A Note on Quantum Errors and Their Correction

Subhash Kak
Department of Electrical & Computer Engineering
Louisiana State University
Baton Rouge, LA 70803-5901; kak@ee.lsu.edu

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Abstract

This note presents a few observations on the nonlocal nature of quantum errors and the expected performance of the recently proposed quantum error-correction codes that are based on the assumption that the errors are either bit-flip or phase-flip or both.

1 Introduction

To disavow an error is to invent retroactively.

—Johann Wolfgang von Goethe

In a classical information system the basic error is represented by a 0 becoming a 1 or vice versa. The characterization of such errors is in terms of an error rate, $\epsilon$, associated with such flips. The correction of such errors is achieved by appending check bits to a block of information bits. The redundancy provided by the check bits can be exploited to determine the location of errors using the method of syndrome decoding. These codes are characterized by a certain capacity for error-correction per block. Errors at a rate less than the capacity of the code are completely corrected.

Now let us look at a quantum system. Consider a single cell in a quantum register. The error here can be due to a random unitary transformation or
by entanglement with the environment. These errors cannot be defined in a
graded sense because of the group property of unitary matrices and the many
different ways in which the entanglements can be expressed. Let us consider
just the first type of error, namely that of random unitary transformation. If
the qubit is the state \(|0\rangle\), it can become \(a|0\rangle + b|1\rangle\). Likewise, the state \(|00\rangle\)
can become \(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle\). In the initialization of the qubit a
similar error can occur\(^2\). If the initialization process consists of collapsing
a random qubit to the basis state \(|0\rangle\), the definition of the basis direction
can itself have a small error associated with it. This error is analog and so,
unlike error in classical digital systems, it cannot be controlled. In almost
all cases, therefore, the qubits will have superposition states, although the
degree of superposition may be very low.

From another perspective, classical error-correction codes map the infor-
mation bits into codewords in a higher dimensional space so that if just a few
errors occur in the codeword, their location can, upon decoding, be identi-
fied. This identification is possible because the errors perturb the codewords,
locally, within small spheres. Quantum errors, on the other hand, perturb
the information bits, in a nonlocal sense, to a superposition of many states, so
the concept of controlling all errors by using a higher dimensional codeword
space cannot be directly applied.

According to the positivist understanding of quantum mechanics, it is
essential to speak from the point of view of the observer and not ask about
any intrinsic information in a quantum state\(^1\). Let’s consider, there
fore, the
representation of errors by means of particles in a register of \(N\) states.

We could consider errors to be equivalent to either \(n\) bosons or fermions.
Bosons, in a superpositional state follow the Bose-Einstein statistics. The
probability of each pattern will be given by

\[
\frac{1}{\binom{N+n-1}{n}}.
\]

(1)

So if there are 3 states and 1 error particle, we can only distinguish
between 3 states: 00, 01 or 10, 11. Each of these will have a probability of
\(\frac{1}{4}\). To the extent this distribution departs from that of classical mechanics,
it represents nonlocality at work.

If the particles are fermions, then they are indistinguishable, and with \(n\)
error objects in $N$ cells, we have each with the probability

$$\frac{1}{\binom{N}{n}}. \quad (2)$$

If states and particles have been identified, these statistics will be manifested by a group of particles. If the cells are isolated then their histories cannot be described by a single unitary transformation.

Like the particles, the errors will also be subject to the same statistics. These statistics imply that the errors will not be independent, an assumption that is basic to the error-correction schemes examined in the literature.

To summarize, important characteristics of quantum errors that must be considered are component proliferation, nonlocal effects and amplitude error. All of these have no parallel in the classical case. Furthermore, quantum errors are analog and so the system cannot be shielded below a given error rate. Such shielding is possible for classical digital systems.

We know that a computation need not require any expenditure of energy if it is cast in the form of a reversible process. A computation which is not reversible must involve energy dissipation. Considering conservation of information+energy to be a fundamental principle, a correction of random errors in the qubits by unitary transformations, without any expenditure of energy, violates this principle.

Can we devise error-correction coding for quantum systems? To examine this, consider the problem of protein-folding, believed to be NP-complete, which is, nevertheless, solved efficiently by Nature. If a quantum process is at the basis of this amazing result, then it is almost certain that reliable or fault-tolerant quantum computing must exist but, paying heed to the above-mentioned conservation law, it appears such computing will require some lossy operations.

