An exact Schwarzschild-like solution in a bumblebee gravity model

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We have obtained an exact vacuum solution from a gravity sector contained in the minimal standard-model extension. The theoretical model assumes a Riemann spacetime coupled to the bumblebee field which is responsible for the spontaneous Lorentz symmetry breaking. The solution achieved in a static and spherically symmetric scenario establishes a Schwarzschild-like black hole. In order to study the effects of the spontaneous Lorentz symmetry breaking, we have investigated some classics tests including the advance of the perihelion, bending of light and Shapiro’s time-delay. Furthermore, we have computed some upper-bounds from which the most stringent one attains a sensitivity at the $10^{-13}$ level.

I. INTRODUCTION

General Relativity (GR) and Standard Model (SM) of particle physics are examples of successfully field theories describing nature. The former describes gravitation at a classical level, and the latter describes particles and the other three fundamental interactions at a quantum level. The unification between these two theories is a fundamental seeking, and this achievement will conduct us necessarily to a deeper understanding of nature.

In the pursuit of this unification some theories of quantum gravity (QG) have already been proposed, but direct tests of their properties are currently beyond the energy scale of current experiments because they would be observed at the Planck scale ($\sim 10^{19}$ GeV). However, it is possible that some signals of QG can emerge at sufficiently low energy scales and its effects could be observed in experiments carried out at current energy scales. One of these signals would be associated with the breaking of Lorentz symmetry.

Lorentz symmetry breaking in nature is an interesting idea to be considered because it arises as a possibility in the context of string theory [1, 2], noncommutative field theories [3] or loop quantum gravity theory [4]. This fact suggests the seeking by evidences of Lorentz violation is a promising way to investigate signals of the existence of an underlying theory of quantum gravity at the Planck scale.

The standard model extension (SME) is an effective field theory describing at low energies the general relativity and the standard model, including in its structures additional terms containing information about the Lorentz violation occurring at the Planck scale [5]. The electromagnetic sector of the SME has been extensively studied in literature [6–17]; the electroweak sector in Refs. [18], some aspects of the strong sector [19] and the hadronic physics [20] as well. Furthermore, some effects of Lorentz violation in the gravitational sector have been studied in [21–25], specifically the case of the gravitational waves were analyzed in Ref. [27].

The aim of the manuscript is to obtain a spherically symmetric exact solution of the Einstein equations in the presence of a spontaneous breaking of Lorentz symmetry because of the nonzero vacuum expectation value of the bumblebee field and its influence in some well-known experimental tests of general relativity: The advance of perihelion of inner planets, the bending of light and time delay effect owing to curvature. All tests we analyzed allow to estimate some upper-bounds for the Lorentz-violating parameter involved. We will adopt the metric signature $(-+++)$ and also all quantities involved are expressed in natural units ($\hbar = c = 1$). When necessary the physical constants will be written explicitly. The manuscript is developed in the following manner: In Sec. II, it is presented a general geometrical framework allowing the existence of nonzero vacuum expectation values promoting the spontaneous breaking of local Lorentz invariance. Furthermore, we have computed the modified Einstein’s equation generated by the bumblebee gravity. In Sec. III, we solve the modified Einstein’s equation seeking for spherically symmetric solutions. In Sec. IV, we study the effects of Lorentz violation in some classical tests of general relativity and we use experimental data to establish some upper-bounds on the Lorentz-violating parameter involved. Finally, in Sec. V, we give our conclusions and remarks.
II. THE THEORETICAL FRAMEWORK

The focus of this work is to study spherically symmetric vacuum solutions in the context of an extended gravitational model including Lorentz-violating terms. Consequently, we study the effects of Lorentz violation in some classical tests of general relativity. For this purpose, we consider the bumblebee model, which is a known example of a gravity model that extends the standard formalism of GR, where under a suitable potential the bumblebee vector field $B_\mu$ acquires a nonvanishing vacuum expectation value (VEV) inducing a spontaneous Lorentz symmetry breaking (LSB).

In order to investigate the spontaneous Lorentz symmetry breaking in the extended gravitational sector, we consider the special class of theories in which the Lorentz violation arises from the dynamics of a single vector $B_\mu$ that acquires a nonzero vacuum expectation value. These theories are called bumblebee models and are among the simplest examples of field theories with spontaneous Lorentz and diffeomorphism violations. It is well-known in the literature that the local LSB is always accompanied by diffeomorphism violation [28]. In this scenario, the spontaneous LSB is triggered by a potential whose functional form possesses a minimum which ensures the breaking of the $U(1)$ symmetry. In general, the action for a single bumblebee field $B_\mu$ coupled to gravity and matter can be written as

$$S_B = \int d^4 x \ L_B = \int d^4 x (L_g + L_g B + L_K + L_V + L_M).$$

In Riemann spacetime, $L_g$ is the pure gravitational Einstein-Hilbert term which may also include the cosmological constant, $L_{gB}$ describes the gravity-bumblebee coupling, $L_K$ contains the bumblebee kinetic and any self-interaction terms, $L_V$ correspond to the potential, which includes terms that trigger the spontaneous Lorentz violation, and $L_M$ defines the matter and other fields contents and their couplings to the bumblebee field. By considering the case of a spacetime with null torsion and null cosmological constant ($\Lambda = 0$), we introduce the following Lagrangian density

$$L_B = \frac{e}{2\kappa} R + \frac{e}{2\kappa} \xi B^\mu B_\mu - \frac{1}{4} e B_{\mu \nu} B^{\mu \nu} - eV(B^\mu) + L_M,$$

being $e \equiv \sqrt{-g}$ the determinant of the vierbein and $\xi$ the real coupling constant (with mass dimension $-1$) which controls the nonminimal gravity-bumblebee interaction. The corresponding bumblebee field strength is defined as

$$B_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,$$

where $B_\mu$ has mass dimensions 1. We point out that some bumblebee models involving nonzero torsion in more general context are investigated in Refs. [5, 28].

