Abstract. We study an inflationary scenario with a vector impurity. We show that the universe undergoes anisotropic inflationary expansion due to a preferred direction determined by the vector. Using the slow roll approximation, we find a formula for determining the anisotropy of the inflationary universe. We discuss possible observable predictions of this scenario. In particular, it is stressed that primordial gravitational waves can be induced from curvature perturbations. Hence, even in low scale inflation, a sizable amount of primordial gravitational waves may be produced during inflation.

Keywords: cosmological perturbation theory, inflation

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1. Introduction

It is often stated that cosmology has entered into a new stage, so-called precision cosmology. Of course, this is referring to developments of the observational side. From a theoretical point of view, however, we have not yet exhausted possible phenomenology on the order of a few per cent. Clearly, it is important to explore qualitatively new scenarios at the per cent level. Here, above all, we would like to point out that an inflationary model with a few per cent of anisotropy yields significant consequences. Indeed, a few per cent does not mean that the consequent effects are negligible. Rather, it provides the leading component of the primordial gravitational waves in low scale inflationary models which are favored by recent model construction in string theory [1].

One may feel that to seek anisotropic inflation goes against the basics of the inflationary scenario. Actually, the isotropy is the most robust property of inflation because of the cosmic no-hair theorem on the isotropization of Bianchi universes [2]. However, it is possible to evade the cosmic no-hair theorem by incorporating the Kalb–Ramond action [3] or considering higher curvature theories of gravity [4, 5]. In spite of the possibility, no one has attempted to construct any inflation models based on these ideas. The apparent other possibility is that of breaking the Lorentz invariance by introducing a condensation of the vector field. The model proposed in [6, 7] seems to be successful; however, it is known to be metastable [8]–[10]. A more natural possibility would be to realize a slow roll phase of vector fields like as inflaton fields in chaotic inflationary scenarios. So far, it has been believed that it is difficult to make the vector field slow roll without fine tuning [11]5. Very recently, the situation was changed by the discovery of a slow roll mechanism for the vector field due to non-minimal coupling [13]. Hence, an apparent difficulty for constructing the anisotropic inflationary scenario has been resolved. At this point, it is important to realize that both scalar and vector fields are ingredients of fundamental particle physics models. Therefore, it is natural to consider both scalar and

5 Recently, a non-standard spinor driven inflation has been proposed [12]. It is interesting to see if the natural slow rolling can be achieved in this case.
vector fields to exist during inflation. Of course, the vector fields should be subdominant in the inflationary dynamics in order to reconcile the scenario with current observational data [14]. In this sense, the vector field should be regarded as an impurity. Nevertheless, the effect of the vector impurity on observables should not be overlooked with the current precision cosmology.

In this paper, we propose an anisotropic inflation model with vector impurity. To the best of our knowledge, this is the first concrete model which realizes anisotropic inflation, exits successfully to the isotropic standard universe, and provides a framework in which to discuss interesting phenomenology. We argue that the anisotropic inflation yields the statistical anisotropy in fluctuations. More importantly, as expected, the primordial gravitational waves could be induced from curvature perturbations through the anisotropic background. Hence, we can expect a correlation between the curvature perturbations and the gravitational waves. In addition to these, we point out that linear polarization of the gravitational waves is created, which should be observed through CMB or direct interferometer observations.

The organization of this paper is as follows. In section 2, we present the model for the anisotropic inflation. In section 3, we analyze the system numerically and show that the anisotropic inflation is realized successfully. In section 4, using the slow roll approximation, we obtain degrees of the anisotropy. In section 5, we discuss phenomenology of the anisotropic inflation induced by the vector impurity. There, we emphasize that the gravitational waves can be produced through the anisotropy of the spacetime. The final section is devoted to a discussion.

2. The model

In this section, we derive the basic equations for the anisotropic inflationary scenario induced by the vector impurity. The setup is similar to those of the previous works [15, 16] where scalar and vector fields are considered. However, back-reaction of the vector field to the background dynamics is neglected there. Here, we consider the back-reaction and find that it makes significant differences.

