LAZARSFELD-MUKAI BUNDLES AND APPLICATIONS. II

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1. Introduction

The notion of Lazarsfeld-Mukai bundle goes back to the 1980’s, when two important problems in algebraic geometry were solved using vector bundle techniques [10, 14]. They were initially defined as vector bundles with particularly nice properties on $K3$ surfaces, and their main applications to date remain within the $K3$ framework. The definition makes sense however in a much larger class of surfaces.

Let $S$ be a surface, $C$ be a smooth curve on $S$, and $A$ be a $g^1_d$ on $C$. A natural question related to this setup is the following: can $A$ be lifted to the surface $S$? The chances for $A$ to be induced by a pencil on $S$ are slim, by the simple fact that we cannot exclude the possibility that $\text{Pic}(S)$ be generated by $C$ itself. This case actually occurs in many situations, for instance on very general $K3$ surfaces. Instead, we can try and lift the pencil $|A|$ to a different object, and in doing so we have to decide what kind of object would that be. If we recall that an element $D$ in the linear system $|A|$ is a sum of points $x_1 + \cdots + x_d$ on the curve $C$, hence points on the surface $S$, then it is plausible that any such collection of points on $S$ be cut out by a section in a rank-two vector bundle. Since the points move in a linear system on $C$, the section would also have move in a two-dimensional space of sections over $S$ in this hypothetical bundle. In other words, we want to erase the curve from the picture, keep the moving points and interpret them as elements in a linear family of zero-dimensional subschemes on $S$, see Figure 1. If this goal is achieved, then the rank-two bundle in question, which comes equipped with a natural two-dimensional space of global sections, is called a Lazarsfeld-Mukai bundle [14, 10, 11]. This strategy works well for regular surfaces [15], and can be extended for higher-dimensional linear systems on $C$, [10, 11]. The details are explained in Section 2.

Lazarsfeld-Mukai bundles proved to be useful in various situations. They appear in the classification of Fano threefolds [14], in classical Brill-Noether problems [10, 11], in higher Brill-Noether theory [8], in syzygy-related questions [17, 18, 4] etc; see [11, 3] for some surveys of this topic.

The specific problem we consider here is the computation of the dimensions of Brill-Noether loci. For curves on regular surfaces, this computation reduces to a dimension calculation of the parameter spaces of Lazarsfeld-Mukai bundles. Beyond the Brill-Noether theoretical interest for this type of problems, the motivation comes from syzygy
The curve is erased, the points still move. Two divisors in the same linear system on the curve.

Figure 1. Divisors in $|A|$ as moving subschemes of $S$. 

theory, see Section 4 for a more detailed discussion. Our goal is twofold. On the one hand, we review the general theory that is generally focused on the case of $K3$ surfaces. This recollection of facts might be of future use. On the other hand, we slightly improve some results obtained so far in the non-$K3$ case.

The present work is a continuation of [3] and its outline is the following. In Section 2 we recall the definition of Lazarsfeld-Mukai bundles on regular surfaces and we prove a general dimension statement, Theorem 2.2: the Brill-Noether loci have the expected dimension if some suitable vanishing conditions are satisfied. In Section 3 we estimate the dimensions of Brill-Noether loci for curves on rational surfaces with an anticanonical bound, Theorem 3.1. It is an extension of the main result of [12]. In Section 4 we discuss some applications to syzygies, based on the main result of [2], Theorem 4.1. An alternate proof of [1, Theorem 8.1] is given in Example 4.2.

2. Lazarsfeld-Mukai bundles on surfaces with $q = 0$

We follow closely the approach from [15, 11]. Let $S$ be a surface with $h^1(O_S) = 0$, $C$ be a smooth connected curve of genus $g$ in $S$ and denote $L = O_S(C)$. The hypothesis $h^1(O_S) = 0$ is needed for technical reasons and ensures that $T_C|L| = H^0(N_C|S)$. Let $A$ be a base-point-free complete $g$-r on $C$ and denote by $M_A$ the kernel of

$$\text{ev}_A : H^0(A) \otimes O_C \to A.$$ 

The evaluation map lifts to a surjective sheaf morphism $H^0(A) \otimes O_S \to A$ on $S$ whose kernel $F_{C,A}$ is a vector bundle of rank $(r + 1)$. Its dual $E_{C,A} = F_{C,A}^*$ is called a Lazarsfeld-Mukai bundle. Dualizing the defining sequence of $E_{C,A}$

$$0 \to F_{C,A} \to H^0(A) \otimes O_S \to A \to 0$$ 

we obtain the defining sequence of $E_{C,A}$

$$0 \to H^0(A)^* \otimes O_S \to E_{C,A} \to N_{C|S} \otimes A^* \to 0.$$ 

The following properties of $E_{C,A}$ and $F_{C,A}$ are obtained by direct computation, using the hypotheses $h^1(O_S) = 0$ and $h^0(C, A) = r + 1$ [11, 15, 12]:

