A Fundamental Upper Bound for Signal to Noise Ratio of Quantum Detectors

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Abstract

Quantum fluctuations yield inevitable noises in quantum detection. We derive an upper bound of signal-to-noise ratios for arbitrary quantum detection described by trace-class operators. The bound is independent of observables to be detected and is computed by quantum fidelity of two initial quantum states. We provide applications of the upper bound.
1 Introduction

Any quantum computation and quantum communication require quantum detection of observables in read-out procedures of the protocols. Hence evaluation of signal-to-noise ratio (SNR) for the quantum detectors is crucial in developing the technology. Of course, classical systematic noises lower SNR. In addition, inevitable noises happen due to quantum fluctuations of the observables and may dominate in quantum regime. Especially in quantum optics, exploration of quantum SNR has been performed in earlier study for individual subjects, including squeezed states [1][2][3], weak measurements [4][5][6], multimode spatial entanglement detection[7], electron-multiplying CCD camera [8], heralded linear amplifier [9], and correlation plenoptic imaging [10]. In this paper, we provide a universal upper bound of quantum SNR for arbitrary quantum detection with trace-class operators. The quantum detectors have the following universal setup: a target system $S$ carrying useful information in a quantum state interacts with a measurement device system $D$ in finite duration. The initial state $\rho_{D,0}$ of $D$ is independent of the information. The information of $S$ is extracted by measuring an observable $\hat{A}_D$ of $D$. Let us consider two initial quantum states $\rho_S(1)$ and $\rho_S(2)$ of $S$. The difference between the two states provides the information of our interest to be detected. After the interaction between $S$ and $D$, the final state of $D$ is computed as

$$\rho_D(s) = \text{Tr}_S \left[ U \left( \rho_S(s) \otimes \rho_{D,0} \right) U^\dagger \right],$$  

(1)

where $s = 1, 2$ and $U$ is time evolution unitary operator of $S + D$ system. We define quantum signal for $A_D$ as

$$S_D = \left| \text{Tr}_D \left[ \hat{A}_D \rho_D(1) \right] - \text{Tr}_D \left[ \hat{A}_D \rho_D(2) \right] \right|$$  

(2)

and its quantum noise as

$$N_D = \sqrt{\text{Tr}_D \left[ \left( \hat{A}_D - \text{Tr}_D \left[ \hat{A}_D \rho_D(1) \right] \right)^2 \rho_D(1) \right]} + \sqrt{\text{Tr}_D \left[ \left( \hat{A}_D - \text{Tr}_D \left[ \hat{A}_D \rho_D(2) \right] \right)^2 \rho_D(2) \right]}.$$  

(3)

The main result of this paper is the following inequality:

$$\frac{S_D}{N_D} \leq \sqrt{\frac{1 - F(\rho_S(1), \rho_S(2))^2}{1 - \sqrt{1 - F(\rho_S(1), \rho_S(2))^2}}},$$  

(4)

where $F(\rho_S(1), \rho_S(2))$ is quantum fidelity of $\rho_S(1)$ and $\rho_S(2)$ which is defined by

\begin{align*}
F(\rho_S(1), \rho_S(2)) &= \frac{1}{2} \text{Tr} \left[ \sqrt{\sqrt{\rho_S(1)} \rho_S(2) \sqrt{\rho_S(1)}} \right],
\end{align*}
\[ F(\rho_S(1), \rho_S(2)) = \text{Tr} \left[ \sqrt{\rho_S(1) \rho_S(2)} \sqrt{\rho_S(1)} \right]. \]

The result includes the standard SNR case as follows. When the initial state of \( D \) is given by \( \rho_{D,0} = |a\rangle\langle a| \) where \( |a\rangle \) is an eigenstate of \( \hat{A}_D \) such that \( \hat{A}_D |a\rangle = a |a\rangle \), and \( U(\rho_S(2) \otimes \rho_{D,0}) U^\dagger = \rho_S(2) \otimes \rho_{D,0} \), \( a = \text{Tr}_D \left[ \hat{A}_D \rho_{D}(2) \right] \) is interpreted as the initial value of \( \hat{A}_D \), and \( \rho_S(2) \) as a no-signal state. Then \( N_D \) becomes the standard noise of \( \hat{A}_D \) for \( \rho_D(1) \) given by

\[ N_D = \sqrt{\frac{\text{Tr}_D \left[ \hat{A}_D - \text{Tr}_D \left[ \hat{A}_D \rho_D(1) \right] \right]^2}{\rho_D(1)}}. \]

