Threshold dynamics and ergodicity of an SIRS epidemic model with Markovian switching

Dan Li\textsuperscript{a}, Shengqiang Liu\textsuperscript{a,}\textsuperscript{*}, Jing’an Cui\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang, 150001, P.R. China
\textsuperscript{b}School of Science, Beijing University of Civil Engineering and Architecture, Beijing, 100044, P.R. China

Abstract
This paper studies the spread dynamics of a stochastic SIRS epidemic model with nonlinear incidence and varying population size, which is formulated as a piecewise deterministic Markov process. A threshold dynamic determined by the basic reproduction number $R_0$ is established: the disease can be eradicated almost surely if $R_0 < 1$, while the disease persists almost surely if $R_0 > 1$. The existing method for analyzing ergodic behavior of population systems has been generalized. The modified method weakens the required conditions and has no limitations for both the number of environmental regimes and the dimension of the considered system. When $R_0 > 1$, the existence of a stationary probability measure is obtained. Furthermore, with the modified method, the global attractivity of the $\Omega$-limit set of the system and the convergence in total variation to the stationary measure are both demonstrated under a mild extra condition.

Keywords: Stochastic SIRS epidemic model; Piecewise deterministic Markov process; Stationary distribution; $\Omega$-limit set; Attractor; Markov switching

AMS subject classifications. 60H10, 93E15, 92D25, 92D30

1. Introduction

Since the seminal work of Kermack and McKendrick \cite{1}, mathematical models have become important tools for understanding the spread and control of infectious diseases. In the real ecological systems, the population dynamics are usually influenced by a random switching in the external environments. For example, the disease transmission rate $\beta$ in epidemic models was modified for meteorological factors because survivals and infectivity of many viruses and bacteria are better in damp conditions with little ultraviolet light \cite{2,3,4}. The population growth rates and the environmental capacities usually fluctuate with the change of food resource abundance, which is in turn dependent upon unpredictable rainfall fluctuations largely \cite{3}. In the literatures, it is a popular way that the random switching of environmental regimes is characterized by the continuous-time Markov chain with values in a finite state space, which drives the changes of the main parameters of population models with state switchings of the Markov chain \cite{6,7,8,9,10,11,12}. The resulting dynamical systems then become regime-switching differential equations, i.e., the

\textsuperscript{*}Corresponding author

Email addresses: 1idanwhy@163.com (Dan Li), sqliu@hit.edu.cn (Shengqiang Liu), cuijingan@bucea.edu.cn (Jing’an Cui)
piecewise-deterministic Markov process. Therefore, based on the biological system subject to stochastic environmental conditions, epidemic models with deterministic parameters are unlikely to be realistic, and it is significant to investigate the effect of the random switching of environmental regimes on the spread dynamics of the disease in the host population.

However, despite the potential importance of the environmental noise, it has received relatively little attention in the epidemiology literatures. Gray, Greenhalgh and Mao et al. [7] are the first ones to propose a piecewise deterministic SIS epidemic model with Markovian switching. By the modeling techniques of time discretization or branching processes, Bacáer et al. [13, 14] obtained very well results concerning the definition of the suitable basic reproduction number for the epidemic model with random switching of environmental regimes. Recently, Hieu and Du et al. [15] and Greenhalgh, Liang and Mao [16] studied the effect of Markovian switching on the deterministic SIRS epidemic models respectively, both of which are special cases of the classic epidemic model as follows

\[
\begin{align*}
\frac{dS(t)}{dt} & = \Lambda - \mu S(t) + \lambda R(t) - \beta S(t)I(t), \\
\frac{dI(t)}{dt} & = \beta S(t)I(t) - (\mu + \alpha + \delta)I(t), \\
\frac{dR(t)}{dt} & = \delta I(t) - (\mu + \lambda)R(t),
\end{align*}
\]

(1.1)

which was presented by Anderson and May [17] to study the influence of infectious diseases (caused by viruses or bacteria, etc.) on the density of host populations. In this model, \( S(t) \), \( I(t) \) and \( R(t) \) are respectively the number (or density) of susceptible, infectious and recovered individuals at time \( t \), and all the parameters are positive. \( \Lambda \) represents a constant recruitment of new susceptibles; \( \mu \) is the per-capita natural mortality rate; \( \beta \) is the transmission rate (i.e., effective per capita contact rate of infective individuals) equal to the product of the contact rate and transmission probability; \( \alpha \) is the mortality caused by the disease; \( \delta \) is the per-capita recovery rate of infected individuals; the recovered hosts are initially immune, but this immunity can be lost at a rate \( \lambda \).

In the existing literatures, there are few researches on the ergodicity of the stochastic epidemic model, which is formulated as the piecewise deterministic Markov process. To the best of our knowledge, Hieu and Du et al. [13] first studied the ergodicity of the regime-switching epidemic model, which was the stochastic edition of a special case of system (1.1). They obtained the threshold between the extinction and persistence of the disease. Using the method of references [11, 12], they described completely the \( \Omega \)-limit set of all positive solutions of the model and established the sufficient condition ensuring that the instantaneous measure converges to the stationary measure in total variation. However, the mortality caused by the disease was not taken into account, which led to the variable population size and then resulted in the challenge of reducing the dimension of the considered system. Moreover, they assumed that there were only two environmental regimes.

In addition, the incidence function plays an important role for ensuring that the epidemic model does give a reasonable description of the disease dynamics [18, 19]. It is traditionally assumed that the incidence rate of disease spread is bilinear with respect to the number of susceptible individuals \( S(t) \) and the number of infective individuals \( I(t) \), e.g., \( \beta SI \) as in the system (1.1). Actually, it is generally difficult to get the details of transmission of infectious diseases because they may vary with the ambient conditions. Moreover, with the general incidence function, the data themselves may flexibly decide the function form of incidence rates in practice [20]. Therefore, in this paper, we will choose the general nonlinear incidence rate of the form \( \beta SG(I) \) to enable the model to be more realistic and have wider application.

Motivated by the facts mentioned above, in this paper, we will study the following stochastic SIRS epidemic model with the general (including both linear and nonlinear) inci-


dS(t)/dt = \Lambda - \mu S(t) + \lambda R(t) - \beta r(t)S(t)G(I(t)),

dI(t)/dt = \beta r(t)S(t)G(I(t)) - (\mu + \alpha + \delta)I(t),

dR(t)/dt = \delta I(t) - (\mu + \lambda)R(t),

(1.2)

where \( G(\cdot) \) is a general function and the transmission rate \( \beta \) is driven by a homogeneous continuous-time Markov chain \( \{r(t), t \geq 0\} \) taking values in a finite state space \( \mathcal{M} = \{1, 2, \ldots, E\} \) representing different environments. As in [21], we let the Markov chain \( r(t) \) be generated by the transition rate matrix \( Q = (q_{e,e'})_{E \times E} \), i.e.,

\[
\mathbb{P}\{r(t + \Delta t) = e' | r(t) = e\} = \begin{cases} 
q_{e,e'}\Delta t + o(\Delta t), & \text{if } e \neq e', \\
1 + q_{e,e'}\Delta t + o(\Delta t), & \text{if } e = e', 
\end{cases}
\]

where \( \Delta t > 0 \) represents a small time increment. Here \( q_{e,e'} \) is the transition rate from state \( e \) to state \( e' \), and \( q_{e,e'} \geq 0 \) if \( e \neq e' \) while \( q_{e,e} = -\sum_{e' \neq e} q_{e,e'} \).

The main aim of this paper is to investigate the extinction and persistence of the disease of system (1.2) and determine the threshold between them. In the case of the disease persistence, we will analyze the ergodic behavior of system (1.2). By skilled techniques, authors in [11, 12] applied the stochastic stability theory of Markov processes in [22, 23, 24, 25] to two-dimensional Kolmogorov systems of competitive type switching between two environmental states. They obtained the sufficient conditions for the global attractivity of the \( \Omega \)-limit set of the system and the convergence of the instantaneous measure to the stationary measure in total variation. However, the method used in [11, 12] is not applicable for our model (1.2). In this paper, we will generalize the method to analyze ergodic behavior of the ecological systems from two environmental regimes to any finite ones. By the theory in [26], we will weaken the condition to ensure the global attractivity of the \( \Omega \)-limit set and the convergence of instantaneous distribution to the stationary distribution. Moreover, the modified techniques avoid the limitations for both the number of environmental regimes and the dimension of the considered system. For instance, the method used by [11, 12] does not deal with the case where the dimension of the system is higher than the number of environmental states of Markov chain, while the modified method can do.

It is worth mentioning that the modified method can be applied to the case where all parameters of the model considered in [7, 15, 16] were derived by the stochastic process \( r(t) \), however, here we only allow the transmission rate \( \beta \) of model (1.2) to be disturbed because in reality it may be more sensitive to environmental fluctuations than other parameters of model (1.1) for human populations. In particular, by the new method, under weaker conditions we can directly extend the results in [11, 12, 15] from two environmental states to any finite ones.

This paper is organized as follows. Section 2 introduces some preliminaries used in the later parts and illustrates our main results. In this section, we investigate the extinction and the persistence in the time mean of the disease and establish the threshold between them. Under the mild conditions, we describe completely the \( \Omega \)-limit set of all positive solutions to the model (1.2), and prove the global attractivity of the \( \Omega \)-limit set and the stochastic stability of the unique invariant measure. In Section 3, we give the proofs of the main results in details. Finally we conclude this paper with further remarks in Section 4.
2. Preliminaries and main results

2.1. Preliminaries

In this paper, unless otherwise specified, let \((\tilde{\Omega}, \mathcal{F}, \mathbb{P})\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions (i.e., it is right continuous and increasing while \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets). Note that we here use \(\tilde{\Omega}\) instead of the usual \(\Omega\) to denote the sample space, still denote by \(\omega\) an element of \(\tilde{\Omega}\), and reserve the notation \(\Omega\) for the notion of omega-limit set to avoid notional conflict. For any initial value \(z_0 = (S(0), I(0), R(0)) \in \mathbb{R}^3_+\) with \(\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_i > 0, i = 1, 2, 3\}\), we denote by \(z(t, \omega, z_0) = (S(t, \omega, z_0), I(t, \omega, z_0), R(t, \omega, z_0))\) the solution to system (1.2) at time \(t\), starting in \(z_0\) (or \(z(t, z_0) = (S(t, z_0), I(t, z_0), R(t, z_0))\)) whenever there is no ambiguity). If \(x \in \mathbb{R}^3\) and \(\epsilon > 0\), the open ball \(B(x, \epsilon)\) with center at \(x\) and radius \(\epsilon\) is defined to be the set of all \(y \in \mathbb{R}^3\) such that \(|y - x| < \epsilon\). Denote \(\mathbb{N} = \{1, 2, \ldots\}\) and \(\mathbb{N}_0 = \{0, 1, 2, \ldots\}\). For any constant sequence \(\{c(e) : e \in \mathcal{M}\}\), define \(c^M = \max_{e \in \mathcal{M}} \{c(e)\}\).

Assume that the Markov chain \(r(t)\) is irreducible. Under this condition, the Markov chain has a unique stationary probability distribution \(\pi = (\pi_1, \ldots, \pi_E) \in \mathbb{R}^{1 \times E}\), which can be determined by solving the following linear equation \(\pi Q = 0\) subject to \(\sum_{e=1}^{E} \pi_e = 1\), and \(\pi_\epsilon > 0, \forall e \in \mathcal{M}\). Let

\[
0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots
\]

be jump times of the Markov chain \(r(t)\), and denote by

\[
\sigma_1 = \tau_1 - \tau_0, \; \sigma_2 = \tau_2 - \tau_1, \; \ldots, \; \sigma_n = \tau_n - \tau_{n-1}, \; \ldots
\]

holding time between adjacent two jumps. Furthermore, we set

\[
\mathcal{F}_0^n = \sigma(\tau_k, k \leq n) \quad \text{and} \quad \mathcal{F}_\infty^n = \sigma(\tau_k - \tau_n, k > n)
\]

for all \(n = 0, 1, \ldots\), then for each \(n\), \(\mathcal{F}_0^n\) is independent of \(\mathcal{F}_\infty^n\).

Throughout this paper, we further assume that

\[
(\textbf{H1}) \quad G(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \quad \text{is a twice-differentiable function,} \quad G(0) = 0 \quad \text{and} \quad 0 < G(I) \leq IG'(0)
\]

holds for all \(I > 0\);

\[
(\textbf{H2}) \quad g(x) \quad \text{is Lipschitz on} \quad [0, \Lambda/\mu], \quad \text{namely, there exists a constant} \quad \vartheta > 0, \quad \text{such that}
\]

\[
|g(x_1) - g(x_2)| \leq \vartheta |x_1 - x_2| \quad \text{for any} \quad x_1, x_2 \in [0, \Lambda/\mu],
\]

where \(g(x) = G(x)/x\).

Notice that the function \(G(I)\) in this paper is not necessarily increasing function with respect to \(I\), e.g., it may firstly increase and then decrease describing some psychological effects: for a very large number of infectives the infection force \(\beta G(I)\) may decrease as \(I\) increases, because in the presence of a very large number of infectives the population may tend to reduce the number of contacts per unit time \([27]\). Our general results can be used in some specific forms for the incidence rate that have been commonly used, for example:

(i) linear type: \(G(I) = I\);

(ii) saturated incidence rate: \(G(I) = I/(1 + aI)\) (e.g., \([27]\));

(iii) non-monotonic incidence rate: \(G(I) = I/(1 + aI^2)\) (e.g., \([28]\));

(iv) incidence rates with “media coverage” effect as shown below:

**Type 1** (see \([29]\)): \(G(I) = I \exp(-mI)\), where \(m\) is a positive constant.

**Type 2** (see \([30]\)): \(\beta G(I) = (\beta - \tilde{\beta} f(I))I\), where \(\beta > \tilde{\beta}\) and the twice-differentiable function \(f(I)\) satisfies \(f(0) = 0, f'(0) \geq 0, \lim_{I \to +\infty} f(I) = 1\), where \(f'(0)\) denotes the derivative of the function \(f(x)\) at \(x = 0\).

To study the dynamics of model (1.2), we need the following two lemmas, the proof of which are straightforward, so are omitted.
Lemma 1. Suppose that the system
\[
\frac{dx(t)}{dt} = F(x(t)), \quad t \geq 0
\]
with \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) has a globally asymptotically stable equilibrium \( x^* \in \mathbb{R}^3 \). Then, for any neighborhood \( U \) of \( x^* \) and any compact set \( C \subset \mathbb{R}^3 \), there exists a \( T > 0 \) such that \( x(t, x_0) \in U \) for all \( t \geq T \) and \( x_0 \in C \).

Lemma 2. For any given initial value \((z_0, r(0)) \in \mathbb{R}_+^3 \times \mathcal{M} \), there exists a unique solution \((S(t, z_0), I(t, z_0), R(t, z_0)) \) of system (1.2) on \( t \geq 0 \) and the solution will always remain in \( \mathbb{R}_+^3 \). Moreover, the solution has the property that for every \( \omega \in \tilde{\Omega} \), there exists \( t_0 = t_0(\omega) \) such that
\[
\frac{\Lambda}{\mu + \alpha} < S(t, \omega, z_0) + I(t, \omega, z_0) + R(t, \omega, z_0) < \frac{\Lambda}{\mu}, \quad \text{for all } t \geq t_0.
\]

Let
\[
X = K \times \mathcal{M},
\]
where \( K = \{ x \in \mathbb{R}_+^3 : \Lambda/(\mu + \alpha) < x_1 + x_2 + x_3 < \Lambda/\mu \} \). From Lemma 2, we get that the set \( X \) is an attraction domain for the system (1.2) in the sense that all sample paths of system (1.2) tend to this set and once they go into it, they will remain there forever. Hence, without loss of generality, we consider only the smaller state-space \( X \) throughout this paper.

2.2. Main results
2.2.1. Extinction and persistence of the disease

In studying epidemic modelings, the most interesting and important issues are usually to establish the threshold condition for the extinction and persistence of the disease, which will be given in this subsection. Let us introduce the basic reproduction number in a random environment as follows
\[
R_0 = \sum_{e \in \mathcal{M}} \pi_e (\Lambda \beta_e G'(0)/\mu) / (\mu + \alpha + \delta).
\]
Here, we refer the readers to [7, 13, 14] for the reason why we label the above formula as the basic reproduction number of our stochastic epidemic model (1.2). Next, we shall show that the position of \( R_0 \) with respect to 1 serves as a threshold between the disease extinction and persistence for our epidemic model (1.2) of SIRS type in a random environment.

Before we prove the results, let us state a proposition which gives an equivalent condition for the position of \( R_0 \) with respect to 1 in terms of the system parameters and the stationary distribution \( \pi \) of the Markov chain \( r(t) \), which drives the switch of environments.

Proposition 3. The following alternative conditions on the value of \( R_0 \) are valid:

(i) \( R_0 < 1 \) if and only if \( \sum_{e \in \mathcal{M}} \pi_e B(e) < 0 \);
(ii) \( R_0 > 1 \) if and only if \( \sum_{e \in \mathcal{M}} \pi_e B(e) > 0 \),

where
\[
B(e) = \frac{\Lambda \beta_e G'(0)}{\mu} - (\mu + \alpha + \delta)
\]
for each \( e \in \mathcal{M} \).
The proof of this proposition is straightforward, so is omitted. We then state our results on the extinction of the disease.

