Universality behind Basquin’s law of fatigue

F. Kun\textsuperscript{1,2},*, H. A. Carmona\textsuperscript{2,3}, J. S. Andrade Jr.\textsuperscript{2,4}, and H. J. Herrmann\textsuperscript{2}

\textsuperscript{1} Department of Theoretical Physics, University of Debrecen, P. O. Box:5, H-4010 Debrecen, Hungary
\textsuperscript{2} Computational Physics, JFB, HIF, E12, ETH, H"{o}nggerberg, 8093 Z"{u}rich, Switzerland
\textsuperscript{3} Centro de Ciências e Tecnologia, Universidade Estadual do Ceará, 60710-903 Fortaleza, Ceará, Brazil
\textsuperscript{4} Departamento de Física, Universidade Federal do Ceará, 60451-970 Fortaleza, Ceará, Brazil

(Dated: February 3, 2008)

One of the most important scaling laws of time dependent fracture is Basquin’s law of fatigue, namely, that the lifetime of the system increases as a power law with decreasing external load amplitude, $t_f \sim \sigma_0^{-\alpha}$, where the exponent $\alpha$ has a strong material dependence. We show that in spite of the broad scatter of the Basquin exponent $\alpha$, the fatigue fracture of heterogeneous materials exhibits intriguing universal features. Based on stochastic fracture models we propose a generic scaling form for the macroscopic deformation and show that at the fatigue limit the system undergoes a continuous phase transition when changing the external load. On the microlevel, the fatigue fracture proceeds in bursts characterized by universal power law distributions. We demonstrate that in a range of systems, including deformation of asphalt, a realistic model of deformation, and a fiber bundle model, the system dependent details are contained in Basquin’s exponent for time to failure, and once this is taken into account, remaining features of failure are universal.

PACS numbers: 46.50.+a, 62.20.Mk, 61.82.Pv

Disordered media subject to sub-critical external loads present a time dependent macroscopic response and typically fail after a finite time \cite{1}. Such time dependent fracture evidently plays a crucial role in a large variety of physical, biological, and geological systems, such as the rupture of adhesion clusters of cells in biomaterials under external stimuli \cite{2}, the sub-critical crack growth due to thermal activation of crack nucleation \cite{3,4}, creep \cite{5} and fatigue fracture of materials \cite{6,7}, and the emergence of earthquake sequences \cite{8}. One of the most important scaling laws of time dependent fracture is the empirical Basquin law of fatigue which states that the lifetime $t_f$ of samples increases as a power law when the external load amplitude $\sigma_0$ decreases, $t_f \sim \sigma_0^{-\alpha}$ \cite{9}. The measured values of the Basquin exponent $\alpha$ typically vary over a broad range indicating a strong dependence on material properties \cite{9,10,11}.

In this Letter we study the fatigue fracture of heterogeneous materials focusing on the underlying microscopic mechanism of the fatigue process and its relation to the macroscopic time evolution. We develop two generic models of time dependent fracture, namely, a fiber bundle model and a discrete element approach, which both capture the most important ingredients of the fatigue failure of disordered materials. Analytic solutions and computer simulations reveal that the models recover the Basquin law of fatigue, whose exponent is determined by the damage process. We show that, as a consequence of healing, a finite fatigue limit emerges at which the system undergoes a continuous phase transition from a regime where macroscopic failure occurs at a finite time to another one exhibiting only partial failure in the system having an infinite lifetime. Based on analytic solutions, we propose a generic scaling form for the macroscopic deformation. On the microlevel the fatigue of the material is accompanied by an avalanche activity where bursts of local breakings are triggered by damage sequences. We demonstrate analytically that the microscopic bursting activity underlying fatigue fracture is characterized by universal power law distributions which implies that the non-universality of the Basquin exponent at the macro-level is solely due to the specific degradation process of the material.

