Almost Kähler manifolds whose antiholomorphic sectional curvature is pointwise constant

M. FALCITELLI – A. FARINOLA – O.T. KASSABOV

Abstract: We prove that an almost Kähler manifold \((M, g, J)\) with \(\dim M \geq 8\) and pointwise constant antiholomorphic sectional curvature is a complex space-form.

1 – Introduction and preliminaries

Let \((M, g, J)\) be a \(2n\)-dimensional almost Hermitian manifold. A 2-plane \(\alpha\) in the tangent space \(T_x M\) at a point \(x\) of \(M\) is antiholomorphic if it is orthogonal to \(J\alpha\).

The manifold \((M, g, J)\) has pointwise constant antiholomorphic sectional curvature (p.c.a.s.c.) \(\nu\) if, at any point \(x\), the Riemannian sectional curvature \(\nu(x) = K_x(\alpha)\) is independent on the choice of the antiholomorphic 2-plane \(\alpha\) in \(T_x M\).

If \((g, J)\) is a Kähler structure, the previous condition means that \((M, g, J)\) is a complex space-form, i.e. a Kähler manifold with constant holomorphic sectional curvature \(\mu = 4\nu\) ([2]). Moreover, the Riemannian curvature tensor \(R\) satisfies:

\[
(1.1) \quad R = \nu(\pi_1 + \pi_2),
\]

\(\nu\) being a constant function and \(\pi_1, \pi_2\) the tensor fields such that:

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\[\pi_1(X, Y, Z, W) = g(X, Z)g(Y, W) - g(Y, Z)g(X, W);\]
\[\pi_2(X, Y, Z, W) = 2g(JX, Y')g(JZ, W') + g(JX, Z)g(JY, W') - g(JY, Z)g(JX, W').\]

According to [16], for any \((0,2)\)-tensor field \(S\), we consider the \((0,4)\)-tensor fields \(\phi(S), \psi(S)\) defined by:

\[\phi(S)(X, Y, Z, W) = g(X, Z)S(Y, W) + g(Y, W)S(X, Z) - g(X, W)S(Y, Z) - g(Y, Z)S(X, W),\]

\[\psi(S)(X, Y, Z, W) = 2g(X, JY)S(Z, JW) + 2g(Z, JW)S(X, JY) + g(X, JZ)S(Y, JW) + g(Y, JW)S(X, JZ) - g(X, JW)S(Y, JZ) - g(Y, JZ)S(X, JW).\]

A generalization of (1.1) is obtained by G. Ganchev ([5]). In fact, he proves that the almost Hermitian manifold \((M, g, J)\) has p.c.a.s.c. \(\nu\) iff

\[R = \frac{1}{2(n+1)}\psi(\rho^*(R)) + \nu \pi_1 - \frac{2(n + 1)\nu + \tau^*(R)}{2(n + 1)(2n + 1)}\pi_2,\]

\(\rho^*(R), \tau^*(R)\) respectively denoting the \(*\)-Ricci tensor and the \(*\)-scalar curvature.

The previous formula allows to relate the symmetric part of \(\rho^*(R)\) to the Ricci tensor \(\rho(R)\) and thus \(\tau^*(R)\) to the scalar curvature \(\tau(R)\).

Indeed, putting

\[L_3R(X, Y, Z, W) = R(JX, JY, JZ, JW),\]

one has:

\[\rho^*(R + L_3R)(X, Y) = \rho^*(R)(X, Y) + \rho^*(R)(Y, X) = \rho^*(R + L_3(R))(JX, JY),\]

and (1.4) implies:

\[\rho^*(R + L_3R) = \frac{2}{3}(n + 1)\rho(R) - \frac{(n + 1)\tau(R) - 3\tau^*(R)}{3n}g,\]

\[8n(n^2 - 1)\nu = (2n + 1)\tau(R) - 3\tau^*(R).\]

Another characterization of the p.c.a.s.c. condition can be obtained regarding the Riemannian curvature tensor as a section of the vector bundle \(\mathcal{R}(M)\) of
the algebraic curvature tensor fields on $M$. According to the splitting $\mathcal{R}(M) = \oplus_{1 \leq i \leq 10} W_i(M)$ considered in [16], the formula (1.4) can be interpreted in terms of the vanishing of suitable $W_i$-projections $p_i(R)$ of $R$.

More precisely, an application of the Theorem 8.1 in [16] yields to the following result.

