PROPERTIES OF THE DIRAC SPECTRUM ON THREE DIMENSIONAL LENS SPACES

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Abstract. We present a spectral rigidity result for the Dirac operator on lens spaces. More specifically, we show that homogeneous lens spaces as well as three dimensional lens spaces $L(q;p)$ with $q$ a prime are completely characterized by their spectra in the class of all lens spaces.

1. Introduction

Let $M$ be a compact Riemannian spin manifold, i.e. a compact oriented Riemannian manifold with a spin structure $P$. Then there is a canonical first order differential operator $D$ on $M$ called the (spin-) Dirac or Atiyah-Singer operator. This operator is elliptic and hence possesses a discrete spectrum $\text{Spec}_D(M)$ consisting only of eigenvalues with finite multiplicities. A typical question in spectral geometry is to what extent the geometry of $M$ is determined by $\text{Spec}_D(M)$ or by the spectrum of any other canonical geometric operator.

In this article, we investigate this question for homogeneous as well as three dimensional lens spaces, quotients of the sphere $S^{2m-1}$ by cyclic groups of isometries.

Spectrally, lens spaces were first examined by A. Ikeda and Y. Yamamoto in [IY79] and [Yam80]. They proved that homogeneous as well as three dimensional lens spaces are completely determined by the spectra of their Laplace-Beltrami operators in the class of all lens spaces.

Spectra are seldom explicitly computable. For a $2m-1$ dimensional lens space $L = L(q;p_1,\ldots,p_m)$ (see Section 2 for notation and definitions), the eigenvalues of the Laplaician are $k(k+2(m-1))$, $k \in \mathbb{N}_0$ with corresponding multiplicities $m_k$ which depend on the lens space $L$. Ikeda and Yamamoto introduced the generating function $F^L$ of the Laplace operator on $L$, defined by

$$F^L(z) = \sum_{k=0}^{\infty} m_k z^k.$$

By definition, this power series encodes the whole spectrum. Hence, two lens spaces $L$ and $L'$ are isospectral if and only if $F^L = F^{L'}$. Ikeda and Yamamoto prove the generating functions to have meromorphic extensions to all of $\mathbb{C}$ of the form

$$F^L(z) = \frac{1}{q} \sum_{k=0}^{q-1} \prod_{j=1}^{m} (\xi^k_{q^j} - z)(\xi^{kq_j} - z)^{-1},$$

where $\xi_q$ denotes the $q$-th root of unity $e^{2\pi i/q}$.

They then use formula (1.1) to prove the spectral rigidity of homogeneous lens spaces by determining the order of the poles of $F^L$. In the non-homogeneous case, a careful analysis of the poles and residues of $F^L$ leads to the following set of equations being true if the three dimensional lens spaces $L(q;p)$ and $L(q;s)$ are isospectral

$$\cot \frac{k(p+1)}{q} \pi - \cot \frac{k(p-1)}{q} \pi + \cot \frac{k(p^*+1)}{q} \pi - \cot \frac{k(p^*-1)}{q} \pi$$

for all $1 \leq k \leq q - 1$ s.t. $k(p \pm 1) \not\equiv 0$ (mod $q$) and $k(s \pm 1) \not\equiv 0$ (mod $q$), where $p^*$ and $s^*$ are multiplicative inverses modulo $q$ of $p$ and $s$ respectively. The solutions of this set of equations are $p \equiv \pm s$ (mod $q$) and $p \cdot s \equiv \pm 1$ (mod $q$) which means precisely that $L(q;p)$ and $L(q;s)$ are isometric.

The key ingredient to solving these equations is the fact that the numbers $\cot \frac{k}{q} \pi$ with $1 \leq k \leq \frac{q}{2}, (k,q) = 1$ are linearly independent over $\mathbb{Q}$ (see e.g. [BBW73]), a fact that goes back to a very interesting problem of Chowla [Cho70].
Inspired by Ikeda’s and Yamamoto’s work, C. Bär introduced the generating functions of the Dirac operator on lens spaces and more generally spherical space forms in his Ph.D. thesis [Bär90] (see also [Bär96]). The eigenvalues of the Dirac operator on the lens space \( L = L(q; p_1, \ldots, p_m) \) with a fixed spin structure are \( \pm \left( \frac{2m-1}{2} + k \right) \), \( k \in \mathbb{N}_0 \) with corresponding multiplicities \( m_k^\pm \). The associated generating functions are

\[
F_{\pm}^L(z) = \sum_{k=0}^{\infty} m_k^\pm z^k.
\]

Bär proves these generating functions to have meromorphic extensions to all of \( \mathbb{C} \). For odd \( q \), these have the form (see Corollary 3.6)

\[
F_{\pm}^L(z) = \frac{2^{m-1}}{q} \sum_{k=0}^{q-1} \sum_{\epsilon_1 \cdots \epsilon_m = (-1)^{m+1}} \xi_{2q}^{(q+1)k \sum_{j} \epsilon_j p_j} - z \cdot \xi_{2q}^{(q+1)k \sum_{j} \epsilon_j p_j} \frac{m}{j=1} \prod_{j=1}^{m} (\xi_{q}^{k p_j} - z)(\xi_{q}^{k p_j} - z),
\]

\[
F_{-}^L(z) = \frac{2^{m-1}}{q} \sum_{k=0}^{q-1} \sum_{\epsilon_1 \cdots \epsilon_m = (-1)^{m+1}} \xi_{2q}^{(q+1)k \sum_{j} \epsilon_j p_j} - z \cdot \xi_{2q}^{(q+1)k \sum_{j} \epsilon_j p_j} \frac{m}{j=1} \prod_{j=1}^{m} (\xi_{q}^{k p_j} - z)(\xi_{q}^{k p_j} - z).
\]

The aim of this article is to shed light onto the relationship between the Dirac spectrum and the geometry of homogeneous as well as three dimensional lens spaces. We will use Bär’s formulas and the path that has been paved by Ikeda and Yamamoto to do so, viz. we will analyse the poles and residues of \( F_{\pm}^L \), which encode spectral information, to deduce geometric properties of \( L \).

There are, however, some differences to the Laplace case. The Dirac spectrum does not only depend on the metric, but also on the orientation and the spin structure. The dependence on the orientation is already inherent in the definition of a spin structure and one can easily see that a change of orientation causes the spectrum to flip around zero. The second dependence is more complicated, i.e. there is no general relation between the spectra associated to inequivalent spin structures, see e.g. [Bär00]. If, however, the Riemannian spin manifold \( M \) has two inequivalent spin structures \( P \) and \( Q \) and an, say orientation preserving, isometry \( f \) that relates the two spin structures, i.e. the differential \( df : SO(M) \rightarrow SO(M) \) as a map of the oriented orthonormal frame \( SO(M) \) of \( M \) lifts to a map \( \tilde{f} : \tilde{P} \rightarrow \tilde{Q} \), then the spectra of the Dirac operators associated to the spin structures \( P \) and \( Q \) coincide. For an orientation reversing isometry \( f \), the definition of \( f \) relating \( P \) and \( Q \) has to be extended via cohomological methods (see Definition 2.5), and the associated spectra will be the same after one is flipped around zero.

Because of the above, we introduce the notion of two Riemannian spin manifolds being \( \varepsilon \)-spin-isometric, i.e. there is an isometry that relates their spin structures and is, according to \( \varepsilon = 1 \) or \( \varepsilon = -1 \), orientation preserving or reversing. We then extend the well-known Theorem 2.3 which states when exactly two lens spaces are isometric, to Theorems 2.7 and 2.8 which state when exactly two lens spaces are \( \varepsilon \)-spin-isometric.

Of course, \( \varepsilon \)-spin-isometric manifolds are \( \varepsilon \)-isospectral; that is for \( \varepsilon = 1 \) their spectra coincide (including multiplicities) whereas for \( \varepsilon = -1 \) their spectra coincide after one is flipped around zero. Isospectrality is then understood as \( \varepsilon \)-isospectrality for an \( \varepsilon \in \{ \pm 1 \} \).

The two main results of this paper, Theorems 3.3 and 4.10 are concerned with the inverse direction. Theorem 3.8 states that if two lens spaces, one of which is homogeneous, are isospectral, they are isometric. In dimension three, this statement is strengthened to state that two \( \varepsilon \)-isospectral lens spaces, one of which is homogeneous, are \( \varepsilon \)-spin-isometric.

