Rendezvous of Agents with Different Speeds

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Abstract

Most cooperative systems are, to some degree, asynchronous. This lack of synchrony is typically considered as an obstacle that makes the achievement of cooperative goals more difficult. In this work, we aim to highlight potential benefits of asynchrony, which is introduced into the system as differences between the speeds of the actions of different entities. We demonstrate the usefulness of this aspect of asynchrony, to tasks involving symmetry breaking. Specifically, in this paper, identical (except for their speeds) mobile deterministic agents are placed at arbitrary locations on a ring of length \( n \) and use their speed difference in order to rendezvous fast. We normalize the speed of the slower agent to be 1, and fix the speed of the faster agent to be some constant \( c > 1 \). (An agent does not know whether it is the slower agent or the faster one.) We present lower and upper bounds on the time of reaching rendezvous in two main cases. One case is that the agents cannot communicate in any way. For this case, we show a tight bound of \( \frac{n}{c^2 - 1} \) for the rendezvous time. The other case is where each agent is allowed to drop a pebble at the location it currently is (and is able to detect whether there is already a pebble there). For this case, we show an algorithm whose rendezvous time is \( \max\{\frac{n}{2(c-1)}, \frac{n}{c}\} \). On the negative side, we show an almost matching lower bound of \( \max\{\frac{n}{2(c-1)}, \frac{n}{c+1}\} \), which holds not only under the “pebble” model but also under the seemingly much stronger “white board” model.

Keywords: rendezvous; asynchrony; heterogeneity; speed; ring; pebble; white board; mobile agents
1 Introduction

1.1 Background and motivation

In this paper, we illustrate some advantages of a quality that is often considered disruptive in real computing systems, namely asynchrony. We consider the manifestation of asynchrony as a difference in action speed between the different constituents of a system; in this initial study, we consider a constant difference, leaving the possible unpredictability of this difference to future study. We demonstrate the usefulness of this manifestation of asynchrony to tasks involving symmetry breaking. More specifically, we show how two mobile agents, identical in every aspect save their speed, can lever their speed difference in order to rendezvous fast.

Symmetry breaking is a major issue in distributed computing that is completely absent from traditional sequential computing. Symmetry can often prevent different processes from reaching a common goal. Well known examples include leader election [3], mutual exclusion [13], agreement [4, 25] and renaming [5]. To address this issue, various differences between processes are exploited. For example, solutions for leader election often rely on unique identifiers assumed to be associated with each entity (e.g., a process) [3]. Another example of a difference is the location of the entities in a network graph. Entities located in different parts of a non-symmetric graph can use this knowledge in order to behave differently; in such a case, a leader can be elected even without using unique identifiers [26]. If no differences exist, breaking symmetry deterministically is impossible (see, e.g., [3, 27]) and one must resort to randomized algorithms, assuming that different entities can draw different random bits [19]. Random bits are often regarded as a scarce resource. We note that distributed systems do have a certain unpredictable source of randomness coming from differences between the speeds of the actions of different entities which may be time dependent. This source is, too, a manifestation of asynchrony, and, possibly, can be exploited in spite of its unpredictability. However, we leave the study of exploiting this part of asynchrony to the future, and focus on the simpler fully deterministic case. That is, we assume a speed heterogeneity that is random but fixed throughout the execution.

We consider mobile agents aiming to rendezvous. See, e.g., [6, 12, 22, 23, 24, 27]. As is the case with other symmetry breaking problems, it is well known that if the agents are completely identical then rendezvous is, in some cases, impossible. We study the case where the agents are identical except for the fact that they have different speeds of motion. To isolate the issue of the speed difference, we remove other possible differences between the agents. For example, the agents, as well as the graph over which the agents walk, are assumed to be anonymous. Moreover, the graph is symmetric (we study a cycle), to avoid solutions of the kind of [26].