**Theorem 4.** Suppose that $R_0 < 1$, then the solution $(S(t), I(t), R(t))$ of system (1.2) with any initial value $(z_0, r(0)) \in X$ has the property that

\[
\begin{align*}
\lim_{t \to +\infty} S(t) &= \Lambda/\mu \quad a.s., \\
\lim_{t \to +\infty} I(t) &= 0 \quad a.s., \\
\lim_{t \to +\infty} R(t) &= 0 \quad a.s.
\end{align*}
\]  

**Proof.** From the second equation of system (1.2), it is easy to see that

\[
\frac{d \log I(t)}{dt} = \beta r(t) \frac{SG(I)}{I} - (\mu + \alpha + \delta).
\]

By Lemma 2 and the assumption (H1): $G(I) \leq IG'(0)$, we have

\[
\frac{d \log I(t)}{dt} \leq \frac{\Lambda \beta r(t) G'(0)}{\mu} - (\mu + \alpha + \delta).
\]

Integrating the above inequality, it is obtained from the Birkhoff Ergodic theorem that

\[
\limsup_{t \to +\infty} \frac{\log I(t)}{t} \leq \sum_{e \in M} \pi_e B(e) \quad a.s.,
\]

this, together with (i) of Proposition 3 implies

\[
\lim_{t \to +\infty} \frac{I(t)}{t} = 0 \quad a.s.
\]

This is the required assertion (2.5). The procedure to prove assertions (2.4) and (2.6) is similar to that given in Theorem 1 in [31], so is omitted. This completes the proof of Theorem 4. □

**Remark 1.** From the proof procedure of Theorem 4 it is obvious to see that when $R_0 < 1$, any positive solution of system (1.2) converges exponentially to the disease-free state $(\Lambda/\mu, 0, 0)$ with probability 1.

We now turn to the persistence of the disease.

**Theorem 5.** Suppose that $R_0 > 1$, then for any initial value $(z_0, r(0)) \in X$, the following statement is valid with probability 1:

\[
\liminf_{t \to +\infty} \frac{1}{t} \int_0^t I(s)ds \geq \frac{\mu^2}{\beta M(\mu \bar{d} + \beta M(G'(0))^2)\Lambda} \sum_{e \in M} \pi_e B(e).
\]

By (ii) of Proposition 3 we hence conclude that the disease is persistent in the time mean with probability 1.
The proof of this theorem is similar to the arguments of Theorem 2 in [31], so we omit it.

**Remark 2.** Indeed, from Theorem 5 it can be seen that the persistence in the time mean implies the following weak persistence. That is, if \( R_0 > 1 \), then for any initial value \((z_0, r(0)) \in X\),

\[
\limsup_{t \to +\infty} I(t) \geq \frac{\mu^2}{\beta M(\mu \vartheta + \beta M'(G'(0))^2)\Lambda} \sum_{e \in \mathcal{M}} \pi_e B(e) > 0 \quad \text{a.s.}
\]

Let us now establish a useful corollary, which indicates that \( R_0 \) is a threshold value determining the disease is extinct or persistent, i.e., the position of the deterministic quantity \( R_0 \) with respect to 1 determines the disease extinction or persistence for system (1.2). We can easily obtain from Theorems 4 and 5 the following corollary.

**Corollary 6.** For any initial value \((z_0, r(0)) \in X\), the solution \((S(t), I(t), R(t))\) of system (1.2) has the property that

(i) if \( R_0 < 1 \), the number of infected individuals \( I(t) \) of system (1.2) tends to zero exponentially almost surely, i.e., the disease dies out with probability one;

(ii) if \( R_0 > 1 \), the disease will be almost surely persistent in the time mean.

2.2.2. \( \Omega \)-limit set and attractor

For each state \( e \in \mathcal{M} \), we denote by \( \pi^e_t(z_0) \) the solution of system (1.2) in the state \( e \) with the initial value \( z_0 \in \mathcal{K} \). As in [11], the \( \Omega \)-limit set of the trajectory starting from an initial value \( z_0 \in \mathcal{K} \) is defined by

\[
\Omega(z_0, \omega) = \bigcap_{T>0} \bigcup_{t>T} (S(t, \omega, z_0), I(t, \omega, z_0), R(t, \omega, z_0)).
\]

We here use the notation “\( \Omega \)-limit set” in lieu of the usual one “\( \omega \)-limit set” in the deterministic dynamical system for avoiding notational conflict with the element notation \( \omega \) in the probability sample space. In this subsection, we shall show that under some appropriate conditions, \( \Omega(z_0, \omega) \) is deterministic, i.e., it is constant almost surely; moreover, it is independent of the initial value \( z_0 \).

Let us recall that the basic reproduction number \( R_0^e \) of the deterministic subsystem of (1.2) corresponding to environmental state \( e \) can be computed by using the next generation matrix approach [32] as

\[
R_0^e = \frac{\Lambda \beta e G'(0)}{\mu (\mu + \alpha + \delta)} \tag{2.7}
\]

In the remainder of this paper, we sometimes need to impose the following assumption:

\((H3)\) If \( R_0^e > 1 \), then there exists a unique and globally asymptotically stable positive equilibrium \( E^*_e = (S^*_e, I^*_e, R^*_e) \) for the corresponding deterministic subsystem of (1.2).

Note that the condition \( R_0 > 1 \) implies that there exists at least one state \( e \in \mathcal{M} \) such that \( B(e) > 0 \), i.e., \( R_0^e > 1 \) corresponding to the subsystem of (1.2) in the state \( e \). We may assume, without any loss of generality, that \( R_0^e > 1 \) in the remainder of this paper if the assumption \((H3)\) is needed. Indeed, Remark 3 indicates that the assumption is reasonable.
Hence, the subsystem in the first state \((e = 1)\) has a globally stable positive equilibrium \(E_1^* = (S_1^*, I_1^*, R_1^*)\).

Now we recall some concepts on the Lie algebra of vector fields \([26, 33]\). Let \(a(x)\) and \(b(x)\) be two vector fields on \(\mathbb{R}^d\). The Lie bracket \([a, b]\) is also a vector field given by

\[
[a, b]_j(x) = \sum_{k=1}^d \left( a_k \frac{\partial b_j}{\partial x_k}(x) - b_k \frac{\partial a_j}{\partial x_k}(x) \right), \quad j = 1, 2, \ldots, d.
\]

In the remainder of this paper, we need the following definition: A point \(z = (S, I, R) \in \mathbb{R}_+^3\) is said to satisfy the condition \((H)\), if vectors \(Y_1(z), \ldots, Y_E(z), [Y_i, Y_j](z)_{i,j\in \mathcal{M}}, [Y_i, [Y_j, Y_k]](z)_{i,j,k\in \mathcal{M}}, \ldots\), span the space \(\mathbb{R}^3\), where for each \(e \in \mathcal{M}\),

\[
Y_e(S, I, R) = \begin{pmatrix}
\Lambda - \mu S + \lambda R - \beta_e SG(I) \\
\beta_e SG(I) - (\mu + \alpha + \delta) I \\
\delta I - (\mu + \lambda) R
\end{pmatrix}.
\]

With this definition, we have

**Theorem 7.** Suppose that \(R_0 > 1\) and the hypotheses \((H3)\) hold. Let

\[
\Gamma = \left\{(S, I, R) = \pi_{p_k}^{p_1} \circ \cdots \circ \pi_{t_1}^{p_1}(E_1^*) : t_1, \ldots, t_k \geq 0 \text{ and } p_1, \ldots, p_k \in \mathcal{M}, k \in \mathbb{N} \right\}.
\]

Then, the following statements are valid:

(a) With \(\overline{\Gamma}\) denoting the closure of \(\Gamma\), \(\overline{\Gamma}\) is a subset of the \(\Omega\)-limit set \(\Omega(z_0, \omega)\) with probability 1.

(b) If there exists a point \(z_* := (S_*, I_*, R_*) \in \Gamma\) satisfying the condition \((H)\), then \(\Gamma\) absorbs all positive solutions in the sense that for any initial value \(z_0 \in \mathcal{K}\), the value

\[
\tilde{T}(\omega) = \inf \left\{ t > 0 : (S(s, \omega, z_0), I(s, \omega, z_0), R(s, \omega, z_0)) \in \Gamma, \forall s > t \right\}
\]

is finite outside a \(\mathbb{P}\)-null set. Consequently, \(\overline{\Gamma}\) is the \(\Omega\)-limit set \(\Omega(z_0, \omega)\) for any \(z_0 \in \mathcal{K}\) with probability 1.

**Remark 3.** If there exists the other state \(e \in \mathcal{M}\) such that \(R_0^e > 1\), we can define

\[
\Gamma_e = \left\{(S, I, R) = \pi_{p_k}^{p_1} \circ \cdots \circ \pi_{t_1}^{p_1}(E_e^*) : t_1, \ldots, t_k \geq 0 \text{ and } p_1, \ldots, p_k \in \mathcal{M}, k \in \mathbb{N} \right\}.
\]

Then, we have the same conclusion that the closure \(\overline{\Gamma}_e\) of \(\Gamma_e\) is the \(\Omega\)-limit set \(\Omega(z_0, \omega)\) with probability 1, which implies that \(\overline{\Gamma}_e = \overline{\Gamma}\).

2.2.3. Global convergence of the distribution in total variation norm

First, we establish the existence of an invariant probability measure.

**Theorem 8.** If \(R_0 > 1\), the Markov process \(((S(t), I(t), R(t)), r(t))\) has an invariant probability measure \(\nu^*\) on the state space \(\mathcal{X}\).
Remark 4. To prove this theorem, we utilize the alternative principle in [24, 36]. In the proceeding of the proof, it is critical to find a compact subset $O$ of $X$ such that the probability that the process $((S(t), I(t), R(t)), r(t))$ falls into $O$ in the time mean is positive, where $X$ is a slightly larger state space compared with the state space $X$. In Section 3 we shall give two alternative methods for constructing such compact subset: one is similar to that given in Theorem 3.1 of [12], while the other one is based on the results of Theorem 5.

We now characterize the invariant probability measure by the following theorem.

Theorem 9. Suppose that $R_0 > 1$ and the hypotheses (H3) hold. Assume further that the hypotheses in (b) of Theorem 7 is satisfied. Then for any initial value $x_0 = (z_0, r(0)) \in X$, the instant distribution of the process $((S(t), I(t), R(t)), r(t))$ satisfies

$$ \lim_{t \to +\infty} \|P(t, x_0, (\cdot, \cdot)) - \nu^*(\cdot, \cdot)\| = 0, \quad (2.8) $$

and

$$ P_{x_0} \left\{ \lim_{t \to +\infty} \frac{1}{t} \int_0^t f(z(s, z_0), r(s)) ds = \sum_{e \in \mathcal{M}} \int_K f(u, e) \nu^*(du, e) \right\} = 1, \quad (2.9) $$

where $f(\cdot, \cdot)$ is any $\nu^*$-integrable function. Moreover, the stationary distribution $\nu^*$ has the density $f^*$ with respect to the product measure $m$ on $X$ and $\text{supp}(f^*) = \Gamma \times \mathcal{M}$.

Remark 5. In [12], it is easy to see that if the condition (4.4) is valid, then the condition (H) holds. In (b) of Theorem 7 and Theorem 9 we therefore have weakened the conditions ensuring the global attractivity of the $\Omega$-limit set of the system and the convergence in total variation of the instantaneous measure to the stationary measure. Moreover, the method in [11, 12] is not applicable to the case where the dimension of the system considered is higher than the number of environmental regimes, while the modified method in proving Theorems 7 and 9 has no limitations for both the number of environmental regimes and the dimension of the system considered. This, together with the following example 1 illustrates that our modified method in this paper has a wider application than the techniques in [11, 12].

Example 1. Let $G(I) = I/(1 + aI^2)$, where $a$ is a positive parameter measuring the psychological or inhibitory effect. The stochastic model (1.2) under regime switching reduces to

$$ \begin{cases} 
  dS(t)/dt = \Lambda - \mu S(t) + \lambda R(t) - \beta r(t) S(t) \frac{I(t)}{1 + aI^2(t)}, \\
  dI(t)/dt = \beta r(t) S(t) \frac{I(t)}{1 + aI^2(t)} - (\mu + \alpha + \delta) I(t), \\
  dR(t)/dt = \delta I(t) - (\mu + \lambda) R(t),
\end{cases} \quad (2.10) $$

which switches from one to the other according to the movement of the right-continuous Markov chain $\{r(t), t \geq 0\}$ taking values in the state space $\mathcal{M} = \{1, 2\}$ with the generator

$$ Q = \frac{1 - 0.5}{365} \cdot \begin{pmatrix} -169 & 169 \\
 196 & -196 \end{pmatrix}. $$

This Markov chain has a unique stationary distribution

$$ \pi = (\pi_1, \pi_2) = \left( \frac{196}{365}, \frac{169}{365} \right). \quad (2.11) $$
For each environmental state \( e \in \mathcal{M} \), Cai and Kang et al. [34] discussed the corresponding deterministic epidemic model:

\[
\begin{align*}
\frac{dS(t)}{dt} &= \Lambda - \mu S(t) + \lambda R(t) - \beta e S(t) \frac{I(t)}{1 + \alpha R(t)}, \\
\frac{dI(t)}{dt} &= \beta e S(t) \frac{I(t)}{1 + \alpha R(t)} - (\mu + \alpha + \delta) I(t), \\
\frac{dR(t)}{dt} &= \delta I(t) - (\mu + \lambda) R(t).
\end{align*}
\]  
(2.12)

The basic reproduction number of the deterministic system (2.12) is

\[
\mathcal{R}_0^e = \frac{\Lambda \beta e G'(0)}{\mu(\mu + \alpha + \delta)} = \frac{\Lambda \beta e}{\mu(\mu + \alpha + \delta)},
\]
which determines the extinction and persistence of the disease. According to Theorem 5.1 in [34], then

- The unique disease-free equilibrium \( E_0 = (\Lambda/\mu, 0, 0) \) is globally asymptotically stable whenever \( \mathcal{R}_0^e \leq 1 \), and it is unstable when \( \mathcal{R}_0^e > 1 \).
- When \( \mathcal{R}_0^e > 1 \), there exists a unique equilibrium \( E^*_e = (S^*_e, I^*_e, R^*_e) \) with \( S^*_e > 0 \), \( I^*_e > 0 \), \( R^*_e > 0 \), which is globally asymptotically stable.

Without any loss of generality, let us assume that \( \mathcal{R}_0^1 > 1 \) and \( \mathcal{R}_0^2 < 1 \) in this example.

The other parameter values used here are mainly taken from the work in [17] investigating the dynamics of Pasteurella muris in colonies of laboratory mice: \( \Lambda = 0.33 \text{ days}^{-1} \), \( \mu = 0.006 \text{ days}^{-1} \), \( \alpha = 0.06 \text{ days}^{-1} \), \( \delta = 0.04 \text{ days}^{-1} \), \( \lambda = 0.021 \text{ days}^{-1} \), \( a = 0.001 \text{ days}^{-1} \), \( \beta_1 = 0.0056 \text{ days}^{-1} \) corresponding to \( \mathcal{R}_0^1 = 2.9057 > 1 \) (the disease is persistent, see (a) in Figure 1), and \( \beta_2 = 0.0013 \text{ days}^{-1} \) corresponding to \( \mathcal{R}_0^2 = 0.6745 < 1 \) (the disease is extinct, see (b) in Figure 1). Combining with (2.11), this implies that the basic reproduction number for the stochastic system (2.10) is

\[
\mathcal{R}_0 = \frac{\pi_1 \Lambda \beta_1}{\mu(\mu + \alpha + \delta)} + \frac{\pi_2 \Lambda \beta_2}{\mu(\mu + \alpha + \delta)} = 1.8726 > 1.
\]

Hence, by Corollary 3, the disease will be almost surely persistent in the time mean (see (c) in Figure 1). For the system (2.10), we define the following set as in Theorem 7

\[
\Gamma = \left\{ (S, I, R) = \pi_{t_k}^{p_k} \cdots \pi_{t_1}^{p_1}(E^*_1) : t_1, \ldots, t_k \geq 0 \text{ and } p_1, \ldots, p_k \in \{1, 2\}, k \in \mathbb{N} \right\},
\]

where \( E^*_1 = (19.0161, 2.8783, 4.2830) \).

For any point \( z = (S, I, R) \in \Gamma \), let

\[
Y_e(z) = \begin{pmatrix}
\Lambda - \mu S + \lambda R - \beta e S G(I) \\
\beta e S G(I) - (\mu + \alpha + \delta) I \\
\delta I - (\mu + \lambda) R
\end{pmatrix}, \quad e = 1, 2.
\]

Because there are only two environmental states for the system (2.10) with three equations, the matrix \( (Y_1(z), Y_2(z)) \) has 3 rows and 2 columns, which indicates that the determinant

\[
\det \left( Y_1(z), Y_2(z) \right)
\]
Hence, the method used in [12] can not be used to judge whether or not the claims that the \( \Omega \)-limit set \( \Gamma \) of the system (2.10) is globally attractive and its instantaneous measure converges in total variation to some stationary measure are valid, while our method in this paper can be feasible.

