EXPONENTIAL STABILITY OF A COUPLED SYSTEM WITH WENTZELL CONDITIONS

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(Communicated by M. Vilmos Komornik)

Abstract. A coupled system of hyperbolic equations in Maxwell/wave with Wentzell conditions in a bounded domain of \( \mathbb{R}^3 \) is considered. Under suitable assumptions, we show the exponential stability of the system. Our method is based on an identity with multipliers that allows to show an appropriate stability estimate.

1. Introduction. In this paper, we shall be concerned with the problem of exponential stabilization of a coupled Maxwell/wave system with Wentzell conditions by linear boundary feedbacks.

\[
\begin{aligned}
\partial_t^2 u - \Delta u + \xi \text{curl} E &= 0 \quad \text{in } Q := \Omega \times ]0, \infty[, \\
\epsilon \partial_t E - \text{curl} H - \xi \text{curl} \partial_t u &= 0 \quad \text{in } Q, \\
\mu \partial_t H + \text{curl} E &= 0 \quad \text{in } Q, \\
\text{div} (\epsilon E) = \text{div} (\mu H) &= 0 \quad \text{in } Q, \\
H \times \nu + \xi \partial_t u \times \nu + (E \times \nu) \times \nu &= 0 \quad \text{on } \Sigma := \Gamma \times ]0, \infty[, \\
\partial_\nu u - \Delta_T u + au + \partial_t u &= 0 \quad \text{on } \Sigma, \\
u(0) = u_0, \; \partial_t u(0) = u_1, \; E(0) = E_0, \; H(0) = H_0 &\quad \text{in } \Omega,
\end{aligned}
\]  

The Wentzell condition on \( \Gamma \) in (1) arises when modelling vibrating bodies with a thin boundary layer of high rigidity. Such boundary conditions are characterized by the presence of tangential differential operators of the same order as the interior operator and appear in several fields of applications such as physics, in diffusion processes [16, 22], in mechanics [4, 15], as well as in wave phenomena [6].

Here \( \Omega \) is a bounded domain of \( \mathbb{R}^3 \) with a boundary \( \Gamma = \partial \Omega \) of class \( C^2 \). In the above system we denote by \( u(x, t) \in \mathbb{R}^3 \) the displacement vector, \( E(x, t), H(x, t) \in \mathbb{R}^3 \) represent respectively the electric and magnetic field at \( x = (x_1, x_2, x_3) \in \Omega \) and \( t \) is the time variable, \( \epsilon \) and \( \mu \) denote the electric permittivity and magnetic permeability, respectively, and we will assume that they are positive real numbers. Also, in system (1) \( \xi \) is the coupling constant, \( a \) is a positive real number, and \( \Delta_T \)
means the tangential Laplace operator on $\Gamma$. As usual $\nu$ is the unit normal of $\Gamma$ pointing toward the exterior of $\Omega$.

For $\xi = 0$, problem (1) decouples into two independent mixed problems for the wave equation with Wentzell boundary conditions

$$
\begin{align*}
&\partial_t^2 u - \Delta u = 0, &\text{in } Q := \Omega \times ]0, \infty[, \\
&\partial_t u - \Delta_T u + au + \partial_t u = 0, &\text{on } \Sigma := \Gamma \times ]0, \infty[, \\
&u(0) = u_0, \partial_t u(0) = u_1, &\text{in } \Omega,
\end{align*}
$$

and for the Maxwell’s system

$$
\begin{align*}
&\epsilon \partial_t E - \text{curl } H = 0, &\text{in } Q := \Omega \times ]0, \infty[, \\
&\mu \partial_t H + \text{curl } E = 0, &\text{in } Q, \\
&\text{div}(\epsilon E) = \text{div}(\mu H) = 0, &\text{in } Q, \\
&H \times \nu + (E \times \nu) \times \nu = 0, &\text{on } \Sigma := \Gamma \times ]0, \infty[, \\
&E(0) = E_0, H(0) = H_0, &\text{in } \Omega.
\end{align*}
$$

There is a large literature devoted to the stabilization of hyperbolic problems set in bounded domains of $\mathbb{R}^d$, $d \geq 1$. Let us mention some of them: Boundary stabilization of the system (2) with nonlinear boundary feedback has been considered in [6] (here $a \in C^1(\Gamma)$ is a nonnegative function), the author showed that the energy tends to zero as $t \to +\infty$ in a bounded smooth domain $\Omega$. However, when $a \equiv 0$, A. Heminna [7, 8] proved that this system is not uniformly exponentially stable. Later on, K. Laoubi and S. Nicaise [14] studied the same problem on the unit square of the plane ($\Omega = (0,1)^2$) and obtained a polynomial rate of the decay of the energy. For Maxwell’s equations with Silver-Müller boundary condition and $\epsilon = \mu = 1$, V. Komornik [11] showed, under suitable geometrical conditions that the energy tends to zero exponentially as time goes to infinity with a precise decay rate estimate. His proof is based on the “standard” identity with multiplier, assuming that $\Omega$ is strictly star-shaped with respect to the origin. Recently, when $\epsilon$ and $\mu$ are piecewise constant with Lipschitz polyhedra $\Omega$, M. Eller et al. [2] gave a necessary and sufficient condition ensuring that the energy of the solutions of (3) decays exponentially. Their method of proof is based on the validity of a kind of stability estimate which is obtained using the multiplier method but leading to a relatively strong assumption on the permittivity and permeability coefficients. This result was later generalized by S. Nicaise [18] in what concerns the electromagneto-elastic system with Dirichlet-Neumann boundary conditions by nonlinear boundary feedbacks. He proved that the energy decays exponentially in the case of linear feedbacks if $\Omega$ is strictly star shaped with respect to the origin, and assuming that $\epsilon$, $\mu$ are constants in the whole domain $\Omega$.

As mentioned before, A. Heminna [7, 8] proved that the natural feedback is not sufficient to guarantee the exponential decay of the energy of the system (2) even if the dissipative feedback is applied to the entire boundary of the domain. A question thus arises of whether the energy of the solutions of the linear coupled system (1) decays exponentially. In other words: are there positive constants $M$, $\omega$ such that

$$
E(t) \leq Me^{-\omega t}E(0), \forall t \geq 0?
$$

Our main interest in this article is exactly to give an answer to this question. Namely, we show that the exponential decay of the energy of the solutions of (1) is
equivalent to the validity of a stability estimate, estimate that can be obtained in some particular cases using the identity with multiplier.

The approach adopted in this paper was inspired in [18], an approach which in turn originated in [2] adapted to our case.

The rest of this paper is organized as follows: Section 2 describes the problem and we introduces some notations and lemmas needed for our work, the main result is stated in Section 3, whereas Section 4 is devoted to the proof of the main result.

