STARLIKENESS AND CONVEXITY OF INTEGRAL OPERATORS INVOLVING MITTAG-LEFFLER FUNCTIONS

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ABSTRACT. In this paper, we shall find the order of starlikeness and convexity for integral operators

\[ F_{\alpha_j, \beta_j, \lambda_j, \zeta}(z) = \left\{ \zeta \int_0^z t^{\delta - 1} \prod_{j=1}^n \left( \frac{E_{\alpha_j, \beta_j}(t)}{t^{1/\lambda_j}} \right)^{1/\zeta} dt \right\}^{1/\zeta}, \]

where the functions \( E_{\alpha_j, \beta_j} \) are the normalized Mittag-Leffler functions.

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1. Introduction and preliminaries

Let \( A \) denote the class of functions of the form:

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1} \]

which are analytic in the open unit disk \( U = \{ z : |z| < 1 \} \). A function \( f(z) \in A \) is said to be starlike of order \( \delta \) if it satisfies

\[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \delta \quad (z \in U) \tag{1.2} \]

for some \( \delta(0 \leq \delta < 1) \). We denote by \( S^*(\delta) \) the subclass of \( A \) consisting of functions which are starlike of order \( \delta \) in \( U \). Clearly \( S^*(\delta) \subseteq S^*(0) = S^* \), where \( S^* \) is the class of functions that are starlike in \( U \). Also, a function \( f(z) \in A \) is said to be convex of order \( \alpha \) if it satisfies

\[ \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \delta \quad (z \in U) \tag{1.3} \]

for some \( \delta(0 \leq \delta < 1) \). We denote by \( C(\delta) \) the subclass of \( A \) consisting of functions which are convex of order \( \alpha \) in \( U \). Clearly \( C(\delta) \subseteq C(0) = C \), the class of functions that are convex in \( U \).

Let \( E_{\alpha}(z) \) be the function defined by

\[ E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an + 1)}, \quad (z \in \mathbb{C}, \Re(\alpha) > 0). \]

The function \( E_{\alpha}(z) \) was introduced by Mittag-Leffler [9] and is, therefore, known as the Mittag-Leffler function. A more general function \( E_{\alpha, \beta} \) generalizing \( E_{\alpha}(z) \) was introduced...
by Wiman \[12, 13\] and defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (z, \alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0).$$ \hspace{1cm} (1.4)

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g. in ([1, 3], [4]-[10]).

Observe that Mittag-Leffler function \(E_{\alpha,\beta}\) does not belong to the family \(\mathcal{A}\). Therefore, we consider the following normalization of the Mittag-Leffler function:

$$E_{\alpha,\beta}(z) = \frac{\Gamma(\beta)}{z} zE_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} z^n, \hspace{1cm} (1.5)$$

where \(z, \alpha, \beta \in \mathbb{C}; \beta \neq 0, -1, -2, \ldots\) and \(\Re(\alpha) > 0\).

Whilst formula (1.5) holds for complex-valued \(\alpha, \beta\) and \(z \in \mathbb{C}\), however in this paper, we shall restrict our attention to the case of real-valued \(\alpha, \beta\) and \(z \in \mathbb{U}\). Observe that the function \(E_{\alpha,\beta}\) contains many well-known functions as its special case, for example, \(E_{2,1}(z) = z \cosh \sqrt{z}, \) \(E_{2,2}(z) = \sqrt{z} \sinh \sqrt{z}, \) \(E_{2,3}(z) = 2[\cosh \sqrt{z} - 1]\) and \(E_{2,4}(z) = 6[\sinh \sqrt{z} - \sqrt{z}]\).

Geometric properties including starlikeness, convexity and close-to-convexity for the Mittag-Leffler function \(E_{\alpha,\beta}\) were recently investigated by Bansal and Prajapat in [2].

