Analytical design of control system mathematical models for mobile robots based on the methods of inverse problems of dynamics and modal PID controllers

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Abstract: Mathematical models of a three-wheeled mobile robot based on the variable state apparatus and in the operator's form are presented. Based on these mathematical models, the synthesis of its adaptive control system is carried out according to the method of constructing Ziegler-Nichols PID controllers, as well as modal PID controllers. The design method of proportional controllers with double differentiation (PDD) of the autonomous robot is justified. The rules of wheel pair control are synthesized on the basis of reference models using a simple gradient scheme of the inverse problems method of dynamics in the formulation of P. D. Krutko for the problem of stabilizing (blanking) the angular velocities for a three-wheeled mobile robot. Simulink-models of robot movement are developed for cases with the use of PID-controllers which are adjusted by the Ziegler-Nichols method at synchronous deviation of wheels. Simulink-models of robot motion have been developed for cases using PID controllers set up according to the Ziegler-Nichols method for synchronous wheel deflection, modal and real PID controllers for differential wheel deflection, as well as double differentiation controllers. The analysis of modeling results is considered.

1. Introduction

There are hardly more scientific publications in the field of the theory and practice of automatic control as compared to those devoted to PID controllers [1, 2] and various correction devices. They are the most widely used industrial controllers as they can provide high control efficiency despite the different dynamic characteristics of an object. In the world about 90 ... 95% of controllers [3, 4], which are currently in operation, use PID algorithms. Nevertheless, the number of journal publications on the theory of PID controllers constantly increases [5-10]. The reason for this is the fact that the problem of synthesizing even ordinary industrial controllers still hasn’t been solved at the level of modern requirements.

At the same time, it is desirable to have any regular procedure for the synthesis of an automatic control system that would allow, given the known structure of the control algorithm, obtain specific laws for changing the parameters of the PID controller or a synthesized correction device based on the results of an experimental study of the time characteristics of an object’s free movement. The most formalized procedures for the synthesis of PID controllers and their comparative analysis on the example of solving a navigation problem (the method of inverse problems of dynamics, the criteria of the optimal module and symmetric optimum, algebraic method) are presented in the works [11, 12]. Modern methods of tuning the parameters of PID controllers based on microcontrollers and ADCs are carried out according to the Ziegler-Nichols and Chien-Hrones-Reswick methods [13, 14], but have a number of significant disadvantages [1, 3, 4].

Despite the long history of development and a large number of publications on the theory of PID controllers, there are still problems of objects control with hysteresis and nonlinearities, automatic tuning and adaptation under the conditions of high-frequency noise. The problems are complicated by the fact that the dynamics of the object is often unknown. The object is multi-linked itself (there
are cross-links between control loops) and measurements are very noisy (filtering of signals is necessary).

In the article on the example of object with one input and one output (e.g., technological modules in the APCS [15-18] and a mobile robot with a synchronous deviation of wheels) is studied modal method of determining the PID controller parameters based on the principle of dynamic compensation [1, 19]. It is required to determine the parameters of the PID controller according to the experimental characteristics of the free movement of the object so that the characteristics of the system closed by the controller correspond to the desired properties of the aperiodic link with the minimum control time. It is assumed that the control object does not contain any non-reducible zeros.

On the example of a multidimensional multiply-connected system - a mobile three-wheeled robot using the inverse problems method of dynamics (REM), a double differentiation controller (PDD controller) is investigated.

Thus, the task is posed to search for the parameters of industrial controllers that are self-adjusting according to the operating modes of a dynamic object. The main idea of this research is design the controller in a way when Modal control zeros approximately corresponded to poles of the approximated object with first and/or second-order polynomials, and at control on REM the trajectory of movement of the robot would precisely follow in the course of time on the appointed trajectory defined by the right parts of reference models (RM).

2. Dynamic model of the mobile robot

To describe the behavior of a mobile robot, it is necessary to develop its mathematical model. This will allow to analyze the work of the control system being created without hardware implementation of the mobile robot and it will allow to make necessary corrections. The mathematical model of the mobile robot consists of kinematic and dynamic models. The kinematic model is the simplest description of the mobile robot's behavior and it allows studying its properties. The dynamic model is a more detailed description of the mobile robot and it takes into account the force moment-related effects produced by the actuators.

The following mathematical description of the mobile robot movement is valid for the case of the robot movement on the horizontal plane.

In order to describe the mobile robot mathematically, it is necessary to introduce a number of conditions formulated in the form of the following assumptions [20]:

- mobile robot will be considered under the condition that its mechanism is tightly coupled;
- wheels are non-deformable and they are in point contact with the surface;
- robot moves without slippage;
- platform is considered as a solid body on which the wheel system is mounted.