2. Statement of problem.

2.1. Notations and results. In this subsection we introduce some few formulas to be invoked in the sequel.

We define Ω, Γ, ν as above. For all x ∈ Γ, we denote by π(x) the orthogonal projection on tangent plane T_{x}(Γ) and for a given vector field v : Ω → ℝ³, we will write (see for instance [5, 4, 15])

∀ x ∈ Γ, v(x) = v_{T}(x) + v_{ν}(x)ν(x),

where v_{T}(x) = π(x)v(x) is the tangential component of v and v_{ν}(x) = v(x) · ν(x). We further denote by ∂_{T} (resp. ∂_{ν}) the tangential (resp. normal) derivative, (∂_{T}ν) the curvature operator on Γ and π∂_{T} the covariant derivative of the field v_{T}. If v is some regular function, the transposed vector of ∂_{T}v_{T} is the tangential gradient of v and is denoted by ∇_{T}v. We have

∇v = ∇_{T}v + ∂_{T}v_{T}ν + ∂_{ν}vν

on Γ. (4)

The proof of the following lemmas can be found in [5], [6], [16] and references therein.

Lemma 2.1. Let f be a function of class C^{2} defined on Γ; we have

π∂_{T}(∂_{T}f) = π∂_{T}(∂_{T}f),

where the bar denotes the transposed of a vector.

Lemma 2.2. Let f be a function of class C^{2} and q_{T} a tangent field of class C^{1} defined on Γ; then

∇_{T}f(∇_{T}f) · q_{T} = ∇_{T}f(π(∂_{T}q_{T}))∇_{T}f + 1/2 ∂_{T}(|∇_{T}f|^{2})q_{T}.

Lemma 2.3. Let q = x - x_{0} = q_{T} + q_{ν}ν and x_{0} is a fixed point in ℝ³, we have

\begin{align*}
\pi\partial_{T}q_{T} + q_{ν}\partial_{T}ν &= i_{2}, \\
\partial_{T}q_{ν} &= q_{T}\partial_{T}ν, \\
\text{div}_{T}q_{T} &= 2 - q_{ν}\text{tr}(\partial_{T}ν),
\end{align*}

where i_{2} is the identity of the tangent plane, and “tr” means the trace of a matrix.

2.2. Well-posedness of (1). We briefly describe the function spaces where we will consider the solution (u, E, H) of problem (1). Here and in what follows we shall use the summation convention for repeated indices. Letting v = ∂_{t}u, we can rewrite system (1) as

\begin{align*}
\partial_{t}U + AU &= 0, \\
U(0) &= U_{0},
\end{align*}

with U = (u, v, E, H), U_{0} = (u_{0}, v_{0}, E_{0}, H_{0}) and

\[ A(u, v, E, H) = \left( -v, -Δu + ξ\text{curl }E, -ε^{-1}(\text{curl }H + ξ\text{curl }v), μ^{-1}\text{curl }E \right). \]
Thus, it is natural to consider the Hilbert spaces (see e.g. [19], [13], [4])
\[ J(\Omega, \epsilon) = \left\{ E \in L^2(\Omega)^3 / \text{div}(\epsilon E) = 0 \text{ in } \Omega \right\}, \]
(7)
\[ \mathcal{U}(\Omega)^3 = \left\{ u \in H^1(\Omega)^3, \ u_\Gamma \in H^1(\Gamma)^3 \right\}, \]
(8)
\[ \mathcal{H} = \mathcal{U}(\Omega)^3 \times L^2(\Omega)^3 \times J(\Omega, \epsilon) \times J(\Omega, \mu), \]
(9)
equipped with the natural norm induced by the inner product:
\[ \langle (u, v, E), (u', v', E') \rangle_{\mathcal{H}} = \langle u, u' \rangle_{\mathcal{U}(\Omega)^3} + \langle v, v' \rangle_{L^2(\Omega)^3} + \langle E, E' \rangle_{J(\Omega, \epsilon)} + \langle H, H' \rangle_{J(\Omega, \mu)}, \quad \forall (u, v, E, H), (u', v', E', H') \in \mathcal{H}, \]
where
\[ \langle E, E' \rangle_{J(\Omega, \epsilon)} = \int_{\Omega} \epsilon(x) E(x) \cdot E'(x) \, dx, \]
\[ \langle H, H' \rangle_{J(\Omega, \mu)} = \int_{\Omega} \mu(x) H(x) \cdot H'(x) \, dx, \]
and
\[ \langle v, v' \rangle_{L^2(\Omega)^3} = \int_{\Omega} v(x) \cdot v'(x) \, dx, \]
and
\[ \langle u, u' \rangle_{\mathcal{U}(\Omega)^3} = \int_{\Omega} \frac{\partial u_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j} \, dx + \int_{\Gamma} u \cdot u' \, d\sigma + \int_{\Gamma} \frac{\partial T u_i}{\partial x_j} \frac{\partial T u'_i}{\partial x_j} \, d\sigma, \]
where we recall that \( \partial T \) denotes the tangential derivative.

Now, let us consider the linear operator \( A : D(A) \subseteq \mathcal{H} \mapsto \mathcal{H} \), with domain
\[ D(A) = \left\{ (u, v, E, H) \in \mathcal{H} / \Delta u, \ \text{curl} \ E, \ \text{curl} \ H \in L^2(\Omega)^3; \ v \in \mathcal{U}(\Omega)^3; \right. \]
\[ \left. E \times \nu, \ H \times \nu \in L^2(\Gamma)^3 \right\} \]
satisfying:
\[ H \times \nu + \xi v \times \nu + (E \times \nu) \times \nu = 0 \text{ on } \Gamma, \]
(10)
\[ \partial_\nu u - \Delta T u + a u + v = 0 \text{ on } \Gamma, \]
(11)
given by
\[ Aw = \left( - w_2, - \Delta w_1 + \xi \text{curl} \ w_3, - \epsilon^{-1}(\text{curl} \ w_4 + \xi \text{curl} \ w_2), \mu^{-1} \text{curl} \ w_3 \right), \]
\[ \forall (w_1, w_2, w_3, w_4) \in D(A). \]

Remark 1. Recall that for a function \( u \in H^1(\Delta, \Omega)^3 \) where \( H^1(\Delta, \Omega)^3 = \left\{ u \in H^1(\Omega)^3 / \Delta u \in L^2(\Omega)^3 \right\} \), then \( \partial_\nu u \) belongs to \( H^{-\frac{1}{2}}(\Gamma)^3 \) and the next Green formula is valid (see for instance section 1.5 of [9])
\[ \int_{\Omega} \Delta u \cdot w \, dx = \langle \partial_\nu u, w \rangle - \int_{\Gamma} \frac{\partial u_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, d\sigma, \quad \forall w \in H^1(\Omega)^3, \]
where \( \langle :, : \rangle \) means the duality pairing between \( H^{-\frac{1}{2}}(\Gamma)^3 \) and \( H^{\frac{1}{2}}(\Gamma)^3 \). Thus, for any \( (u, v, E, H) \in D(A) \), it follows that \( \partial_\nu u - \Delta T u = au + v \in L^2(\Gamma)^3 \) and therefore \( \Delta T u \) belongs to \( H^{-\frac{1}{2}}(\Gamma)^3 \). In addition, since \( u, v \in H^1(\Omega)^3 \) imply that \( H \times \nu + \xi v \times \nu + (E \times \nu) \times \nu \) belongs to \( L^2(\Gamma)^3 \). So, the boundary condition (10) (resp. (11)) has a meaning in \( L^2(\Gamma)^3 \) (resp. in \( H^{-\frac{1}{2}}(\Gamma)^3 \)).
The well-posedness of system (1) is presented in the following theorem, which can be proved, using the semigroup theory [20, 21] as in [2, 18] and the reference therein.