1.2 The Model and the Problem

Consider two identical deterministic agents placed on an anonymous cycle of length n (in some distance units). To ease the description, we name these agents A and B but these names are not known to the agents.
Each agent is initially placed in some location on the cycle by an adversary and both start the execution of the algorithm simultaneously. An agent can move on the ring at each direction. Specifically, at any given point in time, an agent can decide to either start moving, continue in the same direction, stop, or change its direction. The agents have a sense of direction [7], that is, they can distinguish clockwise from the anti-clockwise. The agents’ goal is to rendezvous, namely, to get to be co-located somewhere on the cycle. We consider continuous movement, so this rendezvous can occur at any location along the cycle. An agent can detect the presence of another agent at its location and hence detect a rendezvous.

Although the agents do not hold any direct means of communication, in some cases we do assume that an agent can mark its location by dropping a pebble [9][10]. Somewhat informally, the pebble mechanism is used by an agent to mark a location on the cycle so that this location can be recognized by itself and by the other agent. Both dropping and detecting a pebble are local acts taking place only on the location occupied by the agent. We note that in the case where pebbles can be dropped, our upper bound employs agents that drop a pebble only once and only at their initial location [11][8][24]. On the other hand, our corresponding lower bound holds for any mechanism of (local) pebble dropping. Moreover, this lower bound holds also for the seemingly stronger ”white board” [16][17] model, in which the agent can change a memory associated with its current location such that it could later be read and further manipulated by itself or other agents [20].

Each agent moves at the same constant speed at all times; the speed of an agent $A$, denoted $s(A)$, is the inverse of the time $t_A$ it takes agent $A$ to traverse one unit of length. For agent $B$, the time $t_B$ and speed $s(B)$ are defined analogously. Without loss of generality, we assume that agent $A$ is faster, i.e., $s(A) > s(B)$ but emphasize that this is unknown to the agents themselves. Furthermore, for simplicity of presentation, we normalize the speed of the slower agent $B$ to one, that is, $s(B) = 1$ and denote $s(A) = c$ where $c > 1$. We stress that $c$ can be a function of $n$, e.g., $c = 1 + 1/n$. We assume that the agents know $n$ and $c$. In addition, we assume that each agent has a pedometer that enables it to measure the distance it travels. Specifically, a (local) step of an agent is a movement of one unit of distance (not necessarily all in the same direction, e.g., in one step an agent can move half a step in one direction and the other half in the other direction). Using the pedometer, agents can count the number of steps they took (which is a real number at any given time).

The rendezvous time of an algorithm is defined as the worst case time bound until rendezvous, taken over all pairs of initial placements of the two agents on the cycle.

1.3 Our Results

The results are summarized in Table 1. Upper and lower bounds are given. One case is when no communication is allowed between the agents. For this case, we establish a tight bound of $\frac{n}{c-1}$ for the rendezvous time. We then turn our attention to the more difficult case where pebbles may be employed. For this case, we prove an upper bound of $\max\left\{\frac{n}{2(c-1)}, \frac{n}{c}\right\}$. We also establish an almost tight lower bound of $\max\left\{\frac{n}{2(c-1)}, \frac{n}{c+1}\right\}$. Moreover, this lower bound holds also for the seemingly stronger “white board” model. Observe that for the case where $c \leq 2$ this lower bound is tight.


|                | Lower bound   | Upper bound   |
|----------------|---------------|---------------|
| No pebbles allowed | $\frac{cn}{c^2-1}$ | $\frac{cn}{c^2-1}$ |
| Using pebbles     | $\max\left\{\frac{n}{2(c-1)}, \frac{n}{c+1}\right\}$ | $\max\left\{\frac{n}{2(c-1)}, \frac{n}{c}\right\}$ |

Table 1: Bounds on the advice for given competitiveness

2 Rendezvous without communication

In this section, we show a tight bound for the rendezvous time assuming that the agents cannot use pebbles to mark their location. More generally, the agents cannot communicate in any way (before rendezvous).

Theorem 1. Consider the setting in which no communication is allowed.

- The rendezvous time of any algorithm is, at least, $\frac{cn}{c^2-1}$.
- There exists an algorithm whose rendezvous time is $\frac{cn}{c^2-1}$.

