On fractional Duhamel’s principle and its applications

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Abstract
The classical Duhamel principle, established nearly 200 years ago by Jean-Marie-Constant Duhamel, reduces the Cauchy problem for an inhomogeneous partial differential equation to the Cauchy problem for the corresponding homogeneous equation. Duhamel’s principle is not applicable in the case of fractional order differential equations. In this paper we formulate and prove fractional generalizations of this famous principle directly applicable to a wide class of fractional order differential-operator equations.

1 Introduction
Let $X$ be a reflexive Banach space and $A : D \to X$ a closed linear operator with a domain $D \subset X$. In Section 2 we will introduce a Fréchet type topological vector space $\text{Exp}_{A,G}(X)$ (and its dual $\text{Exp}_{A^*,G^*}(X^*)$), where $G$ is an open subset of the complex plain $\mathbb{C}$. This space represents a modification of the space of entire functions with finite exponential type [6, 8, 21, 26] and its abstract versions. We also introduce a functional calculus in the form $f(A)$, where $f$ is an analytic function defined on $G$. The function $f$ is called the symbol of the operator $f(A)$.

The goal of this paper is to generalize Duhamel’s principle for the Cauchy problem for general inhomogeneous fractional distributed order differential-operator equations of the form

\begin{equation}
L^\Lambda[u] \equiv \int_0^\mu f(\alpha, A)D_\alpha^\alpha u(t)d\Lambda(\alpha) = h(t), \quad t > 0,
\end{equation}

\begin{equation}
 u^{(k)}(0) = \varphi_k, \quad k = 0, ..., m - 1,
\end{equation}

where $\mu \in (m - 1, m]$; $h(t)$ and $\varphi_k$, $k = 0, ..., m - 1$, are given $X$-valued vector-functions; $f(\alpha, A)$ is a family of operators with the symbol $f(\alpha, z)$ continuous in the variable $\alpha \in [0, \mu]$, and analytic in the variable $z \in G \subset \mathbb{C}$; $\Lambda$ is a finite measure defined on $[0, \mu]$; and $D_\alpha^\alpha$ is the operator of fractional differentiation of order $\alpha$ in the sense of Caputo-Djrbashian (see, for example, [5, 12]), i. e.

\[
D_\alpha^\alpha g(t) = \begin{cases} 
\frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{g^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}, & \text{if } n - 1 < \alpha < n, \ n \in \mathbb{N}, \\
\frac{d^n}{dt^n} g(t), & \text{if } \alpha = n \in \{0\} \cup \mathbb{N}.
\end{cases}
\]
Hereafter the integrals are understood in the sense of Bochner if $g(t)$ is a vector-function with values in some topological-vector space for each fixed $t$.

The classical Duhamel principle is not applicable in the case of fractional order differential equations. Its modification combined with some integral transformations can reduce the Cauchy problem for inhomogeneous equation to the Cauchy problem for homogeneous equation. However, this two step process becomes cumbersome for complex equations containing many terms with fractional operators. In this paper we formulate and prove fractional generalizations of Duhamel’s principle applicable directly to the Cauchy problem for inhomogeneous fractional order differential-operator equations, which reduce them to the Cauchy problem for corresponding homogeneous equations. In the particular case of fractional order partial differential equations with a single “fractional” term in the equation (1), a fractional analog of Duhamel’s principle was obtained in [36, 37].

Fractional order differential equations are useful and appropriate mathematical apparatus for modeling problems with memory, and interest in this subject has grown substantially during the last few decades. For instance, probability density functions of a wide class of non-Gaussian diffusion processes satisfy fractional order governing equations with space and time fractional order differential operators (see [1, 2, 15, 24] and references therein). Inhomogeneous fractional order differential equations appear naturally describing the influence of an external force or memory effects. In the study of diffusion processes in complex heterogeneous media with several distinct diffusion modes, even without an external force, the function $h(t)$ embodies memory of the past [22, 38].

There is extensive literature on the Cauchy problem for integer order abstract differential-operator equations (see, e.g. [20, 40]). The first order evolution equations $u'(t) = Au(t)$ in the spaces of abstract exponential vector-functions of a finite type, $\text{Exp}_A(X)$ (and in more general bornological spaces) were studied in [26]. In the case of integer $\alpha_k$, $k = 1, ..., m$, the Cauchy problem for pseudodifferential and differential-operator equations with analytic symbols or with symbols having singularities was studied, for example, in [7, 31, 39], and multipoint value problems in [25, 28, 32, 33, 34].

What concerns fractional order differential-operator equations, Kochubei [17] studied existence and uniqueness of a solution to the abstract Cauchy problem $D^\alpha u(t) = Au(t)$, $u(0) = u_0$, with Caputo-Djrbashian fractional derivative for $0 < \alpha < 1$ and a closed operator $A$ with a dense domain $D(A)$ in a Banach space. El-Sayed [11], Bazhlekov [3] investigated the Cauchy problem for $0 < \alpha < 2$. In the more general case of $\alpha > 0$, Gorenflo et. al. [14] studied existence of solutions in Roumieu-Beurling and Gevrey classes. Kostin [19] proved correctness of the abstract initial value problem (Cauchy type problem) $D^\alpha_+ u(t) = Au(t)$, $D^\alpha_+ t u(0) = \varphi_k$, $k = 1, ..., m$, for $\alpha \in (m - 1, m)$, and with the Riemann-Liouville derivative $D^\alpha_+$. For more information about recent results on the Cauchy problem for abstract fractional differential-operator equations, we refer the reader to [8, 10, 16]; and for a recent mathematical treatment of the distributed fractional order differential equations to papers [18, 23, 35].

The paper is organized as follows. In Section 2 we recall the classic Duhamel
principle, and the basic spaces of elements used in this paper. Since we formulate
a fractional Duhamel principle in the abstract case, we introduce a topological-
vector space on which the corresponding operators act. In Section 3 we formulate
the main result, namely an abstract fractional analog of Duhamel’s principle
and discuss some of its applications.