**Theorem 2.4 (Well-posedness).** The problem (1) is well posed in the space $\mathcal{H}$. In particular:

1. If $(u_0, u_1, E_0, H_0) \in \mathcal{H}$, the problem (1) has a unique (weak) solution which satisfies
   \[
   (u, \partial_t u, E, H) \in C(\mathbb{R}_+, \mathcal{H}).
   \]

2. Furthermore, if $(u_0, u_1, E_0, H_0) \in D(A)$, the problem (1) has a unique (strong) solution satisfying
   \[
   (u, \partial_t u, E, H) \in W^{1, \infty}(\mathbb{R}_+, \mathcal{H}) \cap L^\infty(\mathbb{R}_+, D(A)).
   \]

**Remark 2.** It follows from Remark 1 that $\Delta_T u$ belongs to $H^{-\frac{3}{2}}(\Gamma)^3$ and therefore the standard regularity results imply that $u$ in $H^2(\Omega)^3$, then we would obtain that the solution of (1) satisfying

\[
\begin{align*}
  u &\in L^\infty(\mathbb{R}_+, H^2(\Omega)^3), \\
  E &\in W^{1, \infty}(\mathbb{R}_+, J(\Omega, \epsilon)) \cap L^\infty(\mathbb{R}_+, W_\epsilon), \\
  H &\in W^{1, \infty}(\mathbb{R}_+, J(\Omega, \mu)) \cap L^\infty(\mathbb{R}_+, W_\mu).
\end{align*}
\]

We now define the energy of the problem (1) by

\[
E(t) = \frac{1}{2} \int_\Omega \left( |\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2 \right) dx + \frac{a}{2} \int_\Gamma |u(x,t)|^2 d\sigma \\
+ \frac{1}{2} \int_\Gamma |\nabla u(x,t)|^2 d\sigma + \frac{1}{2} \int_\Omega \left( \epsilon |E(x,t)|^2 + \mu |H(x,t)|^2 \right) dx.
\]

For every solution of (1) in the class (12) the following identity holds

\[
E(T) - E(S) = - \int_S^T \int_\Gamma \left( |E(x,t) \times \nu|^2 + |\partial_t u(x,t)|^2 \right) d\sigma dt,
\]

for all $0 \leq S < T < \infty$, and therefore the energy is a non-increasing function of the time variable $t$.

3. **Main result.** In this section we give the main result of this paper. We first recall the arguments of the beginning of Section 3 of [18] given in the case of the electromagneto-elastic system (see also Section 3 of [2]) and that can be easily extended to our system. Namely, the exponential stability of system (1) when $\Omega$ is strictly star-shaped with respect to the origin, i.e.,

\[
m(x) \cdot \nu(x) = \sum_{i=1}^3 x_i \nu_i > 0, \text{ for all } x \in \Gamma.
\]

Let us now introduce the following definition (see for instance [18])

**Definition 3.1.** We say that $\Omega$ satisfies the stability estimate if there exist $T > 0$ and two non negative constants $C_1$, $C_2$ (which may depend on $T$) with $C_1 < T$ such that

\[
\int_0^T E(t) dt \leq C_1 E(0) + C_2 \int_0^T \int_\Gamma \left( |\partial_t u(t)|^2 + |E(t) \times \nu|^2 \right) d\sigma dt,
\]

for all solution $(u, E, H)$ of (1).
That property admits the following equivalent formulation

**Lemma 3.2 (Observability estimate).** \( \Omega \) satisfies the stability estimate if and only if there exist \( T > 0 \) and a positive constant \( C \) (which depend on \( T \)) such that

\[
\mathcal{E}(T) \leq C \int_0^T \int_{\Gamma} \left( |\partial_t u(t)|^2 + |E(t) \times \nu|^2 \right) d\sigma dt, \tag{18}
\]

for all solutions \((u, E, H)\) of (1).

**Proof.** It’s analogous to the proof of Lemma 3.2 in [2]. \( \Box \)

Now, we recall from [2] a necessary and sufficient condition for the exponential stability.

**Theorem 3.3.** \( \Omega \) satisfies the stability estimate if and only if there exist two positive constants \( M \) and \( \omega \) such that

\[
\mathcal{E}(t) \leq Me^{-\omega t}\mathcal{E}(0), \tag{19}
\]

for all solution \((u, E, H)\) of (1).

**Proof.** The proof is similar to the one in Theorem 3.3 of [2]. \( \Box \)

Applying an argument of [12], we can prove the following lemma.

**Lemma 3.4.** Let \((u, E, H)\) be a strong solution of (1). Then, there exists \( C > 0 \) (depends on \( \alpha \), the domain, the coefficient \( \mu \) and the parameter \( \xi \)) such that for all \( \eta \in (0, 1) \)

\[
\int_0^T \int_{\Gamma} (a|u|^2 + |\nabla_T u|^2) d\sigma dt \leq \frac{C}{\eta} \mathcal{E}(0) + \eta \int_0^T \mathcal{E}(t) dt. \tag{22}
\]

This identity means that

\[
\int_{\Omega} \frac{\partial z_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx = 0, \quad \forall v \in H^1_0(\Omega)^3.
\]

Taking \( v = z - u \) in this identity, we deduce that

\[
\int_{\Omega} \frac{\partial z_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dx = \int_{\Omega} |\nabla z|^2 dx \geq 0. \tag{21}
\]

Using the definition of the energy function and the identity (15), we can find some positive constants \( c_1, c_2 \) such that

\[
\int_{\Omega} |z|^2 dx \leq c_1 \int_{\Gamma} |u|^2 d\sigma \leq c_2 \mathcal{E}(t), \tag{22}
\]

\[
\int_{\Omega} |\partial_t z|^2 dx \leq c_1 \int_{\Gamma} |\partial_t u|^2 d\sigma \leq -c_1 \mathcal{E}'(t), \tag{23}
\]

where \( c_2 = \frac{2c_1}{\theta} \).
For $0 < T < \infty$, we set $Q_T := \Omega \times [0, T]$ and $\Sigma_T := \Gamma \times [0, T]$. Multiplying the first identity of (1) by $z$, invoke Green’s formula and taking into account the boundary conditions in (1) and in (20), we arrive at

$$
\int_{\Sigma_T} \left( |\nabla_T u|^2 + a|u|^2 \right) d\sigma dt = - \int_{\Sigma_T} u \cdot \partial_t u \, d\sigma dt - \int_{Q_T} \left( z \cdot \partial_t^2 u + \frac{\partial z_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} + \xi z \cdot \text{curl} \, E \right) dx dt.
$$

