Dynamical system analysis of interacting dark energy-matter scenarios at the linearized inhomogeneous level

Mohit Kumar Sharma*, and Sourav Sur†

Department of Physics & Astrophysics
University of Delhi, New Delhi - 110 007, India

Abstract

We carry out the dynamical system analysis of interacting dark energy-matter scenarios by examining the critical points and stability for not just the background level cosmological evolution, but at the level of the linear density perturbations as well. While an analysis at the background level can lead to a stable phase space trajectory implying that the universe eventually transpires to a dark energy dominated (de-Sitter) era, a two-fold degeneracy in the spectrum of the critical points is found to arise in the inhomogeneous picture, due to the possible growth and decay of matter density perturbations. Analyzing the phase space dynamics of the growth factor, we show that it turns out to be greater than unity initially, for one of the critical points, and leads to a stable configuration as the fluctuations in the matter density die out asymptotically. As to the growth index, we show that the only trajectory which is physically plausible is the one that evolves mildly at high redshifts and gets steeper as time progresses. However, such a trajectory amounts to the average value of the growth index, throughout the expansion history of the universe, not much deviated from the value $6/11$, corresponding to the background $\Lambda$CDM cosmology.

1 Introduction

In a complicated system such as our universe, which passes through various stages of evolution and is presently undergoing an accelerated expansion, driven supposedly by a dark energy (DE) component $[1][5]$, it is natural to ask: will it ever get stabilized or atleast approach to a quasi-stable stage asymptotically? The best way to address this is to resort to the dynamical analysis that enables one to assert the qualitative description of evolving systems without the prior need of any initial conditions. As is well-known, observations such as that of the Cosmic Microwave Background (CMB) $[6][10]$, large-scale structures (LSS) $[11][12]$, type-Ia Supernovae $[13][14]$, and so on, generally concord to the cosmological constant ($\Lambda$) candidature of the DE, or more specifically the $\Lambda$CDM model (where CDM stands for the cold dark matter). However, because of the severe theoretical problems, viz. fine-tuning and coincidence associated with $\Lambda$ (or the $\Lambda$CDM), one’s focus gets shifted to the dynamical DE, possibly from the scalar fields, such as quintessence $[15][17]$, k-essence $[18][23]$, etc., or from the so-called modified gravity (MG) scenarios $[24][30]$, that go beyond the realm of General Relativity (GR).

* email: mr.mohit254@gmail.com
† email: sourav@physics.du.ac.in, sourav.sur@gmail.com
Apart from the extensive studies of the possible consequences of a dynamically evolving DE, a considerable interest has recently been developed on the interaction(s) or even the unification of the DE component with the matter field(s), from various perspectives [31–46]. Most notable are the MG cosmological scenarios, or the scalar-tensor (ST) equivalents thereof [47–58], which naturally give rise to the DE-matter (DEM) interactions under conformal transformations. An interaction of such a sort can have a significant effect on the cosmic expansion rate as well as on the formation of large-scales structures. Hence, the stability analyses of the corresponding cosmological solutions, on the physical ground (i.e. against the density perturbations) as well as mathematically (i.e. in the phase space), are crucial. While the former has been carried out comprehensively, for a typical (and well-motivated) ST equivalent MG scenario, in a preceding work we focus on performing the latter (i.e. the dynamical analysis) in this paper, by resorting to not just the background cosmological level, but to that of the linear density perturbations as well.

The class of ST theories under consideration is the one with a specific (quadratic) form of the non-minimal coupling between a scalar field \( \phi \) and the Ricci scalar curvature \( R \), with the potential for \( \phi \) being just in the form of a mass \( m \), in the original Jordan frame. The corresponding Lagrangian has its equivalence with that of a wide range of MG scenarios, starting from some variants of \( f(R,\phi) \) to metric-scalar-torsion (MST) formulations, and so on. A conformal transformation to the Einstein frame, followed by a field redefinition \( \phi \rightarrow \varphi \), essentially leads to an exponential coupling of \( \varphi \) with matter (including the CDM), and hence an effective DEM interaction, once \( \varphi \) is considered to be the entity that induces the DE.

Now, the objective of the dynamical system analysis is to determine the critical points (CPs), and examine their nature, or more specifically, the eigenvalues of the linear perturbation matrix \( \mathcal{M} \) corresponding to the autonomous set of equations, which one suitably constructs out of the governing field equations. The essence of the CPs, which are the equilibrium solutions of the autonomous equations, lies in their predictability in asserting whether a system is stable or not. Note however that more the number of autonomous variables (which characterize the mathematical state of the system), more would be the number of phase space trajectories representing the plausible ways in which the system can evolve. In other words, an increased number of autonomous variables would inevitably imply a less restrictive dynamical analysis and a better scope of figuring out a stable configuration to which the system may transpire. In this sense it is much desirable for one to be open to the possibility of the DEM interactions, and also take the cosmological density perturbations in consideration, so as to have additional autonomous variable(s) enhancing the dimensionality of the phase space. [59,61].

As is well known, the LSS can more or less be described correctly just by the matter density perturbations. The scalar field perturbations are generally too small to make any significant contribution, at least at the sub-horizon scales [62,65]. In presence of DEM interaction, the field and matter density perturbations get entangled with each other, such that the overall study of the formation of LSS requires a full-fledged analysis of each perturbed component. Nevertheless, being small in magnitude, one can at least incorporate the on-an-average contribution of field perturbations on the matter density perturbations. Larger is the coupling between the scalar field and matter, larger would be the field contribution on the matter density perturbations. Hence, in view of the high dominance of the latter at the sub-horizon scales, it become reasonable to consider it (or more precisely, the matter density growth factor \( f \)) as a autonomous variable for our dynamical analysis. Suitably defining the other dimensionless autonomous variables, we set up the full set of autonomous equations, considering for simplicity the overall (visible + dark) matter content of the universe to be dust-like, albeit with the interaction with the scalar field \( \varphi \).

With the matter density perturbations taken into consideration, the system is found to possess
more CPs than in the case in which the analysis is restricted to the background level cosmology. We qualitatively distinguish the CPs by examining the eigenvalues of the linear perturbation matrix $M$ for the autonomous set of equations, and solve the latter simultaneously to obtain the phase space trajectories. In addition to having the matter perturbation growth factor $f$ as a dynamical variable, we revert back to the background level physical variables, viz. the matter density parameter $\Omega_{(m)}$ and the DEM interaction parameter (or the $\varphi$-matter coupling parameter) $n$. This then enables us to obtain the $\Omega_{(m)} - f$ phase portraits, for $n = 0$ (ΛCDM), and for a certain fiducial setting $n = 0.1$.

