Bogoliubov transformations and fermion condensates in lattice field theories

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Abstract

We apply generalized Bogoliubov transformations to the transfer matrix of relativistic field theories regularized on a lattice. We derive the conditions these transformations must satisfy to factorize the transfer matrix into two terms which propagate fermions and antifermions separately, and we solve the relative equations under some conditions. We relate these equations to the saddle point approximation of a recent bosonization method and to the Foldy-Wouthuysen transformations which separate positive from negative energy states in the Dirac Hamiltonian.
1 Introduction

In this paper we investigate some properties of Bogoliubov transformations in fermionic lattice field theories in connection with the appearance of fermionic condensates. We are motivated by our study of a new bosonization method [1], but our results have a wider relevance, and relate these transformations to the ones introduced by Foldy-Wouthuysen to separate fermions from antifermions in the Dirac Hamiltonian [2].

Bogoliubov transformations are unitary transformations which mix creation and annihilation operators. They have been introduced in the theory of many-body systems in which have been extensively used for their simplicity, in particular in connection with variational principles, and also as a starting point of more elaborated approximation methods. In their application to superconductivity they reproduce the results of the BCS theory of electron-electron interactions with an easy extension to the case of electrons interacting with phonons [3].

While mixing of creation-annihilation operators was a novelty in the theory of many-body systems, mixing of particles and antiparticles in the Hamiltonian formalism of relativistic field theories is very natural, and in fact generalized Bogoliubov transformations have been used also in this domain. See for example [4] where general bilinear fermionic Hamiltonians are diagonalized, or more recent studies of QCD in the limit of large number of colours [5]. Bogoliubov transformations are a fundamental tool in understanding the black body radiation in the Unruh effect in accelerating reference frame [6] and the black hole Hawking radiation [7]. But the specific difficulties of the renormalization procedure in the Hamiltonian formalism have limited their use in quantum field theory.

Recently Bogoliubov transformations have found a somewhat different application in lattice field theory in the formalism of the transfer matrix, which is close from the physical point of view to the Hamiltonian formalism. They have been used to introduce dynamical composite bosons in fermionic theories. An independent Bogoliubov transformation at each time slice can be performed in the operator form of the partition function. The time-dependent parameters of the transformation are then associated with composite bosonic fields in the presence of fermionic fields (quasi-particles) satisfying a compositeness condition which avoids double counting [8,9]. One thus gets an effective action of composite fields plus quasi-particles, exactly equivalent to the original one, in which ground state and excited states can
be treated on the same footing. Of course, in practical applications, some approximation must be introduced.

A Bogoliubov transformation generates a new vacuum which has the form of a fermion condensate. We studied such a condensate in a first approach to bosonization [1], which can be regarded as an approximation of the method of Ref. [8, 9], in which quasi-particles are altogether neglected. In an application to a four-fermion interaction model with a discrete chiral symmetry (at zero fermion mass), an explicit form of the fermion condensate appearing when the symmetry is spontaneously broken was found in a saddle point approximation. The condensate is made of a composite boson which is a superposition of a symmetry breaking plus a symmetry conserving state. This result is not surprising from the point of view of the renormalization group, because it tells that we have a contribution of two operators of the same dimension which are no longer separated by a symmetry. But looking carefully at this result we found that condensation of a symmetry conserving boson takes place also in the free theory, if we require factorization of the transfer matrix in two terms which propagate particles and antiparticles separately. This requirement gives rise to the same equations as the requirement of extremality of the vacuum energy which is generated by the Bogoliubov transformation, in the same way as in the many-body theory [3].

The saddle point approximation of the effective action we are talking about, under the assumption that the saddle point equations have stationary solutions, equals the effective action obtained after a time independent Bogoliubov transformation by neglecting quasi-particles, without any reference to the bosonization method. We will present our results in this more general perspective, but keeping in mind that, by time independent Bogoliubov transformations, we are investigating the saddle point stationary solutions of the mentioned approximated bosonization method.

At this point the relation of Bogoliubov transformations with Foldy-Wouthuysen transformations should be clear. The latter ones eliminate the mixing between positive and negative energy solutions in the continuum free Dirac Hamiltonian. The explicit form of these transformations has also been found for some interactions, which include minimal [10] and anomalous [11] interaction of spin-1/2 particles with a time-independent magnetic field, anomalous interactions with a time-independent electric field and with a pseudo-scalar field [12]. Other generalizations are discussed in [13]. See also [14] and references therein.

In the second quantization formalism the separation of positive-
negative-energy states is accompanied by the generation of a new vacuum. Of course in the free-field case this vacuum will be unitary-equivalent to the previous one, but more intricate possibilities are opened in the interacting case in presence of a phase transition. We found also interesting to see how, in the correspondence from the first and second quantization, the ambiguities in the Foldy-Wouthuysen transformation [11] appear as multiple solutions of the saddle-point equations for the corresponding Bogoliubov transformation, and they are solved by demanding that the new vacuum energy be minimal.

In this paper we study these transformations and the corresponding fermionic condensates for relativistic field theories regularized on a space-time lattice which are suitable for non-perturbative studies and numerical simulations. The elimination of a mixing between fermions and antifermions corresponds to a factorization of the transfer matrix. One then wonders which transformation, at finite lattice spacing, induces this factorization and how it changes the properties of the vacuum which appears, in general, as a condensate of pairs of the original fermion-antifermions. One way is of course by studying the implementation of symmetries, but for a symmetry which is broken by the regularization, like the chiral one with Wilson fermions, for instance, this can present some difficulty.

The paper is organized in the following way. In Section 2 we define the generalized Bogoliubov transformations and derive the effective action, extending the result of [9] in which quasi-antiparticles were neglected. In this effective action there appear quadratic terms mixing quasi-particles and quasi-antiparticles and a vacuum energy which has the form of a condensate of a composite boson. In Section 3 we derive and solve, at finite lattice spacing, the decoupling equations. They are identical to the ones arising from the requirement of extremality of the vacuum energy, under some simplifying hypothesis. In Section 4 we consider the properties of the condensate in connection with chiral invariance. In Section 5 we discuss the relation with the Foldy-Wouthuysen transformation. In Section 6 we write the effective action at finite temperature in terms the “physical” modes and in Section 7 we present our Conclusions. Many details of our derivations are relegated to the Appendices.
2 Generalized Bogoliubov transformation

