Three dimensional phase diagram of the repulsive Blume-Emery-Griffiths model in the presence of external magnetic field

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For the repulsive Blume-Emery-Griffiths model the phase diagram in the space of three fields, temperature ($T$), crystal field ($\Delta$) and magnetic field ($H$), is computed on a fully connected graph, in the canonical and microcanonical ensembles. When the strength of the biquadratic interaction ($K$) is low, there exists a tricritical point where the three critical lines meet. As $K$ decreases, new multicritical points like the critical end point and bicritical end point arise in the ($T, \Delta$) plane. For $K > -1$, we observe that the two critical lines in the $H$ plane and the multicritical points are different in the two ensembles. At $K = -1$, the two critical lines in the $H$ plane disappear and as $K$ decreases further, we see no phase transition in the $H$ plane. Exactly at $K = -1$ the two ensembles become equivalent. Beyond that for all $K < -1$, there are no multicritical points and there is no ensemble inequivalence in the phase diagram. We also study the transition lines in the $H$ plane for positive $K$ i.e. for attractive biquadratic interaction. We find that the transition lines in $H$ plane are not monotonic in temperature for large positive $K$.

I. INTRODUCTION

The Blume-Emery-Griffiths (BEG) model is the simplest model which incorporates biquadratic interactions\cite{1}. Presence of biquadratic exchange interaction is known to be relevant to understand the properties of the rare-earth compounds. The biquadratic exchange was first suggested by Kittel in the theory of magnetoelastic effect in NiAs type structures\cite{2}, and by Anderson in the superexchange interaction of iron group oxides and fluorides\cite{3}. In rare-earth compounds, the unpaired 4$f$ electrons lie deep inside the 5$d$ and 5$s$ orbital. So these electrons do not experience the strong crystal field generated by other ions in the crystal. Hence their spherically symmetric potential is not completely destroyed. As a result the orbital angular momentum is not entirely quenched. The superexchange between these unquenched orbital momentum gives rise to a biquadratic exchange interaction term in the Hamiltonian\cite{4}. Other interactions such as phonon exchange between ions\cite{5} and the Schrodinger’s spin-one exchange operator\cite{6} can also result in the inclusion of such interaction. Both attractive and repulsive biquadratic interactions are of interest. The requirement of small repulsive exchange interaction in a Hamiltonian was first mentioned by Harris and Owen \cite{7} and Rodbell \emph{et.al.} \cite{8} in order to explain the paramagnetic resonance of the Mn ion pairs which are present as an impurity in the crystals of MgO.

Biquadratic exchange interaction is represented by a term that is fourth order in spin operators. Spin-1 BEG model has been shown to successfully capture the physics of these higher order interactions and has been widely studied. It incorporates an uniform crystal field($\Delta$) and a biquadratic exchange interaction ($K$) along with the bilinear exchange interaction term. This model was first introduced in order to explain the phase separation and superfluidity of $^3\text{He} - ^4\text{He}$ mixture \cite{9}. Apart from this many other physical systems like: metamagnets, liquid crystals, semiconducting alloys, microemulsions, etc can also be mapped to the BEG model. This model has a rich phase diagram depending on the sign and magnitude of the biquadratic term. The special case, $K = 0$ is known as the Blume-Capel model. Blume Capel model was first studied by M.Blume \cite{10} and H.W.Capel \cite{11} in order to explain the first order transition in $U\text{O}_2$. The another extreme case with the zero bilinear exchange was studied by Griffiths \cite{11}.

The attractive($K > 0$) BEG model has been extensively studied. Its phase diagram changes with the value of $K$. For small $K$, there is a transition from a ferromagnetic to paramagnetic phase in the $T - \Delta$ plane. This transition line changes from a continuous to a first order transition line at a tricritical point(TCP). As $K$ increases further, another paramagnetic state emerges and the two paramagnetic states are separated by another first order line. The two first

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Figure 1: Schematic phase diagram of the repulsive BEG model for both canonical and microcanonical ensemble. Solid lines are the critical lines, dashed are the first order line, stars are the BEP, circles are the TCP, and squares are the CEP.

(a) Shows the phase topology for Canonical: $-0.1838 \leq K \leq 0$ and, Microcanonical: $-0.0828 \leq K \leq 0$: In this regime the critical lines($\lambda_{\pm}$) lines meet the $\lambda$ line at the TCP.

(b) Is the phase topology for Canonical: $-1 < K < -0.1838$ and, Microcanonical: $-1 < K < -0.0828$: Here the $\lambda_{\pm}$ move inside the ordered region and meet at the BEP. The $\lambda$ line terminates on the first order line at a CEP.

(c) Shows the phase topology for both Canonical and Microcanonical: $K = -1$: In both the ensemble the wings as well as the BEP and CEP reaches $T = \varepsilon = \Delta = 0$.

(d) Canonical, Microcanonical: $K < -1$: Only the $\lambda$ transition remains. There are no phase transition in the finite $H$ plane. The only transition is from the ferromagnetic($m \neq 0$) state to the paramagnetic state($m = 0$).
is a junction of two critical lines($\lambda \pm$) and hence a BEP. We find that the width of the wings in temperature shrinks as $K$ approaches $K = -1$. At exactly $K = -1$ the BEP as well as the CEP moves to $T = \Delta = 0$ and the wings vanish(see Fig:II(c)). On further reducing $K$, we see no transition in the finite $H$ plane. Only transition occurs from is a ferromagnetic state to a paramagnetic state in the $H = 0$ plane(Fig:II(d)). The area under the $\lambda$ line shrinks as $K$ becomes more and more negative. At $K \rightarrow -\infty$, only the paramagnetic state survives and there is no transition.

The frustration introduced by the repulsive biquadratic interaction makes it a very interesting model to study. In fact we find that the phase diagram for the repulsive($-1 < K \leq 0$) BEG model is similar to the topology of the phase diagram of the Blume Capel model with random crystal field order studied recently for the intermediate and weak disorder. In this paper, we have looked at the ensemble inequivalence not just by looking at the first order line in the $(T, \Delta)$ plane but also by computing the critical lines($\lambda \pm$) in the $H \neq 0$ plane. We find that these two critical lines are different in the two ensembles in general besides the multicritical points. Another interesting observation we have is that for $K \leq -1$ the two ensembles are equivalent. Attractive BEG model has been well studied and we find results similar to as reported in the earlier studies. Since we looked at the phase diagram in the $(T - \Delta - H)$ space, we found a non-monotonic behaviour of the wings in terms of temperature as $K$ becomes greater than $K = 1$. This as far as we know has not been reported earlier.

The plan of the paper is as follows: In Section II we introduce the BEG model and discuss its zero temperature phase diagram. In Section III and Section IV we derive the equations of the critical lines in the $T - \Delta - H$ plane but also by computing the critical lines($\lambda \pm$) in the $H \neq 0$ plane. This as far as we know has not been reported earlier.

In Section V we discuss the ensemble inequivalence in detail. We conclude in Section VI.

