QUANTITATIVE DE GIORGI METHODS IN KINETIC THEORY FOR NON-LOCAL OPERATORS

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Abstract. We derive quantitatively the weak and strong Harnack inequality for kinetic Fokker-Planck type equations with a non-local diffusion operator for the full range of the non-locality exponent $s \in (0,1)$. This implies Hölder continuity. We give novel proofs on the boundedness of the bilinear form associated to the non-local operator and on the construction of a geometric covering accounting for the non-locality to obtain the Harnack inequalities. Our results apply to the inhomogeneous Boltzmann equation in the non-cutoff case.

1. Introduction

1.1. Problem Formulation. We consider non-local kinetic equations of the form

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}f + h, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d, \quad v \in B_{\bar{R}}$$

for some $\bar{R} > 0$, where we assume $h = h(t, x, v)$ is a real scalar field in $L^\infty$, and for a non-negative measurable kernel $K: \mathbb{R} \times \mathbb{R}^d \times B_{\bar{R}} \times \mathbb{R}^d \to [0,+\infty)$ we define

$$\mathcal{L}f(v) := \text{PV} \int_{\mathbb{R}^d} K(t, x, v, w)[f(w) - f(v)] \, dw$$

in the principal value sense. The question we raise is whether solutions to (1.1)-(1.2) satisfy Hölder continuity and Harnack inequalities.

We make the following assumptions on the kernel for $v \in B_{\bar{R}}$. We require the following upper bound for $r > 0$

$$\int_{\mathbb{R}^d \setminus B_r(v)} K(v, w) \, dw \leq \Lambda r^{-2s},$$

and

$$\int_{B_{r} \setminus B_r(v)} K(v, w) \, dv \leq \Lambda r^{-2s}$$

for some constants $0 < \lambda < \Lambda$ with $0 < s < 1$. We also need a coercivity condition

$$\lambda \int_{B_{\bar{R}}} \int_{B_{\bar{R}}} \frac{|f(v) - f(w)|^2}{|v - w|^{d+2s}} \, dv \, dw \leq \int_{B_{\bar{R}}} \int_{B_{\bar{R}}} |f(v) - f(w)|^2 K(v, w) \, dw \, dv.$$

Moreover, instead of the usual symmetry assumption $K(v, w) = K(w, v)$, which corresponds to the cancellation form, we assume the following cancellation

$$\forall v \in \mathbb{R}^d \quad \text{PV} \int_{\mathbb{R}^d} (K(v, w) - K(w, v)) \, dw \leq \Lambda$$

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and if $s \geq \frac{1}{2}$ we assume that for all $R > 0$

\begin{equation}
\forall v \in \mathbb{R}^d \quad \text{PV} \int_{B_R} (v-w)(K(v,w) - K(w,v)) \, dw \leq \Lambda R^{1-2s}.
\end{equation}

We remark that the upper bound (1.3) is equivalent to

\begin{equation}
\int_{B_r(v)} K(v,w)|v-w|^2 \, dw \leq \Lambda r^{2-2s}.
\end{equation}

We also note that (1.6) holds for $s \in (0, \frac{1}{2})$ as a consequence of (1.5) and (1.7). These assumptions lean on the work of Imbert and Silvestre [14]. We want to assume conditions that apply to the Boltzmann collision kernel in the non-cutoff case. We know from section 3 of [14] that the Boltzmann kernel satisfies assumptions (1.3)-(1.6) provided that physically relevant macroscopic quantities stated in Assumption 1.1 of [15] are bounded.

Equation (1.1) is invariant under Galilean transformation, i.e. under the family of transformations $z \to z_0 \circ z = (t_0 + t, x_0 + x + t v_0, v_0 + v)$ where $z_0 = (t_0, x_0, v_0) \in \mathbb{R}^{1+2d}$. It is also invariant under scaling $f_r(t,x,v) = f(r^{2s}t, r^{1+2s}x, rv)$ for $r \in [0,1]$, in the sense that $f_r$ solves a modified equation where the scaled kernel satisfies assumptions (1.3)-(1.6) in a larger radius $\frac{R}{r}$ and the scaled source term is bounded by $h$. This motivates why we consider (1.1) in the cylinder

\[ Q_r(z_0) := \{(t,x,v) : -r^{2s} \leq t-t_0 \leq 0, |v-v_0| < r, |x-x_0-(t-t_0)v_0| < r^{1+2s}\} \]

for $r > 0$ and $z_0 = (t_0, x_0, v_0) \in \mathbb{R} \times \mathbb{R}^d \times B_R$. For later reference, we also introduce the cylinder shifted to the past

\[ Q_r^-(z_0) := Q_r(z_0 - (2r^{2s}, 2r^{2s}v_0, 0)) \]

so that in particular for $z_0 = 0$

\[ Q_r^- := Q_r(-2r^{2s}, 0, 0) = (-3r^{2s}, -2r^{2s}) \times B_{r^{1+2s}} \times B_r. \]

Similarly the cylinder shifted to the future is denoted as

\[ Q_r^+(z_0) := Q_r(z_0 + (2r^{2s}, 2r^{2s}v_0, 0)). \]

Our equation involves a transport term, which transfers some regularity of the velocity variable to the space variable. It also involves a non-local diffusion in the $v$ variable. It is a non-linear equation since the kernel $K$ depends on the solution $f$ in general. Our motivation to study the regularity of this type of equation is linked to the question of well-posedness for smooth classical solutions of the inhomogeneous Boltzmann equation without cut-off. There are linear kinetic equations whose solutions in the hydrodynamic limit are described by a fractional diffusion [2,19,20]. Indeed diffusion limits of the linear Boltzmann equation with a heavy-tailed distribution of infinite variance as an equilibrium distribution give rise to a fractional diffusion equation [20]. Such heavy-tailed distribution functions arise in astrophysical plasmas [21] or also in granular gases through dissipative collision mechanisms [5]. However, the only source of fractional diffusion at the kinetic level stems from long range interactions of the Boltzmann collision kernel.

In the limit case $s \to 1$ equation (1.1) models the local kinetic Fokker-Planck equation, whose study is motivated by applications to the Landau equation [8]. For the local case, there is a non-constructive method discussed in [8]. A constructive proof first appeared in a series of works [28–30] for ultraparabolic equations that has further been developed in [10] to local kinetic Fokker-Planck type equations. The construction is based on a Poincaré-type inequality and Kruzhkov’s method [18]. A novel constructive approach for local kinetic Fokker-Planck type equations has been devised.
by Guerand and Mouhot in [11]. Their method relies on trajectories. For general $s \in (0, 1)$ Cyril Imbert and Luis Silvestre (together with Clément Mouhot in [12,13]) made important contributions in a series of papers [12–14,16,25] that culminated in the final work [15]. In [25] Silvestre proves for a certain range of $s$ that any solution $f$ is a priori essentially bounded provided that the hydrodynamic quantities of mass, energy and entropy satisfy some uniform bounds. In section 3 of [14] they show that these hydrodynamic bounds imply assumptions (1.3)-(1.6) on the kernel. With an additional non-degeneracy assumption, they obtain the weak Harnack inequality with a quantitative argument in case that $s \in (0, \frac{1}{2})$ [14]. Note that their method, which uses barrier functions, can be extended to $s \in (0, 1)$ under the additional symmetry assumption $K(v, v + w) = K(v, v - w)$, cf. section 7 in [14]. In case that $s \in (\frac{1}{2}, 1)$ their methods are non-constructive. In [16] Imbert and Silvestre derive Schauder estimates for kinetic equations, which can then be bootstrapped for the non-cutoff Boltzmann equation to obtain smooth solutions [15]. To achieve global smooth solutions all these local estimates are turned into global ones by a change of variables, see section 5 in [15].

There has also been a work by Stokols [26] where he combines the method of [8] with fractional estimates from [3] to obtain a non-constructive proof of Hölder continuity in the non-local case. The assumptions he poses on the kernel are stronger than ours. Our assumptions coincide with those in [14] apart from the non-degeneracy assumption (Equation (1.4) in [14]) that is required for the construction of the barrier functions in Imbert and Silvestre’s work.

The following work is a constructive proof of Hölder continuity and Harnack inequalities, leaning on the work of Jessica Guerand and Clément Mouhot [11]. We generalise De Giorgi’s method [7] in the kinetic context developed in [11] to the non-local case. These methods were originally established for non-linear elliptic equations by De Giorgi [7]. Moser then showed how to deduce a Harnack inequality [22,23]. The weak and strong Harnack inequality are local regularity results. In particular, the weak Harnack inequality implies Hölder continuity [6]. The assumptions we pose on the kernel are weak enough so that they are satisfied by the Boltzmann collision kernel. By using an argument based on trajectories, we simplify the barrier method used in [14] to obtain Hölder continuity for the non-cutoff Boltzmann equation.

1.2. Contribution. Our contribution consists of a quantitative proof of regularity for fractional Fokker-Planck type equations. Our results are applicable to the non-cutoff Boltzmann equation. We prove Harnack inequalities and Hölder continuity:

**Theorem 1.1** (Harnack inequalities). Let $f$ be a non-negative super-solution of (1.1)-(1.2) in $Q_1$ with a kernel $K$ satisfying (1.3)-(1.6) for $R = 2$. Assume $f$ is essentially bounded in $(-1,0) \times B_1 \times \mathbb{R}^d$. Then there is $C$ and $\zeta > 0$ depending on $s, d, \lambda, \Lambda$ such that the weak Harnack inequality is satisfied:

$$
\left( \int_{\tilde{Q}_{2r_0}^+} f(z)^{\zeta} \right)^{\frac{1}{\zeta}} \leq C \left( \inf_{Q_{2r_0}} f + \|h\|_{L^\infty(Q_1)} \right),
$$

where $r_0 = \frac{1}{12}$ and $\tilde{Q}_{2r_0}^+ := Q_{2r_0}^+ \left((\frac{12}{19})^{2s} r_0^{-2s}, 0, 0\right)$. Moreover, assuming that $f$ is a non-negative weak solution to (1.1)-(1.2), there holds the strong Harnack inequality

$$
\sup_{\tilde{Q}_{r_0}^+} f \leq C \left( \inf_{Q_{2r_0}} f + \|h\|_{L^\infty(Q_1)} \right),
$$

with $C$ depending on $d, s, \lambda, \Lambda$. 

**Theorem 1.2** (Hölder continuity). Let $f$ be a weak solution of (1.1)-(1.2) in $Q_1$ with a kernel $K$ satisfying (1.3)-(1.6) for $R = 2$. Assume $f$ is essentially bounded in $(-1,0] \times B_1 \times \mathbb{R}^d$. Then $f$ is Hölder continuous in $Q_{1/2}$ with Hölder exponent $\alpha \in (0,1)$ depending on $s, d, \lambda, \Lambda, \|h\|_{L^\infty(Q_1)}$ with

$$\|f\|_{C^\alpha(Q_{1/2})} := \sup_{z_1, z_2 \in Q_{1/2}, z_1 \neq z_2} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^{\alpha}} \leq C \left(1 + \|h\|_{L^\infty(Q_1)}\right) \left(\|f\|_{L^2(Q_1)} + \|h\|_{L^\infty(Q_1)}\right),$$

where $C$ depends on $d, s, \lambda$ and $\Lambda$.

1.3. **Structure of the article.** In section 2 we fix the notation and state some preliminary considerations on the definition of weak solutions for (1.1). In particular we state a result on the boundedness in $H^s \times H^s$ of the bilinear form (2.1) corresponding to the non-local diffusion. The theorem was first proved in section 4 of [14]. We give a different proof for the anti-symmetric part of the operator in Theorem 2.1 below. We proceed with the proof in three steps.

For De Giorgi’s first lemma we prove an energy estimate 3.2 in section 3 and improve its regularity by recalling the fundamental solution of the fractional Kolmogorov equation 3.1. We deduce De Giorgi’s first lemma 4.1 by a classical De Giorgi iteration in section 4.

For De Giorgi’s second lemma we start section 5 with a weak Poincaré inequality in $L^1$, which we need in the proof of the intermediate value theorem 5.2. The proof of the former is based on trajectories, as was first employed by [11]. The latter theorem 5.2 then follows just as in [11]. The intermediate value theorem implies together with a direct consequence of De Giorgi’s first lemma 4.2 a measure-to-pointwise estimate 5.3, which is De Giorgi’s second lemma.

As a last step, we deduce Hölder continuity and the Harnack inequalities in section 6. Hölder continuity follows by standard methods. For the Harnack inequalities, we use a covering argument, which we adapt from [11]. The geometric construction for the covering had to account for the fractional diffusion.

2. **Weak Formulation**

2.1. **Notation.** A constant is called universal, if it only depends on the dimension, the fractional exponent $s$ and $\lambda, \Lambda$ in (1.3)-(1.4). We use the notation $a \lesssim b$ if there exists a universal constant $C$ such that $a \leq Cb$. Moreover, we say that $a \lessgtr b$ if $a \leq Cb$ where $C = C(d)$. For a real number $a$ we denote $a_+ = \max(a, 0)$.

For a given domain $\Omega \subset \mathbb{R}^d$ we denote with $\dot{H}^s(\Omega)$ the space that is equipped with the norm

$$\|f\|_{\dot{H}^s(\Omega)}^2 := \int_{\Omega} \int_{\Omega} \frac{|f(v) - f(w)|^2}{|v - w|^{d+2s}} \, dv \, dw.$$

The space $H^s(\Omega)$ is correspondingly equipped with the norm

$$\|f\|_{H^s(\Omega)}^2 := \|f\|_{\dot{H}^s(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2.$$

The space $H^s_0(\Omega)$ is defined as the closure of the space of smooth functions in $\mathbb{R}^d$ with compact support contained in $\Omega$, where the closure is taken with respect to the $H^s(\Omega)$-norm. We denote the dual of $H^s_0(\Omega)$ with $H^{-s}(\Omega)$. 

2.2. Bilinear Form. We introduce the bilinear form associated to the kernel $K$
\[
\mathcal{E}(\varphi, g) = -\int (\mathcal{L}\varphi)(v)g(v)\,dv
\]
(2.1)
\[
= \lim_{\varepsilon \to 0} \int_{|v-w| > \varepsilon} K(t, x, v, w)[\varphi(v) - \varphi(w)]g(v)\,dw\,dv.
\]
In the remainder, we will abuse notation by ignoring the limit as $\varepsilon \to 0$ and understanding some integrals in the principal value sense. The following theorem states the same result as Corollary 5.2 in [14].

**Theorem 2.1.** Let $K : B_R \times \mathbb{R}^d \to \mathbb{R}^d$ be an non-negative kernel satisfying (1.3), (1.5), (1.6). Then for $f, g \in H^s(\mathbb{R}^d)$ supported in $B_{\frac{r}{2}}$ there holds
\[
\mathcal{E}(f, g) \leq C\|f\|_{H^s(\mathbb{R}^d)}\|g\|_{H^s(\mathbb{R}^d)},
\]
where $C$ depends on $s, d$ and $\Lambda$.

**Proof.** By density of $C^\infty$ in Sobolev spaces it suffices to consider smooth $f, g$. We divide the proof into the symmetric and anti-symmetric part of the kernel: we write
\[
\mathcal{E}(f, g) = \frac{1}{2} \int_{B_R} \int_{\mathbb{R}^d} (f(v) - f(w))(g(v) - g(w))K(v, w)\,dw\,dv
\]
\[
=: I_1(f, g)
\]
\[
+ \frac{1}{2} \int_{B_R} \int_{\mathbb{R}^d} (f(v) - f(w))(g(v) + g(w))K(v, w)\,dw\,dv
\]
\[
=: I_2(f, g)
\]
where the integrals are understood in a principal value sense.

For $I_1$ we first note that it suffices to only consider the quadratic form due to the polarisation identity $I_1(f, g) = \frac{1}{2}(I_1(f+g, f+g) - I_1(f-g, f-g))$. Thus we write as in the proof of Lemma 4.2 of [14] for $h = f + g$ or $h = f - g$.

(2.2)
\[
I_1(h, h) = \sum_{k=-\infty}^{+\infty} P(2^k)
\]
where for $r > 0$
\[
P(r) = \int_{\Sigma_r} |h(v) - h(w)|^2 K(v, w)\,dw\,dv
\]
with $\Sigma_r = \{(v, w) \in B_R \times \mathbb{R}^d : r \leq |v-w| < 2r\}$. Let $m = \frac{d+m}{2}$ and consider an intermediate point $u \in B_{\frac{r}{2}}(m)$. We use $|h(v) - h(w)|^2 \leq 2|h(v) - h(u)|^2 + 2|h(u) - h(w)|^2$ to bound
\[
P(r) \leq \frac{1}{r^d} \int_{\Sigma_r} \int_{B_{\frac{r}{2}}(m)} K(v, w)(|h(v) - h(u)|^2 + |h(u) - h(w)|^2)\,du\,dw\,dv
\]
\[
\leq \frac{1}{r^d} \int_{\Sigma_r} |h(v) - h(u)|^2 \left( \int_{\Omega_{v, u}} K(v, w)\,dw \right)\,du\,dv
\]
\[
+ \frac{1}{r^d} \int_{\Sigma_r} |h(u) - h(w)|^2 \left( \int_{\Omega_{u, w}} K(v, w)\,dv \right)\,du\,dw
\]
where we used Fubini’s theorem and write \( \tilde{\Sigma}_r := \{(v, w) \in B_R \times \mathbb{R}^d : \frac{r}{2} \leq |v - w| < \frac{5r}{4} \} \) and \( \Omega_{v,w} \) for the set containing the \( u \) corresponding to any pair \( (v, w) \). We note that \( \Omega_{v,w} \subset B_{2r}(v) \setminus B_r(v) \).

