Soliton resolution for a coupled generalized nonlinear Schrödinger equations with weighted Sobolev initial data

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Abstract
In this work, we employ the $\bar{\partial}$ steepest descent method in order to study the Cauchy problem of the cgNLS equations with initial conditions in weighted Sobolev space $H^{1,1}(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : f', xf \in L^2(\mathbb{R}) \}$. The large time asymptotic behavior of the solution $u(x, t)$ and $v(x, t)$ are derived in a fixed space-time cone $S(x_1, x_2, v_1, v_2) = \{ (x, t) \in \mathbb{R}^2 : x = x_0 + vt, x_0 \in [x_1, x_2], v \in [v_1, v_2] \}$. Based on the resulting asymptotic behavior, we prove the solution resolution conjecture of the cgNLS equations which contains the soliton term confirmed by $|Z(I)|$-soliton on discrete spectrum and the $t^{-\frac{3}{4}}$ order term on continuous spectrum with residual error up to $O(t^{-\frac{3}{2}})$.

Keywords: Integrable system, The coupled generalized nonlinear Schrödinger equations, Infinite conservation laws, Riemann-Hilbert problem, $\bar{\partial}$ steepest descent method, Soliton resolution.

1. Introduction

The classical nonlinear Schrödinger (NLS) equation, i.e.,

$$iu_t + u_{xx} + 2|u|^2u = 0,$$

is a fundamental physical model and has been applied in various fields such as deep water waves, plasma physics, nonlinear optical fibers, etc. Since it plays a significant role in the field of nonlinear science, lots of researchers and scholars do many works on the NLS equation and its extensions. It is worth noting that in nonlinear science such as optical fibers, the NLS equation only describes the propagation of optical solitons in scalar fields of mono-mode fibers. However, because the...
change of nonlinear phase stems from the cross-phase modulation in multi-mode fibers or birefringent fibers, we need take the interaction of some field components at different frequencies or polarisations into consideration. The coupled nonlinear Schrödinger system came into being to describe the characteristics of such soliton. For example, the well-known Manakov system\[12\], i.e.

\[
\begin{align*}
    iu_t &+ \frac{1}{2}u_{xx} + (|u|^2 + |v|^2)u = 0, \nonumber \\
    iv_t &+ \frac{1}{2}v_{xx} + (|u|^2 + |v|^2)v = 0, 
\end{align*}
\]

can be used to describe the intense electromagnetic pulse propagation in birefringent fiber.

In this work, considering that there are more effects of nonlinear factors in practise, we derive a novel integrable coupled generalized nonlinear Schrödinger (cgNLS) equation that reads

\[
\begin{align*}
    iu_t - i\alpha u_x + u_{xx} - 2u^2v + 4\beta^2u^3v^2 + 4i\beta(\nu)v, u + \gamma u &= 0, \nonumber \\
    iv_t - i\alpha v_x - v_{xx} + 2uv^2 - 4\beta^2u^2v^2 + 4i\beta(\nu)xv - \gamma v &= 0, 
\end{align*}
\]

where \(u(x, t)\) and \(v(x, t)\) are complex function of variables \(x, t\), and \(\alpha, \beta, \gamma\) are arbitrary constants. Additionally, we will derive the Lax pair, infinite conservation laws and Hamiltonian function associated with the cgNLS equation. Furthermore, we investigate the soliton resolution of the cgNLS equation with the initial value condition

\[
u(x, 0) = u_0(x), v(x, 0) = v_0(x) \in H^{1,1}(\mathbb{R}), \]

where \(H^{1,1}(\mathbb{R})\) is a weighted Sobolev space, i.e.,

\[
H^{1,1}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f', xf \in L^2(\mathbb{R})\}. \tag{1.3}
\]

In particular, some unique cases of (1.1) can be obtained by fixing the coefficients \(\alpha, \beta\) and \(\gamma\).

- When \(\alpha = \gamma = 0\) and \(\beta \neq 0\), Eq.(1.1) degenerates into the coupled nonlinear Kundu-Eckhaus equation, i.e.,

\[
\begin{align*}
    iu_t + u_{xx} - 2u^2v + 4\beta^2u^3v^2 + 4i\beta(\nu)xu &= 0, \\
    iv_t - v_{xx} + 2uv^2 - 4\beta^2u^2v^2 + 4i\beta(\nu)xv &= 0, 
\end{align*}
\]

which is a completely integrable system and possesses a Lax representation, the Hamiltonian structure and some other properties\[13\]-[15].
• When $\alpha \neq 0$, $\gamma \neq 0$, $\beta = 0$, and $v = -u^*$, Eq. (1.1) degenerates into the modified Landau-Lifshitz (mLL) equation, i.e.,

$$iu_t - i\alpha u_x + u_{xx} + 2|u|^2u + \gamma u = 0,$$

which can be employed to depict the dynamic behavior of local magnetization in electromagnetics. Also, a research for the mLL equation reveals the accumulation of energy plays an important role in the generation of magnetic rogue waves [16]. Furthermore, the long time asymptotic behavior of the mLL equation with nonzero boundary condition have been studied [17].

• When $\alpha = \gamma = \beta = 0$, and $v = -u^*$, Eq. (1.1) degenerates into the classical focusing nonlinear Schrödinger (NLS) equation, i.e.,

$$iu_t + u_{xx} + 2|u|^2u = 0,$$

which has played an important part in nonlinear science.

It is not hard to check that the cgNLS equation is not self-adjoint, so that the soliton solution will emerge for zero boundary conditions. Therefore, it is significant to consider the question that how to obtain the long time asymptotic behavior of the cgNLS equation in the domain of solution solutions.