Computing the Lie bracket of \( Y_1(z) \) and \( Y_2(z) \), we obtain the vector field given by

\[
[Y_1, Y_2](z) = (\beta_1 - \beta_2) \left( \begin{array}{c} (\Lambda + \lambda R)G(I) - (\mu + \alpha + \delta)SIG'(I) \\ - (\Lambda + \lambda R)G(I) - (\alpha + \delta)SG(I) + (\mu + \alpha + \delta)SIG'(I) \end{array} \right).
\]

Hence,

\[
\det(Y_1(z), Y_2(z), [Y_1, Y_2](z)) = -\left[ (\beta_1 - \beta_2)SG(I) \right]^2 \cdot \left[ \mu \delta \left( \frac{\Lambda}{\mu} - (S + I + R) \right) - \alpha (\mu + \lambda)R \right]
\]

\[
= -\left[ \frac{(\beta_1 - \beta_2)SI}{1 + aI^2} \right]^2 \cdot \left[ \mu \delta \left( \frac{\Lambda}{\mu} - (S + I + R) \right) - \alpha (\mu + \lambda)R \right].
\]

from which we can easily find by the numerical method that there exist many points \( z \in \Gamma \) such that \( \det(Y_1(z), Y_2(z), [Y_1, Y_2](z)) \neq 0 \). For instance, \( \det(Y_1(z), Y_2(z), [Y_1, Y_2](z)) \neq 0 \) when \( z = \pi_{100}^{2}(E_{1}^{*}) = (37.3966, 0.0033, 0.4464) \), which implies that at the point \( z = \pi_{100}^{2}(E_{1}^{*}) \), the vectors \( Y_1(z), Y_2(z), [Y_1, Y_2](z) \) span the space \( \mathbb{R}^3 \), i.e., the point \( z = \pi_{100}^{2}(E_{1}^{*}) \) satisfies the condition (H). Therefore, by (b) of Theorem [7] and Theorem [8] we can conclude that the \( \Omega \)-limit set \( \Gamma \) of the system (2.10) is globally attractive and its instantaneous measure converges to some stationary measure in total variation (see, Figures [2] and [3]).

### 3. Proofs of main results

In this section, we first provide some auxiliary definitions and results concerning stability of Markovian processes [22, 23, 24, 36] that we will use later to prove our main results.

Let \( \Phi = \{ \Phi_t : t \geq 0 \} \) be a time-homogeneous Markov process with state space \( (X, \mathcal{B}(X)) \), where the state space \( X \) is a locally compact and separable metric space, and \( \mathcal{B}(X) \) is the Borel \( \sigma \)-algebra on \( X \). The process \( \Phi \) evolves on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}_x) \), where \( x \in X \) is the initial condition of the process, and the measure \( \mathbb{P}_x \) satisfies \( \mathbb{P}_x(\Phi_t \in B) = \mathbb{P}(t, x, B) \) for all \( x \in X, t \geq 0 \) and \( B \in \mathcal{B}(X) \). Assume further that \( \Phi_t \) is a Feller process then it is a Borel right process. Let \( \{\mathbb{P}^t\}_{t \geq 0} \) be the transition semigroup of the process \( \Phi_t \), and the operator \( \mathbb{P}^t \) acts on \( \sigma \)-finite measure \( \nu \) on \( X \) via

\[
\nu \mathbb{P}^t(B) = \int_X \mathbb{P}(t, x, B)\nu(dx), \quad B \in \mathcal{B}(X).
\]

It is easy to see that if \( \nu \) is the initial distribution of the process \( \Phi \) at time 0, then \( \nu \mathbb{P}^t \) is the distribution of \( \Phi \) at time \( t \). A \( \sigma \)-finite measure \( \nu \) on \( \mathcal{B}(X) \), with the property

\[
\nu(B) = \nu \mathbb{P}^t(B) = \int_X \mathbb{P}(t, x, B)\nu(dx)
\]
Figure 1: The paths of $S(t)$, $I(t)$ and $R(t)$ for the systems (2.12) and (2.11) with the same initial values $z_0 = (50, 1, 0)$. (a) the paths of the deterministic system (2.12) in the state 1 with $\beta_1 = 0.0056 \text{ days}^{-1}$ corresponding to $R_{10} = 2.9057 > 1$; (b) the paths of the deterministic system (2.12) in the state 2 with $\beta_2 = 0.0013 \text{ days}^{-1}$ corresponding to $R_{20} = 0.6745 < 1$; (c) the paths of the stochastic system (2.11) switching between the state 1 and the state 2 with $R_0 = 1.8726 > 1$ and $r(0) = 1$. (Color figure online)
Figure 2: A sample orbit of the system (2.10) with the initial values $z_0 = (50, 1, 0)$ and $r(0) = 1$. The red lines and blue lines represent the paths of the system (2.10) in states 1 and 2, respectively. The ◦ denotes the starting point of the orbit, the □ denotes the end of the orbit, and the number of environmental switching is 2000. (Color figure online)

Figure 3: Two-dimension projections of the sample orbit in Figure 2. The orbit of Figure 2 is projected onto the S-I, S-R and I-R coordinate planes from the top to the bottom, respectively. (Color figure online)
for any \( B \in \mathcal{B}(X) \) and \( t \geq 0 \), will be called invariant (or stationary) measure.

For a measurable set \( B \in \mathcal{B}(X) \), let

\[
\tau_B = \inf\{t \geq 0 : \Phi_t \in B\}, \quad \eta_B = \int_0^\infty 1_{\{\Phi_t \in B\}} dt.
\]

We then introduce the standard definition of Harris recurrence as follows. The process \( \Phi \) is called Harris recurrent if either

(a) some nontrivial \( \sigma \)-finite measure \( \phi_1 \) exists such that \( \mathbb{P}_x \{ \tau_B < \infty \} = 1 \) whenever \( \phi_1(B) > 0 \); or

(b) some nontrivial \( \sigma \)-finite measure \( \phi_2 \) exists such that \( \mathbb{P}_x \{ \eta_B = \infty \} = 1 \) whenever \( \phi_2(B) > 0 \),

is valid for any \( x \in X \) and \( B \in \mathcal{B}(X) \). In addition, the process \( \Phi \) is said to be positive Harris recurrent if it is Harris recurrent with a finite invariant measure.

Suppose that \( \alpha \) is a probability measure on \([0, +\infty)\), and define the transition function \( K_\alpha \) of a general sampled Markov chain corresponding to the measure \( \alpha \) as

\[
K_\alpha(x, B) = \int_0^\infty \mathbb{P}(t, x, B) \alpha(dt)
\]

for all \( x \in X \) and \( B \in \mathcal{B}(X) \). We refer the readers to [23] for the detail definition of the sampled Markov chain of the process \( \Phi \). A kernel \( T : X \times \mathcal{B}(X) \to [0, +\infty) \) is called a continuous component of the Markov transition function \( K_\alpha \) if

(i) For each \( B \in \mathcal{B}(X) \) the function \( T(\cdot, B) \) is lower semi-continuous;

(ii) For all \( x \in X \) and \( B \in \mathcal{B}(X) \), the measure \( T(x, \cdot) \) satisfies \( K_\alpha(x, B) \geq T(x, B) \).

The continuous component \( T \) is called everywhere non-trivial if \( T(x, X) > 0 \) for each \( x \in X \). The process \( \Phi \) will be called a \( T \)-process if there exists a probability measure \( \alpha \), such that the corresponding transition function \( K_\alpha \) has an everywhere non-trivial continuous component \( T \).

The process \( \Phi \) is called bounded in probability on average if for each initial condition \( x \in X \) and any \( \varepsilon > 0 \), there exists a compact subset \( C \subset X \) such that

\[
\lim \inf_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{P}_x \{ \Phi_s \in C \} ds \geq 1 - \varepsilon.
\]

For the \( \sigma \)-finite measure \( \phi \), the process \( \Phi \) is called \( \phi \)-irreducible if for any \( x \in X \) and \( B \in \mathcal{B}(X) \),

\[
\mathbb{E}_x[\eta_B] > 0
\]

whenever \( \phi(B) > 0 \), where \( \mathbb{E}_x(\cdot) \) denotes the expectation of the random variable with respect to the probability measure \( \mathbb{P}_x \). In this case, the measure \( \phi \) is called an irreducibility measure. The process \( \Phi \) is usually be called irreducible when the specific irreducibility measure is irrelevant.

For checking the asymptotic stability of the distribution, we need some results concerning the stochastic stability and ergodicity of Markovian processes as follows.

**Theorem 10.** (see, [24, 36]) Assume that \( \Phi \) is a Feller process. Then either

(a) there exists an invariant probability measure on \( X \), or

(b) for any compact set \( C \subset X \),

\[
\lim \sup_{t \to +\infty} \frac{1}{t} \int_0^t \left( \int_X \mathbb{P}(s, x, C) \nu(dx) \right) ds = 0,
\]

where the supremum is taken over all initial distributions \( \nu \) on the state space \( X \).
Theorem 11. (see, Theorem 3.2 in [23]) Suppose that the process $\Phi$ is an irreducible $\mathcal{T}$-process. Then $\Phi$ is positive Harris recurrent if and only if $\Phi$ is bounded in probability on average.

Theorem 12. (see, Theorem 8.1 in [23]) Suppose that the process $\Phi$ is irreducible, and is bounded in probability on average. Assume further that there exists a lattice distribution $a$, such that the corresponding kernel $K_a$ possesses an everywhere non-trivial continuous component. Then the following assertions are valid.

(i) For every $x \in X$, 
$$
\lim_{t \to +\infty} \| P(t, x, \cdot) - \Pi(\cdot) \| = 0,
$$
where $\| \cdot \|$ denotes the total variation norm, and $\Pi$ denotes the invariant probability measure on the state space $X$.

(ii) Moreover, for any $\Pi$-integrable function $f$ and any $x \in X$,
$$
P_x \left\{ \lim_{t \to +\infty} \frac{1}{t} \int_0^t f(\Phi_s) ds = \int_X f(u) \Pi(du) \right\} = 1.
$$

Proposition 13. Suppose that the process $\Phi$ is positive Harris recurrent. If a lattice distribution $a$ exists such that the corresponding kernel $K_a$ admits an everywhere non-trivial continuous component, i.e., $\Phi$ is a $\mathcal{T}$-process, then we have

(i) For every $x \in X$,
$$
\lim_{t \to +\infty} \| P(t, x, \cdot) - \Pi(\cdot) \| = 0,
$$
where $\Pi$ denotes the invariant probability measure on $X$.

(ii) Moreover, for any $\Pi$-integrable function $f$ and any $x \in X$,
$$
P_x \left\{ \lim_{t \to +\infty} \frac{1}{t} \int_0^t f(\Phi_s) ds = \int_X f(u) \Pi(du) \right\} = 1.
$$

3.1. Proof of Theorem 7

To prove Theorem 7, we need the following three lemmas.

Lemma 14. Suppose that $R_0 > 1$. For any initial condition $(z_0, r(0)) \in X$, there exists a $\iota > 0$ such that $\limsup_{t \to +\infty} I(t, z_0) \geq \iota$ a.s.

Proof. By Remark 2, the result of Lemma 14 is immediate. However, we then give another proving method. Let 
$$
\varepsilon = \frac{\mu \sum_{e \in M} \pi_e B(e)}{4 \Lambda \beta^M}.
$$

By the assumptions (H1) and (H2), we get that \( \lim_{x \to 0^+} G(x)/x = G'(0) \), which yields that there exists a positive constant \( \iota \) satisfying

\[
\iota < \frac{\mu^2 \sum_{e \in \mathcal{M}} \pi_e B(e)}{8 \Lambda (\beta^M G'(0))^2},
\]

such that \( G'(0) - G(x)/x < \varepsilon \) if \( 0 < x < \iota \). Next, we shall show that \( \limsup_{t \to +\infty} I(t, z_0) > \iota \) a.s. In the contrary, assume that a measurable set \( B \) with \( \mathbb{P}(B) > 0 \) exists such that \( \limsup_{t \to +\infty} I(t, \omega, z_0) < \iota \) for any \( \omega \in B \). Then, for each \( \omega \in B \), there exists a \( T = T(\omega, \iota) > 0 \) such that \( I(t, \omega, z_0) < \iota \) for all \( t > T \). Note that \( G(I(t, \omega, z_0)) \leq I(t, \omega, z_0)G'(0) \) and \( S(t, \omega, z_0) < \Lambda/\mu \) for all \( t > T \) and \( \omega \in B \), which implies that

\[
\beta_r(t, \omega)S(t, \omega, z_0)G(I(t, \omega, z_0)) \leq \frac{\Lambda \beta^M G'(0) t}{\mu}
\]

for all \( t > T \) and \( \omega \in B \). Hence, it is obtained from the first equation of system (1.2) that for each \( \omega \in B \),

\[
\frac{dS(t, \omega, z_0)}{dt} = \Lambda - \mu S(t, \omega, z_0) + \lambda R(t, \omega, z_0) - \beta_r(t, \omega)S(t, \omega, z_0)G(I(t, \omega, z_0)) \geq \Lambda - \mu S(t, \omega, z_0) - \frac{\Lambda \beta^M G'(0) t}{\mu}
\]

holds for all \( t > T \). By the comparison theorem, one can find a \( T_1 = T_1(\omega, \iota) > T \) satisfying

\[
S(t, \omega, z_0) \geq \frac{\Lambda}{\mu} - \frac{2 \Lambda \beta^M G'(0) t}{\mu^2}
\]

for all \( t > T_1 \) and \( \omega \in B \). Noting that \( G'(0) - (G(I(t, \omega, z_0))/I(t, \omega, z_0)) < \varepsilon \) for all \( t > T_1 \) and \( \omega \in B \), it follows from (3.13) that

\[
\beta_r(t, \omega) \left( \frac{\Lambda G'(0)}{\mu} - \frac{S(t, \omega, z_0)G(I(t, \omega, z_0))}{I(t, \omega, z_0)} \right) = \beta_r(t, \omega) \left( \frac{\Lambda G'(0)}{\mu} - S(t, \omega, z_0)G'(0) \right) + \beta_r(t, \omega) \left( S(t, \omega, z_0)G'(0) - \frac{S(t, \omega, z_0)G(I(t, \omega, z_0))}{I(t, \omega, z_0)} \right) \leq \beta^M G'(0) \left( \frac{\Lambda}{\mu} - S(t, \omega, z_0) \right) + \frac{\Lambda \beta^M}{\mu} \left( G'(0) - \frac{G(I(t, \omega, z_0))}{I(t, \omega, z_0)} \right) \leq \sum_{e \in \mathcal{M}} \pi_e B(e) \frac{\pi_e B(e)}{4} + \sum_{e \in \mathcal{M}} \pi_e B(e) \frac{\pi_e B(e)}{4} = \sum_{e \in \mathcal{M}} \pi_e B(e) \frac{\pi_e B(e)}{2}
\]
for all \( t > T_1 \) and \( \omega \in B \). From the second equation of system \((1.2)\), we then have
\[
d \ln I(t, \omega, z_0) = \left[ \beta_r(t, \omega) \frac{S(t, \omega, z_0)G(I(t, \omega, z_0))}{I(t, \omega, z_0)} - (\mu + \alpha + \delta) \right] dt
\]
\[
= \left( \frac{\Lambda \beta_r(t, \omega)G'(0)}{\mu} - (\mu + \alpha + \delta) \right) dt
\]
\[- \beta_r(t, \omega) \left( \frac{\Lambda G'(0)}{\mu} - \frac{S(t, \omega, z_0)G(I(t, \omega, z_0))}{I(t, \omega, z_0)} \right) dt
\]
\[
\geq \left( \frac{\Lambda \beta_r(t, \omega)G'(0)}{\mu} - (\mu + \alpha + \delta) \right) dt - \frac{\sum_{e \in M} \pi_e B(e)}{2} dt
\]
for all \( t > T_1 \) and \( \omega \in B \). Integrating both sides of the above inequality from \( T_1 \) to \( t \) \((t > T_1)\) yields
\[
\frac{\ln I(t, \omega, z(T_1, \omega, z_0)) - \ln I(T_1, \omega, z_0)}{t} \geq \frac{1}{t} \int_{T_1}^{t} \left( \frac{\Lambda \beta_r(s, \omega)G'(0)}{\mu} - (\mu + \alpha + \delta) \right) ds - \frac{\sum_{e \in M} \pi_e B(e)}{2}
\]
\[(3.14)\]
for all \( t > T_1 \) and \( \omega \in B \), where \( z(T_1, \omega, z_0) = (S(T_1, \omega, z_0), I(T_1, \omega, z_0), R(T_1, \omega, z_0)) \). Since \( I(t, \omega, z(T_1, \omega, z_0)) \) is bounded,
\[
\limsup_{t \to +\infty} \frac{\ln I(t, \omega, z(T_1, \omega, z_0)) - \ln I(T_1, \omega, z_0)}{t} \leq 0.
\]
However, by the Birkhoff Ergodic theorem, it is obtained from the right hand of the inequality \((3.14)\) that
\[
\lim_{t \to +\infty} \left[ \frac{1}{t} \int_{T_1}^{t} \left( \frac{\Lambda \beta_r(s, \omega)G'(0)}{\mu} - (\mu + \alpha + \delta) \right) ds - \frac{\sum_{e \in M} \pi_e B(e)}{2} \right] = \frac{\sum_{e \in M} \pi_e B(e)}{2} > 0
\]
for almost all \( \omega \in B \). This is a contradiction. Thus, \( \limsup_{t \to +\infty} I(t, z_0) > \iota \) a.s. This completes the proof of Lemma \((13)\) \(\square\)