First we consider a mean field model of fatigue fracture, namely, a fiber bundle model (FBM) where fibers fail either due to immediate breaking or to ageing \cite{14}. For the load redistribution after failure events, equal load sharing is assumed so that all the fibers carry the same load \cite{12}. During the evolution of the system, a fiber breaks instantaneously at time $t$ when the load on it $p(t)$ exceeds the local tensile strength $p_{th}^i$ ($i = 1,\ldots,N$). All intact fibers accumulate damage $c(t)$ up to time $t$ that they have experienced and break when $c(t)$ exceeds the local damage threshold $c_{th}^i$ ($i = 1,\ldots,N$). The accumulated damage $c(t)$ up to time $t$ is obtained by integrating over the entire loading history of the specimen $c(t) = a \int_0^t e^{-\frac{t-t'}{\tau}} p(t') \gamma dt'$, where $a > 0$ is a scale parameter, while the exponent $\gamma > 0$ controls the rate of damage accumulation \cite{10,11}. To capture damage recovery in the model due to healing of microcracks \cite{10} or thermally activated rebinding of failed contacts \cite{2,8}, we introduce a memory term in the above damage law of exponential form whose characteristic time scale $\tau$ defines the memory range of the system \cite{2,4,8}. Hence, during

*Electronic address:feri@dtp.atomki.hu
the time evolution of the bundle, the damage accumulated over the time interval \( t' < (t - \tau) \) heals. Assuming independence of the two breaking thresholds \( p_{th} \) and \( c_{th} \), the macroscopic evolution of the system under a constant external load \( \sigma_0 \) can be cast into the form

\[
\sigma_0 = [1 - F(c(t))] [1 - G(p(t))] p(t),
\]

where \( G \) and \( F \) denote the cumulative distributions of \( p_{th} \) and \( c_{th} \), respectively. We solved Eq. (1) analytically obtaining the load \( p(t) \) on the intact fibers at a constant external load \( \sigma_0 < \sigma_c \), with the initial condition \( p(t = 0) = p_0 \), where \( p_0 \) denotes the solution of the constitutive equation \( \sigma_0 = [1 - G(p_0)] p_0 \). Here \( \sigma_c \) denotes the ultimate strength of the material. The most important input parameters of the model calculations are \( a, \gamma \) and \( \tau \), which govern the damage accumulation.

On the macrolevel the process of fatigue is characterized by the evolution of deformation \( \varepsilon(t) \) of the specimen, which is related to \( p(t) \) as \( p(t) = E\varepsilon(t) \), where \( E = 1 \) is the Young modulus of fibers. Neglecting immediate breaking and healing, Eq. (1) can be transformed into a differential equation for the number \( N_b \) of broken fibers as \( dN_b/dt = af(c(t))p^\gamma N \), where \( f(c) = dF/dc \). Using \( p(t) = N\sigma_0/(N - N_b(t)) \), for uniformly distributed threshold values the exact solution of the equation of motion Eq. (1) reads

\[
\varepsilon(t) = \sigma_0 \left[ (t_f - t)/t_f \right]^{-1/(1 + \gamma)} \quad \text{and} \quad t_f = \frac{\sigma_0^{-\gamma}}{a(1 + \gamma)},
\]

where \( t_f \) denotes the lifetime of the system. Equation (2) shows that damage accumulation leads to a finite time singularity where the deformation \( \varepsilon(t) \) of the system has a power law divergence with an exponent determined by

\[
\gamma. \quad \text{It is important to emphasize that } t_f \text{ has a power law dependence on the external load } \sigma_0 \text{ in agreement with Basquin’s law and both models provide a very good fit of the lifetime data of asphalt. The fatigue limit } \sigma_0 \text{ is indicated by the vertical dashed line.}

\[
FIG. 1: \text{(Color online) Scaling plot of deformation-time curves measured experimentally on asphalt specimens} \ [12]. \text{The scaling function } S, \text{ obtained by rescaling the two axis, has a power law dependence on the time-to-failure.}
\]

\[
FIG. 2: \text{(Color online) Characteristic time scales } t_r \text{ and } t_f \text{ of the system. The complete FBM corresponding to Eq. (1) which includes immediate breaking and healing is solved numerically. For } \sigma_0 > \sigma_i \text{ we see Basquin’s law and both models provide a very good fit of the lifetime data of asphalt. The fatigue limit } \sigma_i \text{ is indicated by the vertical dashed line.}
\]

\[
\text{FIG. 1: Scaling plot of deformation-time curves measured experimentally on asphalt specimens [12]. The scaling function } S, \text{ obtained by rescaling the two axis, has a power law dependence on the time-to-failure.}
\]

