Combinatorics via Closed Orbits: Number Theoretic Ramanujan Graphs Are Not Unique Neighbor Expanders

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ABSTRACT

The question of finding expander graphs with strong vertex expansion properties such as unique neighbor expansion and lossless expansion is central to computer science. A barrier to constructing these is that strong notions of expansion could not be proven via the spectral expansion paradigm.

A very symmetric and structured family of optimal spectral expanders (i.e., Ramanujan graphs) was constructed using number theory by Lubotzky, Phillips and Sarnak, and was subsequently generalized by others. We call such graphs Number Theoretic Ramanujan Graphs. These graphs are not only spectrally optimal, but also possess strong symmetries and rich structure. Thus, it has been widely conjectured that number theoretic Ramanujan graphs are lossless expanders, or at least unique neighbor expanders.

In this work we disprove this conjecture, by showing that there are number theoretic Ramanujan graphs that are not even unique neighbor expanders. This is done by introducing a new combinatorial paradigm that we term the closed orbit method.

The closed orbit method allows one to construct finite combinatorial objects with external substructures. This is done by observing that there exist infinite combinatorial structures with external substructures, coming from an action of a subgroup of the automorphism group of the structure. The crux of our idea is a systematic way to construct a finite quotient of the infinite structure containing a simple shadow of the infinite substructure, which maintains its external combinatorial property.

Other applications of the method are to the edge expansion of number theoretic Ramanujan graphs and vertex expansion of Ramanujan complexes. Finally, in the field of graph quantum ergodicity we produce number theoretic Ramanujan graphs with an eigenfunction of small support that corresponds to the zero eigenvalue. This again contradicts common expectations.

The closed orbit method is based on the well-established idea from dynamics and number theory of studying closed orbits of subgroups. The novelty of this work is in exploiting this idea to combinatorial questions, and we hope that it will have other applications in the future.

1 INTRODUCTION

On Ramanujan graphs, and Number Theoretic Ramanujan graphs.

Various combinatorial questions are studied using sparse graphs. Their solution is often based on a spectral analysis of the underlying graph, and in particular on the fact that the given graph is a good expander, meaning that the second eigenvalue of its adjacency matrix is very far from its first eigenvalue ([6]). The strongest spectral expansion condition is the Ramanujan property, which says that the second largest non-trivial eigenvalue in absolute value of the adjacency operator of a $d$-regular graph is bounded by $2\sqrt{d} - 1$.

An explicit family of Ramanujan graphs, which we call LPS graphs, was constructed in the celebrated work of Lubotzky, Phillips, and Sarnak ([12]). This construction is based on number theory and in particular on the theory of automorphic forms, using deep results of Deligne and others. There are various possible variations on the construction (e.g., [15, 19]), including the much earlier work of Ihara ([7]). We will focus on the work of Morgenstern ([18]), who gave another such explicit family which we call Morgenstern graphs (the essential difference between the two works is replacing the field $Q$ by $F_q(t)$, where $F_q$ is the finite field with $q$ elements). We call the graphs resulting from the different variations number theoretic graphs, to distinguish them from other constructions of Ramanujan graphs (e.g., the graphs constructed by [14]). The number theoretic Ramanujan graphs have various other wonderful properties – for example, they are Cayley Graphs and have a very large girth (i.e., the length of the shortest cycle is large).

1.1 Vertex Expansion and the Spectral Method

There are some notoriously hard combinatorial questions about graphs where the spectral theory falls short of proving the desired answer. A notable example is the question of finding a family of explicit $d$-regular graphs which are lossless-expanders. For $X$ a $d$-regular graph, and a subset $Y$ of the vertices of $X$, we define the
expansion ratio of $Y$ as $\frac{\left|N(Y)\right|}{\left|Y\right|}$, where $N(Y)$ is the set of neighboring vertices of the set $Y$, which may include vertices from $Y$ itself. This ratio is obviously bounded by $d$. For $d$ large but fixed as the size of the graph grows to infinity, we say that a family of graphs is a family of lossless expanders if there is a constant $\alpha > 0$ such that for every set $Y \subset X$ of size $|Y| \leq \alpha |X|$, its expansion ratio is $d - o(d)$. There are constructions of graphs satisfying weaker notions ($[1, 4]$), but even going beyond expansion ratio $d/2$ is a major open question ($[6]$).

The best results using the spectral method are due to Kahale ($[8]$). He shows that for a Ramanujan graph the expansion ratio of sets of size bounded by $\alpha |X|$ is at least $d/2 - \beta$, where $\beta \to 0$ as $\alpha \to 0$. Kahale also constructs a family of graphs which are almost-Ramanujan, in the sense that their second largest eigenvalue in absolute value is bounded by $2\sqrt{d-1} + o(1)$, having a subset $Y$ of two vertices which has expansion ratio $d/2$. In particular, he shows that the best expansion ratio that it is possible to get solely by using spectral arguments cannot exceed $d/2$ for linear sized sets.

One of the reasons that the passing $d/2$ barrier is important is that graphs with vertex expansion greater than $d/2$ are also *unique neighbor expanders*, for if a set $Y$ has an expansion ratio that is greater than $d/2$, then there exists a vertex that has a unique neighbor in $Y$. Unique neighbor expanders were constructed by Alon and Capalbo ([1]), but the resulting graphs are not lossless expanders. A weaker desired property is *odd neighbor expansion*, which says that there is a vertex that is connected to an odd number of elements in $Y$. We refer to [6] for a discussion of vertex expansion and its applications from different points of view.

Kahale’s example has a short cycle of length 4. It is also very far from being a Cayley graph. For graphs with large girth, Kahale actually proved that small sets have expansion ratio $d - o(d)$ (see also [16]). For the LPS graphs or Morgenstern graphs, which are the $d$-regular graph that have the best known girth, this implies that sets of size smaller than $|X|^{1/3 - \epsilon}$ have expansion ratio close to $d$.

The fact that Kahale’s construction does not share many of the wonderful properties of the number theoretic construction led various researchers to speculate that Ramanujan number theoretic graphs, which have very large girth, should be lossless expanders, or at least graphs with vertex expansion strictly greater than $d/2$.

We show that this common belief is not true. As a matter of fact, some Morgenstern Ramanujan graphs are not even *odd neighbor expanders*, and therefore not *unique neighbor expanders*. Here is one of our main theorems:

**Theorem 1.1 (Number theoretic Ramanujan graphs that are not odd neighbor expanders).** For every prime power $q$, there exists an infinite family of $(q+1)$-regular number theoretic Ramanujan graphs $X$, and a subset $Y \subset X$, $|Y| = \Theta(\sqrt{|X|})$, such that every $x \in N(Y)$ has precisely 2 neighbors in $Y$. Therefore, $Y$ has no unique neighbors and $|N(Y)| = \frac{q^{2m}}{2} |Y|$.

