Non-existence of 6-dimensional pseudomanifolds with complementarity

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Abstract. In a previous paper ([10]) the second author showed that if \( M \) is a pseudomanifold with complementarity other than the 6-vertex real projective plane and the 9-vertex complex projective plane, then \( M \) must have dimension \( \geq 6 \), and in case of equality - \( M \) must have exactly 12 vertices. In this paper we prove that such a 6-dimensional pseudomanifold does not exist. On the way to proving our main result we also prove that all combinatorial triangulations of the 4-sphere with at most 10 vertices are combinatorial 4-spheres.

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1 Introduction and results

Recall that a simplicial complex is a collection of non-empty finite sets such that every non-empty subset of an element is also an element. For \( i \geq 0 \), the elements of size \( i + 1 \) are called the \( i \)-simplices or \( i \)-faces of the complex. The vertex-set \( V(K) \) of \( K \) is by definition the union of all the faces in \( K \). We identify the 0-faces with the vertices. All the simplicial complexes considered in this paper are with finite vertex-set. For a simplicial complex \( K \), the maximum of \( k \) such that \( K \) has a \( k \)-face is called the dimension of \( K \). An 1-dimensional simplicial complex is also called a graph.

If \( K, L \) are two simplicial complexes, then a simplicial isomorphism from \( K \) to \( L \) is a bijection \( \pi : V(K) \to V(L) \) such that for \( \sigma \subseteq V(K) \), \( \sigma \) is a face of \( K \) if and only if \( \pi(\sigma) \) is a face of \( L \). The complexes \( K, L \) are called (simplicially) isomorphic when such an isomorphism exists. We identify two simplicial complexes if they are isomorphic.

A simplicial complex \( K \) is usually thought of as a prescription for constructing a topological space (polyhedron), called the geometric carrier of \( K \) and denoted by \( |K| \), by pasting together geometric simplices. We say that a simplicial complex \( K \) triangulates a topological space \( X \) (or \( K \) is a triangulation of \( X \)) if \( X \) is homeomorphic to \( |K| \). A simplicial complex \( K \) is said to be connected (respectively, simply connected) if \( |K| \) is connected (respectively, simply connected). Similarly, \( K \) is said to be contractible if \( |K| \) is contractible.

If \( \sigma \) is a face of a simplicial complex \( K \) then the link of \( \sigma \) in \( K \), denoted by \( Lk_K(\sigma) \) (or simply by \( Lk(\sigma) \)), is by definition the complex whose faces are the faces \( \tau \) of \( K \) such that \( \tau \) is disjoint from \( \sigma \) and \( \sigma \cup \tau \) is a face of \( K \).

If the number of \( i \)-simplices of a \( d \)-dimensional simplicial complex \( K \) is \( f_i(K) \) (\( 0 \leq i \leq d \)), then the number \( \chi(K) := \sum_{i=0}^d (-1)^i f_i(K) \) is called the Euler characteristic of \( K \). A simplicial complex \( K \) is called \( k \)-neighbourly if \( f_{k-1}(K) = \binom{f_0(K)}{k} \).
A simplicial complex $K$ is called pure if all the maximal faces of $K$ have the same dimension. A maximal face in a pure simplicial complex is also called a facet. With each $d$-dimensional pure simplicial complex $K$ we associate a graph $\Lambda(K)$ whose vertices are the facets of $K$ and two such vertices are adjacent in $\Lambda(K)$ if and only if the corresponding facets intersect in a $(d - 1)$-face. If $\Lambda(K)$ is connected then $K$ is called strongly connected.

A $d$-dimensional pure simplicial complex $K$ is called a weak pseudomanifold if each $(d - 1)$-face is contained in exactly two $d$-faces of $K$. A strongly connected weak pseudomanifold is called a pseudomanifold. A $d$-dimensional pure simplicial complex $K$ is called a weak pseudomanifold with boundary if each $(d - 1)$-face is contained in one or two $d$-faces of $K$ and there exists a $(d - 1)$-face of $K$ which is in only one $d$-face of $K$.

By a subdivision of a simplicial complex $K$ we mean a simplicial complex $K'$ together with a homeomorphism from $|K'|$ onto $|K|$ which is facewise linear. Two simplicial complexes $K$ and $L$ are called combinatorially equivalent (denoted by $K \approx L$) if they have isomorphic subdivisions. So, $K \approx L$ if and only if $|K|$ and $|L|$ are pl homeomorphic ([15]).

For a set $V$ with $d + 2$ elements, let $S$ be the simplicial complex whose faces are all the non-empty proper subsets of $V$. Then $S$ triangulates the $d$-sphere. This complex is called the standard $d$-sphere and is denoted by $S^d_d(V)$ (or simply by $S^d_d$). A polyhedron is called a pl $d$-sphere if it is pl homeomorphic to $S^d_d$. A simplicial complex $X$ is called a combinatorial $d$-sphere if it is combinatorially equivalent to $S^d_d$. So, $X$ is a combinatorial $d$-sphere if and only if $|X|$ is a pl $d$-sphere.

A simplicial complex $K$ is called a combinatorial $d$-manifold if the link of each vertex is a combinatorial $(d - 1)$-sphere. So, $K$ is a combinatorial $d$-manifold if and only if $|K|$ is a closed pl $d$-manifold ([15]). If $K$ triangulates $X$ and $K$ is a combinatorial manifold then $K$ is called a combinatorial triangulation of $X$. A combinatorial manifold is automatically a weak pseudomanifold. Further, a connected combinatorial manifold is a pseudomanifold.

If $M$ is a triangulation of a 2-manifold then the link of a vertex is a circle and hence $M$ is combinatorial manifold. Again, the link of a vertex in a triangulation of a 3-manifold is a triangulation of the 2-sphere and all triangulations of the 2-sphere are combinatorial 2-spheres. So, any triangulation of a 3-manifold is a combinatorial triangulation.

