BUFFON NEEDLE LANDS IN $\epsilon$-NEIGHBORHOOD OF A 1-DIMENSIONAL SIERPINSKI GASKET WITH PROBABILITY AT MOST $|\log \epsilon|^{-c}$

MATT BOND AND ALEXANDER VOLBERG

Abstract. In recent years, relatively sharp quantitative results in the spirit of the Besicovitch projection theorem have been obtained for self-similar sets by studying the $L^p$ norms of the “projection multiplicity” functions, $f_\theta$, where $f_\theta(x)$ is the number of connected components of the partial fractal set that orthogonally project in the $\theta$ direction to cover $x$. In [4], it was shown that $n$-th partial 4-corner Cantor set with self-similar scaling factor $1/4$ decays in Favard length at least as fast as $\frac{c}{n}$, for $p < 1/6$. In [4], this same estimate was proved for the 1-dimensional Sierpinski gasket for some $p > 0$. A few observations were needed to adapt the approach of [4] to the gasket: we sketch them here. We also formulate a result about all self-similar sets of dimension 1.

1. Definitions and result

Let $E \subset \mathbb{C}$, and let $\text{proj}_\theta$ denote orthogonal projection onto the line having angle $\theta$ with the real axis. The average projected length or Favard length of $E$, $\text{Fav}(E)$, is given by

$$\text{Fav}(E) = \frac{1}{\pi} \int_0^{\pi} |\text{proj}_\theta(E)| d\theta.$$ 

For bounded sets, Favard length is also called Buffon needle probability, since up to a normalization constant, it is the likelihood that a long needle dropped with independent, uniformly distributed orientation and distance from the origin will intersect the set somewhere.

$B(z_0,r) := \{z \in \mathbb{C} : |z - z_0| < r\}$. For $\alpha \in \{-1,0,1\}^n$ let

$$z_\alpha := \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k e^{i\pi[\frac{1}{3} + \frac{2}{3}\alpha_k]}, \quad G_n := \bigcup_{\alpha \in \{-1,0,1\}^n} B(z_\alpha, 3^{-n}).$$

This set is our approximation of a partial Sierpinski gasket; it is strictly larger. We may still speak of the approximating discs as “Sierpinski triangles.”

The main result:
Theorem 1. $\text{Fav}(G_n) \leq \frac{C}{n^c}, c > 0$.

Set $G_n$ is $3^{-n}$ approximation to Besicovitch irregular set (see [2] for definition) called Sierpinski gasket. Recently one detects a considerable interest in estimating the Favard length of such $\epsilon$-neighborhoods of Besicovitch irregular sets, see [5], [6], [4], [3]. In [5] a random model of such Cantor set is considered and estimate $\approx \frac{1}{n}$ is proved. But for non-random self-similar sets the estimates of [5] are more in terms of $\log \cdots \log n$ (number of logarithms depending on $n$) and more suitable for general class of “quantitatively Besicovitch irregular sets” treated in [6].

Let $f_{n,\theta} := \frac{1}{2} \nu_n \ast 3^n \chi_{[-3^{-n}, 3^{-n}]}$, where

$$\nu_n := \prod_{k=1}^n \tilde{\nu}_k$$

and $\tilde{\nu}_k := \frac{1}{3} [\delta_{3^{-k}\cos(\pi/2-\theta)} + \delta_{3^{-k}\cos(-\pi/6-\theta)} + \delta_{3^{-k}\cos(7\pi/6-\theta)}]$.

For $K > 0$, let $A_K := A_{K, n, \theta} := \{x : f_{n, \theta} \geq K\}$. Let $\mathcal{L}_{\theta, n} := \text{proj}_\theta(G_n) = A_{1, n, \theta}$.

For our result, some maximal versions of these are needed:

$f^*_{N, \theta} := \max_{n \leq N} f_{n, \theta}$, $A^*_K := A^*_{K, n, \theta} := \{x : f^*_{n, \theta} \geq K\}$.

Also, let $E := E_N := \{\theta : |A^*_K| \leq K^{-3}\}$ for $K = N^{\epsilon_0}$, $\epsilon_0$.

Later, we will jump to the Fourier side, where the function

$$\varphi_\theta(x) := \frac{1}{3} [e^{-i \cos(\pi/2-\theta)} + e^{-i \cos(-\pi/6-\theta)} + e^{-i \cos(7\pi/6-\theta)}]$$

plays the central role: $\hat{\nu}_n(x) = \prod_{k=1}^n \varphi_\theta(3^{-k}x)$.

2. General philosophy

Fix $\theta$. If the mass of $f_{n, \theta}$ is concentrated on a small set, then $||f_{n, \theta}||_p$ should be large for $p > 1$ - and vice versa. $1 = \int f \leq ||f_{n, \theta}||_p |\mathcal{L}_{\theta, n}|_q$, so $m(\mathcal{L}_{\theta, n}) \geq ||f||_p^q$, a decent estimate. The other basic estimate is not so sharp:

$$m(\mathcal{L}_{\theta, N}) \leq 1 - (K - 1)m(A_{K, N, \theta})$$

However, a combinatorial self-similarity argument of [4] and revisited in [1] shows that for the Favard length problem, it bootstraps well under further iterations of the similarity maps:

Theorem 2. If $\theta \notin E_N$, then $|\mathcal{L}_{\theta, N, K^3}| \leq \frac{C}{K^r}$.

