Laplacian Abelian Projection: 
Abelian dominance and Monopole dominance*

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A comparative study of Abelian and Monopole dominance in the Laplacian and Maximally Abelian projected 
gauges is carried out. Clear evidence for both types of dominance is obtained for the Laplacian projection. 
Surprisingly, the evidence is much more ambiguous in the Maximally Abelian gauge. This is attributed to 
possible “long-distance imperfections” in the maximally abelian gauge fixing.

1. INTRODUCTION

Despite its many successes, the Maximally 
Abelian Gauge (MAG) \cite{1} has the great drawback 
that it is ambiguous. A precise way to phrase this 
ambiguity is as follows: it is in general unlikely, 
and certainly impossible to guarantee, that the 
configuration obtained by the usual local iterative 
minimization algorithm be (arbitrarily) close to 
the desired configuration \( \{ \bar{U}_{\mu,x} = \bar{\Omega}_{x} U_{\mu,x} \bar{\Omega}_{x}^{-1} \} \), 
no matter how high the numerical precision of the 
computer and no matter how long the iteration is 
continued. (The reason is well known: one may 
get stuck in a local minimum.) Here \( \{ \Omega_{x} \} \) is the 
unknown, true (absolute) minimum of the func-
tional

\[
\tilde{S}_{U}(\Omega) = \sum_{x,\mu} \left\{ 1 - \frac{1}{2} \text{Tr} \left[ \sigma_{3} U_{\mu,x}^{(f)} \sigma_{3} U_{\mu,x}^{(f)} + \sigma_{3} U_{\mu,x}^{(f)} \right] \right\}, 
\] (1)

As a consequence, a different result is obtained 
if the procedure is applied to the same configura-
tion several times, starting from a different (ran-
dom) gauge each time \cite{2,3}. In physical terms: 
gauge covariance is lost.

The Laplacian Abelian Gauge (LAG) \cite{4,5} solves this problem. This is a unique and unam-
biguous gauge fixing prescription which can be 
pursued to arbitrarily high precision. Gauge co-
variance is guaranteed, and one has control over 
numerical errors.

In addition, LAG leads to at least as smooth 
configurations as MAG. This is important for a 
reliable extraction of Abelian continuum gauge 
fields \( A_{\mu}(x) \). In fact, in Ref. \cite{5} it was argued 
that the fields in the LAG can be considered 
to be smoother than in the MAG, as physical 
monopoles are treated more “respectfully” in the 
Laplacian gauge.

The present contribution focuses on abelian 
and monopole dominance \cite{7,8} in LAG and MAG.

2. THE LAPLACIAN METHOD

The minimization problem of Eq. (1) can be 
viewed as the minimization of the gauge covariant 
kinetic energy of a real adjoint scalar field \( \phi^{a} \) \( (a = 1, 2, 3) \). In continuum notation \cite{6,4,5}:

\[
\tilde{S}_{A}(\phi) = \int_{V} \frac{1}{2} (D_{\mu}\phi)^{2}, 
\] (2)

The ambiguities in the MAG arise because of the 
constraints \( |\phi(x)| = \sum_{a=1}^{3} (\phi^{a})^{2} = 1 \).

The idea of the Laplacian gauge fixing is to 
relax the latter constraint. Minimization of (2) 
then amounts to determining the lowest mode 
of the covariant Laplacian \( -(D_{\mu})^{2} \). The corre-
sponding lowest eigenvector \( \phi_{0} \) determines the 
gauge transformation to be applied to the gauge 
field configuration. Subsequently, the abelian 
projected fields can be extracted in the standard 
way.

The computation of the lowest eigenmode can 
be done to arbitrary precision, using standard 
sparse matrix routines (Lanczos, Rayleigh-Ritz).

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The only ambiguity arises when the lowest eigenvalue is degenerate. This would signal a true Gribov copy. In practice, however, this never occurs. For further details, see Refs. [4,5].

3. ABELIAN DOMINANCE AND MONOPOLE DOMINANCE

We consider a set of 18 pure SU(2) configurations on a $16^4$ lattice at $\beta = 2.5$. These configurations are gauge fixed using MAG and LAG, and for both gauge fixed configurations the abelian field components are extracted, and the elementary-cube monopoles are identified using the standard prescription. In this way we can compare results in the two gauges on the same set of SU(2) configurations.

Wilson loops and Creutz ratios are calculated from the original (“full”) non-abelian SU(2) configurations, from the abelian fields in both gauges, and from the abelian fields generated by the monopole content only. The Creutz ratios are defined as

$$\chi(R + \frac{1}{2}, T + \frac{1}{2}) = -\ln \frac{\langle W(R, T) \rangle \langle W(R + 1, T + 1) \rangle}{\langle W(R + 1, T + 1) \rangle \langle W(R + 1, T) \rangle},$$

where $W(R, T)$ denotes an $R \times T$ Wilson loop. Statistical errors have been determined by means of a bootstrap analysis on each of the Creutz ratios separately.