Eells and Kuiper defined a manifold like a projective plane to be a cohomology projective plane over reals, complex numbers, quaternions or Cayley numbers. It is well known that the projective planes over reals and complex numbers are the only manifolds like projective planes of dimensions 2 and 4, respectively. In [7], Brehm and Kühnel proved:

**Proposition 1.** Let $M$ be an $n$-vertex combinatorial $d$-manifold ($d > 0$).

(a) If $n < 3[d/2] + 3$ then $M$ is a combinatorial $d$-sphere.

(b) If $n = 3d/2 + 3$ and $M$ is not a combinatorial $d$-sphere then $d = 2, 4, 8$ or 16 and $|M|$ must be a ‘manifold like a projective plane’.

Existence or otherwise of more than one smooth structures on $S^4$ is a long standing open problem. In view of the isomorphism between the group of smooth 4-spheres and the group of pl 4-spheres (see [12, page 201]), this problem is equivalent to the existence problem of a combinatorial triangulation of $S^4$ which is not a combinatorial sphere. By Proposition 1, such a triangulation of $S^4$ must have at least ten vertices. Here we prove:

**Theorem 1.** If $M$ is a 10-vertex combinatorial triangulation of $S^4$ then $M$ is a combinatorial 4-sphere.

This result is an immediate consequence of Lemma 3.4 which we need to prove our main result (Theorem 2).
**Definition 1.** A simplicial complex $K$ is said to satisfy *complementarity* if exactly one of each complementary pair of non-empty subsets of $V(K)$ is a face of $K$. A simplicial complex satisfying complementarity is said to be a *complementary* simplicial complex.

In [2], Arnoux and Marin proved the following:

**Proposition 2.** If $M$ is a combinatorial manifold as in part (b) of Proposition 1 then $M$ satisfies complementarity.

In [9], the second author proved the following converse:

**Proposition 3.** If an $n$-vertex combinatorial $d$-manifold ($d > 0$) satisfies complementarity then $d = 2, 4, 8$ or $16$ and $n = 3d/2 + 3$.

It is easy to see that complementary simplicial complexes are plentiful. Indeed, let $A$ be a family of non-empty subsets of a finite set $X$ such that (a) $A_1, A_2 \in A \Rightarrow A_1 \cup A_2 \neq X$, and (b) $A$ is maximal (w.r.t. set-inclusion) subject to (a). Then $A$ is a complementary simplicial complex. In contrast, complementary (weak) pseudomanifolds are very hard to come by.

**Example 1.** Here is a 7-vertex 3-dimensional complementary weak pseudomanifold with boundary. Its vertex set is $\mathbb{Z}_7$. The facets are $\{i, i + 3, i + 5, i + 6\}$, $i \in \mathbb{Z}_7$. Note that in this example each edge is in exactly two facets and any two facets have exactly one edge in common.

It is well known (see [4, 14]) that there is a unique 6-vertex combinatorial triangulation of $\mathbb{R}P^2$. It is also known (see [3, 5]) that there is a unique 9-vertex combinatorial triangulation of $\mathbb{C}P^2$. In [8], Brehm and Kühnel constructed three 15-vertex combinatorial 8-manifolds which are not combinatorial spheres. By Proposition 2, all these combinatorial manifolds satisfy complementarity. In fact, all the known complementary weak pseudomanifolds are combinatorial manifolds. In [10], the second author proved the following:

**Proposition 4.** Let $M$ be an $n$-vertex $d$-dimensional ($d > 0$) complementary pseudomanifold. If $n \leq d + 6$ or $d \leq 6$ then $d = 2, 4$ or $6$ and $n = 3d/2 + 3$. Moreover, if $d \leq 5$ then $M$ is a combinatorial manifold.

The proofs in [10] show that Proposition 4 remains true even if we assume $M$ is a complementary weak pseudomanifold. In view of this result, we make:

**Conjecture 1.** If an $n$-vertex $d$-dimensional complementary weak pseudomanifold exists then $d$ is even and $n = 3d/2 + 3$.

**Conjecture 2.** If $M$ is a $d$-dimensional complementary weak pseudomanifold then its Euler characteristic $\chi(M)$ is given by:

$$
\chi(M) = \begin{cases} 
1 & \text{if } d \equiv 2 \pmod{4} \\
3 & \text{if } d \equiv 0 \pmod{4}.
\end{cases}
$$

**Remark 1.** If $M$ is any $n$-vertex complementary simplicial complex then the total number $2^{n-1} - 1$ of its faces is odd. Hence $\chi(M)$ is odd. If, further, $n$ is even then it has $2^{n-2}$ even dimensional faces and $2^{n-2} - 1$ odd dimensional faces, and hence $\chi(M) = 1$ in this case.
Remark 2. The referee suggested the following short proof of Proposition 3. Let $M$ be an $n$-vertex complementary combinatorial $d$-manifold. By Remark 1, $|M|$ is not a sphere. Since $M$ has no $(d+1)$-simplex, complementarity implies that $M$ is $(n-d-2)$-neighbourly and hence $|M|$ is $(n-d-4)$-connected. This implies that $n-d-4 \leq (d-1)/2$. Proposition 3 now follows from Proposition 1.

In view of Proposition 4, $d = 6, n = 12$ are the smallest parameters for which the existence of an $n$-vertex $d$-dimensional complementary pseudomanifold was an open problem. In this article we prove:

Theorem 2. There does not exist a $12$-vertex $6$-dimensional weak pseudomanifold with complementarity.

2 Preliminaries and Definitions.

Let $K, L$ be two simplicial complexes with disjoint vertex sets. Then their join $K \ast L$ is the simplicial complex whose faces are those of $K$ and of $L$, and the unions of faces of $K$ with faces of $L$. Clearly, if $K$ and $L$ are (weak) pseudomanifolds, then so is $K \ast L$.

