Dispersion-reduced spline acoustic finite elements for frequency-domain analysis

Takeshi Okuzono1,*, Toru Otsuru2, Reiji Tomiku2 and Noriko Okamoto3

1Faculty of Engineering, Oita University, 700 Dannohara, Oita, 870–1192 Japan
2Department of Architecture and Mechatronics, Architecture Course, Faculty of Engineering, Oita University, 700 Dannohara, Oita, 870–1192 Japan
3Department of Architecture, Ariake National College of Technology, 150 Higashihagio-Machi, Omuta, Fukuoka, 836–8585 Japan

(Received 18 October 2012, Accepted for publication 16 November 2012)

Keywords: Sound field analysis, Finite-element method, Frequency domain, Dispersion error

PACS number: 43.55.Ka, 43.55.Br [doi:10.1250/ast.34.221]

1. Introduction

The frequency-domain finite-element method (FD-FEM) is an indispensable tool for predicting sound fields of rooms with complicated boundaries accurately. However, this method has the drawback of high computational cost for predicting the sound field of a room at high frequencies, making the application of FD-FEM to practical problems difficult. A way to alleviate this difficulty is to reduce the spatial discretization error in FD-FEM, called dispersion error [1], which is known as the difference between the exact wave number and approximate wave number. Because of this error, a spatial discretization requirement known as a rule of thumb is imposed in the mesh generation process to obtain reliable results. For practical problems, this requirement leads to a solution of a linear system of equations with huge degrees of freedom (DOF) at each frequency.

Many methods are available to reduce the dispersion error in FD-FEM and they are well reviewed in [2]. The authors have also proposed an FD-FEM using hexahedral 27-node spline acoustic elements (Spl27) [3] as high-order elements to reduce the dispersion error. The efficiency versus hexahedral 8- and 27-node Lagrange elements has also been presented in the literature. Also, there exists a considerably simple and efficient method, called the modified integration rule (MIR) [4], for reducing the dispersion error, in which significant error reduction is achieved by modifying only numerical integration points of element matrices. Although the MIR was originally developed for first-order elements, it might be possible to derive integration points to reduce the dispersion error for high-order elements, such as Spl27, that can treat a curved surface more precisely.

In this paper, we propose a method of reducing dispersion error in FD-FEM using Spl27 by modifying the numerical integration points of Spl27. Consequently, an efficient FD-FEM using the improved Spl27, which requires fewer DOF than using conventional Spl27 to obtain results of the same accuracy, is proposed.

2. Preliminaries

2.1 FEM for sound field analysis in frequency domain

The discretized matrix equation of sound field \( \Omega \) for a sound source of angular frequency \( \omega \) is formulated by finite-element discretization as

\[
(K - k^2 M + ic) p = i\omega \rho v_0 W,
\]

where \( K \), \( M \), and \( C \) are the global stiffness matrix, global mass matrix, and global dissipation matrix, respectively. Furthermore, \( k \), \( i \), \( p \), \( \rho \), \( v_0 \), and \( W \) respectively represent the wave number, the imaginary unit, sound pressure vector, air density, velocity of vibration, and distribution vector. The element stiffness matrix \( K_e \), element mass matrix \( M_e \), and element dissipation matrix \( C_e \) used for constructing global matrices are defined as follows:

\[
K_e = \int_{\Omega_e} \nabla N^T \nabla N d\Omega,
\]

\[
M_e = \int_{\Omega_e} N^T N d\Omega,
\]

\[
C_e = \frac{1}{z_n^2} \int_{\Gamma_e} N^T N d\Gamma.
\]

Here, \( z_n \) and \( N \) represent the normalized acoustic impedance ratio and shape function, respectively. \( \Omega_e \) and \( \Gamma_e \) are the region and surface areas of an element to be integrated, respectively. The sound pressures at an angular frequency inside the sound field are calculable by solving the linear system of equations given by Eq. (1).

2.2 Spline finite elements

Spl27 proposed by Otsuru and Tomiku [3], which is hexahedral 27-node isoparametric elements using a spline polynomial function for \( N \), is introduced briefly here.