1. $\det(E_{C,A}) = L$,
2. $c_2(E_{C,A}) = d$,
3. $h^0(S, F_{C,A}) = h^1(S, F_{C,A}) = 0$,
4. $\chi(S, F_{C,A}) = h^2(S, F_{C,A}) = (r + 1)\chi(O_S) + g - d - 1$, 
(5) \( h^0(S, E_{C,A}) = r + 1 + h^0(C, N_{C|S} \otimes A^*) \),

(6) \( E_{C,A} \) is generated off the base locus of \( |N_{C|S} \otimes A^*| \) inside \( C \).

Restricting the sequence (1) to the curve \( C \), we obtain a short exact sequence:

\[
0 \to N_{C|S}^* \otimes A \to F_{C,A}|_C \to M_A \to 0
\]

which implies, twisting by \( K_C \otimes A^* \) and using the adjunction formula,

\[
0 \to \omega_S|_C \to F_{C,A} \otimes K_C \otimes A^* \to M_A \otimes K_C \otimes A^* \to 0.
\]

Note that \( H^0(M_A \otimes K_C \otimes A^*) = \ker(\mu_{0,A}) \), where \( \mu_{0,A} : H^0(A) \otimes H^0(K_C \otimes A^*) \to H^0(K_C) \) is the Petri map.

Recall [6] that, for any \( r \) and \( d \), the Brill–Noether loci form a family \( \mathcal{W}_d^r(|L|) \) over the open subset of \( |L| \) corresponding to smooth curves. The next result uses the hypothesis \( h^1(\mathcal{O}_S) = 0 \) and follows from the discussion in [15, p. 197] (see also [4, Lemma 2.3]):

**Lemma 2.1.** If \( (C, A) \in \mathcal{W} \) is a general pair in an irreducible component of \( \mathcal{W}_d^r(|L|) \) dominating over \( |L| \), then the coboundary map \( H^0(C, M_A \otimes K_C \otimes A^*) \to H^1(C, \omega_S|_C) \) vanishes.

Lemma 2.1 exhibits an exact sequence

\[
0 \to H^0(C, \omega_S|_C) \to H^0(C, F_{C,A} \otimes K_C \otimes A^*) \to \ker(\mu_{0,A}) \to 0
\]

for a general choice of a pair \( (C, A) \in \mathcal{W} \). In particular, \( \mathcal{W} \) is a smooth of expected dimension at the point \( (C, A) \) if the following equality holds: \( h^0(C, \omega_S|_C) = h^0(C, F_{C,A} \otimes K_C \otimes A^*) \). The sequence (3) is useful to estimate the dimension of Brill-Noether loci and, in some situations, smoothness follows from appropriate vanishing conditions:

**Theorem 2.2.** Notation as above. Assume that \( h^2(S, L) = 0 \). Let \( (C, A) \) be a general pair in a dominating component \( \mathcal{W} \) such that \( h^2(S, F_{C,A} \otimes E_{C,A}) = h^2(S, \mathcal{O}_S) \) and \( h^2(S, E_{C,A}) = 0 \). Then \( \mathcal{W} \) is of dimension \( \leq \rho(g, r, d) + \dim|L| + (r + 1)h^1(S, E_{C,A}) \) at the point \( (C, A) \). In particular, if \( h^1(S, E_{C,A}) = 0 \), then \( \mathcal{W} \) is smooth of expected dimension \( \rho(g, r, d) + \dim|L| \) at the point \( (C, A) \).

**Proof.** From the long exact sequence associated to the sequence

\[
0 \to \mathcal{O}_S \to L \to N_{C|S} \to 0,
\]

applying the vanishing hypothesis \( h^2(S, L) = 0 \) we obtain an surjection \( H^1(C, N_{C|S}) \to H^2(S, \mathcal{O}_S) \) and hence, since \( K_C \cong N_{C|S} \otimes \omega_S|_C \), we have \( h^0(C, \omega_S|_C) \geq h^2(S, \mathcal{O}_S) \). From the sequence (2) it follows that \( \dim(\ker(\mu_{0,A})) \leq h^0(C, F_{C,A} \otimes K_C \otimes A^*) - h^2(\mathcal{O}_S) \).

