Sharp embedding between Besov-Triebel-Sobolev spaces and modulation spaces

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Abstract

The embedding relations between Besov-Triebel-Sobolev spaces and modulation spaces are determined explicitly. We extend the results in [1], [2], [3] to the most general cases. And we give the sharp embedding between Fourier $L^p$ spaces and modulation spaces.

1 Introduction

The modulation spaces $M^{s}_{p,q}$ are one of the function spaces introduced by Feichtinger [4] in the 1980s using the short-time Fourier transform to measure the decay and the regularity of the function in a different way from the usual $L^p$ Sobolev spaces or Besov-Triebel spaces. The precise definitions will be given in Section 2. Roughly speaking, the Besov-Triebel spaces mostly use the dyadic decompositions of the frequency space while the modulation spaces using the uniform decompositions of the frequency space. Therefore, these spaces may have many different properties in some cases. Also, they have some properties similarly. So, the relationships between the modulation spaces and Besov-Triebel spaces are important.

Many basic properties of the modulation spaces had been studied in [5, 6, 7] such as the Banach spaces properties and the dual spaces of modulation spaces. And the modulation spaces also have some applications in pseudo-differential operators in [8, 9, 10, 11, 12, 13]. Modulation spaces also have many applications in the analysis of partial differential equations. For example, the Schrodinger and wave semigroups which are not bounded on neither $L^p$ or $B^{s}_{p,q}$ for $p \neq 2$, are bounded on $M^{s}_{p,q}$ (see [14]). So, the modulation space is a good space for the initial data of the Cauchy problem for nonlinear dispersive equations (see [15, 2, 16, 17]).

As for embedding properties of modulation spaces, Toft in [18] discussed some of the sufficient conditions of the embedding between modulation spaces and Besov spaces, also, Toft in [13] discussed the embedding between the $L^p$ spaces and the Fourier transform of $L^p$ space with the modulation spaces. Then, Okoudjou in [19] gave some sufficient conditions for some cases of embedding between Sobolev-Besov spaces and modulation spaces by the means of the short-time Fourier transform. Later, Sugmito and Tomita in [1] gave the sharp embedding between modulation spaces and Besov spaces when $1 \leq p, q \leq \infty$ by means of the scaling of modulation spaces, then Wang and Huang in [2] extended the condition to $0 < p, q \leq \infty$ by means of the uniform decomposition.

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of frequency space. But they all discuss the case when the indexes of physics space and frequency space of modulation spaces and Besov spaces are the same. Later, Kobayashi and Sugimoto in [3] studied the sharp embedding between $L^p$-Sobolev space and modulation spaces using the method similar to [20]. As for Triebel spaces, Guo, Wu and Zhao in [21] gave the sharp condition of the embedding $F_{p,r} \hookrightarrow M_{p,q}$ when $0 < p \leq 1$.

In this article, we study the sharp conditions of the embedding between Besov-Triebel-Sobolev spaces and modulation spaces. Let us take Besov spaces for example, in Theorem 4.1 we get the sufficient and necessary conditions of the embedding $B_{s,p,q} \hookrightarrow M_{p,q}$, which is the most general case of $0 < p, q, p_0, q_0 \leq \infty$, also in Theorem 4.2 we get the sufficient and necessary conditions of the embedding of the opposite side. In the case of $0 < p, q < 1$, the dual of Besov spaces and modulation spaces are different, so we cannot just get the opposite side by the method of duality. Instead, we use the method of the uniform decompositions of frequency space to get the result as desired.

We explain the organization of this paper. We give some basic notations and properties of the spaces we need in the next preliminary section. In section 3, we give some lemmas we need in the proof, especially for the case of $0 < p, q \leq \infty$ we discuss in section 4. The embedding between Besov spaces and modulation spaces is considered in section 4. We mainly use the method of uniform decomposition in [2]. Then we study the embedding between $L^p$ Sobolev spaces and modulation spaces in section 5, as we can see, there are four conditions totally. In section 6, we consider the embedding Triebel spaces and modulation spaces which is sometimes equivalent to the embedding of Sobolev spaces by Littlewood-Paley theory. In the last section, we give the sharp embedding relationship between the Fourier $L^r$ spaces and modulation spaces.

2 Preliminaries

2.1 Basic notation

The following notation will be used throughout this article. For $0 < p, q \leq \infty$, we denote

$$\sigma(p, q) := d \left(0 \wedge \left(\frac{1}{q} - \frac{1}{p}\right) \wedge \left(\frac{1}{q} + \frac{1}{p} - 1\right)\right);$$

$$\tau(p, q) := d \left(0 \vee \left(\frac{1}{q} - \frac{1}{p}\right) \vee \left(\frac{1}{q} + \frac{1}{p} - 1\right)\right),$$

where $a \wedge b = \min\{a, b\}, a \vee b = \max\{a, b\}$.

We write $\mathcal{S}(\mathbb{R}^d)$ to denote the Schwartz space of all complex-valued rapidly decreasing infinity differentiable functions on $\mathbb{R}^d$, and $\mathcal{S}'(\mathbb{R}^d)$ to denote the dual space of $\mathcal{S}(\mathbb{R}^d)$, all called the space of all tempered distributions. For simplification, we omit $\mathbb{R}^d$ without causing ambiguity. The Fourier transform is defined by $\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\xi} d\xi$, and the inverse Fourier transform by $\mathcal{F}^{-1} f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\xi} d\xi$. For $1 \leq p < \infty$, we define the $L^p$ norm:

$$\|f\|_p = \left(\int_{\mathbb{R}^d} |f(x)|^p\right)^{1/p}.$$
and \( \|f\|_\infty = \text{ess sup}_{x \in \mathbb{R}^d} |f(x)| \). We also define the \( L^p \) Sobolev norm:

\[
\|f\|_{W^{s,p}} = \left\| (I - \Delta)^{s/2} f \right\|_p
\]

Recall that the Sobolev spaces is defined by \( W^{s,p} = \{ f \in \mathcal{S}' : \|f\|_{W^{s,p}} < \infty \} \). And the Fourier \( L^p \) is defined by \( \mathcal{F}L^p = \left\{ f \in \mathcal{S}' : \|\hat{f}\|_p < \infty \right\} \).

We use the notation \( I \lesssim J \) if there is an independently constant \( C \) such that \( I \leq C J \). Also we denote \( I \approx J \) if \( I \lesssim J \) and \( J \lesssim I \). For \( 1 \leq p \leq \infty \), we denote \( 1/p + 1/p' = 1 \), for \( 0 < p < 1 \), denote \( p' = \infty \).

### 2.2 Modulation spaces

Let \( 0 < p,q \leq \infty \), the short time Fourier transform of \( f \) respect to a window function \( g \in \mathcal{S} \) is defined as (see [4]):

\[
V_g f(x,\xi) = \int_{\mathbb{R}^d} f(t)g(t-x)e^{-it\xi} dt.
\]

And then for \( 1 \leq p, q \leq \infty \), we denote

\[
\|f\|_{M^{s}_{p,q}} = \|V_g f(x,\xi) \langle \xi \rangle^s\|_{L^p_x L^q_\xi},
\]

where \( \langle \xi \rangle = 1 + |\xi| \). Modulation space \( M^{s}_{p,q} \) are defined as the space of all tempered distribution \( f \in \mathcal{S}' \) for which \( \|f\|_{M^{s}_{p,q}} \) is finite.