2 Preliminaries

2.1 Fractional order derivatives.

For a function \( f \) defined on \([0, \infty)\), under some integrability conditions the fractional integral of order \( \beta \) with terminal points \( \tau \) and \( t \), is defined as
\[
\tau J^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_{\tau}^{t} (t-s)^{\beta-1} f(s)ds,
\]
where \( \Gamma(\cdot) \) is Euler’s gamma-function. Obviously, if \( \beta = n \) then \( \tau J^n \) is the \( n \)-fold integral of \( f \) over the interval \([\tau, t]\). By convention, \( \tau J^0 f(t) = f(t) \), i.e. \( \tau J^0 \) coincides with the identity operator. In the notation we do not indicate the upper terminal point \( t \), since in the current paper it is always \( t \).

Further, let \( m \) be a positive integer number. We denote by \( \tau D^\alpha_+ \), \( m-1 < \alpha < 1 \), the fractional derivative of order \( \alpha \) in the sense of Riemann-Liouville, which is defined as
\[
\tau D^\alpha_+ g(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{\tau}^{t} \frac{g(s)ds}{(t-s)^{\alpha+1-m}}, \quad m-1 < \alpha < m,
\]
and \( \tau D^0_+ g(t) = g(t) \). Between this fractional derivative and the Caputo-Djrbashian derivative there is the following relationship [12]:
\[
\tau D^\alpha_* g(t) = \tau D^\alpha_+ g(t) + \sum_{k=0}^{m-1} \frac{g^{(k)}(\tau)}{\Gamma(k+\alpha+1)} (t-\tau)^{\alpha-k}, \quad t > 0.
\]

In the particular case of \( 0 < \alpha < 1 \) one has
\[
\tau D^\alpha_+ g(t) = \tau D^\alpha_* g(t) + g(\tau) \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}, \quad t > 0.
\]

If \( g(\tau) = 0 \), then one obtains the equality \( \tau D^\alpha_+ g(t) = \tau D^\alpha_* g(t) \). Alternative representations via the fractional integral are:
\[
\tau D^\alpha_+ g(t) = \frac{d^m}{dt^m} \tau J^{m-\alpha} g(t) \quad \text{and} \quad \tau D^\alpha_* g(t) = \tau J^{m-\alpha} \frac{d^m g(t)}{dt^m}.
\]

We omit the lower terminal point \( \tau \) if \( \tau = 0 \), writing simply \( D^\alpha_+ \), \( D^\alpha_* \) or \( J^\alpha \). Recall that for the Laplace transform of \( \tau D^\alpha_+ g(t) \) and \( \tau D^\alpha_* g(t) \), where \( \alpha \in (m-1, m] \), the following formulas are valid [12]:
\[
\mathcal{L}[\tau D^\alpha_+ g](s) = s^\alpha \mathcal{L}[g](s) - \sum_{k=0}^{m-1} \frac{d^k}{dt^k} \left(J^{(m-\alpha)} g\right)_{(t=0+)} s^{m-1-k},
\]

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and
\[ L[D^\nu g](s) = s^\nu L[g](s) - \sum_{k=0}^{m-1} g^{(k)}(0+) s^{\alpha-1-k}. \] (6)

Here \( L[g](s) \) denotes the Laplace transform of \( g \).

### 2.2 An operator calculus

In this section we recall some necessary facts about abstract spaces of analytic elements of finite exponential type, and an operator calculus defined on it. See for details [32, 33].

Let \( X \) be a reflexive Banach space with a norm \( \|v\| \), \( v \in X \). Let \( A \) be a closed linear operator with a domain \( D(A) \) dense in \( X \) and a spectrum \( \sigma(A) \subset C \). Assume that \( \sigma(A) \) is not empty and is not bounded.

We will develop an operator calculus \( f(A) \) for analytic functions \( f(\lambda) \) in an open domain \( G \subset C \). If the domain of analyticity of \( f \), \( G \), contains \( \sigma(A) \) then

\[ f(A) = \int_\nu R(\zeta, A) f(\zeta) d\zeta, \] (7)

where \( \nu \) is a contour in \( G \) containing \( \sigma(A) \), and \( R(\zeta, A) \), \( \zeta \in \mathbb{C} \setminus \sigma(A) \), is the resolvent operator of \( A \). However, if \( f \) has singular points in the spectrum of \( A \), then \( f(A) \) can not be defined through the integral (7).

Assume that \( G \) is any open set in \( \mathbb{C} \) not necessarily containing \( \sigma(A) \). Further, let \( 0 < r \leq +\infty \) and \( \nu < r \). Denote by \( \text{Exp}_{A, \nu}(X) \) the set of elements \( v \in \cap_{k\geq 1} D(A^k) \) satisfying the inequalities \( \|A^k v\| \leq C \nu^k \|v\| \) for all \( k = 1, 2, \ldots \), with a constant \( C > 0 \) not depending on \( k \). An element \( v \in \text{Exp}_{A, \nu}(X) \) is said to be a vector of exponential type \( \nu \) [20]. A sequence of elements \( v_n, n = 1, 2, \ldots, \) is said to converge to an element \( v_0 \) in \( \text{Exp}_{A, \nu}(X) \) iff:

1) All the elements \( v_n \) are vectors of exponential type \( \nu < r \), and
2) \( \|v_n - v_0\| \to 0, \ n \to \infty \).

Obviously, \( \text{Exp}_{A, \nu_1}(X) \subset \text{Exp}_{A, \nu_2}(X) \), if \( \nu_1 < \nu_2 \). Let \( \text{Exp}_{A, r}(X) \) be the inductive limit of spaces \( \text{Exp}_{A, \nu}(X) \) when \( \nu \to r \). For basic notions of topological vector spaces including inductive and projective limits we refer the reader to [27]. Set \( A_\lambda = A - \lambda I \), where \( \lambda \in G \), and denote \( \text{Exp}_{A, r, \lambda}(X) = \{ u_\lambda \in X : u_\lambda \in \text{Exp}_{A_\lambda, r}(X) \} \), with the induced topology. Finally, for arbitrary \( G \subset \sigma(A) \), denote by \( \text{Exp}_{A, G}(X) \) the space whose elements are the locally finite sums of elements in \( \text{Exp}_{A, r, \lambda}(X) \), \( \lambda \in G \), \( r < \text{dist}(\lambda, \partial G) \), with the corresponding topology. Namely, any \( u \in \text{Exp}_{A, G}(X) \) has a representation \( u = \sum_{\lambda \in G} u_\lambda \) with a finite sum. It is clear, that \( \text{Exp}_{A, G}(X) \) is a subspace of the space of vectors of exponential type if \( r < +\infty \), and coincides with it if \( r = +\infty \). Moreover, \( \text{Exp}_{A, G}(X) \) is an abstract analog of the space \( \Psi_{G, p}(R^1) \) introduced in [32], where \( A = -i \frac{d}{dx} \), \( G \subseteq R^1 \), \( X = L_p(R^1), 1 < p < \infty \). In the case \( A = -i \frac{d}{dx} \), \( X = L_2(R^1) \), the corresponding space was studied in [17].

