Renormalization group, dimer-dimer scattering, and three-body forces

Boris Krippa$^{1,2}$, Niels R. Walet$^2$, Michael C. Birse$^2$

$^1$Institute for Theoretical and Experimental Physics, Moscow, 117259, Russia
$^2$School of Physics and Astronomy, The University of Manchester, Manchester, M13 9PL, UK

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We study the ratio between the fermion-fermion scattering length and the dimer-dimer scattering length for systems of nonrelativistic fermions, using the same functional renormalisation technique as previously applied to fermionic matter. We find a strong dependence on the cutoff function used in the renormalisation flow for a two-body truncation of the action. Adding a simple three-body term substantially reduces this dependence.

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Ultra-cold Fermi gases provide a fertile ground for research in atomic physics. An important feature of such systems is superfluidity, which is the result of attractive fermion-fermion interactions leading to pairing. Recent advances using Feshbach resonances allow a tuning of the fermion-fermion S-wave scattering length $a_F$. For negative scattering length we get the weak-coupling BCS state. For positive values of $a_F$ bound states of two fermions—“dimers”—form and these can lead to a Bose-Einstein condensate (BEC) [1]. The size of dimers is determined by the fermion-fermion scattering length and their binding energy is of order $1/a_F^2$.

For a sufficiently dilute and cold gas of dimers the main dynamical quantity characterising their interaction is the dimer-dimer scattering length $a_B$. The exact relation between dimer-dimer and fermion-fermion scattering lengths $a_B = 0.6a_F$ was established in Ref. [2] by solving the Schrödinger equation for two composite bosons interacting with an attractive zero-range potential. This method is difficult to extend to the many-body case. Therefore, it is useful to study the ratio $a_B/a_F$ in an approach which can be used both for few and many-body problems.

In this paper we calculate $a_B$ in the framework of the Exact Renormalisation Group (ERG) approach. (For reviews, see Refs. [3, 4].) This technique has been previously used to study a variety of physical systems, from systems of nonrelativistic fermions [3, 5] to quark models [12] and gauge theories [13]. It is based on the scale-dependent average effective action $\Gamma_k$, where $k$ is an auxiliary running scale. The action at scale $k$ contains the effects of field fluctuations with momenta $q$ larger than $k$ only. In the limit $k \to 0$ all fluctuations are included and the full effective action is recovered. In practice one introduces a set of $k$-dependent cutoff functions $R(q)$, which suppress the effect of modes with $q < k$ in the path integral for the action by giving them a large $k$ dependent mass. The functions $R(q)$ should vanish in the limit $k \to 0$ and behave like $k^\alpha$ with $\alpha > 0$ when $q \to 0$.

With this prescription the average effective action at large $k$ is just the classical action of the theory—in our case nonrelativistic fermions with a local interaction. An exact solution of the functional RG equation should be independent of the choice of cutoff for $k \to 0$. However, in practice, truncations of the action inevitably lead to some cutoff dependence of the results. We can use this dependence as a measure of the quality of the truncation. With this tool, we shall see that the standard parametrisation of the effective action containing only two-body terms is insufficient, and that we get better results by including the simplest three-body term.

The flow of the effective action satisfies

$$\partial_k \Gamma = -\frac{i}{2} \text{STr} \left[ (\partial_k R) (\Gamma^{(2)} - R)^{-1} \right]. \quad (1)$$

where $\Gamma^{(2)}$ is the second functional derivative with respect to the fields, and the cutoff functions in the mass-like term $R(k)$ drive the RG evolution. The operation $\text{STr}$ denotes the supertrace [14] taken over energy-momentum variables and internal indices and is defined by

$$\text{STr} \left( \frac{A_{BB}}{A_{BF}} \frac{A_{BF}}{A_{FF}} \right) = \text{Tr}(A_{BB}) - \text{Tr}(A_{FF}). \quad (2)$$

