Mean Field Stochastic Adaptive Control

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Abstract

For noncooperative games the mean field (MF) methodology provides decentralized strategies which yield Nash equilibria for large population systems in the asymptotic limit of an infinite (mass) population. The MF control laws use only the local information of each agent on its own state and own dynamical parameters, while the mass effect is calculated offline using the distribution function of (i) the population’s dynamical parameters, and (ii) the population’s cost function parameters, for the infinite population case. These laws yield approximate equilibria when applied in the finite population.

In this paper, these a priori information conditions are relaxed, and incrementally the cases are considered where, first, the agents estimate their own dynamical parameters, and, second, estimate the distribution parameter in (i) and (ii) above.

An MF stochastic adaptive control (SAC) law in which each agent observes a random subset of the population of agents is specified, where the ratio of the cardinality of the observed set to that of the number of agents decays to zero as the population size tends to infinity. Each agent estimates its own dynamical parameters via the recursive weighted least squares (RWLS) algorithm and the distribution of the population’s dynamical parameters via maximum likelihood estimation (MLE). Under reasonable conditions on the population dynamical parameter distribution, the MF-SAC Law applied by each agent results in (i) the strong consistency of the self parameter estimates and the strong consistency of the population distribution function parameters; (ii) the long run average $L^2$ stability of all agent systems; (iii) a (strong) $\epsilon$-Nash equilibrium for the population of agents for all $\epsilon > 0$; and (iv) the a.s. equality of the long run average cost and the non-adaptive cost in the population limit.

Index Terms

adaptive control, mean field stochastic systems, Nash equilibria, stochastic optimal control

I. INTRODUCTION

Overview

The control and optimization of large-scale stochastic systems is evidently of importance due to their ubiquitous appearance in engineering, industrial, social and economic settings. The complexity of these problems is amplified by the fact that for many such systems the agents involved have conflicting objectives; hence, it is appropriate to consider optimization methodologies based upon individual payoffs or costs. In particular, game theory has been formulated to capture such individual interest seeking behaviour of the agents in many social, economic and

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manmade systems. However, in a large-scale dynamic model, this approach results in an analytic complexity which is in general prohibitively high, and correspondingly leads to few substantive dynamic optimization results.

The optimization of large-scale linear control systems wherein (i) many agents are coupled with each other via their individual dynamics, and (ii) the costs are in an “individual to the mass” form was presented in [1], [2] where the theory of mean field (MF) control (previously termed Nash Certainty Equivalence) was introduced. It is to be noted that the dynamic large-scale cost coupled optimization structure of [2] is motivated by a variety of scenarios, for instance, those analysed in [3]–[6].

In the literature, studies of stochastic dynamic games and team problems may be traced to the 1960s (see e.g. [7]–[9]) while within the optimal control context weakly interconnected systems were studied in [10], and in a two player noncooperative nonlinear dynamic game setting Nash equilibria were analysed in [11], where the coefficients for the coupling terms in the dynamics and costs are required to be small. In contrast to these studies, games with large populations are analyzed in [2], [12], [13]. In [2] the $\epsilon$-Nash equilibrium properties are analysed for a system of competing agents where individual control laws use local information and the average effect of all agents taken together, henceforth referred to as the mass. Overall, the MF methodology for noncooperative LQG games with mean field coupling has been developed in [1], [2], [14] providing decentralized strategies which yield Nash equilibria. A nonlinear extension using McKean-Vlasov Markov process models is also presented in [15].

The central notion of MF theory is that for general classes of large population stochastic dynamic games there exist game theoretic Nash equilibria for the individual agents when each applies certain competitive strategies (i.e. control laws) with respect to the mass effect resulting from all the agents’ strategies. Here each agent is modelled by an individually controlled stochastic system and the systems interact through their individual cost functions and possibly via weak dynamical interaction. The key feedback nature of the mean field solutions is that the individual competitive actions against the mass, plus local feedback control, act so as to collectively reproduce that mass behaviour. The mass effect and associated feedback control laws are calculated offline for the infinite population case and yield approximate equilibria when applied in the finite population case.

For this class of game problems, a related approach has been independently developed in [16], [17], where the notion of oblivious equilibrium by use of a mean field approximation for models of many firm industry dynamics is proposed. The asymptotic equilibrium properties of a market with a large population of agents is studied in [18]. Another related work is [19] where a mean field Nash equilibrium is studied subject to the assumed existence of a factorizing mean field distribution corresponding to the propagation of chaos for the infinite population system. The work in [20] presents mean field control results for a Markov Decision Problem (MDP) formulation of evolutionary games and teams where the basic system hypothesis is the exchangeability of the underlying random processes. 

**Stochastic Adaptive Control**

For discrete time dynamics the long run average (LRA) asymptotically optimal adaptive tracking problem was solved in [21]; subsequently, it was shown in [22] that strongly consistent parameter estimates may be obtained by the use of persistently excited controls. The LRA stochastic (sample path) mean square stability for continuous time linear stochastic adaptive systems was established in [23]. The weighted least squares (WLS) scheme introduced
in [24] was shown in [25] to be convergent without stability and excitation assumption, and a LRA asymptotically optimal solution to the continuous time adaptive LQG control problem under controllability and observability assumptions using the WLS scheme for identification was subsequently obtained in [26] following [27]–[29] and [30].

**MF Stochastic Adaptive Control**

It is important to note that in the non-adaptive MF theory [1], [2] each agent uses its *self state* and *self dynamical parameters* (i.e. its own state and its own dynamical parameters) and statistical information on the dynamical parameters of the population in order to generate the control action. The natural initial problem in the development of adaptive MF stochastic system theory is that where each agent needs to estimate its own dynamical parameters, while its control actions are permitted to be explicit functions of the parameter distribution of the entire population of competing agents [31]. Subsequent problem generalizations are such that (i) each agent also needs to estimate the distribution parameter of the population’s dynamical parameters [32], and (ii) cost function parameters also vary over the population and this distribution parameter is unknown to each agent and hence needs to be estimated [33]. In this paper we provide a solution to the most general problem in this sequence.

The inclusion of learning procedures for the identification by a given agent of the dynamical and cost function parameters of other competing agents in a stochastic dynamic system, or of the statistical distribution of these parameters in a mass of competing agents, introduces new features into the system theoretic MF setup. In this connection we note that in the economics literature the so-called “privacy of information” on dynamical parameters and cost function parameters is an important issue [34]–[36].

This paper presents an MF stochastic adaptive control (SAC) law in which each agent observes a random subset of the population of agents. The MF-SAC Law specifies that the ratio of the cardinality of the observed set of agents to that of the population of agents is chosen so that it decays to zero as the population size tends to infinity. When the MF-SAC Law is applied by each member of the population, each agent estimates its self dynamical parameters via the recursive weighted least squares (RWLS) algorithm and the distribution of the population’s dynamical parameters via maximum likelihood estimation (MLE).

Under reasonable conditions on the population dynamical parameter distribution, the MF-SAC Law results in (i) the strong consistency of the self parameter estimates and the strong consistency of the population distribution function parameters; (ii) the long run average $L^2$ stability of all agent systems; (iii) a (strong) $\epsilon$-Nash equilibrium for the population of agents for all $\epsilon > 0$; and (iv) the a.s. equality of the long run average cost and the non-adaptive cost in the population limit.

**Notation**

We denote the set of nonnegative real numbers by $\mathbb{R}_+$, the set of nonnegative integers by $\mathbb{Z}_+$, and the set of strictly positive integers by $\mathbb{Z}_1$. The norm $\|\cdot\|$ denotes the 2-norm of vectors and matrices, and $\|x\|_Q^2 \triangleq x^T Q x$. $C_b = \{x : x \in \mathbb{C}, \sup_{t \geq 0} \|x(t)\| < \infty\}$ denotes the family of all bounded continuous functions, and for any $x \in C_b$, $\|\cdot\|_\infty$ denotes the supremum norm: $\|x\|_\infty \triangleq \sup_{t \geq 0} \|x(t)\|$. $\text{Tr}(X)$ denotes the trace, and $X^T$ denotes the transpose of a matrix $X$. 

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II. PROBLEM FORMULATION AND MF-SAC LAW SPECIFICATION

A. Review of Non-Adaptive MF Stochastic Control

We consider a large population of $N$ stochastic dynamic agents which (subject to independent controls) are stochastically independent, but which shall be cost coupled, where the individual dynamics are defined by

$$dx_i = [A_i x_i + B_i u_i]dt + Dw_i, \quad t \geq 0, \quad 1 \leq i \leq N,$$

where for agent $A_i$, $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^m$ is the control input, $w_i \in \mathbb{R}^r$ is a standard Wiener process on a sufficiently large underlying probability space $(\Omega, \mathcal{F}, P)$ such that $w_i$ is progressively measurable with respect to $\mathcal{F}^w_i \triangleq \{\mathcal{F}^w_i; t \geq 0\}$. We denote the state configuration by $x = (x_1, \cdots, x_N)^T$, and (with an abuse of notation) the population average state by $x^N = (1/N) \sum_{i=1}^N x_i$.

The long run average (LRA) cost function for the agent $A_i$, $1 \leq i \leq N$, is given by

$$J_i^N(u_i, u_{-i}) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \{\|x_i - m^N\|_Q^2 + \|u_i\|_R^2\} dt,$$

w.p.1, where we assume the cost-coupling to be of the form $m^N(t) \triangleq m(x^N(t) + \eta), \eta \in \mathbb{R}^n$. The coefficients $\theta^T_i \triangleq [A_i, B_i, Q_i] \in \Theta \subset \mathbb{R}^{n(m+n(n+1)/2)}$, will be called the dynamical and cost function parameters. The disturbance weight matrix $D$ and the control action penalizing matrix $R$ are constant matrices, which are assumed to be known by all agents, and assumed to be the same for all agents in the population. The choice of homogeneous parameters for $D$ and $R$ is only for notational brevity; the analysis is similar for varying $D$ and $R$. The function $u_i(\cdot)$ is the control input of the agent $A_i$ and $u_{-i}$ denotes the control inputs of the complementary set of agents $A_{-i} = \{A_j, j \neq i, 1 \leq j \leq N\}$.

For the basic MF control problem, the following assumptions are adopted.

A1: The disturbance processes $w_i$, $1 \leq i \leq N$, are mutually independent and independent of the initial conditions, and $\sup_{t \geq 1}[\text{Tr}\Sigma_i + \mathbb{E}[\|x_i(0)\|^2]] < \infty$, where $\mathbb{E}w_i w_i^\top = \Sigma_i, 1 \leq i \leq N$.

A2: $\hat{\Theta}$ is an open set such that for each $\theta^T = [A_{\theta}, B_{\theta}, Q_{\theta}] \in \hat{\Theta}$, $[A_{\theta}, B_{\theta}]$ is controllable and $[Q_{\theta}^{1/2}, A_{\theta}]$ is observable.

A3: Let the parameter set $\Theta$ be a compact set such that $\Theta \subset \hat{\Theta} \subset \mathbb{R}^{n(m+n(n+1)/2)}$, and $\|R^{-1}\|_1 \int_{\theta \in \Theta} \|Q(\theta)\|_1 \|B(\theta)\|_2^2 (\int_0^\infty \|e^{A(\theta)\tau}\|_2 d\tau)^2 dF_\zeta(\theta) < 1$, where $A_\zeta = A - BR^{-1}B^\top \Pi$, $\zeta$ is the distribution parameter and $\gamma$ is defined in the next hypothesis.

A4: The cost-coupling is of the form: $m^N(\cdot) \triangleq m((1/N) \sum_{k=1}^N x_k + \eta), \eta \in \mathbb{R}^n$, where the function $m(\cdot)$ is Lipschitz continuous on $\mathbb{R}^n$ with a Lipschitz constant $\gamma > 0$, i.e. $\|m(x) - m(y)\| \leq \gamma\|x - y\|$ for all $x, y \in \mathbb{R}^n$.

For dynamics (1) and cost function (2), a production output planning example is provided in [2] that satisfies the assumptions given above. Each agent’s production level $x_i$ is modeled by (1), and each agent’s cost function is of tracking type (2), where the tracked signal is a function of price, which is an averaging function of production levels: $m^N(t) \triangleq m(x^N(t) + \eta), \eta \in \mathbb{R}^n$. 

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Following [2], the long run average (LRA) mean field (MF) problem is formulated in [37]. Each agent $A_i$, $1 \leq i \leq N$, obtains the positive definite solution to the algebraic Riccati equation
\[
A_i^T \Pi_i + \Pi_i A_i - \Pi_i B_i R^{-1} B_i^T \Pi_i + Q_i = 0.
\] (3)

Moreover, for a given mass tracking signal $x^* \in C_0[0,\infty)$ the mass offset function $s_i(t)$ is generated by the differential equation
\[
-\frac{ds_i(t)}{dt} = A_i^T s_i(t) - \Pi_i B_i R^{-1} B_i^T s_i(t) - Q_i x^*(t), \quad t \geq 0.
\] (4)

Then, the optimal tracking control law [38] is given by
\[
u_i(t) = -R^{-1} B_i^T (\Pi_i x_i(t) + s_i(t)), \quad t \geq 0,
\] (5)

where $u_i(\cdot)$ solves $\inf J_i(u_i, x^*)$, which is defined below by an abuse of notation:
\[
J_i(u_i, x^*) \triangleq \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \|x_i - x^*\|_{Q_i}^2 + \|u_i\|_{P_i}^2 \right\} dt \quad \text{w.p.1.}
\]

Note that the procedure above assumes a given mass tracking signal $x^*$. The equation system to calculate $x^*$ will be given subsequently.

We first define the empirical distribution associated with the first $N$ agents:
\[
F^N_\zeta(\theta) = \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{\theta_i < \theta\}}, \theta \in \mathbb{R}^{n+n+n(n+1)/2}, \quad \text{w.p.1}
\]

with the probability distribution $F_\zeta(\theta)$, parameterized by $\zeta \in P \subset \tilde{P} \subset \mathbb{R}^P$, the population dynamical and cost function distribution parameter such that $P$ is compact and $\tilde{P}$ is an open set. Then we employ the following assumption.

**A5:** There exists a family of distribution functions \( \{F_\zeta(\theta); \theta \in \Theta\}, \zeta \in \tilde{P} \), such that $F^N_\zeta(\cdot) \to F_\zeta(\cdot)$ w.p.1 weakly on $\theta \in \Theta$ and uniformly over $\zeta \in P$ as $N \to \infty$.

Each agent solves the equation system below to calculate the mass tracking signal $x^*(\tau, \zeta)$, $t_0 \leq \tau < \infty$, offline, for an infinite population of agents.

**Definition 2.1:** Mean Field (MF) Equation System on $[0, \infty)$:
\[
-\frac{ds_\theta}{d\tau} = (A_\theta^T - \Pi_\theta B_\theta R^{-1} B_\theta^T) s_\theta - Q_\theta x^*(\tau, \zeta),
\]
\[
\frac{d\bar{x}_\theta}{d\tau} = (A_\theta - B_\theta R^{-1} B_\theta^T \Pi_\theta) \bar{x}_\theta - B_\theta R^{-1} B_\theta^T s_\theta,
\]
\[
\bar{x}(\tau, \zeta) = \int_{\Theta} \bar{x}_\theta dF_\zeta(\theta),
\]
\[
x^*(\tau, \zeta) = m(\bar{x}(\tau, \zeta) + \eta), \quad t_0 \leq \tau < \infty.
\] (6)

Under **A1-A4**, the MF Equation System admits a unique bounded solution [2].

**The Global Observation Control Set $U^N_\zeta$**:

For the optimality analysis, we first introduce the global observation control set. The set of control inputs $U^N_\zeta$ consists of all feedback controls adapted to \( \{\theta_j, 1 \leq j \leq N; F_\zeta(\theta); F^N_t, t \geq 0\} \), where $F^N_t$ is the $\sigma$-field generated by the set \( \{x_j(\tau); 0 \leq \tau \leq t, 1 \leq j \leq N\} \).
The Local Observation Control Set $U^{N}_{i,t}$: The local observation control set of agent $A_i$ is the set of control inputs $U^{N}_{i,t}$ which consists of the feedback controls adapted to the set $\{\theta_i; F_{i}(\theta); \mathcal{F}_{i,t}, t \geq 0\}$. The $\sigma$-field $\mathcal{F}_{i,t}$ is generated by $(x_i(\tau); 0 \leq \tau \leq t)$, and $\mathcal{F}^N_i$ is the $\sigma$-field generated by the set $\{x_j(\tau); 0 \leq \tau \leq t, 1 \leq j \leq N\}$.

**Theorem 2.1: Non-Adaptive MF Stochastic Control (SC) Theorem** [37, following [2]]

Let A1-A5 hold. The MF Stochastic Control Law (5) generates a set of controls $U^{N}_{MF} \triangleq \{u^0_i; 1 \leq i \leq N\}$, $1 \leq N < \infty$, with

$$u^0_i(t) = -R^{-1}B_i^T(\Pi_i x_i(t) + s_i(t)), \quad t \geq 0, \quad (7)$$

such that

(i) the MF equations (6) have a unique solution;

(ii) all agent system trajectories $x_i, 1 \leq i \leq N$, are $LRA - L^2$ stable w.p.1;

(iii) $\{U^{N}_{MF}; 1 \leq N < \infty\}$ yields an $\epsilon$-Nash equilibrium for all $\epsilon > 0$, i.e., for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \geq N(\epsilon)$

$$J^N_i(u^0_i, u^0_{-i}) - \epsilon \leq \inf_{u_i \in U^{N}_i} J^N_i(u_i, u^0_{-i}) \leq J^N_i(u^0_i, u^0_{-i}).$$

Conceptually, Theorem 2.1 may be paraphrased to say that individual competitive actions against the mass effect collectively produce the mass behaviour, and hence the $\epsilon$-Nash equilibrium is obtained. In the proof of Theorem 2.1, the results are first established for an infinite population and then are shown to be approximated by a large finite population with the approximation error decaying to zero as the population size goes to infinity; it is this which gives the $\epsilon$-Nash property.

**B. MF Stochastic Adaptive Control (SAC)**

In this section we first present the identification schemes to be used by each agent under the MF Stochastic Adaptive Control (SAC) Law to estimate both the self dynamical parameters and the population dynamical and cost function distribution parameter. In other words, the analysis concerns a family of agents $A_i, 1 \leq i \leq N$, whose control action at any instant is not permitted to be an explicit function of the self dynamical parameters $[A_i, B_i]$ and the dynamical and cost function distribution parameter $\zeta$. At time $t \geq 0$, the self dynamical parameters are estimated from the input-output sample path $\{x_i(\tau), u_i(\tau); 0 \leq \tau \leq t\}$ of $A_i$; in other words, each agent $A_i$ performs the identification based upon observations of its own trajectory. The distribution parameter $\zeta$ is estimated from observations $\{x_j(\tau), u_j(\tau); 0 \leq \tau \leq t, j \in Obs_i(N)\}$ on a random subset of agents $Obs_i(N)$ where $|Obs_i(N)| \rightarrow \infty$, and $|Obs_i(N)|/N \rightarrow 0$ as $N \rightarrow \infty$.

**The Adaptive Agent Control Set $U^{N}_{a,i}$:** We next define the set of control inputs $U^{N}_{a,i}$, the admissible control set of an adaptive agent $A_i$, which consists of all feedback controls adapted to the set $\{\mathcal{F}_{i,t}, \mathcal{F}_{i,t}^{obs}, t \geq 0; Q_i\}$. The $\sigma$-field $\mathcal{F}_{i,t}$ is generated by the agent’s own trajectory and control input, $\{x_i(\tau), u_i(\tau); 0 \leq \tau \leq t\}$, and $\mathcal{F}_{i,t}^{obs}, t \geq 0$, is the observation $\sigma$-field generated by the trajectories and control inputs in the set $Obs_i(N), \{x_j(\tau), u_j(\tau); 0 \leq \tau \leq t, j \in Obs_i(N)\}$. For definiteness in this paper, the identification algorithms employed are recursive weighted least
squares (RWLS) for the self dynamical parameter identification and maximum likelihood estimation (MLE) for the distribution parameter identification. However, any identification scheme which generates consistent estimates w.p.1 (subject to the given hypotheses) will also yield the system asymptotic equilibrium properties to be established.

1) **Self Dynamical Parameter Identification (SDPI):** We denote the self estimate of the matrix $\theta_i$ by $\hat{\theta}_{i,t} = [\hat{A}_{i,t}, \hat{B}_{i,t}, Q_i]$, $t \geq 0$, and the estimate of $\zeta$ by $\hat{\zeta}_{i,t}$, $t \geq 0$, where $N_0 \triangleq |\text{Obs}_i(N)|$, and assume $\hat{\theta}_{i,t}$ and $\hat{\zeta}_{i,t}$ are generated at $t \geq 0$ by the identification algorithm. Note that the self cost function parameter $Q_i$ is in the information set of agent $A_i$, and is therefore not to be estimated. We adopt the notation $\zeta^0 \triangleq \zeta$, $\theta^0 \triangleq \theta$ for the true parameters in the system. At time $t \geq 0$, agent $A_i$ solves the RWLS equations with the measurement variable set as $dx_t$ with the regression vector $[x_t^T, u_t^T]$ in order to obtain the estimates $[\hat{A}_{i,t}, \hat{B}_{i,t}].$ To ensure controllability and observability of the estimates, a projection method is used; the estimates are projected onto the compact set $\Theta_{Q_i} \subset \hat{\Theta}_{Q_i}$, where given $Q_i$, $[A_\theta, B_\theta]$ is controllable and $[Q_i^{1/2}, A_\theta]$ is observable. Note that $\Theta$ is known to all agents in the system.

