Effective spacetime from multi-dimensional gravity

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Abstract
We study the effective spacetimes in lower dimensions that can be extracted from a multidimensional generalization of the Schwarzschild-Tangherlini spacetimes derived by Fadeev, Ivashchuk and Melnikov (Phys. Lett., A 161 (1991) 98). The higher-dimensional spacetime has $D = (4 + n + m)$ dimensions, where $n$ and $m$ are the number of “internal” and “external” extra dimensions, respectively. We analyze the effective $(4 + n)$ spacetime obtained after dimensional reduction of the $m$ external dimensions. We find that when the $m$ extra dimensions are compact (i) the physics in lower dimensions is independent of $m$ and the character of the singularities in higher dimensions, and (ii) the total gravitational mass $M$ of the effective matter distribution is less than the Schwarzschild mass. In contrast, when the $m$ extra dimensions are large this is not so; the physics in $(4 + n)$ does explicitly depend on $m$, as well as on the nature of the singularities in high dimensions, and the mass of the effective matter distribution (with the exception of wormhole-like distributions) is bigger than the Schwarzschild mass. These results may be relevant to observations for an experimental/observational test of the theory.

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1 Introduction

Nowadays, there are a number of theories suggesting that the universe may have more than four dimensions. These arise naturally in supergravity (11D) and superstring theories (10D), which seek the unification of gravity with the interactions of particle physics, and are expected to become important near the horizon of black holes, as “windows” to extra dimensions [1], and during the evolution of the early universe [2]. Also, it appears that black holes will play a crucial role in understanding non-perturbative effects in a quantum theory of gravity [3]. The natural question here is to know which of the properties of black holes are particular only to (3 + 1) dimensions, and which hold more generally.

Higher dimensional extensions of the Schwarzschild black hole metric have been obtained by Tangherlini [4] and generalized by Myers and Perry [3]. Fadeev, Ivashchuk and Melnikov [5] obtained a class of static, spherically symmetric solutions of the Einstein vacuum field equations, which generalize the Tangherlini solution to a chain of several Ricci-flat subspaces. They contain, as particular cases, the solutions previously considered by Yoshimura [6] and Myers [7]. Various extensions of these solutions, as well as a thorough analysis of their singularities and horizons, are provided by Ivashchuk and Melnikov [8]-[10].

Although the problem of finding higher dimensional extensions of the Schwarzschild metric has been thoroughly discussed, the question of how these multidimensional solutions reduce to lower dimensions seems to have been less discussed. In this paper we study this question. In particular, in the case of ordinary 4D spacetime we ask: (i) How does the physics in 4D depend on the number of extra dimensions? (ii) Does the physics in 4D depend on the specific nature of the singularities of the higher-dimensional spacetime? (iii) Does the physics in 4D depend on whether the extra dimensions are compact or large? (iv) Can we provide some specific observational criterion to determining whether the putative extra dimensions are compact or large?

2 Field equations and their solutions

To facilitate the discussion, make the paper self-consistent and introduce our notation we start by reviewing the solution presented in [5]. In this work the spacetime signature is (+, −, −, −); we follow the definitions of Landau and Lifshitz [11]; and the speed of light $c$ is taken to be unity.

Let us write the metric as

$$dS^2 = A^2(r)dt^2 - B^2(r)\left[dr^2 + r^2d\Omega_{(2+n)}^2\right] - \sum_{i=1}^{N} C^2_{(i)}(r)dg^2_{(i)},$$

where $d\Omega_{(n+2)}^2$ is the metric on a unit $(n+2)$-sphere ($n = 0, 1, 2...$) and

$$dg^2_{(i)} = \sum_{a, b=1}^{m_{(i)}} \delta_{ab}(x_{(i)}) \, dx^a_{(i)} dx^b_{(i)}, \quad m_{(i)} \geq 1.$$  

Here $n$ is the number of “internal” dimensions; $N$ is the number of “external” subspaces; $m_{(i)}$ is the dimension of the $i$-th subspace It is assumed that the Ricci tensors $R_{ab}(x_{(i)})$ formed out by the $\delta_{ab}(x_{(i)})$ alone all vanish.