For our purposes, the particular form of the potential $V(B_\mu)$ in Eq. (2) driving its dynamics is irrelevant, but it is important to emphasize that it must be formed from scalar combinations of the bumblebee field $B_\mu$ and the metric $g_{\mu \nu}$. In any case, we choose a potential $V$ providing a nonvanishing VEV for $B_\mu$ which could have the following general functional form

$$V \equiv V(B^\mu B_\mu \pm b^2),$$

where $b^2$ is a positive real constant. Some qualitative features of the symmetry breaking potential have been explored in Refs. [1, 5, 28, 31]. It follows the VEV of the bumblebee field is determined when $V(B^\mu B_\mu \pm b^2) = 0$ implying that the condition

$$B^\mu B_\mu \pm b^2 = 0,$$

must be satisfied. This is solved when the field $B^\mu$ acquires a non null vacuum expectation value

$$\langle B^\mu \rangle = b^\mu,$$

where the vector $b^\mu$ is a function of the spacetime coordinates such that $b^\mu b_\mu = \mp b^2 = \text{const.}$, then the nonnull vector background $b^\mu$ spontaneously breaks the Lorentz symmetry. We note the $\pm$ signs in the potential (4) determine whether the field $b^\mu$ is timelike or spacelike.

On the other hand, the Lorentz-violating contribution to the gravitational sector provided by the minimal SME is

$$S_{LV} = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} (uR + s^{\mu \nu} R_{\mu \nu} + t^{\mu \nu \alpha \beta} R_{\mu \nu \alpha \beta}),$$

where $u$, $s^{\mu \nu}$ and $t^{\mu \nu \alpha \beta}$ are real and dimensionless tensors carrying on information about Lorentz violation. It is possible to establish a correspondence between the bumblebee action (1) and the Lorentz-violating action (7) by considering the following correspondence of the underlying bumblebee field and the metric with the Lorentz-violating tensors $u$, $s^{\mu \nu}$ and $t^{\mu \nu \alpha \beta}$:

$$u = \frac{1}{4} \xi B^\mu B_\mu, \quad s^{\mu \nu} = \xi \left( B^\mu B^\nu - \frac{1}{4} g^{\mu \nu} B_\alpha B^\alpha \right),$$

$$t^{\mu \nu \alpha \beta} = 0,$$

with $s^{\mu \nu}$ being traceless [5, 25, 28, 30].

The next step is to establish the fields equations from the action (1) aiming at finding vacuum solutions in the context of extended gravitational sector.

A. The Field Equations

The Lagrangian density (2) yields the extended Einstein equations

$$G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = \kappa T_{\mu \nu},$$

where $T_{\mu \nu}$ is the energy-momentum tensor of matter.

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where \( G_{\mu \nu} \) is the Einstein tensor and \( T_{\mu \nu} \) is the total energy-momentum tensor arising from the matter sector \( (T_{\mu \nu}^M) \) and the contributions of the bumblebee field \( (T_{\mu \nu}^B) \), so we write
\[
T_{\mu \nu} = T_{\mu \nu}^M + T_{\mu \nu}^B
\tag{11}
\]
with
\[
T_{\mu \nu}^B = -B_{\mu \alpha}B_{\nu}^\alpha - \frac{1}{4}B_{\alpha \beta}B^{\alpha \beta}g_{\mu \nu} - Vg_{\mu \nu} + 2V'B_{\mu}B_{\nu} \\
+\frac{\xi}{\kappa}\left[ \frac{1}{2}B^\alpha B^\beta R_{\alpha \beta}g_{\mu \nu} - B_{\mu}B^{\beta}R_{\alpha \nu} - B_{\nu}B^{\alpha}R_{\alpha \mu} \right] \\
+\frac{1}{2}\nabla_{\alpha}\nabla_{\nu}(B^\alpha B_{\nu}) + \frac{1}{2}\nabla_{\alpha}\nabla_{\nu}(B^\alpha B_{\mu}) \\
-\frac{1}{2}\nabla^2(B_{\mu}B_{\nu}) - \frac{1}{2}g_{\mu \nu}\nabla_{\alpha}\nabla_{\beta}(B^{\alpha}B^{\beta}).
\tag{12}
\]
The prime denotes differentiation with respect to the argument, as usual. Similarly, Eq. (2) provides the following equation of motion for the bumblebee field,
\[
\nabla^\mu B_{\mu \nu} = J_{\nu},
\tag{13}
\]
where \( J_{\nu} = J_{\nu}^B + J_{\nu}^M \), with \( J_{\nu}^M \) being associated with the matter sector (acting as a source of the bumblebee field) and \( J_{\nu}^B \) a partial current that arises from the bumblebee self-interaction given explicitly as
\[
J_{\nu}^B = 2V'B_{\nu} - \frac{\xi}{\kappa}B^{\mu}R_{\mu \nu}.
\tag{14}
\]
Taking the covariant divergence on the extended Einstein equations (10) and using the contracted Bianchi identities \( (\nabla^\mu G_{\mu \nu} = 0) \), this leads to condition
\[
\nabla^\mu T_{\mu \nu} = 0,
\tag{15}
\]
which gives the covariant conservation law for the total energy-momentum tensor \( T_{\mu \nu} \).

The trace of Eq. (10) reads
\[
R = -\kappa T^M + 4\kappa V - 2\kappa V'B_{\mu}B^{\mu} \\
+\frac{\xi}{\kappa}\left[ \frac{1}{2}\nabla^2(B_{\mu}B^{\mu}) + \nabla_{\alpha}\nabla_{\beta}(B^\alpha B^\beta) \right],
\tag{16}
\]
where \( T^M = g^{\mu \nu}T_{\mu \nu}^M \), and substituting it in Eq. (10) we obtain the trace-reversed version
\[
R_{\mu \nu} = \kappa\left( T_{\mu \nu}^M - \frac{1}{2}g_{\mu \nu}T^M \right) + \kappa T_{\mu \nu}^B + 2\kappa g_{\mu \nu}V \\
-\kappa B_{\alpha}B^{\alpha}g_{\mu \nu}V' + \frac{\xi}{4}g_{\mu \nu}\nabla^2(B_{\alpha}B^{\alpha}) \\
+\frac{\xi}{2}g_{\mu \nu}\nabla_{\alpha}\nabla_{\beta}(B^{\alpha}B^{\beta}).
\tag{17}
\]
Note that if both the bumblebee field \( B_{\mu} \) and the potential \( V(B_{\mu}) \) vanishes, Eq. (17) recovers the usual GR equations, as expected.

