SCALING AND ENTROPY FOR THE RG-2 FLOW

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Abstract. Let \((M, g)\) be a closed Riemannian manifold. The second order approximation to the perturbative renormalization group flow for the nonlinear sigma model (RG-2 flow) is given by:

\[
\frac{\partial}{\partial t} g(t) = -2\text{Ric}(t) - \frac{\alpha}{2} \text{Rm}^2(t),
\]

where \(g\) = Riemannian metric, \(\text{Ric} = \text{Ricci curvature}, \text{Rm}^2_{ij} := R_{irml}R^r_{mk}\), and \(\alpha \geq 0\) is a parameter. The flow is invariant under diffeomorphisms, but not under scaling of the metric. We first develop a geometrically defined coupling constant \(\alpha_g\) that leads to an equivalent, scale-invariant flow. We further find a modified Perelman entropy for the flow, and prove local existence of the resulting variational system. The crucial idea is to modify the flow by two diffeomorphisms, the first being the usual DeTurck diffeomorphism the second being strictly related to the geometrical characterization of the coupling constant \(\alpha_g\). Although the modified Perelman entropy is monotonic, the RG-2 flow is not a gradient flow with respect this functional. We discuss this issue in detail, showing how to deform the functional in order to give rise to a gradient flow for a DeTurck modified RG-2 flow.

1. INTRODUCTION: A SCALE-INVARIANT RG-2 FLOW

The RG-2 flow (see e.g. \[1\], \[7\], \[11\], \[12\], \[13\], \[19\]) is the geometric flow associated with the two–loop (i.e. second order) approximation to the perturbative renormalization group flow for nonlinear sigma models \[3\], \[8\] \[9\] given by

\[
\frac{\partial}{\partial t} g_{ij}(t) = -2\text{Ric}_{ij}(t) - \frac{\alpha}{2} \text{Rm}^2_{ij}(t),
\]

\[
g_{ab}(t = 0) = g_{ab},
\]

where \(\text{Ric}(t), \text{Rm}(t)\) denote the Ricci and the Riemann tensor of the evolving metric \(g(t)\), and

\[
\text{Rm}^2_{ij}(t) := \text{Rm}_{iklm}(t)\text{Rm}_{jqr}(t)g^{kp}(t)g^{rq}(t)g^{mr}(t).
\]

Note that the fixed parameter \(\alpha \geq 0\) in \[1\] is dimensionful (it has dimension of a length squared, i.e. \([\alpha] = [L^2]\)) and is typically assumed to be unrelated to the geometry. This immediately implies that the system of partial differential equations \[1\] is not invariant under scalings of the metric: if \(g \rightarrow \lambda g, \lambda \in \mathbb{R}_{>0}\), then \(\text{Rc}(\lambda g) = \text{Rc}(g)\), but \(\text{Rm}^2(\lambda g) = \frac{1}{\lambda}\text{Rm}^2(g)\), and consequently

\[
2\text{Ric}(\lambda g) + \frac{\alpha}{2} \text{Rm}^2(\lambda g) = 2\text{Ric}(g) + \frac{\alpha}{2} \lambda^{-1} \text{Rm}^2(g).
\]

This is at variance with what happens for the Ricci flow, where one has manifestly parabolic space and time scaling symmetry, which are of basic importance in the geometric applications of the theory. The lack of scaling invariance is a source of a number of delicate problems in the analysis of the RG-2 flow. This is already evident when dealing with the condition assuring its (weak)–parabolicity, according to which the flow exists (and is parabolic) provided that \[11\], \[7\], \[19\]

\[
1 + \alpha K(g)[\sigma] > 0, \quad \forall \sigma \in Gr(2)(TM),
\]

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where $\mathcal{K}_g[\sigma]$ denotes the sectional curvature of the initial $(M,g)$ along the plane $\sigma \in \text{Gr}_2(TM)$, and $\text{Gr}_2(TM)$ is the Grassmannian of 2-laes in $TM$; however, under the scaling action $g \rightarrow \lambda g$ we get

\begin{equation}
\mathcal{K}_{\lambda g}[\sigma(X,Y)] = \lambda^{-1} \mathcal{K}_g[\sigma(X,Y)].
\end{equation}

It follows that if we assume that the condition $[\mathfrak{I}]$ holds for the manifold $(M,g)$, then on the rescaled manifold $(M,\lambda g)$ the analogous condition may easily fail as soon as $\mathcal{K}_g[\sigma] < 0$ and $\lambda$ is small enough. The fact that (weak–)parabolicity of $[\mathfrak{I}]$ depends on the size of the manifold is a somewhat unsatisfactory feature. One may argue that from a PDE point of view this behavior cannot be formally ruled out; nonetheless, one would like a deeper rationale for the fact that a geometric flow, driven by local curvatures, changes nature abruptly as a function of the overall size of the manifold. Moreover, the characterization of the coupling $\alpha$ as a quantity unrelated to geometry is physically unjustified from the point of view of the perturbative renormalization group for nonlinear sigma model, where one is forced to attribute the role of true coupling parameter to the normalized metric $\alpha^{-1}g$. This latter remark is made quite clear in D. Friedan’s foundational paper (see [8] pp.324) where (referring to the parameter $\alpha$ as a temperature) he stresses that “... The temperature $T$ in the coupling $T^{-1}g_{ij}$ is not a separate parameter. Multiplying $T$ by a positive constant $c$ while multiplying $g_{ij}$ by $c^{-1}$ leaves the coupling unchanged. The temperature is written separately only to make the expansion parameter visible and appears only in the combination $(Tg^{-1})^{ij}$. ...”. It must be said that, even if physically motivated, it is difficult to implement Friedan’s remark in the geometric flow framework associated with $[\mathfrak{I}]$.

We need a natural mechanism that forces the rescaling of $\alpha$ along with the rescaling of the metric, and this cannot be implemented with the RG-2 flow as it stands. The most obvious candidate for such a mechanism, i.e. setting $(\alpha)_{\lambda}^{\frac{2}{T}} := \int_M d\mu(g)$, where $d\mu(g)$ is the Riemannian volume element, is not a viable prescription since along $[\mathfrak{I}]$ the Riemannian volume is not constant. To develop a solution to this problem, we exploit a natural variant of the Perelman’s strategy [24] by introducing along the RG-2 flow a reference measure, and associating to it a geometrically defined coupling constant $\alpha_g$.

More precisely, let $(M,g, d\omega(g))$ be a closed $n$–dimensional Riemannian manifold $(n \geq 3)$ with density [14], [15], i.e. a smooth orientable manifold without boundary, endowed with a Riemannian metric $g$ and a Borel measure $d\omega(g)$ that is absolutely continuous with respect to the Riemannian volume element $d\mu_g$. We set $d\omega(g) = e^{-f} d\mu_g$ for some smooth function $f \in C^\infty(M, \mathbb{R})$, and denote by

\begin{equation}
(\alpha_g)^\frac{2}{T} := \int_M d\omega(g)
\end{equation}

the total $d\omega(g)$–mass of $(M,g)$, and by

\begin{equation}
\tilde{d}\omega := (\alpha_g)^{-\frac{2}{T}} d\omega(g) = (\alpha_g)^{-\frac{2}{T}} e^{-f} d\mu_g, \quad \int_M \tilde{d}\omega = 1,
\end{equation}

the associated probability measure $d\tilde{\omega}(g)$. Note that under the metric rescaling $g \mapsto \lambda g$, $\lambda \in \mathbb{R}_{>0}$, the parameter $\alpha_g$, associated with $(M,g, d\omega(g))$ scales according to $\alpha_{\lambda g} = \lambda \alpha_g$. Moreover, $\alpha_g$ is defined up to the gauge transformation

\begin{equation}
d\omega \mapsto \tilde{d}\omega := d\omega + \alpha_g L_{\xi_g} d\omega = \left(1 + \alpha_g \text{div}^{(\omega)} \xi_g\right) d\omega,
\end{equation}

where $\xi_g \in C^\infty(M,TM)$ is a smooth vector field (actually, as we shall see, a gradient vector field, hence we assume that it has the dimension of an inverse length, i.e. $[\xi] = [L^{-1}]$), and where the weighted divergence operator $\text{div}^{(\omega)}$ is characterized in terms of the Lie derivative $L_{\xi_g} d\omega$ of $d\omega$. 


along $\xi_g$ according to
\begin{equation}
L_{\xi_g} d\omega = \text{div}^{(\omega)}(\xi_g) d\omega = \nabla_k^{(\omega)} \xi_g^k d\omega := (\nabla_i - \nabla_i f) \xi_g^i d\omega.
\end{equation}

Hence, $\int_M d\bar{\omega} = \int_M \left(1 + \alpha_g \text{div}^{(\omega)}(\xi_g)\right) d\omega = \int_M d\omega = (\alpha_g)^{\frac{2}{n}}$.

In the analysis that follows, it is useful to keep track of the gauge freedom by introducing the following characterization of the scale invariant RG–2 flow.

**Definition 1.** (The scale–invariant RG–2 flow).

Let $[0,1] \ni t \mapsto \xi_{g(t)} \in C^\infty(M,TM)$ be a given choice of a possibly $t$–dependent vector field on $(M,g,d\omega)$. The scale–invariant RG–2 flow associated with the Riemannian manifold with density $(M,g,\omega(t))$ is
\begin{equation}
\frac{\partial}{\partial t} g_{ij}(t) = -2 \text{Ric}_{ij}(t) - \frac{\alpha_g}{2} \text{Rm}_{ij}(t), \quad g_{ij}(t = 0) = g_{ij},
\end{equation}
coupled to the backward Fokker–Planck equation describing the (backward) diffusion of the measure $d\omega(t)$ in presence of the drift generated by the given time–dependent vector field $\xi_{g(t)}$,
\begin{equation}
\frac{\partial}{\partial t} d\omega(t) = -\Delta_{g(t)} d\omega(t) - \text{div}^{(\omega)}(\xi_{g(t)}) d\omega(t),
\end{equation}
where $\Delta_{g(t)}$ denotes the Laplace–Beltrami operator associated with the evolving metric $g(t)$.