We consider the following action for the background gravitational field, the scalar field $\phi$ and the non-minimally coupled massive vector field $A_\mu$:

$$
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} \left( \partial_\mu \phi \right)^2 - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left( m^2 - \frac{R}{6} \right) A_\mu A^\mu \right],
$$

(1)

where $g$ is the determinant of the metric, $R$ is the Ricci scalar, $V(\phi)$ is the scalar potential, $m$ is the mass of the vector field, and we have defined $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The equation of motion for $A_\mu$ from the above action:

$$
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} F^{\mu\nu} \right) = \left( m^2 - \frac{R}{6} \right) A^\nu,
$$

(2)

reduces $A_0(t) = 0$ in the case of $\nu = 0$ because of antisymmetry of $F^{\mu\nu}$. As we find that the vector field has only spatial components, we take the $x$ axis in the direction of the vector,

$$
A_\mu = (0, A_x(t), 0, 0), \quad \phi = \phi(t).
$$

(3)
Now, we will take the metric to be homogeneous but anisotropic Bianchi type I, i.e.,
\[ ds^2 = -N(t)^2 dt^2 + e^{2\alpha(t)} \left[ e^{-4\sigma_+(t)} dx^2 + e^{2\sigma_+(t)} \left( e^{2\sqrt{3}\sigma_-(t)} dy^2 + e^{-2\sqrt{3}\sigma_-(t)} dz^2 \right) \right], \]
where \( N(t) \) is the lapse function. With this ansatz, the background action becomes
\[ S = \int \frac{1}{N} e^{3\alpha} \left\{ \frac{3}{\kappa^2} \left( -\dot{\alpha}^2 + \dot{\sigma}_+^2 + \dot{\sigma}_-^2 \right) + \frac{1}{2} \ddot{\phi}^2 - \mathcal{N}^2 V \right. \\
+ \left. \frac{1}{2} \left( \ddot{X} - 2\dot{\sigma}_+ X \right)^2 - \frac{m^2}{2} \mathcal{N}^2 X^2 + \left( \frac{1}{2} \dot{\sigma}_+^2 + \frac{1}{2} \dot{\sigma}_-^2 - 2\dot{\alpha}\dot{\sigma}_+ \right) X^2 \right\}, \]
where we have introduced a new variable \( X = \exp(-\alpha + 2\sigma_+)A \) and defined \( \cdot = \partial_t \).
In general, this procedure leads to incorrect results. However, for homogeneous Bianchi models, this reduction procedure is valid, as can be checked directly.

The variational equations of motion with respect to \( N, \phi, X, \sigma_-, \sigma_+ \) and \( \alpha \) then become (after setting \( \mathcal{N} = 1 \))
\[ \frac{3}{\kappa^2} \left( -\dot{\alpha}^2 + \dot{\sigma}_+^2 + \dot{\sigma}_-^2 \right) + \frac{1}{2} \ddot{\phi}^2 + V \\
+ \frac{1}{2} \left( \dot{X} - 2\dot{\sigma}_+ X \right)^2 + \frac{m^2}{2} X^2 + \left( \frac{1}{2} \dot{\sigma}_+^2 + \frac{1}{2} \dot{\sigma}_-^2 - 2\dot{\alpha}\dot{\sigma}_+ \right) X^2 = 0, \]
\[ \ddot{\phi} + 3\dot{\phi} V_{\phi} = 0, \]
\[ \ddot{X} + 3\dot{\alpha} \dot{X} + \left( m^2 - 2\dot{\sigma}_+ - 2\dot{\alpha}\dot{\sigma}_+ - 5\dot{\sigma}_+^2 - \dot{\sigma}_-^2 \right) X = 0, \]
\[ e^{3\alpha} \left\{ \left( \frac{6}{\kappa^2} + X^2 \right) \dot{\sigma}_- \right\} = 0, \]
\[ e^{3\alpha} \left\{ \left( \frac{6}{\kappa^2} + 5X^2 \right) \dot{\sigma}_+ - 2X \dot{X} - 2\dot{\alpha} X^2 \right\} = 0, \]
\[ \ddot{\alpha} + 3\dot{\alpha}^2 - \kappa^2 V + \frac{2}{3}\kappa^2 \dot{\sigma}_+ X \dot{X} + \kappa^2 \left( \frac{1}{2} \dot{\sigma}_+ + \dot{\alpha}\dot{\sigma}_+ - \frac{1}{2} m^2 \right) X^2 = 0, \]
where \( V_{\phi} \equiv dV/d\phi \). Notice that the effective mass squared for \( X \), i.e. \( m^2 - 2\dot{\sigma}_+ - 2\dot{\alpha}\dot{\sigma}_+ - 5\dot{\sigma}_+^2 - \dot{\sigma}_-^2 \), in equation (8) might look different from the mass term of \( X \) in the action (5). However, we can identify them by transforming the action (5) into the canonical form for \( X \).