### III. A SPHERICALLY SYMMETRIC SOLUTION IN THE LSB SCENARIO

We focus on a vacuum solution, i.e., the one describing an empty space surrounding a gravitating body by imposing \( T_{\mu \nu}^M = 0 \). We note the potential in (2) vanishes when Eq. (6) is satisfied, which characterizes the vacuum. We will assume this scenario in the remainder of this manuscript.

Specifically, we are interested in vacuum solutions induced by the LSB when the bumblebee field \( B_{\mu} \) remains frozen in its vacuum expectation value \( b_{\mu} \). Similar hypothesis was used in Ref. [32]. In this way, the bumblebee field is fixed to be
\[
B_{\mu} = b_{\mu},
\tag{18}
\]
consequently, we have \( V = 0 \) and \( V' = 0 \), which when substituted in Eq. (17) provides the extended Einstein equations in vacuum
\[
0 = \bar{R}_{\mu \nu} \\
= R_{\mu \nu} + \kappa b_{\mu \alpha}b^{\alpha \nu} + \frac{\kappa}{4}b_{\mu \alpha \beta}b^{\alpha \beta}g_{\mu \nu} + \xi b_{\mu}b^{\alpha}R_{\alpha \nu} \\
+\frac{\xi}{2}b_{\alpha}b^{\alpha}R_{\mu \nu} + \frac{\xi}{2}\nabla_{\alpha}\nabla_{\beta}(b^{\alpha}b^{\beta}) - \frac{\xi}{2}\nabla_{\alpha}\nabla_{\nu}(b^{\beta}b_{\mu}) + \frac{\xi}{2}\nabla^2(b_{\mu}b_{\nu}),
\tag{19}
\]
where \( b_{\mu \nu} = \partial_{\mu}b_{\nu} - \partial_{\nu}b_{\mu} \) is the field strength of the vector \( b_{\nu} \).

In order to obtain a static, spherically symmetric vacuum solution to the extended Einstein equations, we assume a spacetime driven by a Birkhoff metric \( g_{\mu \nu} = \text{diag}(-c^2\gamma, c^2\epsilon^2, r^2, r^2\sin^2\theta) \), with \( \gamma \) and \( \rho \) being functions of \( r \). Hereafter, we consider a spacelike background \( b_{\mu} \) assuming the form
\[
b_{\mu} = (0, b_{r}(r), 0, 0).
\tag{20}
\]

Moreover, once we have assumed a background field in the form (20), it follows that all components of the corresponding field strength vanishes, i.e., \( b_{\mu \nu} = 0 \).

Now by using the condition \( b^{\mu}b_{\mu} = b^2 = \text{const.} \), we determine the explicit form of the radial background field
\[
b_{r}(r) = |b|\epsilon^{\rho},
\tag{21}
\]
It is easy to verify that the background given by Eq. (21) is not covariantly constant, i.e., we have some non-vanishing values for \( \nabla_{\mu}b_{\nu} \). It is worthwhile to point the difference with the proposal analyzed in Ref. [32] which assumes the condition \( \nabla_{\mu}b_{\nu} = 0 \).

Next, we proceed to solve for the functions \( \gamma(r) \) and \( \rho(r) \). For this, we take the extended Einstein equations in vacuum with our metric ansatz to get the following
nonvanishing components for the tensor \( (19) \):

\[
\bar{R}_{tt} = \left( 1 + \frac{\ell}{2} \right) R_{tt} + \frac{\ell}{r} (\partial_t \gamma + \partial_t \rho) e^{2(\gamma - \rho)},
\]

\( (22) \)

\[
\bar{R}_{rr} = \left( 1 + \frac{3\ell}{2} \right) R_{rr},
\]

\( (23) \)

\[
\bar{R}_{\theta\theta} = (1 + \ell) R_{\theta\theta} - \ell \left( \frac{1}{2} \right)^2 e^{-2\rho} R_{rr} + 1,
\]

\( (24) \)

\[
\bar{R}_{\phi\phi} = \sin^2 \theta \bar{R}_{\theta\theta},
\]

\( (25) \)

where we have defined the constant \( \ell = \xi \theta^2 \). The components of the Ricci tensor \( R_{\mu\nu} \) that appear above are given by

\[
R_{tt} = e^{2(\gamma - \rho)} \left[ \partial_t^2 \gamma + (\partial_t \gamma)^2 - \partial_t \gamma \partial_t \rho + \frac{2}{r} \partial_t \gamma \right],
\]

\( (26) \)

\[
R_{rr} = -\partial_t^2 \gamma - (\partial_r \gamma)^2 + \partial_t \gamma \partial_r \rho + \frac{2}{r} \partial_r \rho,
\]

\( (27) \)

\[
R_{\theta\theta} = e^{-2\rho} [r (\partial_r \rho - \partial_r \gamma) - 1] + 1.
\]

\( (28) \)

Note that, as in a scenario without a background field, \( \bar{R}_{\mu\nu} \) also has only three diagonal independent components.

