New methods in conformal partial wave analysis

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Abstract We report on progress concerning the partial wave analysis of higher correlation functions in conformal quantum field theory.

1 Introduction

Partial wave analysis (PWA) is a powerful tool in conformal quantum field theory. It gives not only information about the field content and the operator product expansion (OPE) of a model [8, 7], but can also be used for probing the positivity of the inner product induced by the correlation functions (Wightman positivity) [10].

Positivity is difficult to establish because it is a nonlinear property. It also necessarily involves correlation functions of any number of fields [10]. The most prominent example is the classification of central charges below 1 of the Virasoro algebra. An example in four spacetime dimensions (4D) is the result that conformal scalar fields with global conformal invariance (GCI, [13]) are necessarily Wick squares of free fields [11], and cannot couple in a nontrivial manner to other fields [2].

While conformal PWA for 4-point functions is well understood [6], we intend to develop methods for higher correlation functions. The basic task is to decompose a
correlation function of conformally covariant fields into a sum over partial waves
\[
(O, \phi_1(x_1) \cdots \phi_n(x_n) O) = \sum_{\lambda} \langle O, \phi_1(x_1) \cdots \phi_{k-1}(x_{k-1}) \Pi_{\lambda} \phi_k(x_k) \cdots \phi_n(x_n) O \rangle, \tag{1.1}
\]
where \(\Pi_{\lambda}\) is the projection to the subspace of the Hilbert space which carries the irreducible representation \(\lambda\) of the conformal group. A projection can be inserted in any position within the correlation, so that the \(n\)-point partial waves depend on \(n - 1\) representations, where the first and last projections are redundant because they are fixed by the first and the last field.

In principle, the non-vanishing partial waves give information about the contributions to the OPE of two or more fields [8]. Since a projection is a positive operator, each partial wave contribution of the form
\[
(O, \phi'(x_1) \cdots \phi(x_n) \Pi_{\lambda} \phi(x_{n+1}) \cdots \phi'(x_{2n}) O)
\]
must separately satisfy Wightman positivity (i.e., after smearing with test functions \(f(x_n, \ldots, x_1) f(x_{n+1}, \ldots, x_{2n})\) it must yield a non-negative number which is the norm square of the vector \(\Pi(\phi \otimes \cdots \otimes \phi')(f) O\). More generally, partial waves are subject to Cauchy-Schwartz type inequalities.

Now, partial waves are to a large extent determined by conformal symmetry, being solutions to eigenvalue equations for the Casimir operators of the conformal group. Therefore, the positivity requirement reduces to the positivity of a numerical coefficient, the partial wave amplitude, which multiplies a model-independent partial wave function [10].

Conformal PWA is by now mostly limited to 4-point functions, because the higher partial waves are not sufficiently well known. Even for 4 points, the determination of partial waves in 4D required a considerable effort [6]. Moreover, the decomposition of a given correlation function into a known system of partial waves may not be a straight-forward task without a suitable notion of orthogonality between the partial waves. Some progress was made in [10] giving a systematic expansion formula for scalar 4-point partial waves, and in [14] for a suitable notion of orthogonality.

In this note, we report some further intermediate progress. In Sect. 2 we present a power series representation Eq. (2.4) for general \(n\)-point partial waves in two space-time dimensions (2D) for all \(n\), extending known formulae for \(n \leq 4\). In 4D, however, such an expansion seems unrealistic because of the complicated structure of the higher-order Casimir operators which the partial waves must diagonalize, and because the partial waves are no longer unique.

In Sect. 3 we therefore present an alternative to the actual decomposition Eq. (1.1), which is applicable also in 4D. The idea is a successive reduction of \(n\)-point functions to \(n - 1\)-functions, in terms of local linear maps \(\phi_1(x_1) \phi_2(x_2) O \rightarrow \phi_2(x) O\) selecting each contribution to the OPE of the last two (or the first two) fields in the correlation. Our main result is the characterization of these linear maps as partial differential operators that intertwine the respective representations of the conformal group. This property is encoded in Eq. (3.4), which is subsequently solved. Acting
on the correlation functions, the intertwiners effectuate the desired reduction. As we shall see, this method is applicable only for representations of integer scaling dimension (otherwise, the differential operators would have to be replaced by integral kernels [5], and locality would become a nontrivial issue).

This method is therefore well-suited for QFT with global conformal invariance, where all correlation functions are rational functions [13]. We shall apply it in Sect. 4 to address the problem of positivity of a class of “exotic” higher \((n \geq 6)\) correlation structures of twist 2. The motivation is the following.

Twist-2 contributions in free field theories above the unitarity bound arise from quadratic Wick products such as: \(\phi^2(x_1)\phi(x_2)\) or \(x_{12\mu} \phi(x_1) \psi(x_2)\), in which each factor can be contracted “only once”, so that both variables can only have poles w.r.t. one other variable. In contrast, the exotic structures contain so-called double poles, thus indicating a nontrivial theory. These are strongly constrained by the conservation laws for twist-2 fields [11], allowing for a classification [4]. In particular, they cannot arise in correlations of less than six fields. While the exotic structures satisfy all linear properties, it remains an open problem whether they are compatible with positivity.

First steps of the positivity analysis of the simplest exotic structure will be reported in Sect. 4.

2 Higher chiral partial waves

Irreducible representations \(\lambda\) of the conformal group are eigenspaces of the Casimir operators. Thus, correlation functions with projections onto irreducible subrepresentations inserted:

\[
\langle \Omega, \phi_1(x_1)\Pi_{\lambda_1}\phi_2(x_2)\cdots\Pi_{\lambda_{i-1}}\phi_i(x_i)\Pi_{\lambda_i}\cdots\phi_{n-1}(x_{n-1})\Pi_{\lambda_{n-1}}\phi_n(x_n)\Omega \rangle \tag{2.1}
\]

are eigenvectors of the corresponding differential operators arising by commuting the conformal generators with the fields. Partial waves are, by definition, solutions to the same eigenvalue differential equations, with some standard normalization. These are “universal” in the sense that they are completely determined by conformal symmetry. They depend on the sequence of representations \((\mu_i)_{i=1,\ldots,n}\) of the fields \(\phi_i\) in Eq. (2.1), and on the sequence of representations \((\lambda_i)_{i=1,\ldots,n-1}\) of the projections, where \(\lambda_1 = \mu_1\) and \(\lambda_{n-1} = \mu_n\) are redundant.