The evolution equation for the average effective action has a one-loop structure, but contains a fully dressed, scale-dependent propagator $(\Gamma^{(2)} - R)^{-1}$. Thus, despite its apparently simple form, Eq. (1) is actually a functional differential equation. In the absence of general methods to solve such equations numerically we must resort to approximations. One common approach is to parametrise the effective action with a finite set of terms, turning the evolution into a system of coupled ordinary differential equations for their coefficients. These equations can then be solved numerically. Here we study possible truncations for fermionic few-body systems, and our choice of ansatz for the action is motivated by both ERG studies of many-body systems [3, 5] and effective field theories (EFTs) for few-fermion systems [13]. The technique we use is similar to the one used in Ref. [9] to analyse the scattering in the system of two nonrelativistic fermions.

A rather different approach to the low-energy fermion-dimer scattering was considered in Ref. [8]. There
the Skorniakov-Ter-Matiosian equation \cite{13,15} was derived from the RG flow with energy- and momentum-dependent three-body couplings. Whilst the results obtained in that work demonstrate the formal equivalence between an RG approach and three-body quantum mechanics, it seems to be very difficult to extend the treatment to the more complicated cases of four-body or many-body systems. In the present paper we focus on the effect of the three-body forces in the system of two dimers, keeping in mind possible extensions of the formalism to many-body systems, such as that attempted in Ref. \cite{11}. Therefore we only include the simplest three-body term.

The formation of the trion – the correlated state of three fermions – was also studied in Refs. \cite{6, 7}, where many the technical details can be found, and so we give only a brief account of the formalism to many-body systems, such as that attempted through the system and \cite{17}.

We first examine the case when fermions interact only pairwise. This has previously been considered in Refs. \cite{6, 7}, where many the technical details can be found, and so we give only a brief account of the formalism, concentrating on the dimer-dimer scattering length \( a_B \). We use this to extend the results of Ref. \cite{6} and examine the cutoff dependence of \( a_B \). We then turn to our main task, the inclusion of three-body terms in \( \Gamma \).

As the cutoff scale tends to infinity, we demand the action to be a purely fermionic one containing a contact two-body interaction without derivatives. This kind of interaction has been extensively used in the EFT-based studies of nuclear forces \cite{17}. It is convenient to re-express the theory in terms of an auxiliary composite boson field by making a Hubbard-Stratonovich transformation. This replaces the two-body interaction by a Yukawa-type coupling between the fermions and the auxiliary boson. A kinetic term for the boson is then generated by the RG evolution. The minimal effective action used in previous work is \cite{13, 15}:

\[
\Gamma_{\text{min}}[\psi, \psi^\dagger, \phi, \phi^\dagger, k] = \int d^4x \left[ \int d^4x' \phi^\dagger(x) \Pi(x, x'; k) \phi(x') + \psi^\dagger(x) \left( i \partial_t + \frac{1}{2M} \nabla^2 \right) \psi(x) \right. \\
\left. - \frac{i}{2} g \left( \psi^T(x) \sigma_2 \psi(x) \phi^\dagger(x) - \psi^\dagger(x) \sigma_2 \psi^T(x) \phi(x) \right) \right] \\
- \frac{1}{2} u_2 \left( \phi^\dagger(x) \phi(x) \right)^2.
\]  

Here \( \Pi(x, x', k) \) is the scale-dependent boson self-energy and \( u_2 \) parametrises the boson-boson interaction which can be generated by the evolution. The latter is equivalent to a four-body interaction in terms of the underlying fermions. To this action we add a local three-body interaction, similar to that used in another context in Ref. \cite{17}.