Explicitly, for every odd prime power $q$ and $m$ large enough, there exists a $(q+1)$-regular bipartite Morgenstern Ramanujan graph

\[ X = \text{Cayley}\left(\text{PGL}_2 \left(\mathbb{F}_{q^m}\right), \{\gamma_1, \ldots, \gamma_{q+1}\}\right), \]

with generators $\gamma_1, \ldots, \gamma_{q+1}$, such that the subgroup $\langle \gamma_1^2, \ldots, \gamma_{q+1}^2 \rangle$ is isomorphic to $\text{PGL}_2(\mathbb{F}_{q^m})$. Moreover, the graph

\[ Y = \text{Cayley}\left(\left\{\gamma_1^2, \ldots, \gamma_{q+1}^2\right\}, \left\{\gamma_1^2, \ldots, \gamma_{q+1}^2\right\}\right) \]

is also a $(q+1)$-regular bipartite Morgenstern Ramanujan graph.

The subset $Y \subset X$ is of size $|Y| = \Theta(\sqrt{|X|})$, and every $x \in N(Y)$ has precisely 2 neighbors in $Y$.

The theorem is based on a new idea we introduce to Combinatorics called the *closed orbit method*, which is based on working with the simply connected covering object, a topic we explain in the next subsection. In Subsection 1.3 we describe how the general method applies to the vertex expansion question. We also discuss vertex expansion in Ramanujan complexes. In Subsection 1.4 we discuss the problem of edge expansion in Ramanujan graphs and how our method addresses it. In Subsection 1.5 we describe the surprising application of our method to the field of quantum ergodicity of graphs, where we show in Theorem 1.9 the existence of a concentrated eigenfunction of the adjacency operator, again contradicting a natural belief that such eigenfunctions do not exist for number theoretic graphs. In Subsection 1.6 we explain the closed orbit machinery in more detail, from a group theoretic point of view, and present our main abstract result, Theorem 1.10. In Subsection 1.7 we discuss the inverse situation in which closed orbits do not exist. We pose the conjecture that the non-existence of closed orbits implies lossless expansion of number theoretic graphs.

In Section 2 we prove the explicit part of Theorem 1.1, showing explicit number theoretic Ramanujan graphs with bad vertex expansion for odd prime powers. We assume the results of [16], the proof uses elementary number theory in function fields.

This is a short version of the full paper, which contains the proofs of all the results mentioned here.

1.2 Combinatorics via the Covering Object

In our work, we study finite $d$-regular graphs by understanding new properties that they inherit from their infinite simply-connected cover, the $d$-regular tree $T_d$, together with the action of some group $G$ on it. We will usually call $T_d$ by $B_2$ below, since we think of it with the $G$-action. The infinite covering object is already evident in the definition of a Ramanujan graph -- it is a finite $d$-regular graph that inherits the spectrum of its covering object. Namely, the graph’s non-trivial spectrum is contained in the spectrum of $T_d$.

The action of the group $G$ on $T_d$ is more subtle, but it underlies the number theoretic constructions of Ramanujan graphs. For example, the LPS construction ([12]) is based on the action of the $p$-adic group $G = \text{PGL}_2(\mathbb{Q}_p)$ ($\mathbb{Q}_p$ is the $p$-adic field) on its Bruhat-Tits tree $B_G = T_{q^m+1}$. Using number theory which is related to quaternion algebras, it is possible to construct an arithmetic lattice $\Gamma$ in $G$ such that by taking the quotient of $T_{q^m+1}$ by $\Gamma$ we get a $(q+1)$-regular graph which inherits the spectrum of the infinite tree, namely, a graph with the Ramanujan property.

In our work, we focus on the action of a subgroup $H \leq G$ on $B_G$. An orbit of the $H$-action on $B_G$ gives a substructure $Z \subset B_G$ with various desired properties. This substructure $Z$ is used to solve some combinatorial questions for the infinite cover.

We then look at the projection of $Z$ into the finite quotient graph $X$. We want to understand the image $Y \subset X$ of the map, as it inherits
the properties of $Z$. Usually, this map is very complicated, and in particular, its image $\mathcal{Y}$ is the entire finite graph $X$. However, using the closed orbit method that we introduce in this work, we show that there are special situations when this map is simple, and in particular, its image $\mathcal{Y}$ may be small relative to $X$.

As we explain in Subsection 1.6, the special situations happen if the orbit $\Gamma\backslash H$ in the compact space $\Gamma\backslash G$ is closed, hence the name of the method\footnote{More generally, we are actually interested in a periodic orbit, which is an $H$-orbit supporting a finite $H$-invariant measure. When $\Gamma\backslash G$ is compact, which is the case of interest to combinatorics, both notions are equivalent, and we think that the closed orbit method simply sounds better.}. The notion of a closed orbit is basic in ergodic theory, and has many uses in homogeneous dynamics, number theory, and representation theory. The novelty of our work is exploiting this well-known notion to get a new understanding of finite combinatorial questions.

All the above seems quite abstract, so let us now explain how we apply it to the problem of vertex expansion. This will require a more technical discussion.

### 1.3 Vertex Expansion and the Closed Orbit Method

Using subgroups to find an infinite subgraph with bad vertex expansion. Consider the field $\mathbb{F}_q(t)$ of Laurent series over the finite field $\mathbb{F}_q$ with $q$ elements ($q$ a prime power). This field is analogous to the $q$-adic field $\mathbb{Q}_q$ when $q$ is replaced by the field $\mathbb{F}_q(t)$. As with the group $\text{PGL}_2(\mathbb{Q}_q)$, the group $G = \text{PGL}_2(\mathbb{F}_q(t))$ acts naturally on a $(q + 1)$-regular Bruhat-Tits tree $B_G$. Notice that in this case, there is a subfield $\mathbb{F}_q((t^{2})) \subset \mathbb{F}_q((t))$. This subfield gives rise to a subgroup $H = \text{PGL}_2(\mathbb{F}_q((t^{2}))) \leq G$. Notice that the groups $G$ and $H$ are isomorphic, so $H$ acts on its own $(q + 1)$-regular Bruhat-Tits tree $B_H$.

Next, consider the $H$-action on $B_G$, via the embedding of $H$ in $G$. An orbit of $H$ gives rise to an embedding of the vertices of the $(q + 1)$-regular tree $B_H$, for some $G$-subgroup $B_G$ (see Figure 1 and the discussion in the full version of the paper). The embedding can also be described as an embedding of the $(q + 1, 2)$-biregular subdivision graph of $B_H$ in the $(q + 1)$-regular tree $B_G$.

Notice that the image of the embedding is very thin, in the sense that a large ball in $B_G$ with $n$ vertices will contain $\Theta(\sqrt{n})$ vertices of $B_H$. The following lemma says that the embedded set has bad expansion properties.

**Lemma 1.2.** Let $Z \subset B_G$ be the embedding of the vertices of $B_H$ in $B_G$. Then each vertex $v \in N(Z)$ is a neighbor of precisely two vertices of $Z$.

**Lemma 1.3 (Special case of Theorem 1.10).** It is possible to choose a family of arithmetic lattices $\Gamma \leq G$, such that the projection $Y$ of the set $Z$ into the finite graph $X = \Gamma \backslash B_G$ is of size $|Y| = O\left(\sqrt{|X|}\right)$.

Most of the non-explicit part of Theorem 1.1 follows from Lemma 1.3 and the discussion above. The discussion implies that every vertex $x \in N(Y)$ is connected by at least two edges to $Y$. The fact that $x$ has precisely two neighbors in $Y$ follows from a symmetry trick we explain in Lemma 2.7. This implies that there are number theoretic Ramanujan graphs that are not even odd neighbor expanders.