Theorem 4.10 states that for prime \( q \), if the lens spaces \( L(q; p) \) and \( L(q; s) \) are \( \varepsilon \)-isospectral, then they are \( \varepsilon \)-spin-isometric. The proof starts out for general \( q \) in the same way that Ikeda’s and Yamamoto’s proof did and arrives at the following set of equations in the case that \( q \) is odd (the case \( q \) even is similar, see Corollary 4.7) and both lens spaces are non-homogeneous

\[
\xi_{q}^{\pm 1} k p \left( \cot \frac{k(p+1)}{q} \pi - \cot \frac{k(p-1)}{q} \pi \right) + \xi_{q}^{\pm 1} k p^* \left( \cot \frac{k(p+1)}{q} \pi - \cot \frac{k(p-1)}{q} \pi \right) = \xi_{q}^{\pm 1} k s \left( \cot \frac{k(s+1)}{q} \pi - \cot \frac{k(s-1)}{q} \pi \right) + \xi_{q}^{\pm 1} k s^* \left( \cot \frac{k(s+1)}{q} \pi - \cot \frac{k(s-1)}{q} \pi \right)
\]
for all $1 \leq k \leq q - 1$ s.t. $k(p \pm 1) \not\equiv 0 \pmod q$ and $k(s \pm 1) \not\equiv 0 \pmod q$. Unlike the case of the numbers $\cot \frac{k\pi}{q}$, $(k, q) = 1$, there is no statement about the linear independence of the numbers $\xi^k q \cot \frac{k\pi}{q}$. In fact, it is already hard to formulate such a statement as there are nontrivial linear dependences among these numbers due to dimensional reasons.

However, at the end of [Yam80], Yamamoto gave an alternative proof for the solutions of the equations (1.2) in case $q$ is a prime number. That proof uses techniques from analytic number theory, namely the theory of $\lambda$-adic series in the cyclotomic fields $\mathbb{Q}_q = \mathbb{Q}(\xi_q)$ and it carries over the the case of the Dirac operator, i.e. we will use this technique to solve the equations (1.3).

Unfortunately, that proof does not work for arbitrary $q \in \mathbb{N}$. Numerical calculations for $q$ in a large range suggest that Theorem 4.10 is true if one drops the assumption that $q$ is prime, see Remark 4.13 and Conjecture 4.12, but the method of proof would have to be different.

This paper’s main focus lies on three dimensional lens spaces and one can ask about Dirac isospectrality of higher dimensional lens spaces. This has been investigated in [BL14]. With a representation theoretic approach, explicit formulas for the multiplicities $m^k_q$ have been found and consequently criteria for the isospectrality of lens spaces established. Furthermore, we could construct families of isospectral lens spaces in higher dimensions.

We refer the reader who is new to spin geometry to [LM89] for a comprehensive introduction. A thorough treatment of the Dirac spectrum is given in [Gin09]. In particular, several examples of Dirac isospectral pairs and families are given therein.

The paper is organised as follows. In Section 2 we introduce lens spaces, describe their spin structures and their $+1$-spin-isometry classes. Section 3 contains the description of the spectra of lens spaces via generating families are given therein.

**Dedication**

I dedicate this article to Jörg Schülke, my dear father and first teacher in mathematics.

### 2. Lens Spaces, Spin Structures and Isometry Classes

This section contains the setup for everything to come. We define lens spaces, describe their spin structures and isometry classes.

**Definition 2.1.** Let $q \in \mathbb{N}$, $p_1, \ldots, p_m \in \mathbb{Z}$ with $(q, p_i) = 1$ for $1 \leq i \leq m$. Define $\gamma^p_q$ to be

$$
\gamma^p_q := \gamma^p_{q_1 \cdots q_m} := \text{diag} \left( \begin{array}{c}
\cos \left( \frac{2\pi q_1}{p_1} \right) \sin \left( \frac{2\pi q_1}{p_1} \right) \\
\sin \left( \frac{2\pi q_1}{p_1} \right) \cos \left( \frac{2\pi q_1}{p_1} \right)
\end{array} \right), \ldots, \left( \begin{array}{c}
\cos \left( \frac{2\pi q_m}{p_m} \right) \sin \left( \frac{2\pi q_m}{p_m} \right) \\
\sin \left( \frac{2\pi q_m}{p_m} \right) \cos \left( \frac{2\pi q_m}{p_m} \right)
\end{array} \right) \right) \in SO(2m).
$$

The lens space $L(q; p_1, \ldots, p_m)$ is defined as the quotient

$$L(q; p_1, \ldots, p_m) = \langle \gamma^p_{q_1 \cdots q_m} \rangle \setminus S^{2m-1}.$$

The matrix $\gamma^p_{q_1 \cdots q_m}$ is of order $q$ and has, by assumption, only primitive $q$-th roots of unity as eigenvalues. As such, it generates a free action of a finite group of orientation preserving isometries of $S^{2m-1}$ and $L(q; p_1, \ldots, p_m)$ is thus canonically given the structure of an oriented Riemannian manifold.

The bundle of oriented orthonormal frames $SO(S^{2m-1})$ of $S^{2m-1}$ is $SO(2m)$ with projection onto the last column vector. Spin structures are the fibrewise non-trivial two-sheeted coverings of the bundle of oriented orthonormal frames (see [LMS9] Theorem 1.4], which, in this case, is seen to be $Spin(2m)$. Due to its simply-connectedness, this is the only spin structure of $S^{2m-1}$. As lens spaces $L(q; p_1, \ldots, p_m)$ are quotients of the sphere, their spin structures arise as certain quotients of the sphere’s spin structure $Spin(2m)$ (see [Gin09 Proposition 1.4.2]), which are in one-to-one correspondence with group homomorphisms $\tau : \Gamma \to Spin(2m)$ such that $\Theta \circ \tau = \text{Id}_\tau$, where $\Theta : Spin(2m) \to SO(2m)$ is the universal covering homomorphism.

Spin structures on lens spaces were first classified in [Pra87], though not in the language described above.

**Proposition 2.2.** The lens space $L = L(q; p_1, \ldots, p_m)$ admits a spin structure if and only if $q$ is odd or $m$ is even. If $q$ is odd, the unique spin structure is given by

$$
\tau \left( \left( \gamma^p_q \right)^k \right) := \prod_{j=1}^{m} \left( \cos \frac{k(q+1)p_j\pi}{q} + \sin \frac{k(q+1)p_j\pi}{q} e_{2j-1} e_{2j} \right).
$$
If \( q \) and \( m \) are even, there are precisely two spin structures given by

\[
\tau_h \left( \left( \gamma_q^p \right)^k \right) := (-1)^{h(h+\sum_{j=1}^m)} \prod_{j=1}^m \left( \cos \frac{k p_j \pi}{q} + \sin \frac{k p_j \pi}{q} e_{2j-1} e_{2j} \right). 
\]

for \( h \in \{0, 1\} \), where \( h_q^p := h_{q1} \cdots h_{qm} := \sum_{j=1}^m |L_j^q|. \)

**Proof.** A group homomorphism \( \tau : \langle \gamma_q^p \rangle \rightarrow \text{Spin}(2m) \) with \( \Theta \sigma = \text{Id} \) has to map to one of its preimages under \( \Theta \), which are \( \pm \prod_{j=1}^m \left( \cos \frac{k p_j \pi}{q} + \sin \frac{k p_j \pi}{q} e_{2j-1} e_{2j} \right) \) (see e.g. [BR95, pp. 173]). It thus suffices to determine the order of these elements. By elementary calculations in the group \( \text{Spin}(2m) \) (see e.g. [LM89, Chapter I])

\[
\left( \pm \prod_{j=1}^m \left( \cos \frac{k p_j \pi}{q} + \sin \frac{k p_j \pi}{q} e_{2j-1} e_{2j} \right) \right)^q = (\pm 1)^q \prod_{j=1}^m (\cos (k p_j \pi) + \sin (k p_j \pi) e_{2j-1} e_{2j}) 
= (\pm 1)^q \prod_{j=1}^m (-1)^{p_j} 
= (\pm 1)^q (\sum_{j=1}^m p_j) .
\]

If \( q \) is odd, exactly one of the two preimages has, depending on the parity of \( \sum_j p_j \), order \( q \). If \( q \) is even, the \( p_i \)'s are necessarily odd so that both preimages have order \( q \) if and only if \( m \) is even. For \( \tau_h, h \in \{0, 1\} \) to be invariant modulo \( q \) instead of \( 2q \), we introduce \( h_q^h \).

**Notation 2.3.** If \( q \) is odd, let \( L(q; p_1, \ldots, p_m) \) be equipped with its unique spin structure. If, on the other hand, \( q \) and \( m \) are even, denote by \( L(q; p_1, \ldots, p_m; h) \) the lens space \( L(q; p_1, \ldots, p_m) \) together with the spin structure \( \tau_h \).