Consider a system of fermions interacting with external bosonic fields including gauge fields, regularized on a lattice. The fermionic part of the partition function at finite temperature $T$ and chemical potential $\mu$ can be written

$$Z = \text{Tr}^F \prod_{t=0}^{L_0/s-1} e^{M_t + M_t^\dagger} T_{t,t+1}$$

(2.1)

where $L_0 = T^{-1}$ is the number of links in the temporal direction, $T$ is the fermion transfer matrix and $\text{Tr}^F$ is the trace over the Fock space of fermions. The parameter $s$ takes the value 1 in the Wilson formulation for lattice fermions, but $s = 2$ for the Kogut-Susskind fermions which live on blocks of size twice the lattice spacing. The index $t$ labels the blocks along the “time” direction. The expression of $T$ which we will use was given by Lüscher [15], for Wilson fermions, in the gauge $U_0 = 1$, in which one has to impose the Gauss constraint in the Hilbert space of the system (a Fock space of fermions in which the coefficients of the fermionic states are polynomials of spatial link variables). Here we report a slightly modified form which avoids the Gauss constraint by reinstating the temporal links variables which generate the Gauss constraint

$$T_{t,t+1} := \hat{T}_t^\dagger \hat{V}_t \exp(s \mu \hat{n}) \hat{T}_{t+1}$$

(2.2)

where $\hat{n}$ is the fermion number operator and the sum on all the indices is understood

$$\hat{n} := \hat{u}^\dagger \hat{u} - \hat{\nu}^\dagger \hat{\nu},$$

(2.3)

with $\hat{u}^\dagger$ and $\hat{\nu}^\dagger$, creation operators of fermions and antifermions, obeying canonical anti-commutation relations and

$$\hat{T}_t = \exp[-\hat{u}^\dagger M_t \hat{u} - \hat{\nu}^\dagger M_t^T \hat{\nu}] \exp[\hat{\nu} N_t \hat{u}]$$

$$\hat{V}_t = \exp[\hat{u}^\dagger \ln U_{0,t} \hat{u} + \hat{\nu}^\dagger \ln U_{0,t}^* \hat{\nu}].$$

(2.4)

(2.5)

The matrices $M_t$ ($M_t^T$ being the transposed of $M_t$) and $N_t$ are functions of the spatial link variables at time $t$ and possibly of other bosonic fields. Explicit expressions for Wilson and Kogut-Susskind fermions in the flavor basis are reported in Appendix A. The variables $U_{0,t}$ are matrices in a unitary representation of the gauge group whose elements are the link variables between Euclidean time $t$ and $t + 1$

$$(U_{0,t})_{x_1,x_2} = \delta_{x_1,x_2} U_{0,t}(x_1)$$

(2.6)
where boldface letters, as \( \mathbf{x} \), denote spatial coordinates.

We introduced the notation, which we will use for any matrix \( \Lambda \)

\[
\text{tr}_\pm \Lambda := \text{tr} \left( P_\pm \Lambda \right).
\]

(2.7)

The operators \( P_\pm \) project on the components of the fermion field which propagate forward or backward in time

\[
\begin{align*}
    u &= P_+ \psi \\
    v^\dagger &= P_- \psi
\end{align*}
\]

(2.8)

and their expressions are given in the Appendix A. The symbol “\( \text{tr} \)” denotes the trace over fermion-antifermion intrinsic quantum numbers and spatial coordinates (but not over time).

Let us compare the transfer matrix to the Hamiltonian

\[
H = M \hat{c}^\dagger \hat{c} + N (\hat{c}^\dagger \hat{c}^\dagger + \hat{c} \hat{c})
\]

(2.9)

This Hamiltonian, which contains only one fermion operator, can be diagonalized by a Bogoliubov transformation. We then try to factorize the transfer matrix by its natural generalization. However, we have the transfer matrix in the form of a product of non-commuting operators. Even in the simplest case when \( M_t = \ln U_{0,t} = 0 \), so that

\[
T_{t,t+1} = \exp[\hat{u}^\dagger N_t \hat{v}^\dagger] \exp[\hat{v} N_{t+1} \hat{u}]
\]

(2.10)

it is not possible to solve the problem by diagonalization of each factor. Indeed, \( \hat{v} N_{t+1} \hat{u} \) cannot be diagonalized. To diagonalize the product the strategy is, also in this more complex situation, to perform a generalized Bogoliubov transformation which generates new creation-annihilation operators of “physical” particles

\[
\begin{align*}
    \hat{\alpha} &= R^{\frac{1}{2}} (\hat{u} - \mathcal{F}^\dagger \hat{v}^\dagger) \\
    \hat{\beta} &= (\hat{v} + \hat{u}^\dagger \mathcal{F}^\dagger) \hat{R}^{-\frac{1}{2}} \\
    \hat{\alpha}^\dagger &= (\hat{u}^\dagger - \hat{v} \mathcal{F}) \hat{R}^{\frac{1}{2}} \\
    \hat{\beta}^\dagger &= \hat{R}^{\frac{1}{2}} (\hat{v}^\dagger + \mathcal{F} \hat{u})
\end{align*}
\]

(2.11a, b)

where

\[
\begin{align*}
    R &= (1 + \mathcal{F}^\dagger \mathcal{F})^{-1} \\
    \hat{R} &= (1 + \mathcal{F} \mathcal{F}^\dagger)^{-1}
\end{align*}
\]

(2.12)

and \( \mathcal{F} \) is an arbitrary matrix. The upperscript circle denotes the involution defined by the above equations. The new operators satisfy canonical commutation relations for any choice of the matrix \( \mathcal{F} \). We will let \( \mathcal{F} \) to depend on all
the fields coupled to the fermions in such a way as to respect all symmetries. The vacuum of the new operators is

$$|\mathcal{F}\rangle = \exp \hat{\mathcal{F}}^\dagger |0\rangle$$

(2.13)

where

$$\hat{\mathcal{F}}^\dagger = \hat{u}^\dagger \mathcal{F}^\dagger \hat{v}^\dagger,$$

(2.14)
is a the creation operator of a composite boson. As already said the new vacuum appears as a coherent state of fermion-antifermion pairs.

Usually the trace appearing in the definition of the transfer matrix is evaluated using coherent states of fermions

$$|\alpha,\beta\rangle = \exp(-\alpha \hat{u}^\dagger - \beta \hat{v}^\dagger)|0\rangle,$$

(2.15)

where the $\alpha,\beta$ are Grassmann fields. We will use instead states obtained by applying a Bogoliubov transformation

$$\hat{\mathcal{U}}(\mathcal{F}) |\alpha,\beta\rangle = |\alpha,\beta; \mathcal{F}\rangle$$

(2.16)

$$= \exp(-\alpha \hat{\alpha}^\dagger - \beta \hat{\beta}^\dagger)|\mathcal{F}\rangle$$

(2.17)

$$= \exp \left( \hat{u}^\dagger \mathcal{F}^\dagger \hat{v}^\dagger - a \hat{\alpha}^\dagger - b \hat{\beta}^\dagger - \beta \mathcal{F} \alpha \right) |0\rangle$$

(2.18)

where $|\mathcal{F}\rangle$ is the coherent state of fermion-antifermion pairs defined in (2.13), $a := R^{-\frac{1}{2}} \alpha$ and $b := \beta \hat{R}^{-\frac{1}{2}}$. The explicit definition of the operator $\hat{\mathcal{U}}$ is given in Appendix B.