In the following, we apply the closed orbit method to the Morgenstern Ramanujan graphs $[18]$. This will give explicit number theoretic graphs with bad vertex expansion, and the explicit part of Theorem 1.1.

**Morgenstern graphs**: Explicit number theoretic graphs that are not lossless expanders. Let us describe how the above can be applied to the construction of Ramanujan graphs by Morgenstern ([16]), for $q$ an odd prime power. Morgenstern constructs a lattice $\Gamma \leq \text{PGL}_2(\mathbb{F}_q((t^{2})))$ that acts simply transitively on the Bruhat-Tits tree $B_G$, with generators $\gamma_1, \ldots, \gamma_{q+1}$. If $\Gamma_a$ is a normal subgroup of $\Gamma$, the graph $\Gamma_a \backslash B_G$ is then naturally isomorphic to the Cayley graph $X = \text{Cayley}(\Gamma_a, \{\gamma_1, \ldots, \gamma_{q+1}\})$.

When $\Gamma_a$ is chosen by some explicit congruence conditions we get the Morgenstern graphs, which have plenty of nice properties, described in [18, Theorem 4.13], and are very similar to the celebrated LPS graphs ([10, 12]). In particular, $\Gamma_a \backslash B_G$ is isomorphic to $\text{PGL}_2(\mathbb{F}_{q^m})$, the graphs are Ramanujan graphs, and their girth is at least $4/3 \log q(|X|)$.

The general method applies as follows: It turns out that after a “change of variables”, for $H = \text{PGL}_2(\mathbb{F}_q((t^{2})))$, the subgroup $\Gamma \cap H$ is generated by $\delta_1 = \gamma_1^{\ast}, \ldots, \delta_{q+1} = \gamma_{q+1}^{\ast}$ and is actually also a Morgenstern lattice of $H$, which acts simply transitively on the Bruhat-Tits tree $B_H$. After some computations, we end up with the explicit part of Theorem 1.1.

**Lossless expansion for Ramanujan complexes.** A promising option for graphs with good vertex expansion are the underlying graphs of the Ramanujan complexes constructed in [9, 13].

Unlike $d$-regular graphs, higher dimensional Ramanujan complexes have a rigid local structure, which implies interesting combinatorial properties. For example, many recent works used the Garland method to show that they are high dimensional expanders (see [11] and the references therein). Therefore, one may speculate that the rigid local structure will imply some form of lossless expansion.

However, we show that the underlying graph on the vertices of the complex can have bad vertex expansion:

**Theorem 1.4 (Skeleton graphs of Ramanujan complexes are not unique neighbor expanders).** Let $n$ be prime, $q$ a prime power, $G = \text{PGL}_n(\mathbb{F}_q((t)))$ and $B_G$ be the Bruhat-Tits building of $G$. Then there is an infinite family of Ramanujan quotients $\Gamma \backslash B_G$ such that its underlying graph $X$ has a subset $Y$ of size $|Y| = O\left(|X|^{1/2}\right)$, with no unique neighbors.
The results about vertex expansion we described above have an \( M/\sqrt{|\text{manujan graph}} \), with result of Kahale:

between spectral gap and edge expansion is given in the following for every small set

\( S \) is small. The closed orbit method allows us to find lattices such that

also have no unique neighbors. However, it does not have to be

\( M/\sqrt{|\text{edge expansion}} \) analog for edge expansion. The

Figure 1: Part of the tree of \( \text{PGL}_2 (\mathbb{F}_2 (\langle t^2 \rangle)) \) (left) embedded in part of the tree of \( \text{PGL}_2 (\mathbb{F}_2 (\langle t \rangle)) \) (right). See the full version of

paper for the meaning of the vertex labels.

As with vertex expansion, we can prove that there exist number theoretic graphs with as bad edge expansion as allowed by Kahale’s result:

**Theorem 1.7** (Kahale’s spectral bound for edge expansion is tight). For every prime power \( q \), there exists an infinite family of \( (q^2 + 1) \)-regular number theoretic Ramanujan graphs \( X \), and a \( (q + 1) \)-regular induced subgraph \( Y \subset X \), \( |Y| = O(\sqrt{|X|}) \).

The proof of this theorem is based on applying the general construction to \( G = \text{PGL}_2 (\mathbb{F}_2' (\langle t \rangle)) \) and \( H = \text{PGL}_2 (\mathbb{F}_q (\langle t \rangle)) \). This gives as embedding \( Z \) of \( B_{HT} \) in \( B_G \). The basic property of this embedding is described in Figure 2 and the following Lemma:

**Lemma 1.8.** Every vertex of \( Z \) is connected to \( q + 1 \) other vertices of \( Z \).

When projected to a finite quotient, the image \( Y \) of \( Z \) still has an induced degree of at least \( q + 1 \). The closed orbit method allows us to find an arithmetic lattice such that this projection is small.

### 1.4 Bad Edge Expansion for Number Theoretic Graphs

The results about vertex expansion we described above have an analog for edge expansion. The edge expansion of a set \( S \subset X \) is the ratio \( \frac{M(S,X-S)}{|S|} \), where \( M(S,X-S) \) is the number of edges between \( S \) and its complement in \( X \). Another way of studying this ratio is by looking at the number \( M(S,S) \) of internal edges in \( S \), as \( M(S,X-S) + M(S,S) = d|S| \). Finally, \( \frac{M(S,S)}{|S|} \) is simply the average degree of the induced graph on \( S \), which is the property we will actually study.

A graph is a good edge expander if the average induced degree for every small set \( S \) is small. The best result about the connection between spectral gap and edge expansion is given in the following result of Kahale:

**Theorem 1.6** (Kahale ([8])). Let \( X \) be a \( (q^2 + 1) \)-regular Ramanujan graph, with \( |X| \to \infty \). Then for every subset \( Y \subset X \) with \( |Y| = o(|X|) \), the average degree of the induced graph on \( Y \) is bounded by \( \sqrt{q} + 1 + o(1) \).

As with vertex expansion, we can prove that there exist number theoretic graphs with as bad edge expansion as allowed by Kahale’s result:

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### 1.5 Concentrated Eigenfunctions of Number Theoretic Graphs

The closed orbit machinery could be useful beyond the specific question of expansion. Indeed, we use this idea to construct an eigenfunction with eigenvalue 0, which has small support, in a number theoretic graph.

There is a lot of recent work, initiated by Brooks and Lindenstrauss ([3]), whose aim is to understand eigenfunctions of the adjacency operator on \( (q + 1) \)-regular graphs. Similar to the setting of vertex expansion, eigenfunctions on a \( (q + 1) \)-regular graph with
girth at least $\beta \log_q n$ have support of size at least $\Theta\left(\frac{n^2}{\beta} \right)$ (see \cite[Subsection 1.1]{5}). Therefore, the support of eigenfunctions on the graphs $X$ of Theorem 1.1 is of size at least $n^{1/3-o(1)}$. More generally, Brooks and Lindestrauss ([3]) proved that for a $(q+1)$-regular graph $X$ with girth $\beta \log_q n$, for every $\epsilon > 0$ there is $\delta > 0$ such that if a set $Y$ supports $\epsilon$ of the mass of an eigenfunction $f$ (where the eigenfunction is normalized to $\|f\|_2 = 1$ and the mass is determined by $|f|^2$), then $|Y| \geq \Omega\left(n^\delta\right)$. This was improved by Ganguly and Srivastava ([5]) to $|Y| = \Omega\left(n^{\epsilon^2/4}\right)$.

