Learning Gaussian Graphical Models via Multiplicative Weights

Anamay Chaturvedi
Khory College of Computer Sciences
Northeastern University
chaturvedi.a@northeastern.edu

Jonathan Scarlett
Depts. Computer Science & Mathematics
National University of Singapore
scarlett@comp.nus.edu.sg

Abstract

Graphical model selection in Markov random fields is a fundamental problem in statistics and machine learning. Two particularly prominent models, the Ising model and Gaussian model, have largely developed in parallel using different (though often related) techniques, and several practical algorithms with rigorous sample complexity bounds have been established for each. In this paper, we adapt a recently proposed algorithm of Klivans and Meka (FOCS, 2017), based on the method of multiplicative weight updates, from the Ising model to the Gaussian model, via non-trivial modifications to both the algorithm and its analysis. The algorithm enjoys a sample complexity bound that is qualitatively similar to others in the literature, has a low runtime $O(m^2 p)$ in the case of $m$ samples and $p$ nodes, and can trivially be implemented in an online manner.

1 Introduction

Graphical models are a widely-used tool for providing compact representations of the conditional independence relations between random variables, and arise in areas such as image processing [Geman and Geman, 1984], statistical physics [Glauber, 1963], computational biology [Durbin et al., 1998], natural language processing [Manning and Schütze, 1999], and social network analysis [Wasserman and Faust, 1994]. The problem of graphical model selection consists of recovering the graph structure given a number of independent samples from the underlying distribution.

In the Gaussian setting, the support of the sparse inverse covariance matrix directly corresponds to the graph under which the Markov property holds [Wainwright and Jordan, 2008]. Each node in the graph corresponds to a variable, and any two variables are independent conditioned on a separating subset.

In this paper, we present an algorithm for Gaussian graphical model selection that builds on the multiplicative weights approach recently proposed for (discrete-valued) Ising models [Klivans and Meka, 2017]. This extension comes with new challenges due to the continuous and unbounded nature of the problem, prohibiting the use of several parts of the analysis in [Klivans and Meka, 2017] (as discussed more throughout the paper). Under suitable assumptions on the (inverse) covariance matrix, we provide formal recovery guarantees of a similar form to other algorithms in the literature; see Section 1.2 and Theorem 9.

1.1 Related Work

Learning Gaussian graphical models. The problem of learning Gaussian graphical models (and the related problem of inverse covariance matrix estimation) has been studied using a variety of techniques and assumptions; our overview is necessarily brief, with a focus on those most relevant to the present paper.

Information-theoretic considerations lead to the following algorithm-independent lower bound on the number of samples $m$ [Wang et al., 2010]:

$$m = \Omega \left( \max \left\{ \frac{\log p}{\kappa^2}, \frac{d \log p}{\log(1 + \kappa d)} \right\} \right),$$  

where $p$ is the number of nodes, $d$ the maximal degree of the graph, and $\kappa$ the minimum normalized edge strength (see below). Ideally, algorithmic upper
Early algorithms such as SGS and PC [Kalisch and Buhlmann, 2007; Spirtes et al., 2000; van de Geer et al., 2013] adopted conditional independence testing methods, and made assumptions such as strong faithfulness. A popular line of works studied the Graphical Lasso and related $ℓ_1$-based methods [d’Aspremont et al., 2008; Hsieh et al., 2013; Meinshausen et al., 2006; Ravikumar et al., 2011; Yuan and Lin, 2007; Zhou et al., 2011], typically attaining low sample complexities (e.g., $(d^2 + \kappa^{-2}) \log p$ [Ravikumar et al., 2011]), but only under some-what strong coherence-based assumptions. More recently, sample complexity bounds were given under walk-summability assumptions [Anandkumar et al., 2012; Kelner et al., 2019] and eigenvalue (e.g., condition number) assumptions [Cai et al., 2011; 2016; Wang et al., 2016]. Another line of works has adopted a Bayesian approach to learning Gaussian graphical models [Lopuä-Aho et al., 2017; Mohammadi and Wit, 2015], but to our knowledge, these have not come with sample complexity bounds.

Misra et al. [Misra et al., 2017] provide an algorithm that succeeds with $m = O(\log \frac{p}{\kappa^2})$ without further assumptions, thus coming fairly close to the lower bound (1). However, this is yet to be done efficiently, as the time complexity of $O(d^3)$ in Misra et al. [2017] (see also [Kelner et al., 2019, Thm. 11]) is prohibitively large unless $d$ is small. Very recently, efficient algorithms were proposed for handling general graphs under the additional assumption of attractivity (i.e., only having non-positive off-diagonal terms in the inverse covariance matrix) [Kelner et al., 2019].

**Learning (generalized) Ising models.** Since our focus is on Gaussian models, we only briefly describe the related literature on Ising models, other than a particular algorithm that we directly build upon.

Early works on Ising models relied on assumptions that prohibit long-range correlations [Anandkumar et al., 2012; Bento and Montanari, 2009; Bresler et al., 2008; Jalali et al., 2011; Ravikumar et al., 2010], and this hurdle was overcome in a series of works pioneered by Bresler et al. [Bresler, 2015; Bresler et al., 2014; Hamilton et al., 2017]. Recent developments have brought the sample complexity upper bounds increasingly close to the information-theoretic lower bounds [Santhanam and Wainwright, 2012], using techniques such as interaction screening [Vuffray et al., 2016], multiplicative weights [Klivans and Meka, 2017], and sparse logistic regression [Wu et al., 2019].

The present paper is particularly motivated by Klivans and Meka [2017], in which an algorithm was developed for learning (generalized) Ising models based on the method of multiplicative weights. More specifically, the algorithm constructs the underlying graph with a nearly optimal sample complexity and a low time complexity by using a weighted majority voting scheme to learn neighborhoods variable-by-variable, and updating the weights using Freund and Schapire’s classic Hedge algorithm [Freund and Schapire, 1997]. The proof of correctness uses the regret bound for the Hedge algorithm, as well as showing that approximating the distribution well according to a certain prediction metric ensures accurately learning the associated weight vector (and hence the neighborhood).

### 1.2 Contributions

In this paper, we adapt the approach of Klivans and Meka [2017] to Gaussian graphical models, and show that the resulting algorithm efficiently learns the graph structure with rigorous bounds on the number of samples required, and a low runtime of $O(mp^2)$ when there are $m$ samples and $p$ nodes. As we highlight throughout the paper, each step of our analysis requires non-trivial modifications compared to Klivans and Meka [2017] to account for the continuous and unbounded nature of the Gaussian distribution.

While we do not claim that our sample complexity bound improves on the state-of-the-art, it exhibits similar assumptions and dependencies to existing works that adopt condition-number-type assumptions (e.g., ACLIME [Cai et al., 2016]; see the discussion following Theorem 9). In addition, as highlighted in Klivans and Meka [2017], the multiplicative weights approach enjoys the property of directly applying in the online setting (i.e., samples arrive one-by-one and must be processed, but not stored, before the next sample).

In Appendix A we discuss the runtimes of a variety of the algorithms mentioned in Section 1.1 highlighting the fact that our $O(mp^2)$ runtime is very attractive.

### 2 Problem Statement

Given a Gaussian random vector $X \in \mathbb{R}^p$ taking values in $\mathbb{R}^p$ with zero mean covariance matrix $\Sigma \in \mathbb{R}^{p \times p}$. Our techniques can also handle the non-zero mean setting, but we find the zero-mean case to more concisely convey all of the relevant concepts.
the entries in $\Theta$, and an upper bound $n_{\max}$; e.g., see the example of Misra et al. 2017.

We will sometimes refer to the maximal degree $d$ of the graph in our discussions, but our analysis and final result will not depend on $d$. Rather, one can think of $\lambda$ as implicitly capturing the dependence on $d$.

For the purpose of simplifying our final expression for the sample complexity, we make some mild assumptions on the scaling laws of the above parameters:

- We assume that $\lambda = \Omega(1)$. This is mild since one can verify that $\lambda = \Omega(\kappa d)$, and the typical regimes considered in existing works are $\kappa d = \Theta(1)$ and $\kappa d \to \infty$ (e.g., see Wang et al. 2010).

- We assume that $\lambda$, $\kappa$, $n_{\max}$, and $\theta_{\max}$ are in between $\frac{1}{\text{poly}(p)}$ and poly$(p)$. This is mild since these may be high-degree polynomials, e.g., $p^{10}$ or $p^{100}$.

### 3 Overview of the Algorithm

To recover the graph structure, we are first interested in estimating the inverse covariance matrix $\Theta \in \mathbb{R}^{p \times p}$ of the multivariate Gaussian distribution.

For a zero-mean Gaussian random vector $X$, we have the following well-known result for any index $i$ (see Lemma 2 below):

$$
\mathbb{E}[X_i | X_i] = \sum_{j \neq i} \frac{-\theta_{ij}}{\theta_{ii}} X_j = w^i \cdot X_i,
$$

where $w^i = \left(\frac{-\theta_{ij}}{\theta_{ii}}\right)_{j \neq i}$, $X_j = \langle X_j \rangle_{j \neq i}$, and $a \cdot b$ denotes the dot product. In the related setting of learning Ising models and generalized linear models, the authors of Klivans and Meka, 2017 used an analogous relation to turn the ‘unsupervised’ problem of learning the inverse covariance matrix to a ‘supervised’ problem of learning weight vectors given samples $(x^t, y^t)$, where the $x^t$ are $n$-dimensional tuples consisting of the values of $X_t$, and $y^t$ are the values of $X_t$. In particular, under the standard Ising model, the relationship analogous to (7) follows a logistic (rather than linear) relation.