According to Eq. \((19)\), each of these components vanishes independently so we do the following combination:

\[
r^2 e^{-2\gamma} \bar{R}_{tt} - r^2 e^{-2\rho} \bar{R}_{rr} + 2 \bar{R}_{\theta\theta} = 0,
\]

\( (29) \)

which yields a differential equation for the function \( \rho(r) \),

\[
\partial_r (re^{-2\rho}) (1 + \ell) = 1.
\]

\( (30) \)

It is easy to show the solution is

\[
e^{2\rho} = (1 + \ell) \left( 1 - \frac{\rho_0}{r} \right)^{-1},
\]

\( (31) \)

where \( \rho_0 \) is an arbitrary constant.

With the aim to find the function \( \gamma(r) \) we consider the following combination

\[
r^2 e^{-2\gamma} \bar{R}_{tt} - \left( 1 + \frac{2}{\ell} \right) \bar{R}_{\theta\theta} = 0,
\]

\( (32) \)

which provides

\[
0 = (2 + 3\ell) (1 + \ell) r \partial_r \gamma + (1 + \ell) (2 + \ell) + (\ell^2 - \ell - 2) r \partial_r \rho - (2 + \ell) e^{2\rho}.
\]

\( (33) \)

By substituting Eq. \((31)\), we obtain an explicit differential equation for \( \gamma(r) \),

\[
(2 + 3\ell) \left( (\rho_0 - r) \partial_r \gamma + \frac{\rho_0}{2r} \right) = 0,
\]

\( (34) \)

whose solution, written in a convenient form, is given by

\[
e^{2\gamma} = e^{-2\gamma_0} \left( 1 - \frac{\rho_0}{r} \right),
\]

\( (35) \)

where \( e^{-2\gamma_0} \) is a constant which can be removed by means of the rescaling \( t \to e^{\gamma_0} t \). It can be verified in fact the solutions \((31)\) and \((35)\) actually satisfy the set of Eqs. \((22)-(25)\).

Finally, we write down the LSB spherically symmetric solution

\[
ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + (1 + \ell) \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\]

\( (36) \)

where we have conveniently identified \( \rho_0 \equiv 2M \) \((M = G N m \) is the usual geometrical mass\) such that in the limit \( \ell \to 0 \) \((b^2 \to 0)\) we recover the usual Schwarzschild metric. The metric \((36)\) represents a purely radial LSB solution outside a spherical body characterizing a modified black hole solution. Furthermore, we compute the Kretschmann scalar

\[
R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = \frac{4 (12M^2 + 4\ell Mr + \ell^2 r^2)}{r^6 (\ell + 1)^2},
\]

\( (37) \)

which clearly differs from the Schwarzschild Kretschmann invariant for nonnull \( \ell \). This ensures the metric \((36)\) is a true solution containing Lorentz-violating corrections, i.e., it means there exists no coordinate transformation connecting the metric \((36)\) to the Schwarzschild one, otherwise, the scalar invariant \((37)\) would be the same for both metrics. We observe, for \( r = 2M \), the Kretschmann invariant is finite so such a singularity can be removed (by an adequate coordinate transformation). However, \( r = 0 \) is a physical, or not removable, singularity due to the fact the Kretschmann invariant is divergent. Therefore, we point out the nature of the singularities \( r = 0 \) and \( r = 2M \) (event horizon) remains unchanged.

IV. SOME CLASSICAL TESTS

In this section, we shall study the motion of particles in a spacetime described by the spherically symmetric solution \((36)\). With the aim to impose some upper-bounds for the Lorentz-violating coefficient \( \ell \), we consider the Solar system to study the effects of LV on precession of perihelia of inner planets, the bending of light around the Sun and the Shapiro time-delay effect.

We consider the motion of test particles along the geodesics described by \( x^\mu(\lambda) \) obeying the equation,

\[
\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\nu} \frac{dx^\sigma}{d\lambda} \frac{dx^\nu}{d\lambda} = 0,
\]

\( (38) \)

where \( \lambda \) is an affine parameter. However, due to the metric compatibility, it is always possible to use a constant of motion, \( \chi \), defined by

\[
\chi = g_{\mu\nu} U^\mu U^\nu,
\]

\( (39) \)
where the vector \( u^\mu \) is defined as
\[
U^\mu = \frac{dx^\mu}{d\lambda} = \dot{x}^\mu, \tag{40}
\]
where dot denotes differentiation with respect to the affine parameter. For massive particles, the affine parameter is typically chosen to be the proper time \( \tau \) and \( \chi = +1 \) (timelike geodesics). On the other hand, for massless particles we have \( \chi = 0 \) and the parameter \( \lambda \) is not fixed (null geodesics).

### A. Advance of the perihelion

From the geodesic equation (38), we obtain the equations describing the trajectory of the massive test particle moving in the spacetime (36):
\[
\frac{d}{d\tau} \left[ \left( 1 - \frac{2M}{r} \right) \dot{t} \right] = 0, \tag{41}
\]
\[
\dot{r} + \frac{M (r - 2M)}{r^3 (\ell + 1)} \dot{t}^2 - \frac{M}{r (r - 2M)} \dot{r}^2 - \frac{r - 2M}{\ell + 1} \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) = 0, \tag{42}
\]
\[
\frac{d}{d\tau} \left( r^2 \dot{\theta} \right) - r^2 \sin \theta \cos \theta \dot{\phi}^2 = 0, \tag{43}
\]
\[
\frac{d}{d\tau} \left( r^2 \sin^2 \theta \dot{\phi} \right) = 0. \tag{44}
\]
By considering the initial conditions in \( \tau = \tau_0 \) on the coordinate \( \theta \): \( \theta (\tau_0) = \frac{\pi}{2} \) and \( \theta (\tau_0) = 0 \), from Eq. (43), it follows that \( \ddot{\theta} \) and any other higher order derivatives are equal to zero, so the particle motion is confined to the plane \( \theta = \frac{\pi}{2} \). Therefore, we have a spherically symmetric spacetime with two Killing vectors corresponding to the conserved energy (E) and the conserved angular momentum (L). The timelike Killing vector, \( K^\mu = (\partial_t)^\mu \), is related to the conserved particle energy given by
\[
E = -g_{\mu\nu}K^\mu U^\nu = \left( 1 - \frac{2M}{r} \right) \dot{t}, \tag{45}
\]
The rotational Killing vector, \( \psi^\mu = (\partial_\phi)^\mu \), providing the conserved angular momentum of the particle,
\[
L = g_{\mu\nu} \psi^\mu U^\nu = r^2 \dot{\phi}, \tag{46}
\]
Clearly, the Eqs. (45) and (46) are consistent with the Eqs. (41) and (44), respectively.

