ATTACHING HANDLES TO DELAUNAY NODOÏDS

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Abstract. For all $m \in \mathbb{N} - \{0\}$, we prove the existence of a one dimensional family of genus $m$, constant mean curvature (equal to 1) surfaces which are complete, immersed in $\mathbb{R}^3$ and have two Delaunay ends asymptotic to nodoïdal ends. Moreover, these surfaces are invariant under the group of isometries of $\mathbb{R}^3$ leaving a horizontal regular polygon with $m + 1$ sides fixed.

1. Introduction

Delaunay surfaces are complete, non compact constant mean curvature surfaces of revolution in $\mathbb{R}^3$ which are either embedded or immersed. The embedded Delaunay surfaces are usually referred to as unduloïds. The elements of this family are generated by roulettes of ellipses [2] and they interpolate between a right cylinder $S^1(\frac{1}{2}) \times \mathbb{R} \subset \mathbb{R}^3$ and a singular surface which is constituted by infinitely many tangent spheres of radius 1 which are periodically arranged along the vertical axis. Close to the singular limit, the Delaunay unduloïds can be understood as infinitely many spheres of radius 1 which are disjoint, arranged periodically along the vertical axis; each sphere being connected to its two nearest neighbors by catenoïds whose rotational axis is the vertical axis, which have been scaled by a small factor $\tau > 0$.

The immersed Delaunay surfaces are referred to as nodoïds. The element of this family are generated by roulettes of hyperbola [2]. Again, part of this family converges to infinitely many spheres of radius 1 which are periodically arranged along the vertical axis. In contrast to unduloïds, close to the singular limit, the Delaunay nodoïds can be understood as infinitely many spheres of radius 1 which are either disjoint or slightly overlapping and which are arranged periodically along the vertical axis; each sphere being connected to its two nearest neighbors (with whom it shares a slight overlap) by catenoïds whose axis is the vertical, which have been scaled by a small factor $\tau > 0$.

In this paper, we prove the existence of constant mean curvature surfaces which have two Delaunay ends (of nodoïd type) and finite genus.

Theorem 1.1. For all $m \geq 1$, there exists a one parameter family of genus $m$ constant mean curvature (with mean curvature equal to 1) surfaces which are invariant under the action of the full dihedral group $\text{Dih}_{m+1}^{(3)}$ (the group of isometries of $\mathbb{R}^3$ leaving a horizontal regular polygon with $m + 1$ sides fixed) and which have two Delaunay ends asymptotic to nodoïdal ends.

Let us briefly describe how these surfaces are constructed since this will be the opportunity to give a precise picture of the surfaces themselves. As already mentioned, close to the singular limit, the Delaunay nodoïds can be understood as infinitely many spheres of radius 1 which are either disjoint or slightly overlapping.

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arranged periodically along the vertical axis and which are connected together by catenoids with vertical axis, which are scaled by a small factor $\tau > 0$, these latter are called catenoidal necks. The spheres of radius 1 arranged along the vertical axis can be ordered (by the height of their center) and can be indexed by $j \in \mathbb{Z}$ (without loss of generality, we can assume that the center of the sphere of index $j$ is at height $2j + 1$). In this description, one can check that the distance between the centers of two consecutive spheres can be expanded as

$$d_\tau = 2 + 2 \tau \log \tau + \mathcal{O}(\tau),$$

as $\tau$ tends to 0. In order to obtain the surfaces of Theorem 1.1, instead of connecting the sphere indexed by 0 and the sphere indexed by 1 using one catenoidal neck, we connect these two spheres using $m + 1$ catenoids which are scaled by a factor

$$\tilde{\tau} = \frac{\tau}{m + 1} + \mathcal{O}(\tau^{3/2}),$$

and whose axis are vertical and pass through the vertices of a horizontal regular polygon (with $m + 1$ sides) of size $\rho > 0$. We will show that this construction is successful provided the parameter $\rho$ which measures the size of the polygon, is carefully chosen (as a function of $\tau$) and, in fact, we will find that

$$\rho^2 = \frac{m}{m + 1} \frac{\tau}{2} + \mathcal{O}(\tau^{5/4}).$$

Notice that all the surfaces we construct have the same small vertical flux (we refer to §4 for a definition of the flux of a Delaunay surface).

Our construction is quite flexible and provides many other interesting constant mean curvature surfaces. For example, using similar ideas and proofs, one can also construct singly periodic constant mean curvature surfaces with (infinite) topology: starting with the spheres of radius 1 which are periodically arranged along the vertical axis and which are either disjoint or slightly overlapping, we can choose to connect any two consecutive spheres using $m + 1$ catenoids scaled by a factor $\tilde{\tau}$ whose axis are vertical and pass through the vertices of a small horizontal regular polygon (with $m + 1$ sides) of size $\rho > 0$. More generally, there is strong evidence that the following is true:

It should be possible to construct constant mean curvature surfaces starting from a subset $\mathcal{F} \subset \mathbb{Z}$ and assuming that, for all $j \in \mathbb{Z} - \mathcal{F}$, we decide to connect the sphere of index $j$ to the sphere of index $j + 1$ using one catenoid whose axis is the vertical axis and which is scaled by a factor $\tau$, while, when $j \in \mathcal{F}$, we decide to connect the sphere of index $j$ to the sphere of index $j + 1$ using $m + 1$ catenoids whose axis are vertical and pass through the vertices of a small horizontal regular polygon (with $m + 1$ sides) of size $\rho > 0$ with $\rho^2 \sim \frac{m}{m + 1} \frac{\tau}{2}$, and which are scaled by a factor $\tilde{\tau} \sim \frac{\tau}{m + 1}$. We believe that this configuration can be perturbed into a genuine constant mean curvature surface.

To complete this introduction, let us mention that the present construction is very much inspired by [4] where the construction of minimal surfaces in $\mathbb{R}^3$ which have finite genus and two Riemann type ends is performed. In fact, part of the analysis in the present paper parallels the analysis in [4]. Nevertheless, in the present situation, some extra technical difficulties arise in the construction (see §6) since the points where the connected sum is performed are located at the vertices of a polygon whose size tends to 0 as the parameter $\tau$ tends to 0.
We end this introduction by giving an overview of the paper. In section 2 we recall some well known facts about the mean curvature operator of normal graphs with special emphasize on the differential of the mean curvature operator. Section 3 is concerned with harmonic extensions on half cylinders for which we prove some decay properties. The next section is quite long, it starts with a careful description of the Delaunay nodoids as the Delaunay parameter $\tau$ tends to 0 (i.e. close to the singular limit). Then, we proceed with the analysis of the Jacobi operator about a Delaunay surface as the Delaunay parameter tends to 0. Finally, in section 4.5, we apply the implicit function theorem about a half nodoid (which is a constant mean curvature surface with one boundary and one Delaunay end) to prove the existence of an infinite dimensional family of constant mean curvature surfaces which have one Delaunay end and one boundary. These surfaces are close to the half nodoid we started with and are parameterized by their boundary data. In section 6, we perform a similar analysis starting from the catenoid. As a result, we obtain the existence of an infinite dimensional family of constant mean curvature surfaces which have two boundaries, are close to a truncated catenoid and are parameterized by their boundary data. In section 6, we start with a unit sphere from which we excise one small disc close to the north pole and $m+1$ small discs arranged symmetrically at the vertices of a regular polygon near the south pole. We perturb this surface with $m+2$ boundaries applying the implicit function theorem to obtain an infinite dimensional family of constant mean curvature surfaces which are parameterized by their boundary data. In the final section, we explain how all these pieces can be connected together to produce the surfaces in Theorem 1.1. At this stage, the problem then reduces to be able to chose the boundary data of the different summands so that their union is a $C^1$ surface, since elliptic regularity theory will imply that what we have built is a smooth constant mean curvature surface.

The construction heavily relies on the analysis of elliptic operators on non compact spaces as in [11], [7], [6]. It is true that similar techniques and ideas have already been used in many constructions, but the proofs are usually hard to read for non specialists since they always refer to results which are difficult to find in the literature in the precise form they are needed. This is the reason why we have decided to present here complete proofs based on simple well known tools, hoping that this will help the interested reader to master these technics.

Finally, we mention a problem related to our work. To introduce this problem, we consider $\Sigma$ to be the union of the upper hemisphere of the sphere of radius 1 centered at the points $(0,0,-1)$ and the lower hemisphere of the sphere of radius 1 centered at the points $(0,0,1)$. The existence of unduloïds, nodoids with small Delaunay parameters and the existence of the surfaces we construct in this paper show that, for all $\epsilon > 0$ there exists infinitely many constant mean curvature ($=1$) surfaces which are included in an $\epsilon$-tubular neighborhood of the unduloïd and are not congruent. Obviously a similar result holds for the surface $\Sigma$.

Now, if we consider two radius one spheres tangent at a point. Can one find constant mean curvature ($=1$) surfaces (with no boundary) in any small tubular neighborhood of this configuration? In fact, we can not even answer the (apparently) simpler but striking question. Is there any compact mean curvature ($=1$) surface (with no boundary) near a radius one sphere? More precisely: is there an
\( \epsilon_0 > 0, \) such that if \( \Sigma \) is a mean curvature (= 1) surface in the \( \epsilon_0 \)-tubular neighborhood of a radius one sphere, then \( \Sigma \) is congruent to the sphere? In other words, what is the form of a compact constant mean curvature surface?

2. Generalities

2.1. The mean curvature. We gather some basic material concerning the mean curvature of a surface in Euclidean space. All these results are well known but we feel that it makes the reading of the paper easier if we collected them here. Moreover, this will also be the opportunity to introduce some of the notations we will use throughout the paper. We refer to [1] or [5] for further details.

Let us assume that \( \Sigma \) is a surface which is embedded in \( \mathbb{R}^3 \). We denote by \( g \) the metric induced on \( \Sigma \) by the Euclidean metric \( \tilde{g} \) and by \( h \) the second fundamental form defined by

\[
h(t_1, t_2) = -\tilde{g}(\nabla t_1 N, t_2),
\]

for all \( t_1, t_2 \in T\Sigma \). Here \( N \) is a unit normal vector field on \( \Sigma \). In this paper, we agree that the mean curvature of a surface is defined to be the average of the principal curvatures, or, since we are interested in 2 dimensional surfaces, the half of the trace of the second fundamental form. Hence, the mean curvature of \( \Sigma \) is given by

\[
H := \frac{1}{2} \text{tr} h,
\]

and the mean curvature vector is then given by \( \vec{H} := H N \).

For computational purposes, we recall that the mean curvature appears in the first variation of the area functional. More precisely, given \( w \), a sufficiently small smooth function which is defined on \( \Sigma \) and has compact support, we consider the surface \( \Sigma_w \) which is the normal graph over \( \Sigma \) for the function \( w \). Namely

\[
\Sigma \ni p \mapsto \vec{p} + w(p) N(p) \in \Sigma_w.
\]

We denote by \( A_w \) the area of the surface \( \Sigma_w \) (we assume that this area is finite). Then

\[
DA_{|w=0}(v) = -2 \int_{\Sigma} H v \text{dvol}_{\tilde{g}}.
\]

In the case where surfaces close to \( \Sigma \) are parameterized as graphs over \( \Sigma \) using a vector field \( \tilde{N} \) which is transverse to \( \Sigma \) but which is not necessarily a unit normal vector field, the previous formula has to be modified. Let us denote by \( \tilde{\Sigma}_w \) the surface which is the graph over \( \Sigma \), using the vector field \( \tilde{N} \), for some sufficiently small smooth function \( w \). Namely

\[
\Sigma \ni p \mapsto \vec{p} + w(p) \tilde{N}(p) \in \tilde{\Sigma}_w.
\]

We denote by \( \tilde{A}_w \) the area of this surface. The previous formula has to be changed into

\[
(2.1) \quad D\tilde{A}_{|w=0}(v) = -2 \int_{\Sigma} (\tilde{H} \cdot \tilde{N}) v \text{dvol}_{\tilde{g}}.
\]

In the next result, we give the expression of the mean curvature \( H_w \) of the surface \( \Sigma_w \) in terms of \( w \). Some notations are needed. For \( z \in \mathbb{R} \) small enough, we define \( g_z \) to be the induced metric on the parallel surface

\[
\Sigma_z := \Sigma + z N.
\]
It is given explicitly by
\[ g_z = g - 2zh + z^2k, \]
where the tensor \( k \) is defined by
\[ k(t_1, t_2) := g(\nabla_{t_1} N, \nabla_{t_2} N). \]
for all \( t_1, t_2 \in T\Sigma \). With these notations, we have the :

**Proposition 2.1.** The mean curvature \( H_w \) of the surface \( \Sigma_w \) is given by the formula
\[
H_w = \frac{1}{2} \sqrt{1 + |\nabla g(w)|^2} \text{tr}(h - wk) + \frac{1}{2} \text{div}_{g_w} \left( \frac{\nabla g_w}{\sqrt{1 + |\nabla g_w|^2}} \right) - \frac{1}{2} \frac{1}{\sqrt{1 + |\nabla g_w|^2}} (h - wk) (\nabla g_w, \nabla g_w).
\]

**Proof.** The induced metric \( \tilde{g} \) on \( \Sigma_w \) is given by
\[
\tilde{g} = g_z = dw \otimes dw.
\]
In particular, this implies that
\[
\det \tilde{g} = (1 + |\nabla g(w)|^2) \det g_w.
\]
We can now compute the area of \( \Sigma_w \)
\[
A_w = \int_{\Sigma} \sqrt{1 + |\nabla g(w)|^2} \, d\text{vol}_{g_w},
\]
as well as the differential of this functional with respect to \( w \). In doing so, one should be careful that the function \( w \) appears implicitly in the definition of \( g_w \). We find using an integration by parts
\[
DA_w(v) = -\int_{\Sigma} \text{div}_{g_w} \left( \frac{\nabla g_w}{\sqrt{1 + |\nabla g_w|^2}} \right) v \, d\text{vol}_{g_w} - \frac{1}{2} \int_{\Sigma} \frac{1}{\sqrt{1 + |\nabla g_w|^2}} g_w'(\nabla g_w, \nabla g_w) v \, d\text{vol}_{g_w} + \frac{1}{2} \int_{\Sigma} \sqrt{1 + |\nabla g_w|^2} tr g_w g_w' v \, d\text{vol}_{g_w},
\]
where \( g_w' := \partial_z g |_{z=w} = -2(h - wk) \). To proceed, observe that, if \( N_w \) denotes the normal vector field about \( \Sigma_w \), we have
\[
N_w = \frac{1}{\sqrt{1 + |\nabla g_w|^2}} (N - \nabla g_w),
\]
and hence we get
\[
d\text{vol}_{g_w} = (N_w \cdot N) \, d\text{vol}_{\tilde{g}}.
\]
The result then follows at once from (2.1). \( \Box \)
2.2. Linearized mean curvature operators. Again the material in the section is well known and we refer to [1] and [5] for a more detailed description. The Jacobi operator appears in the linearization of the mean curvature operator when nearby surfaces are parameterized as normal graphs over a given surface. Indeed, we can consider the nonlinear operator \( w \mapsto H_w \) which is defined for example from the space \( \mathcal{C}^2_{loc}(\Sigma) \) into the space \( \mathcal{C}^0_{loc}(\Sigma) \) and it follows from Proposition 2.1 that the differential of this operator with respect to \( w \), computed at \( w = 0 \), is given by

\[
J := DH_{w=0} = \frac{1}{2} (\Delta_g + \text{tr}^g k),
\]

where \( \Delta_g \) is the Laplace-Beltrami operator on \( \Sigma \) and \( \text{tr}^g k \) is the square of the norm of the shape operator.

Finally, we recall that if \( \Sigma \) is a constant mean curvature surface and if \( \Xi \) is a Killing vector field (namely \( \Xi \) generates a one parameter family of isometries) then the function \( N \cdot \Xi \), which is usually referred to as a Jacobi field, satisfies

\[
J(N \cdot \Xi) = 0.
\]

This is probably a good time to recall some elementary facts concerning linearized mean curvature operators when different vector fields are used. As above, we assume that we are given a vector field \( \tilde{N} \) which is transverse to \( \Sigma \), but which is not necessarily a unit normal vector field. Any surface close enough to \( \Sigma \) can be considered either as a normal graph over \( \Sigma \) or as a graph over \( \Sigma \), using the vector field \( \tilde{N} \), hence, we can define two nonlinear operators

\[
w \mapsto H_w, \quad \text{and} \quad w \mapsto \tilde{H}_w,
\]

which are (respectively) the mean curvature of the normal graph of \( w \) and the mean curvature of the graph of \( w \) using the vector field \( \tilde{N} \). The following result gives the relation between the differentials of these two operators at \( w = 0 \) [9].

**Proposition 2.2.** The following relation holds

\[
DH_{|w=0}(v) = D\tilde{H}_{|w=0}((\tilde{N} \cdot N) v + (\nabla H \cdot \tilde{N}) v),
\]

for any \( v \in \mathcal{C}^2_{loc}(\Sigma) \) where \( H \) denotes the mean curvature of \( \Sigma \). In the particular case where \( \Sigma \) has constant mean curvature, this formula reduces to

\[
DH_{|w=0}(v) = D\tilde{H}_{|w=0}((\tilde{N} \cdot N) v).
\]

**Proof.** The implicit function theorem can be applied to the equation

\[
p + t N(p) = q + s \tilde{N}(q),
\]

to express (at least locally) \( p \) and \( t \) as functions of \( q \) and \( s \), namely

\[
p = \Phi(q, s) \quad \text{and} \quad t = \Psi(q, s),
\]

with \( \Phi(q, 0) = q \) and \( \Psi(q, 0) = 0 \). It is easy to check that

\[
\partial_s \Phi(\cdot, 0) = \tilde{N}^T, \quad \text{and} \quad \partial_s \Psi(\cdot, 0) = \tilde{N} \cdot N.
\]

where the superscript \( T \) denotes the projection over \( T\Sigma \).

Differentiation of the identity

\[
H_{\Phi(\cdot, w)}(\Phi(q, w(q))) = \tilde{H}_w(q),
\]
with respect to \( w \), at \( w = 0 \), we find

\[
DH|_{w=0}(\partial_s \Psi(\cdot, 0) \nu) + \nabla H|_{w=0} \cdot \partial_s \Phi \nu = D\tilde{H}|_{w=0}(\nu).
\]

The result then follows from the expression of \( \partial_s \Phi \) and \( \partial_s \Psi \) and the fact that \( \tilde{H}|_{w=0} = H|_{w=0} \). \( \square \)

### 3. Harmonic extensions

For all \( x \in \mathbb{R}^2 \) and all \( r > 0 \) we denote by \( D(x, r) \subset \mathbb{R}^2 \) the open disc of radius \( r \), centered at \( x \) and \( \overline{D}(x, r) \subset \mathbb{R}^2 \) the closed disc of radius \( r \), centered at \( x \). In this section, we study the harmonic extension either in a half cylinder \([0, \infty) \times S^1\), the punctured unit disc \( D^*(0, 1) \) in \( \mathbb{R}^2 \) or the complement of the closed unit disc \( \mathbb{R}^2 - D(0, 1) \), of a function which is defined on the unit circle \( S^1 \). We will use the fact that all these domains are conformal to each other and that the Laplacian is conformally invariant in dimension 2.

Let us assume that we are given a function \( f \in C^{2,\alpha}(S^1) \). We consider \( F \) to be the bounded harmonic extension of \( f \) in the half cylinder, endowed with the cylindrical metric

\[
g_{cyl} = ds^2 + d\theta^2.
\]

In other words, \( F \) is bounded and is a solution of

\[
\Delta_{g_{cyl}} F = 0,
\]

in \([0, \infty) \times S^1\) with \( F = f \) on \( \{0\} \times S^1 \).

Observe that one can use cylindrical coordinates to parameterize the punctured unit disc by

\[
\tilde{X}(s, \theta) = (e^{-s} \cos \theta, e^{-s} \sin \theta),
\]

in which case the function \( \tilde{F} \) defined by \( \tilde{F} \circ \tilde{X} := F \) is the unique bounded solution of

\[
\Delta \tilde{F} = 0,
\]

(where \( \Delta \) denotes the Laplacian in \( \mathbb{R}^2 \)) in the punctured unit disc with \( \tilde{F} = f \) on \( S^1 \). We set

\[
W_{\text{ins}}^f := \tilde{F}.
\]

Also, one can use cylindrical coordinates to parameterize the complement of the unit disc in \( \mathbb{R}^2 \) by

\[
\hat{X}(s, \theta) = (e^s \cos \theta, e^s \sin \theta),
\]

in which case \( \hat{F} \) defined by \( \hat{F} \circ \hat{X} = F \) is the unique bounded solution of

\[
\Delta \hat{F} = 0,
\]

in the complement of the unit disc with \( \hat{F} = f \) on \( S^1 \). We set

\[
W_{\text{out}}^f := \hat{F}.
\]

In particular, all properties of \( F \) will transfer easily to \( \tilde{F} \) and \( \hat{F} \).