Twisting the sequence (2) by \( F_{C,A} \otimes \omega_S \), taking global sections in the sequence

\[
0 \to H^0(A^* \otimes F_{C,A} \otimes \omega_S) \to F_{C,A} \otimes F_{C,A} \otimes \omega_S \to F_{C,A} \otimes K_C \otimes A^* \to 0
\]

applying Serre duality and using the hypothesis: \( h^0(F_{C,A} \otimes \omega_S) = 0 \) and \( h^0(F_{C,A} \otimes E_{C,A} \otimes \omega_S) = h^2(\mathcal{O}_S) \) we obtain the inequality \( h^0(C, F_{C,A} \otimes K_C \otimes A^*) \leq h^2(\mathcal{O}_S) + (r + 1)h^1(S, E_{C,A}) \).

We obtain \( \dim \ker(\mu_{0,A}) \leq (r + 1)h^1(S, E_{C,A}) \) and hence the conclusion follows.
Remark 2.3. The assumption $h^2(S, F_{C,A} \otimes E_{C,A}) = h^2(O_S)$ is natural. For stable bundles, it is a sufficient condition for the smoothness of the moduli space at the point defined by $E_{C,A}$. Absent this condition, we obtain the weaker estimate
\[ \dim(C,A)[W] \leq \rho(g, r, d) + \dim[L] + (r + 1)h^1(S, E_{C,A}) + (h^2(S, F_{C,A} \otimes E_{C,A}) - h^2(O_S)). \]
Note that $O_S$ is a direct summand of $F_{C,A} \otimes E_{C,A}$, its complement is $\text{ad}(E_{C,A})$, the bundle of trace-free endomorphisms, and hence
\[ h^2(S, F_{C,A} \otimes E_{C,A}) - h^2(O_S) = h^2(S, \text{ad}(E_{C,A})). \]

Remark 2.4. If in addition $p_g(S) = h^2(O_S) = 0$ then the vanishing of $h^2(S, E_{C,A})$ follows from $h^2(S, F_{C,A} \otimes E_{C,A}) = 0$ and Serre duality in the sequence (\ref{eq:sequence}). The condition $h^2(S, L) = h^0(S, L^* \otimes \omega_S) = 0$ is also automatic, as $L$ is effective and hence $h^0(S, L^* \otimes \omega_S) \leq h^0(S, \omega_S)$. Furthermore, the sequence (\ref{eq:sequence}) twisted by $\omega_S$ shows that $h^1(S, E_{C,A}) = h^0(C, \omega_S \otimes A)$ in this case.

The hypotheses of Theorem 2.2 are realised in a number of situations, which we enumerate below:

1. **K3 surfaces**, see also \([10, 11]\). In this case, $\omega_S \cong O_S$ and $N_{C|S} = K_C$. The hypothesis $h^2(S, F_{C,A} \otimes E_{C,A}) = h^2(O_S) = 1$ is equivalent to simplicity of $E_{C,A}$. The vanishing of $h^1(S, E_{C,A})$ and of $h^2(S, E_{C,A})$ follows from Serre duality and the vanishing of $h^1(S, F_{C,A})$ and of $h^0(S, F_{C,A})$.

2. **Enriques surfaces**, compare to \([10]\). Assume $S$ is an Enriques surface and consider $X \rightarrow S$ the K3 universal cover of $S$. Suppose that for general pair $(C,A)$ in a dominating component $W$, the associated Lazarsfeld-Mukai bundle $E_{C,A}$ is stable with respect to a given polarization $H$. Since the property of being a Lazarsfeld-Mukai bundle is an open condition, the main result of \([13]\) (see Theorem on page 88) shows that for a general $(C,A)$, the bundle $E_{C,A}$ is not isomorphic to $E_{C,A} \otimes \omega_S$ and hence $h^2(S, F_{C,A} \otimes E_{C,A}) = h^2(O_S) = 0$. As noted in Remark 2.4, the condition $h^2(S, E_{C,A}) = 0$ follows.

