SOME REMARKS ON HEATH-BROWN’S THEOREM ON QUADRATIC FORMS

ANDREY DYMOV, SERGEI KUKSIN, ALBERTO MAIOCCHI, AND SERGEI VLĂDUȚ

Abstract. In his paper from 1996 on quadratic forms Heath-Brown developed a version of circle method to count points in the intersection of an unbounded quadric with a lattice of short period, if each point is given a weight. The weight function is assumed to be $C^\infty_0$–smooth and to vanish near the singularity of the quadric. In out work we allow the weight function to be finitely smooth and not vanish near the singularity, and we give also an explicit dependence on the weight function.

1. Introduction

1.1. Setting and result. Let us consider a non-degenerate and non-sign-definite quadratic form on $\mathbb{R}^d$,

$$F(z) = \frac{1}{2} A z \cdot z,$$

where $A$ is a symmetric matrix. Then for $t \in \mathbb{R}$ the quadric

$$\Sigma_t = \{ z : F^t(z) = 0 \}, \quad F^t = F - t,$$

is an unbounded hyper-surface in $\mathbb{R}^d$. It is smooth if $t \neq 0$, while $\Sigma_0$ is a cone and has a locus at zero.

Let $\mathbb{Z}^d_L$ be the lattice of a small period $L^{-1}$,

$$\mathbb{Z}^d_L = L^{-1} \mathbb{Z}^d, \quad L > 1,$$

and let $w$ be a regular real function on $\mathbb{R}^d$ which means that $w$ and its Fourier transform $\hat{w}(\xi)$ are continuous functions which decay at infinity sufficiently fast:

$$|w(z)| \leq C |z|^{-d-\gamma}, \quad |\hat{w}(\xi)| \leq C |\xi|^{-d-\gamma},$$

for some $\gamma > 0$ and some $C > 0$. Our goal is to study the behaviour of series

$$N_L(w; F, m) = \sum_{z \in \Sigma_m \cap \mathbb{Z}^d_L} w(z),$$

where $m \in \mathbb{R}$ is such that $mL^2$ is an integer.\footnote{E.g., $m = 0$ – this case is the most important for us.}

Obviously,

$$N_L(w; F, m) = N_1(w_L; F, L^2 m) =: N(w_L; F, L^2 m), \quad w_L(z) := w(z/L).$$
To study \( N_L(w; F, L^2 m) \) we closely follow the circle method in the form, given to it by Heath-Brown in \[7\]. We start with a key theorem which expresses the analogue of Dirac’s delta function on integers, i.e. the function \( \delta : \mathbb{Z} \to \mathbb{R} \) such that

\[
\delta(n) := \begin{cases} 
1 & \text{for } n = 0 \\
0 & \text{for } n \neq 0
\end{cases},
\]

through a sort of Fourier representation. This result goes back at least to Duke, Friedlander and Iwaniec \[3\] (cf. also \[8\]), and we state it in the form, given in \[7, Theorem 1\]; basically, it replaces (a major arc decomposition of) the trivial identity

\[
\delta(n) = \int_0^1 e^{2\pi i \alpha n} d\alpha
\]

employed in the usual circle method. In the theorem for \( q \in \mathbb{Z}^* \) we denote by \( e_q \) the exponential function \( e_q(x) := e^{2\pi i x/q} \), and denote by \( \sum^*_a(\text{mod } q) \) the summation over residues \( a \) with \( (a, q) = 1 \), i.e., over all integers \( a \in [1, q-1] \), relatively prime with \( q \).

**Theorem 1.1.** For any \( Q > 1 \), there exists \( c_Q > 0 \) and a smooth function \( h(x, y) : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{R}_{\geq 0} \), such that

\[
\delta(n) = c_Q Q^{-2} \sum_{q=1}^{\infty} \sum_{a(\text{mod } q)}^* e_q(\alpha n) h \left( \frac{q}{Q}, \frac{n}{Q^2} \right).
\]

The constant \( c_Q \) satisfies \( c_Q = 1 + O(N Q^{-N}) \) for any \( N > 0 \), while \( h \) is such that \( h(x, y) \leq c/x \) and \( h(x, y) = 0 \) for \( x > \max(1, 2|y|) \) (so for each \( n \) the sum in \( 5 \) contains finitely many non zero terms).

Since \( N(w; F, t) \) may be written as \( \sum_{\mathbf{z} \in \mathbb{Z}^d} \tilde{w}(\mathbf{z}) \delta(F^t(\mathbf{z})) \), then Theorem 1.1 allows to represent series \( N(\tilde{w}; F, t) \) as an iterated sum. Transforming that sum further using the Poisson summation formula as in \[7, Theorem 2\] we arrive at the following result:

**Theorem 1.2** (Theorem 2 of \[7\]). For any regular function \( \tilde{w} \), any \( t \) and any \( Q > 1 \) we have the expression

\[
N(\tilde{w}; F, t) = c_Q Q^{-2} \sum_{c \in \mathbb{Z}^d} \sum_{q=1}^{\infty} q^{-d} S_q(c) I^0_q(c),
\]

with

\[
S_q(c) := \sum_{a(\text{mod } q)}^* \sum_{b(\text{mod } q)} e_q(aF^t(b) + c \cdot b)
\]

and

\[
I^0_q(c) := \int_{\mathbb{R}^d} \tilde{w}(\mathbf{z}) h \left( \frac{q}{Q}, \frac{F^t(\mathbf{z})}{Q^2} \right) e_q(-\mathbf{z} \cdot \mathbf{c}) d\mathbf{z}.
\]

\[2\] In \[7\] the result below is stated for \( w \in C_0^\infty \). But the argument there, based on the Poisson summation, applies as well to regular functions \( w \).
We will apply Theorem 1.2 to examine \( N(w_L; F, L^2m) = N_L(w; F, m) \) with large \( L \), choosing \( Q = L > 1 \) and estimating explicitly the leading terms in \( L \) of \( S_q(c) \) and \( I^0_q(c) \) as well as the remainders. The answer will be given in terms of the integral

\[
\sigma_{\infty}(w; F, m) = \int_{\Sigma_m} w(z) \mu_{\Sigma_m}^w(dz)
\]

(which is singular if \( m = 0 \)). Here \( \mu_{\Sigma_m}^w(dz) = |Az|^{-1}dz|_{\Sigma_t} \), with \( dz|_{\Sigma_t} \) representing the volume element over \( \Sigma_t \), induced from the standard euclidean structure on \( \mathbb{R}^d \), and \( A \) the symmetric matrix in (1). Another quantity, entering the asymptotic for \( N_L(w; F, m) \), is the infinite product

\[
\sigma(F, m) = \prod_p \sigma_p(F, m),
\]

where \( p \) ranges over all primes, and \( \sigma_p \) is defined by

\[
\sigma_p(F, m) := \sum_{l=0}^{\infty} p^{-dl} S_p^l(0),
\]

where \( S_1 \equiv 1 \) and \( S_q \) is given by (1) with \( t = L^2m \).

Motivated by the continuation [5] of the research [4] we are the most interested in the special case of the quadratic forms \( F \) when \( d \) is an even number, so \( \mathbb{R}^d = \{ z = (x, y), \ x, y \in \mathbb{R}^{d/2} \} \), and

\[
F_0(z) = x \cdot y, \quad A_0(x, y) = (y, x).
\]

Our main result, stated below, specifies Theorem 5 from [7] for \( F = F_0 \) in three respects: firstly, now the function \( w \) has finite smoothness and sufficiently fast decays at infinity, while in [7] \( w \in C^\infty \), secondly, we specify how the remainder depends on \( w \), and thirdly and the most importantly, we remove the imposed in [7] restriction that the support of \( w \) does not contain the origin (this improvement is crucial for us since in [5] the theorem is used in the situation when \( w(0) \neq 0 \)).

We note that a similar specification of the Heath-Brown method was obtained in [1, Section 5], also for the purposes of wave turbulence.

In the theorem and everywhere below for a function \( f \in C^k(\mathbb{R}^N) \) we denote \( \| f \|_{n_1,n_2} = \sup_{x \in \mathbb{R}^N} \max_{|\alpha| \leq n_1} |\partial^\alpha f(x)||z|^{n_2} \), where \( n_1, n_2 \in \mathbb{N} \cup \{0\} \), \( n_1 \leq k \), and \( n_2 \in \mathbb{R}_{\geq 0} \). Here

\[
\langle x \rangle := \max\{1, |x|\} \quad \text{for} \quad x \in \mathbb{R}^l,
\]

for any \( l \geq 1 \). Note that if \( \| w \|_{d+1,d+1} < \infty \), then the function \( w \) is regular. Indeed, the fist relation in (3) is obvious. To prove the second note that for any integer vector \( \alpha \in (\mathbb{N} \cup \{0\})^d \), \( \xi^\alpha \hat{w}(\xi) = \left( \frac{i}{2\pi} \right)^{|\alpha|} \partial^\alpha x w(\xi) \). But if \( |\alpha| = \sum_{j=1}^{d} \alpha_j \leq d + 1 \), then \( |\partial^\alpha x w| \leq C(\langle x \rangle)^{-d-1} \), so \( \partial^\alpha x w \) is an \( L_1 \)-function.

Thus its Fourier transform \( \partial^\alpha x \hat{w} \) is a bounded continuous function for each \( |\alpha| \leq d + 1 \) and the second relation in (3) also holds.
Theorem 1.3. For any $0 < \varepsilon \leq 1$ and any even dimension $d > 4$, there exist constants $N_1(d, \varepsilon)$ and $N_2(d, \varepsilon) \leq N_3(d, \varepsilon)$, such that if $w \in C^{N_1}$ and a real number $m$ satisfies $mL^2 \in \mathbb{Z}$, then
\begin{equation}
|N_L(w; F_0, m) - \sigma_\infty(F_0, w, m)L^{d-2}| \leq CL^{d/2 + \varepsilon}(\|w\|_{N_1, N_2} + \|w\|_{0, N_3}),
\end{equation}
where the constant $C$ depends on $d, \varepsilon, m$. In particular if $\varepsilon = 1/2$, then one can take $N_1 = 2d^2 - 2d$, $N_2 = 7(d+1)$ and $N_3 = N_1 + 3d + 4$.