**Proposition 1.1.** Let $(M, J, g)$ be an almost Hermitian manifold. If $\dim M = 4$, $(M, g, J)$ has p.c.a.s.c. iff $p_3(R) = p_7(R) = p_8(R) = 0$. If $\dim M \geq 6$, then $(M, g, J)$ has p.c.a.s.c. iff $p_3(R) = p_6(R) = p_7(R) = p_8(R) = p_{10}(R) = 0$ and (1.6) holds.

Combining with the Theorem 18 in [4], one has:

**Proposition 1.2.** Let $(M, J, g)$ be an almost Hermitian manifold with p.c.a.s.c. Then $g$ is an Einstein metric iff $(M, g, J)$ has pointwise constant holomorphic sectional curvature.

The classification of the almost Hermitian manifolds with p.c.a.s.c. is still an open problem, even if nowadays several partial results are known.

In [1] V. Apostolov, G, Ganchev and S. Ivanov classify the compact Hermitian surfaces with constant antiholomorphic curvature. Moreover, they construct an example of conformal Kähler surface with p.c.a.s.c. $\nu$, the function $\nu$ being non-constant. Thus, the Schur’s lemma of antiholomorphic type is not valid in the 4-dimensional case.

Furthermore, the third autor of the present paper has already solved the abovementioned problem for $2n$-dimensional, $n \geq 3$, connected, $\mathcal{R}_3$–manifolds, i.e. almost Hermitian manifolds such that $R = L_3R$ (equivalently, $p_8(R) = p_9(R) = p_{10}(R) = 0$).

In fact, any connected $\mathcal{R}_3$-manifold $M$ with p.c.a.s.c. and $\dim M \geq 6$ has constant antiholomorphic sectional curvature ([9]) and turns out to be a real space-form or a complex space-form ([10]).

This result allows the classification of nearly Kähler as well as locally conformal Kähler manifolds with p.c.a.s.c. In fact, any nearly Kähler manifold is a $\mathcal{R}_3$-manifold ([7]). Since for a locally conformal Kähler manifolds the projections $p_9(R)$ vanishes, the locally conformal Kähler manifolds with p.c.a.s.c. turn out to be $\mathcal{R}_3$-manifolds ([3]).

Moreover, combining the results stated in [10] and [13], any connected $\mathcal{R}_3$-almost Kähler manifold with p.c.a.s.c. and $\dim M \geq 6$ turns out to be a complex space-form.
Since the projection $p_0(R)$, a priori, does not vanish in the almost Kähler case the classification of the almost Kähler manifolds with p.c.a.s.c is meaningful.

We recall the almost Kähler condition, i.e.:

\[(1.8) \quad \sigma_{V,X,Y}(\nabla_V \omega)(X,Y) = 0,\]

$\sigma$ denoting the cyclic sum and $\nabla \omega$ the covariant derivative of the fundamental 2-form $\omega$ ($\omega(X,Y) = g(JX,Y)$) with respect to the Levi-Civita connection $\nabla$.

Moreover, \(1.8\) implies:

\[(1.9) \quad (\nabla_X J)Y + (\nabla_J X)JY = 0;\]

\[(1.10) \quad \sum_i (\nabla e_i)J e_i = 0,\]

for any local orthonormal frame $\{e_i\}_{1 \leq i \leq 2n}$.

In this paper we state the following theorem, whose proof is divided into several steps.

**Theorem 1.** Let $(M, g, J)$ be a $2n$-dimensional, $n \geq 4$, connected, almost Kähler manifold. If $(M, g, J)$ has pointwise constant antiholomorphic sectional curvature, then $(M, g, J)$ is a complex space-form.

**2 – Some auxiliary lemmas**

Given a $2n$-dimensional almost Hermitian manifold $(M, g, J)$, the tensor field

\[(2.1) \quad Q = \frac{1}{6} \rho(R) + \frac{1}{4(n + 1)} \rho^*(R - L_3 R)\]

is, in general, neither symmetric nor skew-symmetric, since $\rho(R)$, $\rho^*(R - L_3 R)$ respectively determine its symmetric, skew-symmetric components. Moreover, we assume that $(M, g, J)$ has p.c.a.s.c.; then the formula \(1.6\) implies:

\[(2.2) \quad Q(JX, JY) = Q(Y, X),\]

and thus one has:
(2.3) \( Q((\nabla_V J)X, JY) = Q(Y, (\nabla_V J)X) \); 

(2.4) \((\nabla_V Q)(JX, JY) = (\nabla_V Q)(Y, X) - Q((\nabla_V J)X, JY) - Q(JX, (\nabla_V J)Y)\); 

(2.5) \( \sum_i Q((\nabla_V J)e_i, Je_i) = -\sum_i Q(Je_i, (\nabla_V J)e_i) \), 

for any local orthonormal frame \( \{e_i \}_{1 \leq i \leq 2n} \).

