LATTECE PACKINGS OF CROSS-POLYTOPES FROM REED–SOLOMON CODES AND SIDON SETS

MLADEN KOVACEVIC

Abstract. Two constructions of lattice packings of n-dimensional cross-polytopes (ℓ₁ balls) are described, the density of which exceeds that of any prior construction by a factor of at least \(2^{n/2n(1+o(1))}\) when \(n \to \infty\). The first family of lattices is explicit and is obtained by applying Construction A to a class of Reed–Solomon codes. The second family has subexponential construction complexity and is based on the notion of Sidon sets in finite Abelian groups. The construction based on Sidon sets also gives the highest known asymptotic density of packing discrete cross-polytopes of fixed radius \(r \geq 3\) in \(\mathbb{Z}^n\).

1. Introduction

Dense packings of spheres and other bodies in Euclidean spaces have been objects of mathematical research for centuries [2,8,14], and have also found applications in various fields such as coding theory and physics. In this paper we consider the problem of efficiently packing cross-polytopes and give two simple constructions of lattice packings in \(\mathbb{R}^n\), for arbitrary \(n\), of density larger than that of any prior construction.

An \(n\)-dimensional cross-polytope \(C_n\) is a unit ball in \(\mathbb{R}^n\) with respect to the \(\ell_1\) metric, \(C_n = \{y \in \mathbb{R}^n : \sum_{i=1}^{n} |y_i| \leq 1\}\). A cross-polytope of radius \(r \in \mathbb{R}\) is the body \(rC_n = \{ry : y \in C_n\}\) of volume \(\left(\frac{2r}{n!}\right)^n\). By a discrete cross-polytope of radius \(r \in \mathbb{Z}\) we mean the set \((rC_n) \cap \mathbb{Z}^n\) of cardinality \(\sum_{j \geq 0} 2^j \binom{n}{j} \binom{r}{j}\).

A lattice packing of cross-polytopes of radius \(r\) is an arrangement of these bodies in \(\mathbb{R}^n\) of the form \(\mathcal{L} + rC_n = \{x + y : x \in \mathcal{L}, y \in rC_n\}\), where \(\mathcal{L}\) is a lattice (a discrete additive subgroup of \(\mathbb{R}^n\)) of minimum \(\ell_1\) distance \(\geq 2r\). The density of such a packing is the fraction of space covered by the cross-polytopes; it can be computed as the ratio of the volume of a cross-polytope and the determinant of the lattice \(\mathcal{L}\) (the volume of its fundamental cell), that is \(\frac{(2r)^n}{n! \det \mathcal{L}}\).

Date: April 23, 2022.

2020 Mathematics Subject Classification. Primary: 11H31, 52C17, 05B40. Secondary: 11B83, 11H71, 11T71.

Key words and phrases. Lattice packing, cross-polytope, superball, Lee metric, Manhattan metric, Reed–Solomon code, Sidon set, \(B_h\) sequence.

The author is with the Faculty of Technical Sciences, University of Novi Sad, 21000 Novi Sad, Serbia (email: kmladen@uns.ac.rs; orcid: 0000-0002-2395-7628).

This work was supported by the European Union’s Horizon 2020 research and innovation programme under Grant Agreement number 856967, and by the Secretariat for Higher Education and Scientific Research of the Autonomous Province of Vojvodina through the project number 142-451-2686/2021.
Rush [18] gave a construction of lattice packings of cross-polytopes in $\mathbb{R}^n$, for $n = \frac{p-1}{2}$, $p$ an odd prime, of density
\begin{equation}
\max_{1 \leq t \leq n} \frac{(2t+1)^n}{n! (2n+1)^t}.
\end{equation}
The value of $t$ for which the maximum in (1.1) is attained is [18]
\begin{equation}
t = \frac{n}{\ln(2n+1)} - \frac{1}{2}.
\end{equation}

In the present paper we describe two constructions of lattice packings in arbitrary dimension, one based on Reed–Solomon codes (Section 2) and the other based on the notion of Sidon sets in finite Abelian groups (Section 3), both of which exceed the density in (1.1) by a factor that scales as $2^{\Theta(n \log n)}$ when $n \to \infty$.