For $n \geq 3$, the combinatorial 1-sphere (circle) with $n$ vertices is the unique 1-dimensional $n$-vertex pseudomanifold and is denoted by $S_1^n$.

If $\sigma$ is an $i$-face in a $d$-dimensional weak pseudomanifold $K$ then $\text{Lk}_K(\sigma)$ is a $(d-i-1)$-dimensional weak pseudomanifold. The number of vertices in $\text{Lk}(\sigma)$ is called the degree of $\sigma$ and is denoted by $\text{deg}_K(\sigma)$ (or simply by $\text{deg}(\sigma)$).

We need the next two results later.

Proposition 5 (Brehm and Kühnel [7]). Let $M$ be an $n$-vertex combinatorial $d$-manifold ($d > 0$). If $n \leq 2d+3-i$ for some $i < d/2$ then $|M|$ is $i$-connected in the sense of homotopy.

It follows from the next example that this result is best possible for $i = 0$.

Example 2. For $d \geq 2$, let $K_{2d+3}^d$ be the $d$-dimensional simplicial complex whose vertex set is the vertex set of the circle $S_{2d+3}^1$ and the facets are the sets of $d+1$ vertices obtained by deleting an interior vertex from the $(d+2)$-paths in the circle. The simplicial complex $K_{2d+3}^d$ is a combinatorial $d$-manifold (see [13, 14]) and $|K_{2d+3}^d|$ is not simply connected. The space $|K_3^d|$ is known as the 3-dimensional Klein bottle.

Proposition 6 (Altshuler and Steinberg [1]). If $M$ is a $9$-vertex combinatorial $3$-manifold then either $M$ is a combinatorial 3-sphere or $M$ is isomorphic to $K_3^3$.

A subcomplex $L$ of a simplicial complex $K$ is called an induced (or full) subcomplex of $K$ if $\sigma \in K$ and the vertices of $\sigma$ are in $L$ imply $\sigma \in L$.

Let $L \subseteq K$ be simplicial complexes. The simplicial neighbourhood of $L$ in $K$ is the subcomplex $N(L, K)$ of $K$ whose maximal simplices are those maximal simplices of $K$ which intersect $V(L)$. Clearly, $N(L, K)$ is the smallest subcomplex of $K$ such that its geometric carrier is a topological neighbourhood of $|L|$ in $|K|$. The induced subcomplex $C(L, K)$ on the vertex-set $V(K) \setminus V(L)$ is called the simplicial complement of $L$ in $K$.

Suppose $P' \subseteq P$ are polyhedra and $P = P' \cup B$, where $B$ is a pl $(k+1)$-ball. If $P' \cap B$ is a pl $k$-ball then we say that there is an elementary collapse of $P$ on $P'$. We say that $P$ collapses on $Q$ and write $P \downarrow Q$ if there exists a sequence $P = P_0, P_1, \ldots, P_n = Q$ of polyhedra such that there is an elementary collapse of $P_{i-1}$ on $P_i$ for $1 \leq i \leq n$ ([15]).
A regular neighbourhood of a polyhedron $Q$ in a pl $d$-manifold $M$ is a $d$-dimensional submanifold $N$ with boundary such that $N \setminus Q$ and $N$ is a neighbourhood of $Q$ in $M$.

Let $\tau \subset \sigma$ be two faces of a simplicial complex $K$. We say that $\tau$ is a free face of $\sigma$ if $\sigma$ is the only face of $K$ which properly contains $\tau$. (It follows that $\dim(\sigma) - \dim(\tau) = 1$ and $\sigma$ is a maximal simplex in $K$.) If $\tau$ is a free face of $\sigma$ then $K' := K \setminus \{\tau, \sigma\}$ is a simplicial complex. We say that there is an elementary collapse of $K$ on $K'$. We say $K$ collapses on $L$ and write $K \natural L$ if there exists a sequence $K = K_0, K_1, \ldots, K_n = L$ of simplicial complexes such that there is an elementary collapse of $K_{i-1}$ on $K_i$ for $1 \leq i \leq n$ (see [6]). If $L$ consists of a 0-simplex (a point) we say that $K$ is collapsible and write $K \natural 0$. Clearly, if $K \natural L$ then $|K| \natural |L|$ as polyhedra and hence $|K|$ and $|L|$ have the same homotopy type (see [15]). So, if a simplicial complex $K$ is collapsible then $|K|$ is contractible.

3 Ten-vertex four-sphere.

Lemma 3.1. Let $S$ be a combinatorial triangulation of a simply connected 4-manifold with $\chi(S) = 2$ and $\sigma$ be a facet of $S$. Let $L$ be the induced subcomplex of $S$ on $V(S) \setminus \sigma$. Then

(a) $L$ is contractible.

(b) If, further, $L$ is collapsible then $S$ is a combinatorial sphere.

Proof. Being simply connected, $|S|$ is orientable. Using Poincaré duality, one sees that any simply connected 4-manifold of Euler characteristic 2 is a homotopy sphere. So, $|S|$ is a homotopy 4-sphere.

Let $D = |S| \setminus |\sigma|$. Then, $D$ is homotopic to a point and hence contractible.

If $x \in D \setminus |L|$ then there exists a unique pair $(\alpha, \beta)$, where $\alpha \subseteq \sigma$ and $\beta \in L$ such that $x$ is in the interior of $|\alpha \cup \beta|$. So, there exist a unique pair $(y, z) \in |\alpha| \times |\beta|$ and $0 < s < 1$ such that $x = sy + (1 - s)z$. Then $H: D \times [0, 1] \rightarrow D$, given by

$$H(x, t) = \begin{cases} x & \text{if } x \in |L| \\ (1 - t)y + (1 - s + ts)z & \text{if } x = sy + (1 - s)z \notin |L|, \end{cases}$$

defines a homotopy between $D$ and $|L|$. So, $|L|$ is contractible. This proves (a).

