Distinct Partitions and Some q-Binomial Summation Identities

M.J. Kronenburg

Abstract

The partition functions $P(n, m, p)$, the number of integer partitions of $n$ into exactly $m$ parts with each part at most $p$, and $Q(n, m, p)$, the number of integer partitions of $n$ into exactly $m$ distinct parts with each part at most $p$, are related by double summation identities which follow from their generating functions. From these identities and some identities from an earlier paper, some other identities involving distinct partitions and some q-binomial summation identities are proved, and from these follow some combinatorial identities.

Keywords: q-binomial coefficient, integer partition function.
MSC 2010: 05A17 11B65 11P81

1 Introduction

The following q-binomial summation identities are proved, of which (1.2) is known as identity (1.8) and (1.6) as (1.9) in [5] and both in [8], and (1.3) is known as corollary 4.1 and (1.7) as corollary 3.7 in [8]. Corollaries 2.4 and 3.4 of [8] follow from (1.9) and (1.10).

\[ \sum_{k=0}^{n} (-1)^k q^{k/2} \binom{m+n-k}{m} q^{m+1} k = \delta_{n,0} \]  
(1.1)

\[ \sum_{k=0}^{[n/2]} q^{(n-2k)/2} \binom{m+1}{n-2k} q^{m+k} k^2 = \binom{m+n}{m} q \]  
(1.2)

\[ \sum_{k=0}^{[n/2]} (-1)^k q^{k/2} \binom{m+n-2k}{m} q^{m+1} k^2 = q^n \binom{m+1}{n} q \]  
(1.3)

\[ \sum_{k=0}^{[n/3]} (-1)^k q^{(n-3k)/2} \binom{m+1}{n-3k} q^{m+k} k^3 = \sum_{k=0}^{n} \cos\left(\frac{2k-n}{3}\pi\right) \binom{m+n-k}{m} q^{m+k} k \]  
(1.4)

\[ \sum_{k=0}^{[n/3]} (-1)^k q^{3k/2} \binom{m+n-3k}{m} q^{m+1} k^3 = \sum_{k=0}^{n} \cos\left(\frac{2k-n}{3}\pi\right) q^{(n-k)/2} \binom{m+1}{n-k} q^{m+1} k \]  
(1.5)
\[
\sum_{k=0}^{\lfloor n/4 \rfloor} q^{\lfloor n-4k \rfloor} \binom{m+1}{n-4k} q^{\binom{m+k}{m}} q^4 = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{m+n-2k}{m} q^{\binom{m+k}{m}} q^2 \tag{1.6}
\]
\[
\sum_{k=0}^{\lfloor n/4 \rfloor} (-1)^k q^{k} \binom{m+n-4k}{m} q^{\binom{m+1}{k}} q^4 = \sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{n-2k}{2}} \binom{m+1}{n-2k} q^{\binom{m+1}{k}} q^2 \tag{1.7}
\]
\[
\sum_{k=0}^{n} \sum_{l=0}^{n-k} (-1)^k q^a \binom{n-k-l}{2} \binom{p+n-k-l}{p} q^{\binom{m+1}{k}} q^b \binom{m+l}{m} q^b = q^a \binom{n}{2} \binom{p+n}{p} q^c \tag{1.8}
\]
\[
\sum_{k=0}^{n} \sum_{l=0}^{n-k} (-1)^k q^a \binom{n-k-l}{2} \binom{p+n-k-l}{p} q^b \binom{m+1}{k} q^b \binom{m+l}{m} q^b = q^a \binom{n}{2} \binom{p+n}{p} q^c \tag{1.9}
\]
\[
\sum_{k=0}^{n} \sum_{l=0}^{n-k} (-1)^k q^a \binom{n-k-l}{2} \binom{p+n-k-l}{p} q^c \binom{m+1}{k} q^b \binom{m+l}{m} q^b = q^a \binom{n}{2} \binom{p+n}{p} q^c \tag{1.10}
\]
\[
\sum_{k=0}^{n} \sum_{l=0}^{n-k} (-1)^k q^a \binom{n-k-l}{2} \binom{p+n-k-l}{p} q^c \binom{m+1}{k} q^c \binom{m+l}{m} q^b = q^a \binom{n}{2} \binom{p+n}{p} q^c \tag{1.11}
\]

In the summands of the last four identities, because of the type of double summation, \(k\) and \(l\) can be interchanged, and \(l\) can be replaced by \(n-k-l\). When \(q=1\) these identities give the following combinatorial identities, of which \((1.13)\) and \((1.14)\) are known as (3.24) and (3.25) in [4].

\[
\sum_{k=0}^{n} (-1)^k \binom{m+k}{m} \binom{m+1}{n-k} = \delta_{n,0} \tag{1.12}
\]
\[
\sum_{k=0}^{n} \binom{m+1}{2k} \binom{m+n-k}{m} = \binom{m+2n}{m} \tag{1.13}
\]
\[
\sum_{k=0}^{n} \binom{m+1}{2k+1} \binom{m+n-k}{m} = \binom{m+2n+1}{m} \tag{1.14}
\]
\[
\sum_{k=0}^{n} (-1)^k \binom{m+2k}{m} \binom{m+1}{n-k} = (-1)^n \binom{m+1}{2n} \tag{1.15}
\]
\[
\sum_{k=0}^{n} (-1)^k \binom{m+2k+1}{m} \binom{m+1}{n-k} = (-1)^n \binom{m+1}{2n+1} \tag{1.16}
\]
\[
\sum_{k=0}^{n} (-1)^k \binom{m+1}{3k} \binom{m+n-k}{m} = \sum_{k=0}^{3n} \cos \left( \frac{2k}{3} \pi \right) \binom{m+3n-k}{m} \binom{m+k}{m} \tag{1.17}
\]
\[
\sum_{k=0}^{n} (-1)^k \binom{m+1}{3k+1} \binom{m+n-k}{m} = \sum_{k=0}^{3n+1} \cos \left( \frac{2k-1}{3} \pi \right) \binom{m+3n-k+1}{m} \binom{m+k}{m} \tag{1.18}
\]
\[ \sum_{k=0}^{n} (-1)^{k} \binom{m+1}{3k+2} \binom{m+n-k}{m} = \sum_{k=0}^{3n+2} \cos\left(\frac{2k-2}{3} \pi\right) \binom{m+3n-k+2}{m} \binom{m+k}{m} \]  
(1.19)