Taking into account the accepted assumptions, the nonlinear mathematical model of the autonomous mobile robot is described by the system of differential equations:

\[
\begin{align*}
\dot{x} &= V \cos \phi \\
\dot{y} &= V \sin \phi \\
\dot{\phi} &= \omega \\
\dot{V} &= \frac{1}{\rho m} (\tau_R + \tau_L) \\
\dot{\omega} &= \frac{1}{2 \rho J_C} (\tau_R - \tau_L),
\end{align*}
\]  

(1)

here \(x, y, \phi\) – generalized linear and angular coordinates of the platform position; \(V, \omega\) – Linear and angular velocities of the platform; \(m\) – mass; \(\rho\) – two-wheel drive radius; \(J_C\) – the torque of
inertia of the platform relative to the center of mass; \( \tau_R, \tau_L \) – Electromagnetic moments produced by electric motors (EM).

Further, the equations of motion of the mobile robot are determined taking into account the dynamics of the servo drives.

Consider the case when the wheels of the platform through the gears are directly controlled by DC motors with independent excitation.

Suppose that the account of presence of following drives does not involve principal complexities.

It is known that the equation of voltage balance of one electric motor (EM) of a direct current made according to the II law of Kirchhoff looks like

\[
L \frac{dI}{dt} + RI + K_W \omega_W = U
\]

(2)

here: \( R \) – the inductance of the anchor circuit; \( I \) – the current flowing through the anchor winding; \( R \) – armature resistance; \( \omega_W \) – angular velocity of the EM shaft; \( U \) – armature voltage.

The angular rotation speeds of the ED shaft \( \omega_m \) and the wheels are related by the ratio

\[
\omega_m = \omega_W \frac{p_i}{p_w}
\]

(3)

here \( p_i \) – the gear ratios of the gearboxes.

The expression for the torque \( \tau \) developed by each EM is

\[
\tau = K_M I.
\]

(4)

here \( K_M, K_W \) – the given design constants of EM on the electromagnetic moment and its angular speed of rotation.

Taking into account relations (3), (4) equation (2) takes the form

\[
\tau = \frac{R}{L} I - \frac{K_M K_W}{L} \omega_W + \frac{K_M}{L} U.
\]

(5)

Since two DC EMs are used separately for the right and left wheels:

\[
\tau_L = \frac{R}{L} I - \frac{K_M K_W}{L} \omega_L + \frac{K_M}{L} U_L,
\]

\[
\tau_R = \frac{R}{L} I - \frac{K_M K_W}{L} \omega_R + \frac{K_M}{L} U_R,
\]

(6)

here \( U_L, U_R \) – Voltages to the electric circuits of the right and the left wheel.

Now let’s express angular velocities of rotation of wheels \( \omega_L \) и \( \omega_R \) through phase coordinates.

For this purpose, the following relationships are used

\[
\omega = \frac{1}{l} (V_R - V_L),
\]

\[
V = \frac{1}{2} (V_R + V_L),
\]

(7)

here \( V_L, V_R \) – linear speeds of the right and left wheels; \( l \) – axle length between wheels.

Given the fact that \( V_L = \omega_L \rho \) и \( V_R = \omega_R \rho \), it turns out

\[
\omega_R = \frac{1}{\rho} \left( V + \frac{\omega L}{2} \right),
\]

\[
\omega_L = \frac{1}{\rho} \left( V - \frac{\omega R}{2} \right).
\]

(8)

After substitution of equations (8) in equations (6), there determine the complete mathematical model of mobile robot dynamics taking into account the following drives.
\[ \dot{x} = V \cos \varphi, \]
\[ \dot{y} = V \sin \varphi, \]
\[ \dot{\varphi} = \omega, \]
\[ \dot{V} = \frac{1}{\rho m} (\tau_R + \tau_L), \]
\[ \dot{\omega} = \frac{1}{2\rho J_c} (\tau_R - \tau_L), \]
\[ \dot{\tau}_L = -\frac{R}{L} \tau_L - \frac{K_M K_w i}{\rho L} \left( \frac{V - \omega \ell}{2} \right) + \frac{K_M U_L}{L}, \]
\[ \dot{\tau}_R = -\frac{R}{L} \tau_R - \frac{K_M K_w i}{\rho L} \left( \frac{V + \omega \ell}{2} \right) + \frac{K_M U_R}{L}. \]

The model of dynamics (9) represents a system of nonlinear differential equations of the seventh order with a phase vector \((x, y, V, \varphi, \omega, \tau_L, \tau_R)\) and a control vector \((U_L, U_R)\). It can be used directly when modeling in any algorithmic programming language. However, to synthesize the laws of robot control other forms of representation of its mathematical model are necessary.

