UVW RELATIONS OVER A SUBVARIETY OF A HYPERELLIPTIC JACOBIAN

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ABSTRACT. This article extends relations of Mumford’s UVW-expressions to those in subvariety in a hyperelliptic Jacobian using Baker’s method.

1. Introduction

For a hyperelliptic curve $C_g$ whose affine part is given by $y^2 = \prod_{i=1}^{2g+1}(x - b_i)$, where $b_i$’s are complex numbers, its Jacobian $J_g$ is given as a complex torus $\mathbb{C}/\Lambda$ by the Abel map $\omega$. The Abelian theorem enables us to have a natural morphism from the symmetrical product $\text{Sym}^g(C_g)$ to the Jacobian $J_g \approx \omega[\text{Sym}^g(C_g)]/\Lambda$.

Mumford and his coworkers used UVW expression based upon Jacobi’s considerations $\text{[Mu]}$, which represents the hyperelliptic functions over the Jacobian.

Let $D(z)$ be a certain derivative of the Jacobian, the Bolza polynomial, $F(z) \equiv U(z) := (z - x_1) \cdots (z - x_g)$ for $(x_i, y_i)_{i=1, \ldots, g} \in \text{Sym}^g(C_g)$. Further let $V(z) := D(z)(x_1 + \cdots + x_2)$ and $W(z) := (f(z) - V(z)^2)/F(z)$. Mumford and his coworkers showed $\text{[Mu]}$ Theorem 3.1,

Theorem 1.1.

(1) $D(z_1)F(z_2) = \frac{F(z_2)V(z_1) - V(z_2)F(z_1)}{z_1 - z_2}$.

(2) $D(z_1)V(z_2) = \frac{1}{2} \left( \frac{F(z_2)W(z_1) - W(z_2)F(z_1)}{z_1 - z_2} - F(z_1)F(z_2) \right)$,

(3) $D(z_1)W(z_2) = \frac{W(z_2)V(z_1) - V(z_2)W(z_1)}{z_1 - z_2} + F(z_1)V(z_2)$.

(4) $D(z_1)D(z_2) = D(z_2)D(z_1)$.

There exists an interesting Poisson structure in these relations, which are studied by several authors $\text{[AHP, PV, Mu]}$.

On the other hand, as zeros of an appropriate shifted Riemann theta function over $J_g$, the theta divisor is defined as

$\Theta := \omega[\text{Sym}^{g-1}(C_g)]/\Lambda$

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which is a subvariety of $\mathcal{J}_g$. Similarly, it is natural that we introduce a subvariety

$$\Theta_k := \omega[\text{Sym}^k(C_g)]/\Lambda$$

and a sequence,

$$\Theta_0 \subset \Theta_1 \subset \Theta_2 \subset \cdots \subset \Theta_{g-1} \subset \Theta_g = \mathcal{J}_g$$

Vanhaecke studied the structure of these subvarieties as stratifications of the Jacobian $\mathcal{J}_g$ using the strategies developed in the integrable system $[V1, V2]$. He showed that it is connected with stratifications of the Sato Grassmannian $[V1]$. Further in $[V2]$, he studied Lie-Poisson structure in the Jacobian and showed that invariant manifolds associated with Poisson brackets can be identified with these strata; it implies that these strata are characterized by the Lie-Poisson structure. Further he and Abenda and Fedorov $[AF]$ considered their relations to finite-dimensional integrable systems, i.e., Henon-Heiles system and Neumann systems. Independently the author considered a relation of symmetric functions over $\Theta_k$ as an extension of the study of Weierstrass on al-functions $[Ma]$. The elementary symmetric functions over $\Theta_k$ appear in $[AF, Ma]$ and play the important roles to reveal structure of $\Theta_k$. In the case of the Jacobian, the relations of the elementary symmetric functions over the Jacobian is represented by Theorem 1.1 which is directly related to Neumann system and other many studies on structures, like a Lie-Poisson structure, of hyperelliptic curves $[AHP, PV, Mu]$. Though the structure of these subvarieties was studied using Theorem 1.1, its variant over $\Theta_k$ was not studied.

