Path Integral Approach to Noncommutative Spacetimes

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Abstract

We propose a path integral formulation of noncommutative generalizations of spacetime manifold in even dimensions, characterized by a length scale $\lambda_P$. The commutative case is obtained in the limit $\lambda_P = 0$.

PACS numbers: 04.60.Gw, 04.60.-m, 02.40-k.

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1 Introduction

It is commonly believed that the picture of spacetime as a smooth pseudo–Riemannian manifold should breakdown at most at the Planck scale $\lambda_P \sim 10^{-33}$ cm, due to the effects of quantum gravity. Many arguments on operational limits on position and time measurements have been considered in literature [1] - [4], suggesting that our description of spacetime as a collection of points equipped with suitable topological and metric structures, should be modified in a way which is reminiscent of the quantization of phase space in ordinary quantum mechanics. In this case too, in fact, the classical phase space, seen as the joint spectrum of commuting position and momentum operators, turns into its fuzzy quantum version, where the notion of points is replaced, in a somehow intuitive way, by fundamental cells of area $\hbar$. This quantization procedure can be formalized by adopting a dual point of view, namely by considering the algebra of smooth functions over the phase space, generated by positions and momenta. This algebra contains, using Gelf’and–Naimark reconstruction theorem, all informations on the underlying space. Switching on the Planck constant then amounts to consider the noncommutative algebra still generated by positions and momenta, considered now as non commuting elements and with usual commutation relations.

Quantum phase space is perhaps the most famous example of what is now usually called Noncommutative Geometry. While at the classical level the phase space can be described either geometrically, i.e. defining its structure as a topological manifold, or algebraically, its noncommutative quantum version can only be defined via the algebra of observables generated by positions and momenta, since the notion of ordinary manifold is lost. The algebraic description of noncommutative geometries has recently received new insights in particular due to Connes [5].

In this paper we will assume, as first considered long ago [6], and as suggested by the previous considerations, that the effect of gravity at very short distances is such to render positions and time $X^\mu$ non commutative

$$[X^\mu, X^\nu] = i\lambda_P^2 Q^{\mu\nu}$$

where the antisymmetric tensor $Q^{\mu\nu}$, depending in general on $X^\mu$, should be eventually given by a satisfactory theory of quantum gravity. The spacetime is described by the *-algebra $\mathcal{A}$ generated by regular representations of Eq.(1), whereas the notion of points is now lost. Here and in the following we will use upper case for the elements of $\mathcal{A}$ and lower one for their commutative limit, so for example, $F(X)$ is an element of $\mathcal{A}$ and $f(x)$ is instead an ordinary smooth function over the commutative spacetime manifold. Many ansatzs for Eq.(1) have been considered, in different contexts, [3], [4], [7], [8]. They predict uncertainty relations among position and time operators which in turn imply that quantum field theories built upon these generalized spacetimes have a more regular behaviour at small distances, and, in particular, may be free of ultraviolet divergences. See for example Refs. [3], [8] - [10].

The key observation of the present paper is that the analogy with ordinary quantum mechanics and quantization of phase space can also be pursued to establish a
path integral formulation of these noncommutative geometries, taking the point of view that a class of linear functionals over $A$, which turn into evaluation maps in the commutative limit, can be expressed as integrals, with a suitable measure, of ordinary functions over the classical, commutative spacetime manifold. It is well known in fact that quantization of phase space can be introduced either in the canonical approach, namely by defining a representation for positions and momenta as self-adjoint operators on a separable Hilbert space, or via Feynman sum over paths in classical phase space. In this case, to evaluate matrix elements of time ordered polynomials in position and momenta one should average over all trajectories with fixed boundary conditions, $q(t_0) = q_0$ and $q(t_f) = q_f$, with measure $\exp(iS/\hbar)$. For one degree of freedom

$$< q_f, t_f | T[Q(t_1)P(t_2)\ldots] | q_0, t_0 > = \int D[q] D[p] e^{iS/\hbar} (q(t_1)p(t_2)\ldots)$$

where $T$ denotes time ordering and the action $S$ is the integral over the considered path of the one-form $pdq - H(p,q)dt$. Equation (2) defines a set of linear functionals over the algebra generated by $Q$ and $P$ in the Heisenberg representation. In particular for $q_0 = q_f$, of the two terms in $S$, the first one measures for each closed trajectory $\gamma$ in the phase space the area of the surface $\Gamma$ with boundary $\gamma$ normalized to $\hbar$, i.e. the surface integral of the symplectic form $dp \wedge dq$ over $\Gamma$. Actually, the picture of quantum phase space as composed by fundamental cells of area $\hbar$ is already emerging from this term, independently of which particular dynamical system we are considering, defined by a particular choice for the Hamilton function $H(p,q)$.