Note that in the system of equations (9) only the last four equations are dependent on each other, and the first three equations are related equations. Differentiating the sixth and seventh equations in system (9) and taking into account the substitution of the fourth and fifth equations in them,

\[ \ddot{\tau}_L + a_{11} \ddot{\tau}_L + \left( a_{12} \frac{1}{\rho m} - \frac{l^2}{4\rho J_c} \right) \tau_R + \left( a_{12} \frac{1}{\rho m} + \frac{l^2}{4\rho J_c} \right) \tau_L = b_1 \dot{U}_L, \]
\[ \ddot{\tau}_R + a_{11} \ddot{\tau}_R + \left( a_{12} \frac{1}{\rho m} + \frac{l^2}{4\rho J_c} \right) \tau_R + \left( a_{12} \frac{1}{\rho m} - \frac{l^2}{4\rho J_c} \right) \tau_L = b_1 \dot{U}_R, \]

denote: \( a_{11} = \frac{R}{L}, a_{12} = \frac{K_M K_w i}{\rho L}, b_1 = \frac{K_M}{L}, \dot{U}_L, \dot{U}_R \) — supply voltage change rate (control parameter).

Design parameters of a tricycle autonomous mobile robot (Figure 1) are presented in Table 1.

Figure 1. Robot visual appearance
Table 1. Constructive parameters of the mobile robot

| Parameter | $J_C$ | $\rho$ | $m$ | $l$ | $K_W$ | $R$ | $L$ | $i_p$ |
|-----------|-------|-------|-----|-----|-------|-----|-----|-------|
| Units of measure | kg \cdot cm$^2$ | m | kg | m | - | ohm | H | - |
| Numerical value | $25 \times 10^{-5}$ | 0.0325 | 1 | 0.135 | 0.0080 | 0.36 | $1.23 \times 10^{-3}$ | 55.74 |

The robot’s mathematical model (10) is taken as a control object. The matrix equation system (10) looks like

$$\dot{X} + CX + DX = Bu,$$

where

$$X = (\tau_L, \tau_R)$$  system condition vector;  
$$u = (\dot{U}_L, \dot{U}_R)$$  control vector;

$$C = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{11} \end{bmatrix} = \text{diag}\{a_{11}, a_{11}\}$$  diagonal matrix;  
$$B = \begin{bmatrix} b_{11} \\ b_{11} \end{bmatrix} = \text{diag}\{b_{11}, b_{11}\}$$  diagonal control efficiency matrix;

$$D = \begin{bmatrix} \frac{1}{\rho m} + \frac{l^2}{4\rho J_C} & a_{12} & \frac{1}{\rho m} - \frac{l^2}{4\rho J_C} \\ a_{12} & \frac{1}{\rho m} - \frac{l^2}{4\rho J_C} & \frac{1}{\rho m} + \frac{l^2}{4\rho J_C} \end{bmatrix}$$  symmetrical matrix ($D^{-1} = D^T$);

Thus, the mobile robot as a control object is a multi-linked (number of inputs $m = 2$), multi-dimensional ($n = 2$) system.

3. Mobile robot movement control using PID control method

To apply the technique of PID controller synthesis and Simulink-models construction in Matlab programming environment, it is necessary to bring the robot mathematical model (10) to the operator's view.

a) Common case of a robot operator's model

Suppose that in the initial model (10) the condition of synchronization of wheel movement is satisfied:  $\tau_R = \tau_L$. Then the angular velocity of the robot platform is equal to given one $\omega = \omega_{gy} = \text{const}$, and the angle of rotation of the wheels is equal to $\varphi = \alpha t$. This condition is met if the equations (10) of the robot model include

$$d_{12} = a_{12} - \frac{1}{\rho m} - \frac{l^2}{4\rho J_C} = 0.$$

There are two possible approaches to robot design:

1) Standard DC EMs are used. Then it is advisable to select the axis length of the robot’s front drive wheels from the synchronization condition

$$l = \sqrt{\frac{4K_m^2i_pJ_C}{\rho Lm}}.$$

2) The axle length of the front drive wheels $l$ is rigid and you need to select the DC EMs characteristics. Then the winding factor and the design parameters of the EM are selected from the synchronization condition

$$K_m = K_w = \frac{l^2}{4i_pJ_C}.$$
\[ \dot{U} + a_1 U = \left(\frac{a_{12}}{\rho m} + \frac{l^2}{4\rho I_c}\right) \dot{U} = b_1 \dot{U}, \]

where \( \dot{U} = U_R = \dot{U}_L = \dot{U} \).

Further, the last equation through the change of variables is reduced to the Cauchy normal form \( \tau = x_1, \dot{\tau} = x_2 \)

\[ \dot{x}_1 = x_2, \]
\[ \dot{x}_2 = \left(\frac{a_{12}}{\rho m} + \frac{l^2}{4\rho I_c}\right)x_1 - a_{11}x_2 + b_1 U, \]

or in matrix form
\[ \dot{x} = Ax + Du, \quad (12) \]

here \( x = (x_1, x_2) \), \( A = \begin{bmatrix} 0 & 1 \\ -\left(\frac{a_{12}}{\rho m} + \frac{l^2}{4\rho I_c}\right) & -a_{11} \end{bmatrix} \), \( B = (0, b_1) \), \( u = \dot{U} \).