Remarks. 1) Here and everywhere else the dependence on $m$ is uniform on every compact interval.
2) The values of the constants $N_j(d, \varepsilon)$ in (12), obtained below, is far from optimal since it was not our goal to optimise them.
3) In the theorem and in similar situations below the result does not apply if the r.h.s. in (12) is infinite.
4) Since the theorem’s proof is based on the representation (6), then the function $w$ should be regular (see (8)). But this holds true if $\|w\|_{d+1, d+1} < \infty$, and so is valid if the r.h.s. of (12) is finite, with $N_1, N_2$ sufficiently big. E.g. if $N_1, N_2$ are as big as in the last line of the theorem’s assertion.
5) The quantity $\sigma(F_0, m)$ can be easily evaluated. Let us give this evaluation for $m = 0$. Recall then that $\sigma(F_0, 0) = \sigma(F_0) = \Pi_{\text{prime } p} \sigma_p(F_0)$ for
\[ \sigma_p(d) = \sigma_p(F_0) = \lim_{k \to \infty} \frac{\#\{F_0(x, y) \equiv 0 \mod (p^k)\}}{p^{(2d-1)k}}. \]
However, since for any prime $p$ the hypersurface $\{F_0 \mod p = 0\}$ over $\mathbb{F}_p$ is the affine cone over a smooth projective quadric we see that
\[ \sigma_p(d) = \#\{F_0(x, y) \equiv 0 \mod (p)\}d!^p = N_p(d)p^{1-2d} \]
where $N_p$ is the number of $\mathbb{F}_p$-points on $\{F_0 \equiv 0 \mod p\}$. Thus, $N_p(1) = 2p - 1$ and
\[ N_p(d+1) = \#\{\text{solutions with } x_{d+1} = 0\} + \#\{\text{solutions with } x_{d+1} \neq 0\} \]
\[ = pN_p(d) + (p-1)p^{2d}. \]
Therefore,
\[ \forall d \geq 2, \; N_p(d) = p^{2d-1} + p^d - p^{d-1}, \; \sigma_p(d) = 1 + p^{1-d} - p^{-d}. \]
For $\sigma_d$ we get then that
\[ \sigma_3 = \prod_p (1 + p^{-2} - p^{-3}) = 1.305.., \quad \sigma_4 = \prod_p (1 + p^{-3} - p^{-4}) = 1.100.., \]
whereas
\[ 1 < \sigma_d = \prod_p (1 + p^{1-d} - p^{-d}) < 1 + 2^{2-d} \]
tends to 1 when $d$ grows.

The proof of Theorem 1.3 occupies the rest of the paper and closely follows that of [7, Theorem 5] with additional control how the constants depend on
The only significant difference comes in Sections 3 and 4 where we do not assume that the function $w$ vanishes near the origin, the last assumption being crucial in the analysis of integrals in Sections 6 and 7 of [7]. To cope with this difficulty we have to examine the smoothness at zero of the function

$$m \mapsto \sigma_\infty(F_0, w, m)$$

and its decay at infinity. The corresponding analysis is performed in Appendix, where using the techniques, developed in [4] to study integrals (8), we prove that function (13) is $(d/2 - 2)$-smooth, but for a generic $w(z)$ its derivative of order $(d/2 - 1)$ has a logarithmic singularity at zero. There we also get a (non optimal) estimate for the rate of growth of (13) at infinity.

The approach we use to prove Theorem 1.3 is general and applies to other quadratic forms (1). Moreover, most of the auxiliary results, obtained on the way to prove the theorem, are established for the general $F$. In an extended version [6] of this paper which is now under preparation, the theorem will be proven in its full generality. Namely, there we obtain

**Amplification 1.4.** The assertion of Theorem 1.3 with modified constants $N_j$ and $C$ remain true for any non degenerate and non sign-definite quadratic form (1). Then the constants $N_j$ and $C$ also depend on the minimal and the maximal eigenvalues of the operator $A$.

**1.2. Scheme of the proof of Theorem 1.3.** It is easy to see that if the r.h.s. in (12) is finite, then the function $w$ is regular in the sense of Section 1.1, so Theorem 1.2 applies. Then, according to (6) and (4),

$$N_L(w; F, m) = c_L L^{-2} \sum_{c \in \mathbb{Z}^d} \sum_{q=1}^{\infty} q^{-d} S_q(c) I_q(c),$$

where $S_q$ is given by (7) and

$$I_q(c) := \int_{\mathbb{R}^d} w \left( \frac{Z}{L} \right) h \left( \frac{q}{L}, \frac{FmL^2(z)}{L^2} \right) e_q(-z \cdot c) dz.$$ 

Consider

$$n(c) = n(c; L) = \sum_{q=1}^{\infty} q^{-d} S_q(c) I_q(c),$$

so that $N_L(w; F, m) = c_L L^{-2} \sum_{c \in \mathbb{Z}^d} n(c)$. For an $\gamma_1 \in (0, 1)$ we write

$$N_L(w; F, m) = c_L L^{-2} (J_0 + J_{< \gamma_1} + J_{> \gamma_1}),$$

where

$$J_0 := n(0), \quad J_{< \gamma_1} := \sum_{c \neq 0, |c| \leq L^{\gamma_1}} n(c), \quad J_{> \gamma_1} := \sum_{|c| > L^{\gamma_1}} n(c).$$
Proposition 5.1 (which is a modification of Lemmas 19 and 25 from [7]) implies that

\[ |J_{\gamma_1}^2| \lesssim_{d, \gamma_1, m} \|w\|_{N_0, 2N_0 + d + 1} \]

with \( N_0 := [(d + 1)(1 + 1/\gamma_1)] \) (see Corollary 5.2). In Proposition 6.1, following Lemmas 22 and 28 from [7], we show that

\[ |J_{\gamma_1}^2| \lesssim_{d, \gamma_1, m} L^{d/2 + 2 + \gamma_1(d + 1)} \left( \|w\|_{\bar{N}_0, d + 5} + \|w\|_{0, \bar{N}_0 + 3d + 4} \right), \quad \bar{N} = \lceil d^2/\gamma_1 \rceil - 2d. \]

To analyse \( J_0 \) we write it as \( J_0 = J_0^+ + J_0^- \) with

\[ J_0^+ := \sum_{q > L^{1-\gamma_2}} q^{-d} S_q(0) I_q(0), \quad J_0^- := \sum_{q \leq L^{1-\gamma_2}} q^{-d} S_q(0) I_q(0), \]

for some \( 0 < \gamma_2 < 1 \). Lemma 4.2, which is a combination of Lemmas 16 and 25 from [7], modified using the results from Appendix, implies that

\[ |J_0^+| \lesssim_{d, \gamma_2} L^{d/2 + 2 + \gamma_2(d - 2 - 1)} \|w\|_{L_1} \leq L^{d/2 + 2 + \gamma_2(d - 2 - 1)} \|w\|_{0, d + 1}. \]

Finally Lemma 4.3, which is a combination of Lemma 13 and simplified Lemma 31 from [7] with the results from Appendix, establishes that

\[ J_0^- = L^d \sigma_{\infty}(F, w) \sigma(F, m) + O_{d, \gamma_2, m} \left( \|w\|_{d/2 - 2, d - 1} + \|w\|_{0, d + 1} \right) L^{d/2 + 2 + \gamma_2} \]

(see 8 and 9). Identity (16) together with the estimates above implies the desired result if we choose \( \gamma_2 = \varepsilon/(d/2 - 1) \) and \( \gamma_1 = \varepsilon/(d + 1) \).

**Notation.** We write \( A \lesssim_{a,b} B \) if \( A < CB \), where the constant \( C \) depends on \( a \) and \( b \). Similar, \( O_{a,b} \|w\|_{m_1, m_2} \) stands for a quantity, bounded in norm by \( C(a, b) \|w\|_{m_1, m_2} \). We denote \( e_1(x) = e^{2\pi ix/q} \) and abbreviate \( e_1(x) =: e(x) \).

2. COMPONENTS OF SINGULAR SERIES

In the present section we analyse the sums \( S_q(c) \) entering in the definition of the singular series \( \sigma(F, m), \sigma_p(F, m) \).

**Lemma 2.1.** (25 in [7].) We have \( |S_q(c)| \lesssim_A q^{d/2 + 1}, \) uniformly in \( c \in \mathbb{Z}^d \)

**Proof.** According to (7),

\[
|S_q(c)|^2 \leq \phi(q) \sum_{a \pmod{q}} \sum_{b \pmod{q}} e_q(aF^mL^2(b + c \cdot b))^2 \\
= \phi(q) \sum_{a \pmod{q}} \sum_{u, v \pmod{q}} e_q(a(F^mL^2(u) - F^mL^2(v)) + c \cdot (u - v)),
\]

where \( \phi(q) \) is the Euler totient function. Since \( F^t(z) = \frac{1}{2}Az \cdot z - t \), then

\[ F^mL^2(u) - F^mL^2(v) = (A\mathbf{v}) \cdot \mathbf{w} + F(\mathbf{w}) = \mathbf{v} \cdot Aw + F(\mathbf{w}). \]

So

\[ e_q(F^mL^2(u) - F^mL^2(v)) + c \cdot (u - v)) = e_q(aF(\mathbf{w}) + c \cdot \mathbf{w}) e_q(a \mathbf{v} \cdot Aw). \]
Now we see that the summation over $v$ in (19) produces a zero contribution, unless each component of the vector $Aw$ is divisible by $q$. This property holds for $N_\Delta$ possible values of $w$, where $\Delta = \det A$. Thus,

$$|S_q(c)|^2 \lesssim_A \phi(q) \sum_{a(\mod q)}^* \sum_{v(\mod q)} 1 \leq \phi^2(q) q^d.$$ 

\[ \square \]

We have the following trivial corollary of Lemma 2.1:

**Corollary 2.2.** We have

$$\left| \sum_{q \leq X} q^{-d} S_q(0) \right| \lesssim \frac{A}{X^{d/2+2}}$$

uniformly in $c \in \mathbb{Z}^d$.

Recalling the definition (10) for a prime $p$ we also have

**Lemma 2.3.** For any $d > 4$ we have

$$\sum_{q \leq X} q^{-d} S_q(0) = \prod_p \sigma_p + O_A(X^{-d/2+2}),$$

where the product is taken over all primes.

**Proof.** Let us write

$$\sum_{q \leq X} q^{-d} S_q(0) = \sum_{q=1}^\infty q^{-d} S_q(0) - \sum_{q > X} q^{-d} S_q(0).$$

By definition

$$S_{qq'}(c) = \sum_{a(\mod qq') \leq c}^* \sum_{v(\mod qq')} e_{qq'}(aF(v) + c \cdot v).$$

When $(q, q') = 1$ we can replace the summation on $a$ by a double summation on $aq$ modulo $q$ and $a_{q'}$ modulo $q'$, writing $a = qa_{q'} + q'a_q$, and the summation on $v$ with the double summation on $v_q$ modulo $q$ and $v_{q'}$ modulo $q'$, by writing $v = qq'v_q + q'q'v_q$, where $\bar{q}$ and $\bar{q}'$ are defined through $q\bar{q} = 1 \pmod{q'}$ and $q'\bar{q}' = 1 \pmod{q}$. We substitute in the previous formula and get

$$S_{qq'}(c) = S_q(q'c)S_{q'}(\bar{q}c),$$

whenever $(q, q') = 1$ (cf. Lemma 23 from [7]).