If we introduce the quantity

$$V = \prod_{i=1}^{N} [C_{(i)}]^{m_{(i)}},$$

then, the field equations become

$$\frac{A''}{A'} + (n + 1)\frac{B'}{B} + \frac{n + 2}{r} + \frac{V'}{V} = 0$$

$$\frac{A''}{A} + \frac{(n + 2)}{2} \frac{B''}{B} + \frac{B'}{B} \left[ \frac{n + 2}{r} - \frac{A'}{A} - (n + 2) \frac{B'}{B} \right] - \frac{B'V'}{BV} + \sum_{i=1}^{N} \frac{m_{(i)} \, C'_{(i)}}{C_{(i)}} = 0,$$
\[
\frac{B''}{B} + \frac{B'}{B} \left[ \frac{2n + 3}{r} + \frac{A'}{A} + n \frac{B'}{B} \right] + \frac{A'}{rA} + \frac{V'}{V} \left[ \frac{B'}{B} + \frac{1}{r} \right] = 0, \tag{6}
\]

\[
\frac{A'}{A} + (n + 1) \frac{B'}{B} + \frac{n + 2}{r} + \frac{V'}{V} + \left( \frac{C''(i)}{C'(i)} - \frac{C'(i)}{C(i)} \right) = 0, \tag{7}
\]

Combining equations (4) and (7) we obtain \(C'(i)/C(i) = -\gamma(i) A'/A\), where \(\gamma(i)\) is a constant of integration. Thus, \(C(i) \propto A^{-\gamma(i)}\) and \(V'/V = -\omega A'/A\), where \(\omega = \sum_{i=1}^{N} m(i) \gamma(i)\). Therefore,

\[
V = V_0 A^{-\omega}, \tag{8}
\]

where \(V_0\) is a constant of integration. Now from (4) it follows that \(B^{n+1} \propto A^n r^{n+2} A'\). Substituting these expressions into (5) we obtain an equation for \(A\) whose solution is

\[
A(r) = \left( \frac{ar^{n+1} - 1}{ar^{n+1} + 1} \right)^\alpha, \tag{9}
\]

where \(a\) and \(\alpha\) are constants of integration. Henceforth we assume \(a \neq 0\) and \(\alpha \neq 0\). Consequently, the remaining metric functions are given by:

\[
B^{n+1}(r) = \frac{(ar^{n+1} + 1)^{[\alpha(1-\omega)+1]}}{a^2 r^{2(n+1)} (ar^{n+1} - 1)^{[\alpha(1-\omega)-1]}}, \quad C(i)(r) = \left( \frac{ar^{n+1} + 1}{ar^{n+1} - 1} \right)^{\alpha \gamma(i)}. \tag{10}
\]

Finally, in order to satisfy (3) the constants of integration \(\{\alpha, \gamma(i)\}\) must obey the relation:

\[
\alpha^2 \left[ (\omega - 1)^2 + (\sigma + 1)(n + 1) \right] = n + 2, \quad \omega \equiv \sum_{i=1}^{N} m(i) \gamma(i), \quad \sigma \equiv \sum_{i=1}^{N} m(i) \gamma(i)^2. \tag{11}
\]

In the case where \(n = 0\); \(N = 1\); \(m = 1\), setting \(\alpha = \epsilon k\) and \(\gamma = 1/k\) this reduces to \(\epsilon^2 (k^2 - k + 1) = 1\), which is the consistency relation in Davidson-Owen solution [1].

We note that all the above quantities are invariant under the simultaneous change \(a \rightarrow -a, \alpha \rightarrow -\alpha\). Therefore, the solutions with \(\alpha < 0, a > 0\) duplicate those with \(\alpha > 0, a < 0\). Consequently, in what follows, without loss of generality, we assume \(a > 0\).

The physical and geometrical properties of the solutions depend on the behavior of the metric functions near \(ar^{n+1} \sim 1\). To illustrate this we first consider the physical radius \(R(r)\) of a \((n + 2)\) sphere with coordinate \(r\), which is given by

\[
R(r) = r B(r). \tag{12}
\]

At large distances, i.e. \(ar^{n+1} \gg 1, R \sim r\). However, for \(ar^{n+1} \sim 1\)

\[
a r^{n+1} \sim 2^{[\alpha(1-\omega)+1]} (ar^{n+1} - 1)^{\alpha(\omega-1)+1}. \tag{13}
\]

On the other hand we find

\[
-g_{RR} dR^2 = B^2(r) dr^2 = \frac{\left[ a^2 r^{2(n+1)} - 1 \right]^2 dR^2}{\left[ a^2 r^{2(n+1)} + 2\alpha(\omega - 1)ar^{n+1} + 1 \right]^2}. \tag{14}
\]
Thus, regarding the behavior near $ar^{n+1} = 1$ there are three distinct families of solutions: (i) When $\alpha(\omega - 1) + 1 > 0$ we find that $g_{00} \to 0$, $g_{RR} \to 0$, $R(r) \to 0$ as $ar^{n+1} \to 1^+$. These solutions represent naked singularities; (ii) In the same limit when $\alpha(\omega - 1) + 1 < 0$ we find $g_{00} \to 0$, $g_{RR} \to 0$, $R(r) \to \infty$. In these solutions $dR/dr$ vanishes at some finite value of $r$, say $\bar{r}$. Since $R_{\text{min}} = R(\bar{r}) > 0$, they can be used to generate higher dimensional wormholes similar to those discussed in 5D by Agnese et al [12]; (iii) When $\alpha(\omega - 1) + 1 = 0$ we find that $g_{00} \to 0$, $g_{RR} \to -\infty$, $aR^{n+1}(r) \to 2^{[\alpha(\omega - 1) + 1]}$ as $ar^{n+1} \to 1^+$. These solutions are specially simple because now $\alpha$ is fixed. Namely, either $\alpha = 1$, $\omega = 0$, or $\alpha = 1/(\omega - 1)$. Below we present them separately.

