Weyl’s Formula as the Brion Theorem for
Gelfand-Tsetlin Polytopes

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Abstract

We exploit the idea that the character of an irreducible finite dimensional \( gl_n \)-module is the sum of certain exponents of integer points in a Gelfand-Tsetlin polytope and can thus be calculated via Brion’s theorem. In order to show how the result of such a calculation matches Weyl’s character formula we prove some interesting combinatorial traits of Gelfand-Tsetlin polytopes. Namely, we show that under the relevant substitution the integer point transforms of all but \( n! \) vertices vanish, the remaining ones being the summands in Weyl’s formula.

1. Introduction

In this section we recall the few theoretical notions discussed in the paper.

1.1. Representations of \( gl_n \) and Gelfand-Tsetlin Polytopes

Consider the Lie algebra \( gl_n(C) \) realized as the set of \( n \times n \) complex matrices (\( n \) can be considered fixed across the entire length of the article). Within the Cartan subalgebra of diagonal matrices we will work with the basis composed of matrices with a single nonzero entry on the diagonal equalling 1. This defines our coordinate system of choice in the corresponding weight space.

Dominant integral weights are then precisely those the coordinates of which constitute a nonincreasing sequence of integers. Each such weight provides an irreducible finite-dimensional representation. In each of these representations one may define the Gelfand-Tsetlin basis consisting of weight-vectors indexed by \textit{Gelfand-Tsetlin patterns}. (We will often abbreviate Gelfand-Tsetlin to GT.)

A Gelfand-Tsetlin pattern is a number triangle with \( n \) rows numbered from 0 (top) to \( n - 1 \) (bottom). The \( i \)-th row contains \( n - i \) numbers. Such a triangle is standardly visualised with its elements in the vertices of a triangular plane tiling, the start of each row thus being at a 150 degree bearing from the start of the previous one. The zeroth row is simply the sequence of the highest weight's

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coordinates. In order to compose a GT pattern the elements of the triangle in rows starting with the first have to be integers meeting one simple requirement:

\[ A_{i-1,j} \geq A_{i,j} \geq A_{i-1,j+1}, \]

where \( A_{i,j} \) is the \( j \)-th (starting with 1 this time) element of the \( i \)-th row of GT pattern \( A \). In other words, each value is between the two values immediately above it.

Let us note that the \( i \)-th coordinate of the weight of the vector corresponding to a GT pattern is equal to the total of row \( n-i \) minus the total of row \( n-i+1 \), the first coordinate being equal to the number in the last row.

GT patterns corresponding to a fixed highest weight can be viewed as the integral points of a polytope which is the corresponding Gelfand-Tsetlin polytope. Specifically, if the elements of all rows but row 0 of a pattern are viewed as the coordinates of a point in \( \mathbb{R}^{\frac{n(n-1)}{2}} \) -dimensional real space, then the requirements turn into linear restrictions. These linear restrictions define the facets of the GT polytope. We will identify a pattern with the corresponding integer point.

The above may be found in any introduction to the representation theory of general linear Lie algebras.

1.2. Brion’s Theorem

Consider the space \( \mathbb{R}^m \) with a fixed basis determining the subset \( \mathbb{Z}^m \) of integer points. To each such point we associate its exponent, the Laurent monomial \( \exp(x) = t_1^{x_1} \cdots t_m^{x_m} \) in formal variables \( t_1, \ldots, t_m \). For a subset \( \Sigma \subset \mathbb{R}^m \) we define its characteristic function as the Laurent series

\[ S(\Sigma) = \sum_{x \in \Sigma \cap \mathbb{Z}^m} \exp(x). \]

Let \( \Sigma \) be a rational polyhedron, i.e. the intersection of a finite number of (closed) half-spaces, each defined by a linear inequality with rational coefficients. By induction on dimension one may straightforwardly deduce that \( S(\Sigma) \) may in this case be obtained as a linear combination of characteristic functions of rational simplicial cones – polyhedral cones with rational vertex and finite set of rational linearly independent generators. However, for such a cone \( C \) it is easy to see that there exists a nonzero Laurent polynomial \( \theta \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \) such that \( \theta S(C) \) is also a Laurent polynomial (i.e. has a finite number of nonzero coefficients). Moreover, the rational function \( \frac{\partial S(C)}{\theta} \) does not depend on the choice of \( \theta \).