Further, let \( f(\lambda) \) be an analytic function on \( G \). An arbitrary element \( u \in \text{Exp}_{A, G}(X) \) is represented as a finite sum \( u = \sum_{\lambda \in G} u_\lambda, \ u_\lambda \in \text{Exp}_{A, r, \lambda}(X) \).
Then for \( u \in \text{Exp}_{A,G}(X) \) the operator \( f(A) \) is defined by the formula

\[
f(A)u = \sum_{\lambda \in G} f_\lambda(A)u_\lambda,
\]

where

\[
f_\lambda(A)u_\lambda = \sum_{n=0}^{\infty} \frac{f^{(n)}(\lambda)}{n!} A^n u_\lambda.
\]

(8)

In other words, each \( f_\lambda \) represents locally \( f \) in a neighborhood of \( \lambda \in G \), and for \( u_\lambda \) the operator \( f_\lambda(A) \) is well defined.

Additionally assume that there exists a one-parameter family of bounded invertible operators \( U_\lambda : X \to X \) such that

\[
AU_\lambda - U_\lambda A = \lambda U_\lambda, \quad \lambda \in \sigma(A).
\]

(10)

For example, if \( X = L_2 \equiv L_2(R) \) and \( A = -i \frac{d}{dx} : L_2 \to L_2 \) is the operator of differentiation with domain \( D(A) = \{ v \in L_2 : Av \in L_2 \} \), then for the operator \( U_\lambda : v(x) \to e^{i\lambda x}v(x) \) we have

\[
AU_\lambda v(x) = -\frac{d}{dx}(e^{i\lambda x}v(x)) = \lambda e^{i\lambda x}v(x) - ie^{i\lambda x}\frac{dv}{dx} = \lambda U_\lambda v(x) + U_\lambda Av(x),
\]

obtaining (10). Condition (10) indicates a shift of the spectrum of operator \( A \) to \( \lambda \). This is seen from the relationship \( A - \lambda I = U_\lambda AU_\lambda^{-1} \), which follows from (10) multiplying by \( U_\lambda^{-1} \) from the right. It follows from the latter that

\[
(A - \lambda I)^n = U_\lambda A^n U_\lambda^{-1},
\]

(11)

for all \( n = 1, 2, \ldots \).

Let \( X^* \) denote the conjugate of \( X \), and \( A^* : X^* \to X^* \) be the operator conjugate to \( A \). Further, denote by \( \text{Exp}_{A^*,G}^*(X^*) \) the space of linear continuous functionals defined on \( \text{Exp}_{A,G}(X) \), with respect to weak convergence. Specifically, a sequence \( u_m^* \in \text{Exp}_{A^*,G}^*(X^*) \) converges to an element \( u^* \in \text{Exp}_{A^*,G}^*(X^*) \) if for all \( v \in \text{Exp}_{A,G}(X) \) the convergence \( < u_m^* - u, v > \to 0 \) holds as \( m \to \infty \).

For an analytic function \( f^w \) defined on \( G^* = \{ z \in \mathbb{C} : \bar{z} \in G \} \), we define a weak extension of \( f(A) \) as follows:

\[
<f^w(A^*)u^*, v > = < u^*, f(A)v >, \quad \forall v \in \text{Exp}_{A,G}(X).
\]

**Lemma 2.1.** Let \( X \) be a reflexive Banach space and \( A \) be a closed operator defined on \( D(A) \subset X \). Let \( f \) be an analytic function defined on an open connected set \( G \subset \mathbb{C} \). Then the following mappings are well defined and continuous:

1. \( f(A) : \text{Exp}_{A,G}(X) \to \text{Exp}_{A,G}(X), \)
2. \( f^w(A^*) : \text{Exp}_{A^*,G}^*(X^*) \to \text{Exp}_{A^*,G}^*(X^*). \)
Proof. Notice that $f(A)$ maps $\text{Exp}_{A,G}(X)$ into itself. Let $u \in \text{Exp}_{A,G}(X)$ has a representation $u = \sum_{\lambda} u_{\lambda}$, $u_{\lambda} \in \text{Exp}_{A,\nu}(X)$. Then for $f(A)u$ defined in (8), one has the following estimate

$$\|A^{k}_{\lambda} f_{\lambda}(A) u_{\lambda}\| \leq \sum_{n=0}^{\infty} \frac{|f^{n}(\lambda)|}{n!} \|(A - \lambda I)^{n} A^{k}_{\lambda} u_{\lambda}\| \leq C \nu^{k} \|u_{\lambda}\|. \quad (12)$$

with some $\nu < r$. It follows that $f_{\lambda}(A) u_{\lambda} \in \text{Exp}_{A,\nu}(X)$ with the same $\nu$, and $f(A)u \in \text{Exp}_{A,G}(X)$. The estimate (12) also implies continuity of the mapping $f(A)$ in the topology of $\text{Exp}_{A,G}(X)$.

Now assume that a sequence $u^{n}_{*} \in \text{Exp}_{A^{*},G^{*}}(X^{*})$ converges to 0 in the weak topology of $\text{Exp}_{A^{*},G^{*}}(X^{*})$. Then for arbitrary $u \in \text{Exp}_{A,G}(X)$ we have

$$< f^{w}(A^{*}) u^{n}_{*}, u > = < u^{n}_{*}, f(A)u > = < u^{n}_{*}, v, >$$

where $v = f(A)u \in \text{Exp}_{A,G}(X)$ due to the first part of the proof. Hence, $f^{w}(A^{*}) x^{n}_{*} \to 0$, as $n \to \infty$, in the weak topology of $\text{Exp}_{A^{*},G^{*}}(X^{*})$.

