An angle representation of QCD *

Dieter Stoll
Department of Physics, University of Tokyo
Bunkyo–ku, Tokyo 113, Japan

Abstract

For the sake of eliminating gauge variant degrees of freedom we discuss the way to introduce angular variables in the hamiltonian formulation of QCD. On the basis of an analysis of Gauss' law constraints a particular choice is made for the variable transformation from gauge fields to angular field variables. The resulting formulation is analogous to the one of Bars in terms of corner variables and it is closely related to the hamiltonian lattice QCD formulation. Therefore the corner or angle formulation may constitute an useful starting point for the investigation of the low energy properties of QCD in terms of gauge invariant degrees of freedom.

1 Introduction

One of the long standing problems in contemporary physics is understanding confinement of quarks and gluons from first principles. The difficulty in dealing with the infrared properties of QCD is on the one hand due to the non-linear gluonic interaction and on the other due to the constraints on the dynamics of the fundamental degrees of freedom which originate from the requirement of gauge invariance. In spite of the general belief that the non–linear interaction gives rise to confinement it has been conjectured recently that in fact the non–abelian constraints may be most important [1]. Aiming at an understanding of the low energy properties of QCD we should therefore try to develop approximations to the full QCD dynamics after the gauge variant degrees of freedom have been identified and isolated.

*Invited talk presented at the workshop on "Quantum Infrared Physics", Paris June 6–10, 1994
The aforementioned constraints are given by Gauss’ law operators, which generate a compact group in each point in space, telling us that the gauge variant degrees of freedom are ”angle” variables. Due to the fact that field theory deals with an infinite number of degrees of freedom, the choice of unphysical variables is to a large extent arbitrary. Therefore various decompositions into unphysical ”angle” variables and remaining physical variables are possible to arrive at the desired separation of unphysical degrees of freedom \[1, 2, 3\]. Although successful in that respect the variables chosen in this way to parametrize the physical Hilbert space may be inadequate to account in a simple way for the dynamics relevant for the low energy properties of QCD. For further variable changes, on the other hand, the complexity of the so derived hamiltonians constitutes a basic obstacle.

Therefore our starting point is the assumption that not only the unphysical but all variables are ”angle” variables and the hamiltonian should be expressed first in terms of these angular degrees of freedom before making a separation into gauge variant and gauge invariant ones. To find a definition of ”angle” variables in terms of gauge or electric fields we concentrate on an analysis of Gauss’ law operators. It will be shown that the form of these operators suggests the introduction of ”angles” which are non–locally related to the gauge fields. By a variable transformation the originally quantized gauge fields and electric fields in the Hamiltonian can be replaced by ”angle” and angular momentum operators respectively. The resulting formulation is analogous to the one in terms of corner variables obtained by Bars \[4\]. In contrast to similar approaches \[1, 2\] the separation into gauge variant and gauge invariant degrees of freedom is not made from the outset in the ”angle” or corner variable formulation. Therefore it may constitute an useful starting point for the search of approximations to the full QCD dynamics intended to understand its nonperturbative aspects.

2 The hamiltonian in terms of angle variables

We consider a Hamiltonian formulation of SU(N) gauge theories on a d–dimensional torus. Choosing the Weyl gauge \( A_0 = 0 \) we have the following Hamiltonian density

\[
\mathcal{H} = \sum_i \bar{\psi}(x) \gamma_i (i \partial_i + g A_i) \psi(x) + m \bar{\psi}(x) \psi(x) + \frac{1}{2} \sum_i E_i^a(x) E_i^a(x) + \frac{1}{2} \sum_{ij} \text{Tr} \left\{ F_{ij} F^{ij} \right\}.
\]

