MAXIMUM INDEPENDENT SET OF CLIQUES AND THE GENERALIZED MANTEL’S THEOREM

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ABSTRACT. A complete subgraph of any simple graph $G$ on $k$ vertices is called a $k$-clique of $G$. In this paper, we first introduce the concept of the value of a $k$-clique ($k > 1$) as an extension of the idea of the degree of a given vertex. Then, we obtain the generalized version of handshaking lemma which we call it clique handshaking lemma. The well-known classical result of Mantel states that the maximum number of edges in the class of triangle-free graphs with $n$ vertices is equal to $\frac{n^2}{4}$. Our main goal here is to find an extension of the above result for the class of $K_{\omega+1}$-free graphs, using the ideas of the value of cliques and the clique handshaking lemma.

1. INTRODUCTION

Finding the maximum values of some key invariants in discrete structures with forbidden (finite) family of substructures is an interesting problem in the area of extremal combinatorics with potential applications in theoretical and applied computer science. One of the classical problems of these kind is the well-known Mantel’s theorem which answers the question about the maximum number of edges in any simple graph in which the family of forbidden subgraphs consists of only the triangle graph $K_3$. There are many interesting proofs of the well-known Mantel’s theorem and one of those beautiful proofs are based on the idea of maximum independent set of vertices. Roughly speaking, the basic idea is to partition the vertex set of a given graph $G = (V, E)$ into two sets $A$ and $B$. The first set $A$ is an independent set of maximum size (maximum number of vertices) and $B$ is the rest of vertices. Now, considering the maximality of $A$, the triangle-freeness of $G$ and the well-known handshaking lemma, one can bound (an upper bound) the number of edges based on the sum of degrees of vertices lie in $B$. Finally combining all previous findings with the well-known arithmetic-geometric mean inequality we get the classical Mantel’s theorem.

It seems that we can extend the idea of the independent set of vertices (maximum independent set) to the independent set of edges (maximum matching). Then, using all the previous machinery, one can get a generalization of Mantel’s result for the class of $K_4$-free graphs which we call it Edge Mantel’s theorem. Next, we generalize the concept of the degree of a vertex to a higher $k$-clique ($k > 1$) by introducing the idea of the value of a clique. This simply means that a value of a clique can be defined as the number of common neighbors of it’s vertices. In this direction, we also obtain a higher clique generalization of the handshaking lemma which we call it clique handshaking lemma. Finally, using the same machinery introduced for proving the classical Mantel’s theorem, we obtain the Clique Mantel’s theorem.
Figure 1. The values of edges for the graph $G_1$

2. Basic Definitions and Notations

Throughout this paper, we will assume that our graphs are finite, simple and undirected. For terminologies which are not defined here, one can refer to the book [4].

For a given graph $G = (V, E)$, the vertex set and the edge set will be denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v \in V(G)$, it’s open neighborhood denoted by $N_G(v)$ is the set of vertices adjacent to $v$. A subgraph of $G$ consisting of all those vertices that are pairwise adjacent is called a complete subgraph (clique) of $G$. A complete subgraph with $k$ vertices will be called a $k$-clique. The set of all $k$-cliques in $G$ is denoted by $\Delta_k(G)$. We will also denote the number of $k$-cliques of a graph $G$ by $c_k(G)$. A complete subgraph on three vertices is called a triangle. A subset of vertices with no edges among them is called an independent set of $G$.

A generalization of the concept of the degree of a vertex can be extended to the edge value, as follows.

**Definition 2.1.** For a given graph $G = (V, E)$ and an edge $e = \{u, v\} \in E(G)$, the value of $e$ denoted by $\text{val}_G(e)$ is defined as the number of common neighbors of two end vertices $u$ and $v$ of the edge $e$. More precisely, we have

$$\text{val}_G(e) = |N_G(u) \cap N_G(v)|.$$ 

**Remark 2.2.** It is interesting to note that a definition similar to the edge value has been given in the literature [5]. Indeed, the co-degree of two vertices $u, v \in V(G)$, not necessarily adjacent, is defined as the number of their common neighborhoods.

**Example 2.3.** Let $G = (V, E)$ be a graph depicted in Figure 1. Then, the values of edges are, as follows.

(2.1) \hspace{1cm} \text{val}_G(e_{12}) = \text{val}_G(e_{13}) = \text{val}_G(e_{23}) = 1, \hspace{0.5cm} \text{val}_G(e_{34}) = 0.