Recently, Alon, Ganguly and Srivastava ([2]), extending the results of Ganguly and Srivastava ([5]), constructed a family of $(q+1)$-regular graphs of high girth, with many eigenfunctions of small support, of eigenvalues that are dense in $(-2\sqrt{q}, 2\sqrt{q})$. Their graphs have second eigenvalue bounded by $\frac{1}{\sqrt{2}} \approx 1.2121 \sqrt{q}$, which is close to being Ramanujan.

As for our contribution, let $X, Y$ be the graphs from Theorem 1.1, with $X$ identified with $PGL_2(F_q,m)$ and $Y$ identified with $PGL_2(F_q^m)$. Let $f: X \to \mathbb{C}$ be

$$
 f(x) = \begin{cases} 
+1 & x \in PSL_2(F_q^m) \\
-1 & x \in PGL_2(F_q^m) - PSL_2(F_q^m) \\
0 & x \not\in PGL_2(F_q^m) 
\end{cases}.
$$

Recall that $Y$ is a bipartite graph. The function $f$ is simply the function giving the value $+1$ to one part of $Y$, the value $-1$ to the other part of $Y$, and the value $0$ for the vertices in $X - Y$.

**Theorem 1.9 (Number theoretic Ramanujan graphs with concentrated eigenfunctions).** The function $f \in L^2(X)$ is an eigenfunction of the adjacency operator $A$ of $X$ with eigenvalue $0$.

Therefore, for every odd prime power $q$ there exists a $(q+1)$-regular number theoretic Ramanujan graph $X$ of girth greater than $4/3 \log_q(|X|)$, with an eigenfunction of the adjacency operator of eigenvalue $0$, which is supported on $O(\sqrt{|X|})$ vertices.

**Proof.** Let $x \in X$. Notice that if $y_k x \in Y$, then also $y_k^{-1} x = y_k^{-2} y_k x \in Y$. Moreover, $y_k x$ and $y_k^{-1} x$ are on different parts of $X$, as they differ by $y_k^2$, which is a generator of $Y$ as a Cayley graph, and $Y$ is bipartite. Therefore, the total number of $+1$ contributions to $(Af)(x)$ is equal to the total number of $-1$ contributions to $(Af)(x)$. Therefore $(Af)(x) = 0$. \hfill $\square$

Notice that after normalization our eigenfunction to $\|f\|_2 = 1$, we have $\|f\|_\infty = \Omega\left(n^{-1/4}\right)$. By moving the eigenfunction with the automorphisms of the Cayley graph, we actually get $\Theta\left(\sqrt{n}\right)$ such functions. We are not familiar with any similar construction of an explicit non-trivial eigenfunction on number-theoretic graphs. However, our method is limited to the eigenvalue $0$.

We remark that our method is similar to the work of Milicic about large values of eigenfunctions of arithmetic hyperbolic 3-manifolds ([17]), and also to the earlier work of Rudnick and Sarnak ([21]). Their methods show that an automorphic eigenfunction can have a large supremum-norm at closed orbits of smaller subgroups. The work [17], in particular, uses subgroups coming from field extensions. The main difference is that in our combinatorial setting we can explicitly construct the eigenfunction, and this eigenfunction is not automorphic in the sense that it is not an eigenfunction of the other Hecke operators that act on the space. However, the eigenvalue $0$ can be perhaps explained by the existence of an automorphic lift from the smaller group. It will be interesting to clarify this. Finally, it will be interesting to apply the methods of [17, 21] to graphs, as they may prove the existence of more general eigenfunctions with a large supremum norm.
1.6 The Closed Orbit Method

In the following, we explain the closed orbit method more accurately and state our abstract theorem about it.

Let $G$ be a locally compact group, $H \leq G$ a closed subgroup and $\Gamma \leq G$ a cocompact lattice. For $x \in \Gamma \backslash G$, we may look at the $H$-orbit $GxH \subset \Gamma \backslash G$. The $H$-action defines a map
$$\tilde{F}_x : \Gamma_x \backslash H \to \Gamma \backslash G,$$
where $\Gamma_x \triangleq x^{-1}Gx \cap H$, given by sending $\Gamma_x\cdot h$ to $Gxh$. Its image is $GxH \subset \Gamma \backslash G$.

An $H$-orbit, which is the image of this map, can be quite complicated topologically. However, when $\Gamma_xH$ is a lattice in $H$, the map becomes much simpler, and in particular, it becomes a topological embedding, and its image is closed. We will focus on the case when $x = e \in G$ is the identity, and denote $\Gamma_H = \Gamma_xH = \Gamma \cap H$. For our combinatorial purposes, we move from the group $G$ itself to a discrete space. We assume that $G$ and $H \leq G$ are semisimple $p$-adic groups, and let $K \leq G$ be a compact open subgroup. The space $G/K$ is a discrete space with a $G$-action, which is closely related to the Bruhat-Tits building $B_K$ of $G$. For simplicity, we will work with the space $G/K$ instead of the Bruhat-Tits building $B_K$. The left $H$-action on $G/K$ defines an embedding
$$H/K \to G/K,$$
where $K_H = H \cap K$.

The reader may restrict herself to the case when $G = PGL_n(F_q((t)))$, $H = PGL_n(F_q((t^2)))$, $K = PGL_n(F_q[[t]])$ and $K_H = PGL_n(F_q[[t^2]])$. Then $G/K$ and $H/K_H$ may be identified with the vertices of the Bruhat-Tits buildings $B_K$ and $B_H$.

When we insert $\Gamma$ again into the picture, we get a map of discrete spaces
$$F_\Gamma : \Gamma H \backslash H/K_H \to \Gamma \backslash G/K,$$
When $\Gamma_H$ is a lattice in $H$, this is a map between two finite combinatorial objects.

For the applications, we want two properties: First, $\Gamma_H$ should indeed be a lattice in $H$. Second, we want $\Gamma H \backslash H/K_H$ to be as small as possible relative to $\Gamma \backslash G/K$.

To achieve the two properties we turn to number theory. Our main goal of this section is to prove the explicit part (i.e., for odd prime powers $q$) of Theorem 1.1. The theorem is more than a special case of Theorem 1.10, since the lattices do not come from simply connected groups, as we assume (implicitly) in Theorem 1.10. This allows us to construct very explicit Cayley graphs, but add another layer of complexity, which we resolve using Morgenstern’s results.

2 EXPPLICIT NUMBER THEORETIC GRAPHS WITH BAD VERTEX EXPANSION

The main goal of this section is to prove the explicit part (i.e., for odd prime powers $q$) of Theorem 1.1. The theorem is more than a special case of Theorem 1.10, since the lattices do not come from simply connected groups, as we assume (implicitly) in Theorem 1.10. This allows us to construct very explicit Cayley graphs, but add another layer of complexity, which we resolve using Morgenstern’s results.