Imposing periodic boundary conditions for the gauge and anti–periodic ones for the fermion fields we quantize canonically \( (E_i^a(x) = \partial_0 A_i^a(x)) \)

\[
\left[ E_i^a(x), A_j^b(y) \right] = -i \delta_{a,b} \delta_{ij} \delta^d(x - y); \quad a, b = 1, \ldots, N^2 - 1; \quad i, j = 1, \ldots, d
\]

\[
\left\{ \psi_{k,\alpha}^\dagger(x), \psi_{l,\beta}(y) \right\} = \delta_{k,l} \delta_{\alpha,\beta} \delta^d(x - y); \quad k, l = 1, \ldots, N; \quad \alpha, \beta = \text{Spinor indices}
\]
where it is understood that the $\delta$–functions are periodic, as well. Since we have not fixed the gauge classically, Gauss’ law operator is the quantum mechanical generator of the gauge symmetry. It commutes with the Hamiltonian and therefore physical, i.e. gauge invariant, eigenstates must be annihilated by the generators of the symmetry

$$G^a(x)|\text{phys.}> = 0$$

(2.1)

$$G^a(x) = \sum_i \left[ \partial_i E_i^a(x) + g f^{abc} A_i^b(x) E_i^c(x) \right] + g \psi^\dagger(x) \frac{\lambda^a}{2} \psi(x)$$

(2.2)

$$\left[ G^a(x), G^b(y) \right] = ig f^{abc} G^c(x) \delta^d(x - y).$$

(2.3)

Since the generators obey the Lie algebra of the gauge group it is understood that out of $d\cdot(N^2-1)$ gauge degrees of freedom only a set of $N^2-1$ ”angle” variables in each point in space is changed by gauge transformations. Consequently the constraints are satisfied if these ”angles” have been identified and Gauss’ law operator has been transformed such that it is the angular momentum operator only with respect to these unphysical ”angles”. Physical states then correspond to $s$–wave states which are annihilated by these angular momentum operators and the Hamiltonian after transformation will not contain the unphysical variables anymore.

Since we want to replace gauge and electric fields by ”angles” and angular momenta in such a way that the constraints can easily be implemented, we study the form of Gauss’ law operators first in 1+1 dimensions. In this case the contribution in eq.(2.2)

$$f^{abc} A^b(x) E^c(x)$$

(2.4)

acts locally as an angular momentum operator on $N(N-1)$ ”angle” variables in either the gauge field or the electric field representation. The missing $(N-1)$ ”angle” variables could not be identified, if this was the complete Gauss’ law operator already. Therefore we must conclude that in 1+1 dimensions the full number of $(N^2-1)$ variables in each point in space can only be eliminated due to the presence of $\partial_x E(x)$ in the Gauss’ law operators. This term not only distinguishes the gauge fields as source of the additional unphysical variables but also introduces a non-locality into the Gauss’ law operators. Therefore it seems natural to assume that the ”angle” variables which are unphysical are nonlocally related to the gauge field variables. Although this argument is rigorous only in 1+1 dimensions we assume it to be an useful hypothesis for introducing ”angle” variables in any dimensions.

An expression for the gauge fields satisfying this requirement is

$$A_i(x) = \frac{i}{g} \partial_i V_i^\dagger(x)$$

(no summation)

(2.5)

\footnote{Note that throughout the paper spatial indices are not summed over unless explicitly indicated.}
\[ V_i(x) = P \exp \left[ ig \int_0^{x_i} dz_i A_i(x_i^+, z_i) \right] \]
\[ V_i(x) = \exp [i \xi_i(x)] ; \quad \xi_i(x) = \xi_i^a(x) \frac{\lambda^a}{2}; \quad 0 < x_i \leq L \]

where \( V_i(x) \) is a SU(N) matrix parametrized in terms of "angles" \( \xi_i^a(x) \), \( P \) denotes path ordering and \( x_i^+ \) stands for all coordinates orthogonal to \( x_i \). Since this definition together with the specific choice of paths in eq.\((2.6)\) leads to an unique relation\footnote{With the choice \( 0 < x_i \leq L \) the "angles" are uniquely determined from gauge fields by eq.\((2.6)\) if the derivative is taken from one side only \( \partial_i f(x_i) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} [f(x_i) - f(x_i - \epsilon)] \); \( 0 < x_i \leq L \). In this way it is possible to work with periodic, although not continuous "angles" and angular momenta.} between \( \xi_i(x) \) and \( A_i(x) \) a change of variables from \( A_i(x), E_i(x) \) to \( \xi_i(x) \) and the corresponding angular momenta \( J_i(x) \) becomes feasible. Using eq.\((2.5)\) we rewrite fermionic and magnetic part of the hamiltonian