2) **Population Dynamical and Cost Function Distribution Parameter Identification:**

a) **Population Dynamical Parameter Identification (PDPI):** At $t \geq 0$, agent $A_i$ estimates dynamical parameters $\{[\hat{A}_{j,t}, \hat{B}_{j,t}]; j \in \text{Obs}_i(N)\}$ of the agents in its observation set, $\text{Obs}_i(N)$. The admissible control set of agent $A_i$ is $U_{Ai}$, consisting of observations of the trajectories and control inputs of all the agents in the set $\text{Obs}_i(N).$ Based upon this observation set, agent $A_i$ obtains estimates $\{[\hat{A}_{j,t}, \hat{B}_{j,t}]; j \in \text{Obs}_i(N)\}$ solving the RWLS equations using $\{dx_{j,t}; j \in \text{Obs}_i\}$ as the measurement variable with the regression vector $\{x_{j,t}^T, u_{j,t}^T; j \in \text{Obs}_i\}$.

b) **Population Cost Function Parameter Identification (PCPI):** The solution to the RWLS equations with the inputs described above generates the estimates $\{[\hat{A}_{j,t}, \hat{B}_{j,t}]; j \in \text{Obs}_i(N)\}$. The objective at this point for each agent is to obtain the estimates $\{\hat{Q}_{j,t}; j \in \text{Obs}_i(N)\}$. The RWLS equations are then solved employing the observed control inputs $\{u_j(t); j \in \text{Obs}_i(N)\}$ such that agent $A_i$ calculates $\{- (\hat{B}_{j,t}^T)^{-1} R u_j(t); j \in \text{Obs}_i(N)\}$ and sets as the measurement vector. Note that one needs the following additional assumption.

A6*: $B_\theta$ is invertible (and hence, necessarily, $[A_\theta, B_\theta]$ is controllable) for all $\theta \in \Theta.$

This rather restrictive assumption is only needed for the cost function parameter identification; therefore, PCPI will be given as an optional procedure in the MF-SAC Law. The observed control action is in the form (7); therefore arranging the variables in a certain way to be specified later, agent $A_i$ obtains the estimates $\{\hat{Q}_{j,t}; j \in \text{Obs}_i(N)\}$. Solving the algebraic Riccati equation for $\hat{Q}_{j,t}$ agent $A_i$ obtains its estimates $\{\hat{Q}_{j,t}; j \in \text{Obs}_i(N)\}$. The symmetry of $\{\hat{Q}_{j,t}; j \in \text{Obs}_i(N)\}$ is guaranteed. To ensure the positive definiteness of the obtained estimates $\{\hat{Q}_{j,t}; j \in \text{Obs}_i(N)\}$, $[A, B]$ controllability, $[Q^{1/2}, A]$ observability, and that the requirement in A3 holds, the set $\{\hat{A}_{j,t}, \hat{B}_{j,t}, \hat{Q}_{j,t}, j \in \text{Obs}_i(N)\}$ is projected onto $\Theta$.

c) **Distribution Parameter Identification (DPI):** Once the projected estimates $\hat{\theta}^{[1:N_0]}_{i,t} \triangleq [\hat{A}_{j,t}, \hat{B}_{j,t}, \hat{Q}_{j,t}, j \in \text{Obs}_i(N)], N_0 = |\text{Obs}_i(N)|,$ are obtained, agent $A_i$ forms the scaled log-likelihood-type function

$$L(\hat{\theta}^{[1:N_0]}_{i,t}; \zeta) \triangleq - \frac{1}{N_0} \log \left( \prod_{j \in \text{Obs}_i(N)} f_\zeta(\hat{\theta}_{j,t}) \right).$$
calculates \( \hat{\zeta}_{i,t}^{N_0} \), the estimate of the distribution parameter, solving \( \arg \min_{\zeta} L(\hat{\theta}_{i,t}^{[1:N_0]}, \zeta) \). Note that \( P \) is known to all agents in the system.

Overall using the identification procedures explained above, agent \( A_i \) obtains estimates \( \hat{\mathbf{A}}_{i,t}, \hat{\mathbf{B}}_{i,t} \) and \( \hat{\zeta}_{i,t}^{N_0} \) and forms the self estimated dynamical parameter vector \( \hat{\theta}_{i,t}^{T} = [\hat{\mathbf{A}}_{i,t}, \hat{\mathbf{B}}_{i,t}, \hat{\zeta}_{i,t}^{N_0}] \).

3) Certainty Equivalence Adaptive Control: At time \( t \), employing \( \hat{\zeta}_{i,t}^{N_0} \) agent \( A_i \) solves the MF Equation System (6) to obtain \( x^*(\tau, \hat{\zeta}_{i,t}^{N_0}) \), \( t \leq \tau < \infty \). Then using \( \hat{\theta}_{i,t} \) agent \( A_i \) solves the Riccati equation (3), obtains \( \hat{\mathbf{P}}_{i,t} \equiv \mathbf{P}(\hat{\theta}_{i,t}) \) and solves the mass offset differential equation (4) to obtain \( \hat{s}_i(t) \equiv s(t; \hat{\theta}_{i,t}, \hat{\zeta}_{i,t}^{N_0}) \). The certainty equivalence adaptive control for the admissible control set \( U_{a,i}^{N} \) is then given by \( \hat{u}_i^0(t) \equiv u_i^0(t; \hat{\theta}_{i,t}, \hat{\zeta}_{i,t}^{N_0}) = -R^{-1} \hat{\mathbf{B}}_{i,t}^{T} (\hat{\mathbf{P}}_{i,t} x_i(t) + \hat{s}_i(t)), t \geq 0. \)

To obtain the main MF-SAC result stated in Theorem 2.2, we first establish the strong consistency for the family of estimates \( \{ \hat{\theta}_{i,t}; t \geq 0, 1 \leq i \leq N \} \) and \( \{ \hat{\zeta}_{i,t}^{N_0}; t \geq 0, 1 \leq i \leq N \} \).

4) Control Excitation for Consistent Identification: In order to generate a consistent sequence of estimates \( (\hat{\theta}_{i,t}; t \geq 0) \) w.p.1, a diminishing excitation is added to the adaptive control in (5) to give

\[
\hat{u}_i^0(t) = -R^{-1} \hat{\mathbf{B}}_{i,t}^{T} (\hat{\mathbf{P}}_{i,t} x_i(t) + \hat{s}_i(t)) + \xi_k [e_i(t) - e_i(k)], \quad t \in (k, k+1], \quad k \in \mathbb{N}, \quad 1 \leq i \leq N, \quad (8)
\]

where \( \hat{u}_i^0(0) = 0, (\xi_k^2 = \log k/\sqrt{k}; k \in \mathbb{Z}_+) \), and the process \( (e(t), t \geq 0) \) is an \( \mathbb{R}^m \)-valued standard Wiener process that is independent of \( (u_i(t); t \geq 0) \). The sequence of random processes \( (e(t) - e(k)); t \in (k, k+1], k \in \mathbb{N} \) is assumed to be mutually independent and all members of the set have the same probability law on \( (0,1] \). Since the sequence \( (\xi_k; k \in \mathbb{N}) \) converges to zero at a suitable rate, it will be established following [26] that the diminishing control excitation \( (\xi_k[e(t) - e(k)]; t \in [0,1], k \in \mathbb{N}) \) provides sufficient excitation for almost sure consistent identification and decreases sufficiently rapidly enough not to affect the limiting performance of the system with respect to \( \hat{\theta}_{i,t} = \theta_i; t \geq 0 \), i.e. the non-adaptive case. In other words, the asymptotic performance achieved is equal to the one obtained in the non-adaptive case almost surely. The diminishing control excitation (8) was introduced in [27], [28], and it was shown in [26] to generate strongly consistent parameter estimates via RWLS for dynamical parameters of the system (1) under certainty equivalence adaptive control.

C. The MF Stochastic Adaptive Control (SAC) Law

We observe that the control law (8) has three terms computed from the local state information, the self dynamical parameter estimates and the population distribution parameter estimate. It can be written for each agent \( A_i \), \( 1 \leq i \leq N \), in the form of \( u_i^0(t; \hat{\theta}_{i,t}, \hat{\zeta}_{i,t}^{N_0}) = u_i^0(t; \hat{\theta}_{i,t}) + u_i^{locl}(t; \hat{\theta}_{i,t}, \hat{\zeta}_{i,t}^{N_0}) + u_i^{dit}(t), t \geq 0 \), where \( u_i^{locl}(\cdot) \) is the LQG feedback for the system of agent \( A_i \) based on local information; \( u_i^{pop}(\cdot) \) is the mass offset term based on local information and \( population \) information received from the observed set; and \( u_i^{dit}(\cdot) \) is the locally generated \( dither \) input. In this section we present the MF-SAC Law which generates the feedback control law \( \hat{u}_i^0(t) \equiv u_i^0(t; \hat{\theta}_{i,t}, \hat{\zeta}_{i,t}^{N_0}), t \geq 0 \), that leads to the \( \epsilon \)-Nash equilibrium. The continuous time MF-SAC Law for agent \( A_i \), \( 1 \leq i \leq N \), with parameter \( \theta_i \in \Theta, 1 \leq i \leq N \), is summarized in three major steps in Table I.
Specification of the MF-SAC Law

For agent $A_i, t \geq 0$:

(i) Self parameter $\hat{\theta}_i$ identification:

Solve the RWLS equations (9) for the dynamical parameters:

$$
\begin{align*}
    u_i^T &= [\hat{A}_i, \hat{B}_i], \\
    \psi_i^T &= [x_i^T, u_i^T],
\end{align*}
$$

$$
\begin{align*}
    du_i &= a(t)\Psi_t\psi_i \left[ dx_i - \psi_i^T u_t dt \right], \\
    d\Psi_t &= -a(t)\Psi_t\psi_i \psi_i^T \Psi_t dt,
\end{align*}
$$

and calculate $v_i^p = \arg\min_{\psi_i \in \Theta_i} \| u_i - \psi_i \|, \hat{\theta}_i = [v_i^T, Q_i]$. 

(ii) Population-parameter $\zeta_{N_0}$ identification:

(a) Solve the RWLS equations (9) for the dynamical parameters $\{\hat{A}_{j,t}, \hat{B}_{j,t}, j \in Obs_i(N)\}$.

(b) Solve the RWLS equations (11) for the population-parameter $\hat{\zeta}_{N_0}$: (subscript $j$ suppressed for clarity)

$$
\begin{align*}
    u_i^T &= [\hat{\Pi}_i, \hat{s}_i(t)], \\
    \psi_i^T &= [x_i^T, 1],
\end{align*}
$$

$$
\begin{align*}
    du_i &= a(t)\Psi_t\psi_i \left[ -\left(\hat{B}_i^T\right)^{-1}Ru_t \right] - \psi_i^T u_t, \\
    d\Psi_t &= -a(t)\Psi_t\psi_i \psi_i^T \Psi_t dt,
\end{align*}
$$

solve the algebraic Riccati Equation (12) for $\hat{\theta}_{j,t}$, and calculate

$$
\hat{\theta}_{j,t}^p = \arg\min_{\theta_{j,t}} \| \theta_{j,t} - \psi \|.
$$

(c) Solve the MLE equation (14) at $\hat{\theta}_{i,t}^{[1:N_0]} = [\hat{A}_{j,t}, \hat{B}_{j,t}, \hat{Q}_{j,t}]$, $j \in Obs_i(N)$, to estimate $\zeta^0$ via:

$$
\begin{align*}
    L(\hat{\theta}_{i,t}^{[1:N_0]}, \zeta) &= -\frac{1}{N_0} \log \left( \prod_{j \in Obs_i(N)} f_{\zeta}(\hat{\theta}_{j,t}) \right), \\
    \zeta_{N_0} &= \arg\min_{\zeta \in \mathbb{P}} L(\hat{\theta}_{i,t}^{[1:N_0]}, \zeta), \quad N_0 = |Obs_i(N)|,
\end{align*}
$$

and solve the set of MF Equations (6) for all $\theta \in \Theta$ generating $x^* (\tau, \zeta_{N_0}), t \leq \tau < \infty$.

(iii) Solve the MF Control Law Equation at $\hat{\theta}_{i,t}^p$ and $\zeta_{N_0}$:

(a) $\hat{\Pi}_{i,t}$: Solve the Riccati equation (15) at $\hat{\theta}_{i,t}^p$:

$$
\hat{A}_{i,t}^T\hat{\Pi}_{i,t} + \hat{\Pi}_{i,t}\hat{A}_{i,t} - \hat{\Pi}_{i,t}\hat{B}_{i,t}R^{-1}\hat{B}_{i,t}^T\hat{\Pi}_{i,t} + Q_i = 0.
$$

(b) $\hat{s}_i(t) \equiv s(t; \hat{\theta}_{i,t}^p, \zeta_{N_0})$: Solve the mass offset differential equation (16) at $\hat{\theta}_{i,t}^p$ and $\zeta_{N_0}$:

$$
\frac{d\hat{s}_i(t)}{dt} = (\hat{A}_{i,t}^T \hat{\Pi}_{i,t} + \hat{\Pi}_{i,t} \hat{A}_{i,t} - \hat{\Pi}_{i,t} \hat{B}_{i,t} R^{-1} \hat{B}_{i,t}^T \hat{\Pi}_{i,t} + Q_i) \hat{s}_i(t) - Q_i x^*(\tau, \zeta_{N_0}), \quad t \leq \tau < \infty.
$$

(c) Obtain the Certainty Equivalence Adaptive Control at $\hat{\theta}_{i,t}^p$ and $\zeta_{N_0}$:

$$
\hat{u}_i(t) = -R^{-1} \hat{B}_i^T \left( \hat{\Pi}_{i,t} x_i(t) + \hat{s}_i(t) \right) + \varepsilon \left[ x_i(t) - e_i(k) \right], \quad t \in (k, k + 1], \quad k \in \mathbb{N}.
$$

| TABLE I |
| --- |
| MF-SAC LAW |
The function $a(t), t \geq 0$, in (9) is in the form of $a(t) = 1/f(r(t))$, where $r(t) = \|\Psi_0^{-1}\| + \int_0^t |\psi(s)|^2 ds$, and $f \in \{f : \mathbb{R}_+ \to \mathbb{R}_+, f$ is slowly increasing and $\int_0^\infty 1/(xf(x))dx < \infty; c \geq 0\}$. The function $f(\cdot)$ is slowly increasing if it is increasing and satisfies $f(\cdot) \geq 1$ and $f(x^2) = O(f(x))$ [26].

Note that a positive definite solution to the Riccati equation (15) exists as the projected estimate is in the set of controllable and observable dynamical parameters: $\hat{\theta}_i^{pr} \in \Theta \subset \tilde{\Theta} \subset \mathbb{R}^{n(n+m+(n+1)/2)}$.

D. Asymptotic Properties of the MF-SAC Law

A key feature of the work in this paper is that the state aggregation integration in (6) is performed by use of the estimated distribution $F_{\hat{X}N_{\theta}\theta}(\cdot)$ in place of the true distribution $F_{X_{\theta}\theta}(\cdot)$ (see (18) below). Then the central results of this paper are the following: under the MF-SAC Law, asymptotically as the population tends to infinity, the competitive best response actions of the adaptive agents with no prior information on self dynamical parameters and no prior statistical information on dynamical and cost function parameters of the mass give rise to a unique Nash equilibrium. Moreover, the resulting cost for each agent from the MF-SAC Law is asymptotically almost surely equal to the cost resulting from the non-adaptive MF Stochastic Control Law.

**Theorem 2.2: MF-SAC Theorem**

Let A1-A5, A7, A8 hold. Then, assume each agent $A_i, 1 \leq i \leq N$, is such that it:

(i) observes a random subset $\text{Obs}_i(N)$ of the total population $N$ such that $|\text{Obs}_i(N)| \to \infty$, $|\text{Obs}_i(N)|/N \to 0$, as $N \to \infty$;

(ii) estimates its own parameter $\hat{\theta}_{i,t}$ via the RWLS (9);

(iii) estimates the population dynamical and cost function distribution parameter $\hat{\zeta}_{i,t}^{N_{\theta}}$ via MLE (14); and

(iv) computes $u_i^0(t; \hat{\theta}_{i,t}, \hat{\zeta}_{i,t}^{N_{\theta}})$ via the extended MF equations plus dither.

Then,

(a) $\hat{\theta}_{i,t} \to \theta_i^0$ w.p.1 as $t \to \infty$, $1 \leq i \leq N$ (strong consistency);

(b) $\hat{\zeta}_{i,t}^{N_{\theta}} \to \zeta^0$ w.p.1 as $t \to \infty$, and $N \to \infty$, $1 \leq i \leq N$.

The MF-SAC Law generates a set of controls $\hat{U}_{MF}^N = \{\hat{\omega}_i^0; 1 \leq i \leq N\}, 1 \leq N < \infty$, such that:

(c) all agent system trajectories $x_i, 1 \leq i \leq N$, are LRA – $L^2$ stable w.p.1;

(d) $\epsilon$–Nash Property: $\{\hat{U}_{MF}^N; 1 \leq N < \infty\}$ yields an $\epsilon$-Nash Equilibrium for all $\epsilon$, i.e., for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \geq N(\epsilon)$

$$J_i^N(\hat{\omega}_i^0, \hat{\omega}_{-i}^0) - \epsilon \leq \inf_{u_i \in \mathcal{U}_i^0} J_i^N(u_i, \hat{\omega}_{-i}^0) \leq J_i^N(\hat{\omega}_i^0, \hat{\omega}_{-i}^0) \quad \text{w.p.1,} \quad 1 \leq i \leq N;$$

(e) Equal Adaptive and Non-adaptive ($\theta^0, \zeta^0$) MF Equilibrium Performance:

$$\lim_{N \to \infty} J_i^N(\hat{\omega}_i^0, \hat{\omega}_{-i}^0) = \lim_{N \to \infty} J_i^N(u_i^0, u_{-i}^0) \quad \text{w.p.1,} \quad 1 \leq i \leq N;$$

(f) Adaptive Control Performance Equals Complete Information Performance:

$$\lim_{N \to \infty} J_i^N(\hat{\omega}_i^0, \hat{\omega}_{-i}^0) = \lim_{N \to \infty} \inf_{u_i \in \mathcal{U}_i^0} J_i^N(u_i, \hat{\omega}_{-i}^0) \quad \text{w.p.1,} \quad 1 \leq i \leq N.$$
The proof consists of the unification of the principal Theorems 3.2, 3.3, 4.2 and the Propositions 4.4 and 4.5 that are presented in the remaining sections. The outline of the proof is given in Appendix D.

The technical plan of the paper is presented in three layers. The main theorem of the paper is Theorem 2.2. In the first layer, Propositions 4.4, 4.5 and Theorems 3.2, 3.3 and 4.2 support Theorem 2.2. In the second layer, Lemmas D.1, D.2, D.3, D.4, D.5, Theorem 4.3 and Proposition C.1 support Proposition 4.4 whereas Lemma 3.1 supports Theorem 3.2. In the third layer, Lemmas A.1, A.2, A.3 and Proposition 4.1 support Theorem 4.3.

### III. Convergence Properties of the MF-SAC Parameter Estimates

We show that for self dynamical parameter identification, the RWLS equations for dynamical parameters (9) with the projection method (10) provide strongly consistent, uniformly controllable and observable estimates. The population dynamical and cost function distribution parameter identification is handled in three steps. First, each agent obtains the dynamical parameter estimates for the agents in its observation set solving the RWLS equations (9). It is shown that the RWLS equations (9) with the projection method (10) applied on the observed agents’ controlled trajectories also provide strongly consistent, uniformly controllable and observable estimates. Secondly, another set of RWLS equations (11) are solved using the previously obtained dynamical parameter estimates as inputs; and finally cost function parameter estimates are obtained for the agents in the observation set (12). We show that the estimates obtained are positive definite and uniformly bounded by use of a projection method (13). Finally, we show that the MLE scheme (14) employed using these estimates provides strongly consistent population distribution parameter estimates.

#### A. Asymptotic Convergence of the Dynamical Parameter Estimates

The RWLS algorithm is self-convergent [25], i.e., it converges to a certain random vector almost surely irrespective of the control law design, but there is no guarantee that the estimated dynamical parameters will be controllable and observable, or the cost function estimates will be positive definite. To ensure that the sequence of estimated dynamical parameters are controllable, observable, uniformly bounded and the sequence of estimated cost parameters are positive definite and uniformly bounded we use the projection method [23].

For self dynamical parameter identification, the self dynamical parameter estimates with the cost function parameter $Q_i \in \mathbb{R}^{n(n+1)/2}$, $\hat{\theta}_{1:t}^Y = [\hat{A}_{1:t}, \hat{B}_{1:t}, Q_i] \in \mathbb{R}^{n(n+m+(n+1)/2)}$, $t \geq 0$, ($Q_i$ known by agent $A_i$) is projected (denoted by $\hat{\theta}_{1:t}^{pr}$ in (10)) onto the compact set $\Theta_{|Q_i} \subset \hat{\Theta}_{|Q_i}$, where for the given $Q_i$, $[A_\theta, B_\theta]$ is controllable and $[Q_i^{1/2}, A_\theta]$ is observable.

For the distribution parameter identification, the population dynamical parameter estimates together with the cost function parameter estimates are projected onto the compact subset $\Theta$ of the set of controllable and observable dynamical parameters $\hat{\Theta}$ where, in addition, $Q_\theta, \hat{\theta} \in \hat{\Theta}$, is positive definite (for which the control law generated by (15) necessarily exists and is asymptotically stabilizing).
Lemma 3.1: Let $\Theta$ be a compact set such that $\theta_0^i \in \Theta \subset \dot{\Theta} \subset \mathbb{R}^{n(n+m+(n+1)/2)}$, $1 \leq i \leq N$. Set $\hat{\theta}^i_{t,i} = [\hat{A}_{i,t}, \hat{B}_{i,t}, \hat{Q}_{i,t}]^T$, $t \geq 0$. Let $[\hat{A}_{i,t}, \hat{B}_{i,t}, \hat{Q}_{i,t}]^T$ be the estimate of $[A^0_i, B^0_i]$, obtained by the RWLS equations (9), and let $\hat{Q}_{i,t}$ be the estimate of $Q^0_i$ obtained by the RWLS equations (11) and (12). Assume $\hat{\theta}^i_{t,i} \to \theta_0^i$ w.p.1 as $t \to \infty$, $1 \leq i \leq N$. Then, $\hat{\theta}^{pr}_{i,t} \equiv [A^{pr}_{i,t}, B^{pr}_{i,t}, Q^{pr}_{i,t}] \equiv \arg \min_{\psi \in \Theta} \| \theta - \psi \|$ (together with a coordinate ordering measurable tie breaking rule), satisfies $\hat{\theta}^{pr}_{i,t} \in \Theta$ w.p.1 for all $t \geq 0$, and $\hat{\theta}^{pr}_{i,t} \to \theta_0^i$ w.p.1 as $t \to \infty$. In the SDP case the corresponding result is achieved by setting $\hat{Q}_{i,t} = Q^0_i$ for all $t \geq 0$.

