Almost sixty years since Landauer linked the erasure of information with an increase of entropy, his famous erasure principle and byproducts like reversible computing are still subjected to debates in the scientific community. In this work we use the Liouville theorem to establish three different types of the relation between manipulation of information by a logical gate and the change of its physical entropy, corresponding to three types of the final state of environment. A time-reversible relation can be established when the final states of environment corresponding to different logical inputs are macroscopically distinguishable, showing a path to reversible computation and erasure of data with no entropy cost. A weak relation, giving the entropy change of $k \ln 2$ for an erasure gate, can be deduced without any thermodynamical argument, only requiring the final states of environment to be macroscopically indistinguishable. The common strong relation that links entropy cost to heat requires the final states of environment to be in a thermal equilibrium. We argue in this work that much of the misunderstanding around the Landauer’s erasure principle stems from not properly distinguishing the limits and applicability of these three different relations. Due to new technological advances, we emphasize the importance of taking into account the time-reversible and weak types of relation to link the information manipulation and entropy cost in erasure gates beyond the considerations of environments in thermodynamic equilibrium.
I. INTRODUCTION

The seminal work of Landauer [1] in 1961 on the physical cost of erasing information is still controversial, either from a theoretical or an experimental point of view. There are some experiments in the literature validating [2–5] the Landauer’s erasure principle, but others seem to indicate some type of limitations [6, 7]. At the theoretical level, the situation is even more polarized. For the majority of theoretical scientists it is a masterpiece of science connecting information manipulation with physical entropy [8–20], while for others it is just nonsense [21–29].

An important source of misunderstanding around the Landauer’s erasure principle appears because the historical works presenting the Landauer’s erasure principle [1, 8–11] linked its validity to thermodynamics. In particular, the conversion of entropy into heat requires environments in a thermal equilibrium. Even though environments in thermal equilibrium are typical in many scenarios, other more general types of environment can be envisioned. One can engineer environments with a limited number of degrees of freedom, with Markovian or non-Markovian behavior [15, 16], without satisfying the conditions for thermodynamic equilibrium. Therefore, in this work, we revisit the Landauer’s erasure principle without necessarily linking it to thermodynamics, but to the more fundamental Liouville theorem. We show that depending on the type of final state of environment involved in the computational process, three different types of relation between manipulation of information and entropy change can be established. The original Landauer’s erasure principle is recovered when the final state of environment is in a thermodynamic equilibrium, which we refer to as a strong relation between manipulation of information and entropy change. Therefore, in this paper, we rename the original Landauer’s erasure principle as the strong Landauer’s erasure principle. Alternatively, a weak relation between manipulation of information and entropy change can be deduced for more general environments when the only (macroscopic) condition imposed on the final states of environment is that they have to be macroscopically indistinguishable when different logical inputs are compared. Such a weak relation gives the well known limit $k \ln 2$ of entropy change when applied to an erasure gate. We refer to this result as the weak Landauer’s erasure principle because it can be applied to general types of environments in which the concept of temperature itself may remain undefined, but still the macroscopic indistinguishability condition applies. Finally, for states of environment that can be macroscopically distinguished when originating from different logical inputs, a time-reversible relation between information manipulation and entropy change can be established. Such a relation, which we refer to as the time-reversible relation, requires no entropy change for erasure computations. Although macroscopically distin-
guishable states of environment seem like an idealized scenario, we argue that some intermediate
types of environment states between totally distinguishable and totally indistinguishable are tech-
nologically accessible. The present computational technologies are linked to physical environments
with a number of degrees of freedom small enough to make the thermodynamical limit doubtful
and the strong version of the Landauer’s erasure principle not directly applicable. We show in this
work that the concepts developed by the scientific community in this research field known as “ther-
modynamics of computation” [9] can be directly understood from more generals laws not linked to
thermodynamic systems. Hopefully, the limits, applicability and generalizations of the Landauer’s
erasure principle analysed in this paper could open novel unexplored paths to manipulation of
information and the resulting change of entropy.

The structure of the rest of the paper is as follows. In Section II we define the physical char-
acteristics of a logical gate and the requirements imposed by the Liouville theorem. In Section
III we define three types of relation between manipulation of information and entropy change cor-
responding to three different types of environment: strong, weak and the time-reversible one. In
Section IV in light of these three types of relation, we provide some general remarks and we discuss
some misleading argumentation found in the literature involving the Landauer’s erasure principle.
Finally, we provide some general conclusions in Section V.

II. MICROSCOPIC AND MACROSCOPIC STATES

We are interested in describing, in the most general way, gates that manipulate information.
For simplicity, the discussion in this paper will be done using classical mechanics, but most of
the concepts mentioned in this work can be straightforwardly generalized to quantum systems by
replacing a time evolution in the phase space with a time evolution in the Hilbert space of quantum
states. From a microscopic point of view, the information is encoded in the state of the system S
containing \( N_S \) particles, the positions and momenta of which are denoted by a vector \( x \) with 6\( N_S \)
components, stemming from three position and three momentum components for each of the \( N_S \)
particles. In other words, \( x \) is a point in the 6\( N_S \)-dimensional phase space. In most gates there
are also additional \( N_E = N - N_S \) particles of the environment (with \( N \) being the total number
of particles in the full closed system) whose position and momenta variables are denoted by the
vector \( y \) in the 6\( N_E \)-dimensional phase space. The interaction between all the degrees of freedom is
determined by the Hamiltonian \( H(x, y) \), which describes the physical implementation of the logical
gate. We emphasize that no assumption at all is done yet on the type of environments involved in
From a macroscopic point of view, the initial information encoded in the system $S$ is described by an initial macroscopic state $A$ characterized by some macroscopic properties. Many different microscopic states $x$ correspond to the same macroscopic state $A$, so by $X = \{x\}$ we denote the set of all microscopic states $x$ that describe the same macroscopic state $A$. Similarly, $Y = \{y\}$ is the set of all microscopic states $y$ compatible with the macroscopic state $E_A$ of the environment, coexisting with the system macroscopic state $A$.

Now consider a process $A \rightarrow B$, where $B$ is the final macroscopic state of the system $S$. At the initial time $t_i$, we consider the set $X(t_i) = \{x(t_i)\}$ of all initial microscopic states of $S$ that are compatible with the macroscopic state $A$, together with the set $Y(t_i) = \{y(t_i)\}$ of all initial microscopic environment states compatible with the macroscopic state $E_A$. At the final time $t_f$, the initial set $X(t_i) = \{x(t_i)\}$ evolves into the final set $X(t_f) = \{x(t_f)\}$, where the evolution of each member of the set is determined by the Hamiltonian $H(x,y)$. Hence the microscopic description of the final state in the $A \rightarrow B$ process is given by the final set $X(t_f) = \{x(t_f)\}$ compatible with the final macroscopic state $B$, together with the final set $Y(t_f) = \{y(t_f)\}$ compatible with the final macroscopic state $E_B$.