The MAG fixing was done using a standard iterative algorithm, with alternating cooling and overrelaxation sweeps, and a very tight stopping criterion: the iteration was terminated when $\frac{1}{2} \text{Tr}$ of the gauge transformation matrix connecting subsequent configurations deviated from unity less than $10^{-12}$ at each site.

Abelian dominance. Fig. 1 shows diagonal Creutz ratios $\chi(L + \frac{1}{2}, L + \frac{1}{2})$ against $L + \frac{1}{2}$. Shown are full SU(2) and abelian Creutz ratios in MAG and LAG.

Monopole dominance. Before presenting the results, let us discuss the important point that the monopole gauge field cannot be computed from the usual monopole currents $k_\mu$ alone. (See Ref. [10], Eqs. (24), (30).) In Landau gauge one has

$$A_\mu^{\text{mon}}(x) = -2\pi \sum_y D(x - y) \partial_y \tilde{m}_{\nu\mu}(y),$$

where $\tilde{m}_{\nu\mu}$ is the dual of the “Dirac sheet” field $m_{\nu\mu}$, and $D(x - y)$ is the 4-dimensional Coulomb propagator. It is this gauge field $A_\mu^{\text{mon}}$ which determines the monopole Wilson loop. In addition to the contribution from the monopole currents $k_\mu(x)$, given by

$$k_\mu(x) = \partial_\nu m_{\nu\mu}(x),$$

there is an additional contribution from the zero mode in $m_{\nu\mu}$. If this zero-mode contribution is ignored, a ‘trivial’ $N_s \times N_t$ Wilson loop will not equal unity.
Fig. 2 shows diagonal Creutz ratios $\chi(L + \frac{1}{2}, L + \frac{1}{2})$ against $L + \frac{1}{2}$, in MAG and LAG.

In excellent agreement with the full string tension on the same lattice [9].

The MAG Creutz ratios, on the other hand, show a rising tendency. In view of the small error bars it is difficult to argue that the data are consistent with a plateau, and no asymptotic string tension can be extracted.

A possible explanation is as follows. The MAG algorithm is a local iterative procedure, which does well locally but is unable to do the global optimization well. As a result of this defect in the gauge fixing algorithm, artificial decorrelations in abelian projected or monopole Wilson loops might show up at some intermediate distance, leading to an apparently smaller correlation length, hence an apparently larger string tension. In other words, imperfect gauge fixing leads to abelian projected and monopole string tensions which are larger than the true non-abelian string tension. The rising tendency of the MAG Creutz ratios in Fig. 2 with distance might reflect precisely this effect.

The Laplacian projection, on the other hand, by construction looks at the lowest-momentum, longest-distance eigenmode of the covariant Laplacian, so a similar artificial intermediate-distance decorrelation is expected (and confirmed) to be absent.

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REFERENCES

1. A.S. Kronfeld, M.L. Laursen, G. Schierholz and U.-J. Wiese, Monopole Condensation and Color Confinement, Phys. Lett. 198B (1987) 516.
2. V.G. Bornyakov, E.-M. Ilgenfritz, M.L. Laursen, V.K. Mitrjushkin, M. Müller-Preussker, A.J. van der Sijs and A.M. Zadorozhny, The Density of Monopoles in SU(2) Lattice Gauge Theory, Phys. Lett. 261B (1991) 116.
3. S. Hioki, S. Kitahara, Y. Matsubara, O. Miyamura, S. Ohno and T. Suzuki, Gauge-fixing Ambiguity and Monopole, Phys. Lett. 271B (1991) 201.
4. A.J. van der Sijs, Laplacian Abelian Projection, Nucl. Phys. B (Proc. Suppl.) 53 (1997) 535 [hep-lat/9608041].
5. A.J. van der Sijs, Abelian Projection without Ambiguities, preprint SCSC-TR-98-01 (ETH-Zürich), [hep-lat/9803001], to appear in the Proceedings of the 1997 Yukawa International Seminar (YKIS’97), “Non-Perturbative QCD — Structure of the QCD Vacuum”, Kyoto, Japan, 2–12 December 1997.
6. A.J. van der Sijs, Monopoles and Confinement in SU(2) Gauge Theory, Thesis, University of Amsterdam, February 1991.
7. T. Suzuki and I. Yotsuyanagi, A possible evidence for Abelian dominance in quark confinement, Phys. Rev. D42 (1990) 4257.
8. H. Shiba and T. Suzuki, Monopoles and string tension in SU(2) QCD, Phys. Lett. 333B (1994) 461.
9. A. Hart, J.D. Stack and M. Teper, The string tension in the maximally Abelian gauge after smoothing, [hep-lat/9808050].
10. J. Smit and A.J. van der Sijs, Monopoles and Confinement, Nucl. Phys. B 355 (1991) 603.