Let us first give a short explanation how the explicit construction fits into the general framework. Let $\Gamma = \langle y_1, \ldots, y_{q+1} \rangle$ be a free group with $y_1, \ldots, y_{q+1}$ as generators and their inverses. Let $\Gamma'$ be the subgroup of $\Gamma$ generated by $\delta_1, \ldots, \delta_{q+1}$, where $\delta_i = y_i^2$.

The Cayley graph $T_\Gamma = Cayley(\Gamma, \{y_1, \ldots, y_{q+1}\})$ is a $(q + 1)$-regular tree. Similarly, $T_{\Gamma'} = Cayley(\Gamma', \{\delta_1, \ldots, \delta_{q+1}\})$ is also a $(q + 1)$-regular tree. The embedding of $\Gamma'$ in $T_{\Gamma'}$ gives an embedding of the vertices of $T_{\Gamma'}$ in $T_\Gamma$. Moreover, each edge in $T_{\Gamma'}$ corresponds to two edges in $T_\Gamma$, or alternatively, the embedding extends to a graph embedding of the $(q + 1, 2)$-biregular subdivision graph of $T_{\Gamma'}$ in $T_\Gamma$. We deduce that if $Z \subset T_{\Gamma'}$ is the embedding of the vertices of $T_{\Gamma'}$ into $T_\Gamma$, then every vertex $x \in N(Z)$ is connected to two vertices of $Z$. This is a version of Lemma 1.2 in our case.
Now, let \( \Gamma_n \) be a finite index normal subgroup of \( \Gamma \). Then we may look at the Cayley graph \( X = \text{Cayley}(\Gamma/\Gamma_n, \{\gamma_1, ..., \gamma_{q+1}\}) \) (with the elements being identified with their image in \( \Gamma/\Gamma_n \)). Alternatively, \( X \) can be identified with the quotient of \( T_1 \) by \( \Gamma_n \). There is a natural embedding \( \Gamma'//(\Gamma' \cap \Gamma_n) \to \Gamma/\Gamma_n \). The image \( Y \) of this embedding can be identified with the projection of \( Z \subset T_1 \) to \( X \). We deduce that every neighbor of \( Y \) is also connected to \( Y \) by at least two edges.

The problem is then to find a subgroup \( \Gamma_n \) such that \( |Y| \) will be much smaller than \( |X| \), or alternatively \( [\Gamma' : \Gamma' \cap \Gamma_n] \) will be much smaller than \( [\Gamma : \Gamma_n] \), which will be an explicit version of Lemma 1.3. During the proof, we will show that this holds for the Morgenstern graphs using explicit calculations.

It may be hard to identify the relation between the proof and the general theory, which involves \( p \)-adic groups. After setting some preliminaries we explain some of it in Remark 1, and some more discussion appears in the full version of the paper.

Throughout the proof, we freely use basic number theory in function fields. See [20] for a good introduction to this subject.

We start by recalling the construction of the Morgenstern Ramanujan graphs [18]). Let \( q \) be an odd prime power, with \( \mathbb{F}_q \) the corresponding finite field. Consider the quaternion algebra \( A(\mathbb{F}_q(u)) \), which has a base \( 1, i, j, k \) over \( \mathbb{F}_q(u) \), with relations

\[
    i^2 = e, j^2 = u - 1, ij = -ji,
\]

where \( e \in \mathbb{F}_q \) is a non-square. This algebra has a norm

\[
    N(a + bi + cj + dij) = a^2 - eb^2 + (ed^2 - c^2)(u - 1).
\]

We let \( \mathbb{A}^\times(\mathbb{F}_q(u))/\mathbb{Z}^\times \) be the quotient of the invertible elements of \( A(\mathbb{F}_q(u)) \) by the equivalence condition \( a \sim a' \) if and only if there is a \( a \in \mathbb{F}_q(u) \) with \( aa' = a'' \).

Denote by \( A(\mathbb{F}_q(u)) \) the elements of \( A \) with \( a, b, c, d \in \mathbb{F}_q[u] \). There are \( q + 1 \) elements \( \{y_1', ..., y_{q+1}'\} \subset A(\mathbb{F}_q(u)) \), satisfying

\[
    y_k' = 1 + c_k j + d_k ij,
\]

with \( c_k, d_k \in \mathbb{F}_q \) and \( N(y_k') = u \). Those elements correspond to the \( q + 1 \) solutions of \( ed^2 - c^2 = 1 \).

We let \( S = \{y_1', ..., y_{q+1}'\} \) be the image of those elements in \( \mathbb{A}^\times(\mathbb{F}_q(u))/\mathbb{Z}^\times \). Finally, let \( \Gamma \) be the group generated by \( y_1', ..., y_{q+1}' \).

Theorem 2.1 ([18, Corollary 4.7]). \( \Gamma \) is a free group on \( \frac{q+1}{2} \) generators, with \( y_1', ..., y_{q+1}' \) as generators and their inverses. Moreover, there is a bijection between \( \Gamma' \) and the set

\[
    \left\{ \alpha = a + bi + cj + dij \in A(\mathbb{F}_q[u]) : \exists l \geq 0, N(\alpha) = u^l, u - 1, gcd(a - 1, b), u \not| gcd(a, b, c, d) \right\}.
\]

The bijection is given by choosing for every \( \gamma \in \Gamma \) the unique element in its equivalence class in \( \mathbb{A}^\times(\mathbb{F}_q(u)) \) from the set above.

Now let \( \gamma \in \mathbb{F}_q(u) \) be an irreducible polynomial of degree \( 2m \). Then the congruence subgroup \( \Gamma(\mathfrak{p}) \) of \( \Gamma \) is the set of elements of \( \Gamma \) who have in their equivalence class an element \( a + bi + cj + dij \in A(\mathbb{F}_q[u]) \), satisfying \( q \not| a, g|b, g|c, g|d \).

Recall that the Legendre symbol for \( f, g \in \mathbb{F}_q(u), g \neq 0 \) irreducible is defined as

\[
    \left( \frac{f}{g} \right) = \begin{cases} 
    0 & \text{if } f \text{ is a square } \not\equiv 0 \text{ mod } g \\
    1 & \text{else } \end{cases}
\]

\[
    \equiv f^{-\frac{a+b-1}{2}} \text{ (mod g)}. \]

Theorem 2.2 ([18, Theorem 4.13]). The Cayley graph \( X_\gamma = \text{Cayley}(\Gamma/\Gamma(\mathfrak{p}) \cdot \{y_1', ..., y_{q+1}'\}) \) is a \((q + 1)\)-regular Ramanujan graph.

There are two possibilities, depending on the Legendre symbol \( \left( \frac{u}{g} \right) \):

1. If \( \left( \frac{u}{g} \right) = -1 \) then the group \( \Gamma/\Gamma(\mathfrak{p}) \) is isomorphic to \( \text{PGL}_2(\mathbb{F}_q(u)) \), then the graph \( X_\gamma \) is bipartite and its girth is at least \( \frac{1}{2} \log_q(|X_\gamma|) \).

2. If \( \left( \frac{u}{g} \right) = 1 \) then the group \( \Gamma/\Gamma(\mathfrak{p}) \) is isomorphic to \( \text{PSL}_2(\mathbb{F}_q(u)) \), then the graph \( X_\gamma \) is not bipartite, and its girth is at least \( \frac{3}{2} \log_q(|X_\gamma|) \).