Using (21), we get

$$
\int_{\Sigma_T} \left( |\nabla_T u|^2 + a|u|^2 \right) d\sigma dt \leq - \int_{\Sigma_T} u \cdot \partial_t u \, d\sigma dt - \int_{Q_T} z \cdot \left( \partial_t^2 u + \xi \text{curl} \, E \right) dx dt.
$$

Using the third equation in (1) and integrating by parts in $t$, we obtain

$$
\int_{\Sigma_T} \left( |\nabla_T u|^2 + a|u|^2 \right) d\sigma dt \leq - \int_{\Sigma_T} u \cdot \partial_t u \, d\sigma dt + \int_{Q_T} \partial_t z \cdot \left( \partial_t u - \xi \mu H \right) dx dt - \left( \int_{\Omega} z \cdot \left( \partial_t u - \xi \mu H \right) \right)_0^T.
$$

Using several times (15), (22), (23) and Young’s inequality ($ab \leq \alpha a^2 + \frac{b^2}{4\alpha}$ for all $\alpha > 0$ and all real numbers $a, b$) we can estimate the different integrals of the right-hand side of the above inequality as follows

$$
\left| \int_{\Sigma_T} u \cdot \partial_t u \, d\sigma dt \right| \leq \alpha \int_{\Sigma_T} |u|^2 d\sigma dt + \frac{1}{4\alpha} \int_{\Sigma_T} |\partial_t u|^2 d\sigma dt
$$

$$
\leq \frac{2\alpha}{a} \int_0^T \mathcal{E}(t) dt - \frac{1}{4\alpha} \int_0^T \mathcal{E}'(t) dt
$$

$$
\leq \frac{2\alpha}{a} \int_0^T \mathcal{E}(t) dt + \frac{1}{4\alpha} \mathcal{E}(0).
$$

$$
\left| \int_{Q_T} \partial_t z \cdot \left( \partial_t u - \xi \mu H \right) dx dt \right| \leq \frac{1}{4\alpha} \int_{Q_T} |\partial_t z|^2 dx dt + \alpha \int_{Q_T} |\partial_t u - \xi \mu H|^2 dx dt
$$

$$
\leq \frac{c_1}{a} \int_0^T \mathcal{E}'(t) dt + (1 + \xi^2 \mu) \alpha \int_0^T \mathcal{E}(t) dt
$$

$$
\leq \frac{c_1}{4\alpha} \mathcal{E}(0) + (1 + \xi^2 \mu) \alpha \int_0^T \mathcal{E}(t) dt.
$$

$$
\left| \int_{\Omega} \left[ z \cdot \left( \partial_t u - \xi \mu H \right) \right]_0^T \right| \leq c_3 (\mathcal{E}(0) + \mathcal{E}(T)) \leq 2c_3 \mathcal{E}(0),
$$

for some positive constant $c_3$ (depending on the domain, the $\mu$ and the parameter $\xi$). Using these different estimates, we arrive at the requested estimate by taking $\alpha = \frac{\eta}{(2a + 1 + \xi^2 \mu)}$ and $C = \left( \frac{1 + c_1}{4} + 2c_3 \alpha \right) \left( \frac{2}{a} + 1 + \xi^2 \mu \right)$. \hfill \Box

We now state our main results of this paper.

**Theorem 3.5.** Assume that $\Omega$ is a bounded domain of $\mathbb{R}^3$ strictly star-shaped with respect to the origin and having smooth boundary $\Gamma$ of class $C^2$. Then $\Omega$ satisfies the stability estimate.
Remark 3. This result can be readily extended to the case when \( \epsilon \) and \( \mu \) are positive functions satisfying some regularity and technical conditions, inspiring from [2, 10, 19].

4. Proof of main result.

Proof of Theorem 3.5. We note that it is sufficient to prove that the estimate (17) holds for any strong solution \((u, E, H)\) of (1). The main tool we use is the multiplier method [12, 17].

In order to prove (17) we proceed in two steps.

**Step 1.** We begin with the following identity

\[
0 = -\left[ \int_{\Omega} 2(q : \nabla u) \cdot \partial_1 u dx \right]_0^T + \left[ \int_{\Omega} 2\xi(\mu(q : \nabla u)) \cdot H dx \right]_0^T + \left[ \int_{Q_T} 2\epsilon(E \times H) \cdot q dx \right]_0^T + \int_{Q_T} (\text{div} q) \left( |\nabla u|^2 - |\partial_1 u|^2 \right) dx dt
\]

\[
- 2 \int_{Q_T} \frac{\partial u}{\partial x_j} \frac{\partial u_i}{\partial x_k} \frac{\partial q_k}{\partial x_j} dx dt - \int_{Q_T} \left\{ \epsilon(QE \cdot E + \mu QH \cdot H) \right\} dx dt + 2\xi \int_{Q_T} \mu H \cdot \nabla q \cdot \partial_1 u dx dt + \int_{Q_T} \left( q \cdot \nu \left( |\partial_1 u|^2 + |\partial_2 u|^2 - |\nabla_T u|^2 \right) \right) d\sigma dt
\]

\[
- 2 \int_{\Sigma_T} \frac{\partial q_T}{\partial x_j} \left( \pi \frac{\partial q_T}{\partial x_k} \right) \frac{\partial u_i}{\partial x_k} d\sigma dt + \int_{\Sigma_T} (\text{div} q_T) |\nabla_T u|^2 d\sigma dt
\]

\[
- 2 \int_{\Sigma_T} (\partial u + \partial_1 u) \cdot (q_T \cdot \nabla_T u) d\sigma dt + \int_{\Sigma_T} \left\{ q \cdot \nu (|E|^2 + |H|^2) - 2\epsilon q \cdot E + 2\mu q \cdot H - 2\xi q \cdot H \right\} d\sigma dt,
\]

where \( Q = \text{div} q_T - 2\nabla q \) and \( I \) is a \( 3 \times 3 \) matrix. Here we have used the notation \( q : \nabla u = (q \cdot \nabla u_1, q \cdot \nabla u_2, q \cdot \nabla u_3) \) and \( q_T : \nabla_T u = (q_T \cdot \nabla_T u_1, q_T \cdot \nabla_T u_2, q_T \cdot \nabla_T u_3) \), where \( q_T \) is the tangential component of \( q \) and \( \text{div} q_T \) is the tangential divergence of the field \( q_T \).

