Isolated black holes without $\mathbb{Z}_2$ isometry

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A mechanism to construct asymptotically flat, isolated, stationary black hole (BH) spacetimes with no $\mathbb{Z}_2$ (NoZ) isometry is described. In particular, the horizon geometry of such NoZ BHs does not have the usual north-south (reflection) symmetry. We discuss two explicit families of models wherein NoZ BHs arise. In one of these families, we exhibit the intrinsic horizon geometry of an illustrative example by isometrically embedding it in Euclidean 3-space, resulting in an “egg-like” shaped horizon. This asymmetry leaves an imprint in the NoZ BH phenomenology, for instance in its lensing of light; but it needs not be manifest in the BH shadow, which in some cases can be analytically shown to retain a $\mathbb{Z}_2$ symmetry. Light absorption from an isotropic source surrounding a NoZ BH endows it a non-zero momentum, producing an asymmetry triggered BH rocket effect.

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Introduction. Knowledge of black hole (BH) physics is to some large extent based on exact solutions. Of central importance is the Kerr-Newman (KN) spacetime [1, 2], which according to the uniqueness theorems (see [3] for a review) is the most general, non singular (on and outside an event horizon) stationary, single BH solution of Einstein-Maxwell theory. The spatial sections of the event horizon of KN BHs are, geometrically, squashed spheres [4]. These surfaces, moreover, have an antipodal isometry: the metric is invariant under a parity transformation $(\theta, \varphi) \rightarrow (\pi - \theta, \varphi + \pi)$ in the standard Boyer-Lindquist coordinates [5]. Due to the axi-symmetry of the solution this implies that the spacetime geometry, and in particular the horizon, has a $\mathbb{Z}_2$ symmetry: the northern and southern hemispheres are isometric. There is also an unambiguous equator corresponding to the set of fixed points of the $\mathbb{Z}_2$ isometry. An analogous (with adequate generalisations) antipodal isometry is widely acknowledged to be present in all known isolated BHs in all dimensions (see e.g. [6]).

These observations raise the following question: in four spacetime dimensions, can an isolated, asymptotically flat, stationary BH spacetime, free of singularities on and outside the event horizon, have a non $\mathbb{Z}_2$ isometric horizon? Here we show the answer is yes, unveil a generic mechanism to construct non $\mathbb{Z}_2$ (NoZ) invariant BHs, and discuss how some observables manifest the $\mathbb{Z}_2$ symmetry violation.

Scalar deformations. To search for non NoZ BHs one must go beyond electrovacuum. The simplest additional matter content one may consider are scalar fields. The existence of different examples of BHs with scalar hair, e.g. [7–13], some of which with rather different physical properties than KN (see e.g. [14]), justifies this choice. To establish a proof of principle, we consider deforming the KN geometry through a real scalar field $\phi$. Starting with the test field limit, $\phi$ is nonminimally coupled to the fixed KN background, via some tensorial scalar invariant source term $\mathcal{J}$, constructed from the KN metric and gauge field. Its Lagrangian density is

$$
\mathcal{L}_\phi = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - f(\phi) \mathcal{J}(g; A) \, .
$$

For concreteness we take $f(\phi) = e^{-2\alpha \phi}$, corresponding to an often considered non-minimal coupling, occurring naturally in, say, String Theory [15], and Kaluza-Klein theory [16]. The specific choice of the constant $\alpha \neq 0$ is not central for our discussion. Below we shall also comment on other couplings. Employing the conventions in [17], the (dyonic) KN metric is

$$
d s^2 = \left\{ -\Delta(\omega_t^2 + \sin^2 \theta \omega_\varphi^2) \right\} \Sigma + \Sigma \left( d\tau^2 / \Delta + d\theta^2 \right),
$$

where $\omega_t \equiv dt - a \sin^2 \theta d\varphi$, $\omega_\varphi \equiv adt - (r^2 + a^2) d\varphi$, $\Sigma \equiv r^2 + a^2 \cos^2 \theta$, $\Delta \equiv r^2 - 2Mr + a^2 + Q^2 + P^2$ and $a \equiv J/M \, , \, (M, J; Q, P)$ are the ADM mass, angular momentum, electric and magnetic charges of the BH, respectively. The gauge connection is $A = Qr (\omega_t - P \cos \theta \omega_\varphi) / \Sigma$.