Also, we know another equivalent definition of modulation by uniform decomposition of frequency space (see [6]).

Let \( \sigma \) be a smooth cut-off function adapted to the unit cube \([-1/2, 1/2]^d\) and \( \sigma = 0 \) outside the cube \([-3/4, 3/4]^d\), we write \( \sigma_k = \sigma(\cdot - k) \), and assume that

\[
\sum_{k \in \mathbb{Z}^d} \sigma_k(\xi) \equiv 1, \ \forall \xi \in \mathbb{R}^d.
\]

Denote \( \sigma_k(\xi) = \sigma(\xi - k) \), and \( \Box_k = \mathcal{F}^{-1} \sigma_k \mathcal{F} \), then we have the following equivalent norm of modulation space:

\[
\|f\|_{M^{s}_{p,q}} = \|\langle k \rangle^s \|\Box_k f\|_{L^p_x L^q_\xi} \|_{\ell^2_{\mathbb{Z}^d}}.
\]

We simply write \( M_{p,q} \) instead of \( M^{0}_{p,q} \). One can prove the \( M^{s}_{p,q} \) norm is independent of the choice of cut-off function \( \sigma \). Also \( M^{s}_{p,q} \) is a quasi Banach space and when \( 1 \leq p,q \leq \infty \), \( M^{s}_{p,q} \) is a Banach space. When \( p,q < \infty \), then \( \mathcal{S}' \) is dense in \( M^{s}_{p,q} \). Also, \( M^{s}_{p,q} \) has some basic properties, we list them in the following lemma(see [2, 22]).

**Lemma 2.1.** For \( s, s_0, s_1 \in \mathbb{R}, 0 < p, p_0, p_1, q, q_0, q_1 \leq \infty \),

1. if \( s_0 \leq s_1 \), \( p_1 \leq p_0, q_1 \leq q_0 \), we have \( M^{s_1}_{p_1,q_1} \hookrightarrow M^{s_0}_{p_0,q_0} \).
(2) when \( p, q < \infty \), the dual space of \( M^{s}_{p,q} \) is \( M^{-s}_{p',q'} \);

(3) the interpolation spaces theorem is true for \( M^{s}_{p,q} \), i.e. for \( 0 < \theta < 1 \) when

\[
s = (1 - \theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1},
\]

we have \((M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1})_{\theta} = M^{s}_{p,q}\);

(4) when \( q_1 < q, \ s + d/q > s_1 + d/q_1 \), then we have \( M^{s}_{p,q} \hookrightarrow M^{s_1}_{p,q_1} \).

2.3 Besov-Triebel spaces

Let \( 0 < p, q \leq \infty, s \in \mathbb{R} \), choose \( \psi : \mathbb{R}^d \rightarrow [0, 1] \) be a smooth radial bump function adapted to the ball \( B(0, 2) \): \( \psi(\xi) = 1 \) as \( |\xi| \leq 1 \) and \( \psi(\xi) = 0 \) as \( |\xi| \geq 2 \). We denote \( \varphi(\xi) = \psi(\xi) - \psi(2\xi) \), and \( \varphi_j(\xi) = \varphi(2^{-j}\xi) \) for \( 1 \leq j, j \in \mathbb{Z}, \varphi_0(\xi) = 1 - \sum_{j \geq 1} \varphi_j(\xi) \). We say that \( \Delta_j = \mathcal{F}^{-1}\varphi_j\mathcal{F} \) are the dyadic decomposition operators. The Besov spaces \( B^{s}_{p,q} \) and the Triebel spaces \( F^{s}_{p,q} \) are defined in the following way:

\[
B^{s}_{p,q} = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{B^{s}_{p,q}} = \left\|2^{js} \|\Delta_j f\|_{L^p_{\ell_{\theta}}} \right\|_{\ell^{q}_{\theta}} < \infty \right\},
\]

\[
F^{s}_{p,q} = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{F^{s}_{p,q}} = \left\|2^{js} \|\Delta_j f\|_{L^p_{\ell_{\theta}}} \right\|_{\ell^{q}_{\theta}} < \infty \right\}.
\]

One can prove that the Besov-Triebel norms defined by different dyadic decompositions are all equivalent (see [23]), so without loss of generality, we can assume that when \( 1 \leq j, \varphi_j(\xi) = 1 \) on \( D_j := \{\xi \in \mathbb{R}^d : \frac{2^j}{4} \leq |\xi| \leq \frac{2^j}{4} \} \) for convenience. Also, Besov-Triebel spaces have some basic properties known already (see [23, 7, 22]).

**Lemma 2.2.** Let \( s, s_1, s_2 \in \mathbb{R}, 0 < p, p_1, p_2, q, q_1, q_2 \leq \infty, \)

(1) if \( q_1 \leq q_2 \), we have \( B^{s}_{p,q_1} \hookrightarrow B^{s}_{p,q_2}, F^{s}_{p,q_1} \hookrightarrow F^{s}_{p,q_2} \);

(2) \( \forall \varepsilon > 0 \), we have \( B^{s+\varepsilon}_{p,q_1} \hookrightarrow B^{s}_{p,q_2}, F^{s+\varepsilon}_{p,q_1} \hookrightarrow F^{s}_{p,q_2} \);

(3) \( B^{s}_{p,p\vee q} \hookrightarrow F^{s}_{p,q} \hookrightarrow B^{s}_{p,p\wedge q} \);

(4) if \( p_1 \leq p_2, s_1 - d/p_1 = s_2 - d/p_2 \), we have \( B^{s_1}_{p_1,q} \hookrightarrow B^{s_2}_{p_2,q} \);

(5) if \( p_1 < p_2, s_1 - d/p_1 = s_2 - d/p_2 \), we have \( F^{s_1}_{p_1,q} \hookrightarrow F^{s_2}_{p_2,q} \);

(6) when \( 1 \leq p, q < \infty \), the dual space of \( B^{s}_{p,q} \) is \( B^{-s}_{p',q'} \), the dual space of \( F^{s}_{p,q} \) is \( F^{-s}_{p',q'} \);

(7) the interpolation spaces theorem is true for \( B^{s}_{p,q} \) and \( F^{s}_{p,q} \), i.e. for \( 0 < \theta < 1 \) when

\[
s = (1 - \theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1},
\]

we have \((B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1})_{\theta} = B^{s}_{p,q}, (F^{s_0}_{p_0,q_0}, F^{s_1}_{p_1,q_1})_{\theta} = F^{s}_{p,q} \);

(8) when \( 1 < p < \infty \), \( F^{s}_{p,2} = W^{s,p}, F^{s}_{1,2} \hookrightarrow W^{s,1}, W^{s,\infty} \hookrightarrow F^{s}_{\infty,2} \).
3 Some lemmas

In this section, we give some useful lemmas and propositions proved already, which will be used in our proof.