We now consider the \( q + 1 \) elements

\[
    S' = \{\delta_1, ..., \delta_{q+1}\} \subset A^\times(\mathbb{F}_q(u))/\mathbb{Z}^\times
\]

satisfying \( \delta_k = y_k' \), so explicitly

\[
    \delta_k = 2 - u + 2c_k j + 2d_k ij.
\]

Let \( \Gamma' = \{\delta_1, ..., \delta_{q+1}\} \leq \Gamma \).

To prove Theorem 1.1 we need to understand the group

\[
    \Gamma'/(\Gamma' \cap \Gamma(\mathfrak{p}))
\]

and its Cayley structure relative to the generators \( \delta_1, ..., \delta_{q+1} \).

It is simpler to work with valuations instead of divisions. Recall that the valuations of the field \( \mathbb{F}_q(u) \) are \( \gamma_1/u \) defined by \( \gamma_1/u \left( \frac{a}{b} \right) = \deg_u a - \deg_u b, f, g \in \mathbb{F}_q[x] \), and for every irreducible monic polynomial \( p \in \mathbb{F}_q(u) \) the valuation \( v_p \left( \frac{a}{b} \right) = a, \text{ where } f, g \in \mathbb{F}_q(u) \) are not divisible by \( p \).

Using the language of valuations, \( \Gamma(g) \) contains all the elements of \( \mathbb{A}^\times(\mathbb{F}_q(u))/\mathbb{Z}^\times \), which are in the free group generated by \( \gamma_1', ..., \gamma_{q+1}' \), and further have an element \( \alpha = a + bi + cj + dij \) in their equivalence class satisfying

\[
    v_q(a) = 0, v_q(b) > 0, v_q(c) < 0, v_q(d) > 0.
\]

Next we make the change of variables \( t = \frac{u}{x-u} \). It holds that \( u = \frac{1}{2} + 1 \) and \( 2 - u = \frac{2}{t + 1} \).

When we make a change of variables, the quaternion algebra \( A \) changes to the quaternion algebra \( A_1 = \text{span}(1, i_1, j_1, i_1 j_1) \) over \( \mathbb{F}_q(t) \) with \( i_1^2 = e, j_1^2 = f, i_1 j_1 = -j_1 i_1 \). In the new algebra, we have

\[
    y_k = 1 + c_k j_1 + d_k i_1 j_1
\]

where \( c_k, d_k \in \mathbb{F}_q(u) \) and \( N(y_k) = u \).

Let \( T: \mathbb{F}_q(u) \to \mathbb{F}_q(t), T(f(u)) = f \left( \frac{u}{u-1} \right) \) be the isomorphism of fields defined by the change of coordinates. There is a bijection between valuations \( \sigma \) of \( \mathbb{F}_q(u) \) and valuations \( \sigma \) of \( \mathbb{F}_q(t) \), defined
by $\sigma(f) = \sigma(T(f))$ for every $f \in \mathbb{F}_q(u)$. Let us describe this bijection.

The change of variables is a composition of two simpler operations: A linear transformation $t = au + b, a \neq 0, b \in \mathbb{F}_q$ and an inversion $t = 1/u$.

For $t = au + b$, the bijection is as follows: $u_1/u_2$ corresponds to $\sigma_{t'/t}$, and for $g(u)$ monic irreducible, $\deg g = m'$, let $h(t) = a^{-m'}g(at + b)$. Then $u_\sigma$ corresponds to $\sigma_{h}$.

For $t = 1/u$, the bijection is as follows: $u_1/u_2$ corresponds to $\sigma_t$, $u_3$ corresponds to $\sigma_{t'/t}$, and for $g(u)$ monic irreducible, $\deg g = m'$, $\gamma(t) = a^{-m'}g(1/t) = t^{m'} + a_1t^{m'-1} + \ldots + a_{m'-1}t + a_{m-1}$. Then $u_\gamma$ corresponds to $\sigma_{h}$.

Applying the above to $t = \frac{u}{u-2}, u = \frac{3t}{t+1}$, the bijection is

$\begin{align*}
u_1/u_2 \leftrightarrow & \sigma_{t'/t} \\
u_2-2 \leftrightarrow & \sigma_{t'/t},
\end{align*}$

and for $g(u) \neq \frac{u}{u-2}$ of degree $m'$, let $h(t)$ be the monic polynomial corresponding to $(t + 1)^{m'} g(\frac{2t}{t+1})$. Then $u_\gamma \leftrightarrow \sigma_{h}$.

The other direction of this correspondence is given as follows: For $h(t) \neq t + 1$ of degree $m'$, let $g(t)$ be the monic polynomial corresponding to $(u - 2)^{m'} h(\frac{u}{u-2})$. Then $u_\sigma \leftrightarrow u_\gamma$.

**Lemma 2.3.** Using the correspondence above, for $g(u) \neq 2$ of degree $m'$, it holds that $(\frac{u}{g(u)}) = (\frac{2(t+1)}{t+1})$.

**Proof.** Let the Legendre symbol $(\frac{f(u)}{g(u)})$ for $u_\sigma(f) \geq 0$ be determined by whether the image of $f$ in the finite field is zero, a non-zero square, or neither.

Then $u \equiv 2t$ and $2t$ is the result follows.

Returning to $\Gamma(g)$, let $h(t)$ correspond to $g(u)$ as above. Then after the change of coordinates we have

**Lemma 2.4.** The group $\Gamma(g)$ is isomorphic to a subgroup of $A_{\mathbb{F}_q}^\times /Z^\times$.

The last equality follows from the fact that we work in $A_{\mathbb{F}_q}^\times (\mathbb{F}_q(t))/Z^\times$.

Moving to $A_{\mathbb{F}_q}^\times (\mathbb{F}_q(t))/Z^\times$, $\Gamma$ is generated by

$\gamma_1, \ldots, \gamma_{q+1} \in A_{\mathbb{F}_q}^\times (\mathbb{F}_q(t))/Z^\times$,

while $\gamma(g)$ consists of the elements with an element in their equivalence class satisfying Equation (1).

Therefore $\Gamma'$ is generated by $\delta_1, \ldots, \delta_{q+1} \in A_{\mathbb{F}_q}^\times (\mathbb{F}_q(t))$, and $\Gamma' \cap \Gamma(g)$ are the elements in $\Gamma'$ satisfying Equation (1).

**Remark 1.** Let us stop the proof for a moment and explain the connection between Theorem 1.1 and Theorem 1.10.

**Theorem 1.10.** After the change of variables, look at the group

$G = A_{\mathbb{F}_q}^\times (\mathbb{F}_q(t))/Z^\times \cong \text{PGL}_2(\mathbb{F}_q(t))$.

Morgenstern shows that $\Gamma'$ acts simply transitively on the Bruhat-Tits building $B_{\mathbb{F}_q}$ of $G$, so the Cayley graph of $\Gamma$ with respect to $\gamma_1, \ldots, \gamma_{q+1}$ can be identified with $B_{\mathbb{F}_q}$.

