Gravitational Instability of Perturbations of a Nonlinear Nonstationary Model of a Disklike System.

IV. Formation of Double-ring Structures in Galaxies

K. T. Mirtadjieva1* and K. A. Mannapova2**

1Astronomical Institute, Academy of Science of RUz, 33 Astronomicheskaya St., Tashkent, 100052, Uzbekistan
2National University of Uzbekistan, Vuzgorodok, Tashkent, 100174, Uzbekistan

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Abstract—To find out precise criteria of double-ring structure formation in galaxies, we have studied the problem of gravitational instability of the corresponding structure oscillation modes in the background of a previously described compound disk model [1, 2, 3] with an exact nonlinear law of nonstationarity. Nonstationary analogs have been derived for the dispersion equations for these structure oscillation modes of the model, and the results of their analysis are discussed. A comparative analysis has been carried out for the instability increments of ringlike oscillation modes, aimed at determining the dependence of their characteristic manifestation times on the physical parameters of the model.

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1. INTRODUCTION

Ringlike structures are observed in many types of astrophysical objects, ranging from planets [4] to galaxies (see, e.g., [5] and references therein). This suggests that the origin of these structures is connected, in particular, with certain general physical mechanisms, for example, with the evolution of the eigenmodes of oscillations. In [5] we have developed a classification of ringlike galaxies in order to study their nature and formation mechanisms. At the same time, a physical explanation of the features of ringlike galaxies and the problems of their origin requires the construction of a theoretical model and an analysis of its gravitational instabilities with respect to specific structural oscillation modes.

Assuming that global structures of galaxies can be formed at an early nonstationary stage of their evolution, we have studied [2, 6] the origin of single-ring galaxies against the background of nonlinear nonstationary models of a nonequilibrium self-gravitating disk. These models are a nonlinear nonequilibrium generalization of the Bisnovatyi-Kogan–Zeldovich stationary model [7] to the case of radial pulsations. The criteria for the formation of single-ring galaxies against the background of a composite nonlinear nonstationary mode were studied by us in [2] by investigating the modes of oscillations with the fundamental harmonic index $N = 4$ and the azimuthal wave numbers $m = 0$ and $m = 2$. However, in our classification of ringlike galaxies [5], two-ring galaxies have made up a larger percentage than was expected. Based on that, in this paper we investigate the origin of two-ring galaxies, studying the problems of gravitational instability of the corresponding structural modes of oscillations against the background of a composite disk model.

The two-ring structures are formed in the background of this model as a result of a gravitational instability of the oscillation modes with $N = 6$ and $m = 0$, 2. While at $N = 6$ and $m = 0$ we have pure ring structures, at $N = 6$ and $m = 2$ the rings split into separate lumps. Using the results of this study, we have built critical diagrams for the dependence of the initial virial ratio on the parameters of the compound model. We have also carried out a comparative analysis of the instability increments of the ringlike oscillation modes in order to determine the dependence of the characteristic times of their manifestation on the basic physical parameters of the model.

2. BASIC RELATIONS AND EQUATIONS

In this paper, as in [1–3], we study the gravitational instability problem for the following nonstationary model in a phase description with an anisotropic velocity diagram:

$$
Ψ(\vec{r}, \vec{v}, \Omega, \lambda, \nu, t) = (1 - \nu)Ψ_1(\vec{r}, \vec{v}, \Omega, \lambda, t)
+ \nuΨ_2(\vec{r}, \vec{v}, \Omega, \lambda, t),
$$

(1)
which gives us an opportunity to study intermediate states between two different models, encompassing a wider range of possible initial conditions at an early nonstationary stage of evolution of dislikle self-gravitating systems. Here $\nu$ is a superposition parameter, $\Omega$ is a dimensionless parameter characterizing the rate of rigid disk rotation, and the pulsation amplitude $\lambda = 1 - (2T/[U])_0$ is exactly expressed in terms of the values of the virial ratio at the time instant $t = 0$.

In the compound model (1), we took as $\Psi_1$ and $\Psi_2$ the nonlinearly pulsating disc models [1–3, 6]

$$\Psi_1 = \frac{\sigma_0}{2\pi \Pi \sqrt{1 - \Omega^2}} \left[ \frac{1 - \Omega^2}{\Pi^2} \left( \frac{1 - r^2}{\Pi t} \right)^{-1/2} \right] - (v_r - v_a)^2 - (v_\perp - v_b)^2,$$

$$\Psi_2 = \frac{\sigma_0}{\pi} (1 + \Omega \cdot r \cdot v_\perp) \cdot \chi(D),$$

where $\sigma_0$ is the value of the disk surface density at $t = 0$, $r = 0$,

$$\Pi(\psi) = (1 - \lambda^2)^{-1} \left( 1 + \lambda \cos \psi \right)$$

is the system stretching factor, in which $\psi$ is an auxiliary variable related to time $t$ by