The following Bernstein’s inequality is very useful in time-frequency analysis (see [22]) :

**Lemma 3.1. [Bernstein’s inequality]** Let $0 < p \leq q \leq \infty, b > 0, \xi_0 \in \mathbb{R}^d$. Denote $L^p_{B(\xi,b)} = \{ f \in L^p : \text{supp} \hat{f} \subseteq B(\xi,R) \}$. Then there exists $C(d,p,q) > 0$, such that

$$
\|f\|_q \leq C(d,p,q) R^d(1/p - 1/q) \|f\|_p
$$

holds for all $f \in L^p_{B(\xi,b)}$ and $C(d,p,q)$ is independent of $b > 0$ and $\xi_0 \in \mathbb{R}^d$.

Also, by using the Bernstein’s inequality, we can get the following Young type inequality for $0 < p < 1$:

**Lemma 3.2 ([7]).** Let $0 < p < 1, R_1, R_2 > 0, \xi_1, \xi_2 \in \mathbb{R}^d$, then there exists $C(d,p) > 0$, such that

$$
\| |f| * |g| \|_p \leq C(d,p)(R_1 + R_2)^d(1/p - 1) \|f\| \|g\|
$$

holds for all $f \in L^p_{B(\xi_1,R_1)} , g \in L^p_{B(\xi_2,R_2)}$.

By the definition of $\Box_k$ and $\triangle_j$ operators, we know that they are all convolution type operators, so by the Lemma above and the usual Young’s inequality, we have:

**Corollary 3.3.** For $0 < p \leq \infty, 1 \leq j, \ell, k \in \mathbb{Z}^d$, there exists $C > 0$ independent of $k, j$, such that

$$
\|\triangle_j \Box_k f\| \leq C \|\Box_k f\|_p , \\|\triangle_j \triangle_\ell f\|_p \leq C \|\triangle_j f\|_p .
$$

Finally we recall a result known already, which is the special case of Theorem 4.1 and Theorem 4.2.

**Proposition 3.4 ([2]).** For $0 < p,q \leq \infty, s \in \mathbb{R}$, we have

1. $B^s_{p,q} \hookrightarrow M_{p,q}$ if and only if $s \geq \tau(p,q)$;

2. $M_{p,q} \hookrightarrow B^s_{p,q}$ if and only if $s \leq \sigma(p,q)$.

**Proposition 3.5. ([2])** Let $1 \leq p,q \leq \infty, s \in \mathbb{R}$, then $W^{s,p} \hookrightarrow M_{p,q}$ if and only if one of the following cases is satisfied:

1. $1 < p \leq q, s \geq \tau(p,q)$;

2. $p > q, s > \tau(p,q)$;

3. $p = 1, q = \infty, s \geq \tau(1,\infty)$;

4. $p = 1, q \neq \infty, s > \tau(1,q)$. 


4 Embedding between Besov spaces and modulation spaces

In this section, we consider the embedding with the form $B^{s}_{p_0,q_0} \hookrightarrow M_{p,q}$. If we get the sufficient and necessary condition of this embedding, then the embedding $B^{s_1}_{p_1,q_1} \hookrightarrow M_{p,q}$ can be done in the same way.

Our main result is:

**Theorem 4.1.** For $0 < p, q, p_0, q_0 \leq \infty, s \in \mathbb{R}$, the embedding $B^{s}_{p_0,q_0} \hookrightarrow M_{p,q}$ is true if and only if one of the following two conditions is satisfied:

1. $p_0 \leq p, q_0 \leq q, s \geq \tau(p_0, q)$;
2. $p_0 \leq p, q_0 > q, s > \tau(p_0, q)$.

As for the other side of the embedding between Besov spaces and modulation spaces, we also have:

**Theorem 4.2.** For $0 < p, q, p_1, q_1 \leq \infty, s \in \mathbb{R}$, the embedding $M_{p,q} \hookrightarrow B^{s}_{p_1,q_1}$ is true if and only if one of the following two conditions is satisfied:

1. $p_1 \geq p, q_1 \geq q, s \leq \sigma(p_1, q)$;
2. $p_1 \geq p, q_1 < q, s < \sigma(p_1, q)$.

**Remark 4.3.** If we regard the Besov space as a special $\alpha$-modulation space when $\alpha = 1$ as stated in [24], the result above is coincident with the Theorem 1.2 in [24], where the authors got the full characterization of embedding between $\alpha$-modulation spaces.

For a special case when $p_0 = q_0 = 2$, which means that $B^{s}_{p_0,q_0} = H^s$, the classical Sobolev spaces, then we have

**Corollary 4.4.** Let $0 < p, q \leq \infty, s \in \mathbb{R}$, we have

1. the embedding $H^s \hookrightarrow M_{p,q}$ is true if and only if one of the following two cases is true:
   1. $2 \leq p, 2 \leq q, s > 0$;
   2. $2 \leq p, 2 > q, s > d(1/q - 1/2)$.
2. The embedding $M_{p,q} \hookrightarrow H^s$ is true if and only if the following two cases is true:
   1. $p \leq 2, q \leq 2, s \leq 0$;
   2. $p \leq 2, q > 2, s < d(1/q - 1/2)$.

4.1 Proof of Theorem 4.1

In this subsection, we prove the Theorem 4.1. For sufficiency, by using the lemmas above, we can easily prove it. As for necessity, it may be much more difficult. Here, we mostly use the way in [2] to prove it.
4.1.1 Sufficiency

Use Lemma 2.2 we know that for any case in 4.1 we have \( B_{p_0,q_0}^s \hookrightarrow B_{p_0,q_0}^{T(p_0,q)} \); then by Proposition 3.4 we know that \( B_{p_0,q_0}^{T(p_0,q)} \hookrightarrow M_{p_0,q} \); also by Lemma 2.1 we have \( M_{p_0,q} \hookrightarrow M_{p,q} \).

4.1.2 Necessity

If we have \( B_{p_0,q_0}^s \hookrightarrow M_{p,q} \), then we have

\[
\|f\|_{M_{p,q}} \lesssim \|f\|_{B_{p_0,q_0}^s}, \quad \forall f \in B_{p_0,q_0}^s.
\] (4.1)

Then we can take some different kinds of \( f \) into (4.1) to get some restrictions of indexes.

1. Choose \( \eta \in \mathcal{S} \), such that \( \text{supp} \ \eta \subset [1/8, 1/8]^d \), denote \( f = \mathcal{F}^{-1} \eta \), for \( \forall \lambda \leq 1 \), take \( f_\lambda(x) = f(\lambda x) \) into (4.1). By the choose of \( \eta \), we know that \( \text{supp} \ \mathcal{F} f_\lambda \subset [1/8, 1/8]^d \), which means that

\[
\Box_k f_\lambda = \begin{cases} f_\lambda, & k = 0; \\ 0, & k \neq 0; \end{cases} \quad \triangle f_\lambda = \begin{cases} f_\lambda, & k = 0; \\ 0, & k \neq 0. \end{cases}
\]

So, take \( f_\lambda \) into (4.1), we have \( \|f_\lambda\|_p \lesssim \|f_\lambda\|_{p_0} \), which means that \( \lambda^{-d/p} \lesssim \lambda^{-d/p_0} \). Take \( \lambda \to 0 \), we have \( p_0 \leq p \).

