A REPRESENTATION FORMULA FOR MEMBERS OF SBV DUAL

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Abstract. We give an integral representation formula for members of the dual of $SBV(\mathbb{R}^n)$ in terms of functions that are defined on $\mathbb{R}^n$, an appropriate fiber space that we introduce, consisting of pairs $(x, [E]_x)$ where $[E]_x$ is an approximate germ of an $(n-1)$-rectifiable set $E$ at $x$.

1. Introduction

Let $n \geq 2$ and $\Omega \subseteq \mathbb{R}^n$ be open. The Banach spaces $BV(\Omega)$ and $SBV(\Omega)$ consist, respectively, of functions of bounded variation [2] 3.1 and special functions of bounded variation [2, §4.1]. A long-standing open problem [1, 7.4] is to provide a useful description of the dual of $BV(\Omega)$. This seems to be still open, despite important contributions by N.G. Meyers and W.P. Ziemer [9], N.C. Phuc and M. Torres [10], and N. Fusco and D. Spector [8].

Regarding the dual of $SBV(\Omega)$, the second author contributed the article [5]. In the present paper, we describe the dual of $SBV(\Omega)$ in a way that is “optimal” in some specific universal sense.

For each $u \in SBV(\Omega)$, the distributional gradient $Du$ of $u$ decomposes as $Du = \mathcal{L}^n \subseteq \nabla u + \mathcal{H}^{n-1} \subseteq j_u$, where $\nabla u$ is the pointwise a.e. approximate gradient of $u$, and $j_u$ is a vector field carried on the approximate discontinuity set $S_u$ of $u$, on which we have $j_u = (u^+ - u^-)\nu_u$, $\nu_u$ being a unitary field normal to $S_u$, and $u^+$, $u^-$ the approximate limits of $u$ on either sides of $S_u$. We have

$$
\|u\|_{SBV(\Omega)} = \int_\Omega |u| d\mathcal{L}^n + \int_{S_u} \|j_u\| d\mathcal{H}^{n-1} + \int_\Omega \|\nabla u\| d\mathcal{L}^n.
$$

Thus, $u \mapsto (u, j_u, \nabla u)$ is an isometric embedding

$SBV(\Omega) \to L^1(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \times L^1(\Omega, \mathcal{B}(\Omega), \mathcal{H}^{n-1}; \mathbb{R}^n) \times L^1(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n; \mathbb{R}^n)$.

Since $(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n)$ is $\sigma$-finite, the dual of the corresponding $L^1$ space is, of course, the corresponding $L^\infty$ space. This remark does not apply to the middle measure space $(\Omega, \mathcal{B}(\Omega), \mathcal{H}^{n-1})$. An attempt at understanding the dual of the corresponding $L^1$ space leads to a study of the canonical map

$\Upsilon: L^\infty(\Omega, \mathcal{B}(\Omega), \mathcal{H}^{n-1}) \to L^1(\Omega, \mathcal{B}(\Omega), \mathcal{H}^{n-1})^*$.

This relates to the Radon-Nikodym Theorem. In this respect, we note that:

- $\Upsilon$ is not surjective. If $\mathcal{B}(\Omega)$ is replaced with $\mathcal{M}_{\mathcal{B}^{n-1}}(\Omega)$, whether $\Upsilon$ is surjective or not is undecidable in ZFC.

- $\Upsilon$ is injective. If $\mathcal{B}(\Omega)$ is replaced with $\mathcal{M}_{\mathcal{B}^{n-1}}(\Omega)$, $\Upsilon$ is not injective.

Here, $\mathcal{M}_{\mathcal{B}^{n-1}}(\Omega)$ is the $\sigma$-algebra of $\mathcal{H}^{n-1}$-measurable subsets of $\Omega$. For a detailed treatment of these, see [4]. Our goal now is to stick with the axioms of ZFC.

The problem is with measurability. Let $\varphi \in L^1(\Omega, \mathcal{B}(\Omega), \mathcal{H}^{n-1})$ and let $E \subseteq \Omega$ be Borel such that $\mathcal{H}^{n-1}(E) < \infty$. If $\iota_E: L^1(E, \mathcal{B}(E), \mathcal{H}^{n-1}) \to L^1(\Omega, \mathcal{B}(\Omega), \mathcal{H}^{n-1})$ is

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the obvious injection, then \( \varphi \circ \iota_E \) belongs to the dual of \( L^1(E, \mathcal{B}(E), \mathcal{H}^{n-1}) \) and, by the classical Riesz Theorem, is represented by a Borel measurable function \( g_E : E \to \mathbb{R} \), i.e. 
\[
\langle \varphi \circ \iota_E \rangle(f) = \int_f g_E d\mathcal{H}^{n-1} \quad \text{whenever } f \in L^1(E, \mathcal{B}(E), \mathcal{H}^{n-1}).
\]
The family \( \langle g_E \rangle_E \) is compatible in the sense that \( \mathcal{H}^{n-1}(E \cap E' \cap \{ g_E \neq g_{E'} \}) = 0 \) for all \( E, E' \). There does not exist, in general, a gluing of this family, i.e. a Borel measurable function \( g : \Omega \to \mathbb{R} \) such that \( \mathcal{H}^{n-1}(E \cap \{ g \neq g_E \}) = 0 \) for all \( E \), and whether an \( \mathcal{H}^{n-1} \)-measurable gluing of \( \langle g_E \rangle_E \) exists is undecidable.