If \((g, J)\) is an almost Kähler structure, (1.8) and (1.9) imply also

(2.6) \( \sum_i Q(V, e_i)(\nabla_{e_i} \omega)(Y, X) = Q(V, (\nabla_X J)Y - (\nabla_Y J)X) \); 

(2.7) \( 2\sum_i Q(Je_i, (\nabla_{e_i} J)V) = \sum_i Q(Je_i, (\nabla_V J)e_i) \).

Now we observe that (2.1), (1.6) and (1.7) allow to rewrite (1.4) as follows:

(2.8) \( R = \psi(Q) + \nu \pi_1 - \frac{2n - 1}{3} \nu \pi_2 \).

By means of (2.8) and the second Bianchi identity, we will state some properties of \( Q \) and \( \nabla Q \) useful for the proof of the Theorem 1. 

First of all, from (2.8), one has:

(2.9) \begin{align*}
(\nabla_V R)(X, Y, Z, W) &= 2g(X, JY)\{ (\nabla_V Q)(Z, JW) \\
&+ Q(Z, (\nabla_V J)W) \} + 2g(Z, JW)\{ (\nabla_V Q)(X, JY) \\
&+ Q(X, (\nabla_V J)Y) \} + g(X, JZ)\{ (\nabla_V Q)(Y, JW) \\
&+ Q(Y, (\nabla_V J)W) \} + g(Y, JW)\{ (\nabla_V Q)(X, JZ) \\
&+ Q(X, (\nabla_V J)Z) \} - g(Y, JZ)\{ (\nabla_V Q)(X, JW) \\
&+ Q(X, (\nabla_V J)W) \} - g(X, JW)\{ (\nabla_V Q)(Y, JZ) \\
&+ Q(Y, (\nabla_V J)Z) \} + 2(\nabla_V \omega)(Y, X)Q(Z, JW) \end{align*} \)
\[ +2(\nabla_V \omega)(W, Z)Q(X, JY) + (\nabla_V \omega)(Z, X)Q(Y, JW) \\
+ (\nabla_V \omega)(W, Y)Q(X, JZ) - (\nabla_V \omega)(Z, Y)Q(X, JW) \\
- (\nabla_V \omega)(W, X)Q(JY, JZ) \\
+ V(\nu)(\frac{2n-1}{3}\pi_2)(X, Y, Z, W) \\
- \frac{2n-1}{3}\nu\{2g(X, JY)(\nabla_V \omega)(W, Z) \\
+ 2g(Z, JW)(\nabla_V \omega)(Y, X) + g(X, JZ)(\nabla_V \omega)(W, Y) \\
+ g(Y, JW)(\nabla_V \omega)(Z, X) - g(X, JW)(\nabla_V \omega)(Z, Y) \\
- g(Y, JZ)(\nabla_V \omega)(W, X)\}. \]

**Lemma 2.1.** Let \((M, g, J)\) be a \(2n\)-dimensional \((n \geq 2)\) almost-Kähler manifold with p.c.a.s.c. The covariant derivative \(\nabla Q\) is given by:

\[
2(n+1)(2n-1)(\nabla_V Q)(X, JY) = (2n+3)(Q(Y, (\nabla_X J)V) \\
- Q(X, (\nabla_Y J)V) + (4n+3)Q(V, (\nabla_X J)Y - (\nabla_Y J)X) \\
- Q(Y, (\nabla_Y J)X) - (4n^2 + 2n - 3)Q(X, (\nabla_V J)Y) \\
+ g(X, JY)\left\{2n \sum_i Q(Je_i, (\nabla_V J)e_i) \\
+ \frac{4}{3}(n+1)(n-2)V(\nu) + \frac{2n-1}{6}V(\tau(R))\right\}\}
\\
+ g(X, JV)\left\{\frac{4n-1}{2} \sum_i Q(Je_i, (\nabla_X J)e_i) \\
- \frac{2}{3}(n+1)(2n^2 - 4n + 3)Y(\nu) + \frac{2n-1}{6}Y(\tau(R))\right\}\}
\\
- g(Y, JV)\left\{\frac{4n-1}{2} \sum_i Q(Je_i, (\nabla_X J)e_i) \\
- \frac{2}{3}(n+1)(2n^2 - 4n + 3)X(\nu) + \frac{2n-1}{6}X(\tau(R))\right\}\}
\\
- 2(n+1)\{JX(\nu)g(Y, V) - JY(\nu)g(X, V)\} \\
+ \frac{1}{3}(n+1)(\tau(R) - 2(2n-1)^2\nu)(\nabla_V \omega)(X, Y),
\]
where \( \{e_i\}_{1 \leq i \leq 2n} \) is a local orthonormal frame.