Apply to equations (12) continuous Laplace transformation
\[ A^*(p)x(p) = Bu(p), \quad (13) \]

here \( A^*(p) = (lp - A) \)
\[ A^*(p) = \begin{bmatrix} p & -1 \\ \frac{a_{12}}{\rho m} + \frac{l^2}{4\rho I_c} & p + a_{11} \end{bmatrix}, \quad l - \text{unit size matrix } 2 \times 2. \]

The solution of the system of algebraic operator equations is determined by Kramer equation
\[ x_{ij}(p) = \frac{\Delta_{ij}}{\Delta(p)} u_i(p), \quad i = \overline{1, n}; \quad j = \overline{1, m}. \]

In our case, the system is multidimensional \( (n = 2) \), but single-linked \( (m = 1) \), and its determinants are not equal to zero:
\[ \Delta(p) = \begin{vmatrix} p & -1 \\ \frac{a_{12}}{\rho m} + \frac{l^2}{4\rho I_c} & p + a_{11} \end{vmatrix} = p^2 + a_{11}p + \frac{a_{12}}{\rho m} + \frac{l^2}{4\rho I_c} \neq 0. \]

Therefore, the system has the only solution \((\text{the matrix } A^*(p) \text{ is undegenerate})\).

Private determinants and transfer functions are equal:
\[ \Delta_{11}(p) = \begin{vmatrix} 0 & -1 \\ b_{11} & p + a_{11} \end{vmatrix} = b_{11}, \quad \Delta_{21}(p) = \begin{vmatrix} p & 0 \\ \frac{a_{12}}{\rho m} + \frac{l^2}{4\rho I_c} & b_{11} \end{vmatrix} = pb_{11}, \]
\[ W_{11}(p) = W_{\dot{U}}(p) = b_{11}, \quad W_{21}(p) = W_{\dot{U}}(p) = \frac{b_{11}}{p^2 + a_{11}p + \frac{a_{12}}{\rho m} + \frac{l^2}{4\rho I_c}}, \]
\[ \dot{V} = \frac{2}{\rho m} \dot{\tau}, \]
\[ W_x(p) = \frac{2}{\rho m} b_{11} \left( p^2 + a_{11} p + a_{12} \frac{1}{\rho m} + \frac{l^2}{4 \rho I_c} \right). \]

b) General case of a robot operator’s model

Similarly to the special case, the matrix equations (11) are reduced to normal Cauchy form by replacing the variables \( x = x_1, \dot{x} = x_2 \)

\[
\dot{x}_1 = x_2, \\
\dot{x}_2 = -Dx_1 - Cx_2 + bu .
\]

Here \( x_1 = (\tau_L, \tau_R), \; x_2 = (\dot{\tau}_L, \dot{\tau}_R). \)

In block form, the latest equations take the form
\[
\dot{Y} = AY + Bu, \tag{14}
\]

Here \( Y = (x_1, x_2) \) - generalized vector of state, size 4×1; \( u = (U_L, U_R) \) - control vector, size 2×1;

\[
A = \begin{bmatrix} O_{(2 \times 2)} & I_{(2 \times 2)} \\ -D_{(2 \times 2)} & -C_{(2 \times 2)} \end{bmatrix} - \text{block matrix at state vector, size 4×4}; \; B = \begin{bmatrix} O_{(2 \times 2)} \\ b_{(2 \times 2)} \end{bmatrix} - \text{block matrix at control vector, size 4×2.}
\]

Through the Laplace transform, the system of differential equations is transformed into a system of algebraic equations of the form

\[
A^w(p)Y(p) = Bu(p),
\]

Here \( A^w(p) = (Ip - A) = \begin{bmatrix} I_{(2 \times 2)}p & -I_{(2 \times 2)} \\ D_{(2 \times 2)} & I_{(2 \times 2)}p + C \end{bmatrix} - \text{single matrix.} \)

As \( \Delta(p) = \det A^w(p) \neq 0 \), the system is non-degenerate and has a unique solution defined by the greatest common zero divisor (GCD) [21]

\[
Y_j(p) = \frac{\Delta_j(p) GCD_j(p)}{\Delta(p)} u_j(p), \; i = 1, n; \; j = 1, m.
\]

In proposed case \( n = 4, \; m = 2 \). Therefore, the robot is a multivariate and multi-linked (multi-circuit) control object.

Here, the private identifier is equal to \( \Delta_j(p) = \det A^w_j(p) \), and the matrix \( A^w_j(p) \) is determined by the substitution in the matrix \( A^w(p) \) \( i - \text{th} \) column on \( j - \text{th} \) free-member column. The overall solution for \( m \) controlling impacts is obtained as a superposition of private solutions

\[
x_j(p) = \sum_{i=1}^{m} x_j(p) = \sum_{i=1}^{m} \frac{\Delta_j(p) GCD_j(p)}{\Delta(p)} u_j = \sum_{i=1}^{m} W_j(p) u_j,
\]

Here \( GCD_j(p) \) - largest common denominator by \( j - \text{th} \) control input.