**Proof.** Multiply the first equation in (1) by \( 2(q : \nabla u) \) and integrating over \( Q_T \) and \( \Sigma_T \) by parts, we obtain

\[
\int_{Q_T} 2(q : \nabla u) \cdot \partial_1^2 u dx dt = 2 \left[ \int_{\Omega} (q : \nabla u) \cdot \partial_1 u dx \right]_0^T - \int_{\Sigma_T} q \cdot \nu |\partial_1 u|^2 d\sigma dt + \int_{Q_T} (\text{div} q) |\partial_1 u|^2 d\sigma dt,
\]

and

\[
2 \int_{Q_T} \Delta u \cdot (q : \nabla u) \ dx dt = 2 \int_{\Sigma_T} \partial_1 u \cdot (q : \nabla u) d\sigma dt - 2 \int_{Q_T} \frac{\partial u_i}{\partial x_j} \frac{\partial}{\partial x_j} (q_k \frac{\partial u_i}{\partial x_k}) \ dx dt
\]
\[\begin{align*}
&\quad = -2 \int_{Q_T} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_k} \frac{\partial q_k}{\partial x_j} \, dxdt + \int_{Q_T} \text{div} q |\nabla u|^2 \, dxdt \\
&\quad + 2 \int_{\Sigma_T} \partial_{\nu} u \cdot (q : \nabla u) \, d\sigma dt - \int_{\Sigma_T} q \cdot \nu |\nabla u|^2 \, d\sigma dt.
\end{align*}\]

On \(\Gamma\), we have

\[q : \nabla u = q_T : \nabla_T u + q_{\nu} (\partial_{\nu} u),\]

and

\[|\nabla u|^2 = |\nabla_T u|^2 + |\partial_{\nu} u|^2.\]

In that case (26) becomes

\[\begin{align*}
2 \int_{Q_T} \Delta u \cdot (q : \nabla u) \, dxdt \\
&\quad = -2 \int_{Q_T} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_k} \frac{\partial q_k}{\partial x_j} \, dxdt + \int_{Q_T} \text{div} q |\nabla u|^2 \, dxdt \\
&\quad + 2 \int_{\Sigma_T} \partial_{\nu} u \cdot (q_T : \nabla_T u) \, d\sigma dt + \int_{\Sigma_T} q \cdot \nu (|\partial_{\nu} u|^2 - |\nabla_T u|^2) \, d\sigma dt.
\end{align*}\]

Using the second boundary condition in (1) and integrating by parts on \(\Sigma_T\), we get

\[\begin{align*}
2 \int_{Q_T} \Delta u \cdot (q : \nabla u) \, dxdt \\
&\quad = -2 \int_{Q_T} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_k} \frac{\partial q_k}{\partial x_j} \, dxdt + \int_{Q_T} \text{div} q |\nabla u|^2 \, dxdt \\
&\quad - 2 \int_{\Sigma_T} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_k} (q_T \cdot \nabla_T u) \, d\sigma dt + \int_{\Sigma_T} q \cdot \nu (|\partial_{\nu} u|^2 - |\nabla_T u|^2) \, d\sigma dt \\
&\quad - 2 \int_{\Sigma_T} (q_T : \nabla_T u) \cdot (au + \partial_{\nu} u) \, d\sigma dt.
\end{align*}\]

From Lemma 2.2, we deduce that

\[\begin{align*}
2 \int_{Q_T} \Delta u \cdot (q : \nabla u) \, dxdt \\
&\quad = -2 \int_{Q_T} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_k} \frac{\partial q_k}{\partial x_j} \, dxdt + \int_{Q_T} \text{div} q |\nabla u|^2 \, dxdt \\
&\quad - 2 \int_{\Sigma_T} \frac{\partial u_i}{\partial x_j} \left( \frac{\partial_T q_k}{\partial x_j} \right) \frac{\partial u_i}{\partial x_k} \, d\sigma dt - \int_{\Sigma_T} \frac{\partial_T (|\partial_{\nu} u|^2)}{\partial x_j} q_{\nu_T} \, d\sigma dt \\
&\quad + \int_{\Sigma_T} q \cdot \nu (|\partial_{\nu} u|^2 - |\nabla_T u|^2) \, d\sigma dt - 2 \int_{\Sigma_T} (q_T : \nabla_T u) \cdot (au + \partial_{\nu} u) \, d\sigma dt.
\end{align*}\]

Now, we observe that

\[\int_{\Sigma_T} \partial_T (|\nabla_T u|^2) \cdot q_T \, d\sigma dt = - \int_{\Sigma_T} \text{div}_T q_T |\nabla_T u|^2 \, d\sigma dt.\]
Hence

\[
\int_{Q_T} 2(q : \nabla u) \cdot (\Delta u - \xi \text{curl} E) \, dx \, dt \\
= -2 \int_{Q_T} \partial_{ij} \partial_{ji} q_{kj} \, dx \, dt + \int_{Q_T} \text{div} |\nabla u|^2 \, dx \, dt \\
- 2\xi \int_{Q_T} q : \nabla u \cdot \text{curl} E \, dx \, dt - 2 \int_{Q_T} \partial_{ij} \partial_{ji} (\pi \partial_{ij} q_{kj}) \partial_{ij} q_{ij} \, dx \, dt \\
+ \int_{Q_T} \text{div}(\nabla u \cdot \nabla u) \, dx \, dt - 2 \xi \int_{Q_T} q : \nabla u \cdot \text{curl} E \, dx \, dt \\
- 2 \int_{Q_T} \nabla u : \nabla u \cdot (au + \partial_i u) \, dx \, dt. \\
\] (27)

Combining (25) and (27) we infer

\[
0 = \int_{Q_T} 2(q : \nabla u) \cdot (\partial_{ij} u - \Delta u + \xi \text{curl} E) \, dx \, dt \\
= -2 \int_{Q_T} \partial_{ij} \partial_{ji} q_{kj} \, dx \, dt - 2 \int_{Q_T} \partial_{ij} \partial_{ji} (\pi \partial_{ij} q_{kj}) \partial_{ij} q_{ij} \, dx \, dt \\
+ \int_{Q_T} \text{div}(\nabla u \cdot \nabla u) \, dx \, dt - 2 \xi \int_{Q_T} q : \nabla u \cdot \text{curl} E \, dx \, dt \\
- 2 \int_{Q_T} \nabla u : \nabla u \cdot (au + \partial_i u) \, dx \, dt. \\
\] (28)

Next, multiplying the equation (1)₂ and (1)₃ by $2\mu(q \times H)$ and $2\varepsilon(E \times q) - 2\xi(q : \nabla u)$, respectively, and integrating over $Q_T$ by parts (as in [2]), we get

\[
0 = \int_{Q_T} \left\{2\mu(q \times H) \cdot (\varepsilon \partial_i E - \text{curl} H - \xi \text{curl} \partial_i u)\right\} \, dx \, dt \\
+ \int_{Q_T} \left\{(2\varepsilon(E \times q) - 2\xi(q : \nabla u)) \cdot (\mu \partial_i H + \text{curl} E)\right\} \, dx \, dt \\
= \left[\int_{Q_T} 2\xi \mu H \cdot (q : \nabla u) \, dx\right]_0^T + \left[\int_{Q_T} 2\varepsilon \mu \partial_i E \cdot q \, dx\right]_0^T \\
- \int_{Q_T} (\psi E \cdot E + \mu Q H \cdot H) \, dx \, dt + 2\xi \int_{Q_T} \mu H \cdot \nabla q \cdot \partial_i u \, dx \, dt \\
+ 2\xi \int_{Q_T} (q : \nabla u) \cdot \text{curl} E \, dx \, dt + \int_{Q_T} \left\{q \cdot \nu(\varepsilon |E|^2 + |H|^2) - 2\varepsilon(q \cdot E)(E \cdot u) + 2\xi(q : H)(H \cdot u) - 2\mu(q \cdot H)(H \cdot u) \right\} \, dx \, dt. \\
\] (29)

