Normed Space Of Measurable Functions With Some Of Their Properties

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Abstract Let \( L^0(\Omega,F,\mu) \) be the space of measurable functions defined on measure space \((\Omega,F,\mu)\), where we consider any two functions in which are equal almost everywhere (a.e.). Then \( L^0(\Omega,F,\mu) \) is complete metric space with respect to metric functions defined by \( d(f,g) = \int_{\Omega} \frac{|f-g|}{1+|f-g|} \, d\mu \) for all \( f, g \in L^0(\Omega,F,\mu) \). This paper includes two main parts, the first part we prove this space \( L^0(\Omega,F,\mu) \) in general is not a normed space, and second we prove norm on \( L^0(\Omega,F,\mu) \) achieved if and only if she was \( \Omega \) is the finite union of disjoint atom.

1. Introduction

In the measure theory, we deal with different types of convergence of sequences of measurable functions, especially convergence in measure and convergence almost everywhere (a.e.), and study the relationships between them. For example, is that each sequence convergence measure is convergence (a.e.)? And is the converse is true? And under what condition is that achieved? There are many sources that have studied this topic from them Marczewiski showed in [4] 1955 convergence in measure implies convergence everywhere (a.e) and Thomasian proved in [8] (1957) convergence in metric equivalent to convergence a.s (in probability) if and only if \( \Omega \) is the union of finite of disjoint atoms. Eugene was introduced in [2] 1975 several different definitions for the stochastic on convergence of sequence of random variables. And Jordan was proved in [3] 2015 \( L^0(\Omega,F,\mu) \) is a complete metric space. Noori and Asawer were proved in [6] 2020 \( L^0(\Omega,F,\mu) \) is a complete metric space using another metric function. In this paper, we are discussed the relationship between convergence in measure and convergence almost everywhere (a.e), and what condition that must be set for equivalence to be achieved between them. After that we set with proof the necessary and sufficient condition for the existence of the norm on \( L^0(\Omega,F,\mu) \).

2. Topology of convergence in measure

Let \( L^0(\Omega,F,\mu) \) be the space of measurable functions defined on measure space \((\Omega,F,\mu)\) are equals almost everywhere (a.e). Then \( L^0(\Omega,F,\mu) \) is a linear space under the following addition and scalar multiplication

1. \((f+g)(x) = f(x) + g(x)\) for all \(f,g \in L(\Omega)\)
2. \((\lambda f)(x) = \lambda(x)\) for all \(f \in L(\Omega)\) and for \(\lambda \in R\)
Theorem (2.1)

Let \( L^0(\Omega, F, \mu) \) be the space of measurable functions which is defined on \( \Omega \). Define \( \| \cdot \| : L^0(\Omega, F, \mu) \to \mathbb{R} \) by \( \| f \|_0 = \int_\Omega \frac{|f|}{1+|f|} d\mu \) for all \( f \in L^0(\Omega, F, \mu) \), then

1. \( \| f \|_0 \geq 0 \) for all \( f \in L^0(\Omega, F, \mu) \)
2. \( \| f \|_0 = 0 \) iff \( f = 0 \) \( a.e. \)
3. \( \| f + g \|_0 \leq \| f \|_0 + \| g \|_0 \) for all \( f, g \in L^0(\Omega, F, \mu) \)

Proof:

1. Since \( |f| \geq 0 \) for all \( f \in L^0(\Omega, F, \mu) \), then \( \frac{|f|}{1+|f|} \geq 0 \) for all \( f \in L^0(\Omega, F, \mu) \) \( \Rightarrow \int_\Omega \frac{|f|}{1+|f|} d\mu \geq 0 \)
2. Let \( f \in L^0(\Omega, F, \mu) \)
   - If \( f = 0 \) \( a.e. \), then \( \int_\Omega \frac{|f|}{1+|f|} d\mu = 0 \) \( a.e. \)
3. Let \( f, g \in L^0(\Omega, F, \mu) \)
   - Since \( \frac{|f|}{1+|f|} + \frac{|g|}{1+|g|} \geq \frac{|f|+|g|}{1+|f|+|g|} \), \( \frac{|f|}{1+|f|} \geq \frac{1}{1+|f|+|g|} \)
   - \( \frac{|f|}{1+|f|} \geq \frac{1}{1+|f|+|g|} \)
   - \( f \geq 0 \) \( a.e. \) \( \Rightarrow \int_\Omega \frac{|f|}{1+|f|} d\mu = 0 \) \( a.e. \)

