Order Reductions of “Predictive Dynamical Systems”

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Abstract

It has been recently pointed out that dynamical systems depending on future values of the unknowns may be useful in different areas of knowledge. We explore in this context the extension of the concept of order reduction that has been useful with singular and delay differential equations in electrodynamics and general relativity. We discuss some general properties of order reductions in this new context and explore a method of successive approximations, which among other things is used to check and improve the “extrapolate prediction” and “fixed rate prediction” methods.

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I. INTRODUCTION

T. Ohira has recently proposed and analyzed a formalism and concrete examples of dynamical systems governed by predictions of future states, which he calls “predictive dynamical systems” [1]. Since initial conditions are not sufficient to solve this kind of dynamical system (and make sure the solution is unique), Ohira proposes two methods of predicting the future values of the unknowns necessary to find (numerically) the solution to the system: “fixed rate prediction” and “extrapolate prediction.” Since both methods can at most provide some approximate solution of the dynamical system, it may be interesting to explore other methods that could eventually improve the quality of the approximation.

In classical electrodynamics and general relativity one finds singular differential equations and delay differential equations for which the usual physical initial conditions are not enough to compute the solution. In this context the idea of order reduction has been useful [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

We extend the concept of order reduction to predictive dynamical systems in section II as well as a method of successive approximations to compute the solution. Since we lack a rigorous theory of this method, a simple but illustrative example is analyzed in section III. Numerical results are discussed in section IV for the same dynamical systems discussed in [1]. In section V we briefly explore higher order reductions and compare again our numerical results with those of Ohira’s.

II. ORDER REDUCTIONS OF PREDICTIVE DYNAMICAL SYSTEMS

Although our definition and results can be readily extended to continuous dynamical systems, for simplicity we will consider only discrete dynamical systems in which a physical quantity $x$ is defined only for integer values $n = 0, 1, 2, \ldots$ of “time” according to a law in the form

$$x_{n+1} = M (x_n, x_{n+p})$$

(1)

with some advance $p = 1, 2, \ldots$ It is obvious that an initial condition $x_0$ for, say, $n = 0$ is not enough to predict the future. Even if one could solve (1) for $x_{n+p}$, the resulting dynamical system

$$x_{n+p} = N (x_n, x_{n+1})$$

(2)

would require specifying $p$ initial conditions: $x_0, x_1, \ldots, x_{p-1}$. In consequence more assumptions are necessary to solve (1). In Ohira’s “fixed rate prediction” method [1], one replaces $x_{n+p}$ on the
right hand side of (1) by \( x_n + p (x_n - x_{n-1}) \), which still needs some additional initial condition for \( n = -1 \) or another suitable assumption. In the “extrapolate prediction” method one would substitute for \( x_{n+p} \) the value obtained by applying \( p \) times to \( x_n \) the map (1) with \( p = 0 \). It is clear that, in general, both methods will provide at most approximations to a solution of the original dynamical system.

The idea behind order reductions is that (1) is not the true evolution equation but only a necessary condition every solution to the actual (unknown) dynamical system must satisfy. If the true dynamical system is a deterministic one in the form

\[
x_{n+1} = F(x_n),
\]

knowledge of the future is only necessary because our incomplete theory did not led us to (3) but only to a less restrictive condition in the form (1). Since the latter must be satisfied by every solution to (3), we have the following condition for the unknown \( F \):

\[
F(x) = M(x, F^p(x)), \quad F^p \equiv F \circ F \circ \cdots \circ F.
\]

Of course, in general, one cannot solve (4) for \( F \) (this is the reason it is unknown) but one can try finding good approximations by different methods. In some cases there is a small parameter in the problem, so that the natural way would be to try Taylor expansions with respect to that parameter. But we are here going to explore a general method of successive approximations which have proved useful with singular al delay differential equations [9, 10, 11].

