BOUNDS FOR THE DIFFERENCE BETWEEN TWO ČEBYŠEV FUNCTIONALS

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ABSTRACT. In this work, a generalization of pre-Grüss inequality is established. Several bounds for the difference between two Čebyšev functional are proved.

1. Introduction

It is well known that for a continuous function $f$ defined on $[a, b]$, the integral mean-value theorem (IMVT) guarantees $x \in [a, b]$ such that

$$f(x) = \frac{1}{b-a} \int_a^b f(t) \, dt. \tag{1.1}$$

On the other hand, for a monotonic function $g : [a, b] \to \mathbb{R}$ that does not change sign in the interval $[a, b]$, the weighted IMVT reads that there exists $x \in [a, b]$ such that

$$\int_a^b f(t) g(t) \, dt = f(x) \int_a^b g(t) \, dt. \tag{1.2}$$

If one replaces the value of $f(x)$ in (1.2) by its value in (1.1) then we get

$$\int_a^b f(t) g(t) \, dt = \frac{1}{b-a} \int_a^b f(t) \, dt \int_a^b g(t) \, dt. \tag{1.3}$$

To get weighted values in (1.3) we divide the both sides by the quantity ‘$b - a$’ to get

$$\frac{1}{b-a} \int_a^b f(t) g(t) \, dt = \frac{1}{b-a} \int_a^b f(t) \, dt \cdot \frac{1}{b-a} \int_a^b g(t) \, dt, \tag{1.4}$$

which means in such way that the weighted product of two functions equal to the product of weights of that functions.

The difference between these weights

$$\mathcal{T}_a^b (f, g) = \frac{1}{b-a} \int_a^b f(t) g(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \cdot \frac{1}{b-a} \int_a^b g(t) \, dt. \tag{1.5}$$

is called ‘the Čebyšev functional’, which plays an important role in Numerical Approximations and Operator Theory. For more detailed history see [17].

The most famous bounds for the Čebyšev functional are incorporated in the following theorem:

2000 Mathematics Subject Classification. 26D15.

Key words and phrases. Čebyšev functional, Grüss inequality.
Theorem 1. Let \( f, g : [a, b] \rightarrow \mathbb{R} \) be two absolutely continuous functions, then

\[
|\mathcal{T}_a^b (f, g)| \leq \begin{cases} 
\frac{(b-a)^2}{12} \|f'\|_\infty \|g'\|_\infty, & \text{if } f', g' \in L_\infty[a, b], \text{ proved in [11]} \\
\frac{1}{8} (M_1 - m_1) (M_2 - m_2), & \text{if } m_1 \leq f \leq M_1, \ m_2 \leq g \leq M_2, \text{ proved in [14]} \\
\frac{(b-a)}{\pi^2} \|f'\|_2 \|g'\|_2, & \text{if } f', g' \in L_2[a, b], \text{ proved in [16]} \\
\frac{1}{8} (b - a) (M - m) \|g'\|_\infty, & \text{if } m \leq f \leq M, \ g' \in L_\infty[a, b], \text{ proved in [18]}
\end{cases}
\]

The constants \( \frac{1}{12}, \frac{1}{8}, \frac{1}{\pi^2} \) and \( \frac{1}{8} \) are the best possible.

Many authors were studied the functional (1.5) and therefore various bounds have been implemented, for more new results and generalizations the reader may refer to [1],[2],[6],[7],[9],[12],[15] and [19].

In 2001, Cerone [10] established the following identity for the Čebyšev functional:

Theorem 1. Let \( f, g : [a, b] \rightarrow \mathbb{R} \) be such that \( f \) is of bounded variation and \( g \) is continuous on \([a,b]\). Then, we have the following representation:

\[
\mathcal{T}_a^b (f, g) = \frac{1}{(b-a)^2} \int_a^b \left[ (t-a) \int_t^b g(s) \, ds - (b-t) \int_a^t g(s) \, ds \right] df(t).
\]

In 2007, Dragomir [13] established three equivalent identities that generalized Cerone identity (1.7) for Riemann-Stieljes integrals, in case of Riemann integral Dragomir representation incorporated in the following theorem.

Theorem 2. Let \( f, g : [a, b] \rightarrow \mathbb{R} \) be such that \( f \) is of bounded variation and \( g \) is Lebesque integrable on \([a,b]\). Then,

\[
\mathcal{T}_a^b (f, g) = \frac{1}{(b-a)^2} \int_a^b \left[ (t-a) \int_a^b g(t) \, dt - (b-a) \int_a^t g(s) \, ds \right] df(t).
\]

The absolute difference between two integral means was studied firstly by Barnett et al. in [5] and then by Cerone and Dragomir in [8], we may summarize the obtained results, as follow:

- For an absolutely continuous function \( f \) defined on \([a,b]\) and for all \( a \leq c < d \leq b \), we have

\[
\left| \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{d-c} \int_c^d f(s) \, ds \right| 
\leq \left[ 1 + \left( \frac{(a+b) / 2 - (c+d) / 2}{(b-a) - (d-c)} \right)^2 \right] \|(b-a) - (d-c) \| f' \|_\infty
\leq \frac{1}{2} \| (b-a) - (d-c) \| f' \|_\infty
\]
and

\[
\left| \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{d-c} \int_c^d f(s) \, ds \right| \leq \begin{cases} \frac{(b-a)}{(q+1)^{1/q}} \left[ 1 + \left( \frac{p}{1-p} \right)^q \right]^{1/q} \left[ b^{q+1} + \lambda^{q+1} \right]^{1/q} \|f'\|_p, \\ \frac{1}{2} [1 - \rho + |v - \lambda|] \|f'\|_1, \end{cases} f' \in L_p[a, b], \quad 1 \leq p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1; \]

where \((b - a)v = c - a, (b - a)\rho = d - c\) and \((b - a)\lambda = b - d).

- For a Hölder continuous function \(f\) of order \(r \in (0, 1]\) with constant \(H > 0\) on \([a, b]\), we have

\[
\left| \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{d-c} \int_c^d f(s) \, ds \right| \leq H \frac{(c - a)^{r+1} + (b - d)^{r+1}}{(r + 1) [(b - a) - (d - c)]};
\]

- For a function \(f\) of bounded variation on \([a, b]\), we have

\[
\left| \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{d-c} \int_c^d f(s) \, ds \right| \leq \left\{ \begin{array}{ll} \frac{|b-a-(d-c)|}{2} + \frac{|c+d-a+b|}{2} \frac{\|f\|_L}{b-a}, & \text{if } f \text{ is } L\text{-Lipschitzian} \\
\frac{L}{2(b-a)} (c-a)^2 + (b-d)^2 & \text{if } f \text{ is monotonic nondecreasing} \end{array} \right. \]

where, \(s_0 = \frac{cb-ad}{(b-a)-(d-c)} \in [c, d]\).

For recent results the reader may refer to [3], where the author used (1.8) to obtain several bounds for the Čebyšev functional. Bounds for the difference between two Stieltjes integral means was presented in [4].

