Finite Factorization equations and Sum Rules for BPS correlators in $N = 4$ SYM theory

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A class of exact non-renormalized extremal correlators of half-BPS operators in $N = 4$ SYM, with $U(N)$ gauge group, is shown to satisfy finite factorization equations reminiscent of topological gauge theories. The finite factorization equations can be generalized, beyond the extremal case, to a class of correlators involving observables with a simple pattern of $SO(6)$ charges. The simple group theoretic form of the correlators allows equalities between ratios of correlators in $N = 4$ SYM and Wilson loops in Chern-Simons theories at $k = \infty$, correlators of appropriate observables in topological $G/G$ models and Wilson loops in two-dimensional Yang-Mills theories. The correlators also obey sum rules which can be generalized to off-extremal correlators. The simplest sum rules can be viewed as large $k$ limits of the Verlinde formula using the Chern-Simons correspondence. For special classes of correlators, the saturation of the factorization equations by a small subset of the operators in the large $N$ theory is related to the emergence of semiclassical objects like KK modes and giant gravitons in the dual $ADS \times S$ background. We comment on an intriguing symmetry between KK modes and giant gravitons.

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1. Introduction

In [1] we gave a systematic study of extremal correlators of the most general half BPS operators of the $U(N)$ theory. These include single trace as well as multiple trace operators. Young Diagrams associated with $U(N)$ representations are useful in characterizing the operators. When we use a Young Diagram basis, the correlators take a very simple form. The two point functions are diagonal in this basis and the three point functions are proportional to fusion coefficients of the unitary groups, i.e Littlewood Richardson coefficients. Analogous group theoretic quantities enter the higher point correlators. Extremal correlators are also distinguished in that they obey non-renormalization theorems [2,3]. Half-BPS operators and their non-renormalization theorems have been the subject of many papers [4,5,6,7,8,9,10,2,11,12,3]. A more complete list of references can be found [13][14].

In this paper we will explore several consequences of these results. In the first place the extremal correlators obey finite factorization and fusion equations which are of a form similar to the ones that appear in topological gauge theories, for example three-dimensional Chern-Simons theory, two-dimensional $G/G$ models and two dimensional Yang-Mills. The factorization equations imply bounds on the large $N$ growth of the correlators which guarantee that a probabilistic interpretation of certain correlators in terms of overlaps of incoming and outgoing states is sensible. The group theoretic nature of the correlators also implies that certain sum rules can be written down relating weighted sums of higher point functions to lower point functions.

The similarity of the factorization equations to those of topological gauge theories (TGT’s) leads us to investigate more detailed connections between observables in TGT’s and the SYM4 correlators. We find that there are simple identities between ratios of extremal correlators and appropriate observables. In the SYM4-Chern-Simons correspondence, there are natural relations involving both observables on $S^3$ and observables on $S^2 \times S^1$. After making these identifications, the factorization equations for correlators indeed map to factorization equations in TGT’s. The sum rules we found above turn out to reduce, in simple cases, to relations between Chern-Simons on $S^2 \times S^1$ and on $S^3$. They are also related to the large $k$ limit of the Verlinde formula [15]. These results are section 7.

Many of these basic remarks extend beyond extremal correlators. The basic technical tool here is the fact that perturbative calculations based on free fields involving arbitrary
composites are naturally described in terms of projection operators in tensor spaces, which can be represented in an economical fashion in diagrammatic form. We will call these **projector diagrams**. Examples of projector diagrams are Figures 7, 10, 12, 13. These projectors in tensor spaces can be related to projectors acting in products of irreducible $U(N)$ representations. The latter diagrams in turn can be related to graphs where edges are labelled by irreps and vertices correspond to Clebsch-Gordan coefficients. We will call these **projector graphs**. Examples of such projector graphs are in Figures 11, 18 and 19. Such diagrams, or slight variations thereof, are familiar from various related contexts: two dimensional Rational Conformal field Theory, three-dimensional Chern-Simons theory, integrable lattice models, and two dimensional Yang-Mills theory. Thus we are able to map correlators in SYM4 to observables in Chern-Simons theory, in a very general manner. This map is particularly simple in the cases where the correlators in SYM4 are chosen to have simple spacetime dependence.

The factorization equations can be generalized to an equation we will call **staggered factorization**. The terminology is based on the structure of the projector diagrams or projector graphs which allow such factorization. We will describe the SYM4 correlators which admit such a factorization. We observe that staggered factorization is related to simple operations on Wilson loop expectation values in Chern-Simons or in two-dimensional Yang-Mills theory. In the context of Chern-Simons theory, it is related to connected sums of links (Wilson loops). In the context of two dimensional Yang-Mills, it is related to Wilson loops with tangential intersections. These points are developed in section 8.

The extremal correlators, which were first identified as an interesting set of correlators in the context of studies of non-renormalization theorems, were shown in [1] to have simple dependence on the $U(N)$ representation content of the operators involved, that is, the dependence is entirely in terms of dimensions and fusion coefficients rather than $6J$ symbols or such. This simplicity, given the results of this paper, can be traced to the factorization equations. Since these factorization equations have been generalized to staggered factorization, it is tempting to conjecture that the correlators which can be simplified using staggered factorization, into dimensions and fusion coefficients, are non-renormalized. We will give some very heuristic arguments in favour of such a conjecture in section 8 and leave a detailed exploration of the question to the future.

In section 9 we will argue that the basic factorization equations, derived above using explicit knowledge of the complete space of half-BPS representations and their extremal
correlators, actually follow from general considerations based on the OPE and superalgebra. As such they should work, not just for $U(N)$ but also for the $SU(N)$ theory. We do not have a direct proof in the case of $SU(N)$, but we describe, in section 10, the complete set of half-BPS representations in that case and write down some formulae for the relevant correlators, which should be constrained by the basic factorization. We predict that some clever manipulation of the relevant quantities, which involve sums of products of Littlewood-Richardson coefficients will give a direct proof of the factorization equations here.

In section 11 we make some comments on giant gravitons [16]. These were the original motivation for the current study. The first remark is that the group theoretic formulae for three point functions imply a symmetry between Kaluza-Klein (KK) correlators and giant graviton correlators – it would be fascinating to understand this symmetry from the spacetime side. The second, not unrelated, remark concerns the special properties that the factorization equations have when the extremal correlators involve giant graviton states.

In appendix 1 we derive, starting from some group integrals familiar from lattice gauge theory, some basic properties of projection operators in tensor space, in particular the relation between unitary group projectors and symmetric group projectors. In appendix 2 we derive the relation between fusion multiplicities of unitary groups and branching multiplicities of Symmetric groups from the same point of view. In appendix 3 we give a general argument underlying staggered factorization, which allows a trace in tensor space to be related to a product of traces in tensor space. In terms of projector diagrams, it relates one projector diagram to a product of projector diagrams. Appendix 4 gives some steps in the diagrammatic derivation of correlators of descendants, which are discussed further in section 5. Appendix 5 reviews some $SO(6)$ group theory which is useful in section 8 in describing the class of correlators which obey staggered factorization. In appendix 6, we translate the results of [1] where general extremal correlators were given in the Schur Polynomial basis, to get some extremal correlators of traces. These have appeared recently in the context of pp-waves [17,18,19,20]. The perspective of this paper suggests the existence of interesting connections between pp-wave backgrounds and Chern-Simons theory.
2. Factorization and Unitarity for extremal correlators

In [1] a 1 − 1 correspondence was established between half-BPS representations in $U(N)$ maximally supersymmetric Yang-Mills theory and symmetric polynomials in the eigenvalues of a complex matrix $\Phi$ or Schur Polynomials. $\Phi$ is one combination of the six Hermition scalars of $\mathcal{N} = 4$ SYM given by $\Phi = \phi_1 + i\phi_4$. We recall from [1] that $k \rightarrow l$ multi-point extremal correlation functions are given by

$$
\langle \chi_{R_1}(\Phi(x_1))\chi_{R_2}(\Phi(x_2)) \cdots \chi_{R_k}(\Phi(x_k)) \chi_{S_1}(\Phi^\dagger(0)) \cdots \chi_{S_{l}}(\Phi^\dagger(0)) \rangle
$$

$$
= \sum_S g(R_1, R_2, \cdots R_k; S) \frac{n(S)!D\text{im}\text{S}}{d_S} g(S_1, S_2, \cdots S_{l}; S) \frac{1}{(x_1)^{2n(R_1)} \cdots (x_k)^{2n(R_k)}} \tag{2.1}
$$

where

$$
g(R_1, R_2, \cdots R_k; S) = \int dU \chi_{R_1}(U)\chi_{R_2}(U) \cdots \chi_{R_k}(U)\chi_{S}(U^\dagger) \tag{2.2}
$$

where the characters and measure for $U(N)$ are normalized so that $\int dU \chi_R(U)\chi_S(U^\dagger) = \delta_{RS}$. The sum runs over all irreps of $U(N)$ having $n(S) = n(R_1) + \cdots + n(R_k)$ boxes. $D\text{im}\text{S}$ is the dimension of the $U(N)$ irrep. associated with the Young Diagram $S$ and $d_S$ is the dimension of the symmetric group irrep. associated with the same Young diagram. The $g$ factors can be expressed in terms of the basic fusion coefficient $g(R_1, R_2; S)$ which can be calculated using the Littlewood-Richardson combinatoric rule for combining Young Diagrams [21,22,23].

The four-point extremal correlators satisfy the following factorization condition.

$$
\langle \chi_{R_1}(\Phi(x_1))\chi_{R_2}(\Phi(x_2)) \chi_{S_1}(\Phi^\dagger(0))\chi_{S_2}(\Phi^\dagger(0)) \rangle
$$

$$
= \sum_S \frac{\chi_{R_1}(\Phi(x_1))\chi_{R_2}(\Phi(x_2))\chi_{S_1}(\Phi^\dagger(0))\chi_{S_2}(\Phi^\dagger(0))}{\chi_{S}(\Phi(0))\chi_{S}(\Phi^\dagger(0))} \langle \chi_S(\Phi(y))\chi_{S_1}(\Phi^\dagger(0))\chi_{S_2}(\Phi^\dagger(0)) \rangle \tag{2.3}
$$

where $y \neq 0$ but is otherwise arbitrary.

There is an obvious generalization to higher point functions

$$
\langle \prod_i \chi_{R_i}(\Phi(x_i)) \prod_j \chi_{S_j}(\Phi^\dagger(0)) \rangle
$$

$$
= \sum_S \frac{\prod_i \chi_{R_i}(\Phi(x_i)) \chi_{S}(\Phi^\dagger(0))}{\chi_{S}(\Phi(0))\chi_{S}(\Phi^\dagger(0))} \langle \chi_S(\Phi(y))\prod_j \chi_{S_j}(\Phi^\dagger(0)) \rangle \tag{2.4}
$$

Taking equation (2.3) with $R_1 = S_1$ and $R_2 = S_2$ we see that it can be expressed as saying that the sum of normalized two-point functions is equal to 1.

$$
\sum_S \frac{\langle \chi_{R_1}(\Phi)\chi_{R_2}(\Phi)\chi_{S}(\Phi^\dagger) \rangle}{\sqrt{\langle \chi_{R_1}(\Phi)\chi_{R_2}(\Phi)\chi_{S}(\Phi^\dagger) \chi_{S}(\Phi^\dagger) \rangle}} = 1 \tag{2.5}
$$
This equation guarantees that the normalized two-point functions can be interpreted as amplitudes for transitions between states described by the wavefunctions $\chi_{R_1}(\Phi)\chi_{R_2}(\Phi)$ on the one hand and $\chi_S(\Phi)$ on the other. It shows that when the incoming state is a product of highest weight states of half-BPS representations of the superalgebra, the probability for the outgoing state to be the dual to a highest weight of a half-BPS representation is 1.

2.1. Fusion identities

The result (2.1) also implies that higher point functions can be expressed in terms of lower point functions. Consider, for example the four-point function

$$\langle \chi_{R_1}(\Phi(x_1))\chi_{R_2}(\Phi(x_2))\chi_{R_3}(\Phi(x_3))\chi_S(\Phi^+(0)) \rangle$$

$$= \sum_{S_1} g(R_1, R_2, S_1) g(S_1, R_3, S) \frac{n(S)!\text{Dim}(S)}{d_S} x_1^{-2n_1}x_2^{-2n_2}x_3^{-2n_3}$$

$$= \sum_{S_1} \langle \chi_{R_1}(\Phi(x_1))\chi_{R_2}(\Phi(x_2))\chi_{S_1}(\Phi^+(0)) \rangle \frac{1}{\langle \chi_{S_1}(\Phi(y))\chi_{S_1}(\Phi^+(0)) \rangle}$$

$$\langle \chi_{S_1}(\Phi(y))\chi_{R_3}(\Phi(x_3))\chi_S(\Phi^+(0)) \rangle$$

To obtain the second line we have used (2.2) for $g(R_1, R_2, R_3; S)$ in terms of group integrals and expanded the class function $\chi_{R_1}(U)\chi_{R_2}(U)$ into a sum of irreducible characters of $U(N)$. This yields the coefficients $g(R_1, R_2; S_1)$ which in turn can be re-expressed in terms of three-point functions. The simple spacetime dependences of these extremal correlators allow the functions of $x_i$ on the left and right to match.

The generalization to higher point functions is

$$\langle \prod_{i=1}^{k} \chi_{R_i}(\Phi(x_i))\chi_{S}(\Phi^+(0)) \rangle = \sum_{S_1 \cdots S_{k-2}} \langle \chi_{R_1}(\Phi(x_1))\chi_{R_2}(\Phi(x_2))\chi_{S_1}(\Phi^+(0)) \rangle$$

$$\prod_{i=1}^{k-2} \frac{1}{\langle \chi_{S_i}(\Phi(0))\chi_{S_i}(\Phi^+(y_i)) \rangle} \langle \chi_{S_1}(\Phi(y_1))\chi_{R_{i+2}}(\Phi^+(x_{i+2}))\chi_{S_{i+1}}(\Phi(0)) \rangle$$

(2.7)

Here $S_{k-1} \equiv S$ and $y_{k-1} = 0$.

In section 9, we outline a general derivation of these equations based on properties of the superalgebra symmetry and the OPE. From the OPE we expect such factorization equations to be true, but typically we would expect infinitely many operators to be involved in the intermediate sums. In two-dimensional CFTs finite sums are possible, for general correlators, because there is an infinite conformal algebra symmetry. In the case at hand, such equations would typically involve infinite sums, but the sums are finite when we work with extremal correlators of half-BPS operators.
3. Elements of diagrammatic derivations

The calculation of the finite $N$ extremal correlators in [1] as well as further calculations of finite $N$ non-extremal correlators can be simplified by using a diagrammatic method.

The matrix $\Phi$ is an operator which transforms states of an $N$ dimensional vector space $V$. The matrix elements of $\Phi$ are $\Phi_{i,j}$.

$$\langle e^i | \Phi | e_j \rangle = \Phi^i_j \quad (3.1)$$

We can naturally extend the action of $\Phi$ to the n-fold tensor product $V^\otimes n$, by considering the operator $\Phi \otimes \Phi \cdots \Phi$. The matrix elements are now $\Phi_{i_1,j_1} \Phi_{i_2,j_2} \cdots \Phi_{i_n,j_n}$.

$$\langle e^{i_1} \otimes e^{i_2} \otimes \cdots \otimes e^{i_n} | \Phi \otimes \Phi \otimes \cdots \otimes \Phi | e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n} \rangle = \Phi_{i_1,j_1} \Phi_{i_2,j_2} \cdots \Phi_{i_n,j_n} \quad (3.2)$$

\[\Phi\]

\[I(n)\]

\[\Phi\]

\[J(n)\]

\[\Phi\]

**Fig. 1:** Diagram for $\Phi$ operator in $V^\otimes n$

Sometimes it is convenient to use a multi-index $I(n)$ for an index set $(i_1, i_2, \cdots i_n)$. The equation (3.2) is presented diagrammatically in fig. 1.

The trace of the operator $\Phi$ acting on $V^\otimes n$ is denoted by the diagram in fig. 2. When we do free field contractions, we end up summing over different permutations which describe how we are contracting. It is useful to recall that the matrix element of the permutation $\gamma$ acting on $V^\otimes n$ is

$$(\gamma)_{j_1,j_2,\cdots,j_n}^{i_1,i_2,\cdots,i_n} = \delta_{j_1, i_{\gamma(1)}}^{i_1} \delta_{j_2, i_{\gamma(2)}}^{i_2} \cdots \delta_{j_n, i_{\gamma(n)}}^{i_n} \quad (3.3)$$
The basic correlator can be written as

\[
\langle \Phi_{j_1}^{i_1}(x_1) \cdots \Phi_{j_n}^{i_n}(x_1) \ (\Phi^\dagger)^{k_1}_{l_1}(x_2) \cdots (\Phi^\dagger)^{k_n}_{l_n}(x_2) \rangle \\
= (x_1 - x_2)^{-2n} \sum_\gamma \delta_i^{i_1} \delta_l^{l_1} \cdots \delta_i^{i_n} \delta_l^{l_n} \delta_{j_1}^{k_1} \delta_{j_2}^{k_2} \cdots \delta_{j_n}^{k_n}
\]

(3.4)

A slightly more compact way of writing this is

\[
\langle \Phi_{j(n)}^{I(n)}(x_1) \ (\Phi^\dagger)^{K(n)}_{L(n)}(x_2) \rangle = (x_1 - x_2)^{-2n} \sum_\gamma (\gamma)^{I(n)}_{L(n)} (\gamma^{-1})^{K(n)}_{J(n)}
\]

(3.5)

The notation \( I(n) \) indicates that \( I \) is a multi-index involving \( n \) indices \( i_1, \cdots i_n \). When the number of indices involved is clear from the context, we will not need to make it explicit,
and we can write:

\[
\langle \Phi^I_J(x_1) (\Phi^\dagger)^K_L(x_2) \rangle = (x_1 - x_2)^{-2n} \sum_\gamma (\gamma^I)_L (\gamma^{-1})^K_J.
\] (3.6)

This can be expressed in diagrammatic form as in fig. 3.