Remark: \( \| \cdot \| \) is not norm on \( L^0(\Omega, F, \mu) \), since if \( f \in L^0(\Omega, F, \mu) \), then \( \| g \|_0 = \int_\Omega \frac{|g|}{1+|g|} d\mu \neq \lambda \| f \|_0 \)

In order to discuss the compatibility of convergence in measure and a norm we have to introduce a definition from the theory of summability

Theorem (2.2): [6]
The metric space \( L^0(\Omega, F, \mu) \) is complete

Definition (2.3): [2]
The sequence of real numbers \( \{ x_n \} \) is called Cesaro summable of order 1 to \( x \) and write
\[
x_n \xrightarrow{(c,1)} x \text{ if } \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n x_i = x.
\]
The following result is very important

Lemma (2.4)

Let \( \{ x_n \} \) be a convergent sequence of real numbers if \( x_n \to x \), then \( x_n \xrightarrow{(c,1)} x \). The converse is not true.

Proof:

Let \( \varepsilon > 0 \), since \( x_n \to x \), then is \( k \in \mathbb{Z}^+ \) such that \( |x_n - x| < \frac{\varepsilon}{2} \) for all \( n \geq k \).

Let \( y_n = \frac{1}{n} \sum_{i=1}^n x_i \), then \( y_n - x = \frac{1}{n} \sum_{i=1}^n x_i - x = \frac{1}{n} \sum_{i=k}^n (x_i - x) + \frac{1}{n} \sum_{i=k}^n (x_i - x) \)

Let \( m = \max\{x, \max x_i\} \) and select \( n \) so large that \( \frac{1}{n} < \frac{\varepsilon}{4km} \), then
\[
|y_n - x| < \frac{\varepsilon}{4km} k(2m) + \frac{\varepsilon}{2} \left( \frac{n-k}{n} \right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Therefore \( y_n \to x \).
Example (2.5): \[2\]

Let \( x_n = \frac{1+(-1)^{n-1}}{2} \) for all \( n \in \mathbb{N} \).

Clearly \( x_{2n} = 0 \), \( x_{2n-1} = 1 \) so that the sequence is divergent, but \( x_n \xrightarrow{(c,1)} \frac{1}{2} \).

Remark:
In similar manner we can introduce \((c,1)\)-summability for sequence of measurable functions \((c,1)\) to \( f \),
and write \( f_n \xrightarrow{(c,1)-s} f \) if \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f_i = f \). Here \( f \) may be proper or a degenerate measurable function.

Theorem (2.6):
If \( L^0(\Omega, F, \mu) \) is a normed space which is compatible with \( s \)-convergence, and \( \{f_n\} \) is a sequence in \( L^0(\Omega, F, \mu) \) such that \( f_n \xrightarrow{s} 0 \), then \( f_n \xrightarrow{(c,1)-s} 0 \).

Proof:
Let \( g_n = \frac{1}{n} \sum_{i=1}^{n} f_i \) \( \Rightarrow \|g_n\| = \left\|\frac{1}{n} \sum_{i=1}^{n} f_i\right\| \leq \frac{1}{n} \sum_{i=1}^{n} \|f_i\| \)
Since \( f_n \xrightarrow{s} 0 \) \( \Rightarrow \lim_{i \to \infty} \|f_i\| = 0 \) by theorem (2.4), we have \( \|f_n\| \xrightarrow{(c,1)} 0 \), so that \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \|f_i\| = 0 \) \( \Rightarrow \lim_{n \to \infty} \|g_n\| = 0 \)
Using the equivalent of norm convergence and \( s \)-convergence, we conclude that \( g_n \xrightarrow{s} 0 \), therefore \( f_n \xrightarrow{(c,1)-s} 0 \)

Remark:
We construct an example (example 2.5) of a sequence \( \{f_n\} \) which converge in measure to zero but for which \( f_n \) not converge to zero in \((c,1)\)-\( \mu \). i.e.,
If \( f_n = \frac{1+(-1)^{n-1}}{2} \) for all \( n \in \mathbb{N} \). Then \( f_n \xrightarrow{\mu} 0 \), but \( f_n \) not converge to zero in \((c,1)-\mu \).
We can then use theorem (2.6) to prove the following statement.