2. Denote \( A_j := \{ k \in \mathbb{Z}^d : k + [-3, 3/4]^d \subseteq D_j \} \), for \( \ell \geq 10 \), choose \( k_\ell \in A_\ell \), take \( f_\ell = \mathcal{F}^{-1} \eta(-k_\ell) \), \( \eta \) is just the function in (1). Then we know that \( \text{supp} \ \mathcal{F} f_\ell \subseteq k_\ell + [-1/8, 1/8]^d \subseteq D_\ell \), so we have

\[
\Box_k f_\ell = \begin{cases} f_\ell, & k = k_\ell; \\ 0, & k \neq k_\ell; \end{cases} \quad \triangle f_\ell = \begin{cases} f_\ell, & j = \ell; \\ 0, & 0 < \|j - \ell\|_\infty \leq 3; \\ 0, & \|j - \ell\|_\infty > 3. \end{cases}
\]

Then by Corollary 3.3, we have

\[
\|f_\ell\|_{M_{p,q}} = \|f_\ell\|_p \approx 1
\]

\[
\|f_\ell\|_{B_{p_0,q_0}^s} = \left\|2^j s \|\triangle_j f_\ell\|_{p_0}\right\|_{\ell_0^{p_0} \|f_\ell\|_{L_\infty} \lesssim 3} \lesssim 2^{\ell s} \|f_\ell\|_{p_0} \approx 2^{\ell s}.
\]

Then take the estimate above into (4.1), we have \( 1 \lesssim 2^{\ell s}, \forall \ell \geq 10 \), so we get \( s \geq 0 \).

3. Take \( f_\ell = \mathcal{F}^{-1} \varphi_\ell \), then by Corollary 3.3 we have

\[
\|f_\ell\|_{M_{p,q}} = \left\|\Box_k f_\ell\right\|_{\ell_k} \approx \left\|\Box_k f_\ell\right\|_{\ell_k^{p_0} \|f_\ell\|_{L_\infty} \lesssim 3} \lesssim 2^{\ell d/p};
\]

\[
\|f_\ell\|_{B_{p_0,q_0}^s} = \left\|2^j s \|\triangle_j f_\ell\|_{p_0}\right\|_{\ell_0^{p_0} \|f_\ell\|_{L_\infty} \lesssim 3} \lesssim 2^{\ell s} \|f_\ell\|_{p_0} \approx 2^{\ell (s+d(1/p_0 + 1/q - 1))}.
\]

Take the estimates above into (4.1), let \( \ell \to \infty \), we have \( s \geq d(1/p_0 + 1/q - 1) \).
(4) When \( p_0 \geq 2 \), choose \( \eta \in \mathcal{S} \), such that \( \text{supp } \hat{\eta} \subseteq [-1/8, 1/8]^d \), take \( f_\ell(x) = \sum_{k \in A_\ell} e^{ikx} \eta(x-k) \), then \( \hat{f}_\ell(\xi) = \sum_{k \in A_\ell} e^{-ik(\xi-k)} \hat{\eta}(\xi-k) \), so we know that \( \text{supp } \hat{f}_\ell \subseteq \bigcup_{k \in A_\ell} k + [-1/8, 1/8]^d \subseteq D_\ell \), so by the orthogonality of \( \triangle_j, \Box_k \), we have:

\[
\Box_k f_\ell = \begin{cases} 
e^{ikx} \eta(x-k), & k \in A_\ell; \\ 0, & \text{else;}
\end{cases} \quad \triangle_j f_\ell = \begin{cases} f_\ell, & j = \ell; \\ \triangle_j f_\ell, & 0 < ||j-\ell||_\infty \leq 3; \\ 0, & ||j-\ell||_\infty > 3.
\]

Then we have

\[
\|f_\ell\|_{M_{p,q}} \geq \left\| \Box_k f_\ell \right\|_{\ell^q_{k \in A_\ell}} \approx 2^{d/q},
\]

\[
\|f_\ell\|_{B^s_{p_0,q_0}} = \left\| 2^{js} \| \triangle_j f_\ell \|_{p_0} \right\|_{\ell^q_j} = \left\| 2^{js} \| \triangle_j f_\ell \|_{p_0} \right\|_{\ell^q_j} \lesssim 2^{\ell s} \|f_\ell\|_{p_0} \lesssim 2^{\ell s} \|f_\ell\|^2_{p_0} \|f\|_1^{-2/p_0} .
\]

By the orthogonality of \( L^2 \) space, we have

\[
\|f_\ell\|_2 = \left( \sum_{k \in A_\ell} \|\eta(x-k)\|^2_2 \right)^{1/2} \approx 2^{d/2}.
\]

Also by the fast decay of \( \eta \), we have

\[
|f_\ell(x)| \leq \sum_{k \in A_\ell} |\eta(x-k)| \leq \sum_{k \in \mathbb{Z}^d} (1 + |x-k|)^{-N} \lesssim 1.
\]

Combine the estimates above, we have \( \|f\|_{B^s_{p_0,q_0}} \lesssim 2^{(s+d)/p_0} \). Then take it into (4.11), and put \( \ell \to \infty \), we have \( s \geq d(1/q - 1/p_0) \).

(5) Combine case(2)-(4), we have \( s \geq \tau(p_0, q) \). In contrast with the necessary condition we need in Theorem (4.11) we only need to prove that if \( B^s_{p_0,q_0} \to M_{p,q} \), then we have \( q_0 \leq q \).

(5.1) When \( \tau(p_0, q) = 0 \), take \( f = \sum_\ell a_\ell f_\ell \), where \( f_\ell \) in case (2), then by the same calculation, we have

\[
\|f\|_{M_{p,q}} \approx \|a_\ell\|_{\ell^p}; \quad \|f\|_{B^s_{p_0,q_0}} \lesssim \|a_\ell\|_{\ell^{p_0}} .
\]

Take the estimates into (4.11), we have \( q_0 \leq q \).

(5.2) When \( \tau(p_0, q) = d(1/p_0 + 1/q - 1) \), take \( f = \sum_\ell a_\ell f_\ell \), where \( f_\ell \) in case (3), then by the same calculation, we have

\[
\|f\|_{M_{p,q}} = \left\| \sum_\ell a_\ell \Box_k f_\ell \right\|_{p} \left\| \sum_\ell a_\ell \Box_k f_\ell \right\|_{\ell^q_{k \in A_\ell}} \geq \left\| \sum_\ell a_\ell \Box_k f_\ell \right\|_{p} \left\| \sum_\ell a_\ell \Box_k f_\ell \right\|_{\ell^q_{k \in A_\ell}} \approx \|a\|_{2^{d/q} \ell^q};
\]

\[
\|f\|_{B^s_{p_0,q_0}} = \left\| 2^{j\tau} \sum_\ell a_\ell \triangle_j f_\ell \right\|_{p_0} \leq \sum_{\ell ||\ell||_\infty \leq 3} \|a\|_{2^{j\tau} \ell^q} \|\triangle_j f_{\ell+j}\|_{p_0} \ell^{p_0} .
\]

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Lemma 2.2, we have
\[ a_{t+j} 2^{jd(1-1/p_0)} 2^{j\tau} \leq a_{j} 2^{jd/q} \]

Take the estimates into (4.1), we have \( q_0 \leq q \).