In this note we examine the currently investigated models of quantum error-correction from the point of view of their limitations. We also consider how quantum errors affect a computation in comparison with classical errors.

2 Representing quantum errors

*Sed fugit interea, fugit inreparabile tempus.*
But meanwhile it is flying, irretrievable time is flying.
—Virgil

Every unitary matrix can be transformed by a suitable unitary matrix into a diagonal matrix with all its elements of unit modulus. The reverse also being true, quantum errors can play havoc.

The general unitary transformation representing errors for a qubit is:

$$\frac{1}{\sqrt{||e_1||^2 + ||e_2||^2}} \begin{pmatrix} e_1^* & e_2^* \\ e_2 & -e_1 \end{pmatrix}.$$ (3)

These errors ultimately change the probabilities of the qubit being decoded as a 0 and as a 1. From the point of view of the user, when the quantum state has collapsed to one of its basis states, it is correct to speak of an error rate. But such an error rate cannot be directly applied to the quantum state itself.

Unlike the classical digital case, quantum errors cannot be completely eliminated because they are essentially analog in nature.

The unitary matrix (1) represents an infinite number of cases of error. The error process is an analog process, and so, in general, such errors cannot be corrected. From the point of view of the qubits, it is a nonlocal process.

If it is assumed that the error process can be represented by a small rotation and the initial state is either a 0 or a 1, then this rotation will generate a superposition of the two states but the relative amplitudes will be different and these could be exploited in some specific situations to determine the starting state. But, obviously, such examples represent trivial cases.

The error process may be usefully represented by a process of quantum diffusions and phase rotations.

Shor\cite{4} showed how the decoherence in a qubit could be corrected by a system of triple redundancy coding where each qubit is encoded into nine qubits as follows:

\[ |0\rangle \rightarrow \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \]

\[ |1\rangle \rightarrow \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle). \] (4)
Shor considers the decoherence process to be one where a qubit decays into a weighted amplitude superposition of its basis states. In parallel to the assumption of independence of noise in classical information theory, Shor assumes that only one qubit out of the total of nine decoheres. Using a Bell basis, Shor then shows that one can determine the error and correct it.

But this system does not work if more than one qubit is in error. Since quantum error is analog, each qubit will be in some error and so this scheme will, in practice, not be useful in completely eliminating errors.

The question of decoherence, or error, must be considered as a function of time. One may use the exponential function $\lambda e^{-\lambda t}$ as a measure of the decoherence probability of the amplitude of the qubit. The measure of decoherence that has taken place by time $t$ will then be given by the probability, $p_t$:

$$p_t = 1 - \lambda e^{-\lambda t}.$$ \hfill (5)

In other words, by time $t$, the amplitude of the qubit would have decayed to a fraction $(1 - \lambda e^{-\lambda t})$ of its original value. At any time $t$, there is a 100% chance that the probability amplitude of the initial state will be a fraction $\alpha_k < 1$ of the initial amplitude.

If we consider a rotation error in each qubit through angle $\theta$, there exists some $\theta_k$ so that the probability

$$\text{Prob}(\theta > \theta_k) \rightarrow 1.$$ \hfill (6)

This means that we cannot represent the qubit error probability by an assumed value $p$ as was done by Shor in analogy with the classical case. In other words, there can be no guarantee of eliminating decoherence.

### 3 Recently proposed error-correction codes

*The fox knows many things—the hedgehog one big one.*

—Archilochus

The recently proposed models of quantum error-correction codes assume that the error in the qubit state $a|0\rangle + b|1\rangle$ can be either a bit flip $|0\rangle \leftrightarrow |1\rangle$, a phase flip between the relative phases of $|0\rangle$ and $|1\rangle$, or both $[\text{[ ] [ ] }]$. 
In other words, the errors are supposed to take the pair of amplitudes $(a, b)$ to either $(b, a)$, $(a, -b)$, or $(-b, a)$.

But these three cases represent a vanishingly small subset of all the random unitary transformations associated with arbitrary error. These are just three of the infinity of rotations and diffusions that the qubit can be subject to. The assumed errors, which are all local, do not, therefore, constitute a distinguished set on any physical basis.

