Brans-Dicke equations [24,30]. Clearly, all the results in the sequel can be easily transcribed into those of Brans-Dicke theory and further on, to string theory [29].
A. Massive WHs

A well known class of solutions of the EMS theory is the Janis-Newman-Winnicour (JNW) [39] solution (or a variant of the Wyman [40] solution):

\[ A(x) = \left(1 - \frac{2m}{x}\right)^\gamma, B(x) = \left(1 - \frac{2m}{x}\right)^{-\gamma}, C(x) = x^2 \left(1 - \frac{2m}{x}\right)^{1-\gamma} \]  

\[ \Phi(x) = \sqrt{\frac{1-\gamma^2}{2\kappa}} \ln \left[1 - \frac{2m}{x}\right] \simeq \frac{q}{x} \]  

\[ \gamma = \frac{M}{m} \]  

where \( M \) is the ADM mass given by

\[ M^2 = m^2 - \frac{1}{2}q^2 \]  

\[ q = m \sqrt{\frac{2(1-\gamma^2)}{\kappa}} \]  

is the asymptotic scalar JNW charge. In the field equations (30), we have introduced a new constant parameter \( \kappa \) that does not appear in the observables but facilitates the analysis of the nature of the EMS solutions. With a positive sign on the right hand side of Eq.(30), the stress tensor represents ordinary scalar matter with positive energy density. The solution (32,33) then has a globally strong naked singularity at \( x = 2m \) when \( \gamma < 1 \). However, with a negative sign on the right hand side, the stress tensor represents energy condition violating exotic matter necessary for constructing WHs.

Now, this negative sign can be achieved in two ways: (i) Take \( \kappa = -2 \) (that is, break all the energy conditions “by hand” or assume that this sign comes as an input from another theory) and keep \( \Phi \) real or (ii) Take \( \kappa = 2 \) but make \( \Phi \) imaginary or which the same thing, \( q \) imaginary. The latter case also throws up a negative sign on the right side of Eq.(30) and is completely physically valid as discussed by Armendáriz-Picón [41]. In either case, the solution represents the spacetime of a symmetric traversable wormhole [22,24,27,28]. That there are two asymptotic regions can be best seen by transforming the metric (30) into isotropic coordinates via a radial transformation

\[ x = \rho \left(1 + \frac{m}{2\rho}\right)^2 \]  

in which case the solution reduces to the Buchdahl solution [42] of 1959 given by

\[ A(\rho) = \left(\frac{1 - \frac{mp}{\rho}}{1 + \frac{mp}{\rho}}\right)^{2\gamma}, B(\rho) = \left(1 - \frac{m}{2\rho}\right)^{2(1-\gamma)} \left(1 + \frac{m}{2\rho}\right)^{2(1+\gamma)}, C(\rho) = \rho^2 B(\rho) \]  

\[ \Phi(x) = \sqrt{\frac{2(1-\gamma^2)}{\kappa}} \ln \left[\frac{1 - \frac{mp}{\rho}}{1 + \frac{mp}{\rho}}\right] \simeq \frac{q}{\rho} \]  

The solution is invariant in form under radial coordinate transformation \( \rho = \frac{m^2}{\rho'} \) and hence one asymptotic region occurs at \( \rho = \infty \) and the other at \( \rho' = 0 \), the two coordinate patches meeting at \( \rho = \rho' = \frac{m}{2} \). The WH throat occurs at \( \rho_{th} = \frac{m}{2} \left(\gamma + \sqrt{\gamma^2-1}\right) \) and the requirement that \( \rho_{th} \) be real
and positive demands that $\gamma > 1$. This is the WH condition. The energy density $\rho_D$ and the scalar curvature $R$ for the solutions (38) and (39) become

$$\rho_D = \left(\frac{1}{2}\right) m^2 \left(\frac{1 - \gamma^2}{1 - \frac{m^2}{4\rho^2}}\right)^2 \left(\rho + \frac{m}{2}\right)^{-2(1+\gamma)} \left(\rho - \frac{m}{2}\right)^{-2(1-\gamma)}$$

(40)

$$R = 2m^2 \rho^4 \left(\rho - \frac{m}{2}\right)^{-2(2+\gamma)} \left(\rho - \frac{m}{2}\right)^{-2(2-\gamma)}$$

(41)

Clearly, $\rho_D < 0$ for $\gamma < 1$ so that the Weak Energy Condition (WEC) is violated. For $M \neq 0$, and for the case (i), we have $\gamma = \frac{M}{\sqrt{M^2 - q^2}}$ and for the case (ii), defining $q = iq'$ with $q' > 0$, we have $\gamma = \frac{M}{\sqrt{M^2 - q^2}}$. Thus, $\gamma$ increases beyond unity if $q$ (or $q'$) is non-zero. The photon sphere appears at

$$\rho_{ps} = \frac{m}{2} \left[2\gamma \pm \sqrt{4\gamma^2 - 1}\right]$$

(42)

It is clear that $\rho_{ps} > \rho_{th}$ so long as $\gamma > 1$. All the functions $A(\rho), B(\rho), C(\rho), A'(\rho)$ and $C'(\rho)$ are positive for $\rho > \rho_{ps}$. There is also the so-called static limit at $\rho_s = \frac{M}{2}$ at which $A(\rho_s) = 0$. But the surface $\rho_s = \frac{M}{2}$ is a strong naked singularity. However, $\rho_{ps} > \rho_{th} > \rho_s$ for $\gamma > 1$ which implies that the photon sphere hides the throat and the naked singularity. The situation resembles the lensing scenario by Weakly Naked singularity (WNS) defined by Virbhadra and Ellis [7] to the extent that the naked singularity is hidden under the photon sphere. The occurrence of a throat hiding further the naked singularity is a new feature in the present case. However, the main difference is that the Virbhadra-Ellis choice of $\gamma$ is still less than unity for $q > M$ since they defined $\gamma = \frac{M}{\sqrt{M^2 + q^2}}$