Remark 2.2. It is not hard to see that the above constructions are valid with corresponding specifications in the case of operators with discrete spectrum as well. Note that in this case the space $\text{Exp}_{A,G}(X)$ consists of the root lineals of eigenvectors corresponding to the part of $\sigma(A)$ with nonempty intersection with $G$. If the spectrum $\sigma(A)$ is empty then an additional exploration is required for solution spaces to be non-trivial (for details see, [9]).

As is shown in [33], the space $\text{Exp}_{A,G}(X)$ is invariant with respect to the action of an operator $f(A)$ and this operator acts continuously.

2.3 Two lemmas

The following two lemmas will be useful in proofs of theorems in Section 3.

**Lemma 2.3.** Let $h(t)$ be a continuous differentiable function. Then the equation $J^{\alpha} u(t) = h(t)$, $t > 0$, where $0 < \alpha < 1$, has a unique continuous solution given by the formula

$$u(t) = D_{1}^{1-\alpha} h(t), \quad t > 0. \quad (13)$$

Lemma 2.3 is essentially the well-known result on a solution of Abel’s integral equation of first kind. See [12, 28] for the proof.

**Lemma 2.4.** Suppose $v(t, \tau)$ is a vector-function in a Banach space $X$, defined for all $t \geq \tau \geq 0$ and $k$ times differentiable with respect to the variable $t$. Let $u(t) = \int_{0}^{t} v(t, \tau) d\tau$. Then

$$\frac{d^{k} u(t)}{dt^{k}} = \sum_{j=0}^{k-1} \frac{d^{j}}{dt^{j}} \left[ \frac{\partial^{k-j}}{\partial t^{k-j}} v(t, \tau) \right]_{\tau=t} + \int_{0}^{t} \frac{\partial^{k}}{\partial t^{k}} v(t, \tau) d\tau. \quad (14)$$
Proof. For a fixed \( t > 0 \) and small \( h \) one can easily verify that

\[
\frac{u(t+h) - u(t)}{h} = \frac{1}{h} \left( \int_{0}^{t+h} v(t + h, \tau) d\tau - \int_{0}^{t} v(t, \tau) d\tau \right)
= \frac{1}{h} \int_{t}^{t+h} v(t, \tau) d\tau + \int_{0}^{t} \frac{v(t + h) - v(t)}{h} d\tau
+ \int_{t}^{t+h} \frac{v(t + h) - v(t)}{h} d\tau.
\]  

Making use of the mean value theorem (in the integral form), we obtain

\[
\left\| \frac{1}{h} \int_{t}^{t+h} v(t, \tau) d\tau - v(t, t) \right\| \leq C_1 \| v(t, \tau_*) - v(t, t) \|, \quad t < \tau_* < t + h,
\]

\[
\left\| \int_{0}^{t} \frac{v(t + h) - v(t)}{h} d\tau - \int_{0}^{t} \frac{\partial v(t, \tau)}{\partial t} d\tau \right\| \leq C_2 |h|,
\]

\[
\left\| + \int_{t}^{t+h} \frac{v(t + h) - v(t)}{h} d\tau \right\| \leq C_3 |h|,
\]

where constants \( C_1, C_2, \) and \( C_3 \) do not depend on \( h \). Now, letting \( h \to 0 \), estimates (16)-(18) and equation (15) imply the following formula:

\[
\frac{d}{dt} u(t) = v(t, t) + \int_{0}^{t} \frac{\partial}{\partial \tau} v(t, \tau) d\tau.
\]  

Formula (14) follows from (19) by differentiation repeatedly. 

2.4 Classical Duhamel’s principle.

Duhamel’s principle was formulated first for the Cauchy problem for second order linear inhomogeneous differential equations. Let \( B = B(x, \frac{\partial}{\partial x}, D_x) \), where \( D_x = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}) \), be a linear differential operator with coefficients not depending on \( t \), and containing temporal derivatives of order not higher than 1. Consider the Cauchy problem

\[
\frac{\partial^2 u}{\partial t^2}(t, x) + Bu(t, x) = f(t, x), \quad t > 0, \ x \in \mathbb{R}^n,
\]

with homogeneous initial conditions

\[
u(0, x) = 0, \ \frac{\partial u}{\partial t}(0, x) = 0.
\]

Let a sufficiently smooth function \( v(t, \tau, x), \ t \geq \tau, \ \tau \geq 0, \ x \in \mathbb{R}^n, \) be for \( t > \tau \) a solution of the homogeneous equation

\[
\frac{\partial^2 v}{\partial t^2}(t, \tau, x) + Bv(t, \tau, x) = 0,
\]
satisfying the following conditions:

\[ v(t, \tau, x)|_{t=\tau} = 0, \quad \frac{\partial v}{\partial t}(t, \tau, x)|_{t=\tau} = h(\tau, x). \]

Then a solution of the Cauchy problem (20), (21) is given by means of the integral

\[ u(t, x) = \int_0^t v(t, \tau, x)d\tau. \] (22)

The formulated statement is known as Duhamel’s principle, and the integral in (22) as Duhamel’s integral. A similar statement is valid in the case of the Cauchy problem with a homogeneous initial condition for a first order inhomogeneous partial differential equation

\[ \frac{\partial u}{\partial t}(t, x) + Cu(t, x) = f(t, x), \quad t > 0, \quad x \in \mathbb{R}^n, \]

where \( C = C(x, D_x) \) is a linear differential operator containing only spatial derivatives, and with coefficients not depending on \( t \) (see [4]).

3 Generalizations of Duhamel’s principle

In this section we prove abstract fractional generalizations of Duhamel’s principle and discuss some of their applications.

