UNIVERSALITY IN THE TWO MATRIX MODEL:  
A RIEMANN-HILBERT STEEPEST DESCENT ANALYSIS

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Abstract. The eigenvalue statistics of a pair \((M_1, M_2)\) of \(n \times n\) Hermitian matrices taken random with respect to the measure
\[
\frac{1}{Z_n} \exp \left( -n \operatorname{Tr}(V(M_1) + W(M_2) - \tau M_1 M_2) \right) \, dM_1 dM_2
\]
can be described in terms of two families of biorthogonal polynomials. In this paper we give a steepest descent analysis of a 4 \times 4 matrix-valued Riemann-Hilbert problem characterizing one of the families of biorthogonal polynomials in the special case \(W(y) = y^4/4\) and \(V\) an even polynomial. As a result we obtain the limiting behavior of the correlation kernel associated to the eigenvalues of \(M_1\) (when averaged over \(M_2\)) in the global and local regime as \(n \to \infty\) in the one-cut regular case. A special feature in the analysis is the introduction of a vector equilibrium problem involving both an external field and an upper constraint.

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1. Introduction

1.1. Two matrix model. The two matrix model in random matrix theory is a probability measure

\[
\frac{1}{Z_n} e^{-n \operatorname{Tr} \left(V(M_1)+W(M_2)-\tau M_1 M_2\right)} dM_1 dM_2
\]

defined on pairs \((M_1, M_2)\) of \(n \times n\) Hermitian matrices. Here \(Z_n\) is a normalization constant, \(V\) and \(W\) are two polynomials of even degree and positive leading coefficients, \(\tau \neq 0\) is a coupling constant and \(dM_i\) denotes the product of the Lebesgue measures on the independent entries of \(M_i\) for \(i = 1, 2\). The model was introduced in \([49, 64]\) as a generalization of the unitary one matrix model which allows for a larger class of critical phenomena, see also
An open problem in random matrix theory is to give rigorous asymptotic results on the eigenvalue statistics of $M_1$ and $M_2$ as $n \to \infty$. A natural approach is to use the connection with biorthogonal polynomials. In [66] it is shown that the eigenvalue statistics can be described in terms of two families of polynomials $(p_{k,n})_k$ and $(q_{l,n})_l$, where $p_{k,n}$ and $q_{l,n}$ are monic polynomials of degrees $k$ and $l$, respectively, satisfying

$$\int_{\mathbb{R}^2} p_{k,n}(x) q_{l,n}(y) e^{-n(V(x)+W(y)-\tau xy)} \, dx \, dy = 0, \quad k \neq l.$$  

(1.2)

These polynomials are well-defined and have real and simple zeros [41]. From a complete asymptotic description of the polynomials, it is possible to compute the limiting behavior of the eigenvalue statistics. This has been carried out for the case

$$V(x) = x^2/2 \quad \text{and} \quad W(y) = ay^2/2$$

(1.3)

in [41]. Although heuristic calculations can be found in the physics literature [42], fully rigorous asymptotic results for the biorthogonal polynomials for more general $V$ and $W$ are not known.

The orthogonal polynomial approach to random matrices proved to be successful in the one matrix models. The orthogonal polynomials appearing in these models are characterized by a $2 \times 2$ matrix valued Riemann-Hilbert problem [47]. In [33], [34] the authors applied the Deift/Zhou steepest descent method to this Riemann-Hilbert problem and obtained a complete asymptotic description of the polynomials. As a result it was possible to give a rigorous proof of the universality conjecture for the local eigenvalue correlations.

Inspired by the success of [33], [34], several attempts were made to study the asymptotic behavior of the biorthogonal polynomials by Riemann-Hilbert methods. A number of Riemann-Hilbert problems for the biorthogonal polynomials were formulated [12], [41], [50], [56], but a successful steepest descent analysis has not been carried out so far. It is included in Deift’s list of major open problems in random matrix theory and the theory of integrable systems [31].

In this paper we present the first complete steepest descent analysis for a case beyond the fully quadratic case (1.3). We analyze the Riemann-Hilbert problem for the biorthogonal polynomials $p_{n,n}$ given in [56] for the special case

$$W(y) = y^4/4 \quad \text{and} \quad V \text{ is an even polynomial}.$$  

(1.4)

As a result we are able to compute asymptotics of the eigenvalue correlations of the matrix $M_1$, when averaged over $M_2$. 

[29], [51] and [37], [38] for a survey. For more recent advances in the physics literature, see e.g. [8], [43], [45], [46] and the references cited therein.
1.2. **Unitary ensembles.** Let us first recall some aspects of the unitary ensembles and orthogonal polynomials. In the unitary ensemble one considers $n \times n$ Hermitian matrices taken randomly with respect to the probability measure defined by

$$
\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM,
$$

where $V$ is such that

$$
\lim_{x \to \pm \infty} \frac{V(x)}{\log(x^2 + 1)} = +\infty.
$$

Let $p_{k,n}$ be the unique monic polynomial of degree $k$ satisfying

$$
\int_{-\infty}^{\infty} p_{k,n}(x) x^j e^{-nV(x)} \, dx = 0, \quad j = 0, \ldots, k-1.
$$

Then the eigenvalues of $M$ describe a determinantal point process on $\mathbb{R}$ with kernel $K_n$ defined by

$$
K_n(x, y) = \gamma_{n-1}^2 e^{-nV(x) + V(y)} \frac{p_{n,n}(x)p_{n-1,n}(y) - p_{n,n}(y)p_{n-1,n}(x)}{x-y},
$$

where the constant $\gamma_{n-1}$ is the leading coefficient of the orthonormal polynomial of degree $n-1$. Thus the joint probability density $P$ on the eigenvalues $x_1, \ldots, x_n$ is (up to a constant) equal to the determinant

$$
P(x_1, \ldots, x_n) = \frac{1}{n!} \det (K_n(x_i, x_j))_{i,j=1}^n, \quad i \neq j,
$$

and similarly for the marginal densities for $k = 1, \ldots, n-1$.

$$
\int_{n-k \text{ times}} \cdots \int P(x_1, \ldots, x_n) \, dx_{k+1} \cdots dx_n = \frac{(n-k)!}{n!} \det (K_n(x_i, x_j))_{i,j=1}^k.
$$

In order to compute the asymptotic behavior of the eigenvalue statistics, it is sufficient to obtain asymptotics for the orthogonal polynomials and the kernel $K_n$.

Fokas, Its, and Kitaev \[47\] characterized the orthogonal polynomials $p_{n,n}$ in terms of a Riemann-Hilbert problem (RH problem). It consists of seeking a $2 \times 2$ matrix valued function $Y$ satisfying

$$
\begin{cases}
Y \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, \\
Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-nV(x)} \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R}, \\
Y(z) = (I + O(z^{-1})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \to \infty.
\end{cases}
$$

Here $Y_+$ and $Y_-$ denote the limiting values of $Y$ on $\mathbb{R}$ when $\mathbb{R}$ is approached from above and below, respectively. The unique solution of the RH problem
is given by

\begin{equation}
Y = \left(-2\pi i \gamma_{n-1}^2 p_{n-1,n}^C - 2\pi i \gamma_{n-1}^2 C \left(p_{n-1,n} e^{-nV}\right)\right),
\end{equation}

where \(C f\) denotes the Cauchy transform

\begin{equation}
(C f)(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - z} \, dx, \quad z \in \mathbb{C} \setminus \mathbb{R}.
\end{equation}

In [34] and [33], the Deift/Zhou steepest descent method for RH problems [36] was applied to obtain the asymptotic behavior of \(Y\) as \(n \to \infty\), and hence of the orthogonal polynomials.

A key ingredient in the steepest descent analysis in [33, 34] is an equilibrium measure, see also [30, 71]. This measure is the unique minimizer of the energy functional

\begin{equation}
\iint \log \frac{1}{|x - y|} \, d\mu(x)d\mu(y) + \int V(x)d\mu(x)
\end{equation}

among all Borel probability measures \(\mu\) on \(\mathbb{R}\). If \(V\) is real analytic then the equilibrium measure is supported on a finite number of intervals, has an analytic density in the interior of each interval and vanishes at the endpoints [32]. It is the weak limit of the normalized counting measure on the zeros of \(p_{n,n}\) as \(n \to \infty\). Moreover, it describes the limiting mean eigenvalue distribution of a matrix from (1.5).

The papers [33, 34] had a major impact on the theory of random matrices and orthogonal polynomials. Inspired by these papers, several authors extended the methods of [33, 34] to obtain asymptotics for different types of orthogonal polynomials. For example, for orthogonal polynomials on the half line [74], on the interval [57], and on the unit circle [62, 63]. Another important development is the asymptotic analysis for discrete orthogonal polynomials [5]. In all these cases the orthogonal polynomials can be characterized in terms of a 2 × 2 matrix-valued RH problem and an associated equilibrium measure plays an important role.

### 1.3. Two matrix models and biorthogonal polynomials

Let us now return to the two matrix model (1.1) and the biorthogonal polynomials \(p_{k,n}\) and \(q_{l,n}\) in (1.2). In the one-matrix model the eigenvalues of the random matrix follow a determinantal point process on \(\mathbb{R}\) whose kernel is the reproducing kernel corresponding to the orthogonal polynomials. In the two matrix model we have a similar result but the situation is more complicated.

Define the transformed functions

\begin{align}
Q_{k,n}(x) &= e^{-nV(x)} \int_{-\infty}^{\infty} q_{k,n}(y) e^{-n(W(y) - \tau xy)} \, dy,
\end{align}

\begin{align}
P_{k,n}(y) &= e^{-nW(y)} \int_{-\infty}^{\infty} p_{k,n}(x) e^{-n(V(x) - \tau xy)} \, dx.
\end{align}
and let $h_{k,n}^2$ be defined as

$$h_{k,n}^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{k,n}(x)q_{k,n}(y)e^{-n(V(x)+W(y)-txy)} \, dx \, dy. \tag{1.16}$$

Consider the following kernels

$$K_{11}^{(n)}(x_1, x_2) = \sum_{k=0}^{n-1} \frac{1}{h_{k,n}^2} p_{k,n}(x_1)Q_{k,n}(x_2), \tag{1.17}$$

$$K_{22}^{(n)}(y_1, y_2) = \sum_{k=0}^{n-1} \frac{1}{h_{k,n}^2} p_{k,n}(y_1)Q_{k,n}(y_2), \tag{1.18}$$

$$K_{12}^{(n)}(x, y) = \sum_{k=0}^{n-1} \frac{1}{h_{k,n}^2} p_{k,n}(x)q_{k,n}(y), \tag{1.19}$$

$$K_{21}^{(n)}(y, x) = \sum_{k=0}^{n-1} \frac{1}{h_{k,n}^2} p_{k,n}(y)Q_{k,n}(x) - e^{-n(V(x)+W(y)-txy)}. \tag{1.20}$$

Then the joint probability density function $\mathcal{P}$ for the eigenvalues $x_1, \ldots, x_n$ of $M_1$ and the eigenvalues $y_1, \ldots, y_n$ of $M_2$ is given by $[42, 66]$ (see also $[65, \text{Chapter 23}])$,

$$\mathcal{P}(x_1, \ldots, x_n, y_1, \ldots, y_n) = \frac{1}{(n!)^2} \det \left( \begin{array}{c}
(K_{11}^{(n)}(x_i, x_j))_{i,j=1}^{n} \\
(K_{12}^{(n)}(x_i, y_j))_{i,j=1}^{n}
\end{array} \right) \times \left( \begin{array}{c}
(K_{21}^{(n)}(y_i, x_j))_{i,j=1}^{n} \\
(K_{22}^{(n)}(y_i, y_j))_{i,j=1}^{n}
\end{array} \right). \tag{1.21}
$$

Moreover, the marginal densities also have the determinantal form

$$\int_{n-k+n-l}^{\infty} \cdots \int_{n-k+n-l}^{\infty} \mathcal{P}(x_1, \ldots, x_n, y_1, \ldots, y_n) \, dx_{k+1} \cdots dx_n \, dy_{l+1} \cdots dy_n = \frac{(n-l)!(n-k)!}{(n!)^2} \det \left( \begin{array}{c}
(K_{11}^{(n)}(x_i, x_j))_{i,j=1}^{k} \\
(K_{12}^{(n)}(x_i, y_j))_{i,j=1}^{k}
\end{array} \right) \times \left( \begin{array}{c}
(K_{21}^{(n)}(y_i, x_j))_{i,j=1}^{l} \\
(K_{22}^{(n)}(y_i, y_j))_{i,j=1}^{l}
\end{array} \right). \tag{1.22}
$$

After averaging over the eigenvalues of $M_2$, we see that the eigenvalues of $M_1$ follow a determinantal point process with kernel $K_{11}^{(n)}$. Similarly, the eigenvalues of $M_2$ follow a determinantal point process with kernel $K_{22}^{(n)}$. Both these determinantal point processes are examples of biorthogonal ensembles in the sense of $[24]$.

To determine the asymptotic behavior of the eigenvalues of $M_1$ and $M_2$ it is sufficient to determine asymptotic behavior of the biorthogonal polynomials and the kernels $K_{ij}^{(n)}$.

The biorthogonal polynomials have been studied for many years and many interesting properties have been discovered $[11, 9, 10, 11, 12, 16, 41, 42, 44, 50, 56]$. Although heuristic results on the asymptotic behavior of the
biorthogonal polynomials can be found in [12], rigorous asymptotic results have not yet been obtained.

A first step to an asymptotic analysis by means of RH methods is the formulation of a RH problem for the biorthogonal polynomials [12, 41, 50, 56]. Here we follow [56]. The relationship with the RH problem of [12] has been clarified in [10]. In fact, this relationship will be exploited as part of our analysis.

Assume $W$ is a polynomial of degree $d_W$ and define $d_W - 1$ weights

\begin{equation}
\label{eq:1.23}
w_{j,n}(x) = e^{-nV(x)} \int_{-\infty}^{\infty} y^j e^{-n(W(y) - \tau x y)} \, dy, \quad j = 0, 1, \ldots, d_W - 2.
\end{equation}

Then the RH problem associated with the biorthogonal polynomial $p_{k,n}$ is the following. We look for a $d_W \times d_W$-matrix valued function $Y$ satisfying the following properties

\begin{equation}
\label{eq:1.24}
\begin{align*}
&Y \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, \\
&Y_+(x) = Y_-(x), \\
&Y(z) = (I + O(1/z)) \operatorname{diag} \left( z^k, z^{-k_1}, \ldots, z^{-k_{d_W-1}} \right), \quad z \to \infty,
\end{align*}
\end{equation}

where $k_i$ is the integer part of $(k + d_W - 1 - j)/(d_W - 1)$. In [56] it is proved that this RH problem has a unique solution given by

\begin{equation}
\label{eq:1.25}
Y = \begin{pmatrix}
\begin{pmatrix} p_{k,n}^{(0)} \end{pmatrix}_{k-1,n} & C(p_{k,n}^{(0)} w_{0,n}) & \cdots & C(p_{k,n}^{(0)} w_{d_W-2,n}) \\
\begin{pmatrix} p_{k-1,n}^{(0)} \end{pmatrix} & C(p_{k-1,n}^{(0)} w_{0,n}) & \cdots & C(p_{k-1,n}^{(0)} w_{d_W-2,n}) \\
\vdots & \vdots & \ddots & \vdots \\
\begin{pmatrix} p_{k-1,n}^{(d_W-2)} \end{pmatrix} & C(p_{k-1,n}^{(d_W-2)} w_{0,n}) & \cdots & C(p_{k-1,n}^{(d_W-2)} w_{d_W-2,n})
\end{pmatrix}
\end{pmatrix}
\end{equation}

where $p_{k,n}$ is the monic biorthogonal polynomial of degree $k$ and $p_{k-1,n}^{(j)}$, $j = 0, \ldots, d_W - 2$, are certain polynomials of degree $\leq k - 1$. Here $Cf$ denotes the Cauchy transform as given in [1,12].

In the case $d_W = 2$ the RH problem (1.24) reduces to the $2 \times 2$ matrix valued RH problem for orthogonal polynomials. Then the biorthogonal polynomials $p_{k,n}$ are simply orthogonal polynomials on the real line with respect to a varying exponential weight, see also [41].

For $d_W > 2$ the biorthogonal polynomials $p_{k,n}$ do not reduce to orthogonal polynomials, but instead they are examples of what is known as multiple orthogonal polynomials. Multiple orthogonal polynomials for $r$ weights are characterized by $(r + 1) \times (r + 1)$ matrix valued RH problems [78], and (1.24)
is an example of such a RH problem. The steepest descent analysis has not been applied to the RH problem (1.24).

Certain other systems of multiple orthogonal polynomials and their associated RH problems were successfully analyzed recently. These polynomials were related to matrix models with external source [3, 19, 20, 61] and non-intersecting paths [27, 28, 54]. See also [13, 14] for a $3 \times 3$ matrix valued RH problem describing a Cauchy two matrix model. The steepest descent analysis of the larger size RH problems revealed several new features that are not present in the analysis of the $2 \times 2$ RH problem (1.10).

In our opinion, an important obstacle in the asymptotic analysis of (1.24) is the lack of a tractable equilibrium problem. One of the main contributions of the present paper is a suitable equilibrium problem that we use in the steepest descent analysis of (1.24) for the special case (1.4).

1.4. RH problem for case $W(y) = y^4/4$. In this paper, we will not treat the general two matrix model with polynomial potentials, but we restrict ourselves to the special case (1.4). Due to the assumption $W(y) = y^4/4$ the RH problem (1.24) is of size $4 \times 4$. The assumption that $V$ is an even polynomial introduces a symmetry with respect to the imaginary axis into the problem. This will be important for the steepest descent analysis. We also make the additional assumption that $n$ is a multiple of three. This is not essential and is made only for reasons of exposition, since it simplifies many of the formulas.

Due to these assumptions the RH problem (1.24) characterizing the biorthogonal polynomial $p_{n,n}$ takes the form

\[
\begin{cases}
Y \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, \\
Y_+(x) = Y_-(x) \left( \begin{array}{cccc}
1 & w_{0,n}(x) & w_{1,n}(x) & w_{2,n}(x) \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 
\end{array} \right), & x \in \mathbb{R}, \\
Y(z) = (I + O(1/z)) \left( \begin{array}{cccc}
z^n & 0 & 0 & 0 \\
0 & z^{-n/3} & 0 & 0 \\
0 & 0 & z^{-n/3} & 0 \\
0 & 0 & 0 & z^{-n/3} 
\end{array} \right), & z \to \infty.
\end{cases}
\]

with

\[
w_{j,n}(x) = e^{-nV(x)} \int_{-\infty}^{\infty} y^j e^{-n(y^4/4-\tau xy)} \, dy, \quad j = 0, 1, 2,
\]

This RH problem has a unique solution which is given by (1.25). A central observation in [56] that leads to this result, is that the polynomial $p_{n,n}$ is a multiple orthogonal polynomial with respect to the three weights $w_{j,n}$ on the real line. This means that $p_{n,n}$ is the unique monic polynomial of degree $n$ such that

\[
\int_{-\infty}^{\infty} p_{n,n}(x)x^lw_{j,n}(x) \, dx = 0, \quad l = 0, \ldots, \frac{n}{3} - 1, \quad j = 0, 1, 2.
\]
The polynomials $p_{k,n}$ are multiple orthogonal polynomials of type II. There are also multiple orthogonal polynomials of type I. These appear in the function $Q_{k,n}$ of (1.14), which can be written as

$$Q_{k,n}(x) = \sum_{j=0}^{3} A_{(k,n)}^{(j)}(x) w_{j,n}(x),$$

for certain polynomials $A_{(k,n)}^{(j)}$, called multiple orthogonal polynomials of type I. In [26] a Christoffel-Darboux formula is proved for the reproducing kernel associated to multiple orthogonal polynomials in a general setting. In our case it applies to the kernel $K_{11}^{(n)}$ of (1.17). As a result of the Christoffel-Darboux formula the kernel $K_{11}^{(n)}$ can be expressed in terms of the solution $Y$ to the RH problem (1.26) as follows.

**Proposition 1.1.** We have that

$$K_{11}^{(n)}(x, y) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} 1 \\ 0 \\ w_{0,n}(y) \\ w_{1,n}(y) \\ w_{2,n}(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

for $x, y \in \mathbb{R}$.

**Proof.** See [26].

## 2. Statement of results

Our results deal with the limiting behavior as $n \to \infty$ of the eigenvalues of the matrix $M_1$ in the two matrix model

$$\frac{1}{Z_n} e^{-n \text{Tr} \left( V(M_1) + W(M_2) - \tau M_1 M_2 \right)} dM_1 dM_2$$

with $\tau > 0$ under the following assumptions

(a) $V$ is an even polynomial,

(b) $W(y) = y^4/4$, and

(c) $n$ is a multiple of three.

Guionnet [48] showed that the limiting mean eigenvalue density of the matrix $M_1$ exists. Her result is valid in much greater generality than stated in the next proposition.

**Proposition 2.1.** The limiting mean density of the eigenvalues of the matrix $M_1$ exists. That is, there exists a probability measure $\mu_1$ on $\mathbb{R}$ with respect to Lebesgue measure such that in the sense of weak convergence of measures

$$\lim_{n \to \infty} \frac{1}{n} K_{11}^{(n)}(x, x) dx = d\mu_1.$$ 

**Proof.** See [48].
Guionnet characterized $\mu_1$ by a variational problem. Under the above assumptions (a)-(b), we are going to characterize $\mu_1$ in terms of an equilibrium problem from potential theory. It is not clear to us if the two minimization problems are related to each other.

2.1. The equilibrium problem. As already mentioned, we characterize $\mu_1$ by means of an equilibrium problem from potential theory. Main references for potential theory in the complex plane are [70, 71].

We will work with (non-negative) measures on $\mathbb{C}$, but only on $\mathbb{R}$ and $i\mathbb{R}$. A signed measure is real and can be negative. The support of a measure $\nu$ is denoted by $S(\nu)$. The measure may have unbounded support. If a measure $\nu$ has unbounded support then we assume that

$$\int \log(1 + |x|) \ d\nu(x) < \infty. \tag{2.2}$$

The logarithmic potential of $\nu$ is the function

$$U^{\nu}(z) = \int \log \frac{1}{|z - x|} d\nu(x). \tag{2.3}$$

The logarithmic energy of $\nu$ is defined as

$$I(\nu) = \iint \log \frac{1}{|x - y|} d\nu(x)d\nu(y) \tag{2.4}$$

and $I(\nu) \in (-\infty, +\infty]$. If $\nu_1$ and $\nu_2$ are two measures with finite logarithmic energy, then their mutual logarithmic energy is defined by

$$I(\nu_1, \nu_2) = \iint \log \frac{1}{|x - y|} d\nu_1(x)d\nu_2(y). \tag{2.5}$$

For a given $V$ and $\tau > 0$, we study the energy functional $E_V$ defined by

$$E_V(\nu_1, \nu_2, \nu_3) = \sum_{j=1}^{3} I(\nu_j) - \sum_{j=1}^{2} I(\nu_j, \nu_{j+1}) + \int \left( V(x) - \frac{3}{4} \tau^{4/3} |x|^{4/3} \right) d\nu_1(x) \tag{2.6}$$

where $\nu_1$, $\nu_2$, $\nu_3$ are three measures with finite logarithmic energy. The equilibrium problem is the following.

**Definition 2.2.** The equilibrium problem is to minimize $E_V(\nu_1, \nu_2, \nu_3)$ among all measures $\nu_1$, $\nu_2$, $\nu_3$ such that

(a) the measures have finite logarithmic energy;
(b) $\nu_1$ is a measure on $\mathbb{R}$ with $\nu_1(\mathbb{R}) = 1$;
(c) $\nu_2$ is a measure on $i\mathbb{R}$ with $\nu_2(i\mathbb{R}) = 2/3$;
(d) $\nu_3$ is a measure on $\mathbb{R}$ with $\nu_3(\mathbb{R}) = 1/3$;
(e) $\nu_2$ satisfies the upper constraint $\nu_2 \leq \sigma$ where $\sigma$ is the (unbounded) measure on $i\mathbb{R}$ defined by

$$d\sigma(z) = \frac{\sqrt{3}}{2\pi} \tau^{4/3} |z|^{1/3} |dz|, \quad z \in i\mathbb{R}, \tag{2.7}$$
where $|dz|$ denotes the arclength on $i\mathbb{R}$.