In Klivans and Meka, 2017, the Hedge algorithm of Freund and Schapire, 1997 is adapted to the problem of estimating the coefficients of $w^i$. This is achieved for sparse generalized linear models (with bounded Lipschitz transfer functions) by first finding a vector $v$ that approximately minimizes an expected risk quantity with high probability, which we define analogously.

\[\mathbb{R}^{p \times p}, \text{ and inverse covariance matrix } \Theta \in \mathbb{R}^{p \times p}; \Theta = [\theta_{ij}]_{i,j \in [p]} \text{ (where } p = \{1, \ldots, p\} \text{), we are interested in recovering the graph } G = (V, E) \text{ (with } V = [p] \text{) whose adjacency matrix coincides with the support of } \Theta. \text{ That is, we are interested in learning which entries of } \Theta \text{ are non-zero.}

The graph learning is done using $m$ independent samples $(X^1, \ldots, X^m)$ from $\mathcal{N}(0, \Sigma)$. Given these samples, the estimation algorithm forms an estimate $\hat{G}$ of the graph, or equivalently, an estimate $\hat{E}$ of the edge set, and the error probability is given by

$$
\mathbb{P}(\text{error}) = \mathbb{P}(\hat{G} \neq G).
$$

We are interested in characterizing the worst-case error probability over all graphs within some class (described below). Since our approach is based on neighborhood estimation, and each node has $p-1$ candidate neighbors, it will be convenient to let $n = p-1$.

**Definitions and assumptions.** Similarly to existing works such as Misra et al., 2017, Wang et al. 2010, our results depend on the minimum normalized edge strength, defined as

$$
\kappa = \min_{(i,j) \in E} \left| \frac{\theta_{ij}}{\sqrt{\theta_{ii} \theta_{jj}}} \right|.
$$

Intuitively, the sample complexity must depend on $\kappa$ because weaker edges require more samples to detect.

We introduce some assumptions that are similar to those appearing in some existing works. First, for each $i = 1, \ldots, p$, we introduce the quantity

$$
\lambda_i = \sum_{j \neq i} \left| \frac{\theta_{ij}}{\theta_{ii}} \right|, \quad \text{for } i = 1, \ldots, p
$$

and we assume that $\max_{i \in [p]} \lambda_i$ is upper bounded by some known value $\lambda$. As we discuss following our main result (Theorem 9), this is closely related to an assumption made in Cai et al., 2011, 2016, and as discussed in Kelner et al., 2019, the latter can be viewed as a type of condition number assumption, though eigenvalues do not explicitly appear.

In addition, we define an upper bound $\theta_{\max}$ on the absolute values of the entries in $\Theta$, and an upper bound $n_{\max}$ on the variance of any marginal variable:

$$
\theta_{\max} = \max_{i,j} |\theta_{ij}| = \max_i \theta_{i,i}
$$

$$
n_{\max} = \max_i \text{Var}[X_i].
$$

A $(\theta_{\max} n_{\max})^2$ term appears in our final sample complexity bound (Theorem 9). This can again be viewed as a type of condition number assumption, since matrices with a high condition number may have large $\theta_{\max}$; e.g., see the example of Misra et al. 2017.
for our setting. Their algorithm, referred to as Sparsitron, is shown in Algorithm 1. Note that both \((\tilde{x}^t, \tilde{y}^t)\) will represent suitably-normalized samples \((x^t, y^t)\) to be described below, and \((a^t, b^t)\) will represent further samples with the same distribution as \((\tilde{x}^t, \tilde{y}^t)\).

**Definition 1.** The expected risk of a candidate \(v \in \mathbb{R}^n\) for the neighborhood weight vector \(w^i\) of a marginal variable \(X_i\) is

\[
\varepsilon(v) := \mathbb{E}_X \left[ (v \cdot X_i - w^i \cdot X_i)^2 \right].
\]  

The Sparsitron algorithm uses what can be seen as a simple majority weighted voting scheme. For every possible member \(X_j\) of the neighborhood of node \(i\), the algorithm maintains a weight \(v_j\) (which we think of as seeking to approximate the \(j\)-th entry of \(w^i\)), and updates the weight vector via multiplicative updates as in the Hedge algorithm. After \(T\) such consecutive estimates, the algorithm uses an additional \(M\) samples to estimate the expected risk for each of the \(T\) candidates empirically, and then returns the candidate with the smallest empirical risk.

As in Klivans and Meka, 2017, we assume without loss of generality that \(w_i \geq 0\) for all \(i\); for if not, we can map our samples \((x, y)\) to \((x, -x, y)\) and adjust the weight vector accordingly. We can also assume that \(|w|_1\) equals its upper bound \(\lambda\), since otherwise we can introduce a new coefficient and map our samples to \((x, -x, 0, y)\). If the true norm were \(\lambda' < \lambda\), then the modified weight vector would have a value of \(\lambda - \lambda'\) corresponding to the 0 coefficient.

Once the neighborhood weight vectors have been estimated, we recover the graph structure using thresholding, as outlined in Algorithm 2. Here, \(T\) and \(M\) must satisfy certain upper bounds that we derive later (see Theorem 9). The overall sample complexity is \(m = T + M\), and as we discuss following Theorem 9, the runtime is \(O(mp^2)\). This runtime is compared to the runtimes of various existing algorithms for learning Gaussian graphical models in Appendix A.

### 4 Analysis and Sample Complexity

Our analysis proceeds in several steps, given in the following subsections.

---

Data: \(T + M\) normalized samples \(\{(\tilde{x}^t, \tilde{y}^t)\}_{t=1}^T, \{(a^j, b^j)\}_{j=1}^M\); \(\ell_1\)-norm parameter \(\lambda\); update parameter \(\beta\) (default value \(\frac{1}{1+\sqrt{T}}\))

Result: Estimate of weight vector in \(\mathbb{R}^n\)

Initialize \(v^0 = 1/n\)

for \(t = 1, \ldots, T\) do

- Let \(p^t = \frac{v^{t-1}}{\|v^{t-1}\|_1}\)
- Define \(t^t \in \mathbb{R}^n\) by
  \[ t^t = (1/2) (1 + (\lambda p^t \cdot \tilde{x}^t - \tilde{y}^t) \tilde{z}^t) \]
- Update the weight vectors: For each \(i \in [n]\), set \(v_i^t = v_i^{t-1} \cdot \beta^{t^t}
end

for \(t = 1, \ldots, M\) do

- Compute the expected risk for each \(t\):
  \[
  \hat{\varepsilon}(\lambda p^t) = \frac{\sum_{j=1}^M (\lambda p^t \cdot a^j - b^j)^2}{M}
  \]
end

return \(\lambda p^{t^*}\) for \(t^* = \arg \min_{t \in [T]} \hat{\varepsilon}(\lambda p^t)\)

Algorithm 1: Sparsitron algorithm for estimating a weight vector \(w \in \mathbb{R}^n\). It is assumed here that the true weight vector has only positive weights and \(\ell_1\)-norm exactly \(\lambda\) (see Footnote 2).

---

Data: \(T + M\) samples, tuple \((\nu_{\max}, \theta_{\max}, \lambda, \kappa)\), target error probability \(\delta\)

Result: Estimate of the graph

for \(i = 1, \ldots, p\) do

- Normalize the \(T\) samples as \((\tilde{x}^t, \tilde{y}^t) := \frac{1}{B \sqrt{\nu_{\max} (\lambda + 1)}} (x^t, y^t)\) with \(B = \sqrt{2 \log \frac{2p}{\delta}}\) and similarly normalize
  the final \(M\) samples to obtain \(\{(a^j, b^j)\}_{j=1}^M\)
- Run Sparsitron on the normalized samples to obtain an estimate \(v^i\) of the weight vector \(w^i = (\frac{\lambda a^j}{\nu_{\max}})_{j \neq i}\) of node \(i\)
end

For every pair \(i\) and \(j\), identify an edge between them if \(max\{|v^i_j|, |v^j_i|\} \geq 2\kappa / 3\)

Algorithm 2: Overview of the algorithm for Gaussian graphical model selection.

---

This step is omitted in Algorithm 1 so that we can lighten notation and work with vectors in \(\mathbb{R}^n\) rather than \(\mathbb{R}^{2n+1}\). Formally, it can be inserted as an initial step, and then the resulting length \(2n + 1\) weight vector can be mapped back to a length-\(n\) weight vector by taking the first \(n\) entries and subtracting the second \(n\) entries, while ignoring the final entry. The initial part of our analysis considering Sparsitron can be viewed as corresponding to the case where the weights are already positive and the \(\ell_1\)-norm bound \(\lambda\) already holds with equality, but it goes through essentially unchanged in the general case.
4.1 Preliminary Results

4.1.1 Properties of Multivariate Gaussians

We first recall some results regarding multivariate Gaussian random variables that we will need throughout the analysis.