In the diagrams, we will not make the \(x\) dependences explicit. We can also keep the positions of the indices unchanged with respect to the first term, at the cost of introducing a twist, that is, a permutation acting on \(V^\otimes n \otimes V^\otimes n\) which switches the first \(n\) copies with the last \(n\) copies, compare figures 3 and 4.

\[\text{Fig. 4: Alternative diagram for two-point function}\]

The advantage of the last step is that we can express the key correlator (3.5) in a completely index-free diagrammatic way as in figure 5, with the understanding that when we put back the indices to recover the more familiar formula, there is no reshuffling of multi-indices between left and right hand side of the equations.

\[\text{Fig. 5: Index-free diagram for two-point function}\]
Now we express some of the correlators evaluated in [1] using the diagrammatic notation. Gauge invariant correlators are obtained by contracting the free indices with the $U(N)$ invariant metric $\delta^i_j$. In [1] these contractions were taken in the Schur polynomial basis, that is one contracts the $\Phi(\Phi^\dagger)$ indices with operators

$$\frac{1}{d_R} (P_R)^{J(n)}_{I(n)} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \left( \sigma \right)^{J(n)}_{I(n)} = \frac{\text{Dim} R}{d_R} \int dU \chi_R(U) \left( U^\dagger \right)^{J(n)}_{I(n)}$$

(3.7)

where in the first line we have given the symmetric group form of the projector and in the second line the unitary group integral form of the projector. The equality of these two projectors when acting on tensor space is proved in appendix 1. Indeed it is straightforward to show that $P_R$ is a projection operator satisfying

$$P_R P_S = \delta_{R,S} P_S$$

(3.8)

and the trace condition

$$tr(P_R) = d_R \text{ Dim} R.$$  

(3.9)

To calculate the two-point function $d_R d_S \langle \chi_R(\Phi) \chi_S(\Phi^\dagger) \rangle$ one has to evaluate $\langle tr(P_R \Phi) tr(P_S \Phi^\dagger) \rangle$. This is illustrated in the left of figure 6, with $R, S$ drawn for $P_R, P_S$. The correlator of $\Phi$’s is evaluated using figure 4, to give the middle diagram of figure 6, a sum over $\gamma$ being understood. An obvious diagrammatic manipulation or equivalently an identity relating traces in $V^\otimes n \otimes V^\otimes n$ to traces in $V^\otimes n$ leads to the final diagram in figure 6.

![Diagram of the $U(N)$ two point function.](image)

Fig. 6: Diagram of the $U(N)$ two point function.
Explicitly one has
\[
\langle \chi_R(\Phi) \chi_S(\Phi^\dagger) \rangle = \frac{1}{d_R d_S} \sum_{\gamma \in S_n} \text{tr}(P_R(\gamma^{-1} P_S \gamma)) \tag{3.10}
\]
\[
= \frac{n! \text{Dim}_R}{d_R} \delta_{R,S}
\]
where we have used the properties \((3.8)\) and \((3.9)\) and the fact that \(P_S\) commutes with \(\gamma\) to evaluate \((3.10)\).

More generally for the multi-point correlator we have
\[
\langle \chi_{R_1}(\Phi) \cdots \chi_{R_k}(\Phi) \chi_{S_1}(\Phi^\dagger) \cdots \chi_{S_l}(\Phi^\dagger) \rangle
\]
\[
= \frac{1}{\prod_i d_{R_i}} \frac{1}{\prod_i d_{S_i}} \text{tr}(P_{R_1} \Phi) \cdots \text{tr}(P_{R_k} \Phi) \text{ tr}(P_{S_1} \Phi^\dagger) \cdots \text{tr}(P_{S_l} \Phi^\dagger) \tag{3.11}
\]

As in the case of the two point function, we can express this in a diagram and then use figure 4 to simplify. The resulting diagram can be manipulated into figure 7. To avoid clutter we denote the projection operators \(P_R\) in the projector diagrams by just \(R\).

Written out this is
\[
\langle \chi_{R_1}(\Phi) \cdots \chi_{R_k}(\Phi) \chi_{S_1}(\Phi^\dagger) \cdots \chi_{S_l}(\Phi^\dagger) \rangle = \frac{1}{d_{R_1} \cdots d_{R_k} d_{S_1} \cdots d_{S_l}} \times \sum_{\gamma \in S_n} \text{tr}((P_{R_1} \otimes \cdots \otimes P_{R_k}) \gamma^{-1}(P_{S_1} \otimes \cdots \otimes P_{S_l}) \gamma) \tag{3.12}
\]

The integer \(n\) is given by \(n = n(R_1) + \cdots + n(R_k) = n(S_1) + \cdots + n(S_l)\). To evaluate the trace we proceed as follows. The projection operator \((P_{S_1} \otimes \cdots \otimes P_{S_l})\) projects onto a subspace of \(V^\otimes n\) corresponding to a reducible representation of \(U(N)\). To decompose this reducible representation into irreducible representations we recall the theorem, see eg. [22] [23], that \(V^\otimes n\) can be decomposed as
\[
V^\otimes n \cong \bigoplus_{S} S \otimes s \tag{3.13}
\]
where the sum is over all Young diagrams corresponding to irreducible representations \(S\) of \(U(N)\) with dimension \(\text{Dim}S\) and irreps \(s\) of \(S_n\) with dimension \(d_S\). When \(n\) is smaller than \(N\) \((3.13)\) includes all irreps \(s\) of \(S_n\), but when \(n \geq N\), it only includes those irreps of \(S_n\) which correspond to Young Diagrams with no column of length larger than \(N\). As a representation of \(U(N) \times S_n\), \(V^\otimes n\) decomposes into irreps \(S \otimes s\) with unit multiplicity. Equivalently, \(V^\otimes n\) consists of \(d_S\) copies of each irrep \(S\) of \(U(N)\). This multiplicity of irreps
of $U(N)$ in tensor space has the consequence that Schur Polynomials are related to traces of projectors by a factor $\frac{1}{d_S}$.

$$\chi_S(\Phi) = \frac{1}{d_S} tr(P_S \Phi)$$  \hspace{1cm} (3.14)

We can compute the trace by inserting a complete set of projectors $P_S$ acting in $V^\otimes n$.

$$tr((P_{R_1} \otimes \cdots \otimes P_{R_k}) \gamma^{-1}(P_{S_1} \otimes \cdots \otimes P_{S_l}) \gamma)$$

$$= \sum_S tr((P_{R_1} \otimes \cdots \otimes P_{R_k}) P_S P_S(\gamma^{-1}(P_{S_1} \otimes \cdots \otimes P_{S_l}) \gamma))$$ \hspace{1cm} (3.15)

The $P_S$ can be expressed in terms of a unitary group integral or a symmetric group sum as in (3.7). Now $\sum \gamma^{-1}(P_{S_1} \otimes \cdots \otimes P_{S_l}) \gamma$ commutes with the action of $U(N) \otimes S_n$ in $V^\otimes n$. If we write the projectors $P_{S_i}$ in terms of symmetric groups, it is clear that this operator commutes with $U(N)$. The averaging by $\gamma$ guarantees that it also commutes with $S_n$. By
Schur’s Lemma, and taking advantage of the fact that $P_S$ projects onto a single irreducible representation of $U(N) \times S_M$ we can factor the trace to get

$$\sum_{\gamma} \text{tr} \left( (\tau^{-1}(P_{R_1} \otimes \cdots \otimes P_{R_k})\tau)(\gamma^{-1}(P_{S_1} \otimes \cdots \otimes P_{S_l})\gamma) \right)$$

$$= \sum_{\gamma} \text{tr}( (P_{R_1} \otimes \cdots \otimes P_{R_k}) P_S) \frac{1}{d_S \text{Dim} S} \text{tr}(P_S \gamma^{-1}(P_{S_1} \otimes \cdots \otimes P_{S_l})\gamma)$$

(3.16)

The manipulation has a simple diagrammatic meaning. We have factored the diagram in figure 7 to a pair of diagrams as in figure 8. The factors can now be evaluated.

![Diagram](image)

**Fig. 8**: Diagram illustrating the factored form of fig. 7.

Equivalently we state a relation between tensor products of projection operators in $V^\otimes n_1 \otimes \cdots V^\otimes n_l$ and projectors in $V^\otimes (n_1+n_2+\cdots+n_l)$. Schur’s lemma implies the decomposition

$$\sum_{\gamma} \gamma^{-1}(P_{S_1} \otimes \cdots \otimes P_{S_l})\gamma = \sum_{S} \alpha(S_1, \cdots, S_l; S) P_S$$

(3.17)

To fix the coefficient $\alpha(S_1, \cdots, S_l; S)$ we multiply the expression by a projection operator $P_{S'}$ and trace. The result is

$$\sum_{\gamma} \gamma^{-1}(P_{S_1} \otimes \cdots \otimes P_{S_l})\gamma = d_{S_1} \cdots d_{S_l} \sum_{S} \frac{n_S!}{d_S} g(S_1, \cdots, S_l; S) P_S$$

(3.18)

where the trace has been evaluated to be

$$\sum_{\gamma} \text{tr}(\gamma^{-1}(P_{S_1} \otimes \cdots \otimes P_{S_l})\gamma P_S) = n! d_{S_1} \cdots d_{S_l} g(S_1, \cdots S_l; S) \text{Dim} S.$$  

(3.19)
The trace can be evaluated either directly using the $U(N)$ form of the projection operators given in (3.7) or just by using the fact that we can convert the trace in tensor spaces with projectors $P_{S_1} \otimes \cdots P_{S_l}$ into a trace in the tensor product of irreducible $U(N)$ representations with a factor $d_{S_1} \cdots d_{S_l}$. Further the tensor product $S_1 \otimes \cdots S_l$ contains the representation $S$ with a multiplicity $g(S_1, S_2 \cdots S_l; S)$. Taking the trace gives a factor of $\text{Dim} S$.

Collecting all the factors we get

$$\langle \chi_{R_1}(\Phi) \cdots \chi_{R_k}(\Phi) \chi_{S_1}(\Phi) \cdots \chi_{S_l}(\Phi) \rangle$$

$$= \sum_S g(R_1, \cdots, R_k; S) \frac{n(S)! \text{Dim} S}{d_S} g(S_1, \cdots, S_l; S).$$  \hspace{1cm} (3.20)

which was derived in [1] by using characters rather than projectors.

4. Sum rules for three-point functions

The basic idea for sum rules is that sums of projectors onto Young diagrams in tensor space gives 1, as in (3.13). In terms of diagrams we have figure 9.

![Fig. 9: Basic identity leading to sum rules](image)

Consider the three-point function

$$\langle \chi_{R_1}(\Phi(x)) \chi_{R_2}(\Phi(x)) \chi_{S}(\Phi(0)) \rangle$$

$$= \frac{1}{x^{2n(S)}} g(R_1, R_2; S) \frac{n(S)! \text{Dim} S}{d_S}$$

$$= \frac{1}{x^{2n(S)}} \frac{n(S)!}{d_{R_1} d_{R_2} d_S} \text{tr}((P_{R_1} \otimes P_{R_2}) P_S)$$  \hspace{1cm} (4.1)
The integer $n(S)$, the number of boxes in the Young Diagram of $S$, is related to the numbers of boxes $R_1$ and $R_2$ by $n(S) = n(R_1) + n(R_2)$. When we sum over $R_1$ we get the identity operator in $V^\otimes n(R_1)$. Summing over $R_2$ we get the identity in $V^\otimes n(R_2)$. This leads to $tr_{n(S)}(P_S)$, which is just the dimension of the irrep of $U(N) \times S_n$ associated with the Young Diagram of $S$, i.e $d_SDim(S)$.

\[
\sum_{R_1R_2} d_{R_1}d_{R_2} \langle \chi_{R_1}(\Phi(x))\chi_{R_2}(\Phi(x))\chi_S(\Phi^\dagger(0)) \rangle = \frac{1}{x^{2n(S)}} Dim(S)n(S)! = d_S\langle \chi_S(\Phi(x))\chi_S(\Phi^\dagger(0)) \rangle
\]  

(4.2)

Expressed in terms of $\hat{\chi}_R = d_R\chi_R$ we have

\[
\sum_{R_1R_2} \langle \hat{\chi}_{R_1}(\Phi(x))\hat{\chi}_{R_2}(\Phi(x))\hat{\chi}_S(\Phi^\dagger(0)) \rangle = \langle \hat{\chi}_S(\Phi(x))\hat{\chi}_S(\Phi^\dagger(0)) \rangle
\]  

(4.3)

Another sum rule is obtained from just summing over the single representation $S$.

\[
\sum_S d_S\langle \chi_{R_1}(\Phi(x))\chi_{R_2}(\Phi(x))\chi_S(\Phi^\dagger(0)) \rangle = d_{R_1}d_{R_2} \frac{n(S)!}{n(R_1)!n(R_2)!} \langle \chi_{R_1}(\Phi(x))\chi_{R_1}(\Phi^\dagger(0)) \rangle \langle \chi_{R_2}(\Phi(x))\chi_{R_2}(\Phi^\dagger(0)) \rangle
\]  

(4.4)

In the above we presented the sum rules using properties of projectors. This is indeed the approach that works in general. In the simple extremal case considered above, we may also write down the above equations by recalling the fact that $g(R_1,R_2;S)$ is a fusion coefficient for $U(N)$ so that

\[
\sum_S DimS \ g(R_1, R_2; S) = DimR_1 \ DimR_2
\]  

(4.5)

which leads to (4.4). Since it also has an interpretation as a branching coefficient for $S_{n(R_1)} \times S_{n(R_2)} \to S_n(S)$, we immediately write down

\[
\sum_{R_1,R_2} d_{R_1}d_{R_2}g(R_1, R_2; S) = d_S
\]  

(4.6)

which leads to (4.2). The equality between the fusion coefficient and branching coefficient follows from Schur duality [22]. For completeness a proof is given in appendix 2.

The argument based on projectors generalizes to cases where the original correlator cannot be written in terms of fusion coefficients. These sum rules can also be used to simplify non-extremal three-point functions. The basic fact that leads to sum rules is that once we have expressed the correlator in terms of a trace in tensor space of a sequence of projectors, each of which acts in some subspace of the tensor space, it is natural to expect that the trace will simplify if some projectors are summed to leave the identity as in figure 9.
5. Formulae for non-extremal three-point functions

We will consider two types of departure from extremality. One involves descendants. Another involves generalizing position dependencies while leaving the operators to be highest weights and their duals. A new kind of sum rule will be possible here – where we sum over one projector and then simplify using the staggered factorization of appendix 3.

5.1. Correlators of descendants: \( \sum_i \Delta_i \neq \sum_j \Delta_j \)

As a specific example of a non-extremal correlator we consider the descendant \( \langle E_{21}^k \chi_{R_1}(\Phi_1)Q_{12}^k \chi_{R_2}(\Phi_1)\chi_{R_3}(\Phi_1^\dagger) \rangle \) of the extremal correlator \( \langle \chi_{R_1}(\Phi_1)\chi_{R_2}(\Phi_1)\chi_{R_3}(\Phi_1^\dagger) \rangle \) evaluated in section 2. The operators \( E_{21} \) and \( Q_{12} \) are elements of the Lie algebra of \( SO(6) \) and are defined in appendix 5. For the case at hand \( E_{21} \) converts \( \Phi_1 \) scalars to \( \Phi_2 \) scalars and \( Q_{12} \) converts \( \Phi_1 \) scalars to \( \Phi_2^\dagger \) scalars. Explicitly we have

\[
E_{21}^k \chi_{R_1}(\Phi_1) = \frac{(n_1 + k)!}{n_1!} \frac{\text{Dim} R_1}{d_{R_1}} \int dU_1 \chi_{R_1}(U_1)(tr(U_1^\dagger \Phi_2))^k (tr(U_1^\dagger \Phi_1))^n_1 \tag{5.1}
\]

in terms of unitary group integrals, with a similar expression for \( Q_{12}^k \chi_{R_2}(\Phi_1) \). By the diagrammatic rules we can show that the above non-extremal correlator is equal to a trace of a sequence of projectors acting on tensor space, up to a factor \( \frac{1}{d_{R_1}d_{R_2}d_{R_3}} (n_1 + n_2)! k! \frac{(n_1+k)! (n_2+k)!}{n_1! n_2!} \). The factor \( (n_1 + n_2)!k! \) arises from all possible ways of contracting the \( \Phi_1 \) and \( \Phi_2 \) fields respectively. Equivalently this factor arises from summing over the permutations \( \gamma_1 \) and \( \gamma_2 \) appearing in appendix 4 where the derivation of the correlator is explained diagrammatically. The factors \( (n_1 + k)!/n_1! \) and \( (n_2 + k)!/n_2! \) arise from the number of ways of converting \( k \) \( \Phi_1 \)’s into \( \Phi_2 \)’s and \( \Phi_2^\dagger \)’s respectively as in (5.1). The factor of \( \frac{1}{d_{R_1}d_{R_2}d_{R_3}} \) comes from the relation between projectors and Schur Polynomials given in (3.14).