\[ \tilde{\psi}(x) \{ \gamma_i [i \partial_i + g A_i(x)] + m \} \psi(x) = \tilde{\psi}(x)U_i(x) \{ \gamma_i i \partial_i + m \} \left[ U_i^\dagger(x) \psi(x) \right], \quad (2.8) \]
\[ [D_i, D_j] = [i \partial_i + g A_i, i \partial_j + g A_j] = -U_i \left\{ \partial_i \left[ (U_i^\dagger U_j) \partial_j (U_j^\dagger U_i) \right] \right\} U_i^\dagger, \quad (2.9) \]
\[ \Rightarrow \text{Tr} \{ F_{ij} F_{ij}' \} = \frac{-1}{g^2} \text{Tr} \left\{ \partial_i \left[ \left( (U_i^\dagger U_j) \partial_j (U_j^\dagger U_i) \right) \right] \right\} \left[ \partial_i \left[ \left( (U_i^\dagger U_j) \partial_j (U_j^\dagger U_i) \right) \right] \right\}. \]

In order to reformulate the electric part of the Hamiltonian which contains the conjugate momenta of the gauge fields, we introduce the angular momentum operators \( J_i^a(z) \) \footnote{Since this definition \[ N_i^{ac}(x) = \text{Tr} \left\{ V_i^\dagger(x) \frac{\lambda^a}{2} V_i(x) \lambda^c \right\} \]
\[ J_i^a(z), N_j^{ac}(x) = i f^{abc} N_j^{ae}(x) \delta_{i,j} \delta^d(x-z) \]

one can derive the following expression for the electric part of the hamiltonian density \footnote{In this way it is possible to work with periodic, although not continuous "angles" and angular momenta.}
Collecting all the results the hamiltonian density reads

\[ \mathcal{H} = \sum_i \left[ \bar{\psi}(x) U_i(x) \right] \left[ \gamma_i \partial_i + m \right] \left[ U_i^\dagger(x) \psi(x) \right] \]

\[ + \frac{g^2}{2} \sum_i \int_{x_i}^L dz_i J_i^a(x_i^+, z_i) \int_{x_i}^L dz_i' J_i^b(x_i^+, z_i') \]

\[ + \frac{1}{2g^2} \sum_{ij} \text{Tr} \left\{ \left[ \partial_i \left[ (U_i^\dagger U_j) \partial_j (U_j^\dagger U_i) \right] \right] \left[ \partial_i \left[ (U_i^\dagger U_j) \partial_j (U_j^\dagger U_i) \right] \right] \right\} \]

which is the "angle" representation we have been looking for. Note that the locality of the Hamiltonian has been lost although we have not been fixing the gauge yet. The electric part of the Hamiltonian is non–local and it shows already the linear "potential" \(|z_i - z_i'|\) characteristic for both the axial gauge formulation and the strong coupling limit in lattice gauge theory. We observe also that the Hamiltonian in QED, corresponding to eq.(2.13), is obtained by dropping the summations over color indices which shows the similarity of abelian and non–abelian gauge theories in the "angle" formulation.

Finally we want to consider the form of Gauss' law operator in these variables. We find

\[ g f^{abc} A_i^a(x) E_i^c(x) = -g \left[ \partial_i N_i^{ad}(x) \right] \int dz_i \theta(z_i - x_i) \theta(x_i) J_i^d(x_i^+, z_i) \]

\[ \partial_i E_i^a(x) = g \left[ \partial_i N_i^{ad}(x) \right] \int dz_i \theta(z_i - x_i) \theta(x_i) J_i^d(x_i^+, z_i) \]

\[ + g N_i^{ad}(x) \int dz_i J_i^d(x_i^+, z_i) \partial_i \left[ \theta(z_i - x_i) \theta(x_i) \right] \]

and taking the sum of these two contributions and the charge density operator we obtain for Gauss law the following expression

\[ G^a(x) = -g \sum_i \left[ N_i^{ad}(x) J_i^d(x) - \delta(x_i) \int dz_i J_i^a(x_i^+, z_i) \right] + g \psi^\dagger(x) \frac{\lambda^a}{2} \psi(x). \]