$$t = (1 - \lambda^2)^{-3/2} (\psi + \lambda \sin \psi);$$

$\chi$ is the symbol of Heaviside’s function. We everywhere use the normalization

$$\pi^2 G \sigma_0 = 2R_0 \quad (R_0 = 1),$$

while the quantities $\lambda$ and $\Omega$, as well as $\nu$, take values from the range $[0; 1]$; $v_r$ and $v_\perp$ are the radial and tangential velocity components of a “particle” with the coordinates $\vec{r}(x, y)$, the quantity $D$ is given by

$$D = (1 - r^2/\Pi^2) \left( 1 - \Pi^2 v_\perp^2 \right) - \Pi^2 (v_r - v_a)^2,$$

and

$$v_a = -\lambda \frac{r \sin \psi}{\sqrt{1 - \lambda^2 \Pi^2}}, \quad v_b = \frac{\Omega r}{\Pi^2}.$$

The composite model (1) has the surface density

$$\sigma(\vec{r}, t) = \frac{\sigma_0}{\Pi^2(t)} \sqrt{1 - \frac{r^2}{\Pi^2(t)}},$$

and performs strict radial oscillations with the period

$$P(\lambda) = \frac{2\pi}{(1 - \lambda^2)^{3/2}}.$$

Let us recall that, by analogy with the theory of stability of equilibrium models, to analyze a nonlinear nonequilibrium model and to find the instability criteria, it is necessary to derive a nonstationary analogue of the dispersion equation (NADE). And to obtain the appropriate NADE for the composite model (1), one applies it to a small nonsymmetric perturbation with the potential $\delta \Phi$, and with this in mind, in [1, 2, 6] we have derived the basic equation for the centroid displacement vector $\vec{d}r$

$$\Lambda \vec{d}r = \left[ (1 + \lambda \cos \psi) \frac{d^2}{d\psi^2} + \lambda \sin \psi \frac{d}{d\psi} + 1 \right] \vec{d}r = \Pi^2(\psi) \frac{\partial (\delta \Phi)}{\partial r},$$

where the overbar denotes averaging over the velocity space. The solution to Eq. (7) may be presented in the integral form [1, 2, 6]

$$\vec{d}r = \int_{-\infty}^{\psi} \Pi^2(\psi_1) \cdot S(\psi, \psi_1) \frac{\partial (\delta \Phi)}{\partial r} d\psi_1,$$

where $S(\psi, \psi_1)$ is an analogue of a Green function which is constructed in the standard way from the solution of the homogeneous equation in Eq. (7) and is given by

$$S(\psi, \psi_1) = \left[ \sin \psi \left( \cos \psi_1 + \lambda \right) - \sin \psi_1 \left( \cos \psi + \lambda \right) \right] \left( 1 + \lambda \cos \psi_1 \right)^{-2}.$$

Now it remains to clarify the form of the perturbation $\delta \Phi$. Note that the ring-shaped modes belong to the class of horizontal perturbations which develop only in the plane of the disk $(x, y)$ and do not depend on $z$. Taking into account the nature of the nonstationary model under study (1), by analogy with the stability theory of stationary models [8, 9], these perturbations can be described in the form

$$\delta \Phi = D_{mN}(\psi) \cdot r^{-N-m}(x + iy)^m \left( r = \sqrt{x^2 + y^2} \right),$$

where $D_{mN}(\psi)$ is the function to be found, which, unlike the case of stationary models, is time-dependent.

Now, to derive the NADE, we need to calculate the density perturbation and to compare the results with the potential theory for dislikle self-gravitating systems (DSS). It should be noted that the nonlinear nonstationarity of the model (1) significantly complicates the analysis of its stability as compared to the corresponding equilibrium disk, since it greatly complicates the derivation of the NADE in the general case. That is why it is advisable to study the most interesting perturbation modes separately.
3. NADE DERIVATION FOR TWO-RING-SHAPED PERTURBATION MODES AND THEIR ANALYSIS

3.1. The Case \( m = 0, N = 6 \)

The instability of this mode leads to formation of two rings in the disk. This mode corresponds to the perturbation potential

\[
\delta \Phi = D_{06}(\psi) \cdot (x^2 + y^2)^3, \tag{11}
\]

and using Eq. (8), we obtain the centroid displacement components as

\[
\overline{dx} = 6 \int_{-\infty}^{\psi} \Pi^3(\psi_1)S(\psi, \psi_1) \times D_{06}(\psi_1)x_1(x_1^2 + y_1^2)^2d\psi_1,
\]

\[
\overline{dy} = 6 \int_{-\infty}^{\psi} \Pi^3(\psi_1)S(\psi, \psi_1) \times D_{06}(\psi_1)y_1(x_1^2 + y_1^2)^2d\psi_1. \tag{12}
\]