The sequence of projectors is shown in the fig. 10 below and the derivation is explained in appendix 4. This allows us to write down a group integral for the correlator.

\[
\langle E_{21}^k \chi_{R_1}(\Phi_1(x_1))Q_{12}^k \chi_{R_2}(\Phi_1(x_2))\chi_{R_3}(\Phi_1^\dagger(0)) \rangle = \frac{(n_1 + n_2)! k! (n_1 + k)! (n_2 + k)!}{n_1! n_2!} \\
\times \frac{1}{x_1^{2n_1}} \frac{1}{x_2^{2n_2}} \frac{1}{(x_1 - x_2)^{2k}} \frac{\text{Dim}(R_1)}{d_{R_1}} \frac{\text{Dim}(R_2)}{d_{R_2}} \frac{\text{Dim}(R_3)}{d_{R_3}} \tag{5.2}
\]

\[
\int dU_1 dU_2 dV_1 \chi_{R_1}(U_1^\dagger) \chi_{R_2}(U_2^\dagger) \chi_{R_3}(U_3^\dagger) (tr(U_1 U_3))^{n_1} (tr(U_2 U_3))^{n_2} (tr(U_1 U_2))^k.
\]
Fig. 10: Projector diagram for a correlator of descendants

It is useful to convert the traces in tensor spaces to traces in tensor products of irreps. of $U(N)$

$$
\langle E_{21}^k \chi_{R_1}(\Phi_1(x_1))Q_{12}^k \chi_{R_2}(\Phi_1(x_2))\chi_{R_3}(\Phi_1^\dagger(0)) = \frac{(n_1 + n_2)!k!(n_1 + k)!(n_2 + k)!}{n_1!n_2!}
$$

$$
\times \frac{1}{x_1^{2n_1}} \frac{1}{x_2^{2n_2}} \frac{1}{(x_1 - x_2)^{2k}} \frac{\text{Dim}(R_1)}{d_{R_1}} \frac{\text{Dim}(R_2)}{d_{R_2}} \frac{\text{Dim}(R_3)}{d_{R_3}}
$$

$$
\sum_{S_1,S_2,S_3} \text{Dim}(S_1) \text{Dim}(S_2) \text{Dim}(S_3)
$$

$$
\int dU_2dV_1 \chi_{R_1}(U_1^\dagger) \chi_{R_2}(U_2^\dagger) \chi_{R_3}(U_3^\dagger) \chi_{S_1}(U_1U_3) \chi_{S_2}(U_2U_3) \chi_{S_3}(U_1U_2)
$$

$S_1$ runs over irreps of $U(N)$ with $n_1$ boxes, $S_2$ has $n_2$ boxes and $S_3$ has $k$ boxes.

Now we have a sequence of projectors acting on a tensor product of irreps of $U(N)$

$$
\text{Dim}(R_1)\text{Dim}(R_2)\text{Dim}(R_3)
$$

$$
\int dU_1dU_2dV_1 \chi_{R_1}(U_1^\dagger) \chi_{R_2}(U_2^\dagger) \chi_{R_3}(U_3^\dagger) \chi_{S_1}(U_1U_3) \chi_{S_2}(U_2U_3) \chi_{S_3}(U_1U_2) = \text{tr}_{S_1\otimes S_2 \otimes S_3} (P_{S_1,S_2}^{R_1} P_{S_2,S_3}^{R_2} P_{S_3}^{R_3})
$$

It is useful to draw a diagrammatic representation of this where a projector has been replaced by a sequence of two trivalent vertices. This corresponds, in formulae, to the expansion of projectors in terms of a product of Clebsch-Gordan coefficients.

$$
P_{S_1,S_2}^{R_1} = \sum_{\alpha,m_1,m_2} C_{\alpha,R_1,m_1;S_2,m_2}^{S_1,m_1:S_2,m_2} C_{\alpha,R_1,m_1;S_2,m_2}^{S_1,m_1:S_2,m_2} (5.5)
$$
Here $\alpha$ runs over the $g(S_1, S_2; R_1)$ occurrences of $R_1$ in the irrep. decomposition of $S_1 \otimes S_2$. $m_1, m_2, m$ run over states in the irreps $S_1, S_2$ and $R_1$ respectively. The diagrammatic presentation now makes no reference to tensor space and is a graph with legs labelled by irreps. of $U(N)$ and vertices representing Clebsch-Gordan coefficients.

Fig. 11: projector graph with $U(N)$ irrep labels

Converting the projectors in fig. 5 to products of Clebsch’s and representing the result diagrammatically we arrive at fig. 11. These projector graphs are useful in the mapping of the SYM4 correlators to quantities in Chern-Simons theory as we will discuss in section 7.
5.2. Correlators with general positions

Another class of non-extremal correlators of interest are generalizations of those considered in [1] by allowing arbitrary position dependence. Consider for example the four-point correlator

\[
\langle \chi_{R_1}(\Phi(x_1))\chi_{R_2}(\Phi(x_2))\chi_{R_3}(\Phi^\dagger(x_3))\chi_{R_4}(\Phi^\dagger(x_4)) \rangle
\]

where now the two \(\Phi^\dagger\) operators are evaluated at different locations. From the diagrammatic rules described in section 3 we see that this correlator will involve a sum of traces of products of projection operators corresponding to the diagram shown in fig. 12.

The sum can taken to be over the number \(j\) of fields from \(\chi_{R_1}\) contracted with fields from \(\chi_{R_3}\). For fixed \(j\) this fixes the number of fields of \(\chi_{R_1}\) contracted with fields from \(\chi_{R_4}\) and moreover the number of fields from \(\chi_{R_2}\) contracted with fields from \(\chi_{R_3}\) and \(\chi_{R_4}\) respectively. In fig. 12 we have included a label on each strand denoting the number of contractions occurring as described above. Also we denote the number of boxes in the Young diagram associated to representation \(R_i\) by \(n_i\).

Inserting the character expansions in terms of unitary integrals and evaluating the
contractions one finds

\[
\langle \chi_{R_1}(\Phi(x_1))\chi_{R_2}(\Phi(x_2))\chi_{R_3}(\Phi(x_3))\chi_{R_4}(\Phi(x_4)) \rangle = \left( \prod_{i=1}^{4} \frac{\text{Dim}_{R_i}}{d_{R_i}} \right) \frac{1}{(x_1 - x_4)^{2n_{1}}}
\times \frac{1}{(x_2 - x_4)^{2(n_4-n_1)}(x_2 - x_3)^{2n_3}} \sum_{j=\text{sup}(0,n_1-n_4)}^{\text{inf}(n_1,n_3)} \frac{n!n_2!n_3!n_4!}{j!(n_1-j)!(n_4-n_1+j)!(n_3-j)!}
\times \left( \frac{(x_1-x_4)(x_2-x_3)}{(x_1-x_3)(x_2-x_4)} \right)^{2j} \int \left( \prod_{i=1}^{4} dU_i \chi_{R_i}(U_i) \right) (tr(U_1^\dagger U_3^\dagger))^j (tr(U_1^\dagger U_4^\dagger))^{n_1-j}
\times (tr(U_2^\dagger U_4^\dagger))^{n_4-n_1+j} (tr(U_2^\dagger U_3^\dagger))^{n_3-j}.
\]  

(5.6)

This can be simplified somewhat by expressing it in terms of a trace of projection operators as

\[
\langle \chi_{R_1}(\Phi(x_1))\chi_{R_2}(\Phi(x_2))\chi_{R_3}(\Phi(x_3))\chi_{R_4}(\Phi(x_4)) \rangle = \left( \prod_{i=1}^{4} \frac{1}{d_{R_i}} \right) \frac{1}{(x_1 - x_4)^{2n_{1}}}
\times \frac{1}{(x_2 - x_4)^{2(n_4-n_1)}(x_2 - x_3)^{2n_3}} \sum_{j=\text{sup}(0,n_1-n_4)}^{\text{inf}(n_1,n_3)} \frac{n!n_2!n_3!n_4!}{j!(n_1-j)!(n_4-n_1+j)!(n_3-j)!}
\times \left( \frac{(x_1-x_4)(x_2-x_3)}{(x_1-x_3)(x_2-x_4)} \right)^{2j} tr((P_{R_1} \otimes P_{R_2})(P_{R_3} \otimes_j P_{R_4}))
\]  

(5.7)

where the \( j \) subscript on the tensor product denotes that the projection operator \( P_{R_3} \) acts on the first \( j \) indices and the last \( (n_3 + n_4 - j) \) indices while \( P_{R_4} \) acts on the remaining indices as in fig. [12]. The main difference here as compared to the earlier computation in (3.12) is that there is a nontrivial position dependence in the sum on \( j \). Were it not for this, the correlator could be re-expressed as in (3.12) and evaluated in terms of fusion coefficients and dimensions of representations. As it is however we cannot perform this simplification. This can be generalized in a fairly obvious way to higher point correlators.

6. Sum-rules for non-extremal correlators

We consider in this section sum rules for the various non-extremal correlators evaluated in the previous section. As in section 4, the sum rules follow easily when the correlators are expressed in terms of traces of products of projection operators, using \( \sum_{R} P_{R} = I \).
As an example consider the non-extremal three point correlator (5.3) rewritten in terms of projection operators using (5.4). If we multiply by $d_{R_3}$ and sum over all the $R_3$ representations, then the projector $P_{R_3}$ is replaced by the identity operator resulting in

$$
\sum_{R_3} d_{R_3} \langle E_{21}^{k} \chi_{R_1}(\Phi_1(x_1)) Q_{12}^{k} \chi_{R_2}(\Phi_1(x_2)) \chi_{R_3}(\Phi_1^\dagger(0)) \rangle = \frac{(n_1 + k)! (n_2 + k)! k! (n_1 + n_2)!}{n_1! n_2!}
$$

$$
x \times \frac{1}{d_{R_1} d_{R_2}} \frac{1}{x_1^{2n_1}} \frac{1}{x_2^{2n_2}} \frac{1}{(x_1 - x_2)^{2k}} tr ((P_{R_1} \otimes 1_{n_2})(1_{n_1} \otimes P_{R_2})).
$$

(6.1)

The trace appearing here can be evaluated in terms of just fusion coefficients and dimensions of various representations. Diagrammatically the trace is represented in fig. [13].

This sort of diagram is of the general class discussed in appendix 3 that can be split, or factorized, into a pair of diagrams. The argument for the factorization identity is detailed there, but the important point to note here is that the operators $F_{T_1}$ and $F_{T_2}$ introduced in appendix 3 are given by

$$
F_{T_1} = (tr_{n_1} \otimes 1_k) P_{R_1}
$$

$$
F_{T_2} = (1_k \otimes tr_{n_2}) P_{R_2}
$$

(6.2)

and where $F_{T_1}$ is represented diagrammatically in fig. [14].
Fig. 14: Projector diagram defining the operator $F_{T_1}$ in terms of the projection operator $P_{R_1}$.

In terms of unitary group integrals $F_{T_1}$ is given by

$$F_{T_1} = \text{Dim}R_1 \int dU_1 \chi_{R_1}(U_1)(\text{tr}(U_1^\dagger))^n_1 \rho_k(U_1^\dagger)$$  \hspace{1cm} (6.3)

and similarly for $F_{T_2}$

$$F_{T_2} = \text{Dim}R_2 \int dU_2 \chi_{R_2}(U_2)(\text{tr}(U_2^\dagger))^n_2 \rho_k(U_2^\dagger).$$  \hspace{1cm} (6.4)

Plugging in the explicit forms for $F_{T_1}$ and $F_{T_2}$ into the factorized form (15.5) of the identity derived in appendix 3 leads to

$$\text{tr}((P_{R_1} \otimes 1_{n_2})(1_{n_1} \otimes P_{R_2})) = \sum_{S_2} \frac{1}{\text{Dim}S_2} \text{tr}(P_{R_1}(P_{S_2} \otimes 1_{n_1}))\text{tr}((1_{n_2} \otimes P_{S_2})P_{R_2}).$$  \hspace{1cm} (6.5)

The latter traces are of the form evaluated in [1] and moreover follow easily from relation (3.18) derived in section 3. Specifically one finds that

$$\text{tr}((P_{S_2} \otimes P_{S_1})P_{R_1}) = d_{S_2}d_{S_1}g(S_2, S_1; R_1) \text{ Dim}R_1.$$  \hspace{1cm} (6.6)

To get exactly the traces on the right-hand-side of (6.5) one simply notes that $\sum_{S_1} P_{S_1} = I$. This results in the sum rule

$$\sum_{R_3} d_{R_3} \langle E_{21}^k \chi_{R_1}(\Phi_1(x_1))Q_{12}^k \chi_{R_2}(\Phi_1(x_2))\chi_{R_3}(\Phi_1^\dagger(0)) \rangle = \frac{(n_1 + k)(n_2 + k)!k!(n_1 + n_2)!}{n_1!n_2!} \times \frac{1}{x_1^{2n_1}} \frac{1}{x_2^{2n_2}} \frac{1}{(x_1 - x_2)^{2k}} \frac{\text{Dim}R_1 \text{ Dim}R_2}{d_{R_1}d_{R_2}} \sum_{S_1, S_2, S_3} \frac{d_{S_1}d_{S_2}d_{S_3}}{\text{Dim}S_2} g(S_1, S_2; R_1)g(S_3, S_2; R_2).$$  \hspace{1cm} (6.7)
Finally we note that multiplying by $d_{R_1}$ or $d_{R_2}$ instead and summing would yield a correlator of a similar form to that as in (6.1). In particular the same sort of trace would appear as is shown in the diagram fig. 13.

6.1. Sum rules for Correlators with general positions

![Diagram](image)

Fig. 15: Diagram representing the trace appearing in the sum rule for the correlator (5.7).

The sum rules apply also to correlators with general position dependence. As an example we consider the four-point correlator evaluated in the last section. Multiplying by $d_{R_1}$ say and summing over all representations $R_1$ (as above one could instead perform these same operations on any of the other representations $R_2$, $R_3$, or $R_4$ and obtain similar results) replaces the projection operator $P_{R_1}$ in (5.7) by the identity. The resulting trace that appears is represented diagrammatically in fig. 15.

This diagram is also of the general class analyzed in appendix 3 that can be factorized. In this case one cuts the diagram in two places: (1) along the strand connecting $R_2$ to $R_3$ and (2) along the strand connecting $R_2$ to $R_4$. Considering case (1) first, the operator $F_{T_1}$ defined in appendix 3 will be exactly as in figure fig. 14, or in terms of an equation (6.2), with $R_1$ replaced by $R_3$. The other operator, $F_{T_2}$, defined in appendix 3 will in this case be given by

$$F_{T_2} = (1_{n_3-j} \otimes tr_{n_4-n_1+j} \otimes tr_{n_1-j})(P_{R_2} \otimes 1_{n_1-j})(1_{n_3-j} \otimes P_{R_4}) \quad (6.8)$$
and is represented diagrammatically in fig. 16.

Fig. 16: Projector diagram defining the operator $F_{T_2}$ in terms of the projection operators $P_{R_2}$ and $P_{R_4}$.

Applying the factorization equation (15.5) from appendix 3 along the strand described above in (1) produces

$$
tr((1_{n_1} \otimes P_{R_2})(P_{R_3} \otimes_j P_{R_4})) = \sum_{T_1} \frac{1}{d_{T_1}} \frac{1}{DimT_1} tr(P_{R_3}(1_j \otimes P_{T_1})) \times tr((P_{T_1} \otimes 1_{n_4})(P_{R_2} \otimes 1_{n_1-j})(1_{n_3-j} \otimes P_{R_4})).
$$

(6.9)

Another application of the factorization equation along the strand described in (2) above, or equivalently along the strand connecting $R_2$ and $R_4$ in the second trace appearing in the right-hand-side of (6.9), yields

$$
tr((1_{n_1} \otimes P_{R_2})(P_{R_3} \otimes_j P_{R_4})) = \sum_{T_1, T_2} \frac{1}{d_{T_1}} \frac{1}{DimT_1} \frac{1}{d_{T_2}} \frac{1}{DimT_2} tr(P_{R_3}(1_j \otimes P_{T_1})) \times tr(P_{R_4}(1_{n_1-j} \otimes P_{T_2}))tr(P_{T_1} \otimes P_{T_2}).
$$

(6.10)

All traces appearing on the right-hand-side of (6.10) can now be evaluated using (6.6), we find

$$
tr((1_{n_1} \otimes P_{R_2})(P_{R_3} \otimes_j P_{R_4})) = DimR_2 \frac{1}{DimT_1} \frac{1}{DimT_2} \sum_{T_1, T_2} \sum_{U_1, U_2} d_{U_1} d_{U_2} \frac{d_{T_1} d_{T_2}}{g(T_1, T_2; R_2)g(U_1, T_1; R_3)g(U_2, T_2; R_4)}.
$$

(6.11)
Substituting this into (5.7) gives the final form of the sum rule evaluated in terms of fusion coefficients and dimensions of various representations.

