Geometric Background for Thermal Field Theories

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We study a new spacetime which is shown to be the general geometrical background for Thermal Field Theories at equilibrium. The different formalisms of Thermal Field Theory are unified in a simple way in this spacetime. The set of time-paths used in the Path Ordered Method is interpreted in geometrical terms.

I. INTRODUCTION

Thermal Field Theory (TFT) [1,3] is the generic name denoting Quantum Field Theories (QFT) at finite temperature, such as the Matsubara or Imaginary Time (IT) formalism [1,3], Thermo Field Dynamics (TFD) [1,4], and the Path Ordered Method (POM), which comprises the Closed Time Path (CTP) formalism [1,3] as a particular case. These distinct approaches result from different efforts to introduce a temperature within the framework of quantum field theory.

The IT formalism exploits the analogy of the temperature with imaginary time in calculating the partition function, and within this formalism the two-point Green’s function is given by the Matsubara propagator. In contrast, the POM formalism and TFD both deal with real time. These two last approaches lead to the same matrix structure for the propagator in equilibrium, although their founding ideas are different. In the POM formalism, the temperature is introduced by taking a pure imaginary number to the real time and by choosing a special path in the complex-time plane. The propagators are calculated by taking functional derivatives of a path integral over fields, the so-called finite-temperature generating functional. On the other hand, in TFD, the field algebra is doubled and the temperature is contained explicitly in the resulting “vacuum” state, which is then referred to as the “thermal vacuum”. The propagators are expressed in this last formalism as expectation values of time-ordered products of quantum fields with respect to the thermal vacuum.

An important and interesting question is whether such different formalisms have some roots in common and if their features can be understood in a deeper way so that they appear unified. A clue to an answer may be found in the well-known discovery of Hawking [5] that temperature arises in a quantum theory as a result of a non-trivial background endowed with event-horizon(s), in this case a black-hole spacetime. Rindler spacetime, the spacetime of an accelerated observer, was also shown later to exhibit thermal features [6,7]. There is also a flat back-ground with a non-trivial structure which exhibits thermal features, the so-called $\eta$-$\xi$ spacetime $\mathbb{S}^1 \times \mathbb{R}^3$. Its Lorentzian section is made up of four copies of Minkowski spacetime glued together along some of their past or future null hyperplanes infinities. In Kruskal-like coordinates, which cover the entire $\eta$-$\xi$ spacetime, the metric is singular on these hyperplanes. For this reason they are formally called “event-horizons”. Their existence leads to the doubling of the degrees of freedom of the fields. The vacuum propagator on the Lorentzian section corresponds to the real-time thermal matrix propagator. In the Euclidean section of Kruskal coordinates, the time coordinate is periodic, so that the propagator is equal to the Matsubara propagator there.

In real-time TFTs, there is a freedom in the parameterization of the thermal matrix propagator. In the POM formalism, this parameterization is related to the choice of the path in the complex-time plane going from $t = 0$ to $t = -i\beta$ which is not unique [2] (see Fig. 1). In TFD, different parameterizations of the Bogoliubov thermal matrix are possible. It is important to stress that although the choice of the parameterization is irrelevant in the thermal equilibrium case since it does not affect physical quantities in real-time TFTs, it nevertheless plays a role in the non-equilibrium case, where the choice of a closed time path is the only useful one [4,6].

In this paper we show the relevance of other complex sections of $\eta$-$\xi$ spacetime than the Lorentzian and the Euclidean ones. All possible time paths used in the POM formalism, including the CTP formalism, and the different parameterizations of the Bogoliubov matrices of TFD are interpreted at a purely geometric level in these complex sections. The geometric picture for TFTs is consequently enlarged. We also show how the different for-
nalisms of TFT unify themselves in a very natural way in this framework. The generalization introduced here could be useful in order to extend the geometrical picture of \(\eta, \xi, y, z\) in this section is given from Eq. (1) by

\[ \eta \equiv \text{Euclidean and Lorentzian sections, and their complex extensions. In Section III, we consider explicitly a quantum boson field in } \eta, \xi, y, z \text{ spacetime. Section IV is devoted to the study of the relationships between } \eta, \xi \text{ spacetime and TFTs, and to the unification of TFTs in the framework of } \eta, \xi \text{ spacetime. Section V contains a discussion of some other features of the extended Lorentzian section. Finally, Section VI is devoted to conclusions.} \\

II. \(\eta, \xi\) SPACETIME

\(\eta, \xi\) spacetime \[3\] is a four-dimensional complex manifold defined by the line element

\[ ds^2 = \frac{-dt^2 + d\xi^2}{\alpha^2 (\xi^2 - \eta^2)} + dy^2 + dz^2, \tag{1} \]

where \(\alpha \equiv 2\pi/\beta\) is a real constant and \((\eta, \xi, y, z) \in \mathbb{C}\). We shall use the symbol \(\xi^\mu \equiv (\eta, \xi, y, z)\) to denote as a whole the set of \(\eta, \xi\) coordinates, but for simplicity we shall actually drop the index \(\mu\) when no confusion with the space-like coordinate is possible.

A. Euclidean section

The Euclidean section of \(\eta, \xi\) spacetime is obtained by assuming that \((\sigma, \xi, y, z) \in \mathbb{R}\) where \(\eta \equiv i\sigma\). The metric in this section is given from Eq. (1) by

\[ ds^2 = \frac{d\sigma^2 + d\xi^2}{\alpha^2 (\sigma^2 + \xi^2)} + dy^2 + dz^2. \tag{2} \]

By use of the transformation

\[ \begin{align*}
\sigma &= (1/\alpha) \exp(\alpha x) \sin(\alpha \tau), \\
\xi &= (1/\alpha) \exp(\alpha x) \cos(\alpha \tau),
\end{align*} \tag{3} \]

the metric becomes that of the cylindrical Euclidean flat spacetime,

\[ ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2, \tag{4} \]

where the time \(\tau\) has a periodic structure, i.e. \(\tau \equiv \tau + \beta\).

B. Lorentzian section

In the Lorentzian section the metric is given by Eq. (4), where \((\eta, \xi, y, z) \in \mathbb{R}\). This metric is singular on the two hyperplanes \(\eta = \pm \xi\) which shall be called the “event-horizons”. They divide \(\eta, \xi\) spacetime into four regions denoted by \(R_I, R_{II}, R_{III}\) and \(R_{IV}\) (see Fig. 2).

