A REMARK ON A PAPER OF P. B. DJAKOV AND M. S. RAMANUJAN

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ABSTRACT. Let \( \ell \) be a Banach sequence space with a monotone norm in which the canonical system \((e_n)\) is an unconditional basis. We show that if there exists a continuous linear unbounded operator between \( \ell \)-Köthe spaces, then there exists a continuous unbounded quasi-diagonal operator between them. Using this result, we study in terms of corresponding Köthe matrices when every continuous linear operator between \( \ell \)-Köthe spaces is bounded. As an application, we observe that the existence of an unbounded operator between \( \ell \)-Köthe spaces, under a splitting condition, causes the existence of a common basic subspace.

1. Introduction

Following [2], we denote by \( \ell \) a Banach sequence space in which the canonical system \((e_n)\) is an unconditional basis. The norm \( \| \cdot \| \) is called monotone if \( \| x \| \leq \| y \| \) whenever \( |x_n| \leq |y_n| \), \( x = (x_n) \), \( y = (y_n) \) \( \in \ell \), \( n \in \mathbb{N} \). Let \( \Lambda \) be the class of such spaces with monotone norm. In particular, \( l_p \in \Lambda \) and \( c_0 \in \Lambda \). It is known that every Banach space with an unconditional basis \((e_n)\) has a monotone norm which is equivalent to its original norm. Indeed, it is enough to put

\[
\| x \| = \sup_{|\beta_n| \leq 1} \left| \sum_n e_n'(x)\beta_n e_n \right|
\]

where \( \| \cdot \| \) denotes the original norm, \( (e_n') \) denote the sequence of coefficient functionals.

Let \( \ell \in \Lambda \) and \( \| \cdot \| \) be a monotone norm in \( \ell \). If \( A = (a_n^k) \) is a Köthe matrix, the \( \ell \)-Köthe space \( \lambda^\ell (A) \) is the space of all sequences of scalars \((x_n)\) such that \( (x_n a_n^k) \in \ell \) with the topology generated by the

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seminorms
\[ \| (x_n) \|_k = \| (x_n a_n^k) \| \]

For any linear operator \( T : X \rightarrow Y \) between Fréchet spaces we consider the following operator seminorms
\[ \| T \|_{p,q} = \sup \left\{ \| Tx \|_p : \| x \|_q \leq 1 \right\}, \quad p, q \in \mathbb{N} \]

which may take the value +\( \infty \). In particular, for any one dimensional operator \( T = u \otimes x \), we have
\[ \| T \|_{p,q} = \| u \|_q^* \| x \|_p \]

The operator \( T \) is continuous if and only if for all \( k \) there is \( N(k) \) such that
\[ \| T \|_{k,N(k)} < \infty, \]
\( T \) is bounded if and only if there is \( N \in \mathbb{N} \) such that for all \( r \in \mathbb{N} \),
\[ \| T \|_{r,N} < \infty. \]

We write \( (X, Y) \in B \) if every continuous linear operator on \( X \) to \( Y \) is bounded. Zahariuta [7] obtained that if the matrices \( A \) and \( B \) satisfy the conditions \( d_2 \) and \( d_1 \), respectively, then \( (\lambda^1(A), \lambda^1(B)) \in B \). This phenomenon was studied extensively by Vogt [6] not only for Köthe spaces but also for the general case of Fréchet spaces. In case of \( \ell \)-Köthe spaces, there is no characterization of pairs \( (X, Y) \) with the property \( B \).

For Fréchet spaces \( X \) and \( Y \), in [6], Vogt proved that \( (X, Y) \in B \) if and only if for every sequence \( N(k) \), \( \exists N \in \mathbb{N} \) such that \( \forall r \in \mathbb{N} \) we have \( k_0 \in \mathbb{N} \) and \( C > 0 \) with
\[ \| T \|_{r,N} \leq C \max_{1 \leq k \leq k_0} \| T \|_{k,N(k)} \quad (1.1) \]
for all \( T \in \mathcal{L}(X, Y) \).