- When $\alpha = 1$, the condition $\alpha(\omega - 1) + 1 = 0$ yields $\omega = 0$. Then from (11) it follows that $\sigma = 0$, which in turn requires $\gamma(i) = 0$ for $i = (1, ..., N)$, i.e. $C(i) = C^0(i) = \text{constant}$. In this case (12) reduces to

$$R = \frac{[ar^{n+1} + 1]^{2/(n+1)}}{a^{2/(n+1)}},$$

(15)

The physical meaning of $a$ is obtained from the asymptotic behavior of $g_{00}$. Far away from a stationary source $g_{00} \sim (1 + 2\phi)$, where $\phi$ is the Newtonian gravitational potential which goes as $-M/r^{n+1}$, and $M$ represents the total gravitational mass. In the present case from (9) we find

$$M = \frac{2}{a},$$

(16)

In terms of $R$ the solution becomes

$$dS^2 = \left(1 - \frac{2M}{R^{n+1}}\right) dt^2 - \left(1 - \frac{2M}{R^{n+1}}\right)^{-1} dR^2 - R^2 d\Omega_{(n+2)}^2 - \sum_{i=1}^{N} \left(C^0(i)\right)^2 dg_{(i)}^2,$$

(17)

which, up to the innocuous $\sum_{i=1}^{N} m(i)$ flat extra dimensions, describes the so-called Schwarzschild-Tangherlini black holes with spherical symmetry in $(n+3)$ rather than three spatial dimensions. The radius $R_h$ of the horizon of the black hole is given by $R_h = (4/a)^{1/(n+1)}$, which in isotropic coordinates corresponds to $ar^{n+1} = 1$, as expected. For $n = 0$ they reduce to the conventional Schwarzschild solution of general relativity.

- For $\alpha = 1/(1 - \omega)$, from (11) we find $\omega^2 - 2\omega - \sigma = 0$. Thus, $\omega = 1 \pm \sqrt{1 + \sigma}$. The correct solution is the one that gives $\omega = 0$ when $\sigma = 0$. Consequently, $\omega = 1 - \sqrt{1 + \sigma}$ and

$$M = \frac{2}{a\sqrt{1 + \sigma}},$$

(18)

In terms of the Schwarzschild coordinate (15) the solution becomes

$$dS^2 = F^{1/\sqrt{1 + \sigma}} dt^2 - \frac{dR^2}{F} - R^2 d\Omega_{(n+2)}^2 - \sum_{i=1}^{N} F^{-\gamma(i)/\sqrt{1 + \sigma}} dg_{(i)}^2,$$

(19)

where

$$F = 1 - \frac{2\sqrt{1 + \sigma} M}{R^{n+1}},$$

(20)

For $\sigma = \sum_{i=1}^{N} m(i)\gamma(i)^2 = 0$, we recover (17). It is important to emphasize that (17) and (19)–(20) are the only family of solutions for which $g_{tt} = 0$ and $g_{RR} = -\infty$ in the same region of the spacetime.

For completeness, we briefly examine the radial motion of light towards the center. Assuming $dS = 0$ from (11) we get

$$dt = -\left(\frac{B}{A}\right) dr.$$

(21)

Note that $\alpha = -1$ is not a possible solution of $\alpha(\omega - 1) + 1 = 0$ because this would require $\omega = 2$, for which (11) yields $\sigma = 0$. In turn this requires $\gamma(i) = 0$, i.e. $\omega = 0$ instead of $\omega = 2$. 

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[3] Note that $\alpha = -1$ is not a possible solution of $\alpha(\omega - 1) + 1 = 0$ because this would require $\omega = 2$, for which (11) yields $\sigma = 0$. In turn this requires $\gamma(i) = 0$, i.e. $\omega = 0$ instead of $\omega = 2$. 