Note that the maximal version $f^*_N$ is used here. A stack of $K$ triangles at stage $n$ generally accounts for more stacking per step the smaller $n$ is. For fixed $x \in A^*_{K, N, \theta}$,
the above theorem considers the smallest $n$ such that $x \in A_{K,n,\theta}$, and uses self-similarity and the Hardy-Littlewood theorem to prove its claim by successively refining an estimate in the spirit of (2.1). Of course, now Theorem 1 follows from the following:

**Theorem 3.** Let $\epsilon_0 < 1/11$. Then for $N >> 1$, $|E| < N^{-\epsilon_0}$.

It turns out that $L^2$ theory on the Fourier side is of great use here. It is proved in [4], [1]:

**Theorem 4.** For all $\theta \in E_n$ and for all $n \leq N$, $\|f_n,\theta\|_{L^2}^2 \leq CK_n$.

One can then take small sample integrals on the Fourier side and look for lower bounds as well. Let $K = N^{\epsilon_0}$, and let $m = 2\epsilon_0 \log_3 N$. Theorem 3 easily implies the existence of $\tilde{E} \subset E$ such that $|\tilde{E}| > |E/2|$ and number $n$, $N/4 < n < N/2$, such that for all $\theta \in \tilde{E}$,

$$\int_{3^{n-m}}^{3^n} \prod_{k=0}^{n} |\varphi_\theta(3^{-k}x)|^2 dx \leq \frac{2CKm}{N} \leq 2\epsilon_0 N^{\epsilon_0 - 1} \log N.$$ 

Number $n$ does not depend on $\theta$; $n$ can be chosen to satisfy the estimate in the average over $\theta \in E$, and then one chooses $\tilde{E}$. Let $I := [3^{n-m}, 3^n]$.

Now the main result amounts to this (with absolute constant $A$ large enough):

**Theorem 5.**

$$\theta \in \tilde{E} : \int_I \prod_{k=0}^{n} |\varphi_\theta(3^{-k}x)|^2 dx \geq c3^{m-2Am} = cN^{-2\epsilon_0(2A-1)}.$$ 

The result: $2\epsilon_0 \log N \geq N^{1-\epsilon_0(4A-1)}$, i.e., $N \leq N^*$. Now we sketch the proof of Theorem 5. We split up the product into two parts: high and low-frequency: $P_1,\theta(z) = \prod_{k=0}^{n-m-1} \varphi_\theta(3^{-k}z)$, $P_2,\theta(z) = \prod_{k=n-m}^{n} \varphi_\theta(3^{-k}z)$.

**Theorem 6.** For all $\theta \in E$, $\int_I |P_1,\theta|^2 dx \geq C 3^n$.

Low frequency terms do not have as much regularity, so we must control the damage caused by the set of small values, $SSV(\theta) := \{x \in I : |P_2(x)| \leq 3^{-\ell}\}$, $\ell = \alpha m$ with sufficiently large constant $\alpha$. In the next result we claim the existence of $E \subset \tilde{E}$, $|E| > |E/2|$ with the following property:

**Theorem 7.**

$$\int_E \int_{SSV(\theta)} |P_1,\theta(x)|^2 dx d\theta \leq 3^{2n-\ell/2} \Rightarrow \forall \theta \in E \int_{SSV(\theta)} |P_1,\theta(x)|^2 dx d\theta \leq cK 3^{2m-\ell/2}.$$ 

Then Theorems 6 and 7 give Theorem 5.
3. Locating zeros of $P_2$

We can consider $\Phi(x, y) = 1 + e^{ix} + e^{iy}$. The key observations are

$$|\Phi(x, y)|^2 \geq a(|4 \cos^2 x - 1|^2 + |4 \cos^2 y - 1|^2), \quad \frac{\sin 3x}{\sin x} = 4 \cos^2 x - 1.$$ 

Changing variable we can replace $3 \varphi(x)$ by $\phi_t(x) = \Phi(x, tx)$. Consider $P_{2,t}(x) := \prod_{k=n-m}^n \frac{1}{3} \phi_t(3^{-k}x)$, $P_{1,t}(x) := \prod_{k=0}^{n-m} \frac{1}{3} \phi_t(3^{-k}x)$. We need $SSV(t) := \{ x \in I : |P_{2,t}(x)| \leq 3^{-\ell} \}$. One can easily imagine it if one considers $\Omega := ((x, y) \in [0, 2 \pi]^2 : |P(x, y)| := \prod_{k=0}^m \Phi(3^k x, 3^k y) \leq 3^{m-\ell})$. Moreover, (using that if $x \in SSV(t)$ then $3^{-n} x \geq 3^{-m}$, and using $x dx dt = dxdy$) we change variable in the next integral:

$$\int E \int_{SSV(t)} |P_{1,t}(x)|^2 dx dt = 3^{-2n+2m} \cdot 3^n \int E \int_{3^{-n}SSV(t)} \prod_{k=m}^n \Phi(3^k x, 3^k tx)^2 dx dt \leq 3^{-n+3m} \int_{\Omega} \prod_{k=m}^n \Phi(3^k x, 3^k y)^2 dxdy.$$