This implies the identity

$$\sum_{q=1}^\infty q^{-d} S_q(0) = \prod_p \sigma_p.$$ 

On the other hand, due to Lemma 2.1

$$\left| \sum_{q \geq X} q^{-d} S_q(0) \right| \lesssim \sum_{q \geq X} q^{-d/2+1} \lesssim_A X^{-d/2+2}.$$ 

\[ \square \]
Singular integral

3.1. Properties of \( h(x, y) \). We construct a function \( h(x, y) \) entering Theorem 1.1, starting from the weight function \( w_0 \in C^\infty_0(\mathbb{R}) \), defined as

\[
(20) \quad w_0(x) = \begin{cases} 
\exp \left( \frac{1}{x^2 - 1} \right) & \text{for } |x| < 1 \\
0 & \text{for } |x| \geq 1
\end{cases}.
\]

We denote by

\[
c_0 := \int_{-\infty}^{\infty} w_0(x) \, dx
\]

and introduce the shifted weight function

\[
\omega(x) = 4c_0 w_0(4x - 3).
\]

which belongs, of course, to \( C^\infty_0(\mathbb{R}) \). Obviously, \( 0 \leq \omega \leq 4e^{-1}/c_0 \), \( \omega \) is supported on \((1/2, 1)\), and

\[
\int_{-\infty}^{\infty} \omega(x) \, dx = 1.
\]

The required function \( h \) is defined in terms of \( \omega \) as

\[
h(x, y) := h_1(x) - h_2(x, y),
\]

with

\[
(21) \quad h_1(x) := \sum_j \frac{1}{x^j} \omega(x^j), \quad h_2(x, y) := \sum_j \frac{1}{x^j} \omega \left( \frac{|y|}{x^j} \right).
\]

For any fixed pair \((x, y)\), each sum on \( j \) involved in the definition contains a finite number of nonzero terms, ranging from \( \frac{1}{x^j} \) to \( 1/x \) for the summation in \( h_1 \), and from \( |y|/x \) to \( 2|y|/x \) for \( h_2 \). So \( h \) is a smooth function.

In [7], Section 3, it is shown how to derive Theorem 1.1 from the definition (21). Here we limit ourselves to providing some relevant properties of \( h \), proved in Section 4 of [7]. In particular these properties imply that for small \( x \), \( h(x, y) \) behaves as the Dirac delta function in \( y \).

Lemma 3.1 (Lemma 4 in [7]). We have:

1. \( h(x, y) = 0 \) for \( x \geq 1 \) and \( |y| \leq x/2 \).
2. If \( x \leq 1 \) and \( |y| \leq x/2 \), then \( h(x, y) = h_1(x) \) and for any \( m \geq 0 \)

\[
\frac{\partial^m h(x, y)}{\partial x^m} \lesssim_m \frac{1}{x^{m+1}}.
\]

3. If \( |y| \geq x/2 \), then for any \( m, n \geq 0 \)

\[
\frac{\partial^{m+n} h(x, y)}{\partial x^m \partial y^n} \lesssim_{m,n} \frac{1}{x^{m+1}|y|^n}.
\]

Lemma 3.1 immediately implies

**Corollary 3.2.** For any \( x, y \in \mathbb{R}_+ \times \mathbb{R} \) we have \( |h(x, y)| \lesssim 1/x \).

\[\text{3 Actually, it is proven how the function } h \text{ defined through } (21) \text{ can provide a representation of } \delta(n) \text{ for any weight function } \omega \in C^\infty_0(\mathbb{R}) \text{ supported on } [1/2, 1].\]
Lemma 3.3 (Lemma 5 in [7]). Let $m, n, N \geq 0$. Then
\[ \frac{\partial^{m+n}h(x, y)}{\partial x^m \partial y^n} \lesssim_{N, m, n} \frac{1}{x^{1+m+n}} \left( \delta(n)x^N + \min \left\{ 1, (|y|/n)^N \right\} \right). \]

Lemma 3.4 (Lemma 6 in [7]). Fix $X \in \mathbb{R}_{>0}$ and let $x < C \min \{1, X\}$ for $C > 0$. Then for any $N \geq 0$,
\[ \int_{-X}^{X} h(x, y) \, dy = 1 + O_{N, C} \left( Xx^{N-1} + O_{N, C} \left( \frac{x^N}{X^N} \right) \right). \]

Lemma 3.5 (Lemma 8 in [7]). Fix $X \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$. Let $x < C \min \{1, X\}$ for $C > 0$. Then
\[ \int_{-X}^{X} y^n h(x, y) \, dy \lesssim_{N, C} X^n \left( Xx^{N-1} + \frac{x^N}{X^N} \right). \]

The previous results are used to prove the key Lemma 9 of [7], which can be extended to the following

Lemma 3.6 (Lemma 9 of [7]). Let an integrable function $f$ be $C^{M-1}$-smooth, $M \geq 1$, and be such that $f^{(M-1)}$ is absolutely continuous in $[-1, 1]$. Let $x \leq C$ with $C > 0$, then, for any $\gamma_1 > 0$,
\[ \int_{\mathbb{R}} f(y)h(x, y) \, dy = f(0) + O_{M, C, \gamma_1} \left( x^{M-\gamma_1} \left( \frac{1}{X} \int_{-X}^{X} |f^{(M)}(y)| \, dy + \|f\|_{L^1} \right) \right), \]
where $X := \min \{1, x^{1-\gamma_1/(M+1)}\}$.

Proof. By Lemma 3.3 $h(x, y) \lesssim_M x^M$ if $|y| \geq X$, so that the integral on the tails can be bounded by
\[ \int_{|y| \geq X} f(y)h(x, y) \, dy \lesssim_M x^M \int_{\mathbb{R}} |f(y)| \lesssim_M x^M \|f\|_{L^1}. \]

For the integral for $|y| < X$, instead, we take the Taylor expansion of $f(y)$ around zero and get
\[ \int_{-X}^{X} f(y)h(x, y) \, dy = \sum_{j=0}^{M-1} \frac{f^{(j)}(0)}{j!} \int_{-X}^{X} y^j h(x, y) \, dy + O_M \left( \frac{X^M}{x} \int_{-X}^{X} |f^{(M)}(y)| \, dy \right), \]
by Corollary 3.2. Then, we fix $N$ and use Lemma 3.4 to get
\[ f(0) \int_{-X}^{X} h(x, y) \, dy = f(0) + O_{N, C} \left( \|f\|_{0, 0} \left( Xx^{N-1} + \frac{x^N}{X^N} \right) \right), \]
while, from Lemma 3.5, for any $j > 0$ we have
\[ \left| \frac{f^{(j)}(0)}{j!} \int_{-X}^{X} y^j h(x, y) \, dy \right| \lesssim_{N, j, C} \|f\|_{j, 0} X^j \left( Xx^{N-1} + \frac{x^N}{X^N} \right). \]
Putting together (22)–(25), putting $N = M(M + 1)/\gamma_1$ for $x \leq 1$, $N = M$ for $x > 1$, and using the definition of $X$ we obtain the proof.

3.2. The approximation for $I_q(0)$. We have the following proposition, which replaces Lemmas 11, 13 and Theorem 3 of [7], not assuming that $0 \notin \text{supp } w$.

Proposition 3.7. Let $F$ be the quadric $F_0$, see (11), (so $d$ is an even number). Let $q \leq CL$, with $C > 0$. Then for any $1 \leq M < d/2 − 1$

$$I_q(0) = L^d\sigma_\infty(F, w, m) + O_{M, C, M; d, m} \left( q^M L^{d-M+\gamma_1} \|w\|_{M,d-1} + \|w\|_{0,d+1} \right).$$

If $1 \leq M = d/2 − 1$, $m \neq 0$ and $C \leq m/2$, instead, we have

$$I_q(0) = L^d\sigma_\infty(F, w, m) + |\log m|O_{C, M; d, m} \left( q^M L^{d-M+\gamma_1} \|w\|_{M,d-1} + \|w\|_{0,d+1} \right).$$

Finally, if $1 \leq M = d/2 − 1$ and $m = 0$, we have

$$I_q(0) = L^d\sigma_\infty(F, w, m) + O_{C, M; d, m} \left( q^M L^{d-M+\gamma_1} \|w\|_{M,d-1} + \|w\|_{0,d+1} \right).$$

Proof. Let us write $I_q(c)$ as

$$I_q(c) = L^d\tilde{I}_q(c),$$

where

$$\tilde{I}_q(c) = \int_{\mathbb{R}^d} w(z) h \left( \frac{q}{L}, F^m(z) \right) e_q(-z \cdot cL) \, dz.$$ Applying the co-area formula (see e.g. [2], p.138) to the integral in (28) with $c = 0$ we get that

$$\tilde{I}_q(0) = \int_{\mathbb{R}} I(m+t)h(q/L, t) \, dt,$$

where

$$I(t) = \int_{\Sigma_t} w(z) \mu^{\Sigma_1}(dz)$$

(the measure $\mu^{\Sigma_1}$ is the same as in (8)). Since $q \leq CL$, then on account of Lemma 3.8

$$\int_{\mathbb{R}} I(m+t)h(x, t) \, dt = I(m) + O_{M, C} \left( x^{M-\gamma_1} \left( \frac{1}{X} \int_{-X}^X |I^{(M)}(m+t)| \, dt + \|I\|_{0,2} \right) \right),$$

where $X := \min\{1, x^{1-\gamma_1/(M+1)}\}$. In order to conclude the proof, we make use of Proposition A.3 of Appendix A which guarantees that

$$\int_{-X}^X |I^{(M)}(m+t)| \, dt \lesssim_{d, m} \begin{cases} X\|w\|_{M,d-1} & M < d/2 - 1, \\ X|\log m|\|w\|_{M,d-1} & M = d/2 - 1, m \neq 0, C \leq m/2, \\ X|\log X|\|w\|_{M,d-1} & M = d/2 - 1, \end{cases}$$

and that $\|I\|_{L_1} \lesssim_d \|w\|_{0,d+1}$.
4. The $J_0$ term

In this section we prove the following proposition concerning the term $J_0$ from (16) when $F = F_0$, not assuming that $0 \notin \text{supp} \ w$:

**Proposition 4.1.** Let $F$ be the quadric $F_0$. Assume that $w \in C^{d/2-2}(\mathbb{R}^d)$. Then, for any $0 < \gamma_2 < 1$,

$$|J_0 - L^d \sigma_\infty(w; F, m) \prod_p \sigma_p| \lesssim_{\gamma_2, d, m} L^{d/2+\gamma_2} \left( \|w\|_{d/2-2,d-1} + \|w\|_{0,d+1} \right).$$

**Proof.** To establish Proposition 4.1 we write $J_0$ in the form (18). Then the assertion follows from Lemmas 4.2 and 4.3 below, estimating $J_0^+$ and $J_0^-$ separately, and noting then that $\|w\|_{L_1} \leq \|w\|_{0,d+1}$ for $d \geq 3$. \hfill $\Box$