II. MODEL

The Hamiltonian of the BEG model on a fully connected graph in the presence of external magnetic field is given by:

$$H = -\frac{1}{2N}(\sum_{i} S_i)^2 - \frac{K}{2N}(\sum_{i} S_i^2)^2 + \Delta \sum_{i} S_i^2 - H \sum_{i} S_i$$

where $S_i$ can take three values $\pm 1, 0$, $H$ is a constant external field coupled with the order parameter, $\Delta$ is the crystal field, and $K$ is the biquadratic interaction coefficient. The two order parameters are: magnetization, $x_1 = \sum_{i} \frac{S_i}{N}$ and the density of the $\pm 1$ spins: $x_2 = \sum_{i} \frac{S_i^2}{N}$. For any finite $K$, as $\Delta \rightarrow -\infty$, this model reaches the Ising point at $T = 1$, as the spins take only $\pm 1$ values. As $\Delta$ increases, the number of vacancies increases in the system. For negative $K$, spins are more likely to take value $0$. At finite temperature, when $K < 0$, both the biquadratic term and the crystal field term prefer 0 spins. Hence the $\lambda$ transition occurs at a lower $\Delta$ as $K$ decreases. On the other hand, the presence of positive $K$ chooses the magnetic spins. Hence, when $K > 0$ there is a competition between the biquadratic and crystal field term in the Hamiltonian and the $\lambda$ transition occurs at a higher $\Delta$ for positive $K$.

First let us look at the zero temperature phase diagram of the system. The energy per particle can be written as from the Hamiltonian Eq:II: $\epsilon = -\frac{1}{2}(x_1^2 + Kx_2^2) + \Delta x_2 - Hx_1$. When all the spins are zero the energy is $\epsilon[S = 0] = 0$. Apart from this paramagnetic phase, there are other states which are possible depending on the parameter values. For $-1 \leq K \leq \infty$, the ferromagnetic state, $x_1 = \pm 1$ and $x_2 = 1$ dominates. Energy of this state is $\epsilon[S = \pm 1] = -\frac{1}{2}(1 + K) + \Delta$. If $2\Delta > 1 + K$, then the phase is paramagnetic, for $2\Delta < 1 + K$, the phase is ferromagnetic. At exactly $2\Delta = 1 + K$ there is a first order phase transition. For $K < -1$, the term $-\frac{1}{2}(1 + K)$ in the energy contributes a positive value. So for any $\Delta \geq 0$, paramagnetic phase is the stable state. As $\Delta$ becomes negative, there is another ferromagnetic state with $|x_1| = x_2 < 1$, which becomes stable when $|\Delta| < -\frac{1}{2}(1 + K)$. For $|\Delta| > -\frac{1}{2}(1 + K)$, the state with $x_1 = x_2 = 1$ becomes stable. And there is a first order transition now between these two ferromagnetic state at $\Delta = \frac{1 + K}{2}$.

III. CANONICAL ENSEMBLE

Given the Hamiltonian(Eq:II), the probability of the spin configuration($C_N = \{S_i\}$) for $N$ spins can be expressed as:

$$P(C_N) = \frac{e^{\beta(\frac{1}{N}(\sum_{i} S_i)^2 + \sum_{i} S_i^2 - \Delta \sum_{i} S_i^2 + H \sum_{i} S_i)}}{Z_N}$$

(2)
be obtained by linearizing Eq. [7]. On linearizing we get the equation of the function w.r.t. the two order parameters:

The rate function $I(x_1, x_2)$ can be seen as the full Landau free energy functional for the system. Minimizing the rate function w.r.t $x_1$ and $x_2$ gives the free energy of the system. The detailed calculation of the rate function $I(x_1, x_2)$ is shown in Appendix A. Minimization of the rate function w.r.t $x_1$ and $x_2$ gives the following two coupled equations for the two order parameters:

$$m = \frac{2e^{\beta(Kq-\Delta)}}{1 + 2e^{\beta(Kq-\Delta)}} \sinh \beta(m + H)$$

$$q = \frac{2e^{\beta(Kq-\Delta)}}{1 + 2e^{\beta(Kq-\Delta)}} \cosh \beta(m + H)$$

where $m$ and $q$ are the extremums of $x_1$ and $x_2$. For $m \neq 0$, the two fixed point equations are connected via:

$$q = m \coth \beta(m + H)$$

and the free energy at the fixed point can be written as (putting Eq. [7] in Eq. [A-11]):

$$f(m) = \frac{\beta m^2}{2} + \frac{\beta K m^2 \coth^2 \beta m}{2} + \log(1 + 2e^{-\beta\Delta} \cosh \beta H) - \log(1 + 2e^{\beta(Km \coth \beta m - \Delta)} \cosh \beta(m + H))$$

For $H = 0$, the system has a line of continuous transition ($\lambda$ line) in the $T - \Delta$ plane. The equation of this line can be obtained by linearizing Eq. [7]. On linearizing we get the equation of the $\lambda$ line to be:

$$2(\beta - 1) = e^{\beta\Delta - K}$$

For $H \neq 0$, the system has a line of continuous transition in $H_+$ and $H_-$ planes separating the two magnetic states. To calculate the line of these critical points we instead take $f'(m) = f''(m) = f'''(m) = 0$ and $f''''(m) < 0$, which gives the locus of the transitions. This gives the following equations:

$$f_1 = m - \frac{az}{C} = 0$$

$$f_2 = \frac{1}{za\beta} - \frac{az^2 + \sqrt{y^2 + 1} + yK(\sqrt{y^2 + 1} - \beta mz)}{C^2} = 0$$

$$f_3 = \frac{\beta^3 z}{C^3} \left(azK^2 + \frac{\beta}{y^2 + 1} - K(\frac{2}{y^2 + 1} + y)\left(2 + K + 2az(1 + K)m\beta y\right)
+ \sqrt{y^2 + 1}\left[a(1 + 2K) + 2\beta K^2 m^2 y^3 + \beta^2 az K^2 m^2 y^4\right]
- y\left[2az^2 - 2z^2 + 2\beta K az \beta y - 2K y^2 - 2az \beta K m^2 y^3 + \beta^2 K^2 m^2 y^4\right]\right) = 0$$

where $a = e^{-\beta\Delta}$, $y = \cosech\beta(H + m)$, $z = 2e^{\beta K m \coth \beta(H + m)}$, and $C = \sqrt{y^2 + 1} za + y$. For $K = 0$, the above equations reduces to the following equations for pure Blume-Capel model:

$$m = \pm \frac{2a\sqrt{x^2 - 1}}{2ax + 1}$$

$$\frac{2a + x}{(2ax + 1)^2} = \frac{1}{2a\beta}$$

$$\frac{8a^2 + 2ax - 1}{(2ax + 1)^2} = 0$$
Figure 2: Plot of $f_1$, $f_2$ and $f_3$ in $\beta - \Delta$ plane for (a) $K = -0.4$ at $H = 0$. The intersection of the three derivative lines give the critical point at a non-zero value of $m$, which gives the locus of the BEP at which the $\lambda_\pm$ lines meet at the $H = 0$ plane (b) $K = -2$ at $H = 0$. The three lines never intersect simultaneously for any value of $m$, which shows there are no critical points.

with $x = \cosh(\beta(m + H))$. The solutions of Eq.[14]-[16] are hence given by:

$$x = \cosh(\beta(m + H)) = \frac{\beta - 2}{\sqrt{4 - \beta}}$$

$$a = e^{-\beta \Delta} = \frac{\sqrt{4 - \beta}}{4}$$

and the critical lines for $H \neq 0$ plane are the following:

$$m = \pm \sqrt{\frac{\beta - 3}{\beta}}$$

$$H = \pm \frac{1}{\beta} \log(\beta - 2 + \sqrt{\beta^2 - 3\beta}) - m$$

These lines are the $\lambda_\pm$ lines[1](depending on the sign of $H$). These lines enclose two first order surfaces in $H \neq 0$ plane called the wings.