We will construct a succession of approximations \( F_0, F_1, \ldots \) defined by

\[
F_{m+1}(x) = M(x, F_m^p(x))
\]

along with some suitable initial \( F_0(x) \). It is clear that if the succession is convergent, its limit \( F(x) = \lim_{m \to \infty} F_m(x) \) is a solution of (4). One obvious choice for the initial condition is

\[
F_0(x) = M(x, x),
\]

in which case \( F_1(x) \) is the value \( x_{n+1} \) obtained by means of Ohira’s “extrapolate prediction;” but, although the limit \( F(x_n) \) will be unattainable in practice, the approximation can be improved by computing successive \( F_m(x_n) \) until \( |F_{m+1}(x_n) - F_m(x_n)| \) is below some tolerance value. However, we will see later that different initial conditions may change dramatically the convergence
rate (moreover, the may lead to different order reductions), so that in practice some additional
criterion must be used (for instance, in electrodynamics one can use the limit in which the charge
vanishes to select the right order reduction).

The problem of the existence of the limit $F$ is here posed in too general grounds to have an
answer. Instead of that we will consider an artificial but illustrative problem.

### III. A LINEAR EXAMPLE

Let us first consider the discrete dynamical system

$$x_{n+1} = ax_n + bx_{n+1}, \quad (a \neq 0, \ b \neq 0, \ 1). \tag{7}$$

Of course, this can be written as

$$x_{n+1} = \alpha x_n, \quad \alpha \equiv \frac{a}{1-b}; \tag{8}$$

but let us pretend we do not know that and want to solve (7) by the method of successive ap-
proximations. It is easy to see that for $F_0(x) = (a+b)x$, which corresponds to (6), or for any
$F_0(x) = \alpha_0 x$ with constant $\alpha_0$, we have

$$F_m(x) = \alpha_m x \tag{9}$$

with

$$\alpha_{m+1} = a + b\alpha_m. \tag{10}$$

Since $(\alpha - \alpha_{m+1}) = b(\alpha - \alpha_m)$, whatever $\alpha_0$ is, the recurrence (10) will converge (to $\alpha$) if and
only if $|b| < 1$. In consequence, in this example the method of successive approximations will
converge (to the right dynamical system) when $|b| < 1$ and diverge for $|b| > 1$. One cannot
expect the method to be convergent always, but the example suggests that (as is often the case in
electrodynamics [9, 10, 11]) it may work if some parameter in the theory is small enough.

The following example is

$$x_{n+1} = ax_n + bx_{n+2}, \quad (a \neq 0, \ b \neq 0, \ -1 - a). \tag{11}$$

Also in this case we can solve for $x_{n+2}$ to obtain a two-point recurrence which needs two initial
conditions (say $x_{-1}$ and $x_0$). Instead, we seek an order reduction (3) which only requires one
initial condition and must satisfy

$$F(x) = ax + bF(F(x)). \tag{12}$$
For $4ab \leq 1$ this functional equation has, at least, the following two linear solutions:

$$F(x) = \alpha x, \quad \alpha = \alpha_\pm \equiv \frac{1 \pm \sqrt{1 - 4ab}}{2b}. \quad (13)$$

Starting from any $F_0(x) = \alpha_0 x$ with constant $\alpha_0$ we get again $\alpha$ with

$$\alpha_{m+1} = a + b\alpha_m^2. \quad (14)$$

But this quadratic map is just the logistic map whose properties have been explored in depth in chaos theory [12]. For this reason it is easy to prove that $\alpha_m$ will converge to $\alpha_-$ for any parameter values such that $-3 < 4ab < 1$ provided the initial condition is choosed so that

$$|\alpha_0| \leq \frac{1 + \sqrt{1 - 4ab}}{2|b|}. \quad (15)$$

This is the case for $\alpha_0 = a + b$ —which correspond to (6)— for $|a + b|$ small enough. For other initial conditions or parameters $\alpha_m$ may go to infinity, approach a cycle of any period or change chaotically. Again we see that the method could work for small parameter values, but also that it could never converge to the right solution ($F(x) = \alpha_+ x$, for instance), in which case other methods should be tried (maybe an appropriate series expansion, or a numerical method to solve (1) for $x_{n+p}$).