As pointed out in [18], packings of much higher density can be shown to exist by non-constructive methods such as the Minkowski–Hlawka (MH) theorem: the MH lower bound on the lattice packing density of cross-polytopes is of the form $2^{1-n+o(n)}$ [14] [17], while (1.1) scales as $e^{-n \ln \ln n + O(n)}$. The most efficient known algorithms for constructing lattices that achieve the MH bound (up to lower order terms in the exponent) have complexity $2^{O(n \log n)}$ and are obtained by exhaustive search methods such as the Gilbert–Varshamov bound from coding theory [17]. It is desirable, however, both from the mathematical viewpoint and in applications, to have at one’s disposal more efficient and explicit constructions of packings.

How one defines constructiveness of a packing is to an extent subjective, but one very natural definition [9] is that, for a given family of lattice packings, there exists an algorithm for constructing the lattices (e.g., for producing their basis vectors) whose complexity grows polynomially with the dimension $n$. The first family of packings described in this paper is constructive in this sense. While it is not clear whether the second family can also be constructed in polynomial time, we do show that it can be constructed by using probabilistic algorithms of subexponential complexity $2^{\Theta(\sqrt{n \log n})}$. Furthermore, the second family can also be considered constructive in the (weak) sense that it “arises from other natural mathematical objects”, see the discussion by Litsyn and Tsfasman [9, Section 3]. Finally, as we point out in Section 4, the construction based on Sidon sets is interesting for the following reason as well: when the radius $r$ is fixed, this construction is of polynomial complexity and produces densest known packings of discrete cross-polytopes in $\mathbb{Z}^n$ in the asymptotic regime $n \to \infty$, for any $r \geq 3$.

To conclude the introductory part of the paper, let us mention that dense packings of cross-polytopes also induce reasonably dense packings of superballs ($\ell_\sigma$ balls) for small values of $\sigma$ ($1 \leq \sigma < 2$) by using the trivial method of inscribing a superball inside a cross-polytope, see [18].

1 Regarding the MH theorem, we point the reader to the recent paper [19] and references therein for (subexponential) improvements of this bound in the sphere-packing case, and constructions of complexity $2^{O(n \log n)}$ that achieve it. We also note that, in the case of packing superballs ($\ell_\sigma$ balls), exponential improvements of the MH bound are known when $\sigma > 2$ [17] [4]. In the case of cross-polytopes ($\sigma = 1$), it is conjectured [19] that no such exponential improvement is possible.
2. Construction based on Reed–Solomon codes

Given positive integers \( n, t \) and a prime \( p \) satisfying \( 1 \leq t \leq n < p \), let \( C_{n,t,p}^{\text{RS}} \) be the set of all vectors \( \mathbf{x} \in \mathbb{Z}_p^n \) satisfying the congruences \( \sum_{i=1}^n i^s x_i = 0 \) (mod \( p \)), for \( s = 0, 1, \ldots, t - 1 \). The set \( C_{n,t,p}^{\text{RS}} \) is a (generalized) Reed–Solomon code of length \( n \) over the field \( GF(p) \equiv \mathbb{Z}_p \) [15, Chapter 5]. Both the cardinality, \( |C_{n,t,p}^{\text{RS}}| = p^{n-t} \), and the minimum distance properties of the code are controlled by the parameter \( t \). Namely, the minimum Hamming distance of \( C_{n,t,p}^{\text{RS}} \) is \( t+1 \) [15, Proposition 5.1], while its minimum Lee distance\(^2\) is lower bounded by \( 2t \) [16, Theorem 3].

Let \( L_{n,t,p}^{\text{RS}} = p\mathbb{Z}^n + C_{n,t,p}^{\text{RS}} \) be the lattice obtained by periodically extending the above code to all of \( \mathbb{Z}^n \). Written explicitly,\(^3\)

\[
(L_{n,t,p}^{\text{RS}}) = \left\{ \mathbf{x} \in \mathbb{Z}^n : \sum_{i=1}^n i^s x_i = 0 \pmod{p}, \quad s = 0, 1, \ldots, t - 1 \right\}.
\]

The fact that the minimum Lee distance of the code \( C_{n,t,p}^{\text{RS}} \) is at least \( 2t \) implies that the minimum \( \ell_1 \) distance of the lattice \( L_{n,t,p}^{\text{RS}} \) is at least \( 2t \). This lattice therefore induces a packing of cross-polytopes of radius \( t \). Since \( \det L_{n,t,p}^{\text{RS}} = p^t \), the density of the packing is \( \frac{2^n}{n^t p^t} \). Moreover, the lattice can be constructed efficiently, which is evident from its definition. We have just shown the following.