Let $J = S^3_2(\sigma)$ and $K = S \setminus \{\sigma\}$. Let $X = |L|$ and $N = |K|$.

Then (i) $N$ is a neighbourhood of $X$ in the closed pl manifold $|S|$, (ii) $N$ is a compact pl manifold with boundary (see [15, Corollary 3.14]) and (iii) $(K, L, J)$ is a triangulation of $(N, X, \partial N)$ with $L$ a full subcomplex of $K$, $K = N(L, K)$ and $J = N(L, K) \cap C(L, K)$. Therefore by the Simplicial Neighbourhood Theorem (see [15, page 34]), $N$ is a regular neighbourhood of $X$.

Now, if $L$ is collapsible, then $N$ is a regular neighbourhood of the collapsible polyhedron $X$ and hence (see [15, page 41]) $N$ is a pl 4-ball.

Let $\sigma = 12345$. Then $B := |S^3_6(\{1, \ldots, 6\}) \setminus |\sigma||$ is a pl 4-ball. Let $\varphi: N \rightarrow B$ be a pl-homeomorphism. Let $u$ be a point in the interior of $|\sigma|$. For $x \in |\sigma|$ and $x \neq u$ there exist a unique point $y \in |\sigma| \cap N$ and $0 \leq t < 1$ such that $x = tu + (1 - t)y$. Then $\tilde{\varphi}$, given by

$$\tilde{\varphi}(x) = \begin{cases} u & \text{if } x = u \\ tu + (1 - t)\varphi(y) & \text{if } x \in |\sigma|, x \neq u \text{ and } x = tu + (1 - t)y \\ \varphi(x) & \text{otherwise}, \end{cases}$$
defines a pl-homeomorphism between $|S|$ and $|S^3_6(\{1, \ldots, 6\})|$. This proves (b). \qed
Lemma 3.2. Let $N$ be a 2-dimensional simplicial complex on at most five vertices. Suppose each edge of $N$ is contained in at least two triangles of $N$. Then $N$ contains a combinatorial 2-sphere as a subcomplex.

Proof. Clearly, $\#(V(N)) \geq 4$. In case of equality, $N$ has to be $S^2_4$. So, assume $\#(V(N)) = 5$. Define a binary relation $\sim$ on $V(N)$ by: $x \sim y$ if $V(N) \setminus \{x, y\}$ is not a triangle in $N$. The hypothesis on $N$ implies that $\sim$ is an equivalence relation with at least two equivalence classes. Since $\#(V(N)) = 5$, either there exists an equivalence class $W$ of size 4 or $V(N)$ can be written as $V(N) = V_1 \sqcup V_2$, where $V_1$ is of size 2 and $V_1$ is a union of equivalence classes. Accordingly, $S^2_4(W)$ or $S^0_2(V_1) \ast S^1_3(V_2)$ is a subcomplex of $N$. \qed

Lemma 3.3. If a 5-vertex simplicial complex $L$ is contractible then it is collapsible.

Proof. Let $f_i$ be the number of $i$-faces in $L$. Suppose $\dim(L) \leq 2$. The proof is by induction on the number $f_2$. If $f_2 = 0$ then, being contractible, $L$ is a tree and hence collapsible. So assume $f_2 > 0$ and we have the result for smaller values of $f_2$. Let $N$ be the subcomplex of $L$ consisting of the triangles of $L$ and their faces. If $N$ satisfies the hypothesis of Lemma 3.2, then by Lemma 3.2, $(N$ and hence) $L$ contains a combinatorial 2-sphere as subcomplex. Then $H_2(L) \neq 0$. This is a contradiction, since $L$ is contractible. So we may assume that some edge of $N$ is contained in a unique triangle. Then $L$ is collapsible to a contractible simplicial complex with fewer triangles and hence we are done by induction hypothesis.

Consider the case when $\dim(L) = 3$. Clearly, $f_3 \leq 5$. If $f_3 = 5$ then $L = S^3_5$ and hence not contractible. So, $f_3 \leq 4$. Then there exists a triangle which is in a unique tetrahedron. Then $L$ is collapsible to a contractible simplicial complex with one less tetrahedron. Inductively, $L$ is collapsible to a contractible 2-dimensional simplicial complex and hence $L$ is collapsible by the previous step. Finally, if $\dim(L) = 4$ then $L$ consists of one 4-face and its faces. Clearly, $L$ is collapsible in this case. \qed

Lemma 3.4. If $M$ is a 10-vertex combinatorial triangulation of a simply connected 4-manifold with $\chi(M) = 2$ then $M$ is a combinatorial 4-sphere.

Proof. Let $\sigma$ be a facet of $M$. Let $L$ be the induced subcomplex of $M$ on $V(M) \setminus \sigma$. Then, by Part (a) of Lemma 3.1, $L$ is contractible. Since $L$ has 5 vertices, by Lemma 3.3, $L$ is collapsible. Therefore, by Part (b) of Lemma 3.1, $M$ is a combinatorial sphere. \qed

Proof of Theorem 1. Follows from Lemma 3.4. \qed

4 Twelve-vertex complementary pseudomanifold.

Throughout this section, $M^6_{12}$ will denote a putative (fixed but arbitrary) 12-vertex 6-dimensional complementary weak pseudomanifold.

Lemma 4.1. $M^6_{12}$ has 12 vertices, $\binom{12}{2} = 66$ edges, $\binom{12}{3} = 220$ triangles, $\binom{12}{4} = 495$ tetrahedra, 660 4-faces, 462 5-faces and 132 facets.