It is important to emphasize that, after the process $A \rightarrow B$ is finished, the gate has to be ready to work again for another process $A' \rightarrow B'$ giving the new logical output correctly. This last condition excludes gates designed to work only once. It also excludes gates that, before being able to properly execute the subsequent $A' \rightarrow B'$ process, require resetting of the first process $A \rightarrow B$ by doing this process backwards in time and loosing the output $B$. We notice that when the gate is used the second time, it involves the same macroscopic states of the system, but such macroscopic states are now built from different microscopic degrees of freedom of the system. For example, if the microscopic description of a logical signal is given by electrons that arrive from a battery, then those electrons travel physically from the input to the output of the gate. Let us label the electrons involved during the first gate manipulation by $x_1(t)$. Those electrons provide the output voltage of the gate $x_1(t_f)$ which, in fact, is then used as an input voltage of any other part of the whole circuit. So, if the gate is used the second time, the new electrons coming from the battery should not be labeled by $x_1(t)$ because the initial ones are no longer in the gate, but in other parts of the circuit. Hence we label those second electrons coming from the battery as $x_2(t)$. By contrast, the environment variables $y(t)$ during the first use of the gate are the same variables as those during the second use. So, strictly speaking, the first time we use the gate, the involved degrees of freedom and the Hamiltonian are $H(x_1, y)$, while the second time we use the
gate, they are \( H(x_2, y) \). Nevertheless, since we shall concentrate on discussion of one operation of the gate, we shall simplify the notation by always writing the Hamiltonian as \( H(x, y) \), assuming that the initial environment degrees of freedom (of the macroscopic state \( E_A \)) are not correlated with the system degrees of freedom. This, however, will not be assumed for the final environment macroscopic state \( E_B \), which will be determined by the Hamiltonian causing correlations between the system and environment degrees of freedom. In any case, it is important to emphasize that the initial environment degree of freedom (and its macroscopic state \( E_A \)) are affected by all previous operations of the gate.

Now let us define the Liouville function \( W(x, y, t_i) \) as the probability density in the full phase space. From such a function, we can define the “high-probability” region \( R(t_i) = R_x(t_i) \times R_y(t_i) \) in the phase space with the property \( \int_{R_x(t_i)} dx \int_{R_y(t_i)} dy W(x, y, t_i) = 1 - \epsilon \) with \( \epsilon \ll 1 \). From such a definition of the “high-probability” region \( R(t_i) \), we define the phase space volume assigned to the macroscopic state \( A \) at time \( t_i \) as

\[
V_A = \int_{R_x(t_i)} dx \int_{R_y(t_i)} dy.
\]

This can be generalized to define the phase space volume during the whole process \( A \rightarrow B \) at any time \( t \) as

\[
V_{A \rightarrow B}(t) = \int_{R_x(t)} dx \int_{R_y(t)} dy.
\]

As is well known, the Liouville theorem states that the volume \( V_{A \rightarrow B}(t) \) remains constant during the whole evolution, i.e. does not depend on \( t \). Thus, for any process \( A \rightarrow B \) we have

\[
V_{A \rightarrow B}(t_i) = V_{A \rightarrow B}(t) \ \forall t.
\]

Another important property relevant for the discussion in this paper is a consequence of time reversibility of Hamiltonian dynamics. If two phase space trajectories coincide at one time, then such trajectories are identical at all times. Thus, if two different process, \( A \rightarrow B \) and \( A' \rightarrow B' \), have a null intersection at the initial time, then they have a null intersection at any time. In other words,

\[
\text{if } \{X_{A \rightarrow B}(t_i), Y_{A \rightarrow B}(t_i)\} \cap \{X'_{A' \rightarrow B'}(t_i), Y'_{A' \rightarrow B'}(t_i)\} = \emptyset \\
\text{then } \{X_{A \rightarrow B}(t), Y_{A \rightarrow B}(t)\} \cap \{X'_{A' \rightarrow B'}(t), Y'_{A' \rightarrow B'}(t)\} = \emptyset \ \forall t.
\]

From the definitions (1) and (2) it is clear that \( V_A = V_{A \rightarrow B}(t_i) \). But, in general, \( V_B \neq V_{A \rightarrow B}(t_f) \). To understand this, it is useful to have a concrete example in mind, e.g. a process in which \( A \)
FIG. 1. Schematic representation of the phase space at two different times. The fact that both set of initial trajectories satisfy the Liouville theorem for the $A \rightarrow B$ and $A' \rightarrow B'$ processes does not imply that the phase space volume of the macroscopic state $A$ is equal to the phase space volume of the macroscopic state $B$.

is an unbroken glass while $B$ is a glass broken into many small pieces. If the interactions with the environment are such that the initial $A$ ends up in the final $B$, then all microscopic states $x$ corresponding to the macroscopic state $A$ at time $t_i$ will evolve to the macroscopic state $B$ at time $t_f$. However, there is no reason to impose that all points belonging to the macroscopic state $B$ at time $t_f$ originated from the macroscopic state $A$ at time $t_i$. See Figure 1 for a graphical explanation of this important point. In our concrete example, an unbroken glass will evolve into a broken one, but a state that looks like a broken glass might have never been an unbroken glass in the past. Such an asymmetry, of course, corresponds to the existence of a statistical arrow of time.

Now we define the (Boltzmann) entropy as a property of the macroscopic state $A$. The entropy $S_A$ associated with the macroscopic state $A$ is determined by the phase space volume of $A$ as

$$S_A = k \ln V_A,$$

where $k$ is the Boltzmann constant and $V_A \equiv V_{A\rightarrow B}(t_i)$. Similarly, the entropy $S_B$ is defined as $S_B = k \ln V_B$. Then the change of entropy during the $A \rightarrow B$ process is the entropy associated
with the macroscopic state $B$ minus the entropy associated with the macroscopic state $A$

$$\Delta S = S_B - S_A.$$  \hfill (6)

Notice that the entropy change depends on both the final and initial macroscopic states and cannot be evaluated by knowing only one of them. The Liouville theorem Eq. (3) by itself cannot explain the entropy change. Namely, if we naively defined $V_A \equiv V_{A \to B}(t_i)$ and $V_B \equiv V_{A \to B}(t_f)$, then Eq. (3) together with Eq. (5) would imply $\Delta S = 0$. But this is not true in general because we can expect $V_B \geq V_{A \to B}(t_f)$, to include the possibility of realizing the macroscopic state $B$ by a microscopic state $\{x'(t_f), y'(t_f)\}$ that at the initial time does not correspond to the macroscopic state $A$. See the schematic representation in Figure 1. The remark, together with the Liouville theorem, provides the essential elements of the generalized second law [30] stating that $\Delta S \geq 0$ for any computation. Notice that no thermodynamical argument has been invoked so far. Our macroscopic states, in general, have nothing to do with thermodynamic equilibrium states. Following [30], we refer to the condition $\Delta S \geq 0$ as the generalized second law. In our opinion, it is misleading to call it the second law of thermodynamics, even though it is often called so in the literature, because it applies to non-thermodynamic systems too. Our goal in this paper is to identify what type of entropy change can be expected in different $A \to B$ processes, depending on different physical conditions imposed on the final macroscopic environment state $E_B$.

III. THREE TYPES OF THE RELATION BETWEEN MANIPULATION OF INFORMATION AND ENTROPY CHANGE

In general, following discussions in the literature, we will be interested in two processes; the $0 \to 0$ process and the $1 \to 0$ process. Here the macroscopic states $1$ and $0$ are understood as the logical states corresponding to a bit of classical information. A Hamiltonian designed to accomplish any of the two processes is a Hamiltonian of the erasure gate, $H_{\text{erasure}}(x, y)$. This gate is the simplest example of logical irreversibility, because the final logical result $0$ does not allow us to deduce what was the initial logical state (either $0$ or $1$). We will also mention the processes $1 \to 0$ and $0 \to 1$, namely the inverter gate, accomplished by a Hamiltonian $H_{\text{inverter}}(x, y)$. The inverter gate is an example of logical reversibility, because the final logical state uniquely determines the initial logical state. We distinguish three types of the relation between manipulation of information and entropy change, corresponding to the three types of conditions, $C_1$, $C_2$ and $C_3$, that define the final macroscopic environment state $E_B$. 
A. A time-reversible relation

The first type of relation between manipulation of information and entropy change, that we call time-reversible relation, corresponds to imposing the following condition on the final macroscopic environment state $E_B$:

- **C1 TIME REVERSIBLE CONDITION:** The final macroscopic state $E_B$ of the environment is defined by the states $Y_{A\rightarrow B}(t_f)$ that evolved deterministically from $Y_{A\rightarrow B}(t_i)$.