**Proof.** In fact, by the second Bianchi identity, we have:

\[
\sigma_{(V,X,Y)} \sum_i (\nabla_V R)(X,Y,e_i,Je_i) = 0,
\]

which, combined with (2.9), (2.4) and (1.8), yields to:

\[
(2.11)
\begin{align*}
2(n+1) \sigma_{(V,X,Y)} (\nabla_V Q)(X,JY) &= \sigma_{(V,X,Y)} \{Q(Y,(\nabla_V J)X) \\
&- (2n+3)Q(X,(\nabla_V J)Y) + g(JX,Y)\frac{1}{6}V(\tau(R)) \\
&- \frac{4}{3}(n^2-1)V(\nu) + \sum_i Q(Je_i,(\nabla_V J)e_i)\}.
\end{align*}
\]

Moreover, by the second Bianchi identity, we obtain:

\[
\sum_{i,q}\{2(\nabla_{e_i} R)(V,e_q,Je_i,Je_q) - (\nabla_V R)(e_i,e_q,Je_i,Je_q)\} = 0,
\]

which, combined with (2.9), (2.5) and (2.7) implies:

\[
(2.12)
\begin{align*}
\sum_i (\nabla_{e_i} Q)(V,e_i) &= \frac{4n+1}{4(n+1)} \sum_i Q(Je_i,(\nabla_V J)e_i) \\
&+ \frac{n}{6(n+1)}V(\tau(R)) - \frac{2}{3}(n-1)^2V(\nu).
\end{align*}
\]

This formula, with (2.4), (2.6), (1.10) and the condition

\[
\sum_i \{(\nabla_V R)(e_i,Le_i,X,Y) + 2(\nabla_{e_i} R)(Je_i,V,X,Y)\} = 0
\]

yields to
\[2n(\nabla_V Q)(X, JY) + (\nabla_X Q)(Y, JV) + (\nabla_Y Q)(V, JX) = 2Q(V, (\nabla_X J)Y)\]
\[-3Q(V, (\nabla_Y J)X) - 2nQ(X, (\nabla_Y J)Y) - Q(X, (\nabla_Y J)V)\]
\[+ g(X, JY)\left\{ \frac{2n-1}{2(n+1)} \sum_i Q(Je_i, (\nabla_Y J)e_i) + \frac{2}{3}(2n-3)V(\nu) \right\}\]
\[+ \frac{n-1}{6(n+1)}V(\tau(R)) \right\} + g(V, JY)\left\{ \frac{4n+1}{4(n+1)} \sum_i Q(Je_i, (\nabla_X J)e_i) \right\}\]
\[-\frac{1}{3}(2n^2 - 2n + 1)X(\nu) + \frac{n}{6(n+1)}X(\tau(R)) \right\}\]
\[-g(V, JX)\left\{ \frac{4n+1}{4(n+1)} \sum_i Q(Je_i, (\nabla_Y J)e_i) \right\}\]
\[-\frac{1}{3}(2n^2 - 2n + 1)Y(\nu) + \frac{n}{6(n+1)}Y(\tau(R)) \right\}\]
\[-JX(\nu)g(Y, V) + JY(\nu)g(X, V)\]
\[+ \frac{1}{6}(\tau(R) - 2(2n - 1)^2\nu)(\nabla_V \omega)(X, Y).\]

Thus, combining with (2.11), one proves the statement.

**Lemma 2.2.** In the hypothesis of the Lemma 2.1, when \( n \neq 3 \), one has:

\[(2.13) \sum_i Q((J e_i, (\nabla_V J)e_i) = \frac{4}{3}(n^2 - 1)V(\nu).\]

**Proof.** In fact, the second Bianchi identity and (2.1) give:

\[V(\tau(R)) = 2\sum_i (\nabla_{e_i} \rho(R))(V, e_i) = 6\sum_i \{((\nabla_{e_i} Q)(V, e_i) + (\nabla_{e_i} Q)(e_i, V)\}.\]

Moreover, the formulas (2.10), (2.12), (2.7), (1.8), (1.9), (1.10) imply:

\[\sum_i \{((\nabla_{e_i} Q)(V, e_i) + (\nabla_{e_i} Q)(e_i, V)\} = \frac{1}{6}V(\tau(R))\]
\[+ \frac{n-2}{2n-1}\left\{ \sum_i Q(Je_i, (\nabla_Y J)e_i) - \frac{4}{3}(n^2 - 1)V(\nu) \right\},\]
and then the statement.

