The Spiral of Life

F Cajiao Vélez, J Z Kamiński and K Krajewska
Institute of Theoretical Physics, Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warsaw, Poland
E-mail: Felipe.Cajiao-Velez@fuw.edu.pl

Abstract. High-energy photoionization driven by short and circularly-polarized laser pulses is studied in the framework of the relativistic strong-field approximation. The saddle-point analysis of the integrals defining the probability amplitude is used to determine the general properties of the probability distributions. Additionally, an approximate solution to the saddle-point equation is derived. This leads to the concept of the three-dimensional spiral of life in momentum space, around which the ionization probability distribution is maximum. We demonstrate that such spiral is also obtained from a classical treatment.

1. Introduction
The recent development of laser facilities like ELI [1] or XCELS [2], where short laser pulses can reach power of few petawatts, calls for a better understanding of the high-energy photoionization, where relativistic and spin effects may occur. Under such circumstances, the numerical solution of the wave equations becomes computationally demanding due to a very large number of grid points to be taken into account in the temporal and spatial dimensions. For this reason, other approaches are required.

Recently, we have analyzed the high-energy ionization driven by intense, circularly-polarized, few-cycle laser pulses under the framework of the so-called relativistic strong-field approximation (RSFA) [3, 4]. We have demonstrated that, for laser field intensities larger than $10^{17}$ W/cm$^2$, the resulting photoelectrons can have kinetic energies ranging from tens of keV up to MeV (or even GeV if the laser light is very powerful) [5, 6, 7, 8]. Also, according to our results, the probability distributions of ionization can present two types of structures: interference-free (the so-called supercontinua) or interference-dominated. With the help of the saddle-point analysis of the probability amplitudes, we have located the regions where such structures appear in the energy spectra of photoelectrons. In Refs. [9, 10] we have proved that it is possible to construct an approximate solution to the saddle-point equation. This solution defines the three-dimensional spiral of life in momentum space and contains most of the information related to the high-energy photoionization process. It is the aim of this paper to further analyze the properties of this spiral from the perspective of the RSFA, and to compare them with the predictions arising from a purely classical treatment based on the solution of the Newton-Lorentz equations of motion.

2. Theory
We consider an initially unperturbed hydrogen-like ion in the presence of a strong and short laser pulse, lasting for time $T_p$. The relativistic probability amplitude of photoionization, which
relates to the transition of an electron from the initial bound state $\Psi_i(x)$ of energy $E_0$ to the continuum, is given by

$$A_R = -i \int d^4x e^{-i(E_0/c)x^0} \Psi_i(x) e^{iA(x)} \Psi_1(x). \tag{1}$$

Here, $A(x)$ is the four-vector potential describing the laser pulse and $\Psi_1(x)$ is the scattering state of energy $E_p$. Even though Eq. (1) is exact, it cannot be used for very intense laser fields. The reason being that, with the current computer capabilities, $\Psi_i(x)$ can only be obtained numerically for laser fields of moderate intensities. That is why, in the RSFA [3, 4, 5], the scattering state is expanded with respect to the binding potential in a Born series. If the kinetic energy of the photoelectrons is much larger than the ionization potential of the system, i.e.,

$$\sqrt{(m_e c^2)^2 + (cp)^2} - m_e c^2 \gg m_e c^2 - E_0, \tag{2}$$

the relevant term of the above-mentioned expansion is its zeroth-order. To obtain an analytical form of the scattering state in the lowest order Born expansion, the so-called plane-wave-fronted pulse approximation (PWFA) is also required. In this approximation, the laser field propagating with the speed of light $c$ depends on the time and space variables only through a real phase, $\phi = k \cdot x = \omega (t - \mathbf{n} \cdot \mathbf{x}/c)$. Here, $k = k^0(1, \mathbf{n}) \equiv k^0 n$ is the wave four-vector, $\mathbf{n}$ is the propagation direction of the laser field, $k^0 = \omega/c$ is the wave number, and $\omega = 2\pi/T_p$ is the fundamental frequency of field oscillations. To be more specific, we will describe the laser field through the following vector potential,

$$A(\phi) = A_0 [\varepsilon_1 f_1(\phi) + \varepsilon_2 f_2(\phi)], \tag{3}$$

where $f_i(\phi)$, $i = 1, 2$, are two arbitrary shape functions which vanish for $\phi \leq 0$ and $\phi \geq 2\pi$. Furthermore, in Eq. (3), $\varepsilon_i = (0, \varepsilon_i)$ are two polarization four vectors such that $\varepsilon_i \cdot \varepsilon_j = -\delta_{ij}$ and $k \cdot \varepsilon_i = 0$.