Given a function \( f \) defined on \( S^1 \), we shall frequently assume that one or both of the following assumptions is/are fulfilled

\[
(H1) \quad \int_{S^1} f \, d\theta = 0,
\]

and

\[
(H2) \quad \int_{S^1} \cos \theta \, f \, d\theta = \int_{S^1} \sin \theta \, f \, d\theta = 0.
\]
The following result follows essentially from [3] where a similar result was proven in higher dimensions:

**Lemma 3.1.** There exists a constant $C > 0$ such that, for all $f \in C^{2,\alpha}(S^1)$ satisfying (H1), we have

$$
\|e^s F\|_{C^{2,\alpha}([0,\infty) \times S^1)} \leq C \|f\|_{C^{2,\alpha}(S^1)},
$$

and, if $f$ satisfies (H1) and (H2), we have

$$
\|e^{2s} F\|_{C^{2,\alpha}([0,\infty) \times S^1)} \leq C \|f\|_{C^{2,\alpha}(S^1)}.
$$

Before we proceed with the proof of this result, let us emphasize that the norms in $C^{2,\alpha}([0,\infty) \times S^1)$ are computed with respect to the cylindrical metric $g_{cyl}$.

**Proof.** We consider the Fourier series decomposition of the function $f$

$$
f(\theta) = \sum_{n \in \mathbb{Z}} f_n e^{i n \theta}.
$$

Observe that $f_0 = 0$ when (H1) is fulfilled and $f_{\pm 1} = 0$ when (H2) is fulfilled. For the time being, let us assume that both (H1) and (H2) are satisfied. Then, the (bounded) harmonic extension of $f$ is given explicitly by

$$
F(s, \theta) = \sum_{|n| \geq 2} e^{-|n| s} f_n e^{i n \theta}.
$$

Since

$$
|f_n| \leq \|f\|_{L^\infty(S^1)},
$$

we get the pointwise estimate

$$
|F(s, \theta)| \leq 2 \|f\|_{L^\infty(S^1)} \sum_{n \geq 2} e^{-n s} \leq 2 \|f\|_{L^\infty(S^1)} \frac{e^{-2s}}{1 - e^{-s}},
$$

which implies that

$$
\sup_{[0,\infty) \times S^1} e^{2s} |F(s, \theta)| \leq C \|f\|_{L^\infty(S^1)}.
$$

Increasing the value of $C > 0$ if this is necessary, we can use the maximum principle in the annular region $[0, 1] \times S^1$ to get

$$
\sup_{[0,\infty) \times S^1} e^{2s} |F(s, \theta)| \leq C \|f\|_{L^\infty(S^1)}.
$$

The estimates for the derivatives of $F$ then follow from classical elliptic estimates since Schauder’s estimates can be applied on each annulus $[s, s + 1] \times S^1$, for all $s \geq 0$. This already completes the proof of the result when both (H1) and (H2) are fulfilled. When only (H1) holds, one has to take into account the function $f_{\pm 1} e^{-s} e^{\pm i \theta}$ which accounts for the slower decay of $F$ as $e^{-s}$.

\[\square\]

4. **The Delaunay nodoids**

4.1. **Parameterization and notations.** The Delaunay nodoid $D_\tau$ is a surface of revolution which can be parameterized by

$$
X_\tau(s, \theta) := (\phi_\tau(s) \cos \theta, \phi_\tau(s) \sin \theta, \psi_\tau(s)),
$$

where $(s, \theta) \in \mathbb{R} \times S^1$. Here, the functions $\phi_\tau$ and $\psi_\tau$ depend on the real parameter $\tau > 0$ but, unless this is necessary, we shall not make this apparent in the notation.
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anymore. The function $\phi$ is chosen to be the unique smooth, periodic, non-constant solution of

\begin{equation}
\dot{\phi}^2 + (\phi^2 - \tau)^2 = \phi^2,
\end{equation}

which takes its minimal value at $s = 0$ (we agree that $\cdot$ denotes differentiation with respect to the parameter $s$) and the function $\psi$ is obtained by integration of

\begin{equation}
\dot{\psi} = \phi^2 - \tau,
\end{equation}

with initial condition $\psi(0) = 0$. Observe that $\phi$ is a smooth solution of

\begin{equation}
\ddot{\phi} + 2 \phi (\phi^2 - \tau) = \phi.
\end{equation}

Since $\phi^2 - \tau$ changes sign, the function $\psi$ is not monotone and closer inspection of the solutions shows that $\mathcal{D}_\tau$ is actually not embedded. The Delaunay nodoids also arise as the surface of revolution whose generating curve is a roulette of a hyperbola and we refer to [2] for a description of this construction. The quantity $\frac{1}{4} \tau$ is sometimes referred to as the vertical flux of the Delaunay surface $\mathcal{D}_\tau$ (see Definition 3.1 in [13]).

We define

\begin{equation}
\zeta := \frac{\sqrt{1 + 4 \tau} - 1}{2} \quad \text{and} \quad \tau := \frac{\sqrt{1 + 4 \tau} + 1}{2},
\end{equation}

which, thanks to (4.3) and (4.5), are respectively the minimum and maximum values of $\phi$. As already mentioned, the function $\phi$ is periodic. We agree that $s_\tau$ denotes one half of the fundamental period of $\phi$. Using (4.3), we can write

\begin{equation}
s_\tau = \int_\zeta^\tau \frac{d\zeta}{\sqrt{\zeta^2 - (\zeta^2 - \tau)^2}}.
\end{equation}

In the above parameterization, the induced metric on $\mathcal{D}_\tau$ is given by

\begin{equation}
g_\tau := \phi^2 \left( ds^2 + d\theta^2 \right),
\end{equation}

and it is easy to check that the second fundamental form on $\mathcal{D}_\tau$ is given by

\begin{equation}
h_\tau := (\phi^2 + \tau) ds^2 + (\phi^2 - \tau) d\theta^2,
\end{equation}

when the unit normal vector field is chosen to be

\begin{equation}
N_\tau := \frac{1}{\phi} \left( (\tau - \phi^2) \cos \theta, (\tau - \phi^2) \sin \theta, 1 \right).
\end{equation}

Finally, the tensor $k_\tau$ is given by

\begin{equation}
k_\tau := \left( \frac{\phi + \tau}{\phi} \right)^2 ds^2 + \left( \frac{\phi - \tau}{\phi} \right)^2 d\theta^2.
\end{equation}

In particular, the formula for the induced metric and the second fundamental form implies that the mean curvature of this surface is constant equal to

\begin{equation}
H := \frac{1}{2} \text{tr}^g h = 1.
\end{equation}

In these coordinates, it follows at once from the expression of $g_\tau$ and $k_\tau$, that the Jacobi operator about $\mathcal{D}_\tau$ is given by

\begin{equation}
J_\tau := \frac{1}{2 \phi^2} \left( \partial_s^2 + \partial_\theta^2 + 2 \left( \phi^2 + \frac{\tau^2}{\phi^2} \right) \right).
\end{equation}
4.2. **Structure and refined asymptotics.** The structure of the Delaunay surfaces $D_\tau$ is well understood and it is known that, as the parameter $\tau$ tends to 0, $D_\tau$ converges to the union of infinitely many spheres of radius 1 which are arranged periodically along the vertical axis. To get a better grasp on the structure of $D_\tau$ as $\tau$ tends to 0, we have the following results which were already used in many constructions of constant mean curvature surfaces by gluing [8], [9] and [10]. For the sake of completeness we give here independent proofs of these results.

First, we have the:

**Lemma 4.1.** As $\tau$ tends to 0 the following holds:

(i) The sequence of functions $\phi_\tau(\cdot + s_\tau)$ converges uniformly on compacts of $\mathbb{R}$ to the function $s \mapsto \tanh s$.

(ii) The sequence of functions $\psi_\tau(\cdot + s_\tau) - \psi_\tau(s_\tau)$ converges uniformly on compacts of $\mathbb{R}$ to the function $s \mapsto -1$.

**Proof.** It is easy to check that $\phi_\tau(s_\tau)$ is even and that $\phi_\tau(s_\tau) = \tau$ converges to 1 as $\tau$ tends to 0 (this follows from the fact that the function $\phi_\tau$ achieves its maximal value when $s = s_\tau$). Passing to the limit in (4.3), we conclude that the sequence of functions $\phi_\tau$ converges uniformly on compacts of $\mathbb{R}$ to a function $\phi_0$ which is a solution of

$$\ddot{\phi}_0 + 2 \phi_0^3 = \phi_0.$$

Moreover, the function $\phi_0$ is even and is equal to 1 when $s = 0$. Therefore, necessarily $\phi_0(s) = (\cosh s)^{-1}$. Next, one can pass to the limit in (4.4) to prove that the sequence $\psi_\tau(\cdot + s_\tau) - \psi_\tau(s_\tau)$ converges to a function $\psi_0$ which is a solution of

$$\dot{\psi}_0 = \phi_0^2,$$

and satisfies $\psi_0(0) = 0$. We find that $\psi_0(s) = \tanh s$ and this completes the proof of the result. $\square$

Now, we investigate the behavior of $D_\tau$ close to the origin in $\mathbb{R}^3$. It turns out that the sequence of rescaled surfaces $\frac{1}{\tau} D_\tau$ converges on compacts of $\mathbb{R}^3$ to a catenoid whose axis is the vertical axis. This is the content of the :

**Lemma 4.2.** As $\tau$ tends to 0, the following holds:

(i) The sequence of functions $\frac{1}{\tau} \phi_\tau$ converges uniformly on compacts of $\mathbb{R}$ to the function $s \mapsto \cosh s$.

(ii) The sequence of functions $\frac{1}{\tau} \psi_\tau$ converges uniformly on compacts of $\mathbb{R}$ to the function $s \mapsto -s$.

**Proof.** It is easy to check that $\phi_\tau$ is even and that $\frac{1}{\tau} \phi_\tau(0) = 1/\tau$ converges to 1 as $\tau$ tends to 0 (this follows from the fact that the function $\phi_\tau$ achieves its minimum value when $s = 0$). Passing to the limit in (4.3), we conclude that $\frac{1}{\tau} \phi_\tau$ converges uniformly on compacts of $\mathbb{R}$ to a function $\phi_0$ which is a solution of

$$\dot{\phi}_0 = \phi_0.$$

Moreover, $\phi_0$ is even and is equal to 1 when $s = 0$. Therefore, $\phi_0(s) = \cosh s$. Next, one can pass to the limit in (4.4) to prove that the sequence $\frac{1}{\tau} \psi_\tau$ converges to a function $\psi_0$ which is a solution of

$$\dot{\psi}_0 = -1,$$

and satisfies $\psi_0(0) = 0$. We find that $\psi_0(s) = \tanh s$ and this completes the proof of the result. $\square$
and satisfies $\psi_0(0) = 0$. Therefore, we conclude that $\psi_0(s) = -s$ as desired. \(\Box\)

Geometrically, these results show that, as $\tau$ tends to 0, the Delaunay surface $\mathcal{D}_\tau$ is close to infinitely many spheres of radius 1, which are arranged along the vertical axis (and are slightly overlapping) and each sphere is connected to its two neighbors by small rescaled catenoids.

We will need a refined and more quantitative version of Lemma 4.2. Observe that $\dot{\phi} < 0$ in $(-s, 0)$ and hence $\phi$ is a diffeomorphism from $(-s, 0)$ into $(\mathcal{Z}, \tau)$. We can define the change of variables

$$ r = \phi_\tau(s), $$

to express $s \in (-s, 0)$ as a function of $r \in (\mathcal{Z}, \tau)$, so that the equality

$$ X_\tau(s, \theta) = (r \cos \theta, r \sin \theta, u_\tau(r)), $$

where $r = \phi_\tau(s)$, holds for some function $u_\tau$ defined in an annulus of $\mathbb{R}^2$. Geometrically, this means that the image of $(-s, 0) \times S^1$ by $X_\tau$ is a vertical graph for some function $u_\tau$ which is defined over the annulus

$$ \{x \in \mathbb{R}^2 : \mathcal{Z} < |x| < \tau\}. $$

The next result gives a precise expansion of the function $u_\tau$ as $\tau$ tends to 0.

**Proposition 4.1.** As $\tau$ tends to 0,

$$ u_\tau(r) = \frac{\tau}{\sqrt{1 + 2\tau}} \log \left( \frac{2r}{\tau} \right) + O_C \left( \frac{\tau^3}{r^2} \right) + O_C \left( r^2 \right), $$

for $r \in (2\mathcal{Z}, \frac{1}{2}\tau)$, uniformly as $\tau$ tends to 0.

The notation $f_1 = O_C(f_2)$ means that the function $f_1$ and all its derivatives with respect to the vector fields $r \partial_r$ and $\partial_\theta$ are bounded by a constant (depending on the order of derivation) times the (positive) function $f_2$.

**Proof.** By definition $\tau$ is the minimal value of $\phi$. Hence, we can write

$$ \phi(s) = \mathcal{Z} \cosh(w(s)), $$

for some function $w$ which vanishes at $s = 0$. Plugging this into (4.3) we find that the function $w$ is a solution of

$$ w^2 = 1 + 2\tau - \mathcal{Z}^2(1 + \cosh^2 w). $$

As long as $|w(s) - s| \leq 1$, we can estimate,

$$ w(s) = \sqrt{1 + 2\tau} s + O(\tau^2 \cosh^2 s). $$

In particular, we conclude a posteriori that $|w(s) - s| \leq 1$ holds, and hence that the above estimate is justified, provided $|s| \leq -\log \tau - c$, for some constant $c > 0$ independent of $\tau \in (0, 1)$. In the range of study, we are entitled to consider the change of variable

$$ r = \mathcal{Z} \cosh w(s), $$

and express $s < 0$ as a function of $r$. We find

$$ \sqrt{1 + 2\tau} s = -\log \left( \frac{2r}{\mathcal{Z}} \right) + O \left( \frac{\tau^2}{r^2} \right) + O(r^2). $$

Finally, using (4.4), we can write

$$ \dot{\psi} = -\tau + \mathcal{Z}^2 \cosh^2 w.
Integrating over $s$ we get
$$\psi(s) = -\tau s + O(\tau^2 \cosh^2 s),$$
and the result follows directly from (4.7) together with the fact that, by definition
$$u_\tau(r) = \psi(s).$$
Similar estimates can be obtained for the derivatives of $u_\tau$. □

A close inspection of the proof of Proposition 4.1 also yields the :

Lemma 4.3. As $\tau$ tends to 0, half of the fundamental period of the function $\phi_\tau$ can be expanded as
$$s_\tau = -\log \tau + O(1),$$
and there exists a constant $C > 1$ such that
$$\frac{\tau}{C} \cosh s \leq \phi_\tau \leq C \tau \cosh s,$$
when $s \in (-s_\tau, s_\tau)$; this estimate being uniform as $\tau$ tends to 0.

Proof. The asymptotic of the half period of $\phi$ can also be derived from the formula (4.6). The estimate for $\phi$ follows from the proof of Proposition 4.1. □

4.3. Analysis of the Jacobi operator. We analyze the mapping properties of the Jacobi operator about the Delaunay surface $D_\tau$, paying special attention to what happens when $\tau$ tends to 0. This analysis is very close to the one available in [8] or [4]. Again, we give here a self contained proof which is adapted to the nonlinear argument we will use in the subsequent sections.

We first analyze the behavior, as $\tau$ tends to 0, of the potential which appears in the expression of $J_\tau$. To this aim we assume that we are given for each $\tau > 0$ a real number $t_\tau \in \mathbb{R}$ and we define
$$\xi_\tau := \left(\phi_\tau^2 + \frac{\tau^2}{\phi_\tau^2}\right)(\cdot - t_\tau).$$
We have the :

Lemma 4.4. As $\tau$ tends to 0, a subsequence of the sequence of functions $\xi_\tau$ converges uniformly on compacts of $\mathbb{R}$ either to $s \mapsto (\cosh(s-s_0))^{-2}$, for some $s_0 \in \mathbb{R}$, or to 0.

Proof. We define
$$\zeta_\tau := \left(\phi_\tau + \frac{\tau}{\phi_\tau}\right)(\cdot - t_\tau),$$
and we observe that, using (4.3) we find that $\zeta_\tau$ is a solution of
$$\ddot{\zeta}_\tau = (\zeta_\tau^2 - 2 \tau) (1 + 4 \tau - \zeta_\tau^2),$$
and (4.5) also implies that
$$\ddot{\zeta}_\tau = \zeta_\tau (1 + 6 \tau - 2 \zeta_\tau^2).$$
Now, we can estimate
$$\zeta_\tau^2 = \left(\phi - \frac{\tau}{\phi}\right)^2 + 4\tau \leq 1 + 4\tau,$$
where we have used (4.3) which provides the estimate $(\phi^2 - \tau)^2 \leq \phi^2$. This implies that $\zeta_\tau$ and its derivatives remain bounded as $\tau$ tends to 0. We can then let $\tau$ tend
to 0 and pass to the limit in (4.8) and (4.9) to get that, as \( \tau \) tends to 0, the sequence \( \zeta_\tau \) converges on compacts to a solution of the equation \( \dot{\zeta} = \zeta \left( 1 - 2 \zeta^2 \right) \) which satisfies \( \dot{\zeta}^2 = \zeta^2 \left( 1 - \zeta^2 \right) \). Hence \( \zeta \) is either 0 or a translation of \( z \mapsto (\cosh s)^{-1} \). The result then follows from the identity \( \xi_\tau = \zeta_\tau^2 - 2 \tau \). \hfill \Box 

We denote by \( \pm \delta_j(\tau) \), for \( j \in \mathbb{N} \), the indicial roots of the operator \( J_\tau \). Recall that the indicial roots \( \pm \delta_j \) correspond to the indicial roots of \( J_{\tau,j} := \frac{1}{2} \phi^2 \left( \partial_t^2 - j^2 + 2 \left( \phi^2 + \frac{\tau^2}{\phi^2} \right) \right) \), which appears in the Fourier decomposition of the operator \( J_\tau \) in the \( \theta \) variable. By definition, the indicial roots of \( J_{\tau,j} \) characterize the exponential growth or decay rate at infinity of the solutions of the homogeneous problem \( J_{\tau,j} w = 0 \). In general, it is a very hard problem to determine the exact value of the indicial roots of an operator, in the present case, taking advantage of the geometric nature of the problem, we can prove the following :

**Proposition 4.2.** For all \( \tau > 0 \),

\[
\delta_0(\tau) = \delta_1(\tau) = 0.
\]

Furthermore, for \( j \geq 2 \)

\[
\delta_j(\tau) \geq \sqrt{j^2 - 2 - 4\tau},
\]

provided \( \tau < \sqrt{j^2 - 2} \).

**Proof.** The fact that \( \delta_0(\tau) = 0 \) follows from the observation that the function \( \dot{\phi}/\phi \) is periodic and solves

\[
J_{\tau,0} \left( \frac{\dot{\phi}}{\phi} \right) = 0.
\]

This either follows from direct computation or can be derived from the fact that

\[
\Phi_0^+ := \frac{\dot{\phi}}{\phi},
\]

is the Jacobi field associated to vertical translation (see §2.2). Since the function \( \phi \) is periodic, the homogeneous problem \( J_{\tau,0} w = 0 \) has a bounded solution and this implies that \( \delta_0(\tau) = 0 \).

The fact that \( \delta_1(\tau) = 0 \) follows from the observation that the function \( \phi - \frac{\tau}{\phi} \) is periodic and solves

\[
J_{\tau,1} \left( \phi - \frac{\tau}{\phi} \right) = 0.
\]

Again, this either follows from direct computation or can be derived from the fact that

\[
\Phi_1^+ := \left( \phi - \frac{\tau}{\phi} \right) \cos \theta \quad \text{and} \quad \Phi_{-1}^+ := \left( \phi - \frac{\tau}{\phi} \right) \sin \theta,
\]

are the Jacobi fields associated to translations perpendicular to the axis of the Delaunay surface.