In fact, a “proper” gluing of \( \langle g_E \rangle_E \) at \( x \in \Omega \) depends both on \( x \) and \( E \). However, it will depend on \( (x, E) \) only according to the behavior of \( E \) near \( x \). This prompts us to introduce a notion of approximate germ of \( E \) at \( x \) – finer than the first order tangential behavior in case \( E \) is, for instance, a smooth hypersurface. Carrying out these ideas to describe the dual of \( L^1(\Omega, \mathcal{B}(\Omega), \mathcal{H}^{n-1}) \) presents technical difficulties that we avoid by considering a slightly different measure space. The jump set \( S_u \) of a function of bounded variation being countably rectifiable, one has \( \mathcal{H}^{n-1} \subseteq S_u = \mathcal{J}_{\mathcal{H}}^{n-1} \subseteq S_u \), where \( \mathcal{J}_{\mathcal{H}}^{n-1} \) is an integral geometric measure. The application to studying \( SBV(\Omega) \) is not affected either when we replace \( \mathcal{J}_{\mathcal{H}}^{n-1} \) by its semi-finite version \( \mathcal{J}^{n-1} \). Both measures are described in [2.2].

What we have gained is that each set \( E \in \mathcal{B}(\Omega) \) with \( \mathcal{J}^{n-1}(E) < \infty \) has the property that \( \Theta^{n-1}(E, x) = 1 \) for \( \mathcal{J}^{n-1} \)-almost every \( x \in E \), according to the Structure Theorem. One recognizes a density property. This is an essential ingredient of our proof below and would not hold with \( \mathcal{H}^{n-1} \) in place of \( \mathcal{J}^{n-1} \).

Notice that a family \( \langle g_E \rangle_E \) is, in fact, the same thing as a function \( g(x, E) \) that depends both on \( x \) and \( E \), as in [3], but the compatibility of the family \( \langle g_E \rangle_E \) considered above means that there is then some redundancy in the corresponding space of variables \((x, E)\). The gist of the present paper is to consider a special quotient of the space of pairs \((x, E)\) and to equip it with a structure of a measure space \((\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathcal{J}}^{n-1})\), see [2.2] below. One can of course wonder if the quotient presented here is “optimal”, i.e. the smallest one for the purpose of representing the dual of \( L^1 \), or if we are still left with some redundancy in the variables. The optimality of the measure space \((\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathcal{J}}^{n-1})\) is stated in [3] and established there, in an appropriate categorical setting. Thus, the combination of [3] and the present paper can be considered a solution to describing the dual of \( SBV(\Omega) \), see Theorem 2.10 as explicitly as one possibly can in ZFC.

1.1 (Notations). — Our choice of notation is mostly compatible with that of [6] and [2]. As an exception, we let \( \ominus \) be the set theoretic symmetric difference. In \( \mathbb{R}^n \), the \((n - 1)\)-dimensional Hausdorff measure is \( \mathcal{H}^{n-1} \) and the Lebesgue measure is \( \mathcal{L}^n \). The \( \sigma \)-algebras consisting of Borel subsets and \( \mathcal{H}^{n-1} \)-measurable subsets of \( E \subseteq \mathbb{R}^n \) are respectively denoted by \( \mathcal{B}(E) \) and \( \mathcal{M}_{\mathcal{H}^{n-1}}(E) \). The restriction symbol for measures is \( \mathcal{L}^{-}\) [6 2.1.2]. The lower and upper \((n - 1)\)-dimensional densities of \( E \subseteq \mathbb{R}^n \) at \( x \in \mathbb{R}^n \), denoted \( \Theta^{n-1}(E, x) \) and \( \Theta^+^{n-1}(E, x) \), respectively, are defined in [6 2.10.19] (with \( \mu = \mathcal{L}^n \subseteq \mathcal{L} \)). Both \( x \mapsto \Theta^{n-1}(E, x) \) and \( x \mapsto \Theta^+^{n-1}(E, x) \) are Borel measurable. When they coincide at \( x \), the common value is denoted \( \Theta^{n-1}(E, x) \).