(5.3) When \( \tau(p_0, q) = d(1/q - 1/p_0) \), take \( f = \sum t a_{t} f_{t} \), where \( f_{t} \) in case (4), then by the same calculation, we have

\[
\|f\|_{p, q} = \left\| \sum \tau a_{\ell} \right\|_{p, q, |k|} \approx \left\| \sum a_{\ell} f_{\ell} \right\|_{p, q, |k|} = \left\| a_{j} 2^{jd/q} \right\|_{p, q, |k|}.
\]

\[
\|f\|_{B_{p, q, |k|}} = 2^{j\tau} \left\| \sum \tau a_{\ell} \right\|_{p, q, |k|} \approx \sum a_{j} 2^{jd/p} \left\| f_{\ell} \right\|_{p, q, |k|} \leq \sum a_{j} 2^{jd/|k|} \left\| f_{\ell} \right\|_{p, q, |k|} \leq \left\| a_{j} 2^{jd/q} \right\|_{p, q, |k|}.
\]

Take the estimates into (4.1), we have \( q_0 \leq q \).

4.2 Proof of Theorem 4.2

In this section, we prove the Theorem 4.2. Because we consider the case when \( 0 < p, q \leq \infty \), we can not only use the duality to prove it. Thanks to the wonderful method in [2], also combining the construction in the proof of Theorem 4.1 we can prove this theorem as well.

4.2.1 Sufficiency

By Lemma 2.1 we have \( M_{p, q} \hookrightarrow M_{p_1, q} \); by Proposition 3.1 we have \( M_{p_1, q} \hookrightarrow B^{p}_{p_1, q} \); finally by Lemma 2.2 we have \( B^{p}_{p_1, q} \hookrightarrow B^{s}_{p_1, q} \) for any case in Theorem 4.2.

4.2.2 Necessity

If we have \( M_{p, q} \hookrightarrow B^{s}_{p_1, q} \), then we have

\[
\|f\|_{B_{p_1, q}} \leq \|f\|_{M_{p, q}}.
\]

Just like the proof of Theorem 4.1 we can take some different kinds of \( f \) into 4.2 to get some restrictions of indexes, and in some cases, we can take the same \( f \) as in 4.1.2.

(1) Take \( f_{\lambda} \) in the case (1) in 4.1.2 into 4.2, we can get \( p \leq p_1 \).

(2) Take \( f_{\ell} \) in the case (2) in 4.1.2 we have \( \|f_{\ell}\|_{M_{p, q}} \approx 1 \), and \( \|f_{\ell}\|_{B^{s}_{p_1, q}} \geq 2^{\ell s} \|\Delta_{\ell} f_{\ell}\|_{p_1} = 2^{\ell s} \|f_{\ell}\|_{p_1} \approx 2^{\ell s} \). Then we have \( s \leq 0 \).
Denote \( B_j = \{ k \in \mathbb{Z}^d : k + [-3/4, 3/4]^d \cap D_j \neq \emptyset \} \), choose \( g \in \mathcal{S} \), such that \( \text{supp} \ g \subseteq D_0 \), then denote \( g_\ell(\cdot) = g(2^{-\ell} \cdot) \). Take \( f_\ell = \mathcal{F}^{-1} g_\ell \), so supp \( \mathcal{F} f_\ell \subseteq D_\ell \). So by Corollary 3.3 we have

\[
\| f_\ell \|_{M_{p,q}} = \left\| \| k \|_p \right\|_{\ell_\ell = B_\ell} \leq \left\| \| \mathcal{F}^{-1} \sigma_k \|_p \right\|_{\ell_\ell = B_\ell} \lesssim 2^{d/2};
\]

\[
\| f_\ell \|_{B_{p_1, q_1}} \gtrsim 2^{\| f_\ell \|_{B_{p_1, q_1}}} = 2^{\| f_\ell \|_{p_1}} \approx 2^{d(s+1-1/p_1)}.
\]

Take the estimate into (1.2), let \( \ell \to \infty \), we can get \( s \leq d(1/p_1 + 1/q - 1) \).

(4) Take \( f_\ell(x) = \sum_{k \in A_\ell} e^{ikx} \eta(\frac{|k|}{a}) \), we can choose \( a \ll 1 \) such that \( \| f_\ell \|_{p_1} \gtrsim C2^{d/p_1} \) (see [2]), then by the same calculation of case (4) in [1.2] we have

\[
\| f_\ell \|_{B_{p_1, q_1}} \gtrsim C2^{d(s+1/p_1)}; \quad \| f_\ell \|_{M_{p,q}} \lesssim 2^{d/q}.
\]

Take the estimates above into (1.2) and put \( \ell \to \infty \), we have \( s \leq d(1/q - 1/p_1) \).

(5) Combine case(2)-(4), we have \( s \leq \sigma(p_1, q) \). In contrast with the necessary condition we need in Theorem 1.2 we only need to prove that if \( M_{p,q} \hookrightarrow B_{p_1, q_1} \), then we have \( q_1 \geq q \).

By the same way in the proof of Theorem 4.1 take \( f = \sum a_\ell f_\ell \), \( f_\ell \) in case (2)-(4), we can get

\[
\| a_\ell \|_{q_1} \lesssim \| a_\ell \|_{q},
\]

which means that \( q_1 \geq q \).

5 Embedding between Sobolev spaces and modulation spaces

In this section, we can get the sufficient and necessary conditions of the embedding between \( L^r \)-Sobolev spaces and modulation spaces. We consider the most general case of this kind of embedding with the form like \( W^{s,r} \hookrightarrow M_{p,q} \), which is the general case of Proposition 3.5 where \( r = p \). Also, we consider the case when \( 0 < q \leq \infty \). Unlike the proof in that proposition, we mostly use the uniform decomposition of frequency space and the Theorem 4.1 we get in last section.

Our main result is:

**Theorem 5.1.** Let \( 1 \leq p, r \leq \infty, 0 < q \leq \infty, s \in \mathbb{R} \), then \( W^{s,r} \hookrightarrow M_{p,q} \) if and only if \( r \leq p \) and one of the following conditions is satisfied:

1. \( r > q, s > \tau(r,q) \);
2. \( 1 < r \leq q, s \geq \tau(r,q) \);
3. \( r = 1, q = \infty, s \geq \tau(r,q) \);
4. \( r = 1, 0 < q < \infty, s > \tau(r,q) \);

**Remark 5.2.** As the can see, the sufficient and necessary conditions we got in the theorem above is the same as the result in Proposition 3.4. The index \( p \) here only make sense in \( r \leq p \). The reason of this may be that the modulation space have good embedding property of the index \( p \) like in Lemma 2.1.
Figure 1: the four cases of $(1/r, 1/q)$ in Theorem 5.1

As for the other side of this kind of embedding, we also have:

**Theorem 5.3.** Let $1 \leq p, r \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, then $M_{p,q} \hookrightarrow W^{s,r}$ if and only if $p \leq r$ and one of the following conditions is satisfied:

1. $r < q, s < \sigma(r,q)$;
2. $q \leq r < \infty, s \leq \sigma(r,q)$;
3. $r = \infty, 0 < q = 1, s \leq \sigma(r,q)$;
4. $r = \infty, 1 < q \leq \infty, s < \sigma(r,q)$;

**5.1 Proof of Theorem 5.1**

In this subsection, we prove Theorem 5.1. By the Proposition 3.5, we can easily prove the sufficiency. As for necessity, we mainly construct some functions to the embedding estimates to get some restrictions of the indexes.