When condition C1 is satisfied, the Liouville theorem states that

$$V_{B} = V_{A\rightarrow B}(t_f) = V_{A\rightarrow B}(t_i) = V_A$$

and the entropy change in the process $A \rightarrow B$ is then given by

$$\Delta S = k \ln V_{A\rightarrow B}(t_f) - k \ln V_{A\rightarrow B}(t_i) = 0,$$

which satisfies the lower limit of the generalized second law $\Delta S \geq 0$.

Now let us analyze the meaning of Eq. (7) for an inverter gate, with the $0 \rightarrow 1$ and $1 \rightarrow 0$ processes. We have $X_{1\rightarrow 0}(t_i) \cap X_{0\rightarrow 1}(t_i) = \emptyset$ at the initial time and $X_{1\rightarrow 0}(t_f) \cap X_{0\rightarrow 1}(t_f) = \emptyset$ at the final time. Thus, Eq. (4) is satisfied for any type of environment. In particular, for those satisfying C1. In fact, at least conceptually, we can build the inverter without environment, $N_E = 0$, so that the condition C1 can become irrelevant for a correct functioning of the inverter.

Next let us analyze Eq. (7) for an erasure gate, with the $0 \rightarrow 0$ and $1 \rightarrow 0$ processes. We get, by construction, that $X_{0\rightarrow 0}(t_f) = X_{1\rightarrow 0}(t_f)$ since both final macroscopic system states are 0. We also have $Y_{0\rightarrow 0}(t_i) = Y_{1\rightarrow 0}(t_i)$ since the two initial macroscopic environment states are equal. Then, since $X_{0\rightarrow 0}(t_i) \cap X_{1\rightarrow 0}(t_i) = \emptyset$, one way of satisfying Eq. (4) and Condition C1 is just dealing with two macroscopically different final environment states $E_{0\rightarrow 1,B} \neq E_{1\rightarrow 0,B}$. Thus, we can effectively get $\Delta S = 0$ for an erasure gate. Notice that Condition C1 implies that the initial macroscopic information encoded in $A$ will effectively disappear from the final state of the system, but it will appear in the final environment state. The same conclusion was obtained by Hemmo and Shenker [24], which is based on the fact that, owing to time-reversibility, information can never be erased at the microscopic level in a full closed system.

The fact that this type of erasure gate, giving $\Delta S = 0$, violates the Landauer limit were already well known by Landauer and Bennet. Bennet mentioned what is the problem with a such type of time-reversible erasure gate in his 1973 paper [8]. Let us use a simple example to understand it. We consider a Hamiltonian $H_{\text{erasure}}(x,y)$ with two degrees of freedom, so that $N_S = 1$ and $N_E = 1$. The system variable $x$ can be set in two positions: the left position corresponds to a 1 and the right
position corresponds to a 0. The erasure procedure involves another degree of freedom $y$ prepared to have an elastic collision with the first degree of freedom only when it is on the left, so that the system degree of freedom moves from left to right. The collision will either happen (when the system is initially in 1) or not (when the system is initially in 0), so the final state of environment $E_B$ will reveal the initial value of the gate. It is argued in [8] that such an erasure gate can work perfectly once, but if we want to use it more than once then the environment degree of freedom has to be reset to its initial state to make the erasure gate useful again. Contrary to what happens in the inverter gate, here the environment degree of freedom is a crucial element of the gate. The erasure gate cannot work without the environment degree of freedom in its correct state. To use this erasure gate more than once, we need to reset the final environment state $E_B(t_f)$ to its initial state $E_A(t_i)$. Since the forward computation is time reversible with $\Delta S = 0$, from the knowledge of the final environment state $E_B(t_f)$ we can undo the computation backwards with $\Delta S = 0$ as well and let the environment state to recover its initial state in a process $E_B(t_f) \rightarrow E_A(t_i)$. Following Bennet [8], to avoid that the backward computation destroys the output, we can make a copy of the output data by writing it into a different degree of freedom (a “tape”) before the background computation. In this way we are able to convert a logically irreversible erasure gate (with one system degree of freedom $x$) into a logically reversible one (with two degrees of freedom $x$ and $y$) by converting the environment degree of freedom into a new (control) degree of freedom of the system. Now, at least in principle, we do not need any additional type of degrees of freedom in the environment to work with our reversible erasure gate. This implementation is able to use the same reversible erasure gate $M$ times, as far as we have a “tape” with $M$ empty slots. This implementation is what inspired Bennet to develop reversible computing [8]. The full proposal requires solving one final additional problem about how to use this gate for more than $M$ steps (once we have already “filled” all $M$ slots with recorded information). We omit the ingenious (and a bit complex) solution proposed by Bennet because it is not relevant for the rest of the discussion in this section.

In short, according to Bennet, the possibility of developing an erasure gate in a laboratory with $\Delta S = 0$ is, at least conceptually, perfectly possible as far as it is used only once and the condition $C1$ is satisfied (or the output is reset by doing the process backwards in time as demonstrated experimentally in [7]). The deep reason why these types of gates are disregarded in the subsequent literature is because it is widely accepted that the additional “control” degree of freedom has to be reset to its initial state each time after the gate is used. If such additional “control” degree of freedom is not reset to its initial state, it is argued that the erasure gate cannot work properly.
next time.

We will see that, in a conventional gate, the initial macroscopic state of environment is not equal to the final macroscopic state of environment; the final one contains more heat. So instead of a “tape” environment that receives new data we have a thermal environment that “receives” new heat. At the fundamental microscopic level, there is no any significant difference between receiving data and “receiving” heat, as both can be viewed as mere changes of the microscopic state. The difference is significant only at the macroscopic level, due to which we associate an entropy change $\Delta S > 0$ with a “received” heat. But in both cases the true reason why the erasure gate can be used many times is the change in the environment degrees of freedom. It is mandatory to change the environment each time an erasure process takes place. At the microscopic level, the role of the environment in an erasure gate is to replace an irreversible closed system $x(t)$ (which, in fact, is forbidden because of the time reversibility of Hamiltonian dynamics) with a reversible closed one $x(t), y(t)$ (which is allowed by time reversibility of Hamiltonian dynamics). In this paper we argue that the widely accepted result that a erasure gate with $\Delta S = 0$ cannot work many times is not correct, at least, conceptually. The solution to use such erasure gates with $\Delta S = 0$ is to ask $y(t)$ to be different each time we use the gate, but not too different (this is the same that happens in a conventional erasure gate: although the environment is hotter each time we use the gate, the microscopic states of the environment are always very similar). See Figure 2. For example, the $y$ degree of freedom can have much larger mass than the $x$ degree, so that the interchange of momentum provides a very large change in the velocity of $x$, but a very small one in the velocity of $y$. So small that the $y$ degree of freedom can be used for another process because its position will be almost the same. Notice that we are not violating the time reversibility of Hamiltonian dynamics. The trajectory $\{x_{1\to0}(t), y_{1\to0}(t)\}$ is different from the trajectory $\{x_{0\to0}(t), y_{0\to0}(t)\}$ because $y_{1\to0}(t_f)$ is slightly different from $y_{0\to0}(t_f)$. In this way we can realize irreversible logic with reversible physics. Our simplified erasure gate has a limit on the number of times it can be used (related to the spatial and temporal dimensions of the experiment). But, in principle, it is not different from conventional erasure gates because they also have a limit on the number of times that they can be consecutively used, which is related with the limit on the extra heat that can be “received” by the erasure gate when we take into account that the number $N_E$ of environment particles is not strictly infinite.