Let \(g : [\alpha, \beta] \to \mathbb{R}\) be any integrable function and define \(\Psi : [\alpha, \beta] \to \mathbb{R}\), such that

\[
\Psi_g(t; \alpha, \beta) := \int_{\alpha}^t g(s) \, ds - \frac{t - \alpha}{\beta - \alpha} \int_{\alpha}^\beta g(s) \, ds.
\]

From (1.8), it is easy to observe the following representation of the Čebyšev functional

\[
\mathcal{T}_\alpha^\beta (f, g) := -\frac{1}{\beta - \alpha} \int_{\alpha}^\beta \Psi_g(t; \alpha, \beta) \, df(t).
\]

In this work by utilizing the inequalities (1.9)–(1.12), several new bounds for the absolute Difference between two Čebyšev functional \(\mathcal{T}_\alpha^v (f, g) - \mathcal{T}_\alpha^b (f, g)\), for all \(a \leq u < v \leq b\) are provided.
Let us start by providing the following refinements of pre-Grüss inequality, which states that for any two integrable mappings defined on \([a, b]\), the inequality

\[
\mathcal{T}_a^b (f, g) \leq \left[ \mathcal{T}_a^b (f, f) \right]^{1/2} \cdot \left[ \mathcal{T}_a^b (g, g) \right]^{1/2},
\]

holds and sharp (see [14]). Trivially, by applying AM–GM inequality on the right hand side of (1.13), we get

\[
\left[ \mathcal{T}_a^b (f, f) \right]^{1/2} \cdot \left[ \mathcal{T}_a^b (g, g) \right]^{1/2} \leq \frac{\mathcal{T}_a^b (f, f) + \mathcal{T}_a^b (g, g)}{2}.
\]

We may generalize the pre-Grüss inequality (1.13) as follows:

**Theorem 3.** Let \(f, g : [a, b] \to \mathbb{R}\) be two integrable mappings, then

\[
\begin{align*}
|\mathcal{T}_a^u (f, g) - \mathcal{T}_a^v (f, g)| &
\leq (\mathcal{T}_a^u (f, f))^{1/2} (\mathcal{T}_a^u (g, g))^{1/2} + (\mathcal{T}_a^b (f, f))^{1/2} (\mathcal{T}_a^b (g, g))^{1/2} \\
&\leq \frac{1}{2} \left[ \mathcal{T}_a^u (f, f) + \mathcal{T}_a^u (g, g) + \mathcal{T}_a^b (f, f) + \mathcal{T}_a^b (g, g) \right],
\end{align*}
\]

for all \(a \leq u < v \leq b\). The double inequality is sharp.

**Proof.** Simply using the (1.13), we have

\[
\begin{align*}
|\mathcal{T}_a^u (f, g) - \mathcal{T}_a^v (f, g)|^2 &
\leq (\mathcal{T}_a^u (f, g))^2 + 2\mathcal{T}_a^u (f, g) \cdot \mathcal{T}_a^v (f, g) + (\mathcal{T}_a^b (f, g))^2 \\
&\leq (\mathcal{T}_a^u (f, f)) (\mathcal{T}_a^u (g, g)) + 2\mathcal{T}_a^u (f, g) \cdot \mathcal{T}_a^u (f, g) + (\mathcal{T}_a^b (f, f)) (\mathcal{T}_a^b (g, g)) \\
&\leq \left[ (\mathcal{T}_a^u (f, f))^{1/2} (\mathcal{T}_a^u (g, g))^{1/2} + (\mathcal{T}_a^b (f, f))^{1/2} (\mathcal{T}_a^b (g, g))^{1/2} \right] \\
&\quad \times (\mathcal{T}_a^u (f, f))^{1/2} (\mathcal{T}_a^u (g, g))^{1/2} \\
&\quad + \left[ (\mathcal{T}_a^u (f, f))^{1/2} (\mathcal{T}_a^b (g, g))^{1/2} + (\mathcal{T}_a^b (f, f))^{1/2} (\mathcal{T}_a^u (g, g))^{1/2} \right] \\
&\quad \times (\mathcal{T}_a^b (f, f))^{1/2} (\mathcal{T}_a^b (g, g))^{1/2} \\
&= \left[ (\mathcal{T}_a^u (f, f))^{1/2} (\mathcal{T}_a^u (g, g))^{1/2} + (\mathcal{T}_a^b (f, f))^{1/2} (\mathcal{T}_a^b (g, g))^{1/2} \right]^2
\end{align*}
\]

and this implies the first inequality in (1.15). The second inequality follows by applying the AM–GM inequality. The sharpness follows by letting \(f = g = x\). □

**Remark 1.** We note that (1.15) reduces to (1.13) by setting \(u = a\) and \(v = u + \epsilon\), thus

\[
|\mathcal{T}_a^u (f, g) - \mathcal{T}_a^v (f, g)| \to |\mathcal{T}_a^b (f, g)| \quad \text{as} \quad \epsilon \to 0^+.
\]

Consequently, the right hand of (1.15) \(\to\) the right hand of (1.13).

2. Bounds for bounded variation integrators

The first result regarding bounded variation integrators is presented as follows:
Theorem 4. Let \( f, g : [a, b] \to \mathbb{R} \) be such that \( f \) is of bounded variation on \([a, b]\) and \( g \) is absolutely continuous on \([a, b]\), then

\[
\left| T_a^b (f, g) - T_a^b (f, g) \right| \leq \frac{1}{2} \left[ \frac{(v-a)+(b-u)}{2} + \frac{|b-u|}{2} \right] \| g' \|_{\infty, [a, b]}, \quad \text{if} \quad g' \in L_\infty [a, b];
\]

\[
\frac{1}{2} \left[ \frac{b-a}{2} + \frac{v-a+b}{2} \right] \cdot \| g' \|_{p, [a, b]}, \quad \text{if} \quad g' \in L_p [a, b],
\]

\[
\frac{1}{2} \| g' \|_{1, [a, b]}, \quad \text{if} \quad g' \in L_1 [a, b],
\]

for all \( a \leq u < v \leq b \), where \( \| \cdot \|_p \) are the usual Lebesgue norms, i.e.,

\[
\| h \|_p := \left( \int_a^b |h(t)|^p \, dt \right)^{1/p}, \quad \text{for} \quad p \geq 1
\]

and

\[
\| h \|_{\infty} := \text{ess sup}_{t \in [a, b]} |h(t)|.
\]

Proof. It is known that for a continuous function \( w \) on \([a, b]\) and a bounded variation \( \nu \) on \([a, b]\), one have the inequality

\[
\left| \int_a^b w(t) \, d\nu(t) \right| \leq \sup_{t \in [a, b]} |w(t)| \cdot \sqrt{\nu}.
\]

Employing (2.2) for the Cerone-Dragomir identity

\[
T (f, g) = -\frac{1}{b-a} \int_a^b \left( \int_a^t g(s) \, ds - \frac{t-a}{b-a} \int_a^b g(s) \, ds \right) \, df(t).
\]

