Inner Functions and Inner Factors of Their Derivatives

Konstantin M. Dyakonov

Abstract. For an inner function $\theta$ with $\theta' \in \mathcal{N}$, where $\mathcal{N}$ is the Nevanlinna class, several problems are posed in connection with the canonical (inner-outer) factorization of $\theta'$.

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Let $\mathbb{D}$ stand for the disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T}$ for its boundary. Recall that a function $\theta \in H^\infty$ (i.e., a bounded analytic function $\theta$ on $\mathbb{D}$) is called inner if $\lim_{r \to 1^-} |\theta(r\zeta)| = 1$ for almost all $\zeta \in \mathbb{T}$. Further, a nonvanishing analytic function $F$ on $\mathbb{D}$ is said to be outer if $\log |F|$ agrees with the Poisson integral of an integrable function on $\mathbb{T}$. The Nevanlinna class $\mathcal{N}$ (resp., the Smirnov class $\mathcal{N}^+$) is formed by the functions representable as $u/v$, where $u, v \in H^\infty$ and $v$ is zero-free (resp., outer) on $\mathbb{D}$. Now, for an inner function $\theta$, we want to understand what the derivative $\theta'$ looks like, provided that the latter happens to be in $\mathcal{N}$ (or $\mathcal{N}^+$).

We assume that the reader is familiar with basic concepts and facts of function theory on $\mathbb{D}$. These include the definitions and standard properties of the Hardy spaces $H^p$, Blaschke products, singular inner functions, the canonical factorization in $\mathcal{N}$, $\mathcal{N}^+$ and $H^p$, etc. All of this background material can be found in [7, Chapter II].

The problems raised in this note are in part motivated by the following result; see [4] or [5].

Proposition A. Let $\theta$ be a nonconstant inner function with $\theta' \in \mathcal{N}$. Then $\theta'$ is outer if and only if $\theta$ is a Möbius transformation (i.e., a conformal automorphism of $\mathbb{D}$).

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This, in turn, is a simple consequence of the “reverse Schwarz–Pick type inequality”
\[
\frac{1 - |\theta(z)|^2}{1 - |z|^2} \leq |O(z)|, \quad z \in \mathbb{D};
\]
(1)
here \(O = O_{|\theta'|}\) is the outer function with modulus \(|\theta'|\) on \(\mathbb{T}\), while \(\theta\) satisfies the hypotheses of Proposition A. For (1), the reader is referred to [3, Section 2] or to [4], where a more general version is given.

Now, write \(\mathcal{I}\) for the set of nonconstant inner functions \(\theta\) with \(\theta' \in \mathcal{N}\). For \(\theta \in \mathcal{I}\), we actually have the (seemingly) stronger property that \(\theta' \in \mathcal{N}^+\), a fact established by Ahern and Clark in [1]. In particular, the map \(\theta \mapsto \text{inn}(\theta')\), \(\theta \in \mathcal{I}\), (2)
where \(\text{inn}(\cdot)\) stands for “the inner factor of”, is well defined. Proposition A tells us that \(J := \text{inn}(\theta')\) is nonconstant for every non-Möbius \(\theta \in \mathcal{I}\). In addition, the boundary spectra \(\sigma(\theta)\) and \(\sigma(J)\) of the two inner functions must then agree:
\[
\sigma(\theta) = \sigma(J).
\]
(3)
(By definition, the boundary spectrum \(\sigma(I)\) of an inner function \(I\) is the smallest closed set \(K \subset \mathbb{T}\) such that \(I\) is analytic across \(\mathbb{T} \setminus K\).) A weaker version of (3) is verified in [6]; the full version can be proved by combining inequality (1) with a result from [8].