It should be stressed that the upper bound of quantum SNR in eq. (4) does not depend on the observable \( \hat{A}_D \) and is computed only by use of the initial quantum fidelity \( F(\rho_S(1), \rho_S(2)) \) of \( S \). This paper is organized as follows. In section 2, we prove the upper bound of quantum SNR in eq. (4). In section 3, we discuss applications of our results. In section 4, summary is given. We adopt the natural units \( c = \hbar = 1 \) in this paper.

### 2 Derivation of Universal Upper Bound of Quantum SNR

Let us derive the upper bound for SNR in eq. (4). The dimension of Hilbert space of \( D \) is denoted by \( d \). Eigenvalues of \( \hat{A}_D \) are denoted by \( \{a_i\} \) that satisfy \( a_1 \leq a_2 \leq \cdots \leq a_d \). Let us denote \( |a_i\rangle \) a normalized eigenvector associated with \( a_i \). The spectral decomposition of \( \hat{A}_D \) is given by \( \hat{A}_D = \sum_{i=1}^{d} a_i |a_i\rangle\langle a_i| \).

For a quantum state \( \rho_D(1) \) of \( D \), probability of observing \( a_i \) is computed as \( p_i = \langle a_i | \rho_D(1) | a_i \rangle \). Similarly, \( q_i = \langle a_i | \rho_D(2) | a_i \rangle \) is given for a quantum state \( \rho_D(2) \) of \( D \). Fidelity \( F(\{p_i\}, \{q_i\}) \) and Bures distance \( L_B(\{p_i\}, \{q_i\}) \) between the above classical probability distributions of \( \{p_i\} \) and \( \{q_i\} \) are defined as

\[ F(\{p_i\}, \{q_i\}) = \sum_i \sqrt{p_i q_i}, \]

and

\[ L_B(\{p_i\}, \{q_i\}) = \sqrt{1 - F(\{p_i\}, \{q_i\})} = \sqrt{1 - \sum_i \sqrt{p_i q_i}}. \]

Let first prove a lemma such that

\[ S_D \leq \sqrt{2 - L_B(\rho_D(1), \rho_D(2))^2 L_B(\rho_D(1), \rho_D(2)) \left[ \text{Tr}_D[\hat{A}_D^2 \rho_D(1)] + \text{Tr}_D[\hat{A}_D^2 \rho_D(2)] \right]}. \]

(5)
Let us introduce a real vector $\vec{X}$ whose $i$th component is given by $\sqrt{p_i}$. Similarly, $\vec{Y}$ is defined as a real vector whose $i$th component is $\sqrt{q_i}$. It is noticed that the following relation holds using inner products of the vectors:

$$\vec{X} \cdot \vec{Y} = F(\{p_i\}, \{q_i\}),$$
$$\vec{X} \cdot \vec{X} = 1,$$
$$\vec{Y} \cdot \vec{Y} = 1.$$  

In terms of the vectors $\vec{X}$ and $\vec{Y}$, $S_D$ in eq. (2) is written as

$$S_D = \sum_i a_i (p_i - q_i) = |\vec{X} \cdot A_D \vec{X} - \vec{Y} \cdot A_D \vec{Y}|,$$

where $A_D$ is a $d$-dim matrix given by $A_D = [a_i \delta_{ij}]$. The following relation also holds.

$$\sqrt{\text{Tr}_D [A_D^2 \rho_D(1)]} + \sqrt{\text{Tr}_D [A_D^2 \rho_D(2)]} = \sqrt{\sum_i a_i^2 p_i} + \sqrt{\sum_i a_i^2 q_i} = \sqrt{\vec{X} \cdot A_D^2 \vec{X} + \vec{Y} \cdot A_D^2 \vec{Y}}.$$