Thus, there are generalized controls and matrix transfer functions of the numerator and denominator. The transfer functions from the control actions to the moments of the right and the left wheel are determined by equations:

\[
W_{\tau_x}(p) = \frac{\Delta_{11} GCD_1(p)}{\Delta(p)},
\]

Here \( \Delta_{11} = b_{11} \);
\[ GCD_1(p) = GCD_2(p) = p^2 + a_{11}p + d_{11} - d_{12} = p^2 + a_{11}p + \frac{l^2}{2\rho J_c}. \]

\[ W_{\frac{\Delta_1}{\Delta_2}}(p) = \frac{\Delta_2 GCD_2(p)}{\Delta(p)}, \]

here \( \Delta_1 = b_{11}; \)

\[ GCD_2(p) = GCD_3(p) = p^2 + a_{11}p + d_{11} + d_{12} = p^2 + a_{11}p + \frac{2a_{12}}{2\rho m}, \]

\[ \Delta(p) = GCD_1(p)GCD_2(p) = \left(p^2 + a_{11}p + d_{11} - d_{12}\right)\left(p^2 + a_{11}p + d_{11} + d_{12}\right) = \]

\[ \left(p^2 + \frac{R}{L} p + \frac{2K_w^2 l_p}{\rho^2 mL}\right)\left(p^2 + \frac{R}{L} p + \frac{l^2}{2\rho J_c}\right). \]

\[ W_{11} = W_{\frac{b_{11}}{b_{11}}} = \frac{b_{11}}{p^2 + a_{11}p + d_{11} - d_{12}} = \frac{b_{11}}{p^2 + a_{11}p + \frac{l^2}{2\rho J_c}}, \]

\[ W_{22} = W_{\frac{b_{11}}{b_{11}}} = \frac{b_{11}}{p^2 + a_{11}p + d_{11} + d_{12}} = \frac{b_{11}}{p^2 + a_{11}p + \frac{2K_w^2 l_p}{\rho^2 mL}}. \]

\[ W_{21}(p) = pW_{22}(p). \]

4. Modal control method for the second-order object

On the basis of PID controller transfer functions (TF) with ideal and real differentiating links on dominant characteristics of free movement of the object is enough simply defined its parameters. The method of designing PID controllers is applicable to systems of the first, second and higher orders on the basis of their approximation by systems of the first or second order with dominant poles

\[ G_{PID}(s) = \frac{K_D}{s} \left[ s^2 + \frac{K_P}{K_D} s + \frac{K_I}{K_D} \right] = \frac{K_D}{s} \left[ s^2 + 2\xi \omega_n s + \omega_n^2 \right], \] (15)

\[ G_{PID} = K_P \left[ 1 + \frac{1}{T_I s} + \frac{T_D s}{1 + \frac{T_D s}{N}} \right], \] (16)

here \( \omega_n^2 = \frac{K_I}{K_D}, \ 2\xi \omega_n = \frac{K_P}{K_D}, \) constant \( N \) determines the gain \( K_{HF} \) of PID controller in the high frequency range. The gain \( K_{HF} \) should be limited because the noise measurement signal often contains high-frequency components. As a rule, the divider is selected in the range from 2 to 20 [14].

The TF of the controller (15) is determined after a constant time for the differential and integral components in the following form

\[ G_{PID} = K_P \left[ 1 + \frac{1}{T_I s} + T_D s \right]. \] (17)
Here $T_i = \frac{K_p}{K_I}$ – time constant of the integral component; $T_D = \frac{K_D}{K_P}$ – time constant of the differential component. That's why we have: $K_I = \frac{K_P}{T_i}$, $K_D = K_P T_D$.

It is required to determine the parameters of the PID controller according to the free movement characteristics of the reduced object $\omega_n$, $\xi$ so that the zeros of its PF approximately coincide with the poles of the PF of the reduced object, that is, that the equalities:

$$\omega_n^2 = \frac{K_I}{K_D}, \quad 2\xi\omega_n = \frac{K_P}{K_D}, \quad K_D = X_{giv}.$$  

General standard form of description of second-order systems with a dominant pole in the form of PF oscillator link with a relative attenuation coefficient $\xi$ and the frequency of the object's free oscillations $\omega_n$ is determined by the equation

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \Leftrightarrow G_{PID}(s) = \frac{K_D s^2 + 2\xi\omega_n s + \omega_n^2}{s},$$

which is equivalent to the description of the PF PID controller by the expression (15). If the parameters of the damped oscillations are taken equal to the parameters of free oscillations: $\omega_p = \omega_n$ and $\xi = \xi$, then $G_{PID}(s)G(s) = \frac{KK_D\omega_n^2}{\omega_n^2}$, and the TF of the closed loop system will look like an aperiodic link $F(s) = \frac{1}{Ts + 1}$ with time-constant $T = \frac{1}{KK_D\omega_n^2}$.