Proof. Since $M^6_{12}$ is 6-dimensional, no set of $\geq 8$ vertices forms a face. Therefore, by complementarity, $M^6_{12}$ is 4-neighbourly. Since exactly one set in each of the $\frac{1}{2}\binom{12}{6}$ pairs of complementary 6-sets forms a face, it follows that the number of 5-faces is $\frac{1}{2}\binom{12}{6} = 462$. Since each 5-face is contained in two facets and each facet contains seven 5-faces, an obvious
two-way counting shows that the number of facets is \( \frac{1}{41} \binom{12}{6} = 132 \). Finally, since a set of 5 vertices forms a face if and only if the complementary set is not a facet, it follows that the number of 4-faces is \( \binom{12}{5} - 132 = 660 \).

\( \square \)

**Definition 2.** (cf. [5].) A partition of the vertex set of \( M_{12}^6 \) into three 3-faces \( A_1, A_2, A_3 \) is called an *amicable partition* if the link of each \( A_i \) is \( S^2_6(A_{i+1}) \) (addition in the suffix is modulo 3).

We have:

**Lemma 4.2.** Let \( A \) be a 3-face of \( M_{12}^6 \). Suppose the link of \( A \) is a standard sphere \( S^2_4 \). Then \( A \) belongs to a unique amicable partition of \( M_{12}^6 \).

**Proof.** Put \( A = A_1 \). Let \( A_2 \) be the vertex set of the link of \( A_1 \) and let \( A_3 \) be the set of vertices outside \( A_1 \sqcup A_2 \). Then each \( A_i \) contains 4 vertices. By Lemma 4.1, \( M_{12}^6 \) is 4-neighbourly. In particular, each \( A_i \) is a 3-face of \( M_{12}^6 \). So, to complete the proof, it is sufficient to show that the link of \( A_2 \) (respectively \( A_3 \)) is the standard sphere on \( A_3 \) (respectively \( A_1 \)).

Take any vertex \( x \in A_2 \). Then \( A_3 \cup \{x\} \) is not a face since its complement \( A_1 \cup (A_2 \setminus \{x\}) \) is a face. Thus no vertex of \( A_2 \) belongs to the link of \( A_3 \). Therefore, the vertex set of the link of \( A_3 \) is contained in \( A_1 \). Since this link has at least 4 vertices, it follows that the link of \( A_3 \) is the standard sphere on \( A_1 \). Replacing \( A_1 \) by \( A_3 \) (and hence \( A_2 \) by \( A_1, A_3 \) by \( A_2 \)) this in argument, we see that the link of \( A_2 \) is the standard sphere on \( A_3 \).

\( \square \)

**Lemma 4.3.** The link of any 4-face in \( M_{12}^6 \) is a circle.

**Proof.** If not, then the link (of some 4-face) is a disconnected regular graph of degree two on at most seven vertices. Hence the link is either \( S^1_3 \sqcup S^1_3 \) or \( S^1_3 \sqcup S^1_4 \).

**Case 1.** The link of a 4-face \( \sigma \) is \( S^1_3(V_1) \sqcup S^1_3(V_2) \). Let \( u \) be the unique vertex outside \( \sigma \sqcup V_1 \sqcup V_2 \). Consider \( \alpha_i = V_i \cup \{u\}, i = 1, 2 \). By Lemma 4.1, \( \alpha_i \) is a 3-face. Since \( \sigma \sqcup e \) is a face of \( M_{12}^6 \) for every 2-subset \( e \) of \( V_2 \), it follows by complementarity that \( \text{Lk}(\alpha_1) \) has no vertex in \( V_2 \). Thus \( V(\text{Lk}(\alpha_1)) \subseteq \sigma \). Since \( \text{Lk}(\alpha_1) \) is a 2-dimensional weak pseudomanifold, it follows that \( \text{Lk}(\alpha_1) \) has 4 or 5 vertices. Similarly, \( \text{Lk}(\alpha_2) \) has 4 or 5 vertices.

**Subcase 1.1.** \( \text{Lk}(\alpha_i) \) has 4 vertices for some \( i \), say \( i = 1 \). By Lemma 4.2, \( \alpha_1 \) belongs to a unique amicable partition \( \{\alpha_1, \beta, \gamma\} \) of \( M_{12}^6 \). Clearly, \( \beta \subseteq \sigma \). This implies that \( V(\text{Lk}(\sigma)) \subseteq V(\text{Lk}(\beta)) = \gamma \), a contradiction.

**Subcase 1.2.** \( \text{Lk}(\alpha_i) \) has 5 vertices for \( i = 1, 2 \). Thus \( V(\text{Lk}(\alpha_i)) = \sigma \). Now, since the join \( S^1_3 \ast S^0_2 \) of two standard spheres is the only 5-vertex 2-dimensional weak pseudomanifold (see [4]), it follows that, for \( i = 1, 2 \), \( \text{Lk}(\alpha_i) = S^1_3 \ast S^0_2 \) on the common vertex set \( \sigma \). Now, it is easy to see that given any two copies of \( S^1_3 \ast S^0_2 \) on a common vertex set, there is an edge of one whose complement is a triangle in the other. Thus there must exist an edge \( e \) of \( \text{Lk}(\alpha_2) \) such that \( \sigma \setminus e \) is a triangle of \( \text{Lk}(\alpha_1) \). Then the faces \( \alpha_2 \cup e \) and \( \alpha_1 \cup (\sigma \setminus e) \) of \( M_{12}^6 \) cover the vertex set, contradicting complementarity.

**Case 2.** The link of a 4-face \( \sigma \) is \( S^1_3 \sqcup S^1_2 \). Let \( \alpha \) be the vertex set of the \( S^1_3 \). By Lemma 4.1, \( \alpha \) is a face of \( M_{12}^6 \). Since all the edges of the \( S^1_3 \) occur in \( \text{Lk}(\sigma) \), by complementarity it follows that \( V(\text{Lk}(\sigma)) \subseteq \sigma \). Therefore, \( \text{Lk}(\alpha) \) is either an \( S^2_4 \) or an \( S^1_3 \ast S^0_2 \).

