NERETIN GROUPS ADMIT NO NON-TRIVIAL INVARIANT RANDOM SUBGROUPS

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Abstract. We show that Neretin groups have no non-trivial invariant random subgroups. These groups provide first examples of non-discrete, compactly generated, locally compact groups with this property.

1. Introduction

Let $G$ be a locally compact group and denote by $\text{Sub}(G)$ the space of closed subgroups of $G$ equipped with the Chabauty topology. An invariant random subgroup (IRS), defined in [AGV14], is a Borel probability measure on $\text{Sub}(G)$ which is invariant under conjugation by $G$.

Normal subgroups corresponds to $\delta$-measures on $\text{Sub}(G)$. A subgroup $H$ of $G$ is said to be of finite co-volume, or co-finite for short, if $H$ is closed and $G/H$ carries a $G$-invariant probability measure. A lattice in $G$ is a discrete subgroup of finite co-volume. Co-finite subgroups give rise to IRSs: the pushforward of a $G$-invariant probability measure on $G/H$ under the map $gH \mapsto gHg^{-1}$ is an IRS. Thus IRSs can be viewed as generalizations of both normal subgroups and lattices. It is natural to ask what properties of normal subgroups or lattices can be extended to IRSs.

In the other direction, viewing lattices as elements in the space of IRSs on $\text{Sub}(G)$ turns out to be a powerful tool in studying lattices, see [ABB+17, Gel18a].

Invariant random subgroups are closely related to probability measure preserving (p.m.p.) actions. Given a p.m.p. action $G \actson (X, m)$, the pushforward of the probability measure $m$ under the stabilizer map $x \mapsto St_G(x)$ gives rise to an IRS, which we refer to as the stabilizer IRS of the action $G \actson (X, m)$. It is known that all IRSs arise in this way ([AGV14, ABB+17]), and moreover, an ergodic IRS arises as the stabilizer IRS of an ergodic p.m.p. action ([CP17, Proposition 3.5]).

We say that $G$ has no non-trivial IRSs if every IRS is a convex combination of $\delta_{\{id\}}$ and $\delta_G$. By the characterization of IRSs in terms of stabilizers as cited above, $G$ has no non-trivial IRSs if and only if every non-trivial ergodic p.m.p. action of $G$ is essentially free. Recall that an action is essentially free if there is a full measure subsets consisting of points with trivial stabilizer.

In [ABB+18] it is asked whether there exists a simple, non-discrete locally compact group which does not have non-trivial IRSs; and Neretin groups are proposed as candidates. A more detailed discussion of this question can be found in the survey [Gel18b]. The supporting evidences are that Neretin groups are abstractly simple by [Kap99], and they are first examples of locally compact group which do not admit any lattices by [BCGM12]. Note that many examples of groups with no nontrivial IRSs can be found among countable groups, see for example [DM14].

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First examples of non-discrete locally compact groups with no nontrivial IRSs are constructed in \[\text{LBMB18}\]. The groups constructed in \[\text{LBMB18}\] are not compactly generated.

Let $T$ be a $(d + 1)$-regular unrooted tree. The Neretin group $\mathcal{N}_d$ is the group of almost automorphisms of $T$, or equivalently, the group of spheromorphisms of $\partial T$. The group $\mathcal{N}_d$ is introduced by Neretin in \[\text{Ner92}\] as combinatorial analogues of the group of diffeomorphisms of the circle, with some ideas tracing back to his earlier work \[\text{Ner84}\]. There is a unique group topology on $\mathcal{N}_d$ such that the natural inclusion $\text{Aut}(T) \hookrightarrow \mathcal{N}_d$ is continuous and open, and endowed with this topology, $\mathcal{N}_d$ is locally compact and compactly generated, see \[\text{CDM11}\].

Neretin groups are now fundamental examples in the growing structure theory of totally disconnected locally compact groups, see \[\text{CRW17a, CRW17b}\] and references therein.

The main goal of the present work is to show that Neretin groups admit no nontrivial IRSs. Our argument applies to a generalization of Neretin groups, called coloured Neretin groups, which are introduced and studied recently in \[\text{Led17}\]. We now briefly describe these groups, more precise definitions are recalled in Section 3. For every vertex of $T$, fix a bijection from the edges incident to it to the set of colours $D = \{0, 1, \ldots, d\}$. Given a subgroup $F \leq \text{Sym}(D)$, Burger and Mozes \[\text{BM00}\] constructed a closed subgroup of $\text{Aut}(T)$, denoted by $U(F)$, which is the universal group with local actions at every vertex in $F$. The coloured Neretin group $\mathcal{N}_F$ is defined as the group of $U(F)$-almost automorphisms. It is shown in \[\text{Led17}\] that there is a unique group topology on $\mathcal{N}_F$ such that the inclusion $U(F) \hookrightarrow \mathcal{N}_F$ is continuous and open, and endowed with this topology, $\mathcal{N}_F$ is locally compact and compactly generated.

**Theorem 1.1.** Let $F < \text{Sym}(D)$ be any subgroup. Let $\mu$ be an ergodic IRS of the coloured Neretin group $\mathcal{N}_F$. Then either $\mu = \delta_{\text{id}}$ or $\mu$-a.e. $H$ contains the derived subgroup $\mathcal{N}_F' = [\mathcal{N}_F, \mathcal{N}_F]$. In particular, the Neretin group $\mathcal{N}_d$, corresponding to the case $F = \text{Sym}(D)$, admits no non-trivial IRSs.

For general $F$, by \[\text{Led17}\] $\mathcal{N}_F'$ is simple, open and of finite index in $\mathcal{N}_F$. In particular, Theorem 1.1 implies that $\mathcal{N}_F$ has no lattices, answering \[\text{Led17}\] Question 1.5 by removing the constraints on $F$.

The key step in the proof of Theorem 1.1 is the following statement on containment of rigid stabilizers. Let $G$ be a group acting on a topological space $X$ by homeomorphisms and $U \subseteq X$ an open subset. Denote by $R_G(U)$ the rigid stabilizer of $U$ in $G$, that is, $R_G(U) = \{g \in G : x \cdot g = x \text{ for all } x \in X \setminus U\}$. Given a finite subtree $A$ of $T$, denote by $B_n(A)$ the subtree with vertices within distance $n$ to $A$. Denote by $O^\beta_A$ the subgroup which consists of almost automorphisms which can be represented by a triple $(B_n(A), B_n(A), \varphi)$ for some $n \in \mathbb{N}$, see the precise definition in Section 3. When $A$ consists of a single vertex, $O^\beta_A$ is the same as the group $O$ considered in \[\text{BCGM12, Led17}\]. Note that $O^\beta_A$ is an open subgroup of $\mathcal{N}_F$.

**Proposition 1.2.** Let $A$ be a finite complete subtree of $T$ and $\mu_A$ be an IRS of $O^\beta_A$. Then $\mu_A$-a.e. $H$ satisfies the following: if $H \neq \{\text{id}\}$, then there exists a non-empty open set $U \subseteq \partial T$ such that $[R_{O^\beta_A}(U), R_{O^\beta_A}(U)] < H$.

The statement of Proposition 1.2 is an exact analogue of the double commutator lemma for IRSs of a countable group in \[\text{Zhe19, Theorem 1.2}\]. In the countable
setting, one can show the double commutator lemma for IRSs with rather soft arguments. It seems to be an interesting question to what extent such a result is true in the non-discrete setting. Proposition 1.2 is proved via studying induced IRSs of finite sub-quotients of \( O_A^d \), see more discussion below.

The Higman-Thompson group \( V_{d,d+1} \) embeds in \( N_{d} \) as a dense subgroup, see [CDM11]. It is observed in [Nek04] that the topological full group of the one-sided Bernoulli shift over the alphabet with \( d \) letters is isomorphic to a Higman-Thompson group. More generally, topological full groups of one-sided irreducible shifts of finite type are introduced and investigated in [Mat15]. For the coloured Neretin group \( N_{F} \), it is shown in [Led17] Theorem 3.9 that \( N_{F} \) has a dense subgroup \( V_{F} \), which can be identified as the topological full group of a one-sided irreducible shift of finite type. By [DM14] Corollary 3.9, the countable group \( V'_{F} \) does not have non-trivial IRSs.

Proposition 1.2 allows us to transfer the problem of IRSs of \( N_{F} \) to \( V_{F} \) by considering the intersection map \( H \mapsto H \cap V_{F} \). More precisely, Proposition 1.2 guarantees that almost surely \( H \cap V_{F} \neq \{id\} \), so that known results on IRSs of \( V_{F} \) as cited above can be applied, see Section 1.2.

Most of this paper is devoted to the proof of Proposition 1.2. The basic idea in the proof is that in the finite sub-quotients of \( O_A^d \) considered, if a subgroup does not contain a large finite alternating group, then the probability that a random conjugate of it containing a specific kind of almost automorphisms is small, quantitatively. Then the Borel-Cantelli lemma can be applied to combine the estimates in finite sub-quotients to obtain almost sure statements on the IRS. In Section 2 we formulate two general bounds for IRSs in countable groups in terms of subgroup index (Lemma 2.1) and conjugacy class size (Lemma 2.2). The starting point of the proof in [BCGM12] for absence of lattices in \( O \) is a co-volume estimate in the finite sub-quotients, which is later confronted by the discreteness of the lattice. In some sense the subgroup index Lemma 2.1 is a replacement for co-volume bounds in the context of IRSs, although it is weaker. An outline of the proof of Proposition 1.2 can be found in Section 4 after introducing the necessary objects. We mention that the proof is rather self-contained: the only result on finite symmetric groups invoked is the Praeger-Saxl bound [PS80] on the orders of primitive subgroups.

Following [CRW17a, CRW17b], let \( \mathcal{A} \) be the class of all non-discrete, compactly generated, locally compact groups that are topologically simple. There is an evolving theory which treats \( \mathcal{A} \) as a whole, see the survey [Cap16] and references therein. The class \( \mathcal{A} \) naturally divides into two subclasses, \( \mathcal{A}_{Lie} \) which consists of connected Lie groups in \( \mathcal{A} \); and \( \mathcal{A}_{td} \) which consists of totally disconnected groups in \( \mathcal{A} \). Motivated by the theory of lattices in semisimple Lie groups, it is natural to investigate lattices and more generally, IRSs of groups in the class \( \mathcal{A}_{td} \). It is reasonable to expect that an abundance of examples of non-discrete compactly generated locally compact groups with no non-trivial IRSs can be found in the class \( \mathcal{A}_{td} \): for instances, some topological full groups similar to Neretin type groups and certain simple groups acting on trees with almost prescribed local action introduced and studied in [LB16]. Our proof relies on properties of finite symmetric groups. It is interesting to develop a more conceptual and robust approach that could contribute to the study of \( \mathcal{A}_{td} \).

**Organization of the paper.** In Section 2 we formulate two quantitative bounds for IRSs of countable groups, which might be useful as general tools. Section 3...
contains preliminaries on Neretin type groups. In Section 3 we introduce the induced IRSs in certain sub-quotients and relevant events we consider are introduced. In Section 4 we explain how to deduce Theorem 1.1 from Proposition 1.2. Section 5 contains an auxiliary bound for the probability of two randomly chosen sets to be in the same orbit of some tree automorphism group. In Section 6 we present the proof of Proposition 1.2 when \( F = \text{Sym}(D) \), e.g., is complete. Section 7 explains the additional arguments needed to prove Proposition 1.2 for general \( F \).

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2. Two counting lemmas for IRSs of countable groups

Let \( G \) be a locally compact second countable group and \( \text{Sub}(G) \) the space of closed subgroups of \( G \). Recall that a pre-basis of open sets for the Chabauty topology is given by sets of the form

\[
\{ H \in \text{Sub}(G) : H \cap V \neq \emptyset \}, \quad \{ H \in \text{Sub}(G) : H \cap K = \emptyset \},
\]

where \( V \) is a relatively compact open subset of \( G \), and \( K \) a compact subset of \( G \). The space \( \text{Sub}(G) \) endowed with the Chabauty topology is compact and metrizable.

In this section we formulate two quantitative bounds which exploit the conjugation invariance of an IRS \( \mu \). These bounds are applicable to general countable groups, finite or infinite.

The first lemma bounds the probability that a random subgroup with distribution \( \mu \) intersects a given set, in terms of the size of the set and certain subgroup index. In order to state the bound we introduce some notations. Let \( \Gamma \acts X \) by homeomorphisms and \( U, V \) be two disjoint non-empty open sets in \( X \). Given a subgroup \( H \) of \( \Gamma \), let

\[
H_{U \to V} := \{ h \in H : V = U \cdot h \},
\]

and

\[
\bar{H}_{U \to V} := \{ h|_U : h \in H_{U \to V} \}.
\]

Elements of \( \bar{H}_{U \to V} \) are viewed as partial homeomorphisms with domain \( U \) and range \( V \), denoted by \( h|_U : U \to V \). Let \( \Omega_{U,V} \) be the event that \( H \) contains an element which maps \( U \) to \( V \), that is,

\[
\Omega_{U,V} = \{ H \in \text{Sub}(\Gamma) : H_{U \to V} \neq \emptyset \}.
\]

Recall that \( R_\Gamma(U) \) denotes the rigid stabilizer of \( U \) in \( \Gamma \). In probabilistic expressions involving \( \mathbb{E}_\mu \) or \( \mathbb{P}_\mu \), the symbol \( H \) denotes a random subgroup with distribution \( \mu \). Write \( 1_\Omega \) for the indicator of the set \( \Omega \).