These considerations suggest that a noncommutative structure of spacetime could be introduced starting with the classical commutative manifold equipped with a symplectic form $\Omega$. A path integral formalism similar to Eq.(2) may then provide a way of evaluating a class of linear functionals over polynomials of the deformed noncommutative position and time operators $X^\mu$. Of course this will restrict our analysis to the case of even dimensional spacetime manifolds $M^{2n}$, in order that a nondegenerate, closed two-form $\Omega$ exists. This construction requires the introduction of a parameter with dimension of length, $\lambda_P$, which plays a role analogous of $\hbar$ in ordinary quantum mechanics. In particular in the limit $\lambda_P \to 0$ all functionals we will consider reduce to evaluation maps of the commutative algebra of smooth functions over $M^{2n}$. For finite $\lambda_P$ instead, position and time operators $X^\mu$ satisfy non trivial commutation relations of the form of Eq.(1) with $Q^{\mu\nu}$ related in a simple way to $\Omega$.

## 2 Path Integral over spacetimes

We start considering the pair $(R^{2n}, \Omega)$, with $\Omega$ a symplectic form and the following generating functional

$$Z(x_0, J) = N \int D[\gamma] \exp \left[ i\lambda_P^2 (\Sigma[x_0, \gamma] + (x, J)) \right]$$

where $T$ denotes time ordering and the action $S$ is the integral over the considered path of the one-form $pdq - H(p,q)dt$. Equation (2) defines a set of linear functionals over the algebra generated by $Q$ and $P$ in the Heisenberg representation. In particular for $q_0 = q_f$, of the two terms in $S$, the first one measures for each closed trajectory $\gamma$ in the phase space the area of the surface $\Gamma$ with boundary $\gamma$ normalized to $\hbar$, i.e. the surface integral of the symplectic form $dp \wedge dq$ over $\Gamma$. Actually, the picture of quantum phase space as composed by fundamental cells of area $\hbar$ is already emerging from this term, independently of which particular dynamical system we are considering, defined by a particular choice for the Hamilton function $H(p,q)$.

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## 2 Path Integral over spacetimes

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$$Z(x_0, J) = N \int D[\gamma] \exp \left[ i\lambda_P^2 (\Sigma[x_0, \gamma] + (x, J)) \right]$$

(3)
where $N$ is a normalization constant, $J$ an arbitrary source, the integration is carried over all closed curves $\gamma$ in $R^{2n}$ with base point $x_0 \in R^{2n}$ and we define

$$\Sigma[x_0, \gamma] = \int_\Gamma \Omega_{\mu\nu}(x) dx^\mu \wedge dx^\nu$$

where we have introduced a parameterization, $\gamma: \tau \mapsto x^{\mu}(\tau)$ and $\Gamma$ is any two-dimensional surface with boundary $\gamma$. Finally the antisymmetric matrix $\Omega_{\mu\nu}(x)$ is the representation of the symplectic form in the coordinate basis $x^{\mu}$ and, in general, explicitly depends on $x^{\mu}$. Here we will assume for simplicity that $\Omega_{\mu\nu}(x)$ is a slowly varying function of $x$ on the scale $\lambda_P$, and we consider only zero order terms in Taylor expansion around the base point $x_0$ in integrals (4), $\Omega_{\mu\nu}(x) \sim \Omega_{\mu\nu}(x_0)$. This assumption simplifies the explicit evaluation of $Z(x_0, J)$, which is our main aim in this paper, but is not necessary. The general case will be considered elsewhere. In this case $\Sigma[x_0, \gamma]$ can therefore be written as

$$\Sigma[x_0, \gamma] = \int_\gamma x^{\mu}(\tau) \frac{d}{d\tau} x^{\nu}(\tau) d\tau$$

$Z(x_0, J)$ is invariant under curve reparameterization, as is easy to verify, provided we redefine the arbitrary external source $J$. In the following we will consider $\tau \in [0, 1]$.