2. Results

2.1. — A set \( E \subseteq \mathbb{R}^n \) is called countably \((n - 1)\)-rectifiable whenever there are countably many Lipschitz maps \( f_k : \mathbb{R}^{n-1} \to \mathbb{R}^n \) such that
\[
\mathcal{H}^{n-1}\left(E \setminus \bigcup_{k=0}^{\infty} f_k(\mathbb{R}^{n-1})\right) = 0.
\]
Also, \( E \) is called \((n - 1)\)-rectifiable if it is \( \mathcal{H}^{n-1} \)-measurable, of finite \( \mathcal{H}^{n-1} \)-measure, and countably \((n - 1)\)-rectifiable.

Throughout the paper, we fix an open subset \( \Omega \subseteq \mathbb{R}^n \). For an \((n - 1)\)-rectifiable set \( E \subseteq \Omega \), the following hold:
The set $E = \{ x \in \Omega : \Theta^{n-1}(E, x) = 1 \}$ coincides with $\mathcal{H}^{n-1}$-almost everywhere with $E$, i.e., $\mathcal{H}^{n-1}(\hat{E} \cap E) = 0$. It satisfies:

- $E$ is a Borel measurable $(n-1)$-rectifiable subset of $\Omega$;
- A point $x \in \Omega$ is in $\hat{E}$ if $\Theta^{n-1}(\hat{E}, x) = 1$.

2.2. — For $S \subseteq \Omega$, we let

$$\mathcal{F}^{n-1}(S) = \sup \{ \Theta^{n-1}(S \cap E) : E \subseteq \Omega \text{ is an } (n-1)\text{-rectifiable set} \}.$$

It is easily checked that $\mathcal{F}^{n-1}$ is a Borel regular outer measure on $\Omega$. In fact, the measure $\mathcal{F}^{n-1}$ is the semi-finite version of the Structure Theorem [6, 3.3.14] and [6, 3.2.26]. The latter is a consequence of the Structure Theorem [6, 3.3.14] and [6, 3.2.26]. The measure $\mathcal{F}^{n-1}$ is much more manageable than the Hausdorff measure $\mathcal{H}^{n-1}$, as it ignores purely unrectifiable sets. We notice that $\mathcal{H}^{n-1} \setminus E = \mathcal{F}^{n-1} \setminus E$ for any countably $(n-1)$-rectifiable Borel set $E \subseteq \Omega$. To find an integral representation for the elements of $SBV(\Omega)^*$, it is crucial to describe the dual of $L^1(\Omega, \mathcal{B}(\Omega), \mathcal{F}^{n-1})$. We will do this in Theorem 2.7.

Prior to that, we must construct a particular measure space $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathcal{F}}^{n-1})$.

2.3 (The measure space $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathcal{F}}^{n-1})$). — Let $\mathcal{E}$ be the collection of sets that satisfy conditions (C) and (D) of 2.1. For each $x \in \Omega$, we let $\mathcal{E}_x := \{ E \in \mathcal{E} : x \in E \}$. We define an equivalence relation $\sim_x$ on $\mathcal{E}_x$:

$$E \sim_x E' \iff \Theta^{n-1}(E \cap E', x) = 0 \iff \Theta^{n-1}(E \cap E', x) = 1.$$

The equivalence class $[E]_x$ under $\sim_x$ of some $E \in \mathcal{E}_x$ is called the approximate germ of $E$ at $x$. We will use repeatedly the following consequence of 2.1(A): if $E, E' \in \mathcal{E}$, then $[E]_x = [E']_x$ for $\mathcal{F}^{n-1}$-almost all $x \in E \cap E'$.

We let $\hat{\Omega}$ be the set consisting of the pairs $(x, [E]_x)$, where $x \in \Omega$ and $[E]_x$ is some approximate germ at $x$. This set should be thought of as a fiber space over $\Omega$, with respect to the obvious projection map $p : \hat{\Omega} \to \Omega$ that sends each $(x, [E]_x)$ to $x$. Below, we equip $\hat{\Omega}$ with a $\sigma$-algebra and a measure.

For each $E \in \mathcal{E}$, we let $\gamma_E : E \to \hat{\Omega}$ be the map defined by $\gamma_E(x) := (x, [E]_x)$. We define $\hat{\mathcal{A}}$ to be the finest $\sigma$-algebra on $\hat{\Omega}$ so that all maps $\gamma_E$ are $(\mathcal{M}_{\mathcal{F}^{n-1}}(E), \hat{\mathcal{A}})$-measurable, that is

$$\hat{\mathcal{A}} := \{ A \subseteq \hat{\Omega} : \gamma_E^{-1}(A) \in \mathcal{M}_{\mathcal{F}^{n-1}}(E) \text{ for all } E \in \mathcal{E} \}.$$

For all $E \in \mathcal{E}$, we denote by $\mu_E$ the measure on $(\hat{\Omega}, \hat{\mathcal{A}})$ defined by $\mu_E(A) := \mathcal{F}^{n-1}(\gamma_E^{-1}(A))$. We let $\hat{\mathcal{F}}^{n-1}$ be the least upper bound of the measures $\mu_E$, with $E$ running over $\mathcal{E}$ (see e.g. [2] 1.68)). The measure space $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathcal{F}}^{n-1})$ is complete: a set $N \subseteq \hat{\Omega}$ is negligible iff $\gamma_E^{-1}(N)$ is $\mathcal{F}^{n-1}$-negligible for all $E \in \mathcal{E}$.