The projected correlations Eq. (2.1) are multiples of the partial waves. The coefficients contain model-specific information, and Wightman positivity can be formulated as a system of numerical inequalities on the partial wave coefficients [10].

The conformal Lie algebra in 4D, \(so(4,2)\), has three Casimir operators (quadratic, cubic and quartic in the generators). In contrast, the conformal Lie algebra in 2D factorizes: \(so(2,2) \sim sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})\), and each \(sl(2,\mathbb{R})\) has one quadratic Casimir operator. For this reason, the Casimir eigenvalue differential equations are much simpler (both, to write down and to solve) in 2D.
The relevant positive-energy representations of \( sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R}) \) are parameterized by the chiral scaling dimensions \( d_\pm \), such that \( d_+ + d_- \) is the total scaling dimension, and \( d_+ - d_- \) the helicity.

Because of the chiral factorization of the conformal group, also the partial waves factorize. In the sequel, we display only chiral partial waves as functions of either \( x_+ = t + x \) or \( x_- = t - x \), and suppress the subscript. Thus, a general projected correlation function in 2D has the form of a product of two chiral functions

\[
\langle \Omega, \phi_1(x_1) \Pi_{a_1} \phi_2(x_2) \cdots \Pi_{a_{n-1}} \phi_n(x_n) \rangle \tag{2.2}
\]

where the chiral fields \( \phi_i \) have chiral dimensions \( d_i \), and \( \Pi_a \) are the projections onto the chiral representations with chiral scaling dimension \( a \). In particular, \( a_1 = d_1 \) and \( a_{n-1} = d_n \) are fixed.

The Casimir eigenvalue equation for the projector insertion \( \Pi_i \) reads

\[
\left( \sum_{i<j<k} x_{jk}^2 \partial_j \partial_k + 2 \sum_{i<j,k} d_j(x_{jk} \partial_k) + \sum_{i<k} d_k - \left( \sum_{i<k} d_k \right)^2 \right) \langle \ldots \Pi_i \phi_{i+1}(x_{i+1}) \ldots \rangle = (a_i - a_i^2) \langle \ldots \Pi_i \phi_{i+1}(x_{i+1}) \ldots \rangle,
\]

which is equivalent by conformal invariance to

\[
\left( \sum_{j<k \leq i} x_{jk}^2 \partial_j \partial_k + 2 \sum_{j \leq k \leq i} d_j(x_{jk} \partial_k) + \sum_{k \leq i} d_k - \left( \sum_{k \leq i} d_k \right)^2 \right) \langle \ldots \phi_i(x_i) \Pi_i \ldots \rangle = (a_i - a_i^2) \langle \ldots \phi_i(x_i) \Pi_i \ldots \rangle.
\]

In principle, these equations can be reformulated in terms of \( n - 3 \) independent conformal cross ratios. It turns out convenient to choose

\[
u_k = \frac{\Lambda_{k+1}}{\Lambda_{k+3}} - 1.
\]

We have worked out the invariant differential equations for \( n \leq 6 \) points: Let

\[
(2.2) = \frac{f(u_1, u_2, u_3)}{u_1^{d_1 + d_2 - d_3} u_2^{d_1 + d_3 - d_2} u_3^{d_2 + d_3 - d_1} u_4^{d_4 + d_5 + d_6 - d_1 - d_2 - d_3} u_5^{d_4 + d_5 + d_6 - d_2 - d_3} u_6^{d_4 + d_5 + d_6 - d_3 - d_1}}.
\]

Then (with the Euler operators \( E_i = u_i \partial_{u_i} \))

\[
(E_1 + d_3 - a_2)(E_1 + d_3 + a_2 - 1)f = u_1(E_1 + E_2)(E_1 + d_1 - d_2 + d_3)f,
\]

\[
(E_2 - a_3)(E_2 + a_3 - 1)f = u_2(E_2 + E_1)(E_2 + E_3)f,
\]

\[
(E_3 + a_4)(E_3 + d_4 + a_4 - 1)f = u_3(E_3 + E_2)(E_3 + d_6 - d_5 + d_4)f.
\]  

(The cases \( n < 6 \) are covered by admitting the trivial field \( 1 \) of dimension 0.) This system is obviously symmetric under hermitean conjugation \( 1, 2, \ldots, 6 \to 6, 5, \ldots, 1 \). It can be recursively solved as a power series with leading powers \( u_1^{d_2 - d_3} u_2^{d_3 - d_2} u_3^{d_4 - d_5} \).
From the solution with \( n \leq 6 \), we have extrapolated the general power series expansion for all \( n \), as follows. By default, we put \( a_0 = a_n := 0 \), and \( \ell_0 = \ell_{n-2} := 0 \).

**Proposition 1:** The general chiral \( n \)-point partial wave is

\[
\sum_{\ell_1, \ldots, \ell_{n-3} \geq 0} \frac{\prod_{j=1}^{n-2} x_{j+1}^{d_{j+1}-a_{j+1} - a_j + 1} (a_j + a_{j+1} - d_{j+1} + 1) \ell_{j+1} - \ell_j}{\prod_{i=1}^{n-4} x_{i+1}^{d_i + d_{i+1} - a_{i+1} - a_i + 1}} \frac{n-3}{\ell_k} \frac{u_k}{(2k+1)\ell_k}. \tag{2.4}
\]

This formula has a remarkable “short-range” feature: It involves only coordinate distances \( x_{ij} \) with \( j = i + 1 \) or \( i + 2 \). The powers of \( x_{ii+1} \) and \( x_{ii+2} \) depend only on the dimensions of the fields \( \phi_i, \phi_{i+1} \), respectively \( \phi_{i+1} \), and their adjacent projections, apart from the summation indices \( \ell \). The same is true for the numerical coefficients.

For \( n = 3 \) points, this is just the 3-point function. For \( n = 4, 5, 6 \) points, we have derived this formula by solving the differential equations Eq. (2.3) for the Casimir eigenvalues. For \( n = 4 \), the sum is a hypergeometric series, and Eq. (2.4) coincides with well-known formulas.

One way to prove the Eq. (2.4) for all \( n \) is an application of the method discussed in the next section. There, we introduce “intertwining” differential operators \( t \circ \hat{E}_h \) with the distinguishing property that they annihilate all partial waves carrying the “wrong” representation \( a \neq h \), and reduce the \( n \)-point partial wave carrying the representation \( a = h \) to an \((n-1)\)-point partial wave with the first pair of fields replaced by \( \phi_0 \) of dimension \( h \).