**Lemma 4.2.** Assume that $w \in L_1(\mathbb{R}^d)$ and $d > 2$. Then we have the bound

$$|J_0^+| \lesssim A \sum_{q \leq L^{1-\gamma_2}} q^{-d/2} I_q(0).$$

**Proof.** Since according to Lemma 2.1 $|S_q(0)| \lesssim A q^{d/2+1}$, then

$$|J_0^+| \lesssim A L^{d+1} \|w\|_{L_1} \sum_{q \leq L^{1-\gamma_2}} q^{-d/2} \lesssim A L^{d+1} \|w\|_{L_1} L^{-d/2+1}(1-\gamma_2) = L^{d/2+\gamma_2(2/2-1)} \|w\|_{L_1}.$$ 

**Lemma 4.3.** Let $F$ be the quadric $F_0$. Assume that $w \in C^{d/2-2}(\mathbb{R}^d)$. Then

$$J_0^- = L^d \sigma_\infty(w; F, m) \prod_p \sigma_p + O_{\gamma_2, d, m} \left( \|w\|_{d/2-2,d-1} + \|w\|_{0,d+1} \right) L^{d/2+\gamma_2}.$$ 

**Proof.** Inserting (26) into the definition of the term $J_0^-$, we get $J_0^- = I_A + I_B$, where

$$I_A := L^d \sigma_\infty(w; F, m) \sum_{q \leq L^{1-\gamma_2}} q^{-d} S_q(0),$$

$$|I_B| \lesssim_{M, \delta, d, m} L^{d-M+\delta} \left( \|w\|_{M,d-1} + \|w\|_{0,d+1} \right) \sum_{q \leq L^{1-\gamma_2}} S_q(0) q^{-d+M},$$
for $M \leq d/2 - 2$ and any $\delta > 0$. Lemma 2.3 implies that
\[
\sum_{q \leq L^{1-\gamma_2}} q^{-d} S_q(0) = \prod_p \sigma_p + O(L^{(1-d/2+2)(1-\gamma_2)}),
\]
so
\[
I_A = L^d \sigma_\infty(w; F, m) \prod_p \sigma_p + O(\sigma_\infty(w; F, m)L^{d/2+2+\gamma_2(d/2-2)}),
\]
whereas $|\sigma_\infty(w; F, m)| = |I(m)| \leq \|w\|_{0,d-1}$ on account of Proposition A.4.

As for the term $I_B$, Lemma 2.1 implies that
\[
|I_B| \lesssim M, \delta, d, m \frac{L}{q} \left(\|w\|_{d/2-2,d-1} + \|w\|_{0,d+1} \right) \sum_{q \leq L^{1-\gamma_2}} q^{-d/2+1+M}. \]
Choosing $M = d/2 - 2$ and $\delta = \gamma_2/2$, we get
\[
|I_B| \lesssim_{\delta, d, m} \left(\|w\|_{d/2-2,d-1} + \|w\|_{0,d+1} \right) L^{d/2+2+\gamma_2}.
\]

5. The $J_{\gamma_1}^0$ Term

We provide here an estimate of the term $J_{\gamma_1}^0$ defined in (17). The key point of the proof is an adaptation of Lemma 19 of [7] to our case; we recall the notation (27).

**Proposition 5.1.** For any $N > 0$ and $w \in C^N(\mathbb{R}^d)$

\begin{equation}
|\tilde{I}_q(c)| \lesssim_{d,N,A,m} \frac{L}{q} \|c\|^{-N} \|w\|_{N,2N+d+1}
\end{equation}

**Proof.** We call $f_q(z) := w(z) h \left(\frac{q}{L}, F^m(z)\right)$. Since
\[
\frac{i q}{2\pi L} |c|^{-2} (c \cdot \nabla_z) e_q(-z \cdot c L) = e_q(-z \cdot c L),
\]
then integrating by parts $N$ times (28) we get that
\[
|\tilde{I}_q(c)| \leq \left(\frac{q}{2\pi L} |c|^{-2}\right)^N \int_{\mathbb{R}^d} |c \cdot \nabla_z|^N f_q(z) \, dz
\]
\[
\lesssim_{d,N,A} \left(\frac{q}{L}\right)^N |c|^{-N} \sum_{0 \leq l \leq N} \max_{0 \leq l \leq n/2} \left| \frac{\partial^{n-l}}{\partial y^{n-l}} h \left(\frac{q}{L}, F^m(z)\right) \right|
\]
\[
\times |z|^{n-2l} \left| \nabla_z^{N-n} w(z) \right| \, dz,
\]
where $\frac{\partial}{\partial y} h$ stands for the derivative of $h$ with respect to the second argument.
Let us distinguish then two cases. When $q \leq L$, Lemma 3.3 (with $N = 0$) implies that
\[
\max_{0 \leq l \leq n/2} \left| \frac{\partial^{n-l} h}{\partial y^{n-l}} \left( \frac{q}{L}, F^m(z) \right) \right| |z|^{n-2l} \left| \nabla_{\mathbf{z}}^{N-n} w(z) \right| \leq (L/q)^{n+1} \langle \mathbf{z} \rangle^{-d-1} \|w\|_{N-n,n+d+1},
\]
from which (29) follows since $n \leq N$. When $q > L$, because of Lemma 3.1, point 1, $h$ is different from zero only if
\[
2 |F^m(z)| > \frac{q}{L}.
\]
Then for such $z$ and for $l \leq n$, point 3 of Lemma 3.1 implies that

$$\left| \frac{\partial^{n-l}}{\partial y^{n-l}} h \left( \frac{q}{L}, F^n(z) \right) \right| \lesssim_n \frac{L}{q} \frac{1}{|F^n(z)|^{n-t}} \lesssim_n \left( \frac{L}{q} \right)^{n-l+1}.$$  

So

$$\max_{0 \leq l \leq n/2} \left| \frac{\partial^{n-l}}{\partial y^{n-l}} h \left( \frac{q}{L}, F^n(z) \right) \right| |z|^{n-2l} |\nabla_z^{N-n} w(z)| \leq \max_{0 \leq l \leq n} \left( \frac{L}{q} \right)^{n-l+1} \|w\|_{N-n,2N-n+d+1}.$$  

Since from (30) we have that $q/L \lesssim_{A,m} (z)^2$, then the first fraction above is bounded by $(L/q)^{N+1}$, and again (29) follows.

As a corollary, we can infer the desired estimate for $J^{n_1}_r$:

**Corollary 5.2.** For $J^{n_1}_r$ defined in (17) and $d > 2$ we have

$$|J^{n_1}_r| \lesssim_{d, \gamma_1, A, m} \|w\|_{N_0, 2N_0 + d+1},$$

where $N_0 := [(d+1)(1+1/\gamma_1)]$.

**Proof.** By the definition of $J^{n_1}_r$ we have

$$|J^{n_1}_r| \lesssim_d \sum_{s \geq L^{\gamma_1}} s^{d-1} \sum_{q=1}^{\infty} q^{-d} \sup_{||e||_1 = s} |S_q(c)||I_q(c)|$$

$$\lesssim_{d, A} \sum_{s \geq L^{\gamma_1}} s^{d-1} \sum_{q=1}^{\infty} q^{1-d/2} L^d \sup_{||e||_1 = s} |\tilde{I}_q(c)|$$

$$\lesssim_{d, N, A, m} \sum_{s \geq L^{\gamma_1}} s^{d-1} \sum_{q=1}^{\infty} q^{-d/2} s^{-N} L^{d+1} \|w\|_{N, 2N+d+1},$$

where the second line follows through Lemma 2.1, while the third one via Proposition 5.1. We choose $N = [(d+1)(1+1/\gamma_1)]$ and get that

$$\sum_{s \geq L^{\gamma_1}} s^{d-1} s^{-N} L^{d+1} \leq \sum_{s \geq L^{\gamma_1}} s^{-2} \lesssim 1,$$

while the sum in $q$ is bounded, too. This concludes the proof.  

6. The $J^{n_1}_r$ term

6.1. **The estimate.** Our next (and final) goal is to estimate the term $J^{n_1}_r$ from (16).

**Proposition 6.1.** Let $w \in C^{N}(\mathbb{R}^d)$, where $N = N(d, \gamma_1) := [d^2/\gamma_1] - 2d$, and $0 < \gamma_1 < d/2 - 1$. Then,

$$|J^{n_1}_r| \lesssim_{A, d, \gamma_1, m} L^{d/2+2+\gamma_1(d+1)} \left( \|w\|_{N, d+5} + \|w\|_{0, N+3d+4} \right).$$

Proposition 6.1 follows from the next lemma which is a modification of Lemma 22 in [7].

1ANDREY DYMOV, SERGEI KUKSIN, ALBERTO MAIOCCHI, AND SERGEI VLADUT
Lemma 6.2. For $|c| \leq L^{\gamma_1}$, $c \neq 0$,
\[ |I_q(c)| \lesssim_{A,d,\gamma_1,m} L^{d/2+1+\gamma_1} q^{d/2-1} \left( \|w\|_{\tilde{N},d+5} + \|w\|_{0,\tilde{N}+3d+4} \right), \]
where $\tilde{N}$ and $\gamma_1$ are the same as above.