If both $\vec{X}$ and $\vec{Y}$ are eigenvectors associated with eigenvalue 0 of $A_D$, the problem becomes trivial and the relation in eq. (5) is satisfied. In the following, let us consider other nontrivial cases. One of $\vec{X}$ and $\vec{Y}$ is not an eigenvector associated with eigenvalue 0 of $A_D$. In the case we are able to define $f_{A_D}(\vec{X}, \vec{Y})$ as follows:

$$f_{A_D}(\vec{X}, \vec{Y}) = \left( \frac{\vec{X} \cdot A_D \vec{X} - \vec{Y} \cdot A_D \vec{Y}}{\sqrt{\vec{X} \cdot A_D^2 \vec{X} + \vec{Y} \cdot A_D^2 \vec{Y}}} \right)^2.$$

Note that the inner products, $\vec{X} \cdot A_D \vec{X}$, $\vec{Y} \cdot A_D \vec{Y}$, $\vec{X} \cdot A_D^2 \vec{X}$ and $\vec{Y} \cdot A_D^2 \vec{Y}$ are invariant in the following coordinate transformation:

$$\vec{X}' = R \vec{X}, \quad \vec{Y}' = R \vec{Y}, \quad A' = R A_D R^T,$$

where $R$ is an arbitrary orthogonal matrix satisfying $RR^T = I$. Thus $f_{A_D}(\vec{X}, \vec{Y})$ is also invariant. Using the above symmetry of $f_{A_D}(\vec{X}, \vec{Y})$, without loss of generality, we are able to fix $R$ to a specific matrix so that only the first two components of two vectors $\vec{X}$ and $\vec{Y}$ are nonvanishing as

$$\vec{X}' = [x_1 \ x_2 \ 0 \ \ldots \ 0]^T, \quad \vec{Y}' = [y_1 \ y_2 \ 0 \ \ldots \ 0]^T.$$

Here, we define two dimensional real vectors as follows:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$
Note that $\vec{x} \cdot \vec{y} = F(\{p_i\}, \{q_i\})$ is satisfied. Let us define a two dimensional matrix $B$ to be the submatrix of $A'$ as

$$B = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix}.$$

Then we obtain

$$s_B(x, y) = \left( \frac{\vec{x} \cdot \vec{B} \vec{x} - \vec{y} \cdot \vec{B} \vec{y}}{\sqrt{\vec{x} \cdot \vec{B}^2 \vec{x} + \vec{y} \cdot \vec{B}^2 \vec{y}}} \right)^2 = f_{A'}(\vec{X}', \vec{Y}') \quad (6)$$

Since $A_D$ is a real symmetric matrix, $B$ is also a real symmetric matrix. Eigenvalues of $B$ are denoted by $b_1$ and $b_2$ satisfying $|b_1| \geq |b_2|$. Using a trivial scale invariance of $s_B(x, y)$, $s_{cB}(x, y) = s_B(x, y)$, it can be assumed that the matrix $B$ has eigenvalues 1 and $b = b_2 / b_1$ where $|b| \leq 1$. The corresponding eigenvector for the eigenvalue 1 is denoted by $\vec{u}_1$ and the corresponding eigenvector for the eigenvalue $b$ is $\vec{u}_2$ respectively. Spectral decomposition of $B$ is given by

$$B = \vec{u}_1 \vec{u}_1^T + b \vec{u}_2 \vec{u}_2^T.$$

Defining $P = |\vec{u}_1 \cdot \vec{x}|$ and $Q = |\vec{u}_1 \cdot \vec{y}|$, the function $s_B(x, y)$ is represented by a function $g(b, P, Q)$ of $b$, $P$ and $Q$ as

$$s_B(x, y) = g(b, P, Q) = \frac{(1 - b)^2 (P - Q)^2}{\left(\sqrt{P + b^2 (1 - P)} + \sqrt{Q + b^2 (1 - Q)}\right)^4}.$$ \quad (7)