Denote $H = A_{\mathbb{F}_q}^\times (\mathbb{F}_q(t))/Z^\times \cong \text{PGL}_2(\mathbb{F}_q(t))$ which is a closed subgroup of $G$. It is not hard to see that $\Gamma' = \Gamma \cap H$. The calculation above implies that $\Gamma'$ acts simply transitively on the Bruhat-Tits building $B_{\mathbb{F}_q}$ of $G$, and its Cayley graph with respect to $\delta_1, \ldots, \delta_{q+1}$ is isomorphic to $B_{\mathbb{F}_q}$. Therefore, $\Gamma' = \Gamma \cap H$ is a lattice in $H$, which is far from obvious. As we explain in the full version of the paper, this fact also follows from the fact that $A_2$ is actually defined over $\mathbb{F}_q(s)$.

Let us stop the proof for a moment and explain the connection between Theorem 1.1 and Theorem 1.10.

**Theorem 1.1.** After the change of variables, look at the group

$G = A_{\mathbb{F}_q}^\times (\mathbb{F}_q(t))/Z^\times \cong \text{PGL}_2(\mathbb{F}_q(t))$.

Morgenstern shows that $\Gamma'$ acts simply transitively on the Bruhat-Tits building $B_{\mathbb{F}_q}$ of $G$, so the Cayley graph of $\Gamma$ with respect to $\gamma_1, \ldots, \gamma_{q+1}$ can be identified with $B_{\mathbb{F}_q}$.

The embedding of the group $\Gamma'$ in $\Gamma$ is the same as the embedding of $B_{\mathbb{F}_q}$ in $\Gamma$. This implies that the embedding of $\Gamma'/(\Gamma' \cap \Gamma(g))$ is the same as the embedding of $(\Gamma' \cap \Gamma(g))/\Gamma(g)$ in $\text{PGL}_2(\mathbb{F}_q)$. In the next part of the proof we study the growth of $[\Gamma' : (\Gamma' \cap \Gamma(g))]$ relative to $[\Gamma : \Gamma(g)]$ as in Lemma 1.3 or Theorem 1.10. We will actually understand a bit more than that.

Continuing the proof, we may assume that $\Gamma' \subset A_{\mathbb{F}_q}^\times (\mathbb{F}_q(s))/Z^\times$. However, we still need to handle the conditions of Equation (1) to understand $\Gamma' \cap \Gamma(g)$.

There are two cases: the "good case" $h \in \mathbb{F}_q(t^2) = \mathbb{F}_q(s)$ (i.e., $h$ only has $t$ to an even power) and the "bad case" $h \notin \mathbb{F}_q(t^2) = \mathbb{F}_q(s)$.

In the good case, let $\hat{h}(s) \in \mathbb{F}_q(s)$ be polynomial satisfying $\hat{h}(t^2) = \hat{h}(t)$. Notice that $\deg(\hat{h}) = \deg(h)/2 = \deg(g)/2 = m$.

In the bad case, let $\hat{h}(s) \in \mathbb{F}_q(s)$ be the polynomial satisfying $\hat{h}(t^2) = \hat{h}(t)$.

In both cases, $\hat{h}(s) = \mathbb{F}_q(s) = \mathbb{F}_q(t^2)$ is an irreducible polynomial (or prime), which lies below the irreducible polynomial $h(t) \in \mathbb{F}_q(t)$ in the extension of $\mathbb{F}_q(s)$ to $\mathbb{F}_q(t)$. In other words, it holds that $h(t) \mathbb{F}_q[t] \cap \mathbb{F}_q[s] = \hat{h}(s) \mathbb{F}_q[s]$ and for $f \in \mathbb{F}_q(s)$, $v_{\hat{h}}(f(t^2)) = v_f(f(s))$.

This means that $\hat{h}$ as a subgroup of

$A_{\mathbb{F}_q}^\times (\mathbb{F}_q(s))/Z^\times$, 433
generated by $\delta_k = 1 + c_k j_2 + d_k i_2 j_2$, and $\Gamma' \cap \Gamma (g)$ as its subgroup of elements with an element in the equivalence class satisfying $v = 0, v_h(b) = 0, v_h(c) > 0, v_h(d) > 0$.

The final description is exactly the description of the Morgenstern graph, with $u$ replaced by $s$ and $g(u)$ replaced by $\hat{h}(s)$. Therefore:

**Theorem 2.5.** $Y_g = \text{Cayley}(\Gamma' / (\Gamma' \cap \Gamma (g)), \{\delta_1, ..., \delta_{q+1}\})$ is isomorphic to the Morgenstern graph $X_{\hat{h}}$.

In particular, the subgroup $\Gamma' / (\Gamma' \cap \Gamma (g)) \leq \Gamma / \Gamma (g)$ is isomorphic to either $\text{PSL}_2(\mathbb{F}_{q^m} \hat{h})$ or $\text{PGL}_2(\mathbb{F}_{q^m} \hat{h})$.

Our next goal is to understand which of the two cases, PSL$_2$ or PGL$_2$ happens. In the "good case", $\hat{h}(s) \in \mathbb{F}_q(s) = \mathbb{F}_q(t^2)$ remains irreducible in the extension $\mathbb{F}_q(t)$. In other words, it is inert in the extension. Since this is a quadratic extension, it is well known that it happens if and only if $\frac{s}{h(s)} = -1$. In this case, by Theorem 2.2, $Y_g$ is a bipartite Cayley graph on $\text{PGL}_2(\mathbb{F}_{q^m} \hat{h}) = \text{PGL}_2(\mathbb{F}_{q^m})$.

In the "bad case", $\hat{h}(s) \in \mathbb{F}_q(s) = \mathbb{F}_q(t^2)$ splits in the extension to $\mathbb{F}_q(t)$. This happens if and only if $\frac{s}{h(s)} = 1$. In this case, by Theorem 2.2, $Y_g$ is a non-bipartite Cayley graph on $\text{PSL}_2(\mathbb{F}_{q^m} \hat{h}) = \text{PSL}_2(\mathbb{F}_{q^m})$.

For Theorem 1.1, we need to prove that the good case may happen, and to understand $X_g$ in this case. For this we notice that we may first choose $\hat{h} \in \mathbb{F}_q(s)$ of degree $m$, which is inert in the field extension to $\mathbb{F}_q(t)$, then get $h(t) = \hat{h}(t^2)$ and finally get $g(u)$ from it as the irreducible monic corresponding to $(u - 2)^m h \frac{u}{\hat{h}(u)}$.

We recall quadratic reciprocity in $\mathbb{F}_q(s)$ ([20, Theorem 3.3]), which states that for $f, g \in \mathbb{F}_q[s]$ irreducible

$$\left(\frac{f}{g}\right) = (-1)^{\frac{1}{2}(\deg f)(\deg g)} \frac{g}{f}.$$  

Then

$$\frac{s}{h(s)} = (-1)^{\frac{1}{2}(\deg h)(\deg \hat{h})} \frac{\hat{h}(s)}{s} \frac{\hat{h}(0)}{q}.$$  

The last element is the usual Legendre symbol in $\mathbb{Z}$.