Equation (3) is clearly inspired by the expression of path integral in phase space with external sources. However it is worth pointing out that there is one main difference with ordinary path integral approach to quantum mechanics of Eq.(2), even for $q_0 = q_f$. In this case, in fact, the variables $q$ and $p$ are treated on different grounds, since integration is carried over all curves with boundary conditions on positions only, while momenta run over all possible values. In case of Eq.(3), instead, all curves $\gamma$ should have equal endpoints $x^{\mu}_0$ for all $x^{\mu}$'s, $\mu = 1, \ldots, 2n$.

The generating functional $Z(x_0, J)$ defines a one parameter family of (noncommutative) algebras $A(\tau)$ generated by $2n$ elements $X^{\mu}(\tau)$, implicitly defined via the introduction of a set of linear functionals $\rho_{x_0}$, as follows

$$\rho_{x_0}(P[X^{\mu}(\tau_1)\ldots X^{\nu}(\tau_k)]) \equiv NZ(x_0, 0)^{-1} \int D[\gamma] e^{i\Sigma[x_0, \gamma]/\lambda_P^2} X^{\mu}(\tau_1)\ldots x^{\nu}(\tau_k)$$

$$= (-i\lambda_P^2)^k Z(x_0, 0)^{-1} \frac{\delta}{\delta J^{\mu}(\tau_1)} \ldots \frac{\delta}{\delta J^{\nu}(\tau_k)} Z(x_0, J) \bigg|_{J=0}$$

where $P$ stands for path ordering, defined analogously to time ordering in quantum mechanics. Once defined on all polynomials in the $X^{\mu}$, Eq.(7) can be applied to the entire algebra of continuous functions $F(X)$ in the weak topology.

To better understand Eq.(7) we consider the classical limit $\lambda_P \to 0$. The integral over curves $\gamma$ is then dominated by the contribution at stationary points satisfying
$$\delta \Sigma[x_0, \gamma] = 0$$, i.e. \( \Omega_{\mu\nu}(x_0)dx^\nu/d\tau = 0 \). Since \( \Omega \) is not degenerate, the leading contribution satisfying \( x^\mu(0) = x^\mu(1) = x^\mu_0 \), is therefore given by the curve \( x^\mu(\tau) = x^\mu_0 \), \( \forall \tau \), so that we get

$$\rho_{x_0}(P[X^\mu(\tau_1)\ldots X^\nu(\tau_k)]) \rightarrow x^\mu \ldots x^\nu|x_0 + \mathcal{O}(\lambda_P^2)$$

(8)

Thus, the maps \( \rho_{x_0} \) reduce to the evaluation maps of the commutative algebra of smooth functions over \( R^{2n} \), \( \rho_{x_0} : f(x) \mapsto f(x_0) \), independently of \( \tau \), and thus completely reconstruct \( R^{2n} \) as a topological manifold. For finite \( \lambda_P \), instead, all closed curves whose projection on the \( x^\mu - x^\nu \) plane enclose surfaces with area of the order \( \Omega_{\mu\nu}(x_0)^{-1}\lambda_P^2 \) give a relevant contribution to r.h.s. of Eq.(7), and evaluation maps are smeared over cells in \( R^{2n} \) with projection on the \( x^\mu - x^\nu \) plane of this order of magnitude. It is worth mentioning that stationary phase method requires Hessian to be non degenerate at stationary points. As for the case of wave packets of zero mass particles, this aspect deserves a careful analysis in the case under consideration, since the self-adjoint extension of the operator \( \Omega_{\mu\nu}(x_0)d/d\tau \) admits zero modes. We will deal with this problem by using in the following a customary \( i\epsilon \) prescription which renders the above operator invertible.