In order to obtain the long time behavior of nonlinear evolution equations, researchers have done a series work. In 1974, using the inverse scattering method, the long time behavior of nonlinear wave equation was first carried out by Manakov [18]. In 1976, the long time asymptotic solutions of nonlinear Schrödinger (NLS) equation with decaying initial value was obtained by Zakharov and Manakov [19]. In 1993, a nonlinear steepest descent method was developed to study the modified Korteweg-de Vries (MKdV) equation and obtained the solution rigorously by Defit and Zhou [20]. Then, this approach has been applied widespread. However, it has been shown in literature [21, 22] that if the initial value is smooth and decay fast enough then the error term is $o\left(\frac{\log t}{t}\right)$. Then, after the hard work of many researchers [23], the error term becomes $O\left(t^{-\left(1+\frac{1}{4}\right)}\right)$ for any $0 < \epsilon < \frac{1}{4}$ with the condition that the initial value belongs to the weighted Sobolev space $(1.3)$.

In recent years, McLaughlin and Miller have made contributions to develop the method for the long time asymptotic analysis of Riemann-Hilbert problem (RHP). Combining steepest descent with $\tilde{\partial}$-problem, they presented a $\tilde{\partial}$ steepest descent method which has been used to analyse the long time asymptotic behavior of series equations [24, 25]. Since then, lots of significant works have been done by applying this method.
For example, with the assumption of finite mass initial data, Dieng and McLaughlin studied the defocusing NLS equation [26]; with finite density initial data, the defocusing NLS equation was investigated by Cuccagna and Jenkins [27]; Borghese and Jenkins studied the Cauchy problem for the focusing NLS equation via applying this method [28]. Of course, there are a series of great work about this $\bar{\partial}$ steepest descent method [29]-[33]. Compared with the nonlinear steepest descent method, it is not necessary to study the delicate estimates involving $L^p$ estimates of Cauchy projection operators by applying the $\bar{\partial}$ steepest descent method. Also, it improves the error estimates, i.e., if the initial value belongs to the weighted Sobolev space (1.3), the error becomes $o(t^{-\frac{3}{4}})$ which has been shown in [26].

In this work, we are going to use the $\bar{\partial}$ steepest descent method to study the long time asymptotic behavior of the cgNLS equation with the initial value $u_0(x)$ and $v_0(x)$ which belong to the weighted Sobolev space (1.3).

The outline of this work is as follows. In section 2, based on the Lax pair of the cgNLS equation, we introduce two kinds of eigenfunction and scattering matrix. Also, the analytical, symmetries and asymptotic properties are analyzed. Then the Riemann-Hilbert problem is constructed for the cgNLS equation with initial problem. In section 3, we introduce the matrix function $T(z)$ to separate the jump matrix which is defined near the phase point $z_0 = -\frac{1}{4}(\xi + \alpha)$. In section 4, we make the continuous extension of the jump matrix off the real axis by introducing a matrix function $R^{(2)}(z)$ and get a mixed $\bar{\partial}$-Riemann-Hilbert(RH) problem. In section 5, we decompose the mixed $\bar{\partial}$-RH problem into two parts which are a model RH problem with $\bar{\partial}R^{(2)} = 0$ and a pure $\bar{\partial}$-RH problem with $\bar{\partial}R^{(2)} \neq 0$, i.e., $M_{RHP}$ and $M^{(3)}$. In section 6, we solve the model RH problem $M_{RHP}$ via an outer model $M^{(out)}(z)$ for the soliton part and an inner model near the phase point $z_0$ which can be solved by a parabolic cylinder model problem. Also, the error function $E(z)$ with a small-norm RH problem is obtained. In section 7, the pure $\bar{\partial}$-RH problem is studied. Finally, in section 8, we obtain the soliton resolution and long time asymptotic behavior of the cgNLS equation.

2. The spectral analysis of cgNLS equation

The Lax pair of the equation (1.1) reads

$$\psi_x = U\psi, \quad \psi_t = V\psi,$$

$$\psi = (\psi_1, \psi_2)^T,$$  \hspace{1cm} (2.1)
where \( \psi_i, (i = 1, 2) \) are eigenfunctions, and

\[
U = -iz\sigma_3 - i\beta u v \sigma_3 + U_0, \\
V = -2iz^2\sigma_3 + a_1\sigma_3 + 2zU_0 - i(U_0)\sigma_3 - 2\beta U_0^3 + \alpha U_0
\]  
(2.2)

with \( U_0 = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( a_1 = \beta(u_x v - uv_x) + 4i\beta^2 u^2 v^2 - i(1 - \alpha\beta)uv \).

We can check that \( U, V \) satisfy the zero curvature equation \( U_t - V_x + [U, V] = 0 \), which is the compatibility condition of (1.1).

Considering the initial conditions that \( u_0(x), v_0(x) \in H^{1,1}(\mathbb{R}) \), letting \( x \to \pm\infty \), we can get an asymptotic scattering problem and construct the two Jost solutions, i.e., \( \psi_{\pm} \sim e^{-\frac{i}{2}(zx + \frac{1}{2})\gamma(\sigma_3)} \).

Then, we introduce \( m_{\pm}(x, t; z) \) which satisfies

\[
m_{\pm}(x, t; z) = \psi_{\pm}(x, t; z) e^{i(zx + \frac{1}{2})\gamma(\sigma_3)}. 
\]  
(2.3)

The equivalent Lax pair of Eq. (2.4) can be derived as

\[
m_x + iz[\sigma_3, m] = (-i\beta u v \sigma_3 + U_0)m \equiv U_1m, \\
m_t + i(2z^2 + az - \frac{1}{2}\gamma)(\sigma_3, m) = (a_1\sigma_3 + 2zU_0 - i(U_0)\sigma_3 - 2\beta U_0^3 + \alpha U_0)m \equiv V_1m,
\]  
(2.4)

which can be written in full derivative form, i.e.,

\[
d(e^{i(zx + \frac{1}{2})\gamma(\sigma_3)}m) = e^{i(zx + \frac{1}{2})\gamma(\sigma_3)}(U_1dx + V_1dt)m.
\]  
(2.5)