**Lemma 2.** Given a zero-mean multivariate Gaussian $X = (X_1, \ldots, X_p)$ with inverse covariance matrix $\Theta = [\theta_{ij}]$, and given $T$ independent samples $(X^1, \ldots, X^T)$ with the same distribution as $X$, we have the following:

1. For any $i \in [p]$, we have $X_i = \eta_i + \sum_{j \neq i} (-\frac{\theta_{ij}}{\eta_i})X_j$, where $\eta_i$ is a Gaussian random variable with variance $\frac{1}{\eta_i}$, independent of all $X_j$ for $j \neq i$.

2. $E[X_i|X_i] = \sum_{j \neq i} (-\frac{\theta_{ij}}{\eta_i})X_j = w^i \cdot X_i$, where $w^i = (-\frac{\theta_{ij}}{\eta_i})_{j \neq i} \in \mathbb{R}^n$ (with $n = p - 1$).

3. Let $\lambda$ and $\nu_{\text{max}}$ be defined as in (4) and (5), set $B := \sqrt{2 \log \frac{2\nu_{\text{max}}}{\delta}}$, and define $(\tilde{x}^i, \tilde{y}^i) := \frac{1}{B\sqrt{\nu_{\text{max}}} + 1}(x^i, y^i)$, where $(x^i, y^i) = (X^i_1, X^i_2)$ for an arbitrary fixed coordinate $i$. Then, with probability at least $1 - \delta$, $\tilde{y}^i$ and all entries of $\tilde{x}^i$ ($t = 1, \ldots, T$) have absolute value at most $\frac{1}{\sqrt{\lambda + 1}}$.

**Proof.** These properties are all standard and/or use standard arguments; see Appendix B for details.

4.1.2 Loss Guarantee for Sparsitron

Recall that $n = p - 1$. In the proof of [Klivans and Meka, 2017, Theorem 3.1], it is observed that the Hedge regret guarantee implies the following.

**Lemma 3.** (Klivans and Meka, 2017) For any sequence of loss vectors $l^t \in [0,1]^n$ for $t = 1, \ldots, T$, the Sparsitron algorithm guarantees that

$$\sum_{t=1}^{T} p^t \cdot l^t \leq \min_{i \in [n]} \sum_{t=1}^{T} l^t_i + O(\sqrt{T \log n + \log n}). \quad (10)$$

To run the Sparsitron algorithm, we need to define an appropriate sequence of loss vectors in $[0,1]^n$. Let

$$l^t = (1/2)(1 + (\lambda p^t \cdot \tilde{x}^t - \tilde{y}^t)\tilde{x}^t), \quad (11)$$

where $1$ is the vector of ones, and $\lambda p^t$ is Sparsitron’s estimate at the beginning of the $t$-th iteration, formed using samples $1, \ldots, t-1$. To account for the fact that the Hedge algorithm requires bounded losses for its regret guarantee, we use the high probability scaling in the third part of Lemma 2. Since $p^t \in [0,1]^n$ and $\sum_{t=1}^{T} p_t = 1$, we have that $|\lambda p^t \cdot \tilde{x}^t - \tilde{y}^t| < \sqrt{\lambda + 1}$, and that consequently $(\lambda p^t \cdot \tilde{x}^t - \tilde{y}^t)\tilde{x}^t \in [-1,1]^n$. It then follows that $l^t$, as defined in (11), lies in $[0,1]^n$. Hence, Lemma 3 applies with probability at least $1 - \delta$ when we use $(\tilde{x}^t, \tilde{y}^t) := \frac{1}{B\sqrt{\nu_{\text{max}}} + 1}(x^t, y^t)$.

4.1.3 Concentration Bound for Martingales

Unlike the analysis of Ising (and related) models in [Klivans and Meka, 2017], here we do not have the liberty of assuming bounded losses. In the previous subsection, we circumvented this issue by noting that the losses are bounded with high probability, and such an approach is sufficient for that step due to the fact that the Hedge regret guarantee applies for arbitrary (possibly adversarially chosen) bounded losses. However, while such a “truncation” approach was sufficient above, it will be insufficient (or at least challenging to make use of) in later parts of the analysis that rely on the Gaussianity of the samples.

In this subsection, we present a concentration bound that helps to overcome this difficulty, and serves as a replacement for the Azuma-Hoeffding martingale concentration bound used in [Klivans and Meka, 2017].

Specifically, we use [van de Geer, 1993, Lemma 2.2], which states that given a martingale $M_t$, if we can establish Bernstein-like inequalities on the ‘sums of drifts’ of certain higher order processes, then we can establish a concentration bound on the main process. Here we state a simplified version for discrete-time martingales that suffices for our purposes (in [van de Geer, 1993, continuous-time martingales are also permitted). This reduction from [van de Geer, 1993, Lemma 2.2] is outlined in Appendix C.

**Lemma 4.** (van de Geer, 1993) Let $M_t$ be a discrete-time martingale with respect to a filtration $\mathcal{F}_t$ such that $E[M^2_t] < \infty$ for all $t$, and define $\Delta M_t = M_t - M_{t-1}$ and $V_{m,t} = \sum_{j=1}^{m} E[|\Delta M_j|^m | \mathcal{F}_{j-1}]$. Suppose that for all $t$ and some $0 < K < \infty$, we have

$$V_{m,t} \leq \frac{m!}{2} K^{m-2} R_t, \quad m = 2, 3, \ldots \quad (12)$$

for some process $R_t$ that is measurable with respect to $\mathcal{F}_{t-1}$. Then, for any $a, b > 0$, we have

$$\mathbb{P}(M_t \geq a \text{ and } R_t \leq b^2 \text { for some } t) \leq \exp \left( -\frac{a^2}{2abK + b^2} \right). \quad (13)$$
4.2 Bounding the Expected Risk

For compactness, we subsequently write \( w \) as a shorthand for the weight vector \( w^t \in \mathbb{R}^n \) of the node \( i \) whose neighborhood is being estimated. We recall the choice of \( t^j \) in \((11)\), and make use of the following definitions from [Klivans and Meek 2017]:

\[
Q^t := (p^t - w/\lambda) \cdot t^t \\
Z^t := Q^t - \mathbb{E}_{t-1}[Q^t],
\]

where here and subsequently, we use the notation \( \mathbb{E}_{t}[\cdot] := \mathbb{E}[\cdot | (x^1, y^1), \ldots, (x^t, y^t)] \) to denote conditioning on the samples up to time \( t \). The analysis proceeds by showing that \( \sum_{j=1}^T Z^j \) is concentrated around zero, upper bounding the expected risk in terms of \( \mathbb{E}_{t-1}[Q^t] \), and applying Sparsitron’s guarantee from Lemma 3.

We first use Lemma 4 to obtain the following result.

**Lemma 5.** \( | \sum_{j=1}^T Z^j | = O \left( \sqrt{T \log \frac{1}{\delta}} \right) \) with probability at least \( 1 - \delta \).

**Proof.** The proof essentially just requires substitutions in Lemma 4. The martingale process is \( M_t = \sum_{j \leq t} Z^j \), and we obtain \( \Delta M_t = Z^t \), along with

\[
V_{m,t} = \sum_{j=1}^t \mathbb{E}_{j-1}[|Z^j|^m].
\]

The rest of the proof entails unpacking the definitions and using standard properties of Gaussian random variables to show that the Bernstein-like requirements are satisfied for the concentration bound in Lemma 4. The details are provided in Appendix D.

**Lemma 6.** If Sparsitron is run with \( T \geq \log n \), then

\[
\min_{t \in [T]} \varepsilon(\lambda p^t) = O \left( \frac{\lambda(\lambda + 1)\nu_{\max} \log \frac{n}{\delta}}{T} \sqrt{T \log \frac{1}{\delta}} \right)
\]

with probability at least \( 1 - \delta \).

**Proof.** From the definition of \( Q^t \) in \((14)\), we have that

\[
\mathbb{E}_{t-1}[Q^t]
= \mathbb{E}_{t-1}[(p^t - (1/\lambda)w) \cdot t^t]
= \mathbb{E}_{t-1}[(p^t - (1/\lambda)w) \cdot (1/2)(1 + (\lambda p^t \cdot \bar{x}^t - \bar{y}^t)\bar{x}^t)]
\]

where \((19)\) uses the definition of \( t^j \) in \((11)\), \((21)\) uses \( \|w\| = \lambda \) (see Footnote 2), \((23)\) follows by noting that \( p^t \) is a function of \( \{x^i, y^i\}_{i=1}^{t-1} \) and computing the expectation over \( y^t \) first (using the second part of Lemma 3), and \((24)\) uses the definition of expected risk in \((8)\), along with \( \bar{x}^t = \frac{1}{\sqrt{\nu_{\max}(\lambda + 1)n}}x^t \).