The calculation of the strong field limit coefficients becomes awkward in the isotropic coordinates and it is more convenient to use the metric (32) which is in standard coordinates. Then the photon sphere appears at $x_{ps} = m(2\gamma + 1)$. Without involving any loss of rigor, all that we need to do is to take the WH range of $\gamma$ from from the foregoing analysis. Now, in the case of Schwarzschild lensing, the value $u - u_{ps} = 0.003$ involves an error of only 0.4% from the exact position of the outer image [8]. Taking this value as the starting point and using $u = \theta D_{OL}$, the coefficients become

$$\pi = 1$$

(43)

$$b = -\pi + b_R + \ln \left[(2\gamma + 1)^\gamma - (2\gamma - 1)^\gamma\right]^{2(2\gamma + 1)}$$

(44)

$$b_R = 0.9496 - 0.1199(\gamma - 1) + O(\gamma - 1)^2$$

(45)

$$u_{ps} = \frac{(2\gamma + 1)^\gamma + \frac{1}{2}}{2(2\gamma - 1)^\gamma}$$

(46)

$$\beta_{ps} = \frac{[(2\gamma + 1)^\gamma - (2\gamma - 1)^\gamma]^2}{4\gamma^2(4\gamma^2 - 1)^\gamma - 1}$$

(47)

$$\alpha(\gamma) = -\pi \ln \left(\frac{0.003}{u_{ps}}\right) + \bar{b}$$

(48)

It was shown in Ref.[8] that the deflection angle $\alpha(\gamma)$ decreases from the Schwarzschild value in the range of naked singularity ($\gamma < 1$). In contrast, the deflection angle $\alpha(\gamma)$ actually increases from the Schwarzschild value with the value of increasing $\gamma$ in the WH range ($\gamma > 1$) as will be seen in the Table I below. This behavior is markedly different from the case of naked singularity or the Schwarzschild BH.

**B. Massless WHs**
Consider a WH for which the ADM mass $M = 0$. The cases (i) and (ii) mentioned in Sec.IIIA respectively give $0 = m^2 + q^2$ and $0 = m^2 + q'^2$ which imply that both $m = 0$ and $q = q' = 0$. This is a trivial case. But we can also have $M = m\gamma = 0$ by putting $\gamma = 0$, $m \neq 0$. However, we must remember that, physically, the solar system tests fix $\gamma \sim 1$ while one is free to choose $m = 0$ to achieve $M = 0$. This notwithstanding, we consider the reverse case ($\gamma = 0$, $m \neq 0$) here only as a mathematically interesting possibility. Thus, taking $\kappa = 2$, we have from Eq.(35) that

$$q^2 = m^2$$

(49)

implying that the gravitational stresses due to $m$ and non-gravitational stresses due to $q$ exactly balance each other. This is an extremal situation. Though $M = 0$, and we should not expect any deflection at all, the spacetime is not flat. It is conceptually a classic example of Wheeler’s “charge without charge” [43], and it is a stable WH [41].

Due to our present choice of $\kappa$, $q^2 < 0$ as argued before, and so $m^2 < 0$. Let us take $m = -im'$. Then, we have $q'^2 = m'^2$ and moreover the energy density $\rho_D$ and scalar curvature $R$ at the throat $\rho_{th} = \frac{m'}{2} = \frac{q'}{2}$ are given by

$$\rho_D = -\frac{1}{2q'^2}, R = -\frac{2}{q'^2}$$

(50)

It would be interesting to analyze the effect of this massless curvature on the light rays. Also, as the radial variable $\rho \rightarrow \infty$ in Eqs.(40) and (41), both $\rho_D$ and $R \rightarrow 0$ implying that the zero mass WH solution is asymptotically flat and also perfectly nonsingular everywhere without a horizon. The solutions (38,39) become

$$A(\rho) = 1, B(\rho) = \left(1 + \frac{q'^2}{4\rho^2}\right)^2, C(\rho) = \rho^2 \left(1 + \frac{q'^2}{4\rho^2}\right)^2, \Phi(\rho) \simeq \frac{q'}{\rho}$$

(51)

and the throat occurs at $\rho_{th} = \frac{q'}{2}$. It also represents the Ellis [44] “drainhole” particle model. The Eqs.(51) can be expressed in proper distance $l = \rho - \frac{q'^2}{4\rho}$ in a quite familiar form

$$ds^2 = dt^2 - dl^2 - (l^2 + q'^2) d\Omega^2, \Phi(l) = ArcTan \left(\frac{l + \sqrt{l^2 + q'^2}}{q'}\right)$$

(52)

where we have used the identity $ArcTan(x) = \frac{1}{2} \ln \left[\frac{1 + ix}{1 - ix}\right]$ in $\Phi(\rho)$ of Eq.(39). The photon sphere exists and it appears at $\rho_{ps} = \frac{q'}{2}$ which coincides with the nonsingular throat radius $\rho_{th}$. This is an extremal situation. Looking at the necessary conditions, we see that (a) is satisfied even though $A \neq 0$ anywhere. This only points to the fact that (c) is not a necessary prerequisite. The condition (b) is marginally satisfied since all the desired functions are positive and nonzero for $\rho > \frac{q'}{2}$ except that $A' = 0$. [Note incidentally that in the standard coordinates of the metric (1), the throat is at $x_{th} = m(1 + \gamma)$ for $\gamma \neq 0$ and therefore $x_{th} = m$ for $\gamma = 0$. The photon sphere occurs also at $x_{ps} = m$, but the difficulty is that $C(x) \neq 0$ for $x > m$ in violation of the condition (b)]. Because $A' = 0$, the functions $R(z, x_0)$ and $R(0, x_m)$ diverge, and consequently do the coefficients $\pi$ and $b_R$. Since the object is massless, there is no possibility to consider the Schwarzschild lensing as a starting point in the strong field analysis as we did before.