Varying the value of $b$ so as to find the maximum of $g(b, P, Q)$ for fixed $P$ and $Q$ yields the following equation:

$$\frac{\partial g(b, P, Q)}{\partial b} = \frac{-2(1 - b)(P - Q)^2}{\left(\sqrt{P + b^2 (1 - P)} + \sqrt{Q + b^2 (1 - Q)}\right)^3} \left[ \frac{P + b(1 - P)}{\sqrt{P + b^2 (1 - P)}} + \frac{Q + b(1 - Q)}{\sqrt{Q + b^2 (1 - Q)}} \right] = 0.$$ \quad (8)

It is easy to check that the above equation has a trivial solution $P = Q$, though this provide the minimum and we do not have interest now. Thus we focus on the case with $P \neq Q$ later. When $P \neq 1$ and $Q \neq 1$, the nontrivial solution of the above equation is given by

$$b = b^* = -\sqrt{\frac{PQ}{(1 - P)(1 - Q)}}.$$ \quad (8)

If $|b^*| \leq 1$ holds, $g(b, P, Q)$ takes the maximum $g(b^*, P, Q) = 1 - (\sqrt{PQ} + \sqrt{(1 - P)(1 - Q)})^2$ at $b = b^*$. This constraint requires $P + Q \leq 1$. Since the following inequality is satisfied:

$$\sqrt{PQ} + \sqrt{(1 - P)(1 - Q)} \geq F(\{p_i\}, Q\{q_i\}),$$ \quad (9)
we get
\[ g(b, P, Q) = \left( \frac{\sum a_i (p_i - q_i)}{\sqrt{\sum a_i^2 p_i} + \sqrt{\sum a_i^2 q_i}} \right)^2 \leq 1 - F(\{p_i\}, \{q_i\})^2. \] (10)

It is stressed that the relation in eq. (10) generally holds even in the case that \( P + Q > 1 \) since \( g(b, P, Q) \) monotonically decreases in \(-1 \leq b \leq 1\) and \( g(b, P, Q) \) takes the maximum value \((P - Q)^2\) at \( b = -1\). Even when \( P = 1 \) or \( Q = 1 \), eq. (10) trivially holds.

From the above computation, the relation holds for a fixed basis of \( \hat{A}_D \). Let us arbitrarily choose the basis \( \{|a_i\}\}. \) The classical fidelity \( F(P, Q) \) is greater than or equal to quantum fidelity \( F(\rho_D(1), \rho_D(2)) \) for any \( \{|a_i\}\}. \) Thus we obtain
\[ 1 - F(\{p_i\}, \{q_i\})^2 \leq 1 - F(\rho_D(1), \rho_D(2))^2 = (2 - L_B(\rho_D(1), \rho_D(2))^2) L_B(\rho_D(1), \rho_D(2))^2. \] (11)

From (10) and (11), it is possible to derive
\[ \left( \frac{\sum a_i (p_i - q_i)}{\sqrt{\sum a_i^2 p_i} + \sqrt{\sum a_i^2 q_i}} \right)^2 \leq (2 - L_B(\rho_D(1), \rho_D(2))^2) L_B(\rho_D(1), \rho_D(2))^2 \]
for arbitrary \( \hat{A}_D \). This yields
\[ S_D \leq \sqrt{2 - L_B(\rho_D(1), \rho_D(2))^2} L_B(\rho_D(1), \rho_D(2)) \left[ \sqrt{\text{Tr}[\hat{A}_D^2 \rho_D(1)]} + \sqrt{\text{Tr}[\hat{A}_D^2 \rho_D(2)]} \right]. \] (12)