Summing both sides above over \( t = 1, \ldots, T \), we have the following with probability \( 1 - O(\delta) \).

\[
\frac{1}{2\lambda(\lambda + 1)\nu_{\max} B^2} \sum_{t=1}^T \varepsilon(\lambda p^t)
\]

where \((19)\) uses the definition of \( t^j \) in \((11)\), \((21)\) uses \( \|w\| = \lambda \) (see Footnote 2), \((23)\) follows by noting that \( p^t \) is a function of \( \{x^i, y^i\}_{i=1}^{t-1} \) and computing the expectation over \( y^t \) first (using the second part of Lemma 3), and \((24)\) uses the definition of expected risk in \((8)\), along with \( \bar{x}^t = \frac{1}{\sqrt{\nu_{\max}(\lambda + 1)n}}x^t \).

In the analysis, we apply multiple results that each hold with probability at least \( 1 - \delta \). More precisely, \( \delta \) should be replaced by \( \delta/L \) when applying a union bound over \( L \) events, but since \( L \) is finite, this only amounts to a change in the constant of the \( O(\cdot) \) notation in \((17)\).
= O\left(\sqrt{T \log n + \log n + \sqrt{T \log \frac{1}{\delta}}}\right), \quad (30)

and substituting \( B = \sqrt{2 \log \frac{2(n+1)T}{\delta}} \) gives

\[
\min_{t \in [T]} \varepsilon(\lambda p_t^t) = O\left(\frac{\lambda(\lambda + 1)v_{\max} \log \frac{2T}{\delta}}{T}\right) \times \left(\sqrt{T \log n + \log n + \sqrt{T \log \frac{1}{\delta}}}\right), \quad (31)
\]

where we also lower bounded \( \sum_{j=1}^{T} \varepsilon(\lambda p_t^t) \) by \( T \) times the minimum value. When \( T \geq \log n \), the above bound simplifies to \( (17) \), as desired. \hfill \Box

Having ensured that that the minimal expected risk is small, we need the algorithm to identify a candidate whose expected risk is also sufficiently close to that minimum. Spursitron does this by using an additional \( M \) samples to estimate the expected risk empirically.

**Lemma 7.** For \( \gamma > 0 \), \( \rho \in (0, 1] \), and fixed \( v \in \mathbb{R}^n \) satisfying \( \|v\|_1 \leq \lambda \), there is some \( M = O\left((\lambda + 1)\frac{\log(1/\rho)}{\gamma}\right) \) such that

\[
P\left(\frac{1}{M} \sum_{j=1}^{M} \left((v \cdot a^j - b^j)^2 - \Xi\right) - \varepsilon(v) \geq \gamma \right) \leq \rho, \quad (32)
\]

where \( \{(a^j, b^j)\}_{j=1}^{M} \) are the normalized samples defined in Algorithm 3 and \( \Xi = \mathbb{E}[\text{Var}(b^j | a^j)] \).

**Proof.** The high-level steps of the proof are to first establish the equality

\[
\mathbb{E}[(v \cdot a^j - b^j)^2] = \varepsilon(v) + \Xi, \quad (33)
\]

and then use Bernstein’s inequality to bound the deviation of \( \sum_{j=1}^{M} \left((v \cdot a^j - b^j)^2 - \Xi\right) \) from its mean value \( \varepsilon(v) \). The details are given in Appendix F. \hfill \Box

### 4.3 Graph Recovery and Sample Complexity

We complete the analysis of our algorithm in a sequence of three steps, given as follows.

**An \( \ell_\infty \) bound.** We show that if our estimate \( v \) approximates the true weight vector \( w \in \mathbb{R}^n \) well in terms of the expected risk, then it also approximates it in the \( \ell_\infty \) norm. In [Klivans and Meka, 2017], this was done using a property termed the ‘\( \delta \)-unbiased condition’, whose definition relies on the underlying random variables being binary. Hence, we require a different approach, given as follows.\(^4\)

\[4\]This quantity is the same for all values of \( j \).

**Lemma 8.** Under the preceding setup, if we have \( \varepsilon(v) \leq \epsilon \), then we also have \( \|v - w\|_\infty \leq \sqrt{\theta_{\max}} \), where \( \theta_{\max} \) is a uniform upper bound on the diagonal entries of \( \Theta \).

**Proof.** The proof uses a direct calculation to establish that \( \text{Var}(v \cdot w \cdot X_i) \geq |v_i - w_i|^2 \text{Var}(\eta_i) \) for a fixed index \( i^* \); the details are given in Appendix F. \hfill \Box

Suppose that we would like to recover the true weight vector with a maximum deviation of \( \epsilon^\prime \) in any coordinate with probability at least \( 1 - \delta \). By Lemma 8 we require \( \epsilon \) to be no more than \( (\epsilon^\prime)^2 / \theta_{\max} \). We know from Lemma 6 that

\[
\min_{t \in [T]} \varepsilon(\lambda p_t^t) = O\left(\frac{\lambda(\lambda + 1)v_{\max} \log \frac{2T}{\delta}}{T}\right), \quad (34)
\]

from which we have that with \( T = O\left(\lambda^2(\lambda + 1)v_{\max}^2 \log^3 \frac{n}{\delta}\right) \), the minimum expected risk is less than \( \epsilon/2 \) with probability at least \( 1 - \delta/2 \).

From Lemma 7 with \( \rho = \delta/(2T) \) and \( \gamma = \epsilon/2 \), we observe that we can choose \( M \) satisfying

\[
M \leq O\left(\frac{(\lambda + 1)\log(T/\delta)}{\epsilon}\right) \leq O\left(\frac{\lambda(\lambda + 1)v_{\max} \log^{3/2} \frac{n}{\delta}}{\epsilon^\prime}\right), \quad (35)
\]

and estimate \( \varepsilon(\lambda p_t^t) + \Xi \) (note that the second term doesn’t affect the arg-min) of the \( T \) candidates \( \lambda p_t^t \) within \( \epsilon/4 \) with probability at least \( 1 - \frac{\delta}{\sqrt{T}} \). By the union bound (which blows the \( \frac{\delta}{\sqrt{T}} \) up to \( \frac{\delta}{2T} \)), the same follows for all \( T \) candidates simultaneously. We then have that the candidate with the lowest estimate has expected risk within \( \epsilon/2 \) of the candidate with the lowest expected risk, and that the latter candidate’s expected risk is less than \( \epsilon/2 \), so in sum the vector returned by the candidate has an expected risk less than \( \epsilon \) with probability at least \( 1 - \delta \). Moreover, the sample complexity is

\[
T + M = O\left(\frac{\lambda^2(\lambda + 1)^2v_{\max} \log^{3} \frac{n}{\delta}}{\epsilon^2}\right) + O\left(\frac{(\lambda + 1)\log(\lambda(\lambda + 1)v_{\max} \log^{3/2} \frac{n}{\delta})}{\epsilon^\prime}\right) \leq O\left(\frac{\lambda^4v_{\max}^2 \log^{3} \frac{n}{\delta}}{\epsilon^2}\right), \quad (37)
\]

\[6\]The removal of \( T \) in the logarithm \( \log^{3/2} \) can be justified by the assumption that all parameters are polynomially bounded with respect to \( p \) (see Section 2).

\[7\]See also [Keller et al., 2013, Thm. 17] for similar considerations under a different set of assumptions.
Learning Gaussian Graphical Models via Multiplicative Weights

where the simplification comes by recalling from Section 2 that \( \lambda = \Omega(1) \) and all parameters are polynomially bounded with respect to \( n \). While the sample complexity (38) corresponds to probability at least \( 1 - \delta \) for the algorithm of only a single \( i \in [p] \), we can replace \( \delta \) by \( \delta/p \) and apply a union bound to conclude the same for all \( i \in [p] \); since \( p = n + 1 \), this only amounts to a chance in the constant of the \( O(\cdot) \) notation.

**Recovering the graph.** Recall from Lemma 8 that an expected risk of at most \( \epsilon \) translates to a coordinate-wise deviation of at most \( \epsilon' = \sqrt{\epsilon \theta_{\max}} \). We set \( \epsilon = \frac{\kappa^2}{\theta_{\max}} \), so that \( \epsilon' = \frac{\kappa}{3} \).

We observe that if \( X_i \) and \( X_j \) are neighbors, then (39) yields the following lower bound:

\[
\frac{\theta_{ij}^2}{\theta_{ii}\theta_{jj}} \geq \kappa^2
\]  

This ensures that at least one of the two values \(|\theta_{ij}/\theta_{ii}|\) and \(|\theta_{ij}/\theta_{jj}|\) must be greater than or equal to \( \kappa \). On the other hand, if they are not neighbors, then the true value of both of these terms must be 0. Since we have estimated all weights to within \( 2\kappa/3 \), it follows that any estimate of at least \( 2\kappa/3 \) must arise from a true neighborhood relation (with high probability). Conversely, if there is a neighborhood relation, then at least one of the two factors \( \theta_{ij}/\theta_{ii} \) and \( \theta_{ij}/\theta_{jj} \) must have been found to be at least \( 2\kappa/3 \).

The method for recovering the graph structure is then as follows: For each possible edge, the weight estimates \( v_i^j \) and \( v_j^i \) are calculated; if either of them is found to be greater than \( 2\kappa/3 \), then the edge is declared to lie in the graph, and otherwise it is not.