We now turn our attention to the isometry classes of lens spaces. The following theorem is well-known.

**Theorem 2.4.** If \( L = L(q; p_1, \ldots, p_m) \) and \( L' = L(q; s_1, \ldots, s_m) \), the following assertions are equivalent:

1. \( \tau \) and \( \tau' \) are isometric.
2. There is a number \( \ell \in \mathbb{Z} \), a permutation \( \sigma \in S_m \), and there are numbers \( \epsilon_i \in \{\pm 1\} \) such that
   \[ \ell p_\sigma(i) \epsilon_\sigma(i) \equiv s_i \pmod{q} \]
   for every \( 1 \leq i \leq m \).

**Proof.** (1) → (2): If \( L \) and \( L' \) are isometric, they are certainly homeomorphic. Now [Coh73, §31] asserts (2).

To prove (2) → (1), we associate to the data \( \ell, \sigma \) and \( \epsilon_\sigma, 1 \leq i \leq m \) the \( S^{2m-1}\)-isometry

\[
\Psi(x_1, y_1, \ldots, x_m, y_m) := \left( x_{\sigma(1)}, \epsilon_{\sigma(1)} y_{\sigma(1)}, \ldots, x_{\sigma(m)}, \epsilon_{\sigma(m)} y_{\sigma(m)} \right).
\]

One easily checks that

\[
(\gamma_q^p)^\ell = \Psi^{-1} \circ \gamma_q^{\sigma} \circ \Psi
\]
so that \( \Psi \) induces an isometry \( \psi : L \rightarrow L' \) by \( [x] \mapsto [\Psi(x)] \).

The spin structures of a spin manifold \( M \) are in one-to-one correspondence with the elements of \( H^1(M; \mathbb{Z}_2) \) (see e.g. [LM89, Theorem 1.7]). Let \( M \) and \( N \) be spin manifolds with spin structures \( P \) and \( Q \) corresponding to the elements \( \mu \in H^1(M; \mathbb{Z}_2) \) and \( \nu \in H^1(N; \mathbb{Z}_2) \) respectively and let \( f : M \rightarrow N \) be a diffeomorphism.

**Definition 2.5.** (1) The diffeomorphism \( f \) relates the spin structures \( P \) and \( Q \) if \( f^*(\nu) = \mu \).

2. A smooth map between two oriented Riemannian manifolds is said to be a \(+1\)-isometry if it is an orientation preserving isometry. It is said to be a \(-1\)-isometry if it is an orientation reversing isometry.

3. Two Riemannian spin manifolds are spin-isometric if there is an isometry between them that relates their spin structures.

4. Let \( \varepsilon \in \{\pm 1\} \). Two Riemannian spin manifolds are \( \varepsilon\)-spin-isometric if there is an \( \varepsilon \)-isometry between them relating their spin structures.

**Theorem 2.6.** Let \( \varepsilon \in \{\pm 1\} \). If \( L = L(q; p_1, \ldots, p_m) \) and \( L' = L(q; s_1, \ldots, s_m) \), the following assertions are equivalent:

1. \( L \) and \( L' \) are \( \varepsilon \)-isometric.
(2) There is a number \( \ell \in \mathbb{Z} \), a permutation \( \sigma \in S_m \) and there are numbers \( \varepsilon_i \in \{ \pm 1 \} \) such that
\[
lp_{\sigma(i)} \varepsilon_{\sigma(i)} \equiv s_i \pmod{q} \quad 1 \leq i \leq m
\]
\[
\prod_{i=1}^{m} \varepsilon_i = \varepsilon.
\]

Proof. (1) \( \Rightarrow \) (2): We assume \( q > 2 \). Let \( f : L \to L' \) be an \( \varepsilon \)-isometry. Identify \( \pi_1(L) \) and \( \pi_1(L') \) with \( \langle \gamma^p \rangle \) and \( \langle \gamma^s_q \rangle \) respectively and denote by \( f_\# : \pi_1(L) \to \pi_1(L') \) the induced map on fundamental groups. Define \( \ell \in \mathbb{Z} \) as the smallest non-negative integer satisfying
\[
f_\# \left( \langle \gamma^p_q \rangle^\ell \right) = \gamma^s_q.
\]
By \([\text{Coh} 73, 30.1]\), there are numbers \( \varepsilon_i \in \{ \pm 1 \} \) and a permutation \( \sigma \in S_m \) such that
\[
\ell p_{\sigma(i)} \varepsilon_{\sigma(i)} \equiv s_i \pmod{q} \quad 1 \leq i \leq m,
\]
It remains to show that \( \prod \varepsilon_i = \varepsilon \). Suppose not. By \([\text{Coh} 73, 29.6]\), the isometry induced by \( (2.1) \) is homotopy-equivalent through homotopy-equivalences to \( f \). In particular, \( f \) and \( \psi \) have the same orientational behaviour, which is a contradiction.

To proof (2) \( \Rightarrow \) (1), we note that \( \Psi \) given by \( (2.1) \) is an \( \varepsilon \)-isometry. \( \square \)

Remark 2.7. Note that for odd \( q \), the lens spaces \( L(q;p_1,\ldots,p_m) \) have a unique spin structure. In this case (1) of Theorem 2.6 is equivalent to the statement that \( L \) and \( L' \) are \( \varepsilon \)-spin-isometric.

Theorem 2.8. Let \( q \) be even and \( \varepsilon \in \{ \pm 1 \} \). If \( L = L(q;p_1,\ldots,p_m;h) \) and \( L' = L(q;s_1,\ldots,s_m;h') \), the following assertions are equivalent:

(1) \( L \) and \( L' \) are \( \varepsilon \)-spin-isometric.
(2) There is a number \( \ell \in \mathbb{Z} \), a permutation \( \sigma \in S_m \) and there are numbers \( \varepsilon_i \in \{ \pm 1 \} \) such that
\[
lp_{\sigma(i)} \varepsilon_{\sigma(i)} \equiv s_i \pmod{q} \quad 1 \leq i \leq m
\]
\[
\prod_{i=1}^{m} \varepsilon_i = \varepsilon,
\]
\[
h + h' + h^p_q + h^s_q \equiv \frac{1}{q} \sum_{i=1}^{m} \left( lp_{\sigma(i)} \varepsilon_{\sigma(i)} - s_i \right) \pmod{2}.
\]

Proof. The statement about the isometry and the orientation is clear from the previous Theorem. Since the mapping of spin structures is a homotopy invariant of a map (in the class of homotopy equivalences), we can work with the isometry \( \psi : L \to L' \). Its lift \( \Psi \in O(2m) \) is covered by two elements \( \pm \Psi \in \text{Pin}(2m) \). Now the relation \( (2.2) \) lifts to \( \text{Spin}(2m) \) as
\[
\tau_h \left( \langle \gamma^p_q \rangle^\ell \right) = \Psi^{-1} \tau_{h'} \langle \gamma^s_q \rangle \cdot \Psi
\]
if and only if \( h + h' + h^p_q + h^s_q \equiv \frac{1}{q} \sum_{i=1}^{m} \left( lp_{\sigma(i)} \varepsilon_{\sigma(i)} - s_i \right) \pmod{2} \).

For any lens space \( L(q;p_1,\ldots,p_m) \) we can always, either by one of the two last theorems or simply by choosing another generator \( \langle \gamma^p_q \rangle^k \), \( k \in \mathbb{Z} \) with \( (q,k) = 1 \), go to a \( (+1) \)-spin-isometric representation \( L(q;1,s_2,\ldots,s_m) \). In particular, every three dimensional lens space can be written as \( L(q;1,p) \), which we will abbreviate as \( L(q;p) \) from now on. Theorems 2.6 and 2.8 then take on the following form.