Note also that if the effective mass squared of \( X \) in equation (8) is negative, the system will be tachyonic. To avoid this, we require the effective mass squared to be positive, \( m^2 - 2\dot{\sigma}_+ - 2\dot{\alpha}\dot{\sigma}_+ - 5\dot{\sigma}_+^2 - \dot{\sigma}_-^2 > 0 \), that is, \( m \neq 0 \) and \( \dot{\sigma}_\pm \) has to be sufficiently small. As we will see later in equation (17), this latter condition implies that the amplitude of the vector field \( X \) should be small. In this sense, the vector field is a kind of impurity. In the limit \( \kappa X \rightarrow 0 \), the above set of equations reduce to those of conventional inflation scenarios. Also, taking a look at equations (6)–(11), we see that the deviation from the conventional slow roll dynamics comes in at of the order \( X^2 \).

Although we have assumed that the direction of the vector field does not change in time and considered only its \( x \) component for simplicity in equation (3), we could also incorporate other spatial components of the vector field. In that case, each component
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of the vector field obeys a similar equation to $X$. The vector field turns out to start to rotate under arbitrary initial conditions. If the slow roll conditions are satisfied for each, the vector will spiral down slowly. It is interesting to study the consequence of this complicated dynamics. However, we leave this for future work.

### 3. Anisotropic inflation

In this section, we consider the evolution of the background spacetime driven by the scalar field $\phi$, which we refer to as the inflaton below, in the presence of the vector impurity.

Let us consider the universe after a sufficient expansion, $\alpha \to \infty$. It is straightforward in this limit to integrate equations (9) and (10) to find

$$\dot{\sigma}_+ = \frac{1}{6/\kappa^2 + 5X^2} \left( 2X \dot{X} + 2\dot{\alpha}X^2 \right).$$

(12)

We find that the anisotropy in the $y$–$z$ plane, $\dot{\sigma}_-$, will disappear. This is reminiscent of the cosmic no-hair theorem of Wald [2]. However, as long as the vector field exists, $X \neq 0$, the anisotropy in the $x$ direction, $\dot{\sigma}_+$, will still remain even if the universe undergoes a period of inflation. In the proof of the cosmic no-hair theorem for Bianchi models, the strong and dominant energy conditions are assumed. The reason why the anisotropy does not disappear in our model is that the non-minimal coupling breaks the strong energy condition. Thus, we can restrict our metric to the following form:

$$ds^2 = -dt^2 + e^{2\alpha(t)} \left[ e^{-4\sigma_+(t)} dx^2 + e^{2\sigma_+(t)} \left( dy^2 + dz^2 \right) \right].$$

(13)

The above result might be generalized to the modified version of the cosmic no-hair theorem. It is interesting to examine other Bianchi type models in the presence of the vector impurity. In particular, it is intriguing to prove the cosmic no-hair theorem for these cases.

Now, we numerically solve equations (6)–(11) by setting $\sigma_- = 0$. We will use new variables $H = \dot{\alpha}$ and $\Sigma = \dot{\sigma}_+$ below. We take $V = \mu^2 \phi^2/2$ as the potential for the inflaton $\phi$. The parameter of the system is the ratio $m/\mu$. For this calculation, we set $\kappa = 1$, $\mu = 10^{-5}$ and $m = 2\sqrt{2} \times 10^{-5}$. And we took the initial values $\phi_0 = 10$ and $H_0 = 3 \times 10^{-4}$ for all figures.