\[ \sum_{k=0}^{n} (-1)^{k} \binom{m+3k}{m} \binom{m+1}{n-k} = \sum_{k=0}^{3n} \cos\left(\frac{2k-1}{3} \pi\right) \binom{m+1}{m} \binom{m+1}{m} \]  
(1.20)

\[ \sum_{k=0}^{n} (-1)^{k} \binom{m+3k+1}{m} \binom{m+1}{n-k} = \sum_{k=0}^{3n+1} \cos\left(\frac{2k-1}{3} \pi\right) \binom{m+1}{m} \binom{m+1}{m+1} \]  
(1.21)

\[ \sum_{k=0}^{n} (-1)^{k} \binom{m+3k+2}{m} \binom{m+1}{n-k} = \sum_{k=0}^{3n+2} \cos\left(\frac{2k-2}{3} \pi\right) \binom{m+1}{m} \binom{m+1}{m+1} \]  
(1.22)

\[ \sum_{k=0}^{n} (-1)^{k} = \binom{m+2k}{m} \binom{m+2n-k}{m} \]  
(1.23)

\[ \sum_{k=0}^{n} (-1)^{k} = \binom{m+2k+1}{m} \binom{m+2n-k}{m} \]  
(1.24)

\[ \sum_{k=0}^{n} (-1)^{k} = \binom{m+2k}{m} \binom{m+2n-k+1}{m} \]  
(1.25)

\[ \sum_{k=0}^{n} (-1)^{k} = \binom{m+2k+1}{m} \binom{m+2n-k+1}{m} \]  
(1.26)

\[ \sum_{k=0}^{n} (-1)^{k} = \binom{m+2k}{2k} \binom{m+1}{2n-k} \]  
(1.27)

\[ \sum_{k=0}^{n} (-1)^{k} = \binom{m+2k+1}{2k+1} \binom{m+1}{2n-k} \]  
(1.28)

\[ \sum_{k=0}^{n} (-1)^{k} = \binom{m+2k+2}{2k} \binom{m+1}{2n-k} \]  
(1.29)

\[ \sum_{k=0}^{n} (-1)^{k} = \binom{m+2k+3}{2k+1} \binom{m+1}{2n-k} \]  
(1.30)

\[ \sum_{k=0}^{n} (-1)^{k} = \binom{m+2k+1}{2k+1} \binom{m+1}{2n-k+1} \]  
(1.31)

\[ \sum_{k=0}^{n} (-1)^{k} = \binom{m+2k+2}{2k} \binom{m+1}{2n-k+1} \]  
(1.32)

\[ \sum_{k=0}^{n} (-1)^{k} = \binom{m+2k+3}{2k+1} \binom{m+1}{2n-k+1} \]  
(1.33)
\[
\sum_{k=0}^{n} \sum_{l=0}^{n-k} (-1)^l \binom{p}{n-k-l} \binom{m+1}{k} \binom{m+l}{m} = \binom{p}{n}
\]  

(1.34)

2 Definitions and Basic Identities

Let the coefficient of a power series be defined as:

\[
[q^n] \sum_{k=0}^{\infty} a_k q^k = a_n
\]

(2.1)

Let \( P(n) \) be the number of integer partitions of \( n \), let \( Q(n) \) be the number of integer partitions of \( n \) into distinct parts, let \( P(n,m) \) be the number of integer partitions of \( n \) into exactly \( m \) parts, and let \( Q(n,m) \) be the number of integer partitions of \( n \) into exactly \( m \) distinct parts. Let \( P(n,m,p) \) be the number of integer partitions of \( n \) into exactly \( m \) parts, each part at most \( p \), and let \( P^*(n,m,p) \) be the number of integer partitions of \( n \) into at most \( m \) parts, each part at most \( p \), which is the number of Ferrer diagrams that fit in a \( m \) by \( p \) rectangle:

\[
P^*(n,m,p) = \sum_{k=0}^{m} P(n,k,p)
\]

(2.2)

Let the following definition of the q-binomial coefficient, also called the Gaussian polynomial, be given.

Definition 2.1. The q-binomial coefficient is defined by \([1,3]\):

\[
\binom{m+p}{m}_q = \prod_{j=1}^{m} \frac{1 - q^{p+j}}{1 - q^j}
\]

(2.3)

The q-binomial coefficient is the generating function of \( P^*(n,m,p) \) \([1]\):

\[
P^*(n,m,p) = [q^n] \binom{m+p}{m}_q
\]

(2.4)

In the earlier paper \([7]\) it was proved that:

\[
P^*(n,m,p) = P(n + m, m, p + 1)
\]

(2.5)

For the q-binomial coefficient there is the following symmetry identity \([7]\):

\[
\binom{m+p}{m}_q = \binom{m+p}{p}_q
\]

(2.6)

3 Formulas Involving Distinct Partitions

Let \( Q(n,m,p) \) be the number of integer partitions of \( n \) into exactly \( m \) distinct parts with each part at most \( p \). In the earlier paper \([7]\) it was proved that:

\[
Q(n,m,p) = P(n - m(m-1)/2, m, p - m + 1)
\]

(3.1)
The generating functions for $P(n, m, p)$ and $Q(n, m, p)$ are identities (7.3) and (7.4) in [2]:

$$\prod_{j=1}^{p} \frac{1}{1 - zq^j} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, p) q^n z^m$$  \hspace{1cm} (3.2)$$

$$\prod_{j=1}^{p} (1 + zq^j) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, p) q^n z^m$$  \hspace{1cm} (3.3)$$

**Theorem 3.1.**

$$P(n, m, p) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} Q(n - 2k, m - 2l, p) P(k, l, p)$$  \hspace{1cm} (3.4)$$