**Theorem 2.1.** For every \( n \geq 1 \), the cross-polytope can be constructively lattice packed in \( \mathbb{R}^n \) with density

\[
\max_{1 \leq t \leq n} \frac{(2t)^n}{n! p(n)^t},
\]

where \( p(n) \) is the smallest prime larger than \( n \).

The value of \( t \) that maximizes the expression in (2.2) also maximizes \( n \ln(2t) - t \ln p(n) \), and by differentiating the latter we find this value to be

\[
t = \frac{n}{\ln p(n)}.
\]

The same method of constructing lattices from codes was used in [18] (the so-called Construction A from [2, Chapter 5]), but the starting point therein was the Berlekamp’s negacyclic code [15, Chapter 10.6] of length \( n = \frac{p-1}{2} \), where \( p \) is an odd prime, and minimum Lee distance \( 2t+1 \).

The main advantages of the construction (2.1) with respect to [18] are the following: (a) the packing (2.1) is defined in all dimensions, (b) the construction (2.1) is explicit, while that from [18] requires finding a primitive element in the field \( GF(p) \), and (c) the packing density of (2.1) is larger by a factor that scales as \( 2^{\frac{n}{2t+1}(1+o(1))} \) when \( n \to \infty \).

To justify (c), suppose that \( n = \frac{p-1}{2} \) for an odd prime \( p \), and note that the ratio of the densities from (2.2) and (1.1) equals \( \left(\frac{2}{2t+1}\right)^n (\frac{2n+1}{p(n)})^t \), where \( t \approx \frac{n}{\ln n} \) (see (1.2) and (2.3)). Recalling that \( p(n) = n + o(n) \) for sufficiently large \( n \) [3, Section 1.4.1], our claim follows.

\(^2\)Lee distance is essentially the \( \ell_1 \) distance defined on the torus \( \mathbb{Z}_m^n \). A code in \( \mathbb{Z}_m^n \) having minimum Lee distance \( d \) can therefore be thought of as a packing of discrete cross-polytopes (\( \ell_1 \) balls) of radius \( \frac{d-1}{2} \) in the torus, see [6].
3. Construction based on Sidon sets

A collection of elements $b_1, b_2, \ldots, b_n$ from an Abelian group $(G, +)$ having the property that the sums $b_{i_1} + b_{i_2} + \cdots + b_{i_h}$, $1 \leq i_1 \leq i_2 \leq \cdots \leq i_h \leq n$, are all different is called a Sidon set of order $h$.\footnote{Or a Sidon sequence of order $h$, or a $B_h$ sequence. These objects have been studied quite extensively [13]. For more on their connection to lattice packing problems, see [11][12].} An equivalent way of expressing this property is that the sums

$$\sum_{i=1}^{n} r_i b_i, \quad \text{where } r_i \in \mathbb{Z}, r_i \geq 0, \quad \sum_{i=1}^{n} r_i = h,$$

are all different.

Here $r_i b_i$ represents the sum of $r_i$ copies of the element $b_i \in G$. Two elegant constructions of Sidon sets were described by Bose and Chowla in [1], one of which is repeated next for completeness.

For a prime power $n$, let $\alpha_1 = 0, \alpha_2, \ldots, \alpha_n$ be the elements of the Galois field $GF(n)$, and $\beta$ a primitive element of the extended field $GF(n^h)$ (i.e., a generator of its multiplicative group). Let $b_1, b_2, \ldots, b_n$ be the numbers from the set $\{1, 2, \ldots, n^h - 1\}$ defined by

$$\beta^{b_i} = \beta + \alpha_i, \quad i = 1, \ldots, n.$$

Then the numbers $b_1 = 1, b_2, \ldots, b_n$, thought of as elements of the cyclic group $(\mathbb{Z}_{n^h}, +)$, satisfy the condition (3.1). To see that they do, suppose, for the sake of contradiction, that $b_{i_1} + b_{i_2} + \cdots + b_{i_h} = b_{j_1} + b_{j_2} + \cdots + b_{j_h}$ for two different sets of indices $\{i_k\}, \{j_k\}$. Then it would follow from (3.2) that

$$(\beta + \alpha_{i_1})(\beta + \alpha_{i_2}) \cdots (\beta + \alpha_{i_h}) = (\beta + \alpha_{j_1})(\beta + \alpha_{j_2}) \cdots (\beta + \alpha_{j_h})$$

and, after canceling the $\beta^{b_i}$ terms, that $\beta$ is a root of a polynomial of degree $< h$ with coefficients in $GF(n)$, which is not possible. When the desired cardinality $n$ is not a prime power, one can use the same method to produce a Sidon set $b_1, b_2, \ldots, b_{\tilde{p}(n)}$, where $\tilde{p}(n)$ is the smallest prime power greater than or equal to $n$, and keep any $n$ of its $\tilde{p}(n)$ elements (a subset of a Sidon set is also a Sidon set).