Very recently, Srivastava et al. \[11\] introduced a new integral operator \(F_{\alpha_j,\beta_j,\lambda_j,\zeta}\) involving Mittag-Leffler functions given by

$$F(z) = F_{\alpha_j,\beta_j,\lambda_j,\zeta}(z) = \left\{ \zeta \int_0^z t^{\zeta-1} \prod_{j=1}^n \left( \frac{E_{\alpha_j,\beta_j}(t)}{t} \right)^{1/\lambda_j} dt \right\}^{1/\zeta}, \hspace{1cm} (1.6)$$

where the functions \(E_{\alpha_j,\beta_j}\) are the normalized Mittag-Leffler functions defined by

$$E_{\alpha_j,\beta_j}(z) = \Gamma(\beta_j) zE_{\alpha_j,\beta_j}(z).$$

and the parameters \(\lambda_1, \lambda_1, \ldots, \lambda_n\) and \(\zeta\) are are positive real numbers such that the integrals in (1.6) exist. Here and throughout in the sequel every many-valued function is taken with the principal branch.

In the present paper, we will find the order of starlikeness and convexity for the above integral defined by (1.6).

In order to prove our main results, we recall the following lemmas.
Lemma 1.1. (8). Let $\Phi(u, v)$ be a complex valued function, 
$$\Phi : \mathbb{D} \to \mathbb{C}, \quad (\mathbb{D} \subset \mathbb{C}^2)$$
and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that the function $\Phi(u, v)$ satisfies

(i) $\Phi(u, v)$ is continuous in $\mathbb{D}$;
(ii) $(1, 0) \in \mathbb{D}$ and $\text{Re}(\Phi(1, 0)) > 0$;
(iii) $\text{Re}(\Phi(iu_2, v_1)) \leq 0$ for all $(iu_2, v_1) \in \mathbb{D}$ and such that $v_1 \leq -(1 + u_2^2)/2$.

Let $p(z) = 1 + p_1z + p_2z^2 + \cdots$ be analytic in $U$ such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in U$. If $\text{Re}(\Phi(p(z), zp'(z))) > 0$ $(z \in U)$, then $\text{Re}(p(z)) > 0$ $(z \in U)$.

Lemma 1.2. (2) Let $\alpha \geq 1$ and $0 \leq \eta < 1$. Suppose also that
$$\Psi(\eta) = (3 - \eta) + \sqrt{5\eta^2 - 18\eta + 17}.$$ 
If $\beta \geq \Psi(\eta)$, then $E_{\alpha,\beta}$ is starlike function of order $\eta$.

Lemma 1.3. (11) Let $\alpha \geq 1$ and $\beta \geq 1$. Then
$$\left| \frac{zE'_{\alpha,\beta}(z)}{E_{\alpha,\beta}(z)} - 1 \right| \leq \frac{2\beta + 1}{\beta^2 - \beta - 1}, \quad (z \in U). \quad (1.7)$$

2. Main Results

Our first result provides the order of starlikeness for integral operator of the type (1.6).

Theorem 2.1. Let $\alpha_j \geq 1, 0 \leq \eta_j < 1$, and
$$\beta_j \geq \frac{(3 - \eta_j) + \sqrt{5\eta_j^2 - 18\eta_j + 17}}{2(1 - \eta_j)},$$
for all $j = 1, 2, 3, \ldots, n$. Suppose also that $\lambda_1, \lambda_2, \ldots, \lambda_n, \zeta$ are positive real numbers such that
$$\sum_{j=1}^n \frac{1 - \eta_j}{\lambda_j} \leq \zeta,$$ 
then $F(z) \in S^*(\delta)$, where
$$\delta = \frac{-\left(\sum_{j=1}^n \frac{2(1-\eta_j)}{\lambda_j} - 2\zeta + 1\right) + \sqrt{\left(\sum_{j=1}^n \frac{2(1-\eta_j)}{\lambda_j} - 2\zeta + 1\right)^2 + 8\zeta}}{4\zeta}, \quad 0 \leq \delta < 1. \quad (2.1)$$

Proof. Define the function $p(z)$ by
$$\frac{zF'(z)}{F(z)} = \delta + (1 - \delta)p(z), \quad (2.2)$$
where $\delta$ as given in (2.1).
Then $p(z) = 1 + b_1 z + b_2 z + \cdots$ is analytic in $U$. It follows from (1.6) and (2.2) that
\[
z^\z \prod_{j=1}^n \left( \frac{\mathbb{E}_{\alpha_j,\beta_j}(z)}{z} \right)^{1/\lambda_j} = \delta + (1 - \delta)p(z).
\] (2.3)