Next, we generalize the above idea for any $k$-clique $q_k \in \Delta_k(G)$ ($k > 1$) of $G$.

**Definition 2.4 (Value of a Clique).** Let $G = (V, E)$ be a simple graph and $q_k$ be a $k$-clique of $G$. Then, we define the value of $q_k$ denoted by $\text{val}_G(q_k)$, as follows

(2.2) \hspace{1cm} \text{val}_G(q_k) = \left| \bigcap_{v \in V(q_k)} N_G(v) \right|.

As an extension of the well-known handshaking lemma, we have the following key result.
Lemma 2.5 (Clique Handskaing Lemma). For a simple graph $G = (V, E)$, we have

$$\sum_{q_k \in \Delta_k(G)} \text{val}_G(q_k) = (k + 1)c_{k+1}(G), \quad (k \geq 1).$$

One can given several proofs of the above lemma. Here, we present a proof which based on the idea of double counting.

Proof. Let $G = (V, E)$ be any simple graph. Define the set $I_k(G)$, as follows.

$$(2.4) \quad I_k(G) = \{(q_k, q_{k+1}) \in \Delta_k(G) \times \Delta_{k+1}(G) \mid q_k \text{ is a subgraph of } q_{k+1}\}.$$  

The proof proceeds by counting the set $I_k(G)$ in two ways.

Case I. We first fix the clique $q_k$. Then, it is clear that the number of such $(k + 1)$-cliques is exactly $\text{val}_G(q_k)$. Now, summing over all those $k$-cliques $q_k$ will result in

$$\sum_{q_k \in \Delta_k(G)} \text{val}_G(q_k).$$

Case II. Next, we fix the $(k+1)$-clique $q_{k+1}$. Then, it is obvious that the number of such $k$-cliques which are the subgraph of $q_{k+1}$ is equal to $k + 1$. Thus, by summing over all $(k + 1)$-cliques, we get $(k + 1)c_{k+1}(G)$.

Finally, the proof is complete by the double-counting technique. □

Remark 2.6. It is worthy to note that the above lemma is called the transfer equations by Knill in [5] which is even true for the generalized discrete structures like simplicial complexes. The transfer equations are used to obtain a graph-theoretical version of the well-known Gauss-Bonnet formula.

3. Main Results

In this section, we will use the idea of the clique value to generalize the following clique-counting inequality due to Mantel [1].

Theorem 3.1 (Mantel’s Theorem for Triangle-free graphs). For a given triangle-free graph $G = (V, E)$ with $n$ vertices and $m$ edges, we have

$$m \leq \frac{n^2}{4}.$$  

The motivation of this paper originates from the proof of the above classical result which is based on the idea of maximality. Thus, we also include the proof.

Proof. Let $A \subseteq V(G)$ be an independent set of maximum size (a maximum independent set). Next, we put $B = V(G) - A$. Since $G$ is triangle-free, the open neighborhood of any arbitrary vertex $v \in V(G)$ is an independent set. Hence, by the maximality of $A$, we immediately conclude that

$$\deg_G(v) = |N_G(v)| \leq |A|, \quad \forall v \in V(G).$$

On the other hand, since $A$ is an independent set of vertices, we obviously have

$$\sum_{v \in A} \deg_G(v) \leq m.$$

Considering the well-known handshaking lemma, we also get

$$\sum_{v \in A} \deg_G(v) + \sum_{v \in B} \deg_G(v) = 2m.$$
From identities (3.2) and (3.3), we conclude that

\[ m \leq \sum_{v \in B} \deg_G(v). \] (3.4)

Thus, from relations (3.1), (3.4) and the arithmetic-geometric mean inequality, we finally obtain

\[ m \leq \sum_{v \in B} \deg_G(v) \leq \sum_{v \in B} |A| |B| \leq \left( \frac{|A| + |B|}{2} \right)^2 \leq \frac{n^2}{4}, \]

as required. \(\square\)

**Remark 3.2.** It is important to note that from the arithmetic-geometric mean inequality in the above proof it immediately follows that the extremal graph for Mantel’s classical result is the balanced complete bipartite graph \(K_{n^2} \sim n^2\). But here, we are only interested in extremal bounds (inequalities) and not the extremal graphs themselves.