where we recall that $Q = \text{div} q - 2
\n\nCombining (28) and (29) we deduce (24). So, Lemma 4.1 is proved. □
Lemma 4.3. The following identity holds

\[ 0 = \left[ \int_{\Omega} -2(m : \nabla u) \cdot \partial_t u \, dx \right]^T_0 + \left[ \int_{\Omega} 2 \xi \mu H \cdot (m : \nabla u) \, dx \right]^T_0 \\
+ \left[ \int_{\Omega} 2 \xi (E \times H) \cdot m \, dx \right]^T_0 + \int_{Q_T} (|\nabla u|^2 - 3|\partial_t u|^2) \, dx \, dt \\
- \int_{Q_T} \{ \epsilon |E|^2 + \mu |H|^2 \} \, dx \, dt + \int_{Q_T} \mu H \cdot \partial_t u \, dx \, dt \\
+ \int_{\Sigma_T} m \cdot \nu \{ |\partial_t u|^2 + |\partial_{\nu} u|^2 - |\nabla_T u|^2 \} \, d\sigma \, dt - 2 \int_{\Sigma_T} \partial_{\nu} u_i (\eta \partial_{\nu} m_k u_i \partial_{\nu} u_j \partial_{x_k}) \, d\sigma \, dt \\
+ \int_{\Sigma_T} \{ m \cdot \nu (\epsilon |E|^2 + \mu |H|^2) - 2\epsilon (m \cdot E)(E \cdot \nu) - 2 \mu (m \cdot H)(H \cdot \nu) \} \, d\sigma \, dt. \]  

(30)

Lemma 4.2. We have the following identity

\[ 0 = \left[ \int_{\Omega} -2u \cdot \partial_t u \, dx \right]^T_0 + 2 \xi \left[ \int_{\Omega} \mu H \cdot u \, dx \right]^T_0 + 2 \int_{Q_T} (|\partial_t u|^2 - |\nabla u|^2) \, dx \, dt \\
- 2 \xi \int_{Q_T} \mu H \cdot \partial_t u \, dx \, dt - 2 \int_{\Sigma_T} |\nabla_T u|^2 \, d\sigma \, dt - 2 \int_{\Sigma_T} (au + \partial_t u) \cdot u \, d\sigma \, dt. \]  

(31)

Proof. We multiply the equation (1) and (3) by \( 2u \) and \( 2\xi u \), respectively, and integrating by parts on \( Q_T \) and \( \Sigma_T \), we obtain (31).

Combining (30) and (31) we obtain the following Lemma

Lemma 4.3. The following identity holds

\[ 0 = \left[ \int_{\Omega} -2(m : \nabla u + u) \cdot \partial_t u \, dx \right]^T_0 + \left[ \int_{\Omega} 2 \xi \mu H \cdot (m : \nabla u + u) \, dx \right]^T_0 \\
+ \left[ \int_{\Omega} 2 \xi \mu (E \times H) \cdot m \, dx \right]^T_0 - \int_{Q_T} (|\nabla u|^2 + |\partial_t u|^2) \, dx \, dt \\
- \int_{Q_T} \{ \epsilon |E|^2 + \mu |H|^2 \} \, dx \, dt + \int_{\Sigma_T} m \cdot \nu \{ |\partial_t u|^2 + |\partial_{\nu} u|^2 - |\nabla_T u|^2 \} \, d\sigma \, dt \\
- 2 \int_{\Sigma_T} \partial_{\nu} u_i (\eta \partial_{\nu} m_k u_i \partial_{\nu} u_j \partial_{x_k}) \, d\sigma \, dt + \int_{\Sigma_T} (\text{div}_T m_T - 2)|\nabla_T u|^2 \, d\sigma \, dt \\
+ \int_{\Sigma_T} (au + \partial_t u) \cdot (m_T : \nabla_T u + u) \, d\sigma \, dt + \int_{\Sigma_T} \{ m \cdot \nu (\epsilon |E|^2 + \mu |H|^2) \} \, d\sigma \, dt - 2 \epsilon (m \cdot E)(E \cdot \nu) - 2 \mu (m \cdot H)(H \cdot \nu) \}

\[ \cdot (m \cdot \partial_t u) \, d\sigma \, dt. \]  

(32)

It therefore follows from (32) that

\[ 2 \int_0^T \mathcal{E}(t) \, dt = I_1 + I_2 + I_3 + I_4, \]  

(33)
where we have set
\[
I_1 = - \left[ \int_{\Omega} 2(m : \nabla u + u) \cdot \partial_t u \, dx \right]_0^T + \left[ \int_{\Omega} 2 \xi \mu (m : \nabla u + u) \cdot H \, dx \right]_0^T + 2 \left[ \int_{\Omega} \epsilon \mu (E \times H) \cdot m \, dx \right]_0^T,
\]
\[
I_2 = \int_{\Sigma_T} \left\{ m \cdot \nu |\partial_t u|^2 + |\nabla_T u|^2 + a|u|^2 \right\} \, d\sigma dt,
\]
\[
I_3 = \int_{\Sigma_T} \left\{ m \cdot \nu \left( \epsilon |E|^2 + \mu |H|^2 \right) - 2 \epsilon (m \cdot E)(E \cdot \nu) - 2 \mu (m \cdot H)(H \cdot \nu) - 2 \xi \mu (H \cdot \nu)(m \cdot \partial_t u) \right\} \, d\sigma dt,
\]
\[
I_4 = -2 \int_{\Sigma_T} \left[ \frac{\partial_T u_i}{\partial x_j} \frac{\partial_T q_{kj}}{\partial x_j} \frac{\partial T}{\partial x_j} \right] \, d\sigma dt + \int_{\Sigma_T} (\text{div}_T m_T - 2 - m_\nu)|\nabla_T u|^2 \, d\sigma dt + \int_{\Sigma_T} m_\nu |\partial_t u|^2 \, d\sigma dt - 2 \int_{\Sigma_T} (m_T : \nabla_T u + u) \cdot (a_\nu + \partial_t u) \, d\sigma dt.
\]
where \( m_\nu = m \cdot \nu \) and \( m_T = m - m_\nu \).

**Step 2.** It remains to estimate each term \( I_i \) \( (i = 1, \ldots, 4) \).

From Lemma 3.2 of [12] (see also [1]) and the identity (15), there exist two positive constants \( k_1 \) and \( k_2 \) such that
\[
\left[ \int_{\Omega} 2(m : \nabla u + u) \cdot \partial_t u \, dx \right]_0^T \leq k_1 \mathcal{E}(0),
\]
and
\[
\left[ \int_{\Omega} 2 \mu (m : \nabla u + u) \cdot H \, dx \right]_0^T \leq k_2 \mathcal{E}(0).
\]
As in [12, 2], we can show that there exists a constant \( k_3 > 0 \) such that
\[
\left[ \int_{\Omega} 2 \epsilon \mu (E \times H) \cdot m \, dx \right]_0^T \leq k_3 \mathcal{E}(0).
\]
Hence
\[
I_1 \leq k_4 \mathcal{E}(0),
\]
where \( k_4 \) is a positive constant independent of \( T \).