Expressed in terms of the boson field this has the form

\[
\Gamma[\psi, \psi^\dagger, \phi, \phi^\dagger, k] = \Gamma_{\text{min}}[\psi, \psi^\dagger, \phi, \phi^\dagger, k] - \lambda \int d^4x \psi^\dagger(x) \phi^\dagger(x) \phi(x) \psi(x).
\]  

We concentrate first on the two-body part of Eq. \( \ref{eq:3} \). The evolution of the boson self-energy is given by

\[
\partial_k \Pi(x, x', k) = \frac{\delta^2}{\delta \phi^\dagger(x') \delta \phi(x)} \partial_k \Gamma|_{\phi = 0},
\]  

although from now on we shall express all evolution in momentum space. Note that only fermion loops contribute to the evolution of the boson self-energy in vacuum. These depend on the fermionic cutoff function \( R_F \), for which we take the form \( \ref{eq:17} \)

\[
R_F(q, k) = \frac{k^2 - q^2}{2M} \theta(k - q).
\]  

We impose the boundary condition that the scattering amplitude in the physical limit \( k \to 0 \) reproduces the fermion-fermion scattering length,

\[
\frac{1}{\Gamma(p)} = \frac{1}{g^2} \Pi(P_0, P, 0) = \frac{M}{4\pi a_{BF}}.
\]  

Here \( P_0 \) (\( P \)) denote the total energy (momentum) flowing through the system and \( p = \sqrt{2MP_0 - P^2/2} \) is the relative momentum of the two fermions. Integrating the resulting ERG equation gives

\[
\Pi(P_0, P, k) = g^2 M \left[ -\frac{3}{4} k + \frac{\pi}{a_F} + \frac{16}{3k} \left( MP_0 - \frac{P^2}{2} \right) \right] - \frac{P^3}{24k^2} + \ldots
\]  

Using a gradient expansion of the action, we can define boson wave-function and mass renormalisation factors by

\[
Z_\phi(k) = \left. \frac{\partial}{\partial P_0} \Pi(P_0, P, k) \right|_{P_0 = \xi_D, P = 0},
\]  

and

\[
\frac{1}{4M} Z_m(k) = - \frac{\partial}{\partial P_2} \Pi(P_0, P, k) \bigg|_{P_0 = \xi_D, P = 0},
\]  

where \( \xi_D = -1/(Ma_F^2) \) denotes the bound-state energy of a pair of fermions. Note that these renormalisation factors are only identical in vacuum for a limited subset of cutoff functions (which, like \( \xi_D \) must preserve Galilean invariance to lowest order), otherwise the identity \( Z_\phi(k) = Z_m(k) \) holds only in the physical limit \( k \to 0 \) and the evolution of these renormalisation factors should be calculated separately.

The evolution of the boson-boson scattering amplitude follows from

\[
-\frac{2}{(2\pi)^4} \partial_k u_2(\xi_D, k) = \frac{\delta^4}{\delta \phi^\dagger(\xi_D, 0) \delta \phi(\xi_D, 0)} \partial_k \Gamma|_{\phi = 0}.
\]  

\[\text{(11)}\]
This equation can be separated into fermionic and bosonic contributions containing $\partial_k R_F$ and $\partial_k R_B$, respectively. We first look at the mean-field result, where bosonic contributions are neglected. The evolution of $u_2$ is then given by

$$\partial_k u_2 = -\frac{3g^4}{4} \int \frac{d^3q}{(2\pi)^3} \frac{\partial_k R_F}{[(E_{FR}(q,k) - E_D/2]^2}$$

(12)

where $E_{FR}(q,k) = 1/2\pi q^2 + R_F(q,k)$. Integrating this gives

$$u_2(0) = \frac{1}{16\pi} M^3 g^4 a_F^3$$

(13)

where we have again used the sharp cutoff function of Eq. (6). The scattering amplitude at threshold is

$$T_{BB} = \frac{8\pi}{2M} a_B = \frac{2u_2(0)}{Z_\phi^2} = \frac{8\pi a_F}{M}$$

(14)

giving the well-known mean-field result $a_B = 2a_F$ which is far from the exact value $a_B = 0.6a_F$. This implies that beyond-mean-field effects such as dimer-dimer rescattering are important be considered.