The estimate from below for the other indicial roots follows from the fact that, according to (4.3)

\[
2 \left( \phi^2 + \frac{\tau^2}{\phi^2} \right) = 2 \left( \phi - \frac{\tau}{\phi} \right)^2 + 4 \tau \leq 2 + 4 \tau.
\]
This in particular implies that the potential in $\partial_s^2 - j^2 + 2 \left( \phi^2 + \frac{r^2}{\sigma^2} \right)$ can be estimated from below by $\delta_j^2$, where

$$\delta_j := \sqrt{j^2 - 2 - 4 \tau}.$$  

The result then follows from the maximum principle and standard ODE arguments since the function $s \mapsto e^{\delta_j s}$ can be used as a barrier to prove the existence of two positive solutions of $J_{\tau,j} w = 0$ which are defined on $(0, \infty)$, one of which being bounded from above by $e^{-\delta_j s}$ and the other one being bounded from below by $e^{\delta_j s}$. In particular, this implies that $\delta_j \geq \delta_j$ and hence, this completes the proof of the result. □

For all $\delta \in \mathbb{R}$, we define the operator

$$L_\delta : e^{\delta s} C^{2,\alpha}(\mathbb{R} \times S^1) \to e^{\delta s} C^{0,\alpha}(\mathbb{R} \times S^1)$$

where the norms in the function spaces $C^{k,\alpha}(\mathbb{R} \times S^1)$ are computed with respect to the cylindrical metric $g_{cyl}$. Observe that the Jacobi operator has been multiplied by the conformal factor $\phi^2$ and hence

$$\phi^2 J_{\tau} = \frac{1}{2} \left( \partial_s^2 + \partial_\theta^2 + 2 \left( \phi^2 + \frac{r^2}{\sigma^2} \right) \right).$$

Also, this operator depends on the parameter $\tau$. We now study the mapping properties of $\phi^2 J_{\tau}$ as the parameter $\tau$ tends to 0. The following result selects a range of weights for which the norm of the solution of $L_\delta w = f$ is controlled, uniformly as $\tau$ tends to 0.

**Proposition 4.3.** Assume that $|\delta| > 1$, $\delta \notin \mathbb{Z}$ is fixed. Then, there exist $\tau_\delta > 0$ and $C > 0$, only depending on $\delta$, such that, for all $\tau \in (0, \tau_\delta)$ and for all $w \in e^{\delta s} C^{2,\alpha}(\mathbb{R} \times S^1)$, we have

$$\| e^{-\delta s} w \|_{C^{2,\alpha}(\mathbb{R} \times S^1)} \leq C \| e^{-\delta s} L_\delta w \|_{C^{0,\alpha}(\mathbb{R} \times S^1)}.$$  

**Proof.** Observe that, thanks to Schauder’s elliptic estimates, it is enough to prove that

$$\| e^{-\delta s} w \|_{L^\infty(\mathbb{R} \times S^1)} \leq C \| e^{-\delta s} \phi_\tau^2 J_{\tau} w \|_{L^\infty(\mathbb{R} \times S^1)},$$

provided $\tau$ is close enough to 0. The proof of this estimate is by contradiction. Assume that, for some sequence $\tau_n$ tending to 0 there exists a sequence of functions $w_n$ such that

$$\| e^{-\delta s} w_n \|_{L^\infty(\mathbb{R} \times S^1)} = 1$$  

and

$$\lim_{j \to \infty} \| e^{-\delta s} \phi_{\tau_j}^2 J_{\tau_j} w_n \|_{L^\infty(\mathbb{R} \times S^1)} = 0.$$  

Pick a point $s_n \in \mathbb{R}$ such that $\| e^{-\delta s_n} w_n(s_n, \cdot) \|_{L^\infty(S^1)} \geq 1/2$ and define the rescaled sequence

$$\bar{w}_n(s, \theta) := e^{-\delta s_n} w(s + s_n, \theta).$$

We still have

$$\| e^{-\delta s} \bar{w}_n \|_{L^\infty(\mathbb{R} \times S^1)} = 1$$  

and

$$\lim_{j \to \infty} \| e^{-\delta s} \bar{L}_n \bar{w}_n \|_{L^\infty(\mathbb{R} \times S^1)} = 0,$$
where by definition $\bar{L}_n$ is defined by
\[
\bar{L}_n := \partial_s^2 + \phi_\theta^2 + 2 \left( \phi_{s_n}^2 + \phi_{s_n}^2 \right) ( \cdot + s_n ).
\]

Elliptic estimates and Ascoli-Arzela’s theorem allows one to extract some subsequence and pass to the limit as $n$ tends to $\infty$ to get a function $w_\infty$ which is a nontrivial solution to either
\[
(4.10) \quad (\partial_s^2 + \partial_\theta^2) w_\infty = 0,
\]
or
\[
(4.11) \quad \left( \partial_s^2 + \partial_\theta^2 + \frac{2}{\cosh^2(\cdot + s_n)} \right) w_\infty = 0,
\]
according to the different cases described in Lemma 4.4. To simplify the notations, we assume that $s_n = 0$, straightforward modifications are needed to handle the general case. Observe that we also have
\[
\|e^{-\delta s} w_\infty\|_{L^\infty(\mathbb{R} \times S^1)} \leq 1,
\]
and $\|w_\infty(0, \cdot)\|_{L^\infty(S^1)} \geq 1/2$.

We decompose $w_\infty$ as
\[
w_\infty(s, \theta) = \sum_{j \in \mathbb{Z}} w^{(j)}(s) e^{ij\theta}.
\]

It is easy to prove that for any solution of (4.10), $w^{(j)}$ is a linear combination of $e^{\pm js}$ and none is bounded by a constant times $e^{\delta s}$ unless $\delta$ is an integer (which we have assumed not to be the case).

Similarly, if $w_\infty$ is a solution of (4.11), we find that $w^{(j)}$ is a solution of
\[
(4.12) \quad \left( \partial_s^2 - j^2 + \frac{2}{\cosh^2 s} \right) w^{(j)} = 0,
\]
and is either asymptotic to $e^{js}$ or to $e^{-js}$ at $\pm \infty$. Inspection of the behavior of (4.12) at infinity then implies that there is no such solution which is bounded by a constant times $e^{\delta s}$ if $|j| < |\delta|$. When $|j| > |\delta|$, inspection of the behavior of (4.12) at infinity implies that any such solution is necessarily bounded and the maximum principle then implies that this solution is identically 0 (observe that in this case $j^2 > 2$ since $|j| > |\delta| > 1$ and hence the potential in (4.12) is negative).

When $j = 0$, all solutions of
\[
\left( \partial_s^2 + \frac{2}{\cosh^2 s} \right) w^{(0)} = 0,
\]
are linear combination of the functions $\tanh s$ and $1 - s \tanh s$ and none is bounded by a constant times $e^{\delta s}$ unless $\delta = 0$ (which is not the case).

Finally, and this is the reason why we had to choose $|\delta| > 1$, when $j = 1$, all solutions of
\[
\left( \partial_s^2 - 1 + \frac{2}{\cosh^2 s} \right) w^{(1)} = 0,
\]
are linear combination of the functions $(\cosh s)^{-1}$ and $s (\cosh s)^{-1} + \sinh s$ and none is bounded by a constant times $e^{\delta s}$ unless $|\delta| \leq 1$ (which is contrary to our assumption). Again we have reached a contradiction. Having reached a contradiction in all cases, the proof of the Proposition is complete. □
Thanks to the previous result, we can now describe the mapping properties of $\phi^2 J_\tau$ for the range of weights $\delta$ of interest for our problem.

**Proposition 4.4.** Assume that $|\delta| > 1$, $\delta \notin \mathbb{Z}$ is fixed. Then, there exist $\tau_\delta > 0$ and $C > 0$, only depending on $\delta$, such that, for all $\tau \in (0, \tau_\delta)$, the operator $L_\delta$ is an isomorphism the norm of whose inverse is bounded independently of $\tau$.

**Proof.** Injectivity follows at once from Proposition 4.3. As far as surjectivity is concerned, we give here a simple self contained proof in the case where $\delta \in (1, \sqrt{2})$ (or $\delta \in (-\sqrt{2}, -1)$). We will then sketch a general proof.

To fix the ideas, let us assume that $\delta \in (1, \sqrt{2})$. We first assume that $f \in C^0,\alpha(\mathbb{R} \times S^1)$ has compact support and we decompose it as

$$f(s, \theta) = f_0(s) + f_{\pm 1}(s) e^{\pm i \theta} + \bar{f}(s, \theta),$$

where, by definition,

$$\bar{f} := \sum_{j \neq 0, \pm 1} f_j(s) e^{ij\theta}.$$

Observe that, if we restrict our attention to functions $\bar{w}$ whose Fourier decomposition in the $\theta$ variable is of the form

$$\bar{w}(s, \theta) = \sum_{j \neq 0, \pm 1} w_j(s) e^{ij\theta},$$

we have

$$\int_{\mathbb{R} \times S^1} \left( |\partial_s \bar{w}|^2 + |\partial_\theta \bar{w}|^2 - 2 \left( \phi^2 + \frac{\tau^2}{\phi^2} \right) \bar{w}^2 \right) ds d\theta \geq (2 - 4\tau) \int_{\mathbb{R} \times S^1} \bar{w}^2 ds d\theta. \tag{4.13}$$

This follows at once from the estimate of the potential involved in the expression of $J_\tau$ which has been obtained in the proof of Proposition 4.2, namely

$$2 \left( \phi^2 + \frac{\tau^2}{\phi^2} \right) \leq 2 + 4 \tau, \tag{4.14}$$

together with the fact that

$$\int_{\mathbb{R} \times S^1} |\partial_\theta \bar{w}|^2 ds d\theta \geq 4 \int_{\mathbb{R} \times S^1} \bar{w}^2 ds d\theta.$$

Thus, if we assume that $\sqrt{2} \tau < 1$, this inequality implies that we can solve

$$\phi^2 J_\tau \bar{w} = f,$$

in $H^1(\mathbb{R} \times S^1)$. Elliptic estimates then imply that $\bar{w} \in C^{2,\alpha}(\mathbb{R} \times S^1)$. Finally, the solvability of

$$\phi^2 J_\tau (w_j e^{ij\theta}) = f_j e^{ij\theta},$$

for $j = 0, \pm 1$, follows easily from integration of the associated second order ordinary differential equation starting from $-\infty$, hence $w_j \equiv 0$ when $s$ is close to $-\infty$. Obviously, the function

$$w := w_0 + w_{\pm 1} e^{\pm i \theta} + \bar{w},$$

is a solution of the equation $\phi^2 J_\tau w = f$.

We claim that, provided $\tau$ is chosen small enough, $w \in C^{2,\alpha}(\mathbb{R} \times S^1)$. Assuming that the claim is already proven, the result of Proposition 4.3 applies and we get

$$\|e^{-\delta s} w\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} \leq C \|e^{-\delta s} f\|_{C^{0,\alpha}(\mathbb{R} \times S^1)}.$$
for any function $f$ having compact support. The general result, when $f$ does not necessarily have compact support, follows from a standard exhaustion argument. We choose a sequence of functions $f^{(n)} \in C^{0,\alpha}(\mathbb{R} \times S^1)$ having compact support converging on compacts to a given function $f \in e^{\delta s} C^{0,\alpha}(\mathbb{R} \times S^1)$. Moreover, without loss of generality, we can assume that
\[
\|e^{-\delta s} f^{(n)}\|_{C^{0,\alpha}(\mathbb{R} \times S^1)} \leq C \|e^{-\delta s} f\|_{C^{0,\alpha}(\mathbb{R} \times S^1)},
\]
for some constant $C > 0$ independent of $n \geq 0$. Thanks to the above, we have a sequence of solutions of $\phi^2 J_{\tau} \bar{w}^{(n)} = f^{(n)}$ satisfying
\[
\|e^{-\delta s} \bar{w}^{(n)}\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} \leq C \|e^{-\delta s} f\|_{C^{0,\alpha}(\mathbb{R} \times S^1)}.
\]
Extracting some subsequence and passing to the limit, one gets the existence of $w \in e^{\delta s} C^{0,\alpha}(\mathbb{R} \times S^1)$, a solution of $\phi^2 J_{\tau} w = f$ satisfying
\[
\|e^{-\delta s} w\|_{C^{0,\alpha}(\mathbb{R} \times S^1)} \leq C \|e^{-\delta s} f\|_{C^{0,\alpha}(\mathbb{R} \times S^1)}.
\]
The result then follows from Schauder’s estimates.

It remains to prove the claim. We keep the notations introduced above. We first prove that $\bar{w}$ tends to 0 exponentially fast at infinity. Indeed, away from the support of $\bar{f}$, we can multiply the equation $\phi^2 J_{\tau} \bar{w} = \bar{f}$ by $\bar{w}$ and integrate over $S^1$ to get
\[
\frac{1}{2} \frac{d^2}{ds^2} \left( \int_{S^1} \bar{w}^2 \, d\theta \right) = \int_{S^1} \left( |\partial_s \bar{w}|^2 + |\partial_\theta \bar{w}|^2 - \left( \phi^2 + \frac{\tau^2}{\phi^2} \right) \bar{w}^2 \right) \, d\theta.
\]
But
\[
\int_{S^1} |\partial_\theta \bar{w}|^2 \, d\theta \geq 2 \int_{S^1} \bar{w}^2 \, d\theta,
\]
and we conclude from (4.14) that
\[
\frac{d^2}{ds^2} \left( \int_{S^1} \bar{w}^2 \, d\theta \right) \geq 4 (1 - 2 \tau) \int_{S^1} \bar{w}^2 \, d\theta.
\]
Since we have assumed that $\delta \in (1, \sqrt{2})$, we can assume that $\tau > 0$ is small enough so that $\delta^2 \leq 2 (1 - 2\tau)$, and using the fact that $\bar{w}$ is bounded, we conclude that there exists $C > 0$, such that
\[
\int_{S^1} \bar{w}^2 \, d\theta \leq C (\cosh s)^{-2\delta}.
\]
This shows that $\bar{w} \in (\cosh s)^{-\delta} L^2(\mathbb{R} \times S^1)$ and, by elliptic regularity, this implies that $\bar{w} \in (\cosh s)^{-\delta} C^{2,\alpha}(\mathbb{R} \times S^1)$.

Finally, it remains to check that the functions $w_0$ and $w_{\pm 1}$ are at most linearly growing at $+\infty$. This follows at once from the fact that, for $s$ large enough, these functions are solutions of second order homogeneous ordinary differential equations
\[
\left( \partial_s^2 - j^2 + 2 \left( \phi^2 + \frac{\tau^2}{\phi^2} \right) \right) w_j = 0.
\]
For $j = 0, 1$, this ordinary differential equation whose potential is periodic, has one solution which is periodic (see the proof of Proposition 4.2) and standard result imply that the other linearly independent solution of this ordinary differential equation is at most linearly growing (see Appendix 1). In particular, $w_j \in e^{\delta s} C^{2,\alpha}(\mathbb{R})$ and this completes the proof of the claim.
We briefly explain how the proof of the general result can be obtained. The idea is to solve the equation \( \phi^2 J \bar{w} s_0 = \bar{f} \) in \([-s_0, s_0] \times S^1\) with 0 boundary conditions, this can be done using the coercivity inequality (4.13). Then, the proof of Proposition 4.3 can be adapted to prove that

\[
\|e^{-\delta s} \bar{w} s_0\|_{C^{2,\alpha}([-s_0, s_0] \times S^1)} \leq C \|e^{-\delta s} \bar{f}\|_{C^{0,\alpha}(\mathbb{R} \times S^1)},
\]

for some constant \( C > 0 \) independent of \( s_0 > 1 \) (observe that we use the fact that the Fourier decomposition of the function \( \bar{w} \) in the \( \theta \) variable does not have any component over 1 and \( e^ {\pm i \theta } \)). It then remains to pass to the limit in the sequence \( w_{s_0} \) as \( s_0 \) tends to \( \infty \) to prove the existence of \( \bar{w} \) solution of \( \phi^2 J \bar{w} = \bar{f} \) in \( \mathbb{R} \times S^1 \) which satisfies the correct estimate.

Using similar arguments, one can give a direct proof of the following general result, which will not be needed in this paper:

**Theorem 4.1.** Assume that \( \delta \neq \pm \delta_j (\tau) \), for all \( j \in \mathbb{N} \), then \( L_\delta \) is an isomorphism.

The proof of this result follows from the general theory developed in [12] (see Theorem 10.2.1 on page 61 and Proposition 12.2.1 on page 81) or in [11], [7], ...

In what follows we will restrict our attention to functions which are invariant under some symmetries. More precisely, we will assume that the functions are invariant under the action on \( S^1 \) of the dihedral group \( \text{Dih}_{m+1}^{(2)} \) of isometries of \( \mathbb{R}^2 \) which leave a regular polygon with \( m + 1 \) sides fixed. The operator associated to \( \phi^2 J \tau \), acting on the weighted space of functions which are invariant under these symmetries, will be denoted by \( L_{\delta}^{\tau} \). This time \( L_{\delta}^{\tau} \) is an isomorphism provided \( \delta \neq \pm \delta_j \) for all \( j \in \mathbb{Z} \) for which there exist eigenfunctions of \( \partial_{\theta}^2 \) which are invariant under the action of \( \text{Dih}_{m+1}^{(2)} \), namely, \( j \notin m \mathbb{Z} \). Observe that, when \( j = 1 \), there are no such eigenfunctions and hence, working equivariantly allows us to extend the range in which the weight parameter \( \delta \) can be chosen.

Close inspection of the previous proof shows that the range in which the weight \( \delta \) can be chosen so that the inverse of \( L_{\delta}^{\tau} \) remains bounded as \( \tau \) tends to 0 can be enlarged if we work equivariantly. Even though we will not use it, we state here the corresponding result for the sake of completeness.

**Proposition 4.5.** Assume that \( \delta \notin m \mathbb{Z} \) is fixed. Then, there exist \( \tau_\delta > 0 \) and \( C > 0 \), only depending on \( \delta \), such that, for all \( \tau \in (0, \tau_\delta) \) and for all \( w \in e^{\delta s} C^{2,\alpha} (\mathbb{R} \times S^1) \) which is invariant under the action of \( \text{Dih}_{m+1}^{(2)} \), we have

\[
\|e^{-\delta s} w\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} \leq C \|e^{-\delta s} L_{\delta}^{\tau} w\|_{C^{0,\alpha}(\mathbb{R} \times S^1)}.
\]

4.4. The mean curvature of normal graphs over \( \mathcal{D}_\tau \). In this section, we investigate the mean curvature of a surface which is a normal graph over \( \mathcal{D}_\tau \). Given a smooth function \( w \) (small enough) defined on \( \mathcal{D}_\tau \), we consider the surface parameterized by

\[
\bar{X}(s, \theta) = X_\tau(s, \theta) + w(s, \theta) N_\tau(s, \theta).
\]

We have the following technical result:
Lemma 4.5. The mean curvature of the surface parameterized by $\tilde{X}$ is given by

$$H(w) = 1 + J_\tau w + \frac{1}{\phi} Q_\tau \left( \frac{w}{\phi} \right),$$

where the second order differential nonlinear operator $Q_\tau$ depends on $\tau$ and satisfies

$$\|Q_\tau(v_2) - Q_\tau(v_1)\|_{C^0([s,s+1] \times S^1)} \leq c \left( \|v_1\|_{C^2([s,s+1] \times S^1)} + \|v_2\|_{C^2([s,s+1] \times S^1)} \right) \|v_2 - v_1\|_{C^2([s,s+1] \times S^1)},$$

for some constant $c > 0$ independent of $s$ and $\tau \in (0,1)$, for all functions $v_1, v_2$ satisfying $\|v_i\|_{C^2([s,s+1] \times S^1)} \leq 1$.

Proof. This follows at once from Proposition 2.1 together with the fact that the functions $\phi$, $\tau \phi$ and $\frac{\partial}{\partial s} w \phi$ as well as the derivatives of these functions are uniformly bounded as $\tau$ tends to 0. Indeed, we have

$$g_w = g - \frac{1}{2} w h + w^2 k$$

$$= \phi^{-2} \left( 1 - \left( \phi + \frac{\tau}{\phi} \right) \frac{w}{\phi} \right)^2 ds^2 + \left( 1 - \left( \phi - \frac{\tau}{\phi} \right) \frac{w}{\phi} \right)^2 d\theta^2.$$}

Hence $\phi^{-2} g_w$ has coefficients which are bounded functions of $\frac{w}{\phi}$. Similarly, the tensor $\phi^{-1}(h - w k)$ also has coefficients which are bounded functions of $\frac{w}{\phi}$. Using this, it is straightforward to check that the nonlinear terms in $H(w)$ are a function of $\partial_s w \phi$ and $\partial_s \phi \phi$, for $k + \ell = 0, 1, 2$ with coefficients bounded by $\frac{1}{\phi}$. Finally, observe that

$$\partial_s w \phi = \partial_s \left( \frac{w}{\phi} \right) + \phi \frac{\partial}{\partial s} w \phi$$

and hence, any expressions of the form $\partial_s w$ can also be expressed as a linear combination (with coefficients bounded uniformly as $\tau$ tends to 0) of the function $\frac{w}{\phi}$ and its derivatives. We leave the details to the reader.