The next result is a slight generalization of the Mantel’s theorem and is based on the idea of the edge values and the maximum matchings in graphs.

**Theorem 3.3.** [Edge Mantel’s Theorem] For a given \(K_4\)-free graph \(G = (V, E)\) with \(m\) edges and \(t\) triangles, we have

\[ t \leq \frac{m^2}{8}. \]

**Proof.** Let \(A\) be the maximum independent set of edges (the maximum matching); that is, the edges with no common end points. Put \(B = E(G) - A\). Since \(G\) is a \(K_4\)-free graph, \(N_G(u) \cap N_G(v)\) is an independent set of edges for any edge \(e = uv \in E(G)\). Therefore, by maximality of \(A\), we have

\[ \text{val}_G(e) = |N_G(u) \cap N_G(v)| \leq |A|. \] (3.5)

Next, we observe that since \(A\) is an independent set of edges, we clearly have

\[ \sum_{e \in A} \text{val}_G(e) \leq t. \] (3.6)

On the other hand, considering the clique handshaking lemma (2.3) for \(\omega(G) = 3\), we have

\[ \sum_{e \in A} \text{val}_G(e) + \sum_{e \in B} \text{val}_G(e) = 3t. \] (3.7)

Now, form formulas (3.6) and (3.7), we conclude that
Finally considering the arithmetic-geometric mean inequality and the inequalities (3.5) and (3.8), we get

\[ t \leq \frac{1}{2} \sum_{e \in B} \text{val}_G(e) \]

\[ \leq \frac{1}{2} \left( |A| + |B| \right)^2 \]

\[ = \frac{m^2}{8}, \]

which completes the proof.

Now, considering the idea of the value of a clique and the clique handshaking lemma using similar arguments as above, we obtain the following generalization of Theorem 3.3.

**Theorem 3.4.** [Clique Mantel’s Theorem] For a given \( K_{\omega+1} \)-free graph \( G = (V, E) \) with \( c_{\omega-1}(G) \) cliques of size \( \omega(G) - 1 \) and \( c_\omega(G) \) cliques of size \( \omega(G) \), we have

\[ c_\omega(G) \leq \frac{1}{\omega(G) - 1} \cdot \frac{c_{\omega-1}^2(G)}{4}. \]

### 4. Concluding Remarks and Future Works

In this paper, we obtain an upper bound for the number \( k \)-clique in the class of \((k+1)\)-cliques-free graphs; that is, the class of those graphs not containing any complete subgraph of on \( k+1 \) vertices. The basic ideas were maximality, clique handshaking identity and using the arithmetic-geometric mean inequality.

Our future project is to consider a more general class of graphs that we will call \( \mathcal{H} \)-free graphs. We recall that the increasing family \([3]\) of graphs \( \mathcal{H} \) is the following

\[ \mathcal{H} = \{ H_1, H_2, \ldots, H_k, H_{k+1}, \ldots \}, \]

in which \( H_1 = K_1 \) and each \( H_i \) is an induced subgraph of \( H_{i+1} \), for all \( i \). Our main goal is to find an upper bound similar to that of Theorem 3.4 for the maximum number of copies of \( H_k \) in the class of those graphs not containing any subgraph isomorphic to \( H_{k+1} \) (for any integer \( k > 1 \)). To achieve this goal, we need two main steps. We have to first define a similar notion of the value of a clique for any graph \( H_k \) in \( \mathcal{H} \). Then, we need to find an analogue of our key lemma; the clique handshaking lemma. We will call this \( \mathcal{H} \)-handshaking lemma. The following result, due to Kelly \([2]\), will play an essential role.
Proposition 4.1. Let $G = (V, E)$ be an $n$-vertex graph with no isolated vertices. Then for any graph $H$ on $k$ vertices, we have

$$(n-k)s(H,G) = \sum_{v \in V} s(H,G-v),$$

where $s(H,G)$ denotes the number of subgraphs of $G$ isomorphic to $H$.

Note that in particular case where $H_k$ is a $k$-clique, it is not hard to show that Proposition 4.1 is equivalent to our clique handshaking lemma. We can also recover all the results of this paper in this special case. More details will appear in our sequel paper [5].

References

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