Qubit decoherence due to detector switching

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We provide insight into the qubit measurement process involving a switching type of detector. We study the switching-induced decoherence during escape events. We present a simple method to obtain analytical results for the qubit dephasing and bit-flip errors, which can be easily adapted to various systems. Within this frame we investigate potential of switching detectors for a fast but only weakly invasive type of detection. We show that the mechanism that leads to strong dephasing, and thus fast measurement, inverts potential bit flip errors due to an intrinsic approximate time reversal symmetry.

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Noise-activated switching out of a metastable state is a common phenomenon in a wide range of physical systems, including Josephson junctions, nanomechanical devices, chemical reactions \cite{1,2}. Starting with Kramers seminal work \cite{3}, such processes have been studied close to equilibrium \cite{4}, as well as in driven systems \cite{5}. The activated escape paths have been studied theoretically and observed experimentally \cite{6,7}.

Recently, noise-activated switching has gained attention due to its role in quantum measurement, in particular for qubit detection. Examples of switching detectors include the superconducting quantum interference device (SQUID) \cite{8,9,10}, where switching occurs between the superconducting and dissipative state. The Josephson bifurcation amplifier \cite{11,12,13} has been recently employed in the delicate task of detecting a qubit state in a minimally invasive fashion \cite{14}. In this case, the detector can switch between different, weakly dissipative, dynamical states. Using an appropriate choice of a reference frame, switching between such dynamical states can also be described as escape from a static metastable potential well \cite{15,16}.

Switching is a highly nonlinear phenomenon, driven by large, non-equilibrium environmental fluctuations, so this type of detection is far from the weak measurement scenario. Some understanding for the switching type of detectors has been provided by numerical studies \cite{17,18} and in a simplified two-state detector version in Ref. \cite{19}. However, a full description of the qubit decoherence during, and induced by the switching event is still missing.

In this paper we propose a simple and analytical method to investigate qubit decoherence due to a switching type of detector. We model the detector as a classical, overdamped particle trapped in a metastable potential. The escape of the particle is driven by large, rare fluctuations in the environment. We investigate the qubit dephasing and bit flip errors induced by the switching, during the escape event. This allows novel insights into the measurement process and reveals the specific conditions during the switching event that lead to a combination of strong coherence loss and low error rate. These are desirable qualities for a qubit detector. The overdamped classical particle performs Brownian motion according to

$$\dot{x} = K(x) + f(t),$$ \hfill (1)

where $K$ is the deterministic force experienced by the particle due to the metastable potential and $f$ is white Gaussian noise with a probability density functional given by \cite{20}

$$P[f(t)] = \exp \left( -\int_0^t f(t)^2/(2D) dt \right),$$ \hfill (2)

where we assume, \cite{21} the noise intensity $D$ to be small compared to the barrier height $\Delta U$, see Fig. 1. The probability density functional for a noise driven trajectory can be obtained by expressing $f(t)$ in terms of $x(t)$ using

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Prehistory probability density $P(x,t)$ for metastable potential $U(x)$, where $U'(x) = -K(x)$, and optimal trajectory $x_{opt}(t)$. Here $x_{m,i,M}$ are the positions of the minimum, maximum and inflexion points of the potential.}
\end{figure}
Eq. (1)  
\[ P[x(t)] = \exp \left( -\frac{S[x(t)]}{D} \right), \quad S[x(t)] = \frac{1}{2} \int_0^t dt (\dot{x} - K(x))^2. \]  

For the study of qubit decoherence during a switching event one will need expectation values of the type
\[ O(t_0) = \langle \exp (\lambda \phi[x(t), s(t, t_0)]) \rangle_{sw}, \tag{4} \]
\[ \phi[x(t), s(t, t_0)] = \int_0^{t_f} x(t)s(t, t_0)dt, \tag{5} \]
were \( s(t, t_0) \) is a time dependent modulation of \( x(t) \). We are interested in the qubit decoherence during switching. Thus, the averaging \( \langle \rangle_{sw} \) is performed only over switching trajectories of the detector, which satisfy the boundary conditions \( x(0) = x_m \) and \( x(t_f) = x_f \), with \( x_m \) inside and \( x_f \) outside the metastable well. By choosing \( s(t, t_0) = 0 \) at \( t_f > t > t_0 \), the average becomes post-conditioned by a switching event taking place at the final time \( t_f \).