4.5. A first fixed point argument. We assume that we are given $\tau > 0$. We define $s_0 \in (-\tau,0)$ by the identity

$$\phi_\tau(s) = \tau^{3/4}.$$

Observe that $s$ depends on $\tau$, even though we have chosen not to make this apparent in the notation. Moreover, it follows from the proof of Proposition 4.1 that

$$s = \frac{1}{4} \log \tau + O(1),$$

as $\tau$ tends to 0. We define the truncated nodoid $D_\tau^+$ to be the image of $[s_0, +\infty) \times S^1$ by $X_\tau$. Observe that this surface has a boundary and, thanks to Proposition 4.1, close to this boundary, it can be parameterized as the vertical graph of the function

$$x \mapsto \tau \log \left( \frac{2|x|}{\tau} \right) + O_\tau (\tau^{3/2}),$$

over the annulus $D(0, \tau^{3/4}) - D(0, \tau^{3/4}/2)$. Moreover, $D_\tau^+$ has one end in the upper half space.
In this section, we apply the implicit function theorem (or to be more precise, a fixed point argument for a contraction mapping) to produce an infinite dimensional family of constant mean curvature surfaces which are close to $\mathcal{D}_\tau^+$, have one boundary which can be described using a function $f : S^1 \to \mathbb{R}$.

**Proposition 4.6.** Assume that we are given $\kappa > 0$ large enough (the value of $\kappa$ will be fixed later on). Then, for all $\tau > 0$ small enough and for all functions $f$ invariant under the action of $\text{Dih}^{(2)}_{m+1}$, satisfying (H1) (observe that (H2) is automatically satisfied) and

\[
\|f\|_{C^2(S^1)} \leq \kappa \tau^{3/2},
\]

there exists a constant mean curvature surface $\mathcal{D}_{\tau,f}$ with mean curvature equal to 1, which is a graph over $\mathcal{D}_\tau^+$, has one Delaunay end asymptotic to the end of $\mathcal{D}_\tau^+$ and one boundary. When $f = 0$, $\mathcal{D}_{\tau,0} = \mathcal{D}_\tau^+$ and, close to its boundary, the surface $\mathcal{D}_{\tau,f}$ is a vertical graph over the annulus

\[
\left\{ x \in \mathbb{R}^2 : \frac{1}{2} \tau^{3/4} \leq |x| \leq \tau^{3/4} \right\},
\]

for the function $x \mapsto U_{\tau,f}^1(\tau^{-3/4} x)$ which can be expanded as follows

\[
U_{\tau,f}^1(x) = \tau \log \left( \frac{2}{|x|^{1/2}} \right) + \tau \log |x| - W_f^{\text{ins}}(x) + \bar{U}_{\tau,f}^1(x),
\]

where we recall that $W_f^{\text{ins}}$ denotes the bounded harmonic extension of $f$ in the punctured unit disc and where

\[
\|\bar{U}_{\tau,0}^1\|_{C^2(\overline{D}(0,1) - D(0,1/2))} \leq C \tau^{3/2}.
\]

Moreover, the nonlinear operator

\[
C^2(S^1) \ni f \mapsto \bar{U}_{\tau,f}^1 \in C^2(\overline{D}(0,1) - D(0,1/2)),
\]

is Lipschitz and, given $\delta \in (-2,-1)$, we have

\[
\|\bar{U}_{\tau,f}^1 - \bar{U}_{\tau,f}^0\|_{C^2(\overline{D}(0,1) - D(0,1/2))} \leq C \tau^{(2+\delta)/4} \|f' - f\|_{C^2(S^1)},
\]

for some constant $C > 0$, independent of $\kappa$, $\tau$ and $f, f'$. Finally, $\mathcal{D}_{\tau,f}$ is invariant under the action of the dihedral group $\text{Dih}^{(2)}_{m+1}$.

Before we proceed with the proof of the Proposition, one comment is due. Observe that we have chosen to describe the surface near its boundary as the graph of the function $x \mapsto U_{\tau,f}^1(\tau^{-3/4} x)$ and as a consequence the function $U_{\tau,f}^1$ is defined over the annulus $\overline{D}(0,1) - D(0,1/2)$. Alternatively, we could have chosen not to scale the coordinates and have a function defined over the annulus $\mathcal{D}(0, \tau^{3/4}) - D(0, \tau^{3/4}/2)$, which would be more natural. However, with this latter choice, we would have to consider in (4.17) and (4.18), function spaces where partial derivatives are taken with respect to the vector fields $r \partial_r$ and $\partial_\theta$ to evaluate the norm of these functions, while with the former choice, the Hölder spaces are the usual ones.

**Proof.** The proof of this result is fairly technical but by now standard. To begin with, in the annular region which is the image of $(\bar{s} - 2, \bar{s} + 2) \times S^1$ by $X_\tau$, we modify the unit vector field $N_\tau$ into $\bar{N}_\tau$ in such a way that $\bar{N}_\tau$ is equal to $-e_3$, the downward pointing unit normal vector field on the image of $(\bar{s} - 1, \bar{s} + 1) \times S^1$ by

\[
\bar{N}_\tau = -e_3.
\]
coefficients are smooth, have support in \([\bar{s}, \bar{s} + 2]\) times \(\tau^{1/2}\). This estimate comes from the fact that

\[ N \cdot (-e_3) = 1 + O(\tau^{1/2}), \]

on the image of \([\bar{s} - 2, \bar{s} + 2] \times S^1\) by \(X_\tau\).

We assume that we are given a function \(f \in C^{2,\alpha}(S^1)\) satisfying both (H1), (H2) and (5.26) and we denote by \(F\) the harmonic extension of \(f\) in \((\bar{s}, \infty) \times S^1\).

Given these data, we would like to solve the nonlinear equation

\[ (4.19) \quad \phi^2 J_\tau(F + w) + \ell_\tau(F + w) + \phi \bar{Q}_\tau \left( \frac{F + w}{\phi} \right) = 0, \]

in \((\bar{s}, \infty) \times S^1\). Provided \(w\) is small enough and decays exponentially at infinity, this will then provide constant mean curvature surfaces which are close to a half nodoid \(\mathcal{Q}_\tau^+\).

We choose

\[ \mathcal{E}_\tau : C^{0,\alpha}([\bar{s}, \infty) \times S^1) \rightarrow C^{0,\alpha}(\mathbb{R} \times S^1), \]

an extension operator such that

\[
\begin{align*}
\mathcal{E}_\tau(\psi) &= \psi \quad \text{in} \quad [\bar{s}, \infty) \times S^1 \\
\mathcal{E}_\tau(\psi) &= 0 \quad \text{in} \quad (-\infty, \bar{s} - 1) \times S^1,
\end{align*}
\]

and

\[ \|\mathcal{E}_\tau(\psi)\|_{C^{0,\alpha}([\bar{s} - 1, \bar{s} + 1] \times S^1)} \leq C \|\psi\|_{C^{0,\alpha}([\bar{s}, \bar{s} + 1] \times S^1)}. \]

We rewrite (4.19) as

\[ (4.20) \quad \phi^2 J_\tau w = -\mathcal{E}_\tau \left( \phi^2 J_\tau(F + w) + \ell_\tau F + \phi \bar{Q}_\tau \left( \frac{F + w}{\phi} \right) \right), \]

where, this time, the function \(w\) is defined on all \(\mathbb{R} \times S^1\) (to be more precise, one should say that, on the right hand side, we consider the restriction of \(w\) to \([\bar{s}, \infty) \times S^1\)).

The following estimates follow easily if one uses the fact that

\[ \frac{C}{\tau} \cosh s \leq \phi \leq C \tau \cosh s \quad \text{in} \quad (-s_\tau, s_\tau), \]

for some \(C > 1\), and also that \(\phi\) is periodic of period \(2s_\tau\). We assume that \(\delta \in (-2, -1)\) is fixed. It is easy to check that there exists a constant \(c > 0\) (independent of \(\kappa\)) and a constant \(c_\kappa > 0\) (depending on \(\kappa\)) such that such that

\[ \|e^{-\delta s} \mathcal{E}_\tau \left( \left( \phi^2 + \frac{\tau^2}{\phi^2} \right) F \right)\|_{C^{0,\alpha}(\mathbb{R} \times S^1)} \leq c \tau^{1/2} \|f\|_{C^{2,\alpha}(S^1)}, \]

\[ \|e^{-\delta s} \mathcal{E}_\tau (\ell_\tau (F + w))\|_{C^{0,\alpha}(\mathbb{R} \times S^1)} \]

\[ \leq c \tau^{1/2} \left( \|e^{-\delta s} w\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} + \tau^{-\delta/4} \|f\|_{C^{2,\alpha}(S^1)} \right), \]

where

\[ Q_\tau \]
and we also have
\[
\left\| e^{-\delta s} E_{\tau} \left( \phi Q_{\tau} \left( \frac{w' + F'}{\phi} \right) - \phi Q_{\tau} \left( \frac{w + F}{\phi} \right) \right) \right\|_{C^{0,\alpha}(R \times S^1)} \\
\leq c_{\kappa} \left( \tau^{3/4} \left\| e^{-\delta s} (w' - w) \right\|_{C^{2,\alpha}(R \times S^1)} + \tau^{(3-\delta)/4} \left\| f' - f \right\|_{C^{2,\alpha}(S^1)} \right),
\]
provided \( w \) and \( w' \) satisfy
\[
\left\| e^{-\delta s} w \right\|_{C^{2,\alpha}(R \times S^1)} + \left\| e^{-\delta s} w' \right\|_{C^{2,\alpha}(R \times S^1)} \leq C_{\kappa} \tau^2,
\]
for some fixed constant \( C_{\kappa} > 0 \). Here \( F \) and \( F' \) are respectively the harmonic extensions of the boundary data \( f \) and \( f' \).

At this stage, we make use of the result of Proposition 4.4 (or more precisely its equivariant version) to rephrase (6.45) as a fixed point problem in \( e^{\delta s} C^{2,\alpha}(R \times S^1) \). The estimates we have just derived are precisely enough to solve this nonlinear problem using a fixed point argument for contraction mappings in the ball of radius \( C_{\kappa} \tau^2 \) in \( e^{\delta s} C^{2,\alpha}(R \times S^1) \), where \( C_{\kappa} \) is a constant which is fixed large enough. Therefore, for all \( \tau > 0 \) small enough, we find a solution \( w \) of (6.45) satisfying
\[
\left\| e^{-\delta s} w \right\|_{C^{2,\alpha}(R \times S^1)} \leq C_{\kappa} \tau^2.
\]
In addition, it follows from the above estimates that
\[
\left\| e^{-\delta s} (w' - w) \right\|_{C^{2,\alpha}(R \times S^1)} \leq C_{\kappa} \tau^{1/2} \left\| f' - f \right\|_{C^{2,\alpha}(S^1)},
\]
where \( w \) (resp. \( w' \)) is the solution associated to \( f \) (resp. \( f' \)).

To complete the result, it is enough to change coordinates \( r = \phi(s) \) in the range where \( \frac{1}{2} \tau^{3/4} \leq r \leq 2 \tau^{3/4} \) and \( |s - \bar{s}| \leq 1 \). There is no real difficulty in deriving the estimates (4.17) and (4.18) which follow from Proposition 4.3 and the estimate for \( w \). One should be aware that there is some subtlety here, since one should pay attention to the fact that if we change variables \( r = \phi(s) \) for \( \frac{1}{2} \tau^{3/4} \leq r \leq 2 \tau^{3/4} \) and \( |s - \bar{s}| \leq 1 \), then \( F(s) \) is not equal to \( W^\text{ins}_f(\phi(s)) \) (because \( s \) does not correspond to the cylindrical coordinate \( r = e^t \) in \( \mathbb{R}^2 - \{0\} \))! In fact we have
\[
F(s) = W^\text{ins}_f(e^{\bar{s} - s}),
\]
and \( \tau^{-3/4} r = \phi(s)/\phi(\bar{s}) \) and is not equal to \( e^{\bar{s} - s} \). Nevertheless, using the expansion of \( \phi \) derived it is easy to check that
\[
\left\| F(\bar{s} - \log \phi(s) + \log \phi(\bar{s})) - F(s) \right\|_{C^{2,\alpha}([\bar{s}, \bar{s} + 2] \times S^1)} \leq c \tau^{1/2} \left\| f \right\|_{C^{2,\alpha}(S^1)},
\]
for some constant \( c > 0 \) independent of \( \tau \). \( \square \)

5. The catenoid

5.1. Parameterization and notations. We recall some well known facts about catenoids in Euclidean space. The normalized catenoid \( \mathcal{C} \) is the minimal surface of revolution which is parameterized by
\[
Y_0(s, \theta) := \left( \cosh s \cos \theta, \cosh s \sin \theta, s \right),
\]
where \( (s, \theta) \in \mathbb{R} \times S^1 \). The induced metric on \( \mathcal{C} \) is given by
\[
g_0 := (\cosh s)^2 (ds^2 + d\theta^2),
\]
and it is easy to check that the second fundamental form is given by
\[
h_0 := ds^2 - d\theta^2,
\]
when the unit normal vector field is chosen to be

\[ N_0 := \frac{1}{\cosh s} (\cos \theta, \sin \theta, -\sinh s). \]

In particular, the formula for the induced metric and the second fundamental form implies that the mean curvature of the surface \( \mathcal{C} \) is constant equal to 0.

In the above defined coordinates, the Jacobi operator about the catenoid is given by

\[ J_0 := \frac{1}{2 \cosh^2 s} \left( \partial_s^2 + \partial_\theta^2 + \frac{2}{\cosh^2 s} \right). \]

5.2. **Refined asymptotics.** We are interested in the parameterization of the catenoid as a (multivalued) vertical graph over the horizontal plane. We consider for example the lower part of the catenoid as the graph, over the complement of the unit disc in the horizontal plane for the function \( u_0 \). Namely, \( u_0 \) is the negative function defined by

\[ u_0(\cosh s) = s, \]

for all \( s \leq 0 \). It is easy to check that

\[ s = -\log(2r) + O_{\mathcal{C}^{\infty}}(r^{-2}), \]

and hence the lower end of the catenoid can also be parameterized as a vertical graph over \( \mathbb{R}^2 - D(0,1) \) by

\[ (r, \theta) \mapsto (r \cos \theta, r \sin \theta, u_0(r)). \]

With little work, one proves the :

**Lemma 5.1.** The following expansion holds

\[ u_0(r) = -\log(2r) + O_{\mathcal{C}^{\infty}}(r^{-2}), \]

in \( \mathbb{R}^2 - D(0,2) \).

5.3. **Mapping properties of the Jacobi operator about the catenoid.** The functional analysis of the Jacobi operator about the catenoid is well understood and some results can be found for example in [9]. Again the indicial roots of \( J_0 \) characterize the asymptotic behavior of the solutions of the homogeneous problem \( J_{0,j} w = 0 \) where

\[ J_{0,j} := \frac{1}{2 \cosh^2 s} \left( \partial_s^2 - j^2 + \frac{2}{\cosh^2 s} \right), \]

It is easy to see that the indicial roots of \( J_{0,j} \) are equal to \( \pm j \) and hence the indicial roots of \( J_0 \) are equal to \( \pm j \), for \( j \in \mathbb{N} \).

For all \( \delta \in \mathbb{R} \), we define the operator

\[ \mathcal{L}_\delta : (\cosh s)^\delta \mathcal{C}^{2,\alpha} (\mathbb{R} \times S^1) \rightarrow (\cosh s)^\delta \mathcal{C}^{0,\alpha} (\mathbb{R} \times S^1) \]

\[ w \mapsto (\cosh s)^2 J_0 w, \]

where, as usual, the norms in the function spaces \( \mathcal{C}^{k,\alpha} (\mathbb{R} \times S^1) \) are computed with respect to the cylindrical metric \( g_{cyl} \).

Paralleling what we have proven in §4, we have the:
Proposition 5.1. Assume that \( \delta \in (-2,2) \). Then, there exists \( C > 0 \), only depending on \( \delta \), such that, for all \( \bar{w} \in (\cosh s)^{\delta} C^{2,\alpha}(\mathbb{R} \times S^1) \), we have
\[
\| (\cosh s)^{-\delta} \bar{w}\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} \leq C \| (\cosh s)^{-\delta} \mathcal{L}_\delta \bar{w}\|_{C^{0,\alpha}(\mathbb{R} \times S^1)},
\]
provided
\[
(5.21) \quad \int_{S^1} \bar{w}(s,\theta) \, d\theta = \int_{S^1} \bar{w}(s,\theta) e^{\pm i\theta} \, d\theta = 0,
\]
for all \( s \in \mathbb{R} \).

Proof. The proof is really parallel to the proof of Proposition 4.3 and is left to the reader. \( \square \)

The following result follows from the general theory developed in [12] (see Theorem 10.2.1 on page 61 and Proposition 12.2.1 on page 81) or [6], [11], [7], . . . For the sake of completeness we provide a self-contained proof.

Theorem 5.1. Assume that \( \delta \in (1,2) \) then \( \mathcal{L}_\delta \) is surjective and has a 6-dimensional kernel.

Proof. The proof follows the proof of Proposition 4.4.

Recall that the action of rigid motions and dilations provides many Jacobi fields. For example,
\[
(5.22) \quad J_0 \left( \frac{1}{\cosh s} e^{\pm i\theta} \right) = 0 \quad \text{and} \quad J_0 \left( 1 - s \tanh s \right) = 0,
\]
which either follows from direct computation or from the fact that these are the Jacobi fields associated to the group of vertical translations and the group of dilations centered at the origin.

Similarly
\[
(5.23) \quad J_0 \left( \frac{1}{\cosh s} e^{\pm i\theta} \right) = 0 \quad \text{and} \quad J_0 \left( \left( \sinh s + \frac{1}{\cosh s} \right) e^{\pm i\theta} \right) = 0,
\]
which again either follows from direct computation or from the fact that these are the Jacobi fields associated to the group of horizontal translations and the group of rotations about the vertical axis, centered at the origin.

This already shows that, when \( \delta \in (1,2) \), the kernel of \( \mathcal{L}_\delta \) is at least 6-dimensional.

We first assume that \( f \in C^{0,\alpha}(\mathbb{R} \times S^1) \) has compact support and we decompose it as \( f(s,\theta) = f_0(s) + f_{\pm 1}(s) e^{\pm i\theta} + \bar{f}(s,\theta) \) where, by definition,
\[
\bar{f} := \sum_{j \neq 0, \pm 1} f_j(s) e^{ij\theta}.
\]

If we restrict our attention to functions \( \bar{w} \) whose Fourier decomposition in the \( \theta \) variable is of the form
\[
\bar{w}(s,\theta) = \sum_{j \neq 0, \pm 1} w_j(s) e^{ij\theta},
\]
we have
\[
(5.24) \quad \int_{\mathbb{R} \times S^1} \left( |\partial_s \bar{w}|^2 + |\partial_\theta \bar{w}|^2 - \frac{2}{\cosh^2 s} \bar{w}^2 \right) \, ds \, d\theta \geq 2 \int_{\mathbb{R} \times S^1} \bar{w}^2 \, ds \, d\theta.
\]

Therefore, we can solve
\[
(\cosh s)^2 J_0 \bar{w} = \bar{f},
\]
in \( H^1(\mathbb{R} \times S^1) \). Elliptic estimates then imply that \( \bar{w} \in C^{2,\alpha}(\mathbb{R} \times S^1) \).
Obviously $\bar{w} \in (\cosh s)^{\delta} C^{2,\alpha}(\mathbb{R} \times S^1)$, since $\delta > 0$. The result of Proposition 5.1 applies and we get

$$\|(\cosh s)^{-\delta} \bar{w}\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} \leq C \|(\cosh s)^{-\delta} f\|_{C^{0,\alpha}(\mathbb{R} \times S^1)},$$

for any function $f$ having compact support. The general result, when $f$ does not necessarily have compact support, follows from a standard exhaustion argument.