3.1 Duhamel’s principle: \( \Lambda = \sum_{k=0}^m \delta_{\alpha_k} \) with \( \alpha_k = k, \ k = 1, \ldots, m. \)

Suppose the measure \( \Lambda \) in (11) has the form \( \Lambda = \sum_{k=0}^m \delta_k \), where \( \delta_k \) denotes Dirac’s delta with mass on \( a \). Suppose also that \( f(m, A) = I \), the identity operator. Then the Cauchy problem (11), (2) takes the form

\[ u^{(m)}(t) + \sum_{k=0}^{m-1} f_k(A)u^{(k)}(t) = h(t), \quad t > 0, \] (23)

\[ u^{(k)}(0) = \varphi_k, \quad k = 0, \ldots, m - 1. \] (24)

The operators \( f_k(A) = f(k, A), k = 0, \ldots, m - 1, \) are understood in the sense of the functional calculus introduced in Section 2.2. In the following theorem we assume that the vector-functions \( U(t, \tau) \) and \( h(t) \) are \( \text{Exp}_{A, G}(X) \)-, or \( \text{Exp}_{A^*, G^*}(X^*) \)-valued. In this abstract case Duhamel’s principle is formulated as follows.
Theorem 3.1. Let a vector-function \( U(t, \tau) \) for all \( \tau : 0 \leq \tau < t \) be a solution of the Cauchy problem for a homogeneous equation

\[
\frac{\partial^m U}{\partial t^m}(t, \tau) + \sum_{k=0}^{m-1} f_k(A) \frac{\partial^k U}{\partial t^k}(t, \tau) = 0, \quad t > \tau, \tag{25}
\]

\[
\frac{\partial^k U}{\partial t^k}(t, \tau)|_{t=\tau+0} = 0, \quad k = 0, \ldots, m - 2, \tag{26}
\]

\[
\frac{\partial^{m-1} U}{\partial t^{m-1}}(t, \tau)|_{t=\tau+0} = h(\tau), \tag{27}
\]

where \( h(t) \) is a continuous vector-function. Then a solution of the Cauchy problem for the inhomogeneous equation

\[
u^{(m)}(t) + \sum_{k=0}^{m-1} f_k(A) u^{(k)}(t) = h(t), \tag{28}
\]

\[
u^{(k)}(0) = 0, \quad k = 0, \ldots, m - 1. \tag{29}\]

is represented via Duhamel’s integral

\[
u(t) = \int_0^t U(t, \tau)d\tau. \tag{30}\]

Proof. Obviously \( \nu(0) = 0 \). Further, for the first order derivative of \( \nu(t) \), using Lemma 2.4, one has

\[
\frac{d\nu}{dt}(t) = U(t, t) + \int_0^t \frac{\partial U}{\partial t}(t, \tau)d\tau,
\]

By virtue of (26) the latter implies that \( \frac{d\nu}{dt}(0) = 0 \). Further, differentiating,

\[
\frac{d^k\nu}{dt^k}(t) = \frac{\partial^{k-1} U}{\partial t^{k-1}}(t, t) + \int_0^t \frac{\partial^k U}{\partial t^k}(t, \tau)d\tau,
\]

which due to condition (26) implies that

\[
\frac{d^k\nu}{dt^k}(0) = 0, \quad k = 2, \ldots, m - 1.
\]

Therefore, the function \( \nu(t) \) in (30) satisfies initial conditions (29). Moreover, substituting (30) to (28), and taking into account (27), we have

\[
u^{(m)}(t) + \sum_{k=0}^{m-1} f_k(A) u^{(k)}(t) = \frac{\partial^m U}{\partial t^m}(t, \tau) + \sum_{k=0}^{m-1} f_k(A) \frac{\partial^k U}{\partial t^k}(t, \tau)\int_0^t U(t, \tau)d\tau
\]

\[
= \frac{\partial^{m-1} U}{\partial t^{m-1}}(t, t) + \int_0^t \frac{\partial^m U}{\partial t^m}(t, \tau)d\tau + \sum_{k=0}^{m-1} f_k(A) \int_0^t \frac{\partial^k U}{\partial t^k}(t, \tau)d\tau
\]

\[
= h(t) + \int_0^t \left[ \frac{\partial^m U}{\partial t^m}(t, \tau) + \sum_{k=0}^{m-1} f_k(A) \frac{\partial^k U}{\partial t^k}(t, \tau) \right] d\tau = h(t).
\]

Hence, \( \nu(t) \) in (30) satisfies equation (28) as well. \( \blacksquare \)
Remark 3.2. It is not hard to see that Theorem 3.1 holds with generic closed operators $B_k$ (with dense domain $D(B_k)$ and commuting with $\frac{d}{dt}$) instead of $f_k(A)$. In this case we assume that $h(t) \in X$ and $\frac{\partial^k U(t,\tau)}{\partial t^k} \in D(B_k), k = 0, ..., m - 1$.

3.2 Fractional Duhamel’s principle: $\Lambda = \delta_\mu + \lambda$ with $\mu \in (m - 1, m]$.

Let $\Lambda = \delta_\mu + \lambda$, where $\mu$ is a number such that $m - 1 < \mu < m$, and $\lambda$ is a finite measure with $\text{supp } \lambda \subset [0, m - 1)$. Consider the operator

$$\tau L^{(\mu, \lambda)}[u](t) \equiv \tau D_\mu^\mu u(t) + \int_0^{m-1} f(\alpha, A) \tau D_\alpha^\mu u(t) \lambda(\alpha),$$

acting on $m$-times differentiable vector-functions $u(t), t \geq \tau \geq 0$. If $\tau = 0$, then instead of $0 L^{(\mu, \lambda)}[u](t)$ we write $L^{(\mu, \lambda)}[u](t)$.

Theorem 3.3. Suppose that $V(t, \tau), t \geq \tau \geq 0$, is a solution of the Cauchy problem for the homogeneous equation

$$\tau L^{(\mu, \lambda)}[V(\cdot, \tau)](t) = 0, \quad t > \tau, \quad \tag{32}$$

$$\frac{\partial^k V}{\partial t^k}(t, \tau)|_{t=\tau+0} = 0, \quad k = 0, ..., m - 2, \quad \tag{33}$$

$$\frac{\partial^{m-1} V}{\partial t^{m-1}}(t, \tau)|_{t=\tau+0} = D_+^{m-\mu} h(\tau), \quad \tag{34}$$

where $h(t)$ is a given vector-function. Then Duhamel’s integral

$$u(t) = \int_0^t V(t, \tau)d\tau \quad \tag{35}$$

solves the Cauchy problem for the inhomogeneous equation

$$L^{(\mu, \lambda)}[u](t) = h(t), \quad t > 0, \quad \tag{36}$$

with the homogeneous Cauchy conditions

$$u^{(k)}(0) = 0, \quad k = 0, ..., m - 1. \quad \tag{37}$$

Proof. First notice that since $m - 1 < \mu < m$, and therefore $0 < m - \mu < 1$, due to Lemma 2.3, the equation $J^{m-\mu} g(t) = h(t)$ has a unique solution

$$g(t) = D_+^{m-\mu} h(t). \quad \tag{38}$$

Let $V(t, \tau)$ as a function of the variable $t$ be a solution to Cauchy problem 32, 33 for any fixed $\tau$. We verify that $u(t) = \int_0^t V(t, \tau)d\tau$ satisfies equation

