Linear Half-Space Problems in Kinetic Theory: Abstract Formulation and Regime Transitions

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Abstract: Half-space problems in the kinetic theory of gases are of great importance in the study of the asymptotic behavior of solutions of boundary value problems for the Boltzmann equation for small Knudsen numbers. In this work a generally formulated half-space problem, based on generalizations of stationary half-space problems in one spatial variable for the Boltzmann equation - for hard-sphere models of monatomic single species and multicomponent mixtures - is considered. The number of conditions on the indata at the interface needed to obtain well-posedness is investigated. Exponential fast convergence is obtained "far away" from the interface. In particular, the exponential decay at regime transitions - where the number of conditions on the indata needed to obtain well-posedness changes - for linearized kinetic half-space problems related to the half-space problem of evaporation and condensation in kinetic theory are considered. The regime transitions correspond to the transition between subsonic and supersonic evaporation/condensation, or the transition between evaporation and condensation. Near the regime transitions, slowly varying modes might occur, preventing uniform exponential speed of convergence there. By imposing extra conditions on the indata at the interface, the slowly varying modes can be eliminated near a regime transition, giving rise to uniform exponential speed of convergence near the regime transition.

Values of the velocity of the flow at the far end, for which regime transitions take place are presented for some particular variants of the Boltzmann equation: for monatomic and polyatomic single species and mixtures, and the quantum variant for bosons and fermions.

1 Introduction with motivating examples

Half-space problems in the kinetic theory of gases are of great importance in the study of the asymptotic behavior of solutions of boundary value problems for the Boltzmann equation for small Knudsen numbers; providing boundary conditions for the fluid-dynamic-type equations and Knudsen-layer corrections to solutions of the fluid-dynamic-type equations in a neighborhood of the boundary [31, 32]. The steady half-space problem for the Boltzmann equation in a slab symmetry [19, 31, 32] reads

$$ \begin{cases} \frac{\partial F}{\partial x} = Q(F, F), & F = F(x, v), \\ F(0, v) = M_B(v) \text{ for } v > 0, \\ F \to M_\infty \text{ as } x \to \infty, \end{cases} \tag{1} $$
where \( x \in \mathbb{R}_+ \) and \( \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3 \), with \( v = v_1 \). Here \( F = F(x, \mathbf{v}) \) denotes the distribution function of molecules with velocity \( \mathbf{v} \in \mathbb{R}^3 \) at distance \( x \) from a (planar) interface - typically between a gas and its condensed phase - and the Gaussian \( M_\infty = M_\infty(\mathbf{v}) = \frac{\rho}{(2\pi T)^{3/2}} e^{-|\mathbf{v} - \mathbf{u}|^2/(2T)} \) denotes the equilibrium, or, Maxwellian, distribution - approached far away from the interface; as \( x \to \infty \).

Moreover, \( \rho \in \mathbb{R}_+ \), \( \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3 \), with \( u = u_1 \), and \( T \in \mathbb{R}_+ \), relate (are equal, or, at least proportional) to the density, bulk velocity, and temperature at the far end, respectively. The impinging molecules are absorbed at the interface, while the emerging molecules are desorbed according to a (given) Maxwellian distribution of the interface (condensed phase). The collision integral \( Q(F, F) \), acting on the distribution function \( F = F(x, \mathbf{v}) \) only with respect to the velocity dependence, is quadratic in \( F \). The bilinear operator \( Q = Q(F, G) \) is assumed to be symmetric, such that \( Q(F, G) = Q(G, F) \) \cite[11]. After a shift in the velocity space, we obtain, due to invariance of the collision operator \( Q \) under such a transformation,

\[
\begin{align*}
\left\{ \begin{array}{l}
(v + u) \frac{\partial \tilde{F}}{\partial x} = Q\left( \tilde{F}, \tilde{F} \right), \\
\tilde{F}(0, \mathbf{v}) = M_B(\mathbf{v} + \mathbf{u}) \text{ for } v + u > 0, \\
\tilde{F} \to M \text{ as } x \to \infty,
\end{array} \right.
\end{align*}
\]

where \( \tilde{F}(x, \mathbf{v}) = F(x, \mathbf{v} + \mathbf{u}) \) and

\[
M = M(\mathbf{v}) = M_\infty(\mathbf{v} + \mathbf{u}) = \frac{\rho}{(2\pi RT)^{3/2}} e^{-|\mathbf{v}|^2/(2RT)}.
\]

A suitable linearization, \( \tilde{F} = M + \sqrt{Mf} \), around the non-drifting Maxwellian \( M = M_\infty(\mathbf{v} + \mathbf{u}) \), results, after discarding the quadratic terms, in

\[
\begin{align*}
\left\{ \begin{array}{l}
(v + u) \frac{\partial f}{\partial x} + \mathcal{L}f = 0, \\
f = f(x, \mathbf{v}), \\
f(0, \mathbf{v}) = M^{-1/2}(M_B(\mathbf{v} + \mathbf{u}) - M) \text{ for } v + u > 0, \\
f \to 0 \text{ as } x \to \infty,
\end{array} \right.
\end{align*}
\]

where \( \mathcal{L}f = -2M^{-1/2}Q(M, M^{1/2}f) \). This problem has been extensively studied in the literature, see e.g. \( [3, 19, 22, 25, 26, 4] \).

More general boundary conditions, where the distribution function for the emerging molecules (for which \( v > 0 \) and \( v + u > 0 \), in problem \( \text{(1)} \) and \( \text{(2)} \), respectively, for an interface at rest) may depend (partly, or, completely) on the distribution function for the impinging molecules (for which \( v < 0 \) and \( v + u < 0 \), in problem \( \text{(1)} \) and \( \text{(2)} \), respectively), can be considered at the interface \( x = 0 \) \cite{20, 21, 31, 32}.

Considering the real Hilbert space \( \mathfrak{h}_1: = L^2(d\mathbf{v}) \), and viewing the molecules as hard spheres, the linearized operator \( \mathcal{L} \) is a nonnegative, self-adjoint Fredholm operator on \( \mathfrak{h}_1 \), with domain

\[
D(\mathcal{L}) = L^2((1 + |\mathbf{v}|)d\mathbf{v})
\]
and kernel
\[
\ker L = \text{span}\left\{ \sqrt{M}, \sqrt{Mv}, \sqrt{Mv_2}, \sqrt{Mv_3}, \sqrt{M|v|^2} \right\}.
\]

A Fredholm operator \( L \) on a Hilbert space \( H \) is a closed operator with finite dimensional kernel and cokernel, and a closed range. The latter assumption - that the range is closed - is, in fact, redundant, since any closed operator with a finite dimensional cokernel has a closed range \cite{24}. Moreover, a self-adjoint operator is, by definition, densely defined, and, hence, the adjoint operator - and so the operator itself - is linear and closed \cite{24}. Also, the orthogonal complement of the range of a closed operator is equal to the kernel of the adjoint operator \cite{24, 28}. Hence, for any self-adjoint operator \( L \) on \( H \) with a closed range, the cokernel is equal to the kernel:
\[
coker L = H/\text{Im} L = (\text{Im} L)^\perp = \ker L^* = \ker L.
\]

The linearized operator \( L \) can be split into a positive multiplication operator \( \nu = \nu(|v|) \) minus a compact operator \( K \) on \( H \):
\[
(Lf)(v) = \nu(|v|) f(v) - K(f)(v), \quad f \in D(L), \tag{4}
\]
such that for some constants \( 0 < \nu_- < \nu_+ \)
\[
\nu_-(1 + |v|) \leq \nu(|v|) \leq \nu_+(1 + |v|) \quad \text{for all } v \in \mathbb{R}^3, \tag{5}
\]
and for some constant \( 0 < \lambda < 1 \)
\[
\int_{\mathbb{R}^3} (f(Lf))(v) \, dv \geq \lambda \int_{\mathbb{R}^3} \nu(|v|) f^2(v) \, dv \geq \lambda \nu_- \int_{\mathbb{R}^3} (1 + |v|) f^2(v) \, dv. \tag{6}
\]

Turning our attention to the Boltzmann equation for a mixture of \( s \geq 2 \) (\( s = 1 \) corresponds to the case of single species considered above) species \( \alpha_1, ..., \alpha_s \), with masses \( m_{\alpha_1}, ..., m_{\alpha_s} \), respectively, the distribution functions (in problem \( \text{(1)} \)) will be of the form
\[
F = (F_1, ..., F_s), \quad \text{where } F_i = F_i(x, v).
\]

Consider the real Hilbert space \( H_s := (L^2(dv))^s \), with inner product
\[
(f, g) = \sum_{i=1}^s \int_{\mathbb{R}^d} f_i g_i \, dv, \quad f, g \in (L^2(dv))^s,
\]
and make (after a possible shift in the velocity space, cf. problem \( \text{(2)} \)) a linearization, cf. problem \( \text{(3)} \), around a non-drifting Maxwellian
\[
M = (M_{\alpha_1}, ..., M_{\alpha_s}), \quad \text{with } M_{\alpha_i} = n_{\alpha_i} \left( \frac{m_{\alpha_i}}{2\pi T} \right)^{3/2} e^{-m_{\alpha_i}|v|^2/(2T)},
\]
where \( \{n_{\alpha_i}, ..., n_{\alpha_s}\} \subset \mathbb{R}_+ \) and \( T \in \mathbb{R}_+ \) relate (are equal, or, at least proportional) to the number densities of the species \( \alpha_1, ..., \alpha_s \) and the temperature,
respectively. Then the linearized operator $L$ is (considering the molecules to be hard spheres) a nonnegative, self-adjoint Fredholm operator on $h_s$ with domain
\[ D(L) = \left( L^2 \left( (1 + |v|) \, dv \right) \right)^s \]
and kernel
\[ \ker L = \text{span} \left\{ \sqrt{M} \alpha_1 e_1, ..., \sqrt{M} \alpha_s e_s, \sqrt{M} v, \sqrt{M} v_2, \sqrt{M} v_3, \sqrt{M} |v|^2 \right\}, \]
where $M = (m_1^2, M_1, ..., m_s^2, M_s)$ and $\{e_1, ..., e_s\}$ is the standard basis of $\mathbb{R}^s$. Moreover, the linearized collision operator $L = (L_1, ..., L_s)$ can be decomposed, correspondingly to equations (4)-(6), into a positive multiplication operator $\nu = \nu(|v|) = \text{diag} (\nu_1(|v|), ..., \nu_s(|v|))$ minus a compact operator $K = (K_1, ..., K_s)$ on $h_s$:
\[ (Lf)(v) = \nu(|v|)f(v) - K(f)(v), \quad f \in D(L), \]
with
\[ \nu_-(1 + |v|) \leq \nu_i(|v|) \leq \nu_+(1 + |v|) \quad \text{for all } v \in \mathbb{R}^3 \text{ and } i \in \{1, ..., s\}, \]
for some constants $0 < \nu_- < \nu_+$ and for some constant $0 < \lambda < 1$
\[ (f | Lf) \geq \lambda (f | \nu(|v|)f) \geq \lambda \nu_- (f | (1 + |v|) f). \]

The remaining of this paper is organized as follows. In Section 2 we formulate an abstractly formulated half-space problem, motivated by e.g. the two examples above. The boundary conditions at the interface (at rest) considered include in addition to those of complete absorption presented above, much more general ones; cf. the boundary conditions for the linearized Boltzmann equation presented in [20, p. 164]. The main results, including an existence result, Theorem 1, which tells that, for a certain number of conditions on the indata at the interface, there exists a unique solution converging at exponential speed as $x \to \infty$, are presented in Section 3. In Section 4, a related penalized problem, which is proved to have a unique solution for any indata, is considered. Sequentially, in Section 5, it is proved that under a certain number of conditions on the indata at the interface, the unique solution to the penalized problem is also a solution to the original problem. Regime transitions, related to the half-space problem of evaporation and condensation of gases, are considered in Section 6. The regime transitions, corresponding to the transition between subsonic and supersonic evaporation/condensation or the transition between evaporation and condensation, take place at some degenerate values (zero, or, plus/minus "speed of sound") of the parameter $u$ (the velocity of the flow - in $x$-direction - at the far end), where the number of conditions, needed to be imposed on the indata for existence of a unique solution (stated in Theorem 1), changes. In general, the exponential decay is not uniform in any neighborhood of a degenerate value, since, slowly varying modes may occur as the flow velocity $u$ approaches the degenerate value (from below). However, by posing
some extra condition(s) on the indata at the interface (before the flow velocity \( u \) reaches the degenerate value), such that the number of conditions on the indata to obtain existence of a unique solution is the same in a neighborhood of the degenerate value, the slowly varying modes can be eliminated. Then a uniform exponential decay in some neighborhood of the degenerate value (Theorems \([2, 3]\)) is obtained. In the Appendix the degenerate values of the parameter \( u \), including the speed of sound, where the regime transitions take place, together with some important orthogonal basis of the kernel of the linearized operator (cf. the orthogonality properties \([13]\) below), are presented for some particular variants of the Boltzmann equation: for monatomic single species and mixtures, as well as, corresponding cases for polyatomic molecules, and also the quantum variant for bosons and fermions. The linearized Boltzmann collision operator for monatomic single species, as well as, mixtures satisfies, for hard spheres, the assumed properties on the linearized operator in the abstract problem \([27, 20, 17, 12]\). Some recent corresponding results for polyatomic molecules can be found in \([12, 13]\).

2 Abstract formulation of the problem

Let \( \mathfrak{h} \) be a real Hilbert space, with inner product \((\cdot, \cdot)\) and denote by \( \mathfrak{L}(\mathfrak{h}) \) the set of all linear operators on \( \mathfrak{h} \). Consider the steady equation

\[
B \frac{\partial f}{\partial x} + L f = S,
\]

where \( f \equiv f(x, \cdot) \in \mathfrak{h} \) for \( x > 0 \), \( S = S(x, \cdot) \in L^2(\mathbb{R}_+; \mathfrak{h}) \), \( S = S(x, \cdot) \in \text{Im} L \) belongs to the range \( \text{Im} L \) of \( L \) for all \( x \in \mathbb{R}_+ \). Furthermore, the linear operators \( L \) and \( B \), \( \{L, B\} \subset \mathfrak{L}(\mathfrak{h}) \), are assumed to satisfy properties \( \text{H1-H3} \) below.

**H1** The linear operator \( L \) is a nonnegative self-adjoint Fredholm operator, or, equivalently, \( L \) is a nonnegative self-adjoint (and, hence, densely defined) operator with a finite dimensional kernel \( \ker L \) and a closed range \( \text{Im} L \).

