An analysis of the maximum likelihood estimates for the Lomax distribution

Tanujit Chakraborty

Abstract
The Lomax distribution is a popularly used heavy tailed distribution that has application in actuarial sciences, reliability, operations research, income and wealth distributions among many others. But the analytical behavior of the maximum likelihoods for the Lomax distribution is anomalous and it causes difficulty in real life application of the model for practical purposes. In this work, we provide precise arguments for the anomalous behavior of the log-likelihood surface of the Lomax distribution. A relative variability measure, namely the coefficient of variation (CV) is introduced to characterize the profile log-likelihood of the Lomax distribution. A sufficient condition for the existence of the global maximum of the maximum likelihood estimates and asymptotic results for statistical inferences are also discussed in this paper. An application on a reliability dataset describes the practical relevance of these results.

Key Words: Lomax distribution; Maximum Likelihood estimation; Asymptotic properties.

1 Introduction
The Lomax distribution (also called Pareto type II distribution) is a widely used distribution that has applications in the field of actuarial science, reliability modeling, life testing, economics, network analysis, and operations research \[21, 17, 9, 7\]. It was first introduced for modeling business failures \[21\] but latter applied to solve prediction problems arising in the fields of biological sciences, business economics, and internet traffic modeling as well

*Corresponding author’s E-mail: tanujit_r@isical.ac.in

\(^{a}\) Statistical Quality Control and Operations Research Unit, Indian Statistical Institute, Kolkata.
A Lomax distribution is one of the well known distribution that is used to fit heavy tailed data. It is essentially a Pareto type II distribution that has been shifted so that its support begins at zero $[21, 1]$. The Lomax distribution can be motivated in a number of ways. For example, the Lomax distribution was shown to be the limiting distribution of the residual lifetime at great age $[5]$ and it can also be derived as a special case of a particular compound gamma distribution $[11]$. Even mixture of the Lomax distribution with Poisson distribution was proposed by $[2]$ and the record values of the Lomax distribution were studied by $[1]$.

Although the Lomax distribution can be used quite effectively to analyze data sets but there is enough evidence that numerical methods used for maximum likelihood estimation do not always work well for small or moderate samples $[20, 8, 3, 19]$. To avoid these limitations, a recent work proposed to make Bayesian inferences for the parameters of the Lomax distribution using non-informative priors $[13]$. This paper deals with the problem from frequentist point of view. We will precisely provide mathematical arguments to explain the anomalous behavior of the likelihood surface for the Lomax distribution. A sufficient condition for the existence of the global maximum of the maximum likelihood estimates is given using the notion of the coefficient of variation, a relative variability measure for the empirical data. This finding will be useful for practitioners since calculation of the coefficient of variation from the empirical data is simple. Also, we establish the asymptotic results for the maximum likelihood estimates of the parameters for the Lomax distribution using estimates from modified log-likelihood function. In order to practically validate these results, this paper provides illustrative examples on a real dataset related to the active repair times (in hours) for an airborne communication transceiver.

The rest of the paper is organized as follows. In Section 2, we discuss the basic definition and the model. The sufficient condition for the existence of the global maximum of the maximum likelihood estimates for the Lomax distribution is given in Section 3. The asymptotic results for statistical inferences are discussed in Section 4. An application to the reliability data is presented in Section 5. Finally, we conclude the paper and identify future scope of this work in Section 6.
2 Preliminaries

The Lomax distribution was proposed by Lomax (1954) for modeling business failures [21]. It has been widely studied in the variety of contexts like income or wealth inequality, internet traffic modeling, actuarial science, medical and biological science, and reliability and life testing to mention a few [4, 9, 15, 12]. Various modifications and extensions to the distribution were also studied in the past literature, one can refer to [23, 14].