By definition [1, 2, 6],

\[
\overline{x_1} = xH_\alpha + \overline{\pi H}_\beta,
\]

\[
\overline{y_1} = yH_\alpha + \overline{\eta H}_\beta, \tag{14}
\]

where \( u \) and \( v \) are the \( x- \) and \( y- \) velocity components, respectively, and

\[
H_\alpha = \frac{(\lambda + \cos \psi_1) \cos \psi + \sin \psi \sin \psi_1}{1 + \lambda \cos \psi},
\]

\[
H_\beta = (1 - \lambda^2)^{-3/2} \left[ (\lambda + \cos \psi) \sin \psi_1 - (\lambda + \cos \psi_1) \sin \psi \right]. \tag{15}
\]

Then, in accord with (14), we have

\[
x_1(x_1^2 + y_1^2) = x(x^2 + y^2)D_5^2H_\alpha^5
+ [4x(x^2 + y^2)(xu + y\overline{v}) + (x^2 + y^2)^2]\overline{H}_\alpha^3H_\beta
+ (2y(x^2 + y^2)(u^2 + \overline{v}^2)
+ 4x(x^2u^2 + 2xyuv + y^2\overline{v}^2)
+ 4(x^2 + y^2)(xu + y\overline{v})H_\alpha^3H_\beta
+ 4(y^2(x^2 + y^2)^2 + yu\overline{v}^2)\overline{H}_\alpha^3 + (2x^2 + y^2)(u^2 + \overline{v}^2)
+ 4(x^2u^2 + 2xyuv + y\overline{v}^2)\overline{H}_\alpha^3 + (2y(x^2 + y^2)^2 + yu\overline{v}^2)\overline{H}_\alpha^3,
\]

\[
y_1(x_1^2 + y_1^2) = x(x^2 + y^2)D_5^2H_\alpha^5
+ [4y(x^2 + y^2)(xu + y\overline{v}) + (x^2 + y^2)^2]\overline{H}_\alpha^3H_\beta
+ (2y(x^2 + y^2)(u^2 + \overline{v}^2)
+ 4y(x^2u^2 + 2xyuv + y^2\overline{v}^2)\overline{H}_\alpha^3H_\beta
+ 4(y^2(x^2 + y^2)^2 + yu\overline{v}^2)\overline{H}_\alpha^3 + (2x^2 + y^2)(u^2 + \overline{v}^2)
+ 4(x^2u^2 + 2xyuv + y\overline{v}^2)\overline{H}_\alpha^3 + (2y(x^2 + y^2)^2 + yu\overline{v}^2)\overline{H}_\alpha^3.
\]

Averaging over the velocity space, for example, for the \( u \) component, is defined as follows:

\[
u^k = \frac{1}{\sigma(r, t)} \int \int u^k \Psi du d\vartheta. \tag{16}
\]

Then we calculate the response of the density,

\[
\delta \sigma = -\frac{\partial (\overline{\sigma x})}{\partial x} - \frac{\partial (\overline{\sigma y})}{\partial y}. \tag{17}
\]

On the other hand, it is known from the theory of disk potential that the density perturbation

\[
\delta \sigma = \sigma_0 \Pi^{-1} P_0^m(\xi) e^{im\varphi} \tag{18}
\]

corresponds to the following perturbation of the potential [8, 9]:

\[
\delta \Phi = 2\Pi^2 \cdot \frac{(N + m - 1)!!(N - m - 1)!!}{(N + m)!!(N - m)!!} \times P_0^m(\xi) e^{im\varphi}, \tag{19}
\]

Now, comparing (18) at \( m = 0, N = 6 \) with the calculated result (17), taking into account the expressions (11) and (19) for \( \delta \Phi \), and passing over from the integral form to the differential one, we obtain the following NADE for the present mode:

\[
\Lambda \ell_\tau(\psi) = \frac{525}{64} D_{06}(\psi) \cdot (\lambda + \cos \psi)^5 - \tau \sin \tau \psi \quad (\tau = 0 - 5), \tag{20}
\]

where the function \( D_{06}(\psi) \) is presented in Appendix A. The resulting NADE (20) is a set of six second-order differential equations. It does not admit an analytical consideration, and therefore we have studied it numerically using the method of the stability of periodic solutions [10].

In the course of numerical calculations, changing the values of the rotation parameters \( \Omega \), and the superposition of \( \nu \) and the amplitude of pulsation \( \lambda \) in the interval from 0 to 1, we found the critical values of the initial virial ratio \( 2T/U_0 \), beginning with which the model (1) becomes unstable relative to the perturbation mode (0; 6). The results of the calculations are
presented in the form of the dependences of the critical value of $(2T/U)_0$ on $\Omega$ and $\nu$ (Fig. 1).