Since the exact trajectory between the initial and final point remain unknown, we average over all possible paths
\[ O(t_0) = \int_{(x_m, 0)}^{(x_f, t_f)} D x(t) \exp \left( \lambda \phi[x(t), s(t, t_0)] - \frac{S[x(t)]}{D} \right) \times P(x_m, 0|x_f, t_f)^{-1}, \tag{6} \]
where the total switching probability is
\[ P(x_m, 0|x_f, t_f) = \int_{(x_m, 0)}^{(x_f, t_f)} D x(t) \exp (-S[x(t)]/D). \tag{7} \]

The switching trajectories form a narrow tube in the phase space centered around the optimal trajectory \[22, 23\] which minimizes \( S \), and for the present case satisfies
\[ \ddot{x}_{opt} = K(x_{opt})K'_{opt}(x_{opt}), \quad x_{opt}(0) = x_m, \quad x_{opt}(t_f) = x_f. \tag{8} \]

Thus \( S[x(t)] = S[x_{opt}(t)] + S_2[x(t) - x_{opt}(t)] \) and we approximate
\[ S_2[x(t)] \approx \frac{1}{2} \int_0^{t_f} dt (\dot{x}(t)^2 - \Lambda(t)^2 x(t)^2), \tag{9} \]
where \( \Lambda(t)^2 = -(K''(x)^2 + K(x)K''_x(x))|_{x=x_{opt}(t)} \). Divergences due to the emergence of a slow mode on the barrier top \[21, 24\] are avoided by the appropriate choice of the initial (kinetic) energy \( 0 < \dot{x}(0)^2/2 \ll \Delta U \) which satisfies the boundary conditions \[8\]. Thus, the switching event takes place, with non-vanishing probability, within a finite time \( t_f \). One can show that
\[ O(t_0) = \exp \left( \lambda \phi[x_{opt}(t) + x_0(t)/2, s(t, t_0)] \right), \tag{10} \]
where \( x_0 \) is the solution of
\[ \ddot{x_0} + \Lambda^2 x_0 + D\lambda s(t, t_0) = 0, \quad x_0(0) = x_0(t_f) = 0. \tag{11} \]

The two linearly independent solutions of the homogeneous part of Eq. (11) are
\[ x_1(t) = \dot{x}_{opt}(t), \quad x_2(t) = \dot{x}_{opt}(t) \int_0^t dt '\dot{x}_{opt}(t ')^{-2}, \tag{12} \]
and the full \( x_0(t) = x_1(t)c_1(t) + x_2(t)c_2(t) \) can be determined by variation of constants.

We consider the case of a metastable potential described by \( U(x)/\Omega = x^2/2 - x^4/6 \), which can represent a Josephson junction DC-biased at half the critical current. The characteristic frequency of the detector is given by \( \Omega = K''(x_m) \).

Pure dephasing: We assume a qubit Hamiltonian of the form
\[ \hat{H} = \hbar \omega \hat{\sigma}_z + \eta \hat{\sigma}_x x(t), \tag{13} \]
where \( x(t) \) is the coordinate of the classical particle. The only effect of the environment in this case is the irreversible decay of the phase coherence \( C(t_0) = O(t_0) \), see Eq. (4), for the specific value of \( \lambda = \eta/(i\hbar) \) and
\[ s(t, t_0) = \begin{cases} 1, & t < t_0 \\ 0, & t > t_0. \end{cases} \tag{14} \]

![Fig. 2: Qubit coherence during a switching event (\( t_0 < t_f \)) (a), optimal noise trajectory \( f_{opt}(t_0) \) (b) and most probable switching trajectory \( x_{opt}(t_0) \) (c) for various values of \( t_f \).](image)
coherence remains at an almost constant value. We observe that the optimal noise becomes stronger for shorter values of $t_I$. However, the strongest drop in qubit coherence was observed for the longer $t_I$. In this case the optimal trajectory spends more time close to the barrier top, where the motion is diffusive, driven by low amplitude noise.

**Bit flip errors**: We consider a qubit-environment coupling which allows for energy exchange, and can induce bit flip errors

$$\dot{\hat{H}} = \hbar \omega \hat{\sigma}_z + \eta x(t) \hat{\sigma}_x. \quad (15)$$

The probability of noise induced errors during a switching event, at $t_0 < t_I$ is given by

$$P_{\text{err}}(t_0) = |\langle \downarrow | \hat{U}(t_0) | \uparrow \rangle|^2 P(x_m, 0|x_1, t_1)^{-1}, \quad (16)$$

where in the limit of short time and weak coupling

$$\hat{U}(t_0) = \mathcal{T} \exp \left( \int_{0}^{t_0} dt \frac{\hat{H}_I(t)}{i \hbar} \right) \approx 1 + \int_{0}^{t_0} dt \frac{\hat{H}_I(t)}{i \hbar},$$

$$\hat{H}_I(t) = \eta \hat{U}_0(t) \hat{\sigma}_x \hat{U}_0(t) x(t), \quad (17)$$

and $\hat{U}_0$ describes the free qubit evolution. We obtain

$$P_{\text{err}}(t_0) = \lim_{\delta x \to 0} \frac{\partial^2}{\partial x \partial x} \left( \exp (\lambda \phi(x(t), s(t, t_0)) \mathcal{R}(t)) \right) \exp (\lambda \phi(x(t), s(t, t_0)) \mathcal{R}(t)) \mathcal{R}(t) + i \mathcal{I}(t) = \langle \downarrow | \hat{U}_0^{-1}(t) \hat{\sigma}_x \hat{U}_0(t) | \uparrow \rangle. \quad (18)$$