**Subcase 2.1.** \( \text{Lk}(\alpha) = S^2_4 \). Then, by Lemma 4.2, \( \alpha \) belongs to a unique amicable partition \( \{\alpha, \beta, \gamma\} \) of \( M_{12}^6 \). Clearly, \( \beta \subseteq \sigma \) and hence \( V(\text{Lk}(\sigma)) \subseteq V(\text{Lk}(\beta)) = \gamma \), contradiction.
Subcase 2.2. \( \text{Lk}(\alpha) = S^4_3 \setminus S^1_2 \). Thus \( V(\text{Lk}(\alpha)) = \sigma \). Let \( \text{Lk}(\sigma) = S^4_3(V_1) \cup (S^2_2(\{z, y\}) \setminus S^4_3(\{x, y\})) \). Choose one of the four vertices \( x, y, z, w \), say we choose \( x \). Consider the 3-face \( \beta = V_1 \cup \{x\} \). Since \( yz \) and \( yw \) are edges in \( \text{Lk}(\sigma) \), complementarity implies that \( z \) and \( w \) are not vertices of \( \text{Lk}(\beta) \). So, \( V(\text{Lk}(\beta)) \subseteq \sigma \cup \{y\} \). Thus \( \text{Lk}(\beta) \) is a 2-dimensional weak pseudomanifold on \( \leq 6 \) vertices. Let \( \{a, b\} \subseteq \sigma \) be the unique non-edge in \( \text{Lk}(\alpha) \). Since \( \text{Lk}(\alpha) \) contains three triangles through \( a \), complementarity implies that at least three edges through \( b \) (contained in \( \sigma \)) are missing in \( \text{Lk}(V_1) \) and hence in \( \text{Lk}(\beta) \). Since any vertex in a 2-dimensional weak pseudomanifold on \( \leq 6 \) vertices can be on at most two non-edges, it follows that \( b \) is not a vertex of \( \text{Lk}(\beta) \). Similarly, \( a \) is not a vertex of \( \text{Lk}(\beta) \). Therefore \( V(\text{Lk}(\beta)) \subseteq \sigma \cup \{y\} \setminus \{a, b\} \) and hence \( \text{Lk}(\beta) = S^2_1(\sigma \cup \{y\} \setminus \{a, b\}) \). In particular, \( \sigma \setminus \{a, b\} \) is a face of \( \text{Lk}(\beta) \). Hence \( V_1 \cup (\sigma \setminus \{a, b\}) \cup \{x\} \) is a facet of \( M^6_{12} \). Since this argument goes through with any of the four vertices of \( \alpha \) in place of \( x \), this shows that the 5-face \( V_1 \cup \sigma \setminus \{a, b\} \) is contained in (at least) four facets of \( M^6_{12} \). This is a contradiction since \( M^6_{12} \) is a weak pseudomanifold.

\textbf{Lemma 4.4.} \textit{The link of any tetrahedron in \( M^6_{12} \) is a connected combinatorial 2-manifold.}

\textbf{Proof.} Let \( L \) be the link of a 3-face \( \sigma \). Then \( L \) is a 2-dimensional weak pseudomanifold on at most 8 vertices. By Lemma 4.3, the link in \( L \) of each vertex is a circle. Hence \( L \) is a combinatorial 2-manifold. If \( L \) were disconnected, (since any 2-dimensional weak pseudomanifold has \( \geq 4 \) vertices with equality only for \( S^1_2 \) \( L \) would have to be \( S^1_2 \cup S^1_2 \). Say, \( L = S^1_2(\alpha) \cup S^1_2(\beta) \). By complementarity, \( \text{Lk}(\alpha) = S^2_1(\sigma) = \text{Lk}(\beta) \). This contradicts Lemma 4.2. \( \square \)

\textbf{Lemma 4.5.} \textit{The link of any tetrahedron in \( M^6_{12} \) is a combinatorial 2-sphere.}

\textbf{Proof.} For \( i \geq 4 \), let \( c_i \) be the number of 3-faces of degree \( i \) in \( M^6_{12} \). Counting the total number of 3-faces in \( M^6_{12} \) and the total number of pairs \((\alpha, \beta)\) where \( \alpha \subseteq \beta, \alpha \) is a 3-face and \( \beta \) is a 4-face of \( M^6_{12} \) we get (in view of Lemma 4.1):

\[ \sum c_i = 495 \quad \text{and} \quad \sum ic_i = 660 \times 5 = 3300. \quad (1) \]

Now, for any 3-face \( \alpha \) of \( M^6_{12} \), the link of \( \alpha \) is a connected combinatorial 2-manifold (by Lemma 4.4) and hence has Euler characteristic \( \leq 2 \). If this link has \( i \) vertices then it follows that it has \( \geq 2i - 4 \) triangles, i.e., \( \alpha \) is contained in at least \( 2i - 4 \) facets. Hence

\[ \sum (2i - 4)c_i \]

is a lower bound on the number of pairs \((\alpha, \gamma)\), where \( \alpha \subseteq \gamma, \alpha \) is a 3-face and \( \gamma \) is a facet of \( M^6_{12} \). But by Lemma 4.1, the number of such pairs is \( 132 \times 35 \). Therefore we get

\[ \sum (2i - 4)c_i \leq 132 \times 35 = 4620. \]

By (1), equality holds in this inequality. Therefore, equality holds throughout the above argument. Thus the link of each 3-face is a combinatorial 2-manifold of Euler characteristic 2, and hence it is a combinatorial 2-sphere. \( \square \)

\textbf{Lemma 4.6.} \textit{The link of any triangle in \( M^6_{12} \) is a connected combinatorial 3-manifold on 9 vertices.}

\textbf{Proof.} The link is a combinatorial 3-manifold by Lemma 4.5. It has 9 vertices by Lemma 4.1. It is connected since any disconnected combinatorial 3-manifold needs at least \( 5 + 5 = 10 \) vertices. \( \square \)
Lemma 4.7. If δ is a triangle in $M_{12}^6$ such that $L = \text{Lk}(\delta)$ is 2-neighbourly then $L$ does not have any induced $S^2_4$.