**5.1.1 Sufficiency**

When $1 \leq q \leq \infty$, by Proposition 3.5 in any case of (1)-(4), we have $W^{s,r} \hookrightarrow M_{r,q}$; so, if $r \leq p$, by Lemma 2.1 we have $M_{r,q} \hookrightarrow M_{p,q}$. 
When $0 < q < 1$, which could only happen in case (2) or (4), then we have $s > \tau(r, q) = \tau(r, 1) - d + d/q$. Take $\varepsilon > 0$ small enough, such that $s > \tau(r, 1) - d + d/q + \varepsilon$, then by Proposition \ref{prop:3.5} we have $W^{s, r} \hookrightarrow M^d_{r, 1}$. Then by (4) in Lemma \ref{lem:2.1} we have $M^d_{r, 1} \hookrightarrow M^{d/q - d + \varepsilon}_{r, 1}$. So we have $\|f\|_{M^{d/q - d + \varepsilon}_{r, 1}} \lesssim \|f\|_{W^{s, r}}$. Then by (5.1) we have $\|f\|_{M^{d/q - d + \varepsilon}_{r, 1}} \lesssim \|f\|_{W^{s, r}}$. Therefore, when $r = \infty$, we know that $p = \infty$, this is the case in Proposition \ref{prop:3.5}.

5.1.2 Necessity

If we have $W^{s, r} \hookrightarrow M_{p, q}$, then we have

$$\|f\|_{M_{p, q}} \lesssim \|f\|_{W^{s, r}}. \quad (5.1)$$

(i) Choose $\eta \in \mathcal{S}$, such that $\text{supp } \eta \subset [-1/8, 1/8]^d$, denote $f = \mathcal{F}^{-1}\eta$, for $\forall \lambda \leq 1$, take $f_\lambda(x) = f(\lambda x)$ into (5.1), we have $\|f_\lambda\|_r \approx \|f_\lambda\|_{W^{s, r}} \lesssim \|f_\lambda\|_{M_{p, q}} \approx \|f_\lambda\|_p$, so we have $r \leq p$. Therefore, when $r = \infty$, we know that $p = \infty$, this is the case in Proposition \ref{prop:3.5}.

(ii) When $1 \leq r < \infty$, by Lemma \ref{lem:2.2} we have $B^{s}_{r, r \wedge 2} \hookrightarrow F^s_{r, 2} \hookrightarrow W^{s, r} \hookrightarrow M_{p, q}$, so when $r \leq 2$, we have $B^s_{r, r} \hookrightarrow M_{p, q}$, by Theorem \ref{thm:4.1} we know that if $r \leq q$, then $s \geq \tau(r, q)$; if $r > q$, then $s > \tau(r, q)$.

When $r > 2$, we have $B^s_{r, 2} \hookrightarrow M_{p, q}$, also by Theorem \ref{thm:4.1} we know that if $2 \leq q$, then $s \geq \tau(r, q)$; if $2 > q$, then $s > \tau(r, q)$.

(iii) We prove that $W^{\tau(r, q), r} \hookrightarrow M_{p, q}$ is not true for any $2 \leq q < r \leq p$, in which case $\tau(r, q) = d(1/q - 1/r)$, for convenience we also denote it by $\tau$.

If not, we have $W^{\tau(r, q), r} \hookrightarrow M_{p, q}$ which is equivalent to $L^r \hookrightarrow M^{-\tau}_{p, q}$. So we have

$$\|f\|_{M^{-\tau}_{p, q}} \lesssim \|f\|_r. \quad (5.2)$$

Choose $\eta \in \mathcal{S}$, such that $\text{supp } \hat{\eta} \subset [-1/8, 1/8]^d$, take $f(x) = \sum_{k \in \mathbb{Z}^d} a_k e^{ikx} \eta(x - k)$, then $\hat{f}(\xi) = \sum_{k \in \mathbb{Z}^d} a_k e^{ik(\xi - k)} \hat{\eta}(\xi - k)$. Then by the orthogonality of $\Box_k$, we have

$$\Box_k f = \begin{cases} a_k e^{ikx} \eta(x - k); & k = k'; \\ 0; & k \neq k'. \end{cases}$$

So, we have $\|f\|_{M^{-\tau}_{p, q}} \approx \|k^{-\tau} a_k\|_{\ell^q_k}$.

By the orthogonality of $L^2$ space, we know that

$$\|f\|_2 = \left( \sum_{k \in \mathbb{Z}^d} \|a_k e^{ikx} \eta(x - k)\|_2^2 \right)^{1/2} \approx \|a_k\|_{\ell^2_k}.$$

By the rapidly decrease of $\eta$, we know that

$$|f(x)| \lesssim \sum_{k \in \mathbb{Z}^d} |a_k| |\eta(x - k)| \lesssim \|a_k\|_{\ell^\infty_k} \sum_{k \in \mathbb{Z}^d} (1 + |x - k|)^{-N} \lesssim \|a_k\|_{\ell^\infty_k}.$$

So we have $\|f\|_\infty \lesssim \|a_k\|_{\ell^\infty_k}$, then by interpolation, we have $\|f\|_r \lesssim \|a_k\|_{\ell^r_k}$. 

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We prove that when \( 0 < q < r \), the condition (2)(3); the necessity of condition (4) can be got by (ii,iv); as for condition (1), when \( \sigma \leq r \), \( \tau > d(1/q - 1/r) \), which is a contradiction.

(iv) We prove that when \( r = 1 \leq p, 0 < q < \infty, s = \tau(1, q) = d/q, W^{s,1} \hookrightarrow M_{p,q} \) is not true.

If not, we have \( W^{d/q,1} \hookrightarrow M_{p,q} \hookrightarrow M_{\infty,q} \) which is equivalent to \( L^1 \hookrightarrow M_{\infty,q}^{d/q} \), i.e.

\[
\|f\|_{M_{\infty,q}^{d/q}} \lesssim \|f\|_1.
\]

(5.3)

Choose \( \eta \in \mathcal{S} \) such that \( \hat{\eta} = 1 \) on \([-1,1]^d\), for \( 0 < t < 1 \), take \( f(x) = t^{-d}\hat{\eta}(x/t) \), then \( \hat{f} = 1 \) on \( \frac{1}{t}[-1,1]^d \), denote \( K_t = \{k \in \mathbb{Z}^d : k + [-3/4, 3/4]^d \subseteq \frac{1}{t}[-1,1]^d\} \), then we have

\[
\|f\|_{M_{\infty,q}^{d/q}} = \left\| \langle k \rangle^{-d/q} \|\nabla_k f\|_\infty \right\|_{\ell_q^q} \geq \left\| \langle k \rangle^{-d/q} \|\nabla_k f\|_\infty \right\|_{\ell_q^q} \approx \left\| \langle k \rangle^{-d/q} \right\|_{\ell_q^q}.
\]

Also, \( \|f\|_1 = \|\eta\|_1 \approx 1 \). So, take the two estimates into (5.3), we have \( \|\langle k \rangle^{-d/q}\|_{\ell_q^q} \lesssim 1 \), take \( t \to 0 \), we have \( \|\langle k \rangle^{-d/q}\|_{\ell_q^q} \lesssim 1 \), which is a contraction.