By Lemma 3.4, we find that
\[
I_2 \leq \frac{C}{\eta} \mathcal{E}(0) + R \int_{\Sigma_T} |\partial_t u|^2 \, d\sigma dt + \eta \int_0^T \mathcal{E}(t) \, dt.
\]
where \( R = \sup_{x \in \Omega} |m(x)|. \)
Since we have
\[
2 \epsilon (m \cdot E)(E \cdot \nu) = 2 \epsilon (m_T \cdot E_T)E_\nu + 2 \epsilon m \cdot \nu |E_\nu|^2,
\]
and
\[
2 \mu (m \cdot H)(H \cdot \nu) = 2 \mu (m_T \cdot H_T)H_\nu + 2 \mu m \cdot \nu |H_\nu|^2.
\]
Then the term $I_3$ becomes
\[
I_3 = \int_{\Sigma_T} \left\{ m \cdot \nu (\epsilon |E_T|^2 + \mu |H_T|^2) - \epsilon m \cdot \nu |E_\nu|^2 - \mu m \cdot \nu |H_\nu|^2 - 2(\nu \cdot \nu) E_\nu \right\} d\sigma dt,
\]
where we recall that $E_\nu = E \cdot \nu$ and $E_T = E - E_\nu$. 

By Young's inequality, there exists $k_5 > 0$ such that for all $\theta > 0$
\[
I_3 \leq \int_{\Sigma_T} \left\{ \left( m \cdot \nu + \frac{k_5}{\theta} \right) (\epsilon |E \times \nu|^2 + \mu |H \times \nu|^2) + (\theta - m \cdot \nu) \epsilon |E_\nu|^2 
+ (20 - m \cdot \nu) \mu |H_\nu|^2 + \frac{k_5}{\theta} |\partial_t u|^2 \right\} d\sigma dt.
\]  
Using the first and third equations in Lemma 2.3, we may rewrite $I_4$ as
\[
I_4 = -\int_{\Sigma_T} (2 + m_\nu + m_\nu tr(\partial_t \nu))|\nabla_T u|^2 d\sigma dt + 2 \int_{\Sigma_T} m_\nu \partial_T u_i (\frac{\partial T v_k}{\partial x_j}) \frac{\partial T u_i}{\partial x_k} d\sigma dt
+ \int_{\Sigma_T} m_\nu |\partial_t u|^2 d\sigma dt - 2 \int_{\Sigma_T} (m_T : \nabla_T u + u) \cdot (au + \partial_t u) d\sigma dt.
\]  
Next, we are going to estimate each term of the right-hand side of (37). From now on we will denote by $C$ various positive constants which may be different at different occurrences and is independent of $u$ and large enough. Since $m_\nu$ and $\partial_T \nu$ are bounded (see e.g. [1]), we obtain
\[
\left| \int_{\Sigma_T} (2 + m_\nu + m_\nu tr(\partial_t \nu))|\nabla_T u|^2 d\sigma dt \right| \leq C \int_{\Sigma_T} |\nabla_T u|^2 d\sigma dt,
\]
\[
2 \int_{\Sigma_T} m_\nu \partial_T u_i (\frac{\partial T v_k}{\partial x_j}) \frac{\partial T u_i}{\partial x_k} d\sigma dt \leq C \int_{\Sigma_T} |\nabla_T u|^2 d\sigma dt.
\]
Since $u \cdot (m_T : \nabla_T u) = \frac{1}{2} \nabla_T (|u|^2) \cdot m_T$, the last term of $I_4$ is equivalent to
\[
2 \int_{\Sigma_T} (m_T : \nabla_T u + u) \cdot (au + \partial_t u) d\sigma dt
= a \int_{\Sigma_T} m_T \cdot \nabla_T (|u|^2) d\sigma dt + 2 \int_{\Sigma_T} (m_T : \nabla_T u) \cdot \partial_t u d\sigma dt
+ 2a \int_{\Sigma_T} |u|^2 d\sigma dt + 2 \int_{\Sigma_T} u \cdot \partial_t u d\sigma dt.
\]
We set
\[
I_1 = a \int_{\Sigma_T} m_T \cdot \nabla_T (|u|^2) d\sigma dt,
I_2 = 2 \int_{\Sigma_T} (m_T : \nabla_T u) \cdot \partial_t u d\sigma dt,
I_3 = 2a \int_{\Sigma_T} |u|^2 d\sigma dt,
I_4 = 2 \int_{\Sigma_T} u \cdot \partial_t u d\sigma dt.
\]
By Green's formula, we obtain
\[
I_4 = -a \int_{\Sigma_T} div_T m_T |u|^2 d\sigma dt,
\]
and therefore
\[ |\mathcal{I}_1| \leq C \int_{\Sigma_T} |u|^2 \, d\sigma dt. \]

The terms \( \mathcal{I}_2 \) and \( \mathcal{I}_4 \) being estimated with the help of Young's inequality, we get
\[ |\mathcal{I}_2| \leq \theta \int_{\Sigma_T} |\partial_t u|^2 \, d\sigma dt + \frac{C}{\theta} \int_{\Sigma_T} |\nabla_T u|^2 \, d\sigma dt, \]
\[ |\mathcal{I}_4| \leq \theta \int_{\Sigma_T} |\partial_t u|^2 \, d\sigma dt + \frac{C}{\theta} \int_{\Sigma_T} |u|^2 \, d\sigma dt, \]
for all \( \theta > 0 \). We readily check that
\[ |\mathcal{I}_3| \leq C \int_{\Sigma_T} |u|^2 \, d\sigma dt. \]

Combining these estimates, we find that
\[ \left| 2 \int_{\Sigma_T} (m_T \cdot \nabla_T u + u)(au + \partial_t u) \, d\sigma dt \right| \leq \frac{C}{\theta} \int_{\Sigma_T} |\nabla_T u|^2 \, d\sigma dt + \left( 2C + \frac{C}{\theta} \right) \int_{\Sigma_T} |u|^2 \, d\sigma dt + 2\theta \int_{\Sigma_T} |\partial_t u|^2 \, d\sigma dt. \tag{40} \]

We now estimate the term \( \int_{\Sigma_T} m \cdot \nu |\partial_t u|^2 \, d\sigma dt \). We then have
\[ \left| \int_{\Sigma_T} m \cdot \nu |\partial_t u|^2 \, d\sigma dt \right| \leq C \int_{\Sigma_T} |\partial_t u|^2 \, d\sigma dt. \]

Since \( \Gamma \) is of class \( C^2 \), then there exists a vector field \( h \in (W^{1,\infty}(\Omega))^3 \) such that \( h = \nu \) on \( \Gamma \).