Proof of Proposition 6.1. Accordingly to Lemma 2.1
\[
|J_{\gamma_1}^c| \lesssim_A \sum_{c \neq 0, |c| \leq L^{\gamma_1}} \sum_{q=1}^{\infty} q^{-d} q^{d/2+1} |I_q(c)| \lesssim_{A,d} L^{d\gamma_1} \max_{c \neq 0; |c| \leq L^{\gamma_1}} |I_q(c)| \sum_{q=1}^{\infty} q^{-d/2+1} = L^{d\gamma_1} \left( \sum_{q<L} + \sum_{q \geq L} \right) q^{-d/2+1} \max_{c \neq 0; |c| \leq L^{\gamma_1}} |I_q(c)| =: J_-. \]

Corollary 3.2 together with (27), (28) implies
\[
(31) \quad |I_q(c)| \lesssim \frac{L^{d+1}}{q} \|w\|_{L^1},
\]
so that
\[
J_+ \lesssim L^{d\gamma_1} L^{d+1} \|w\|_{L^1} \sum_{q \geq L} q^{-d/2} \lesssim L^{d\gamma_1 + d/2 + 2} \|w\|_{L^1}.
\]

By Lemma 6.2 we get
\[
J_- \lesssim_{A,d,\gamma_1,m} L^{d\gamma_1} \left( \|w\|_{\tilde{N},d+5} + \|w\|_{0,\tilde{N}+3d+4} \right) \sum_{q<L} L^{d/2+1+\gamma_1} \\
\leq \left( \|w\|_{\tilde{N},d+5} + \|w\|_{0,\tilde{N}+3d+4} \right) L^{\gamma_1(d+1)+d/2+2}.
\]

\[ \square \]

6.2. Proof of Lemma 6.2. We begin with

6.2.1. An application of the inverse Fourier transform. Note that the proof is nontrivial only for $q \lesssim L$ since for any $\alpha > 0$, the bound (31) implies that for $q \geq \alpha L$ we have
\[
|I_q(c)| \lesssim \alpha L^d \|w\|_{L^1} \lesssim_{A,d} L^{d/2+1} q^{d/2-1} \|w\|_{L^1}.
\]

Let us take a small enough $\alpha = \alpha(d,\gamma_1, A) \in (0,1)$ and assume that $q < \alpha L$. Consider the positive function $w_2(x) = 1/(1 + x^2)$ and set
\[
\hat{w}(z) := \frac{w(z)}{w_2(Fm(z))} = w(z)(1 + F^m(z)^2).
\]

Let
\[
p(t) := \int_{-\infty}^{+\infty} w_2(v) h(q/L, v) e(-tv) \, dv, \quad e(x) := e_1(x) = e^{2\pi ix}.
\]

This is the Fourier transform of the function $w_2(\cdot) h(q/L, \cdot)$. Then, expressing $w_2h$ via $p$ by the inverse Fourier transform, we find that
\[
w(z) h(q/L, F^m(z)) = \hat{w}(z) \int_{-\infty}^{+\infty} p(t) e(tF^m(z)) \, dt.
\]
Inserting this representation into (28) we get
\[ \tilde{I}_q(c) = \int_{-\infty}^{\infty} dt \ p(t) e(-tm) \int_{\mathbb{R}^d} dz \tilde{w}(z) e(tF(z) - u \cdot z), \]
where we define
\[ u := c L/q. \]
Note that \(|u| \geq \alpha^{-1} > 1\). Now let us denote \( W_0(x) = c_0^{-d} \prod_{i=1}^d w_0(x_i) \) (see (20)). Then \( W_0 \in C_0^\infty(\mathbb{R}^d) \), \( W_0 \geq 0 \) and
\[ \text{supp} \ W_0 = [-1, 1]^d \subset \{|x| \leq \sqrt{d}\}, \quad \int W_0 = 1. \]
Let us set \( \delta = |u_0|^{-1/2} \) and write \( \tilde{w} \) as
\[ \tilde{w}(z) = \delta^{-d} \int W_0(\frac{z-a}{\delta}) \tilde{w}(z) da. \]
Then setting \( b := \frac{z-a}{\delta} \) we get that
\[ |\tilde{I}_q(c)| \leq \int_{\mathbb{R}^d} da \int_{-\infty}^{\infty} dt \ |p(t)| |I_{a,t}|, \]
where
\[ I_{a,t} := \int_{\{|b| \leq \sqrt{d}\}} W_0(b) \tilde{w}(z) e(tF(z) - u \cdot z) \ d b, \quad z := a + \delta b \]
(in virtue of (33)). Consider the exponent in the integral \( I_{a,t} \):
\[ f(b) = f_{a,t}(b) := tF(a + \delta b) - u \cdot (a + \delta b). \]
At the next step we will estimate integral \( I_{a,t} \), regarding \((a, t)\) as a parameter, and distinguishing two cases:
1. \((a, t)\) belongs to the "good" domain \( S_R \), where
\[ S_R = \{(a, t): |\nabla f(0)| = \delta |t| A a - u| \geq R(t/|u|) = R(\delta^2 t)\}. \]
2. \((a, t)\) belongs to the "bad" set \( S_R^c = (\mathbb{R}^d \times \mathbb{R}) \setminus S_R \).
Here
\[ 1 \leq R \leq |u|^{1/3} \]
is a parameter to be chosen later.

6.2.2. Integrals over \( S_R \) and \( S_R^c \). We consider first the integral over the good set \( S_R \):

**Lemma 6.3.** For any \( N \geq 0 \) and \( R \geq 2\|A\| \sqrt{d} \) we have
\[ \int_{S_R} da \ dt |p(t)| |I_{a,t}| \lesssim_{d,N,m} L \frac{R^{-N}}{q} R^{N} ||w||_{N,d+5}. \]
Proof. Let \( I := \nabla f(0)/|\nabla f(0)| \) and \( \mathcal{L} = 1 \cdot \nabla_b \). Then for \((a, t) \in S_R\), (35)
\[
|\mathcal{L} f(b)| \geq |\nabla f(0)| - \delta^2 |t| A b | \geq R(\delta^2 t) - \delta^2 |t| A \frac{R}{2 \|A\|} \geq \frac{1}{2} R(\delta^2 t) \geq R/2.
\]
Since \((2 \pi i \mathcal{L} f(b))^-1 \mathcal{L} e(f(b)) = e(f(b))\), then integrating by parts \( N \) times in the integral for \( I_{a,t} \) we get that
\[
|I_{a,t}| \lesssim_{d,N} \sup_{|b| \leq 1} \max_{0 \leq k \leq N} \left| \mathcal{L}^{N-k} \tilde{w}(\delta b + a) \left( \frac{\mathcal{L}^2 f(b)}{\mathcal{L} f(b)} \right)^k \right|,
\]
where we have used that \( \mathcal{L}^m f(b) = 0 \) for every \( m \geq 3 \). Since \( |\mathcal{L}^2 f(b)| \leq \delta^2 |t||1 \cdot A| \leq \delta^2 |t| A\), then in view of (35)
\[
\left| \frac{\mathcal{L}^2 f(b)}{\mathcal{L} f(b)} \right| \leq \frac{\delta^2 |t| A}{\frac{1}{2} R(\delta^2 t)} = \frac{2 \|A\|}{R} \leq \frac{1}{\sqrt{d}}.
\]
So using that \( \left| \frac{1}{\mathcal{L} f(b)} \right| \leq \frac{2}{R} \) by (35), we find
\[
|I_{a,t}| \lesssim_{d,N} R^{-N} \sup_{|b| \leq 1} \max_{0 \leq k \leq N} \left| \mathcal{L}^k \tilde{w}(\delta b + a) \right|.
\]
Thus, denoting by \( 1_{S_R} \) the indicator function of the set \( S_R \), we have
\[
\int_{R^d} |I_{a,t}| 1_{S_R} da \lesssim_{d,N} R^{-N} \int_{R^d} \frac{da}{(a)^{d+1}} \left( \sup_{|b| \leq 1} \max_{0 \leq k \leq N} \left| \mathcal{L}^k \tilde{w}(\delta b + a) \right| \right)
\]
\[
\lesssim_{d,N} R^{-N} \|\tilde{w}\|_{N,d+1} \lesssim_{d,N,m} R^{-N} \|w\|_{N,d+5},
\]
for every \( t \). Then
\[
\text{l.h.s. of (33)} \lesssim_{d,N,m} R^{-N} \|w\|_{N,d+5} \int_{-\infty}^{+\infty} |p(t)| dt.
\]
It remains to show that
\[
|p(t)| \lesssim_{-\infty}^{\infty} |p(t)| dt \lesssim L/q.
\]
In virtue of Lemma 3.3,
\[
\left| \frac{\partial^k}{\partial v^k} h(x, v) \right| \lesssim_k x^{-k-1} \min\{1, x^2/v^2\}, \quad k \geq 1,
\]
and by Corollary 3.2 \( |h(x, v)| \lesssim x^{-1} \). Then, a simple integration by parts in (32) shows that, for any \( M \geq 0 \),
\[
|p(t)| \lesssim_{M} \left( \int_{-\infty}^{\infty} |w_2^{(M)}(v)| x^{-1} dv \right)
\]
\[
+ \max_{1 \leq k \leq M} \int_{-\infty}^{\infty} |w_2^{(M-k)}(v)| x^{-k-1} \min\{1, x^2/v^2\} dv,
\]
where \( x := q/L < \alpha \). Writing the latter integral as a sum \( \int_{|v| \leq x} + \int_{|v| > x} \) we see that
\[
\int_{|v| \leq x} = x^{-k-1} \int_{|v| \leq x} |w_2^{(M-k)}(v)| \, dv \lesssim_k x^{-k}
\]
and
\[
\int_{|v| > x} = x^{-k+1} \int_{|v| > x} |w_2^{(M-k)}(v)| \, dv \lesssim_k x^{-k}.
\]
Then, since \( x = q/L < \alpha \),
\[
|p(t)| \lesssim_M \left( \frac{q}{L} |t| \right)^{-M}, \quad M \geq 0.
\]
Choosing \( M = 2 \) when \( |t| > L/q \) and \( M = 0 \) when \( |t| \leq L/q \) we get \( 37 \).

Then we study the integral over the bad set \( S_R^c \).

**Lemma 6.4.** For any \( 1 \leq R \leq |u|^{1/3} \) and \( 0 < \beta < d^2/2 \) we have
\[
\int_{S_R^c} \, d \mathbf{a} \, dt |p(t)||I_{a,t}| \lesssim_{A,d,m} R^d |u|^{-d/2+1+\beta} \|w\|_{0,K(d,\beta)},
\]
where \( K(d,\beta) = d + \lceil d^2/2\beta \rceil + 4 \).

**Proof.** On \( S_R^c \) we use for \( I_{a,t} \) the easy upper bound
\[
|I_{a,t}| \lesssim_d \sup_{|b| \leq 1} |\tilde{w}(\delta \mathbf{b} + \mathbf{a})| \leq \|\tilde{w}\|_{0,0}.
\]
The fact that \((\mathbf{a}, t) \in S_R^c\) implies that the integration in \( d \mathbf{a} \) for a fixed \( t \) is restricted to the region where
\[
\left|A \mathbf{a} - \frac{\mathbf{u}}{t}\right| \leq \frac{R}{\delta |t|} \left( t/|u| \right),
\]
or
\[
\left| \mathbf{a} - \frac{A^{-1} \mathbf{u}}{t} \right| \leq \|A^{-1} \| \frac{R}{\delta |t|} \left( t/|u| \right).
\]
We first consider the case \(|t| \geq |u|^{1-\beta/d} \). Since \( \delta = |u|^{-1/2} \) and \(|u| > 1\), then
\[
\left( \frac{R}{\delta |t|} \right)^t (t/|u|) \leq R \max(|u|^{-1/2+\beta/d}, |u|^{-1/2}) = R |u|^{-1/2+\beta/d}.
\]
In view of \( 39 \) - \( 41 \),
\[
\left| \int_{\mathbb{R}^d} |I_{a,t}| \mathbf{1}_{S_R^c}(\mathbf{a}, t) \, d \mathbf{a} \right| \lesssim_{A,d} R^d |u|^{-d/2+\beta} \|\tilde{w}\|_{0,0} \lesssim_{A,d,m} R^d |u|^{-d/2+\beta} \|w\|_{0,4}.
\]
Since by \( 37 \) \( \int_{|t| \geq |u|^{1-\beta/d}} |p(t)| \, dt \lesssim \frac{L}{q} \leq |u| \), then
\[
\int_{|t| \geq |u|^{1-\beta/d}} dt \int_{\mathbb{R}^d} d \mathbf{a} |p(t)||I_{a,t}| \mathbf{1}_{S_R^c}(\mathbf{a}, t) \lesssim_{A,d,m} R^d |u|^{-d/2+1+\beta} \|w\|_{0,4}.
\]
Now let $|t| \leq |u|^{1-\beta/d}$. Then the r.h.s. of (42) is bounded by $\|A^{-1}\| R/(\delta |t|)$. So