The scalar field equation derived from (1) is:

$$
\square \phi + 2\alpha e^{-2\alpha \phi} \mathcal{J} = 0 \, .
$$

We may now observe the following generic mechanism. For a $\mathbb{Z}_2$ invariant background, such as KN, the d’Alembertian operator is $\mathbb{Z}_2$ even. But not all scalar invariants constructed from the KN metric and gauge field are $\mathbb{Z}_2$ even; some are $\mathbb{Z}_2$ odd and some are not $\mathbb{Z}_2$ eigenstates. Choosing one non-$\mathbb{Z}_2$ odd term as the source $\mathcal{J}$, the scalar field (and its energy-momentum tensor) will be neither even nor odd under the $\mathbb{Z}_2$ transformation, and when considering its back reaction, if regular on and outside the horizon, it will lead to a NoZ BH (with scalar hair).

An electromagnetic source. Let us illustrate this mechanism with an electromagnetic source. The simplest KN electromagnetic invariant that is not a $\mathbb{Z}_2$ eigenstate is

$$
\mathcal{J} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \equiv \frac{F^2}{4} \, ,
$$

where $F_{\mu\nu}$ is the Maxwell tensor. Indeed, for the KN solution:

$$
F^2 = \frac{16}{\Sigma^4} \left[ b \left( \frac{\Sigma^2}{8} - \Sigma r^2 + r^4 \right) + d \cos \theta \left( \Sigma - 2r^2 \right) \right],
$$

$b \equiv Q^2 - P^2$, $d \equiv JPQ/M$. Whereas the first term in the square brackets is $\mathbb{Z}_2$ even, the second is $\mathbb{Z}_2$ odd. Thus, in the generic case with all $(M, J; Q, P)$ non-zero, the $F^2$ invariant...
is not an eigenstate of the $Z_2$ isometry of the KN background $\theta \to \pi - \theta$. If this invariant sources a scalar field via (2), such scalar field is not a $Z_2$ eigenstate, for any $\alpha \neq 0$. Moreover, this source allows non-singular solutions of the scalar field on and outside the event horizon, as confirmed by the existence of regular (fully non-linear) Einstein-Maxwell-dilaton BHs [18].

In Fig. 1 we plot the amplitude of the scalar field on a KN background with $(J;Q,P) = (0.6; 0.4, 0.6)$ [19]. This was obtained by solving numerically (2) with (3)-(4) and $\alpha = 1$. Such analysis confirms the solution is regular everywhere on and outside the horizon, vanishes asymptotically and it is not a $Z_2$ eigenstate, as it is clear from the plot. The failure to be a $Z_2$ eigenstate can be analytically checked in the far-field, where the scalar field solution reads $\phi(r) = Q_s/r + (e + f \cos \theta)/r^2 + \ldots$; the constants $Q_s, e, f$ depend on the background parameters. The sub-leading $1/r^2$ term, for instance, is generically not a $Z_2$ eigenstate.

The solution plotted in Fig. 1 is obtained in the test field limit. To establish the existence of No$Z$ BHs the scalar field backreaction must be considered. To obtain such fully non-linear solutions, and the corresponding deformed BHs, we couple the scalar field Lagrangian to the gravitational action. For simplicity, we use the electromagnetic source (3). Thus, we consider the model described by the action

$$S = \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{e^{-2\alpha \phi}}{4} F_{\mu\nu} F^{\mu\nu} \right].$$

(5)

For the main question here, qualitative difference are not expected for any $\alpha \neq 0$. Thus, as an example, we focus on the special case with $\alpha = \sqrt{3}$. This is the well known Kaluza-Klein Einstein-Maxwell-dilaton model, wherein the generalisation of the KN solution, a rotating dyonic BH, is known [20–22] (see also [23]). This BH is characterised by the same four parameters $(M, J; Q, P)$ as the KN solution, but KN is not a special case of this BH (whereas the Kerr solution is).

According to our previous discussion, these BHs should fail to have (in the generic case, when all parameters are non-vanishing) a $Z_2$ isometry. To establish this hitherto unnoticed fact, it is enough to observe that the metric and the matter functions of this solution are combinations of building blocks of the generic form $U(r, \theta) = a_0 + a_1 \cos \theta + a_2 \cos^2 \theta$, with $a_1 = a_1(r; M, J, Q, P)$. As such, the solution is not, generically, $Z_2$ isometric.