Then \( L \) has a kernel

\[
\ker L = \text{span} \{\phi_1, \ldots, \phi_n\}, \quad \text{dim} (\ker L) = n,
\]

for some \( \{\phi_1, \ldots, \phi_n\} \subset \mathfrak{h} \); and a domain \( \text{D}(L) \), such that

\[
L^* = L \geq 0, \quad \overline{\text{D}(L)} = \mathfrak{h} = \ker L \oplus \text{Im} L.
\]

**H2** The linear operator \( B \) is a self-adjoint non-singular operator, such that the domain of \( L \) is a subset of the domain of \( B \);

\[
B^* = B, \quad \ker B = \{0\}, \quad \text{D}(L) \subseteq \text{D}(B).
\]

Denote by \( E \equiv E(d\lambda) \) the spectral measure of \( B \) and introduce the projections

\[
P_+ := \int_0^\infty E(d\lambda), \quad P_- := \int_{-\infty}^0 E(d\lambda).
\]
Then the following decomposition may be introduced
\[ B = B^+ - B^-, \quad |B| = B^+ + B^-, \quad \text{with} \quad B^\pm = \pm BP_\pm. \]

Furthermore, denote
\[ (\cdot, \cdot)_\pm = (\cdot, \cdot)|_{\mathfrak{h}_\pm}, \quad \text{where} \quad \mathfrak{h}_\pm = P_\pm \mathfrak{h}. \]

Remind that, for any closed linear operator \( T \) with a closed range, there exists a positive number \( \mu > 0 \) ("the reduced minimum modulus of \( T \" [28, IV-§5.1]) such that \([24, 28]\)
\[ (Th|Th) \geq \mu (h|h) \quad \text{for all} \quad h \in (\ker T)^\perp \cap D(T). \]

Hence, by assumptions \( \text{H1-H2} \), we obtain (by letting \( T = \mathcal{L}^{1/2} \) - note that \( \mathcal{L}^{1/2} \) is a self-adjoint linear operator with common kernel with \( \mathcal{L} \) and a domain containing the domain of \( \mathcal{L} \); \( \ker \mathcal{L} = \ker \mathcal{L}^{1/2} \) and \( D(\mathcal{L}) \subseteq D(\mathcal{L}^{1/2}) \) [28]) that there exists a positive number \( \mu > 0 \), such that
\[ (\mathcal{L}h|h) \geq \mu (h|h) \quad \text{for all} \quad h \in \text{Im}\mathcal{L} \cap D(\mathcal{L}). \]  

However, here it will be assumed that the operator \( \mathcal{L} \) satisfies an, in general, even stronger condition:

**H3** There exists a positive number \( \gamma > 0 \), such that \( \mathcal{L} \geq \gamma (1 + |B|) \) on \( \text{Im}\mathcal{L} \cap D(\mathcal{L}); \)
\[ (\mathcal{L}h|h) \geq \gamma ((1 + |B|) h|h) \quad \text{for all} \quad h \in \text{Im}\mathcal{L} \cap D(\mathcal{L}). \]  

**Remark 1** Assumptions \( \text{H1-H3} \) are fulfilled for the linearized Boltzmann collision operator for hard spheres, for (monatomic) single species, as well as, for (monatomic) multicomponent mixtures, see Section [1] with \( B = v + u \), where the velocity is given by \( v = (v_1, v_2, v_3) \in \mathbb{R}^3 \) and \( u \in \mathbb{R} \) is fixed [3]. Assumptions \( \text{H1-H3} \) are also fulfilled for the linearized Boltzmann collision operators for hard potentials (including hard spheres) if the operator \( B \) is bounded, while this is not the case for soft potentials (the range of the linearized Boltzmann collision operator for soft potentials is not closed).

**Remark 2** If the operator \( B \) is bounded, then assumption [8] follows directly by property [7]. Furthermore, \( D(\mathcal{L}) \subseteq \mathfrak{h} = D(B) \).

We will consider two different types of boundary conditions at \( x = 0 \). Introduce operators \( P \) and \( R \) defined in either of the following two ways (only considering cases for which such operators exist):

**H4(a)** \( P \in \mathfrak{L}(\mathfrak{h}_-; \mathfrak{h}_+) \) is a bijective linear operator from \( \mathfrak{h}_- \) to \( \mathfrak{h}_+ \), and \( R \in \mathfrak{L}(\mathfrak{h}_+) \) is a linear operator on \( \mathfrak{h}_+ \), such that
\[
\begin{align*}
(B|g|h)_- &= (BPg|Ph)_+ , \\
(Rh|Bg)_+ &= (Bh|Rg)_+ , \\
(Rg|BRg)_+ &\leq (g|Bg)_+ .
\end{align*}
\]
Here and below, we use the simplified notations (for this case):

\[ Pg = PP_- g, \quad Rg = RP_+ g. \]

**H4(b)** \( P = 1_{h_-} \) is the identity operator on \( h_- \), while \( R = 0 \in \mathcal{L}(h_-; h_+) \).

The general formulation of the steady half-space problem of our interest reads:

\[
\begin{aligned}
B \frac{\partial f}{\partial x} + \mathcal{L} f &= S \\
P_+ f(0, \cdot) &= RPP_+ f(0, \cdot) + f_b
\end{aligned}
\]

for some given \( f_b \in h_+ \cap D(\mathcal{L}) \), where \( e^{\sigma x} f = e^{\sigma x} f(x, \cdot) \in L^2(\mathbb{R}_+, h) \) and \( e^{\sigma x} S(x, \cdot) \in L^2(\mathbb{R}_+, h) \) for some positive number \( \sigma > 0 \), and \( S = S(x, \cdot) \in \text{Im} \mathcal{L} \) for all \( x \in \mathbb{R}_+ \). Substituting \( f = e^{-\sigma x} g \), in problem (10) and introducing the operator

\[
\tilde{R} := P_+ - RPP_- \in \mathcal{L}(h, h_+)
\]

we obtain

\[
\begin{aligned}
B \frac{\partial g}{\partial x} + \mathcal{L} g - \sigma Bg &= e^{\sigma x} S \\
\tilde{R} g(0, \cdot) &= f_b
\end{aligned}
\]

for some given \( f_b \in h_+ \cap D(\mathcal{L}) \), where \( g = g(x, \cdot) \in L^2(\mathbb{R}_+, h) \), \( S = S(x, \cdot) \in \text{Im} \mathcal{L} \) for all \( x \in \mathbb{R}_+ \), and \( e^{\sigma x} S(x, \cdot) \in L^2(\mathbb{R}_+, h) \) for some positive number \( \sigma > 0 \).

**Remark 3** Typically, for Boltzmann(-type) equations (cf. Section 1 and Remark 1), \( B = v + u \), while \( f = f(x, v) \), with \( v = (v, v_2, ..., v_d) \in \mathbb{R}^d \) and fixed \( u \in \mathbb{R} \).

Then \( h_\pm = h|_{v+u \geq 0} \) - where, typically, \( h = \{ L^2(\mathbb{R}) \}^s \) for some positive integer \( s \geq 1 \) - \( Pf(x, v) = f(x, v_-) \), with \( v_- = v - (2(v + u), 0, ..., 0) \), while the linear operator \( R = R_u \) fulfills (here, properties H4(a)-(b) can be combined), cf. [20, p. 164] for boundary conditions of linearized Boltzmann equation (cf. also [22, p. 164])

\[
(Rh| (v + u) g)_+ = ((v + u) h| Rg)_+,
(Rg| (v + u) Rg)_+ \leq (g| (v + u) g)_+ , \quad (\cdot|\cdot)_+ = (\cdot|\cdot)_{u+v>0}.
\]

Note that property H4(b) corresponds to complete absorption at the interface \( x = 0 \).

### 3 Main results

Let \((k^+, k^-, l)\) be the signature of the restriction of the quadratic form \((B\phi|\phi)\) to the kernel of \( \mathcal{L} \); \( k^+ \), \( k^- \), and \( l \) denote the numbers of positive, negative, and
zero eigenvalues of a symmetric $n \times n$ matrix $K$ with elements $k_{ij} = (B\phi_i | \phi_j)$ for any basis $\{\phi_1, ..., \phi_n\}$ of the kernel of $L$. Due to Sylvester's law of inertia, these numbers are independent of the choice of basis of the kernel $\ker L$. There exists an orthonormal basis

$$\{\phi_1, ..., \phi_{n-l}, \psi_1, ..., \psi_l\}$$

of the kernel $\ker L$, such that

$$\begin{align*}
(\phi_i | \phi_j) &= \delta_{ij}, \quad (\psi_r | \psi_s) = \delta_{rs}, \quad (\phi_i | \psi_r) = 0, \\
(B\phi_i | \phi_j) &= \beta_i \delta_{ij}, \quad (B\psi_s | \psi_r) = (B\phi_i | \psi_r) = 0,
\end{align*}$$

with $\beta_1, ..., \beta_{k^+} > 0$ and $\beta_{k^++1}, ..., \beta_{n-l} < 0$, cf. [16]. Decompose the kernel of $L$ in the following way

$$\begin{align*}
\ker L &= Z_+ \oplus Z_- \oplus Z_0, \text{ where } Z_+ := \text{span} \{\phi_1, ..., \phi_{k^+}\}, \\
Z_- := \text{span} \{\phi_{k^++1}, ..., \phi_{n-l}\}, \quad Z_0 := \text{span} \{\psi_1, ..., \psi_l\}.
\end{align*}$$

For $\psi \in \text{D}(B)$, there exists $\varphi \in \text{D}(L)$, such that $L\varphi = B\psi$, if and only if

$$B\psi \in \text{Im}L = (\ker L)^\perp.$$ 

Hence, there exists $\varphi_r \in \text{D}(L)$ for each $\psi_r, r \in \{1, ..., l\}$, such that

$$L\varphi_r = B\psi_r.$$ 

Without loss of generality, cf. [16], it can be assumed that

$$B\varphi_r \in Z_+^\perp \cap Z_-^\perp, \quad (B\psi_r | \varphi_s) = (L\varphi_r | \varphi_s) = \alpha_r \delta_{rs} \text{ with } \alpha_r > 0.$$ 

**Theorem 1** Assume that $S = S(x, \cdot) \in \text{Im}L$ for all $x \in \mathbb{R}_+$, $e^{\bar{\sigma}x}S(x, \cdot) \in L^2(\mathbb{R}_+; \mathfrak{h})$ for some $\bar{\sigma} > 0$, $\bar{R}Z_+ \cup \bar{R}Z_0 \subseteq \text{D}(L)$, and $\dim (\mathfrak{h}_+, \text{D}(L)) > k^+ + l$. Then there exists a unique solution $f$ of the problem (10) such that

$$e^{\sigma x}f(x, \cdot) \in L^2(\mathbb{R}_+; \mathfrak{h}),$$

for some $\sigma > 0$, assuming $k^+ + l$ conditions on $f_\mathfrak{h} \in \mathfrak{h}_+ \cap \text{D}(L)$. 

Related problems, in some literature referred to as the Milne and Kramer problems, see [3] for the Boltzmann equation, can also be considered:

**Corollary 1** Assume that $S = S(x, \cdot) \in \text{Im}L$ for all $x \in \mathbb{R}_+$, $e^{\bar{\sigma}x}S(x, \cdot) \in L^2(\mathbb{R}_+; \mathfrak{h})$ for some $\bar{\sigma} > 0$, $\bar{R}Z_+ \cup \bar{R}Z_0 \subseteq \text{D}(L)$, and $\dim (\mathfrak{h}_+, \text{D}(L)) > k^+ + l$. Then there exists a unique solution $f$ of the problem (10) such that

$$e^{\sigma x}f(x, \cdot) - f_\infty \in L^2(\mathbb{R}_+; \mathfrak{h}),$$

with $f_\infty = \lim_{x \to \infty} f(x, \cdot) \in \ker L$, for some $\sigma > 0$, if the $k^- = n - k^+ - l$ parameters

$$\{(f_\infty | \phi_{k+1}), ..., (f_\infty | \phi_{n-l})\},$$

are prescribed.
Corollary 2 Let $S \equiv 0$ and assume that $\tilde{R}Z_+ \cup \tilde{R}Z_0 \subseteq D(\mathcal{L})$ and $\dim (\mathfrak{h}_+, D(\mathcal{L})) > k^+ + l$. Then there is a unique solution $f$ of the problem (16) such that

\[ e^{\sigma x} (f(x, \cdot) - f_\infty) \in L^2(\mathbb{R}_+; \mathfrak{h}), \quad f_\infty = \tilde{f}_\infty + xf'_\infty, \]

\[ f'_\infty = \lim_{x \to -\infty} f'(x, \cdot), \quad \tilde{f}_\infty = \lim_{x \to -\infty} (f(x, \cdot) - xf'_\infty) \in \ker \mathcal{L}, \]

for some $\sigma > 0$, if the $k^- + l = n - k^+$ parameters

\[ \{(f_\infty | \phi_{k+1}), ..., (f_\infty | \phi_{n-1}), (f'_\infty | \varphi_1), ..., (f'_\infty | \varphi_l)\}, \]

are prescribed.

Letting the linear operator $B$ being the identity operator on $\mathfrak{h}$, implies that $\mathfrak{h}_+ = \mathfrak{h}$ and $\mathfrak{h}_- = \{0\}$. Then, replacing $x$ by $t$ and putting $f_0 = f_0$, we obtain a spatially homogeneous Cauchy problem:

\[
\begin{aligned}
\frac{\partial f}{\partial t} + \mathcal{L} f &= S, \quad t > 0, \\
f(0, \cdot) &= f_0
\end{aligned}
\]

(17)

where $f(t, \cdot) = f(t, \cdot)$, $S = S(t, \cdot)$, $f_0 \in D(\mathcal{L})$.

Here property (8) follows by property (7), and the following result follows:

Corollary 3 Let $\mathcal{L}$ be a nonnegative self-adjoint operator with a closed range and a finite dimensional kernel

\[ \ker \mathcal{L} = \text{span} \{\phi_1, ..., \phi_n\}, \]

for some $\phi_1, ..., \phi_n \in \mathfrak{h}$. Furthermore, let $S = S(t, \cdot) \in \text{Im} \mathcal{L}$ for all $t \in \mathbb{R}_+$, $e^{\sigma t} S(t, \cdot) \in L^2(\mathbb{R}_+; \mathfrak{h})$ for some $\sigma > 0$, and $f_0 \in D(\mathcal{L})$. Then the linearized spatially homogeneous initial value problem (17) has a unique solution $f$ such that $e^{\sigma t} f(t, \cdot) \in L^2(\mathbb{R}_+; \mathfrak{h})$ for some $\sigma > 0$ if and only if

\[ (f_0 | \phi_1) = ... = (f_0 | \phi_n) = 0. \]

For (variants of) the Boltzmann equation, cf. Remark 3 a typical steady half-space problem reads

\[ (v + u) \frac{\partial f_u}{\partial x} + \mathcal{L} f_u = S_u, \quad f_u = f_u(x, v), \quad S_u = S_u(x, v), \]

\[ f_u(0, v) = R f_u(0, v_-) + f_{bu}(v) \] for $v + u > 0$, $v_- = v - (2(v + u), 0, ..., 0)$,

\[ e^{\sigma u} f_u(x, v) \in L^2(\mathbb{R}_+; (L^2(\mathfrak{h}))^s) \text{ and } \]

\[ e^{\sigma u} S_u(x, v) \in L^2(\mathbb{R}_+; (L^2(\mathfrak{h}))^s) \text{ for some } \sigma > 0, \]

\[ S_u = S_u(x, v) \in \text{Im} \mathcal{L} \text{ for all } x \in \mathbb{R}_+ \]

for some fixed positive integer $s \geq 1$, $v = (v_1, ..., v_d) \in \mathbb{R}_d$, and $u \in \mathbb{R}$. Assuming that the linear operator $R = R_u$ fulfills property (12) in Remark 3
and properties $\textbf{H}1$ and $\textbf{H}3$ being fulfilled, see Remark 1 for some important cases, Theorem 1 is applicable, as well as, Corollaries 1 and 2.

The theory could be applied to steady half-space problems of the Boltzmann equation for hard spheres, for monatomic single species \cite{3,33,25}, as well as, for monatomic binary mixtures \cite{1,5} (note that the stated results in \cite{5} are for two species with equal mass), which in a natural way can be extended to monatomic multicomponent mixtures (cf. \cite{17,18,23,12}). It can also be applied for quantum Boltzmann equations as the one for excitations near a Bose condensate \cite{2}. Recent results for polyatomic molecules where the polyatomicity is modelled by either a discrete, or a continuous, internal energy variable, show that for some particular collision kernels the linearized collision operators fulfill the assumed properties also in those cases \cite{12,13}. Favorable applications to discrete velocity Boltzmann models for single species and multicomponent mixtures (of monatomic molecules, as well as polyatomic molecules), and quantum extensions for Bosons, Fermions, as well as anyons, cf. \cite{16,6,7,15,8,9,10,11}, can be stressed as well.