Using (1.3) we get

\[
P(r) \lesssim \frac{\Lambda}{r^{d+2s}} \left( \int_{\tilde{\Sigma}_r} |h(v) - h(u)|^2 \, du \, dv + \int_{\tilde{\Sigma}_r} |h(u) - h(w)|^2 \, du \, dw \right)
\]

\[
\leq \Lambda \left( \int_{\tilde{\Sigma}_r} \frac{|h(v) - h(u)|^2}{|v - u|^{d+2s}} \, du \, dv + \int_{\tilde{\Sigma}_r} \frac{|h(u) - h(w)|^2}{|u - w|^{d+2s}} \, du \, dw \right)
\]

\[
\lesssim \Lambda \|h\|^2_{H^s}.
\]

We can apply this final estimate to each term in the sum (2.2), use polarisation again and Cauchy-Schwarz inequality to obtain

\[
I_1(f, g) \lesssim \Lambda \|f\|_{H^s(\mathbb{R}^d)} \|g\|_{H^s(\mathbb{R}^d)}.
\]

For \( I_2 \) we distinguish the far and near part for \( 0 < R < \bar{R} \) to be determined below

\[
I_2(f, g) = \int_{B_R} \int_{B_R} \ldots + \int_{B_R} \int_{\mathbb{R}^d \setminus B_R} \ldots + \int_{B_{\bar{R}}} \int_{B_R} \int_{\mathbb{R}^d} \ldots
\]

We rewrite \( I_{21} \) with Fubini’s theorem

\[
I_{21}(f, g) = \int_{B_R} \int_{B_R} \left( K(v, w) - K(w, v) \right) f(v) g(v) \, dv \, dw + \int_{B_R} \int_{B_R} \left( K(v, w) - K(w, v) \right) f(v) g(v) \, dv \, dw
\]

Then for \( I_{21} \) we use the cancellation assumption (1.5) and (1.6). We get

\[
I_{211}(f, g) \lesssim \int_{B_R} \text{PV} \int_{B_R} \left( K(v, w) - K(w, v) \right) \, dw \left| f(v) g(v) \right| dv
\]

\[
\lesssim \Lambda \int_{B_R} f(v) g(v) \, dv
\]

\[
\lesssim \Lambda \|f\|_{L^2(B_R)} \|g\|_{L^2(\mathbb{R}^d)}.
\]

For \( I_{212} \) we use Taylor’s theorem to write

\[
I_{212}(f, g) = \int_{B_R} \int_{B_R} \left( K(v, w) - K(w, v) \right) f(v) g(v) + \nabla g(v) \cdot (w - v) + D^2 g(u) \frac{(w - v)^2}{2} \, dv \, dw
\]
for some \( u \) between \( v \) and \( w \). We can bound the first term as above in (2.4). For the second term we use (1.6)

\[
\int_{B_R} \left| \int_{B_R} (K(v, w) - K(w, v))(w - v) \, dv \right| f(v) \nabla g(v) \, dv
\]

\[
\leq \lambda R^{1-2s} \int_{B_R} f(v) \nabla g(v) \, dv
\]

\[
\leq \lambda R^{1-2s} \| f \|_{L^2(B_R)} \| \nabla g \|_{L^2(B_R)}
\]

\[
\leq \lambda R^{1-2s} \| f \|_{L^2(B_R)} \| g \|_{L^2(B_R)} \| \nabla g \|_{L^2(B_R)}
\]

\[
\leq \| f \|_{L^2(B_R)} \| g \|_{L^2(B_R)} \| g \|_{H^2(B_R)}.
\]

We used Corollary 2.5 in [17] and chose \( R^2 = \frac{|g|_{L^2}}{|D^2g|_{L^2}} \). With this choice of \( R \) we get for the last term

\[
\int_{B_R} \left| \int_{B_R} (K(v, w) - K(w, v))(w - v)^2 \, dv \right| f(v) D^2 g(u) \, dv
\]

\[
\leq \lambda R^{2-2s} \| f \|_{L^2} \| D^2 g \|_{L^2}
\]

\[
\leq \lambda \| f \|_{L^2(B_R)} \| g \|_{L^2(B_R)} \| g \|_{H^2(B_R)}.
\]

Hence we obtain

\[ (2.5) \quad I_{21}(f, g) \leq \lambda \| f \|_{L^2(B_R)} \| g \|_{L^2(B_R)} + \| f \|_{L^2(B_R)} \| g \|_{L^2(B_R)} \| g \|_{H^2(B_R)}. \]

For the far part \( I_{22} \) we have for \( u \) between \( v, w \)

\[
I_{22}(f, g) = \int_{B_R} \int_{\mathbb{R}^d \setminus B_R} (f(v) - f(w))(g(v) + g(w)) K(v, w) \, dv \, dw
\]

\[
= \int_{B_R} \int_{\mathbb{R}^d \setminus B_R} (f(v) - f(w)) g(v) K(v, w) \, dv \, dw
\]

\[
+ \int_{B_R} \int_{\mathbb{R}^d \setminus B_R} (f(v) - f(w))(g(v) + \nabla g(v) \cdot (w - v) + D^2 g(u) \frac{(w - v)^2}{2}) K(v, w) \, dv \, dw
\]

\[
\leq \lambda \int_{B_R} \left( \int_{\mathbb{R}^d \setminus B_R} (f(v) - f(w))^2 K(v, w) \, dv \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d \setminus B_R} (w - v)^2 K(v, w) \, dv \right)^{\frac{1}{2}} g(v) \, dv
\]

\[
+ \lambda \int_{B_R} \left( \int_{\mathbb{R}^d \setminus B_R} (f(v) - f(w))^2 K(v, w) \, dv \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d \setminus B_R} (w - v)^2 K(v, w) \, dv \right)^{\frac{1}{2}} \nabla g(v) \, dv
\]

\[
+ \lambda \int_{B_R} \left( \int_{\mathbb{R}^d \setminus B_R} (f(v) - f(w))^2 K(v, w) \, dv \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d \setminus B_R} (w - v)^2 K(v, w) \, dv \right)^{\frac{1}{2}} D^2 g(u) \, dv
\]

\[
\leq \lambda R^{-s} \| f \|_{H^s} \| g \|_{L^2} + \lambda R^{1-s} \| f \|_{H^s} \| \nabla g \|_{L^2} + \lambda R^{2-s} \| f \|_{H^s} \| D^2 g \|_{L^2}
\]

\[
\leq \lambda \| f \|_{H^s} \| g \|_{L^2} \| g \|_{H^2}.
\]

We used Taylor’s theorem, the same proof as for the symmetric part to deduce the third inequality, repeatedly the Cauchy Schwarz inequality, the upper bound (1.3), Corollary 2.5 in [17] and the
choice of $R$ as above. Finally for $I_{23}$ we can use the exact same estimates as for $I_{22}$ when we first apply Fubini’s theorem and expand $g(v)$ instead of $g(w)$:

\[
I_{23}(f, g) = \text{PV} \int_{\mathbb{R}^4} \int_{B_R \setminus B_R} (f(v) - f(w))(g(v) + g(w))K(v, w) \, dw \, dv \\
= \text{PV} \int_{\mathbb{R}^4} \int_{B_R \setminus B_R} (f(v) - f(w))(g(v) + g(w))K(v, w) \, dv \, dw \\
= \text{PV} \int_{\mathbb{R}^4} \int_{B_R \setminus B_R} (f(v) - f(w))g(w)K(v, w) \, dv \, dw \\
+ \text{PV} \int_{\mathbb{R}^4} \int_{B_R \setminus B_R} (f(v) - f(w))(g(w) + \nabla g(w) \cdot (v - w) + D^2 g(u) \frac{(v - w)^2}{2})K(v, w) \, dv \, dw \\
\lesssim \text{PV} \int_{\mathbb{R}^4} \left( \int_{B_R \setminus B_R} (f(v) - f(w))^2 K(v, w) \, dv \right)^{\frac{1}{2}} \left( \int_{B_R \setminus B_R} K(v, w) \, dv \right)^{\frac{1}{2}} g(w) \, dw \\
+ \text{PV} \int_{\mathbb{R}^4} \left( \int_{B_R \setminus B_R} (f(v) - f(w))^2 K(v, w) \, dv \right)^{\frac{1}{2}} \left( \int_{B_R \setminus B_R} (v - w)^2 K(v, w) \, dv \right)^{\frac{1}{2}} \nabla g(w) \, dw \\
+ \text{PV} \int_{\mathbb{R}^4} \left( \int_{B_R \setminus B_R} (f(v) - f(w))^2 K(v, w) \, dv \right)^{\frac{1}{2}} \left( \int_{B_R \setminus B_R} (v - w)^4 K(v, w) \, dv \right)^{\frac{1}{2}} D^2 g(u) \, dw \\
\lesssim \Lambda R^{-s} \|f\|_{H^s} \|g\|_{L^2} + \Lambda R^{1-s} \|f\|_{H^s} \|
abla g\|_{L^2} + \Lambda R^{2-s} \|f\|_{H^s} \|D^2 g\|_{L^2} \\
\leq \Lambda R^{-s} \|f\|_{H^s} \|g\|_{L^2} + \Lambda R^{1-s} \|f\|_{H^s} \|g\|_{L^2} \frac{2}{1} \|D^2 g\|_{L^2} + \Lambda R^{2-s} \|f\|_{H^s} \|D^2 g\|_{L^2} \\
\leq \Lambda \|f\|_{H^s} \|g\|_{L^2} \frac{1}{2} \|g\|_{H^{s}}^2.
\]

Now we use Littlewood-Paley theory inspired from the proof of Theorem 4.1 in [14]. We denote with $\Delta_i$ the Littlewood-Paley projectors. We decompose $f = \sum_{i=0}^{\infty} \Delta_i f$, where we use the convention that all low modes are contained in $\Delta_0$ so that the index $i \geq 0$. Note that for $s \geq 0$

\[
\|\Delta_i f\|_{H^s} \approx 2^{si} \|\Delta_i f\|_{L^2}.
\]

Moreover, we bound as in [14] (for a justification see [1])

\[
\|\Delta_i g\|_{L^2}^{1-s} \|\Delta_i g\|_{H^2} \lesssim 2^{si} \|\Delta_i g\|_{H^s}.
\]

and

\[
\|\Delta_i g\|_{L^2}^{1-s} \|\Delta_i g\|_{H^2} \lesssim \|\Delta_i g\|_{H^s}.
\]
Then
\[ I_2(f, g) = \sum_{i,j} I_2(\Delta_i f, \Delta_j g) \]
\[
\lesssim \sum_{j \leq i} \left( \| \Delta_i f \|_{L^2} \| \Delta_j g \|_{L^{2^{-s}}} \right) + \sum_{i \leq j} \left( \| \Delta_j g \|_{L^2} \| \Delta_i f \|_{L^{2^{-s}}} \right)
\]
\[
\lesssim \sum_{i,j} \left( 2^{-s(i-j)} \| \Delta_i f \|_{H^s} \| \Delta_j g \|_{H^s} + \| \Delta_i f \|_{H^s} \| \Delta_j g \|_{H^s} \right) + \| f \|_{L^2} \| g \|_{L^2}
\]
(2.6)
\[
\lesssim \sum_{k=0}^{\infty} 2^{-sk} \left( \sum_{i=0}^{\infty} \| \Delta_i f \|_{H^s} \| \Delta_{i+k} g \|_{H^s} + \| \Delta_{i+k} f \|_{H^s} \| \Delta_i g \|_{H^s} \right) + \| f \|_{H^s} \| g \|_{H^s}
\]
\[
\lesssim \sum_{k=0}^{\infty} 2^{1-sk} \left( \sum_{i=0}^{\infty} \| \Delta_i f \|_{H^s} \right) + \left( \sum_{i=0}^{\infty} \| \Delta_i g \|_{H^s} \right) + \| f \|_{H^s} \| g \|_{H^s} + \| f \|_{L^2} \| g \|_{L^2}
\]
\[
\lesssim \| f \|_{H^s} \| g \|_{H^s}.
\]

Note that splitting the sum into \( j \leq i \) and \( i < j \) works since the adjoints of the corresponding integral operators satisfy the same bounds as in (2.5). Thus we conclude with (2.3) and (2.6). \( \Box \)

To motivate the following definition of weak solutions, we recall Lemma 5.6 of [14], which states that the operator \( L \) is bounded from the space \( L^\infty(\mathbb{R}^d \setminus \bar{B}_R) + H^s(\mathbb{R}^d) \) to \( H^{-s}(B_R) \), where the former space is equipped with the norm
\[
\| f \|_{L^\infty(\mathbb{R}^d \setminus \bar{B}_R) + H^s(\mathbb{R}^d)} = \inf \left\{ \| f_1 \|_{L^\infty(\mathbb{R}^d \setminus \bar{B}_R)} + \| f_2 \|_{H^s(\mathbb{R}^d)} : f = f_1 + f_2, \ f_1 = 0 \text{ in } B_{\frac{R}{4}} \right\}.
\]

**Lemma 2.2.** Let \( \varphi \in H^s(\mathbb{R}^d) \) with \( \text{supp } \varphi \subset B_{\frac{R}{2}} \). Let \( K \) be a kernel with (1.3), (1.5), (1.6). Then for \( f \in L^\infty(\mathbb{R}^d \setminus \bar{B}_{\frac{R}{2}}) + H^s(\mathbb{R}^d) \) there exists \( C \) depending on \( s, d, \Lambda \) and \( \text{supp } \varphi \) so that
\[
\mathcal{E}(f, \varphi) \leq C \| f \|_{L^\infty(\mathbb{R}^d \setminus \bar{B}_{\frac{R}{2}}) + H^s(\mathbb{R}^d)} \| \varphi \|_{H^s(\mathbb{R}^d)}.
\]

**Proof.** The proof comes from Lemma 5.6 in [14]. It suffices to consider smooth \( f, \varphi \) just as above. We write \( f = f_1 + f_2 \) with \( f_1 \) and \( f_2 \) as in the definition of the norm for the space \( L^\infty(\mathbb{R}^d \setminus \bar{B}_{\frac{R}{2}}) + H^s(\mathbb{R}^d) \). By Theorem 2.1 we have \( \mathcal{E}(f_2, \varphi) \leq C \| f_2 \|_{H^s(\mathbb{R}^d)} \| \varphi \|_{H^s(\mathbb{R}^d)} \). For \( \mathcal{E}(f_1, \varphi) \) we get
\[
\mathcal{E}(f_1, \varphi) = \lim_{\varepsilon \to 0} \int_{|w-v| > \varepsilon} \int_{\mathbb{R}^d} \left[ f_1(w) - f_1(v) \right] \varphi(v) K(v, w) \, dw \, dv
\]
\[
= \lim_{\varepsilon \to 0} \int_{\text{supp } \varphi} \int_{\mathbb{R}^d \setminus B_{\varepsilon}(v)} f_1(w) K(v, w) \, dw \varphi(v) \, dv
\]
\[
= \int_{\text{supp } \varphi} \int_{\mathbb{R}^d \setminus B_{\varepsilon}(v)} f_1(w) K(v, w) \, dw \varphi(v) \, dv
\]
where $\delta = \text{dist} (\text{supp } \varphi, \mathbb{R}^d \setminus B_{\hat{R}})$. Using the upper bound on the kernel, we find

$\mathcal{E}(f_1, \varphi) \leq \Lambda \delta^{-2s} \| f_1 \|_{L^\infty(\mathbb{R}^d \setminus B_{\hat{R}})} \int_{\text{supp } \varphi} \varphi(v) \, dv \leq C \| f_1 \|_{L^\infty(\mathbb{R}^d \setminus B_{\hat{R}})} \| \varphi \|_{H^s}.$

\[ \square \]

**Definition 2.3 (Weak Solutions).** Assume $K$ satisfies (1.3), (1.5), (1.6). We say that $f : (-(\frac{\delta}{2})^{2s}, 0) \times B_{\hat{R}} \to \mathbb{R}$ is a weak sub-solution of (1.1)-(1.2) in $Q_{\hat{R}} = I \times \Omega_x \times \Omega_v$ if

(i) $f \in C^0(I, L^2(\Omega_x \times \Omega_v)) \cap L^2(I \times \Omega_x, L^\infty(\mathbb{R}^d \setminus \Omega_x) + H^s(\mathbb{R}^d))$

(ii) $Tf \in L^2(I \times \Omega_x, H^{-s}(\Omega_v))$

(iii) for all non-negative $\varphi \in L^2(I \times \Omega_x, H^s(\mathbb{R}^d))$ such that for every $t, x$ the support of $\varphi(t, x, \cdot)$ is compactly contained in $\Omega_v$ there holds

$$\int_{Q_{\hat{R}}} (Tf) \varphi \, dz + \int_{\Omega_v} \int_I \mathcal{E}(f, \varphi) \, dt \, dx - \int_{Q_{\hat{R}}} h \varphi \, dz \leq 0.$$

A function $f$ is a super-solution of (1.1)-(1.2) in $Q_{\hat{R}}$ if $-f$ is a sub-solution in $Q_{\hat{R}}$. It is a solution if it is a sub- and super-solution.