Therefore, it is sufficient to show that this is true for our “candidate” partial waves Eq. (2.3). With Eq. (3.18), we have to apply the differential operator

\[
\hat{E}_h \equiv E_h \circ x_1^{d_1 + d_2} = \left( \sum_{p+q = h} \frac{(q-b)^p (p+b)^q}{p! q!} \partial_1^p (-\partial_2)^q \right) \circ x_1^{d_1 + d_2},
\]

where \( b = d_1 - d_2 \), to Eq. (2.4), and then equate \( x_1 = x_2 \). The result must be \( \delta_{n,2} \) times the reduced partial wave.

To do this, we have to exhibit all terms that involve \( x_1 \) or \( x_2 \). Eq. (2.4) can be arranged as \( x_1^{d_1 - d_2} \) times the sum \( \sum_{\ell_2, \ldots, \ell_{n-3}} \) over

\[
\left( \begin{array}{c} x_{12} \vspace{1em} \\ x_{13} x_{24} \end{array} \right) \left( \begin{array}{c} x_{23} \vspace{1em} \\ x_{13} x_{24} \end{array} \right) \left( \begin{array}{c} x_{23} \vspace{1em} \\ x_{13} x_{24} \end{array} \right) \sum_{\ell \geq 0} \frac{(a+b)\ell (a+c)\ell^u}{\ell!(2\ell)!} \times \text{remaining factors}, \tag{2.5}
\]

where \( a = a_1, b = d_1 - d_2, c = a_3 - d_3 + \ell_2 \). Notice that for each \( \ell_2 \), the sum over \( \ell \) is a 4-point partial wave where the 4\textsuperscript{th} field has dimension \( a_3 + \ell_3 \). Thus, knowing that Eq. (2.4) correctly reproduces the 4-point partial waves, and that \( t \circ \hat{E}_h \) reduces 4-point partial waves to 3-point partial waves, the same must be true for the higher partial waves.

However, we have not been able to evaluate the result of \( E_0^{d_1, d_2} \) on the power series Eq. (2.5), and verify this indirect conclusion by a direct computation. Only for \( n = 3 \) this can be done by the following argument. For \( n = 3 \), one has \( c = 0 \) in Eq. (2.5), only \( \ell = 0 \) contributes, and there are no “remaining factors”. Then
(i) Because $E_h^{d_1,d_2}$ is a differential operator of order $h$, it annihilates the 3-point function whenever $h < a$, due to the surviving factors of $x_{12}$.

(ii) Writing $\frac{x_{12}^{a}}{x_{13}x_{23}} = \frac{1}{x_{23}} - \frac{1}{x_{13}}$ and performing the binomial expansion of its powers, $E_h^{d_1,d_2}$ can easily be applied. It is then seen by inspection that the resulting series is symmetric under the exchange $a \leftrightarrow h$. Therefore, it also vanishes whenever $h > a$.

(iii) When $h = a$, all derivatives must hit the factor $x_{12}^a$. That the result is the 2-point function, is then obvious.

For $n > 3$, the Leibniz rule produces multiples sums which are not easy to handle. But a trick helps: The sum in Eq. (2.5) equals $\,^2F_1(a+b,a+c;2a;u)$. We then use the identity

$$\frac{x_{34}^{2a-1}}{(x_{13}x_{24})^a} \left( \frac{x_{23}}{x_{13}} \right)^a \left( \frac{x_{23}}{x_{24}} \right)^c \,^2F_1(a+b,a+c;2a;u) = \frac{\Gamma(2a)}{\Gamma(a+c)\Gamma(a-c)} \int_{x_3}^{x_4} dx (x_1-x)^{-a+b}(x_2-x)^{-a+c}(x_3-x)^{a+c-1}(x-x_4)^{a-c-1},$$

which can be established by direct computation: namely, the change of variables $t = \frac{x_3(x_4-x)}{x_{34}(x_1-x)}$ yields precisely the standard integral representation of the hypergeometric function.

Therefore, each term Eq. (2.5) is, as far as its dependence on $x_1$ and $x_2$ is concerned, an integral over a 3-point function. Thus, we only have to evaluate $E_h^{d_1,d_2}$ on a 3-point function, which can be done as before. The remaining integral is again of the hypergeometric type (after the change of variables $t = -\frac{x_3-x}{x_3 x_{12}}$), and reproduces precisely the necessary "leading" factors for the $(n-1)$-point partial wave Eq. (2.4).

From this, we conclude that Eq. (2.4) indeed is the correct power series expansion of general $n$-point chiral partial waves.

### 3 Intertwining differential operators

Let $\phi_1$ and $\phi_2$ be two conformal fields transforming in representations $\mu_1$ and $\mu_2$.

We shall determine differential operators $\tilde{E}_\lambda$ w.r.t. $x_1$ and $x_2$ such that

$$\phi_\lambda(x) := t_x \circ \tilde{E}_\lambda \phi_1(x_1)\phi_2(x_2)$$

transforms like a conformal field in the representation $\lambda$. Here, $t_x$ is the evaluation map $t_x(f) = f(x_1,x_2)|_{x_1=x_2=x} = f(x,x)$.

It will become clear below that such operators exist only when the scaling dimensions satisfy $d_\lambda - d_1 - d_2 \in \mathbb{Z}$. They can therefore be expected to be exhaustive (w.r.t. $\lambda$) only in a globally conformal invariant (GCI) theory.

Such operators have been presented previously [9, Sect. VI.B] for the special case of $\phi_1$ and $\phi_2$ being two (complex conjugate) canonical scalar massless Klein-Gordon fields of dimension 1, in order to extract the current, the stress-energy tensor and
higher conserved symmetric traceless tensor fields from \( \phi^* \phi \). The same operators actually can be used also for scalar biharmonic bifields \( V(x_1, x_2) \) which collect the twist-2 contribution in any product of two scalar fields of equal dimension \[11\], where biharmonicity, i.e., the wave equation w.r.t. both arguments is exploited in an essential way. We shall reproduce these operators, but there will be additional terms including the wave operators, so that Eq. (3.1) is true without using the equation of motion, or biharmonicity.