$$|a| \gtrsim_A \frac{|u| - R\sqrt{|u|}}{|t|} \geq (1 - |u|^{-1/6}) \frac{|u|}{|t|} \geq \frac{1}{2} \frac{|u|}{|t|} \geq \frac{1}{2} |u|^{\beta/d}$$

since $|u|^{-1} \leq \alpha$, if $\alpha$ is so small that $1 - \alpha^{1/6} \geq 1/2$. Then $1_{S_R^e}(a, t) \lesssim A, d |u|^{-d/2+\beta/d} |a|^{d^2/2\beta - 1}$, and we deduce from (39) that for such values of $t$

$$\left| \int_{\mathbb{R}^d} |I_{a,t}| 1_{S_R^e}(a, t) da \right| \lesssim_{A, d} |u|^{-d/2+\beta/d} \int_{\mathbb{R}^d} |a|^{d^2/2\beta - 1} \sup_{|b| \leq 1} \|\tilde{w}(\delta b + a)\| da$$

$$\lesssim_{A, d, m} |u|^{-d/2+\beta/d} \|w\|_{0,K(d, \beta)},$$

where $K(d, \beta) = d + \lceil d^2/2\beta \rceil + 4$. On the other hand, by (38) with $M = 0$,

$$\int_{|t| \leq |u|^{1-\beta/d}} |p(t)| dt \lesssim |u|^{1-\beta/d},$$

from which we obtain

(43)

$$\int_{|t| \leq |u|^{1-\beta/d}} \int_{\mathbb{R}^d} |a| |p(t)| |I_{a,t}| 1_{S_R^e}(a, t) \lesssim_{A, d, m} |u|^{-d/2+1} \|w\|_{0,K(d, \beta)}.$$

Putting together (42) and (43) we get the assertion. □

6.2.3. End of the proof. In order to complete the proof of Lemma 6.2 we combine Lemmas 6.3 and 6.4 to get that

$$|\tilde{I}_q(c)| \lesssim_{A, d, N, m} \left( \frac{L}{q} R^{-N} + R^d |u|^{-d/2+1+\beta} \right) \left( \|w\|_{N, d+5} + \|w\|_{0,K(d, \beta)} \right).$$

We fix here $\beta = \gamma_1/2$, $R = |u|^{2\gamma_1/\gamma_2} \leq |u|^{\gamma_1/\gamma_2}$ and pick $N = \lceil d/2 \rceil - 2d > 0$ (notice that $R > 2|A|\sqrt{d}$ if $\alpha$ is small enough). Then $K(d, \beta) = N + 3d + 4$, $R^{-N} \leq |u|^{-d/2+\gamma_1} \leq (L/q)^{-d/2+\gamma_1}$ since $-d/2 + \gamma_1 < 0$ and $|u| \geq L/q$. Moreover, $R^d |u|^{-d/2+1+\beta} = |u|^{-d/2+1+\gamma_1} \leq (L/q)^{-d/2+1+\gamma_1}$. This concludes the proof. □

**Acknowledgements**

The authors thank Professor Heath-Brown for his advice concerning his paper.

**Appendix A. Function $\tilde{I}(t)$.**

In this appendix we assume that $d$ is an even number, $d = 2d_1$, and the quadratic form $F$ has the form $F_0$. 
A.1. The two measures on $\Sigma_t$. Recall that

$$\Sigma_t = \{z = (x, y) : x \cdot y = t\} \subset \mathbb{R}^{d_1}_x \times \mathbb{R}^{d_1}_y = \mathbb{R}^{2d_1}_z$$

is a smooth manifold if $t \neq 0$, while $\Sigma_0$ has a locus at $(0, 0)$. For $t \neq 0$ we denote by $dz|_{\Sigma_t}$ the volume element on $\Sigma_t$. If $t = 0$, then we fist set $dz|_{\Sigma_0}$ to be the volume element on $\Sigma_0 \setminus \{(0, 0)\}$ and then extend it to a Borel measure on $\Sigma_0$ which assigns zero measure to the locus $(0, 0)$.

We start with a convenient disintegration of the measure $dz|_{\Sigma_t}$. Let us set $\Sigma^x_t = \Sigma_t \setminus \{(x, y) : y = 0\}$. Then for each $t$ the mapping

$$\pi^x_t : \Sigma^x_t \rightarrow \mathbb{R}^{d_1} \setminus \{0\}, \quad (x, y) \mapsto x,$$

is a smooth affine euclidean vector–bundle. For $t \neq 0$ the set $\Sigma^x_t$ equals $\Sigma^x$, but not for $t = 0$, and the quadric $\Sigma_0$ is important for what follows. Denote by $\sigma_t(x)$ the fibers of $\pi^x_t$, $\sigma_t(x) = (\pi^x_t)^{-1}(x)$. Then

$$\sigma_t(x) = (x, x^+ + tx|x|^2) \quad \forall x \in \mathbb{R}^{d_1} \setminus \{0\}, \forall t,$$

where $x^\perp$ stands for the orthogonal complement to $x$ in $\mathbb{R}^{d_1}$.

**Proposition A.1.** For any $t$ the measure $dz|_{\Sigma_t}$, restricted to $\Sigma^x_t$, disintegrates as follows:

$$dz|_{\Sigma_t} = |x|^{-1}dx|z|d\sigma_t(x)y,$$

where $d\sigma_t(x)y$ is the volume element on the affine hyper space $\sigma_t(x)$.

We recall that equality (44) means that for any bounded continuous function $f$ on $\Sigma^x_t$

$$\int_{\Sigma^x_t} f(z)dz|_{\Sigma_t} = \int_{\mathbb{R}^{d_1} \setminus \{0\}} |x|^{-1}dx \int_{\sigma_t(x)} f(z) |z| d\sigma_t(x)y.$$

**Proof.** The argument below follows the proof of Theorem 3.6 in [1]. It suffices to verify (45) for all continuous functions $f$, supported by a compact set $K$, for any $K \in \Sigma_t \cap (\mathbb{R}^{d_1} \setminus \{0\}) \times \mathbb{R}^{d_1}$. For $x' \in \mathbb{R}^{d_1} \setminus \{0\}$ we denote $r' = |x'| > 0$ and set $U_{x'} = \{x : |x - x'| < \frac{1}{2}r'\}$. Since any $K$ as above can be covered by a finite system of domains $U_{x'} \times \mathbb{R}^{d_1}_y$, it suffices to prove (45) for any set $U_{x'} \times \mathbb{R}^{d_1}_y = : U \subset \Sigma^x_t$ and any $f \in C_0(U \cap \Sigma^x_t)$, where $C_0(O)$ stands for the space of continuous compactly supported functions on $O$.

Now we construct explicitly a trivialisation of the linear bundle $\pi^x_t$ over $U_{x'}$. To do this we fix in $\mathbb{R}^{d_1}$ a coordinate system, corresponding to a frame $(e_1, \ldots, e_d)$ such that the ray $\mathbb{R}_+ e_1$ intersects $U_{x'}$. Then

$$x_1 \geq \kappa > 0 \quad \text{for any} \quad x = (x_1, x_2, \ldots, x_d) =: (x, \bar{\bar{x}}) \in U_{x'}.$$  

Next we construct a linear in the second argument $\bar{\bar{\eta}}$ coordinate mapping $\Phi_t : U_{x'} \times \mathbb{R}^{d_1-1} \rightarrow (\pi^x_t)^{-1}U \cap \Sigma^x_t = U \cap \Sigma^x_t$ of the form

$$\Phi_t(x, \bar{\bar{\eta}}) = (x, \Phi^x_t(\bar{\bar{\eta}})), \quad \phi_t(\bar{\bar{\eta}}) = (\phi_t(x, \bar{\bar{\eta}}), \bar{\bar{\eta}}).$$

The function $\phi_t$ should be such that $\Phi^x_t(\bar{\bar{\eta}}) \in \sigma_t(x)$. That is, it should satisfy $x \cdot \Phi^x_t(\bar{\bar{\eta}}) = x_1 \phi_t + \bar{\bar{x}} \cdot \bar{\bar{\eta}} = t$. From here we find that $\phi_t = \frac{t-x \cdot \bar{\bar{\eta}}}{x_1}$. Thus...
obtained mapping $\Phi^x_t$ is affine in $\bar{\eta}$, and the image of $\Phi_t$ equals $U \cap \Sigma^x_t$. So $\Phi_t$ provides the required trivialisation of $\pi^x_t$ over $U \cap \Sigma^x_t$.

In the coordinates $(x, \bar{\eta}) \in U_{x'} \times \R^{d_1-1}$ the hypersurface $\Sigma^x_t$ is embedded in $\R^{2d_1}$ as a graph of the function $(x, \bar{\eta}) \mapsto \phi_t$. Accordingly, in these coordinates the volume element $dz|_{\Sigma_t}$ on $\Sigma^x_t$ reads $dz|_{\Sigma_t} = \tilde{p}_t(x, \bar{\eta}) dx d\bar{\eta}$, where

$$\tilde{p}_t(x, \bar{\eta}) = (1 + |\nabla \phi_t(x, \bar{\eta})|^2)^{1/2} = \left(1 + x_1^{-2} (x - \bar{x} \cdot \bar{\eta})^2 + |\bar{\eta}|^2 + |\bar{e}|^2\right)^{1/2}.$$

So

$$\int_{U \cap \Sigma^x_t} f(z) dz|_{\Sigma_t} = \int_{U_{x'}} \left( \int_{\R^{d_1-1}} f(x, \Phi^x_t(\bar{\eta})) \tilde{p}_t(x, \bar{\eta}) d\bar{\eta} \right) dx.$$

Passing from the variable $\bar{\eta}$ to $y = \Phi^x_t(\bar{\eta}) \in \sigma_t(x)$ we write the measure $\tilde{p}_t(x, \bar{\eta})d\bar{\eta}$ as $p_t(z)d\sigma_t(x,y)$ with

$$p_t(x, y) = \tilde{p}_t(x, (\Phi^x_t)^{-1}(y)) | \det \Phi^x_t|^{-1}$$

(determinant of an affine map is understood as that of its linear part, mapping $\R^{d_1-1}$ to $\sigma_t(x) = \R^{d_1-1}$). Then

$$\int_{U \cap \Sigma^x_t} f(z) dz|_{\Sigma_t} = \int_{U_{x'}} \left( \int_{\sigma_t(x)} f(z) p_t(z)d\sigma_t(x,y) \right) dx.$$ 

The smooth function $p_t$ in the integral above is defined on $U \cap \Sigma^x_t$ in a unique way and does not depend on the trivialisation of $\Pi$ over $U_{x'}$, used to obtain it. Indeed, if $p_1(z)$ is another continuous function on $U \cap \Sigma^x_t$ such that (46) holds with $p_t := p_1$, then $\int dx \int_{x'} f(z) (p_t(z) - p_1(z)) d\sigma_t(x,y) = 0$ for any $f \in C_0(U \cap \Sigma^x_t)$, which obviously implies that $p_1 = p_t$.