Substituting \( \epsilon = \frac{\kappa^2}{\theta_{\max}} \) into (39), and recalling our notation \( n = p - 1 \), we deduce the final sample complexity, stated as follows.

**Theorem 9.** For learning graphs on \( p \) nodes with minimum normalized edge strength \( \kappa \), under the additional assumptions stated in Section 4 with parameters \( (\lambda, \nu_{\max}, \theta_{\max}) \), the algorithm described above attains \( \mathbb{P}(\text{error}) \leq \delta \) with a sample complexity of at most

\[
m = O\left( \frac{\lambda^4 \nu_{\max}^2 \theta_{\max}^2 \log \frac{p}{\delta}}{\kappa^3} \right). 
\]

We can compare this guarantee with those of existing algorithms: As discussed in [Kivel et al., 2019, Remark 8], the \( \ell_1 \)-based ACLIME algorithm [Cai et al., 2016] can be used for graph recovery with \( m = O\left( \frac{\lambda^3}{\nu_{\max}} \right) \) samples, where \( \lambda \) is an upper bound on the \( \ell_1 \) norm of any row of \( \Theta \). An algorithm termed HybridMB in [Kivel et al., 2019] achieves the same guarantee, and a greedy pruning method in the same paper attains a weaker \( m = O\left( \frac{\lambda^3}{\nu_{\max}} \right) \) bound.

The quantities \( \lambda \) and \( \lambda \) are closely related; for instance, in the case that \( \theta_{ii} = 1 \) for all \( i \), we have \( \lambda = 1 + \lambda \). More generally, if \( \nu_{\max} \) and \( \theta_{\max} \) behave as \( \Theta(1) \), then our bound can be written as \( O\left( \frac{\lambda}{\nu_{\max} \theta_{\max}} \right) \), which is qualitatively similar to the bounds of [Cai et al., 2016, Kelner et al., 2019] but with an extra \( \left( \lambda \log \frac{p}{\delta} \right)^2 \) term.

We again highlight that our main goal is not to attain a state-of-the-art sample complexity, but rather to introduce a new algorithmic approach to Gaussian graphical model selection. The advantages of this approach, as highlighted in [Kivel and Meka, 2017], are low runtime and direct applicability to the online setting. In addition, as we discuss in the following section, we expect that there are parts of our analysis that could be refined to bring the sample complexity down further.

**Runtime.** The algorithm enjoys a low runtime similar to the case of Ising models [Kivel and Meka, 2017]: Sparsitron performs \( m = T + M \) iterations that each require time \( O(n) = O(p) \), for an overall runtime of \( O(mp) \). Since this is done separately for each \( i = 1, \ldots, p \), the overall runtime is \( O(mp^2) \).

5 Conclusion

We have introduced a novel adaptation of the multiplicative weights approach to graphical model selection [Kivel and Meka, 2017] to the Gaussian setting, and established a resulting sample complexity bound under suitable assumptions on the covariance matrix and its inverse. The algorithm enjoys a low runtime compared to existing methods, and can directly be applied in the online setting.

The most immediate direction for further work is to seek refinements of our algorithm and analysis that can further reduce the sample complexity and/or weaken the assumptions made. For instance, we normalized the samples to ensure a loss function in \([0,1]\) with high probability, and this is potentially more crude then necessary (and ultimately yields the \( \log^3 p \) dependence). One may therefore consider using an alternative to Hedge that is more suited to unbounded rewards. In addition, various steps in our analysis introduced \( \theta_{\max} \) and \( \nu_{\max} \), and the individual estimation of diagonals of \( \Sigma \) and/or \( \Theta \) (e.g., as done in [Cai et al., 2016]) may help to avoid this.
Acknowledgment

This work was supported by the Singapore National Research Foundation (NRF) under grant number R-252-000-A74-281.

Bibliography

A. Anandkumar, V. Y. F. Tan, F. Huang, and A. S. Willsky. High-dimensional Gaussian graphical model selection: Walk summability and local separation criterion. *J. Mach. Learn. Res.*, 13:2293–2337, 2012.

J. Bento and A. Montanari. Which graphical models are difficult to learn? In *Conf. Neur. Inf. Proc. Sys. (NeurIPS)*. 2009.

S. Boucheron, G. Lugosi, and P. Massart. *Concentration inequalities: A nonasymptotic theory of independence*. Oxford university press, 2013.

G. Bresler. Efficiently learning Ising models on arbitrary graphs. In *ACM Symp. Theory Comp. (STOC)*, 2015.

G. Bresler, E. Mossel, and A. Sly. Reconstruction of Markov random fields from samples: Some observations and algorithms. In *Appr., Rand. and Comb. Opt. Algorithms and Techniques*, pages 343–356. Springer Berlin Heidelberg, 2008.

G. Bresler, D. Gamarnik, and D. Shah. Structure learning of antiferromagnetic Ising models. In *Conf. Neur. Inf. Proc. Sys. (NeurIPS)*. 2014.

T. Cai, W. Liu, and X. Luo. A constrained $\ell_1$ minimization approach to sparse precision matrix estimation. *J. Amer. Stat. Assoc.*, 106(494):594–607, 2011.

T. T. Cai, W. Liu, and H. H. Zhou. Estimating sparse precision matrix: Optimal rates of convergence and adaptive estimation. *Ann. Stats.*, 44(2):455–488, 04 2016.

A. d’Aspremont, O. Banerjee, and L. El Ghaoui. First-order methods for sparse covariance selection. *SIAM J. Matrix Analysis and Apps.*, 30(1):56–66, 2008.

R. Durbin, S. R. Eddy, A. Krogh, and G. Mitchison. *Biological sequence analysis: Probabilistic models of proteins and nucleic acids*. Cambridge Univ. Press, 1998.

Y. Freund and R. E. Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. *J. Comp. Sys. Sci.*, 55(1):119–139, 1997.

S. Geman and D. Geman. Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. *IEEE Trans. Patt. Analysis and Mach. Intell.*, (6):721–741, 1984.

R. J. Glauber. Time-dependent statistics of the Ising model. *J. Math. Phys.*, 4(2):294–307, 1963.

L. Hamilton, F. Koehler, and A. Moitra. Information theoretic properties of Markov random fields, and their algorithmic applications. In *Conf. Neur. Inf. Proc. Sys. (NeurIPS)*, pages 2463–2472, 2017.

C.-J. Hsieh, M. A. Sustik, I. S. Dhillon, P. K. Ravikumar, and R. Poldrack. BIG & QUIC: Sparse inverse covariance estimation for a million variables. In *Conf. Neur. Inf. Proc. Sys. (NeurIPS)*, pages 3165–3173, 2013.

A. Jalali, C. C. Johnson, and P. K. Ravikumar. On learning discrete graphical models using greedy methods. In *Conf. Neur. Inf. Proc. Sys. (NeurIPS)*, 2011.

M. Kalisch and P. Bühlmann. Estimating high-dimensional directed acyclic graphs with the PC-algorithm. *J. Mach. Learn. Research*, 8(Mar):613–636, 2007.

J. Kelner, F. Koehler, R. Meka, and A. Moitra. Learning some popular Gaussian graphical models without condition number bounds. https://arxiv.org/abs/1905.01282, 2019.

A. Klivans and R. Meka. Learning graphical models using multiplicative weights. In *Symp. Found. Comp. Sci. (FOCS)*, pages 343–354. IEEE, 2017.

J. Leppä-Aho, J. Pensar, T. Roos, and J. Corander. Learning Gaussian graphical models with fractional marginal pseudo-likelihood. *Int. J. Approximate Reasoning*, 83:21–42, 2017.

R. Liptser and A. N. Shiryaev. *Theory of martingales*, volume 49. Kluwer, Dordrecht, 1989.

C. D. Manning and H. Schütze. *Foundations of statistical natural language processing*. MIT press, 1999.

N. Meinshausen, P. Bühlmann, et al. High-dimensional graphs and variable selection with the lasso. *Ann. Stats.*, 34(3):1436–1462, 2006.

S. Misra, M. Vuffray, and A. Y. Lokhov. Information theoretic optimal learning of Gaussian graphical models. https://arxiv.org/abs/1703.04886, 2017.

A. Mohammadi and E. C. Wit. Bayesian structure learning in sparse Gaussian graphical models. *Bayesian Analysis*, 10(1):109–138, 2015.

P. Ravikumar, M. J. Wainwright, J. D. Lafferty, and B. Yu. High-dimensional Ising model selection using $\ell_1$-regularized logistic regression. *Ann. Stats.*, 38(3):1287–1319, 2010.

P. Ravikumar, M. J. Wainwright, G. Raskutti, and B. Yu. High-dimensional covariance estimation
Learning Gaussian Graphical Models via Multiplicative Weights

by minimizing $\ell_1$-penalized log-determinant divergence. *Elec. J. Stats.*, 5:935–980, 2011.

N. Santhanam and M. Wainwright. Information-theoretic limits of selecting binary graphical models in high dimensions. *IEEE Trans. Inf. Theory*, 58(7): 4117–4134, July 2012.