For $K \neq 0$ solving Eq.[10]-[12] is not possible analytically. Hence we have used the graphical methods to get the co-ordinates of the critical points in the $T - \Delta - H$ plane for a given $K$. We plot $f_1$, $f_2$ and $f_3$ in $\beta - \Delta$ plane for $m$, fixing $K$ and changing different values of $H$. The value of $m$ for a fixed value of $H$ and $K$ at which three equations meet gives the co ordinates of the critical point. If we now take $H = 0$ in Eq.[10] [11] [12], then we will get the coordinates of the point of intersection of the $\lambda_\pm$ lines in the $T - \Delta$ plane. We can hence use this to locate the multicritical points(TCP and BEP) in the $T - \Delta$ plane. We use this to obtain the phase diagram for various values of $K$. For example: Fig[2](a) is the contour plot of $f_1$, $f_2$ and $f_3$ in the $\beta - \Delta$ plane at $K = -0.4$ and $H = 0$. The intersection of the three functions gives the co ordinates of the critical point. We find that for $K > -0.1838$ the functions intersects only for $m = 0$, which is the point where the $\lambda_\pm$ lines meet the $\lambda$ line. Hence this point is the TCP. For the range $-0.1838 > K > -1$ we find that the intersection occurs for $m \neq 0$. This $m \neq 0$ solution gives the locus of the point where the $\lambda_\pm$ lines meet the $H = 0$ plane in this regime of $K$. Hence this point is no longer a TCP, it becomes a BEP. Interestingly, we find that for $K < -1$ the three functions never intersect at the same point for any $m$. For example in Fig[2](b) we plot three functions for $K = -2$. We will discuss these results in detail in the next section(Se[III A]).
Fig. 3: The behaviour of temperature (T) and the crystal field (∆) as a function of H along the \( \lambda_+ \) line in canonical ensemble for \( K = -0.6 \). The main plot shows that the temperature decreases exponentially with H and it saturates towards a certain temperature (\( T_{\text{sat}} \)) for high magnetic field. The inset shows how \( \Delta \) increases linearly with H.

Fig. 4: The width of the wings in temperature (\( T_w \)) as a function of K for the repulsive BEG model. The main plot shows that as K decreases, the width in temperature goes to zero. The inset is the semi-log plot for the same.

A. Repulsive Blume-Emery-Griffiths model

In this section we analyse the results of the repulsive BEG model. We find that for \( 0 \geq K \geq -0.1838 \) two critical lines (\( \lambda_\pm \)) at \( H \neq 0 \) meets the \( \lambda \) line (at \( H = 0 \)) at the TCP. Temperature decreases exponentially and \( \Delta \) increases linearly with increasing \( H \) along the \( \lambda_\pm \) lines, as shown in Fig. 3. As \( K \) becomes more negative (for \(-0.1838 > K > -1\) ), the (\( \lambda_\pm \)) lines no longer meet the \( \lambda \) line, instead they enter into the ordered region and meet at the first order surface (\( H = 0 \)) at a BEP. The \( \lambda \) line in this case terminates on the first order line at a CEP.

As \( K \) approaches \(-1\), the wing width in temperature (which is the difference between the temperature at the BEP (\( T_{BEP} \)) and the saturation value of the temperature (\( T_{\text{sat}} \)) above which the wings extends to infinity) denoted as \( T_w \), starts to shrink. At exactly \( K = -1 \) the BEP, CEP, and the \( T_w \) reach zero. As we decrease \( K \) further, we find that there are no transition in the \( H \) plane. This is also supported by the fact that now there are no multicritical points in the \( T - \Delta \) plane. Hence we conclude that for \( K < -1 \) the wing surfaces completely disappear. The phase diagram consists of only a continuous transition line (\( \lambda \) line) from ferromagnetic phase to paramagnetic phase in the \( H = 0 \) plane. For large negative \( K \), the area enclosed by the \( \lambda \) line in the \( T - \Delta \) plane shrinks. At \( K \to -\infty \), there is no phase transition, only the \( S = 0 \) state dominates. The decreasing width of the wings with decreasing \( K \) is shown in Fig. 4. We also observe that the \( T_{\text{sat}} \) can be approximated numerically as \( T_{\text{sat}} \approx (K + 1)/4 \). This will be discussed in more detail in Sec. IV A where we obtain the same results in the microcanonical ensemble. The values of the \( T_w \), \( T_{\text{sat}} \) and the co-ordinates of the multicritical points (TCP, BEP) for the repulsive BEG model are listed on Table I.

Absence of phase transition for \( K < -1 \) in the \( H \) plane can also be seen by looking at the magnetization and
TCP / BEP \[\approx 0.2\]

\[\approx 0.1\]

\[0.100192068\]

\[0.03127376\]

\[\approx \]

\[T\]

\[0.2857142\]

\[0.072843316\]

\[0.391831\]

\[0.4\]

\[0.149998\]

\[0.01253394\]

\[0.2\]

\[\approx \]

\[1\]

\[0.15446222\]

\[0.2312737\]

\[0.33333\]

\[0.346377\]

\[0.07500018\]

\[0.17499956\]

\[\approx \]

\[0.25\]

\[0.075009189\]

\[\approx \]

\[0.298727\]

\[0.462098\]

\[0.431268\]

\[0.8\]

\[0.199958\]

\[0\]

\[\approx \]

\[0.6\]

\[T\]

\[0.237501\]

\[0.1875335\]

\[0.3103448\]

\[0.2250124\]

\[0.14999925\]

\[\approx \]

\[0.0833333\]

\[= -0.5\]

\[0.00446297\]

\[0.44741\]

\[0.000009009\]

\[0.5\]

\[0.000192068\]

\[0.7569096\]

\[0.079999875\]

\[\approx \]

\[0.049999875\]

\[\approx \]

\[0.000000375\]

\[-0.8\]

\[0.05000025\]

\[0.149998\]

\[0.07500018\]

\[0.000009009\]

\[-0.9\]

\[0.024999968\]

\[0.024999968\]

\[\approx \]

\[0.000000001\]

\[-0.999\]

\[0.000025\]

\[0.000025\]

\[0.00\]

Table I: Coordinates of the TCP and BEP for different K’s. \(T_{sat}\) is the saturation value of the temperature above which the wings become infinite. \(T_w\) is the width of the wing lines for different K.