Up to here, our discussion has been only conceptual. How realistic is an environment subjected to Condition C1 from a technological point of view? In theory, environments are assumed to have an infinite number of degrees of freedom (the so called reservoirs in the literature). Often,
FIG. 2. Schematic representation of $1 \rightarrow 0$ (top) and $0 \rightarrow 0$ (bottom) processes in a two-dimensional space for an erasure gate with a time-reversible environment. The system (orange) degree of freedom starts in the 1 or 0 state. Notice that the environment (violet) state $y_{1\rightarrow0}(t_f)$ is not much different from $y_{0\rightarrow0}(t_f)$, so that the gate can be used successfully many times without any special reset. Yet, $y_{1\rightarrow0}(t_f)$ and $y_{0\rightarrow0}(t_f)$ are not strictly equal to each other, so that the whole system is time-reversible even if $x_{1\rightarrow0}(t_f) = x_{0\rightarrow0}(t_f)$.

such infinite environments are assumed to be in a state of thermal equilibrium, called bath or thermal bath. However, in recent times, prototypes in laboratories are developed that try to minimize the number of degrees of freedom that interact with the system, or at least to have some control over them. A paradigmatic example is the development of quantum computers, where an initially prepared quantum state is required to suffer several unitary time-reversible (closed system) evolutions until a final measurement is done. The problem of decoherence in quantum computing, for which error correction techniques are under development, is an example of a technological difficulty of controlling the environment as we wanted in Condition C1. We quote here a statement by Landauer [31] about seminal work of Benioff [32] on quantum computing and the attempt of developing Hamiltonians with only information-bearing degrees of freedom: “You invoke a Hamiltonian (or a unitary time evolution) which causes the information bearing degrees...
of freedom to interact, and to evolve with time, as they do in a computer. You introduce no other parts or degrees of freedom. I was too engineering oriented to see that possibility; I assumed that you had to describe the apparatus and not just the Hamiltonian.” To be fair, in present-day real computers, where the environment has an enormously large number of degrees of freedom \(N_E \gg 10^{23}\) belonging to all parts of the computer, ensuring that we have the ability to distinguish environments belonging to the \(0 \to 0\) or \(1 \to 0\) process, as stated by the \(C_1\) condition, seems totally unrealistic. In any case, regarding the conceptual discussion done here, there is a crucial difference between being very difficult from a technological point of view and being impossible from a fundamental physical point of view.

## B. Weak relation

The second type of relation between manipulation of information and entropy change can be obtained by changing the definition of the final macroscopic state \(E_B\):

- **C2 WEAK CONDITION**: For two different processes involved in a gate with two initial environment states which are macroscopically equal (e.g. \(E_{1 \to 0, A}(t_i) = E_{0 \to 0, A}(t_i)\)), the two final environment states are macroscopically equal as well (e.g. \(E_{1 \to 0, B}(t_f) = E_{0 \to 0, B}(t_f)\)).

Notice that we are not imposing that the initial environment state is macroscopically equal to the final environment state in a given process, but only that the two final environment states of the different processes involved in a gate are macroscopically equal to each other. We note that macroscopic states are defined by macroscopic properties that are relative to the resolution of the observer (or measuring apparatus). Nevertheless, this resolution is, of course, objective and is related to the Hamiltonian that defines the gate. In other words, for the same logical gate, the Hamiltonian \(H_{C1}(x, y)\) (and hence the dynamics of \(x(t), y(t)\)) that satisfies the previous Condition \(C1\) is different from the Hamiltonian \(H_{C2}(x, y)\) (and hence from the respective dynamics) that satisfies the new Condition \(C2\).

For an inverter, after repeating the steps done in section IIIA, the Condition \(C2\) can be trivially satisfied since the Liouville theorem does not imply any extra restrictions on what type of environment is acceptable. In particular, even if the environment states are indistinguishable from a macroscopic point of view, the whole (system plus environment) processes are clearly macroscopically distinguishable, so that the entropy change in the process from the macroscopic state \(A\) to
the macroscopic state $B$ can again be

$$\Delta S_{0\rightarrow 0} = 0.$$  \hfill (8)

The fact that there is no entropy limit for reversible logic is well known and even tested experimentally \cite{5}.

Now let us analyze an erasure gate. From C2, the final macroscopic states of environment in the $0 \rightarrow 0$ and $1 \rightarrow 0$ process are identical. This implies that the two final sets of states $Y_1 \rightarrow 0(t_f)$ and $Y_0 \rightarrow 0(t_f)$ of the environment have the same macroscopic properties that characterize the macroscopic state $E_B$. In other words, the sets $Y_1 \rightarrow 0(t_f)$ and $Y_0 \rightarrow 0(t_f)$ belong to the same macrostate $E_B$. This is an example of a relation between initial and final macroscopic states depicted in Figure 1. Since we know that the processes $1 \rightarrow 0$ and $0 \rightarrow 0$ belong to different sets, we must have $Y_1 \rightarrow 0(t_f) \cap Y_0 \rightarrow 0(t_f) = \emptyset$ so the final phase space volume $V_B$ of the macroscopic state $E_B$ is the sum of phase space volumes $V_B = V_1 \rightarrow 0(t_f) + V_0 \rightarrow 0(t_f)$. From Eq. (3) we know that

$$\Delta S_{0\rightarrow 0} = k \ln(V_A + V_A') - k \ln V_A = k \ln \left(1 + \frac{V_A'}{V_A}\right).$$  \hfill (9)

By similar arguments, for the $1 \rightarrow 0$ process we get

$$\Delta S_{1\rightarrow 0} = k \ln(V_A + V_A') - k \ln V_A' = k \ln \left(1 + \frac{V_A}{V_A'}\right).$$  \hfill (10)

If we compute the average with equal \textit{a priori} probabilities for the $1 \rightarrow 0$ and $0 \rightarrow 0$ processes, then the average entropy change is

$$\Delta S = \frac{1}{2} \Delta S_{0\rightarrow 0} + \frac{1}{2} \Delta S_{1\rightarrow 0} = k \ln \left(\frac{V_A + V_A'}{\sqrt{V_A V_A'}}\right).$$  \hfill (11)

The minimum of $\Delta S$ appears when we impose a natural assumption $V_A' = V_A$, in which case we get the well known result

$$\Delta S = k \ln 2.$$  \hfill (12)

This result was already indicated \cite{1} by Landauer himself. Notice, however, that we have made no reference to thermodynamics at all in the present development. We only invoke the Liouville theorem and macroscopic properties of \textit{typical} environments. For this reason, we refer to the result Eq. (12) as the weak Landauer’s erasure principle, because it is more general than the original Landauer limit which implicitly assumed that all the entropy increase was due to a production of heat. Notice that, from Eq. (9) and Eq. (10), one can engineer systems with asymmetric phase-space volumes for the initial $1$ and $0$ (or the associated environments) so that the entropy change
FIG. 3. Schematic representation of $1 \rightarrow 0$ (top) and $0 \rightarrow 0$ (bottom) processes in a two-dimensional space for an erasure gate with a weak condition for the environment. The system (orange) degrees of freedom are identical to those in Figure 2, but now we have three degrees of freedom for the environment: the one (violet) used in Figure 2, plus a double pendulum (blue), the motion of which is chaotic. The chaotic motion implies that we are not able to reproduce the initial state of the environment from its final state. In this sense, although the whole physical system is time-reversible at the fundamental microscopic level, it is time-irreversible at the macroscopic level. The two final environment states are indistinguishable at the macroscopic level, so that the Condition $C_2$ is satisfied. The important point here is that such time-irreversible physics implies an increase of entropy, but not a production of heat.

can be lower in one of the processes than the value predicted by Landauer [33]. In any case, these exotic results (validated experimentally in [6]) do not contradict the general Landauer’s principle, because the Landauer’s principle refers to Eq. (11), which cannot be lower than Eq. (12).