We conclude this section with the following useful bound, where the integrals are understood in the principal value sense.

**Proposition 2.4.** Let $v \in B_R$ and assume $K$ is a non-negative kernel satisfying (1.3). Then for $r > 0$ there holds

$$PV \int_{B_r(v)} (f(w) - f(v)) K(v, w) \, dw \leq \Lambda \left( PV \int_{B_r(v)} \frac{|f(w) - f(v)|}{|v - w|^{d+2s}} \, dw \right).$$

**Proof.** Consider the set $\Sigma_R := \{ w \in B_r(v) : R < |v - w| < 2R \} \subset B_r(v) \cap B_{2R}(v) \setminus B_R(v)$. Then we write

$$\int_{B_r(v)} (f(w) - f(v)) K(v, w) \, dw = \sum_{k=\infty}^{\infty} \int_{\Sigma_{2k}} (f(w) - f(v)) K(v, w) \, dw.$$

(2.7)
For \( w \in \Sigma_R \), choose \( \varepsilon > 0 \) such that \( B_\varepsilon (w) \subset \Sigma_R \). We consider \( u \in B_\varepsilon (w) \) and estimate with (1.3)

\[
\int_{\Sigma_R} \left( f(w) - f(v) \right) K(v, w) \, dw \\
\leq \int_{\Sigma_R} |f(w) - f(u)| K(v, w) \, dw + \int_{\Sigma_R} |f(u) - f(v)| K(v, w) \, dw \\
\leq \frac{1}{R^d} \int_{\Sigma_R} \int_{B_R \setminus B_R(w)} |f(w) - f(u)| K(v, w) \, dw \, dw + \frac{\Lambda}{R^{2s}} |f(u) - f(v)| \\
\leq \frac{\Lambda}{R^{2s+d}} \int_{\Sigma_R} |f(w) - f(u)| \, dw + \frac{\Lambda}{R^{2s+d}} \int_{\Sigma_R} |f(u) - f(v)| \, dw \\
\leq \frac{\Lambda}{R^{2s+d}} \int_{\Sigma_R} |f(w) - f(v)| \, dw + \frac{2\Lambda}{R^{2s+d}} \int_{\Sigma_R} |f(u) - f(v)| \, dw \\
\leq \frac{\Lambda}{R^{2s+d}} \int_{\Sigma_R} |f(w) - f(v)| \, dw + \frac{2\Lambda}{R^{2s+d}} \int_{\Sigma_R} |f(w) - f(v)| \, dw \\
\lesssim \Lambda \int_{\Sigma_R} \frac{|f(w) - f(v)|}{|v - w|^{d+2s}} \, dw.
\]

We repeatedly use the non-negativity of the integrands. We can use this estimate for each summand in (2.7) to conclude. \( \square \)

3. Integral Estimates

3.1. Kolmogorov’s fundamental solutions. In this subsection, we consider the fractional Kolmogorov equation given by

\[
(3.1) \quad \partial_t f + v \cdot \nabla_x f + (-\Delta)^s f = h - m,
\]

for some \( h \in L^2([\tau, 0] \times \mathbb{R}^d, H^{-s}(\mathbb{R}^d)) \) and some non-negative measure \( m \in M^1([\tau, 0] \times \mathbb{R}^d) \) with finite mass. Then there exists \( h_1, h_2 \in L^2([\tau, 0] \times \mathbb{R}^d) \) so that \( h = h_1 + (-\Delta)^{\frac{s}{2}} h_2 \) and

\[
\|h\|_{L^2([\tau, 0] \times \mathbb{R}^d, H^{-s}(\mathbb{R}^d))} \approx \|h_1\|_{L^2([\tau, 0] \times \mathbb{R}^d)} + \|h_2\|_{L^2([\tau, 0] \times \mathbb{R}^d)}.
\]

Proposition 3.1. Let \( 0 \leq f \) solve (3.1) in \([\tau, 0] \times \mathbb{R}^d\) for \( h = h_1 + (-\Delta)^{\frac{s}{2}} h_2 \) with \( h_1, h_2 \in L^1 \cap L^2([\tau, 0] \times \mathbb{R}^d) \), \( 0 \leq m \in M^1([\tau, 0] \times \mathbb{R}^d) \) and with \( f(0, x, v) = f_0(x, v) \in L^1 \cap L^2(\mathbb{R}^{2d}) \).

Then

\[
(3.2) \quad \|f\|_{L^p([\tau, 0] \times \mathbb{R}^{2d})} \lesssim \|h_1\|_{L^2([\tau, 0] \times \mathbb{R}^d)} + \|h_2\|_{L^2([\tau, 0] \times \mathbb{R}^d)} + \|f_0\|_{L^2(\mathbb{R}^{2d})},
\]

where \( \frac{1}{p} > \frac{1}{p^*} - \frac{1}{2} \) for \( p^* = \frac{2d(1+s)+2s}{2d(1+s)+d} \in (1, 2) \). Moreover, for \( s \in \left[ 0, \frac{s}{2(d+s)} \right) \)

\[
(3.3) \quad \|f\|_{L^1_{t,x} W^{s,1}_x([\tau, 0] \times \mathbb{R}^{2d})} \lesssim \|h_1\|_{L^1([\tau, 0] \times \mathbb{R}^d)} + \|h_2\|_{L^1([\tau, 0] \times \mathbb{R}^d)} + \|m\|_{M^1([\tau, 0] \times \mathbb{R}^d)} + \|f_0\|_{L^1(\mathbb{R}^{2d})}.
\]
We remark that the modified convolution satisfies the usual Young inequality independent of $r$

We define as in Section 2.4 of [14] the modified convolution

Equation (3.1) admits a fundamental solution, see for example Theorem 1.1 in [24] or Section 2.4 in [14], given by

$$f(t, x, v) = \int_{\mathbb{R}^{2d+1}} [h(t', x', v') - m(t', x', v')] J(t - t', x - x' - (t - t')v', v - v') \, dt' \, dx' \, dv'$$

$$+ \int_{\mathbb{R}^{2d}} f_0(x', v') J(t, x - x' - (t - t')v', v - v') \, dx' \, dv',$$

where $(t, x, v) \in (0, \tau) \times \mathbb{R}^{2d}$ and

$$J(t, x, v) = \frac{C_d}{t^{d+1}} \hat{J} \left( \frac{x}{t^{1/2}}, \frac{v}{t^{1/2}} \right),$$

$$\hat{J} (\varphi, \xi) = \exp \left( \int_0^t |\xi - \tau \varphi|^2 \, d\tau \right).$$

Since $f$ and $J$ are non-negative, we deduce that

$$0 \leq f(t, x, v) \leq \int_{\mathbb{R}^{2d+1}} h(t', x', v') J(t - t', x - x' - (t - t')v', v - v') \, dt' \, dx' \, dv'$$

$$+ \int_{\mathbb{R}^{2d}} f_0(x', v') J(t, x - x' - (t - t')v', v - v') \, dx' \, dv'.$$

We remark that for any $r \geq 1$ there holds for $t > 0$

$$\| J(t, \cdot, \cdot) \|_{L^r(\mathbb{R}^{2d})} = t^{-d(1 + \frac{1}{r})(1 - \frac{1}{p})} \| J \|_{L^r(\mathbb{R}^{2d})},$$

$$\| (-\Delta)^{\frac{r}{2}} J(t, \cdot, \cdot) \|_{L^r(\mathbb{R}^{2d})} = t^{-d(1 + \frac{1}{r})(1 - \frac{1}{p}) - \frac{1}{2}} \| (-\Delta)^{\frac{r}{2}} J \|_{L^r(\mathbb{R}^{2d})}.$$

In particular for $r = p^*$ we deduce

$$\| J(t, \cdot, \cdot) \|_{L^{p^*}(\mathbb{R}^{2d})} = t^{\frac{1}{2} - \frac{d}{p^*}} \| J \|_{L^{p^*}(\mathbb{R}^{2d})},$$

$$\| (-\Delta)^{\frac{r}{2}} J(t, \cdot, \cdot) \|_{L^{p^*}(\mathbb{R}^{2d})} = t^{- \frac{d}{p^*}} \| (-\Delta)^{\frac{r}{2}} J \|_{L^{p^*}(\mathbb{R}^{2d})}.$$

We define as in Section 2.4 of [14] the modified convolution

$$f \ast_t g(x, v) := \int_{\mathbb{R}^{2d}} f(x', v') g(x - x' - tv', v - v') \, dx' \, dv'.$$

We remark that the modified convolution satisfies the usual Young inequality independent of $t$:

$$\| f \ast_t g \|_{L^{p_{t,v}}} \leq \| f \|_{L^{p_{t,v}}} \| g \|_{L^{q_{t,v}}}$$

for $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Following the proof of Proposition 2.2 in [14], we split $f = \tilde{f}_0 + f_1 + f_2$ with

$$\tilde{f}_0(t, \cdot, \cdot) := f_0 \ast_t J(t, \cdot, \cdot),$$

$$f_1(t, \cdot, \cdot) := \int_0^t h_1 \ast_{t-t'} J(t - t', \cdot, \cdot) \, dt',$$

$$f_2(t, \cdot, \cdot) := \int_0^t h_2 \ast_{t-t'} (-\Delta)^{\frac{r}{2}} J(t - t', \cdot, \cdot) \, dt'.$$
Let $q \in [1, p^*)$ be such that $\frac{1}{p} = \frac{1}{q} - \frac{1}{2}$. By Young’s inequality we get for $\alpha = d(1 + \frac{1}{s})(1 - \frac{1}{q}) + \frac{1}{2} < \frac{1}{p^*} < \frac{1}{q}$

$$
\| \tilde{f}_0(t, \cdot, \cdot) \|_{L^p(\mathbb{R}^{2d})} \leq \| f_0 \|_{L^2(\mathbb{R}^{2d})} \| J(t, \cdot, \cdot) \|_{L^q(\mathbb{R}^{2d})} = \| f_0 \|_{L^2(\mathbb{R}^{2d})} \| \mathcal{J} \|_{L^q(\mathbb{R}^{2d})} t^{-\alpha} 
$$

$$
\| f_1(t, \cdot, \cdot) \|_{L^p(\mathbb{R}^{2d})} \leq \int_0^t \| h_1(t') \|_{L^2(\mathbb{R}^{2d})} \| J(t - t', \cdot, \cdot) \|_{L^q(\mathbb{R}^{2d})} \, dt' 
= \int_0^t \| h_1(t') \|_{L^2(\mathbb{R}^{2d})} \| \mathcal{J} \|_{L^q(\mathbb{R}^{2d})} (t - t')^{-\alpha} \, dt' 
$$

$$
\| f_2(t, \cdot, \cdot) \|_{L^p(\mathbb{R}^{2d})} \leq \int_0^t \| h_2(t') \|_{L^2(\mathbb{R}^{2d})} \| (\Delta)^{\frac{\tau}{2}} J(t - t', \cdot, \cdot) \|_{L^q(\mathbb{R}^{2d})} \, dt' 
= \int_0^t \| h_2(t') \|_{L^2(\mathbb{R}^{2d})} \| (\Delta)^{\frac{\tau}{2}} \mathcal{J} \|_{L^q(\mathbb{R}^{2d})} (t - t')^{-\alpha} \, dt' 
$$

Since $p(\frac{1}{2} - \alpha) > -1$ we get that $\tilde{f}_0 \in L^p([-\tau, 0] \times \mathbb{R}^{2d})$ and

$$
\| \tilde{f}_0 \|_{L^p([-\tau, 0] \times \mathbb{R}^{2d})} \leq C \| f_0 \|_{L^2(\mathbb{R}^{2d})},
$$

For $f_1$ and $f_2$ we apply Young’s inequality again and get

$$
\| f_1 \|_{L^p([-\tau, 0] \times \mathbb{R}^{2d})} \leq C \| h_1 \|_{L^2([-\tau, 0] \times \mathbb{R}^{2d})} \| \mathcal{J} \|_{L^q(\mathbb{R}^{2d})} t^{-\alpha + \frac{\tau}{2}},
$$

$$
\| f_2 \|_{L^p([-\tau, 0] \times \mathbb{R}^{2d})} \leq C \| h_2 \|_{L^2([-\tau, 0] \times \mathbb{R}^{2d})} \| \mathcal{J} \|_{L^q(\mathbb{R}^{2d})} t^{-\alpha}.
$$

This implies (3.2). To prove (3.3) we follow the idea of Lemma 10 in [11]. We split for $(t, x, v) \in (0, \tau) \times \mathbb{R}^{2d}$

$$
J = J_\varepsilon + J_\varepsilon' \quad \text{with} \quad J_\varepsilon(t, x, v) := \eta\left(\frac{t}{\varepsilon}\right) J(t, x, v)
$$

where $\varepsilon > 0$ and $\eta$ is a smooth function on $\mathbb{R}_+$ such that $0 \leq \eta \leq 1$, equal to 1 in $[0, 1]$ and 0 on $[2, +\infty)$ (we assume without loss of generality $\tau \geq 1$). Then we estimate for $l \in \mathbb{N}$

$$
|\nabla_x^l J_\varepsilon(t, x, v)| \lesssim \varepsilon^{-(d+\frac{3}{2})} e^{-l(1+\frac{1}{\varepsilon})} |\nabla_y^l J_\varepsilon(y, w)|,
$$

$$
|(-\Delta)^{\frac{\tau}{2}} \nabla_x^l J_\varepsilon(t, x, v)| + t|\nabla_x \nabla_x^l J_\varepsilon| \lesssim \varepsilon^{-(d+\frac{3}{2})} e^{-l(1+\frac{1}{\varepsilon})} \left[\varepsilon^{\frac{1}{2}} |\nabla_y \nabla_x^l J_\varepsilon(y, w)| + e^{-\frac{1}{2} l} |(-\Delta)^{\frac{\tau}{2}} \nabla_y^l J_\varepsilon(y, w)|\right] \lesssim \varepsilon^{-(d+\frac{3}{2})} e^{-l(1+\frac{1}{\varepsilon}) - \frac{1}{2}} \left[|\nabla_y \nabla_x^l J_\varepsilon(y, w)| + |(-\Delta)^{\frac{\tau}{2}} \nabla_y^l J_\varepsilon(y, w)|\right].
$$

Now assuming $\varepsilon < 1$ these estimates yield

$$
\| J_\varepsilon \|_{L^1_t([-\tau, 0] \times \mathbb{R}^{2d}; W^{1,1}_x(\mathbb{R}^d))} + \|(-\Delta)^{\frac{\tau}{2}} J_\varepsilon \|_{L^1_t([-\tau, 0] \times \mathbb{R}^{2d}; W^{1,1}_x(\mathbb{R}^d))} + \| t \nabla_x J_\varepsilon \|_{L^1_t([-\tau, 0] \times \mathbb{R}^{2d}; W^{1,1}_x(\mathbb{R}^d))} \lesssim \tau \varepsilon^{-l(1+\frac{1}{\varepsilon}) - \frac{1}{2}},
$$

$$
\| J_\varepsilon \|_{L^1_t([-\tau, 0] \times \mathbb{R}^{2d})} + \|(-\Delta)^{\frac{\tau}{2}} J_\varepsilon \|_{L^1_t([-\tau, 0] \times \mathbb{R}^{2d})} + \| t \nabla_x J_\varepsilon \|_{L^1_t([-\tau, 0] \times \mathbb{R}^{2d})} \lesssim \tau \varepsilon^{\frac{1}{2}}.
$$
The splitting on $J$ yields a splitting on the solution $f = f_x + f^\perp_x$. Young’s convolution inequality and the convolution inequality on $M^1 \ast L^1 \to L^1$ imply

$$
\|f^\perp_x\|_{L^1_t([\tau,0] \times \mathbb{R}^d)} \lesssim \tau \varepsilon^{-(1+\frac{1}{2})-\frac{1}{2}} (\|h_1\|_{L^1([\tau,0] \times \mathbb{R}^d)} + \|h_2\|_{L^1([\tau,0] \times \mathbb{R}^d)} + \|m\|_{M^1([\tau,0] \times \mathbb{R}^d)} + \|f_0\|_{L^1(\mathbb{R}^d)}),
$$

$$
\|f_x\|_{L^1([\tau,0] \times \mathbb{R}^d)} \lesssim \tau \varepsilon^{\frac{1}{2}} (\|h_1\|_{L^1([\tau,0] \times \mathbb{R}^d)} + \|h_2\|_{L^1([\tau,0] \times \mathbb{R}^d)} + \|m\|_{L^1([\tau,0] \times \mathbb{R}^d)} + \|f_0\|_{L^1(\mathbb{R}^d)}).
$$

The decomposition above holds for all $\varepsilon > 0$; thus we can conclude the proof with the same justification as in the proof of Lemma 10 in [11]: Let $\sigma \in [0, \frac{\varepsilon}{1+2d}])$. Using the notation $\langle \zeta \rangle := (1 + |\zeta|^2)^{\frac{1}{2}}$ we can decompose

$$
(1 - \Delta_x)\frac{x}{\varepsilon} f(t, x, v) = \int_{\mathbb{R}^d} e^{i\langle x - \varphi \rangle} \langle \zeta \rangle^\sigma f(t, \varphi, v) d\varphi d\zeta
$$