**Proof.** Using $(1 + zq^j)(1 - zq^j) = 1 - z^2 q^{2j}$:

$$\prod_{j=1}^{p} (1 - zq^j) = \prod_{j=1}^{p} (1 + zq^j) \prod_{j=1}^{p} (1 - z^2 q^{2j})$$  \hspace{1cm} (3.5)$$

Substituting the generating functions (3.2) and (3.3):

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, p) q^n z^m = (\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, p) q^n z^m)(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, p) q^{2n} z^{2m})$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} Q(n_1, m_1, p) P(n_2, m_2, p) q^{n_1 + 2n_2} z^{m_1 + 2m_2}$$  \hspace{1cm} (3.6)$$

The coefficients on both sides must be equal, so $n_1 + 2n_2 = n$ and $m_1 + 2m_2 = m$, which is equivalent to $n_1 = n - 2n_2$ and $m_1 = m - 2m_2$:

$$P(n, m, p) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} Q(n_1, m_1, p) P(n_2, m_2, p)$$

$$= \sum_{n_1=0}^{\lfloor n/2 \rfloor} \sum_{n_2=0}^{\lfloor m/2 \rfloor} Q(n - 2n_2, m - 2m_2, p) P(n_2, m_2, p)$$  \hspace{1cm} (3.7)$$

□

Let $Q^*(n, m, p)$ be the number of integer partitions of $n$ into at most $m$ distinct parts with each part at most $p$, which is defined like $P^*(n, m, p)$ in (2.2):

$$Q^*(n, m, p) = \sum_{k=0}^{m} Q(n, k, p)$$  \hspace{1cm} (3.8)$$

From the previous theorem a relation between $Q^*(n, m, p)$ and $P(n, m, p)$ can be derived.
Theorem 3.2.

\[ P(n + m, m, p + 1) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} Q^*(n - 2k, m - 2l, p)P(k, l, p) \]  \hfill (3.9)

Proof. Using the previous theorem and (2.5):

\[ P^*(n, m, p) = P(n + m, m, p + 1) = \sum_{h=0}^{m} P(n, h, p) \]

\[ = \sum_{h=0}^{m} \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor h/2 \rfloor} Q(n - 2k, h - 2l, p)P(k, l, p) \]

\[ = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} \sum_{h=2l}^{m} Q(n - 2k, h - 2l, p)P(k, l, p) \]

\[ = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} Q^*(n - 2k, m - 2l, p)P(k, l, p) \]

Let \( P_{\text{most}}(n, p) \) be the number of integer partitions of \( n \) with each part at most \( p \). From (2.2), (2.5) and conjugation of Ferrer diagrams [7]:

\[ P_{\text{most}}(n, p) = \sum_{k=0}^{n} P(n, k, p) = P^*(n, n, p) = P(2n, n, p + 1) = P(n + p, p) \]  \hfill (3.11)

Obviously \( P^*(n, m, p) = P_{\text{most}}(n, p) \) when \( m \geq n \). Let \( Q_{\text{most}}(n, p) \) be the number of integer partitions of \( n \) into distinct parts with each part at most \( p \), for which \( Q^*(n, m, p) = Q_{\text{most}}(n, p) \) when \( m \geq n \). When taking \( m = n \) in this theorem for nonzero summands \( l \leq k \) and therefore \( n - 2l \geq n - 2k \):

\[ P(n + p, p) = \sum_{k=0}^{\lfloor n/2 \rfloor} Q_{\text{most}}(n - 2k, p)P(p + k, p) \]  \hfill (3.12)

From this identity follows as a special case when taking \( p = n \), and using \( P(2n, n) = P(n) \) and from [6] \( P(n, m) = P(n - m) \) if \( 2m \geq n \) and therefore \( P(n + k, n) = P(k) \) if \( n \geq k \):

\[ P(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} Q(n - 2k)P(k) \]  \hfill (3.13)

Theorem 3.3.

\[ Q(n, m, p) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^{l}P(n - 2k, m - 2l, p)Q(k, l, p) \]  \hfill (3.14)
Proof. Using \((1 + izq^j)(1 - izq^j) = 1 + z^2q^{2j}\):
\[
\prod_{j=1}^p(1 + izq^j) = \frac{\prod_{j=1}^p(1 + z^2q^{2j})}{\prod_{j=1}^p(1 - izq^j)}
\tag{3.15}
\]

Substituting the generating functions \(3.2\) and \(3.3\):
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, p)izq^n z^m = (\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P(n, m, p)izq^n z^m)(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Q(n, m, p)q^{2n}z^{2m})
\]
\[
= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} P(n_1, m_1, p)Q(n_2, m_2, p)izq^{n_1+2n_2}z^{m_1+2m_2}
\]
\[
= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} i^{n_1}P(n_1, m_1, p)Q(n_2, m_2, p)
\tag{3.16}
\]

The coefficients on both sides must be equal, so \(n_1 + 2n_2 = n\) and \(m_1 + 2m_2 = m\), which is equivalent to \(n_1 = n - 2n_2\) and \(m_1 = m - 2m_2\):
\[
i^{n_1}Q(n, m, p) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} i^{m_1}P(n_1, m_1, p)Q(n_2, m_2, p)
\]
\[
= \sum_{n_2=0}^{\infty} \sum_{m_2=0}^{\infty} i^{m_2}P(n - 2n_2, m - 2m_2, p)Q(n_2, m_2, p)
\]

With \(i^{-2m_2} = (-1)^{m_2}\) the theorem is proved. \(\square\)

Using a similar derivation as in theorem \(3.2\) gives:
\[
Q^*(n, m, p) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^lP(n + m - 2(k + l), m - 2l, p + 1)Q(k, l, p)
\tag{3.18}
\]

Using a similar reasoning as above:
\[
Q(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor n/2 \rfloor} (-1)^lP(n - 2k)Q(k, l)
\tag{3.19}
\]

**Theorem 3.4.**
\[
\sum_{k=0}^{\lfloor n/3 \rfloor} \sum_{l=0}^{\lfloor m/3 \rfloor} (-1)^lQ(n - 3k, m - 3l, p)P(k, l, p) = \sum_{k=0}^{n} \sum_{l=0}^{m} \cos\left(\frac{2l - m}{3}\pi\right)P(n - k, m - l, p)P(k, l, p)
\tag{3.20}
\]