We start evaluating \( Z(x_0, J) \) for the case \( n = 1 \). Use of Darboux theorem allows for a straightforward generalization to arbitrary \( n \). In two dimensions we write \( \Omega_{\mu\nu}(x_0) \) as \( \omega(x_0)\epsilon_{\mu\nu} \), with \( \omega(x_0) \) a non vanishing function on \( R^2 \). Furthermore all closed curves \( x^\mu(\tau) \) with endpoints at \( x_0^\mu \) can be written as \( x^\mu(\tau) = x_0^\mu + y^\mu(\tau) \), where \( y^\mu(0) = y^\mu(1) = 0 \). Substituting in (3) we get

$$Z(x_0, J) = N \int D[y] \exp \left[ i\lambda_P^{-2} \left( \omega(x_0) \int_0^1 y^\mu(\tau)\epsilon_{\mu\nu} \frac{d}{d\tau} y^\nu(\tau) d\tau + (x_0, J) + (y, J) \right) \right]$$

(9)

The operator \( \epsilon_{\mu\nu}d/d\tau \) is self-adjoint with respect to the scalar product (3) on the domain of periodic two component functions \( y^\mu(\tau) \). Its spectrum is given by \( \mu_m = 2m\pi, m \in Z \). To each eigenvalue correspond the two eigenfunctions, normalized to \( \lambda_P \)

$$y^{(+)}_m(\tau) = \lambda_P \left( \begin{array}{c} \cos(\mu_m \tau) \\ \sin(\mu_m \tau) \end{array} \right) \quad , \quad y^{(-)}_m(\tau) = \lambda_P \left( \begin{array}{c} \sin(\mu_m \tau) \\ -\cos(\mu_m \tau) \end{array} \right)$$

(10)

Expanding the functions \( y^\mu(\tau) \) in the basis (10), \( Z(x_0, J) \) becomes an usual pseudo-gaussian integral over the Fourier components \( c_m^{(\pm)} = \lambda_P^{-1}(y_m^{\pm}, y) \). As mentioned before, we will introduce a \( i\epsilon \) prescription as a regularization of the singularity due to the presence of a zero mode, namely shifting in the complex plane \( \mu_m \rightarrow \mu_m + i\epsilon \). In this way we get, integrating over \( c_m^{(\pm)} \)

$$Z(x_0, J) = N' \exp \left[ i\lambda_P^{-2} \int_0^1 J_\mu(\tau) d\tau - (4\epsilon\lambda_P^2)^{-1} \delta^{\mu\nu} \int_0^1 J_\mu(\tau) d\tau \int_0^1 J_\nu(\tau') d\tau' \right] \\
- i \sum_{m \neq 0} (4\mu_m \omega(x_0) + i\epsilon)^{-1} \left( (J^{(+)}_m)^2 + (J^{(-)}_m)^2 \right)$$

(11)
where \( J_m^{(\pm)} = \lambda_p^{-1}(x_m^{(\pm)}, J) \) are the Fourier components of \( J_\mu(\tau) \) and notice that the zero mode does not contribute to the sum, so we can safely neglect the \( i\epsilon \) term in this case. Using (10) and performing the sum over \( m \) we finally obtain

\[
Z(x_0, J) = N' \exp \left[ i\lambda_p^{-2} x_0^{\mu} \int_0^1 J_\mu(\tau) d\tau - (4\epsilon\lambda_p^2)^{-1} \delta^{\mu\nu} \int_0^1 J_\mu(\tau) d\tau \int_0^1 J_\nu(\tau') d\tau' + i\lambda_p^{-2} \int_0^1 d\tau \int_0^1 d\tau' \Omega^{-1\mu\nu}(x_0) J_\mu(\tau) J_\nu(\tau') \Delta(\tau - \tau') \right]
\]

(12)

where the kernel \( \Delta(t) \) is given by

\[
8 \Delta(t) = \begin{cases} 
2t - 1 & t \in [0, 1] \\
2t + 1 & t \in [-1, 0]
\end{cases}
\]

(13)

which is discontinuous at \( t = 0 \). We can now proceed to explicitly evaluate the action of the maps \( \rho_{x_0} \) on path ordered polynomials of \( X^\mu \). Up to second order polynomials we get

\[
\rho_{x_0}(X^\mu(\tau)) = x_0^{\mu} \quad \rho_{x_0}(P[X^\mu(\tau)X^\nu(\tau')]) = x_0^{\mu}x_0^{\nu} + \frac{\lambda_p^2}{2\epsilon} \delta^{\mu\nu} - 2i\lambda_p^2 \Omega^{-1\mu\nu}(x_0) \Delta(\tau - \tau')
\]

(14)

Actually, since \( Z(x_0, J) \) only contains linear and quadratic terms in \( J \), all other higher order polynomials can be evaluated, up to combinatorial factors, in terms of these expressions.