We have derived the lemma (5). Next we consider a constant shift of the origin of eigenvalues of \( A_D \) by \( \alpha \) and define \( \hat{A}_D = \hat{A}_D - \alpha I \). This generates a tighter inequality than (12) to by optimizing \( h(\alpha) = \sqrt{\text{Tr}[\hat{A}_D^2 \rho_D(1)]} + \sqrt{\text{Tr}[\hat{A}_D^2 \rho_D(2)]} \). The function \( h(\alpha) \) attains the minimum at
\[ \alpha^* = \frac{\text{Tr}[\hat{A}_D \rho_D(1)] \delta_{\hat{A}_D}(\rho_D(2)) + \text{Tr}[\hat{A}_D \rho_D(2)] \delta_{\hat{A}_D}(\rho_D(1))}{\delta_{\hat{A}_D}(\rho_D(1)) + \delta_{\hat{A}_D}(\rho_D(2))}, \]
where \( \delta_{\hat{A}_D}(\rho) = \sqrt{\text{Tr}[\hat{A}_D^2 \rho]} - \left( \text{Tr}[\hat{A}_D \rho] \right)^2 \).

Let us take \( \alpha = \text{Tr}[\hat{A}_D \rho_D(1)] \). Then the following relations are obtained:
\[ S_D = \left| \text{Tr}[\hat{A}_D \rho_D(1)] - \text{Tr}[\hat{A}_D \rho_D(2)] \right| \]
\[ \leq \sqrt{2 - L_B(\rho_D(1), \rho_D(2))^2} L_B(\rho_D(1), \rho_D(2)) h(\text{Tr}[\hat{A}_D \rho_D(1)]) \]
\[ = \sqrt{2 - L_B(\rho_D(1), \rho_D(2))^2} L_B(\rho_D(1), \rho_D(2)) \left[ \delta_{\hat{A}_D}(\rho_D(1)) + \sqrt{\delta_{\hat{A}_D}(\rho_D(2))^2 + S_D^2} \right] \]
\[ \leq \sqrt{2 - L_B(\rho_D(1), \rho_D(2))^2} L_B(\rho_D(1), \rho_D(2)) \left[ \delta_{\hat{A}_D}(\rho_D(1)) + \delta_{\hat{A}_D}(\rho_D(2)) + S_D \right]. \]
Note that a similar inequality appears in [11]. But the above inequality is more stringent. This result can be rewritten as

\[
\frac{S_D}{N_D} \leq \frac{L_B (\rho_D(1), \rho_D(2)) \sqrt{2 - L_B (\rho_D(1), \rho_D(2))^2}}{1 - L_B (\rho_D(1), \rho_D(2)) \sqrt{2 - L_B (\rho_D(1), \rho_D(2))^2}}.
\]

By using the relation between quantum fidelity and Bures distance as

\[L_B (\rho, \rho') = \sqrt{1 - F (\rho, \rho')},\]

we get an upper bound for SNR such that

\[
\frac{S_D}{N_D} \leq \sqrt{1 - F (\rho_D(1), \rho_D(2))^2} \frac{1}{1 - \sqrt{1 - F (\rho_D(1), \rho_D(2))^2}}.
\]

Note that the fidelity \(F (\rho, \rho')\) obeys monotonicity property under quantum channel \(\Gamma\) [12];

\[F (\Gamma [\rho], \Gamma [\rho']) \geq F (\rho, \rho').\]

By using the monotonicity, it turns out that

\[F (\rho_D(1), \rho_D(2)) \geq F (\rho_S(1), \rho_S(2))\] (14)

holds for the quantum states \(\rho_S(1), \rho_S(2)\) of \(S\) via eq. (1) as follows:

\[
F (\rho_D(1), \rho_D(2)) = F \left( \text{Tr}_S \left[ U (\rho_S(1) \otimes \rho_D, 0) U^\dagger \right] \right) \geq F (\rho_S(1) \otimes \rho_D, 0, \rho_S(2) \otimes \rho_D, 0) = F (\rho_S(1), \rho_S(2)).
\]

Here we have used the fact that taking partial trace as a quantum channel increases the fidelity. Thus we obtain eq. (14). Note that \(h(x) = \sqrt{1 - x^2} / (1 - x^2)\) is a monotonically deceasing function for \(x \in [0, 1]\). From eq. (13) and eq. (14), our main result in eq. (4) is derived.