Remark 4 The theory is also applicable for a BGK-like hard-sphere model of the Boltzmann equation, cf. \cite{20, p. 96-97, 208} with the collision frequency \( \nu = \nu(|v|) \) for hard spheres, for which

\[
L f = \nu(|v|) \left( f - \sum_{i=1}^{d+2} (\nu f \phi_i) \phi_i \right),
\]

where \( \phi_1, \ldots, \phi_{d+2} \) is an orthonormal, with respect to \( (\nu \phi_i \phi_j) \), basis of \( \{ \sqrt{M}, \sqrt{M} v, \sqrt{M} v_2, \ldots, \sqrt{M} v_d, \sqrt{M} |v|^2 \} \):

\[
\text{span} \{ \phi_1, \ldots, \phi_{d+2} \} = \text{span} \left\{ \sqrt{M}, \sqrt{M} v, \sqrt{M} v_2, \ldots, \sqrt{M} v_d, \sqrt{M} |v|^2 \right\},
\]

\[
(\nu \phi_i | \phi_j) = \delta_{ij}.
\]

As noted, Theorem 1 is applicable for problem (18). However, in general \( \sigma_u \) will depend on \( u \): Theorem 1 will provide us with existence and an exponential speed of convergence for fixed \( u \), while the exponential speed of convergence will, in general, not be uniform in \( u \). Nevertheless, on any bounded interval, whose closure does not contain any degenerate value \( u = u_0 \), i.e. on any bounded interval, such that \( l = 0 \) for all \( u \) in its closure, \( \sigma_u \) can be chosen uniformly. This will still remain true if the lower end point of the interval will be a degenerate value \( u = u_0 \), i.e. with \( l > 0 \) for \( u = u_0 \). Contrary, if the closure of a bounded interval contains a degenerate value \( u = u_0 \) (other than the lower end point), there will, in general, be no uniform exponential speed of convergence on the interval. As \( u \) tends to the degenerate value \( u_0 \) from below, \( u \rightarrow u_0 \), then, in general (without imposing some additional conditions on \( f_{bu} \) \( \sigma_u \) will tend to zero. For \( u \) sufficiently close to a degenerate value \( u_0 \) from below, it may occur some slowly varying mode(s) (see e.g. \cite{14} for a more explicit presentation in a case of \( l = 1 \); corresponding from the transition from between condensation and
evaporation). The slowly varying mode(s), can be cancelled by posing some - more precisely \(l\) - additional condition(s) at the interface (for \(u\) less than \(u_0\)). The following result can be obtained:

**Theorem 2** Let \(u = u_0\) be a degenerate value of \(u\), i.e. such that \(l > 0\) for \(u = u_0\), assume that \(\dim(\mathfrak{h}_+L) > k_0^+ + l - k_0^+\) equals \(k^+\) for \(u = u_0\), and assume that for all \(u \in \mathbb{R}^d\), \(S_u \in S_u(x,v) \in \text{ImL} \) for all \(x \in \mathbb{R}^d\), \(e^{\sigma x}S_u(x,v) \in L^2(\mathbb{R}^d; (L^2(dx))^2)\) for some \(\sigma > 0\) and \(\mathfrak{R}_u \mathbb{Z}_0 \cup \mathfrak{R}_u \mathbb{Z}_0 \subset D(\mathcal{L}_u^{1+ > 0})\), with \(\mathfrak{R}_u = 1_{u+v > 0} - \mathfrak{R}_u\), where \(\mathfrak{R}_u(x,v) = f(x,v)\), while \(\mathbb{Z}_0\) and \(\mathbb{Z}_0\) are defined in (14) (with \(B = v + u_0\)).

Then there exists a positive number \(\delta(u_0) > 0\), such that by imposing \(k_0^+ + l\) conditions on \(f_{bu} \in (L^2((1 + |v|)1_{u+v > 0}dv))^{\mathbb{S}} \cap D(\mathcal{L}_u^{1+ > 0})\), there exists a family \(\{f_u\}_{|u-u_0| \leq \delta(u_0)}\) of unique solutions \(f_u\) of the problem (18) such that

\[ e^{\alpha x}f_u(x,v) \in L^2(\mathbb{R}; (L^2(dx))^2) \]

for some positive number \(\alpha > 0\), independent of \(u\), if \(|u - u_0| \leq \delta(u_0)\).

4 Penalized problem

Let \(\{\phi_1, ..., \phi_{n-l}, \psi_1, ..., \psi_l\}\) be an orthonormal basis of the kernel of \(\mathcal{L}\), such that relations (13) are satisfied, and let \(\{\varphi_1, ..., \varphi_l\} \subset D(\mathcal{L})\), be such that

\[ \mathcal{L}\varphi_r = B\psi_r, \]

with relations (16) satisfied. Without loss of generality, it may be assumed that

\[ (B\varphi_r | \varphi_s) = 0 \text{ for all } \{r, s\} \subseteq \{1, ..., l\}. \quad (19) \]

Indeed, if the contrary, replace \(\{\varphi_1, ..., \varphi_l\}\) with \(\{\tilde{\varphi}_1, ..., \tilde{\varphi}_l\}\), where

\[
\tilde{\varphi}_r = \varphi_r - \sum_{s=r+1}^{l} \frac{(B\varphi_r | \varphi_s)}{\alpha_s} \psi_s - \frac{(B\varphi_r | \varphi_r)}{2\alpha_r} \psi_r \text{ for } r \in \{1, ..., l\}, \]

\[
\alpha_i = (B\psi_i | \phi_i) = (L\phi_i | \phi_i) > 0,
\]

and relations (16),(19) will be satisfied. Relations (19) will be of use in the forthcoming section.

If \(l \neq 0\), denote

\[ \psi = (\psi_1, ..., \psi_l), \varphi = (\varphi_1, ..., \varphi_l), \]

and introduce the symmetric \(l \times l\) matrix \((\psi \otimes \psi)_{B^2}\) with elements

\[ ((\psi \otimes \psi)_{B^2})_{rs} = (B^2\psi_r | \psi_s) = (B\psi_r | B\psi_s) = (\mathcal{L}\varphi_r | \varphi_s). \]
The matrix $\langle \psi \otimes \psi \rangle_{B^2}$ is symmetric, and hence, there exists an orthogonal $l \times l$ matrix $U$, such that, cf. [16],

$$
\langle \tilde{\psi} \otimes \tilde{\psi} \rangle_{B^2} = \langle U \psi \otimes U \psi \rangle_{B^2} = U^T \langle \psi \otimes \psi \rangle_{B^2} U = \text{diag}(\gamma_1, \ldots, \gamma_l),
$$

for some real numbers $\gamma_1, \ldots, \gamma_l$, or, equivalently,

$$
\left( B^2 \tilde{\psi}_r \big| \tilde{\psi}_s \right) = \left( B \tilde{\psi}_r \big| \tilde{\psi}_s \right) = (L \tilde{\varphi}_r \big| L \tilde{\varphi}_s) = \gamma_r \delta_{rs},
$$

with $\tilde{\psi} := U \psi = (\tilde{\psi}_1, \ldots, \tilde{\psi}_l)$ and $\tilde{\varphi} := U \varphi = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_l)$, (20)

where, without loss of generality,

$$
\gamma_1 \geq \ldots \geq \gamma_l > 0.
$$

It may be stressed that, by construction,

$$
\tilde{\psi}^T \tilde{\psi} = \psi^T U^T U \psi = \psi^T \psi = I_l,
$$

where $I_l$ denote the $l \times l$ identity matrix, or, in other words, \{\tilde{\psi}_1, \ldots, \tilde{\psi}_l\} is an orthonormal basis of $Z_0$.

Otherwise, if $l = 0$, let $\gamma_1 = 0$.

Define the linear operators $\Pi_+ \in \mathcal{L}(\mathfrak{h}; Z_+)$ and $\Pi_0 \in \mathcal{L}(\mathfrak{h}; Z_0)$ by

$$
\Pi_+ := \sum_{i=1}^{k^+} (\cdot | \phi_i) \phi_i; \quad \Pi_0 := \sum_{r=1}^{l} (\cdot | \varphi_r) \varphi_r,
$$

and consider the penalized problem, cf. [33, 25, 14]:

$$
\begin{cases}
B \frac{\partial g}{\partial x} + \mathcal{L}g - \sigma Bg = e^{\sigma x} S - \alpha \Pi_+ (Bg) - \beta B \Pi_0 (Bg), \quad x > 0, \\
\tilde{R}g(0, \cdot) = f_b, \quad \tilde{R} = P_+ - RPP_-,
\end{cases}
$$

or, equivalently, by introducing the linear operator

$$
\Lambda := \mathcal{L} - \sigma B + \alpha \Pi_+ B + \beta B \Pi_0 B,
$$

$$
\begin{cases}
B \frac{\partial g}{\partial x} + \Lambda g = e^{\sigma x} S, \quad x > 0, \\
\tilde{R}g(0, \cdot) = f_b, \quad \tilde{R} = P_+ - RPP_-.
\end{cases}
$$

**Lemma 1** For appropriately chosen positive constants $\alpha$, $\beta$, and $\sigma$ the operators $\Lambda$ and $\Lambda^*$ are coercive on $D(\mathcal{L})$. Indeed, there exists a positive number $\mu = \mu(\alpha, \beta, \sigma) > 0$ such that

$$(\Lambda^* f | f) = (Af | f) \geq \mu (f | f) \quad \text{for all } f \in D(\mathcal{L}).$$
Proof. Firstly, decompose $f$ orthogonally as
\[ f = h + q, \quad q \in \ker \mathcal{L}, \quad h \in \text{Im} \mathcal{L}, \]
and then, cf. decomposition (14), $q$ as
\[ q = q_+ + q_- + q_0, \quad q_{\pm} \in \mathcal{Z}_{\pm}, \quad q_0 \in \mathcal{Z}_0. \]
The operators $\mathcal{L}$ and $B$, as well as $\Pi_0$ and $\Pi_+$, are all self-adjoint, and hence, the adjoint operator of $\Lambda$ (24) equals
\[ \Lambda^* = \mathcal{L} - \sigma B + \alpha B \Pi_+ + \beta B \Pi_0 B. \]
Let $0 < \varepsilon_1, \varepsilon_2 < 1$. Then
\[
2 (h|Bq_0) = \frac{1}{\varepsilon_1^2} (h|h) - \left( \frac{h}{\varepsilon_1} \right) (h|Bq_0) \right) + \varepsilon_1^2 (Bq_0|Bq_0)
\leq \frac{1}{\varepsilon_1^2} (h|h) + \varepsilon_1^2 (Bq_0|Bq_0)
\]
and
\[
(\Pi_0 (B (h + q_0))|B (h + q_0))
= \left( 1 - \frac{1}{\varepsilon_2^2} \right) (\Pi_0 (Bh)|Bh) + \left( \Pi_0 \left( B \left( \frac{h}{\varepsilon_2} + \varepsilon_2 q_0 \right) \right) \right) + (1 - \varepsilon_2^2) (\Pi_0 (Bq_0)|Bq_0)
\geq (1 - \varepsilon_2^2) (\Pi_0 (Bq_0)|Bq_0) - \frac{1 - \varepsilon_2^2}{\varepsilon_2^2} (\Pi_0 (Bh)|Bh).
\]
Therefore, with $\alpha = 2\sigma$ and $0 < \varepsilon_1, \varepsilon_2 < 1$,
\[
(\Lambda^* f | f) = (\Lambda f | f)
= (\mathcal{L} h | h) - \sigma (Bh|h) + (\alpha - 2\sigma) (Bh|q_+) + (\alpha - \sigma) (Bq_+|q_+) - 2\sigma (Bh|q_-)
- \sigma (Bq_-|q_-) - 2\sigma (h|Bq_0) + \beta (\Pi_0 (B (h + q_0))|B (h + q_0))
\geq (\mathcal{L} h | h) - \sigma (Bh|h) + \sigma (Bq_+|q_+) - 2\sigma (Bh|q_-) - \sigma (Bq_-|q_-) - \frac{\sigma}{\varepsilon_1} (h|h)
\]
\[
- \sigma \varepsilon_1^2 (Bq_0|Bq_0) - \frac{1 - \varepsilon_2^2}{\varepsilon_2^2} \beta (\Pi_0 (Bh)|Bh) + (1 - \varepsilon_2^2) \beta (\Pi_0 (Bq_0)|Bq_0).
\]
(26)
Using construction (20), (22), $q_0$ can be decomposed in the following two ways:
\[ q_0 = \sum_{r=1}^{l} a_r \psi_r = \sum_{s=1}^{l} \tilde{a}_s \tilde{\psi}_s, \quad (q_0|q_0) = \sum_{r=1}^{l} a_r^2 = \sum_{s=1}^{l} \tilde{a}_s^2, \quad a_r, \tilde{a}_s \in \mathbb{R}. \]
Then
\[ Bq_0 = \sum_{s=1}^{l} \tilde{a}_s B\tilde{\psi}_s = \sum_{r=1}^{l} a_r B\psi_r = \sum_{r=1}^{l} a_r \mathcal{L}\varphi_r. \]
By relations \((20), (21)\), follows that

\[
(Bq_0|Bq_0) = \sum_{j=1}^{l} \gamma_j \tilde{a}_j^2 \leq \gamma_1 \sum_{j=1}^{l} \tilde{a}_j^2 = \gamma_1 (q_0|q_0),
\]

while

\[
\left(\Pi_0 (Bq_0)|Bq_0\right) = \sum_{r=1}^{l} a_r^2 \left(\frac{B\psi_r|\varphi_r}{(\varphi_r|\mathcal{L}\varphi_r)^2}\right) \leq \sum_{r=1}^{l} a_r^2 = (q_0|q_0).
\]

Furthermore, applying the Cauchy-Schwarz inequality gives that

\[
\left(\Pi_0 (Bh)|Bh\right) \leq \left(\varphi_r|L\varphi_r\right) \sum_{r=1}^{l} a_r^2 = (q_0|q_0).
\]

Moreover, \(q_-\) can be decomposed as

\[
q_- = \sum_{i=k^+1}^{n-l} b_i \phi_i, \text{ with } b_i \in \mathbb{R},
\]

resulting in

\[
\frac{1}{2} (Bq_-|q_-) + 2 (Bh|q_-) = \sum_{i=k^+1}^{n-l} \left(\frac{b_i^2}{2} (B\phi_i|\phi_i) + 2b_i (Bh|\phi_i)\right)
\]

\[
= \sum_{i=k^+1}^{n-l} \left(\frac{b_i^2}{2} (B\phi_i|\phi_i) + 2b_i (Bh|\phi_i)\right)
\]

Let \(0 < \varepsilon < 1\). Then, for \(i \in \{k^+ + 1, \ldots, n - l\}\),

\[
\frac{b_i^2}{2} (B\phi_i|\phi_i) + 2b_i (Bh|\phi_i) = \pm \frac{1}{\varepsilon^2} (Bh|h) \pm \frac{2\epsilon^2}{\varepsilon} b_i^2 (B\phi_i|\phi_i) \pm
\]

\[
= \pm \frac{1}{\varepsilon^2} (Bh|h) \pm \frac{2\epsilon^2}{\varepsilon} b_i^2 (B\phi_i|\phi_i) \pm \leq \frac{1}{\varepsilon^2} (Bh|h) \pm \frac{2\epsilon^2}{\varepsilon} b_i^2 (B\phi_i|\phi_i) \pm
\]

For \(n > k^+ + l\), denote

\[
\hat{\beta}_{\text{max}} = \max_{k^+ + 1 \leq i \leq n - l} \left\{ \frac{\beta_i^-}{|\beta_i^+|} \right\} \geq 1, \quad \beta_i^- = (|B|\phi_i|\phi_i)_-,
\]

\[
\beta_i = (B\phi_i|\phi_i) = (B\phi_i|\phi_i)_+ - \beta_i^- < 0.
\]
For \( i \in \{ k^+ + 1, \ldots, n - l \} \) follows, since \( \beta_{k^+ + 1}, \ldots, \beta_{n - l} < 0 \), that
\[
\frac{\beta_i}{2} + \frac{\beta_i + 2\beta^-}{4 \left( \frac{\beta^-}{|\beta_i|} - \frac{1}{2} \right)} = \beta_i^- \frac{\beta_i + |\beta_i|}{2\beta^- - |\beta_i|} = 0,
\]
and furthermore, by inequality (30), for \( 0 < \varepsilon \leq \frac{1}{2 \sqrt{\beta_{\max}}} \),
\[
\frac{1}{2} (Bq_- q_-) + 2 (B h q_-) \leq \sum_{i=k^+ + 1}^{n-l} \left[ \frac{1 + 2 \varepsilon^2}{2} b_i^2 (\beta_i + \beta_i^-) - \frac{1 - 2 \varepsilon^2}{2} b_i^2 \beta_i^- + \frac{1}{\varepsilon^2} (|B| h |h|) \right] = \frac{k^-}{\varepsilon^2} (|B| h |h|) + \sum_{i=k^+ + 1}^{n-l} b_i^2 \left( \frac{\beta_i}{2} + \frac{\beta_i + 2\beta_i^-}{4 \left( \frac{\beta_i^-}{|\beta_i|} - \frac{1}{2} \right)} \right) \leq \frac{k^-}{\varepsilon^2} (|B| h |h|) , \text{ where } k^- = n - k^+ + l > 0. \tag{32}
\]
Otherwise, if \( k^- = n - k^+ + l = 0 \), let \( \varepsilon = 1 \).