A similar procedure can be followed while the growth factor is formulated in terms of the growth index $\gamma$. However, for the interacting system one finds an inconsistency with the parametrization $f = [\Omega_{(m)}]^\gamma$ commonly used in the literature [66–76]. So, to cope with this, we use an alternative parametrization which we have proposed in the earlier paper [60]. From the corresponding phase portraits, we find that there can be only one physically plausible trajectory which exhibits a very mild time-evolution of $\gamma$ at high redshifts, which picks up gradually near the present epoch and beyond. For this trajectory, the growth index is not much deviated from the ΛCDM value 6/11, throughout the expansion history of the universe, which implies that for whatever interacting picture we resort to, the overall (and observationally favoured) ΛCDM cosmology is not much distorted.

Conventions and Notations: Throughout this paper, we use metric signature $(-, +, +, +)$ and natural units, with the speed of light $c = 1$. We denote and the gravitational coupling factor by $\kappa = \sqrt{8\pi G_N}$, where $G_N$ is the Newton’s constant, the metric determinant by $g$, and the values of parameters or functions at the present epoch by an affixed subscript ‘0’.

2 Interacting DE-matter framework

Let us begin with a recourse to the DEM interaction(s) in the standard Friedmann-Robertson-Walker (FRW) cosmological framework, limiting our attention to the scenarios in which such interactions arise naturally. Note that the very formulation of a gravitational theory, such as GR, leaves the scope of such interactions. To be specific, the conservation of the total energy-momentum tensor, does not insinuate to that for each individual component in a multi-component system, such as the universe, composed of baryons, CDM, DE, radiation etc. Since the late-time evolution of the universe is effectively driven by two main constituents, viz. the DE and the (visible + dark) matter, any breach of their individual self-conservation (due to interactions) can affect the dynamical stability of the universe. It is then natural to ask: what happens to such an interacting system if one slightly perturbs it in the space of the autonomous variables about the critical points?

Consider now a DE component, induced by a scalar field $\varphi$, is interacting with matter. Let $T_{\mu\nu}^{(m)}$ and $T_{\mu\nu}^{(\varphi)}$ denote the respective energy-momentum tensors. One has

\[ \nabla^\mu \left[ T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(\varphi)} \right] = 0. \]  

This in general implies

\[ \nabla^\mu T_{\mu\nu}^{(m)} = - \nabla^\mu T_{\mu\nu}^{(\varphi)} = Q_\nu, \]  

where $Q_\nu$ is an interaction four-vector.

Let us note that an interaction of type $Q_\nu$ although seems to be arbitrary in Eq. (1), arises naturally in the ST equivalent formulations of MG theories. In other words, the non-minimally coupling of a scalar field with the Ricci scalar $R$ in the Jordan frame induces a coupling between scalar field and
matter in the Einstein frame. In this paper, we consider a class of such ST formulation for $Q_\nu$ takes the form \cite{60}:

$$Q_\nu = \kappa n \rho^{(m)} \varphi_{,\mu},$$

such that $\kappa \phi = e^{n \kappa \varphi}$, \hspace{1cm} (3)

where $\kappa^2 = 8\pi G$, $n$ is a coupling parameter and $\rho^{(m)}$ is the matter density.

For a background cosmological evolution described by the spatially flat metric, and driven by a DEM interaction given by Eq. (3), the corresponding Friedman and the Klein-Gordon equation for $\varphi$ are given as

$$H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa^2}{3} \left[ \rho^{(m)} + \rho^{(\varphi)} \right],$$

$$\ddot{\varphi} + 3H \dot{\varphi} + U_{,\varphi} = \frac{3n}{\kappa} H^2 \Omega^{(m)} \dot{\varphi},$$

where an overdot represents derivative with respect to the cosmic time $t$, $H$ is the Hubble parameter, $\rho^{(\varphi)}$ is the energy-density of field $\varphi$ and $\Omega^{(m)} := \kappa^2 \rho^{(m)}/(3H^2)$ is the matter density parameter. Note that for our analysis we consider an exponential potential of the form $U^{(\varphi)} = \Lambda e^{-2\kappa n \varphi}$ (where $\Lambda$ is the value of $U$ at present epoch $t = t_0$), which lead to exact background solutions, as pointed out in ref. \cite{60}.

Now, we know that a given matter density perturbation $\delta \rho^{(m)}$ plays the all-important role in the LSS formation. However, in presence of an interaction between matter and the DE induced by a field $\varphi$, a small but not necessarily negligible effect of $\varphi$ on the LSS may result. This may be perceived from the equations of motion of the matter density contrast $\delta^{(m)} := \delta \rho^{(m)}/\rho^{(m)}$ given by \cite{60},

$$\delta^{(m)}_{,NN} + \left[ 2(1 - 2n^2) - \frac{3\Omega^{(m)}}{2} \right] \delta^{(m)}_{,N} = \frac{3(1 + 2n^2)\Omega^{(m)}}{2} \delta^{(m)},$$

where $N(t) = \ln a(t)$ is the number of e-foldings. Note that the rightmost term, which is nothing but the source term for $\delta^{(m)}$ gets enhanced by a factor of $2n^2$, regardless of its contribution in the drag-force due to the background expansion (middle term). Let us emphasize that this effect is not just crucial from the perspective of parametric estimations but also from the dynamical behavior of the system even up to the perturbative level, as more solutions are expected than in the case when the effects of DE perturbations are ignorable.