Spl27 uses the natural cubic spline polynomial \( S_i \) of degree 3 to construct the shape function. By using \( S_i \), the shape function in three dimensions for Spl27 is defined as

\[
N_m(\xi, \eta, \zeta) = S_i(\xi)S_i(\eta)S_i(\zeta) \quad (m = 1, 2, \cdots, 27),
\]
Fig. 1 A FE mesh used to evaluate dispersion error in
FE analysis using three-node line elements.

where $\xi$, $\eta$, $\zeta$ are the coordinates of a hexahedron in a local
coordinate system. Furthermore, the spline function $S_i$ is
defined as

\[
S_i(\xi) = \begin{cases} 
0.25\xi^3 + 0.75\xi^2 + 0.5\xi & : \xi \in [-1, 0] \\
-0.25\xi^3 + 0.75\xi^2 + 0.5\xi & : \xi \in [0, 1] 
\end{cases}
\]

(6)

Here, $\xi_i$ is the local corner coordinate of the hexahedron in the
$\xi$-direction. The function forms of $S_i$ for the $\eta$-
and $\zeta$-directions are identical.

3. Method of reducing dispersion error in FEM using
Spl27

3.1. Derivation of dispersion relation in one dimension

To derive the dispersion relation between the exact wave
number $k$ and approximate wave number $k^h$ in FD-FEM using
spline elements, a dispersion error analysis in one dimension
is performed. The resulting dispersion relation is used to
design an improved Spl27 with low dispersion in the next
section.

The dispersion error is defined as

\[
e_{\text{dis}} = \left| \frac{k^h - k}{k} \right|.
\]

(8)

In Eq. (8), $k^h$ can be derived analytically in the following
way. We consider a finite-element mesh that consists of three-
node spline line elements of nodal distance $d$, as shown in
Fig. 1. The element matrices $K_e$ and $M_e$ defined by Eqs. (2)
and (3) are calculated using the Gauss-Legendre rule with
three integration points as below:

\[
K_e = \sum_{i=1}^{3} W_i \nabla N_i(\xi_i^k)^T \nabla N_i(\xi_i^k) \det(J),
\]

(9)

\[
M_e = \sum_{i=1}^{3} W_i N_i(\xi_i^M)^T N_i(\xi_i^M) \det(J),
\]

(10)

where $J$ is a Jacobian matrix. $W_i$, $\xi_i^k$ and $\xi_i^M$ are the weight
and the local coordinates of the $i$ th integration point for $K_e$
and $M_e$, respectively. $N$ for three-node spline elements is
defined as

\[
N_m(\xi) = S_i(\xi) \quad (m = 1, 2, 3).
\]

(11)

Owing to the symmetry of integration points, we assume that
$W_1 = W_3, \xi_1^k = -\xi_3^k = -\alpha_K, \xi_2^k = 0, \xi_1^M = -\xi_3^M = -\alpha_M,$ and

$\xi_2^M = 0$. Neglecting the dissipation term and the source term,
the element coefficient matrix $(A_e = K_e - k^2M_e)$ of Eq. (1)
with three-node spline elements is given by

\[
A_e = \begin{bmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{22} & a_{12} & a_{11}
\end{bmatrix},
\]

(12)

where

\[
a_{11} = \frac{W_1[4 + 9\alpha_K^2(\alpha_K - 2)^2] + 2W_2}{8} - \frac{k^2dW_1(\alpha_K^2 + \alpha_M^2 - 3\alpha_M^2)}{8}.
\]

(13)

Furthermore, the approximate solution $p_x^h$ at location $x$ is
given by

\[
p_x^h = e^{i\omega x}.
\]

(14)

The finite-element equation at node $x$ is calculated as

\[
a_{13}(p_{x-2d}^h + p_{x+2d}^h) + a_{12}(p_{x-d}^h + p_{x+d}^h) + 2a_{11}p_x^h = 0.
\]

(15)

Similarly, the finite-element equations at nodes $x - d$ and $x + d$ are

\[
a_{12}p_{x-2d}^h + a_{22}p_{x-d}^h + a_{12}p_x^h = 0,
\]

(16)

\[
a_{12}p_{x+d}^h + a_{22}p_{x+d}^h + a_{12}p_{x+2d}^h = 0.
\]

(17)

Combining Eqs. (15)–(17) and substitution of Eq. (14) into
the resulting equation yields

\[
a_{13}\cos(2k^h d) + a_{11} - \frac{a_{12}^2}{a_{22}} (1 + \cos(2k^h d)) = 0.
\]