From the definition of the $\sigma$-algebra $\hat{\mathcal{A}}$, it is easy to show that a map $g : \hat{\Omega} \to Z$ with values in a measurable space $(Z, \mathcal{C})$ is $(\hat{\mathcal{A}}, \mathcal{C})$-measurable iff all maps $g \circ \gamma_E$ are $(\mathcal{M}_{\mathcal{F}^{n-1}}(E), \mathcal{C})$-measurable. As a consequence, the projection map $p$ is $(\hat{\mathcal{A}}, \mathcal{B}(\hat{\Omega}))$-measurable, for the maps $p \circ \gamma_E$ are the inclusion maps $E \to \hat{\Omega}$ and, therefore, are $\mathcal{F}^{n-1}$-measurable.

2.4. LEMMA. — The following hold:

1. For all $A \in \hat{\mathcal{A}}$, one has $\hat{\mathcal{F}}^{n-1}(A) = \sup \{ \mu_E(A) : E \in \mathcal{E} \}$.
2. $\mu_n \hat{\mathcal{F}}^{n-1} = \mathcal{F}^{n-1}$.
3. For $E \in \mathcal{E}$, we have $\hat{\mathcal{F}}^{n-1}(p^{-1}(E) \setminus \gamma_E(E)) = 0$.

Proof. (1) Letting $\nu(A)$ denote the right hand side of the claimed equality, it suffices to check that $A \mapsto \nu(A)$ defines a measure. Clearly, $\nu$ is non-decreasing and $\sigma$-subadditive.
It remains to check that \( \nu(A_1 \cup A_2) \geq \nu(A_1) + \nu(A_2) \) for any pair of disjoint \( A_1, A_2 \in \mathcal{E} \). Let \( E_1, E_2 \in \mathcal{E} \). Recalling the discussion in [2.1], there is a set \( E \in \mathcal{E} \) that coincides \( \mathcal{H}^{n-1} \)-almost everywhere with \( E_1 \cup E_2 \). For \( i \in \{1, 2\} \), we have \( [E]_x = [E_i]_x \) for \( \mathcal{H}^{n-1} \)-almost all \( x \in E \cap E_i \). From this, it follows that

\[
\gamma_{E_i}^{-1}(A_i) \in \mathcal{H}^{n-1}(E_i \cap \{x \in E \cap E_i : [E]_x \neq [E_i]_x\})
\]
is \( \mathcal{H}^{n-1} \)-negligible. Hence, \( \mu(E(A_i) \geq \mu(E(A_i)) \). As \( A_1 \) and \( A_2 \) are disjoint, one has \( \mu(E(A_1 \cup A_2) = \mu(E(A_1)) + \mu(E(A_2)) \geq \mu(E(A_1)) + \mu(E(A_2)) \). The proof is then completed by taking the supremum over \( E_1, E_2 \in \mathcal{E} \).

(2) For all \( B \in \mathcal{B}(\Omega) \), one has (1)

\[
\hat{\gamma}^{-1}(p^{-1}(B)) = \sup \left\{ \mathcal{H}^{n-1}(\gamma_{E}^{-1}(p^{-1}(B)) : E \in \mathcal{E} \right\}
\]

which is \( \mathcal{F}^{n-1}(B) \).

(3) Let \( E' \in \mathcal{E} \). The set

\[
\gamma_{E}^{-1}(p^{-1}(E) \setminus \gamma_{E}(E)) = \{ x \in E \cap E' : [E]_x \neq [E']_x \}
\]
has \( \mathcal{H}^{n-1} \) measure zero. By the arbitrariness of \( E' \), it follows that \( p^{-1}(E) \setminus \gamma_{E}(E) \in \mathcal{E} \) and \( \hat{\gamma}^{-1}(p^{-1}(E) \setminus \gamma_{E}(E)) = 0 \).

2.5 (Density points). — We recall the following well-known facts regarding the existence of Lebesgue density points for functions defined on rectifiable sets. Let \( E \) be an \( (n-1) \)-rectifiable set and \( h : E \to \mathbb{R} \) be a function in \( L^1(E, \mathcal{M}_{\mathcal{H}^{n-1}(E)}(E), \mathcal{H}^{n-1} \mathcal{L}_E \) ), we have

\[
\frac{1}{(n-1)^{n-1}} \int_{E \cap B(x, r)} h \, d\mathcal{H}^{n-1} \underset{r \to 0}{\longrightarrow} h(x)
\]
at \( \mathcal{H}^{n-1} \)-almost every \( x \in E \). When the limit exists, \( x \) is called a density point of \( h \). The proof of the next lemma is elementary.