Proof. Let us first show the first part of the theorem, namely, the lower bound. Given an algorithm, let $\hat{t}$ denote the rendezvous time of the algorithm, that is, the maximum time (over all initial placements and all cycles of length $n$) for the agents (executing this algorithm) to reach rendezvous. Without loss of generality, we assume that the direction an agent starts walking is clockwise.

Consider, first, two identical cycles $C_A$ and $C_B$ of length $n$ each. Let us mark a location $v \in C_A$ and a location $u \in C_B$. Let us examine the (imaginary) scenario in which agent $A$ (respectively, $B$) in placed on the cycle $C_A$ (respectively, $C_B$) alone, that is, the other agent is not on the cycle. Furthermore, assume that agent $A$ is placed at $v$ and agent $B$ is placed at $u$. In this imaginary scenario, agents $A$ and $B$ start executing the protocol at the same time, separately, on each of the corresponding cycles. Viewing $u$ as homologous to $v$, a homologous location $h(x) \in C_B$ can be defined for each location in $x \in C_A$ in a natural way (in particular, $h(v) = u$).

For each time $t \in [0, \hat{t}]$, let $d(t)$ denote the clockwise distance between the location $b_t \in C_B$ of the slower agent $B$ at time $t$ and the homologous location $h(a_t) \in C_B$ of the location $a_t \in C_A$ of the faster agent $A$ at time $t$. Note that $d(t)$ is a real value in $[0, n)$. Initially, we assume that all reals in $[0, n)$ are colored white. As time passes, we color the corresponding distances by black, that is, at every time $t$, we color the distance $d(t) \in [0, n)$ by black. Note that the size of the set of black distances is monotonously non-decreasing with time.

We first claim that, by time $\hat{t}$, the whole range $[0, n)$ is colored black. To prove by contradiction, assume that there is a real $d \in [0, n)$ that remains white. Now, consider the execution on a single cycle of length
n, where agent \( A \) is initially placed at anti-clockwise distance \( d \) from agent \( B \). In such a case, rendezvous is not reached by time \( \hat{t} \), contrary to the assumption that it is. This implies that the time \( T \) it takes until all reals in \([0, n)\) are colored black in the imaginary scenario, is a lower bound on the rendezvous time, that is, \( T \leq \hat{t} \). It is left to analyze the time \( T \).

With time, the change in the distance \( d(t) \) has to follow two rules:

1. After one time unit, the distance can change by at most \( 1 + c \) (the sum of the agents’ speeds).
2. At time \( x \), the distance, in absolute value is, at most, \( x(c - 1) \).

To see why Rule 2 holds, recall that the programs of the agents are identical. Hence, if agent \( A \) is at some point \( a \in C_A \), at \( A \)’s \( k \)th step, then agent \( B \) is at point \( b = h(a) \) at its own \( k \)th step. This happens for agent \( B \) at time \( k \) and for agent \( A \) at time \( k/c \). Since \( A \)’s speed is \( c \), the maximum it can get away from point \( a \) during the time period from \( k/c \) until \( c \) is \( c(k - k/c) = k(c - 1) \).

Consider the path \( P := d(t) \) covering the range \([0, n)\) in time \( T \). First, note that this path \( P \) may go several times through the zero point. At a given time \( s \), we say that the path is on the right side, if the last time it left the zero point before time \( s \) was while going towards 1. Similarly, the path is on the left, if the last time it left the zero point before time \( s \) was while going towards \( n - 1 \). Let \( x \) denote the largest point on the range \([0, n)\) reached by the path while the path was on the right side. Let \( y = n - x \). By time \( T \), path \( P \) had to go left to distance \( y \) from the zero point. Assume, w.l.o.g. that \( x < y \). (In particular, \( x \) may be zero.) The fastest way to color these two points (and all the rest of the points, since those lie between them), would be to go from zero to the right till reaching \( x \), then return to zero and go to distance \( y \) on the left. Hence, \( T \) will be at least: \( T \geq \frac{x}{c-1} + \frac{n}{c+1} \). Indeed, Rule 2, applied to the time of reaching distance \( x \), proceeds with the distance reaching the zero point and ends when the distance reaches \( y \) when going left. Since \( c > 1 \), we obtain

\[
x \leq \left( T - \frac{n}{c+1} \right) (c - 1).
\]

On the other hand, applying Rule 2 to the final destination \( y \), we have \( T(c - 1) \geq y = n - x \). This implies that:

\[
x \geq n - T(c - 1).
\]

Combining Equations 1 and 2 we get \( T \geq \frac{cn}{c-1} \). This establishes the first part of the theorem.