In the zeroth-order of the Born expansion and using the PWFA, the electron scattering state $\Psi_i(x)$ in Eq. (1) is approximated by the so-called Volkov solution [11]; here, denoted as $\psi_{p(+)}^{i}(x)$. It describes the quantum-mechanical behavior of a free electron of asymptotic momentum $p$, the energy $E_p = \sqrt{(m_e c^2)^2 + (cp)^2}$, and the spin degree of freedom $\lambda = \pm$ in the laser pulse (3). Therefore, in the RSFA, the probability amplitude of photoionization can be written as [5, 7, 8]

$$A_{\lambda_1\lambda}(p) = -i \int \frac{d^3q}{(2\pi)^3} \int d^4x e^{-iq\cdot x} \psi_{p\lambda}^{i}(x)e^{iA(x)}\tilde{\Psi}_i(q), \tag{4}$$

where $q = (E_0/c, q)$ and $\tilde{\Psi}_i(q)$ is the Fourier transform of the initial state $\Psi_i(x)$. Also, $\lambda_1 = \pm$ is the spin degree of freedom of the initial electron. At this point, we would like to emphasize that the RSFA and, in particular, Eq. (4) are valid provided that the high-energy photoionization is considered, i.e., that the relation (2) holds.

Next, introducing the light cone coordinates (for their definition, see Refs. [12, 13]), most integrals in Eq. (4) can be performed exactly. In doing so, we obtain [10]

$$A_{\lambda_1\lambda}(p) = N \int_0^{2\pi} d\phi \exp \left[ i \int_0^\phi du G'(u) \right] \tilde{B}_{p\lambda}(\phi) \cdot \tilde{\Psi}_i(Q), \tag{5}$$

where $N$ contains all irrelevant factors, $Q = p + (q^0 - p^0)\mathbf{n}$, and

$$G'(u) = g_0 + g_1 f_1(u) + g_2 f_2(u) + h (f_1^2(u) + f_2^2(u)). \tag{6}$$

Here,

$$g_0 = \frac{p^0 - q^0}{k^0}, \quad h = \frac{(m_e c\mu)^2}{2k \cdot p}, \quad g_j = -m_e c\mu \frac{\varepsilon_j \cdot p}{k \cdot p} \quad \text{for} \quad j = 1, 2, \tag{7}$$
and \( \mu = |eA_0|/(mc) \) is the normalized amplitude of the vector potential. Note that, in Eq. (5), the exponential factor oscillates fast as a function of the phase \( \phi \). This is in contrast to the term \( \tilde{B}_{p\lambda}(\phi) \), which is slow-oscillating in the interval \( 0 \leq \phi \leq 2\pi \). (Even though we do not present here the explicit form of the function \( \tilde{B}_{p\lambda}(\phi) \), it is given in Eq. (11) of Ref. [10]). This allows us to approximate the integral in (5) using the saddle-point method. In doing so, we obtain that the spin-resolved probability amplitude of ionization, in the RSFA and in the saddle-point approximation, is

\[
A_{\lambda\lambda}(p) = N \sum_s e^{iG(\phi_s)} \sqrt{2\pi i} G''(\phi_s) \tilde{B}_{p\lambda}(\phi_s) \cdot \tilde{\Psi}(Q). \tag{8}
\]

The sum in (8) is carried out over all relevant complex saddle points; here, denoted as \( \phi_s \). The latter are calculated from the equation

\[
G'(\phi_s) \equiv \frac{dG(\phi)}{d\phi} \big|_{\phi_s} = 0, \tag{9}
\]

with \( \text{Im} \ G(\phi_s) > 0 \) or, equivalently, \( \text{Im} \ \phi_s > 0 \). Also, it is assumed that \( \text{Re} \ \phi_s \) determines the time of photoelectron emission, i.e., the moment at which the electron appears in the continuum.