**Proof.** Using \((1 - zq^j)(1 - (-1)^{2/3}zq^j)(1 + (-1)^{1/3}zq^j) = 1 - z^3q^{3j}\):
\[
\frac{\prod_{j=1}^p(1 + (-1)^{1/3}zq^j)}{\prod_{j=1}^p(1 - z^3q^{3j})} = \frac{1}{\prod_{j=1}^p(1 - zq^j)}\prod_{j=1}^p(1 - (-1)^{2/3}zq^j)
\tag{3.21}
\]
Substituting the generating functions (3.2) and (3.3):

\[
\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} Q(n_1, m_1, p) P(n_2, m_2, p) (-1)^{m_1/3} q^{n_1 + 3n_2} z^{m_1 + 3m_2} = \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{m_3=0}^{\infty} \sum_{m_4=0}^{\infty} P(n_3, m_3, p) P(n_4, m_4, p) (-1)^{2m_3/3} q^{n_3 + n_4} z^{m_3 + m_4}
\]  

(3.22)

Taking the coefficients on both sides equal to \( q^n z^m \), then \( n_1 + 3n_2 = n_3 + n_4 = n \) and \( m_1 + 3m_2 = m_3 + m_4 = m \), which is equivalent to \( n_1 = n - 3n_2, n_3 = n - n_4, m_1 = m - 3m_2 \) and \( m_3 = m - m_4 \), gives:

\[
\sum_{k=0}^{[n/3]} \sum_{l=0}^{[m/3]} (-1)^k Q(n - 3k, m - 3l, p) P(k, l, p) = \sum_{k=0}^{n} \sum_{l=0}^{m} (-1)^{m-2l} P(n-k, m-l, p) P(k, l, p) 
\]  

(3.23)

Equating the imaginary parts of this identity gives:

\[
\sum_{k=0}^{n} \sum_{l=0}^{m} \sin\left(\frac{m-2l}{3} \pi\right) P(n-k, m-l, p) P(k, l, p) = 0
\]  

(3.24)

Changing \( k \) into \( n-k \) and \( l \) into \( m-l \) changes the sign of the summand, and therefore this identity is trivial. Equating the real parts of the previous identity and using \( \cos(x) = \cos(-x) \) gives the theorem.

\[\square\]

**Theorem 3.5.**

\[
\sum_{k=0}^{[n/3]} \sum_{l=0}^{[m/3]} (-1)^k P(n - 3k, m - 3l, p) Q(k, l, p) = \sum_{k=0}^{n} \sum_{l=0}^{m} \cos\left(\frac{2l-m}{3} \pi\right) Q(n-k, m-l, p) Q(k, l, p)
\]  

(3.25)

**Proof.** In the previous theorem replacing \( z \) by \( -z \) leads to:

\[
\prod_{j=1}^{p} \frac{1 + z^3 q^{2j}}{1 + (-1)^{1/3} z q^3} = \prod_{j=1}^{p} (1 + z q^j) \prod_{j=1}^{p} (1 + (-1)^{2/3} z q^j)
\]  

(3.26)

When comparing this with the previous theorem it is clear that \( P \) and \( Q \) are interchanged, which gives:

\[
\sum_{k=0}^{[n/3]} \sum_{l=0}^{[m/3]} (-1)^k P(n - 3k, m - 3l, p) Q(k, l, p) = \sum_{k=0}^{n} \sum_{l=0}^{m} (-1)^{m-2l} Q(n-k, m-l, p) Q(k, l, p)
\]  

(3.27)

Equating the imaginary parts of this identity gives:

\[
\sum_{k=0}^{n} \sum_{l=0}^{m} \sin\left(\frac{m-2l}{3} \pi\right) Q(n-k, m-l, p) Q(k, l, p) = 0
\]  

(3.28)

As in the previous theorem this identity is trivial. Equating the real parts of the previous identity and using \( \cos(x) = \cos(-x) \) gives the theorem.

\[\square\]
Theorem 3.6.
\[
\sum_{k=0}^{\lfloor n/4 \rfloor} \sum_{l=0}^{\lfloor m/4 \rfloor} Q(n-4k, m-4l, p)P(k, l, p) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^{l}P(n-2k, m-2l, p)P(k, l, p) \tag{3.29}
\]

Proof. Using \((1 + izq^j)(1 - izq^j)(1 - z^4q^{2j}) = 1 - z^4q^{4j}\):
\[
\prod_{j=1}^{p}(1 + izq^j) = \prod_{j=1}^{p}(1 - z^4q^{4j})
\]

The proof is similar to the previous proofs. \(\Box\)

Theorem 3.7.
\[
\sum_{k=0}^{\lfloor n/4 \rfloor} \sum_{l=0}^{\lfloor m/4 \rfloor} (-1)^{l}P(n-4k, m-4l, p)Q(k, l, p) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} Q(n-2k, m-2l, p)Q(k, l, p) \tag{3.31}
\]

Proof. Using \((1 + (-1)^{1/4}zq^j)(1 - (-1)^{1/4}zq^j)(1 + iz^2q^{2j}) = 1 + z^4q^{4j}\):
\[
\prod_{j=1}^{p}(1 + (-1)^{1/4}zq^j) = \prod_{j=1}^{p}(1 + (-1)^{1/4}z^2q^{2j}) \tag{3.32}
\]

The proof is similar to the previous proofs. \(\Box\)

Theorem 3.8.
\[
\sum_{k=0}^{n} \sum_{l=0}^{m} (-1)^{l}P(n-k, m-l, p)Q(k, l, p) = \delta_{n,0}\delta_{m,0} \tag{3.33}
\]

Proof.
\[
\prod_{j=1}^{p}(1 + (-z)q^j) = 1 \tag{3.34}
\]

\[
\sum_{n, m=0}^{\infty} \delta_{n,0} \delta_{m,0} q^{n}z^{m} = \left( \sum_{n, m=0}^{\infty} P(n, m, p)q^{n}z^{m} \right) \left( \sum_{n, m=0}^{\infty} Q(n, m, p)(-1)^{m}q^{n}z^{m} \right) = \prod_{n=0}^{\infty} \prod_{m=0}^{\infty} P(n, m, p)Q(n, m, p)(-1)^{m}q^{n}z^{m} \tag{3.35}
\]