It follows that there exists a nonzero Laurent polynomial \( \theta \) such that \( \theta S(\Sigma) \) is also a Laurent polynomial and the rational function \( \frac{\partial S(\Sigma)}{\theta} \) does not depend on the choice of \( \theta \). Now let \( \mathcal{V} \) be the subspace of Laurent series spanned by the characteristic functions of all rational polyhedra. We have just shown that \( \mathcal{V} \subset \mathcal{M} \), the subspace of series which may be multiplied by a nonzero polynomial yielding another polynomial. However, there is a unique linear map

\[ \sigma : \mathcal{M} \to \mathbb{C}(x_1, \ldots, x_n) \]
sending series \( S \) to \( \frac{\theta S}{\theta} \), where \( \theta \) and \( \theta S \) are polynomials. As before it is easy to see that this map is well defined. It is linear because if one considers a linear combination of series from \( \mathcal{M} \) and the same linear combination of corresponding rational functions, then there exists such a nonzero polynomial \( \theta \) that both of these expressions turn into the same polynomial after multiplication by \( \theta \). This means that whenever the linear combination of series is zero so is the linear combination of rational functions, rendering \( \sigma \) linear. Furthermore, \( \sigma \) is obviously a homomorphism of modules over the ring of Laurent polynomials.

This remarkably simple argument proves the following important result.

**Theorem 1.1** (A.V. Pukhlikov, A.G. Khovanskii, [1]). There exists a unique homomorphism of \( \mathbb{C}[t_{\pm 1}, \ldots, t_{\pm m}] \)-modules

\[
\sigma : \mathcal{V} \rightarrow \mathbb{C}(x_1, \ldots, x_n)
\]

sending 1 to 1.

For a polyhedron \( \Sigma \) we will abbreviate \( \sigma(S(C)) \) to \( \sigma(C) \) and refer to it as the integer-point transform (IPT for short) of \( C \), following [4]. (The latter book also contains a detailed and well-written discussion of Brion’s theorem and related topics.)

It easy to define \( \sigma \) explicitly in two basic cases. First of all, if \( \Sigma \) is bounded (is a polytope) then \( \sigma(\Sigma) \) is simply \( S(\Sigma) \). Second, let \( C \) be a simplicial cone with vertex \( v \) and linearly independent set of integral generators \( u_1, \ldots, u_l \). Each of these generators is chosen to be minimal, i.e. have setwise coprime coordinates.

The set \( P \) of points

\[
v + \sum_{i=1}^{l} \alpha_i u_i, \quad \text{all } \alpha_i \in [0, 1)
\]

is the fundamental parallelepiped of \( C \). In these terms one has

\[
\sigma(C) = \frac{S(P)}{(1 - \exp u_1) \ldots (1 - \exp u_l)}.
\] (1)

Let us now move on to (a generalized version of) Brion’s theorem.

For any vertex (zero-dimensional face) of a rational polyhedron \( \Sigma \) one may consider the corresponding vertex cone. Vertex cone \( C_i \) is the intersection of half-spaces corresponding to the facets (maximal proper faces) of \( \Sigma \) containing vertex \( i \).

**Theorem 1.2** (A.V. Pukhlikov, A.G. Khovanskii, [2]). In the above notation the identity

\[
\sigma(\Sigma) = \sum_i \sigma(C_i)
\]

holds in the field of rational functions for any rational polyhedron \( 0 \).

Note that if \( \Sigma \) contains an affine line, then it has no vertices. In this case the above theorem is just the fundamental fact that \( \sigma(\Sigma) = 0 \). For the case of
bounded rational polyhedra (rational polytopes) this theorem was proved in [3] and is referred to as Brion’s theorem.