Note that when we couple (11) to (10), we can write $\alpha_{g(t)}$ in place of $\alpha_g$ since under the flows (10) and (11) the coupling parameter $\alpha_{g(t)}$ remains constant. Explicitly, let us introduce the $d\omega$–weighted Laplacian on $(M,g,\omega)$ according to
\begin{equation}
\Delta_g^{(\omega)} \varphi := e^f \Delta_g \left( e^{-f} \varphi \right) = \Delta_g \varphi - g^{ik} \nabla_i f \nabla_k \varphi, \quad \varphi \in C^\infty(M,\mathbb{R}).
\end{equation}
From the relation
\begin{equation}
\Delta_g d\omega = \Delta_g \left( e^{-f} d\mu_g \right) = -\left( \Delta_g f - |\nabla f|^2_g \right) e^{-f} d\mu_g =: -\Delta_g^{(\omega)} f d\omega,
\end{equation}
we compute
\begin{equation}
\frac{d}{dt} \left( \alpha_{g(t)} \right)^{\frac{2}{n}} = \frac{d}{dt} \int_M d\omega(t) = \int_M \left[ -\Delta_{g(t)} d\omega(t) - \text{div}^{(\omega)}(\xi_{g(t)}) d\omega(t) \right]
\end{equation}
\begin{equation}
= -\int_M \Delta_{g(t)} (d\omega(t)) = \int_M \Delta_{g(t)}^{(\omega)} f(t) d\omega(t) = 0.
\end{equation}
In a sense, the evolution (11) is a time–dependent version of the gauge transformation. Its Fokker–Planck nature immediately follows from the relation
\begin{equation}
\frac{\partial}{\partial t} d\omega(t) = -\Delta_{g(t)} d\omega(t) - \text{div}^{(\omega)}(\xi_{g(t)}) d\omega(t) = -\Delta_{g(t)} d\omega(t) + \nabla_k \left( \xi_{g(t)}^k d\omega(t) \right),
\end{equation}
which allows us to interpret the given time–dependent vector field $\xi_{g(t)}$ as the generator of a drift acting on the (backward) diffusion of the measure $d\omega(t)$. The first set of results we present in this paper are the local existence for the initial value problem associated with the coupled system (10) and (11), and the characterization of its scaling properties. The second set of results concerns the proof of existence of a monotonic functional, which plays for the RG-2 flow the same role that Perelman’s $F$–energy has in standard Ricci flow theory. The main theorems are Theorem 5 and Theorem 7.
2. LOCAL EXISTENCE AND SCALE INVARIANCE

Local existence for the initial value problem associated with the coupled system (10) and (11) directly follows from an obvious adaptation of the conditions [7], [11] for weak–parabolicity for the standard RG-2 flow (1). However, at variance with respect to its non–scaling behavior, we now have manifestly parabolic space and time scaling symmetry.

Theorem 2. Let \((M, g, d\omega(g))\) be a closed \(n\)–dimensional Riemannian manifold \((n \geq 3)\) with density, and denote by \(\text{Gr}_{(2)}(TM)\) the Grassmannian of \(2\)–planes in \(TM\). If the parameter \(\alpha_g\) and the initial metric \(g\) are such that

\[
1 + \alpha_g K_P(M, g) > 0, \quad \forall P \in \text{Gr}_{(2)}(TM),
\]

where \(K_P(M, g)\) is the sectional curvature of \((M, g)\) along the plane \(P \in \text{Gr}_{(2)}(TM)\), then the initial value problem associated with (10) is weakly-parabolic, and there exists a unique solution

\[
(t, g) \mapsto g(t),
\]

on some time interval \([0, T)\). Let \(T_0 < T\) and set \(\eta := T_0 - t\). Then along the time–reversed flow \(\eta \mapsto g(\eta), \eta \in [0, T_0]\), the evolution (17) of the measure \(\eta \mapsto d\omega(t = T_0 - \eta) = e^{-f(\eta)} d\mu_{g(\eta)}\), in the gauge defined by the chosen vector field \(\eta \mapsto \xi_\alpha(t = T_0 - \eta)\), is governed by the Fokker–Planck equation

\[
\frac{\partial}{\partial \eta} d\omega(\eta) = \Delta_{g(\eta)} d\omega(\eta) + \nabla_k \left( \xi_g^k(\eta) d\omega(\eta) \right).
\]

The resulting evolution \([0, T_0] \ni t \mapsto (g(t), d\omega(T_0 - t))\) induces on the solution space of (10) and (11) the parabolic space and time scaling symmetry

\[
(g(t), \xi_{g(t)} d\omega(T_0 - t)) \mapsto \left( \lambda g(t/\lambda), \lambda \xi_{g(t/\lambda)}, \lambda^2 d\omega \left( g((T_0 - t)/\lambda) \right) \right),
\]

for \(t \in [0, T_0]\), and \(\forall \lambda \in \mathbb{R}_{>0}\).

Proof. Since the coupling parameter \(\alpha_g\) in (10) refers to the given initial metric \(g\), the proof of local existence of (10) follows directly from the conditions [7], [11], for weak–parabolicity for the standard RG-2 flow (1) with \(\alpha \equiv \alpha_g\). Before proceeding with the analysis of the evolution (11) of the measure \(t \mapsto d\omega(t)\) along the solution \((t, g) \mapsto g(t)\) of (10), and of the associated scaling properties (19), we need to further explore the nature of the gauge choice associated with the drift vector field \(\xi_g\). Note that (13) is an infinitesimal version of Moser’s theorem [18], and we can exploit the Helmholtz decomposition of the vector field \(\xi_g\), to provide a finer resolution of the gauge freedom (8). To this end, let \(L^2(M, d\omega)\) be the \(L^2\) inner product on \(M\) with respect to the measure \(d\omega\), and let \(W^{p,q}(M)\) and \(W^{p,q}(TM)\) respectively denote the corresponding space of functions and vector fields of Sobolev class \((p, s)\) with \(p > 1, s > n/p + 2\). Then, for any \(1 \leq q \leq s\), we have the weighted Helmholtz decomposition

\[
W^{p,q-1}(TM) = \text{grad} \left( W^{p,q}(M) \right) \oplus \text{Ker} \left( \text{div}(\omega) \right)
\]

(this is an obvious adaptation to \((M, g, d\omega)\) of the standard Helmholtz decomposition (see e.g. [2]), which holds as long as the metric and \(f\) are smooth enough, say in the usual \(W^{p,q}\)). Hence, we can write

\[
\xi_g = \nabla \psi + \xi_\perp,
\]

where the vector field \(\xi_\perp\) is such that \(\text{div}(\omega)(\xi_\perp) = 0\), and the scalar function \(\psi \in W^{p,q}(M)\) is the solution of the elliptic PDE

\[
\Delta_g^{(\omega)} \psi = \text{div}(\omega) \xi_g,
\]
where $\Delta_g^{(\omega)}$ is the $d\omega$-weighted Laplacian \cite{21} on $(M, g, d\omega)$. Note that \cite{22} can be interpreted as the Otto parametrization \cite{21, 22} of the tangent vectors

\begin{equation}
T_\omega \mathrm{Prob}_{ac}(M, g) := \left\{ h \in C^\infty(M) \mid \int_M h d\overline{\omega} = 0 \right\} ,
\end{equation}

to the space of absolutely continuous probability measures $\mathrm{Prob}_{ac}(M, g)$ on $(M, g)$. It follows that the gauge transformation \cite{8} can be equivalently rewritten as

\begin{equation}
d\omega \mapsto d\overline{\omega} = \left( 1 + \alpha_g \Delta_g^{(\omega)} \psi \right) d\omega ,
\end{equation}
in terms of the scalar function $\psi$. This gauge freedom is clearly defined up to the residual gauge characterized by the transformation $\nabla \psi \mapsto \nabla \psi + \xi_\perp$, with $\nabla g^{(\omega)} \xi_\perp = 0$. Formally this can be justified on more sophisticated grounds by using the isomorphism between $T_\omega \mathrm{Prob}_{ac}(M, g)$ and $C^\infty(M)/\mathbb{R}$. This goes as follows. Let $h \in C^\infty(M)$ with $\int_M h d\overline{\omega} = 0$ be the scalar function representing a tangent vector to $\mathrm{Prob}_{ac}(M, g)$ (see \cite{23}). Any such $h$ generates a gauge transformation \cite{8} according to

\begin{equation}
d\omega \mapsto d\overline{\omega} := d\omega - h d\overline{\omega} ,
\end{equation}
where we have used the normalized probability measure \cite{7} associated with $d\omega$, and the minus sign in front of $h$ is for later convenience. We can parametrize any $h \in T_\omega \mathrm{Prob}_{ac}(M, g)$ in terms of the solution $\varphi$ of the (Otto) elliptic partial differential equation

\begin{equation}
\Delta_g^{(\omega)} \varphi = -h ,
\end{equation}
under the equivalence relation identifying any two such solutions differing by an additive constant. It is relatively easy to prove (\cite{21} see also \cite{17}) that the map so defined,

\begin{equation}
T_\omega \mathrm{Prob}_{ac}(M, g) \rightarrow C^\infty(M, \mathbb{R})/\mathbb{R} ,
\end{equation}

is an isomorphism. Hence we can always write the general gauge transformation \cite{25} in the form

\begin{equation}
d\omega \mapsto d\overline{\omega} := d\omega + \Delta_g^{(\omega)} \varphi d\overline{\omega} ,
\end{equation}
which, up to the constant normalization factors related to $\alpha_g$, is exactly \cite{24}.