| K     | TCP / BEP | \(T_{sat}\) | \(T_w\) |
|-------|-----------|--------------|---------|
| 0     | 0.33333   | 0.462098     | 0.25    |
| -0.05 | 0.3103448 | 0.44741      | 0.237501|
| -0.1  | 0.285714  | 0.431268     | 0.2250124|
| -0.2  | 0.2312737 | 0.391831     | 0.2     |
| -0.3  | 0.1875335 | 0.346377     | 0.17499956|
| -0.4  | 0.15446222| 0.298727     | 0.14999925|
| -0.6  | 0.100192  | 0.199998     | 0.1     |
| -0.7  | 0.0750018 | 0.149998     | 0.07500018|
| -0.8  | 0.0500025 | 0.1       | 0.049999875|
| -0.9  | 0.02499968| 0.05         | 0.024999968|
| -0.999| 0.00025   | 0.0005       | 0.00025 |

Figure 5: Magnetization(m) and magnetic susceptibility(\(\chi\)) as a function of \(\Delta\) for (a) \(K = -0.6\), \(T = 0.1\) and \(H = 0.5\). This shows that the \(m\) goes to zero continuously around \(\Delta = 0.7\). Also the \(\chi\) has a singularity at the same \(\Delta\) which suggests that there is a second order transition in the \(H \neq 0\) plane. (b) \(K = -1.2\), \(T = 0.025\) and \(H = 0.5\). Both the \(m\) and \(\chi\) changes continuously as a function of \(\Delta\). Magnetic susceptibility(\(\chi\)) shows no singularity or discontinuity and there is no phase transition in the finite \(H\) plane.

susceptibility. We find that the magnetic susceptibility diverges around the expected critical point for \(K > -1\). On the other hand for \(K \leq -1\) magnetic susceptibility is finite in the entire \(H \neq 0\) plane. We have plotted the magnetization and the susceptibility for \(H = 0.5\) for \(K = -0.6\) and \(K = -1.2\). In Fig 5(a) we plot them as a function of \(\Delta\) for \(K = -0.6\), by fixing \(T = 0.1\). The susceptibility shows singular behaviour at \(\Delta = 0.7\). The point of divergence matches with the coordinates of the transition obtained in Sec III. On the other hand for \(K = -1.2\), we find no such divergence. In Fig 5(b) for \(T = 0.025\) we plot the magnetization and find that it changes continuously along \(\Delta\) and the susceptibility shows a cusp but does not diverge. Though we have plotted only for a fixed \(T\), we have checked the entire plane by changing the values of \(T\). Magnetic susceptibility has no divergence for any \(T\).

### B. Attractive Blume-Emery-Griffiths model

The attractive BEG model has been extensively studied earlier by various authors [1]-[14] and the topology of the phase diagram is known as a function of \(K\) in the \(T - \Delta\) plane. We observe the similar topology of the phase diagram. We study the \(T - \Delta - H\) phase diagram and find that the topology of the phase diagram for different \(K\)’s are similar to [14]. To recap we find: For \(0 < K \leq 2.78\), the phase diagram is similar to what we find for \(0 \geq K \geq -0.1838\). The \(\lambda_{\pm}\) meets at the TCP. For \(2.78 < K < 3\) a new first order surface appears separating two paramagnetic states: \(P2(m = 0, q_- < 0.5)\) and \(P1(m = 0, q_+ > 0.5)\). This surface meets the first order line(at \(H = 0\)) at a triple point, a point meeting of three first order lines. This new first order surface terminates on a line of critical points(at \(H \neq 0\) plane).
Figure 6: Plot for the non-monotonic behaviour of temperature($T$) as a function of $H$ along the $\lambda_+$ line for $K = 2.89$. The inset shows that for lower magnetic field $T$ decreases with $H$ like before, but for higher $H$ it increases and saturates to a higher value($T_{sat}$) shown in the main plot.

As $K$ changes from $K = 2.78$, this line of critical points in the paramagnetic region moves higher in temperature and at exactly $K = 3$ it intersects the $\lambda_\pm$ lines and then extends to infinity. For $3 < K \leq 3.8$, the $\lambda_\pm$ lines terminates at the first order surface which separates the P1 and P2 phase, and becomes finite. Triple point still exists. For $K > 3.8$, the triple point reaches near the TCP and the $\lambda$ line terminates at this triple point, before reaching TCP and becomes a CEP, and thus the wings vanish.

IV. MICROCANONICAL ENSEMBLE

In order to analyze the system in the microcanonical ensemble, we need to express the energy in terms of the number of particles with spin $\pm 1$ and 0. Let us assume the number of particles with $\pm 1$ spin are $N_\pm$ and the number of particles with zero spins are $N_0$, such that $N = N_+ + N_- + N_0$, where $N$ is the total number of particles in the system. The energy of the system can thus be written as,

$$E = \Delta Q - \frac{1}{2N} M^2 - \frac{K}{2N} Q^2 - HM$$

(20)

where $M = N_+ - N_-$ is the total magnetization and $Q = N_+ + N_-$ is the spin density of the system. In terms of $m(= M/N)$ and $q(= Q/N)$, the expression for energy will be,

$$\epsilon = \Delta q - \frac{1}{2} m^2 - \frac{K}{2} q^2 - Hm$$

(21)

where, $\epsilon = \frac{E}{N}$ is the energy per particle, $m$ and $q$ are the single site magnetization and density(as mentioned in Sec.II). The total number of microstates of the system can be written in terms of $N$, $N_+$, $N_-$ and $N_0$ as,

$$\Omega = \frac{N!}{N_+!N_-!N_0!}$$

(22)
In the limit when \(N_+, N_- \), \(N_0\) are large, the expression for entropy, i.e., \(S = k_B \ln(\Omega)\) can be written by using Stirling approximation as,

\[
s = \frac{S}{k_B N} = q \ln(2) - (1 - q) \ln(1 - q) - \frac{1}{2} (q + m) \ln(q + m) - \frac{1}{2} (q - m) \ln(q - m)
\]  

(23)

where, \(s\) is the entropy per particle of the system. The equilibrium entropy can be obtained by maximizing the entropy of Eq. 23 with respect to \(m\) and \(q\). We can express \(q\) in terms of \(m\) and the other variables as,

\[
q_{\pm} = \frac{\Delta}{K} \pm \gamma^{1/2}
\]  

(24)

where, \(\gamma = \left(\frac{\Delta}{K}\right)^2 - \frac{2q}{K} - \frac{m^2}{K} - \frac{2Hm}{K}\). For \(K = 0\), the expression has a much simpler form, \(q = \frac{1}{3} (\epsilon + \frac{1}{2} m^2 + Hm)\).

Since, there are two values of \(q\), the one which is in the range \([0, 1]\) will be accepted. There is also a possibility that both the \(q\) values are in the range \([0, 1]\), then the equilibrium entropy will be the one with maximum value at its corresponding equilibrium \(m\). We find that for \(K < 0\), only \(q_-\) is acceptable, however, for \(K > 0\), both the \(q_{\pm}\) solutions are acceptable.