By conformal covariance, the assumed transformation behaviour of Eq. (5.1) implies

\[
t_v \circ \tilde{E}_\lambda (\phi_\mu (y) \Omega, \phi_1 (x_1) \phi_2 (x_2) \Omega) = \delta_\lambda,\mu (\phi_\mu (y) \Omega, \phi_\lambda (x) \Omega),
\]

i.e., the operator annihilates all 3-point functions with fields in the “wrong” representation. In particular, if applied to the vacuum operator product expansion \[8\]

\[
\phi_1 (x_1) \phi_2 (x_2) \Omega = \sum_\lambda \int dx K^{\mu_1 \mu_2}_\mu (x_1, x_2; x) \phi_\mu (x) \Omega,
\]

where \( K^{\mu_1 \mu_2}_\mu \) are certain integral kernels, it will annihilate all contributions \( \mu \neq \lambda \), and if applied to a correlation function, it will annihilate all partial waves with \( \mu \neq \lambda \) in the 1-2-channel, and reduce the contribution with \( \mu = \lambda \) to an \( n - 1 \)-point partial wave. Thanks to the latter feature, one can perform a partial wave analysis without actually knowing the partial waves, cf. Sect. \[4\].

Let us now proceed to determine the differential operators.

For definiteness, we specialize to \( \mu_1 = \mu_2 \) to be scalar representations of dimension \( d_1 = d_2 = d \). In this case, only symmetric traceless tensor representations \( \lambda \) can occur \[8\]. It is convenient to write \( \lambda = (\kappa, L) \) where \( L \) is the tensor rank, and 2\( \kappa \) the “twist”, such that the scaling dimension is \( d = 2\kappa + L \). The unitarity bound requires \( \kappa \geq 0 \) for \( L = 0 \), and \( \kappa \geq 1 \) for \( L > 0 \). We write a symmetric traceless tensor as \( T(v) = T^{\mu_1 \ldots \mu_L} v_{\mu_1} \ldots v_{\mu_L} \) which is a homogeneous polynomial of degree \( L \) in the polarization vector \( v \). Tracelessness is equivalent to the harmonic equation \( \Box T(v) = 0 \). Eq. (3.1) implies that \( \tilde{E}_{\kappa L} \) is a harmonic homogeneous polynomial of degree \( L \) in the polarization vector \( v \). The harmonic part of any polynomial in \( v \) is uniquely determined \[3\], so it is sufficient to know \( \tilde{E}_{\kappa L} \) up to terms involving \( v^2 \).

Let \( T = P_\mu, D, M_{\mu \nu}, K_\mu \) be the generators of translations, dilations, Lorentz and special conformal transformations, respectively, and

\[
i[T, \phi (x)] = t^\kappa_\lambda \phi (x)
\]

the commutation relations with covariant (“quasiprimary”) fields, where \( t^{\kappa L} = \partial \) for the translations, \( = (x^\partial + d_\lambda) \) for the scale transformations, \( = x \wedge \partial + v \wedge \partial_v \) for the Lorentz transformations, and \( = 2x(x^\partial) - x^2 \partial + 2(v(x^\partial_v) - (xv) \partial_v) + 2d_\lambda x \) for the special conformal transformations. For the tensor representations, \( d_\lambda = 2\kappa + L \), while for the scalar representations \( \mu_1 = \mu_2 \) the \( v \)-terms are absent and \( d_\mu = d \).
Commuting the generators with Eq. (3.1), the assumption that \( \phi_2 \) transforms in the representation \( \lambda \) is equivalent to the intertwining relations

\[
t \circ \hat{E}_\lambda \circ (t_{\lambda_1} + t_{\lambda_2}) = t_{\lambda} \circ \hat{E}_\lambda.
\]

In the case at hand, we make an ansatz

\[
\hat{E}_\lambda = E_{KL}(x, \partial, v) \circ (x_{12}^2)^d.
\]

Notice that by virtue of the pole bounds \([13]\), any correlation function of \( \phi_1(x_1) \phi_2(x_2) \) is not more singular than \( (x_{12}^2)^{-d} \), so that the differential operators \( E_{KL} \) act on a regular function, and the subsequent evaluation \( t_\lambda \) is possible (provided \( E_{KL} \) is regular).

Next, we evaluate the intertwining relations Eq. (3.2). They tell us in turn:

Translations: \((\partial_1 + \partial_2)E_{KL} = 0\). Thus the differential operators do not involve the coordinate \( x_1 + x_2 \). Since \( E_{KL} \) is followed by the evaluation map \( t_\lambda \), we may also assume that it does not involve the difference coordinate \( x_1 - x_2 \), hence \( E_{KL} \) involves only derivatives and the polarization vector \( v \). Let us denote by \( \nabla_i \) the derivatives with respect to the “variables” \( \partial_i \) of \( E_{KL}(\partial_1, \partial_2, v) \).

Scale transformations: \((\partial_1 \nabla_1 + \partial_2 \nabla_2)E_{KL} = (2\kappa + L)E_{KL}\). Thus, \( E_{KL} \) is homogeneous of degree \( 2\kappa + L \) in the derivatives \( \partial_i \).

Lorentz transformations: \((\partial_1 \wedge \nabla_1 + \partial_2 \wedge \nabla_2 + v \wedge \partial_i)E_{KL} = 0\). Thus, \( E_{KL} \) is a Lorentz scalar. It is therefore a function of \((\partial_1 \partial_2), (v \partial_1)\) and \( v^2 \). Together with the known homogeneities in \( v \) and in \( \partial_i \), it can be a polynomial in the derivatives only if \( \kappa \) is an integer. This is in perfect agreement with GCI because tensor-scalar-scalar 3-point functions are rational only if the twist \( 2\kappa \) is even.

Special conformal transformations: While the previous intertwining conditions gave information about the gross structure of \( E_{KL} \), the special conformal transformations yield a differential equation that specifies the operators completely.