**Remark 3.** According to these remarks, it would appear simpler to use directly $\nabla \psi$ rather than $\xi_g$. However, the characterization $\psi$ along the solution $(t, g) \mapsto g(t)$ of \cite{14} requires the solution of the (non-uniformly, since $g(t)$ is time-dependent) elliptic PDE \cite{22} coupled with the backward evolution \cite{14} of the measure $d\omega$ (which depends on $\xi_g$). Hence, by design, the prescription we adopt is to assign along $(t, g) \mapsto g(t)$ the gauge vector field $\xi_{g(t)}$, (possibly satisfying a gauge condition of choice), and then solve the evolution \cite{14} as a backward parabolic equation. Explicitly, the local existence result for \cite{14} implies that for any $T_0 < T$ we can consider the time-reversed flow $\eta \mapsto g(\eta), \eta := T_0 - t$, decorated by the given drift-generating time-dependent vector field $\xi_{g(t)}$ according to $\xi_{g(\eta)} := \xi_{g(t=T_0-\eta)}$. The data $(g(\eta), \xi_{g(\eta)})$ so defined characterize the evolution \cite{14} of the measure $d\omega$ as the solution $\eta \mapsto d\omega[g(\eta)] = e^{-f(\eta)} d\mu_{g(\eta)}$ of the forward parabolic Fokker–Planck equation

\begin{equation}
\frac{\partial}{\partial \eta} d\omega[g(\eta)] = \Delta_{g(\eta)} d\omega[g(\eta)] + \nabla_k \left( \xi_{g(\eta)}^k d\omega[g(\eta)] \right) ,
\end{equation}

along the flow $\eta \mapsto (g(\eta), \xi_{g(\eta)}), \eta := T_0 - t$.

It is in such a framework that the solution, $t \mapsto (g(t), d\omega[g(T_0 - t)])$, $t \in [0, T_0]$, of the coupled flows \cite{10} and \cite{11} exhibits a scale invariance as in the case of the Ricci flow. Explicitly, for the given drift vector field $\xi_{g(t)}$ and its $\xi$-weighted Laplacian (12) on $(M, g, d\omega)$, the local existence result for (10) implies that for any $T > 0$ (condition of choice), and then solve the evolution (11) as a backward parabolic equation. Explicitly, adopt is to assign along the evolution (11) of the measure $d\omega$ the gauge transformation (8) according to $g \mapsto T_\omega \mathrm{Prob}_{ac}(M, g)$, decorated by the given drift-generating time–dependent vector field $\xi_{g(t)}$, (possibly satisfying a gauge condition of choice), and then solve the evolution (11) as a backward parabolic equation. Explicitly, the local existence result for (11) implies that for any $T_0 < T$ we can consider the time-reversed flow $\eta \mapsto g(\eta), \eta := T_0 - t$, decorated by the given drift-generating time–dependent vector field $\xi_{g(t)}$ according to $\xi_{g(\eta)} := \xi_{g(t=T_0-\eta)}$. The data $(g(\eta), \xi_{g(\eta)})$ so defined characterize the evolution (11) of the measure $d\omega$ as the solution $\eta \mapsto d\omega[g(\eta)] = e^{-f(\eta)} d\mu_{g(\eta)}$ of the forward parabolic Fokker–Planck equation

\begin{equation}
\frac{\partial}{\partial \eta} d\omega[g(\eta)] = \Delta_{g(\eta)} d\omega[g(\eta)] + \nabla_k \left( \xi_{g(\eta)}^k d\omega[g(\eta)] \right) ,
\end{equation}

along the flow $\eta \mapsto (g(\eta), \xi_{g(\eta)}), \eta := T_0 - t$.
be a solution of the coupled RG-2 flows (10) and (11) in the forward $\tilde{t}$ and backward $\tilde{\eta} : = \tilde{T}_0 - \tilde{t}$ evolution times, i.e.,

\begin{equation}
\frac{\partial}{\partial \tilde{t}} g_{ij}(\tilde{t}) = -2\text{Ric}_{ij}(g(\tilde{t})) - \frac{\alpha g}{2} \text{Rm}_{ij}^2(\tilde{t}) , \quad g_{ab}(\tilde{t} = 0) = g_{ab} ,
\end{equation}

and

\begin{equation}
\frac{\partial}{\partial \tilde{\eta}} d\omega[g(\tilde{\eta})] = \Delta_{g(\tilde{\eta})} d\omega[g(\tilde{\eta})] + \nabla_k \left( \xi^k_{g(\tilde{\eta})} d\omega[g(\tilde{\eta})] \right) .
\end{equation}

Let us rescale $\tilde{t}$ according to $\tilde{t} = \frac{\tilde{t}}{\tilde{\lambda}}$, $\lambda \in \mathbb{R}_{>0}$. We get

\begin{equation}
\frac{\partial}{\partial \tilde{t}} \lambda g_{ij}(t/\lambda) = -2\text{Ric}_{ij}(g(t/\lambda)) - \frac{\alpha g}{2} \text{Rm}_{ij}^2(\tilde{t}) , \quad g_{ab}(t/\lambda = 0) = g_{ab} ,
\end{equation}

and

\begin{equation}
\frac{\partial}{\partial \tilde{\eta}} \lambda d\omega[g(\eta/\lambda)] = \Delta_{g(\eta/\lambda)} d\omega[g(\eta/\lambda)] + \nabla_k \left( \xi^k_{g(\eta/\lambda)} d\omega[g(\eta/\lambda)] \right) ,
\end{equation}

where we have rewritten $\partial/\partial (t/\lambda)$ as $\lambda \partial/\partial t$ (similarly for $\partial/\partial (\eta/\lambda)$). We start by discussing the scaling properties of (33). Since under the rescaling $g(\eta/\lambda) \mapsto \lambda g(\eta/\lambda)$ the Laplace–Beltrami operator $\Delta_{g(\eta/\lambda)}$ and the measure $d\omega[g(\eta/\lambda)]$ scale as

\begin{equation}
\Delta_{g(\eta/\lambda)} = \lambda \Delta_{\lambda g(\eta/\lambda)} ,
\end{equation}

\begin{equation}
d\omega[g(\eta/\lambda)] = \lambda^{-\frac{n}{2}} d\omega[\lambda g(\eta/\lambda)] ,
\end{equation}

it follows that if along $t \mapsto g(t)$ we rescale the metric and the gauge drift according to $g_{ab}(\eta) \mapsto \lambda g_{ab}(\frac{\eta}{\lambda})$ and $\xi_{g(\eta)} \mapsto \lambda \xi_{\lambda g(\frac{\eta}{\lambda})}$, respectively, then (33) reduces to

\begin{equation}
\frac{\partial}{\partial \eta} d\omega[\lambda g(\eta/\lambda)] = \Delta_{\lambda g(\eta/\lambda)} d\omega[\lambda g(\eta/\lambda)] + \nabla_k \left( \xi^k_{\lambda g(\eta/\lambda)} d\omega[\lambda g(\eta/\lambda)] \right) .
\end{equation}

According to (33) and (14), the scaling relation (36) for the measure $d\omega$ also implies that

\begin{equation}
\alpha_{\lambda g} = \alpha_{\lambda g(\eta/\lambda)} = \lambda \alpha_{g(\eta/\lambda)} = \lambda \alpha_g .
\end{equation}

If we take into account this latter result and the Riemannian scaling relations $\text{Ric} \left( g \left( \frac{\cdot}{\lambda} \right) \right) = \text{Ric} \left( \lambda g \left( \frac{\cdot}{\lambda} \right) \right)$, $\text{Rm}^2 \left( \lambda g \left( \frac{\cdot}{\lambda} \right) \right) = \lambda^{-1} \text{Rm}^2 \left( g \left( \frac{\cdot}{\lambda} \right) \right)$, it easily follows that the rescaled metric $(\lambda g_{(0)}, t) \mapsto \lambda g \left( \frac{\cdot}{\lambda} \right)$, $t \in [0, \lambda T]$ is a space and time rescaled solution of the RG-2 flow

\begin{equation}
\frac{\partial}{\partial \tilde{t}} \lambda g_{ij} \left( \frac{\cdot}{\lambda} \right) = -2\text{Ric}_{ij} \left( \lambda g \left( \frac{\cdot}{\lambda} \right) \right) - \frac{\alpha \lambda g}{2} \text{Rm}_{ij}^2 \left( \lambda g \left( \frac{\cdot}{\lambda} \right) \right) ,
\end{equation}

\begin{equation}
\lambda g_{ab} \left( \frac{\cdot}{\lambda} \right) = 0 , \quad t \in [0, \lambda T] .
\end{equation}

This, together with (35), implies that the solution of the coupled system (10) and (11) has the parabolic space and time scaling symmetry

\begin{equation}
(g(t), \xi_{g(t)}, d\omega(T_0 - t)) \mapsto (\lambda g \left( \frac{\cdot}{\lambda} \right), \lambda \xi_{\lambda g(t/\lambda)}, \lambda^\frac{n}{2} d\omega \left( \frac{(T_0 - t)/\lambda)}{\lambda} \right) ,
\end{equation}

$t \in [0, T_0], \forall \lambda \in \mathbb{R}_{>0}$, as stated. \hfill \Box
We briefly elaborate on the consequences of the scale-invariant flow to solitons. In [10], the authors investigated soliton structures for the RG-2 flow, and (without the scale-invariant \( \alpha \)) concluded that homothetically expanding solitons were quite restricted; for example, they are Einstein manifolds. Quite remarkably this behavior holds also for the scale invariant RG-2 flow.