To include such effects we must take into account the boson loops. After some algebra, we find

$$\partial_k u_2|_B = \frac{u_2^2(k)}{2Z_\phi^3} \int \frac{d^3q}{(2\pi)^3} \frac{\partial_k R_B}{[E_{BR}(q,k) - E_D]^2}$$

(15)

where

$$E_{BR}(q,k) = \frac{1}{4M} q^2 + \frac{u_1(k)}{Z_\phi(k)} + \frac{R_B(q,k)}{Z_\phi(k)}$$

(16)

and

$$u_1(k) = -\Pi(E_D,0,k).$$

(17)

We choose the bosonic cutoff function to be as close as possible to the fermionic one,

$$R_B(q,k) = Z_\phi \frac{(c_Bk)^2 - q^2}{4M} \theta(c_Bk - q),$$

(18)

apart from the addition of a parameter $c_B$, which sets the relative scale of the fermionic and bosonic regulators, and a factor of $Z_\phi$. The latter has the important advantage of leading to a consistent scaling behaviour, so that all contributions to a single evolution equation decay with the same power of $k$ for large $k$. Moreover it also gives $a_F$-scaling, where all terms in a single equation have the same dependence on $a_F$.

The mean-field result is recovered for $c_B = 0$ while the opposite limit $c_B \to \infty$ leads to $a_B \to 0$. Using $c_B = 1$ gives a ratio of $a_B/a_F = 1.13$. Taking $c_B = \sqrt{2}$ as in Ref. [6] results in $a_B/a_F = 0.75$. In general, the results show a rather strong dependence on the relative scale parameter $c_B$, as can be seen below in Fig. 2. Such dependence of a physical result on the choice of cutoff must be an artifact of our truncation of the action, since it should vanish for an exact solution of the functional RG equation. It indicates that the minimal ansatz used for the effective action needs to be extended.

We now consider the effect of adding the local three-body force of Eq. (4). The evolution equations for $u_1$ and $Z_\phi$ remain unchanged but the one for $u_2$ now becomes

$$\partial_k u_2 = -\frac{3g^4}{4} \int \frac{d^3q}{(2\pi)^3} \frac{\partial_k R_F}{[E_{FR}(q,k) - E_D/2]^2}$$

$$-2\lambda g^2 \int \frac{d^3q}{(2\pi)^3} \frac{\partial_k R_F}{[E_{FR}(q,k) - E_D/2]^3}$$

$$+ \frac{u_2^2}{2Z_\phi} \int \frac{d^3q}{(2\pi)^3} \frac{\partial_k R_B}{[E_{BR}(q,k) - E_D]^2}$$

(19)

The evolution equation for $\lambda$ is defined by an expansion about the energy of the bound state pole for bosons, and half that energy for fermions,

$$\partial_k \lambda = \frac{i}{2} \frac{\delta^4 ST[\partial_k R(\Gamma^{(2)} - R)^{-1}]}{\delta \phi(\Gamma^D,0) \delta \phi(\Gamma^D,0) \delta \psi(\Gamma^D/2,0) \delta \psi(\Gamma^D/2,0)}.$$  

(20)

There are three distinct contributions to the running of $\lambda$, coming from ladder, triangle and box diagrams, as shown in Fig. 1. We denote the corresponding driving terms as $D_l$, $D_t$ and $D_b$, splitting the last two into their fermionic and bosonic contributions. After evaluation of traces and contour integrals, we get

$$D_l = \lambda^2 \int \frac{d^3q}{(2\pi)^3} \frac{\partial_k (R_F Z_\phi)}{E_{FR,D} Z_\phi + E_{BR,D}^2},$$

(21)

$$D_t^F = g^2 \lambda \int \frac{d^3q}{(2\pi)^3} \frac{\partial_k R_F(E_{FR,D} Z_\phi + E_{BR,D})}{E_{FR,D}^2 E_{FR,D}^2 Z_\phi + E_{BR,D}^2},$$