We see from (14) that \( \rho_{x_0} \) simply associates to \( X^\mu(\tau) \) the value \( x_0^{\mu} \), independently of \( \tau \). In fact the only dependence on the \( \tau \) variable is contained in the function \( \Delta(\tau - \tau') \), appearing for path ordered products of \( X^1(\tau)X^2(\tau') \). As long as arbitrary polynomials \( X^\mu(\tau_1)...X^\mu(\tau_k) \) in only one of the \( X^\mu \) operators are considered, the dependence on the parameters \( \tau_i \) drops out. We also notice that we get for the uncertainty functional

\[
\rho_{x_0}(X^\mu(\tau)X^\mu(\tau')) - \rho_{x_0}(X^\mu(\tau))\rho_{x_0}(X^\mu(\tau')) = \frac{\lambda_p^2}{2\epsilon} \quad \forall \tau, \tau'
\]

(15)

These results have a customary interpretation by looking at \( \rho_{x_0} \) as corresponding to states \( |x_0> \) in a Hilbert space with respect to which \( X^\mu(\tau) \) have mean value \( x_0^{\mu} \) and squared uncertainty as in (15). For the equal \( \tau \) commutator we find

\[
\rho_{x_0}([X^\mu, X^\nu](\tau)) = \lim_{\delta \to 0^+} \left( \rho_{x_0}(X^\mu(\tau + \delta)X^\nu(\tau)) - \rho_{x_0}(X^\nu(\tau + \delta)X^\mu(\tau')) \right) = \frac{i}{2} \lambda_p^2 \Omega^{-1\mu\nu}(x_0), \forall \tau
\]

(16)

As expected the commutator is proportional to the inverse symplectic form \( \Omega^{-1} \) and using notation of Eq.(10), we see that \( X^\mu(\tau) \) generates a noncommutative algebra with, for any \( \tau, \rho_{20}(Q^\mu(\tau)) = \Omega^{-1\mu\nu}(x_0)/2 \).
The uncertainties (13) diverge if one let the regularizing parameter \( \epsilon \) to go to zero. This means that in this limit the states \( |x_0> \) are no more in the domain of \( X^\mu \). However the \( 1/\epsilon \) singularity can be removed if one define a renormalized set of linear functionals \( \tilde{\rho}_{x_0} \) which satisfy the first of Eqs. (14), Eq. (16), and have minimal squared uncertainties compatible with Eq. (16), and equals for both \( X^1(\tau) \) and \( X^2(\tau) \), namely \\
\[ \lambda_\mu^2(4\omega(x_0))^{-1} \] \\
Equation (14) then becomes \\
\[ \tilde{\rho}_{x_0}(X^\mu(\tau)) = x_0^\mu , \tilde{\rho}_{x_0}(P[X^\mu(\tau)X^\nu(\tau')]) = x_0^\mu x_0^\nu + \frac{\lambda_\mu^2}{4\omega(x_0)} \delta^{\mu\nu} - 2i\lambda_\mu^2 \Omega^{-1\mu\nu}(x_0) \Delta(\tau-\tau') \] \\
(17)

Accordingly, the corresponding generating functional \( \tilde{Z}(x_0, J) \) is obtained from Eq. (12) by substituting \( \epsilon \) with \( 2\omega(x_0) \). The states corresponding to this new set of linear functionals would be the analogous of coherent states for ordinary quantum mechanics. They would also correspond to the states of maximal localization introduced in Ref. [8]. Notice also that \( \tilde{\rho}_{x_0} \) give finite results when applied to arbitrary higher order polynomials in \( X^\mu(\tau) \), which, as already stressed, can all be expressed in terms of (17).

The above results can be easily generalized to the \( 2n \) dimensional case. We start again with Eqs. (13)-(14), written now on \( R^{2n} \). As before we will consider in Eq. (14) only zero terms in Taylor expansion of \( \Omega(x) \). By Darboux theorem it is always possible reduce \( \Omega(x_0) \), for any \( x_0 \), to the canonical form

\[ \omega = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \] \\
(18)

with a suitable coordinate transformation \( z^\mu = (A^{-1})^\mu_\nu(x_0) y^\nu \). In these new coordinates and within the approximation \( \Omega(x) \sim \Omega(x_0) \), \( \Sigma[x_0, \gamma] \) is simply the sum of \( n \) bidimensional integrals like the first term in Eq. (13) and \( Z(x_0, J) \) factorizes into the product of \( n \) bidimensional generating functionals depending on the sources \( J'_\mu = (A^T)^\nu_\mu(x_0) J_\nu \), with \( A^T \) the transpose of \( A \). Carrying out the pseudogaussian integrals and transforming back the result in terms of \( J_\mu \) we therefore obtain