In terms of the transformed coherent states the partition function can be written

$$Z = \text{Tr} \left[ F^{\frac{L_0}{s} - 1} \prod_{t=0}^{L_t} e^{\mathcal{M}_t + \mathcal{M}_t^\dagger} \hat{\mathcal{P}} \mathcal{T}_t^\dagger \hat{V}_t \epsilon^{\mu \nu \tilde{n}} \mathcal{T}_{t+1} \right]$$

(2.19)

where

$$\hat{\mathcal{P}} := \int \mathcal{D}[\alpha^*,\alpha,\beta^*,\beta] \frac{|\alpha\beta; \mathcal{F}\rangle \langle \alpha\beta; \mathcal{F}|}{\langle \alpha\beta; \mathcal{F}| \alpha\beta; \mathcal{F}\rangle}$$

(2.20)

is a representation of the identity. More explicitly

$$Z = \int \mathcal{D}[\alpha^*,\alpha,\beta^*,\beta] \prod_{t} e^{M_t + M_t^\dagger} \frac{\langle \alpha_t \beta_t; \mathcal{F}_t | \hat{T}_t^\dagger \hat{V}_t \epsilon^{\mu \nu \tilde{n}} \hat{T}_{t+1} | \alpha_{t+1} \beta_{t+1}; \mathcal{F}_{t+1} \rangle}{\langle \alpha_t \beta_t; \mathcal{F}_t | \alpha_t \beta_t; \mathcal{F}_t \rangle^{-1}}$$

(2.21)
where the Grassmann variables $\alpha^*, \alpha, \beta^*, \beta$ satisfy antiperiodic boundary conditions in time. Evaluating the trace as outlined in the Appendix C we get

$$Z = \int D[\alpha^*, \alpha, \beta^*, \beta] e^{-S_0(\mathcal{F}) - S_F(\alpha, \beta; \mathcal{F})}.$$ (2.22)

In the above equation $S_0(\mathcal{F})$ is a term independent of the Grassmann variables which can be interpreted as a vacuum energy

$$S_0(\mathcal{F}) := -\sum_{t=0}^{L_0/s-1} \text{tr}_+ \ln (R_t \ U_{0,t} \ E_{t+1,t})$$ (2.23)

where

$$E_{t+1,t} := (\mathcal{F}_{N,t+1})^\dagger e^{M_{t+1} U_{0,t}^\dagger e^{M_t} \mathcal{F}_{N,t}} + \mathcal{F}_{t+1}^\dagger e^{-M_{t+1} U_{0,t}} e^{-M_t} \mathcal{F}_{t},$$ (2.24)

with

$$\mathcal{F}_{N,t} := 1 + N_t^\dagger \mathcal{F}_t.$$ (2.25)

The other term is the action of quasi-particles

$$S_F(\alpha, \beta; \mathcal{F}) = -s \sum_{t=0}^{L_0/s-1} \left[ \beta_t I_t^{(2,1)}_t + \alpha_t I_t^{(1,2)}_t \right]$$

written in terms of lattice covariant derivatives

$$\nabla_t := s^{-1} \left( e^{s\mu U_{0,t} - T_0^{(-)}} \right)$$ (2.27)

$$\nabla_t^\circ := s^{-1} \left( e^{-s\mu U_{0,t}^\dagger - T_0^{(+)}} \right)$$ (2.28)

and fermion-antifermion Hamiltonians which are given by

$$\mathcal{H}_t := s^{-1} e^{s\mu} \left( U_{0,t} - R_t^{-\frac{3}{2}} E_{t+1,t}^{-1} R_t^{-\frac{1}{2}} \right)$$ (2.29)

$$\mathcal{H}_t^\circ := s^{-1} e^{-s\mu} \left( U_{0,t}^\dagger - R_t^{-\frac{1}{2}} E_{t+1,t}^{\circ -1} R_t^{-\frac{1}{2}} \right)$$ (2.30)

plus the unwanted terms which mix quasi-particles with quasi-antiparticles whose coefficients are

$$I_t^{(2,1)} := s^{-1} R_t^\frac{1}{2} \left[ R_t^\circ - \circ E_{t+1,t}^{-1} \mathcal{F}_{N,t+1}^\circ e^{M_t} U_{0,t-1} e^{M_{t-1}} \right] \mathcal{F}_t^{-1} R_t^\frac{1}{2}$$ (2.31)

$$I_t^{(1,2)} := s^{-1} R_t^\frac{1}{2} \mathcal{F}_t^{-1} \left[ \circ R_t^\circ - e^{M_t} U_{0,t} e^{M_{t+1}} \left( \mathcal{F}_{N,t+1}^\circ \right)^\dagger E_{t+1,t}^{-1} \right] R_t^\frac{1}{2}$$ (2.32)
$T_0^{(\pm)}$ are the forward and backward translation operators of one block, that is $s$ lattice spacing, in the “time” direction

$$[T_0^{(\pm)}]_{t_1,t_2} = \delta_{t_2,t_1 \pm 1} \quad (2.33)$$

and the definitions of the other new symbols are

$$\hat{E}_{t+1,t} := \hat{\mathcal{F}}_{N,t} e^{M_t} U_{0,t} e^{M_{t+1}} \left( \hat{\mathcal{F}}_{N,t+1} \right)^\dagger + \mathcal{F}_t e^{-M_t} U_{0,t} e^{-M_{t+1}} \mathcal{F}_{t+1}^\dagger \quad (2.34)$$

$$\hat{\mathcal{F}}_{N,t} := 1 + \mathcal{F}_t N_t^\dagger. \quad (2.35)$$

### 3 Factorization of the transfer matrix

In the study of the factorization of the transfer matrix we will consider both fermion regularizations, but since the main motivation is to disentangle unphysical condensates from condensates related to symmetry breaking, the Kogut-Susskind regularization is more interesting because it conserves a form of chiral invariance.

The fermion-antifermion mixing can be eliminated whenever $\mathcal{F}$ can be chosen in such a way that the equations

$$I_t^{(2,1)} = I_t^{(1,2)} = 0 \quad (3.1)$$

are satisfied. Therefore for these particular values of $\mathcal{F}$ the transformations (2.11) factorize the transfer matrix.

It is remarkable that the requirement that the vacuum energy be minimal (namely extremality of the vacuum energy with respect to $\mathcal{F}; \mathcal{F}^\dagger$) gives rise to the same equations, in full analogy with the Bogoliubov transformation in the BCS theory [3].