### 3 Critical points and their stability conditions

In order to perform the dynamical system analysis, let us define following dimensionless quantities:

$$X := \kappa \frac{\dot{\varphi}}{\sqrt{6H}}, \hspace{0.5cm} Y := \kappa \frac{\sqrt{\Lambda} e^{-\kappa n \varphi}}{\sqrt{3H}}, \hspace{0.5cm} f := \frac{\delta^{(m)}_{,N}}{\delta^{(m)}},$$

such that from Eq. (4) it follows that

$$\Omega^{(m)} = 1 - X^2 - Y^2,$$

$$\Omega^{(m)} = 1 - \frac{\delta^{(m)}_{,N}}{\delta^{(m)}},$$

(8)
The coupled set of first-order differential autonomous equations can be worked out as

\[ X_N = -\frac{1}{2} \left( \sqrt{6n} - 3X \right) \left( X^2 - Y^2 - 1 \right), \]  
\[ Y_N = -\frac{1}{2} Y \left( 2\sqrt{6n}X - 3X^2 + 3Y^2 - 3 \right), \]  
\[ f_N = -\frac{1}{2} \left[ 2f^2 - f \left( 3(2X^2 - 2) - 1 + 2\sqrt{6n}X \right) + 3(1 + 2n^2)(X^2 + Y^2 - 1) \right]. \]  

where we have used Eqs. (4), (5) and (6) and relation \( dN = Hdt \). Taking the derivative of Eq. (8) and using Eqs. (9) and (10), we find that

\[ \Omega_{c,N}^{(m)} = \left( X^2 + Y^2 - 1 \right) \left( \sqrt{6n}X - 3X^2 + 3Y^2 \right) \]  

The solutions or the critical points (CP) of above set of Eqs. (9), (10) and (11) can be obtained by simultaneously putting \( \frac{dX}{dN} = \frac{dY}{dN} = \frac{df}{dN} = 0 \).

The obtained CPs: \( (X_c, Y_c, f_c) \) are enlisted in table 1. In that table, we have shown all the CPs together with their corresponding field energy density parameter \( \Omega_{\phi} \) and its equation of state parameter \( w_{\phi} := p_{\phi}/\rho_{\phi} \). Also, due to the invariance in the form of autonomous Eqs. (9), (10) and (11) under the exchange of \( Y \rightarrow -Y \), we represent both positive and negative \( Y_c \) in a single CP. In other words, both signs will give identical set of eigenvalues and hence stability condition(s).

In order to obtain the stability condition(s) for CPs given in table 1, one requires to add small perturbations \( (|\zeta| < 1) \) to each one of them and to check whether the system still remains intact or gets deviated from its original configuration. Let us consider small perturbations around \( X_c, Y_c \) and \( f_c \):

\[ X = X_c + \delta X, \quad Y = Y_c + \delta Y \quad \text{and} \quad f = f_c + \delta f, \]  

such that \( \zeta \equiv \{ \delta X, \delta Y, \delta f \} \) satisfies the following relation:

\[ \frac{d\zeta}{dN} = \mathcal{M} \zeta, \]  

where \( \mathcal{M}_{(3 \times 3)} \) is the Jacobian matrix evaluated at \( \{X_c, Y_c, f_c\} \). Note that the sign of each eigenvalue of a matrix \( \mathcal{M} \) for a given CP determines the stability of autonomous system of equations. In particular, if all the eigenvalues are negative (positive), then that CP is stable (unstable) or saddle otherwise, provided that each eigenvalue is real-valued and non-zero. In table 2, we have shown all the obtained eigenvalues together with their stability condition(s). Let us now examine the stability criteria of all CPs one by one:

- **CP(a):** It is valid for all real values of \( n \). It is either unstable or saddle but its nature cannot be determined for \( n = -\sqrt{3/2} \) and \( 1/6 \). It corresponds to a scenario where the DE density parameter \( \Omega_{\phi} = 1 \) and the growth of matter perturbations \( f = 0 \). But due to the steep-fluid like behavior of DE i.e. \( w_{\phi} = 1 \) the cosmic acceleration is not possible. Hence, it is non-physical.

- **CP(b):** It is valid for all real values of parameter \( n \). This point is either unstable or saddle but its nature cannot be determined for \( n = -\sqrt{1/6} \) and \( 3/2 \). It corresponds to a state of the universe when \( \Omega_{\phi} = 1 \) and \( f = 0 \). However, due to the steep fluid-like nature of DE and hence absence of cosmic acceleration, it is also not physical. Hence, it does not give physical description of the universe.
Table 1: Critical points $X_c, Y_c$ and $f_c$ with their corresponding DE density and equation of state parameter of field $\varphi$ and total equation of state parameter of system.

| CP | $X_c$ | $Y_c$ | $f_c$ | $\Omega(\varphi)$ | $w(\varphi)$ | $w$ |
|----|------|------|------|------------------|-------------|-----|
| (a) | $-1$ | $0$ | $0$ | $1$ | $1$ | $1$ |
| (b) | $1$ | $0$ | $0$ | $1$ | $1$ | $1$ |
| (c) | $-1$ | $0$ | $1 - \sqrt{6n}$ | $1$ | $1$ | $1$ |
| (d) | $1$ | $0$ | $1 + \sqrt{6n}$ | $1$ | $1$ | $1$ |
| (e) | $\sqrt{\frac{3}{2}n^2} \pm \sqrt{\frac{3}{2}n^2 - 1}$ | $2 - \sqrt{6n^2 - 2 - \frac{9}{2}n^2}$ | $\frac{3}{n^2} - 1$ | $n^2/3 - n^2$ | $1$ |
| (f) | $\sqrt{\frac{3}{2}n^2} \pm \sqrt{\frac{3}{2}n^2 - 1}$ | $2 + \sqrt{6n^2 - 2 - \frac{9}{2}n^2}$ | $\frac{3}{n^2} - 1$ | $n^2/3 - n^2$ | $1$ |
| (g) | $\sqrt{\frac{3}{2}n}$ | $0$ | $n^2 - \frac{3}{2}$ | $2n^2/3$ | $1$ | $2n^2/3$ |
| (h) | $\sqrt{\frac{3}{2}n}$ | $0$ | $1 + 2n^2$ | $2n^2/3$ | $1$ | $2n^2/3$ |
| (i) | $\sqrt{\frac{3}{2}n}$ | $\pm \sqrt{1 - 2n^2/3}$ | $0$ | $1$ | $4n^2/3 - 1$ | $4n^2/3 - 1$ |
| (j) | $\sqrt{\frac{3}{2}n}$ | $\pm \sqrt{1 - 2n^2/3}$ | $-2(1 - 2n^2)$ | $1$ | $4n^2/3 - 1$ | $4n^2/3 - 1$ |

- CP(c): It is valid for all real values of parameter $n$. This point could be stable, unstable or saddle but its nature cannot be determined for $n = -\sqrt{1/6}$ and $\sqrt{3/2}$. Although, for small value of $n$ i.e. within the saddle parametric regime, this CP could be considered to lie in the deep-matter dominated era when $f \leq 1$. However, due to dominated DE i.e. $\Omega(\varphi) = 1$ and $w = 1$ it is not physically possible. Hence, it does not give physical description of the universe.