\[ Z(x_0, J) = N' \exp \left[ i\lambda_P^{-2} x_0^\mu \int_0^1 J_\mu(\tau) d\tau - (4\epsilon\lambda_P^2)^{-1}(AA^T)^{\mu\nu}(x_0) \int_0^1 J_\mu(\tau) d\tau \cdot \int_0^1 J_\nu(\tau') d\tau' + i\lambda_P^{-2} \int_0^1 d\tau \int_0^1 d\tau' \Omega^{-1\mu\nu}(x_0) J_\mu(\tau) J_\nu(\tau') \Delta(\tau-\tau') \right] \] \\
(19)

which gives

\[ \rho_{x_0}(X^\mu(\tau)) = x_0^\mu \] \\
(20)

\[ \rho_{x_0}(P[X^\mu(\tau)X^\nu(\tau')]) = x_0^\mu x_0^\nu + \frac{\lambda_P^2}{2\epsilon} (AA^T)^{\mu\nu}(x_0) - 2i\lambda_P^2 \Omega^{-1\mu\nu}(x_0) \Delta(\tau-\tau') \] \\
(21)

and for the equal \( \tau \) commutator again Eq. (16).
It is worth noticing that a constant $\Omega$ in four dimensions corresponds to the case considered in Ref. [3], where the following algebra has been studied

\[
[X^\mu, X^\nu] = i\lambda^2 Q^{\mu\nu}, \quad [X^\mu, Q^{\rho\sigma}] = 0,
\]
\[Q_{\mu\nu}Q^{\mu\nu} = 0, \quad \left(\frac{1}{8}e_{\mu\nu\rho\sigma}Q^{\mu\nu}Q^{\rho\sigma}\right)^2 = I \quad (22)
\]

Actually the condition $\rho^{x_0}(\langle X^\mu, Q^{\mu\rho}\rangle(\tau)) = 0, \forall \tau$ can be easily verified using Eq.(19), while the constraint $\rho^{x_0}(e_{\mu\nu\rho\sigma}Q^{\mu\nu}Q^{\rho\sigma}(\tau)) = \pm 8$ is equivalent, up to a suitable normalization, to require $\Omega$ to be non degenerate. The procedure of describing a non-commutative geometry with a path integral approach outlined in this paper cover however more general cases, since the commutator may depend on $x_0$, as clear from Eqs.(16) and (19). Moreover, if one does not assume that $\Omega(x)$ is a slowly varying function of $x$ and that for each $x_0$ one can approximate $\Omega(x) \sim \Omega(x_0)$ in the phase $\Sigma[x_0, \gamma]$, one may consider, at least in principle, a completely general case, though the corresponding generating functional could be difficult to handle. Nevertheless, even if $Z(x_0, J)$ is not exactly solvable for more involved choices of $\Omega$, applications of perturbation techniques, as in ordinary path integral in quantum mechanics, may provide some information on the underlying noncommutative spacetime.

3 Path Integral over spacetimes with non-trivial topologies

In the previous section we have considered the path integral construction of noncommutative generalizations of spacetimes for the case of $R^{2n}$ as the underlying classical manifold. When the method outlined there is applied to manifolds $M^{2n}$ with non trivial topologies, in particular when the first and/or second homology groups are non trivial, there are however two aspects which one should take into account. First of all, if $H^1(M^{2n}) \neq \{0\}$, there are closed curves contributing to the sum over paths which are not boundaries of any surface, so the surface integral of the assigned symplectic form is not defined for these paths. In these cases the introduction of a one-form $A$, such that locally $\Omega = dA$ is a better starting point. Moreover, if $H^2(M^{2n}) \neq \{0\}$, the integral $\Sigma[x_0, \gamma]$ for any $\gamma$ is in general a multivalued function, since the integral of $\Omega$ over two surfaces $\Gamma$ and $\Gamma'$, both with boundary $\gamma$, may be different if $\Gamma - \Gamma'$ is a non trivial 2-cycle (of course this problem is absent if $\Omega$ is exact). The ambiguity in the choice of the surface, however, can be removed by requiring quantization conditions on the integral of $\Omega$ over a set of generators of $H^2(M^{2n})$ such that $\exp(i\Sigma[x_0, \gamma]/\lambda^2)$ is single valued, analogous to the one introduced to quantize the motion of a charged particle in the field of a magnetic charge. We will illustrate these points in two bidimensional simple examples, the sphere $S^2$ and the torus $S^1 \times S^1$, and discuss the generalization to general cases.
Let us consider the pair \((S^2, \Omega)\). Any closed curve \(\gamma\) will divide \(S^2\) into two surfaces, and therefore the value of \(\Sigma[x_0, \gamma]\) of Eq.(4) will be defined only up to the integral of \(\Omega\) over \(S^2\). The measure in the path integral will be nevertheless single valued if the latter is a multiple of \(2\pi\), in unit \(\lambda_P^2\)