Denote
\[
\beta_{\min} = \min_{1 \leq i \leq n-l} \{|\beta_i|\} > 0 , \beta_i = (B \phi_i | \phi_i) , \tag{33}
\]
let \( \beta = \sigma \beta_{\min} + 2 \gamma \varepsilon_1^2 \) and
\[
\sigma = \frac{\gamma}{\max \left( \frac{1 + \frac{k^-}{\varepsilon^2}}{\varepsilon_1^2}, \frac{\beta_{\min} + 2 \gamma \varepsilon_1^2}{\varepsilon_2^2} \sum_{r=1}^l \frac{(B \varphi_r | B \varphi_r)}{\alpha_r^2}, \frac{2 \gamma \left( 1 - \varepsilon_2^2 \right)}{\beta_{\min} + 2 \gamma \varepsilon_1^2} \right)} , \tag{34}
\]
where \( \gamma \) is given in relation H3 [8] and \( \alpha_r = (\varphi_r | \mathcal{L} \varphi_r) \) for \( r \in \{ 1, \ldots, l \} \). Note that the latter argument of the maximum in equality (34) is in fact not used here, but will be of use first in Lemma 3 in the forthcoming section.

The lemma follows, by inequalities (26), (29), (32), and equality (28):
\[
(\Lambda^* f | f) = (\Lambda f | f) - \sigma \left(1 + \frac{k^-}{\varepsilon^2}\right) (|B|h|h) - \sigma (Bq_+ | q_+ ) \]
\[
- \frac{\sigma}{2} (Bq_- | q_-) + (\frac{(1 - \varepsilon_2)}{\varepsilon_2^2} \beta - \gamma_1 \sigma \varepsilon_2^2) (q_0 | q_0 )
\]
\[
- \left(\frac{\sigma}{\varepsilon_1^2} + \frac{1 - \varepsilon_2^2}{\varepsilon_2^2} \beta \sum_{r=1}^{l} \left(\frac{B\varphi_r | B\varphi_r }{|\varphi_r | \mathcal{L} \varphi_r |^2}ight)\right) (h | h)
\]
\[
\geq \gamma \frac{2}{\varepsilon_1^2} (h | h) + \left(\gamma - \sigma \left(1 + \frac{k^-}{\varepsilon^2}\right) \right) (|B|h|h) + \frac{\sigma \beta_{\text{min}}}{2} (q | q )
\]
\[
+ \left(\frac{\gamma}{2} - \sigma \frac{1}{\varepsilon_1^2} + \frac{\beta_{\text{min}} + 2 \gamma_1 \varepsilon_2^2}{2 \varepsilon_2^2} \sum_{r=1}^{l} \frac{B\varphi_r | B\varphi_r }{\alpha_r^2}\right) (h | h)
\]
\[
\geq \gamma \frac{2}{\varepsilon_1^2} (h | h) + \frac{\sigma \beta_{\text{min}}}{2} (q | q ) \geq \mu (f | f), \text{ with } 2\mu = \min (\gamma, \sigma \beta_{\text{min}}) > 0. \]

Define
\[
Tg := B\frac{\partial g}{\partial x} + \Lambda g,
\]
with domain, we remind that \(\bar{R} = P_+ - RPP_-\).
\[
D(T) := \left\{ g(x, \cdot), B\frac{\partial g}{\partial x}(x, \cdot), Lg(x, \cdot) \right\} \subset L^2(\mathbb{R}^+; h), \bar{R}g(0, \cdot) = 0 \}
\]Then, it follows that
\[
T^* g := -B\frac{\partial g}{\partial x} + \Lambda^* g, \Lambda^* = \mathcal{L} - \sigma B + \alpha B\Pi_+ + \beta B\Pi_0 B,
\]
with domain
\[
D(T^*) := \left\{ g(x, \cdot), B\frac{\partial g}{\partial x}(x, \cdot), Lg(x, \cdot) \right\} \subset L^2(\mathbb{R}^+; h), \bar{R}^* g(0, \cdot) = 0 \}
\]
where
\[
\bar{R}^* = RP_+ - P P_-.
\]

**Lemma 2** For appropriately chosen positive constants \(\alpha, \beta, \) and \(\sigma\), there exists a positive number \(\mu = \mu (\alpha, \beta, \sigma) > 0\) such that
\[
\| Tg \|_{L^2(\mathbb{R}^+; h)} \geq \mu \| g \|_{L^2(\mathbb{R}^+; h)} \text{ for all } g \in D(T),
\]
\[
\| T^* g \|_{L^2(\mathbb{R}^+; h)} \geq \mu \| g \|_{L^2(\mathbb{R}^+; h)} \text{ for all } g \in D(T^*).
\]
In particular, \(\ker T = \{0\}\) and \(\text{Im} T = L^2(\mathbb{R}^+; h).\)
**Proof.** Let \( g \in \text{D}(T) \). Then \( Bg \in L^2(\mathbb{R}^+; \mathfrak{h}) \). Hence, there exists a sequence \( (s_n)_{n=1}^\infty \) of positive real numbers such that \( s_n \to \infty \) and \( Bg(s_n, \cdot) \to 0 \) in \( L^2(\mathbb{R}^+; \mathfrak{h}) \) as \( n \to \infty \).

By inequality (4),

\[
(Bg(0, \cdot)|g(0, \cdot)) = (Bg(0, \cdot)|g(0, \cdot))_+ - (|B|g(0, \cdot)|g(0, \cdot))_-
\]

\[
= (BRPg(0, \cdot)|RPg(0, \cdot))_+ - (|B|Pg(0, \cdot)|Pg(0, \cdot))_+
\]

\[
\leq 0,
\]

where the last inner product is chosen to match the choice of \( P \) and \( R \) in the boundary conditions \( \text{H4(a)-(b)} \); indeed, choose \((\cdot, \cdot)_+\) for \( \text{H4(a)} \), while \((\cdot, \cdot)_-\) is to be chosen for \( \text{H4(b)} \), where \( R = 0 \).

Thus, by Lemma 1, there exists a positive number \( \mu = \mu(\alpha, \beta, \sigma) > 0 \) such that

\[
\|Tg\|_{L^2(\mathbb{R}^+; \mathfrak{h})} \|g\|_{L^2(\mathbb{R}^+; \mathfrak{h})} \geq \int_0^{s_n} (Tg(x, \cdot)|g(x, \cdot)) \, dx
\]

\[
= (Bg(s_n, \cdot)|g(s_n, \cdot)) - (Bg(0, \cdot)|g(0, \cdot)) + \int_0^{s_n} (\Lambda g(x, \cdot)|g(x, \cdot)) \, dx
\]

\[
\geq (Bg(s_n, \cdot)|g(s_n, \cdot)) + \mu \int_0^{s_n} (g(x, \cdot)|g(x, \cdot)) \, dx.
\]

Taking the limit as \( n \to \infty \), results in the inequality

\[
\|Tg\|_{L^2(\mathbb{R}^+; \mathfrak{h})} \|g\|_{L^2(\mathbb{R}^+; \mathfrak{h})} \geq \mu \|g\|_{L^2(\mathbb{R}^+; \mathfrak{h})}^2,
\]

or, equivalently,

\[
\|Tg\|_{L^2(\mathbb{R}^+; \mathfrak{h})} \geq \mu \|g\|_{L^2(\mathbb{R}^+; \mathfrak{h})}.
\]

The analogous statement for \( T^* \) is proved in a similar way.

The first inequality in the statement of the lemma implies that \( \ker T = \{0\} \), and the second one that \( \text{Im} T = L^2(\mathbb{R}^+; \mathfrak{h}) \). ■

**Proposition 1** Let \( e^{\sigma x}S(x, \cdot) \in L^2(\mathbb{R}^+; \mathfrak{h}) \) and assume that \( f_0 \in \mathfrak{h}_+ \cap \text{D}(\mathcal{L}) \). Then there exists a unique solution \( g(x, \cdot) \in L^2(\mathbb{R}^+; \mathfrak{h}) \) of the penalized problem \( 25 \), such that

\[
\mu \|g\|_{L^2(\mathbb{R}^+; \mathfrak{h})} \leq \|e^{\sigma x}S\|_{L^2(\mathbb{R}^+; \mathfrak{h})} + \frac{1}{\sqrt{2\sigma}} \|\Lambda f_0\|_{\mathfrak{h}} + \sqrt{\frac{\sigma}{2}} \|Bf_0\|_{\mathfrak{h}}.
\]

**Proof.** Let \( h = g(x, \cdot) - f_0 e^{-\sigma x} \). Then \( h \in \text{D}(T) \) if and only if \( g \in \overline{\text{D}(T)} \), where

\[
\overline{\text{D}(T)} = \left\{ \left. g(x, \cdot), \frac{\partial g}{\partial x}(x, \cdot), \mathcal{L}g(x, \cdot) \right| g(x, \cdot) \in L^2(\mathbb{R}^+; \mathfrak{h}), \mathcal{R}g(0, \cdot) = f_0 \right\}.
\]

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and
\[ T h = S + \sigma B f e^{-\sigma x} - e^{-\sigma x} \Lambda f_b \in L^2(\mathbb{R}_+; h) \]
if and only if \( g \) is a solution of the penalized problem \( (25) \). However, by Lemma \( 2 \), this problem has a unique solution in \( L^2(\mathbb{R}_+; h) \). Hence, there exists a unique solution of the penalized problem \( (25) \).

Moreover,
\[
\mu \|g\|_{L^2(\mathbb{R}_+; h)} \leq \|S\|_{L^2(\mathbb{R}_+; h)} + \left( \|\Lambda f_b\|_h + \sigma \|B f_b\|_h \right) \|e^{-\sigma x}\|_{L^2} \\
\leq \|S\|_{L^2(\mathbb{R}_+; h)} + \frac{1}{\sqrt{2\sigma}} \|\Lambda f_b\|_h + \frac{\sigma}{\sqrt{2}} \|B f_b\|_h.
\]

\[ \blacksquare \]

## 5 Removal of the penalization

Denote, with \( \varphi_1, ..., \varphi_l \) given by equations \( (15), (16), (19) \),
\[
W_0 := \text{span} \{ \varphi_1, ..., \varphi_l \},
\]
and let \( S = S(x, \cdot) \in \text{Im} \mathcal{L} \) for all \( x \in \mathbb{R}_+ \).

An initial step is to transform the problem \( (11) \), in a way such that \( e^{\sigma x} S \) is replaced with \( \bar{S} = \bar{S}(x, \cdot) \in \text{Im} \mathcal{L} \cap W_0^\perp \). Indeed, substituting
\[
\bar{g}(x, \cdot) = g(x, \cdot) + e^{\sigma x} \sum_{r=1}^l \psi_r \int_x^\infty \frac{S(\tau, \cdot)|\varphi_r)}{(\varphi_r|\mathcal{L}\varphi_r)} d\tau
\]
in problem \( (11) \) results in
\[
\begin{cases}
B \frac{\partial \bar{g}}{\partial x} + \mathcal{L} \bar{g} - \sigma B \bar{g} = \bar{S}, \ x > 0, \\
\bar{R} \bar{g}(0, \cdot) = f_b, \ \bar{R} = P_+ - \mathcal{R} \mathcal{P}_-,
\end{cases}
\]
where, under the assumption that \( \bar{R} \mathcal{Z}_0 \subseteq \text{D}(\mathcal{L}) \),
\[
\bar{S}(x, \cdot) = e^{\sigma x} \left( S(x, \cdot) - \sum_{r=1}^l B \psi_r \frac{(S(x, \cdot)|\varphi_r)}{(\varphi_r|\mathcal{L}\varphi_r)} \right) \in \text{Im} \mathcal{L} \cap W_0^\perp \text{ for all } x \in \mathbb{R}_+
\]
\[
\bar{S}(x, \cdot) \in L^2(\mathbb{R}_+; h), \ g_b = f_b + \sum_{r=1}^l \bar{R} \psi_r \int_0^\infty \frac{(S(\tau, \cdot)|\varphi_r)}{(\varphi_r|\mathcal{L}\varphi_r)} d\tau \in \mathfrak{h}_+ \cap \text{D}(\mathcal{L}).
\]

Therefore, we may, without loss of generality, consider the problem \( (11) \), as well as the penalized problem \( (25) \), assuming that \( S \in \text{Im} \mathcal{L} \cap W_0^\perp \) for all \( x \in \mathbb{R}_+ \).

Denote by \( I : \mathfrak{h}_+ \rightarrow \mathfrak{h} \) the solution operator
\[
I(f_b) = g(0, \cdot),
\]

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where $g(x, \cdot) \in L^2(\mathbb{R}_+; h)$ is the unique solution of the penalized problem \(25\) in Proposition 1, and by $I : h_+ \rightarrow h$ the linear solution operator

$$
I(f_b) = g(0, \cdot),
$$

in the particular case where $g(x, \cdot) \in L^2(\mathbb{R}_+; h)$ is the unique solution in Proposition 1 of the homogeneous penalized problem

$$
\begin{cases}
B \frac{\partial g}{\partial x} + \Lambda g = 0, & x > 0, \\
\tilde{R} g(0, \cdot) = f_b, & \tilde{R} = P_+ - RPP_-
\end{cases}
$$

Observe the relation

$$
I(f_b + \tilde{f}_b) = I(f_b) + I(\tilde{f}_b)
$$

for $\{f_b, \tilde{f}_b\} \subset h_+ \cap D(L)$. 