Evaluating a solution of Eq. (2.5) with the form

\[
m = X + \frac{m_1}{z} + \frac{m_2}{z^2} + o(z^{-3}), \quad z \to \infty,
\]  
(2.6)

where \( X, m_1 \) and \( m_2 \) are independent of variable \( z \). Substituting Eq. (2.6) into Eq. (2.4), and comparing the coefficients of the same power of \( z \), we show that

\[
X_x = (-i\beta u v \sigma_3)X, \\
X_t = (\beta(u_x v - uv_x) + 4i\beta^2 u^2 v^2 - i\alpha\beta uv)\sigma_3 X.
\]  
(2.7)

We note that Eq. (1.1) possesses the conservation law

\[
(-i\beta u v)_x = (\beta(u_x v - uv_x) + 4i\beta^2 u^2 v^2 - i\alpha\beta uv)_x.
\]  
(2.8)

Therefore, if we define

\[
X(x, t) = e^{i \int_{-\infty}^{t} X_x d\tau},
\]  
(2.9)
where
\[
\Delta(x, t) = -\beta uv dx + (-i\beta(u, v - uv_x) + 4\beta^2 u^2 v^2 - \alpha \beta uv) dt, \quad (2.10)
\]
the two equations in (2.7) for \( X \) are consistent and are both satisfied. Now, introducing a new function \( \mu \)
\[
m(x, t, z) = e^{\int \Delta(x, t) dt} \mu(x, t, z). \quad (2.11)
\]
It is obvious that
\[
\mu = I + O(1/z), \quad z \to \infty. \quad (2.12)
\]
Then, from Eq. (2.5), we can derive that
\[
d(e^{(\psi + (2z^2 + az - \frac{1}{2} y)\sigma_3) \mu}) = e^{(\psi + (2z^2 + az - \frac{1}{2} y)\sigma_3)(U_2dx + V_2dt)\mu}. \quad (2.13)
\]
where
\[
U_2 = e^{-i \int \Delta(x, t) dt} \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix},
\]
\[
V_2 = e^{-i \int \Delta(x, t) dt} \begin{pmatrix} iuv & 2zu + iu_x - 2\beta u^2v + uv \\ 2v - iv_x - 2\beta u^2v + av & -iuv \end{pmatrix}.
\]
Also, the Lax pair (2.4) can be transformed into
\[
\mu_x + iz[\sigma_3, \mu] = U_2\mu, \quad (2.14)
\]
\[
\mu_t + i(2\beta \sigma_2 + \alpha z - \frac{1}{2} \gamma) [\sigma_3, \mu] = V_2\mu. \quad (2.15)
\]
Based on the Eq. (2.13), we can select two special integration paths i.e., \((-\infty, t) \to (x, t)\) and \((+\infty, t) \to (x, t)\), and obtain the following Volterra type integrals
\[
\mu_\pm(x, t; z) = \frac{3}{2} + \int_{-\infty}^{x} e^{-i(x-y)\sigma_3} [U_2(y, t; z)\mu_\pm(y, t; z)] dy, \quad (2.16)
\]
from which we can derive the analytical properties of \( \mu_\pm \).

- It is assumed that \( u(x) - u_0, v(x) - v_0 \in H^{1.1}(\mathbb{R}) \). Then, \( \mu_{-1}, \mu_{+2} \) are analytic in \( C_+ \) and \( \mu_{-2}, \mu_{+1} \) are analytic in \( C_- \) and they can be recorded as \( \mu_{-1}^+, \mu_{+2}^+, \mu_{-2}^-, \mu_{+1}^- \), respectively. The \( \mu_{\pm,j} \) \((j = 1, 2)\) mean the \( j \)-th column of \( \mu_\pm \).

For \( z \in \mathbb{R} \) both \( \psi_+ \) and \( \psi_- \) are two fundamental matrix solutions of the scattering problem. Therefore,
\[
\psi_+(x, t; z) = \psi_-(x, t; z)\tilde{S}(z), \quad x, t \in \mathbb{R}, \quad z \in \mathbb{R}, \quad (2.17)
\]
where $S(z)$ is a $2 \times 2$ matrix and independent of the variable $x$ and $t$. According to the Abel’s theorem and the Lax pair, we know that $(\det \psi)_x = (\det \psi)_t = 0$. Furthermore, we have $\det \psi_\pm = \det \mu_\pm = 1$. Then, we can derive that $\psi_\pm$ are reversible. Therefore, we obtain that

\begin{align}
    s_{11}(z) &= \text{Wr}(\psi_{+,1}, \psi_{-,2}), & s_{22}(z) &= \text{Wr}(\psi_{-,1}, \psi_{+,2}), \\
    s_{12}(z) &= \text{Wr}(\psi_{+,2}, \psi_{-,2}), & s_{21}(z) &= \text{Wr}(\psi_{-,1}, \psi_{+,1}),
\end{align}

(2.18)

where $\psi_{\pm,j}$ mean the $j$-column of $\psi_\pm$, respectively, and Wr infers to the Wronskians determinant. Then, according to the analytical properties of $\mu_\pm$ and the relationship between $\psi_\pm$ and $\mu_\pm$, we obtain that $s_{11}$ is analytic in $\mathbb{C}^+$, and $s_{22}$ is analytic in $\mathbb{C}^-$. It is easy to check that if $\mu(x, t; z)$ is the solution of Eq. (2.14) then $-\sigma \mu^*(x, z^*) \sigma$ also is the solution of Eq. (2.14) and it follows

\[ \mu_\pm(z) = -\sigma \mu_\mp^*(z^*) \sigma. \]

(2.19)

Then, it is obvious that

\[ S(z) = -\sigma S^*(z^*) \sigma. \]

(2.20)

Based on Eq. (2.12) and Eq. (2.17), we can derive that $S(z) \to I, z \to \infty$.