The coefficients on both sides must be equal, so \(n_1 + n_2 = n\) and \(m_1 + m_2 = m\), which is equivalent to \(n_1 = n - n_2\) and \(m_1 = m - m_2\), which gives the theorem. \(\Box\)

4 Some q-Binomial Summation Identities

From theorems 3.1, 3.3, 3.4, 3.5, 3.6, 3.7 and 3.8 the following q-binomial summation identities are proved.
In theorem 3.1 it was used that:

\[ k \]

so the summation over \( k \) can be done:

\[ \sum_{k=0}^{\lfloor n/2 \rfloor} q^{(n-2k)/2} \binom{m+1}{n-2k} q^{m+k} q^2 = \binom{m+n}{m} q \]  

(4.1)

**Proof.** From theorem 3.1 using (2.5) and (3.1):

\[ P^*(n-m, m, p-1) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} P^*(n-2k-(m-2l)(m-2l+1)/2, m-2l, p-m+2l) P^*(k-l, l, p-1) \]  

(4.2)

Using (2.4):

\[ [q^{n-m}] \binom{m+p-1}{m} = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} [q^{n-2k}] q^{-(m-2l)(m-2l+1)/2-2k} \binom{p}{m-2l} q^{k-l} \binom{p+l-1}{l} q \]  

(4.3)

In theorem 3.1 it was used that:

\[ \sum_{k=0}^{\lfloor n/2 \rfloor} a_{n-2k} b_k = [q^n] \left( \sum_{k=0}^{\infty} a_k q^k \right) \left( \sum_{k=0}^{\infty} b_k q^{2k} \right) \]  

(4.4)

so the summation over \( k \) can be done:

\[ [q^n] q^m \binom{m+p-1}{m} = [q^n] \sum_{l=0}^{\lfloor m/2 \rfloor} q^{-(m-2l)(m-2l+1)/2} \binom{p}{m-2l} q^{2l} \binom{p+l-1}{l} q^2 \]  

(4.5)

Because all coefficients \([q^n]\) are equal, the polynomials must be equal, and cancelling some powers of \( q \):

\[ \binom{m+p-1}{m} = \sum_{l=0}^{\lfloor m/2 \rfloor} q^{-(m-2l)(m-2l-1)/2} \binom{p}{m-2l} q^{2l} \binom{p+l-1}{l} q^2 \]  

(4.6)

Replacing \( m \) by \( n \) and \( l \) by \( k \) and \( p \) by \( m+1 \) and using (2.4) gives the theorem.

Taking \( q = 1 \) and replacing \( n \) by \( 2n \) and \( k \) by \( n-k \) this is combinatorial identity (3.24) in [4]:

\[ \sum_{k=0}^{n} \binom{m+1}{2k} \binom{m+n-k}{m} = \binom{m+2n}{m} \]  

(4.7)

and replacing \( n \) by \( 2n+1 \) and \( k \) by \( n-k \) this is combinatorial identity (3.25) in [4]:

\[ \sum_{k=0}^{n} \binom{m+1}{2k+1} \binom{m+n-k}{m} = \binom{m+2n+1}{m} \]  

(4.8)
Theorem 4.2.

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{k(\binom{n}{2})} \binom{m + n - 2k}{m} \binom{m + 1}{k} q^{\binom{n}{2}} \binom{m + 1}{n} q \tag{4.9}
\]

Proof. From theorem 4.1 using (2.5) and (3.1):

\[
P^*(n - m(m + 1)/2, m, p - m)
\]

\[
= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l P^*(n - 2k - m + 2l, m - 2l, p - 1) P^*(k - l(l + 1)/2, l, p - l)
\tag{4.10}
\]

Using (2.4) and (2.6):

\[
[q^{n-m(m+1)/2}] \binom{p}{m} q = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l [q^{n-m-2k+2l}] \binom{m + p - 2l - 1}{m - 2l} q^{k-(l(l+1)/2)} \binom{p}{l} q
\]

\[
= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l [q^{n-2k}] q^{m-2l} \binom{m + p - 2l - 1}{p - 1} q^{k-l(l+1)/2} \binom{p}{l} q
\tag{4.11}
\]

As in theorem 4.1, the sum over \(k\) can be done:

\[
[q^n] q^{m(m+1)/2} \binom{p}{m} q = [q^n] \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l q^{m-2l} \binom{m + p - 2l - 1}{p - 1} q^{l(l+1)/2} \binom{p}{l} q^2
\tag{4.12}
\]

Because all coefficients \([q^n]\) are equal, the polynomials must be equal, and cancelling some powers of \(q\):

\[
q^{m(m-1)/2} \binom{p}{m} q = \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l q^{l(l-1)} \binom{m + p - 2l - 1}{p - 1} q^{l(l+1)/2} \binom{p}{l} q^2
\tag{4.13}
\]

Replacing \(m\) by \(n\) and \(l\) by \(k\) and \(p\) by \(m + 1\) gives the theorem.