**Proposition 2:** Given the previous specifications of \( E_{KL}(\partial_1, \partial_2, v) \) in Eq. (3.2) as homogeneous polynomials (of degrees depending on the parameters \( \kappa \) and \( L \)), the intertwining condition Eq. (3.2) is equivalent to

\[
(2(\partial_1 \nabla_1)\nabla_1 - \partial_1 \nabla_1^2 + 2(\partial_2 \nabla_2)\nabla_2 - \partial_2 \nabla_2^2)E_{KL}(\partial_1, \partial_2, v) = 0.
\]

One may directly solve these equations with a polynomial ansatz for \( E_{KL} \) with the specified homogeneities. A more systematic way is to write

\[
E_{KL}(\partial_1, \partial_2, v) = (\partial_1 \partial_2)^\kappa \cdot \left[ (v \partial_1) + (v \partial_2) \right]^L \cdot e_{KL}(p, q, r)|_0
\]

where \( p = \frac{\partial_1^2}{\partial_1 \partial_2}, q = \frac{\partial_2^2}{\partial_1 \partial_2}, \) and \( r = \frac{(v \partial_1) - (v \partial_2)}{|v|} \). Clearly, \( e_{KL} \) must be a polynomial of degree at most \( L \) in \( r \) and degree of at most \( \kappa \) in \( p \) and \( q \). The notation \( |P(v)|_0 \) stands for the harmonic part of the polynomial \( P(v) \). The variable \( v^2 \) does not appear explicitly, because the harmonic part \( [v^2Q(v)]_0 = 0 \) for any polynomial \( Q \).
With this ansatz, the differential equation Eq. (3.6) turns into the system of three PDE for \( e_{KL}(p, q, r) \):

\[
(L(L-1) + (1-r^2)\partial_r^2 + 2\kappa(L-r\partial_r) + 2(p\partial_p - q\partial_q)\partial_r) e_{KL} = 0,
\]

\[
[4(p\partial_p - 1)\partial_p - q(\kappa - p\partial_p - q\partial_q)(\kappa - 1 - p\partial_p + q\partial_q) + 2(\kappa - p\partial_p - q\partial_q)(\kappa - 1 - p\partial_p + q\partial_q) + (r-1)\partial_r)] e_{KL} = 0,
\]

\[
[4(q\partial_q - 1)\partial_q - p(\kappa - p\partial_p - q\partial_q)(\kappa - 1 - p\partial_p + q\partial_q) + 2(\kappa - p\partial_p - q\partial_q)(\kappa - 1 + p\partial_p - q\partial_q) + (r+1)\partial_r)] e_{KL} = 0.
\]

One may repeat the same strategy in 2D. In this case, the intertwining operators factorize into two chiral operators, labelled by the chiral dimensions \( h_\pm \). These are polynomial functions in the chiral (one-dimensional) partial derivatives \( \partial_1 \) and \( \partial_2 \). Following the same line of arguments as in 4D, one finds the chiral intertwining condition

\[
(\partial_1 \nabla_1^2 + \partial_2 \nabla_2^2) E_h(\partial_1, \partial_2) = 0,
\]

where \( E_h(\partial_1, \partial_2) \) is a homogeneous polynomial of degree \( h \). Writing \( E_h = (\partial_1 + \partial_2)^h \cdot e_h(\frac{\partial_1}{\partial_1 + \partial_2}) \), this reduces to the differential equation for \( e_h(r) \)

\[
(h(h-1) + (1-r^2)\partial_r^2) e_h(r) = 0,
\]

which is exactly the same as the case \( \kappa = 0, L = h \) of Eq. (3.6).

Notice that in 4D, representations \( (0, L) \) with \( L \neq 0 \) are below the unitarity bound. Such representations must not contribute to a correlation function. Thus, any admissible correlation function must be annihilated by the operators \( t \circ \hat{E}_{0L} \). The solution for \( \kappa = 0 \) is

\[
e_{0L}(r) = (1-r^2)\partial_r P_{L-1}(r),
\]

where \( P_n \) are the Legendre polynomials. Using Eq. (3.5), this gives

\[
E_{0L}(\partial_1, \partial_2, v) = \sum_{p+q=L} \frac{(q)_p (p)_q}{p! q!} \left[ (v\partial_1)^p (-v\partial_2)^q \right]_0,
\]

or (in the chiral case)

\[
E_h(\partial_1, \partial_2) = \sum_{p+q=h} \frac{(q)_p (p)_q}{p! q!} \partial_1^p (-\partial_2)^q.
\]

For \( \kappa > 0 \), we may expand

\[
e_{KL}(p, q, r) = \sum_{m,n \geq 0, m+n \leq \kappa} p^m q^n e_{KL,mn}(r).
\]

Then Eq. (3.6) must hold for each term \( p^m q^n e_{KL,mn}(r) \) separately, giving

\[
((1-r^2)\partial_r^2 - 2\kappa r\partial_r + 2(m-n)\partial_r + L(L+2-1)) e_{KL,mn}(r) = 0.
\]
This equation involves only the difference $m - n =: \delta$. It is solved by polynomials of degree $L$ with the symmetry $f_{KL,\delta}(r) = (-1)^{L}f_{KL,-\delta}(-r)$:

$$f_{KL,\delta}(r) = (\kappa - \delta)L \cdot 2F_{1}
\left(-L,L + 2\kappa - 1;\kappa - \delta;\frac{1-r}{2}\right).$$

(3.12)

Thus, to solve Eq. (3.11) it remains to determine only the coefficients in

$$e_{KL,mn}(r) = c_{KL,mn} \cdot f_{KL,m-n}(r).$$

(3.13)

Indeed, the remaining Eq. (3.7) and Eq. (3.8) turn into the recursive system

$$4(m^{2} - 1)c_{KL,m+1,n} + 2(\kappa - m - n)(L + \kappa - 1 - m + n)c_{KL,m,n}
- (\kappa - m - n)(\kappa - m - n + 1)c_{KL,m,n-1} = 0,$$

(3.14)

$$4(n^{2} - 1)c_{KL,m,n+1} + 2(\kappa - m - n)(L + \kappa - 1 + m - n)c_{KL,m,n}
- (\kappa - m - n)(\kappa - m - n + 1)c_{KL,m-1,n} = 0.$$  

(3.15)

Here, we have used the fact ([1] Eqs. 15.2.14 and 15.2.16) that the differential operators

$$A_{KL,\delta}^{\pm} := \frac{(r \mp 1)\partial \mp \kappa - 1 \mp \delta}{L + \kappa - 1 \mp \delta}$$

act as raising and lowering operators for the parameter $\delta$:

$$A_{KL,\delta}^{\pm} f_{KL,\delta} = f_{KL,\delta \pm 1}.$$  

(3.16)

We conclude:

**Proposition 3:** The intertwining differential operators in Eq. (3.7) are given by

$$\hat{E}_{KL} = \sum_{m+n \leq \kappa} c_{KL,mn}(\partial_{1} \partial_{2})^{\kappa-m-n} \square_{1}^{m} \square_{2}^{n} \left[(v \partial_{1} + v \partial_{2})^{L}f_{KL,m-n}((v \partial_{1} - v \partial_{2})^{L}f_{KL,m-n}(v \partial_{1} + v \partial_{2}))^{L} \circ (x_{12}^{2})^{d} \right]$$

where $[...]_{0}$ stands for the harmonic part with respect to $v \in \mathbb{R}^{1,3}$, the polynomials $f_{KL,m-n}$ are given by Eq. (3.12), and the coefficients $c_{KL,mn}$ solve the recursion Eq. (3.7, 3.8).