\[
\text{FIG. 2: Characteristic time scales } t_r \text{ and } t_f \text{ of the system. The complete FBM corresponding to Eq. (1) which includes immediate breaking and healing is solved numerically. For } \sigma_0 > \sigma_i \text{ we see Basquin’s law and both models provide a very good fit of the lifetime data of asphalt. The fatigue limit } \sigma_i \text{ is indicated by the vertical dashed line.}
\]
take place during cyclic loading of rocks with a stress amplitude increasing from one cycle to the next [10]. It is important to note that approaching the fatigue limit \( \sigma_l \) from either side, the characteristic time scale diverges. Figure 3 shows that both the relaxation time \( t_r \) and the lifetime \( t_f \) follow a power law as a function of the difference from the fatigue limit with distinct exponents: \( t_r \sim (\sigma_l - \sigma_0)^{-1/3} \) and \( t_f \sim (\sigma_0 - \sigma_l)^{-2/3} \). We stress that the exponents neither depend on the disorder distributions (\( F \) and \( G \)) nor on the details of the damage law (\( a, \gamma \) and \( \tau \)), i.e., they are universal implying a continuous phase transition at the fatigue limit \( \sigma_l \) between partial failure and macroscopic fracture (see Fig. 3).

Our calculations revealed that the Basquin law of lifetime emerges on the macrolevel as a consequence of the competition between the two microscopic failure mechanisms of fibers. Rewriting Eq. 1 in the form of the constitutive equation of simple FBM as \( \sigma_0 / [1 - F(c(t))] = [1 - G(p(t))] p(t) \) it can be seen that the slow damage process on the left hand side quasi-statically increases the load on the system: ageing fibers accumulate damage and break slowly one-by-one in the increasing order of their damage thresholds \( c_{th}^i \). After a number \( \Delta_d \) of damage breakings, the emerging load increment on the remaining intact fibers can trigger a burst of immediate breakages. Since load redistribution and immediate breaking occur on a much shorter time scale than damage accumulation, the entire fatigue process can be viewed on the microlevel as a sequence of bursts of immediate breakages triggered by a series of damage events happening during waiting times \( T \), i.e., the time intervals between the bursts. The microscopic failure process is characterized by the size distribution of bursts \( P(\Delta) \), damage sequences \( P(\Delta_d) \), and by the distribution of waiting times \( P(T) \). At small loads \( \sigma_0 < \sigma_c \) most of the fibers break in long damage sequences, because the resulting load increments do not suffice to trigger bursts. Consequently, the burst size distribution \( P(\Delta) \) has a rapid exponential decay. Increasing \( \sigma_0 \) the total number of bursts \( n_b \) increases linearly \( n_b \sim \sigma_0 \) and a power law regime of burst sizes emerges \( P(\Delta) \sim \Delta^{-\xi} \) with the well-known mean field exponent of FBM \( \xi = 5/2 \) [13]. When macroscopic failure is approached \( \sigma_0 \sim \sigma_c \), the failure process accelerates such that the size \( \Delta_f \) and duration \( T \) of damage sequences decrease, while they trigger bursts of larger sizes \( \Delta \), and finally macroscopic failure occurs as a catastrophic burst of immediate failures. Since in the limiting case of \( \sigma_0 \sim \sigma_c \), there is a large number of weak fibers breaks in the initial burst, we found that the distribution \( P(\Delta) \) has a crossover to a smaller exponent \( \xi = 3/2 \), in agreement with Ref. [13]. After the linear increase, the number of bursts \( n_b \) has a maximum at \( \sigma_0/\sigma_c \approx 0.4 \) and rapidly decreases to 1 as \( \sigma_0 \) is approached. All these results are independent of \( \gamma \), \( a \), and \( \tau \).

Since damage events increase the load on the remaining intact fibers until an immediate breaking is triggered, the size of damage sequences \( \Delta_d \) is independent of the damage characteristics \( c(t) \) and \( F(c_{th}) \) of the material, instead, it is determined by the load bearing strength distribution \( G(p_{th}) \) of fibers. Under broad conditions this mechanism leads to an universal power law form with an exponential cutoff \( P(\Delta_d) \sim \Delta_d^{-1} \exp(-\Delta_d/\langle \Delta_d \rangle) \), where \( \langle \Delta_d \rangle \sim \sigma_0^{-1} \). The damage law \( c(t) \) of the material controls the time scale of the process of fatigue fracture through the temporal sequence of single damage events. In damage sequences fibers break in the increasing order of their damage thresholds \( c_{th} \) which determine the time intervals \( \Delta t \) between consecutive fiber breakings. Analytic calculations showed that \( P(\Delta t) \) has an explicit dependence on \( \gamma \) as \( P(\Delta t) \sim \Delta t^{-(1/\gamma)} \), however, the duration of sequences \( T = \sum_{j=1}^{\Delta_d} \Delta t_j \), i.e., the waiting times between bursts follow an universal power law distribution \( P(T) \sim T^{-1} \exp(-T/\langle T \rangle) \), where only the cutoff has \( \gamma \)-dependence \( \langle T \rangle \sim \sigma_0^{-1+\gamma} \) (see Fig. 4).