**Assumption 2.1.** In the following analysis, we make the assumption to avoid the many pathologies possible, i.e.,

- For $z \in \mathbb{R}$, no spectral singularities exist, i.e., $s_{11}(z) \neq 0$;
- Suppose that $s_{11}(z)$ has $N$ zero points, denoted as $\mathcal{Z} = \{(z_j, \text{Im}z_j > 0)\}_{j=1}^N$. So that $s_{22}(z)$ has $N$ zero points $\mathcal{Z}^* = \{(z_j^*, \text{Im}z_j^* < 0)\}_{j=1}^N$.
- The discrete spectrum is simple, i.e., if $z_0$ is the zero of $s_{11}(z)$, then $s_{11}'(z_0) \neq 0$.

Furthermore, based on Eq. (2.18) and Eq. (2.20), there exists norming constants $c_j$ such that

\[ \mu_{-1}(z_j) = e^{-2i\theta(z)} \mu_{+2}(z_j); \quad \mu_{-2}(z_j^*) = -e^{2i\theta(z^*)} \mu_{+1}(z_j^*), \]

where $\theta(z) = (z \frac{d}{dz} + 2z^2 + az - \frac{1}{2} \gamma)$.

Now, we introduce a sectionally meromorphic matrices

\[ M(x, t; z) = \begin{cases} 
    M^+(x, t; z) = \left( \frac{\mu_{-1}(x, t; z)}{s_{11}(z)}, \frac{\mu_{+2}(x, t; z)}{s_{11}(z)} \right), & z \in \mathbb{C}^+, \\
    M^-(x, t; z) = \left( \frac{\mu_{+1}(x, t; z)}{s_{22}(z)}, \frac{\mu_{-2}(x, t; z)}{s_{22}(z)} \right), & z \in \mathbb{C}^-,
\end{cases} \]

(2.21)
where \( M^+(x, t; z) = \lim_{\varepsilon \to 0^+} M(x, t; z \pm i\varepsilon) \), \( \varepsilon \in \mathbb{R} \), and reflection coefficients

\[
\begin{align*}
    r(z) &= \frac{s_{12}(z)}{s_{11}(z)}, \quad \frac{s_{12}(z)}{s_{22}(z)} = -\frac{s_{11}(z^*)}{s_{21}(z^*)} = -r^*(z^*) = -r^*(z), \quad z \in \mathbb{R}. \tag{2.22}
\end{align*}
\]

Based on the above analysis, the matrix function \( M(x, t; z) \) admits the following matrix RHP.

**Riemann-Hilbert Problem 2.2.** Find an analysis function \( M(x, t; z) \) with the following properties:

- \( M(x, t; z) \) is meromorphic in \( C \setminus \mathbb{R} \).
- \( M^+(x, t; z) = M^-(x, t; z)G(x, t; z), \quad z \in \mathbb{R}, \) where
  \[
  G(x, t; z) = \begin{pmatrix}
  1 + |r(z)|^2 & r'(z)e^{-2i\theta(z)} \\
  r(z)e^{2i\theta(z)} & 1
  \end{pmatrix}; \tag{2.23}
  \]
- \( M(z) = I + O(z^{-1}) \) as \( z \to \infty \);
- At each \( z_j \in \mathbb{Z} \) and \( z_j^* \in \mathbb{Z}^* \), \( M(z) \) satisfies the residue condition, i.e.,
  \[
  \text{Res } M = \lim_{z \to z_j} M \begin{pmatrix}
  0 & 0 \\
  c_j e^{2i\theta} & 0
  \end{pmatrix}, \text{Res } M = \lim_{z \to z_j^*} M \begin{pmatrix}
  0 & -c_j^* e^{-2i\theta} \\
  0 & 0
  \end{pmatrix}. \tag{2.24}
  \]

**Remark 2.3.** By referring to the Zhou’s vanishing lemma, the existence of the solutions of RHP [2.2] for \( (x, t) \in \mathbb{R}^2 \) is guaranteed. According to a consequence of Liouville’s theorem, we know that if a solution exists, it is unique.

Then, expanding this solution as \( z \to \infty \) and combining with Eq. (2.14), we can reconstruct the solution of cgNLS equation as

\[
\begin{align*}
    u(x, t) &= 2i \lim_{z \to \infty} (zM)_{12} e^{2i \int_{\infty}^{z)} \Delta}, \quad v(x, t) = 2i \lim_{z \to \infty} (zM)_{21} e^{-2i \int_{\infty}^{z)} \Delta}. \tag{2.25}
\end{align*}
\]

3. Conjugation

In this section, in order to renormalize the Riemann-Hilbert problem [2.2], we introduce a function to establish a transformation \( M \to M^{(1)} \).

In jump matrix (2.23), the oscillation term is \( e^{2i\theta(z)} \), from which we find a phase point

\[
    z_0 = -\frac{1}{4} \left( \frac{x}{t} + \alpha \right).
\]
Consequently, $\theta(z)$ can be written as
\[
\theta(z) = 2z^2 - 4z_0z - \frac{1}{2}y = 2(z - z_0)^2 - 2z_0^2 - \frac{1}{2}y. \tag{3.1}
\]
And we have $Re(2i\theta) = -8(Rez - z_0)Imz$. Then, we derive the decaying domains of the oscillation term.

To make the following analysis more convenient, we introduce some notations.