- CP(d): It is valid for all real values of parameter $n$. This point could be stable, unstable or saddle but its nature cannot be determined for $n = -\sqrt{1/6}$ and $\sqrt{3/2}$. Although, it is mathematically stable for $n > \sqrt{3/2}$ but due to the similar reasons mentioned for CP (c) it has no physical relevance.

- CP(e): It is valid for all values of $n$ except when $n = 0$. This point is only saddle but its nature cannot be determined for $n = \pm \sqrt{3/2}$ and $\pm \sqrt{(1 + 2\sqrt{7})/6}$. Since $\Omega(\varphi) = -1 + 3/n^2$, therefore, one requires $\sqrt{3/2} < |n| < \sqrt{3}$ for a physically allowed region $0 < \Omega(\varphi) < 1$, which is too far from allowed observational constraints. Also, since it lacks the cosmic acceleration due to the stiff nature of DE it does not give a physical description of the universe.

- CP(f): It is valid for all values of $n$ excluding $n = 0$. This point is only saddle but its nature cannot be determined for $n = \pm \sqrt{3/2}$ and $\pm \sqrt{(1 + 2\sqrt{7})/6}$. Similar, to the reasons mentioned for CP (e), it is also not a physically relevant point.

- CP(g): It is valid for all values of $n$. It is only a saddle point but its nature cannot be determined for $n = \pm \sqrt{3/2}$. Due to its saddle behavior, it may correspond to the growth of matter.
| CP | Eigenvalues | CP nature |
|----|-------------|-----------|
| (a) | $1 - \sqrt{6} n$ | $3 + \sqrt{6} n$ | $3 + \sqrt{6} n$ | Unstable or saddle |
| (b) | $1 + \sqrt{6} n$ | $3 - \sqrt{6} n$ | $3 - \sqrt{6} n$ | Unstable or saddle |
| (c) | $-1 + \sqrt{6} n$ | $3 + \sqrt{6} n$ | $3 + \sqrt{6} n$ | Stable or unstable or saddle |
| (d) | $-(1 + \sqrt{6} n)$ | $3 - \sqrt{6} n$ | $3 - \sqrt{6} n$ | Stable or unstable or saddle |
| (e) | $\sqrt{24n^2 - 8 - 15n^2} - \frac{\sqrt{3}}{2} (\frac{3}{2} n - 2n)$ | $-\sqrt{\frac{3}{2}} (\frac{3}{2} n - 2n)$ | Saddle |
| (f) | $-\sqrt{24n^2 - 8 - 15n^2} - \frac{\sqrt{3}}{2} (\frac{3}{2} n - 2n)$ | $-\sqrt{\frac{3}{2}} (\frac{3}{2} n - 2n)$ | Saddle |
| (g) | $\frac{5}{2} + n^2$ | $\frac{3}{2} - n^2$ | $-(\frac{3}{2} - n^2)$ | Saddle |
| (h) | $-(\frac{5}{2} + n^2)$ | $\frac{3}{2} - n^2$ | $-(\frac{3}{2} - n^2)$ | Saddle |
| (i) | $-2(1 - 2n^2)$ | $-3 + 2n^2$ | $-3 + 2n^2$ | Stable or unstable or saddle |
| (j) | $2(1 - 2n^2)$ | $-3 + 2n^2$ | $-3 + 2n^2$ | Stable or saddle |

Table 2: Eigenvalues of critical points of table (1) and their corresponding nature.

perturbations, lets say, near the recombination epoch for $n > \sqrt{3/2}$. Also, for $n \ll \sqrt{3/2}$, it corresponds to matter dominated epoch with $w \to 0$, but then $f$ becomes negative, which results in decay of matter perturbations. Hence, this point does support growth of matter perturbations. Therefore, it is not a physically relevant point.

- **CP(h):** It is valid for all values of $n$. It is only a saddle point but its nature cannot be determined for $n = \pm \sqrt{3/2}$. Due to its saddle behavior, it may correspond to the growth of matter perturbations, lets say, near the recombination epoch while also providing a physical description of the background at that epoch i.e. $w \ll 1$ for $n < \sqrt{3/2}$. Interestingly, this point indeed show that $f \leq 1$, which is one of the important feature of an interacting DEM scenario. Therefore, this point is of physical relevance in large redshifts.

- **CP(i):** It is valid for all values of $n$. It could be stable, unstable or saddle but its nature cannot be determined for $n = \pm \sqrt{3/2}$ and $\pm \sqrt{1/2}$. For $n \in (-\sqrt{1/2}, \sqrt{1/2})$ this point actually determines the state of the universe in which DE becomes dominated and no evolution in the matter density perturbations can be observed. This point is a late-time attractor. For $n = 0$, it corresponds to a perfect de-Sitter universe in the future. This point is of physical relevance in far future.

- **CP(j):** It is valid for all values of $n$. It is either stable or saddle but its nature cannot be determined for $n = \pm \sqrt{3/2}$ and $\pm \sqrt{1/2}$. Although, at the background level, it correctly describe the DE dominated universe but shows a negative growth of matter perturbations for
n ∈ (−√1/2, √1/2). This results decay of the matter density contrast which is not possible. Hence, it is not physical.

3.1 Numerical solutions of state variables

Now in order to depict the evolutionary behavior of dimensionless variables X, Y and f, we numerically solve the coupled system of Eqs. (9), (10) and (11). However, before proceeding ahead let us first note that the free parameter n are very tightly constrained by both background as well as at the perturbative level observations. In fact, we have shown in [60] that from the OHD and RSD-GOLD data, n ≤ 0.1 upto 1σ confidence interval. Therefore, in order to be compatible with observational constraints, we take fiducial values: n = 0 and 0.1. Moreover, we take the initial conditions at N = −7 or z = 1095:

\[ X(−7) = 0.01, \quad Y(−7) = 4 \times 10^{-5}, \quad f(−7) = 1. \]  

(16)

The obtained evolutionary profiles are shown in Fig. 1. In that figure one observes that f tends to cross the value of unity in large enough redshifts. One also finds that variable X is quite small (≪ 1) and least evolving variable, whereas, Y increases monotonically.