In figure 1, we depicted the phase flow in the $X$–$\dot{X}$ plane. We set the initial value $\Sigma_0 = 0$ and determined $\dot{\phi}_0$ using the constraint equation (6). We see that the slow roll phase of the vector field is an attractor when the amplitude of the vector field is sufficiently small. For the appropriate parameter $m/\mu$, a small value such as $X^2 \sim 0.1$ is easily attainable. Both the mass and the non-minimal coupling play important roles for realizing the slow roll. They affect the coefficient of $X$ in equation (8). In detail, the mass is important for the system to be stable and the non-minimal coupling cancels out the term $\dot{\alpha}^2$ in the effective mass squared of $X$ which enables the vector field to slow roll. Thus, both mass and non-minimal coupling help the vector field to slow roll. Thanks to the slow roll for the vector field, we have enough anisotropic era. After the vector field goes to the minimum of its potential, the anisotropy disappears. The question of when the anisotropy disappears depends on the parameter $m/\mu$ and the initial amplitude of $X$. The complete analysis of the initial conditions is possible, as done in the case of pure vector models [17].
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Figure 1. The phase flow in the $X\dot{X}$ plane is depicted. For various initial conditions with small amplitude of $X$, we have plotted the trajectories. Every trajectory converges to the slow roll attractor.

In figure 2, the $\phi\dot{\phi}$ phase flow is plotted with the same initial conditions as for figure 1. The trajectories are almost the same irrespective of the initial conditions for $X$, and show the slow roll phase. After slow rolling, the inflaton field gets its damped oscillations. Therefore, the anisotropic inflation ends with reheating as in the standard inflation.

In figure 3, we have plotted $H$ as a function of e-folding number $N \equiv \alpha - \alpha_0$. Actually, we plotted four lines for different initial conditions $\Sigma_0/H_0 = 0, 0.05, 0.1, 0.15$ with fixed initial conditions $X_0 = 0.5$ and $\dot{X}_0 = 1.5 \times 10^{-5}$. In figure 3, all lines are degenerate. Irrespective of the initial conditions, we see the slow roll phase from this figure. As for figure 2, the qualitative behavior does not depend on the initial conditions for $X$ as long as its amplitude is small.

In figure 4, we have plotted $\Sigma/H$ as a function of the e-folding number with the same initial conditions as for figure 3. We see that the anisotropy remains sizable for some period and then is reduced to zero around $N \sim 20$ irrespective of its initial condition. The duration of this phase depends on the mass of the vector field and other parameters. We find that a larger initial condition for $\Sigma$ makes smaller anisotropy later. This can be understood from equations (8) and (12). Suppose the initial anisotropy is larger than the value calculated from equation (12) using the initial data for $X, \dot{X}, H$; then the anisotropy has to decrease rapidly to adjust to the value of the attractor. During this phase, the effective mass in equation (8) becomes large due to the term $-\dot{\sigma}_+$. Then, the amplitude of the vector $X$ before entering the slow roll phase decreases more than for the case of a smaller initial condition. From equation (12), the small amplitude of $X$ implies small anisotropy during the slow roll phase. Thus, we have understood the behavior in figure 4. In the case of the smaller initial anisotropy, the opposite happens.
Figure 2. The phase flow in the $\phi - \dot{\phi}$ plane is depicted. For the same initial conditions as figure 1, we have plotted the trajectories. Every trajectory is degenerate irrespective of the initial conditions for $X$. As we can see, the inflaton rolls down the potential slowly. The inflaton begins to oscillate around the minimum of the potential and the inflation ends with the reheating. Here we have plotted a part of the graph for the sake of visualization, though we started from $\phi_0 = 10$.

Figure 3. The Hubble parameter $H$ is plotted as a function of e-folding number $N$. We see a slow roll phase clearly. We have e-folding number $N \sim 40$ for this case.
Previously, the conventional inflationary scenario in the Bianchi type I model has been considered [18]–[20]. In those cases, the anisotropy decays exponentially fast. In the recent works [21,22], the anisotropic universe has been investigated in the context of dark energy. It would be interesting to apply our framework to the dark energy problem.

4. Slow roll approximation

Now we consider the regime where both of the vector and the scalar field are slow rolling. Thanks to the slow roll for the vector field which breaks the strong energy condition, we have enough anisotropic era. For the reason explained in the previous section, we can exclude $\sigma_-$ from the consideration. In the slow roll approximation, we can write equations (6)–(8) and (10)–(11) as

$$3(-H^2 + \Sigma^2) + \kappa^2 V + 2\kappa^2 \Sigma^2 X^2 + \frac{\kappa^2}{2} \left( m^2 - \Sigma^2 - 4H\Sigma \right) X^2 = 0,$$