In Fig. 3 (a) we observe, similar to the pure dephasing case, a sharp feature in $P_{\text{err}}(t_0)$ at the point in time where the most probable trajectory (c) reaches the steepest point on the potential barrier, and the most probable noise (b) reaches its maximum strength. Despite the optimal noise being strongest for short switching time $t_I$, the peak in $P_{\text{err}}(t_0)$ is higher for longer $t_I$. Another notable feature is the quasi-reversibility of the bit flip error which occurs at this point. This feature cannot be explained by the single, deterministic trajectory $x_{\text{opt}}$ alone. It causes only the steady increase of $P_{\text{err}}(t_0)$.

**Prehistory density distribution**: The results presented above can be understood from the distribution of switching trajectories. We calculate the probability $P_h(x, t)$ for the classical particle to occupy the position $x$ at time $t$ during a switching event, in the form of a prehistory density distribution

$$P_h(x, t) = \frac{P(x_m, 0|x, t) P(x, t|x_1, t_I)}{P(x_m, 0|x_1, t_I)}, \quad (19)$$

Within the approximation $|\mathcal{R}|$, the probability for a transition between any pair of points $(x_1, t_1)$ and $(x_2, t_2)$, with $t_{1,2} < t_I$ reads

$$P(x_1, t_1|x_2, t_2) = \int_{0}^{(\delta x_1, t_2)} D x(t) \exp \left( -\frac{S[x_{\text{opt}}(t)]^2 + S_2[x(t)]^2}{D} \right) \mathcal{R}(f(t_1)|t_2), \quad (20)$$

where $S[x(t)]^2$ implies that the time integral is taken between $t_1$ and $t_2$ and $\delta x_{1,2} = x_{1,2} - x_{\text{opt}}(t_{1,2})$. One can show that

$$P(x_1, t_1|x_2, t_2) = \exp \left( -\frac{S[x_{\text{opt}}(t)]^2 + S_2[x_b(t)]^2}{D} \right) \cdot \mathcal{R}(f(t_1)|t_2), \quad (21)$$

where $\bar{x}_b + \Lambda^2(t)x_b(t) = 0$ and $x_b(t_{1,2}) = \delta x_{1,2}$ and

$$F(t_1|t_2) = \int_{0}^{(\delta x_1, t_2)} D x(t) \exp \left( -\frac{S_2[x(t)]}{D} \right) \left( 2\pi D x_{\text{opt}}(t_1) x_{\text{opt}}(t_2) \int_{t_1}^{t_2} x_{\text{opt}}(t)^{-2} dt \right)^{-1/2}. \quad (22)$$

We obtain a Gaussian distribution, centered around $x_{\text{opt}}(t)$

$$P_h(x, t) = \frac{1}{\sqrt{\pi w(t)}} \exp \left( -\frac{(x - x_{\text{opt}}(t))^2}{w(t)} \right), \quad (23)$$

where $w(t) = F(0|t_I)^2/(F(0|t)^2 F(t|t_I)^2)$.

Fig. 3 shows a narrow tube of trajectories close to the bottom of the well. This is followed by a strong widening of the distribution in the process of climbing up the potential barrier. This event is driven by a sharp noise pulse. On the barrier top we see again a fairly localized density distribution, driven by low-amplitude noise. The tube narrows even more on the outer side of the barrier. These results are in agreement with the findings of Ref. [25], for a different system.
We observed that the trajectory tube width, $\text{P}_\text{h}(x,t)$, exhibits two distinct effects: strong widening at $t \approx t_1$, and a sharp drop near $t = t_f$, which we attribute to the quasi-reversibility of the escape process, leading to fast decoherence. This behavior is evident in Fig. 1, where the probability distribution $\text{P}_\text{h}(x,t)$ is plotted as a function of $t$.

The strong widening of the prehistory distribution, $\text{P}_\text{h}(x,t)$, at $t \approx t_1$, is due to the interference of multiple escape trajectories, leading to coherence loss. This effect is more pronounced at higher $\Omega f$ values, as shown in Fig. 1. The observed increase in decoherence rate is consistent with theoretical predictions and experimental observations.

We note that the widening of the trajectory tube at $t \approx t_1$ originates in the strong widening of the prehistory probability distribution approach and for such incoherent behavior causes decoherence.

In conclusion, we have presented a switching detector based on the quasi-reversibility of escape trajectories, which shows high sensitivity and low decoherence. This work has been supported by NSERC Discovery Grants and Quantum Works.

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