Theorem (2.7): [2]
Convergence in measure is in general incompatible with the existence of a norm.

The reason is that the existence of a norm which is compatible with converge in measure is possible if the basic measure space \((\Omega, F, \mu)\) has a certain property.

3. The necessary and sufficient condition for the existence of the norm on \( L^0(\Omega, F, \mu) \)

Definition (3.1): [4]
1. A set \( A \subset F \) is called an atom, i.e. there no proper subset \( B \) of \( A \) such that \( B \subset F \).
2. An atom of a measure space \((\Omega, F, \mu)\) is a set \( A \subset F \) with \( \mu(A) > 0 \) such that \( B \subset A \) and \( B \subset F \) imply that either \( \mu(B) = 0 \) or \( \mu(B) = \mu(A) \). i.e.
A set \( A \subset F \) is called atom of \( \mu \) if \( 0 < \mu(A) < \infty \) and for every \( B \subset A \) with \( B \subset F \) either \( \mu(B) = 0 \) or \( \mu(B) = \mu(A) \).
A set \( A \subset F \) is called atom of \( \mu \) if \( \mu(A) > 0 \) and for any \( B \in F \) and \( B \subset A \) with \( \mu(A) < \mu(B) \), then \( \mu(B) = 0 \).
3. A measure without any atoms is called nonatomic (or atomless or diffuse). In other words
A measure \( \mu \) is called nonatomic or diffuse, if there are no atoms.
A measure \( \mu \) is nonatomic if for any \( A \in F \) with \( \mu(A) > 0 \) there exists \( B \in F \) and \( B \subset A \) such that \( \mu(A) > \mu(B) > 0 \).
A measure \( \mu \) is nonatomic if there are no atoms for \( \mu \). This means that every measurable set of
positive measure can be split into two disjoint measurable sets, each having positive measure.

4. μ is called purely atomic or simply atomic if every measurable set of positive measure contains an atom. In other words, a measure space (Ω, F, μ) or the measure μ is called purely atomic if there is a family g of atoms of μ such that for each A ∈ F, μ(A) is the sum of the numbers of μ(B) for all B ∈ g such that μ(A ∩ B) = μ(A).

5. Let (Ω, F, μ) be a measure space such that all singletons {x} ∈ F. A point x ∈ Ω is called an atom for the measure μ if μ({x}) > 0.

Example (3.2): [4]

1. Let Ω = {1, 2, 3, ..., 10} and let F = P(X) be the power set of Ω. Define the measure μ of a set to be cardinality, that is, the number of elements in the set. Then, each of the singletons {x} for x ∈ Ω is an atom.

2. The singleton {x} with positive finite measure are atoms of μ.

3. If A ∈ F is an atom for μ and μ(A ∩ B) > 0, then A ∩ B is also an atom for μ.

4. A set of positive finite measure is an atom if its only measurable subsets are itself and ∅.

Here is a less trivial atom.

5. Let Ω be an uncountable set and let F be family of sets which either countable, with μ(A) = 0 or have countable complement, with μ(A) = 1. Then μ is a measure and Ω is an atom.

6. Lebesgue measure is nonatomic.

7. If μ is a finite measure, the set of atoms of μ is countable.

8. The zero measure is the only measure which is both purely atomic and nonatomic.

Theorem (3.3):

Let (Ω, F, μ) be a measure space.

1. A measurable function is a.e. constant on an atom.

2. There is a decomposition of Ω into disjoint sets, Ω = ∪∞n=0 An where A0 is either empty or an atomless set of positive measure, and each of the sets A1, A2, ... is either a point or an atom.