2.6. LEMMA. — Let \( E, E' \) be two Borel measurable \( (n-1) \)-rectifiable subsets of \( \Omega \), \( h \in L^\infty(E, \mathcal{M}_{\mathcal{H}^{n-1}(E)}(E), \mathcal{H}^{n-1} \mathcal{L}_E \) ), \( h' \in L^\infty(E', \mathcal{M}_{\mathcal{H}^{n-1}(E')}(E'), \mathcal{H}^{n-1} \mathcal{L}_E \) ) be two functions, and \( x \in E \cap E' \). Assume that

(A) \( h = h' \mathcal{H}^{n-1} \)-almost everywhere on \( E \cap E' \);
(B) \( \Theta^{n-1}(x, E) = \Theta^{n-1}(x, E') = \Theta^{n-1}(E \cap E', x) = 1 \);
(C) \( x \) is a density point of \( h \).

Then \( x \) is a density point of \( h' \) and

\[
\lim_{r \to 0} \frac{1}{\alpha(n-1)^{n-1}} \int_{E \cap B(x, r)} h' \, d\mathcal{H}^{n-1} = \lim_{r \to 0} \frac{1}{\alpha(n-1)^{n-1}} \int_{E' \cap B(x, r)} h' \, d\mathcal{H}^{n-1}.
\]

We now come to our main result.

2.7. THEOREM. — The map \( \mathcal{Y} : L^\infty(\tilde{\Omega}, \tilde{\mathcal{E}}, \hat{\mathcal{F}}^{n-1}) \to L^1(\Omega, \mathcal{B}(\Omega), \mathcal{F}^{n-1})^* \) defined by

\[
\mathcal{Y}(f) = \int_{\Omega} g(f \circ p) \, d\hat{\mathcal{F}}^{n-1}
\]
is an isometric isomorphism.

Proof. Let \( g \in L^\infty(\tilde{\Omega}, \tilde{\mathcal{E}}, \hat{\mathcal{F}}^{n-1}) \) and \( f \in L^1(\Omega, \mathcal{B}(\Omega), \mathcal{F}^{n-1}) \).

Then

\[
|\mathcal{Y}(f)()| \leq \|g\|_\infty \int_{\Omega} |f \circ p| \, d\hat{\mathcal{F}}^{n-1} = \|g\|_\infty \int_{\hat{\mathcal{F}}^{n-1}} \|f \circ p\|_\mathcal{F}^{n-1} = \|g\|_\infty \|f\|_1,
\]

by Lemma [2.4]. This shows that \( \mathcal{Y} \) is well-defined and \( \|\mathcal{Y}\| \leq 1 \).

Let \( \epsilon > 0 \). The set \( A := \{ |g| \geq \|g\|_\infty - \epsilon \} \) has positive \( \mathcal{F}^{n-1} \) measure, hence we infer the existence of an \( (n-1) \)-rectifiable set \( E \in \mathcal{E} \) such that \( \mathcal{H}^{n-1}(\gamma_E^{-1}(A)) > 0 \). The function \( g \circ \gamma_E \) is \( \mathcal{H}^{n-1} \)-measurable. As the measure space \( (E, \mathcal{M}_{\mathcal{H}^{n-1}(E)}(E), \mathcal{H}^{n-1} \mathcal{L}_E \) ) is the
completion of \((E, \mathcal{B}(E), \mathcal{H}^{n-1} \upharpoonright E)\), we can find a Borel measurable function \(g_E : E \to \mathbb{R}\) that coincides almost everywhere with \(g \circ \gamma_E\). We let \(f \in L^1(\Omega, \mathcal{B}(\Omega), \mathcal{F}^{n-1})\) be the map
\[
 f : x \mapsto \begin{cases} 
 1 & \text{if } x \in E \text{ and } g_E(x) > \|g\|_\infty - \varepsilon \\
 -1 & \text{if } x \in E \text{ and } g_E(x) < -\|g\|_\infty - \varepsilon \\
 0 & \text{otherwise}
\end{cases}
\]
of norm \(\|f\|_1 = \mathcal{H}^{n-1}(\gamma_E^{-1}(A))\). As \(p \circ \gamma_E\) is the inclusion map \(E \to \Omega\), we have \(\gamma_E(p(z)) = z\) for all \(z \in \gamma_E(E)\). By Lemma 2.43, it then follows that \(\gamma_E(p(z)) = z\) for \(\mathcal{F}^{n-1}\)-almost all \(z \in p^{-1}(E)\). Hence, \(g\) and \(g \circ \gamma_E \circ p\) coincide \(\mathcal{F}^{n-1}\)-everywhere on \(p^{-1}(E)\). Moreover, we have
\[
p^{-1}(E) \cap \{g \circ \gamma_E \circ p \neq g_E \circ p\} \subseteq p^{-1}\{g \circ \gamma_E \neq g_E\}
\]
By the Borel regularity of \(\mathcal{F}^{n-1}\) and Lemma 2.42, we also infer that \(g \circ \gamma_E \circ p = g_E \circ p\) almost everywhere on \(p^{-1}(E)\). Whence
\[
 Y(g)(f) = \int_{p^{-1}(E)} g(f \circ p) \, d\mathcal{F}^{n-1} = \int_{p^{-1}(E)} (g_E \circ p)(f \circ p) \, d\mathcal{F}^{n-1} = \int_E g_E f \, d\mathcal{F}^{n-1},
\]
Thus, by construction of \(f\), we have \(Y(g)(f) \geq (\|g\|_\infty - \varepsilon)\|f\|_1\). This shows, as \(\varepsilon > 0\) can be taken arbitrarily small, that \(\|Y(g)\| = 1\) and \(Y\) is an isometry.