An electrostatic interpretation of the equilibrium problem for the energy functional (2.6) is the following. Consider three types of charged particles. The particles of the first type are put on $\mathbb{R}$ and have total charge 1. The particles of the second type are put on $i\mathbb{R}$ and have total charge $2/3$. The particles of the third type are put on $\mathbb{R}$ and have total charge $1/3$. Particles of the same type repel each other. The particles of the first and the second types attract each other with a strength that is half the strength of the repulsion of particles of the same type. So do the particles of the second and the third types. Particles of the first and the third types do not interact directly. Particles of the first type are influenced by an external field depending on $V$ and $\tau$.

In the equilibrium problem the particles distribute themselves in order to minimize their energy under the extra condition that the particle density of the second type particles does not exceed the density of $\sigma$, where $\sigma$ is the given measure (2.7). Thus $\sigma$ acts as an upper constraint on the second measure.

Equilibrium problems for a vector of measures with mutual interaction as in (2.6) arise for Nikishin systems [68] in the theory of rational approximation, see also the survey [2], and the more recent papers [4, 6, 40, 54, 60, 73].

Equilibrium problems with constraint appeared before in asymptotic results for discrete orthogonal polynomials [5, 7, 39, 58, 69], singular limits of integrable systems [35], and convergence results for Krylov methods in numerical linear algebra [52].

We prove the following.

**Theorem 2.3.** Let $V$ be an even polynomial and $\tau > 0$. Then there is a unique minimizer $(\mu_1, \mu_2, \mu_3)$ of $E_V$ subject to the conditions (a)-(e) in the equilibrium problem. Moreover,

(a) The measure $\mu_1$ is supported on a finite number of disjoint intervals $\bigcup_{j=1}^N [a_j, b_j]$ with a density of the form

$$
\frac{d\mu_1(x)}{dx} = h_j(x) \sqrt{(b_j - x)(x - a_j)}, \quad x \in [a_j, b_j],
$$

where $h_j$ is real analytic and non-negative on $[a_j, b_j]$, for $j = 1, \ldots, N$.

(b) $S(\mu_2) = i\mathbb{R}$ and there exists a constant $c > 0$ such that

$$
S(\sigma - \mu_2) = i\mathbb{R} \setminus (-ic, ic).
$$

The measure $\sigma - \mu_2$ has an analytic density on $i\mathbb{R} \setminus (-ic, ic)$ which vanishes as a square root at $\pm ic$. Moreover, the logarithmic potential $U^{\mu_2}$ is such that

$$
\frac{3}{4} \tau^{4/3} |x|^{4/3} + U^{\mu_2}(x)
$$

is real analytic on $\mathbb{R}$.

(c) $S(\mu_3) = \mathbb{R}$ and $\mu_3$ has a density which is analytic in $\mathbb{R} \setminus \{0\}$. 
(d) all three measures $\mu_1$, $\mu_2$ and $\mu_3$ are symmetric in the sense that $\mu_j(-A) = \mu_j(A)$ for $j = 1, 2, 3$, and for every Borel set $A$.

The symmetry property in part (d) of Theorem 2.3 is a direct consequence of the uniqueness of the minimizer and the fact that $V$ is even. We state it explicitly here, since it will be used many times in what follows, often without explicit mentioning it.

For given $\nu_2$, $\nu_3$, we can minimize $E_V(\nu_1, \nu_2, \nu_3)$ with respect to $\nu_1$ only. Then we look for the minimizer of the energy functional

$$I(\nu_1) + \int \left( V(x) - \frac{3}{4} \tau^{4/3}|x|^{4/3} - U^{\nu_2}(x) \right) \, d\nu_1(x),$$

among all probability measures $\nu_1$ on $\mathbb{R}$. This is a usual equilibrium problem with external field [30, 71]. The external field is analytic in $\mathbb{R} \setminus \{0\}$, but possibly not at 0. However, due to (2.9) the external field is analytic at 0 if $\nu_2 = \mu_2$ is part of the minimizer $(\mu_1, \mu_2, \mu_3)$ of the full equilibrium problem. Thus $\mu_1$ is the minimizer of an energy functional with analytic external field on $\mathbb{R}$. The statements in Theorem 2.3 about the structure of the measure $\mu_1$ then follow from results of [32].

If we minimize only with respect to $\nu_3$, with $\nu_1$ and $\nu_2$ fixed, then we minimize the energy functional

$$I(\nu_3) - \int U^{\nu_2}(x) \, d\nu_3(x),$$

among all $\nu_3$ on $\mathbb{R}$ with total mass $1/3$. Again this is an equilibrium problem with external field, but the external field $-U^{\nu_2}(x)$ is only slowly growing at infinity. In [72] such an external field is called weakly admissible. It is a consequence of the slow growth that the support $S(\nu_3)$ of the minimizer will be unbounded, and in fact it will be the full real line.

Finally, if we fix $\nu_1$ and $\nu_3$, and minimize with respect to $\nu_2$, then we look for the minimizer of the energy functional

$$I(\nu_2) - \int (U^{\nu_1}(x) + U^{\nu_3}(x)) \, d\nu_2(x),$$

among all $\nu_2$ on $i\mathbb{R}$ with total mass $2/3$ satisfying the constraint $\nu_2 \leq \sigma$. Here we again have a weakly admissible external field, but in addition there is the upper constraint. Again, we will have that the support of the minimizer is unbounded, and indeed it is the full imaginary axis. The constraint $\sigma$ is active on a symmetric interval $[-ic, ic]$ around the origin.

2.2. Variational conditions and regular/singular behavior. From the discussion above we see that each of $\mu_1$, $\mu_2$, $\mu_3$ is the minimizer for an equilibrium problem with external field and/or upper constraint, and as such they are characterized by the following set of variational conditions.

Proposition 2.4. The measures $\mu_1$, $\mu_2$ and $\mu_3$ satisfy

$$2U^{\mu_1}(x) = U^{\mu_2}(x) - V(x) + \frac{3}{4} \tau^{4/3}|x|^{4/3} + \ell, \quad x \in S(\mu_1),$$

(2.13)
\[2U^{\mu_1}(x) \geq U^{\mu_2}(x) - V(x) + \frac{3}{4}x^{4/3}|x|^{4/3} + \ell, \quad x \in \mathbb{R} \setminus S(\mu_1),\]

for some constant \(\ell\),

\[2U^{\mu_2}(x) = U^{\mu_1}(x) + U^{\mu_3}(x), \quad x \in S(\sigma - \mu_2),\]

\[2U^{\mu_2}(x) < U^{\mu_1}(x) + U^{\mu_3}(x), \quad x \in \mathbb{R} \setminus S(\sigma - \mu_2),\]

and

\[2U^{\mu_3}(x) = U^{\mu_2}(x), \quad x \in S(\mu_3) = \mathbb{R}.\]

Note that the inequality in (2.16) is strict. By contrast the inequality in (2.14) need not be strict. Then as in the case of the one-matrix model and orthogonal polynomials \([33]\) we can distinguish between regular and singular cases as follows.

**Definition 2.5.** The measure \(\mu_1\) is called regular (otherwise singular) if the functions \(h_j\) in (2.8) satisfy \(h_j > 0\) on \([a_j, b_j]\) for every \(j = 1, \ldots, N\), and if the inequality in (2.14) is strict.

A point \(x^* \in \mathbb{R}\) at which one of these conditions fails is a singular point. There are three types of singular points, namely

- \(x^*\) is a singular end point if \(x^* \in \{a_j, b_j\}\) for some \(j = 1, \ldots, N\) and \(h_j(x^*) = 0\),
- \(x^*\) is a singular interior point if \(x^* \in (a_j, b_j)\) for some \(j = 1, \ldots, N\) and \(h_j(x^*) = 0\),
- \(x^*\) is a singular exterior point if \(x^* \in \mathbb{R} \setminus S(\mu_1)\) and equality holds for \(x = x^*\) in (2.14).

We see that the structure for the measure \(\mu_1\) in the global regime is the same as what is known for the one-matrix model in a real analytic external field \([33]\). It is known that the regular case holds generically in one-matrix models \([55]\) and all three types of singular behavior can occur. The same is true in the present two matrix model, but the maybe surprising fact is that no other type of singular behavior can occur. We emphasize that this is related to our assumption that \(W(y) = y^4/4\).

We conclude with an easy to check convexity result, which is analogous to the well-known fact that for a convex external field the equilibrium measure is supported on one interval. The following is a similar result for the equilibrium problem of Definition 2.2.

**Proposition 2.6.** Suppose that \(V\) is even and that \(x \mapsto V(\sqrt{x})\) is convex for \(x > 0\). Then \(S(\mu_1)\) consists of one or two intervals.

The proposition applies to all even quartic potentials \(V(x) = ax^4 + bx^2\) with \(a > 0\).

2.3. **One-cut regular case.** The core of the present paper is a Deift/Zhou steepest descent analysis for the RH problem for the biorthogonal polynomials. As a particular result we are able to obtain the limiting behavior of
the kernel $K_{11}^{(n)}$ as $n \to \infty$. The steepest descent analysis simplifies in the one-cut regular case, that is, the measure $\mu_1$ is regular and supported on one interval. We will only deal with this case in this paper, but our methods can be extended to the non-regular and multi-cut situations.

The non-regular cases for unitary ensembles have been treated recently, see [18, 23, 24] for singular interior points, [25] for singular endpoints and [15, 22, 67] for singular exterior points. All three singular cases can appear in our situation, and can be implemented within our methods to obtain the limiting behavior of the polynomials $p_{n,n}$ and the kernel $K_{11}^{(n)}$ for the singular cases as well. We will not go into details here.

As part of our analysis we will express the minimizer $(\mu_1, \mu_2, \mu_3)$ in terms of a meromorphic functions defined on a four sheeted Riemann surface. The genus of the Riemann surface equals $N - 1$ where $N$ is the number of intervals in the support of $\mu_1$. Hence the genus is non-zero in the multi-cut case, which complicates the construction of the outside parametrix in the steepest descent analysis. For reasons of exposition we only deal here with the one-cut case and plan to return to the multi-cut case in future work.

2.4. Global eigenvalue regime. Our first main theorem states that the measure $\mu_1$ that we obtain as part of the minimizer $(\mu_1, \mu_2, \mu_3)$ in the equilibrium problem for $E_V$ is the limiting mean distribution for the eigenvalues of $M_1$. We prove this in this paper for the one-cut regular case.

**Theorem 2.7.** Let $W(y) = y^4/4$, let $V$ be an even polynomial and $\tau > 0$. Let $(\mu_1, \mu_2, \mu_3)$ be the minimizer of the equilibrium problem described in Subsection 2.1 above. Suppose that the measure $\mu_1$ is one-cut regular.

Then we have as $n \to \infty$ with $n \equiv 0 \pmod{3}$

\[
\lim_{n \to \infty} \frac{1}{n} K_{11}^{(n)}(x, x) = \frac{d\mu_1}{dx}(x),
\]

uniformly for $x \in \mathbb{R}$.

2.5. Local eigenvalue regime. In the local eigenvalue regime we obtain the same universal limiting behaviors that are known from one-matrix models. This is in agreement with the universality conjecture in random matrix theory, which says that local eigenvalue correlations in random matrix models with unitary symmetry do not depend on the particular features of the model, but only on the global regime.

We again restrict ourselves to the one-cut regular case. So we assume that the measure $\mu_1$ from the solution of the equilibrium problem is supported on one interval and we write $S(\mu_1) = [-a, a]$.

**Theorem 2.8.** Let $W(y) = y^4/4$, let $V$ be an even polynomial and $\tau > 0$. Assume that the measure $\mu_1$ is one-cut regular.
Let \( x^* \in (-a,a) \) and define

\[
\rho := \frac{d\mu_1}{dx}(x^*) > 0.
\]

Then, as \( n \to \infty \) with \( n \equiv 0(\text{mod } 3) \) we have for every \( k \in \mathbb{N} \) that

\[
\lim_{n \to \infty} \det \left( \frac{1}{\rho^n} K_{11}(n) \left( x^* + \frac{u_i}{\rho^n}, x^* + \frac{u_j}{\rho^n} \right) \right)_{i,j=1}^k = \det \left( \frac{\sin \pi (u_i - u_j)}{\pi (u_i - u_j)} \right)_{i,j=1}^k.
\]

uniformly for \((u_1, \ldots, u_k)\) in compact subsets of \( \mathbb{R}^k \).

(b) Let \( \rho > 0 \) be such that

\[
\frac{d\mu_1(x)}{dx} = \frac{\rho}{\pi} (a - x)^{1/2} (1 + O(a - x)), \quad \text{as } x \to a.
\]

Then, as \( n \to \infty \) with \( n \equiv 0(\text{mod } 3) \) we have for every \( k \in \mathbb{N} \) that

\[
\lim_{n \to \infty} \det \left( \frac{1}{(\rho n)^{2/3}} K_{11}^{(n)} \left( a + \frac{u_i}{(\rho n)^{2/3}}, a + \frac{u_j}{(\rho n)^{2/3}} \right) \right)_{i,j=1}^k = \det \left( \frac{\text{Ai}(u_i) \text{Ai}'(u_j) - \text{Ai}'(u_i) \text{Ai}(u_j)}{u_i - u_j} \right)_{i,j=1}^k,
\]

uniformly for \((u_1, \ldots, u_k)\) in compact subsets of \( \mathbb{R}^k \). Here \( \text{Ai} \) denotes the usual Airy function.

In Theorem 2.8 we only deal with the one-cut regular case. The non-regular cases that can occur in our situation, also appear in the unitary ensembles. In these cases, we also obtain the same limits for the kernel \( K_{11}^{(n)} \) as in the unitary ensembles. For example, if the density of the measure \( \mu_1 \) vanishes quadratically at an interior point of the support, then the limiting behavior of the kernel \( K_{11}^{(n)} \) in a double scaling limit, is expressed in terms of the \( \Psi \)-functions associated to the Hastings-McLeod solution of the Painlevé II equation, see also [18, 23].

2.6. Overview of the rest of the paper. The proof of Theorem 2.8 follows from a Deift/Zhou steepest descent analysis applied to the RH problem for biorthogonal polynomials. The Deift/Zhou steepest descent analysis consists of a sequence of explicit and invertible transformations

\[
Y \mapsto X \mapsto U \mapsto T \mapsto S \mapsto R.
\]

The RH problem for \( Y \) is not in a suitable form for an immediate application of the equilibrium measures from the equilibrium problem. In the first transformation \( Y \mapsto X \) we transform the RH problem for \( Y \) to a RH problem for \( X \) that is more suitable for further asymptotic analysis. This transformation depends on a method that we learned from the authors of
The construction involves Pearcey integrals and is presented in Section 3.

In the second transformation $X \mapsto U$ we will use the equilibrium measures $\mu_1$, $\mu_2$ and $\mu_3$ from the equilibrium problem of Definition 2.2 and their corresponding $g$-functions. In Section 4 we discuss the equilibrium problem and we prove Theorem 2.3 and Propositions 2.4 and 2.6.

The properties of the $g$-functions are conveniently expressed in terms of functions that come from a four sheeted Riemann surface, which in the one-cut case has genus zero. We introduce the Riemann surface in Section 5. The second transformation of the RH problem $X \mapsto U$ is given in Section 6.

In the transformation $U \mapsto T \mapsto S$ in Section 7 we open lenses. In the transformation $U \mapsto T$ we open the lens around the supports of $\mu_2$ and $\mu_3$. These supports are unbounded so special care has to be taken at infinity. In the transformation $T \mapsto S$ we open the lenses around the support of $\mu_1$.

The next step is the construction of a parametrix in Section 8. Here the Riemann surface is used again. We give a rational parametrization which allows us to give explicit formulas for the outside parametrix. In the multi-cut case the Riemann surface has higher genus and the construction of the outside parametrix is more complicated. We construct local Airy parametrices around the branch points of the Riemann surface. These are the endpoints $\pm \alpha$ of the support of $\mu_1$ and the endpoints $\pm i \alpha$ of the support of $\sigma - \mu_2$. The final transformation $S \mapsto R$ is also given in Section 8. It leads to a RH problem for $R$ with jump matrices that are uniformly close to the identity matrix. The RH problem for $R$ is also normalized at infinity.

The proofs of Theorems 2.7 and 2.8 are in the final Section 9.

3. The first transformation $Y \mapsto X$

The starting point is the RH problem (1.26) with functions (1.27).

In this section we introduce the first transformation $Y \mapsto X$ which makes use of the special form of the weight functions $w_{j,n}$.

3.1. The main idea. The main idea behind the first transformation $Y \mapsto X$ can be found in an unpublished manuscript of Bertola, Harnad and Its [16]. The starting point is the observation that $w_{0,n}(x)e^{nV(x)}$ satisfies a scaled version of the so-called Pearcey differential equation

\[ p'''(z) = zp(z). \]  

Special solutions to the equation (3.1) are given by the Pearcey integrals

\[ p_j(z) = \int_{\Gamma_j} e^{-s^2/4 + s z} \, ds, \quad j = 0, \ldots, 5, \]  

where the contours $\Gamma_j$ are

\[ \Gamma_0 = (-\infty, \infty), \quad \Gamma_1 = (i\infty, 0] \cup [0, \infty), \]
\[ \Gamma_2 = (i\infty, 0] \cup [0, -\infty), \quad \Gamma_3 = (-i\infty, 0] \cup [0, -\infty), \]
\[ \Gamma_4 = (-i\infty, 0] \cup [0, \infty), \quad \Gamma_5 = (-i\infty, i\infty), \]  

Figure 3.1. Contours $\Gamma_j$ in the definition of the Pearcey integrals.

or homotopic deformations such as the ones shown Figure 3.1. Each $\Gamma_j$ is equipped with an orientation as shown in Figure 3.1. Since the Pearcey equation (3.1) is a linear third order equation, there must be a linear relation between any four solutions. Indeed, from the integral representations (3.2) one can find for example,

$$p_5(z) = p_4(z) - p_1(z).$$

(3.4)

The functions $w_{j,n}$ from (1.27) are expressed in terms of the Pearcey integral $p_0$ and its derivatives as follows

$$w_{0,n}(z) = n^{-1/4}e^{-nV(z)}p_0(n^{3/4}\tau z),$$

(3.5)

$$w_{1,n}(z) = n^{-1/2}e^{-nV(z)}p_0'(n^{3/4}\tau z),$$

(3.6)

$$w_{2,n}(z) = n^{-3/4}e^{-nV(z)}p_0''(n^{3/4}\tau z).$$

(3.7)

Now we can illustrate the main idea behind the transformation $Y \mapsto X$. Consider the matrix-valued function $\tilde{X}$ defined by

$$\tilde{X}(z) = Y(z) \begin{pmatrix} 1 & 0 \\ 0 & D_n \hat{P}^{-t}(n^{4/3}\tau z) \end{pmatrix},$$

(3.8)

where

$$D_n = \begin{pmatrix} n^{1/4} & 0 & 0 \\ 0 & n^{1/2} & 0 \\ 0 & 0 & n^{3/4} \end{pmatrix}, \quad \hat{P} = \begin{pmatrix} p_0 & p_j & p_k \\ p_0' & p_j' & p_k' \\ p_0'' & p_j'' & p_k'' \end{pmatrix},$$

(3.9)

and $j$ and $k$ are such that $p_0$, $p_j$, and $p_k$ are linearly independent solutions of the Pearcey equation, so that $\hat{P}$ is indeed invertible. To compute the jumps
for $\tilde{X}$ we note that (3.5)–(3.7) can be written as

$$D_n^{-1} P(n^{4/3} \tau z) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = e^{nV(z)} \begin{pmatrix} w_{0,n}(z) \\ w_{1,n}(z) \\ w_{2,n}(z) \end{pmatrix},$$

so that for $x \in \mathbb{R}$, by (1.26) and (3.8),

$$\tilde{X}_-(x)^{-1} \tilde{X}_+(x) = \begin{pmatrix} 1 & w_{0,n}(x) & w_{1,n}(x) & w_{2,n}(x) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} D_n^{-1} P^{-1}(n^{4/3} \tau x).$$

The jump matrix has only one non-trivial upper triangular $2 \times 2$ block. This $2 \times 2$ block has a varying exponential in its 12 entry and so is precisely of the form that appears in the RH problem for orthogonal polynomials on the real line with respect to varying exponential weight [30, 33, 47]. So the jump matrix for $\tilde{X}$ has a form that seems very promising for asymptotic analysis.

The complication however, is that the asymptotic behavior of $\tilde{X}$ becomes rather involved due to the nontrivial asymptotic behavior of the Pearcey integrals $p_j(z)$ as $z \to \infty$. We also have to deal with the Stokes phenomenon which implies that the asymptotic behavior may be different as $z \to \infty$ in different sectors in the complex plane. For example, by classical steepest descent methods one can prove as in [20] that

$$p_0(z) = \sqrt{\frac{2\pi}{3}} z^{-1/3} e^{\frac{2}{3} z^{4/3}} (1 + O(z^{-4/3}))$$

as $z \to \infty$ with $\text{Re} \, z > 0$, and

$$p_0(z) = \sqrt{\frac{2\pi}{3}} (-z)^{-1/3} e^{\frac{2}{3} (-z)^{4/3}} (1 + O(z^{-4/3}))$$

as $z \to \infty$ with $\text{Re} \, z < 0$. Here and throughout the paper we use principal branches of the fractional powers, that is the branch cut is along the negative real axis. So $p_0$ has a different asymptotic behavior as $z \to \infty$ in the right half-plane and in the left half-plane. Also the asymptotic formulas (3.12)–(3.13) are only uniformly valid if we stay away from the imaginary axis.

To deal with the different asymptotic behaviors we define $P$ differently in each of the quadrants and define the transformation $Y \mapsto X$ accordingly. This procedure will introduce a new constant jump on the imaginary axis, while still simplifying the jump matrix on the real axis. Also the asymptotic condition in the RH problem for $X$ takes a nice form.
3.2. A RH problem for Pearcey integrals. We denote the four quadrants in the complex plane by I, II, III and IV, respectively. We define $P$ in the different quadrants by

$$
P = \begin{pmatrix}
  p_0 & -p_2 & -p_5 \\
  p_0'' & -p_2 & -p_5 \\
  p_0'' & -p_2 & -p_5
\end{pmatrix} \quad \text{in I,} \quad P = \begin{pmatrix}
  p_0 & -p_1 & -p_5 \\
  p_0'' & -p_1 & -p_5 \\
  p_0'' & -p_1 & -p_5
\end{pmatrix} \quad \text{in II,}
$$

$$
P = \begin{pmatrix}
  p_0 & -p_4 & -p_5 \\
  p_0'' & -p_4 & -p_5 \\
  p_0'' & -p_4 & -p_5
\end{pmatrix} \quad \text{in III,} \quad P = \begin{pmatrix}
  p_0 & -p_3 & -p_5 \\
  p_0'' & -p_3 & -p_5 \\
  p_0'' & -p_3 & -p_5
\end{pmatrix} \quad \text{in IV.}
$$

Hence each column of $P$ contains a particular Pearcey integral $p_j$, see \eqref{3.2}, and its derivatives. We use $p_0$ in the first column and $p_5$ in the last column. In the middle column we use different Pearcey integrals in the different quadrants.

Then $P$ is analytic in $\mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ with a jump on $\mathbb{R} \cup i\mathbb{R}$. By linear relations such as \eqref{3.4} one can easily obtain jump conditions for $P$. In the following lemma we show the RH problem that is satisfied by this function $P$.