Definition 1 A random variable $X$ follows Lomax distribution with parameters $\alpha, \gamma$, and denoted by $LM(\alpha, \gamma)$ if the cumulative distribution function (CDF) is of the form:

$$F(x; \alpha, \gamma) = 1 - \left(1 + \frac{x}{\gamma}\right)^{-\alpha}; \text{ with support } x \geq 0,$$

where $\alpha (> 0)$ be the shape parameter (real) and $\gamma (> 0)$ is a scale parameter (real).

The probability distribution function (PDF) is given by

$$f(x; \alpha, \gamma) = \frac{\alpha}{\gamma} \left(1 + \frac{x}{\gamma}\right)^{-\alpha-1}; \text{ } x \geq 0 \quad (1)$$

The quantile function is given by

$$Q(u; \alpha, \gamma) = \gamma \left[(1 - u)^{-1/\alpha} - 1\right]; \quad (2)$$

and the mean function is

$$E(X) = \gamma \frac{\alpha}{\alpha - 1}, \text{ for } \alpha > 1$$

and the variance function is

$$V(X) = \frac{\alpha \gamma^2}{(\alpha - 1)^2(\alpha - 2)}, \text{ for } \alpha > 2.$$

The expression for the $r^{th}$ order moment of the Lomax distribution is as follows [16]:

$$E \left[X^r\right] = \frac{\alpha \gamma^r \Gamma (r + 1) \Gamma (\alpha - r)}{\Gamma (\alpha + 1)}; \text{ } \alpha > r, \text{ } r = 1, 2, 3, \ldots \quad (3)$$
The median is given by $\gamma \left(2^{1/\alpha} - 1\right)$ and the mode is zero. The hazard function is given by

$$h(x; \alpha, \gamma) = \frac{\alpha}{\gamma} \left(1 + \frac{x}{\gamma}\right)^{-1}; \ x \geq 0$$ (4)

which is a decreasing function of $x$. This makes the Lomax distribution model suitable for components that age with time. The survival function is given by

$$S(x; \alpha, \gamma) = \left(1 + \frac{x}{\gamma}\right)^{-\alpha}; \ x \geq 0$$ (5)

We note that the Lomax distribution can be expressed as a scale mixture of distributions using the following hierarchical form as in [13]

$$X \mid \gamma, \lambda \sim \text{Exponential} \left(\frac{\lambda}{\gamma}\right) \text{ and } \lambda \mid \gamma \sim \text{Gamma}(\alpha, 1).$$

We also notice that as the shape parameter $\alpha \to \infty$, the Lomax distribution approaches the Dirac delta function $\delta(x)$ [10] which is defined as follows.

**Definition 2** The Dirac-delta function, which was first introduced by physicist Dirac (1958) is a function that is equal to zero everywhere except for zero with an integral of one over the entire domain. It is defined as follows.

$$\delta(x) = \begin{cases} +\infty & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

which is constrained by $\int_{-\infty}^{\infty} \delta(x)dx = 1$.

Below we present in figures the behavior of the Lomax distribution PDF and CDF for various values of the shape parameter $\alpha (= 0.5, 1, 2, 3, 10000)$ and scale parameter $\gamma = 1$. It is clear from Figure[11] that Lomax distribution will be a good choice for modeling heavy tail behaviors.

## 3 Main Results

Given a set of samples $\{x_i\}$ of size $n$, the log-likelihood function for the Lomax distribution, after dividing it by the sample size $n$, is given by

$$\ell(\alpha, \gamma) = \log \alpha - \log \gamma - \frac{(\alpha + 1)}{n} \sum_{i=1}^{n} \log \left(1 + \frac{x_i}{\gamma}\right)$$ (6)
Differentiating (6) w.r.t. $\alpha$ and $\gamma$, respectively, we have:

$$\frac{\partial \ell(\alpha, \gamma)}{\partial \alpha} = \frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^{n} \log \left(1 + \frac{x_i}{\gamma}\right)$$  \hspace{1cm} (7)

$$\frac{\partial \ell(\alpha, \gamma)}{\partial \gamma} = -\frac{1}{\gamma} + \frac{(1 + \alpha)}{n\gamma} \sum_{i=1}^{n} \left(\frac{x_i}{\gamma + x_i}\right)$$  \hspace{1cm} (8)