Corollary 2.9. Let \( \varepsilon \in \{ \pm 1 \} \). The lens spaces \( L(q;p) \) and \( L(q;s) \) are \( \varepsilon \)-isometric if and only if
\[
p \equiv \varepsilon s \pmod{q} \quad \text{or} \quad p \cdot s \equiv \varepsilon \pmod{q}.
\]

Corollary 2.10. Let \( q \) be even and \( \varepsilon \in \{ \pm 1 \} \). The lens spaces \( L(q;p;h) \) and \( L' = L(q;s;h') \) are \( \varepsilon \)-spin-isometric if and only if
\[
\varepsilon p \equiv s \pmod{q} \quad \text{and} \quad h + h' + h^p_q + h^s_q \equiv \frac{p - \varepsilon s}{q} \pmod{2}
\]
or
\[
\varepsilon p \cdot s \equiv 1 \pmod{q} \quad \text{and} \quad h + h' + h^p_q + h^s_q \equiv \frac{ps - \varepsilon}{q} \pmod{2}.
\]
To facilitate the understanding of Corollaries 2.9 and 2.10 we provide the following
Example 2.11. The representation of lens spaces \( L(q; p) \) by integers \( p, q \in \mathbb{Z} \) with \( (q, p) = 1 \) is not unique and one can see that \( L(7; 2) \) is +1-spin-isometric to \( L(7; 4) \) because \( 2 \cdot 4 \equiv 1 \pmod{7} \). In this way, we can also see that \( L(17; 4) \) has a symmetric spectrum since it is \(-1\)-spin-isometric to itself. Proceeding, we see that \( L(8; 3; 1) \) is +1-spin-isometric to \( L(8; 3; 1) \) since \( 3^2 \equiv 1 \pmod{8} \) and \( 0 + 1 + 0 + 0 = \frac{3 \cdot 3 - 1}{8} \pmod{2} \). Analogously, \( L(10; 3; 0) \) is \(-1\)-spin-isometric to \( L(10; 3; 1) \). Note that there is a \(-1\)-spin-isometry of \( L(q; p; h) \), \( h \in \{0, 1\} \) if and only if \( p^2 \equiv -1 \pmod{2} \) and \( \frac{p^2 + 1}{4} \) is even, which is never the case.

At last, we cite a theorem that states precisely when a lens space is a Riemannian homogeneous space.

Theorem 2.12 \((\text{Wol84}, \text{Corollary 2.7.2})\). The lens space \( L(q; p_1, \ldots, p_m) \) is homogeneous if and only if \( p_i \equiv \pm p_j \pmod{q} \) for all \( 1 \leq i < j \leq m \). In particular, two homogeneous lens spaces of the same dimension and volume are isometric.

3. The Spectrum of the Sphere and Its Quotients

In this section, we describe the spectrum of the Dirac operator on the sphere and its quotients, the spherical space forms. We then specialise the formulas to lens spaces and end the section with a theorem about the spectral rigidity of homogeneous lens spaces.

The spectrum of the Dirac operator on the three dimensional sphere was first calculated in \([\text{Hit74}]\), though the round metric was only one among a large class of metrics for which Hitchin calculated the spectrum. In \([\text{Sul79}]\), S. Sulanke calculated the spectrum on \( S^n \) for all \( n \geq 2 \) with a representation theoretic approach. C. Bär found an alternative and shorter method to calculate the spectrum in \([\text{Bär90}] \) (cf. \([\text{Bär90}] \)) using Killing spinors which also paved the way for the description of the spectrum on the spherical space forms. It is this approach which we will follow and use in this and the sections to come.

Theorem 3.1. The Eigenvalues of the Dirac operator on the round sphere \( S^n \) are

\[
\pm \left( \frac{n}{2} + k \right), \; k \in \mathbb{N}_0
\]

with corresponding multiplicities

\[
\text{mult}_{S^n} \left( \pm \left( \frac{n}{2} + k \right) \right) = 2^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n + k - 1}{k}.
\]

We now pass to spherical space forms, quotients \( \Gamma \backslash S^{2m-1} \) of the sphere, where \( \Gamma \subseteq \text{SO}(2m) \) is a finite and freely acting group of orientation preserving isometries of \( S^{2m-1} \). Restricting to odd dimensions is no loss of generality since the only spherical space forms in even dimensions are the sphere \( S^{2m} \) itself and the real projective space \( \mathbb{P}R^{2m} \), which is not orientable and in particular not spin. Suppose that the quotient \( \Gamma \backslash S^{2m-1} \) is spin and let a spin structure be given by \( \tau : \Gamma \to \text{Spin}(2m) \). The spinor fields on \( \Gamma \backslash S^{2m-1} \) can be identified with the \( \Gamma \)-invariant spinor fields on \( S^{2m-1} \) by an unitary isomorphism (see \([\text{Gin09}] \) Proposition 1.4.2). In particular, the eigenspinor fields of the Dirac operator on \( \Gamma \backslash S^{2m-1} \) can be identified with the \( \Gamma \)-invariant eigenspinor fields of the Dirac operator on \( S^{2m-1} \).

It follows that the eigenvalues of the Dirac operator on \( \Gamma \backslash S^{2m-1} \) are

\[
\pm \left( \frac{2m - 1}{2} + k \right), \; k \geq 0
\]

with corresponding multiplicities

\[
0 \leq \text{mult}_{(\Gamma \backslash S^{2m-1}, \tau)} \left( \pm \left( \frac{2m - 1}{2} + k \right) \right) \leq \text{mult}_{S^{2m-1}} \left( \pm \left( \frac{2m - 1}{2} + k \right) \right).
\]

We weave these multiplicities into two power series.

Definition 3.2. Let \( \Gamma \backslash S^{2m-1} \) be a spherical space form equipped with a spin structure \( \tau : \Gamma \to \text{Spin}(2m) \). The generating functions of (the spectrum of the Dirac operator on) \( \Gamma \backslash S^{2m-1} \) are

\[
P_\pm(\Gamma \backslash S^{2m-1}, \tau)(z) = \sum_{k=0}^{\infty} \text{mult}_{(\Gamma \backslash S^{2m-1}, \tau)} \left( \pm \left( \frac{2m - 1}{2} + k \right) \right) z^k.
\]

Using the multiplicities on \( S^{2m-1} \), a standard argument shows that these power series converge absolutely for \( |z| < 1 \). In accordance with Definition 2.7 we now make the

Definition 3.3. Let \( M \) and \( N \) be compact Riemannian spin manifolds. Then \( M \) and \( N \) are +1-isospectral if the spectra of their Dirac operators \( D_M \) and \( D_N \) coincide, where each eigenvalue is counted with its multiplicity. The Riemannian spin manifolds \( M \) and \( N \) are \(-1\)-isospectral if the following condition is met: \( \lambda \in \mathbb{R} \) is an eigenvalue of \( D_M \) with multiplicity \( m = \text{mult}_M(\lambda) \) if and only if \( -\lambda \) is an eigenvalue of \( D_N \) with multiplicity \( m = \text{mult}_N(\lambda) \). The manifolds \( M \) and \( N \) are isospectral if they are \( \varepsilon \)-isospectral for some \( \varepsilon \in \{ \pm 1 \} \).
Remark 3.7. Heat kernel methods for the Dirac Laplacian \( \Box \) can be found on [BtD85, p. 290].

Proposition 3.4. Let \( \Gamma \setminus S^{2m-1} \) and \( \Gamma' \setminus S^{2m-1} \) be spherical space forms with spin structures \( \tau : \Gamma \to \text{Spin}(2m) \) and \( \tau' : \Gamma' \to \text{Spin}(2m) \) respectively. Then \( \Gamma \setminus S^{2m-1} \) and \( \Gamma' \setminus S^{2m-1} \) are \((+1)\)-isospectral if and only if \( F^\tau(\Gamma \setminus S^{2m-1}) = F^{\tau'}(\Gamma' \setminus S^{2m-1}) \) and \((-1)\)-isospectral if and only if \( F^\tau(\Gamma \setminus S^{2m-1}) = F^{\tau'}(\Gamma' \setminus S^{2m-1}) \).

For the following Theorem, denote by \( \chi^\pm : \text{Spin}(2m) \to \mathbb{C} \) the positive and negative half-spin characters respectively.

Theorem 3.5 ([Bar96, Theorem 2]). Let \( \Gamma \setminus S^{2m-1} \) be a spherical space form equipped with a spin structure \( \tau : \Gamma \to \text{Spin}(2m) \). Then the eigenvalues of the Dirac operator are \( \pm \left( \frac{2m-1}{2} + k \right) \), \( k \geq 0 \) with multiplicities determined by

\[
F^\tau(\Gamma \setminus S^{2m-1}, \tau)(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\chi^\tau(\tau(\gamma)) - z \cdot \chi^\tau(\tau(\gamma))}{\det(\text{Id} - z \cdot \gamma)}. 
\]

Denote by \( \xi = \xi_n \) the \( n \)-th root of unity \( e^{2\pi i/n} \). Then, for lens spaces, Theorem 3.5 takes on the following form.