We show this in the case that \( g_0 \) has constant curvature. Let \((M, g_0, d\omega(g_0) = e^{-f_0} d\mu(g_0))\) be the Riemannian manifold that we use as initial datum (in the sense specified by Theorem 2) for the coupled system (10) and (11) defining the scale-invariant RG-2 flow, and let us assume that \( g_0 \) is a constant curvature metric, i.e. \( R_{ijkl}(g_0) = K (g_0)_{ij} (g_0)_{kl} - (g_0)_{ik} (g_0)_{jl} \) for some constant \( K \). Say one has a solution \( t \mapsto (g(t), d\omega(t) = e^{-f(t)} d\mu(g(t))) \) of (10) and (11) where the metric part evolves by scaling, i.e., \( g(t) = \sigma(t) g_0 \), with \( \sigma(t) \in \mathbb{R}_{>0}, \sigma(t = 0) = 1 \). It is important to stress that if such scaling evolution of the RG-2 flow is possible, then according to Definition 1 (see (14)), the coupling \( \alpha_{g_0} \) still remains constant, i.e. \( \alpha_{g_0} = \alpha_{g(t)} \). As counterintuitive as it may appear in the face of the assumption \( g(t) = \sigma(t) g_0 \), we cannot write \( \alpha_{g(t)} = \sigma(t) \alpha_{g_0} \). Indeed, \( \alpha \) only satisfies the requisite scaling \( \alpha_{g_0} = \lambda \alpha_{g_0} \) for constant \( \lambda \). This is because \( \sigma(t) \), via the relation \( \sigma(t) = \left( \frac{\text{Vol}(M, g(t))}{\text{Vol}(M, g_0)} \right)^{2/n} \), can be identified with the time-dependent Riemannian volume of \((M, g(t))\) and, as stressed in our analysis, this is not a viable prescription for \( \alpha \) since it leads to a time-dependent nature of the coupling constant.

The fact that \( \alpha_{g(t)} \neq \sigma(t) \alpha_{g_0} \) follows explicitly by observing that

\[
(\alpha_{g(t)})^{\frac{2}{n}} = \int_M d\omega(t) = \int_M e^{-f(t)} d\mu(g(t)) = \sigma(t)^{\frac{2}{n}} \int_M e^{-f(t)} d\mu(g_0) \\
= \sigma(t)^{\frac{2}{n}} \int_M e^{-f_0} d\mu(g_0) = \sigma(t)^{\frac{2}{n}} (\alpha_{g_0})^{\frac{2}{n}},
\]

since, in general, under the evolution (11) \( f(t) \neq f_0 \). If we take into account these remarks, then

\[
-2\text{Ric}(g(t)) - \frac{\alpha_{g(t)}}{2}\text{Rm}^2(g(t)) = -2\text{Ric}(g_0) - \frac{\alpha_{g_0}}{2} \sigma(t)^{-1} \text{Rm}^2(g_0),
\]

where, besides \( \alpha_{g_0} = \alpha_{g(t)} \), we have used \( \text{Ric}(g(t)) = \text{Ric}(g_0) \) and \( \text{Rm}^2(g(t)) = \sigma(t)^{-1} \text{Rm}^2(g_0) \). Since \( g_0 \) is of constant curvature, we can write \( R_{ij}(g_0) = K (n-1) (g_0)_{ij} \) and \( \text{Rm}^2_{ij}(g_0) = 2K^2 (n-1) (g_0)_{ij} \). Hence, corresponding to the assumed scaling evolution for the metric, the RG-2 flow takes the form (factorizing out a common \( g_0 \))

\[
\frac{d}{dt} \sigma(t) = -2K (n-1) - \alpha_{g_0} \sigma(t)^{-1} K^2 (n-1),
\]

which is implicitly solved by the Lambert W function construction that was developed in [12], i.e., by scaling factors \( \sigma(t) \) which satisfy

\[
\sigma(t) = -2K (n-1)t + 1 + \frac{\alpha_{g_0} K}{2} \ln \left( \frac{2\sigma(t) + \alpha_{g_0} K}{2 + \alpha_{g_0} K} \right).
\]

These remarks explicitly show that the variegated nature of the soliton structures for the standard RG-2 flow is not caused by the fact that the flow is not scale invariant. The complex structure is indeed found also for the scale-invariant RG-2 flow. It is the geometric interplay with the \( \alpha_{g_0} \text{Rm}^2 \) term that claims responsibility for that.

3. ENTROPIES

A second set of results we prove concerns the existence of a monotonic functional which plays for (10) the same role Perelman’s \( F \)-energy [24] has in standard Ricci flow. This is a very delicate issue which has two distinct aspects. One concerns to what extent the Perelman’s functional \( F \) may be used to control also the RG-2 flow, an issue that in the physics literature has been addressed at various levels by A. Tseytlin [24] and by T. Oliynyk, V. Suneeta, and E. Woolgar [20], in connection
with A. Zamolodchikov’s c-theorem \[25\]. The other issue concerns the possibility of extending Perelman’s technique for constructing explicitly a monotonic functional with respect to which the RG-2 flow is monotonic. An entropy for a (normalized) RG-2 flow on surfaces with positive curvature was found by V. Branding \[1\], by generalizing R. Hamilton’s entropy for the Ricci flow on surfaces with positive curvature \[16\]; however, as in the Ricci flow case, this RG-2 flow surface entropy does not generalize to higher dimensional manifolds. It is interesting to note that in \[5\], B. Chow and P. Lu considered an approach to entropy for the RG-2 flow in general dimensions, with the hope that it would apply, in some recursive way, also to higher loops corrections (see \[5\], equation (17.32)). The functional that they consider is the natural analog of Perelman’s functional. They were able to derive a quantity based on this functional which, at a given fixed time, is instantaneously monotonic if one considers the sum of the instantaneous and synchronous variation of the Perelman functional along the Ricci flow direction and along a \(R^2\) flow direction.

In what follows, we derive an entropy that is a natural generalization of Perelman’s entropy by exploiting the gauge freedom related to the \(d\omega\)–drift vector field \(\xi_g\). Although our strategy emphasizes, as in Perelman’s analysis of the Ricci flow \[23\], the interplay between the diffeomorphism group and the RG-2 flow, it has aspects that are in the spirit of Chow and Lu’s suggestion. We replace their two–flows splitting with the full RG-2 flow coupled to a corresponding auxiliary flow governing the gauge drift vector field \(\xi_g\) associated to the measure \(d\omega\). It is the latter that allows to take into account the contribution of the \(R^2\) term to the entropy.

To begin, let us recall that in the Ricci flow case, Perelman’s energy functional is constructed by considering, along the Ricci flow metric \([0, T_0] \ni t \mapsto h(t)\), solution of

\[
\frac{\partial}{\partial t} h(t) = -2 \text{Ric}(h(t)) , \quad h(0) = h_0 ,
\]

a (probability) measure \(d\pi(t) := e^{-m(t)}d\mu_t\), \(m(t) \in C^\infty(M, \mathbb{R})\), evolving according to the backward heat equation

\[
\frac{\partial}{\partial t} d\pi(t) = -\Delta_{h(t)} d\pi(t) .
\]

To the resulting \(t \mapsto (M, h, d\pi)\) we associate the \(\mathcal{F}(h(t), m(t))\)–energy functional

\[
\mathcal{F}(h(t), m(t)) := \int_M \left[ R(h(t)) + |\nabla m(t)|^2_{h(t)} \right] d\pi(t) = \int_M R_{\text{Per}}(h(t)) d\pi(t) ,
\]

where

\[
R_{\text{Per}} := R + 2\Delta_h f - |\nabla f|_{h}^2 = R + 2\Delta_\omega f + |\nabla f|_{h}^2
\]

denotes the Perelman’s modified scalar curvature associated with the Riemannian manifold with density \((M, h, d\pi)\). \(\mathcal{F}(h(t), m(t))\) is the entropy production functional \(\frac{4}{\beta} N(h(t), d\pi(t))\) of the relative entropy (Nash entropy)

\[
N(h(t), d\pi(t)) := -\int_M \log \left( \frac{d\pi(t)}{d\mu_{h(t)}} \right) d\pi(t) ,
\]

associated with the coupled evolution \[43\] and \[44\]. As a consequence of the time–dependence of the metric \(h(t)\), the Nash entropy is not a monotonic quantity, whereas the entropy production functional \(\mathcal{F}(h(t), m(t))\) turns out to enjoy a subtle monotonicity property of great geometrical relevance. This was one of Perelman’s fundamental discoveries \[23\]. If, along the flow \(t \mapsto (h(t), d\pi(t))\), \(t \in [0, T_0]\), one considers the 1–parameter family of diffeomorphisms \(\varphi(t) : M \rightarrow M\) solution of the system of ODE \(\frac{d}{dt} \varphi(t) = -\nabla_{h(t)} m(t)\), \(\varphi(t = 0) = \text{id}_M\), then the pulled back
metric and measure, \( \overline{\gamma}(t) \) := \( \varphi(t)^* \gamma(t) \) and \( d\pi(t) := \varphi(t)^* d\pi(t) \) satisfy the system
\[
\frac{\partial}{\partial t} \overline{\gamma}(t) = -2 \left( \text{Ric}(\overline{\gamma}(t)) + \nabla_{\gamma(t)} \nabla_{\gamma(t)} m(t) \circ \varphi(t) \right), \quad \overline{\gamma}(0) = \gamma_0 ,
\]
(48)
\[
\frac{\partial}{\partial t} d\pi(t) = 0 ,
\]
which appears as the gradient flow of the functional \( (55) \). Note that by diffeomorphism invariance, one easily shows that \( F(h(t), m(t)) \) is monotonic along the original coupled flows \( (43) \) and \( (44) \),
\[
\frac{d}{dt} F(h(t), m(t)) = 2 \int_M \left[ \text{R}(h(t)) + |\nabla m(t)|_{h(t)}^2 \right] d\pi(t) ,
\]
(49)
The minimization of \( F(h(t), m(t)) \) over all possible (absolutely continuous) probability measures \( d\pi \) provides Perelman’s nondecreasing functional \( F[h] \) along the Ricci flow.

4. MONOTONICITY OF THE NASH ENTROPY

Not surprisingly, the situation described above is significantly more complex for the RG–2 flow \( [10] \). To begin with, if we choose the gauge vector field \( \xi_g(t) \equiv 0 \) for all \( \beta \in [0, T] \) (actually, it is sufficient to assume \( \text{div}^{(\omega)} \xi_g = 0 \)), then from the parabolicity requirement for the RG-2 flow, we have monotonicity for a modified Nash entropy.