Next,

$$\frac{2t(t+1)}{h(t)} = \frac{2}{h(t)} \left(\frac{t}{h(t)} \frac{t+1}{h(t)}\right).$$

Since $h(t)$ is of even degree, $\mathbb{F}_q(t) / h(t) \mathbb{F}_q(t)$ contains $\mathbb{F}_q^2$. Therefore, every $a \in \mathbb{F}_q$ has a square root in it and $\frac{2t(t+1)}{h(t)} = 1$. It holds by quadratic reciprocity, since the degree of $h(t)$ is even, that

$$\left(\frac{t}{h(t)}\right) = \left(\frac{h(t)}{t}\right) = \frac{h(0)}{t} = \left(\frac{\hat{h}(0)}{q}\right);$$

$$\left(\frac{t+1}{h(t)}\right) = \left(\frac{h(t)}{t+1}\right) = \frac{h(-1)}{q} = \left(\frac{\hat{h}(1)}{q}\right).$$

The last two elements in each row are the Legendre symbol in $\mathbb{Z}$. Therefore, $\left(\frac{2t(t+1)}{h(t)}\right) = \left(\frac{\hat{h}(0)}{q}\right) \left(\frac{\hat{h}(1)}{q}\right)$ determines whether $X_g$ is PGL or PSL.

We conclude that by determining $\hat{h}(0), \hat{h}(1)$ we can make $Y_g$ be isomorphic to PGL$_2(\mathbb{F}_{q^m})$ and $X_g$ to be isomorphic to either of PGL$_2(\mathbb{F}_{q^m})$ or PSL$_2(\mathbb{F}_{q^m})$. Finally, for $m$ large enough, we may freely choose $\hat{h}(0), \hat{h}(1)$ while keeping $\hat{h}$ irreducible by Chaboterev’s density theorem ([20, Theorem 4.7 and Theorem 4.8]).

We collect our findings in the following lemma:

**Lemma 2.6.** For every $m$ large enough, we may find a monic irreducible polynomial $g$ of degree $2m$ such that:

1. The Morgenstern Ramanujan Cayley graph

$X_g = \text{Cayley}(\Gamma / \Gamma (g), \{y_1, ..., y_{q+1}\})$

is bipartite, $\Gamma / \Gamma (g) \cong \text{PGL}_2(\mathbb{F}_{q^m})$ and its girth is greater than $4/3 \log_q |X_g|$.

2. The subgroup $\Gamma' / (\Gamma' \cap \Gamma (g)) \leq \Gamma / \Gamma (g)$ that is generated by $\{Y_1^2, ..., Y_{q+1}^2\}$ is isomorphic to PGL$_2(\mathbb{F}_{q^m})$. Moreover, $Y_g = \text{Cayley}(\Gamma' / (\Gamma' \cap \Gamma (g)), \{Y_1^2, ..., Y_{q+1}^2\})$ is also a Morgenstern Ramanujan graph.

Notice that the lemma implies that

$$|Y_g| = |\text{PGL}_2(\mathbb{F}_{q^m})| = O\left(\sqrt{|\text{PGL}_2(\mathbb{F}_{q^m})|}\right) = O(\sqrt{|X_g|}).$$

By the discussion at the beginning of this section, we conclude that Lemma 2.6 is an explicit version of Lemma 1.3.

We need however another result to complete the explicit part of the proof of Theorem 1.1.

**Lemma 2.7.** Every vertex $x \in N(Y_g)$ is connected to exactly two vertices in $Y_g$.

**Proof.** If $x \in N(Y_g)$, then $x = y_i y_j$ for $y \in Y_g$, and $y_i$ a generator. Therefore, $x$ is connected in $X_g$ to both $y = y_i^{-1} x$ and $y = y_i x \in Y_g$. Therefore, every neighbor of $Y_g$ is connected to at least two vertices of it (we discussed this part of the proof at the beginning of the section).

Assume by contradiction that $x \in N(Y_g)$ is connected to more than two vertices of $Y_g$, then by applying the automorphism of the Cayley graph $X_g$ defined by subgroup $\Gamma' / (\Gamma' \cap \Gamma (g))$, we would get $O(|Y_g|)$ other neighbors of $Y_g$ which are connected to more than 2 vertices of $Y_g$. Then there is some $\delta > 0$ such that on average a neighbor of $Y_g$ is connected to 2+ vertices in $Y_g$. This contradicts Kahle’s vertex expansion Theorem.

**Remark 2.** Assume that we choose $\hat{h}(s)$, deg $\hat{h}(s) = m$, such that it splits in the extension to $\hat{h}(t^2) = h(t) h(-t)$. Then a similar construction still works – we look at $g(u) = g_1(u) g_2(u)$, where $g_1(u)$ corresponds to $h(t)$ and $g_2(u)$ corresponds to $h(-t)$. Then $\Gamma' \cap \Gamma (g)$, $\Gamma (g)$ defines a Cayley graph $X_g = \text{Cayley}(\Gamma / \Gamma (g), \{y_1, ..., y_{q+1}\})$.

There is an embedding $F : Y_g \cong X_g$ onto $X_g$, which extends to a graph map on the subdivision graph $Y'_g$ of $Y_g$.  

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In this case $Y_q$ will be a Cayley graph on $\text{PSL}_2(F_{q^m})$, while $X_q$ will be a Cayley graph on some subgroup between $\text{PSL}_2(F_{q^m}) \times \text{PSL}_2(F_{q})$ and $\text{PGL}_2(F_{q^m}) \times \text{PGL}_2(F_{q})$. So again we get a similar map from a small $(q + 1, 2)$-biregular graph and a big $(q + 1)$-regular Ramanujan graph.

2.1 Explicit Generators

We shortly describe how to construct our graphs explicitly. Assume we are given monic irreducible $h(s) \in F_q(s)$, $\deg h = m$, which is inert (remains irreducible) in the extension to $F_q(t)$. Let $h(t) = \tilde{h}(t^2)$ be the irreducible polynomial of degree $2m$ in $F_q(t)$ above $h(s)$. Let $e \in F_q$ be a non-square and let $i \in F_q(t)/h(t)F_q(t)$ be a square root of $e$ (which exists since $h$ is of even degree). Consider the following elements in $\text{PGL}_2(F_q(t)/h(t)F_q(t)) \cong \text{PGL}_2(F_{q^m})$,

$\gamma_k = \left( \begin{array}{cc} t + 1 & (c_k - d_k i) \\ (c_k + d_k i)(t^2 - 1) & t + 1 \end{array} \right)$

$\delta_k = \left( \begin{array}{cc} 1 & (c_k - d_k i) \\ (c_k + d_k i)(t^2 - 1) & 1 \end{array} \right)$

(we work modulo center, so it makes sense). Recall that $(c_k, d_k)$ are the $(q + 1)$ solutions to $e d^2 - c^2 = 1$. Then $\gamma_1, \ldots, \gamma_{q+1}$ generate a Ramanujan Cayley graph isomorphic to the Morgenstern Cayley graph of the monic polynomial corresponding to $(u - 2)^{2m} h(\frac{1}{2u})$. The elements $\{\gamma_1, \ldots, \gamma_{q+1}\}$ generate $\text{PGL}_2(F_{q^m})$ if and only if $\frac{2(t+1)}{h(t)} = \left( \frac{h(1)}{q} \right) \left( \frac{h(1)}{q} \right) = -1$. The elements $\delta_k$ generate a Ramanujan Cayley graph isomorphic to $\text{PGL}_2(F_{q^m})$. This is simplest to see when $m$ itself is even, since then $e \in F_q(t^2)/h(t^2)F_q(t^2) \subset F_q(t)/h(t)F_q(t)$, and the $\delta_k$ are the generators given in [18, Equation (14)].

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