Let $g^s(h, n)$ denote the size of the smallest Abelian group containing a Sidon set of order $h$ and cardinality $n$. From the Bose–Chowla construction just described we know that $g^s(h, n) < \tilde{p}(n)^h$.

**Theorem 3.1.** For every $n \geq 1$, the cross-polytope can be constructively (in the weak sense) lattice packed in $\mathbb{R}^n$ with density

$$\max_{t \geq 1} \frac{(2t)^{n-1}}{n! g^s(t-1, n)} > \max_{t \geq 1} \frac{(2t)^{n-1}}{n! \tilde{p}(n)^{t-1}},$$

where $\tilde{p}(n)$ is the smallest prime power greater than or equal to $n$.

The value of $t$ that maximizes the expression on the right-hand side of the inequality (3.4) is

$$t = \frac{n - 1}{\ln \tilde{p}(n)}.$$
Proof of Theorem 3.1. Given a Sidon set $B = \{b_1, b_2, \ldots, b_n\}$ of order $t - 1$ in an Abelian group $G$, define the following lattice:

$$\mathcal{L}_{n,t:B}^s = \left\{ x \in \mathbb{Z}^n : \sum_{i=1}^n x_i = 0 \pmod{2t}, \sum_{i=1}^n x_ib_i = 0 \right\}.$$  

Here $x_ib_i$ represents the sum of $|x_i|$ copies of the element $b_i \in G$ (resp. $-b_i \in G$) if $x_i \geq 0$ (resp. $x_i < 0$). We claim that the minimum $\ell_1$ distance of the points in this lattice is $2t$. To see this, note that any two points $x, y \in \mathcal{L}_{n,t:B}^s$ with $\sum_{i=1}^n x_i \neq \sum_{i=1}^n y_i$ satisfy $\sum_{i=1}^n (x_i - y_i) = 2tk$ for a nonzero integer $k$. They must be at distance at least $2t$ because $\sum_{i=1}^n |x_i - y_i| \geq |\sum_{i=1}^n (x_i - y_i)| = 2t|k|$. Now consider $x, y \in \mathcal{L}_{n,t:B}^s$, $x \neq y$, with $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Suppose that, for two such points, $\sum_{i=1}^n |x_i - y_i| \leq 2(t - 1)$ (the distance is in this case necessarily even). Then one can write $x + r = y + s$, for some $r, s \in \mathbb{Z}^n$, $r \neq s$, with $r_i \geq 0$, $s_i \geq 0$, and $\sum_{i=1}^n r_i = \sum_{i=1}^n s_i = t - 1$. This, together with the fact that $\sum_{i=1}^n x_ib_i = \sum_{i=1}^n y_ib_i = 0$ (see (3.6)), implies $\sum_{i=1}^n r_ib_i = \sum_{i=1}^n s_ib_i$. As this contradicts (3.1) (with $h = t - 1$), our assumption that $\sum_{i=1}^n |x_i - y_i| \leq 2(t - 1)$ must be wrong. Therefore, as claimed, the minimum $\ell_1$ distance of the lattice $\mathcal{L}_{n,t:B}^s$ is $2t$, implying that it induces a packing of cross-polytopes of radius $t$. Since $\det \mathcal{L}_{n,t:B}^s = 2t |G|$, the obtained packing density is $\frac{2(t)\ln n}{n^2}$. Furthermore, by the result from [1] cited above we may take $G = \mathbb{Z}_{\tilde{p}(n)^{t-1}}$, implying the lower bound in (3.4).

The density obtained in Theorem 3.1 is comparable to that from Theorem 2.1. The former is also larger than the density obtained in [18] by a factor that scales as $2\sqrt{\ln n}/(1+o(1))$ when $n \to \infty$, see (1.1) and (3.4). An additional advantage of the construction based on Sidon sets, compared to both Theorem 2.1 and [18], is that the packing is defined for every $t$, i.e., the dimension and the radius are independent variables in this approach.