\[
\int_{S^2} \Omega = 2n\pi \lambda_P^2 \tag{23}
\]

A path integral over \(S^2\) was already considered in \([11]\), as a way to quantize a classical particle with spin.

Incidentally, we observe that condition (23) could have an intriguing relationship with the idea that black hole horizon area is quantized and its spectrum is uniformly spaced \([12]\). In our approach this quantization, in unit of \(\lambda_P^2\), would be intimately related to the fact that positions and time are noncommuting operators and their commutator is of the form reported in Eq.(1). Actually black hole physics is one of the best scenarios where the noncommutative nature of spacetime at small scales, otherwise unobservable, could be felt well above the Planck length.

The ambiguity in the definition of the measure in path integral and the necessity of a quantization condition (23) is due to the fact that the second homology group of \(S^2\) is non trivial. A similar ambiguity in defining the action \(\Sigma[x_0, \gamma]\) corresponding to a path \(\gamma\) on a manifold \(M^{2n}\), and therefore the generating functional \([8]\), will arise whenever \(H^2(M^{2n}) \neq \{0\}\). In particular, if its 2-Betti number is equal to \(b_2(M^{2n}) = k\), and denoting with \(\Gamma_i, i = 1, \ldots, k\), a set of generators of \(H^2(M^{2n})\), the measure \(\exp(i\Sigma[x_0, \gamma]\lambda_P^2)\) will be single valued if \(\Omega\) satisfies the following conditions

\[
\int_{\Gamma_i} \Omega = 2n_i\pi \lambda_P^2 \quad , \quad i = 1, \ldots, k \tag{24}
\]

which of course do not depend on the choice for the elements of the basis \(\Gamma_i\).

As a second example of a non trivial topological manifold, we consider the bidimensional torus \(T^2 = S^1 \times S^1\). In this case there are closed curves which are not boundaries of any surface and therefore the construction outlined in the previous section cannot be applied. Nevertheless it is still possible to construct a path integral over \(T^2\) and the generating functional \(Z(x_0, J)\) starting with a one–form \(A\) such that (locally) \(\Omega = dA\). Let us denote with \(\gamma_i(x_0)\), \(i = 1, 2\) two generators of the \(T^2\) 1-homology group passing through the point \(x_0\) and define

\[
I_i(x_0) = \lambda_P^{-2} \int_{\gamma_i(x_0)} A \tag{25}
\]

The path integral measure we will introduce soon will be single valued only if the values of \(I_i(x_0)\) only differ for a factor \(2n\pi\) when the curve \(\gamma_i\) is continuously moved around the manifold and go back to the original path after a complete cycle. For example, in the angular coordinates \(\theta\) and \(\phi\) on \(T^2\), taking \(A = r_1r_2(\theta d\phi - \phi d\theta)\), with \(r_1, r_2\) the two radii, and choosing \(\gamma_1(\theta_0, \phi_0) : \{\theta = \theta_0\}\) and \(\gamma_2(\theta_0, \phi_0) : \{\phi = \phi_0\}\), the
values of $I_1(x_0)$ for the identified points $\theta_0$ and $\theta_0 + 2\pi$, and analogously for $I_2(x_0)$, will be the same up to a factor $2n\pi$ provided that

$$\int_{T^2} dA = 4\pi^2 r_1 r_2 = 2n\pi \lambda_P^2$$

(26)