We now prove the second part of the theorem, namely, the upper bound. First, we show that a trivial algorithm, called Distributed Race (DR), already ensures an upper bound that is rather close to the lower bound given above. Later, we revise the algorithm to show a tight bound. Under DR, an agent starts moving clockwise, and continues to walk in that direction until reaching rendezvous. Let \( d \) denote the clockwise distance from the initial location of \( A \) to that of \( B \). The rendezvous time \( t \) will thus satisfy \( t \cdot s(A) = t \cdot s(B) + d \). Hence, we obtain the following.
Observation 2.1 The rendezvous time of DR is \( d/(c-1) < n/(c-1) \).

We now revise DR and consider an algorithm which consists of two stages. At the first stage, the agent walks clockwise for \( k = \frac{cn}{c^2-1} \) steps. Subsequently (if rendezvous hasn’t occurred yet), the agent executes the second stage of the algorithm: it turns around and goes in the other direction until rendezvous.

Assume by contradiction, that rendezvous hasn’t occurred by time \( k \). By that time, agent \( B \) took \( k \) steps. Agent \( A \) took those \( k \) steps by time \( k/c \). At that time, agent \( A \) turns. (However, agent \( B \) will turn only at time \( k \)). Hence, between those two turning times, there is a time duration \( k(1 - \frac{1}{c}) \) where the two agents walk towards each other. Hence, at each time unit they shorten the distance between them by \( 1 + \frac{1}{c} \). Let \( d' \) denote the maximum distance between the agents at time \( k/c \). It follows that \( d' > k(1 - \frac{1}{c})(1 + c) = n \). A contradiction. This establishes the second part of the theorem.

3 Rendezvous using Pebbles

3.1 Lower bound

The following lower bound actually holds even in a model that is stronger than that of pebbles.

Theorem 2. Any rendezvous algorithms in the “white board” model requires \( \max\{ \frac{n}{2(c-1)}, \frac{n}{c+1} \} \) time.

Proof. We first show that any algorithm in the “white board” model requires \( \frac{n}{2(c-1)} \) time. Consider the case that the adversary locates the agents at symmetric locations of the cycle, i.e., they are at distance \( n/2 \) apart. Now consider any algorithm used by the agents. Let us fix any \( c' \) such that \( 1 < c' < c \), and define the interval

\[
I := \left[ 0, \frac{nc'}{2(c-1)} \right].
\]

For every \( i \), let us define the following (imaginary) scenario \( S_i \). In scenario \( S_i \), each agent executes the algorithm for \( i \) steps and terminates.\(^1\) We claim that for every \( i \in I \), the situation at the end of scenario \( S_i \) is completely symmetric: that is, the white board at symmetric locations contain the same information and the two agents are at symmetric locations. We prove this claim by induction. The basis of the induction, the case \( i = 0 \), is immediate. Let us assume that the claim holds for scenario \( S_i \), for \( i \in I \), and consider scenario \( S_{i+\epsilon} \) for any positive \( \epsilon \) such that \( \epsilon < \frac{n}{4c}(1 - \frac{1}{c}) \). Our goal is to show that the claim holds for scenario \( S_{i+\epsilon} \).