Then, from the conserved quantities in Eq. (39) for timelike geodesics it yields a single differential equation for the coordinate \( r \) in terms of the proper time \( \tau \),
\[
(1 + \ell) \ddot{r}^2 + \left( 1 - \frac{2M}{r} \right) \left( \frac{L^2}{r^2} + 1 \right) = E^2. \tag{47}
\]
We now introduce the variable \( u = r^{-1} \), such that
\[
\dot{r} = \frac{dr}{d\phi} = -L \frac{du}{d\phi}. \tag{48}
\]
By substituting it in Eq. (47) we obtain
\[
(1 + \ell) \left( \frac{du}{d\phi} \right)^2 + u^2 = \frac{E^2}{L^2} + \frac{2M}{L^2} u + 2Mu^3. \tag{49}
\]
As usually done in this treatment, it is preferable to solve the second-order equation which is obtained by differentiating the above equation with respect to \( \phi \), providing
\[
(1 + \ell) \frac{d^2u}{d\phi^2} + u - \frac{M}{L^2} - 3Mu^2 = 0. \tag{50}
\]
It only presents LV contributions into the coefficient of the first term maintaining the total structure of the one obtained in the context of GR. In order to solve perturbatively the Eq. (50) and due to the fact we are assuming the LV parameter \( \ell \ll 1 \), it is still valid to consider the last term as a relativistic correction when compared with the Newtonian case. The perturbative solution is defined in terms of a small parameter \( \epsilon = 3M^2/L^2 \):
\[
u ≃ u^{(0)} + \epsilon u^{(1)}. \tag{51}
\]
The differential equation at zeroth-order in \( \epsilon \) yields
\[
(1 + \ell) \frac{d^2u^{(0)}}{d\phi^2} + u^{(0)} - \frac{M}{L^2} = 0, \tag{52}
\]
whose solution is given by
\[
u^{(0)} = \frac{M}{L^2} \left[ 1 + \epsilon \cos \left( \frac{\phi}{\sqrt{1+\ell}} \right) \right]. \tag{53}
\]
It is analogous to the Newtonian result. Here, the integration constants we have considered are the orbital eccentricity \( e \) (considered small as that of GR) and the initial value \( \phi_0 = 0 \).

The differential equation at first-order in \( \epsilon \) is
\[
(1 + \ell) \frac{d^2u^{(1)}}{d\phi^2} + u^{(1)} - \frac{L^2}{M} (u^{(0)})^2 = 0, \tag{54}
\]
which admits an approximated solution of the form
\[
u^{(1)} ≃ \frac{M}{L^2} e^{-\frac{\phi}{\sqrt{1+\ell}}} \sin \left( \frac{\phi}{\sqrt{1+\ell}} \right) + \frac{M}{L^2} \left[ 1 + \frac{\epsilon^2}{2} - \frac{\epsilon^2}{6} \cos \left( \frac{2\phi}{\sqrt{1+\ell}} \right) \right]. \tag{55}
\]
For our purposes, the second term can be neglected once it consists of a constant displacement and a quantity that oscillates around zero.

Therefore, the perturbative solution (51) reads
\[
u ≃ \frac{M}{L^2} \left[ 1 + e \cos \left( \frac{\phi}{\sqrt{1+\ell}} \right) + \epsilon e \frac{\phi}{\sqrt{1+\ell}} \sin \left( \frac{\phi}{\sqrt{1+\ell}} \right) \right]. \tag{56}
\]
Because $ε \ll 1$, the perturbative solution (56) can be rewritten in the form of an ellipse equation,

$$u \simeq \frac{M}{L^2} \left[ 1 + ε \cos \left( \phi (1 - ε) \right) \right]. \quad (57)$$

Despite of the presence of Lorentz violation, the orbit remains periodic with period $Φ$,

$$Φ = \frac{2π\sqrt{1 + ℓ}}{1 - ℓ} \approx 2π + ΔΦ. \quad (58)$$

The advance of the perihelion ($ΔΦ$) is obtained by taking the lowest order in the $ε$ and $ℓ$ expansion. Then, $ΔΦ$ is given by the following expression

$$ΔΦ = 2πε + πℓ = ΔΦ_{GR} + ΔΦ_{LV}, \quad (59)$$

where $ΔΦ_{GR}$ is the prediction of GR

$$ΔΦ_{GR} = 2πε = \frac{6πG_Nm}{c^2 (1 - ℓ^2) a}. \quad (60)$$

We use the observational error in experimental data to compute some upper-bounds for the LV parameter $ℓ$.

For example, for the motion of Mercury around the Sun, the observational error is $0.003° \text{C}^{-1}$ (or $72.3 × 10^{-7}$ arcseconds per orbit). So, we suppose the contribution of the Lorentz violation $ΔΦ_{LV}$ is less than the observational error. Such a procedure allows us to estimate an upper-bound at the level of $ΔΦ_{LV} < 1.1 × 10^{-11}$. By applying the same procedure for the other planets in Table I, we have achieved the set of estimates of attainable experimental sensitivities (upper-bounds) presented in Table II. Thus, we observe the best upper-bound attained from the advance of the perihelion is $ℓ < 10^{-12}$.