Corollary 3.6. Let \( L = L(q; p_1, \ldots, p_m) \). If \( q \) is odd, the generating functions of \( L \) are

\[
F^\tau(\Gamma \setminus S^{2m-1}, \tau)(z) = \sum_{k=0}^{m-q-1} \sum_{\epsilon_1 \cdots \epsilon_m = (-1)^{m+1}} \frac{\xi^{k+1} \left( \sum_{j} \epsilon_j p_j \right) - z \cdot \xi^{k+1} \left( \sum_{j} \epsilon_j p_j \right)}{\prod_{j=1}^{m} (\xi^{kp_j} - z)(\xi^{kp_j} - z)}.
\]

If \( q \) and \( m \) are even and \( L \) is equipped with the spin structure \( \tau_h \), then the generating functions of \( L \) are

\[
F^\tau(\Gamma \setminus S^{2m-1}, \tau_h)(z) = \sum_{k=0}^{m-q-1} \sum_{\epsilon_1 \cdots \epsilon_m = (-1)^{m+1}} \frac{\xi^k \left( \sum_{j} \epsilon_j p_j \right) - z \cdot \xi^k \left( \sum_{j} \epsilon_j p_j \right)}{\prod_{j=1}^{m} (\xi^{kp_j} - z)(\xi^{kp_j} - z)}.
\]

Proof. The values of the half-spin characters on the image of the homomorphisms inducing the spin structures can be found on [BrDS5, p. 290].

Remark 3.7. Heat kernel methods for the Dirac Laplacian \( D^2 \) on an arbitrary compact spin manifold show that the dimension as well as the volume of \( M \) are spectrally determined (see [BGG04]). Thus, for a lens space \( L(q; p_1, \ldots, p_m) \), \( m \) and \( q \) are spectrally determined.

The first application of the preceding corollary is the following spectral rigidity result for homogeneous lens spaces.

Theorem 3.8. Let \( L = L(q; p_1, \ldots, p_m) \) and \( L' = L(q; s_1, \ldots, s_m) \) be lens spaces with fixed spin structures. Assume \( L \) is homogeneous. If \( L' \) is isospectral to \( L \), \( L' \) is homogeneous as well and so in particular isometric to \( L \). Moreover, if \( m = 2 \), \( q \geq 2 \) and \( L' \) is \( \varepsilon \)-isospectral to \( L \), then \( L' \) and \( L \) are \( \varepsilon \)-spin-isometric and carry the same spin structure.

Proof. Let \( F\pm \) be the generating functions of \( L \), \( F\pm' \) those of \( L' \). Formulas (3.2) - (3.5) show that \( F\pm \) have poles at and only at the \( q \)-th roots of unity and that these are at most of order \( 2m \). In fact, the term for \( k = 0 \) generates a pole of order \( 2m - 1 \) at \( z = 1 \) and in case \( q \) is even, the term for \( k = q/2 \) generates a pole of order \( 2m - 1 \) at \( z = -1 \). By the homogeneity assumption, the denominator of every term for \( k \neq 1, q/2 \) has a zero of order \( m \). In case \( m \) is even, the denominators of these terms have real coefficients, hence, no zeros cancel the ones
from the denominator. Let \( m \) be odd and fix \( 1 < k_0 < q \). Since the coefficients of the nominator of the term for \( k = k_0 \) are complex conjugate to the coefficients of the nominator of the term for \( k = q - k_0 \), at least one of these terms has a pole of order \( m \) at \( z = \xi_k^{p_s} \). If \( L \) and \( L' \) are isospectral, then, by Proposition 3.3, \( F^\pm_\alpha \) must have a pole of order \( m \) at every \( q \)-th root of unity that is not 1 or \(-1\), hence \( s_i \equiv \pm s_j \pmod{q} \) for all \( 1 \leq i < j \leq m \).

Now let \( m = 2 \) and \( q > 2 \). If \( q \) is odd, the multiplicities of the first positive and respectively negative eigenvalue of the Dirac operator on \( L \) are \( F(L,\tau_1)(0) = 2 \) and \( F(L,\tau_1)(0) = 0 \). If \( q \) is even, then \( F(L,\tau_1)(0) = 2 \) and \( F(L,\tau_1)(0) = 0 \), whereas \( F(L,\tau_1)(0) = 0 = F(L,\tau_1)(0) \). To distinguish the positive from the negative spectrum when \( L \) is endowed with the spin structure \( \tau_1 \), we note that \( \lim_{z \to -1} (1 + z)^3 F(L,\tau_1)^{(1)}(z) = \frac{2}{q} = -\lim_{z \to -1} (1 + z)^3 F(L,\tau_1)^{(1)}(z) \). \( \square \)

Remark 3.9. Let \( L = L(2;1) = \mathbb{RP}^3 \), then \( F(L,\tau_1)^{(1)} = \frac{(1 - z)^3}{1 + z} = F(L,\tau_1)^{(1)} \). This symmetry is generated by the isometry of \( \mathbb{RP}^3 \) corresponding to the choices \( \ell = 1 \), \( \varepsilon_1 = 1 \) and \( \sigma = (12) \) (see Theorem 2.8). Thus, by Theorem 3.8 the spin manifold \( \mathbb{RP}^3 \) is the only three dimensional homogeneous lens space for which the spectrum is invariant by a simultaneous change of orientation and spin structure.

4. Three Dimensional Lens Spaces

In this section we consider only three dimensional lens spaces. We denote by \( q \) a positive integer, by \( p \) and \( s \) integers that are coprime to \( q \) and \( p, s \not\equiv \pm 1 \pmod{q} \). Furthermore, let \( p^*, s^* \in \mathbb{Z} \) be any integers such that \( p \cdot p^* \equiv s \cdot s^* \equiv 1 \pmod{q} \).

We note that in the three dimensional case, the generating functions of \( L(q;p) \) simplify to

\[
F_\pm(z) = 2 \sum_{k=0}^{q-1} \cos \frac{k(p+1)(q+1)}{q} \pi z - \cos \frac{k(p+1)(q+1)}{q} \pi \xi_k - z \xi_k^{-k} - z
\]

for odd \( q \) whereas those of \( L(q;p,h) \) simplify to

\[
F_\pm(z) = 2 \sum_{k=0}^{q-1} (-1)^{k(h+h_p^s)} \cos \frac{k(p+1)}{q} \pi z - \cos \frac{k(p+1)}{q} \pi \xi_k - z \xi_k^{-k} - z
\]

We do not give proofs from Lemma 4.1 up to Corollary 4.3 as the statements are the same as those of the corresponding Lemmata and Corollaries in [1Y79] and the proofs go through with at most minor modifications.

Lemma 4.1.

\[
(q, p+1) = (q, p^* + 1)
(q, p-1) = (q, p^* - 1)
\]

Corollary 4.2. Let \( k \) be an integer such that \( k(p \pm 1) \not\equiv 0 \pmod{q} \).

Then \( k(p^* \pm 1) \not\equiv 0 \pmod{q} \).

Lemma 4.3. If \( L(q;p) \) and \( L(q;s) \) are isospectral, then

\[
(q, p-1) = (q, s-1)
(q, p+1) = (q, s+1)
\]

or

\[
(q, p-1) = (q, s+1)
(q, p+1) = (q, s-1).
\]

In particular \(((q, p-1), (q, p+1)) = \begin{cases} 1 & \text{if } q \text{ is odd} \\ 2 & \text{if } q \text{ is even}. \end{cases} \)
Corollary 4.4. If \( L(q; p) \) and \( L(q; s) \) are isospectral and \( k \) is an integer satisfying
\[
k(p \pm 1) \not\equiv 0 \pmod{q}.
\]
Then
\[
k(s \pm 1) \not\equiv 0 \pmod{q}.
\]

Proposition 4.5. Let \( k \in \mathbb{Z} \) such that \( k(p \pm 1) \not\equiv 0 \pmod{q} \). If \( q \) is odd, the residue of the generating functions \( F_{\pm k} \) of the lens space \( L(q; p) \) at \( z = \xi_q^k \) are
\[
\begin{aligned}
-\frac{2i}{q} & \frac{\xi_q^k}{(1 - \xi_q^{2k})^2} \\
\left( \frac{\cos \frac{k(p \mp 1)(q + 1)}{q} \pi - \xi_q^k \cos \frac{k(p \pm 1)(q + 1)}{q} \pi}{\pi} \right) & \frac{\cot \frac{k(p - 1)}{q} \pi - \cot \frac{k(p + 1)}{q} \pi}{\pi} + \\
\left( \frac{\cos \frac{k(p^* \mp 1)(q + 1)}{q} \pi - \xi_q^k \cos \frac{k(p^* \pm 1)(q + 1)}{q} \pi}{\pi} \right) & \frac{\cot \frac{k(p^* - 1)}{q} \pi - \cot \frac{k(p^* + 1)}{q} \pi}{\pi}.
\end{aligned}
\]

If \( q \) is even, the residue of the generating functions \( F_{\pm k} \) of the lens space \( L(q; p; h) \) at \( z = \xi_q^k \) are
\[
\begin{aligned}
-\frac{2i}{q} & \frac{\xi_q^k}{(1 - \xi_q^{2k})^2} \\
(-1)^{k(h + h_q^*)} & \left( \frac{\cos \frac{k(p \mp 1)}{q} \pi - \xi_q^k \cos \frac{k(p \pm 1)}{q} \pi}{\pi} \right) \frac{\cot \frac{k(p - 1)}{q} \pi - \cot \frac{k(p + 1)}{q} \pi}{\pi} + \\
(-1)^{k(p^* (h + h_q^*) + u(p, p^*))} & \left( \frac{\cos \frac{k(p^* \pm 1)}{q} \pi - \xi_q^k \cos \frac{k(p^* \pm 1)}{q} \pi}{\pi} \right) \frac{\cot \frac{k(p^* - 1)}{q} \pi - \cot \frac{k(p^* + 1)}{q} \pi}{\pi}
\end{aligned}
\]
where \( u(p, p^*) \) is defined by \( p \cdot p^* = u(p, p^*)q + 1 \).