For higher point correlators of the form
\[
\langle \chi_{R_1}(\Phi(x_1)) \cdots \chi_{R_n}(\Phi(x_n)) \chi_{S_1}(\Phi^\dagger(y_1)) \cdots \chi_{S_m}(\Phi^\dagger(y_m)) \rangle
\]  
(6.12)
where \( n, m > 2 \), it is no longer true that multiplying by a single dimension \( d_{R_i} \) say and summing over representations \( R_i \) yields a correlator expressible just in terms of fusion coefficients and dimensions of representations. In such a case the trace of projection operators does not simplify to a form that is factorizable. To get such traces one needs to multiply by \( (n-1) \) of the \( d_{R_i} \)'s and sum or by \( (m-1) \) of the \( d_{S_j} \)'s and sum to get a factorizable diagram that can be evaluated just in terms of fusion coefficients.

7. Comparison to 3D Chern-Simons theory, 2D G/G theory and 2D YM theory

Highly supersymmetric correlators in super Yang-Mills theories often capture a topological phase of the theory, as for example in [24]. Often the connection to topology proceeds via instantons. However, the extremal correlators do not receive instanton corrections. So the standard route to establishing deductively a connection between them and topological gauge theories is not available. Nevertheless we remarked in previous sections that factorization equations, fusion relations and sum rules can be written down for the extremal correlators which have analogs in topological gauge theories. The group theoretic character of the correlators allows equalities between ratios of correlators and appropriate observables in topological gauge theories. By exploiting known exact answers for topological theories in lower dimensions, in particular Chern-Simons theory in the \( k \to \infty \) limit and two-dimensional Yang-Mills theory in the zero area limit, we describe these relations. We will be using results from the literature on three-dimensional Chern-Simons theory and knot invariants [25,26,27,28,29,30], as well as that on two-dimensional Yang-Mills [31,32,33,34,35].

For the extremal correlator we have
\[
\frac{\langle \chi_{R_1}(\Phi(x)) \chi_{R_2}(\Phi(x)) \chi_{S}(\Phi^\dagger(0)) \rangle_{\text{SYM4}}}{\langle \chi_{S}(\Phi(x)) \chi_{S}(\Phi^\dagger(0)) \rangle_{\text{SYM4}}} = g(R_1, R_2; S)
\]
\( = Z_{CS}(S^2 \times S^1; R_1, R_2, \bar{S}) \)
\( = \langle \chi_{R_1}(g) \chi_{R_2}(g) \chi_{S}(g) \rangle_{G/G} \)
\( = Z_{2DYM}(R_1; W_{R_2}; \bar{S}) \)  
(7.1)
Fig. 17: Diagram illustrating observable in 2DYM.

The first line is a special extremal correlator in four-dimensional $U(N)$ super-Yang-Mills theory, where the insertion points of $R_1$ and $R_2$ are taken to coincide. The second line is the Littlewood-Richardson coefficient, i.e. the number of times the tensor product $R_1 \otimes R_2$ contains $S$. The third line is a correlator of $U(N)$ Chern-Simons theory on $S^2 \times S^1$ with two Wilson lines carrying representations $R_1$ and $R_2$ winding in one direction along the $S^1$ and a third Wilson line in the $S$ representation winding in the opposite direction along the circle. By the standard relation between Chern-Simons theory in three dimensions with gauge group $G$ and $G/G$ topological field theory, this correlator is equal to a correlator for the $G/G$ model with insertions at three points on $S^2$ which can be viewed as the intersection of the three Wilson lines in Chern-Simons with an $S^2$ at a fixed point of $S^1$ on $S^2 \times S^1$. The fifth line gives a correspondence between the ratio of 4D correlators and topological two-dimensional Yang-Mills theory with $U(N)$ gauge group on a cylinder. Topological two-dimensional Yang-Mills is the zero area limit of standard two-dimensional Yang-Mills. Evaluating the path integral for two-dimensional Yang-Mills on the cylinder requires the specification of a boundary holonomy, or more generally the holonomies can be weighted by the character in a representation $R$. The relevant observable has boundary holonomies specified by $R_1$ on the left and $\bar{S}$ on the right, and there is a Wilson loop insertion in the irrep $R_2$ parallel to $R_1$. The diagram in fig. 17 illustrates the relevant observable in two-dimensional Yang-Mills.

This generalizes to

$$\frac{\langle \prod_{i=1}^{k} \chi_{R_i}(\Phi(x)) \chi_{S}(\Phi^\dagger(0)) \rangle_{\text{SYM4}}}{\langle \chi_{S}(\Phi(0)) \chi_{S}(\Phi^\dagger(x)) \rangle} = Z_{CS}(S^2 \times S^1 : R_1, R_2, \cdots R_n; \bar{S})$$

$$= \prod_{i=1}^{k} \chi_{R_i}(g) \chi_{\bar{S}}(g)_{G/G}$$

$$= Z_{2\text{DYM}}(R_1; W_{R_2}, W_{R_3} \cdots W_{R_k}, W_{\bar{S}_1}, \cdots W_{\bar{S}_{l-1}}; \bar{S}_l)$$

(7.2)
It is also worth noting that for two point functions we have
\[
\langle \chi_R(\Phi(x))\chi_S(\Phi^\dagger)(0) \rangle_{SYM4} = x^{-2n} \delta_{RS} \frac{n! \text{Dim}R}{d_R}
\]
\[
= x^{-2n} \frac{n!}{d_R} Z_{CS}(S^2 \times S^1 : R, S) \frac{Z_{CS}(S^3 : R)}{Z_{CS}(S^3 : \emptyset)}
\]
\[
= x^{-2n} Z_{2DYM}(S^1 \times I : R, \bar{S}) \frac{n! \text{Dim}R}{d_R}
\]
(7.3)

The relations in (7.3) are useful in deriving the previous equations but, with the \(x\) dependences still present, do not convincingly exhibit the topological character of the correlators the way the earlier equations do.

7.1. Converting graphs to Wilson loops in \(S^3\)

Equation (7.1) uses only a special class of extremal correlators, those where all the holomorphic operators are at the same point and all the anti-holomorphic operators are also at a fixed point. The correlator is still extremal and non-renormalized if the anti-holomorphic operators are fixed at one point and holomorphic operators are at different points. It is also possible to find ratios involving such correlators which map to correlators in topological gauge theories.

\[
\frac{\langle \chi_{R_1}(\Phi(x_1))\chi_{R_2}(\Phi(x_2))\chi_S(\Phi^\dagger(0)) \rangle_{SYM4}}{\langle \chi_{R_1}(\Phi(0))\chi_{R_2}(\Phi^\dagger(x_1)) \rangle_{SYM4} \langle \chi_{R_2}(\Phi(0))\chi_{R_2}(\Phi^\dagger(x_2)) \rangle_{SYM4}}
= \frac{(n_1 + n_2)!}{n_1!n_2!} \frac{d_{R_1}d_{R_2}}{d_S} g(R_1, R_2; S) \frac{\text{Dim}S}{\text{Dim} R_1 \text{Dim} R_2}
\]
(7.4)

In the last line of (7.4) we have taken advantage of the fact that group theoretic graphs with legs labelled by irreps and vertices by Clebsch Gordans are large \(k\) limits of the normalized expectation values of the corresponding intersecting Wilson loops in Chern-Simons theory. \(W(R_1, R_2, S)\) denotes the Wilson loop shown in fig. [8].

\[
\frac{Z_{CS}(S^3 : W(R_1, R_2, S))}{Z_{CS}(S^3 : \emptyset)} = g(R_1, R_2; S) \text{Dim} S
\]
(7.5)

As explained in [26] the vertices can be labelled by a choice of invariant tensor in \(R_1 \otimes R_2 \otimes \bar{S}\). Here we are summing over all the choices with equal weight to get the fusion multiplicity \(g(R_1, R_2; S)\) and \(\text{Dim} S\) appears because of the trace. The second equation just gives the expectation value for two disconnected Wilson loops.
Fig. 18: Diagram illustrating observable in Chern-Simons.

For the more general case of the $k \to l$ multipoint function (2.1) we can

\[
\frac{\langle \chi_{R_1}(\Phi(x_1))\chi_{R_2}(\Phi(x_2)) \cdots \chi_{R_k}(\Phi(x_k)) \chi_{S_1}(\Phi^+(0)) \cdots \chi_{S_l}(\Phi^+(0)) \rangle}{\langle \chi_{R_1}(\Phi(x_1))\chi_{R_1}(\Phi^+(0)) \cdots \chi_{R_k}(\Phi(x_k))\chi_{R_k}(\Phi^+(0)) \rangle} = \frac{1}{n(R_1)!n(R_2)! \cdots n(R_k)!} \frac{1}{\text{Dim}R_1 \text{Dim}R_2 \cdots \text{Dim}R_k} \frac{1}{d_{S_1} \cdots d_{S_l}} \times \sum_{S} \sum_{\gamma} \text{tr} \left( (P_{S_1} \otimes P_{S_2} \cdots P_{S_l}) P_{S} \gamma^{-1} (P_{R_1} \otimes P_{R_2} \cdots P_{R_k}) \gamma P_{S} \right)
\]

\[
= \frac{n!}{n(R_1)!n(R_2)! \cdots n(R_k)!} \frac{1}{\text{Dim}R_1 \text{Dim}R_2 \cdots \text{Dim}R_k} \times \sum_{S} \text{tr}_{S_1 \otimes \cdots \otimes S_l} (P_{S}(P_{R_1} \otimes \cdots \otimes P_{R_k})P_{S})
\]

\[
= \frac{n!}{n(R_1)!n(R_2)! \cdots n(R_k)!} \sum_{S} \frac{Z_{CS}(S^3; W(R_1, R_2, \cdots R_k; S; S_1, S_2 \cdots S_l))}{Z_{CS}(S^3; R_i)}
\]

where $Z_{CS}(S^3; R_i)$ is the expectation value of a Wilson loop in the $R_i$ representation in $k = \infty$ three dimensional $U(N)$ Chern-Simons theory, and $Z_{CS}(S^3; W(R_1, R_2, \cdots R_k; S; S_1, S_2 \cdots S_l)$ is the expectation value of the Wilson loop shown in fig. [19].

The relation we used in (7.3) for the relation between Chern-Simons expectation values in $S^3$ and knotted group theoretic graphs is quite general, and can also be applied to
quantities like the one in fig. 11 which appears in non-extremal correlators. The extremal correlators only involve the simple graphs in fig. 18 and fig. 19 which when evaluated group theoretically gives fusion coefficients and dimensions. The fusion coefficient is also the expectation value in non-intersecting Wilson loops in $S^2 \times S^1$ as we saw in (7.1). This is also true for the extremal $SU(N)$ correlators as we will see later, which involves products of the simple graph. The non-extremal correlators are of course renormalized at finite coupling $g_{YM}^2$, so it appears that correlators which are associated with complicated graphs have non-trivial coupling dependence, while those associated with simple graphs have trivial coupling dependence. We will return to this point in section 9.
7.2. Sum Rules, Relations between $S^2 \times S^1$ and $S^3$ and Verlinde formula.

In sections 4 and 6 we have written down sum rules for both extremal and non-extremal correlators. Here we will consider the extremal case and show how they are equivalent to identities we expect from the mapping to Chern-Simons theory. The basic group theoretic identity (4.5) leads to (4.4). Using the map to Chern-Simons correlators given in (7.1) and (7.3) this translates into

$$\sum_S \text{Dim} S \ Z_{CS}(S^2 \times S^1; R_1, R_2, \bar{S}) = \frac{Z_{CS}(S^3 : R_1, R_2)}{Z_{CS}(S^3 : \emptyset)}$$

(7.7)

This is indeed the large $k$ limit of the relation between expectation values in $S^3$ and in $S^2 \times S^1$ which holds for any observable $O$ inserted on $S^3$ [25]

$$Z_{CS}(S^3 : O) = \sum_S S_{0S} \ Z_{CS}(S^2 \times S^1 : O, S)$$

(7.8)

Here $S_{0S}$ is the matrix element of the modular $S$ matrix between the identity rep and $S$, and it obeys

$$S_{00} = Z_{CS}(S^3 : \emptyset)$$

$$\frac{S_{0R}}{S_{00}} = \text{Dim} R$$

(7.9)

At finite $k$ the right hand side of the second line is the quantum dimension, but at large $k$ it is the ordinary dimension. Using (7.9) it is clear that (7.7) is the same as (7.8).

It is also of interest to compare the identity (4.5) underlying a class of sum rules (4.4) with the large $k$ limit of the Verlinde formula [15]. This is expressed in terms of $S_{R_1 R_2}$ which in the large $k$ limit is proportional to $\text{Dim}(R_1) \text{Dim}(R_2)$. It can be expressed [25] as:

$$\frac{S_{R_1 R_2} S_{R_1 R_3}}{S_{0 R_1}} = \sum_S S_{R_1 S} \ g(R_2, R_3; S)$$

(7.10)

Substituting the appropriate product of dimensions for each matrix element of the modular $S$-matrix, (7.10) implies

$$\text{Dim} R_1 \text{Dim} R_2 \text{Dim} R_3 = \sum_S \text{Dim} R_1 \text{Dim} S \ g(R_2, R_3; S)$$

(7.11)

which reduces to the (4.3).

In section 3 we saw that there are two kinds of sum rules involving the fusion coefficient, as a result two kinds of sum rules for extremal three point functions. This arises
from the fact that $g(R_1, R_2; R_3)$ is both a fusion number for unitary groups and a branching number for symmetric groups. The fusion number interpretation leads to (4.5) which we related to Chern Simons in different ways. The symmetric group interpretation leads to (4.6). It would be interesting to find the Chern-Simons interpretation of this latter sum rule. In section 6, it was observed that the sum rules can also be applied to non-extremal correlators. In a sense then, these more general sum rules are generalizations of the large $k$ limit of the Verlinde formula. Given the rich geometry of the Verlinde formula in terms of dimensions of spaces of sections of holomorphic bundles over moduli spaces of complex structures in the context of two-dimensional rational CFT or equivalently in topological $G/G$ models, the above remarks may be viewed as a hint that the correlators of four-dimensional $N = 4$ super-Yang-Mills theory might have an analogous geometrical meaning in four dimensions, although the precise geometric objects relevant here are unknown. Higher dimensional versions of the Verlinde formula have been discussed in [36] and it would be interesting to explore any possible connections of [36] with $N = 4$ SYM.

8. Staggered Factorization

Fig. 20: Picture of contraction pattern

Fig. 21: Factorized contraction pattern
We now describe a class of factorization equations obeyed by correlators which are obtained by a simple patterns of contractions. To describe the contraction patterns, for each operator insertion we draw a circle. We draw a single line connecting two circles if there is a set of $\Phi$’s being contracted between them. If the contraction diagram can be disconnected by cutting a line, we can write a factorized expression. These contraction diagrams thus have to be “one-particle reducible” to be factorizable. We have used ordinary Feynman diagram language here, although each line describes a set of propagators rather than a single propagator. A simple factorizable pattern of contractions is shown in fig. 20. After factorization we get a sum over products of correlators of the form in fig. 21. It is in fact simplest to describe the class of relevant correlators by the kind of projector diagrams that calculates their expectation value. This class of diagrams is described in appendix 3, and given the form of the relevant projector diagrams we will call this factorization property “staggered factorization” which is a generalization of the basic factorization discussed in sections 2 and 3.