**Proposition 2.1.** In the hypothesis of the Lemma 2.1, if \( n \geq 3 \), one has:

\[
4(2n - 3)(Q(X, (\nabla Y)JW) - Q(Y, (\nabla X)JW)
\]

\[
+ 4n(Q(X, (\nabla W)JY) - Q(Y, (\nabla W)JX))
\]

\[
- 4(n - 3)Q(W, (\nabla X)JY - (\nabla Y)JX) + \tau(R)(\nabla W \omega)(X, Y)
\]

(2.14)

\[
- \frac{8}{3}(2n^2 - 4n + 3)(X(\nu)g(JY, W)
\]

\[
- Y(\nu)g(JX, W) + 2W(\nu)g(X, JY)
\]

\[
+ 8n(n - 2)(JX(\nu)g(Y, W)
\]

\[
- JY(\nu)g(X, W)) = 0 .
\]

**Proof.** The Lemmas 2.1 and 2.2, the formula (2.4) and the condition:

\[
\sum_i\{(\nabla e_i R)(X, Y, e_i, W) + (\nabla Y R)(e_i, X, e_i, W) - (\nabla X R)(e_i, Y, e_i, W)\} = 0
\]

imply the vanishing of the tensor field \( S \) defined by:

\[
S(W, X, Y) = 4(2n^2 - 3){Q(X, (\nabla Y)JW) - Q(Y, (\nabla X)JW)}
\]

\[
- 4n\{Q((\nabla Y)JW, X) - Q((\nabla X)JW, Y)\}
\]

\[
+ 2(2n^2 + 3n + 3){Q(X, (\nabla W)JY) - Q(Y, (\nabla W)JX)}
\]

\[
- 2(n + 3){Q((\nabla W)JY, X) - Q((\nabla W)JX, Y)}
\]

\[
+ 2(n - 3){Q(W, (\nabla X)JY - (\nabla Y)JX) - (2n + 3)Q(\nabla X)JY
\]

\[
- (\nabla Y)JX, W)\} + (n + 1)\tau(R)(\nabla W \omega)(X, Y)
\]

\[
- 4(n + 1)(2n^2 - 2n - 3){X(\nu)g(JY, W) - Y(\nu)g(JX, W)}
\]

\[
+ \frac{4}{3}(n + 1)(4n^2 - 4n + 3){JX(\nu)g(Y, W)
\]

\[
- JY(\nu)g(X, W) - 2W(\nu)g(X, JY)\} .
\]

In particular, by means of (1.9) and (2.3), the conditions:

\[
S(W, X, Y) - S(W, JX, JY) - S(JW, JX, Y) - S(JW, X, JY) = 0 ;
\]
\[ S(W, X, Y) - S(W, JX, JY) + S(JW, JX, Y) + S(JW, X, JY) = 0, \]

turn out to be equivalent to:

\[
2(n - 3)\{ Q(W, (\nabla_X J) Y - (\nabla_Y J) X) + Q((\nabla_Y J) X) \\
- (\nabla_Y J) X, W) \} - 2(2n - 3)\{ Q(X, (\nabla_Y J) W) \\
+ Q((\nabla_X J) W, X) - Q(Y, (\nabla_X J) W) - Q((\nabla_X J) W, Y) \}
- 2n\{ Q(X, (\nabla_W J) Y) + Q((\nabla_W J) Y, X) \\
- Q(Y, (\nabla_W J) X) - Q((\nabla_W J) X, Y) \\
- \tau(R)(\nabla_W \omega)(X, Y) = 0; \tag{2.15} \]

Thus, if \( n = 3 \), the statement follows from (2.15) combined with the condition \( S = 0 \). If \( n > 3 \), (2.16) implies also, with suitable change of the involved variables, the relation:

\[
(n - 3)\{ Q(W, (\nabla_Y J) X - (\nabla_X J) Y) - Q((\nabla_Y J) X) \\
- (\nabla_X J) Y, W) + \frac{2}{3}(n + 1)\{ X(\nu)g(JY, W) \\
- Y(\nu)g(JX, W) + JX(\nu)g(Y, W) - JY(\nu)g(X, W) \} \} = 0. \tag{2.16} \]