In the first two regions, one defines two sets of tortoise-like coordinates \(x_{I,II} \in \mathbb{R}^4\) by \(x_{I,II} = (t_{I,II}, x_{I,II}, y, z)\) where

\[ \begin{align*}
\eta &= + (1/\alpha) \exp(\alpha x) \sin(\alpha t_I), \\
\xi &= + (1/\alpha) \exp(\alpha x) \cosh(\alpha t_I),
\end{align*} \tag{5} \]

\[ \begin{align*}
\eta &= - (1/\alpha) \exp(\alpha x) \sin(\alpha t_{II}), \\
\xi &= - (1/\alpha) \exp(\alpha x) \cosh(\alpha t_{II}).
\end{align*} \tag{6} \]

Similar transformations can be defined to cover \(R_{III}\) and \(R_{IV}\) (see Refs. [8,11]) but it shall be sufficient for our purposes to consider only the first two regions. The metric given in Eq. (4) becomes the Minkowski metric in these regions and in the new coordinates,

\[ ds^2 = -dt_{I,II}^2 + dx_{I,II}^2 + dy^2 + dz^2. \tag{7} \]

Regions I to IV are copies of the Minkowski spacetime glued together along the “event-horizons” making up the Lorentzian section of \(\eta, \xi\) spacetime.

Although Eqs. (5) and (6) are formally Rindler transformations \[4\], the \((t_{I,II}, x_{I,II}, y, z)\) coordinates should not be confused with Rindler coordinates since in those coordinates the metric is the Minkowski one. An observer whose motion is described for example by \((x_i(t_i), y_i(t_i), z(t_i)) = (x_0, y_0, z_0)\) is subjected to an inertial motion, and thus cannot be identified with an accelerated observer as in the Rindler case. In Eqs. (5) and (6), the role of the inertial and non-inertial coordinates are actually reversed with respect to the Rindler case \[3\].

We now study the analytical properties of the transformations given in Eqs. (5) and (6). If \(\xi^\pm = \eta \pm \xi\), we rewrite them in the form

\[ \begin{align*}
\xi^+ (t_I, x_I) &= + (1/\alpha) \exp[+\alpha (t_I + x_I)], \\
\xi^- (t_I, x_I) &= - (1/\alpha) \exp[-\alpha (t_I - x_I)],
\end{align*} \tag{8} \]

\[ \begin{align*}
\xi^+ (t_{II}, x_{II}) &= - (1/\alpha) \exp[+\alpha (t_{II} + x_{II})], \\
\xi^- (t_{II}, x_{II}) &= + (1/\alpha) \exp[-\alpha (t_{II} - x_{II})],
\end{align*} \tag{9} \]

where we have added the subscripts < and > to the variables \(\xi^\pm\) to indicates their ranges, i.e. one has \(\xi^+_I > 0\) and \(\xi^-_I < 0\). The reciprocals of these transformations are...
Equations (10) and (11) are defined in regions $R_i$ and $R_μ$ respectively. We now would like to extend analytically these expressions to obtain the functions $t_i(ξ, x_i) , t_μ(ξ)$ and $x_i(ξ), x_μ(ξ)$ defined in $R_i \cup R_μ$. This amounts to extend these expressions from positive or negative values of $ξ$ to their negative or positive values respectively. In order to do this, we choose to analytically extend the expressions above in the lower half-planes of both the $ξ^+$ and $ξ^-$ complex planes for reasons which will become clear below. In other words we assume that $−\pi ≤ arg ξ^± < \pi$, or equivalently that the cuts in the $ξ^\pm$ complex planes are given by $R_{−+} + i 0^+$.

It is not possible to perform the analytic extensions with respect to the two variables $ξ^\pm$ at once, otherwise an erroneous result would be obtained. To fix the ideas, we choose to perform the extension first in the $ξ^+$ variable and then in $ξ^−$ (the choice of the opposite order gives the same result for our particular purposes). If $ξ_±$ is the analytic continuation of $ξ^±$ from negative to positive values, one has

$$\ln (−ξ^\pm) = \ln (+ξ^\pm) + i \pi,$$

$$\ln (+ξ^\pm) = \ln (−ξ^\pm) − i \pi.$$  \hspace{1cm} (12)

This implies

$$\ln \left(\frac{ξ^+}{ξ^-}\right) = \ln \left(\frac{−ξ^−}{ξ^+}\right) + i 2 \pi,$$

$$\ln \left(−α^2ξ^±ξ^-\right) = \ln \left(−α^2ξ^±ξ^+\right),$$ \hspace{1cm} (15)

which means that these expressions are the analytic continuations of each others. In consequence, by using the variable $ξ$ one obtains in $R_i \cup R_μ$,

$$\left\{ \begin{array}{l}
  t_μ(ξ) = t_i(ξ) + i \beta/2, \\
  x_μ(ξ) = x_i(ξ).
\end{array} \right.$$ \hspace{1cm} (16)

C. Extended Lorentzian section

We now consider a class of complex sections of $η,ξ$ spacetime obtained from the Lorentzian section by shifting the Minkowski time coordinates in the imaginary direction only in region $R_μ$.

$$\left\{ \begin{array}{l}
  t_μ(ξ) = t_i(ξ) + i \beta/2, \\
  x_μ(ξ) = x_i(ξ).
\end{array} \right.$$ \hspace{1cm} (16)

$$\left\{ \begin{array}{l}
  t_μ(ξ) = t_i(ξ) + i \beta, \\
  x_μ(ξ) = x_i(ξ).
\end{array} \right.$$  \hspace{1cm} (17)

In terms of the real $η,ξ$ variables, we have

$$\left\{ \begin{array}{l}
  η_s = −(1/α) \exp (αx_I) \sinh \left[ a(η_I + i β) \right], \\
  ξ_s = −(1/α) \exp (αx_II) \cosh \left[ a(η_I + i β) \right].
\end{array} \right.$$ \hspace{1cm} (18)

The time shift induces thus a rotation in the $(η, iξ)$ plane of $R_μ$. The metric becomes in terms of the rotated coordinates

$$ds^2 = \frac{−dη^2 + dξ^2}{α^2(ξ^2 − η^2)} + dy^2 + dz^2,$$ \hspace{1cm} (21)

and is thus unchanged by the time shift which is therefore an isometry of the four-dimensional complex $η,ξ$ spacetime. The equations (14) become after the time shift

$$\left\{ \begin{array}{l}
  t_μ(ξ) = t_i(ξ) + i \beta (1/2 + s), \\
  x_μ(ξ) = x_i(ξ).
\end{array} \right.$$ \hspace{1cm} (22)