Now we turn to establishing the surjectivity of \(Y\). For any \(E \in \mathcal{E}\), we define a map \(\alpha_E \in L^1(E, \mathcal{B}(E), \mathcal{H}^{n-1} \upharpoonright E)\) that is the localized version of \(\alpha\): for an integrable function \(f \in L^1(E, \mathcal{B}(E), \mathcal{H}^{n-1} \upharpoonright E)\), we set \(\alpha_E(f) := \alpha(f)\), where \(\alpha\) is the extension by zero of \(f\) to \(\Omega\). Since \((E, \mathcal{B}(E), \mathcal{H}^{n-1} \upharpoonright E)\) is a finite measure space, the standard duality between \(L^1\) and \(L^\infty\) spaces holds, so there is a function \(g_E \in L^\infty(E, \mathcal{B}(E), \mathcal{H}^{n-1} \upharpoonright E)\) such that
\[
 \alpha_E(f) = \int_E g_E f \, d\mathcal{H}^{n-1} \text{ for all } f \in L^1(E, \mathcal{B}(E), \mathcal{H}^{n-1} \upharpoonright E).
\]
Obviously, for any \(E, E' \in \mathcal{E}\), we have
\[
 \int_{E \cap E'} g_E f \, d\mathcal{H}^{n-1} = \int_{E \cap E'} g_{E'} f \, d\mathcal{H}^{n-1}
\]
for any integrable function \(f\) on \(E \cap E'\), which implies that \(g_E \equiv g_{E'}\) almost everywhere on \(E \cap E'\).

Let \(g : \hat{\Omega} \to \mathbb{R}\) be the function partially defined by
\[
 g(x, [E]_x) := \lim_{r \to 0} \frac{1}{\alpha(n-1)r^{n-1}} \int_{E \cap B(x,r)} g_E \, d\mathcal{H}^{n-1}
\]
whenever \(x\) is a density point of \(g_E\). The compatibility between the local Radon-Nikodým derivatives \(g_E, E \in \mathcal{E}\), together with Lemma 2.6, guarantees that \(g\) is well-defined. Denoting by \(N \subseteq \hat{\Omega}\) the set of points at which \(g\) is not defined, we readily see that, for any \(E \in \mathcal{E}\), the set \(\gamma_E^{-1}(N)\) consists of the \(x \in E\) that are not density points of \(g_E\). We readily have \(\mathcal{H}^{n-1}(\gamma_E^{-1}(N)) = 0\), and this shows that \(\mathcal{F}^{n-1}(N) = 0\). We can extend \(g\) to \(\hat{\Omega}\) by sending the elements of \(N\) to an arbitrary value. Then \(g\) is \(\mathcal{F}\)-measurable. Indeed, for any \(E \in \mathcal{E}\) the function \(g \circ \gamma_E\) coincides \(\mathcal{H}^{n-1}\)-almost every with \(g_E\), and therefore is \(\mathcal{H}^{n-1}\)-measurable.

Finally, we claim that \(\alpha = Y(g)\). Let \(f \in L^1(\Omega, \mathcal{B}(\Omega), \mathcal{F}^{n-1})\). Choose \(\varepsilon > 0\) and set \(E := \{|f| > \varepsilon\}\). As \(\mathcal{F}^{n-1}(E) < \infty\), we can suppose, up to a modification of \(f\) on a negligible set, that \(E \in \mathcal{E}\). Let \(f_E\) be the restriction of \(f\) to \(E\). Then, recalling that \(\mathcal{H}^{n-1} \upharpoonright E = \mathcal{F}^{n-1} \upharpoonright E\), we have
\[
 \alpha(\mathbb{1}_{\{|f| > \varepsilon\}} f) = \alpha_E(f_E) = \int_E g_E f_E \, d\mathcal{F}^{n-1}.
\]
By Lemma 2.7(2), we deduce that
\[
\alpha(\mathbb{I}_{\{|f| > \varepsilon\}}) = \int_{p^{-1}(E)} (g_E \circ p)(f_E \circ p) \, d\hat{\nu}^{n-1}.
\]
Clearly, \( f_E \circ p = f \circ p \) on \( p^{-1}(E) \) and \( g_E \circ p = g \) almost everywhere on \( \gamma_E(E) \). By Lemma 2.4.3, this ensures that \( g_E \circ p = g \circ p \) almost everywhere on \( p^{-1}(E) \). Thus,
\[
\alpha(\mathbb{I}_{\{|f| > \varepsilon\}}) = \int_{p^{-1}(E)} g(f \circ p) \, d\hat{\nu}^{n-1} = \int \mathbb{I}_{\{|f| > \varepsilon\}} g(f \circ p) \, d\hat{\nu}^{n-1}.
\]
Letting \( \varepsilon \to 0 \) yields \( \alpha(f) = \Upsilon(g(f)) \).