3. **Rational surface with an anti-canonical pencil**, \([12]\). Suppose that $E_{C,A}$ is stable with respect to a given polarization $H$ and $A \not\cong \omega_S^2|C$. Since $\omega_S^2$ is effective, it follows that there are no non-zero morphisms from $E_{C,A} \otimes \omega_S^2$ to $E_{C,A}$, hence $h^2(S, F_{C,A} \otimes E_{C,A}) = h^2(O_S) = 0$. As pointed out in \([12]\), the existence of an anti-canonical pencil implies $h^1(S, E_{C,A}) = 0$. The vanishing of $h^2(S, E_{C,A}) = h^0(S, F_{C,A} \otimes \omega_S)$ follows from the vanishing of $h^0(S, F_{C,A})$.

3. **Lazarsfeld-Mukai bundles of rank two on rational surfaces**

It was pointed out in Remark 2.4 that if $p_g(S) = p_g(S) = 0$, some of the hypotheses of Theorem 2.2 are superfluous. Under this assumption, since there are fewer conditions, Theorem 2.2 can be substantially refined. We shall consider the case of rank-two Lazarsfeld-Mukai bundles, i.e. associated to base-point-free complete $g^1_d$'s, on rational surfaces. The main result is the following:

**Theorem 3.1.** Let $S$ be a rational surface, $L \geq 0$ be an effective line bundle on $S$ and denote by $k \geq 3$ the maximal gonality of smooth curves on $S$ and by $g$ their genus. Let $H$ be an ample line bundle on $S$ and $C$ be a smooth curve on $S$ of gonality $k$. Suppose
that \( \omega_S \cdot H \leq 0 \), \(-\omega_S \cdot C \geq k \) and that \( C \) is of Clifford dimension one. Then, for any integer \( d \) such that \( k \leq d \leq g - 2k + 2 \), and any component \( W \) of \( W^d_2(L) \) that dominates the linear system \( |L| \), we have

\[
\dim(W) \leq \dim|L| + (d - k). \tag{7}
\]

Note that of \( C \) is an ample divisor, then the polarisation \( H \) can be chosen to be \( O_S(C) \). If the anticanonical bundle is effective, then the polarisation \( H \) can be arbitrarily chosen.

Lelli-Chiesa proved this result for rational surfaces \( S \) with \( h^0(S, \omega_S^2) \geq 2 \). Note that this condition implies automatically \(-\omega_S \cdot C \geq k \), as an anticanonical pencil will restrict to a pencil on \( C \). For sake of completeness, we present here a full proof covering also the case of surfaces with an anticanonical pencil \([12]\), using the strategy from \([4]\); the really new case compared to \([12]\) is Subcase I.c.

**Proof.** We proceed by induction on \( d \geq k \). There are several possible cases, according to the behaviour of the \( g^1_d \)'s and of their associated Lazarsfeld-Mukai bundles.

**Case I.** Assume that, for \((C, A) \in W \) general, \( A \) is base-point-free and complete.

**Subcase I.a.** Assume that, for \((C, A) \in W \) general, \( h^1(S, E_{C,A}) = 0 \) and \( E_{C,A} \) is \( H \)-stable. Since \( \omega_S^2 \cdot H \geq 0 \), the stability implies that there is no non-zero morphism form \( E_{C,A} \otimes \omega_S^2 \) to \( E_{C,A} \), i.e. \( h^2(S, F_{C,A} \otimes E_{C,A}) = h^0(S, F_{C,A} \otimes E_{C,A} \otimes \omega_S) = 0 \). We apply Remark \([2,4]\) and Theorem \([2,2]\).

**Subcase I.b.** Assume that, for \((C, A) \in W \) general, \( h^1(S, E_{C,A}) = 0 \) and \( E_{C,A} \) is not \( H \)-stable. If \( h^2(S, F_{C,A} \otimes E_{C,A}) = 0 \), we apply again Theorem \([2,2]\) hence we may assume that there is a non-zero morphism from \( E_{C,A} \otimes \omega_S \) to \( E_{C,A} \). The bundle \( E_{C,A} \) has a maximal destabilising subsheaf \( M \), which induces an extension

\[
0 \rightarrow M \rightarrow E_{C,A} \rightarrow N \otimes I_\xi \rightarrow 0, \tag{8}
\]

with \( M \cdot H \geq N \cdot H \), where \( \xi \) is a zero-dimensional locally complete intersection subscheme of length \( \ell = \ell(\xi) = M \cdot N - d \). Note that if \( E_{C,A} \not\cong M \otimes N \) then we have (compare to \([4]\) Lemma 3.4):

\[
\dim \Hom(E_{C,A}, N_{C|S} \otimes A^*) = h^0(S, F_{C,A} \otimes E_{C,A}) = \dim \Hom(N \otimes I_\xi, M) + 1. \tag{9}
\]