Differentiating (2.3) logarithmically, we obtain
\[
\sum_{j=1}^n \frac{1}{\lambda_j} \left( \frac{z \mathbb{E}'_{\alpha_j,\beta_j}(z)}{\mathbb{E}_{\alpha_j,\beta_j}(z)} \right) = \zeta(1 - \delta)p(z) + \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)} + \sum_{j=1}^n \frac{1 - \eta_j}{\lambda_j} - \zeta(1 - \delta).
\] (2.4)

From Lemma 1.1, $\mathbb{E}_{\alpha_j,\beta_j}$ is starlike function of order $\eta_j$ for all $j = 1, 2, 3, \ldots, n$, therefore we have
\[
\sum_{j=1}^n \frac{1}{\lambda_j} \text{Re} \left( \frac{z \mathbb{E}'_{\alpha_j,\beta_j}(z)}{\mathbb{E}_{\alpha_j,\beta_j}(z)} \right)
= \text{Re} \left\{ \zeta(1 - \delta)p(z) + \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)} + \sum_{j=1}^n \frac{1 - \eta_j}{\lambda_j} - \zeta(1 - \delta) \right\} > 0.
\] (2.5)

If we define the function $\Phi(u, v)$ by
\[
\Phi(u, v) = \zeta(1 - \delta)u + \frac{(1 - \delta)v}{\delta + (1 - \delta)u} + \sum_{j=1}^n \frac{1 - \eta_j}{\lambda_j} - \zeta(1 - \delta)
\] (2.6)

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$, then

(i) $\Phi(u, v)$ is continuous in $\mathbb{D} = \mathbb{C}^2$;

(ii) $(1, 0) \in \mathbb{D}$ and $\text{Re}(\Phi(1, 0)) = \sum_{j=1}^n \frac{1 - \eta_j}{\lambda_j} > 0$;

(iii) For all $(iu_2, v_1) \in \mathbb{D}$ and such that $v_1 \leq -(1 + u_2^2)/2$,
\[
\text{Re}(\Phi(iu_2, v_1)) = \frac{\delta(1 - \delta)v_1}{\delta^2 + (1 - \delta)^2u_2^2} + \sum_{j=1}^n \frac{1 - \eta_j}{\lambda_j} - \zeta(1 - \delta)
\leq \frac{A + Bu_2^2}{C}
\] (2.7)

where
\[
A = \delta \left( 2\zeta \delta^2 + \sum_{j=1}^n \frac{2(1 - \eta_j)}{\lambda_j} - 2\zeta + 1 \right) \delta - 1,
\]
\[
B = (1 - \delta)^2 \left( \sum_{j=1}^n \frac{2(1 - \eta_j)}{\lambda_j} - 2\zeta(1 - \delta) \right) - \delta(1 - \delta),
\]
and
\[
C = 2\delta^2 + 2(1 - \delta)^2u_2^2.
\]

The right hand side of (2.7) is negative if $A \leq 0$ and $B \leq 0$. From $A \leq 0$, we have the value of $\delta$ given by (2.1) and from $B \leq 0$, we have $0 \leq \delta < 1$. Therefore, the function
\( \Phi(u, v) \) satisfies the conditions in Lemma 1.1. Thus we have \( \text{Re}(\rho(z)) > 0 \) \((z \in \mathbb{U})\), that is \( F(z) \in S^*(\delta). \)

Let \( n = 1, \alpha_1 = \alpha, \beta_1 = \beta, \lambda_1 = \lambda \) and \( \eta_1 = 0 \) in Theorem 2.1, we have the following result.

**Corollary 2.2.** Let \( \alpha \geq 1 \) and \( \beta \geq \frac{3 + \sqrt{17}}{2} \). Then \( F_{\alpha,\beta,1,1}(z) = \int_0^z \left( E_{\alpha,1}(t) \right)^{1/\lambda} dt \) is starlike of order \( 1/2 \) in \( \mathbb{U} \).