**Theorem 4.** Let \( T_0 < T \) and, along the flow \( [0, T_0] \ni t \mapsto (g(t), d\omega(t); \xi_g(t) = 0) \) solution of the RG–2 flow \( [10] \), define the extended Nash entropy functional
\[
N(g(t), d\omega(t)) := -\int_M \log \left( \frac{d\omega(t)}{d\mu_{g(t)}} \right) d\omega(t) - n(n-1) \alpha_g^{-1} t
\]
(50)
\[
= -\int_M \left( f(t) + \frac{n(n-1)t}{\alpha_g} \right) e^{-f(t)} d\mu_{g(t)} .
\]
Then, as long as \( 1 + \alpha_g K_{\mathcal{P}}(t) > 0, \forall P \in Gr_{(2)}(TM) \), we have
\[
\frac{d}{dt} N(g(t), d\omega(t))
\]
(51)
\[
= \int_M \left[ \text{R}^{\text{Per}}(g(t)) + \frac{\alpha_g}{4} |\text{Rm}(g(t))|_{g(t)}^2 + \frac{n(n-1)}{\alpha_g} \right] e^{-f(t)} d\mu_{g(t)} \geq 0 .
\]

**Proof.** The gauge choice \( \xi_g(t) = 0 \) (or \( \text{div}^{(\omega)} \xi_g(t) = 0 \)) uncouples the evolution of the measure \( d\omega(\beta) \) from \( \xi_g \), and if we compute, along \( [10] \) and \( [11] \), the derivative \( \frac{d}{dt} N(g(t), d\omega(t)) \) of the relative entropy functional defined by \( (50) \), we get
\[
\frac{d}{dt} N(g(t), d\omega(t)) = -\frac{d}{dt} \int_M f e^{-f} d\mu_g + \frac{n(n-1)}{\alpha_g} \int_M e^{-f} d\mu_g
\]
(52)
\[
= -\int_M \frac{\partial f}{\partial t} e^{-f} d\mu_g - \int_M f \frac{\partial}{\partial t} \left( e^{-f} d\mu_g \right) + \frac{n(n-1)}{\alpha_g} \int_M e^{-f} d\mu_g ,
\]
where we dropped all \( t \)-dependence, since notation wants to travel light. From \( (13) \), one recovers the standard relation
\[
\frac{\partial f}{\partial t} = -\Delta_g f + |\nabla f|^2_{\gamma_g} + \frac{1}{2} g^{ab} \partial g_{ab} \frac{\partial g_{ab}}{\partial t} = -\Delta^{(\omega)} f + \frac{1}{2} g^{ab} \partial g_{ab} \frac{\partial g_{ab}}{\partial t} ,
\]
(53)
where $\Delta_g^{(\omega)}$ is the $d\omega(t)$–weighted Laplacian on $(M, g(t), d\omega(t))$. We also need the identity (integration by parts)

$$
\int_M f \Delta_g \left( e^{-f} \right) d\mu_g = \int_M \Delta_g f e^{-f} d\mu_g = \int_M \left( \Delta_g^{(\omega)} f + |\nabla f|^2_g \right) e^{-f} d\mu_g
$$

Introducing these expressions in (52) we get

$$
\frac{d}{dt} N(g(t), d\omega(t), t) = \int_M \left[ -\frac{1}{2} g^{ab} \frac{\partial g_{ab}}{\partial t} + |\nabla f|^2_g + \frac{n(n-1)}{\alpha_g} \right] e^{-f} d\mu_{g(t)},
$$

where everything depends on $t$, and which along the RG–2 flow (10) provides

$$
\frac{d}{dt} N(g(t), d\omega(t), t) = \int_M \left[ R + \alpha_g |\text{Rm}|^2_g + \frac{n(n-1)}{\alpha_g} \right] e^{-f} d\mu_{g(t)}.
$$

Let us now observe that at any given point $x \in M$ we can rewrite the scalar curvature $R(x, t)$ in terms of the sectional curvatures $K_P(x, t)$ of $(M, g(t))$ as the 2–planes $P$ vary in the Grassmannian $Gr(2)(T_x M)$. To this end, if we let $\{ e(a) \}_{a=1}^{n}$ denote an orthonormal basis for $T_x M$, and denote by $P(a, b)$ the 2–plane in $Gr(2)(T_x M)$ generated by $e(a) \wedge e(b)$, with $a \neq b$, then

$$
R(x, t) = \sum_{a, b=1, a \neq b}^{n} K_{P(a, b)}(x, t)
$$

Since $\sum_{a, b=1, a \neq b}^{n} 1 = n(n-1)$ we can write

$$
R(x, t) + \frac{n(n-1)}{\alpha_g} = \sum_{P(a, b)} \frac{1 + \alpha_g K_{P(a, b)}(x, t)}{\alpha_g}.
$$

Hence, as long as $1 + \alpha_g K_{P(a, b)}(x, t) > 0$, we have

$$
R(x, t) + \frac{n(n-1)}{\alpha_g} > 0,
$$

and the theorem follows.

5. AN EXTENDED PERELMAN’S ENERGY FUNCTIONAL

The monotonicity of the Nash entropy functional is a rather weak result since it requires that the curvature condition $1 + \alpha_g K_P(t) > 0, \forall P \in Gr(2)(TM)$, imposed on the initial metric $g$, holds along the evolution of the RG–2 flow, a property that is very difficult to establish. If we direct our attention to the behavior of the Perelman’s energy functional $^{23}$

$$
\mathcal{F}(g(t), f(t)) := \int_M \left[ R(g(t)) + |\nabla f(t)|^2_{g(t)} \right] d\omega(t) = \int_M \text{R}^{Per}(g(t)) d\omega(t),
$$

the situation, hard to handle in the standard RG–2 flow (11), improves considerably along the scale–invariant RG–2 flow $[0, T_0] \ni t \mapsto (g(t), d\omega(t); \xi_{g(t)})$, defined by (10). We can exploit the freedom in choosing the drift vector field $\xi_g$ for controlling the vagaries of the $\text{Rm}^2$ term and obtain monotonicity for a natural variant of $\mathcal{F}(g(t), f(t))$. We start by recalling the expression of the pointwise evolution of Perelman’s modified scalar curvature $\text{R}^{Per}(g(t))$ which appears in (60). This is indeed instrumental for the characterization of Perelman’s $\mathcal{F}$–energy for the Ricci flow, and plays a basic role in the RG–2 flow case.
In full generality, let us consider the generic (germ of) curve of metrics $[0, 1] \ni t \mapsto g(t)$, with tangent vector provided by a smooth ($t$–dependent) symmetric bilinear form $v \in C^\infty(M, \otimes^2_{\text{sym}} T^* M)$,
\begin{equation}
\frac{\partial}{\partial t} \aja_{jk}(t) = v_{jk} .
\end{equation}
Along (61), we have (for a detailed derivation see [4], Chapter 6, Exercise 6.84, p. 274)
\begin{equation}
\frac{\partial}{\partial t} R_{\text{Per}}(g(t)) = \nabla_j \nabla_k v_{jk} + v_{jk} R_{jk} - 2 \nabla_j f \nabla_k v_{jk}
\end{equation}
\begin{equation}
+ v_{jk} \nabla_j f \nabla_k f + 2 (\Delta_g - \nabla_k f \nabla_k) \left( \frac{\partial f}{\partial t} - \frac{1}{2} \text{tr}_g(v) \right)
\end{equation}
\begin{equation}
- 2 v_{jk} (R_{jk} + \nabla_j \nabla_k) f .
\end{equation}
After performing the sign-sensitive derivatives of the $g^{-1}$s, we adopt here a lower index notation, where the convention of summation over repeated indices is understood. It is useful to write the rather complicated expression (62) in terms of the weighted covariant derivative $\nabla(\omega)$ associated with the measure $d\omega$ (see [4]). We extend it to a generic tensor field $T$ over $M$ according to
\begin{equation}
\nabla(\omega) T := e^f \nabla \left( e^{-f} T \right) = \nabla T - \nabla f \otimes T ,
\end{equation}
where $\nabla$ is the Levi-Civita connection on $(M, g, d\omega)$, (or when time-dependent, $(M, g(t), d\omega(t)))$. $\nabla(\omega)$ is a natural differential operator on the Riemannian manifold with density $(M, g, d\omega = e^{-f} d\mu_g)$. To rewrite (62) in terms of $\nabla(\omega)$, let us apply the easily proven relations
\begin{equation}
\nabla(\omega) \nabla(\omega) v_{jk} := e^f \nabla \left[ e^{-f} e^f \nabla_k \left( e^{-f} v_{jk} \right) \right] = e^f \nabla_j \nabla_k \left( e^{-f} v_{jk} \right)
\end{equation}
\begin{equation}
= \nabla_j \nabla_k v_{jk} - 2 \nabla_j f \nabla_k v_{jk} + v_{jk} \nabla_j f \nabla_k f - v_{jk} \nabla_j \nabla_k f .
\end{equation}
Then
\begin{equation}
\nabla_j \nabla_k v_{jk} - 2 \nabla_j f \nabla_k v_{jk} + v_{jk} \nabla_j f \nabla_k f = \nabla(\omega) \nabla(\omega) v_{jk} + v_{jk} \nabla_j \nabla_k f .
\end{equation}
By introducing this latter expression in (62), and recalling that $\Delta_g - \nabla_k f \nabla_k := \Delta_g(\omega)$, we eventually get
\begin{equation}
\frac{\partial}{\partial t} R_{\text{Per}}(g(t)) = \nabla(\omega) \nabla(\omega) v_{jk} - R_{\text{BE}}^{jk} v_{jk} + 2 \Delta_g(\omega) \left( \frac{\partial f}{\partial t} - \frac{1}{2} \text{tr}_g(v) \right) ,
\end{equation}
where
\begin{equation}
\text{Ric}^{\text{BE}}(g) := \text{Ric}(g) + \nabla \nabla f ,
\end{equation}
denotes the Bakry–Emery Ricci tensor associated with $(M, g(t), d\omega(t))$.