**Lemma 2.1 (Subgroup index Lemma).** Let \( \Gamma \) be a countable group acting faithfully on a topological space \( X \) by homeomorphisms. Let \( U, V \) be two disjoint non-empty open sets such that \( \Gamma_{U \to V} \neq \emptyset \). Let \( \mu \) be an IRS of \( \Gamma \). Then for any finite subset \( A \) of partial homeomorphisms in \( \bar{H}_{U \to V} \), we have that \( \mu \)-a.e. \( H \) with \( \bar{H}_{U \to V} \cap A \neq \emptyset \)
satisfies \(|R_1(U) : (\bar{H}_{U \to U} \cap R_1(U))| < \infty\). Moreover,

\[
(2.3) \quad P_\mu (\bar{H}_{U \to V} \cap A \neq \emptyset) \leq E_\mu \left[ \min \left\{ \frac{|A|}{|R_1(U) : (\bar{H}_{U \to U} \cap R_1(U))|}, 1 \right\} 1_{\Omega_{U,V}}(H) \right].
\]

In the statement of the previous lemma it is understood that in the expression \(|R_1(U) : (\bar{H}_{U \to U} \cap R_1(U))|\), both \(R_1(U)\) and \(\bar{H}_{U \to U}\) are viewed as groups of homeomorphisms of \(U\). The second lemma is in the setting of product of two groups. It bounds the probability that a random subgroup contains a given set of group elements \(B\), in terms of the size of the conjugacy class of some coset associated with \(B\). Given a subset \(B \subseteq \Gamma\) of a subgroup \(W < \Gamma\), denote by \(Cl_W(B)\) the collection \(W\)-conjugates of \(B\), that is

\[
Cl_W(B) = \{ g^{-1}Bg : g \in W \}.
\]

**Lemma 2.2 (Conjugacy class size Lemma).** Suppose \(G\) is a subgroup of the product \(L_1 \times L_2\), where \(L_1, L_2\) are countable. Denote by \(\pi_i\) the projection \(L_1 \times L_2 \to L_i\), \(i = 1, 2\). Let \(\mu\) be an IRS of \(\Gamma\). Then for any subset \(B \subseteq \Gamma\), we have that \(\mu\)-a.e. \(H\) with \(H \supseteq B\) satisfies \(|Cl_{\pi_1}(\pi_1(B)H_1)| < \infty\). Moreover,

\[
(2.4) \quad P_\mu (H \supseteq B) \leq E_\mu \left[ \frac{1}{|Cl_{\pi_1}(\pi_1(B)H_1)|} 1_{\{\pi_2(A) \leq \pi_2(H)\}} \right],
\]

where \(H_1 = H \cap (L_1 \times \{id_{\Gamma_2}\})\) and \(N_1\) is the normalizer of \(H_1\) in \(\pi_1(\Gamma)\).

Lemma 2.1 and 2.2 can be used in conjunction as follows. Start with a pair of open sets \(U, V\) with \(U \cap V = \emptyset\) and \(P \subseteq \Gamma_{U \to V}\) such that \(\mu(\bar{H}_{U \to V} \cap P \neq \emptyset) > 0\). Then Lemma 2.1 provides information on \(\bar{H}_{U \to U}\), and moreover, those \(H\) with large index \(|R_1(U) : (\bar{H}_{U \to U} \cap R_1(U))|\) make small contribution to the probability \(\mu(\bar{H}_{U \to V} \cap P \neq \emptyset)\). Next consider the induced IRS in \(\Gamma_{U \to U}\), which is a subgroup of the product \(L_1 \times L_2\), where \(L_1 = \pi_U(\Gamma_{U \to U})\) and \(L_2 = \pi_U(\Gamma_{U \to U})\). Then Lemma 2.2 provides information on sizes of conjugacy classes in the quotient group \(\bar{H}_{U \to U}/R_H(U)\). Such information can be useful towards showing that \(R_H(U)\) must contain certain subgroups.

Given a non-discrete t.d.l.c. group, to apply such estimates towards understanding its IRSs, one first needs to choose a collection of finite sub-quotients and consider the induced IRSs. For Neretin groups, unlike countable groups discussed in [Zhe19], Lemma 2.1 and 2.2 applied to induced IRSs are useful, but far from being sufficient to conclude containment of rigid stabilizers. We will need additional probability estimates in the finite sub-quotients in the next sections. Such estimates heavily depend on the properties of finite symmetric groups.

The rest of this section is devoted to the proof of the two lemmas. We follow notations of regular conditional distributions in the book [Par67, Chapter V.8]. Let \((X, B), (Y, \mathcal{C})\) be two Borel spaces, \(\mathbb{P}\) a probability measure on \(B\) and \(\pi : X \to Y\) a measurable map. Let \(Q = \mathbb{P} \circ \pi^{-1}\) be probability measure on \(\mathcal{C}\) which is the pushforward of \(\mathbb{P}\). A regular conditional distribution given \(\pi\) is a mapping \(y \mapsto \mathbb{P}(y, \cdot)\) such that

(i) for each \(y \in Y\), \(\mathbb{P}(y, \cdot)\) is a probability measure on \(B\);

(ii) there exists a set \(N \in \mathcal{C}\) such that \(Q(N) = 0\) and for each \(y \in Y \setminus N\), \(\mathbb{P}(y, X \setminus \pi^{-1}({y})) = 0\);
(iii) for any \( A \in \mathcal{B} \), the map \( y \mapsto \mathbb{P}(y, A) \) is \( \mathcal{C} \)-measurable and

\[
\mathbb{P}(A) = \int_y \mathbb{P}(y, A) d\mathbb{Q}(y).
\]

We will refer to these three items as properties (i),(ii),(iii) of a regular conditional distribution.

Recall that a measure space \((X, \mathcal{B})\) is called a standard Borel space if it is isomorphic to some Polish space equipped with the Borel \( \sigma \)-field. It is classical that if \((X, \mathcal{B})\) and \((Y, \mathcal{C})\) are standard Borel spaces and \( \pi : X \to Y \) is measurable, then there exists such a regular conditional distribution \( y \mapsto \mathbb{P}(y, \cdot) \) with properties (i),(ii),(iii); and moreover it is unique: if \( \mathbb{P}'(y, \cdot) \) is another such mapping, then \( \{ y : \mathbb{P}'(y, \cdot) \neq \mathbb{P}(y, \cdot) \} \) is a set of \( \mathcal{Q} \)-measure 0, see \cite[Theorem 8.1]{Par67}.

In the proofs below, the outline is the same as in \cite{Zhe19}. We keep track of the subgroup index and conjugacy class sizes which appear in the argument, which naturally lead to the bounds stated in Lemma \ref{lem:2.1} and \ref{lem:2.2}.

**Proof of Lemma \ref{lem:2.1}** For \( U, V \) such that \( \mu(\Omega_{U,V}) = 0 \), the statement of the lemma is trivially true. Take a pair of \( U, V \) such that \( \mu(\Omega_{U,V}) > 0 \) and consider the random variables \( \bar{H}_{U\to V}, \bar{H}_{U\to V} \) and \( \bar{H}_{U\to U} \) as defined in \ref{eq:2.1}, \ref{eq:2.2}. Denote by \( \mathbb{P}_{U,V}^\mu(\bar{H}_{U\to U}, \cdot) \) the regular conditional distribution of \( (\bar{H}_{U\to V}, \bar{H}_{U\to U}) \) given \( \bar{H}_{U\to V} \), where \( H \) has distribution \( \mu_{U,V} = \mu(\Omega_{U,V}) \) on \( \Omega_{U,V} \).

Since \( \bar{H}_{U\to V} \) is a coset of \( \bar{H}_{U\to U} \) and \( \Gamma \) is countable, we have that \( \mathbb{P}_{U,V}^\mu(\bar{H}_{U\to U}, \cdot) \) is a probability measure on a countable set. Conjugation invariance of \( \mu \) implies that any \( g \in G_{U\to V} \) and \( \gamma \in R_1(V) \),

\[
\mathbb{P}_{U,V}^\mu \left( \bar{H}_{U\to U}, \left\{ (\bar{H}_{U\to U} g|_U, \bar{H}_{U\to U}) \right\} \right) = \mathbb{P}_{U,V}^\mu \left( \bar{H}_{U\to U}, \left\{ (\bar{H}_{U\to U} g|_V \gamma|_V, \bar{H}_{U\to U}) \right\} \right),
\]

see \cite[Lemma 2.3]{Zhe19}. For \( \mu \)-a.e. \( H \in \Omega_{U,V} \), there must exist a coset \( \bar{H}_{U\to U} | \sigma|_U \), \( \sigma \in G_{U,V} \) depending on \( \bar{H}_{U\to U} \), such that \( \mathbb{P}_{U,V}^\mu(\bar{H}_{U\to U}, \left\{ (\bar{H}_{U\to U} | \sigma|_U, \bar{H}_{U\to U}) \right\}) > 0 \). If the number of right cosets \( \bar{H}_{U\to U} | \sigma|_V \gamma|_V \) depends on \( \bar{H}_{U\to U} \), then the probability measure \( \mathbb{P}_{U,V}^\mu(\bar{H}_{U\to U}, \cdot) \) cannot be invariant under right multiplication as in \ref{eq:2.3}. Therefore there are only finitely many cosets of \( \bar{H}_{U\to U} | \sigma|_U \) in this collection. Denote by \( \ell(\bar{H}_{U\to U} | \sigma|_U) \) the number of cosets

\[
\ell(\bar{H}_{U\to U} | \sigma|_U) = \left| \left\{ (\bar{H}_{U\to U} | \sigma|_V \gamma|_V, \gamma \in R_1(V) \right\} \right|.
\]

In other words, there are \( \ell = \ell(\bar{H}_{U\to U} | \sigma|_U) \) representatives \( \gamma_1, \ldots, \gamma_\ell \) in \( R_1(V) \) such that for any \( \gamma \in R_1(V) \), we have \( \bar{H}_{U\to U} | \sigma|_V \gamma|_V = \bar{H}_{U\to U} | \sigma|_U \gamma_k|_V \) for exactly one \( k \in \{1, \ldots, \ell\} \). It follows that for any \( \gamma \in R_1(V) \), there is a representative \( \gamma_k \), \( k \in \{1, \ldots, \ell\} \), such that \( \bar{H}_{U\to U} \) contains \( \sigma|_V(\gamma_k^{-1})|_V \sigma|_U^{-1} \). Consider the subgroup \( R_1 \) of \( R_1(V) \) generated by the collection \( \gamma_k^{-1} \), where \( \gamma \in R_1(V) \) and \( \gamma_k \) is its corresponding representative. It’s clear by definition of \( R_1 \) that \( \bigcup_{j=1}^\ell R_1 \gamma_j \neq R_1(V) \), therefore \( R_1 \) is a subgroup of \( R_1(V) \) with index at most \( \ell \). Recall that \( \sigma \) maps \( U \) to \( V \), therefore \( \sigma R_1(V) \sigma^{-1} = R_1(U) \). Let \( T_1 = \sigma R_1 \sigma^{-1} \), it is a subgroup of \( R_1(U) \) of index at most \( \ell \). Elements of \( T_1 \) satisfy the property that \( \bar{H}_{U\to U} = \bar{H}_{U\to U} \gamma|_U \), in other words, \( \bar{H}_{U\to U} \gamma|_U \leq H_{U\to U} \). Note that we have bounds on the index

\[
\left| R_1(U) : (R_1(U) \cap \bar{H}_{U\to U}) \right| \leq |\pi_U(\bar{R}_1(V)) : \pi_U(T_1)| \leq |R_1(V) : T_1| \leq \ell(\bar{H}_{U\to U} | \sigma|_U).
\]
The first statement on finite index follows. Now we proceed to prove (2.3). Take any \( g \in \Gamma_U \times V \). Then by property (iii) of regular conditional probability, we have

\[
\mathbb{P}_\mu (\bar{H}_U \cap A \neq \emptyset)
= \mu (\Omega_U \times V) \mathbb{E}_{\mu_U \times V} \left[ \sum_{(\bar{H}_U \cap g) \cap A \neq \emptyset} \mathbb{P}^\mu_{U,V} (\bar{H}_U \cap (\bar{H}_U \cap g \cap A) \cap A \neq \emptyset) \right],
\]

where the summation is over those cosets in \( \{ \bar{H}_U \cap g : g \in \Gamma_U \times V \} \) with non-empty intersection with \( A \). By the same reasoning as in the previous paragraph, translation invariance (2.5) implies that for each coset,

\[
\mathbb{P}^\mu_{U,V} (\bar{H}_U \cap (\bar{H}_U \cap g \cap A) \cap A \neq \emptyset)
\leq \frac{1}{\ell(g)}.
\]

where \( \ell(g) \) is the number of cosets defined in (2.6) and in the last step we plugged in (2.7). Since the cosets are disjoint, there are at most \( |A| \) of them that intersect \( A \). It follows that

\[
\mathbb{P}_\mu (\bar{H}_U \cap A \neq \emptyset)
\leq \mu (\Omega_U \times V) \mathbb{E}_{\mu_U \times V} \left[ |\{ (\bar{R}_U (\bar{H}_U \cap g) \cap A) \cap g \} | \leq |A| \right] + \frac{|A|}{\ell(g)}.
\]

The statement follows.

\[ \square \]

Proof of Lemma 2.3 Denote by \( A_B \) the event \( \{ H \in \text{Sub}(\Gamma) : \pi_2(B) \subset \pi_2(H) \} \). If \( \mu (A_B) = 0 \) then the statement is trivially true. We may assume \( \mu (A_B) > 0 \). Recall that for any \( H < L_1 \times L_2 \), there is an isomorphism \( \varphi_H : \pi_2(B) / H \rightarrow \pi_1(H) / H_1 \), given by the map \( h_2 \rightarrow \{ h_1 \in \pi_1(H) : (h_1, h_2) \in H \} \), see [Zhe19] Fact 3.1. We refer to \( \varphi \) as the paring in \( H \) between the two coordinates. Denote by \( \mathbb{P}^\mu_B \) the regular conditional distribution of \( (\varphi_H (\pi_2(B))) \), \( H_1 \) given the random variable \( H_1 \), where \( H \) has distribution \( \mu (\cdot | A_B) \) on \( A_B \). Then the conjugation invariance property of \( \mu \) implies that

\[
\mathbb{P}^\mu_B (H_1, \{ (\pi_1(B) H_1) \}) = \mathbb{P}^\mu_B (H_1, \{ (g^{-1} \pi_1(B) H_1 g, H_1) \}),
\]

for any \( g \in N_1 \), see [Zhe19] Lemma 3.3. It follows that

\[
\mathbb{P}^\mu_B (H_1, \{ (\pi_1(B) H_1) \}) \leq \frac{1}{C_{N_1} (\pi_1(B) H_1)}.
\]

In order for \( B \) to be contained in \( H \), it is necessarily that \( \pi_2(B) \) is paired with \( \pi_1(B) \) under \( \varphi_H \). Thus, by property (iii) of regular conditional distribution, we have

\[
\mathbb{P}_\mu (B \subset H) \leq \mathbb{P} (H \in A_B) \mathbb{P} (\varphi_H (\pi_2(B)) = \pi_1(B) H_1)
= \mu (A_B) \mathbb{E}_{\mu (\cdot | A_B)} [ \mathbb{P}^\mu_B (H_1, \{ (\pi_1(B) H_1, B) \}) ]
\leq \mathbb{E}_{\mu} \left[ \frac{1}{C_{N_1} (\pi_1(B) H_1)} \right] A_B (H).
\]

\[ \square \]
3. Preliminaries on Neretin-type groups

Terminologies and notations in this section follow [Led17].