Explicitly we get

$$\mathcal{F}_{t+1} = N_{t+1} + e^{-M_{t+1}} U_{0,t} e^{-M_{t}} \mathcal{F}_t \left( \mathcal{F}_{N,t} \right)^{-1} e^{-M_t} U_{0,t} e^{-M_{t+1}} \quad (3.2)$$

$$\mathcal{F}_t^\dagger = N_t^\dagger + e^{-M_{t}} U_{0,t} e^{-M_{t+1}} \left( \mathcal{F}_{N,t+1} \right)^{-1} \mathcal{F}_{t+1}^\dagger e^{-M_{t+1}} U_{0,t} e^{-M_t} \quad (3.3)$$

We notice that the form of these equations does not depend on the chemical potential $\mu$, but they must be solved under the constraint of a given fermion density which does depend on the chemical potential. This point will be discussed in Section 6.
For $U_0 = 1$, $M = M^\dagger$ and time-independent external fields, we can look for time-independent solutions, so that the previous equations become

$$F = N + e^{-2M}F^\dagger e^{-2M},$$
$$F^\dagger = N^\dagger + e^{-2M}(F^\dagger_N)^{-1}F^\dagger e^{-2M}.$$  \hspace{1cm} (3.4)

Now we make an ansatz for the solutions of the form

$$\overline{F} = NA.$$  \hspace{1cm} (3.5)

The matrix $A$ satisfies a quadratic equation. Assuming the commutation relations

$$[N, M] = 0$$  \hspace{1cm} (3.6)

and

$$[N^\dagger N, A] = 0$$  \hspace{1cm} (3.7)

we can solve for $A$ in a basis in which it is diagonal. We have then in general two solutions for each diagonal element. This multiplicity of solutions for $A$ can be related to the well known ambiguity in the Foldy-Wouthuysen transformation [11]. In the second quantization formalism this ambiguity can be solved by requiring that the vacuum energy be minimal. As a consequence we find

$$A = (2N^\dagger N)^{-1} \left[ -Y + \sqrt{Y^2 + 4N^\dagger N} \right]$$  \hspace{1cm} (3.8)

where

$$Y = 1 - N^\dagger N - e^{-4M}.$$  \hspace{1cm} (3.9)

For Kogut-Susskind fermions $M = 0$ and

$$N^\dagger N = 4 H^2$$  \hspace{1cm} (3.10)

where $H$ is the lattice Hamiltonian

$$H^2 = m^2 - \Delta$$  \hspace{1cm} (3.11)

with

$$\Delta = \frac{1}{4} \sum_{i=1,3} \left( T_i^{(+)} + T_i^{(-)} - 2 \right).$$  \hspace{1cm} (3.12)

Then [1]

$$A = (2H)^{-1} \left( H + \sqrt{1 + H^2} \right)$$  \hspace{1cm} (3.13)
and using this expression we derive that

\[ \mathcal{H} = e^{s\mu} H \left( \sqrt{1 + H^2} - H \right) \]
\[ \mathcal{H}^* = e^{-s\mu} H \left( \sqrt{1 + H^2} - H \right) \]

so that in the formal continuum limit

\[ \mathcal{H} \approx \mathcal{H}^* \approx H = \sqrt{m^2 - \Delta} \tag{3.14} \]

both approach the same value.

## 4 Chiral and vector symmetries

Now we want to verify that the condensation generated by the factorization of the transfer matrix does not break any symmetry, namely the vacuum \( |\mathcal{F}\rangle \) remains invariant. For this purpose we can consider only the Kogut-Susskind formulation in which there is a residual chiral invariance. In this case chiral and vector transformations are

\[ \psi' = \exp(i\theta_5 \gamma_5 \otimes t_5 + i\theta)\psi \]
\[ \overline{\psi'} = \exp(i\theta_5 \gamma_5 \otimes t_5 - i\theta) \overline{\psi} \tag{4.1} \]

so that

\[ u' = \exp(i\theta_5 \gamma_5 \otimes t_5 + i\theta)u \]
\[ v^\dagger' = \exp(i\theta_5 \gamma_5 \otimes t_5 + i\theta)v^\dagger \tag{4.2} \]

The operator which implements chiral transformations is

\[ \Lambda = \exp(i\theta_5 \gamma_5 \otimes t_5) \tag{4.3} \]

and therefore

\[ \Lambda^\dagger N \Lambda = -2im\gamma_0 \otimes I \exp(2i\theta_5 \gamma_5 \otimes t_5). \tag{4.4} \]

Given the form of \( \mathcal{F} \) for massive spinors

\[ \Lambda |\mathcal{F}\rangle \neq |\mathcal{F}\rangle \tag{4.5} \]
Neither the original theory nor the new vacuum are chiral invariant.

When we consider the case of massless spinors, instead, $\Lambda$ commutes with $N$, so that
\[
\Lambda |F\rangle = |F\rangle .
\] (4.6)

Both the original theory and the new vacuum are chiral invariant.

This particular case of massless spinors allows a factorization of the transfer matrix by using the chiral components
\[
\chi_{\pm} := \Pi_{\pm} \psi, \quad \bar{\chi}_{\pm} := \overline{\psi} \Pi_{\pm}
\] (4.7)
where the chiral projectors are defined as
\[
\Pi_{\pm} := \frac{1}{2}(1 \pm \gamma_5 \otimes t_5).
\] (4.8)

The chiral components transform according to
\[
\chi'_{\pm} = \exp(\pm i\theta_5 + i\theta) \chi_{\pm}, \quad \bar{\chi}'_{\pm} = \bar{\chi}_{\pm} \exp(\mp i\theta_5 - i\theta).
\] (4.9)

5 Connection with the Foldy-Wouthuysen transformation

There must be a relation between the Bogoliubov transformation, which factorizes the transfer matrix, and the Foldy-Wouthuysen one which separates particles from antiparticles in the Dirac Hamiltonian. In particular one might expect that in the formal continuum limit they should coincide.