The Lemma is proved in Appendix B.

Now, given the projection method lemma, we show that the RWLS equations for dynamical parameters (9) and the RWLS equations for cost function parameters (11) generate strongly consistent estimates.

Theorem 3.2: Let hypotheses A1-A3 hold, $x^* \in C_b[0,\infty)$, and let $([\hat{A}_{i,t}, \hat{B}_{i,t}, \theta_0^i]; t \geq 0)$, $1 \leq i \leq N$, be the process of estimates obtained by the RWLS equations (9), and $(\hat{Q}_{i,t}; t \geq 0)$ be the process of estimates obtained by (12) along the controlled trajectory $((x_{i,t}, \hat{u}_{i,t}^0); t \geq 0)$, generated by the control $(\hat{u}_{i,t}^0; t \geq 0)$ according to the MF-SAC Law (17). Furthermore, let $(\hat{\theta}_{i,t}^{pr} = [\hat{A}_{i,t}^{pr}, \hat{B}_{i,t}^{pr}, \hat{Q}_{i,t}^{pr}]; t \geq 0)$, be the projected estimates according to Lemma 3.1. Then,

(i) the input process given in (17) is well defined,
(ii) $[\hat{A}_{i,t}, \hat{B}_{i,t}] \to [A^0_i, B^0_i]$ w.p.1 as $t \to \infty$, $1 \leq i \leq N$,
(iii) with the optional assumption A6’, $\hat{Q}_{i,t} \to Q^0_i$ w.p.1 as $t \to \infty$, $1 \leq i \leq N$.

The theorem is proved in Appendix B using the methodology of [26], which establishes the convergence of the RWLS estimates (9) with diminishing excitation in the controls (17). The required uniform controllability and observability of the estimates is a consequence of Lemma 3.1 since $\hat{\theta}^{pr}_{i,t} \in \Theta$, $t \geq 0$.

B. Asymptotic Convergence of the Population Distribution Parameter Estimates

The MF-SAC Law specifies that the distribution parameter identification is such that each agent $A_i$, $1 \leq i \leq N$, observes the control and state trajectories of a random subset of agents $O_{bs_i}(N)$, $1 \leq i \leq N$, and at each time iteration applies (9) to obtain the dynamical parameter estimates of each agent in its set. The MLE scheme (14) is then applied to these estimated parameters of the agents $O_{bs_i}(N)$, $1 \leq i \leq N$, for $t \geq 0$, to obtain an estimate of the distribution parameter. To obtain the strong consistency of the distribution parameter estimates we adopt the hypotheses A7 and A8 below.

A7: There exists a bounded continuous (on $\Theta \times \hat{P}$) family of densities $f_\zeta \equiv \{ f_\zeta(\theta); \theta \in \Theta, \zeta \in \hat{P} \}$ for the family of dynamical and cost function parameter distributions $\{ F_\zeta(.) ; \zeta \in \hat{P} \}$. Further, the distribution function $f_\zeta(\theta)$ is bounded away from 0 uniformly over $\Theta \times \hat{P}$, i.e., $f_\zeta(\theta) \geq \delta$ for some $\delta > 0$ for all $\theta \in \Theta$ and $\zeta \in \hat{P}$. Moreover, for each $j$, $1 \leq j \leq p$, $(\partial f_\zeta / \partial \zeta_j)(\theta)$ exists for all $\zeta \in \hat{P}$, and is uniformly bounded on $\Theta \times \hat{P}$, except possibly on a Lebesgue null set independent of $\zeta \in \hat{P}$.

For (14), let $f(\theta^{[1:N_0]}; \zeta)$ be the likelihood function of $f_\zeta$ at $\theta^{[1:N_0]} \equiv \{ A_j, B_j, Q_j, j \in O_{bs_i}(N), N_0 = |O_{bs_i}(N)| \}$, and let $L(\theta^{[1:N_0]}; \zeta)$ be the continuously differentiable monotonically decreasing function of $f(\theta^{[1:N_0]}; \zeta)$.

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given by the scaled log-likelihood function \(- (1/N) \log f(\theta^{[1:N_0]}; \zeta)\).

\[ \text{A8: } \{ f_\zeta(\cdot); \; \zeta \in P \} \text{ satisfies:} \]
\[ \mathbb{E}_{\zeta_0}[\log f_\zeta(\theta)] = \mathbb{E}_{\zeta_0}[\log f_{\zeta'}(\theta)] \Leftrightarrow \zeta = \zeta', \]

for all \( \zeta, \zeta', \zeta^0 \in P \), where \( \zeta^0 \) is the true parameter.

**Theorem 3.3:** Let \textbf{A1-A3, A7, A8} hold; let \( |\text{Obs}_i(N)| \to \infty \) and \( |\text{Obs}_i(N)|/N \to 0 \) as \( N \to \infty \), \( 1 \leq i \leq N \), and let \( \hat{\zeta}^i_{N_00} (\theta_N^0) \), \( t \geq 0 \), \( N_0 = |\text{Obs}_i(N)| \), be the MLE process given by (14) along the controlled trajectories of the observed set of agents \( \{(x_{j,t}, \hat{a}_{j,t}^0); \; t \geq 0, \; j \in \text{Obs}_i(N)\} \) generated by the controls \( \hat{a}_{j,t}^0; \; t \geq 0, \; j \in \text{Obs}_i(N) \) (17). Then, \( \hat{\zeta}^i_{N_00} \) is strongly consistent at \( \zeta^0 \), that is, \( \lim_{N \to \infty} \lim_{t \to \infty} \hat{\zeta}^i_{N_00}(\theta_N^0) = \zeta^0 \) w.p.1, \( 1 \leq i \leq N \).

The proof is given in Appendix B.

**IV. THE PRINCIPAL ASYMPTOTIC RESULTS**

**A. Asymptotic Behaviour of the MF Equations**

The MF Equations (6) that permit the calculation of the mass tracking signal \( x^*(\tau, z), \; t_0 \leq \tau < \infty \), are dependent on the population distribution parameter \( \zeta \). Correspondingly, the MF Equations of the MF-SAC Law on \([t, \infty), \; t \geq 0\), with the strongly consistent distribution parameter estimate \( \hat{\zeta}^i_{N_00}, \; t \geq 0, \; 1 \leq i \leq N \), are given below.

**Definition 4.1:** MF-SAC Equation System on \([t, \infty)\):

\[
\begin{align*}
- \frac{ds_\theta}{d\tau} &= (A_\theta^\top - \Pi_\theta B_\theta R^{-1}B_\theta^\top) s_\theta - Q_\theta x^*(\tau, \hat{\zeta}^i_{N_00}), \\
\frac{d\bar{x}_\theta}{d\tau} &= (A_\theta - B_\theta R^{-1}B_\theta\Pi_\theta) \bar{x}_\theta - B_\theta R^{-1}B_\theta^\top s_\theta, \\
\bar{x}(\tau, \hat{\zeta}^i_{N_00}) &= \int \bar{x}_\theta \, dF_{\zeta^0_i}(\theta), \\
x^*(\tau, \hat{\zeta}^i_{N_00}) &= m(\bar{x}(\tau, \hat{\zeta}^i_{N_00}) + \eta), \quad t \leq \tau < \infty.
\end{align*}
\]

**Proposition 4.1:** For the system (1) let \textbf{A1-A4, A7, A8} hold. For agent \( \bar{A}_i, \; 1 \leq i \leq N \), let: (i) \( \hat{\theta}_{i,t}^{pr} \) be the solution to (10), \( \hat{\zeta}^i_{N_00} \) be the solution to (14) in the MF-SAC Law; (ii) \( x^*(\tau, \hat{\zeta}^i_{N_00}), \; t \leq \tau < \infty \), be the solution to the MF-SAC Equation System (18); \( x^*(\tau, \zeta^0), \; t \leq \tau < \infty \), be the solution to the MF Equation System (6); (iii) \( s(t; \hat{\theta}_{i,t}^{pr}, \hat{\zeta}^i_{N_00}) \) be the solution to (16) in the MF-SAC Law; and \( s(t; \theta^0_i, \zeta^0) \) be the solution to the mass offset function differential equation (4). Then,

(i) \( \lim_{N \to \infty} \lim_{t \to \infty} x^*(\tau, \hat{\zeta}^i_{N_00}) = x^*(\tau, \zeta^0) \) w.p.1, \( t \leq \tau < \infty, \; 1 \leq i \leq N \),

(ii) \( \lim_{N \to \infty} \lim_{t \to \infty} s(t; \hat{\theta}_{i,t}^{pr}, \hat{\zeta}^i_{N_00}) = s(t; \theta^0_i, \zeta^0) \) w.p.1, \( 1 \leq i \leq N \),

(iii) The input process given in (17) is well defined and is given at \( \hat{\theta}_{i,t}^{pr} \) and \( \hat{\zeta}^i_{N_00} \) by

\[ u^0_i(t; \hat{\theta}_{i,t}^{pr}, \hat{\zeta}^i_{N_00}) = -R^{-1}B_{i,t}^\top (\bar{\Pi}_{i,t} x_{i,t} + s(t; \hat{\theta}_{i,t}^{pr}, \hat{\zeta}^i_{N_00})) + \xi_k [\epsilon_i(t) - \epsilon_i(k)]. \]

The result is proved in Appendix C.
B. Asymptotic Behaviour of System Trajectories

We show that under the hypotheses that the self-dynamical parameter estimates and the population distribution parameter estimates converge to their true values, the trajectories of adaptive individual agents are stable in the $L^2 - \text{LRA}$ sense. Moreover, these trajectories and the corresponding control actions converge to the non-adaptive values obtained with the true parameters.

Recall that $\hat{U}_{MF}^N = \{u_i^0; 1 \leq i \leq N\}$ is the set of controls generated by the non-adaptive MF Stochastic Control Law, while $\hat{U}_{MF}^N = \{\hat{u}_i^0; 1 \leq i \leq N\}$ is the set of controls generated by the MF-SAC Law.

Using the notation $\hat{\theta}_{i,t}^0 \triangleq (\hat{\theta}_{i,t}, 0 \leq \tau \leq t)$, and $\hat{\zeta}_{i,t}(N_0) \triangleq (\hat{\zeta}_{i,t}^N, 0 \leq \tau \leq t)$, let $\hat{x}_i^0 \triangleq x_i^0(t; \hat{\theta}_{i,t}^0, \hat{\zeta}_{i,t}^N)$ be the state trajectory of agent $A_i$, $1 \leq i \leq N$, under the control law $u_i^0(t; \hat{\theta}_{i,t}^0, \hat{\zeta}_{i,t}^N) \in \hat{U}_{MF}^N$, and $\hat{x}_i^0 \triangleq x_i(t; \theta_i^0, \zeta_i^0)$ be the state trajectory of agent $A_i$ under the control law $u_i^0 \triangleq u_i^0(t; \theta_i^0, \zeta_i^0) \in \hat{U}_{MF}^N$, where $\hat{\theta}_{i,t}$ is the solution to (10), and $\hat{\zeta}_{i,t}^N$ is the solution to (14).

**Theorem 4.2:** Let A1-A4 hold; then, the process $(\hat{x}_i^0(t); t \geq 0)$, $1 \leq i \leq N$, is stable in the sense that

$$\sup_{N \geq 1} \max_{1 \leq i \leq N} \lim_{T \to \infty} \frac{1}{T} \int_0^T \|\hat{x}_i^0(t)\|^2 dt < \infty \quad \text{w.p.1.}$$

**Proof:** It has been shown in Theorem 3.2 that $\hat{\theta}_{i,t} \to \theta_i^0$ w.p.1, and in Theorem 3.3 that $\hat{\zeta}_{i,t}^N \to \zeta_i^0$ w.p.1 as $t \to \infty$ and $N \to \infty$, $1 \leq i \leq N$. Moreover, it has already been shown in Proposition 4.1 that the tracking signal $x^*(\tau, \hat{\zeta}_t^N) \in C_b[0, \infty)$, and the input process is well defined. All the hypotheses in [26] are satisfied, and Theorem 1 in [26] proves the claim.

**Theorem 4.3:** For the system (1), under A1-A4, A7, A8

$$\lim_{N \to \infty} \lim_{T \to \infty} \frac{1}{T} \int_0^T \|\hat{x}_i^0 - \hat{x}_i^0\|^2 dt = 0 \quad \text{w.p.1, } 1 \leq i \leq N.$$

The result is proved in Appendix C.

C. Asymptotic Behaviour of Cost Functions

In the population limit, the asymptotic cost of an agent performing the MF-SAC Law in a system within which all of the agents are adaptive is almost surely equal to the cost of an agent in a system of agents all of which are performing the non-adaptive MF-SC Law. This is shown in Proposition 4.4 whose proof is given in Appendix D. Moreover, Proposition 4.5 shows that in the population limit, the best response of an agent in a population of agents performing the MF-SAC Law is almost surely equal to the best response of an agent in a population of agents performing the non-adaptive MF-SC Law. The proof is given in Appendix D.

**Proposition 4.4:** For the system (1), let A1-A4, A7, A8 hold, let $u_i^0 \in \hat{U}_{MF}^N$, $1 \leq i \leq N$, be the set of controls generated by the non-adaptive $(\theta_i^0, \zeta_i^0)$ MF Stochastic Control Law, and let $\hat{u}_i^0 \in \hat{U}_{MF}^N$, $1 \leq i \leq N$, be the set of controls generated by the MF-SAC Law. Then,

$$\lim_{N \to \infty} J_i^N(\hat{u}_i^0, \hat{u}_{-i}^0) = \lim_{N \to \infty} J_i^N(u_i^0, u_{-i}^0) \text{ w.p.1, } 1 \leq i \leq N. \quad (19)$$
Proposition 4.5: For the system (1), under A1-A4, A7, A8, \( u_i \in U_g^N, \hat{u}_i^0 \in U_{MF}^N, 1 \leq i \leq N \), the following holds:

\[
\lim_{N \to \infty} \inf_{u_i \in U_g^N} J_i^N(u_i, \hat{u}_i^0) = \lim_{N \to \infty} \inf_{u_i \in \hat{U}_i^N} J_i^N(u_i, u_i^0) \quad \text{w.p.1,} \quad 1 \leq i \leq N.
\]

V. SIMULATIONS

Consider a system of 400 agents where each agent is modeled by a 2 dimensional system. All agents apply the MF-SAC Law; each of 400 agents observes its own 20 randomly chosen agents’ outputs and control inputs, as well as its own trajectory. Rapid convergence of the state trajectories of all agents to the steady state values can be seen in Fig. 1 where ‘x’ and ‘y’ represent the two dimensions of each agent’s state and ‘t’ denotes time. In order to plot the convergence of the self identification of dynamical parameters \( A_i, 1 \leq i \leq N \), we plot the norm trajectories of the estimates in Fig. 2. The symbol ‘*’ denotes the true value of the parameter for each agent. Only 10 randomly chosen agents are shown in Fig. 2 for clarity of presentation. In Fig. 3, we depict each agent’s estimate of the mean of the dynamical parameter \( A \) (i.e., the mean of the random variable \( A \)), and we display 10 randomly chosen agents’ estimate trajectories for clarity. Again, the norm of the estimates and the true values are displayed in this diagram. The resulting parameter estimate is different for each agent due to the fact that each agent only observes 20 randomly chosen agents out of a system of 400 agents.

VI. CONCLUSION

This paper presents a study of the mean field stochastic adaptive control problem where the cost functions of the agents in a population are coupled, and each agent estimates its own dynamical parameters based upon observations of its own trajectory, and furthermore estimates the distribution parameter of the population’s dynamical and cost function parameters by observing a randomly chosen fraction of the population. This work makes a contribution to the mean field literature by extending the established \( \epsilon \)-Nash equilibrium results of a large population of egoistic agents to a large population of adaptive egoistic agents. The information requirement for each agent is kept limited in the sense that the distribution parameter is estimated only through an observed set of agents, where the ratio of the cardinality of the observed set to the number of agents in the population becomes negligible as the population size grows to infinity. The strong consistency of the self parameter estimates and the distribution parameter estimates, the stability of the all agent systems, and an \( \epsilon \)-Nash Equilibrium property are all established in the paper.

Future research directions include: (i) investigation of the influence of various rates of observed population fraction decay and rates of convergence on the results in this paper, together with (ii) the extension of adaptive MF theory to (a) the currently developing areas of distance dependent cost function influence among agents [39], (b) altruist and egoist MF theory [40] and (c) problems involving partially observed systems.
APPENDIX A

Preparatory Lemmas on Asymptotic Dynamics and Dither Inputs

Four basic properties to be used in the sequel are given in the following lemmas.

Lemma A.1: Let $\mathbf{A}(\omega)$ be an asymptotically stable random matrix on $p^s$ for all $\omega \in \Omega$ except on a $P$–null set $\mathcal{N}$, and $\mathbf{A}_t(\omega), t \geq 0$, be a bounded random matrix function of $t \geq 0$. If for all $\omega \in \Omega \setminus \mathcal{N}$ and all $\epsilon > 0$, there exists $T_\omega = T_\omega(\epsilon)$ such that $t > T_\omega$ implies $\|\mathbf{A}_t - \mathbf{A}\| < \epsilon$, i.e. $\mathbf{A}_t \to \mathbf{A}$ w.p.1 as $t \to \infty$, then $(\mathbf{A}_t, t \geq 0)$ is an exponentially stable time varying matrix w.p.1, in the sense that its fundamental matrix satisfies the estimate

Fig. 1. State Trajectories

Fig. 2. Self Dynamical Parameter Identification

Fig. 3. Population Parameter Identification
\[\|\Phi_{t,s}\| \leq \beta e^{-\rho(t-s)}\] for \(t_0 \leq s \leq t\), where \(\rho(\omega) > 0\) and \(0 < \beta(\omega) < \infty\).

**Proof:**

Suppressing mention of \(\omega \in \Omega \setminus \mathcal{N}\), whenever possible for simplicity of notation we consider,

\[
\dot{x}(t) = Ax(t), \quad t \geq 0; \quad x(0) = x_0 \in \mathbb{R}^n, \quad \text{and}
\]

\[
\dot{x}_a(t) = A_t x_a(t), \quad t \geq 0; \quad x_a(0) = x_a \in \mathbb{R}^n.
\]

(21) (22)

Since \(A\) is asymptotically stable, we may form the Lyapunov function \(V(x) = x^\top \Pi x\), where \(\Pi > 0\) satisfies

\[
\Pi A + A^\top \Pi = -Q\]

for some \(Q > 0\). Now,

\[
\dot{V}(x(t)) = \dot{x}^\top(t) \Pi x(t) + x^\top(t) \Pi \dot{x}(t)
\]

(23)

\[
= x^\top(t)(\Pi A + A^\top \Pi)x(t)
\]

(24)

\[
= -x^\top(t)Qx(t), \quad t \geq 0.
\]

(25)

Then writing

\[
A_t^\top \Pi + \Pi A_t = (A_t - A)^\top \Pi + A^\top \Pi + \Pi(A_t - A) + \Pi A, \quad t \geq 0,
\]

(26)

we see that for all \(\omega \in \Omega \setminus \mathcal{N}\) there exists sufficiently large \(T_\omega\) such that for all \(t > T_\omega\),

\[
A_t^\top \Pi + \Pi A_t < -Q + \frac{Q}{2} = -\frac{Q}{2}.
\]

(27)

Therefore,

\[
\dot{V}_a(x_a(t)) = \frac{d}{dt} \left[ x_a^\top(t) \Pi x_a(t) \right] < -x_a^\top(t) \frac{Q}{2} x_a(t) < - \left( \frac{\lambda_{\max}(Q)}{2\lambda_{\min}(\Pi)} \right) (x_a^\top(t) \Pi x_a(t)) < 0,
\]

(28)

which implies \(\dot{V}_a(x_a(t)) < -\alpha V_a(x_a(t))\), where \(\alpha \equiv \left( \frac{\lambda_{\max}(Q)}{2\lambda_{\min}(\Pi)} \right)\), which gives

\[
V_a(x_a(t)) \leq V_a(x_a(t_0)) e^{-\alpha(t-t_0)}.
\]

(29)

Now, for the fundamental matrix, we have

\[
\|\Phi(t,t_0)\|_0 = \sup_{x_{t_0} \neq 0} \frac{\|\Phi(t,t_0)x_{t_0}\|}{\|x_{t_0}\|} = \sup_{x_{t_0} \neq 0} \sqrt{\frac{\|x_t\|^2}{\|x_{t_0}\|^2}}.
\]

(30)
Without loss of generality, take \( \|x_{t_0}\| = 1 \). Then,

\[
|\Phi(t, t_0)|_0 = \sup_{x_{t_0}} \sqrt{\|x_t\|^2} 
\]

\[
\leq \sup_{x_{t_0}} \left( \frac{V(x_t)}{\lambda(\Pi)_{\min}} \right)^{\frac{1}{2}} 
\]

\[
\leq \frac{1}{\sqrt{\lambda(\Pi)_{\min}}} \sup_{x_{t_0}} (V(x_t))^{\frac{1}{2}} 
\]

\[
\leq \frac{1}{\sqrt{\lambda(\Pi)_{\min}}} \sup_{x_{t_0}} e^{-a(t-t_0)} V(x_{t_0}) \sup_{x_{t_0}} (x_{t_0}^T \Pi x_{t_0})^{\frac{1}{2}} 
\]

\[
\leq e^{-\rho(t-t_0)} \left( \frac{\lambda(\Pi)_{\max}}{\lambda(\Pi)_{\min}} \right)^{\frac{1}{2}} 
\]

\[
\leq \beta e^{\rho(t-t_0)}, \quad t \geq t_0; \quad \text{when } \rho = \frac{\alpha}{2} \quad \text{and } \beta = \sqrt{\frac{\lambda(\Pi)_{\max}}{\lambda(\Pi)_{\min}}}. 
\]

\[\blacksquare\]

Lemma A.2: Let \((A_t, t \geq 0)\) be a random bounded matrix sequence on \((\Omega, \mathcal{F}, P)\), which converges almost surely to the asymptotically stable matrix \(A^0\) as \(t \to \infty\); let \(\Psi_{t_0,t_0}\) be defined by \(\frac{d}{dt} \Psi_{t_0,t} = A^0 \Psi_{t_0,t_0}\), i.e. \(\Psi_{t_0,t_0} = e^{A^0(t-t_0)}\), and let \(\frac{d}{dt} \Phi_{t_0,t_0} = A_t \Phi_{t_0,t_0}\) with \(\Phi_{t_0,t_0} = \Phi_{t_0,t_0} = I\). Then the following limit holds:

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \|\Phi_{t,t_0} - \Psi_{t,t_0}\|^2 dt = 0 \quad \text{w.p.1.} 
\]

Proof:

The proof is given in four steps below.