III. FIELDS IN $η,ξ$ SPACETIME

We now consider a free scalar field in the $η,ξ$ spacetime (for the case of a fermion field see Ref. [13]). The “global” scalar field in $η,ξ$ coordinates shall be denoted by $Φ(ξ)$. It satisfies to the Klein-Gordon equation

$$ (□ + m^2)Φ(ξ) = 0,$$ \hspace{1cm} (23)

where $□ = ∇_μ ∇^μ$ if $∇_μ$ is the covariant derivative. The scalar product of two fields is given by

$$⟨Φ_1, Φ_2⟩ = −i \int dΣ ||g(ξ)||^{1/2} Φ_1(ξ) n^ρ \Phi_2^*(ξ),$$ \hspace{1cm} (24)

where $g$ is the determinant of the metric $g_{μν}$, $Σ$ is any space-like surface and $n^ρ$ an orthonormal vector to this surface.
A. Euclidean section

In the Euclidean section we have $\Phi = \Phi(\sigma, \xi, y, z)$, and the field in the $t$-$x$ coordinates defined in Eq. (3) shall be denoted by $\phi = \phi(\tau, x, y, z)$. These two fields are related by

$$\phi(\tau, x, y, z) = \Phi(\sigma(\tau, x), \xi(\tau, x), y, z).$$  

(25)

Because of the periodic nature of the time $\tau$ and by imposing single valuedness we have necessarily

$$\phi(\tau, x, y, z) = \phi(\tau + \beta, x, y, z).$$  

(26)

B. Lorentzian section

In the Lorentzian section, as we have seen, we have four different regions, each of them being a complete Minkowski spacetime. Since we are interested only in regions $R_i$ and $R_\eta$, we shall consider the quantum field over these two regions only. Our aim is to find an expansion for the global field $\Phi$ in the joining $R_i \cup R_\eta$.

We start by defining the “local” fields $\phi^i(x, \xi)$ and $\phi^\mu(x, \eta)$ by

$$\Phi(\xi) = \begin{cases} 
\phi^i(x, \xi), & \text{when } \xi \in R_i, \\
\phi^\mu(x, \eta), & \text{when } \xi \in R_\eta.
\end{cases}$$  

(27)

They have support in $R_i$ and $R_\eta$ respectively. By choosing the particular surface $\eta = a\xi$ where $a$ is a constant satisfying $-1 < a < 1$, one shows from Eq. (24) that the global scalar product is given by

$$(\Phi_1, \Phi_2) = <\phi_1, \phi_2^i > + <\phi_1^\mu, \phi_2^\mu >,$$  

(28)

where $<$, $>$ is the local scalar product in Minkowski spacetime

$$<\phi_1, \phi_2^i > = -i \int_{\mathbb{R}^3} d^3x \phi_1(x) \bar{\phi}_2(x).$$  

(29)

In $\eta$-$\xi$ spacetime covered by $t$-$x$ coordinates given in Eqs. (3) and (3), the mode solutions of the Klein-Gordon equation are just local plane waves restricted to a given region. They are given by

$$u_k(x_i) = (4\pi\omega_k)^{-\frac{3}{2}} e^{i(\omega_k t_i + k \cdot x_i)},$$  

(30)

$$v_k(x_\eta) = (4\pi\omega_k)^{-\frac{3}{2}} e^{i(\omega_k t_\eta + k \cdot x_\eta)},$$  

(31)

where $\omega_k = \sqrt{k^2 + m^2}$. From these Minkowski modes, one defines the two wave functions $U_k(\xi)$ and $V_k(\xi)$ with support in $R_i$ and $R_\eta$ respectively by

$$U_k(\xi) = \begin{cases} 
u_k(x_i(\xi)), & \text{when } \xi \in R_i, \\
0, & \text{when } \xi \in R_\eta.
\end{cases}$$  

(32)

$$V_k(\xi) = \begin{cases} 0, & \text{when } \xi \in R_i, \\
u_k(x_\eta(\xi)), & \text{when } \xi \in R_\eta.
\end{cases}$$  

(33)

Their power spectrum with respect to the momenta conjugated to $\xi^+$ and $\xi^-$ contains negative contributions, which are furthermore not bounded by below. Consequently, the sets of functions $\{U_k(\xi), U^*_{-k}(\xi)\}_{k \in \mathbb{R}^3}$ and $\{V_k(\xi), V^*_{-k}(\xi)\}_{k \in \mathbb{R}^3}$ defined on $R_i$ and $R_\eta$ respectively are both over-complete since the same energy contribution (i.e. momentum contribution conjugate to $\eta$) can appear twice in these sets. In other words, the energy spectrum of $U_k$ and $U^*_{-k}$ overlap, and so do the ones of $V_k$ and $V^*_{-k}$. These sets can thus not be used as a basis in their respective regions, and the joining of these sets is clearly not a basis in $R_i \cup R_\eta$.

To construct a basis in $R_i \cup R_\eta$, we could solve the Klein-Gordon equations in $\eta$-$\xi$ coordinates to obtain the field modes in these coordinates. However, the Bogoliubov transformations resulting from this basis choice are rather complicated. So instead of doing this, we shall construct from the wave functions $u_k(x_i(\xi))$ and $v^*_{-k}(x_\eta(\xi))$ basis elements having positive energy spectrum.

We shall demand these basis elements to be analytical functions in the lower complex planes of $\xi^+$ and $\xi^-$, so that their spectrum contain only positive contributions of the momenta conjugate with respect to $\xi^+$ and $\xi^-$. In consequence, they shall have positive energy spectra.