3 The connection with the lattice QCD hamiltonian

The task in this section is to show that the similarity between the variables in our formulation and those of the lattice hamiltonian approach to QCD can be made explicit. To this extend we replace the continuous spatial variables \(x_i\) by lattice points \(x_i^m\) and differential operators are replaced by finite differences. In order not to violate gauge invariance, we have to guarantee that the derivative operators only act on gauge
invariant functions. This is the case for the fermionic part of the hamiltonian. Using the simplest prescription for the derivative operation we find

\[ \partial_1 \left[ V_i^\dagger(x) \psi(x) \right] \rightarrow \frac{1}{a^{5/2}} \left[ V_i^\dagger(x_i^+, x_i^m) \psi(x_i^+, x_i^m) - V_i^\dagger(x_i^+, x_i^m - a) \psi(x_i^+, x_i^m - a) \right]. \]  

(3.1)

Introducing the link variables in terms of ”angles” \( \xi_i^m(x_i^+) \)

\[ L_i(x_i^{m-1}, x_i^+) = \mathcal{P} \exp \left[ ig \int_{x_i^{m-}\alpha}^{x_i^+} dz_i A_i(x_i^+, z_i^+) \right] = \exp \left[ i \xi_i^m(x_i^+) \right] \]  

(3.2)

\[ V_i(x_i^+, x_i^m) = L_i(x_i^{m-1}, x_i^+) L_i(x_i^{m-2}, x_i^+) \ldots L_i(0, x_i^+) \]

the fermionic hamiltonian can be cast into the familiar form

\[ \mathcal{H}_{\text{ferm}} = \frac{i}{2a^4} \sum_i \left[ \bar{\psi}(x_i^+, x_i^m) \gamma_i L_i^\dagger(x_i^{m-1}, x_i^+) \psi(x_i^+, x_i^m) \right. \]

\[ \left. - \bar{\psi}(x_i^+, x_i^m) \gamma_i L_i(x_i^{m-1}, x_i^+) \psi(x_i^+, x_i^{m-1}) \right] + \frac{m}{a^3} \bar{\psi}(x_i^+, x_i^m) \psi(x_i^+, x_i^m) \]  

(3.3)

Defining angular momentum operators \( \mathcal{J}_i^a(x_i^+, x_i^m) \) with respect to the link ”angles” \( \xi_i^m(x_i^+) \) obeying the following commutation relations

\[ \left[ \mathcal{J}_i^a(x_i^+, x_i^m), L_j(z_j^{n-1}, z_j^+) \right] = \delta_{i,j} \delta_{n,m} \delta_{x_i^+, z_i^+} L_j(z_j^{n-1}, z_j^+) \frac{\lambda^a}{2} \]

\[ \left[ \mathcal{J}_i^a(x_i^+, x_i^m), \mathcal{J}_j^b(z_j^+, z_j^m) \right] = \delta_{i,j} \delta_{x_i^+, z_i^+} \delta_{n,m} \delta_{j,a} \mathcal{J}_i^c(x_i^+, x_i^m) \]  

(3.4)

we arrive at the following expression for the integral over ”angular momentum” operators appearing in eq.(2.13)

\[ \int_{x_i}^{L} dz_i \mathcal{J}_i^c(x_i^+, z_i) \rightarrow \frac{1}{a^2} \sum_{k=m}^{N} J_i^c(x_i^+, x_{i+k}) = \frac{1}{a^2} N_i^a(x_i^+, x_i^{m-1}) \mathcal{J}_i^c(x_i^+, x_i^m) \]  

(3.5)

The electric part of the hamiltonian eq.(2.13) then becomes

\[ \mathcal{H}_{\text{elec}} = \frac{g^2}{2a^4} \sum_i \sum_{k=m}^{N} \sum_{l=m}^{N} J_i^a(x_i^+, x_k^i) J_i^a(x_i^+, x_l^i) = \frac{g^2}{2a^4} \sum_i \mathcal{J}_i^a(x_i^+, x_i^m) \mathcal{J}_i^a(x_i^+, x_i^m) \]  