The macroscopic lifetime \( t_f \) of a finite system can be related to characteristic quantities of the microscopic failure process as \( t_f = \sum_{i=1}^{n_b} T_i \), from which the average lifetime can be obtained in the form \( t_f \approx \langle n_b \rangle \langle T \rangle \). In the load regime where the generic scaling laws of the distributions \( P(\Delta) \), \( P(\Delta_d) \), and \( P(T) \) prevail, this leads to the form \( t_f \sim \sigma_0^{-\gamma} \) in agreement with the Basquin law Eq. 2 of the system. The results demonstrate that the Basquin law of lifetime on the macro-scale is a fingerprint of the scale-free microscopic bursting activity, with the material dependence entering only through the damage law determining the waiting times between bursts. Experimentally, the microscopic fracture process underlying fatigue can be monitored by the acoustic emission tech-
technique and by direct optical observations [1, 2, 10, 11]. Sub-critical cracking has recently been found to produce a power law distribution of step sizes of the advancing crack in agreement with our predictions on the size distribution of bursts [2].

In order to study the effect of stress concentration and crack growth in fatigue fracture, we also developed a discrete element model (DEM) [13] in which we discretize a two-dimensional disc-shaped specimen in terms of randomly shaped convex polygons connected by elastic beams. The beams fail either due to immediate breaking or damage which are coupled in a single failure variable $q(t) = p(t) + \int_t^\tau e^{-(t-t')/\tau} p(t') \gamma dt'$. Here $p(t)$ describes the deformation state of the beam taking into account both stretching and bending $p(t) = (\varepsilon/\varepsilon_l)^2 + \max(|\Theta_1|, |\Theta_2|)/\Theta_{th}$, being $\varepsilon$ the longitudinal deformation, $\Theta_1$ and $\Theta_2$ the bending angles at the two ends of the beam, and $\varepsilon_l$ and $\Theta_{th}$ denote the threshold values a beam can sustain under stretching and bending, respectively. As a consequence, the parameters $a$, $\gamma$, and $\tau$ play the same role as their counterparts in our FBM. The time evolution of the system is followed by numerically solving the equations of motion of polygons. The breaking criterion $q(t) > 1$ is evaluated at each time step and beams which fulfill the condition are removed [13]. We study the fatigue fracture under diametric compression of discs with constant stress $\sigma_0$ (Brazil test). Figure 2 shows that DEM provides also an excellent fit of the lifetime data of asphalt specimens [14]. DEM simulations revealed that in the presence of stress concentrations bursts are spatially correlated and they can be identified as sudden advancements of slowly growing cracks. DEM results on burst characteristics also show power law behavior as the mean field FBM, but with different exponents due to the two-dimensionality of the model. The localized stress concentration built up around cracks gives rise to higher values of the exponents of the size distribution of bursts $P(\Delta) \sim \Delta^{-2.7}$, and of damage sequences $P(\Delta q) \sim \Delta_q^{-1.8}$, while for the waiting time distribution $P(T)$ the DEM exponent falls very close to the mean field value (see Fig. 4). The results proved to be independent of the value of $\gamma$.

Although the exponent of Basquin’s law depends on the microscopic damage accumulation, we found an astonishing spectrum of universal features hidden behind this originally empirical law. On one hand we discovered in the experimentally relevant situation of finite damage memory a continuous phase transition between partial failure and macroscopic rupture. On the microscopic level of individual breaking events we showed that the separation of time scales of the two competing failure mechanisms leads to a bursting activity, where we disclosed several universal scaling laws in the distributions and determined their exponents as well in mean-field as in two dimensions. In summary our approach provides a direct connection between the microscopic mechanisms constituting the main ingredients of the model (i.e., immediate breaking, damage accumulation and healing of microcracks) and the macroscopic behavior of the fatigue process. The (macroscopic) exponent from Basquin’s law coincides with the (microscopic) exponent of the degradation law, namely $\alpha = \gamma$. Following a slightly different pathway, our methodology is also capable to show explicitly the bridge between the (universal) mechanism related with the scale-free bursting activity at the microscale and the (non-universal) lifetime law of the material at the macro-scale.