A key ingredient of the argument below is another identity, closely related to Brion’s theorem. For any positive-dimensional face \( X \) of polyhedron \( \Sigma \) one may analogously define the associated cone as the intersection of half-spaces corresponding to facets of \( \Sigma \) containing \( X \).

**Theorem 1.3.** Consider a rational polyhedral cone \( C \) with the cones \( C_u \) defined at each of its edges (1-dimensional faces) \( u \). Let \( \beta \) be a hyperplane containing the vertex of \( C \) but none of its interior points. Let \( D \) stand for the half-space bounded by \( \beta \) and containing \( C \). Then

\[
\sigma(C) = \sum_{u \not\subset \beta} \sigma(C_u \cap D).
\]

This result and a more general version of it are proved in [5]. However, let us now show that it is immediate from theorem 1.2.

**Proof.** Let \( C' \) be a cone containing \( C \) and obtained from it by some nonzero rational shift, the latter being small enough for \( S(C') = S(C) \) to hold. We then also have \( S(C' \cap D) = S(C) \) and, consequently, \( \sigma(C' \cap D) = \sigma(C) \).

Now, for edge \( u \) of \( C \) let \( u' \) and \( C'_u \) be obtained from \( u \) and \( C_u \) by our shift (i.e. be the corresponding objects for \( C' \)). Then it is easy to see that the vertices of \( C' \cap D \) are precisely \( \{ u' \cap \beta, u \not\subset \beta \} \) and by theorem 1.2 we have

\[
\sigma(C) = \sigma(C' \cap D) = \sum_{u \not\subset \beta} \sigma(C'_u \cap D).
\]

However, for a small enough shift \( S(C'_u \cap D) = S(C_u \cap D) \) for all \( u \), completing the proof.

2. **The Main Result**

Since the integer points of the GT polytope parametrize the vectors of a basis in the corresponding irreducible module, the character of this module is in fact a certain sum over these points. Furthermore, since the weight of a vector depends linearly on the elements of the corresponding pattern, the contribution of each point is in fact its exponent submitted to a certain substitution!

Namely, let the formal variable \( t_{i,j} \) correspond to the \( j \)-th element of the \( i \)-th row (\( 1 \leq i \leq n - 1 \) and \( 1 \leq j \leq n - i \)). Exponents of points in GT polytopes are Laurent monomials in these \( t_{i,j} \). On the other hand, let \( x_i \) be the exponent of the weight with \( i \)-th coordinate 1 and the rest 0 (\( 1 \leq i \leq n \)). The \( x_i \) are then the generators of the character ring. Now, fix a dominant integral weight \( \lambda = (\lambda_1, \ldots, \lambda_n) \). Let \( L_\lambda \) be the corresponding irreducible module and \( GT_\lambda \) the corresponding GT polytope. The information provided in the introduction lets one then easily obtain the following.
Theorem 2.1. The character $\text{char} L_\lambda$ is equal to $S(GT_\lambda)$ submitted to the substitution

$$t_{i,j} \rightarrow x_{n-i}x_{n-i+1}^{-1}$$

and then multiplied by $x_{n}^{\lambda_1}+...+\lambda_n$.

We will denote $F(X)$ the result of application of (2) to expression $X$.

Of course, char $L_\lambda$ is the corresponding Schur polynomial and can be expressed via the Weyl character formula. On the other hand, $S(GT_\lambda)$ is provided by Brion’s theorem. Our aim is to show why these two approaches can be said to give the same result. This is especially true in the case of a strictly dominant weight $\lambda$, i.e., a weight for which the sequence $(\lambda_i)$ decreases strictly (also known as regular dominant weight). Let us formulate this statement precisely.