We thus extend analytically the two wave functions $u_k(x_i(\xi))$ and $v^*_{-k}(x_\eta(\xi))$ in the lower complex planes of $\xi^+$ and $\xi^-$ (as in Section 11B, the cut in the complex planes is given by $\Re > 0^+$). By applying formula (14) we get directly

$$u_k(x_i(\xi)) = e^{-\frac{\beta}{2} + i k \cdot x_i} u^*_{-k}(x_\eta(\xi)),$$  

(34)

$$v_k(x_\eta(\xi)) = e^{-\frac{\beta}{2} - i k \cdot x_\eta} u^*_{-k}(x_i(\xi)).$$  

(35)

The expressions on the right and left hand sides of these last equations are analytic continuations of each others. In this way we are led to introduce the two normalized linear combinations

$$\left\{ \begin{array}{ll}
F_k(\xi) = (1 - f_k)^{-\frac{1}{2}} \left[ U_k(\xi) + f_k^{\frac{1}{2}} V^*_{-k}(\xi) \right], \\
\bar{F}_k(\xi) = (1 - f_k)^{-\frac{1}{2}} \left[ V_k(\xi) + f_k^{\frac{1}{2}} U^*_{-k}(\xi) \right],
\end{array} \right.$$  

(36)

where $f_k = e^{-\beta \omega_k}$, and where $U_k(\xi)$ and $V_k(\xi)$ are defined in Eqs. (32) and (33). These wave functions are still solutions of the Klein-Gordon equation. They are analytical in $R_i \cup R_\eta$ and in particular at the origin $\xi^+ = \xi^- = 0$. Since they are analytical complex functions in the lower complex planes of $\xi^+$ and $\xi^-$, their
spectrum has only positive energy contributions. The set \((F_k, F^*_k, \bar{F}_k, \bar{F}^*_k)\) in \(\mathbb{R}^3\) is thus complete but not over-complete over the joining \(R_t \cup R_{\delta t}\). Furthermore it is an orthogonal set since
\[
(F_k, F_p) = (\bar{F}^*_k, F_p) = +\delta^3(k-p),
\]
\[
(F^*_k, F^*_p) = (\bar{F}_k, \bar{F}_p) = -\delta^3(k-p),
\]
with all the other scalar products vanishing.

On one hand, the local scalar fields can be expanded in the Minkowski modes given in Eqs. (39) and (40),
\[
\phi^I(x_I) = \int d^3k \left[ a_k^I u_k(x_I) + a_k^I u^*_k(x_I) \right],
\]
\[
\phi^{II} (x_{II}) = \int d^3k \left[ a_k^{II} v_k(x_{II}) + a_k^{II} v^*_k(x_{II}) \right].
\]
And on the other hand, the global scalar field can be expanded in terms of the “global” modes given in Eqs. (36) as
\[
\Phi(\xi) = \int d^3k \left[ b_k F_k(\xi) + \bar{b}_k \bar{F}_k(\xi) \right] + \delta_k \bar{F}_k(\xi) + \bar{b}_k \bar{F}^*_k(\xi) \right].
\]
These three expansions define the local and global creation and annihilation operators, which are related by Bogoliubov transformations. To obtain these, we recall the definition (27) relating the local and global fields and we use the field expansions (38), (39) and (40). We obtain
\[
\begin{aligned}
  b_k &= a_k^I \cosh \theta_k - a_k^{II} \sinh \theta_k, \\
  \bar{b}_k &= a_k^I \cosh \theta_k - a_k^{II} \sinh \theta_k,
\end{aligned}
\]
where \(\sinh^2 \theta_k = n(\omega_k) = (e^{\beta \omega_k} - 1)^{-1}\).

C. Extended Lorentzian section

By following the above procedure, we shall now construct a set of positive energy modes defined in the extended Lorentzian section introduced in Section 1C. We start by considering region \(R_{\delta t}\), where the set \(\{v_k(x_{II}), v^*_k(x_{II})\}\) is a plane wave basis. One has
\[
\begin{aligned}
v_k(x_{II}) &= (4\pi \omega_k)^{-\frac{1}{2}} e^{i(\omega_k t + k \cdot x_{II})}, \\
v^*_k(x_{II}) &= (4\pi \omega_k)^{-\frac{1}{2}} e^{i(-\omega_k t + k \cdot x_{II})}.
\end{aligned}
\]
Under the time shift, Eq. (17), this set is transformed into \(\{v_k(x_{II}), v^*_k(x_{II})\}\) in \(\mathbb{R}^3\), where we have replaced the symbol * by \(\sharp\) because \(v^*_k(x_{II})\) is no longer the complex conjugate of \(v_k(x_{II})\). Indeed, one has
\[
v_k(x_{II}) = e^{-\beta \omega_k \delta} v_k(x_{II}),
\]
\[
v^*_k(x_{II}) = e^{+\beta \omega_k \delta} v^*_k(x_{II}).
\]
The complex conjugation and the time shift are not commutative. We notice that \(v^*_k(x_{II})\) can be obtained from \(v_k(x_{II})\) by complex conjugation and by the replacement \(\delta \rightarrow -\delta\). This rule actually defines the \(\sharp\)-conjugation.

In the same way as in Eq. (27) one defines
\[
\Phi(\xi) = \left\{ \begin{array}{ll}
\phi^I (x_I(\xi)), & \text{when } \xi \in R_t, \\
\phi^{II}(x_{II}(\xi)), & \text{when } \xi \in R_{\delta t}.
\end{array} \right.
\]
The global scalar product, Eq. (28), is modified and becomes in \(R_t \cup R_{\delta t}\),
\[
(\Phi_1, \Phi_2) = \langle \phi_1^I, \phi_2^I \rangle + \langle \phi_1^{II}, \phi_2^{II} \rangle.
\]
where the local Minkowski scalar product \(<, >\) in region \(R_{\delta t}\) is given by
\[
< \phi_1, \phi_2 >_{\delta t} = -i \int_{\mathbb{R}^3} d^3x_{II} \phi_1(x_{II}) \partial_t \phi_2^\sharp (x_{II}).
\]
Equations (54), (55), (44) and (45) imply
\[
\begin{aligned}
u_k(x_{II}(\xi_\delta)) &= e^{-\beta \omega_k (1/2+\delta)} v^\sharp_k(x_{II}(\xi_\delta)), \\
v_k(x_{II}(\xi_\delta)) &= e^{-\beta \omega_k (1/2-\delta)} v^\sharp_k(x_{II}(\xi_\delta)).
\end{aligned}
\]
Consequently one also has
\[
\begin{aligned}
u^\sharp_k(x_{II}(\xi_\delta)) &= e^{-\beta \omega_k (1/2+\delta)} v^\sharp_k(x_{II}(\xi_\delta)), \\
v^\sharp_k(x_{II}(\xi_\delta)) &= e^{-\beta \omega_k (1/2-\delta)} v^\sharp_k(x_{II}(\xi_\delta)).
\end{aligned}
\]
The expressions on the left and right hand sides of these equations are thus analytic continuations of each others.

