Angular measures and Birkhoff orthogonality in Minkowski planes

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Abstract. Let \( x \) and \( y \) be two unit vectors in a normed plane \( \mathbb{R}^2 \). We say that \( x \) is Birkhoff orthogonal to \( y \) if the line through \( x \) in the direction \( y \) supports the unit disc. A \( B \)-measure (Fankhaüel in Beitr Algebra Geom 52(2):335–342, 2011) is an angular measure \( \mu \) on the unit circle for which \( \mu(C) = \pi/2 \) whenever \( C \) is a shorter arc of the unit circle connecting two Birkhoff orthogonal points. We present a characterization of the normed planes that admit a \( B \)-measure.

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1. Introduction

Let \( K \) be an origin-symmetric convex body in the plane, that is, a compact convex set with non-empty interior in \( \mathbb{R}^2 \), and consider the normed plane \( (\mathbb{R}^2, \|\cdot\|_K) \), where \( \|x\|_K = \min \{ \lambda > 0 : x \in \lambda K \} \) for any \( x \in \mathbb{R}^2 \). Then \( K \) is the unit ball of the norm, and its boundary \( \text{bd} K \) the unit circle.

Let \( x, y \in \text{bd} K \) be two unit vectors in \( \mathbb{R}^2 \). We say that \( x \) is Birkhoff orthogonal to \( y \), and denote it by \( x \perp y \), if \( \|x\|_K \leq \|x + ty\|_K \) for all \( t \in \mathbb{R} \). Geometrically, this means that the line through the point \( x \) in the direction \( y \) supports the unit ball \( K \). In general, Birkhoff orthogonality is not a symmetric relation. Normed planes where Birkhoff orthogonality is symmetric are called

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\( \{ \text{Birkhäuser} \)
Radon planes and the boundaries of their unit balls Radon curves (see the survey [5]).

A Borel measure $\mu$ on $\partial K$ is called an angular measure, if $\mu(\partial K) = 2\pi$, $\mu(X) = \mu(-X)$ for every Borel subset $X$ of $\partial K$, and $\mu$ is continuous, that is, $\mu(\{x\}) = 0$ for every $x \in \partial K$. There always exists an angular measure on $\partial K$, such as the one-dimensional Hausdorff measure on $\partial K$ normalized to $2\pi$, but an arbitrary angular measure does not necessarily have any relation to the geometry of $(\mathbb{R}^2, \|\cdot\|_K)$. A natural problem then is to find angular measures with interesting geometric properties. For instance, Brass [2] showed that whenever the unit ball is not a parallelogram, there is an angular measure in which the angles of any equilateral triangle are equal. This type of angular measure is very useful in studying packings of unit balls [2,8]. Angular measures with other properties have been proposed; see the survey [1, Section 4] for an overview. An angular measure $\mu$ is called a B-measure [3] if $\mu(C) = \pi/2$ for every closed arc $C$ of $\partial K$ that contains no opposite points of $\partial K$, and whose endpoints $x$ and $y$ satisfy $x \leftrightarrow y$.

The main result of this note (Theorem 1) is a characterization of the normed planes $(\mathbb{R}^2, \|\cdot\|_K)$ which admit a B-measure. In order to formulate this theorem, we need to introduce two subsets of $\partial K$.

We call a point $x$ in $\partial K$ an Auerbach point, if there is a $y \in \partial K$ such that $x \leftrightarrow y$ and $y \leftrightarrow x$. In this case we say that $x$ and $y$ form an Auerbach pair. It is well known that Auerbach points exist for any norm [9, Section 3.2]. We denote the set of Auerbach points of $K$ by $A(K)$. Note that $A(K)$ is a closed subset of $\partial K$. We denote the union of open non-degenerate line segments contained in $\partial K$ by $E(K)$.

**Theorem 1.** Let $K$ be an origin-symmetric convex body in $\mathbb{R}^2$. Then there is a B-measure on $\partial K$ if, and only if, the set $A(K) \setminus E(K)$ is uncountable.