Theorem 2.2. For a regular dominant integral weight $\lambda$ the polytope $GT_\lambda$ has exactly $n!$ simplicial vertices. After the application of substitution (2) the IPTs of all nonsimplicial vertices vanish. The IPTs of the simplicial vertices, after the substitution and multiplication by $x_{n}^{\lambda_1}+...+\lambda_n$, become exactly the $n!$ summands in Weyl’s character formula.

Most of what follows will be devoted to the proof of theorem 2.2. The case of a singular highest weight will be briefly discussed in the end of the paper.

3. Proof of the Main Theorem

Fix a regular dominant integral weight $\lambda$. For all vertices $A$ of $GT_\lambda$ we are to calculate the expressions $F(\sigma(C_A))$, where $C_A$ is the vertex cone.

We will start off by exploring some of the structure of $GT_\lambda$. From the fact that vertices are, in general, precisely the points of a polytope contained in a maximal subset of facets, one easily deduces the following characterization of the vertices of $GT_\lambda$.

Proposition 3.1. A GT pattern $A$ is a vertex of $GT_\lambda$ iff $A_{i,j}$ is equal to at least one of $A_{i-1,j}$ and $A_{i-1,j+1}$ (the two elements above) for every $1 \leq i \leq n - 1$ and $1 \leq j \leq n - i$.

The following notion turns out to be exceptionally convenient. With each vertex $A$ of $GT_\lambda$ let us associate a graph $\Gamma_A$. The vertices of the graph are all of the corresponding pattern’s elements. We connect two elements with an edge iff they are equal and in consecutive rows. The two figures below are examples.
The edges correspond to facets containing the vertex or, put differently, facets of the vertex cone. (This uses the obvious fact that none of the inequalities follow from the rest and thus each one actually provides a facet of $GT_{\lambda}$.)

Consider now the connected components of $\Gamma_A$. There is exactly $n$ of them, one for each value in the 0-th row. Let us describe the set of all possible components of all the graphs $\Gamma_A$. Consider $T$, a graph the vertices of which are positioned like the elements of a GT pattern, forming a triangle of height $n$ and wherein every vertex outside the top row is joined with the two immediately above. The components in question can then be any full(!) connected subgraph $\Delta$ of $T$ with a single vertex in the top row possessing the following property.

A. If $\Delta$ contains two consecutive vertices of a row, $\Delta$ necessarily contains all (either one or two) vertices which are adjacent to both of the former vertices (adjacent in $T$ and hence in $\Delta$).

The vertices of such a $\Delta$ are, of course, again divided into rows, which we will enumerate in the same way.

Each such a graph $\Delta$ defines a cone $C_\Delta$. This cone is contained in the coordinate space which has a coordinate for each vertex of $\Delta$ other than the one in the top row. We place the vertex of the cone at the origin for convenience. Facets, in complete analogy to the vertex cones of GT polytopes, are defined as follows. Each edge connecting two vertices neither in the top row imposes the restriction that one of the corresponding coordinates is less than or equal to the other, the sign depending on whether the lower vertex is to the left or to the right of the upper one. An edge from the top vertex adds the restriction that the coordinate corresponding to the lower vertex is either nonpositive or nonnegative, the sign being determined as before.

Note that the IPT of such a cone $C_\Delta$ can be naturally viewed as an expression in the variables $t_{i,j}$ introduced above.

We consider these cones for one simple reason. For any vertex $A$ the cone $C_A$ is, visibly, the Minkowski sum of its vertex and the cones $C_\Delta$ for all components $\Delta$ of $\Gamma_A$. (With the spaces containing the $C_\Delta$ being embedded into $C_A$’s space in the obvious way.) This directly implies that $F(\sigma(C_A))$ is the product of $F(\sigma(C_\Delta))$ over all the components and $F(\exp A)$. This observation shows that it is, in fact, sufficient to calculate the expressions $F(\sigma(C_\Delta))$.

In order to be able employ theorem 1.3 we must describe the set of edges of an arbitrary cone $C_\Delta$. Here is such a description in terms of the edges’
generators. Since in this case the vertex is the origin, a generator's coordinates coincide with those of the edge's closest point to the origin.