To clarify this point we study the formal continuum limit of the expression the Bogoliubov transformation takes at the saddle point. We observe that with both Wilson and Kogut-Susskind regularizations the following relations hold true
\[
\mathcal{F} = \mathcal{F}^\dagger
\] (5.1)
and
\[
P_{\pm} \mathcal{F} = \mathcal{F} P_{\mp}
\] (5.2)
so that $\mathcal{F}$ is odd under the transformation $P_+ - P_-$
\[
(P_+ - P_-) \mathcal{F} (P_+ - P_-) = -\mathcal{F}.
\] (5.3)
Then in both cases the Bogoliubov transformation at the saddle point is

$$\psi' = R^{\frac{1}{2}} \left[1 + \mathcal{F} (P_+ - P_-)\right] \psi.$$  \hfill (5.4)

With Wilson regularization

$$\mathcal{F} \approx \frac{i \sigma \cdot \nabla}{m + \sqrt{m^2 - \Delta}} = -\gamma_0 \frac{\vec{\gamma} \cdot \vec{\nabla}}{m + \sqrt{m^2 - \Delta}}$$  \hfill (5.5)

so that

$$R \approx \frac{m + \sqrt{m^2 - \Delta}}{2\sqrt{m^2 - \Delta}}.$$  \hfill (5.6)

We used the \(\vec{\gamma}\)-matrices in terms of the Pauli matrices using a convention different from that of Lüscher [15]: our \(\vec{\gamma}\)-matrices have opposite sign

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & -i \vec{\sigma} \\ i \vec{\sigma} & 0 \end{pmatrix}. \hfill (5.7)$$

It easy to check that the transfer matrix becomes the exponential of minus the continuum Hamiltonian times the temporal lattice spacing

$$\hat{T} \approx \exp \left[-a \overline{\psi} (m + \vec{\gamma} \cdot \vec{\nabla}) \psi\right]$$  \hfill (5.8)

and the Bogoliubov transformation coincides with the Foldy-Wouthuysen transformation for wave functions

$$\psi' \approx \left[2\sqrt{m^2 - \Delta} (\sqrt{m^2 - \Delta} + m)\right]^{-1} \left[\sqrt{m^2 - \Delta} + m + \vec{\gamma} \cdot \vec{\nabla}\right] \psi.$$  \hfill (5.9)

The transformed Hamiltonian

$$\overline{\psi}' \left(m + \vec{\gamma} \cdot \vec{\nabla}\right) \psi' = \overline{\psi}' \gamma_0 \sqrt{m^2 - \Delta} \psi'$$  \hfill (5.10)

is free of fermion-antifermion mixing and the states of positive/negative energy are those which propagate forward/ backward in time.

For Kogut-Susskind fermions

$$\mathcal{F} \approx -\gamma_0 \frac{m + \vec{\gamma} \cdot \vec{\nabla}}{\sqrt{m^2 - \Delta}}$$  \hfill (5.11)
so that
\[ R \approx 2 . \] (5.12)

The transfer matrix is approximately the exponential of minus the continuum Hamiltonian times the temporal block lattice spacing
\[ \hat{T} \approx \exp \left[ -2a \bar{\psi} (m + \vec{\gamma} \cdot \vec{\nabla}) \psi \right] \] (5.13)

and the Bogoliubov transformation
\[ \psi' \approx \left[ 2 \sqrt{m^2 - \Delta} \right]^{-1} \left[ \sqrt{m^2 - \Delta} + \gamma_5 \otimes t_5 t_0 (m + \vec{\gamma} \cdot \vec{\nabla}) \right] \psi \] (5.14)

does not have the form of the standard Foldy-Wouthuysen transformation. The transformed Hamiltonian in this case is
\[ \bar{\psi}' \left( m + \vec{\gamma} \cdot \vec{\nabla} \right) \psi' = \bar{\psi}' \gamma_0 \gamma_5 \otimes t_0 t_5 \sqrt{m^2 - \Delta} \psi' . \] (5.15)

This is also free of fermion-antifermion mixing, and the states of positive (resp. negative) energy are those which propagate forward (resp. backward) in time on the lattice.

We remark that, in both regularizations, \( \mathcal{F} \), in the continuum limit, is proportional to the part of the Hamiltonian odd under \( P_+ - P_- \).

6 Finite temperature

In this Section we shall consider the consequences of the identification of the physical modes in the case in which the temperature is finite. To this end we first perform the integration over the quasi-particles fields getting the effective action
\[ S_{\text{eff}}(\mathcal{F}) = - \sum_{t=0}^{L_0/s-1} \text{tr}_+ \ln (R_t U_{0,t} E_{t+1,t}) \]
\[ - \text{Tr}_+ \ln \left[ (\mathcal{H}_t - \nabla_t) T_0^{(+)} \right] - \text{Tr}_- \ln \left[ T_0^{(-)} \left( \mathcal{H}_t - \nabla_t \right) \right] \] (6.1)

where at variance with “tr” symbol “Tr” is the trace over all fermion quantum numbers and time labels.
For time-independent bosonic fields and for $U_{0,t} = I$ the effective action can be computed by using the saddle point solution $\mathcal{F}$, which does not depend on time and thus on temperature, and we get

$$S_{\text{eff}}(\mathcal{F}) = -\frac{L_0}{s} \text{tr}_+ \ln (RE) - \text{Tr}_+ \ln \left[ 1 - e^{s\mu} (RE)^{-1} T_0^{(+)} \right] - \text{Tr}_- \ln \left[ 1 - e^{-s\mu} (RE)^{-1} T_0^{(-)} \right].$$

(6.2)

After time-Fourier transformation

$$S_{\text{eff}}(\mathcal{F}) = -\frac{L_0}{s} \text{tr}_+ \ln (RE) - \sum_{n=0}^{L_0/s-1} \text{tr}_+ \left[ \ln \left( 1 - e^{s\mu} (RE)^{-1} e^{i\omega_n} \right) + \ln \left( 1 - e^{-s\mu} (RE)^{-1} e^{-i\omega_n} \right) \right].$$

(6.3)

where

$$\omega_n = \frac{s\pi}{L_0} (2n + 1),$$

(6.4)

with $n = 0, \ldots, L_0/s - 1$, are the Matsubara frequencies. The corresponding sum, performed in Appendix D gives

$$S_{\text{eff}}(\mathcal{F}) = \text{tr}_+ \left\{ \frac{L_0}{s} \ln \left( 1 - e^{-s\mu} s \mathcal{H} \right) - \ln \left[ 1 + e^{L_0\mu} \left( 1 - e^{-s\mu} s \mathcal{H} \right)^{\frac{L_0}{s}} \right] - \ln \left[ 1 + e^{-L_0\mu} \left( 1 - e^{s\mu} \mathcal{H}^\circ \right)^{\frac{L_0}{s}} \right] \right\}.$$

(6.5)

In the last equations we used the invariance of the trace under the involution \([2.12]\) and the relations

$$(RE)^{-1} = 1 - e^{-s\mu} s \mathcal{H} = 1 - e^{s\mu} s \mathcal{H}^\circ.$$

(6.6)

In the continuum limit

$$S_{\text{eff}}(\mathcal{F}) \approx -\text{tr}_+ \left[ L_0 H + \ln \left( 1 + e^{-L_0(H-\mu)} \right) + \ln \left( 1 + e^{-L_0(H+\mu)} \right) \right].$$