$$3H \dot{\phi} + V_{,\phi} = 0,$$

$$3H \dot{X} + \left( m^2 - 2H\Sigma - 5\Sigma^2 \right) X = 0,$$

$$\Sigma = \frac{2X^2}{6/k^2 + 5X^2} H,$$

$$3H^2 = \kappa^2 V + \frac{\kappa^2}{2} \left( m^2 - 2H\Sigma \right) X^2,$$

$\Sigma/H = 0$

$\Sigma_0/H = 0.05$

$\Sigma_0/H = 0.1$

$\Sigma_0/H = 0.15$

**Figure 4.** The ratio $\Sigma/H$ is plotted as a function of e-folding number. In spite of the rapid expansion of the universe, the anisotropy remains sizable for some period relevant to observations.
where we have ignored all second derivatives with respect to $t$ and square terms of the first derivatives with respect to $t$ except for $H^2$, $\Sigma^2$ and $H\Sigma$. Since we are interested in the situation where the anisotropy is larger than the slow roll parameter, we have kept the anisotropy $\Sigma^2$ in the above equations. Adding equations (14) and (18), we obtain equation (17). Hence, equation (14) is redundant.

From equation (17), we see that $X$ should be small in order for the anisotropy to be small as is mentioned at the end of the section 2. Eliminating $\Sigma$ from equations (17) and (18), we can deduce the Friedmann equation for this model

$$H^2 = \frac{6/\kappa^2 + 5X^2}{(6/\kappa^2 + X^2)(3/\kappa^2 + 2X^2)} \left( V + \frac{m^2}{2}X^2 \right).$$

(19)

Note that if there is no scalar field, i.e. no potential, this equation tells us that $H \leq m$, no matter what value the vector field takes. This contradicts the condition for the slow roll: $H \gg m$. Thus we find that the inflation does not occur only with the vector field.

From equation (17) and (19), we find

$$\Sigma = \frac{2X^2}{\sqrt{(6/\kappa^2 + 5X^2)(6/\kappa^2 + X^2)(3/\kappa^2 + 2X^2)}} \sqrt{V + \frac{m^2}{2}X^2}.$$  

(20)

Here we have taken the positive sign because $H > 0$. Inserting these results into equations (15) and (16), we obtain a set of differential equations which determines $\phi$ and $X$.

Before proceeding to the next section, we would like to clarify the scenario that we envisage here. We are supposing the existence of inflaton and vector impurity. The inflation is driven by the inflaton. In the case of chaotic inflation, initial conditions for the inflaton are arbitrary as long as the energy of the inflaton does not exceed the Planck energy. The larger initial amplitude for the inflaton gives the longer inflationary period. One of the initial conditions should correspond to our present universe. Similarly, initial conditions for the vector field are arbitrary. However, as we have seen in section 2, the large amplitude of $X$ induces the tachyonic instability. Hence, this does not correspond to our universe. The criterion for the stability condition $2H\Sigma \ll m^2$ in the slow roll phase gives

$$\kappa\phi \ll \frac{m}{\mu} \frac{3}{\kappa X},$$

(21)

where we used $\Sigma/H \sim \kappa^2 X^2/3$ and $H^2 \sim \kappa^2 \mu^2 \phi^2/6$ from equations (17) and (18). We find that the inflaton field has the upper limit but still has enough parameter range for the anisotropic inflationary universe. For example, the parameters such as $\kappa\phi_0 = 10$, $\kappa X_0 = 0.5$ and $m/\mu \sim 2\sqrt{2}$, which we used in the numerical calculations, satisfy this condition.

5. Implications for primordial fluctuations

Although mixing between tensor and scalar perturbations in anisotropic spacetime has been studied in the literature (see, for example, [19]), there has been no concrete realization in the context of an inflationary scenario. In this section, we would like to revisit various effects with our concrete anisotropic inflation model. Here, we do not show the explicit
numbers. However, from the structure of the background equations, we guess possible important consequences of vector impurity. We especially stress the gravitational wave production due to the mixing in the context of low scale inflation. Detailed calculations will appear in the follow-up paper [23].