10
Lemma 2.4 and conditions (33) imply that

Further, put

Again due to Lemma 2.4 and condition (34), we have

Now equations (39), (42), (44), and (45) imply that

Therefore the first term on the right hand side of (39) takes the form

Further, put \( g(t) = D_{+}^{m-\mu}h(t) \). Then by virtue of (38),

Now equations (39), (42), (44), and (45) imply that

\[ L^{(\mu, \lambda)}[u](t) = h(t) + \frac{1}{\Gamma(m-\mu)} \int_{0}^{t} (t-s)^{m-\mu-1} \int_{0}^{s} \frac{\partial^{m}}{\partial s^{m}} V(s, \tau) d\tau ds \]

\[ + \sum_{k=1}^{m-1} f(\alpha, A) \frac{1}{\Gamma(k-\alpha)} \int_{0}^{t} (t-s)^{k-\alpha-1} \int_{0}^{s} \frac{\partial^{k}}{\partial s^{k}} V(s, \tau) d\tau ds \lambda(d\alpha). \]
Changing the order of integration (Fubini is allowed) in (46) we get

$$L^{(\mu, \lambda)}[u](t) = h(t) + \int_0^t \int_\tau^t \frac{1}{\Gamma(m-\mu)}(t-s)^{m-\mu-1}\frac{\partial^m}{\partial s^m}V(s, \tau)ds d\tau$$

$$+ \sum_{k=1}^{m-1} \int_0^t \int_\tau^{t-\tau} f(\alpha, A) \int_\tau^t \frac{1}{\Gamma(k-\alpha)}(t-s)^{k-\alpha-1}\frac{\partial^k}{\partial s^k}V(s, \tau)ds \lambda(d\alpha) d\tau$$

$$= h(t) + \int_0^t \tau D^\alpha V(t, \tau) d\tau + \int_0^t \int_0^{t-\tau} f(\alpha, A) \tau D^\alpha V(t, \tau) \lambda(d\alpha) d\tau$$

$$= h(t) + \int_0^t \tau L^{(\mu, \lambda)}[V(\cdot, \tau)](t) d\tau = h(t).$$

Finally, using the relations (41) it is not hard to verify that $u(t)$ in (35) satisfies initial conditions (37) as well.

If the vector-function $h$ satisfies the additional condition $h(0) = 0$ then condition (34) in accordance with the relationship (4) can be replaced by

$$\frac{\partial^{m-1}V}{\partial t^{m-1}}(t, \tau)|_{t=\tau} = D^{m-\mu}_t h(\tau),$$

with the Caputo-Djrbashian derivative $D^{m-\mu}_t$ of order $m-\mu$. As a consequence the formulation of the fractional Duhamel’s principle takes the form:

**Theorem 3.4.** Suppose that for all $\tau : 0 < \tau < t$ a function $V(t, \tau)$, is a solution to the Cauchy problem for the homogeneous equation

$$\tau L^{(\mu, \lambda)}[V(\cdot, \tau)](t) = 0, \quad t > \tau,$$

$$\frac{\partial^k V}{\partial t^k}(t, \tau)|_{t=\tau+0} = 0, \quad k = 0, ..., m - 2,$$

$$\frac{\partial^{m-1}V}{\partial t^{m-1}}(t, \tau)|_{t=\tau+0} = D^{m-\mu}_t h(\tau),$$

where $h(t)$ is a given vector-function such that $h(0) = 0$. Then

$$u(t) = \int_0^t V(t, \tau) d\tau$$

is a solution of the Cauchy problem for the inhomogeneous equation

$$L^{(\mu, \lambda)}[u](t) = h(t), \quad t > 0,$$

with the homogeneous Cauchy conditions

$$u^{(k)}(0) = 0, \quad k = 0, ..., m - 1.$$

**Remark 3.5.** 1. Lemma 2.3 can be extended to absolutely continuous functions $h(t)$ with an appropriate meaning of solution in equation (13) (see, [28]). It is also known [28] that the fractional derivative $D^{k-\mu}_\tau h(t), k -$
1 < \mu < k, k = 1, ..., m, exists a.e., if \( h(t) \) is an absolutely continuous function on \([0; T]\) for any \( T > 0 \). These two facts imply that generalized Duhamel’s principles proved above hold true for absolutely continuous functions \( h(t) \).

2. In Theorems 3.3 and 3.4 we assumed that \( f(\mu, A) \) is the identity operator (see equation (31)). In the general case, with appropriate selection of \( G \) we can assume that the inverse operator \( [f(\mu, A)]^{-1} \) exists. Then with the condition

\[
\frac{\partial^{m-1} V(t, \tau)}{\partial t^{m-1}} \bigg|_{t=\tau} = [f(\mu, A)]^{-1} D_{+}^{m-\mu} h(\tau)
\]

instead of (31), Theorems 3.3 and 3.4 remain valid.

### 3.3 Fractional Duhamel’s principle with Riemann-Liouville derivative

The operator \( \tau L^\Lambda \) in Theorem 3.3 is defined via the fractional derivative in the sense of Caputo-Djrbashian. A fractional generalization of Duhamel’s principle is also possible when this operator is defined via the Riemann-Liouville fractional derivative. In this section we briefly discuss this important case proving the corresponding theorem in the simple particular case

\[
\tau L[u](t) = \tau D_{+}^{\alpha} u(t) + Bu(t), \quad 0 < \alpha < 1, \quad \text{and} \quad B \text{is a closed operator with a domain } \mathcal{D}(B) \text{ dense in } X.
\]

The general case can be treated in a similar manner.