If we now define
\[
\begin{aligned}
U_k(\xi_\delta) &= \left\{ \begin{array}{ll}
u_k(x_{II}(\xi_\delta)), & \text{when } \xi_\delta \in R_t, \\
0, & \text{when } \xi_\delta \in R_{\delta t},
\end{array} \right.
\end{aligned}
\]
\[
\begin{aligned}
V_k(\xi_\delta) &= \left\{ \begin{array}{ll}
u_k(x_{II}(\xi_\delta)), & \text{when } \xi_\delta \in R_t, \\
v_k(x_{II}(\xi_\delta)), & \text{when } \xi_\delta \in R_{\delta t},
\end{array} \right.
\end{aligned}
\]
the global modes in the extended Lorentzian section \(R_t \cup R_{\delta t}\) are given by
\[
\begin{aligned}
G_k(\xi_\delta) &= (1 - f_k)^{-\frac{1}{2}} \left[ U_k(\xi_\delta) + f_k^{1/2+\delta} V^\sharp_k(\xi_\delta) \right], \\
\tilde{G}_k(\xi_\delta) &= (1 - f_k)^{-\frac{1}{2}} \left[ V_k(\xi_\delta) + f_k^{1/2+\delta} U^\sharp_k(\xi_\delta) \right],
\end{aligned}
\]
where $f_k = e^{-\beta \omega_k}$. From Eqs. \([13]\) to \([21]\), we see that the global modes are analytic in $R_j \cup R_{\mu}$, in particular at the origin $\xi_k^+ = \xi_k^- = 0$. Since they are analytical complex functions on the lower complex planes of $\xi_k^+$ and $\xi_k^-$, their spectrum has only positive energy contributions. When $\delta = 0$, the above combinations consistently reduce to the expressions of Eq. \([56]\).

We emphasize that the global modes $G_k^+$ and $G_k^-$ are not analytic in the extended Lorentzian section, contrary to the non-hermitian combinations $G_k^+ \nonumber$ and $G_k^-$ in Ref. \([19]\) and \([21]\) it is shown the necessity of the so-called Osterwalder-Schrader conjugation as opposed to the hermiticity property in Euclidean field theories even when the temperature vanishes.

The set \{
$G_k^+, G_k^-, \tilde{G}_k^+, \tilde{G}_k^-$\}$_{k \in \mathbb{R}^3}$ is thus complete over $R_i \cup R_{\eta \mu}$. It is furthermore an orthogonal set since

\[
\begin{align*}
\langle G_k, G_p \rangle & = \langle \tilde{G}_k^+, \tilde{G}_p^+ \rangle = +\delta^3(k - p), \\
\langle G_k^+, G_p^+ \rangle & = \langle \tilde{G}_k^-, \tilde{G}_p^- \rangle = -\delta^3(k - p),
\end{align*}
\]

where all the other scalar products vanish.

On one hand, the expansions of the local fields in the Minkowski modes over regions $R_j$ and $R_{\mu}$ are given respectively by

\[
\phi^I(x_i) = \int d^3k \left[ a_k^I u_k(x_i) + a_k^I u_k^+(x_i) \right],
\]

\[
\phi^{\mu I}(x_{\mu}) = \int d^3k \left[ a_k^{\mu I} c_k(x_{\mu}) + a_k^{\mu I} c_k^+(x_{\mu}) \right].
\]

On the other hand, the expansion of the global field in the $G$ modes over the joining $R_i \cup R_{\eta \mu}$ is

\[
\Phi(\xi) = \int d^3k \left[ c_k G_k(\xi) + \tilde{c}_k \tilde{G}_k(\xi) \right. \\
\left. + \tilde{c}_k G_{-k}(\xi) + \tilde{c}_k \tilde{G}_{-k}(\xi) \right].
\]

From these last expansions and by using Eqs. \([53]\) and \([61]\), one finds the Bogoliubov transformations,

\[
\begin{align*}
\phi_k & = (1 - f_k)^{-\frac{1}{2}} \left( a_k^I - f_k^{\frac{1}{2}+\delta} a_k^{\mu I} \right), \\
\tilde{c}_k & = (1 - f_k)^{-\frac{1}{2}} \left( a_k^{\mu I} - f_k^{\frac{1}{2}+\delta} a_k^I \right),
\end{align*}
\]

and their $\xi$-conjugate equivalent,

\[
\begin{align*}
\phi_k & = (1 - f_k)^{-\frac{1}{2}} \left( a_k^I - f_k^{\frac{1}{2}+\delta} a_k^{\mu I} \right), \\
\tilde{c}_k & = (1 - f_k)^{-\frac{1}{2}} \left( a_k^{\mu I} - f_k^{\frac{1}{2}+\delta} a_k^I \right).
\end{align*}
\]

Again the transformations given in Eq. \([11]\) are recovered when $\delta = 0$.

### IV. THERMAL FIELD THEORIES IN $\eta$-$\xi$ SPACETIME

We now show the equivalence of Quantum Field Theory in $\eta$-$\xi$ spacetime with Thermal Field Theories by considering the scalar field. We shall see that in several sections of $\eta$-$\xi$ spacetime QFT naturally reproduces the known formalisms of TFTs as insofar as the correct thermal Green functions are recovered there.

On one hand, we shall show that in the Euclidean section of $\eta$-$\xi$ spacetime QFT corresponds to the imaginary time formalism, and that the Green functions are the Matsubara Green functions in this section. On the other hand, we shall show that in the extended Lorentzian section QFT reproduces the two known formalisms of TFTs with real time, namely the POM formalism and TFD. Furthermore we shall fix the parameter $\delta$ of the extended Lorentzian section with respect to the parameter $\sigma$ of the POM formalism and of TFD. The most general thermal matrix propagator shall be obtained in the framework of $\eta$-$\xi$ spacetime. We shall need below the covariant Lagrangian for the scalar field with a source term $J$. It is given by

\[
\mathcal{L}[\Phi, J] = \sqrt{-g} \left( \frac{1}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi + \frac{m^2}{2} \Phi^2 - V(\Phi) - J \Phi \right).
\]

#### A. Imaginary time formalism

The generating functional for the Green functions in the Euclidean section of $\eta$-$\xi$ spacetime is

\[
Z_E[J] = N \int [d\Phi] \exp \left\{ - \int d\sigma d\xi dy dz \mathcal{L}_{\sigma,\xi}[\Phi, J] \right\},
\]

where

\[
\mathcal{L}_{\sigma,\xi}[\Phi, J] = \frac{1}{2} \left( \partial_{\sigma} \Phi \right)^2 + \left( \partial_{\xi} \Phi \right)^2 \right)^2
\]

\[
\frac{1}{\alpha^2 (\sigma^2 + \xi^2)} \left( \frac{1}{2} \left( \nabla_{\perp} \Phi \right)^2 + \frac{m^2}{2} \Phi^2 + V(\Phi) - J \Phi \right).
\]