(See Remark 3.2 Chap I of [17] for the construction of this vector field). So, applying identity (24) with \( q = h \), we deduce that
\[
\int_{\Sigma_T} |\partial_t u|^2 \, d\sigma dt = \left[ \int_{\Omega} 2(q : \nabla u) \cdot \partial_t u \, dx \right]_0^T - \left[ \int_{\Omega} 2\xi \mu H \cdot (q : \nabla u) \, dx \right]_0^T \\
- \left[ \int_{\Omega} 2\mu \mu (E \times H) \cdot q \, dx \right]_0^T + \int_{Q_T} \text{div} q (|\partial_t u|^2 - |\nabla u|^2) \, dx \, dt \\
+ 2 \int_{Q_T} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \, dx \, dt + \int_{Q_T} \left\{ \mu Q \cdot E + \mu Q H \cdot H \right\} \, dx \, dt \\
- 2\xi \int_{Q_T} \mu H \cdot \nabla q \cdot \partial_t u \, dx \, dt + \int_{\Sigma_T} (|\nabla_T u|^2 - |\partial_t u|^2) \, d\sigma dt \\
+ \int_{\Sigma_T} \left\{ \epsilon |E_n|^2 + \mu |H_n|^2 + 2\xi \mu (H \cdot \nu)(\partial_t u \cdot \nu) - (\epsilon |E_t|^2 + \mu |H T|^2) \right\} \, d\sigma dt.
\]

Proceeding as above, we can easily deduced the existence of some constant \( C > 0 \) such that
\[
\int_{\Sigma_T} |\partial_t u|^2 \, d\sigma dt \leq C E(0) + C \int_0^T \mathcal{E}(t) \, dt - 2\xi \int_{Q_T} \mu H \cdot \nabla q \cdot \partial_t u \, dx \, dt \\
+ \int_{\Sigma_T} |\nabla_T u|^2 \, d\sigma dt + \int_{\Sigma_T} \left\{ \epsilon |E_n|^2 + \mu |H_n|^2 + 2\xi \mu (H \cdot \nu)(\partial_t u \cdot \nu) \right\} \, d\sigma dt.
\]
The remaining terms are estimated by Young’s inequality to obtain for all $\beta$, $\theta > 0$
\[
\int_{\Sigma_T} |\partial_\nu u|^2 d\sigma dt \leq C\mathcal{E}(0) + C(\beta) \int_0^T \mathcal{E}(t) dt + \frac{C}{\theta} \int_{\Sigma_T} |\partial_\mu u|^2 d\sigma dt
\]
\[
+ \int_{\Sigma_T} |\nabla_T u|^2 d\sigma dt + \int_{\Sigma_T} \left\{ \epsilon |E_\nu|^2 + (1 + \theta) \mu |H_\nu|^2 \right\} d\sigma dt,
\] (41)
where $C(\beta) = C + \max(\beta, \frac{C}{\theta})$.

Finally, with (38), (39), (40) and (41), we obtain
\[
I_4 \leq C\mathcal{E}(0) + C(\beta) \int_0^T \mathcal{E}(t) dt + k_6 \int_{\Sigma_T} \left( |\nabla_T u|^2 + a|u|^2 \right) d\sigma dt
\]
\[
+ \left(2\theta + \frac{C}{\theta}\right) \int_{\Sigma_T} |\partial_\mu u|^2 d\sigma dt + \int_{\Sigma_T} \left\{ \epsilon |E_\nu|^2 + (1 + \theta) \mu |H_\nu|^2 \right\} d\sigma dt,
\]
where $k_6 = \max \left\{ \frac{1}{2} \left(2C + \frac{C}{\theta}\right), (2C + \frac{C}{\theta} + 1) \right\}$.

By Lemma 3.4, we conclude that
\[
I_4 \leq \frac{k_7}{\eta} \mathcal{E}(0) + (k_6 \eta + C(\beta)) \int_0^T \mathcal{E}(t) dt + \left(2\theta + \frac{C}{\theta}\right) \int_{\Sigma_T} |\partial_\mu u|^2 d\sigma dt
\]
\[
+ \int_{\Sigma_T} \left\{ \epsilon |E_\nu|^2 + (1 + \theta) \mu |H_\nu|^2 \right\} d\sigma dt.
\] (42)

Substituting the estimates (34), (35), (36) and (42) into (33), we obtain
\[
\left[ 2 - \left( \eta(k_6 + 1) + C(\beta) \right) \right] \int_0^T \mathcal{E}(t) dt \leq \left( k_4 + \frac{k_7 + C}{\eta} \right) \mathcal{E}(0)
\]
\[
\left( m \cdot \nu + \frac{k_2}{\theta} \right) \int_{\Sigma_T} \left( \epsilon |E \times \nu|^2 + \mu |H \times \nu|^2 \right) d\sigma dt
\]
\[
+ \int_{\Sigma_T} \left\{ \theta + 1 - m \cdot \nu \right\} \epsilon |E_\nu|^2 dt + \left( 2\theta + 1 - m \cdot \nu \right) \mu |H_\nu|^2 \right\} d\sigma dt
\]
\[
+ \left(2\theta + \frac{k_5 + C}{\theta} + R\right) \int_{\Sigma_T} |\partial_\mu u|^2 d\sigma dt,
\]
where $k_4$, $k_5$, $k_6$ and $k_7$ are positive constants (depending on $\theta$, $\eta$, the domain, the coefficient $\mu$ and the parameter $\xi$).

Next, by picking $0 < \theta < \frac{k_m - 1}{3}$ where $k_m = \min \{m \cdot \nu, \ m \in \Gamma \} > 1$, we obtain
\[
\left[ 2 - \left( \eta(k_6 + 1) + C(\beta) \right) \right] \int_0^T \mathcal{E}(t) dt \leq \left( k_4 + \frac{k_7 + C}{\eta} \right) \mathcal{E}(0)
\]
\[
\left( m \cdot \nu + \frac{k_5}{\theta} \right) \int_{\Sigma_T} \left( \epsilon |E \times \nu|^2 + \mu |H \times \nu|^2 \right) d\sigma dt
\]
\[
+ \left(2\theta + \frac{C + k_5}{\theta} + R\right) \int_{\Sigma_T} |\partial_\mu u|^2 d\sigma dt.
\]
By using the first boundary condition in (1), we get finally
\[
\int_0^T \mathcal{E}(t) dt \leq C_1 \mathcal{E}(0) + C_2 \int_0^T \left( |\partial_\nu u(t)|^2 + |E(t) \times \nu|^2 \right) d\sigma dt,
\]
where $C_1 = \frac{k_4 + (k_7 + C)\eta}{2 - \eta(1 + k_6) + C(\beta)}$, $C_2 = \frac{k_9}{2 - \eta(1 + k_6) + C(\beta)}$ (with the same dependence as above but does not depend on $T$) and choosing $0 < \eta < \frac{2 - C(\beta)}{(1 + k_6)}$ and $C(\beta) < 2$ (i.e., we choose $\beta$ large enough such that $\beta < 2 - C$ with $C < 2$). The proof of Theorem 3.5 is now complete.

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Received February 2016; revised April 2016.

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