Having dispensed with these preliminary remarks, let us consider the scale–invariant RG-2 flow for which, according to Theorem 2 we have short time existence on some interval $t \in [0, T)$. For $T_0 < T$, let us choose along $[0, T_0] \ni t \mapsto g(t)$ a corresponding gauge drift vector field $t \mapsto \xi_g(t)$ by requiring that its divergence $\text{div}_g(t) \xi_g(t) := \nabla_k \xi_g^k(t)$ evolves, starting from a given initial condition $\xi_{g(0)}$ for $\xi_g(t)$, that satisfies
\begin{equation}
\frac{\partial}{\partial t} \text{div}_g(t) \xi_g(t) = \Delta_g(t) \left( \text{div}_g(t) \xi_g(t) \right) - \text{L}_{\xi_g(t)} \left( \text{div}_g(t) \xi_g(t) \right) - \frac{\alpha_g^2}{32} |\text{Rm}^2(t) (g(t), \xi_g(t))|_{g(t)}^2 ,
\end{equation}
\footnote{As usual, in what follows we adopt the convention that $\nabla$, when acting on a time-dependent vector or tensor field, denotes the covariant derivative with respect to $(M, g(t))$.}
where $\mathbf{L}_{\xi_g(t)} \left( \text{div}_{g(t)} \xi_g(t) \right)$ denotes the Lie derivative along $\xi_g(t)$ of the scalar function $\text{div}_{g(t)} \xi_g(t)$, and

$$\alpha^2 g \left| \text{Rm}^2(g(t), \xi_g(t)) \right|^2$$

is the squared norm of the drift-modified squared curvature:

$$\text{(69)} \quad \alpha_g \text{Rm}^2(g(t), \xi_g(t)) := \alpha_g \text{Rm}^2(g(t)) - 2 \mathbf{L}_{\xi_g} g(t).$$

For a given initial condition, (68) is a (forward) parabolic PDE which admits a unique solution for $t \in [0, T_0]$. To the resulting evolution $[0, T_0] \ni t \mapsto (g(t), \xi_g(t))$ defined by (10) and (68), we can associate the time–reversed flow $0, T_0 \ni \eta \mapsto (g(\eta), \xi_g(\eta))$, $\eta := T_0 - t$ and, according to Theorem 2, consider the corresponding parabolic equation (11)

$$\text{(70)} \quad \frac{\partial}{\partial \eta} d \omega(\eta) = \Delta_{g(\eta)} d \omega(\eta) + \text{div} \omega_t \xi_g(\eta) d \omega(\eta), \quad d \omega(\eta = 0) = d \omega(0),$$

whose solution defines the evolution $\eta \mapsto d \omega(\eta)$. This forward/backward parabolic see–saw game characterizes the flow $t \mapsto (d \omega(t = T_0 - \eta) = e^{-f(t)} d \mu_g(\xi_g(t))$ and, according to Theorem 2, consider the corresponding parabolic equation (11)

$$\text{(71)} \quad W(t) := - \left( \nabla \frac{\partial}{\partial t} \phi_t \right), \quad t \in [0, T_0].$$

The next step is to consider the action on (10), (11), and (68) of the family of diffeomorphisms (71)

$$\text{(72)} \quad \frac{\partial}{\partial t} \phi_t(p) = W(\phi_t(p), t), \quad \phi_{t=0} = \text{id}_M.$$  

We prove the following result:

**Theorem 5.** If we denote by $\overline{\gamma}(t) := \phi_t^* (g(t))$, $d \overline{\omega}(t) := \phi_t^* (d \omega(t))$, and $\overline{\xi}_g(t) := \phi_t^* (\xi_g(t))$, and $\overline{\nabla} := \nabla_{\overline{\gamma}}$ the relevant pullbacks under the action of the one–parameter family of diffeomorphisms $\phi_t$ solution of (72), then the corresponding (DeTurck) modified scale–invariant RG-2 flow associated to the action of $\phi_t$ on (10), (11) and (68) is provided by

$$\frac{\partial}{\partial t} \overline{\gamma}(t) = - 2 \text{Ric}^{BE}_{\overline{\gamma}(t)} - \frac{\alpha^2}2 \text{Rm}^2(\overline{\gamma}(t), \overline{\xi}_g(t)),$$

$$\frac{\partial}{\partial t} d \overline{\omega}(t) = 0,$$

$$\frac{\partial}{\partial t} \text{div}_{\overline{\gamma}(t)} \overline{\xi}_g(t) = \Delta_{\overline{\gamma}(t)} \left( \text{div}_{\overline{\gamma}(t)} \overline{\xi}_g(t) \right) - \frac{\alpha^2}{32} \left| \text{Rm}^2(\overline{\gamma}(t), \overline{\xi}_g(t)) \right|^2 \overline{\gamma}(t)^2,$$

and along $t \mapsto \left( M, \overline{\gamma}(t), d \overline{\omega}(t), \overline{\xi}_g(t) \right)$, we have

$$\frac{d}{dt} \int_M \left( \text{Ric}_{\overline{\gamma}(t)} - \text{div}_{\overline{\gamma}(t)} \overline{\xi}_g(t) \right) d \overline{\omega}(t)$$

$$\text{(74)} \quad = \frac{1}{2} \int_M \left| \text{Ric}^{BE}_{\overline{\gamma}(t)} + \frac{\alpha^2}8 \text{Rm}^2(\overline{\gamma}(t), \overline{\xi}_g(t)) \right|^2 \overline{\gamma}(t)^2 d \overline{\omega}(t).$$

**Proof.** Since $M$ is compact, (72) defines a one–parameter family of diffeomorphisms as long as the solutions of (10), (11), and (68) exist; in particular, we may assume that $\left\{ \phi_t \in \text{Diff}(M) \mid t \in [0, T_0] \right\}$, and we consider the relevant pullbacks $\overline{\gamma}(t) := \phi_t^* (g(t))$, $d \overline{\omega}(t) := \phi_t^* (d \omega(t))$, $\overline{\nabla} := \nabla_{\overline{\gamma}}$, and $\overline{\xi}_g(t) := \phi_t^* (\xi_g(t))$. In the latter, we used the notation $\phi_t^* (\xi_g(t)) := (\phi_t)^{-1}_* (\xi_g(t))$. Starting with $\phi_t^* (d \omega(t))$, we have
\[
\frac{\partial}{\partial t} \varphi_t^*(d\omega(t)) = \frac{\partial}{\partial s}|_{s=0}(\varphi_{t+s}^*d\omega(t+s)) = \varphi_t^*\left(\frac{\partial}{\partial t}d\omega(t)\right) + \frac{\partial}{\partial s}|_{s=0}(\varphi_{t+s}^*d\omega(t)) = \varphi_t^*\left(-\triangle_g d\omega(t) - \text{div}^\omega \xi_g(t) d\omega(t)\right) + L_{(\varphi_t^{-1})^*W(t)}(\varphi_t^*d\omega(t)).
\]

As usual, we calculate

\[
L_{(\varphi_t^{-1})^*W(t)}(\varphi_t^*d\omega(t)) = \varphi_t^*(L_{W(t)}d\omega(t)) = \varphi_t^*\left(L_{\xi_t} d\omega(t) - L_{\nabla f} d\omega(t)\right) = \varphi_t^*\left(\text{div}^\omega \xi_g(t) d\omega(t) - \text{div}^\omega \nabla f(t) d\omega(t)\right) = \varphi_t^*\left(\text{div}^\omega \xi_g(t) d\omega(t) - \Delta^\omega f(t) d\omega(t)\right) = \varphi_t^*\left(\text{div}^\omega \xi_g(t) d\omega(t) + \triangle_g d\omega(t)\right),
\]

where we have used the characterization \[89\] of the weighted divergence \(\text{div}^\omega\) in terms of the Lie derivative of the measure \(d\omega(t)\), and the relation \(\triangle_g d\omega = -\Delta^\omega f d\omega\) (see \[13\]). Introducing this result into \[75\], we get

\[
\frac{\partial}{\partial t} d\omega(t) := \frac{\partial}{\partial t} \varphi_t^*(d\omega(t)) = 0.
\]

Similarly, from \[88\] we compute

\[
\frac{\partial}{\partial t}(\varphi_t^*(\text{div}_g(t)\xi_g(t))) = \varphi_t^*\left(\frac{\partial}{\partial t}(\text{div}_g(t)\xi_g(t)) + \frac{\partial}{\partial s}|_{s=0}(\varphi_{t+s}^*(\text{div}_g(t)\xi_g(t)))\right) = \varphi_t^*\left(\triangle_g(t) \left(\text{div}_g(t)\xi_g(t)\right) - L_{\xi_t}(\text{div}_g(t)\xi_g(t)) - \frac{\alpha_g^2}{32} |\text{Rm}^2(g(t), \xi_g(t))|^2_{g(t)} + L_{W(t)}(\text{div}_g(t)\xi_g(t))\right).
\]