IV. NUMERICAL RESULTS

Successive approximations to the order reduction can be numerically computed in any programming language. For instance, the Mathematica code [13] in Table II will compute and display $x_n$ (for $n = 0, 1, \ldots, 10$) by using the second approximation $F_2$, in the case of the “sigmoid function” discussed in reference [1]:

$$M(x, y) = (1 - \mu)x + \frac{2}{1 + e^{-\beta y}} - 1. \quad (16)$$

We have use that code for Figure II where the values $x_0, x_1, \ldots, x_{10}$ obtained with $F_1, F_2, F_4$ and $F_5$ are displayed for $\mu = 0.5, \beta = 0.8, p = 5$ and initial guess (6). The dots in the upper polygonal have been computed with $F_1$ and, thus, are the same obtained by Ohira’s “extrapolate prediction.” We can see there is room for improvement, for the values with $F_2$ are rather smaller, while those obtained with $F_4$ and $F_5$ are indistinguishable in the figure, proving they are very near those one would obtain with the limit $F$. We can see in Figure II the importance of a good guess.
for $F_0$: selecting $F_0(x) = M(x, x_0)$ leads to a much slower convergence and even $F_8$ is not a good approximation.

In Figure 3 one can see that convergence is faster for the “Mackey-Glass function” of reference [1],

$$M(x, y) = (1 - \mu)x + \frac{\beta y}{1 + y^s},$$

with $\mu = 0.5$, $\beta = 0.8$, $s = 10$, $p = 5$ and initial guess (6): solutions with $F_2$ and $F_3$ are already very close.

One can also have the program compute at each step $x_n$ successive approximations $F_m(x_n)$ until the difference between two consecutive approximations is below some maximum relative error, which is called tol in the code in Table III for the “Mackey-Glass function” of reference [1]. For more complex calculations this code can (must) be improved in many ways, including a better storage management (here every computed value is stored) and using a compiled programming language.

V. HIGHER ORDER REDUCTIONS

To keep things simple we have reduced (1) to the first-order dynamical system (3), which only needs $x_0$ to identify each solution. In some cases we might have theoretical reasons to think that the true dynamical system is of second order,

$$x_{n+1} = G(x_n, x_{n-1}),$$

with

$$G(x, y) = M\left(x, G^{(p)}(x, y)\right),$$

and

$$G^{(0)}(x, y) \equiv x, \quad G^{(1)}(x, y) \equiv G(x, y), \quad G^{(p+1)}(x, y) \equiv G\left(G^{(p)}(x, y), G^{(p-1)}(x, y)\right),$$

so that $x_{-1}$ and $x_0$ must be specified. Notice that in the corresponding scheme of successive approximations,

$$G_{m+1}(x, y) = M\left(x, G^m_{G_m}(x, y)\right),$$

one could use Ohira’s “fixed rate prediction” [1] to provide the following starting guess:

$$G_0(x, y) = M\left(x, x + p(x - y)\right).$$
We have used the code in Table III to compute the results in Figure 4, where the results for $G_2$, $G_4$, $G_6$ and $G_8$ are displayed for $\mu = 0.5$, $\beta = 0.8$, $p = 5$, $x_0 = x_{-1} = 0.5$ and initial guess (22). We see that the successive approximations converge slowly to the same solution displayed in Figure 1; in particular this means that in this example (3) is also an order reduction of (18).

VI. FINAL COMMENTS

We have extended the concept of order reduction to predictive dynamical systems and discussed some examples in which it can be used to construct good approximations to exact solutions of those systems. In particular, we have shown that a method of successive approximations may be used to check and improve the accuracy of Ohira’s extrapolate prediction [1]. We are not claiming that the method will work always, but that, as happens with singular and delay differential equations, there may be interesting cases in which it can be used to construct solutions to predictive dynamical systems. In other cases one must have to resort to other approximation scheme, such as series expansions, backward integration, shooting methods, (or a root finding routine to solve for $x_{n+p}$ at each step), etc.