(i) Integral Representation \(I^T\):

For almost all \(\omega \in \Omega\), we have \(A_t(\omega) \to A^0(\omega)\) as \(t \to \infty\), restricting attention to the probability 1 subset of \(\Omega_0 \subset \Omega\) on which a unique solution exists. Since

\[
\frac{d}{dt} \Psi_{t_0,t} = A^0 \Psi_{t_0,t} \quad \text{with} \quad \Psi_{t_0,t_0} = I, \quad \text{and} \quad \frac{d}{dt} \Phi_{t_0,t} = A_t \Phi_{t_0,t_0} \quad \text{with} \quad \Phi_{t_0,t_0} = I, 
\]

we have

\[
\frac{d}{dt} (\Phi_{t_0,t} - \Psi_{t_0,t}) = A_t (\Phi_{t_0,t} - A^0 \Psi_{t_0,t}) 
\]

\[
= A_t \Phi_{t_0,t} - (A^0 - A_t) \Psi_{t_0,t} - A_t \Psi_{t_0,t_0} 
\]

\[
= A_t (\Phi_{t_0,t} - \Psi_{t_0,t}) - (A^0 - A_t) \Psi_{t_0,t_0}. 
\]

Integrating we obtain,

\[
\Phi_{t_0,t} - \Psi_{t_0,t} = \Phi_{t_0,t} (\Phi_{t_0,t_0} - \Psi_{t_0,t_0}) - \int_{t_0}^t \Phi_{s,t} (A^0 - A_s) e^{A^0(t-t_0)} ds, 
\]
with the initial condition $\Phi_{t_0} - \Psi_{t_0} = I - I = 0$. Therefore,
\[
\Phi_{t,t_0} - \Psi_{t,t_0} = - \int_{t_0}^t \Phi_{t,s}(A^0 - A_s)e^{A^0(s-t_0)}ds, \quad t \geq t_0, \tag{43}
\]
and so,
\[
I^T = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^T \| (\Phi_{t,t_0} - \Psi_{t,t_0}) \|^2 dt = \frac{1}{T} \int_{t_0}^T \int_{t_0}^t \| \Phi_{t,s}(A^0 - A_s)e^{A^0(s-t_0)}ds \|^2 dt. \tag{44}
\]

(ii) $I^T = I^T_1 + I^T_2$: Convergence of $I^T_1$:

Let us split the integrals in (44) as follows:
\[
I^T = \frac{1}{T} \left( \int_{t_0}^{T_\omega} \| \cdot \|^2 dt + \int_{T_\omega}^T \| \cdot \|^2 dt \right) = : I^T_1 + I^T_2, \tag{45}
\]
where the inner integrals are defined by \( \cdot \) for brevity in this definition and $T_\omega > t_0$ is a random instant whose value is to be determined later.

We take the norm inside the integral in $I^T_1$; then by use of the Cauchy Schwarz Inequality (henceforth termed CS) we may bound $I^T_1$ above as in
\[
I^T_1 \leq \frac{1}{T} \int_{t_0}^{T_\omega} \left( (t - t_0) \int_{t_0}^t \Phi_{t,s}e^{A^0(s-t_0)}\|\tilde{A}_s\|^2 ds \right) dt, \quad t_0 \leq t \leq T_\omega, \tag{46}
\]
where $\tilde{A}_s = (A^0 - A_s), s \geq 0$.

Next, we may bound $\| \tilde{A}_s \|$ above by some $M_{T_\omega}^T$ for $t_0 \leq s \leq T_\omega$, and we may bound $\| e^{A^0(s-t_0)} \|$ by $\beta_0 e^{-\rho A^0(s-t_0)}$, for some $\beta_0 > 0$. Moreover $\| \Phi_{t,s} \| \leq M_{T_\omega}^T$, for all $t_0 \leq s \leq t \leq T_\omega$, for some $M_{T_\omega}^T < \infty$, by the continuity of solutions to (38).

Then,
\[
\limsup_{T \to \infty} I^T_1 \leq \limsup_{T \to \infty} \frac{\beta_0^2}{T} \left( \left( M_{T_\omega}^T \right)^2 \left( M_{T_\omega}^T \right)^2 \int_{t_0}^{T_\omega} (t - t_0) \left( \int_{t_0}^t e^{-2\rho A^0(s-t_0)} ds \right) dt \right) =: \lim_{T \to \infty} \frac{1}{T} \kappa g(t_0, T_\omega), \tag{47}
\]
where $\kappa = \beta_0^2 M_{T_\omega}^T M_{T_\omega}^T < \infty$ and $g(\cdot)$ is a bounded continuous function of $t_0$ and $T_\omega$. Hence for a fixed $T_\omega$, \( \limsup_{T \to \infty} \frac{1}{T} \kappa g(t_0, T_\omega) = 0 \). Therefore $I^T_1$ tends to 0 as $T$ tends to $\infty$.

(iii) $I^T = I^T_1 + I^T_2$: Convergence of $I^T_2$:

For the second integral $I^T_2$ in (45), we have
\[
I^T_2 = \frac{1}{T} \int_{T_\omega}^T \left\| \int_{t_0}^t \Phi_{t,s}(A^0 - A_s)e^{A^0(s-t_0)} ds \right\|^2 dt \tag{48}
\]
\[
\leq \frac{1}{T} \int_{T_\omega}^T \left( (t - t_0) \int_{t_0}^t \left\| \Phi_{t,s}e^{A^0(s-t_0)} \right\|^2 \| \tilde{A}_s \|^2 ds \right) dt, \quad t_0 \leq T_\omega \leq t \leq T, \tag{49}
\]
\[
= \frac{1}{T} \int_{T_\omega}^T \left( (t - t_0) \int_{t_0}^{T_\omega} \| \cdot \|^2 ds + (t - t_0) \int_{T_\omega}^T \| \cdot \|^2 ds \right) dt =: I^{T_1}_{21} + I^{T_2}_{22}, \tag{50}
\]
where we split the inner integral and use the (\( \cdot \)) notation for brevity.

Using the semi-group property of the state transition matrix, we may write $\Phi_{t,s} = \Phi_{t,T_\omega} \Phi_{T_\omega,s}$ for all $t_0 \leq s \leq T_\omega < T$. But we have $\sup_{t_0 \leq s \leq T_\omega} \| \Phi_{T_\omega,s} \| =: M_{T_\omega}^T < \infty$, and we have $M_{T_\omega}^T := \sup_{t_0 \leq s \leq T_\omega} \| \tilde{A}_s \|$. Therefore,
\[ I_{21}^T \leq \frac{1}{T} \left( M_{Tw}^T \right)^2 \left( M_{\Phi}^T \right)^2 \int_{T_{w}}^{T} \left( e^{\beta_1(s-t_0)} e^{\beta(s-t_0)} \right) ds \, dt, \quad t_0 \leq s \leq T. \] (51)

Concerning \( I_{22}^T \), the random time \( T_w \) is chosen so that for \( s \geq T_w \) (increasing the value over that used in (47) without affecting that argument), \( \| \tilde{A}_s \| < \epsilon \). Hence,

\[ I_{22}^T \leq \frac{1}{T} \epsilon^2 \int_{T_{w}}^{T} \left( e^{\beta_1(s-t_0)} e^{\beta(s-t_0)} \right) ds \, dt, \quad t_0 \leq T_w \leq s \leq t \leq T. \] (52)

From Lemma A.1, \( \Phi_{t,t_0} \) satisfies the bound \( \| \Phi_{t,t_0} \| \leq \beta_1 e^{-\rho(t-t_0)}, t \geq t_0 \), where \( \beta_1 = \beta_1(\omega), \rho = \rho(\omega) \).

Finally, bounding \( \| e^{\beta(s-t_0)} \| \) by \( \beta_0 e^{-\rho_0(s-t_0)} \) yields

\[ I_{21}^T \leq \frac{\beta_0^2}{T} \left( M_{Tw}^T \right)^2 \left( M_{\Phi}^T \right)^2 \int_{T_{w}}^{T} \left( e^{-2\rho_0(t-T_w)} e^{-2\rho_0(s-t_0)} \right) ds \, dt, \quad t_0 \leq s \leq T \leq t. \] (53)

For simplicity, in the following we use \( \rho = \min [\rho_\Phi, \rho_\Lambda^0] \) and \( \beta = \max [\beta_0, \beta_1] \); then,

\[ I_{21}^T \leq \frac{1}{T} \kappa' \int_{T_{w}}^{T} \left( e^{-2\rho_0(t-s)} e^{-2\rho_0(s-t_0)} \right) ds \, dt, \quad t_0 \leq s \leq T_w \leq t \leq T, \] (54)

where \( \kappa' = \beta^4 \left( M_{Tw}^T \right)^2 \left( M_{\Phi}^T \right)^2 \),

\[ I_{21}^T \leq \limsup_{T \to \infty} \frac{1}{T} \kappa'' e^{2\rho T} \left( T e^{2\rho T} + t_0 e^{2\rho T} + t_0 e^{2\rho T} + T_w e^{2\rho T} \right), \] (55)

for a suitable constant \( \kappa'' \) independent of \( T_w \), which tends to 0 as \( T \to \infty \).

(iv) \( I_T^2 = I_{21}^T + I_{22}^T \): Convergence of \( I_T^2 \): Employing the hypothesis \( A_t \to A^0 \) w.p.1 as \( t \to \infty \), we shall fix \( T_w \) such that \( \| \tilde{A}_s \| < \epsilon \).

For \( I_{22}^T \), applying Lemma A.1 for \( T_w \leq s \leq t \leq T \), we obtain

\[ I_{22}^T \leq \frac{1}{T} \epsilon^2 \beta^4 \int_{T_{w}}^{T} \left( e^{-2\rho_0(t-s)} e^{-2\rho_0(s-t_0)} \right) ds \, dt, \quad t_0 \leq T_w \leq s \leq t \leq T, \] (57)

\[ \leq \frac{1}{T} \epsilon^2 \beta^4 \int_{T_{w}}^{T} \left( e^{-2\rho(t-t_0)} \right) ds \, dt, \quad \rho := \min [\rho_\Phi, \rho_\Lambda^0], \] (58)

\[ \leq g \left( \frac{T_w}{T} \right) + \epsilon^2 \beta^2 \exp(-2\rho(T-T_w)) \frac{2\rho^2 T^2 + 2\rho T + 1}{4\rho^4 T}, \] (59)

where \( g(\cdot) \) is a bounded continuous function. Then, \( \limsup_{T \to \infty} I_{22}^T = 0 \). Therefore \( \limsup_{T \to \infty} I_T^2 \leq \limsup_{T \to \infty} I_{21}^T + \limsup_{T \to \infty} I_{22}^T = 0 \).

Since we have established that \( I_1^T \to 0, I_2^T \to 0 \) w.p.1 as \( T \to \infty \), we obtain \( \limsup_{T \to \infty} I_T^2 \leq \limsup_{T \to \infty} I_1^T + \limsup_{T \to \infty} I_2^T = 0 \).

Hence, we have proved that

\[ \lim_{T \to \infty} \frac{1}{T} \int_{T_{w}}^{T} \| \Phi_{t,t_0} - \psi_{t,t_0} \|^2 dt = 0, \ w.p.1. \] (60)
Lemma A.3: [26, Duncan et al. (1999)] Assume the process \( (e(t), t \geq 0) \) is an \( \mathbb{R}^m \)-valued standard Wiener process that is independent of \( (w(t), t \geq 0) \), and assume the countable set of random processes \( \{(e(t+k) - e(k), t \in (0,1); k \in \mathbb{N}\) to be mutually independent and all members of the set have the same probability law on \( (0,1] \).

Then, for all \( f(\cdot) \in L^\infty[0,\infty) \):

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \left\| \sum_{k=0}^{\lfloor T/t \rfloor} \int_k^{\min\{T,k+1\}} f(\tau) \xi_k [e(\tau) - e(k)] d\tau \right\|^2 dt = 0. \quad \text{w.p.1.}
\]

The proof is given in [26, Lemma 5].

Lemma A.4: [28, Chen and Guo (1991)] Let \( A \in \mathbb{R}^{n \times n} \) be an asymptotically stable matrix, and let \( D \in \mathbb{R}^{n \times r} \). Then

\[
\limsup_{T \to \infty} \frac{1}{T} \int_0^T \left\| \int_0^t e^{A(t-\tau)} D dw(\tau) \right\|^2 dt = \int_0^\infty \text{Tr}(e^{AT}DD^T e^{AT}dt).
\]

Proof (after [28]):

Consider the stochastic differential equation

\[
dx_t = Ax_t dt + Dw_t, \quad t \geq 0.
\]  

(61)

Since \( A \) is asymptotically stable, there exists a positive definite matrix \( \Pi > 0 \) such that

\[
\Pi A + A^\top \Pi = -I.
\]  

(62)

Following [28], applying the Itô formula to the Lyapunov function \( x_t^\top \Pi x_t, t \geq 0 \),

\[
d[x_t^\top \Pi x_t] = x_t^\top (\Pi A + A^\top \Pi) x_t dt + \text{Tr}(\Pi DD^\top) dt + 2 x_t^\top \Pi D dw_t
\]  

(63)

\[
= -\|x_t\|^2 dt + \text{Tr}(\Pi DD^\top) dt + 2 x_t^\top \Pi D dw_t.
\]  

(64)

Integrating (64) and using the result in Lemma 4 of Christopeit [41] to estimate the third term on the RHS of (63), we obtain

\[
x_t^\top \Pi x_t \leq -\int_0^t \|x_s\|^2 ds + \text{Tr}(\Pi DD^\top) t + O(1), \quad \text{where } 0 < \eta < \frac{1}{2},
\]  

(65)

and hence, \( \int_0^t \|x_s\|^2 ds = O(t) \) \quad \text{w.p.1.}

We apply the Itô formula to the outer product \( x_t x_t^\top, t \geq 0 \),

\[
d[x_t x_t^\top] = x_t x_t^\top A^\top dt + A x_t x_t^\top dt + DD^\top dt + Dw_t x_t^\top + x_t dw_t^\top D^\top.
\]  

(66)

Integrating the outer product \( x_t x_t^\top \) from \( t = 0 \) yields

\[
x_t x_t^\top = \left( \int_0^t x_s x_s^\top ds \right) A^\top + A \left( \int_0^t x_s x_s^\top ds \right) + (DD^\top) t + \int_0^t (D dw_s x_s^\top) + \int_0^t (x_s dw_s^\top D^\top).
\]  

(67)

A “Lyapunav integral move” yields

\[
\int_0^t x_s x_s^\top ds = \int_0^t e^{A(t-s)} (DD^\top) s e^{A^\top(t-s)} ds + \int_0^t e^{A(t-s)} \left( \int_s^t \left( (D dw_t x_t^\top) + (x_t dw_t^\top D^\top) \right) \right) e^{A^\top(t-s)} ds.
\]  

(68)
We deal with the second term of RHS of (68). Using Christopeit’s [41] estimate again we write,
\[
\left\| \int_0^t e^{A(t-s)} \left( \int_0^s \{ (Ddw_r x_r^T) + (x_r dw_r^T D^T) \} d\tau \right) e^{A(t-s)} ds \right\| 
\leq \left( \int_0^t e^{-2\rho(t-s)} \left( o \left\{ \int_0^s \| x_r \|^2 d\tau \right\}^{1/2+\eta} + O(1) \right) ds \right), \quad \text{where } 0 < \eta < \frac{1}{2},
\]
(69)
\[
\int_0^t e^{-2\rho(t-s)} o \left( s^{1/2+\eta} \right) ds = o \left( t^{1/2+\eta} \right), \quad \eta > 0.
\]
(70)
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t e^{-2\rho(t-s)} ds = \int_0^\infty e^{-2\rho \tau} d\tau,
\]
As \( \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{-2\rho(t-s)} ds = \int_0^\infty e^{-2\rho \tau} d\tau \), we take the time average limit of (68) and get
\[
\lim_{t \to \infty} \frac{1}{T} \int_0^T \left\| \int_0^t e^{A(t-s)} Ddw(\tau) \right\|^2 dt = \int_0^\infty \text{Tr}(e^{A\tau} D D^T e^{A^T\tau}) dt.
\]
(72)
Thus we obtain the desired result.

\section*{APPENDIX B}

\textbf{Proof of Lemma 3.1}

We drop the subscript \( i \) for clarity. By definition, when \( \hat{\theta}_t \) is the solution to RWLS equations (9), \( \hat{\theta}_{pr}^i \) satisfies \( \hat{\theta}_{pr}^i \triangleq \arg \min_{\psi \in \Theta} \| \hat{\theta}_t - \psi \| \), employing a co-ordinate ordering measurable tie breaking rule, if necessary. Since \( \hat{\theta}^i_t \in \mathbb{R}^{n(n+m+(n+1)/2)}, \hat{\theta}_{pr}^i \in \Theta \) and \( \theta^0 \in \Theta \), the definition of \( \hat{\theta}_{pr}^i \) gives \( \| \hat{\theta}_t - \hat{\theta}_{pr}^i \| \leq \| \hat{\theta}_t - \theta^0 \| \). But by hypothesis, \( \| \hat{\theta}_t - \theta^0 \| \to 0 \) w.p.1 as \( t \to \infty \); therefore, \( \hat{\theta}_{pr}^i \to \theta^0 \) w.p.1 as \( t \to \infty \).

\textbf{Proof of Theorem 3.2}

(i) Since the solution \( \Pi_{\theta} \in \mathbb{R}^n \), \( \theta \in \mathbb{R}^{n(n+m+(n+1)/2)} \), to the algebraic Riccati equation (3) parametrized by \( \theta \in \Theta \) is a smooth function of \( \theta \) (see [42]), and since \( \hat{\theta}_{pr}^i \in \Theta, t \geq 0, 1 \leq i \leq N, \Pi(\hat{\theta}_{pr}^i, t \geq 0, \) satisfies \( \Pi(\hat{\theta}_{pr}^i) \prec \infty \) w.p.1 for all \( t \geq 0 \). It is given that \( x^* \in C_b(0, \infty) \); therefore, \( s(t; \hat{\theta}_{pr}^i, \hat{\xi}_{N_t}^i) \prec \infty \) w.p.1 for \( t \geq 0 \) evaluated along \( \hat{\theta}_{pr}^i, t \geq 0 \). Hence, \( \hat{s}_i(t; \hat{\theta}_{pr}^i, \hat{\xi}_{N_t}^i) \) given in (17) is well defined.

(ii) The strong consistency of the dynamical parameter estimates \( \hat{A}_{i,t}, \hat{B}_{i,t}, t \geq 0, 1 \leq i \leq N, \) is shown in [26] under the controllability and observability of the true parameters \( (A_1, A_2) \) and the uniform controllability and observability of the estimates (Definition 1 in [26]). In our work, the controllability and observability assumptions are satisfied since \( [A_1, A_2] \) is controllable and \( [Q_1/2, A_0] \) is observable for all \( \theta \in \Theta \) by \( A_2 \), and moreover, the uniform controllability and observability of the estimates are satisfied due to Lemma 3.1.

(iii) Dropping the subscript \( i \) for clarity we set the estimation vector \( \psi_t^i = [\Pi_t, \hat{s}(t)] \) and the regression vector as \( \psi_t^i = [x_t^T, 1] \). The persistence of excitation is satisfied since
\[
\lim_{T \to \infty} \frac{1}{T} \lambda_{\min} \left( \int_0^T \psi_t \psi_t^T dt \right) > 0.
\]
Setting the measurement vector to be \((- (\hat{\mathbf{B}}^T)^{-1} \mathbf{R}_{n_t})^T\) and employing \(\text{A}6\) we get \([\hat{\mathbf{P}}_t, \hat{s}(t)] \to [\mathbf{P}_0^0, s_0(t)]\) w.p.1 as \(t \to \infty\). Also, as shown in Part (i) \([\hat{\mathbf{A}}_t, \hat{\mathbf{B}}_t] \to [\mathbf{A}_0^0, \mathbf{B}_0^0]\) w.p.1 as \(t \to \infty\). Each estimated parameter in \(\hat{\mathbf{Q}}_t = - \hat{\mathbf{A}}_t^T \hat{\mathbf{P}}_t^T - \hat{\mathbf{P}}_t \hat{\mathbf{A}}_t + \hat{\mathbf{P}}_t^T \hat{\mathbf{B}}_t \mathbf{R}^{-1} \hat{\mathbf{B}}_t^T \hat{\mathbf{P}}_t\) converges to its true value as \(t \to \infty\). Hence, \(\hat{\mathbf{Q}}_t \to \mathbf{Q}_0^0\) w.p.1 as \(t \to \infty\).

We observe that instead of the random regularization method used in [25] and [26], we employ here the projection method (Lemma 3.1), which guarantees the uniform controllability and observability of the estimates.