(22)

$$D_t^B = g^2 \lambda \int \frac{d^3q}{(2\pi)^3} \frac{\partial_k R_B(E_{FR,D} Z_\phi + E_{BR,D})}{E_{FR,D}^2 E_{FR,D}^2 Z_\phi + E_{BR,D}^2},$$

(23)

$$D_b^F = \frac{g^4}{4} \int \frac{d^3q}{(2\pi)^3} \frac{\partial_k R_F(2E_{BR,D} + 3Z_\phi E_{FR,D})}{E_{FR,D}^2 E_{FR,D}^2 Z_\phi + E_{BR,D}^2},$$

(24)

$$D_b^B = \frac{g^4}{4} \int \frac{d^3q}{(2\pi)^3} \frac{\partial_k R_B(2E_{FR,D} Z_\phi + E_{BR,D})}{E_{FR,D}^2 E_{FR,D}^2 Z_\phi + E_{BR,D}^2},$$

(25)
The full behaviour of $a_B/a_F$ as a function of $c_B$ is presented in Fig. 2. This shows that, as well as reducing the overall size of the ratio $a_B/a_F$, the inclusion of the three-body force significantly weakens its dependence on the relative scale $c_B$. We expect the qualitative features of this picture to remain correct for any bosonic regulator although the quantitative details will depend on the particular functional form used.

Note that for large $c_B$, the dominant contributions to $a_B/a_F$ come from the boson-loop terms in the equation for $u_2$. Since these do not depend on the three-body coupling $\lambda$, the two curves approach each other. Moreover, this limit corresponds to integrating out the fermions first, which generates a non-zero value for $u_2$ at the start of the bosonic integration. In the limit $c_B \to \infty$, this coupling is driven to the trivial fixed point, $u_2 = 0$, since we have no terms to cancel the linearly divergent boson-boson loop diagram and the diagrams with three-body couplings are too weak to alter this behaviour.

On the other hand, the main contributions for small $c_B$ come from the fermion and mixed fermion-boson loops, the latter arising in the three-body coupling. In particular, the mixed boson-fermion loop diagrams containing the fermionic cut-off contribute to the evolution of the three-body coupling, even when the bosonic degrees of freedom have been integrated out. As a result, inclusion of $\lambda$ leads to a significant deviation from the mean-field result, $a_B/a_F = 2$, that survives in the limit $c_B \to 0$.

These results for very large or very small values of $c_B$ should not be taken too seriously. Arguments based on “optimisation” of the cut-off function, see Ref. [19], indicate that one should choose the cut-off to try to maximise the rate of convergence for our expansion of the action. Although a precise criterion has not yet been defined for nonrelativistic theories, such arguments suggest a choice where bosons and fermion cut-offs run at roughly the same rate, i.e. $c_B$ is of the order of 1.

In spite of this clear improvement over calculations that include two-body interactions only, adding the simplest possible three-body term is not enough to ensure that the results are completely independent of the parameter $c_B$ in the region $c_B \sim 1$. It is worth emphasising again that, as long as any truncation of the effective action is made, the results will never be completely independent of the choice of cutoff. It seems likely that further extensions of the effective action will result in stability of the results with respect to the variations of $c_B$ in wider region. Such extensions could include both four-body interactions as well as energy and/or momentum dependent three-body forces. Work along these lines is now in progress.

In summary, we have performed an ERG analysis of the boson-boson scattering length in a system of nonrelativistic fermions. Our study indicates that, while a simple ansatz with only local two-body interactions can yield results that are close to the exact value, these results are sensitive to the value of a parameter controlling the relative scales of the fermionic and bosonic cutoffs. We show that the inclusion of a local three-body interaction brings the scattering length closer to the exact value and significantly reduces its sensitivity to the relative scale parameter.

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