Consider a correlator corresponding to fig. 20

$$\langle \chi_{R_1}(\Phi_3(x_1))Q_{13}^{n_3}E_{21}^{n_2}\chi_{R_2}(\Phi_1(x_2))Q_{12}^{n_2}\chi_{R_3}(\Phi_1(x_3))\chi_{R_4}(\Phi_1^\dagger(x_4)) \rangle \quad (8.1)$$

Here $n(R_1) = n_3, n(R_2) = n_2 + n_3, n(R_3) = n_1 + n_2,$ and $n(R_4) = n_1.$ We would like to describe the $SO(6)$ transformation properties of the operators involved. One way to specify the content is to list the numbers $(n_1, n_2, n_3)$ which count the number of $\Phi_1, \Phi_2,$ and $\Phi_3$ fields respectively. We count $\Phi_i^\dagger$ as contributing negatively to these three charges. Here, the charges are

$$\vec{v}_1 = (0, 0, n_3)$$
$$\vec{v}_2 = (0, n_2, -n_3)$$
$$\vec{v}_3 = (n_1, -n_2, 0)$$
$$\vec{v}_4 = (-n_1, 0, 0) \quad (8.2)$$

The correlator in (8.1) is

$$\frac{n_2!n_3!(n_1 + n_2)!(n_2 + n_3)!}{d_{R_1}d_{R_2}d_{R_3}d_{R_4}} tr \left( (P_{R_1} \otimes 1_{n_{R_3}})(P_{R_2} \otimes 1_{n_{R_4}})(1_{n_{R_1}} \otimes P_{R_3})(1_{n_{R_2}} \otimes P_{R_4}) \right)$$

$$= \frac{n_2!n_3!(n_1 + n_2)!(n_2 + n_3)!}{d_{R_1}d_{R_2}d_{R_3}d_{R_4}} tr \left( (P_{R_1} \otimes P_{R_3})(P_{R_2} \otimes P_{R_4}) \right) \quad (8.3)$$
Fig. 22: Diagram corresponding to the trace appearing in correlator

where the trace is represented diagrammatically in fig. 22. Using the group integral version of projectors, we can express the trace as:

$$
tr((P_{R_1} \otimes P_{R_3})(P_{R_2} \otimes P_{R_4})) = DimR_1 \ DimR_2 \ DimR_3 \ DimR_4 \\
\int dU_1 dU_2 dU_3 dU_4 \ \chi_{R_1}(U_1^\dagger)\chi_{R_2}(U_2^\dagger) \ \chi_{R_3}(U_3^\dagger) \ \chi_{R_4}(U_4^\dagger) \\
tr_{n_3}(U_1U_2) \ tr_{n_2}(U_2U_3) \ tr_{n_1}(U_4U_3) \\
= \sum_S d_{R_1} \ DimR_2 \ d_S \ DimR_3 \ d_{R_4} \\
\int dU_1 dU_2 \chi_{R_2}(U_1^\dagger) \ \chi_{R_3}(U_2^\dagger)\chi_{R_1}(U_1)\chi_{S}(U_1U_2)\chi_{R_4}(U_2)
$$

(8.4)

The diagram in fig. 22 is of the general form considered in appendix 3 that can be factorized. Specifically we can cut the diagram along the line connecting $R_2$ and $R_3$ by inserting a complete set of irreducible projectors. In detail, the operators $F_{T_1}$ and $F_{T_2}$ introduced in appendix 3 are given by

$$
F_{T_1} = (tr_{n_3} \otimes 1_{n_2})(P_{R_1} \otimes 1_{n_2})P_{R_2} \\
F_{T_2} = (1_{n_2} \otimes tr_{n_1})(1_{n_2} \otimes P_{R_4})P_{R_3}
$$

(8.5)

Using the factorization property

$$
tr_{n_2}(P_S F_{T_1} F_{T_2}) = \frac{1}{d_S DimS} tr_{n_2}(P_S F_{T_1}) tr_{n_2}(P_S F_{T_2})
$$

(8.6)
proved in appendix 3 yields the relation

\[
\text{tr} \left( (P_{R_1} \otimes P_{R_3})(P_{R_2} \otimes P_{R_4}) \right) \\
= \sum_S \frac{1}{d_S \dim S} \text{tr} \left( (P_{R_1} \otimes P_S)P_{R_2} \right) \text{tr} \left( P_{R_3}(P_S \otimes P_{R_4}) \right) \\
= \sum_S \frac{1}{d_S \dim S} d_S d_{R_1} d_{R_2} g(R_1, S; R_2) g(S, R_4; R_3) \dim R_2 \dim R_3
\] (8.7)

The factorized product of traces is shown diagrammatically in fig. 23.

These steps lead to the following result

\[
\langle \chi_{R_1}(\Phi_3(x_1)) Q_{13}^{n_3} E_{21}^{n_2} \chi_{R_2}(\Phi_1(x_2)) Q_{12}^{n_2} \chi_{R_3}(\Phi_1(x_3)) \chi_{R_4}(\Phi_1^\dagger(x_4)) \rangle \\
= n_2! n_3! (n_2 + n_3)! (n_1 + n_2)! \frac{1}{(x_1 - x_2)^{2n_2}} \frac{1}{(x_2 - x_3)^{2n_3}} \frac{1}{(x_3 - x_4)^{2n_1}} \] (8.8)

\[
\times \sum_S \frac{\dim R_2 \dim R_3}{\dim S} \frac{d_S}{d_{R_2} d_{R_3}} g(R_1, S; R_2) g(S, R_4; R_3)
\]

Since we now have fusion coefficients which are reminiscent of three-point functions, it is natural to convert (8.8) into a statement about correlators as indicated by fig. 21. This is indeed possible, and when the RHS of (8.8) is expressed in terms of the appropriate
correlator with general positions given by (5.7). Assume that but picking out the desired contraction patterns. Consider as an example the four-point correlator

\[ \langle \chi R_1 (\Phi_3(x_1)) Q_{13}^{n_3} E_{21}^{n_2} \chi R_2 (\Phi_1(x_2)) Q_{12}^{n_2} \chi R_3 (\Phi_1(x_3)) \chi R_4 (\Phi_1^\dagger(x_4)) \rangle \]

\[ = \sum_S \langle \chi R_1 (\Phi_3(x_1)) Q_{13}^{n_3} E_{21}^{n_2} \chi R_2 (\Phi_1(x_2)) \chi S (\Phi_2^\dagger(x_3)) \rangle \]

\[ \times \langle \chi S (\Phi_2(x_2)) Q_{12}^{n_2} \chi R_3 (\Phi_1(x_3)) \chi R_4 (\Phi_1^\dagger(x_4)) \rangle \frac{1}{\chi S (\Phi_2(x_2)) \chi S (\Phi_2^\dagger(x_3))} \]

(8.9)

The key to this factorization equation is that the correlator is obtained from a simple pattern of contractions. In the above, we fixed the pattern of contractions by choosing the \(SO(6)\) quantum numbers of the operators. We can also fix the contraction patterns by working with just the highest weight or lowest weight states under the \(SO(6)\) action, but picking out the desired contraction patterns. Consider as an example the four-point correlator with general positions given by (5.7). Assume that \(n_1 \leq n_3\) (and therefore \(n_4 \leq n_2\) since \(n_1 + n_2 = n_3 + n_4\) for this correlator to be non-zero) and extract the \(j = n_1\) term from the sum over \(j\) by a contour integral,

\[ \int dy_1 dy_2 dy_3 \frac{y_3^{2(n_1-n_1)-1} y_4^{2(n_3-n_1)-1}}{y_1 y_2 y_3 y_4} \chi R_1 (\Phi(x)) \chi R_2 (\Phi(x)) \chi R_3 (\Phi(x)) \chi R_4 (\Phi(x)) \]

(8.10)

where \(y_1 = x_1 - x_3\), \(y_2 = x_2 - x_3\), and \(y_3 = x_2 - x_4\). We find then the exact same trace as appears on the left-hand-side of (8.7) (only with \(R_2\) and \(R_3\) exchanged). Consequently we can factorize this contribution to the general position four-point correlator as above.

This approach to picking out simple contraction patterns clearly generalizes to higher point functions as well. For example, consider the following contour integral of a correlator involving five insertions

\[ \int dy_1 dy_2 dy_3 dy_4 \frac{y_3^{2(n_2-1)-1} y_4^{2(n_3+1)-1}}{y_1 y_2 y_3 y_4} \chi R_1 (\Phi(x)) \chi R_2 (\Phi(x)) \chi R_3 (\Phi(x)) \chi R_4 (\Phi(x)) \chi R_5 (\Phi(x)) \]

(8.11)

where the projector diagram with the contractions of interest is shown below. The \(y\)'s are differences of \(x\) coordinates \(y_1 = x_1 - x_2\), \(y_2 = x_2 - x_3\), \(y_3 = x_3 - x_4\), \(y_4 = x_4 - x_5\) and the numbers of boxes \(n(R_i)\) in the Young diagrams for the representations \(R_i\) are chosen to satisfy \(n(R_1) = n_1\), \(n(R_2) = n_1 + n_2\), \(n(R_3) = n_2 + n_3\), \(n(R_4) = n_3 + n_4\), and \(n(R_5) = n_4\) guaranteeing that a contraction pattern of the form shown in fig. [24] exists. This diagram is clearly also factorizable using the results of appendix 3.

It will be interesting to see if these integrated correlators satisfy non-renormalization theorems.
8.1. *Staggered factorization and tangential crossings in YM2*

Staggered factorization has a simple meaning in two-dimensional Yang Mills at zero area. The rules for constructing the partition function for manifolds with boundary and arbitrary Wilson loop insertions are derived in [31, 32] and reviewed in [35].

Consider the partition function of topological two-dimensional Yang-Mills associated with the above diagram. Circles lined by dash-dotted lines are boundaries, with the dash-dots being in the interior of the two-dimensional manifolds. The holonomies on the boundaries are weighted by a character in the irrep shown next to the boundary. The boundary irreps are $S, R_1, R_4$. In calculating the partition function we assign some weights for each region. The regions have been labelled $A_1, A_2, A_3$. The appropriate factors are

$$
A_1 \rightarrow \chi_S(U_1U_2) \\
A_2 \rightarrow \chi_{R_1}(U_1) \\
A_3 \rightarrow \chi_{R_4}(U_2)
$$

There are also Wilson loop insertions $\chi_{R_2}(U_1^\dagger)$ and $\chi_{R_3}(U_2^\dagger)$ along the contours in fig. 25 labelled by $U_1$ and $U_2$ respectively.
The expectation value of this observable is

$$Z = \int dU_1 dU_2 \chi_{R_2}(U_1^\dagger) \chi_{R_3}(U_2^\dagger) \chi_{R_1}(U_1) \chi_{S}(U_1 U_2) \chi_{R_4}(U_2)$$  \hspace{1cm} (8.13)

This is precisely the expression that entered in (8.4). The staggered factorization argument leads to a simpler group integral.

$$Z = \int dU_1 dU_2 \chi_{R_2}(U_1^\dagger) \chi_{R_3}(U_2^\dagger) \chi_{R_1}(U_1) \frac{\chi_{S}(U_1) \chi_{S}(U_2)}{\text{Dim}S} \chi_{R_4}(U_2)$$  \hspace{1cm} (8.14)

The latter expression is just the expression we would write, using the rules of [31,32] for the Wilson loop expectation value where the two tangentially intersecting Wilson loops are slid away so that they are not touching anymore. The factor $\frac{1}{\text{Dim}S}$ arises because the region $A_2$ now has Euler character $-1$ as opposed to 0. In a topological theory, we may
naturally expect such a sliding move on tangential intersections to leave the expectation value invariant, and the rules of [31][32] show that this is indeed true in the zero area limit of two-dimensional Yang-Mills. To summarize then, staggered factorization of correlators in 4D SYM is mapped to deformations of Wilson Loops in 2DYM which disentangle tangential intersections.

![Multiple tangential intersections of Wilson loops in 2DYM](image)

This remark is quite general. For example, the factorizable five-point correlator associated with the trace diagram in fig. [24] is related to the 2DYM observable with tangential crossings given in fig. [26]. In this case there are Wilson loop insertions $\chi_{R_2}(U_1^\dagger)\chi_{R_3}(U_2^\dagger)\chi_{R_4}(U_3^\dagger)$.

8.2. Staggered factorization and connected sums of links in Chern Simons
Consider the projector diagram in fig. 22 which is a sequence of traces in tensor space. It is related to a sum over irreps $S$ when a complete set of irreducible projectors is inserted in $tr_{n_2}$. The diagram is now given in fig. 26. Up to factors of $d_{R_1}d_Sd_{R_4}$ it can be converted to a trace in the tensor product of irreducibles of $U(N)$ given by $R_1 \otimes S \otimes R_4$. Replacing the projectors $R_2$ and $R_3$ by products of Clebsh-Gordan coefficients, it is given by a formula which has a diagrammatic representation in terms of a graph with trivalent vertices as explained in section 5. As such it is related to the expectation value of a knotted graph in $U(N)$ Chern-Simons. The same procedure applied to fig. 23 gives a pair of graphs. The geometrical operation which takes the pair of graphs to the single graph is to form a connected sum of the pair of graphs.

These projector graphs have a $q$-deformation relevant to finite $k$ Chern-Simons and link invariants. The relation to standard link invariants is clearer if instead of the projectors $R_2, R_3$, we have $\sigma R$ where $R$ is the universal R-matrix for $U_q(N)$ and $\sigma$ is a permutation.
\(\sigma R\) commutes with \(U_q(N)\) (see for example [30], [28]) and hence is related to a sum of projectors. In fact, in the example of fig. 23, if we replace \(R_2\) and \(R_3\) by \((\sigma R)^2\) we get exactly the example of a connected sum discussed in [25]. Since the derivation of the staggered factorization in appendix 3 relied on properties of projectors acting on tensor products, it generalizes to the quantum group case, and hence to finite \(k\) Chern-Simons.

To summarize, we have described a way to map correlators in SYM4 to expectation values of Wilson loops in CS3. The correlators in SYM4 can be simplified by a generalization of the basic factorization equation (2.4) when they map to Wilson loops which are connected sums of simpler Wilson loops.

### 8.3. Remarks on Staggered factorization and Non-renormalization Theorems

The extremal correlators have simple spacetime dependence and a simple coupling dependence. The proof of the NR theorems indeed relies on relating spacetime and coupling dependences [3]. The result of [1] and the first few sections of this paper is that the extremal correlators can be expressed in terms of simple group theoretic quantities like fusion coefficients and dimensions. The reason this is possible is given by the basic factorization and fusion equations we have discussed. More general correlators are given by more intricate group theory invariants such as those described in section 5.

The basic factorization was then generalized to staggered factorization which allowed a more general class of correlators to be expressed in terms of products of dimensions and fusion coefficients. So there is a more general class of correlators which have simple colour dependence. It is natural to speculate that these more general correlators would also have simple spacetime and coupling dependence. Given that dimensions and fusion coefficients are related to two and three-point correlators [11], this line of reasoning is similar to [11,12]. There non-renormalization properties of correlators, and in some cases pieces of correlators which are expressed in terms of products of two and three-point functions at zero coupling, were explored. The next-to-extremal correlators considered in [11] can also be evaluated in terms of dimensions and fusion coefficients using the techniques of this paper. We shall however defer the details to future work.

Some support for this line of thinking is given by the fact that the dependence on the representation content translates into a dependence on the wavefunctions of giant gravitons on the \(S^5\) of the \(ADSS_5 \times S^5\) dual. Now in a supersymmetric theory governed by a superalgebra \(SU(2,2|4)\) where the symmetries of the \(S^5\) and the \(ADSS_5\) are unified in a single algebra, we might expect that simple dependence on the \(S^5\) would be related to simple
dependence on $\text{ADS}_5$. By the existing arguments relating coupling and spacetime dependence, this would imply simple coupling dependence, and perhaps no coupling dependence at all.

It would be very interesting to develop this line of reasoning further and give a proof of the generalized non-renormalization theorems suggested by the staggered factorization property.

9. General derivation of Basic Factorization equation and Fusion equation

Consider the four point function in (2.3). This is given by a path integral on $R^4$ with four copies of $B_4$ (four-ball) removed and having four $S^3$ boundaries. We can think of cutting along an $S^3$ which separates the two chiral operators and the two anti-chiral operators, and inserting a complete set of states. We now have two four-manifolds with three boundaries each. One contains two chiral operators and we would like to argue that only chiral operators can flow through.

The leading term in the OPE of a field operator which creates a highest weight of the superalgebra with another similar operator is also a highest weight. We are taking advantage of the fact that a tensor product of two highest weight states of short reps of the superalgebra is itself a highest weight state of a short rep. Sub-leading terms in the OPE will, of course, contain descendants. If the highest weight operators are all coincident then only the leading term is relevant. This leading highest weight is the only term which will contribute to correlators when the coincident highest weight operators are in a correlator involving only lowest weight operators as the remaining operators. This leads us to expect, in general, an equation of the form (2.4) once the complete set of highest weight states have been described, as we have done already for the case of $U(N)$.

With this heuristic understanding of the basic factorization equation we can expect that it will also hold true in the case of $SU(N)$ (and for any gauge group) where the explicit formulae for the correlators are more intricate. In the next section we will describe the complete set of highest weights in the case of $SU(N)$. We will also give a general formula for the extremal correlators. Explicit checks of the factorization equations should be possible, but do not look trivial since the Schur polynomial basis which diagonalized the two-point functions for $U(N)$ no longer does so for $SU(N)$.
10. Observables and Correlators for $SU(N)$

10.1. Classification of half-BPS operators

We start by making some remarks about the classification of operators. In the case of $U(N)$ we could convert from the conjugacy class to the Schur basis. This was very useful since the two-point correlators in the Schur basis were orthogonal, and the three-point correlators were directly related to fusion coefficients. Also when $n$ becomes comparable or bigger than $N$, the Schur basis provides a useful way to characterize the independent gauge invariant operators.

In the case of $SU(N)$ it is still true that we can change basis to the Schur polynomials, but as we shall see they are not orthogonal. Moreover they obey several relations. We can solve the relations but the independent observables are most easily described in terms of conjugacy classes, more precisely in terms of those conjugacy classes corresponding to permutations with no cycles of length 1. The triviality of the operator $tr_n(\sigma \Phi)$ where $\sigma \in S_n$ and contains a cycle (or cycles) of length 1 follows from the $SU(N)$ condition that $tr(\Phi) = 0$.

When $n$ is larger than or comparable to $N$ the exact characterization of the invariants becomes more intricate. We consider the Schur polynomial basis. For small $n$ we imposed identities for each conjugacy class with one or more cycles of length 1. When $n \geq N$ we also have to impose the trace relations, eg., for $n = N + 1$ we have the relation $tr(\Phi^{N+1}) = tr(\Phi)tr(\Phi^N) + \cdots$ and similarly for larger $n$. Once we have guaranteed the vanishing of conjugacy classes $[n_1, n_2 \cdots, 1 \cdots]$ with $n_1 \geq n_2 \geq \cdots$ with $n$’s reaching and including $N$, we don’t need to impose extra conditions. For example, the vanishing of $[N+1, 1]$ is guaranteed by the vanishing of $[N, 1, 1]$ and others, because $[N+1]$ can be expressed in terms of $[N, 1]$ and other terms all involving no traces with powers of $\Phi$ greater than $N$.

