Examples and Comments Related to Relativity Controversies

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Abstract

Recently Mansuripur has called into question the validity of the Lorentz force in connection with relativistic electromagnetic theory. Here we present some very simple point-charge systems treated through order \(v^2/c^2\) in order to clarify some aspects of relativistic controversies both old and new. In connection with the examples, we confirm the validity of the relativistic conservation laws. The relativistic examples make clear that external forces may produce a vanishing torque in one inertial frame and yet produce a non-zero torque in another inertial frame, and that the conservation of angular momentum will hold in both frames. We also discuss a relativistic point-charge model for a magnetic moment and comment on the interaction of a point charge and a magnetic moment. Mansuripur’s claims of incompatibility between the Lorentz force and relativity are seen to be invalid.
I. INTRODUCTION

A. Mansuripur’s Controversial Claim

The Lorentz force on point charges, Maxwell’s equations for the electromagnetic fields, and appropriate boundary conditions on the differential equations—these three aspects form the foundations of relativistic classical electron theory appearing in textbooks of classical electrodynamics. Therefore it seems surprising when an article by Mansuripur, published in the prominent physics research journal Physical Review Letters, claims to present "incontrovertible theoretical evidence of the incompatibility of the Lorentz law with the fundamental tenets of special relativity." The claim of incompatibility arises in connection with the interaction of a point charge and a magnetic moment and is based on the statement, "The appearance of this torque in the xyz frame in the absence of a corresponding torque in the x'y'z' frame is sufficient proof of the inadequacy of the Lorentz law." Mansuripur’s claim is highlighted by an article in Science under the title "Textbook Electrodynamics May Contradict Relativity." The surprising claim of incompatibility arises in connection with 1) torques and involving 2) the interaction of a point charge and a magnetic moment. Now these two aspects are familiar in long-lasting controversies related to relativity. The issue of torques, which appeared in some inertial frames but not others, formed the basis for the experiments of Trouton and Noble back in 1903 and is still discussed as a relativity paradox. The interaction of a magnetic moment with an electric field (such as provided by a point charge) is at the heart of the controversies involving "hidden momentum," the Aharonov-Bohm effect, and the Aharonov-Casher effect. Indeed, recently Griffiths provided a review of some of the clashing points of view in his resource letter on electromagnetic momentum.

In this article we point out that Mansuripur’s bold claim of relativistic incompatibility for the Lorentz force is invalid for two basic reasons. First, it is a fact of relativity that the absence of a torque in one inertial frame does not preclude the existence of a torque due to the same forces as seen in a second inertial frame. Second, the interaction of a point charge and a magnetic moment has never been described at the level of detail required for a relativistic understanding, and therefore this interaction cannot be used as a basis for a claim of inconsistency. Specifically, there has never been a discussion of even the nonrelativistic
internal electromagnetic stresses required for the stability of a magnetic moment in the presence of an electric field. This same lack of necessary relativistic detail is present in many of the discussions listed in Griffiths’ resource letter.\[10\]

In fairness to Mansuripur, we should note that his real interest is in the relativistic description of polarized and magnetized materials; his incompatibility claim arises from attempts to understand this complicated problem involving materials. In the present article, we have a far more modest aim. We will deal only with very simple point charge systems which can be understood in relativistic detail. We present three simple charged-particle systems which are relativistic (to order $v^2/c^2$) and which are related to past and present relativistic controversies. These systems highlight some basic aspects of relativity and remind us that Mansuripur’s claim of incompatibility can not be taken seriously.

\textbf{B. Outline of the Presentation}

Before introducing the point-charge examples, we deal with some relativistic preliminaries. We remind the reader of the relativistic conservation laws and of the relativistic force transformation laws which will be used in the examples. Our analysis makes use of the Darwin Lagrangian which involves interacting charged point masses and is known to represent the relativistic interactions of classical electrodynamics through order $v^2/c^2$. We review the expressions for energy, momentum, angular momentum, and the equations of motion which follow from the Darwin Lagrangian.

Our first example involves two point charges held at rest by external forces. This system is related to the old ”4/3-problem” for the classical model of the electron,\[11\] where stabilizing forces must be present to maintain the electrostatic configuration. The point charge system is also related to the Trouton-Noble experiment and to Mansuripur’s implied assertion that the absence of a torque in one inertial frame means its absence in all inertial frames. We note that when external forces are present, electromagnetic energy and momentum do not transform as a Lorentz 4-vector. To obtain a physical appreciation of this situation, we consider two charged point particles of equal mass $m$ and charge $e$ which are approaching each other symmetrically in their center-of-energy inertial frame. When the charges come instantaneously to rest in this center-of-energy frame, external forces are applied. However, the application of the external forces which is simultaneous in one inertial frame need not
be simultaneous in a relatively moving frame; furthermore, the external forces need not be along a common line of action. Thus we find that in the moving frame the external forces may introduce energy, linear momentum, and angular momentum.

Second we consider a model for a magnetic moment which consists of two point particles of charges $e$ and $-e$ and masses $m$ and $M$ which are in uniform circular motion in an inertial frame. We find the associated system magnetic moment. We consider the aspects of the magnetic moment as seen in a second, relatively moving inertial frame and note the appearance of a non-zero electric dipole moment in the frame where the center of energy of the magnetic dipole is moving.

Thirdly, we consider the introduction of an additional point charge $q$ into the inertial frame with the magnetic moment model. We point out that the magnetic moment model is no longer stable, not even to zero order in $v/c$. Any description of the interaction of the magnetic moment and the point charge must account explicitly for the forces or accelerations which are associated with the response of the magnetic moment model to the presence of an external electric field across the magnetic moment. Although such an account has been given for the case when external forces are applied to the charges of the magnetic moment model, a convincing account of the interaction has never been given for a system where relativistic internal stresses are assumed to exist. In the absence of such a detailed relativistic description, Mansuripur’s claims of an incompatibility with relativity can not be taken seriously.

II. RELATIVISTIC PRELIMINARIES

A. Relativistic Conservation Laws

In the examples to follow, we will turn repeatedly to the relativistic conservation laws. These include the familiar conservation laws for energy, linear momentum, and angular momentum. However, the last (and only specifically relativistic) conservation law involves the uniform motion of the center of energy. In the presence of an external force $F_{\text{ext},i}$ on particle $i$ at position $r_i$ with velocity $v_i$, the conservation laws for a Lorentz-invariant mechanical system or field theory take the following forms. The sum of the external
forces on the system gives the time rate of change of the system linear momentum $P$

$$\sum_i F_{ext,i} = \frac{dP}{dt}. \quad (1)$$

The total power delivered to the system by the external forces gives the time rate of change of the system energy $U$

$$\sum_i F_{ext,i} \cdot v_i = \frac{dU}{dt}. \quad (2)$$

The sum of the external torques gives the time rate of change of system angular momentum $L$

$$\sum_i r_i \times F_{ext,i} = \frac{dL}{dt}. \quad (3)$$

The external power-weighted position equals the time rate of change of (the total energy $U$ times the center of energy $\vec{X}$) minus $c^2$ times the system momentum:

$$\sum_i (F_{ext,i} \cdot v_i) r_i = \frac{d}{dt}(U \vec{X}) - c^2 P. \quad (4)$$

The center of energy $\vec{X}$ is defined so that

$$U \vec{X} = \sum_i U_i r_i + \int d^3 r \, u(r) \, r, \quad (5)$$

where $U_i$ is the mechanical energy (rest energy plus kinetic energy) of the $i$th particle and $u(r)$ is the continuous system energy density at position $r$. This last conservation law in Eq. (4) expresses the continuous flow of energy in Lorentz-invariant systems. In an isolated system where no external forces are present, the linear momentum $P$, energy $U$, and angular momentum $L$ are all constants in time, and the center of energy $\vec{X}$ moves with constant velocity $d\vec{X}/dt = c^2 P/U$, because the energy $U$ and momentum $P$ in Eq. (4) are both constant.