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4
To study the motion near the singularity we introduce the coordinate \( \xi = ar^{n+1} - 1 \) and consider an expansion about \( \xi = 0 \). Then (21) becomes

\[
d\tau = a^{1/(n+1)}dt \sim -\xi^\sigma d\xi, \quad \kappa = \frac{\alpha(\omega - 2 - n) + 1}{n+1}.
\]

The time required to move from a point \( \xi = \xi_0 \) in the neighborhood of the singularity to \( \xi = 0 \) is finite for every \( \kappa \neq -1 \). However, for \( \kappa = -1 \) we get \( \xi \sim e^{-\tau} \) which means that there is a horizon. Substituting \( \kappa = -1 \) into the compatibility equation (11) we get \( \sigma(n+2) + \omega^2 = 0 \) whose solution is \( \sigma = \omega = 0 \) (recall that \( \sigma \geq 0 \)). The conclusion is that only the Schwarzschild-Tangherlini spacetimes possess a horizon, all the rest are naked singularities.

### 3 Dimensional reduction for compact extra dimensions

First we study the dimensional reduction of the solutions in the case where the external coordinates are rolled up to a small size. To put the discussion in perspective, let us recall that when \( n = 0 \) the curvature scalar \( R_{(D)} \) associated with the metric

\[
dS^2 = \gamma_{\mu\nu}(x)dx^\mu dx^\nu - \sum_{i=1}^{m} H_i^2(x)dy_i^2, \quad m = \sum_{i=1}^{N} m_i,
\]

can be expressed as (see, e.g., [13])

\[
\sqrt{|g_{(D)}R_{(D)}|} = \sqrt{|g_{(4)}^{\text{eff}}|} \left[ R_{(4)} - \frac{m}{a=1} \frac{\partial_a H_a \partial^a H_a}{N_a^2} - \frac{1}{2} \sum_{a=1}^{m} \sum_{b=1}^{m} \left( \frac{\partial_a H_a}{H_a} \right) \left( \frac{\partial_b H_b}{H_b} \right) + \Delta_{(4)} \ln V \right].
\]

Here \( R_{(4)} \) is the four-dimensional curvature scalar calculated from the effective 4D metric tensor

\[
g_{\mu\nu}^{\text{eff}} = \gamma_{\mu\nu} \prod_{i=1}^{m} H_i,
\]

\( g_{(D)} \) and \( g_{(4)}^{\text{eff}} \) denote the determinants of the \( D \)-dimensional metric (23) and effective 4D metric (25), respectively; \( V \) is the function defined in (3), which in the notation of (23) becomes \( V = \prod_{i=1}^{m} H_i \), and \( \Delta_{(4)} \) is the Laplace-Beltrami operator corresponding to \( g_{\mu\nu}^{\text{eff}} \).

The choice of the factor \( \prod_{i=1}^{m} H_i \) in (25) assures that the effective action in 4D contains the exact Einstein Lagrangian, with a fixed effective gravitational constant \([1, 14]\). The second and third expressions in (24) are proportional to a Lagrangian for an effective energy-momentum tensor (EMT) \( T_{\mu\nu} \), while the last one gives rise to a boundary term in the effective action and vanishes when the field equations \( R_{AB} = 0 \) are imposed. Thus, for the higher-dimensional metric (9)-(10) with \( n = 0 \) all the physics in 4D is concentrated in the effective line element \( g_{\mu\nu}^{\text{eff}} = V \gamma_{\mu\nu} \).

Let us now go back to the case where \( n > 0 \). We have found that the effective metric in \( (4 + n) \)-dimensions, say \( g_{\mu\nu}^{\text{eff}} \), is obtained from \( \gamma_{\mu\nu} \), the \( (4 + n) \) part of the metric in \( D \)-dimensions, as

\[
g_{\mu\nu}^{\text{eff}} = \gamma_{\mu\nu} \prod_{i=1}^{m} H_i^{p(n)}, \quad p(n) = \frac{2}{n+2},
\]

which for \( n = 0 \) reduces to (25). Similar to the above discussion, with this choice the gravitational action has the standard form

\[
S_{(4+n)} = -\frac{1}{16\pi G_{(4+n)}} \int \sqrt{|g_{(4+n)}^{\text{eff}}|} R_{(4+n)} d^{4+n}x,
\]

in any number of dimensions, where \( G_{(4+n)} \) is the gravitational constant in \( (4 + n) \). If \( L_{(j)} \) represents the size of the \( j \)-th external coordinate, then
\[
\frac{1}{G_{(4+n)}} = \frac{\prod_{j=1}^{m} L_{(j)}}{G_D}.
\]

(28)

In what follows, to simplify the notation we set \(G_{(4+n)} = 1\).