This is a strengthening of a result of Fankhäl [3, Theorem 1], where the existence of a B-measure is shown under the condition that $A(K) \setminus E(K)$ contains an arc. (Fankhäl does not explicitly exclude line segments, but it is clear that they have to be excluded, as line segments in $A(K)$ necessarily have measure 0 for any B-measure; see Lemma 3.) We prove Theorem 1 in Section 2, where we also present a smooth, strictly convex, centrally symmetric planar body $K$ such that $A(K)$ is the union of two disjoint copies of the Cantor set and a countable set of isolated points (Example 4). Thus, $A(K)$ is of Lebesgue measure zero and yet, by Theorem 1, there is a B-measure on $\partial K$.

We recall that a subset of a topological space is called perfect if it is closed and has no isolated point. Recall that the support $\text{supp}(\mu)$ of a Borel measure $\mu$ on a topological space $X$ is the set of all $x \in X$ such that all open sets containing $x$ have positive $\mu$-measure. It is easy to see that the support of any continuous measure is a perfect set. In the proof of Theorem 1, we rely on the following converse for $X = [0,1]$. 

**Theorem 1.** Let $K$ be an origin-symmetric convex body in $\mathbb{R}^2$. Then there is a B-measure on $\partial K$ if, and only if, the set $A(K) \setminus E(K)$ is uncountable.
Proposition 2. Let $H \subset [0,1]$ be a non-empty, perfect set. Then there is a continuous probability measure on $[0,1]$ whose support is $H$.

This is a well-known result holding more generally for any separable complete metric space [6, Chapter II, Theorem 8.1], but for the convenience of the reader we present an explicit construction for this special case in Section 3. It is well known that every non-empty perfect set is uncountable [7, Theorem 2.43] and every uncountable Borel set contains a perfect set [4, Section 6B]. (There is an even larger class, the analytic sets, with this property [4], but we will only need it for $F_\sigma$ sets).

2. The Auerbach set and B-measure

Given two non-opposite points $a, b \in \text{bd} K$, we denote by $\langle a,b \rangle$ the closed arc from $a$ to $b$ that does not contain any opposite pairs of points. We denote the closed line segment with endpoints $a, b \in \mathbb{R}^2$ by $[a,b]$.

Lemma 3. Let $K$ be an origin-symmetric convex body in $\mathbb{R}^2$ and $\mu$ be a B-measure on $\text{bd} K$. Then $\text{supp}(\mu) \subseteq A(K) \setminus E(K)$.

Proof. Let $x \in E(K)$. Then $x \in [x^-,x^+] \subset \text{bd} K$ for some $x^-,x^+$ with $x^-\neq x^+ \neq x^-$ distinct. Let $y \in \text{bd} K$ be parallel to $[x^-,x^+]$. Since $x^-,x^+\parallel y$, we have $\mu([x^+,y]) = \mu([x^-,y]) = \pi/2$, hence $\mu([x^-,x^+]) = 0$ and $x \notin \text{supp}(\mu)$.

Next, let $x \in \text{bd} K \setminus A(K)$. Let $y_1, y_2 \in \text{bd} K$ such that $x \parallel y_1$ and $y_2 \parallel x$. Then $y_1 \neq y_2$. By possibly replacing $y_2$ by $-y_2$, we assume without loss of generality that $y_1$ and $y_2$ are in the same open half plane bounded by the line $ox$. By possibly replacing $x$ by $-x$, we may also assume without loss of generality that $y_2$ and $x$ are in the same open half plane bounded by $oy_1$. Let $x_1$ and $x_2$ be points on the same side of $oy_1$ as $x$ such that $x_1 \parallel x_1$ and $x_2 \parallel y_2$. Then $x_1, x_2 \neq x$. Because $y_2$ is between $x$ and $y_1$, we have that $x_1$ and $x_2$ are in opposite open half planes bounded by $ox$. As above, since $\mu$ is a B-measure, $\mu(\langle x_1, x_2 \rangle) = \mu(\langle x, x_1 \rangle) = \mu(\langle x, x_2 \rangle) = 0$, hence $x \notin \text{supp}(\mu)$. \hfill \Box

Proof of Theorem 1. Let $\mu$ be a B-measure on $\text{bd} K$. Then $\text{supp}(\mu)$ is a perfect set, hence uncountable, and Lemma 3 gives that $A(K) \setminus E(K)$ is uncountable.