### B. Lorentz Transformation of Forces

Although the 4-vector Lorentz transformations for spacetime displacements and for energy-momentum are familiar to students, the force transformations are usually less familiar. The Lorentz transformation of a force $\mathbf{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z$ acting on a point
Particle moving with velocity \( \mathbf{u} \) in the \( S \) inertial frame is given in the \( S' \) inertial frame (moving with velocity \( \mathbf{V} = \hat{\mathbf{i}}V \) along the \( x \)-axis of the \( S \) frame) by

\[
F'_x = \frac{F_x - \mathbf{u} \cdot \mathbf{F} V/c^2}{1 - u_x V/c^2}
\]

\[
F'_y = \frac{F_y}{\gamma(1 - u_x V/c^2)}, \quad F'_z = \frac{F_z}{\gamma(1 - u_x V/c^2)}
\]

These transformation formulae follow from use of the Lorentz force and the Lorentz transformations for the electric and magnetic fields.

C. The Darwin Lagrangian

In the examples to follow, we will not discuss electromagnetic interactions to all orders in \( v/c \), since such an analysis can become quite complicated. Rather for simplicity sake, we will restrict our analysis to order \( v^2/c^2 \).

The electromagnetic interaction of two point particles of charges \( e_1, e_2 \), and masses \( m_1, m_2 \) can be described through order \( v^2/c^2 \) by the Lagrangian

\[
\mathcal{L} = -m_1 c^2 + \frac{1}{2} m_1 \left( v_1^2 + \frac{1}{4} \frac{(v_1^2)^2}{c^2} \right) - m_2 c^2 + \frac{1}{2} m_2 \left( v_2^2 + \frac{1}{4} \frac{(v_2^2)^2}{c^2} \right) - \frac{e_1 e_2}{|r_1 - r_2|} + \frac{e_1 e_2}{2c^2} \left[ \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|r_1 - r_2|} + \frac{\mathbf{v}_1 \cdot (r_1 - r_2)\mathbf{v}_2 \cdot (r_1 - r_2)}{|r_1 - r_2|^3} \right]
\]

first introduced by C. G. Darwin. Here the terms involving the masses arise from the relativistic mechanical Lagrangian

\[
\mathcal{L}_{\text{mech}} = -\frac{mc^2}{(1 - v^2/c^2)^{1/2}} \approx -mc^2 + \frac{1}{2} m_1 \left( v_1^2 + \frac{1}{4} \frac{(v_1^2)^2}{c^2} \right)
\]

and the terms involving the charges \( e_1, e_2 \) correspond to electromagnetic field contributions \(-\int d^3r (E^2 - B^2)\) with the divergent self-energy integrals omitted. The total momentum is the sum \( \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \) where the canonical momentum \( \mathbf{p}_i \) of the \( i \)th particle is given by

\[
\frac{\partial \mathcal{L}}{\partial \mathbf{v}_i} = \mathbf{p}_i = m_i \left( 1 + \frac{v_i^2}{2c^2} \right) \mathbf{v}_i + \frac{e_i e_j}{2c^2} \left( \frac{\mathbf{v}_j \cdot (r_i - r_j)(r_i - r_j)}{|r_i - r_j|^3} \right)
\]

which includes the \( v^2/c^2 \) contribution from the mechanical momentum of the \( i \)th particle

\[
\mathbf{p}_{\text{mech}, i} = \frac{m_i \mathbf{v}_i}{(1 - v_i^2/c^2)^{1/2}} \approx m_i \left( 1 + \frac{v_i^2}{2c^2} \right) \mathbf{v}_i
\]
and also electromagnetic field momentum associated with the electrostatic field of the \(i\)th particle and the magnetic field of the \(j\)th particle. The total angular momentum is the sum \(L = L_1 + L_2\) where the canonical angular momentum \(L_i\) of the \(i\)th particle is given by

\[
L_i = r_i \times p_i = r_i \times \left[ m_i (1 + \frac{v_i^2}{2c^2}) v_i + e_i e_j \left( \frac{v_j}{|r_i - r_j|} + \frac{v_j \cdot (r_i - r_j)(r_i - r_j)}{|r_i - r_j|^3} \right) \right]
\]

The total system energy \(U\) including the rest energy is given by

\[
U = m_1 c^2 + \frac{1}{2} m_1 \left( \frac{v_1^2}{1 - v_1^2/c^2} + \frac{3 (v_1^2)^2}{4 c^2} \right) + m_2 c^2 + \frac{1}{2} m_2 \left( \frac{v_2^2}{1 - v_2^2/c^2} + \frac{3 (v_2^2)^2}{4 c^2} \right) + \frac{e_1 e_2}{|r_1 - r_2|} + \frac{e_1 e_2}{2 c^2} \left[ \frac{v_1 \cdot v_2}{|r_1 - r_2|} + \frac{v_1 \cdot (r_1 - r_2)v_2 \cdot (r_1 - r_2)}{|r_1 - r_2|^3} \right]
\]

and the center of energy \(\bar{X}\) is given by

\[
U \bar{X} = \left[ m_1 c^2 + \frac{1}{2} m_1 \left( \frac{v_1^2}{1 - v_1^2/c^2} + \frac{3 (v_1^2)^2}{4 c^2} \right) \right] r_1 + \left[ m_2 c^2 + \frac{1}{2} m_2 \left( \frac{v_2^2}{1 - v_2^2/c^2} + \frac{3 (v_2^2)^2}{4 c^2} \right) \right] r_2 + \left\{ \frac{e_1 e_2}{|r_1 - r_2|} + \frac{e_1 e_2}{2 c^2} \left[ \frac{v_1 \cdot v_2}{|r_1 - r_2|} + \frac{v_1 \cdot (r_1 - r_2)v_2 \cdot (r_1 - r_2)}{|r_1 - r_2|^3} \right] \right\} \frac{(r_1 + r_2)}{2}
\]

Thus there are both mechanical and field contributions to the energy, linear momentum, angular momentum, and center of energy. The equations of motion for the \(i\)th particle follow from the Lagrangian as

\[
\frac{d}{dt} \left[ \frac{m_i v_i}{(1 - v_i^2/c^2)^{1/2}} \right] \approx \frac{d}{dt} \left[ m_i \left( 1 + \frac{v_i^2}{2c^2} \right) v_i \right] = e_i E_j (r_i, t) + e_i \frac{v_i}{c} \times B_j (r_i, t)
\]

where the electric and magnetic fields due to the \(j\)th particle are given through order \(v^2/c^2\) by

\[
E_j (r, t) = e_j \frac{(r - r_j)}{|r - r_j|^3} \left[ 1 + \frac{1}{2} \frac{v_j^2}{c^2} - \frac{3}{2} \left( \frac{v_j \cdot (r - r_j)}{c|r - r_j|} \right)^2 \right]
\]

\[
- \frac{e_j}{2c^2} \left( \frac{a_j}{|r - r_j|} + \frac{a_j \cdot (r - r_j)(r - r_j)}{|r - r_j|^3} \right)
\]

\[
B_j (r, t) = e_j \frac{v_j}{c} \times \frac{(r - r_j)}{|r - r_j|^3}
\]
We note that, in general, accelerations appear on both the left- and right-hand sides of Eq. (16) since the electric field $\mathbf{E}_j(r_i, t)$ at the position $r_i$ of the $i$th particle due to the charge $e_j$ of the $j$th particle depends upon the acceleration of the $j$th particle as in Eq. (17).