**Example 2.4.** Let \( E_{2,4}(z) = 6[\sinh \sqrt{z} - \sqrt{z}] / \sqrt{z} \), then \( \int_0^z 6[\sinh \sqrt{z} - \sqrt{z}] \left( t^{1/2} - t^{1/2} \right) dt \) is starlike of order \( 1/2 \) in \( \mathbb{U} \).

Making use Lemma 1.3, we determine the order of convexity for integral operator of the type (1.6).

**Theorem 2.5.** Let \( \alpha_1, \alpha_2, \ldots, \alpha_n \geq 1, \beta_1, \beta_2, \ldots, \beta_n \geq \frac{1}{2}(1 + \sqrt{5}) \) and consider the normalized Mittag-Leffler functions \( E_{\alpha_j,\beta_j} \) defined by

\[
E_{\alpha_j,\beta_j}(z) = \Gamma(\beta_j)zE_{\alpha_j,\beta_j}(z).
\]

Let \( \beta = \min\{\beta_1, \beta_2, \ldots, \beta_n\} \) and \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be nonzero positive real numbers. Moreover, suppose that these numbers satisfy the following inequality

\[
0 \leq 1 - \frac{2\beta + 1}{\beta^2 - \beta - 1} \sum_{j=1}^n \frac{1}{\lambda_j} < 1.
\]

Then the function \( F_{\alpha_j,\beta_j,\lambda_j} \) defined by

\[
F_{\alpha_j,\beta_j,\lambda_j}(z) = \int_0^z \prod_{j=1}^n \left( \frac{E_{\alpha_j,\beta_j}(t)}{t^{1/\lambda_j}} \right) dt,
\]

is in \( C(\delta) \), where

\[
\delta = 1 - \frac{2\beta + 1}{\beta^2 - \beta - 1} \sum_{j=1}^n \frac{1}{\lambda_j}.
\]
Proof. We observe that \( E_{\alpha_j, \beta_j} \in \mathcal{A} \), i.e. \( E_{\alpha_j, \beta_j}(0) = E_{\alpha_j, \beta_j}'(0) - 1 = 0 \), for all \( j \in \{1, 2, \ldots, n\} \). On the other hand, it is easy to see that

\[
E'_{\alpha_j, \beta_j, \lambda_j}(z) = \prod_{j=1}^{n} \left( \frac{E_{\alpha_j, \beta_j}(z)}{z} \right)^{1/\lambda_j}
\]

and

\[
\frac{zE''_{\alpha_j, \beta_j, \lambda_j}(z)}{E'_{\alpha_j, \beta_j, \lambda_j}(z)} = \sum_{j=1}^{n} \frac{1}{\lambda_j} \left( \frac{zE'_{\alpha_j, \beta_j}(z)}{E_{\alpha_j, \beta_j}(z)} - 1 \right),
\]

or, equivalently,

\[
1 + \frac{zE''_{\alpha_j, \beta_j, \lambda_j}(z)}{E'_{\alpha_j, \beta_j, \lambda_j}(z)} = \sum_{j=1}^{n} \frac{1}{\lambda_j} \left( \frac{zE'_{\alpha_j, \beta_j}(z)}{E_{\alpha_j, \beta_j}(z)} - 1 \right) + 1 - \sum_{j=1}^{n} \frac{1}{\lambda_j}. \tag{2.11}
\]

Taking the real part of both terms of (2.11), we have

\[
\text{Re} \left\{ 1 + \frac{zE''_{\alpha_j, \beta_j, \lambda_j}(z)}{E'_{\alpha_j, \beta_j, \lambda_j}(z)} \right\} = \sum_{j=1}^{n} \frac{1}{\lambda_j} \text{Re} \left( \frac{zE'_{\alpha_j, \beta_j}(z)}{E_{\alpha_j, \beta_j}(z)} \right) + \left( 1 - \sum_{j=1}^{n} \frac{1}{\lambda_j} \right). \tag{2.12}
\]