Proof of Theorem 3.3

Recall that \(\theta^{[1:N_0]} = \{\theta_j; j \in \text{Obs}(N)\}\) is an independently selected subset of \(\theta^{[1:N]}\) of cardinality \(N_0(N)\), and \(\theta_i, 1 \leq i \leq N\), is an independently distributed sequence with each \(\theta_i\) having the density \(f_\zeta(\theta)\), and hence \(\theta^{[1:N_0]}\) possesses a density in product form. Consequently, the scaled log-likelihood function of \(\zeta\) at \(\theta^{[1:N_0]}\) is given by \(L_{N_0}(\zeta) = L\left(\theta^{[1:N_0]}; \zeta\right) = - \frac{1}{N_0} \log \left(\prod_{j \in \text{Obs}(N)} f_\zeta(\theta_j)\right)\). Note that the subscript \(i\) is suppressed for clarity. The maximum likelihood estimate of \(\zeta\) given \(\theta^{[1:N_0]}\) is then given by \(\hat{\zeta}_{N_0} = \arg \min_{\zeta \in \mathbb{P}} L(\theta^{[1:N_0]}; \zeta)\). Now, it has been established in Theorem 3.2 that \(\hat{\theta}^{[1:N_0]}_t; t \geq 0\) for each \(N_0(N), N \in \mathbb{Z}_1\), constitutes a strongly consistent estimate of \(\theta^{[1:N_0]}\), i.e., \(\lim_{t \to \infty} \hat{\theta}^{[1:N_0]}_t = \theta^{[1:N_0]}\) w.p.1. Based upon this, the proof of the theorem consists of an analysis of the convergence (as \(N \to \infty\) and hence \(N_0(N) \to \infty\), and \(t \to \infty\)) of the likelihood function \(L(\theta^{[1:N_0]}; \zeta)\) with \(\hat{\theta}^{[1:N_0]}_t\) substituted for \(\theta^{[1:N_0]}\) and hence of the associated sequence of estimators \((\hat{\zeta}_{N_0}^N; N_0 \in \mathbb{Z}_1)\) to \(\zeta^0\).

First we present two lemmas that will be used in the sequel for the proof of Theorem 3.3.

Convergence of the Likelihood Functions \(L(\theta^{[1:N_0]}; \zeta)\):

Lemma B.1: Subject to \(\text{A}7, \text{A}8\) we have

\[L_{N_0}(\zeta) \triangleq L\left(\theta^{[1:N_0]}; \zeta\right) \to L_{\zeta^0}(\zeta) \triangleq -E_{\zeta^0} \log f_\zeta(\theta)\text{ w.p.1,}\]

as \(N_0 \to \infty\) uniformly over \(\zeta \in \mathbb{P}\).

The proof of Lemma B.1 is given later in Appendix B.

Lemma B.2: \(L_{\zeta^0}(\zeta^0) \leq L_{\zeta^0}(\zeta)\) for all \(\zeta \in \mathbb{P}\), with equality holding if and only if \(\zeta = \zeta^0\).

The proof of Lemma B.2 follows a standard argument. A typical treatment can be found in [43].

Convergence of the Functions \(L(\hat{\theta}^{[1:N_0]}_t; \zeta)\): Now \(P\) is a compact set, so it is sequentially compact [44], and the sequence \((\hat{\zeta}_{N_0}^N; N_0 \in \mathbb{Z}_1)\) has a convergent subsequence \((\hat{\zeta}^{NM}_t; M \in \mathbb{Z}_1)\) for all \(t \geq 0\), for which \(\lim_{t \to \infty} \lim_{M \to \infty} \hat{\zeta}^{NM}_t = \zeta^* \in \mathbb{P}\), in the topology of \(P\). Further, we observe that \(\zeta^*\) is a \(B\)-measurable \(P\)-valued random variable. We will adopt the notation \((\hat{\zeta}^{N_0}_t; N_0 \in \mathbb{Z}_1) \triangleq (\lim_{t \to \infty} \hat{\zeta}_{N_0}^N_t; N_0 \in \mathbb{Z}_1)\) in order to denote the sequence of MLE estimates indexed by the size of the population.

We will show that \(L_{\zeta^0}(\zeta^*) \leq L_{\zeta^0}(\zeta^0)\) for any \(\zeta^0 \in \mathbb{P}\). This, together with Lemma B.2 with \(\zeta\) set equal to \(\zeta^*\) gives \(L_{\zeta^0}(\zeta^*) \leq L_{\zeta^0}(\zeta^0) \leq L_{\zeta^0}(\zeta^*)\). The Identifiability Condition \(\text{A}8\) gives \(\zeta^0 = \zeta^*\) w.p.1 and we conclude that all subsequential limits of \((\hat{\zeta}^{N_0}_t; N_0 \in \mathbb{Z}_1)\) equal \(\zeta^0\) w.p.1, and hence \(\lim_{N \to \infty} \lim_{t \to \infty} \hat{\zeta}_{N_0}^N_t = \zeta^0\) w.p.1.
(i) To show \( L_{C^{0}}(\zeta^*) \leq \lim_{t \to \infty} L_{C^{0}}(\hat{\zeta}_{t}^{N_{0}}) + \epsilon/3 \): For any \( \epsilon > 0 \), there exists an almost surely finite random integer \( N_{1}(\omega, \epsilon) \) so that for all \( M \) such that \( N_{M} > N_{1}(\omega, \epsilon) \), the estimate \( \hat{\zeta}^{N_{M}} \triangleq \lim_{t \to \infty} \hat{\zeta}_{t}^{N_{M}} \) lies in a neighbourhood \( N_{C^{*}}(\zeta) \) of \( \zeta^* \) for which \( |L_{C^{0}}(\zeta) - L_{C^{0}}(\zeta^{*})| < \epsilon/3 \), for all \( \zeta \in N_{C^{*}}(\zeta) \), by the continuity of \( L_{C^{0}}(\cdot) \) on \( P \). The uniform continuity of \( L_{C^{0}} \) on \( P \) is shown as follows: pick arbitrary \( \zeta, \zeta' \in P \subset \hat{P} \) such that \( \zeta' \in P \) lies in a coordinate neighborhood \( N_{\delta}(\zeta) \) of \( \zeta \in P \). We have
\[
|L_{C^{0}}(\zeta) - L_{C^{0}}(\zeta')| = | -E_{C^{0}} \log f_{\zeta}(\theta) + E_{C^{0}} \log f_{\zeta'}(\theta) |
\leq \int_{\Theta} |\log f_{\zeta}(\theta) - \log f_{\zeta'}(\theta)| f_{C^{0}}(\theta) d\theta.
\]
Hence for some \( \zeta'' \in \hat{P} \) in the line segment \( \{\lambda \zeta + (1 - \lambda)\zeta' ; \lambda \in (0, 1)\} \), the Mean Value Theorem yields
\[
|L_{C^{0}}(\zeta) - L_{C^{0}}(\zeta')| \leq \int_{\Theta} \frac{1}{f_{C^{0}}(\theta)} \|f_{C^{0}}'(\theta)\| \|\zeta' - \zeta\| f_{C^{0}}(\theta) d\theta. \tag{73}
\]
But by A7, \( f_{C^{0}}(\theta) > 0 \) for all \( \zeta'' \in \hat{P} \). Then by (73), for each \( \epsilon > 0 \), there exists \( \delta_{\epsilon} > 0 \) such that for all \( \zeta, \zeta' \in P \), \( \|\zeta' - \zeta\| < \delta_{\epsilon} \) implies \( |L_{C^{0}}(\zeta) - L_{C^{0}}(\zeta')| < \epsilon \). Hence, \( L_{C^{0}}(\zeta) \) is uniformly continuous over \( P \).

(ii) To show \( \lim_{t \to \infty} L_{C^{0}}(\hat{\zeta}_{t}^{N_{0}}) \leq \lim_{t \to \infty} L(\hat{\theta}_{t}^{[1:N_{0}]}; \hat{\zeta}_{t}^{N_{0}}) + \epsilon/3 \) for all \( N \geq N_{2}(\omega, \epsilon) \) for some random \( N_{2}(\omega, \epsilon) \in \mathbb{Z}_{1} \): Lemma B.1 assures us that we can pick an almost surely finite random integer \( N_{2}(\omega, \epsilon) \in \mathbb{Z}_{1} \) so that for all \( N > N_{2}(\omega, \epsilon) \), we have \( |L(\hat{\theta}_{t}^{[1:N_{0}]}; \zeta) - L_{C^{0}}(\zeta)| < \epsilon/3 \) for all \( \zeta \in P \), where \( N_{0} = |\text{Obs}(N)|, N_{0} \to \infty \), as \( N \to \infty \), and where \( \theta_{t}^{[1:N_{0}]} = \lim_{t \to \infty} \hat{\theta}_{t}^{[1:N_{0}]} \). But, from the continuity of \( L_{N_{0}}(\cdot) \), \( N_{0} \in \mathbb{Z}_{1} \), we have \( \lim_{t \to \infty} L(\hat{\theta}_{t}^{[1:N_{0}]}; \zeta) = L(\theta_{t}^{[1:N_{0}]}; \zeta) \) for all \( \zeta \in P \), therefore the inequality holds.

(iii) To show \( \lim_{t \to \infty} L(\hat{\theta}_{t}^{[1:N_{0}]}; \hat{\zeta}_{t}^{N_{0}}) \leq \lim_{t \to \infty} L(\hat{\theta}_{t}^{[1:N_{0}]}; \hat{\zeta}_{t}^{N_{0}}), \forall \zeta \in P \): This follows from (14) since we have \( L(\theta_{t}^{[1:N_{0}]}; \hat{\zeta}_{t}^{N_{0}}) \leq L(\theta_{t}^{[1:N_{0}]}; \zeta) \) for all \( \zeta \in P \), where \( \lim_{t \to \infty} L(\hat{\theta}_{t}^{[1:N_{0}]}; \hat{\zeta}_{t}^{N_{0}}) = L(\theta_{t}^{[1:N_{0}]}; \hat{\zeta}_{t}^{N_{0}}) \), and \( \lim_{t \to \infty} L(\hat{\theta}_{t}^{[1:N_{0}]}; \zeta) = L(\theta_{t}^{[1:N_{0}]}; \zeta) \).

(iv) To show \( \lim_{t \to \infty} L(\hat{\theta}_{t}^{[1:N_{0}]}; \zeta) \leq L_{C^{0}}(\zeta) + \epsilon/3, \forall \zeta \in P \): Again, we employ Lemma B.1: pick an almost surely finite random integer \( N_{3}(\omega, \epsilon) \in \mathbb{Z}_{1} \) so that for all \( N > N_{3}(\omega, \epsilon) \) we have \( |L(\hat{\theta}_{t}^{[1:N_{0}]}; \zeta) - L_{C^{0}}(\zeta)| < \epsilon/3 \) for all \( \zeta \in P \), and let \( \hat{\theta}_{t}^{[1:N_{0}]} \to \theta_{t}^{[1:N_{0}]} \) as \( t \to \infty \).

Combining (i)-(iv), yields
\[
L_{C^{0}}(\zeta^*) \leq \lim_{t \to \infty} L_{C^{0}}(\hat{\zeta}_{t}^{N_{0}}) + \frac{\epsilon}{3} \leq \lim_{t \to \infty} L(\hat{\theta}_{t}^{[1:N_{0}]}; \hat{\zeta}_{t}^{N_{0}}) + \frac{2\epsilon}{3}
\leq \lim_{t \to \infty} L(\hat{\theta}_{t}^{[1:N_{0}]}; \zeta) + \frac{2\epsilon}{3}
\leq L_{C^{0}}(\zeta) + \epsilon \quad \text{w.p.1 for all } \zeta \in P,
\]
for all \( N_{M} > \max(N_{1}, N_{2}, N_{3})(\omega, \epsilon) \).

Convergence of the Sequence of Estimators \( (\hat{\zeta}_{t}^{N_{0}}; N_{0} \in \mathbb{Z}_{1}) \): Evaluating the relation above at \( \zeta = \zeta_{0}^{*} \) yields \( L_{C^{0}}(\zeta^{*}) \leq L_{C^{0}}(\zeta^{0}) + \epsilon \) w.p.1. But this expression is independent of \( N_{M} \), and \( \epsilon \) is arbitrary. So, \( L_{C^{0}}(\zeta^{*}) \leq L_{C^{0}}(\zeta^{0}) \) w.p.1 for all \( \zeta \in P \). However, as stated in Lemma B.2, \( L_{C^{0}}(\zeta^{0}) \leq L_{C^{0}}(\zeta) \) w.p.1 for all \( \zeta \in P \), with equality holding if and only if \( \zeta = \zeta^{0} \). Therefore, \( L_{C^{0}}(\zeta^{*}) \leq L_{C^{0}}(\zeta^{0}) \leq L_{C^{0}}(\zeta^{*}) \) w.p.1, implying \( \zeta^{0} = \zeta^{*} \) w.p.1 by the Identifiability Condition A8, and so all subsequential limits of \( (\hat{\zeta}_{t}^{N_{0}}; N_{0} \in \mathbb{Z}_{1}) = (\lim_{t \to \infty} \hat{\zeta}_{t}^{N_{0}}; N_{0} \in \mathbb{Z}_{1}) \) equal \( \zeta^{0} \) w.p.1, or equivalently \( \lim_{N \to \infty} \lim_{t \to \infty} \hat{\zeta}_{t}^{N_{0}} = \zeta^{0} \) w.p.1.
Proof of Lemma B.1

By A7 the family of densities \( f_\zeta \triangleq \{ f_\zeta(\theta) ; \theta \in \Theta, \zeta \in P \} \) exists for the family of dynamical and cost function parameter distributions \( \{ F_\zeta(\cdot) ; \zeta \in P \} \). Let \( f(\theta^{[1:N_0]} ; \zeta) \), where \( N_0 \triangleq |\text{Obs}(N)| \), be the likelihood function of \( f_\zeta \) at \( \theta^{[1:N_0]} \) and let \( L_{N_0}(\zeta) \triangleq L(\theta^{[1:N_0]} ; \zeta) \) be the continuously differentiable function of \( f(\theta^{[1:N_0]} ; \zeta) \), given by the scaled log-likelihood function

\[
L_{N_0}(\zeta) \triangleq L(\theta^{[1:N_0]} ; \zeta) \triangleq - (1/N_0) \log f(\theta^{[1:N_0]} ; \zeta) \equiv -(1/N_0) \log \left( \prod_{j \in \text{Obs}(N)} f_\zeta(\theta_j) \right),
\]

\( N_0 \triangleq |\text{Obs}(N)| \), where \( \theta^{[1:N_0]} = \{ \theta_j ; j \in \text{Obs}(N) \} \).

The random sequence \( L(\zeta) \triangleq (L(\theta^{[1:N_0]} ; \zeta) ; N_0 \in \mathbb{Z}_1) \) converges w.p.1 for each \( \zeta \in P \) [43], where \( P \) is a compact set by A7. Then, in order for the almost sure convergence of \( L(\zeta) \) to be uniform over \( P \), it is sufficient that the process \((\partial L_{N_0}/\partial \zeta)(\zeta) ; N_0 \in \mathbb{Z}_1) \) exists as a sequence of random variables which is w.p.1 bounded uniformly over \( P \), where \( L_{N_0}(\zeta) \triangleq L(\theta^{[1:N_0]} ; \zeta) \). This may be shown as follows by the Mean Value Theorem:

\[
\tilde{L}_{K_0,L_0}(\zeta') \triangleq L_{K_0,L_0}(\zeta') - L_{L_0}(\zeta') = \tilde{L}_{K_0,L_0}(\zeta) + \frac{\partial \tilde{L}_{K_0,L_0}(\zeta)}{\partial \zeta}(\zeta'')(\zeta' - \zeta),
\]

where \( \zeta' \in P \) lies in an \( \epsilon \)-coordinate neighbourhood \( N(\zeta) \) of \( \zeta \in P \) and \( \zeta'' \) lies on the line segment \( \{ \lambda \zeta + (1 - \lambda)\zeta' ; \lambda \in (0, 1) \} \). Consequently,

\[
|\tilde{L}_{K_0,L_0}(\zeta')| \leq |\tilde{L}_{K_0,L_0}(\zeta)| + \left\| \frac{\partial \tilde{L}_{K_0,L_0}(\zeta''(\zeta'))}{\partial \zeta} \right\| \| \zeta' - \zeta \| \quad \text{with } \| \zeta' - \zeta \| < \epsilon,
\]

where the differentiability of \((L_{N_0}; N \in \mathbb{Z}_1) \) follows from its definition. Let each such \( N(\zeta) \subset \hat{P} \), choosing a smaller \( \epsilon = \epsilon(\zeta) \), possibly depending upon \( \zeta \), if necessary. Then take an open cover of the compact set \( \hat{P} \) by these \( \epsilon \)-neighbourhoods and let \( \{ N_i(\zeta) ; 1 \leq i \leq M \} \) be a finite subcover. By A7 for each \( j, 1 \leq j \leq p \), \((\partial f_\zeta/\partial \zeta_j)(\theta ; \zeta) \) is bounded uniformly for all \( \theta \in \Theta \). Therefore, \( \sup_{\zeta \in \hat{P}} \left( \| (\partial L_{N_0}/\partial \zeta)(\zeta) \| ; N_0 \in \mathbb{Z}_1 \right) < K \). Moreover, by the convergence of the sequences \((L_{N_0}(\zeta) ; 1 \leq i \leq M) \) w.p.1 and the boundedness of \( \| (\partial L_{N_0}/\partial \zeta)(\zeta) \| ; N_0 \in \mathbb{Z}_1 \) by \( K \) uniformly over \( \zeta \in \hat{P} \), we obtain \( |\tilde{L}_{K_0,L_0}(\zeta')| \leq \epsilon + 2K \epsilon \) w.p.1 for all \( \zeta' \in P \), for all \( K_0, L_0 \geq N(\omega, \epsilon) \) for some random \( N(\omega, \epsilon) \in \mathbb{Z}_1 \). But this shows that \((L_{N_0}(\zeta) ; N_0 \in \mathbb{Z}_1) \) satisfies the Cauchy convergence criterion w.p.1 uniformly over \( P \). Therefore \( L_{N_0} \triangleq L(\theta^{[1:N_0]}; \zeta) \to L_{\zeta^0}(\zeta) \triangleq -\mathbb{E}_{\zeta^0} \log f_\zeta(\theta) \) w.p.1, as \( N_0 \to \infty \) uniformly over \( P \), where \( N_0 \triangleq |\text{Obs}(N)| \), and hence as \( N \to \infty \) uniformly over \( P \).

APPENDIX C

Proof of Proposition 4.1

1) Proof of \( \lim_{N \to \infty} \lim_{t \to \infty} x^*(\tau, \hat{\zeta}^{N_0}_t) = x^*(\tau, \zeta^0) \) w.p.1, \( t \leq \tau < \infty \):

Recall that \( x^*(\tau, \zeta^0) \), \( t \leq \tau < \infty \), is the solution to the MF Equation System (6), and \( x^*(\tau, \hat{\zeta}^{N_0}_t), t \leq \tau < \infty \), is the solution to the MF-SAC Equation System (18). Note that the subscript \( i \) is suppressed for clarity. A contraction mapping argument together with A1-A4 ensure the existence and uniqueness of \( x^*(\tau, \zeta^0) \in \mathcal{C}_0[0, \infty) \) (see [2]). A1-A4 also hold for \( x^*(\tau, \hat{\zeta}^{N_0}_t), t \leq \tau < \infty, t \geq 0 \), by Lemma 3.1; therefore, the existence and uniqueness properties
also hold for $x^*(\tau, \hat{\zeta}^{N,0}_t)$ for $t \geq 0$. Since $x^*(\cdot, \zeta)$ is a continuous function of $\zeta$ on $P$, and by Theorem 3.3, 
\lim_{N \to \infty} \lim_{t \to \infty} \hat{\zeta}^{N,0}_t = \zeta^0$ w.p.1, \lim_{t \to \infty} \sup_{\tau \in \tau < \infty} \left\| x^*(\tau, \hat{\zeta}^{N,0}_t) - x^*(\tau, \zeta^0) \right\| = O(\epsilon_1(N))$, where $\epsilon_1(N) \to 0$ as $N \to \infty$. Therefore, 
\lim_{N \to \infty} \lim_{t \to \infty} \left\| x^*(\tau, \hat{\zeta}^{N,0}_t) - x^*(\tau, \zeta^0) \right\| = 0 \text{ w.p.1, } t \leq \tau < \infty.

2) Proof of $\lim_{N \to \infty} \lim_{t \to \infty} s(t; \theta^0, \hat{\zeta}^{N,0}_t) = s(t; \theta^0, \zeta^0)$ w.p.1:

The solution to the differential equation (4) is given by 
\[ s(t; \theta^0, \zeta^0) = -\int_t^\infty \Psi_{t,\tau}^{-1}(t, \tau, \theta^0)Q(\theta^0)x^*(\tau, \zeta^0)d\tau, \]
where $\frac{d}{d\tau} \Psi_{t,\tau} = A_s(\theta^0)\Psi_{t,\tau}, \Psi_{t,\tau} = I$, and $x^*$ is generated by the MF equation system (6), and $A_s \triangleq (A - BR^{-1}B^T \Pi)$. For the certainty equivalence offset function $s(t; \hat{\theta}_t, \hat{\zeta}_t)$ generated by the MF-SAC Law, we have 
\[ s(t; \hat{\theta}_t, \hat{\zeta}_t) = -\int_t^\infty \Phi_{t,\tau}^{-1}(t, \tau, \theta^0, \hat{\theta}_t, \hat{\theta}_\tau)Q(\hat{\theta}_t)x^*(\tau, \hat{\zeta}_t^N)d\tau, \]
where $\frac{d}{d\tau} \Phi_{t,\tau} = A_s(\hat{\theta}_t)\Phi_{t,\tau}, \Phi_{t,\tau} = I$. We adopt the notation $\Phi_{t,\tau} \triangleq \Phi(t, \tau, \theta^0, \hat{\theta}_t, \hat{\theta}_\tau), \Psi_{t,\tau} \triangleq \Psi(t, \tau, \theta^0)$ and obtain 
\[ \left\| s(t; \hat{\theta}_t, \hat{\zeta}_t^N) - s(t; \theta^0, \zeta^0) \right\| = \left\| \int_t^\infty \Phi_{t,\tau}^{-1}Q(\hat{\theta}_t)x^*(\tau, \hat{\zeta}_t^N)d\tau - \int_t^\infty \Phi_{t,\tau}^{-1}Q(\theta^0)x^*(\tau, \zeta^0)d\tau \right\|. \]

Adding and subtracting $\int_t^\infty \Phi_{t,\tau}^{-1}Q(\hat{\theta}_t)x^*(\tau, \zeta^0)d\tau$ and $\int_t^\infty \Phi_{t,\tau}^{-1}Q(\theta^0)x^*(\tau, \zeta^0)d\tau$, and using the triangle inequality we get 
\[ \left\| s(t; \hat{\theta}_t, \hat{\zeta}_t^N) - s(t; \theta^0, \zeta^0) \right\| \leq \left\| \int_t^\infty \Phi_{t,\tau}^{-1}Q(\hat{\theta}_t)x^*(\tau, \hat{\zeta}_t^N)d\tau - \int_t^\infty \Phi_{t,\tau}^{-1}Q(\hat{\theta}_t)x^*(\tau, \zeta^0)d\tau \right\| + \left\| \int_t^\infty \Phi_{t,\tau}^{-1}Q(\theta^0)x^*(\tau, \zeta^0)d\tau \right\| \]
\[ + \left\| \int_t^\infty \Phi_{t,\tau}^{-1}Q(\theta^0)x^*(\tau, \zeta^0)d\tau - \int_t^\infty \Phi_{t,\tau}^{-1}Q(\theta^0)x^*(\tau, \zeta^0)d\tau \right\| =: I_1^{N,t} + I_2 + I_3. \]

(i) Convergence of $I_1^{N,t}$ and $I_2$: $\lim_{t \to \infty} I_1^{N,t} = O(\epsilon_1(N))$, where $\epsilon_1(N) \to 0$, as $N \to \infty$, and $\lim_{t \to \infty} I_2 = 0$ follows from Lemma A.1 and Part 1 of the proof.