The main point we want to stress here is that erasure gates with $\Delta S = k \ln 2$ do not necessarily need to dissipate heat. An example is presented in Figure 3. There we consider the same system degree of freedom as in the erasure gate of Figure 2, but now we have three degrees of freedom for the environment: one of them (violet) is the same as in Figure 2, while two additional ones (blue) constitute a double pendulum. We still keep the condition from the previous example that the violet degree of freedom has a mass much larger than the other degrees, so that it has a very
slow dynamics in comparison with all other (environment and system) degrees of freedom. We take \( X_{1 \rightarrow 0}(t_i) = x_{1 \rightarrow 0}(t_i) \) and \( X_{0 \rightarrow 0}(t_i) = x_{0 \rightarrow 0}(t_i) \) as fixed initial conditions for the system with \( x_{1 \rightarrow 0}(t_i) \neq x_{0 \rightarrow 0}(t_i) \) and \( x_{1 \rightarrow 0}(t_f) = x_{0 \rightarrow 0}(t_f) \). (Here, with a slight abuse of notation, \( X = \{ x \} \) denotes that the set \( X \) contains only one member \( x \), to be distinguished from \( X = \{ x \} \) which denotes a set \( X \) with many different members \( x \).) We also assume that \( Y_{1 \rightarrow 0}(t_i) = \{ y_{1 \rightarrow 0}(t_i) \} \) to account for the uncertainty of the initial conditions of the pendulum in repetitions of the same experiment. The initial environment states for the two processes are assumed to be macroscopically identical, \( Y_{1 \rightarrow 0}(t_i) = Y_{0 \rightarrow 0}(t_i) \), and not correlated with the system. Here we do not want to solve the dynamical equations of motion (in fact, we have not explicitly defined the Hamiltonian), but we want to use some general properties, Eq. (3) and Eq. (4), to make some predictions about the state of environment at the final time \( t_f \). The double pendulum is known to be a chaotic system, in the sense that a very small perturbation in the initial conditions is sufficient to cause a dramatic change in the evolution. The interaction of the double pendulum with the violet degree of freedom can be viewed as such a small perturbation. In each experiment, at the final time \( t_f \) the pendulum will be found in a state \( y \pm \Delta y \), where \( \Delta y \) is a small measurement error. Due to the chaotic behavior, the small uncertainty \( \Delta y \) implies that we cannot determine whether the input system state was 1 or 0. In this sense, we can say that the phase space points of \( Y_{1 \rightarrow 0}(t_f) \) are similar to the phase space points \( Y_{0 \rightarrow 0}(t_f) \) when the processes \( 1 \rightarrow 0 \) and \( 0 \rightarrow 0 \) are repeated with different initial conditions of the pendulum. But we know from Eq. (4) that such repetitions must imply \( y_{1 \rightarrow 0}(t_f) \neq y_{0 \rightarrow 0}(t_f) \). How can these phase space points be similar and different at the same time? The set \( Y_{1 \rightarrow 0}(t) \) “spreads” with time, in the sense that, if the initial phase-space distance between two points in the set is small, then the distance grows during the evolution governed by the Hamiltonian. A similar “spread” happens for the phase space points \( Y_{0 \rightarrow 0}(t) \), satisfying the conditions that the two sets of points are similar, but different. In this sense \( Y_{1 \rightarrow 0}(t) \) and \( Y_{0 \rightarrow 0}(t) \) belong to the same macrostate \( E_B \) of the environment. This is the Condition \textbf{C2}, which implies that the phase space volume of this macrostate is, at least, the double of the initial volume of the environment macrostate for each of the processes, which leads to the result Eq. (12). But what does the increase of entropy in the gate in Figure 3 mean physically? It means that the double pendulum will tend to take new positions and/or velocities at \( t_f \) that were not reachable at the initial time \( t_i \). But, and this is the important point, the double pendulum does not increase its temperature, because the concept of temperature itself is not applicable in such a mechanical system with only three degrees of freedom.

There are some researchers who dislike to refer to the relation of information manipulation and
entropy change as “thermodynamics of computation”. See, for example, Refs. [25–29]. Among other arguments, they say that any useful electronic device is, in fact, a device outside of thermodynamic equilibrium because of the battery which polarizes the device [25–27]. Those who are discomforted with the concept of “thermodynamics of computation” will find in our development of the weak Landauer’s erasure principle an argument in favor of their point of view, because we have reached the result $\Delta S = k \ln 2$ without referring to thermodynamics. The confusion is also encoded in the terminology used in the scientific literature, because what we call the generalized second law in this paper is usually called the second law of thermodynamics in the literature. The fact is that $\Delta S \geq 0$ is a law that can by applied to any system where one wants to account for microscopic states that are compatible with a given macroscopic property, not only to systems in thermodynamic equilibrium. See the excellent paper [30] which clarifies this point with simple and elegant examples. We emphasize that, after accepting that the result $\Delta S = k \ln 2$ has, in general, nothing to do with heat or temperature, in discussions of the behavior of erasure gates one can look for new types of entropy different from thermodynamic entropy. Such new possibilities will violate the original Landauer’s erasure principle in terms of heat and temperature, without violating Eq. (12) when $C2$ is assumed.

Finally, we want to mention that there is an important difference between conditions $C1$ and $C2$. As discussed in Section III A, the condition $C1$ is not natural for large environments and it requires a substantial effort to “engineer” an environment with a limited number of $N_E$ particles to satisfy $C1$. On the contrary, the condition $C2$ is basically a definition of a natural large environment, rather than an ad hoc imposed condition. In any case, there is no fundamental law stating that only environments satisfying $C2$ are possible. Thus, even if the result $\Delta S = 0$ implicit in the condition $C1$ is difficult to achieve in practice, it shows the possibility to look for new Hamiltonians that, while working as an erasure gate, are also able to provide an entropy change that interpolates between the present result $\Delta S = k \ln 2$ and the previous one $\Delta S = 0$.

C. Strong relation

The strong relation between manipulation of information and entropy change leads to the original Landauer’s erasure principle. To arrive to it, we invoking the following condition on the final state of environment $E_B$: 

- **C3 STRONG CONDITION:** The final states of environment of the different process of a gate (e.g. $Y_{1\rightarrow0}(t_f)$ and $Y_{0\rightarrow0}(t_f)$) are described by the same thermal bath.
This condition should be understood as a supplement to C2, i.e. in Condition C3 we assume that Condition C2 is already satisfied. We are not only imposing that the final states of environment are macroscopically identical, but also that the final states of environment can be described by a state in a thermodynamic equilibrium with a well defined temperature.

Here, from the well defined macroscopic property called temperature \( T \), it is easy to understand how the conditions \( Y_{1 \to 0}(t_f) \cap Y_{0 \to 0}(t_f) = \emptyset \) and \( E_{0 \to 0,B} = E_{1 \to 0,B} \) can be satisfied simultaneously. The first refers to microscopic variables (particle positions and momenta) in the phase space, while the second refers to the macroscopic temperature. In statistical mechanics, there are many different microscopic states corresponding to the same temperature. For an environment in thermodynamic equilibrium (which, as we have discussed in Section III B, is just an approximation for a realistic device), it is well known that the increment of heat \( \Delta Q \) is related to the increment of entropy \( \Delta S \) through the thermodynamic relation \( \Delta Q = T \Delta S \). Hence, since the increment of entropy is given by Eq. (12), we finally have

\[
\Delta Q = kT \ln 2. \tag{13}
\]

This is exactly the original Landauer’s erasure principle [1], which we call the strong Landauer’s erasure principle to be distinguished from the weak Landauer’s erasure discussed in Section III B. The universality of the strong Landauer’s erasure principle in Eq. (13) is based on the assumption that all final states of environments are indeed thermal baths implicit in Condition C3. Following the arguments done in previous subsections, the Condition C3 is a good approximation for most real environments in Nature, but not necessarily for all of them. The possibility of classical computation with few atoms, as well as the possibility of quantum computation, seem to indicate that a new relation between manipulation of information and entropy change (beyond the one described by a thermal bath) require a detailed study. Such a study, indeed, is what we have done in the two previous subsections III A and III B.