**Proposition 3.2.** The coordinates of the generator of an edge of $C_\Delta$ can only equal 0, 1 or $-1$ with exactly one of the last two values occurring. The full subgraph of $\Delta$ whose vertices are all the vertices corresponding to coordinates taking some specific value is connected and has property A introduced above.

Here are a few examples of such generators for various $\Delta$ with solid lines representing edges of the mentioned subgraphs and dotted lines representing other edges of $\Delta$. These examples will also be useful to us later on.

![Figure 3](image.png)
![Figure 4](image.png)
![Figure 5](image.png)
![Figure 6](image.png)

**Proof.** Assume $\Delta$ contains at least two vertices, i.e. $C_\Delta$ is not a point and actually has edges. The proof utilizes the characterization of an edge of a cone as any ray with endpoint at the vertex for which the subset of facets containing it is maximal (amongst such rays).

First of all, if at least three different values of coordinates are present, changing all the coordinates with a certain value to take another certain occurring value will provide a ray which is contained in a strictly larger set of facets. Now, $\Delta$ includes at least one vertex in row 1 (below the top zeroth row). If two different nonzero values are present then the coordinate from row 1 is equal to one of them. Change it and all the coordinates of the same value to zero, again obtaining a ray contained in a larger set of facets.

Minimality of the generator requires coprimeness of the coordinates, completing the proof of the statement regarding the possible sets of values of coordinates.

Property A follows simply from the fact that the vector gives a point in the cone. A violation of this property by one of these graphs immediately entails the violation of one of the linear restrictions defining $C_\Delta$ by the generator’s endpoint.

The vertex cones can be now divided into simplicial and nonsimplicial ones in a pleasantly simple way.

**Proposition 3.3.** A cone $C_\Delta$ is simplicial iff graph $\Delta$ is acyclic. Consequently, a vertex cone $C_A$ is simplicial iff $\Gamma_A$ is acyclic.
Thus figure 1 on page 6 is an example of a nonsimplicial vertex while figure 2 is an example of a simplicial one.

Proof. \( C_\Delta \) is simplicial iff it has as many facets as the dimension of its affine hull. The number of facets is the number of edges in \( \Delta \). The dimension is the number of vertices minus one. Since \( \Delta \) is connected the equality is equivalent to \( \Delta \) being acyclic. \( \square \)

The first statement of theorem 2.2 has visibly been reduced to the following.

**Theorem 3.1.** If \( \Delta \) is acyclic, then \( F(\sigma(C_\Delta)) = 0 \).

Proof. We will proceed by induction on the number of vertices in \( \Delta \). In order to carry out the step we will consider two cases, the base of the induction being proved within the second case.

**Case 1.** Row 1 (second from the top!) contains only one vertex of \( \Delta \).

Consider the coordinate corresponding to this vertex. This case is simple because there is only one edge for which this coordinate of the generator is nonzero. This is the vector \( v \) all of whose coordinates are the same. It easily seen that any other vector for which the coordinate in question is nonzero is contained in a smaller set of facets. Figure 8 is an example of such a vector \( v \).

This means that the \( \sigma(C_\Delta) \) is the product of \( (1 - \exp v)^{-1} \) and \( \sigma(C') \), where \( C' \) is the section of \( C \) by the hyperplane where the coordinate in the top row is zero. We show that \( F(\sigma(C')) \) vanishes due to the induction hypothesis.

Indeed, let \( \Delta' \) be obtained from \( \Delta \) by removing the vertex in the top row (number 0) and then shifting the whole graph up one row (with the horizontal shift, evidently, being insignificant). On one hand, \( F(\sigma(C_{\Delta'})) = 0 \) by induction hypothesis. On the other, one sees that the cones \( C' \) and \( C_{\Delta'} \) can be identified in a natural way. This identification shows that \( F(\sigma(C')) \) is obtained from \( F(\sigma(C_{\Delta'})) \) by the change of variables corresponding to the "downward" shift, namely \( x_i \to x_{i-1} \) for all \( i \). This completes the induction step.