Conversely, assume that $\tilde{A} := A(K) \setminus E(K)$ is uncountable. We next find an appropriate perfect subset of $\tilde{A}$ and use Proposition 2 to define a B-measure on $\text{bd} K$. We first need to define an auxiliary map $\phi : A \rightarrow A(K)$ by setting $\phi(x)$ to be the first $y \in A(K)$ in the positive direction along $\text{bd} K$ from $x$ so that $x \parallel y$ and $y \parallel x$. Then $\phi$ is monotone, but not necessarily injective. However, if $\phi(x_1) = \phi(x_2)$, then $x_1 \parallel y$ and $x_2 \parallel y$, as well as $x_1$ and $x_2$ being on the same side of line $oy$. Thus $[x_1, x_2]$ is a line segment on $\text{bd} K$. Since the set $E'(K) := \{y \in \text{bd} K : K \text{ has more than one supporting line at } y\}$
is countable, it follows that for any given $y \in A(K)$, there are at most two values of $x \in \tilde{A}$ such that $\phi(x) = y$, and there are at most countably many $y \in A(K)$ for which there is more than one $x \in \tilde{A}$ such that $\phi(x) = y$. In particular, $\phi$ is a Borel measurable map.

We next find an appropriate arc $\angle(a, b)$ such that $\angle(a, b) \cap \tilde{A}$ is uncountable. For any $x \in \text{bd} K$, let $x^+$ denote the first element of $\tilde{A}$ in the positive direction from $x$, and let $x^-$ be the first element of $\tilde{A}$ in the negative direction from $x$. (If $x \in \tilde{A}$ then $x = x^- = x^+$).

Let $E(K)$ denote the union of the closed line segments on $\text{bd} K$. Then $E(K)$ is the union of $E(K')$ with a countable set. Observe that for any $p \in \text{bd} K$, the set $\phi^{-1}(p)$ contains at most two points. Thus, $\phi^{-1}(E(K'))$ is countable. Moreover, $\phi^{-1}(E(K))$ is countable, since $\phi$ takes at most one value on an open line segment on $\text{bd}(K)$. Fix an element $a \in A(K) \setminus [E(K) \cup E'(K) \cup \phi^{-1}(E(K) \cup E'(K))]$, and let $b = \phi(a)$. Since $a \notin E'(K)$, the only two points of $\text{bd}(K)$ that form an Auerbach pair with $a$ are $\pm b$. Since $a \notin E(K)$, the only two points of $\text{bd}(K)$ that form an Auerbach pair with $b$ are $\pm a$. Since $a \notin \phi^{-1}(E(K))$, we have $b \in A(K) \setminus E(K)$. It follows that $\phi(b) = -a, \phi(-a) = -b$ and $\phi(-b) = a$.

We also obtain that $\angle(a, b) \cap \tilde{A}$ or $\angle(b, -a) \cap \tilde{A}$ is uncountable. Thus we may assume without loss of generality that $\angle(a, b) \cap \tilde{A}$ is uncountable, where $\phi(a) = b$ and $\phi(b) = -a$, so it contains a perfect set, and by Proposition 2 there is a continuous probability measure $\nu$ on the Borel sets of $\text{bd} K$ with $\text{supp}(\nu) \subseteq \angle(a, b) \cap \tilde{A}$. We use $\nu$ to define the B-measure as follows. For any Borel set $S \subseteq \text{bd} K$, let

$$\mu(S) := \frac{\pi}{2} \left[ \nu(S) + \nu(-S) + \nu(\phi^{-1}(S)) + \nu(\phi^{-1}(-S)) \right].$$