Let $\mathcal{T} = \mathcal{T}_{d+1}$ be a (unrooted) regular tree of degree $d+1$. Denote by $\text{Aut}(\mathcal{T})$ the group of automorphisms of $\mathcal{T}$, equipped with the topology of pointwise convergence. We fix, once and for all, a reference point $v_0 \in \mathcal{T}$, and a legal colouring of (geometric) edges of $\mathcal{T}$. Recall that a legal colouring is a map $\text{col}$ from (geometric) edges of $\mathcal{T}$ to the set $D = \{0, 1, \ldots, d\}$, such that at every vertex the edges incident to it have different colours.

Denote by $\partial \mathcal{T}$ the boundary of $\mathcal{T}$, which consists of all infinite geodesic rays starting at $v_0$. Given a vertex $v \in \mathcal{T}$, denote by $C_v$ the subset of $\partial \mathcal{T}$ which consists of infinite geodesic rays that starts at $v_0$ and passes through $v$. As usual, $\partial \mathcal{T}$ is equipped with the topology generated by the basis $\{C_v\}_{v \in \mathcal{T}}$.

Let $A$ be a finite subtree of $\mathcal{T}$. The subtree $A$ is called complete if it contains the reference point $v_0$ and if a vertex $v \in A$ is not a leaf, then all of its children are contained in $A$. Denote by $\partial A$ the set of leaves of $A$. By $\mathcal{T} \setminus A$ we mean the subgraph $\bigsqcup_{v \in \partial A} T_v$, that is the disjoint union (forest) of subtrees rooted at leaves of $A$.

An almost automorphism of $\mathcal{T}$ is represented by a triple $(A, B, \varphi)$, where $A, B \subseteq \mathcal{T}$ are complete finite subtrees such that $|\partial A| = |\partial B|$, and $\varphi : T \setminus A \to T \setminus B$ is a forest isomorphism. Two such triples are equivalent if up to enlarging the subtrees $A, B$ they coincide. An almost automorphism is the equivalence class of such a representing triple. An almost automorphism of $\mathcal{T}$ induces a homeomorphism of $\partial \mathcal{T}$, called a spheromorphism of $\partial \mathcal{T}$. The Neretin group $\mathcal{N}_d$ is defined as the group of all almost automorphisms of $\mathcal{T}$. Equivalently, $\mathcal{N}_d$ is the group of all spheromorphisms of $\partial \mathcal{T}$. For more detailed exposition see for example [GLI18].

The group $\mathcal{N}_d$ can be viewed as the topological full group of $\text{Aut}(\mathcal{T}) \curvearrowright \partial \mathcal{T}$. Given a group $G$ acting on a topological space $X$, the topological full group of $G \curvearrowright X$ consists of all homeomorphisms $\varphi$ of $X$ such that for any $x \in X$, there exists a neighborhood $U$ of $x$ and an element $g \in G$ such that $\varphi|_U = g|_U$. The topology on $\mathcal{N}_d$ is defined such that the inclusion $\text{Aut}(\mathcal{T}) \hookrightarrow \mathcal{N}_d$ is open and continuous.

In [CDM11], it is shown that $\mathcal{N}_d$ is compactly generated: indeed it contains a dense copy of the Higman-Thompson group $V_{d,d+1}$, which is finitely generated. We now describe the embedded Higman-Thompson group following [CDM11]. For general reference on Higman-Thompson groups, see for instance [Bro97]. Let $T_{d,k}$ be the rooted tree where the root $v_0$ has $k$ children and all the other vertices have $d$ children. For each vertex $v$, fix a local order $<_v$, which is a total order on the children of $v$. Such a collection of total orders $\{<_v\}$ is referred to as a plane order, as it specifies an embedding of the tree $T_{d,k}$ in $\mathbb{R}^2$, where the root $v_0$ is drawn at the origin, and the children of a vertex are drawn below the parent, arranged from left to right according to the order. An almost automorphism is locally order preserving if it can be represented by a triple $(A, B, \varphi)$ where for each vertex $v \in T_{d,k} \setminus A$, the restriction of $\varphi$ on the children of $v$ preserves the order. The subgroup of $\text{AAut}(T_{d,k})$ which consists of locally order preserving elements is the Higman-Thompson group $V_{d,k}$. Returning to the $(d+1)$-regular tree $\mathcal{T}$, we have that a plane order on $\mathcal{T}$ gives an embedding of the group $V_{d,d+1}$ as a dense subgroup of $\mathcal{N}_d$.

Coloured Neretin groups are introduced and investigated in [Led17]. Take a closed subgroup $G < \text{Aut}(\mathcal{T})$ and let $\text{E}(G)$ be the topological full group of the
action $G \cap \partial \mathcal{T}$. When $G$ has Tits’ independence property, there exists a unique group topology on $F(G)$ such that the inclusion $G \hookrightarrow F(G)$ is open and continuous, see [Led17] Proposition 2.22). Equipped with this topology, $F(G)$ is a t.d.l.c. group containing $G$ as an open subgroup.

Consider the case where $G$ is a universal group acting on $\mathcal{T}$ with a prescribed local action in the sense of Burger-Mozes [BM00]. Recall that we have fixed a legal colouring of the tree $\mathcal{T}$. Given a subgroup $F < \text{Sym}(D)$, the Burger-Mozes’ universal group $U(F)$ is defined as the subgroup of $\text{Aut}(\mathcal{T})$ which consists of elements whose local action at every vertex is in $F$. More precisely, at any vertex $v$ of $\mathcal{T}$, an automorphism $g \in \text{Aut}(\mathcal{T})$ induces a bijection $g_v : E(v) \to E(g(v))$, where $E(v)$ denotes edges incident to $v$. The bijection $g_v$ gives rise to a local permutation of colours given by $\sigma(g,v) = \text{col}^{-1}_v \circ g_v \circ \text{col}_v$ in $\text{Sym}(D)$. The group $U(F)$ consists of all automorphisms $g \in \text{Aut}(\mathcal{T})$ such that $\sigma(g,v) \in F$ for all $v \in \mathcal{T}$.

Denote by $\mathcal{N}_F$ the topological full group of the action $U(F) \backslash \partial \mathcal{T}$, equipped with the unique group topology such that $U(F) \hookrightarrow \mathcal{N}_F$ is open and continuous. We refer to $\mathcal{N}_F$ as the coloured Neretin group associated with $F$. Elements of $\mathcal{N}_F$ are called $U(F)$-almost automorphisms and each element $g \in \mathcal{N}_F$ can be represented by a triple $(A, B, \varphi)$, where $A, B$ are complete finite subtrees with $|\partial A| = |\partial B|$ and $\varphi$ is a forest isomorphism $\mathcal{T} \backslash A \to \mathcal{T} \backslash B$ such that for each leaf $v \in \partial A$, there exists an element $h_v \in U(F)$ such that $\varphi|_{\mathcal{T}_v} = h_v|_{\mathcal{T}_v}$.

Consider the case where $\mathcal{T}$ is a forest isomorphism $\mathcal{T} \backslash A \to \mathcal{T} \backslash B$ such that for each leaf $v \in \partial A$, there exists an element $h_v \in U(F)$ such that $\varphi|_{\mathcal{T}_v} = h_v|_{\mathcal{T}_v}$. It is an open compact subgroup of $\mathcal{N}_F$. Let $O^A_F$ denote the subgroup which consists of elements in $\mathcal{N}_F$ which can be represented by a triple $(B_n(A), B_n(A), \varphi)$, where $\varphi$ is a forest isomorphism $\mathcal{T} \backslash B_n(A) \to \mathcal{T} \backslash B_n(A)$ such that for each leaf $v \in \partial B_n(A)$, there exists an element $h_v \in U(F)$ such that $\varphi|_{\mathcal{T}_v} = h_v|_{\mathcal{T}_v}$. It is an open compact subgroup of $\mathcal{N}_F$. Let $O^A_F$

4. INDUCED IRSs AND EVENTS IN SUB-QUOTIENTS

Let $A$ be a complete finite subtree of $\mathcal{T}$. Recall that $A$ is called complete if it contains the root $v_0$ and if a vertex $v \in A$ is not a leaf, then all of its children are contained in $A$. Denote by $B_n(A)$ the subtree with vertices within distance $n$ to $A$, in other words it is the subtree which contains $A$ and trees of height $n$ rooted at the leaves of $A$.

Denote by $O^A_F(n)$ the subgroup which consists of elements in $\mathcal{N}_F$ which can be represented by a triple $(B_n(A), B_n(A), \varphi)$, where $\varphi$ is a forest isomorphism $\mathcal{T} \backslash B_n(A) \to \mathcal{T} \backslash B_n(A)$ such that for each leaf $v \in \partial B_n(A)$, there exists an element $h_v \in U(F)$ such that $\varphi|_{\mathcal{T}_v} = h_v|_{\mathcal{T}_v}$. It is an open compact subgroup of $\mathcal{N}_F$. Let $O^A_F$
be the increasing union

\[ O^A_F := \bigcup_{n=0}^{\infty} O^A_F(n). \]

For instance, when \( A = \{v_0\} \), the corresponding group \( O^\{v_0\}_F \) is the open subgroup \( O \) considered in BCGM12 Led17.

The group \( O^A_F(n) \) permutes the leaves of the subtree \( B_n(A) \) and the kernel of this action is the pointwise stabilizer of \( B_n(A) \) in \( U(F) \), which we denote by

\[ U^A_F(n) := \{ g \in U(F) : x \cdot g = x \text{ for all } x \in B_n(A) \}. \]

Note that \( U^A_F(n) \) is open and compact. Denote by \( S^A_F(n) \) the quotient group \( O^A_F(n)/U^A_F(n) \) and \( \pi_n \) the projection

\[ \pi_n : O^A_F(n) \to S^A_F(n). \]

When \( F \) is transitive on \( D \), the quotient \( S^A_F(n) \) is isomorphic to the symmetric group \( \text{Sym}(\partial B_n(A)) \). For general \( F \), recall that we denote by \( \{D^{(0)}, \ldots, D^{(i)}\} \) the \( F \)-orbits on \( D \) and \( \ell_F \) be the labeling associated to \( F \)-orbits on \( T \setminus \{v_0\} \). Then the quotient group \( S^A_F(n) \) is isomorphic to \( \prod_{i=0}^{n} \text{Sym} \left( D^{(i)}_{n,A} \right) \), where \( D^{(i)}_{n,A} = \{ v \in \partial B_n(A) : \ell_F(v) = D^{(i)} \} \).

Given a finite complete subtree \( A \) and two distinct leaves \( u, v \) of \( A \) with \( \ell_F(u) = \ell_F(v) \), consider the event in \( \text{Sub}(N_F) \)

\[ \Theta^A_{u,v} := \{ H \in \text{Sub}(N_F) : \exists h \in H \cap O^A_F(0) \text{ s.t. } v = u \cdot \pi_0(h) \}. \]

Since the set of \( \{ g \in O^A_F(0) : v = u \cdot \pi_0(g) \} \) is open in \( N_F \), we have that \( \Theta^A_{u,v} \) is an open subset in \( \text{Sub}(N_F) \). This event is similar to the event \( \Omega_{U,V} \) considered in Section 2.

Let \( \mu \) be an ergodic IRS of \( N_F, \mu \neq \delta_{(id)} \). We will verify that for \( \mu \)-a.e. \( H \), there exists a finite complete tree \( A \) and two distinct leaves \( u, v \in \partial A \) such that \( H \in \Theta^A_{u,v} \), see Lemma 5.1. Thus in what follows we focus on these complete finite subtree \( A \) and leaves \( u \neq v \) in \( \partial A \) such that \( \mu \left( \Theta^A_{u,v} \right) > 0 \).

Given a leaf \( u \) of a finite complete subtree \( A \), denote by \( C^u_n \) the set of vertices \( x \in T \) such that \( u \) lies on the geodesic from the root \( v_0 \) to \( x \) and \( d_T(u,x) = n \). In other words, \( C^u_n \) consists of vertices of distance \( n \) to \( u \) in the subtree \( T_u \) rooted at \( u \).