(3.4)

$$
= \sum_{k \geq -1} \int_{\mathbb{R}^d} e^{i\langle x - \varphi \rangle a_k(\zeta)} f(t, \varphi, v) d\varphi d\zeta
$$

$$
+ \sum_{k \geq -1} \int_{\mathbb{R}^d} e^{i\langle x - \varphi \rangle b_k(\zeta)(1 - \Delta_x)\frac{x}{\varepsilon}} f(t, \varphi, v) d\varphi d\zeta,
$$

where $a_k(\zeta) := \langle \zeta \rangle^\sigma \phi_k$ and $b_k(\zeta) := \langle \zeta \rangle^{\sigma - 1} \phi_k$ with $\phi_k(\zeta) := [\eta(2^{-k} \zeta) - \eta(2^{-k+1} \zeta)]$ for $k \geq 0$ where $\eta$ is a smooth function such that $0 \leq \eta \leq 1$ and $\eta = 1$ in $B_1(0)$ and 0 outside $B_2(0)$. For $k = -1$ we define $\phi_{-1}(\zeta) := \sum_{k \geq -1} \eta(2^{-k} \zeta) - \eta(2^{-k+1} \zeta)]$. Note that for a given function $F = F(\varphi)$ there holds

$$
\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i\langle x - \varphi \rangle a_k(\zeta)} F(y) d\varphi d\zeta \right| dx \lesssim 2^{k \sigma} \|F\|_{L^1},
$$

$$
\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i\langle x - \varphi \rangle b_k(\zeta)(1 - \Delta_x)\frac{x}{\varepsilon}} F(y) d\varphi d\zeta \right| dx \lesssim 2^{k(\sigma - 1)} \|F\|_{W^{1,1}}.
$$

Therefore we find for the solution $f$ to the fractional Kolmogorov equation

$$
\|f\|_{L^1_t([\tau,0] \times \mathbb{R}^d)} = \|(1 - \Delta_x)\frac{x}{\varepsilon} f\|_{L^1([\tau,0] \times \mathbb{R}^d)}
$$

$$
= \int_{[\tau,0] \times \mathbb{R}^d} \left| \sum_{k \geq -1} \int_{\mathbb{R}^d} e^{i\langle x - \varphi \rangle a_k(\zeta)} f_x(t, \varphi, v) d\varphi d\zeta + \int_{\mathbb{R}^d} e^{i\langle x - \varphi \rangle b_k(\zeta)(1 - \Delta_x)\frac{x}{\varepsilon}} f^\perp_x(t, \varphi, v) d\varphi d\zeta \right| dt dv
$$

$$
\lesssim \sum_{k \geq -1} \tau \left( 2^{k \sigma} \varepsilon^{\frac{1}{2}} + 2^{(k(\sigma - 1))} \varepsilon^{-(1+\frac{1}{2})-\frac{1}{2}} \right) \left( \|h_1\|_{L^1} + \|h_2\|_{L^1} + \|f_0\|_{L^1} + \|m\|_{M^1} \right)
$$

$$
\lesssim \tau \left( \|h_1\|_{L^1([\tau,0] \times \mathbb{R}^d)} + \|h_2\|_{L^1([\tau,0] \times \mathbb{R}^d)} + \|f_0\|_{L^1(\mathbb{R}^d)} + \|m\|_{M^1([-\tau,0] \times \mathbb{R}^d)} \right)
$$

where we choose $\sigma = \frac{\varepsilon}{2(2s+1)} - \delta \in [0, \frac{\varepsilon}{2(2s+1)})$, $\varepsilon = 2^{-2k\delta(\frac{2}{2s+1} - \frac{1}{2})}$ and $l > \frac{3\delta(2s+1)+4\delta}{(2s+1)(2s+1)+2-2s}$ for some small $\delta > 0$. This concludes the proof.

\[ \square \]
3.2. Energy estimates. The following two lemmas are the analogue of Proposition 9 and Proposition 11 in [11] respectively. The proofs are technically more involved due to the non-locality.

Lemma 3.2 (Local energy estimate). Let  be a non-negative sub-solution to (1.1)-(1.2) in for some  where 0 ≤ K satisfies (1.3)-(1.6). Let  for some  with 0 < r < R ≤ be given. Assume 0 ≤ f ≤ 1 a.e. in (−R²s + t₀, t₀] × B₀ × R²s. Then there holds

\[
\sup_{\tau \in (-r² + t₀, t₀)} \int_{Q_{r}(z₀)} f²(τ, x, v) \, dx \, dv + \int_{Q_r(z₀)} \int_{B_r(z₀)} \frac{|f(w) - f(v)|²}{|v - w|^{d+2s}} \, dw \, dz \\
\lesssim \lambda C(s, R, v₀) \int_{Q_R(z₀)} f² \, dz + \int_{Q_R(z₀)} h² \, dz
\]

where  = (t₀, v₀, z₀),  = \{(x, v) ∈ R²d : (τ, x, v) ∈ Merr(z₀)\} ⊂ B₀ × R²s and

\[
C(s, R, v₀) := \left(1 + \frac{1}{R²s - r²} + \frac{|v₀| + R}{(R²s - r²)²} + \frac{1}{R²s}\right).
\]

Proof. Let be a smooth function such that 0 ≤ φ ≤ 1 equal to 1 in  and 0 outside . We integrate (1.1) against  ≥ 0 up to time  \[
\int_{(t ≤ τ) × R²d} (T f) fφ² \, dz
\]

\[
\leq \int_{(t ≤ τ) × R²d} \int_{R²d} K(v, w)[f(w) - f(v)](fφ²)(v) \, dw \, dz + \int_{(t ≤ τ) × R²d} hfφ² \, dz
\]

\[
= \int_{(t ≤ τ) × Q₀^d(z₀) \setminus B₀(v₀)} \int_{R²d \setminus B₀(v₀)} \ldots + \int_{(t ≤ τ) × B₀ \setminus B₀(z₀)} \int_{R²d} \ldots =: I₁ + I₂
\]

\[
+ \int_{(t ≤ τ) × Q₀^d(z₀) \setminus B₀(v₀)} \int_{R²d} \ldots + \int_{(t ≤ τ) × R²d} hfφ² \, dz =: I₃.
\]

First, we note that by choice of φ we have that  = 0. Let us now deal with  = 0. We will split it into three parts

\[
I₃ = \int_{(t ≤ τ) × Q₀^d(z₀) \setminus B₀(v₀)} \int_{B₀(v₀)} \ldots + \int_{(t ≤ τ) × B₀ \setminus B₀(v₀)} \int_{B₀(v₀)} \ldots =: I₁ + I₂
\]

\[
+ \int_{(t ≤ τ) × Q₀^d(z₀) \setminus B₀(v₀)} \int_{B₀(v₀)} \ldots \]

where  = 0.
Then for $I_{31}$ we get with Proposition 2.4

$$I_{31} = \int_{\{t \leq \tau\} \times Q_\lambda'(z_0)} \text{PV} \int_{B_r(v_0)} K(v, w) \left[ f(w) - f(v) \right] (f \varphi^2)(v) \, dw \, dz$$

$$\leq \Lambda \int_{\{t \leq \tau\} \times Q_\lambda'(z_0)} \int_{B_r(v_0)} \frac{|f(w) - f(v)|}{|v - w|^{d+2s}} f(v) \, dw \, dz$$

$$= \frac{\Lambda}{2} \int_{\{t \leq \tau\} \times Q_\lambda'(z_0)} \int_{B_r(v_0)} \frac{|f(w) - f(v)| f(v)}{|v - w|^{d+2s}} \, dw \, dz.$$

Equivalently

$$\int_{\{t \leq \tau\} \times \mathbb{R}^{2d}} (T f) f \varphi^2 \, dz + \frac{\Lambda}{2} \int_{\{t \leq \tau\} \times Q_\lambda'(z_0)} \int_{B_r(v_0)} \frac{|f(w) - f(v)|}{|v - w|^{d+2s}} f(v) \, dw \, dz$$

$$\leq \frac{\Lambda}{2} \int_{\{t \leq \tau\} \times Q_\lambda'(z_0)} \int_{B_r(v_0)} \frac{|f(w) - f(v)| f(v)}{|v - w|^{d+2s}} \, dw \, dz$$

$$+ I_1 + I_{32} + I_{33} + \int_{\{t \leq \tau\} \times \mathbb{R}^{2d}} h f \varphi^2 \, dz.$$

Now for the left hand side we find on the one hand

$$\frac{\Lambda}{2} \int_{\{t \leq \tau\} \times Q_\lambda'(z_0)} \int_{B_r(v_0)} \frac{|f(w) - f(v)| f(v)}{|v - w|^{d+2s}} \, dw \, dz$$

$$= \frac{\Lambda}{4} \int_{\{t \leq \tau\} \times Q_\lambda'(z_0)} \int_{B_r(v_0)} \frac{|f(w) - f(v)|^2}{|v - w|^{d+2s}} \, dw \, dz.$$

For the right hand side we estimate using Cauchy Schwarz and Young’s inequality

$$\frac{\Lambda}{2} \int_{\{t \leq \tau\} \times Q_\lambda'(z_0)} \text{PV} \int_{B_r(v_0)} \frac{|f(w) - f(v)|}{|v - w|^{d+2s}} f(v) \, dw \, dt \, dx \, dv$$

$$= \frac{\Lambda}{2} \int_{\{t \leq \tau\} \times Q_\lambda'(z_0)} \text{PV} \int_{B_r(v_0)} \frac{|f(w) - f(v)|}{|v - w|^{d+2s}} \, dw f(v) \, dz$$

$$\leq \frac{\Lambda}{2r^s} \int_{\{t \leq \tau\} \times Q_\lambda'(z_0)} \left( \int_{B_r(v_0)} \frac{|f(w) - f(v)|^2}{|v - w|^{d+2s}} \, dw \right)^{\frac{1}{2}} f(v) \, dz$$

$$\leq \frac{\Lambda}{8} \int_{\{t \leq \tau\} \times Q_\lambda'(z_0)} \int_{B_r(v_0)} \frac{|f(w) - f(v)|^2}{|v - w|^{d+2s}} \, dw \, dz$$

$$+ \frac{\Lambda}{2r^2 s} \int_{\{t \leq \tau\} \times Q_\lambda'(z_0)} f^2 \, dz.$$
Similarly, we estimate the first term we absorb on the left hand side. Now we consider the error terms. We find

\[
I_{32} = \int_{\{t \leq \tau\} \times Q^t(z_0)} \int_{B_R(v_0) \setminus B_r(v_0)} K(v, w) \left[ f(w) - f(v) \right] (f \varphi^2)(v) \, dw \, dz
\]

\[
\leq \int_{\{t \leq \tau\} \times Q^t(z_0)} \int_{B_R(v_0) \setminus B_r(v_0)} K(v, w) \left[ f(w) - f(v) \right] f(v) \, dw \, dz
\]

\[
\leq \int_{\{t \leq \tau\} \times Q^t(z_0)} \left( \int_{B_R(v_0) \setminus B_r(v_0)} K(v, w) \, dw \right) \sup_{w \in B_R(v_0) \setminus B_r(v_0)} \left[ f(w) - f(v) \right] f(v) \, dz
\]

\[
\leq \Lambda r^{-2s} \sup_{(t, x, v) \in \{t \leq \tau\} \times Q^t(z_0)} \sup_{w \in B_R(v_0) \setminus B_r(v_0)} \left[ f(w) - f(v) \right] \int_{\{t \leq \tau\} \times Q^t(z_0)} f \, dz
\]

\[
\lesssim \Lambda r^{-2s} \left( \int_{\{t \leq \tau\} \times Q^t(z_0)} f^2 \, dz \right)^{\frac{1}{2}}
\]

\[
\lesssim \Lambda^{2r^{-4s}} + \int_{\{t \leq \tau\} \times Q^t(z_0)} f^2 \, dz.
\]

We used 0 \leq f \leq 1 almost everywhere. For \(I_{3,2}\) we get

\[
I_{32} = \int_{\{t \leq \tau\} \times Q^t(z_0) \setminus Q^t(z_0)} \int_{B_R(z_0)} K(v, w) \left[ f(w) - f(v) \right] (f \varphi^2)(v) \, dw \, dz
\]

\[
\leq \int_{\{t \leq \tau\} \times Q^t(z_0) \setminus Q^t(z_0)} \int_{B_R(z_0)} K(v, w) \left[ f(w) - f(v) \right] f(v) \, dw \, dz
\]

\[
= \int_{\{t \leq \tau\} \times B_R(z_0) \setminus B_r(z_0) \times B_R(v_0)} \int_{B_R(v_0) \setminus B_r(v_0)} \left[ f(w) - f(v) \right] f(v) \, dv \, dt \, dx \, dw
\]

\[
\leq \Lambda r^{-2s} \int_{\{t \leq \tau\} \times B_R(z_0) \setminus B_r(z_0) \times B_R(v_0)} \sup_{w \in B_R(v_0) \setminus B_r(v_0)} \left[ f(w) - f(v) \right] f(v) \, dt \, dx \, dw
\]

\[
\leq \Lambda r^{-2s} \sup_{(t, x, v) \in \{t \leq \tau\} \times B_R(z_0) \setminus B_r(z_0) \times B_R(v_0)} \sup_{w \in B_R(v_0) \setminus B_r(v_0)} \left[ f(w) - f(v) \right] \int_{\{t \leq \tau\} \times Q^t(z_0)} f \, dz
\]

\[
\lesssim \Lambda r^{-2s}.
\]

Similarly, we estimate \(I_1\). We obtain

\[
I_1 = \int_{\{t \leq \tau\} \times Q^t(z_0) \setminus \mathbb{R}^d \setminus B_R(v_0)} K(v, w) \left[ f(w) - f(v) \right] (f \varphi^2)(v) \, dw \, dz
\]

\[
\leq \int_{\{t \leq \tau\} \times Q^t(z_0) \setminus \mathbb{R}^d \setminus B_R(v_0)} K(v, w) \left[ f(w) - f(v) \right] f(v) \, dw \, dz
\]

\[
\leq \Lambda r^{-2s} \int_{\{t \leq \tau\} \times Q^t(z_0) \setminus \mathbb{R}^d \setminus B_R(v_0)} \sup_{w \in \mathbb{R}^d \setminus B_R(v_0)} \left[ f(w) - f(v) \right] f(v) \, dw \, dz
\]

\[
\leq \Lambda r^{-2s} \sup_{(t, x, v) \in \{t \leq \tau\} \times Q^t(z_0) \setminus \mathbb{R}^d \setminus B_R(v_0)} \sup_{w \in \mathbb{R}^d \setminus B_R(v_0)} \left[ f(w) - f(v) \right] \int_{\{t \leq \tau\} \times Q^t(z_0)} f \, dz
\]

\[
\lesssim \Lambda^{2r^{-4s}} + \int_{\{t \leq \tau\} \times Q^t(z_0)} f^2 \, dz.
\]
On the other hand, we integrate the transport term by parts
\[
\int_{\{t \leq \tau\} \times \mathbb{R}^d} (Tf) f \varphi^2 \, dz = \frac{1}{2} \int_{\mathbb{R}^d} f^2(t, x, v) \, dx \, dv - \int_{\{t \leq \tau\} \times \mathbb{R}^d} f^2(\partial_t \varphi + v \cdot \partial_x \varphi) \, dz.
\]
Putting everything together we have
\[
\frac{1}{2} \int_{\mathbb{R}^d} f^2(t, x, v) \, dx \, dv + \frac{\Lambda}{8} \int_{\{t \leq \tau\} \times \mathbb{R}^d} \frac{|f(w) - f(v)|^2}{|v - w|^{d+2s}} \, dw \, dz \\
\leq \int_{\{t \leq \tau\} \times \mathbb{R}^d} f^2(\partial_t \varphi + v \cdot \partial_x \varphi) \, dz + \int_{\{t \leq \tau\} \times \mathbb{R}^d} hf \varphi^2 \, dz \\
+ \left( \frac{\Lambda}{2R^{2s}} + 3 \right) \int_{\{t \leq \tau\} \times Q'(z_0)} f^2 \, dz + 3 \Lambda^2 r^{-4s} \\
\lesssim \int_{\{t \leq \tau\} \times Q_R(z_0)} f^2 \left( \frac{1}{R^{2s} - r^{2s}} + \frac{|v_0| + R}{(R^{2s} - r^{2s})r} \right) \, dz \\
+ \left( \frac{\Lambda}{2R^{2s}} + 4 \right) \int_{\{t \leq \tau\} \times Q'_R(z_0)} f^2 \, dz + \int_{\{t \leq \tau\} \times Q'_R(z_0)} h^2 \, dz
\]
We used the bounds $|\partial_t \varphi| \lesssim \frac{1}{R^{2s} - r^{2s}}$, $|\nabla_x \varphi| \lesssim \frac{1}{(R^{2s} - r^{2s})r}$. Finally we conclude by taking the supremum in $\tau \in (-r^2 + t_0, t_0)$.