We suppose that \( E_{C,A} \) is indecomposable. Fix \( N \) and \( \ell \) and denote by \( \mathcal{P}_{N,\ell} \) the parameter space of bundles \( E \) with Chern classes \( c_1(E) = L \) and \( c_2(E) = d \) given by extensions of type \([3]\), and by \( \mathcal{G}_{N,\ell} \) the Grassmann bundle over \( \mathcal{P}_{N,\ell} \) whose fibre over \( E \) is \( G(2, H^0(E)) \). If we assume that \( \mathcal{P}_{N,\ell} \) contains Lazarsfeld-Mukai bundles corresponding to \( W \), then we have a rational map

\[
\pi_{N,\ell}: \mathcal{G}_{N,\ell} \rightarrow W
\]

whose fibre over \( E_{C,A} \) is the projectivisation of \( \Hom(E_{C,A}, N_{C|S} \otimes A^*) \). Since \( \Pic(S) \) is discrete and the Lazarsfeld-Mukai condition is open, it follows that for a given \( N \) and \( \ell \), the map \( \pi_{N,\ell} \) is dominant. Hence, using \([4]\), it suffices to prove that

\[
\dim \mathcal{G}_{N,\ell} - \dim \Hom(N \otimes I_\xi, M) \leq \dim|L| + (d - k). \tag{10}
\]

Similarly to \([4]\) Lemma 3.10] and \([12]\) Lemma 4.1], we obtain the inequality

\[
M \cdot N + \omega_S \cdot N + 2 \geq k. \tag{11}
\]
We have
\[ \dim \mathcal{G}_{N, \ell} \leq \dim G(2, H^0(E_{C,A})) + \dim S^\ell + (\dim \text{Ext}^1(N \otimes \mathcal{I}_\xi, M) - 1), \]
and hence, using Serre duality and observing that \( M \cdot H \geq N \cdot H \geq (N + \omega_S) \cdot H \geq 0 \) implies \( \text{Ext}^2(N \otimes \mathcal{I}_\xi, M) = \text{Hom}(M, N \otimes \mathcal{I}_\xi \otimes \omega_S) = 0 \), we obtain the estimate:
\[ \dim \mathcal{G}_{N, \ell} - \dim \text{Hom}(N \otimes \mathcal{I}_\xi, M) \leq \dim G(2, H^0(E_{C,A})) + 2\ell - \chi(S, M^* \otimes N \otimes \omega_S \otimes \mathcal{I}_\xi) - 1. \]

By the assumption \( h^1(S, E_{C,A}) = 0 \), we have \( h^0(C, \omega_S \otimes A) = 0 \), Remark \[2.4\]. It follows that \( h^0(C, N_C|S \otimes A^*) = g - d - 1 - \omega_S \cdot C \). Since \( h^0(S, E_{C,A}) = 2 + h^0(C, N_C|S \otimes A^*) \) it implies
\[ \dim G(2, H^0(E_{C,A})) = 2(g - d - 1 - \omega_S \cdot C) \]
The conclude the proof of inequality \[10\] we compute by the Riemann-Roch theorem (compare to \[12\]) \[
\chi(S, M^* \otimes N \otimes \omega_S \otimes \mathcal{I}_\xi) = g - 2N \cdot M - \omega_S \cdot M - \ell.
\]
and use \[11\] and the inequality:
\[ h^0(S, L) \geq g - \omega_S \cdot C = \chi(S, L); \]

note that \( h^2(S, L) = 0 \), since \( L \) is effective on a surface with \( p_g = 0 \).

Subcase I.c. Assume that, for all \((C, A) \in \mathcal{W}, h^1(S, E_{C,A}) > 0 \). From Remark \[2.4\] we are in the situation \( h^0(C, \omega_S \otimes A) > 0 \), in particular, \( A \in \{\omega_S^2|C\} + W^0_{d+\omega_S \cdot C}(C) \). Since \( \omega_S \cdot C \leq -k \), it follows that \( A \) moves in a family of dimension \( \leq d - k \).