Thus the purpose of this article is an extension of the relations in Theorem 1.1 to similar variants of elementary symmetric functions over $\Theta_k$ as in our main theorem 3.1. We believe that Theorem 3.1 has an effects on these studies.

Of course, in this stage our theorem 3.1 is not connected with such a finite integrable system directly though Abenda and Fedorov $[AF]$ studied similar subjects, and E. Previato suggested the author that there might be a connection between Theorem 3.1 and a finite integrable system. We expect that our results shed some light on these studies.

Furthermore our strategy in this article is based upon Baker’s method in $[Ba]$, which is a direct application of the reciprocity laws for differentials over a curve to a relation over there. Thus we believe that we should re-evaluate Baker’s method using modern expressions of the reciprocity $[BP]$ in future. If we could, it is expected that we would have a modern language to expresses the subvarieties in Jacobians.

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2. Hyperelliptic Curve

Hyperelliptic Curve: This article deals with a hyperelliptic curve $C_g$ of genus $g$ ($g > 0$) given by the affine equation,

$$y^2 = f(x) = \lambda_{2g+1} x^{2g+1} + \lambda_{2g} x^{2g} + \cdots + \lambda_2 x^2 + \lambda_1 x + \lambda_0 = b_0 (x - b_1)(x - b_2) \cdots (x - b_{2g+1}),$$

where $\lambda_j$'s and $b_j$'s are complex numbers.

For a point $(x_i, y_i) \in C_g$, the unnormalized differentials of the first kind are defined by,

$$d u_1^{(i)} := \frac{dx_i}{2y_i}, \quad d u_2^{(i)} := \frac{x_i dx_i}{2y_i}, \quad \cdots, \quad d u_g^{(i)} := \frac{x_{i}^{g-1} dx_i}{2y_i}.$$

For positive integer $k (\leq g)$, the Abel map from $k$-th symmetric product of the curve $C_g$ to $C_g$ is defined by,

$$u := (u_1, \cdots, u_g) : \text{Sym}^k(C_g) \longrightarrow C^g, \quad u_k := \sum_{i=1}^{k} \int_{\infty}^{(x_i, y_i)} d u_k^{(i)}.$$

The Jacobian is defined by

$$J_g := C^g / \langle \text{lattice} \rangle.$$

Let its image quotient by the lattice denoted by $\Theta_k$.

$$(2.1) \quad \{0\} \subset C_g \equiv \Theta_1 \subset \cdots \subset \Theta_{g-1} \subset \Theta_g \equiv J_g.$$

Let us fix $k \leq g$ and $((x_1, y_1), (x_2, y_2), \cdots, (x_k, y_k)) \in \text{Sym}^k(C_g)$. We will introduce a variant in $\Theta_k$ of UVW-expression, which also appears in [AF, Ma, Mu].

Definition 2.1. We define

1. $F^{(k)}(z) := (z - x_1)(z - x_2) \cdots (z - x_k)$ and for brevity we denote it by $F(z)$ if there is no confusion.

2. Let $k$ be an integer such that $k \leq m \leq g$ and natural number $n := m - k + 1$.

$$D(z) := \frac{1}{2z^n} \sum_{i=n}^{m} z^i \partial x_i.$$

3. $V(z) := D(z)(x_1 + x_2 + \cdots + x_k)$.

4. $W(z) := (f(z) - [V(z)z^{n-1}]^2)/F(z)z^{2n-2}$.

Simple consideration gives the following proposition:

Proposition 2.1. (1)

$$D(z) = \sum_{i=1}^{k} \frac{y_i F(z)}{F'(x_i)(z - x_i)x_i^n} \partial x_i.$$

(2)

$$V(z) = \sum_{i=1}^{k} \frac{y_i F(z)}{F'(x_i)(z - x_i)x_i^n}.$$

(3) $y_i = V(x_i)x_i^n$. 