Now suppose \( A, u, v \) are such that \( \mu \left( \Theta^A_{u,v} \right) > 0 \). Expand the subtree \( A \) to \( B_n(A) \) and consider in the finite group \( S^A_F(n) \) the event

\[ \Theta^A_{u,v} := \{ H \leq S^A_F(n) : \exists h \in H \cap \pi_n \left( O^A_F(0) \right) \text{ s.t. } C^u_n = C^v_n \cdot h \}. \]

It’s clear by definitions of the events that for any \( n \geq 0 \),

\[ \Theta^A_{u,v} \subseteq \{ H \in \text{Sub}(N_F) : \pi_n \left( H \cap O^A_F(n) \right) \in \Theta^A_{u,v} \}. \]

Note that the maps \( H \mapsto H \cap O^A_F(n) \) and \( H \mapsto \pi_n \left( H \cap O^A_F(n) \right) \) are continuous. Denote by \( \tilde{\mu}^A_n \) the induced IRS in the finite group \( S^A_F(n) \), that is, the pushforward of \( \mu \) under \( H \mapsto \pi_n \left( H \cap O^A_F(n) \right) \). Suppose \( \mu \left( \Theta^A_{u,v} \right) > 0 \), then we have for all \( n \),

\[ \tilde{\mu}^A_n \left( \Theta^A_{u,v} \right) \geq \mu \left( \Theta^A_{u,v} \right) > 0. \]

This uniform lower bound, independent of \( n \), on the probability of containing a specific kind of almost automorphisms, is the starting point of our argument. We will show, by combining general lemmas in Section 2 and properties of finite symmetric
groups, that this lower bound forces the finitary IRS $\mu_n^A$ to charge groups that contain a "large" alternating subgroup of $S^2_n(n)$. Proposition 1.2 will be shown by applying the Borel-Cantelli lemma to combine the estimates in each level $n$.

5. Proof of Theorem 1.1 given Proposition 1.2

In this section we explain how to deduce Theorem 1.1 from Proposition 1.2. Recall that in the previous section, we have defined the countable collection of events $\{\Theta^A_{u,v}\}$ as in (4.1), where $u,v$ goes over all pairs of distinct vertices in $\partial A$ and $A$ goes over all finite complete subtrees of $T$.

To proceed, we first show that if $\mu$ is an ergodic IRS of $N_F$ such that $\mu \neq \delta_{\{id\}}$, then for $\mu$-a.e. $H$, there exists some $A$ such that $H \cap O_F^A \neq \{id\}$. When $H$ is of finite co-volume, this is obvious because $O_F^A$ has infinite Haar measure. For essentially the same reason, an ergodic IRS that is concentrated at the trivial group cannot intersect all $O_F^A$ trivially:

Lemma 5.1. Let $\mu$ be an ergodic IRS of $N_F$, $\mu \neq \delta_{\{id\}}$. Then for $\mu$-a.e. $H$, there exists a finite complete subtree $A$ such that $H \cap O_F^A \neq \{id\}$.

Proof. Suppose $H$ is such that $H \cap O_F^A = \{id\}$ for all $A$. It follows that for any $A,B$ with $|\partial A| = |\partial B|$, there is at most one element in $H$ that can be represented by a triple of the form $(A,B,\varphi)$. Indeed, if there are two distinct elements $h_1,h_2$ of this form, then $h_1h_2^{-1}$ would be a non-identity element in $H \cap O_F^A$, contradicting $H \cap O_F^A = \{id\}$.

Note that the union $\bigcup_A \{H : H \cap O_F^A \neq \{id\}\}$, over all finite complete subtrees $A$, is invariant under conjugation by $N_F$. Thus by ergodicity the $\mu$-measure of this union is either 0 or 1. We argue by contradiction and assume from now on

\begin{equation}
\mu \left( \bigcup_A \{H : H \cap O_F^A \neq \{id\}\} \right) = 0.
\end{equation}

Given two finite complete subtrees $A,B$ with $|\partial A| = |\partial B|$, a point $u \in \partial A$ and $v \in \partial B$, denote by $W(A,B : u,v)$ the set of $U(F)$-almost automorphisms that can be written as a product $\Psi(A,B,\sigma)$, where $\Psi \in O_F^A$ with $u \cdot \Psi = u$ for all $u \in \partial A$ and $\sigma$ is a locally order preserving forest isomorphism $T \setminus A \to T \setminus B$ with $uv = v$. Take the collection $\{W(A,B : u,v)\}$ where the pairs $u,v$ are such that $u \in \partial A$, $v \in \partial B$ and $C_u \cap C_v = \emptyset$; and $A,B$ go over all finite complete subtrees with $|\partial A| = |\partial B|$. The corresponding collection $\{H \in \text{Sub}(N_F) : H \cap W(A,B : u,v) \neq \emptyset\}$ form an open cover of $\text{Sub}(N_F) \setminus \{\{id\}\}$. Indeed, to verify this claim, let $g \in N_F$ be any non-identity element and $(A_0,B_0,\varphi_0)$ be a representing triple for $g$. Then there exists disjoint clopen subsets $V_1,V_2 \subseteq \partial T$ such that $V_2 = V_1 \cdot g$. Expand the trees $A_0$ and $B_0$ to sufficiently large levels, we may represent $g$ by a triple $(A_1,B_1,\varphi_1)$ such that there exists a vertex $u \in \partial A_1$ with $C_u \subseteq V_1$. It follows that $g \in W(A_1,B_1 : u,\varphi_1(u))$ where $C_u \cap C_{\varphi_1(u)} = \emptyset$.

Since the cover of $\text{Sub}(N_F) \setminus \{\{id\}\}$ in the previous paragraph is countable, there must exist some $A, B, u, v$ such that

$\mu (\{H : H \cap W(A,B : u,v) \neq \emptyset\}) > 0$.

Recall the fact that if $H \cap W(A,B : u,v) \neq \emptyset$ and $H \cap O_F^A = \{id\}$, then the intersection $H \cap W(A,B : u,v)$ consists of a unique element. In this case, we write the unique element of $H \cap W(A,B : u,v)$ as $(\Psi^H_x)_{x \in \partial A} \sigma^H$, where $(\Psi^H_x)_{x \in \partial A} \in O_F^A$.
and $\sigma^H$ is a bijection from $\partial A$ to $\partial B$ with $\sigma^H(u) = v$. Recall that we have assumed \[5.1\]. Denote by $\eta_u$ the conditional distribution of the element $\Psi^H_u$ given that $H \cap W(A, B : u, v) \neq \emptyset$. We now repeat the same kind of argument as in the proof of Lemma 2.1. Take any element $g \in R_{O^F}(C_v)$, then under conjugation by $g$, we have
\[
\Psi^{-1}_u H_g = \Psi_u g \sigma^{-1}.
\]
Note that the set $\{H : H \cap W(A, B : u, v) \neq \emptyset\}$ is invariant under conjugation by $R_{O^F}(C_v)$. It follows from conjugacy invariance of $\mu$ and \[5.2\] that the distribution $\eta_u$ is invariant under right translation by $R_{O^F}(C_v)$. However this is impossible because $R_{O^F}(C_v)$ does not admit a finite right Haar measure. We have reached a contradiction and therefore \[5.1\] is false. Instead, we have

$$
\mu \left( \bigcup_A \{H : H \cap O^F_A \neq \{id\} \} \right) = 1.
$$

\[\square\]

Now we assume Proposition 1.2 and show Theorem 1.1 by reducing the problem to the Higman-Thompson type group $V_F$ densely embedded in $\mathcal{N}_F$.

Since $V_F$ is countable, the Chabauty space $\text{Sub}(V_F)$ is equipped with the topology inherited from the product topology on $\{0, 1\}^{V_F}$. The intersection map $\text{Sub}(\mathcal{N}_F) \to \text{Sub}(V_F)$ given by $H \mapsto H \cap V_F$ is Borel measurable. Thus given an IRS $\mu$ of $\mathcal{N}_F$, we can consider its pushforward $\mu_{V_F}$ under $H \mapsto H \cap V_F$.

A priori, an ergodic IRS of $\mathcal{N}_F$ may intersect with $V_F$ trivially. However this cannot happen unless $H = \{id\}$ because of Lemma 5.1 and Proposition 1.2. The intersection $V_F \cap R_{O^F} U$ is nontrivial for any choice of $A$ and open set $U$, indeed, $V_F \cap R_{O^F} U$ is dense in $R_{O^F} U$, where $R_{O^F} U$ is equipped with the subspace topology inherited from the natural inclusion $R_{O^F} U \hookrightarrow \mathcal{N}_F$. Thus we complete the proof of Theorem 1.1 by combining Proposition 1.2 and the result that $V'_F$ has no non-trivial IRSs.

**Proof of Theorem 1.1 given Proposition 1.2.** Let $\mu$ be an ergodic IRS of $\mathcal{N}_F$, $\mu \neq \delta_{\{id\}}$. By Lemma 5.1 and Proposition 1.2 (when $F$ is transitive on $D$ one can use the special case Proposition 1.3), we have that the collection of subgroups $H$ with the property that there exists a complete finite tree $A$ and a non-empty open set $U \subseteq \partial T$ such that $H \geq R_{O^F} U$ has $\mu$-measure 1. Since $R_{O^F}(U)'$ contains non-trivial locally order preserving $U(F)$-almost automorphisms, we have that $V'_F \cap H \neq \{id\}$.

Consider the induced IRS $\mu_{V_F}$ which is the pushforward of $\mu$ under the intersection map $H \mapsto H \cap V_F$. Then by [DM14, Corollary 3.9], the ergodic decomposition of $\mu_{V_F}$ is of a convex combination of the form

$$
\mu_{V_F} = \lambda_0 \delta_{\{id\}} + \sum_{i=1}^p \lambda_i \delta_{L_i},
$$

where $p$ is finite and $L_i$ is a normal subgroup of $V_F$ that contains $V'_F$, see also [Zic19, Corollary 5.4] for an alternative proof. As explained in the previous paragraph, Lemma 5.1 and Proposition 1.2 implies that $\mu$-a.e. $H$ satisfies $V'_F \cap H \neq \{id\}$. Therefore $\lambda_0 = 0$ and $\mu$-a.e. $H$, $V_F \cap H \geq V'_F$. Since $H$ is closed, it follows that
$H$ contains the closure of $V'_p$. Since the closure of $V'_p$ contains the commutator subgroup $N'_p$, we have proved the statement.

The proof of Proposition 6.2 occupies the next three sections.

6. TREE AUTOMORPHISM ORBITS VERSUS RANDOM PERMUTATIONS

Consider a rooted tree $T_{d,q}$, $d,q \geq 2$, where the root $o$ has $q$ children and the rest of the vertices have $d$ children. Denote by $W = \Aut(T_{d,q})$ the group of rooted tree automorphisms of $T_{d,q}$. Note that $W$ has the structure of a semi-direct product

$$W = (\oplus_{v \in L_1} \Aut(T_v)) \rtimes \Sym(L_1),$$

where $T_v$ is the subtree rooted at $v$ and $\Aut(T_v)$ is the group of rooted tree automorphisms of $T_v$. The font $\mathbb{T}$ is used in this section to emphasize that the tree is rooted and $\Aut(T_v)$ is the group of rooted automorphisms.

Write $L_n$ for the level $n$ vertices with respect to the root $o$, that is, $L_n$ consists of vertices of $T_{d,q}$ such that $d(o,v) = n$. A vertex in $L_n$ is encoded as a string $v = v_1 \ldots v_n$, where $v_1 \in \{0, \ldots, q-1\}$ and $v_j \in \{0, \ldots, d-1\}$ for $2 \leq j \leq n$. For two subset $E,F$ of $L_n$, we write

$$E \sim_W F$$

if $F$ is in the $W$-orbit of $E$, that is, $E \sim_W F$ if and only if there exists $g \in W$ such that $F = E \cdot g$.

In this auxiliary section we estimate the probability that two randomly chosen subsets are in the same orbit of $W$. Such estimates will be useful in the next sections to rule out certain cases of intransitivity or imprimitivity. As in the previous section, denote by $C^m_n$ the vertices in the subtree rooted at $u$ of distance $n$ to $u$.

**Lemma 6.1.** In the rooted tree $T_{d,q}$, let $u,v$ be two distinct vertices in $L_1$. Let $n,k$ be integers such that $n \geq 2$ and $2 \leq k \leq d^n/2$. Choose a set $E$ of size $k$ uniformly random from $C^m_n$ and independently choose a set $F$ of size $k$ uniformly random from $C^m_n$. Then for any $\delta > 0$, there exists constants $C,c > 0$ only depending on $\delta,d$, such that for all such $n,k$,

$$\Pr(E \sim_W F) \leq C \exp\left(-ck\frac{n-1}{n}\delta\right).$$

The lemma is shown by recursion down the tree. We use the following well-known basic probability estimates. For $p,q \in (0,1)$, denote by $H(q||p)$ the relative entropy (also called the Kullback–Leibler divergence) between the Bernoulli distribution with parameter $q$ and the Bernoulli distribution with parameter $p$, that is,

$$H(q||p) = q \log \frac{q}{p} + (1-q) \log \frac{1-q}{1-p}.

The relative entropy $H(q||p)$ is always non-negative and is zero if and only if $q = p$.

**Fact 6.2.** Let $X$ be a finite set, $\sigma$ a uniformly random permutation in $\Sym(X)$. Let $U$ and $K$ be two non-empty subset of $X$ and write $p = |U|/|X|$, $k = |K|$. Then for any $x > 0$,

$$\Pr(|(K \cdot \sigma) \cap U| > (p + x)k) \leq e^{-H(p+x||p)k}, \quad (6.1)$$

$$\Pr(|(K \cdot \sigma) \cap U| < (p - x)k) \leq e^{-H(p-x||p)k}. \quad (6.2)$$
Moreover, suppose $k \leq |X|/2$, there exists an absolute constant $C > 0$ such that for $i \in [pk/2, \frac{3}{2}pk]$,

\[
P((K \cdot \sigma) \cap U = i) \leq C \sqrt{\frac{k}{i(k-i)}}.
\]

We include a proof for Fact 6.2 for the convenience of the reader. Recall Stirling’s approximation:

\[
1 \leq \frac{n!}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} \leq \frac{e}{\sqrt{2\pi}} \text{ for all } n \geq 1.
\]

**Proof of Fact 6.2.** List the elements of $K$ as $x_1, \ldots, x_k$ and write $Z_j = \sum_{i=1}^{k} 1_{U}(x_i \cdot \sigma)$. Then $Z_k = |(K \cdot \sigma) \cap U|$. The moment generating function of $Z_k$ satisfies that for $\lambda > 0$,

\[
\mathbb{E}[e^{\lambda Z_k}] = \mathbb{E}[e^{\lambda Z_{k-1}} \mathbb{E}[e^{\lambda (Z_k - Z_{k-1}) | Z_{k-1}}]] \\
= \mathbb{E}[e^{\lambda Z_{k-1}} (e^{\lambda (|U| - Z_{k-1})_+} + |X| - Z_{k-1} - (|U| - Z_{k-1})_+)] \\
\leq \mathbb{E}[e^{\lambda Z_{k-1}} (e^{\lambda p + 1 - p})].
\]

Iterate this inequality we have $\mathbb{E}[e^{\lambda Z_k}] \leq (e^{\lambda p + 1 - p})^k$. By the Chernoff bound, we have

\[
P(Z_k \geq (p + x)k) \leq e^{-\lambda (p+x)} \mathbb{E}[e^{\lambda Z_k}] \leq e^{-\lambda (p+x)} (e^{\lambda p + (1 - p)})^k.
\]

Optimize the choice of $\lambda$ we obtain the first inequality. Similarly, $P(Z_k \leq (p - x)k) = P((K \cdot \sigma) \cap U^c \geq (1 - p + x)k)$. Apply the first inequality to the set $U^c$, we obtain the second inequality.