**Proof.** If possible let $S^2_4(X)$ be an induced sub-complex of $L$. Put $Y = V(L) \setminus X$. By complementarity, $Y$ is a 4-face of $M_{12}^6$ and $\text{Lk}(Y)$ does not contain any vertex of $X$. This implies that $\text{Lk}(Y) = S^2_4(\delta)$. Therefore, for each vertex $x$ in $\delta$, $Y \cup \delta \setminus \{x\}$ is a facet of $M_{12}^6$. Hence, by complementarity, $x \not\in \text{Lk}(X)$. Since this holds for each $x \in \delta$, $\delta \cap V(\text{Lk}(X)) = \emptyset$. So, if $\tau$ is a facet of $M_{12}^6$ containing $X$ then $\tau$ is disjoint from $\delta$. Thus $\alpha = V(M_{12}^6) \setminus \tau$ is a set of 5 vertices containing $\delta$ and, by complementarity, $\alpha$ is not a face. Thus $\alpha \setminus \delta$ is not an edge of $\text{Lk}(\delta)$. This contradicts the assumption that $\text{Lk}(\delta)$ is 2-neighbourly.

Lemma 4.8. The link of any triangle in $M_{12}^6$ is a combinatorial 3-sphere on 9 vertices.

**Proof.** By Proposition 6, there is a unique 9-vertex combinatorial 3-manifold which is not a combinatorial 3-sphere, namely the 3-dimensional Klein bottle $K^3_9$. So, in view of Lemma 4.6, it is enough to prove that $K^3_9$ can not occur as the link of a triangle of $M_{12}^6$. Now, $K^3_9$ is 2-neighbourly and has an induced subcomplex isomorphic to $S^2_4$. Indeed any 4-path in the underlying $S^2_4$ induces an $S^2_3$ in $K^3_9$. The result now follows from Lemma 4.7.

Lemma 4.9. $M_{12}^6$ is a pseudomanifold.

**Proof.** By complementarity, two facets of $M_{12}^6$ can not cover its vertex set. Hence any two of the facets have at least a triangle in common. Since by Lemma 4.6, the link of any triangle is strongly connected, it follows that $M_{12}^6$ is strongly connected and hence is a pseudomanifold.

Lemma 4.10. Let $\alpha$ be a facet of $M_{12}^6$. For $0 \leq j \leq 2$ let $e_j$ be the number of $(6 - j)$-faces meeting $\alpha$ in exactly $6 - j$ vertices. Then $e_0 = 7$, $e_1 = 51$ and $e_2 = 139$.

**Proof.** Let $\mathcal{A}$ be the set of all 3-faces of $M_{12}^6$ disjoint from $\alpha$. For $\gamma \in \mathcal{A}$, let $d_j(\gamma)$ be the number of $j$-faces in $\text{Lk}(\gamma)$, $0 \leq j \leq 2$. Clearly $\sum_{\gamma \in \mathcal{A}} d_j(\gamma)$ counts the number of pairs $(\gamma, \delta)$ where $\gamma \in \mathcal{A}$ and $\delta$ is a $(4 + j)$-face of $M_{12}^6$ containing $\gamma$. Since $\alpha$ is a facet of $M_{12}^6$, by complementarity, $V(\text{Lk}(\gamma)) \subseteq \alpha$ for $\gamma \in \mathcal{A}$. It follows that $\gamma = \delta \setminus \alpha$ for any such pair $(\delta, \gamma)$. Therefore, the number of such pairs $(\gamma, \delta)$ equals the number of $(4 + j)$-faces $\delta$ of $M_{12}^6$ meeting $\alpha$ in $j + 1$ vertices. By complementarity, this equals the number of vertex sets of size $7 - j$ meeting $\alpha$ in $6 - j$ vertices, which do not form a face of $M_{12}^6$. Out of the $5 \times \binom{7}{j + 1}$ sets of size $7 - j$ meeting $\alpha$ in $6 - j$ vertices, exactly $e_j$ are faces of $M_{12}^6$. So we get

$$e_j = 5 \times \binom{7}{j + 1} - \sum_{\gamma \in \mathcal{A}} d_j(\gamma), \quad 0 \leq j \leq 2. \quad (2)$$

Now, for each fixed $\gamma \in \mathcal{A}$, $d_j(\gamma)$ is the numbers of $j$-faces in $\text{Lk}(\gamma)$. Since $\text{Lk}(\gamma)$ is an $S^2$ (Lemma 4.5), we have

$$d_1(\gamma) = 3(d_0(\gamma) - 2), \quad d_2(\gamma) = 2(d_0(\gamma) - 2). \quad (3)$$

Adding (3) over all $\gamma \in \mathcal{A}$, and plugging the result into (2) we get $e_1 = 3e_0 + 30$ and $e_2 = 125 + 2e_0$. Since $M_{12}^6$ is a 6-dimensional weak pseudomanifold, we have $e_0 = 7$ and hence $e_1 = 51$ and $e_2 = 139$.

\[9\]
Lemma 4.11. Given any facet of $M_{12}^6$ the number of facets meeting the given facet in 3, 4, 5, 6 vertices equals 36, 58, 30 and 7 respectively.

Proof. Fix a facet $\alpha$ of $M_{12}^6$. Clearly exactly 7 facets meet $\alpha$ in 6 vertices.