Now, by using the inequality (1.7) for each \( \beta_j \), where \( j \in \{1, 2, \ldots, n\} \), we obtain

\[
\text{Re} \left\{ 1 + \frac{zE''_{\alpha_j, \beta_j, \lambda_j}(z)}{E'_{\alpha_j, \beta_j, \lambda_j}(z)} \right\} = \sum_{j=1}^{n} \frac{1}{\lambda_j} \text{Re} \left( \frac{zE'_{\alpha_j, \beta_j}(z)}{E_{\alpha_j, \beta_j}(z)} \right) + \left( 1 - \sum_{j=1}^{n} \frac{1}{\lambda_j} \right) > \sum_{j=1}^{n} \frac{1}{\lambda_j} \left( 1 - \frac{2\beta_j + 1}{\beta_j^2 - \beta - 1} \right) + \left( 1 - \sum_{j=1}^{n} \frac{1}{\lambda_j} \right) = 1 - \frac{2\beta + 1}{\beta^2 - \beta - 1} \sum_{j=1}^{n} \frac{1}{\lambda_j}
\]

for all \( z \in \mathbb{D} \) and \( \beta_1, \beta_2, \ldots, \beta_n \geq \frac{1}{2}(1 + \sqrt{5}) \). Here we used that the function \( \varphi : (\frac{1}{2}(1 + \sqrt{5}), \infty) \to \mathbb{R} \), defined by

\[
\varphi(x) = \frac{2x + 1}{x^2 - x - 1},
\]

is decreasing. Therefore, for all \( j \in \{1, 2, \ldots, n\} \) we have

\[
\frac{2\beta_j + 1}{\beta_j^2 - \beta - 1} \leq \frac{2\beta + 1}{\beta^2 - \beta - 1}. \tag{2.13}
\]

Because \( 0 \leq 1 - \frac{2\beta + 1}{\beta^2 - \beta - 1} \sum_{j=1}^{n} \frac{1}{\lambda_j} < 1 \), we get \( F_{\alpha_j, \beta_j, \lambda_j}(z) \in \mathcal{C}(\delta) \), where \( \delta = 1 - \frac{2\beta + 1}{\beta^2 - \beta - 1} \sum_{j=1}^{n} \frac{1}{\lambda_j} \). This completes the proof. \( \square \)

Let \( n = 1 \), \( \alpha_1 = \alpha \), \( \beta_1 = \beta \) and \( \lambda_1 = \lambda \) in Theorem 2.1 we have the following result.
Corollary 2.6. Let $\alpha \geq 1$, $\beta \geq \frac{1}{2}(1 + \sqrt{5})$ and $\lambda > 0$. Moreover, suppose that these numbers satisfy the following inequality

$$0 \leq 1 - \frac{2\beta + 1}{\lambda(\beta^2 - \beta - 1)} < 1.$$ 

Then the function $F_{\alpha, \beta, \lambda}$ defined by

$$F_{\alpha, \beta, \lambda}(z) = \int_0^z \left( \frac{E_{\alpha, \beta}(t)}{t} \right)^{1/\lambda} dt,$$

is in $C(\delta)$, where

$$\delta = 1 - \frac{2\beta + 1}{\lambda(\beta^2 - \beta - 1)}.$$

Example 2.7. (i) If $0 \leq 1 - \frac{5}{\lambda} < 1$, then $\int_0^z \left( \frac{\sinh \sqrt{t}}{\sqrt{t}} \right)^{1/\lambda} dt \in C(\delta); \delta = 1 - \frac{5}{\lambda}; \lambda \geq 5$.

(ii) If $0 \leq 1 - \frac{7}{5\lambda} < 1$, then $\int_0^z \left( \frac{2[\cosh \sqrt{t-1}]}{t} \right)^{1/\lambda} dt \in C(\delta); \delta = 1 - \frac{7}{5\lambda}; \lambda \geq 7/5$.

(iii) If $0 \leq 1 - \frac{9}{11\lambda} < 1$, then $\int_0^z \left( \frac{6[\sinh \sqrt{t-\sqrt{t}}]}{t^{3/2}} \right)^{1/\lambda} dt \in C(\delta); \delta = 1 - \frac{9}{11\lambda}; \lambda \geq 9/11$.

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