(6.7)
we recognize the contribution from the condensate energy and those from
the Fermi statistics of the quasi-particles and quasi-antiparticles.
For $L_0 \to \infty$ we recover the zero temperature result

$$
\lim_{L_0 \to \infty} \frac{1}{L_0} S_{\text{eff}}(\mathcal{F}) = \frac{1}{s} \text{tr} \left[ \ln \left( 1 - e^{-s\mu} s \mathcal{H} \right) - \theta \left( e^{s\mu} - 1 - s \mathcal{H} \right) \ln \left( e^{s\mu} - s \mathcal{H} \right) - \theta \left( e^{-s\mu} - 1 - s \mathcal{H} \right) \ln \left( e^{-s\mu} - s \mathcal{H} \right) \right]
$$

which reduces to the one obtained in [1] by neglecting the contribution of
quasi-particles, and to the one obtained in [9] by neglecting only the contribu-
tion from antiquarks, which is given by the last term, an approximation
justified for low values of temperature. Indeed, we can see the effect of the
selection of a given number of fermions, by using the constra-int

$$
- \frac{1}{L_0} \frac{\partial}{\partial \mu} S_{\text{eff}}(\mathcal{F}) = n_F
$$

we get at zero temperature

$$
\text{tr} \theta \left( e^{s\mu} - 1 - s \mathcal{H} \right) - \text{tr} \theta \left( e^{-s\mu} - 1 - s \mathcal{H} \right) = n_F.
$$

For non-negative $\mu$, the second $\theta$-function is always zero, meaning that quasi-
antiparticles do not contribute. Let us call $\sigma$ the smallest mass in $H$. Near
the continuum limit, for $\mu < \sigma$, $n_F = 0$. For $\mu > \sigma$, quasi-fermions occupy
the states from zero energy up to a maximum energy depending on the fermion
number $n_F$. The effect of a finite fermion density is to deplete the condensate,
namely to reduce the number of fermionic states in the condensed boson
structure function, without altering its form for the remaining states.

7 Conclusion

We have shown that the transfer matrix of relativistic field theories of
fermions, whose Lagrangian can be written in a form quadratic in the fermion
fields, can be factorized whenever a certain classical equation has solutions.
In general there exists a multiplicity of solutions which can be restricted by
the requirement that the vacuum energy be minimal.
Restricting our analysis to the coupling of fermions with time independent bosonic fields we have determined such solutions. They imply condensation of a composite boson which does not break any symmetry.

Our main interest is in connection with our approach to bosonization, which can be used when the Fermi system is dominated at low energy by excitations which can be described in terms of bosonic composites. In such a case bosonization is achieved by performing independent Bogoliubov transformations at each time slice, and associating the time dependent parameters of the transformations to dynamical bosonic fields. The results of the present paper can then become relevant when a saddle point expansion exists for such a bosonized system. In the presence of spontaneous breaking of a symmetry, the structure function of the condensed bosons is in general a superposition of a symmetry breaking and a symmetry conserving state, whose relative weight depends on temperature and chemical potential.

We have also shown that factorization of the transfer matrix is related to the elimination of fermion-antifermion mixing in the Dirac Hamiltonian. Again there are many solutions to this latter problem [11], which can be restricted considering it in second quantization and choosing the ones of minimal energy. It would be very interesting to explore the possibility of determining the Bogoliubov transformations corresponding to the Foldy-Wouthuysen ones for fermions interacting with gauge fields.

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A The matrices $M, N$ of the transfer matrix

In this Appendix we report the expressions of the matrices $M, N$ appearing in the definition of the transfer matrix for the Kogut-Susskind and the Wilson regularization. Their common feature is that they depend only on the spatial link variables.

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A.1 Kogut-Susskind’s regularization

Kogut-Susskind fermions in the flavor basis are defined on hypercubes whose sides are twice the basic lattice spacing. While in the text intrinsic quantum numbers and spatial coordinates were comprehensively represented by one index $i$, here we distinguish the spinorial index $\alpha = \{1, \ldots, 4\}$, the taste index $a = \{1, \ldots, 4\}$ and the flavour index $i = \{1, \ldots, N_f\}$, while $x = \{t, x_1, \ldots, x_3\}$ is a 4-vector of even integer coordinates ranging in the intervals $[0, L_t - 1]$ for the time component and $[0, L_s - 1]$ for each of the spatial components. We distinguish summations over basic lattice and hypercubes according to

$$\sum x' := 2^d \sum x.$$  \hspace{1cm} (A.1)

The projection operators over fermions-antifermion states are

$$P_{\pm} = \frac{1}{2} \left( \mathbb{1} \otimes \mathbb{1} \mp \gamma_0 \gamma_5 \otimes t_5 t_0 \right).$$  \hspace{1cm} (A.2)

In the presence of gauge fields, neglecting an irrelevant constant, $M = 0$, while $N$ is [16]

$$N = -2 \left\{ m \gamma_0 \otimes \mathbb{1} + \sum_{j=1}^3 \gamma_0 \gamma_j \otimes \mathbb{1} \left[ P_j^{(-)} \nabla_j^{(+)} + P_j^{(+)} \nabla_j^{(-)} \right] \right\}$$  \hspace{1cm} (A.3)

where

$$\nabla_j^{(+)} = \frac{1}{2} \left( U_j T_j^{(+)} - 1 \right)$$  \hspace{1cm} (A.4)

$$\nabla_j^{(-)} = \frac{1}{2} \left( 1 - T_j^{(-)} U_j^\dagger \right)$$  \hspace{1cm} (A.5)

are the lattice covariant derivative and

$$P_j^{(\pm)} = \frac{1}{2} \left( \mathbb{1} \otimes \mathbb{1} \mp \gamma_j \gamma_5 \otimes t_5 t_0 \right).$$  \hspace{1cm} (A.6)

The eigenvalues of $H^2$ are the fermion energies

$$E_q^2 = m^2 + \tilde{p}^2,$$  \hspace{1cm} (A.7)

where momentum component $\tilde{p}_i^2$ is

$$\tilde{p}_i^2 = \frac{1}{2} (1 - \cos 2p_i)$$  \hspace{1cm} (A.8)
and
\[ \vec{p}^2 = \sum_{i=1}^{3} \vec{p}_i^2 \] (A.9)

### A.2 Wilson’s regularization

The projection operators over fermions-antifermions are
\[ P_\pm = \frac{1}{2} (1 \pm \gamma_0) . \] (A.10)

The matrices \( M, N \) are
\[ M = \frac{1}{2} \ln \left( \frac{B}{2K} \right) \] (A.11)
\[ N = 2K B^{-\frac{1}{2}} c B^{-\frac{1}{2}} , \] (A.12)

where
\[ B = 1 - K \sum_{j=1}^{3} \left( U_j T^{(+)}_j + T^{(-)}_j U_j^\dagger \right) \] (A.13)

\( K \) is the hopping parameter and
\[ c = \frac{1}{2} \sum_{j=1}^{3} i \left( U_j T^{(+)}_j - T^{(-)}_j U_j^\dagger \right) \sigma_j . \] (A.14)

### B The unitary operator of the Bogoliubov transformation

In the framework of the BCS theory, the relation between the BCS wave function and the Bogoliubov transformation was shown by Yosida [3, 17] who constructed a unitary operator which transforms fermion creation-destruction operators into quasi-particle operators. It can be of some interest to construct the corresponding operator for our generalized Bogoliubov transformation.