A numerical analysis of the NADE (20) shows that, against the background of the composite model, the mode $(0; 6)$ has instabilities of both vibrational and aperiodic nature at all values of the rotation parameter $\Omega$. The critical diagram of this mode (Fig. 1a) for a nonrotating composite model is very similar to the diagram of the single-ring mode $(0; 4)$, which we have studied in [2]. But only here the extremal points A and B are more distant from each other, and, moreover, the additional narrow branch of the instability region, observed in the case of the $(0; 4)$ mode, here forms a separate peninsula with a vertex at the point $S(0.550; 0.588)$, without connection with the main instability zone.

Let us also note that against the background of a nonrotating composite model, the $(0; 6)$ mode is completely unstable up to the value $\nu \leq 0.114642$. Then the instability region sharply decreases, and when the superposition parameter takes the value $\nu = 0.404447$, this region once again occupies the entire range of possible values of the initial virial ratio. And after that, in the range $0.405 < \nu \leq 0.614$, the instability region again decreases up to the value $(2T/U)_0 \approx 0.302$, and when the superposition parameter tends to its maximum value, it begins to slowly increase again.

The critical diagram at $\Omega = 0.5$ has a peculiar form (Fig. 1b). In the range $0.0 \leq \nu \leq 0.58$, the unstable region gradually decreases from $(2T/U)_0 = 0.994$ to $(2T/U)_0 = 0.399$, and at $0.6 < \nu \leq 1$, it is almost stable. Moreover, we see here two islands of instability: $\alpha(0.379 \leq \nu \leq 0.54); 0.595 \leq (2T/U)_0 \leq 0.715$ and $\gamma(0.56 \leq \nu \leq 0.71); 0.596 \leq (2T/U)_0 \leq 0.636$), one long peninsula $\beta(0.45 \leq \nu \leq 1.0); 0.393 \leq (2T/U)_0 \leq 0.741$, and one stability island inside the instability region in the form of a “spherical” triangle $\delta(0.589 \leq \nu \leq 0.87); 0.293 \leq (2T/U)_0 \leq 0.393$.

At the maximum value of the rotation parameter $\Omega$, the critical diagram (Fig. 1c) contains two resonant points $(\nu_1 = 0.650$ and $\nu_2 = 0.740)$, and up to the value $\nu \leq 0.46$, the unstable region occupies the entire range of possible values assumed by the initial virial ratio. Next, the stability and instability regions alternate at $(2T/U)_0 \geq 0.254$. When the superposition parameter approaches its maximum value, stability islands are observed: $\alpha(0.8984 \leq \nu \leq 0.962); 0.221 \leq (2T/U)_0 \leq 0.418); \beta(0.9086 \leq \nu \leq 0.935); 0.360 \leq (2T/U)_0 \leq 0.437), and a small peninsula $\delta(0.963 \leq \nu \leq 1.0); 0.368 \leq (2T/U)_0 \leq 0.447)$.

![Fig. 1. Critical dependences of the initial virial ratio on the superposition parameter at different values of the rotation parameter for the mode $(0; 6)$: (a) $\Omega = 0$, $A(0.220; 0.259), B(0.614; 0.302), S(0.550; 0.588), \alpha = 0.404447,$ (b) $\Omega = 0.5$, (c) $\Omega = 1.0$, $A(0.46; 0.405), B(0.739999; 0.254), S1(0.801; 0.78), S2(0.855; 0.863), S3(0.877103; 0.98), a = 0.650, b = 0.740$]
3.2. The Case \( m = 2; N = 6 \)

This mode is also responsible for the formation of two rings, but those consisting of separate thickenings. In this case, the perturbation potential has the form

\[
\delta \Phi = D_{26}(\psi)(x^2 + y^2)^2(x + iy)^2,
\]

(21)

Then the centroid displacement components in the perturbed system are determined, according to (8), in the form

\[
\bar{\delta}x = 2 \int_{-\infty}^{\psi} \Pi^3(\psi_1)S(\psi, \psi_1)D_{26}(\psi_1)\nonumber \times \left[ 2x_1(x_1^2 + y_1^2)(x_1 + iy_1)^2 \right.\nonumber + \left. (x_1^2 + y_1^2)^2(x_1 + iy_1) \right] d\psi_1,
\]

\[
\bar{\delta}y = 2 \int_{-\infty}^{\psi} \Pi^3(\psi_1)S(\psi, \psi_1)\nonumber \times D_{26}(\psi_1)\left[ 2y_1(x_1^2 + y_1^2)(x_1 + iy_1)^2 \right.\nonumber + \left. i(x_1^2 + y_1^2)^2(x_1 + iy_1) \right] d\psi_1.
\]

These expressions for the centroid displacement components show that, in addition to the results presented above for the averages of different combinations of these components over the velocity space, it is also necessary to calculate, by the same method, the averages \( w^3\vartheta, w^2\vartheta^2, w\vartheta^3, w^4, \vartheta^4, w^3\vartheta^2, w^2\vartheta^3, w\vartheta^4, \vartheta^2, \vartheta^3, \vartheta^4 \), and then, passing over to calculating the density response and comparing the obtained result with its theoretical expression, we obtain as a final result for the NADE mode (2; 6) against the background of the composite model (1):

\[
\Lambda \mu_{r}(\psi) = \frac{105}{256} D_{26}(\psi)(\lambda + \cos \psi)^{5-\tau}\sin^{\tau}\psi
\]

(\( \tau = 0 - 5 \)),

(22)

where the function \( D_{26}(\psi) \) is presented in Appendix B.