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Proof. Let us introduce quantities and matrices:

\[
    W := \begin{pmatrix}
        \chi_{1,0} & \chi_{1,1} & \cdots & \chi_{1,k-1} \\
        \chi_{2,0} & \chi_{2,1} & \cdots & \chi_{2,k-1} \\
        \vdots & \vdots & \ddots & \vdots \\
        \chi_{k,0} & \chi_{k,1} & \cdots & \chi_{k,k-1}
    \end{pmatrix},
    \quad
    \mathcal{Y} := \begin{pmatrix}
        y_1/x_1^{-1} \\
        y_2/x_2^{-1} \\
        \vdots \\
        y_k/x_k^{-1}
    \end{pmatrix},
\]

\[
    \mathcal{F} := \begin{pmatrix}
        F'(x_1) \\
        F'(x_2) \\
        \vdots \\
        F'(x_k)
    \end{pmatrix},
    \quad
    \mathcal{V} := \begin{pmatrix}
        1 & 1 & \cdots & 1 \\
        x_1 & x_2 & \cdots & x_k \\
        x_1^2 & x_2^2 & \cdots & x_k^2 \\
        \vdots & \vdots & \ddots & \vdots \\
        x_1^{k-1} & x_2^{k-1} & \cdots & x_k^{k-1}
    \end{pmatrix},
\]

where \( F'(x) := dF(x)/dx \). Then we have

\[
    \begin{pmatrix}
        du_n \\
        \vdots \\
        du_m
    \end{pmatrix} = \frac{1}{2} \mathcal{V}^{-1} \begin{pmatrix}
        dx_1 \\
        \vdots \\
        dx_k
    \end{pmatrix}.
\]

and \( \mathcal{V}^{-1} = \mathcal{F}^{-1} W \). By letting \( \partial_{u_i} := \partial/\partial u_i \) and \( \partial_{z_i} := \partial/\partial x_i \), we obtain

\[
    \begin{pmatrix}
        \partial_{u_n} \\
        \partial_{u_{n+1}} \\
        \vdots \\
        \partial_{u_m}
    \end{pmatrix} = t(2\mathcal{V}\mathcal{F}^{-1} W) \begin{pmatrix}
        \partial_{z_1} \\
        \partial_{z_2} \\
        \vdots \\
        \partial_{z_k}
    \end{pmatrix}.
\]

Hence \( D(z) \) is given by (1). (2) and (3) are obvious from (1). \( \square \)

3. Relations of \( UVW \) in \( \Theta_k \)

Now we will give our main theorem as follows:

**Theorem 3.1.** Assume \( \lambda_0 \neq 0 \).

(1)

\[
    D(z_1) F(z_2) = \frac{F(z_2) V(z_1) - V(z_2) F(z_1)}{z_1 - z_2}.
\]

(2)

\[
    D(z_1) V(z_2) = \frac{1}{2} \left[ \frac{F(z_2) W(z_1) - W(z_2) F(z_1)}{z_1 - z_2} - F(z_1) F(z_2) H(\lambda, z_1, z_2) \right],
\]

where \( H(\lambda, z_1, z_2) := H_0(\lambda, z_1, z_2) + H_\infty(\lambda, z_1, z_2) \),

\[
    H_0(\lambda, z_1, z_2) := \frac{1}{(2n - 3)!} \left[ \frac{\partial^{2n-3}}{\partial x^{2n-3}} F(x)^2 (z_1 - x)(z_2 - x) \right]_{x=0},
\]

\[
    H_\infty(\lambda, z_1, z_2) := \frac{1}{(4g - 4m)!} \left[ \frac{\partial^{4g-4m}}{\partial t^{4g-4m}} F(1/t)^2 (1 - z_1 t^2)(1 - z_2 t^2) \right]_{t=0}.
\]
Proof. Using Proposition 2.1 (1) is directly shown: $D(z_1)F(z_2)$ is equal to
\[
- \sum_{i=1}^{k} \frac{F(z_1)F(z_2)y_i}{F'(x_i)x_i^{n-1}(z_1-x_i)(z_2-x_i)}
\]
which is the right hand side of (1) from the definition of $V$.