Then the parameters of the PID controller $K_p$, $K_I$, $K_D$ can be found by the equation (15) through the relative attenuation coefficient $\xi$ and the frequency of free oscillations $\omega_n$ of the object:

$$\omega_n^2 = \frac{K_I}{K_D} \Rightarrow K_D = \frac{K_I \omega_n^2}{\omega_n^2}, \quad \quad (18)$$

$$2\xi\omega_n = \frac{K_P}{K_D} \Rightarrow K_D = \frac{K_P}{2\xi\omega_n}. \quad \quad (19)$$

Assuming in proportional component of the PID regulation law the gain equal to one ($K_p = 1$), from the expressions (18), (19) the gains $K_I$, $K_D$ of the integral and differential components are determined:

$$\frac{K_p}{2\xi\omega_n} = \frac{K_I}{\omega_n^2} \Rightarrow \frac{1}{2\xi\omega_n} = \frac{K_I}{\omega_n^2} \Rightarrow K_I = \frac{\omega_n^2}{2\xi\omega_n} \Rightarrow K_I = \frac{\omega_n}{2\xi}, \quad \quad (20)$$

$$K_D = \frac{K_p}{2\xi\omega_n} \Rightarrow K_D = \frac{1}{2\xi\omega_n}.$$  

However, calculation of PID controller coefficients by formulas (20) leads to initial overshooting and to the transition process tightening, because for a stable system the frequency of undamped (free) oscillations $\omega_n$ determines the number of these fluctuations and has a direct impact on the quality of transition processes. Therefore, in order to speed up the attenuation of processes and reduce overregulation, smoothing coefficient $\varepsilon = \frac{\xi}{\xi}$ and damping $\alpha = \frac{\omega_n}{\omega_n}$, with the help of which the parameters of the integral (coefficient $K_I$) and less often the differential (coefficient $K_D$) components of the PID controller are corrected.

Thus, all three components of the PID controller parameters can be selected through the free movement properties of the object:
\[ K_p = 1, \; K_D = \alpha \frac{1}{2\xi\omega_n}, \; K_I = \varepsilon \frac{\omega_n}{2\xi}. \] (21)

Researches have shown that the introduction of smoothing coefficient \( \varepsilon \) into the integral component allows to eliminate the oscillation of the transition process in the range of its change from 0.1 to 2. Further, increasing the value \( \varepsilon \) will reduce the time of regulation, but will lead to some oscillatory transient process. With the help of the setting damping coefficient \( \alpha \) in the differential component the minimum time of process control is provided. The summary equations for calculations and the limits of the parameters of the PID controllers are shown in Table 2.

Similar formulas for calculating parameters of PID controllers through time constants \( T_D \) and \( T_I \) are obtained from the equations (16), (17):

\[ T_I = \frac{K_D}{K_I} = \frac{2\xi K_D}{\omega_n} = \frac{2\xi}{\omega_n}, \; T_D = \frac{K_D}{K_P} = \frac{\alpha}{2\xi\omega_n} = \frac{\alpha}{2\xi} \]

Table 2. Equations for calculation of PID controllers coefficients

| Object | Parameters of PID controller |
|--------|-----------------------------|
| \( \xi \) | \( \omega_n \) | 1 | \( \frac{\omega_n}{2\xi} \) | \( \frac{1}{2\xi\omega_n} \) | \( \frac{1}{2\xi\omega_n} \) | \( \frac{2\xi}{\omega_n} \) | 2 \( \div \) 20 |

Limits | 1 | \( \frac{\varepsilon\omega_n}{2\xi} \) | \( \frac{1}{2\xi\omega_n} \) | \( \frac{\alpha}{2\xi\omega_n} \) | \( \frac{2\xi}{\omega_n} \) |

\( \varepsilon = 0.1 \ldots 1.2 \) \( \alpha = 0.58 \ldots 1.5 \)

5. **Mobile robot control algorithm using the inverse dynamics problem method**

Since in the equation system (9) the first three equations are coupling equations, it is sufficient to use the last four equations in formula (9) to study the robot motion dynamics.

By differentiating the sixth and seventh equations in system (9) and after substituting the fourth and fifth equations in them, equations (10) are obtained.

In the synthesis of control laws by the method of inverse problems of dynamics, the mathematical model of the robot (10) is taken as the control object

\[ \ddot{\tau} + \alpha_1 \dot{\tau}_L + d_{11} \tau_L + d_{12} \tau_R = b_1 \ddot{U}_L, \]

\[ \ddot{\tau} + \alpha_1 \dot{\tau}_R + d_{11} \tau_R + d_{12} \tau_L = b_1 \ddot{U}_R, \] (22)

Therefore, the mobile robot as a control object is a multi-linked (\( m = 2 \)) and multidimensional (\( n = 2 \)) system.

The problem of synthesizing controls for stabilizing (damping) the robot's angular velocities in the time interval \([t_0, t_1]\) (\( I_k \) - non-fixed value) is formulated as follows.

Suppose that the current time \( t = t_0 = 0 \) torques-controlled \( \tau_1 = \tau_L, \; \tau_2 = \tau_R \) system state (22) is determined by the values of \( \tau_i(0) = \tau_{i0}, \; i = 1, 2 \).