### TABLE I. Theoretical and observed values of perihelion shifts given in arcseconds per century ($° \text{C}^{-1}$).

| Planets         | GR prediction$^a$ | Observed           | Error estimates$^b$ |
|-----------------|-------------------|--------------------|---------------------|
| Mercury ($\varpi$) | 42.9814           | 42.9794 ± 0.0030   | −0.0020 ± 0.0030    |
| Venus ($\varpi$)  | 8.6247            | 8.6273 ± 0.0016    | 0.0026 ± 0.0016     |
| Earth ($\oplus$)  | 3.83877           | 3.8396 ± 0.00019   | 0.00019 ± 0.00019   |
| Mars ($\sigma$)   | 1.350938          | 1.350918 ± 0.000037| −0.000020 ± 0.000037|
| Jupiter ($\oplus$)| 0.0623            | 0.121 ± 0.0283     | 0.0587 ± 0.0283     |
| Saturn ($\oplus$) | 0.01370           | 0.01338 ± 0.00047  | −0.00032 ± 0.00047  |
| Icarus           | 10.1              | 9.8 ± 0.8          | −0.3 ± 0.8          |

$^a$ Computed from the data base of Refs.[35, 36].
$^b$ From the Refs.[39, 40].

### TABLE II. Some upper bounds obtained from the observational error of the perihelion shifts.

| Parameters LSB | Bounds               |
|----------------|----------------------|
| $ℓ_{\varpi}$   | $1.1 × 10^{-11}$     |
| $ℓ_π$          | $1.5 × 10^{-11}$     |
| $ℓ_⊕$          | $2.9 × 10^{-12}$     |
| $ℓ_{\sigma}$   | $1.1 × 10^{-12}$     |
| $ℓ_⊕$          | $5.2 × 10^{-9}$      |
| $ℓ_{\oplus}$   | $2.1 × 10^{-10}$     |
| $ℓ_{\text{Icarus}}$ | $1.3 × 10^{-8}$    |
B. Bending of light

Unlike the previous case, we now have massless test particles whose trajectories correspond to null geodesics so \( \chi = 0 \) in Eq. (39) which after substituting the conserved quantities becomes

\[
(1 + \ell) \dot{r}^2 + \frac{1 - 2M}{r} \frac{L^2}{r^2} = E^2,
\]

where the dot now denotes differentiation with respect to some affine parameter.

Again, we consider \( u = r^{-1} \) with \( r \equiv r(\phi) \) and the differentiation with respect to \( \phi \) in (62) which results

\[
(1 + \ell) \frac{d^2u}{d\phi^2} + u - 3M u^2 = 0.
\]

We observe that in the limit \( \ell \to 0 \) Eq. (63) recovers the corresponding GR result providing the deflection of light rays, as expected. In analogy with the previous subsection, we use a perturbative method to achieve a solution by considering the quantity \( Mu \) as sufficiently small. Thus, we write the approximate solution of the form

\[
u \simeq u^{(0)} + 3M u^{(1)},
\]

which, after replacement in Eq. (63), gives the following differential equation for \( u^{(0)} \),

\[
(1 + \ell) \frac{d^2u^{(0)}}{d\phi^2} + u^{(0)} = 0,
\]

whose solution is

\[
u^{(0)} = \frac{1}{D} \sin \left( \frac{\phi}{\sqrt{1 + \ell}} \right),
\]

where \( D \) is a constant of integration and we have considered the initial angle \( \phi_0 = 0 \), for convenience. This result corresponds to the equation of a straight-line which is analogous to the Newtonian prediction.

The differential equation for \( u^{(1)} \), in turn, becomes

\[
(1 + \ell) \frac{d^2u^{(1)}}{d\phi^2} + u^{(1)} - \frac{1}{D^2} \sin^2 \left( \frac{\phi}{\sqrt{1 + \ell}} \right) = 0,
\]

and its solution is written as

\[
u^{(1)} = \frac{1}{3D^2} \left[ 1 + A \cos \left( \frac{\phi}{\sqrt{1 + \ell}} \right) + \cos^2 \left( \frac{\phi}{\sqrt{1 + \ell}} \right) \right].
\]

Hence, a general solution for \( u(\phi) \) assume the following form

\[
u \simeq \frac{1}{D} \sin \left( \frac{\phi}{\sqrt{1 + \ell}} \right) + \frac{M}{D^2} \left[ 1 + A \cos \left( \frac{\phi}{\sqrt{1 + \ell}} \right) + \cos^2 \left( \frac{\phi}{\sqrt{1 + \ell}} \right) \right],
\]

with \( A \) being an arbitrary constant.

Since we are interested in determining the angle of deflection for a light ray, the boundary conditions are determined by assuming: (i) The source is located in \( r \to \infty \) such that \( u(r \to \infty) \to 0 \) and \( \phi = -\delta_1 \), and (ii) the observer is localized in \( r \to \infty \) such that \( u(r \to \infty) \to 0 \) and \( \phi = +\delta_2 \), so the total angle of deflection is given by \( \delta = \delta_1 + \delta_2 \). By using these boundary conditions in Eq. (69) and taking in consideration \( \ell \ll 1 \) and \( \delta_1, \delta_2 \ll 1 \), the first-order equation provides

\[
\delta_1 = \frac{M}{D} \left( 2 + A \right),
\]

\[
\delta_2 = \frac{M}{D} \left( 2 - A \right) + \frac{\pi \ell}{2}.
\]

Hence, the light-ray deflection angle in the metric (36) is

\[
\delta = \delta_{GR} + \delta_{LV} = \frac{4Gm}{c^2 D} + \frac{\pi \ell}{2},
\]

with \( m \) being the mass of the deflecting body and \( D \) the so-called impact parameter (defined as the distance of closest approach of the light ray to the center of mass of the deflecting body). The first term \( \delta_{GR} \),

\[
\delta_{GR} = \frac{4Gm}{c^2 D},
\]

gives the usual deviation of light predicted by the GR. The second term \( \delta_{LV} \),

\[
\delta_{LV} = \frac{\pi \ell}{2},
\]

is the correction coming from the LSB effects. Of course, taking the limit \( \ell \to 0 \) in Eq. (72) we recover the usual result established by GR for the bending of light.