**Lemma 3** Let $S = S(x, \cdot) \in \text{Im}L \cap W^1_0$ for all $x \in \mathbb{R}_+$ and $e^{\tilde{\sigma} x} S(x, \cdot) \in L^2(\mathbb{R}_+; h)$ for some $\tilde{\sigma} > 0$, and assume that $f_b \in h_+ \cap D(L)$. Then the solution of the penalized problem \(25\) is a solution of the problem \(11\) if and only if

$$
\Pi_+ (BI(f_b)) = \Pi_0 (BI(f_b)) = 0,
$$

or, equivalently, if and only if

$$
\Pi_+ (BI(f_b)) = \bar{\Pi}_0 (BI(f_b)) = 0,
$$

where the projection $\bar{\Pi}_0 \in \mathfrak{L}(h; Z_0)$ on $Z_0$ is the linear operator

$$
\bar{\Pi}_0 = \sum_{s=1}^{l} (\cdot | \psi_s) \psi_s.
$$

**Proof.** Notice that a solution of the penalized problem \(25\) is a solution of problem \(36\) if and only if

$$
\Pi_+ (Bg) = \Pi_0 (Bg) = 0.
$$

Assume that $g = g(x, \cdot) \in L^2(\mathbb{R}_+; h)$ is a solution of the penalized problem \(25\). Then, with $\alpha = 2\sigma$ and $\sigma$ given by expression \(34\),

$$
\frac{\partial}{\partial x} (Bg|\phi_i) + \sigma (Bg|\phi_i) = 0, \quad i \in \{1, \ldots, k^+\},
$$

$$
\frac{\partial}{\partial x} (Bg|\psi_s) - \sigma (Bg|\psi_s) + \beta \left( \frac{Bg|\varphi_s}{\varphi_s| L \varphi_s} \right) = 0, \quad s \in \{1, \ldots, l\},
$$

$$
\frac{\partial}{\partial x} (Bg|\varphi_s) + (Bg|\psi_s) + 2\sigma \sum_{i=1}^{k^+} (Bg|\phi_i) (\phi_i| \varphi_s) - \sigma (Bg|\varphi_s) = 0, \quad s \in \{1, \ldots, l\},
$$

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or

\[(B\gamma|\phi_i) = e^{-\sigma x} (B\gamma(0,\cdot)|\phi_i)\]

for \(i \in \{1, \ldots, k^+\}\), while for \(s \in \{1, \ldots, l\}\)

\[(B\gamma|\psi_s) = \beta \int_x^\infty e^{\sigma(x-\tau)} \frac{(B\gamma(\tau,\cdot)|\phi_s)}{(\phi_s|L\phi_s)} d\tau, \text{ and} \]

\[(B\gamma|\varphi_s) = \int_x^\infty e^{\sigma(x-\tau)} \left(2\sigma \sum_{i=1}^{k^+} (B\gamma(\tau,\cdot)|\phi_i) (\phi_i|\varphi_s) + (B\gamma (\tau,\cdot)|\psi_s)\right) d\tau.\]

However, assuming that \((B\gamma|\phi_1) = \ldots = (B\gamma|\phi_{k^+}) = 0\), the substitution

\[-\beta \frac{(B\gamma|\varphi_s)}{(\varphi_s|L\varphi_s)} = \frac{\partial}{\partial x} (B\gamma|\psi_s) - \sigma (B\gamma|\varphi_s)\]

in system (42), results in

\[\left(\frac{\partial^2}{\partial x^2} - 2\sigma \frac{\partial}{\partial x} + \sigma^2 - \frac{\beta}{(\varphi_s|L\varphi_s)}\right) (B\gamma|\psi_s) = 0, \ s \in \{1, \ldots, l\},\]

or, correspondingly, the substitution

\[-(B\gamma|\psi_s) = \frac{\partial}{\partial x} (B\gamma|\varphi_s) - \sigma (B\gamma|\varphi_s)\]

in system (41), results in

\[\left(\frac{\partial^2}{\partial x^2} - 2\sigma \frac{\partial}{\partial x} + \sigma^2 - \frac{\beta}{(\varphi_s|L\varphi_s)}\right) (B\gamma|\varphi_s) = 0, \ s \in \{1, \ldots, l\}.\]

Then, with \(\beta = \sigma \frac{\beta_{\text{min}} + 2\gamma_1 \epsilon_1^2}{2(1-\epsilon_2^2)}\), it follows that

\[(B\gamma (x,\cdot)|\psi_s) = e^{\left(\sigma - \sqrt{\frac{\beta_{\text{min}} + 2\gamma_1 \epsilon_1^2}{2(1-\epsilon_2^2)(\varphi_s|L\varphi_s)}}\right)x} (B\gamma (0,\cdot)|\psi_s), \ s \in \{1, \ldots, l\},\]

and, correspondingly,

\[(B\gamma (x,\cdot)|\varphi_s) = e^{\left(\sigma - \sqrt{\frac{\beta_{\text{min}} + 2\gamma_1 \epsilon_1^2}{2(1-\epsilon_2^2)(\varphi_s|L\varphi_s)}}\right)x} (B\gamma (0,\cdot)|\varphi_s), \ s \in \{1, \ldots, l\}.\]

That is, \(\Pi_+ (B\gamma) = \Pi_0 (B\gamma) = 0\) if and only if

\[\Pi_+ (B\mathcal{I}(f_b)) = \Pi_0 (B\mathcal{I}(f_b)) = 0\]

or, equivalently, if and only if

\[\Pi_+ (B\mathcal{I}(f_b)) = \tilde{\Pi}_0 (B\mathcal{I}(f_b)) = 0.\]
Note that in the notations of (36): \( \Pi_+ (Bg(x, \cdot)) = \Pi_+ (B\tilde{g}(x, \cdot)) \) and \( \tilde{\Pi}_0 (Bg(x, \cdot)) = \tilde{\Pi}_0 (B\tilde{g}(x, \cdot)) \), why \( \Pi_+ (BI(f_b)) = \Pi_+ (B\tilde{I}(g_b)) \) and \( \tilde{\Pi}_0 (BI(f_b)) = \tilde{\Pi}_0 (B\tilde{I}(g_b)) \), where \( I : h_+ \to h \) and \( \tilde{I} : h_+ \to \tilde{h} \) and the solution operators (37) for right hand side \( S \) and \( \tilde{S} \), respectively. Therefore, we can state the following Corollary.

**Corollary 4** Let \( S = S (x, \cdot) \in \text{Im}\mathcal{L} \) for all \( x \in \mathbb{R}_+ \) and \( e^{\tilde{\sigma}x}S(x, \cdot) \in L^2 (\mathbb{R}_+; h) \) for some \( \tilde{\sigma} > 0 \), and assume that \( f_b \in h_+ \cap D(\mathcal{L}) \). Then the solution of the penalized problem (25) is a solution of the problem (11) if and only if

\[
\Pi_+ (BI(f_b)) = \tilde{\Pi}_0 (BI(f_b)) = 0,
\]

where \( \tilde{\Pi}_0 \in \mathfrak{L}(h; Z_0) \) is the projection (10).

**Theorem 3** Let \( S = S (x, \cdot) \in \text{Im}\mathcal{L} \) for all \( x \in \mathbb{R}_+ \) and \( e^{\tilde{\sigma}x}S(x, \cdot) \in L^2 (\mathbb{R}_+; h) \) for some \( \tilde{\sigma} > 0 \), \( \tilde{R}Z_+ \cup \tilde{R}Z_0 \subseteq D(\mathcal{L}) \), and \( \dim (h_+ \cap D(\mathcal{L})) > k^+ + l \). Then there exists a unique solution \( g (x, \cdot) \in L^2 (\mathbb{R}_+; h) \) of the problem (11), such that

\[
\mu \|g\|_{L^2(\mathbb{R}_+; h)} \leq \|S\|_{L^2(\mathbb{R}_+; h)} + \frac{1}{\sqrt{2\sigma}} \|\mathcal{L}g_b\|_h + \sqrt{2\sigma} \|B\mathcal{L}g_b\|_h,
\]

for some \( \sigma > 0 \), assuming

\[
\mathsf{codim} \left( \{ f_b \in h_+ \cap D(\mathcal{L}) \mid \Pi_+ (BI(f_b)) = \tilde{\Pi}_0 (BI(f_b)) = 0 \} \right) = k^+ + l
\]

conditions on \( f_b \in h_+ \cap D(\mathcal{L}) \).

**Proof.** Denote

\[
\mathcal{P} = \left\{ f_b \in h_+ \cap D(\mathcal{L}) \mid \Pi_+ (BI(f_b)) = \tilde{\Pi}_0 (BI(f_b)) = 0 \right\}.
\]

Since \( \Pi_+ \in \mathfrak{L}(h; Z_+) \) and \( \tilde{\Pi}_0 \in \mathfrak{L}(h; Z_0) \),

\[
\mathsf{codim} (\mathcal{P}) \leq \dim Z_+ + \dim Z_0 = k^+ + l.
\]

Since, by assumption, \( \dim (h_+ \cap D(\mathcal{L})) > k^+ + l \)

\[
\widetilde{\mathcal{P}}_S = \left\{ f_b \in h_+ \cap D(\mathcal{L}) \mid \Pi_+ (BI(f_b)) = \tilde{\Pi}_0 (BI(f_b)) = 0 \right\}
\]

is non-empty for any right hand side \( \tilde{S} \), such that \( e^{\tilde{\sigma}x}S(x, \cdot) \in L^2 (\mathbb{R}_+; h) \) for some \( \tilde{\sigma} > 0 \). Let \( \alpha = 2\sigma \) and let \( g_0 = g_0 (x, \cdot) \) be the unique solution of the penalized problem (25), and therefore of the undamped problem (11), as well, for some \( f_b = g_0 \in \mathcal{P}_S \).

Note that \( f_{b1} - f_{b2} \in \mathcal{P} \) for \( \{f_{b1}, f_{b2}\} \subseteq \widetilde{\mathcal{P}}_S \), while \( f_{b1} + f_{b2} \in \widetilde{\mathcal{P}}_S \) if \( f_{b1} \in \widetilde{\mathcal{P}}_S \) and \( f_{b2} \in \mathcal{P} \), and consequently,

\[
\mathcal{P}_S = g_{b0} + \mathcal{P}.
\]
For $i \in \{1, \ldots, k^+\}$: let $S_i(x, \cdot) = 2\sigma e^{-2\sigma x}(B\phi_i - \beta_i\phi_i) \in L^2(\mathbb{R}_+; h)$, with $\beta_i = (B\phi_i|\phi_i) > 0$, and $g_{bi} \in P_{S_i}$. Moreover, let $g_i'(x, \cdot) \in L^2(\mathbb{R}_+; h)$ be the unique solution of the problem (25) with $S = S_i$ and $f_b = g_{bi}$:

\[
\begin{cases}
B\frac{\partial g_i'}{\partial x} + \Lambda g_i' = 2\sigma e^{-\sigma x} (B\phi_i - \beta_i\phi_i), \ x > 0, \\
\bar{R}g_i'(0, \cdot) = g_{bi}.
\end{cases}
\]

By simple calculations, it can be verified that

\[h_i(x, \cdot) := g_i'(x, \cdot) + e^{-\sigma x}\phi_i \in L^2(\mathbb{R}_+; h),\]

is the unique solution of the homogeneous penalized problem (39) with $f_b = g_{bi} + \bar{R}\phi_i$, while

\[g_i(x, \cdot) := g_0(x, \cdot) + h_i(x, \cdot) = g_0(x, \cdot) + g_i'(x, \cdot) + e^{-\sigma x}\phi_i \in L^2(\mathbb{R}_+; h),\]

is the unique solution of the penalized problem (25) with $f_b = g_0 + g_{bi} + \bar{R}\phi_i$. It follows that

\[
\Pi_+(Bg_i(0, \cdot)) = \Pi_+(Bh_i(0, \cdot)) = \beta_i\phi_i, \ \beta_i = (B\phi_i|\phi_i) > 0,
\]

\[
\Pi_0(Bg_i(0, \cdot)) = \Pi_0(Bh_i(0, \cdot)) = 0.
\]

For $r \in \{1, \ldots, l\}$: let $\tilde{S}_r(x, \cdot) = e^{-2\sigma x}(4\sigma^2\alpha_r - 1)B\psi_r \in L^2(\mathbb{R}_+; h)$, $\alpha_r = (B\psi_r|\varphi_r) = (L\varphi_r|\varphi_r) > 0$ and $\tilde{g}_{br} \in P_{\tilde{S}_r}$. Moreover, let $\tilde{g}_r'(x, \cdot) \in L^2(\mathbb{R}_+; h)$ be the unique solution of the problem (25) with $S = \tilde{S}_r$ and $f_b = \tilde{g}_{br}$:

\[
\begin{cases}
B\frac{\partial \tilde{g}_r'}{\partial x} + \Lambda \tilde{g}_r' = e^{-\sigma x}\left(\frac{4\sigma^2\alpha_r}{\beta} - 1\right)B\psi_r, \ x > 0, \\
\bar{R}\tilde{g}_r'(0, \cdot) = \tilde{g}_{br}.
\end{cases}
\]

Again by simple calculations, it can be verified that

\[\tilde{h}_r(x, \cdot) := \tilde{g}_r'(x, \cdot) + e^{-\sigma x}\left(\varphi_r + \frac{2\sigma\alpha_r}{\beta}\psi_r\right) \in L^2(\mathbb{R}_+; h),\]

is the unique solution of the homogeneous penalized problem (39) with $f_b = \tilde{g}_{br} + \bar{R}\left(\varphi_r + \frac{2\sigma\alpha_r}{\beta}\psi_r\right)$, while

\[\tilde{g}_r(x, \cdot) := g_0(x, \cdot) + \tilde{h}_r(x, \cdot) = g_0(x, \cdot) + \tilde{g}_r'(x, \cdot) + e^{-\sigma x}\left(\varphi_r + \frac{2\sigma\alpha_r}{\beta}\psi_r\right) \in L^2(\mathbb{R}_+; h),\]

is the unique solution of the penalized problem (25) with $f_b = g_0 + \tilde{g}_{br} + \bar{R}\left(\varphi_r + \frac{2\sigma\alpha_r}{\beta}\psi_r\right)$. It follows that

\[
\Pi_+(B\tilde{g}_r(0, \cdot)) = \Pi_+(B\tilde{h}_r(0, \cdot)) = 0,
\]

\[
\Pi_0(B\tilde{g}_r(0, \cdot)) = \Pi_0(B\tilde{h}_r(0, \cdot)) = \alpha_r\psi_r.
\]
Consequently, 
\[
\text{codim}(\mathcal{P}) = k^+ + l,
\]
following by the uniqueness of solutions to the homogeneous penalized problem \([39]\) and the linear independence of \(\{\phi_1, ..., \phi_{k^+}, \psi_1, ..., \psi_l\}\).

6 Half-space problem of evaporation and condensation - regime transitions

This section concerns Boltzmann(-type) equations \([18]\), assuming properties H1 and H3, as well as H4 in the form \([12]\), cf. Remark\([3]\). Therefore, Theorem\([1]\) is applicable.