(ii) Convergence $I_3$: From the proof of Lemma A.2, 
\[ I_3 = \left\| \int_t^\infty \left( \Phi_{t,\tau}^{-1} - \Psi_{t,\tau}^{-1} \right)Q(\theta^0)x^*(\tau, \zeta^0)d\tau \right\| \]
\[ = \left\| \left\| \int_t^\infty \Phi_{t,\tau}^{-1} \left( A_s(\theta^0) - A_s(\hat{\theta}_s) \right)e^{A_s(\theta^0)(s-t)}ds \right\| \right\| \]
\[ \times \| Q(\theta^0)x^*(\tau, \zeta^0)d\tau \|. \]
$t \leq s \leq \tau < \infty$. Lemma A.1 yields the bound $\|\Phi_{t,s}\| \leq \beta_0 e^{-\rho_k(\tau-s)}$, $0 \leq s \leq \tau$, and for the time invariant case, $\|e^{A_0(t-s)}\| \leq \beta_1 e^{-\rho_{A_0}(t-s)}$, $0 \leq t \leq s$. For simplicity, set $\rho = \min[\rho_k, \rho_{A_0}]$ and let $T_\omega(\epsilon_2)$ be such that $\|A_\epsilon(\theta_0) - A_\epsilon(\hat{\theta}_s)\| < \epsilon_2$, $s > T_\omega(\epsilon_2)$. Then for $T_\omega(\epsilon_2) < t \leq s \leq \tau < \infty$ we obtain

$$I^s_t \leq \int_t^\infty \left[ \int_t^\tau \|\Phi_{t,s}\| \|A_\epsilon(\theta^0) - A_\epsilon(\hat{\theta}_s)\| \right.$$ 

$$\left. \left\|e^{A_\epsilon(\theta^0)^T(s-t)}\right\| ds \right] \|Q^0\| \|x^*(\tau, \zeta^0)\| d\tau$$ 

$$\leq \beta_0 \beta_1 \lambda_0 \epsilon_2 \|Q^0\| \int_t^\infty \left( \int_t^\tau e^{-\rho(t-s)} e^{-\rho(s-t)} ds \right) d\tau,$$

where $Q^0 \triangleq Q(\theta^0)$. The term (74) is satisfied for all arbitrarily small $\epsilon_2 > 0$ for all sufficiently large $t \geq T_\omega(\epsilon_2)$ by use of the bounds $\|\Phi_{t,s}\| \leq \beta_0 e^{-\rho(t-s)}$, $\|e^{A_0(t-s)}\| \leq \beta_1 e^{-\rho_{A_0}(t-s)}$, and $\lambda_0 = \sup_{\tau} \|x^*(\tau, \zeta^0)\|$. Hence, $I^s_t \leq \beta_0 \beta_1 \lambda_0 \epsilon_2 \|Q^0\| \int_t^\infty (\tau - t) e^{-\rho(t-s)} d\tau = \kappa \epsilon_2., \text{w.p.1., where} \kappa = \beta_0 \beta_1 \lambda_0 \|Q^0\|/\rho^2$. By Theorem 3.2 $\|A_\epsilon(\theta^0) - A_\epsilon(\hat{\theta}_s)\| \to 0$ as $t \to \infty$; therefore, as $t \to \infty$, $\epsilon_2 \to 0$. Hence, we obtain $\lim_{t\to\infty} I^s_t = 0 \text{ w.p.1}.$

In conclusion we have shown that $\lim_{t\to\infty} I^N_t = O(\epsilon_1(N))$, and therefore $\lim_{N\to\infty} \lim_{t\to\infty} I^{N,T}_t = 0 \text{ w.p.1.}$ In addition, $\lim_{t\to\infty} I^s_t = 0$ and $\lim_{t\to\infty} I^s_t = 0$. Therefore, $\lim_{N\to\infty} \lim_{t\to\infty} I^{N,T}_t = 0 \text{ w.p.1.}$ Hence, $\lim_{t\to\infty} s(t; \hat{\theta}_t^{pr}, \hat{\zeta}_t^{N_0}) - s(t; \theta^0, \zeta^0) = O(\epsilon_1(N))$, and $\lim_{t\to\infty} \lim_{t\to\infty} s(t; \hat{\theta}_t^{pr}, \hat{\zeta}_t^{N_0}) - s(t; \theta^0, \zeta^0) = 0 \text{ w.p.1.}$

3) Proof of $\hat{u}^0(t; \hat{\theta}_t^{pr}, \hat{\zeta}_t^{N_0}) = -R^{-1} \hat{B}_t^T (\hat{P}_t x_t + s(t; \hat{\theta}_t^{pr}, \hat{\zeta}_t^{N_0})) + \xi_k [\epsilon(t) - \epsilon(k)]$:

The solution $\Pi_{\theta} \in \mathbb{R}^{n^2}$, $\theta \in \mathbb{R}^{n(n+m+(n+1)/2)}$, to the algebraic Riccati equation (3) parametrized by $\theta \in \Theta$ is a smooth function of $\theta$ (see [42]). Hence, $(\Pi(\theta^0); t \geq 0)$ satisfies $\Pi(\theta^0) < \infty \text{ w.p.1 for all } t \geq 0 \text{ since } \hat{\theta}_t^{pr} \in \Theta, t \geq 0$. It is shown in Part 1 of the proof that the mass signal $x^*(\tau, \hat{\zeta}_t^{N_0}) \in C_\theta(0, \infty), t \leq \tau < \infty$; therefore, $s(t; \hat{\theta}_t^{pr}, \hat{\zeta}_t^{N_0}) < \infty \text{ w.p.1 for all } t \geq 0$, evaluated along $\hat{\theta}_t^{pr}, t \geq 0$. Hence, $u^0(t; \hat{\theta}_t^{pr}, \hat{\zeta}_t^{N_0})$ is well defined.

**Proof of Theorem 4.3**

We recall the following notation and basic assumptions: $\theta^0_\epsilon$ denotes the true dynamical parameter of agent $A_i$ in $\Theta$ that parametrizes the matrices $[A_i, B_i, Q_i] \in \Theta$, which are to be estimated by agent $A_i$, and $\hat{\theta}_{i,t} = [\hat{A}_{i,t}, \hat{B}_{i,t}, \hat{Q}_i]$ is the estimated parameter of agent $A_i$ at time $t$. Note that $Q_i$ is in the information set of agent $A_i$, therefore does not need to be estimated. We set $\hat{\theta}^{pr}_{i,t} = (\hat{\theta}^s_t, \tau \leq s \leq t)$, the sample path of the estimated parameter matrices from time $\tau$ to time $t$. The population distribution parameter denoted by $\zeta^0 \in P$, where $P$ is the parameter set for $F_\zeta(\cdot)$, parametrizes $F_\zeta(\cdot)$. Further, the estimated population distribution parameter of agent $A_i$ is denoted as $\hat{\zeta}^{N_0}_{i,t}$, and $\hat{\zeta}^{pr}_{i,t} (N_0) \triangleq (\hat{\zeta}^{N_0}_{i,s}, \tau \leq s \leq t)$, is the sample path of the estimated distribution parameter from time $\tau$ to $t$. As shown in Theorem 3.2, under $A1$-$A3$, on the probability space $p^\theta$, $(\hat{\theta}_{i,t}^0; t \geq 0)$ converges w.p.1 to $\theta^0_\epsilon$ as $t \to \infty$, and by Theorem 3.3, under $A7$ and $A8$, $(\hat{\zeta}^{N_0}_{i,t}; t \geq 0)$ converges w.p.1 to $\zeta^0$ as $t \to \infty$ and $N \to \infty$, $1 \leq i \leq N$. Note that for the optional PCPI, $A6'$ also needs to be employed. In the sequel, $A_{\hat{\theta}_{i,t}}, B_{\hat{\theta}_{i,t}}$ will be used to denote the estimated dynamical parameters whereas $\Pi_{\hat{\theta}_{i,t}}$ denotes the solution to (15). Since the solution $\Pi_\theta \in \mathbb{R}^{n^2}$, $\theta \in \mathbb{R}^{n(n+m+(n+1)/2)}$, to the algebraic Riccati equation (3) parametrized by $\theta \in \Theta$, is a smooth function of $\theta$ (see [42]), $\Pi_{\hat{\theta}_{i,t}}, t \geq 0$, satisfies $\|\Pi_{\hat{\theta}_{i,t}} - \Pi_{\theta^0_\epsilon}\| \to 0 \text{ w.p.1 as } t \to \infty, 1 \leq i \leq N$. To
establish the theorem we first observe $x_i^0(t; \hat{\theta}_{i,t}^0, \hat{\theta}_{i,t}^t(N_0))$, $t \geq 0$, $1 \leq i \leq N$, is the state of the system subject to the dithered MF Adaptive control law computed from the sum
\begin{equation}
  u_i^0(t; \hat{\theta}_{i,t}^0, \hat{\theta}_{i,t}^t) = u_i^{loc}(t; \hat{\theta}_{i,t}) + u_i^{pop}(t; \hat{\theta}_{i,t}, \hat{\theta}_{i,t}^t) + u_i^{diff}(t), \quad t \geq 0, \quad 1 \leq i \leq N,
\end{equation}
where the control input due to the MF-SAC Law is given by
\begin{equation}
  u_i^0(t; \hat{\theta}_{i,t}^0, \hat{\theta}_{i,t}^t(N_0)) = -R^{-1}\mathbf{B}_{\theta_i}^T \Pi_{\theta_i,i} x_i(t) - R^{-1}\mathbf{B}_{\theta_i}^T s(t; \hat{\theta}_{i,t}, \hat{\theta}_{i,t}^t) + \xi_k [\epsilon_i(t) - \epsilon_i(k)], \quad t \geq 0.
\end{equation}
The term $\|x_i^0(t; \hat{\theta}_{i,t}^0, \hat{\theta}_{i,t}^t(N_0)) - x_i^0(t; \theta_0^0, \xi_0^0)\|^2$ will be decomposed into four parts, and convergence properties will be established for each term. We have
\begin{equation}
  d\Phi_i(t, \tau, \theta_0^0) = [A_{\theta_0^0} - B_{\theta_0^0}R^{-1}B_{\theta_0^0}^T \Pi_{\theta_0^0}] \Phi_i(t, \tau, \theta_0^0) dt,
\end{equation}
and
\begin{equation}
  d\Phi_i(t, \tau, \theta_0^0, \hat{\theta}_{i,t}^t) = [A_{\theta_0^0} - B_{\theta_0^0}R^{-1}B_{\theta_0^0}^T \Pi_{\theta_0^0}] \Phi_i(t, \tau, \theta_0^0, \hat{\theta}_{i,t}^t) dt,
\end{equation}
Also, in the sequel for clarity we will suppress the subscript $i$ and adopt the notation: $\Phi_{t,s} \triangleq \Phi(t, s, \theta_0^0, \hat{\theta}_{t,s})$, $\Psi_{t,s} \triangleq \Psi(t, s, \theta_0^0)$, $A_0 \triangleq A_{\theta_0^0}$, $B_0 \triangleq B_{\theta_0^0}$, $I_0 \triangleq I_{\theta_0^0}$, $s(t) \triangleq s(t; \theta_0^0, \xi_0^0)$, $x_0^0(t) \triangleq x_0^0(t; \theta_0^0, \xi_0^0)$. Displaying the dependence of the fundamental matrix on the parameter estimate trajectory, we use the integral representation and by use of the Cauchy Schwarz Inequality (henceforth termed CS)’s Inequality we obtain
\begin{equation}
  \frac{1}{T} \int_0^T \left\| x_i^0(t; \hat{\theta}_{i,t}^0, \hat{\theta}_{i,t}^t(N_0)) - x_i^0(t; \theta_0^0, \xi_0^0) \right\|^2 dt \leq \frac{4}{T} \int_0^T \left\| \Phi_{t,0} x(0) - \Psi_{t,0} x(0) \right\|^2 dt
\end{equation}
\begin{equation}
  + \frac{4}{T} \int_0^T \left\| \int_0^t \Phi_{t,\tau} B_0^0 R^{-1} B_{\theta_0^0}^T \Pi_{\theta_0^0} s(t) dt - \int_0^t \Psi_{t,\tau} B_0^0 R^{-1} B_{\theta_0^0}^T s(t) dt \right\|^2 dt
\end{equation}
\begin{equation}
  + \frac{4}{T} \int_0^T \left\| \Phi_{t,0} \int_0^t \Phi_{t,\tau} Dw(\tau) - \Psi_{t,0} \int_0^t \Psi_{t,\tau} Dw(\tau) \right\|^2 dt
\end{equation}
\begin{equation}
  + \frac{4}{T} \int_0^T \left\| \left\{ \int_{k}^{\min\{t,k+1\}} \Phi_{t,\tau} B_{\theta_0^0} \xi_k [\epsilon(\tau) - \epsilon(k)] d\tau \right\} - \left\{ \int_{k}^{\min\{t,k+1\}} \Psi_{t,\tau} B_{\theta_0^0} \xi_k [\epsilon(\tau) - \epsilon(k)] d\tau \right\} \right\|^2 dt
\end{equation}
\begin{equation}
  =: I_1^T + I_2^{N,T} + I_3^T + I_4^T.
\end{equation}
We will show one by one that the limit supremums ($\limsup_{T \to \infty}$) of $I_1^T$ (76), $I_2^{N,T}$ (77) and $I_4^T$ (79) are all 0 with probability 1, and the limit supremum ($\limsup_{N \to \infty} \limsup_{T \to \infty}$) of $I_2^{N,T}$ (77) is equal to 0 with probability 1.

(i) Convergence of $I_1^T$ follows from Lemma A.2.

(ii) Convergence of $I_2^{N,T}$ (77): Adding and subtracting $\Phi(t, \tau, \theta_0^0, \hat{\theta}_{t,s}) B_0^0 R^{-1} B_{\theta_0^0}^T s(t; \theta_0^0, \xi_0^0)$ and $\Phi(t, \tau, \theta_0^0, \hat{\theta}_{t,s}) B_0^0 R^{-1} B_{\theta_0^0}^T s(t; \theta_0^0, \xi_0^0)$ using Lemma A.1, Lemma A.2, and $\sup_{\tau \geq t} \|s(t; \hat{\theta}_{t,s}, \hat{\theta}_{t,s}) - s(t; \theta_0^0, \xi_0^0)\| \leq \epsilon_1(N)$, from Proposition 4.1 we get $\limsup_{T \to \infty} I_2^{N,T} \leq O((\epsilon_1(N))^2)$ w.p.1, which implies
\begin{equation}
  \limsup_{N \to \infty} \limsup_{T \to \infty} I_2^{N,T} = 0 \quad \text{w.p.1.}
\end{equation}
(iii) Convergence of $I_3^T$ (78): We have

$$I_3^T = \frac{4}{T} \int_0^T \left\| \Phi(t, 0, \theta^0, \hat{\theta}_r(t)) \right\|_{\mathbb{E}}^2 dt,$$

where $x_0 = y_0 < \infty$ by A1. The difference $z_t = x_t - y_t$, satisfies

$$z_t = \Phi_{t,0} \int_0^t \Phi_{\tau,0} \mathbf{D} dw(\tau) - \Psi_{t,0} \int_0^t \Psi_{\tau,0} \mathbf{D} dw(\tau), \quad t \geq 0.$$ 

Alternatively, one can write $dz_t = A_s(\theta^0)z_t dt + [A_s(\theta^0) - A_s(\hat{\theta}_s)]y_t dt$, giving $z_t = \Psi_{t,0} z_0 + \int_0^t \Psi_{t,s} (A_s(\theta^0) - A_s(\hat{\theta}_s)) y_s ds$. Hence we can write (80) as $I_3^T = \frac{4}{T} \int_0^T \left\| \Psi_{t,0} z_0 + \int_0^t \Psi_{t,s} (A_s(\theta^0) - A_s(\hat{\theta}_s)) y_s ds \right\|_{\mathbb{E}}^2 dt$, and use the CS Inequality to obtain

$$I_3^T \leq \frac{8}{T} \int_0^T \left\| \Psi_{t,0} z_0 \right\|_{\mathbb{E}}^2 dt + \frac{8}{T} \int_0^T \left\| \int_0^T \Psi_{t,s} (A_s(\theta^0) - A_s(\hat{\theta}_s)) y_s ds \right\|_{\mathbb{E}}^2 dt.$$ 

Now $z_0 = x_0 - y_0 = 0$, since $x_0 = y_0$; therefore, $I_3^T \leq \frac{8}{T} \int_0^T \left\| \int_0^T \Psi_{t,s} (A_s(\theta^0) - A_s(\hat{\theta}_s)) y_s ds \right\|_{\mathbb{E}}^2 dt$. Let $T_\omega(\epsilon_2)$ be such that $t \geq T_\omega(\epsilon_2)$ implies $\|\hat{\theta}_t - \theta^0\| < \epsilon_2$ then $I_3^T \leq \frac{8}{T} \int_0^{T_\omega} \left\| \int_0^T \Psi_{t,s} (A_s(\theta^0) - A_s(\hat{\theta}_s)) y_s ds \right\|_{\mathbb{E}}^2 dt = I_{31}^T + I_{32}^T$. Following Lemma A1 we write $\left\| \Psi_{t,s} \right\| \leq \beta_0 e^{-\rho_s(t-s)}$ for $t \geq s \geq 0$. We also use the CS Inequality, and let $\epsilon_2 \rightarrow 0$ as $t \rightarrow \infty$. We get $\lim \sup_{T \rightarrow \infty} I_{31}^T = 0$ and $\lim \sup_{T \rightarrow \infty} I_{32}^T = 0$ w.p.1. Therefore $\lim \sup_{T \rightarrow \infty} I_3^T = 0$ w.p.1.

(iv) Convergence of $I_4^T$ (79): We have

$$\lim \sup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left\| \left( \int_k \int_k^{\min(t,k+1)} \left( \Phi_{t,\tau} - \Psi_{t,\tau} \right) \mathbf{B}^0 \xi_k |e(\tau) - e(k)| \right) dt \right\|_{\mathbb{E}}^2 dt.$$ 

This term is treated by a direct application of Lemma A3; therefore, the limit of the time average integral tends to 0.

Overall, we have shown that $\lim \sup_{T \rightarrow \infty} I_1^T = 0$, $\lim \sup_{T \rightarrow \infty} I_2^{N,T} = O((\epsilon_1(N))^2)$, $\lim \sup_{T \rightarrow \infty} I_3^T = 0$ and $\lim \sup_{T \rightarrow \infty} I_4^T = 0$. This implies $\lim \sup_{T \rightarrow \infty} (1/T) \int_0^T \left\| x_0(t; \theta^0, \xi^0) - x_0(t; \theta^0, \xi^0) \right\|_{\mathbb{E}}^2 = O((\epsilon_1(N))^2)$ w.p.1. Consequently, $\lim_{N \rightarrow \infty} \lim_{T \rightarrow \infty} (1/T) \int_0^T \left\| \hat{x}_0(t; \theta^0, \xi^0) - x_0(t; \theta^0, \xi^0) \right\|_{\mathbb{E}}^2 = 0$ w.p.1, $1 \leq i \leq N$.