We can now focus on the horizon geometry of one of these BHs [24]. The spatial sections of the event horizon have an induced metric of the form $d\sigma^2 = g_{\theta\theta}(\theta)d\theta^2 + [g_{\theta\theta}(\theta)]^2 \sin^2 \theta d\varphi^2/g_{\theta\theta}(\theta)$, where $g_{\theta\theta}(\theta) = \sqrt{(\sum_{i=0}^2 b_i \cos^i \theta)(\sum_{j=0}^2 c_j \cos^j \theta)}$, and $b_i, c_j$ have cumbersome expressions in terms of $(M, J, Q, P)$. It is obvious that no $Z_2$ isometry exists in the generic case, the usual north-south symmetry being lost. This is illustrated in Fig. 2, where the embedding of the induced horizon metric in Euclidean 3-space of some solutions are shown. The construction of these embeddings follows a standard procedure [4]. One observes this intrinsic geometry has an “egg-like” shape.
BHs occur, we again start with a test field analysis, but specialise for simplicity the KN background to the uncharged limit (Kerr, \( Q = 0 = P \)) and consider a scalar-tensor model (no Maxwell field). Whereas the Kretschmann invariant, \( R^{\mu
u\alpha\beta}R_{\mu
u\alpha\beta} \), is \( \mathbb{Z}_2 \) even, thus unsuitable to be the source for the purpose in sight, the Pontryagin density is \( \mathbb{Z}_2 \) odd. Consequently, we take the latter as the source term:

\[
\mathcal{J} = \xi * R^\mu_{\nu} \alpha^\beta R^\nu_{\mu\alpha\beta}, \quad * R^\mu_{\nu} \alpha^\beta \equiv \frac{1}{2} \epsilon^{\alpha\beta\sigma\tau} R^\mu_{\nu\sigma\tau},
\]

(6)

where \( \xi \) is a dimensionful coupling constant (\( [\xi] = \text{Length}^2 \)), and \( \epsilon^{\alpha\beta\sigma\tau} \) are the components of the Levi-Civita tensor. For Kerr, the Pontryagin density is compactly written as

\[
* R^\mu_{\nu} \alpha^\beta R^\nu_{\mu\alpha\beta} = \frac{96aM^2r \cos \theta}{\Sigma^6} (3\Sigma^2 - 16r^2\Sigma + 16r^4),
\]

(7)

confirming it is \( \mathbb{Z}_2 \) odd. Again, this source allows non-\( \mathbb{Z}_2 \) symmetric, non-singular solutions of the scalar (test) field on and outside the event horizon, similar to those in Fig. 1.

In this example, unlike the case of the electromagnetic source, fully non-linear solutions in closed analytic form are unknown and, likely, do not exist. Such solutions can, however, be constructed numerically, following [25], in the corresponding non-linear model:

\[
S = \int d^4x \sqrt{-g} \left[ \frac{R}{4} - \frac{\partial_\mu \phi \partial^\mu \phi}{2} - \frac{\xi}{4} e^{-2\phi} * R^\mu_{\nu} \alpha^\beta R^\nu_{\mu\alpha\beta} \right].
\]

(8)

This model is a variation of the usual dynamical Chern-Simons modified gravity, wherein the coupling of the scalar field to the Pontryagin density is of the form \( \phi * R^\mu_{\nu} \alpha^\beta R^\nu_{\mu\alpha\beta} \) [26, 27]. In the latter, therefore, the source term (7) gives rise to a \( \mathbb{Z}_2 \) odd scalar field, which is compatible with a \( \mathbb{Z}_2 \) even geometry [28–30].

We have confirmed that the \( \mathbb{Z}_2 \) symmetry is lost for BH solutions of (8) with any \( \alpha \neq 0 \). Using the same coordinate system as in [25], we define the “equatorial” plane as corresponding to the value of \( \theta = \theta_0 \) which maximizes the proper length \( L_\theta = \int_{0}^{2\pi} d\phi \sqrt{g_{\phi\phi}(r_H, \theta)} \) of a \( \theta = \text{const.} \) circle on the induced horizon metric. For the KN metric (or the solutions in [25]), \( L_\theta \) is maximized for \( \theta_0 = \pi/2 \). This is not the case for the generic BH solutions of (8). We define the following measure for the \( \mathbb{Z}_2 \) symmetry violation \( \epsilon \equiv 1 - L_p^{(N)} / L_p^{(S)} \), where \( L_p^{(N)} \) (\( L_p^{(S)} \)) is the proper length from the north (south) pole to the “equatorial” plane, \( L_p^{(N)} = \int_{0}^{\pi} d\theta \sqrt{g_{\theta\theta}(r_H, \theta)} \), \( L_p^{(S)} = \int_{\pi}^{2\pi} d\theta \sqrt{g_{\theta\theta}(r_H, \theta)} \).