III. TWO PARTICLES HELD AT REST BY EXTERNAL FORCES

A. Two Point Charges Held at Rest

The first system which we consider is two point particles, both of charge $e$ and mass $m$, which are held at rest at positions $r_1$ and $r_2$ with separation $|r_1 - r_2| = l$ in an inertial frame $S$. The external forces $\mathbf{F}_1 = -\mathbf{F}_2$ holding the particles at rest must balance the electrostatic repulsion between the charges $F_1 = e^2/l^2 = F_2$. In the frame $S$, the system has total energy $U = 2mc^2 + e^2/l$ (including contributions from the rest mass and the electrostatic energy), has total momentum $\mathbf{P} = 0$, and total angular momentum $\mathbf{L} = 0$. We will assume that the particles are placed symmetrically about the origin of coordinates so that the center of energy $\mathbf{X}$ is also zero, $U\mathbf{X} = mc^2 r_1 + mc^2 r_2 + (e^2/l)(r_1 + r_2)/2 = 0$ since $r_1 = -r_2$.

B. Lorentz Transformation of Energy and Momentum in the Presence of External Forces

1. The Lorentz-4-Vector Expectation

Many students expect energy and momentum to transform as a Lorentz 4-vector. However, this expectation may not be true when external forces are present. If we consider an inertial frame $S'$ moving with velocity $\mathbf{V} = \hat{\mathbf{i}}V$ along the $x$-axis of the inertial frame $S$, then naive 4-vector expectation would suggest that in $S'$ the energy and momentum are

$$U' = \gamma_V U \approx \left(1 + \frac{V^2}{2c^2} + \frac{3V^4}{8c^4}\right) \left(2mc^2 + \frac{e^2}{l}\right)$$

$$= 2mc^2 + 2m \left(\frac{V^2}{2} + \frac{3V^4}{8c^2}\right) + \left(1 + \frac{V^2}{2c^2}\right) \frac{e^2}{l} \quad \text{expected 4-vector form} \quad (19)$$

$$\mathbf{P}' = -\frac{\gamma_V \mathbf{V} U}{c^2} \approx -2m \left(1 + \frac{V^2}{2c^2}\right) \mathbf{V} - \frac{e^2}{lc^2} \mathbf{V} \quad \text{expected 4-vector form} \quad (20)$$

since

$$\gamma_V = \frac{1}{\sqrt{1 - V^2/c^2}} \approx 1 + \frac{V^2}{2c^2} + \frac{3V^4}{8c^4} + ... \quad (21)$$

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In the inertial frame \( S' \), the particles are moving with velocities \(-\hat{i}V\) so that the mechanical contributions to the energy and momentum are indeed of the 4-vector form

\[
U'_{\text{mech}} = 2mc^2\gamma_V = \gamma_V U_{\text{mech}} \approx 2mc^2 + 2m \left( \frac{V^2}{2} + \frac{3V^4}{8c^4} \right)
\]  

(22)

\[
P'_{\text{mech}} = -2mV\gamma_V = -\gamma_V VU_{\text{mech}}/c^2 \approx -2m \left( 1 + \frac{V^2}{2c^2} \right) V
\]

(23)

However, the electromagnetic field contributions depend upon the relative orientation between the velocity \( V \) and the particle separation \( r_1 - r_2 \). From Eqs. (13) and (10), we have

\[
U_{\text{em}} = \frac{e_1 e_2}{|r_1 - r_2|} + \frac{e_1 e_2}{2c^2} \left[ \frac{v_1 \cdot v_2}{|r_1 - r_2|} + \frac{v_1 \cdot (r_1 - r_2)v_2 \cdot (r_1 - r_2)}{|r_1 - r_2|^3} \right]
\]

(24)

\[
P_{\text{em}} = \frac{e_1 e_2}{2c^2} \left( \frac{v_1 + v_2}{|r_1 - r_2|} + \frac{(v_1 + v_2) \cdot (r_1 - r_2)(r_1 - r_2)}{|r_1 - r_2|^3} \right)
\]

(25)

If the particles are separated along the \( y \)-axis in \( S \), then they will be separated along the \( y \)-axis in \( S' \) (separation perpendicular to the velocity) and the electromagnetic energy and momentum contributions are

\[
U'_{\text{em}} = \frac{e^2}{l} + \frac{e^2V^2}{2c^2l} \quad \text{particle separation perpendicular to velocity}
\]

(26)

\[
P'_{\text{em}} = -\frac{e^2}{lc^2} V
\]

(27)

since \( v_1 \cdot (r_1 - r_2)v_2 \cdot (r_1 - r_2) = \left[ (-\hat{i}V) \cdot \hat{j}l \right] \left[ (-\hat{i}V) \cdot \hat{j}l \right] = 0 \). These terms in Eqs. (26) and (27) agree with the terms in the expected 4-vector form in Eqs. (19) and (20). On the other hand, if the charges are separated along the \( x \)-axis (separation parallel to the velocity), then

\[
 v_1 \cdot (r_1 - r_2)v_2 \cdot (r_1 - r_2) = V^2l^2
\]

and the electromagnetic contributions to the energy and momentum are doubled becoming

\[
U'_{\text{em}} = \frac{e^2}{l} + \frac{e^2V^2}{c^2l} \quad \text{particle separation parallel to velocity}
\]

(28)

\[
P'_{\text{em}} = -\frac{2e^2}{lc^2} V
\]

(29)

which disagrees with the naive 4-vector form in Eqs. (19) and (20) by a factor of 2 in the \( 1/c^2 \) terms. For the spherical classical model of the electron, the errant factor is the famous 4/3 for the momentum; the 4/3 is intermediate between the factors of 1 and 2 found for the point charge examples above and corresponds to averaging over the relative orientations for the spherical charge distribution. 

[18]
2. Validity of Relativistic Conservation Laws

We should note that this failure of the naive 4-vector energy-momentum transformation is fully consistent with the relativistic conservation laws. In the presence of external forces, we must use the relativistic conservation laws given above in Eqs. (1)-(4). Since the external forces $F_1$ and $F_2$ are equal in magnitude and opposite in direction, there is no net energy or linear momentum introduced by the external forces. The angular momentum situation will be discussed below. Since $F_A = -F_B$, the fourth conservation law (4) in frame $S'$ here gives

$$F_1' \cdot (-V)(r_{1'} - r_{2'})/c^2 = U'(V)/c^2 - P'$$

(30)

If the charges are separated along the $y$-axis perpendicular to the velocity, then $F_1'$ is perpendicular to $V$ and the left-hand side of Eq. (30) vanishes giving $P' = (U/c^2)(-V)$ which indeed holds in connection with Eqs. (22), (23), (26) and (27). If the charges are separated along the $x$-direction parallel to the velocity, then the left-hand side of Eq. (30) can be found from the force transformation Eq. (6) (and the Lorentz contraction in the direction of motion) $F_A' \cdot (-V)(r_A' - r_B')/c^2 = \hat{i}(e^2/l^2)(Vl/c^2) = \hat{i}(e^2/l)(V/c^2)$ which is exactly the additional term needed to connect Eqs. (22), (23), (28) and (29) through Eq. (30).