The results of the numerical calculation of the NADE (22) for the mode (2; 6) are presented in Fig. 2 in the form of marginal dependences of the initial virial ratio on the superposition parameter for different values of the rotation parameter. Using the results of the numerical analysis of NADE (22), we can conclude that the mode (2; 6), by the nature of its instability, reminds us of the case of the single-ring mode (2; 4) in [2]. Namely, against the background of a nonrotating composite model (1), there are both oscillatory and aperiodic instabilities, but when the
Fig. 3. Comparison of the instability increments of ringlike oscillation modes against the background composite model for different values of the rotation parameter $\Omega$ and the superposition parameter $\nu$. 

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model begins to rotate, an instability is only observed having an oscillatory nature.

In the absence of rotation of the composite model (Fig. 2a), the mode (2; 6) has a more stable behavior as compared to the other ring-shaped modes considered by us. There are stability peninsulas and Islands: \( \alpha(0 \leq \nu \leq 0.0077); \ 0.320 \leq (2T/|U|)_0 \leq 0.325), \ \beta(0 \leq \nu \leq 0.06); \ 0.292 \leq (2T/|U|)_0 \leq 0.308), \) and \( \delta(0.035 \leq \nu \leq 1.0; \ 0.293 \leq (2T/|U|)_0 \leq 0.473), \ \gamma(0.5413 \leq \nu \leq 0.653); \ 0.227 \leq (2T/|U|)_0 \leq 0.247). \)

Let us note that for \( \Omega = 0.5 \) the critical diagram (Fig. 2b) of this mode repeats the picture for the case of \( \Omega = 0 \) of the single-ring mode (2; 4) [2]. But only here the peninsula is relatively short, and it begins from the point \( S(0.84; 0.552) \). Here we can also see a slow increase of the instability region at \( \nu > 0.6 \), accompanied by an additional branch, \( S1(0.66; 0.621) \). And finally, when the rotation parameter of the composite model takes its maximum value, the critical diagram (Fig. 2c) of this mode (2; 6) does not differ qualitatively from the previously considered ring modes.

Figure 3 shows plots intended for comparing the values of the instability increments of the ring-shaped modes against the background of the composite model (1) for specific values of the rotation parameter \( \Omega \) and the superposition parameter \( \nu \). Figure 3 demonstrates that for \( \Omega \neq 1 \), the single-ring modes are always leading, but only in the case of \( \Omega = \nu = 0 \) the two-ring mode (0; 6) is dominant. Conversely, when the rotation parameter takes its maximum value, the superiority of the two-ring modes is observed, but only at \( \nu = 1 \) the single-ring modes are leading.

4. CONCLUSION

Let us enumerate the main results and conclusions obtained in the present paper.

1. The NADE have been obtained for two-ring perturbation modes superimposed on a nonequilibrium model of a self-gravitating disk (1), which is constructed as a superposition of two nonstationary phase densities of the DSS with isotropic and anisotropic velocity diagrams.

2. The marginal dependences between the physical parameters of the model have been constructed, such as the critical dependences between the values of the initial virial ratio, the degrees of rotation, the instability increments, and the superposition parameter.

3. It has been found that, against the background of the composite model, the mode (0; 6) has instabilities of both oscillatory and aperiodic nature at all values of the rotation parameter \( \Omega \). However, in the case of the mode (2; 6) against the background of a nonrotating composite model (1), there are both oscillatory and aperiodic instabilities, but when the model starts to rotate, an instability is only observed with an oscillatory nature.

4. It has been found that the superposition of the two models leads to the emergence of a resonant effect, as a result of which the instability region is stretched to the value \( (2T/|U|)_0 \approx 1 \) at certain values of the superposition parameter.

5. It has been found that as the rotation increases, increases as well the range of the initial virial ratio at which the structures under study can form.

6. It has been proved that, in the nonrotating model (1), the formation rate of the structure corresponding to the mode (0; 4) is higher then that of the mode (2; 4). When the model begins to rotate, the picture changes.

7. It has been shown that, in the absence of rotation of the composite model, the mode (2; 6) behaves in a more stable manner as compared to the other ring-shaped modes considered here. This means that if the system has no rotation, the probability of forming two rings consisting of separate thickenings, is very small.