It should be noted, however, that the construction complexity of the family of lattices (3.6) is higher. In particular, constructing a Sidon set of cardinality $n$ and order $t - 1$ that was described in [1] (see the second paragraph of this section) involves: (i) finding a primitive element $\beta$ in the field $GF(\tilde{p}(n)^{t-1})$, and (ii) finding solutions in \{1, 2, \ldots, \tilde{p}(n)^{t-1} - 1\} to $n$ equations of the form (3.2). Note that, when $t \sim \frac{n}{\ln n}$ (see (3.5)), the required field size is exponential in the dimension, $\tilde{p}(n)^{t-1} = e^{n+o(n)}$. The problem (ii) is an instance of the discrete logarithm problem which, as recent advances have shown [12], can be solved in expected quasi-polynomial time $2^{O(\log^2 n)}$. As for problem (i) – finding a primitive element in $GF(\tilde{p}(n)^{t-1})$, $t \sim \frac{n}{\ln n}$ – a result of Shoup [20] implies that it can be reduced in time polynomial in $n$ to the problem of (i) testing whether a given element $\beta$ is primitive. A classical way of solving (i) is by factoring $\tilde{p}(n)^{t-1} - 1$ (the order of the multiplicative group of $GF(\tilde{p}(n)^{t-1})$) and checking whether $\beta^x = 1$ for any non-trivial factor $x$. Factoring numbers of this magnitude can be performed in expected time $2^{O(\sqrt{n \log \tilde{p}(n)})}$ [3] Chapter 6]. In conclusion, the complexity of constructing (3.6) is dominated by the problem of finding primitive elements in large finite fields, and can be upper bounded by $2^{O(\sqrt{n \log \tilde{p}(n)})}$.

---

4Faster methods are known, such as the number field sieve, but they are not rigorous and rely on heuristics [3] Chapter 6].
4. Lattice packings of discrete cross-polytopes in $\mathbb{Z}^n$

A lattice packing of discrete cross-polytopes of radius $t \in \{1, 2, \ldots\}$ is an arrangement of discrete cross-polytopes in $\mathbb{Z}^n$ of the form $\mathcal{L} + (tC_n \cap \mathbb{Z}^n)$, where $\mathcal{L}$ is a sublattice of $\mathbb{Z}^n$ with minimum $\ell_1$ distance $> 2t$ (note the strict inequality here). The density of such a packing – the fraction of points in $\mathbb{Z}^n$ covered by the cross-polytopes – is $\frac{|(tC_n \cap \mathbb{Z}^n)|}{\det \mathcal{L}}$.

Apart from being interesting on their own, packings in $\mathbb{Z}^n$ are useful for producing packings in $\mathbb{R}^n$, as we have seen in the previous two sections. In fact, most lattices described in the literature for the purpose of packing various convex bodies in $\mathbb{R}^n$, are sublattices of $\mathbb{Z}^n$. Moreover, one can show that optimal lattice packings of, e.g., cross-polytopes in $\mathbb{R}^n$, for any fixed $n$, can be obtained via optimal lattice packings of discrete cross-polytopes of radius $t \to \infty$ in $\mathbb{Z}^n$ [11, Remark 2.2]. Discrete packings are also of interest in coding theory where they frequently represent the underlying geometric problem. For example, an appropriate finite restriction of a lattice $\mathcal{L} \subseteq \mathbb{Z}^n$ with minimum $\ell_1$ distance $\geq 2t + 1$ can be interpreted as a code correcting $t$ errors of certain type [12].

We state below the discrete version of Theorem 3.1, as it may be of separate interest. The construction based on Sidon sets appears to produce very dense packings of discrete cross-polytopes, at least in the case when the radius $t$ is fixed and $n \to \infty$. Furthermore, in this regime the construction is of polynomial complexity.

**Theorem 4.1.** Fix an arbitrary positive integer $t$. For every $n \geq 1$, the discrete cross-polytope of radius $t$ can be constructively lattice packed in $\mathbb{Z}^n$ with density

$$(4.1) \quad \frac{\sum_{j \geq 0} 2^j \binom{n}{j} \binom{t}{j}}{(2t + 1) g^*(t, n)} > \frac{\sum_{j \geq 0} 2^j \binom{n}{j} \binom{t}{j}}{(2t + 1) \tilde{p}(n)^t},$$

where $\tilde{p}(n)$ is the smallest prime power greater than or equal to $n$.