To establish (45) it remains to verify that in (46)

$$p(x_s, y_s) = |x_s|^{-1}|z_s| \quad \forall z_s = (x_s, y_s) \in U \cap \Sigma_t.$$

To prove this equality let us choose in $\R^{d_1}$ the euclidean coordinates, corresponding to a frame with the first basis vector $e_1 = x_s/|x_s|$. Then $x_s = (|x_s|, 0)$ and $y_s = (t|x_s|, \bar{y}_s)$ for some $\bar{y}_s \in \R^{d_1-1}$. Repeating the calculation above in this coordinate system we readily see that $\Phi^x_t(\bar{\eta}) = (t|x_s|, \bar{\eta})$. So $\phi_t(x_s, \bar{\eta}) = t/|x_s|$, and

$$\tilde{p}_t(x_s, \bar{\eta}) = |x_s|^{-1} \left( |x_s|^2 + t^2 |x_s|^2 + |\bar{\eta}|^2 \right)^{1/2}, \quad \bar{\eta} \in \R^{d_1-1}.$$

Since $\Phi^x_t(\bar{\eta}) = (t|x_s|, \bar{\eta}) \in \sigma_t(x_s)$, then $\det \Phi^x_t = 1$. Thus we have that

$$\tilde{p}_t(x_s, \bar{\eta}) = p_t(x_s, y_s) = |x_s|^{-1} \left( |x_s|^2 + |y_s|^2 \right)^{1/2} = |x_s|^{-1} |z_s|,$$

and (47) is obtained. This proves (45).

Considering the projection $\pi^y_t : \Sigma_t \ni (x, y) \mapsto y$ instead of $\pi^x_t$ we see that the volume element $dz|_{\Sigma_t}$, restricted to the domain $\Sigma^y_t = \{(x, y) \in \Sigma_t : y \neq 0\}$, disintegrates as

$$dz|_{\Sigma^y_t} = |y|^{-1} dy |z| d(\pi^y_t)^{-1} x, \quad y \in \R^{d_1} \setminus \{0\}.$$
Since $\Sigma^+_t \cap \Sigma^-_t$ is the empty set if $t \neq 0$ and is $(0,0)$ if $t = 0$, then from here we obtain that

$$\Sigma^+_t \text{ has the full volume in } \Sigma_t.$$  

As $d \geq 2$, then by (44) the function $|z|^{-1}$ is locally integrable on $\Sigma^+_t$; so also on $\Sigma_t$. Accordingly, the measure $d\mu_t = |z|^{-1}dz \mid_{\Sigma_t}$ is well defined and is equivalent to $dz|_{\Sigma_t}$. Using (44) and (48) we get:

**Corollary A.2.** For any $t$ the measure $\mu_t$ disintegrates as

$$(\mu_t \mid_{\Sigma^+_t})(dz) = |x|^{-1}dx \, d_{\sigma_t(x)}y,$$

and as

$$(\mu_t \mid_{\Sigma^-_t})(dz) = |y|^{-1}dy \, d_{\sigma_t(y)}^{-1}x.$$  

A.2. Integrating over $\Sigma_t$. Using the embedding $\Sigma_t \to \mathbb{R}^{2d_t}$ we regard $\mu_t$ both as an atomless Borel measure on $\Sigma_t$ and on $\mathbb{R}^{2d_t}$. Now our goal is to study integrals

$$(52) \quad \mathcal{I}(t) = \mathcal{I}(t; f) := \int_{\Sigma_t} f(z) \mu_t(dz),$$

where $f$ is a $k_f$–smooth function on $\mathbb{R}^{2d_t}$, $0 \leq k_f \leq \infty$, decaying at infinity. Due to (49), in (52) we may replace the integrating over $\Sigma_t$ by that over $\Sigma^+_t$ or over $\Sigma^-_t$. So we may use the disintegrations (50) and (51) to study $\mathcal{I}(t)$. To do this, note that for any $t$ the mapping

$$L_t = \Sigma^+_0 \to \Sigma^+_t, \quad (x, y) \mapsto (x, y + t|x|^{-2}x),$$

defines an affine isomorphism of the bundles $\pi_0$ and $\pi_t$, which we will also denote as $L_t$, and that $L_t$ preserve the volume of the fibers. So due to (50) the mapping $L_t$ sends the measure $\mu_0$ to $\mu_t$, and

$$(53) \quad \mathcal{I}(t; f) = \int_{\Sigma_0} f(L_t(z)) \mu_0(dz) = \int_{\mathbb{R}^{2d_t}} |x|^{-1}dx \int_{\mathbb{R}^d} f(x, y + t|x|^{-2}x) \, d_x \, y,$$

since $\sigma_0(x) = x^\perp$. This equality suggests that the main difficulty to study $\mathcal{I}(t)$ comes from the integrating over small $x$'s. To separate it from the effects, coming from the integrating over the vicinity of infinity we will split function $f$ in a some of three functions. Firstly, taking any smooth function $\phi \geq 0$ on $\mathbb{R}$ which vanishes for $|t| \geq 2$ and equals one for $|t| \leq 1$, we write

$$f = f_{00} + f_1, \quad f_{00} = \phi(|t|^2)f.$$  

Denoting $B_r(\mathbb{R}^n) = \{x \in \mathbb{R}^n : |x| \leq r\}$ and $B^r(\mathbb{R}^n) = \{x \in \mathbb{R}^n : |x| \geq r\}$ we see that $\text{supp } f_{00} \subset B_{\sqrt{t}}(\mathbb{R}^{2d_t})$ and $\text{supp } f_1 \subset B^1(\mathbb{R}^{2d_t})$. So

$$(x, y) \in \text{supp } f_1 \Rightarrow |x| > 1/\sqrt{2} \text{ or } |y| \geq 1/\sqrt{2}.$$
Then, setting \( f_{10} = f_1(z) \phi(4|x|^2) \) and \( f_{11} = f_1(z)(1 - \phi(4|x|^2)) \), we get that 
\[ f = f_{00} + f_{10} + f_{11}, \]
where
\[
\text{supp } f_{11} \subseteq B^{1/2}(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_1},
\]
\[
\text{supp } f_{10} \subseteq \left( (B_{1/\sqrt{2}}(\mathbb{R}^{d_1}) \times \mathbb{R}^{d_1}) \cap \text{supp } f_1 \right) \subseteq \mathbb{R}^{d_1} \times B^{1/\sqrt{2}}(\mathbb{R}^{d_1}).
\]

We denote \( I_{ij}(t) = \mathcal{I}(t; f_{ij}) \). Then \( \mathcal{I}(t) = \mathcal{I}_{00}(t) + \mathcal{I}_{10}(t) + \mathcal{I}_{11}(t) \), and we will estimate the three integrals in the r.h.s.

For a function \( F \in C^k(\mathbb{R}^N) \), \( n_1 \in \mathbb{N} \cup \{0\}, n_1 \leq k \) and \( n_2 \in \mathbb{R} \) we set 
\[ \|F\|_{n_1,n_2} = \sup_x \max_{|\alpha| \leq n_1} |\partial_\alpha F(x)| |x|^{n_2}. \]
Assume that \( \|f\|_{k_f,n} < \infty \) for some \( n \geq 0 \). Then obviously,
\[
\|f_{00}\|_{k_f,n} \leq C_n \|f\|_{k_f,0} \quad \|f_{1j}\|_{k_f,n} \leq C_{k_f,n} \|f\|_{k_f,n} \quad j = 0,1.
\]

We start with the integral \( \mathcal{I}_{00}(t) \). It follows immediately from (50) that this is a continuous function. Since for any \((x,y) \in \text{supp } f_{00}\) we have \(|(x,y)| \leq \sqrt{2}\), \( |x| \leq |y| \leq \frac{1}{2}(|x|^2 + |y|^2) \leq 1 \). So \( \mathcal{I}_{00}(t) \) vanishes for \(|t| \geq 1\). Now let \(|t| < 1\). From (53) we get that
\[
\partial^k \mathcal{I}_{00}(t) = \int_{\mathbb{R}^{d_1}} |x|^{-1} dx \int_{x^+} (dx/dt)^k f_{00}(x, y + t|x|^2) d_x y
\]
\[
= \int_{\mathbb{R}^{d_1}} |x|^{-1} dx \int_{x^+} d_x^k f_{00}(x, y + t|x|^2) (|x|^2) d_x y.
\]
Since \( y \in x^+ \), then \((x,y+t|x|^2)\) \(= |x|^2 + |y|^2 + t^2|x|^2 \). So the integrand of the internal integral is supported by the ball
\[
Q_t(|x|) = \{ y \in x^+ : |y|^2 \leq h_t(|x|^2) \}, \quad h_t(\rho) = 2 - \rho - t^2 \rho^{-1} \leq 2 \quad \text{for } \rho > 0.
\]
The function \( h_t \) of argument \( \rho > 0 \) is concave. For \(|t| < 1\) it has two positive zeroes
\[
\rho_{1,2}(t) = 1 \pm \sqrt{1 - t^2}, \quad 0 < \rho_1 = t^2/2 + O(t^4) < \rho_2 = 2 - t^2/2 + O(t^4) < 2
\]
and is positive between them. Since \( h_t \leq 2 \), then \( \text{Vol } Q_t(|x|) \leq C_{d_1 - 2(d_1 - 1)/2} = C' \). The internal integral in (55) is non-zero only when \( h \) is positive, i.e. when
\[
\rho_1 < |x|^2 < \rho_2.
\]
Since \( \rho_1 > 0 \) for \( t \neq 0 \), then we see from (55) that \( \mathcal{I}_{00} \) is as smooth a function of \( t \neq 0 \) as \( f_{00} \) is; so \( \mathcal{I}_{00} |_{\mathbb{R} \setminus \{0\}} \in C^{k_f} \). To study the behaviour of \( \mathcal{I}_{00} \) at zero we note that since \( \|d_y^k f_{00}\| \leq C_k \|f\|_{k,0} \) and \( \text{Vol } Q_{t}(|x|) \leq C' \), then by (55) and (56)
\[
|\partial^k \mathcal{I}_{00}(t)| \leq C_k \|f\|_{k,0} \int_0^{\sqrt{\rho_2}} r^{-1+d_1-1-k} dr.
\]
As $\sqrt{\rho_1} \geq c|t|$ and $\rho_2 < 2$, then

$$|\partial^k \mathcal{I}_{00}(t)| \leq C_k\|f\|_{k,0} \int_{|c|}^{\sqrt{2}} r^{d_1 - 2 - k} \leq C_k\|f\|_{k,0}(1 + |t|^{d_1-k-1}),$$

where we make an agreement that $|t|^0 := \max(\ln |t|^{-1},1)$.