In fact, we already get the result as desired. To be specific, from (i,ii) we can get the necessity of the condition (2)(3); the necessity of condition (4) can be got by (ii,iv); as for condition (1), when \( q < r \leq 2 \) or \( r > 2 > q \), the necessity is proved in (ii), when \( 2 \leq q < r \); the necessity is proved in (iii).

### 5.2 Proof of Theorem 5.3

When \( 1 \leq q \leq \infty \), we can get the result as desired just by duality of Theorem 5.1 so we only need to consider the case \( 0 < q < 1 \), which could only happen in case (2) or (3). Also, we know that \( \sigma(r,q) = 0 \) in these cases.

Sufficiency: by Lemma 2.1 and dual of Proposition 3.5, we can get \( M_{p,q} \hookrightarrow W^{s,1} \hookrightarrow W^{s,r} \) when \( s \leq 0 \).

Necessity: if we have \( M_{p,q} \hookrightarrow W^{s,r} \) which is equivalent to \( M_{p,q}^{-s} \hookrightarrow L^r \). So, we have

\[
\|f\|_r \lesssim \|f\|_{M_{p,q}^{-s}}.
\]

For any \( k \in \mathbb{Z}^d \), take \( f(x) = e^{ikx}\eta(x) \) into the equation above, where \( \eta \in \mathcal{S} \) and supp \( \hat{\eta} \subseteq [-1/8,1/8]^d \). Then we have \( 1 \lesssim \langle k \rangle^{-s} \), which means that \( s \leq 0 \).
6 Embedding between Triebel spaces and modulation spaces

In this section, we mainly study the embedding $F^s_{p,2} \hookrightarrow M_{p,q}$. We first consider a special case like $F^s_{p,2} \hookrightarrow M_{p,q}$, which is just the same as Proposition 3.5 in some cases. And using this embedding and interpolation method we can get the result we need.

As for special case, we have:

**Theorem 6.1.** Let $0 < p, q \leq \infty$, then $F^s_{p,2} \hookrightarrow M_{p,q}$ if and only if one of the following conditions is satisfied:

1. $q \geq p, s \geq \tau(p, q)$;
2. $q < p, s > \tau(p, q)$.

As for the other side of this kind of embedding, we also have:

**Theorem 6.2.** Let $0 < p, q \leq \infty$, then $M_{p,q} \hookrightarrow F^s_{p,2}$ if and only if one of the following conditions is satisfied:

1. $q \leq p, s \leq \sigma(p, q)$;
2. $q > p, s < \sigma(p, q)$.

Then, our main result is:

**Theorem 6.3.** Let $0 < p, p_0, q \leq \infty$, then $F^s_{p_0,2} \hookrightarrow M_{p,q}$ if and only if $p_0 \leq p$ and one of the following conditions is satisfied:

1. $p_0 \leq q, s \geq \tau(p_0, q)$;
2. $p_0 > q, s > \tau(p_0, q)$.

As for the other side of this kind of embedding, we also have:

**Theorem 6.4.** Let $0 < p, p_1, q \leq \infty$, then $M_{p,q} \hookrightarrow F^s_{p_1,2}$ if and only if $p_1 \geq p$ and one of the following conditions is satisfied:

1. $p_1 \geq q, s \leq \sigma(p_1, q)$;
2. $p_1 < q, s < \sigma(p_1, q)$.

### 6.1 Proof of Theorem 6.1

For $1 < p < \infty$, we know that $F^s_{p,2} \approx W^{s,p}$, so by Theorem 5.1 we get the result as desired.

For $0 < p \leq 1$, we know that $F^s_{p,2} = h_p$ the local Hardy space, so by the result in [20], we get the result as desired.

For $p = \infty$, if we have $F^s_{\infty,2} \hookrightarrow M_{p,q}$, then by Lemma 2.2 we have $W^{s,p} \hookrightarrow F^s_{\infty,2} \hookrightarrow M_{p,q}$, then by Theorem 5.1 we have the condition as desired. As for sufficiency, in condition (1), by Lemma 2.2 we have $F^s_{\infty,2} \hookrightarrow B^{0}_{\infty,\infty}$, then by Proposition 3.4 we have $B^{0}_{\infty,\infty} \hookrightarrow M_{\infty,\infty}$; in condition (2), by Lemma 2.2 we have $F^s_{\infty,2} \hookrightarrow B^s_{\infty,\infty} \hookrightarrow B^{d/q}_{\infty,q}$, then by Proposition 3.4 we have $B^{d/q}_{\infty,q} \hookrightarrow M_{\infty,q}$. 

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6.2 Proof of Theorem 6.2

By the same discussion in the proof of Theorem 6.1, we only need to consider the case when \( p = \infty \). Also, when \( 1 \leq q \leq \infty \), by duality, we can get the result as desired. When \( 0 < q < 1, p = \infty \), we have \( \sigma(p, q) = 0 \).

Sufficiency: by Lemma 2.1, we have \( M_{\infty, q} \hookrightarrow M_{1, 1} \). By Theorem 6.1, we have \( F_{1, 2} \hookrightarrow M_{1, \infty} \), by duality, we have \( M_{\infty, 1} \hookrightarrow F_{\infty, 2} \), so we have \( M_{\infty, q} \hookrightarrow F_{\infty, 2} \hookrightarrow F_{s, 2}^s \) for \( s \leq 0 \).

Necessity: if we have \( M_{\infty, q} \hookrightarrow F_{\infty, 2}^s \), then by Lemma 2.2, we have \( F_{s, 2}^s \hookrightarrow B_{s, \infty}^{s} \). By Theorem 4.2, we have \( s \leq \sigma(\infty, q) \).

6.3 Proof of Theorem 6.3

6.3.1 Sufficiency

For condition (1): by Lemma 2.2, we have \( F_{p_0, q}^s \hookrightarrow B_{p_0, q}^s \), then by Theorem 4.1, we have \( B_{p_0, q}^s \hookrightarrow M_{p, q} \).

For condition (2): by Lemma 2.2, we have \( F_{p_0, q}^s \hookrightarrow B_{p_0, p_0}^s \hookrightarrow B_{p_0, q}^{\tau(p_0, q) + \varepsilon} \) for some \( 0 < \varepsilon \ll 1 \), then by Theorem 4.1, we have \( B_{p_0, q}^{\tau(p_0, q) + \varepsilon} \hookrightarrow M_{p, q} \).

6.3.2 Necessity

If we have \( F_{p_0, q}^s \hookrightarrow M_{p, q} \), then we have

\[
\|f\|_{M_{p, q}} \lesssim \|f\|_{F_{p_0, q}^s} .
\] (6.1)
(i) By the same method as in subsection 5.1.2, we can get $p_0 \leq p$.

(ii) By Lemma 2.2 we have $B_{p_0,p_0}^s \hookrightarrow F_{p_0,q}^s \hookrightarrow M_{p,q}$, then by Theorem 4.1 we have $s \geq \tau(p_0,q)$.

(iii) When $p_0 > q \geq 2$, by Lemma 2.2 we have $W^{s,p_0} \approx F_{p_0,2}^s \hookrightarrow F_{p_0,q}^s \hookrightarrow M_{p,q}$, then by Theorem 4.1, we have $s \geq \tau(p_0,q)$.