**Case 2.** Row 1 contains two vertices.

This step is similar, however, it invokes theorem 1.3. We apply the theorem to cone \( C_\Delta \) and the hyperplane \( \beta \) obtained by fixing the coordinate corresponding to the left vertex in row 1. We now need a description of the edges for whose generator this coordinate is nonzero (and thus equal to 1).

For the most part, the description provided by proposition 3.2 suffices. For one of the generators in consideration \( v \) let \( \Delta_1 \) and \( \Delta_2 \) be full subgraphs of \( \Delta \) defined the following way. \( \Delta_1 \) contains the vertex in the top row, the left vertex in row 1 and all the other vertices for which \( v \)'s corresponding coordinate is 1. Meanwhile, \( \Delta_2 \) contains the vertex in the top row and all vertices from other rows not contained in \( \Delta_1 \), or, in other words, all the vertices for which \( v \)'s corresponding coordinate is 0. Figures 4, 5 and 6 demonstrate splittings into \( \Delta_1 \) and \( \Delta_2 \).

In terms of theorem 1.3 the contribution of the edge generated by \( v \) is the IPT of a certain cone. Namely, the cone bounded by \( \beta \) and all facets containing this edge. This cone, however, visibly coincides with the product of \( \Delta_1 \) and
Due to proposition 3.2 these two graphs are connected and satisfy property A, thus allowing us to conclude that \(\sigma(C_\Delta) = \sigma(C_{\Delta_1})\sigma(C_{\Delta_2})\). If either one of the graphs is cyclic as in figure 4, induction hypothesis will thus provide \(F(\sigma(C_\Delta)) = 0\).

It remains to deal with the generators for which neither \(\Delta_1\) nor \(\Delta_2\) is cyclic. In view of property A, this is only possible if \(\Delta\) contains no more than two vertices from any row and in case there are two, the left one belongs to \(\Delta_1\) while the right one to \(\Delta_2\). Consequently, if \(\Delta\) only has one vertex in some row (other than the top one), then all the lower rows must contain no more than one vertex as well. Otherwise one of \(\Delta_1\) and \(\Delta_2\) would fail to be connected.

This means that if there are at all edges for which \(\Delta_1\) and \(\Delta_2\) are both acyclic, then there are exactly two of them. They differ by whether all those vertices which are alone in their row belong to \(\Delta_1\) or \(\Delta_2\) (i.e. whether the corresponding coordinates of the generator are 1 or 0). The contributions of these two edges then add up to zero, which is verified by a straightforward calculation. Figures 5 and 6 present such a pair of edges. Note that this fact also provides our induction’s base, the case of \(\Delta\) being a single cycle with four vertices.

Let us now move on to the part of theorem 2.2 concerning the simplicial vertices of \(\Gamma_A\). For \(C_A\) to be simplicial the graph \(\Gamma_A\) must be acyclic. Due to property A of the components, this in turn is possible only if the values in each row of \(A\) are pairwise distinct. In such a pattern each row is necessarily obtained from the previous by simply removing one of the elements. The pattern is defined by the order in which the elements \((\lambda_1, \ldots, \lambda_n)\) are removed, a permutation of the set \(\{1, \ldots, n\}\). Thus there are indeed \(n!\) simplicial vertices which are parametrized by the permutation group \(S_n\). In order to comply with the standard notations we will associate with a simplicial vertex the permutation \(w\) such that \(\lambda_{w(i)}\) occurs in row \(n - i\) but not row \(n - i + 1\). Thus the weight of the vector corresponding to the pattern associated with permutation \(w\) is equal to \((\lambda_{w(1)}, \ldots, \lambda_{w(n)})\).

The proof of theorem 2.2 will be complete with the proof of this proposition.