Perlick [45] has discussed detailed lensing properties of the Ellis drainhole given by the metric (52). He discussed, in terms of an exact lens equation, the cases that observer and light sources are (i) on different sides and (ii) on the same side of the WH’s throat. If the observer is closer to the throat as the light source, the behavior of the bending angle is similar in both the cases. In terms of the metric form (51), we can integrate the deflection angle, Eq.(6), as

$$\alpha(\rho) = -\pi + 4 ArcTan \left(\frac{\rho}{\rho_0}\right)$$

(53)
where \( \rho = \rho_0 \) is the closest approach. The function \( \alpha(\rho) \) is real only in the range \( |\rho| < \rho_0 \) and this simply means that a light ray that starts from one asymptotic end and passes through the throat of the WH can not go back to the same asymptotic end from which it has started. It is very well possible that the light ray goes to the other asymptotic end, as already discussed by Ellis [44]. [The plot of \( \alpha(\rho) \) vs \( \rho \) for \( \rho_0 = \rho_{th} = \frac{q'}{2} \) (with units \( q' = 1 \)) is same as the fig.8 in Ref.[45] derived earlier.] Thus, there are two classes of light rays that start from one asymptotic end: Members of the first class turn around before they reach the neck, members of the second class pass through the throat and proceed to the other asymptotic end. The borderline cases between the two are light rays that asymptotically spiral towards the photon sphere at the throat. These features are qualitatively similar to the light trajectories starting at infinity in the Schwarzschild spacetime: Members of the first class turn around before they reach the photon sphere, members of the second class pass through the photon sphere and proceed to the horizon. The borderline trajectories are those that spiral towards the photon sphere.

It is also possible to calculate the deflection angle, Eq.(6) using the familiar proper form of the drainhole metric, Eq.(52), considered in Ref.[45]. The minimum surface area 4\( \pi q'^2 \) appears at the throat \( l = 0 \) which is the same as \( \rho_{ps} = \frac{q'}{2} \). The extremal situation \( \rho_{th} = \rho_{ps} = \frac{q'}{2} \) now translates into \( l_{th} = l_{ps} = 0 \) Taking the closest approach at \( l = l_0 = a \) (say), we get, for the exact deflection, an elliptic function

\[
\alpha(a) = -\pi + \frac{2\sqrt{1 + a^2} \text{EllipticK}\left[-\frac{a'^2}{a^2}\right]}{a} \tag{54}
\]

where \( \text{EllipticK}(x) \) is a particular case of hypergeometric function. With units in which \( q' = 1 \), we immediately find that \( \alpha(a) \to \infty \) (capture) as \( a \to 0 \). The plot \( \alpha(a) \) vs \( a \) in the range \( 0 < a < \infty \) again shows that the stable massless WH acts qualitatively like a Schwarzschild deflector. What is interesting is that the Keplerian mass \( M \) is zero, yet light rays coming from the source respond to this configuration and images the source. The strong deflection limit around \( a \sim 0 \) in Bozza’s formalism [8] is given by

\[
\alpha(a) \simeq -\pi - 2 \ln(a - \frac{q'}{2}) + 2 \ln(2q') \tag{55}
\]

In terms of the distance \( OL \), the same is given by \( \alpha \simeq -\pi - 2 \ln(OL) + 2 \ln(4q') \). These plots approximate the exact deflection pattern perfectly well.

**IV. BH/WHs in the brane theory**

The 5-D Weyl tensor when projected onto the brane produces a trace-free tensor \( E_\mu^\nu \) appearing in the Shiromizu-Maeda-Sasaki brane field equations [46]

\[
G_\mu^\nu = -\Lambda_4 \delta_\mu^\nu - \kappa_4^2 T_\mu^\nu - \kappa_5^4 \Pi_\mu^\nu - E_\mu^\nu \tag{56}
\]

\[
\Pi_\mu^\nu = \frac{1}{2}[T_\mu^\alpha T_\nu^\alpha - T T_\mu^\nu - \delta_\mu^\nu (T^{\alpha \beta} T_{\alpha \beta} - \frac{1}{2} T^2)] \tag{57}
\]

\[
\Lambda_4 \equiv \frac{1}{2} \kappa_5^2 \left( \Lambda_5 + \frac{1}{6} \kappa_5^2 \lambda^2 \right) \tag{58}
\]

\[
\kappa_4^2 \equiv 8\pi G_N \equiv \kappa_4^2 \lambda / 6\pi \tag{59}
\]

where \( G_N \) is the Newtonian Gravitational constant (we had earlier put \( 8\pi G_N = 1 \)), \( \Lambda_4 \) and \( \Lambda_5 \) are, respectively, the 4-D and 5-D cosmological constants, \( \lambda \) is the brane tension. Visser and Wiltshire [47] worked out an algorithm for finding solutions when matter fields are present (\( T_\mu^\nu \neq 0 \)) on the brane. To separate the observable effects of pure bulk gravity from those due to ordinary matter on the brane, we set \( T_\mu^\nu = 0 \). As we are interested in the local self-gravitating objects, we can ignore the
cosmological $\Lambda_4$ term. The trace of the vacuum brane field equations then simply gives $R = 0$ where $R$ is the Ricci scalar. This equation is solved to derive different classes of brane world BH/WHs.

The general class of 4-D solutions is given by the metric of the form (2) where $A(x)$ and $B(x)$ are two well behaved positive functions for $x > x_o$ and have a simple zero at $x = x_h$ defining the horizon. The singularities, if any, of the BH solutions when propagated off the brane into the 5-D bulk may make the AdS horizon singular (“black cigar” [48]). However, several classes of nonsingular, static, spherically symmetric solutions of the brane world model have been proposed almost simultaneously by Germani and Maartens [49] and by Casadio, Fabbri and Mazzacurati (GMCFM) [50] and some quantum properties have also been investigated [51]. Under certain assumptions on the behavior of the metric functions, Bronnikov, Melnikov and Dehnen [52] have shown that the generic solutions can have $R \times S^2$ topology of spatial sections. Assuming asymptotic flatness at large $x$, the global causal structure of such solutions coincides with a section of the Kerr-Newman nonextremal solutions. We shall specifically consider below two important GMCFM classes (I and II) of solutions.