3. Particles Approaching Each other Along the $x$-Axis

A physical understanding of the failure of the naive energy-momentum 4-vector transformation when external forces are present can be obtained by considering the formation of the system when the particles are sent towards each other from spatial infinity, and then the external forces are applied as the particles reach equilibrium positions.

First we consider two point particles, both of charge $e$ and mass $m$ approaching the origin symmetrically along the $x$-axis in the inertial frame $S$. The left-hand particle has velocity $v_1 = \hat{i}v_0$ and the right-hand particle has velocity $v_2 = -\hat{i}v_0$ so that the system linear momentum and angular momentum about the origin are both zero

$$P_0 = 0, \quad L_0 = 0$$

(31)

The particles start out at spatial infinity with speed $v_0$, and then the particles slow down as they repel each other and the kinetic energy is converted into electric potential energy.
In the order of approximation $v^2/c^2$ of the Darwin Lagrangian, there is no radiation energy loss due to the acceleration of the charges. At time $t = 0$ in the $S$ inertial frame, both the particles come to rest at a separation $l$, $r_1(0) = -\hat{i}l/2 = - r_2(0)$, where the electrostatic potential energy equals the initial kinetic energy $KE_0 = U_0 - 2mc^2$,

$$KE_0 = 2 \left[ \frac{1}{2} m \left( 1 + \frac{3 v_0^2}{4 c^2} \right) v_0^2 \right] = \frac{e^2}{l}$$  \hspace{1cm} (32)

When the particles come to rest, external forces $F_1 = -\hat{i}e^2/l^2 = - F_2$ are applied simultaneously to the left- and right-hand particles respectively, so as to balance the electrostatic repulsion and to keep the particles at rest. Since these particles are applied at the same time in the $S$ frame when the particles are at rest, the forces introduce neither energy nor linear momentum nor angular momentum into the system of the two point particles. Thus in the $S$ frame, after the external forces are applied, the final energy $U_F = U_0$, the final momentum equals the initial momentum, $P_F = P_0 = 0$, and the final angular momentum equals the initial angular momentum, $L_F = L_0 = 0$.

However, now consider this same situation from the point of view of an observer in an inertial frame $S'$ moving with velocity $V = \hat{i}V$ relative to the inertial frame $S$. When the particles are far apart, the velocities of the particles in the $S'$ frame are along the $x'$-axis

$$v'_1 = \frac{v_0 - V}{1 - v_0 V/c^2} \approx (v_0 - V)(1 + v_0 V/c^2)$$  \hspace{1cm} (33)

$$v'_2 = \frac{-v_0 - V}{1 + v_0 V/c^2} \approx (-v_0 - V)(1 - v_0 V/c^2)$$  \hspace{1cm} (34)

through order $v^2/c^2$. Then in the $S'$ frame when the particles are far apart, the total energy and momentum are found to be

$$U'_0 = 2mc^2 + \frac{1}{2} m \left( v'_1^2 + \frac{3 (v'_1)^2}{c^2} \right) + \frac{1}{2} m \left( v'_2^2 + \frac{3 (v'_2)^2}{c^2} \right)$$

$$= 2mc^2 + \frac{1}{2} m \left\{ \left[(v_0 - V) \left(1 + \frac{v_0 V}{c^2} \right) \right]^2 + \frac{3}{4c^2} \left[(v_0 - V) \left(1 + \frac{v_0 V}{c^2} \right) \right]^4 \right\}$$

$$+ \frac{1}{2} m \left\{ \left[(-v_0 - V) \left(1 - \frac{v_0 V}{c^2} \right) \right]^2 + \frac{3}{4c^2} \left[(-v_0 - V) \left(1 - \frac{v_0 V}{c^2} \right) \right]^4 \right\}$$

$$= \left(1 + \frac{1}{2} \frac{V^2}{c^2} + \frac{3V^4}{8c^4} \right) 2 \left[ mc^2 + \frac{1}{2} m \left( v_0^2 + \frac{3 (v_0^2)^2}{c^2} \right) \right] \approx \gamma V U_0 - \gamma V P_0$$

$$= \left(1 + \frac{1}{2} \frac{V^2}{c^2} + \frac{3V^4}{8c^4} \right) \left( 2mc^2 + \frac{e^2}{l} \right)$$  \hspace{1cm} (35)
and

\[ \mathbf{P}' = m \left( 1 + \frac{v_A^2}{2c^2} \right) \mathbf{v}_A + m \left( 1 + \frac{v_B^2}{2c^2} \right) \mathbf{v}_B \]

\[ = \hat{i}m \left\{ 1 + \frac{1}{2c^2} \left[ (v_0 - V) \left( 1 + \frac{v_0 V}{c^2} \right) \right]^2 \right\} (v_0 - V) \left( 1 + \frac{v_0 V}{c^2} \right) \]

\[ + \hat{i}m \left\{ 1 + \frac{1}{2c^2} \left[ (-v_0 - V) \left( 1 - \frac{v_0 V}{c^2} \right) \right]^2 \right\} (-v_0 - V) \left( 1 - \frac{v_0 V}{c^2} \right) \]

\[ = - \left( 1 + \frac{V^2}{2c^2} \right) V \frac{U_0}{c^2} \approx \hat{i} \gamma \gamma V P_0 - \frac{\gamma V U_0}{c^2} \]

\[ \approx -2m \left( 1 + \frac{V^2}{2c^2} \right) \frac{V}{c^2} \]

(36)

where in the last lines we have used Eq. (32) connecting the initial kinetic energy and the final potential energy is \( S \). Thus when the particles are far apart, the total energy and momentum of the system transform as a Lorentz four-vector between the \( S \) and the \( S' \) inertial frames.

In the absence of external forces, the energy and momentum are constant in each inertial frame. Therefore the four-vector transformation character between the frames \( S \) and \( S' \) is retained until the application of the external forces. However, when the external forces are applied simultaneously in the \( S \) frame, they are not applied simultaneously in the \( S' \) frame. Rather the force \( \mathbf{F}'_1 \) is applied at time \( t'_1 = \gamma V [0 + V l/(2c^2)] \approx V l/(2c^2) \) while the force \( \mathbf{F}'_2 \) is applied at time \( t'_2 = \gamma V [0 - V l/(2c^2)] \approx -V l/(2c^2) \), where we have used the Lorentz transformations through order \( v^2/c^2 \). Thus in the \( S' \) frame the application of the forces is not simultaneous but rather the right-hand force is applied first and acts alone during a time interval \( \delta t' = t'_2 - t'_1 = V l/c^2 \). Thus during the time interval \( \delta t' \) there is a net energy \( \delta U' \) delivered by the unbalanced force \( \mathbf{F}'_2 \)

\[ \delta U' = \int \mathbf{F}'_2 \cdot \mathbf{v}_2 dt' = \frac{e^2 V l}{c^2} \]

(37)

and a net momentum

\[ \delta \mathbf{P}' = \int \mathbf{F}'_2 dt' = -\hat{i} \frac{e^2 V l}{c^2} \]

(38)

Thus after both external forces are applied, the final energy \( U'_F \) is given by

\[ U'_F = U'_0 + \delta U' = \gamma V U_0 + \frac{e^2 V^2}{lc^2} = 2mc^2 \left( 1 + \frac{V^2}{2c^2} + \frac{3V^4}{8c^4} \right) + \left( 1 + \frac{V^2}{2c^2} \right) \frac{e^2}{l} + \frac{e^2 V^2}{lc^2} \]

(39)
the final momentum $\mathbf{P}_F'$ is given by

$$
\mathbf{P}_F' = \mathbf{P}_0' + \delta \mathbf{P}' = \hat{i} \left( -\frac{\gamma V U_0}{c^2} \right) - \gamma \frac{e^2 V}{lc^2} = \hat{i} \left( 1 + \frac{V^2}{2c^2} \right) \left( 2m + \frac{e^2}{lc^2} \right) V - i \frac{e^2 V}{lc^2}
$$

(40)

The extra contributions $\delta U'$ and $\delta \mathbf{P}'$ in Eqs. (37) and (38) correspond exactly to the changes between Eqs. (26), (27) and Eqs. (28), (29). The final angular momentum is still zero. Thus the system energy and momentum in the $S'$ frame are changed on the non-simultaneous application of the external forces, and the energy and momentum in $S'$ are no longer related by four-vector Lorentz transformation to the unchanged energy and momentum in the $S$ frame. Nevertheless the conservation laws connecting the external forces to the changes of energy, of linear momentum, and of angular momentum are valid in each inertial frame. This example provides a physical understanding as to why the system energy and momentum may not transform as a Lorentz four-vector between inertial frames when there are external forces present.