Proof. The condition on \( k \) ensures that \( F_{\pm k} \) have a pole of order one at \( z = \xi_q^k \). There are precisely four terms that contribute to the residue of \( F_{\pm k} \) at \( z = \xi_q^k \). Let \( q \) be odd. Then we calculate straightforwardly
\[
\lim_{z \to \xi_q^k} (\xi_q^k - z) F_{\pm k} = \frac{2}{q} \lim_{z \to \xi_q^k} (\xi_q^k - z) \sum_{l=0}^{q-1} \frac{\cos \frac{((p \mp 1)(q + 1)) \pi}{q} - z \cos \frac{((p \pm 1)(q + 1)) \pi}{q}}{\xi_q^l - z} \frac{\xi_q^{lp} - z}{(\xi_q^k - \xi_q^l)} \frac{\xi_q^{lp} - z}{(\xi_q^k - \xi_q^l)}
\]
\[
= 4 \left( \frac{\cos \frac{k(p \mp 1)(q + 1)}{q} \pi - \xi_q^k \cos \frac{k(p \pm 1)(q + 1)}{q} \pi}{\xi_q^l - \xi_q^k} \right) \frac{\xi_q^l - \xi_q^k}{(\xi_q^k - \xi_q^l)} \frac{\xi_q^{lp} - z}{(\xi_q^k - \xi_q^l)} + \frac{\cos \frac{k(p^* \mp 1)(q + 1)}{q} \pi - \xi_q^k \cos \frac{k(p^* \pm 1)(q + 1)}{q} \pi}{\xi_q^l - \xi_q^k} \frac{\xi_q^l - \xi_q^k}{(\xi_q^k - \xi_q^l)} \frac{\xi_q^{lp} - z}{(\xi_q^k - \xi_q^l)}
\]
\[
= 4 \left( \frac{\cos \frac{k(p \mp 1)(q + 1)}{q} \pi - \xi_q^k \cos \frac{k(p \pm 1)(q + 1)}{q} \pi}{\xi_q^l - \xi_q^k} \right) \frac{\xi_q^{lp} - z}{(\xi_q^k - \xi_q^l)} + \frac{\cos \frac{k(p^* \mp 1)(q + 1)}{q} \pi - \xi_q^k \cos \frac{k(p^* \pm 1)(q + 1)}{q} \pi}{\xi_q^l - \xi_q^k} \frac{\xi_q^{lp} - z}{(\xi_q^k - \xi_q^l)}
\]
\[
\frac{1}{(1 - \xi_q^{k(1-p)}) (1 - \xi_q^{k(1+p)})} = \frac{1}{\xi_q^{k(p+1)} (1 - \xi_q^{-k(p-1)}) (1 - \xi_q^{-k(p+1)})} = \frac{1}{\xi_q^{k(p+1)} (1 - \xi_q^{-k(p-1)}) (1 - \xi_q^{-k(p+1)})} = \frac{1}{\xi_q^{k(p+1)} (1 - \xi_q^{-k(p-1)}) (1 - \xi_q^{-k(p+1)})}
\]
\[
\frac{1}{1 - \xi_q^{2k}} = \frac{1}{\xi_q^{2k} (1 - \xi_q^{-2k})} = \frac{1}{\xi_q^{2k} (1 - \xi_q^{-2k})} = \frac{1}{\xi_q^{2k} (1 - \xi_q^{-2k})}
\]
\[
\frac{1}{1 - \xi_q^{2k}} = \frac{1}{\xi_q^{2k} (1 - \xi_q^{-2k})} = \frac{1}{\xi_q^{2k} (1 - \xi_q^{-2k})} = \frac{1}{\xi_q^{2k} (1 - \xi_q^{-2k})}
\]
\[
\frac{1}{q} \left( \frac{\cot \frac{k(p - 1)}{q} \pi - \cot \frac{k(p + 1)}{q} \pi}{\pi} \right)
\]
\[
\frac{1}{q} \left( \frac{\cot \frac{k(p - 1)}{q} \pi - \cot \frac{k(p + 1)}{q} \pi}{\pi} \right)
\]
\[
\frac{1}{q} \left( \frac{\cot \frac{k(p - 1)}{q} \pi - \cot \frac{k(p + 1)}{q} \pi}{\pi} \right)
\]
\[
\frac{1}{q} \left( \frac{\cot \frac{k(p - 1)}{q} \pi - \cot \frac{k(p + 1)}{q} \pi}{\pi} \right)
\]
\[
\frac{1}{q} \left( \frac{\cot \frac{k(p - 1)}{q} \pi - \cot \frac{k(p + 1)}{q} \pi}{\pi} \right)
\]
\[
\frac{1}{q} \left( \frac{\cot \frac{k(p - 1)}{q} \pi - \cot \frac{k(p + 1)}{q} \pi}{\pi} \right)
\]
\[
\frac{1}{q} \left( \frac{\cot \frac{k(p - 1)}{q} \pi - \cot \frac{k(p + 1)}{q} \pi}{\pi} \right)
\]
\[
\frac{1}{q} \left( \frac{\cot \frac{k(p - 1)}{q} \pi - \cot \frac{k(p + 1)}{q} \pi}{\pi} \right)
\]
\[
\frac{1}{q} \left( \frac{\cot \frac{k(p - 1)}{q} \pi - \cot \frac{k(p + 1)}{q} \pi}{\pi} \right)
\]
whereas if \( q \) is even, define
\[
I_q^{p;k,h} = (-1)^k(h + h_q^p) \xi_{2q}^{kp} \left( \cot \frac{k(p - 1)}{q} \pi - \cot \frac{k(p + 1)}{q} \pi \right)
\]
for \( h \in \mathbb{Z} \).

For the next Corollary, denote by \( \sigma : \mathbb{C} \to \mathbb{C} \) complex conjugation.

**Corollary 4.7.** Assume \( q \) is odd. If the lens spaces \( L(q;p) \) and \( L(q;s) \) are \( \varepsilon \)-isospectral, then
\[
I_q^{p;k} + I_q^{p^*;k} = \sigma \left( \frac{1}{2} \left( I_q^{k;h} + I_q^{k^*;h} \right) \right)
\]
for every \( k \in \mathbb{Z} \) s.t. \( k(p \pm 1) \neq 0 \) (mod \( q \)). Now assume \( q \) is even. If the lens spaces \( L(q;p;h) \) and \( L(q;s;h') \) are \( \varepsilon \)-isospectral, then
\[
I_q^{p;k,h} + (-1)^k(u(p;p^*) + h_q^p + h_q^{p^*}) I_q^{p^*;k,h} = \sigma \left( \frac{1}{2} \left( I_q^{k;h} + (-1)^k(u(s,s^*) + h_q^s + h_q^{s^*}) I_q^{s^*;k;h'} \right) \right)
\]
for every \( k \in \mathbb{Z} \) s.t. \( k(p \pm 1) \neq 0 \) (mod \( q \)).