Equating to zero the derivative of $\ell(\alpha, \gamma)$ w.r.t. $\alpha$ in (7), we obtain $\hat{\alpha} = \alpha(\gamma)$ as follows:

$$\hat{\alpha} = \alpha(\gamma) = \frac{n}{\sum_{i=1}^{n} \log \left(1 + \frac{x_i}{\gamma}\right)}$$  \hspace{1cm} (9)

Differentiating (9) w.r.t. $\gamma$ we have,

$$\alpha'(\gamma) = \frac{\hat{\alpha}^2}{n\gamma} \sum_{i=1}^{n} \frac{x_i}{\gamma + x_i}$$  \hspace{1cm} (10)
It is important to note that there is no closed form solution to the likelihood based on (7) and (8), and a suitable numerical algorithm (for example, Newton-Raphson method) can be employed to obtain the maximum likelihood estimates (MLEs) of the $\alpha$ and $\gamma$. Different estimation procedures of the MLEs have been discussed in previous literature, for example see [15]. But for small or medium-sized samples, anomalous behavior of the likelihood surface can be encountered when sampling from the Lomax distribution. In this paper, we characterize the profile log-likelihood function in terms of the coefficient of variation (CV), defined as follows:

**Definition 3** The CV is the ratio of the standard deviation $(s)$ to the mean $(\mu)$,

\[
CV = \frac{s}{\mu},
\]

where $\mu = \frac{1}{n} \sum_{i=1}^{n} x_i$ and $s = \sqrt{\frac{1}{n} \sum_{i=1}^{n} x_i^2 - \mu^2}$.

Using standard notation, the profile log-likelihood function is given by

\[
\ell_p(\gamma) = \sup \ell(\hat{\alpha}, \gamma) = \log(\alpha(\gamma)) - \log\gamma - 1 - \frac{1}{\alpha(\gamma)} \tag{11}
\]

Differentiating (11) w.r.t. $\gamma$, we have the following:

\[
\ell_p'(\gamma) = \frac{\alpha'(\gamma)}{\alpha(\gamma)} - \frac{1}{\gamma} + \frac{\alpha'(\gamma)}{[\alpha(\gamma)]^2} \tag{12}
\]

Below we present the following lemmas which will be useful to find the sufficient condition for the existence for the global maximum of the profile log-likelihood function (11).

**Lemma 1** The following limit holds:

(a) \( \lim_{\gamma \to \infty} \gamma \log \left( 1 + \frac{x}{\gamma} \right) = x; \)

(b) \( \lim_{\gamma \to \infty} \frac{\gamma x}{\gamma + x} = x; \)

(c) \( \lim_{\gamma \to \infty} \gamma^2 \left( \log \left( 1 + \frac{x}{\gamma} \right) - \frac{x}{\gamma + x} \right) = \frac{x^2}{2}. \)
Proof: The proof is elementary and can easily be done using series expansions.

**Lemma 2**  The following limit holds:

(a) \( \lim_{\gamma \to \infty} \frac{1}{\alpha(\gamma)} = 0; \)

(b) \( \lim_{\gamma \to \infty} \frac{\alpha(\gamma)}{\gamma} = \frac{1}{\bar{x}}, \) where \( \bar{x} \) is the sample mean;

(c) \( \ell_0 \equiv \lim_{\gamma \to \infty} \ell_p(\gamma) = \log \left( \frac{1}{\bar{x}} \right) - 1. \)

Proof: The proofs are straightforward and can be done using Lemma (1).