We write the operator in the form
\[ \hat{U} := e^\hat{S} \] (B.1)
where
\[
\hat{S} := \hat{u}^\dagger X^\dagger \hat{v}^\dagger - \hat{v} X \hat{u}.
\] (B.2)

First we determine the action of the operator \(\hat{S}\) on creation and destruction operators
\[
\begin{align*}
[\hat{S}, \hat{u}] &= -X^\dagger \hat{v}^\dagger, & [\hat{S}, \hat{v}^\dagger] &= X \hat{u} \\
[\hat{S}, \hat{S}, \hat{u}] &= -X^\dagger X \hat{u}, & [\hat{S}, \hat{S}, \hat{v}^\dagger] &= -XX^\dagger \hat{v}^\dagger.
\end{align*}
\] (B.3)

Using the identity
\[
e^A B e^{-A} = \sum_{n=0}^\infty \frac{1}{n!} [A, [A, \cdots [A, B] \cdots]]
\] (B.5)

for \(e^A = e^{\hat{S}}, B = \hat{u}, \hat{v}^\dagger\) we find the transformations of creation destruction operators
\[
\begin{align*}
e^{\hat{S}} \hat{u} e^{-\hat{S}} &= \sum_{n=0}^\infty \frac{(-X^\dagger X)^n}{(2n)!} \left[ \hat{u} - \frac{1}{2n+1} X^\dagger \hat{v} \right] \\
&= \cos \left( \sqrt{X^\dagger X} \right) \left[ \hat{u} - \operatorname{tg} \left( \sqrt{X^\dagger X} \right) \left( X^\dagger X \right)^{-\frac{1}{2}} X^\dagger \hat{v} \right],
\end{align*}
\] (B.6)

\[
\begin{align*}
e^{\hat{S}} \hat{v}^\dagger e^{-\hat{S}} &= \sum_{n=0}^\infty \frac{(-XX^\dagger)^n}{(2n)!} \left[ \hat{v}^\dagger + \frac{1}{2n+1} X \hat{u} \right] \\
&= \cos \left( \sqrt{XX^\dagger} \right) \left[ \hat{v}^\dagger + \operatorname{tg} \left( \sqrt{XX^\dagger} \right) \left( X^\dagger X \right)^{-\frac{1}{2}} X \hat{u} \right].
\end{align*}
\] (B.7)

They coincide with the transformations (2.11) after the identification
\[
\frac{1}{\sqrt{X^\dagger X}} \operatorname{tg} \sqrt{X^\dagger X} X^\dagger = \mathcal{F}^\dagger.
\] (B.10)

Next we determine the transformation of the vacuum. To this end it is convenient to write the operator \(\hat{S}\) in the form
\[
\hat{S} = \sum_i \hat{u}_i^\dagger \hat{\Lambda}_i^\dagger - \hat{\Lambda}_i \hat{u}_i = \sum_i \hat{\sigma}_i,
\] (B.11)

where
\[
\hat{\Lambda}_i := \sum_j \hat{\nu}_j X_{ji} \quad \hat{\sigma}_i := (\hat{u}_i^\dagger \hat{\Lambda}_i^\dagger - \hat{\Lambda}_i \hat{u}_i).
\] (B.12)
It follows that \( \{ \hat{u}_j, \hat{\Lambda}_i \} = 0 \) and
\[
[\sigma_i, \sigma_j] = \hat{u}_j^\dagger (X^\dagger \cdot X)_{ij} \hat{u}_i - \hat{u}_i^\dagger (X^\dagger \cdot X)_{ij} \hat{u}_j. \tag{B.13}
\]
The matrix \( X^\dagger X \) can be diagonalized by the transformation \( \hat{u} \mapsto (O\hat{u})_k =: \tilde{u}_k \). The new operators satisfy the relation \( (OX^\dagger XO^{-1})_{k,k'} = \delta_{k,k'}(X^\dagger X)_k \), so that the commutator in the above equation vanishes. Therefore
\[
e^S = \prod_k e^{\tilde{\sigma}_k} \tag{B.14}
\]
and writing \( \tilde{\sigma}_i := (\tilde{u}_i^\dagger \tilde{\Lambda}_i^\dagger - \tilde{\Lambda}_i \tilde{u}_i) \), \( \tilde{\Lambda}_i = (\Lambda \cdot O^{-1})_i \) we have
\[
\tilde{\sigma}_k^2 = -\left( \tilde{\Lambda}_k^\dagger \bar{\Lambda}_k \tilde{u}_k \tilde{u}_k \bar{\Lambda}_k^\dagger \tilde{u}_k \right)
\]
\[
\tilde{\sigma}_k^{2n} = (-1)^n \left[ \left( \tilde{\Lambda}_k^\dagger \tilde{\Lambda}_k \right)^n \tilde{u}_k \tilde{u}_k \right]
\]
\[
e^{\tilde{\sigma}_k} = \sum_n \frac{(-1)^n}{(2n)!} \left[ 1 + \frac{\tilde{\sigma}_k}{2n + 1} \right] \left[ (\tilde{\Lambda}_k^\dagger \tilde{\Lambda}_k)^n \tilde{u}_k \tilde{u}_k \right] \tag{B.15}
\]
\[
e^{\tilde{\sigma}_k} |0\rangle = \sum_n \frac{(-1)^n}{(2n)!} \left[ 1 + \frac{\tilde{\sigma}_k}{2n + 1} \right] \left( \tilde{\Lambda}_k^\dagger \tilde{\Lambda}_k \right)^n \tilde{u}_k \tilde{u}_k |0\rangle. \tag{B.16}
\]
Observing that \( \tilde{\Lambda}_k \tilde{\Lambda}_k^\dagger |0\rangle = (\tilde{X}^\dagger \tilde{X})_{k,k} |0\rangle \) the last equation can be rewritten
\[
e^{\tilde{\sigma}_k} |0\rangle = \cos \sqrt{\tilde{X}_k^\dagger \tilde{X}_k} e^{\tilde{\sigma}_k} \tilde{X}_k \tilde{X}_k^{\dagger} |0\rangle. \tag{B.17}
\]
Using this equation in Eq. (B.14) we have
\[
\prod_k e^{\tilde{\sigma}_k} |0\rangle = \left( \prod_k \cos \sqrt{\tilde{X}_k^\dagger \tilde{X}_k} \right) e^{\sum_k \tilde{\sigma}_k} \sqrt{\tilde{X}_k^\dagger \tilde{X}_k} (\tilde{X}_k \tilde{X}_k) \tilde{u}_k^\dagger \tilde{\Lambda}_k^\dagger |0\rangle \tag{B.18}
\]
so that in the original basis
\[
e^S |0\rangle = \det_+ \cos \sqrt{X^\dagger X} e^{\tilde{u}_k^\dagger \tilde{u}_k} (X^\dagger X)^{\frac{1}{2}} |0\rangle \tag{B.19}
\]
\[
= \det_+ \cos \sqrt{X^\dagger X} e^{\tilde{u}_k^\dagger \tilde{u}_k} |0\rangle \tag{B.20}
\]
where we used the identification in Eq. (B.10).
C  Evaluation of the trace in the partition function

In this section, we sketch the evaluation of the matrix element of the transfer matrix in the quasi-particles coherent state base Eq. (2.16), and the effective action in Eq. (2.22). During the computation we use the property that in the transfer matrices and in the quasi-particles coherent state, the fermionic operators appears always in the exponential; so we can evaluate the matrix element in the coherent state bases performing some Berezin integral.