**Proposition C.1:** For the system (1), let A1-A4, A7, A8 hold. Let $\hat{w}_0^i \in \hat{U}_M^N$ be the MF-SAC input (17) and $u_0^i \in U_M^N$ be the non-adaptive MF-SC input. Then,

$$\lim \sup_{N \rightarrow \infty} \lim \sup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left\| u_0^i - u_0^i \right\|_{\mathbb{E}}^2 dt = 0 \quad \text{w.p.1}, \quad 1 \leq i \leq N.$$ 

**Proof:** We have the term $I_{N,T} = \frac{1}{T} \int_0^T \left\| u_0^i(t; \theta^0, \xi^0) - u_0^i(t; \theta^0, \xi^0) \right\|_{\mathbb{E}}^2 dt$, which we separate into two parts as $I_{N,T} = \frac{1}{T} \int_0^{T_\omega} \left\| u_0^i(t; \theta^0, \xi^0) - u_0^i(t; \theta^0, \xi^0) \right\|_{\mathbb{E}}^2 dt =: I_1^{N,T} + I_2^{N,T}$, where $T_\omega$ is a random instant to be determined later. We will only establish $\lim_{N \rightarrow \infty} \lim_{T \rightarrow \infty} I_{1}^{N,T} = 0$ w.p.1 here, as $\lim_{N \rightarrow \infty} \lim_{T \rightarrow \infty} I_{1}^{N,T} = 0$ w.p.1 is a simpler case of the same argument.
Convergence of $I_{2}^{N,T}$: We have

$$I_{2}^{N,T} = \frac{1}{T} \int_{T_{\omega}}^{T} \|u_{i}^{0}(t; \hat{\theta}_{i,t}, \hat{\zeta}_{i,t}) - u_{i}^{0}(t; \theta_{0}, \zeta_{0})\|^2 dt =$$

$$= \frac{1}{T} \int_{T_{\omega}}^{T} \|R_{i}^{-1}B_{t}^{T} \Pi_{x}x_{i}^{0}(t; \theta_{0}, \zeta_{0}) + R_{i}^{-1}\dot{B}_{t} s(t; \hat{\theta}_{i,t}, \hat{\zeta}_{i,t}) - R_{i}^{-1}B_{t}^{T} \Pi_{x}x_{i}^{0}(t; \theta_{0}, \zeta_{0})\|^2 dt,$$

Dropping the subscript $i$, adopting the notation $B_{0}^{0} = B(\theta_{0}), \dot{B}_{i} = B(\hat{\theta}_{i}), \Pi_{x}^{0} = \Pi(\theta_{0}), \Pi_{x}^{i} = \Pi(\hat{\theta}_{i}), u_{0}^{0} = u_{0}(t; \theta_{0}, \zeta_{0}), \hat{u}_{0}^{0} = u_{0}(t; \hat{\theta}_{i}, \hat{\zeta}_{i})$, $x_{0}^{0} = x_{0}(t; \theta_{0}, \zeta_{0}), \hat{x}_{0}^{0} = x_{0}(t; \theta_{0}, \zeta_{0})(N_{0}), s(t) = s(t; \theta_{0}, \zeta_{0}), \hat{s}(t) = s(t; \hat{\theta}_{i}, \hat{\zeta}_{i})$, and using the CS Inequality, we obtain

$$I_{2}^{N,T} \leq \frac{2}{T} \int_{T_{\omega}}^{T} \|R_{i}^{-1}B_{t}^{T} \Pi_{x}x_{i}^{0}(t) - R_{i}^{-1}B_{t}^{T} \Pi_{x}x_{i}^{0}(t)\|^2 dt$$

$$+ \frac{2}{T} \int_{T_{\omega}}^{T} \|R_{i}^{-1}\dot{B}_{t} s(t; \hat{\theta}_{i}, \hat{\zeta}_{i}) - R_{i}^{-1}B_{t}^{T} s(t; \theta_{0}, \zeta_{0})\|^2 dt$$

$$= I_{21}^{N,T} + I_{22}^{N,T}.$$ We set $T_{\omega}$ to be the random instant such that $t \geq T_{\omega}$ implies $\|\hat{x}_{i}^{0}(t) - x_{i}^{0}(t)\| < \epsilon_{1}(N)$ and $\|\hat{s}(t) - s(t)\| < \epsilon_{1}(N)$. We obtain $\limsup_{T \to \infty} I_{21}^{N,T} = O(\epsilon_{1}(N)^{2})$ and $\limsup_{T \to \infty} I_{22}^{N,T} = O(\epsilon_{1}(N)^{2})$ from Section C 2.(i), which implies

$$\limsup_{N \to \infty} \limsup_{T \to \infty} I_{21}^{N,T} + \limsup_{N \to \infty} \limsup_{T \to \infty} I_{22}^{N,T} = 0 \text{ w.p.1.}$$

**APPENDIX D**

The following five lemmas will be used to prove Proposition 4.4 and Proposition 4.5. We use the notation $m((x^{N})^{0}(t; \theta^{[1:N]}, \zeta^{0})) \triangleq m((1/N) \sum_{k=1}^{N} x_{k}^{0}(t; \theta_{0}, \zeta_{0}) + \eta)$, $m((x^{N})^{0}(t; \hat{\theta}^{[1:N]}, \hat{\zeta}^{[1:N]})) \triangleq m((1/N) \sum_{k=1}^{N} x_{k}^{0}(t; \hat{\theta}_{k}, \hat{\zeta}_{k}) + \eta)$, where $m(\cdot)$ is defined in A4.

**Lemma D.1:** Let Assumptions A1-A4 hold. For the system (1), the MF control law $u_{i}(t; \theta_{0}, \zeta_{0})$ (7) and its corresponding closed-loop solution $x_{i}^{0}(t; \theta_{0}, \zeta_{0})$ satisfy

$$\sup_{N \geq 1} \max_{1 \leq i \leq N} \limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \left( \|x_{i}^{0}(t; \theta_{0}, \zeta_{0})\|^2 + \|u_{i}^{0}(t; \theta_{0}, \zeta_{0})\|^2 \right) dt < \infty.$$  \hspace{1cm} (81)

**Proof:**

The same result has been shown to hold in [37] (Theorem 4.1) for control action in the form of $u_{i}^{0}(t) = u_{i}^{loc}(t) + u_{i}^{pop}(t)$ using the notation defined in (75). We are going to repeat this result here for completeness.
(i) \( \limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \| x_i^0(t; \theta_i^0, \zeta^0) \|^2 dt \leq K_1 < \infty: \)

\[
\| x_i^0(t; \theta_i^0, \zeta^0) \|^2 = \left\| e^{A_i(\theta_i^0)t}x_i(0) - \int_{0}^{t} e^{A_i(\theta_i^0)(t-\tau)}B_i^0R^{-1}B_i^{1\top}s(\tau; \theta_i^0, \zeta^0)d\tau + \int_{0}^{t} e^{A_i(\theta_i^0)(t-\tau)}Dw_i(\tau) \right\|^2. \tag{82}
\]

Using the CS Inequality, we obtain the inequality:

\[
\limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \| x_i^0(t; \theta_i^0, \zeta^0) \|^2 dt \leq \limsup_{T \to \infty} \frac{3}{T} \int_{0}^{T} \left\| e^{A_i(\theta_i^0)t}x_i(0) \right\|^2 dt + \limsup_{T \to \infty} \frac{3}{T} \int_{0}^{T} \left\| \int_{0}^{t} e^{A_i(\theta_i^0)(t-\tau)}B_i^0R^{-1}B_i^{1\top}s(\tau; \theta_i^0, \zeta^0)d\tau \right\|^2 dt + \limsup_{T \to \infty} \frac{3}{T} \int_{0}^{T} \left\| \int_{0}^{t} e^{A_i(\theta_i^0)(t-\tau)}Dw_i(\tau) \right\|^2 dt, \tag{84}
\]

\[
\leq \limsup_{T \to \infty} I_1^T + \limsup_{T \to \infty} I_2^T + \limsup_{T \to \infty} I_3^T. \tag{87}
\]

For A1 and A2 hold, we obtain \( \limsup_{T \to \infty} I_1^T = 0 \) w.p.1.

It is shown in [37] (Theorem 4.1) that \( \limsup_{T \to \infty} I_2^T \leq \kappa_1 < \infty \) uniformly for all \( \theta_i^0 \in \Theta \).

Using Lemma A.4 we write

\[
\limsup_{T \to \infty} I_3^T = 3 \int_{0}^{\infty} \text{Tr} \left( e^{A_i(\theta_i^0)(t-\tau)}DD^{\top}e^{A_i(\theta_i^0)(t-\tau)} \right). \tag{88}
\]

We use \( \sup_{\theta_i^0 \in \Theta} \| e^{A_i(\theta_i^0)(t-\tau)} \| \leq \beta e^{-\rho(t-s)} \) as shown in Lemma A.1 and get

\[
\limsup_{T \to \infty} I_3^T \leq 3\|D\|^2 \beta^2 / 2\rho = \kappa_2 < \infty. \tag{89}
\]

Therefore,

\[
\sup_{N \geq 1} \max_{1 \leq i \leq N} \limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \| x_i^0(t; \theta_i^0, \zeta^0) \|^2 dt \leq \kappa_1 + \kappa_2 = K_1 < \infty \text{ w.p.1.} \tag{90}
\]

(ii) \( \limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \| u_i^0(t; \theta_i^0, \zeta^0) \|^2 dt \leq K_2 < \infty: \) We have the MF Control Law

\[
u_i^0(t; \theta_i^0, \zeta^0) = \nu_i^{loc}(t; \theta_i^0) + \nu_i^{pop}(t; \theta_i^0, \zeta^0)
= -R^{-1}B_i^{1\top}(\Pi_i^0x_i(t) + s(t; \theta_i^0, \zeta^0)), \quad t \geq 0. \tag{91}
\]

Also, the mass offset function is

\[
s(t; \theta_i^0, \zeta^0) = -e^{-A_i^\top(\theta_i^0)t} \int_{t}^{\infty} e^{A_i^\top(\theta_i^0)s}Q_i x^*(\tau, \zeta^0)d\tau. \tag{93}
\]

We employ A1 and obtain \( M_{x^*} = \sup_{\tau \geq 1} \| x^* \|, M_B = \sup_{\theta_i \in \Theta} \| B_\theta \|, M_{\Pi} = \sup_{\theta_i \in \Theta} \| \Pi_\theta \|, \) and \( M_Q = \sup_{\theta_i \in \Theta} \| Q_\theta \|. \) Then we obtain

\[
\sup_{\theta \in \Theta} \| s_i \| \leq M_Q M_{x^*} \beta / \rho \triangleq M_s, \quad 1 \leq i \leq N. \tag{94}
\]

Using (90), and the bounds given above, we write

\[
\sup_{N \geq 1} \max_{1 \leq i \leq N} \limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \| u_i^0(t; \theta_i^0, \zeta^0) \|^2 dt \leq \| \nu_i^0 \| M_B M_{\Pi} K_1 + \| \nu_i^0 \| M_B M_s = K_2 \text{ w.p.1.} \tag{95}
\]
Consequently, we get $\sup_{N \geq 1} \max_{1 \leq i \leq N} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left( \| x_i^0(t; \theta_i^0, \zeta_0) \|^2 + \| u_i^0(t, \theta_i^0) \|^2 \right) dt \leq K_1 + K_2 < \infty$. As both $K_1$ and $K_2$ are independent of $1 \leq i \leq N$ and $N \geq 1$, we obtain (81).

**Lemma D.2:** Let Assumptions A1–A4 hold. For the system (1), the closed loop solution $x_i^0(t; \theta_i^0, \zeta_0)$ with the control law $u_i^0(t; \theta_i^0, \zeta_0)$ and the cost-coupling function $m((x^N)^0(t; \theta^{[1:N]}, \zeta_0))$ satisfy

$$\sup_{N \geq 1} \max_{1 \leq i \leq N} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left\| x_i^0(t; \theta_i^0, \zeta_0) - m((x^N)^0(t; \theta^{[1:N]}, \zeta_0)) \right\|^2 dt < \infty.$$  

We recall the definition $J_i(u_i, x^*) \triangleq \limsup_{T \to \infty} \frac{1}{T} \int_0^T \{ \| x_i - x^* \|_Q^2 + \| u_i \|_R^2 \} dt$ w.p.1, where $x^* \in \mathcal{C}_b[0, \infty)$ is the solution to the MF Equation System (6).

**Proof:**

Using the CS Inequality we write

$$\frac{1}{T} \int_0^T \left\| x_i^0(t; \theta_i^0, \zeta_0) - m((x^N)^0(t; \theta^{[1:N]}, \zeta_0)) \right\|^2 dt \leq \frac{1}{T} \int_0^T \left\| x_i^0(t; \theta_i^0, \zeta_0) \right\|^2 dt + \frac{1}{T} \int_0^T \left\| m((x^N)^0(t; \theta^{[1:N]}, \zeta_0)) \right\|^2 dt$$

$$\leq I_1^T + I_2^{N,T}. \tag{96}$$

Using Lemma D.1 we get $\limsup_{T \to \infty} I_1^T \leq 2K_1$, where $K_1$ is given in (90).

For $I_2^{N,T}$ we employ A4, and LHS of (96) can be further bounded by

$$I_2^{N,T} = \frac{2}{T} \int_0^T \left\| \frac{1}{N} \sum_{k=1}^N x_k^0(t; \theta_k^0, \zeta_0) + \eta \right\|^2 dt \leq \frac{2\gamma^2}{T} \int_0^T \left\| \frac{1}{N} \sum_{k=1}^N x_k^0(t; \theta_k^0, \zeta_0) + \frac{\eta}{N} \right\|^2 dt. \tag{99}$$

Using the CS Inequality we write

$$I_2^T \leq \frac{4\gamma^2}{T^2N} \int_0^T \left\| \sum_{k=1}^N x_k^0(t; \theta_k^0, \zeta_0) \right\|^2 dt + \frac{4\gamma^2}{T} \int_0^T \eta^2 dt \leq I_2^{N,T} + I_2^T, \tag{100}$$

We have $I_2^T = 4\gamma^2\eta^2$. For $I_2^{N,T}$ using the CS Inequality again we get

$$I_2^{N,T} \leq \frac{4\gamma^2N}{T^2N} \int_0^T \| x_i^0(t; \theta_i^0, \zeta_0) \|^2 dt. \tag{101}$$

We have shown in Lemma D.1 that $\sup_{N \geq 1} \max_{1 \leq i \leq N} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| x_i^0(t; \theta_i^0, \zeta_0) \|^2 dt \leq K_1$. Therefore we get the bound

$$I_2^{N,T} \leq \frac{4\gamma^2K_1}{N} + 4\gamma^2 \eta^2. \tag{104}$$
We have shown that \( \limsup_{T \to \infty} I^T \leq 2K_1 \). Now we have shown that \( \limsup_{T \to \infty} I^{N,T}_2 \leq 4\gamma^2 K_1/N + 4\gamma^2 \eta^2 \).

Finally we define \( K_3 \triangleq 2K_1 + 4\gamma^2 K_1/N + 4\gamma^2 \eta^2 \) and finish the proof:

\[
\sup_{N \geq 1} \max_{1 \leq t \leq N} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left\| x_i^o(t; \theta_i^0, \zeta_i^0) - m((x^N)_t^o(0; \theta^{[1:N]}, \zeta^0)) \right\|^2 dt \leq K_3 < \infty. \tag{105}
\]

\[\blacksquare\]

**Lemma D.3**: For the system (1) subject to **A1-A5**, when all agents apply the control generated by (7), the cost function \( J_i(u_i^0, u^0_{-i}) \) (2) satisfies

\[
\lim_{N \to \infty} J_i^N(u_i^0, u^0_{-i}) = J_i(u_i^0, x^*) \quad \text{w.p.1, } 1 \leq i \leq N.
\]

**Proof:**

From (2) we have the cost function

\[
J_i^N(u_i^0, u^0_{-i}) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \| x_i^o(t; \theta_i^0, \zeta_i^0) - m((x^N)_t^o(0; \theta^{[1:N]}, \zeta^0)) \|_Q^2 + \| u_i^0(t; \theta_i^0, \zeta_i^0) \|_R^2 \right\} dt. \tag{106}
\]

Adding and subtracting \( x^*(t, \zeta^0), 0 \leq t \leq T, \) to the first integrand on the RHS, we get

\[
J_i^N(u_i^0, u^0_{-i}) \leq J_i^\infty(u_i^0, x^*) \tag{107}
\]

\[\begin{aligned}
&+ \limsup_{T \to \infty} \frac{2}{T} \int_0^T \left\{ (x_i^o(t; \theta_i^0, \zeta_i^0) - x^*(t, \zeta^0))^T \begin{pmatrix} Q & m((x^N)_t^o(0; \theta^{[1:N]}, \zeta^0)) \end{pmatrix} - m((x^N)_t^o(0; \theta^{[1:N]}, \zeta^0)) \right\} dt \\
&+ \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| m((x^N)_t^o(0; \theta^{[1:N]}, \zeta^0)) - x^*(t, \zeta^0) \|_Q^2 dt,
\end{aligned} \tag{108}
\]

\[=: I_1 + I_2 + I_3, \tag{110}\]

where,

\[
J_i^\infty(u_i^0, x^*(t, \zeta^0)) \triangleq \limsup_{T \to \infty} \int_0^T \left\{ \| x_i^o(t; \theta_i^0, \zeta_i^0) - x^*(t, \zeta^0) \|_Q^2 + \| u_i^0(t; \theta_i^0, \zeta_i^0) \|_R^2 \right\} dt. \tag{111}
\]

It is shown in [37, Lemma 6.3] that \( I_2^N = O(\epsilon_2(N)) \) and \( I_3^N = o(\epsilon_2(N)) \) where \( \epsilon_2(N) \to 0 \) as \( N \to \infty \). Therefore,

\[
J_i^N(u_i^0, u^0_{-i}) \leq J_i^\infty(u_i^0, x^*) + O(\epsilon_2(N)). \tag{112}
\]

Adding and subtracting \( m((x^N)_t^o(0; \theta^{[1:N]}, \zeta^0)) \) to \( J_i^N(u_i^0, x^*) \), and following the same steps above one obtains

\[
J_i(u_i^0, x^*) \leq J_i(u_i^0, u^0_{-i}) + O(\epsilon_2(N)). \tag{113}
\]

Hence, one gets

\[
\lim_{N \to \infty} J_i^N(u_i^0, u^0_{-i}) = J_i(u_i^0, x^*) \quad \text{w.p.1, } 1 \leq i \leq N. \tag{114}
\]

\[\blacksquare\]

**Lemma D.4**: Under **A1-A5**, the set of controls \( U^N_{MF} = \{ u_i^0, 1 \leq i \leq N \} \) is such that when \( u_i \in U^N_g \) is any control adapted to \( \mathcal{F}^N \),

\[
\lim_{N \to \infty} \inf_{u_i \in U^N_g} J_i^N(u_i, u^0_{-i}) = \lim_{N \to \infty} J_i^N(u_i^0, u^0_{-i}) \quad \text{w.p.1, } 1 \leq i \leq N.
\]
Proof:

Let \( u_i \triangleq u_i(t; \theta_i^0, \zeta^0) \in \mathcal{U}_i^N \) be a feedback control action and \( x_i \triangleq x_i(t; \theta_i^0, \zeta^0) \) be the corresponding closed loop solution. Then, from (2) we have the cost function

\[
J_i^N(u_i, x^*) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \| x_i(t) - x^*(t, \zeta^0) \|_Q^2 + \| u_i(t) \|_R^2 \right\} dt.
\]  

Adding and subtracting \( m^N_{i, u_{-i}}(t) \triangleq m(x_i(t; \theta_i^0, \zeta^0), x^0_{j \neq i}(t; \theta^{[1:N]}, \zeta^0)) \) we get

\[
J_i^N(u_i, x^*) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \| x_i(t) - m^N_{i, u_{-i}}(t) \|_Q^2 + \| u_i(t) \|_R^2 \right\} dt 
\leq \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| x^*(t, \zeta^0) - m^N_{i, u_{-i}}(t) \|_Q^2 dt 
+ \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| m^N_{i, u_{-i}}(t) - x^*(t, \zeta^0) \|_Q dt 
+ \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| x_i(t) - m^N_{i, u_{-i}}(t) \|_Q \| m^N_{i, u_{-i}}(t) - x^*(t, \zeta^0) \|_Q dt 
= J_i^N(u_i, 0^i_{-i}) + J^N_i + I^N_2.
\]  

It is shown in [37, Lemma 6.3] that \( I^N_1 = o(\epsilon_2(N)) \) where \( \epsilon_2(N) \to \infty \) as \( N \to \infty \).

For \( I^N_2 \):

We add and subtract \( (m^N)^0 \triangleq \langle (x^N)^0(t; \theta^{[1:N]}, \zeta^0) \rangle \) and obtain

\[
I^N_2 = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left\{ (x_i(t) - m^N_{i, u_{-i}}(t))^\top Q (m^N_{i, u_{-i}}(t) - (m^N)^0(t) - x^*(t, \zeta^0)) \right\} dt 
\leq \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left\{ (x_i(t) - m^N_{i, u_{-i}}(t))^\top Q (m^N_{i, u_{-i}}(t) - (m^N)^0(t)) \right\} dt 
+ \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left\{ (x_i(t) - m^N_{i, u_{-i}}(t))^\top Q ((m^N)^0(t) - x^*(t, \zeta^0)) \right\} dt 
= I^N_{21} + I^N_{22}.
\]  

It is shown in [37, Lemma 6.4] that \( |I^N_{21}| = O(\epsilon_2(N)) \) and it is shown in [37, Lemma 6.4] that \( |I^N_{22}| = O(1/N) \).