(iv) When $2 \geq p_0 > q$, by a similar proof of Proposition 3.2 in [21], we have $\|2^{j/p_0}a_j\|_{L^q(p_0)} \lesssim \|2^{j(s-d(1-p_0)/p_0)}a_j\|_{L^q}$, then we have $s > d(1/q + 1/p_0 - 1)$.

(v) We prove that $F_{p_0,q}^s \hookrightarrow M_{p,q}$ is not true for any $p_0 > q, q < 2, p_0 > 2$, in which case $\tau(p_0,q) = d(1/q - 1/p_0)$.

If not, we have $F_{p_0,q}^{d(1/q-1/p_0)} \hookrightarrow M_{p,q}$, also by sufficiency, we have $F_{p_0,p_0}^q \hookrightarrow M_{p_0,p_0} \hookrightarrow M_{p,p_0}$, then by interpolation in Lemma 2.2 and 2.1 we have $F_{p_0,2}^{d(1/2-1/p_0)} \hookrightarrow M_{p,2}$ which is contradiction with case (iii).

Combine the case (i), (ii), we get the condition (1); combine the case (i), (iii), (iv), (v), we get the condition (2) as desired.

Remark 6.5.

1. As for case (v) in the proof above, we prove it by contradiction using the method of interpolation. Also, we can prove it directly by take some good function into the norm inequality. One can modify the proof of Proposition 3.1 in [21] to get the result.

2. The proof of Theorem 6.4 is similar to the proof above. For the reader’s convenience, we give the outline of the proof here. The sufficiency can get by the relation of Triebel spaces and Besov spaces, then use Theorem 4.2 can get the result. As for necessity, we can use a revised version of Proposition 3.1, 3.2 in [21] directly to get the result as desired.

3. As we can see, the embedding we consider above is not the most general case like $F_{p_0,q}^s \hookrightarrow M_{p,q}$. I believe that for the general case the sufficient and necessary is the similar to Theorem 6.3. But I have not proven it yet. The most difficult case left is the same as the open problem in [21].

7 Embedding between Fourier $L^p$ spaces and modulation spaces

In this section, we consider the embedding $M_{p,q}^s \hookrightarrow \mathcal{F}L^r$. Because of the orthogonality of $\sigma_k(\sigma_h)$, we can calculate the $\mathcal{F}L^r$ norm easily. Then, we can get the result as follow. Recall that when $0 < p < 1$, we denote $p' = \infty$.

Theorem 7.1. Let $0 < p, q, r \leq \infty, s \in \mathbb{R}$, then we have $M_{p,q}^s \hookrightarrow \mathcal{F}L^r$ if and only if one of the following conditions is satisfied:

(1) $p \leq 2, q \leq r \leq p', s \geq 0$;
(2) \( p \leq 2, r \leq p', r < q, s > d(1/r - 1/q). \)

As for the other side of this kind of embedding, we also have:

**Theorem 7.2.** Let \( 0 < p, q, r \leq \infty, s \in \mathbb{R}, \) then we have \( \mathcal{F}L^r \hookrightarrow M^s_{p,q} \) if and only if one of the following conditions is satisfied:

(1) \( p \geq 2, p' \leq r \leq q, s \leq 0; \)
(2) \( p \geq 2, p' \leq r, r > q, s < d(1/r - 1/q). \)

### 7.1 Proof of Theorem 7.1

#### 7.1.1 Sufficiency

If we have condition (1), when \( 1 \leq p \leq \infty, \) by the orthogonality of \( \sigma_k, \) Hölder’s inequality of \( L^p \) spaces and Hausdorff-Young’s inequality of Fourier transform, we have

\[
\|\hat{f}\|_{r} \approx \|\sigma_k \hat{f}\|_{r} \leq \|\sigma_k \hat{f}\|_{p'} \leq \|\square_k f\|_{p} \leq \|\square_k f\|_{p'} \approx \|f\|_{M^s_{p,q}}.
\]

When \( 0 < p < 1, \) by Lemma 3.1 we have \( \|\sigma_k \hat{f}\|_{p'} \leq \|\square_k f\|_{1} \leq C \|\square_k f\|_{p}, \) then by the same estimates we can get the result as desired.

If we have condition (2), by Lemma 2.1 we have \( M^s_{p,q} \hookrightarrow M_{p,r}, \) then by the sufficiency of condition(1), we have \( M_{p,r} \hookrightarrow \mathcal{F}L^r. \)

#### 7.1.2 Necessity

If we have \( M^s_{p,q} \hookrightarrow \mathcal{F}L^r, \) then we have

\[
\|\hat{f}\|_{r} \lesssim \|f\|_{M^s_{p,q}}. \tag{7.1}
\]

The conditions we need are contained the following cases.

(i) For any \( f \in \mathcal{S} \) with \( \text{supp } \hat{f} \subseteq [-1/8, 1/8]^d, \) in this case we have \( \|f\|_{M^s_{p,q}} \approx \|f\|_{p}. \) So, (7.1) means that

\[
\|\hat{f}\|_{r} \lesssim \|f\|_{p}.
\]

Then by Hausdorff-Young’s inequality on compact set (see Tao’s note of Math 254B, one can extend the result to the case of \( 0 < p < 1 \) with the same proof), we have \( r \leq p', p \leq 2. \)

(ii) For any \( k \in \mathbb{Z}^d, \) take \( f(x) = e^{ik \cdot x} \eta(x), \) where \( \eta \in \mathcal{S}, \text{supp } \hat{\eta} \subseteq [-1/8, 1/8]^d, \) then we know that \( \text{supp } \hat{f} \subseteq k + [-1/8, 1/8]^d. \) So, we have

\[
\|f\|_{M^s_{p,q}} = \|k\|^s \|\eta\|_{p} \approx \|k\|^s; \quad \|f\|_{\mathcal{F}L^r} = \|\hat{\eta}\|_{r} \approx 1.
\]

Take the estimates into (7.1), we have \( 1 \lesssim \|k\|^s, \) which means that \( s \geq 0, \) which is the result in condition (1).
(iii) When \( r < q \), take \( f(x) = \sum_{k \in \mathbb{Z}^d} a_k e^{ikx} \eta(x) \), \( \eta \) is the same as case (ii). Then we have \( \hat{f}(\xi) = \sum_{k \in \mathbb{Z}^d} a_k \hat{\eta}(\xi - k) \). So, we have

\[
\|f\|_{M^p_q} \approx \|\langle k \rangle^s a_k\|_{\ell^p_k}; \quad \|\hat{f}\|_r \approx \|a_k\|_{\ell^r_k}.
\]

Take the estimates above into (7.1), we have

\[
\|a_k\|_{\ell^r_k} \lesssim \|\langle k \rangle^s a_k\|_{\ell^q_k}; \quad \text{then we have} \quad \|\langle k \rangle^{-sr} b_k\|_{\ell^1_k} \lesssim \|b_k\|_{\ell^{q/r}_k},
\]

which means that \( \{\langle k \rangle^{-sr}\} \in \ell^{q(r)} \). So, we have \( sr(q/r) > d \), i.e. \( s > d(1/r - 1/q) \), which is the result in condition (2).

**Remark 7.3.** The proof of Theorem 7.2 is similar to the proof above, we omit it here.

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