Then $\mu$ is clearly an angular measure. Showing that $\mu$ is a B-measure is somewhat technical, mainly because $\angle$ is not in general a symmetric relation. Let $x, y \in \text{bd} K$ with $x \parallel y$. We have to show that $\mu(\angle(x, y)) = \pi/2$. After possibly replacing $x$ by $-x$ and $y$ by $-y$, we may assume that $x \in \angle(a, b) \cup \angle(b, -a)$ and $y \in \angle(a, b) \cup \angle(b, -a)$.

Case 1: $x \in \angle(a, b)$. Then either $y \in \angle(a, b)$ or $y \in \angle(b, -a) \setminus \{b\}$.

Case 1.1: $y \in \angle(a, b)$. There are two cases depending on the relative position of $x$ and $y$.

Case 1.1.1: $x \in \angle(a, y)$. Since $a \notin E(K)$, we obtain $x = a$, and since $a \notin E'(K)$, we obtain $y = b$. Hence, $\mu(\angle(x, y)) = \pi/2$ as required.

Case 1.1.2: $x \in \angle(y, b)$. Since $b \notin E'(K)$, we obtain $y = a$, and since $b \notin E(K)$, we obtain $x = b$, and again $\mu(\angle(x, y)) = \pi/2$.

Case 1.2: $y \in \angle(b, -a) \setminus \{b\}$. In order to show that $\mu(\angle(x, y)) = \pi/2$, it will be sufficient to show that $\phi^{-1}(\angle(b, y))$ equals $\angle(a, x) \cap \tilde{A}$ up to $\nu$-measure 0. In fact, we show that
First, let \( p \in \phi^{-1}(\vartriangle(b, y^+) \cup (\{x\} \cap \tilde{A}) \cup \phi^{-1}(E(K)) \)\). Then \( \phi(p) \in \vartriangle(b, y^+) \) and \( p \in \tilde{A} \). Without loss of generality, \( \phi(p) \neq b, y^+ \), and we want to show that \( p \in \vartriangle(a, x) \). Clearly, \( p \in \vartriangle(a, b) \). Suppose that \( p \in \vartriangle(x, b) \) and \( p \neq x \). It follows from \( p \vdash \phi(p) \) and \( x \vdash y \) that \( \phi(p) \notin \vartriangle(b, y) \{y\} \), since otherwise \( p = x \). Therefore, \( \phi(p) \in \vartriangle(y, y^+) \). However, since \( \phi(p), y^+ \in \tilde{A} \), we obtain the contradiction \( \phi(p) = y^+ \). Therefore, \( p \notin \vartriangle(x, b) \{x\} \), and it follows that \( p \in \vartriangle(a, x) \), which finishes the proof of the \( \subseteq \)-inclusion of (2).

For the opposite inclusion, we assume without loss of generality that \( p \in \vartriangle(a, x) \cap \tilde{A} \) and \( \phi(p) \neq b, y^+ \). Suppose that \( \phi(p) \notin \vartriangle(b, y^+) \). Then \( y^+ \in \vartriangle(b, \phi(p)) \{\phi(p)\} \). By considering \( p \vdash \phi(p) \) and \( x \vdash y \), we obtain that \( p = x \), so \( p \in \{x\} \cap \tilde{A} \). This proves the \( \supseteq \)-inclusion of (2).

**Case 2:** \( x \in \vartriangle(b, -a) \). This case is very similar to Case 1 and we only summarize the argument.

**Case 2.1:** \( y \in \vartriangle(b, -a) \). As in Case 1.1, we use \( a, b \notin E'(K) \cup \overline{E}(K) \) to obtain that \( \{x, y\} = \{a, b\} \).