4. Particles Approaching Each Other Along the $y$-Axis

We can also consider the case where the two charged particles approach each other symmetrically along the $y$-axis, with initial velocities given by $\mathbf{v}_1 = \hat{j} v_0$, $\mathbf{v}_2 = -\hat{j} v_0$, in the $S$ frame, while the inertial frame $S'$ still has velocity $\mathbf{V} = \hat{i} V$ along the $x$-axis of the $S$ frame. In the $S$ frame at time $t = 0$, the particles again come to rest at a separation $l$, $\mathbf{r}_1(0) = -\hat{j} l/2 = -\mathbf{r}_2(0)$, and the external forces $\mathbf{F}_1 = \hat{j} e^2/l^2 = -\mathbf{F}_2$ are applied simultaneously so as to maintain the particles at rest in the $S$ frame. In this case, the external forces are also applied simultaneously in the $S'$ frame, and so do not introduce net energy or linear momentum or angular momentum in the $S'$ frame. Thus for this situation, the energy and momentum of the system are connected as a Lorentz four-vector between the $S$ and $S'$ inertial frames both before and after the application of the external forces. This is in agreement with the result of Eqs. (26) and (27).

5. Particles Approaching Each Other Along the Line $x = y$

Finally, we consider the two charged particles approaching each other symmetrically along the line $x = y$, $z = 0$ in the $S$ inertial frame. Initially the particles are very far apart
and have velocities $\mathbf{v}_1 = (\hat{i} + \hat{j}) v_0 / 2^{1/2} = -\mathbf{v}_2$. Once again, at time $t = 0$, the particles come instantaneously to rest at separation $l$ with $\mathbf{r}_1(0) = -(\hat{i} + \hat{j}) l / 2^{1/2} = -\mathbf{r}_2(0)$. The external forces $\mathbf{F}_1 = (\hat{i} + \hat{j}) e^2 / (2^{1/2} l^2) = -\mathbf{F}_2$ are applied simultaneously at $t = 0$, and so, in $S$, the system energy, linear momentum, and angular momentum are not changed by the simultaneous application of the forces. The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are along the $45^\circ$-line joining the particles and so do not introduce any angular momentum. However, in the $S'$ frame, the forces are not applied simultaneously, so that the $x$-components of the forces introduce energy and momentum in the $S'$ frame. However, in this case where the particles are separated along the line $x = y$, there is a new aspect involving torques and angular momentum. In this case, the forces in $S'$ are not colinear with the line joining the particles. It is this last aspect, the non-colinearity of the forces, which we wish to emphasize at this point.

C. Angular Momentum and Torques in Different Inertial Frames

1. Two Point Charges Held Along the Line $x = y$

Let us consider the situation of the two point charges along the line $x = y$, $z = 0$ after the application of the external forces. In the center-of-energy frame, the particles are at rest and the forces are colinear with the line separating the particles so in this frame, the angular momentum of the system vanishes, $\mathbf{L} = 0$, and the the net torque due to external forces vanishes. The conservation law for angular momentum (3) holds trivially in the $S$ inertial frame.

The positions $\mathbf{r}'_1$ and $\mathbf{r}'_2$ of the two particles in the $S'$ frame can be found by Lorentz transformation from the known positions $\mathbf{r}_1(0) = -(\hat{i} + \hat{j}) l / 2^{1/2} = -\mathbf{r}_2(0)$ in the $S$ inertial frame. Thus we find from $x = \gamma_V (x' + V t')$ and $y = y'$

$$\mathbf{r}'_1 = \hat{i} x'_1 + \hat{j} y'_1 = \hat{i} \left( -V t' + \frac{x_1}{\gamma_V} \right) + \hat{j} y_1 \approx \hat{i} \left( -V t' + \left( 1 - \frac{V^2}{2 c^2} \right) \frac{-l}{2^{1/2}} \right) + \hat{j} \frac{-l}{2^{1/2}} \quad (41)$$

$$\mathbf{r}'_2 = \hat{i} x'_2 + \hat{j} y'_2 = \hat{i} \left( -V t' + \frac{x_2}{\gamma_V} \right) + \hat{j} y_2 \approx \hat{i} \left( -V t' + \left( 1 - \frac{V^2}{2 c^2} \right) \frac{l}{2^{1/2}} \right) + \hat{j} \frac{l}{2^{1/2}} \quad (42)$$

The external forces $\mathbf{F}'_1$ and $\mathbf{F}'_2$ on the particles in the $S'$ frame can be found by Lorentz
transformation from Eqs. (6) and (7),
\[ \mathbf{F}'_1 = \left( \hat{i} + \frac{\hat{j}}{\gamma} \right) e^2 \approx \left[ \hat{i} + \hat{j} \left( 1 + \frac{V^2}{2c^2} \right) \right] \frac{e^2}{2^{1/2}l^2} = -\mathbf{F}'_2 \] (43)

However, these forces are not along the line joining the particles. Thus in the \( S' \) inertial frame, the external forces \( \mathbf{F}'_1 \) and \( \mathbf{F}'_2 \) give a net torque, whereas there is no torque in the \( S \) frame. Specifically, the net external torque in the \( S' \) frame is given by
\[
\mathbf{r}'_A \times \mathbf{F}'_A + \mathbf{r}'_B \times \mathbf{F}'_B \\
= \left\{ \hat{i} \left[ -\frac{l}{2^{1/2}} \left( 1 - \frac{V^2}{2c^2} \right) - Vt' \right] + \hat{j} \left( \frac{-l}{2^{1/2}} \right) \right\} \times \left[ \hat{i} + \hat{j} \left( 1 + \frac{V^2}{2c^2} \right) \right] \frac{e^2}{2^{1/2}l^2} \\
+ \left\{ \hat{i} \left[ \frac{l}{2^{1/2}} \left( 1 - \frac{V^2}{2c^2} \right) - Vt' \right] + \hat{j} \left( \frac{l}{2^{1/2}} \right) \right\} \times \left[ -\hat{i} - \hat{j} \left( 1 + \frac{V^2}{2c^2} \right) \right] \frac{e^2}{2^{1/2}l^2} \\
= \hat{k} \frac{e^2V^2}{2c^2l} \] (44)