For $I = [a, b]$, define
\[
\mathcal{Z}(I) = \{z_k \in \mathbb{Z} : Re z_k \in I\},
\]
\[
\mathcal{Z}^-(I) = \{z_k \in \mathbb{Z} : Re z_k < a\},
\]
\[
\mathcal{Z}^+(I) = \{z_k \in \mathbb{Z} : Re z_k > b\}.
\]

For $z_0 \in I$, define
\[
\Delta^-_{z_0}(I) = \{k \in \{1, \cdots, N\} : a \leq Re z_k < z_0\},
\]
\[
\Delta^+_{z_0}(I) = \{k \in \{1, \cdots, N\} : a < Re z_k \leq z_0\}.
\]

Next, we introduce the function
\[
\delta(z) = \exp[i \int_{-\infty}^{\infty} \frac{\nu(s)}{s - z} ds], \quad \nu(s) = -\frac{1}{2\pi} \log(1 + |r(s)|^2).
\]
and
\[
T(z) = \prod_{k \in \Delta_{z_0}} \frac{z - z_k^*}{z - z_k} \delta(z), \tag{3.3}
\]
which has the following properties.
Proposition 3.1. $T(z)$ admits that

(a) $T$ is meromorphic in $C \setminus (-\infty, z_0]$;

(b) For $z \in C \setminus (-\infty, z_0)$, $T^*(z^*) = \frac{1}{T(z)}$;

(c) For $z \in (-\infty, z_0]$, the boundary values $T_\pm$ satisfy

$$T_+(z)/T_-(z) = 1 + |r(z)|^2, z \in (-\infty, z_0];$$  \hfill (3.4)

(d) As $|z| \to \infty$ with $|\arg(z)| \leq c < \pi$,

$$T(z) = 1 + \frac{i}{z} \left[ 2 \sum_{k \in \Lambda_{z_0}} \text{Im} z_k - \int_{-\infty}^{z_0} \nu(s) ds \right] + O(z^{-2});$$ \hfill (3.5)

(e) As $z \to z_0$ along any ray $z_0 + e^{i\phi} R$, with $|\phi| \leq c < \pi$

$$|T(z - z_0) - T_0(z_0)(z - z_0)^{i\nu(z_0)}| \leq C \| r \|_{H^1(B)} |z - z_0|^\frac{1}{2},$$ \hfill (3.6)

where $T_0(z_0)$ is the complex unit

$$T_0(z_0) = \prod_{k \in \Lambda_{z_0}} \left( \frac{z_0 - z^*_k}{z_0 - z_k} \right) e^{\beta(z_0, z_k)},$$ \hfill (3.7)

$$\beta(z, z_0) = -\nu(z_0) \log(z - z_0 + 1) + \int_{-\infty}^{z_0} \frac{\nu(s) - \chi(s)\nu(z_0)}{s - z} ds,$$

with $\chi(s) = 1$ as $s \in (z_0 - 1, z_0)$, and $\chi(s) = 0$ as $s \in (-\infty, z_0 - 1]$.

Proof. For part (a), (b) and (c), it is obvious to obtain their results from the definition of $T(z)$ shown in (3.3). For part (d), we expand the $\prod_{k \in \Lambda_{z_0}} \frac{z - z^*_k}{z - z_k}$ and $\delta(z)$ for large $z$. Then, a direct calculation can show the result. For part (e), we rewrite (3.3) as

$$T(z) = T(z, z_0) = \prod_{k \in \Lambda_{z_0}} \left( \frac{z - z^*_k}{z - z_k} \right) \exp \left( i \int_{z_0 - 1}^{z_0} \frac{\nu(z_0)}{s - z} ds + i \int_{-\infty}^{z_0} \frac{\nu(s) - \chi(s)\nu(z_0)}{s - z} ds \right)$$

$$= \prod_{k \in \Lambda_{z_0}} \left( \frac{z - z^*_k}{z - z_k} \right) (z - z_0)^{i\nu(z_0)} \exp(i\beta(z, z_0)).$$

Then, according to the fact that

$$|(z - z_0)^{i\nu(z_0)}| \leq |e^{i\nu(z_0)}| = \sqrt{1 + |r(z_0)|^2},$$

and the Lemma shown in [34], i.e.,

$$|\beta(z, z_0) - \beta(z_0, z_0)| \leq c \| r \|_{H^1(B)} |z - z_0|^\frac{1}{2},$$

the result of (e) can be obtained directly. \hfill $\Box$
Now, using the function $T(z)$, we establish a transformation

$$M_1(z) = M(z)T(z)^{-\sigma_3},$$  \hspace{1cm} (3.8)

which admits the following matrix RHP.

**Riemann-Hilbert Problem 3.2.** Find an analysis function $M^{(1)}$ with the following properties:

- $M^{(1)}$ is meromorphic on $C \setminus R$;
- $M_0^{(1)}(z) = I + O(z^{-1})$ as $z \to \infty$;

- $M_{\pm}^{(1)}(z)$ satisfy the jump relationship $M_{+}^{(1)}(z) = M_{-}^{(1)}(z)V^{(1)}(z)$, where

$$V^{(1)} = \begin{pmatrix}
1 & r'(z)T(z)^2e^{-2iz\Theta} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
r(z)T(z)^2e^{2iz\Theta} & 1
\end{pmatrix} L \approx W_LW_R, z \in (z_0, \infty),$$

$$L \approx U_LU_R, z \in (-\infty, z_0).$$ \hspace{1cm} (3.9)

- $M^{(1)}(z)$ has simple poles at each $z_k \in Z$ and $z_k \in Z$ at which

$$\text{Res} \ M^{(1)}_{z_k} = \begin{pmatrix}
\lim_{z \to z_k^-} M^{(1)}(z) \\
\lim_{z \to z_k^+} M^{(1)}(z)
\end{pmatrix}
\begin{pmatrix}
0 & c_{\kappa}^{-1}(\hat{T}(z_k)^{-2}e^{-2iz\Theta}) \\
0 & 0
\end{pmatrix}, k \in \Delta_0^-$$

$$\text{Res} \ M^{(1)}_{z_k^*} = \begin{pmatrix}
\lim_{z \to z_k^*^-} M^{(1)}(z) \\
\lim_{z \to z_k^*^+} M^{(1)}(z)
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
-\tilde{c}_{\kappa}^{-1}(\hat{T}(z_k^*)^{-2}e^{2iz\Theta}) & 0
\end{pmatrix}, k \in \Delta_0^+$$ \hspace{1cm} (3.10)

**Proof.** Firstly, according to the definition of $M^{(1)}$, Proposition 3.1 and the properties of $M(x, t; z)$, we can easily obtain the analyticity, jump matrix and asymptotic behavior of $M^{(1)}$. Then, we consider the residue conditions. When $k \in \Delta_0^+$, $T(z)$ is analytic at the points $z_k, z_k^*$. Thus, at these points, the residue conditions can be obtained directly from (2.24) and (3.8). When $k \in \Delta_0^-$, $z_k$ is the pole of $T(z)$ and $T^{-1}(z)$ is analytic at $z_k$. 