Lemma 15. If \( \mathcal{R}_0 > 1 \), then there exists a \( \iota > 0 \) such that for each \( i \in \mathcal{M} \), the event
\[
E_i = \{ \text{for infinite many } k \in \mathbb{N}_0, (S(\tau_k), I(\tau_k), R(\tau_k)) \in H_\iota \text{ occur with } r(\tau_k) = i \}
\]
occurs with probability one, where
\[
H_\iota = \{ (S, I, R) : \frac{\Lambda}{\mu + \alpha} \leq S + I + R \leq \frac{\Lambda}{\mu}, I \geq \iota \}.
\]
PROOF. Fix a $T > 0$, by Lemma 13 one can define almost surely finite stopping times with respect to filtration $\mathcal{F}_t$:

$$
\eta_1 = \inf \left\{ t > 0 : I(t) \geq 1 \right\}, \\
\eta_2 = \inf \left\{ t > \eta_1 + T : I(t) \geq 1 \right\}, \\
\ldots \\
\eta_n = \inf \left\{ t > \eta_{n-1} + T : I(t) \geq 1 \right\}, \ldots .
$$

Let $\tau(\eta_k) = \inf \left\{ t > \eta_k : r(t) \neq r(\eta_k) \right\}$, $\tilde{\sigma}(\eta_k) = \tau(\eta_k) - \eta_k$ and $A_k = \{ \tilde{\sigma}(\eta_k) < T \}$, $k \in \mathbb{N}$. Then $A_{k+1}$ is in the $\sigma$-algebra generated by $\{ r(\eta_k + s) : s \geq 0 \}$ while $A_k \in \mathcal{F}_{\eta_k+1}$. From the strong Markov property of the process $((S(t), I(t), R(t)), r(t))$, it is obtained that for any $A_k^c$, $k \in \mathbb{N}$,

$$
P(A_k^c) = \sum_{e \in \mathcal{M}} P(A_k^c | r(\eta_k) = e) P(r(\eta_k) = e)
$$

and

$$
P(A_k^c \cap A_{k+1}^c) = \sum_{e \in \mathcal{M}} P(A_k^c \cap A_{k+1}^c | r(\eta_{k+1}) = e) P(r(\eta_{k+1}) = e)
$$

where $\mathcal{M} := \max_{e \in \mathcal{M}} \{ P(\tilde{\sigma}(0) \geq T | r(0) = e) \} < 1$. This implies

$$
P\left( \bigcap_{i=1}^{\infty} A_k \right) = 1 - P\left( \bigcup_{i=1}^{\infty} A_k \right) = 1,
$$

that is, the events $A_k$, $k \in \mathbb{N}$ occur infinitely often a.s. Since

$$
\frac{dI(t)}{dt} = \beta_r(t)S(t)G(I(t)) - (\mu + \alpha + \delta)I(t) \geq -(\mu + \alpha + \delta)I(t)
$$

(3.15)
with $I(\eta_k) \geq \nu$, it follows from the comparison theorem that $I(\eta_k + t) \geq \nu e^{-(\mu+\alpha+\delta)t}$ for all $t \geq 0$. Thus, with probability one the events $\{(S(\eta_k + S(\eta_k)), I(\eta_k + S(\eta_k)), R(\eta_k + S(\eta_k))) \in H_\beta\}$, $k \in \mathbb{N}$ occur infinitely often with $\beta := \nu e^{-(\mu+\alpha+\delta)T}$, which yields that the events $\{(S(\tau_k), I(\tau_k), R(\tau_k)) \in H_\beta\}$, $k \in \mathbb{N}_0$ occur infinitely often a.s.

Let us fix $i \in \mathcal{M}$. Introduce the following sequence of stopping times with respect to filtration $\mathcal{F}_0^n$:

$$
\begin{align*}
\xi_1 &= \inf\left\{k \in \mathbb{N}_0 : (S(\tau_k), I(\tau_k), R(\tau_k)) \in H_\beta\right\}, \\
\xi_2 &= \inf\left\{k \in \mathbb{N}_0 : \tau_k > \xi_1 + T, (S(\tau_k), I(\tau_k), R(\tau_k)) \in H_\beta\right\}, \\
&\quad \ldots \\
\xi_n &= \inf\left\{k \in \mathbb{N}_0 : \tau_k > \xi_{n-1} + T, (S(\tau_k), I(\tau_k), R(\tau_k)) \in H_\beta\right\}, \ldots,
\end{align*}
$$

Define the events

$$
\tilde{A}_k = \left\{\right. \text{for some } s \in (0, T], r((\tau_{\xi_k} + s) -) \neq r(\tau_{\xi_k} + s) \text{ and } r(\tau_{\xi_k} + s) = i \left. \right\},
$$

$k \in \mathbb{N}$. Next, to obtain the assertion that $\mathbb{P}\{E_i\} = 1$, by (3.15) we need only to prove that with probability one the events $\tilde{A}_k$, $k \in \mathbb{N}$ occur infinitely often provided we set $\rho := \tilde{\beta}e^{-(\mu+\alpha+\delta)T}$. From the strong Markov property of the process $((S(t), I(t), R(t)), r(t))$, it then follows that for any $\tilde{A}_k^c$, $k \in \mathbb{N}$,

$$
\begin{align*}
\mathbb{P}\left(\tilde{A}_k^c\right) &= \sum_{n=0}^{\infty} \mathbb{P}\left(\tilde{A}_k^c \mid \xi_k = n\right) \mathbb{P}\left(\xi_k = n\right) \\
&= \sum_{n=0}^{\infty} \mathbb{P}\left(\xi_k = n\right) \cdot \sum_{e \in \mathcal{M}} \mathbb{P}\left(\tilde{A}_k^c \mid \xi_k = n, r(\tau_{\xi_k}) = e\right) \mathbb{P}\left(r(\tau_{\xi_k}) = e \mid \xi_k = n\right) \\
&= \sum_{n=0}^{\infty} \mathbb{P}\left(\xi_k = n\right) \cdot \sum_{e \in \mathcal{M}} \mathbb{P}\left(\tilde{A}_k^c \mid \xi_k = n, r(\tau_n) = e\right) \mathbb{P}\left(r(\tau_{\xi_k}) = e \mid \xi_k = n\right) \\
&= \sum_{n=0}^{\infty} \mathbb{P}\left(\xi_k = n\right) \cdot \sum_{e \in \mathcal{M}} \mathbb{P}\left(E \mid r(0) = e\right) \mathbb{P}\left(r(\tau_{\xi_k}) = e \mid \xi_k = n\right) \\
&\leq \tilde{\rho}_1 \sum_{n=0}^{\infty} \mathbb{P}\left(\xi_k = n\right) \cdot \sum_{e \in \mathcal{M}} \mathbb{P}\left(r(\tau_{\xi_k}) = e \mid \xi_k = n\right) \\
&= \tilde{\rho}_1
\end{align*}
$$

and

$$
\begin{align*}
\mathbb{P}\left(\tilde{A}_k^c \cap \tilde{A}_{k+1}^c\right) &= \sum_{n=0}^{\infty} \mathbb{P}\left(\tilde{A}_k^c \cap \tilde{A}_{k+1}^c \mid \xi_{k+1} = n\right) \mathbb{P}\left(\xi_{k+1} = n\right)
\end{align*}
$$

19
\[= \sum_{n=0}^{\infty} \mathbb{P}(\xi_{k+1} = n) \cdot \sum_{e \in M} \mathbb{P}(\tilde{A}_k \cap \tilde{A}_k^c \mid \xi_{k+1} = n, r(\tau_{\xi_{k+1}}) = e) \mathbb{P}(r(\tau_{\xi_{k+1}}) = e \mid \xi_{k+1} = n)\]

\[= \sum_{n=0}^{\infty} \mathbb{P}(\xi_{k+1} = n) \cdot \sum_{e \in M} \mathbb{P}(\tilde{A}_k \mid \xi_{k+1} = n, r(\tau_{\xi_{k+1}}) = e) \mathbb{P}(\tilde{A}_k^c \mid \xi_{k+1} = n, r(\tau_{\xi_{k+1}}) = e)\]

\[= \sum_{n=0}^{\infty} \mathbb{P}(\xi_{k+1} = n) \cdot \sum_{e \in M} \mathbb{P}(\tilde{A}_k \mid \xi_{k+1} = n, r(\tau_{\xi_{k+1}}) = e) \mathbb{P}(\tilde{A}_k^c \mid \xi_{k+1} = n, r(\tau_{\xi_{k+1}}) = e)\]

\[\leq \tilde{p}_1 \sum_{n=0}^{\infty} \mathbb{P}(\xi_{k+1} = n) \cdot \sum_{e \in M} \mathbb{P}(\tilde{A}_k \mid \xi_{k+1} = n, r(\tau_{\xi_{k+1}}) = e) \mathbb{P}(\tilde{A}_k^c \mid \xi_{k+1} = n, r(\tau_{\xi_{k+1}}) = e)\]

\[\leq \tilde{p}_1 \sum_{n=0}^{\infty} \mathbb{P}(\tilde{A}_k \mid \xi_{k+1} = n) \mathbb{P}(\xi_{k+1} = n)\]

\[\leq \tilde{p}_1^2,\]

where \(\tilde{p}_1 := \max_{e \in M} \{\mathbb{P}(E \mid r(0) = e)\} < 1\) with the event

\[E = \left\{\text{for all } s \in (0, T], \text{if } r(s-) \neq r(s) \text{ then } r(s) \neq i\right\}.

Hence, by induction we have

\[\mathbb{P}\left(\bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} \tilde{A}_k \right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{\infty} \bigcap_{k=i}^{\infty} \tilde{A}_k \right) = 1,

which means that the events \(\tilde{A}_k, k \in \mathbb{N}\) occur infinitely often a.s. The proof of Lemma 15 is completed. \(\square\)

**Lemma 16.** Fix any two distinct states \(e_1, e_2 \in \mathcal{M}\) satisfying the probability that the state \(e_2\) can be accessible from the state \(e_1\) through only one step is positive, i.e., \(q_{e_1,e_2} > 0\). Let \(\{\xi_n\}_{n=1}^{\infty}\) be a sequence of strictly increasing finite stopping times with respect to filtration \(\mathcal{F}_0^n\) satisfying \(r(\tau_{\xi_n}) = e_1\) for all \(n \in \mathbb{N}\). Suppose that \(B\) is a bounded Borel subset of \([0, +\infty)\) with positive Lebesgue measure. Then, the events \(A_k = \{r(\tau_{\xi_{k+1}}) = e_2, \sigma_{\xi_{k+1}} \in B\}, k \in \mathbb{N}\) occur infinitely often a.s., that is,

\[\mathbb{P}\left(\bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k \right) = 1.

**Proof.** By the strong Markov property of the process \((\langle S(t), I(t), R(t) \rangle, r(t))\), it is obtained that for each \(k \in \mathbb{N}\),

\[\mathbb{P}(A_k) = \sum_{n=0}^{\infty} \mathbb{P}(A_k \mid \xi_k = n) \mathbb{P}(\xi_k = n)\]
Thus, this completes the proof of Lemma 16. □

This implies that $\mathbb{P}(A_k^c) = 1 - \mathbb{P}(A_k) = 1 - \tilde{p} < 1$. Moreover,

$$\mathbb{P}(A_k^c \cap A_{k+1}^c)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(A_k^c \cap A_{k+1}^c \mid \xi_{k+1} = n) \mathbb{P}(\xi_{k+1} = n)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(\xi_{k+1} = n) \cdot \sum_{e \in M} \mathbb{P}(A_k^c \cap A_{k+1}^c \mid \xi_{k+1} = n, r(\tau_{\xi_{k+1}}) = e) \mathbb{P}(r(\tau_{\xi_{k+1}}) = e \mid \xi_{k+1} = n)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(\xi_{k+1} = n) \mathbb{P}(A_k^c \mid \xi_{k+1} = n, r(\tau_{\xi_{k+1}}) = e_1) \mathbb{P}(e_1 \mid \xi_{k+1} = n)$$

$$= (1 - \tilde{p}) \sum_{n=0}^{\infty} \mathbb{P}(\xi_{k+1} = n) \mathbb{P}(A_k^c \mid \xi_{k+1} = n, r(\tau_{\xi_{k+1}}) = e_1) \mathbb{P}(e_1 \mid \xi_{k+1} = n)$$

$$= (1 - \tilde{p}) \sum_{n=0}^{\infty} \mathbb{P}(\xi_{k+1} = n) \mathbb{P}(A_k^c \mid \xi_{k+1} = n)$$

$$= (1 - \tilde{p})^2$$

for all $k \in \mathbb{N}$. Hence, by induction we have

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} \bigcap_{k=1}^{\infty} A_k^c \right) = 0.$$

Thus,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_k \right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_k^c \right) = 1.$$

This completes the proof of Lemma 16. □
Proof of Theorem 7. (a) Let us divide the following proof of the assertion (a) into three steps.

Step 1. We shall show that $E_1^* \in \Omega(z_0, \omega)$ a.s. From Lemma 15, without loss of generality, we assume that the event $E_1$ in Lemma 15 occurs. Specifically, there exists a sequence of strictly increasing finite stopping times $\{\xi_n\}_{n=1}^\infty$ with respect to filtration $\mathcal{F}_0^n$ satisfying $r(\tau_{\xi_n}) = 1$ and $(S(\tau_{\xi_n}), I(\tau_{\xi_n}), R(\tau_{\xi_n})) \in H_\delta$ for all $n \in \mathbb{N}$ and almost all $\omega \in \tilde{\Omega}$. In addition, for any $\tilde{\delta} > 0$, we can conclude that one can find a $T > 0$ such that $(S(t, z), I(t, z), R(t, z)) \in B(E_1^*, \tilde{\delta})$ is valid for all $t \geq T$ and $z \in H_\delta$. If $\mathcal{F}_\delta = H_\delta \cup \{ (S, I, R) : S = 0, \Lambda/(\mu + \alpha) \leq I + R \leq \Lambda/\mu \}$, then by Lemma 15 it follows that with probability one, the events

$$
\{(S(\tau_{\xi_n+1}), I(\tau_{\xi_n+1}), R(\tau_{\xi_n+1})) \in B(E_1^*, \tilde{\delta}) \}\text{, }k \in \mathbb{N}
$$

will occur infinitely often with $\sigma_{\xi_n+1} \in [T, 2T]$. This means that $E_1^* \in \Omega(z_0, \omega)$ a.s.

Step 2. Let

$$
\tilde{\Gamma} = \left\{ (S, I, R) = \pi_{t_1} \circ \cdots \circ \pi_{t_k}(E_1^*) : t_1, \ldots, t_k > 0 \text{ and } p_1, \ldots, p_k \in \mathcal{M}, k \in \mathbb{N} \right\},
$$

with $q_{1,p_1}, q_{p_1,p_1+1} > 0$ and $p_i \neq p_{i+1}$, $i = 1, \ldots, k - 1$. We now prove that $\tilde{\Gamma} \subset \Omega(z_0, \omega)$ a.s. To begin with, we shall show that for any $t_1 > 0$, $\pi_{t_1}(E_1^*) \in \Omega(z_0, \omega)$ a.s. Denote $\tilde{\zeta} = \pi_{t_1}(E_1^*)$. For any $\tilde{\delta}_1 > 0$, by the continuous dependence of the solutions on the time and initial conditions, one can find two numbers $\tilde{\varepsilon}_1, \tilde{\delta}_1 > 0$ such that if $z \in B(E_1^*, \tilde{\varepsilon}_1)$, then $\pi_{t_1}(z) \in B(\tilde{\zeta}, \tilde{\delta}_1)$ for all $t \in B(t_1, \tilde{\delta}_1)$. Moreover, by Step 1 and Lemma 16, it follows that for such number $\tilde{\varepsilon}_1$, there exists a sequence of strictly increasing finite stopping times $\{\eta_n^1\}_{n=1}^\infty$ with respect to filtration $\mathcal{F}_0^n$ satisfying

$$
r(\tau_{\eta_0^1}) = p_1 \text{ and } (S(\tau_{\eta_0^1}), I(\tau_{\eta_0^1}), R(\tau_{\eta_0^1})) \in B(E_1^*, \tilde{\varepsilon}_1)
$$

for all $n \in \mathbb{N}$ and almost all $\omega \in \tilde{\Omega}$. Hence, by Lemma 16 again, we can obtain a sequence of strictly increasing finite stopping times $\{\eta_n^1\}_{n=1}^\infty$ with respect to filtration $\mathcal{F}_0^n$ satisfying

$$
r(\tau_{\eta_0^1}) = p_2 \text{ and } (S(\tau_{\eta_0^1}), I(\tau_{\eta_0^1}), R(\tau_{\eta_0^1})) \in B(\tilde{\zeta}, \tilde{\delta}_1)
$$

for all $n \in \mathbb{N}$ and almost all $\omega \in \tilde{\Omega}$. This implies that $\pi_{t_1}(E_1^*) \in \Omega(z_0, \omega)$ a.s. Therefore, by induction one can conclude that $\tilde{\Gamma} \subset \Omega(z_0, \omega)$ a.s.