**Theorem 3.6.** Suppose that \( V(t, \tau), \ t \geq \tau \geq 0, \) is a solution of the Cauchy type problem for the homogeneous equation

\[
\tau D_{+}^{\alpha} V(t, \tau) + BV(t, \tau) = 0, \quad t > \tau,
\]

\[
\tau J^{1-\alpha} V(t, \tau) \big|_{t=\tau} = h(\tau),
\]

where \( h(\tau), \ \tau > 0, \) is a continuous vector-function. Then Duhamel’s integral

\[
u(t) = \int_0^t V(t, \tau) d\tau
\]

solves the Cauchy type problem for the inhomogeneous equation

\[
\tau D_{+}^{\alpha} u(t) + Bu(t) = h(t), \quad t > 0,
\]

with the homogeneous initial condition \( J^{1-\alpha} u(0) = 0 \).

**Proof.** Let \( V(t, \tau) \) satisfy the conditions of the theorem. Then for Duhamel’s integral (49), by virtue of Lemma 2.4 we have

\[
\tau D_{+}^{\alpha} u(t) + Bu(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{V(s, \tau) ds}{(t-s)^\alpha} - \int_0^t BV(t, \tau) d\tau
\]

\[
= \frac{d}{dx} \int_0^t \tau J^{1-\alpha} V(t, \tau) d\tau + \int_0^t BV(t, \tau) d\tau
\]

\[
= \tau J^{1-\alpha} V(t, \tau) \big|_{t=\tau} + \int_0^t [\tau D_{+}^{\alpha} V(t, \tau) + BV(t, \tau)] d\tau = h(t).
\]
On the other hand, changing the order of integration and using the mean value theorem, we obtain
\[
\|J^{1-\alpha}u(t)\| = \bigg\| \int_0^t \tau J^{1-\alpha}V(t, \tau) d\tau \bigg\| \leq t \| J^{1-\alpha}V(t, \tau) \|. \tag{52}
\]
Condition (48) implies that \( \lim_{t \to 0^+} \tau J^{1-\alpha}V(t, \tau) = h(0) \) in the norm of \( X \). It follows from (52) that \( \lim_{t \to 0^+} J^{1-\alpha}u(t) = 0 \) in the norm of \( X \).  

### 3.4 Applications. Existence and uniqueness theorems

Theorems 3.1 and 3.3 lead to generalization of the existence and uniqueness results obtained in papers [13, 30] for the abstract Cauchy problems. Let \( L^\Lambda \) be the distributed fractional order abstract differential-operator defined in (31) with \( \tau = 0 \), and via the characteristic function
\[
\Delta(s, z) = s^\mu + \int_0^{m-1} f(\alpha, z) s^\alpha d\lambda,
\]
where \( \mu \in (m-1, m] \), \( \lambda \) is a finite measure with \( \text{supp} \lambda \subset [0, m-1] \), and \( f(\alpha, z) \) is a function continuous in \( \alpha \) and analytic in \( z \in G \subset \mathbb{C} \). Denote by \( \hat{v}(s) = L[v](s) \) the Laplace transform of a vector-function \( v(t) \), namely
\[
L[v](s) = \int_0^\infty e^{-s t} v(t) dt, \quad s > s_0,
\]
where \( s_0 \geq 0 \) is a real number. It is not hard to verify that if \( v(t) \in \text{Exp}_{A, G}(X) \) for each \( t \geq 0 \) and satisfies the condition \( \|v(t)\| \leq C e^{\gamma t}, t \geq 0 \), with some constants \( C > 0 \) and \( \gamma \), then \( \hat{v}(s) \) exists and
\[
\| A^k \hat{v}(s) \| \leq \frac{C s}{s - \gamma} \mu^k, \quad s > \gamma,
\]
implying \( \hat{v}(s) \in \text{Exp}_{A, G}(X) \) for each fixed \( s > \gamma \). The lemma below gives a formal representation formula for a solution of the general abstract Cauchy problem
\[
L^\Lambda[u](t) = h(t), \quad t > 0, \tag{53}
\]
\[
u^{(k)}(0^+) = \varphi_k, \quad k = 0, \ldots, m-1. \tag{54}
\]

#### Lemma 3.7
Let \( c_\beta(t, z) = L^{-1}\left[ \frac{s^\beta}{\Delta(s, z)} \right](t) \), \( z \in G \subset \mathbb{C} \), where \( L^{-1} \) stands for the inverse Laplace transform, and
\[
S_k(t, z) = c_{\mu-k-1}(t, z) + \int_k^{m-1} f(\alpha, z) c_{\alpha-k-1}(t, z) \lambda(d\alpha).
\]
Then \( u_k(t) = S_k(t, A)\varphi_k \) solves the Cauchy problem
\[
L[u] = 0, \quad u^{(j)}(0) = \delta_{j,k} \varphi_j, \quad j = 0, \ldots, m-1,
\]
where \( \delta_{j,k} = 1 \) if \( j = k \), and \( \delta_{j,k} = 0 \) if \( j \neq k \).
Corollary 3.8. Let $S_k(t, A), k = 0, \ldots, m - 1$, be the collection of solution operators with the symbols $S_k(t, z)$ defined in Lemma 3.7. Then the solution of the Cauchy problem

$$L[u] = 0, \quad u^j(0) = \varphi_j, \quad j = 0, \ldots, m - 1. \quad (55)$$

is given by the following representation formula

$$u(t) = \sum_{k=0}^{m-1} S_k(t, A)\varphi_k. \quad (56)$$

Proof of Lemma. Applying formula (6) we have

$$\mathcal{L}[L[u]](s) = s^\mu \hat{u}(s) - \sum_{i=0}^{m-1} u^i(0)s^{\mu-i-1} + \sum_{k=1}^{m-1} \int_{k-1}^{k} f(\alpha, A)(s^\alpha \hat{u}(s) - \sum_{j=0}^{k-1} u^j(0)s^{\alpha-j-1})\lambda(d\alpha) = 0.$$ 

Due to the initial conditions $u^j(0) = \delta_{j,k}\varphi_j, \quad j = 0, \ldots, m - 1$, the latter reduces to

$$\Delta(s, z)\hat{u}(s) = \varphi_k \left( s^{\mu-k-1} + \int_{k}^{m-1} f(\alpha, z)s^{\alpha-k-1}\lambda(d\alpha) \right).$$

Now it is easy to see that the solution in this case is represented $u_k = S_k(t, A)\varphi_k$.