**Case 2.2:** \( y \in \vartriangle(a, b) \). In an almost identical way as in Case 1.2, we can show that

\[
\phi^{-1}(\vartriangle(b, x^+)) \cup (\{y\} \cap \tilde{A}) \cup \phi^{-1}(E(K)) = (\vartriangle(a, y) \cap \tilde{A}) \cup \phi^{-1}(\{b, x^+\}) \cup \phi^{-1}(E(K)),
\]

from which it follows that \( \nu(\vartriangle(b, x)) = \nu(\vartriangle(a, y)) \), hence \( \mu(\vartriangle(x, y) = \pi/2 \) by (1).

This completes the proof of Theorem 1. \( \square \)

**Example 4.** We present a smooth, strictly convex, origin-symmetric planar body \( K \) such that \( A(K) \) is the union of two disjoint copies of the Cantor set and a countable set of isolated points.

First, let \( D \) denote the Euclidean unit disk centered at the origin, and let \( C \) be the shorter arc connecting the two points whose angles with the positive \( x \) axis are \(-\pi/4 \) and \( \pi/4 \). Let \( C_0 \) denote the Cantor set in \( C \). Now, \( C_0 \) can be written as

\[
C_0 = C \setminus \bigcup_{n=1}^{\infty} I_n,
\]

where the \( I_n \) are disjoint open arcs in \( C \).

For each \( n \in \mathbb{Z}^+ \), we construct a smooth and strictly convex curve \( C_n \) connecting the two endpoints of \( I_n \) with the following properties.

1. \( C_n \) has the same tangents at the endpoints as \( D \);
2. \( C_n \) is contained in \( \text{conv} \, I_n \);
3. For any point $x$ of $C_n$, the tangent of $C_n$ at $x$ is orthogonal (in the Euclidean sense) to $x$ if, and only if, $x$ is the midpoint or an endpoint of $C_n$.

Consider the bump function

$$
\Psi(x) = \begin{cases} 
\exp\left(-\frac{1}{1-x^2}\right) & \text{if } x \in (-1, 1), \\
0 & \text{otherwise.}
\end{cases}
$$

It is well known that $\Psi$ is non-negative, smooth, its support is $[-1, 1]$, and the only points in its support where the derivative is zero are $-1, 1$ and $1/2$.

Let the endpoints of $I_n$ be $(\cos \alpha_n, \sin \alpha_n)$ and $(\cos \beta_n, \sin \beta_n)$, where $\alpha_n < \beta_n$. Let $C_n$ be the curve

$$
\varphi \mapsto \left(1 - \varepsilon \Psi\left(\frac{2}{\beta_n - \alpha_n} \left[\varphi - \frac{\alpha_n + \beta_n}{2}\right]\right)\right)(\cos \varphi, \sin \varphi), \quad \varphi \in [\alpha_n, \beta_n],
$$

for some small $\varepsilon > 0$.

Clearly, $C_n$ is a smooth curve, and if $\varepsilon$ is sufficiently small, then it is also strictly convex. Moreover, $C_n$ satisfies Property 1, as $\Psi'(-1) = \Psi'(1) = 0$. If $\varepsilon$ is sufficiently small, then $C_n$ satisfies Property 2 as well. Finally, to verify Property 3, observe that the tangent of $C_n$ is orthogonal to $(\cos \varphi, \sin \varphi) \in C_n$ if, and only if, the derivative of

$$
\varphi \mapsto 1 - \varepsilon \Psi\left(\frac{2}{\beta_n - \alpha_n} \left[\varphi - \frac{\alpha_n + \beta_n}{2}\right]\right)
$$

vanishes at $\varphi$. However, this is only the case at the midpoint and two endpoints of $C_n$.