Lemma 3.3 (Local gain of integrability). Let $f$ be a non-negative sub-solution of (1.1)-(1.2) with (1.3)-(1.6) in $Q_\frac{3}{\sigma}$. Let $Q_r(z_0) \subset Q_R(z_0) \subset Q_\frac{3}{\sigma}$ for some $0 < r < R \leq 1$. Assume $0 \leq f \leq 1$ a.e. in $(-R^{2s} + t_0, t_0] \times B_{R^{1+2s}} \times \mathbb{R}^d$. Given $p$ such that $\frac{1}{p} > \frac{1}{p'} - \frac{1}{2}$ with $p'$ from Proposition 3.1 and $\sigma \in \left[0, \frac{s}{2s+1}\right)$, $f$ satisfies
\[
\|f\|_{L^p(Q_r(z_0))} \lesssim C'(s, \frac{r + R}{2}, R, v_0) \|f\|_{L^2(Q_R(z_0))} + \|h\|_{L^2(Q_R(z_0))}
\]
\[
\|f\|_{L^1_{\text{loc}} W^{1,1}_{\infty}(Q_r(z_0))} \lesssim C''(s, \frac{r + R}{2}, R, v_0) \|f\|_{L^2(Q_R(z_0))} + \|h\|_{L^2(Q_R(z_0))}
\]
where $C'(s, r, R, v_0) := \left( \frac{1}{r^2} + 1 \right) C(s, r, R, v_0) + \frac{1}{r^{2s} - r^{2s}} + \frac{|v_0| + R}{(R^{2s} - r^{2s})r}$ with $C$ from Lemma 3.2 and $C'' = C + C'$.

Proof. Let $\varphi$ be a smooth function with values in $[0, 1]$ such that $\varphi = 1$ on $Q_r(z_0)$ and $\varphi = 0$ outside $Q_{r + R}(z_0)$. Consider $g := f \varphi$. Then there is some non-negative measure $\tilde{m}$ such that
\[
\partial_t g + v \cdot \nabla_g + (-\Delta)^s_{\tilde{m}} g = (Tf) \varphi + f(T\varphi) + (-\Delta)^s_{\tilde{m}} g \\
= (Lf) \varphi + h_1 \varphi - \tilde{m} \varphi + f(T\varphi) + (-\Delta)^s_{\tilde{m}} g \\
=: h_1 + (-\Delta)^s_{\tilde{m}} h_2 - m,
\]
where $h_1 := (Lf) \varphi + h_1 \varphi + f(T\varphi)$, $h_2 := (-\Delta)^s_{\tilde{m}} g$ and $m := \tilde{m} \varphi \geq 0$. Note that we have by the energy estimate
\[
\|g\|_{L^2_{\text{loc}}(\dot{B}^s_{\infty, \infty} (\mathbb{R} \times \mathbb{R}^{2d})))} = \|f\|_{L^2_{\text{loc}}(\dot{B}^s_{\infty, \infty}(Q_{r+R}(z_0)))} \lesssim C(s, \frac{R + r}{2}, R, v_0) \|f\|_{L^2(Q_R)}.
\]
with $C$ from Lemma 3.2. For $L f$ we have for $v \in B_{\frac{r}{2}}$

$$\varphi L[f](v) = L[\varphi f](v) - f L[\varphi](v)$$

$$= \int_{\mathbb{R}^d} (f(w)\varphi(w) - f(v)\varphi(v)) K(v, w) dw - f(v) \int_{\mathbb{R}^d} (\varphi(w) - \varphi(v)) K(v, w) dw$$

$$= \int_{\mathbb{R}^d} (f(w)\varphi(w) - f(v)\varphi(w)) K(v, w) dw$$

$$= \int_{B_{\frac{r}{2}}(z_0)} (f(w) - f(v)) K(v, w) dw$$

$$\lesssim \frac{1}{(r + \frac{R + R - r}{2})^s} \left( \int_{B_{\frac{r}{2}}(z_0)} \frac{|f(w) - f(v)|^2}{|v - w|^{2s+2s}} \frac{d v}{d w} \right)^{\frac{1}{2}},$$

where in the last step we use Proposition 2.4 and Cauchy-Schwarz inequality. Therefore we get using the energy estimate 3.2

$$\|h_1\|_{L^2(\mathbb{R} \times \mathbb{R}^{2d})} + \|h_2\|_{L^2(\mathbb{R} \times \mathbb{R}^{2d})}$$

$$\leq \frac{\Lambda}{(R + \frac{R + R - r}{2})^s} \left( \frac{1}{R^{2s} - (\frac{R + R - r}{2})^{2s}} + \frac{|v_0| + R}{(R^{2s} - (\frac{R + R - r}{2})^{2s})(\frac{R + R - r}{2})} \right) \|f\|_{L^2(Q_R(z_0))}$$

$$\lesssim \lambda, C' \left( s, \frac{r + R}{2}, R, v_0 \right) \|f\|_{L^2(Q_R(z_0))} + \|h\|_{L^2(Q_R(z_0))}.$$

Using Lemma 3.1 we deduce

$$\|g\|_{L^p(\mathbb{R} \times \mathbb{R}^{2d})} \lesssim \|h_1\|_{L^2(\mathbb{R} \times \mathbb{R}^{2d})} + \|h_2\|_{L^2(\mathbb{R} \times \mathbb{R}^{2d})}$$

$$\lesssim C' \left( s, \frac{r + R}{2}, R, v_0 \right) \|f\|_{L^2(Q_R(z_0))} + \|h\|_{L^2(Q_R(z_0))}.$$

and

$$\|g\|_{L^1_{t,v} W^{s,1}_{t,v}(\mathbb{R} \times \mathbb{R}^{2d})} \lesssim \|h_1\|_{L^1(\mathbb{R} \times \mathbb{R}^{2d})} + \|h_2\|_{L^1(\mathbb{R} \times \mathbb{R}^{2d})} + \|m\|_{M^1(Q_{\frac{r}{2}}(z_0))}$$

$$\lesssim \|h_1\|_{L^2(\mathbb{R} \times \mathbb{R}^{2d})} + \|h_2\|_{L^2(\mathbb{R} \times \mathbb{R}^{2d})} + \|m\|_{M^1(Q_{\frac{r}{2}}(z_0))}$$

$$\lesssim C' \left( s, \frac{r + R}{2}, R, v_0 \right) \|f\|_{L^2(Q_R(z_0))} + \|h\|_{L^2(Q_R(z_0))} + \|m\|_{M^1(Q_{\frac{r}{2}}(z_0))}.$$

To conclude the proof, we consider a second smooth cut-off function $0 \leq \tilde{\varphi} \leq 1$ such that $\tilde{\varphi} = 1$ in $Q_{\frac{r}{2}}(z_0)$ and $\tilde{\varphi} = 0$ outside $Q_R(z_0)$. Integrating (1.1) against $\tilde{\varphi}$ yields just as in the energy estimate

$$\|m\|_{M^1(Q_{\frac{r}{2}}(z_0))} \lesssim \|m\tilde{\varphi}\|_{M^1(\mathbb{R} \times \mathbb{R}^{2d})}$$

$$\lesssim C \left( s, \frac{R + r}{2}, R, v_0 \right) \|f\|_{L^2(Q_R(z_0))} + \|h\|_{L^2(Q_R(z_0))}$$

so that

$$\|g\|_{L^1_{t,v} W^{s,1}_{t,v}(\mathbb{R} \times \mathbb{R}^{2d})} \leq C'' \left( s, \frac{r + R}{2}, R, v_0 \right) \|f\|_{L^2(Q_R(z_0))} + \|h\|_{L^2(Q_R(z_0))}$$

for $C'' = C + C'.$
Lemma 4.1. Let $Q_r(z_0) \subset Q_R(z_0)$ with $0 < r < R \leq 1$ and $t_0 < r^{2s}$. Let $f$ be a non-negative sub-solution of (1.1)-(1.2) for $h = 0$ with (1.3)-(1.6) in $Q_R(z_0)$. Assume $0 \leq f \leq 1$ a.e. in $(-R^{2s} + t_0, t_0) \times B_{R^{1+2s}}(x_0 + (t - t_0)v_0) \times \mathbb{R}^d$ and there exists $\varepsilon_0 > 0$ so that

$$
\int_{Q_R(z_0)} f^2 \, dz \leq \varepsilon_0.
$$

Then we have

$$
f \leq \frac{1}{2} \quad \text{a.e. in } Q_r(z_0).
$$

As a consequence, we get the following result.

Lemma 4.2. Let $Q_r(z_0) \subset Q_R(z_0)$ with $0 < r < R \leq 1$ and $t_0 < r^{2s}$. Let $f$ be a non-negative sub-solution of (1.1)-(1.2) with (1.3)-(1.6) in $Q_R(z_0)$. Assume $0 \leq f \leq 1$ a.e. in $(-R^{2s} + t_0, t_0) \times B_{R^{1+2s}}(x_0 + (t - t_0)v_0) \times \mathbb{R}^d$. Then for $\zeta > 0$ there holds

$$
\|f\|_{L^\infty(Q_r(z_0))} \leq \|f\|_{L^5(Q_R(z_0))} + \|h\|_{L^\infty(Q_R(z_0))}.
$$

Proof. We can apply Lemma 4.1 to $g := \frac{\sqrt{\varepsilon_0}}{(\|f\|_{L^2(Q_R(z_0))} + \|h\|_{L^\infty(Q_R(z_0))})(f - (t - t_0)\|h\|_{L^\infty(Q_R(z_0))})}$ where $t \in (-R^{2s} + t_0, t_0)$. Then $0 \leq g \leq 1$ is a sub-solution of (1.1)-(1.2) with $h = 0$ in $Q_R(z_0)$. Moreover, we have

$$
\int_{Q_R(z_0)} g^2 \, dz = \frac{1}{(\|f\|_{L^2(Q_R(z_0))} + \|h\|_{L^\infty(Q_R(z_0))})^2} \int_{Q_R(z_0)} (f - (t - t_0)\|h\|_{L^\infty(Q_R(z_0))})^2 \, dz
$$

$$
= \frac{1}{(\|f\|_{L^2(Q_R(z_0))} + \|h\|_{L^\infty(Q_R(z_0))})^2} \frac{\varepsilon_0}{(\|f\|_{L^2(Q_R(z_0))} + \|h\|_{L^\infty(Q_R(z_0))})^2} \times \int_{Q_R(z_0)} (f^2 + (t - t_0)^2\|h\|_{L^\infty(Q_R(z_0))}^2 - 2f(t - t_0)\|h\|_{L^\infty(Q_R(z_0))}) \, dz
$$

$$
\leq \frac{\varepsilon_0}{\|f\|_{L^2(Q_R(z_0))}^2} \int_{Q_R(z_0)} f^2 \, dz + \frac{\varepsilon_0}{\|h\|_{L^\infty(Q_R(z_0))}^2} \int_{Q_R(z_0)} (t - t_0)^2\|h\|_{L^\infty(Q_R(z_0))}^2 \, dz
$$

$$
+ \frac{\varepsilon_0}{\|f\|_{L^2(Q_R(z_0))}^2} \int_{Q_R(z_0)} f(t - t_0)\|h\|_{L^\infty(Q_R(z_0))} \, dz
$$

$$
\leq \varepsilon_0 + \varepsilon_0 R^{8s} + \varepsilon_0 \left( \int_{Q_R(z_0)} (t - t_0)^2 \, dz \right)^{1/2}
$$

$$
\leq \varepsilon_0
$$

where we used $R \leq 1$. Thus Lemma 4.1 gives us

$$
\|f\|_{L^\infty(Q_r(z_0))} \leq \|f - (t - t_0)h\|_{L^\infty(Q_r(z_0))} \leq \|f\|_{L^2(Q_R(z_0))} + \|h\|_{L^\infty(Q_R(z_0))}
$$

where for the first inequality we note that $t - t_0 \leq 0$. 

4. First Lemma of DeGiorgi
Now, we can interpolate and use Young’s inequality to get for $\zeta \in (0, 2)$

$$
\|f\|_{L^\infty(Q_{r_k}(z_0))} \lesssim \|f\|_{L^\infty(Q_{r_k}(z_0))}^{1-\frac{2}{\zeta}} \|f\|_{L^\zeta(Q_{r_k}(z_0))}^{\frac{2}{\zeta}} + \|h\|_{L^\infty(Q_{r_k}(z_0))}
$$

(4.2)

Then

$$
\leq \frac{1}{2} \|f\|_{L^\infty(Q_{r_k}(z_0))} + C \|f\|_{L^\zeta(Q_{r_k}(z_0))} + \|h\|_{L^\infty(Q_{r_k}(z_0))}.
$$

Following [11] we define an ascending sequence of radii $r_k = R$ and $k = r_{k-1} + \delta k^{-2}$ for $\delta = \frac{1}{2}\left(\sum l \geq 1 \frac{1}{l^2}\right)^{-1}(R - r)$. Then we get by induction

$$
\|f\|_{L^\infty(Q_{r_k}(z_0))} \leq \frac{1}{2} \|f\|_{L^\infty(Q_{r_{k+1}}(z_0))} + Ck^2 \|f\|_{L^\zeta(Q_{r_k}(z_0))} + \|h\|_{L^\infty(Q_{r_k}(z_0))}.
$$

Thus

$$
\|f\|_{L^\infty(Q_{r_k}(z_0))} \leq \left(\frac{1}{2}\right)^k \|f\|_{L^\infty(Q_{r_{1+1}}(z_0))} + C \sum_{l=1}^{k} \frac{1}{2^l} \|f\|_{L^\zeta(Q_{r_l}(z_0))} + \|h\|_{L^\infty(Q_{r_k}(z_0))}.
$$

Taking the limit $k \to \infty$ yields

$$
\|f\|_{L^\infty(Q_{r_0}(z_0))} \leq C \|f\|_{L^\zeta(Q_{r_0}(z_0))} + \|h\|_{L^\infty(Q_{r_0}(z_0))},
$$

which we insert into (4.2) to obtain

$$
\|f\|_{L^\infty(Q_{r_0}(z_0))} \lesssim \|f\|_{L^\zeta(Q_{r_0}(z_0))} + \|h\|_{L^\infty(Q_{r_0}(z_0))}.
$$

Proof of Lemma 4.1. We do a De Giorgi iteration, as was done in Lemma 6.6 of [14] and Lemma 3.8 of [9]. For an intuitive description of this technique, we refer the reader to [4]. Let $\hat{r} = r_0 - t_0$. Then $0 < \hat{r} < \hat{\tau}$. Consider $l_k = \frac{1}{2} - 2^{-k-1}$, $r_k = r + 2^{-k}(R - r)$, and $t_k = \hat{r} - 2^{-k}(\hat{\tau} - \hat{r})$. Define $Q_k(z_0) = (t_k, t_0) \times B_{r_k}(x_0 + (t - t_0)v_0) \times B_{r_k}(v_0)$

$$
A_k := \int_{Q_k(z_0)} (f - l_k)^2 \, dz.
$$

Then $A_0 \leq \varepsilon_0$ by assumption. We want to prove that $A_k \to 0$ as $k \to +\infty$. This then proves the lemma.

Note that $(f - l_{k+1})_+$ is a sub-solution of (1.1)-(1.2) for $h = 0$ such that $0 \leq (f - l_{k+1})_+ \leq 1$ a.e. in $(-R^{2s} + t_0, t_0) \times B_{R^{1+2s}}(x_0 + (t - t_0)v_0) \times \mathbb{R}^d$. We apply Lemma 3.3 to find

$$
\left(\int_{Q_k(z_0)} (f - l_{k+1})_+^p \, dz\right)^{\frac{1}{p}} \leq \frac{2^{k+2}(6s+2)(|v_0| + 1)}{(R - r)^{6s+2}} \int_{Q_k(z_0)} (f - l_{k+1})_+^2 \, dz
$$

$$
\leq \frac{2^{k+2}(6s+2)(|v_0| + 1)}{(R - r)^{6s+2}} \int_{Q_k(z_0)} (f - l_k)_+^2 \, dz
$$

$$
\lesssim_{s, v_0, R} 2^{8k} A_k,
$$

where we used $l_{k+1} \geq l_k$. 
Using $Q_{k+1} \subset Q_k$ and Chebyshev’s inequality we get
\[
A_{k+1} \leq \left( \int_{Q_{k+1}(z_0)} (f - l_{k+1})^2 \, dz \right)^{\frac{1}{p}} \left| \{ f > l_{k+1} \} \cap Q_{k+1}(z_0) \right|^{1 - \frac{2}{p}}
\]
\[
\lesssim_{s,v_0,R} 2^{kh} A_k \left\{ f > l_{k+1} \right\} \cap Q_k(z_0)^{1 - \frac{2}{p}}
\]
\[
= C(s,v_0,R) 2^{kh} A_k \{ (f - l_k)_+ > 2^{-k-2} \} \cap Q_k(z_0)^{1 - \frac{2}{p}}
\]
\[
\lesssim_{s,v_0,R} 2^{kh} A_k (2^{2k+1} A_k)^{1 - \frac{2}{p}}
\]
\[
\lesssim_{s,v_0,R} 2^{10k} A_k^{1 + \frac{1}{p^2}}
\]
Since $\frac{2-2}{p} > 0$ this implies $A_k \to 0$ as $k \to +\infty$. \hfill \Box

5. Second Lemma of DeGiori

5.1. Weak Poincaré. This is where we introduce trajectories in order to obtain a hypoelliptic Poincaré-type inequality with an error term. The idea comes from Guerand and Mouhot [11].