For a ray grazing the Sun we have \( m = M_{\odot} \) and \( D \approx R_{\odot} \). Using, for example, the values of Ref. [36], one can verify that GR predicts an angle given by \( \delta_{GR} = 4Gm M_{\odot} c^2 D \approx 43.7516687^\prime \). Therefore, if there is, indeed, Lorentz violation in nature, the effects arising from the LV term (\( \delta_{LV} \)) must be smaller than the observational errors. The error bars obtained in recent measurements for the deflection of light by the Sun [41–44] allows us to provide an interesting sensitivity for Lorentz violation. Specifically, a detailed analysis of the observational data for the bending of light, adopting the values from Ref. [43] as example, yields an error bar of order \( \sim 0.0001051^\prime \). Taking this value, we set the upper-bound from the inequality: \( \delta_{LV} < 0.0001051^\prime \). A quickly calculation allows us to found \( \ell < 3.2 \times 10^{-10} \); it is similar but not better than those found in the previous test.

C. Time delay of light

A further measurable relativistic phenomenon involving light rays is the Shapiro time-delay effect [45]. The
solar-system tests involving this effect can yield interesting sensitivities to Lorentz violation. For this purpose we will derive an expression involving the Lorentz-violating corrections for time-delay effect from the result already obtained in the subsection IV B. Namely, we are interested in an equation providing the change in the round trip travel time of light to an object due to the influence of a massive body such as the Sun.

By considering the motion of light in the equatorial plane ($\theta = \pi/2$) and because it travels along a null geodesic in the spacetime (36), i.e., the condition $ds^2 = 0$ is satisfied, we can write

$$- \left(1 - \frac{2M}{r}\right) dt^2 + (1 + \ell) \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\phi^2 = 0.$$  

(75)

Next, we consider the zero-order solution (66) characterizing the straight-line approximation,

$$r \sin \left(\frac{\phi}{\sqrt{1 + \ell}}\right) = D,$$  

(76)

and we use it to establish the following relation,

$$r^2 d\phi^2 = (1 + \ell) \left(\frac{D^2}{r^2 - D^2}\right) dr^2.$$  

(77)

Thus, Eq. (75) can be rewritten as

$$dt^2 = \frac{1 + \ell}{1 - \frac{2M}{r}} \left(1 - \frac{1}{1 - \frac{2M}{r}} \frac{D^2}{r^2 - D^2}\right) dr^2.$$  

(78)

Expanding it in terms of $M/r$ and considering the contributions at first-order we get

$$dt \simeq \pm \sqrt{1 + \ell} \left(1 + \frac{2M}{r} \frac{MD^2}{r^3}\right) rdr.$$  

(79)

The setup for Shapiro delay effect involves two stations at large distances from the massive source (or curvature source). By assuming a light ray (or radar signal) from an emitter located at $r_E$ traveling to a receiver at $r_R$, the travel time is given by

$$t = t_0 \sqrt{1 + \ell} + 2M \sqrt{1 + \ell} \ln \left[\frac{r_E + (r_E^2 - D^2)^{1/2}}{D}\right]$$

$$+ 2M \sqrt{1 + \ell} \ln \left[\frac{r_R + (r_R^2 - D^2)^{1/2}}{r_E + (r_E^2 - D^2)^{1/2}}\right]$$

$$- M \sqrt{1 + \ell} \left[\frac{(r_R^2 - D^2)^{1/2}}{r_R} + \frac{(r_E^2 - D^2)^{1/2}}{r_E}\right],$$  

(80)

where $t_0$ represents the travel time in flat spacetime,

$$t_0 = (r_R^2 - D^2)^{1/2} + (r_E^2 - D^2)^{1/2},$$  

(81)

It should be noted that in the absence of the Lorentz violation, $\ell = 0$, the Eq. (80) recovers the expression predicted by GR, as expected. It is evident that the first term in (80) stands the travel of radar signal along a straight-line including the effects due to Lorentz violation in a flat spacetime (special relativity). The other terms represent the delay produced by the curved spacetime. Such a delay may be interpreted as an effective increase in the distance between the emitter and receiver of the radar signal.

The result (80) can be applied to the solar-system by considering the reflection of a signal from a planet or spacecraft. We take, for example, the spacetime near the Sun ($m = M_\odot$) and a radar signal emitted from Earth ($r_E = r_\oplus$) traveling to a planet or spacecraft located at ($r_R$). For simplicity, we consider both the Earth and the planet as stationary relative to the Sun. The time-delay is a maximum when the planet is at superior conjunction and the radar signal just grazes the Sun’s surface such that the radius of closest approach is $D \approx R_\odot$ satisfying the condition $D \ll r_\oplus, R_\odot$. Therefore, from Eq. (80), the total round-trip time for a signal traveling from the Earth to an other planet (or spacecraft) and returning is

$$T \approx T_0 \sqrt{1 + \ell} + \frac{4GNM_\odot}{c^3} \ln \left[\frac{4r_\oplus r_R}{R_\odot^2}\right] - 1 \sqrt{1 + \ell},$$  

(82)

where $T_0$ represents the total travel time in flat spacetime,

$$T_0 = \frac{2}{c} (r_R^2 - R_\odot^2)^{1/2} + \frac{2}{c} (r_E^2 - R_\odot^2)^{1/2}.$$  

(83)

Similarly to what is done in GR, from Eq. (82), we define, in this Lorentz violating framework, the total excess-delay by

$$\delta T = T - T_0 = \delta T_{GR} + \delta T_{LV},$$  

(84)

where, taking only first order terms in $\ell \ll 1$, the quantity $\delta T_{GR}$ is given by

$$\delta T_{GR} \approx \frac{4GNM_\odot}{c^3} \ln \left[\frac{4r_\oplus r_R}{R_\odot^2}\right],$$  

(85)

and represents the excess-delay due to pure GR. The other term, $\delta T_{LV}$, is the contribution of Lorentz violation to the excess-delay,

$$\delta T_{LV} = \frac{\ell}{2} \left(\delta T_{GR} + T_0\right) \approx \frac{\ell}{2} T_0,$$  

(86)

and we shall use it to obtain estimates of sensitivities to Lorentz violation.