Next, we aim to find the second order transition line in the \((\Delta, \epsilon, H)\) plane. In the \(H = 0\) plane, the value of the magnetization \(m\) on the line of continuous transition is zero, however, for any nonzero \(H\), the magnetization \(m\) will have a nonzero value on the continuous transition line. In order to obtain this continuous transition line, we need to equate the first three derivatives of \(s\) (with respect to \(m\)) to zero, with the constraint that the fourth derivative will be positive. The first four derivatives of the entropy \(s\) are:

\[
\frac{\partial s}{\partial m} = q' \ln \left( \frac{2(1-q)}{\sqrt{q^2 - m^2}} \right) - \ln \sqrt{\frac{q + m}{q - m}}
\]  

(25)

\[
\frac{\partial^2 s}{\partial m^2} = q'' \ln \left( \frac{2(1-q)}{\sqrt{q^2 - m^2}} \right) - \frac{q'^2}{1-q} - \frac{1}{2} \left( \frac{(q' + 1)^2}{q + m} + \frac{(q' - 1)^2}{q - m} \right)
\]  

(26)

\[
\frac{\partial^3 s}{\partial m^3} = q''' \ln \left( \frac{2(1-q)}{\sqrt{q^2 - m^2}} \right) - \frac{q'^3}{(1-q)^2} + \frac{3}{2} \left( \frac{(q' + 1)^3}{q + m} + \frac{(q' - 1)^3}{q - m} \right) - 3 \left( \frac{2q'q''}{1-q} + \frac{q''(q' + 1)}{q + m} + \frac{q''(q' - 1)}{q - m} \right)
\]  

(27)

\[
\frac{\partial^4 s}{\partial m^4} = q'''' \ln \left( \frac{2(1-q)}{\sqrt{q^2 - m^2}} \right) - 2q''' \left( \frac{q' + 1}{q + m} + \frac{q' - 1}{q - m} \right) - \frac{2q'^2}{1-q} + 3 \left( \frac{(q' + 1)^2}{(q + m)^2} + \frac{(q' - 1)^2}{(q - m)^2} - \frac{2q'^2}{1-q} \right)
\]

(28)

where, \(q', q'' \ldots\) are partial derivatives of \(q\) w.r.t. \(m\). We solve the above first three equations numerically and obtain a set of physical solutions \((\Delta, \epsilon, m)\), such that the fourth derivative is positive. We then calculate the temperature, using the relation \(\beta = \frac{\partial s}{\partial q}\), which gives,

\[
\beta = \mp \frac{1}{K\gamma^{1/2}} \ln \left( \frac{2(1-q_{\pm})}{\sqrt{q_{\pm}^2 - m^2}} \right)
\]  

(29)

A. Repulsive Blume-Emery-Griffiths Model

In this section, we show our results for repulsive BEG model in the microcanonical ensemble in the \((\Delta, \epsilon, H)\) space. In the absence of magnetic field, this model has been recently studied in \([12, 22]\). We find that for \(-0.0828 \leq K \leq 0\) the phase diagram consists of a TCP where the \(\lambda_+\) lines meet the \(\lambda\) line in the \(H = 0\) plane. As the \(K\) decreases further, for \(-1 < K < -0.0828\), it was reported earlier in \([23]\) that a critical point (CP) appears in the ordered region.
The main plot shows the $\lambda_+$ line in the $\Delta - H$ plane, where the value of $\Delta$ increases almost linearly with $H$. From our numerical data, the variation of this line comes out to be $\Delta \approx (K+1)/2 + H$. **Bottom Inset:** The $\lambda_+$ line in the $\epsilon - H$ plane. The value of $\epsilon$ decreases and finally saturates at some value $\epsilon_{sat}$, which is numerically predicted to be $(K+1)/8$. **Top Inset:** The $\lambda_+$ line in the $T - H$ plane, showing similar qualitative behaviour as in the $\epsilon - H$ plot. The value of $T$ saturates for large $H$ at $(K+1)/4$.

### Table I

| $K$ | $\Delta$ | $\epsilon$ | $T$ | $\epsilon_{sat}$ | $T_{sat}$ | $\epsilon_w$ | $T_w$ |
|-----|----------|----------|-----|-----------------|-----------|-------------|-------|
| 0   | 0.46240  | 0.15275  | 0.33033 | 0.12502         | 0.25007   | 0.02773     | 0.8026 |
| -0.05 | 0.44741 | 0.14125  | 0.31032 | 0.11875         | 0.23750   | 0.0225      | 0.0782 |
| -0.1  | 0.43079 | 0.12556  | 0.27964 | 0.11250         | 0.22500   | 0.01306     | 0.05464|
| -0.2  | 0.39100 | 0.10343  | 0.22454 | 0.10000         | 0.20000   | 0.00334     | 0.02454|
| -0.3  | 0.34624 | 0.08837  | 0.18564 | 0.08750         | 0.17500   | 0.00087     | 0.01064|
| -0.4  | 0.29871 | 0.07519  | 0.15401 | 0.07500         | 0.15000   | 0.00019     | 0.00401|
| -0.5  | 0.24968 | 0.06259  | 0.126145| 0.06250         | 0.12500   | 0.00029     | 0.001145|
| -0.6  | 0.19996 | 0.050025 | 0.1001879| 0.0500000     | 0.100000  | 0.000025    | 0.0001879|
| -0.7  | 0.15000 | 0.0375000| 0.07509205| 0.0375000    | 0.075000  | 0.0000061   | 0.0000692|
| -0.75 | 0.12500 | 0.0312500| 0.06250008| 0.0312500   | 0.062500  | 0.00000035  | 0.0000078|
| -0.8  | \pm 0.1 | \pm 0.02500000103 | \pm 0.05000000192 | \pm 0.025 | \pm 0.05 | \pm 0.00000000103 | \pm 0.00000000192 |

Table II: Coordinates of the multicritical points (TCP, BEP), saturation values of $\epsilon$, $T$ and the width of the wings for $-1 < K \leq 0$.

Figure 7: The behaviour of crystal field ($\Delta$) and temperature ($T$) along the $\lambda_+$ line at $K = -0.4$ in the microcanonical ensemble. The values of $\epsilon$ and $T$ decrease with $H$ and saturate for large $H$. We note that the variation of $\Delta$ in the large $H$ limit is of the type, $\Delta \approx (K+1)/2 + H$. Also, the saturation values are, $\epsilon_{sat} \approx (K+1)/8$ and $T_{sat} \approx (K+1)/4$. The values of $\Delta$, $\epsilon$, $T$ for BEP and the saturation values of $\epsilon$ and $T$ are listed in Table II.