Before closing this section, we comment on the attainability of our upper bound. When the Hilbert space dimension of \(D\) is equal to the Hilbert space dimension of \(S\), we can take a SWAP operator \(U\) between \(D\) and \(S\) as an interaction. Then the upper bound is attained for two commutable quantum states \(\rho_1\) and \(\rho_2\) such that \([\rho_1, \rho_2] = 0\) and

\[
\overrightarrow{y} = \left[ \frac{\overrightarrow{u}_1 \cdot \overrightarrow{y}}{\sqrt{\rho}} \right], \overrightarrow{y} = \left[ \frac{\overrightarrow{u}_1 \cdot \overrightarrow{y}}{\sqrt{\rho}} \right] = \left[ \frac{\sqrt{\rho}}{\sqrt{1 - \rho}} \right]
\]

which attains the equality of eq. (9). In that case, the physical observable \(\hat{A}_D\) which achieves the bound has a very complicated form, but in principle it is
fixed by eq. (8). Here we show two examples which attains the equality. First example is as follows:

$$\rho_D(1) = \cos^2 \theta |0\rangle \langle 0| + \sin^2 \theta |1\rangle \langle 1|,$$
$$\rho_D(2) = \sin^2 \theta |0\rangle \langle 0| + \cos^2 \theta |1\rangle \langle 1|,$$

where $|0\rangle$ and $|1\rangle$ are eigenstates associated with eigenvalues 0 and 1 of number operator of harmonic oscillators respectively. $\theta$ is a real parameter which satisfies $0 \leq \theta < 2\pi$. We define $\omega$ as the angular frequency and $a$ as creation operator and $a^\dagger$ as annihilation operator. When $\hat{A}_D = \hbar \omega (a^\dagger a - \frac{1}{2})$, the equality of eq.(12) is attained. In this case, $\rho_D(1)$, $\rho_D(2)$ and $\hat{A}_D$ commute with each other.

Next we consider more nontrivial case where two states $\rho_D(1)$ and $\rho_D(2)$ do not commute with each other:

$$\rho_D(1) = p|0\rangle \langle 0| + (1 - p)|1\rangle \langle 1|,$$
$$\rho_D(2) = \sigma_z |\pm\rangle \langle +|,$$

where $|0\rangle$ and $|1\rangle$ are eigenstates associated with eigenvalues 1, $-1$ of pauli $z$ matrix $\sigma_z$ respectively and $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. $p$ is a real number that satisfies $0 \leq p \leq 1$. When $\hat{A}_D = \pm (\sigma_x - 1)$, where $\sigma_x$ is pauli $x$ matrix, the upper bound of eq. (12) is achieved. In this case two states $\rho_D(1)$ and $\rho_D(2)$ do not commute with each other, but $\hat{A}_D$ and $\rho_D(2)$ are commutable.

### 3 Applications

In this section, we discuss applications of the universal upper bound of quantum SNR. In subsection 3.1, we consider in a case where two initial states of $S$ are pure. In subsection 3.2, we derive a fundamental upper bound of power consumption to perform quantum switching using quantum SNR. In subsection 3.3, an application for the fidelity estimation is shown.

#### 3.1 Pure state case

Let us consider a case of pure initial states of $S$:

$$\rho_S(1) = |\psi(1)\rangle \langle \psi(1)|,$$
$$\rho_S(2) = |\psi(2)\rangle \langle \psi(2)|.$$

The upper bound in eq.(1) becomes:

$$\frac{S_A}{N_A} \leq \frac{\sqrt{1 - |\langle \psi(1)|\psi(2)\rangle|^2}}{1 - \sqrt{1 - |\langle \psi(1)|\psi(2)\rangle|^2}}.$$