From the analysis of Eq. (8), the following conclusions can be drawn. First, the saddle points contributing the most to the probability amplitude of ionization are those characterized by a positive imaginary part which is as close to zero as possible. Second, if there are two or more of such points in a given range of photoelectron kinetic energies, strong interference effects are expected. Third, if only one saddle point dominates, interference-free structures (supercontinua) should be observed in the energy spectra of photoelectrons. Those properties have been studied in Refs. [5, 7, 8, 9, 10] and are going to be analyzed here in relation to the spiral of life (Sec. 3.1).

Once the probability amplitude of ionization is calculated, the spin-independent probability distribution \( P(p) \), in atomic units, can be determined according to (for the details see, Refs [7, 8])

\[
P(p) = \frac{\alpha^2 m_e V E_p}{2(2\pi)^3} \sum_{\lambda_1, \lambda_2 = \pm} |p| \cdot |A_{\lambda_1, \lambda_2}(p)|^2. \tag{10}
\]

Here, \( \alpha \) is the fine structure constant and \( V \) is the quantization volume. Note that, as \( N \) in Eq. (5) contains a factor \( 1/\sqrt{V} \), \( P(p) \) is independent of \( V \). In the numerical analysis presented in this paper, the probability distributions of photoionization will be calculated from Eq. (10) where \( A_{\lambda_1, \lambda_2}(p) \) is the exact probability amplitude in the RSFA [Eq. (5)]. The saddle-point analysis and, in particular, Eq. (8) will be only used to interpret those results.

### 2.1. Spiral of life: Approximate solution to the saddle-point equation

The azimuthal angle \( \varphi \), i.e., the angle in the plane perpendicular to the laser pulse propagation direction, can be defined in terms of the vector potential \( A(\phi) \). Namely,

\[
\varphi = \text{arg}(eA(\phi) \cdot \epsilon_1 + ieA(\phi) \cdot \epsilon_2), \tag{11}
\]

where \( \text{arg}(z) \in [0, 2\pi] \) is the argument of the complex number \( z \). Conversely, it is possible to define the vector potential as a function of the azimuthal angle, i.e., \( A(\varphi) \equiv A(\phi) \). Note, however, that the relation between \( \phi \) and \( \varphi \) is not one-to-one and two or more phases can lead to the same values of \( \varphi \). This, in turn, means that the curve \( A(\varphi) \) can intersect itself at certain azimuthal angles for laser pulses comprising of more than one field oscillation [9, 10]. In order
to proceed, we also introduce the time-dependent ponderomotive energy of an electron in the laser field,

$$U_p(\phi) = \frac{e^2 A^2(\phi)}{2m_e} = U_p(\varphi). \quad (12)$$

In the same way as the vector potential, the curve $U_p(\varphi)$ can intersect itself.

As it was mentioned before, the phases $\phi_s$ which possess imaginary parts close to zero are the ones that contribute the most to the sum in (8). Therefore, as a first approximation, we can look for a real phase $\phi_p$ which satisfies Eq. (9),

$$G'(\phi_p) = \frac{p^0 - q^0}{k^0} - \frac{eA(\phi_p) \cdot p}{k \cdot p} + \frac{e^2 A^2(\phi_p)}{2k \cdot p} = 0. \quad (13)$$

In order to solve this equation, we define the parallel and perpendicular components of the asymptotic momentum $p$ with respect to the propagation direction of the laser field as $p^\parallel = p \cdot n$ and $p^\perp = p - p^\parallel n$, respectively. Assuming that \cite{9, 10}

$$p^\perp = eA(\phi_p), \quad (14)$$

we find out that the saddle-point equation (13) requires

$$p^\parallel = p^0 - q^0 = \frac{e^2 A^2(\phi_p)}{2q^0}. \quad (15)$$

Hence, we conclude that the photoelectrons with momentum $p(\phi_p)$ given by

$$p(\phi_p) = eA(\phi_p) + \frac{e^2 A^2(\phi_p)}{2q^0} n \quad (16)$$

are detected with maximum probability. This equation defines the spiral of life in ionization (also called the momentum spiral). However, it is worth mentioning that the four-momentum associated with $p(\phi_p)$ is not exactly on the mass shell, as $p \cdot p = E^2_0/c^2$. The ground state energy of a hydrogen-like ion depends on the atomic number $Z$ according to $E_0 = m_e c^2 \sqrt{1 - Z^2 \alpha^2}$ \cite{14}.