Step 3. Now, we shall prove that the assertion (a) is valid. Let us first select any state $\tilde{\zeta} \in \mathcal{M}$. If $q_{1,\tilde{\varepsilon}} > 0$, then by Steps 1 and 2 we have $\pi_{t_1}(E_1^*) \in \Omega(z_0, \omega)$ a.s. for all $t \geq 0$. If $q_{1,\tilde{\varepsilon}} = 0$, we can also claim that $\pi_{t_1}(E_1^*) \in \Omega(z_0, \omega)$ a.s. for all $t > 0$. In fact, fix any $t > 0$ and let $\tilde{\delta}_2$ be a any positive constant. By the continuous dependence of the solutions on the time and the initial conditions, one can find a constant $\tilde{\varepsilon}_2 > 0$ such that $\pi_{t_1}(z) \in B(\tilde{\zeta}, \tilde{\delta}_2)$ if $z \in B(E_1^*, \tilde{\varepsilon}_2)$. On the other hand, it is obtained from the irreducibility of the generator $Q$ that a positive integer $n_0$ $(1 \leq n_0 \leq E - 1)$ exists such that

$$
q_{l_0,l_1,l_2} \cdots q_{l_{n_0-1},l_{n_0}} > 0
$$
with \( \{k : 0 \leq k \leq n_0\} \subset \mathcal{M} \) and \( l_0 = 1, l_{n_0} = \tilde{c} \). Thus, from the continuous dependence of the solutions on the initial conditions, it follows that there exist \( s_1, s_2, \ldots, s_{n_0-1} > 0 \) such that
\[
\pi_t^\varepsilon \circ \pi_{s_{n_0-1}}^{l_{n_0-1}} \circ \cdots \circ \pi_{s_1}^{l_1}(E_1^*) \in B \left( \pi_t^\varepsilon(E_1^*), \tilde{\delta}_2 \right).
\]
Because \( B(\pi_t^\varepsilon(E_1^*), \tilde{\delta}_2) \) is a open set in \( \mathbb{R}_+^d \), a sufficiently small \( \tilde{\delta}_3 > 0 \) exists such that
\[
B(\tau^+, \tilde{\delta}_3) \subset B \left( \pi_t^\varepsilon(E_1^*), \tilde{\delta}_2 \right),
\]
where \( \tau^* := \pi_t^\varepsilon \circ \pi_{s_{n_0-1}}^{l_{n_0-1}} \circ \cdots \circ \pi_{s_1}^{l_1}(E_1^*) \). Hence, it is obtained from Step 2 that there exists a sequence of strictly increasing finite stopping times \( \{\eta_n^2\}_{n=1}^\infty \) with respect to filtration \( \mathcal{F}_0^n \), such that
\[
(S(\tau_{\eta_n^2}^*), I(\tau_{\eta_n^2}^*), R(\tau_{\eta_n^2}^*)) \in B(\tau^*, \tilde{\delta}_3) \subset B \left( \pi_t^\varepsilon(E_1^*), \tilde{\delta}_2 \right)
\]
for all \( n \in \mathbb{N} \) and almost all \( \omega \in \tilde{\Omega} \). This implies that \( \pi_t^\varepsilon(E_1^*) \in \Omega(z_0, \omega) \) a.s. for any \( t > 0 \). By the similar arguments, we can conclude that \( \Gamma \subset \Omega(z_0, \omega) \) a.s. Also because \( \Omega(z_0, \omega) \) is a close set, we have \( \tilde{\Gamma} \subset \Omega(z_0, \omega) \) a.s. This completes the proof of the assertion (a).

(b) We shall prove that the second assertion is valid. By Theorem 1 of Chapter 3 in [26], the claim that the point \( \tau^* \) satisfies the condition (H) means that there exist elements \( \hat{p}_1, \ldots, \hat{p}_i \) in \( \mathcal{M} \) and \( \hat{t} \in \mathbb{R}^d \) with positive coordinates \( \hat{t}_1, \ldots, \hat{t}_i \) such that the function \( (t_1, \ldots, t_i) \mapsto \pi_t^{\hat{p}_1} \circ \cdots \circ \pi_{t_i}^{\hat{p}_i}(\tau^*) \) has a tangent map, the rank of which at \( \hat{t} \) is equal to the dimension of \( \mathbb{R}^3 \). That is, the rank of matrix \( [\partial \varphi / \partial t_1, \ldots, \partial \varphi / \partial t_i]_{(t_1, \ldots, t_i)} \) is equal to 3. Without loss of generality, we assume that
\[
\det \left( \frac{\partial \varphi}{\partial t_1}, \frac{\partial \varphi}{\partial t_2}, \frac{\partial \varphi}{\partial t_3} \right)_{(t_1, \ldots, t_i)} \neq 0. \tag{3.16}
\]
By the existence and continuous dependence on the initial conditions of the solutions, the function
\[
\psi(s, t, u) = \pi_t^{\hat{p}_1} \circ \cdots \circ \pi_{t_i}^{\hat{p}_i}(\tau^*)
\]
is defined and continuously differentiable in some small domain \( (-a, a) \times (-b, b) \times (-c, c) \subset \mathbb{R}^3 \). From (3.16), it follows that
\[
\det \left( \frac{\partial \psi}{\partial s}, \frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial u} \right)_{(0,0,0)} \neq 0.
\]
By the inverse function theorem, it is obtained that there exist \( 0 < a_1 < a, 0 < b_1 < b, \) and \( 0 < c_1 < c \) with \( a_1, b_1, c_1 < \min\{\hat{t}_1, \hat{t}_2, \hat{t}_3\} \), such that the function \( \psi(s, t, u) \) is a diffeomorphism between \( V = (-a_1, a_1) \times (-b_1, b_1) \times (-c_1, c_1) \) and \( U = \psi(V) \). Hence, for each point \( (u_1, u_2, u_3) \in U \), a point \( (\bar{s}, \bar{t}, \bar{u}) \in V \) exists such that
\[
(u_1, u_2, u_3) = \psi(s, t, u) = \psi(\bar{s}, \bar{t}, \bar{u}) \in \Gamma,
\]
which, together with the assertion (a), implies that \( U \subset \Gamma \subset \Omega(z_0, \omega) \) a.s. Note that \( U \) is a open subset of \( \mathcal{K} \subset \mathbb{R}^3_+ \). Therefore, a almost surely finite stopping time \( \tilde{T} \) with respect
to filtration $\mathcal{F}_t$ exists such that $(S(\bar{T}), I(\bar{T}), R(\bar{T})) \in U$ a.s. And because $\Gamma$ is a positive invariant set for the solutions of system (1.2) and $U \subset \Gamma$, we can get that for almost all $\omega \in \tilde{\Omega}$, $(S(t), I(t), R(t)) \in \Gamma$ is valid for all $t > \bar{T}$. This also implies that $\Omega(z_0, \omega) \subset \Gamma$ a.s. Thus, combining with the claim (a) of this theorem, one can get $\Gamma = \Omega(z_0, \omega)$ a.s. This completes the proof of Theorem 7. □

3.2. Proof of Theorem 8

To prove Theorem 8, we need the following two lemmas.

**Lemma 17.** For any $\varepsilon > 0$, there exists a $\bar{T} = \bar{T}(\varepsilon) > 0$ and a subset $A \in \mathcal{F}_\infty$ with $\mathbb{P}(A) > 1 - \varepsilon$ such that for any $t > \bar{T}$ and $S_0 \in [0, \Lambda/\mu]$, holds for every $\omega \in A$, where $\tilde{S}(s, S_0)$ is the solution to system

$$\frac{d\tilde{S}(t)}{dt} = \Lambda - \mu \tilde{S}(t)$$

with the initial condition $\tilde{S}(0) = S_0$.

**Proof.** Consider the equation

$$\frac{d\tilde{S}(t)}{dt} = \Lambda - \mu \tilde{S}(t).$$

Let $\varepsilon_1$ be any positive constant satisfying $\varepsilon_1 < \varepsilon/(8G'(0) \sum_{e \in \mathcal{M}} \pi_e \beta_e)$. By the compactness of the interval $[0, \Lambda/\mu]$ and the continuous dependence of solutions on initial conditions, there exists a positive number $T_1 = T_1(\varepsilon_1) = T_1(\varepsilon)$ such that

$$|\tilde{S}(t, S_0) - \frac{\Lambda}{\mu}| < \varepsilon_1$$

for all $t > T_1$ and $S_0 \in [0, \Lambda/\mu]$. Let us first fix a $S_0^* \in [0, \Lambda/\mu]$. For every $\omega \in \tilde{\Omega}$ and $t > T_1$, we then have

$$\left| \frac{1}{t} \int_0^t \left( \beta_{r(s, \omega)} \tilde{S}(s, S_0^*) G'(0) - (\mu + \alpha + \delta) \right) ds - \sum_{e \in \mathcal{M}} \pi_e B(e) \right|$$

$$\leq \left| \frac{1}{t} \int_0^t \beta_{r(s, \omega)} \left( \tilde{S}(s, S_0^*) - \frac{\Lambda}{\mu} \right) G'(0) ds \right|$$

$$+ \left| \frac{1}{t} \int_0^t \left( \beta_{r(s, \omega)} \frac{\Lambda G'(0)}{\mu} - (\mu + \alpha + \delta) \right) ds - \sum_{e \in \mathcal{M}} \pi_e B(e) \right|$$

$$\leq \frac{1}{t} \int_0^{T_t} \beta_{r(s, \omega)} \left| \tilde{S}(s, S_0^*) - \frac{\Lambda}{\mu} \right| G'(0) ds + \frac{1}{t} \int_{T_t}^t \beta_{r(s, \omega)} \left| \tilde{S}(s, S_0^*) - \frac{\Lambda}{\mu} \right| G'(0) ds$$

$$+ \left| \frac{1}{t} \int_0^t \left( \beta_{r(s, \omega)} \frac{\Lambda G'(0)}{\mu} - (\mu + \alpha + \delta) \right) ds - \sum_{e \in \mathcal{M}} \pi_e B(e) \right|. \quad (3.20)$$
Since $\beta_{r(s,\omega)}|\overline{S}(s, S_0^*) - \frac{\Lambda}{\mu}G'(0) \leq \frac{\Lambda \beta_\mu G'(0)}{\mu}$ for all $s \in [0, T_1]$ and $\omega \in \Omega$, there exists a $T_2 = T_2(\varepsilon) \geq T_1$ such that for $t > T_2$,

$$\frac{1}{t} \int_{0}^{T_1} \beta_{r(s,\omega)} \mid \overline{S}(s, S_0^*) - \frac{\Lambda}{\mu}G'(0)ds < \frac{\varepsilon}{3}. \quad (3.21)$$

Moreover, it is obtained from (3.19) that

$$\frac{1}{t} \int_{T_1}^{t} \beta_{r(s,\omega)} \mid \overline{S}(s, S_0^*) - \frac{\Lambda}{\mu}G'(0)ds \leq \frac{\varepsilon_1 G'(0)}{t - T_1} \int_{T_1}^{t} \beta_{r(s,\omega)} ds. \quad (3.22)$$

From the Birkhoff Ergodic theorem, there exists a subset $A_1 \in \mathcal{F}_{\infty}$ satisfying $\mathbb{P}(A_1) = 1$ such that for each $\omega \in A_1$, a $T_3 = T_3(\omega, \varepsilon) \geq T_1$ exists, and when $t > T_3$,

$$\frac{1}{t} \int_{T_1}^{t} \beta_{r(s,\omega)} \mid \overline{S}(s, S_0^*) - \frac{\Lambda}{\mu}G'(0)ds \leq 2\varepsilon_1 G'(0) \sum_{e \in \mathcal{M}} \pi_e \beta_e < \frac{\varepsilon}{3} \quad (3.23)$$

and

$$\left| \frac{1}{t} \int_{0}^{t} \left( \beta_{r(s,\omega)} \frac{\Lambda G'(0)}{\mu} - (\mu + \alpha + \delta) \right) ds - \sum_{e \in \mathcal{M}} \pi_e B(e) \right| < \frac{\varepsilon}{3}. \quad (3.24)$$

Hence, substituting (3.21), (3.22) and (3.23) into (3.20), it is obtained that for $t > T_4(\omega, \varepsilon) = \max\{T_2, T_3\}$,

$$\left| \frac{1}{t} \int_{0}^{t} \left( \beta_{r(s,\omega)} \overline{S}(s, S_0^*)G'(0) - (\mu + \alpha + \delta) \right) ds - \sum_{e \in \mathcal{M}} \pi_e B(e) \right| < \varepsilon.$$

holds for each $\omega \in A_1$. This implies

$$\lim_{t \to +\infty} \frac{1}{t} \int_{0}^{t} \left( \beta_{r(s)} \overline{S}(s, S_0^*)G'(0) - (\mu + \alpha + \delta) \right) ds = \sum_{e \in \mathcal{M}} \pi_e B(e) \quad a.s.$$
for any $S_0 \in [0, \Lambda/\mu]$, $t > T_0$ and $\omega \in A_3$. Let $A = A_2 \cap A_3$, then $\mathbb{P}(A) > 1 - \varepsilon$. For any $t > T(\varepsilon) = \max\{T_0, T_0\}$ and $S_0 \in [0, \Lambda/\mu]$,

$$\frac{1}{t} \int_0^t \left( \beta_r(s, \omega) \tilde{S}(s, S_0) G'(0) - (\mu + \alpha + \delta) \right) ds - \sum_{e \in M} \pi_e B(e) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

holds for each $\omega \in A$. This completes the proof of Lemma 17. □

**Lemma 18.** For any $\varepsilon > 0$, there exists a $\gamma = \gamma(\varepsilon) \in (0, \Lambda/(\mu + \alpha)]$ such that the following statements are valid:

(i) there exist two positive constants $\tilde{T}_1 = \tilde{T}_1(\varepsilon)$ and $\tilde{T}_2 = \tilde{T}_2(\varepsilon)$, such that for any $t > \tilde{T}_1 + \tilde{T}_2$, if $I(s, \omega, z_0) < \gamma$ for all $s \in [0, t]$, then

$$\left| S(s, \omega, z_0) - \tilde{S}(s, S_0) \right| < \frac{\varepsilon}{3L}$$

holds for all $s \in [\tilde{T}_1 + \tilde{T}_2, t]$;

(ii) for any $t > 0$, if $I(s, \omega, z_0) < \gamma$ for all $s \in [0, t]$, then

$$\left| \frac{G(I(s, \omega, z_0))}{I(s, \omega, z_0)} - G'(0) \right| < \frac{\varepsilon}{3L}$$

holds for all $s \in [0, t]$, where $L = \max\{\beta^M G'(0), (\Lambda \beta^M)/\mu\}$ and $\tilde{S}(s, S_0)$ is the solution of system (3.18) with the initial condition $\tilde{S}(0) = S_0 \in [0, \Lambda/\mu]$.

**Proof.** (i) Let $\varepsilon_1 = \varepsilon \varepsilon/6L$. Consider the following equation

$$\frac{du^1(t)}{dt} = -\mu u^1(t) + \varepsilon_1$$

(3.24)

with the initial value $u^1(0) = u^1_0 \in \mathbb{R}$. It is easy to see that the system (3.24) has a globally asymptotically stable equilibrium $u^*_1 = \varepsilon_1/\mu$. By Lemma 11, it follows that a constant $\tilde{T}_1 > 0$ exists such that $u^1(t) \leq 2\varepsilon_1/\mu$ for all $t \geq \tilde{T}_1$ and $u^1_0 \in [-\Lambda/\mu, \Lambda/\mu]$.