Finally, the solvability of

$$(\cosh s)^{2} J_{0} (w_{j} e^{ij\theta}) = f_{j} e^{ij\theta},$$

for $j = 0, \pm 1$, follows easily from integration of the associated second order ordinary differential equation starting from 0 (with initial data and initial velocity equal to 0). We have explicitly

$$w_{j} = A_{j}^{+} \int_{0}^{s} A_{j}^{-}(t) f_{j}(t) \, dt - A_{j}^{-} \int_{0}^{s} A_{j}^{+}(t) f_{j}(t) \, dt,$$

where $A_{j}^{\pm}$ are the two independent solutions of

$$\left( \partial_{s}^{2} - j^2 + \frac{2}{\cosh^2 s} \right) A_{j}^{\pm} = 0,$$

which are given in (5.22) and (5.23) and are normalized so that their Wronskian is equal to 1. Direct estimates imply that

$$\|(\cosh s)^{-\delta} w_{j}\|_{C^{2,\alpha}(\mathbb{R})} \leq C \|(\cosh s)^{-\delta} f\|_{C^{0,\alpha}(\mathbb{R} \times S^1)},$$

provided $\delta > 1$ (more precisely, $\delta > 0$ is needed to derive the estimate for $w_{0}$ and $\delta > 1$ is needed to derive the estimate for $w_{\pm 1}$). We set $w = w_{0} + w_{\pm 1} e^{\pm i\theta} + \bar{w}$. This completes the proof of the fact that the operator $L_{\delta}$ is surjective when $\delta \in (1, 2)$. The fact that this operator, restricted to the space of functions satisfying the orthogonality conditions (5.21) is injective follows from the result of Proposition 5.1. Hence the kernel of $L_{\delta}$ is 6-dimensional. \(\square\)

### 5.4. The mean curvature of normal graphs over the catenoid

We consider in this section the mean curvature of a surface which is a normal graph over $\mathcal{C}$. Hence, for some smooth (small enough) function $w$ defined on $\mathcal{C}$, we consider the surface parameterized by

$$Y(s, \theta) = Y_{0}(s, \theta) + w(s, \theta) N_{0}(s, \theta).$$

We have the following technical result :

**Lemma 5.2.** The mean curvature of the surface parameterized by $Y$ is given by

$$H(w) = J_{0} w + \frac{1}{\cosh s} Q_{0} \left( \frac{w}{\cosh s} \right),$$

where the nonlinear second order differential operator $Q_{0}$ satisfies

$$\|Q_{0}(v_{2}) - Q_{0}(v_{1})\|_{C^{0,\alpha}([s,s+1] \times S^1)} \leq c \left( \|v_{1}\|_{C^{2,\alpha}([s,s+1] \times S^1)} + \|v_{2}\|_{C^{2,\alpha}([s,s+1] \times S^1)} \right) \|v_{2} - v_{1}\|_{C^{2,\alpha}([s,s+1] \times S^1)},$$

for some constant $c > 0$ independent of $s$ and $\tau \in (0, 1)$, for all functions $v_{1}$, $v_{2}$ satisfying $\|v_{i}\|_{C^{2,\alpha}([s,s+1] \times S^1)} \leq 1$. 
Proof. This result is already proven in [8]. In any case, a simple proof follows easily from Proposition 2.1 together with the fact that
\[ g_w = \cosh^2 s \left( \left( 1 - \frac{w}{\cosh^2 s} \right)^2 ds^2 + \left( 1 + \frac{w}{\cosh^2 s} \right)^2 d\theta^2 \right). \]
We leave the details to the reader. \( \square \)

5.5. A second fixed point argument. Assume that \( \tau, \tilde{\tau} > 0 \) are chosen small enough and satisfy
\begin{equation}
\left| \frac{\tilde{\tau} - \tau}{m + 1} \right| \leq \kappa \tau^{3/2},
\end{equation}
where the constant \( \kappa > 0 \) is fixed large enough (the value of \( \kappa \) will be fixed in the last section of the paper). The rationale for this estimate will also be explained in the last section of the paper. We define \( \tilde{s} > 0 \) by
\[ \tilde{\tau} \cosh \tilde{s} = \frac{\tau^3}{4}. \]
Observe that \( \tilde{s} \) depends on both \( \tau \) and \( \tilde{\tau} \) even though we have chosen not to make this apparent in the notation. It is easy to check that \( \tilde{s} = -\frac{1}{3} \log \tau + O(1) \). We define the truncated catenoid \( \mathcal{C}_{\tau} \) to be the image of \([-\tilde{s}, \tilde{s}] \times S^1 \) by \( \tilde{\tau} Y_0 \) (to simplify the notations, we do not write the dependence of this surface on the parameter \( \tau \)).

Building on the previous analysis, we prove the existence of constant mean curvature surfaces which are close to the truncated catenoid \( \mathcal{C}_{\tau} \) and have two boundaries which can be described by some function \( f : S^1 \to \mathbb{R} \). We also require that the surfaces are invariant under the action of the symmetry with respect to the horizontal plane. More precisely, we have the following:

Proposition 5.2. Assume we are given \( \kappa > 0 \) large enough (the value of \( \kappa \) will be fixed later on). For all \( \tau, \tilde{\tau} > 0 \) small enough satisfying (5.25) and for all functions \( f \) invariant under the action of the \( \text{Dih}_{m+1}^{(2)} \), satisfying \( (H1) \) (notice that \( (H2) \) is automatically satisfied) and
\begin{equation}
\| f \|_{C^{2,\alpha}(S^1)} \leq \kappa \tau^{3/2},
\end{equation}
there exists a constant mean curvature surface \( \mathcal{C}_{\tau,f} \) which is close to \( \mathcal{C}_{\tau} \) and has two boundaries. The surface \( \mathcal{C}_{\tau,f} \) is invariant under the action of \( S_3 \), the symmetry with respect to the horizontal plane \( x_3 = 0 \), \( S_2 \), the symmetry with respect to the plane \( x_2 = 0 \), and, close to its lower boundary, the surface \( \mathcal{C}_{\tau,f} \) is a vertical graph over the annulus
\[ \left\{ x \in \mathbb{R}^2 : \frac{1}{2} \tau^{3/4} \leq |x| \leq \tau^{3/4} \right\}, \]
for some function \( x \mapsto U_{\tau,f}(\tau^{-3/4} x) \) which can be expanded as follows
\begin{equation}
U_{\tau,f}(x) = -\tilde{\tau} \log \left( \frac{2 \tau^{3/4}}{\tilde{\tau}} \right) - \tilde{\tau} \log |x| + W_{f}^{\text{ins}}(x) + \bar{U}_{\tau,f}(x),
\end{equation}
where we recall that \( W_{f}^{\text{ins}} \) denotes the bounded harmonic extension of \( f \) in the punctured unit disc and where
\begin{equation}
\| \bar{U}_{\tau,f} \|_{C^{2,\alpha}(\overline{D(0,1)} - D(0,1/2))} \leq C \tau^{3/2}.
\end{equation}
Moreover, the nonlinear mapping
\[ C^{2,\alpha}(S^1) \ni f \mapsto \bar{U}_{\tau,f} \in C^{2,\alpha}(\overline{D(0,1)} - D(0,1/2)), \]
we modify the unit vector field $N$ shall only insist on the main differences.

Proof. The proof of this result is very close to the proof of Proposition 4.6 so we continuously on $\tilde{\tau}$ the image of $(\tilde{s}, 2) > 0$ for some constant $\tau$ and are bounded (in the smooth topology) by a constant, independent of the upper half of the catenoid, on the image of $(\tilde{s}, 2, \tilde{s} + 2) \times S^1$ by $Y_0$, so that our construction remains invariant under the action of the symmetry with respect to the horizontal plane. In this case, using Proposition 2.2, one can check that the expression of the mean curvature given in Lemma 4.5 has to be altered into

$$H(w) = J_0 w + \frac{\cosh\bar{s}}{s} \ell_0 w + \frac{1}{\cosh s} Q_0 \left( \frac{w}{\cosh s} \right),$$

where $Q_0$ enjoys properties which are similar to the properties enjoyed by $Q_0$ and where $\ell_0$ is a linear second order partial differential operator in $\partial_s$ and $\partial_\theta$ whose coefficients are smooth, supported in $(-\tilde{s}, \tilde{s}) \times S^1$ and in $(\tilde{s}, 2, \tilde{s} + 2) \times S^1$ and are bounded (in the smooth topology) by a constant, independent of $\tau$, times $\tau^{1/2}$.

We assume that $f$ is chosen to satisfy (H1), (H2) and

$$\|f\|_{C^{2,\alpha}(S^1)} \leq \kappa \tau^{1/2}.$$  

Observe that the norm of the boundary data $f$ is bounded by a constant times $\tau^{1/2}$ and not $\tau^{3/2}$. The reason being that we are going to perturb the image of $[-\tilde{s}, \tilde{s}] \times S^1$ by $Y_0$ and then scale the surface we obtain by a factor $\tilde{\tau}$ instead of perturbing $\xi_2$ which is the image of $[0, \tilde{s}] \times S^1$ by $Y_0$. Also this is the reason why the equation we will solve is $H(w) = \tilde{\tau}$ and not $H(w) = 1$.

We denote by $F$ the harmonic extension of $f$ in $(-\infty, \tilde{s}) \times S^1$ and we set

$$\tilde{F}(s, \theta) := F(s, \theta) + F(-s, \theta)$$

which is well defined in $[-\tilde{s}, \tilde{s}] \times S^1$. One should be aware that the boundary data of $F$ is not exactly equal to $f$ but the difference between $F$ and $f$ on the boundary tends to 0 as $\tau$ tends to 0. More precisely we have

$$\|F - \tilde{F}\|_{C^{2,\alpha}([-\tilde{s}, \tilde{s}] \times S^1)} \leq C \tau \|f\|_{C^{2,\alpha}(S^1)}.$$  

We would like to solve the equation

$$(\cosh s)^2 J_0 (\tilde{F} + w) + \ell_0 (\tilde{F} + w) + \cosh s Q_0 \left( \frac{\tilde{F} + w}{\cosh s} \right) = (\cosh s)^2 \tilde{\tau}.$$  

in $(-\tilde{s}, \tilde{s}) \times S^1$. This will provide constant mean curvature surfaces with mean curvature equal to $\tilde{\tau}$, which are close to the truncated catenoid. Again, the solvability of this nonlinear problem will follow from a fixed point theorem for a contraction mapping.

We choose

$$\mathcal{E}_\tau : C^{0,\alpha}([-\tilde{s}, \tilde{s}] \times S^1) \rightarrow C^{0,\alpha}(\mathbb{R} \times S^1)$$
an extension operator such that

\[
\begin{aligned}
\bar{\mathcal{E}}_\tau(\psi) &= \psi \quad \text{in} \quad [-\bar{s}, \bar{s}] \times S^1 \\
\tilde{\mathcal{E}}_\tau(\psi) &= 0 \quad \text{in} \quad ((-\infty, -\bar{s} - 1] \cup [\bar{s} + 1, \infty)) \times S^1,
\end{aligned}
\]

and

\[
\|\bar{\mathcal{E}}_\tau(\psi)\|_{C^{0,\alpha}([-\bar{s} - 1, -\bar{s} + 1] \times S^1)} \leq C \|\psi\|_{C^{0,\alpha}([-\bar{s}, \bar{s} + 1] \times S^1)}.
\]

and

\[
\|\tilde{\mathcal{E}}_\tau(\psi)\|_{C^{0,\alpha}([-\bar{s} - 1, -\bar{s} + 1] \times S^1)} \leq C \|\psi\|_{C^{0,\alpha}([-\bar{s}, \bar{s} + 1] \times S^1)}.
\]

We rewrite (5.30) as

\[
(cosh s)^2 J_0 w = \tilde{\mathcal{E}}_\tau \left( ((cosh s)^2 (\tilde{\tau} - J_0 \tilde{F}) - \ell_0 (\tilde{F} + w) - \cosh s Q_0 \left( \frac{\tilde{F} + w}{cosh s} \right) \right),
\]

Again, on the right hand side it is understood that we consider the image by \( \tilde{\mathcal{E}}_\tau \) of the restriction of the functions to \([-\bar{s}, \bar{s}] \times S^1 \).

We assume that \( \delta \in (1, 2) \) is fixed. It is easy to check that there exists a constant \( c > 0 \) (independent of \( \kappa \)) and a constant \( \kappa_0 > 0 \) (depending on \( \kappa \)) such that

\[
\|((cosh s)^{-\delta} \bar{\mathcal{E}}_\tau (cosh s^2 \tilde{\tau}))\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} \leq c \tau^{(2+\delta)/4},
\]

\[
\|((cosh s)^{-\delta} \tilde{\mathcal{E}}_\tau \left( \frac{\tilde{F}}{\cosh s^2} \right))\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} \leq c \tau^{1/2} \|f\|_{C^{2,\alpha}(S^1)},
\]

\[
\|((cosh s)^{-\delta} \tilde{\mathcal{E}}_\tau \left( \ell_0 (\tilde{F} + w) \right))\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} \leq c \tau^{1/2} (\|f\|_{C^{2,\alpha}(S^1)} + \tau^{\delta/4} \|(cosh s)^{-\delta} w\|_{C^{2,\alpha}(\mathbb{R} \times S^1)}),
\]

and

\[
\|((cosh s)^{-\delta} \tilde{\mathcal{E}}_\tau \left( \cosh s Q_0 \left( \frac{w' + \tilde{F}'}{\cosh s} \right) - \cosh s Q_0 \left( \frac{w + \tilde{F}}{\cosh s} \right) \right))\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} \leq C \kappa \left( \tau^{3/4} \|(cosh s)^{-\delta} (w' - w)\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} + \tau^{(1+\delta)/4} \|f' - f\|_{C^{2,\alpha}(S^1)} \right),
\]

provided \( w \) and \( w' \) satisfy

\[
\|((cosh s)^{-\delta} w\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} + \|(cosh s)^{-\delta} w'\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} \leq C \tau^{(2+\delta)/4},
\]

for some fixed constant \( C > 0 \) (independent of \( \kappa, \tau \) and \( f \)). Here \( \tilde{F} \) and \( \tilde{F}' \) are respectively the harmonic extensions of the boundary data \( f \) and \( f' \).

Now, we make use of the result of Theorem 5.1 to rephrase the problem as a fixed point problem and the previous estimates are precisely enough to solve this nonlinear problem using a fixed point argument for contraction mappings in the ball of radius \( C \tau^{(2+\delta)/4} \) in \((cosh s)^{\delta} C^{2,\alpha}(\mathbb{R} \times S^1)\), where \( C > 0 \) is fixed large enough independent of \( \kappa \) provided \( \tau \) is small enough. Then, for all \( \tau > 0 \) small enough, we find that there exists a constant \( C > 0 \) (independent of \( \kappa \)) \( C_\kappa > 0 \) (depending on \( \kappa \)) such that, for all functions \( f \) satisfying (H1), (H2) and (5.26), there exists a unique solution \( w \) of (6.45) satisfying

\[
\|(cosh s)^{\delta} w\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} \leq C \tau^{(2+\delta)/4},
\]

In addition, we have the estimate

\[
\|e^{-\delta} (w' - w)\|_{C^{2,\alpha}(\mathbb{R} \times S^1)} \leq C \kappa \tau^{1/2} \|f' - f\|_{C^{2,\alpha}(S^1)},
\]
where \( w \) (resp. \( w' \)) is the solution associated to \( f \) (resp. \( f' \)).

To complete the result, we simply shrink the surface we have obtained by a factor \( \tilde{\tau} \) to get a surface whose mean curvature is constant and equal to 1. The description of this surface close to its boundaries follows from the arguments already used in the proof of Proposition 4.6. Observe that, the solution of (6.45) is obtained through a fixed point theorem for contraction mappings, and it is classical to check that the solution we obtain depends continuously on the parameters of the construction. In particular, the constant mean curvature surface we obtain depends continuously on \( \tilde{\tau} \) (in fact one can also prove that the surface depends smoothly on \( \tilde{\tau} \)), but we shall not use this property.

To prove that, near its lower boundary, the surface we have obtained is a vertical graph for some function which enjoys the decomposition (5.27), we make use of the expansion in Lemma 5.1 and we follow the steps of the construction. Notice that \( \bar{U}^1_{\tilde{\tau},j} \) collects many remainders: the one coming from the expansion in Lemma 5.1, the difference between \( F \) and \( \tilde{F} \), the function \( w \) solution of the fixed point problem and also the change of coordinates which takes into account that the variable \( s \) does not correspond to the cylindrical coordinates in \( \mathbb{R}^2 - \{0\} \).

\[ \square \]

6. The unit sphere

6.1. Notations and definitions. We denote by \( x_1, x_2, x_3 \) the coordinates in \( \mathbb{R}^3 \). We agree that \( S_j \) denotes the symmetry with respect to the \( x_j = 0 \) plane, and, for all \( m \in \mathbb{N} \) and that \( R_{m+1} \) denotes the rotation of angle \( \frac{2\pi}{m+1} \) about the \( x_3 \)-axis. With slight abuse of notations, we will keep the same notation to denote the restriction of these isometries to the horizontal plane.

We define \( z_0, \ldots, z_m \in S^1 \) to be vertices of a regular polygon with \( m + 1 \) edges in the plane. Without loss of generality, we can choose

\[ z_0 := (1, 0) = e_1 \in \mathbb{R}^2, \]

and, for \( j = 1, \ldots, m - 1 \), \( z_{j+1} \in \mathbb{R}^2 \) is the image of \( z_j \) by \( R_{m+1} \). In other words, if we identify the horizontal plane with \( \mathbb{C} \), the vertices of the polygon are exactly the \((m + 1)\)-th roots of unity. Recall that the dihedral group of symmetries of \( \mathbb{R}^2 \) leaving this polygon fixed has been denoted by \( \text{Dih}^{(2)}_{m+1} \). It is generated by \( R_{m+1} \) and \( S_2 \).

Let \( S^2 \) be the unit sphere in \( \mathbb{R}^3 \). The upper half hemisphere of \( S^2 \) can be parameterized by

\[ X^+(x) := \left( x, \sqrt{1 - |x|^2} \right), \]

while the lower hemisphere is parameterized by

\[ X^-(x) := \left( x, -\sqrt{1 - |x|^2} \right) , \]

where, in both cases, \( x \in D(0, 1) \).

For all \( \tau > 0 \) small enough, we set

\[ B^\tau := X^+(D(0, \tau^{3/4})), \]

and, for all \( \rho > 0 \) satisfying

\[ \frac{1}{C} \tau \leq \rho^2 \leq C \tau, \]
for some fixed constant $C > 1$, we define
\[ B_j := X(D(\rho z_j, \tau^{3/4})), \]
for $j = 0, \ldots, m$. We also define
\[ p := X(0), \]
to be the north pole of $S^2$ and, for $j = 0, \ldots, m$, we define the points
\[ p_j := X(\rho z_j), \]
which are $m + 1$ points arranged symmetrically near the south pole of $S^2$. By construction $p_{j+1}$ is the image of $p_j$ by $R_{m+1}$.

**Definition 6.1.** We define $S_{\tau, \rho}$ to be the surface obtained by excising from $S^2$, the sets $B^1$ and $B_j^1$, for $j = 0, \ldots, m$.

Observe that, provided $\tau$ is chosen small enough, the surface $S_{\tau, \rho}$ has $m + 2$ boundaries. Moreover, this surface has been constructed in such a way that it is invariant under the action of the dihedral group $\text{Dih}_{m+1}^{(2)}$.

### 6.2. The mean curvature of vertical graphs

We recall some well known facts about the mean curvature of vertical graphs in $\mathbb{R}^3$. The mean curvature of the graph of the function $u \in C^2_{loc}(\mathbb{R}^2)$, namely the surface parameterized by
\[ X(x) := (x, u(x)) \in \mathbb{R}^3, \]
where $x$ belongs to some open domain in $\mathbb{R}^2$, is given by
\[ M(u) := \frac{1}{2} \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right). \]
Recall that the mean curvature is defined to be the average of the principal curvatures and this explains the factor $1/2$.

It follows from this formula that the linearized mean curvature operator about the graph of $u$ is given by
\[
DM(u)(v) = \frac{\Delta v}{2W} + \frac{3}{2W^5} (\nabla u \cdot \nabla v) D^2 u [\nabla u, \nabla u] - \frac{1}{2W^3} \left( (\nabla u \cdot \nabla v) \Delta u + D^2 v [\nabla u, \nabla u] + 2D^2 u [\nabla u, \nabla v] \right),
\]
where
\[ W := \sqrt{1 + |\nabla u|^2}, \]
and where $D^2 f [\cdot, \cdot]$ is the second order differential of the function $f$. One should be aware that $DM(u)$ is not the Jacobi operator $J_u$ about the graph of the function $u$ since nearby surfaces are not parameterized as normal graphs but are parameterized as vertical graphs over the horizontal plane. As explained in §2, this operator and the Jacobi operator are conjugate and in fact, assuming the the vertical graph is oriented to so that unit normal vector field points upward, we have the relation
\[ X^*(J_u w) = DM(u)(W X^*w), \]
for any function defined on the graph of $u$.

Of interest will be the case where, for example,
\[ u(x) = \pm \sqrt{1 - |x|^2}, \]
where $x = (x_1, x_2) \in \mathbb{R}^2$. According to the sign chosen, the graph of $u$ is the lower or the upper hemisphere of the sphere of radius 1 centered the origin. In this case, we have

$$
\nabla u(x) = \mp \frac{x}{\sqrt{1 - |x|^2}}, \quad \nabla^2 u(x) = \mp \frac{(1 - |x|^2) \text{Id} + x \otimes x}{(1 - |x|^2)^{3/2}},
$$

and

$$
\Delta u(x) = \mp \frac{2 - |x|^2}{(1 - |x|^2)^{3/2}}.
$$

Using these, we find that the explicit expression of $DH(u)$ is given by

$$
\text{(6.33)} \quad DM(u) w = \frac{1}{2} (1 - |x|^2)^{1/2} \left( \Delta w - \nabla^2 w(x,x) - 4 (x \cdot \nabla w) \right),
$$

in $D(0,1)$.