(a) \( \lim_{\gamma \to \infty} \frac{1}{\alpha(\gamma)} = \lim_{\gamma \to \infty} \frac{1}{n} \sum_{i=1}^{n} \log \left( 1 + \frac{x_i}{\gamma} \right) = \lim_{\gamma \to \infty} O \left( \frac{1}{\gamma} \right) = 0. \)

(b) \( \lim_{\gamma \to \infty} \frac{\alpha(\gamma)}{\gamma} = \lim_{\gamma \to \infty} \frac{n}{\gamma \sum_{i=1}^{n} \log \left( 1 + \frac{x_i}{\bar{x}} \right)} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}. \)

(c) \( \lim_{\gamma \to \infty} \ell_p(\gamma) = \lim_{\gamma \to \infty} \left[ \log \left( \frac{\alpha(\gamma)}{\gamma} \right) - 1 - \frac{1}{\alpha(\gamma)} \right] = \log \left( \frac{1}{\bar{x}} \right) - 1. \)

A sufficient condition for monotonic increasing (decreasing) for the profile log-likelihood function is presented in Theorem (1) below, for sufficiently large \( \gamma \). Also, we present the sufficient condition for the global maximum of the likelihood function for the Lomax distribution to be at a finite point in Corollary (1).

**Theorem 1** Let \( X \) follows \( LM(\alpha, \gamma) \) distribution with \( \alpha, \gamma > 0 \). A sufficient condition for \( \ell_p(\gamma) \) to be monotonically decreasing function is \( CV > 1 \) for \( \gamma \to \infty \), and if \( CV < 1 \), it is monotonically increasing.

Proof: Using (9) and (10) in Eqn. (12), we can write \( \ell'_p(\gamma) \) as:

\[
\ell'_p(\gamma) = -\frac{1}{\gamma} \left[ \frac{\sum_{i=1}^{n} \log \left( 1 + \frac{x_i}{\gamma} \right) - \sum_{i=1}^{n} \frac{x_i}{\gamma + x_i}}{\sum_{i=1}^{n} \log \left( 1 + \frac{x_i}{\gamma} \right)} \right] + \frac{1}{n \gamma} \sum_{i=1}^{n} \frac{x_i}{\gamma + x_i} \quad (13)
\]
Using the limits of Lemma 1 in Eqn. (13), we have

$$- \lim_{\gamma \to \infty} \gamma^2 \ell_p'(\gamma) = \frac{1}{2} \times \frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i} - \bar{x}.$$  \hspace{1cm} (14)

Finally, we note that $$- \lim_{\gamma \to \infty} \gamma^2 \ell_p'(\gamma) > 0$$ when the R.H.S of Eqn.(14) is strictly greater than 0. Alternatively, the likelihood function is monotonic decreasing when $$\frac{1}{2n} \sum_{i=1}^{n} x_i^2 - \bar{x}^2 > 0$$, or, equivalently, $$CV > 1$$.

In a similar way, we can show that if $$CV < 1$$, then the $$\ell_p(\gamma)$$ is monotonic increasing function for sufficiently large $$\gamma$$. ■

**Remark 1** As a consequence of Theorem 1, it can be immediately concluded that $$\ell_p(\gamma)$$ tends to $$\ell_0$$ based on Lemma 2 and $$\ell_p(\gamma)$$ is a monotonic function for sufficiently large $$\gamma$$. The value of $$CV$$ as a measure that can be useful to determine when $$\ell_p(\gamma)$$ will be monotonic increasing or decreasing function for sufficiently large $$\gamma$$.

**Corollary 1** Given a set of samples $$\{x_i\}$$ of (+)ve numbers with $$CV > 1$$, the profile likelihood function for the LM($$\alpha, \gamma$$) distribution has a global maximum at a finite point.