Suppose that $S$ is a free quantum scalar field $\phi(t, x)$ in 3+1 dimensions:

$$\phi(t, x) = \int \frac{d^3k}{\sqrt{(2\pi)^32E_k}} \{\hat{a}(k)e^{-ik\cdot x} + \hat{a}^\dagger(k)e^{ik\cdot x}\},$$
where \( E_k = \sqrt{k^2 + m^2} \), \( \hat{a}^\dagger(k) \) is creation operator and \( \hat{a}(k) \) is annihilation operator. The vacuum state \( |0\rangle \) is defined by \( \hat{a}(k) |0\rangle = 0 \). The coherent state is given as follows:

\[
|c\rangle = \exp \left( -\frac{1}{2} \int d^3k |c(k)|^2 \right) \exp \left( \int d^3k c(k) \hat{a}^\dagger(k) \right) |0\rangle,
\]

where \( c(k) \) is a complex function of \( k \). Let us take \( \rho_S(1) = |0\rangle\langle 0| \), \( \rho_S(2) = |c(k)\rangle\langle c(k)| \).

In this case the upper bound of SNR is

\[
\frac{S_A}{D_A} \leq \frac{2}{1 - \exp \left( -\int d^3k |c(k)|^2 \right)} \left( 1 - \exp \left( -\int d^3k |c(k)|^2 \right) \right). \tag{15}
\]

In quantum cryptography experiments, small amplitude coherent states with \( \int d^3k |c(k)|^2 \) being small are often used. Then eq. (15) provides a severe upper bound such that

\[
\frac{S_A}{D_A} \leq 2 \int d^3k |c(k)|^2 + O \left( \left( \int d^3k |c(k)|^2 \right)^2 \right). \tag{16}
\]

### 3.2 Fundamental Upper Bound of Power Consumption to Perform Quantum Switching

Let us consider an application to fundamental upper bound of power consumption of rapid quantum switching in short time duration \( \tau \). Quantum switches are consisted of control system \( C \) and target system \( T \). We consider two different initial states of \( C \), \( \rho_C(\text{on}) \) and \( \rho_C(\text{off}) \). The initial state of \( T \) is represented by \( \rho_T(0) \). When the initial state of \( C \) is \( \rho_C(\text{on}) \), the target system is switched \( \rho_T(0) \) to \( \rho_T(\tau, \text{on}) \). Let us assume that when the initial state of \( C \) is \( \rho_C(\text{off}) \), the state of \( T \) is unchanged. The total Hamiltonian is denoted by \( H = H_T + H_C + V_{CT} \), where \( H_T \) and \( H_C \) are free Hamiltonians of each system, \( V_{CT} \) represents the interaction between \( C \) and \( T \). When the initial state of \( C \) is \( \rho_C(\text{on}) \), the time evolution of \( T \) is given by

\[
\rho_T(\tau, \text{on}) - \rho_T(0) = -i\tau \text{Tr}_C[H, \rho_T(0) \otimes \rho_C(\text{on})] + O(\tau^2).
\]

The energy cost of switching is

\[
\text{Tr}_T[H_T(\rho_T(\tau, \text{on}) - \rho_T(0))] = -i\tau \text{Tr}(H_T[H_T(0) \otimes \rho_C(\text{on})])).
\]

From the cyclic rule of trace and the property that \( H_T \) commutes with \( H_C \),
\n\[
\text{Tr}(H_T[H_T(0) \otimes \rho_C(\text{on})]) = \text{Tr}([H_T \otimes I_C, H](\rho_T(0) \otimes \rho_C(\text{on})) = \text{Tr}(H_T \otimes
\]

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I_C, V_{TC}([\rho_T(0) \otimes \rho_C(on)]) holds. The power consumption to switching is defined by

\[ P = \frac{\text{Tr}[H_T \rho_T(\tau, \text{on}) - \rho_T(0) \otimes \rho_C(\text{on})]}{\tau}. \]

Note that in case the initial state of C is \( \rho_C(\text{off}) \), the state of T is unchanged, so the energy cost and the power consumption are zero. Therefore the power consumption needed to switch becomes as follows:

\[ P = -i \text{Tr}(H_T \otimes I_C, V_{TC}([\rho_T(0) \otimes (\rho_C(\text{on}) - \rho_C(\text{off}))]). \]