An operator \( T : \lambda^\ell(A) \rightarrow \lambda^\ell(B) \) is called quasi-diagonal if there exists \( k : \mathbb{N} \rightarrow \mathbb{N} \) and constants \( m_n \) such that
\[ T e_n = m_n \tilde{e}_{k(n)}, \quad n \in \mathbb{N} \]

Following [4], a pair of Köthe spaces \( (\lambda^\ell(B), \lambda^\ell(A)) \) satisfies the condition \( S \) if,
\[ \forall p \quad \exists q, k \quad \forall s, l \quad \exists r, C : \frac{b^s_m}{a^s_n} \leq C \max \left\{ \frac{b^r_l}{a^r_n}, \frac{b^r_m}{a^r_n} \right\} \quad (1.2) \]

In [3] it was proved that the existence of an unbounded continuous linear operator from nuclear \( l_1 \)-Köthe space to another implies the existence of a continuous unbounded quasi-diagonal operator. Also, if the both Köthe spaces are nuclear, in [5], Nurlu and Terzioğlu proved that
the existence of an unbounded continuous linear operator on \( \lambda^1(A) \) to \( \lambda^1(B) \) implies, under some conditions, the existence of a common basic subspaces of \( \lambda^1(A) \) and \( \lambda^1(B) \). Djakov and Ramanujan generalized these results by omitting nuclearity condition \([1]\).

Let \( X = \lambda^\ell(A) \) and \( Y = \lambda^\ell(B) \) be the \( \ell \)-\( \text{Köthe} \) spaces. Here, we modify Proposition 1 in \([1]\) for \( \ell \)-\( \text{Köthe} \) spaces and using it we obtain a necessary and sufficient condition in terms of corresponding Köthe matrices when \((X,Y) \in B\). Also we observe a common basic subspace between \( \ell \)-\( \text{Köthe} \) spaces \( X \) and \( Y \) when \((X,Y) \not\in B\) and \((Y,X) \in S\) following the same lines in \([1]\).

2. Bounded and unbounded operators in \( \ell \)-\( \text{Köthe} \) spaces

Let \( \lambda^\ell(A), \lambda^\ell(B) \) be \( \ell \)-\( \text{Köthe} \) spaces. As in \([1]\) we obtain the following.

**Proposition 2.1.** Let \( \lambda^\ell(A) \) and \( \lambda^\ell(B) \) be \( \ell \)-\( \text{Köthe} \) spaces. If there exists a continuous linear unbounded operator \( T \colon \lambda^\ell(A) \to \lambda^\ell(B) \), then there exists a continuous unbounded quasi-diagonal operator on \( \lambda^\ell(A) \) to \( \lambda^\ell(B) \).

**Proof.** Let \( T : \lambda^\ell(A) \to \lambda^\ell(B) \) be continuous and unbounded. We may assume without loss of generality that

\[
\|Tx\|_k \leq \frac{1}{2^k} \|x\|_k, \quad \forall x \in \lambda^\ell(A)
\]

\[
\sup_n \frac{\|Te_n\|_{k+1}}{\|e_n\|_k} = \infty, \quad k \in \mathbb{N}.
\]

Indeed, one may obtain these by using appropriate multipliers and passing to a subsequence of seminorms, if necessary. Let \((k_j)\) be a sequence of integers such that each \( k \in \mathbb{N} \) appears in it infinitely many times and choose an increasing subsequence \((n_j)\) such that

\[
\frac{\|Te_{n_j}\|_{k_j+1}}{\|e_{n_j}\|_{k_j}} \geq 2^j, \quad \forall j
\]
Let us remind that \( \| \tilde{e}_v \|_k = b^k_v \) and \( \| e_n \|_k = a^k_n \) and let \( T e_n = \sum_v \theta_{nv} \tilde{e}_v \).

Note that,

\[
\sup_{|\alpha_v| \leq 1} \left| \sum_v \theta_{nv} \alpha_v \left( \sup_k \frac{b^k_v}{a^k_n} \right) \tilde{e}_v \right| \leq \sum_k \left( \frac{b^k_v}{a^k_n} \right) \left( \sup_{|\alpha_v| \leq 1} \left| \sum_v \theta_{nv} \alpha_v \tilde{e}_v \right| \right) \leq \sum_k \frac{1}{a^k_n} \sup_{|\alpha_v| \leq 1} \left| \sum_v \theta_{nv} \alpha_v b^k_v \tilde{e}_v \right| \leq \sum_k \frac{1}{2^k} \leq 1
\]

Therefore we obtain that

\[\tag{2.1} \sup_{|\alpha_v| \leq 1} \left| \sum_v \theta_{nv} \alpha_v \left( \sup_k \frac{b^k_v}{a^k_n} \right) \tilde{e}_v \right| \leq \leq \sup_{|\alpha_v| \leq 1} \left| \sum_v \theta_{nv} \alpha_v b^k_v \tilde{e}_v \right| \leq \sum_k \frac{1}{2^k} \leq 1 \]

So there is a \( v_j \) such that

\[ t_j := \sup_k \frac{b^k_v}{a^k_n} \leq \frac{1}{2^j} \frac{b_j}{a_j} \]

Otherwise we obtain a contradiction to (2.1) by monotonicity of \( \| \cdot \| \).