We now perform the change of coordinates given in Eq. \([3]\) in the generating functional. In the $\tau-x$ coordinates, the sum over fields is taken over all periodic fields
satisfying to the periodic constraint in Eq. (21),
\[ Z_E[J] = N \int_{\beta - \text{periodic}} [d\phi] \exp \left\{ - \int_0^\beta dt \int_{\mathbb{R}^3} dx dy dz \mathcal{L}_{t,x}[\phi, J] \right\} \]  
(66)
where \( J(\tau, x, y, z) = J(\sigma, \xi, y, z) \) and
\[ \mathcal{L}_{t,x}[\phi, J] = \frac{1}{2} \left[ (\partial_x \phi)^2 + (\nabla \phi)^2 + m^2 \phi^2 \right] + V(\phi) - J\phi. \]  
(67)
By differentiation this last equation with respect to the source \( J \) we obtain the Matsubara propagator whose Fourier transforms is
\[ G_\beta(k, \omega_n) = \frac{1}{\omega_n^2 + k^2 + m^2}, \]  
(68)
where the Matsubara frequencies \( \omega_n \) are given for bosons by \( \omega_n = 2\pi n / \beta \) (\( n \in \mathbb{N} \)).

B. Real time formalism: Path Ordered Method

Let us now consider the extended Lorentzian section of \( \eta - \xi \) spacetime. The generating functional is given by
\[ Z[J] = N \int [d\Phi] \exp \left\{ i \int d\eta d\xi dxdydz \mathcal{L}_{\eta, \xi}[\Phi, J] \right\}, \]  
(69)
where
\[ \mathcal{L}_{\eta, \xi}[\Phi, J] = \frac{1}{2} \left[ \left( \partial_\eta \Phi \right)^2 - \left( \partial_\xi \Phi \right)^2 \right] \]  
(70)
+ \[ \frac{1}{\alpha^2(\eta_0^2 - \xi_0^2)} \left\{ -\frac{1}{2} (\nabla \Phi)^2 - \frac{m^2}{2} \Phi^2 - V(\Phi) + J\Phi \right\}. \]
Since we shall be interested only in the propagators whose spacetime arguments belong only to the joining \( R_r \cup R_{i_0} \), we can set the source to zero in regions \( R_m \) and \( R_n \),
\[ J(x) = 0, \quad \text{when} \ x \in R_m \cup R_n. \]  
(71)
Then the expression in Eq. (69) reduces to
\[ Z[J] = N \int [d\Phi] \exp \left\{ i \int_{R_r \cup R_{i_0}} d\eta d\xi dxdydz \mathcal{L}_{\eta, \xi}[\Phi, J] \right\}. \]  
(72)
We now express the fields in regions \( R_r \) and \( R_{i_0} \) in terms of the local Minkowskian coordinates by using the transformations given in Eq. (13). We obtain
\[ Z[J] = N \int [d\phi] \exp \left\{ i \int dt dx dy dz \mathcal{L}_{t,x}[\phi, J] \right\} \]  
(73)
+ \[ i \int dt_{i_0} dx_{i_0} dy dz \mathcal{L}_{t,x}[\phi, J] \],
where the integration is taken over the Minkowski spacetime, \( \phi \) is the local field and where
\[ \mathcal{L}_{t,x}[\phi, J] = \frac{1}{2} \left[ (\partial_t \phi)^2 - (\nabla \phi)^2 - m^2 \phi^2 \right] + V(\phi) + J\phi. \]  
(74)
We now use the relations (12) and obtain
\[ Z[J] = N \int [d\phi] \exp \left\{ i \int dt dx dy dz \left[ \mathcal{L}_{t,x}[\phi, J] (t, x) - \mathcal{L}_{t,x}[\phi, J] (t + i\beta, x) \right] \right\}, \]  
(75)
where in the last step we have dropped the subscript \( I \) and where we have taken into account the fact that the direction of time in \( R_{i_0} \) is opposite to the one in \( R_r \), resulting in a minus sign in the second integration.

We now consider the expression for the generating functional as given in the POM formalism [2]. It is given by
\[ Z_{\text{POM}}[J] = N \int [d\phi] \exp \left\{ i \int_{C} dt dx dy dz \mathcal{L}_{t,x}[\phi, J] \right\}, \]  
(76)
where the time path \( C \) is shown in Fig. 1. The contribution from the vertical parts of the contour may be included in the normalization factor \( N \) and neglected when calculating the real-time Green functions. The generating functionals in Eq. (75) and (76) can then be identified provided that
\[ \delta = \sigma - 1/2. \]  
(77)
We thus see that the type of time path in the POM formalism is related directly to the “rotation angle” between the two regions \( R_r \) and \( R_{i_0} \) of \( \eta - \xi \) spacetime. In the free field case, from the above generating functionals we obtain the well-known thermal matrix propagator [1,2],
\[ D_{11}(k) = \frac{i}{k^2 - m^2 + i0^+} + 2\pi n(k_0) \delta(k^2 - m^2), \]  
(78)
\[ D_{22}(k) = D_{11}^*(k), \]
\[ D_{12}(k) = e^{\pi\delta k_0} \left[ n(k_0) + \theta(-k_0) \right] 2\pi \delta(k^2 - m^2), \]
\[ D_{21}(k) = e^{-\pi\delta k_0} \left[ n(k_0) + \theta(k_0) \right] 2\pi \delta(k^2 - m^2), \]
where \( n(k_0) = (e^{\beta k_0} - 1)^{-1} \). The parameter \( \sigma \) appears explicitly only in the off-diagonal components of the matrix propagator.