Proposition 5.1. Let $f$ be a non-negative sub-solution to (1.1)-(1.2) in $Q_1$ with assumptions (1.3)-(1.6) for $R = 6$. Assume $0 \leq f \leq 1$ a.e. in $(-3^2,0] \times B_{3^2+2^2} \times \mathbb{R}^d$. Given any $\varepsilon \in (0,1)$ and $0 < \sigma < \frac{1}{2^+}$ there holds
\[
\| (f - \langle f \rangle_{Q_1}^-) + \|_{L^1(Q_1)} \leq \frac{1}{\varepsilon^2} \int_{Q_1} \left( \int_{B_3} \frac{|f(v) - f(v')|^2}{|v - v'|^{d+2s}} \, dv' \right)^{\frac{1}{2}} \, dv + \varepsilon \sigma \| f \|_{L^2(Q_2)} + \| h \|_{L^2(Q_3)}
\]
where $Q_1^- := Q_1(-2,0) = (-3,-2] \times B_{12^{-s+1}} \times B_1$ and $\langle f \rangle_{Q_1^-} = \int_{Q_1^-} f = \frac{1}{|Q_1^-|} \int_{Q_1^-} f$.

Proof. The method for this proof is taken from Proposition 13 in [11]. Consider for $\varepsilon \in (0,1)$ a small function $\varphi_\varepsilon = \varphi_\varepsilon(y,w)$, $0 \leq \varphi_\varepsilon \leq 1$ with support in $B_{12^{-s+1}} \times B_1$, so that $\varphi_\varepsilon = 1$ in $B_{(1-\varepsilon)2^{-s+1}} \times B_{(1-\varepsilon)}$ and $|\nabla_y \varphi_\varepsilon| \lesssim \frac{1}{\varepsilon^2} \| \nabla w \varphi_\varepsilon \| \lesssim \frac{1}{\varepsilon}$. We then split
\[
\| (f - \langle f \rangle_{Q_1}^-) + \|_{L^1(Q_1)} \lesssim \| (f - \langle f \varphi_\varepsilon \rangle_{Q_1}^-) + \|_{L^1(Q_1)}
\]
\[
\lesssim \int_{Q_1} \left\{ \int_{Q_1^-} \left[ f(t,x,v) - f(s,y,w) \right] \varphi_\varepsilon(y,w) \, ds \, dy \, dw \right\} + \, dt \, dx \, dv
\]
\[
+ \| f \|_{L^1(Q_1)} \int_{Q_1^-} \left( 1 - \varphi_\varepsilon(y,w) \right) \, ds \, dy \, dw
\]
\[
\lesssim \int_{Q_1} \left\{ \int_{Q_1^-} \left[ f(t,x,v) - f(s,y,w) \right] \varphi_\varepsilon(y,w) \, ds \, dy \, dw \right\} + \, dt \, dx \, dv
\]
\[
+ \varepsilon d(1+1) \| f \|_{L^2(Q_1)}
\]
where we used $\langle f \varphi_\varepsilon \rangle_{Q_1^-} \leq \langle f \rangle_{Q_1^-}$ and Cauchy-Schwarz inequality.
Consider the first term. For fixed $t, x, v$ we decompose the trajectory $(t, x, v) \rightarrow (s, y, w)$ into four sub-trajectories in $Q_3$ as follows:

$$(t, x, v) \rightarrow (t, x + \varepsilon w, v) \rightarrow \left(t, x + \varepsilon w, \frac{x + \varepsilon w - y}{t - s}\right) \rightarrow \left(s, y, \frac{x + \varepsilon w - y}{t - s}\right) \rightarrow (s, y, w).$$

Thus we split the integrand along these trajectories

$$f(t, x, v) - f(s, y, w) = \left[ f(t, x, v) - f(t, x + \varepsilon w, v) \right]$$

$$+ \left[ f(t, x + \varepsilon w, v) - f \left(t, x + \varepsilon w, \frac{x + \varepsilon w - y}{t - s}\right) \right]$$

$$+ \left[ f \left(t, x + \varepsilon w, \frac{x + \varepsilon w - y}{t - s}\right) - f \left(s, y, \frac{x + \varepsilon w - y}{t - s}\right) \right]$$

$$+ \left[ f \left(s, y, \frac{x + \varepsilon w - y}{t - s}\right) - f(s, y, w) \right].$$

We integrate against $\varphi_\varepsilon(y, w)$ on $Q_1$ yielding

$$I_1(t, x, v) := \int_{Q_1}^{} \left[ f(t, x, v) - f(t, x + \varepsilon w, v) \right] \varphi_\varepsilon(y, w) \, ds \, dy \, dw$$

$$I_2(t, x, v) := \int_{Q_1}^{} \left[ f(t, x + \varepsilon w, v) - f \left(t, x + \varepsilon w, \frac{x + \varepsilon w - y}{t - s}\right) \right] \varphi_\varepsilon(y, w) \, ds \, dy \, dw$$

$$I_3(t, x, v) := \int_{Q_1}^{} \left[ f \left(t, x + \varepsilon w, \frac{x + \varepsilon w - y}{t - s}\right) - f \left(s, y, \frac{x + \varepsilon w - y}{t - s}\right) \right] \varphi_\varepsilon(y, w) \, ds \, dy \, dw$$

$$I_4(t, x, v) := \int_{Q_1}^{} \left[ f \left(s, y, \frac{x + \varepsilon w - y}{t - s}\right) - f(s, y, w) \right] \varphi_\varepsilon(y, w) \, ds \, dy \, dw$$
For $I_2$ we find

$$
\int_{Q_1} I_2 \, dt \, dx \, dv = \int_{Q_1} \int_{Q_1} \left[ f(t, x + \varepsilon w, v) - f\left(t, x + \varepsilon w, \frac{x + \varepsilon w - y}{t - s}\right) \right] \varphi_{\varepsilon}(y, w) \, dy \, dw \, dt \, dx \, dv
$$

$$
\leq \int_{Q_1} \int_{Q_1} \left| f(t, x + \varepsilon w, v) - f\left(t, x + \varepsilon w, \frac{x + \varepsilon w - y}{t - s}\right) \right| 
\times \left| v - \left( \frac{x + \varepsilon w - y}{t - s} \right) \right|^{\frac{d+2s}{2}} \, ds \, dy \, dw \, dt \, dx \, dv
$$

$$
\leq \int_{(-1,0) \times B_2 \times B_1} \int_{(-3,-2) \times B_3} \left| f(t, X, v) - f(t, X, \frac{X - y}{t - s}) \right| 
\times \left| v - \left( \frac{X - y}{t - s} \right) \right|^{\frac{d+2s}{2}} \, ds \, dy \, dw \, dt \, dX \, dv
$$

$$
\leq \int_{(-1,0) \times B_2 \times B_1} \int_{(-3,-2) \times B_3} \left( \int_{B_3} \frac{|f(t, X, v) - f(t, X, V)|^2}{|v - V|^{d+2s}} \, dV \right)^{\frac{1}{2}} 
\times \left( \int_{(-3,-2) \times B_3} \left| v - V \right|^{\frac{d+2s}{2}} \left( \frac{t - s}{\varepsilon} \right)^{2d} \, ds \, dV \right)^{\frac{1}{2}} \, dt \, dX \, dv
$$

$$
\leq \varepsilon^{-d} \int_{Q_3} \left( \int_{B_3} \frac{|f(t, X, v) - f(t, X, V)|^2}{|v - V|^{d+2s}} \, dV \right)^{\frac{1}{2}} \, dt \, dX \, dv
$$

where we used $0 \leq \varphi_{\varepsilon} \leq 1$, the change of variables $x \to X = x + \varepsilon w$ and $w \to V = \frac{x + \varepsilon w - y}{t - s}$ and the Cauchy-Schwarz inequality.
For $I_4$ we can proceed similarly with the change of variables $x \to \tilde{V} = \frac{x + \varepsilon w - y}{t - s}$, Fubini and Cauchy-Schwarz:

\[
\int_{Q_3} I_4 \, dt \, dx \, dv = \int_{Q_1} \int_{Q_1^1} \left[ f\left( s, x, \frac{x + \varepsilon w - y}{t - s} \right) - f(s, y, w) \right] \varphi_\varepsilon(y, w) \, ds \, dy \, dw \, dt \, dx \, dv
\]

\[
\int_{Q_1} \int_{Q_1^1} \frac{|f(s, y, \frac{x + \varepsilon w - y}{t - s}) - f(s, y, w)|}{\left| \left( \frac{x + \varepsilon w - y}{t - s} \right) - w \right|^{d+2s}} \times \left| \left( \frac{x + \varepsilon w - y}{t - s} \right) - w \right|^{d+2s} \, ds \, dy \, dw \, dt \, dx \, dv
\]

\[
\int_{Q_3} \left( \int_{B_3} \frac{|f(s, y, \tilde{V}) - f(s, y, w)|^2}{|\tilde{V} - w|^{d+2s}} \, d\tilde{V} \right)^{\frac{1}{2}} \left( \int_{B_3} |\tilde{V} - w|^{d+2s} \, d\tilde{V} \right)^{\frac{1}{2}} \, ds \, dy \, dw
\]

For $I_3$ we use a Taylor formula between $(t, x + \varepsilon w)$ and $(s, y)$ along $\mathcal{T}$ and the equation (1.1) satisfied by $f$. We obtain

\[
I_3(t, x, v) = \int_{Q_1} \left[ f\left( t, x + \varepsilon w, \frac{x + \varepsilon w - y}{t - s} \right) - f\left( s, y, \frac{x + \varepsilon w - y}{t - s} \right) \right] \varphi_\varepsilon(y, w) \, ds \, dy \, dw
\]

\[
\int_{Q_1} \int_{0}^{1} (t-s)\mathcal{T} f\left( \tau t + (1-\tau)s, \tau(x + \varepsilon w) + (1-\tau)y, \frac{x + \varepsilon w - y}{t - s} \right) \varphi_\varepsilon(y, w) \, d\tau \, ds \, dy \, dw
\]

\[
\int_{Q_1} \int_{0}^{1} (t-s)\mathcal{L} f\left( \tau t + (1-\tau)s, \tau(x + \varepsilon w) + (1-\tau)y, \frac{x + \varepsilon w - y}{t - s} \right) \varphi_\varepsilon(y, w) \, d\tau \, ds \, dy \, dw
\]

\[
+ \int_{Q_1} \int_{0}^{1} (t-s)\psi \left( \tau t + (1-\tau)s, \tau(x + \varepsilon w) + (1-\tau)y, \frac{x + \varepsilon w - y}{t - s} \right) \varphi_\varepsilon(y, w) \, d\tau \, ds \, dy \, dw
\]

\[
:= I_{31} + I_{32}.
\]

We perform the change of variables $y \to V = \frac{x + \varepsilon w - y}{t - s}$, $x \to X = x + \varepsilon w - (1 - t)(t - s)V$, $s \to s' = t - s$ and $t \to t' = t - (1 - t)s'$:

\[
\int_{Q_1} I_{31}(t, x, v) \, dt \, dx \, dv \lesssim \int_{(-4,0] \times B_x \times B_t} \mathcal{L} f(t', X, V) \varphi_\varepsilon(V, w) \, dV \, dt' \, dX \, dv
\]

\[
= \int_{(-4,0] \times B_3 \times B_1} \int_{B_3 \times B_1} \int_{\mathbb{R}^d} \left[ f(V') - f(V) \right] K(V, V') \, dV' \varphi_\varepsilon(V, w) \, dV \, dt' \, dX
\]

\[
= \int_{(-4,0] \times B_3 \times B_1} \int_{B_3 \times B_1} \ldots + \int_{(-4,0] \times B_3 \times B_1} \int_{B_3 \times B_1} \ldots
\]

\[
= I_{311} + I_{312}.
\]
Then we bound

\[
I_{312} = \int_{(-4,0] \times B_3} \int_{B_3 \times B_1} \int_{B_3} \left[ f(V') - f(V) \right] K(V, V') \varphi_\varepsilon(V, w) \, dV' \, dw \, dt' \, dX \\
\leq \Lambda^{-2s} \int_{(-4,0] \times B_3} \int_{B_3 \times B_1} \int_{B_3} \left[ f(V') - f(V) \right] \, dV' \, dw \, dt' \, dX \\
\lesssim \Lambda^{-s} \sup_{(t, X, V) \in (-4,0] \times B_3 \times B_3} \left[ f(V') - f(V) \right] \\
\lesssim \Lambda \, 1.
\]

Now for \( I_{311} \) we have

\[
I_{311} = \int_{(-4,0] \times B_3} \int_{B_3 \times B_1} \int_{B_3} \left[ f(V') - f(V) \right] K(V, V') \varphi_\varepsilon(V, w) \, dV' \, dV \, dt' \, dX \\
\leq \Lambda \int_{(-4,0] \times B_3} \int_{B_3 \times B_1} \int_{B_3} \frac{|f(V') - f(V)|}{|V - V'|^{d+2s}} \varphi_\varepsilon(V, w) \, dV' \, dV \, dt' \, dX \\
= \frac{1}{2} \int_{(-4,0] \times B_3} \int_{B_3 \times B_1} \int_{B_3} \frac{|f(V') - f(V)|}{|V - V'|^{d+2s}} \\
\times \left[ \varphi_\varepsilon(V, w) - \varphi_\varepsilon(V', w) \right] \, dV' \, dV \, dt' \, dX \\
\lesssim \Lambda \int_{(-4,0] \times B_3} \int_{B_3 \times B_1} \left( \int_{B_3} \frac{|f(V') - f(V)|^2}{|V - V'|^{d+2s}} \, dV \right)^{\frac{1}{2}} \\
\times \left( \int_{B_3} \frac{\varphi_\varepsilon(V, w) - \varphi_\varepsilon(V', w)^2}{|V - V'|^{d+2s}} \, dV \right)^{\frac{1}{2}} \, dV' \, dV \, dt' \, dX \\
\lesssim \||\varphi_\varepsilon||c^2 \int_{Q_3} \left( \int_{B_3} \frac{|f(V') - f(V)|^2}{|V - V'|^{d+2s}} \, dV \right)^{\frac{1}{2}} \, dV' \, dt' \, dX \\
\lesssim \frac{1}{\varepsilon^2} \int_{Q_3} \left( \int_{B_3} \frac{|f(V') - f(V)|^2}{|V - V'|^{d+2s}} \, dV \right)^{\frac{1}{2}} \, dV' \, dt' \, dX.
\]

We used Proposition 2.4, the symmetry of the fractional Laplacian, Fubini’s theorem, the Cauchy-Schwarz inequality, some rescaling in the time variable and the bounds on \( \nabla_y \varphi_\varepsilon \) and \( \nabla_w \varphi_\varepsilon \). With the same change of variables and rescaling in time, we also deduce

\[
\int_{Q_1} I_{52}(t, x, v) \, dt \, dx \, dv \leq \int_{(-4,0] \times B_3 \times B_3} h(t', X, V) \, dt' \, dX \, dV \lesssim \int_{Q_3} |h| \, dt' \, dX \, dV.
\]
Finally, for $I_1$ we estimate
\[ 
\int_{Q_1} I_1 \, dt \, dx \, dv \lesssim \int_{Q_1} \int_{Q_1} [f(t, x, v) - f(t, x + \varepsilon w, v)] \varphi_c(y, w) \, ds \, dy \, dt \, dx \, dv 
\]
where we used that $0 \leq \varphi_c \leq 1$, the change of variables $w \rightarrow x' = x + \varepsilon w$ and (3.6). Putting everything together yields the claim, when we notice that $\varepsilon > \varepsilon^{d(1+\varepsilon)}$. \hfill \Box

5.2. Intermediate Value Lemma.

**Theorem 5.2.** Given $\delta_1, \delta_2 \in (0, 1)$, there is $r_0 < 1, \theta \sim \frac{(\delta_1 \delta_2)^{1/2} + 1}{1 + \|h\|_{L^\infty(Q_1)^2}^{2d(1+\varepsilon) + 2}}$ and $\nu \gtrsim \frac{(\delta_1 \delta_2)^{1/2} + 1}{1 + \|h\|_{L^\infty(Q_1)^2}^{2d(1+\varepsilon) + 2}}$, such that any sub-solution $f : Q_1 \rightarrow \mathbb{R}$ to (1.1)-(1.2) under assumptions (1.3)-(1.6) for $\bar{R} = 2$ with $f \leq 1$ in $(-2^{-2s}, 0) \times B_{2^{-(1+2s)} + 2} \times \mathbb{R}^d$ and so that (5.2)
\[ |\{f \leq 0\} \cap Q_{r_0}^-| \geq \delta_1 |Q_{r_0}^-| \quad \text{and} \quad |\{f \geq 1 - \theta\} \cap Q_{r_0}^-| \geq \delta_2 |Q_{r_0}^-| \]

satisfies
\[ |\{0 \leq f \leq 1 - \theta\} \cap Q_{\frac{1}{2}}| \gtrsim \nu |Q_{\frac{1}{2}}|. \]

Recall $Q_{r_0}^- := Q_{r_0}(-2r_0^{2s}, 0, 0) = (-3r_0^{2s}, -2r_0^{2s}) \times B_{r_0^{1+2s}} \times B_{r_0}.$