It is necessary to define the control actions \( \ddot{U}_1(\ddot{U}_1 = \ddot{U}_L, \ddot{U}_2 = \ddot{U}_R) \), at which the system (22) switches from the state \( \tau_{i0} \) to the new state \( \tau_{igiv} \), where \( \tau_{igiv} = \tau_{igiv}(t) \) - the set torques of rotation of the system (22) must be defined. It is necessary that the moments of controlled motion on each degree of freedom with required accuracy followed the moments defined by the reference model

\[ \ddot{X}_{ref} + C_i \dot{X}_{ref} + D_i X_{ref} = D_i \tau_{igiv}, \; i = 1, 2, \ldots \] (23)
with the previously calculated coefficients $C_i$ and $D_i$.

The specified change in supply voltage at the terminals of the armature winding of the right and left electric motor determines the specified reference movement

$$B_i U_{igiv} = D_i \tau_{igiv}, \ i = 1, 2, 3, \ldots$$  \hspace{1cm} (24)

The PWM controller is used to change the supply voltage. The reference movement is stable $X_{refi} \rightarrow \tau_{igiv}$ for $t \rightarrow \infty$.

Let’s consider a reference model (23) in scalar form

$$\ddot{\tau}_{ref L} + a_1 \dot{\tau}_{ref L} + d_1 \tau_{ref L} = d_1 \dot{\tau}_{Lgiv}, \hspace{1cm} \ddot{\tau}_{ref R} + a_1 \dot{\tau}_{ref R} + d_1 \tau_{ref R} = d_1 \dot{\tau}_{Rgiv}, \hspace{1cm} (25)$$

From equation (24) the change of supply voltage on the armature clamps of the right and left wheel winding in scalar form will be written as

$$\dot{U}_{Lgiv} = \frac{d_1}{b_{11}} \tau_{Lgiv}, \hspace{1cm} \dot{U}_{Rgiv} = \frac{d_1}{b_{11}} \tau_{Rgiv}. \hspace{1cm} (26)$$

By minimizing the functional

$$G(\dot{U}) = \frac{1}{2} \sum_{i=1}^{2} [\dot{X}_{refi} - \dot{X}_i(t, \dot{U}_i)]^2 \hspace{1cm} (27)$$

the degree of approximation of the controlled process to the reference model is assessed $x_{refi}(t) \rightarrow \tau_{igiv}$.

Functional characterizes the change of the moment of rotation of the mobile robot wheels in the vicinity of the change of the standard moment of rotation.

The gradient method scheme defines the laws of control $\dot{U}_i(x)$ for each degree of freedom of rotation of wheels

$$\frac{\partial \dot{U}_i(X)}{\partial t} = \dot{U}_i(x) = -\sum_{j=1}^{2} r_{ij} \frac{\partial G(\dot{U})}{\partial U_j}, \ r_{ij} = \text{const}. \hspace{1cm} (28)$$

From equations (23), (27), the gradient components of (28) are defined

$$\frac{\partial G(\dot{U})}{\partial U_j} = -b_{11}(\dot{X}_{ref j} - \dot{X}_j), \ j = 1, 2, \ldots \hspace{1cm} (29)$$

It follows that

$$\dot{U}_i(x) = \sum_{j=1}^{2} r_{ij} b_{11}(\ddot{X}_{ref j} - \ddot{X}_j). \hspace{1cm} (30)$$

The required control law (voltage regulation in the circuit of the armature winding of motor) is determined by integrating over time of both parts of the expression (30)

$$U_i(x) = \sum_{j=1}^{2} r_{ij} b_{11}(X_{ref j} - X_j). \hspace{1cm} (31)$$

Here, the required torque values $X_{ref j}$ are calculated using the equation (23).

The laws of control (31) of the robot’s wheel set in scalar form look like:

$$U_L = r_1 b_{11}(\tau_{ref L} - \tau_L) + r_2 b_{11}(\tau_{ref R} - \tau_R), \hspace{1cm} U_R = r_2 b_{11}(\tau_{ref R} - \tau_R) + r_2 b_{11}(\tau_{ref R} - \tau_R), \hspace{1cm} (32)$$

and in matrix form they define a known procedure of analytical design of controllers in the next form

$$U = -R B \Delta \tau. \hspace{1cm} (33)$$
here  \( \Delta \tau = \begin{bmatrix} \tau_L - \tau_{ref \, L} \\ \tau_R - \tau_{ref \, R} \end{bmatrix} \) – vector of change of moments developed by electric motors; 
\( B = \text{diag}(b_1, b_1) \) – diagonal matrix. The structure of matrix is determined by the type of cross-linkages in a variable \( x_j \) in equations (30) or (31), \( R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \), moreover \( r_{12} = r_{21} \).