C. Real time formalism: Thermo Field Dynamics

Another formalism for real-time TFT is Thermo Field Dynamics [1,13]. In this approach a central role is played
by the Bogoliubov transformation, relating the zero-
temperature annihilation and creation operators with the
thermal ones. In TFD the field algebra is doubled and
one then considers two commuting field operators \( \phi \) and
\( \tilde{\phi} \) given by

\[
\phi(x) = \int \frac{d^4k}{(2\pi)^2} \left[ a_k e^{i(\omega_k t + kx)} + \tilde{a}_k^\dagger e^{-i(\omega_k t - kx)} \right],
\]

\[
\tilde{\phi}(x) = \int \frac{d^4k}{(2\pi)^2} \left[ \tilde{a}_k e^{i(\omega_k t - kx)} + a_k^\dagger e^{-i(\omega_k t + kx)} \right].
\]

(79)

(80)

In the original formulation of Takahashi and Umezawa
[4], the thermal Bogoliubov transformation is given by

\[
\begin{cases}
\sigma_k = \alpha_k \cosh \theta_k - \sigma_k^\dagger \sinh \theta_k, \\
\tilde{\sigma}_k = \tilde{\alpha}_k \cosh \theta_k - \sigma_k^\dagger \sinh \theta_k,
\end{cases}
\]

\[
\begin{cases}
\gamma_k = (1 - f_k)^{-\frac{1}{2}} \left( a_k - f_k^{-1}\sigma_k a_k^\dagger \right), \\
\tilde{\gamma}_k = (1 - f_k)^{-\frac{1}{2}} \left( \tilde{a}_k - f_k^{-1}\sigma_k \tilde{a}_k^\dagger \right),
\end{cases}
\]

(81)

(82)

where \( \sinh^2 \theta_k = n(\omega_k) \). These operators annihilate the
"thermal vacuum" \( |0(\beta)\rangle \). Thermal averages are calculated as expectation values with respect to \( |0(\beta)\rangle \).

The form of the thermal Bogoliubov matrix is however
not unique. It is possible to generalize the above transforma-
tion to a non-hermitian superposition of the form

\[
\begin{cases}
\gamma_k = (1 - f_k)^{-\frac{1}{2}} \left( a_k - f_k^{-1}\sigma_k a_k^\dagger \right), \\
\tilde{\gamma}_k = (1 - f_k)^{-\frac{1}{2}} \left( \tilde{a}_k - f_k^{-1}\sigma_k \tilde{a}_k^\dagger \right),
\end{cases}
\]

(83)

which give the correct canonical commutators, \( [\gamma_p, \gamma_p^\dagger] = \delta^3(\mathbf{k} - \mathbf{p}) \), etc. Here the \( \tilde{\gamma} \) conjugation corresponds to the usual hermitian conjugation \( \dagger \) together with the replacement \( \sigma \to 1 - \sigma \). The hermitian representation in
Eq. (81) is recovered when \( \sigma = 1/2 \).

Thermal averages are now expressed as \[13, 14\]

\[
\langle A \rangle = \frac{\langle 0(\beta) | A | 0(\beta) \rangle_L}{\langle 0(\beta) | 0(\beta) \rangle_L},
\]

(84)

where \( A \) is an observable, and where \( |0(\beta)\rangle_L \) and \( \langle 0(\beta) | \) are the left and right vacuum states defined by

\[
\begin{cases}
\gamma_k^\dagger |0(\beta)\rangle_L = 0 = \langle 0(\beta) | \sigma_k \end{cases}
\]

(85)

The thermal propagator for a scalar field is calculated in
TFD as follows:

\[
D^{(ab)}(x, y) = \left( T [\phi^a(x) \phi^b(y)\dagger] \right),
\]

(86)

where

\[
\phi^a \equiv \left( \begin{array}{c} \phi \\ \phi^\dagger \end{array} \right).
\]

(87)

This propagator coincides with the one of Eqs. (28), as
can be easily checked by using the definitions given above.
The connection of TFD to the geometrical picture of
\( \eta-\xi \) spacetime is immediate if we make the identification

\[
\left( \begin{array}{c} \phi \\ \phi^\dagger \end{array} \right) \equiv \left( \begin{array}{c} \phi_L \\ \phi_R \end{array} \right)
\]

(88)

under the constraint of Eq. (77) (see also Eq. (46)). Then the Bogoliubov transformations in Eqs. (61), (62) and those in Eqs. (82), (83) are identical.

V. OTHER FEATURES OF THE EXTENDED
LORENTZIAN SECTION

In this Section we discuss other features of the \( \eta-\xi \) spacetime, which are influenced by the rotation of
Eq. (19). Along the lines of Ref. [20], we consider the an-
alytic continuation of the imaginary time thermal propagator
to real times in the context of \( \eta-\xi \) spacetime. In Ref. [20] it was shown that the geometric structure of \( \eta-\xi \) spacetime plays a crucial role in obtaining the matrix real-time propagator from the Matsubara one.

To see how this works, it is sufficient to consider
the simple case of a massless free scalar field in two-
dimensions. In the Euclidean section of \( \eta-\xi \) spacetime, the equation for the propagator is

\[
\left( \frac{\partial^2}{\partial \sigma^2} + \frac{\partial^2}{\partial \xi^2} \right) D_I(A - A') = -(g_E)^{-\frac{1}{2}} \delta^2(A - A'),
\]

(89)

where \( (A, A') \) denotes a couple of points, \( g_E \) is the
determinant of the metric in the Euclidean section, and
where \( D_I \) is the imaginary time thermal propagator. Now we continue Eq. (89) to the extended Lorentzian section. This is achieved by first replacing \( \sigma \) by \( i\eta \) and then performing the rotation of Eq. (19). If \( D \) is the real time propagator, we have

\[
\left( -\frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \xi^2} \right) D(A - A') = -(g_L)^{-\frac{1}{2}} \delta^2(A - A'),
\]

(90)

where \( g_L \) stands for the determinant of the metric in the
Lorentzian section. Because of the presence of different
disconnected regions in the Lorentzian section, the prop-
gator has a matrix structure, since now the points \( A \) and
$A'$ can belong either to region $R_j$ or $R_{jk}$ ($R_{ii}$ and $R_{ij}$ are excluded since they are space-like with respect to $R_j$ and $R_{jk}$). The delta function with complex arguments $A$ and $A'$ is defined to be zero when the points belong to different regions.

In Minkowski coordinates, Eq. (11) reads

$$\left( -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \right) D(A - A') = -\delta_C(A - A').$$  \hspace{1cm} (91)

The delta function containing the time variable in the last equation needs to be defined on an appropriate path since the time variable is complex. This path coincides with the POM time path of Fig. 1 when the identification of Eq. (77) is made. By use of Eq. (22) and by following the procedure of Ref. [20] we obtain, for example for the component $D_{12}$, the equation

$$\left( -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \right) D(t - t' + i\sigma\beta, x - x') = -\delta_C(t - t' + i\sigma\beta) \delta(x - x'),$$  \hspace{1cm} (92)

which has the $D_{12}$ propagator given in Eq. (78) as solution.