A. GMCFM I solution

In the units such that $2m = 1$, the metric components are [49,50]:

$$A(x) = \left(\lambda + \lambda^2 \sqrt{1 - \frac{1}{x}}\right)^2, \quad B(x) = \left(1 - \frac{1}{x}\right)^{-1}, \quad C(x) = x^2$$

(60)

in which $\lambda, \lambda$ are arbitrary constants. Lensing in this spacetime has already been investigated by Whisker [37], but some additional observations seem to be in order. This is actually a self-dual solution of $R = 0$ spacetimes with two asymptotic regions. For different domains of the constants, this solution represents a Schwarzschild BH, naked singularity and traversable wormholes [53]. Only for $\lambda = 0$ and $\lambda = 1$, we have a Schwarzschild BH. In isotropic coordinates $x = \rho(1 + \frac{1}{4\rho})^2$, the above metric becomes

$$A(\rho) = \left[\lambda + \lambda^2 \left(\frac{1 - \frac{1}{4\rho}}{1 + \frac{1}{4\rho}}\right)^2\right], \quad B(\rho) = \left(1 + \frac{1}{4\rho}\right)^4, \quad C(\rho) = B(\rho)\rho^2$$

(61)

The equation $A(\rho_s) = 0$ gives $\rho_s = \frac{1 + \lambda - \sqrt{\lambda^2 + 4}}{2\lambda}$. But at $\rho = \rho_s$, there appears a naked singularity as can be seen from the following

$$\rho_D = 0, \quad p_\rho = -\frac{2816\lambda\rho^3}{(1 + 4\rho)^6 \sqrt{A(\rho)}}, \quad p_\perp = \frac{1408\lambda\rho^3}{(1 + 4\rho)^6 \sqrt{A(\rho)}}$$

(62)

where $\rho_D$ is the density, $p_\rho$, $p_\perp$ are the radial and cross radial components of pressure. The equation of state is of the so-called “dark radiation” given by $\rho_D - (p_\rho + 2p_\perp) = 0$. To get into Whisker’s notation, one has only to identify $\lambda = 1 + \epsilon$ and $\lambda = -\epsilon$ so that $\rho_s = \frac{1 + 4\epsilon}{2\epsilon}$. Due to the negative sign before $p_\rho$, Averaged Null Energy Condition (ANEC), which is the weakest, is violated. The metric (61) then represents a traversable symmetric WH with the throat occurring at $\rho = \frac{1}{2\epsilon}$. In order that $\rho_h > \rho_s$, we must have $\epsilon < \frac{1}{4\epsilon}$. This is the condition for traversability though this condition is not strictly needed in the strong field lensing calculation because the light rays are assumed to travel only up to the photon sphere and not up to the throat. Since both the throat and the naked singularity are hidden below $\rho_p$, we have here a situation like the EMS case investigated above.

However, it can be shown [27] that the total amount of ANEC violating matter is $\Omega_{ANEC} = -\lambda \ln \rho |_{\rho_h}$ which diverges logarithmically with $\rho$. Therefore, unless some technical modifications to the solution are made (e.g., as in Ref.[54]), the only way to remove this divergence is to set $\lambda = 0$ which then produces the trivial Schwarzschild BH solution. However, this divergence is not a problem as the total gravitating mass is positive and finite as explicit calculations will show in Sec.VC.
B. GMCFM II solution

There another solution, described below, that also represents Schwarzschild BH, naked singularity and traversable WHs which in the unit $2m = 1$ has the form $[49,50]$.

$$A(x) = 1 - \frac{1}{x}, \quad B(x) = \frac{(1 - \frac{3}{4x})}{(1 - \frac{1}{2})(1 - \frac{\eta}{x})}, \quad C(x) = x^2$$

The nature of the solution depends on various choices of a constant adjustable parameter $\eta$ interpreted as a bulk induced “tidal charge” - a Weyl tensor projection from the 5-D bulk into the brane. The 4-D effective stress tensor components are:

$$\rho_D = \frac{4a - 3}{x^2(4x - 3)^2}, \quad p_x = -\frac{4a - 3}{x^2(4x - 3)^2}, \quad p_{\perp} = \frac{(2x - 1)(4a - 3)}{x^2(4x - 3)^2}$$

For further interesting aspects of this spacetime, see Sec.V below. The horizon appears at $x_h = 1$. The spacetime structure depends on the parameter $\eta = a - \frac{3}{4}$. Let us state the various cases $[52]$: (i) If $\eta < 0$ or $0 < a < \frac{3}{4}$, the structure is that of a Schwarzschild BH with a spacelike singularity at $x_s = \frac{3}{4}$. (ii) If $\eta > 0$ or $\frac{3}{4} < a < 1$, then the solution describes a non-singular BH with a WH throat at $x_{th} = a$. The causal structure is that of the $(1 + 1)$ dimensional subspace of a nonextremal Kerr BH solution. (iii) If $a = 1$, then we have a double horizon at $x_h = 1$ with a timelike curvature singularity at $x_s = \frac{3}{4}$. The global structure is that of an extreme Reissner-Nordström BH and finally (iv) The range $a > 1$ corresponds to a symmetric traversable WH with its throat occurring at either $x_{th} = 1$ or $x_{th} = a$. For $a = \frac{3}{4}$, one recovers the Schwarzschild solution.

The photon sphere and the minimum impact parameter are given by

$$x_{ps} = \frac{3}{2}$$

$$u_{ps} = \frac{3\sqrt{3}}{2}$$

which are independent of the tidal charge $a$. Thus, so long as $a \leq 1$ as in the cases (i)-(iii), we see that the photon sphere covers the surfaces of event horizon or singularities. In case (iv), too, the same situation occurs if $1 < a < \frac{3}{2}$. Thus, all physically meaningful solutions satisfying the conditions (a)-(c) of Sec. IIA are contained in the range $0 \leq a < \frac{3}{2}$. We can not take $a \geq \frac{3}{2}$ because in this case, the WH throat radius $x_{th} = a$ exceeds that of the photon sphere $x_{ps}$. The relevant coefficients work out to:

$$\alpha_1 = 2 - \frac{3}{x_{ps}}$$

$$\beta_1 = \frac{3}{x_{ps}} - 1$$

$$R(0, x_{ps}) = \sqrt{\frac{3 - 4x_{ps}}{a - x_{ps}}}$$

$$\bar{\alpha} = \left(\frac{1}{2}\right) \sqrt{\frac{x_{ps}(3 - 4x_{ps})}{(a - x_{ps})(3 - x_{ps})}}$$

$$\bar{b} = -\pi + b_R + \frac{1}{2} \sqrt{\frac{x_{ps}(3 - 4x_{ps})}{(a - x_{ps})(3 - x_{ps})}} \ln \left[\frac{6 - 2x_{ps}}{x_{ps} - 1}\right]$$

$$b_R = \int_0^1 g(z, x_{ps})dz + O(x_0 - x_{ps})$$

$$g(z, x_{ps})dz$$
\[ g(z, x_{ps}) = -\sqrt{6} \frac{\sqrt{z}}{z\sqrt{3 - 2a}} + 3\sqrt{2} \frac{z + 1}{z(3 - 2z)(3 + 2a(z - 1))} \] (73)