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Consequently, we have
\[
\text{Res}_z M_1^{(1)}(z) = \text{Res}_z (M_1 T^{-1}) = 0,
\]
\[
M_1^{(1)}(z_k) = \lim_{z \to z_k} (M_1(z) T^{-1}(z)) = c_k e^{2i\theta(z_k)} M_2(z_k) \left( \frac{1}{T} \right)'(z_k),
\]
\[
\text{Res}_z M_2^{(1)}(z) = \text{Res}_z (M_2 T) = M_2(z_k) \lim_{z \to z_k} T(z)(z - z_k)
\]
\[
= c_k e^{-2i\theta(z_k)} M_1^{(1)}(z_k) \left( \frac{1}{T} \right)'(z_k))^{-1},
\]
from which the first formula shown in (3.10) can be obtained. Similarly, it is easy to derive the residue condition at \( z^*_k \) with \( k \in \Delta^-. \)

\[\square\]

4. Continuous extension to a mixed \( \bar{\partial} \)-RH problem

In this section, we are going to make the continuous extension of the jump matrix off the real axis. It should be pointed that the extension is not necessary and the oscillation term along the new contours are decaying. To achieve this, we define the contours
\[
\Sigma_j = -\frac{1}{4} \left( \frac{\chi}{r} + \alpha \right) + e^{i(2j-1)\pi/4} \mathbb{R}_+, \quad j = 1, 2, 3, 4,
\]
\[
\Sigma_R = R \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4.
\]

Moreover, define
\[
\rho = \frac{1}{2} \min_{\lambda, \zeta \in \mathbb{Z} \cup \mathbb{Z}^* \setminus \mu} |\lambda - \zeta|,
\]
and \( \chi \in C^\omega_0(C, [0, 1]) \) which is supported near the discrete spectrum such that
\[
\chi(z) = \begin{cases} 
1, & \text{dist}(z, \mathbb{Z} \cup \mathbb{Z}^*) < \rho/3, \\
0, & \text{dist}(z, \mathbb{Z} \cup \mathbb{Z}^*) > 2\rho/3.
\end{cases}
\]

It is easy to check that \( \text{dist}(\mathbb{Z} \cup \mathbb{Z}^*, R) > \rho, k = 1, 2, \cdots, N. \)

Next, we need to extend the jump matrix onto the new contours along which oscillation term are decaying. Therefore, we introduce a transformation
\[
M^{(2)} = M^{(1)} R^{(2)},
\]
where \( R^{(2)} \) is selected to admit some conditions.

- The purpose of the transformation is to deform the contour \( R \) to the contour \( \Sigma^{(2)} \) where \( \Sigma^{(2)} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4 \). So on the real axis, \( M^{(2)} \) must have no jump.
• To guarantee that the $\bar{\partial}$-contribution to the large-time asymptotic solution of $u(x,t)$ and $v(x,t)$, the norm of $R_2^{(2)}$ should be controlled.

• The introduced transformation need to have no impact on the residue condition.

Thus, we define $R_2^{(2)}$ as

$$R_2^{(2)} = \begin{cases} 
\left( \begin{array}{cc} 1 & 0 \\
R_1 e^{2i\theta} & 1 \end{array} \right)^{-1} & \text{in } \Omega_1, \\
\left( \begin{array}{cc} 1 & R_3 e^{-2i\theta} \\
0 & 1 \end{array} \right)^{-1} & \text{in } \Omega_3, \\
\left( \begin{array}{cc} 1 & 0 \\
R_4 e^{2i\theta} & 1 \end{array} \right)^{-1} & \text{in } \Omega_4, \\
\left( \begin{array}{cc} 1 & R_6 e^{-2i\theta} \\
0 & 1 \end{array} \right)^{-1} & \text{in } \Omega_6, \\
\left( \begin{array}{cc} 1 & 0 \\
0 & 1 \end{array} \right) & \text{in } \Omega_2 \cup \Omega_5,
\end{cases}$$

(4.5)

where $R_j, j = 1, 3, 4, 6$ are defined in the following proposition.

**Proposition 4.1.** There exists functions $R_j : \Omega_j \to C, j = 1, 3, 4, 6$ such that

$$R_1(z) = \begin{cases} 
\Gamma(z)T^{-2}(z), & z \in (z_0, \infty), \\
\lim_{z \to z_0} \frac{r(z)}{T(z)} & z \in (-\infty, z_0),
\end{cases}$$

(4.6)

$$R_3(z) = \begin{cases} 
\Gamma(z)T^2(z), & z \in (z_0, \infty), \\
\lim_{z \to z_0} \frac{r(z)}{T(z)} & z \in (-\infty, z_0),
\end{cases}$$

(4.7)

$$R_4(z) = \begin{cases} 
\Gamma(z)T^2(z), & z \in (z_0, \infty), \\
\lim_{z \to z_0} \frac{r(z)}{T(z)} & z \in (-\infty, z_0),
\end{cases}$$

(4.8)