Let us finally consider the tilde conjugation in the context of $\eta$-$\xi$ spacetime. The tilde conjugation rules are postulated in TFD in order to connect the physical and the tilde operators. Due to the geometrical structure of $\eta$-$\xi$ spacetime, these rules are there seen as coordinate transformations. This was first discussed in Ref. [21]. We extend here the result to the extended Lorentzian section of $\eta$-$\xi$ spacetime.

Let us recall the tilde rules as defined in TFD (we restrict for simplicity to bosonic operators):

$$\tilde{(AB)} = \tilde{A}\tilde{B}, \quad \tilde{(C_1A + C_2B)} = \tilde{C_1A} + \tilde{C_2B}, \quad \tilde{(A)} = A, \quad \tilde{(A)}^\dagger = (A^\dagger)^\sim,$$  \hspace{1cm} (93)

where $A, B$ are operators and $C_1, C_2$ are c-numbers. In order to reproduce this operation in the extended Lorentzian section of $\eta$-$\xi$ spacetime, let us first introduce the following $M$ operation as defined in Ref. [21].

$$M\Phi(\eta, \xi) M^{-1} \equiv \Phi(-\eta, -\xi).$$  \hspace{1cm} (94)

Next we perform a rotation by an angle $\delta$ transforming the $\eta, \xi$ coordinates according to Eqs. (19) and the field becomes then

$$R_\delta \Phi(\eta, \xi) R_\delta^{-1} \equiv \Phi(\eta_\delta, \xi_\delta).$$  \hspace{1cm} (97)

Finally we introduce a $\delta$ conjugation operation, which is similar to a charge conjugation, by

$$C_\delta \phi(t, x) C_\delta^{-1} \equiv \phi^\dagger(t - i(1 - \sigma)\beta, x),$$  \hspace{1cm} (98)

where the change $\delta \rightarrow -\delta$ (or equivalently $\sigma \rightarrow 1 - \sigma$) has to be performed together with usual charge conjugation.

The combined action of the three operations results in the tilde conjugation. By defining for simplicity the notation $G_\delta \equiv C_\delta R_\delta M$, we get

$$G_\delta \phi(t, x) G_\delta^{-1} = \phi^\dagger(t - i\sigma\beta, x),$$

$$G_\delta [G_\delta \phi(t, x) G_\delta^{-1}] \equiv M\Phi(\eta, \xi) M^{-1} \equiv \Phi(-\eta, -\xi).$$

We then get by omitting the space dependence for simplicity

$$G_\delta \phi_1(t) \phi_2(t') G_\delta^{-1} = \phi_1^\dagger(t - i\sigma\beta) \phi_2^\dagger(t' - i\sigma\beta),$$

$$G_\delta [G_\delta \phi(t) G_\delta^{-1}] \equiv \phi^\dagger \equiv M\Phi(\eta, \xi) M^{-1} \equiv \Phi(-\eta, -\xi).$$

The c-numbers are conjugated since the $M$ operation is anti-linear. The second of the above relations is derived as follows

$$G_\delta [G_\delta \phi(t) G_\delta^{-1}] \equiv M\Phi(\eta, \xi) M^{-1} \equiv \Phi(-\eta, -\xi).$$

The tilde rules of Eq. (93) are thus reproduced.

**VI. DISCUSSION AND CONCLUSIONS**

We have discussed a section of $\eta$-$\xi$ spacetime which represent the general geometric background for real-time thermal field theories at equilibrium. This section is complex and can be regarded as an extension of the usual Lorentzian section of $\eta$-$\xi$ spacetime by means of a rotation of region $R_{ii}$ with respect to $R_j$ in the complex $\eta$-$\xi$ spacetime. In terms of Minkowski coordinates the rotation is equivalent to a constant time shift, leaving the metric invariant.
The angle between the two regions turns out to be related to the \( \sigma \) parameter of the time path as used in the POM formalism. It also reproduces the parameter present in the Bogoliubov thermal matrix of TFD, when the relation between modes belonging to different regions is considered. The general form of the thermal matrix propagator containing the parameter \( \sigma \) has been obtained by use of this particular geometrical background. Finally, we have discussed the analytic continuation of imaginary time propagator to real time matrix propagator and the tilde rule in the context of \( \eta, \xi \) spacetime.

In the geometrical background of \( \eta, \xi \) spacetime, it is possible to understand the differences between the various formalisms of TFT in a simple way. In particular, with regards to the real-time methods, i.e. the POM formalism and TFD, the geometric picture is the following. In the Lorentzian section of \( \eta, \xi \) spacetime we have two different regions \( R_I \) and \( R_{I \delta} \), over which the field is defined. For a global observer this field is a free field. However when one does restrict to one of the two regions (say \( R_I \)), temperature arise as a consequence of the loss of informations (increase in entropy) about the other region. In order to calculate the propagator, one needs then to compare the fields defined in different regions. This can be done essentially in two ways:

1) By analytically continuing the field \( \phi^{(I)}(x, t) \) defined in region \( R_{I \delta} \) to region \( R_I \). Then the time argument gets shifted by \( i\beta(1/2 + \delta) \), as described in Section III. One thus ends up with one field and two possible time arguments, which can be either \( t \) or \( t - i\beta(1/2 + \delta) \). The generating functional defined in \( \eta, \xi \) spacetime by following this procedure turns out to be the same of the one defined in the POM formalism.

2) One can attach the information about the region to the field operator rather than putting it in the time argument. Thus the identification \( \phi^I(x) \equiv \phi(x) \) and \( \phi^{I \delta}(x) \equiv \tilde{\phi}(x) \) can be made and one obtains the formalism of TFD, which consists of two commuting field operators and a single time argument.

Of course, the two pictures give the same physics, i.e. the same propagator, since they are only different “viewpoints” of local observers in the context of \( \eta, \xi \) spacetime. It is an interesting question to ask if such an equivalence can be extended for situation which are out of thermal equilibrium. Work is in progress along this direction.

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FIG. 1. The time path used in POM. The parameter $\sigma$ ranges from the value $\sigma = 0$ (Closed Time Path) to $\sigma = 1$.

FIG. 2. Lorentzian section of $\eta$-$\xi$ spacetime: the solid lines represent the singularities at $\xi^2 - \eta^2 = 0$. On the straight lines time is constant, while on the hyperbolas the Minkowski coordinate $x$ is constant. Note that time flows in opposite directions in regions I and II.