Taking \(q = 1\) and replacing \(n\) by \(2n\) and \(k\) by \(n - k\) gives the combinatorial identity:

\[
\sum_{k=0}^{n} (-1)^k \binom{m + 2k}{m} \binom{m + 1}{n - k} = (-1)^n \binom{m + 1}{2n}
\tag{4.14}
\]

and replacing \(n\) by \(2n + 1\) and \(k\) by \(n - k\) gives the combinatorial identity:

\[
\sum_{k=0}^{n} (-1)^k \binom{m + 2k + 1}{m} \binom{m + 1}{n - k} = (-1)^n \binom{m + 1}{2n + 1}
\tag{4.15}
\]

Theorem 4.3.

\[
\sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k q^{k(n-3k)/2} \binom{m + 1}{m} \binom{m + k}{n-3k} q^{m+k} q^3 = \sum_{k=0}^{n} \cos\left(\frac{2k - n}{3} \pi\right) \binom{m + n - k}{m} q^k \binom{m + k}{m} q
\tag{4.16}
\]
Proof. From theorem 3.4 using (2.5) and (3.1):

$$\sum_{k=0}^{n} \sum_{l=0}^{m} (-1)^l P^*(n-3k-(m-3l)(m-3l-1)/2, m-3l, p-m+3l) P^*(k-l, l, p-1)$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{m} \cos(\frac{2l-m}{3}\pi) P^*(n-k-m+l, m-l, p-1) P^*(k-l, l, p-1)$$

(4.17)

Using (2.4) and (2.6):

$$\sum_{k=0}^{n} \sum_{l=0}^{m} (-1)^l q^{n-3k} q^{(m-3l)(m-3l+1)/2} \binom{p}{m-3l} q^{l} [q^{k}] q^{l} \binom{p+l-1}{p-1} q$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{m} \cos(\frac{2l-m}{3}\pi) [q^{k}] q^{l} \binom{p+l-1}{p-1} q$$

(4.18)

In theorem 3.4 it was used that:

$$\sum_{k=0}^{n} a_{n-3k} b_k = [q^n] \sum_{k=0}^{\infty} a_k q^k (\sum_{k=0}^{\infty} b_k q^{3k})$$

(4.19)

so as in the previous theorems the summation over \( k \) can be done:

$$\sum_{l=0}^{m} (-1)^l q^{m-3l} q^{(m-3l)(m-3l+1)/2} \binom{p}{m-3l} q^{l} \binom{p+l-1}{p-1} q^3$$

(4.20)

Cancelling some powers of \( q \) and replacing \( m \) by \( n \) and \( l \) by \( k \) and \( p \) by \( m+1 \) gives the theorem. \( \square \)

Taking \( q = 1 \) and replacing \( n \) by \( 3n \) and in the left side replacing \( k \) by \( n-k \) and using \( \cos(\alpha - n\pi) = (-1)^n \cos(\alpha) \) gives the combinatorial identity:

$$\sum_{k=0}^{n} (-1)^k \binom{m+1}{3k} \binom{m+n-k}{m} = \sum_{k=0}^{3n} \cos(\frac{2k}{3}\pi) \binom{m+3n-k}{m} \binom{m+k}{m}$$

(4.21)

Replacing \( n \) by \( 3n+1 \) or \( 3n+2 \) gives similar identities.

**Theorem 4.4.**

$$\sum_{k=0}^{n} (-1)^k q^{n-k} \binom{m+n-3k}{m} q \binom{m+1}{k} q^3 = \sum_{k=0}^{n} \cos(\frac{2k-n}{3}\pi) q^{(n-k)+\binom{k}{2}} q \binom{m+1}{k}$$

(4.22)
Proof. From theorem 3.5 using (2.5) and (3.1):

\[
\sum_{k=0}^{[n/3]} \sum_{l=0}^{[m/3]} (-1)^l P^*(n-3k-m+3l, m-3l, p-1) P^*(k-l(l+1)/2, l, p-l)
\]

\[
= \sum_{k=0}^{n} \sum_{l=0}^{m} \cos\left(\frac{2l-m}{3}\pi\right) P^*(n-k-(m-l)(m-l+1)/2, m-l, p-m+l)
\]

\[
\cdot P^*(k-l(l+1)/2, l, p-l)
\]  

Using (2.4) and (2.6):

\[
\sum_{k=0}^{[n/3]} \sum_{l=0}^{[m/3]} (-1)^l [q^{n-3k}] q^{m-3l} \left(\frac{m+p-3l-1}{p-1}\right) q^{3l(l+1)/2} \left(\frac{p}{l}\right) q
\]

\[
= \sum_{k=0}^{n} \sum_{l=0}^{m} \cos\left(\frac{2l-m}{3}\pi\right) [q^{n-k}] q^{(m-l)(m-l+1)/2} \left(\frac{p}{m-l}\right) q^{3l(l+1)/2} \left(\frac{p}{l}\right) q
\]  

As in the previous theorem the summation over \(k\) can be done:

\[
\sum_{l=0}^{[m/3]} (-1)^l q^{m-3l} \left(\frac{m+p-3l-1}{p-1}\right) q^{3l(l+1)/2} \left(\frac{p}{l}\right) q^3
\]

\[
= \sum_{l=0}^{m} \cos\left(\frac{2l-m}{3}\pi\right) q^{(m-l)(m-l+1)/2} \left(\frac{p}{m-l}\right) q^{3l(l+1)/2} \left(\frac{p}{l}\right) q
\]  

Cancelling some powers of \(q\) and replacing \(m\) by \(n\) and \(l\) by \(k\) and \(p\) by \(m+1\) gives the theorem. \(\Box\)

Taking \(q = 1\) and replacing \(n\) by \(3n\) and in the left side replacing \(k\) by \(n-k\) and using \(\cos(\alpha - n\pi) = (-1)^n \cos(\alpha)\) gives the combinatorial identity:

\[
\sum_{k=0}^{n} (-1)^k \left(\frac{m+3k}{m}\right) \left(\frac{m+1}{n-k}\right) = \sum_{k=0}^{3n} \cos\left(\frac{2k}{3}\pi\right) \left(\frac{m+1}{3n-k}\right) \left(\frac{m+1}{k}\right)
\]  

Replacing \(n\) by \(3n+1\) or \(3n+2\) gives similar identities.