**Lemma 3.1.** We have that $P$ satisfies the following RH problem:

$$
P \text{ is analytic in } \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}),
$$

$$
P_+(z) = P_-(z) \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & -1 & 1
\end{pmatrix}, \quad z \in \mathbb{R},
$$

$$
P_+(z) = P_-(z) \begin{pmatrix}
  1 & -1 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}, \quad z \in i\mathbb{R},
$$

$$
P(z) = \sqrt{2\pi}(I + O(z^{-2/3})) \begin{pmatrix}
  z^{-1/3} & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & z^{1/3}
\end{pmatrix} B_j \Theta_j(z),
$$

as $z \to \infty$ in the $j$th quadrant,

where the constant matrices $B_j$ are given by (we use $\omega = e^{2\pi i/3}$),

$$
B_1 = \frac{i}{\sqrt{3}} \begin{pmatrix}
  1 & -\omega^2 & -\omega \\
  1 & -1 & -1 \\
  1 & -\omega & -\omega^2
\end{pmatrix}, \quad B_2 = \frac{i}{\sqrt{3}} \begin{pmatrix}
  -\omega^2 & -1 & -\omega \\
  -1 & -1 & -1 \\
  -\omega & -1 & -\omega
\end{pmatrix},
$$

$$
B_3 = \frac{i}{\sqrt{3}} \begin{pmatrix}
  -\omega & -1 & \omega^2 \\
  -1 & -1 & 1 \\
  -\omega^2 & -1 & \omega
\end{pmatrix}, \quad B_4 = \frac{i}{\sqrt{3}} \begin{pmatrix}
  1 & -\omega & \omega^2 \\
  1 & -1 & 1 \\
  1 & -\omega^2 & \omega
\end{pmatrix},
$$

and the diagonal matrices $\Theta_j(z)$ for $j = 1, \ldots, 4$ are given by

$$
\Theta_1(z) = \begin{pmatrix}
  e^{\theta_1(z)} & 0 & 0 \\
  0 & e^{\theta_1(z)} & 0 \\
  0 & 0 & e^{\theta_2(z)}
\end{pmatrix}, \quad \Theta_2(z) = \begin{pmatrix}
  e^{\theta_1(z)} & 0 & 0 \\
  0 & e^{\theta_3(z)} & 0 \\
  0 & 0 & e^{\theta_2(z)}
\end{pmatrix}
$$
(3.20) \[ \Theta_3(z) = \begin{pmatrix} e^{\theta_2(z)} & 0 & 0 \\ 0 & e^{\theta_3(z)} & 0 \\ 0 & 0 & e^{\theta_1(z)} \end{pmatrix}, \quad \Theta_4(z) = \begin{pmatrix} e^{\theta_3(z)} & 0 & 0 \\ 0 & e^{\theta_2(z)} & 0 \\ 0 & 0 & e^{\theta_1(z)} \end{pmatrix} \]

with functions \( \theta_j \) defined as

(3.21) \[ \theta_j(z) = \frac{3}{4} \omega_j z^{4/3}, \quad j = 1, 2, 3, \quad -\pi < \arg z < \pi. \]

The asymptotics for \( P \) are uniform as \( z \to \infty \) in any region provided we stay away from the axis, that is, it is uniform as \( z \to \infty \) in the region

(3.22) \[ \text{dist} \left( \arg z, \frac{\pi}{2} \right) > \varepsilon, \]

for some \( \varepsilon > 0 \).

Proof. In [20] a detailed discussion of a RH problem for Pearcey integrals can be found. The lemma is proved by following these arguments. However, we do wish to make a few remarks on the asymptotics of \( P \). From classical steepest descent arguments one can prove as in [20] that

(3.23) \[ p_0(z) = \sqrt{\frac{2\pi}{3}} z^{-1/3} e^{\frac{3}{4} z^{1/3}} (1 + O(z^{-4/3})) \]

(3.24) \[ p_2(z) = \omega^2 \sqrt{\frac{2\pi}{3}} z^{-1/3} e^{\frac{3}{4} \omega z^{1/3}} (1 + O(z^{-4/3})) \]

(3.25) \[ p_5(z) = \omega \sqrt{\frac{2\pi}{3}} z^{-1/3} e^{\frac{3}{4} \omega^2 z^{1/3}} (1 + O(z^{-4/3})) \]

as \( z \to \infty \) in the first quadrant. In [20] these formulas (with a different numbering of the functions \( p_j \)) are stated with an error term \( O(z^{-2/3}) \). However, our case is a special case of the more general case considered in [20]. By analyzing the proof for our special case we obtain that the error term is indeed of order \( O(z^{-4/3}) \). From these asymptotics (and the corresponding ones for the derivatives of the Pearcey integrals) we obtain by (3.14) that

(3.26) \[ P(z) = \sqrt{\frac{2\pi}{3}} \begin{pmatrix} z^{-1/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{1/3} \end{pmatrix} \begin{pmatrix} 1 + O(z^{-4/3}) \end{pmatrix} B_1 \Theta_1(z), \]

as \( z \to \infty \) in the first quadrant. Since

(3.27) \[ \begin{pmatrix} z^{-1/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{1/3} \end{pmatrix} \begin{pmatrix} 1 + O(z^{-4/3}) \end{pmatrix} = \begin{pmatrix} z^{-1/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{1/3} \end{pmatrix} \begin{pmatrix} 1 + O(z^{-2/3}) \end{pmatrix}, \quad \text{as } z \to \infty, \]
we obtain the asymptotic behavior of $P$ in the first quadrant. The other quadrants can be dealt with in a similar way.

Finally, because of the Stokes phenomenon, the asymptotics for $p_0$ are only uniform if we stay away from the imaginary axis. Similarly, the asymptotics for $p_5$ are uniform if we stay away from the real axis. Hence the asymptotic behavior is uniform if we stay away from both the real and imaginary axis and let $z \to \infty$ so that it remains in the region given by (3.22) for some $\varepsilon > 0$. □

3.3. The inverse transpose of $P$. In the transformation $Y \mapsto X$ we will use

\[(3.28) \quad Q = P^{-t}.\]

Then $Q$ satisfies a RH problem which is easily obtained from the RH problem satisfied by $P$.

**Lemma 3.2.** Define $Q = P^{-t}$. Then $Q$ satisfies the following RH problem:

\[
\begin{cases}
Q \text{ is analytic in } \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}), \\
Q_+(z) = Q_-(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R}, \\
Q_+(z) = Q_-(z) \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in i\mathbb{R}, \\
Q(z) = \frac{1}{\sqrt{2\pi}}(I + O(z^{-2/3})) \begin{pmatrix} z^{1/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1/3} \end{pmatrix} A_j \Theta_j(z)^{-1}, \quad \text{as } z \to \infty \text{ in the } j\text{th quadrant,}
\end{cases}
\]

where $A_j = B_j^{-t}$, and the $B_j$ and $\Theta_j$, $j = 1, 2, 3, 4$, are given by (3.17)–(3.20).

The asymptotics for $Q$ are uniform as $z \to \infty$ in any quadrant such that (3.22) holds for some $\varepsilon > 0$.

**Proof.** This follows immediately from (3.28) and Lemma 3.1. □

The constant matrices $A_j$ are given explicitly by (with $\omega = e^{2\pi i/3}$),

\[
A_1 = \frac{i}{\sqrt{3}} \begin{pmatrix} -1 & \omega & \omega^2 \\ -1 & 1 & 1 \\ -1 & \omega^2 & \omega \end{pmatrix}, \quad A_2 = \frac{i}{\sqrt{3}} \begin{pmatrix} \omega & 1 & \omega^2 \\ 1 & 1 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix},
\]

\[
A_3 = \frac{i}{\sqrt{3}} \begin{pmatrix} \omega^2 & 1 & \omega \\ 1 & 1 & 1 \\ \omega & 1 & \omega^2 \end{pmatrix}, \quad A_4 = \frac{i}{\sqrt{3}} \begin{pmatrix} -1 & \omega^2 & \omega \\ -1 & 1 & 1 \\ -1 & \omega & \omega^2 \end{pmatrix}.
\]

The prefactor $i/\sqrt{3}$ is so that all matrices $A_j$ have determinant 1.
3.4. The transformation $Y \mapsto X$. For clarity we define the transformation $Y \mapsto X$ in two steps. First we define $Y \mapsto \hat{X}$ and then $\hat{X} \mapsto X$.

We define $\hat{X}$ by
\begin{equation}
\hat{X}(z) = \begin{pmatrix} 1 & 0 \\ 0 & C_n \end{pmatrix} Y(z) \begin{pmatrix} 1 & 0 \\ 0 & D_n Q(n^{3/4} \tau z) \end{pmatrix},
\end{equation}
where $D_n$ and $Q$ are given in (3.9), (3.28), and the constant prefactor $C_n$ is given by
\begin{equation}
C_n = \sqrt{\frac{2\pi}{n}} \begin{pmatrix} 2^{-1/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau^{1/3} \end{pmatrix}.
\end{equation}

We also define
\begin{equation}
X(z) = \hat{X}(z) \begin{pmatrix} 1 \\ 0 & \Theta_j(n^{3/4} \tau z) \end{pmatrix}
\end{equation}
for $z$ in the $j$th quadrant, where $\Theta_j$ is given by (3.19)–(3.20).

The $4 \times 4$ matrix valued functions $\hat{X}$ and $X$ are analytic in each of the four quadrants with jumps across the real and imaginary axes.

Lemma 3.3. We have that $\hat{X}$ satisfies the following RH problem:
\begin{equation}
\begin{cases}
\text{$\hat{X}$ is analytic in } \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}), \\
\hat{X}_+(z) = \hat{X}_-(z) \begin{pmatrix} 1 & e^{-nV(z)} \\ 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & z \in \mathbb{R}, \\
\hat{X}_+(z) = \hat{X}_-(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, & z \in i\mathbb{R}, \\
\hat{X}(z) = (I + \mathcal{O}(z^{-2/3})) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & z^{-\frac{n+1}{3}} & 0 & 0 \\ 0 & 0 & z^{-\frac{n}{3}} & 0 \\ 0 & 0 & 0 & z^{-\frac{n-1}{3}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ A_j \Theta_j^{-1}(n^{3/4} \tau z) \\ \frac{1}{n^{3/4} \tau z} \end{pmatrix}
\end{cases}
\end{equation}

as $z \to \infty$ in the $j$th quadrant.

The asymptotics for $\hat{X}$ are uniform as $z \to \infty$ in any quadrant provided we stay away from the axis, that is, it holds uniformly as $z \to \infty$ with (3.22) for some $\varepsilon > 0$.

Proof. Let $z \in \mathbb{R}$. Then we obtain from (1.26) and (3.32) that
\begin{equation}
\hat{X}_-(z) \hat{X}_+(z) = \begin{pmatrix} 1 & \left( w_{0,n}(z) \ w_{1,n}(z) \ w_{2,n}(z) \right) D_n Q_+(n^{3/4} \tau z) \\ 0 & Q_-(n^{3/4} \tau z)^{-1} Q_+(n^{3/4} \tau z) \end{pmatrix}.
\end{equation}
Since $Q = P^{-t}$, we can use (3.11) to find

$$(w_{0,n}(z) \quad w_{1,n}(z) \quad w_{2,n}(z)) D_n Q_+(n^{3/4} \tau z) = \left( e^{-nV(z)} \quad 0 \quad 0 \right).$$

Using also the jump condition on the real axis in the RH problem (1.26) for $Q$, we obtain the jump matrix for $\hat{X}$ on the real line as given in the RH problem (3.35).

The proof of the jump on the imaginary axis is similar (even simpler).

Finally we check the asymptotic formula for $\hat{X}$. From the asymptotic behavior of $Q$ given in (3.29) and the definitions (3.9) and (3.33), of $D_n$ and $C_n$, we obtain as $z \to \infty$ in the $j$th quadrant,

$$(3.37) \quad D_n Q(n^{3/4} \tau z) = C_n^{-1}(I + \mathcal{O}(z^{-2/3})) \begin{pmatrix} z^{1/3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1/3} \end{pmatrix} A_j \Theta_j^{-1}(n^{3/4} \tau z).$$

From the asymptotic condition in the RH problem (1.26) for $Y$ and the definition (3.32), it follows that

$$(3.38) \quad \hat{X}(z) = \begin{pmatrix} 1 & 0 \\ 0 & C_n \end{pmatrix} (I + \mathcal{O}(z^{-1})) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & z^{-n/3} & 0 & 0 \\ 0 & 0 & z^{-n/3} & 0 \\ 0 & 0 & 0 & z^{-n/3} \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 \\ 0 & D_n Q(n^{3/4} \tau z) \end{pmatrix}.$$

Combining this with (3.37) gives the asymptotic condition for $\hat{X}$ in (3.35).

The transformation (3.34) from $\hat{X}$ to $X$ has the effect of simplifying the asymptotic condition in the RH problem, but the jumps on the real and imaginary axis are more complicated.

**Lemma 3.4.** We have that $X$ satisfies the following RH problem:

$$(3.39) \quad \begin{cases} X \text{ is analytic in } \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}), \\ X_+(z) = X_-(z)J_X(z), \quad z \in \mathbb{R} \cup i\mathbb{R}, \\ X(z) = (I + \mathcal{O}(z^{-2/3})) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & z^{-n+1/3} & 0 & 0 \\ 0 & 0 & z^{-n/3} & 0 \\ 0 & 0 & 0 & z^{-n+1/3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A_j \end{pmatrix} \end{cases} \text{ as } z \to \infty \text{ in the } j\text{th quadrant}$$
where the jump matrices $J_X$ are given by

$$J_X(z) = \begin{pmatrix} 1 & e^{-n(V(z) - \frac{1}{2} \tau \frac{4}{3} |z|^{4/3})} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{\pm n \frac{3\sqrt{3}}{4} \tau \frac{4}{3} |z|^{4/3}} & 1 \\ 0 & 0 & 0 & e^{\pm n \frac{3\sqrt{3}}{4} \tau \frac{4}{3} |z|^{4/3}} \end{pmatrix},$$

for $z \in \mathbb{R}_+$, and

$$J_X(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\pm n \frac{3\sqrt{3}}{4} \tau \frac{4}{3} |z|^{4/3}} & 0 & 0 \\ 0 & 1 & e^{\pm n \frac{3\sqrt{3}}{4} \tau \frac{4}{3} |z|^{4/3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $z \in i\mathbb{R}_\pm$.

The asymptotics for $X$ are uniform as $z \to \infty$ in any quadrant provided we stay away from the axis, that is, it holds uniformly as $z \to \infty$ with (3.22) for some $\varepsilon > 0$.

Proof. The asymptotic condition is clear from (3.34) and the asymptotic condition in the RH problem (3.35) for $\hat{X}$.

Let $z \in \mathbb{R}_+$. Then it follows from (3.34), the jump condition in the RH problem (3.35), and the expressions for $\Theta_1$ and $\Theta_4$ from (3.19)–(3.20) that

$$X^{-1}(z)X_+(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \Theta_4^{-1}(n^{3/4}z) & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \Theta_1(n^{3/4}z) & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
in the jump matrix (3.40) on \( \mathbb{R} \) and the measure \( \mu_3 \) will handle the lower right block. The measure \( \mu_2 \) is supported on the imaginary axis and will handle the middle \( 2 \times 2 \) block in the jump matrix (3.41) on the imaginary axis.

Thus \( \mu_1 \) will act on the first and second rows and columns, \( \mu_2 \) and the second and third, and \( \mu_3 \) and the third and fourth. Then we also see that \( \mu_1 \) and \( \mu_2 \) interact in some way, since they both act on the second row and column. Similarly, \( \mu_2 \) and \( \mu_3 \) will interact. On the other hand, the measures \( \mu_1 \) and \( \mu_3 \) do not interact with each other, since they act on different rows and columns. This is reflected in the equilibrium problem where we have that the mutual logarithmic energies \( I(\mu_1, \mu_2) \) and \( I(\mu_2, \mu_3) \) are present, but not the mutual logarithmic energy of \( \mu_1 \) and \( \mu_3 \).

The potential \( V(z) \) in (3.40) comes with an extra term \(-\frac{3}{4} \tau^{4/3} |z|^{4/3}\). Together they appear in the equilibrium problem as the external field on \( \mu_1 \). The lower \( 2 \times 2 \) block in (3.40) and the middle \( 2 \times 2 \) block in (3.41) have rapidly oscillating entries on the diagonal. This is reminiscent of what happens in the steepest descent analysis of the RH problem for orthogonal polynomials after the application of the \( g \)-functions [33].

We will explain at a later stage why the restriction \( \mu_2 \leq \sigma \) plays a role. Here we can only say that it is partly explained by the triangular structure of the jump matrices (3.41) and (3.40). Indeed the non-trivial \( 2 \times 2 \) block in (3.41) is lower triangular, while the \( 2 \times 2 \) blocks in (3.40) are upper triangular.

4. Analysis of the equilibrium problem

To prepare for the next transformation of the RH problem we need the equilibrium problem for the energy functional \( E_V \). In this section we prove Theorem 2.3 which gives existence and properties of the equilibrium measures \( \mu_1, \mu_2 \) and \( \mu_3 \). We also prove Propositions 2.4 and 2.6. A basic tool is the balayage (sweeping out) of a measure.

4.1. Balayage. The balayage of a finite measure \( \nu \) onto a closed set \( K \) with positive capacity is a positive measure \( \tilde{\nu} \) on \( K \) with \( \|\nu\| = \|\tilde{\nu}\| \) and with the property that there exists a constant \( C \) such that

\[
U^\nu(x) = U^{\tilde{\nu}}(x) + C, \quad \text{q.e. } x \in K,
\]

where q.e. means quasi-everywhere, that is, with the exception of a set of zero capacity. We will be interested in cases where \( K \) is either the real or imaginary axis, or \( K \) consists of two unbounded intervals along the imaginary axis

\[
K_c = (-i\infty, -ic] \cup [ic, i\infty), \quad c > 0.
\]

In these cases we have \( C = 0 \) for the constant in (4.1).
We use the notation \( \hat{\nu} \) for the balayage of \( \nu \) if \( K \) is understood. To emphasize \( K \) we write
\[
\hat{\nu} = \text{Bal}(\nu, K).
\]

Each balayage measure can be written as the integral over balayage measures of Dirac-delta measures as follows,
\[
\text{Bal}(\nu, K) = \int \text{Bal}(\delta_z, K) \, d\nu(z).
\]
Hence it is sufficient (for our purposes) to calculate \( \text{Bal}(\delta_z, K) \) for several \( z \) and \( K \).

**Example 4.1.** If \( K = \mathbb{R} \) is the real line, and \( y \neq 0 \), then the balayage measure of \( \delta_y \) onto \( \mathbb{R} \) has the Cauchy density
\[
\frac{d}{dx} \text{Bal}(\delta_y, \mathbb{R})(x) = \frac{1}{\pi} \frac{|y|}{x^2 + y^2}, \quad x \in \mathbb{R}.
\]

For \( K = i\mathbb{R} \) and \( x \neq 0 \) we have similarly
\[
\frac{d}{dz} \text{Bal}(\delta_y, i\mathbb{R})(z) = \frac{1}{\pi} \frac{|z|}{|z|^2 + x^2}, \quad z \in i\mathbb{R}.
\]

These results are well-known. They are also the limiting case \( c \to 0^+ \) of (4.8) in the next example.

**Example 4.2.** Let \( K_c = (-i\infty, -ic] \cup [ic, i\infty) \) with \( c > 0 \). If \( y \in (-c, c) \) then the balayage of \( \delta_y \) onto \( K_c \) has the density
\[
\frac{d}{dz} \text{Bal}(\delta_y, K_c)(z) = \frac{1}{\pi} \frac{\sqrt{c^2 - y^2}}{|z| - iy\sqrt{|z|^2 - c^2}}, \quad z \in K_c.
\]

If \( x \in \mathbb{R} \) then the balayage of \( \delta_x \) onto \( K_c \) has the density
\[
\frac{d}{dz} \text{Bal}(\delta_x, K_c)(z) = \frac{1}{\pi} \frac{|z\sqrt{c^2 + x^2}}{(|z|^2 + x^2)^{1/2}\sqrt{|z|^2 - c^2}}, \quad z \in K_c.
\]

In all cases we have that the constant \( C \) in (4.1) vanishes.

These expressions can be proved by simple contour integration. We omit the proofs. It will be important for us that the densities (4.7) and (4.8) are decreasing as \( |z| \) increases, with a decay rate \( O(|z|^{-2}) \) as \( |z| \to \infty \).

In the following we will see how the notion of balayage is related to the equilibrium problem.

### 4.2. Equilibrium problem for \( \nu_3 \)

If we fix two of the measures \( \nu_j, j = 1, 2, 3 \) we can consider the equilibrium problem with respect to the remaining measure only. If \( \nu_1 \) and \( \nu_2 \) are given, then the equilibrium problem for \( \nu_3 \) is to minimize
\[
I(\nu) - \int U^{\nu_3} \, d\nu
\]
with respect to all measures \( \nu \) on \( \mathbb{R} \) with total mass \( 1/3 \). This problem only depends on \( \nu_2 \). Assuming that \( \nu_2 \) has finite logarithmic energy and total mass \( 2/3 \) (as in conditions (a) and (c) of the equilibrium problem), the minimizer for \( (4.9) \) is given by \( \nu_3 = \frac{1}{2} \text{Bal}(\nu_2, \mathbb{R}) \) since it satisfies \( U^{\nu_3} = \frac{1}{2} U^{\nu_2} \) on \( \mathbb{R} \), which is the variational condition for \( (4.9) \), and it has the correct total mass \( 1/3 \). So we have

**Lemma 4.3.** Suppose \( \nu_1 \) and \( \nu_2 \) are fixed so that conditions (a), (b), and (c) of the equilibrium problem are satisfied. Then the measure \( \nu_3 \) that minimizes \( E_V(\nu_1, \nu_2, \nu_3) \) subject to the condition (d) is given by

\[
(4.10) \quad \nu_3 = \frac{1}{2} \text{Bal}(\nu_2, \mathbb{R}).
\]

Thus by (4.5) and (4.4), we see that \( \nu_3 \) has a density given by

\[
\frac{d\nu_3}{dx}(x) = \frac{1}{\pi} \int \frac{|z|}{x^2 + |z|^2} d\nu_2(z).
\]

### 4.3. Equilibrium problem for \( \nu_1 \)

For given \( \nu_2 \) and \( \nu_3 \) having finite logarithmic energy, the equilibrium problem for \( \nu_1 \) is to minimize

\[
(4.11) \quad I(\nu) + \int \left( V(x) - \frac{3}{4} r^{4/3}|x|^{4/3} - U^{\nu_2}(x) \right) d\nu(x)
\]

among all probability measures \( \nu \) on \( \mathbb{R} \). This is a usual equilibrium problem on \( \mathbb{R} \) with external field

\[
V(x) - \frac{3}{4} r^{4/3}|x|^{4/3} - U^{\nu_2}(x).
\]

The term \(-U^{\nu_2}\) in the external field attracts the measure \( \nu \) towards 0.

**Lemma 4.4.** Suppose \( \nu_2 \) and \( \nu_3 \) are fixed so that conditions (a), (c), (d) and (e) of the equilibrium problem are satisfied. Then the measure \( \nu_1 \) that minimizes \( E_V(\nu_1, \nu_2, \nu_3) \) subject to the condition (b) is the minimizer for \( (4.11) \).

It has the following properties:

(a) The support of \( \nu_1 \) is contained in the convex hull of the support of the equilibrium measure on \( \mathbb{R} \) in the external field \( V(x) - \frac{3}{4} r^{4/3}|x|^{4/3} \).

(b) If \( \nu_2 \) is so that there exists a constant \( c > 0 \) such that \( \mathcal{S}(\sigma - \nu_2) \subset K_c \) where \( K_c \) is given by (4.2), then

\[
\frac{3}{4} r^{4/3}|x|^{4/3} + U^{\nu_2}(x)
\]

is real analytic on \( \mathbb{R} \).

**Proof.** (a) We use the fact that the support of the equilibrium measure with external field \( Q \) on \( \mathbb{R} \) is the closure of those points where weighted polynomials \( e^{-nQ}P_n^2 \), \( \deg P_n \leq n, n \in \mathbb{N} \), take their maximum modulus [71, Chapter III.2].