Thus, applying (2.17) and (2.16), the relation (2.15) yields to:

\[
Q((\nabla_W J) X, Y) - Q((\nabla_W J) Y, X) = Q((\nabla_X J) W, Y) \\
- Q((\nabla_Y J) W, X) + Q(Y, (\nabla_W J) X - (\nabla_X J) W) \\
- Q(X, (\nabla_W J) Y - (\nabla_Y J) W) + \frac{2}{3}(n + 1)\{ X(\nu)g(Y, JW) \\
- Y(\nu)g(X, JW) + 2W(\nu)g(X, JY) \\
+ JX(\nu)g(Y, W) - JY(\nu)g(X, W) \}. \tag{2.17} \]
Moreover, via (2.17), (2.18) and (2.16), with a direct computation, one has:

\[
S(W, X, Y) = \frac{n(n+1)}{n-1} \left\{ 4(2n-3)(Q(X, (∇_Y J)W) - Q(Y, (∇_X J)W)) \\
- Q(Y, (∇_X J)W) + 4n(Q(X, (∇_W J)Y) \\
- Q(Y, (∇_W J)X) - 4(n-3)Q(W, (∇_X J)Y - (∇_Y J)X) \\
+ \tau(R)(∇_W ω)(X, Y) + \frac{4}{3}(n+1)((2n-3)X(ν)g(JY, W) \\
- Y(ν)g(JX, W) - 2nW(ν)g(X, JY)) \\
- 4(n+1)\{JX(ν)g(Y, W) - JY(ν)g(X, W)\}\right\}.
\]

Therefore, the vanishing of \(S\) implies the statement.

**Proposition 2.2.** In the hypothesis of the Lemma 2.1, if \(n \geq 4\), one has:

\[
Q(X, (∇_Y J)V) - Q((∇_Y J)V, X) = \frac{2}{3}((2n - 1)(Y(ν)g(JV, X)
+ JY(ν)g(V, X)) + (n - 2)(V(ν)g(JY, X)
+ JV(ν)g(Y, X))\}.
\]

**Proof.** We consider the (0,3)-tensor field \(T\) such that:

\[
T(V, X, Y) = Q(V, (∇_X J)Y - (∇_Y J)X).
\]

Since \(T\) satisfies:

\[
T(V, X, Y) = -T(V, Y, X) = -T(V, JX, JY),
\]
can be regarded as a section of the vector bundle \( \mathcal{W}(M) \) whose fibre, at any point \( x \) of \( M \), is the linear space \( \mathcal{W}_x \) considered in [8].

According to the splitting \( \mathcal{W}(M) = \bigoplus_{1 \leq i \leq 4} \mathcal{W}_i(M) \) defined in [8], we define by \( q_1(T) \) the \( \mathcal{W}_1 \)-projection of \( T \); it is the skew-symmetric tensor field such that:

\[
6q_1(T)(V, X, Y) = \sigma_{(V, X, Y)} (T(V, X, Y) - T(JV, JX, Y)).
\]

Since \( n \geq 4 \), applying (2.16) and then (2.14), one obtains:

\[
3q_1(T)(V, X, Y) = \sigma_{(V, X, Y)} (Q(V, (\nabla_X J)Y - (\nabla_Y J)X)
+ \frac{2}{3}(n + 1)V(\nu)g(X, JY))
= \frac{1}{n} \left\{ 3Q(V, (\nabla_X J)Y - (\nabla_Y J)X)
+ 3(n - 1)(Q(X, (\nabla_Y J)V)
- Q(Y, (\nabla_X J)V) + \frac{1}{4} \tau(R)(\nabla \omega)(X, Y)
+ 2n(n - 2)(JX(\nu)g(V, Y) - JY(\nu)g(V, X))
+ 2(n^2 - n + 1)(X(\nu)g(JV, Y) - Y(\nu)g(JV, X))
- 2(n - 1)(n - 2)V(\nu)g(X, JY) \right\}.
\]

Then, the condition: \( q_1(T)(V, X, Y) + q_1(T)(JY, JX, Y) = 0 \) combined with (1.9), (2.3), (2.16) proves the statement.