So, if we have $Z \alpha \ll 1$ (light ions) then $E_0 \approx m_e c^2$ and $p \cdot p \approx m_e^2 c^2$. This reflects the approximate nature of our treatment \cite{9, 10}.

Let us now go back to Eq. (14). This equation implies that the azimuthal angle of electron detection, $\varphi_p$, is given by

$$\varphi_p = \text{arg}(eA(\phi_p) \cdot \varepsilon_1 + ieA(\phi_p) \cdot \varepsilon_2). \quad (17)$$

This means that, for a given $\varphi_p$, one or more phases $\phi_p$ are obtained. In this case, Eq. (15) determines the photoelectron energy at which the probability of photoionization is maximum. Namely,

$$E_p = E_0 + \frac{e^2 A^2(\phi_p)}{2E_0}, \quad (18)$$

which for a light ion takes the form $E_p - m_e c^2 \approx U_p(\phi_p)$, where $U_p(\phi)$ is the ponderomotive energy given in (12). Thus, if we interpret $\phi_p$ as the phase of the electron birth (real part of the saddle point), we can conclude that the final kinetic energy of the photoelectron is equal to its ponderomotive energy at the moment of appearance in the continuum. This will be analyzed from a classical perspective in Sec. 4.
Finally, from Eq. (15) and for light ions, the polar angle of electron detection, $\theta_p$, is given by

$$\cos \theta_p = \frac{p^0}{|p|} \approx \sqrt{\frac{E_p - m_e c^2}{E_p + m_e c^2}}. \quad (19)$$

Note that, for small kinetic energies ($E_p \approx m_e c^2$) the polar angle is close to $\pi/2$ and it decreases with increasing $E_p$. This is a consequence of the radiation pressure exerted by the laser field on the photoelectrons [5, 15].

In this Section we have shown that the three-dimensional spiral of life (16) determines the maxima of the probability distribution of high-energy photoelectrons. At the given azimuthal angle, the phase $\phi_p$ predicts the kinetic energy for which the electrons will be observed with the largest probability. Furthermore, from Eq. (19), the polar angle of detection can be calculated directly. Another interesting feature is that the three-dimensional spiral can also intersect itself. Such self-intersections are characterized by two different phases $\phi_p$ (i.e., two contributing saddle-points) leading to the same asymptotic momentum. Therefore, the self-intersections of the curve $p(\phi_p)$ determine the directions in space and the kinetic energies at which the interference-dominated patterns can be observed. Even though the momentum spiral is a very powerful construction, its predictions are valid provided that the ions are light ($Z \alpha \ll 1$) and the photoelectrons are fast [i.e., if the relation (2) is also fulfilled]. In the next Section, the last statement will be quantified.

3. Numerical calculations

In the following, we consider a hydrogen-like ion with $Z = 2$ (He$^+$) in the presence of a circularly-polarized, intense, and short laser pulse propagating along the $z$-direction ($\mathbf{n} = e_z$). The pulse comprises of five field oscillations ($N_{osc} = 5$) within a $\sin^2$ envelope and has a carrier-envelope phase $\chi = \pi$. The explicit form of the corresponding shape functions can be found in Eqs. (40) to (44) of Ref. [5]. For our numerical analysis, we choose the carrier frequency $\omega_L = N_{osc}\omega$ of a Ti-Sapphire laser field ($\omega_L = 1.5498$ eV) and the time-averaged intensity $I = 10^{17}$ W/cm$^2$.

3.1. Probability of ionization along the spiral

As it was mentioned above, the three-dimensional spiral of life determines the momenta for which the probability of ionization is maximal. Therefore, in this Section, the behavior of the saddle points and the probability distributions along the spiral are going to be analyzed.