Theorem 4.5.

\[
\sum_{k=0}^{[n/4]} q \binom{n-4k}{2} \binom{m+1}{n-4k} q^{\binom{m+k}{m}} q^4 = \sum_{k=0}^{[n/2]} (-1)^k \binom{m+n-2k}{m} q^{\binom{m+k}{m}} q^2
\]  

Proof. From theorem 3.6 using (2.5) and (3.1):

\[
\sum_{k=0}^{[n/4]} \sum_{l=0}^{[m/4]} P^*(n-4k - (m-4l)(m-4l+1)/2, m-4l, p-m+4l) P^*(k-l, l, p-1)
\]

\[
= \sum_{k=0}^{[n/2]} \sum_{l=0}^{[m/2]} (-1)^l P^*(n-2k-m+2l, m-2l, p-1) P^*(k-l, l, p-1)
\]  

13
Using (2.4) and (2.6):

\[
\sum_{k=0}^{\lfloor n/4 \rfloor} \sum_{l=0}^{\lfloor m/4 \rfloor} [q^{n-4k}] q^{(m-4l)(m-4l+1)/2} \left( \frac{p}{m-4l} \right) q^k \qbin{n}{l} q^{p+1-l} \qbin{p}{p-1} q
\]

\[
= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l [q^{n-2k}] q^{m-2l} \left( \frac{m+p-2l-1}{p-1} \right) q^k \qbin{n}{l} q^{p+1-l} \qbin{p}{p-1} q
\]

(4.29)

As in the previous theorem the summation over \( k \) can be done:

\[
\sum_{k=0}^{\lfloor n/4 \rfloor} \sum_{l=0}^{\lfloor m/4 \rfloor} q^{(m-4l)(m-4l+1)/2} \left( \frac{p}{m-4l} \right) q^k \qbin{n}{l} q^{p+1-l} \qbin{p}{p-1} q^4
\]

\[
= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} (-1)^l q^{m-2l} \left( \frac{m+p-2l-1}{p-1} \right) q^k \qbin{n}{l} q^{p+1-l} \qbin{p}{p-1} q^2
\]

(4.30)

Cancelling some powers of \( q \) and replacing \( m \) by \( n \) and \( l \) by \( k \) and \( p \) by \( m+1 \) gives the theorem. \( \square \)

Taking \( q = 1 \) and replacing \( n \) by \( 4n \) and in the left side replacing \( k \) by \( n-k \) and in the right side \( k \) by \( 2n-k \) gives the combinatorial identity:

\[
\sum_{k=0}^{n} \left( \frac{m+1}{4k} \right) \left( \frac{m+n-k}{m} \right) = \sum_{k=0}^{2n} (-1)^k \left( \frac{m+2k}{m} \right) \left( \frac{m+2n-k}{m} \right)
\]

(4.31)

Replacing \( n \) by \( 4n+1 \) or \( 4n+2 \) or \( 4n+3 \) gives similar identities.

**Theorem 4.6.**

\[
\sum_{k=0}^{\lfloor n/4 \rfloor} \sum_{l=0}^{\lfloor m/4 \rfloor} (-1)^k q^k \qbin{n}{l} q^4 \qbin{m}{k} q^4 \left( \frac{m+n-4k}{k} \right) \left( \frac{m+1}{m} \right)
\]

\[
= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} q^{(n-2k)/2} \left( \frac{m+1}{n-2k} \right) \qbin{m}{k} q^2
\]

(4.32)

**Proof.** From theorem 3.7 using (2.5) and (3.1):

\[
\sum_{k=0}^{\lfloor n/4 \rfloor} \sum_{l=0}^{\lfloor m/4 \rfloor} (-1)^l P^*(n-4k-m+4l, m-4l, p-1) P^*(k-l(l+1)/2, l, p-l)
\]

\[
= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} P^*(n-2k-(m-2l)(m-2l+1)/2, m-2l, p-m+2l) P^*(k-l(l+1)/2, l, p-l)
\]

(4.33)

Using (2.4) and (2.6):

\[
\sum_{k=0}^{\lfloor n/4 \rfloor} \sum_{l=0}^{\lfloor m/4 \rfloor} (-1)^l [q^{n-4k}] q^{m-4l} \left( \frac{p+m-4l-1}{p-1} \right) q^k \qbin{n}{l} q^{p+(l+1)/2} \left( \frac{p}{l} \right) q
\]

\[
= \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} [q^{n-2k}] q^{(m-2l)(m-2l+1)/2} \left( \frac{p}{m-2l} \right) q^k \qbin{n}{l} q^{p+(l+1)/2} \left( \frac{p}{l} \right) q
\]

(4.34)
Replacing $n$ in the previous theorem the summation over $k$ can be done:

\[
\sum_{k=0}^{\lfloor n/4 \rfloor} \sum_{l=0}^{\lfloor m/4 \rfloor} (-1)^i q^{m-4l} \binom{p+m-4l-1}{p-1} q^{2l(i+1)} \binom{p}{l} q^4 = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{l=0}^{\lfloor m/2 \rfloor} q^{m-2l(m-2l+1)/2} \binom{p}{m-2l} q^{l(i+1)} \binom{p}{l} q^2 \tag{4.35}
\]

Cancelling some powers of $q$ and replacing $m$ by $n$ and $l$ by $k$ and $p$ by $m+1$ gives the theorem. \(\square\)

Taking $q = 1$ and replacing $n$ by $4n$ and in the left side replacing $k$ by $n-k$ and in the right side $k$ by $2n-k$ gives the combinatorial identity:

\[
\sum_{k=0}^{n} (-1)^k \binom{m+4k}{m} \binom{m+1}{n-k} = (-1)^n \sum_{k=0}^{2n} \binom{m+1}{2k} \binom{m+1}{2n-k} \tag{4.36}
\]

Replacing $n$ by $4n+1$ or $4n+2$ or $4n+3$ gives similar identities.