For the last bound, write $u = |U|$, $x = |X|$. We have that by Stirling’s approximation [6.3],

\[
P((K \cdot \sigma) \cap U = i) = \binom{x}{i} \left(\frac{x-u}{k-i}\right)^i \left(\frac{k}{1-k}ight)^{k-i} \\
\leq C \left(\frac{1-k/x}{1-i/u} \frac{1}{(1-(k-i)/(x-u))} \frac{k}{i(k-i)}\right)^{1/2} \\
\cdot e^{-H(i/k)p} k - H \left(\frac{u-i}{x-k}\right)p (x-k).
\]

The statement follows.

**Proof of Lemma 6.7.** Consider recursively down the subtrees rooted at $u, v$. In order to have an element in $W$ which maps $E \to F$, it is necessary that there exists a permutation $\gamma \in \text{Sym} \{\{0, \ldots, d-1\}\} = \text{Sym}(d)$ such that for each child $ui$ of $u$, $E \cap C_{ui}^{n-1} \sim_W F \cap C_{v\gamma(i)}^{n-1}$. Note that since the sets are chosen uniformly at random independently, conditioned on the event $|E \cap C_{ui}^{n-1}| = |F \cap C_{v\gamma(i)}^{n-1}| = r$, the distribution of $E \cap C_{ui}^{n-1}$ and $F \cap C_{v\gamma(i)}^{n-1}$ are independent, where $E \cap C_{ui}^{n-1}$ is distributed uniformly on subsets of size $r$ of $C_{ui}^{n-1}$ and $F \cap C_{v\gamma(i)}^{n-1}$ is distributed uniformly on subsets of size $r$ of $C_{v\gamma(i)}^{n-1}$. 

14
The number \( n \) is fixed throughout the calculation. For two distinct vertices \( y, z \) in level \( \ell \), choose a set \( A_1 \) of size \( r \) uniformly from \( C_n^{n-\ell} \), and independently a set \( A_2 \) of size \( r \) uniformly from \( C_n^{n-\ell} \). It is clear that the probability that \( A_1 \sim_W A_2 \) depends only on the level \( \ell \) and the size \( r \), and we denote it by
\[
(6.5) \quad a(\ell, r) = P(A_1 \sim_W A_2).
\]

Take a small constant \( \epsilon < 1/d^2 \). We have
\[
P(A_1 \sim_W A_2) \leq P\left( \exists i \in \{0, \ldots, d-1\} \left| A_1 \cap C_y^{n-\ell-1} \right| - \frac{r}{d} > \epsilon r \right)
+ P\left( A_1 \sim_W A_2 \text{ and } \left| A_1 \cap C_y^{n-\ell-1} \right| - \frac{r}{d} \leq \epsilon r \text{ for all } i \right)
:= I + II.
\]

Write
\[
\mathfrak{h}(\epsilon) = \max \left\{ H\left( \frac{1}{d} - \epsilon \frac{1}{d} \right), H\left( \frac{1}{d} + \epsilon \frac{1}{d} \right) \right\}.
\]

Then by (6.2), we have
\[
I \leq 2d \exp (-\mathfrak{h}(\epsilon)r).
\]

Then one step recursion to the children of \( y \) and \( z \) as in the previous paragraph shows that
\[
(6.6) \quad II \leq \sum_{\gamma \in \text{Sym}(d)} \sum_{|r_i - r/d| \leq \epsilon r} \sum_{r_i = r} P\left( \bigcap_{i=0}^{d-1} \left| A_1 \cap C_y^{n-\ell-1} \right| = \left| A_2 \cap C_z^{n-\ell-1} \right| = r_i \right) \prod_{i=0}^{d-1} a(\ell+1, r_i).
\]

By independence we have
\[
P\left( \bigcap_{i=0}^{d-1} \left| A_1 \cap C_y^{n-\ell-1} \right| = \left| A_2 \cap C_z^{n-\ell-1} \right| = r_i \right)
= P\left( \bigcap_{i=0}^{d-1} \left| A_1 \cap C_y^{n-\ell-1} \right| = r_i \right) P\left( \bigcap_{i=0}^{d-1} \left| A_2 \cap C_z^{n-\ell-1} \right| = r_i \right)
\leq P\left( \bigcap_{i=0}^{d-1} \left| A_1 \cap C_y^{n-\ell-1} \right| = r_i \right) C_{d, \epsilon} r^{-\frac{d-\epsilon}{d}},
\]

where in the second line we applied the bound (6.3) \( d-1 \) times and the constant \( C_{d, \epsilon} \) is \( C_d - (d-1)\epsilon \). Note that if \( r > d^{n-\ell}/2 \), then we should replace \( r \) by \( d^{n-\ell} - r \) in \( a(\ell, r) \). For \( r \leq d^{n-\ell}/2 \), set
\[
\tilde{a}(\ell, r) = \sup_{r \leq s \leq d^{n-\ell}/2} a(\ell, s).
\]

Plugging back in (6.6), we have
\[
II \leq d! C_{d, \epsilon} r^{-\frac{d-\epsilon}{2}} \tilde{a}\left( \ell + 1, \frac{r}{d} - \epsilon r \right)^d.
\]

Combine part I and II, we have
\[
\tilde{a}(\ell, r) \leq d \exp (-\mathfrak{h}(\epsilon)r) + d! C_{d, \epsilon} r^{-\frac{d-\epsilon}{2}} \tilde{a}\left( \ell + 1, \frac{r}{d} - \epsilon r \right)^d.
\]
Using the bound \((x + y)^n \leq 2^{n-1}(x^n + y^n)\) we can iterate this inequality. Start with \(r\) where \(d^{n-\ell} > 2r\) and iterate for \(s\) steps, where \(s\) is such that

\[
(6.7) \quad d! C_{d,\epsilon} r^{-\frac{d-1}{2}} \leq \left(\frac{1}{4} \left(\frac{1}{d} - \epsilon\right)^s\right)^d,
\]

then summing up the terms we have

\[
a(\ell, r) \leq C_1 \exp \left(-\delta(\epsilon) r (1 - d\epsilon)^{s-1}\right) + 2^{-d^\ell-1},
\]

where \(C_1\) is a constant depending only on \(d, \epsilon\). Given a \(\delta > 0\), choose \(\epsilon\) sufficiently small and \(s\) the largest integer satisfying (6.7), we conclude that for \(r \leq d^{n-\ell}/2\),

\[
(6.8) \quad a(\ell, r) \leq C \exp \left(-c r^{\frac{d-1}{2}}\right).
\]

The statement is given by taking \(\ell = 1\) in (6.8). \(\square\)

We deduce two corollaries from Fact 6.2 and Lemma 6.1, which will be used in the next section.

**Corollary 6.3.** In the rooted tree \(T_{d,q}\), let \(u, v\) be two distinct vertices in \(L_1\). Let \(\sigma\) be a uniform random permutation in \(\text{Sym}(L_n)\). Let \(K\) be a subset of \(L_n\) with \(k = |K| \leq |L_n|/2\). Then for any \(\delta > 0\), we have

\[
\mathbb{P}\left( (K \cdot \sigma) \cap C_u^n \sim_W (K \cdot \sigma) \cap C_v^n \right) \leq C \exp \left(-c k^{\frac{d-1}{2}}\right),
\]

where \(C, c > 0\) are constants that only depend on \(d, \delta\) and \(q\).

**Proof.** In order to have an element in \(W\) which maps \((K \cdot \sigma) \cap C_u^n \to (K \cdot \sigma) \cap C_v^n\), it is necessary that they are of equal sizes. Since \(\sigma\) is uniform, conditioned on the event \(|(K \cdot \sigma) \cap C_u^n| = |(K \cdot \sigma) \cap C_v^n| = r\), the distribution of the sets \((K \cdot \sigma) \cap C_u^n\) and \((K \cdot \sigma) \cap C_v^n\) are independent, where each \((K \cdot \sigma) \cap C_x^n\) is distributed uniformly on subsets of size \(r\) of \(C_x^n\), \(x \in \{u, v\}\). Therefore we have

\[
\mathbb{P}\left( (K \cdot \sigma) \cap C_u^n \sim_W (K \cdot \sigma) \cap C_v^n\right) = \sum_{r \leq k/2} \mathbb{P}\left( |(K \cdot \sigma) \cap C_u^n| = |(K \cdot \sigma) \cap C_v^n| = r\right) p(r, n),
\]

where \(p(r, n) = \mathbb{P}(E \sim_W F)\), the set \(E\) is a uniformly random subset of size \(r\) in \(C_u^n\) and \(F\) is an independent uniformly random subset of size \(r\) in \(C_v^n\).

By Fact 6.2, the size of \(|(K \cdot \sigma) \cap C_u^n|\) is concentrated around \(k/q\). Thus, apply Fact 6.2 and Lemma 6.1, we have for any \(\epsilon > 0\),

\[
\mathbb{P}\left( (K \cdot \sigma) \cap C_u^n \sim_W (K \cdot \sigma) \cap C_v^n\right) \leq \exp\left(-H(1/q + \epsilon\|1/q\)k) + \exp\left(-H(1/q - \epsilon\|1/q\)k) + C \exp \left(-c \left\{\frac{k}{q} - \epsilon k\right\}^{\frac{d-1}{2}}\right)\right).
\]

Choosing for example \(\epsilon = \frac{1}{2q}\), we obtain the statement. \(\square\)
Corollary 6.4. In the rooted tree $T_{d,q}$, let $u, v$ be two distinct vertices in $L_1$. Let $\sigma$ be a uniform random permutation in $\text{Sym}(L_n)$. Let $K_1, K_2$ be two disjoint subsets of $L_n$ with $|K_1| = |K_2| = k$. Then

$$\mathbb{P}((K_1 \cdot \sigma) \cap C^n_u \sim_W (K_2 \cdot \sigma) \cap C^n_v) \leq \exp\left(-c_\delta k \frac{d^4}{\delta^2} \right),$$

where $c_\delta > 0$ is a constant that only depends on $\delta, d$ and $q$.

Proof. The proof is similar to Corollary 6.3. Since $\sigma$ is uniform, conditioned on the event $|(K_1 \cdot \sigma) \cap C^n_u| = |(K_2 \cdot \sigma) \cap C^n_v| = r$, the distributions of $(K_1 \cdot \sigma) \cap C^n_v$ and $(K_2 \cdot \sigma) \cap C^n_v$ are independent and uniform in sets of size $r$ in $C^n_u$ and $C^n_v$ respectively. Let $p(r, n)$ be as in the proof of Corollary 6.3. Then we have

$$\mathbb{P}((K_1 \cdot \sigma) \cap C^n_u \sim_W (K_2 \cdot \sigma) \cap C^n_v) = \sum_r \mathbb{P}(|(K_1 \cdot \sigma) \cap C^n_u| = |(K_2 \cdot \sigma) \cap C^n_v| = r) \cdot p(r, n) \leq \mathbb{P}(\{|(K_1 \cdot \sigma) \cap C^n_u| \not\in [k/q - \epsilon k, k/q + \epsilon k]\} + \max_{r \in [k/q - \epsilon k, k/q + \epsilon k]} p(r, n)).$$

The statement follows from Fact 6.2 and Lemma 6.1.

\[ \square \]

7. Containment of rigid stabilizers when $F$ is transitive on $D$

The goal of this section is to prove Proposition 7.5 assuming $F$ is transitive. The case of intransitive $F$ brings in the complication that the quotient $S^F_k(n)$ is a product of symmetric groups instead of $\text{Sym}(\partial B_n(A))$. This is not hard to handle (see the next section), but for clarity we present the argument for the transitive case first.

Throughout this section $F$ is assumed to be transitive on $D$. Let $\mu$ be an IRS of $\mathcal{N}_F$. Recall the setting and notations in Section 6. Suppose $A, u, v$ are such that $\mu(\Theta^n_{u,v}) > 0$, where $A$ is a finite complete subtree, $u, v$ are two distinct vertices in $\partial A$, and the event $\Theta^n_{u,v}$ is defined in (4.1). Fix such a triple $A, u, v$. Go down $n$ more levels and consider the induced IRS $\mu_n^A$ in the finite group $S^F_k(n) = \text{Sym}(\partial B_n(A))$, that is, $\mu_n^A$ is the pushforward of $\mu$ under the map $H \mapsto \pi_n(H \cap O^F_k(n))$. For $\Gamma \subset \text{Sym}(\partial B_n(A))$, let $\nu_T$ be the IRS of $\text{Sym}(\partial B_n(A))$ which is uniform on conjugates of $\Gamma$. Denote the ergodic decomposition of $\mu_n^A$ by

$$\mu_n^A = \sum_{i=1}^{I_n} \lambda_i \nu_{\Gamma_i},$$

where $\nu_{\Gamma_i}$ is the IRS associated with the subgroup $\Gamma_i \subset \text{Sym}(\partial B_n(A))$ and $I_n$ is a finite indexing set.