This condition also ensures, as we have seen for the $S^2$ case, that for all curves which are instead boundaries of surfaces the corresponding contribution to the generating functional is single valued. Defining

$$\Sigma[x_0, \gamma] = \int_{\gamma} A$$

(27)

the contribution to $Z(x_0, J)$ of a generic path $\gamma$ on $T^2$ can be written as

$$\exp \left[ i\lambda_P^2 \Sigma[x_0, \gamma] \right] = \exp \left[ in_1 I_1(x_0) + in_2 I_2(x_0) + i\lambda_P^{-2} \int_{\gamma'} A \right]$$

(28)

for some integers $n_1$ and $n_2$ and with $\gamma'$ a boundary of some bidimensional surface, so the generating functional takes the form

$$Z(x_0, J) = N \sum_{n_1, n_2} \int D[\gamma] \exp (in_1 I_1(x_0) + in_2 I_2(x_0)) \exp \left( i\lambda_P^{-2} \int_{\Gamma} dA \right) \exp \left( i\lambda_P^{-2} (x, J) \right)$$

(29)

where the integration is over all 1–boundaries $\gamma'$ with base point $x_0$ and $\partial\Gamma = \gamma'$.

This expression can be now easily generalized to an arbitrary manifold $M^{2n}$ with Betti numbers $b_1 = p$, $b_2 = q$. Consider the pair $(M^{2n}, A)$, a set of basis $\gamma_i(x_0)$ and a base $\Gamma_j$, for $H^1(M^{2n})$ and $H^2(M^{2n})$ respectively. The fact that the path measure should be single valued enforces $q$ integral conditions on $\Omega = dA$

$$\int_{\Gamma_j} \Omega = 2m_j \pi \lambda_P^2, \quad j = 1, \ldots, q$$

(30)

and defining, as in (25)

$$I_i(x_0) = \lambda_P^{-2} \int_{\gamma_i(x_0)} A, \quad i = 1, \ldots, p$$

(31)

we may introduce the generating functional as follows

$$Z(x_0, J) = N \prod_{i=1}^{p} \sum_{n_i} \int D[\gamma'] \exp (i n_i I_i(x_0)) \exp \left( i\lambda_P^{-2} \int_{\Gamma} \Omega \right) \exp \left( i\lambda_P^{-2} (x, J) \right)$$

(32)

where $\partial\Gamma = \gamma'$. Equation (32) represents the generalization of Eq.(3) to a generic topological manifold $M^{2n}$. 

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4 Conclusions and outlook

In this paper we have described a path integral approach to noncommutative geometries. In particular, using the analogy with ordinary quantum mechanics we have described a way of expressing a set of linear functionals over noncommutative generalizations, in even dimensions, of the algebra generated by position and time operators in terms of ordinary functions over the classical manifold, which represents its commutative classical geometry. In particular all these linear functionals reduce in the limit $\lambda_P = 0$ to evaluation maps over the commutative algebra of smooth functions over the manifold.

There are several points related to this approach which deserve further studies.

We have considered in more details the simple case of a classical starting manifold with trivial first and second homology groups. It would be interesting to perform a deeper study of the properties of the generating functional over manifolds with more involved topological properties, which has been discussed in section 3, and of the corresponding deformed noncommutative geometries.

Another important issue is that this approach may give the possibility of introducing a differential calculus over noncommutative generalizations of the algebra of functions over classical spacetimes using all customary results for a commutative geometry. In particular, quite relevant is to look for a consistent definition of a metric structure, connections and curvature. Of course, when $\lambda_P$ is switched on, one expects that dramatic changes affect all these structures. Nevertheless, to have a simple way of relating the algebra generated by the $X^\mu$’s to its commutative counterpart, could be a powerful way to look for their generalization.

The role of the parameter $\tau$ should be better understood. While it has a clear meaning as the parameter of closed curves over $M^{2n}$, its role for the noncommutative algebras $A(\tau)$ is not transparent. Actually all dependence on $\tau$ is contained in the kernel $\Delta(\tau - \tau')$. One possibility is of course that each of these algebras, for fixed $\tau$, is already sufficient to describe a noncommutative spacetime, so that the role of the parameter $\tau$ in this case would be only auxiliary. On the other hand the analogy with quantum phase space we started with, suggests that it could also have an intrinsic non trivial meaning.

5 Acknowledgments

I am pleased to thank F. Lizzi, G. Miele and G. Sparano for discussions and comments.

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