Since the expansion is anisotropic, we expect statistically anisotropic fluctuations. In fact, in the slow roll phase \((\dot{H}, \dot{\Sigma} \simeq \text{const})\), we have the metric
\[ds^2 = -dt^2 + e^{2Ht} \left[ e^{-4\Sigma t} \, dx^2 + e^{2\Sigma t} \left( dy^2 + dz^2 \right) \right].\] (22)

Now, let us consider a test scalar field \(\psi\) in this background metric. According to [6], deviations from isotropy come in to the power spectrum of the form
\[P_\psi(k) = P_0(k) \left[ 1 + \mathcal{O}(\kappa^2 X^2) (\hat{k} \cdot \mathbf{n})^2 \right],\] (23)
where \(P_\psi\) is the power spectrum of the scalar field \(\psi\), \(P_0(k)\) is its isotropic part, \(\mathbf{n}\) is the unit vector in the direction of \(x\) and \(\hat{k}\) is the unit vector along the direction of the wavenumber vector \(k\). Here, in evaluating the deviations, we used the approximate relation \(\Sigma/H \sim \kappa^2 X^2\) derived from equation (17). Hence, if \(P_0(k)\) has a flat spectrum, the total spectrum should also be flat, even though it is anisotropic. In the case of the curvaton scenario, the scalar field \(\psi\) decays after reheating and converts to curvature perturbations. For this case, the fluctuation of \(\psi\) gives CMB fluctuations directly. Even for our more complicated case where inflaton also fluctuates, we expect a similar result to equation (23). This spectrum yields the anomaly in the CMB observations. In paper [14], it is pointed out that about 10% deviation from the isotropic statistic can be detected by WMAP and 2% by PLANCK. Hence, the interesting number is around \(\mathcal{O}(\kappa^2 X^2) \sim 0.1\).

In addition to this apparent effect, we can expect much more interesting ones. First of all, we should recall that the primordial gravitational waves are created quantum mechanically in the standard inflationary scenario. Hence, their amplitude should be of the order of \(H/M_{\text{pl}}\) (\(M_{\text{pl}}\) is the Planck mass). On the other hand, here, we have another new component of tensor fluctuations induced by curvature perturbations through the anisotropic background expansion. This can be understood from the background equation (24):
\[\frac{d}{dt} \left[ \sigma_+ - \frac{1}{3} \kappa^2 X^2 \alpha \right] = 0,\] (24)
where we have used the fact that \(\kappa^2 X^2 \ll 1\) and assumed that \(X\) is almost constant. If we define the following quantity which is supposed to be the gravitational wave:
\[H_+ = \delta \sigma_+ - \frac{1}{3} \kappa^2 X^2 \delta \alpha,\] (25)
we find that this is a gauge invariant quantity in the long wavelength limit. The background equation (24) implies that \(H_+\) is a conserved quantity, namely, \(dH_+/dt = 0\). In the conventional isotropic case, this equation reduces to \(d\delta \sigma_+/dt = 0\), which is equivalent to the conservation law for gravitational waves \(\dot{h}^+ = 0\) in the long wavelength limit. Here, the polarized component of the transverse traceless tensor, \(h^+\), goes to \(\delta \sigma_+\) in the long wavelength limit, that is, \(h^+ \xrightarrow{k \to 0} \delta \sigma_+\). Now, we have an extra term \(\delta \alpha\) in the gauge invariant gravitational waves, so \(\delta \sigma_+\) itself is not conserved in the long wavelength limit. Since the curvature perturbation is dominant during the inflation, the
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gauge invariant gravitational waves are generated first through $\delta \alpha$ which is supposed to be the curvature perturbation. Thus, we have the relation $H_+ = -\frac{1}{3} \kappa^2 X^2 \delta \alpha$. As the vector field goes to zero, $\delta \sigma_+$ starts to appear because of the conservation law. Finally $\delta \sigma_+$ becomes dominant after $X$ vanishes. Thus, we have the relation $H_+ = \delta \sigma_+$. Because of the conservation law, we have $\delta \sigma_+|_f = -\frac{1}{3} \kappa^2 X^2 \delta \alpha|_k$, where the left-hand side is evaluated at the final time and the right-hand side is evaluated at the horizon crossing time for the mode with wavenumber $k$. Now, let us define the curvature perturbation, $\mathcal{R}$, which goes to $\delta \alpha$ in the long wavelength limit, that is, $\mathcal{R} \xrightarrow{k \to 0} \delta \alpha$. Then, we obtain the final result