This work opens up new experimental challenges. Our scaling relation of the macroscopic deformation should be verified on various types of materials, after which it could help to extract the relevant information from fatigue life measurements. For instance, it would be interesting to check our theoretical predictions with fatigue measurements performed at very low external loads, i.e., for $\sigma_0 \approx \sigma_l$. More precisely, in the infinite lifetime limit, $\sigma_0 \lesssim \sigma_l$, the experimental confirmation of the power-law variability with load of the relaxation time should certainly provide some considerable insight on the role of healing in the entire fatigue process. For similar reasons, it would be also interesting to verify the distinct lifetime behavior obtained from the model in the other limit of low external loads, $\sigma_0 \gtrsim \sigma_l$. Finally, another interesting outcome from our study is the statistical behavior related with the bursting activity during the fatigue process of a given material. According to our analysis, both the size $\Delta_q$ of damage sequences and magnitude $T$ of waiting times between bursts should obey universal power-law distributions that might reflect the intrinsic features of the typical restructuring events taking place at the microscopic level. As a possible monitoring technique, acoustic emission measurements could be conducted in conjunc-

![Figure 4](image_url)
tion with fatigue experiments to confirm our claim for universality behind Basquin’s law.

We thank the Brazilian agencies CNPq, CAPES, FUN-CAP and FINEP, and the Max Planck prize for financial support. F. Kun was supported by OTKA T049209.

[1] M. Alava, P. K. Nukala, and S. Zapperi, Adv. Phys. 55, 349 (2005).
[2] T. Erdmann and U. Schwarz, Phys. Rev. Lett. 92, 108102 (2004).
[3] S. Santucci et al., Phys. Rev. Lett. 93, 095505 (2004).
[4] D. Sornette and G. Ouillon, Phys. Rev. Lett. 94, 038501 (2005).
[5] H. Nechad et al., Phys. Rev. Lett. 94, 045501 (2005); R. C. Hidalgo, F. Kun, and H. J. Herrmann, Phys. Rev. E 65, 032502 (2002).
[6] D. Sornette and C. Vanneste, Phys. Rev. Lett. 68, 612 (1992); H. J. Herrmann, J. Kertész, and L. de Arcangelis, Europhys. Lett. 10, 147 (1989).
[7] D. Farkas, M. Willemann, and B. Hyde, Phys. Rev. Lett. 94, 165502 (2005); D. Sornette, T. Magnin, and Y. Brechet, Europhys. Lett. 20, 433 (1992).
[8] C. Marone, Nature 391, 69 (1998).
[9] O. H. Basquin, Proc. ASTM 10, 625 (1910).
[10] D. Krajcinovic, Damage Mechanics, (Elsevier, Amsterdam, 1996); S. Suresh, Fatigue of Materials, (Cambridge University Press, 2006).
[11] A. Rinaldi et al., Int. J. Fatigue 28, 1069 (2006); M. E. Biancolini et al., ibid, 1820.
[12] J. V. Andersen, D. Sornette, and K. Leung, Phys. Rev. Lett. 78, 2140 (1997); F. Kun, S. Zapperi, and H. J. Herrmann, Eur. Phys. J. B 17, 269 (2000); R. C. Hidalgo et al., Phys. Rev. Lett. 89, 205501 (2002).
[13] H. A. Carmona et al., Phys. Rev. E 75, 046115 (2007).
[14] F. Kun et al., J. Stat. Mech. P02003 (2007); M. J. Alava, J. Stat. Mech. N04001 (2007).
[15] M. Kloster, A. Hansen, and P. Hemmer, Phys. Rev. E 56, 2615 (1997); S. Pradhan, A. Hansen, and P. C. Hemmer, Phys. Rev. Lett. 95, 125501 (2005).
[16] A. Lavrov, Int. J. Rock Mech. Min. Sci. 40, 151 (2003).