**Proof.** Assume \( q \) is odd and that the lens spaces \( L = L(q;p) \) and \( L' = L(q;s) \) are \( +1 \)-isospectral. By Proposition 4.5 the generating functions of these lens spaces satisfy \( F_L^q = F_{L'}^q \). In particular, the residues of their poles at every \( z = \xi^q_k \) with \( k \) such that \( k(p + 1) \neq 0 \) (mod \( q \)) coincide. Dividing the residues \( \{(4.1)\} \) by their common factor \(-2i/q \cdot \xi^q_k/(1 - \xi^{2k}_q)^2 \), considering the imaginary parts of the resulting equations and dividing again by the common factor \((2\pi ik/q)\), we obtain the set of equations
\[
\cos \frac{k(p + 1)(q + 1)}{q} \pi \left( \cot \frac{k(p - 1)}{q} \pi - \cot \frac{k(p + 1)}{q} \pi \right) + \cos \frac{k(p + 1)(q + 1)}{q} \pi \left( \cot \frac{k(p + 1)}{q} \pi - \cot \frac{k(p - 1)}{q} \pi \right) = 0.
\]
Addition and subtraction of the equations ”+” and ”-” yield, up to a common factor, the real and imaginary parts of equation (4.4). The cases of \(-1\)-isospectrality as well as when \( q \) is even are very similar. \( \square \)

**Remark 4.8.** One can show by either straightforward calculations or using Corollary 4.7 that the term \((-1)^k(u(p,p^*) + h_q^p + h_q^{p^*})\) in the equations (4.4) is an invariant of the \(+1\)-spin-isometry class of \( L(q;p;h) \equiv L(q;p^*;h + h_q^p + h_q^{p^*} + u(p,p^*)) \).

We denote by \( \mathbb{Q}_q \) the \( q \)-th cyclotomic field \( \mathbb{Q}({\xi}_q) \). Its ring of integers is \( \mathbb{Z}[\xi_q] \). Let \( \lambda := 1 - \xi_q \in \mathbb{Q}_q \), then \( \lambda \) is a prime ideal in \( \mathbb{Z}[\xi_q] \) and we have \((q) = (\lambda)^{q-1} \). From now on, we consider on \( \mathbb{Q}_q \) the \( \lambda \)-adic valuation and the corresponding metric. This gives us a notion of convergence (see e.g. [Rib01 Chapter 17.1.2]).

**Lemma 4.9.** Let \( q \geq 5 \) be prime. Then for all \( 1 \leq l, k \leq q - 1 \), the number
\[
\frac{\lambda}{1 - \xi_q^k} \in \mathbb{Z}[\xi_q]
\]
has the convergent power series expansion
\[
\frac{\lambda}{1 - \xi_q^k} = \frac{1}{k} \left( 1 + \frac{-1 - 2l + k}{2} \lambda \right)
+ \frac{-1 + k^2 - 6kl + 6l^2}{12} \lambda^2
+ \frac{-1 + k^2 - 6kl - 2k^2l + 6l^2 + 6kl^2 - 4l^3}{24} \lambda^3 + \ldots
\]
whose coefficients are all elements of \( \mathbb{Z}(q) \), the localization of \( \mathbb{Z} \) by the maximal ideal \((q)\).

**Proof.** To prove the first claim, choose \( 1 \leq k^* \leq q - 1 \) such that \( kk^* \equiv 1 \) (mod \( q \)). Then
\[
\frac{\lambda}{1 - \xi_q^k} = \frac{1 - \xi_q}{1 - \xi_q^k} = \frac{1 - (\xi_q^k)^{k^*}}{1 - \xi_q^k} = 1 + \xi_q^k + \xi_q^{2k} + \ldots + \xi_q^{k(k^* - 1)}.
\]
To proof the statement about the power series, we expand the individual parts separately. The first part is

\[ \xi_q^t = (1 - \lambda)^t = \sum_{j=0}^{t} \binom{t}{j}(-\lambda)^j \]

\[ = 1 - t\lambda + \frac{l(l - 1)}{2}\lambda^2 + \frac{l(l - 1)(l - 2)}{6}\lambda^3 + \ldots . \]

Next, we expand \( 1 - \xi_q^k \) into the power series

\[ 1 - \xi_q^k = 1 - (1 - \lambda)^k = \lambda k \sum_{j=0}^{k-1} \binom{k - 1}{j}(-\lambda)^j \]

\[ = \lambda k \left( 1 - \frac{k - 1}{2}\lambda + \frac{(k - 1)(k - 2)}{6}\lambda^2 - \frac{(k - 1)(k - 2)(k - 3)}{24}\lambda^3 + \ldots . \right) . \]

Because of \( 1 \leq k \leq q - 1 \), all the coefficients of this power series are elements of \( \mathbb{Z}/q \). The same is then true for the series of \( \lambda/(1 - \xi_q^k) \) and its Cauchy product with the series of \( 1 - \xi_q^t \), which is sufficient for the claimed series to converge. □

**Theorem 4.10.** Let \( q \in \mathbb{N} \) be prime. If two lens spaces, whose fundamental groups have order \( q \), are \( \epsilon \)-isospectral, they are \( \epsilon \)-spin-isometric.

**Proof.** The case \( q = 2 \) was discussed in Remark 3.9. For \( q = 3, 5, 7 \) there are at most two different isometry classes, one of which is homogeneous. Thus, these are spectrally determined by Theorem 3.8. Let \( q \geq 11 \) and \( L = L(q; p) \) and \( L' = L(q; s) \) be two \(+1\)-isospectral lens spaces. By the same theorem we can assume \( p \neq \pm 1 \) (mod \( q \)) and thus by assumption \( s \neq \pm 1 \) (mod \( q \)). By Corollary 4.4 we have

\[ I_q^{p+1} + I_q^{s-1} = I_q^{p+1} + I_q^{s-1} . \]

Using the formula \( \cot \theta = \frac{1}{i\pi} - i \) and Lemma 4.9, we see that \( \frac{i}{\sqrt{2\lambda}}(I_q^{p+1} + I_q^{s-1}) = \frac{i}{\sqrt{2\lambda}}(I_q^{p+1} + I_q^{s-1}) \in \mathbb{Z}^q \).

Let \( R = \{0, 1, \ldots , q - 1\} \) and

\[ \frac{i}{\sqrt{2\lambda}}(I_q^{p+1} + I_q^{s-1}) = \sum_{n=0}^{\infty} g_n \lambda^n , \]

\[ \frac{i}{\sqrt{2\lambda}}(I_q^{p+1} + I_q^{s-1}) = \sum_{n=0}^{\infty} g'_n \lambda^n \]

be the unique power series with \( g_n, g'_n \in R \) for all \( n \in \mathbb{N} \) (see e.g. [Rib99] Chapter 2.2 O). By assumption, \( g_n = g'_n \) for all \( n \in \mathbb{N} \). Furthermore, by Lemma 4.9

\[ g_3 \equiv \frac{1}{24} \sum_{(l, k)} \left( -\frac{1}{k} + \frac{k}{\Pi} - \frac{6l}{\Pi \Pi} - \frac{2k}{\Pi \Pi} + \frac{6l^2}{\Pi \Pi \Pi} + \frac{4l^3}{\Pi \Pi \Pi \Pi} \right) \]

\[ \equiv \frac{1}{24} \sum_{(l', k')} \left( -\frac{1}{k'} + \frac{k'}{\Pi'} - \frac{6l'}{\Pi \Pi'} - \frac{2k'}{\Pi \Pi'} + \frac{6l'^2}{\Pi \Pi \Pi'} + \frac{4l'^3}{\Pi \Pi \Pi \Pi'} \right) \ \text{(mod \( q \))} . \]

where we sum over \( (l, k) \in \{(\frac{q+1}{2}p, p - 1), (\frac{q+1}{2}p, p - 1)\} \) and \( (l', k') \in \{(\frac{q+1}{2}p, p + 1), (\frac{q+1}{2}p, p + 1)\} . \)