**Proposition 3.4.** Let \(A\) be the simplicial vertex corresponding to permutation \(w\). Then \(x_{\lambda_1}^{\lambda_1} \cdots + x_n^{\lambda_n} F(\sigma(C_A))\) is equal to

\[
\frac{(-1)^w \prod_{1 \leq i < j \leq n} (1 - x_i x_j) \prod_{1 \leq i < j \leq n} (1 - x_i^{\lambda_{w(1)}} + (n+1-w(1))-n \ldots x_i^{\lambda_{w(n)}} + (n+1-w(n))-1)}{x_1^{\lambda_{w(1)}} + x_n^{\lambda_{w(n)}} - \sum x_i^{\lambda_{w(i)}} x_j^{\lambda_{w(j)}}},
\]

which is clearly equal to a summand in Weyl’s formula.

**Proof.** First of all, note that \(x_{\lambda_1}^{\lambda_1} \cdots + x_n^{\lambda_n} F(\exp A)\) is simply equal to \(x_1^{\lambda_{w(1)}} \cdots x_n^{\lambda_{w(n)}}\).

Now consider the components of \(\Gamma_A\) each of which is a path graph. Proposition 3.2 lets us work out the following about the set of generators of all \(\frac{n(n-1)}{2}\) edges of \(C_A\). For each pair \(1 \leq i \leq j \leq n-1\) there is exactly one edge which has one nonzero coordinate in each row from \(i\) to \(j\) and all other coordinates 0.
The coordinates are \(-1\) if \(w(n+1-i) > w(n-j)\) and 1 otherwise. With the nature of substitution (2) taken into account, this means that

\[
\frac{F(\sigma(C_A))}{F(\exp A)} = (-1)^w X \prod_{1 \leq i < j \leq n} (1 - x_i^{-1} x_j),
\]

where \(X\) is a monomial defined by the following rule. The power of \(x_i\) is equal to the quantity of \(j > i\) such that \(w(j) < w(i)\) minus the quantity of \(j < i\) such that \(w(j) > w(i)\). This number is known to equal

\[
i - w(i) = (n+1 - w(i)) - (n+1 - i),
\]

which completes the proof of the proposition and the main theorem.

\[\qed\]

4. The Case of a Singular Weight

Recall that in this case the sequence \((\lambda_i)\) is not strictly decreasing. Here we may define the graphs \(\Gamma_A\) in the exact same way. However, the presence of equal coordinates in \(\lambda\) leads to the components \(\Delta\) not necessarily having just one vertex in the top row. They are still, nonetheless, full subgraphs with property A and their edges still satisfy the description in proposition 3.2.

The idea is that in this case the set of \(GT_\lambda\)'s vertices is smaller and the vertices are obtained by some of the vertices from the regular case "gluing" together. This is formalized as follows. One sees that for any two regular dominant weights \(\lambda_1\) and \(\lambda_2\) there is a one-to-one correspondence between the vertices of \(GT_{\lambda_1}\) and \(GT_{\lambda_2}\), matching vertices with the same graphs \(\Gamma_A\). Let \(\Gamma\) be the set of all the graphs \(\Gamma_A\) occurring in the regular case. Similarly, for our singular dominant weight \(\lambda\) let \(\Gamma_\lambda\) be the set of \(\Gamma_A\) for all vertices \(A\) of \(GT_\lambda\). Since some of \(\lambda_i\)'s consecutive coordinates have become equal, a graph \(\Gamma_A\) from the regular case corresponds to the graph from \(\Gamma_\lambda\) obtained by joining components corresponding to equal coordinates by all possible edges. More formally, there is a map \(f_\lambda : \Gamma \to \Gamma_\lambda\) sending each graph to the only graph which it is a subgraph of. This map is easily understood to be surjective.

The relation between the character formula and Brion's theorem in this case is shown by this theorem.