The coherent states for particles and antiparticles are defined in Eq. (2.15), we can write the identity by this state as follow

$$\hat{I}d = \int D\omega^* D\omega D\varphi^* D\varphi \frac{|\omega, \varphi\rangle \langle \omega, \varphi|}{\langle \omega, \varphi|\omega, \varphi\rangle}.$$ (C.1)

Then we insert $\hat{I}d$ between each factor of the transfer matrix

$$\langle \alpha_t \beta_t; \phi_t|\hat{T}_t^\dagger \hat{V}_t e^{\mu B_t} \hat{T}_{t+1} |\alpha_{t+1} \beta_{t+1}; \phi_{t+1}\rangle =$$

$$= \langle \alpha_t \beta_t; \phi_t|\hat{T}_t^\dagger \hat{I}d \hat{V}_t e^{\mu B_t} \hat{I}d \hat{T}_{t+1} |\alpha_{t+1} \beta_{t+1}; \phi_{t+1}\rangle$$ (C.2)

and between the two factors of $\hat{T}_t^\dagger$ and $\hat{T}_{t+1}$. The presence of coherent state permit us to substitute every operators with the Grassmann variable. So we can arrive at the value of the matrix element of transfer matrix performing the Berezin integral respect the Grassmannian variables appearing in $\hat{I}d$. Some intermediate results are

$$\langle \rho\sigma | e^{\mu N u} |\alpha\beta; \phi\rangle = \langle \rho\sigma | e^{\mu N u} \hat{I}d |\alpha\beta; \phi\rangle =$$

$$\det \mathcal{F}_N^\dagger \exp \left[ -\beta \mathcal{F}_t \alpha + b N (\mathcal{F}_N^\dagger)^{-1} \alpha + -b (\mathcal{F}_N^\dagger)^{-1} \sigma + \rho^\dagger (\mathcal{F}_N^\dagger)^{-1} \sigma^\dagger \right]$$

$$\langle \omega, \varphi | e^{-\gamma A \delta - \delta B \hat{t}} |\rho, \sigma\rangle = \exp \left( \omega e^{-A} \rho + \varphi e^{-B} \sigma \right).$$ (C.3)

Where we defined $a := R^{-\frac{1}{2}} \alpha$ and $b := \beta R^{-\frac{1}{2}}$. The integration over $\rho$ and $\sigma$ and their h.c. of the product of the matrix element, in the last equations, gives

$$\langle \omega\varphi | \hat{V}_t e^{\mu \alpha^\dagger \beta} \hat{T}_{t+1} |\alpha\beta; \phi\rangle = \det \mathcal{F}_{N_{t+1}}^\dagger \exp \left[ -\beta \mathcal{F}_{t+1} \alpha + b N_{t+1} (\mathcal{F}_{N_{t+1}}^\dagger)^{-1} \alpha \right.$$

$$+ \omega^\dagger e^{\mu U_0 U_{t+1} \mathcal{F}_{N_{t+1}}^\dagger} e^{-M_{t+1}} \mathcal{F}_{N_{t+1}}^\dagger e^{-M_{t+1}} U_{0, t} \varphi^\star +$$

$$+ \omega^\dagger U_{0, t} e^{-M_{t+1}} (\mathcal{F}_{N_{t+1}}^\dagger)^{-1} \mathcal{F}_{t+1}^\dagger e^{-M_{t+1}} U_{0, t} \varphi^\star \right].$$ (C.4)
In the last equations we highlight the time index. The last equation, its Hermitian adjoint and an integral over the complex variable $\omega$ and $\varphi$ permit us to write the partition function in path integral formalism; the effective action can be split in two contribute $S_0(\mathcal{F})$ defined in Eq. (2.23) and

$$S_F(\alpha, \beta; \mathcal{F}) = \sum_{t=0}^{L_0/2-1} \left[ -a_t^* L_t a_t + b_t \bar{L}_t b_t^* - b_t I_t^{(2,1)} a_t - a_t^* I_t^{(1,2)} b_t^* \right]$$  \hspace{1cm} (C.5)$$

where

$$L_t := -R_t + E_{t+1,t}^{-1} e^{s \mu} T^{(+)} \hspace{1cm} (C.6)$$

$$L_t := -\bar{R}_t + e^{-s \mu} T^{(-)} \bar{E}_{t+1,t}^{-1} \hspace{1cm} (C.7)$$

It can be put in a more transparent form as in Eq. (2.26).

### D Sum over the Matsubara frequencies

We want to prove the following equation

$$\sum_{n=0}^{N-1} \ln \left( 1 - \frac{e^{i\omega_n}}{c} \right) = \ln \left( 1 + \frac{1}{c^N} \right).$$ \hspace{1cm} (D.1)$$

where

$$\omega_n = \frac{\pi}{N} (2n + 1).$$ \hspace{1cm} (D.2)$$

Taking the exponential of the left hand side we get

$$\prod_{n=0}^{N-1} \left( 1 - \frac{e^{i\omega_n}}{c} \right) = \frac{e^{i\pi}}{c^N} \prod_{n=0}^{N-1} \left( c e^{-i\frac{\pi}{N}} - e^{i\frac{2\pi}{N} n} \right).$$ \hspace{1cm} (D.3)$$

Now we use the identity

$$\prod_{n=0}^{N-1} \left( A - e^{i\frac{2\pi}{N} n} \right) = A^N - 1$$ \hspace{1cm} (D.4)$$

(which an obvious consequence of the fact that the $e^{i\frac{2\pi}{N} n}$ are the $N$-th roots of 1) valid for arbitrary $N$ and $A$, to get the final result

$$\prod_{n=0}^{N-1} \left( 1 - \frac{e^{i\omega_n}}{c} \right) = 1 + \frac{1}{c^N}. \hspace{1cm} (D.5)$$
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