Provided (2), (3) is proved as follows: Noting that $D(z_1)$ is a direction differential operator and its action obeys the Leibniz rule. Using the fact $D(z_1)f(z_2) = 0$, and the Leibniz rule, $D(z_1)W(z_2)$ is given by
\[
-\frac{1}{F'(z_2)^2z_2^{2n-2}}[D(z_1)F(z_2)](f(z_2) - V(z_2)^2z_2^{2n-2}) - \frac{2}{F(z_2)}[V(z_2)D(z_1)V(z_2)].
\]
Using (1) and (2), it becomes
\[
-\frac{1}{F(z_2)^2z_2^{2n-2}}(f(z_2) - V(z_2)^2z_2^{2n-2})V(z_2)F(z_1) - \frac{2}{F(z_2)}V(z_2)\left[\frac{1}{2} F(z_2)W(z_1) - W(z_2)F(z_1) - \frac{1}{2} F(z_1)F(z_2)H(\lambda, z_1, z_2)\right],
\]
which is the right hand side of (3) from the definition of $W$.

(4) is roughly proved due to the fact $D(z_1)V(z_2) = D(z_2)V(z_1)$. Thus we consider (4) after proving (2).

Let us consider the formula (2). The strategy is essentially the same as [Ba]. First we translate the words of the Jacobian into those of the curves in Sym$^k(C_g)$; we rewrite the differentials in the Jacobian in terms of the differentials over curves in Sym$^k(C_g)$ by (1). We count the residue of an integration and use a combinatorial trick. Then we will obtain (2):

\[
D(z_1)V(z_2) = \sum_{j=1, i=1}^{k} \frac{F(z_1)y_i}{F'(x_i)x_i^{n-1}(z_1-x_i)} \frac{\partial}{\partial x_i} \frac{F(z_2)y_j}{F'(x_j)x_j^{n-1}(z_2-x_j)}.
\]
Let us decompose the summation into $j = i$ and $j \neq i$ parts. Further we will note the derivative of $F(x)$:
\[
\frac{\partial}{\partial x_i} \left( \left[ \frac{\partial}{\partial x} F(x) \right]_{x=x_i} \right) = \frac{1}{2} \left[ \frac{\partial^2}{\partial x^2} F(x) \right]_{x=x_i}.
\]
Then we obtain the formulation that $D(z_1)V(z_2)$ is equal to

$$
\sum_{i=1}^{k} \frac{F(z_1)F(z_2)}{F'(x_i)x_i^{2n-2}(z_2-x_i)(z_1-x_i)} \frac{1}{2} \left( \frac{f'(x_i)}{F'(x_i)} - \frac{f(x_i)F''(x_i)}{F'(x_i)} - 2(n-1) \frac{f(x_i)}{x_i F'(x_i)} \right)
+ \sum_{i \neq j} \frac{F(z_1)F(z_2)y_iy_j}{F'(x_i)F'(x_j)x_i^{n-1}x_j^{n-1}(z_2-x_i)(z_1-x_j)} \left( \frac{1}{x_j-x_i} - \frac{1}{z_2-x_i} \right)
+ \sum_{i \neq j} \frac{F(z_1)F(z_2)y_iy_j}{2 F'(x_i)F'(x_j)x_i^{2n-2}} \left( \frac{1}{x_j-x_i}^2 - \frac{1}{(z_1-x_i)^2} \right).
$$

We will consider each term in the formulae; we refer the first and second terms $[DV]_1(z_1, z_2)$ and the third term $[DV]_2(z_1, z_2)$. The proof of (2) finishes due to the following Lemma because $[DV]_1(z_1, z_2)$ is given by

$$
\frac{1}{2} \frac{F(z_2)f(z_1)/z_2^{2n-2}F(z_1) - f(z_2)F(z_1)/z_1^{2n-2}F(z_2)}{z_1-z_2} - F(z_1)F(z_2)H(\lambda, z_1, z_2)
+ \frac{1}{2} \frac{1}{z_1-z_2} \sum_{i=1}^{k} \frac{F(z_1)F(z_2)}{F'(x_i)x_i^{2n-2}} \left( \frac{1}{(z_2-x_i)^2} - \frac{1}{(z_1-x_i)^2} \right)
$$

whereas $[DV]_2(z_1, z_2)$ is equal to

$$
\frac{1}{2} \frac{1}{z_1-z_2} \sum_{i \neq j} \frac{F(z_1)F(z_2)y_iy_j}{F'(x_i)F'(x_j)x_i^{n-1}x_j^{n-1}} \left( \frac{1}{(z_1-x_j)(z_1-x_i)} - \frac{1}{(z_2-x_i)(z_2-x_j)} \right).
$$