By the way, eq. (13) can be rewritten as follows:

\[ |P| \leq \sqrt{1 - F(\rho_{D(1)}, \rho_{D(2)})^2} \left[ \delta_A(\rho_{D(1)}) + \delta_A(\rho_{D(2)}) \right], \]

where \( \delta_A(\rho) \) is the standard deviation of \( \hat{A}_D \) defined in Sec. 2. Suppose that the system D is regarded as a composite system \( C + T \) and \( \hat{A}_D = -i[H_T \otimes I_C, V_{TC}] \). We substitute \( \rho_D(1) = \rho_{C+T(\text{on})} = \rho_T(0) \otimes \rho_C(\text{on}) \) and \( \rho_D(2) = \rho_{C+T(\text{off})} = \rho_T(0) \otimes \rho_C(\text{off}) \). Then we find

\[ |P| \leq \sqrt{1 - F(\rho_{C(\text{on})}, \rho_{C(\text{off})})^2} \times \left[ \delta_A(\rho_{C+T(\text{on})}) + \delta_A(\rho_{C+T(\text{off})}) \right]. \]

This inequality implies that the rapid quantum switching has a tight constraint from quantum fluctuation of certain physical observable \( \hat{A}_D \). Similar inequalities are proven in [13], [14]. Since their results contain spectrum norms of target Hamiltonian \( \|H_T\| \) which diverges in infinite dimensional systems including harmonic oscillators, we cannot apply their results to such a system. However our result eq. (17) is written by non-divergent quantum fluctuations, we are able to give nontrivial upper bounds for infinite dimensional systems.

### 3.3 Fidelity Estimation

An application for the fidelity estimation is possible. From eq. (12) can be rewritten as follows:

\[ F^2(\rho_D(1), \rho_D(2)) \leq 1 - \left( \frac{S_D}{\sqrt{\text{Tr}[\hat{A}_D^2 \rho_D(1)]} + \sqrt{\text{Tr}[\hat{A}_D^2 \rho_D(2)]}} \right)^2. \]

Suppose we want to know an approximate value of fidelity between two states \( \rho_D(1), \rho_D(2) \). The upper bound of fidelity is given by the right hand side of eq. (18). This aspect is different from that of the result in [15].
\[ F^2(\rho_D(1), \rho_D(2)) \leq \text{Tr}[\rho_D(1)\rho_D(2)] + \sqrt{(1 - \text{Tr}[\rho_D(1)^2])(1 - \text{Tr}[\rho_D(2)^2])}. \]  

The right hand side of eq. (19) can be fixed by performing controlled-SWAP test [16]. On the other hand, our bound is easily measurable since it can be fixed by measuring the observable \( \hat{A}_D \). It is worth stressing that our bound sometimes gives more stringent upper bounds. For example, consider the case where matrix representations of two states are

\[ \rho_D(1) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \]

\[ \rho_D(2) = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}, \]

and the observable \( \hat{A}_D \) is fixed as

\[ \hat{A}_D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

Our bound becomes \( \frac{29}{30} \simeq 0.967 \). On the other hand, their bound is \( \frac{35}{36} \simeq 0.972 \). Therefore our upper bound is tighter in this case.

4 Summary

We proved the fundamental upper bound for SNR of quantum detectors in eq. (4). The above bound is computed using the information of the signal system \( S \). This bound is independent of the interaction between \( S \) and the detector system \( D \) and its observable \( \hat{A}_D \). Moreover, our result is more stringent than the previous result in [11]. In Sec. 3, we have shown applications of eq. (4) and eq. (5). In sub. sec. 3.1, the upper bound of quantum SNR has been demonstrated in the case where two initial states of \( S \) are pure. In sub. sec. 3.2, we derived the power consumption bound of rapid quantum switching in eq. (17). This bound can be applied to infinite dimensional systems including harmonic oscillators and quantum fields. Finally, the application for the fidelity estimation has been provided in sub. sec. 3.3. Our upper bound of fidelity (18) is easily measurable and sometimes becomes more stringent than previous result [15].

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