It may be interesting to note that $f_{KL,0}$ are multiples of derivatives of Legendre polynomials (cf. [1] Eqs. 15.2.2, 15.4.4):

$$f_{KL,0}(r) = \frac{2^{\kappa-1}L_{L}}{(L + \kappa)_{\kappa-1}} \cdot \partial_{r}^{\kappa-1}P_{L+\kappa-1}(r).$$

(3.17)

so that, by Eq. (3.16), all functions $f_{KL,mn}(r)$ are derivatives of the Legendre polynomials $P_{L+\kappa-1}(r)$. E.g., for twist $2$ ($\kappa = 1$), we have

$$e_{1L}(p,q,r) = \left(1 + \frac{p}{2}(r - 1)\partial_{r} + \frac{q}{2}(1 + r)\partial_{r}\right)P_{L}(r).$$
The next task is to relax the assumption $\mu_1 = \mu_2 = \text{scalar}$, and to find and solve the analogue of Eq. (3.4) in the general case. This will be necessary in order to compute the contributions from all insertions of projectors as in Eq. (1.1) by successive reduction according to Eq. (3.1).

For two scalar fields of different dimensions, $d_1 \neq d_2$, the ansatz $\hat{E}_\lambda = E_{\kappa L} \circ (x_{12}^2)^{d_1 + d_2}/2$ is solved by a scalar polynomial $E_{\kappa L}(\partial_1, \partial_2, \nu)$, homogeneous of degree $2\kappa + L$ in $\partial_i$, homogeneous of degree $L$ and harmonic in $\nu$, as before, but now satisfying the differential equation

$$(2(\partial_1 \nabla_1) \nabla_1 - \partial_1 \nabla_1^2 + 2(\partial_2 \nabla_2) \nabla_2 - \partial_2 \nabla_2^2 + (d_1 - d_2)(\nabla_1 - \nabla_2))E_{\kappa L}(\partial_1, \partial_2, \nu) = 0.$$ 

Note that the homogeneity conditions require that $\kappa$ is an integer, and that in a GCI theory, fields with even twist $2\kappa$ can arise in the OPE only if $d_1 - d_2$ is even. One would therefore have to modify the ansatz when $d_1 - d_2$ is odd.

Similarly, in the chiral case, the ansatz $\hat{E}_{d_1, d_2} = E_{d_1, d_2} \circ (x_{12}^2)^{d_1 + d_2}$ implies that $E_{d_1, d_2}$ is homogeneous of degree $h$ in $\partial_i$ and satisfies the differential equation

$$(\partial_1 \nabla_1^2 + \partial_2 \nabla_2^2 + (d_1 - d_2)(\nabla_1 - \nabla_2))E_{d_1, d_2}(\partial_1, \partial_2) = 0.$$ 

This is solved by

$$E_{d_1, d_2}(\partial_1, \partial_2) = \sum_{p+q=h} \frac{(q-d_1 + d_2)p (p+d_1 - d_2)q}{p! q!} \partial_1^p (-\partial_2)^q. \quad (3.18)$$

### 4 Application: Test of positivity of a 6-point structure

Recall the positivity problem for the exotic scalar 6-point structures addressed in the introduction. We consider here only the simplest example of such a structure, which has double poles and is consistent with the constraints due to the requirement that the OPE in both the first and last pair of fields starts with twist 2 [11]. More general double pole structures have been classified in [4].

In [11], the leading part of this structure was displayed. In [12], its “tetraharmonic completion” (i.e., the biharmonic completion in both pairs of variables $x_1, x_2$ and $x_5, x_6$) was presented in terms of a transcendental function $g(s, t)$. The tetraharmonic completion is precisely the twist-2 part in both channels. Unfortunately, however, due to a wrong resummation factor, this function $g(s, t)$ was incorrectly computed in [12]. We shall display the correct function below.

The leading part of the exotic structure for four scalar fields $\phi_1, \phi_2, \phi_5, \phi_6$ of dimension $d$ and two scalar fields $\phi_3, \phi_4$ of dimension $d'$ is given by

$$E(x_1, \ldots, x_6) = \frac{(x_{13}^2 x_{36}^2 x_{54}^2 - 2x_{13}^2 x_{32}^2 x_{46}^2 - 2x_{14}^2 x_{34}^2 x_{56}^2)}{(x_{12}^2)^{d-1} \cdot x_{13}^2 x_{34}^2 x_{23}^2 x_{24}^2 \cdot (x_{34}^2)^{d-3} \cdot x_{35}^2 x_{36}^2 x_{45}^2 x_{46}^2 \cdot (x_{56}^2)^{d-4}}, \quad (4.1)$$
where \((\cdot)_{[k,l]}\) stands for antisymmetrization. Without loss of generality, we choose \(d = d' = 3\). For comparison, we also introduce the following 6-point structure with the same symmetries as \(E\), but which has no double poles and appears as part of the 6-point function of six cubic Wick products of a complex massless scalar free field:

\[
B(x_1, \ldots, x_6) = \frac{1}{(x_1^2)^2} \cdot \left( \frac{1}{x_{14} x_{23}^2} \right)_{[1,2]} \cdot \frac{1}{x_{34}^2} \cdot \left( \frac{1}{x_{36} x_{45}^2} \right)_{[5,6]} \cdot \frac{1}{(x_5^2)^2}.
\]

The structure \(B\) is separately biharmonic in both the 1-2 and 5-6 channels. It turns out that the tetraharmonic completion \(H\) of \(B - \frac{E}{2}\) can be written more compactly than that of \(E\) given in [12], namely

\[
H(x_1, \ldots, x_6) = \left( B - \frac{E}{2} \right) \cdot g(s,t)g(s',t'),
\]

where \(s = \frac{x_{12}^2}{x_{34}^2}, t = \frac{x_{13}^2}{x_{34} x_{45}}, \) and \(s' = \frac{x_{23}^2}{x_{35}^2}, t' = \frac{x_{23} x_{45}}{x_{35} x_{56}}\). The condition of biharmonicity amounts to the differential equation [12]

\[
\left[ (1 - t \partial_t)(1 + t \partial_t + s \partial_s) - ((1 - t \partial_t) + t(2 + t \partial_t + s \partial_s)) \partial_t \right] g = 0
\]

for the function \(g(s,t)\). The expansion in a power series in \(s, \ g(s,t) = \sum_n \frac{a_n}{\pi^2} g_n(t)\), gives the recursion \((1 + (n + 1)t - t(1 - \partial_t)g_{n-1})(1 - t \partial_t)g_n = 0\) with \(g_0(t) = 1\). This can be solved in terms of hypergeometric functions, giving

\[
g(s,t) = \sum_n \frac{n! (n+1)!(n+1)^{n+1}}{(2n+2)!} \cdot {}_2 \!F_1(n,n+1;2n+2;1-t).
\]