P. Spirtes, C. N. Glymour, R. Scheines, D. Heckerman, C. Meek, G. Cooper, and T. Richardson. *Causation, prediction, and search*. MIT press, 2000.

S. van de Geer. Exponential inequalities for martingales, with application to maximum likelihood estimation for counting processes. *Ann. Stats.*, pages 1779–1801, 1995.

S. van de Geer, P. Bühlmann, et al. $\ell_0$-penalized maximum likelihood for sparse directed acyclic graphs. *Ann. Stats.*, 41(2):536–567, 2013.

M. Vuffray, S. Misra, A. Lokhov, and M. Chertkov. Interaction screening: Efficient and sample-optimal learning of Ising models. In *Conf. Neur. Inf. Proc. Sys. (NeurIPS)*, pages 2595–2603, 2016.

M. J. Wainwright and M. I. Jordan. Graphical models, exponential families, and variational inference. *Found. Trend. Mach. Learn.*, 2008.

L. Wang, X. Ren, and Q. Gu. Precision matrix estimation in high dimensional Gaussian graphical models with faster rates. In *Int. Conf. Art. Intel. Stats. (AISTATS)*, 2016.

W. Wang, M. Wainwright, and K. Ramchandran. Information-theoretic bounds on model selection for Gaussian Markov random fields. In *IEEE Int. Symp. Inf. Theory (ISIT)*, 2010.

S. Wasserman and K. Faust. *Social network analysis: Methods and applications*, volume 8. Cambridge Univ. Press, 1994.

S. Wu, S. Sanghavi, and A. G. Dimakis. Sparse logistic regression learns all discrete pairwise graphical models. In *Conf. Neur. Inf. Proc. Sys. (NeurIPS)*, 2019.

M. Yuan and Y. Lin. Model selection and estimation in the Gaussian graphical model. *Biometrika*, 94(1): 19–35, 2007.

S. Zhou, P. Rütimann, M. Xu, and P. Bühlmann. High-dimensional covariance estimation based on Gaussian graphical models. *J. Mach. Learn. Res.*, 12:2975–3026, Nov. 2011.
A Comparison of Runtimes

Recall that $p$ denotes the number of nodes, $d$ denotes the maximal degree, $\kappa$ denotes the minimum normalized edge strength, and $m$ denotes the number of samples. The runtimes of some existing algorithms in the literature for Gaussian graphical model selection (see Section 1.1 for an overview) are outlined as follows:

- The only algorithms with assumption-free sample complexity bounds depending only on $(p, d, \kappa)$ have a high runtime of $O(p^{O(d)})$, namely, $O(p^2d + 1)$ in [Misra et al., 2017], and $O(p^d + 1)$ in [Kelner et al., 2019, Thm. 11].

- A greedy method in [Kelner et al., 2019, Thm. 7] has runtime $O((d \log \frac{1}{\kappa})^3 mp^2)$. The sample complexity for this algorithm is $O(d \kappa^2 \cdot \log \frac{1}{\kappa} \cdot \log n)$, but this result is restricted to attractive graphical models.

- To our knowledge, $\ell_1$-based methods [Cai et al., 2011, 2016, d’Aspremont et al., 2008, Meinshausen et al., 2006, Ravikumar et al., 2011, Wang et al., 2016, Yuan and Lin, 2007] such as Graphical Lasso and CLIME do not have precise time complexities stated, perhaps because this depends strongly on the optimization algorithm used. We expect that a general-purpose solver would incur $O(p^3)$ time, and we note that [Kelner et al., 2019, Table 2] indeed suggests that these approaches are slower.

- In practice, we expect BigQUIC [Hsieh et al., 2013] to be one of the most competitive algorithms in terms of runtime, but no sample complexity bounds were given for this algorithm.

- Under the local separation condition and a walk summability assumption, the algorithm of [Anandkumar et al., 2012] yields a runtime of $O(p^2 + \eta)$, where $\eta > 0$ is an integer specifying the local separation condition.

Hence, we see that our runtime of $O(mp^2)$ is competitive among the existing works – it is faster than other algorithms for which sample complexity bounds have been established.

B Proof of Lemma 2 (Properties of Multivariate Gaussians)

We restate the lemma for ease of reference.

**Lemma 2.** Given a zero-mean multivariate Gaussian $X = (X_1, \ldots, X_p)$ with inverse covariance matrix $\Theta = [\theta_{ij}]$, and given $T$ independent samples $(X_1^1, \ldots, X_T^1)$ with the same distribution as $X$, we have the following:

1. For any $i \in [p]$, we have $X_i = \eta_i + \sum_{j \neq i} \left( -\frac{\theta_{ij}}{\theta_{ii}} \right) X_j$, where $\eta_i$ is a Gaussian random variable with variance $\frac{1}{\theta_{ii}}$, independent of all $X_j$ for $j \neq i$.

2. $E[X_i|X_i] = \sum_{j \neq i} \left( -\frac{\theta_{ij}}{\theta_{ii}} \right) X_j = w^i \cdot X_i$, where $w^i = \left( -\frac{\theta_{ij}}{\theta_{ii}} \right)_{j \neq i} \in \mathbb{R}^n$ (with $n = p - 1$).

3. Let $\lambda$ and $\nu_{\max}$ be defined as in (4) and (5), set $B := \sqrt{2 \log \frac{2eT}{\delta}}$, and define $(\tilde{x}^t, \tilde{y}^t) := \frac{1}{B \sqrt{\nu_{\max}(\lambda + 1)}} (x^t, y^t)$, where $(x^t, y^t) = (X_i^t, X_i^t)$ for an arbitrary fixed coordinate $i$. Then, with probability at least $1 - \delta$, $\tilde{y}^t$ and all entries of $\tilde{x}^t$ ($t = 1, \ldots, T$) have absolute value at most $\frac{1}{\sqrt{\lambda + 1}}$. 

Proof. The first claim is standard in the literature (e.g., see Zhou et al. 2011, Eq. (4)), and the second claim follows directly from the first.

For the third claim, let $N$ be a Gaussian random variable with mean 0 and variance 1. We make use of the standard (Chernoff) tail bound
\[
P(|N| > x) \leq 2e^{-x^2/2}.
\] (41)

By scaling the standard Gaussian distribution, recalling the definition of $\nu_{\text{max}}$ in (6), and using $B = \sqrt{2 \log \frac{2pT}{\delta}}$, it follows that
\[
P(|x^t_i| > \sqrt{\nu_{\text{max}}B}) \leq P\left(|N| > \sqrt{2 \log \frac{2pT}{\delta}}\right)
\] (42)
\[
\leq 2 \exp\left(- \log \frac{2pT}{\delta}\right)
\] (43)
\[
\leq \frac{\delta}{pT},
\] (44)

and hence
\[
P\left(|x^t_i| > \frac{1}{\sqrt{\lambda + 1}}\right) \leq \frac{\delta}{pT}.
\] (45)

The same high probability bound holds similarly for $\tilde{y}^t$. By taking the union bound over these $p$ events, and also over $t = 1, \ldots, T$, we obtain the desired result. $\square$

### C Establishing Lemma 4 (Martingale Concentration Bound)

Here we provide additional details on attaining Lemma 4 from a more general result in van de Geer, 1995. While the latter concerns continuous-time martingales, we first state some standard definitions for discrete-time martingales. Throughout the appendix, we distinguish between discrete time and continuous time by using notation such as $M_t, F_t$ for the former, and $\tilde{M}_t, \tilde{F}_t$ for the latter.

**Definition 10.** Given a discrete-time martingale $\{M_t\}_{t=0,1,\ldots}$ with respect to a filtration $\{F_t\}_{t=0,1,\ldots}$, we define the following:

1. The **compensator** of $\{M_t\}$ is defined to be
   \[
   V_t = \sum_{j=1}^{t} \mathbb{E}[M_j - M_{j-1} | F_{j-1}].
   \] (46)

2. A **discrete-time process** $\{W_t\}_{t=1,2,\ldots}$ defined on the same probability space as $\{M_t\}$ is said to be **predictable** if $W_t$ is measurable with respect to $F_{t-1}$.

3. We say that $\{M_t\}$ is locally square integrable if there exists a sequence of stopping times $\{\tau_k\}_{k=1}^{\infty}$ with $\tau_k \to \infty$ such that $\mathbb{E}[M_{\tau_k}^2] < \infty$ for all $k$.

In the continuous-time setup of van de Geer, 1995, Lemma 2.2, the preceding definitions are replaced by generalized notions, e.g., see Liptser and Shiryaev, 1989. Note that the notion of a compensator in the continuous-time setting is much more technical, in contrast with the explicit formula (46) for discrete time.