**IV. FINAL REMARKS**

In this section, we want to emphasize several relevant points about the relation between manipulation of information and entropy change. Some of them are clarifications of what, in our opinion, are confusing arguments found in the literature about the interpretation and consequences of the Landauer’s erasure principle. The others are just comments to emphasize some key aspects of the developments done in this work.
A. In an erasure gate, are $0 \rightarrow 0$ and $1 \rightarrow 0$ independent processes?

Obviously, each time when a classical erasure gate is working, it is working either as a $0 \rightarrow 0$ or a $1 \rightarrow 0$ process. The two processes cannot work simultaneously. From this, one could erroneously conclude that the two processes are independent. The process $0 \rightarrow 0$ is indeed different from $1 \rightarrow 0$, but they are not independent. There are at least two major sources of a relation between them. The first relation appears because of the condition given by Eq. (4), which is satisfied in any (time-reversible, weak or strong) erasure gate. It is just due to the fact that both processes, $0 \rightarrow 0$ and $1 \rightarrow 0$, share the same Hamiltonian $H(x, y)$. The second relation between the two processes appears in the weak and strong Landauer’s erasure principle through the macroscopic Condition C2. Those two relations between $1 \rightarrow 0$ and $0 \rightarrow 0$ processes are the heart of the Landauer’s erasure principle. We emphasize that, by construction, all erasure gates have to satisfy Eq. (4), but, as we have discussed, it is possible to envision environments that do not satisfy C2.

B. Why the dissipation of energy on a gate can depend on whether the final environment states are distinguished?

The dissipation on a gate is a physical process that, obviously, happens independently of whether the humans observe it. The fact that conditions C1, C2 and C3 are macroscopic conditions adapted to "anthropomorphic" perceptions on the state of the final environment states does not mean that those conditions are subjective or depending on human observations. Macroscopic conditions on physical systems are objective (physical) conditions which have to be satisfied by the microscopic evolutions (independently of human observations). Thus, macroscopic conditions have to be understood as a way to determine which system’s and environment’s trajectories, \{x(t), y(t)\}, are allowed. Specifying the whole set of allowed evolutions \{X(t), Y(t)\} in a system is equivalent to specifying the type of the Hamiltonian $H(x, y)$. In other words, the Hamiltonian $H_{C1}(x, y)$ satisfying C1 is different from the Hamiltonian $H_{C2}(x, y)$ and both are different from $H_{C3}(x, y)$. The three Hamiltonians, by construction, are designed to satisfy the same logical input/output table corresponding to the logical gate encoded in the system degrees of freedom, but they are physically different in the way they manipulate the environment degrees of freedom. Thus, each of the Hamiltonians can have a different dissipation effect, even if they provide the same logical table.
C. Is time reversible mapping sufficient for a demonstration of the weak Landauer limit?

In the literature, one can find developments leading to $\Delta S = k \ln 2$ invoking only the time-reversible mapping inherent to classical or quantum mechanics [14]. Such “demonstrations” are based on a simple argumentation. For an erasure gate, we get, by construction, that $X_{1 \rightarrow 0}(t_f) = X_{0 \rightarrow 0}(t_f)$, since both final macroscopic states are 0. We can also assume that the initial environment states are identical $Y_{1 \rightarrow 0}(t_i) = Y_{0 \rightarrow 0}(t_i)$. Then, since $X_{1 \rightarrow 0}(t_i) \cap X_{0 \rightarrow 0}(t_i) = \emptyset$, we conclude from Eq. (4) that $Y_{1 \rightarrow 0}(t_f) \cap Y_{0 \rightarrow 0}(t_f) = \emptyset$. Up to here, we have just repeated what we have deduced in subsection III A. The “demonstration” is ended by saying that since we need the doubled number of states in the final environment as compared to the initial state of environment, the final environment phase space volume has to be the double of the initial phase space volume. In this way one easily gets the result $\Delta S = k \ln 2$. This “demonstration”, although very simple, is incorrect (or at least incomplete) if it is not complemented with Condition C2. One can easily imagine a final environment set of states $Y_{1 \rightarrow 0}(t_f)$ with all particles in the left part of the environment and $Y_{0 \rightarrow 0}(t_f)$ with all particles in the right part of the environment. Those two final environment set of states perfectly satisfy $Y_{1 \rightarrow 0}(t_f) \cap Y_{0 \rightarrow 0}(t_f) = \emptyset$, but the phase space volume for each of the processes does not need to double. We are arguing here that classical or quantum mechanics alone cannot explain $\Delta S = k \ln 2$. Indeed, this conclusion has already been mentioned by Hemmo and Shenker in [24]. We argue in this work that if macroscopic conditions for the environment different from C2 are invoked, then different limits for $\Delta S$ can be envisioned.

D. Is Shannon information entropy proportional to the physical entropy?

There are many “demonstrations” of the Landauer’s erasure principle in the literature that invoke the idea of identifying the change of Shannon information entropy with the change of negative thermodynamic entropy. A typical argument in the literature says that, for one bit of information, the initial Shannon entropy [34] (before erasing the information) is

$$H(t_i) = -p_0(t_i) \ln p_0(t_i) - p_1(t_i) \ln p_1(t_i) = \ln 2,$$  \hspace{1cm} (14)

where $p_0(t_i) = 1/2$ is the probability of having 0 at the input gate and $p_1(t_i) = 1/2$ is the probability of having 1. The final Shannon entropy (after erasing the information) is

$$H(t_f) = -p_0(t_f) \ln p_0(t_f) - p_1(t_f) \ln p_1(t_f) = 0,$$  \hspace{1cm} (15)
where we have \( p_0(t_f) = 1 \) and \( p_1(t_f) = 0 \) because there is no uncertainty on the final logical state. Thus, the change of Shannon entropy during the erasure process is \( \Delta H = H(t_f) - H(t_i) = -\ln 2 \).

By identifying the change of information entropy with the change of negative thermodynamic entropy, we get \( \Delta S = -k\Delta H = k\ln 2 \) (where the Boltzmann constant \( k \) is just the conversion factor needed because Shannon entropy and Boltzmann entropy are expressed in different units).

Although the final result might look like an extremely simple (even magical) demonstration of Eq. (12), it is very misleading because it suggests a wrong conclusion that dissipation by a physical gate only depends on the logical (not physical) gate.