All the quantities above are well defined for \( a < \frac{3}{2} \). The integral \( b_R \) has no divergence on \([0, 1] \) but its analytic evaluation in closed form is rather unwieldy. However, we can easily expand \( g(z, x_{ps}) \) in powers of \( z \):

\[ g(z, x_{ps}) = b_1(4a + 3) + \frac{b_1(64a - 30a^2 - 33)}{8(3 - 2a)^2} z + O(z^2) \] (74)

where \( b_1 = 3\sqrt{2}(9 - 6a)^{-\frac{1}{2}} \) which shows that \( g(z, x_{ps}) \) is perfectly regular at \( z = 0 \). Since the solution under consideration resembles that of Schwarzschild in many ways, especially, the photon sphere appears exactly at the same value, we can, up to a good accuracy, consider photon orbits for \( u - u_{ps} = 0.003 \). We can then find the corresponding value of \( z \) by employing the expression [8]

\[ u - u_{ps} = c(x_0 - x_{ps})^2 \] (75)

where \( c \) is a constant. It turns out that

\[ c = \beta_{ps} \left[ A \sqrt{C^2 - 2(1 - A)^2} \right]_{x=x_{ps}} = 1 \] (66)

and so \( x_0 = 1.554 \). From the definition that \( z = 0 \) at \( x_0 = x_{ps} \), we can write

\[ z = \frac{A(x_0) - A(x_{ps})}{1 - A(x_{ps})} \] (77)

which gives \( z = z_{min} = 0.035 \) corresponding to \( u - u_{ps} = 0.003 \). By a Taylor expansion around the Schwarzschild value \( a = \frac{3}{4} \), we now obtain

\[ b_R = \int_{z_{min}}^{1} g(z, x_{ps}) \mid_{a=\frac{3}{4}} dz + \left( a - \frac{3}{4} \right) \int_{z_{min}}^{1} \frac{\partial g}{\partial a} \mid_{a=\frac{3}{4}} dz + O \left( a - \frac{3}{4} \right)^2 \] (78)

Therefore

\[ b_R \approx 0.9496 - \left( a - \frac{3}{4} \right) \times 1.565 + O \left( a - \frac{3}{4} \right)^2 \] (79)

The neglected higher order terms are smaller due to the gradually diminishing factors in the powers of \( (a - \frac{3}{4}) \) for \( 0 \leq a < \frac{3}{2} \). The deflection \( \alpha(x_0) \) as a function of the closest approach distance \( x_0 \) now works out to

\[ \alpha(x_0) = b_R - \pi + \frac{1}{2} \ln \left[ \frac{(18 - 6x_0)\sqrt{3(x_0 - 1)}}{2(x_0 - 1)\sqrt{x_o^2 - 3(x_0 - 1)}} \right] \] (80)

where

\[ \Omega \equiv \sqrt{\frac{x_0(3 - 4x_0)}{(3 - x_0)(a - x_0)}} \] (81)

Using the Schwarzschild value \( \theta_{\infty} = 16.87 \mu \text{ arcsec} \), the expressions for \( \pi, \beta, r, s \) and \( u \) as a function of closest approach \( x_0 \) turn out to be

\[ \pi = \frac{\Omega}{2} \] (82)

\[ \beta = b_R - \pi + \frac{\Omega}{2} \ln \left[ \frac{6 - 2x_0}{x_0 - 1} \right] \] (83)
\[ r = \exp \left[ \frac{4\pi}{\Omega} \right] \]  
\[ (84) \]

\[ s = -\left[ \frac{33.74(x_0 - 3)}{x_0 - 1} \right] \exp \left[ \frac{2(b_R - 3\pi)}{\Omega} \right] \]  
\[ (85) \]

\[ u = \sqrt{\frac{x_0^2}{x_0 - 1}} \]  
\[ (86) \]

From the above expressions, it is evident that, for the tidal charge value \( a \simeq \frac{3}{2} \), the values for \( \alpha(x_0) \), \( \pi, \bar{b}, r \), and \( s \) differ significantly from other choices of \( a \) within the chosen range, especially near the photon sphere, \( x_0 \simeq x_{ps} \). At \( x_{ps} = \frac{3}{2} \), the relevant expressions become

\[ \pi = \sqrt{\frac{3}{6 - 4a}} \]  
\[ (87) \]

\[ \bar{b} = -\pi + 2.123 - 1.565a + 2.194\sqrt{\frac{1}{3 - 2a}} \]  
\[ (88) \]

\[ r = \exp \left[ 2\pi \left( \frac{\sqrt{3}}{3}(3 - 2a) \right) \right] \]  
\[ (89) \]

\[ s = 101.22 \times \exp \left[ (-7.301 - 1.565a) \sqrt{\frac{6 - 4a}{3}} \right] \]  
\[ (90) \]

Defining \( u = \theta D_{OL} \), the deflection angle \( \alpha(\theta) \) can be rewritten as

\[ \alpha = -\pi \ln \left( \frac{u - u_{ps}}{u_{ps}} \right) + \bar{b} + O(u - u_{ps}) \]  
\[ (91) \]

that works out to

\[ \alpha(a) = -\pi + 2.123 - 1.565a + 10.478\sqrt{\frac{1}{3 - 2a}} \]  
\[ (92) \]

The values of the observables are tabulated below. We see that the values of \( \pi, \bar{b} \) continue to increase from the Schwarzschild values \( (\pi = 1, \bar{b} = -0.4009) \) as we increase the tidal charge. We also observe that the deflection angle \( \alpha(a) \) increases from the Schwarzschild value as the tidal charge is increased as opposed to the decrease caused by ordinary scalar fields (e.g., JNW scalar field) [8]. This difference due to the tidal charge \( a \) is particularly manifest in the WH region corresponding to \( 1 < a < \frac{3}{2} \). For \( a \sim \frac{3}{2} \), the deflection angle \( \alpha(a) \) increases more than three times compared to the value \( \alpha(\frac{3}{2}) \) for the Schwarzschild BH. Such behavior could be interpreted as a signature for a WH as well as effect of the extra dimension or tidal charge. The behavior of the observables \( r \) and \( s \) too are very different from the Schwarzschild BH (or the JNW scalar field configuration) for different values of \( a \), especially at \( a \sim \frac{3}{2} \).