The setup of van de Geer, 1995 is as follows: Let $\{\tilde{M}_t\}_{t \geq 0}$ be a locally square integrable continuous-time martingale with respect to to a filtration $\{\tilde{F}_t\}_{t \geq 0}$ satisfying right-continuity ($\tilde{F}_t = \cap_{s \geq t} \tilde{F}_s$) and completeness ($\tilde{F}_0$ includes all sets of null probability). For each $t > 0$, the martingale jump is defined as $\Delta \tilde{M}_t = \tilde{M}_t - \tilde{M}_{t-}$, where $t_-$ represents an infinitesimal time instant prior to $t$. For each integer $m \geq 2$, a higher-order variation process $\{\sum_{s \leq t} |\Delta \tilde{M}_s|^m\}$ is considered, and its compensator is denoted by $\tilde{V}_{m,t}$. Then, we have the following.
Lemma 11. [van de Geer, 1996, Lemma 2.2] Under the preceding setup for continuous-time martingales, suppose that for all $t \geq 0$ and some $0 < K < \infty$, it holds that

$$\hat{V}_{m,t} \leq \frac{m!}{2} K^{m-2} \hat{R}_t, \quad m = 2, 3, \ldots,$$

for some predictable process $\hat{R}_t$. Then, for any $a, b > 0$, we have

$$\mathbb{P}(\hat{M}_t \geq a \text{ and } \hat{R}_t \leq b^2 \text{ for some } t) \leq \exp \left( -\frac{a^2}{2aK + b^2} \right).$$

While Lemma 11 is stated for continuous-time martingales, we obtain the discrete-time version in Lemma 4 by considering the choice $\hat{M}_t = M_{[t]}$, where $\{M_t\}_{t=0,1,...}$ is the discrete-time martingale. Due to the floor operation, the required right-continuity condition on the continuous-time martingale holds. Moreover, the definition of a compensator in (16) applied to the higher-order variation process with parameter $m$ yields

$$V_{m,t} = \sum_{j=1}^{t} \mathbb{E}[|\Delta M_j|^m | \mathcal{F}_{j-1}]$$

with $\Delta M_t = M_t - M_{t-1}$, in agreement with the statement of Lemma 4. Finally, since we assumed that $\mathbb{E}[M_t^2] < \infty$ for all $t$ in Lemma 4, the locally square integrable condition follows by choosing the trivial sequence of stopping times, $\tau_k = k$.

D Proof of Lemma 5 (Concentration of $\sum_j Z^j$)

Lemma 5 is restated as follows.

Lemma 5. $|\sum_{j=1}^{T} Z^j| = O \left( \sqrt{T \log \frac{1}{\delta}} \right)$ with probability at least $1 - \delta$.

Proof. Recall that $\mathbb{E}_{t-1}[:]$ denotes expectation conditioned on the history up to index $t-1$. Using the notation of Lemma 4, we let $M_t = \sum_{j \leq t} Z^j$, which yields $\Delta M_t = Z^j$. The definition of $Z^j$ in (15) ensures that $\mathbb{E}_{t-1}[Z^j] = 0$, so that $M_t$ is a martingale. In addition, we have

$$V_{m,t} = \sum_{j=1}^{t} \mathbb{E}_{j-1}[|\Delta M_j|^m] = \sum_{j=1}^{t} \mathbb{E}_{j-1}[|Z^j|^m].$$

To use Lemma 4, we need to bound $\sum_{j=1}^{t} \mathbb{E}_{j-1}[|Z^j|^m]$ for some appropriate choices of $K$ and $R_t$ in (12). The conditional moments of $|Z^j|$ are the central conditional moments of $Q^j$:

$$\mathbb{E}_{j-1}[|Z^j|^m] = \mathbb{E}_{j-1}[|Q^j - \mathbb{E}_{j-1}[Q^j]|^m] \leq \mathbb{E}_{j-1}[2^m(|Q^j|^m + |\mathbb{E}_{j-1}[Q^j]|^m)] \leq 2^{m+1} \mathbb{E}_{j-1}[|Q^j|^m],$$

where (51) follows from the definition of $Z^j$ in (15), (52) uses $|a - b| \leq 2 \max\{|a|, |b|\}$, and (53) follows from Jensen’s inequality ($\mathbb{E}[|Q^j|^m] \leq \mathbb{E}[|Q^j|^m]$). Furthermore, we have that

$$\mathbb{E}_{j-1}[|Q^j|^m] = \mathbb{E}_{j-1}[|\langle \lambda p^j \cdot \tilde{x}^j - \tilde{y}^j \rangle (p^j - w/\lambda) \cdot \tilde{x}^j|^m] \leq \mathbb{E}_{j-1}[|\langle \lambda p^j \cdot \tilde{x}^j - \tilde{y}^j \rangle |^{2m}]^{1/2} \mathbb{E}_{j-1}[|p^j - w/\lambda| \cdot \tilde{x}^j|^{2m}]^{1/2},$$

where (54) uses the definition of $Q^j$ in (12), and (55) follows from the Cauchy-Schwartz inequality. Both of the averages in (55) contain Gaussian random variables (with $p^j$ fixed due to the conditioning); we proceed by
establishing an upper bound on the variances. Since \((\tilde{x}^j, \tilde{y}^j) = \frac{1}{B\sqrt{\nu_{\max}(\lambda + 1)}(x^j, y^j)}\), the definition of \(\nu_{\max}\) (see (60)) implies that each coordinate has a variance of at most \((\frac{1}{B\sqrt{\nu_{\max}(\lambda + 1)}})^2\). Then, using that \(\sum_i p_i^j = 1\), we have

\[
\text{Var}(\lambda p^j \cdot \tilde{x}^j - \tilde{y}^j) \leq (\lambda + 1)^2 \max_{z \in \tilde{x}_1, \ldots, \tilde{x}_n} \text{Var}(z) \leq \frac{\lambda + 1}{B^2}.
\]

(57)

and similarly, using \(\sum_i p_i^j = 1\) and \(\|w\| = \lambda\) (see Footnote 2),

\[
\text{Var}(p^j - w/\lambda) \cdot \tilde{x}^j) \leq \frac{4}{(\lambda + 1)B^2}.
\]

(58)

Next, we use the standard fact that if \(N\) is a Gaussian random variable with mean 0 and variance \(\sigma^2\), then

\[
\mathbb{E}[N^p] = \begin{cases} 
0 & \text{if } p \text{ is odd} \\
\sigma^p(p - 1)!! & \text{if } p \text{ is even}.
\end{cases}
\]

(59)

It then follows from (63) and (57)–(59) that

\[
\mathbb{E}_{\tilde{z}}[|Z^j|^m] \leq 2^{m-1} \mathbb{E}_{\tilde{z}}[(\lambda p^j \cdot \tilde{x}^j - \tilde{y}^j)[2^m]^{1/2}\mathbb{E}_{\tilde{z}}[(p^j - w/\lambda) \cdot \tilde{x}^j[2^m]^{1/2}}
\]

\[
\leq 2^{m-1} \left( \left( \frac{\lambda + 1}{B^2} \right)^{2m} (2m - 1)!! \left( \frac{4}{(\lambda + 1)B^2} \right)^{2m} (2m - 1)!! \right)^{1/2}
\]

\[
= 2^{m-1} \left( \frac{4^m}{B^{4m}} (2m - 1)!! \right)
\]

(60)

\[
= 2^{m-1} \left( \frac{4^m}{B^{4m}} (1 \cdot 3 \cdot \ldots \cdot (2m - 1)) \right)
\]

(61)

\[
\leq 2^{m-1} \left( \frac{4^m}{B^{4m}} (2 \cdot 4 \cdot \ldots \cdot 2m) \right)
\]

(62)

\[
= 2 \cdot 2^{m-1} \left( \frac{4^m}{B^{4m}} \right) m!
\]

(63)

\[
= m! \left( \frac{16}{B^4} \right)^{m-2} \frac{2^{10}}{B^8},
\]

(64)

and summing over \(j = 1, \ldots, t\) gives

\[
\sum_{j=1}^t \mathbb{E}_{\tilde{z}}[|Z^j|^m] \leq \frac{m!}{2} \left( \frac{16}{B^4} \right)^{m-2} \frac{2^{10} t}{B^8}.
\]

(65)

(66)

Hence, using the notation of Lemma 4 it suffices to set \(K = \frac{16}{B^4}\) and \(R_t = \frac{2^{10} t}{B^8}\). Plugging everything in, we get

\[
\mathbb{P} \left( \sum_{j=1}^T Z^j > a \right) < \exp \left( -\frac{a^2}{32 \cdot \frac{1}{B^4} + \frac{2^{10} t}{B^8}} \right).
\]

(67)

(68)

Let \(a = 2^{10} \sqrt{T \log \frac{1}{\delta}}\). Then, since \(B = \sqrt{2 \log \frac{2eT}{\delta}}\) is always greater than \(\sqrt{\log \frac{1}{\delta}}\), we obtain

\[
\mathbb{P} \left( \sum_{j=1}^T Z^j > 2^{10} \sqrt{T \log \frac{1}{\delta}} \right) \leq \frac{\delta}{2}.
\]

(69)

By replacing \(Z^j\) by \(-Z^j\) above, we get a symmetric lower bound on \(\sum_j Z^j\), as all the moments used above remain the same. Applying the union bound, we get that \(\mathbb{P}(\sum_{j=1}^T Z^j) = O(\sqrt{T \log \frac{1}{\delta}})\) with probability at least \(1 - \delta\).
Lemma 7 is restated as follows.