As \( u_i^0(t; \theta_i^0, \zeta^0) \) is the optimal tracking solution to tracking signal \( x^*(t, \zeta^0) \) (5), we obtain

\[
J_i(u_i^0, x^*) \leq \inf_{u_i \in \mathcal{U}_i^0} J_i^N(u_i, u_{-i}^0) + o(\epsilon_2(N)) + O(\epsilon_2(N)) + O(1/N).
\]  

Adding and subtracting \( x^*(t, \zeta^0) \) to \( J_i^N(u_i, u_{-i}^0) \), and following the same steps above one obtains

\[
\inf_{u_i \in \mathcal{U}_i^0} J_i^N(u_i, u_{-i}^0) \leq J_i(u_i^0, x^*) + o(\epsilon_2(N)) + O(\epsilon_2(N)) + O(1/N).
\]
Hence, $$\lim_{N \to \infty} \inf_{u_i \in \mathcal{U}_i^N} J_i^N(u_i, u_{-i}^0) = J_i(u_i^0, x^*)$$ w.p.1, $$1 \leq i \leq N$$. It is shown in Lemma D.3 that $$\lim_{N \to \infty} J_i^N(u_i^0, u_{-i}^0) = J_i(u_i^0, x^*)$$ w.p.1, $$1 \leq i \leq N$$. Therefore, one gets

$$\lim_{N \to \infty} \inf_{u_i \in \mathcal{U}_i^N} J_i^N(u_i, u_{-i}^0) = \lim_{N \to \infty} J_i^N(u_i^0, u_{-i}^0) \text{ w.p.1, } 1 \leq i \leq N. \quad (129)$$

Lemma D.5: Under the MF-SAC Law and A1-A4, A7, A8

$$\lim_{N \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \|m((x^N)^0(t; \theta^{[1:N]}, \zeta^0)) - m((x^N)^0(t; \hat{\theta}^{[1:N]}, \hat{\zeta}^{[1:N]}))\|^2 dt = 0 \text{ w.p.1, } 1 \leq i \leq N.$$ 

Proof: We have the equation $$I_{N,T} = \frac{1}{T} \int_0^T \|m((x^N)^0(t; \theta^{[1:N]}, \zeta^0)) - m((x^N)^0(t; \hat{\theta}^{[1:N]}, \hat{\zeta}^{[1:N]}))\|^2 dt,$$ where $$\theta^{[1:N]} \triangleq \{\theta_i^0, 1 \leq i \leq N\}, \hat{\theta}^{[1:N]} \triangleq \{\hat{\theta}_i^0, 0 \leq \tau \leq T, 1 \leq i \leq N\},$$ and $$\zeta^{[1:N]} \triangleq \{\zeta_t^i(N), 0 \leq \tau \leq T, 1 \leq i \leq N\}.$$ Employing A4, we get the inequality $$I_{N,T} \leq \frac{1}{T} \int_0^T \left\| \frac{N}{N} \sum_{i=1}^N x_i^0(t; \theta_i^0, \zeta^0) - \frac{1}{N} \sum_{i=1}^N x_i^0(t; \hat{\theta}_i^{0,t}, \hat{\zeta}_i^{0,t}(N)) \right\|^2 dt.$$ 

Using the CS Inequality we get $$I_{N,T} \leq \frac{1}{N} \sum_{i=1}^N \left\{ \limsup_{T \to \infty} \frac{1}{T} \int_0^T \|x_i^0(t) - \hat{x}_i^0(t)\|^2 dt \right\}. \text{ It is shown in Theorem 4.3 that } \limsup_{T \to \infty} \frac{1}{T} \int_0^T \|x_i^0(t) - \hat{x}_i^0(t)\|^2 dt = O(\epsilon_1(N)^2) \text{ w.p.1; hence, we get } \limsup_{T \to \infty} I_{N,T} = O(\epsilon_1(N)^2) \text{ w.p.1, which implies } \limsup_{N \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \|m((x^N)^0(t; \theta^{[1:N]}, \zeta^0)) - m((x^N)^0(t; \hat{\theta}^{[1:N]}, \hat{\zeta}^{[1:N]}))\|^2 dt = 0 \text{ w.p.1.}
(\hat{m}^N)^0 = m((x^N)^0(t; \hat{\theta}^{1:N}, \hat{\zeta}^{1:N})), \ u^0_i(t; \theta_i^0; \zeta^0), \ \hat{u}^0_i(t; \theta_i^0; \zeta^0), \ and \ get \ the \ inequality

\[ J_i^N(\hat{u}^0_i, \hat{u}^0_{-i}) \leq \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| \hat{x}^0_i - x^0_i \|^2 dt + \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| (m^N)^0_j - (\hat{m}^N)^0_j \|^2 dt + \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| u^0_i - \hat{u}^0_i \|^2 dt + \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| u^0_i - \hat{u}^0_i \|^2 dt \]

Therefore, we have

\[ \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| (m^N)^0 - (\hat{m}^N)^0 \|^2 dt + \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| u^0_i - \hat{u}^0_i \|^2 dt + \limsup_{T \to \infty} \frac{1}{T} \int_0^T \| u^0_i - \hat{u}^0_i \|^2 dt \]

(i) Convergence of \( I_1^{N,T} \): We show in Theorem 3.2 that \( \hat{\theta}_i(t) \to \theta_i^0 \) w.p.1 as \( t \to \infty, \ 1 \leq i \leq N \), and in Theorem 3.3 that \( \hat{\zeta}_{i,t}^N \to \zeta^0 \) w.p.1, as \( t \to \infty \) and \( N \to \infty, \ 1 \leq i \leq N \); therefore the hypotheses for Theorem 4.3 are satisfied and \( \limsup_{T \to \infty} I_1^{N,T} = O(\epsilon_1(N)^2) \) w.p.1.

(ii) \( I_2^T \) \& \( I_5^T \): \( I_2^T + I_5^T \) equals to the non-adaptive MF cost function; i.e., \( J_i(u^0_i, u^0_{-i}) = \limsup_{T \to \infty} (I_2^T + I_5^T) \) w.p.1.

(iii) Convergence of \( I_3^{N,T} \): We have

\[ I_3^{N,T} = \frac{\| Q_i \|}{T} \int_0^T \| (m^N)^0(t) - (\hat{m}^N)^0(t) \|^2 dt =: \| Q_i \| I_{31}^{N,T}. \]

From Lemma D.5 we have \( \limsup_{T \to \infty} I_{31}^{N,T} = O(\epsilon_1(N)^2) \). Therefore,

\[ \limsup_{T \to \infty} I_3^{N,T} = \limsup_{T \to \infty} \| Q_i \| I_{31}^{N,T} = O(\epsilon_1(N)^2) \] w.p.1.

(iv) Convergence of \( I_4^{N,T} \): We have

\[ I_4^{N,T} = \frac{2\| Q_i \|}{T} \int_0^T (\hat{x}^0_i(t) - \hat{x}^0_i(t))^T (x^0_i(t) - (m^N)^0(t)) dt. \]
Applying the CS Inequality we obtain

\[ I_{4}^{N,T} \leq 2\|Q_{i}\| \left( \frac{1}{T} \int_{0}^{T} \|\hat{x}_{i}^{0}(t) - x_{i}^{0}(t)\|^{2} dt \right)^{1/2} \]

\[ =: 2\|Q_{i}\| I_{41}^{N,T} \times I_{42}^{T}. \]

We prove in Lemma D.2 that \( \limsup_{T \to \infty} I_{42}^{T} \leq K_{3} \text{ w.p.1} \). It is proved in Theorem 3.2 that \( \hat{\theta}_{i}(t) \to \theta_{i}^{0} \) w.p.1 as \( t \to \infty \) and \( \hat{\zeta}_{i,0} \to \zeta^{0} \) w.p.1 as \( t \to \infty \) and \( N \to \infty \). Hence, we get \( \limsup_{T \to \infty} I_{41}^{N,T} = O(\epsilon_{1}(N)) \) w.p.1. Therefore,

\[ \limsup_{T \to \infty} I_{4}^{N,T} \leq 2\|Q_{i}\| \left( \limsup_{T \to \infty} I_{41}^{N,T} \right) \left( \limsup_{T \to \infty} I_{42}^{T} \right) = O(\epsilon_{1}(N)). \]

Hence, \( \limsup_{T \to \infty} I_{4}^{N,T} = O(\epsilon_{1}(N)) \).

(v) **Convergence of** \( I_{5}^{N,T} \): We have the equation

\[ I_{5}^{N,T} = \frac{2\|Q_{i}\|}{T} \int_{0}^{T} (\hat{x}_{i}^{0}(t) - x_{i}^{0}(t))^{\top} \left( (m^{N})^{0}(t) - (\hat{m}^{N})^{0}(t) \right) dt. \]

Applying the CS Inequality we obtain

\[ I_{5}^{N,T} \leq 2\|Q_{i}\| \left( \frac{1}{T} \int_{0}^{T} \|\hat{x}_{i}^{0}(t) - x_{i}^{0}(t)\|^{2} dt \right)^{1/2} \]

\[ =: 2\|Q_{i}\| I_{51}^{N,T} \times I_{52}^{N,T}. \]

We have shown in Theorem 3.2 that \( \hat{\theta}_{i}(t) \to \theta_{i}^{0} \) w.p.1 as \( t \to \infty \), and \( \hat{\zeta}_{i,0} \to \zeta^{0} \) as \( t \to \infty \) and \( N \to \infty \) w.p.1. Hence, we get \( \limsup_{T \to \infty} I_{51}^{N,T} = O(\epsilon_{1}(N)) \) w.p.1. The convergence of \( I_{52}^{N,T} \) was shown as \( \limsup_{T \to \infty} I_{52}^{N,T} = O(\epsilon_{1}(N)) \) w.p.1 in Lemma D.5. Therefore,

\[ \limsup_{T \to \infty} I_{5}^{N,T} \leq 2\|Q_{i}\| \left( \limsup_{T \to \infty} I_{51}^{N,T} \right) \left( \limsup_{T \to \infty} I_{52}^{N,T} \right) = O(\epsilon_{1}(N)^{2}). \]

Hence, \( \limsup_{T \to \infty} I_{5}^{N,T} = O(\epsilon_{1}(N)^{2}) \).

(vi) **Convergence of** \( I_{6}^{N,T} \): We have the equation

\[ I_{6}^{N,T} = \frac{2\|Q_{i}\|}{T} \int_{0}^{T} \left( x_{i}^{0}(t) - (m^{N})^{0}(t) \right)^{\top} \left( (m^{N})^{0}(t) - (\hat{m}^{N})^{0}(t) \right) dt. \]
Applying the CS Inequality we obtain

\[
I_{6}^{N,T} \leq 2\|Q_{i}\| \left( \frac{1}{T} \int_{0}^{T} \left\| x_{0}^{0}(t) - (m_{N})^{0}(t) \right\|^{2} dt \right)^{1/2} \left( \frac{1}{T} \int_{0}^{T} \left\| (m_{N})^{0}(t) - (\hat{m}_{N})^{0}(t) \right\|^{2} dt \right)^{1/2}.
\]

\[
= 2\|Q_{i}\| I_{61}^{T} \times I_{62}^{N,T}.
\]

Using Lemma D.2, we get \(\limsup_{T\to\infty} I_{61}^{T} \leq K_{3}\) w.p.1. The convergence of \(I_{62}^{N,T}\) was shown as

\[
\limsup_{T\to\infty} I_{62}^{T} = O(\epsilon_{1}(N))\text{ w.p.1}
\]

in Lemma D.5. Therefore,

\[
\limsup_{T\to\infty} I_{6}^{N,T} \leq 2\|Q_{i}\| (\limsup_{T\to\infty} I_{61}^{T}) \times (\limsup_{T\to\infty} I_{62}^{N,T}) = O(\epsilon_{1}(N))\text{ w.p.1.}
\]

Hence, \(\limsup_{T\to\infty} I_{6}^{N,T} = O(\epsilon_{1}(N))\) w.p.1.

(vii) Convergence of \(I_{7}^{N,T}\): We can bound \(I_{7}^{N,T}\) from above as

\[
I_{7}^{N,T} \leq \frac{\|R\|}{T} \int_{0}^{T} \left\| \hat{u}_{0}^{0}(t) - u_{0}^{0}(t) \right\|^{2} dt =: \|R\| I_{71}^{N,T}.
\]

From Proposition C.1 we get \(\limsup_{T\to\infty} I_{71}^{N,T} = O(\epsilon_{1}(N)^{2})\) w.p.1. Therefore,

\[
\limsup_{T\to\infty} I_{7}^{N,T} = \limsup_{T\to\infty} \|R\| I_{71}^{N,T} = O(\epsilon_{1}(N)^{2})\text{ w.p.1.}
\]

(viii) Convergence of \(I_{9}^{N,T}\): We have the equation

\[
I_{9}^{N,T} = \frac{2\|Q_{i}\|}{T} \int_{0}^{T} (\hat{u}_{i}^{0}(t) - u_{i}^{0}(t))^{\top} (u_{i}^{0}(t)) dt.
\]

Applying the CS Inequality we obtain

\[
I_{9}^{N,T} \leq 2\|Q_{i}\| \left( \frac{1}{T} \int_{0}^{T} \left\| \hat{u}_{i}^{0}(t) - u_{i}^{0}(t) \right\|^{2} dt \right)^{1/2} \left( \frac{1}{T} \int_{0}^{T} \left\| u_{i}^{0}(t) \right\|^{2} dt \right)^{1/2} =: 2\|Q_{i}\| I_{91}^{N,T} \times I_{92}^{T}.
\]

It is shown in Proposition C.1 that \(\limsup_{T\to\infty} I_{91}^{N,T} = O(\epsilon_{1}(N))\) w.p.1. We obtain \(\limsup_{T\to\infty} I_{92}^{T} \leq K_{2}\) w.p.1 as shown in Lemma D.1. Therefore,

\[
\limsup_{T\to\infty} I_{9}^{N,T} \leq 2\|Q_{i}\| (\limsup_{T\to\infty} I_{91}^{N,T}) \times (\limsup_{T\to\infty} I_{92}^{T}) = O(\epsilon_{1}(N))\text{ w.p.1.}
\]

Hence, \(\limsup_{T\to\infty} I_{9}^{N,T} = O(\epsilon_{1}(N))\) w.p.1.

Overall we have shown that

\[
\limsup_{T\to\infty} I^{N,T} \leq \limsup_{T\to\infty} (I_{2}^{T} + I_{8}^{T}) + O(\epsilon_{1}(N))\text{ w.p.1.}
\]
Using the same decomposition technique applied in (130) we also show that

\[ J^N_i(u^0_i, u^0_{-i}) \leq J^N_i(\hat{u}^0_i, \hat{u}^0_{-i}) + O(\epsilon_1(N)) \quad \text{w.p.1.} \]

Consequently,

\[ \lim_{N \to \infty} J^N_i(\hat{u}^0_i, \hat{u}^0_{-i}) = \lim_{N \to \infty} J^N_i(u^0_i, u^0_{-i}) \text{ w.p.1, } 1 \leq i \leq N. \]

**Proof of Proposition 4.5**

Let \( u_i \triangleq u_i(t; \theta^0_i, \zeta^0) \in \mathcal{U}_o^N \) be a feedback control action and \( x_i \triangleq x_i(t; \theta^0_i, \zeta^0) \) be the corresponding closed loop solution. LHS of (20) is written as

\[
J^N_i(u_i, u^0_{-i}) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \left\| x_i(t) - m^N_{|u_i, u^0_{-i}}(t) \right\|_{\mathcal{Q}_i}^2 + \| u_i(t) \|_{\mathcal{R}}^2 \right\} dt,
\]

where \( m^N_{|u_i, u^0_{-i}}(t) \triangleq m(x_i(t; \theta^0_i, \zeta^0), x^0_{i \neq j}(t; \theta^{[1:N]}, \zeta^{[1:N]})). \) By adding and subtracting

\[
m^N_{|u_i, u^0_{-i}}(t) \triangleq m(x^0_i(t; \theta^0_i, \zeta^0), x^0_{i \neq j}(t; \theta^{[1:N]}, \zeta^{[1:N]}))
\]

to the integrand, we get

\[
J^N_i(u_i, u^0_{-i}) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \| x_i(t) - m^N_{|u_i, u^0_{-i}}(t) \|_{\mathcal{Q}_i}^2 + \| u_i(t) \|_{\mathcal{R}}^2 \right\} dt. \tag{131}
\]

Expanding (131), we get

\[
J^N_i(u_i, u^0_{-i}) = \limsup_{T \to \infty} \left\{ \frac{1}{T} \int_0^T \| x_i(t) - m^N_{|u_i, u^0_{-i}}(t) \|_{\mathcal{Q}_i}^2 dt 
+ \frac{1}{T} \int_0^T \left\| m^N_{|u_i, u^0_{-i}}(t) - m^N_{|u_i, u^0_{-i}}(t) \right\|_{\mathcal{Q}_i}^2 dt 
+ \frac{2}{T} \int_0^T \left( x_i(t) - m^N_{|u_i, u^0_{-i}}(t) \right)^\top \mathcal{Q}_i (m^N_{|u_i, u^0_{-i}}(t) 
- m^N_{|u_i, u^0_{-i}}(t)) dt 
+ \frac{1}{T} \int_0^T \| u_i(t) \|_{\mathcal{R}}^2 dt \right\} \tag{132}
= : \limsup_{T \to \infty} \{ I_1^{N,T} + I_2^{N,T} + I_3^{N,T} + I_4^{N,T} \}
J^N_i(u_i, u^0_{-i}) \leq \limsup_{T \to \infty} \{ I_1^{N,T} + I_4^{N,T} \} + \limsup_{T \to \infty} I_2^{N,T} + \limsup_{T \to \infty} I_3^{N,T} \text{ w.p.1.}
\]

We have \( \limsup_{T \to \infty} \{ I_1^{N,T} + I_4^{N,T} \} = J^N_i(u_i, \hat{u}^0_{-i}); \) therefore,

\[
J^N_i(u_i, u^0_{-i}) \leq J^N_i(u_i, \hat{u}^0_{-i}) + \limsup_{T \to \infty} I_2^{N,T} + \limsup_{T \to \infty} I_3^{N,T} \tag{133}
\]

w.p.1.

(i) **Convergence of** \( I_2^{N,T} \): Lemma D.5 states that

\[
\limsup_{T \to \infty} I_2^{N,T} = O((\epsilon_1(N))^2) \quad \text{w.p.1.}
\]
(ii) **Convergence of \( I_{3}^{N,T} \):** Applying the CS Inequality we obtain,

\[
I_{3}^{N,T} \leq 2\|Q\| \left( \frac{1}{T} \int_{0}^{T} \left\| x_{i}(t) - m_{n_{i}, o_{i}, t}^{N}(t) \right\|^{2} dt \right)^{1/2}
\]

where

\[
\left\| \frac{1}{T} \int_{0}^{T} \left( \left| m_{n_{i}, o_{i}, t}^{N}(t) \right| - \left| m_{n_{i}, o_{i}, t}^{N}(t) \right| \right)^{2} dt \right\|^{1/2}
\]

\[
= 2\|Q\| \limsup_{T \to \infty} I_{31}^{N,T} \times I_{32}^{N,T}.
\]

Using Lemma D.2 we obtain \( \limsup_{T \to \infty} I_{31}^{N,T} \leq K_{4} \) w.p.1 and using Lemma D.5, we get

\[
\limsup_{T \to \infty} I_{32}^{N,T} = \mathcal{O}(\epsilon_{1}(N)) \text{ w.p.1.}
\]

Therefore,

\[
\limsup_{T \to \infty} I_{3}^{N,T} \leq 2\|Q\| \left( \limsup_{T \to \infty} I_{31}^{N,T} \right) \times \left( \limsup_{T \to \infty} I_{32}^{N,T} \right)
\]

\[
= \mathcal{O}(\epsilon_{1}(N)).
\]

Hence, \( \limsup_{T \to \infty} I_{3}^{N,T} = \mathcal{O}(\epsilon_{1}(N)) \).

Repeating (133) here for ease of reference we see that

\[
J_{i}^{N}(u_{i}, u_{i}^{0}) \leq J_{i}^{N}(u_{i}, \hat{u}_{i}) + \limsup_{T \to \infty} \left( I_{2}^{N,T} + I_{3}^{N,T} \right) \text{ w.p.1,}
\]

where

\[
\limsup_{T \to \infty} \left( I_{2}^{N,T} + I_{3}^{N,T} \right) = \mathcal{O}(\epsilon_{1}(N)).
\]

Hence, \( J_{i}^{N}(u_{i}, u_{i}^{0}) \leq J_{i}^{N}(u_{i}, \hat{u}_{i}) + \mathcal{O}(\epsilon_{1}(N)) \) w.p.1. Applying the decomposition technique in (132) for \( J_{i}^{N}(u_{i}, \hat{u}_{i}) \), one can also get \( J_{i}^{N}(u_{i}, \hat{u}_{i}) \leq J_{i}^{N}(u_{i}, u_{i}) + \mathcal{O}(\epsilon_{1}(N)) \) w.p.1, which implies the claim that \( \lim_{N \to \infty} J_{i}^{N}(u_{i}, u_{i}^{0}) = \lim_{N \to \infty} J_{i}^{N}(u_{i}, \hat{u}_{i}) \) w.p.1, \( 1 \leq i \leq N \). Therefore,

\[
\lim_{N \to \infty} \inf_{u_{i} \in U_{n}^{N}} J_{i}^{N}(u_{i}, \hat{u}_{i}) = \lim_{N \to \infty} \inf_{u_{i} \in U_{n}^{N}} J_{i}^{N}(u_{i}, u_{i}) \text{ w.p.1, } 1 \leq i \leq N.
\]

\[
(134)
\]

**Proof of Theorem 2.2**

First, it is evident that Theorem 3.2 gives (a), Theorem 3.3 gives (b), and Theorem 4.2 gives (c). Second, using a technique similar to that used in [37, Theorem 6.2], it is shown in Proposition 4.4 that

\[
J_{i}^{N}(\hat{u}_{i}^{0}, \hat{u}_{i}^{0}) \leq J_{i}^{N}(u_{i}^{0}, u_{i}^{0}) + \mathcal{O}(\epsilon_{1}(N)) \text{ w.p.1,}
\]

where \( \epsilon_{1}(N) \to 0 \) as \( N \to \infty \). Then, Lemma D.4 gives

\[
J_{i}^{N}(u_{i}^{0}, u_{i}^{0}) \leq \inf_{u_{i} \in U_{n}^{N}} J_{i}^{N}(u_{i}, u_{i}^{0}) + o(\epsilon_{2}(N)) + \mathcal{O}(\epsilon_{2}(N)) + \mathcal{O}(\epsilon_{2}(N)) \text{ w.p.1,} \quad 1 \leq i \leq N,
\]

where \( \epsilon_{2}(N) \to 0 \) as \( N \to \infty \). Finally, Proposition 4.5 states that

\[
\inf_{u_{i} \in U_{n}^{N}} J_{i}^{N}(u_{i}, u_{i}^{0}) \leq \inf_{u_{i} \in U_{n}^{N}} J_{i}^{N}(u_{i}, \hat{u}_{i}^{0}) + \mathcal{O}(\epsilon_{1}(N)),
\]

w.p.1, \( 1 \leq i \leq N \), where \( \epsilon_{1}(N) \to 0 \) as \( N \to \infty \).
Equations (135), (136), (137) together then give the first inequality in

$$J_i^N(\hat{u}_{0i}, \hat{u}_{0i}^-) - \epsilon(N) \leq \inf_{u_i \in U_i^g} J_i^N(u_i, \hat{u}_{0i}^-) \leq J_i^N(\hat{u}_{0i}, \hat{u}_{0i}^-),$$

w.p.1, $1 \leq i \leq N$, while the second is immediate, where $\epsilon(N) = O(\epsilon_1(N)) + O(\epsilon_2(N)) + o(\epsilon_2(N)) + O(1/N)$. This concludes the proof for (d).

Claim (e) restates Proposition 4.4, and claim (f) is a consequence of the $\epsilon$-Nash property (d), with the existence of the limits given by (19).

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