Assume the equilibrium measure in the external field \( Q(x) = V(x) - \frac{3}{4} r^{4/3}|x|^{4/3} \) is supported in \([-X, X]\) for some \( X > 0 \). Hence the maximum
modulus of $e^{-nQ} P_n^2$, deg $P_n \leq n$, is attained only in $[-X, X]$. Since $U^{\nu_2}(x) = \int \log \frac{1}{|x-z|} d\nu_2(z)$ is even on $\mathbb{R}$ and strictly decreasing as $|x|$ increases, it then follows that the maximum modulus of $e^{nU^{\nu_2}} e^{-nQ} P_n^2 = e^{-n(Q-U^{\nu_2})} P_n^2$, deg $P_n \leq n$ is attained on $[-X, X]$ only, which proves part (a) of the lemma.

(b) Suppose $\nu_2$ and $c > 0$ are such that $S(\sigma - \nu_2) \subset K_c$. Then

\begin{equation}
U^{\nu_2}(x) = -\int_{-ic}^{ic} \log |x-z| d\sigma(z) - \int_{|z| \geq c} \log |x-z| d\nu_2(z),
\end{equation}

and the second term is real analytic for $x \in \mathbb{R}$. For the first term in (4.12) we have by the definition (2.7) of $\sigma$,

\begin{equation}
-\int_{-ic}^{ic} \log |x-z| d\sigma(z) = -\frac{\sqrt{3}}{2\pi} \int_0^c \log(x^2 + y^2) y^{1/3} dy,
\end{equation}

whose derivative with respect to $x$ is

\begin{equation}
-\frac{\sqrt{3}}{\pi} \frac{1}{r^{4/3}} \int_0^c \frac{xy^{1/3}}{x^2 + y^2} dy = -\frac{\sqrt{3}}{\pi} \int_0^\infty \frac{xy^{1/3}}{x^2 + y^2} dy + \frac{\sqrt{3}}{\pi} \int_c^\infty \frac{xy^{1/3}}{x^2 + y^2} dy.
\end{equation}

The second term on the right-hand side of (4.14) is real analytic on $\mathbb{R}$. In the first term we introduce the change of variables $y = |x| s$ to obtain

\begin{equation}
\int_0^\infty \frac{xy^{1/3}}{x^2 + y^2} dy = x|x|^{-2/3} \int_0^\infty \frac{s^{1/3}}{1 + s^2} ds.
\end{equation}

Because of the standard integral

\begin{equation}
\int_0^\infty \frac{s^p}{1 + s^2} ds = \frac{\pi}{2 \cos(p\pi/2)}, \quad -1 < p < 1,
\end{equation}

which for $p = 1/3$ gives the value $\frac{1}{3}\sqrt{3}\pi$, it then follows that

\begin{equation}
-\frac{\sqrt{3}}{\pi} \int_0^\infty \frac{xy^{1/3}}{x^2 + y^2} dy = -\tau^{4/3} x|x|^{-2/3}.
\end{equation}

Combining this with (4.12), (4.13), and (4.14), we obtain

\begin{equation}
\frac{d}{dx} U^{\nu_2}(x) = -\tau^{4/3} x|x|^{-2/3} + \text{“real analytic function”},
\end{equation}

which after integration completes the proof of part (b) of the lemma.

Lemmas 4.3 and 4.4 show how the solutions of the equilibrium problems for $\nu_1$ and $\nu_3$ are determined by the measure $\nu_2$. Next we study the converse: given $\nu_1$ and $\nu_3$, what are the properties of the solution of the equilibrium problem for $\nu_2$?
4.4. Equilibrium problem for \( \nu_2 \). Now suppose that we are given \( \nu_1 \) and \( \nu_3 \) satisfying the conditions (a), (b), (d) of the equilibrium problem and we wish to minimize \( E_V(\nu_1, \nu_2, \nu_3) \) with respect to \( \nu_2 \). The equilibrium problem for \( \nu_2 \) is to minimize

\[
I(\nu) - \int (U^{\nu_1} + U^{\nu_3}) \, d\nu
\]

with respect to all measures \( \nu \) on \( i\mathbb{R} \) with total mass \( 2/3 \) and satisfying the constraint \( \nu \leq \sigma \).

If the constraint \( \nu \leq \sigma \) were not present, then simple balayage arguments would give that the minimizer is given by

\[
\frac{1}{2} \text{Bal} (\nu_1 + \nu_3, i\mathbb{R})
\]

Note that this measure has indeed the correct total mass \( 2/3 \). However, this measure will violate the constraint. Indeed, the density of the balayage of any measure on \( \mathbb{R} \) onto \( i\mathbb{R} \) has its maximum at 0, and is strictly decreasing for \( z \in i\mathbb{R} \) if \( |z| \) increases, see also [4.10]. Since the density of \( \sigma \) [2.7] vanishes at \( z = 0 \) and is strictly increasing if \( |z| \) increases, it is clear that the density of the balayage measure lies strictly above the density of \( \sigma \) on a non-empty symmetric interval around 0 on \( i\mathbb{R} \).

**Lemma 4.5.** Suppose that \( \nu_1 \) and \( \nu_3 \) are fixed so that conditions (a), (b), and (d) of the equilibrium problem are satisfied. Then the measure \( \nu_2 \) that minimizes \( E_V(\nu_1, \nu_2, \nu_3) \) subject to the conditions (c) and (e) exists, and there is a constant \( c > 0 \) such that

\[
\text{supp}(\sigma - \nu_2) = K_c := (-ic, -ic] \cup [ic, ic)
\]

Moreover, \( \nu_2 \) satisfies

\[
\begin{cases}
U^{\nu_2}(z) = U^{\nu_1}(z) + U^{\nu_3}(z), & z \in i\mathbb{R} \setminus (-ic, ic) \\
U^{\nu_2}(z) < U^{\nu_1}(z) + U^{\nu_3}(z), & z \in (-ic, ic).
\end{cases}
\]

and \( \sigma - \nu_2 \) vanishes as a square root at \( \pm ic \).

**Proof.** We use the iterated balayage algorithm of [53]. Write

\[
\nu_{2,0} := \frac{1}{2} \text{Bal}(\nu_1 + \nu_3, i\mathbb{R})
\]

As already noted above, there exists \( c_0 \) such that

\[
S((\nu_{2,0} - \sigma)^+) = [-ic_0, ic_0]
\]

where \( (\nu_{2,0} - \sigma)^+ \) denotes the positive part of the signed measure \( \nu_{2,0} - \sigma \). It then follows from the saturation principle of [39] that the minimizer \( \nu_2 \) (if it exists) is equal to the constraint \( \sigma \) on \( [-ic_0, ic_0] \).

Then we define

\[
\nu_{2,1} := \min(\sigma, \nu_{2,0}) + \text{Bal} ((\nu_{2,0} - \sigma)^+, K_{c_0})
\]

In other words, we balayage the part of \( \nu_{2,0} \) that lies above \( \sigma \) onto the part of the imaginary axis \( K_{c_0} \) where the constraint is not (yet) active. Then \( \nu_{2,1} \)
has a density for \( z \in K_c \) that is strictly decreasing as \(|z|\) increases, see also (4.7). Then there exists \( c_1 > c_0 \) so that
\[
S((\nu_{2,1} - \sigma)^+) = [-ic_1, -ic_0] \cup [ic_0, ic_1]
\]
and we define
\[
\nu_{2,2} := \min(\sigma, \nu_{2,1}) + \text{Bal}((\nu_{2,1} - \sigma)^+, K_{c_1}).
\]
Continuing this way we find an increasing sequence \((c_k)\) and a sequence \((\nu_{2,k})\) of measures of total mass 2/3. Clearly, the \( c_k \) must be such that
\[
\sigma([-ic_k, ic_k]) \leq \frac{2}{3}.
\]
Hence the sequence converges and we denote the limit with \( c \). It follows as in [53] that \((\nu_{2,k})\) converges weakly to a measure \( \nu_2 \). In fact, after each step in the iterated balayage process we have that
\[
\begin{align*}
U^\nu_{2,k+1}(x) &= U^\nu_{2,k}(z), \quad z \in i\mathbb{R} \setminus (-ic_k, ic_k), \\
U^\nu_{2,k+1}(x) &< U^\nu_{2,k}(z), \quad z \in (-ic_k, ic_k).
\end{align*}
\]
From these equations and the fact that \( 2U^{\nu_{2,0}}(z) = U^{\nu_1}(z) + U^{\nu_3}(z) \) for all \( z \in i\mathbb{R} \), we can easily deduce (4.16). Hence \( \nu_2 \) satisfies the variational conditions (with strict inequality) that uniquely characterize the minimizer.

From the above arguments it also follows that the density of \( \nu_2 \) for \( z \in K_c \) is strictly decreasing as \(|z|\) increases. Therefore the density of \( \nu_2 \) lies strictly below the density of \( \sigma \) on \((-i\infty, -ic) \cup (ic, i\infty)\) and there is a square root vanishing of the density of \( \sigma - \nu_2 \) at \( \pm ic \). \( \square \)

4.5. Proof of uniqueness of the minimizer. We start by rewriting the energy functional as
\[
E_V(\nu_1, \nu_2, \nu_3) = \frac{1}{3}I(\nu_1) + \frac{1}{3}I(2\nu_1 - 3\nu_2) + \frac{1}{4}I(\nu_2 - 2\nu_3) + \int \left(V(x) - \frac{3}{4}x^{4/3}\right)d\nu_1(x).
\]

If two measures \( \nu \) and \( \mu \) have finite logarithmic energy and \( \int d\nu = \int d\mu \), then
\[
I(\nu - \mu) \geq 0,
\]
with equality if and only if \( \nu = \mu \). This is a well-known result if \( \nu_1 \) and \( \nu_2 \) have compact supports [71]. For measures with unbounded support, this is a more recent result of Simeonov [72], who obtained this from a very elegant integral representation for \( I(\nu - \mu) \).

It follows from (4.22) that the energy functional (4.21) is strictly convex. The minimizer is therefore uniquely characterized by the Euler-Lagrange variational conditions listed in Proposition [74].
Proof of the existence of the minimizer. We rewrite $E_V$ again as in (4.21). From (4.22) it follows that

$$I(2\nu_1 - 3\nu_2) > 0 \text{ and } I(\nu_2 - 2\nu_3) > 0.$$  

Combining this with (4.21) leads to

$$E_V(\nu_1, \nu_2, \nu_3) \geq \frac{1}{3} I(\nu_1) + \int \left( V(x) - \frac{3}{4} x^{4/3} |x|^{4/3} \right) d\nu_1(x),$$

and this implies that $E_V$ is bounded from below.

A natural approach to prove the existence of the minimizer is now the following. Since $E_V$ is bounded from below we can approach the infimum by a sequence of measures satisfying the conditions of the equilibrium problem. If we can prove that this sequence is tight, then we have that the sequence converges weakly to the minimizer of the $E_V$. However, it seems complicated to follow this procedure. The main difficulty is the fact that the measures $\nu_2$ and $\nu_3$ will have unbounded supports with rather fat tails. It turns out that their densities behave like $O(|z|^{-5/3})$ as $|z| \to \infty$.

Our method of proof consists of restricting the measure $\nu_2$ by an additional constraint

$$\nu_2 \leq K\sigma_p$$

where $\sigma_p$ is the measure on $\mathbb{R}$ with density

$$d\sigma_p(z) = \frac{1}{|z|^p} |dz|, \quad z \in \mathbb{R},$$

with $p \in (1, 5/3)$ and $K > 0$ is a suitable constant that will be determined later on. So we consider the equilibrium problem with extra constraint (4.25) and we prove that for the minimizer the extra constraint is not active.

The special role of the value $p = 5/3$ is made clear in the following lemma.

Lemma 4.6. Let $1 < p < 2$. If $\nu_2 \leq K\sigma_p$ and $\nu_3 = \frac{1}{2} \text{Bal}(\nu_2, \mathbb{R})$ then

$$\frac{d\nu_3}{dx} \leq C_p \frac{K}{|x|^p}, \quad x \in \mathbb{R},$$

with constant

$$C_p = \frac{1}{2 \sin(p\pi/2)}.$$  

We have $C_p < 1$ if and only if $p < 5/3$.

Proof. By direct calculation

$$\frac{d\nu_3}{dx} = \frac{1}{2\pi} \int \frac{|z|}{x^2 + |z|^2} d\nu_2(z) \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|y|}{x^2 + y^2} \frac{K}{|y|^p} dy$$

$$= \frac{K}{\pi} \int_0^\infty \frac{y^{1-p}}{x^2 + y^2} dy, \quad x \in \mathbb{R}.$$
Make the change of variables \( y = |x|s \) and use (4.15) to obtain
\[
\frac{d\nu_2}{dx} \leq \frac{K}{\pi|x|^p} \int_0^\infty \frac{s^{1-p}}{1 + s^2} ds = \frac{K}{\pi|x|^p} \frac{\pi}{2 \cos((1 - p)\pi/2)}
\]
\[
= \frac{K}{|x|^p} \frac{1}{2 \sin(p\pi/2)}, \quad x \in \mathbb{R}.
\]
This proves the lemma. Note that indeed \( C_p < 1 \) if and only if \( p < 5/3 \). \( \square \)

We may iterate the above argument and we obtain

**Corollary 4.7.** If \( \nu_2 \leq K\sigma_p \) and \( \nu_3 = \frac{1}{2} \text{Bal}(\nu_2, \mathbb{R}) \), then
\[
(4.29) \quad \frac{1}{2} \text{Bal}(\nu_3, i\mathbb{R}) \leq C_p^2 K\sigma_p.
\]

We fix \( p \in (1, 5/3) \) and we show first what constant \( K > 0 \) to take. Let \( X > 0 \) be such that the support of the equilibrium measure in the external field \( V(x) - \frac{2}{3} x^{4/3}|x|^{1/3} \) is contained in \([-X, X]\).

From the explicit formulas (4.7) and (4.8) for the balayage of point masses in \([-ic, ic] \cup \mathbb{R} \) onto \( K_c \), we can easily find the bounds
\[
\frac{d \text{Bal}(\delta_y, K_c)}{|dz|} \leq \frac{1}{\pi} \frac{c}{|z|^2 - c^2}, \quad z \in K_c,
\]
and
\[
\frac{d \text{Bal}(\delta_x, K_c)}{|dz|} \leq \frac{1}{\pi} \frac{\sqrt{c^2 + x^2}}{|z|^2 - c^2}, \quad z \in K_c.
\]
From this it follows that
\[
(4.30) \quad \frac{d \text{Bal}(\rho, K_c)}{|dz|} \leq \frac{1}{\pi} \frac{\sqrt{c^2 + X^2}}{|z|^2 - c^2}, \quad z \in K_c,
\]
for every probability measure \( \rho \) on \([-X, X] \cup [-ic, ic]\).

Let \( c_0 \) be such that \( \sigma([-ic_0, ic_0]) = 2/3 \). It follows from (2.7) and (4.30) that there exists a constant \( L = L_X \) so that
\[
(4.31) \quad \min \left( \frac{d\sigma}{|dz|}, \frac{d \text{Bal}(\rho, K_{c_0})}{|dz|} \right) \leq \frac{L}{|z|^p}, \quad z \in K_{c_0}
\]
for every probability measure \( \rho \) on \([-X, X] \cup [-ic_0, ic_0]\). Since \( C_p < 1 \) we can then find a constant \( K \) large enough so that
\[
(4.32) \quad C_p^2 K + \frac{2}{3} L < K.
\]
This is how we choose \( K \) and this determines the extra constraint \( K\sigma_p \).

We now claim that there exists a vector of measures \( (\nu_1^*, \nu_2^*, \nu_3^*) \) that minimizes \( E_V(\nu_1, \nu_2, \nu_3) \) subject to the conditions \((a) - (e)\) of the equilibrium problem of Definition 2.2 and subject to the additional constraint
\[
(4.33) \quad \nu_2 \leq K\sigma_p
\]
with \( K \) satisfying (4.32).
We already found that $E_V$ is bounded from below. Therefore there exists a sequence $(\nu_{1,n}, \nu_{2,n}, \nu_{3,n})$ such that

\begin{equation}
E_V(\nu_{1,n}, \nu_{2,n}, \nu_{3,n}) \leq \frac{1}{n} + \inf E_V
\end{equation}

where both the sequence and the infimum are taken with respect to conditions (a)-(e) of the equilibrium problem and (4.33). By Lemma 4.3 we may as well assume that

\begin{equation}
\nu_{3,n} = \frac{1}{2} \text{Bal}(\nu_{2,n}, \mathbb{R}).
\end{equation}

Moreover, we take $\nu_{1,n}$ as the minimizer of $E(\nu_1, \nu_{2,n}, \nu_{3,n})$ subject to condition (b). By (4.25) and Lemma 4.6 the sequences $\nu_{2,n}$ and $\nu_{3,n}$ are tight. Moreover, by Lemma 4.4 and the choice of $X$ we have that the measures $\nu_{1,n}$ are supported in $[-X, X]$. Therefore there exists a subsequence $(\nu_{1,kn}, \nu_{2,kn}, \nu_{3,kn})$ that converges weakly to a vector of measures $(\nu_1^*, \nu_2^*, \nu_3^*)$. This is the minimizer that we seek for and hence we proved the claim.

Our final goal is to show that the extra upper constraint $K\sigma_p$ is not active.

Given the measures $\nu_1^*$ and $\nu_3^*$ from the minimizer of the equilibrium problem with extra constraint, we denote by $\nu_2^{**}$ the measure that minimizes the energy functional $E_V(\nu_1^*, \nu_2, \nu_3^*)$ subject to the constraint $\nu_2 \leq \sigma$ only. By Lemma 4.5 we then have that $\nu_2^{**}$ is equal to $\sigma$ on some interval $[-ic, ic]$. We clearly have $c < c_0$. It also follows from properties of balayage that

\begin{equation}
\nu_2^{**}|_{K_{c_0}} \leq \frac{1}{2} \text{Bal}(\nu_1^* + \nu_3^*, K_{c_0})
\end{equation}

where

\begin{equation}
\rho = \frac{1}{2} \nu_1^* + \frac{1}{2} \text{Bal}(\nu_3^*, i\mathbb{R})|_{[-ic_0, ic_0]}
\end{equation}

is a measure on $[-X, X] \cup [-ic_0, ic_0]$ with total mass $\leq 2/3$. Recalling also that

\begin{equation}
\nu_2^{**} \leq \sigma
\end{equation}

we find by Corollary 4.7 (4.31), and (4.36) that

\begin{equation}
\frac{d\nu_2^{**}}{|dz|} \leq C^2_p K \frac{2}{3} \frac{L}{|z|^p} \quad z \in i\mathbb{R}.
\end{equation}

Then it follows from (4.32) that

\begin{equation}
\frac{d\nu_2^{**}}{|dz|} < \frac{K}{|z|^p}, \quad z \in i\mathbb{R}.
\end{equation}

Thus $\nu_2^{**}$ would also be the minimizer if we impose the extra constraint $\nu_2 \leq K\sigma_p$. It follows that

\begin{equation}
\nu_2^* = \nu_2^{**}
\end{equation}

and since the inequality in (4.37) is strict, it follows that $\nu_2^*$ does not feel the extra constraint.
Thus \((\nu_1^*, \nu_2^*, \nu_3^*)\) is also the minimizer of the equilibrium problem of Definition 2.2 (without extra constraint), and existence of the minimizer is now proved.

4.7. Proof of Theorem 2.3 and proof of Proposition 2.4.

Proof. We have shown that a unique minimizer \((\mu_1, \mu_2, \mu_3)\) for the equilibrium problem exists.

Proposition 2.4 and parts (a)-(c) of Theorem 2.3 follow by taking the minimizer \((\mu_1, \mu_2, \mu_3)\) for the vector \((\nu_1, \nu_2, \nu_3)\) in Lemmas 4.5, 4.3 and 4.4.

Part (d) of Theorem 2.3 follows from the symmetry in the equilibrium problem (note that \(V\) is even) and the uniqueness of the minimizer.

4.8. Proof of Proposition 2.6.

Proof. The measure \(\mu_1\) is the unique minimizer of the energy functional given in (4.11). Since the external field is symmetric, it follows from [71, Theorem IV.1.10 (f)] that \(d \mu_1(t) = d \tilde{\mu}(t^2)/2\) where \(\tilde{\mu}\) is the unique minimizer of the energy functional

\[
\iint \log \frac{1}{|x-y|} d\nu(x) d\nu(y) + 2 \int \left( V(\sqrt{x}) - \tau^{4/3} x^{2/3} - U^{\mu_2}(\sqrt{x}) \right) d\nu(x),
\]

among all probability measures \(\nu\) with support on \([0, \infty)\). From

\[
-U^{\mu_2}(\sqrt{x}) = -\frac{1}{2} \int \log(x + |z|^2) d\mu_2(z)
\]

it follows that \(x \mapsto -U^{\mu_2}(\sqrt{x})\) is a convex function on \([0, \infty)\). Also \(x \mapsto -\tau^{4/3} x^{2/3}\) is convex on \([0, \infty)\). Hence, since we assumed that \(x \mapsto V(\sqrt{x})\) is convex as well, the external field in the equilibrium problem for \(\tilde{\mu}\) is convex. Thus the support of \(\tilde{\mu}\) consists of one interval \([a, b]\) with \(0 \leq a < b\), see e.g. [71, Theorem IV.1.10]. Hence the support of \(\mu_1\) is either one or two intervals, depending on whether \(a = 0\) or \(a > 0\).

5. A Riemann surface

5.1. A four-sheeted Riemann surface. We use the solution of the equilibrium problem to construct a Riemann surface as follows. Start with the four sheets \(\mathcal{R}_j\) defined as

\[
\mathcal{R}_1 = \mathbb{C} \setminus S(\mu_1), \quad \mathcal{R}_2 = \mathbb{C} \setminus (S(\mu_1) \cup S(\sigma - \mu_2)), \\
\mathcal{R}_3 = \mathbb{C} \setminus (S(\sigma - \mu_2) \cup S(\mu_3)), \quad \mathcal{R}_4 = \mathbb{C} \setminus S(\mu_3).
\]

The four sheets are connected as follows: \(\mathcal{R}_1\) is connected to \(\mathcal{R}_2\) via \(S(\mu_1)\), \(\mathcal{R}_2\) is connected to \(\mathcal{R}_3\) via \(S(\sigma - \mu_2)\) and \(\mathcal{R}_3\) is connected to \(\mathcal{R}_4\) via \(S(\mu_3)\), every connection is in the usual crosswise manner. See also Figure 5.1 for a picture of the Riemann surface in the case where the support of \(\mu_1\) consists of one interval.

The Riemann surface is compactified by adding two points at infinity: one is on the first sheet, and the other point at infinity is common to the other
three sheets. The Riemann surface has genus $N - 1$ if $N$ is the number of
intervals in the support of $\mu_1$.

5.2. The Cauchy transforms. The Cauchy transform $F$ of a measure $\mu$
is defined as

$$F(z) = \int \frac{1}{z - x} \, d\mu(x), \quad z \in \mathbb{C} \setminus S(\mu).$$

Note that we slightly abuse the notion of Cauchy transform since

$$C_\mu(z) = \frac{1}{2\pi i} \int \frac{1}{x - z} \, d\mu(x) = -\frac{1}{2\pi i} F(z)$$

is also called the Cauchy transform of $\mu$.