Let us discuss an example of how, of course, the dissipation have to be linked to the type of physical Hamiltonian used to design the logical gate. Let us imagine that we use a NAND gate, the Hamiltonian of which is \( H_{NAND,C3}(x,y) \) and satisfies the following logical irreversible table:

\[
\begin{align*}
00 & \rightarrow 1, \\
10 & \rightarrow 1, \\
01 & \rightarrow 1, \\
11 & \rightarrow 0.
\end{align*}
\]

It is straightforward to show that it is a dissipative gate when \( C_3 \) is assumed. But let us imagine that we use this NAND gate with the second input bit always fixed to \( 1 \). So now, by looking at the non-fixed first input bit and at the output bit, we have an inverter \( 0 \rightarrow 1, 1 \rightarrow 0 \) (which in the language of the original NAND gate means that we only take into account the \( 01 \rightarrow 1 \) and \( 11 \rightarrow 0 \) cases). The logical operation is now an inverter, which is a reversible logical gate. Then, according to the Shannon entropy which only looks at the logic, not at the physics, our initial entropy in expression (14) will be zero, giving \( \Delta H = 0 \). Thus, if we accept that \( \Delta S = -k\Delta H = 0 \), we will conclude that there is no dissipation. But in this logical inverter gate, constructed from a physical NAND gate with the Hamiltonian \( H_{NAND,C3}(x,y) \), we know that there is dissipation. The mistake is obviously in the identification \( \Delta S = -k\Delta H \), which is wrong. Clearly, if we want to construct an inverter, we can always construct a new Hamiltonian \( H_{inverter,C3}(x,y) \) that may or may not produce dissipation. But if we use the original Hamiltonian \( H_{NAND,C3}(x,y) \) to perform the inversion, then dissipation will be unavoidable (whatever the Shannon information entropy says). Having said this, one could still argue that the Shannon entropy is related to the Boltzmann entropy under the assumption that one only considers physical gates that take the minimum value of entropy change allowed by the weak Landauer’s principle. Such an argument would be correct, but it would seem as useless as saying that the Shannon entropy is related to the Boltzmann entropy whenever such a relation is valid.

Moreover, we have shown in Section III A that we can develop a reversible erasure gate with \( \Delta S = 0 \) that perfectly works \( M \) times. Each time we have the Shannon result \( \Delta H = -\ln 2 \), clearly showing that, in general, \(-k\Delta H \) cannot be identified with \( \Delta S \).
At this point, it may be instructive to recapitulate the essence of the problem of describing an *irreversible* logic gate with a physical system. The problem arises from the fact that the known microscopic laws of physics are time-reversible, so, from a fundamental microscopic point of view, there is no such thing as information erasure. Information only looks erased when looked from a coarse grained macroscopic point of view. The apparently erased information is in fact encoded in fine details that, for one reason or another, we do not perceive as such in practice. There are, in fact, two types of possible reasons for not perceiving those details; either because we choose to not pay attention to them, or because the details are well hidden so that, in practice, we cannot resolve them even if we want to. The first possibility corresponds to time-reversible erasure, because we can restore information simply by changing our focus and choosing to pay attention to the previously ignored details. The second possibility can be split into two sub-types; either the fine details are in a state close to a thermal equilibrium, or they aren’t. If they are close to a thermal equilibrium, we talk of a strong version of the Landauer’s erasure principle. If they aren’t, we can still talk of a weak version of the Landauer’s principle, which is a generalization of the strong one.

**E. Can we develop logical erasure gates without environment by using one initial time \( t_i \) and two final times \( t_{f,0 \rightarrow 0} \) and \( t_{f,1 \rightarrow 0} \)?**

All developments done in this paper are, at the end of the day, based on time-reversibility of microscopic trajectories in the phase space. If we try to develop an erasure gate without environment degrees of freedom, we have to look for two different trajectories: \( x_{0 \rightarrow 0}(t) \) and \( x_{1 \rightarrow 0}(t) \). We want them to be different at the initial time \( t_i \) but equal at the final time \( t_f \), which, however, is physically impossible in a closed system because it violates time-reversibility. This is exactly the reason why the erasure gate (or any other irreversible logical gate) needs an environment. Not because the environment cannot be avoided, but just the contrary. We want the environment degrees of freedom to convert a (closed) time-reversible system into an (open) reversible one, so that we can justify the physical effective irreversibility of the system by saying that the system is open due to the existence of the disregarded environment.

However, one could attempt to avoid a need for environment by considering a gate with two different final times. One time \( t_{f,1 \rightarrow 0} \) is related to the trajectory \( x_{1 \rightarrow 0}(t) \) and another time \( t_{f,0 \rightarrow 0} \) is related to the trajectory \( x_{0 \rightarrow 0}(t) \). Then, in principle, we can satisfy the requirements \( x_{1 \rightarrow 0}(t_i) \neq x_{0 \rightarrow 0}(t_i) \) and \( x_{1 \rightarrow 0}(t_{f,1 \rightarrow 0}) = x_{0 \rightarrow 0}(t_{f,0 \rightarrow 0}) \) without violating the time-reversibility. However, such an attempt does not work for time independent Hamiltonians \( H(x) \). If it is time-independent,
then it is invariant under time translations so the solutions of the Hamilton equations of motion are also invariant under time translations. Therefore, if $x_{1\rightarrow 0}(t_{f,1\rightarrow 0}) = x_{0\rightarrow 0}(t_{f,0\rightarrow 0})$ where $t_{f,1\rightarrow 0} - t_{f,0\rightarrow 0} \equiv T$, then $x_{1\rightarrow 0}(t + T) = x_{0\rightarrow 0}(t)$ for all $t$. In particular, at $t = t_i$ this implies $x_{1\rightarrow 0}(t_i + T) = x_{0\rightarrow 0}(t_i)$, which is not satisfactory for a properly working logical gate. If, on the other hand, one considered a time dependent Hamiltonian, this, in fact, would imply the existence of environment degrees of freedom that one attempted to avoid. In any case, the possibility of operating with many consecutive gates, each with a different computational time depending on its logical input value, would not be a valid strategy for typical synchronous digital circuits.

F. Do the weak and strong Landauer’s erasure principles take into account the noise margins?

The value of the Landauer erasure noise margins are the “engineering” way of specifying the macroscopic state $A$ of the input information. In present day devices, the input information is linked to a value of an electrostatic potential. It is accepted that if the value of the input electrostatic potential is $A \pm \Delta A$, the Hamiltonian will provide the correct out value (within the margins) $B \pm \Delta B$. The important point is that another different input potential $A' \pm \Delta A'$ has to be clearly distinguishable from $A \pm \Delta A$. One typical condition for distinguishability is $\Delta A = \Delta A' = |A - A'|/3$. This implies that there are many initial microscopic states that belong neither to the macrostate 1 nor to 0. We denote the whole set of those microstates as $X_{A'' \rightarrow B''}(t_i)$, belonging to the process $A'' \rightarrow B''$. Thus there are at least three types of input macroscopic states, namely $X_{1\rightarrow 0}(t_i), X_{0\rightarrow 0}(t_i)$ and $X_{A'' \rightarrow 0}(t_i)$. If we assume that all macroscopic states evolve to 0 at the final time, we have $X_{A'' \rightarrow 0}(t_f) = X_{0\rightarrow 0}(t_f) = X_{1\rightarrow 0}(t_f)$. Then, under the assumption C2 for the three processes, we arrive at the conclusion that the dissipated entropy in the environment of a more realistic erasure gate is

$$\Delta S_{0\rightarrow 0} = k \ln(V_A + V_{A'} + V_{A''}) - k \ln V_A = k \ln \left(1 + \frac{V_{A'}}{V_A} + \frac{V_{A''}}{V_A}\right).$$

(16)

The relevant point now is to evaluate the ratio $V_{A''}/V_A$, which takes a value of several orders of magnitude if we realize that we are discussing states in the $N_S$-dimensional phase space where some important restrictions are needed to define the macroscopic states $A = 1$ and $A' = 0$, while the definition of $A''$ is free from such restrictions. The conclusion is that the weak Landauer’s erasure principle is just the lower limit, arising from many assumptions made in its derivation. Among others, we point out that the weak and strong Landauer’s erasure principle do not take
into account the presence of noise margins. Notice that the time-reversible result developed in Section III A is free from this approximation (as far as it is reasonable to model the final state of environment with C1 that can distinguish the three final states of environment.)