Consider scenario \( S_{i+\epsilon} \). During the time interval \([0, \frac{1}{c}]\), both agents perform the same actions as they do in the corresponding time interval in scenario \( S_i \). Let \( a \) denote the location of agent \( A \) at time \( i/c \). Now,\(^2\) Note that for some \( i \in I \) and some \( \epsilon < \frac{n}{4c}(1 - \frac{1}{c}) \), we may have that \( i + \epsilon \notin I \). Our inductive proof will show that the claim for \( S_{i+\epsilon} \) holds also in such cases. However, since we wish to show that the claim holds for \( S_j \), where \( j \in I \), we are not really interested in those cases, and are concerned only with the cases where \( i + \epsilon \in I \).

\(^1\)We can think of this scenario as if each agent executes another algorithm \( B \), in which it simulates precisely \( i \) steps of the original algorithm and then terminates.

\(^2\)Note that for some \( i \in I \) and some \( \epsilon < \frac{n}{4c}(1 - \frac{1}{c}) \), we may have that \( i + \epsilon \notin I \). Our inductive proof will show that the claim for \( S_{i+\epsilon} \) holds also in such cases. However, since we wish to show that the claim holds for \( S_j \), where \( j \in I \), we are not really interested in those cases, and are concerned only with the cases where \( i + \epsilon \in I \).
during the time period \([\frac{i}{c}, \frac{i + \epsilon}{c}]\), agent \(A\) performs some movement all of which is done at distance at most \(\epsilon\) from \(a\) (during this movement it may write information at various locations it visits); then, at time \(\frac{i + \epsilon}{c}\), agent \(A\) terminates.

Let us focus on agent \(B\) (in scenario \(S_{i+\epsilon}\)) during the time period \([\frac{i}{c}, i]\). We claim that during this time period, agent \(B\) is always at distance at least \(\epsilon\) from \(a\). Indeed, as long as it is true that agent \(B\) is at distance at least \(\epsilon\) from \(a\), it performs the same actions as it does in scenario \(S_i\) (because it is unaware of any action made by agent \(A\) in scenario \(S_{i+\epsilon}\), during the time period \([\frac{i}{c}, \frac{i + \epsilon}{c}]\)). Therefore, if at some time \(t' \in [\frac{i}{c}, i]\), agent \(B\) is in (scenario \(S_{i+\epsilon}\)) at distance less than \(\epsilon\) from \(a\) then the same is true also for scenario \(S_i\). However, in scenario \(S_i\), by the induction hypothesis, agent \(B\) was at time \(i\) at \(\bar{a}\), the symmetric location of \(a\), that is, at distance \(n/2\) from \(a\). Thus, to get from a distance less than \(\epsilon\) from \(a\) to \(\bar{a}\), agent \(B\) needs to travel a distance of \(n/2 - \epsilon\), which takes \(n/2 - \epsilon\) time. This is impossible since

\[
    i - i/c \leq \frac{nc'}{2c} < \frac{n}{2} - \epsilon,
\]

where the first inequality follows from the definition of \(I\) and the second follows from the definition of \(\epsilon\).

It follows that during the time period from \([\frac{i}{c}, i]\), agent \(B\) behaves (in scenario \(S_{i+\epsilon}\)) the same as it does in the corresponding time period in scenario \(S_i\). Therefore, according to the induction hypothesis, at the corresponding \(i'\)th steps in scenario \(S_{i+\epsilon}\), agent \(B\) is at distance \(n/2\) from where agent \(A\) is at its \(i'\)th step (recall, agent \(A\) is at \(a\) at its \(i'\)th step), and the cycle configuration (including the white boards) is completely symmetric. Now, since \(\epsilon < n/4\), during the time period \([i, i + \epsilon]\), agent \(B\) is still at a distance more than \(\epsilon\) from \(a\) and remains unaware of any action made by agent \(A\), during the time period \([\frac{i}{c}, \frac{i + \epsilon}{c}]\). (Similarly, agent \(A\), during the time period \([\frac{i}{c}, \frac{i + \epsilon}{c}]\), is unaware of any action made by agent \(B\) during this time period.) Hence, at each time \(i' \in [i, i + \epsilon]\), agent \(B\) takes the same action as agent \(A\) in the corresponding time \(i'/c\). This establishes the induction proof. To sum up, we have just shown that for any \(i \in I\), the cycle configuration at the end of scenario \(S_i\) is completely symmetric.