Naturally estimate (57) remains true for $\mathcal{I}(t; f)$ if $f$ is a $C^k$-function with a compact support. In general it cannot be improved for $C^k_0$-functions:

**Example A.3.** Let $f$ be supported by the ball $B_1(\mathbb{R}^{2d_1})$, so that $f_{00} = f$. Further on, let $f = F(|x|^2)g(|y|^2)$, where $F$ and $g$ are non-negative $k_f$-smooth functions, supported by $[-1/2,1/2]$, and such that $0 \notin \text{supp} g$. Then

$$f_t(x,y) := F(|x|^2)g(|y + tx||x|^{-2}|^2) = F(|x|^2)g(|y|^2 + t^2|x|^2).$$

So $\partial_t f_t = F(|x|^2)g(|y|^2 + t^2|x|^{-2})2t|x|^{-2}$, and

$$\partial^k_t f_t = F(|x|^2)\sum_{l=1}^{k} C_l g^{(l)}(|y|^2 + t^2|x|^{-2})t^l|x|^{-l-k},$$

where the coefficients $C_l$ are non-negative and some of them are zero. Then the internal integral in (55) equals

$$F(|x|^2)\sum_{l=1}^{k} C_l t^l|x|^{-l-k}J_l(|x|^2), \quad J_l(|x|^2) = \int g^{(l)}(|y|^2 + t^2|x|^{-2})d_+y.$$ 

Now we see from (55) that $\partial^k \mathcal{I}(t; f)$ equals to the sum in $l$ of the integrals

$$\Upsilon_l := C_l|t|^l \int_{\mathbb{R}^d} r^{-1+d_1-1-l-k}F(r^2)J_l(r^2) dr \sim \pm (1 + |t|^{d_1-k-1}).$$

For generic $F$ and $g$ the numbers $\Upsilon_l$, corresponding to non-zero $C_l$, do not vanish and do not cancel each other (see below), so in general case estimate (57) cannot be improved.

**Sub-example.** Let $d \neq 3$ and $k = 1$, so also $l = 1$ (if $d = 3$ we may consider $k = 2$ and argue as below). Denoting $A = t^2|x|^{-2}$ we have

$$J_1(|x|^2) = C_{d_1-1} \int_0^\infty R^{d_1-2}g(R^2 + A) dR = \frac{1}{2C_{d_1-1}} \int_0^\infty R^{d_1-3} \frac{d}{dR}g(R^2 + A) dR = -\frac{C_{d_1-1}(d_1 - 3)}{2} \int_0^\infty R^{d_1-4}g(R^2 + A) dR,$$

which is $> 0$ or $< 0$, depending on $d$. Accordingly the integral in the expression for $\Upsilon_1$ does not vanish.

It remains to consider functions $\mathcal{I}_{11}$ and $\mathcal{I}_{10}$. Let us start with $\mathcal{I}_{11}$, assuming that

$$n > 2d_1 - 2.$$
In view of (54), \( \partial^k I_{11} \) may be written in the form (55) with \( f_{00} \) replaced by \( f_{11} \), where the integrating in \( dx \) is taken over \( B^{1/2}(\mathbb{R}^{d_1}) \). So \(|x| \geq 1/2\) everywhere on the support of the integrand, and the function \( I_{11} \) is \( k_f \)-smooth. Since in the integral

\[
\| \partial_y^k f_{11} \| \leq C(1 + |x|^2 + |y|^2 + t^2|x|^{-2})^{-n/2}\| f \|_{k,n}
\]

and \(|x| \geq 1/2\), then

\[
|\partial^k I_{11}(t)| \leq C\| f \|_{k,n} \int_{|x| \geq 1/2} |x|^{-1-k} dx \int_x^\infty (1 + |x|^2 + |y|^2 + t^2|x|^{-2})^{−\frac{3}{2}} dy
\]

\[
\leq C\| f \|_{k,n} \int_0^\infty r^{−1−k+d_1−1} dr \int_0^\infty (1 + r^2 + t^2 r^{-2} + \rho^2)^{-\frac{n}{2}} \rho^{d_1−2} d\rho.
\]

Denoting \( 1 + r^2 + t^2 r^{-2} = A^2 \) we write the internal integral as

\[
\int_0^\infty (A^2 + \rho^2)^{-\frac{n}{2}} \rho^{d_1−2} d\rho = A^{−n−1+d_1} \int_0^\infty (1+|x|^2)^{-\frac{n}{2}} x^{d_1−2} dx = C_n A^{−n−1+d_1}
\]

(we recall (58)). So

\[
|\partial^k I_{11}(t)| \leq C\| f \|_{k,n} \int_0^\infty r^{d_1−k−2} (1 + r^2 + t^2 r^{-2})^{−(n+1−d_1)/2} dr.
\]

For \(|t| \leq 1\) the integral in the r.h.s. obviously is bounded by a \( t \)-independent constant. Now let \(|t| \geq 1\). Since by Young’s inequality

\[
(A + B)^{−1} \leq C' \| A^{−a} B^{a−1} \| \quad 0 < a < 1,
\]

for any \( A, B > 0 \), and as \( r \geq 1/2 \), then

\[
(1 + r^2 + t^2 r^{-2})^{−1} \leq C_a r^{−2a} 2^{(a−1)} r^{−2(a−1)} = C_a r^{−2(2a−1)} |t|^{2(a−1)}.
\]

Denoting \( 2a−1 = b \in (−1, 1) \) we get that

\[
|\partial^k I_{11}(t)| \leq C_{k,a} \| f \|_{k,n} |t|^{−\frac{b}{2}(n+1−d_1)} \int_0^\infty r^{d_1−k−2−b(1+n−d_1)} dr.
\]

Consider \( b_* = \frac{d_1−k−1}{n+1−d_1} < 1 \). Taking any \( b \in (b_*, 1) \) we achieve that the exponent for \( r \) in the integral above is \(< −1\), so the integral converges. Denote \( \kappa(b) = \frac{1−b}{2}(n+1−d_1) \). Then \( \kappa(b_*) = \frac{1}{2}(n+k−2(d_1−1)) > 0 \) and any \( 0 < \kappa < \kappa(b_*) \) equals \( \kappa(b) \) for some \( b \in (b_*, 1) \).

We have seen that for every \( 0 < \kappa < \kappa(b_*) \),

\[
|\partial^k I_{11}(t)| \leq C(k, n, \kappa) \| f \|_{k,n} |t|^{−\kappa} \quad \forall t.
\]

For integral \( I_{10} \) we use disintegration (51) to get an analogy of estimate (59) with \( x \) and \( y \) swapped. Since by (54) \(|y| \geq 1/\sqrt{2}\) for \((x, y)\) in the support of \( f_{10} \), then repeating the argument above we get that \( \partial^k I_{10} \) also satisfies estimate (59) with a modified constant \( C_k \). We have proved
Proposition A.4. Assume that \( \|f\|_{k,n} < \infty \) for some \( k \in \mathbb{N} \cup \{0\} \) and that \( n > 2d_1 - 2 \). Then for \( 0 < |t| \leq 1 \)
\[
|\partial^k \mathcal{I}(t; f)| \leq C_k \|f\|_{k,n}(1 + |t|^{d_1-k-1})
\]
(we recall the notation \( |t|^0 := \max(\ln |t|^{-1}, 1) \)), while for \( |t| \geq 1 \)
\[
|\partial^k \mathcal{I}(t; f)| \leq C \|f\|_{k,n}|t|^{-\kappa}
\]
for every \( \kappa < \frac{1}{2}(n + k - 2(d_1 - 1)) \) with a suitable \( C \), depending on \( k, n, \kappa \).

We have seen that for a smooth \( f \), decaying at infinity, the function \( \mathcal{I}(t; f) \) is \((d_1-2)\)-smooth, is smooth outside zero and decays at infinity. But for a generic \( f \) its derivative of order \( d_1 - 1 \) has at zero a logarithmic singularity.

References

[1] T. Buckmaster P. Germain Z. Hani J. Shatah, Effective Dynamics of the Nonlinear Schrödinger Equation on Large Domains, Comm. Pure Appl. Math. 71 1407–1460, (2018).
[2] I. Chavel, Riemannian Geometry: a Modern Introduction, CUP 2006.
[3] W. Duke, J. Friedlander and H. Iwaniec, Bounds for automorphic L-function, Invent. Math., 112 (1993), 1-8.
[4] A. Dymov, S. Kuksin, Formal expansions in stochastic model for wave turbulence 1: kinetic limit, Comm. Math. Physics. 382 (2021), 951-1014.
[5] A. Dymov, S. Kuksin, A. Maiocchi, S. Vlăduţ, The large-period limit for the equations of discrete turbulence, MS under preparation.
[6] A. Dymov, S. Kuksin, A. Maiocchi, S. Vlăduţ, A refinement of Heath-Brown theorem on quadratic forms, MS under preparation.
[7] D. R. Heath–Brown, A new form of the circle method, and its application to quadratic forms, J. Reine Angew. Math. 481 (1996), 149-206.
[8] H. Iwaniec, The circle method and the Fourier coefficients of modular forms, in: Number theory and related topics, 47–55 (Tata Institute of Fundamental Research, Bombay, 1989).

Andrey Dymov, Steklov Mathematical Institute of RAS, Moscow 119991, Russia & National Research University Higher School of Economics, Moscow 119048, Russia

Email address: dymov@mi-ras.ru

Sergei Kuksin, Université Paris-Diderot (Paris 7), UFR de Mathématiques - Batiment Sophie Germain, 5 rue Thomas Mann, 75205 Paris, France & School of Mathematics, Shandong University, Jinan, PRC

Email address: Sergei.Kuksin@imj-prg.fr

Alberto Maiocchi, Università degli Studi di Padova, Dipartimento di Matematica, Padova, Italy

Email address: alberto.maiocchi@unipd.it

Sergei Vlăduţ, Aix Marseille Université, CNRS, Centrale Marseille, 12M UMR 7373, 13453, Marseille, France and IITP RAS, 19 B. Karetnyi, Moscow, Russia

Email address: sergei.vladuts@univ-amu.fr