**Proof:** For small or moderate values of $$\gamma$$, using (9), we have

$$\lim_{\gamma \to 0} \alpha(\gamma) = \lim_{\gamma \to 0} \frac{n}{\sum_{i=1}^{n} \log \left(1 + \frac{x_i}{\gamma}\right)} = 0.$$  \hspace{1cm} (15)

Now, using (15) in (11) we have the following:

$$\lim_{\gamma \to 0} \ell_p(\gamma) = -\infty.$$  \hspace{1cm} (16)

Since $$\ell_p(\gamma)$$ is a continuous and monotonic decreasing function for sufficiently large $$\gamma$$ (as in Theorem 1 and using (16)), we can conclude that a global maximum exists at a finite point when $$CV > 1$$. ■

**Remark 2** A sufficient condition for the existence of MLEs for small or moderate samples are given in Corollary 1. It is shown that the likelihood function for the Lomax distribution has a global maximum for the samples $$\{x_i\}$$ with $$CV > 1$$ at a finite point. The calculation of $$CV$$ is completely based on available empirical data. This finding is useful from practitioner’s point of view as well and simple to compute numerically.
4 Asymptotic Properties for the MLE

We observe a random sample \( x_1, x_2, \ldots, x_n \) from (5) and denote by \( x^{(1)} < x^{(2)} < \ldots < x^{(n)} \) the ordered values of \( x_i \)'s. As discussed in the previous section, the MLEs of \( \alpha \) and \( \gamma \) can be obtained by maximizing the profile log-likelihood \( \ell_p(\gamma) \) w.r.t. \( \gamma \). A sufficient condition for the existence of MLEs for small or moderate samples are given in Corollary (1). Next, we discuss about the asymptotic properties for the MLEs for the Lomax distribution. Asymptotic results for Generalized Weibull distribution, Generalized extreme value distributions, and for a class of non-regular cases are already available in the previous literature [24, 22].

The most natural way to the unknown parameters is to first estimate the scale parameter \( \gamma \) by its consistent estimator \( \hat{\gamma} = x^{(n)} \). Then, the modified log-likelihood function based on the \((n - 1)\) observations after ignoring the largest observation and replacing \( \hat{\gamma} = x^{(n)} \) is as follows.

\[
\tilde{\ell}(\alpha, \hat{\gamma}) = (n - 1) \log \alpha - (n - 1) \log x^{(n)} - (\alpha + 1) \sum_{i=1}^{n-1} \log \left(1 + \frac{x_i}{x^{(n)}}\right) \tag{17}
\]

Differentiating (17) w.r.t. \( \alpha \) and equating to zero, we obtain the modified MLE of \( \alpha \) as

\[
\hat{\alpha} = \frac{n - 1}{\sum_{i=1}^{n-1} \log \left(1 + \frac{x_i}{x^{(n)}}\right)} \tag{18}
\]

Below we describe the asymptotic marginal distribution of \( \hat{\gamma} \) and the asymptotic distribution of \( \hat{\alpha} \) given \( \hat{\gamma} \).

**Theorem 2**

\( a \) The marginal distribution of \( \hat{\gamma} = x^{(n)} \) is given by

\[
\mathbb{P}[\hat{\gamma} \leq t] = \left[1 - \left(1 + \frac{t}{\gamma}\right)^{-\alpha}\right]^n.
\]

\( b \) Asymptotically, as \( n \to \infty \),

\[
n^{-\frac{1}{\alpha}} \left[1 + \frac{X^{(n)}}{\gamma}\right] \to Z^{-\frac{1}{\alpha}} \quad \text{in laws},
\]

where \( Z \) denotes the standard exponential random variable.
Proof:

(a) It is straightforward from the CDF of $LM(\alpha, \gamma)$ distribution due to $
\tilde{\gamma} = x(n)$.