3. If μ is atomless, then every A ∈ F, and every c with 0 < c < μ(A), there is a set B ∈ F such that B ⊆ A and μ(B) = 0.

4. If μ is atomless and μ(Ω) = 1, then for every sequence Pn with 0 ≤ Pn ≤ 1, there exists a sequence {An} of stochastically independent sets with μ(An) = Pn.

Proof:

1. let (Ω, F, μ) be a measurable space and f : Ω → ℝ be a measurable function.

2. If A ∈ F is called an atom of μ, then f is constant on A.

3. If y ∈ ℝ and μ({x ∈ A : f(x) < y}) = 0, then μ({x ∈ A : f(x) < z}) = 0 for all z ≤ y.

Let k = sup {y ∈ ℝ : μ({x ∈ A : f(x) < y})} = 0. Then μ({x ∈ A : f(x) < k}) = μ(∪{x ∈ A : f(x) ≥ r}) = 0.

If y > k, then μ({x ∈ A : f(x) < y}) > 0, hence μ({x ∈ A : f(x) ≥ y}) = 0 since A is an atom of A. Thus μ({x ∈ A : f(x) ≥ k}) = μ(∪{x ∈ A : f(x) ≥ r}) = 0.

It follows that f = k a.e. on Ω.

4. Let A0 = A, A2 = Ω for every A ⊆ Ω.

By (3), there is a set A1 with μ(A1) = P1. If Ai are already defined for i ≤ n, and if they are stochastically independent sets with μ(Ai) = Pi, then there is, in view of (3), a set An+1 such that

(∩i=1n+1 Aki) = Pn+1μ(∩i=1n+1 Aki) for every system k1, k2, ..., kn of number 0 and 1. It is easy that {An} is the required sequence.
Definition (3.4) [7]
Let \( \{A_n\} \) be a sequence of subsets of a set \( \Omega \). The set of all points which belong to infinitely many sets of the sequence \( \{A_n\} \) is called the upper limit (or limit superior) of \( \{A_n\} \) and is denoted by \( A^* \) and defined by \( A^* = \lim_{n \to \infty} \sup A_n = \{ x \in A_n : \text{for infinitely many } n \} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \lim_{n \to \infty} \bigcup_{k=n}^{\infty} A_k \)

Thus \( x \in A_n \) iff for all \( n \), then \( x \in A_k \) for some \( k \geq n \)
The lower limit (or limit inferior) of \( \{A_n\} \), denoted by \( A_* \) is the set of all points which belong to almost all sets of the sequence \( \{A_n\} \), and defined by
\[ A_* = \lim_{n \to \infty} \inf A_n = \{ x \in A_n : \text{for all but finitely many } n \} = \bigcap_{n=1}^{\infty} \bigcup_{n=k}^{\infty} A_k = \lim_{n \to \infty} \bigcup_{n=k}^{\infty} A_k \]
Thus \( x \in A_* \) iff for some \( n \), then \( x \in A_n \) for all \( K \geq n \)

Definition (3.5): [1]
A sequence \( \{A_n\} \) of subsets of a set \( \Omega \) is said to
1. Converge if \( \lim_{n \to \infty} \sup A_n = \lim_{n \to \infty} \inf A_n = A \), and \( \Lambda \) is said to be the limit of \( \{A_n\} \), we write
\[ A = \lim_{n \to \infty} A_n \]
2. Converges in measure to set \( A \), write \( A_n \Rightarrow A \) if \( I_{A_n} \Rightarrow I_A \)
3. Converges a.e to set \( A \), write \( A_n \Rightarrow A \) if \( I_{A_n} \Rightarrow I_A \)

Theorem (3.6)
1. If \( f_n \xrightarrow{\mu} f \), then \( f_n \xrightarrow{a.e} f \) on every atom set \( A \) of \( \mu \)
2. If \( \mu \) is atomless, then there is a sequence \( \{A_n\} \) of measurable sets convergence to the void set in measure such that \( \lim_{n \to \infty} \inf A_n = \emptyset \), \( \lim_{n \to \infty} \sup A_n = \Omega \).
3. If the sequence convergence in measure on measurable sets implies their convergence a.e., then \( \mu \) is purely atomic.