Sustained \( R = R^T > 0 \). Then \( G(\dot{U}) < 0 \) and closed loop system (31) (22) asymptotically stable in Lyapunov \( \dot{X}_{ref} \rightarrow \dot{X}_j \) at \( t \rightarrow \infty \).

6. Modeling and discussing

a) PID controllers tuned according to Ziegler-Nicols methodology

On the basis of mathematical model (9) and obtained transfer functions, Simulink-models of controlled mobile robot with parameters of table 1 were developed. For a particular case of the Simulink robot operator’s model, the model is shown in Figure 2.

**Figure 2.** Simulink-model at synchronous deviation of mobile robot wheels

PID controller coefficients were selected according to the Ziegler-Nicols method: \( K_p = 9.5681 \), \( K_i = 6.3948 \), \( K_d = 0.6863 \). Wheel deflection angle \( \varphi = 30^\circ \).

The control torque \( \tau \) transient graphs from time to time \( t \) are shown in Figure 3.

Speed coordinate change graph \( V \) of the robot and the robot’s track \( S \), depending on the time \( t \), are shown in Figures 4 and 5.

**Figure 3.** Control torque \( \tau \) of the mobile robot's wheel set
Figure 4. Changing the speed of the robot \( V \) depending on the time

Figure 5. Changing the track \( S \), passed by the robot depending on the time

On the basis of the above method of modal PID control Simulink-model (Figure 6) of three-wheeled robot as a multi-dimensional multi-linked object \( (n = 4, m = 2) \) for the cases of stabilization of moments with the help of ideal and real (with a real differentiating link) PID controllers was constructed. The stability of the control system was ensured by the introduction of external negative feedback on the angular velocity of the robot's platform, as its initial dynamic model contains two consecutively included integrative links (structural sign of instability).

The nominal robot platform angular velocity values specified in the technical specifications were used as the determining factors, converted into changes in the supply voltage of the electric circuit of the wheel set.

Figure 6. Simulink-model of mobile robot using modal method of PID controller adjustment

The simulation results are shown in Figures 7-16 for two cases:
1) Using ideal modal PID controllers:
PID1 coefficients for the left wheel: \( K_p = 1, \ K_I = 0.113, \ K_D = 0.003 \);
PID2 coefficients for the right wheel: \( K_p = 1, \ K_I = 0.119, \ K_D = 0.003 \).
2) Using real modal PID controllers:
PID3 coefficients for the left wheel: \( K_p = 1, \ K_I = 0.113, \ K_D = 0.003, \ N = 2.5 \);
PID4 coefficients for the right wheel: \( K_p = 1, \ K_I = 0.119, \ K_D = 0.003, \ N = 2.5 \).
**Figure 7.** Left wheel torque (ideal PID1)

**Figure 8.** Left wheel torque (real PID3)

**Figure 9.** Right wheel torque (ideal PID2)

**Figure 10.** Right wheel torque (real PID4)

**Figure 11.** Change of x coordinate (ideal PID controllers)

**Figure 12.** Change of x coordinate (real PID controllers)
Analysis of the results of the simulation shows that the experimental characteristics of the free movement of the robot can be used to determine the parameters of the PID controllers in such a way that the characteristics of the closed loop adaptive control system correspond to the desired properties of the aperiodic link with the minimum control time. At the same time, it is quite predictable that the direct indicators of quality (time of regulation) become worse for the control system with PID controllers containing real differentiating links due to inertia of the forming high-frequency filter.

b). Double differentiation controllers (PDD)

On the basis of the equations (22) of inverse problems of dynamics method [22, 23, 24, 25] which is described above, the mathematical model of the control system of the autonomous robot is made. The general structure of Simulink-model of robot control is presented in Figure 17. The model consists of the following main blocks:
1) a block that implements the robot's angular position and trajectory motion (Figure 17);
2) a block of multivariate mathematical model of robot angular motion (Figure 18);
3) a block implementing a double differentiation controller (PDD controller) based on reference wheel set models (Figure 19).
Figure 17. General structure of Simulink-model of mobile robot control

Figure 18. Multivariate mathematical model of robot angular motion

Figure 19. PDD controller based on a reference model
Analysis of modeling results shows that the synthesis of the control system on the basis of PDD-controller provides high synchronism of the wheelset deflection. According to the fifth equation of the system (9), the angular velocity in the mode of stabilization of robot movement \( \omega_{\text{giv}} = 0 \) also tends to zero, and the robot moves in a straight line.

7. Conclusion

Thus, there were obtained the following main results:

1) Mathematical models of the mobile robot in the state space and in the operator's form have been developed.

2) Synthesized algorithms of robot motion control on the basis of modal PID method - control at synchronous and differential deviation of the wheel set.

3) On the basis of the principle of dynamic compensation the method of construction of PDD controllers of the autonomous robot synthesized on the basis of reference models by a method of inverse problems of dynamics is proved.

4) Simulink-models of the controlled mobile robot are developed on the basis of the method of modal PID control at synchronous and differential deviation of the wheel set, and on the method of inverse problems of dynamics.

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