A complete set of observables could alternatively be expressed in terms of traces and their products. Just take polynomials in $tr(\Phi^2), tr(\Phi^3) \cdots tr(\Phi^N)$. These correspond to conjugacy classes with no entry of length 1 or any entry of length greater than $N$.

10.2. Correlators
We now discuss the extremal correlators for the gauge group $SU(N)$. The story here is similar to the one developed for the gauge group $U(N)$ in [1] except that there are now further $1/N$ corrections arising from the modified two-point function, i.e., for $SU(N)$ one has

$$\langle \Phi^i_j(x)(\Phi^{\dagger})^k_l(y) \rangle = \frac{1}{(x-y)^2} (\delta_i^j \delta_k^l - \frac{1}{N} \delta_i^j \delta_k^l). \quad (10.1)$$

To evaluate the correlators of interest here we make use of the multi-point correlator given diagrammatically in fig. 28.

---

1 We drop the position dependencies in writing this formula. For the extremal correlators all $\Phi^\dagger$ (or $\Phi$) operators will be evaluated at the same spacetime point, so there is no loss in generality in dropping all position dependencies for such correlators as it will only contribute an overall factor and can be reinstated trivially.
Translating this into an explicit formula:

\[
\langle \Phi^J_{(n)} (\Phi^\dagger)^K_{(n)} \rangle = \sum_{\gamma \in S_n} \prod_{p=1}^{n} (\delta_{\gamma(p)}^{j(p)} \delta_{\gamma(p)}^{k(p)}) - \frac{1}{N} \sum_{(p)} (\delta_{\gamma(p)}^{j(p)} \delta_{\gamma(p)}^{k(p)})
\]

\[
= \frac{1}{n!} \sum_{F=0}^{n} (-1)^F \binom{n}{F} \sum_{\gamma_1, \gamma_2 \in S_n} \delta_{\gamma_1(n-F+1)}^{j_1} \delta_{\gamma_1(n-F+1)}^{k_1} \cdots \delta_{\gamma_1(n-F+1)}^{j_{\gamma_1}} \delta_{\gamma_1(n-F+1)}^{k_{\gamma_1}} 
\times \delta_{\gamma_2(n-F+1)}^{j_2} \delta_{\gamma_2(n-F+1)}^{k_2} \cdots \delta_{\gamma_2(n-F+1)}^{j_{\gamma_2}} \delta_{\gamma_2(n-F+1)}^{k_{\gamma_2}} 
\times \delta_{\gamma_1}^{i_1} \delta_{\gamma_2}^{i_2} \cdots \delta_{\gamma_1}^{i_{\gamma_1}} \delta_{\gamma_2}^{i_{\gamma_2}}
\]

(10.2)

The first line of (10.2) is obtained by carrying out all contractions using the two-point function (IO.1). The second line is obtained using the following reasoning. We want to consider all terms of order \((1/N)^F\). Consider in particular the term \(\delta_{l_1}^{i_1} \delta_{l_2}^{i_2} \cdots \delta_{l_{\gamma_1}}^{i_{\gamma_1}} \delta_{l_1}^{j_1} \delta_{l_2}^{j_2} \cdots \delta_{l_{\gamma_2}}^{j_{\gamma_2}}\) which appears at this order. It is easy to see that all other terms arising at this order can be generated from this term by simply permuting the subscripts 1, ..., \(n\), i.e., by replacing \(i \rightarrow \gamma_1(i)\) and summing over all permutations \(\gamma_1 \in S_n\). However, this overcounts by the factor \(F!(n - F)!\), the number of elements in the stabilizer subgroup \(S_{n-F} \times S_F \in S_n\) that preserve a given product of Kronecker delta-functions, like the one given above. Dividing out by this factor we arrive at the second line. The third line then follows by redefining the summation index \(\gamma_2 = \gamma_1\).

To compute the correlators of interest we project the correlator (10.2) onto the appropriate basis. For example, for a correlator such as

\[
\langle \chi_{R_1} (\Phi) \cdots \chi_{R_k} (\Phi) \chi_{S_1} (\Phi^\dagger) \cdots \chi_{S_l} (\Phi^\dagger) \rangle
\]

(10.3)

we project (10.2) onto the Schur polynomial basis by contracting the free indices of (10.2) with projection operators of the form given in (3.7) for each operator \(\chi_{R_i} (\Phi)\) and \(\chi_{S_j} (\Phi^\dagger)\) appearing in the correlator.

After contracting the \(\Phi\)'s using fig. 28 we get fig. 29 which can be rearranged to give fig. 30. In terms of formulae we find

\[
\langle \chi_{R_1} (\Phi) \cdots \chi_{R_k} (\Phi) \chi_{S_1} (\Phi^\dagger) \cdots \chi_{S_l} (\Phi^\dagger) \rangle = \frac{1}{d_{R_1} \cdots d_{R_k} d_{S_1} \cdots d_{S_l}} \sum_{F=0}^{n} \sum_{\gamma_1, \gamma_2 \in S_n} \left(-\frac{1}{N}\right)^F
\times \binom{n}{F} \text{tr} \left( (1_F \otimes (\gamma_1^{-1}(P_{R_1} \otimes \cdots \otimes P_{R_k}) \gamma_1)) (\gamma_2^{-1}(P_{S_1} \otimes \cdots \otimes P_{S_l}) \gamma_2) \otimes 1_F \right).
\]

(10.4)
Fig. 29: Projection diagram arising in the $SU(N)$ multi-point correlator.

This can be simplified using the decomposition (3.18) derived earlier in the $U(N)$ case. That is, the conjugated tensor product of projectors is replaced by a sum of projectors onto irreducible representations to arrive at

$$\langle \chi_{R_1}(\Phi) \cdot \cdot \cdot \chi_{R_k}(\Phi) \chi_{S_1}(\Phi^\dagger) \cdot \cdot \cdot \chi_{S_l}(\Phi^\dagger) \rangle = n! \sum_{F=0}^{n} \left(-\frac{1}{N}\right)^F \binom{n}{F} \times \sum_{T_1; T_2} \frac{g(R_1, \cdot \cdot \cdot , R_k; T_1)}{d_{T_1}} \frac{g(S_1, \cdot \cdot \cdot , S_l; T_2)}{d_{T_2}} \text{tr}((1_F \otimes P_{T_1})(P_{T_2} \otimes 1_F)).$$

Diagrammatically we have simply replaced the boxes corresponding to the tensor product of projection operators by projection operators onto irreducible representations. This
Fig. 30: Rearrangement of the diagram fig. 29 illustrating the multi-point $SU(N)$ correlator.

diagram is in fact just fig. [13] in section 6. The factorization argument there applies word for word here and yields

$$tr((1_F \otimes P_{T_1})(P_{T_2} \otimes 1_F)) = \sum_T \frac{1}{d_T DimT} tr ((1_F \otimes P_T)P_{T_1}) tr ((1_F \otimes P_T)P_{T_2}). \quad (10.6)$$

Finally the traces on the right-hand-side are evaluated as in (6.6) to arrive at the final
expression

\[
\langle \chi_{R_1}(\Phi) \cdots \chi_{R_k}(\Phi) \chi_{S_1}(\Phi^\dag) \cdots \chi_{S_l}(\Phi^\dag) \rangle = n! \sum_{F=0}^{n} \left( \frac{-1}{N} \right)^F \sum_{T_1, T_2} \sum_{U_1, U_2} \sum_{T} \frac{d_{U_1} d_{U_2} d_T}{DimT} \\
\times \frac{DimT_1 DimT_2}{d_{T_1} d_{T_2}} g(U_1; T; T_1) g(U_2; T; T_2) g(R_1, \cdots, R_n; T_1) g(S_1, \cdots, S_m; T_2).
\]

(10.7)

Because of the SU(N) condition \(\text{tr}(\Phi) = 0\), the correlator (10.7) must satisfy some non-trivial constraints. We can trade in a character in the above correlator for a trace by recalling the identity

\[
\sum_R \chi_R(\sigma) \chi_R(\tau) = \sum_{\gamma \in S_n} \delta(\sigma^{-1} \gamma \tau \gamma^{-1}).
\]

(10.8)

Multiplying (10.7) by \(\sum_R \chi_R(\tau)\), for example, for some fixed \(i\), will replace \(\chi_{R_i}(\Phi)\) by \(\sum_\gamma \text{tr}(\gamma^{-1} \tau \gamma \Phi)\) after applying (10.8). If \(\tau\) has a cycle of length 1 then the correlator must vanish. Consider the case that \(k = 1\) and multiply the correlator by \(\sum_{R_1} d_{R_1}\) corresponding to \(\tau\) being the identity permutation. On the right-hand-side of (10.7) the fusion coefficient \(g(R_1; T_1) = \delta_{R_1, T_1}\) so that the sum over \(T_1\) is trivial and moreover the sum over \(R_1\) just involves the terms

\[
\sum_{R_1} Dim R_1 g(U_1; T; R_1) = DimU_1 \text{ Dim} T.
\]

(10.9)

The sum over \(U_1\) now just becomes \(\sum U_1 d_{U_1} \text{Dim} U_1 = N^F\) and the sums over \(U_2\) and \(T\) become \(\sum U_2, T d_{U_2} d_T g(U_2; T; T_2) = d_{T_2}\). This implies that the \(T_2\) sum can be done producing \(\sum_{T_2} g(S_1, \cdots, S_l; T_2) \text{Dim} T_2 = \text{Dim} S_1 \cdots \text{Dim} S_l\). The only \(F\)-dependent term produced in all these sums was \(N^F\), cancelling the \(1/N^F\) in (10.7), yielding for the \(F\) sum

\[
\sum_{F=0}^{n} \binom{n}{F} (-1)^F = 0.
\]

(10.10)

More generally this correlator must vanish for any permutation \(\tau\) containing a 1 cycle and also for \(k \geq 1\). This condition gives rise to some non-trivial constraints on the correlator as evidenced by the above argument for the simplest case of \(\tau = e\) and \(k = 1\).
10.3. Factorization and fusion identities for $SU(N)$

The general arguments for factorization given in section 9 suggests that the $SU(N)$ correlators should satisfy factorization equations similar to those satisfied by the $U(N)$ correlators. A natural guess for the four-point function evaluated in the previous subsection would be

$$\langle \chi_{R_1}(\Phi)\chi_{R_2}(\Phi)\chi_{S_1}(\Phi^\dagger)\chi_{S_2}(\Phi^\dagger) \rangle = \sum_{T_1,T_2} \langle \chi_{R_1}(\Phi)\chi_{R_2}(\Phi)T_1(\Phi^\dagger) \rangle G_{T_1,T_2} \langle T_2(\Phi)\chi_{S_1}(\Phi^\dagger)\chi_{S_2}(\Phi^\dagger) \rangle. \ (10.11)$$

The operators $T_1(\Phi)$ and $T_2(\Phi)$ are either polynomials in $tr(\Phi^2),...,tr(\Phi^N)$ containing exactly $n(R_1)+n(R_2)=n(S_1)+n(S_2)$ $\Phi$’s each, or equivalently can be expressed in terms of Schur polynomials subject to the constraints discussed earlier in the previous subsection. $G_{T_1,T_2}$ denotes the inverse of the two-point function $\langle T_1(\Phi)T_2(\Phi^\dagger) \rangle$ (recall that this is not diagonal in the Schur basis used in the previous subsection) and the sum over $T_1,T_2$ runs over a complete basis of such operators. Using (10.7) this factorization equation can be written in terms of products of fusion coefficients. We can therefore get some combinatoric identities which should be explicitly testable.

11. Remarks on Giant gravitons

11.1. A symmetry between giant gravitons and KK Modes

Young diagrams with a small number $n \ll N$ of boxes give rise to operators with a small number of $\Phi$’s which are related to KK modes in the $AdS_5 \times S^5$ dual, \[37\] \[38\] \[39\]. On the other hand Young diagrams which have a few very long columns, of length close to $N$, are all giant graviton states \[40\] \[1\]. Recall that the three-point function depends on the fusion coefficient \[1\] as in

$$\langle \chi_{R_1}(\Phi)(x)\chi_{R_2}(\Phi)(x)\chi_{S}(\Phi^\dagger)(0) \rangle = x^{-2n_1-2n_2}g(R_1,R_2,S)\frac{n!DimS}{d_S} \ (11.1)$$

and appropriate ratios give exactly the fusion coefficient

$$\frac{\langle \chi_{R_1}(\Phi)(x)\chi_{R_2}(\Phi)(x)\chi_{S}(\Phi^\dagger)(0) \rangle}{\langle \chi_{S}(\Phi(x))\chi_{S}(\Phi^\dagger(0)) \rangle} = g(R_1,R_2,S). \ (11.2)$$

When $R_1$ and $R_2$ are small compared to $N$, $S$ must also be small. This just follows from the fact that $n(S) = n(R_1) + n(R_2)$. 

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When $R_1$ and $R_2$ are both in the region of sphere giants, that is, they have a few columns of length comparable to $N$, the selection rule $n(S) = n(R_1) + n(R_2)$ would in principle allow $S$ to be a few rows of length comparable to $N$, for example, which by \[\text{[1]}\] are related to $ADS$ giants \[\text{[11]} \text{[12]} \text{[13]}\]. The actual three point functions \[\text{(11.1)}\] for such $R_1$, $R_2$, and $S$ are zero because of the properties of $g(R_1,R_2;S)$. If $R_1$ and $R_2$ are in the sphere giant region specified above, their complex conjugates $\bar{R}_1$ and $\bar{R}_2$ are small Young Diagrams. These fuse only into small Young Diagrams $\bar{S}$. The fusion coefficients satisfy the duality property
\[
g(R_1,R_2;S) = g(\bar{R}_1,\bar{R}_2;\bar{S}). \tag{11.3}\]
The conjugate of a small $\bar{S}$ is again in the region of sphere giants. This shows that the fusion rules for KK modes are identical to those for giant graviton modes. It would be very interesting to understand this symmetry from the point of view of three-brane world volumes coupled to gravity.

11.2. Saturating the factorization equation : Sphere giants

Another consequence of this relation between fusions of sphere giants and KK modes is that when we consider an extremal four point function involving only sphere giants, the intermediate states which enter the factorization equation \[\text{(2.4)}\] are again sphere giants.

Indeed in \[\text{(2.4)}\] if $R_1, R_2, S_1, S_2$ are all large antisymmetric reps., then a representation $S$ will only contribute to the factorization sum if $g(R_1,R_2;S)g(S_1,S_2:S)$ is non-zero. Now if $R_1, R_2, S_1, S_2$ are all large anti-symmetric reps then their conjugates $\bar{R}_1, \bar{R}_2, \bar{S}_1, \bar{S}_2$ are all small representations. For these it is clear that $g(\bar{R}_1,\bar{R}_2;\bar{S})g(\bar{S}_1,\bar{S}_2;\bar{S}) \neq 0$ requires that $\bar{S}$ be a small rep. This means that $S$ is a large antisymmetric rep. This proves the claim that if $R_i, S_i$ are related to giant gravitons, then the factorization equations saturate on sphere giants.

Since sphere giants are related to moduli spaces of holomorphic maps \[\text{[44]} \text{[45]}\], this means that the factorization equations are saturated by operators which are related to these moduli spaces. We expect the observables of a topological theory formulated on these moduli spaces to obey factorization equations which would descend from the basic factorization equation of $N = 4$ SYM.

We expect some analogous results should hold true for large symmetric irreps, i.e., Young diagrams with a few rows of length order $N$, related to $AdS$ giants according to \[\text{[1]}\]. The arguments presented above for the sphere giants do not immediately generalize to
this case as the conjugate of a large symmetric irrep consists of a large number of boxes. Nevertheless the Littlewood-Richardson rule tells us which irreps occur when fusing a pair of large symmetric irreps. One finds that the allowed Young diagrams again contain only a few rows, at least one of which will have order $N$ boxes, but some rows containing a small number of boxes are also allowed. Such a Young diagram corresponds to a perturbation of an $AdS$ giant. Hence, it follows that the factorization equations for $AdS$ giants also saturates on $AdS$ giants and perturbations thereof and do not contain, for example, sphere giants. It seems to be a general property, true for KK modes, AdS giants and sphere giants, that the saturation of factorization equations on a small set of operators in the large $N$ theory is related to the existence of semiclassical objects admitting some form of perturbative $1/N$ analysis.

12. Summary and Outlook

We started with the general extremal correlators derived in [1]. It was shown there that there is a one-one correspondence between, on the one hand, operators in $U(N)$ maximally supersymmetric gauge theory in four dimensions transforming in half-BPS representations of the superalgebra and, on the other hand, irreducible representations of $U(N)$. Operators belonging to half-BPS representations of the superalgebra containing a highest weight state of $R$-charge $2n$ are mapped to representations of $U(N)$ built from $n$ copies of the fundamental. The extremal correlators were expressed in terms of $U(N)$ and $S_n$ group theoretic data such as dimensions of representations and fusion or branching coefficients.