Now assume by contradiction that the rendezvous time \(t\) is less than the claimed one, that is, \(t < \frac{n}{2(c-1)}\). At time \(t\), both agents meet at some location \(u\). Since \(t \in I\), the above claim holds for \(S_i\). Hence, at time \(t/c\), agent \(A\) is at \(\bar{u}\), the symmetric location of \(u\). Since rendezvous happened at time \(t\), this means that agent \(A\) traveled from \(\bar{u}\) to \(u\) (i.e., a distance of \(n/2\)) during the time period \([\frac{i}{c}, t]\). Therefore \(t(1 - \frac{1}{c})c \geq n/2\), contradicting the assumption that \(t < \frac{n}{2(c-1)}\). This establishes the fact that any algorithm requires \(\frac{n}{2(c-1)}\) time.

We now show the simpler part of the theorem, namely, that the rendezvous time of any algorithm in the “white board” model is at least \(\frac{n}{c+1}\). Let us represent the cycle as the reals modulo \(n\), that is, we view the cycle as the continuous collection of reals \([0, n]\), where \(n\) coincides with \(0\). Assume that the starting point of agent \(A\) is \(0\). Consider the time period \(T = [0, \frac{n}{c+1} - \epsilon]\), for some small positive \(\epsilon\). In this time period, agent \(A\) moves a total length of less than \(\frac{nc}{c+1}\). Let \(r\) (and \(\ell\), correspondingly) be the furthest point from \(0\) on the cycle that \(A\) reached while going clockwise (or anti-clockwise, correspondingly), during that time period. Note that there is a gap of length larger than \(n - \frac{nc}{c+1} = \frac{n}{c+1}\) between \(\ell\) and \(r\). This gap corresponds to an arc not visited by agent \(A\) during this time period. On the other hand, agent \(B\) walks a total distance
of less than \( \frac{n}{c+1} \) during the time period \( T \). Hence, the adversary can locate agent \( B \) initially at some point in the gap between \( r \) and \( \ell \), such that during the whole time period \( T \), agent \( B \) remains in this gap. This establishes the \( \frac{n}{c+1} \) time lower bound, and concludes the proof of the theorem. 

\[ \square \]

### 3.2 Upper Bound

In view of Theorem 2, the following theorem establishes a tight bound for the case where \( c \leq 2 \).

**Theorem 3.** There exists an algorithm that may drop pebbles whose rendezvous time is \( \max\{ \frac{n}{2(c-1)}, \frac{n}{c} \} \). Moreover, this algorithm uses only one pebble and drops it only once: at the initial position of the agent.

**Proof.** Consider the following algorithm. Each agent (1) leaves a pebble at its initial position and then starts walking clockwise while counting the distance travelled. If (2) an agent reaches a location with a pebble for the first time and (3) the distance it walked is strictly less than \( \tau := \min\{ \frac{n}{2}, \frac{n}{c} \} \), then (4) the agent turns around and walks continuously in the other direction. Let \( d \) be the initial clockwise distance from \( A \) to \( B \).

Consider three cases.

1. \( d = \tau \).
   
   Here, no agent turns around. In other words, they behave exactly as in DR. If \( d = \frac{n}{2} \), Observation 2.1 implies that the rendezvous is reached earlier (by time \( \frac{n}{c(c-1)} \)). Otherwise, \( c > 2 \) and \( d = \frac{n}{c} \). By Observation 2.1, the rendezvous is reached earlier (by time \( \frac{n}{c(c-1)} \)).

2. \( d < \tau \).
   