**Proof.** This proof follows Theorem 3 in [11]. Let $\delta_1, \delta_2 \in (0, 1)$, and $f : Q_1 \rightarrow \mathbb{R}$ be a sub-solution to (1.1)-(1.2) satisfying (5.2). We define $g := f - \min \left\{ t + (3r_0)^{2s} \|h\|_{L^\infty(Q_1)^2} \right\}$. Then $0 \leq g_+ \leq 1$ is a sub-solution to (1.1)-(1.2) in $Q_{3r_0}$ with zero source term. We set $r_0 = \frac{1}{12}$ for $\hat{h} \neq 0$ and $r_0 = \frac{1}{12}$ if $\hat{h} = 0$. Applying (5.1) to $g_+$ for some $\varepsilon$ to be determined yields
\[ 
\int_{Q_{r_0}} (g_+ - (g_+) Q_{r_0}^-)_+ \lesssim \frac{1}{\varepsilon^d} \int_{Q_{3r_0}} \left( \int_{B_{3r_0}} \frac{|g_+(v) - g_+(v')|^2}{|v - v'|^{d+2s}} \, dv' \right)^{1/2} dz + \varepsilon \sigma \left( \int_{Q_{3r_0}} g_+^2 \, dz \right)^{1/2} 
\]
(5.4)
\[ 
\lesssim \frac{1}{\varepsilon^d} \int_{Q_{3r_0}} \left( \int_{B_{3r_0}} \frac{|g_+(v) - g_+(v')|^2}{|v - v'|^{d+2s}} \, dv' \right)^{1/2} dz + \varepsilon \sigma 
\]
\[ 
\lesssim \int_{Q_{3r_0}} \left( \int_{B_{3r_0}} \frac{|g_+(v) - g_+(v')|^2}{|v - v'|^{d+2s}} \, dv' \right)^{1/2} dz + \varepsilon \sigma, 
\]
where we used that $0 \leq g_+ \leq 1$. \hfill \Box
On the one hand, by assumption (5.2) we have

\[
\langle g+ \rangle_{Q_\epsilon} = \int_{Q_\epsilon} \{ f(t', y, w) - [t' + (3\epsilon)^2] \| h \|_{L^\infty(Q_1)} \}_+ dt' dy dw \leq \frac{|\{ f > 0 \} \cap Q_\epsilon^-|}{|Q_\epsilon^-|} \leq 1 - \delta_1
\]

and

\[
\int_{Q_\epsilon} (g+ - \langle g+ \rangle_{Q_\epsilon})_+ \geq \frac{1}{|Q_\epsilon|} \int_{Q_\epsilon} \{ f(t, x, v) - [t + (3\epsilon)^2] \| h \|_{L^\infty(Q_1)} - (1 - \delta_1) \}_+ dz \\
\geq \frac{1}{|Q_\epsilon|} \int_{Q_\epsilon} \left[ f(t, x, v) - (3\epsilon)^2 \| h \|_{L^\infty(Q_1)} - (1 - \delta_1) \right]_+ dz \\
\geq \frac{1}{|Q_\epsilon|} \int_{\{ f \geq \theta \} \cap Q_\epsilon} \left( \frac{\delta_1}{2} - \theta \right)_+ dz \\
\geq \delta_2 \left( \frac{\delta_1}{2} - \theta \right).
\]

On the other hand, we find

\[
\int_{Q_3\epsilon} \left( \int_{B_{3\epsilon}} \frac{|g_+(v) - g_+(v')|^2}{|v - v'|^{d+2s}} dv' \right)^{\frac{1}{2}} dz \\
\leq \int_{Q_3\epsilon} \left( \int_{B_{3\epsilon}} \frac{|f_+(v) - f_+(v')|^2}{|v - v'|^{d+2s}} dv' \right)^{\frac{1}{2}} dz \\
\leq \int_{\{ f = 0 \} \cap Q_3\epsilon} \cdots + \int_{\{ 0 < f < 1 - \theta \} \cap Q_3\epsilon} \cdots + \int_{\{ f \geq 1 - \theta \} \cap Q_3\epsilon} \cdots \\
:= I_1 + I_2 + I_3.
\]

Then \( I_1 = 0 \) since \( f = 0 \).

For \( I_2 \) we get

\[
I_2 \leq |\{ 0 < f < 1 - \theta \} \cap Q_{3\epsilon}|^{\frac{1}{4}} \left( \int_{Q_{3\epsilon}} \int_{B_{3\epsilon}} \frac{|f_+(v) - f_+(v')|^2}{|v - v'|^{d+2s}} dv' dz \right)^{\frac{1}{4}} \\
\lesssim |\{ 0 < f < 1 - \theta \} \cap Q_{\frac{1}{2}}|^{\frac{1}{4}} \left( \int_{Q_{\frac{1}{2}}} f_+^2(t, x, v) dz \right)^{\frac{1}{4}} \\
\lesssim |\{ 0 < f < 1 - \theta \} \cap Q_{\frac{1}{2}}|^{\frac{1}{4}}
\]
where we used the Cauchy-Schwarz inequality, Lemma 3.2 and \(0 < f_+ < 1\). Similarly we find for \(I_3\)

\[
I_3 = \int_{Q_{3r_0}} \left( \int_{B_{3r_0}} \frac{|(f - (1 - \theta))(v) - (f - (1 - \theta))(v')|^2}{|v - v'|^{d+2s}} \, dv' \right)^{\frac{1}{2}} \, dz \\
\leq \left( \int_{Q_{3r_0}} \int_{B_{3r_0}} \frac{|(f - (1 - \theta))(v) - (f - (1 - \theta))(v')|^2}{|v - v'|^{d+2s}} \, dv' \, dz \right)^{\frac{1}{2}} \\
\leq \left( \int_{Q_{\frac{3}{2}}} \left[ f(t, x, v) - (1 - \theta) \right]^2 \, dz + \int_{Q_{\frac{3}{2}}} \left[ f(t, x, v) - (1 - \theta) \right] \, h \right)^{\frac{1}{2}} \\
\lesssim \theta + \theta^{\frac{1}{2}} \| h \|_{L^\infty(Q_1)} \\
\lesssim \theta^{\frac{1}{2}} \left( 1 + \| h \|_{L^\infty(Q_1)} \right).
\]

Therefore we can estimate the right hand side of (5.4) as

\[
\frac{1}{\varepsilon^d r_0^{2d(1+2s)}} \int_{Q_{3r_0}} \left( \int_{B_{3r_0}} \frac{|g_+(v) - g_+(v')|^2}{|v - v'|^{d+2s}} \, dv' \right)^{\frac{1}{2}} \, dz + \varepsilon^\sigma \\
\leq \frac{1}{\varepsilon^d r_0^{2d(1+2s)}} \int_{Q_{3r_0}} \left( \int_{B_{3r_0}} \frac{|f_+(v) - f_+(v')|^2}{|v - v'|^{d+2s}} \, dv' \right)^{\frac{1}{2}} \, dz + \varepsilon^\sigma \\
\lesssim \frac{1}{\varepsilon^d r_0^{2d(1+2s)}} \left[ \{0 < f < 1 - \theta \} \cap Q_{\frac{3}{2}} \right]^{\frac{1}{2}} + \theta^{\frac{1}{2}} \left( 1 + \| h \|_{L^\infty(Q_1)} \right) + \varepsilon^\sigma
\]

Together with (5.5) this gives

\[
\delta_2 (\delta_1 - \theta) \leq C \left( \frac{1}{\varepsilon^d r_0^{2d(1+2s)}} \left[ \{0 < f < 1 - \theta \} \cap Q_{\frac{3}{2}} \right]^{\frac{1}{2}} + \theta^{\frac{1}{2}} \left( 1 + \| h \|_{L^\infty(Q_1)} \right) \right) + \varepsilon^\sigma
\]

for some universal constant \(C\). Now we choose \(\varepsilon < 1\) and \(\theta\) so that \(C \varepsilon^\sigma \leq \frac{\delta_1 \delta_2}{4}\) and \(\delta_2 \theta + \frac{C \theta^{\frac{1}{2}} (1 + \| h \|_{L^\infty(Q_1)})}{\varepsilon^d r_0^{2d(1+2s)}} \leq \frac{\delta_1 \delta_2}{4}\). We pick

\[
\varepsilon = \left( \frac{\delta_1 \delta_2}{8C} \right)^{\frac{1}{2}}, \quad \theta = \frac{\delta_1 \delta_2}{4 \left( \delta_2 + \frac{C (1 + \| h \|_{L^\infty(Q_1)})}{\varepsilon^d r_0^{2d(1+2s)}} \right)}.
\]

Then we conclude

\[
\nu |Q_{\frac{3}{2}}| \leq \{0 < f < 1 - \theta \} \cap Q_{\frac{3}{2}}
\]

for

\[
\nu = \frac{1}{|Q_{\frac{3}{2}}|} \left( \frac{\delta_1 \delta_2 \left( \frac{\delta_1 \delta_2}{8C} \right)^{\frac{1}{2}} r_0^{2d(1+2s)}}{2C} \right)^2 \geq (\delta_1 \delta_2)^{6d + 4d + 4}.
\]

□
5.3. **Measure-to-Pointwise estimate.** The following lemma and its proof is taken from Lemma 16 in [11].

**Lemma 5.3.** Let \( \delta \in (0, 1) \), define \( r_0 = \left( \frac{\delta}{288} \right)^{\frac{1}{2}} \) if \( h \neq 0 \) and \( r_0 = \frac{1}{12} \) if \( h = 0 \). Then there is \( \mu = \mu(\delta) \sim \delta^{2(1+\delta-6d-\frac{4d}{d-4})} > 0 \) such that any sub-solution \( f \) to (1.1)-(1.2) in \( Q_1 \) with (1.3)-(1.6) for \( R = 2 \) so that \( f \leq 1 \) in \( (-2^{-2s}, 0) \times B_2^{-1(1+2s)} \times \mathbb{R}^d \) and

\[
\text{(5.9)} \quad | \{ f \leq 0 \} \cap Q_{r_0}^- | \geq \delta |Q_{r_0}^- |
\]

satisfies \( f \leq 1 - \mu \) in \( Q_{r_0}^- \)

where \( Q_{r_0}^- := Q_{r_0}(-2^{1+2s}0, 0, 0) \) as before.

**Proof.** Due to the scaling invariance, we can assume with no loss of generality that \( \| h \|_{L^\infty(Q_1)} \leq 1 \). By Lemma 4.2 we know that there is a \( \delta_0 > 0 \) such that for any \( r > 0 \) any sub-solution \( f \) on \( Q_{2r} \) satisfying \( \int_{Q_r} f^2 \leq \delta_0 |Q_r| \) and \( \| h \|_{L^\infty(Q_1)} \leq 1 \), there holds \( f \leq \frac{1}{2} \) in \( Q_{r}^- \). Therefore, we define \( \nu, \theta > 0 \) as in (5.7)-(5.8) with \( \delta_1 = \delta \) and \( \delta_2 = \delta_0 \).

We consider \( f_k := \theta^{-k} f - (1 - \theta^k) \) for \( k \geq 0 \). Note that \( f_k \) is a sub-solution to (1.1)-(1.2) for each \( k \geq 0 \) with (1.3)-(1.6) and \( \| h \|_{L^\infty(Q_1)} \leq 1 \) as long as \( k \leq 1 + \frac{1}{\theta} \). Moreover, \( \{ 0 < f_i < 1 - \theta \} \cap \{ 0 < f_j < 1 - \theta \} = 0 \) for all \( i \neq j \) and each \( f_k \) satisfies (5.9).

In case that \( \int_{Q_{r_0}} (f_k)^2 \leq \delta_0 |Q_{r_0}| \) there holds \( f_k \leq \frac{1}{2} \) in \( Q_{r_0}^- \), and we conclude \( f \leq 1 - \mu \) with \( \mu = \frac{\theta}{12} \).

Thus it suffices to consider \( 1 \leq k_0 \leq 1 + \frac{1}{\theta} \) such that \( \int_{Q_{r_0}} (f_k)^2 \geq \delta_0 |Q_{r_0}| \) for any \( 0 \leq k \leq k_0 \). Then for \( 0 \leq k \leq k_0 - 1 \) there holds

\[
| \{ f_k \geq 1 - \theta \} \cap Q_{r_0}^- | = | \{ f_{k+1} \geq 0 \} \cap Q_{r_0}^- | \geq \int_{Q_{r_0}^-} (f_{k+1})^2 > \delta_0 |Q_{r_0}^- |
\]

\[
| \{ f_k \leq 0 \} \cap Q_{r_0}^- | \geq | \{ f \leq 0 \} \cap Q_{r_0}^- | \geq \delta |Q_{r_0}^- |
\]

where we used the fact that \( f \leq 0 \) implies \( f_k \leq 0 \) for all \( k \geq 0 \). Thus we can apply Theorem 5.2 for our choice of \( \delta_1, \delta_2 \) and for \( r_0 = \left( \frac{\delta}{288} \right)^{\frac{1}{2}} \) so that we get

\[
| \{ 0 \leq f_k \leq 1 - \theta \} \cap Q_{\frac{1}{2}} | \geq \nu |Q_{\frac{1}{2}} |
\]

Since these sets are all disjoint, we get by summing these estimates

\[
|Q_{\frac{1}{2}} | \geq \sum_{k=0}^{k_0-1} | \{ 0 \leq f_k \leq 1 - \theta \} \cap Q_{\frac{1}{2}} | \geq k_0 \nu |Q_{\frac{1}{2}} |
\]

Thus \( k_0 \leq \frac{1}{\theta} \) and we deduce with \( f_{k_0+1} \leq \frac{1}{2} \) in \( Q_{r_0}^- \)

\[
f \leq 1 - \frac{\theta^{k_0+1}}{2} \leq 1 - \frac{\theta^{1+\theta}}{2} \quad \text{in } Q_{r_0}^- .
\]

This yields the claim for \( \mu(\delta) := \frac{\theta^{1+\theta}}{2} \sim \delta^{2(1+\delta-6d-\frac{4d}{d-4})} \). \( \square \)
6. Hölder Continuity and Harnack Inequalities

6.1. Harnack Inequalities. We follow section 4.1 in [11]. We consider a non-negative super-solution $f$ to (1.1)-(1.2) for $h = 0$ on $Q_1$ so that (1.3)-(1.6) holds for $R = 2$. Then Lemma 5.3 applied to the sub-solution $g := 1 - \frac{f}{M}$ implies for any $\delta \in (0, 1)$ and $M \sim \delta^{-2(1+\delta-\frac{6d-4d}{1-2})}$ that

$$\forall Q_r(z) \subset Q_1 \text{ so that } Q_f^+ \subset Q_1, \quad \left| \frac{\{f > M\} \cap Q_r(z)}{|Q_r(z)|} \right| > \delta \implies \inf_{Q_f^+(z)} f \geq 1,$$

where we recall $Q_f^+(z) = Q_r(z + (2r^{2s}, 2r^{2s}v, 0))$ for $z = (t, x, v)$. This implies using the layer-cake representation that if $\inf_{Q_f^+} f < 1$, then

$$\left| \frac{\{f > M\} \cap Q_r^{-}}{|Q_r^{-}|} \right| \lesssim \delta(M) = \left( \frac{1}{\ln(1+M)} \right)^{\frac{1}{6d+4d/1-2}} \implies \int_{Q_r^{-}} (\ln(1+f))^{\frac{1}{6d+4d/1-6}} \lesssim 1.$$

We can improve this logarithmic integrability as follows. We pick $r_0 = \frac{1}{12}$ and consider the sequence of cylinders

$$Q^k := Q_{r_0} + \alpha_k \left( -\frac{5}{2}r_0^{2s} + \frac{1}{2}r_0^{2s} + \alpha_k \right)^2(0, 0) \quad \text{where } \alpha_k := \frac{r_0}{2^{1-k}}.$$

Then $Q^1 = Q_{r_0}$ and $Q^k \to \hat{Q}_{r_0}^k := Q_{r_0} \left( (-\frac{5}{8}r_0^{2s}, 0, 0) \right)$ as $k \to \infty$. Note that these cylinders satisfy $\hat{Q}_{r_0}^k \subset Q^k \subset \hat{Q}_{r_0}^k \subset \hat{Q}_{r_0}^k \subset \hat{Q}_{r_0}^{-1} \subset Q_{r_0}$ for all $k \geq 1$. We claim as in [11] that for $\delta_0 > 0$ to be determined, for any non-negative super-solution $f$ with $\inf_{\hat{Q}_{r_0}^k} f < 1$, there holds

$$\forall k \geq 1, \quad \left| \frac{\{f > M^k\} \cap Q_k^k}{|Q_k^k|} \right| \leq \delta_0 \left( \frac{1}{35m} \right)^{(2d+2s(d+1))k}$$

where $M \sim \delta^{-2(1+\delta-\frac{6d-4d}{1-2})}$ with $\delta := \delta_0 \left( \frac{1}{35m} \right)^{2d+2s(d+1)}$ and $m \geq 3$. If we can show that (6.3) holds, then we deduce with the layer-cake representation that there exists $\zeta \gtrsim \delta_0^{6d+4d/1-6} > 0$ such that $\int_{\hat{Q}_{r_0}^k} \hat{f}^\zeta \, dz \lesssim 1$, which in turn yields by linearity

$$\left( \int_{\hat{Q}_{r_0}^k} \hat{f}(z)^\zeta \, dz \right)^{\frac{1}{\zeta}} \lesssim \inf_{\hat{Q}_{r_0}^k} \hat{f}. $$