In this paper we generalized our results and showed how to relate more general correlators, including non-extremal ones, to more general $U(N)$ group theoretic data. Recalling that fusion coefficients of $U(N)$ can be expressed in terms of an integral of a product of characters, the more general group theoretic data can be viewed as more general group integrals involving, generically, many $U(N)$ group variables. Such integrals occur for example in lattice gauge theory or continuum two dimensional Yang-Mills theory. The group theoretic quantities can also be described in terms of sequences of projectors acting on tensor spaces or on irreducible representations of $U(N)$. These sequences are conveniently described in terms of diagrams we called projector diagrams. By using the expression of projectors as products of Clebsch-Gordan coefficients, the projector diagrams can be converted to projector graphs where the vertices of the graphs represent Clebsch-Gordan coefficients, and the projector graph itself is a quantity analogous to $6J$ symbols.
It was found that extremal correlators and some generalizations thereof are associated with simple graphs. When the graphs are simple, the correlator can be reduced entirely to fusion coefficients and dimensions. The simple form of the extremal correlators lead to some factorization equations and fusion identities which were described in section 2. The relation to projectors leads to sum rules. These sum rules can quite generally be used to simplify any projector diagram. The factorization and fusion identities, on the other hand, only work for certain classes of diagrams.

The simple group theoretic form of the extremal correlators, and in particular the factorization and fusion identities, suggest relations to topological gauge theories. We described concretely such relations in section 7. The topological gauge theories we discussed were three dimensional $U(N)$ Chern-Simons at large $k$ and two dimensional $U(N)$ Yang-Mills at zero area. A corollary of the Chern-Simons connection is that there are relations to $G/G$ models in two dimensions.

Beyond the case of extremal correlators, since we still have the projector graphs associated to the correlator, we can use that to systematically map to Wilson Loops in Chern Simons. The projector graph, which begins as a device to summarize a group theoretic quantity, emerges as a, possibly intersecting, Wilson loop for Chern Simons on $S^3$. There is also a connection to intersecting Wilson loops in two-dimensional Yang-Mills. The basics of these relations were described in Section 7. The basic factorization equation was found to have a generalization which we called staggered factorization. Section 8 described the physics of this generalization and shows that, under the maps to topological gauge theories defined in section 7, the sum rules are related to the Verlinde formula, and staggered factorization is related to connected sums of Wilson loops in Chern-Simons and tangentially intersecting Wilson loops in two dimensional Yang-Mills. Section 8 includes some speculations on generalizations of known non-renormalization theorems beyond the case of extremal correlators, which are motivated by staggered factorization. Section 10 began a discussion of $SU(N)$ correlators. In section 11 we remarked that there is an intriguing symmetry between correlators of giant gravitons and correlators of KK modes in the $ADS_5 \times S^5$ dual to $N = 4$ SYM. We also suggested that the truncation of the factorization sums is related to the existence of semiclassical objects described by the correlators of the $N = 4$ SYM.

We now discuss some avenues for the future. Since we can write down the expressions for correlators in terms of Clebsch-Gordan coefficients, we can get a q-deformed version by converting the Clebsch-Gordan to q-Clebsch Gordans. These q-deformed formulae
will have similar factorization properties. We saw that the factorization equations were related to unitarity and supersymmetry. This leads one to suspect that there might be a deformation of SYM preserving at least some SUSY, which has correlators which are these $q$-deformations. It would be interesting to look for such a $q$-deformation, perhaps using recent results on deformations of $N = 4$ SYM [46][47]. It is worth noting that, in general, the $q$-deformation of the correlators is not unique. Whenever we have a tensor product of irreps of $U(N)$, we can project the product onto irreps by using $\Delta$ or $\Delta'$, two consistent ways for the quantum group to act on the tensor product space, related by a permutation. For all the factorizable correlators which can be reduced to fusion coefficients, there is no ambiguity, since fusion coefficients are independent of whether we choose $\Delta$ or $\Delta'$. For the cases where there is an ambiguity, one would hope that it can be interpreted physically, for example in terms of a choice of regulator in defining the deformed theory.

We expect these factorization equations to be true on more general manifolds than $R^4$. Indeed if we consider $R^3 \times S^1$, for instance, we can write down the two-point functions by summing over images. This will modify the two point function. But the form of the answers will remain the same. It might be interesting to make this explicit.

While the connections to Chern-Simons were established by directly analyzing the correlators, a natural question is to find a physical rather than technical proof of these connections. One approach might be to use field theory techniques, e.g., those of [48] relating 4D Donaldson to 3D Floer theory or [49] to relate 3D Chern-Simons to 2D $G/G$ models. Here we would like to relate the (quasi-) topological sector of a four dimensional gauge theory to a topological three dimensional theory. Correlation functions in a four dimensional theory are related to overlaps of wavefunctions which are functionals of three-dimensional fields living on boundaries of four dimensional manifolds. The wavefunctions are defined by path integrals on the four-manifold with boundary [48]. Making this wave-function approach explicit in the case of extremal correlators of $N = 4$ SYM may be expected to lead to relations between the four dimensional theory and three dimensional theories. Another approach to a physical proof of the connections exhibited in section 7 may be to use stringy dualities, perhaps along the lines of [50][51][52].

A puzzle raised by this work is to explain why there are relations between unitary group integrals and correlation functions of Higgs fields. We have shown using Schur Duality that general extremal correlation functions of half-BPS operators can be expressed in terms of simple unitary group integrals. Non-extremal correlators involve more complicated group integrals. It would be interesting to find a physical explanation for the
appearance of group integrals. Is there some way of mapping the action involving the Higgs scalars into an action involving gauge fields, where the appearance of $U(N)$ group integrals would be more direct, the way it happens in lattice gauge theory for example?

We have found quantities in $N = 4$ SYM which are related to $G/G$ field theories on two dimensional closed manifolds. The system of equations satisfied by the observables of $G/G$ have a generalization to the case where two-dimensional manifolds with boundary are included. This open-closed topological theory is related to $K$ theory $[53]$. It would be of interest to find observables related to $N = 4$ SYM and its $ADS \times S$ dual which would map to this open-closed topological set-up.

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13. Appendix 1 : Schur Duality and group integrals, $P_R = P_r$

Consider n-fold tensor space, $V^\otimes n$ where $V$ is the fundamental representation of $U(N)$. This space has an $S_n$ action permuting the vectors in different factors of the tensor product. The theorem described in $[3,13]$ has some powerful implications for relations between unitary and symmetric group actions on tensor space. These relations are useful at various points in this paper.

Consider a fixed Young Diagram. There is an irrep. $R$ of $U(N)$ and an irrep $r$ of $S_n$ associated with it. For an irrep $r$ consider the following element of the group algebra of $S_n$ :

$$P_r = d_r \frac{1}{n!} \sum_\sigma \chi_r (\sigma) \sigma$$

(13.1)

In this sum $\sigma$ is in $S_n$. Instead of $\chi_r (\sigma)$ we could equally well have written $\chi_r (\sigma^{-1})$ since a permutation and its inverse are in the same conjugacy class of $S_n$. Using character identities, reviewed for example in $[1]$, one can prove that

$$P_r P_s = \delta_{rs} P_r$$

(13.2)
This means that $P_r$ is a projector. One further checks that $tr(P_r) = d_r \ Dim R$. Similarly

$$ P_R = Dim R \int dU \chi_R(U^\dagger)U $$

(13.3)

is a projector for $U(N)$, using the standard normalization of the measure where

$$ \int dU \chi_R(U) \chi_S(U^\dagger) = \delta_{RS}. $$

Again one can check that $tr(P_R) = d_r \ Dim R$. Inserting either projector in tensor space gives zero on any state which is not in the $R \otimes r$ subspace and 1 on any state in $R \otimes r$. This is essentially the content of (3.13).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure31}
\caption{Basic $U(N)$ group integral}
\end{figure}

It is instructive to give a derivation of this in physics language by using $U(N)$ group integrals familiar from lattice gauge theory and two-dimensional Yang-Mills literature \cite{54,55,34}. The result we will need can be expressed in diagrammatic notation, as for example in \cite{56}. The basic unitary group integral is expressed diagrammatically in fig. 31 where a sum over the element $\gamma$ in $S_n$ is understood.
In this formula $\Omega^{-1}$ is an element of the group algebra of $S_n$ with coefficients which are functions of $N$ and is defined by the equations

$$
\begin{align*}
Dim R &= \frac{N^n}{n!} \chi_r(\Omega) \\
\Omega \Omega^{-1} &= 1 \\
\chi_r(\Omega^{-1}) &= \frac{d_r^2}{\chi_r(\Omega)}
\end{align*}
$$

(13.4)

It is a useful fact that $\Omega$ is actually in the center of the group algebra of $S_n$. Further details and uses of this object can be found in [34].

\[\begin{array}{c}
\int dU U U^\dagger
\end{array}\]

\[\begin{array}{c}
\gamma^{-1} \Omega^{-1} N^n
\end{array}\]

Fig. 32: Diagrammatic manipulation for projector relations

Consider $\rho_n(P_R)$, i.e the operator in $End(V^\otimes n)$ given by acting with $P_R$.

$$
\begin{align*}
\rho_n(P_R) &= DimR \int dU \chi_R(U) \rho_n(U^\dagger) \\
&= DimR \sum_{\sigma} \frac{\chi_r(\sigma)}{n!} \int dU \ tr_n(\sigma U) \rho_n(U^\dagger) \\
&= DimR \sum_{\sigma} \frac{\chi_r(\sigma)}{n!} \int dU \ (tr_n \otimes 1) \rho_n(\sigma U) \otimes \rho_n(U^\dagger) \\
&= DimR \sum_{\sigma} \frac{\chi_r(\sigma)}{n!} X
\end{align*}
$$

(13.5)
In the second line we used the expansion of $\chi_R(U)$ in terms of traces. In the last line we defined a quantity $X$, which is conveniently manipulated diagrammatically as illustrated in fig. 32.

The diagrammatic steps show that

$$X \equiv (\text{tr}_n \otimes 1) \int dU \, \rho_n(\sigma U) \otimes \rho_n(U^\dagger)$$

$$= N^{-n} n! \, \rho_n(\Omega^{-1}\sigma)$$

The $n!$ comes from doing the sum over $\gamma$ in $S_n$. Collecting terms in (13.5) we write

$$\rho_n(P_R) = \frac{\text{Dim } R}{N^n} \sum_{\sigma} \chi_r(\sigma^{-1}) \, \rho_n(\Omega^{-1}\sigma)$$

$$= \frac{\text{Dim } R}{N^n} \sum_{\sigma} \chi_r(\sigma^{-1}\Omega^{-1}) \, \rho_n(\sigma)$$

$$= \frac{\text{Dim } R}{d_R} \frac{1}{N^n} \chi_R(\Omega^{-1}) \sum_{\sigma} \chi_r(\sigma^{-1}) \rho_n(\sigma)$$

$$= \frac{d_R}{n!} \sum_{\sigma} \chi_r(\sigma) \, \rho_n(\sigma)$$

where we have used (13.4) to obtain the final answer.

The upshot is the simple equation

$$\rho_n(P_R) = \rho_n(P_r)$$

(13.8)

which also follows from (3.13). Taking advantage of this relation we easily convert from the unitary group form of the projector to the symmetric group form. We sometimes use a noncommittal notation $R$ for both projectors, and correspondingly $\chi_R$ for both symmetric group characters and unitary group characters or their extension to complex matrices. The result (13.8) holds for all values of $n$ and $N$. When $n \geq N$, and the Young Diagram considered is not an admissible one as an irrep of $U(N)$, i.e., it has columns of length larger than $N$, then $P_R$ is zero, and therefore $\rho_n(P_R)$ is zero. In that case $P_r$ is not zero, but $\rho_n(P_r)$ is still zero.

14. Appendix 2 : Schur Duality and group integrals – Fusion coefficients and Branching coefficients

The basic group integral in fig. 31 can also be used to derive another useful relation between unitary and symmetric groups. The Littlewood-Richardson coefficient $g(R_1, R_2; R_3)$
is the fusion coefficient for $U(N)$, i.e. the number of times the representation corresponding to the Young Diagram $R_3$ appears in the tensor product $R_1 \otimes R_2$. It is also the number of times the representation $R_1 \otimes R_2$ of $S_{n_1} \times S_{n_2}$ appears when the representation $R_3$ of $S_n$ is decomposed into irreps of the subgroup $S_{n_1} \times S_{n_2}$, i.e., it is a branching coefficient. Here $R_3$ has $n$ boxes and $R_1, R_2$ have $n_1, n_2$ respectively with $n = n_1 + n_2$. These facts, proved for example in [22], were used in [11] to derive the relation between three-point functions and $g(R_1, R_2; R_3)$. In terms of characters, the above facts are stated as

$$g(R_1, R_2; R_3) = \int dU \chi_{R_1}(U) \chi_{R_2}(U) \chi_{R_3}(U^\dagger)$$

$$= \sum_{\sigma_1, \sigma_2} \chi_{R_3}(\sigma_1, \sigma_2) \frac{\chi_{R_1}(\sigma_1)}{n_1!} \frac{\chi_{R_2}(\sigma_2)}{n_2!} \frac{\chi_{R_3}(\sigma_3)}{n_3!} tr_{n_1+n_2}(\sigma_1 \circ \sigma_2 U) tr_n(\sigma_3 U^\dagger)$$

$$= \sum_{\sigma_1, \sigma_2} \chi_{R_3}(\sigma_1, \sigma_2) \frac{\chi_{R_1}(\sigma_1)}{n_1!} \frac{\chi_{R_2}(\sigma_2)}{n_2!} \frac{1}{d_{R_3}} tr_n(\sigma_1 \circ \sigma_2 U) tr_n(P_{R_3} U^\dagger)$$

$$= \sum_{\sigma_1, \sigma_2} \chi_{R_3}(\sigma_1, \sigma_2) \frac{\chi_{R_1}(\sigma_1)}{n_1!} \frac{\chi_{R_2}(\sigma_2)}{n_2!} \frac{1}{d_{R_3}} Y.$$  \hspace{1cm} (14.1)

The last line is a definition of $Y$, an object which we will write and manipulate diagrammatically using the rules in section 3.

The diagrammatic manipulations show

$$Y = n! N^{-n} tr_n(P_{R_3}(\sigma_1 \circ \sigma_2) \Omega^{-1})$$

$$= n! N^{-n} Dim R_3 \chi_{R_3}(\sigma_1 \circ \sigma_2) \frac{1}{d_{R_3}} \chi_{R_3}(\Omega^{-1})$$

$$= d_{R_3} \chi_{R_3}(\sigma_1 \circ \sigma_2)$$  \hspace{1cm} (14.2)

where in the diagram we have written $R_3$ for the projector $P_{R_3}$ and have used the fact that $\gamma$ commutes with action of $P_{R_3}$ to perform the sum over $\gamma$ and obtain the factor of $n!$. Since the irrep $R_3$ of $S_n$ occurs in tensor space with a multiplicity $Dim R_3$, using (3.13), we have further

$$Y = n! N^{-n} Dim R_3 \chi_{R_3}(\sigma_1 \circ \sigma_2) \Omega^{-1})$$

$$= n! N^{-n} Dim R_3 \chi_{R_3}(\sigma_1 \circ \sigma_2) \frac{1}{d_{R_3}} \chi_{R_3}(\Omega^{-1})$$

$$= d_{R_3} \chi_{R_3}(\sigma_1 \circ \sigma_2)$$  \hspace{1cm} (14.3)

where we have used (13.4). Combining this expression for $Y$ with (14.2) we have the proof that the integral in (14.1) is the same as the sum in (14.1), so that $g$ is indeed both a fusion coefficient for unitary groups and a branching coefficient for Symmetric groups.
15. Appendix 3: Derivation of Staggered Factorization

In this appendix we discuss the factorization of a diagram of the form shown in fig. 34. The dots in the figure denote that the diagram can be as complicated as one likes to the left and right of the strand shown, but that there is one strand that flows through two
operators denoted $T_1$ and $T_2$ and is then contracted with itself, and moreover there are no free strands anywhere else in the diagram. In terms of a formula the operators $T_1$ and $T_2$ are expressed in terms of unitary integrals as

$$F_{T_1} := \int dU_1 G_{T_1}(U_1, U_1^\dagger) \rho_n(U_1^\dagger)$$

$$F_{T_2} := \int dU_2 G_{T_2}(U_2, U_2^\dagger) \rho_n(U_2^\dagger).$$

(15.1)

The diagram fig. 34 then corresponds to the expression

$$tr(F_{T_1} F_{T_2}) = \int dU_1 dU_2 G_{T_1}(U_1, U_1^\dagger) G_{T_2}(U_2, U_2^\dagger) (tr(U_1^\dagger U_2^\dagger))^n$$

(15.2)

where the number of lines in the strand is taken to be $n$. In expressing the diagram in terms of $U(N)$ group integrals with $U_{1(2)}$ corresponding to the $T_{1(2)}$ operator insertion, the factors $G_{T_1}$ and $G_{T_2}$ contain all the other dependence of the graph from operator insertions and from tracing over other strands on the left and right respectively. The only constraint is that both factors are $U(N)$ invariant.

![Diagram](image)

**Fig. 35:** Diagram illustrating that the projector $P_R$ commutes with the operator $T_2$.

To factorize the diagram we insert the unit operator $I = \sum_R P_R$ along the strand between the $T_1$ and $T_2$ operators, i.e., we write

$$tr(F_{T_1} F_{T_2}) = \sum_R tr(P_R F_{T_1} F_{T_2}).$$

(15.3)

We now claim that the projection operator $P_R$ commutes with $F_{T_1}$ and $F_{T_2}$. This is expressed diagrammatically in fig. 35 for $T_2$. 