Here, let us emphasize that although in fig. 1 it seems that X is constant, but in general it is not true. In fact, we have explicitly shown in insets that for any given initial conditions, X always tends to settle to a value of the order of n. In other words, changing or perturbing its initial conditions does not affect its evolutionary profile. Hence, no matter what initial condition for X one will start with it always finds it near(or equal to) the magnitude of n for any given epoch. It is also interesting to note that Xc for this stable CP(i) agrees with our previous obtained background solution of \( \varphi(N) \) [60] i.e.,

\[ \varphi(N) = \frac{2nN}{\kappa}, \quad \text{which implies,} \quad X = X_c = \sqrt{\frac{2}{3}} n. \]  

(17)

Hence, it confirms that this unique solution is both stable physically and dynamically of combined background and perturbative system.
3.2 Phase space dynamics of matter perturbations

In order to depict the phase space dynamics of above system comprised of $X, Y$ and $f$ variables, it is suitable to replace $X, Y$ and their derivatives by $\Omega^{(m)}$ and $\Omega_{,N}^{(m)}$, respectively, by using Eq. (8) and (12). In this way, not just our phase space dynamics will get reduced to two-dimensions but it will also be more reasonable to have a phase space dynamics between two physically observable quantities i.e., $\Omega^{(m)}$ and $f$. Using above solution (17) together with Eq. (9), (10) and (11), one obtains a set of equations:

$$\Omega_{,N}^{(m)} = -3 \left(1 - \frac{2}{3}n^2 - \Omega^{(m)}\right) \Omega^{(m)},$$  \hspace{1cm} (18)

$$f_{,N} = \frac{3}{2} \Omega^{(m)} (1 + 2n^2 + f) - f (2 - 4n^2 + f).$$  \hspace{1cm} (19)

Solving them simultaneously by putting $\Omega_{,N}^{(m)} = f_{,N} = 0$, we obtain the phase space dynamics between matter density parameter $\Omega^{(m)}$ and growth factor $f$. In fig. (2), we show the phase portrait diagram for $\Lambda$CDM and $n = 0.1$. In this figure, the nature of CPs are given as: (i) $CP(P_1)$ corresponds to the stable de-Sitter solution, (ii) : $CP(P_2)$ is a saddle point which may belongs to the deep matter-dominated era, (iii) : $CP(P_3)$ is a saddle, and (iv) : $CP(P_4)$ is an unstable point. Since, both $P_3$ and $P_4$ corresponds to the negative growth of matter perturbations and hence the matter density contrast will behave as: $\delta^{(m)} \propto e^{-cN}$ (where $c$ is some positive integration constant), the LSS formation will not occur. Hence, they are physically irrelevant. Therefore, one only expect those trajectories that starts with $P_2$ and ends at $P_1$. In fact, these are the only trajectories which makes the universe to evolve in such a way that growth of matter perturbations declines as matter density parameter decreases with time.

After depicting the possible evolutionary profile of $f$ with $\Omega^{(m)}$, it is also necessary to figure out how does the evolutionary profile of growth index ($\gamma$) follows. Let us recall that, in literature, the growth index is related to the growth factor $f$ via the relation: $f = [\Omega^{(m)}]^{\gamma}$. In general, $\gamma$ is a time-variation function and depends on the background parameters such as $\Omega^{(m)}$ and coupling parameters ($n$.
in our case). As it can be noticed from fig. 11 that \( f \) crosses the upper barrier of unity in presence of \( n \), which is not possible in the above parametrization of \( f \) as it is only limited to the minimally coupled scalar field scenarios, such as quintessence. In order to incorporate this feature, the above parametrization of \( f \) requires modification. In fact, in [60], we have shown that a suitable modification in the parametrization of \( f \) can take the following form:

\[
f(N) = (1 + 2n^2) \left[ \Omega^{(m)} \right]^{\gamma(N)}.
\]  

Using this in the Eq. (19), we obtain a first-order differential equation for \( \gamma(N) \):

\[
\gamma, N = \frac{-4 - 3(\Omega^{(m)})^{1-\gamma} + 2(\Omega^{(m)})^{\gamma} + 6\gamma(\Omega^{(m)} - 1) + 4n^2 [(\Omega^{(m)})^{\gamma + \gamma - 2} - 3\Omega^{(m)}]}{2 \log(\Omega^{(m)})}.
\]  

By putting \( \gamma, N = \Omega^{(m)} = 0 \), and then solving Eqs. (18) and (21) simultaneously we obtain the phase space diagram between \( \Omega^{(m)} \) and \( \gamma \) (shown in fig. 3) for both \( \Lambda \)CDM and \( n = 0.1 \). In that figure, the green vector-field (thick) is the only solution as it remains finite over the entire range of \( \Omega^{(m)} \in [0, 1] \). All other solutions get diverge which either implies extensive amount of growth of matter perturbations (when \( \gamma \ll 0.555 \)) or no structure formation at all (when \( \gamma \gg 0.555 \)). This solution, thus, giving rise to consistent matter perturbations growth at the present epoch is also consistent with the theoretically calculated value \( \gamma = 6/11 \) in the deep matter-dominated epoch. It is interesting to note that this behavior of increasing \( \gamma \) with \( N \) agrees with our fitting function of \( \gamma \) in previous paper [60], which also depicts sharp increase in the value of \( \gamma \) in future by remaining mildly dynamic atleast up to the present epoch.

4 Conclusion

We have performed a thorough dynamical analysis considering both the cosmological background as well as linear perturbative effects in order to find out the stability of the entire system. The
system under consideration involves an interaction between DE and matter which has generated as a consequence of a non-minimal coupling (exponential in nature) between scalar field and matter sector in the Einstein frame. Since this type of coupling can be obtained in various sort of modified gravity theories such as $f(\phi, R)$, metric-scalar-torsion theories, etc., which although have different theoretical motivations but due to the degeneracy at the solution level, they land up to an identical dynamical system. Therefore, it becomes important to perform a dynamical system not just considering the cosmological background but also the linear perturbative level which can incorporate a wide variety of modified gravity theories. Keeping that as an objective we have carried out our dynamical analysis in this paper.