Some more notation will be needed. Given a set \(\mathcal{F} \subset \mathcal{N}^+\), we write \(\text{inn}(\mathcal{F}) = \{\text{inn}(f) : f \in \mathcal{F}\}\), and we denote by \(\text{div inn}(\mathcal{F})\) the collection of all inner functions that arise as divisors of those in \(\text{inn}(\mathcal{F})\). Thus, an inner function \(I\) is in \(\text{div inn}(\mathcal{F})\) if and only if \(J/I \in H^\infty\) for some \(J \in \text{inn}(\mathcal{F})\). Clearly,
\[
\text{inn}(\mathcal{F}) \subset \text{div inn}(\mathcal{F}).
\]
The set \(\mathcal{F}\) that chiefly interests us is
\[
\mathcal{J}' = \{\theta' : \theta \in \mathcal{I}\},
\]
so that \(\text{inn}(\mathcal{J}')\) is the range of the map (2). Also, let \(\mathcal{B}\) (resp., \(\mathcal{S}\)) stand for the set of Blaschke products (resp., singular inner functions) lying in \(\mathcal{I}\). We may then look at the classes \(\mathcal{B}'\) and \(\mathcal{S}'\), defined as the images of \(\mathcal{B}\) and \(\mathcal{S}\) under differentiation, and ask about the inner factors (and their divisors) associated with them.

The case of \(\mathcal{B}'\), however, leads to no new problem. Indeed, for a given \(\theta \in \mathcal{I}\), Frostman’s theorem (see [7, Chapter II]) provides us with an \(\alpha \in \mathbb{D}\) such that the function
\[
B_\alpha := \frac{\theta - \alpha}{1 - \bar{\alpha}\theta}
\]
is a Blaschke product. Differentiating, we see that \(B_\alpha' \in \mathcal{N}\) (whence \(B_\alpha \in \mathcal{B}\)) and \(\text{inn}(B_\alpha') = \text{inn}(\theta')\). Therefore, the inner factors that occur for functions in \(\mathcal{B}'\) are the same as those for \(\mathcal{J}'\). The case of \(\mathcal{S}'\) is more delicate, though.

**Problem 1.** Characterize the sets \(\text{inn}(\mathcal{J}')\) and \(\text{inn}(\mathcal{S}')\).
We do not know whether the former set actually coincides with the collection of all inner functions. Thus, we ask in particular whether every inner function \( J \) can be written as \( J = \text{inn}(\theta') \) for some \( \theta \in \mathcal{I} \). If not, we seek to describe the \( J \)'s that arise in this way.

As to the other set, \( \text{inn}(\mathcal{S}') \), it turns out to be strictly smaller than \( \text{inn}(\mathcal{I}') \) and hence nontrivial. To see why, take \( I(z) = zS_1(z) \), where

\[
S_1(z) := \exp \left( \frac{z + 1}{z - 1} \right), \quad z \in \mathbb{D}. \tag{4}
\]

Letting \( a = 1 - \sqrt{2} \) and \( b(z) = (z - a)/(1 - az) \), one verifies by a straightforward calculation that

\[
(bS_1)'(z) = \frac{-4I(z)}{(1 - az)^2(1 - z)^2}
\]

(in doing so, the identity \( a^2 - 2a - 1 = 0 \) should be used). Therefore, \( I = \text{inn}((bS_1)') \) and hence \( I \in \text{inn}(\mathcal{I}') \).

On the other hand, if we had \( I = \text{inn}(S') \) for some \( S \in \mathcal{G} \), then (3) (with \( S \) and \( I \) in place of \( \theta \) and \( J \), respectively) would tell us that \( \sigma(S) = \sigma(I) = \{1\} \), and so \( S \) would have to coincide with \( S_\gamma(z) := \exp \left( \frac{\gamma z + 1}{z - 1} \right) \) for some \( \gamma > 0 \). However, since \( \text{inn}(S'_\gamma) = S_\gamma \), we see that \( I \) does not agree with the inner factor of \( S'_\gamma \) for any \( \gamma > 0 \). Consequently, \( I \notin \text{inn}(\mathcal{G}') \).

**Problem 2.** Characterize the set \( \text{div inn}(\mathcal{I}') \).

While this set contains \( \text{inn}(\mathcal{I}') \), one may well ask whether the inclusion is proper and/or whether every inner function is in \( \text{div inn}(\mathcal{I}') \).

We leave these questions open, but we do observe that \( \mathcal{I} \subset \text{div inn}(\mathcal{I}') \). Indeed, if \( \theta \in \mathcal{I} \), then \( \theta \) divides the inner factor of \( (\theta^2)' = 2\theta\theta' \); this last formula also shows that \( \theta^2 \) is in \( \mathcal{I} \).