Proof:
1. Let \( \{f_n\} \) be a sequence of measurable sequence defined on \( (\Omega, F, \mu) \) such that \( f_n \xrightarrow{\mu} f \)
Let \( A \) be an atom of \( \mu \), then there is an atom \( A^* \subseteq A \) such that \( \mu(A|A^*) = 0 \) and that \( f_1, f_2, f_3, \ldots \) are constant on \( A^* \), then \( f(x) = c \), \( f_n(x) = c_n \) for \( x \in A^* \)
That \( f_1, f_2, f_3, \ldots \) are constant on \( A^* \), then \( f(x) = c \), \( f_n(x) = c_n \) for \( x \in A^* \)
Let \( \varepsilon > 0 \), since \( f_n \xrightarrow{\mu} f \), then there is \( k \in \mathbb{Z}^+ \) such that \( |f_n(x) - f(x)| < \varepsilon \) for all \( n > k \) outside a set \( Z_n \) with \( \mu(Z_n) < \mu(A^*) \)
Consequently \( |c_n - c| < \varepsilon \) for all \( n > k \) which implies \( f_n \xrightarrow{a.e} f \) on \( A \).
2. Without any loss of generally we may suppose that \( \mu(\Omega) = 1 \).
By (4) theorem (3.3), there exists a sequence \( \{B_n\} \) of stochastically independent sets with
\( \mu(B_n) = \frac{1}{2} \) for all \( n \).
The sequence
\[ A_1 = B_1, \quad A_2 = B_1^C, \quad A_3 = B_1 \cap B_2, \quad A_4 = B_1^C \cap B_2, \quad A_5 = B_1 \cap B_1^C, \quad A_6 = B_1^C \cap B_2^C, \quad A_7 = B_1 \cap B_2 \cap B_3, \quad A_8 = B_1^C \cap B_2 \cap B_3, \ldots \]
Obviously satisfies the conditions of (2).

Theorem (3.7)
If \( \mu \) is finite , then \( \Omega = A \cup \left( \bigcup_{n=1}^{\infty} A_n \right) \), where all of the sets in the decomposition are disjoint and each \( A_n \) is the empty set or an atom , and for every measurable subset \( B \) of \( A \), \( \mu \) takes every value between 0 and \( \mu(B) \) for measurable subset of \( B \).

**Proof :**

There is only a countable numbers of \( \mu \) – equivalence classes of such \( A_l \) of these classes and let \( B \subseteq A = \bigcup_{l=1}^{\infty} A_l \). Select representation inductively sets \( C_n \) \( \subseteq \) \( g_n \) such \( \mu(C_n) > \sup \mu(C) - \frac{1}{n} \) for all \( C \subseteq g_n \), where \( g_n \) is the class of all \( C \subseteq B \cup_{l=1}^{n-1} A_l \) for with \( \mu(C) \leq \mu(\bigcup_{l=1}^{n-1} A_l) \). Then \( \mu(C) = c \), for \( C \subseteq \bigcup_{n=1}^{\infty} C_n \).

**Definition(3.8)[1]**

1. Converge in norm is said to be equivalent to convergence a.e. if for every sequence \( \{ f_n \} \) in \( L^0(\Omega, F, \mu) \), \( \| f_n \| \to 0 \) iff \( f_n \overset{a.e.}{\to} 0 \).
2. Converge in norm is said to be equivalent to convergence in measure if , for every sequence \( \{ f_n \} \) in \( L^0(\Omega, F, \mu) \), \( \| f_n \| \to 0 \) iff \( f_n \overset{\mu}{\to} 0 \).

**Theorem(3.9)**

If \( (\Omega, F, \mu) \) is finite measure. Then there exists a norm on \( L^0(\Omega, F, \mu) \) which is compatible with convergence in measure iff \( \Omega \) is the finite union of disjoint atoms.

**Proof:**

Suppose there exists a norm \( \| . \| \) on \( L^0(\Omega, F, \mu) \) which is compatible with convergence in measure

Assume that \( \Omega \) is not finite union of disjoint atoms.