In the left panel of Fig. 1 we present the three-dimensional spiral $p(\phi_p)$ [Eq. (16)] for the laser pulse described above. This curve is obtained by changing the phase $\phi = \phi_p$ from 0 to $2\pi$. Here, the blue line relates to the ramp-up portion of the pulse ($0 \leq \phi < \pi$) and the cyan line corresponds to its ramp-down portion ($\pi \leq \phi < 2\pi$). As it can be seen, the momentum spiral has several self-intersections which, according to our previous analysis, determine the phases for which interference-dominated patterns can be observed. In the right panel of the same figure, we present the one-dimensional spin-independent probability distribution $P(p)$ [Eq. (10)] calculated along the path followed by the spiral of life, i.e., we show the function $P(\phi) \equiv P(p(\phi))$ for $\phi \in [0, 2\pi]$. This distribution acquires very small values at $\phi < 0.3\pi$ and $\phi > 1.7\pi$, i.e., when the laser field strength is small. Furthermore, in the region from $\phi \approx 0.6\pi$ to $1.4\pi$, $P(\phi)$ presents long plateaux between well-defined interference-dominated structures. The latter are located at phases which coincide with the self-intersections of the spiral. (Note that, in a coherent process, constructive interference would lead to probability distributions with peaks which are four times larger than the interference-free surroundings, which is actually the case.) Also, a soft valley in $P(\phi)$, which occurs when the field strength is maximum, is evidenced. Such behavior could be a signature of the so-called stabilization against ionization (see, e.g., Ref. [16, 17, 18, 19]), as the valley becomes deeper with increasing the laser field intensity.
Figure 1. Three-dimensional spiral of life $p(\phi)$ [Eq. (16)] (left panel) and the one-dimensional spin-independent probability distribution along such path, $P(p(\phi))$ [see, Eq. (10)] (right panel). The laser pulse is described in the text and the parameters used here are $I = 10^{17} \text{ W/cm}^2$, $N_{\text{osc}} = 5$, and $\omega_L = 1.5498 \text{ eV}$. Both curves are obtained by changing the phase $\phi = \phi_p$ from 0 to $2\pi$. For visual purposes, in the left panel the blue line relates to the ramp-up portion of the pulse and the cyan line to its ramp-down portion.

Now, in order to understand better the general features of the $P(\phi)$ curve, we analyze the saddle points along the spiral. To this end, we calculate such points from Eq. (9) as a function of the phase, i.e., we are interested in the quantity $\phi_s(\phi)$. In the upper-left panel of Fig. 2 we show the imaginary part of the saddle point which contributes the most to the probability amplitude. It can be seen that $\text{Im} \phi_s(\phi)$ is small at the middle of the pulse (i.e., in the region where the laser field strength is high). In the upper-right panel, we show the difference between the real part of the saddle point ($\text{Re} \phi_s$) and $\phi_p = \phi$. Such difference is negligible in the mid-portion of the pulse but its absolute value increases at the beginning and at the end of the pulse. Hence, we conclude that $\phi_s \approx \phi_p$ is a good approximation provided that the phase $\phi$ is not close to zero or to $2\pi$, i.e., when the strength of the pulse is large enough. In order to quantify this statement, in the lower-right panel of the same figure we present the time-dependent ponderomotive energy $U_p(\phi)$ [Eq. (12)] divided by the ionization potential, $I_p = m_e c^2 - E_0$. Even though $U_p(\phi)$ acquires very small values at the beginning and at the end of the pulse, it increases rapidly and can be larger than $10^2 I_p$ at $\phi = \pi$. By comparing this curve with the upper panels, we conclude that our approximation $\phi_s \approx \phi_p$ holds (i.e., $\text{Im} \phi_s \approx 0$ and $\text{Re} \phi_s \approx \phi_p$) when $U_p(\phi) > 10 I_p$. Therefore, assuming that Eq. (18) is applicable in high-energy ionization, it can be said that the predictions arising from the three-dimensional spiral of life are valid provided that the photoelectron kinetic energies are larger than ten times the ionization potential (for the He$^+$ ion this value is around 0.54 keV), given the laser field parameters considered here. Such condition agrees with the range of applicability of the RSFA [see, Eq. (2)]. Finally, in the lower-left panel of Fig. 2 we present the imaginary part of the function $G(\phi_s)$ for the two most contributing saddle points. Note that, while the blue curve decreases in the middle part of the interval, the red one is considerably larger, except at well defined phases. Their points of contact coincide with the self-intersections of the three-dimensional spiral and appear at the exact same phases where the interference-dominated structures are located (see, the right panel of Fig. 1). Now, by comparing the behavior of $\text{Im} G(\phi_s)$ (blue curve) and the one-dimensional probability distribution shown in Fig. 1, it can be seen that the phase intervals where the former increases coincide with the regions where $P(\phi)$ is small. This is in agreement with the analysis which follows from the saddle-point treatment of Eq. (5).
Figure 2. Behavior of the most relevant saddle points contributing to the probability amplitude of ionization along the three-dimensional spiral of life \( p(\phi) \) [Eq. (16)]. The laser field parameters are the same as in Fig. 1. In the upper panels we present the imaginary part of \( \phi_s \) as a function of \( \phi \) (left) and the difference \( \phi - \text{Re} \phi_s \) (right). The time-dependent ponderomotive curve \( U_p(\phi) \) [Eq. (12)] in terms of the ionization potential, \( I_p = m_e c^2 - E_0 \), is also shown (lower-right panel). Finally, the function \( \text{Im} G(\phi_s) \) [see, Eq. (6)] evaluated at the two most contributing saddle points is presented (lower-left panel).