In the case under consideration \(g_{\mu \nu}^{\text{eff}} = V^{2/(n+2)} \gamma_{\mu \nu}\). Thus the effective gravity in \((4 + n)\) is determined by the line element

\[
ds^2 = \left(\frac{a r^{n+1} - 1}{a r^{n+1} + 1}\right)^{2\varepsilon} dt^2 - \frac{1}{(a r^{n+1})^{4/(n+1)}} \left(\frac{a r^{n+1} + 1}{a r^{n+1} - 1}\right)^{2(\varepsilon+1)/(n+1)} \left[dr^2 + r^2 d\Omega^2_{(n+2)}\right],
\]

(29)

where

\[
\varepsilon = \frac{\alpha(n + 2 - \omega)}{n + 2}.
\]

(30)

We note that \(\varepsilon^2 \leq 1\). In fact, substituting (30) into (11) we find

\[
1 - \varepsilon^2 = \frac{(n + 1) \left[\omega^2 + \sigma(n + 2)\right]}{(n + 2) \left[\omega - 1 + \sigma + 1\right] \left(n + 1\right)} \geq 0.
\]

(31)

An observer in \((4 + n)\), who is not aware of the existence of external extra dimensions, interprets the metric functions as if they were governed by an effective energy-momentum tensor (EMT) \(T_{\mu \nu}\) determined by the Einstein field equations \(G_{AB} = 8\pi T_{AB}\). In the present case, using (29) we find

\[
8\pi T_0^0 = \frac{2(1 - \varepsilon^2)(n + 1)(n + 2) a^{2(n+3)/(n+1)} r^{2(2+n)} (a r^{n+1} - 1)^{2(\varepsilon+1)/(n+1)}}{(a r^{n+1} + 1)^{2(\varepsilon+1)}}, \quad T_1^1 = -T_0^0, \quad T_2^2 = T_3^3 = \cdots = T_{n+3}^n = T_0^0.
\]

(32)

By virtue of (31), the effective energy density \(T_0^0\) results to be automatically non-negative. When \(\varepsilon = 1\), both \(\omega\) and \(\sigma\) must vanish, which in turn implies \(\gamma_{(i)} = 0\). Consequently, (29) reduces to Schwarzschild-Tangherlini’s spacetimes in isotropic coordinates, as expected.

The relationship between the components of the EMT suggest that the source can be interpreted as a neutral massless scalar field

\[
\Psi = \int \sqrt{-g} g_{\mu \nu} T_0^0 \, dr.
\]

(33)

After integration we find

\[
\Psi(r) = \frac{1}{2} \sqrt{\frac{(n + 2)(1 - \varepsilon^2)}{2\pi(n + 1)}} \ln \left|\frac{ar^{n+1} - 1}{ar^{n+1} + 1}\right|.
\]

(34)

It is not difficult to verify that \(\Psi\) satisfies the Klein-Gordon equation

\[
\left(\sqrt{-g} g^{\mu \nu} \Psi_{,\mu}\right)_{,\nu} = 0,
\]

which is consistent with our interpretation. It should be noted that (24), (22), (21) for \(n = 0\) are equivalent to the static, spherically symmetric solution of the coupled Einstein-massless scalar field equations originally discovered by Fisher [15] and rediscovered by Janis, Newman and Winicour [16]. Thus, the above equations generalize Fisher’s solution to \((4 + n)\) dimensions.

The coordinate transformation

\[
z = \frac{a}{2(n + 1)} \ln \left|\frac{ar^{n+1} - 1}{ar^{n+1} + 1}\right|.
\]

(35)
renders the metric \( g_{zz}(z) = -g_{tt}(z)g_{\theta\theta}(z) \). In terms of \( z \) the solution of the Klein-Gordon equation is \( \Psi = g_z \), where \( q \) is interpreted as the scalar charge \([17]\). Thus, from (34) we find the scalar charge in the present case as

\[
q = \frac{M}{2\varepsilon} \sqrt{\frac{(n+1)(n+2)(1-\varepsilon^2)}{2\pi}}, \quad M = \frac{2\varepsilon}{a},
\]

where \( M \) is the total mass measured by an observer located at spatially infinity. If we denote as \( M_{ST} \) the total Schwarzschild-Tangherlini mass (\( \varepsilon = 1 \)), then the above implies

\[
M^2 = M_{ST}^2 - \frac{8\pi q^2}{(n+1)(n+2)}.
\]

Thus, \( M \leq M_{ST} \). We note that \( M \to 0 \) as \( \varepsilon \to 0^+ \) and \( M \to M_{ST} \) as \( \varepsilon \to 1^- \).