In general, the exponential speed of convergence - \(\sigma_u > 0\) in problem \([18]\) - depends on \(u\). Theorem\([1]\) provides existence of a unique solution and an exponential speed of convergence for fixed \(u\). However, on an interval of \(u\) the exponential speed of convergence may not be uniform - there might occur slowly varying modes in some regions of \(u\) (cf. \([14]\) and references therein). Nevertheless, on any bounded interval, whose closure does not contain any degenerate value \(u = u_0\), i.e. on any interval such that \(l = 0\) for all \(u\) in the closure, there is a uniform exponential speed of convergence - \(\sigma_u > 0\) can be uniformly determined. Remind that \((k^+, k^-, l)\) denotes the signature of the restriction of the quadratic form \(( (v + u)\phi | \phi )\) to the kernel of \(\mathcal{L}\). Moreover, based on the arguments in the previous sections one can prove that, the slowly varying modes can be eliminated, if they occur, by imposing extra conditions on the indata at the interface. The following result can be obtained:

**Theorem 4** Let \(u = u_0\) be a degenerate value of \(u\), i.e. such that \(l > 0\) for \(u = u_0\), assume that \(\dim(\mathfrak{h}_+, \text{D}(\mathcal{L})) > k^+_0 + l - \mathfrak{h}_+ = (L^2 ((1 + |v|) \mathbf{1}_{v+u>0} dv))^s \cap \text{D}(\mathcal{L}1_{v+u>0}), \) while \(k^+_0 = k^+\) for \(u = u_0\) - and assume that for all \(u\) in a neighborhood of \(u_0\): \(S_u = S_u(x, v) \in \text{Im} \mathcal{L}\) for all \(x \in \mathbb{R}_+, e^{\sigma x} S_u(x, v) \in L^2 ((\mathbb{R}_+; (L^2 (dv))^s))\) for some \(\sigma > 0\), and \(R_u Z_\pm \cup R_u Z_0 \subseteq \text{D}(\mathcal{L}1_{v+u>0}), \) with \(R_u = \mathbf{1}_{v+u>0} - R_u P\), where \(P f(x, v) = f(x, v_-)\), while \(Z_\pm\) and \(Z_0\) are defined in \([14]\) - for \(B = v + u_0\). Moreover, let \(\Pi^u_+\) and \(\Pi^u_0\) denote the linear operators \(\Pi^u_+ [23]\) and \(\Pi^u_0 [10]\) for \(u = u_0\), \(k^+ = k^+_0\) for \(u = u_0\), and \(\mathcal{L}\) be the linear solution operator \([18]\). Then there exists a positive number \(\delta(u_0) > 0\), such that by posing

\[
\text{codim} \left( \left\{ f_{bu} \in \mathfrak{h}_+ \bigg| \Pi^u_0 ((v + u) \mathbb{I}(f_{bu})) = \Pi^u_0 ((v + u) \mathbb{I}(f_{bu})) = 0 \right\} \right) = k^+_0 + l,
\]

conditions on \(f_{bu} \in \mathfrak{h}_+\), there exists a family \(\{f_u\}_{|u-u_0| \leq \delta(u_0)}\) of unique solutions \(f_u = f_u(x, v)\) of the problem \([18]\), such that

\[
e^{\sigma x} f_u(x, v) \in L^2 ((\mathbb{R}_+; (L^2 (dv))^s))
\]

for some positive number \(\sigma > 0\), independent of \(u\), if \(|u - u_0| \leq \delta(u_0)\).
Remark 5 Let \( u = u_0 \) be a degenerate value of \( u \) of order \( l \), and let \( k^+ = k_0^+ \) for \( u = u_0 \), with
\[
\phi_i = \phi_{i0} \text{ for all } i \in \{1, \ldots, k_0^+ = k^+\},
\]
\[
\psi_s = \psi_{s0} \text{ for all } s \in \{1, \ldots, l\}.
\]
Then, by the orthogonality relations (13),
\[
((v + u) \phi_{i0} | \phi_{j0}) = (\beta_i^0 + u - u_0) \delta_{ij}, \quad \beta_i^0 = ((v + u) \phi_{i0} | \phi_{i0}) > 0,
\]
\[
((v + u) \psi_{s0} | \psi_{r0}) = (u - u_0) \delta_{rs}, \quad ((v + u) \phi_{i0} | \psi_{r0}) = 0.
\]
Hence, for all \( u > u_0 \), such that there is no degenerate value on the interval \((u_0, u]\), \( k^+ = k_0^+ + l \), and it is possible to choose
\[
\phi_i = \phi_{i0} \text{ for all } i \in \{1, \ldots, k_0^+ = k^+ - l\},
\]
\[
\phi_{k^+-l+s} = \psi_{s0} \text{ for all } s \in \{1, \ldots, l\}.
\]
That is, we impose no extra conditions - compared to the number of conditions imposed in Theorem 1 - on the boundary data as \( u > u_0 \). On the other hand, for all \( u < u_0 \), such that there is no degenerate value on the interval \([u, u_0)\), \( k^+ = k_0^+ \), and it is possible to choose
\[
\phi_i = \phi_{i0} \text{ for all } i \in \{1, \ldots, k_0^+\},
\]
\[
\phi_{k^++s} = \psi_{s0} \text{ for all } s \in \{1, \ldots, l\}.
\]
That is, we impose \( l \) extra conditions - compared to the number of conditions imposed in Theorem 1 - on the boundary data as \( u < u_0 \).

Note that in the notations of Lemma 1 any solution of the undamped problem (11) - cf. the proof of Lemma 3 - must satisfy \( q_+ = q_0 = 0 \). Repeating the arguments of the proof of Lemma 1 assuming that \( q_+ = q_0 = 0 \), will remove any smallness assumptions on \( \sigma \) or \( \mu \) - obtained through large \( \tilde{b}_{\max} \)\( \tilde{b}_{\max} \gg 1 \), or small \( b_{\min} \)\( b_{\min} \ll 1 \) - for \( u \geq u_0 \) as long as \( l = 0 \) for any \( u \in (u_0, u]\). However, this is not the case as \( u \to u_0^- \). For \( u \neq u_0 \): \( \tilde{\beta}_i(u) := ((v + u) \phi_{i0} | \phi_{i0}) = \beta_i^0 + u - u_0 \) for \( i \in \{1, \ldots, k_0^+\} \), while \( \tilde{\beta}_i(u) := ((v + u) \phi_{i0} | \psi_{s0}) = u - u_0 \) for \( i \in \{k_0^+ + 1, \ldots, k_0^+ + l\} \); implying that \( \tilde{\beta}_{k_0^++1}, \ldots, \tilde{\beta}_{k_0^++l} \) tend to zero as \( u \) tends to \( u_0 \), while this will not be the case for \( \tilde{\beta}_{1}, \ldots, \tilde{\beta}_{k_0^+} \). That is, \( \tilde{\beta}_i \to 0^- \) as \( u \to u_0^- \) for all \( i \in \{1, \ldots, k_0^+\} \), while \( \tilde{\beta}_i \to 0 \) as \( u \to u_0^- \) for all \( i \in \{1, \ldots, k_0^+\} \). The fact that \( \tilde{\beta}_i \to 0^- \) as \( u \to u_0^- \) for all \( i \in \{k_0^+ + 1, \ldots, k_0^+ + l\} \) is - not taking the indata \( f_{bu} \) into consideration - equivalent to that there will occur \( l \) slowly varying modes as \( u \to u_0^- \). However, assuming that \( \tilde{\Pi}_{0}^0 ((v + u) \mathcal{I}(f_{bu})) = 0 \), will in the view of Lemma 3 be equivalent to - again in the notations of Lemma 1 - that \( q_+ = \sum_{i=k_0^++l+1}^n b_i \phi_i \) \( (b_{k_0^++1}^+ = \ldots = b_{k_0^++1}^+ = 0) \) also for \( u < u_0 \) as long as \( l = 0 \) for any \( u \in [u, u_0) \). Then any smallness assumptions on \( \sigma \) or \( \mu \) can be removed; removing the slowly
varying modes in a neighborhood of $u_0$, and hence, resulting in that a uniform exponential speed of convergence $\sigma_u$ in a neighborhood $U$ of $u_0$ can be obtained. Indeed, in the view of Theorem 1, this means to impose $l$ extra conditions on the indata for $u$ less than $u_0$.

For the Boltzmann equation, the case of complete absorption at the interface, i.e. with $R = R_u \equiv 0$, is well studied in the literature, and especially, when the disturbed particles are assumed to be distributed in accordance to the Maxwellian $M_B = M_{Bu}(v)$ of the interface - of the condensed phase - see e.g. the review [4] and references therein. Linearizing around a Maxwellian $M = M(v)

\[ F = M + \sqrt{Mf} \]

- neglecting quadratic terms - one obtain a system of the form (18), where (with $R = R_u \equiv 0$)

\[ f_{Bu} = f_{Bu}(v) = M^{-1/2}(M_{Bu} - M) \]

Let $M_B = M_B(v)$ be an arbitrary Maxwellian for a mixture of $s$ species, and denote $M_{Bu} = M_B(v + u)$ after a shift $v \rightarrow v + u$, with $u = (u,u_2,u_3)$, in the velocity space. Then $f_{Bu} = M^{-1/2}(M_{Bu} - M) \in (L^2((1 + |v|)^1u + v > 0 \, dv))^s$ has $d + s + 1$ parameters: \{ $n_{0,1}, \ldots, n_{0,s} \}$ $\subset \mathbb{R}_+$, $u_0 - u \in \mathbb{R}^d$, $T_0 \in \mathbb{R}_+$ - the number densities of the $s$ species, the bulk velocity (after the shift in the velocity space), and the temperature at the boundary, respectively. By Theorem 3, if imposing $k_u^+ + l_u$ (where the subindex $u$ indicates the dependence on $u$) conditions on $f_{Bu}$ there is a unique solution (for a fixed $u$) to problem (10) for each fixed $f_{Bu}$ satisfying the conditions. Accordingly, there will be (at least $k_u^-$ (again the subindex $u$ indicates the dependence on $u$) free parameters of $f_{Bu}$ left. Furthermore, by Theorem 4 to have a unique solution with a uniform exponential speed of convergence on a closed interval $U$ of $u$, it is needed to impose $\max_{u \in U} (k_u^+ + l_u)$ conditions on $f_{Bu}$. Then there will be (at least $2$) $\min_{u \in U} (k_u^-)$ free parameters of $f_{Bu}$ left. The corresponding problem for monatomic single species in a neighborhood of $u = 0$, in the nonlinear context, is considered in more details in [14], see also [30]. It seems most likely that our results can be extended to the weakly nonlinear case, by assuming additional conditions similar to (4)-(6) on the linearized operator, as well as some reasonable conditions on the nonlinear part, applying methods similar to the ones in [14, 25]. Though, this will be a topic for future studies.

7 Appendix

This appendix concerns the degenerate values of the flow velocity - the values for which $l > 0$, and orthogonal bases 1 for some important particular variants

1Depending on how many of the conditions that will be of actual relevance on the subset \{ $f_{Bu}| f_{Bu} = M^{-1/2}(M_{Bu} - M); \{n_{0,1}, \ldots, n_{0,s} \} \subset \mathbb{R}_+, u_0 \in \mathbb{R}^d, T_0 \in \mathbb{R}_+ \}$ of $h_+ \cap D(L)$. Indeed, it is our firm belief that all of them will be, and that at least can be removed, but this still remains to be proven.

2See comment in the footnote above.
of the Boltzmann equation. In fact, the first, second, and fourth cases below are particular cases of the fifth and last one, but are still presented separately, due to their importance on their own.

**Monatomic single species** For the Boltzmann equation for monatomic single species - in dimension $d$ - let $\mathcal{H} = L^2(dv)$, with the inner product

$$(f|g) = \int_{\mathbb{R}^d} fg \, dv, \ f, g \in L^2(dv).$$

Making the linearization

$$F = M + \sqrt{M} f$$

around a non-drifting Maxwellian

$$M = \frac{\rho m^{d/2}}{(2\pi T)^{d/2}} e^{-m|v|^2/(2T)},$$

an orthogonal basis \[^{13}\] of the kernel

$$\ker \mathcal{L} = \text{span} \left\{ \sqrt{M}, \sqrt{M} v, \sqrt{M} v_2, \ldots, \sqrt{M} v_d, \sqrt{M} |v|^2 \right\}$$

of the linearized operator $\mathcal{L}$ is \[^{22}\]

$$\begin{align*}
\phi_1 &= \sqrt{\frac{m}{2\pi T}} \sqrt{M} \left( \sqrt{\frac{m}{a(d+2)T}} |v|^2 + v \right) \\
\phi_2 &= \sqrt{\frac{m}{2\pi T}} \sqrt{M} \left( \sqrt{\frac{m}{a(d+2)T}} |v|^2 - v \right) \\
\phi_{3+j} &= \sqrt{\frac{m}{2\pi T}} \sqrt{M} v_{2+j}, \ j \in \{0, \ldots, d-2\}, \\
\phi_{d+2} &= \sqrt{\frac{d+2}{2\pi}} \sqrt{M} \left( \frac{m}{a(d+2)T} |v|^2 - 1 \right).
\end{align*}$$

Indeed, this can be obtained by - in addition to the fact that odd components of the integrands can be discarded, due to symmetry of the integration domain - the equalities

$$\left( e^{-a|v|^2} \right) \left| v \right|^2 = \left( \frac{\pi}{a} \right)^{d/2},$$

$$d \left( e^{-a|v|^2} \right) v = \left( e^{-a|v|^2} \right) |v|^2 = \frac{d}{2a} \left( \frac{\pi}{a} \right)^{d/2}, \ \text{and}$$

$$d \left( e^{-a|v|^2} \left| v \right|^2 \right) = \frac{d(d+2)}{4a^2} \left( \frac{\pi}{a} \right)^{d/2}, \ a > 0 \small{(44)}.$$

The degenerate values of $u$ become \[^{22}\]

$$u_0 = 0 \ \text{and} \ u_\pm = \pm \sqrt{\frac{T}{m}} \sqrt{\frac{d+2}{d}}.$$
and the values of the signature \((k^+, k^-, l)\) of the restriction of the quadratic form \((B\phi|\phi)\) to the kernel of \(L\) depending on the parameter \(u\) are given by:

| \(u\) | \((-\infty, u_-)\) | \((u_-, 0)\) | \((0, u_+)\) | \((u_+, \infty)\) |
|-------|----------------|----------------|----------------|----------------|
| \(k^+\) | 0 | 0 | 1 | 1 |
| \(k^-\) | \(d + 2\) | \(d + 1\) | \(d + 1\) | \(d + 2\) |
| \(l\) | 0 | 1 | 0 | 0 |

(Multicomponent monatomic mixtures) For the Boltzmann equation - in dimension \(d\) - for a mixture of \(s\) species \(\alpha_1, \ldots, \alpha_s\), with masses \(m_{\alpha_1}, \ldots, m_{\alpha_s}\), respectively, let \(\mathfrak{B} = (L^2 (dv))^s\), with the inner product

\[
(f | g) = \sum_{i=1}^s \int_{\mathbb{R}^d} f_i g_i dv, \quad f, g \in (L^2 (dv))^s.
\]