Suppose that for a fixed constant $\gamma_1 > 0$, $I(s, \omega, z_0) < \gamma_1$ for all $s \in [0, t]$ with $t > \tilde{T}_1 + \tilde{T}_2$, where $\gamma_1 = \gamma_1(\varepsilon)$ and $\tilde{T}_2 = \tilde{T}_2(\gamma_1)$ will be determined later. It is obtained from the third equation of system (1.2) that

$$\frac{dR(t, \omega, z_0)}{dt} = \delta I(t, \omega, z_0) - (\mu + \lambda) R(t, \omega, z_0) \leq \delta \gamma_1 - (\mu + \lambda) R(t, \omega, z_0)$$
for all $s \in [0,t]$. Moreover, for the system

$$\frac{du^2(t)}{dt} = \delta \gamma_1 - (\mu + \lambda)u^2(t)$$

(3.25)

with the initial value $u^2(0) = \Lambda/\mu$, one can easily find a constant $\bar{T}_2 = \bar{T}_2(\gamma_1) > 0$ such that $u^2(t) < 2\delta \gamma_1/(\mu + \lambda)$ if $t \geq \bar{T}_2$. By the comparison theorem, it follows from (3.25) that $R(t,\omega, z_0) < 2\delta \gamma_1/(\mu + \lambda)$ when $t \geq \bar{T}_2$. Define the functions $F_e(S, I, R) = \Lambda - \mu S + \lambda R - \beta_e SG(I)$, $e \in \mathcal{M}$. Due to the continuity of $F_e$ with respect to variables $I$ and $R$, one can find a sufficiently small $\gamma_1 = \gamma_1(\varepsilon) \in (0, \Lambda/(\mu + \alpha))$ such that whenever $0 \leq I(s, \omega, z_0) \leq \gamma_1$ and $0 \leq S(s, \omega, z_0) \leq \Lambda/\mu$ for all $s \in [0,t]$,

$$\Lambda - \mu S(s, \omega, z_0) - \varepsilon_1 \leq \frac{dS(s, \omega, z_0)}{ds} \leq \Lambda - \mu S(s, \omega, z_0) + \varepsilon_1$$

for all $s \in [\bar{T}_2, t]$. Introduce the following equation

$$\frac{du(s)}{ds} = \Lambda - \mu u(s) - \varepsilon_1,$$

(3.26)

with the initial condition $u(\bar{T}_2) = S(\bar{T}_2, \omega, z_0)$. From the comparison theorem, it is easy to see that

$$S(s, \omega, z_0) \geq u(s, u(\bar{T}_2)) \text{ for all } s \in [\bar{T}_2, t].$$

(3.27)

From (3.18) and (3.26), it is obtained that for $s \in [\bar{T}_2, t]$

$$\frac{d}{ds} (\bar{S}(s) - u(s)) = -\mu(\bar{S}(s) - u(s)) + \varepsilon_1$$

with the initial value $\bar{S}(\bar{T}_2) - u(\bar{T}_2) \in [-\Lambda/\mu, \Lambda/\mu]$. Hence, it is obtained from (3.24) and (3.27) that for $s \in [\bar{T}_1 + \bar{T}_2, t]$,

$$\bar{S}(s, S_0) - S(s, \omega, z_0) = (\bar{S}(s, S_0) - u(s, u(\bar{T}_2))) + (u(s, u(\bar{T}_2)) - S(s, \omega, z_0))$$

$$\leq \frac{2\varepsilon_1}{\mu} < \frac{\varepsilon}{3L}.$$ 

Using the similar arguments, we also have

$$S(s, \omega, z_0) - \bar{S}(s, S_0) < \frac{\varepsilon}{3L} \text{ for all } s \in [\bar{T}_1 + \bar{T}_2, t].$$

Hence, for all $s \in [\bar{T}_1 + \bar{T}_2, t]$, we have

$$\left| S(s, \omega, z_0) - \bar{S}(s, S_0) \right| < \frac{\varepsilon}{3L}.$$ 

(ii) On the other hand, by the continuity of the function $g(I) = G(I)/I$ with respect to variable $I$ and $\lim_{I \to 0^+} g(I) = G'(0)$, it follows that there exists a sufficiently small $\gamma_2 = \gamma_2(\varepsilon) \in (0, \Lambda/(\mu + \alpha)]$, such that whenever $0 \leq I(s, \omega, z_0) \leq \gamma_2$,

$$\left| \frac{G(I(s, \omega, z_0))}{I(s, \omega, z_0)} - G'(0) \right| < \frac{\varepsilon}{3L}.$$ 

Let us choose $\gamma = \gamma(\varepsilon) = \min\{\gamma_1, \gamma_2\}$. This completes the proof of Lemma 18. □
Proof of Theorem 8. Let \( \varepsilon \) be any positive constant with \( \varepsilon < \left( \sum_{e \in \mathcal{M}} \pi_e B(e) \right) / 2 \). Consider the process \( ((S(t), I(t), R(t)), r(t)) \) on a larger state space \( \overline{X} = \overline{K} \times \mathcal{M} \), where \( \overline{K} = \overline{K} \setminus \{(S, I, R) \in \mathbb{R}_+^3 : I = 0 \} \) and \( \overline{K} \) is the closure of \( K \). Note that the process \( ((S(t), I(t), R(t)), r(t)) \) is a time-homogeneous Markov process with the Feller property. According to Theorem 10, one can get the existence of an invariant probability measure \( \nu^* \) for the process \( ((S(t), I(t), R(t)), r(t)) \) on the state space \( \overline{X} \), provided that a compact set \( O \subset \overline{X} \) exists such that

\[
\liminf_{t \to +\infty} \frac{1}{t} \int_0^t \left( \int_X \mathbb{P}(s, x, O) \nu(dx) \right) ds = \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{P}(s, x_0, O) ds > 0, \tag{3.28}
\]

for some initial distribution \( \nu = \delta_{x_0} \) with \( x_0 \in \overline{X} \), where \( \mathbb{P}(s, \cdot, \cdot, \cdot) \) is the transition probability function and \( \delta \) is the Dirac function. On the other hand, it is easy to see that for any initial value \( x_0 = (z_0, r(0)) \in X \), when \( t > 0 \) the solution \((S(t), I(t), R(t))\) of system (1.2) do not reach the boundary \( \partial \mathcal{K} \) of the region \( \mathcal{K} \) under the condition that \( R_0 > 1 \). Hence, we have \( \nu^*(\partial \mathcal{K} \times \mathcal{M}) = 0 \), which obviously implies that \( \nu^* \) is also the invariant probability measure of the process \((S(t), I(t), R(t), r(t))\) on the state space \( X \). Consequently, to complete the proof of Theorem 8, it is sufficient to find a compact set \( O \subset \overline{X} \) satisfying (3.28).

Fix a \( T > \max\{T, \tilde{T}_1 + \tilde{T}_2, 6AL(T_1 + \tilde{T}_2)/\mu \varepsilon\} \), and set \( \chi_n(\omega) = 1_A(\theta^n T \omega), \) \( n \in \mathbb{N}_0 \), where \( A \) is as in Lemma 17. \( 1_A(\cdot) \) is the indicator function, and the shift operators \( \theta^t \) on \( \Omega \) are defined by

\[
(\theta^t \omega)_s = \omega_{s+t}, \quad s, t \geq 0.
\]

Let

\[
\chi_n^1(\omega) = \begin{cases}
1, & \text{if } \chi_n(\omega) = 1 \text{ and } I(t, \omega, z_0) < \gamma \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
\chi_n^2(\omega) = \begin{cases}
1, & \text{if } \chi_n(\omega) = 1 \text{ and there exists some } t \in [nT, (n+1)T] \\
& \text{such that } I(t, \omega, z_0) \geq \gamma, \\
0, & \text{otherwise}
\end{cases}
\]

where \( n \in \mathbb{N}_0 \). It is easy to see that \( \chi_n(\omega) = \chi_n^1(\omega) + \chi_n^2(\omega) \). Denote \( \chi_n^3(\omega) = 1 - \chi_n(\omega) \). Since \( r(t+s, \omega) = r(t, \theta^s \omega) \) for any \( s, t > 0 \), we have

\[
z(t+s, \omega, z_0) = z(t, \theta^s \omega, z(s, \omega, z_0)) \quad \text{for any } s, t > 0.
\]

This implies that if \( \chi_n^1(\omega) = 1 \) then \( I(t+nT, \omega, z_0) < \gamma \) holds for all \( 0 \leq t \leq T \), i.e., \( I(t, \theta^{nT} \omega, z(nT, \omega, z_0)) < \gamma \) holds for all \( 0 \leq t \leq T \). In addition, \( \chi_n^1(\omega) = 1 \) implies \( \chi_n(\omega) = 1 \), i.e., \( \theta^{nT} \omega \in A \). Hence, it is obtained from Lemmas 17 and 18 that

\[
\left| \frac{1}{T} \int_0^T \left( \beta_{r(t, \theta^t \omega)} \tilde{S}(t, S(nT, \omega, z_0)) G'(0) - (\mu + \alpha + \delta) \right) dt - \sum_{e \in \mathcal{M}} \pi_e B(e) \right| < \varepsilon, \tag{3.29}
\]

28
\[
\frac{1}{T - (\bar{T}_1 + \bar{T}_2)} \int_{\bar{T}_1 + \bar{T}_2}^{T} \left| S(t, \theta^{nT} \omega, z(nT, \omega, z_0)) - \bar{S}(t, S(nT, \omega, z_0)) \right| dt < \frac{\varepsilon}{3L} \quad (3.30)
\]

and
\[
\frac{1}{T} \int_{0}^{T} \left| \frac{G(I(t, \theta^{nT} \omega, z(nT, \omega, z_0)))}{I(t, \theta^{nT} \omega, z(nT, \omega, z_0))} - G'(0) \right| dt < \frac{\varepsilon}{3L}.
\]

For the second equation of system [1.2], we compute that
\[
\frac{1}{T} \left( \ln I((n + 1)T, \omega, z_0) - \ln I(nT, \omega, z_0) \right)
= \frac{1}{T} \int_{nT}^{(n+1)T} \left( \beta_{r(t, \omega)} \frac{S(t, \theta^n \omega, z(nT, \omega, z_0))}{I(t, \omega, z_0)} - (\mu + \alpha + \delta) \right) dt
= \frac{1}{T} \int_{0}^{T} \left( \beta_{r(t, \theta^n \omega, \omega)} \frac{S(t, \theta^n \omega, z(nT, \omega, z_0))}{I(t, \omega, z_0)} - (\mu + \alpha + \delta) \right) dt
= \frac{1}{T} \left( \beta_{r(t, \theta^n \omega, \omega)} \bar{S}(t, S(nT, \omega, z_0)) G'(0) - (\mu + \alpha + \delta) \right) dt + \frac{1}{T} \int_{0}^{T} \beta_{r(t, \theta^n \omega, \omega)} \left( S(t, \theta^n \omega, z(nT, \omega, z_0)) - \bar{S}(t, S(nT, \omega, z_0)) \right) \frac{G(I(t, \theta^n \omega, z(nT, \omega, z_0)))}{I(t, \theta^n \omega, z(nT, \omega, z_0))} dt
+ \frac{1}{T} \int_{0}^{T} \beta_{r(t, \theta^n \omega, \omega)} \bar{S}(t, S(nT, \omega, z_0)) \left( \frac{G(I(t, \theta^n \omega, z(nT, \omega, z_0)))}{I(t, \theta^n \omega, z(nT, \omega, z_0))} - G'(0) \right) dt
\geq \frac{1}{T} \int_{0}^{T} \left( \beta_{r(t, \theta^n \omega, \omega)} \bar{S}(t, S(nT, \omega, z_0)) G'(0) - (\mu + \alpha + \delta) \right) dt
- L \int_{0}^{\bar{T}_1 + \bar{T}_2} \left( S(t, \theta^n \omega, z(nT, \omega, z_0)) - \bar{S}(t, S(nT, \omega, z_0)) \right) dt
- \frac{L}{T - (\bar{T}_1 + \bar{T}_2)} \int_{\bar{T}_1 + \bar{T}_2}^{T} \left( S(t, \theta^n \omega, z(nT, \omega, z_0)) - \bar{S}(t, S(nT, \omega, z_0)) \right) dt
- \frac{L}{T} \int_{0}^{T} \left( \frac{G(I(t, \theta^n \omega, z(nT, \omega, z_0)))}{I(t, \theta^n \omega, z(nT, \omega, z_0))} - G'(0) \right) dt.
\]

Substituting (3.29), (3.30) and (3.31) into the right side of the above inequality, it is obtained that when \( \chi_n^1(\omega) = 1 \), we have
\[
\frac{1}{T} \left( \ln I((n + 1)T, \omega, z_0) - \ln I(nT, \omega, z_0) \right) \geq \sum_{e \in \mathcal{M}} \pi_e B(e) - 2\varepsilon.
\]

(3.32)
Let
\[ \Theta = \max \left\{ \left\| \frac{SG(I)}{I} - (\mu + \alpha + \delta) \right\| : 0 \leq S, I \leq \frac{\Lambda}{\mu}, e \in \mathcal{M} \right\}. \]

If \( \chi_n^2(\omega) = 1 \) or \( \chi_n^3(\omega) = 1 \), then
\[
\frac{1}{T} \left( \ln I((n+1)T, \omega, z_0) - \ln I(nT, \omega, z_0) \right) = \frac{1}{T} \int_{nT}^{(n+1)T} \left( \beta_r(t, \omega) \frac{S(t, \omega, z_0) G(I(t, \omega, z_0))}{I(t, \omega, z_0)} - (\mu + \alpha + \delta) \right) dt \geq -\Theta.
\]

(3.33)

Summing up both sides of (3.32) and (3.33) respectively yields
\[
\frac{1}{T} \left( \ln I((n+1)T, \omega, z_0) - \ln I(nT, \omega, z_0) \right) \geq \left( \sum_{e \in \mathcal{M}} \pi_e B(e) - 2\varepsilon \right) \chi_n^1(\omega) - \Theta \left( \chi_n^2(\omega) + \chi_n^3(\omega) \right),
\]
which implies
\[
\frac{1}{kT} \left( \ln I(kT, \omega, z_0) - \ln I(0) \right) \geq \frac{1}{k} \left( \left( \sum_{e \in \mathcal{M}} \pi_e B(e) - 2\varepsilon \right) \sum_{n=0}^{k-1} \chi_n^1(\omega) - \Theta \sum_{n=0}^{k-1} \left( \chi_n^2(\omega) + \chi_n^3(\omega) \right) \right)
\]
holds for any \( k \in \mathbb{N} \). Noting that \( I(t) \leq \Lambda/\mu \) for all \( t \geq 0 \), we then have
\[
\limsup_{k \to +\infty} \frac{1}{k} \left( \left( \sum_{e \in \mathcal{M}} \pi_e B(e) - 2\varepsilon \right) \sum_{n=0}^{k-1} \chi_n^1(\omega) - \Theta \sum_{n=0}^{k-1} \left( \chi_n^2(\omega) + \chi_n^3(\omega) \right) \right) \leq 0. \tag{3.34}
\]

Using the arguments similar to that given in Theorem 3.1 of [12], we have
\[
\lim_{k \to +\infty} \frac{1}{k} \sum_{n=0}^{k-1} \chi_n(\omega) = \mathbb{P}(A) \quad a.s.,
\]
which implies
\[
\lim_{k \to +\infty} \frac{1}{k} \sum_{n=0}^{k-1} \chi_n^3(\omega) = 1 - \mathbb{P}(A) \leq \varepsilon \quad a.s., \tag{3.35}
\]
and
\[
\lim_{k \to +\infty} \frac{1}{k} \sum_{n=0}^{k-1} \left( \chi_n^1(\omega) + \chi_n^2(\omega) \right) = \lim_{k \to +\infty} \frac{1}{k} \sum_{n=0}^{k-1} \chi_n(\omega) \geq 1 - \varepsilon \quad a.s. \tag{3.36}
\]
Multiplying both sides of (3.35) by $\Theta$ and multiplying both sides of (3.36) by $-\sum_{e \in M} \pi_e B(e) - 2\varepsilon)$, then adding to (3.34) yields

\[
\limsup_{k \to +\infty} \frac{1}{k} \left[ \Delta \sum_{n=0}^{k-1} \chi_n^1(\omega) - \Theta \sum_{n=0}^{k-1} \chi_n^2(\omega) - \Delta \sum_{n=0}^{k-1} \left( \chi_n^1(\omega) + \chi_n^2(\omega) \right) \right]
\]

\[
= \limsup_{k \to +\infty} \frac{1}{k} \left[ - (\Theta + \Delta) \sum_{n=0}^{k-1} \chi_n^2(\omega) \right]
\]

\[
\leq -\Delta (1 - \varepsilon) + \Theta \varepsilon \quad a.s.,
\]

where $\Delta = \sum_{e \in M} \pi_e B(e) - 2\varepsilon > 0$. Hence, it is obtained that for $\varepsilon$ sufficiently small,

\[
\liminf_{k \to +\infty} \frac{1}{k} \sum_{n=0}^{k-1} \chi_n^2(\omega) \geq \frac{\Delta (1 - \varepsilon) - \Theta \varepsilon}{\Theta + \Delta} := \kappa > 0 \quad a.s. \quad (3.37)
\]

Denote by $D(s)$ the closure of the region

\[
\left\{ (S, I, R) \in \mathbb{R}_+^3 : \frac{\Lambda}{\mu + \alpha} \leq S + I + R \leq \frac{\Lambda}{\mu}, I \geq s, 0 < s < \frac{\Lambda}{\mu} \right\}.
\]

For the system (1.2), one can find a $\overline{\gamma} = \overline{\gamma}(\gamma, T) > 0$ satisfying $z(t, \omega, z_0) \in D(\overline{\gamma})$ for all $t \in [0, T]$ and $z_0 \in \mathcal{K}$, provided that an $s \in [0, T]$ exists such that $z(s, \omega, z_0) \in D(\gamma)$. Hence, for each $\omega \in \overline{\Omega}$ with $\chi_n^2(\omega) = 1$ and any $z_0 \in \mathcal{K}$, we have

\[
\int_{nT}^{(n+1)T} 1_{\{z(t, \omega, z_0) \in D(\overline{\gamma})\}} dt = T,
\]

which, combined with (3.37), implies

\[
\liminf_{t \to +\infty} \frac{1}{t} \int_0^t 1_{\{z(s, \omega, z_0) \in D(\overline{\gamma})\}} ds \geq \kappa > 0 \quad a.s.
\]

By Fatou’s lemma, it then follows that

\[
\liminf_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{P}\left\{ z(s, \omega, z_0) \in D(\overline{\gamma}) \right\} ds \geq \kappa > 0.
\]

Let $O = D(\overline{\gamma}) \times \mathcal{M} \subset \overline{X}$, we then have

\[
\liminf_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{P}(s, x_0, O) ds = \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{P}\left\{ z(s, \omega, z_0) \in D(\overline{\gamma}) \right\} ds \geq \kappa > 0.
\]

This completes the proof of Theorem 8. □

Remark 6. The method used in the proof of Theorem 8 is similar to that given in Theorem 3.1 of [12]. In the proceeding of the proof, it is most important to find a compact set $O \subset \overline{X}$.
satisfying (3.28). Indeed, according to the results of Theorem 5, there exists a simpler method for getting such compact set \( O \subset \mathcal{X} \) as follows. By Theorem 5, we have

\[
\liminf_{t \to +\infty} \frac{1}{t} \int_0^t I(s) ds \geq \frac{\mu^2}{\beta M(\mu \theta + \beta M G'(0))^2} \sum_{e \in \mathcal{M}} \pi_e B(e) := \kappa_1 > 0 \quad \text{a.s.}
\]

if \( R_0 > 1 \). Since

\[
\frac{1}{t} \int_0^t I(s) ds = \frac{1}{t} \int_0^t I(s) \cdot \mathbf{1}_{\{I(s) < \frac{\kappa_1}{2}\}} ds + \frac{1}{t} \int_0^t I(s) \cdot \mathbf{1}_{\{I(s) \geq \frac{\kappa_1}{2}\}} ds
\]

\[
\leq \frac{\kappa_1}{2} + \frac{\Lambda}{\mu} \cdot \frac{1}{t} \int_0^t \mathbf{1}_{\{I(s) \geq \frac{\kappa_1}{2}\}} ds,
\]

we have

\[
\liminf_{t \to +\infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{I(s) \geq \frac{\kappa_1}{2}\}} ds \geq \frac{\mu \kappa_1}{2 \Lambda} \quad \text{a.s.}
\] (3.38)

By Fatou’s lemma, it then follows from (3.38) that

\[
\liminf_{t \to +\infty} \frac{1}{t} \int_0^t I(s) ds = \mathbb{E} \left[ \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{I(s) \geq \frac{\kappa_1}{2}\}} ds \right] \geq \frac{\mu \kappa_1}{2 \Lambda}.
\]

Let \( O = D(\frac{\kappa_1}{2}) \times \mathcal{M} \subset \mathcal{X} \). It then follows that

\[
\liminf_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{P}(s, x_0, O) ds = \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{P} \left\{ \mathbf{1}_{\{I(s) \geq \frac{\kappa_1}{2}\}} ds \right\} \geq \frac{\mu \kappa_1}{2 \Lambda},
\]

which implies that (3.28) holds.