Case II. Assume that, for any \((C, A) \in \mathcal{W}, A \) has base-points. Then we apply the inductive argument and reduce to the previous case. Note that this case cannot occur for \( d = k \). \( \square \)

4. SYZYgies of curves

In recent curve theory a lot of effort has been put into understanding the relations between syzygies of canonical curves (algebraic objects) and the existence of special linear series (geometric objects). The interest in clarifying these deep relationships between algebraic and geometric properties is high, as failure of vanishing of syzygies produces interesting determinantal cycles on various moduli spaces, and the canonical case is the most natural and basic situation. The precise relationship is predicted by Green’s conjecture: the ideal of a non-hyperelliptic curve is generated by quadrics, and the Clifford dimension controls the number of steps up to which the syzygies are linear.

In the language of Koszul cohomology using duality \[9\], it amounts to the following relation
\[ K_{p,1}(C, K_C) = 0 \text{ for all } p \geq g - c - 1, \]
for any curve \( C \) of genus \( g \) and Clifford index \( c \); for the precise definitions of the objects involved in the statement, we refer to \[9\]. Green’s conjecture is known to be true for general curves, \[17, 18\], and moreover the dimension computations of Brill-Noether loci, in particular conditions similar to \[17\], are related to syzygies of canonical curves. This relationship is explained in \[2, 3\] and has been used in \[4\] for curves on \( K3 \) surfaces. In our case, Green’s conjecture is satisfied for general curves, in the linear system \[L\].
which verify the hypotheses of Theorem 3.1. However, under stronger hypotheses, we can prove Green’s conjecture for every curve in the corresponding linear system:

**Theorem 4.1.** Under the assumptions of Theorem 3.1, suppose moreover that
\[ g - k \geq h^0(S, \omega_S^\otimes 2(C)) \].

Then any smooth curve in \(|L|\) is of gonality \(k\), Clifford dimension one, and satisfies Green’s conjecture.

**Proof.** The long exact sequence associated to
\[ 0 \to \omega_S \to \omega_S \otimes L \to K_C \to 0 \]
suggests that the restriction morphism
\[ H^0(S, \omega_S \otimes L) \to H^0(C, K_C) \]
is an isomorphism and provides us with a long exact sequence on Koszul cohomology [9, (1.d.4)]
\[ 0 = K_p,1(S, -C, \omega_S \otimes L) \to K_p,1(S, \omega_S \otimes L) \to K_p,1(C, K_C) \to K_{p-1,2}(S, -C, \omega_S \otimes L) \to \cdots \]
Green’s vanishing Theorem [9] (3.a.1) implies
\[ K_{p-1,2}(S, -C, \omega_S \otimes L) = 0 \]
for \(p \geq h^0(S, \omega_S^\otimes 2(C)) + 1\), in particular, for any \(C\) and any \(p \geq g - k + 1\) we obtain an isomorphism
\[ K_p,1(S, \omega_S \otimes L) \sim K_p,1(C, K_C) \].

Since Green’s conjecture is valid for general curves \(C \in |L|\) which have gonality \(k\) and Clifford dimension one, we infer that \(K_p,1(S, \omega_S \otimes L) = 0\) for any \(p \geq g - k + 1\). In particular, \(K_p,1(C, K_C) = 0\) for any \(p \geq g - k + 1\) and any smooth curve \(C\). From the Green-Lazarsfeld non-vanishing Theorem [9, Appendix], it follows that any smooth curve \(C\) must have gonality \(k\) and Clifford dimension one. \(\square\)

**Example 4.2** (Smooth curves on Hirzebruch surfaces). The hypotheses of Theorem 4.1 are realised for curves on Hirzebruch surfaces \(S = \Sigma_e\) with \(e \geq 2\), hence we obtain an alternate proof of the results of [4].

Indeed, denote by \(C_0\) the minimal section and by \(F\) the class of a fibre, and let \(C \equiv kC_0 + mF\) with \(m \geq ke\) by a smooth curve on \(S\). The gonality of \(C\) is \(k\) and the genus of \(C\) is computed by the formula
\[ g = (k - 1) \left( m - 1 - \frac{ke}{2} \right). \]

The first condition \(-\omega_S \cdot C \geq k\), from Theorem 3.1 is easily verified, as \((2C_0 + (e + 2)F) \cdot (kC_0 + mF) = -ke + 2m + 2k\). The second condition \(g - k \geq h^0(S, \omega_S^\otimes 2(C))\), from Theorem 4.1 is verified by direct computation, using the vanishing of \(h^1\) of \(\omega_S^\otimes 2(C) \equiv (k - 4)C_0 + (m - 2e - 4)F\) and applying the Riemann-Roch Theorem.

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