8. It has turned out that, at \( \Omega \neq 1 \) against the background of a composite model, the single-ring modes are always leading, but only in the case of \( \Omega = \nu = 0 \), the two-ring mode (0; 6) is dominant. Conversely, when the rotation parameter takes its maximum value, a superiority of the two-ring modes is observed, but only at \( \nu = 1 \) the single-ring modes are leading.

Appendix A

EXPRESSION FOR THE FUNCTION \( D_{06}(\psi) \)

In (20) the function \( D_{06}(\psi) \) is

\[
D_{06}(\psi) = \left[ \nu \left( A_{06}^*(\psi) - I_{06}^*(\psi) \right) + I_{06}(\psi) \right],
\]

where
$I_{06}(\psi) = \left( g_1 \cos^5 \psi - g_2 \cos^4 \psi \sin \psi + g_3 \cos^3 \psi \sin^2 \psi - g_4 \cos^2 \psi \sin^3 \psi + g_5 \cos \psi \sin^4 \psi - g_6 \sin^5 \psi \right) l_0(\psi) + \left[ 5g_1 \cos^4 \psi \sin \psi + g_2 (q \cos^4 \psi - 4 \cos^3 \psi \sin^2 \psi) + g_3 (3 \cos^2 \psi \sin^3 \psi - 2q \cos^3 \psi \sin \psi) + g_4 (3q \cos^2 \psi \sin^2 \psi - 2 \cos \psi \sin^4 \psi) + g_5 (\sin^5 \psi - 4q \cos \psi \sin^3 \psi) + 5g_6 q \sin^4 \psi \right] l_1(\psi) + \left[ 10g_1 \cos^3 \psi \sin^2 \psi + g_2 (4q \cos^3 \psi \sin \psi) - 6 \cos^2 \psi \sin^3 \psi \right] l_2(\psi) + \left[ 10g_1 \cos^2 \psi \sin^3 \psi + g_2 (6q \sin \psi \sin^4 \psi) - 4q \sin^4 \psi \right] l_3(\psi) + \left[ 5g_1 \cos \psi \sin^4 \psi + g_2 (4q \cos \psi \sin^3 \psi \sin \psi - 5q \sin^5 \psi) + g_3 (3q^2 \cos^2 \psi \sin \psi - 6q \cos \psi \sin^3 \psi \sin \psi - 3q^2 \cos^2 \psi \sin \psi - 3q \sin^5 \psi) \right] l_4(\psi) + \left[ (g_1 \sin^5 \psi + g_2 q \sin^4 \psi + g_3 q^2 \sin^3 \psi \sin \psi - g_4 q^3 \sin^2 \psi + g_5 q^4 \sin \psi + g_6 q^5) l_5(\psi), \right.

A^{06}_c(\psi) = \frac{h_1^{10}}{8} \left[ q(8q^4 - 20e^2q^2 \sin^2 \psi + 5e^4 \sin^4 \psi) l_0(\psi) + 5e^2 \sin \psi (e^4 \sin^4 \psi - 16e^2q^2 \sin^2 \psi + 16q^4) l_1(\psi) + 10e^2 q (23e^2q^2 \sin^2 \psi - 2q^4 - 8e^4 \sin^4 \psi) l_2(\psi) + 10e^4 \sin \psi (23e^2q^2 \sin^2 \psi - 2e^4 \sin^4 \psi - 8q^4) l_3(\psi) + 5e^4 q (q^4 - 16e^2q^2 \sin^2 \psi + 16q^4) \sin \psi l_4(\psi) + e^6 \sin \psi (8e^4 \sin^4 \psi - 20e^2q^2 \sin^2 \psi + 5q^4) l_5(\psi) \right], \nonumber

and

$$h_1 = (1 + \lambda \cos \psi)^{-1}, \quad c = \frac{\lambda \sin \psi}{\sqrt{1 - \lambda^2}}$$

$$q = \lambda + \cos \psi; \quad e = \sqrt{1 - \lambda^2},$$

$$g_1 = h_1^5; \quad g_2 = -5e\sqrt{1 - \lambda^2} h_1^6;$$

$$g_3 = 2(3\Omega^2 + 5e^2 - 2)(1 - \lambda^2) h_1^2;$$

$$g_4 = 2(6 - 9\Omega^2 - 5e^2)(1 - \lambda^2)^3 h_1^8;$$

$$g_5 = \frac{1}{5}(8 - 36\Omega^2 + 33\Omega^4 - 60e^2 + 90e^2\Omega^2 + 25c^4)(1 - \lambda^2)^2 h_1^9,$$

where $g_6 = \frac{c}{5} (20c^2 - 30c^2\Omega^2 - 5c^4 - 8 + 36\Omega^2 - 33\Omega^4)(1 - \lambda^2)^2 h_1^{10};$

$$l_4(\psi) = \int_{-\infty}^{\psi} (1 + \lambda \cos \psi_1) S(\psi_1, \psi_1) \times D_{06}(\psi_1)(\lambda + \cos \psi_1)^{5-\tau} \sin^\tau \psi_1 d\psi_1.$$