**Theorem 4.1.** Let \(\lambda_1\) be some regular dominant integral weight. For a vertex \(A\) of \(GT_\lambda\) let \(\{A_i\}\) be the set of vertices of \(GT_{\lambda_1}\) such that \(f_\lambda(\Gamma_{A_i}) = \Gamma_A\) for all \(A_i\). Then

\[
\sigma(C_A) = \sum_{A_i} \exp (A - A_i) \sigma(C_{A_i}).
\]

In other words, \(\sigma(C_A)\) is equal to the sum of IPTs of cones obtained from the \(C_{A_i}\) by shifting their vertex to \(A\). (Clearly, this does not depend on the choice of \(\lambda_1\).)

The results in previous sections show that if one writes out Brion’s formula, applies substitution (2) and then decomposes each \(F(\sigma(C_A))\) in accordance with the above formula, one obtains the Weyl character formula for \(\lambda\).
Proof. Consider the edges of $\Gamma_A$. Each such edge provides a hyperplane which intersects $C_A$ in a face of positive dimension (due to the singularity of our weight the intersection is not necessarily a facet). $C_A$ is then the intersection of the corresponding half-spaces. The idea is to shift each of these half-spaces slightly "outwards", i.e. slightly weaken each of the inequalities. This may be done in such a way that the vertices of polyhedron $P$ which is the intersection of the shifted half-spaces are in one-to-one correspondence with the set $f_A^{-1}(\Gamma_A)$, the correspondence being given by considering the set of the shifted hyperplanes containing some vertex. Furthermore, if the shifts are small enough we have $S(P) = S(C_A)$ and $\exp(A - A_i)S(C_{A_i})$ is equal to the characteristic function of $P$'s corresponding vertex. Applying theorem 1.2 to $P$ then yields the desired identity.

For sake of completeness let us give an explicit construction of $P$. Let us weaken every inequality defining $GT_\lambda$ by the same number. That is for a positive rational number $\varepsilon$ we substitute the inequality $A_{i,j} - A_{k,l} \leq 0$ by $A_{i,j} - A_{k,l} \leq \varepsilon$, inequality $A_{i,j} \leq \alpha$ by $A_{i,j} \leq \alpha + \varepsilon$ and inequality $A_{i,j} \geq \alpha$ by $A_{i,j} \geq \alpha - \varepsilon$. Denote the obtained polytope $GT_\varepsilon$. If one shifts $GT_\varepsilon$ by adding $(i + 2(n - 1 - j))\varepsilon$ to the $j$-th coordinate in row $i$ one comes up with polytope

$$GT_{(\lambda_1 + 2(n-1)\varepsilon, \lambda_2 + 2(n-2)\varepsilon, \ldots, \lambda_n)}$$

(which is defined just in the same way as for an integral weight). This implies that for a small enough $\varepsilon$ the relevant set of facets of $GT_\varepsilon$ bound a polyhedron which may be taken for $P$.

It is worth noting that the cumbersome argument in the last part of [5] can be replaced by a more transparent one similar to the above.

5. Remark

Although the polytopes considered in paper [5] are infinite-dimensional, one cannot help but notice the similarities they share with the GT polytopes. That being the fact that under a specific substitution (in these two cases of representation-theoretic nature) the IPTs of all the nonsimplicial vertices vanish.

Furthermore, there’s an additional fact which holds in both situations but isn’t employed directly in either of them. Substitution $F$ corresponds to a linear map $\varphi : \mathbb{R}^{n(n-1)/2} \rightarrow \mathbb{R}^m$ such that $F(\exp A) = \exp(\varphi(A))$ for all points $A$. It turns out that the simplicial vertices of $GT_\lambda$ (for which the IPT doesn’t vanish under $F$) are precisely those vertices which are mapped into vertices of polytope $\varphi(GT_\lambda)$. This means that in our case we have

$$F(\sigma(C_A)) = 0 \iff \sigma(\varphi(C_A)) = 0$$
for any vertex cone $C_A$. The analogous statements also hold for the polytopes in [5].

It would be very interesting to obtain some more general description of situations in which these effects take place. This was, to some extent, the goal of the unanswered MathOverflow question [6].

**References**

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