In presence of an interacting DE, not just matter density ceases to evolve as a standard dust i.e. $a^{-3}(t)$ but its perturbations also experience additional effects both from homogeneous (at background level) as well as the inhomogeneous (at perturbative level) DE. Similarly, the evolution of DE perturbations experience both homogeneous and inhomogeneous contribution from the matter sector. As a consequence, the evolution of cosmological perturbations becomes non-trivial. In fact, one observes following characteristic features in presence of an interacting DEM scenario: (i) enhancement of growth of matter perturbations, (ii) oscillations of DE perturbations about non-zero mean value, (iii) DE perturbations sources matter density perturbations, etc. These features being absent in the standard minimally coupled scalar field scenarios, such as quintessence, demands to re-perform a dynamical analysis under given notable effects.

Our dynamical analysis consists of an interacting DEM FRW background and linear matter density perturbations over it. Since the observable large scale structure and its evolution lies in the deep sub-horizon regime, therefore, for our analysis we consider effects of cosmological perturbations in that regime only. It is also beneficial in a sense that matter density perturbations becomes scale-independent which makes dynamical analysis independent of the perturbation scale. Hence, we have three dimensionless variables in total: two for the background which is constructed out of the kinetic and potential term of the scalar field in the Friedmann equation, and one for perturbations which consists of scale-independent matter density perturbations with an on-an-average effect of field perturbations. By simultaneously solving equations for our dimensionless quantities, we obtain fourteen critical points, out of which four are identical which give rise to same mathematical and physical description of our universe. Hence, in total we have ten distinct critical points which is of course more than one would obtain in case of minimally coupled scenario.

We find that out of these ten distinct critical points only four is mathematically stable under linear perturbations in the solution space. However, from physical point of view only one point is stable i.e. CP(i) of table 1, which is supposed to lie in the far future when field energy density gets completely dominated and growth of structure formation reduces to zero. Also, we find that all points could be saddle for a certain parametric regime of coupling parameter $n$, however, physically only one such point is possible i.e. CP(h) which only gives rise to a matter dominated description of the universe with a large growth of matter perturbations at high redshifts. Moreover, by simultaneously solving the set of autonomous equations, we find that the late-time behavior of dimensionless variables agrees with the stable CP(i). In particular, it also verify our previous found solution of $\varphi$ in [60], in which we have shown that this is the only real solution for a coupled set of background differential equations.

Moreover, by formulating background dimensionless variables in terms of matter density parameter we have constructed a phase space solution between two observable $\Omega^{(m)}$ and $f$. This we have done by fixing background dimensionless variables to that of the value corresponds to stable CP(i). In that we find that only one physical trajectory is possible which leads to matter dominated to DE dominated era with decreasing $f$. However, a similar trajectory which also leads to the same configuration
starts with negative $f$, which is not of physical importance. Further, in view of the necessity in the modifications in the analytical growth factor ansatz, we have also obtained a phase space dynamics of growth index $\gamma$ with $\Omega(m)$. In that we find that there lies only one such solution which is weakly dynamical. While all other solutions get diverge in a finite interval of time, it correctly describe the high redshift approximation of $\gamma$ i.e. $6/11$. Also, a speed up evolution of $\gamma$ is observed near the present epoch till the asymptotic future.

Acknowledgments

The work of MKS was supported by the Council of Scientific and Industrial Research (CSIR), Government of India.

References

[1] E. J. Copeland, M. Sami and S. Tsujikawa, Dynamics of dark energy, Int. J. Mod. Phys. D 15 (2006) 1753, e-Print: hep-th/0603057.

[2] J. A. Frieman, M. S. Turner and D. Huterer, Dark energy and the accelerating universe, Ann. Rev. Astron. Astrophys. 46 (2008) 385, e-Print: 0803.0982[astro-ph].

[3] L. Amendola and S. Tsujikawa, Dark Energy: Theory and Observations, Cambridge University Press, United Kingdom (2010).

[4] G. Wolschin, Lectures on Cosmology: Accelerated expansion of the Universe, Springer, Berlin, Heidelberg (2010).

[5] S. Matarrese, M. Colpi, V. Gorini and U. Moschella, Dark Matter and Dark Energy: A Challenge for Modern Cosmology, Springer, The Netherlands (2011).

[6] G. F. Hinshaw et. al., Nine-year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results, Astrophys. J. Suppl. 208 (2013) 19, e-Print: 1212.5226[astro-ph.CO].

[7] C. L. Bennett et. al., Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Final Maps and Results, Astrophys. J. Suppl. 208 (2013) 20, e-Print: 1212.5225[astro-ph.CO].

[8] P.A.R Ade et. al., Planck 2015 results, XIII. Cosmological parameters, Astron. & Astrophys. 594 (2016) A13, e-Print: 1502.01589[astro-ph.CO].

[9] P.A.R Ade et. al., Planck 2015 results, XIV. Dark energy and modified gravity, Astron. & Astrophys. 594 (2016) A14, e-Print: 1502.01590[astro-ph.CO].

[10] Planck Collaboration: N. Aghanim et. al., Planck 2018 results. VI. Cosmological parameters, e-Print: 1807.06209[astro-ph.CO].

[11] R. Neveux et al., The completed SDSS-IV extended Baryon Oscillation Spectroscopic Survey: BAO and RSD measurements from the anisotropic power spectrum of the quasar sample between redshift 0.8 and 2.2, MNRAS 499 210-229 (2020).
J. Hou et al., MNRAS The completed SDSS-IV extended Baryon Oscillation Spectroscopic Survey: BAO and RSD measurements from anisotropic clustering analysis of the quasar sample in configuration space between redshift 0.8 and 2.2, 500 1201-1221 (2021).

M. Betoule et. al., Improved cosmological constraints from a joint analysis of the SDSS-II and SNLS supernova samples, Astron. & Astrophys. 568 (2014) A22, e-Print: 1401.4064[astro-ph.CO].

D. M. Scolnic et. al., The Complete Light-curve Sample of Spectroscopically Confirmed SNe Ia from Pan-STARRS1 and Cosmological Constraints from the Combined Pantheon Sample, Astrophys. J. 859 (2018) 2, 101, e-Print: 1710.00845[astro-ph.CO].

R. R. Caldwell, R. Dave and P. J. Steinhardt, Cosmological imprint of an energy component with general equation of state, Phys. Rev. Lett. 80 (1998) 1582, e-Print: astro-ph/9708069.