One has as \( f \) is of bounded variation on \([a, b]\),

\[
\left| T_a^b (f, g) - T_a^b (f, g) \right| = \int_a^b \left( \int_a^t g(s) \, ds - \frac{t-a}{v-a} \int_a^v g(s) \, ds \right) \, df(t)
\]

\[
- \frac{1}{b-u} \int_u^b \left( \int_u^r g(s) \, ds - \frac{r-u}{b-u} \int_u^b g(s) \, ds \right) \, df(t)
\]

\[
\leq \left| \int_a^v \left( \int_a^t g(s) \, ds - \frac{t-a}{v-a} \int_a^v g(s) \, ds \right) \, dt \right| \cdot \sqrt{\nu} \cdot (f)
\]

\[
+ \frac{1}{b-u} \sup_{v \leq t \leq u} \left| \int_a^v \left( \int_a^t g(s) \, ds - \frac{t-a}{v-a} \int_a^v g(s) \, ds \right) \, dt \right| \cdot \sqrt{\nu} \cdot (f)
\]

(2.4)
In the inequality (1.9), setting \( d = t, c = a \) and then \( d = r, c = u \), we get

\[
(2.5) \quad \left| \frac{1}{t-a} \int_a^t g(s) \, ds - \frac{1}{v-a} \int_a^v g(s) \, ds \right| \leq \frac{1}{2} (v-t) \|g'\|_{\infty, [a,v]}
\]

and

\[
(2.6) \quad \left| \frac{1}{r-u} \int_u^r g(s) \, ds - \frac{1}{b-u} \int_u^b g(s) \, ds \right| \leq \frac{1}{2} (b-r) \|g'\|_{\infty, [u,b]}.
\]

Substituting (2.5) and (2.6) in (2.4), we get

\[
|T_a^v (f,g) - T_u^b (f,g)| \leq \frac{1}{v-a} \cdot \frac{1}{2} \|g'\|_{\infty, [a,v]} \sup_{t \in [a,v]} \{(t-a) (v-t)\} \bigg\{ \frac{(v-a) v}{2} + \frac{1}{2} \|g'\|_{\infty, [a,b]} \bigg\}
\]

\[
= \frac{1}{8} (v-a) \|g'\|_{\infty, [a,v]} \bigg\{ \frac{(v-a) v}{2} + \frac{1}{2} \|g'\|_{\infty, [a,b]} \bigg\}
\]

\[
\leq \frac{1}{8} \max \{(v-a), (b-u)\} \|g'\|_{\infty, [a,b]} \bigg\{ \frac{(v-a) v}{2} + \frac{1}{2} \|g'\|_{\infty, [a,b]} \bigg\}
\]

where we used the fact that \( \sup_{t \in [\alpha,\beta]} \{(t-\alpha) (\beta-t)\} \), occurs at \( t = \frac{\alpha + \beta}{2} \), therefore,

\[
\sup_{t \in [\alpha,\beta]} \{(t-\alpha) (\beta-t)\} = \frac{1}{4} (\beta-\alpha)^2. \]

Also, we note that the last inequality holds since

\[
\|g'\|_{\infty, [a,v]} \leq \|g'\|_{\infty, [a,b]} \bigg\{ \frac{v}{2} \|f\|_a \leq \|f\|_a \} \text{ and } \bigg\{ \frac{b}{2} \|f\|_a \leq \|f\|_a \}
\]

which proves the first inequality in (2.1).

In the inequality (1.10), replace \( r, u \) instead of \( d, c \); respectively and then \( t, a \) instead of \( d, c \); respectively, we find that

\[
(2.8) \quad \left| \frac{1}{r-a} \int_a^r g(s) \, ds - \frac{1}{v-a} \int_a^v g(s) \, ds \right| \leq \frac{(v-r)^q}{(q+1) q (v-a)^q} \bigg\{ (r-a)^q + (v-r)^q \bigg\} \|g'\|_{p, [a,v]}, \quad g' \in L_p [a,v],
\]

\[
\leq \frac{v-a}{v-a} \|g'\|_{1, [a,v]}, \quad g' \in L_1 [a,v].
\]
which proves the second and the third inequalities in (2.1).

Substituting (2.8) and (2.9) in (2.4), we have respectively

\[
\frac{1}{v-a} \sup_{r \in [a,v]} (r-a) \left| \frac{1}{v-a} \int_r^a g(s) \, ds - \frac{1}{v-a} \int_a^b g(s) \, ds \right| \leq \left\{ \begin{array}{cl}
\frac{(b-t)^\frac{q}{q+1}}{(q+1)^{1/q}(b-u)^{1/q}} \| g' \|_{p,[u,b]} & , \quad g' \in L_p [u,b], \\
\frac{b-t}{v-a} \| g' \|_{1,[u,b]} & , \quad g' \in L_1 [u,b]
\end{array} \right.
\]

Substituting (2.8) and (2.9) in (2.4), we have respectively

\[
\frac{1}{v-a} \sup_{r \in [a,v]} (r-a) \left| \frac{1}{v-a} \int_r^a g(s) \, ds - \frac{1}{v-a} \int_a^b g(s) \, ds \right| \leq \left\{ \begin{array}{cl}
\frac{(b-t)^\frac{q}{q+1}}{(q+1)^{1/q}(v-a)^{1/q}} \sup_{t \in [a,v]} \left\{ (r-a) \left( v-r \right)^\frac{1}{q} \left( (r-a)^q + (v-r)^q \right)^{1/q} \right\} , & g' \in L_p [a,v], \\
\frac{1}{v-a} \frac{(v-a)}{4(q+1)^{1/q}} \| g' \|_{p,[a,v]} , & g' \in L_p [a,v], \\
\frac{1}{v-a} \frac{1}{4} \| g' \|_{1,[a,v]} & , \quad g' \in L_1 [a,v]
\end{array} \right.
\]

and similarly, we have

\[
\frac{1}{b-u} \sup_{r \in [u,b]} (r-u) \left| \frac{1}{b-u} \int_u^r g(s) \, ds - \frac{1}{b-u} \int_u^b g(s) \, ds \right| \leq \left\{ \begin{array}{cl}
\frac{(b-t)^\frac{q}{q+1}}{(q+1)^{1/q}(b-u)^{1/q}} \| g' \|_{p,[u,b]} & , \quad g' \in L_p [u,b], \\
\frac{1}{4} \| g' \|_{1,[u,b]} & , \quad g' \in L_1 [u,b]
\end{array} \right.
\]