The line element (29) acquires a more familiar form in terms of the radial coordinate \( R \) defined by

\[
R = r \left( 1 + \frac{1}{ar^{n+1}} \right)^{2/(n+1)}.
\]

Indeed, it becomes

\[
ds^2 = \left( 1 - \frac{2M/\varepsilon}{R^{n+1}} \right)^{\varepsilon} dt^2 - \left( 1 - \frac{2M/\varepsilon}{R^{n+1}} \right)^{-(n+\varepsilon)/(n+1)} dr^2 - R^2 \left( 1 - \frac{2M/\varepsilon}{R^{n+1}} \right)^{(1-\varepsilon)/(n+1)} d\Omega^2_{(n+2)}.
\]

In addition, the effective EMT is now given by

\[
8\pi T^0_0 = \frac{(n+1)(n+2)(1-\varepsilon^2)M^2}{2\varepsilon^2 R^{2(n+2)}} \left( 1 - \frac{2M/\varepsilon}{R^{n+1}} \right)^{(\varepsilon-n-2)/(n+1)}, \quad T^1_1 = -T^0_0, \quad T^2_2 = T^3_3 = \cdots = T^{n+3}_{n+3}.
\]

It should be noted that the dimensional reduction eradicates the geometrical and physical differences between the three families of higher-dimensional solutions derived in (13)-(14). Specifically, the effective (4 + \( n \)) spacetime shows no evidence of the different nature of the singularity of \( g_{RR} \) near \( a^{n+1} = 1 \) in higher dimensions. The fact is that all solutions in (9)-(10) generate the same effective spacetime in (4 + \( n \)), regardless of their specific properties.

From (10) it follows that \( T = (n+2)T^0_0 \). As a consequence all the components of the Ricci tensor, except for \( R_{11} \), are zero. We now recall that in the case of a constant, asymptotically flat, gravitational field there is an expression for the total energy of matter plus field, which is an integral of \( R^0_0 \) over the volume \( V \), and is given by the expression \([11]\), viz., \( M = \kappa \int \sqrt{|g|} R^0_0 dV \), where the constant of proportionality \( \kappa \) depends on the number of dimensions, e.g., \( \kappa = 1/4\pi \) in 4D. In conventional general relativity this expression is known as the Tolman-Wittaker formula. Since \( R^0_0 = 0 \), it follows that the gravitational mass of any spherical shell is just zero. This conclusion holds for any \( n \) and \( \varepsilon \), including the Schwarzschild-Tangherlini black holes, as well as the familiar Schwarzschild solution of general relativity.

We note that \( R^0_0 = 0 \) implies

\[
(n+1)\rho + p_r + (n+2)p_\perp = 0, \quad p_r = -T^1_1, \quad p_\perp = -T^2_2,
\]

which generalizes to \( n \) dimensions the well-known equation of state \( (\rho + p_r + 2p_\perp) = 0 \) for non-gravitating matter in 4D, which in turn generalizes to anisotropic matter the equation of state \( (\rho + 3p) = 0 \) for a perfect fluid that has no effect on gravitational interactions \([18]-[21]\).

The case where \( n = 0 \) is especially important because it corresponds to our spacetime with spherical symmetry in the three usual spatial dimensions,
\[
\begin{align*}
\text{ds}^2 = & \left(1 - \frac{2M/\varepsilon}{R}\right)^\varepsilon \, dt^2 - \frac{dR^2}{\left(1 - \frac{2M/\varepsilon}{R}\right)^{1-\varepsilon}} - R^2 \left(1 - \frac{2M/\varepsilon}{R}\right)^{1-\varepsilon} \left(d\theta^2 + \sin^2 \theta d\phi^2\right). \\
\text{(42)}
\end{align*}
\]

It has two distinctive properties, which do not hold for any other \( n \neq 0 \): (i) The effective spacetime is Schwarzschild-like in the sense that the 4D line element is in the “gauge” \( g_{00} = g_{11} = -1 \), which has a number of important properties and applications \([22, 24]\); (ii) At large distances from the origin, to first order in \( \varepsilon \), \( \varepsilon \)[25], \( \varepsilon \). As a consequence, \( Z \) is compatible with the (weak) equivalence principle, for all values of \( \varepsilon \). In addition, in the weak-field approximation, it is consistent with Newtonian physics \([25, 26]\).

Horizons and singularities in static spherically symmetric configurations in 4D were recently discussed and classified by Bronnikov et al \([23, 24]\) in terms of the quantity

\[
\hat{Z} \propto \frac{R_{12}^{12} - R_{02}^{02}}{g_{00}},
\]

which characterizes the magnitude of tidal forces in a freely falling reference system near the spacetime region where \( g_{tt} = 0 \). In the present case it yields

\[
\hat{Z} \propto \frac{(1 - \varepsilon^2)M^2}{\varepsilon^2 R^4 (1 - 2M/\varepsilon R)^2},
\]

Thus, \( \hat{Z} = 0 \) for \( \varepsilon = 1 \) (the Schwarzschild black hole) and \( \hat{Z} \to \infty \), as \( g_{00} \to 0 \), for any other \( \varepsilon \). Consequently, in the classification given in \([23]\) for \( \varepsilon \neq 1 \) the singularity at \( R = 2M/\varepsilon \) is a “truly naked” one.