E. J. Copeland, A. R. Liddle and D. Wands, Exponential potentials and cosmological scaling solutions, Phys. Rev. D 57 (1998) 4686, e-Print: gr-qc/9711068.

S. Tsujikawa, Quintessence: A Review, Class. Quant. Grav. 30 (2013) 214003, e-Print: 1304.1961[gr-qc].

C. Armendariz-Picon, V. Mukhanov and P. J. Steinhardt, A Dynamical Solution to the Problem of a Small Cosmological Constant and Late-time Cosmic Acceleration, Phys. Rev. Lett. 85 (2000) 4438, e-Print: astro-ph/0004134.

C. Armendariz-Picon, V. Mukhanov and P. J. Steinhardt, Essentials of k-essence, Phys. Rev. D 63 (2001) 103510, e-Print: astro-ph/0006373.

M. Malquarti, E. J. Copeland, A. R. Liddle and M. Trodden, A New view of k-essence, Phys. Rev. D 67 (2003) 123503, e-Print: astro-ph/0302279.

R. J. Scherrer, Purely kinetic k-essence as unified dark matter, Phys. Rev. Lett. 93 (2004) 011301, e-Print: astro-ph/0402316.

S. Sur and S. Das, Multiple kinetic k-essence, phantom barrier crossing and stability, JCAP 0901 (2009) 007, e-Print: 0806.4368[astro-ph].

S. Sur, Crossing the cosmological constant barrier with kinetically interacting double quintessence, e-Print: 0902.1186[astro-ph.CO].

S. Nojiri and S. D. Odintsov, Introduction to modified gravity and gravitational alternative for dark energy, Int. J. Geom. Methods Mod. Phys. 04 (2007) 115, e-Print: hep-th/0601213.

S. Tsujikawa, Modified gravity models of dark energy, Lect. Notes Phys. 800 (2010) 99, e-Print: 1101.0191[gr-qc].

T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Modified Gravity and Cosmology, Phys. Rept. 513 (2012) 1, e-Print: 1106.2476[astro-ph.CO].

J.H. He, B. Wang, and E. Abdalla, Deep connection between f(R) gravity and the interacting dark sector model, Phys. Rev. D 84 no. 12 (2011) 123521, e-Print: 1109.1730[gr-qc].
[28] E. Papantonopoulos, *Modifications of Einstein’s Theory of Gravity at Large Distances*, Lecture Notes in Physics, Springer, Switzerland (2015).

[29] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, *Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution*, Phys. Rept. 692 (2017) 1, e-Print: 1705.11098[gr-qc].

[30] S. Sur and A. S. Bhatia, *Weakly dynamic dark energy via metric-scalar couplings with torsion*, JCAP 1707 (2017) 039, e-Print: 1611.00654[gr-qc].

[31] L. Wang and P. J. Steinhardt, *Cluster abundance constraints for cosmological models with a time-varying, spatially inhomogeneous energy component with negative pressure*, Astrophys. J. 508 (1998) 483 e-Print: astro-ph/9804015.

[32] L. Amendola, *Coupled quintessence*, Phys. Rev. D 62 (2000) 043511, e-Print: astro-ph/9908023.

[33] D. Comelli, M. Pietroni, and A. Riotto, *Dark energy and dark matter*, Phys. Lett. B 571 (2003) 115, e-Print: hep-ph/0302080.

[34] G. R. Farrar and P. J. E. Peebles, *Interacting dark matter and dark energy*, Astrophys. J. 604 (2004) 1, e-Print: astro-ph/0307316.

[35] R. G. Cai and A. Wang, *Cosmology with interaction between phantom dark energy and dark matter and the coincidence problem*, JCAP 03 (2005) 002, e-Print: hep-th/0411025.

[36] S. Campo, R. Herrera, G. Olivares and D. Pavon, *Interacting models of soft coincidence*, Phys. Rev. D 74 (2006) 023501, e-Print: astro-ph/0606520.

[37] S. Campo, R. Herrera and D. Pavon, *Toward a solution of the coincidence problem*, Phys. Rev. D 78 (2008) 021302, e-Print: 0806.2116[astro-ph].

[38] S. Campo, R. Herrera and D. Pavon, *Interacting models may be key to solve the cosmic coincidence problem*, JCAP 01 (2009) 020, e-Print: 0812.2210[gr-qc].

[39] L. Amendola et. al., *Cosmology and fundamental physics with the Euclid satellite*, Liv. Rev. Rel. 21 (2018) 1, 2, e-Print: 1606.00180[astro-ph.CO].

[40] S. Sinha and N. Banerjee, *Density perturbation in an interacting holographic dark energy model*, Eur. Phys. J. Plus 135 (2020) 779, e-Print: 1911.06520[gr-qc].

[41] D. Bertacca, N. Bartolo and S. Matarrese, *Unified Dark Matter Scalar Field Models*, Adv. Astron. 2010 (2010) 904379, e-Print: 1008.0614[astro-ph.CO].

[42] D. Bertacca, M. Bruni, O. F. Piattella and D. Pietrobon, *Unified Dark Matter scalar field models with fast transition*, JCAP 1102 (2011) 018, e-Print: 1011.6669[astro-ph.CO].

[43] E. Guendelman, E. Nissimov and S. Pacheva, *Unified Dark Energy and Dust Dark Matter Dual to Quadratic Purely Kinetic K-Essence*, Eur. Phys. J. C 76 (2016) 90, e-Print: 1511.07071[gr-qc].

[44] E. G. M. Ferreira, G. Franzmann, J. Khoury and R. Brandenberger, *Unified Superfluid Dark Sector*, JCAP 08 (2019) 027, e-Print: 1810.09474[astro-ph.CO].
[45] H. Ramo Chothe, A. Dutta and S. Sur, *Cosmological Dark sector from a Mimetic-Metric-Torsion perspective*, Int. J. Mod. Phys. D 28 (2019) 15, 1950174, e-Print: 1907.12429[gr-qc].

[46] S. Sur, A. Dutta and H. Ramo Chothe, *Mimetic-Metric-Torsion with induced Axial mode and Phantom barrier crossing*, Eur. Phys. J. C 81 (2021) No.4, e-Print: 2007.04906[gr-qc]

[47] A. S. Bhatia and S. Sur, *Dynamical system analysis of dark energy models in scalar coupled metric-torsion theories*, Int. J. Mod. Phys. D 26 (2017) 1750149, e-Print: 1702.01267[gr-qc].