(18)

After simplification, we obtain the approximate wave number
$k^h$ as

\[
k^h = \frac{1}{2d} \cos^{-1} \left( -\frac{a_{11} + \frac{a_{12}^2}{a_{22}}}{a_{13} - \frac{a_{12}^2}{a_{22}}} \right).
\]

(19)

Using Eq. (19), the dispersion error $e_{\text{dis}}$ in FE analysis using
three-node spline elements is calculable, and the equation also
represents the dispersion relation between $k$ and $k^h$. As an
example, $k^h$ for three-node spline elements using the conventional
Gauss rule ($\alpha_K = \alpha_M = \sqrt{3}/5$, $W_1 = 5/9$ and $W_2 = 8/9$) is given as

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The corresponding DOFs are 132,651, 226,981, and 357,911, integration points are minimized at the corresponding Fig. 2. As expected, dispersion errors for the modified and conventional integration points are shown in
are the most effective values.
d\_K^2 = \frac{d}{C21}=
These integration points are calculated from the conditions derived under each condition, the presented integration points
upper limit frequency. Although other integration points are
d\_K = \frac{d}{C11}=
These integration points are calculated from the conditions that minimize the dispersion error at an arbitrary nondimensional wave number kd on a one-dimensional mesh. Using Eq. (19) to represent the dispersion relation in one dimension, the modified integration points that satisfy \( k^d = k \) at an arbitrary nondimensional wave number kd, are derived under the assumption that \( \alpha_K = \alpha_M \). For simplicity, weights for \( i \) th integration points have conventional values, i.e., \( W_1 = 5/9 \) and \( W_2 = 8/9 \).

3.3. Comparison of dispersion error with that of conventional method in one dimension

The dispersion error in FD-FEM using three-node spline elements in the proposed method is compared with that in FD-FEM using conventional three-node spline elements to show the effectiveness of the proposed method. For improved spline elements in the proposed method, three modified integration points are designed: \( \alpha_K = \alpha_M = 0.870, 0.8669, \) and \( 0.8558 \). These integration points are calculated from the conditions that minimize the dispersion error at \( kd \) corresponding to \( \lambda/d = 4.8, 5.5, \) and \( 6.8 \), where \( \lambda \) is the wavelength of the upper limit frequency. Although other integration points are derived under each condition, the presented integration points are the most effective values.

The dispersion errors calculated using Eq. (19) for modified and conventional integration points are shown in Fig. 2. As expected, dispersion errors for the modified integration points are minimized at the corresponding \( \lambda/d \), and spline elements with the modified integration points outperform the conventional elements with smaller dispersion errors in one dimension.

4. Numerical experiment

The accuracy of FD-FEM using Spl27 with the proposed method for three-dimensional analysis is demonstrated with the help of a numerical example, in which sound pressures calculated by FD-FEM using the conventional and improved Spl27 are compared with those calculated by modal summation (MS). A problem, B0-1F Task B, of the benchmark platform on computational methods for architectural/environmental acoustics [5] is used for the investigation. It is a problem of computing the frequency response of three receiving points R2, R3, and R4 inside a cubic cavity of 1.0 m\(^3\) (Fig. 3) with rigid surfaces at frequencies from 20 Hz to 4 kHz. In the investigation, the frequencies are analyzed with 1 Hz intervals. The speed of sound and air density are 343.7 m/s and 1.205 kg/m\(^3\), respectively. Three FE meshes with different spatial resolutions \( \lambda/d \) are used and the respective values of \( \lambda/d \) are 4.29, 5.16, and 6.0 at 4 kHz.

The corresponding DOFs are 132,651, 226,981, and 357,911, respectively. An absolute diagonal scaled COCG iterative solver with convergence tolerance of \( 10^{-6} \) is used for solving the linear system of equations given by Eq. (1). This is an efficient solver for FD-FEM using Spl27 [6] from the viewpoint of convergence. For the improved Spl27, we used the modified integration points described in the previous section, that is, \( \alpha_K = \alpha_M = 0.870, 0.8669, \) and \( 0.8558 \).