**Proof.** A lattice of minimum $\ell_1$ distance $\geq 2t + 1$ can be obtained by using the same construction as in (3.6), with two minor modifications: we require that $\sum_{i=1}^n x_i = 0$ (mod $2t + 1$) (instead of mod $2t$), and that $B$ is a Sidon set of order $t$ (instead of $t - 1$). The lattice can be efficiently constructed because the radius is fixed and hence the group containing a Sidon set is of polynomial size, $O(n^t)$. \[\blacksquare\]

For a fixed radius $t$ and $n \to \infty$, the asymptotic value of the expression on the right-hand side of (4.1) is

$$(4.2) \quad \frac{2^t}{t!(2t + 1)}.$$  

For $t = 1, 2$, this lower bound can be improved. For $t = 1$, the maximum possible density of 1 can be achieved for every $n$, as perfect packings of discrete cross-polytopes of radius 1 exist (and are easily constructed) in all dimensions [6]. For $t = 2$, the construction from [18] yields the asymptotic density $\frac{1}{3} = \frac{1}{2}$, while the expression in (4.2) equals $\frac{2}{5}$. For $t \geq 3$, the asymptotic density in (4.2) is, to the best of our knowledge, the highest known.
ACKNOWLEDGEMENTS

The author would like to thank the referees for their thorough reading and constructive comments which greatly improved the manuscript, and Benjamin Wesolowski for clarifying several points about [10].

REFERENCES

[1] R. C. Bose, S. Chowla, “Theorems in the Additive Theory of Numbers,” Comment. Math. Helv., 37(1) (1962), 141–147.
[2] J. H. Conway, N. J. A. Sloane, Sphere Packings, Lattices and Groups, 3rd ed., Springer, 1999.
[3] R. Crandall, C. Pomerance, Prime Numbers, A Computational Perspective, 2nd ed., Springer, 2005.
[4] N. D. Elkies, A. M. Odlyzko, J. A. Rush, “On the Packing Densities of Superballs and Other Bodies,” Invent. Math., 105(1) (1991), 613–639.
[5] N. Gargava, V. Serban, “Dense Packings via Lifts of Codes to Division Rings,” https://doi.org/10.48550/arXiv.2111.03684.
[6] S. W. Golomb, L. R. Welch, “Perfect Codes in the Lee Metric and the Packing of Polyominoes,” SIAM J. Appl. Math., 18(2) (1970), 302–317.
[7] R. Granger, T. Kleinjung, J. Zumbrügel, “On the Discrete Logarithm Problem in Finite Fields of Fixed Characteristic,” Trans. Amer. Math. Soc., 370(5) (2018), 3129–3145.
[8] P. M. Gruber, C. G. Lekkerkerker, Geometry of Numbers, 2nd ed., North-Holland, 1987.
[9] S. N. Litsyn, M. A. Tsfasman, “Constructive High-Dimensional Sphere Packings,” Duke Math. J., 54(1) (1987), 147–161.
[10] T. Kleinjung, B. Wesolowski, “Discrete Logarithms in Quasi-polynomial Time in Finite Fields of Fixed Characteristic,” J. Amer. Math. Soc., 35(2) (2022), 581–624.
[11] M. Kovacević, V. Y. F. Tan, “Improved Bounds on Sidon Sets via Lattice Packings of Simplices,” SIAM J. Discrete Math., 31(3) (2017), 2269–2278.
[12] M. Kovacević, V. Y. F. Tan, “Codes in the Space of Multisets—Coding for Permutation Channels with Impairments,” IEEE Trans. Inform. Theory, 64(7) (2018), 5156–5169.
[13] K. O’Bryant, “A Complete Annotated Bibliography of Work Related to Sidon Sequences,” Electron. J. Combin., #DS11 (2004), 39 p. (electronic).
[14] C. A. Rogers, Packing and Covering, Cambridge University Press, 1964.
[15] R. M. Roth, Introduction to Coding Theory, Cambridge University Press, 2006.
[16] R. M. Roth, P. H. Siegel, “Lee-Metric BCH Codes and Their Application to Constrained and Partial-Response Channels,” IEEE Trans. Inform. Theory, 40(4) (1994), 1083–1096.
[17] J. A. Rush, “A Lower Bound on Packing Density,” Invent. Math., 98 (1989), 499–509.
[18] J. A. Rush, “Constructive Packings of Cross Polytopes,” Mathematika, 38(2) (1991), 376–380.
[19] J. A. Rush, “A Bound, and a Conjecture, on the Maximum Lattice-Packing Density of a Superball,” Mathematika, 40(1) (1993), 137–143.
[20] V. Shoup, “Searching for Primitive Roots in Finite Fields,” Math. Comp., 58(197) (1992), 369–380.