Recall the event

$$\Theta^n_{u,v} = \{H \leq \text{Sym}(\partial B_n(A)) : \exists h \in H \cap \pi_n(O^F_k(0)) \text{ s.t. } C^n_v = C^n_u \cdot h\}$$

as defined in (4.2) and the fact that

$$\mu_n^A (\Theta^n_{u,v}) \geq \mu(\Theta^n_{u,v}).$$

Given a subgroup $\Gamma \subset \text{Sym}(\partial B_n(A))$, consider the probability $\nu_T(\Theta^n_{u,v})$. We want to show that if $\Gamma$ does not contain a large alternating subgroup, then $\nu_T(\Theta^n_{u,v})$ is small.
One ingredient that goes into the bounds is the following direct consequence of the subgroup index Lemma 2.1 which is useful to subgroups of relatively small index.

**Lemma 7.1.** Suppose \( Q \) is a subset of \( \{ \gamma \in \text{Sym}(X) : U \cdot \gamma = V \} \). Let \( \Gamma < \text{Sym}(X) \) be any subgroup. Then

\[
P_{\nu_\Gamma}(H \cap Q \neq \emptyset) \leq \frac{|\Gamma| \cdot |Q_U|}{|\text{Sym}(U)|}
\]

where \( Q_U = \{ \gamma |_U : \gamma \in Q \} \).

**Proof.** The rigid stabilizer of \( U \) in \( \text{Sym}(X) \) is \( \text{Sym}(U) \). Apply Lemma 2.1 to the IRS \( \nu_\Gamma \), we have

\[
P_{\nu_\Gamma}(H \cap Q \neq \emptyset) \leq \mathbb{E}_{\nu_\Gamma} \left[ \frac{|Q_U|}{|\text{Sym}(U) : (H_{U \rightarrow U} \cap \text{Sym}(U))|} \right].
\]

Since \( |H_{U \rightarrow U}| \leq |H| = |\Gamma| \), we have

\[
|\text{Sym}(U) : (H_{U \rightarrow U} \cap \text{Sym}(U))| \geq |\text{Sym}(U)| / |\Gamma|.
\]

The statement follows. \( \square \)

Write

\[
q = |\partial A| \text{ and } k_n = q d^n = |\partial B_n(A)|.
\]

The group \( O_F^A(0) \) is a subgroup of the semi-direct product \( W = (\oplus_{v \in \partial A} \text{ Aut}(T_v)) \rtimes \text{ Sym}(\partial A) \) as in the setting of Section 6. We suppress reference to \( A \) in the notations \( q, k_n \) and \( W \), understanding that \( A \) is fixed through the calculations. Denote by \( Q_{u,v}^{A,n} \) the subset

\[
Q_{u,v}^{A,n} = \{ g \in \pi_n (O_F^A(0)) : C_v^n = C_u^n \cdot g \}.
\]

The size of the set of partial homeomorphisms \( \{ g|_{C_v^n} : g \in Q_{u,v}^{A,n} \} \) is bounded by \( |F|^{d^n} \). Then by Lemma 7.1 we have

\[
(7.1) \quad \nu_\Gamma (\Theta_{u,v}^{A,n}) = P_{\nu_\Gamma} (H \cap Q_{u,v}^{A,n} \neq \emptyset) \leq \frac{|\Gamma| \cdot |F|^{d^n}}{(d^n)!}.
\]

This shows that \( \nu_\Gamma (\Theta_{u,v}^{A,n}) \) is small, if the size of \( \Gamma \) is much smaller than \( (d^n)!/|F|^{d^n} \). Recall that the size of \( \text{Sym}(\partial B_n(A)) \) is \( (qd^n)! \). As remarked earlier in the Introduction, this kind of bound is similar to, but weaker than, the co-volume estimate used in the proof of absence of lattices in \[BCGM12\].

Now consider in more detail the structure of \( \Gamma \). The bounds for \( \nu_\Gamma (\Theta_{u,v}^{A,n}) \) are divided into three cases below. The estimates we show here are far from being sharp, but sufficient for the purpose of proving Proposition 7.5.

To apply bounds in Section 6 we fix a number in \((0, \frac{4d}{2d^2})\), for instance, let

\[
\alpha = \frac{d - 1}{4d}.
\]

In what follows, \( \sigma \) denotes a random permutation with uniform distribution in \( \text{Sym}(\partial B_n(A)) \). Denote by \( t_1, \ldots, t_r \) the sizes of transitive components of \( \Gamma \) on \( \partial B_n(A) \) and denote by \( t_\Gamma \) the maximum of \( t_1, \ldots, t_r \).
Lemma 7.2 (Case I: intransitive without giant component). Suppose $\Gamma$ is not transitive and denote by $t_f$ the maximum of sizes of transitive components. Then there exists a constant $c, C > 0$ depending only on $d, q$ such that

$$\nu_T (\Theta_{u,v}^{A,n}) \leq C \exp (-ck_n + t_f) + C \exp \left( -ck_n^{\alpha/2q} \right).$$

Proof. Denote by $Y_1, \ldots, Y_f$ the transitive components of $\Gamma$, with $|Y_i| = t_f$. The size of $\Gamma$ is at most $t_1! \cdots t_f!$. Then by (7.1) and Stirling’s approximation (6.4), we have that if $t_f \leq k_n^{1/2q}$, then there exists a constant $C > 0$ depending only on $q, d$ such that

$$\nu_T (\Theta_{u,v}^{A,n}) \leq e^{-\frac{1}{2q} k_n \log \frac{k_n}{t_f}}.$$

If $k_n^{1/2q} < t_f \leq k_n/2$, then in order for $\sigma^{-1} \Gamma \sigma$ to contain an element $h \in \pi_n(O_{k_n}^{(0)})$ with $C_n = C_u \cdot h$, it is necessary that for each transitive component $Y_i \cdot \sigma$ of $\sigma^{-1} \Gamma \sigma$, the intersections with $C_u$ and $C_v$ are in the same $O_{k_n}^{(0)}$-orbit, that is, for each $1 \leq i \leq r$,

$$(Y_i \cdot \sigma) \cap C_n \sim O_{k_n}^{(0)} (Y_i \cdot \sigma) \cap C_n.$$

Since $O_{k_n}^{(0)} \leq W$, it is necessary then they are in the same $W$-orbit. Thus by Corollary 6.3 applied to the maximum component $Y_1 \cdot \sigma$, we have

$$\nu_T (\Theta_{u,v}^{A,n}) \leq \mathbb{P} ((Y_1 \cdot \sigma) \cap C_n \sim W (Y_1 \cdot \sigma) \cap C_n) \leq C \exp (-ct_n) \leq C \exp \left( -ck_n^{\alpha/2q} \right).$$

If $t_f > k_n/2$, then consider the compliment of $Y_1$ and the same reasoning as above implies that the intersection of $\partial B_n(A) \setminus Y_1 \cdot \sigma$ with $C_u$ and $C_v$ must be in the same $W$-orbit, therefore by Corollary 6.3 we have

$$\nu_T (\Theta_{u,v}^{A,n}) \leq \mathbb{P} (|\partial B_n(A) \setminus Y_1 \cdot \sigma| \sim O_n |\partial B_n(A) \setminus Y_1 \cdot \sigma| \cap C_v) \leq C \exp (-c(k_n - t_f)^\gamma).$$

The statement is obtained by combining these three cases.

Now consider the case where $t_f$ is large and in particular $t_f > k_n/2$. We refer to the largest transitive component as the giant component and denote it by $Y_1$. Denote by $\hat{\Gamma}$ the projection of $\Gamma$ to permutations of the giant component. If $\hat{\Gamma}$ is primitive but does not contain $\text{Alt}(Y_1)$, then the size of $\hat{\Gamma}$ is small and we can apply Lemma 7.1 again. For our purposes it suffices to use Praeger-Saxl’s bound [PS80]: if $L \leq \text{Sym}(X)$ is primitive but does not contain $\text{Alt}(X)$, then $|L| \leq 4^{|X|}$. Stronger bounds which are sub-exponential in $|X|$ are due to Babai [Bab81, Bab82]. Note that these results do not rely on classification of finite simple groups.

Lemma 7.3 (Case II: primitive in the giant component but doesn’t contain $\text{Alt}$). Suppose $t_f > \left( 1 - \frac{1}{2q} \right) k_n$ and the projection $\hat{\Gamma}$ in the giant component $Y_1$ is primitive but doesn’t contain $\text{Alt}(Y_1)$. Then there exists a constant $C$ only depending on $q$ such that

$$\nu_T (\Theta_{u,v}^{A,n}) \leq e^{-\frac{1}{2q} k_n \log \frac{k_n}{t_f}}.$$

Proof. Under the assumptions of the lemma, by the Praeger-Saxl’s bound, we have

$$|\hat{\Gamma}| \leq (k_n - t_f)! \cdot 4^{t_f}.$$


The statement follows then from (7.1). □

It remains to consider the case of imprimitive \( \bar{\Gamma} \). Given \( \bar{\Gamma} \), which is transitive on the giant component \( Y_1 \), let \( Z_1, \ldots, Z_p \) be the sets in the system of imprimitivity for \( \bar{\Gamma} \).

**Lemma 7.4** (Case III: imprimitive in the giant component). Suppose \( t_F > (1 - \frac{1}{2q})k_n \) and \( \bar{\Gamma} \) is imprimitive in the giant transitive component \( Y_1 \). Then

\[
\nu_L (\Theta_{u,v}^{A,n}) \leq Ck_n \exp \left( -ck_n^{\alpha/3q} \right),
\]

where \( c, C \) are constants only depending on \( q \) and \( d \).

**Proof.** Denote by \( p_F \) the number of sets in the system of imprimitivity. Write \( b = t_F / p_F \), that is, the cardinality of the block (domain of imprimitivity) \( Z_i \). The size of \( \bar{\Gamma} \) is at most \((b!)^{p_F} p_F!\). Thus by (7.1) and Stirling’s approximation, we have that if \( 3q \leq b \leq k_n^{3/3q} \), then there exists a constant \( C > 0 \) only depending on \( q \) such that

\[
\nu_L (\Theta_{u,v}^{A,n}) \leq e^{-\frac{1}{3q}k_n \log \frac{4k_n}{c}}.
\]

Next consider the case \( k_n^{3/3q} < b \leq t_F / 2 \). For \( \sigma^{-1} \Gamma \sigma \) to contain an element \( h \in \pi_n(O(F)_{u,v}(0)) \) that sends \( C_u^n \to C_v^n \), it is necessary that for each block \( Z_i \cdot \sigma \), either \( Z_i \cdot \sigma \cap C_u = \emptyset \), or there exists a block \( Z_j \cdot \sigma \) such that \( Z_i \cdot \sigma \cap C_u \sim_W Z_j \cdot \sigma \cap C_v \).

It is allowed that \( i = j \). The case of empty intersection can be viewed as a special instance of \( Z_i \cdot \sigma \cap C_u \sim_W Z_i \cdot \sigma \cap C_v \). For \( i \neq j \), apply Corollary 6.3 to \( Z_i, Z_j \); and for \( i = j \), apply Corollary 6.3 to \( Z_i \). Then for \( Z_i \cdot \sigma \), take a union bound over \( j \), we have that in this case

\[
\nu_L (\Theta_{u,v}^{A,n}) \leq \sum_{1 \leq j \leq p_F} \mathbb{P} (Z_1 \cdot \sigma \cap C_u \sim_W Z_j \cdot \sigma \cap C_v) \leq p_F \cdot C \exp \left( -cb^\alpha \right) \leq C p_F \exp \left( -ck_n^{\alpha/3q} \right).
\]

Finally consider the case \( b \leq 3q \), that is, the blocks are of bounded size. Consider the blocks which are completely contained in \( C_u \), and the blocks which are completely contained in \( C_v \). Denote by \( M_u (\sigma) \) the union of \( Z_i \cdot \sigma \in C_u \), \( x \in \{ u, v \} \). For \( \sigma^{-1} \Gamma \sigma \) to contain an element \( h \in \pi_u(O(F)_{u,v}(0)) \) that sends \( C_u^n \to C_v^n \), it is necessary that \( M_u (\sigma) \sim_W M_v (\sigma) \). Since \( \sigma \) is uniform, we have that conditioned on \( |M_u (\sigma)| = |M_v (\sigma)| = r \), the distribution of \( M_u (\sigma) \) and \( M_v (\sigma) \) are independent and we are in the situation of Lemma 6.1. The probability that a block \( Z_i \cdot \sigma \subseteq C_u^n \) is bounded from below by \( \left( \frac{1 + bq - u^2}{bq - u^2} \right)^b \). The Chernoff bound as in Fact 6.2 implies

\[
\mathbb{P} (|M_u (\sigma)| \leq E|M_u (\sigma)| - cp_F) \leq e^{-c \alpha p_F},
\]

where \( c \) is a constant depending on \( q, \alpha \). It follows from Lemma 6

\[
\mathbb{P} (M_u (\sigma) \sim_W M_v (\sigma)) \leq C e^{-cp_F} + C e^{-cp_F},
\]

where the constants \( c, C > 0 \) only depend on \( q \) and \( d \). The statement follows from combining the three cases. □
Next we combine these three cases. The right hand side of the bounds in Case II and Case III are summable in \( n \), while in the first case the bound depends on the size \( k_n - tr_\gamma \). Choose a sequence of increasing numbers \( (\Delta_n) \) such that \( \sum_{n} e^{-c\Delta_n} < \infty \), where \( c \) is the constant in Lemma 7.2. For instance, we can take \[ \Delta_n = \left(\frac{2}{c} \log n\right)^{1/\alpha}. \]

In the finite symmetric group \( \text{Sym}(\partial B_n(A)) \), denote by \( \Xi_n \) the collection of subgroups \[ \Xi_n := \bigcup\{ L \leq \text{Sym}(U) \times \text{Sym}(U^c) : \text{Alt}(U) \times \{id\} \leq L \}, \]
where the union is taken over all subsets \( U \subseteq \partial B_n(A) \) such that \( |U| \geq k_n - \Delta_n \). Note that the collection \( \Xi_n \) is invariant under conjugation by \( \text{Sym}(\partial B_n(A)) \).