We calculate the summands in pairs. Obviously, \( \Pi - \Pi' \equiv \Pi - \Pi' \equiv 0 \) (mod \( q \)). Furthermore we have

\[ \Pi - \Pi' \equiv -\frac{1}{p - 1} - \frac{1}{p^* - 1} + \frac{1}{p - 1} + \frac{1}{p^* - 1} \equiv 2 \ \text{(mod \( q \))} , \]

\[ \Pi - \Pi' \equiv -\frac{1}{p - 1} - \frac{1}{p^* + 1} + \frac{1}{p^* + 1} + \frac{1}{p - 1} \equiv 2 \ \text{(mod \( q \))} , \]

\[ \Pi - \Pi' \equiv -\frac{1}{p - 1} - \frac{1}{p - 1} + \frac{1}{p^* - 1} + \frac{1}{p^* - 1} \equiv -4 \ \text{(mod \( q \))} . \]
\[
IV - IV' = -2 \frac{q+1}{2} p(p-1) - 2 \frac{q+1}{2} p^*(p^*-1) + 2 \frac{q+1}{2} p(p+1) + 2 \frac{q+1}{2} p^*(p^* + 1) \\
\equiv (q+1)(-p^2 + p - (p^*)^2) + p^* + p^2 + p + (p^*)^2 + p^* = 2(q+1)(p+p^*) \pmod q ,
\]

\[
V - V' = \frac{3}{2} (q+1)^2 \left( \frac{p^2}{p-1} + \frac{(p^*)^2}{p^*-1} - \frac{p^2}{p+1} - \frac{(p^*)^2}{p^* + 1} \right) \\
\equiv \frac{3}{2} (q+1)^2 \left( \frac{p^2}{p-1} - \frac{p^*}{p-1} - \frac{p^2}{p+1} + \frac{p^*}{p+1} \right) \\
\equiv \frac{3}{2} (q+1)^2 \left( \frac{p^2 - p^*}{p-1} + \frac{p^2 + p^*}{p+1} \right) \\
\equiv \frac{3}{2} (q+1)^2 \left( \frac{p^3 + p^2 - 1 - p^3 - 1 + p^2 + p^*}{p^2 - 1} \right) \\
\equiv \frac{3}{2} (q+1)^2 \left( \frac{2p^2 - 2}{p^2 - 1} \right) \equiv 3(q+1)^2 \pmod q ,
\]

\[
VII - VII' = -\frac{(q+1)^3}{2} \left( \frac{p^3}{p-1} + \frac{(p^*)^3}{p^*-1} - \frac{p^3}{p+1} - \frac{(p^*)^3}{p^* + 1} \right) \\
\equiv -\frac{(q+1)^3}{2} \left( \frac{p^3 - (p^*)^2}{p-1} - \frac{p^3 + (p^*)^2}{p+1} \right) \\
\equiv -\frac{(q+1)^3}{2} \left( \frac{p^4 - p^* + p^3 - (p^*)^2 - p^4 - p^* + p^3 + (p^*)^2}{p^2 - 1} \right) \\
\equiv -\frac{(q+1)^3}{2} \left( \frac{2p^3 - 2p^*}{p^2 - 1} \right) \\
\equiv -(q+1)^3 \left( \frac{p^2 - (p^*)^2}{p - p^*} \right) \equiv -(q+1)^3 (p + p^*) \pmod q .
\]

Putting everything together, we obtain

\[
g_3 \equiv \frac{1}{24} (2 - 4 + 2(q+1)(p + p^*) + 3(q+1)^2 - (q+1)^3 (p + p^*)) \\
\equiv \frac{1}{24} (-2 + 3(q+1)^2 - (p + p^*) (q + 1) (q^2 + 2q - 1)) \pmod q .
\]

Analogously,

\[
g'_3 \equiv \frac{1}{24} (-2 + 3(q+1)^2 - (s + s^*) (q + 1) (q^2 + 2q - 1)) \pmod q .
\]

Since, by assumption, \( g_3 = g'_3 \), we have

(4.5) \quad p + p^* \equiv s + s^* \pmod q .

Squaring both sides of (4.5) yields

\[
p^2 + 2 + (p^*)^2 \equiv s^2 + 2 + (s^*)^2 \pmod q
\]

which, after subtracting 4 on both sides, leads to

\[
(p - p^*)^2 \equiv (s - s^*)^2 \pmod q .
\]

This means

(4.6) \quad (p - p^*) \equiv \pm (s - s^*) \pmod q .

Addition and subtraction of equations (4.5) and (4.6) yields

\[
p \equiv s \pmod q \text{ or } p \equiv s^* \pmod q
\]
which means, by Corollary 2.9 that $L$ and $L'$ are +1-isometric. Since for odd $q$ the lens spaces $L(q; p)$ posses only one spin structure, $L$ and $L'$ are trivially +1-spin-isometric.

**Remark 4.11.** A nontrivial question is whether the assumption that $q$ be prime in Theorem 4.10 is necessary for the conclusion to hold. High precision computer calculations of the numbers $I^k_q$ and $I^{p,k;h}_q$ for $2 \leq q \leq 10^4$, $p$ from a full set of representatives of $\mathbb{Z}_q$, $1 \leq k < q, k(p+1) \neq 0 \pmod q$ and $h = 0,1$ suggest that the condition could be dropped. These calculations were verified with a computer and the exact method from [BL14] for all parameters in the same range, so that it is reasonable to

**Conjecture 4.12.** Two three dimensional lens spaces which are $\varepsilon$-isospectral, are $\varepsilon$-spin-isometric.

**Remark 4.13.** Note that Theorem 4.10 implies that for all odd primes $q$, $L(q; p)$ has symmetric spectrum if and only if $p^2 \equiv -1 \pmod q$. If Conjecture 4.12 was true, then this would hold for all odd $q$. Furthermore, under this assumption, $L(q; p; 0)$ would have the same spectrum as $L(q; p; 1)$ if and only if $p^2 \equiv 1 \pmod q$ and $(p^2 - 1)/q$ was odd. Expanding on Remark 5.9, $L(q; p; 0)$ would be $-1$-isometric to $L(q; p; 1)$ if and only if $p^2 \equiv -1 \pmod q$ and $(p^2 + 1)/q$ was odd.

5. THE $\eta$-INARIANT

The $\eta$-invariant measures the asymmetry of the Dirac spectrum in dimension $n \equiv 3 \pmod 4$ and appears in the Atiyah-Patodi-Singer-index Theorem (cf. [Gin09, Chapter 8.7]). We present here a formula for the $L$ under this assumption, $p^2 \equiv -1 \pmod q$. The following Lemma 5.2 shows that this statement does not generalize to all $q \in \mathbb{N}$. Note that (modulo the $\varepsilon$-statement) both these results are already known (see [Kat87]), though the method of proof is new.

**Theorem 5.1 (Bar00 Theorem 5.2).** Let $\Gamma \setminus S^{2m-1}$ be a spherical space form with a spin structure given by $\tau : \Gamma \to \text{Spin}(2m)$. The $\eta$-invariant of $\Gamma \setminus S^{2m-1}$ is

$$\eta = \frac{2}{|\Gamma|} \sum_{\gamma \in \Gamma \setminus \{\text{Id}_{2m}\}} \frac{(\chi^- - \chi^+)(\tau(\gamma))}{\det(\text{Id}_{2m} - \gamma)}.$$ 

**Corollary 5.2.** Let $L = L(q; p_1, \ldots, p_m)$ be a lens space. If $q$ is odd, the $\eta$-invariant of the Dirac operator on the lens space $L$ is given by

$$\eta^L = (-1)^{m/2+1} \frac{1}{q^{2m-1}} \sum_{j=1}^{q-1} \prod_{k=1}^{m} \csc \frac{(q+1)kp_j}{q} \pi.$$ 

If $q$ and $m$ are even and $L$ is equipped with the spin structure $\tau_h$, the eta-invariant of $L$ is

$$\eta^L = (-1)^{m/2+1} \frac{1}{q^{2m-1}} \sum_{j=1}^{q-1} (-1)^{k+h+c} \prod_{j=1}^{m} \csc \frac{kp_j}{q} \pi.$$ 

**Theorem 5.3.** Let $q \in \mathbb{N}$ be a prime and $L = L(q; p)$, $L' = L(q; p)$ be lens spaces such that $\eta^L = \varepsilon \eta^{L'}$ for $\varepsilon \in \{\pm 1\}$. Then $L$ and $L'$ are $\varepsilon$-spin-isometric.

**Proof.** The case $q = 2$ was already discussed in Remark 3.3. Let $q \geq 3$. Represent $\csc x$ as $2i e^{-ix} \frac{1}{1-e^{-2ix}}$. A simple generalization of Lemma 4.9 to the case of products of the form $\zeta_q^\lambda \frac{1}{1-\zeta_q^\lambda}$ yields convergent power series in $\lambda$ for $\eta^L$ and $\eta^{L'}$, along with expressions for the coefficients of $\lambda^0$. Comparing these yields $p \equiv \varepsilon s \pmod q$ or $p \equiv \varepsilon s \pmod q$.

**Example 5.4.** There are six isometry classes of three dimensional lens spaces whose fundamental group have order 25, representatives are $L(25; 1)$, $L(25; 2)$, $L(25; 3)$, $L(25; 4)$, $L(25; 7)$ and $L(25; 9)$. The corresponding $\eta$-invariants are $22/25$, $-\frac{2}{5}$, $\frac{4}{5}$, $-\frac{2}{5}$, 0 and $-\frac{2}{5}$. In particular, Theorem 5.3 does not generalize to all $q \in \mathbb{N}$.

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