### 6.3. Green’s function.

Let $N_0$ denote the inward pointing unit normal vector field on $S^2$. We consider an inward pointing vector field $N^\flat_0$ which is equal to $N_0$ close to the (horizontal) equator of $S^2$ and which is equal to a vertical unit vector field close to the north and south pole of $S^2$ (still pointing inward).

We define $L$ to be the linearized mean curvature operator using the vector field $N^\flat_0$. According to the analysis of §2, we can write

$$
\text{(6.34)} \quad L w := \frac{1}{2} \left( \Delta_{S^2} + 2 \right) (N_0 \cdot N^\flat_0 w).
$$

We let $\Gamma_\rho$ be the unique solution of

$$
\text{(6.35)} \quad L \Gamma_\rho = -\pi \delta_{\rho^1} - \frac{\pi}{\sqrt{1 - \rho^2}} \frac{1}{m + 1} (\delta_{\rho^0} + \ldots + \delta_{\rho^m}),
$$

which satisfies the orthogonality conditions

$$
\int_{S^2} x_i \Gamma_\rho \, d\text{vol}_{S^2} = 0,
$$

for $i = 1, 2$ and 3. Here $\delta_q$ is the Dirac mass at the point $q$. The existence of $\Gamma_\rho$ is guarantied by the fact that the distribution on the right hand side of (6.35) is orthogonal to the cokernel of $L$. Indeed, the Jacobi operator is self-adjoint and its kernel and cokernel are equal and spanned by the restriction of the coordinate functions to the unit sphere. Thanks to (6.34), we conclude that the cokernel of $L$ is also spanned by the restriction of the coordinate functions to the unit sphere, multiplied by the factor $N_0 \cdot N^\flat_0$. For the sake of simplicity, we define the functions

$$
\tilde{x}_i := x_1 N_0 \cdot N^\flat_0,
$$

for $i = 1, 2, 3$, which are obtained by multiplication of the coordinate functions by $N_0 \cdot N^\flat_0$.

Now

$$
\langle x_1, \delta_{\rho^1} \rangle_{D,D'} = \langle x_2, \delta_{\rho^1} \rangle_{D,D'} = 0,
$$

since both $x_1$ and $x_2$ vanish at the north pole of $S^2$ and

$$
\langle x_1, \delta_{\rho^j} \rangle_{D,D'} = \rho \cos \left( \frac{2\pi}{m+1} j \right) \quad \text{and} \quad \langle x_2, \delta_{\rho^j} \rangle_{D,D'} = \rho \sin \left( \frac{2\pi}{m+1} j \right).
$$
Since
\[ \sum_{j=0}^{m} \cos \left( \frac{2\pi}{m+1} j \right) = \sum_{j=0}^{m} \sin \left( \frac{2\pi}{m+1} j \right) = 0, \]
we conclude that the distribution on the right hand side of (6.35) is orthogonal to the coordinate functions \( x_1 \) and \( x_2 \). Geometrically, this is simply a consequence of the fact that the points \( p_j \) are symmetrically arranged around the \( x_3 \)-axis. Finally, we have
\[ \langle x_3, \delta_{p_j} \rangle_{D',D} = -\sqrt{1 - \rho^2}, \]
and, again, we conclude that the distribution on the right hand side of (6.35) is orthogonal to the coordinate function \( x_3 \) thanks to the choice of the constant in front of the Dirac masses at the points \( p_j \). Geometrically, this will have some interesting consequence and can be interpreted as a conservation of the vertical flux of the surfaces we try to construct. We shall return to this point later on.

Finally, observe that \( \Gamma_{\rho} \) is invariant under the action of the elements of \( \text{Dih}_{m+1}^{(2)} \).

The following result provides the expansion of the function \( \Gamma_{\rho} \) close to the north pole of \( S^2 \).

**Lemma 6.1.** The following expansion holds
\[ X^\dagger \Gamma_{\rho}(x) = -\log |x| + a^\dagger + O_{\mathcal{C}^\infty}( |x|^2 ), \]
in a fixed neighborhood of 0, where the constant \( a^\dagger \in \mathbb{R} \) depends smoothly on \( \rho \) and is bounded as \( \rho \) tends to 0. Moreover, the estimate on \( O_{\mathcal{C}^\infty}( |x|^2 ) \) is uniform as \( \rho \) tends to 0.

**Proof.** We define the function \( \Gamma_0 \) on the upper hemisphere by
\[ X^\dagger \Gamma_0(x) = -\log |x|, \]
and, using (6.33), we compute
\[ X^\dagger \Gamma_0(x) = -\frac{3}{2} \sqrt{1 - |x|^2}. \]
This immediately implies that, close to \( p_1 \), the function \( \Gamma_{\rho} - \Gamma_0 \) is smooth. In particular, this function can be expanded as
\[ X^\dagger \Gamma_0(x) = a^\dagger + b^\dagger \cdot x + O_{\mathcal{C}^\infty}( |x|^2 ), \]
where \( a^\dagger \in \mathbb{R} \) and \( b^\dagger \in \mathbb{R}^2 \) depend smoothly on \( \rho \) and remain bounded as \( \rho \) tends to 0. Since the function \( \Gamma_{\rho} \) is also invariant under the action of the elements of \( \text{Dih}_{m+1}^{(2)} \), we conclude that necessarily \( b^\dagger = 0 \). This completes the proof of the result. \( \square \)

Near the other poles, the function \( \Gamma_{\rho} \) also has an expansion which we now describe. As can be suspected, this later description relies on the expansion of the function
\[ G(x) := -\sum_{j=0}^{m} \log |x - \rho z_j|, \]
at any of its singularities. Since this is a key point in our construction, we spend some time to derive this expansion carefully. By symmetry, it is enough to expand
this function at \( \rho z_0 \). We change variables and write \( x = \rho z_0 + y \). We then expand
\[
\log |y - \rho (z_j - z_0)| = \log \rho + \log |z_j - z_0| + \frac{1}{\rho} \frac{z_0 - z_j}{|z_0 - z_j|^2} \cdot y + \mathcal{O}\left(\frac{|y|^2}{\rho^2}\right).
\]
Hence we find
\[
G(\rho z_0 + y) = -\log |y| - m \log \rho - \sum_{j=1}^{m} \log |z_j - z_0| - \frac{1}{\rho} \sum_{j=1}^{m} \frac{z_0 - z_j}{|z_0 - z_j|^2} \cdot y + \mathcal{O}\left(\frac{|y|^2}{\rho^2}\right).
\]

It is easy to check that the following identity holds
\[
\sum_{j=1}^{m} \frac{z_0 - z_j}{|z_0 - z_j|^2} = \frac{m}{2} z_0.
\]

Setting
\[
a_0 := \sum_{j=1}^{m} \log |z_j - z_0|,
\]

we can write
\[
G(\rho z_0 + y) = -\log |y| - m \log \rho - a_0 - \frac{m}{2} \rho z_0 \cdot y + \mathcal{O}\left(\frac{|y|^2}{\rho^2}\right).
\]

Similar estimates can be obtained for the partial derivatives of \( G \). Finally, observe that
\[
\Delta G = -2 \pi (\delta_{\rho z_0} + \ldots + \delta_{\rho z_m}).
\]

Finally, observe that
\[
\sum_{j=1}^{m} \frac{z_0 - z_j}{|z_0 - z_j|^2} = \frac{m}{2} z_0.
\]

Setting
\[
a_0 := \sum_{j=1}^{m} \log |z_j - z_0|,
\]
we can write
\[
G(\rho z_0 + y) = -\log |y| - m \log \rho - a_0 - \frac{m}{2} \rho z_0 \cdot y + \mathcal{O}\left(\frac{|y|^2}{\rho^2}\right).
\]

We now prove that, at \( p_j \), the expansion of the function \( X^{\star} \Gamma_{\rho} \) is (in some sense to be made precise) close to the expansion of \( G \) near \( \rho z_j \). This is the content of the following:

**Lemma 6.2.** The following expansion holds
\[
X^{\star} \Gamma_{\rho}(\rho z_j + y) = -\frac{1}{m+1} \left( \log |y| + m \log \rho + a_{0,\rho} + \frac{m}{2} \rho z_j \cdot y \right) + \mathcal{O}_{\mathcal{C}^{\infty}}(r^{1/2}),
\]

for \( |y| \in \left[ \frac{1}{2} r^{3/4}, 2 r^{3/4} \right] \). Here \( a_{0,\rho} \in \mathbb{R} \) smoothly depends on \( \rho > 0 \) and is uniformly bounded as \( \rho \) tends to 0.

**Proof.** Thanks to the invariance with respect to the action of \( \text{Dih}_{m+1}^{(2)} \), it is enough to describe this expansion near the point \( p_j \). As in the proof of the previous lemma we show that, near the south pole of \( S^2 \), the function \( X^{\star} \Gamma_{\rho} \) is not too far from \( G \).

To this aim, we define \( \tilde{\Gamma}_{\rho} \) on the lower hemisphere of \( S^2 \) by
\[
X^{\star} \tilde{\Gamma}_{\rho} = G,
\]

and, thanks to (6.33), we can compute
\[
X^{\star}(\mathbb{L} \tilde{\Gamma}_{\rho} + \pi \sqrt{1 - \rho^2} (\delta_{p_0} + \ldots + \delta_{p_m})) = \frac{1}{2} \sqrt{1 - |x|^2} \sum_{j=0}^{m} \left( 3 - 2 \rho \frac{z_j \cdot (x - \rho z_j)}{|x - \rho z_j|^2} + \frac{\rho^2}{|x - \rho z_j|^2} \left( 1 - 2 \frac{(z_j \cdot (x - \rho z_j))^2}{|x - \rho z_j|^2} \right) \right).
\]
Observe that the right hand side contains three terms which have different regularity properties. The first one is a smooth function which depends smoothly on \( \rho \) and which is invariant by rotation. The second function has a singularity of order 1 at each \( \rho z_j \) and is bounded by a constant times \( \rho |x - \rho z_j|^{-1} \). Finally, the third function has a singularity of order 2 at each \( \rho z_j \) and is bounded by a constant times \( \rho^2 |x - \rho z_j|^{-2} \).

As a consequence, \( X^1 \ast (\tilde{\Gamma}_\rho - \Gamma_\rho) \) can be decomposed into the sum of three functions which can be analyzed independently. The first one \( f_\rho^{(1)} \) is smooth in a fixed neighborhood of 0 and depends smoothly on \( \rho \). This implies that, near each \( \rho z_j \), this function has a Taylor’s expansion with coefficients smoothly depending on \( \rho \). Hence

\[
f_\rho^{(1)}(x) = f_\rho^{(1)}(\rho z_0) + \nabla f_\rho^{(1)}(\rho z_0) \cdot (x - \rho z_0) + \mathcal{O}(|x - \rho z_0|^2).
\]

Observe that \( \nabla f_\rho^{(1)}(0) = 0 \) and hence \( |\nabla f_\rho^{(1)}(\rho z_0)| \leq C \rho \). We conclude that \( f_\rho^{(1)}(x) = f_\rho^{(1)}(\rho z_0) + \mathcal{O}(\tau^{5/4}) \) when \( |x - \rho z_0| \in \left[ \frac{1}{2} \tau^{3/4}, 2 \tau^{3/4} \right] \).

Since \( \sum_j |x - z_j|^{-1} \in L^p(D(0, 1/2)) \) for all \( p \in (1, 2) \), we find that the second function \( f_\rho^{(2)} \) is in \( W^{2,p}(D(0, 1/2)) \) and hence that it is continuous near \( \rho \) and \( f_\rho^{(2)}(x) - f_\rho^{(2)}(\rho z_0) \) is bounded by a constant times \( \rho \sum_j |x - \rho z_j|^{\nu} \), for any given \( \nu < 1 \). In particular, \( f_\rho^{(2)}(x) = f_\rho^{(2)}(\rho z_0) + \mathcal{O}(\tau^{(2+3\nu)/4}) \) when \( |x - \rho z_0| \in \left[ \frac{1}{2} \tau^{3/4}, 2 \tau^{3/4} \right] \).

Finally, using the result of Proposition 6.2, the third function \( f_\rho^{(3)} \) is bounded by a constant times \( \rho^2 \sum_j |x - \rho z_j|^\mu \), for any \( \mu \in (-1, 0) \).

In particular, when \( |x - \rho z_0| \in \left[ \frac{1}{2} \tau^{3/4}, 2 \tau^{3/4} \right] \), we find that the sum of these function can be decomposed as the sum of a constant function (smoothly depending on \( \rho \)) and a function which is bounded by a constant times \( \tau^{1/2} \) (chose \( \nu = 1/2 \) and \( \mu = -1/2 \)). The statement then follows at once.

It is interesting to observe that \( \Gamma_\rho \) depends on \( \rho > 0 \) since the points \( p_j^\rho \) do and, as \( \rho \) tends to 0, the sequence \( \Gamma_\rho \) converges on compacts to the unique solution of

\[
L \Gamma_0 = -\pi \left( \delta_p + \delta_{\rho p} \right),
\]

which is \( L^2 \)-orthogonal to the smooth kernel of \( \Delta_{S^2} + 2 \). Recall that \( p_j^\rho \) denotes the north pole of \( S^2 \) and we now agree that \( p_j^\rho \) denotes the south pole of \( S^2 \).

**Remark 6.1.** For later use, it will be important to notice that all solutions of \( L w = 0 \) which are defined in \( S^2 - \{ p_j^\rho \} \), are invariant under the action of \( \text{Dih}_{m+1} \) and are bounded by a constant times \( \text{dist}(\cdot, \{ p_j^\rho \})^\nu \) for some \( \nu \in (-1, 0) \), are linear combinations of the functions \( x_3 \) and \( \Gamma_0 \).

We now summarize the above analysis. We set

\[
u^1(x) := \sqrt{1 - |x|^2}, \quad \text{and} \quad \nu^1(x) := -\sqrt{1 - |x|^2}.
\]

Observe that, thanks to the previous results, we see that near 0, the graph of the function

\[
v^1 := \nu^1 + \tau X^1 \ast \Gamma_\rho,
\]

can be expanded

\[
v^1(x) = 1 + \tau (m \log \rho + a^1) + \tau \log |x| + \mathcal{O}_{\infty}(\tau^{3/2}),
\]
for \(|x| \in [\frac{1}{2} \tau^{3/4}, 2 \tau^{3/4}]\), where \(a^j \in \mathbb{R}\) smoothly depends on \(\rho\). Moreover, we see that near \(\rho z_j\), the graph of the function

\[ v^j := u^j + \tau X^j \Gamma_\rho, \]

can be expanded

\[ v^j(\rho z + y) = -\sqrt{1 - \rho^2} - \tau \left( m \log \rho + a^j \right) - \frac{\tau}{m + 1} \log |y| \]

\[ - \left( \rho - \frac{m}{m + 1} \frac{\tau}{2\rho} \right) z_j \cdot y + O_{C^\infty}(\tau^{3/2}), \]

for \(|y| \in [\frac{1}{2} \tau^{3/4}, 2 \tau^{3/4}]\), where \(a^j \in \mathbb{R}\) smoothly depends on \(\rho\). The key point in our construction is that the constant in front of \(z_j \cdot y\) can be adjusted by choosing \(\rho\) appropriately. Indeed, if we define \(\rho_0 > 0\) by the identity

\[ 2 (m + 1) \rho_0^2 = m \tau, \]

then, when \(\rho = \rho_0\), the constant in front of \(z_j \cdot y\) in the last expansion is exactly 0 while choosing \(\rho \neq \rho_0\) slightly larger or smaller allows one to prescribe any value of this constant, close enough to 0.

### 6.4. Mapping properties of the Jacobi operator about a punctured sphere.

To begin with we define on \(S^2\), the distance function to the punctures \(p^1, p_0, \ldots, p_m\) by

\[ d := \text{dist}_{S^2} \left( \cdots \{p^1, p_0, \ldots, p_m\} \right). \]

Even though this is not apparent in the notations, the function \(d\) depends implicitly on \(\rho\) since it depends on the location of the points \(p^j\) which themselves do depend on \(\rho\). We can define some weighted spaces on

\[ S^* := S^2 \setminus \{p^1, p_0, \ldots, p_m\}. \]

For all \(\nu \in \mathbb{R}\) and \(k \in \mathbb{N}\) we define \(C^{k,\alpha}_\nu(S^*)\) to be the space of functions \(w \in C_{\text{loc}}^k(S^*)\) for which the following norm is finite

\[ \|w\|_{C^{k,\alpha}_\nu(S^*)} := \sum_{j=0}^k \sup_{p \in S^*} d^{-\nu - j}(p) \|\nabla^j w(p)\|_{S^2} \]

\[ + \sup_{\zeta \in (0, \pi/2)} \left( \sup_{d(p), d(q) \in [\zeta, 2\zeta]} \zeta^{-k+\alpha} \frac{||\nabla^k w(p) - \nabla^k w(q)||_{S^2}}{\text{dist}_{S^2}(p, q)^\alpha} \right). \]

We further assume that the functions in \(C^{k,\alpha}_\nu(S^*)\) are invariant under the action of \(\text{Dih}_m^{(2)}\). Again, observe that the weighted spaces \(C^{k,\alpha}_\nu(S^*)\) do implicitly depend on \(\rho\).

We consider the operator

\[ \mathbb{L}_\nu : C^{2,\alpha}_\nu(S^*) \to C^{0,\alpha}_{\nu-2}(S^*), \]

\[ w \mapsto \mathbb{L}_\nu w. \]

It is easy to check that \(\mathbb{L}_\nu\) is well defined.

Recall that \(L\) is conjugate to \(\Delta_{S^2} + 2\). When acting on smooth function defined on \(S^2\), the mapping properties of \(\Delta_{S^2} + 2\) are well understood and we recall that the kernel of this operator is spanned by the restriction to \(S^2\) of the linear forms on \(\mathbb{R}^3\). Since we are assuming that the functions we consider are invariant under
the action of the dihedral group $\text{Dih}_{m+1}^{(2)}$, this implies that the bounded kernel of $L$ has dimension 1. We now investigate the mapping properties of $L$ (or alternatively $\Delta_{S^2} + 2$) when acting on functions belonging to the weighted spaces we have just defined. We start with the:

**Proposition 6.1.** Assume that $\nu \in (-1, 0)$, then there exist constants $C, \rho_0 > 0$ only depending on $\nu$ such that, for all $\rho \in (0, \rho_0)$, we have

$$\|w\|_{H^{2, \alpha}_{\nu}(S^2)} \leq C \|Lw\|_{H^{0, \alpha}_{\nu}(S^2)},$$

for all functions $w$ in the $L^2(S^2)$-orthogonal complement of the functions $\tilde{x}_3$ and $\Gamma_{\rho}$.

**Proof.** As usual, thanks to Schauder’s estimates, it is enough to prove that

$$\|d^{-\nu} w\|_{L^\infty(S^2)} \leq C \|d^{2-\nu} Lw\|_{L^\infty(S^2)},$$

for all $\rho$ small enough.

As usual, the proof of this estimate is by contradiction. Assume that the estimate is not true, then, there would exist a sequence $\rho_n$ tending to 0 and a sequence of functions $w_n$ such that

$$\|d^{-\nu} w_n\|_{L^\infty(S^2)} = 1 \quad \text{and} \quad \lim_{n \to \infty} \|d^{2-\nu} Lw_n\|_{L^\infty(S^2)} = 0.$$

Moreover $w_n$ is invariant under the action of $\text{Dih}_{m+1}^{(2)}$ and is $L^2$-orthogonal to $\tilde{x}_3$ and $\Gamma_{\rho_n}$ (recall that $\Gamma_{\rho} = \Gamma_{\rho_n}$ depends on $\rho_n$). Hence,

$$(6.36) \quad \int_{S^2} \tilde{x}_3 w_n \, d\text{vol}_{S^2} = 0,$$

and

$$(6.37) \quad \int_{S^2} \Gamma_{\rho_n} w_n \, d\text{vol}_{S^2} = 0.$$

We choose a point $q_n \in S^2$ such that

$$|w_n(q_n)| \geq 1/2 \, d'(q_n),$$

and we distinguish various cases according to the behavior of the sequence $d(q_n)$. In each case, we rescale coordinates (using the exponential map) by $1/d(q_n)$ and we use elliptic estimates together with Ascoli-Arzelà’s theorem to extract from the sequence $\tilde{w}_n := d^{-\nu}(q_n) w_n$ convergent subsequences. Finally, we pass to the limit in the equation satisfied by $\tilde{w}_n$. If, for some subsequence, $d(q_n)$ remains bounded away from 0, we get in the limit a non trivial solution of

$$L w = 0,$$

which is defined in $S^2 - \{p, p^1 \}$, where we recall that $p^1$ denotes the north pole and $p^1$ denotes the south pole of $S^2$. Moreover, $w$ is bounded by a constant times $(\text{dist}(p, \{p^1, p^1\}))^\nu$ and $w$ is invariant under the action of $\text{Dih}_{m+1}^{(2)}$. Finally, we can pass to the limit in (6.36) and (6.37) and check that $w$ is $L^2$-orthogonal to $\tilde{x}_3$ and $\Gamma_0 := \lim_{n \to \infty} \Gamma_{\rho_n}$. It is easy to check (see Remark 6.1) that this implies that $w \equiv 0$, which is a contradiction.