Table I
V. Some features of the WH solutions

A. Massive EMS WH

The EMS solutions (38,39) correspond to an equation of state \( p_D + p_p + 2p_\perp = 0 \) for the WH case \( \gamma > 1 \) since \( p_D = p_p \) and \( -p_p = p_\perp \). The equation is the limiting case of the dark equation of state \( p = w \rho \) where \( w < -\frac{1}{3} \). (The phantom equation of state is more stringent as it requires \( w < -1 \) which is certainly not the case here.) The first observation is that the total asymptotic gravitating mass \( M = m\gamma \) is positive. It can be calculated in various ways: by the ADM calculation [24] or from the Einstein energy complex or even directly from the Eddington-Robertson expansion of the centrally symmetric metric in isotropic coordinates [55]

\[
\begin{align*}
\alpha \quad (\text{rad.}) & \quad 6.36 & 6.53 & 6.72 & 6.87 & 6.96 & 6.58 & 7.89 & 10.63 & 20.22 \\
\beta \quad (\mu \text{arcsec}) & \quad 16.87 & 20.56 & 26.02 & 31.36 & 35.00 & 16.87 & 16.87 & 16.87 & 16.87 \\
\sigma \quad (\mu \text{arcsec}) & \quad 0.0211 & 0.0205 & 0.0197 & 0.0189 & 0.0185 & 0.0261 & 0.0726 & 0.3047 & 3.1618 \\
r_m \quad (magn.) & \quad 6.82 & 6.82 & 6.82 & 6.82 & 6.82 & 6.59 & 5.57 & 4.31 & 2.50 \\
\frac{u_{ps}}{r} & \quad 2.59 & 3.16 & 4.00 & 4.82 & 5.38 & 2.59 & 2.59 & 2.59 & 2.59 \\
\frac{\delta}{r} & \quad -0.4009 & -0.4292 & -0.4692 & -0.5073 & -0.5321 & -0.4163 & -0.3895 & -0.0641 & 1.6963
\end{align*}
\]

Thus, the scalar field effect is already contained in the metric functions \( A(\rho), B(\rho) \) in terms of \( M = m\gamma \). This mass \( M \) is the gravitating mass and the test particles respond to it \textit{per se}; there is in fact no way of measuring the bare \( m \) if a scalar field gravitationally couples to it. The Eddington-Robertson parameters for the Buchdahl solution (38) are \( \alpha_1 = \beta_1 = \gamma_1 = 1 \) and the post-PPN parameter \( \delta_1 = \frac{4}{3} - \frac{1}{3\gamma} \). The Buchdahl PPN parameters \( \alpha_1, \beta_1, \gamma_1 \) are exactly the same as those in the Schwarzschild solution and at this level EMS theory is indistinguishable from it. However, the deviation appears at the post-PPN level and only finer and second order deflection measurements can reveal the value of \( \delta_1 \). It is known that \( \delta_1 = 1 \) (or, \( \gamma = 1 \)) corresponds to Schwarzschild solution while \( \delta_1 \neq 1 \) would indicate a genuine deviation from it. The second order effect in deflection (\textit{albeit} still in the weak field) can be easily calculated by using the metric (93) involving \( M \) and \( \delta_1 \) and Eq.(6).

However, due to the nonlinearity of the field equations, the total amount of WEC violating scalar matter \( \Omega_{AWEC} \) in spacetime is slightly different from \( -q \) as a result of the generalized Gauss theorem in curved spacetime. The exact difference can be seen from the volume AWEC integral, which, for two sides of the WH becomes

\[
\Omega_{AWEC} = 2 \times \left( \frac{1}{8\pi} \right) \int_0^{2\pi} \int_0^{\pi} \int_0^{\rho_h} \rho_D \sqrt{-g} \sin \theta d\rho d\theta d\varphi
\]

\[
= -m(\gamma^2 - 1) \ln \left( \frac{\gamma + 1}{\gamma - 1} \right) \approx -m \sqrt{\gamma^2 - 1} \left( 1 - \frac{1}{2\gamma^2} \right)
\]
As $\gamma \to 1$ (Schwarzschild case), $\Omega_{AWEC} \to 0$, as expected. Using the WH range, $\gamma > 1$, we see that $\Omega_{AWEC} < 0$. To express $\Omega_{AWEC}$ in terms of scalar charge $q$, recall the two cases (i) and (ii) discussed in Sec. IIIA: If we take $\kappa = -2$, $\Phi$ real or $q = m\sqrt{\gamma^2 - 1}$, then $\Omega_{AWEC} \simeq -q(1 - \frac{1}{\gamma^2})$, with $q > 0$ while for $\kappa = +2$, $\Phi$ imaginary or $q = im\sqrt{\gamma^2 - 1}$, one has $\Omega_{AWEC} \simeq -iq\left(1 - \frac{1}{\gamma^2}\right)$. If we integrate from $\rho_{ps}$ to $\infty$, we get $\Omega_{AWEC} \simeq -q\left(1 - \frac{1}{8\gamma^2}\right)$ and similarly the imaginary version. The positive correction term proportional to $q\gamma^2$ slightly diminishes the quantity $\Omega_{AWEC}$ away from the value $-q$, but this is due to nonlinear effects. For either of the two values of $\kappa$ and $q$, we see that $M^2 = m^2 + q^2 = m^2\gamma^2$. The situation is the following: the WH is attractive and test particles, after being pulled into the throat from both the mouths, attain zero acceleration there. They can re-emerge into the other mouth by maintaining extra outward radial accelerations from being pulled in again [56]. However, light rays are captured at $\rho = \rho_{ps}$ but rays that pass close to it suffer higher deflection angles than those due to Schwarzschild BH.

B. Massless EMS WH

The zero mass WH with its metric given by Eqs.(51) or (52) is a stable configuration (see Ref.[41] for details). The scalar field satisfies, to first order in $\Phi(\simeq q^2 \rho)$, a sourceless equation $\frac{\partial^2 \Phi}{\partial \rho^2} = 0$, yet the observers at a finite asymptotic $\rho$ measures a flux $4\pi q^2$ of the scalar charge though, in reality, there is no source. That is why the configuration is called a “charge without charge”.