   In this case, Agent \( A \) will reach \( B \)'s starting point \( v_B \), at time \( \frac{d}{c} \), before \( B \) reaches \( A \)'s starting point \( v_A \). Moreover, \( B \) does not turn, since its initial distance to \( A \)'s starting point is at least \( \tau \). At time \( \frac{d}{c} \), agent \( B \) is at distance \( \frac{d}{c} \) clockwise from \( v_B \). By the algorithm, Agent \( A \) then turns and walks anti-clockwise. The anti-clockwise distance from \( A \) to \( B \) is then \( n - \frac{d}{c} \). Their relative speed is \( c+1 \). Hence, they will rendezvous in an additional time of \( \frac{n-d/c}{1+c} \), since no agent may turn around after time \( \frac{d}{c} \). Hence, the total time for reaching rendezvous is at most

\[ \frac{d}{c} + \frac{n-d/c}{1+c} = \frac{d+n}{1+c}. \]

This function is maximized when \( d = \tau \) where it is \( \frac{\tau+n}{1+c} \). Now, if \( c \leq 2 \), we have \( \tau = \frac{n}{2} \) and the rendezvous time is therefore \( \frac{\tau+n}{2(c-1)} \). Since \( c \leq 2 \), the later bound is at most \( \frac{n}{2(c-1)} \). On the other hand, if \( c > 2 \), we have \( \tau = \frac{n}{c} \) and the rendezvous time is \( \frac{n}{c} \).

3. \( d > \tau \).
   
   In this case, \( A \) doesn’t turn when it hits \( B \)'s initial position. Consider the following sub-cases.
(a) The agents meet before \( B \) reaches \( A \)'s initial position.
In this case, the rendezvous time (as in DR) is \( d/(c-1) \). On the other hand, the rendezvous time is at most \( n-d \) since \( B \) did not reach \( A \)'s initial position. So \( d/(c-1) \leq n-d \). A simple calculation now implies that the rendezvous time \( d/(c-1) \) is at most \( n/c \).

(b) Agent \( B \) reaches \( A \)'s initial position before rendezvous.
In this case, Agent \( B \) walks for \( d' = n-d \) time to first reach \( A \)'s initial position. We first claim that \( d' < \tau \). One case is that \( c \leq 2 \), and thus, \( \tau = n/2 \). Since \( d > \tau \), we have \( d' < n/2 = \tau \).
The other case is that \( c > 2 \), so \( \tau = n/c \). We claim that also in this case, we have \( d' < \tau \).
Otherwise, we would have had \( d' \geq n/c \), which would have meant that the faster agent \( A \) would have had, at least, \( n/c \) time before \( B \) reached the initial position of \( A \). So much time would have allowed it to cover the whole cycle. This contradicts the assumption that \( B \) reached \( A \)'s initial position before rendezvous.
This establishes the fact that, regardless of the value of \( c \), we have \( d' < \tau \). This fact implies that when agent \( B \) reaches \( A \)'s initial position, it turns around and both agents go towards each other. By the time \( B \) turns around, \( A \) has walked a distance of \( cd' \). Hence, at that point in time, they are \( n - cd' \) apart. This implies to the following rendezvous time:

\[
\frac{d'}{1+c} + \frac{n - cd'}{1+c} = \frac{2n - d}{1+c}.
\]
Now recall that we are in the case that agent \( B \) reaches \( A \)'s initial position before they rendezvous. This implies that \( n - d < n/c \). Hence, the running time is at most

\[
\frac{2n - d}{1+c} \leq \frac{n + \frac{n}{c}}{1+c} = \frac{n}{c}.
\]

\( \square \)

4 Discussion and future work

We showed how asynchrony could be useful for achieving efficient rendezvous. Our algorithms could be considered as a means to harness the (unknown) heterogeneity between individuals in a cooperative population towards more efficient functionality.

There are many natural ways of further exploring this idea in future work. For example, the “level” of asynchrony considered in this paper is very limited: the ratio \( c \) between the speeds of the agents is the same throughout the execution, and is known to the agents. Exploiting a higher level of asynchrony should also be studied, for example, the case that the speed difference is stochastic and changes through time. Moreover, we have studied the exploitation of asynchrony for a specific kind of problems. It seems that it can be useful for other symmetry breaking problems as well. Even for the rendezvous problem, we have studied a very limited case. For example, it may be interesting to study the uniform case [21], in which \( n \) and \( c \) are not
known to agents. Another direction is to generalize the study to multiple agents (more than two, see, e.g., [14,18]) and other graph classes.

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