Now, consider the quasi-diagonal operator \( D : \lambda^f(A) \rightarrow \lambda^f(B) \) defined by

\[ D e_n = t_j^{-1} e_{v_j}, \quad j \in \mathbb{N} \]

\[ D e_n = 0 \quad \text{if} \quad n \neq n_j \]

Let \( x = \sum_j x_n e_n \in \lambda^f(A) \). So, \( D x = \sum_j x_n t_j^{-1} e_{v_j} \). Since \( x_n t_j^{-1} b^k_v \leq x_n a^k_{n_j} \), by monotonicity we obtain that \( \left\| \left( x_n t_j^{-1} b^k_v \right) \right\| \leq \left\| \left( x_n a^k_{n_j} \right) \right\| \), i.e.,

\[ \| D x \|_k \leq \| x \|_k \quad \forall k \]

Hence, \( D \) is continuous.

Similarly, it is easy to see that \( D \) is unbounded since for a fixed \( k \), there is a subsequence \( (j_m) \) such that \( k_{j_m} = k \), \( m \in \mathbb{N} \) and

\[ \frac{\| D e_{n_{j_m}} \|_{k+1}}{\| e_{n_{j_m}} \|_k} \geq 2^{j_m} \rightarrow \infty \]

as \( m \rightarrow \infty \). This completes the proof. \( \square \)
Proposition 2.1 enables us to prove the sufficiency part of the following theorem. Notice that sufficiency can not be obtained directly for a general linear map.

**Theorem 2.2.** Let $\lambda^\ell(A)$ and $\lambda^\ell(B)$ be $\ell$-Köthe spaces. $(\lambda^\ell(A), \lambda^\ell(B)) \in B$ if and only if for every sequence $N(k) \uparrow \infty$ there exists $N \in \mathbb{N}$ such that for each $r \in \mathbb{N}$ we have $k_o \in \mathbb{N}$ and $C > 0$ with

$$\frac{b^r_v}{a^i_v} \leq C \max_{1 \leq k \leq k_0} \frac{b^k_v}{a^N(k)}$$

for all $v \in \mathbb{N}, i \in \mathbb{N}$.

**Proof.** Suppose $(\lambda^\ell(A), \lambda^\ell(B)) \in B$. Consider $T : \lambda^\ell(A) \to \lambda^\ell(B)$ with $T = e_i' \otimes e_v$ where $e_i'(x) = x_i$ for all $x \in \lambda^\ell(A)$.

Since $T$ is the operator of rank one, we note that

$$\|T\|_{k,N(k)} = \|e_i'\|_{N(k)} \|e_v\|_k = \frac{b^k_v}{a^N_j}$$

Similarly $\|T\|_{r,N} = \frac{b^r_v}{a^{N}_v}$. The result follows from (1.1).

Conversely we want to show that every continuous linear quasi-diagonal operator is bounded. Let $T : \lambda^\ell(A) \to \lambda^\ell(B)$ be a continuous quasi-diagonal operator defined by $T(e_i) = t_i \tilde{e}_{z(i)}$. By continuity, $\exists N(k)$ such that

$$\sup_i \|T e_i\|_k = \sup_i \frac{|t_i| b^k_{z(i)}}{a^N_i} = C(k) < \infty.$$ 

Thus for this $N(k)$, $\exists N \in \mathbb{N}$ such that $\forall r \in \mathbb{N}$ we have $k_o \in \mathbb{N}$ and $C > 0$ with

$$\frac{|t_i| b^r_{z(i)}}{a^{N}_i} \leq C \max_{1 \leq k \leq k_0} \frac{|t_i| b^k_{z(i)}}{a^N_i} \leq C \max_{1 \leq k \leq k_0} C(k).$$

Hence $\|T\|_{r,N} < \infty$, i.e., $T$ is bounded. In view of Proposition 2.1 we obtain the result. □ 

$\lambda^\ell(A)$ and $\lambda^\ell(B)$ have a common basic subspace if there is a quasi-diagonal operator $T : X \to Y$ such that the restriction of $T$ to some infinite dimensional basic subspace of $X$ is an isomorphism. We observe the following extension of Proposition 3 in [1] to the $\ell$-Köthe space case. The proof is the same as in [1].

**Corollary 2.3.** If $(\lambda^\ell(B), \lambda^\ell(A)) \in S$ and there exists a continuous unbounded operator $T : \lambda^\ell(A) \to \lambda^\ell(B)$, then $\lambda^\ell(A)$ and $\lambda^\ell(B)$ have a common basic subspace.
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