Now notice that by our key observations $\Omega \subset \{ (x, y) \in [0, 2 \pi]^2 : |\sin 3^{m+1} x|^2 + |\sin 3^{m+1} y|^2 \leq a^{-m} 3^{2m-2\ell} \leq 3^{-\ell} \}$. The latter set $Q$ is the union of $4 \cdot 3^{2m+2}$ squares $Q$ of size $3^{m-\ell/2} \times 3^{-m-\ell/2}$. Fix such a $Q$ and estimate

$$\int_{Q} \prod_{k=m}^n \Phi(3^k x, 3^k y)^2 dxdy \leq 3^\ell \int_{Q} \prod_{k=m+\ell/2}^n \Phi(3^k x, 3^k y)^2 dxdy \leq 3^\ell (3^{m-\ell/2})^2 \int_{[0, 2 \pi]^2} \prod_{k=0}^{n-m-\ell/2} \Phi(3^k x, 3^k y)^2 dxdy \leq 3^\ell (3^{m-\ell/2})^2 3^{n-m-\ell/2} = 3^{-2m} 3^{n-m-\ell/2}.$$

Therefore, taking into account the number of squares $Q$ in $Q$ and the previous estimates we get

$$\int E \int_{SSV(t)} |P_{1,t}(x)|^2 dx dt \leq 3^{2m-\ell/2}.$$ 

Theorem 7 is proved.

To prove Theorem 6 we need the following simple lemma.

**Lemma 8.** Let $C$ be large enough. Let $j = 1, 2, \ldots, k$, $c_j \in \mathbb{C}$, $|c_j| = 1$, and $\alpha_j \in \mathbb{R}$. Let $A := \{ \alpha_j \}_{j=1}^k$. Suppose

$$\int_{\mathbb{R}} \left( \sum_{\alpha \in A} \chi_{[\alpha-1, \alpha+1]}(x) \right)^2 dx \leq S. \text{ Then } \int_0^1 \left| \sum_{\alpha \in A} c_\alpha e^{i\alpha y} \right|^2 dy \leq C S.$$
Some key facts useful for its proof:

\[
\int_0^1 \left| \sum_{\alpha \in A} c_\alpha e^{i \alpha y} \right|^2 dy \leq e \int_0^\infty \left| \sum_{\alpha \in A} c_\alpha e^{i (\alpha + i) y} \right|^2 dy = e \int_{\mathbb{R}} \left| \sum_{\alpha \in A} \frac{c_\alpha}{\alpha + i - x} \right|^2 dx,
\]

and the fact that \( H^2(C_+) \) is orthogonal to \( H^2(C_+) \), so one can pass to the Poisson kernel.

4. THE GENERAL CASE

Let us have \( k \) closed disjoint discs of radii \( 1/k \) located in the unit disc. We build \( k^n \) small discs of radii \( k^{-n} \) by iterating \( k \) linear maps from small discs onto the unit disc. Call the resulting union \( S_k(n) \). We would like to show that exactly as in the case of \( k = 3 \) considered above and in a very special case of \( k = 4 \) considered in [4] \( \text{Fav}(S_k(n)) \leq C n^{-c} \), \( c > 0 \). However, presently we can prove only a weaker result.

**Theorem 9.**

\[
\text{Fav}(S_k(n)) \leq C e^{-c \left( \log n \right)^{1/2}}, \quad c > 0.
\]

**References**

[1] M. Bond, A. Volberg, *The Power Law For Buffon’s Needle Landing Near the Sierpinski Gasket*, arXiv:math.0911.0233v1, 2009, pp. 1-34.

[2] K. J. Falconer, The geometry of fractal sets. Cambridge Tracts in Mathematics, 85. C.U.P., Cambridge–New York, (1986).

[3] I. Laba, K. Zhai, *Favard length of product Cantor sets*, arXiv:0902:0964v1, Feb. 5 2009.

[4] F. Nazarov, Y. Peres, A. Volberg *The power law for the Buffon needle probability of the four-corner Cantor set*, arXiv:0801.2942 2008, pp. 1–15.

[5] Y. Peres and B. Solomyak, *How likely is Buffon’s needle to fall near a planar Cantor set?* Pacific J. Math. 204, 2 (2002), 473–496.

[6] T. Tao, *A quantitative version of the Besicovitch projection theorem via multiscale analysis*, pp. 1–28, arXiv:0706.2446v1 [math.CA] 18 Jun 2007.

Matt Bond, Department of Mathematics, Michigan State University, bondmatt@msu.edu

Alexander Volberg, Department of Mathematics, Michigan State University, volberg@math.msu.edu