In Fig. 3 (top panel) we exhibit the deformation \( \epsilon \) for a subset of solutions of (8). In the region where these numerical solutions are enough accurate, \( \epsilon \) is no larger than a few percent, and the most significant \( \mathbb{Z}_2 \) symmetry violations are found for near extremal BHs. By comparison, \( \epsilon \sim 0.2 \) for the solution in Fig. 2. Thus, within the domain analysed, \( \mathbb{Z}_2 \) deformations within model (8) are barely visible in an embedding diagram. Still, the \( \mathbb{Z}_2 \) symmetry violation is clear, as shown in Fig. 3 (bottom panel) where the contour lines of the scalar field amplitude are shown for a particular, fully non linear solution. We anticipate BHs larger values of \( \epsilon \) exist in this model. Their construction, however, is challenging.

**Phenomenology.** What could be the phenomenological impact of the absence of a \( \mathbb{Z}_2 \) isometry? A simple diagnosis can be performed using null geodesics as phenomenological probes. To maximise the \( \mathbb{Z}_2 \) violation effects we analyse model (5) rather than (8). We have performed ray tracing (following [14], see also [31]) in an illustrative No\( \mathbb{Z}_2 \) dyonic rotating BH, and exhibit in Fig. 4 its gravitational lensing.

Each point’s colour in Fig. 4 encodes the geodesics’s initial position: green (light blue) color denote geodesics that have their origin on the north (south) hemisphere of a far-away celestial sphere, enclosing both the BH and the observer. The first relevant feature of the image is that, even though the observer is placed on the (would be) equatorial plane surface \( \theta = \pi/2 \), the colored pattern is not \( \mathbb{Z}_2 \) symmetric, as can be apparent by interchanging the green/light blue colors. This further implies that \( \theta = \pi/2 \) is not a totally geodesic submanifold, as it would be in the case of spacetimes with a \( \mathbb{Z}_2 \) isometry. The black region in Fig. 4 is associated to null geodesics that would have their origin on the event horizon surface, and forms the BH shadow [32]. The latter is a direct probe of the geometry close to the BH, wherein the fundamen-
null for by the inset of Fig. 4. This can be shown analytically. The Killing tensor \([36, 37]\). The null geodesic equations formal fourth Carter-like constant of motion shadow edge displays a \((Z)\) contour and its \(Z_2\) reflection. Both curves coincide. Thus, despite the lack of \(Z_2\) of the lensing, the shadow is \(Z_2\) symmetric.

Yet, the \(Z_2\) symmetry of the shadow (at any observation process) does not guarantee the overall light absorption by the BH is north-south symmetric, since the shadow seen at the north and south poles has a different size (an analogous effect should occur for time-like particles). As a consequence of this the BH may acquire a thrust due to the asymmetric momentum absorption, leading to a BH rocket.

Consider that each point of the celestial sphere is a (quasi-)isotropic radiation source of light rays with \(L = 0\). Each light ray is parametrised by the inclination angle \(\alpha \in [-\pi/2, -\pi/2]\), with \(\alpha = 0\) pointing to the BH center - Fig. 5. The \(z\)-momentum flux \(P_{z}^{\text{em}}\) emitted at the source is \(P_{z}^{\text{em}} = \zeta \int \frac{d^{2}P_{z}}{dNdA} \, d\alpha \, dA\), where \(\zeta = dN/d\alpha\) is the (constant) flux density of photons per unit angle and \(dA = R^{2} \sin \theta \, d\theta \, d\varphi\) is the area element of the emitting sphere at large radius \(R\). By symmetry \(P_{z}^{\text{em}} = 0\). To compute the \(z\)-momentum

\[
\text{FIG. 4. Lensing and shadow due to a NoZ dyonic BH with } (J, Q, P) \simeq (0.22, 0.15, 1.5). \text{ The green (light blue) colour light is emitted by the north (south) far away celestial sphere. The image is for an observer on the } \theta = \pi/2 \text{ plane. The inset shows the shadow contour and its } Z_2 \text{ reflection. Both curves coincide. Thus, despite the lack of } Z_2 \text{ of the lensing, the shadow is } Z_2 \text{ symmetric.}
\]

\[
\text{FIG. 5. Illustration of the thrust imparted on a NoZ BH due to an isotropic light source (celestial sphere).}
\]