3 – The proof of the Theorem 1

To the Riemannian curvature of a manifold satisfying the hypothesis of the Theorem 1, we apply the second Bianchi identity in the form:

\[
\begin{align*}
\sigma_{(V, X, Y)} & \left\{ (\nabla_V R)(X, Y, Z, W) + (\nabla_V R)(X, Y, JZ, JW) \right\} \\
+ \sigma_{(V, JX, JY)} & \left\{ (\nabla_V R)(JX, JY, Z, W) \\
+ (\nabla_V R)(JX, JY, JZ, JU) \right\} = 0.
\end{align*}
\]

The complete expression of the first member in (3.1), evaluated by means of (2.9), is a tensor field which contains four blocks of terms, respectively depending on \( g \otimes (\nabla Q + Q(., \nabla J)) \), \( \nabla \omega \otimes Q \), \( d\nu \otimes (\pi_1 - \frac{2n-1}{3}\pi_2) \), \( g \otimes \nabla \omega \).
Since \((g, J)\) is an almost Kähler structure, the whole term in \(g \otimes \nabla \omega\) vanishes, while only the skew-symmetric component of \(Q\), i.e. \(\rho^*(R - L_3R)\), is involved in the block depending on \(\nabla \omega \otimes Q\).

After a quite long computation, applying the Lemmas 2.1 and 2.2 and then the Proposition 2.2, the whole expression in \(g \otimes (\nabla Q + Q(\cdot, \nabla J))\) turns out to depend only on \(d\nu \otimes g \otimes g\). Thus, the condition (3.1) is equivalent to

\[
\frac{3}{4(n + 1)} \{\rho^*(R - L_3R)(X, Z)(\nabla_W \omega)(JY, V) \\
- \rho^*(R - L_3R)(Y, Z)(\nabla_W \omega)(JX, V) \\
- \rho^*(R - L_3R)(JX, Z)(\nabla_W \omega)(Y, V) \\
+ \rho^*(R - L_3R)(JY, Z)(\nabla_W \omega)(X, V) - \rho^*(R - L_3R)(X, W) \times \\
\times (\nabla_{Z \omega})(JY, V) + \rho^*(R - L_3R)(Y, W)(\nabla_{Z \omega})(JX, V) \\
+ \rho^*(R - L_3R)(JX, W)(\nabla_{Z \omega})(Y, V) - \rho^*(R - L_3R)(JY, W) \times \\
\times (\nabla_{Z \omega})(X, V) \} - X(\nu)\{\pi_1(V, Y, Z, W) + \pi_1(V, Y, JZ, JW) \\
+ 2g(Y, JV)g(Z, JW)\} + Y(\nu)\{\pi_1(V, X, Z, W) \\
+ \pi_1(V, X, JZ, JW) + 2g(X, JV)g(Z, JW)\} \\
- JX(\nu)\{\pi_1(V, JY, Z, W) \\
- \pi_1(JV, Y, Z, W) + 2g(Y, V)g(Z, JW)\} \\
+ JY(\nu)\{\pi_1(V, JX, Z, W) \\
- \pi_1(JV, X, Z, W) + 2g(X, V)g(Z, JW)\} \\
+ 2V(\nu)\{\pi_1(X, Y, Z, W) \\
+ \pi_1(X, Y, JZ, JW) - 2g(X, JY)g(Z, JW)\} \\
+ 2W(\nu)\{\pi_1(X, Y, Z, V) \\
+ \pi_1(X, Y, JZ, JV) - 2g(X, JY)g(Z, JV)\} \\
- 2Z(\nu)\{\pi_1(X, Y, W, V) \\
+ \pi_1(X, Y, JW, JV) - 2g(X, JY)g(W, JV)\} \\
- 2JW(\nu)\{\pi_1(X, Y, Z, JV) \\
- \pi_1(X, Y, JZ, V) + 2g(X, JY)g(Z, V)\} \\
+ 2JZ(\nu)\{\pi_1(X, Y, W, JV) \\
- \pi_1(X, Y, JW, V) + 2g(X, JY)g(W, V)\}\} = 0.
\]
First of all, this formula implies that \( \nu \) is a constant function.

Indeed, given a vector field \( V \), let \( Y \) be a vector field such that \( g(V,Y) = 0 \) in an open set. Putting in (3.2) \( X = Z = JV, W = Y \), one has:

\[
\frac{8}{3}(n+1)V(\nu)g(Y,Y)g(V,V) = \rho^*(R - L_3R)(JV,Y)(\nabla_V\omega)(V,Y) - \rho^*(R - L_3R)(V,Y)(\nabla_V\omega)(V,JY).
\]

Therefore, \( V(\nu) = 0 \), if \( (\nabla_VJ)V = 0 \).