Another observation is that, in contrast with the *proper* inclusion

\[
\text{inn}(\mathcal{G}') \subset \text{inn}(\mathcal{I}')
\]

(see above), we now have

\[
\text{div inn}(\mathcal{G}') = \text{div inn}(\mathcal{I}'). \tag{5}
\]

In particular, this implies that \( \text{div inn}(\mathcal{G}') \) is strictly larger than \( \text{inn}(\mathcal{G}') \).

To check (5), suppose \( J \) is a divisor of \( \text{inn}(\theta') \) for some \( \theta \) in \( \mathcal{I} \). Now fix a function \( S \in \mathcal{G} \) (e.g., take \( S = S_1 \), where \( S_1 \) is the “atomic” singular inner function given by (1)) and put \( \varphi := S \circ \theta \). Then \( \varphi \) is a singular inner function, and since

\[
\varphi'(z) = S'(\theta(z)) \cdot \theta'(z), \quad z \in \mathbb{D},
\]

it follows that \( \varphi' \in \mathcal{N} \) (whence \( \varphi \in \mathcal{G} \)) and that \( J \) divides \( \text{inn}(\varphi') \). This proves the nontrivial inclusion

\[
\text{div inn}(\mathcal{G}') \supset \text{div inn}(\mathcal{I}').
\]
between the two sides of (5) and thereby establishes the equality.

A similar argument allows us to generalize (5) as follows. Suppose $E$ is a subset of $\mathbb{D}$ such that the class

$$S_E := \{ \theta \in \mathcal{I} : \theta(\mathbb{D}) \cap E = \emptyset \}$$

is nonempty. (A result from [1] tells us that $E$ must be countable.) Then

$$\text{div inn}(S_E') = \text{div inn}(\mathcal{I}') \quad (6)$$

Since $\mathcal{S} = \mathcal{S}_{\{0\}}$, we see that (5) is indeed a consequence of (6).

**Problem 3.** For $0 < p < 1$, let $\mathcal{I}^p$ denote the set of all nonconstant inner functions $\theta$ with $\theta' \in H^p$, and let $\mathcal{S}^p := \mathcal{I}^p \cap \mathcal{S}$. Solve the two problems above for $\mathcal{I}^p$ and $\mathcal{S}^p$ in place of $\mathcal{I}$ and $\mathcal{S}$.

**Problem 4.** What, if anything, is the Bergman space counterpart of Proposition A? In other words, what happens for “Bergman-inner” functions?

To be more precise, recall that the Bergman space $A^p$, $0 < p < \infty$, is defined as the set of analytic functions lying in $L^p(\mathbb{D})$ with respect to area measure. Recall also that a unit-norm function $\psi \in A^p$ is said to be $A^p$-inner if $|\psi|^p$ annihilates every monomial $z^n$ with $n = 1, 2, \ldots$. Furthermore, one defines $A^p$-outer functions in the appropriate way and proves that every $f \in A^p$ has a factorization $f = \psi g$, where $\psi$ is $A^p$-inner and $g$ is $A^p$-outer (see [2, Chapter 9] for these notions and results). However, the factorization is not unique. Now, the question is whether Proposition A extends to the $A^p$ setting, once suitable adjustments are made.

We conclude with a somewhat vaguer problem, which concerns further possible extensions of Proposition A beyond the case of an inner function $\theta$.

**Problem 5.** Find reasonably sharp conditions on a function $f \in H^\infty$ (say, with $f'$ in $N$ or $N^+$) that ensure the presence of a nontrivial inner factor for $f'$, provided that $f$ itself enjoys a similar property. Furthermore, assuming that $f$ obeys those conditions and letting $I$ and $J$ be the inner factors of $f$ and $f'$, respectively, study the interrelationship between $\sigma(I)$ and $\sigma(J)$.

Results to that effect could be viewed as descendants of the classical Gauss–Lucas theorem on the critical points of a polynomial. In fact, some steps in this direction have already been made in [6]. The functions $f$ considered there are “locally inner” in the sense that $\|f\|_\infty = 1$ and $|f| = 1$ on a set of positive measure (on $\mathbb{T}$). However, these hypotheses seem to be too restrictive, and the results in [6] are far from being completely satisfactory, so there is plenty of room for improvement and further development.

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Konstantin M. Dyakonov
ICREA and Universitat de Barcelona
Departament de Matemàtica Aplicada i Anàlisi
Gran Via, 585
E-08007 Barcelona
Spain
e-mail: konstantin.dyakonov@icrea.cat