Adding (2.10) and (2.11), we get

\[
\left| T_a^u (f,g) - T_b^u (f,g) \right| \leq \left\{ \begin{array}{cl}
\frac{(v-a)}{4(q+1)^{1/q}} \| g' \|_{p,[a,v]} \nabla_a^u f + \frac{(b-u)}{4(q+1)^{1/q}} \| g' \|_{p,[u,b]} \nabla_b^u (f) , & g' \in L_p [u,b], \\
\frac{1}{4} \| g' \|_{1,[a,v]} \nabla_a^u f & , \quad g' \in L_1 [a,v]
\end{array} \right.
\]

which proves the second and the third inequalities in (2.1) \( \square \).
Corollary 1. Under the assumptions of Theorem 4, we have

\[
\left| T_u^a (f, g) - T_u^b (f, g) \right| \leq \int_a^b (f) \cdot \begin{cases} 
\frac{1}{8} \left[ \frac{b-a}{2} + \left| u - \frac{a+b}{2} \right| \right] \| g' \|_{\infty, [a,b]}, & \text{if } g' \in L_\infty [a,b]; \\
\frac{1}{2(2q+1)^{1/q}} \left[ \frac{b-a}{2} + \left| u - \frac{a+b}{2} \right| \right] \| g' \|_{p, [a,b]}, & \text{if } g' \in L_p [a,b], \\
\frac{1}{2} \| g' \|_{1, [a,b]}, & \text{if } g' \in L_1 [a,b]. 
\end{cases}
\]

for all \( a \leq u \leq b \). In particular case if \( u = \frac{a+b}{2} \), we get

\[
\left| T_a^a (f, g) - T_a^b (f, g) \right| \leq H \left( \frac{b-a}{2} \right)^p \left[ \frac{b-a}{2} + \left| u - \frac{a+b}{2} \right| \right] \| g' \|_{\infty, [a,b]}, \quad \text{if } g' \in L_\infty [a,b]; \\
\frac{1}{2} \left( \frac{b-a}{2} \right)^p \| g' \|_{1, [a,b]}, \quad \text{if } g' \in L_1 [a,b].
\]

Proof. In Theorem 4, let \( \epsilon > 0 \) and set \( v = u + \epsilon \) so as \( \epsilon \to 0^+ \) we get the required result. \( \square \)

Another result when \( g \) is of \( r \)-H–Hölder type is as follows:

Theorem 5. Let \( f, g : [a, b] \to \mathbb{R} \) be such that \( f \) is of bounded variation on \([a,b]\) and \( g \) is of \( p \)-H–Hölder type on \([a,b]\), for \( p \in (0,1] \) and \( H > 0 \) are given. Then

\[
\left| T_u^a (f, g) - T_u^b (f, g) \right| \leq H \left( \frac{v-a}{2} \right)^p \left( \frac{b-u}{2} \right)^p \sqrt{\frac{b-a}{2}}(f),
\]

and

\[
\left| T_u^a (f, g) - T_u^b (f, g) \right| \leq \frac{H}{2^p (p+1)} \left[ \frac{v-a}{2} + \frac{b-u}{2} \right]^p \sqrt{\frac{b-a}{2}}(f),
\]

for all \( a \leq u < v \leq b \).
Proof: We repeat the proof of Theorem 4. So as \( f \) is of bounded variation and \( g \) is of \( p \)-Hölder type on \([a, b]\), then we have

\[
|T^v_a (f, g) - T^b_u (f, g)| \\
\leq \frac{1}{v-a} \sup_{r \in [a, v]} \left| (r-a) \left[ \frac{1}{r-a} \int_a^r g(s) \, ds - \frac{1}{v-a} \int_a^v g(s) \, ds \right] \right| \int_a^v (f) \\
+ \frac{1}{b-u} \sup_{t \in [u, b]} \left| (t-u) \left[ \frac{1}{t-u} \int_u^t g(s) \, ds - \frac{1}{b-u} \int_u^b g(s) \, ds \right] \right| \int_u^b (f)
\]

\[
\leq \frac{1}{v-a} \frac{H}{p+1} \sup_{r \in [a, v]} (r-a) (v-r)^p \int_a^v (f) + \frac{1}{b-u} \frac{H}{p+1} \sup_{t \in [u, b]} (t-u) (b-t)^p \int_u^b (f)
\]

\[
= H \frac{(v-a)^p}{2^{p+1} (p+1)} \int_a^v (f) + H \frac{(b-u)^p}{2^{p+1} (p+1)} \int_u^b (f)
\]

\[
\leq H \frac{(v-a)^p + (b-u)^p}{2^{p+1} (p+1)} \int_a^b (f),
\]

which proves the first inequality. To obtain the second inequality from the above inequality we may obtain that

\[
|T^v_a (f, g) - T^b_u (f, g)|
\]

\[
\leq H \frac{(v-a)^p}{2^{p+1} (p+1)} \int_a^v (f) + H \frac{(b-u)^p}{2^{p+1} (p+1)} \int_u^b (f)
\]

\[
\leq H \frac{1}{2^{p+1} (p+1)} \max \{(v-a)^p, (b-u)^p\} \left[ \int_a^v (f) + \int_u^b (f) \right]
\]

\[
\leq H \frac{1}{2^{p+1} (p+1)} \left[ \frac{(v-a) + (b-u)}{2} + \left| \frac{v-a}{2} - \frac{b-u}{2} \right| \right]^p \int_a^b (f),
\]

which proves (2.15), and thus the proof is completed. \( \square \)

**Corollary 2.** Under the assumptions of Theorem 5, we have

\[
|T^u_a (f, g) - T^b_u (f, g)| \leq H \frac{(u-a)^p + (b-u)^p}{2^{p+1} (p+1)} \int_a^b (f),
\]

and

\[
|T^u_a (f, g) - T^b_u (f, g)| \leq H \frac{b-a}{2^p (p+1)} \left[ \frac{b-a}{2} + \left| \frac{u-a+b}{2} \right| \right]^p \int_a^b (f),
\]

for all \( a \leq u \leq b \). In particular case if \( u = \frac{a+b}{2} \), then the both inequalities (2.16) and (2.17) gives the same inequality, that is

\[
|T^u_a (f, g) - T^b_u (f, g)| \leq H \frac{(b-a)^p}{2^{2p} (p+1)} \int_a^b (f).
\]

**Proof.** In Theorem 5, let \( \epsilon > 0 \) and set \( v = u + \epsilon \) so as \( \epsilon \to 0^+ \) we get the required result. \( \square \)
Theorem 6. Let \( f, g : [a, b] \rightarrow \mathbb{R} \) be such that \( f \) is of bounded variation on \([a, b]\) and \( g \) is monotonic nondecreasing on \([a, b]\), then

\[
(2.19) \quad \left| \mathcal{T}_v^u (f, g) - \mathcal{T}_u^b (f, g) \right| \leq \frac{1}{4} \left\{ \frac{|g(v) - g(a)| + |g(b) - g(u)|}{2} + \frac{|g(v) + g(u)|}{2} - \frac{g(a) + g(b)}{2} \right\} \cdot \int_a^b \mathcal{V}(f),
\]

for all \( a \leq u < v \leq b \).