The sum can be performed when the integral representation of the hypergeometric functions [1] Eq. 15.3.1] is inserted, and \(s, t\) are expressed in terms of the \textit{“chirality variables”} \(u_\pm\) such that \(s = u_+ u_-\) and \(t = (1 - u_+)(1 - u_-)\). Then

\[
g(s,t) = \sum_n (n+1) s^n \int_0^1 dx x^n (1-x)^n \left( 1 - (1-t)x \right)^{-n}
\]

\[
= \int_0^1 dx \left[ 1 - (u_+ + u_+ u_-) x \right] \left( 1 - (1-t)x \right)^{-2}
\]

\[
= 1 + 2u_+ u_- \cdot \frac{(1-u_+)(1-u_-) \cdot \log \frac{1-u_+ u_-}{1-u_+ u_-} + u_+ - u_- - \frac{1}{2}u_+^2 + \frac{1}{2}u_-^2}{(u_+ - u_-)^3}
\]

\[
= 1 + \sum_{a,b \geq 0} \frac{2ab}{a+b} \cdot \frac{(a+b)(a+b)^2-1}{(a+b)^3} u_+^a u_-^b
\]

\[
= \left( \frac{1-u_+}{u_+ - u_-} \right) \cdot \sum_{a,b \geq 0} \frac{a-b}{a+b} \cdot u_+^a u_-^b.
\]

(In the first line, we corrected a wrong factor of \(n!\), whose presence in [12] and spoiled the subsequent expressions.)
Because the twist-2 part is obtained by inserting projections, it must separately satisfy Wightman positivity. Of course, we would like to apply the twist-2 intertwiners $E_{1L}$ of Sect. 3 in both channels, so that the issue reduces to the positivity of tensor-scalar-scalar-tensor 4-point functions. Applying successively the unknown intertwiners for the resulting tensor-scalar channels, the problem would be reduced to the positivity of the resulting 2-point function, i.e., to the positivity of the numerical amplitude.

Since we know the intertwiners $E_{1L}$, the first step can in principle be done. Notice that it is sufficient to act on the leading part, because it differs from the twist-2 part by contributions of higher twist, that are annihilated by $E_{1L}$. Notice also that $B$ has the form of a product of two 4-point functions in the variables $x_1, x_2, x_3, x_4$ and in the variables $x_3, x_4, x_5, x_6$. Therefore, the application of the intertwining differential operators in the 1-2 channel and in the 5-6 channel also factorizes. The same, however, is not true for $E$.

Thus, even the first step at present seems to be too involved to be carried out in practice. The second step is at present not possible because we have not yet determined the tensor-scalar intertwiners.

For this reason, we decided to perform only a weaker test of positivity. Namely, we restrict the twist-2 structure to 2D, by setting two spatial coordinates to 0. Since this essentially amounts to a smaller class of test functions, Wightman positivity must still be preserved; but notice that 2D positivity after the restriction is necessary but not sufficient to ensure positivity in 4D.

The intertwining operators in 2D are at our disposal Eq. (3.10), and we have computed all coefficients (see below). It turns out that the partial wave amplitudes of the restricted exotic twist-2 structure $B - \frac{1}{2} E$ differ from those of the non-exotic structure $B$ only by certain signs. This means that $E$ has the same partial wave amplitudes as $4B$, except that some of them are absent.

The non-exotic structure $B$ may itself be indefinite, but we know that it occurs in a free-field model, and therefore can be dominated by other positive free-field structures, because free fields are manifestly positive. This seems to indicate that the restricted exotic structure as well can be dominated by positive free-field structures. Thus positivity at the 6-point level alone would not forbid the appearance of this structure as part of a 6-point correlation function.

Let us indicate some details of the actual computations.

Upon restriction to 2D, $u_+$ and $u_-$ turn into the chiral cross ratios $u = \frac{x_1 x_{14}}{x_{12} x_{24}}$. Moreover, the function $B - \frac{1}{2} E$ drastically simplifies:

$$B - \frac{1}{2} E = \frac{1}{2D} \left( \frac{1}{(x_{12}^2)^2 x_{13}^2 x_{24}^2 x_{34}^2 x_{35}^2 x_{46}^2 (x_{56}^2)^2} \right) \cdot \frac{(u_+ - u_-)}{(1 - u_+)(1 - u_-)} \cdot \frac{(u'_+ - u'_-)}{(1 - u'_+)(1 - u'_-)}.$$  

After multiplication with $g(s, t)g(s', t')$, using Eq. (4.4), we have

$$H = \frac{1}{2D} \left( \frac{1}{(x_{12}^2)^2 x_{13}^2 x_{24}^2 x_{34}^2 x_{35}^2 x_{46}^2 (x_{56}^2)^2} \right) \cdot \sum_{a,b \geq 0, a+b \geq 0} \frac{a-b}{a+b} u^a u^b \cdot \sum_{a,b \geq 0, a+b \geq 0} \frac{a-b}{a+b} u'^a u'^b.$$  


For the non-exotic structure $B$, one has instead

$$B = \frac{2D}{(x_{12}^2)^2} \chi_{12}^4 \sum_{a,b} a_u^b u^b_{+} u^b_{-}.$$  

Because the sums factorize, the evaluations of the chiral intertwining differential operators $t \circ E_{hl} (\partial_k, \partial_l) \circ (x_{kl}^*)^d$ in the 1-2 channel ($k, l = 1, 2$) and in the 5-6 channel, with $E_{hl}$ given by Eq. (3.10), completely decouple. Actually, because all structures of interest are of order $x_{12}^a$, and therefore only chiral dimensions $h \geq 1$ will occur, we found it more efficient to work with chiral intertwining operators $t \circ D_{hl} \circ (x_{kl}^*)^d$ where $D_{hl}(\partial_k, \partial_l) = \nabla_k - \nabla_l) E_{hl}(\partial_k, \partial_l),$ and adopt a normalization different from Eq. (3.10):

$$D_{hl}(\partial_1, \partial_2) = \frac{1}{(h-1)!} \sum_{p+q=h-1} \frac{\partial^p (\partial^q)}{p! q!}.$$  