Making the linearization (13) around a non-drifting Maxwellian

\[
M = (M_{\alpha_1}, \ldots, M_{\alpha_s}), \quad M_{\alpha_i} = n_{\alpha_i} \left(\frac{m_{\alpha_i}}{2\pi T}\right)^{d/2} e^{-m_{\alpha_i}|v|^2/(2T)},
\]

an orthogonal basis (13) of the kernel

\[
\ker L = \text{span}\{\sqrt{M_{\alpha_1}}e_1, \ldots, \sqrt{M_{\alpha_s}}e_s, \sqrt{M}v, \sqrt{M}v_2, \sqrt{M}v_3, \sqrt{M}|v|^2\},
\]

\[
\overline{M} = (m_{\alpha_1}^2 M_{\alpha_1}, \ldots, m_{\alpha_s}^2 M_{\alpha_s}), \quad e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \text{ for } i \in \{1, \ldots, s\}
\]

of the linearized operator \(L\) is

\[
\begin{align*}
\phi_{1, \alpha_i} &= \frac{m_{\alpha_i}}{\sqrt{2\rho}} \sqrt{M_{\alpha_i}} \left(\frac{\rho}{\pi(d+2)nT} \right)^{1/2} \left(|v|^2 + v\right) \\
\phi_{2, \alpha_i} &= \frac{m_{\alpha_i}}{\sqrt{2\rho}} \sqrt{M_{\alpha_i}} \left(\frac{\rho}{\pi(d+2)nT} \right)^{1/2} \left(|v|^2 - v\right) \\
\phi_{3+j, \alpha_i} &= \frac{m_{\alpha_i}}{nT} \sqrt{M_{\alpha_i}} v_{2+j}, \quad j \in \{0, \ldots, d - 2\}, \\
\phi_{d+1+j} &= \frac{\bar{\phi}_{d+1+j}}{\|\bar{\phi}_{d+1+j}\|}, \quad j \in \{1, \ldots, s\}, \\
\end{align*}
\]

Again this can be obtained by equalities (44). The degenerate values of \(u\) become

\[
u_0 = 0 \quad \text{and} \quad u_{\pm} = \pm \sqrt{\frac{nT \rho}{d}} \sqrt{\frac{d + 2}{d}}.
\]
and the values of the signature \((k^+, k^-, l)\) of the restriction of the quadratic form \((B\phi|\phi)\) to the kernel of \(\mathcal{L}\) depending on the parameter \(u\) are given by:

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
u & (-\infty, u_-) & u_- & (u_-, 0) & 0 & (0, u_+) & (u_+, \infty) \\
\hline
k^+ & 0 & 0 & 1 & 1 & d + s & d + s & d + s + 1 \\
\hline
k^- & d + s + 1 & d + s & d + s & 1 & 1 & 0 & 0 \\
\hline
l & 0 & 1 & 0 & d + s - 1 & 0 & 1 & 0 \\
\hline
\end{array}
\]

(46)

**Nordheim-Boltzmann equation**  For the Nordheim-Boltzmann equation for monatomic single species - in dimension \(d\) (with \(d \geq 2\)) - let \(\mathfrak{h} = L^2(d\mathbf{p})\), with the inner product

\[
(f|g) = \int_{\mathbb{R}^d} fg \, d\mathbf{p}, \ f, g \in L^2(d\mathbf{p}),
\]

and linearize by

\[F = P_\varepsilon + (P_\varepsilon(1 + \varepsilon P_\varepsilon))^{1/2} f,\]

with \(\varepsilon = 1\) for bosons, and \(\varepsilon = -1\) for fermions, around a non-drifting Planckian

\[P_\varepsilon = P_\pm = \frac{1}{e^{\mathbf{p}^2/(2T)} + 1}, \ \mathbf{p} = (p_1, \ldots, p_d).\]

For bosons we consider the restriction \(|\mathbf{p}| \geq \lambda \sqrt{2T}\), for some \(\lambda > 0\), cf. \[2, 7\] (to avoid a singularity at \(\mathbf{p} = 0\)). Then an orthogonal basis \[13\] of the kernel of the linearized operator \(\mathcal{L}\) is

\[
\ker \mathcal{L} = \text{span} \left\{ \sqrt{\mathcal{R}_\pm}, \sqrt{\mathcal{R}_\pm p_1}, \ldots, \sqrt{\mathcal{R}_\pm p_d}, \sqrt{\mathcal{R}_\pm |\mathbf{p}|^2} \right\},
\]

with

\[
\mathcal{R}_\pm = P_\pm(1 \pm P_\pm),
\]

of the linearized operator \(\mathcal{L}\) is

\[
\begin{align*}
\phi_{1+}^+ &= \left(\frac{2T}{(2T)^{d/2}}\right)^{-1/2} \sqrt{\mathcal{R}_\pm(1 \pm P_\pm)} \left(\frac{1}{\sqrt{J_{d+1}(d+2)^2 T}} |\mathbf{p}|^2 + \sqrt{\frac{1}{J_d} p_1}\right), \\
\phi_{2+}^+ &= \left(\frac{2T}{(2T)^{d/2}}\right)^{-1/2} \sqrt{\mathcal{R}_\pm(1 \pm P_\pm)} \left(\frac{1}{\sqrt{J_{d+1}(d+2)^2 T}} |\mathbf{p}|^2 - \sqrt{\frac{1}{J_d} p_1}\right), \\
\phi_{3+j}^+ &= \left(\frac{2T}{(2T)^{d/2}}\right)^{-1/2} \sqrt{\mathcal{R}_\pm(1 \pm P_\pm)} p_{2+j}, \ j \in \{0, \ldots, d-2\}, \\
\phi_{d+2}^+ &= \left(\frac{2T}{(2T)^{d/2}}\right)^{-1/2} \sqrt{\mathcal{R}_\pm(1 \pm P_\pm)} \left(\frac{J_{d+2}(d+2)^2 T}{J_{d+2}(d+2)^2 T - p_1}\right)^{1/2} |\mathbf{p}|^2 - 1), \\
\end{align*}
\]

with

\[
J_s = \frac{2}{\Gamma(s/2 + 1)} \int_0^\infty \frac{y^{s+1}e^{y^2}}{(e^{y^2} + 1)^2} \, dy = \eta(s/2) \quad \text{for} \ s \geq 0,
\]
where $\Gamma$ is the gamma-function and $\eta = \frac{1}{\Gamma(s)} \int_0^\infty \frac{r^{s-1}}{e^r + 1} \, dr$ is the Dirichlet eta-function (alternating zeta-function), for fermions, while, for $s \geq 0$,

$$J_s^\pm = \frac{2}{\Gamma(s/2 + 1)} \int_\lambda^\infty \frac{r^{s+1}e^r}{(e^r - 1)^2} \, dr$$

$$= \frac{1}{\Gamma(s/2 + 1)} \frac{\lambda^s}{e^{\lambda^2} - 1} + \frac{1}{\Gamma(s/2)} \int_\lambda^\infty \frac{r^{s/2-1}}{e^r - 1} \, dr$$

for bosons. Indeed, this can be obtained by - in addition to the fact that odd components of the integrands can be discarded, due to symmetry of the integration domain - the equalities

$$\left( \frac{e^{a|p|^2}}{(e^{a|p|^2} + 1)^2} \right)^1 = S_{d-1} \frac{\Gamma(d/2)}{2a^{d/2}} J_{d-2}^\pm = \left( \frac{\pi}{a} \right)^{d/2} J_{d-2}^\pm,$$

$$d \left( \frac{e^{a|p|^2}}{(e^{a|p|^2} + 1)^2} |p|^1 \right) = \left( \frac{e^{a|p|^2}}{(e^{a|p|^2} + 1)^2} |p|^2 \right)$$

$$= S_{d-1} \frac{\Gamma(d/2 + 1)}{2a^{d+2}/2} J_{d+2}^\pm = \frac{d}{4a^2} \left( \frac{\pi}{a} \right)^{d/2} J_{d+2}^\pm$$

for $a > 0$,

while $\Gamma(s + 1) = s\Gamma(s)$ for $s > 0$,

where $S_n$ is the surface area of the $n$-sphere $S^n$, i.e. $S_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.

Then the degenerate values of $u$ become [9]

$$u_0 = 0 \text{ and } u_\pm = \pm \sqrt{\eta(d/2 + 1)}/\eta(d/2) \sqrt{T} \sqrt{d + 2}$$

for fermions, and

$$u_0 = 0 \text{ and } u_\pm = \sqrt{J_{d+2}/J_d} \sqrt{T} \sqrt{d + 2}$$

for bosons. Note that, for $0 \leq s \leq 2$, $J_s^+ \to \infty$ as $\lambda \to 0$, while, for $s \geq 2$, $J_s^+ \to \zeta(s/2)$ as $\lambda \to 0$, where $\zeta = \frac{1}{\Gamma(s)} \int_0^\infty \frac{r^{s-1}}{e^r - 1} \, dr$ is the zeta-function (note

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that $\zeta(1)$ is infinite); and hence, $u_\pm \to \pm \sqrt[\zeta(d/2 + 1)]{\zeta(d/2)} \sqrt{T} \sqrt{\frac{d + 2}{d}}$ as $\lambda \to 0$.

The values of the signature $(k^+, k^-, l)$ of the restriction of the quadratic form $(B\phi|\phi)$ to the kernel of $L$ depending on the parameter $u$ are given by table (45), with $u_\pm$ in (47) and (48), respectively.

**Single species of polyatomic molecules**

i) For the Boltzmann equation - in dimension $d$ - for a polyatomic single species with $r$ different internal energy levels $E_1, ..., E_r$, let $\mathcal{B} = (L^2(dv))^r$, with the inner product

$$(f|g) = \sum_{i=1}^{r} \int_{\mathbb{R}^d} f_i g_i dv, \quad f, g \in (L^2(dv))^r.$$ 

Making the linearization (43) around a non-drifting Maxwellian

$$M = (M_1, ..., M_r), \quad M_i = \frac{\rho \varphi_i M_{d/2}^i}{(2\pi T)^{d/2}} e^{-(m|v|^2 + 2E_i)/(2T)};$$

$$Q_j = \sum_{i=1}^{r} \varphi_i E_j^i e^{-E_i/T} \text{ for } j \in \{0, 1, 2\},$$

where $\varphi_i = \varphi(E_i)$ for $i \in \{1, ..., r\}$ are given weights; an orthogonal basis (13) of the kernel

$$\ker L = \text{span} \left\{ \sqrt{M}, \sqrt{Mv}, \sqrt{M}v_2, ..., \sqrt{M}v_d, \sqrt{M}|v|^2 + 2E_M \sqrt{M} \right\},$$

$$E_M = \text{diag} (E_1, ..., E_r),$$

of the linearized operator $L$ is

$$\begin{cases}
\phi_{1,i} = \frac{1}{\sqrt{2\pi}} \sqrt{M_i} (\Psi_i + \sqrt{mv}) \\
\phi_{2,i} = \frac{1}{\sqrt{2\pi}} \sqrt{M_i} (\Psi_i - \sqrt{mv}) \\
\phi_{3+j,i} = \frac{m}{\sqrt{2\pi}} \sqrt{M_i}v_{2+j}, \quad j \in \{0, ..., d-2\}, \\
\phi_{d+2,i} = \frac{\sqrt{d+2+\kappa}}{2\pi} \sqrt{M_i} \left( \sqrt{\frac{(d+\kappa)}{T}} - 1 \right).
\end{cases}$$

with

$$\Psi_i = \frac{1}{\sqrt{(d+\kappa)(d+2+\kappa)T}} \left( m|v|^2 + 2E_i + \kappa T - 2\frac{Q_1}{Q_0} \right)$$

$$\kappa = \frac{2}{T^2} \frac{Q_0 Q_2 - Q_1^2}{Q_0^2}. \quad (49)$$
This can be obtained by equalities (44), indeed, it follows that
\[
\begin{align*}
(\sqrt{M} | \sqrt{M}) &= \rho, \\
(\sqrt{M} v | \sqrt{M} v) &= (\sqrt{M} v_i | \sqrt{M} v_i) = \frac{\rho}{m} T, \ i \in \{2, \ldots, d\}, \\
(\sqrt{M} | e_M) &= \frac{\rho}{m} \left(dT + \frac{2Q_1}{Q_0}\right), \\
(\sqrt{M} v^2 | e_M) &= \frac{\rho}{m^2} \left((d + 2) T^2 + 2T \frac{Q_1}{Q_0}\right), \text{ and} \\
(e_M | e_M) &= \frac{\rho}{m^2} \left(d (d + 2) T^2 + 4dT \frac{Q_1}{Q_0} + 4 \frac{Q_2}{Q_0}\right),
\end{align*}
\]
with \( e_M = \sqrt{M} | v^2 + 2E_M \sqrt{M} \) (50).

Then the degenerate values of \( u \) become (8)
\[
\begin{align*}
u_0 &= 0 \text{ and } u_{\pm} = \pm \sqrt{\frac{T}{m}} \frac{d + \kappa}{d + \kappa},
\end{align*}
\]
and the values of the signature \((k^+, k^-, l)\) of the restriction of the quadratic form \((B\phi | \phi)\) to the kernel of \( \mathcal{L} \) depending on the parameter \( u \) are given by table (45), with \( u_{\pm} \) in (51).

ii) For the Boltzmann equation - in dimension \( d \) - for a polyatomic single species, where the polyatomicity is modelled by a continuous internal energy variable \( I \), let
\[
\mathcal{L} = L^2(dv dI), \text{ with the inner product}
\]
\[(f | g) = \int_0^1 \int_{\mathbb{R}^d} f_i g_i dv dI, \ f, g \in L^2(dv dI).\]
Making the linearization (43) around a non-drifting Maxwellian
\[
M = \frac{\rho \varphi(I) m^{d/2}}{(2\pi T)^{d/2} Q} e^{-(m|v|^2 + 2I)/(2T)}, \text{ with } Q = \int_0^\infty \varphi(I) e^{-I/T} dI,
\]
with \( \varphi(I) = I^{\delta/2 - 1} \) - where \( \delta \) is the number of internal degrees of freedom; an orthogonal basis (13) of the kernel \( \ker \mathcal{L} = \text{span} \left\{ \sqrt{M}, \sqrt{M} v, \sqrt{M} v_2, \ldots, \sqrt{M} v_d, \sqrt{M} \left(m |v|^2 + 2I\right) \right\} \)

of the linearized operator \( \mathcal{L} \) is
\[
\begin{align*}
\phi_1 &= \frac{\sqrt{m}}{\sqrt{2\rho T}} \sqrt{M} \left(\frac{\sqrt{m}}{\sqrt{(d+\delta)(d+2+\delta)}} \left(\frac{|v|^2 + \frac{2I}{m}}{m} + v\right)\right) \\
\phi_2 &= \frac{\sqrt{m}}{\sqrt{2\rho T}} \sqrt{M} \left(\frac{\sqrt{m}}{\sqrt{(d+\delta)(d+2+\delta)}} \left(\frac{|v|^2 + \frac{2I}{m}}{m} - v\right)\right) \\
\phi_{3+j} &= \frac{\sqrt{m}}{\sqrt{2\rho T}} \sqrt{M} \left(\frac{\sqrt{m}}{\sqrt{(d+\delta)(d+2+\delta)}} \left(\frac{|v|^2 + \frac{2I}{m}}{m}\right) - 1\right), \\
\phi_{d+2} &= \frac{\sqrt{m}}{\sqrt{2\rho T}} \sqrt{M} \left(\frac{\sqrt{m}}{\sqrt{(d+\delta)(d+2+\delta)}} \left(\frac{|v|^2 + \frac{2I}{m}}{m}\right) - 1\right).
\end{align*}
\]
This can be obtained by observing that, cf. (49),
\[
\frac{2}{T^2} \int_0^\infty \varphi(I) T^2 e^{-I/T} dI - \left( \int_0^\infty \varphi(I) I e^{-I/T} dI \right)^2 = \delta
\]
(52)
since
\[
\int_0^\infty \varphi(I) T^2 e^{-I/T} dI = \frac{T(\delta + 2)}{2} \int_0^\infty \varphi(I) I e^{-I/T} dI = \frac{T^2(\delta + 2)\delta}{4} Q.
\]
Then the degenerate values of \( u \) become [8]
\[
u_0 = 0 \text{ and } u_{\pm} = \pm \sqrt{\frac{T}{m}} \frac{d + 2 + \delta}{d + \delta}.
\]
(53)
and the values of the signature \((k^+, k^-, l)\) of the restriction of the quadratic form \((B \varphi, \varphi)\) to the kernel of \(L\) depending on the parameter \(u\) are given by table [13], with \(u_{\pm}\) in [33].