We are interested in particular in the Cauchy transforms of the measures
$\mu_1$, $\mu_2$ and $\mu_3$ that are the solution of the equilibrium problem of Definition

We denote their Cauchy transforms by $F_1$, $F_2$, $F_3$, respectively. We
use the Cauchy transforms $F_j$ to construct a meromorphic function on the
Riemann surface $\mathcal{R}$. We recall the Sokhotski-Plemelj formulas according to
which

$$F_{j,+}(z) + F_{j,-}(z) = 2PV \int \frac{1}{z - x} \, d\mu_j(x),$$

$$F_{j,+}(z) - F_{j,-}(z) = -2\pi i \frac{d\mu_j}{dz},$$

where PV denotes the Cauchy principal value.
Lemma 5.1. The function \( \xi : \bigcup_{j=1}^{4} \mathcal{R}_j \to \mathbb{C} \) defined by

\[
(5.3) \quad \xi(z) = \begin{cases} 
V'(z) - F_1(z), & z \in \mathcal{R}_1, \\
F_1(z) - F_2(z) + \tau^{4/3}z^{1/3}, & z \in \mathcal{R}_2, \ \Re z > 0, \\
F_1(z) - F_2(z) - \tau^{4/3}(-z)^{1/3}, & z \in \mathcal{R}_2, \ \Re z < 0, \\
F_2(z) - F_3(z) - \tau^{4/3}(-z)^{1/3}, & z \in \mathcal{R}_3, \ \Re z > 0, \\
F_2(z) - F_3(z) + \tau^{4/3}z^{1/3}, & z \in \mathcal{R}_3, \ \Re z < 0, \\
F_3(z) + e^{4\pi i/3}\tau^{4/3}z^{1/3}, & z \in \mathcal{R}_4, \ \Im z > 0, \\
F_3(z) + e^{2\pi i/3}\tau^{4/3}z^{1/3}, & z \in \mathcal{R}_4, \ \Im z < 0,
\end{cases}
\]

has an extension to a meromorphic function (also denoted by \( \xi \)) on \( \mathcal{R} \). The meromorphic function has a pole of order \( \deg \nu - 1 \) at infinity on the first sheet, and a simple pole at the other point at infinity.

Proof. The analyticity of \( \xi \) on \( \mathcal{R}_1 \) and \( \mathcal{R}_4 \) is immediate. To obtain the analyticity of \( \xi \) on \( \mathcal{R}_2 \) and \( \mathcal{R}_3 \) we note that from the Sokhotski-Plemelj formula and from the fact that \( \mu_2 = \sigma \) on \([-ic,ic]\) it follows that

\[
(5.4) \quad F_{2,+}(z) - F_{2,-}(z) = -2\pi i \frac{d\mu_2(z)}{dz} = -2\pi i \frac{d\sigma(z)}{dz} = -\tau^{4/3} \sqrt{3}|z|^{1/3} = -\tau^{4/3}(z^{1/3} + (-z)^{1/3})
\]

for \( z \in (-ic,ic) \). By the definition of \( \xi \) this implies that \( \xi \) has an analytic continuation to \((-ic,ic)\) on both \( \mathcal{R}_2 \) and \( \mathcal{R}_3 \). Thus \( \xi \) is analytic on each individual sheet \( \mathcal{R}_j \).

It remains to check the analyticity if we cross a cut and move from one sheet to another. That this holds, follows from the variational conditions for the equilibrium problem and the symmetry in the problem. We will only show that \( \xi \) is analytic when one crosses \( S(\mu_1) \) since the other cases follow from similar arguments.

Differentiating \( U^{\mu_1} \) gives

\[
(5.5) \quad 2 \frac{d}{dx} U^{\mu_1}(x) = -2PV \int \frac{1}{x-s} d\mu_1(s), \quad x \in S(\mu_1),
\]

which by the Sokhotski-Plemelj formulas can be written as

\[
(5.6) \quad -2 \frac{d}{dx} U^{\mu_1}(x) = F_{1,+}(x) + F_{1,-}(x), \quad x \in S(\mu_1).
\]

On the other hand, we can differentiate the right-hand side of the variational condition \((2.13)\) to obtain

\[
(5.7) \quad 2 \frac{d}{dx} U^{\mu_1}(x) = \frac{d}{dx} U^{\mu_2}(x) - V'(x) + \tau |x|^{1/3} \text{sgn} x
\]

\[
= -\frac{d}{dx} \int \log |x-s| \text{d}\mu_2(s) - V'(x) + \tau |x|^{1/3} \text{sgn} x
\]

\[
= -\frac{1}{2} \frac{d}{dx} \int \log(x^2 - s^2) \text{d}\mu_2(s) - V'(x) + \tau |x|^{1/3} \text{sgn} x
\]
for $x \in S(\mu_1)$. By symmetry of $\mu_2$ we obtain
\begin{equation}
\frac{1}{2} \int \frac{1}{x-s} \text{d} \mu_2(s) + \frac{1}{2} \int \frac{1}{x+s} \text{d} \mu_2(s) = \int \frac{1}{x-s} \text{d} \mu_2(s) = F_2(x)
\end{equation}
for $x \in \mathbb{R}$. Combining (5.8), (5.7) and (5.6) leads to
\begin{equation}
F_{1,+}(x) + F_{1,-}(x) = F_2(x) + V'(x) - \tau |x|^{1/3} \text{sgn} \ x,
\end{equation}
for $x \in S(\mu_1)$. This is exactly what is needed to prove that $\xi$ is analytic when one crosses $S(\mu_1)$ and moves from $\mathcal{R}_1$ to $\mathcal{R}_2$. \hfill \Box

We use $\xi_j$ to denote the restriction of $\xi$ to $\mathcal{R}_j$.

As a corollary to the lemma we find the following result about the asymptotic behavior of the Cauchy transforms.

**Corollary 5.2.** There exists a constant $\alpha$ such that (with $\omega = e^{2\pi i/3}$)
\begin{align}
F_1(z) &= \frac{1}{z} + \mathcal{O}(z^{-2}), \\
F_2(z) &= \begin{cases} 
\frac{2}{3z} + \frac{\alpha}{z^{9/3}} + \mathcal{O}(z^{-2}), & \text{Re} \ z > 0, \\
\frac{2}{3z} - \frac{\alpha}{z^{9/3}} + \mathcal{O}(z^{-2}), & \text{Re} \ z < 0, \ \text{Im} \ z > 0, \\
\frac{2}{3z} - \frac{\alpha}{z^{9/3}} + \mathcal{O}(z^{-2}), & \text{Re} \ z < 0, \ \text{Im} \ z < 0,
\end{cases} \\
F_3(z) &= \begin{cases} 
\frac{1}{3z} + \frac{\alpha}{z^{5/3}} + \mathcal{O}(z^{-2}), & \text{Im} \ z > 0, \\
\frac{1}{3z} + \frac{\alpha}{z^{5/3}} + \mathcal{O}(z^{-2}), & \text{Im} \ z < 0,
\end{cases}
\end{align}

uniformly for $z \to \infty$.

**Proof.** The statement about $F_1$ is obvious, since $\mu_1$ has a compact support.

For $z$ large enough define $\Xi$ as
\[ \Xi(z) = \xi_3(z^3) \quad \frac{-\pi}{6} < \arg z < \frac{\pi}{6}, \]
and then continued analytically into other sectors. Explicit calculations show that
\begin{align}
\Xi(z) = \begin{cases} 
\xi_2(z^3), & \begin{cases} 
-\pi/6 \leq \arg z \leq \pi/6, \\
5\pi/6 \leq \arg z \leq \pi, \\
\pi/3 \leq \arg z \leq 2\pi/3, \\
\pi/3 \leq \arg z \leq 2\pi/3, \\
\end{cases} \\
\xi_3(z^3), & \begin{cases} 
-\pi/3 \leq \arg z \leq -\pi/6, \\
2\pi/3 \leq \arg z \leq 5\pi/6, \\
-5\pi/6 \leq \arg z \leq -2\pi/3, \\
\pi/3 \leq \arg z \leq 2\pi/3, \\
-2\pi/3 \leq \arg z \leq -\pi/3.
\end{cases}
\end{cases}
\end{align}
Then $\Xi$ is analytic for $|z| > R$ for $R$ large enough with a Laurent expansion
\begin{equation}
\Xi(z) = \tau^{4/3} z + \sum_{j=0}^{\infty} c_j z^{-j}.
\end{equation}

Since $\xi_j(-z) = -\xi_j(z)$ we also have $\Xi(-z) = -\Xi(z)$, and hence the even powers vanish in this expansion.

Next we claim that we can reconstruct the $\xi_j$, $j = 2, 3, 4$ out of $\Xi$ again in the following way
\begin{align}
\xi_2(z) &= \begin{cases} 
\Xi(z^{1/3}), & \text{Re } z > 0, \\
\Xi(z^{1/3} \omega), & \text{Re } z < 0, \text{ Im } z > 0, \\
\Xi(z^{1/3} \omega^2), & \text{Re } z < 0, \text{ Im } z < 0,
\end{cases} \\
\xi_3(z) &= \begin{cases} 
\Xi(z^{1/3}), & \text{Re } z < 0, \\
\Xi(z^{1/3} \omega), & \text{Re } z > 0, \text{ Im } z > 0, \\
\Xi(z^{1/3} \omega^2), & \text{Re } z > 0, \text{ Im } z < 0,
\end{cases} \\
\xi_4(z) &= \begin{cases} 
\Xi(z^{1/3} \omega^2), & \text{Im } z > 0, \\
\Xi(z^{1/3} \omega), & \text{Im } z < 0.
\end{cases}
\end{align}

We leave the verification to the reader. Since we have that by \[5.3\]
\begin{equation}
F_2(z) = -\xi_1(z) - \xi_2(z) + V'(z) + \tau^{4/3} z^{1/3}
\end{equation}
for $\text{Re } z > 0$, we obtain
\begin{equation}
F_2(z) = -\frac{c_1}{z^{1/3}} + \frac{1 - c_3}{z} - \frac{c_5}{z^{5/3}} + \mathcal{O}(z^{-1}),
\end{equation}

for $z \to \infty$ and $\text{Re } z > 0$. Since $F_2(z) = \int \frac{1}{z} d\mu_2(x)$ and since $\mu_2$ has total mass $2/3$, we see that $c_1 = 0$ and $c_3 = 1/3$. This proves the asymptotic behavior of $F_2$ with $\alpha = c_5$ for $\text{Re } z > 0$. The asymptotics for the other sectors and other Cauchy transforms follow from similar arguments.

The solution to the equilibrium problem can be described in terms of the functions $\xi_j$ as described in the following proposition.

**Proposition 5.3.** With $\xi_j = \xi|_{\mathcal{R}_j}$, we have that
\begin{align}
d\mu_1(z) &= \frac{1}{2\pi i} (\xi_{1,+}(z) - \xi_{1,-}(z)) \, dz, \quad z \in \mathbb{R}, \\
d\mu_2(z) &= \frac{1}{2\pi i} (\xi_{2,+}(z) - \xi_{2,-}(z)) \, dz + d\sigma(z), \quad z \in i\mathbb{R}, \\
d\mu_3(z) &= \frac{1}{2\pi i} (\xi_{3,+}(z) - \xi_{3,-}(z)) \, dz - \frac{\tau^{4/3} \sqrt{3}}{2\pi} |z|^{1/3} dz, \quad z \in \mathbb{R}.
\end{align}

**Proof.** This follows by Lemma 5.1 and the Sokhotski-Plemelj formula
\begin{equation}
-\frac{1}{2\pi i} (F_{j,+}(z) - F_{j,-}(z)) = \frac{d}{dz} \mu_j(z)
\end{equation}
for $z \in S(\mu_j)$. □
6. The second transformation $X \mapsto U$

From now on, we assume that $\mu_1$ is one-cut regular. That is, $\mu_1$ is supported on one interval, and there are no singular points in the sense of Definition 2.5. We write

$$S(\mu_1) = [-a, a]$$

with $a > 0$.

6.1. Definition of the transformation. In the second transformation in the steepest descent analysis, we use the $g$-functions associated with the measures $\mu_j$ that minimize the energy functional $E_V$. For $j = 1$ and $j = 3$, we define $g_j$ as

$$g_j(z) = \int \log(z - s) \, d\mu_j(s), \quad j = 1, 3,$$

with the principal branch of the logarithm

$$\log(z - s) = \log|z - s| + i\arg(z - s), \quad \arg(z - s) \in (-\pi, \pi).$$

We also define $g_2$ by a similar integral, but since $\mu_2$ is supported on the imaginary axis, we make a different choice for the branch of the logarithm. We define $g_2$ by

$$g_2(z) = \int \log(z - s) \, d\mu_2(s).$$

where we take the logarithm such that

$$\log(z - s) = \log|z - s| + i\arg(z - s), \quad \arg(z - s) \in (-\pi/2, 3\pi/2),$$

for $z \in \mathbb{C} \setminus i\mathbb{R}$ and $s \in i\mathbb{R}$. Thus $g_2$ is defined and analytic in $\mathbb{C} \setminus i\mathbb{R}$.

The transformation $X \mapsto U$ is now defined by

$$U(z) = L^{-n}X(z)G(z)^nL^n,$$

where $G$ is given by

$$G(z) = \text{diag} \left( e^{-g_1(z)} \quad e^{g_1(z) - g_2(z)} \quad e^{g_2(z) - g_3(z)} \quad e^{g_3(z)} \right)$$

and $L$ is given by

$$L = \text{diag} \left( e^{-\ell} \quad 1 \quad 1 \quad 1 \right),$$

with $\ell$ the variational constant from (2.13).

Since $X$ and the $g_j$ are analytic in $\mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, it is clear that $U$ is also analytic in $\mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$. The jump matrices $J_U$ in the RH problem for $U$ are obtained from the jump matrices $J_X$ by

$$J_U = L^{-n}G_-^{-n}J_XG_+^nL^n,$$

and we have to calculate them on the various parts of the real and imaginary axis. We will see that the jump matrices take a simple form that is suitable for further analysis.
Also the behavior at infinity simplifies under the transformation. However, in contrast to other works (see e.g. [33]), the RH problem is not normalized at infinity after the use of the \( g \)-functions. So \( U(z) \) does not tend to the identity matrix as \( z \to \infty \), but instead there is a more complicated behavior at infinity, which however does not depend on \( n \) anymore.

Before we state the RH problem that is satisfied by \( U \), we first collect the properties of the functions \( g_j \) that will be needed.

### 6.2. The \( g \)-functions.

Observe that the functions \( g_j \) are anti-derivatives of the Cauchy transforms \( F_j \), i.e.,

\[
\frac{d}{dz} g_j(z) = F_j(z).
\]

The asymptotic behavior of \( g_j \) can be obtained by integrating the asymptotic behavior of \( F_j \) as given in Corollary 5.2.

**Lemma 6.1.** With the constant \( \alpha \) as in Corollary 5.2, we have that

\[
\begin{align*}
g_1(z) &= \log z + O(z^{-2}), \\
g_2(z) &= \begin{cases}
\frac{2}{3} \log z - \frac{2\alpha}{2z^{2/3}} + O(z^{-1}), & \text{Re } z > 0, \\
\frac{2}{3} \log z + \frac{4\alpha}{3z^{1/3}} - \frac{3\omega^2 \alpha}{3z^{2/3}} + O(z^{-1}), & \text{Re } z < 0, \text{ Im } z > 0, \\
\frac{1}{3} \log z + \frac{3\omega^2 \alpha}{2z^{2/3}} + O(z^{-1}), & \text{Re } z < 0, \text{ Im } z < 0,
\end{cases} \\
g_3(z) &= \begin{cases}
\frac{1}{3} \log z - \frac{3\omega^2 \alpha}{2z^{2/3}} + O(z^{-1}), & \text{Im } z > 0, \\
\frac{1}{3} \log z + \frac{3\omega^2 \alpha}{2z^{2/3}} + O(z^{-1}), & \text{Im } z < 0,
\end{cases}
\end{align*}
\]

as \( z \to \infty \) uniformly. In (6.9)–(6.11) every \( \log z \) is defined with the principal branch.

**Proof.** The asymptotics (6.9) for \( g_1 \) follows from the fact that \( \mu_1 \) is a symmetric probability measure with compact support.

The asymptotics (6.10)–(6.11) for \( g_2 \) and \( g_3 \) can be found by integrating the asymptotic results for \( F_2 \) and \( F_3 \) as given in Corollary 5.2. The constants of integration vanish, which can be seen from the fact that

\[
\int \log(z - s) d\mu_j(s) = \log(z) ||\mu_j|| + \int \log(1 - s/z) d\mu_j(s)
\]

and \( \log(1 - s/z) d\mu_j(s) \to 0 \) as \( z \to \infty \).

The additional constant \( \frac{4\pi i}{3} \) in (6.10) in the third quadrant comes from the fact that we used the branch (6.3) of the logarithm to define \( g_2 \) which does not correspond to the principal branch of \( \log z \) as \( z \to \infty \) in the third quadrant. \( \square \)

### 6.3. The \( \phi \)-functions.

In the statement of the RH problem for \( U \) it turns out to be convenient to use certain functions \( \phi_j \). We recall that \( \xi_j \) is defined as the restriction of \( \xi \) to the sheet \( R_j \) of the Riemann surface. We define

\[
\phi_j : \mathbb{C} \setminus (\Re \cup ( -i\infty, -i\epsilon] \cup [i\epsilon, i\infty)) \to \mathbb{C},
\]

\[
\begin{align*}
\phi_1(z) &= \frac{2}{3} \log z + O(z^{-1}), \\
\phi_2(z) &= \begin{cases}
\frac{2}{3} \log z - \frac{2\alpha}{2z^{2/3}} + O(z^{-1}), & \text{Re } z > 0, \\
\frac{2}{3} \log z + \frac{4\alpha}{3z^{1/3}} - \frac{3\omega^2 \alpha}{3z^{2/3}} + O(z^{-1}), & \text{Re } z < 0, \text{ Im } z > 0, \\
\frac{1}{3} \log z + \frac{3\omega^2 \alpha}{2z^{2/3}} + O(z^{-1}), & \text{Re } z < 0, \text{ Im } z < 0,
\end{cases} \\
\phi_3(z) &= \begin{cases}
\frac{1}{3} \log z - \frac{3\omega^2 \alpha}{2z^{2/3}} + O(z^{-1}), & \text{Im } z > 0, \\
\frac{1}{3} \log z + \frac{3\omega^2 \alpha}{2z^{2/3}} + O(z^{-1}), & \text{Im } z < 0,
\end{cases}
\end{align*}
\]
as

\( (6.13) \quad \phi_1(z) = -\frac{1}{2} \int_a^z (\xi_1(y) - \xi_2(y)) \, dy, \)

\( (6.14) \quad \phi_2(z) = \begin{cases} -\frac{1}{2} \int_{i\epsilon}^z (\xi_2(y) - \xi_3(y)) \, dy, & \text{Im } z > 0, \\ -\frac{1}{2} \int_{-i\epsilon}^z (\xi_2(y) - \xi_3(y)) \, dy, & \text{Im } z < 0, \end{cases} \)

\( (6.15) \quad \phi_3(z) = \frac{\pi i}{6} - \frac{1}{2} \int_0^z (\xi_3(y) - \xi_4(y)) \, dy. \)

All paths of integration lie entirely (except for their starting values) in \( \mathbb{C} \setminus \left( \mathbb{R} \cup (-\infty, -i\epsilon] \cup [i\epsilon, i\infty) \right) \).

The importance of the \( \phi \)-functions is that jump properties of the function \( g_j \) can be conveniently written in terms of the function \( \phi_j \). We start with the jump properties of \( g_1 \) and the connection with \( \phi_1 \).

**Lemma 6.2.**  
(a) We have for \( z \in [-a, a] \),

\( (6.16) \quad g_{1,+}(z) - g_{1,-}(z) = 2\phi_{1,+}(z) = -2\phi_{1,-}(z), \)

and

\( (6.17) \quad g_{1,+}(z) + g_{1,-}(z) - g_2(z) - V(z) + \frac{3}{4} \tau^{4/3}|z|^{4/3} + \ell \)

\( = \begin{cases} 0, & z \in (0, a], \\ -2\pi i/3, & z \in [-a, 0). \end{cases} \)

(b) For \( z \in \mathbb{R} \setminus [-a, a] \), we have

\( (6.18) \quad g_{1,+}(z) - g_{1,-}(z) = \begin{cases} 0, & z \in (a, \infty), \\ 2\pi i, & z \in (-\infty, -a). \end{cases} \)

\( (6.19) \quad g_{1,+}(z) + g_{1,-}(z) - g_2(z) - V(z) + \frac{3}{4} \tau^{4/3}|z|^{4/3} + \ell \)

\( = \begin{cases} 2\phi_1(z), & z \in (a, \infty), \\ 2\phi_{1,+}(z) - 8\pi i/3, & z \in (-\infty, -a). \end{cases} \)

**Proof.** (a) Let \( z \in [-a, a] \). Then we have

\( (6.20) \quad g_{1,+}(z) - g_{1,-}(z) = i \int (\arg_+(z - s) - \arg_-(z - s)) \, d\mu_1(s) \)

\( = 2\pi i \int_z^a d\mu_1(s). \)

By \( (5.20) \) , \( (6.13) \), and the fact that \( \xi_{1,-}(z) = \xi_{2,+}(z) \) we obtain

\( (6.21) \quad g_{1,+}(z) - g_{1,-}(z) = \int_z^a (\xi_{1,+}(y) - \xi_{1,-}(y)) \, dy \)

\( = \int_z^a (\xi_{1,+}(y) - \xi_{2,+}(y)) \, dy. \)
which proves (6.16).

For \( z \in [-a, a] = S(\mu_1) \), it follows from the variational condition (2.13) that

\[
(6.22) \quad \text{Re} \left( g_{1,+}(z) + g_{1,-}(z) - g_2(z) - V(z) + \frac{3}{4} r^{4/3} |z|^{4/3} + \ell \right) = 0
\]

and hence

\[
(6.23) \quad g_{1,+}(z) + g_{1,-}(z) - g_2(z) - V(z) + \frac{3}{4} r^{4/3} |z|^{4/3} + \ell
= i \int (\arg_+(z-s) + \arg_-(z-s)) d\mu_1(s) - i \int_{\mathbb{R}} \arg(z-s) d\mu_2(s),
\]

where by the choice of branches in the definition of \( g_1 \) and \( g_2 \), we have to use \(-\pi < \arg(z-s) < \pi \) in the first integral and \(-\pi/2 < \arg(z-s) < 3\pi/2 \) in the second integral. Clearly,

\[
(6.24) \quad \arg_+(z-s) + \arg_-(z-s) = 0
\]

for \( z, s \in \mathbb{R} \). From the symmetry of \( \mu_2 \) with respect to the real axis, we obtain

\[
(6.25) \quad \int_{\mathbb{R}} \arg(z-s) d\mu_2(s) = \int_0^{i\infty} (\arg(z-s) + \arg(z+s)) d\mu_2(s).
\]

Since by (6.3)

\[
(6.26) \quad \arg(z-s) + \arg(z+s) = \begin{cases} 0 & \text{if } z > 0, \\ 2\pi & \text{if } z < 0, \end{cases}
\]

for \( s \in (0, i\infty) \), and since \( \mu_2(i\mathbb{R}^+) = 1/3 \), we get from (6.25)

\[
(6.27) \quad \int_{\mathbb{R}} \arg(z-s) d\mu_2(s) = \begin{cases} 0 & \text{if } z > 0, \\ 2\pi/3 & \text{if } z < 0. \end{cases}
\]

Inserting (6.27) and (6.24) into (6.23) leads to (6.17).

(b) Let \( z \in \mathbb{R} \setminus [-a, a] \). Then we continue to have (6.20), and from this the relations (6.18) immediately follow. By (6.8) and (5.3) we further have

\[
(6.28) \quad \frac{d}{dz} \left( g_{1,+}(z) + g_{1,-}(z) - g_2(z) - V(z) + \frac{3}{4} r^{4/3} |z|^{4/3} + \ell \right) = \xi_2(z) - \xi_1(z).
\]

We obtain (6.19) from (6.13), (6.28), and the fact that equality holds in (6.17) for \( z = \pm a \). Note also that \( \phi_{1,+}(-a) = \pi i \).

For the functions \( g_2 \) and \( \phi_2 \) we have in a similar way.