**Appendix B**

**EXPRESSION FOR THE FUNCTION D_{26}(\psi)**

In (22) the function $D_{26}(\psi)$ is

$$D_{26}(\psi) = \left[ \nu \left( A^{06}_c(\psi) - I_{26}^*(\psi) \right) + I_{26}^*(\psi) \right],$$

where

$$I_{26}^*(\psi) = \left( b_1 \cos^5 \psi - b_2 \cos^4 \psi \sin \psi + b_3 \cos^3 \psi \sin^2 \psi - b_4 \cos^2 \psi \sin^3 \psi + b_5 \cos \psi \sin^4 \psi - b_6 \sin^5 \psi \right) \mu_0(\psi) + \left[ 5b_1 \cos^4 \psi \sin \psi + b_2 (q \cos^4 \psi - 4 \cos^3 \psi \sin^2 \psi) + b_3 (3 \cos^2 \psi \sin^3 \psi \sin \psi - 3q \cos^2 \psi \sin \psi - 2q \cos \psi \sin^4 \psi) + 5b_6 q \sin^4 \psi \right] l_1(\psi) + \left[ 10b_1 \cos^3 \psi \sin^2 \psi + b_2 (4q \cos^3 \psi \sin \psi) - 6 \cos^2 \psi \sin^3 \psi \right] l_2(\psi) + \left[ 10b_1 \cos^2 \psi \sin^3 \psi + b_2 (6q \sin \psi \sin^4 \psi) - 4q \sin^4 \psi \right] l_3(\psi) + \left[ 5b_1 \cos \psi \sin^4 \psi + b_2 (4q \cos \psi \sin^3 \psi \sin \psi - 5q \sin^5 \psi) + b_3 (3q^2 \cos^2 \psi \sin \psi - 6q \cos \psi \sin^3 \psi \sin \psi - 3q^2 \cos^2 \psi \sin \psi - 3q \sin^5 \psi) \right] l_4(\psi) + \left[ (b_1 \sin^5 \psi + b_2 q \sin^4 \psi + b_3 q^2 \sin^3 \psi \sin \psi - b_4 q^3 \sin^2 \psi + b_5 q^4 \sin \psi + b_6 q^5) l_5(\psi), \right.$
− 2q \cos^3 \psi \sin \psi + b_4(3q \cos^2 \psi \sin^2 \psi - 2 \cos \psi \sin^4 \psi) + b_5(\sin^5 \psi - 4q \cos \psi \sin^3 \psi) + 5b_6q \sin^4 \psi \right] \mu_1(\psi) + \left[ 10b_1 \cos^3 \psi \sin^2 \psi + b_2(4q \cos^3 \psi \sin \psi) - 6 \cos^2 \psi \sin^3 \psi \right. \\
+ b_3(q^2 \cos^3 \psi - 6q \cos^2 \psi \sin^2 \psi + 3 \cos \psi \sin^4 \psi) + b_4(6q \cos \psi \sin^3 \psi - 3q^2 \cos^2 \psi \sin \psi - \sin^5 \psi) \\
+ b_5(6q^2 \cos \psi \sin^2 \psi - 4q \sin^4 \psi) - 10b_6q^2 \sin^3 \psi \right] \mu_2(\psi) + \left[ 10b_1 \cos^2 \psi \sin^3 \psi + b_2(6q \cos^2 \psi \sin^2 \psi - 4 \cos \psi \sin^4 \psi) + b_3(3q^2 \cos^2 \psi \sin \psi - 6q \cos \psi \sin^3 \psi + \sin^5 \psi) + b_4(q^3 \cos^2 \psi - 6q^2 \cos \psi \sin^2 \psi \\
+ 3q \sin^4 \psi) + b_5(6q^2 \sin^3 \psi - 4q^3 \cos \psi \sin \psi) + 10b_6q^2 \sin^2 \psi \right] \mu_3(\psi) + \left[ 5b_1 \cos \psi \sin^4 \psi \\
+ b_2(4q \cos \psi \sin^3 \psi - \sin^5 \psi) + b_3(3q^2 \cos \psi \sin^2 \psi - 2q \sin^4 \psi) + b_4(2q^3 \cos \psi \sin \psi - 3q^2 \sin^3 \psi) \\
+ b_5(q^3 \cos \psi - 4q^2 \sin^2 \psi) - 5b_6q^4 \sin \psi \right] \mu_4(\psi) + (b_1 \sin^6 \psi + b_2q \sin^4 \psi + b_3q^2 \sin^3 \psi) + b_4q^3 \sin^2 \psi \\
+ b_5q^4 \sin \psi + b_6q^5 \right] \mu_5(\psi), \]