At this point we consider the conservation law (3) connecting this net torque due to external forces to the rate of change of angular momentum in the \( S' \) inertial frame. The time rate of change of the system angular momentum follows from Eq. (12) as
\[
\frac{d}{dt'} [\mathbf{r}'_1 \times \mathbf{p}'_1 + \mathbf{r}'_2 \times \mathbf{p}'_2] \\
= \mathbf{v}'_1 \times \mathbf{p}'_1 + \mathbf{v}'_2 \times \mathbf{p}'_2 = \mathbf{v}'_1 \times (\mathbf{p}'_1 + \mathbf{p}'_2) \\
= -\hat{i}V \times \left\{ -\hat{i}2m \left( 1 + \frac{V^2}{2c^2} \right) V + 2 \frac{V}{2c^2} \left[ -\hat{i}V \right] \frac{l^2}{l^2} + \hat{i}V \cdot (\hat{i} + \hat{j})l/2^{1/2} \right\} \\
= \hat{k} \frac{e^2V^2}{2c^2l} \] (45)

Thus indeed the conservation law \( \sum_i \mathbf{r}_i \times \mathbf{F}_{ext,i} = \frac{d\mathbf{L}}{dt} \) relating the external torque to the rate of change of angular momentum holds in both the \( S \) and the \( S' \) inertial frames, even though there is zero external torque in one frame and non-zero torque in the other.

In Mansuripur’s article, it is implied that the absence of a torque in one inertial frame is inconsistent with the existence of a torque in a second inertial frame.\[3\] However, precisely this situation is involved in the Trouton-Noble situation which is considered in our example. There is no inconsistency with relativity or with conservation laws for this situation.
2. Comments on Stresses on a Rod Holding the Charges at Fixed Positions

In the examples above, we have considered external forces applied to point particles. In this case, the relativistic (through order $v^2/c^2$) analysis can be carried out in a transparent fashion. The original analysis by Trouton and Noble considered charged capacitor plates (in place of the point charges used here) which are held in place by a connecting bar. Such a bar could also be introduced into the point charge analysis given in this article.\[19\] In the $S$ inertial frame, the stresses in this bar would be of nonrelativistic order to balance the electrostatic forces between the charges. However, the detailed treatment of these stresses throughout the bar does not appear to be an easy problem. Furthermore, the relativistic transformation of these unknown stresses over to the $S'$ frame does not seem trivial. Thus the transparent relativistic treatment given above involving external forces must give way to a far more detailed and complicated treatment if the bar (with its internal stresses) is included as part of the system.

In any case, our simple examples do give accurate relativistic insights. In particular, Marsuripur’s implication that a vanishing torque in one inertial frame is inconsistent with a non-zero torque in another inertial frame is not valid.

IV. MODEL FOR A MAGNETIC MOMENT RELATIVISTIC TO ORDER $v^2/c^2$

A. Magnetic Moment in its Center-of-Energy Frame

We now turn away from the torque aspect of Mansuripur’s claims of "incompatibility" over to a consideration of the magnetic moment aspect. To start, we introduce a model of a magnetic moment which is transparently relativistic to order $v^2/c^2$.

A relativistic model for a magnetic moment can be given in terms of the uniform circular motion of two charged point particles of charges $\pm e$ and masses $m$ and $M$ moving about the center of energy of the system. In an inertial frame $S$, the first particle $m$ is located at $r_m(t) = \hat{i}r_m \cos(\omega t + \phi) + \hat{j}r_m \sin(\omega t + \phi)$ and the second particle $M$ is at $r_M(t) = -\hat{i}r_M \cos(\omega t + \phi) - \hat{j}r_M \sin(\omega t + \phi)$. In this frame, the total linear momentum $P$ of the magnetic moment system includes both mechanical momentum and electromagnetic field.
momentum as given above in Eq. (10)

\[ P = \left[ m \left( 1 + \frac{r_m^2 \omega^2}{2c^2} \right) - \frac{e^2}{2c^2 (r_m + r_M)} \right] \left[ -\dot{r}_m \sin(\omega t + \phi) + \dot{r}_m \cos(\omega t + \phi) \right] \]

\[ - \left[ M \left( 1 + \frac{r_M^2 \omega^2}{2c^2} \right) - \frac{e^2}{2c^2 (r_m + r_M)} \right] \left[ -\dot{r}_M \sin(\omega t + \phi) + \dot{r}_M \cos(\omega t + \phi) \right] \] (46)

In this case, the electromagnetic field momentum terms involving \( v_j \cdot (r_i - r_j) \) vanish because the velocity is orthogonal to the relative displacement of the particles.

In the \( S \) inertial frame, the linear momentum is assumed to vanish, \( P = 0 \), giving the condition

\[ \left[ m \left( 1 + \frac{r_m^2 \omega^2}{2c^2} \right) - \frac{e^2}{2c^2 (r_m + r_M)} \right] r_m = \left[ M \left( 1 + \frac{r_M^2 \omega^2}{2c^2} \right) - \frac{e^2}{2c^2 (r_m + r_M)} \right] r_M \] (47)

In the terms of order \( v^2/c^2 \), we may substitute the zero-order (nonrelativistic) conditions

\[ m r_m = M r_M \] (48)

and the zero-order (nonrelativistic) equations of motion

\[ m \frac{v_m^2}{r_m} = m \omega^2 r_m = \frac{e^2}{(r_m + r_M)^2} = M \omega^2 r_M = M \frac{v_M^2}{M} \] (49)

When we replace the \( v^2/c^2 \) terms involving mass in Eq. (47) by using the equations of motion in Eq. (49), we find that Eq. (47) simplifies to exactly the nonrelativistic condition (48). Evidently this condition holds not only nonrelativistically but also relativistically through order \( v^2/c^2 \).

The angular frequency \( \omega \) is determined by the equation of motion (16) for the mass \( m \) in the electric and magnetic fields of the charged particle \( M \)

\[ m \gamma \frac{v^2}{r_m} \approx m \left( 1 + \frac{r_m^2 \omega^2}{2c^2} \right) r_m \omega^2 \]

\[ = \frac{e^2}{(r_m + r_M)^2} \left( 1 + \frac{r_M^2 \omega^2}{2c^2} \right) - \frac{e^2 \omega^2 r_M}{c^2 (r_m + r_M)} + \frac{e^2 r_m \omega r_M \omega}{c^2 (r_m + r_M)^2} \] (50)

Once again we may use the non-relativistic equations of motion (49) in the \( v^2/c^2 \) term involving the mass \( m \). The condition (50) then simplifies to

\[ \omega^2 = \frac{e^2}{m r_m (r_m + r_M)^2 + e^2 (r_m^2 + r_M^2)/(2c^2)} \] (51)

Since we have \( m r_m = M r_M \) from the analysis above, the expression (51) for \( \omega \) is invariant if we interchange \( m \) and \( M \), as the expression should be. We can also write the condition
in the symmetrical form

\[ \omega^2 = \frac{2e^2}{(mr_m + Mr_M)(r_m + r_M)^2 + e^2(r_m^2 + r_M^2)c^2} \]  

(52)

For this magnetic moment model, the angular momentum includes both mechanical and electromagnetic field contributions. From Eq. (12), we find that all the angular momentum is in the direction perpendicular to the plane of the circular motion,

\[ L = \hat{k} \left[ m \left( 1 + \frac{r_m^2\omega^2}{2c^2} \right) r_m^2\omega + M \left( 1 + \frac{r_M^2\omega^2}{2c^2} \right) r_M^2\omega - \frac{e^2}{2c^2} \frac{(r_m^2 + r_M^2}\omega^2}{(r_m + r_M)^2} \right] \]  

(53)