Proof. As \( f \) is of bounded variation on \([a, b]\) and \( g \) is monotonic nondecreasing on \([a, b]\) (which implies that \( \Psi_g (t; a, b) \) is absolutely continuous on \([a, b]\)), by (2.4) we have

\[
(2.20) \quad \left| \mathcal{T}_v^u (f, g) - \mathcal{T}_u^b (f, g) \right| \leq \frac{1}{v - a} \sup_{r \in [a, v]} \left( r - a \right) \left| \frac{1}{r - a} \int_a^r g(s) \, ds - \frac{1}{v - a} \int_a^v g(s) \, ds \right| \cdot \mathcal{V}(f)
\]

Employing the third part of (1.12), setting \( d = r, t \) and \( c = a, u \), respectively we get

\[
(2.21) \quad \left| \frac{1}{r - a} \int_a^r g(s) \, ds - \frac{1}{v - a} \int_a^v g(s) \, ds \right| \leq \frac{v - r}{v - a} \left[ g(v) - g(a) \right],
\]

and

\[
(2.22) \quad \left| \frac{1}{t - u} \int_u^t g(s) \, ds - \frac{1}{b - u} \int_u^b g(s) \, ds \right| \leq \frac{b - t}{b - u} \left[ g(b) - g(u) \right].
\]

Substituting (2.21) and (2.22) in (2.20), we get

\[
\left| \mathcal{T}_v^u (f, g) - \mathcal{T}_u^b (f, g) \right| \leq \frac{1}{(v - a)^2} \sup_{r \in [a, v]} \{ (r - a) (v - r) \} \cdot \left[ g(v) - g(a) \right] \cdot \mathcal{V}(f)
\]

\[
+ \frac{1}{(b - u)^2} \sup_{t \in [a, b]} \{ (t - u) (b - t) \} \cdot \left[ g(b) - g(u) \right] \cdot \mathcal{V}(f)
\]

\[
= \frac{1}{4} [g(v) - g(a)] \mathcal{V}(f) + \frac{1}{4} [g(b) - g(u)] \mathcal{V}(f)
\]

\[
= \frac{1}{4} \max \{ g(v) - g(a), g(b) - g(u) \} \cdot \mathcal{V}(f)
\]

\[
\leq \frac{1}{4} \left\{ \frac{|g(v) - g(a)| + |g(b) - g(u)|}{2} + \frac{|g(v) + g(u)|}{2} - \frac{g(a) + g(b)}{2} \right\} \cdot \mathcal{V}(f),
\]

and thus the proof is finished. \( \square \)
Corollary 3. Under the assumptions of Theorem 6, we have

\[
\left| T_a^u (f, g) - T_b^u (f, g) \right| \leq \left\{ \frac{g(b) - g(a)}{2} + \left| g(u) - \frac{g(a) + g(b)}{2} \right| \right\} \cdot \sqrt{a(f)},
\]

for all \( a \leq u \leq b \). In particular case if \( u = \frac{a+b}{2} \), then the both inequalities (2.23) gives the same inequality, that is

\[
\left| T_{\frac{a+b}{2}}^u (f, g) - T_{\frac{a+b}{2}}^b (f, g) \right| \leq \left\{ \frac{g(b) - g(a)}{2} + \left| g\left(\frac{a+b}{2}\right) - \frac{g(a) + g(b)}{2} \right| \right\} \cdot \sqrt{f},
\]

Proof. In Theorem 5, let \( \epsilon > 0 \) and set \( v = u + \epsilon \) so as \( \epsilon \rightarrow 0^+ \) we get the required result. □

3. Bounds for Lipschitzian integrators

Theorem 7. Let \( f, g : [a, b] \rightarrow \mathbb{R} \) be such that \( f \) is \( L \)-Lipschitzian on \([a, b]\) and \( g \) is an absolutely continuous on \([a, b]\), then

\[
\left| T_a^u (f, g) - T_b^u (f, g) \right| \leq L \left\{ \frac{g(b) - g(a)}{2} + \left| g\left(\frac{a+b}{2}\right) - \frac{g(a) + g(b)}{2} \right| \right\} \cdot \sqrt{f},
\]

where, \( p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. Using the fact that for a Riemann integrable function \( p : [c, d] \rightarrow \mathbb{R} \) and \( L \)-Lipschitzian function \( \nu : [c, d] \rightarrow \mathbb{R} \), one has the inequality

\[
\left| \int_c^d p(t) \, d\nu (t) \right| \leq L \int_c^d |p(t)| \, dt. 
\]
As \( f \) is \( L \)-Lipschitzian on \([a, b]\), by (3.2) we have

\[
|\mathcal{T}_a^v (f, g) - \mathcal{T}_a^b (f, g)|
\]

\[
\leq \frac{L}{v-a} \int_a^v (r-a) \left[ \frac{1}{r-a} \int_a^r g(s) \, ds - \frac{1}{v-a} \int_a^v g(s) \, ds \right] \, dr
+ \frac{L}{b-u} \int_u^b (t-u) \left[ \frac{1}{t-u} \int_u^t g(s) \, ds - \frac{1}{b-u} \int_u^b g(s) \, ds \right] \, dt
\]

\[
\leq \frac{1}{2} L \|g'\|_\infty \left[ \frac{1}{v-a} \int_a^v (r-a) (v-r) \, dr + \frac{1}{b-u} \int_u^b (t-a) (b-t) \, dt \right]
\]

\[
= \frac{1}{6} L \|g'\|_\infty \left[ (v-a)^2 + (b-a)^2 \right]
\]

\[
= L \frac{|(b-a) - (v-u)|}{6} \left[ \frac{1}{4} + \left( \frac{a+b - u+v}{2(v-a)} \right)^2 \right] \|g'\|_\infty,
\]

where we used the inequality (1.9), with \( d = r, t \) and \( c = a, u \); respectively.

To obtain the second inequality, setting \( d = r, t \) and \( c = a, u \); respectively, in (1.10), we get

\[
\frac{1}{r-a} \int_a^r g(s) \, ds - \frac{1}{v-a} \int_a^v g(s) \, ds
\]

\[
\leq \frac{(v-r)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}} (v-a)^{\frac{1}{q}}} \left[ (r-a)^q + (v-r)^q \right]^{\frac{1}{q}} \|g'\|_{p,[a,v]}
\]

and

\[
\frac{1}{t-u} \int_u^t g(s) \, ds - \frac{1}{b-u} \int_u^b g(s) \, ds
\]

\[
\leq \frac{(b-t)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}} (b-u)^{\frac{1}{q}}} \left[ (t-u)^q + (b-t)^q \right]^{\frac{1}{q}} \|g'\|_{p,[u,b]}
\]

Substituting (3.4) and (3.5) in (3.3), we get

\[
|\mathcal{T}_a^v (f, g) - \mathcal{T}_a^b (f, g)|
\]

\[
\leq L \frac{\|g'\|_{p,[a,v]}}{(q+1)^{\frac{1}{q}} (v-a)^{1+\frac{1}{q}}} \int_a^v (r-a) \left( b-r \right)^{\frac{1}{q}} \left[ (r-a)^q + (v-r)^q \right]^{\frac{1}{q}} \, dr
\]

\[
+ L \frac{\|g'\|_{p,[u,b]}}{(q+1)^{\frac{1}{q}} (b-u)^{1+\frac{1}{q}}} \int_u^b (t-u) \left( b-t \right)^{\frac{1}{q}} \left[ (t-u)^q + (b-t)^q \right]^{\frac{1}{q}} \, dt
\]

\[
\leq L \frac{\|g'\|_{p,[a,v]}}{(q+1)^{\frac{1}{q}} (v-a)^{1+\frac{1}{q}}} \sup_{r \in [a,v]} \left[ (r-a)^q + (v-r)^q \right]^{\frac{1}{q}} \int_a^v (r-a) \left( v-r \right)^{\frac{1}{q}} \, dr
\]

\[
+ L \frac{\|g'\|_{p,[u,b]}}{(q+1)^{\frac{1}{q}} (b-u)^{1+\frac{1}{q}}} \sup_{t \in [u,b]} \left[ (t-u)^q + (b-t)^q \right]^{\frac{1}{q}} \int_u^b (t-u) \left( b-t \right)^{\frac{1}{q}} \, dt
\]
\[ = L \frac{(v - a)^2}{(q + 1)^{1/q}} B \left(2, 1 + \frac{1}{q}\right) \cdot \|g'\|_{p,[a,v]} + L \frac{(b - u)^2}{(q + 1)^{1/q}} B \left(2, 1 + \frac{1}{q}\right) \cdot \|g'\|_{p,[u,b]} \]

\[ \leq L \frac{\|g'\|_{p,[a,b]}}{(q + 1)^{1/q}} B \left(2, 1 + \frac{1}{q}\right) \cdot (v - a)^2 + (b - u)^2 \]

\[ = \frac{2 \left[(b - a) - (v - u)\right]}{(q + 1)^{1/q}} B \left(2, 1 + \frac{1}{q}\right) \cdot \|g'\|_{p,[a,b]} \]

which proves the second inequality in (3.1). \[\square\]