**Theorem 4.7.**

\[
\sum_{k=0}^{n} (-1)^k q^{k^2} \binom{m+n-k}{m} q^{(m+1)} \binom{m+1}{k} = \delta_{n,0} \tag{4.37}
\]

**Proof.** From theorem 4.8 using (2.30) and (3.1):

\[
\sum_{k=0}^{n} \sum_{l=0}^{m} (-1)^i P^*_{n-k-m,l-m-l,p-1} P^*(k-l(l+1)/2, l, p) \tag{4.38}
\]

Replacing $m$ by $n$ and $l$ by $k$ and $p$ by $m+1$ gives the theorem. \(\square\)

Taking $q = 1$ and replacing $k$ by $n-k$ gives the combinatorial identity:

\[
\sum_{k=0}^{n} (-1)^k \binom{m+k}{m} \binom{m+1}{n-k} = \delta_{n,0} \tag{4.39}
\]

**Theorem 4.8.**

\[
\sum_{k=0}^{n} \sum_{l=0}^{n-k} (-1)^k F(k+l) q^{k^2} \binom{m+1}{k} q^{(m+l)} \binom{m+l}{m} q = F(0) \tag{4.40}
\]

\[
\sum_{k=0}^{n} \sum_{l=0}^{n-k} (-1)^k F(k+l) q^{k^2} \binom{m+1}{k} q^{(m+l)} \binom{m+l}{m} q = F(0) \tag{4.41}
\]
Proof. The double summation over \( k \) and \( l \) is over a triangle, and the sum over each diagonal \( k + l = c \) is zero because of the previous theorem with \( n = c \), except at the origin \( k = l = c = 0 \), where the summand is the right side of the identity.

The following four identities are an application of this theorem.

\[
\sum_{k=0}^{n-k} \sum_{l=0}^{n-k-1} (-1)^k q^{a(n-k-l)} + b_{\frac{k}{2}} \binom{p+n-k-l}{p} q^c \binom{m-k}{k} q^b \binom{m+l}{m} q^b = q^{a\binom{p+n}{p}} q^c
\]  

(4.42)

\[
\sum_{k=0}^{n-k} \sum_{l=0}^{n-k-1} (-1)^l q^{a(n-k-l)} + b_{\frac{k}{2}} \binom{p+n-k-l}{p} q^c \binom{m-k}{k} q^b \binom{m+l}{m} q^b = q^{a\binom{p+n}{p}} q^c
\]  

(4.43)

\[
\sum_{k=0}^{n-k} \sum_{l=0}^{n-k-1} (-1)^k q^{a(n-k-l)} + b_{\frac{k}{2}} \binom{p}{n-k-l} q^c \binom{m-k}{k} q^b \binom{m+l}{m} q^b = q^{a\binom{p}{n}} q^c
\]  

(4.44)

\[
\sum_{k=0}^{n-k} \sum_{l=0}^{n-k-1} (-1)^l q^{a(n-k-l)} + b_{\frac{k}{2}} \binom{p}{n-k-l} q^c \binom{m-k}{k} q^b \binom{m+l}{m} q^b = q^{a\binom{p}{n}} q^c
\]  

(4.45)

In the summands of the last four identities, because of the type of double summation, \( k \) and \( l \) can be interchanged and \( l \) can be replaced by \( n-k-l \). Some pairs of these identities or identities derived from them in this way have identical summands if \((-1)^k\) is replaced by \((-1)^l\) and vice versa and have identical right sides. In these cases a linear combination of the two identities can be taken such that in the summand of the new identity:

\[
\frac{1}{2} \left[ (-1)^k + (-1)^l \right] = \begin{cases} 
(-1)^k & \text{if } k + l \text{ is even} \\
0 & \text{if } k + l \text{ is odd}
\end{cases}
\]  

(4.46)

where the right side of the new identity is identical to the right sides of the original identities, and:

\[
\frac{1}{2} \left[ (-1)^k - (-1)^l \right] = \begin{cases} 
(-1)^k & \text{if } k + l \text{ is odd} \\
0 & \text{if } k + l \text{ is even}
\end{cases}
\]  

(4.47)

where the right side of the new identity is zero. For example taking identity (4.43) with \( a = 0 \), \( b = c = 1 \) and \( p = m \) and for the first identity interchanging \( k \) and \( l \) and replacing \( l \) by \( n-k-l \), and for the second identity additionally interchanging \( k \) and \( l \) again, and taking linear combination (4.40) gives corollary 2.4 in [3]:

\[
\sum_{k=0}^{n-k} \sum_{l=0}^{n-k-1} (-1)^k q^{a\binom{n-k-l}{2}} \binom{m+k}{m} q^c \binom{m+l}{m} q^b \binom{m+1}{m} q^b = \binom{m+n}{m} q
\]  

(4.48)

and taking identity (4.44) with \( a = b = c = 1 \) and \( p = m+1 \) and for the first identity replacing \( l \) by \( n-k-l \), and for the second identity additionally interchanging \( k \) and \( l \), and taking linear combination (4.46) gives corollary 3.4 in [3]:

\[
\sum_{k=0}^{n-k} \sum_{l=0}^{n-k-1} (-1)^k q_{\frac{k}{2}} \binom{m+1}{k} q^c \binom{m+1}{l} q^b \binom{m+n-k-l}{m} q = q^{a\binom{m+1}{n}} q
\]  

(4.49)
References

[1] G.E. Andrews, *The Theory of Partitions*, Cambridge University Press, 1984.

[2] G.E. Andrews, K. Eriksson, *Integer Partitions*, Cambridge University Press, 2004.

[3] G.E. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge University Press, 1999.

[4] H.W. Gould, *Combinatorial Identities*, rev. ed., Morgantown, 1972.

[5] V.J.W. Guo, D.-M. Yang, A q-analogue of some binomial coefficient identities of Y. Sun, *Electron. J. Combin.* 18 (2011) P78

[6] M.J. Kronenburg, Computation of P(n,m), the Number of Integer Partitions of n into Exactly m Parts, [arXiv:2205.04988](https://arxiv.org/abs/2205.04988) [math.NT]

[7] M.J. Kronenburg, Computation of q-Binomial Coefficients with the P(n,m) Integer Partition Function, [arXiv:2205.15013](https://arxiv.org/abs/2205.15013) [math.CO]

[8] M. Merca, Generalizations of two identities of Guo and Yang, *Quaest. Math.* 41 (2018) 643-652.

[9] E.W. Weisstein, *q-Binomial Coefficient*. From Mathworld - A Wolfram Web Resource. [https://mathworld.wolfram.com/q-BinomialCoefficient.html](https://mathworld.wolfram.com/q-BinomialCoefficient.html)

[10] Wikipedia, *Gaussian binomial coefficient*, [https://en.wikipedia.org/wiki/Gaussian_binomial_coefficient](https://en.wikipedia.org/wiki/Gaussian_binomial_coefficient)