And $R_j$ admit that

$$|R_j(z)| \leq c_1 \sin^2(\arg z) + c_2 \langle \text{Re}z \rangle^{-1/2},$$

(4.9)

$$|\bar{\partial}R_j(z)| \leq c_3 \bar{\partial}\chi(z) + c_2 |z - z_0|^{-1/2} + c_3 |r'(\text{Re}z)|,$$

(4.10)

$$\bar{\partial}R_j(z) = 0, z \in \Omega_2 \cup \Omega_5, \text{or } \text{dist}(z, Z \cup Z') \leq \rho/3.$$
where $\langle \text{Re}z \rangle = \sqrt{1 + (\text{Re}z)^2}$.

\[ \mathcal{R}^{(2)} = U_R^{-1} \]

\[ \mathcal{R}^{(2)} = U_L \]

\[ \mathcal{R}^{(2)} = W_R \]

\[ \mathcal{R}^{(2)} = W_L \]

\[ \Omega_1 \]

\[ \Omega_2 \]

\[ \Omega_3 \]

\[ \Omega_4 \]

\[ \Omega_5 \]

\[ \Omega_6 \]

\[ \Sigma_1 \]

\[ \Sigma_2 \]

\[ \Sigma_3 \]

\[ \Sigma_4 \]

\[ \Sigma_5 \]

\[ \Sigma_6 \]

**Figure 2.** Definition of $\mathcal{R}^{(2)}$ in different domains.

Then, using the transformation (4.4) and the definition of $\mathcal{R}^{(2)}$, we derive that $\mathcal{M}^{(2)}$ admits the following mixed $\bar{\partial}$-RH problem.

**Riemann-Hilbert Problem 4.2.** Find a matrix value function $\mathcal{M}^{(2)}$, admitting

- $\mathcal{M}^{(2)}(x, t, z)$ is continuous in $\mathbb{C} \backslash (\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{Z}^*)$.
- $\mathcal{M}^{(2)}(x, t, z) = \mathcal{M}^{(2)}(x, t, z) V^{(2)}(x, t, z), \quad z \in \Sigma^{(2)}$, where the jump matrix $V^{(2)}(x, t, z)$ satisfies
  \[
  V^{(2)}(z) = (\mathcal{R}^{(2)})^{-1} V^{(1)} \mathcal{R}^{(2)} = I + (1 - \chi(z)) \bar{V}^{(2)},
  \]
  with
  \[
  \bar{V}^{(2)}(z) = \begin{cases}
  0 & z \in \Sigma_1, \\
  \frac{r(z_0)T_0(z_0)^{-2}(z - z_0)^{-2i\nu(z_0)e^{2i\theta}}}{1 + r(z_0)^2} & z \in \Sigma_2, \\
  \frac{r(z_0)T_0(z_0)^{-2}(z - z_0)^{-2i\nu(z_0)e^{2i\theta}}}{1 + r(z_0)^2} & z \in \Sigma_3, \\
  0 & z \in \Sigma_4.
  \end{cases}
  \]

- $\mathcal{M}^{(2)}(x, t, z) \to I, \quad z \to \infty.$
• For \( C \setminus (\Sigma(2) \cup Z \cup Z^*) \), \( \bar{\partial}M^{(2)} = M^{(2)} \bar{\partial}R^{(2)}(z) \), where

\[
\bar{\partial}R^{(2)}(z) = \begin{cases} 
0 & 0, 
-\partial R e^{2in\theta} & 0, 
0 & -\partial R e^{-2in\theta}, 
0 & 0, 
\end{cases}
\]

\[
\bar{\partial}R^{(2)} = \begin{cases} 
0 & 0, 
0 & 0, 
0 & 0, 
0 & 0,
\end{cases}
\]

\( z \in \Omega_1, \)
\( z \in \Omega_3, \)
\( z \in \Omega_4, \)
\( z \in \Omega_2 \cup \Omega_5. \)

• \( M^{(2)} \) admits the residue conditions at poles \( z_k \in Z \) and \( z_k^* \in Z^* \), i.e.,

\[
\text{Res} M^{(2)} = \begin{cases} 
\lim_{z \to z_k^-} M^{(2)} \left( \frac{1}{c_k^{-1}(\frac{1}{k})} (z_k)^{-2} e^{-2in\theta(z_k)} \right), & k \in \Delta_{\Omega}^- \\
\lim_{z \to z_k^+} M^{(2)} \left( c_k^{-1}T(z_k)^{-2} e^{2in\theta(z_k)} \right), & k \in \Delta_{\Omega}^+ \\
\lim_{z \to z_k^-} M^{(2)} \left( -(c_k^*)^{-1}T(z_k)^{-2} e^{-2in\theta(z_k)} \right), & k \in \Delta_{\Omega}^- \\
\lim_{z \to z_k^+} M^{(2)} \left( c_k^*T(z_k)^{-2} e^{2in\theta(z_k)} \right), & k \in \Delta_{\Omega}^+ 
\end{cases}
\]

Then, from Eq. (2.25) and the transformation (3.8) and (4.4), we can reconstruct the solution of cgNLS equation as

\[
u(x, t) = 2i \lim_{z \to \infty_1} (z M^{(2)})_{12} e^{2i \lambda (x, \infty_1)}, \quad v(x, t) = 2i \lim_{z \to \infty_2} (z M^{(2)})_{21} e^{-2i \lambda (x, \infty_2)}. \quad (4.12)
\]
5. Decomposition of the mixed $\bar{\partial}$-RH problem

In this section, we are going to decompose the mixed $\bar{\partial}$-RH problem, i.e., RHP 4.2 into two parts which include a model RH problem with $\bar{\partial}R^{(2)} = 0$ and a pure $\bar{\partial}$-RH problem. We denote $M_{RHP}$ as the solution when $\bar{\partial}R^{(2)} = 0$ in the mixed $\bar{\partial}$-RH problem 4.2. Then, if the existence of the solution of $M_{RHP}$ can be proved and its asymptotic expansion for large $t$ can be constructed, the mixed $\bar{\partial}$-RH problem 4.2 can be reduced to a pure $\bar{\partial}$-RH problem. Now, we first show the pure $\bar{\partial}$-RH problem. The proof of the existence and asymptotic of $M_{RHP}$ will be shown in the following analysis.