4. Spiral of life in classical electrodynamics

It is the aim of this Section to study the dynamics of photoelectrons in the laser field from the perspective of a purely classical theory. As we will show, the Newton-Lorentz equations determine a series of invariant quantities of motion which, in turn, lead to similar expressions as the ones obtained from the approximated saddle-point analysis (Sec. 2.1). Even though the results presented here are well-known (see, for instance, Refs. [20, 21, 22]), we will reformulate them in the spirit of this paper.

Let us start by considering a free electron traveling with velocity \( \mathbf{v} \) and momentum \( \mathbf{p} \) in the presence of a laser pulse. The latter is characterized by the oscillating electric and magnetic fields \( \mathbf{E}(x,t) \) and \( \mathbf{B}(x,t) \), respectively, and it propagates along the direction \( \mathbf{n} \). For transverse waves, such as light, the relation \( \mathbf{n} \cdot \mathbf{E}(\phi) = \mathbf{n} \times \mathbf{B}(\phi) = 0 \) is always fulfilled. The classical equation of motion, which governs the dynamics of the electron, is

\[
\frac{d\mathbf{p}}{dt} = e\mathbf{E}(x,t) + ev \times \mathbf{B}(x,t). \tag{20}
\]

Now, we assume that the electromagnetic field has the plane wave front, i.e., it depends on the phase, \( \phi = \omega (t - \mathbf{n} \cdot \mathbf{x} / c) \). In what follows, we also assume that the field vanishes for \( \phi \leq 0 \) and \( \phi \geq 2\pi \). We choose a gauge in which the vector potential \( \mathbf{A} \) depends only on \( \phi \). Namely, we define it as

\[
\mathbf{A}(\phi) = -\frac{1}{\omega} \int_0^\phi \mathbf{E}(\phi')d\phi'. \tag{21}
\]

Note that this quantity fulfills the conditions \( \mathbf{A}(0) = \mathbf{A}(2\pi) = \mathbf{0} \). Furthermore, as the electric and magnetic fields obey the relation \( \mathbf{B}(\phi) = \mathbf{n} \times \mathbf{E}(\phi)/c \), or, \( \mathbf{E}(\phi) = c\mathbf{B}(\phi) \times \mathbf{n} \), the vector...
potential can also be written in terms of $B(\phi)$,

$$A(\phi) = -\frac{c}{\omega} \int_0^\phi d\phi' B(\phi') \times n.$$  (22)

In the following, we assume that the particle four-momentum is on the effective mass shell, i.e., $p \cdot p = (p^0)^2 - p^2 = (m_{\text{eff}} c)^2$. This implies that the velocity and momentum of the electron are related according to $v = c \frac{p}{p^0}$. With this in mind, we rewrite the magnetic part of the Lorentz force exerted by the laser field as

$$v \times B(\phi) = \frac{1}{p^0} p \times [n \times E(\phi)] \equiv \frac{1}{p^0} \left( [p \cdot E(\phi)] n - (p \cdot n) E(\phi) \right).$$  (23)

Hence, the Newton-Lorentz equation of motion takes the form

$$\frac{dp}{d\phi} \equiv \frac{dp^\perp}{d\phi} + \frac{dp^\parallel}{d\phi} n = \frac{1}{\omega} e E(\phi) + \frac{1}{\omega(p^0 - p^\parallel)} \left[ e E(\phi) \cdot p^\perp \right] n,$$  (24)

where $\phi$ is considered the independent variable. This is the starting point of our further analysis.