The $\lambda_+$ line, the variation of $\Delta$ and $\epsilon$ (or $T$) in the limit $H \to \infty$ can be explained in a simple way. In the limit $H \to \infty$, we can safely assume that there are no particles with spin $-1$, or in other words, $N_- = 0$. Thus, $q$ will be equal to $m$. In this limit, the entropy of the system (per particle) can be written as, $s = -(1-m)\ln(1-m) - m\ln(m)$, having a maximum at $m = 1/2$. Now, the energy per particle, in this limit, turns out to be, $\epsilon \to [(\Delta - H)/2 - (K+1)/8]$. In order for the energy (per particle) to be finite on the transition line, $\Delta$ should also increase linearly with $H$. We indeed get the linear variation of $\Delta$ with $H$ on the $\lambda_+$ line. If we use the variation of $\Delta$ as approximated numerically, i.e., $\Delta \approx (K+1)/2 + H$, we can estimate the saturation value of $\epsilon \to \epsilon_{sat} \approx (K+1)/8$. Using these values in the expression for calculating the temperature (Eq.,[29]), it can be easily shown that the saturation of $T$ will be $T_{sat} \approx (K+1)/4$. From the above observation, it is clear that the saturation values of $\epsilon$ and $T$ will become zero for $K = -1$. 

of the system along with a CEP. In this topology, as we switch on the field $H$, we note that the $\lambda_+$ lines meet at the proposed CP. Thus, the CP is actually a BEP. We show our results for $K = -0.4$ in the $(\Delta, H)$, $(\epsilon, H)$ and $(T, H)$ plane in Fig[7]. Here, we show the behaviour of the $\lambda_+$ line for positive $H$. We note that the value of $\Delta$ on the $\lambda_+$ line increases with $H$ almost linearly in the large $H$ limit. The values of $\epsilon$ and the $T$ decreases with $H$ and saturates for large $H$. We note that the variation of $\Delta$ in the large $H$ limit is of the type, $\Delta \approx (K+1)/2 + H$. Also, the saturation values are, $\epsilon_{sat} \approx (K+1)/8$ and $T_{sat} \approx (K+1)/4$. The values of $(\Delta, \epsilon, T)$ for BEP and the saturation values of $\epsilon$ and $T$ are listed in Table II.
To explain this behaviour we can separate the expression into two parts: \( \Delta \) in the TCP, and becomes a CEP. The first order line continues to exist in the paramagnetic region and becomes a surface line. For \( K \geq -1 \), the variation in \( \lambda \) meets with the phase diagram is similar to the case \( \Delta \). The attractive BEG model has been studied earlier in microcanonical ensemble in \([12]\), in the \( \epsilon - \Delta \) plane. The full phase diagram \( (\Delta - \epsilon - H) \) was not studied before for the microcanonical ensemble as of our knowledge. In this section, we present results for the attractive BEG model in the \( (\Delta - \epsilon - H) \) space. We find that: in the range \( 0 < K < 3 \), the phase diagram is similar to the case \( 0 > K \geq -0.0828 \), the \( \lambda \) line meets the first order line at a TCP, where the \( \lambda \) lines meet with the \( \lambda \) line. For \( K > 3 \), the \( \lambda \) line truncates on the first order line in the \( H = 0 \) plane before reaching the TCP, and becomes a CEP. The first order line continues to exist in the paramagnetic region and becomes a surface in the \( \Delta - \epsilon - H \) space which separates two paramagnetic phases P1 and P2 (discussed before in Sec[I][3] and the wings no longer exists.

For small positive \( K \), the variation of \( \epsilon \) is monotonic with \( H \) on the \( \lambda \) lines, similar to negative \( K \). For large positive \( K (\geq 1) \), however, the variation in \( \epsilon \) is non-monotonic on the transition line. Fig[I][a] we summaries this behaviour. To explain this behaviour we can separate the expression into two parts: \( \epsilon = \epsilon_1 + \epsilon_2 \), where, \( \epsilon_1 = \Delta q - \frac{1}{2} m^2 - H m \) and \( \epsilon_2 = -K q^2 \). We note that the variation of \( \epsilon_1 \) remains similar for small as well as large \( K \), however, the variation of \( \epsilon_2 \) is different for small and large \( K \): it decreases with \( H \) for small \( K \) while increases with \( H \) for large \( K \) (see Fig[I][b]). The variation in \( \epsilon_2 \) is mainly due to the variable \( q \), which itself shows such behaviour. In \( \epsilon_1 \) also, we have the variable \( q \), but it appears with other terms, the variation of which dominates over the variation in \( q \), and \( \epsilon_1 \) does not change its qualitative behaviour when we change \( K \). For small \( K \), since both the \( \epsilon_1 \) and \( \epsilon_2 \) decreases with \( H \), the sum also decreases with \( H \). For large \( K \), there is a competition between \( \epsilon_1 \) and \( \epsilon_2 \). In the small \( H \) regime, the variation in \( \epsilon_2 \) dominates, which gives rise to an increase in \( \epsilon \) with \( H \). For large \( H \), the variation in \( \epsilon_1 \) starts dominating and \( \epsilon \) decreases with \( H \). For very large \( H \), both the \( \epsilon_1 \) and \( \epsilon_2 \) will finally saturate and also \( \epsilon \). The saturation values of \( \epsilon \) follows similar relationship with \( K \) as obtained for the negative \( K \). The variation of \( T \) also shows similar non-monotonic behaviour in the same range of \( K \). We made similar observation in the canonical ensemble in Sec[I][3].
We find that not just the multicritical points, the continuous transition lines and other multicritical points are different for canonical and microcanonical ensembles for a given value of first order line and the multicritical points are known to be located differently [36–39]. It was reported that the TCP or BEP for canonical and microcanonical ensembles in the limit $K > 0$ for both the ensembles become close to each other for large $H$. In the inset of Fig. 10, we plot the difference in the value of $\Delta$ for the two ensembles for a given $K$, and plot it as a function of $H$ in the inset of Fig. 11. We note that this difference reaches to zero almost exponentially as $H$ increases. Thus, in the limit $H \rightarrow \infty$, these critical lines for both the ensembles (iii): $K=1.0$

Figure 9: Non-monotonic variation of the $\epsilon$ as a function of $H$ along the $\lambda_+$ line for positive $K$. (a) The variation of $\epsilon$ along the $\lambda_+$ line for various $K$. For small $K$, the curve is monotonic; $\epsilon$ decreases with $H$ and then saturates. For large $K$, $\epsilon$ varies non-monotonically with $H$. (b) Variation of $\epsilon_1$ and $\epsilon_2$ for $K = 0.20$ and $K = 2.0$. The variation of $\epsilon_1$ is similar for both the small and large $K$ values, however, the qualitative nature in the variation of $\epsilon_2$ is different for small and large $K$, that causes the non-monotonic variation in $\epsilon$.

V. ENSEMBLE INEQUIVALENCE

The inequivalence of different ensembles in the Blume-Emery-Griffiths model has been reported earlier in [12, 22] in the absence of magnetic field. In the $(T - \Delta)$ plane, while the $\lambda$ line equation is same in both the ensembles, the first order line and the multicritical points are known to be located differently [36–39]. It was reported that the TCP and other multicritical points are different for canonical and microcanonical ensembles for a given value of $K$ [12, 22].