This could be done from the passive radar measurements of the inner planets and or active ranging experiments of interplanetary spacecrafts. For the Shapiro delay with passive radar measurements of the inner planets, such as Mercury or Venus, used as passive reflectors of the radar signals, it has been obtained (taking
the planet Venus as example) an excess-delay prediction by general relativity to well within the experimental uncertainty of 20% [46] and, subsequently, within 2% [47]. We take this latter as an upper bound for Lorentz violating effects which would correspond to a sensitivity of \( \ell_\text{Q} < 5.0 \times 10^{-9} \).

Time delay have also been measured with artificial satellites, such as Mariner 6 and 7 spacecrafts in orbit around the Sun, used as active retransmitters of the radar signals. An analysis of the Mariners 6 (M6) and 7 (M7) data suggest that a realistic estimate of the total uncertainty, for both cases, is perhaps less than 3% [48] so that the estimates to sensitivities for Lorentz violation parameters \( \ell_\text{M6} \) and \( \ell_\text{M7} \) are \( 2.2 \times 10^{-9} \) and \( 1.6 \times 10^{-9} \), respectively.

Another major advance was made using an active transmitter on a spacecraft stationed on a planet. An example can be given by experiments conducted during the mission of Viking spacecraft to Mars. This consisted of space probes that orbited Mars, equipped with a lander to study the planet from its surface. The measurement from the Viking Mars (VM) landers resulted in an estimated accuracy of 0.1% [49] which allows us to establish a sensibility of \( \ell_\text{VM} < 1.8 \times 10^{-10} \).

The most precise measurement of the Shapiro time-delay from spacecraft measurements so far has been made from the Cassini mission during its cruise to Saturn [50]. Performing a detailed analysis of the data obtained in the 2002 superior conjunction of Cassini, it is verified the resulting measurement error must be within at most 0.0012% of unity. From this value, we obtain an attainable sensitivity of \( \ell_\text{Cassini} < 6.2 \times 10^{-13} \).

V. CONCLUSIONS AND REMARKS

We have investigated a static and spherically symmetric vacuum solution which is obtained in the context of a Lorentz-violating modified gravity contained into the framework of a Riemannian bumblebee gravity model. We have found a new spherically symmetric solution which is very similar to the Schwarzschild one, however, its Kretschmann invariant (37) guarantees they are very different.

The implications of the theoretical results obtained are studied for some existing classical gravitational experiments, including the advance of the perihelion, bending of light and Shapiro’s time-delay. These tests present an interesting feature: even in the absence of a massive source of curvature we still have corrections coming purely from the Lorentz violation. Indeed, this is compatible with the Kretschmann invariant (37), which is nonvanishing in the limit \( M \to 0 \). This result could indicate the background carrying the Lorentz violating effects also deforms slightly the spacetime, which actually should be approximately flat because of those are significantly small. The smallness of LV effects have allowed to compute some upper-bounds on the parameter \( \ell \), which are all summarized in Table III.

In the case of the relativistic perihelion advance, the induced effect by Lorentz violation can be interpreted as a correction to usual result of GR, which may be recovered in the limit \( \ell \to 0 \). With the corresponding Lorentz-violating terms at hand, we could estimate attainable sensitivities for some inner planets of the solar system. In this particular scenario, the most stringent upper bound provides \( \ell < 10^{-12} \).

| Parameters | Advance perihelion | Bending light | Time delay |
|------------|--------------------|---------------|------------|
| \( \ell_\text{LSB} \) | \( 10^{-11} \) | \( \cdots \) | \( \cdots \) |
| \( \ell_\text{Q} \) | \( 10^{-11} \) | \( \cdots \) | \( 10^{-9} \) |
| \( \ell_\odot \) | \( 10^{-12} \) | \( \cdots \) | \( \cdots \) |
| \( \ell_\oplus \) | \( 10^{-12} \) | \( \cdots \) | \( \cdots \) |
| \( \ell_\odot \) | \( 10^{-9} \) | \( \cdots \) | \( \cdots \) |
| \( \ell_\oplus \) | \( 10^{-10} \) | \( \cdots \) | \( \cdots \) |
| \( \ell_\text{Icarus} \) | \( 10^{-8} \) | \( \cdots \) | \( \cdots \) |
| \( \ell \) | \( \cdots \) | \( 10^{-10} \) | \( \cdots \) |
| \( \ell_\text{M6} \) | \( \cdots \) | \( \cdots \) | \( 10^{-9} \) |
| \( \ell_\text{M7} \) | \( \cdots \) | \( \cdots \) | \( 10^{-9} \) |
| \( \ell_\text{VM} \) | \( \cdots \) | \( \cdots \) | \( 10^{-10} \) |
| \( \ell_\text{Cassini} \) | \( \cdots \) | \( \cdots \) | \( 10^{-13} \) |

The calculation for the bending of light also provides a Lorentz violating correction to the GR result. Such a LV term allows to establish an interesting upper bound for the parameter \( \ell \). For this purpose, the analysis was carried out by using the very long baseline interferometry (VLBI) data [43] providing an attainable sensitivity at the level of \( \ell < 10^{-10} \).

The relativistic effect involving the Shapiro time-delay of light also yields, as well as the other cases, Lorentz violating corrections to GR. It follows that, among the data used for the estimates sensitivities of Shapiro’s time-delay effect that might be attainable, the spacecraft Cassini has provided the most precise measurement at present, yielding an upper-bound of \( \ell < 10^{-13} \).

Within the context developed in the manuscript, we are exploring other possible vacuum configurations of bumblebee field producing Lorentz violating solutions. In addition, there exist the possibility of exploring the effect of bumblebee field on black hole solutions such as the charged and rotating ones. The results of these research will be reported elsewhere.

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