The second case we have to consider is the case where $\lim_{n \to \infty} d(q_n) = 0$ and $\lim_{n \to \infty} d(q_n)/\rho_n = +\infty$ or the case where $\lim_{n \to \infty} d(q_n)/\rho_n = 0$. In either case, we obtain a nontrivial solution of

$$\Delta w = 0,$$
in $\mathbb{R}^2 - \{0\}$ which is bounded by a constant times dist$(\cdot, \{0\})^\nu$. It is easy to check that $w \equiv 0$ since $\delta \notin \mathbb{Z}$, which is again a contradiction.

Finally, we consider the case where $\lim_{n \to \infty} d(q_n)/\rho_n$ exists. In this case, we obtain a nontrivial solution of

$$\Delta w = 0,$$

in $\mathbb{R}^2 - \{r_0 z_0, \ldots, r_0 z_m\}$, for some $r_0 > 0$. Moreover, we know that this solution is bounded by a constant times (dist$(\cdot, \{r_0 z_0, \ldots, r_0 z_m\})^\nu$ and $w$ is also invariant under the action of $\text{Dih}_{m+1}^{(2)}$. Inspection of the behavior of $w$ at the points $r_0 z_j$ together with the fact that $\nu > -1$ and $w$ is invariant with respect to the action of $\text{Dih}_{m+1}^{(2)}$, implies that $w$ is a solution in the sense of distributions of

$$\Delta w = a \sum_{j=0}^{m} \delta_{r_0 z_j},$$

for some $a \in \mathbb{R}$. Then, inspection of $w$ at infinity together with the fact that $\nu < 0$, implies that necessarily $a = 0$ and hence $w \equiv 0$. This is again a contradiction. Having reached a contradiction in each case, this completes the proof of the result.

□

Thanks to the previous result, we can prove the:

**Proposition 6.2.** Assume that $\nu \in (-1, 0)$ is fixed. Then the operator $\mathbb{L}_\nu$ is surjective and has a 2 dimensional kernel spanned by the functions $\tilde{x}_3$ and $\Gamma_{\rho}$. Moreover, the right inverse of $\mathbb{L}_\nu$ which is chosen so that its image is in the $L^2$-orthogonal complement of the kernel of $\mathbb{L}_\nu$, has norm which is bounded independently of $\rho$ small enough.

**Proof.** The existence of a right inverse follows from the general theory developed for example in [12]. Nevertheless, we give here a self-contained proof.

Let us assume that we are given a function $f \in C^{0,\alpha}(S^*)$ which has compact support in $S^*$. Recall that the functions we are interested in are invariant under the action of $\text{Dih}_{m+1}^{(2)}$. We choose $a \in \mathbb{R}$ so that $f - a \delta_p$ is orthogonal to the function $\tilde{x}_3$. In particular, this implies that we can solve

$$\mathbb{L} \hat{w} = f - a \delta_p,$$

and, choosing the constant $b \in \mathbb{R}$ appropriately, we can assume that $w := \hat{w} - b \Gamma_{\rho}$ is $L^2$-orthogonal to the function $\tilde{x}_3$ and $\Gamma_{\rho}$. Observe that

$$\mathbb{L} w = f,$$

in $S^*$ and also that $w \in C^{2,\alpha}_\nu(S^*)$. In particular the result of Proposition 6.1 applies and we have

$$\|w\|_{C^{2,\alpha}_\nu(S^*)} \leq C \|\mathbb{L} w\|_{L^2(S^*)}. $$

The general result, when $f$ is not assumed to have compact support in $S^*$ can be handled as usual using a sequence of functions having compact support and converging on compacts to a given function in $C^{0,\alpha}_\nu(S^*)$. □
6.5. A third fixed point argument. Assume that we are given \( \tau, \tilde{\tau} > 0 \) small enough and satisfying

\[
|\tilde{\tau} - \tau| \leq \kappa \tau^{3/2},
\]

where the constant \( \kappa > 0 \) is fixed large enough and will be fixed in the last section of the paper. We also assume that \( \rho > 0 \) satisfies

\[
\left| \rho - \frac{m}{m+1} \frac{\tau}{2\rho} \right| \leq \kappa \tau^{3/4}.
\]

We prove the existence of an infinite dimensional family of constant mean curvature surfaces which are close to \( S_{\tau, \rho} \) are parameterized by their boundary values described by two functions \( f^1 : S^1 \to \mathbb{R} \) and \( f^1 : S^1 \to \mathbb{R} \). The surfaces also depend on \( \tilde{\tau} \) and \( \rho \) satisfying the above estimates.

Proposition 6.3. Assume we are given \( \kappa > 0 \) large enough (the value of \( \kappa \) will be fixed later on). For all \( \tau, \tilde{\tau} > 0 \) small enough satisfying (6.38) and for all functions \( f^1 \) which are invariant under the action of the dihedral group \( \text{Dih}_{m+1}^{(2)} \) and \( f^1 \), which are invariant under the action of \( S_2 \), both satisfying (H1) and

\[
\|f\|_{C^2(S^1)} \leq \kappa \tau^{3/2},
\]

there exists a constant mean curvature surface \( S_{\tilde{\tau}, \rho, f^1, f^1} \) which is a graph over \( S_{\tau, \rho} \), has \( m + 2 \) boundaries (one boundary close to the north pole and \( m + 1 \) boundaries close to the south pole) and is invariant under the action the dihedral group \( \text{Dih}_{m+1}^{(2)} \).

Close to the upper boundary, the surface \( S_{\tilde{\tau}, \rho, f^1, f^1} \) a vertical graph over the annulus

\[
\{ x \in \mathbb{R}^2 : \tau^{3/4} \leq |x| \leq 2 \tau^{3/4} \},
\]

for some function \( x \to V^1_{\tilde{\tau}, \rho, f^1, f^1}(\tau^{-3/4} x) \) which can be expanded as follows

\[
V^1_{\tilde{\tau}, \rho, f^1, f^1}(x) = 1 + \tilde{\tau} (m \log \rho + a^i_{\tilde{\tau}, \rho, f^1, f^1}) + \frac{3}{4} \tilde{\tau} \log \tau + \tilde{\tau} \log |x| - W^\text{out}_f(x)
\]

where \( a^i \in \mathbb{R} \), \( W^\text{out}_f \) denotes the bounded harmonic extension of \( f \) in \( \mathbb{R}^2 - \overline{D}(0, 1) \) and where

\[
\|V^1_{\tilde{\tau}, \rho, 0, 0}\|_{C^2(\overline{D}(0, 2) - D(0, 1))} \leq C \tau^{3/2},
\]

and, given \( \nu \in (-1, 0) \),

\[
\|V^1_{\tilde{\tau}, \rho, f^1, f^1} - V^1_{\tilde{\tau}, \rho, f^1, f^1}\|_{C^2(\overline{D}(0, 1) - D(0, 1/2))} \leq C \tau^{(1+\nu)/4} \left( \|f^1 - f^1\|_{C^2(S^1)} + \|f^1 - f^1\|_{C^2(S^1)} \right),
\]

for some constant \( C > 0 \) independent of \( \kappa \), \( \tilde{\tau} \) and \( f^1, f^1, f^1, f^1 \).

Near one of the lower boundaries the surface \( S_{\tilde{\tau}, \rho, f^1, f^1} \) is a vertical graph over the annulus

\[
\{ x \in \mathbb{R}^2 : \tau^{3/4} \leq |x - \rho z_0| \leq 2 \tau^{3/4} \}
\]
for some function $x \mapsto V_{\tilde{\tau}, \rho, f^1}^1(\tau^{3/4} (x - \rho z_0))$ which can be expanded as follows (6.43)

$$
V_{\tilde{\tau}, \rho, f^1}^1(x) = \sqrt{1-\rho^2} - \frac{\tilde{\tau}}{m+1} \left( m \log \rho + a_{\tilde{\tau}, \rho, f^1} \right) - \frac{3\tilde{\tau}}{4(m+1)} \log \tau \\
- \frac{\tilde{\tau}}{m+1} \log |x| - \tau^{3/4} \left( \rho - \frac{m}{m+1} \frac{\tilde{\tau}}{2\rho} \right) z_0 \cdot x \\
+ W_{\text{out}}^1 f^1(x) + V_{\tilde{\tau}, f^1}^1(x),
$$

where $V_{\tilde{\tau}, \rho, f^1}^1$ enjoys properties similar to the one described above for $V_{\tilde{\tau}, \rho, f^1}^1$. Moreover, both depend continuously on $\tilde{\tau}$ and $\rho$.

Proof. Again the arguments of the proof are close to the one already performed in the previous sections. The equation we try to solve can be written formally as (6.44)

$$
\mathbb{L}(\tilde{\tau} \Gamma + \hat{F} + w) = Q(\tilde{\tau} \Gamma + \hat{F} + w)
$$

where $Q$ collects all the nonlinear terms. Here $\hat{F}$ is a function which can be described as follows: near the north pole $p$:

$$
X^1 \cdot \hat{F}(x) = \chi W_{\text{out}}^1 (\tau^{-3/4} x),
$$

where $\chi$ is a cutoff function identically equal to 1 in $D(0, 1/4)$ and identically equal to 1 outside $D(0, 1/2)$. Near the south pole $p$:

$$
X^1 \cdot \hat{F}(x) = \sum_{j=0}^m \bar{\chi} \left( \frac{x - \rho z_j}{\rho} \right) W_{f^1}^1 (x - \rho z_j),
$$

where $\bar{\chi}$ is a cutoff function identically equal to 1 in $D(0, c)$ and identically equal to 1 outside $D(0, c/2)$. Here $c = \sin(\pi/(m+1))$ so that the balls of radius $c$ centered at the points $z_j$, for $j = 0, \ldots, m$ are disjoint.

We choose

$$
\tilde{\mathcal{E}}_{\tau} : C^{0, \alpha}(\mathcal{S}_{\tau, \rho}) \longrightarrow C^{0, \alpha}(S^*),
$$

an extension operator such that

$$
\begin{align*}
\tilde{\mathcal{E}}_{\tau}(\psi) &= \psi \quad \text{in} \quad \mathcal{S}_{\tau, \rho} \\
\tilde{\mathcal{E}}_{\tau}(\psi) &= 0 \quad \text{in} \quad X^1(D(0, \tau^{3/4}/2)) \cup \bigcup_{j=0}^m X^1(D(\rho z_j, \tau^{3/4}/2)),
\end{align*}
$$

and

$$
\|\tilde{\mathcal{E}}_{\tau}(\psi)\|_{C^{0, \alpha}(S^*)} \leq C \|\psi\|_{C^{0, \alpha}(\mathcal{S}_{\tau, \rho})}.
$$

By definition, the norm in the space $C^{0, \alpha}(\mathcal{S}_{\tau, \rho})$ is defined exactly as the norm in $C^{0, \alpha}(S^*)$ but points are restricted to $\mathcal{S}_{\tau, \rho}$ instead of $S^*$.

We rewrite (6.44) as

$$
\mathbb{L} w = \tilde{\mathcal{E}}_{\tau} \left( -\mathbb{L} \hat{F} + Q(\tilde{\tau} \Gamma + \hat{F} + w) \right).
$$

Observe that, by construction $\mathbb{L}(\tilde{\tau} \Gamma) = 0$ away from the singular points.

Again, on the right hand side it is understood that we consider the image by $\tilde{\mathcal{E}}_{\tau}$ of the restriction of the functions to $\mathcal{S}_{\tau, \rho}$. 

We assume that $\nu \in (-1,0)$ is fixed. It is easy to check that there exists a constant $c > 0$ (independent of $\kappa$) and a constant $c_\kappa > 0$ (depending on $\kappa$) such that
\[
\left\| \hat{\mathcal{E}} (Q (\tilde{\tau} \Gamma_\rho)) \right\|_{C^{2,\alpha}_{\nu-2}(S^*)} \leq c \tau^{(6-3\nu)/4},
\]
\[
\left\| \hat{\mathcal{E}} (\Gamma \hat{F}) \right\|_{C^{2,\alpha}_{\nu-2}(S^*)} \leq c \tau^{(1-2\nu)/4} \left( \|f^1\|_{C^{2,\alpha}(S^1)} + \|f^1\|_{C^{2,\alpha}(S^1)} \right),
\]
and
\[
\left\| \hat{\mathcal{E}} (Q (\tilde{\tau} \Gamma_\rho + \hat{F}' + w')) - Q (\tilde{\tau} \Gamma_\rho + \hat{F} + w) \right\|_{C^{2,\alpha}_{\nu-2}(S^*)} \leq c_\kappa \left( \tau \|w' - w\|_{C^{2,\alpha}_{\nu-1}(S^*)} 
+ \tau^{(4-3\nu)/4} \left( \|f^1\|_{C^{2,\alpha}(S^1)} + \|f^1\|_{C^{2,\alpha}(S^1)} \right) \right),
\]
provided $w$ and $w'$ satisfy
\[
\|w\|_{C^{2,\alpha}(S^*)} + \|w'\|_{C^{2,\alpha}(S^*)} \leq C \tau^{(6-3\nu)/4},
\]
for some fixed constant $C > 0$ independent of $\kappa$. Here $\hat{F}$ and $\hat{F}'$ are respectively associated to the harmonic extensions of the boundary data $f^1, f^1$ and $f^1, f^1$.

Now, we make use of the result of Proposition 6.2 to rephrase the problem as a fixed point problem and the previous estimates are precisely enough to solve this nonlinear problem using a fixed point argument for contraction mappings in the ball of radius $C_\kappa \tau^{(6-3\nu)/4}$ in $C^{2,\alpha}_\nu(S^*)$, where $C_\kappa$ is fixed large enough. Then, for all $\tau > 0$ small enough, we find that there exists a constant $C_\kappa > 0$ (depending on $\kappa$) such that, for all functions $f^1, f^1$ satisfying the above hypothesis, there exists a solution $w$ of (6.44) satisfying
\[
\|w\|_{C^{2,\alpha}_\nu(S^*)} \leq C \tau^{(6-3\nu)/4}.
\]
In addition, we have the estimate
\[
\|w' - w\|_{C^{2,\alpha}_\nu(S^*)} \leq C_\kappa \tau^{(1-2\nu)/4} \left( \|f^1\|_{C^{2,\alpha}(S^1)} + \|f^1\|_{C^{2,\alpha}(S^1)} \right),
\]
for some constant $C > 0$, which does not depend on $\kappa$ or $\tau$, where $w$ (resp. $w'$) is the solution associated to $f^1, f^1$ (resp. $f^1, f^1$).

The solution of (6.44) is obtained through a fixed point theorem for contraction mappings, and it is classical to check that the solution we obtain depends continuously on the parameters of the construction. In particular, the constant mean curvature surface we obtain depends continuously on $\tilde{\tau}$ and $\rho$. \hfill \Box

7. Connecting the pieces together

We keep the notations of the previous sections. We assume that $\kappa > 0$ is fixed large enough (the value will be decided shortly) and assume that $\tau > 0$ is chosen small enough so that all the results proven so far apply.

For all $\tilde{x} \in \mathbb{R}^2$, we define the annuli
\[
A_\tau^{out}(\tilde{x}) := \{ x \in \mathbb{R}^2 : \tau^{3/4} \leq |x - \tilde{x}| \leq 2 \tau^{3/4} \},
\]
and
\[
A_\tau^{ins}(\tilde{x}) := \{ x \in \mathbb{R}^2 : \frac{1}{2} \tau^{3/4} \leq |x - \tilde{x}| \leq \tau^{3/4} \}.
\]
Recall that a function $f$ defined on $S^1$ is said to satisfy (H1) if
\[ \int_{S^1} f(\theta) \, d\theta = 0. \]
and it is said to satisfy (H2) if
\[ \int_{S^1} f(\theta) \cos \theta \, d\theta = \int_{S^1} f(\theta) \sin \theta \, d\theta = 0. \]
Also recall that a function $f$ defined on $S^1$ is invariant under the action of $\text{Dih}_{m+1}^{(2)}$ if
\[ f\left(\theta + \frac{2\pi}{m+1}\right) = f(\theta) \]
for all $\theta \in S^1$ and $f$ is invariant under the action of the symmetry $S_2$ if
\[ f(-\theta) = f(\theta) \]
for all $\theta \in S^1$.

We now describe the different pieces of constant mean curvature surfaces we have at hand.

(i) Assume that we are given $f \mid C^{2,\alpha}(S^1)$ which is invariant under the action of $\text{Dih}_{m+1}^{(2)}$, satisfies (H1) and
\[ \|f\|_{C^{2,\alpha}(S^1)} \leq \kappa \tau^{3/2}. \]
The result of Proposition 4.6 provides a constant mean curvature (equal to 1) surface $D_{\tau,f}$ which is invariant under the action of the dihedral group $\text{Dih}_{m+1}^{(2)}$, has one end asymptotic to the end of $D_{\tau}$ and which, close to its boundary, can be parameterized as the vertical graph of $x \mapsto -U\mid_{(\tau - 3/4)x}$ over $A_{\infty}(0)$, where
\[ U\mid_{x}(x) = c^1 + \tau \log |x| - W_{x}(x) + \tilde{U}\mid_{x}(x), \]
where
\[ c^1 := \tau \log \left(\frac{2}{\tau^{1/4}}\right) \in \mathbb{R}, \]
and where $\tilde{U}$ satisfies (4.17) and (4.18). To simplify the notations we have not mentioned the data $\tau, f$ in the notation for $U\mid_{x}$ and $\tilde{U}$.

(ii) Next, we assume that we are given $\tau_1 > 0$ satisfying
\[ |\tau_1 - \tau| \leq \kappa \tau^{3/2}, \]
and $\rho_1 > 0$ satisfying
\[ \left|\rho_1 - \frac{m}{m+1} \frac{\tau}{2\rho_1}\right| \leq \kappa \tau^{3/4}. \]
Further assume that we are given a function $f_1 \in C^{2,\alpha}(S^1)$ invariant under the action of the dihedral group $\text{Dih}_{m+1}^{(2)}$ and a function $f_\downarrow \in C^{2,\alpha}(S^1)$, invariant under the action of the symmetry $S_2$, both satisfying (H1) and
\[ \|f_1\|_{C^{2,\alpha}(S^1)} \leq \kappa \tau^{3/2} \quad \text{and} \quad \|f_\downarrow\|_{C^{2,\alpha}(S^1)} \leq \kappa \tau^{3/2}. \]
The result of Proposition 6.3 provides a constant mean curvature (equal to 1) surface $\mathcal{S}_{\tau_1,\rho_1,f_1,f_\downarrow}$ which is invariant under the action of the dihedral
group $\text{Dih}_m^{(2)}$ and which, close to its upper boundary can be parameterized as the vertical graph of $x \mapsto V^1(\tau-3/4 \ x)$ over $A_{\tau}^{\text{out}}(0)$ where

$$V^1(x) = 1 + d^l + \tau_1 \log |x| - W_{f_l^1}^{\text{out}}(x) + \bar{V}^1(x),$$

$$d^l := \tau_1 \left( m \log \rho_1 + a_{1,\rho,f_l^1}^1 + \frac{3}{4} \log \tau \right) \in \mathbb{R},$$

and where $\bar{V}^1$ satisfies (6.41) and (6.42). Close to one of its lower boundaries, this surface can be parameterized as a vertical graph for some function $x \mapsto V^1(\tau-3/4 \ (x - \rho_1 z_0))$ over $A_{\tau}^{\text{out}}(\rho_1 z_0)$ which can be expanded as

$$V^1(x) = -1 + c^l - \frac{\tau_1}{m+1} \log |x| - \tau^{3/4} \left( \rho_1 - \frac{m}{m+1} \frac{\tau_1}{2\rho} \right) z_0 \cdot x + W_{f_l^1}^{\text{out}}(x) + \bar{V}^1(x),$$

$$c^l := 1 - \sqrt{1 - \frac{\tau^2}{\tau^2}} - \frac{\tau_1}{m+1} \left( m \log \rho_1 + a_{1,\rho,f_l^1}^1 - \frac{3}{4} \log \tau \right) \in \mathbb{R},$$

and where $\bar{V}^1$ satisfies estimates of the form (6.41) and (6.42). Again, to simplify the notations we have not mentioned the parameters $\tau_1, \rho_1, f_l^1, f_l^1$ in the notation for $V^1, \bar{V}^1, V^1$ and $\bar{V}^1$.