**Lemma 7.** For $\gamma > 0$, $\rho \in (0, 1]$, and fixed $v \in \mathbb{R}^n$ satisfying $\|v\|_1 \leq \lambda$, there is some $M = O((\lambda + 1)\log(1/\rho))$ such that

$$
P\left(\frac{1}{M} \sum_{j=1}^{M} (\langle v \cdot a^j - b^j \rangle^2 - \Xi) - \varepsilon(v) \right) \geq \gamma \leq \rho,\tag{32}
$$

where $\{(a^j, b^j)\}_{j=1}^{M}$ are the normalized samples defined in Algorithm 2 and $\Xi = E[\text{Var}[b^j | a^j]]$.

**Proof.** We first derive a simple equality:

$$
E[(v \cdot a^j - b^j)^2] = E[E[(v \cdot a^j - b^j)^2 | a^j]]
$$

$$
= E[(E[v \cdot a^j - b^j | a^j])^2 + \text{Var}[b^j | a^j]]
$$

$$
= E[(v \cdot a^j - w \cdot a^j)^2] + E[\text{Var}[b^j | a^j]]
$$

$$
= \varepsilon(v) + \Xi,\tag{73}
$$

where (71) uses $\text{Var}[Z] = E[Z^2] - (E[Z])^2$, (72) uses the second part of Lemma 2, and (73) uses the definitions of $\varepsilon(v)$ and $\Xi$.

In the following, we recall Bernstein’s inequality.

**Lemma 12.** [Boucheron et al., 2013, Corollary 2.11] Let $Z_1, \ldots, Z_n$ be independent real-valued random variables, and assume that there exist positive numbers $\vartheta$ and $c$ such that

$$
\sum_{i=1}^{n} E[Z_i^2] \leq \vartheta \tag{74}
$$

$$
\sum_{i=1}^{n} E[Z_i^2] \leq \frac{q^4}{2} \cdot \vartheta \cdot c^3 - 2;\tag{75}
$$

where $(x)_+ = \max\{x, 0\}$. Letting $S = \sum_{i=1}^{n} (Z_i - E[Z_i])$, we have for all $t > 0$ that

$$
P(S \geq t) \leq \exp\left(-\frac{t^2}{2(\vartheta + ct)}\right).\tag{76}
$$

We would like to use Bernstein’s inequality to bound the deviation of

$$
\frac{1}{M} \sum_{j=1}^{M} (\langle v \cdot a^j - b^j \rangle^2 - \Xi) - \varepsilon(v)\tag{77}
$$

from its mean value 0. To do so, we need to find constants $\vartheta$ and $c$ as described in the statement of Bernstein’s inequality above.

Recall that $\nu_{\text{max}}$ upper bounds the variance of any marginal variable in each unnormalized sample, and that $(a^j, b^j)$ are samples normalized by $B\sqrt{\nu_{\text{max}}(\lambda + 1)}$ with $B = \sqrt{2\log \frac{2n^4}{\delta}} \geq 1$. Hence, the entries of $(a^j, b^j)$ have variance at most $\frac{1}{\lambda + 1}$, and since $\|v\|_1 \leq \lambda$, this implies that $v \cdot a^j - b^j$ has variance at most $\lambda + 1$.

Using the expression for the moments of a Gaussian distribution (see (59)), it follows that

$$
E[(v \cdot a^j - b^j)^4] \leq 8(\lambda + 1)^2,\tag{78}
$$

This quantity is the same for all values of $j$. 

---

Anamay Chaturvedi, Jonathan Scarlett
Specifically, for an arbitrary index $i$, Lemma 8 is restated as follows, and refers to the setup described in Section 4.

\[ \text{Var}(\sum_{i=1}^{M} (v \cdot a^i - b^j)^2 - \Xi - \varepsilon(v)) \geq \gamma M \]
\[ \leq \exp \left( \frac{-\gamma^2 M^2}{2(8\lambda(1+\lambda)^2 + 2(\lambda+1)\gamma M)} \right). \]

where (80) is established in the same way as (65). Since $(v \cdot a^i - b^j)^2$ is a non-negative random variable, the non-central moments bound the central moments from above. Hence, it suffices to let $\vartheta = 8(\lambda+1)^2$ and $c = 2(\lambda+1)$, and we obtain from Bernstein’s inequality that

\[ \text{P} \left( \left| \sum_{j=1}^{M} (v \cdot a^i - b^j)^2 - \Xi - \varepsilon(v) \right| \geq \gamma M \right) \leq \exp \left( \frac{-\gamma^2 M^2}{2(8\lambda(1+\lambda)^2 + 2(\lambda+1)\gamma M)} \right). \]

To simplify the notation, we let $M_0$ be such that $M = (\lambda+1)M_0$, which yields

\[ \text{P} \left( \left| \frac{1}{M} \sum_{j=1}^{M} (v \cdot a^i - b^j)^2 - \Xi - \varepsilon(v) \right| \geq \gamma \right) \leq \exp \left( \frac{-\gamma^2 M_0^2}{16 + 2\gamma M_0} \right). \]

If $\gamma M_0 \geq 1$, then the right hand side is less than or equal to $\exp \left( \frac{-\gamma M_0}{18} \right)$. Otherwise, if $\gamma M_0 < 1$, then the right hand side is less than $\exp \left( \frac{-\gamma^2 M_0^2}{18} \right)$. It follows that to have a deviation of $\gamma$ with probability at most $\rho$, it suffices to set $M_0 = \frac{18\log(1/\rho)}{\gamma}$ Recalling that $M = (\lambda+1)M_0$, it follows that with $M = 18(\lambda+1)\log(1/\rho)$, we attain the desired target probability $\rho$.

\[ \square \]

## F Proof of Lemma 8 (Low Risk Implies an $\ell_\infty$ Bound)

Lemma 8 is restated as follows, and refers to the setup described in Section 4.

**Lemma 8.** Under the preceding setup, if we have $\varepsilon(v) \leq \epsilon$, then we also have $\|v - w\|_{\infty} \leq \sqrt{\epsilon \theta_{\max}}$, where $\theta_{\max}$ is a uniform upper bound on the diagonal entries of $\Theta$.

**Proof.** Recall that $\varepsilon(v) = \mathbb{E}[(v - w) \cdot X_i^2]$, where $w = \left(\frac{-\theta_{ij}}{\theta_{ii}}\right)_{j \neq i}$ is the neighborhood weight vector of the node $i$ under consideration, and $X_i = (X_{j})_{j \neq i}$. To motivate the proof, note from Lemma 2 that $X_i = \eta_i + \sum_{j \neq i} (-\theta_{ij}/\theta_{ii}) X_j$, where $\eta_i$ is a $N(0, 1/\rho)$ random variable independent of $\{X_j\}_{j \neq i}$, from which it follows that $\text{Var}(X_i) \geq \text{Var}(\eta_i) = 1/\theta_{ii}$. In the following, we apply similar ideas to $(v - w) \cdot X_i^2$.

Specifically, for an arbitrary index $i^* \neq i$, we can lower bound the expected risk $\varepsilon(v)$ as follows:

\[ \mathbb{E}[(v - w) \cdot X_i^2] = \text{Var}((v - w) \cdot X_i) \]
\[ = \text{Var}\left( \sum_{j \neq i} (v_j - w_j) X_j \right) \]
\[ = \text{Var}\left( (v_{i^*} - w_{i^*}) X_i + \sum_{j \notin \{i, i^*\}} (v_j - w_j) X_j \right) \]
\[ = \text{Var}\left( (v_{i^*} - w_{i^*}) \eta_i - (v_{i^*} - w_{i^*}) \frac{\theta_{i^* i}}{\theta_{i^* i^*}} X_i + \sum_{j \notin \{i, i^*\}} \left( (v_j - w_j) - (v_{i^*} - w_{i^*}) \frac{\theta_{i^* j}}{\theta_{i^* i^*}} \right) X_j \right) \]
\[ = \text{Var}((v_{i^*} - w_{i^*}) \eta_i) + \text{Var}\left( - (v_{i^*} - w_{i^*}) \frac{\theta_{i^* i}}{\theta_{i^* i^*}} X_i + \sum_{j \notin \{i, i^*\}} \left( (v_j - w_j) - (v_{i^*} - w_{i^*}) \frac{\theta_{i^* j}}{\theta_{i^* i^*}} \right) X_j \right) \]
\[ \geq \text{Var}((v_{i^*} - w_{i^*}) \eta_i) \]
\[ = |v_{i^*} - w_{i^*}|^2 \text{Var}(\eta_i), \]
where (84) follows since \( \mathbb{E}[X_i] = 0 \), (87) follows from the first part of Lemma 2 applied to node \( i^* \), and (88) uses the independence of \( \eta_{i^*} \) and \( X_{i^*} \). Since \( \text{Var}(\eta_{i^*}) = \frac{1}{\theta_{i^*}} \) and \( \varepsilon(v) \leq \epsilon \), this gives \( |v_{i^*} - w_{i^*}| \leq \sqrt{\epsilon \theta_{i^*}} \leq \sqrt{\epsilon \theta_{\max}} \).

Then, since this holds for all \( i^* \neq i \), we deduce that \( \|v - w\|_{\infty} \leq \sqrt{\epsilon \theta_{\max}} \), as desired. \( \square \)