3.3. Proof of Theorem 9

**Proof.** We shall complete the proof of this theorem by three steps.

**Step 1.** We shall prove that the process \(((S(t), I(t), R(t)), r(t))\) is positive Harris recurrent. Since the point \( \bar{\psi}_* \in \Gamma \) satisfies the condition (H), without loss of generality, we can assume by the similar arguments in the proof of (b) in Theorem 7 that the function

\[
\psi(s, t, u) = \pi_{\hat{t}_1}^\hat{p}_1 \circ \cdots \circ \pi_{\hat{t}_4}^\hat{p}_4 \circ \pi_{\hat{t}_3}^\hat{p}_3 \circ \pi_{\hat{t}_2}^\hat{p}_2 \circ \pi_{\hat{t}_1}^\hat{p}_1 (\bar{\psi}_*)
\] (3.39)
is defined and continuously differentiable in \((-c, c)^3 \subset \mathbb{R}^3\) with

\[
\det \left( \frac{\partial \psi}{\partial s}, \frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial u} \right) \neq 0 \quad (3.40)
\]

for any \((s, t, u) \in (-c, c)^3\), where the states \(\hat{p}_k \in \mathcal{M}_k, \hat{t}_k > 0, k = 1, \ldots, i,\) and \(c\) is some positive constant. From the definition of the set \(\Gamma\), without loss of generality we can assume that there exist \(\hat{p}_1, \ldots, \hat{p}_j \in \mathcal{M}\) and \(\bar{t}_1, \ldots, \bar{t}_j > 0\) such that \(\bar{z}_* = \pi_{t_j}^{\hat{p}_j} \circ \cdots \circ \pi_{t_3}^{\hat{p}_3} (E_1^*)\).

Because \(E_1^*\) is the equilibrium of system (1.2) in the state 1, for any \(w_1 > 0\) we then have \(\pi_{w_1}^1 (E_1^*) = E_1^*\). Hence, \(\bar{z}_*\) can be rewritten as

\[
\bar{z}_* = \pi_{t_j}^{\hat{p}_j} \circ \cdots \circ \pi_{t_3}^{\hat{p}_3} \circ \pi_{w_1}^1 (E_1^*). \quad (3.41)
\]

Substituting (3.41) into (3.39), it is obtained that for any \(w_1 > 0\),

\[
\psi_{w_1}(s, t, u) = \psi(s, t, u) = \bar{F}(\bar{t}) \circ \pi_{w_1}^1 (E_1^*),
\]

where \(\bar{t} = (\bar{t}_4, \ldots, \bar{t}_i), \bar{t} = (\bar{t}_1, \ldots, \bar{t}_j)\) and \(\bar{F}(\bar{t}) = \pi_{t_4}^{\bar{t}_4} \circ \cdots \circ \pi_{t_3}^{\bar{t}_3} \circ \pi_{t_2}^{\bar{t}_2} \circ \pi_{t_1}^{\bar{t}_1} \circ (E_1^*)\). For each \(z \in \mathbb{R}^3\), we introduce the function as follows

\[
\Psi_{z,w_1}(s, t, u) = \bar{F}(\bar{t}) \circ \pi_{w_1}^1 (E_1^*),
\]

with the domain

\[
\mathcal{D} = \left\{ (s, t, u) : |s|, |t|, |u| < \bar{c} := \frac{b}{24} \min \{\bar{t}_1, \bar{t}_2, \bar{t}_3, c\} \right\},
\]

where the positive constant \(b\) \((b < 1)\) will be determined later. In particular, if we select \(z = E_1^*\), then

\[
\Psi_{E_1^*,w_1}(s, t, u) = \psi_{w_1}(s, t, u) = \psi(s, t, u),
\]

which, together with (3.40), implies

\[
\det \left( \frac{\partial \Psi_{E_1^*,w_1}}{\partial s}, \frac{\partial \Psi_{E_1^*,w_1}}{\partial t}, \frac{\partial \Psi_{E_1^*,w_1}}{\partial u} \right) \bigg|_{(0,0,0)} \neq 0. \quad (3.42)
\]

Through a small perturbation for the function \(\Psi_{z,w_1}(s, t, u)\), we obtain the following function

\[
\Psi_{z,w}(s, t, u) = \bar{F}(\bar{t} + w_3) \circ \pi_{w_1}^1 \circ \pi_{t_1}^{\bar{t}_1} \circ \pi_{t_2}^{\bar{t}_2} \circ \pi_{w_1}^1 (E_1^*)
\]

with the domain \((s, t, u) \in \mathcal{D}, z \in \mathbb{R}^3\) and \(w = (w_1, w_2, w')\), where \(w_2 = (x_1, \ldots, x_j), w' = (y_1, \ldots, y_{i-4}), (w'_j, \hat{K} - \sum_{k=1}^{i-4} x_k - \sum_{k=1}^{i-4} y_k - w_1 - (s + t + u))\) satisfying

\[
0 < w_1, x_k, y_l < \frac{b}{12(i + j - 3)},
\]

\(k = 1, \ldots, j, l = 1, \ldots, i - 4,\) and here \(\hat{K} = b/3\).
Note that for some fixed values of \( s, t, u \) and \( w \), the point \( \Psi_{z,w}(s,t,u) \) may be not necessarily accessible from any point \( z \in \mathbb{R}^3 \) by the system (1.2), because two adjacent elements \( \bar{e}_1, \bar{e}_2 \) of the ordered state set \( \bar{M} := \{1, \bar{p}_1, \ldots, \bar{p}_j, \hat{p}_1, \ldots, \hat{p}_i\} \) may be not accessible directly, i.e., the case where \( q_{\bar{e}_1, \bar{e}_2} = 0 \) may occur. However, owing to the irreducibility of the transition rate matrix \( Q \) and the continuous dependence of the solutions on the time and initial conditions, without loss of generality one can assume that the ordered state set \( \bar{M} \) in the function \( \Psi_{z,w} \) satisfies that \( q_{\bar{e}_1, \bar{e}_2} > 0 \) for any two adjacent elements \( \bar{e}_1, \bar{e}_2 \in \bar{M} \), i.e., the point \( \Psi_{z,w}(s,t,u) \) is accessible from any point \( z \in \mathbb{R}^3 \) by the system (1.2). Thus, in the remainder of the proof we will always assume that this fact is valid for the function \( \Psi_{z,w} \) considered above.

Let us define the domain

\[
W = \left\{ x \in \mathbb{R}^{i+j-3} : 0 < x_k < \frac{b}{12(i+j-3)}, k = 1, \ldots, i+j-3 \right\}.
\]

By the continuous dependence of the solutions on the time and initial conditions, it is obtained from (3.42) that there exist sufficiently small \( \varepsilon > 0, b \) such that

\[
\det \left( \frac{\partial \Psi_{z,w}}{\partial s}, \frac{\partial \Psi_{z,w}}{\partial t}, \frac{\partial \Psi_{z,w}}{\partial u} \right) \bigg|_{(0,0,0)} \neq 0
\]

for any \( z \in B(E^*_1, \varepsilon) \) and \( w \in W \). Thus, for any fixed \( z \in B(E^*_1, \varepsilon) \) and \( w \in W \), a small neighborhood \( U_{z,w} \subset D \) of the point \((0,0,0)\) exists such that the mapping \( \Psi_{z,w} \) is a diffeomorphism between \( U_{z,w} \) and \( \Psi_{z,w}(U_{z,w}) \). Note that the function \( \Psi_{z,w} \) is continuously differentiable with respect to the parameter \((z,w)\). Due to the modified slightly inverse function theorem, for sufficiently small \( \varepsilon \) and \( b \), one can find a open subset \( \bar{U} \subset \mathbb{R}^3 \) such that for any \( z \in B(E^*_1, \varepsilon) \) and \( w \in W \), we always have \( U_{z,w} \subset D \) and \( \bar{U} = \Psi_{z,w}(U_{z,w}) \) with

\[
d_0 := \inf_{z \in B(E^*_1, \varepsilon), w \in W, \bar{w} \in \bar{U}} \left| J_{z,w}(\bar{w}) \right| > 0,
\]

where \( J_{z,w}(\bar{w}) \) is the determinant of the Jacobian matrix of \( \Psi_{z,w}^{-1} \) at \( \bar{w} \).

Define the event

\[
\mathbf{E} = \left\{ r(0) = 1, \text{the states of the process } r(t) \text{ appear in the order of } \{\bar{p}_1, \ldots, \bar{p}_j, \hat{p}_1, \ldots, \hat{p}_i\} \right\}.
\]

Denote by \( \sigma_0, \sigma_{\bar{p}_1}, \ldots, \sigma_{\bar{p}_j}, \sigma_{\hat{p}_1}, \ldots, \sigma_{\hat{p}_i} \) the sojourns of the process \( r(t) \) in the environmental states \( 1, \bar{p}_1, \ldots, \bar{p}_j, \hat{p}_1, \ldots, \hat{p}_i \), respectively. According to the preceding assumption, we then have \( d_1 := \mathbb{P}(\mathbf{E}) > 0 \) if \( r(0) = 1 \). Note that given that the event \( \mathbf{E} \) occurs, random variables \( \sigma_0, \sigma_{\bar{p}_1}, \ldots, \sigma_{\bar{p}_j}, \sigma_{\hat{p}_1}, \ldots, \sigma_{\hat{p}_i} \) are independent. Hence,

\[
d_2 := \mathbb{P}\left\{(\sigma_0, \sigma_{\bar{p}_1} - \hat{t}_1, \ldots, \sigma_{\bar{p}_j} - \hat{t}_j, \sigma_{\hat{p}_1} - \hat{t}_4, \ldots, \sigma_{\hat{p}_{i-1}} - \hat{t}_{i-1}) \in W, \sigma_{\hat{p}_i} > \tilde{K} + \hat{t}_i \mathbf{E}\right\}
\]

\[
= \mathbb{P}\left\{w \in W, \sigma_{\hat{p}_i} > \tilde{K} + \hat{t}_i \mathbf{E}\right\} > 0.
\]
Let $K = \sum_{k=1}^{i} \tilde{t}_k + \sum_{k=1}^{j} \tilde{t}_k + \bar{\tilde{K}}$. For any $z \in B(E_1^*, \varepsilon)$ and any Borel set $B \subset \overline{U}$, we have
\[
P(K, z, 1, B \times \{\hat{p}_1\})
\geq d_1 \cdot P\left\{ (\sigma_0, \sigma_{\tilde{p}_1} - \tilde{t}_1, \ldots, \sigma_{\tilde{p}_j} - \tilde{t}_j, \sigma_{\tilde{p}_4} - \tilde{t}_4, \ldots, \sigma_{\tilde{p}_{i-1}} = \tilde{t}_i) = w \in W, \ E \right\}
\geq d_1 \cdot P\left\{ (s, t, u) \in \Psi^{-1}(B, \tilde{K} + \tilde{t}_i) \right\} \cdot P\left\{ w \in W, \sigma_{\tilde{p}_i} > \tilde{K} + \tilde{t}_i \right\}
\geq d_1 d_2 \cdot P\left\{ (s, t, u) \in \Psi^{-1}(B, \tilde{K} + \tilde{t}_i) \right\}.
\]

Let $h(x_1, x_2, x_3)$ be the probability density function of random variables $(\sigma_{\tilde{p}_1}, \sigma_{\tilde{p}_2}, \sigma_{\tilde{p}_3})$ under the condition that the event $E$ occurs. Since $\sigma_{\tilde{p}_1}, \sigma_{\tilde{p}_2}, \sigma_{\tilde{p}_3}$ are independent exponential random variables given that the event $E$ occurs, the function $h(s + \tilde{t}_1, t + \tilde{t}_2, u + \tilde{t}_3)$ is a smooth function and $d_3 := \inf_{(s, t, u) \in D} h(s + \tilde{t}_1, t + \tilde{t}_2, u + \tilde{t}_3) > 0$. Thus,
\[
P(K, z, 1, B \times \{\hat{p}_1\}) \geq d_1 d_2 \cdot \int_{U_{z,w}} h(s + \tilde{t}_1, t + \tilde{t}_2, u + \tilde{t}_3) \chi_{\Psi^{-1}(B)}(s, t, u) \, ds \, dt \, du
\geq d_1 d_2 d_3 \cdot \int_{U_{z,w}} \chi_{\Psi^{-1}(B)}(s, t, u) \, ds \, dt \, du
= d_1 d_2 d_3 \cdot \int_{\Omega} \chi_{B}(u_1, u_2, u_3) \, J_{z,w}(u_1, u_2, u_3) \, du_1 \, du_2 \, du_3
\geq d_0 d_1 d_2 d_3 \cdot \tilde{m}(B) = d_4 \cdot m(B \times \{\hat{p}_1\}),
\]

where $d_4$ is some positive constant. By Lemma 13 and the similar arguments in Step 2 of the proof in Theorem 7, it is obtained that for any initial value $(z_0, e) = ((0, 0), (0, 0), e) \in X$, there exists with probability one a sequence of strictly increasing finite stopping times $\{\varsigma_n\}_{n=1}^{\infty}$ with respect to filtration $\mathcal{F}_0^n$, such that $r(\varsigma_n) = 1$ and $(S(\varsigma_n), I(\varsigma_n), R(\varsigma_n)) \in B(E_1^*, \varepsilon)$ for all $n \in \mathbb{N}$. Combining this property and (3.43), one can obtain by the strong Markov property of the process $((S(t), I(t), R(t)), r(t))$ that for $d := P(K, z, 1, B \times \{\hat{p}_1\}) > 0$,
\[
P(\tilde{E}) \geq d + (1 - d) d + (1 - d)^2 d + \cdots
\geq \frac{d}{1 - (1 - d)} = 1,
\]

where $\tilde{E}$ represents the event that the process $((S(t), I(t), R(t)), r(t))$ will enter $B \times \{\hat{p}_1\}$ at some finite moment. Hence, if $m(B \times \{\hat{p}_1\}) > 0$ with $B \subset \overline{U}$, then $P(\tilde{E}) = 1$, which implies that the condition (a) in the definition of Harris recurrent is satisfied with the measure $\phi_1(A) = m(A \cap (B \times \{\hat{p}_1\}))$ for any $A \in \mathcal{B}(X)$. Note that from Theorem 8, we have got that the process $((S(t), I(t), R(t)), r(t))$ has an invariant probability measure $\nu^*$ on the state space $X$. Consequently, the process $((S(t), I(t), R(t)), r(t))$ is positive Harris recurrent.

**Step 2.** Next, we shall show that a lattice distribution $\mathbf{a}$ exists such that the corresponding kernel $\mathbf{K}_a$ admits an everywhere non-trivial continuous component, which implies that
the stochastic process \( ((S(t), I(t), R(t)), r(t)) \) is a \( \mathcal{T} \)-process. Combining with the result in Step 1, it then follows from Proposition 13 that for any initial value \( x_0 \in X \), the assertions \( 2.8 \) and \( 2.9 \) are valid. Let us first consider two cases as follows.