(b) Let $U(n)$ denotes the largest order statistic of a sample of size $n$ from $Uniform(0, 1)$ distribution. Then using the quantile function as defined in Eqn. (2), we obtain the following:

$$X(n) \overset{d}{=} Q(U(n)) = \gamma \left[ (1 - U(n))^{-1/\alpha} - 1 \right].$$

After simplification, we have

$$n^{-1/\alpha} \left[ 1 + \frac{X(n)}{\gamma} \right] = \left[ n \left( 1 - U(n) \right) \right]^{-1/\alpha}.$$

But $n \left( 1 - U(n) \right)$ converges in laws to the standard exponential random variables as $n \to \infty$. Hence, we have the theorem.

\begin{flushright}
\blacksquare
\end{flushright}

**Theorem 3** As $n \to \infty$, the conditional asymptotic distribution of $\tilde{\alpha}$ given $\tilde{\gamma} = x(n)$ of the modified log likelihood function (17) is univariate normal distribution with mean $\alpha$ and covariance matrix $I^{-1}_\alpha$, the inverse of the Fisher information matrix.

\begin{flushright}
\blacksquare
\end{flushright}

**Proof:** An analogous result for non-regular case is available at [24, 22]. Since the modified log-likelihood in our case (Eqn. 17) satisfies all the regularity conditions for the asymptotic normality of the MLE ($\tilde{\alpha}$) as in [24, 22], hence the result follows. \(\blacksquare\)

5 Application

We consider a dataset from the field of reliability related to the active repair times (in hours) for an airborne communication transceiver where the equipment was only noticed during active operation time. The dataset is available and well described in [25]. The dataset contains 46 observations having $CV = 1.370$ ($>1$) are as follows.
**Dataset:** 0.2, 0.3, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

The estimated values of the parameters using MLE for the Lomax distribution is \( \hat{\alpha} = 1.5 \) and \( \hat{\gamma} = 10.3 \). The experiments are performed using the R statistical software with the ‘optim’ function. In order to discriminate among the other survival models like Weibull and Gamma, we consider the results of the Akaike information criterion (AIC) and corrected AIC (AICc) for these models. From Table 1, it is clear that the Lomax distribution has shown the ‘best’ fit having lower AIC and AICc values. Therefore, we can conclude that when the empirical coefficient of variation is greater than 1, the MLE estimates are easy to compute. This is very much useful for practitioners to work with the Lomax distribution in business problems. We can also see that the Lomax distribution with the parameters \( \alpha \) and \( \gamma \) can well describe the active repair time (in hours) data for an airborne communication transceiver.

**Table 1**
Comparison of all the fitted distributions for the airborne communication transceiver dataset.

| Distribution and values | AIC  | AICc |
|-------------------------|------|------|
| Lomax                   | 209.80 | 210.40 |
| Weibull                 | 212.24 | 213.18 |
| Gamma                   | 213.36 | 214.04 |

### 6 Discussion

One needs numerical algorithms to maximize the log-likelihood for the Lomax distribution. We have theoretically shown the behavior of the likelihood function in terms of the empirical coefficient of variation (CV) and it was proved that the likelihood function attains a global maximum at a finite point when empirical \( CV > 1 \). But when empirical \( CV < 1 \) and the data is heavy-tailed, an alternative model will be any truncated version of the Lomax distribution. The practical application to the reliability data set also supports the theoretical results. The future extensions of this paper will be to prove or disprove the necessary condition of the Theorem 1 and suggest...
some generalized version of the Lomax distribution which can be analytically trackable when $CV < 1$ for the dataset at hand.

References

[1] M Ahsanullah. Record values of the lomax distribution. *Statistica Neerlandica*, 45(1):21–29, 1991.

[2] SA Al-Awadhi and ME Ghitany. Statistical properties of poisson-lomax distribution and its application to repeated accidents data. *Journal of Applied Statistical Science*, 10(4):365–372, 2001.

[3] Marek Arendarczyk, Tomasz J Kozubowski, and Anna K Panorska. A bivariate distribution with lomax and geometric margins. *Journal of the Korean Statistical Society*, 47(4):405–422, 2018.