Once again we may use the nonrelativistic results in Eq. (49) when evaluating the terms in order \( v^2/c^2 \) so that the angular momentum can be rewritten as

\[ L = \hat{k}\omega \left[ mr_m^2 + Mr_M^2 - \frac{e^2}{2c^2} \frac{(r_m^2 + 2r_M^2)r_M}{(r_m + r_M)^2} \right] \]  

(54)

The magnetic moment for our model is usually evaluated by time averaging and using the expression for the magnetic dipole moment of a steady-state current distribution \( \vec{\mu} = \frac{1}{2c} \int d^3r \vec{r} \times \vec{J} \). We can also think of averaging over an ensemble of systems with varying phases \( \phi \). Thus here we obtain

\[ \vec{\mu} = \left\langle \frac{1}{2c} \int d^3r \vec{r} \times \sum_i e_i \delta^3(\vec{r} - \vec{r}_i) \right\rangle = \frac{\hat{k}e\omega}{2c} (r_m^2 - r_M^2) \]  

(55)

We notice that if the masses are equal, \( m = M \), then the orbital radii are equal, \( r_m = r_M \), and the magnetic moment vanishes, \( \vec{\mu} = 0 \). On the other hand, in the limit where the mass \( M \) is very large, \( M >> m \), and accordingly \( r_M = r_m(m/M) \) is very small, we find the usual connection between the magnetic moment and the nonrelativistic angular momentum,

\[ \vec{\mu} = \frac{eL}{2mc} \]  

(56)

through order \( v^2/c^2 \).

B. Appearance of an Electric Dipole Moment in a Moving Frame

We can also examine the appearance of our magnetic moment model when viewed from a frame \( S' \) moving with velocity \( V = \hat{i}V \) relative to \( S \), the center-of-energy frame for the magnetic moment. The \( x \)-coordinates are related by Lorentz transformation as \( x_m = \gamma_V (x'_m + Vt') \), giving through order \( 1/c^2 \)

\[ x'_m = -Vt' + (1 - V^2/c^2)^{1/2}x_m \approx -Vt' + [1 - V^2/(2c^2)]r_m \cos(\omega t + \phi) \]  

(57)
The time coordinates are related by Lorentz transformation as $t' = \gamma_V (t - V x_m/c^2)$ so that

$$t = V x_m/c^2 + (1 - V^2/c^2)^{1/2} t' = V x_m/c^2 + [1 - V^2/(2c^2)] t'$$

(58)

Then we can simplify the expression in Eq. (57) through order $v^2/c^2$ as

$$x'_m = -Vt' + r_m \cos(\omega t + \phi) - [V^2 r_m/(2c^2)] r_m \cos(\omega t + \phi)$$

$$= -Vt' + r_m \cos\{\omega[1 - V^2/(2c^2)]t' + \phi + \omega V x_m/c^2\} - [V^2 r_m/(2c^2)] r_m \cos(\omega t' + \phi)$$

$$= -Vt' + r_m \cos\{\omega[1 - V^2/(2c^2)]t' + \phi\} \cos(\omega V x_m/c^2)$$

$$- r_m \sin\{\omega[1 - V^2/(2c^2)]t' + \phi\} \sin(\omega V x_m/c^2) - [V^2 r_m/(2c^2)] r_m \cos(\omega t' + \phi)$$

$$= -Vt' + r_m \cos\{\omega[1 - V^2/(2c^2)]t' + \phi\} - r_m(\omega V x_m/c^2) \sin\{\omega[1 - V^2/(2c^2)]t' + \phi\}$$

$$- [V^2 r_m/(2c^2)] r_m \cos(\omega t' + \phi)$$

$$= -Vt' + r_m \cos\{\omega[1 - V^2/(2c^2)]t' + \phi\} - r_m^2(\omega V/c^2) \cos(\omega t' + \phi) \sin(\omega t' + \phi)$$

$$- [V^2 r_m/(2c^2)] r_m \cos(\omega t' + \phi)$$

(59)

where we have noted that for small $\delta$, $\cos \delta \approx 1$ and $\sin \delta \approx \delta$. The expression for $x'_M$ is exactly analogous with $r_M$ replacing $r_m$ in Eq. (59). Similarly, we can evaluate the $y$-coordinates through order $v^2/c^2$ as

$$y'_m = y_m = r_m \sin(\omega t + \phi) = r_m \sin\{\sigma[1 - V^2/(2c^2)]t' + \phi + \omega V x_m/c^2\}$$

$$= r_m \sin\{\omega[1 - V^2/(2c^2)]t' + \phi\} \cos(\omega V x_m/c^2)$$

$$+ r_m \cos\{\omega[1 - V^2/(2c^2)]t' + \phi\} \sin(\omega V x_m/c^2)$$

$$= r_m \sin\{\omega[1 - V^2/(2c^2)]t' + \phi\} + r_m(\omega V x_m/c^2) \cos\{\omega[1 - V^2/(2c^2)]t' + \phi\}$$

$$= r_m \sin\{\omega[1 - V^2/(2c^2)]t' + \phi\} + (r_m^2 \omega V/c^2) \cos^2(\omega t' + \phi)$$

(60)

Also, the expression for $y_M$ is analogous with $r_M$ replacing $r_m$ in Eq. (60). If we (ensemble) average the expressions (59) and (60) over the phase angle $\phi$ at a single time $t'$, we find that $< x_m > = -Vt'$ while $< y_m > = [r_m^2 \omega V/(2c^2)]$. Thus in the inertial frame $S'$, our model for a magnetic moment has an electric dipole moment $\vec{p}$ perpendicular to the direction of motion of the center of energy of the system

$$\vec{p} = je(r_m^2 - r_M^2)\omega V/(2c^2) = (-\vec{V}/c) \times \vec{t}$$

(61)

where $-\vec{V}$ is the velocity of the center of energy of the magnetic dipole in the $S'$ frame. Thus our magnetic moment model has on average an electric dipole moment of order $v^2/c^2$.
as seen in the $S'$ inertial frame, although the charge distribution in the $S$ inertial frame has no average electric dipole moment.

V. COMMENTS ON THE INTERACTION OF A MAGNETIC DIPOLE AND ELECTRIC CHARGE

The examples above illustrate some relativistic aspects (through order $v^2/c^2$) of very simple point-charge systems. One system which arose great controversy involves a magnetic moment and a point charge. It is the system invoked in the recent work by Mansuripur. The interaction of a point charge and a magnetic moment seems quite complicated. It is evident that if a point charge were to be introduced some distance away from our relativistic (to order $v^2/c^2$) magnetic dipole model, the charges of the dipole would not continue in circular motion. Rather there would be nonrelativistic (zero-order in $v/c$) interactions which would alter the magnetic moment. The behavior of our model (in the limit $M >> m$) has been discussed by Solem[20] in his article "The Strange Polarization of the Classical Atom." The circular orbits become elliptical and lead to fascinating unexpected forces back on the distant point charge. [21] The magnetic moment model discussed here seems to have provided the basis for the only relativistically accurate account of a magnetic moment and a point charge. [21] Of course, the physics literature is full of accounts claiming to describe the interaction of a magnetic moment and a point charge, and these are listed in Griffiths' resource letter.[10] These accounts included counter rotating disks carrying charges[7] or tubes carrying charges[22] or rotating rigid cylinders carrying charges[23]. What is common to all these accounts is the assumption that the path taken by the charges which provide the magnetic moment is rigid and unchanged by the introduction of the external charge or external electric field. An example of exactly this fixed-path point of view is given in a fine undergraduate text book on electromagnetism where the example claims to calculate the "hidden momentum" associated with a magnetic moment.[24] In the example, the "hidden momentum" or order $v^2/c^2$ is calculated without ever discussing the role played by the forces which hold the moving charges in the prescribed path. All these fixed-path accounts[25] are suspect because there is no discussion of the (large) nonrelativistic stresses which must be present to maintain the fixed path in which the (small) relativistic corrections appear for the moving charges. [26] The large nonrelativistic stresses may produce small relativistic
effects which are totally different from the small mechanical relativistic effects calculated from the fixed-path point of view. Indeed exactly this situation appears in the accurate relativistic calculations\(^\text{[21]}\) based upon the point charge magnetic-moment model used in the present article. The nonrelativistic electromagnetic behavior of the magnetic moment leads to surprising relativistic forces back on the distant point charge\(^\text{[21]}\) which are denied in the mainstream physics literature and are exactly of the sort to account for the Aharonov-Bohm phase shift as arising from classical electromagnetic effects.