Case 1. \( r(0) = 1 \in \mathcal{M} \). For each \( k \in \mathbb{N} \), it is obtained from Lemma II and the globally asymptotic stability of the equilibrium \( E_1^* \) of system \( 1.2 \) in the state 1 that there exists some \( n_k^* \in \mathbb{N} \) such that if \( n \geq n_k^* \), \( \pi_{nK}^1(z) \in B(E_1^*, \varepsilon) \) for all \( z \in [k^{-1}, k] \cap \mathcal{K} \).

Case 2. \( r(0) = e \in \mathcal{M} \) but \( e \neq 1 \). Due to the connectivity between distinct states of the Markov chain \( r(t) \), we can get that a positive integer \( l_e \) \( (1 \leq l_e \leq E - 1) \) exists such that

\[
q_{h_0^e, h_1^e, h_2^e, \ldots, h_{l_e-1}^e, h_{l_e}^e} > 0,
\]

where \( \{h_k^e : 0 \leq k \leq l_e\} \subset \mathcal{M} \) and \( h_0^e = e, h_{l_e}^e = 1 \). From Lemma II and the globally asymptotic stability of the equilibrium \( E_1^* \), it then follows that for each \( k \in \mathbb{N} \), some \( n_k^e \in \mathbb{N} \) exists such that if \( n \geq n_k^e \),

\[
\pi_{nK-(s_0 + \ldots + s_{l_e-1})}^h \circ \pi_{s_{l_e-1}}^h \circ \cdots \circ \pi_{s_0}^h(z) \in B(E_1^*, \varepsilon)
\]

for all \( z \in [k^{-1}, k] \cap \mathcal{K} \) and \( (s_0, \ldots, s_{l_e-1}) \in [0, K/l_e]^l_e \).

Let us select a sequence of strictly increasing \( \{n_k\}_{k=1}^{\infty} \) with \( n_k = \max_{e \in \mathcal{M}} \{n_k^e\} \). Denote by \( \sigma_{h_0^e}, \ldots, \sigma_{h_{l_e}^e} \) the sojourn times of the process \( r(t) \) in the states \( h_0^e, \ldots, h_{l_e}^e \), respectively. For each \( k \in \mathbb{N} \), we then have:

1. if \( r(0) = 1 \),
   \[
   p_k^1 = \mathbb{P}\left\{ \tilde{\sigma} > n_kK \mid r(0) = 1 \right\} > 0,
   \]
   where \( \tilde{\sigma} \) denotes the sojourn time of the process \( r(t) \) in the states 1;

2. if \( r(0) = e \in \mathcal{M} \setminus \{1\} \),
   \[
   p_k^e = \mathbb{P}\left\{ \left( \sigma_{h_0^e}, \ldots, \sigma_{h_{l_e}^e} \right) \in [0, K/l_e]^l_e, \sigma_{h_0^e} + \cdots + \sigma_{h_{l_e}^e} > n_kK \mid r(0) = e \right\} > 0.
   \]

Letting \( p_k = \min_{e \in \mathcal{M}} \{p_k^e\} \) for each \( k \in \mathbb{N} \), we can get

\[
\mathbb{P}\left\{ n_kK, z, e, B(E_1^*, \varepsilon) \times \{1\} \right\} \geq p_k
\]

for any \( z \in [k^{-1}, k] \cap \mathcal{K} \) and \( e \in \mathcal{M} \). By the Kolmogorov-Chapman equation, this, together with \( 3.43 \), implies that

\[
\mathbb{P}\left\{ (n_k + 1)K, z, e, B \times \{1\} \right\} \geq p_k d_4 \cdot m(B \times \{1\})
\]

for any \( z \in [k^{-1}, k] \cap \mathcal{K} \), \( e \in \mathcal{M} \) and any Borel set \( B \subset \overline{U} \), where we here use the fact that in the proof of Step 1 in this theorem, we can assume that the state \( \hat{p}_i = 1 \) without loss of generality. As in the proof of Theorem 4.2 in \( 12 \), we construct a lattice distribution \( \alpha(nK) = 2^{-n}, n \in \mathbb{N} \), and the corresponding Markov transition function \( K_n(z, e, A) = \sum_{n=1}^{\infty} 2^{-n} \mathbb{P}(nK, z, e, A) \) for any \( (z, e) \in X \) and \( A \in \mathcal{B}(X) \). Moreover, the kernel \( K_n \) has an everywhere non-trivial continuous component \( \mathcal{T} : X \times \mathcal{B}(X) \to [0, +\infty) \) defined by

\[
\mathcal{T}(z, e, A) = 2^{-(n_{k+1} + 1)} p_{k+1} d_4 \cdot m\left(A \cap \overline{U} \times \{1\}\right)
\]

36
when \( z \in \left( (k+1)^{-1}, k+1 \right)^3 \setminus (k^{-1}, k)^3 \cap \mathcal{K}, k \in \mathbb{N} \). Thus, the process \( ((S(t), I(t), R(t)), r(t)) \) is a \( \mathcal{T} \)-process.

**Step 3.** We now shall that the stationary distribution \( \nu^* \) of the process \( ((S(t), I(t), R(t)), r(t)) \) has a density \( f^* \) with respect to the product measure \( m \) on \( X \) and \( \text{supp}(f^*) = \Gamma \times \mathcal{M} \). Since the argument is similar to that of \([11]\), we here only sketch the proof to point out the difference with it.

Noting that \( \Gamma \) is a positive invariant set of system \((1.2)\), it follows from \((b)\) of Theorem \([7]\) that \( \lim_{t \to +\infty} \mathbb{P}\{z(t) \in \Gamma\} = 1 \), which implies that \( \nu^*(\Gamma \times \mathcal{M}) = 1 \). Applying the Lebesgue decomposition theorem, there exist unique probability measures \( \nu^a_\kappa, \nu^s_\kappa \) and \( \kappa \in [0, 1] \) such that \( \nu^* = (1 - \kappa)\nu^a_\kappa + \kappa\nu^s_\kappa \), where \( \nu^a_\kappa \) is absolutely continuous with respect to \( m \) and \( \nu^s_\kappa \) is singular. It is easy to see that if \( \kappa = 0 \), then \( \nu^* = \nu^*_a \), i.e., \( \nu^* \) is absolutely continuous with respect to \( m \). Suppose \( \kappa \neq 0 \). By the similar arguments in Proposition 3.1 of \([11]\), one can get that \( \nu^*_s \) is also a stationary distribution, and a measurable subset \( \Gamma_0 \subset \Gamma \) exists with \( \tilde{m}(\Gamma_0) = 0 \) and \( \nu^*_s(\Gamma_0 \times \mathcal{M}) = 1 \). From the proof in \((b)\) of Theorem \([7]\) and Step 1 of the proof in this theorem, it is obtained by the continuous dependence of the solutions on the time and initial conditions that for sufficiently small \( \varepsilon, b \) in the Step 1, a open set \( \overline{U} \subset \Gamma \) exists such that

\[
\mathbb{P}\left( K, z, 1, B \times \{ \tilde{p}_i \} \right) \geq d_4 \cdot m(\overline{U} \times \{ \tilde{p}_i \})
\]

for any Borel set \( B \subset \overline{U} \), where \( z \in B(E_1^1, \varepsilon) \). Noting that \( m((A \setminus C) \times \overline{\mathcal{M}}) = m(A \times \overline{\mathcal{M}}) \) for any \( \overline{\mathcal{M}} \subset \mathcal{M} \) and any Lebesgue measurable subsets \( A, C \) of \( \mathcal{K} \) with \( \tilde{m}(C) = 0 \), we then have

\[
\mathbb{P}\left( K, z, 1, (\Gamma \setminus \Gamma_0) \times \mathcal{M} \right) \geq \mathbb{P}\left( K, z, 1, (\overline{U} \setminus \Gamma_0) \times \mathcal{M} \right) \\
\geq \mathbb{P}\left( K, z, 1, (\overline{U} \setminus \Gamma_0) \times \{ \tilde{p}_i \} \right) \\
\geq d_4 \cdot m((\overline{U} \setminus \Gamma_0) \times \{ \tilde{p}_i \}) \\
= d_4 \cdot m(\overline{U} \times \{ \tilde{p}_i \}) > 0.
\]

Through the similar arguments in Proposition 3.1 in \([11]\), it follows that \( \nu^*_s \mathbb{P}^K((\Gamma \setminus \Gamma_0) \times \mathcal{M}) > 0 \), which is contradic with

\[
\nu^*_s \mathbb{P}^K((\Gamma \setminus \Gamma_0) \times \mathcal{M}) = \nu^*_s((\Gamma \setminus \Gamma_0) \times \mathcal{M}) = 0
\]

due to the stationary distribution property. Thus, \( \kappa = 0 \), which implies that \( \nu^* \) is absolutely continuous with respect to \( m \) with the density \( f^* \).

Finally, by the similar arguments of Proposition 3.1 in \([11]\), one can also conclude that \( \int_V f^* dm > 0 \) for all open set \( V \subset \Gamma \times \mathcal{M} \), which implies that \( \text{supp}(f^*) = \Gamma \times \mathcal{M} \). This completes the proof of Theorem \([9]\). \( \square \)

### 4. Discussion

In this paper, we have established the threshold dynamics of the disease extinction and persistence for the system \((1.2)\). That is, the disease can be eradicated almost surely if \( R_0 < 1 \), while the disease persists almost surely if \( R_0 > 1 \). Moreover, we also have given the ergodic analyzing of this epidemic model. The global attractivity of the \( \Omega \)-limit set of the
system and the convergence in total variation of the instantaneous measure to the stationary measure were obtained under the weakened conditions. From two environmental regimes to any finite ones, the method presented in this paper is a generalization of the techniques used by [11, 12] for the ergodic analyzing of the piecewise deterministic Markov process. For analyzing the ergodicity of the considered system, the generalized method requires weaker conditions and is applicable to the multi-dimension system with any number of environmental regimes.

Note also that only the transmission rate $\beta$ of model (1.2) is disturbed because in reality it is more sensitive to environmental fluctuations than other parameters for human populations. Nevertheless, it is easy to see that the method used in this paper can be easily extended to the case where other parameters of model (1.2), such as the recruitment rate $\Lambda$ and the natural mortality $\mu$, can also change with the switching of environmental regimes, which may be more reasonable for the wildlife population. In addition, by this new method, under weaker conditions, we can directly extend the results in [11, 12, 15] from two environmental states to any finite ones.

However, in the case when $R_0 > 1$, the extra condition that some point in the $\Omega$-limit subset $\Gamma$ satisfying the condition (H) exists is required for ensuring the global attractivity of the $\Omega$-limit set of system (1.2) and the convergence in total variation of the instantaneous measure to the stationary measure. From a biological point of view, what is the biological meaning of this condition? To obtain the ergodicity of system (1.2), is it enough that only the condition that $R_0 > 1$ is satisfied? We leave these questions for future work.

Acknowledgments

The authors would extend their thanks to Professors Jianhua Huang and Weiming Wang for their valuable comments. D. Li and S. Liu are supported by the National Natural Science Foundation of China (No.11471089). J. Cui is supported by the National Natural Science Foundation of China (No.11371048).

References

[1] W. O. Kermack, A. G. McKendrick, A Contribution to the mathematical theory of epidemics, Proc. Roy. Soc. Lond. Ser. A 115 (1927) 700–721.

[2] A. V. Arundel, E. M. Sterling, J. H. Biggin, T. D. Sterling, Indirect health effects of relative humidity in indoor environments, Environ. Health Perspect. 65 (1986) 351–361.

[3] S. M. Minhaz Ud-Dean, Structural explanation for the effect of humidity on persistence of airborne virus: Seasonality of influenza, J. Theor. Biol. 264 (2010) 822–829.

[4] M. J. Keeling, P. Rohani, Modeling Infectious Diseases in Human and Animals, Princeton University Press, New Jersey, 2008.

[5] N. Dexter, Stochastic models of foot and mouth disease in feral pigs in the Australian semi-arid rangelands, J. Appl. Ecol. 40 (2003) 293–306.

[6] M. Liu, K. Wang, Persistence and extinction of a stochastic single-specie model under regime switching in a polluted environment, J. Theor. Biol. 264 (2010) 934–944.

[7] A. Gray, D. Greenhalgh, X. Mao, J. Pan, The SIS epidemic model with Markovian switching, J. Math. Anal. Appl. 394 (2012) 496–516.
[8] J. Bao, J. Shao, Permanence and extinction of regime-switching predator-prey models, SIAM J. Math. Anal. 48 (2016) 725–739.

[9] L. Zu, D. Jiang, D. O’Regan, Conditions for persistence and ergodicity of a stochastic Lotka-Volterra predator-prey model with regime switching, Commun. Nonlinear Sci. Numer. Simulat. 29 (2015) 1–11.

[10] X. Zhang, D. Jiang, A. Alsaeedi, T. Hayat, Stationary distribution of stochastic SIS epidemic model with vaccination under regime switching, Appl. Math. Lett. 59 (2016) 87–93.

[11] N. H. Du, N. H. Dang, Dynamics of Kolmogorov systems of competitive type under the telegraph noise, J. Differential Equations 250 (2011) 386–409.

[12] N. H. Dang, N. H. Du, G. Yin, Existence of stationary distributions for Kolmogorov systems of competitive type under telegraph noise, J. Differential Equations 257 (2014) 2078–2101.

[13] N. Bacaër, M. Khaladi, On the basic reproduction number in a random environment, J. Math. Biol. 67 (2013) 1729–1739.

[14] N. Bacaër, A. Ed-Darraz, On linear birth-and-death processes in a random environment, J. Math. Biol. 69 (2014) 73–90.

[15] N. T. Hieu, N. H. Du, P. Auger, N. H. Dang, Dynamical behavior of a stochastic SIRS epidemic model, Math. Model. Nat. Pheno. 10 (2015) 56–73.

[16] D. Greenhalgh, Y. Liang, X. Mao, Modelling the effect of telegraph noise in the SIRS epidemic model using Markovian switching, Physica A 462 (2016) 684–704.

[17] R. M. Anderson, R. M. May, Population biology of infectious diseases: Part I, Nature 280 (1979) 361–367.

[18] V. Capasso, Mathematical Structure of Epidemic Systems, in: Lecture Notes in Biomathematics, vol. 97, Springer, Berlin, 1993.

[19] S. A. Levin, T. G. Hallam, L. J. Gross, Applied Mathematical Ecology, Springer, New York, 1989.

[20] Y. Xia, J. R. Gog, B. T. Grenfell, Semiparametric estimation of the duration of immunity from infectious disease time series: influenza as a case-study, J. Roy. Stat. Soc. Ser. C 54 (2005) 659–672.

[21] X. Mao, Stability of stochastic differential equations with Markovian switching, Stochastic Process. Appl. 79 (1999) 45–67.

[22] S. P. Meyn, R. L. Tweedie, Stability of Markovian processes I: criteria for discrete-time chains, Adv. Appl. Prob. 24 (1992) 542–574.

[23] S. P. Meyn, R. L. Tweedie, Stability of Markovian processes II: Continuous-time processes and sampled chains, Adv. Appl. Prob. 25 (1993) 487–517.

[24] S. P. Meyn, R. L. Tweedie, Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes, Adv. Appl. Prob. 25 (1993) 518–548.
[25] S. P. Meyn, R. L. Tweedie, Markov Chains and Stochastic Stability, Second Edition, Cambridge University Press, New York, 2009.

[26] V. Jurdjevic, Geometric Control Theory, Cambridge Stud. Adv. Math., vol.52, Cambridge University Press, 1997.

[27] V. Capasso, G. Serio, A generalization of the Kermack-McKendrick deterministic epidemic model, Math. Biosci. 42 (1978) 43–61.

[28] D. Xiao, S. Ruan, Global analysis of an epidemic model with nonmonotone incidence rate, Math. Biosci. 208 (2007) 419–429.

[29] J. Cui, Y. Sun, H. Zhu, The impact of media on the control of infectious diseases, J. Dyn. Differ. Equ., 20 (2008) 31–53.

[30] J. Cui, X. Tao, H. Zhu, An SIS infection model incorporating media coverage, Rocky. Mt. J. Math. 38 (2008) 1323–1334.

[31] D. Li, J. Cui, M. Liu, S. Liu, The evolutionary dynamics of stochastic epidemic model with nonlinear incidence rate, Bull. Math. Biol. 77 (2015) 1705–1743.

[32] F. Brauer, P. van den Driessche, J. Wu, Mathematical epidemiology, Springer-Verlag, Berlin, 2008.

[33] R. Rudnicki, K. Pichór, M. Tyran-Kamińska, Markov semigroups and their applications, In: P. Garbaczewski, R. Olkiewicz (Eds.), Lecture Notes in Physics: Dynamics of Dissipation, Springer-Verlag, Berlin, 2002, pp. 215–238.

[34] Y. Cai, Y. Kang, M. Banerjee, W. Wang, A stochastic SIRS epidemic model with infectious force under intervention strategies, J. Differential Equations 259 (2015) 7463–7502.

[35] A. N. Shiryaev, Probability, Translated by R. P. Boas, Springer-Verlag, New York, 1984.

[36] L. Stettner, On the existence and uniqueness of invariant measure for continuous time Markov processes, LCDS Report No. 86-18, Brown University, Providence, April 1986.

[37] M. Sharpe, General Theory of Markov Processes, Academic Press, New York, 1988.