### 4 Splitting procedure for large extra dimensions

We now assume that the external dimensions are large. In order to make contact with previous works, we first consider \( N = 1, m = 1 \). To perform the splitting of the spacetime into \( (4 + n) + 1 \) we introduce a unit vector \( \chi^A \) tangent to the extra dimension, and assume that the \( (4 + n) \) manifold is locally orthogonal to the large extra dimension. As a consequence the metric induced on \( (4 + n) \) is given by \( g_{CD} = \gamma_{CD} - \chi_C \chi_D \), which in the present case is just the \( (4 + n) \) part of \( \chi \)[1] with \( A(r) \) and \( B(r) \) given by \( \chi \)[9] and \( \chi \)[10], respectively, viz.,

\[
\begin{align*}
\text{ds}^2 = & \frac{\left(\frac{ar^{n+1} - 1}{ar^{n+1} + 1}\right)^{2\omega}}{(ar^{n+1})^{(1/2)(n+1)} \left(\frac{ar^{n+1} + 1}{ar^{n+1} - 1}\right)^{2\omega(1-\omega)/(n+1)}}
\times \left[dr^2 + r^2 d\Omega^2_{(n+2)}\right].
\end{align*}
\]

Following a splitting procedure similar to the one used in \([27]\) we obtain an effective EMT in \( (4 + n) \)

\[
8\pi T_{AB} = \frac{C_A B}{C},
\]

where \( C_A = \partial C/\partial x^A \). In the present case \( N = 1, m = 1 \) \( \omega = \gamma, \sigma = \gamma^2 \). Therefore \( C \) satisfies the Klein-Gordon equation constructed with the metric \([15]\). Consequently, the trace of the EMT vanishes and the Ricci scalar is zero. The nonvanishing components of the EMT are\(^6\)

\[
\begin{align*}
8\pi T^0_0 &= \frac{2(n+1)(n+2)}{2^{(n+3)/(n+1)} \left[ar^{2(n+1)} - 1\right]^2} \left\{ \frac{\left(\frac{ar^{n+1} - 1}{ar^{n+1} + 1}\right)^{2\omega(1-\omega)/(n+1)}}{(ar^{n+1} + 1)^{2\omega(1-\omega)/(n+1)}} \right\}, \\
T^1_1 &= -T^0_0 \left( 1 + \frac{\alpha \gamma}{ar^{n+1} \left[1 - \alpha^2 (\gamma - 1)^2\right]} \right), \\
T^2_2 &= -\frac{T^0_0 + T^1_1}{n+2}.
\end{align*}
\]

\(^6\)It is worth mentioning that the effective matter quantities do not have to satisfy the regular energy conditions because they involve terms of geometric origin \([28]\).
From the compatibility condition (11) we find

\[ 1 - \alpha^2(\gamma - 1)^2 = \frac{2\gamma\alpha^2(n + 1)}{(n + 2)}. \]  

(48)

Thus, (i) If \( \gamma = 0 \), then \( \alpha^2 = 1 \) and the EMT vanishes. Consequently, the metric reduces to the Schwarzschild-Tanherlini spacetime with mass \( M_{ST} = 2/\alpha \); (ii) If \( \gamma > 0 \), then \( \alpha > 0 \) and \( 1 - \alpha(1 - \gamma) > 0 \) for any \( n \). As a consequence \( g_{22} \to 0 \) as \( R^{n+1} \to 2M/\alpha \); (ii) If \( \gamma < 0 \), then \( \alpha < 0 \) and \( 1 - \alpha(1 - \gamma) < 0 \). Thus the metric has wormhole-like structure in the sense that \( g_{22} \to -\infty \) as \( R^{n+1} \to 2M/\alpha \), similar to those discussed for \( n = 0 \) in [12]; (iii) For \( 0 < \gamma < 2/(n + 2) \), we find \( \alpha > 1 \). In this range

\[ M_{ST} \leq M \leq \frac{M_{ST}(n + 2)}{\sqrt{n^2 + 4n + 3}}. \]  

(49)

In five-dimensional Kaluza-Klein theory \( (n = 0, N = 1, m = 1) \), the line element (15) plays a central role in the discussion of many important observational problems, which include the classical tests of relativity, as well as the geodesic precession of a gyroscope and possible departures from the equivalence principle [29]-[31]. In the context of the induced-matter approach, the configuration of matter (17) is interpreted as describing extended spherical objects called solitons [32] (for a recent discussion see Ref. [33] and references therein). Solitons and black holes are alike in one important aspect: they contain a curvature singularity at the center of ordinary space. However, (1) solitons do not have an event horizon; and (2) they have an extended matter distribution rather than having all their matter compressed into the central singularity [33].