4.1. Invariants of motion

As presented above, the Newton-Lorentz equations form a system of three first-order differential equations and, therefore, there should be at most three independent invariants of motion. In this Section, we will analyze the properties of (24), which lead to the determination of those invariants. Let us start by examining the parallel component of the momentum, $p^\parallel = p \cdot n$. It follows directly from Eq. (24) that

$$\frac{dp^\parallel}{d\phi} = \frac{dp}{d\phi} \cdot n = \frac{e E(\phi) \cdot p^\perp}{\omega(p^0 - p^\parallel)}.$$  (25)

Furthermore, as the derivative of $p^0$ with respect to the phase $\phi$ is given by

$$\frac{dp^0}{d\phi} = \frac{p}{p^0} \cdot \frac{dp}{d\phi} = \frac{e E(\phi) \cdot p^\perp}{\omega(p^0 - p^\parallel)},$$  (26)

and, according to Eq. (25), we obtain

$$\frac{d}{d\phi} (p^0 - p^\parallel) = 0.$$  (27)

This leads to the conclusion that the quantity $p^0 - p^\parallel$ does not change during the motion of the electron in the laser field. Therefore, the first invariant, denoted as $I_1$, is

$$I_1 = p^0 - p^\parallel.$$  (28)

Let us now go back to Eq. (24). From there, it is evident that the first derivative of the perpendicular component of momentum with respect to $\phi$ is given by

$$\frac{dp^\perp}{d\phi} = \frac{e E(\phi)}{\omega},$$  (29)

which, in terms of the vector potential, reads

$$\frac{d}{d\phi} [p^\perp + e A(\phi)] = 0.$$  (30)
Thus, the second invariant of motion is defined as
\[ I_2 = p^\perp + eA(\phi) = p^\perp - \frac{e}{\omega} \int_0^\phi d\phi' E(\phi'). \] (31)

This quantity is the perpendicular part of the canonical momentum \( p + eA(\phi) \).

Finally, the next invariant is obtained from the fact that the relation
\[ \frac{d}{d\phi} \left[ \frac{1}{2} (p^\perp)^2 - p^\parallel (p^0 - p^\parallel) \right] = 0 \] always holds. Therefore, we define \( I_3 \) as
\[ I_3 = p^\parallel (p^0 - p^\parallel) - \frac{1}{2} (p^\perp)^2. \] (33)

Note, however, that \( I_2 \) represents two independent quantities and, as a consequence, some of the invariants should be related. This is indeed the case, as one can check that \( I_1^2 + 2I_3 = m_e^2 c^2 \). Hence, \( I_1 \) and \( I_3 \) define the effective mass of the electron.

4.2. Physical consequences arising from the invariants of motion

The invariants of motion defined in the previous Section can be used to understand the dynamics of a charged particle in the laser field, at least from a classical perspective. For instance, starting from Eq. (31) and remembering that the vector potential vanishes at the beginning and at the end of the pulse, we can conclude that the initial and final perpendicular components of the momentum of a free electron are the same, i.e., \( p^\perp(\phi = 0) = p^\perp(\phi = 2\pi) \). Furthermore, since \( p^\parallel - p^\parallel(\phi = 0) = p^\parallel(\phi = 2\pi) \) is another invariant of motion, we also obtain \( p^\parallel(\phi = 0) = p^\parallel(\phi = 2\pi) \). This actually means that a free electron will not be accelerated or decelerated by the plane-wave-fronted pulse, as its initial and final four-momenta are the same. Such observation is in agreement with the Lawson-Woodward (no acceleration) theorem [23, 24].

Now, let us assume that for some phase \( \phi_s \in [0, 2\pi] \) an electron is created with initial momentum \( p(\phi_s) = 0 \). Then, after the pulse is over \( (\phi = 2\pi) \), the invariant \( I_2 \) implies that
\[ p^\perp(\phi = 2\pi) = eA(\phi_s). \] (34)

On the other hand, \( I_1 \) and \( I_3 \) determine \( p^\parallel \),
\[ p^\parallel(\phi = 2\pi) = \frac{(p^\perp)^2}{2m_e c} = \frac{e^2 A^2(\phi_s)}{2m_e c} \] (35)

Moreover, at the end of the pulse, the zeroth-component of the four-momentum is given by
\[ p^0(\phi = 2\pi) = m_e c + p^\parallel(\phi = 2\pi) = m_e c + \frac{e^2 A^2(\phi_s)}{2m_e c}. \] (36)

As we are considering a purely classical treatment, we have that \( m_e = m_e \).