Then there exists a sequence \( \{ A_n \} \) in \( \Omega \) with \( \mu(A_n) = 0 \) for all \( n \).

Let \( f_n \) be the in indicator function of the set \( A_n \), i.e. \( f_n = I_{A_n} \).

If \( \| f_n \| = 0 \), then \( f_n \overset{\mu}{\to} 0 \), contradicting \( \mu(A_n) > 0 \). Then \( \| f_n \| \neq 0 \) for all \( n \).

Since \( \| f_n \| = 1 \) for all \( n \), so that the sequence of measurable function \( \{ g_n \} = \frac{f_n}{\| f_n \|} \) cannot converge to 0 in measure. However, it must, because \( \mu(A_n) \to 0 \).

Conversely suppose that \( \Omega \) is the finite union of disjoint atoms.

Define \( \| . \| : L^0(\Omega, F, \mu) \to \mathbb{R} \) by \( \| f \| = \int f \, d\mu \) for all \( f \in L^0(\Omega, F, \mu) \).

In clear \( \| . \| \) is a norm on \( L^0(\Omega, F, \mu) \).

**Theorem(3.10)**

If \( (\Omega, F, \mu) \) is finite measure. Then convergence in measure implies almost everywhere convergence for all sequence in \( L^0(\Omega, F, \mu) \) iff \( \Omega \) is the union of countable number of disjoint atoms.

**Proof:**

Suppose that convergence in measure implies almost everywhere convergence for all sequence in \( L^0(\Omega, F, \mu) \).

Assume that \( \Omega \) is not finite union of disjoint atoms.

Thus in the decomposition of theorem (2.4), \( (A) > 0 \) and for each \( n \), \( A = \bigcup_{k=1}^{\infty} A_{nk} \), where \( \mu(A_{nk}) = \frac{1}{n} \mu(A) \) for \( k=1,2,\ldots,n \), and \( A_{n1}, A_{n2}, \ldots, A_{nk} \) are disjoint.

Let \( f_{nk} \) be the indicator function of the set \( A_{nk} \). The sequence of measurable functions \( \{ f_{nk} \} \) converge to 0 in measure but not a.e. This contradiction.

Conversely: suppose that \( \Omega \) is the finite union of disjoint atoms.

Let \( \{ f_n \} \) in \( L^0(\Omega, F, \mu) \) such that \( f_n \overset{\mu}{\to} 0 \). To prove \( f_n \overset{a.e.}{\to} 0 \).
By theorem (3.9), there exists a norm on $L^0(\Omega, F, \mu)$ which is compatible with convergence in measure. If $\|f_n\| \not\to 0$ then there exists a subsequence $\{f_{nk}\}$ and an $\varepsilon > 0$ such that $\|f_{nk}\| > \varepsilon$. But $f_{nk} \not\to 0$ so that it has a subsequence $f_{nk} \xrightarrow{a.e.} 0$. Thus $\|f_{nk}\| \xrightarrow{a.e.} 0$ contradicting $\|f_{nk}\| > \varepsilon$. Therefore $\|f_n\|$ must converge to 0, hence $f_n \xrightarrow{a.e.} 0$.

3. References

[1] Bilingsley ,p .(1968).Convergence of Probability Measures .New York.
[2] Eugene Lukacs, (1975)"Stochastic convergence" ,Second edition New York.
[3] Jordan Bell,(2015)"L^0,Convergence in measure, equi-integrability , the Vitali convergence theorem ,and the de la Valle'e-Poussin criteriYork .
[4] Marczewski, E.,(1955) Remarks on the convergence of measurable set and measurable functions , Colloq. Math.(3),118-124
[5] M. Loewe(1963)"Probability Theory " ,4th edn Springer ,New York
[6] Noori F. and Asawer (2020)" Some Properties Related With $L^0(\Omega, F, \mu)$ Space ,AL-Qadisiyah journal Of Pure Science .
[7] Paul R.Halmos , (1970)" Measure Theory" Springer -Verlag New York
[8] Thomasian .A.J.,(1957),Metrics and norms on space of random variables ,Ann. Math. Statist.(28),512-514.