The closed curve

$$
L := (\text{bd } D \setminus (C \cup -C)) \cup (C_0 \cup -C_0) \cup \bigcup_{n=1}^{\infty} (C_n \cup -C_n)
$$

is the boundary of a smooth, strictly convex, origin-symmetric planar body $K$, say. In order to identify the Auerbach points of $K$, first observe that if $x, y \in L$ form an Auerbach pair in $K$, then $x$ and $y$ are orthogonal in the Euclidean sense. (The converse does not hold, of course.) By this observation and Property 3, for each $n \in \mathbb{Z}^+$, the only Auerbach point in the relative interior of the arc $C_n$ is the midpoint of $C_n$. The same holds for $-C_n$. Again by the observation, all points of $C_0 \cup -C_0$ are Auerbach points. Finally, again by the observation, the set of Auerbach points of $(\text{bd } D \setminus (C \cup -C))$ is the rotation of the previously described set of Auerbach points in $(C_0 \cup -C_0) \cup \bigcup_{n=1}^{\infty} (C_n \cup -C_n)$ by an angle of $\pi/2$. It follows that $A(K)$ is the union of two disjoint copies of the Cantor set and a countable set of isolated points. □
3. Proof of Proposition 2

We may assume that $0, 1 \in H$. Enumerate the components of $\mathbb{R} \setminus H$ as $I_0, I_1, \ldots$, where $I_0 := (-\infty, 0)$ and $I_1 := (1, \infty)$. We will recursively assign a real number $y_n$ to each open interval $I_n$. Let $y_0 := 0$ and $y_1 := 1$.

If $y_k$ has already been defined for all $k < n$, let

$$y_n := \frac{1}{2} \left( \max_{\ell < n} y_\ell + \min_{\ell < n} y_\ell \right),$$

that is, we consider the two intervals with indices less than $n$ just below and just above $I_n$, and $y_n$ is the average of the two values assigned to these two intervals.

We define a function $f$ on $\mathbb{R}$ as follows. First, on $\mathbb{R} \setminus H$, let $f|_{I_n} = y_n$. To extend $f$ to $\mathbb{R}$, we set

$$a_x := \sup(-\infty, x) \setminus H, \quad \text{and} \quad b_x := \inf(x, \infty) \setminus H. \quad (3)$$

If $x \in H$ and $a_x = b_x$, then the left limit, $f(a_x-)$, of $f$ at $a_x$ clearly equals the right limit $f(b_x+)$. Thus, the function

$$f(x) := \begin{cases} 
  y_n & \text{if } x \in I_n; \\
  f(a_x-) = f(b_x+) & \text{if } x \in H \text{ and } x = a_x = b_x; \\
  f(a_x-) \frac{b_x - x}{b_x - a_x} + f(b_x+) \frac{x - a_x}{b_x - a_x} & \text{if } x \in H \text{ and } a_x < b_x
\end{cases}$$

is continuous, strictly increasing on $H$, and locally constant on $\mathbb{R} \setminus H$.

Finally, let $\mu_0$ denote the Lebesgue-Stieltjes measure corresponding to $f$, and $\mu_1$ the measure $\mu_1(A) = \lambda(A \cap H)$, where $\lambda$ is Lebesgue measure. Then $\mu = \mu_0 + \mu_1$ is a continuous measure, and $\text{supp} \mu \subseteq H$.

To show the reverse inclusion, let $I$ be an open interval and assume that $I \cap H \neq \emptyset$. If $I \cap H$ is of positive Lebesgue measure, then $\mu(I) > \mu_1(I) > 0$. Otherwise, $I$ is intersected by at least two $I_k$. Indeed, if only one $I_k$ intersected $I$, then $I \cap H$ would be the union of at most two intervals, contradicting that $H$ is perfect and of Lebesgue measure zero.

Since the values of $f$ on distinct intervals $I_k$ are distinct, $f$ is not constant on $I$, and hence, $\mu(I) > \mu_0(I) > 0$, completing the proof of Proposition 2.

The total measure is $\mu(\mathbb{R}) = \mu_0(\mathbb{R}) + \mu_1(\mathbb{R}) = 1 + \lambda(H) \in [1, 2]$, and thus $\nu = \mu/\mu(\mathbb{R})$ is a probability measure with the desired properties. □

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