To keep things simple we have only considered discrete dynamical systems; but the concepts explored here can be extended in an obvious way both to reductions of higher order and to differential-difference equations of advanced type (with the meaning defined in reference [14]).

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FIG. 1: $x_0, x_1, \ldots, x_{10}$ obtained with $F_1, F_2, F_4$ and $F_5$, in the case of the “sigmoid function.”

FIG. 2: Same as Fig. [1] but with $F_0(x) = M(x, x_0)$. 
FIG. 3: $x_0, x_1, \ldots, x_{10}$ obtained with $F_1, F_2$ and $F_3$, in the case of the “Mackey-Glass function.”

FIG. 4: $x_0, x_1, \ldots, x_{10}$ obtained with $G_2, G_4, G_6$ and $G_8$, in the case of the “sigmoid function.”
Clear[F]

(* Forget previous calculation *)

F[m_, x_] := F[m, x] = M[x, Nest[F[m-1, #]&, x, p]] (* Recurrence *)

F[0, x_] := F[0, x] = M[x, x] (* Initial guess *)

M[x_, y_] := (1-0.5)x+2/(1+Exp[-0.8 y])-1 (* Map *)

p = 5; (* Advance *)

x0 = 0.5; (* Initial condition *)

m = 2; (* Approximation *)

ListPlot[NestList[F[m, #]&, x0, 10]]; (* Plot F_m(x_n) *)

TABLE I: Mathematica program to compute and display $F_m(x_n)$. 

...
Clear[F] (* Forget previous calculation *)

F::"iterations" = "Too many iterations."

F[x_] := F[x] = Module[{m}, (* Successive approximations *)
    For[m = 1, Abs[(F[m,x]-F[m-1,x])/(F[m,x]+0.001)] > tol, m++,
    If [m > mmax, Message[F::"iterations"]; Break[]]
    ];
    F[m,x]
]

F[m_,x_] := F[m,x] = M[x,Nest[F[m-1,#]&,x,p]] (* Recurrence *)

F[0,x_] := F[0,x] = M[x,x] (* Initial guess *)

M[x_,y_] := (1-0.5)x+0.8y/(1+y^10) (* Map *)

p = 8; (* Advance *)
x0 = 0.5; (* Initial condition *)
tol = 10^-5; (* Maximum relative error *)
mmax = 10; (* Maximum value of m *)
ListPlot[NestList[F,x0,10]]; (* Plot F(x_n) *)

**TABLE II:** Mathematica program to compute and display an approximation to $F(x_n)$. 

12
Clear[G, x]

(* Forget previous calculation *)

G[0, m_, x_, y_] := x

(* Recurrence *)

G[p_, m_, x_, y_] := G[p, m, x, y] = G[1, m, G[p-1, m, x, y], G[p-2, m, x, y]]

G[1, m_, x_, y_] := G[1, m, x, y] = M[x, G[p, m-1, x, y]]

G[1, 0, x_, y_] := G[1, 0, x, y] = M[x, x+p(x-y)]

(* Initial guess *)

M[x_, y_] := (1-0.5)x+2/(1+Exp[-0.8 y])-1

(* Map *)

p = 5;

(* Advance *)

x[m_, -1] := 0.5;

(* Initial conditions *)

x[m_, 0] = 0.5;

x[m_, n_] := x[m, n] = G[1, m, x[m, n-1], x[m, n-2]]

(* Order reduction *)

m = 1;

(* Approximation *)

ListPlot[Table[x[m, n], {n, 0, 50}]],

(* Plot solution *)

TABLE III: Mathematica program to compute and display $G_m(x_n,x_{n-1})$. 

13