2.8. Remark. — The above proof exhibits a construction of the density \( g \), by gluing the local Radon-Nikodým derivatives \( g_E \) in the measure space \((\Omega, \mathcal{B}, \mathcal{F}^{n-1})\). The gluing is not possible within the measure space \((\Omega, \mathcal{B}(\Omega), \mathcal{F}^{n-1})\), as it is not localizable (if \( \Omega \) is non empty). It was proven in [3, Section 11] that \((\hat{\Omega}, \mathcal{B}, \mathcal{F}^{n-1})\) is the “minimal” measure space in which a compatible family of functions defined on subsets of \( \Omega \) can be glued. Using the concept introduced in [3], \((\hat{\Omega}, \mathcal{B}, \hat{\mathcal{F}}^{n-1})\) is the strictly localizable version of the completion of \((\Omega, \mathcal{B}(\Omega), \mathcal{F}^{n-1})\).

Assuming that \((\hat{\Omega}, \mathcal{B}, \hat{\mathcal{F}}^{n-1})\) is localizable, we can give an alternate proof of Theorem 2.7, the map \( i : L^1(\Omega, \mathcal{B}(\Omega), \mathcal{F}^{n-1}) \to L^1(\hat{\Omega}, \mathcal{B}, \hat{\mathcal{F}}^{n-1}) \) that sends \( f \) to \( f \circ p \) is an isometry, as \( \|f \circ p\| = \|f\| \) by Lemma 2.4.3. By the localizability of \((\hat{\Omega}, \mathcal{B}, \hat{\mathcal{F}}^{n-1})\), the dual of \( L^1(\hat{\Omega}, \mathcal{B}, \hat{\mathcal{F}}^{n-1}) \) is \( L^\infty(\hat{\Omega}, \mathcal{B}, \hat{\mathcal{F}}^{n-1}) \), in such a way that \( \Upsilon \) is just the adjoint map of \( \iota \). Reasoning as before, \( \Upsilon \) is injective, which entails that \( \iota \) and then \( \Upsilon \) are isometric isomorphisms.

2.9 (Approximate tangent cone of an approximate germ). — Let \( E \in \mathcal{E} \) and \( x \in E \). The approximate tangent cone \( \text{Tan}^{n-1}(E, x) \) (see [6, 3.16]) is unchanged if we substitute \( E \) with any \( E' \in \mathcal{E}_x \) such that \( E \sim_x E' \). Because of this, we can define the approximate tangent cone \( \text{Tan}^{n-1}(\{E\}, x) := \text{Tan}^{n-1}(E, x) \) of an approximate germ. We recall that \( \text{Tan}^{n-1}(E, x) \) is an \((n-1)\)-plane at \( \mathcal{H}^{n-1} \)-almost every point \( x \in E \), and that the function \( x \mapsto \text{Tan}^{n-1}(E, x) \in \mathcal{G}(n-1, n) \), defined almost everywhere on \( E \), is \( \mathcal{H}^{n-1} \)-measurable. Here \( \mathcal{G}(n-1, n) \) denotes the Grassmann manifold of hyperplanes in \( \mathbb{R}^n \).

2.10. Theorem. — Let \( \varphi : \text{SBV}(\Omega) \to \mathbb{R} \) be a continuous linear functional. There are fields \( f \in L^\infty(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \), \( g \in L^\infty(\hat{\Omega}, \mathcal{B}, \hat{\mathcal{F}}^{n-1}, \mathbb{R}^n) \) and \( h \in L^\infty(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n, \mathbb{R}^n) \) such that
\[
\varphi(u) = \int_\Omega f u \, d\mathcal{L}^n + \int_\hat{\Omega} g \cdot (j_u \circ p) \, d\hat{\nu}^{n-1} + \int_\Omega h \cdot \nabla u \, d\mathcal{L}^n
\]
where the distributional gradient \( Du \) of \( u \) is decomposed into its Lebesgue part \( \mathcal{L}^n \subset \nabla u \) and its jump part \( D^\text{Jump} u = j_u \mathcal{H}^{n-1} \subset \nabla u \). Moreover, one can select \( g \) in such a way that \( g(x, [E]_x) \) is normal to \( \text{Tan}^{n-1}(\{E\}, x) \) for \( \hat{\mathcal{F}}^{n-1} \)-almost all \( x \), \( [E]_x \in \hat{\Omega} \).