Since

\[
L_{W(t)}(\text{div}_g(t)\xi_g(t)) = L_{\xi_t}(\text{div}_g(t)\xi_g(t)) - L_{\nabla f}(\text{div}_g(t)\xi_g(t)) = L_{\xi_t}(\text{div}_g(t)\xi_g(t)) - \nabla^g f(t) \nabla_k (\text{div}_g(t)\xi_g(t)) ,
\]

and by definition of the weighted Laplacian

\[
\Delta_g(t) \left(\text{div}_g(t)\xi_g(t)\right) - \nabla^g f(t) \nabla_k (\text{div}_g(t)\xi_g(t)) = \Delta^\omega \xi_g(t) \left(\text{div}_g(t)\xi_g(t)\right) ,
\]

we eventually get for the evolution of \(\text{div}_{\overline{g}(t)}\overline{\xi}_g(t)\) the expression

\[
\frac{\partial}{\partial t}(\text{div}_{\overline{g}(t)}\overline{\xi}_g(t)) := \frac{\partial}{\partial t}(\varphi_t^*(\text{div}_g(t)\xi_g(t))) = \Delta^\omega \xi_g(t) \left(\text{div}_{\overline{g}(t)}\overline{\xi}_g(t)\right) - \frac{\alpha_g^2}{32} |\text{Rm}^2(\overline{g}(t), \overline{\xi}(t))|^2_{\overline{g}(t)}.
\]
Finally, for the pulled–back metric we have the standard DeTurck computation
\[
\frac{\partial}{\partial t} \tilde{g} := \frac{\partial}{\partial t} \varphi_t^* (g(t)) = \frac{\partial}{\partial s} |_{s=0} (\varphi_{t+s}^* g(t + s))
\]
\[
= \varphi_t^* \left( \frac{\partial}{\partial t} g(t) \right) + \frac{\partial}{\partial s} |_{s=0} (\varphi_{t+s}^* g(t))
\]
\[
= \varphi_t^* \left( -2\text{Ric}(t) - \frac{\alpha}{2} \text{Rm}^2 \right) + L_{(\varphi_t^{-1})} W(t)(\varphi_t^* g(t))
\]
\[
= -2\text{Ric} (g(t)) - \frac{\alpha}{2} \text{Rm}^2 (g(t)) - 2\nabla \nabla f + L_{\xi_g} g(t) .
\]

Putting these results together, we find that the flow \([0,T_0] \ni t \mapsto (\bar{g}(t), d\bar{\omega}(t), \bar{\xi}_g(t))\) is a solution of the following system:

\begin{align}
\frac{\partial}{\partial t} \bar{g} &= -2\text{Ric}^{BE} (\bar{g}(t)) - \frac{\alpha}{2} \text{Rm}^2 (\bar{g}(t), \bar{\xi}_g(t)) \\
\frac{\partial}{\partial t} d\bar{\omega}(t) &= 0 \\
\frac{\partial}{\partial t} (\text{div}_{\bar{g}(t)} \bar{\xi}_g(t)) &= \Delta (\omega) \left( \text{div}_{\bar{g}(t)} \bar{\xi}_g(t) \right) - \frac{\alpha^2}{32} \left| \text{Rm}^2 (\bar{g}(t), \bar{\xi}_g(t)) \right|^2_{\bar{g}(t)},
\end{align}

as stated. Here \(\text{Ric}^{BE}\) is the Bakry–Emery Ricci tensor associated with the Riemannian manifold with density \((M, \bar{g}(t), d\bar{\omega}(t))\) and \(\alpha \text{Rm}^2 (\bar{g}(t), \bar{\xi}_g(t))\) is the corresponding short-hand notation for the drift-modified squared curvature.

Along the flow \([0,T_0] \ni t \mapsto (\bar{g}(t), d\bar{\omega}(t), \bar{\xi}_g(t))\), (81), (82), and (83), the pointwise evolution of the Perelman modified scalar curvature reduces to

\[
\frac{\partial}{\partial t} \text{R}^{\text{Per}} (\bar{g}(t)) = \nabla_j (\omega) \nabla_k (\omega) \text{R} G_{jk}(t) - \text{R}^{BE} (\bar{g}(t)) \text{R} G_{jk}(t),
\]

where

\[
\text{R} G_{jk}(t) := -2\text{Ric}^{BE} (\bar{g}(t)) - \frac{\alpha}{2} \text{Rm}^2 (\bar{g}(t), \bar{\xi}_g(t))
\]

is the generator of the RG-2 flow \((\bar{g}(t), d\bar{\omega}(t), \bar{\xi}_g(t))\), and where the measure–variation term \(2 \Delta (\omega) \left( \frac{\partial f}{\partial t} - \frac{1}{2} \text{tr}_g (v) \right)\) present in the expression (66) vanishes because the pulled–back measure \(d\bar{\omega}(t)\) is preserved along the evolution (81), (82), (83). This preservation of the measure \(d\bar{\omega}(t)\) also implies that we can integrate over \((M, \bar{g}(t), d\bar{\omega}(t))\) to obtain

\[
\frac{d}{dt} \int_M \text{R}^{\text{Per}} (\bar{g}(t)) d\bar{\omega}(t) = -\int_M \text{R}^{BE} (\bar{g}(t)) \text{R} G_{jk}(t) d\bar{\omega}(t),
\]

where we have integrated away the divergence term \(\nabla_j (\omega) \nabla_k (\omega) \text{R} G_{jk}(t)\). We have

\[
-\int_M \text{R}^{BE} (\bar{g}(t)) \text{R} G_{jk}(t) d\bar{\omega}(t)
\]
\[
= 2 \int_M \left[ \left| \text{Ric}^{BE} (\bar{g}(t)) \right|^2_{\bar{g}(t)} + \frac{\alpha^2}{4} \text{R}^{BE} (\bar{g}(t)) \text{Rm}^2 (\bar{g}(t), \bar{\xi}_g(t)) \right] d\bar{\omega}(t),
\]

which, by completing the square, can be written as

\[
2 \int_M \left| \text{Ric}^{BE} (\bar{g}(t)) + \frac{\alpha^2}{8} \text{Rm}^2 (\bar{g}(t), \bar{\xi}_g(t)) \right|^2_{\bar{g}(t)} d\bar{\omega}(t) - \frac{\alpha^2}{32} \int_M \left| \text{Rm}^2 (\bar{g}(t), \bar{\xi}_g(t)) \right|^2_{\bar{g}(t)} d\bar{\omega}(t).
On the other hand, from the evolution equation \cite{83}, we get (again by integrating away the weighted divergence)

\begin{equation}
\frac{d}{dt} \int_M \text{div}_{\overline{g}(t)} \xi_{g(t)} \ d\omega(t) = - \frac{\alpha_g}{32} \int_M \left| \text{Rm}^2(\overline{g}(t), \xi_{g(t)}) \right|^2_{\overline{g}(t)} \ d\omega(t),
\end{equation}

so that we can eventually write

\begin{equation}
\frac{d}{dt} \int_M R^\text{Per}(\overline{g}(t)) \ d\omega(t) = \left[ \int_M \left| \text{Ric}^\text{BE}(\overline{g}(t)) \right|^2_{\overline{g}(t)} \ d\omega(t) \right] + \frac{d}{dt} \int_M \text{div}_{\overline{g}(t)} \xi_{g(t)} \ d\omega(t),
\end{equation}

from which the theorem immediately follows. \hfill \Box

Remark 6. The structure of the above proof strongly suggests that a similar monotonicity result should work also for the geometric flow associated with the higher loop approximations to the perturbative renormalization group flow for non-linear sigma model. This is a largely uncharted territory, and already proving a local existence result for the geometric flows associated to the 3-loop and 4-loop curvature contributions, (where explicit curvature expressions are available–see e.g. \cite{24}) is an extremely demanding task. Were this possible, one could presumably use the parabolic Fokker–Planck evolution \cite{83} (with the quadratic source term \(\frac{\alpha_g}{32} \left| \text{Rm}^2(\overline{g}(t), \xi_{g(t)}) \right|^2_{\overline{g}(t)}\) replaced by the corresponding \(k\)–th order curvature terms present at the given loop approximation) in order to control the monotonicity of the associated energy functional.

6. IS THE RG–2 FLOW A GRADIENT FLOW?

The expression \cite{74} directly shows that although the modified Perelman entropy

\begin{equation}
\mathcal{F}(1)(\overline{g}, d\omega, \xi_{\overline{g}}) := \int_M \left( R^\text{Per}(\overline{g}(t)) - \text{div}_{\overline{g}(t)} \xi_{g(t)} \right) \ d\omega(t),
\end{equation}

is monotonic, the (modified) RG–2 flow \cite{73} is not a gradient flow with respect this functional. Actually, the functional with respect to which the (DeTurck modified) RG–2 flow is gradient is a rather non–trivial modification of \(\mathcal{F}(1)\):

\begin{equation}
\mathcal{F}(2)(\overline{g}, d\omega, \xi_{\overline{g}}) := \int_M \left[ R^\text{Per}(\overline{g}(t)) + \frac{\alpha_g}{8} \left| \text{Rm}(\overline{g}(t)) \right|^2_{g(\beta)} - \text{div}_{\overline{g}(t)} \xi_{g(t)} \right] \ d\omega(t).
\end{equation}

In order to simplify the computation of \(\frac{d}{dt} \mathcal{F}(2)(\overline{g}, \overline{f}, \xi)\) and avoid the annoying overlines, \(\overline{..}\), induced by pulling back the RG–2 flow metric back and forth, we abuse notation and drop the overlines, with the proviso that everything refers to the DeTurck modified RG–2 flow

\begin{equation}
\frac{\partial}{\partial t} \overline{g}(t) = - 2 \text{Ric}^\text{BE}(\overline{g}(t)) - \frac{\alpha_g}{2} \text{Rm}^2(\overline{g}(t), \xi_{\overline{g}(t)}).
\end{equation}

(See \cite{73}. Obviously the pull–back in \(\overline{\mathcal{F}(2)}\) is not generated by the same \(f\) and \(\xi_g\) featuring in \cite{73}).