In this work, we have looked at all the three continuous transition lines($\lambda, \lambda_+, \lambda_-$) and the first order surfaces. We find that not just the multicritical points, the continuous transition lines $\lambda_+$ and $\lambda_-$ are also different in the two ensembles. In fact, the ensemble inequivalence of the two ensembles can be seen as a consequence of this inequivalence. For $K = 0$, which corresponds to the Blume-Capel model, in Fig. 11 we plot the locus of the $\lambda_+$ line in two ensembles and one can see that they are different ($\lambda_-$ line also behaves in a similar way). We plot the product of $\beta \Delta$ on the $\lambda_+$ line as a function of $H$ for both the ensembles, and note that for $H \rightarrow 0$, these lines meet at different points, which is the TCP of their corresponding ensembles. For canonical ensemble, these $\lambda_\pm$ lines meet at $\beta \Delta_{TCP} \approx 1.3863$, while for microcanonical ensemble, these lines meet at $\beta \Delta_{TCP} \approx 1.3998$ (see Fig. 10). We also note that the $\lambda_+$ lines for the two ensembles become close to each other for large $H$. We plot the difference in the value of $\beta \Delta$ for the two ensembles for a given $K$, and plot it as a function of $H$ in the inset of Fig. 11. We note that this value decreases exponentially to zero as $H$ becomes large.

For non-zero $K$, the $\lambda_\pm$ lines meet the $\lambda$ line at the TCP. This topology persists for $0 \geq K \geq -0.1838$ in the canonical ensemble, whereas for microcanonical ensemble this topology occurs for $0 \geq K \geq -0.0828$. As $K$ decreases further(for canonical ensemble $-0.1838 < K < -1$ and for microcanonical ensemble $-0.0828 < K < -1$), the $\lambda_\pm$ lines move inside the ordered region and meet at BEP in the $H = 0$ plane. Interestingly, we find that the difference in the position of BEP and CEP decreases with decreasing $K$ and for $K = -1$ the two ensembles become equivalent. In Fig. 11 (a), we plot the value of $\beta \Delta$ at the BEP for both the ensembles. We note that the value of $\beta_{BEP} \Delta_{BEP}$ for the two ensembles becomes closer as $K \rightarrow -1$. In the inset of Fig. 11 (a), we also plot the difference in the value of $\beta_{BEP} \Delta_{BEP}$ for microcanonical and canonical ensembles, and note that this difference decreases exponentially as $K \rightarrow -1$. Thus, for $K \leq -1$, we find that there is no ensemble inequivalence in the $H = 0$ plane.

We have shown in Sec. III A and Sec. V A that for $K \leq -1$, there is no phase transition for finite magnetic field in either of the ensembles and hence there are no wings. Thus there is no inequivalence in the $H \neq 0$ plane as well. For $K > -1$, however, we do have wings and the continuous transition lines $\lambda_+$ and $\lambda_-$, meet at its corresponding TCP or BEP for canonical and microcanonical ensembles in the limit $H \rightarrow 0$. Thus, the critical lines in the $H$ plane are different for the two ensembles for $K > -1$. In Fig. 11 (b), we plot the value of $\beta \Delta$ on the continuous transition line for $K = -0.3$, as a function of $H$ for both the ensembles. We note that the two lines are different for small $H$, however, these lines tend to meet each other for large $H$. We measure the difference between the value of $\beta \Delta$ for the two ensembles for a given $H$, and plot it as a function of $H$ in Fig. 11 (b) inset. We note that this difference reaches to zero almost exponentially as $H$ increases. Thus, in the limit $H \rightarrow \infty$, these critical lines for both the ensembles...
Figure 10: Ensemble inequivalence in the Blume-Capel model ($K = 0$). We show the locus of the $\lambda_+$ line (product of $\beta\Delta$) as a function of $H$, which is different for the two ensembles. In the inset, we plot the difference in the value of $\beta\Delta$ for the two ensembles for a given $H$, as a function of $H$. This value decreases to zero almost exponentially.

Figure 11: Ensemble inequivalence in the Blume-Emery-Griffiths model ($K \neq 0$). (a) The product of $\beta\Delta$ at the BEP, as a function of $K$. We note that the difference in the $\beta_{\text{BEP}}\Delta_{\text{BEP}}$ decreases to zero as $K \to -1$. (b) The locus of $\lambda_+$ line (product of $\beta\Delta$) for $K = -0.3$, for the two ensembles. These lines are different in the two ensembles in the small $H$ regime, however, the lines tend to becomes closer as $H$ increases. In the inset, we plot the difference in the value of $\beta\Delta$ for a given $H$, for the two ensembles. This difference is large in the small $H$ regime, however, it tends to zero as $H$ increases.

From the above discussion, it is clear that the ensemble inequivalence is observed for $K > -1$ with small $H$ values, however, for $K \leq -1$, the phase diagrams in the two ensembles become equivalent. In previous literature [37, 40–42], where ensemble inequivalence with long-range interactions are studied, it was found that whenever the two ensembles (either micro-canonical/canonical or canonical/grand-canonical) have a continuous transition, the transition occurs at the same critical point. The phase diagrams of the two ensembles can however be different from each other when the phase transition becomes first order in one of the ensembles. This type of behavior is observed in many systems such as the spin-1 Blume-Emery-Griffiths (BEG) model [40, 41], the ABC model [37, 42] etc. However, it is also shown that the different ensembles may exhibit different critical points. In a generalized ABC model in Ref. [43], the canonical and grand-canonical ensembles are found to exhibit a second-order phase transition at different points in the phase space. In [36], a general statement is provided to check the possibility of ensemble inequivalence for continuous transition using Landau theory. The transition is observed for a system undergoing phase transition governed by some order parameter, $'\mu_1'$ (say) in a given ensemble. This parameter can be the average magnetization in the case of a magnetic transition, or the difference in the density of the two phases for a liquid-gas phase transition. Then the model is considered within a ‘higher’ ensemble, where a certain thermodynamic variable, denoted by $'\mu_2'$, is allowed to fluctuate, (within the ‘lower’ ensemble, $'\mu_2'$ was kept at a fixed value). In the case where $\mu_2$ is the energy, the two ensembles would correspond to the canonical and micro canonical ensembles, while in the case when $\mu_2$ is the particle
density, they would correspond to the grand-canonical and canonical ensembles. The system is thus described by the Landau free energy denoted by \( f(\mu_1, \mu_2) \). They found that if \( f(\mu_1, \mu_2) = f(-\mu_1, \mu_2) \), the two ensemble should be equivalent, when any of those show a continuous transition, if on the other hand, \( f(\mu_1, \mu_2) = f(-\mu_1, -\mu_2) \), the system will show ensemble inequivalence even for continuous transition.

In our case, \( \mu_1 \) is the magnetization \( m \), and \( \mu_2 \) is the energy \( \epsilon \) of the system. The lower ensemble in our case is thus the microcanonical ensemble and the higher one is the canonical. If we check the above symmetries, we note that neither of the conditions studied in \cite{36} is satisfied. When we add a magnetic field term, the symmetry of the problem is broken and we find that we have ensemble inequivalence even when the two ensembles show second order transition.