$$h^+ = \mathcal{O}(\kappa^2 X^2) \mathcal{R}, \quad (26)$$

where the right-hand side of equation (26) is evaluated at the horizon crossing time. For the other polarization mode, we have the standard conservation law $\dot{h}^- = 0$ in the long wavelength limit. Here, we have the correspondence $h^- \xrightarrow{k \to 0} \delta \sigma_-$. The above discussion tells us that the scalar perturbation $\mathcal{R}$, which comes from the fluctuations of the trace part of metric (13), induces the tensor perturbation $h^+$, which is the one of the traceless part of the metric. Therefore, the tensor power spectrum $P_{h^+}$ should be related to the curvature power spectrum $P_{\mathcal{R}}$ as

$$P_{h^+} = \mathcal{O}(\kappa^4 X^4) P_{\mathcal{R}}, \quad (27)$$

where the relation (26) is used. The other power spectrum $P_{h^-}$ coming from $\delta \sigma_-$ has no correlation with the curvature perturbations because there is no conversion from the curvature perturbations in this case. In other words, we expect linear polarization in the primordial gravitational waves. This could be detected through CMB or directly by DECIGO [30].

In papers [24]–[29], the gravitational waves generated from the second-order perturbations have been studied. It should be stressed that the mechanism for producing the gravitational waves that we are discussing is a first-order effect in the anisotropic inflationary universe. The primordial fluctuations produced by the anisotropy are not the conventional ones created from quantum fluctuations directly. They are induced from the curvature perturbations created from quantum fluctuations. Since the scalar perturbations are always larger than the tensor perturbations, this anisotropy induced mechanism is very efficient. This is an extremely important result, because this implies the existence of the primordial gravitational waves even in low scale inflationary scenarios. In fact, supposing that we detect the anisotropy $\kappa^2 X^2 \sim 0.1$, we would detect the primordial gravitational waves with the tensor–scalar ratio $r \sim 0.01$.

Since at least parts of the tensor perturbations are induced from the curvature perturbations, there exists a correlation between the curvature and the tensor perturbations

$$\langle h^+ \mathcal{R} \rangle \sim \mathcal{O}(\kappa^2 X^2) \langle \mathcal{R} \rangle,$$  \quad (28)

where we used the relation (26). The correlation should give non-zero TB correlation at the 10% level in the case of $\kappa^2 X^2 \sim 0.1$. In other words, the normalized correlation function $\langle \text{TB} \rangle / \sqrt{\langle \text{TT} \rangle \langle \text{BB} \rangle} = \mathcal{O}(1)$ does not vanish. Surely, this should be searched for observationally.

Although it is necessary to calculate precise numbers for prediction, we have uncovered the new possibility of generating the primordial gravitational waves on dimensional
6. Conclusion

We have explored the cosmological implications of the vector impurity. It turned out that the vector impurity affects the cosmic no-hair theorem. Consequently, the anisotropic accelerating universe is possible in the presence of the vector impurity. To prove this, we have numerically solved the equations and shown the phase flow from which we see that the anisotropic inflation successfully occurs and ends with oscillation. We found the explicit formula for determining the anisotropy of the inflationary universe by using the slow roll approximation.

We have also discussed possible consequences of the anisotropic inflation on the cosmological fluctuations. Since rotational invariance is violated, the statistical isotropy of CMB temperature fluctuations cannot be expected. It is intriguing to seek for the relation to the large scale anomaly discovered in CMB by WMAP [31]–[33]. More interestingly, the tensor perturbations could be induced from the curvature perturbations through the anisotropy of the background spacetime. One immediate consequence is the correlation between the curvature perturbations and the tensor perturbations. This correlation should be detected through the analysis of temperature–$B$-mode correlation in CMB. This new possibility implies that, even in the low scale inflation, we can expect primordial gravitational waves. This is an important result for future observational planning, because there has been worry that string cosmology tends to suggest low scale inflation. Moreover, because of the anisotropy, there might be linear polarization in the primordial gravitational waves. This polarization can be detected either through the CMB observations or direct interferometer observations. These predictions can be checked by future observations.

To make these predictions more precise, we need to develop the perturbative analysis [18]–[20], [34]. This is now under investigation [23]. The calculation of the perturbations is much more complicated due to the violation of rotational invariance. However, since the anisotropic inflationary universe is smoothly connected to the isotropic radiation dominant phase, the interpretation of the results should be clear. The implications for primordial magnetic fields and the structure formation of the universe [35] should also be studied in future work.

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