(iii) Assume that we are given $\tau_2 > 0$ satisfying

$$\left| \tau_2 - \frac{\tau}{m+1} \right| \leq \kappa \tau^{3/2},$$

and a function $f_2^l \in C^{2,\alpha}(S^1)$ which satisfies both (H1), (H2) and

$$||f_2^l||_{C^{2,\alpha}(S^1)} \leq \kappa \tau^{3/2}.$$  

The result of Proposition 5.2 provides a constant mean curvature (equal to 1) surface $\Phi_{\tau_2,f_2^l}$ which is invariant under the action of $S_3$, the symmetry with respect to the horizontal plane $x_3 = 0$ and is also invariant under the action of $S_2$, the symmetry with respect to the plane $x_2 = 0$. Moreover, close to its lower boundary, this surface can be parameterized as the vertical graph of $x \mapsto U^1(\tau-3/4 \ x)$ over $A_{\tau}^{\text{mns}}(0)$, where

$$U^1(x) = d^l - \tau_2 \log |x| + W_{f_2^l}^{\text{mns}}(x) + \bar{U}^1(x),$$

where

$$d^l := -\tau_2 \log \left( \frac{2 \tau^{3/4}}{\tau_2} \right) \in \mathbb{R},$$

and where $\bar{U}^1$ satisfies (5.28) and (5.29). To simplify the notations we have not mentioned the data $\tau_2, f_2^l$ in the notation for $U^1$ and $\bar{U}^1$.

Let us emphasize that the functions $f_1^l, f_2^l$ and $f^l, f_1^l$ are all assumed to satisfy (H1). Hence they have no constant term in their Fourier series. The function $f_2^l$ is also assumed to satisfy (H2). Now, the functions $f^l$ and $f_1^l$ are assumed to be invariant under the action of the dihedral group $\text{Dih}_m^{(2)}$, and, as already mentioned, this implies that both functions also satisfies (H2) since its Fourier series not contain any term of the form $z \cdot x$. Therefore, $f_1^l$ is the only function which does
not satisfy (H2). Since \( f_1 \) is assumed to be invariant under the action of \( S_2 \), we can decompose it as
\[
\frac{f_1}{1} = \lambda_1 \cdot x + \frac{f_1}{1},
\]
where \( \lambda_1 \in \mathbb{R} \) and where \( \frac{f_1}{1} \) satisfies both (H1) and (H2).

We denote by \( \mathcal{C}^{(0)}_{\tau_2, f_2^j, \rho_1} \) the surface \( \mathcal{C}_{\tau_2, f_2^j} \) which has been translated by \( \rho_1 \cdot z_0 \). For \( j = 1, \ldots, m \),
\[
\mathcal{C}^{(0)}_{\tau_2, f_2^j, \rho_1} := \mathcal{C}_{\tau_2, f_2^j} + \rho_1 \cdot z_0.
\]
Moreover, the image of \( \mathcal{C}^{(0)}_{\tau_2, f_2^j, \rho_1} \) under the rotation \( (R_{m+1})^j \) will be denoted by \( \mathcal{C}^{(j)}_{\tau_2, f_2^j, \rho_1} \)
\[
\mathcal{C}^{(j)}_{\tau_2, f_2^j, \rho_1} := (R_{m+1})^j \left( \mathcal{C}^{(0)}_{\tau_2, f_2^j, \rho_1} \right).
\]
In particular, the collection of surfaces \( \mathcal{C}^{(0)}_{\tau_2, f_2^j, \rho_1}, \ldots, \mathcal{C}^{(m)}_{\tau_2, f_2^j, \rho_1} \) constitute \( m + 1 \) constant mean curvature surfaces which are symmetric with respect to the dihedral group \( \text{Dih}_{m+1}^{(3)} \).

Given \( t_1 \in \mathbb{R} \) small enough, we denote by \( \mathcal{G}_{(\tau_1, \rho_1, f_1^1, f_1^1, t_1)} \) the surface \( \mathcal{G}_{(\tau_1, \rho_1, f_1^1, f_1^1, t_1)} \) which has been translated in the vertical direction by \( (1 - c_1 + d_1 + t_1) \cdot e_3 \)
\[
\mathcal{G}_{(\tau_1, \rho_1, f_1^1, f_1^1, t_1)} := \mathcal{G}_{(\tau_1, \rho_1, f_1^1, f_1^1)} + (1 - c_1 + d_1 + t_1) \cdot e_3.
\]
This is a constant mean curvature surface which is symmetric with respect to the dihedral group \( \text{Dih}_{m+1}^{(2)} \). Observe that the lower boundaries of \( \mathcal{C}^{(0)}_{\tau_2, f_2^j, \rho_1}, \ldots, \mathcal{C}^{(m)}_{\tau_2, f_2^j, \rho_1} \) are close to the lower boundaries of \( \mathcal{G}_{(\tau_1, \rho_1, f_1^1, f_1^1, t_1)} \).

Finally, given \( t \in \mathbb{R} \) small enough, we denote by \( \mathcal{D}_{(\tau, f_1, t)}^+ \) the surface \( \mathcal{G}_{(\tau, f_1, t)}^+ \) which has been translated in the vertical direction by \( (2 - c_1 + d_1 - c_1 + d_1 + t_1 + t) \cdot e_3 \)
\[
\mathcal{D}_{(\tau, f_1, t)}^+ := \mathcal{D}_{(\tau, f_1)}^+ + (2 - c_1 + d_1 - c_1 + d_1 + t_1 + t) \cdot e_3.
\]
This is a constant mean curvature surface which is symmetric with respect to the dihedral group \( \text{Dih}_{m+1}^{(2)} \). Observe that the boundary of \( \mathcal{D}_{(\tau, f_1, t)}^+ \) is close to the upper boundary of \( \mathcal{G}_{(\tau_1, \rho_1, f_1^1, f_1^1, t_1)} \).

To complete the proof of the main theorem, it remains to adjust the free parameters of our construction, namely \( t, t_1, \tau_1, \tau_2, \rho_1 \in \mathbb{R} \), and the functions \( f_1^1, f_2^j, f_1^1 \) and \( f_1^1 \) defined on \( S^1 \), so that
\[
\mathcal{C}^{(0)}_{\tau_2, f_2^j, \rho_1} \cup \ldots \cup \mathcal{C}^{(m)}_{\tau_2, f_2^j, \rho_1} \cup \mathcal{G}_{(\tau_1, \rho_1, f_1^1, f_1^1, t_1)} \cup \mathcal{D}_{(\tau, f_1, t)}^+,
\]
constitute a \( C^1 \) surface which can be extended by reflection through the horizontal plane as a \( C^1 \) surface which is complete, non compact and has two ends of Delaunay type (asymptotic to a nodoid end). Observe that the surface is invariant under the action of the dihedral group \( \text{Dih}_{m+1}^{(1)} \) and that there is still one free parameter, namely \( \tau \) which determines the Delaunay type end and hence the vertical flux of the surface.

This surface is in fact piecewise smooth and has constant mean curvature equal to 1 away from the boundaries where the connected sum is performed. Since all pieces have constant mean curvature identically equal to 1, elliptic regularity theory then implies that this surface is in fact a smooth surface. Indeed, near one of the boundaries where the connected sum is performed, the surface is a graph of a
function, say $u^{\text{ins}}$ defined over $A^\tau_{\text{ins}}$ and another function, say $u^{\text{out}}$ defined over $A^\tau_{\text{out}}$. The functions $u^{\text{ins}}$ and $u^{\text{out}}$ are $C^{2,\alpha}$ and solve the mean curvature equation

$$\frac{1}{2} \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 1$$

(7.46)

on their respective domain of definition (for the sake of simplicity, we have assumed that the mean curvature vector is upward pointing near the boundary we are interested in). Moreover, $u^{\text{ins}} = u^{\text{out}}$ and $\partial_r u^{\text{ins}} = \partial_r u^{\text{out}}$ on $\partial A^\tau_{\text{ins}} \cap \partial A^\tau_{\text{out}}$. This implies that the function $u$ defined on $A^\tau_{\text{ins}} \cup A^\tau_{\text{out}}$ by $u := u^{\text{ins}}$ on $A^\tau_{\text{ins}}$ and $u := u^{\text{out}}$ on $A^\tau_{\text{out}}$ belongs to $C^{1,1}$ and is a weak solution of (7.46) on $A^\tau_{\text{ins}} \cup A^\tau_{\text{out}}$. Elliptic regularity implies that $u$ is $C^{2,\alpha}$ and hence the surface we have obtained is a smooth constant mean curvature surface.

Therefore, to complete the proof, it remains to explain how to find the parameters $t, t_1, \tau_1, \tau_2, \rho_1 \in \mathbb{R}$, and the functions $f^1_1, f^1_2, f^1_1$ defined on $S^1$, so that the following system of equations is fulfilled

$$U^1 - c^1 + t = V^1 - 1 - d^1 \quad \text{and} \quad \partial_r (V^1 - U^1) = 0$$

(7.47)

on $S^1$ and

$$V^1 + 1 - c^1 + t_1 = U^1 - d^1 \quad \text{and} \quad \partial_r (V^1 - U^1) = 0$$

(7.48)

on $S^1$. Recall that, even though this is not apparent in the notations, all functions and constants depend on the parameters and boundary data. The rest of this section is devoted to the proof that the above system has indeed a solution, provided $\tau$ is small enough. We will prove the :

**Proposition 7.1.** There exists $\kappa > 0$ such that, for all $\tau > 0$ small enough there exists parameters $t, t_1, \tau_1, \tau_2, \rho_1$, and functions $f^1_1, f^1_2, f^1_1$ defined on $S^1$ and satisfying the above symmetries and estimates, such that the system (7.47) and (7.48) is satisfied.

**Proof.** First we make use of the results of Propositions 4.6, Propositions 5.2 and Propositions 6.3 to get the expansion of the functions $U^1, V^1, U^1$ and $V^1$. Recalling that we have to restrict all those functions to $S^1$, it is easy to check, using (4.16) and (6.40) that the first two equations of the system we have to solve read

$$\begin{align*}
& t + f^1_1 - f^1_1 = \bar{V}^1 - \bar{U}^1 \\
& (\tau_1 - \tau) + \partial_r \left( W^{\text{out}}_{f^1_1} - W^{\text{ins}}_{f^1_1} \right) = \partial_r (\bar{V}^1 - \bar{U}^1),
\end{align*}$$

(7.49)

while, using (5.27) and (6.43), we see that the next two equations are given by

$$\begin{align*}
& t_1 - \tau^{3/4} \left( \rho_1 - \frac{m}{m + 1} \frac{\tau_1}{2 \rho_1} \right) z_0 \cdot x + f^1_1 - f^1_2 \\
& = \bar{U}^1 - \bar{V}^1 \\
& - \left( \frac{\tau_1}{m + 1} - \tau_2 \right) - \tau^{3/4} \left( \rho_1 - \frac{m}{m + 1} \frac{\tau_1}{2 \rho_1} \right) z_0 \cdot x + \partial_r \left( W^{\text{out}}_{f^1_1} - W^{\text{ins}}_{f^1_2} \right) \\
& = \partial_r (\bar{U}^1 - \bar{V}^1),
\end{align*}$$

(7.50)

In writing this system one has to be a bit careful about the invariance of the functions we are interested in. Indeed, in (7.49), all functions are invariant under
the action of $\text{Dih}_2^{(2)}$, while in (7.50), all functions are invariant under the action of the symmetry $S_2$.

Let us denote by $\Pi^0$ the $L^2(S^1)$-orthogonal projection over the space of constant functions, $\Pi^1$ the $L^2(S^1)$-orthogonal projection over the space spanned by the function $x \mapsto z_0 \cdot x$ and let us denote by $\Pi^\perp$ denotes $L^2(S^1)$-orthogonal projection over the orthogonal complement of the space spanned by the constant function and the function $x \mapsto z_0 \cdot x$.

We project this system over the $L^2(S^1)$-orthogonal complement of the constant function and the function $x \mapsto z_0 \cdot x$. We obtain the coupled system

$$f_1^t - f_1^r = \Pi^\perp (\bar{V}^t - \bar{U}^t)$$

$$\partial_r \left( W_{\text{out}}^f f_1^t - W_{\text{ins}}^f f_1^r \right) = \Pi^\perp \partial_r (\bar{V}^t - \bar{U}^t)$$

$$f_2^1 - f_2^3 = \Pi^\perp (\bar{U}^1 - \bar{V}^1)$$

$$\partial_r \left( W_{\text{out}}^f f_2^1 - W_{\text{ins}}^f f_2^3 \right) = \Pi^\perp \partial_r (\bar{U}^1 - \bar{V}^1).$$

(7.51)

where we recall that we have decomposed $f_1^1 = \lambda_1 z_0 \cdot x + f_1^1 \perp$.

The projection of the system (7.49)-(7.50) over the space of constant functions leads to the coupled system

$$t = \Pi^0 (\bar{V}^t - \bar{U}^t)$$

$$\tau_1 - \tau = \Pi^0 \partial_r (\bar{V}^t - \bar{U}^t)$$

$$t_1 = \Pi^0 (\bar{U}^1 - \bar{V}^1)$$

$$\tau_2 - \frac{\tau_1}{m + 1} = \Pi^0 \partial_r (\bar{U}^1 - \bar{V}^1).$$

(7.52)

Finally, the projection of the system (7.49)-(7.50) over the space of functions spanned by $x \mapsto z_0 \cdot x$ leads to the coupled system

$$\left( \begin{array}{c}
\lambda_1 - \tau^{3/4} \\
-\lambda_1 - \tau^{3/4}
\end{array} \right) \left( \begin{array}{c}
\rho_1 - \frac{m}{m + 1} \frac{\tau_1}{2 \rho_1} \\
\rho_1 - \frac{m}{m + 1} \frac{\tau_1}{2 \rho_1}
\end{array} \right) z_0 \cdot x = \Pi^1 (\bar{U}^1 - \bar{V}^1)$$

(7.53)

To obtain the second equation, we have used the fact that

$$W_{\text{out}}^f f_1^t = \lambda_1 \frac{z_0 \cdot x}{|x|^2} + W_{\text{out}}^f f_1^1 \perp.$$

Observe that the right hand sides of (7.51), (7.52) and (7.53) does not depend on $t$ and $t_1$. Hence, the first and third equations in (7.52) will give us the values of $t$ and $t_1$, once the rest of the equations are solved.

For all $\tau$ small enough, we will solve (7.51) using some fixed point theorem for contraction mappings to obtain a solution $(f_1^t, f_1^r, f_1^1 \perp, f_1^3)$ continuously depending on the parameters $\tau_1, \tau_2, \rho_1, \lambda_1$ (and $\tau$). Then, we introduce the corresponding solution in (7.52) and (7.53) to get a nonlinear system in $\tau_1, \tau_2$ and $\rho_1$, which we will solve using Browder’s fixed point theorem.

To begin with, we explain how (7.51) can be rewritten in diagonal form. This makes use of the following result whose proof can be found, for example, in [8]:
Proposition 7.2. The operator
\[ C^{2,\alpha}(S^1)^\perp \ni f \mapsto \partial_r (W_f^{\text{ins}} - W_f^{\text{out}}) \big|_{r=1} \in C^{1,\alpha}(S^1)^\perp \]
is an isomorphism. Here \( C^{k,\alpha}(S^1)^\perp \) denote the image of \( C^{k,\alpha}(S^1) \) under \( \Pi^\perp \).

Proof. The Fourier decomposition of a function \( f \in C^{k,\alpha}(S^1)^\perp \) is given by
\[ f(\theta) = \sum_{n \neq 0, \pm 1} f_n e^{in\theta} \]
in which case
\[ W_f^{\text{out}} = \sum_{n \neq 0, \pm 1} f_n r^{-|n|} e^{in\theta}, \text{ and } W_f^{\text{ins}} = \sum_{n \neq 0, \pm 1} f_n r^{|n|} e^{in\theta}, \]
Therefore,
\[ \partial_r (W_f^{\text{ins}} - W_f^{\text{out}}) \big|_{r=1} = 2 \sum_{n \neq 0, \pm 1} f_n |n| e^{in\theta}, \]
is equal to twice the Dirichlet to Neumann map for the Laplace operator in the unit disc. This is a well defined, self-adjoint, first order elliptic operator which is injective and elliptic regularity theory implies that it is an isomorphism. \( \square \)

Using this result, the system (7.51) can be rewritten as
\[ (f^1, f_f^1, f_f^{1,\perp}, f_f^2) = N_{\tau_1, \tau_2, \rho_1, \lambda_1}^{\perp}(f_f^1, f_f^{1,\perp}, f_f^2), \]
where the nonlinear operator \( N_{\tau_1, \tau_2, \rho_1, \lambda_1}^{\perp} \) satisfies
\[ (7.54) \quad ||N_{\tau_1, \tau_2, \rho_1, \lambda_1}^{\perp}(f_f^1, f_f^{1,\perp}, f_f^2)||_{C^{2,\alpha}(S^1)}^4 \leq C\tau^{3/2} \]
for some constant \( C > 0 \) independent of \( \kappa > 0 \), provided \( \tau \) is chosen small enough. This last estimate follows directly from (4.17) in Proposition 4.6, (5.28) in Proposition 5.2 and (6.41) in Proposition 6.3. Moreover, thanks to (4.18), (5.29) and (6.42), provided \( \kappa > 0 \) is fixed larger than the constant \( C \) which appears in (7.54), we can use a fixed point theorem for contraction mapping in the ball of radius \( \kappa \tau^{3/2} \) in \( (\Pi^\perp C^{2,\alpha}(S^1))^4 \) to get the existence of a solution of (7.54), for all \( \tau > 0 \) small enough. This solution depends continuously on \( \tau_1, \tau_2, \rho_1 \) and \( \lambda_1 \), since \( N_{\tau_1, \tau_2, \rho_1, \lambda_1}^{\perp} \) does (observe that \( N_{\tau_1, \tau_2, \rho_1, \lambda_1}^{\perp} \) depends implicitly on \( \tau \)). We now insert this solution in (7.52) and (7.53). With simple manipulations, we conclude that it remains to solve the nonlinear system
\[ (7.55) \quad \left( \tau_1 - \tau_2 - \frac{\tau}{m+1}, \tau^{3/4} \left( m - \frac{\tau}{2\rho_1} \right) \lambda_1 \right) = N^{0}(\tau_1, \tau_2, \rho_1, \lambda_1), \]
where \( N^{0} \) satisfies
\[ ||N^{0}(\tau_1, \tau_2, \rho_1, \lambda_1)||_{\mathbb{R}^4} \leq C\tau^{3/2} \]
for some constant \( C > 0 \) independent of \( \kappa > 0 \), provided \( \tau \) is chosen small enough. Moreover, \( N^{0} \) depends continuously on the parameters \( \tau_1, \tau_2, \rho_1 \) and \( \lambda_1 \) (observe that \( N^{0} \) depends implicitly on \( \tau \)). The equation (7.55) can then be solved using a simple degree argument (Browder’s fixed point theorem). This completes the proof of the result. \( \square \)
8. Appendix 1

We discuss the elementary result in the theory of second order ordinary differential equations which is used at the end of the proof of Proposition 4.4. Assume that we are given a function $s \mapsto p(s)$ which is periodic (say of period $S > 0$). Further assume that the homogeneous problem $(\partial_x^2 + p) w^+ = 0$ has a nontrivial periodic solution of period $S$. Without loss of generality, we can assume that $w^+(0) = 1$ and $\partial_x w^+(0) = 0$ (just choose the origin so that 0 coincides with a point where $w^+$ achieves its maximum). Let $w^-$ be the unique solution of $(\partial_x^2 + p) w^- = 0$ such that $w^-(0) = 0$ and $\partial_x w^-(0) = 1$. The Wronskian of $w^+$ and $w^-$ being constant, we conclude that

$$\partial_x w^-(S) = \partial_x w^-(S) w^+(S) - \partial_x w^+(S) w^-(S)$$

$$= \partial_x w^-(0) w^+(0) - \partial_x w^+(0) w^-(0)$$

$$= 1.$$

We define

$$v(s) := w^-(S + s) - w^-(S) w^+(s).$$

It is clear that $v$ is a solution $(\partial_x^2 + p) v = 0$ and further observe that $\partial_x v(0) = 1$ and $v(0) = 0$. Therefore, $v = w^-$. This proves that

$$w^-(S + s) = w^-(s) + w^-(S) w^+(s),$$

and hence $w^-$ is at most linearly growing in the sense that $|w^-(s)| \leq C (1 + |s|)$ for some constant $C > 0$.

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