Thus, we apply $t \circ D_{hl} \circ (x_{12}^2)^2 = t \circ \left[ D_{hl} \circ (x_{12}^2)^2 \otimes D_{hl} \circ (x_{12}^2)^2 \right].$ We find

$$t \circ D_{hl} \left[ \frac{1}{x_{13} x_{24}^2} \sum_{a,b} a_u^b u^b_{+} u^b_{-} \right] = \chi_{12}^4 \cdot t \circ D_{hl} \left[ \frac{x_{12}^2}{x_{13} x_{24}}^{a+1} \right] = (-1)^{h-1} c_{a,h} \cdot \chi_{34}^{h-1} \left( \frac{x_{13}^h}{x-x_3} \right) \left( \frac{x_{24}^h}{x-x_4} \right)$$

where $c_{a,h} = \frac{(h)_{a}(1-h)_a}{a!}$. Multiplying the two chiral factors and performing the sum over $a$ and $b$ gives for the structure $B$

$$t \circ D_{hl} \left[ \frac{1}{x_{13} x_{24}} \sum_{a,b} a_u^b u^b_{+} u^b_{-} \right] = C_B(h_+, h_-) \cdot \frac{x_{34}^{h-1}}{(x-x_3)^{h_+} (x-x_4)^{h_-} \left[ + \to - \right]}$$

where, by virtue of $F(z) = \sum a c_{a,h} z^a = P_{h-1}(1-2z)$ and $P_L(-1) = (-1)^L$,

$$C_B(h_+, h_-) = 2 \chi_{\text{odd}}(h), \quad (4.5)$$

where $\chi_{\text{odd}}(h) = 1$ if the helicity $h = h_+ - h_-$ is odd, and zero otherwise. To perform the corresponding computation for the sum weighted with $\frac{a-b}{a+b}$, as in the structure $H$, one may for $b > 0$ put $G_b(z) = \sum a b c_{a,h} z^a,$ solve the equation $z G' + b G = z F' - b F$ by $G(z) = F(z) - 2b z^{-h} \int_0^1 \frac{F(t)}{t} dt$, and use the orthogonality of the Legendre polynomials to conclude $G(1) = F(1) = (-1)^{h-1}$ if $h_+ > h_-$. One finds

$$C_H(h_+, h_-) = \text{sign}(h) \cdot 2 \chi_{\text{odd}}(h), \quad (4.6)$$

The same factors arise in the 5-6 channels. Thus, when the 6-point structures $B$ and $H$ are reduced in both channels by means of $(t \circ D_{hl, h'} \circ (x_{12}^2)^2) \otimes (t \circ D_{hl, h'} \circ (x_{12}^2)^2)$, the result is always a multiple of the same 4-point function

$$W_{h_+, h_2, h_3, h_4} (x, x_3, x_4, x') = \frac{h_+ + h_3 - 3}{(x-x_3)^{h_+} (x-x_4)^{h_3}} \left[ x \to - \right] \cdot \chi_{34}^{h_+} \chi_{12}^{h_3} \left[ x \to - \right].$$
The respective coefficients for the structures $B$ and $H$ are

$$C_B(h_+, h_-) C_B(h'_+, h'_-) = 4 \chi_{odd}(h) \chi_{odd}(h') ,$$
$$C_H(h_+, h_-) C_H(h'_+, h'_-) = \text{sign}(h) \text{sign}(h') \cdot 4 \chi_{odd}(h) \chi_{odd}(h') , \quad (4.7)$$

where $h = h_+ - h_-$, $h' = h'_+ - h'_-$ are the helicities.

Because $H$ is the twist-2 part of $B - \frac{1}{2}E$, we conclude that (after 2D restriction) all partial waves with helicities of equal sign in the 1-2-channel and in the 5-6-channel, that are present in $B$, are absent in the twist-2 part of $E$, while those with helicities of opposite sign arise in the twist-2 part of $E$ with 4 times the coefficient in $B$.

It remains to perform the partial wave expansion of the 4-point functions $W_{h_+, h_-; h'_+, h'_-}$. Here one may use standard methods, e.g., [15, 6, 10]. Namely,

$$W_{h_+, h_-; h'_+, h'_-}(x, x_3, x_4, x') = \sum_{k_+, k_-} B^{k_+, k_-}_{h_+, h_-; h'_+, h'_-} \cdot W^{k_+, k_-}_{h_+, h_-; h'_+, h'_-}(x, x_3, x_4, x') ,$$

where $W^{k_+, k_-}$ is the partial wave for the insertion of a projection on the representation with scaling dimensions $(k_+, k_-)$. It turns out that only $k_\pm \in \frac{1}{2} + \mathbb{N}_0$ contribute. Because of chiral factorization of $W_{h_+, h_-; h'_+, h'_-}$, one has $B^{k_+, k_-}_{h_+, h_-; h'_+, h'_-} = B^{k_+, k_-}_{h_+, h_-; h'_+, h'_-} \cdot W^{k_+, k_-}_{h_+, h_-; h'_+, h'_-}$, where the chiral coefficients are determined by the expansion

$$1 = \sum_{k = \frac{1}{2} + n} B^{k}_{h, h'} \cdot u^n \binom{2}{n} F_1(n + h, n + h'; 2n + 3; u).$$

The problem of Wightman positivity of the (2D-restricted) structures $B$ and $H$ has now been reduced to the positivity of linear combinations of matrices of the form

$$P_{\pm} P_{odd} \left[ B^{k_+} \otimes B^{k_-} \right] P_{odd} P_{\pm} ,$$

where $P_{odd}$ and $P_{\pm}$ are the projections on the odd resp. positive or negative helicities.

To be admissible in a QFT, the exotic structure does not need to be separately positive, but must only be dominated by other, non-exotic structures that contribute to a full 6-point function. Thus, if positivity should fail for $H$ (it certainly does for the twist-2 part of $E$ because in this case all diagonal matrix elements vanish), one would have to establish a bound for the negative part of the above matrices by positive matrices of partial wave amplitudes arising from other structures.

We have not completed this analysis yet.

To conclude: the tools are available to test Wightman positivity of 6-point correlation functions. If a 6-point function involving the exotic structure Eq. (4.1) passes the test, then it could be a candidate for a nontrivial 4D conformal QFT.

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