C. GMCFM I solution

The GMCFM I solution is well discussed in the literature. Its asymptotic physical mass can be found from the Einstein complex as follows [56]

$$M = \text{Lim}_{\rho \to \infty} \frac{\rho A(B - 1)}{\sqrt{2AB}}$$

which works out to $M = m(\kappa + \lambda)$ which is finite and positive. The WH is sustained entirely by the negative pressures as $\rho_D = 0$. With the identifications $\lambda = 1 + \epsilon$ and $\kappa = -\epsilon$, we immediately find that $M = m$. Therefore, energetically, it is still like the same Schwarzschild spacetime while, kinematically, the null geodesics reveal that the strong field behavior is different from that of the Schwarzschild, as the observables obtained by Whisker [37] show.

D. GMCFM II solution

The GMCFM II solution exhibits certain remarkable features. Let us suspend the unit $2m = 1$ and restore $m$ for better comparison. First, we see that the constituent matter is that of dark radiation given by $\rho_D - (\rho_x + 2\rho_\perp) = 0$. Second, it is impossible to ascertain the bulk effect directly from $A(x)$ as it does not contain $a$. Therefore, we proceed as follows. The integration of the Einstein complex of energy gives the asymptotic physical mass of the solution [57]:

$$M = \frac{1}{4}(2a + m)$$

which contains the bulk effect. From the Newtonian limit of $g_{00} = A(x) = 1 - \frac{2m}{x}$, the Keplerian mass is always $m$ but the asymptotic mass $M$, except in the special case $a = \frac{3}{2}m$, is different. This feature is unlike the Buchdahl solution where both masses are the same $M = m\gamma$. When we interpret Eq.(96) as a relation of the type of Eq.(35)

$$M^2 = \frac{1}{16}(m^2 + 4a^2 + 4am)$$

15
we see that, unlike the EMS case, there is an extra interaction term $4am$ between the mass and the Weyl charge contributing to the total mass $M$. We see that the Weyl charge is additive to $m$, unlike the scalar charge. Therefore, $a$ has the dimension of mass. Expressing $a$ in units of $M$ such that $a = \varepsilon M$, and using again the PPN expansion in standard coordinates for a central mass $M$:

$$ds^2 = \left(1 - \frac{2\alpha_1 M}{x} + \frac{2(\beta_1 - \alpha_1 \gamma_1)M^2}{x^2} + \ldots\right)dt^2 - \left(1 + \frac{2\gamma_1 M}{x} + \frac{4\delta_1 M^2}{x^2} + \ldots\right)dx^2 - x^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

(98)

and identifying the metric (63) with it, we find that $\alpha_1 = \beta_1 = 2(2 - \varepsilon)$, $\gamma_1 = 1$, $\delta_1 = \frac{1}{2}(2\varepsilon^2 - 7\varepsilon + 8)$. At $\varepsilon = \frac{3}{7}$, one recovers the Schwarzschild values $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 1$, as expected. When $a = \frac{3}{2}M$, we have $M = m$ from Eq.(96). It turns out that first order tests can provide a value of $\varepsilon = \frac{3}{7}$ and can still be indistinguishable from GR, but (weak field) second order deflection would check if $\delta_1 = 1$. If the measured value of $\delta_1$ differs from unity, then the brane model of stars would stand as a possible contender to that in the EMS theory.

The total amount of energy in spacetime due of bulk stress is

$$\Omega_{T,total} = 2 \times \left(\frac{1}{8\pi}\right) \int_0^{2\pi} \int_0^{\pi} \int_{\frac{3m}{2}}^{\infty} \rho_D \sqrt{-g} \sin \theta dx d\theta d\varphi = m - \sqrt{\frac{2am}{3}}$$

(99)

which is negative for the WH range $\frac{3m}{2} < a < 3m$. (Recall that $x_{th} = a$ lies below $x_{ps} = 3m$). The integral is zero either for $a = \frac{3}{2}m$ or, for $a = \frac{3}{2}M$ since, in both cases, $m = M$. The AWEC by definition is

$$\Omega_{AWEC} = 2 \times \left(\frac{1}{8\pi}\right) \int_0^{2\pi} \int_0^{\pi} \int_{x_{th} = a}^{\infty} \rho_D \sqrt{-g} \sin \theta dx d\theta d\varphi = m$$

(100)

which is independent of $a$! This shows that the entire negative energy $-\sqrt{\frac{2am}{3}}$ is concentrated below the throat $0 < x < a$. However, there are some restrictions on the values of $a$. The pointwise WEC and the AWEC are satisfied here for $a > \frac{3m}{2}$. For $a < \frac{3m}{2}$, WEC is violated as $\rho_D < 0$ but $\Omega_{AWEC} = -\infty$ which is unphysical. The energetics of the brane model requires further study but we see that the situation is very unlike the massive EMS WHs in which both WEC and AWEC are violated for $\gamma > 1$, the value of $\Omega_{AWEC}$ being proportional to the scalar charge $-q$, as shown before.

Third, for $a < \frac{3m}{2}$, we find that the gravitating mass $M$ is decreased from the Schwarzschild value $m$, the latter occurring at $a = \frac{3m}{2}$. For $a > \frac{3m}{2}$, we find that $M > m$, which suggests that the presence of the positive Weyl charge $a$ strengthens the attractive force beyond that due to the Schwarzschild BH. This explains why there is an enhancement in the deflection angle. The two surfaces $x_{ps} = 3m$ and $x_{th} = a$ coincide when $a = 3m$. At this extremal situation, there occurs photon capture as the divergence in the deflection angle $\alpha(a)$ at $a = 3m$ (which is the same as $a = \frac{3}{2}$) show in Eq.(92). At nonextremal situations that we have considered, the throat lies below $x_{ps}$, and the light rays do not reach the throat. We are interested, as mentioned before, in the range $\frac{3m}{2} < a < 3m$ and not in the range $a \geq 3m$ as, in this case, it is the throat that covers the photon sphere, not the other way around.