**Proof.** For each \( u \in \text{SBV}(\Omega) \), we have
\[
\|u\|_{\text{SBV}(\Omega)} = \int_\Omega |u| \, d\mathcal{L}^n + \int_{\mathcal{S}_0} \|j_u\| \, d\mathcal{H}^{n-1} + \int_\Omega \|\nabla u\| \, d\mathcal{L}^n.
\]
Here, the approximate discontinuity set \( \mathcal{S}_0 \) (see [2, 3.63]) is Borel measurable and countable (\( n-1 \))-rectifiable by [2, 3.64(a) and 3.78]. We recall that \( j_u = 0 \) outside \( \mathcal{S}_0 \), whereas \( j_u = (u^+ - u^-)u \) almost everywhere on \( \mathcal{S}_0 \). In particular, \( \mathcal{H}^{n-1} \subset j_u \mathcal{H}^{n-1} \subset j_u \). Therefore, \( u \mapsto (u, j_u, \nabla u) \) is an isometric embedding
\[
\text{SBV}(\Omega) \to L^1(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \times L^1(\Omega, \mathcal{B}(\Omega), \mathcal{F}^{n-1}, \mathbb{R}^n) \times L^1(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n, \mathbb{R}^n).
\]
Hence, \( \varphi \) can be split into three parts \( \varphi : u \mapsto \varphi_1(u) + \varphi_2(j_u) + \varphi_3(\nabla u) \), where \( \varphi_1 \in L^1(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n) \), \( \varphi_2 \in L^1(\Omega, \mathcal{B}(\Omega), \mathcal{F}^{n-1}, \mathbb{R}^n) \), and \( \varphi_3 \in L^1(\Omega, \mathcal{B}(\Omega), \mathcal{L}^n, \mathbb{R}^n) \). The duality theorem provides an integral representation for the term \( \varphi_2 \) that acts on the jump part of the derivative and yields the formula (1).
Regarding the second assertion, we claim that \( j_u \circ p(x, [E]_x) = j_u(x) \) is normal to \( \Tan^{n-1}([E]_x) \) for almost all \( (x, [E]_x) \in \hat{\Omega} \). For this, we need to show that, for all \( E \in \mathcal{E} \),

\[
\{ x \in E : j_u(x) \text{ not normal to } \Tan^{n-1}(E, x) \} = \{ x \in E \cap S_u : j_u(x) \text{ not normal to } \Tan^{n-1}(E, x) \}
\]

is \( \mathcal{H}^{n-1} \)-negligible. This is clear, once we remark that \( \Tan^{n-1}(E, x) = \Tan^{n-1}(S_u, x) \) for almost every \( x \in E \cap S_u \) and that \( j_u \) is normal to \( S_u \) almost everywhere on \( S_u \). Hence, we can replace the vector field \( g \) by its normal part \( \tilde{g} \) (that is, \( \tilde{g}(x, [E]_x) \) is the orthogonal projection of \( g(x, [E]_x) \) onto \( \Tan^{n-1}([E]_x) \)). The measurability of \( \tilde{g} \) follows from the discussion in Paragraph 2.9.

**References**

1. Some open problems in geometric measure theory and its applications suggested by participants of the 1984 AMS summer institute, Geometric measure theory and the calculus of variations (Arcata, Calif., 1984) (J. E. Brothers, ed.), Proc. Sympos. Pure Math., vol. 44, Amer. Math. Soc., Providence, RI, 1986, pp. 441–464. MR 840292

2. L. Ambrosio, N. Fusco, and D. Pallara, Functions of bounded variation and free discontinuity problems, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. MR 1857292

3. Ph. Bouafia and Th. De Pauw, Localizable locally determined measurable spaces with negligibles, submitted.

4. Th. De Pauw, Undecidably semilocalizable metric measure spaces, submitted.

5. Th. De Pauw, On SBV dual, Indiana Univ. Math. J. 47 (1998), no. 1, 99–121. MR 1631541

6. H. Federer, Geometric measure theory, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969. MR 0257325 (41 #1976)

7. D. H. Fremlin, Measure theory. Vol. 2, Torres Fremlin, Colchester, 2003, Broad foundations, Corrected second printing of the 2001 original. MR 2462280

8. N. Fusco and D. Spector, A remark on an integral characterization of the dual of BV, J. Math. Anal. Appl. 457 (2018), no. 2, 1370–1375. MR 3705358

9. N. G. Meyers and W. P. Ziemer, Integral inequalities of Poincaré and Wirtinger type for BV functions, Amer. J. Math. 99 (1977), no. 6, 1345–1360. MR 507433

10. N. C. Phuc and M. Torres, Characterizations of signed measures in the dual of BV and related isometric isomorphisms, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 17 (2017), no. 1, 385–417. MR 3676052

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