[4] N Balakrishnan and M Ahsanullah. Relations for single and product moments of record values from lomax distribution. *Sankhyā: The Indian Journal of Statistics, Series B*, pages 140–146, 1994.

[5] August A Balkema and Laurens De Haan. Residual life time at great age. *The Annals of probability*, pages 792–804, 1974.

[6] Maurice C Bryson. Heavy-tailed distributions: properties and tests. *Technometrics*, 16(1):61–68, 1974.

[7] Aaron Childs, N Balakrishnan, and Mohamed Moshref. Order statistics from non-identical right-truncated lomax random variables with applications. *Statistical Papers*, 42(2):187–206, 2001.

[8] Gauss M Cordeiro, Edwin MM Ortega, and Božidar V Popović. The gamma-lomax distribution. *Journal of Statistical computation and Simulation*, 85(2):305–319, 2015.

[9] Erhard Cramer and Anja Bettina Schmiedt. Progressively type-ii censored competing risks data from lomax distributions. *Computational Statistics & Data Analysis*, 55(3):1285–1303, 2011.

[10] Robert H Dicke. Dirac’s cosmology and mach’s principle. *Nature*, 192(4801):440–441, 1961.
[11] Satya D Dubey. Compound gamma, beta and f distributions. *Metrika*, 16(1):27–31, 1970.

[12] Abu Seif Mohammad Fares and Vajjha Venkata Hara Gopal. The generalized double lomax distribution with applications. *Statistica*, 76(4):341–352, 2016.

[13] Paulo H Ferreira, Eduardo Ramos, Pedro L Ramos, Jhon FB Gonzales, Vera LD Tomazella, Ricardo S Ehlers, Eveliny B Silva, and Francisco Louzada. Objective bayesian analysis for the lomax distribution. *Statistics & Probability Letters*, 159, 2019.

[14] ME Ghitany, FA Al-Awadhi, and LA1122 Alkhalfan. Marshall–olkin extended lomax distribution and its application to censored data. *Communications in Statistics - Theory and Methods*, 36(10):1855–1866, 2007.

[15] David E Giles, Hui Feng, and Ryan T Godwin. On the bias of the maximum likelihood estimator for the two-parameter lomax distribution. *Communications in Statistics-Theor and Methods*, 42(11):1934–1950, 2013.

[16] IS Gradshteyn and IM Ryzhik. Handbook of mathematical functions, 1965.

[17] Amal S Hassan and Amani S Al-Ghamdi. Optimum step stress accelerated life testing for lomax distribution. *Journal of Applied Sciences Research*, 5(12):2153–2164, 2009.

[18] Oliver Holland, Assen Golaup, and AH Aghvami. Traffic characteristics of aggregated module downloads for mobile terminal reconfiguration. *IEE Proceedings-Communications*, 153(5):683–690, 2006.

[19] Sang Gil Kang, Woo Dong Lee, and Yongku Kim. Posterior propriety of bivariate lomax distribution under objective priors. *Communications in Statistics - Theory and Methods*, pages 1–9, 2019.

[20] Artur J Lemonte and Gauss M Cordeiro. An extended lomax distribution. *Statistics*, 47(4):800–816, 2013.

[21] KS Lomax. Business failures: Another example of the analysis of failure data. *Journal of the American Statistical Association*, 49(268):847–852, 1954.
[22] Govind S Mudholkar, Deo Kumar Srivastava, and Georgia D Kollia. A generalization of the weibull distribution with application to the analysis of survival data. *Journal of the American Statistical Association*, 91(436):1575–1583, 1996.

[23] Tapan Kumar Nayak. Multivariate lomax distribution: properties and usefulness in reliability theory. *Journal of Applied Probability*, 24(1):170–177, 1987.

[24] Richard L Smith. Maximum likelihood estimation in a class of nonregular cases. *Biometrika*, 72(1):67–90, 1985.

[25] William H Von Alven. *Reliability engineering*. Prentice Hall, 1964.