VI. CLOSING REMARKS

The present article was provoked by Mansuripur’s recent article claiming that the Lorentz force law is incompatible with relativity.\(^\text{[27]}\) His article is the latest entry in a set of perennial problems mentioned in Griffiths’ valuable resource letter on Electromagnetic Momentum.\(^\text{[10]}\) Apparently there are a number of aspects of classical electromagnetism which are sufficiently unfamiliar that physicists rediscover them and then comment on what seems an unusual situation. One of the oldest conundrums is that of the ”4/3 problem” for the classical model of the electron, involving the failure of 4-vector Lorentz transformation properties for energy-momentum when external forces are present. Another is the controversy between the Abraham and the Minkowski tensors for momentum carried by materials. The Trouton-Noble situation involving inertial-frame-dependent torques is another perennial problem. Finally, the interaction of a point charge and a magnetic moment has puzzled physicists for years, and has generated controversy involving ”hidden momentum” and in connection with the Aharonov-Bohm and Aharonov-Casher effects. Mansuripur’s article has comments relevant to the last two controversies on this list. In the present article, we have tried to discuss some very simple point-charge systems from a relativistic point of view through order \(v^2/c^2\) in order to provide a basic physical understanding of what is involved in some of the controversies. Both the ”4/3 problem” and the Trouton-Noble situations are well, though perhaps not widely understood; they are clearly illustrated by our simple examples. The Abraham-Minkowski controversy continues to receive attention in the research literature but rarely in the teaching literature; it is not treated in the present article. The interaction of a point charge and a magnetic moment has been discussed repeatedly in both the teaching and the research literature with many references to ”hidden momentum.” However, we have
suggested in this article that the behavior of a magnetic moment in an electric field due to external charges is still not properly appreciated, not even at a nonrelativistic level.

[1] D. J. Griffiths, *Introduction to Electrodynamics* 3rd edn (Prentice-Hall, Upper Saddle River, NJ 1999).

[2] J. D. Jackson, *Classical Electrodynamics* 3rd edn (Wiley, New York 1999).

[3] M. Mansuripur, "Trouble with the Lorentz law of force: Incompatibility with special relativity and momentum conservation," Phys. Rev. Lett. 108, 023807 (2012)

[4] A. Cho, "Textbook electrodynamics may contradict relativity," Science 336, 404 (2012).

[5] F. T. Trouton and H. R. Noble, Phil. Trans. A202, 165- (1903); Proc. Roy. Soc. 72, 132- (1903).

[6] See, for example, T. Ivezic, Am. J. Phys. 72, 1484 (2004); J. D. Jackson, Am. J. Phys. 72, 1484 (2004); S. A. Teukolsky, Am. J. Phys. 64 1104 (1996).

[7] W. Shockley and R. P. James, "'Try simplest cases' discovery of 'hidden momentum forces on magnetic currents'," Phys. Rev. Lett. 18, 876-879 (1967).

[8] Y. Aharonov and D. Bohm, "Significance of electromagnetic potentials in quantum theory," Phys. Rev. 115, 485-491 (1959).

[9] Y. Aharonov and A. Casher, "Topological quantum effects for neutral particles," Phys. Rev. Lett. 53, 319-321 (1984).

[10] D. J. Griffiths, Resource Letter EM-1: Electromagnetic Momentum," 80, 7-18 (2012).

[11] See the list of reference in Griffiths’ resource letter (ref. 10) under the heading "D. Momentum and Mass."

[12] T. H. Boyer, "Interaction of a Point Charge and a Magnet: Comments on "Hidden Mechanical Momentum Due to Hidden Nonelectromagnetic Forces", arXiv:0708.3367v1 (2007).

[13] T. H. Boyer, "Illustrations of the relativistic conservation law for the center of energy," Am. J. Phys. 73, 953-961 (2005). The generator of Lorentz transformations is the system energy times the center of energy.

[14] T. H. Boyer, "Illustrating some implications of the conservation laws in relativistic mechanics," Am. J. Phys. 77, 562-569 (2009).

[15] See ref. 2, pp. 596-598. C. G. Darwin was the grandson of the famous Darwin of evolution.
The fields are given by L. Page and N. I. Adams, “Action and reaction between moving charges,” Am. J. Phys. 13, 141–147 (1945). I sometimes assign undergraduate projects which combine what students have learned in both mechanics and electromagnetism by asking students to prove that Maxwell’s equations hold through order $v^2/c^2$ for the fields in Eqs. (17) and (18), and also to evaluate the Poisson brackets and to show that various conservation laws hold for the Darwin Lagrangian in Eq. (8).

Aspects of this example were discussed earlier, T. H. Boyer, “Lorentz-transformation properties for energy and momentum in electromagnetic systems,” Am. J. Phys. 53, 167-171 (1985).

T. H. Boyer, "Classical model of the electron and the definition of electromagnetic field momentum," Phys. Rev. D 25, 3246-3250 (1982).

Recently J. Franklin mentioned a "string" holding two like charges in place, "The lack of rotation in the Trouton-Noble experiment" arXiv:physics/0603110v3, 1-8 (2006). However, a "string" may bring to mind properties which contradict the requirements of special relativity.

J. C. Solem, "The strange polarization of the classical atom," Am. J. Phys. 55, 906-909 (1987).

T. H. Boyer, "Darwin-Lagrangian analysis for the interaction of a point charge and a magnet: Considerations related to the controversy regarding the Aharonov-Bohm and Aharonov-Casher phase shifts," J. Phys. A: Math. Gen. 39, 3455-3477 (2006).

L. Vaidman, "Torque and force on a magnetic dipole," Am. J. Phys. 58, 978-983 (1990).

M. Peshkin, I Talmi, and L. J. Tassie, "the quantum mechanical effects of magnetic fields confined to inaccessible regions," Ann. Phys., NY 12, 426-435 (1961), especially section 5.

See ref. 1, p. 520.

See the list of references in Griffiths’ resource letter (ref. 10) under the heading "C. Hidden Momentum."

Indeed, some authors seem to feel that simply making reference to "hidden momentum" excuses them from properly accounting for the flow of energy and momentum in an electromagnetic system.

Replies to Mansuripur’s assertions have been given also by K. T. McDonald, "Mansuripur’s Paradox," [http://www.physics.princeton.edu/~mcdonald/examples/mansuripur.pdf](http://www.physics.princeton.edu/~mcdonald/examples/mansuripur.pdf) (2012); and by D. A. T. Vanzella, "Comment on 'Trouble with the Lorentz law of force: Incompatibility with special relativity and momentum conservation," arXiv:1205.1502v1 (2012).