By Lemma 4.10 (with $j = 2$), exactly 139 4-faces of $M_{12}^6$ meet $\alpha$ in 4 vertices. Therefore, by complementarity, out of the 5 × $\binom{7}{4}$ sets of size 7 meeting $\alpha$ in 3 vertices, exactly 139 are not facets. So, $5 \times \binom{7}{4} - 139 = 36$ facets meet $\alpha$ in a triangle.

Let $B$ denote the set of all 4-faces contained in $\alpha$. For $\gamma \in B$ and $0 \leq j \leq 1$, let $d_j(\gamma)$ be the number of $j$-faces in Lk($\gamma$). Lemma 4.10 (with $j = 1$) shows that $\sum_{\gamma \in B} (d_0(\gamma) - 2) = 51$. That is, $\sum_{\gamma \in B} d_0(\gamma) = 51 + 2 \times \binom{7}{2} = 93$. Since Lk($\gamma$) is a circle, $d_0(\gamma) = d_1(\gamma)$. Hence $\sum_{\gamma \in B} d_1(\gamma) = 93$, and therefore $\sum_{\gamma \in B} (d_1(\gamma) - 3) = 93 - 3 \times \binom{7}{4} = 30$. But this last sum counts the number of ordered pairs $(\gamma, e)$ where $\gamma \in B$ and $e$ is an edge of Lk($\gamma$) disjoint from $\alpha$. Since the number of such pairs equals the number of facets $\beta = \gamma \cup e$ meeting $\alpha$ in exactly 5 vertices, we see that there are 30 such $\beta$’s.

Since, by Lemma 4.1, there are 131 facets other than $\alpha$, we find by subtraction that $131 - 36 - 30 - 7 = 58$ facets meet $\alpha$ in 4 vertices.

Lemma 4.12. (a) Each edge of $M_{12}^6$ is contained in exactly 42 facets.

(b) Each vertex of $M_{12}^6$ is contained in exactly 77 facets.

Proof. For $i \geq 0$, let $a_i$ be the number of edges of $M_{12}^6$ which are contained in exactly $i$ facets. Counting in two ways (i) total number of edges, (ii) total number of pairs $(e, \alpha)$ where $e$ is an edge, $\alpha$ is a facet and $e \subseteq \alpha$ and (iii) total number of triples $(e, \alpha_1, \alpha_2)$ where $e$ is an edge, $\alpha_1, \alpha_2$ are distinct facets and $e \subseteq \alpha_1 \cap \alpha_2$, we get (in view of Lemmas 4.1 and 4.11):

$$\sum a_i = \binom{12}{2} = 66,$$

$$\sum i a_i = 132 \times \binom{7}{2} = 2772,$$

$$\sum i(i-1) a_i = 132 \times \left(36 \times \binom{3}{2} + 58 \times \binom{4}{2} + 30 \times \binom{5}{2} + 7 \times \binom{6}{2}\right) = 113652.$$ 

Hence we find

$$\sum (i-42)^2 a_i = 0.$$ 

That is, $a_i = 0$ for $i \neq 42$. So every edge of $M_{12}^6$ is contained in exactly 42 facets.

Next, fix a vertex $x$ of $M_{12}^6$. Counting in two ways the number of pairs $(e, \alpha)$ where $e$ is an edge containing $x$ and $\alpha$ is a facet containing $e$, we find that the number $r$ of facets through $x$ satisfies $r \times 6 = 11 \times 42$. So, $r = 77$. Thus each vertex is in 77 facets.

Lemma 4.13. The link of any edge in $M_{12}^6$ is a 10-vertex simply connected combinatorial 4-manifold of Euler characteristic 2.

Proof. Let $L$ be the link of an edge. By Lemma 4.1, $L$ is a 10-vertex 2-neighbourly weak pseudomanifold. By Lemma 4.8, $L$ is a combinatorial 4-manifold. Being 2-neighbourly, $L$ is connected. By Proposition 5, $|L|$ is simply connected.

If $e$ is an edge of $M_{12}^6$ then, by Lemma 4.12, $77 \times 2 - 42 = 112$ facets intersect $e$ and hence $132 - 112 = 20$ facets are disjoint from $e$. Therefore, by complementarity, each edge of $M_{12}^6$ is contained in $\binom{10}{3} - 20 = 100$ 4-faces.

For $0 \leq i \leq 4$, let $f_i$ be the number of $i$-faces of $L$. Since $L$ is 2-neighbourly with 10 vertices, we have $f_0 = 10$ and $f_1 = \binom{10}{2} = 45$. By Lemma 4.12, $f_4 = 42$ and, by the above argument, $f_2 = 100$. An obvious two way counting yields $f_3 = 5f_4/2 = 105$. Hence the Euler characteristic of $L$ is $10 - 45 + 100 - 105 + 42 = 2$. 

\[\square\]
Lemma 4.14. $M_{12}^6$ is a combinatorial manifold.

**Proof.** By Lemmas 4.13 and 3.4, the link of each edge in $M_{12}^6$ is a combinatorial 4-sphere. Then, by Lemma 4.1, the link of any vertex is an 11-vertex combinatorial 5-manifold. Now Part (a) of Proposition 1 implies that the link of any vertex is a combinatorial 5-sphere. Hence $M_{12}^6$ is a combinatorial 6-manifold. □

**Proof of Theorem 2.** By Remark 1 (or by Lemma 4.1) $\chi(M_{12}^6) = 1$. Since $M_{12}^6$ is 3-neighbourly, $|M_{12}^6|$ is simply connected and hence orientable. This contradicts Lemma 4.14 since the Euler characteristic of an orientable closed manifold of dimension $\equiv 2 \pmod{4}$ is even (see [11, Corollary 26.11]). (Lemma 4.14 also contradicts Proposition 3 and Proposition 1 (b).) □

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