Finally, fourth, the WEC ($\rho_D \geq 0$) is locally preserved for $a > \frac{3m}{2}$ and so is AWEC which we saw to be independent of $a$. What about the ANEC violation? Let us consider the volume ANEC integral

$$\Omega_{ANEC} = 2 \times \left(\frac{1}{8\pi}\right) \int_0^{2\pi} \int_0^{\pi} \int_{x_{th} = a}^{\infty} (\rho_D + p_x) \sqrt{-g} \sin \theta dx d\theta d\varphi$$

(101)

$$= -\int_a^{\infty} \frac{2(x - 2m)(2a - 3m)}{x^2(2x - 3m)^2} \sqrt{-g} dx$$

(102)
which contains only the radial pressure $p_r$. (We have not included the transverse components of pressure as inequalities associated with $p_{\perp}$ refer only to ordinary matter [54]). That is, the WH is maintained by negative radial pressure [58]. Now, pointwise NEC violation $\rho_D + p_r < 0$ occurs when $\frac{3m}{2} < a < 3m$, $2m < x < \infty$. Unfortunately, the ANEC integral diverges logarithmically on $[a, \infty)$. Such a divergence seems to be a generic feature of $R = 0$ traversable WHs [27]. However, it turns out that $\Omega_{ANEC} < 0$ for $a < x \leq N$ where $N$ is any arbitrarily large but finite number. But as $N$ increases to infinity, so does $\Omega_{ANEC}$ though not as rapidly. Thus, to have a reasonable WH with finite amount of ANEC violating matter, one perforce needs to join the WH to the exterior vacuum Schwarzschild spacetime at a certain value of the coordinate radius $x$, as is actually done with self-dual $R = 0$ WHs [54]. This peculiarity distinguishes the ANEC violation due to the bulk effect from the violation due to sign reversed scalar field. In the latter case, the $\Omega_{ANEC}$ tapers off smoothly at the asymptotic region.

VI. Summary

Gravitational lensing by WHs in the strong field limit is a new possibility that has not been explored so far though pioneering theoretical works on lensing by black holes or naked singularities exist in the recent literature. On the other hand, WH solutions are physically important, many of their properties have been widely discussed and applied to interpret several outstanding problems in astrophysics [59-62]. It is thus only natural that their analytic lensing properties be investigated as well. Several static, spherically symmetric WH solutions are known, both in the EMS theory and in the brane theory. Some of the brane world solutions represent a synthesis of BH and WH spacetimes thereby providing a more advanced and richer premise for the strong field lensing analysis. We have undertaken a moderately comprehensive investigation here. Certain intrinsic features of the lensing objects in question are also analyzed.

The sign reversed kinetic term in Eq.(30) yields regular, symmetric WH solutions for the range of values $\gamma > 1$ (The two options for $\kappa$ have been spelled out in the text.) For this range, the presence of the scalar charge increases the Schwarzschild mass $m$, that is, the ADM mass $M = m\gamma > m$. The WH throat surface lies inside the photon sphere which has been stipulated to play the limit of the strong field. That is, we have been considering situations in which light rays pass very close to the exterior of the photon sphere but obviously do not reach the throat. It was shown that massive WHs in the EMS theory ($M \neq 0$) produce significantly different values of deflection angles, and other observables as tabulated in Table I. In contrast, in the case of naked singularity, $\Phi$ real, $\kappa > 0$, $\gamma < 1$, there is a decrease from the Schwarzschild mass, that is, $M < m$. This explains why, in this case, the deflection angles always show lesser values than those in the Schwarzschild case [8]. The strong field lensing results thus show that the EMS scalar field exerts stronger gravitational pull to light than that by the Schwarzschild BH. It should however be remarked that an increase in ADM mass does not generally imply an increase in the deflection angle. This follows from the fact that the bending features are unaffected by a conformal factor whereas a conformal factor does change the ADM mass.

The massless WH ($M = m\gamma = 0$) corresponds to $\gamma = 0$ but $m \neq 0$. This is just a mathematically admissible possibility. The spacetime is asymptotically flat at the two mouths. Lensing by these objects is interesting due to the fact that it can reveal the presence of the geometric curvature caused by the scalar field alone. It turns out that such configurations also possess a photon sphere and behave like ordinary deflectors.

The GMCFM I solution is treated as a brane world BH in the literature [37], but it is actually a traversable WH. One recalls [52] that such class of solutions can be a BH only when it is trivially Schwarzschild. Otherwise, it is either a naked singularity or a WH. However, the throat radius $x_{th} = 2m$ is hidden under the photon sphere justifying the application of the strong field analysis. The GMCFM II solution has been investigated in detail here. Though the minimum impact parameter $u_{ps}$ is exactly the same as that in Schwarzschild BH, the spacetime itself is intrinsically very different.
Table I affords a comparison of the values of strong field observables for WHs with those of Schwarzschild BH. It shows that the separation ($s$) between the first image and the rest increases from that due to Schwarzschild BH with increasing $\gamma$ ($>1$) but the increase is more spectacular in the brane world WHs, especially in the region $a \sim 1.4$. This suggests that the outermost image would be better visible in this case. The flux ratio ($r_m$) or relative magnification always remains the same in the EMS WHs but is considerably more than those in the brane world WHs. These features are peculiar enough to observationally distinguish the lensing sources under consideration.

Lensing phenomena in the WH environment offers a good possibility that one might detect the presence not only of a WH, which is by itself interesting, but also of the presence of naturally occurring exotic matter much advocated on galactic or cosmological scales. VLBI observations of a clean system that is devoid of intervention by accretion phenomena, can help us pick up the right model or at least set limits on the observables. Still, it is not unlikely that observations will favor just the usual Schwarzschild BH more than any other advanced generalized solution considered here. Again, we note that several astrophysical phenomena (like $\gamma$-ray bursts) can also be explained by invoking new inputs (like negative energy fields or exotic matter) [62]. Thus, assuming that the center of our galaxy hosts, instead of a BH, a WH threaded by exotic matter, situated at a distance $D_{OL} = 8.5 \text{ kpc}$ from the center of the Sun, then the angular position of the set of relativistic images in the limit $n \to \infty$ would be $\theta_\infty \sim 17 \mu\text{arcsec}$. We have used this value as a basis for calculating $s$ in Table I. However, due to considerable demagnification of relativistic images, one would need a resolution of the order of $0.01 \mu\text{arcsec}$ and if this refinement is technologically attained in future, then the observational limits can either accommodate or rule out WH candidates as possible lensing sources.

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