The first term in $[DV]_1(z_1, z_2)$ is equal to parts of $W$ in the first term in (2) and the second term in (2). The remainder is given by the second term in $[DV]_1(z_1, z_2)$ and $[DV]_2(z_1, z_2)$, which is

$$
\frac{1}{2} \frac{1}{z_1-z_2} \sum_{i \neq j} \frac{F(z_1)F(z_2)y_iy_j}{F'(x_i)F'(x_j)x_i^{n-1}x_j^{n-1}} \left( \frac{1}{(z_1-x_j)(z_1-x_i)} - \frac{1}{(z_2-x_i)(z_2-x_j)} \right).
$$

It is obvious that they forms the right hand side of (2).

Finally we will consider the commutativity of $D(z_a)$. Since $D(z_1)D(z_2)$ is decomposed to the form

$$
\sum_{i=1}^{k} [DD]_{i}(z_1, z_2) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{k} [DD]_{ij}(z_1, z_2) \frac{\partial^2}{\partial x_i \partial x_j},
$$

and obviously $[DD]_{ij}(z_1, z_2) = [DD]_{ij}(z_2, z_1)$ we must check the first term. Let $[DV]_1(z_1, z_2)$ and $[DV]_2(z_1, z_2)$ be denoted by

$$
[DV]_1(z_1, z_2) = \sum_{i=1}^{k} [DV]_{1,i}(z_1, z_2), \quad [DV]_2(z_1, z_2) = \sum_{i \neq j} [DV]_{2,ij}(z_1, z_2),
$$
following the definition in straightforward way. Then we have

\[
\sum_{i=1}^{k} [DD]_i(z_1, z_2) \frac{\partial}{\partial x_i} = \sum_{i=1}^{k} [DV]_{1,i}(z_1, z_2) \frac{\partial}{\partial x_i} + \sum_{i,j}^{k} [DV]_{2,ij}(z_1, z_2) \frac{\partial}{\partial x_j}.
\]

Let us consider \(D(z_1)D(z_2) - D(z_2)D(z_1)\). Since it is clear that \([DV]_{1,i}(z_1, z_2) = [DV]_{1,i}(z_2, z_1),\)

\[
\sum_{i,j}^{k} [DV]_{2,ij}(z_1, z_2) \frac{\partial}{\partial x_j} - \sum_{i,j}^{k} [DV]_{2,ij}(z_2, z_1) \frac{\partial}{\partial x_j}
\]

remains. Using the appropriate symmetric quantities \(K_{2,ij}(z_1, z_2) = K_{2,ij}(z_2, z_1) = K_{2,j1}(z_1, z_2),\) it becomes

\[
\sum_{i,j}^{k} K_{2,ij}(z_1, z_2) \left[ \frac{1}{z_1 - x_i} \frac{1}{z_2 - x_j} \left( \frac{1}{x_j - x_i} - \frac{1}{z_2 - x_i} \right) \right] \frac{\partial}{\partial x_j}.
\]

Noting

\[
\frac{1}{z_1 - x_i} \frac{1}{z_2 - x_j} \left( \frac{1}{x_j - x_i} - \frac{1}{z_2 - x_i} \right) = \frac{1}{z_1 - x_i} \frac{1}{z_2 - x_j} \frac{1}{x_j - x_i},
\]

this vanishes. Hence (4) is proved. \(\square\)

**Lemma 3.1.** Following relations hold:

1. \[
\sum_{i=1}^{k} \frac{1}{F'(x_i)} \left[ \frac{\partial}{\partial x} \left( \frac{f(x)}{(x - z_1)(x - z_2)^{2n-2}F(x)} \right) \right]_{x=x_i} = \frac{1}{(4g - 4m)!} \left[ \frac{\partial^{4g-4m}}{\partial t^{4g-4m}} F(1/t^2) \right]_{t=0} - \frac{1}{(2n-3)!} \left[ \frac{\partial^{2n-3}}{\partial t^{2n-3}} F(x)^2 \right]_{x=0}.
\]