Remark 3.9. 1. Corollary 3.8 can easily be extended to the operator $\tau L$ in (55) as well with the initial conditions $u^j(\tau) = \varphi_j$, maintaining the shift $t' = t - \tau$;

2. A particular case of Lemma 3.7 when $\Lambda = \sum_{k=0}^{m} \delta_{\alpha_k, k} - 1 < \alpha_k < k$, is proved in [13].

Further, denote by $C^{(m)}[t > 0; \text{Exp}_A, G(X)]$ and by $AC[t > 0; \text{Exp}_A, G(X)]$ the space of $m$-times continuously differentiable functions and the space of absolutely continuous functions on $(0; +\infty)$ with values ranging in the space $\text{Exp}_A, G(X)$, respectively. A vector-function $u(t) \in C^{(m)}[t > 0; \text{Exp}_A, G(X)] \cap C^{(m-1)}[t \geq 0; \text{Exp}_A, G(X)]$ is called a solution of the problem (53), (54) if it satisfies the equation (53) and the initial conditions (54) in the topology of $\text{Exp}_A, G(X)$.

Theorem 3.3 and Corollary 3.8 imply the following results.

Theorem 3.10. Let $\varphi_k \in \text{Exp}_A, G(X), k = 0, \ldots, m - 1, \quad h(t) \in AC[0 \leq t \leq T; \text{Exp}_A, G(X)]$ for any $T > 0$, and $D_t^{m-\alpha}h(t) \in C[0 \leq t \leq T; \text{Exp}_A, G(X)]$. Then the Cauchy problem (55), (54) has a unique solution. This solution is given by

$$u(t) = \sum_{k=0}^{m-1} S_k(t, A)\varphi_k + \int_{0}^{t} S_{m-1}(t - \tau, A)D_t^{m-\alpha}h(\tau)d\tau. \quad (57)$$
Proof. We split the Cauchy problem (53), (54) into two Cauchy problems

\[ L^A[u](t) = 0, \ t > 0, \]  
\[ u^{(k)}(0+) = \varphi_k, \ k = 0, ..., m - 1, \]  
and

\[ L^A[v](t) = h(t), \ t > 0, \]  
\[ v^{(k)}(0+) = 0, \ k = 0, ..., m - 1. \]

Due to Corollary 3.8 the unique solution to (58), (59) is given by

\[ u(t) = \sum_{k=0}^{m-1} S_k(t, A) \varphi_k. \]  

For the Cauchy problem (60), (61), in accordance with the fractional Duhamel's principle (Theorem 3.3), it suffices to solve the Cauchy problem for the homogeneous equation:

\[ \tau L^A[V(t, \tau)](t) = 0, \ t > \tau, \]  
\[ \frac{\partial^k V(t, \tau)}{\partial t^k} \bigg|_{t=\tau^+} = 0, \ k = 0, ..., m - 2, \]  
\[ \frac{\partial^{m-1} V(t, \tau)}{\partial t^{m-1}} \bigg|_{t=\tau^+} = D_{+}^{m-\mu} h(\tau). \]

The solution of this problem, again using Corollary 3.8 (with the note in Remark 3.9), has the representation

\[ V(t, \tau) = S_{m-1}(t - \tau, A) D_{+}^{m-\mu} h(\tau). \]  

Thus, Duhamel's integral of the latter and representation (62) lead to formula (57). The uniqueness of a solution also follows from the obtained representation (57) (see 3.8). ◻

The duality immediately implies the following theorem.

**Theorem 3.11.** Let \( \varphi_k^* \in \text{Exp}^1_{A^*, G^*}(X^*), k = 0, ..., m - 1 \), \( h^*(t) \in AC[t \leq t \leq T; \text{Exp}^1_{A^*, G^*}(X^*)] \) and \( D_{+}^{m-\alpha} h^*(t) \in C[t \leq t \leq T; \text{Exp}^1_{A^*, G^*}(X^*)] \).

Assume also that \( \text{Exp}_{A^*, G^*}(X) \) is dense in \( X \). Then the Cauchy problem (53), (54) (with \( A \) switched to \( A^* \)) is meaningful and has a unique weak solution. This solution is given by

\[ u^*(t) = \sum_{k=0}^{m-1} S_k(t, A^*) \varphi_k^* + \int_0^t S_{m-1}(t - \tau, A^*) D_{+}^{m-\mu} h^*(\tau) d\tau. \]
Assume that $\text{Exp}_{A, G}(X)$ is densely embedded into $X$. Besides, let the solution operators $S_k(t, A)$ for each $k = 0, ..., m - 1$, satisfy the estimates

$$
\|S_k(t, A)\varphi\| \leq C\|\varphi\|, \quad \forall \ t \in [0, T],
$$

(67)

where $\varphi \in \text{Exp}_{A, G}(X)$, and $C > 0$ does not depend on $\varphi$. Then there exists a unique closure $\bar{S}_k(t)$ to $X$ of the operator $S_k(t, A)$ which satisfies the estimate

$$
\|\bar{S}_k(t)u\| \leq \|u\| \quad \text{for all} \ u \in X. \quad \text{Using the standard technique of closure (see [32, 33]), we can prove the following theorem.}
$$

**Theorem 3.12.** Let $\varphi_k \in X$, $k = 0, ..., m - 1$, $h(t) \in AC[0 \leq t \leq T; X]$ for any $T > 0$, and $D^{m-\alpha}_+ h(t) \in C[0 \leq t \leq T; X]$. Further let $\text{Exp}_{A, G}(X)$ be densely embedded into $X$, and the estimates (67) hold for solution operators $S_k(t, A), k = 0, ..., m - 1$. Then the Cauchy problem (53), (54) has a unique solution $u(t) \in C^m[0 < t \leq T; X]$. This solution is given by

$$
u(t) = \sum_{k=0}^{m-1} \bar{S}_k(t)\varphi_k + \int_0^t \bar{S}_{m-1}(t-\tau)D^{m-\mu}_+ h(\tau) d\tau.
$$

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