**Lemma 6.3.** (a) For \( z \in S(\sigma - \mu_2) \), we have

\[
(6.29) \quad g_{2,+}(z) + g_{2,-}(z) - g_1(z) - g_3(z) = \begin{cases} 0, & \text{if } z \in [i\epsilon, i\infty), \\ 4\pi i/3, & \text{if } z \in (-i\infty, -i\epsilon], \end{cases}
\]
and for $z \in S(\sigma - \mu_2) \cap i\mathbb{R}_\pm$,

$$g_{2,+}(z) - g_{2,-}(z) \pm \frac{i3\sqrt{3}}{4}r^{4/3}|z|^{4/3} = 2\phi_{2,+}(z) + 2\pi i/3 = -2\phi_{2,-}(z) + 2\pi i/3. \tag{6.30}$$

(b) For $z \in i\mathbb{R} \setminus S(\sigma - \mu_2) = (-ic, ic)$, we have

$$g_{2,+}(z) - g_{2,-}(z) \pm \frac{i3\sqrt{3}}{4}r^{4/3}|z|^{4/3} = 2\pi i/3, \quad z \in i\mathbb{R}_\pm, \tag{6.31}$$

and

$$g_{2,+}(z) + g_{2,-}(z) - g_1(z) - g_3(z) = \begin{cases} 2\phi_2(z) & \text{if } z \in (0, ic), \\ 2\phi_2(z) + 4\pi i/3 & \text{if } z \in (-ic, 0). \end{cases} \tag{6.32}$$

Proof. (a) Let $z \in S(\sigma - \mu_2)$. From the variational condition (2.15) it then follows that

$$\text{Re} \left( g_{2,+}(z) + g_{2,-}(z) - g_1(z) - g_3(z) \right) = 0. \tag{6.33}$$

Hence

$$g_{2,+}(z) + g_{2,-}(z) - g_1(z) - g_3(z)
= i \int \left( \text{arg}_+ (z - s) + \text{arg}_- (z - s) \right) d\mu_2(s)
= -i \int (\text{arg}(z - s)) d(\mu_1 + \mu_3)(s). \tag{6.34}$$

By symmetry we find

$$\int \text{arg}(z - s)d(\mu_1 + \mu_3)(s) = \int_0^{\infty} (\text{arg}(z - s) + \text{arg}(z + s)) d(\mu_1 + \mu_3)(s). \tag{6.35}$$

Since $\text{arg}(z - s) + \text{arg}(z + s) = \pm \pi$ for $z \in i\mathbb{R}_\pm$ and $s \in \mathbb{R}_+$, and since $(\mu_1 + \mu_3)(\mathbb{R}_+) = 2\pi i/3$, we then obtain

$$\int \text{arg}(z - s)d(\mu_1 + \mu_3)(s) = \pm 2\pi i/3, \quad z \in i\mathbb{R}_\pm. \tag{6.36}$$

Moreover,

$$\int (\text{arg}_+ (z - s) + \text{arg}_- (z - s)) d\mu_2(s) = 2\pi i/3, \quad z \in i\mathbb{R}. \tag{6.37}$$

Inserting (6.36) and (6.37) into (6.34) leads to (6.29).

By symmetry we have $\mu_2(i\mathbb{R}_+) = 1/3$. Then by the definition of $g_2$ and (5.21) we obtain

$$g_{2,+}(z) - g_{2,-}(z) = 2\pi i \int_z^{i\infty} d\mu_2(y) = \frac{2\pi i}{3} - 2\pi i \int_0^z d\mu_2(y).$$
Then (6.31) follows by (6.41) and (6.42).

where we used the definition (6.14) of $F$ or $z$

leads to (6.30).

Lemma 6.4. We have for $z \in \mathbb{R}$,

\begin{equation}
(6.38) \quad g_2,+(z) - g_2,-(z) = \frac{2\pi i}{3} - \int_0^z (\xi_2,+(y) - \xi_2,-(y)) \, dy - 2\pi i \int_0^z d\sigma(y).
\end{equation}

By (2.7) we have

\begin{equation}
(6.39) \quad 2\pi i \int_0^z d\sigma(y) = \pm \frac{i3\sqrt{3}}{4} \tau^{4/3} |z|^{4/3}, \quad \text{for } z \in i\mathbb{R}_\pm.
\end{equation}

For $z \in S(\sigma - \mu_2) \cap i\mathbb{R}_\pm$, we also find since $\xi_2$ is analytic in $(-ic, 0) \cup (0, ic)$

\begin{align*}
\int_0^z (\xi_2,+(y) - \xi_2,-(y)) \, dy &= \int_{\pm ic}^z (\xi_2,+(y) - \xi_2,-(y)) \, dy \\
&= \int_{\pm ic}^z (\xi_3,+(y) - \xi_3,-(y)) \, dy \\
&= -2\phi_2,+(z) = 2\phi_2,-(z),
\end{align*}

where we used the definition (6.14) of $\phi_2$. Putting (6.39) and (6.40) into (6.38) leads to (6.30).

(b) Let $z \in (-ic, ic)$. Then as before

\begin{equation}
(6.41) \quad g_2,+(z) - g_2,-(z) = 2\pi i \int_{z}^{\infty} d\mu_2(y) = \frac{2\pi i}{3} - 2\pi i \int_0^z d\mu_2(y).
\end{equation}

Now we have

\begin{equation}
\int_0^z d\mu_2(y) = \int_0^z d\sigma(y),
\end{equation}

and so by (6.39)

\begin{equation}
2\pi i \int_0^z d\mu_2(y) = \pm \frac{i3\sqrt{3}}{4} \tau^{4/3} |z|^{4/3}, \quad \text{for } z \in (-ic, ic) \cap i\mathbb{R}_\pm.
\end{equation}

Then (6.31) follows by (6.41) and (6.42).

Next, from (6.8) and (5.3) we obtain

\begin{equation}
(6.43) \quad \frac{d}{dz} (g_2,+(z) + g_2,-(z) - g_1(z) - g_3(z))
\quad = F_2,+(z) + F_2,-(z) - F_1(z) - F_3(z) = -\xi_2(z) + \xi_3(z).
\end{equation}

Integrating (6.43) and noting that the equality (6.29) holds for $z = \pm ic$, we obtain (6.32) by (6.14).

Finally, for $g_3$ and $\phi_3$ we have the following results.

Lemma 6.4. We have for $z \in \mathbb{R}$,

\begin{equation}
(6.44) \quad g_3,+(z) + g_3,-(z) - g_2(z) = \begin{cases} 0, & z > 0, \\ -2\pi i/3, & z < 0. \end{cases}
\end{equation}

and

\begin{equation}
(6.45) \quad g_3,+(z) - g_3,-(z) + \frac{i3\sqrt{3}}{4} \tau^{4/3} |z|^{1/3}
\quad = 2\phi_3,+(z) = -2\phi_3,-(z) + 2\pi i/3, \quad \text{for } z \in \mathbb{R}_\pm.
\end{equation}
Lemma 6.5. \[ \text{Re} \left( g_{3,+}(z) + g_{3,-}(z) - g_2(z) \right) = 0, \]
and hence
\[ g_{3,+}(z) + g_{3,-}(z) - g_2(z) = \int (\arg_+(z - s) + \arg_-(z - s)) \, d\mu_3(s) \]
\[ - i \int \arg(z - s) \, d\mu_2(s). \]
Combining this with (6.24) and (6.27) gives (6.44).

Proof. From the variational condition (2.17) it follows that for \( z \in \mathbb{R} \),
\[ \text{Re} \left( g_{3,+}(z) + g_{3,-}(z) - g_2(z) \right) = 0, \]
and hence
\[ g_{3,+}(z) + g_{3,-}(z) - g_2(z) = 2\pi i \int z \, d\mu_3(s) = \frac{\pi i}{3} - 2\pi i \int z \, d\mu_3(s). \]
Combining this with (6.15) leads to (6.45).

Further, by symmetry we find \( \mu_3(\mathbb{R}_+) = 1/6 \) and so
\[ g_{3,+}(z) - g_{3,-}(z) = 2\pi i \int z \, d\mu_3(s) = \frac{\pi i}{3} - 2\pi i \int z \, d\mu_3(s). \]
Now inserting (5.22) and using \( \xi_{3,-}(z) = \xi_{4,+}(z) \) gives
\[ g_{3,+}(z) - g_{3,-}(z) = \frac{\pi i}{3} - \int \left( \xi_{3,+}(y) - \xi_{3,-}(y) - i\sqrt{3} \tau^{4/3} |y|^{1/3} \right) dy \]
\[ = \frac{\pi i}{3} - \int \left( \xi_{3,+}(y) - \xi_{4,+}(y) \right) dy \pm \frac{i3\sqrt{3}}{4} \tau^{4/3} |z|^{1/3} \]
for \( z \in \mathbb{R}_+ \). Combining this with (6.15) leads to (6.45). \( \square \)

6.4. The RH problem for \( U \). In the following lemma we state the RH problem that is satisfied by \( U \).

Lemma 6.5. We have that \( U \) satisfies the RH problem
\[ \begin{cases} U \text{ is analytic in } \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \\ U_+(z) = U_-(z)J_U(z), \quad z \in \mathbb{R} \cup i\mathbb{R}, \\ U(z) = (I + \mathcal{O}(z^{-1/3})) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{1/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-1/3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A_j \end{pmatrix}, \end{cases} \]
as \( z \to \infty \) in the \( j \)th quadrant
where the \( 3 \times 3 \) matrices \( A_j \) are given in (3.30) and (3.31). The asymptotics for \( U \) are uniform as \( z \to \infty \) in any region such that (3.22) holds for some \( \varepsilon > 0 \).

The jump matrix \( J_U \) is given by
\[ J_U = \begin{pmatrix} e^{-2n\phi_1,+} & \frac{1}{d} & 0 & 0 \\ 0 & e^{-2n\phi_1,-} & 0 & 0 \\ 0 & 0 & e^{-2n\phi_3,+} & 1 \\ 0 & 0 & 0 & e^{-2n\phi_3,-} \end{pmatrix}, \quad \text{on } S(\mu_1), \]
\[ J_U = \begin{pmatrix} 1 & e^{2n\phi_1,+} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-2n\phi_3,+} & 1 \\ 0 & 0 & 0 & e^{-2n\phi_3,-} \end{pmatrix}, \quad \text{on } \mathbb{R} \setminus S(\mu_1), \]
after a simple calculation three, gives the equalities of the upper left blocks in (6.49) and (6.50).

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{2\phi_2} & 0 & 0 \\
0 & 1 & e^{2\phi_2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \text{on } S(\sigma - \mu_2),
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & e^{-2\phi_2} & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \text{on } \mathbb{R} \setminus S(\sigma - \mu_2),
\]

6.5. **Proof of the jump matrices** $J_U$ **in the RH problem for** $U$.

*Proof.* From (3.40) we see that the jump matrix for $X$ on the real line can be decomposed into $2 \times 2$ blocks. The jump matrix (6.7) for $U$ can also be decomposed into $2 \times 2$ blocks, with two non-trivial diagonal blocks. The upper left block is given by

\[
\begin{pmatrix}
e^{n(g_{1,-}(z)+\ell)} & 0 \\
0 & e^{n(g_{2,-}(z)-g_{1,-}(z))}
\end{pmatrix}
\begin{pmatrix}
e^{-n(V(z)-\frac{3}{4}r^{4/3}|z|^{4/3})} & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
e^{-n(g_{1,+}(z)+\ell)} & 0 \\
0 & e^{-n(g_{2,+}(z)-g_{1,+}(z))}
\end{pmatrix}
\begin{pmatrix}
e^{-n(g_{1,+}(z)-g_{1,-}(z))} & e^{n(g_{2,+}(z)+g_{1,-}(z)-g_{2}(z)-V(z)+\frac{3}{4}r^{4/3}|z|^{4/3})} \\
0 & e^{n(g_{1,+}(z)-g_{1,-}(z))}
\end{pmatrix},
\]

which by the relations in Lemma 6.2 and the fact that $n$ is a multiple of three, gives the equalities of the upper left blocks in (6.49) and (6.50).

For the lower right block in the jump matrix (6.7) on the real line we find after a simple calculation

\[
\begin{pmatrix}
e^{-n(g_{3,+}(z)-g_{3,-}(z)-\frac{3}{4}r^{4/3}|z|^{4/3})} & e^{n(g_{3,+}(z)+g_{3,-}(z)-g_{2}(z))} \\
0 & e^{n(g_{3,+}(z)-g_{3,-}(z)-\frac{3}{4}r^{4/3}|z|^{4/3})}
\end{pmatrix}
\begin{pmatrix}
e^{-n(g_{3,+}(z)-g_{3,-}(z)+\frac{3}{4}r^{4/3}|z|^{4/3})} & e^{n(g_{3,+}(z)+g_{3,-}(z)-g_{2}(z))} \\
0 & e^{n(g_{3,+}(z)-g_{3,-}(z)+\frac{3}{4}r^{4/3}|z|^{4/3})}
\end{pmatrix},
\]

for $z \in \mathbb{R}_+$ and

\[
\begin{pmatrix}
e^{-n(g_{3,+}(z)-g_{3,-}(z)+\frac{3}{4}r^{4/3}|z|^{4/3})} & e^{n(g_{3,+}(z)+g_{3,-}(z)-g_{2}(z))} \\
0 & e^{n(g_{3,+}(z)-g_{3,-}(z)+\frac{3}{4}r^{4/3}|z|^{4/3})}
\end{pmatrix}
\begin{pmatrix}
e^{-n(g_{3,+}(z)-g_{3,-}(z)-\frac{3}{4}r^{4/3}|z|^{4/3})} & e^{n(g_{3,+}(z)+g_{3,-}(z)-g_{2}(z))} \\
0 & e^{n(g_{3,+}(z)-g_{3,-}(z)-\frac{3}{4}r^{4/3}|z|^{4/3})}
\end{pmatrix},
\]

for $z \in \mathbb{R}_-$. This proves the equality of the lower right blocks in (6.49) and (6.50) in view of Lemma 6.4. It is important again that $n$ is a multiple of three.

We finally come to the middle block in the jump matrix on the imaginary axis. By (3.31) and (6.7) we find after a calculation

\[
\begin{pmatrix}
e^{-n(g_{2,+}(z)-g_{2,-}(z)+\frac{3}{4}r^{4/3}|z|^{4/3})} & 0 \\
e^{-n(g_{2,+}(z)+g_{2,-}(z)-g_{1}(z)-g_{3}(z))} & e^{n(g_{2,+}(z)-g_{2,-}(z)+\frac{3}{4}r^{4/3}|z|^{4/3})}
\end{pmatrix}
\begin{pmatrix}
e^{-n(g_{2,+}(z)-g_{2,-}(z)+\frac{3}{4}r^{4/3}|z|^{4/3})} & 0 \\
e^{-n(g_{2,+}(z)+g_{2,-}(z)-g_{1}(z)-g_{3}(z))} & e^{n(g_{2,+}(z)-g_{2,-}(z)+\frac{3}{4}r^{4/3}|z|^{4/3})}
\end{pmatrix}
\]
for $z \in i\mathbb{R}_+$ and
\begin{equation}
(6.57) \begin{pmatrix}
e^{-n(g_{2,+}(z)-g_{2,-}(z)-\frac{3\pi i}{4}z^{4/3})}
& 0 \\
e^{-n(g_{2,+}(z)+g_{2,-}(z)-g_1(z))}
& e^{n(g_{2,+}(z)-g_{2,-}(z)-\frac{3\pi i}{4}z^{4/3})}
\end{pmatrix}
\end{equation}
for $z \in i\mathbb{R}_-$. Then by Lemma 6.3 and the fact that $n$ is a multiple of three, we indeed obtain (6.51) and (6.52). □

6.6. Proof of the asymptotics for $U$.

Proof. We deal with the asymptotic condition in the RH problem for $U$. Define matrices
\begin{equation}
(6.58) H_1 = \frac{3\alpha}{2} \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^2
\end{pmatrix}, \quad H_2 = \frac{3\alpha}{2} \begin{pmatrix}
\omega & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \omega^2
\end{pmatrix},
\end{equation}
\begin{equation}
(6.59) H_3 = \frac{3\alpha}{2} \begin{pmatrix}
\omega^2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \omega
\end{pmatrix}, \quad H_4 = \frac{3\alpha}{2} \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega^2 & 0 \\
0 & 0 & \omega
\end{pmatrix}.
\end{equation}

From Lemma 6.1 and (6.5) we obtain
\begin{equation}
(6.60) G(z) = \begin{pmatrix}
z^{-1} & 0 & 0 & 0 \\
0 & z^{1/3} & 0 & 0 \\
0 & 0 & z^{1/3} & 0 \\
0 & 0 & 0 & z^{1/3}
\end{pmatrix}
(I + \begin{pmatrix}
1 & 0 \\
0 & H_j
\end{pmatrix} z^{-2/3} + O(z^{-1}))
\end{equation}
for $z \to \infty$ and in the $j$th quadrant with $j = 1, 3, 4$. In the second quadrant the situation is slightly different. Then we have
\begin{equation}
(6.61) G(z) = \begin{pmatrix}
z^{-1} & 0 & 0 & 0 \\
0 & \omega^2 z^{1/3} & 0 & 0 \\
0 & 0 & \omega z^{1/3} & 0 \\
0 & 0 & 0 & z^{1/3}
\end{pmatrix}
(I + \begin{pmatrix}
1 & 0 \\
0 & H_2
\end{pmatrix} z^{-2/3} + O(z^{-1}))
\end{equation}
for $z \to \infty$ in the second quadrant. The extra factors $\omega^2$ in the $(2, 2)$ entry and $\omega$ in the $(3, 3)$ entry are due to the extra term $-\frac{4\pi i}{3}$ in the asymptotic behavior of $g_2$ in the second quadrant as given in (6.10). However these extra factors play no role in the asymptotic behavior of $G(z)^n$, since $n$ is a multiple of three.

Then by the asymptotics (3.39) for $X$,
\begin{equation}
(6.62) X(z)G(z)^n = (I + O(z^{-2/3})) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & z^{-1/3} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & z^{1/3}
\end{pmatrix}
x \begin{pmatrix}
1 & 0 \\
0 & A_j
\end{pmatrix} \begin{pmatrix}
I + n \begin{pmatrix}
1 & 0 \\
0 & H_j
\end{pmatrix} z^{-2/3} + O(z^{-1})
\end{pmatrix}
\end{equation}
\[
(I + \mathcal{O}(z^{-2/3})) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-1/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{1/3} \end{pmatrix} 
\times \left( I + n \begin{pmatrix} 1 & 0 \\ 0 & A_j H_j A_j^{-1} \end{pmatrix} z^{-2/3} + \mathcal{O}(z^{-1}) \right) \begin{pmatrix} 1 & 0 \\ 0 & A_j \end{pmatrix}
\]

as \( z \to \infty \) in the \( j \)th quadrant. A simple calculation then shows that

\[ A_j H_j A_j^{-1} = \frac{3\alpha}{2} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad j = 1, 2, 3, 4, \]

and hence

\[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-1/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{1/3} \end{pmatrix} \]

\[ \times \left( I + n \begin{pmatrix} 1 & 0 \\ 0 & A_j H_j A_j^{-1} \end{pmatrix} z^{-2/3} \right) \]

\[ = (I + \mathcal{O}(z^{-2/3})) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-1/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{1/3} \end{pmatrix} \]

as \( z \to \infty \). Combining this with (6.62) leads to

\[ X(z)G(z)^n = (1 + \mathcal{O}(z^{-1/3})) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-1/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{1/3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A_j \end{pmatrix} \]

as \( z \to \infty \) in the \( j \)th quadrant. Then the asymptotic condition in the RH problem (6.48) for \( U \) follows by (6.4) and the fact \( L \) is a diagonal matrix of the form (6.6). This completes the proof of Lemma 6.5. \( \square \)

Let us summarize the progress we made so far. Although the RH problem for \( U \) is not normalized at infinity, an important step is that the asymptotic behavior of \( U \) does not depend on \( n \) anymore.

The jump structure for \( U \) is somewhat involved. Let us take a closer look at the \( 2 \times 2 \) lower right blocks of the jump matrix \( J_U \) on \( \mathbb{R} \), that is, at the lower right blocks of (6.49) and (6.50). We see that the diagonal entries are highly oscillating. Indeed, by (6.15), the fact that \( \xi_{4,+}(z) = \xi_{3,-}(z) \) for \( z \in S(\sigma - \mu_2) \) and (5.22) we obtain

\[ \phi_{3,+}(z) = -\phi_{3,-}(z) = -\pi i \int_0^z d\mu_3 + \frac{i3\sqrt{3}}{8} \tau A^{4/3}|z|^{4/3} \]

for \( z \in \mathbb{R} \), and hence \( \phi_{3,+} \) is purely imaginary on \( \mathbb{R} \).

An important step in the steepest analysis is the so-called opening of the lens. In this step jump matrices with highly oscillatory diagonal entries,
are turned into a constant (or otherwise nice behaved) jump matrix on the original contour and jump matrices with exponentially decaying off-diagonal entries on new contours on the lips of the lens. We do not go into details here, but let us mention that this construction works under the condition

\[ \frac{d}{dx} \text{Im}(\phi_{3,+}(x)) < 0 \]  

which holds in our case by (6.66) and the fact that \( \mu_3 \) is a measure with a density that is strictly positive on \( \mathbb{R} \).

From (6.51) we see that the jump matrix for \( U \) on the the part \( S(\sigma - \mu_2) = (-i\infty, -ic] \cup [ic, i\infty) \) of the imaginary axis has highly oscillating terms on the diagonal. Indeed, by (6.41), the fact that \( \xi_{3,+}(z) = \xi_{2,-}(z) \) for \( z \in S(\sigma - \mu_2) \) and (5.21) we obtain

\[ \phi_{2,+}(z) = -\pi i \int_{ic}^{z} d(\mu_2 - \sigma) \]  

for \( z \in S(\sigma - \mu_2) \cap i\mathbb{R} \). In order to open the lenses successfully we now must have the condition

\[ \frac{d}{dy} \text{Im} \phi_{2,+}(iy) > 0, \]  

for \( y > c \) and \( y < c \). The difference in sign in (6.69) when compared to (6.67) is due to the different triangularity structure of the jump matrices. Indeed the middle 2 \times 2 block in (6.51) is lower triangular, in contrast to the non-trivial blocks in (6.49) and (6.50), which are upper triangular. From (6.68) we see that

\[ \frac{d}{dy} \text{Im} \phi_{2,+}(iy) = -\pi \frac{d\mu_2}{|dz|}(iy) + \pi \frac{d\sigma}{|dz|}(iy) \]  

for \( y \in \mathbb{R} \). And since \( iy \in S(\sigma - \mu_2) \) for \( y > c \) and \( y < c \) we have that condition (6.69) is satisfied. Here we see the importance of the upper constraint \( \mu_2 \leq \sigma \).

7. The third and fourth transformations \( U \mapsto T \mapsto S \)

7.1. Definition of the transformation \( U \mapsto T \). The next step in the Deift/Zhou steepest descent analysis is the opening of lenses. The aim of this step is to turn the oscillating diagonal entries in the jump matrices into exponentially small off-diagonal entries.

We have to open a lens around each of the sets \( S(\mu_1), S(\sigma - \mu_2) \) and \( S(\mu_3) \). The latter two are unbounded and we will treat the opening of lenses around these two sets in this section. The opening of the lens around the bounded set \( S(\mu_1) \) is more standard and it is deferred to the next section.

The lens around \( S(\sigma - \mu_2) \) is opened as follows. In the discussion at the end of the last section we have seen that (6.69) holds for every \( y > c \) and \( y < -c \). Then it follows from the Cauchy-Riemann equations that
Re $\phi_2(z) < 0$ for $z$ in region around $(-i\infty, -ic) \cup (ic, i\infty)$. In particular, one can show that, for some $r > 0$, it contains a cone
\begin{equation}
\{ z = x + iy \in \mathbb{C} \mid 0 < |x| < 2r(|y| - c) \}
\end{equation}
in its interior.

Then we take the contour $\Sigma_2$ around $S(\sigma - \mu_2)$ as shown in Figure 7.1, so that
- $\text{Re} \phi_2(z) < 0$ for $z \in \Sigma_2 \setminus \{ \pm ic \}$, and
- there exists an $r > 0$ such that $|x| > r(|y| - c)$ for every $z = x + iy \in \Sigma_2 \setminus \{ \pm ic \}$.