In general, for any \( N \) and \( m(i) \) we can proceed in a similar way. However, now the trace of the effective EMT does not longer vanish, except in the case where \( \omega^2 = \sigma \). The metric and the effective EMT in terms of the radial coordinate \( R \) introduced in (38) are given by

\[ ds^2 = \left(1 - \frac{2M/\alpha}{R^{n+1}}\right)^\alpha dt^2 - \left(1 - \frac{2M/\alpha}{R^{n+1}}\right)^{n + \alpha(1 - \omega)/(1 + n)} dR^2 - R^2 \left(1 - \frac{2M/\alpha}{R^{n+1}}\right)^{[1 - \alpha(1 - \omega)]/(n + 1)} d\Omega_{n+2}, \]  

(50)

where \( M = 2\alpha/\alpha \); and

\[ 8\pi T^0_0 = \frac{(n + 1)(n + 2)\left[1 - \alpha^2(1 - \omega)^2\right] M^2}{2\alpha^2 R^{2(n+2)}} \quad \left(1 - \frac{2M/\alpha}{R^{n+1}}\right)^{[(\alpha + 1) - n - 2]/(n + 1)}, \]

\[ T_1^1 = -T^0_0 \frac{1 + (\omega^2 - 1)\alpha^2 + \omega\alpha a R^{n+1} - 2}{1 - \alpha^2(\omega - 1)^2}, \]

\[ T_2^2 = -T^0_0 \frac{[n(1 + \omega^2) + 2]\alpha^2 - \omega\alpha a R^{n+1} - 2 - (n + 2)}{(n + 2)(1 - \alpha^2(\omega - 1)^2)}. \]  

(51)

The effective energy density is positive in the range \( (\alpha - 1)/\alpha < \omega < (\alpha + 1)/\alpha \). Wormhole-like solutions are obtained in the region \( \omega < (\alpha + 1)/\alpha \).

The trace of the EMT is

\[ T = T^0_0 + T^1_1 + (n + 2)T^2_2 = T^0_0 \frac{(n + 2)(\sigma - \omega^2)}{2\omega + \sigma - \omega^2}, \]

\[ = \frac{(\sigma - \omega^2)(n + 1)^2(n + 2)M^2}{16\pi R^{2(n+2)}} \left(1 - \frac{2M/\alpha}{R^{n+1}}\right)^{[(\alpha + 1) - n - 2]/(n + 1)}, \]  

(52)

where we have used (11) to eliminate \( \alpha \). Thus, the effective matter quantities satisfy the equation of state

\[ \rho = \frac{2\omega + \sigma - \omega^2}{2\omega - (n + 1)(\sigma - \omega^2)} \left[p_r + (n + 2) p_\perp\right]. \]  

(53)
As it was mentioned above $T = 0$ only if $\sigma = \omega^2$. Besides, when $\omega = 0$, but $\sigma \neq 0$, the above reduces to the massless scalar field given by (39)-(40), in agreement with the fact that the factor $V$ given by (8) is constant in this case. When $\omega = 0$ and $\alpha = 1$ from (11) we find $\sigma = 0$, i.e. $\gamma(i) = 0$. Thus we recover Schwarzchild-Tangherlini spacetimes.

Observations suggest that Kaluza-Klein corrections to general relativity should be small. This means that in practice we should expect $\sigma \approx \omega^2$; $\sigma \approx 0$. Therefore we can expand (11) about $\omega = 0$,

$$\alpha = 1 + \frac{\omega}{n+2} + O(\omega^2). \quad (54)$$

In this approximation $T^0_0 > 0$ requires $\omega > 0$. Since $M = \alpha M_{ST}$, it follows that for positive effective density $M > M_{ST}$ (for wormhole-like distributions $M < M_{ST}$).

5 Summary

At this point we see that the answers to the questions posed in the introduction crucially depend on whether the external extra dimensions are compact or large. For compact external extra dimensions: (i) The physics in $4D$, which is governed by the metric (42), is independent of $N$ and $m$; (ii) Since $0 < \varepsilon < 1$, the reduction procedure flattens out the rich diversity of higher dimensional solutions. In particular, the energy density is always positive and $g_{22} = 0$ at $ar = 1$. For large extra dimensions, the situation is much more elaborated. Firstly, from (53) we see that the effective matter quantities satisfy an equation of state that explicitly depends on $\omega$ and $\sigma$, i.e. the number of extra dimensions. Secondly, the effective spacetime in $4D$ does inherit the singularities of its higher-dimensional counterpart. However, in the exceptional case where $\omega = \sum_{i=1}^{N} m(i)\gamma(i) = 0$ both compact and large extra (external) dimensions yield the same physics in $4D$.

An important result here is that the total gravitational mass $M$ is different in both cases. Namely, it is less than the Schwarzshild mass for compact extra dimensions, but bigger than the Schwarzshild mass for large extra dimensions (with the exception of wormhole-like distributions). This result may be relevant to observations for an experimental/observational test of the theory.

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