2. \[
\frac{1}{z_1 - x_i} \frac{1}{z_2 - x_i} \left( \frac{1}{x_j - x_i} + \frac{1}{z_2 - x_i} \right) = \frac{1}{z_2 - z_1} \left( \frac{1}{(z_2 - x_i)^2} - \frac{1}{(z_1 - x_i)^2} \right).
\]

3. For symmetric term \(K(i, j) = K(j, i),\)

\[
I := \sum_{i,j}^{k} K(i, j) \frac{1}{z_1 - x_i} \frac{1}{z_2 - x_j} \left( \frac{1}{x_j - x_i} - \frac{1}{z_2 - x_i} \right).
\]

is expressed by

\[
I = \frac{1}{2} \sum_{i,j}^{k} K(i, j) \left( \frac{1}{(z_1 - x_i)(z_1 - x_j)} - \frac{1}{(z_2 - x_i)(z_2 - x_j)} \right).
\]
Proof: (1) will be proved by the following residual computation. Let $\partial C_o$ be the boundary of a polygon representation $C_o$ of $C_g$,

$$\oint_{\partial C_o} \omega = 0, \quad \omega := \frac{f(x)}{(x - z_1)(x - z_2)x^{2n-2}F(x)^2} dx.$$  

The divisor of the integrand is

$$(\omega) = 3 \sum_{i=1}^{2g+1} (b_i, 0) - 2 \sum_{i=1}^{k} (x_i, ey_i) - \sum_{a=1}^{2} (z_a, ey(z_a))$$

$$- (2n - 2) \sum_{\epsilon=\pm} (0, ey(0)) - (4g - 4m + 1) \infty.$$  

(i) Using the fact that the local parameter $t$ at $\infty$ is $x = 1/t^2$, we have

$$\text{res}_\infty \omega = -2 \frac{1}{(4g - 4m)!} \left[ \frac{\partial^{4g-4m} t^{4g-4k+1} f(1/t^2)}{\partial t^{4g-4m}} F(1/t^2)^2 (1 - z_1 t^2)(1 - z_2 t^2) \right]_{t=0}.$$  

(ii) Since the local parameter at $x = 0$ is $x$ itself, we have

$$\text{res}_{(0, \pm y(0))} \omega = \frac{1}{(2n-1)!} \left[ \frac{\partial^{2n-1}}{\partial x^{2n-1}} f(x) (z_1 - x)(z_2 - x) \right]_{x=0}.$$

(iii) Noting that the local parameter $t$ at $(x_k, \pm y_k)$ is $t = x - x_k$, we have

$$\text{res}_{(x_k, \pm y_k)} \omega = \frac{1}{F'(x_k)} \left[ \frac{\partial}{\partial x} f(x) \left( \frac{1}{(x - z_1)(x - z_2)x^{2n-2}F'(x)} \right) \right]_{x=x_k}.$$  

(iv) Using the fact that the local parameter $t$ at $z_a$ is $t = x - z_a$, we have

$$\text{res}_{(z_a, \pm y(z_a))} \omega = \frac{f(z_a)}{(z_a - z_b) z_a^{2n-2} F(z_a)^2}.$$  

where $z_b = z_2$ for $a = 1$ and $z_1$ for $a = 2$.

By arranging them, we obtain (1).

On the other hand, (2) can be proved by using a trick:

$$\frac{1}{(z_1 - x)(z_2 - x)} = \frac{1}{z_1 - z_2} \left( \frac{1}{z_2 - x} - \frac{1}{(z_1 - x)} \right).$$

(3) can be evaluated by

$$I = 2 \times \text{right hand side} - I.$$

\[ \square \]

Proposition 3.1.  

(1) $m = g$ case:

$$H_\infty = 1.$$  

(2) $n = 1$ case:

$$H_0 = 0.$$  

(3) $n = 2$ case:

$$H_0 = -2 \frac{2}{(x_1 \cdots x_k)^2 z_1 z_2} \left[ \lambda_1 + \lambda_0 \left( \sum_{i=1}^{k} \frac{2}{x_i} + \frac{1}{z_1} + \frac{1}{z_2} \right) \right].$$  

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