Since \( \Delta_n \ll k_n \), if \( \Gamma < \text{Sym}(\partial B_n(A)) \) is such that its giant transitive component \( Y_1 \) has size at least \( k_n - \Delta_n \) and moreover its projection to \( \text{Sym}(Y_1) \) contains the alternating group \( \text{Alt}(Y_1) \), then \( \Gamma \) contains \( \text{Alt}(Y_1) \times \{id\} \), where \( id \) is the identity element of \( \text{Sym}(\partial B_n(A) \setminus Y_1) \). Thus the three lemmas above implies that if \( \Gamma_i \) is not in \( \Xi_n \), then the contribution of \( \nu_{\Gamma_i} (\Theta_{A,n}^u) \) to \( \tilde{\mu}_n (\Theta_{A,n}^u) \) is small. Taking into account the tree structure, by the Borel-Cantelli lemma we obtain the following.

**Proposition 7.5.** Let \( \mu \) be an IRS of \( N_F \) where \( F \) is transitive on \( D \). Then for any finite complete subtree \( A \) and \( u, v \in \partial A \) two distinct leaves, there exists a subset \( \Theta_{A,v}^u \subseteq \Theta_{A,v}^u \) with \( \mu (\Theta_{A,v}^u) = \mu (\Theta_{A,v}^u) \) such that the following holds. For any \( H \in \Theta_{A,v}^u \), there exists a non-empty open set \( U \subseteq \partial T \) such that \[ \left[R_{O_{+}^A}(U), R_{O_{-}^A}(U)\right] \leq H. \]

**Proof.** Recall that we write \( \tilde{\mu}_n^A = \sum_{i=1}^{I_n} \lambda_i \nu_{\Gamma_i} \) for the ergodic decomposition of \( \tilde{\mu}_n^A \), where each \( \Gamma_i \in \text{Sym}(\partial B_n(A)) \). Denote by \( \tilde{H}_n \) the sub-quotient \[ \tilde{H}_n = \pi_n \left(H \cap O_{+}^A(n)\right). \]

Then we have \[ \mathbb{P}_\mu \left(\tilde{H}_n \notin \Xi_n \text{ and } \tilde{H}_n \in \Theta_{A,v}^u \right) = \tilde{\mu}_n^A \left(\Xi_n \cap \Theta_{A,v}^u \right) = \sum_{\Gamma_i \notin \Xi_n} \lambda_i \nu_{\Gamma_i} (\Theta_{A,v}^u), \]
where the second equality means in the ergodic decomposition, only those \( \Gamma_i \) that are not in \( \Xi_n \) contribute to the probability of the event \( \Xi_n \cap \Theta_{A,v}^u \). By Lemma 7.2, 7.3 and 7.4 we have that if \( \Gamma_i \notin \Xi_n \), then there is a constant \( C, c > 0 \) only depending on \( q, d \) such that \[ \nu_{\Gamma_i} (\Theta_{A,v}^u) \leq C \exp (-c\Delta_n^\alpha) + Ck_n \exp \left(-ck_n^{\alpha/3}\right). \]

It follows that \[ \mathbb{P}_\mu \left(\tilde{H}_n \notin \Xi_n \text{ and } \tilde{H}_n \in \Theta_{A,v}^u \right) \leq C \exp (-c\Delta_n^\alpha) + Ck_n \exp \left(-ck_n^{\alpha/6}\right). \]

Recall that \( \Delta_n = \left(\frac{2}{c} \log n\right)^{1/\alpha} \) is chosen so that the sequence \( \exp(-c\Delta_n^\alpha) \) is summable in \( n \). Then we have \[ \sum_{n=0}^{\infty} \mathbb{P}_\mu \left(\tilde{H}_n \notin \Xi_n \text{ and } \tilde{H}_n \in \Theta_{A,v}^u \right) < \infty. \]
Therefore, by the Borel-Cantelli lemma, we have \( P_\mu \left( H_n \in \Xi \cap \Theta_{\Delta_o} \right) = 0 \), where i.o. stands for infinitely often. 

Now consider the event \( \Theta_{\Delta_o} \), as in [2.1], 

\[ \Theta_{\Delta_o} = \{ H \in \text{Sub} (N_F) : \exists h \in H \cap O_F^0(0) \text{ s.t. } v = u \cdot \pi_0(h) \} . \]

For \( H \in \Theta_{\Delta_o} \), it follows from the definitions that \( H_n \in \Theta_{\Delta_o} \) for all \( n \). Therefore \( P_\mu \left( H_n \in \Xi \cap \Theta_{\Delta_o} \right) = 0 \) implies that the subset \( \{ H \in \Theta_{\Delta_o} : H_n \notin \Xi \text{ i.o.} \} \) has \( \mu \)-measure 0. Denote by \( \tilde{\Theta}_{\Delta_o} \) the complement of this subset in \( \Theta_{\Delta_o} \), that is, 

\[ \tilde{\Theta}_{\Delta_o} = \{ H \in \Theta_{\Delta_o} : \exists n_0(H) \text{ s.t. } H_n \in \Xi \text{ for all } n \geq n_0(H) \} . \]

The reasoning above shows that (7.2) implies \( \mu \left( \tilde{\Theta}_{\Delta_o} \right) = \mu \left( \Theta_{\Delta_o} \right) \).

Next we relate back to the tree structure. Take a subgroup \( H \in \tilde{\Theta}_{\Delta_o} \). For \( n \geq n_0(H) \), denote by \( Y_H(n) \) the subset of \( \partial B_n(A) \) associated with \( H_n \) as in the definition of \( \Xi \), that is, \( Y_H(n) \) is the giant transitive component of \( H_n \). Note that this subset \( Y_H(n) \) is well-defined and one can recognize whether a vertex \( x \) is in \( Y_H(n) \) based on the size of the orbit \( x \cdot H_n \). Recall that \( \Delta_n = (e \log n)^{1/\alpha} \) is very small compared to the size of \( \partial B_n(A) \), the latter being \( qd^n \).

**Claim 7.6.** Let \( H \in \tilde{\Theta}_{\Delta_o} \). Then there is a constant \( n_0 = n_0(H) \) such that for all \( n \geq n_0 \), a vertex \( x \in \partial B_n(A) \) is in \( Y_H(n) \) if and only if all of its children are in \( Y_H(n+1) \).

**Proof of the Claim.** Let \( n_0 \) be the constant depending on \( H \) such that for all \( n \geq n_0 \), \( H_n \in \Xi \).

For the "if" direction, denote by \( V_H(n) \) the set which consists of vertices in \( \partial B_n(A) \) such that all of their children are in \( Y_H(n+1) \). Note that the set \( V_H(n) \) has cardinality at least \( v_n - \Delta_{n+1} \). In the next level \( n+1 \), \( \text{Alt} (d) \cdot V_H(n) \) is a subgroup of \( \text{Alt} (Y_H(n+1)) \cap \text{Alt} (O_F^0(n)) \). It follows that \( H_n \) is transitive on \( V_H(n) \). Since the cardinality of \( V_H(n) \) is at least \( k_n - \Delta_{n+1} \gg \Delta_n \), \( V_H(n) \) has to be contained in the giant transitive component \( Y_H(n) \).

For the "only if" direction, suppose \( x \) is in \( Y_H(n) \), then its orbit under \( H_n \) has size \( |Y_H(n)| \). Then for any of its children \( xi, i \in \{0, \ldots, d-1\} \), the orbit of \( xi \) under \( \pi_{n+1}(H \cap O_F^0(n)) \) is at least \( |Y_H(n)| \). It follows that 

\[ |(x_i) \cdot \pi_{n+1}(H \cap O_F^0(n+1))| \geq |Y_H(n)| \geq k_n - \Delta_n. \]

Since \( k_n - \Delta_n \gg \Delta_{n+1} \), we conclude that \( xi \) must be in the giant component \( Y_H(n+1) \).

We return to the proof of the proposition. By the Claim above, for \( H \in \tilde{\Theta}_{\Delta_o} \), we have that if a vertex \( x \) is in \( Y_H(n) \), where \( n \geq n_0(H) \), then for the subtree rooted at \( x \), all vertices of distance \( \ell \) to \( x \) are contained in \( Y_H(n+\ell) \). In particular, \( \text{Alt} (O_F^0) \times \{id_{\text{Sym}(\partial B_n(A))} \} < \pi_{n+\ell}(H \cap O_F^0(n+\ell)) \) for all \( \ell \geq 0 \). Since \( H \) is a closed subgroup of \( N_F \), we conclude that \( H \) contains the derived subgroup \( [O_F^0(C_x), O_F^0(C_x)] \).

At this moment the proof for Theorem [1.1] when \( F \) is transitive on \( D \) is completed. In the next section we explain the additional arguments needed to cover the general case of \( F \). In particular, we will make use of Lemma [2.2].
8. Proof of Proposition 1.2 for general $F$

We continue to use the notations in Section 3 and 4. Denote by $\{D^{(0)}, \ldots, D^{(l)}\}$ the $F$-orbits in $D = \{0, 1, \ldots, d\}$ and $\ell_F$ the labeling on the vertices of $T \setminus \{v_0\}$ induced by $F$. Let $A$ be a given complete finite subtree of $T$. The quotient $S^A_F(n) = O^A_F(n)/U^A_F(n)$, which acts on the set $\partial B_u(A)$ preserving the label $\ell_F$, is

$$S^A_F(n) = \prod_{i=0}^l \text{Sym} \left( D^{(i)}_{n,A} \right),$$

where $D^{(i)}_{n,A} = \{v \in \partial B_u(A) : \ell_F(v) = D^{(i)}\}$. For each $i \in \{0, \ldots, l\}$, denote by $\vartheta_i$ the natural projection

$$\vartheta_i : \prod_{i=0}^l \text{Sym} \left( D^{(i)}_{n,A} \right) \to \text{Sym} \left( D^{(i)}_{n,A} \right).$$

As before, $C^u_n$ consists of vertices in the subtree rooted at $u$ of distance $n$ to $u$. The size of $C^u_n \cap D^{(i)}_{n,A}$ can be calculated as follows. Denote by $M_F$ the $(l+1) \times (l+1)$ matrix whose $k$-th row has constant entries $(|D^{(k-1)}|, \ldots, |D^{(k-1)}|)$. Then

$$(8.1)\quad \begin{pmatrix} |C^u_n \cap D^{(0)}_{n,A}| \\ \vdots \\ |C^u_n \cap D^{(l)}_{n,A}| \end{pmatrix} = (M_F - I)^n - \begin{pmatrix} |D^{(0)}| - \delta_0(\ell_F(u)) \\ \vdots \\ |D^{(l)}| - \delta_l(\ell_F(u)) \end{pmatrix},$$

where $I$ is the $(l+1) \times (l+1)$ identity matrix, $\delta_i(\ell_F(u)) = 1$ if $\ell_F(u) = D^{(i)}$ and is 0 otherwise. In particular, asymptotically we have

$$|C^u_n \cap D^{(i)}_{n,A}| \sim \frac{1}{l+1} d^n.$$

To proceed, we need a labeled version of Lemma 6.1, which follows from the same kind of proof. Recall that the notation $E_1 \sim_{O^A_F(0)} E_2$ means there exists an element $g \in O^A_F(0)$ such that $E_2 = E_1 \cdot g$.

**Lemma 8.1.** Let $u, v$ be two distinct vertices in $\partial A$ such that $\ell_F(u) = \ell_F(v)$ and $i \in \{0, \ldots, l\}$. Choose a set $E_1$ of size $k$ uniformly random from $C^u_n \cap D^{(i)}_{n,A}$ and independently choose a set $E_2$ of size $k$ uniformly random from $C^v_n \cap D^{(i)}_{n,A}$. Then for any $\delta > 0$, there exists constants $c, C > 0$ only depending on $\delta, d$, such that for all $2 \leq k \leq |C^u_n \cap D^{(i)}_{n,A}|/2$,

$$\mathbb{P} \left( E_1 \sim_{O^A_F(0)} E_2 \right) \leq C \exp \left( -ck^{\frac{k}{l+1} - \delta} \right).$$

**Proof.** Consider recursively down the subtrees rooted at $u, v$. For convenience of notation, we write $O_0 = O^A_F(0)$ through the proof. In order to have an element in $O_0$ which maps $E_1$ to $E_2$, it is necessary that there exists a label preserving permutation $\gamma$ such that for each child $v_j$ of $u$, $E_1 \cap C^{n-1}_{u_j} \sim_{O_0} E_2 \cap C^{n-1}_{v^{\gamma}(j)}$.