58
\[ \sum_R \frac{1}{d_R \dim R} \]

\[ \begin{array}{c}
\vdots \\
R \\
\end{array} \quad \begin{array}{c}
T_1 \\
R \\
\end{array} \quad \begin{array}{c}
T_2 \\
\cdots \\
R \\
\end{array} \]

Fig. 36: The diagram corresponding to the factorized form of fig. 34.

This statement follows since the symmetric group and unitary group actions on the vector space \( V^\otimes n \) commute, and \( P_R \) is a sum of symmetric group elements acting on \( V^\otimes n \) while \( F_{T_1} \) and \( F_{T_2} \) have been expressed in terms of unitary group integrals. Because \( P_R \) projects onto a single irreducible representation of the product group \( U(N) \times S_n \), we can now apply Schur’s lemma to factor the trace appearing on the right-hand-side of (15.3),

\[ \text{tr}(P_R F_{T_1} F_{T_2}) = \frac{1}{d_R \dim R} \text{tr}(P_R F_{T_1}) \text{tr}(P_R F_{T_2}), \] (15.4)

where the factor of \( d_R \dim R \) arises from the fact that \( \text{tr}(P_R) = d_R \dim R \). The final factorization equation is then simply

\[ \text{tr}(F_{T_1} F_{T_2}) = \sum_R \frac{1}{d_R \dim R} \text{tr}(P_R F_{T_1}) \text{tr}(P_R F_{T_2}). \] (15.5)

The diagrammatic interpretation is that the original diagram in fig. 34 has been split into a sum of products of two diagrams as illustrated in fig. 36.

Alternatively, one may expand the factor of \( \text{tr}(U_1^\dagger U_2^\dagger)^n \) appearing in the unitary integral (15.2) in terms of unitary group characters to obtain

\[
\sum_R d_R \int dU_1 dU_2 \chi_R(U_1^\dagger U_2^\dagger) G_{T_1}(U_1, U_1^\dagger) G_{T_2}(U_2, U_2^\dagger) \\
= \sum_R \frac{d_R}{\dim R} (\int dU_1 \chi_R(U_1^\dagger) G_{T_1}(U_1, U_1^\dagger))(\int dU_2 \chi_R(U_2^\dagger) G_{T_2}(U_2, U_2^\dagger)) \\
= \sum_R \frac{1}{d_R \dim R} \text{tr}(P_R F_{T_1}) \text{tr}(P_R F_{T_2}).
\] (15.6)

In the second line we have used the fact that \( F_{T_1} \) and \( F_{T_2} \) commute with the action of the unitary group. To see this let, for example, \( F_{T_1} \) act on \( \rho_n(U) \). By redefining the integration variable \( U_1 \to UU_1 U^\dagger \), the commutativity property follows since the measure and \( G_{T_1} \) are both invariant. Taking the unitary group operator \( F_{T_2} \) inside the character \( \chi_R \) as \( \chi_R(F_{T_2} U_2^\dagger) \) and applying Schur’s lemma now to the group \( U(N) \) yields the second line in (15.6). This time the factor of \( \dim R \) arises because a single copy of the \( U(N) \) irrep \( R \) has this dimension. Finally the last line follows by rewriting the integrals as traces in \( V^\otimes n \).
Here we present the diagrammatic derivation of the non-extremal correlator of Schur polynomials considered in section 5. Let $E_{21}$ be a generator of $SO(6)$ which converts $\Phi_1$ to $\Phi_2$ and let $Q_{12}$ be a generator which converts $\Phi_1$ to $\Phi_2^\dagger$, see appendix 5 for explicit formulae for Lie algebra elements. Let us consider the correlator

$$\langle E_{21}^k \chi_{R_1}(\Phi_1) Q_{12}^k \chi_{R_2}(\Phi_1) \chi_{S_1}(\Phi_1^\dagger) \rangle.$$ (16.1)

$R_1$ is an irrep associated with a Young Diagram with $(n_1 + k)$ boxes, $R_2$ an irrep associated with a Young Diagram having $(n_2 + k)$ boxes, and $S_1$ an irrep associated with a Young diagram with $(n_1 + n_2)$ boxes. After acting with the $k$ powers of $E_{21}$ and $Q_{12}$, we get $k$ copies of $\Phi_2$ from the first set of $\Phi_1$’s and $k$ copies of $\Phi_2^\dagger$ from the second set. Irrespective of the position of the $\Phi_2$ and $\Phi_2^\dagger$, the correlator to be computed can be put in the form shown in the diagram below. There is a combinatoric factor of $\frac{(n_1+k)!}{n_1!} \frac{(n_2+k)!}{n_2!}$ which has to multiply this diagram.

![Diagram of correlator](image)

**Fig. 37:** Diagram of correlator

Now we evaluate the correlator using fig. 3. The $\Phi_2$ contractions involve a permutation $\gamma_1$ in $S_k$. The $\Phi_1$ contractions involve a permutation $\gamma_2$ in $S_{n_1+n_2}$.
Now inspection of this diagram shows that it can be simplified to a three strand diagram of the form given in fig. 3.

17. Appendix 5: \textit{SO}(6) algebra

We need to describe with some care the \textit{SO}(6) algebra in order to show that the operators satisfying simple identities are indeed directly related to half-BPS operators.

Take the standard action of \textit{SO}(6) on \(x_1, \cdots x_6\). Let us form combinations

\begin{align*}
z_1 &= x_1 + ix_4 \\
z_2 &= x_2 + ix_5 \\
z_3 &= x_3 + ix_6
\end{align*}

(17.1)

The Cartan subalgebra is spanned by

\begin{align*}
H_1 &= z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \\
H_2 &= z_2 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \\
H_3 &= z_3 \frac{\partial}{\partial z_3} - \bar{z}_3 \frac{\partial}{\partial \bar{z}_3}
\end{align*}

(17.2)
Additional generators of the $SO(6)$ Lie algebra are, for $i \neq j$ running from 1 to 3:

$$E_{ij} = z_i \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_i}$$ (17.3)

We also take, for $i < j$,

$$P_{ij} = z_i \frac{\partial}{\partial \bar{z}_j} - \bar{z}_j \frac{\partial}{\partial z_i}$$

$$Q_{ij} = -\bar{z}_i \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial z_i}$$ (17.4)

It is easy to check that the above operators preserve $z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3$.

A monomial $z^n_1$ has weights $H_i(z^n_1) = (n, 0, 0)$. We can convert $z_1$'s to $\bar{z}_3$'s by acting with $Q_{13}$ to obtain

$$Q_{13}(z_1) = \bar{z}_3$$
$$Q_{13}(\bar{z}_3) = 0$$
$$Q_{13}^k(z_1^n) = \frac{n!}{(n-k)!} \bar{z}_3^k z_1^{n-k}. \quad (17.5)$$

Similarly we can convert $z_1^n$ to a combination of $z_2$ and $z_1$ by observing:

$$E_{21}(z_1) = z_2$$
$$E_{21}(\bar{z}_2) = 0$$
$$E_{21}^k(z_1^n) = \frac{n!}{(n-k)!} \bar{z}_2^k z_1^{n-k}.$$. (17.6)

Combining these operations and noting that $E_{21}$ and $Q_{13}$ commute, then we find that

$$Q_{13}^k E_{21}^l(z_1^n) = \frac{n!}{(n-k_1-k_2)!} \bar{z}_3^k z_2^{k_2} z_1^{n-k_1-k_2},$$ (17.7)

a fact which is used in section 8.

18. Appendix 6: Extremal Correlators of traces - A Change of Basis

In this appendix we consider various extremal correlators of traces of operators (as opposed to correlators of operators evaluated in the Schur basis that we have been using). As this just involves a change of basis we can use the previous results of [1] (which were recalled in section 3) for the extremal correlators of operators evaluated in the Schur basis. Since traces and multi-traces can be used to give an alternative basis to the Schurs in the space of gauge invariant polynomials built from $\Phi$. Our results on factorization and
fusion equations and sum rules can be restated in this alternative basis. Correlators of
single traces have been of interest recently in regard to the pp-wave limit of the $AdS/CFT$
correspondence. The two- and three-point correlators of traces have also been evaluated
exactly in the recent papers of [18,20].

As a first example we consider the two-point function $\langle Tr(c_1 \Phi)Tr(c_2 \Phi^\dagger)\rangle$. Using the
identity
\begin{equation}
\sum_R \chi_R(\tau)\chi_R(\sigma) = \sum_\gamma \delta(\sigma^{-1}\gamma\tau\gamma^{-1}) \quad (18.1)
\end{equation}
one may rewrite this correlator in terms of the Schur basis as
\begin{equation}
\langle Tr(c_1 \Phi)Tr(c_2 \Phi^\dagger)\rangle = \sum_{R,S} \chi_R(c_1)\chi_S(c_2)\langle \chi_R(\Phi)\chi_S(\Phi^\dagger)\rangle. \quad (18.2)
\end{equation}

Plugging in the previous result for the two-point function (3.10) yields the expression
\begin{equation}
\langle Tr(c_1 \Phi)Tr(c_2 \Phi^\dagger)\rangle = \sum_R \frac{n!(DimNR)}{d_R} \chi_R(c_1)\chi_R(c_2). \quad (18.3)
\end{equation}

This can now be generalized in the obvious way to higher point correlators of traces
of the form $\langle (\prod_{k=1}^K Tr(c_k \Phi))(\prod_{l=1}^L Tr(d_l \Phi^\dagger))\rangle$ where now the cycles $c_k$ are of length $n_k$
and the cycles $d_l$ are of length $m_l$ where $n = n_1 + \cdots + n_K = m_1 + \cdots + m_L$. Specifically
applying the identity (18.1) one finds
\begin{equation}
\langle (\prod_{k=1}^K Tr(c_k \Phi))(\prod_{l=1}^L Tr(d_l \Phi^\dagger))\rangle = \langle Tr((\prod_{k=1}^K c_k \Phi))Tr((\prod_{l=1}^L d_l \Phi^\dagger))\rangle
= \frac{1}{(n!)^2} \sum_{\gamma,\rho \in S_n} \langle Tr((\gamma^{-1}(c_1 \cdots c_K)\gamma)\Phi)Tr(\rho^{-1}(d_1 \cdots d_L)\rho\Phi^\dagger)\rangle
= \sum_R \frac{n!(DimNR)}{d_R} \chi_R(c_1 \cdots c_K)\chi_R(d_1 \cdots d_L). \quad (18.4)
\end{equation}

analogously to the two-point function.

We have so far dropped the position dependence of the scalars. Providing that all $\Phi$
(or $\Phi^\dagger$) operators are evaluated at the same position, then it is a trivial matter to put the
coordinate dependence back in, resulting only in an overall coordinate dependent factor. If
however the correlator has a more general coordinate dependence, then the correlators will
no longer be extremal and we will get more complicated projector diagrams as in section 5.

The sums in the two-point function (18.3) and the three-point function following from (18.4) can actually be done explicitly. To see this we need a few facts from the theory of symmetric groups, see eg. [22]. The first fact is that the character \( \chi_R(c_1) \) for \( c_1 \in S_n \) an \( n \)-cycle is simply \((-1)^s\) for a representation \( R \) corresponding to a Young diagram with partition \((n-s, 1, \ldots, 1)\) and zero otherwise. We will refer to such Young diagrams as “hooks”. Combined with the branching formula (see also appendix 2 where this is related to unitary group characters)

\[
\chi_R(c_1 \circ c_2) = \sum_{R_1, R_2} g(R_1, R_2; R)\chi_{R_1}(c_1)\chi_{R_2}(c_2)
\]  

we can also compute the character \( \chi_R(c_1 \circ c_2) \) for representations \( R \) corresponding to hook Young diagrams. This follows from the Littlewood-Richardson rule which tells us that hook representations \( R \) are only contained in tensor products \( R_1 \otimes R_2 \) provided that \( R_1 \) and \( R_2 \) also correspond to hooks. Another result that we use is that the factor \( n!(\text{Dim}_N R)/d_R \) appearing in both (18.3) and (18.4) is given by

\[
\frac{n!(\text{Dim}_N R)}{d_R} = \frac{(N + n - s - 1)!}{(N - s - 1)!}
\]  

for the hook representation \( R \) described above. Using this information along with the following sum [57],

\[
\sum_{k=m}^{n} \binom{a}{k} \binom{b}{k}^{-1} = \frac{b + 1}{b - a + 1} \left[ \binom{a}{m} \left( \frac{b + 1}{m} \right)^{-1} - \binom{a}{n+1} \left( \frac{b + 1}{n+1} \right)^{-1} \right]
\]  

we find for the two-point function (18.3) of traces

\[
\langle Tr(c_1 \Phi) Tr(c_1 \Phi^\dagger) \rangle = \frac{N}{n + 1} \left( \frac{(N + n)!}{N!} - \frac{(N - 1)!}{(N - n - 1)!} \right)
\]  

and for the three-point function following from (18.4)

\[
\langle Tr(c_1 \Phi) Tr(c_2 \Phi) Tr(c_3 \Phi^\dagger) \rangle = \frac{1}{n + 1} \left( \frac{(N + n)!}{(N - 1)!} - \frac{(N + n_2)!}{(N - n_1 - 1)!} - \frac{(N + n_1)!}{(N - n_2 - 1)!} \right) + \frac{N!}{(N - n - 1)!}
\]  

(18.9)
in agreement with (18.20) where these expressions were derived from a complex matrix model.

More generally the same kinds of manipulations can be applied to correlators of traces involving not just $\Phi_1$ and its conjugate but also the fields $\Phi_2$ and $\Phi_3$ and their conjugates. For example, consider the two-point function

$$\langle E_{31}^{k_1} E_{21}^{k_2} tr(c_1 \Phi_1) E_{13}^{k_1} E_{12}^{k_2} tr(c_2 \Phi_1^\dagger) \rangle$$  \hspace{1cm} (18.10)$$

where the permutations $c_1$ and $c_2$ are $n$-cycles in $S_n$ and the $E_{ij}$ operators are defined in appendix 5. Their purpose here is simply to convert $\Phi_1$’s to $\Phi_2$’s and $\Phi_3$’s and similarly for the conjugate fields. This two-point function differs from the one without the $E_{ij}$ operators only by an overall factor. The derivatives bring down a factor of $(n!/(n - k_1 - k_2)!)^2$. The contractions give factors of $k_1!, k_2!,$ and $(n - k_1 - k_2)!$ from the $\Phi_3$, $\Phi_2$, and $\Phi_1$ contractions respectively. Without the $E_{ij}$ operators one instead would just get an $n!$ from the $\Phi_1$ contractions. Up to this difference in the overall factor however, the resulting projector diagrams for the two-point function (18.10) and (18.4) are identical. As a result one finds

$$\langle E_{31}^{k_1} E_{21}^{k_2} tr(c_1 \Phi_1) E_{13}^{k_1} E_{12}^{k_2} tr(c_2 \Phi_1^\dagger) \rangle = \frac{n!k_1!k_2!}{(n - k_1 - k_2)!} \langle tr(c_1 \Phi_1) tr(c_2 \Phi_1^\dagger) \rangle. \hspace{1cm} (18.11)$$

The same argument holds for the more general multi-point correlator derived in (18.4) and its generalization to include $\Phi_2$ and $\Phi_3$ fields, although we shall not attempt to give a general formula.

18.1. Normalizations

There are two different natural ways to normalize the correlators discussed above: (1) as a multi-point correlator where one divides by the norm of each single trace operator, and (2) as an overlap of states where one divides by the norms of the complete $\Phi$ and $\Phi^\dagger$ operators. The latter normalization leads to a correlator whose magnitude is bounded above by one, a fact which follows from the Schwarz inequality.

To be more explicit, consider the normalized multi-point correlator

$$\frac{\langle (\prod_{k=1}^K Tr(c_k \Phi)) (\prod_{l=1}^L Tr(d_l \Phi^\dagger)) \rangle}{\| \prod_{k=1}^K Tr(c_k \Phi) \| \| \prod_{l=1}^L Tr(d_l \Phi) \|} \hspace{1cm} (18.12)$$
where the norms in the denominator are defined as \( \| \prod_{k=1}^K \text{Tr}(c_k \Phi) \| = \langle \prod_{k=1}^K \text{Tr}(c_k \Phi) \prod_{k=1}^K \text{Tr}(c_k \Phi^\dagger) \rangle \).

The numerator is given in (18.4). The denominator also follows from (18.4) by replacing the \( d_l \)'s by \( c_k \)'s or vice versa. That is

\[
\| \prod_{k=1}^K \text{Tr}(c_k \Phi) \|^2 = \sum_R \frac{n!(\text{Dim}_N R)}{d_R} \chi_R(c_1 \circ \cdots \circ c_K)^2
\]  

(18.13)

with a similar expression for \( \| \prod_{l=1}^L \text{Tr}(d_l \Phi) \| \). The Schwarz inequality bounds the inner product \((u, v)\) for two vectors \( u \) and \( v \) by

\[
| (u, v) | \leq \sqrt{ \| u \| + \| v \| } .
\]

(18.14)

Applied to the normalized correlator \((18.12)\) we see that it is bounded above by one.
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