[48] A. S. Bhatia and S. Sur, *Phase Plane Analysis of Metric-Scalar Torsion Model for Interacting Dark Energy*, e-Print: 1611.06902[gr-qc].

[49] Y. Fujii and K. Maeda, *The Scalar-Tensor Theory of Gravitation*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, United Kingdom (2003).

[50] V. Faraoni, *Cosmology in Scalar-Tensor Gravity*, Kluwer Academic Publishers (2004).

[51] N. Bartolo and M. Pietroni, *Scalar-tensor gravity and quintessence*. Phys. Rev. D 61 (2000) 023518, e-Print: hep-ph/9908521

[52] B. Boisseau, G. Esposito-Farese, D. Polarski and A. A. Starobinsky, *Reconstruction of a scalar tensor theory of gravity in an accelerating universe*, Phys. Rev. Lett. 85 (2000) 2236, e-Print: gr-qc/0001066

[53] S. Tsujikawa, K. Uddin, S. Mizuno, R. Tavakol and J. I. Yokoyama, *Constraints on scalar-tensor models of dark energy from observational and local gravity tests*, Phys. Rev. D 77 (2008) 103009.

[54] E. Elizalde, S. Nojiri and S. D. Odintsov, *Late-time cosmology in (phantom) scalar-tensor theory: Dark energy and the cosmic speed-up*, Phys. Rev. D 70 (2004) 043538, e-Print: hep-th/0405034

[55] S. Campo, R. Herrera and P. Labrana, *Emergent universe in a Jordan-Brans-Dicke theory*, JCAP 0711 (2007) 030, e-Print: 0711.1559[gr-qc].

[56] B. Boisseau, H. Giacomini and D. Polarski, *Bouncing Universes in Scalar-Tensor Gravity Around Conformal Invariance*, JCAP 1605 (2016) 048, e-Print: 1603.06648[gr-qc].

[57] E. N. Saridakis and M. Tsoukalas, *Cosmology in new gravitational scalar-tensor theories*, Phys. Rev. D 93 (2016) 12, 124032, e-Print: 1601.06734[gr-qc].

[58] R. Kase and S. Tsujikawa, *Weak cosmic growth in coupled dark energy with a Lagrangian formulation*, Phys. Lett. B (2020), 135400, e-Print: 1911.02179[gr-qc].

[59] S. Basilakos, G. Leon, G. Papagiannopoulos, and E. N. Saridakis, *Dynamical system analysis at background and perturbation levels: Quintessence in severe disadvantage comparing to ΛCDM*, Phys. Rev. D 100 (2019) 043524, e-Print: 1904.01563[gr-qc].

[60] M. K. Sharma and S. Sur, *Imprints of interacting dark energy on cosmological perturbations*, e-Print: 2112.08477 [astro-ph.CO].

[61] R. G. Landim, *Cosmological perturbations and dynamical analysis for interacting quintessence*, Eur. Phys. J C 79 (2019) 1, e-Print: 1908.03657[gr-qc].
T. Koivisto, *Growth of perturbations in dark matter coupled with quintessence*, Phys. Rev. D 72 (2005) 043516, e-Print: astro-ph/0504571.

C. Di Porto and L. Amendola, *Observational constraints on the linear fluctuation growth rate*, Phys. Rev. D 77 (2008) 083508, e-Print: 0707.2686[astro-ph].

D. Polarski and R. Gannouji, *On the growth of linear perturbations*, Phys. Lett. B 660 (2008) 439, e-Print: 0710.1510[astro-ph].

R. Gannouji and D. Polarski, *The growth of matter perturbations in some scalar-tensor DE models*, JCAP 05 (2008) 018, e-Print: 0802.4196[astro-ph].

P. Wu, H. Yu and X. Fu, *A parametrization for the growth index of linear matter perturbations*, JCAP 06 (2009) 019, e-Print: 0905.3444[gr-qc].

C. Di Porto, L. Amendola and E. Branchini, *Growth factor and galaxy bias from future redshift surveys: a study on parametrizations*, Mon. Not. Roy. Astron. Soc. 419 (2011) 985, e-Print: 1101.2453[astro-ph.CO].

A.B. Belloso, J. Garcia-Bellido and D. Sapone, *A parametrization of the growth index of matter perturbations in various Dark Energy models and observational prospects using a Euclid-like survey*, JCAP 10 (2011) 010, e-Print: 1105.4825[astro-ph.CO].

S. Basilakos and A. Pouri, *The growth index of matter perturbations and modified gravity*, Mon. Not. Roy. Astron. Soc. 423 (2012) 3761, e-Print: 1203.6724[astro-ph.CO].

H. Steigerwald, J. Bel and C. Marinoni, *Probing non-standard gravity with the growth index: a background independent analysis*, JCAP 05 (2014) 042, e-Print: 1403.0898[astro-ph.CO].

R. C. Batista, *The impact of dark energy perturbations on the growth index*, Phys. Rev. D 89 (2014) 123508, e-Print: 1403.2985[astro-ph.CO].

M. Malekjani, S. Basilakos, A. Mehrabi, Z. Davari and M. Rezaei, *Agegraphic dark energy: growth index and cosmological implications*, Mon. Not. Roy. Astron. Soc. 464 (2016) 1192, e-Print: 1609.01998[astro-ph.CO].

D. Polarski, A. A. Starobinsky and H. Giacomini, *When is the growth index constant?*, JCAP 12 (2016) 037, e-Print: 1610.00363[astro-ph.CO].

S. Basilakos and F. K. Anagnostopoulos, *Growth index of matter perturbations in the light of Dark Energy Survey*, Eur. Phys. J. C 80 (2020) 212, e-Print: 1903.10758[astro-ph.CO].

A. F. Heavens, T. D. Kitching, and L. Verde, *On model selection forecasting, dark energy and modified gravity*, MNRAS 380 (2007) 1029-1035, e-Print: arXiv:astro-ph/0703191.

M. K. Sharma and S. Sur, *Growth of Matter Perturbations in an Interacting Dark Energy scenario emerging from Metric-Scalar-Torsion couplings*, Phys. Sci. Forum 2(1) (2021) 51, e-Print: 2102.01525[gr-qc].