Figure 4 shows comparisons of frequency responses of R2 at higher frequencies from 3 kHz to 4 kHz obtained by MS and by FD-FEM with conventional and modified integration points for a mesh with \( \lambda/d = 4.29 \). Here, for modified integration points, we show only the result of \( \alpha_K = \alpha_M = 0.870 \) as an example. It is clear that FD-FEM with modified integration points outperforms the conventional method. The agreement of frequencies, at which peaks and dips occur, is much better between MS and FD-FEM with modified integration points than that with conventional elements. We
obtained similar results for other numerical experiments conducted concurrently.

To quantitatively estimate the accuracy of FD-FEM with modified integration points, the following error in sound pressure level between MS and FD-FEM is computed for FD-FEM with both integration points at frequencies from 20 Hz to 4 kHz:

$$e_L = \frac{1}{N_l} \sum_{j=1}^{N} e(f_j),$$  \hspace{1cm} (21)$$

with

$$e(f) = \sqrt{\frac{\sum_{j=1}^{N} (L_{\text{Ana}}(f, x_j) - L_{\text{FEM}}(f, x_j))^2}{\sum_{j=1}^{N} L_{\text{Ana}}(f, x_j)^2}}.$$  \hspace{1cm} (22)$$

where \(N_l\) and \(N\) respectively represent the number of frequencies and the number of receiving points. \(L_{\text{Ana}}(f, x_j)\) and \(L_{\text{FEM}}(f, x_j)\) respectively represent the sound pressure level of frequency \(f\) at receiving point \(x_j\) calculated by MS and FD-FEM with conventional and modified integration points. Figure 5 shows the results for all meshes. Results show that the use of modified integration points gives lower error than does the use of conventional points for all meshes. The errors of modified integration points are less than half those of conventional ones for a mesh with the same \(\lambda/d\).

Finally, as a reference, the total number of iterations of the COCG method for the improved method increases slightly compared with that of the conventional method for a mesh with the same \(\lambda/d\). However, the increase amounts to only 1.09 times the number in the conventional method at maximum. The numerical evidence clearly shows the efficiency of the proposed method in three-dimensional analysis.

5. Conclusions

A method of reducing dispersion error in FD-FEM using Spl27 was proposed. This method increases the accuracy of FD-FEM using Spl27 by modifying the numerical integration points of element matrices on the basis of a one-dimensional dispersion relation, and the modified integration points are calculated under a condition that minimizes the dispersion error at an arbitrary nondimensional wave number on a one-dimensional mesh. Numerical results in three dimensions revealed that FD-FEM using the improved Spl27 yielded more accurate results than did the conventional method, with fewer DOF. As a further advantage of the proposed method, it is directly applicable to time-domain FEM using Spl27 [7]. Its accuracy and efficiency will be presented in the future.

Acknowledgment

This research was partially supported by the Research Fund at the Discretion of the President of Oita University and by a Grant-in-Aid for Science Research (B) 24360238 from the Japan Society for the Promotion of Science.

References

[1] I. Harari, “Dispersion, pollution, and resolution,” in Computational Acoustics of Noise Propagation in Fluids, S. Marburg and B. Nolte, Eds. (Springer-Verlag, Berlin/Heidelberg, 2008), Chap. 1, pp. 37–56.
[2] L. L. Thompson, “A review of finite-element methods for time-harmonic acoustics,” J. Acoust. Soc. Am., 119, 1315–1330 (2006).
[3] T. Otsuru and R. Tomiku, “Basic characteristics and accuracy of acoustic element using spline function in finite element sound field analysis,” Acoust. Sci. & Tech., 21, 87–95 (2000).
[4] M. N. Guddati and B. Yue, “Modified integration rules for reducing dispersion error in finite element methods,” Comput. Methods Appl. Mech. Eng., 193, 275–287 (2004).
[5] T. Otsuru, T. Sakuma and S. Sakamoto, “Constructing a database of computational methods for environmental acoustics,” Acoust. Sci. & Tech., 26, 221–224 (2005).
[6] N. Okamoto, R. Tomiku, T. Otsuru and Y. Yasuda, “Numerical analysis of large-scale sound fields using iterative methods Part II: Application of Krylov subspace methods to finite element analysis,” J. Comput. Acoust., 15, 473–493 (2007).
[7] T. Okuzono, T. Otsuru, R. Tomiku and N. Okamoto, “Fundamental accuracy of time domain finite element method for sound-field analysis of rooms,” Appl. Acoust., 71, 940–946 (2010).