and

\[
A_6^*(\psi) = \frac{h_1}{64} \left[ 152q(8q^4 - 20e^2q^2 \sin^2 \psi \right. \\
+ 5e^4 \sin^4 \psi) + 37t \Omega e \sin \psi(16e^2q^2 \sin^2 \psi \\
- 16q^4 - e^4 \sin^4 \psi) \right] \mu_0(\psi) + \left[ 760e^2 \sin \psi(e^4 \sin^4 \psi \\
- 16e^2q^2 \sin^2 \psi + 16e^4) + 37t \Omega e q(16q^4 \\
+ 37e^4 \sin^4 \psi - 112e^2q^2 \sin^2 \psi) \right] \mu_1(\psi) \\
+ \left[ 1520e^2q(23e^2q^2 \sin^2 \psi - 2q^4 - 8e^4 \sin^4 \psi) \\
+ 74t \Omega e^3 \sin \psi(56q^4 - 101e^2q^2 \sin^2 \psi \\
+ 8e^4 \sin^4 \psi) \right] \mu_2(\psi) + \left[ 1520e^4 \sin \psi(23e^2q^2 \\
\times \sin^2 \psi - 2e^4 \sin^4 \psi - 8q^4) + 74t \Omega e^3 q(101e^2q^2 \\
\times \sin^2 \psi - 56e^4 \sin^4 \psi - 8q^4) \right] \mu_3(\psi) \\
+ \left[ 760e^4 q(q^4 - 16e^2q^2 \sin^2 \psi + 16e^4 \sin^4 \psi) \\
+ 37t \Omega e^5 \sin \psi(112e^2q^2 \sin^2 \psi - 16e^4 \sin^4 \psi \\
- 37q^4) \right] \mu_4(\psi) + \left[ 152e^6 \sin \psi(8e^4 \sin^4 \psi - 20e^2q^2 \\
\times \sin^2 \psi + 5q^4) + 37t \Omega e^5 q(16e^4 \sin^4 \psi \\
- 16e^2q^2 \sin^2 \psi + q^4) \right] \mu_5(\psi), \]

with

\[
b_1 = 19h_1^5, \]
\[
b_2 = (-95c + 37t \Omega) \sqrt{1 - \lambda^2 h_1^6}, \]
\[
b_3 = [2(33t \Omega^2 + 95c^2 - 32) - 148t \Omega c](1 - \lambda^2)h_1^7, \]
\[
b_4 = 2[c(96 - 99t \Omega^2 - 95c^2) + i\Omega(111c^2 \\
+ 750t \Omega^2 - 56)](1 - \lambda^2)^2 h_1^5, \]
\[
b_5 = [16 - 33t \Omega^2 - 192e^2 + 198c^2 \Omega^2 + 95c^4 \\
+ 4r \Omega c(37c^2 - 75t \Omega^2 + 56)](1 - \lambda^2)^2 h_1^9, \]
\[
b_6 = [c(64c^2 - 66c^2 \Omega^2 - 19c^4 - 16 + 33t \Omega^4) \\
+ i\Omega(37c^4 + 150c^2 \Omega^2 - 112c^2 + 16 - 48c^2 \Omega^2 \\
+ 33t \Omega^4)](1 - \lambda^2)^2 h_1^{10}, \]
\[
\mu_\tau(\psi) = \int_{-\infty}^{\psi} (1 + \lambda \cos \psi \lambda)^3 S(\psi, \psi_1) \\
\times D_2(\psi_1)(\lambda + \cos \psi_1)^{5-\tau} \sin^7 \psi_1 d\psi_1. \]

**REFERENCES**

1. K. T. Mirtadjieva, Grav. Cosmol. 15, 278, (2009).
2. K. T. Mirtadjieva, Grav. Cosmol. 18, 6 (2012).
3. K. T. Mirtadjieva, Grav. Cosmol. 18, 249 (2012).
4. J. A. Burns, Bull. Amer. Astron. Soc. 36, 1404 (2004).
5. K. T. Mirtadjieva and S. N. Nuritdinov, Astron. Astrophy. Transacnces 29, 305 (2016).
6. S. N. Nuritdinov, K. T. Mirtadjieva, and Mariam Sul-tana, Astrophysics 51, 410 (2008).
7. G. S. Binsnovat-Kogan and Ya. B. Ze’lovitch, Astrophyzika 6, 387 (1970).
8. V. A. Antonov, Uchyonye Zapiski LGU 32, 79 (1976).
9. A. M. Fridman and V. L. Polyachenko, Physics of Gravitating Systems (Springer, New-York, 1984).
10. I. G. Malkin, Theory of the Stability of Motion (Nauka, Moscow, 1967, in Russian).