flux \(P_{z}^{\text{abs}}\) absorbed by the BH take \(p_{z} = d^{2}P_{z}/(dNdA) = -\mathcal{P} (\cos \alpha \cos \theta - \sin \theta \sin \alpha) \, \Theta(\alpha)\), where \(\mathcal{P}\) is the momentum density per photon, per area and \(\Theta\) is a step function which is 1 (0) if \(\alpha\) is part (not part) of the BH shadow. Since the shadow is symmetric for every static observer the step function restricts the \(\alpha\) domain into \(\alpha \in [-\alpha_o, \alpha_o]\), where \(\alpha_o(\theta)\) computes the shadow size for each \(\theta\). The BH absorbed momentum is then \(P_{z}^{\text{abs}} = -\mathcal{P} (4\pi R^2 \zeta \mathcal{P}) \int_{\alpha_o}^{\alpha_o} d\theta \sin \theta \cos \theta \sin \alpha_o\), which, in general, may be non-vanishing. Using the Carter constant of the \(L = 0\) spherical photon orbit, and \(q_1, q_2\) are geodesic separation constants of the \(r\) and \(\theta\) sectors. If \(q_1 = 0\), then \(P_{z}^{\text{abs}} = 0\) due to the \(Z_2\) reflection symmetry. Defining the north and south fluxes as \(P_{N} = \int_{0}^{\frac{\pi}{2}} d\xi \, \mathcal{X}, P_{S} = \int_{-\frac{\pi}{2}}^{0} d\xi \, \mathcal{X}\) for the BH in Fig. 2 \(|P_{S}|\) is \(\sim 7\%\) larger than \(|P_{N}|\), triggering a “thrust” in the positive \(z\) direction, which is consistent with a larger shadow seen from the south pole, \(\text{cf. Fig. 5}\). For other BH solutions of model (5) this value can be as high as \(\sim 19\%\).

\textbf{Final Remarks.} In the presence of a negative cosmological constant static BHs exist without any spatial continuous symmetry \([41]\), since asymptotically Anti-de-Sitter spacetimes allow boundary conditions that anchor horizon deformations. This is not so in asymptotically flat spacetimes. Still, the construction herein shows that non-minimal couplings allow isolated, regular on and outside an event horizon, stationary BHs
with uncommon deformations, such as the loss of the usual north-south $\mathbb{Z}_2$ isometry. A similar construction can be made taking the scalar coupling $f(\phi)$ in (1) to be a more general function. In particular, if this is a function of $\phi^2$ only, then KN (or Kerr) is also a solution of the fully non-linear model, coexisting with the scalarised BH [10–13], which is not the case in the models (5), (8).