The lens around $S(\mu_3) = \mathbb{R}$ is opened as follows. Since (6.67) holds for every $x \in \mathbb{R}$, we have that $\text{Re} \phi_3(z) > 0$ for $z$ in a region around $\mathbb{R}$. The region is unbounded and it can be shown that it contains the cone
\begin{equation}
\{ z = x + iy \in \mathbb{C} \mid 0 < |y| < 2r(|x| + 1) \}
\end{equation}
for some $r > 0$. We take the contour $\Sigma_3$, as shown in Figure 7.1, such that
- $\text{Re} \phi_3(z) > 0$ for $z \in \Sigma_3$, and
- there exists an $r > 0$ so that $|y| > r(|x| + 1)$ for every $z = x + iy \in \Sigma_3$, see also Figure 7.1. We may (and do) assume that $\Sigma_2$ and $\Sigma_3$ do not intersect.

The contours $\Sigma_2$ and $\Sigma_3$ give rise to a partitioning of the complex plane as in Figure 7.1. The inner part of the lens around $S(\sigma - \mu_2)$ enclosed by
the contour $\Sigma_2$ is denoted by $\Omega_2$, and the inner part of the lens around $\mathbb{R}$
enclosed by $\Sigma_3$ is denoted by $\Omega_3$.

We define the $4 \times 4$ matrix valued function $T$ by

(7.2)   \[ T(z) = U(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & e^{2n\phi_2(z)} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z \in \Omega_2, \quad \Re z > 0, \]

(7.3)   \[ T(z) = U(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -e^{2n\phi_2(z)} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z \in \Omega_2, \quad \Re z < 0, \]

(7.4)   \[ T(z) = U(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -e^{-2n\phi_3(z)} & 1 \end{pmatrix}, \quad z \in \Omega_3, \quad \Im z > 0, \]

(7.5)   \[ T(z) = U(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & e^{-2n\phi_3(z)} & 1 \end{pmatrix}, \quad z \in \Omega_3, \quad \Im z < 0, \]

and

(7.6)   \[ T(z) = U(z) \quad \text{elsewhere}. \]

Then the $4 \times 4$ matrix valued function $T$ is defined and analytic in $\mathbb{C} \setminus \Sigma_T$
where $\Sigma_T$ is given by

(7.7)   \[ \Sigma_T = \mathbb{R} \cup i\mathbb{R} \cup \Sigma_2 \cup \Sigma_3. \]

7.2. RH problem for $T$. In the following lemma we state the RH problem
that is satisfied by $T$.

Lemma 7.1. $T$ is the unique solution of the following RH problem

\[ \begin{cases} \text{$T$ is analytic in $\mathbb{C} \setminus \Sigma_T$,} \\ T_+(z) = T_-(z)J_T(z), \quad z \in \Sigma_T, \\ T(z) = (I + \mathcal{O}(z^{-1/3})) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{1/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-1/3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A_j \end{pmatrix} \to \begin{pmatrix} \frac{1}{e^{2n\phi_2(z)}} & 0 \\ 0 & \frac{1}{e^{2n\phi_3(z)}} \end{pmatrix} \quad \text{as } z \to \infty \text{ in the } j\text{th quadrant.} \end{cases} \]

The matrices $A_j$ are given by (3.30) and (3.31), and the asymptotic condi-
tion in (7.8) holds uniformly as $z \to \infty$ in each quadrant.
The jump matrix $J_T$ is given by

\begin{align*}
J_T &= \begin{pmatrix}
e^{-2n\phi_1^+} & 1 & 0 & 0 \\
0 & e^{-2n\phi_1^-} & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}, \quad \text{on } S(\mu_1), \\
J_T &= \begin{pmatrix}
1 & e^{2n\phi_1^+} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}, \quad \text{on } \mathbb{R} \setminus S(\mu_1), \\
J_T &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & e^{-2n\phi_3} & 1
\end{pmatrix}, \quad \text{on } \Sigma_3, \\
J_T &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \text{on } S(\sigma - \mu_2), \\
J_T &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & e^{2n\phi_2} & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \text{on } \Sigma_2,
\end{align*}

and

\begin{align*}
J_T &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & e^{-2n\phi_2} & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \text{on } (-ic, ic) \setminus \Omega_3, \\
J_T &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & e^{-2n(\phi_2 + \phi_3)} & 1 & 0 \\
0 & e^{-2n\phi_3} & 0 & 1
\end{pmatrix}, \quad \text{on } (0, ic) \cap \Omega_3, \\
J_T &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & e^{-2n\phi_2} & 1 & 0 \\
0 & -e^{-2n(\phi_2 + \phi_3)} & 0 & 1
\end{pmatrix}, \quad \text{on } (-ic, 0) \cap \Omega_3.
\end{align*}

**Proof.** Each of the jump matrices (7.9)–(7.16) follows from straightforward calculations based on the definitions (7.2)–(7.5) and the jump matrices in the RH problem for $U$. Then the jump matrices (7.9)–(7.11) are based on the factorization

\begin{align*}
\begin{pmatrix}
e^{-2n\phi_3^+} & 0 \\
e^{2n\phi_3^-}
\end{pmatrix}
&= \begin{pmatrix}1 & 0 \\
e^{-2n\phi_3^-}
\end{pmatrix}
\begin{pmatrix}1 & 0 \ \ 0 & 1 \\
0 & 1 \\
-1 & 0 \\
e^{-2n\phi_3^+}
\end{pmatrix}
\begin{pmatrix}1 & 0 \\
e^{2n\phi_3^-}
\end{pmatrix}.
\end{align*}
of the $2 \times 2$ lower right block in the jump matrix $J_U$ on $\mathbb{R}$, see (6.49) and (6.50). The jump matrices (7.12)--(7.13) are similarly based on the factorization
\[ (e^{2n\phi_2,-} 0) = (1 e^{2n\phi_2,-}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (1 e^{2n\phi_2,+}) \]
of the $2 \times 2$ middle block in the jump matrix $J_U$ on $S(\sigma - \mu_2)$, see (6.51).

The jump matrix (7.14) is the same as the corresponding jump matrix (6.52) in the RH problem for $U$. The jump matrix (7.15) follows from conjugating (6.52) with either (7.4) or (7.5).

The asymptotic condition in the RH problem for $T$ follows from the definition (7.2)--(7.5), the asymptotic condition in the RH problem for $U$ in Lemma 6.5, and the fact that $\text{Re} \phi_2 < 0$ in $\Omega_2$ and $\text{Re} \phi_3 > 0$ in $\Omega_3$. For the latter facts, see also Lemmas 7.2 and 7.3 below.

A more detailed analysis would show that the asymptotics for $T$ actually holds uniformly up to the axes, in contrast to the asymptotics for $U$. To show this we would have to trace back the transformations $Y \mapsto X \mapsto U \mapsto T$ to see in particular what combination of Pearcey integrals is actually involved in the regions $\Omega_2$ and $\Omega_3$ near the axes. It turns out that the asymptotics of the relevant combinations of Pearcey integrals is uniformly valid up to the axes. We will not give details here. □

### 7.3. Large $n$ behavior of $J_T$. In the following lemmas we state result about the sign of $\text{Re} \phi_2$ and $\text{Re} \phi_3$ on various parts of the contour $\Sigma_T$.

**Lemma 7.2.** For every neighborhoods $U_{ic}$ and $U_{-ic}$ around $ic$ and $-ic$ there exists an $\varepsilon_2 > 0$ such that

(a) $\text{Re} \phi_2(z) < -\varepsilon_2 |z|^{1/3}$ for $z \in \Sigma_2 \setminus U_{\pm ic}$,

(b) $\text{Re} \phi_2(z) > \varepsilon_2$ for $z \in [-ic, ic] \setminus U_{\pm ic}$.

**Proof.** (a) By definition of $\Sigma_2$ we have $\text{Re} \phi_2(z) < 0$ for $z \in \Sigma_2 \setminus \{ \pm ic \}$. Let us consider the behavior for $z \in \Sigma_2 \cap I$ near infinity. Then

\[ \phi_2(z) = -\frac{1}{2} \int_{ic}^{\infty} (\xi_2(y) - \xi_3(y)) \, dy = \frac{3(\omega - 1)^{4/3}}{8} z^{4/3} + \mathcal{O}(\log(z)), \]

as $z \to \infty$ remaining in the first quadrant. Since $\Omega_2$ contains the cone (7.1) for some $r > 0$, we have that there exists a $\delta > 0$ such that

\[ 0 < \text{arg } z < \pi/2 - \delta \]

for large enough $z \in \Sigma_2 \cap I$. Then we have from (7.17) that

\[ \text{Re} \phi_2(z) = \frac{3}{8} r^{4/3} |z|^{4/3} \text{Re} \left( (\omega - 1)e^{i\text{arg } z/3} \right) \left( 1 + \mathcal{O}(|z|^{-4/3} \log |z|) \right) \]

as $z \to \infty$ and $z \in \Sigma_2 \cap I$. By (7.18) we have that

\[ \text{Re} \left( (\omega - 1)e^{i\text{arg } z/3} \right) = \sqrt{3} \sin \left( \frac{4}{3} (\text{arg } z - \pi/2) \right) \]
for $z \in \Sigma_2 \cap I$ large enough. Hence there exists an $\varepsilon_2 > 0$ such that

$$\text{Re} \phi_2(z) < -\varepsilon_2|z|^{4/3}$$

(7.21)

for $z \in \Sigma_2 \cap I$ large enough. By similar arguments in the other quadrants, we see that we can choose $\varepsilon_2$ such that (7.21) holds for $z \in \Sigma_2$ large enough. Finally, by continuity we can choose $\varepsilon_2$ small enough such that (7.21) holds for $z \in \Sigma_2 \setminus U_{\pm ic}$. This proves the first property.

(b) The second property follows from the variational conditions for $\mu_2$.

From (5.3) and (2.16) we have

$$\text{Re} \phi_2(z) = \text{Re}(2g_2(z) - g_1(z) - g_3(z)) = -2U^{\mu_2}(z) + U^{\mu_1}(z) + U^{\mu_3}(z) > 0$$

(7.22)

for $z \in (-ic, ic)$. It is also clear that $\text{Re} \phi_2$ is continuous on $(-ic, ic)$. Hence the statement follows.

**Lemma 7.3.** There exists an $\varepsilon_3 > 0$ such that $\text{Re} \phi_3(z) > \varepsilon_3|z|^{4/3}$ for all $z \in \Sigma_3$.

**Proof.** The statement can be proved in the same way as we proved property (a) in the proof of Lemma 7.2. □

By Lemma 7.2 we see that the jump matrices $J_T$ in (7.13) and (7.14) converge pointwise to the identity matrix $I$ at an exponential rate as $n \to \infty$. In fact, it shows that this convergence is uniform as long as we stay away from the points $\pm ic$. Since $\text{Re} \phi_3 > 0$ in $\Omega_3$ we also have that (7.15) and (7.16) converge uniform to the identity matrix $I$ at an exponential rate as $n \to \infty$. By Lemma 7.3, we see that the jump matrix $J_T$ in (7.11) converges uniformly to the identity matrix $I$ at an exponential rate as $n \to \infty$.

**7.4. The fourth transformation $T \mapsto S$.** In the next transformation we also open the lens $\Omega_1$ around $S(\mu_1) = [-a, a]$.

We use $\Sigma_1$ to denote the outer boundary of $\Omega_1$ so that $\Sigma_1$ consists of two contours from $-a$ to $a$, one in the upper half-plane and one in the lower half-plane. Both contours are oriented from $-a$ to $a$. See also Figure 7.2.

Note that

$$\phi_{1,+}(z) = -\pi i \int_a^z d\mu_1, \quad z \in [-a, a],$$

(7.23)

Since we assume that $\mu_1$ is regular we have

$$\frac{d}{dx} \text{Im} \phi_{1,+}(x) < 0 \quad \text{and} \quad \frac{d}{dx} \text{Im} \phi_{1,-}(x) > 0,$$

(7.24)
for \( x \in (-a, a) \), and we see that by the Cauchy-Riemann equations, there exists a region around \((-a, a)\) so that \( \text{Re} \phi(z) > 0 \) for every \( z \not\in (-a, a) \) in that region. Hence the contours \( \Sigma \) can be taken such that

- \( \text{Re} \phi(z) > 0 \) for \( z \in \Sigma \setminus \{-a, a\} \),
- \( \Sigma \) is contained in \( \Omega \).

We define the 4 \( \times \) 4 matrix valued function \( S \) by

\[
S(z) = T(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ -e^{-2n\phi(z)} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z \in \Omega_1, \ \text{Im} z > 0,
\]

\[
S(z) = T(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ e^{-2n\phi(z)} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z \in \Omega_1, \ \text{Im} z < 0,
\]

and

\[
S(z) = T(z) \quad \text{elsewhere.}
\]

Then \( S \) is defined and analytic in \( \mathbb{C} \setminus \Sigma \) where

\[
\Sigma = \mathbb{R} \cup i\mathbb{R} \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3,
\]

see Figure 7.2 for a sketch of \( \Sigma \). The next lemma gives the RH problem that is satisfied by \( S \).

**Lemma 7.4.** \( S \) is the unique solution of the following RH problem

\[
S(z) = \begin{cases} S(z) = \frac{1}{z} \frac{0}{z} \frac{0}{z} \frac{0}{z} J_S(z), & z \in \Sigma_S, \\
S(z) = (I + \mathcal{O}(z^{-1/3})) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{1/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-1/3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A_j \end{pmatrix} & \text{uniformly as } z \to \infty \text{ in the } j\text{th quadrant.}
\end{cases}
\]

The matrices \( A_j \) are given by (3.30) and (3.31) and the jump matrices \( J_S \) are given by

\[
J_S(z) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad z \in S(\mu_1),
\]

\[
J_S(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -e^{-2n\phi(z)} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z \in \Sigma,
\]

\[
J_S(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ e^{-2n\phi(z)} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z \in \Sigma_1.
\]
Proof. Since $S$ differs from $T$ only in a bounded domain, it is clear that the asymptotic behavior of $S$ is the same as that of $T$. Hence the asymptotic condition in $\eqref{7.28}$ follows from the asymptotic condition in the RH problem $\eqref{7.8}$.

The calculations that lead to the jump matrices are based on the factorization

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-e^{-2n(\phi_1(z)+\phi_2(z))} & e^{-2n\phi_2(z)} & 1 & 0 \\
-e^{-2n(\phi_1(z)+\phi_2(z)+\phi_3(z))} & e^{-2n(\phi_2(z)+\phi_3(z))} & 0 & 1
\end{pmatrix},
\]

for $z \in (0, ic) \cap \Omega_1$.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
e^{-2n(\phi_1(z)+\phi_2(z))} & e^{-2n\phi_2(z)} & 1 & 0 \\
-e^{-2n(\phi_1(z)+\phi_2(z)+\phi_3(z))} & e^{-2n(\phi_2(z)+\phi_3(z))} & 0 & 1
\end{pmatrix},
\]

for $z \in (-ic, 0) \cap \Omega_1$, and

\[
J_S(z) = J_T \quad \text{elsewhere.}
\]

Proof. Since $S$ differs from $T$ only in a bounded domain, it is clear that the asymptotic behavior of $S$ is the same as that of $T$. Hence the asymptotic condition in $\eqref{7.28}$ follows from the asymptotic condition in the RH problem $\eqref{7.8}$.

The calculations that lead to the jump matrices are based on the factorization

\[
\begin{pmatrix}
e^{-2n\phi_{1,+}} & 1 \\
0 & e^{-2n\phi_{1,-}}
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix} 1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix} 1 & 0 \\
0 & 1
\end{pmatrix}
\]

of the $2 \times 2$ upper left block of $\eqref{7.9}$. We will not give further details here as this step in the RH steepest descent analysis is similar to the corresponding step in the RH analysis for orthogonal polynomials considered in $33$. □

The following lemma deals with the sign of $\Re \phi_1$ at parts of the contour $\Sigma_S$.

Lemma 7.5. Let $U_{-a}$ be a neighborhood of $-a$ and $U_a$ a neighborhood of $a$. Then there exists an $\varepsilon_1 > 0$ such that

(a) $\Re \phi_1(z) > \varepsilon_1$ for $z \in \Sigma_1 \setminus (U_{-a} \cup U_a)$,

(b) $\Re \phi_{1,+}(x) < -\varepsilon_1(|x| + 1)$ for $x \in \mathbb{R} \setminus (S(\mu_1) \cup U_{-a} \cup U_a)$.

Proof. (a) This follows immediately from the continuity of $\phi_1$ and the fact that $\Re \phi_1 > 0$ on $\Sigma_1^{(k)} \setminus \{a_k, b_k\}$.

(b) Part (b) follows from the variational condition $\eqref{2.14}$. Indeed, by $\eqref{6.19}$ we have that

\[
\Re \phi_{1,+}(x) = \frac{1}{2} \Re \left(2g_1(x) - g_2(x) - V(x) + \frac{3}{4}x^{4/3} + k_1 \right)
\]

for $x \in \mathbb{R} \setminus S(\mu_1)$. By $\eqref{2.14}$ and regularity of the measure we see that $\Re \phi_1(x) < 0$ for $x \in \mathbb{R} \setminus S(\mu_1)$. Moreover, for large $x$ we have that the
dominant term at the right-hand side of (7.34) is \(-V(x)\), which is a polynomial of even degree. Combining this with the continuity of \(\text{Re } \phi_1\) we see that there exists an \(\varepsilon_1 > 0\) such that

\[
\text{Re } \phi_{1,+}(x) < -\varepsilon_1(|x| + 1)
\]

for all \(x \in \mathbb{R} \setminus S(\mu_1)\). □

By Lemma 7.5 the matrix \(J_T\) in (7.30) converges to the identity matrix \(I_4\) at an exponential rate as \(n \to \infty\). If one stays away from the endpoints of \(S(\mu_1)\), the convergence is uniform. Combining Lemmas 7.5, 7.2 and 7.3 we also see that the jump matrix \(J_T\) given in (7.31) and (7.32) converge uniformly to the identity matrix \(I_4\) at an exponential rate as \(n \to \infty\).

We also see that the \((1, 2)\) entry of \(J_T\) in (7.10) converges to zero at an exponential rate as \(n \to \infty\). Again if we stay away from the endpoints of \(S(\mu_1)\) the convergence is uniform.

8. Construction of parametrices and the transformation \(S \mapsto R\)

8.1. The RH problem for \(M\). If we ignore all exponentially small entries in the jump matrices \(J_S\) in the RH problem for \(S\), then we obtain the
following model RH problem for a $4 \times 4$ matrix valued function $M$.

$$
\begin{align*}
M(z) &= (I + O(z^{-1}))
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & z^{1/3} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & z^{-1/3}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & A_j
\end{pmatrix}, \\
\text{uniformly as } z \to \infty \text{ in the } j\text{th quadrant.}
\end{align*}
$$

Here, the matrices $A_j$ are given by (3.30) and (3.31).

The solution of (8.1) is not unique. To ensure uniqueness, we impose the additional conditions

$$
\begin{align*}
M(z) &= O((z \mp a)^{-1/4}), \quad z \to \pm a, \\
M(z) &= O((z \mp ic)^{-1/4}), \quad z \to \pm ic.
\end{align*}
$$

The RH problem for $M$ can be solved in the one-cut case by using a rational parametrization of the Riemann surface, which provides a conformal map to the Riemann sphere.

We have $S(\mu_1) = [-a, a]$ and $S(\sigma - \mu_2) = i\mathbb{R} \setminus (-ic; ic)$. The Riemann surface thus depends on two parameters $a$ and $c$. It has genus zero. An explicit rational parametrization is given by the equation

$$
\begin{align*}
w + \frac{s^2 - t^2}{w} + \frac{s^2 t^2}{3w^3} = z,
\end{align*}
$$

where the constants $s > 0$, $t > 0$ are the unique positive solutions of the equations

$$
\begin{align*}
2s - \frac{2t^2}{3s} &= a, \\
2t - \frac{2s^2}{3t} &= -c.
\end{align*}
$$

In this parametrization the branch points at $z = \pm a$ and $z = \pm ic$ correspond to $w = \pm s$ and $w = \mp it$ respectively.

We introduce the following function

$$
w \mapsto ((w^2 - s^2)(w^2 + t^2))^{1/2},
$$

8.2 Construction of the outside parametrix $M$. The RH problem for $M$
defined and analytic in the complex $w$-plane cut along
\begin{equation}
\Gamma_N := w_{1,+}(S(\mu_1)) \cup w_{2,-}(S(\sigma - \mu_2)) \cup w_{3,+}(S(\mu_3)).
\end{equation}

The square root is taken such that \((8.5)\) behaves like $w^2$ as $w \to \infty$, and such that it changes sign when crossing the cuts. See also Figure 8.1.

**Proposition 8.1.** The RH problem \((8.1), (8.2)\) for $M$ has a unique solution.

*Proof.* The uniqueness of the solution follows by standard arguments for uniqueness of RH problems, see e.g. [30]. We have to prove existence of a solution.

Choose any basis $Q_k, \ k = 1, \ldots, 4$ of the space of polynomials of degree $\leq 3$, and define
\begin{equation}
N_k(w) = \frac{Q_k(w)}{w((w^2 - s^2)(w^2 + t^2))^{1/2}}.
\end{equation}

Then each $N_k$ is analytic in $\mathbb{C} \setminus \Gamma_N$ and satisfies
\begin{align*}
N_{k,+}(w) &= -N_{k,-}(w), \quad w \in \Gamma_N, \\
N_k(w) &= \mathcal{O}(1), \quad \text{as } w \to \infty, \\
N_k(w) &= \mathcal{O}(w^{-1}), \quad \text{as } w \to 0, \\
N_k(w) &= \mathcal{O}((w \mp s)^{-1/2}), \quad \text{as } w \to \pm s, \\
N_k(w) &= \mathcal{O}((w \mp it)^{-1/2}), \quad \text{as } w \to \pm it.
\end{align*}

Then the $4 \times 4$ matrix valued function $z \mapsto \hat{M}(z)$ with entries
\begin{equation}
\hat{M}_{kj}(z) = N_k(w_j(z)), \quad k, j = 1, \ldots, 4,
\end{equation}
where \( w_j, j = 1, \ldots, 4, \) are the mapping functions for (8.3), is analytic in \( \mathbb{C} \setminus (\mathbb{R} \cup S(\sigma - \mu_2)) \), satisfies the jump conditions in the RH problem for \( M \) (due to the property (8.8) of \( N_k \)), as well as the fourth root condition (8.2) (due to (8.11)–(8.12)). In addition we have as \( z \to \infty \),

\[
\tilde{M}_{kj}(z) = \begin{cases} 
\mathcal{O}(1), & \text{for } j = 1, \\
\mathcal{O}(z^{1/3}), & \text{for } j = 2, 3, 4,
\end{cases}
\]

which follows from (8.9)–(8.10) and (8.13).

Since the jump matrices have determinant one, it follows by standard arguments that \( z \to \det \tilde{M}(z) \) extends to an entire function. From (8.14) it follows that \( \det \tilde{M}(z) = \mathcal{O}(z) \) as \( z \to \infty \), so that

\[
\det \tilde{M}(z) = \alpha z + \beta,
\]

for some constants \( \alpha \) and \( \beta \). Since the polynomials \( Q_k, k = 1, \ldots, 4 \) are linearly independent, \( \alpha \) and \( \beta \) cannot both be zero.

Define

\[
A(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{1/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-1/3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A_j \end{pmatrix}
\]

for \( z \) in the \( j \)th quadrant. Then \( A \) is analytic in \( \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \) and from the formulas (3.30) and (3.31) for the matrices \( A_j \) it easily follows that on \( \mathbb{R} \cup i\mathbb{R} \) it satisfies the jump conditions

\[
A_+(z) = A_-(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad z \in \mathbb{R},
\]

\[
A_+(z) = A_-(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z \in i\mathbb{R}.
\]

Comparing this with the jumps in the RH problem for \( M \), we see that \( A \) and \( \tilde{M} \) satisfy the same jump conditions for \( z \) large enough. Then \( \tilde{M}A^{-1} \) is analytic in a neighborhood of infinity.