G. Do the time-reversible, weak and strong relations take into account the physical processes occurring inside the device during the time interval \( t_f - t_i \)?

At room temperature, according to the Landauer’s erasure principle, the erasure of a one bit implies a heat dissipation \( \Delta Q = T\Delta S = kT \ln 2 = 3 \cdot 10^{-21} \) Joules. This value is several orders of magnitude smaller than the actual dissipation in the state-of-the-art devices produced today. Certainly, the Landauer’s erasure principle has to be understood as a lower limit on the entropy change when information is manipulated by a gate. One of the reasons is the fact that, in this paper, we only discussed the entropy at the initial time \( t_i \) and the final time \( t_f \). In principle, one could integrate the entropy change along the whole time interval in the evolution from 1 towards 0. However, such a procedure would require to solve an important conceptual problem of how to define macroscopically distinguishable intermediate states the entropy of which is to be computed for each intermediate time \( t \). Conceptually, the problem of dissipation during the time interval \( t_f - t_i \) can be better tackled with a purely mechanical analysis, e.g. by a simulation of evolution of \( N \) degrees of freedom on a computer. In this way one can study types of environment which are much more complex than those discussed in this paper. In any case, with the use of the Liouville theorem, Eq. (3), satisfied at all times, we have been able to identify some properties of possible evolutions without actually computing them. This allowed us to establish some minimum value of the entropy change in a process involving a manipulation of information.

V. CONCLUSIONS

There are still debates on correctness of the Landauer’s erasure principle [1]. For the majority of theoretical scientists it is a masterpiece of science connecting information manipulation with physical entropy [8,20], while for others it is just nonsense [21–29]. In this paper, we discussed the limitations, applicability and generalizations of the original Landauer’s erasure principle [1]. We argued that in some scenarios (those where the environment cannot be treated as a thermal bath assumed in Condition C3), the (original) strong Landauer’s erasure principle has to be substituted by a (more general) weak Landauer’s erasure principle (that satisfies Condition C2) where the
entropy change does not need to be related to heat dissipation. We also argued that even the weak Landauer’s erasure principle does not apply in more exotic environments accomplishing Condition C1. Having those limitations, applicability and generalizations of the original Landauer’s erasure principle[1], we want to emphasize the extraordinary merit of the Landauer’s contribution to clarifying the fundamental physical limits between manipulation of information and entropy change.

We end this paper with a sentence of Landauer [31]. “The path to understanding in science is often difficult. If it were otherwise, we would not be needed. This field [fundamental physical limits of information handling], however, seems to have suffered from an unusually convoluted path”. What we find especially unfortunate during the recent developments in this field is linking the result to thermodynamics, through the so called “thermodynamics of computation” [9]. After all, electronic devices (including the system and its environment) are non-equilibrium systems when a bias is applied. But the main reason why it is unfortunate is because it is an unnecessary link. The weak Landauer’s erasure principle can be deduced, as we have done here, without invoking thermodynamics at all. Linking the development of computing gates only to thermal environments has had the undesired effect of unnecessarily limiting the imagination of many researchers. The original sin is the widespread mistake in many textbooks and scientific literature of referring to the generalized second law as a second law of thermodynamics. We hope that this paper can help in correcting this mistake and open new possibilities for engineering environments that satisfy Condition C2 involving entropy change without heat dissipation, or even approaching Condition C1 where the entropy change can be reduced significantly.

ACKNOWLEDGMENTS

X.O. acknowledges financial support from Spain’s Ministerio de Ciencia, Innovación y Universidades under Grant No. RTI2018-097876-B-C21 (MCIU/AEI/FEDER, UE), the European Union’s Horizon 2020 research and innovation programme under grant agreement No Graphene Core2 785219 and under the Marie Skodowska-Curie grant agreement No 765426 (TeraApps). The work of H.N. was supported by the European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (KK.01.1.1.06).
REFERENCES

[1] R. Landauer, IBM Journal of Research and Development 5, 183 (1961).
[2] A. Bérut, A. Arakelyan, A. Petrosyan, S. Ciliberto, R. Dillenschneider, and E. Lutz, Nature 483, 187 (2012).
[3] J. Hong, B. Lambson, S. Dhuey, and J. Bokor, Science Advances 2, e1501492 (2016).
[4] L. Yan, T. Xiong, K. Rehan, F. Zhou, D. Liang, L. Chen, J. Zhang, W. Yang, Z. Ma, and M. Feng, Physical Review Letters 120, 210601 (2018).
[5] A. O. Orlov, C. S. Lent, C. C. Thorpe, G. P. Boechler, and G. L. Snider, Japanese Journal of Applied Physics 51, 06FE10 (2012).
[6] M. Gavrilov and J. Bechhoefer, Physical Review Letters 117, 200601 (2016).
[7] M. Lopez-Suarez, I. Neri, and L. Gammaitoni, Nature Communications 7, 1 (2016).
[8] C. H. Bennett, IBM Journal of Research and Development 17, 525 (1973).
[9] C. H. Bennett, International Journal of Theoretical Physics 21, 905 (1982).
[10] C. H. Bennett, IBM Journal of Research and Development 32, 16 (1988).
[11] C. H. Bennett, Studies In History and Philosophy of Science Part B: Studies In History and Philosophy of Modern Physics 34, 501 (2003).
[12] E. Lutz and S. Ciliberto, Phys. Today 68, 30 (2015).
[13] M. P. Frank, in International Conference on Reversible Computation (Springer, 2018) pp. 3–33.
[14] K. Jacobs, arXiv preprint quant-ph/0512105 (2005).
[15] M. Pezzutto, M. Paternostro, and Y. Omar, New Journal of Physics 18, 123018 (2016).
[16] D. Reeb and M. M. Wolf, New Journal of Physics 16, 103011 (2014).
[17] K. Maruyama, F. Nori, and V. Vedral, Reviews of Modern Physics 81, 1 (2009).
[18] C. H. Bennett, Physical Review Letters 53, 1202 (1984).
[19] P. Benioff, Physical Review Letters 53, 1203 (1984).
[20] T. Toffoli, Physical Review Letters 53, 1204 (1984).
[21] J. D. Norton, Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 36, 375 (2005).
[22] J. D. Norton, Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics 42, 184 (2011).
[23] J. D. Norton, Philosophy of Science 80, 1182 (2013).
[24] M. Hemmo and O. Shenker, Entropy 15, 3297 (2013).
[25] L. B. Kish and D. K. Ferry, Journal of Computational Electronics 17, 43 (2018).
[26] W. Porod, R. Grondin, D. Ferry, and G. Porod, Physical Review Letters 52, 232 (1984).
[27] W. Porod, R. Grondin, D. Ferry, and G. Porod, Physical Review Letters 53, 1206 (1984).
[28] L. B. Kish and C. G. Granqvist, Europhysics Letters 98, 68001 (2012).

[29] L. B. Kish, C. G. Granqvist, S. P. Khatri, and H. Wen, in International Journal of Modern Physics: Conference Series, Vol. 33 (World Scientific, 2014) p. 1460364.

[30] E. T. Jaynes, American Journal of Physics 33, 391 (1965).

[31] R. Landauer, in Proceedings Workshop on Physics and Computation. PhysComp’94 (IEEE, 1994) pp. 54–59.

[32] P. Benioff, Journal of Statistical Physics 22, 563 (1980).

[33] T. Sagawa and M. Ueda, Physical Review Letters 102, 250602 (2009).

[34] C. E. Shannon, Proceedings of the IRE 37, 10 (1949).