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[1] R. P. Kerr, Phys.Rev.Lett. 11, 237 (1963).
[2] E. T. Newman, R. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, J. Math. Phys. 6, 918 (1965).
[3] P. T. Chrusciel, J. L. Costa, and M. Heusler, Living Rev.Rel. 15, 7 (2012), arXiv:1205.6112 [gr-qc].
[4] L. Smarr, Phys. Rev. D7, 289 (1973).
[5] R. H. Boyer and R. W. Lindquist, J. Math. Phys. 8, 265 (1967).
[6] G. W. Gibbons, Proceedings, 20th International Fall Workshop on Geometry and Physics (IFWGP 2011): Madrid, Spain, August 31-September 3, 2011, AIP Conf. Proc. 1460, 90 (2012), arXiv:1201.2340 [gr-qc].
[7] C. A. R. Herdeiro and E.Radu, Phys.Rev.Lett. 112, 221101 (2014), arXiv:1403.2757 [gr-qc].
[8] T. P. Sotiriou and S.-Y. Zhou, Phys. Rev. D90, 124063 (2014), arXiv:1408.1698 [gr-qc].
[9] C. A. R. Herdeiro and E. Radu, Proceedings, 7th Black Holes Workshop 2014: Aveiro, Portugal, December 18-19, 2014, Int. J. Mod. Phys. D24, 1542014 (2015), arXiv:1504.08209 [gr-qc].
[10] D. D. Doneva and S. S. Yazadjiev, Phys. Rev. Lett. 120, 131103 (2018), arXiv:1711.01187 [gr-qc].
[11] H. O. Silva, J. Sakstein, L. Gualtieri, T. P. Sotiriou, and E. Berti, Phys. Rev. Lett. 120, 131104 (2018), arXiv:1711.02080 [gr-qc].
[12] G. Antoniou, A. Bakopoulos, and P. Kanti, Phys. Rev. Lett. 120, 131102 (2018), arXiv:1711.03390 [hep-th].
[13] C. A. R. Herdeiro, E. Radu, N. Sanchis-Gual, and J. A. Font, (2018), arXiv:1806.05190 [gr-qc].
[14] P. V. P. Cunha, C. A. R. Herdeiro, E. Radu, and H. F. Runarsson, Phys. Rev. Lett. 115, 211102 (2015), arXiv:1509.00021 [gr-qc].
[15] M. B. Green, J. H. Schwartz, and E. Witten, SUPERSTRING THEORY. Vol. 1: INTRODUCTION, Cambridge Monographs on Mathematical Physics (1988).
[16] T. Appelquist, A. Chodos, and P. G. O. Freund, eds., MODERN KALUZA-KLEIN THEORIES (1987).
[17] P. K. Townsend, (1997), arXiv:gr-qc/9707012 [gr-qc].
[18] D. Garfinkle, G. T. Horowitz, and A. Strominger, Phys. Rev. D43, 3140 (1991), [Erratum: Phys. Rev.D45,3888(1992)].
[19] Here and below we use units with $M = 1$.
[20] D. Rasheed, Nucl. Phys. B454, 379 (1995), arXiv:hep-th/9505038 [hep-th].
[21] T. Matos and C. Mora, Class. Quant. Grav. 14, 2331 (1997), arXiv:hep-th/9610013 [hep-th].
[22] F. Larsen, Nucl. Phys. B575, 211 (2000), arXiv:hep-th/9909102 [hep-th].
[23] B. Kleihaus, J. Kunz, and F. Navarro-Lerida, Phys. Rev. D69, 081501 (2004), arXiv:gr-qc/0309082 [gr-qc].
[24] The explicit form of the near horizon extremal solution can be found in Ref. [42].
[25] T. Delsate, C. Herdeiro, and E. Radu, (2018), arXiv:1806.06700 [gr-qc].
[26] R. Jackiw and S. Y. Pi, Phys. Rev. D68, 104012 (2003), arXiv:gr-qc/0308071 [gr-qc].
[27] S. Alexander and N. Yunes, Phys. Rept. 480, 1 (2009), arXiv:0907.2562 [hep-th].
[28] N. Yunes and F. Pretorius, Phys. Rev. D79, 084043 (2009), arXiv:0902.4669 [gr-qc].
[29] K. Yagi, N. Yunes, and T. Tanaka, Phys. Rev. D86, 044037 (2012), [Erratum: Phys. Rev.D89,049902(2014)], arXiv:1206.6130 [gr-qc].
[30] L. C. Stein, Phys. Rev. D90, 044061 (2014), arXiv:1407.2350 [gr-qc].
[31] L. Amarilla and E. F. Eiroa, Phys. Rev. D87, 044057 (2013), arXiv:1301.0532 [gr-qc].
[32] H. Falcke, F. Melia, and E. Agol, Astrophys.J. 528, L13 (2000), arXiv:astro-ph/9912263 [astro-ph].
[33] P. V. P. Cunha, C. A. R. Herdeiro, and E. Radu, Phys. Rev. D96, 024039 (2017), arXiv:1705.05461 [gr-qc].
[34] A. E. Broderick, T. Johannsen, A. Loeb, and D. Psaltis (Perimeter Institute for Theoretical Physics), Astrophys.J. 784, 7 (2014), arXiv:1311.5564 [astro-ph.HE].
[35] C. Goddi et al., Proceedings, 14th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories (MG14) (In 4 Volumes): Rome, Italy, July 12-18, 2015, Int. J. Mod. Phys. D26, 1730001 (2016), [1,863(2017)], arXiv:1606.08879 [astro-ph.HE].
[36] A. N. Aliev and G. D. Esmer, Phys. Rev. D87, 084022 (2013), arXiv:1303.1705 [gr-qc].
[37] N. M. J. Woodhouse, Communications in Mathematical Physics 44, 9 (1975).
[38] E. Teo, General Relativity and Gravitation 35, 1909 (2003).
[39] J. M. Bardeen, Timelike and null geodesics in the Kerr metric (C. Witt and B. Witt, editors, Black Holes, pp.215, 1973).
[40] P. V. P. Cunha, C. A. R. Herdeiro, and M. J. Rodríguez, Phys. Rev. D97, 084020 (2018), arXiv:1802.02675 [gr-qc].
[41] C. A. Herdeiro and E. Radu, Phys. Rev. Lett. 117, 221102 (2016), arXiv:1606.02302 [gr-qc].
[42] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen, and S. P. Trivedi, JHEP 10, 058 (2006), arXiv:hep-th/0606244 [hep-th].