Assuming that \( (\nabla_VJ)V \) does not vanish at some point, we consider an open set where \( (\nabla_VJ)V \) never vanishes and apply (3.3) to a local vector field \( Y \) orthogonal to \( V, JY, (\nabla_VJ)V, J(\nabla_VJ)V \). Then, we obtain again: \( V(\nu) = 0 \).

Therefore, one has: \( d\nu = 0 \); hence, since \( M \) is connected, \( \nu \) is a constant function.

Now, the condition (3.2) turns out to be equivalent to the vanishing of the tensor field \( H \) defined by:

\[
H(V, X, Y, Z, W) = \rho^*(R - L_3R)(X, Z)(\nabla_W\omega)(JY,V) - \rho^*(R - L_3R)(Y, Z)(\nabla_W\omega)(JX,V) - \rho^*(R - L_3R)(JX, Z)(\nabla_W\omega)(Y,V) + \rho^*(R - L_3R)(JY, Z)(\nabla_W\omega)(X,V) - \rho^*(R - L_3R)(X, W)(\nabla_Z\omega)(JY,V) + \rho^*(R - L_3R)(Y, W)(\nabla_Z\omega)(JX,V) + \rho^*(R - L_3R)(JX, W)(\nabla_Z\omega)(Y,V) - \rho^*(R - L_3R)(JY, W)(\nabla_Z\omega)(X,V).
\]

This implies also the vanishing of the tensor field \( H' \) defined by:

\[
H'(V, X, Y, Z, W) = 2(H(V, X, Y, Z, W) + H(V, Z, W, X, Y)) - H(V, Y, Z, X, W) - H(V, X, W, Y, Z) - H(V, Z, Y, W) - H(V, Y, W, Z, X).
\]

Then, combining the conditions:

\[
H'(V, X, Y, Z, W) = 0 \quad H'(V, JX, JY, Z, W) = 0 \\
H'(JV, X, Y, Z, JW) = 0
\]

and using (1.9), one has:
\[\rho^*(R - L_3 R)(X, Z)(\nabla_V \omega)(JY, W)\]
\[-\rho^*(R - L_3 R)(Y, Z)(\nabla_V \omega)(JX, W)\]
\[-\rho^*(R - L_3 R)(JX, Z)(\nabla_V \omega)(Y, W)\]
\[+ \rho^*(R - L_3 R)(JY, Z)(\nabla_V \omega)(X, W) = 0.\]

This implies the Kähler condition, i.e. \(\nabla J = 0\). Indeed, if \(\nabla J \neq 0\), we consider vector fields \(Y, V\) such that \((\nabla_V J)Y\) never vanishes in an open set.

Putting in (3.4) \(W = Y, X = (\nabla_V J)Y\), one obtains, for any \(Z\), \(\rho^*(R - L_3 R)(Y, Z) = 0\) and also \(\rho^*(R - L_3 R)(Y, Z) = -\rho^*(R - L_3 R)(JY, JZ) = 0\). Thus, (3.4) reduces to:

\[\rho^*(R - L_3 R)(X, Z)(\nabla_V \omega)(JY, W) - \rho^*(R - L_3 R)(JX, Z)(\nabla_V \omega)(Y, W) = 0,\]

or, equivalently, to:

\[\rho^*(R - L_3 R)(X, Z)J((\nabla_V J)Y) + \rho^*(R - L_3 R)(JX, Z)((\nabla_V J)Y) = 0.\]

Therefore, \(\rho^*(R - L_3 R)\) vanishes. According to [16], this means the vanishing of the projection \(p_9(R)\). Since also \(p_8(R) = p_{10}(R) = 0\) (see also the Proposition 1.1), \((M, g, J)\) turns out to be a \(\mathcal{R}_3\)-almost Kähler manifold with p.c.a.s.c. Since \(\dim M \geq 8\), a direct application of the classification theorem in [10] implies that \((M, g, J)\) is a Kähler manifold with constant holomorphic sectional curvature. This contradicts the condition \(\nabla J \neq 0\).

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INDIRIZZO DEGLI AUTORI:
M. Falcitelly – A. Farinola – Dipartimento di Matematica – Università – Via E. Orabona, 4 - 70125 Bari, Italia

O. T. Kassabov – Higher Transport School (BBTY) – “T. Kableshkov” – section of Math. – Slatina 1574 Sofia, Bulgaria