Let us now compare the classical results obtained in this Section [Eqs. (34), (35), and (36)] with the set of equations derived from the analytical approximation of the saddle points [Eqs. (14), (15), and (18)]. In doing so, it is important to remember that the real part of the saddle points are related to the time of birth of the electron in the continuum, i.e., those are the phases for which an electron is created with zero momentum. First, one can see that Eq. (14) is the exact same as its classical counterpart (34), with the ‘phase of birth’ \( \phi_s \) replaced by the approximated saddle point \( \phi_p \). Moreover, Eqs. (15) and (18) differ from (35) and (36) only by the definition of the effective mass (while in the classical treatment \( m_{e} = m_{e} \), in the saddle-point approximation \( m_{e} = q^0 / c \equiv E_0 / c^2 \)). We, therefore, conclude that the three-dimensional spiral of life presented in Sec. 2.1 is recovered from classical considerations provided that \( q^0 / c \approx m_e \), i.e., for light ions.
5. Conclusions
We have shown that the three-dimensional spiral of life, which arises from an approximate solution to the saddle-point equation, is a very powerful tool in understanding the high-energy photoionization. It determines the possible electron momenta for which the probability of ionization is non-negligible. Also, the shape of the spiral can be used to predict the formation (or absence) of interference-dominated structures by locating its self-intersections. Finally, we demonstrated that this three-dimensional curve is also obtained from a purely classical theory, provided that the ions are light.

Acknowledgments
This work is supported by the National Science Centre (Poland) under Grant No. 2014/15/B/ST2/02203.

References
[1] https://eli-laser.eu
[2] www.xcels.iapras.ru
[3] Reiss H R 1990 Phys. Rev. A 42 1476
[4] Reiss H R 1990 J. Opt. Soc. Am. B 7 574
[5] Krajewska K and Kamiński J Z 2016 Phys. Rev. A 94 013402
[6] Cajaio Vélez F, Kamiński J Z and Krajewska K 2017 J. Phys. B: At. Mol. Opt. Phys. https://doi.org/10.1088/1361-6455/aaa056
[7] Cajaio Vélez F, Kamiński J Z and Krajewska K 2017 J. Phys.: Conf. Ser. 826 012010
[8] Krajewska K, Cajaio Vélez F and Kamiński J Z 2017 Proc. of SPIE 10241 102411J
[9] Kamiński J Z, Cajaio Vélez F and Krajewska K 2017 Laser Phys. Lett. 14 075301
[10] Krajewska K, Cajaio Vélez F and Kamiński J Z 2017 EPL 119 13001
[11] Wolkom D M 1935 Z. Phys. 94 250
[12] Krajewska K and Kamiński J Z 2012 Phys. Rev. A 85 062102
[13] Krajewska K and Kamiński J Z 2012 Phys. Rev. A 86 052104
[14] Bjorken J D and Drell S D 1964 Relativistic Quantum Mechanics (New York: McGraw-Hill)
[15] Krajewska K and Kamiński J Z 2015 Phys. Rev. A 92 043419
[16] Gavrila M and Kamiński J Z 1984 Phys. Rev. Lett. 52 613
[17] Fedorov M V and Movsesian A M 1988 J. Phys. B 21 L155
[18] Popov A M, Tikhonova O V and Volkova E A 2003 J. Phys. B 36 R125
[19] Bogatskaya A V and Popov A M 2015 Laser Phys. Lett. 12 043303
[20] Landau L D and Lifshitz E M 1975 The Classical Theory of Field (New York: Pergamon)
[21] Ortner J and Rylyuk V M 2000 Phys. Rev. A 61 033403
[22] Wooten R E and Macek J H 2004 Am. J. Phys. 72 998
[23] Woodward P M 1946 J. Inst. Electr. Eng., Part III A 93 1554
[24] Lawson J D 1979 IEEE Trans. Nucl. Sci. 26 4217