In one proposed error-correction code, each of the states $|0\rangle$ or $|1\rangle$ is represented by a 7-qubit code, where the strings of the codewords represent the codewords of the single-error correcting Hamming code, the details of which we don’t need to get into here. The code for $|0\rangle$ has an even number of 1s and the code for $|1\rangle$ has an odd number of 1s.

$$|0\rangle_{\text{code}} = \frac{1}{\sqrt{8}}(|0000000\rangle + |0001111\rangle + |0110011\rangle + |0111100\rangle + |1010101\rangle + |1011010\rangle + |1100110\rangle + |1101001\rangle), \quad (7)$$

$$|1\rangle_{\text{code}} = \frac{1}{\sqrt{8}}(|1111111\rangle + |1110000\rangle + |1001100\rangle + |1001100\rangle + |0101010\rangle + |0100101\rangle + |0011001\rangle + |0010110\rangle). \quad (8)$$

As mentioned before, the errors are assumed to be either in terms of phase-flips or bit-flips. Now further ancilla bits—three in total—are augmented that compute the syndrome values. The bit-flips, so long as limited to one in each group, can be computed directly from the syndrome. The phase-flips are likewise computed, but only after a change of the bases has been performed.

Without going into the details of these steps, which are a straightforward generalization of classical error correction theory, it is clear that the assumption of single phase and bit-flips is very restrictive.

In reality, errors in the 7-qubit words will generate a superposition state of 128 sequences, rather than the 16 sequences of equations (5) and (6), together with 16 other sequences of one-bit errors, where the errors in the amplitudes are limited to the phase-flips mentioned above. *All kinds of bit-flips*, as well as modifications of the amplitudes will be a part of the quantum state.
We can represent the state, with the appropriate phase shifts associated with each of the 128 component states, as follows:

\[
|\phi\rangle = e^{i\theta_1}a_1|0000000\rangle + e^{i\theta_2}a_2|0000001\rangle + \ldots + e^{i\theta_N}a_N|1111111\rangle
\] (9)

While the amplitudes of the newly generated components will be small, they would, nevertheless, have a non-zero error probability. These components, cannot be corrected by the code and will, therefore, contribute to an residual error probability.

The amplitudes implied by (7) will, for the 16 sequences of the original codeword after the error has enlarged the set, be somewhat different from the original values. So if we speak just of the 16 sequences the amplitudes cannot be preserved without error.

Furthermore, the phase errors in (7) cannot be corrected. These phases are of crucial importance in many recent quantum algorithms.

It is normally understood that in classical systems if error rate is smaller than a certain value, the error-correction system will correct it. In the quantum error-correction systems, this important criterion is violated. Only certain specific errors are corrected, others even if smaller, are not.

In summary, the proposed models are based on a local error model while real errors are nonlocal where we must consider the issues of component proliferation and amplitude errors. These codes are not capable of completely correcting small errors that cause new entangled component states to be created.

4 The sensitivity to errors

The nonlocal nature of the quantum errors is seen clearly in the sensitivity characteristics of these errors.

Consider that some data sets related to a problem are being simultaneously processed by a quantum machine. Assume that by some process of phase switching and diffusion the amplitude of the desired solution out of the entire set is slowly increased at the expense of the others. Nearing the end of the computation, the sensitivity of the computations to errors will increase dramatically, because the errors will, proportionately, increase for the smaller amplitudes. To see it differently, it will be much harder to reverse
the computation if the change in the amplitude or phase is proportionally greater.

This means that the “cost” of quantum error-correction will depend on the state of the computing system. Even in the absence of errors, the sensitivity will change as the state evolves, a result, no doubt, due to the nonlocal nature of quantum errors.

These errors can be considered to be present at the stage of state preparation and through the continuing interaction with the environment and also due to the errors in the applied transformations to the data. In addition, there may exist nonlocal correlations of qubits with those in the environment. The effect of such correlations will be unpredictable.

Quantum errors cannot be localized. For example, when speaking of rotation errors, there always exists some $\theta_k > 0$ so that $\text{prob}(\theta > \theta_k) \to 1$.

When doing numerical calculations on a computer, it is essential to have an operating regime that provides reliable, fault-tolerant processing. Such regimes exist in classical computing. But the models currently under examination for quantum computing cannot eliminate errors completely.

The method of syndrome decoding, adapted from the theory of classical error-correcting codes, appears not to be the answer to the problem of fault-tolerant quantum computing. New approaches to error-correction need to be investigated.

## 5 Conclusions

Nonlocality, related both to the evolution of the quantum information system and errors, defines a context in which error-correction based on syndrome decoding will not work.

How should error-correction be defined then? Perhaps through a system akin to associative learning in spin glasses.

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