**Corollary 4.** Under the assumptions of Theorem 7, then

\[ \begin{align*}
&\left| T_a^u (f,g) - T_b^v (f,g) \right|
&\leq L \begin{cases}
\frac{6}{7} \|g'\|_\infty \left[\frac{(a - u)^2 + (b - u)^2}{2}\right], & g' \in L_\infty [a,b]; \\
\frac{(a - a)^2 + (b - u)^2}{2(q + 1)^{1/q}} B \left(2, 1 + \frac{1}{q}\right) \cdot \|g'\|_{p,[a,b]}, & g' \in L_p [a,b],
\end{cases}
\end{align*} \]

where, \( p > 1 \) and \( p + \frac{1}{q} = 1 \). In particular case, if \( u = \frac{a + b}{2} \) then

\[ \begin{align*}
&\left| T_a^{\frac{a + b}{2}} (f,g) - T_b^{\frac{a + b}{2}} (f,g) \right|
&\leq L \begin{cases}
\frac{6}{7} \|g'\|_\infty, & g' \in L_\infty [a,b]; \\
\frac{(b - a)^2}{2(q + 1)^{1/q}} B \left(2, 1 + \frac{1}{q}\right) \cdot \|g'\|_{p,[a,b]}, & g' \in L_p [a,b],
\end{cases}
\end{align*} \]

**Theorem 8.** Let \( f, g : [a, b] \to \mathbb{R} \) be such that \( f \) is \( L \)-Lipschitzian on \([a, b]\) and \( g \) is of \( p-H \)-Hölder type on \([a, b]\) where \( p \in (0, 1) \) and \( H > 0 \) are given, then

\[ \begin{align*}
&T_a^v (f,g) - T_b^u (f,g)
&\leq \frac{LH}{(p + 1)^2(p + 2)} \left[\frac{(b - a) + (v - u)}{2} + \frac{|u + v|}{2} \frac{a + b}{2}\right]^{p+1}.
\end{align*} \]

**Proof.** We repeat the proof of Theorem 7. As \( f \) is \( L \)-Lipschitzian and \( g \) is of \( p-H \)-Hölder type on \([a, b]\), by (1.11) we have

\[ \begin{align*}
&T_a^v (f,g) - T_b^u (f,g)
&\leq \frac{L}{v - a} \int_a^v \left|[r - a] \left[\frac{1}{r - a} \int_a^r g(s) ds - \frac{1}{v - a} \int_a^v g(s) ds\right]\right| \, dr \\
&+ \frac{L}{b - u} \int_u^b \left|[t - u] \left[\frac{1}{t - u} \int_u^t g(u) du - \frac{1}{b - u} \int_u^b g(u) du\right]\right| \, dt \\
&\leq \frac{LH}{(p + 1)(v - a)} \int_a^v (r - a)(v - r)^p \, dr + \frac{LH}{(p + 1)(b - u)} \int_u^b (t - u)(b - t)^p \, dt \\
&= \frac{LH(v - a)^{p+1}}{(p + 1)^2(p + 2)} + \frac{LH(b - u)^{p+1}}{(p + 1)^2(p + 2)}
\end{align*} \]
\[
LH \leq \frac{LH}{(p + 1)^2(p + 2)} \cdot \left[ \frac{(b - a) + (v - u)}{2} + \left| \frac{u + v}{2} - \frac{a + b}{2} \right| \right]^{p+1},
\]

where for the last inequality a simple calculation yields that
\[
\int_a^b (t - a) (b - t)^p dt = (b - a)^{p+2} \int_0^1 (1 - t)^p dt = \frac{(b - a)^{p+2}}{(p + 1)(p + 2)},
\]

which completes the proof. \(\square\)

**Corollary 5.** Let \(f, g\) be two Lipschitzian mappings on \([a, b]\) with Lipschitz constants \(L_f, L_g > 0\), then

\[
|\mathcal{T}_a^u (f, g) - \mathcal{T}_a^b (f, g)| \leq \frac{L_f L_g}{12} \cdot \left[ \frac{(b - a) + (v - u)}{2} + \left| \frac{u + v}{2} - \frac{a + b}{2} \right| \right]^{2}.
\]

Moreover,

\[
|\mathcal{T}_a^u (f, g) - \mathcal{T}_u^b (f, g)| \leq \frac{L_f L_g}{12} \cdot \left[ \frac{b - a}{2} + \left| \frac{u + v}{2} - \frac{a + b}{2} \right| \right]^{2},
\]

for all \(a \leq u \leq b\). In particular case if \(u = \frac{a+b}{2}\), we have

\[
|\mathcal{T}_{\frac{a+b}{2}}^u (f, g) - \mathcal{T}_{\frac{a+b}{2}}^b (f, g)| \leq \frac{1}{24} L_f L_g (b - a)^2.
\]

**Proof.** In (3.8), let \(p = 1\) we get (3.9). The inequality (3.10) can be obtained by setting \(v = u + \epsilon, \epsilon > 0\), and letting \(\epsilon \to 0^+\). \(\square\)

**Theorem 9.** Let \(f, g : [a, b] \to \mathbb{R}\) be two absolutely continuous on \([a, b]\). If \(f' \in L_\alpha[a, b], \alpha, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1\), then

\[
|\mathcal{T}_a^u (f, g) - \mathcal{T}_u^b (f, g)|
\]

\[
\leq \begin{cases} 
\left( \frac{(v-a)^{\frac{1}{p}} + (b-u)^{\frac{1}{q}}}{2} \right)^{\frac{1}{p} + \frac{1}{q}} \cdot B_{\frac{1}{p} + \frac{1}{q}} \left( \beta + 1, \beta + 1 \right) \cdot \|f'\|_{L_\alpha[a, b]} \cdot \|g'\|_{L_\beta[a, b]} & \text{if } g' \in L_\infty[a, b] \\
\left( \frac{(v-a)^{\frac{1}{p}} + (b-u)^{1+\frac{1}{q}}}{(q+1)^{1/q}} \right)^{\frac{1}{p} + \frac{1}{q}} \cdot B_{\frac{1}{p} + \frac{1}{q}} \left( \beta + 1, \beta + 1 \right) \cdot \|g'\|_{L_p[a, b]} \cdot \|f'\|_{L_\alpha[a, b]} & \text{if } g' \in L_p[a, b] \\
\left( (v-a)^{1+\frac{1}{p}} + (b-u)^{1+\frac{1}{q}} \right)^{\frac{1}{p} + \frac{1}{q}} \cdot B_{\frac{1}{p} + \frac{1}{q}} \left( \beta + 1, \beta + 1 \right) \cdot \|g'\|_{L_1[a, b]} \cdot \|f'\|_{L_\alpha[a, b]} & \text{if } g' \in L_1[a, b]
\end{cases}
\]
Proof. Taking the absolute value in (1.8) and utilizing the triangle inequality. As $f' \in L_\alpha([a, b])$, by Hölder inequality we have