VI. CONCLUSION

The repulsive and attractive BEG model in canonical and in microcanonical ensemble has been studied earlier. This model is known to exhibit many multicritical points along with the first and second order line of transition. Earlier the model was studied on the \( T - \Delta \) plane \cite{1, 12–32} and the ensemble inequivalence was reported \cite{12, 22}. The full phase diagram in the \( (T - \Delta - H) \) space was studied only for the attractive BEG model in the presence of external field in the canonical ensemble \cite{14}. We revisited the model in order to study the full phase diagram in the \( T - \Delta - H \) space in both the ensembles on a fully connected graph. Though we explored the phase diagram for the entire range of \( K \), we mainly focused on the repulsive BEG model. We found that for small negative \( K \), the model exhibits a TCP where the \( \lambda \pm \) lines meet. For the canonical ensemble the range of \( K \) for such a topology was \( 0 \geq K \geq -0.1838 \), whereas for microcanonical ensemble it was \( 0 \geq K \geq -0.0828 \). As \( K \) decreases further, the wings meet inside the ordered phase at a BEP (for canonical \(-0.1838 > K > -1 \) and for microcanonical \(-0.0828 > K > -1 \)). This point was identified as an ordered critical point in the earlier studies \cite{22}. We also observed that as \( K \rightarrow -1 \), the width of the wings decreases. At exactly \( K = -1 \), the wing width in temperature becomes zero along with the CEP and BEP reaching \( T = \Delta = 0 \) in both canonical and microcanonical ensemble. For \( K \geq -1 \), we observe that the \( \lambda \pm \) lines are different in the two ensembles and they meet at different multicritical points in the \( H = 0 \) plane. For \( K \leq -1 \), we find that there is no phase transition in the \( H \) plane for both the ensembles.

Absence of transition in the \( H \) plane for \( K \leq -1 \) can be argued by looking at the energy. The energy of the system in terms of the order parameters can be written as: \( \epsilon = -\frac{1}{2}(m^2 + Kq^2) + \Delta q - Hm \). For low temperatures we can take \( m \approx q \). In that case, only for \( K > -1 \) the first term can lower the energy and can take over the entropy at sufficiently small temperatures. Hence the transition in \( H \) plane is likely only for \( K > -1 \).

Disorder, in general, is known to smoothen the first order transition and has been known to convert a TCP into a BEP \cite{34}. We studied a pure BEG model here. We found that the frustration induced by negative \( K \) affects the phase diagram in a similar manner. It would be interesting to see if there is a similarity in the phase diagram of the two problems even in finite dimensions. Also, earlier work on frustrated BEG on bipartite lattices \cite{23, 26} shows that two new ordered phases, namely antiquadrupolar and ferrimagnetic occur for repulsive BEG. These phases are not possible on a fully connected graph. Hence it would be interesting to study the effect of large negative \( K \) on these two phases especially in the mean field limit on a bipartite lattice.

VII. ACKNOWLEDGEMENT

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Appendix A: Calculation of Free energy functional

In order to solve the Hamiltonian(Eq \( \text{(1)} \)), we take the non-interacting Hamiltonian: \( H \sum_i S_i - \Delta \sum_i S_i^2 \), with the probability measure:
\[ P(1) = \frac{e^{\beta(H-\Delta)}}{1 + 2e^{-\beta\Delta \cosh \beta H}} \]  
(A-1)

\[ P(-1) = \frac{e^{-\beta(H+\Delta)}}{1 + 2e^{-\beta\Delta \cosh \beta H}} \]  
(A-2)

\[ P(0) = \frac{1}{1 + 2e^{-\beta\Delta \cosh \beta H}} \]  
(A-3)

The scaled Cumulant generating function (CGF) is:

\[ \Lambda(k_1, k_2) = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}^{P} \left[ e^{N(x_1k_1 + x_2k_2)} \right] \]

\[ = \log(1 + 2e^{k_2-\beta\Delta \cosh(k_1 + \beta H)}) - \log(1 + 2e^{-\beta\Delta \cosh \beta H}) \]  
(A-5)

The rate function \( R \) for the non-interacting Hamiltonian then can be evaluated using Gartner Ellis theorem[35] and is given by:

\[ R(x_1, x_2) = \sup_{k_1, k_2} [x_1k_1 + x_2k_2 - \Lambda(k_1, k_2)] \]

\[ = \sup_{k_1, k_2} [x_1k_1 + x_2k_2 - \log(1 + 2e^{k_2-\beta\Delta \cosh(k_1 + \beta H)})] + \log(1 + 2e^{-\beta\Delta \cosh \beta H}) \]  
(A-6)

Minimising the above equation w.r.t \( k_1 \) and \( k_2 \) gives the following relations:

\[ \Rightarrow x_1 = \frac{2e^{k_2-\beta\Delta \sinh k_1^*}}{1 + 2e^{k_2-\beta\Delta \cosh k_1^*}} \]  
(A-7)

\[ \Rightarrow x_1 = \frac{2e^{k_2-\beta\Delta \cosh k_1^*}}{1 + 2e^{k_2-\beta\Delta \cosh k_1^*}} \]  
(A-8)

where \( k_1^* \) and \( k_2^* \) are the minimums of \( k_1 \) and \( k_2 \). This implies:

\[ \frac{x_1}{x_2} = \tanh k_1^* \]  
(A-9)

The interacting part of the Hamiltonian is \(-\frac{1}{2}x_1^2 - \frac{K}{2}x_2^2\). Now the full rate function of the total Hamiltonian can be obtained by making use of the tilted LDP. Tilted LDP is one of the properties of the large deviation principle (LDP), using which one can generate a new large deviation principle (LDP) from an old LDP by a change of measure. It states that if \( P_n \) is the probability measure on \( \mathcal{H} \) which satisfies LDP with rate function \( I \) and \( \Phi \) be a continuous function mapping \( \mathcal{H} \) into \( \mathbb{R} \) which is bounded above. Let \( W_n \) be a sequence of a random variable taking values from \( \mathcal{H} \) and a subset \( A \) of \( \mathcal{H} \) we define the probability measures \( Q_n, \Phi = 1 \int_A e^{[\Phi(x)]} P_n \{ W_n \in dx \} \)

where \( Z_n \) denotes the normalizing constant. Then the sequence of probability measures \( \{Q_n, \Phi, n \in N\} \) satisfies LDP on \( \mathcal{H} \) with rate function

\[ I_\Phi(x) = [I(x) - \Phi(x)] - \inf_{y \in \mathcal{H}} \{I(y) - \Phi(y)\} \]

In our system we get the full rate function as:

\[ I(x_1, x_2) = x_1k_1^* + x_2k_2^* - \Lambda(k_1^*, k_2^*) - \frac{\beta x_1^2}{2} - \frac{\beta K x_2^2}{2} - \inf_{k_1, k_2} [R(k_1, k_2) - \frac{\beta k_1^2}{2} - \frac{\beta K k_2^2}{2}] \]  
(A-10)
Minimisation of which w.r.t $x_1$ and $x_2$ gives the following free energy functional:

$$\tilde{f}(m, q) = \frac{\beta m^2}{2} + \frac{\beta K q^2}{2} - \log(1 + 2e^{\beta(Kq - \Delta)} \cosh(\beta(m + H))) + \log(1 + 2e^{-\beta\Delta} \cosh(\beta H))$$

(A-11)

where the minimums are denoted as: $x_1^* = m, x_2^* = q$, which gives the Eq[8] at the fixed points in Section[11]
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