(3.6)

which is the usual lattice form of this contribution to the hamiltonian. In particular the non–locality has disappeared and the existence of the linear ”potentials” \( |z_i - z_j^r| \) has become less obvious. The operator which remains to be discretized is the magnetic part of the hamiltonian. Introducing the following modification

\[ V_i^\dagger(x_i, x_j) V_j^\dagger(x_i, x_j) \rightarrow U_{ij}^\dagger(x_i, x_j) = V_j^\dagger(x_i = 0, x_j) V_i^\dagger(x_i, x_j) V_j^\dagger(x_i, x_j) V_i(x_i, x_j = 0) \]
it can be rewritten such that the derivatives are acting on gauge invariant functions. Therefore we obtain

\[ \partial_t \left[ U_{ji}(x_i, x_j) \partial_j U^\dagger_{ji}(x_i, x_j) \right] \rightarrow \frac{1}{a^2} \left[ U_{ji}(x_i - a, x_j) U^\dagger_{ji}(x_i - a, x_j - a) \right. \]

\[ \left. - U_{ji}(x_i, x_j) U^\dagger_{ji}(x_i, x_j - a) \right] . \] (3.7)

Multiplying this result with its hermitian conjugate and using the link matrices \( L_i \) defined in eq.(3.2) the result for the magnetic part of the Hamiltonian eq.(2.13) becomes

\[ \mathcal{H}_{magn} = -\frac{1}{2g^2a^4} \sum_{ij} \text{Tr} \left\{ L_i^\dagger(x_i^{m-1}, x_j^n)L_j(x_i^m, x_j^{n-1}) L_i(x_i^{m-1}, x_j^{n-1}) L_j^\dagger(x_i^{m-1}, x_j^{n-1}) \right\} \]

\[ + L_i^\dagger(x_i^{m-1}, x_j^n)L_j(x_i^m, x_j^{n-1})L_i(x_i^{m-1}, x_i^n)L_j(x_i^{m-1}, x_j^{n-1}) - 2 \right\} \]

\[ = \frac{1}{2g^2a^4} \sum_{ij} \text{Tr} \left\{ 2 - U_\Box(x_i^{m-1}, x_j^n) - U_\Box^\dagger(x_i^{m-1}, x_j^{n-1}) \right\} \] (3.8)

where \( U_\Box(x_i^{m-1}, x_j^n) \) is seen to be the unitary matrix obtained from the path ordered integral along the boundary of a plaquette in the (i,j)–plane starting from the point \((x_i^{m-1}, x_j^n)\). Thus collecting the results eq.(3.3,3.6,3.8) we find the standard Kogut–Susskind lattice hamiltonian density [3]

\[ \mathcal{H}_{lattice} = \frac{1}{a^4} \left\{ \frac{g^2}{2} \sum_i \mathcal{J}_i^a(x) \mathcal{J}_i^a(x) + \frac{1}{2g^2} \sum_{ij} \text{Tr} \left\{ 2 - U_\Box(x_i^{m-1}, x_j^n) - U_\Box^\dagger(x_i^{m-1}, x_j^{n-1}) \right\} \right\} \]

\[ + \frac{i}{2} \sum_i \left[ \bar{\psi}(x_i^+, x_i^{m-1}) \gamma_i L_i^\dagger(x_i^{m-1}, x_i^+) \psi(x_i^+, x_i^m) \right. \]

\[ \left. - \bar{\psi}(x_i^+, x_i^m) \gamma_i L_i(x_i^{m-1}, x_i^+) \psi(x_i^+, x_i^{m-1}) \right] + (ma) \bar{\psi}(x_i^+, x_i^m) \psi(x_i^+, x_i^m) \}\] . (3.9)

To complete the comparison with the lattice formulation we determine also the functional integration measure in our formulation. For this calculation we have to evaluate the Jacobian of the transformation from the gauge fields \( A_i \) to the "angles" \( \xi_j \) for which we find

\[ \mathcal{R}_{ij}^{ab}(x, y) = \frac{\delta A^a_i(x)}{\delta \xi_j^b(y)} = \frac{1}{g} \delta_{ij} \delta^{d-1}(x_i^+ - y_i^+) M_i^{bc}(y) N_i^{ac}(x) \partial_i \delta(x_i - y_i) . \] (3.10)