\[
(3.13) \quad |T^v_a (f, g) - T^b_u (f, g)| \\
\leq \frac{1}{v-a} \left| \int_a^v (r-a) \left[ \frac{1}{r-a} \int_a^r g(s) \, ds - \frac{1}{v-a} \int_a^v g(s) \, ds \right] |f'(r)| \, dr \right| + \frac{1}{b-u} \left| \int_u^b (t-u) \left[ \frac{1}{t-u} \int_u^t g(s) \, ds - \frac{1}{b-u} \int_u^b g(s) \, ds \right] |f'(t)| \, dt \right|
\]

\[
\leq \frac{1}{v-a} \left( \int_a^v |r-a|^{\beta} \left| \frac{1}{r-a} \int_a^r g(s) \, ds - \frac{1}{v-a} \int_a^v g(s) \, ds \right| \, dr \right)^{1/\beta} \times \left( \int_a^v |f'(r)|^{\alpha} \, dr \right)^{1/\alpha}
\]

\[
+ \frac{1}{b-u} \left( \int_u^b |t-u|^{\beta} \left| \frac{1}{t-u} \int_u^t g(s) \, ds - \frac{1}{b-u} \int_u^b g(s) \, ds \right| \, dt \right)^{1/\beta} \times \left( \int_u^b |f'(t)|^{\alpha} \, dt \right)^{1/\alpha}
\]

Now, in (1.9) put $d = r, t$ and $c = a, u$; respectively, then

\[
\left| \frac{1}{r-a} \int_a^r g(s) \, ds - \frac{1}{v-a} \int_a^v g(s) \, ds \right| \leq \frac{v-r}{2(v-a)} \cdot \|g'\|_{\infty, [a, v]}
\]

and

\[
\left| \frac{1}{t-u} \int_u^t g(s) \, ds - \frac{1}{b-u} \int_u^b g(s) \, ds \right| \leq \frac{b-t}{2(b-u)} \cdot \|g'\|_{\infty, [u, b]}
\]

Substituting these inequalities in (3.13) we get

\[
|T^v_a (f, g) - T^b_u (f, g)|
\]

\[
\leq \frac{1}{2(v-a)^2} \cdot \|f'\|_{\alpha, [a, v]} \cdot \|g'\|_{\infty, [a, v]} \left( \int_a^v (r-a)^\beta (v-r)^\beta \, dr \right)^{1/\beta} + \frac{1}{2(b-u)^2} \cdot \|f'\|_{\alpha, [u, b]} \cdot \|g'\|_{\infty, [u, b]} \left( \int_u^b (t-u)^\beta (b-t)^\beta \, dt \right)^{1/\beta}
\]

\[
= \frac{(v-a)^{\frac{\beta}{2}}}{2} \cdot B^\frac{\beta}{2} (\beta + 1, \beta + 1) \cdot \|f'\|_{\alpha, [a, v]} \cdot \|g'\|_{\infty, [a, v]} + \frac{(b-u)^{\frac{\beta}{2}}}{2} \cdot B^\frac{\beta}{2} (\beta + 1, \beta + 1) \cdot \|f'\|_{\alpha, [u, b]} \cdot \|g'\|_{\infty, [u, b]}
\]

\[
\leq \frac{(v-a)^{\frac{\beta}{2}} + (b-u)^{\frac{\beta}{2}}}{2} \cdot B^\frac{\beta}{2} (\beta + 1, \beta + 1) \cdot \|f'\|_{\alpha, [a, b]} \cdot \|g'\|_{\infty, [a, b]}
\]

which prove the first inequality in (3.12).
To prove the second and third inequalities in (3.12), we apply (1.10) by setting 
\(d = r, t\) and \(c = a, u\); respectively, then we get
\[
\left| \left( \int_a^v \frac{|r - a|^\beta}{r - a} \left\{ \frac{1}{r - a} \int_a^r g(s) ds - \frac{1}{v - a} \int_a^v g(s) ds \right\}^\beta dr \right| \right.
\]
\[
\leq \left\{ \begin{array}{ll}
\|g'\|_{p,v}^{\beta} \int_a^v (r - a)^\beta (v - r)^\beta \left[(r - a)^q + (v - r)^q\right]^{\beta/q} dr, & \mbox{if } g' \in L_p[a, v]; \\
 p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\
\end{array} \right.
\]
\[
\left\{ \begin{array}{ll}
\|g'\|_{p,v}^{\beta} \int_a^v (r - a)^\beta (v - r)^\beta dr, & \mbox{if } g' \in \mathbb{L}_1[a, v]. \\
 p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\
\end{array} \right.
\]
\[
\left\{ \begin{array}{ll}
\|g'\|_{1,v}^{\beta} \cdot (v - a)^{2\beta + 1} B(\beta + 1, \beta + 1), & \mbox{if } g' \in L_1[a, v]. \\
\end{array} \right.
\]
\begin{equation}
(3.14)
\end{equation}

\[
\left\{ \begin{array}{ll}
\frac{(v - a)^{2 + \frac{1}{p}}}{(g + 1)^{\beta/q}(v - a)^{\beta/q}} \cdot B\left(\beta + 1, \frac{\beta}{q} + 1\right) \left\|g'\right\|_{p,v}^{\beta} & \mbox{if } g' \in L_p[a, v]; \\
 p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\
\end{array} \right.
\]
\[
\left\{ \begin{array}{ll}
\|g'\|_{1,v}^{\beta} \cdot (v - a)^{2\beta + 1} B(\beta + 1, \beta + 1), & \mbox{if } g' \in L_1[a, v]. \\
\end{array} \right.
\]
\begin{equation}
(3.15)
\end{equation}

Similarly, we have
\[
\left| \left( \int_u^b |t - u|^\beta \left\{ \frac{1}{t - u} \int_u^t g(s) ds - \frac{1}{b - u} \int_u^b g(s) ds \right\}^\beta dt \right| \right.
\]
\[
\leq \left\{ \begin{array}{ll}
\frac{(b - u)^{2 + \frac{1}{p}}}{(g + 1)^{\beta/q}(b - u)^{\beta/q}} \cdot B\left(\beta + 1, \frac{\beta}{q} + 1\right) \left\|g'\right\|_{p,v}^{\beta} & \mbox{if } g' \in L_p[u, b]; \\
 p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\
\end{array} \right.
\]
\[
\left\{ \begin{array}{ll}
\|g'\|_{1,v}^{\beta} \cdot (b - u)^{2\beta + 1} B(\beta + 1, \beta + 1), & \mbox{if } g' \in L_1[u, b]. \\
\end{array} \right.
\]
\begin{equation}
(3.16)
\end{equation}