**Multicomponent mixtures of polyatomic molecules**

i) For the Boltzmann equation - in dimension \(d\) - for a mixture of polyatomic molecules, consisting of \(s\) different species \(\alpha_1, \ldots, \alpha_s\), with masses \(m_{\alpha_1}, \ldots, m_{\alpha_s}\), and with, for each species \(\alpha_i\), \(r_i\) different internal energy levels \(E_{1i}^{\alpha_i}, \ldots, E_{r_i}^{\alpha_i}\), let \(h = (L^2 (dv))^\tilde{r}\), where \(\tilde{r} = \sum_{i=1}^s r_i\), with the inner product
\[(f|g) = \sum_{i=1}^s \sum_{j=1}^{r_i} \int_{R^d} f_{i,j} g_{i,j} dv, f, g \in (L^2 (dv))^\tilde{r}, \tilde{r} = \sum_{i=1}^s r_i\]
Making the linearization [13] around a non-drifting Maxwellian
\[
M = (M_{\alpha_1}, \ldots, M_{\alpha_s}), \quad M_{\alpha_i} = (M_{\alpha_{i,1}}, \ldots, M_{\alpha_{i,r_i}}),
\]
\[
M_{\alpha_{i,j}} = \frac{n_{\alpha_i} \varphi_{\alpha_{i,j}} m_{\alpha_i}^{d/2}}{(2\pi T)^{d/2}} e^{-\left(m_{\alpha_i} |v|^2 + 2E_{j}^{\alpha_i}\right)/2T},
\]
\[
Q_{k}^{\alpha_i} = \sum_{j=1}^{r_i} \varphi^{\alpha_i, j} \left(E_{j}^{\alpha_i}\right)^k e^{-E_{j}^{\alpha_i}/T}, \quad k \in \{0, 1, 2\},
\]
while \(\varphi^{\alpha_{i,j}} = \varphi^{\alpha_{i, j_i}}, j_i \in \{1, \ldots, r_i\}, i \in \{1, \ldots, s\}\), are given weights; an orthogonal basis [13] of the kernel
\[
\ker L = \text{span} \left\{ \sqrt{M_{\alpha_1}}, \ldots, \sqrt{M_{\alpha_s}}, m \sqrt{M v_1}, \ldots, m \sqrt{M v_{r_s}}, \mathcal{E}_M \right\},
\]
\[
m = \text{diag}(m_{\alpha_1}, \ldots, m_{\alpha_s}, m_{\alpha_1}, \ldots, m_{\alpha_s}), \quad M_{\alpha_i} = (0, \ldots, 0, M_{\alpha_i}, 0, \ldots, 0),
\]
\[
\mathcal{E}_M = \left( m |v|^2 + 2E_M \right) \sqrt{M}, \quad \mathcal{E}_M = \text{diag} \left( E_{1}^{\alpha_1}, \ldots, E_{r_1}^{\alpha_1}, \ldots, E_{1}^{\alpha_s}, \ldots, E_{r_s}^{\alpha_s} \right),
\]
32
of the linearized operator $L$ is

$$
\begin{cases}
\phi_{1,\alpha_i,j} = \frac{1}{\sqrt{2\rho}} \sqrt{M_{\alpha_i,j}} (\Psi_{\alpha_i,j} + m_{\alpha_i} v) \\
\phi_{2,\alpha_i,j} = \frac{1}{\sqrt{2\rho}} \sqrt{M_{\alpha_i,j}} (\Psi_{\alpha_i,j} - m_{\alpha_i} v) \\
\phi_{3+k,\alpha_i,j} = \frac{m_{\alpha_i}}{\sqrt{\pi T}} \sqrt{M_{\alpha_i,j}} v_{2+k}, \ k \in \{0, ..., d - 2\}, \\
\phi_{d+1+k} = \frac{\sqrt{2\rho}}{\phi_{d+1+k}}, \ k \in \{1, ..., s\},
\end{cases}
$$

with

$$
\Psi_{\alpha_i,j} = \sqrt{\frac{\rho}{(d + \pi) (d + 2 + \pi) nT}} (m_{\alpha_i} |v|^2 + 2E_{\alpha_i}^0 + \pi T - 2Q_{\alpha_i}^0) Q_{\alpha_i}^0 - (Q_{\alpha_i}^0)^2 n_{\alpha_i}.
$$

Again this can be obtained by equalities (44), cf. also equalities (50). Then the degenerate values of $u$ become

$$
u_0 = 0 \quad \text{and} \quad u_{\pm} = \pm \sqrt{\frac{nT}{\rho} \frac{d + 2 + \pi}{d + \pi}}. \quad (54)
$$

and the values of the signature $(k^+, k^-, l)$ of the restriction of the quadratic form $(B\phi, \phi)$ to the kernel of $L$ depending on the parameter $u$ are given by Table (46), with $u_{\pm}$ in (54).

ii) For the Boltzmann equation - in dimension $d$ - for a polyatomic multi-component mixture of $s$ species with masses $m_{\alpha_1}, ..., m_{\alpha_s}$, respectively, where the polyatomicity is modelled by a continuous internal energy variable $I$, let $\mathfrak{h} = (L^2 (dv dI))^s$, with the inner product

$$(f | g) = \sum_{i=1}^s \int_0^1 \int_{\mathbb{R}^d} f_i g_i \ dv \ dI, \ f, g \in (L^2 (dv dI))^s.$$  

Making the linearization (43) around a non-drifting Maxwellian

$$
M = (M_{\alpha_1}, ..., M_{\alpha_s}), \ M_{\alpha_i} = \frac{n_{\alpha_i} \varphi_{\alpha_i}(I) m_{\alpha_i}^{d/2}}{(2\pi T)^{d/2}} e^{-(m_{\alpha_i} |v|^2 + 2I)/(2T)}, \\
Q_{\alpha_i} = \int_0^\infty \varphi_{\alpha_i}(I) e^{-I/T} dI,
$$

33
with \( \varphi_{\alpha_i}(I) = I^{n_{\alpha_i}/2-1} \), where \( n_{\alpha_i} \) is the number of internal degrees of freedom for species \( \alpha_i \); an orthogonal basis \( \{ \text{span} \{ \sqrt{M_{\alpha_1}}, e_1, \ldots, \sqrt{M_{\alpha_s}}, e_s, m\sqrt{M}v, m\sqrt{M}v_2, \ldots, m\sqrt{M}v_d, e_M \} \) of the linearized operator \( L \)

\[
\ker L = \text{span} \left\{ \sqrt{M_{\alpha_1}}e_1, \ldots, \sqrt{M_{\alpha_s}}e_s, m\sqrt{M}v, m\sqrt{M}v_2, \ldots, m\sqrt{M}v_d, e_M \right\},
\]

with \( m = \text{diag} (m_{\alpha_1}, \ldots, m_{\alpha_s}) \) and \( e_M = m\sqrt{M} |v|^2 + 2I\sqrt{M} \), of the linearized operator \( L \) is

\[
\left\{ \begin{array}{ll}
\phi_{1,\alpha_i} &= \frac{1}{\sqrt{2T}} \sqrt{M_{\alpha_i}} (\Psi_{\alpha_i} + m_{\alpha_i}v) \\
\phi_{2,\alpha_i} &= \frac{1}{\sqrt{2T}} \sqrt{M_{\alpha_i}} (\Psi_{\alpha_i} - m_{\alpha_i}v) \\
\phi_{3+j,\alpha_i} &= \frac{m}{\sqrt{nT}} \sqrt{M_{\alpha_i}} v_{2+j}, \ j \in \{0, \ldots, d-2\}, \\
\phi_{d+1+j} &= \frac{\tilde{\phi}_{d+1+j}}{\|\tilde{\phi}_{d+1+j}\|}, \ j \in \{1, \ldots, s\},
\end{array} \right.
\]

with

\[
\Psi_{\alpha_i} = \sqrt{\frac{\rho}{(d+\bar{\delta})(d+2+\bar{\delta}) nT}} \left( m_{\alpha_i} |v|^2 + 2I + (\bar{\delta} - n_{\alpha_i}) T \right)
\]

\[
\tilde{\phi}_{d+1+j} = \hat{\phi}_{d+1+j} - \sum_{k=2}^{j} \left( \frac{\hat{\phi}_{d+1+j}}{\|\hat{\phi}_{d+1+j}\|} \right) \phi_{d+k},
\]

\[
\hat{\phi}_{d+1+j,\alpha_i} = \sqrt{M_{\alpha_i}} \left( \frac{d+\bar{\delta}}{(d+2+\bar{\delta}) mT} \rho n_{\alpha_i} \Psi_{\alpha_i} - \delta_{ij} \right),
\]

\[
\rho = \sum_{k=1}^{s} m_{\alpha_k} n_{\alpha_k}, \ n = \sum_{k=1}^{s} n_{\alpha_k}, \ \bar{\delta} = \frac{\sum_{i=1}^{s} \delta_{\alpha_i} n_{\alpha_i}}{n}.
\]

We stress that \( \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \) is the Kronecker delta, and not connected to the numbers \( n_{\alpha_i} \) of internal degrees of freedom. This can be obtained by observing that, cf. (52),

\[
2 \frac{T^2}{Q_{\alpha_i}} \int_{0}^{\infty} \varphi_{e_{\alpha_i}(I)} I e^{-l/T} dI - \left( \int_{0}^{\infty} \varphi_{e_{\alpha_i}(I)} I e^{-l/T} dI \right)^2 = \delta_{\alpha_i}, \ i \in \{1, \ldots, s\}.
\]

Then the degenerate values of \( u \) become

\[
u_0 = 0 \quad \text{and} \quad u_{\pm} = \pm \sqrt{\frac{nT}{\rho}} \sqrt{\frac{d+2+\bar{\delta}}{d+\bar{\delta}}}, \quad (55)
\]

and the values of the signature \( (k^+, k^-) \) of the restriction of the quadratic form \( (B\varphi, \varphi) \) to the kernel of \( L \) depending on the parameter \( u \) are given by table (46), with \( u_{\pm} \) in (55).
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References

[1] K. Aoki, C. Bardos, and S. Takata, *Knudsen layer for gas mixtures*, J. Stat. Phys., 112 (2003), pp. 629–655.

[2] L. Arkeryd and A. Nouri, *A Milne problem from a Bose condensate with excitations*, Kinet. Relat. Models, 6 (2013), pp. 671–686.

[3] C. Bardos, R. E. Caflisch, and B. Nicolaenko, *The Milne and Kramers problems for the Boltzmann equation of a hard sphere gas*, Commun. Pure Appl. Math., 39 (1986), pp. 323–352.

[4] C. Bardos, F. Golse, and Y. Sone, *Half-space problems for the Boltzmann equation: A survey*, J. Stat. Phys., 124 (2006), pp. 275–300.

[5] C. Bardos and X. Yang, *The classification of well-posed kinetic boundary layer for hard sphere gas mixtures*, Commun. Partial Differ. Equ., 37 (2012), pp. 1286–1314.

[6] N. Bernhoff, *On half-space problems for the linearized discrete Boltzmann equation*, Riv. Mat. Univ. Parma, 9 (2008), pp. 73–124.

[7] __________, *Half-space problems for a linearized discrete quantum kinetic equation*, J. Stat. Phys., 159 (2015), pp. 358–379.

[8] __________, *Discrete velocity models for multicomponent mixtures and polyatomic molecules without nonphysical collision invariants and shock profiles*, AIP Conf. Proc., 1786 (2016), p. 040005.

[9] __________, *Boundary layers for discrete kinetic models: multicomponent mixtures, polyatomic molecules, bimolecular reactions, and quantum kinetic equations*, Kinet. Relat. Models, 10 (2017), pp. 925–955.

[10] __________, *Discrete velocity models for polyatomic molecules without nonphysical collision invariants*, J. Stat. Phys., 172 (2018), pp. 742–761.

[11] __________, *Discrete quantum Boltzmann equation*, AIP Conf. Proc., 2132 (2019), pp. 130011:1–9.

[12] __________, *Linearized Boltzmann collision operator: I. Polyatomic molecules modeled by a discrete internal energy variable and multicomponent mixtures*, arXiv: 2201.01365, (2022).
[13] Linearized Boltzmann collision operator: II. Polyatomic molecules modeled by a discrete internal energy variable and multicomponent mixtures, arXiv: 2201.01377, (2022).

[14] N. Bernhoff and F. Golse, On the boundary layer equations with phase transition in the kinetic theory of gases, Arch. Ration. Mech. Anal., 240 (2021), pp. 51–98.

[15] N. Bernhoff and M. C. Vinerean, Discrete velocity models for multicomponent mixtures without nonphysical collision invariants, J. Stat. Phys., 165 (2016), pp. 434–453.

[16] A. V. Bobylev and N. Bernhoff, Discrete velocity models and dynamical systems, in Lecture Notes on the Discretization of the Boltzmann Equation, N. Bellomo and R. Gatignol, eds., World Scientific, 2003, pp. 203–222.

[17] L. Boudin, B. Grec, M. Pavić, and F. Salvarani, Diffusion asymptotics of a kinetic model for gaseous mixtures, Kinet. Relat. Models, 6 (2013), pp. 1375–157.

[18] M. Briant and E. S. Daus, The Boltzmann equation for a multi-species mixture close to global equilibrium, Arch. Ration. Mech. Anal., 222 (2016), pp. 1367–1443.

[19] C. Cercignani, Half-space problems in the kinetic theory of gases, in Trends in Applications of Pure Mathematics to Mechanics, E. Kroner and E. Kirchgassner, eds., Springer-Verlag, 1986, pp. 35–50.

[20] ——, The Boltzmann equation and its applications, Springer-Verlag, 1988.

[21] ——, Rarefied Gas Dynamics, Cambridge University Press, 2000.

[22] F. Coron, F. Golse, and C. Sulem, A classification of well-posed kinetic layer problems, Commun. Pure Appl. Math., 41 (1988), pp. 409–435.

[23] E. S. Daus, A. Jungel, C. Mouhot, and S. Zamponi, Hypocoercivity for a linearized multispecies Boltzmann system, SIAM J. Math. Anal., 48 (2016), pp. 538–568.

[24] S. Goldberg, Unbounded Linear Operators, McGraw-Hill Book Company, 1966.

[25] F. Golse, Analysis of the boundary layer equation in the kinetic theory of gases, Bull. Inst. Math. Acad. Sin., 3 (2008), pp. 211–242.

[26] F. Golse and F. Poupaud, Stationary solutions of the linearized Boltzmann equation in a half-space, Math. Methods Appl. Sci., 11 (1989), pp. 483–502.

[27] H. Grad, Asymptotic theory of the Boltzmann equation II, in Rarefied Gas Dynamics Vol 1, Academic Press, 1963, pp. 26–59.
[28] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, 1980.

[29] Q. Li, J. Lu, and W. Sun, *Half-space kinetic equations with general boundary conditions*, Math. Comp., 86 (2017), pp. 1269–1301.

[30] T.-P. Liu and S.-H. Yu, *Invariant manifolds for steady Boltzmann flows and applications*, Arch. Rational Mech. Anal., 209 (2013), pp. 869–997.

[31] Y. Sone, *Kinetic Theory and Fluid Dynamics*, Birkhauser, 2002.

[32] ———, *Molecular Gas Dynamics*, Birkhauser, 2007.

[33] S. Ukai, T. Yang, and S.-H. Yu, *Nonlinear boundary layers of the Boltzmann equation: I. Existence*, Commun. Math. Phys., 236 (2003), pp. 373–393.