Take a vertex $y$ in the subtree rooted at $u$ of distance $\ell$ to $u$ and a vertex $z$ in the subtree rooted at $v$ of distance $\ell$ to $v$ such that $\ell_F(y) = \ell_F(z)$. Choose a set $A_1$ of size $r$ uniformly from $C^u_n \cap D^{(i)}_{n,A}$, and independently a set $A_2$ of size $r$ uniformly from $C^v_n \cap D^{(i)}_{n,A}$. The probability that $A_1 \sim_{O_0} A_2$ depends only on the level $\ell$,
the size $r$ and the label $\ell_F(y)$. Denote by $a(\ell, r)$ the maximum of the probability $\Pr(A_1 \sim_{\Theta_n} A_2)$ over all $y \in C_{u, v}$. Write for $r \leq \frac{1}{2} \min_{z \in C_u} |C_z^{n-\ell} \cap D_{n, A}^{(i)}|$

$$\tilde{a}(\ell, r) = \max \left\{ a(\ell, s) : r \leq s \leq \frac{1}{2} \max_{z \in C_u} |C_z^{n-\ell} \cap D_{n, A}^{(i)}| \right\}. $$

Since the size of $C_y^{n-\ell} \cap D_{n, A}^{(i)}$ satisfies the equation (7.1), we have that there is a constant $\lambda_0 < 0, 1)$ depending on the matrix $M_E - I$, such that for any children $y_j$ of $y$,

$$\left| \frac{C_{y_j}^{n-\ell-1} \cap D_{n, A}^{(i)}}{C_y^{n-\ell} \cap D_{n, A}^{(i)}} \right| \leq \frac{1}{d} \lambda_0^{n-\ell-1}.$$ 

Take a small constant $\epsilon > 0$. Then for level $\ell$ such that $\lambda_0^{n-\ell-1} < \epsilon/2$, the same one step recursion to children of vertices in level $\ell$ as in the proof of Lemma 6.1 implies

$$\tilde{a}(\ell, r) \leq d \exp \left( -c_d, \epsilon r \right) + C_d, \epsilon r^{-\frac{d-1}{d}} \tilde{a} \left( \ell + 1, \frac{r}{d} - \epsilon r \right),$$

where the constants $c_d, \epsilon$ and $C_d, \epsilon$ depend only on $d$ and $\epsilon$. Start with $r$ where $d^{n-\ell} > 2r$ and iterate for $s$ steps, where $s$ is such that

$$C_d \epsilon r^{-\frac{d-1}{d}} \leq \left( \frac{1}{4} \left( \frac{1}{d} - \epsilon \right)^s \right)^d,$$

then summing up the terms we have

$$a(\ell, r) \leq C_1 \exp \left( -c(\epsilon) r (1 - d \epsilon)^{s-1} \right) + 2^{-d^{s-1}},$$

where $C_1$ is a constant depending only on $d, \epsilon$. Given a $\delta > 0$, choose $\epsilon$ sufficiently small and $s$ the largest integer satisfying (8.2), we conclude that there are constants $C, \delta$ only depending on $d, \delta$ such that

$$\tilde{a}(\ell, r) \leq C \exp \left( -c r^{\frac{d-1}{2d} - \delta} \right).$$

Let $\Gamma$ be a subgroup of $S_\Delta^A(n)$ and $\nu_\Gamma$ be the IRS which is the uniform measure on $S_\Delta^A(n)$-conjugates of $\Gamma$. With Lemma 8.1 one can repeat the argument in Section 7 to each component $\vartheta_1(\Gamma)$. Then we use the conjugacy class size Lemma 2.2 to show that if $\Gamma$ does not contain the product of the large alternating subgroups then $\nu_\Gamma \left( \Theta_{u,v}^A \right)$ is small. More precisely, let $(\Delta_n)$ be an increasing sequence of positive numbers, $\Delta_n \ll n$. Denote by $\Pi_n^A$ be the collection of subgroups of $S_\Delta^A(n)$ where $L \in \Pi_n^A$ if and only if there exists a subset $U_i \subseteq D_{n, A}^{(i)}$ for each $i \in \{0, \ldots, l\}$ with $|U_i| \geq |D_{n, A}^{(i)}| - \Delta_n$, such that $L \leq \Pi_{i=0}^l \left( \text{Sym}(U_i) \times \text{Sym} \left(D_{n, A}^{(i)} \setminus U_i\right) \right)$ and $L \geq \Pi_{i=0}^l \left( \text{Alt}(U_i) \times \{ id \} \right)$.

As in the previous section, write

$$q = |\partial A|, k_n = q d^n = |\partial B_n(A)| \quad \text{and} \quad \alpha = \frac{d-1}{4d}.$$
Lemma 8.2. Suppose that \( \Gamma \leq S_F^A(n) \) is such that \( \Gamma \notin \Pi_A^4 \). Then there are constants only depending on \( q, d \) such that

\[ \nu_T (\Theta_{u,v}^{A,n}) \leq C \exp (-c\Delta_n^a) + Ck_n \exp (-ck_n^{a/6q}) . \]

Proof. Denote by \( \vartheta_i (\Pi_A^4) \) the collection of subgroups of \( \text{Sym} \left( D_{n,A}^{(i)} \right) \) that can be obtained as projection of some \( L \in \Pi_A^4 \). This collection is similar to \( \Xi_n \) considered in the previous section.

First consider the projection \( \vartheta_i (\Gamma) \) in the symmetric group \( \text{Sym} \left( D_{n,A}^{(i)} \right) \), where \( i \in \{0, ..., l\} \). In order for \( \sigma^{-1} \Gamma \sigma \) to be in \( \Theta_{u,v}^{A,n} \), it is necessary that \( \vartheta_i (\sigma^{-1} \Gamma \sigma) \) contains an element \( g \) such that \( \left( C_u^n \cap D_{n,A}^{(i)} \right) \cdot g = C_v^n \cap D_{n,A}^{(i)} \) and there exists \( h \in \pi_n (O_F^A(0)) \) such that the restriction of \( g \) to \( C_v^n \cap D_{n,A}^{(i)} \) coincides with the restriction of \( h \). Apply the same arguments in Section 7 to a random conjugate of \( \vartheta_i (\Gamma) \) in \( \text{Sym} \left( D_{n,A}^{(i)} \right) \), with Lemma 6.1 replaced by Lemma 8.1, we obtain that if \( \Gamma \) is such that there exists \( i \in \{0, ..., l\} \) with \( \vartheta_i (\Gamma) \notin \vartheta_i (\Pi_A^4) \), then

\[ \nu_T (\Theta_{u,v}^{A,n}) \leq C \exp (-c\Delta_n^a) + Ck_n \exp (-ck_n^{a/6q}) , \]

where the constants \( C, c > 0 \) depends only on \( q, d \).

Denote by \( \Gamma_i \) the following normal subgroup of \( \Gamma \):

\[ \Gamma_i = \{ \gamma \in \Gamma : \vartheta_j (\gamma) = \text{id} \text{ for all } j \neq i \} . \]

Next we consider the case where \( \vartheta_i (\Gamma) \in \vartheta_j (\Pi_A^4) \) for every \( j \in \{0, ..., l\} \), but \( \Gamma \notin \Pi_A^4 \). Then there must exist an index \( i \in \{0, ..., l\} \) and subset \( U_i \subseteq D_{n,A}^{(i)} \) such that \( \Gamma_i \) does not contain \( \text{Alt}(U_i) \), but

\[ \text{Alt}(U_i) \times \{ \text{id} \} \subseteq \vartheta_i (\Gamma) \subseteq \text{Sym}(U_i) \times \text{Sym} \left( D_{n,A}^{(i)} \setminus U_i \right) . \]

Since \( \Gamma_i \) is normal in \( \vartheta_i (\Gamma) \), the assumption that \( \Gamma_i \) does not contain \( \text{Alt}(U_i) \) implies that

\[ \Gamma_i \leq \{ \text{id}_{\text{Sym}(U_i)} \} \times \text{Sym} \left( D_{n,A}^{(i)} \setminus U_i \right) . \]

Regard \( S_F^A(n) \) as the product of \( L_1 = \text{Sym}(D_{n,A}^{(i)}) \) and \( L_2 = \prod_{j \neq i} \text{Sym}(D_{n,A}^{(j)}) \). Now take an element \( g \in O_F^A(0) \) and apply Lemma 2.2 to the IRS \( \nu_T \), then we have

\[ \mathbb{P}_{\nu_T} (H \ni g) \leq \mathbb{E}_{\nu_T} \left[ \frac{1}{|C_{N_1} (\vartheta_i(g)H_i)|} \right] , \]

where \( N_1 \) is the normalizer of \( H_i \) in \( L_1 = \text{Sym}(D_{n,A}^{(i)}) \), \( H_i = \{ h \in H : \vartheta_j (h) = \text{id} \text{ for all } j \neq i \} \).

We now estimate the conjugacy class size which appears in (8.5) for \( g \in \pi_n (O_F^A(0)) \) with \( C_v^n = C_v^n \cdot g \). For \( \sigma \in L_1 \), we have \( (\sigma^{-1} \Gamma \sigma) \mid \leq \{ \text{id}_{\text{Sym}(U_j \cdot \sigma)} \} \times \text{Sym} \left( D_{n,A}^{(i)} \setminus U_i \cdot \sigma \right) \) and the associated normalizer \( N_1 \supseteq \text{Sym}(U_i \cdot \sigma) \times \{ \text{id} \} \). Note that for any \( \sigma \in L_1 \),

\[ |(U_i \cdot \sigma) \cap C_v^n| \geq |C_v^n \cap D_{n,A}^{(i)}| - \Delta_n . \]

We claim that if \( H = \sigma^{-1} \Gamma \sigma \) contains the element \( g \), then the map from the rigid stabilizer of \( (U_i \cdot \sigma) \cap C_v^n \) in \( L_1 \) to conjugacy classes of \( \vartheta_i(g)H_i \), given by conjugation

\[ \gamma \mapsto \gamma^{-1} \vartheta_i(g)H_i, \]

Page 25
is injective. Indeed the set of partial homeomorphisms \( \{ h_{|U_i \cdot \sigma \cap C_n^u} : h \in \vartheta_i(g)H_i \} \)
consists of a unique element, which is \( g|_{U_i \cdot \sigma \cap C_n^u} \). After the conjugation by \( \gamma \) in the
rigid stabilizer of \( (U_i \cdot \sigma) \cap C_n^u \), the set \( \{ h_{|U_i \cdot \sigma \cap C_n^u} : h \in \gamma^{-1}\vartheta_i(g)\gamma H_i \} \)
consists of a unique element \( g|_{U_i \cdot \sigma \cap C_n^u} \gamma \). The claim on injectivity follows. We conclude from
(8.5) that
\[
\mathbb{P}_{\nu_L}(H \ni g) \leq \frac{1}{(|C_v^u \cap D_{n,A}^{(i)}| - \Delta_n)!}.
\]
Take a union bound over \( g \in \pi_n \left( O_{F_0}^A(0) \right) \) with \( C_v^u = C_n^u : g \), we have that
\[
(8.9) \quad \nu_L \left( \Theta_{u,v}^{A,n} \right) \leq \frac{|\partial A||F|^d}{\left( |C_v^u \cap D_{n,A}^{(i)}| - \Delta_n \right)!} \leq C \exp \left( -cd^n \log \frac{d}{C} \right).
\]
The statement follows from combining the two cases (8.5) and (8.9).

Now we conclude the proof of Proposition 1.2 stated in the Introduction.

Proof of Proposition 1.2. Let \( A_0 \) be a given finite complete tree and \( \mu = \mu_{A_0} \) an IRS of \( O_{F_0}^A \).

We first prove that for every finite complete tree \( A \supseteq A_0 \), and two distinct vertices \( u, v \in \partial A \) with \( \ell_F(u) = \ell_F(v) \), there is a subset \( \tilde{\Theta}_{u,v}^A \) of \( \Theta_{u,v}^A \) such
that \( \mu \left( \tilde{\Theta}_{u,v}^A \right) = 0 \) and for every \( H \in \tilde{\Theta}_{u,v}^A \), there exists a vertex in \( T \setminus A \)
satisfying \( R_{O_{F_0}^A}(C_x) \prec H \).

Take the sequence \((\Delta_n)\) to be \( \Delta_n = (\frac{2}{\log n})^{1/n} \) such that the RHS of (8.4)
is summable in \( n \). Then the same reasoning as in the proof of Proposition 7.2 shows that Lemma 8.2 and the Borel-Cantelli Lemma imply
\[
(8.10) \quad \mathbb{P}_\mu \left( \pi_n \left( H \cap O_{F_0}^A(n) \right) \in \Theta_{u,v}^{A,n} \cap \left( \Pi_{A,n}^A \right)^c \text{ i.o.} \right) = 0.
\]
Denote by \( \tilde{\Theta}_{u,v}^A \) the subset of \( \Theta_{u,v}^A \) which consists of those \( H \) with the property that there exists some constant \( n_0 = n_0(H) \) such that \( \pi_n \left( H \cap O_{F_0}^A(n) \right) \in \Pi_{A,n}^A \) for all \( n \).

Then (8.10) implies \( \mu \left( \tilde{\Theta}_{u,v}^A \right) = \mu \left( \Theta_{u,v}^A \right) \).

For a subgroup \( \Gamma \in \Pi_{A,n}^A \), there are well-defined subsets \( U_i \subseteq D_n, i \in \{0, \ldots, l\} \),
associated with \( \Gamma \), such that for each \( i \), the set \( U_i \) is the gigantic transitive component of \( \Gamma \) on label \( i \) vertices. For \( H \in \tilde{\Theta}_{u,v}^A \), denote by \( Y_H^l(n) \) the set \( U_i \) associated with \( \pi_n \left( H \cap O_{F_0}^A(n) \right) \), \( n \geq n_0(H) \). By the same argument as in Claim 7.2 a vertex \( x \) is in \( \cup_{i=0}^l Y_H^l(n) \) if and only if all of its children are in \( \cup_{i=0}^l Y_H^l(n+1) \). Thus if \( n \)
is a level with \( n \geq n_0(H) \) and \( x \in \cup_{i=0}^l Y_H^l(n) \), then in the subtree rooted at \( x \), for
every level \( k \in \mathbb{N} \) we have
\[
\prod_{i=0}^l \text{Alt} \left( C_x^k \cap D_{n+k,A}^{(i)} \right) \leq \pi_{n+k} \left( H \cap O_{F_0}^A(n+k) \right).
\]
Since \( H \) is closed, we conclude that \( H \) contains the derived subgroup of \( R_{O_{F_0}^A}(C_x) \).

Note that since \( x \in T \setminus A \) and \( A \) is an expansion of \( A_0 \), we have \( R_{O_{F_0}^A}(C_x) = R_{O_{F_0}^{A_0}}(C_x) \). Finally, since the collection of events \( \Theta_{u,v}^A \), where \( A \) goes over all
expansions of \( A_0 \) and \( u, v \) go over all distinct vertices in \( \partial A \) of the same \( \ell_F \)-label,
form an open cover of $\text{Sub} \left( O^A_{F^0} \right) \setminus \{id\}$, we have that the union of $\Theta^A_{u,v}$ over all such triples $A, u, v$ gives the full measure subset in the statement.

□

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