Substituting (3.14) and (3.15) in (3.13), we get
\[
\left| T^e_a (f, g) - T^b_a (f, g) \right|
\]
\[
\leq \left\{ \begin{array}{ll}
\frac{(v - a)^{1 + \frac{1}{p}} + (b - u)^{1 + \frac{1}{p}}}{(g + 1)^{\beta/q}(b - u)^{\beta/q}} \cdot B\left(\beta + 1, \frac{\beta}{q} + 1\right) \left\|g'\right\|_{p,v}^{\beta} \left\|f'\right\|_{\alpha,v}^{\beta} & \mbox{if } g' \in L_p[a, b]; \\
 & \mbox{for all } p, q, \alpha, \beta > 1 \mbox{ with } \frac{1}{p} + \frac{1}{q} = 1 \mbox{ and } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \mbox{ which proves the second and the third inequalities in (3.12).} \quad \square
\end{array} \right.
\]
Corollary 6. Under the assumptions of Theorem 9, we have

\[
\left| T^u_a (f, g) - T^b_u (f, g) \right|
\]

\[
\leq \begin{cases} 
\frac{(u-a)^{1+\frac{1}{q} + (b-u)^{1+\frac{1}{q}}}}{2^{1+\frac{1}{q}}} \cdot B^\frac{1}{q} \left( \frac{1}{q}, 1 \right) \cdot \| f' \|_{\alpha, [a,b]} \cdot \| g' \|_{\alpha, [a,b]} & \text{if } g' \in L_{\infty} [a,b] \\
\frac{(u-a)^{1+\frac{1}{q} + (b-u)^{1+\frac{1}{q}}}}{(q+1)^{1/q}} \cdot B^\frac{1}{q} \left( \frac{1}{q}, 1 \right) \cdot \| g' \|_{p, [a,b]} \cdot \| f' \|_{\alpha, [a,b]} & \text{if } g' \in L_p [a,b] \\
\end{cases}
\]

In particular case, if \( u = \frac{a+b}{2} \) we get

\[
\left| T^u_a (f, g) - T^b_u (f, g) \right|
\]

\[
\leq \begin{cases} 
\frac{(b-a)^{1+\frac{1}{q}}}{2^{1+\frac{1}{q}}} \cdot B^\frac{1}{q} \left( \frac{1}{q}, 1 \right) \cdot \| f' \|_{\alpha, [a,b]} \cdot \| g' \|_{\alpha, [a,b]} & \text{if } g' \in L_{\infty} [a,b] \\
\frac{2^{1+\frac{1}{q}}}{(b-a)^{1+\frac{1}{q}}} \cdot B^\frac{1}{q} \left( \frac{1}{q}, 1 \right) \cdot \| g' \|_{p, [a,b]} \cdot \| f' \|_{\alpha, [a,b]} & \text{if } g' \in L_p [a,b] \\
\end{cases}
\]

Remark 1. For the second inequality in (3.12) we have the following particular cases:

(1) If \( \alpha = p \) and \( \beta = q \), then we have

\[
\left| T^u_a (f, g) - T^b_u (f, g) \right|
\]

\[
\leq \frac{(v-a)^{1+\frac{1}{q} + (b-u)^{1+\frac{1}{q}}}}{(q+1)^{1/q}} \cdot B^\frac{1}{q} \left( q+1, 2 \right) \cdot \| g' \|_{p, [a,b]} \cdot \| f' \|_{p, [a,b]} 
\]

Therefore, as \( v \rightarrow u^+ \) we have

\[
\left| T^u_a (f, g) - T^b_u (f, g) \right|
\]

\[
\leq \frac{(u-a)^{1+\frac{1}{q} + (b-u)^{1+\frac{1}{q}}}}{(q+1)^{1/q}} \cdot B^\frac{1}{q} \left( q+1, 2 \right) \cdot \| g' \|_{p, [a,b]} \cdot \| f' \|_{p, [a,b]} 
\]
and for \( u = \frac{a+b}{2} \) we have

\[
\left| \mathcal{T}^{\frac{a+b}{2}}_a (f, g) - \mathcal{T}^{\frac{b-a}{2}}_b (f, g) \right| \\
\leq \frac{(b-a)^{1+\frac{p}{q}}}{2^{1+\frac{p}{q}} (q+1)^\frac{p}{q}} \cdot B^\beta (q+1, 2) \cdot B^\alpha \left( p + 1, \frac{p}{q} + 1 \right) \cdot \frac{\| g' \|_{p,[a,b]} \cdot \| f' \|_{p,[a,b]} \cdot}{\| u \|_{p,[a,b]}}.
\]

(2) If \( \alpha = q \) and \( \beta = p \), then we have

\[
\left| \mathcal{T}^u_a (f, g) - \mathcal{T}^v_b (f, g) \right| \\
\leq \frac{(v-a)^{1+\frac{p}{q}} + (b-u)^{1+\frac{p}{q}}}{(q+1)^\frac{p}{q}} \cdot \frac{\| g' \|_{p,[a,b]} \cdot \| f' \|_{q,[a,b]} \cdot}{\| u \|_{p,[a,b]}}.
\]

Similarly, as \( v \to u^+ \), we have

\[
\left| \mathcal{T}^u_a (f, g) - \mathcal{T}^v_b (f, g) \right| \\
\leq \frac{(u-a)^{1+\frac{p}{q}} + (b-u)^{1+\frac{p}{q}}}{(q+1)^\frac{p}{q}} \cdot \frac{\| g' \|_{p,[a,b]} \cdot \| f' \|_{q,[a,b]} \cdot}{\| u \|_{p,[a,b]}}.
\]

and for \( u = \frac{a+b}{2} \) we have

\[
\left| \mathcal{T}^{\frac{a+b}{2}}_a (f, g) - \mathcal{T}^{\frac{b-a}{2}}_b (f, g) \right| \\
\leq \frac{(b-a)^{1+\frac{p}{q}}}{2^{1+\frac{p}{q}} (q+1)^\frac{p}{q}} \cdot \frac{\| g' \|_{p,[a,b]} \cdot \| f' \|_{q,[a,b]} \cdot}{\| u \|_{p,[a,b]}}.
\]

for all \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Remark 2.** In this work, all obtained bounds for the difference between two Čebyšev functional were taken under the assumption that \([a, v] \cap [u, b] = [u, v] \). The same bounds hold with a few changes in the case that \([u, v] \subset [a, b] \). Namely, replace every ‘a’ (in the obtained results) by ‘u’; every ‘u’ (in the obtained results) by ‘a’ and accordingly the differences \((v-u), (b-a)\) instead of \((v-a), (b-u)\).

**Remark 3.** All obtained bounds hold for the Čebyšev functional \( \left| \mathcal{T}^b_a (f, g) \right| \), this can be done by noting that \( \left| \mathcal{T}^b_a (f, g) - \mathcal{T}^v_b (f, g) \right| \to \left| \mathcal{T}^b_a (f, g) \right| \) as \( v = u \to a \).

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