To begin, we remark that along a generic (germ of) curve of metrics \([0, 1] \ni t \mapsto g(t)\) with tangent vector \(v \in C^\infty(M, \otimes^2_{\text{sym}} T^* M)\)

\begin{equation}
\frac{\partial}{\partial t} g_{jk}(t) = v_{jk},
\end{equation}

we have

\begin{equation}
\frac{\partial}{\partial t} \left| \text{Rm}(t) \right|^2 = - 4 R_{ijkl} \nabla^i \nabla^l v^{jk} - 2 R_{jk}^2 v^{jk}.
\end{equation}
We have, from (95) and (66), the pointwise evolution
\[
\frac{\partial}{\partial t} \left[ R_{\text{Per}}(g(t)) + \frac{\alpha_g}{8} |Rm(t)|^2 \right] = \nabla_j^{(\omega)} \nabla_k^{(\omega)} v_{jk} - \frac{\alpha_g}{2} R_{ijkl} \nabla_i \omega_{vjk}
\]  
(96)

\[- \left( R_{jk}^{BE} + \frac{\alpha_g}{4} Rm_{jk}^2 \right) v_{jk} + 2 \Delta_g^{(\omega)} \left( \frac{\partial f}{\partial t} - \frac{1}{2} \text{tr}_g(v) \right).\]

If in line with the preservation of the pull-back measure \(d\varpi(\beta)\) we assume the measure preserving condition \(\frac{d}{dt} d\omega(t) = 0\), and take into account the integration by parts formula
\[
\int_M R_{ijkl} \nabla_i \omega_{vjk}(\beta) \, d\omega = \int_M \nabla_i^{(\omega)} R_{ijkl} v_{jk}(\beta) \, d\omega,
\]
then we easily get from (96)
\[
\frac{d}{dt} F_2(g, f, \xi) = -\int_M \left( R_{jk}^{BE}(g(t)) + \frac{\alpha_g}{4} Rm_{jk}(g(t), \xi(t)) \right) v_{jk} \, d\omega(t) 
\]  
(97)
\[- \frac{1}{2} \int_M g^{ab}(t) \left[ \frac{\partial}{\partial t} L_{g(t)} g_{ab} + \alpha_g \nabla_a^{(\omega)} \nabla_i^{(\omega)} R_{ijkl}(g(t)) v_{jk} \right] \, d\omega(t).\]

Since \(\int_M g^{ab} \Delta_g^{(\omega)} L_{\xi(t)} g_{ab}(t) \, d\omega(t) = 0\), we can conveniently rewrite this expression as
\[
\frac{d}{dt} F_2(g, f, \xi) = -\int_M \left( R_{jk}^{BE}(g(t)) + \frac{\alpha_g}{4} Rm_{jk}(g(t), \xi(t)) \right) v_{jk} \, d\omega(t) 
\]  
(98)
\[- \frac{1}{2} \int_M g^{ab}(t) \left[ \frac{\partial}{\partial t} L_{\xi(t)} g_{ab} - \Delta_g^{(\omega)} L_{\xi(t)} g_{ab}(t) + \alpha_g \nabla_a^{(\omega)} \nabla_i^{(\omega)} R_{ijkl}(g(t)) v_{jk} \right] \, d\omega(t),\]

which directly implies the following result, where we have set \(RG := -2\text{Ric}^{BE}(g) - \frac{\alpha_g}{2} Rm^2(g, \xi).

**Theorem 7 (Entropy).** The coupled DeTurck RG-2 flow \([0, T_0] \ni t \mapsto (g(t), d\omega(t), \xi(t))\) solution of
\[
\frac{\partial}{\partial t} g_{ij}(t) = -2\text{Ric}_{ij}^{BE}(g(t)) - \frac{\alpha_g}{2} Rm_{ij}(g(t), \xi(t)),
\]
\[
\frac{\partial}{\partial t} d\omega(t) = 0,
\]
(100)
\[
\frac{\partial}{\partial t} L_{\xi} g_{ab} = \Delta_g^{(\omega)} L_{\xi(t)} g_{ab}(t) - \alpha_g \nabla_a^{(\omega)} \nabla_i^{(\omega)} R_{ijkl}(g(t)) \, RG_{jk},
\]
is the gradient flow of the functional \(F_2(g, f, \xi)\).

Since the term \(\nabla_a^{(\omega)} \nabla_i^{(\omega)} R_{ijkl}\) in (100) gives rise to such a strong non–linear coupling among the \(g(t)\) and \(\xi(t)\) evolution, it would seem difficult to explicitly characterize a diffeomorphism that pulls back the solution of (100) to a standard RG-2 flow.

To better understand the geometric nature of this latter remark, let us introduce Hamilton’s Harnack quadric (see [5] p. 32)

\[
H_{\nabla f} = e^f \cdot (\text{div} \circ \text{div} + \text{Ric} + \nabla \nabla f)_{1,4} \left( e^{-f} Rm \right)
\]  
(101)
\[= e^f \left( \nabla^i \nabla^i + R^{ij} + \nabla^i \nabla^i f \right) \left( e^{-f} R_{ijkl} \right),\]
where the subscript \((\ldots)_{1,4}\) denote the components of the Riemann tensor on which the operator between brackets is acting. Since

\[
e^f \nabla^l \nabla^i \left( e^{-f} R_{ijkl} \right) = e^f \nabla^l \left[ e^{-f} e^f \nabla^i \left( e^{-f} R_{ijkl} \right) \right] = \nabla^l (\omega) \nabla^i (\omega) R_{ijkl} ,
\]

we can equivalently write (101) as

\[
\nabla^i (\omega) \nabla^i (\omega) R_{ijkl} = (H \nabla f)_{jk} - R^{BE}_{li} R_{ijkl} .
\]

Note that a long but straightforward computation provides

\[
\nabla^i (\omega) \nabla^i (\omega) R_{ijkl} = \Delta^i (\omega) R^{BE}_{kj} - R^{BE}_{kl} R^{BE}_{lj} + R_{ijkl} R^{BE}_{il} - \frac{1}{2} L_X g_{kj} ,
\]

where \(X_h := \frac{1}{2} \nabla h R^{Per} \). In particular we can rewrite the Harnack quadric (103) as

\[
(H \nabla f)_{jk} = \Delta^i (\omega) R^{BE}_{kj} - R^{BE}_{kl} R^{BE}_{lj} + 2 R_{ijkl} R^{BE}_{il} - \frac{1}{2} \nabla j \nabla k R^{Per} ,
\]

which, for \(f = 0\), reduces to the standard expression

\[
M_{jk} := \Delta R_{kj} - R_{kl} R_{lj} + 2 R_{ijkl} R_{il} - \frac{1}{2} \nabla j \nabla k R ,
\]

featuring (up to the term \((2t)^{-1} R_{kj}\) in the analysis of Hamilton’s Harnack inequality. Note also that if \((M, g, d\omega)\) is a Ricci soliton, \textit{i.e.} if

\[
R^{BE}_{kl} := R_{kl} + \nabla k \nabla l f = \frac{\varepsilon}{2} g_{kl} , \quad \varepsilon \in \mathbb{R} ,
\]

we have

\[
\nabla^i (\omega) \nabla^i (\omega) R_{ijkl} = 0 .
\]

Indeed, for a Ricci soliton one easily computes

\[
\Delta^i (\omega) R^{BE}_{kj} - R^{BE}_{kl} R^{BE}_{lj} + R_{ijkl} R^{BE}_{il} = - \frac{\varepsilon^2}{4} g_{kj} + \frac{\varepsilon}{2} R_{kj} = - \frac{\varepsilon^2}{4} g_{kj} + \frac{\varepsilon}{2} \left( \frac{\varepsilon}{2} g_{kj} - \nabla k \nabla j f \right)
\]

\[
= - \frac{\varepsilon}{2} \nabla k \nabla j f .
\]

On the other hand, for \(R^{BE}_{kl} = \frac{\varepsilon}{2} g_{kl}\) one has

\[
X_j := \nabla^i (\omega) R^{BE}_{ij} = - \frac{\varepsilon}{2} \nabla j f ,
\]

so that

\[
\frac{1}{2} L_X g_{kj} = - \frac{\varepsilon}{2} \nabla k \nabla j f ,
\]

and

\[
\nabla^i (\omega) \nabla^i (\omega) R_{ijkl} = \left[ \Delta^i (\omega) R^{BE}_{kj} - R^{BE}_{kl} R^{BE}_{lj} + R_{ijkl} R^{BE}_{il} - \frac{1}{2} L_X g_{kj} \right]_{Ric^{BE}} = \frac{\varepsilon}{2} g = 0 .
\]

From these remarks it directly follows that it is the \textit{extended} Harnack term \(\nabla^i (\omega) \nabla^i (\omega) R_{ijkl}\) that makes the gradient flow nature of the RG–2 flow so complex. This is quite manifest if we set \(\xi_g = 0\) in (99) to get

\[
\frac{d}{dt} F_{(2)}(g, f) = - \int_M \left[ R_{ijkl}^g (g(t)) + \frac{\alpha g}{4} Rm^g_{jk} (g(t)) - \frac{\alpha g}{2} \nabla^i (\omega) \nabla^i (\omega) R_{ijkl} (g(t)) \right] v_{jk} d\omega(t) ,
\]
which is monotonic along Ricci solitons, and also shows in a rather direct way that it is the (fourth-order) flow
\[
\frac{\partial}{\partial t} g_{jk}(t) = -2R_{jk}^{BE}(g(t)) - \frac{\alpha_g}{2} Rm_{jk}(g(t)) + \alpha_g \nabla_i^{(\omega)} \nabla_i^{(\omega)} R_{ijkl}(g(t))
\]
that is formally the gradient flow of the functional \(F_{(2)}(g, f, \xi)\) for \(\xi = 0\). It is only by taming the Harnack term \(\nabla_i^{(\omega)} \nabla_i^{(\omega)} R_{ijkl}(g(t))\) by introducing the evolution of \(\xi\) provided by the inhomogeneous heat equation
\[
\frac{\partial}{\partial t} L_{\xi g_{ab}} = \Delta_g^{(\omega)} L_{\xi g_{ab}}(t) - \alpha_g \nabla_a^{(\omega)} \nabla^i_{\xi g}(g(t)) R_{ijkl}(g(t)) RG_{jk},
\]
(see (100)) that one can make manifest the gradient-like nature of the RG-2 flow with respect to \(F_{(2)}(g, f, \xi)\).

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