Due to (8.14) and (8.16), we have \( \tilde{M}(z)A^{-1}(z) = \mathcal{O}(z^{2/3}) \) as \( z \to \infty \). Therefore, by analyticity at infinity, we have

\[
\tilde{M}(z)A^{-1}(z) = C + \mathcal{O}(z^{-1}), \quad z \to \infty,
\]

for a constant matrix \( C \). Then det \( \tilde{M}(z) = \det C + \mathcal{O}(z^{-1}) \), so that by (8.15), \( \alpha = 0 \) and \( \beta = \det C \). Since \( \alpha \) and \( \beta \) are not both zero, we find \( \det C \neq 0 \). Thus \( C^{-1} \) exists, and then it easily follows that

\[
M = C^{-1}\tilde{M}
\]
satisfies all conditions in the RH problem for $M$, as well as the fourth root condition (8.2).

This completes the construction of a solution of the RH problem for $M$ in the one-cut case.

□

8.3. Construction of local parametrices. The next step is the construction of local parametrices near the branch points. Since we are in the one-cut regular case, the density of $\mu_1$ vanishes as a square root at the endpoints $\pm a$. As in the case of orthogonal polynomials \[33, 34\] the local parametrix will then be constructed with the help of Airy functions. Also for larger size RH problems Airy parametrices have been constructed, see e.g. \[3, 19, 28, 59\]. The situation in the present case is similar, and so we will not give all details of the construction here.

8.3.1. The model RH problem: Airy functions. Airy functions solve a model RH problem. Let $y_0, y_1$ and $y_2$ be defined by

$$y_0(s) = \text{Ai}(s), \quad y_1(s) = \omega \text{Ai}(\omega s), \quad y_2(s) = \omega^2 \text{Ai}(\omega^2 s).$$

where $\text{Ai}$ is the Airy function and $\omega = e^{2\pi i/3}$. Consider the $2 \times 2$ matrix valued function $\Psi$

$$\Psi(s) = \begin{pmatrix} y_0(s) & -y_2(s) \\ y'_0(s) & -y'_2(s) \end{pmatrix}, \quad \text{arg } s \in (0, 2\pi/3),$$

$$\Psi(s) = \begin{pmatrix} -y_1(s) & -y_2(s) \\ -y'_1(s) & -y'_2(s) \end{pmatrix}, \quad \text{arg } s \in (2\pi/3, \pi),$$

$$\Psi(s) = \begin{pmatrix} -y_2(s) & y_1(s) \\ -y'_2(s) & y'_1(s) \end{pmatrix}, \quad \text{arg } s \in (-\pi, -2\pi/3),$$

$$\Psi(s) = \begin{pmatrix} y_0(s) & y_1(s) \\ y'_0(s) & y'_1(s) \end{pmatrix}, \quad \text{arg } s \in (-2\pi/3, 0),$$

Then $\Psi$ is analytic in the complex $s$-plane with a jump discontinuity along the rays $\text{arg } s = 0$, $\text{arg } = \pm 2\pi/3$ and $\text{arg } s = \pi$. We equip these rays with an orientation as shown in Figure 8.2. This figure also shows the jump matrices, which can be easily obtained from the definition (8.22)–(8.25) and the linear relation $y_0 + y_1 + y_2 = 0$.

The asymptotic behavior of $\Psi(s)$ is given by

$$\Psi(s) \approx \frac{1}{2\sqrt{\pi}} \begin{pmatrix} s^{-1/4} & 0 \\ 0 & s^{1/4} \end{pmatrix} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \left(I + \mathcal{O}(s^{-2/3})\right) \begin{pmatrix} e^{-\frac{2}{3}s^{3/2}} & 0 \\ 0 & e^{\frac{2}{3}s^{3/2}} \end{pmatrix},$$

as $s \to \infty$.

8.3.2. The local parametrix near $\pm a$. The local parametrix $P_a$ is constructed in a small disk $U_a$ around $a$. It should satisfy the jump conditions in the RH problem for $S$ exactly and match with $M$ on the boundary $\partial U_a$. 

Recall the function $\phi_1$ from (6.13), which behaves near $z = a$ as
\begin{equation}
-\frac{3}{2} \phi_1(z) = \rho(z - a)^{3/2} \left(1 + O(z - a)\right), \quad z \to a,
\end{equation}
where $\rho > 0$ is such that
\begin{equation}
\frac{d}{dx} \mu_1(x) = \frac{\rho}{\pi} (a - x)^{1/2} \left(1 + O((a - x))\right), \quad x \not\to a.
\end{equation}
Then
\begin{equation}
f_1(z) = \left(-\frac{3}{2} \phi_1(z)\right)^{2/3}, \quad z \in U_a,
\end{equation}
is a conformal map from $U_a$ onto a neighborhood of the origin, so that $f_1(z) > 0$ for $z \in U_a \cap [a, \infty)$. We use the freedom we have in opening the lens around $[-a, a]$ so that $f_1$ maps the part of $\Sigma_1 \cap U_a$ in the upper half-plane into the ray $\arg s = 2\pi/3$ and the part in the lower half-plane into the ray $\arg s = -2\pi/3$.

Now define $P_a$ by
\begin{equation}
P_a(z) = E_a(z) \begin{pmatrix} \Psi (n^{2/3} f_1(z)) & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} e^{-n \phi_1(z) \sigma_3} & 0 \\ 0 & I_2 \end{pmatrix},
\end{equation}
for $z \in U_a$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $E_a$ is given by
\begin{equation}
E_a(z) = \sqrt{\pi} M(z) \begin{pmatrix} 1 & -1 & 0 \\ -i & -i & 0 \\ 0 & 0 & I_2 \end{pmatrix} \begin{pmatrix} (n^{2/3} f_1(z))^{\sigma_3/4} & 0 \\ 0 & I_2 \end{pmatrix}.
\end{equation}
for $z \in U_a$. 

**Figure 8.2.** The contour and the jump matrices in the RH problem for $\Psi$. 

- **Matrix $A_0$:**
  - Upper triangle: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
  - Lower triangle: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

- **Matrix $A_1$:**
  - Upper triangle: $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$
  - Lower triangle: $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

- **Matrix $A_2$:**
  - Upper triangle: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
  - Lower triangle: $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

- **Contour:**
  - Clockwise with a $2\pi/3$ rotation.

- **Jump Matrices:**
  - $A_0$ to bottom right $A_1$
  - $A_1$ to bottom left $A_2$
In the following lemma we prove that $P_a$ exactly solves the jump condition in the RH problem for $S$ in $U_a$ and matches with the outside parametrix $M$ on $\partial U_a$.

**Lemma 8.2.** The function $P_a$ satisfies

\begin{equation}
\begin{cases}
P_a \text{ is analytic in } U_a \setminus J_S, \\
P_a(z) = P_{a_-}(z) J_S(z), \quad z \in U_a \cap J_S, \\
P_a(z) = M(z)(1 + O(1/n)), \quad \text{as } n \to \infty, \text{ uniformly for } z \in \partial U_a.
\end{cases}
\end{equation}

**Proof.** The proof is standard. \hfill \Box

The construction of the local parametrix $P_{-a}$ around $-a$ can be done similarly. We can also use the symmetry to define $P_{-a}$ directly in terms of $P_a$ as follows

\begin{equation}
P_{-a}(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} P_a(-z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{equation}

**8.3.3. The local parametrix near $\pm ic$.** The local parametrices near the branch cuts $\pm ic$ are constructed in essentially the same way. We will focus on $-ic$.

The jump matrices in the RH problem for $S$ that are relevant near $-ic$ (7.12), (7.13) and (7.14). They are non-trivial only in the $2 \times 2$ middle block.

We again construct the local parametrix by means of the Airy model RH problem and a conformal map that maps a disk $U_{-ic}$ around $-ic$ onto a neighborhood of the origin. There is a small difference in the fact that the $2 \times 2$ middle block in (7.13) is upper triangular, whereas the jump matrix for $\Psi$ on the rays $\arg s = \pm 2\pi/3$ is lower triangular. We can deal with the different triangularity structure by using

\begin{equation}
\Phi = \Psi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{equation}

instead of $\Psi$ in the construction of $P_{-ic}$.

The conformal map $f_2$ is constructed out of $\phi_2$, which behaves like

\begin{equation}
\frac{3}{2} \phi_2(z) = \bar{\rho}(z + ic)^{3/2} e^{-3\pi i/4}(1 + O(z + ic)),
\end{equation}

as $z \to -ic$. The fractional power is defined here with a branch cut along $(-ic, -\infty)$ and so that $\phi_2$ takes positive values on $(-ic, 0)$. The number $\bar{\rho} > 0$ is such that (see also part (b) of Theorem (2.3))

\begin{equation}
\frac{d(\sigma - \mu_2)}{|dz|} = \frac{\bar{\rho}}{\pi} |y + c|^{1/2}(1 + O(y + c)), \quad y \nearrow -c.
\end{equation}
Then $f_2$ is defined on a small enough disk $U_{-ic}$ around $-ic$ by

$$f_2(z) = \left( \frac{3}{2} \phi_2(z) \right)^{2/3},$$

with the $2/3$-root taken so that $f_2(z) > 0$ for $z \in (-ic, 0) \cap U_{-ic}$. Then $f_2$ is a conformal map from $U_{-ic}$ onto a neighborhood of zero. We adjust the definition of the lens around $(-ic, -ic]$ so that the part of $\Sigma_2 \cap U_{-ic}$ in the left half-plane is mapped into the ray $\arg s = 2\pi/3$ and the part in the right half-plane into the ray $\arg s = -2\pi/3$.

Now we define $P_{-ic}$ by

$$P_{-ic}(z) = E_{-ic}(z) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & \Phi(n^{2/3}f_2(z)) & 0 \\ 0 & 0 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & e^{-n\phi_2(z)\sigma_3} & 0 \\ 0 & 0 & 1 \end{array} \right)$$

with $\Phi$ as in (8.34), and $E_{-ic}$ given by

$$E_{-ic}(z) = \sqrt{\pi} M(z) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -i & -i \\ 0 & 1 & -1 \end{array} \right) \left( \begin{array}{ccc} 1 & 0 \\ 0 & (n^{2/3}f_2(z))^{-\sigma_3/4} \\ 0 & 0 \end{array} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{array} \right)$$

The fractional power $(f_2(z))^{-1/4}$ is defined with a branch cut along $(-ic, -ic]$ and so that it is real and positive for $z \in (-ic, 0) \cap U_{-ic}$.

Then we have the following result, whose proof is again omitted.

**Lemma 8.3.** The function $P_{-ic}$ satisfies

$$P_{-ic} \text{ is analytic in } U_{-ic} \setminus J_S,$$

$$P_{-ic,+}(z) = P_{-ic,-}(z) J_S(z), \quad z \in U_{-ic} \cap J_S,$$

$$P_{-ic}(z) = M(z)(1 + \mathcal{O}(1/n)), \quad \text{as } n \to \infty \ \text{uniformly for } z \in \partial U_{-ic}.$$}

The local parametrices $P_{ic}$ around $ic$ can be constructed in a similar way. We can also use the symmetry in the problem and define $P_{ic}$ in terms of $P_{-ic}$ as follows

$$P_{ic}(z) = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) P_{-ic}(-z) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right).$$

**8.4. The final transformation $S \mapsto R$.** Having constructed the outside parametrices $M$ and the local parametrices $P_{\pm a}$ and $P_{-ic}$, we define

$$P(z) = \begin{cases} M(z), & z \in \mathbb{C} \setminus \left( \overline{U_{\pm a}} \cup \overline{U_{\pm ic}} \cup \Sigma_S \right), \\
P_{\pm a}(z), & z \in U_{\pm a} \setminus \Sigma_S, \\
P_{\pm ic}(z), & z \in U_{\pm ic} \setminus \Sigma_S. \end{cases}$$
Recall that the $3 \times 3$ matrix-valued function (3.14)–(3.15) constructed out of the Pearcey integrals was also denoted by $P$. Since this function will not play a role anymore, we trust that the double use of the symbol $P$ will not lead to any confusion. From now on $P$ will always refer to the function defined in (8.42).

Define the final transformation $S \mapsto R$ by

$$R = S P^{-1}. \tag{8.43}$$

Then $R$ is defined and analytic in $\mathbb{C} \setminus (\Sigma_S \cup \partial U_{\pm a} \cup \partial U_{\pm ic})$, and has an analytic continuation to $\mathbb{C} \setminus \Sigma_R$, with

$$\Sigma_R = (\partial U_{\pm a} \cup \partial U_{\pm ic} \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup (-ic, ic) \cup \mathbb{R}) \setminus ([-a, a] \cup U_{\pm a} \cup U_{\pm ic}), \tag{8.44}$$

see also Figure 8.3. The circles $\partial U_{\pm a}$ and $U_{\pm ic}$ are oriented in the counterclockwise direction.

We obtain the following RH problem for $R$.

**Lemma 8.4.** We have that $R$ satisfies the RH problem

$$\begin{cases}
R \text{ is analytic in } \mathbb{C} \setminus \Sigma_R, \\
R_+(z) = R_-(z) J_R(z), \quad z \in \Sigma_R, \\
R(z) = I + O(1/z), \quad \text{uniformly as } z \to \infty,
\end{cases} \tag{8.45}$$
where the jump matrix $J_R$ is given on the various parts of $\Sigma_R$ as follows

(8.46) \[ J_R(z) = M(z)P_{\pm a}(z)^{-1}, \quad z \in \partial U_{\pm a}, \]

(8.47) \[ J_R(z) = M(z)P_{\pm ic}(z)^{-1}, \quad z \in \partial U_{\pm ic}, \]

(8.48) \[ J_R(z) = M_{-}(z) \begin{pmatrix} 1 & e^{2\alpha_1(z)} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} M_{+}(z)^{-1}, \quad z \in \mathbb{R} \setminus ([-a, a] \cup U_{\pm a}) \]

(8.49) \[ J_R(z) = M(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ e^{-2\alpha_1(z)} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M(z)^{-1}, \quad z \in \Sigma_1 \setminus U_{\pm a}, \]

(8.50) \[ J_R(z) = M(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & e^{2\alpha_2(z)} & 1 \end{pmatrix} M(z)^{-1}, \quad z \in \Sigma_2 \setminus U_{\pm ic}, \]

(8.51) \[ J_R(z) = M(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & e^{2\alpha_3(z)} & 1 \end{pmatrix} M(z)^{-1}, \quad z \in \Sigma_3, \]

and on the various pieces of the segment $(-ic, ic)$ we have

(8.52) \[ J_R(z) = M(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M(z)^{-1}, \quad z \in (-ic, ic) \setminus (\Omega_3 \cup U_{\pm ic}), \]

(8.53) \[ J_R(z) = M(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & e^{-2\alpha_2(z)} & 1 & 0 \\ 0 & e^{-2\alpha_3(z) + \phi_2(z)} & 0 & 1 \end{pmatrix} M(z)^{-1}, \quad z \in (-ic, ic) \cap (\Omega_3 \setminus \Omega_1), \]

(8.54) \[ J_R(z) = M(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -e^{-2\alpha_3(z) + \phi_2(z)} & e^{-2\alpha_3(z)} & 1 & 0 \\ -e^{-2\alpha_1(z) + \phi_2(z) + \phi_3(z)} & e^{-2\alpha_2(z) + \phi_3(z)} & 0 & 1 \end{pmatrix} M(z)^{-1}, \quad z \in (-ic, ic) \cap \Omega_1. \]
Proof. The formulas for $J_R$ follow by straightforward calculations. From the asymptotic condition in the RH problems for $S$ and $M$ in (7.28) and (8.1), respectively, we obtain
$$R(z) = S(z)M(z)^{-1} = I + O(z^{-1/3}) \quad \text{as } z \to \infty.$$ Since the jump matrices $J_R(z)$ are exponentially close to the identity matrix as $z \to \infty$ (see (8.56), (8.58), (8.60) below) the improved error term $R(z) = I + O(z^{-1})$ will follow. □

From Lemmas 7.2, 7.3, 7.5, and the matching conditions for the local parametrices $P_{\pm a}$ and $P_{\pm i\epsilon}$ it follows that all jump matrices are close to the identity matrix if $n$ is large.

In fact, by (8.46) and (8.47) and the matching conditions in (8.32) and (8.40) it follows that there exists a constant $C_0 > 0$ such that
$$\|J_R(z) - I\| \leq \frac{C_0}{n}, \quad z \in \partial U_{\pm ic} \cup \partial U_{\pm a}.$$ Here we can use any matrix norm $\| \cdot \|$. The other jump matrices are exponentially close to the identity matrix as $n \to \infty$. By Lemma 7.5 we have that
$$\|J_R(z) - I\| \leq C_1 \exp(-n\epsilon_1(|z| + 1)), \quad z \in \mathbb{R} \setminus (S(\mu_1) \cup U_{\pm a})$$ and
$$\|J_R(z) - I\| \leq C_1 \exp(-n\epsilon_1), \quad z \in \Sigma_1 \setminus U_{\pm a}$$ for some constants $C_1 > 0$ and $\epsilon_1 > 0$. By Lemma 7.2 we have
$$\|J_R(z) - I\| \leq C_2 \exp(-n\epsilon_2(|z| + 1)), \quad z \in \Sigma_2 \setminus U_{\pm ic}$$ for some constants $C_2 > 0$ and $\epsilon_2 > 0$. From the fact that $\text{Re} \phi_1(z) > 0$ for $z \in \Omega_1$, $\text{Re} \phi_3 > 0$ for $z \in \Omega_3$ and by Lemma 7.2 we have
$$\|J_R(z) - I\| \leq C_2 \exp(-n\epsilon_2), \quad z \in (-ic, ic) \setminus U_{\pm ic}$$ where we can use the same constants $C_2$ and $\epsilon_2 > 0$. Finally, by Lemma 7.3 we have
$$\|J_R(z) - I\| \leq C_3 \exp(-n\epsilon_3|z|^{4/3}), \quad z \in \Sigma_3$$ for some constants $C_3 > 0$ and $\epsilon_3 > 0$.

This leads to the following result.

Proposition 8.5. There exists a constant $C > 0$ such that for large enough $n$,
$$\|R(z) - I\| \leq \frac{C}{n(|z| + 1)}, \quad z \in \mathbb{C} \setminus \Gamma_R.$$ Proof. The proposition follows from the above estimates on $\|J_R(z) - I\|$ from the arguments as used in [34, Theorem 7.10]. □

This concludes the steepest descent analysis of the RH problem for $Y$. 

9. Proofs of Theorems 2.7 and 2.8

In this section we prove our main results Theorems 2.7 and 2.8. We assume that $\mu_1$ is one-cut regular.

9.1. The kernel $K_{11}^{(n)}$. The kernel $K_{11}^{(n)}$ is expressed in terms of $Y$ by formula (1.30). By following the sequence of transformations $Y \mapsto X \mapsto U \mapsto T \mapsto S \mapsto R$ we obtain expressions for $K_{11}^{(n)}$ in terms of the solutions of the other RH problems.

First transformation $Y \mapsto X$. From the transformation $Y \mapsto X$ (3.32)–(3.34), it follows that the right-hand side of (1.30) can be written as

\[ K_{11}^{(n)}(x, y) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} 0 & w_{0,n}(y) & w_{1,n}(y) & w_{2,n}(y) \\ 0 & D_n Q_+ (n^{3/4} \tau y) \Theta_{1,+} (n^{3/4} \tau y) \end{pmatrix} X_-(y)^{-1} X_+(x) \times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \Theta_{1,+} (n^{3/4} \tau x)^{-1} Q_+ (n^{3/4} \tau x)^{-1} D_n^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \]

for $x > 0$ and $y > 0$. By arguments as in (3.10) and (3.19) we can rewrite (9.1) as

\[ K_{11}^{(n)}(x, y) = \frac{1}{2\pi i(x - y)} \begin{pmatrix} 0 & -e^{-n(V(y)-\frac{3}{4}\tau^4/3|y|^{4/3})} & 0 & 0 \end{pmatrix} X_+(y)^{-1} X_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \]

for $x > 0$ and $y > 0$. Similar calculations show that (9.2) also holds for general $x, y \in \mathbb{R}$.

Second transformation $X \mapsto U$. Applying (6.4) and using (6.5) and (6.6), we next rewrite (9.2) as

\[ K_{11}^{(n)}(x, y) = \frac{e^{n g_{1,+}(x)}}{2\pi i(x - y)} \begin{pmatrix} 0 & -e^{-n(V(y)-\frac{3}{4}\tau^4/3|y|^{4/3})} & 0 & 0 \end{pmatrix} X_+(y)^{-1} X_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} U_+^{-1}(y) U_+(x), \]

for $x, y \in \mathbb{R}$. 
Then by \((6.17)\) and \((6.16)\) in case \(y \in S(\mu_1)\), and by \((6.19)\) in case \(y \in \mathbb{R} \setminus S(\mu_1)\), we have

\[
V(y) - \frac{3}{4}T^{4/3}|y|^{4/3} - g_1(+y) + g_2(y) - k_1 \equiv -2\phi_1(\mathcal{y}) + g_1(\mathcal{y}) \quad \text{(mod } 2\pi i/3),
\]

Substituting this into \((9.3)\), we arrive at

\[
(9.4) \quad K_{11}^{(n)}(x, y) = \frac{e^{n(g_1,+)(x)-g_1,+)(y)}}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{2n\phi_1,+}(y) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times U_+^{-1}(y)U_+(x), \quad x, y \in \mathbb{R}.
\]

Third transformation \(U \mapsto T\). The transformation \((7.4)\) acts only on the lower right \(2 \times 2\) block. It does not affect the kernel \((9.4)\). So we have

\[
(9.5) \quad K_{11}^{(n)}(x, y) = \frac{e^{n(g_1,+)(x)-g_1,+)(y)}}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{2n\phi_1,+}(y) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times T_+^{-1}(y)T_+(x), \quad x, y \in \mathbb{R}.
\]

Fourth transformation \(T \mapsto S\). Combining \((7.25)\) and \((7.26)\) with \((9.5)\) we obtain

\[
(9.6) \quad K_{11}^{(n)}(x, y) = \frac{e^{n(g_1,+)(x)-g_1,+)(y)}}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{2n\phi_1,+}(y) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times S_+^{-1}(y)S_+(x) \begin{pmatrix} 1 & 0 & 0 & 0 \\ \chi_{[-a,a]}(x)e^{2n\phi_1,+}(x) & 0 & 0 & 0 \end{pmatrix}, \quad x, y \in \mathbb{R},
\]

where \(\chi_{[-a,a]}\) is the characteristic function of the interval \([-a, a]\).

Final transformation \(S \mapsto R\). Since \(S = RP\), we obtain

\[
(9.7) \quad K_{11}^{(n)}(x, y) = \frac{e^{n(g_1,+)(x)-g_1,+)(y)}}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{2n\phi_1,+}(y) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \times P_+^{-1}(y)R^{-1}_+(y)R_+(x)P_+(x) \begin{pmatrix} 1 & 0 & 0 & 0 \\ \chi_{[-a,a]}(x)e^{2n\phi_1,+}(x) & 0 & 0 & 0 \end{pmatrix}, \quad x, y \in \mathbb{R}.
\]

for \(x, y \in \mathbb{R}\). This is the final formula.
9.2. Proofs of Theorems 2.7 and 2.8

Proof. Recall that $R(z) = I + O(1/n)$ as $n \to \infty$ uniformly for $z \in \mathbb{C} \setminus \Sigma_R$, which we use in (9.7). Theorems 2.7 and 2.8 then follow by calculations that are similar to the ones in e.g. [19, 28].

Note that the factor $e^{n(g_1+(x)-g_1+(y))}$ in (9.7) disappears for $x = y$, and it also drops out of the determinantal formulas in (2.20) and (2.21). □

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