This implies the weak Harnack inequality (1.8) for any non-negative super-solution $\hat{f}$ to (1.1)-(1.2) by applying the previous estimate to $f = \hat{f} + (1 + t)||h||_{L^\infty}$. For the strong Harnack inequality (1.9) we apply (4.1) in Lemma 4.2 to a non-negative solution $f$

$$\sup_{\hat{Q}_{r_0}^k} f \lesssim \left( \int_{\hat{Q}_{r_0}^k} f(z)^\zeta \, dz \right)^{\frac{1}{\zeta}} \lesssim \inf_{\hat{Q}_{r_0}^k} f \lesssim \inf_{\hat{Q}_{r_0}^k} f.$$

We now prove (6.3) inductively. The case $k = 1$ holds by (6.2). We define

$$A_{k+1} := \{f > M^{k+1}\} \cap Q^{k+1}$$

and denote
\[ \mathcal{C}_r[z] := z \circ Q_{2r}\left(\frac{1}{2}(2r)^2s, 0, 0\right) = z \circ \left(-\frac{1}{2}(2r)^2s, \frac{1}{2}(2r)^2s\right) \times B_{(2r)^2s+1} \times B_{2r}. \]

Just as in [11] we construct \( z_l = (t_l, x_l, v_l) \in \mathcal{Q}^{k+1} \) and \( r_l > 0, l \geq 1, m \geq 3, n \geq 1 \) so that
\[
\begin{align*}
\forall l \geq 1, r_l &\in \left(0, \frac{\alpha+1}{5m \cdot n}\right), \\
\forall l \geq 1, |A_{k+1} \cap \mathcal{C}_{5m \cdot r_l}[z_l]| &\leq \delta_0|\mathcal{C}_{5m \cdot r_l}[z_l]|, \\
\forall l \geq 1, |A_{k+1} \cap \mathcal{C}_{r_l}[z_l]| &> \delta_0|\mathcal{C}_{r_l}[z_l]|, \\
\mathcal{C}_{r_l}, l \geq 1, &\text{ are disjoint cylinders,} \\
A_{k+1} &\text{ is covered by the family } (\mathcal{C}_{5m \cdot r_l}[z_l])_{l \geq 1}.
\end{align*}
\]

For these cylinders, we have \( \mathcal{C}_{r_l}^+[z_l] := z_l \circ Q_{2r_l}\left(\frac{1}{2}(2r_l)^2s, 0, 0\right) = z_l \circ \left(\frac{1}{2}(2r_l)^2s, \frac{1}{2}(2r_l)^2s\right) \times B_{(2r_l)^2s+2} \times B_{2r_l}. \) We remark that for \( m \geq \frac{5}{2} \), there holds \( \mathcal{C}_{r_l}[z_l]^+ \subset \mathcal{C}_{m \cdot r_l}[z_l] \). Moreover, we pick \( n \geq 1 \) so that \( \forall k \geq 1 \)
\[
(\tau^k + \tau)^{2s} - (\tau^{k+1})^{2s} \geq \left(\frac{2}{n}\right)^{2s}.
\]

Then by choice of \( z_l \) and \( r_l \) we have that \( \mathcal{C}_{5m \cdot r_l}[z_l] \subset \mathcal{Q}^{k} \). Note that for example, if \( s \geq \frac{1}{2} \) we can choose \( n = 1 \).

As in [11], we prove that the family of cylinders \( \mathcal{C}_r[z] \) with \( z \in \mathcal{Q}^{k+1}, r \in \left(0, \frac{\alpha+1}{5m \cdot n}\right) \), so that \( |A_{k+1} \cap \mathcal{C}_{5m \cdot r}[z]| \leq \delta_0|\mathcal{C}_{5m \cdot r}[z]| \) and \( |A_{k+1} \cap \mathcal{C}_r[z]| > \delta_0|\mathcal{C}_r[z]| \) cover \( A_{k+1} \). By induction hypothesis there holds for all \( r \in \left(\frac{5m \cdot n}{\alpha+1}, \alpha+1\right) \)
\[
(6.4) \quad |A_{k+1} \cap \mathcal{C}_r[z]| \leq |A_k \cap \mathcal{C}_r[z]| \leq |A_k \cap \mathcal{Q}^{k}| \leq \delta_0\left(\frac{1}{5m\cdot n}\right)^{(2d+2s(d+1))k} |\mathcal{Q}^{k}| \leq \delta_0|\mathcal{C}_r[z]|.
\]

If \( z \in A_{k+1} \) is not covered by \( \mathcal{F} \) then the continuous positive function \( \varphi(r) = \frac{|A_{k+1} \cap \mathcal{C}_r[z]|}{|\mathcal{C}_r[z]|} \) on \( (0, +\infty) \) satisfies \( \varphi(r) \leq \delta_0 \) or \( \varphi(5m \cdot n \cdot r) > \delta_0 \) for all \( r \in \left(0, \frac{\alpha+1}{5m \cdot n}\right) \). By (6.4) and by continuity, we must have \( \varphi(r) \leq \delta_0 \) for all \( r \in \left(0, \frac{\alpha+1}{5m \cdot n}\right) \). Lebesgue’s differentiation theorem implies in the limit \( r \to 0 \) there holds \( z \notin A_{k+1} \). Thus \( \mathcal{F} \) covers \( A_{k+1} \).

In particular, \( A_{k+1} \) is covered by the family \( \mathcal{F}' \) of cylinders \( \mathcal{C}_{m \cdot r}[z] \) with \( z \in \mathcal{Q}^{k+1}, r \in \left(0, \frac{\alpha+1}{5m \cdot n}\right) \), so that \( |A_{k+1} \cap \mathcal{C}_{m \cdot r}[z]| \leq \delta_0|\mathcal{C}_{m \cdot r}[z]| \) and \( |A_{k+1} \cap \mathcal{C}_r[z]| > \delta_0|\mathcal{C}_r[z]| \). Vitali’s covering lemma gives us the existence of a countable sub-family, denoted \( (\mathcal{C}_{r_l}[z_l])_{l \geq 1} \), such that \( (\mathcal{C}_{5m \cdot r_l}[z_l])_{l \geq 1} \) covers \( A_{k+1} \) and the \( (\mathcal{C}_{m \cdot r_l}[z_l])_{l \geq 1} \) are disjoint.

This finishes the construction of the covering with the desired properties. We can then apply Lemma 5.3 to each \( \mathcal{C}_{r_l}[z_l] \) to obtain \( \mathcal{C}_{r_l}[z_l]^+ \subset A_k \) and the \( (\mathcal{C}_{r_l}[z_l]^+))_{l \geq 1} \) are disjoint since \( \mathcal{C}_{r_l}[z_l]^+ \subset \mathcal{C}_{m \cdot r_l}[z_l] \).
This yields for the right choice of $\delta_0$, i.e. $\delta_0 \leq \left( \frac{4}{1225m^2} \right)^{2d+2s(d+1)}$, $\mu$
\[|A_{k+1}| \leq \sum_{l \geq 1} |A_{k+1} \cap C_{s_m^l} | \leq \delta_0 \sum_{l \geq 1} |C_{s_m^l}| \leq (5m)^{2d(s+1)+2s\delta_0} \sum_{l \geq 1} |C_{s_m^l}| \leq (5m)^{2d(s+1)+2s\delta_0} \left( \frac{1}{35m} \right)^{(2d+2s(d+1))k} |Q_k| \leq \delta_0 \left( \frac{1}{35m} \right)^{(2d+2s(d+1))(k+1)} |Q_{k+1}|.\]

This proves (6.3) and thus concludes the proof.

6.2. Hölder Continuity. Obtaining (1.10) is a standard argument, see [27] and [11]. Let $f$ be a weak solution to (1.1)-(1.2). For given $r_0 = \frac{1}{M}$ Lemma 5.3 implies
\[\text{osc}_{Q_{r_0}} f \leq \left( 1 - \frac{\mu}{2} \right) \max \left\{ \text{osc}_{Q_1} f, e^{2(1+2^{d+\frac{d+4}{4}})} \Vert h \Vert_{L^\infty(Q_2)} \right\} \]

To see this we apply Lemma 5.3 rescaled to $Q_2$ to either $F$ or $-F$ depending on which of these satisfy (5.9), where $F$ is defined by
\[F := \frac{2f - (\sup_{Q_1} f + \inf_{Q_1} f)}{\max \left\{ \text{osc}_{Q_1} f, e^{2(1+2^{d+\frac{d+4}{4}})} \Vert h \Vert_{L^\infty(Q_2)} \right\}}.\]

We want to prove $\forall z_0 \in Q_1, \forall r \in (0, r_0),$
\[\text{osc}_{Q_r(z_0)} f \leq r^\alpha e^{2(1+2^{d+\frac{d+4}{4}})} \left( 1 + \Vert h \Vert_{L^\infty(Q_2)} \right) \max \left\{ \text{osc}_{Q_1} f, e^{2(1+2^{d+\frac{d+4}{4}})} \Vert h \Vert_{L^\infty(Q_2)} \right\},\]

where we chose $\alpha \in (0,1)$ so that $1 - \frac{\mu}{2} = r_0^\alpha$. From there we can deduce Hölder regularity by choosing $z, z' \in Q_1$ so that $|z - z'| \leq r_0$.

In order to prove (6.6) we proceed iteratively. We define for $n \in \mathbb{N} \setminus \{0\}$ a sequence of solutions to (1.1)-(1.2) in $Q_1$ by
\[f_n(s, y, w) := \frac{2(1 - \frac{\mu}{2})^{1-n} f(t_0 + r_0^{2s_n} s_0 - r_0^{2s_n} s_0 + r_0^{1+2s_n} y_0, v_0 + r_0^{w} w)}{\max \left\{ \text{osc}_{Q_1} f, e^{2(1+2^{d+\frac{d+4}{4}})} \Vert h \Vert_{L^\infty(Q_2)} \right\}}.\]

We prove by induction on $n \geq 1$ that $\text{osc}_{Q_1} f_n \leq 2e^{2(1+2^{d+\frac{d+4}{4}})} \left( 1 + \Vert h \Vert_{L^\infty(Q_2)} \right)$. The case $n = 1$ is clear by definition of $f_1$. Using (6.5) and the induction hypothesis, we get
\[\text{osc}_{Q_1} f_n = \left( 1 - \frac{\mu}{2} \right)^{-1} \text{osc}_{Q_{r_0}} f_{n-1} \leq \max \left\{ \text{osc}_{Q_1} f_{n-1}, e^{2(1+2^{d+\frac{d+4}{4}})} \Vert h \Vert_{L^\infty(Q_2)} \right\} \leq 2e^{2(1+2^{d+\frac{d+4}{4}})} \left( 1 + \Vert h \Vert_{L^\infty(Q_2)} \right).\]
This implies (6.6) if we choose $\alpha \in (0, 1)$ so that $1 - \frac{n}{2} = r_0^\alpha$. Indeed for $z_0 \in Q_1$ and $r \in (0, r_0)$ we write $r = r_0^n$ for some $n \in \mathbb{N} \setminus \{0\}$. Then

$$\text{osc}_{Q_{r(z_0)}} f = \frac{1}{2} r_0^{n(n-1)} \max \left\{ \text{osc}_{Q_0} f, e^{2(1+2^{3d+4}))n\|h\|_{L^\infty(\mathbb{R}^d)}} \right\} \text{osc}_{Q_{r(z_0)}} f_n$$

$$= \frac{1}{2} r_0^{n(n-1)} r_0^n \max \left\{ \text{osc}_{Q_{z_0}} f, e^{2(1+2^{3d+4}))n\|h\|_{L^\infty(\mathbb{R}^d)}} \right\} \text{osc}_{Q_1} f_{n+1}$$

$$\leq r_0^{n(n-1)} e^{2(1+2^{3d+4}))n\|h\|_{L^\infty(\mathbb{R}^d)}} (1 + \|h\|_{L^\infty(\mathbb{R}^d)}) \max \left\{ \text{osc}_{Q_1} f, e^{2(1+2^{3d+4}))n\|h\|_{L^\infty(\mathbb{R}^d)}} \right\}$$

$$= r_0^{n(n-1)} e^{2(1+2^{3d+4}))n\|h\|_{L^\infty(\mathbb{R}^d)}} (1 + \|h\|_{L^\infty(\mathbb{R}^d)}) \max \left\{ \text{osc}_{Q_1} f, e^{2(1+2^{3d+4}))n\|h\|_{L^\infty(\mathbb{R}^d)}} \right\}.$$ 

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**References**

[1] Hajer Bahouri. The Littlewood-Paley theory: a common thread of many works in nonlinear analysis. *Eur. Math. Soc. News.*, 112:15–23, 2019.

[2] Emeric Bouin and Clément Mouhot. Quantitative fluid approximation in transport theory: a unified approach. 2020.

[3] Luis Caffarelli, Chi Hin Chan, and Alexis Vasseur. Regularity theory for nonlinear integral operators. *J. Amer. Math. Soc.*, 24(3):849–869, 2011.

[4] Emeric Bouin and Clément Mouhot. Quantitative fluid approximation in transport theory: a unified approach. 2021.

[5] M. H. Ernst and R. Brito. Scaling Solutions of Inelastic Boltzmann Equations with Over-Populated High Energy Tails. *Journal of Statistical Physics*, 109(3/4):407–432, 2002.

[6] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Berlin: Springer-Verlag, 1983.

[7] Ennio De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino Cl. Sci. Fis. Math. Nat.*, 3:25–43, 1957.

[8] François Golse, Cyril Imbert, Clément Mouhot, and Alexis F. Vasseur. Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 19(1):253–295, 2019.

[9] Jessica Guerand. Quantitative regularity for parabolic De Giorgi classes. *Communications in Mathematical Sciences*, 2020. To appear.

[10] Jessica Guerand and Cyril Imbert. Log-transform and the weak Harnack inequality for kinetic Fokker-Planck equations, 2021.

[11] Jessica Guerand and Clément Mouhot. Quantitative de Giorgi Methods in Kinetic Theory. *Journal de l’École polytechnique*, 2021. To appear.

[12] Cyril Imbert, Clément Mouhot, and Luis Silvestre. Décroissance aux grandes vitesse pour les solutions de l’équation de Boltzmann sans troncature angulaire. *J. Éc. Polytech.*, 7:143–184, 2020.

[13] Cyril Imbert, Clément Mouhot, and Luis Silvestre. Gaussian Lower Bounds for the Boltzmann Equation without Cut-off. *SIAM J. Math. Anal.*, 52(3):2930–2944, 2020.

[14] Cyril Imbert and Luis Silvestre. The weak Harnack inequality for the Boltzmann equation without cut-off. *J. Eur. Math. Soc. (JEMS)*, 22(2):507–592, 2020.

[15] Cyril Imbert and Luis Silvestre. Global regularity estimates for the Boltzmann equation without cut-off. *J. Amer. Math. Soc.*, 2021.

[16] Cyril Imbert and Luis Silvestre. The Schauder estimate for kinetic integral equations. *Anal. PDE*, 14(1):171–204, 2021.
[17] Sergiu Klainerman and Igor Rodnianski. A geometric approach of Littlewood-Paley theory. *Geom. Funct. Anal.*, 16(1):126–163, 2006.

[18] Stanislav Nikolaevich Kružkov. A priori bounds for generalized solutions of second-order elliptic and parabolic equations. *Dokl. Akad. Nauk SSSR*, 150:748–751, 1963.

[19] Antoine Mellet. Fractional diffusion limit for collisional kinetic equations: a moments method. *Indiana Univ. Math. J.*, 59(4):1333–1360, 2010.

[20] Antoine Mellet, Stéphane Mischler, and Clément Mouhot. Fractional diffusion limit for collisional kinetic equations. *Archive for Rational Mechanics and Analysis*, 199(2):493–525, Aug 2010.

[21] D. A. Mendis and M. Rosenberg. Cosmic dusty plasma. *Ann. Rev. Astron. Astrophys.*, 32:419–463, 1994.

[22] Jürgen Moser. On Harnack’s theorem for elliptic differential equations. *Comm. Pure Appl. Math.*, 14:577–591, 1961.

[23] Jürgen Moser. A Harnack inequality for parabolic differential equations. *Commun. Pure Appl. Math.*, 17:101–134, 1964.

[24] Lukas Niebel and Rico Zacher. Kinetic maximal $L^2$-regularity for the (fractional) Kolmogorov equation. *J. Evol. Equ.*, 21:3585–3612, 2021.

[25] Luis Silvestre. A new regularization mechanism for the Boltzmann equation without cut-off. *Comm. Math. Phys.*, 348(1):69–100, 2016.

[26] Logan F. Stokols. Hölder continuity for a family of nonlocal hypoelliptic kinetic equations. *SIAM J. Math. Anal.*, 51(6):4815–4847, 2019.

[27] Alexis F. Vasseur. The De Giorgi method for elliptic and parabolic equations and some applications. In *Lectures on the analysis of nonlinear partial differential equations. Part 4*, volume 4 of *Morningside Lect. Math.*, pages 195–222. Int. Press, Somerville, MA, 2016.

[28] Wendong Wang and Liqun Zhang. The $C^\alpha$ regularity of a class of non-homogeneous ultraparabolic equations. *Sci. China Ser. A.*, 52:1589–1606, 2009.

[29] Wendong Wang and Liqun Zhang. The $C^\alpha$ regularity of weak solutions of ultraparabolic equations. *Discrete Contin. Dyn. Syst.*, 29:1261–1275, 2011.

[30] Wendong Wang and Liqun Zhang. $C^\alpha$ regularity of weak solutions of non-homogenous ultraparabolic equations with drift terms, 2017.

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