The Jacobian is obtained from the determinant of \( \mathcal{R} \) which can be written as

\[ \text{det}(\mathcal{R}) = \text{det}(\mathcal{R}^{-1}) = \prod_i \text{det} [M_i(z)] . \] (3.11)
The expression eq.(3.11) is just the volume of the gauge group and after discretization this result gives the usual integration measure in the lattice formulation (see e.g. [3]). We note that in establishing the relation between the lattice and the angle formulation of QCD an expansion in powers of the lattice spacing $a$ only occurs due to the necessity to replace differential operators by finite differences. As a consequence we naturally obtain the change from the flat integration measure in the space of gauge fields $A_i(x)$ to the group integration measure in the space of unitary matrices $V_i(x)$ or links $L_i(x^m_i, x^+_i)$. It also implies that continuum operators can be translated into lattice operators. In general this will not lead to simple expressions in terms of plaquettes (e.g. in the case of $\bar{\psi}\sigma_{\mu\nu}F^{\mu\nu}\psi$ or $\text{Tr} \{\epsilon_{\mu\nu\rho\tau}F^{\mu\nu}F^{\rho\tau}\}$) unlike in the special combination $(\text{Tr} \{F_{ij}(x)F_{ij}(x)\})$ appearing in the Hamiltonian. With regard to the widely discussed differences between compact and non-compact formulations of lattice gauge theories we clearly observe that the existence of the strong coupling limit is related to the replacement of an unbounded by a bounded magnetic operator

$$U_{ji}(x_i, x_j)\partial_j U^\dagger_{ji}(x_i, x_j) \rightarrow \left| \frac{1}{a} \left[ 1 - U_{ji}(x_i, x_j)U^\dagger_{ji}(x_i, x_j - a) \right] \right| \leq \frac{2}{a}.$$ 

This suggests to use a non-compact continuum formulation, separate gauge variant degrees of freedom and introduce a lattice for the gauge invariant degrees of freedom as a procedure leading to a meaningful non-compact lattice formulation of non-abelian gauge theories.

### 4 Summary and conclusion

In summary we have arrived at a continuum formulation of non-abelian gauge theories entirely in terms of angular degrees of freedom. The objective for doing so was the wish to have a formulation that leaves us freedom in choosing appropriate unphysical "angle" variables. This may prove important for being able to develop useful approximations to understand the low energy properties of QCD. The resulting formulation can be shown to be equivalent in a finite volume to Bars corner variable formulation and is like Bars formulation [7] closely related to the lattice Hamiltonian approach to QCD [3] as we have shown explicitly. In addition to this it is a formulation which has the built-in freedom in selecting unphysical variables. Furthermore it has been shown for the axial gauge representation of QCD [3] already that this reformulation leads to great technical simplifications in the elimination of unphysical variables [9]. All these advantages together may render the "angle" formulation an useful starting point for investigating non-perturbative aspects of QCD in terms of gauge invariant degrees of freedom.
Acknowledgements

We thank Profs. K.Ohta, M.Thies and K.Yazaki for helpful discussions and acknowledge gratefully the financial support by the "Japan Society for the Promotion of Science".

References

[1] K.Johnson, L.Lellouch and J.Polonyi, Nucl. Phys. B 367 (1991) 675
K.Johnson, Invited talk at the 1992 Aachen conference on "QCD, 20 Years Later"; eds.: P.M.Zerwas and H.A.Kastrup, World Scientific 1993

[2] J.Goldstone and R.Jackiw, Phys. Lett. 74 B (1978) 81
V.Baluni and B.Grossmann, Phys. Lett 78 B (1978) 226
A.G.Izergin, M.A.Semenov and L.D.Fadeev, Theor. Mat. Fiz. 38 (1979) 3
M.Creutz, I.J.Muzinich and T.N.Tudron, Phys. Rev. D 19 (1979) 531
N.H.Christ and T.D.Lee, Phys.Rev. D 22 (1980) 939
Yu.A.Simonov, Sov. J. Nucl. Phys. 41 (1985) 1311

[3] F.Lenz, H.W.L.Naus and M.Thies, The axial gauge representation of QCD, to appear in Ann. Phys.

[4] I.Bars, Phys. Rev. Lett. 40 (1978) 688
I.Bars, Nucl. Phys. B 148 (1979) 445

[5] J.B.Kogut and L.Susskind, Phys. Rev. D 11 (1975) 395

[6] M.Creutz, Quarks, gluons and lattices; Cambridge University press 1990

[7] I.Bars, Nucl. Phys. B 149 (1979) 39

[8] D.Stoll, The hamiltonian formulation of QCD in terms of angle variables, preprint

[9] D.Stoll, Gauge fixing by unitary transformations in QCD, preprint