Riemann-Hilbert Problem 5.1. Find a matrix value function $M_{RHP}$, admitting

- $M_{RHP}$ is analytical in $\mathbb{C}\setminus(\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{Z}^*)$;
- $M_{RHP,+}(x,t,z) = M_{RHP,-}(x,t,z)V^{(2)}(x,t,z)$, $z \in \Sigma^{(2)}$, where $V^{(2)}(x,t,z)$ is the same with the jump matrix appears in RHP 4.2;
- As $z \to \infty$, $M_{RHP}(x,t,z) = I + o(z^{-1})$;
- $M_{RHP}$ possesses the same residue condition with $M^{(2)}$.

Now, using the $M_{RHP}$, we construct a transformation

$$M^{(3)}(z) = M^{(2)}(z)M_{RHP}(z)^{-1}. \quad (5.1)$$

Then, we can get the following pure $\bar{\partial}$-RH problem.
Riemann-Hilbert Problem 5.2. Find a matrix value function $M^{(3)}$, admitting

- $M^{(3)}$ is continuous with sectionally continuous first partial derivatives in $\mathbb{C}\setminus(\Sigma^{(2)}\cup \mathbb{Z}\cup \mathbb{Z}^*)$;
- For $z \in \mathbb{C}$, we obtain $\bar{\partial}M^{(3)}(z) = M^{(3)}(z)W^{(3)}(z)$, where
  \[
  W^{(3)} = M^{(2)}_{\text{RHP}}(z)\bar{\partial}R^{(2)}(z)M^{(2)}_{\text{RHP}}(z)^{-1};
  \]
- As $z \to \infty$,
  \[
  M^{(3)}(z) = I + o(z^{-1}).
  \]

Proof. Based on the properties of the $M_{\text{RHP}}$ and $M^{(2)}$ that has been shown in RHP 5.1 and (4.2), we can derive the analyticity and asymptotic properties of $M^{(3)}$ easily. According to the construction of the $M_{\text{RHP}}$, we know that $M_{\text{RHP}}$ possesses the same jump matrix with $M^{(2)}$. Consequently, we obtain that
\[
M^{(3)}_-(z)^{-1}M^{(3)}_+(z) = M_{\text{RHP},-}(z)M^{(2)}_-(z)^{-1}M^{(2)}_+(z)M_{\text{RHP},+}(z)^{-1}
= M_{\text{RHP},-}(z)V^2(z)(M_{\text{RHP},-}(z)V^2(z))^{-1} = I,
\]
from which we know that $M^{(3)}$ has no jump. Next, we further explain that there exists no pole in $M^{(3)}$. We use $N_k$ to denote nilpotent matrix which appears in the left side of the residue condition of RHP 4.2 and RHP 5.1. Then, we obtain the Laurent expansions
\[
M^{(2)}(z) = C(z_k)\left[\frac{N_k}{z - z_k} + I\right] + o(z - z_k),
\]
\[
M_{\text{RHP}}(z) = \hat{C}(z_k)\left[\frac{N_k}{z - z_k} + I\right] + o(z - z_k),
\]
where $C(z_k)$ and $\hat{C}(z_k)$ are constant terms. Then, we can derive that
\[
M^{(2)}(z)M_{\text{RHP}}(z)^{-1} = o(1),
\]
which means has removable singularities at $z_k$. Finally, based on the definition of $M^{(3)}$, we have
\[
\bar{\partial}M^{(3)}(z) = \bar{\partial}(M^{(2)}(z)M_{\text{RHP}}(z)^{-1}) = \bar{\partial}M^{(2)}(z)M_{\text{RHP}}(z)^{-1} = M^{(2)}(z)\bar{\partial}R^{(2)}(z)M_{\text{RHP}}(z)^{-1}
= M^{(2)}(z)M_{\text{RHP}}(z)^{-1}(M_{\text{RHP}}(z)\bar{\partial}R^{(2)}(z)M_{\text{RHP}}(z)^{-1}) = M^{(3)}(z)W^{(3)}(z),
\]
i.e., the second condition of RHP 5.2. \qed
6. The pure RH problem

In this section, we construct the solution \( M_{\text{RHP}} \) of RHP 5.1. Define
\[
\mathcal{U}_{z_0} = \{z : |z - z_0| < \rho/2\},
\]
and we decompose \( M_{\text{RHP}} \) into two parts
\[
M_{\text{RHP}}(z) = \begin{cases} E(z)M^{(\text{out})}(z), & z \in \mathbb{C} \setminus \mathcal{U}_{z_0}, \\ E(z)M^{(\text{out})}(z)M^{(\text{pc})}(z_0, r_0), & z \in \mathcal{U}_{z_0}, \end{cases}
\]
where \( M^{(\text{out})} \) solve a model RHP, \( M^{(\text{pc})} \) is a known parabolic cylinder model and \( E(z) \) is an error function which is a solution of a small-norm Riemann-Hilbert problem.

6.1. Outer model RH problem: \( M^{(\text{out})} \)

Considering the construction of \( M_{\text{RHP}} \), we know that \( M_{\text{RHP}} \) admits a pure Riemann-Hilbert problem. Additionally, for the jump matrix \( V^{(2)} \), we have the following estimate.
\[
||V^{(2)} - I||_{L^\infty(\Sigma^{(2)})} = \begin{cases} o(e^{-4|z-z_0|^2}), & z \in \Sigma^{(2)} \setminus \mathcal{U}_{z_0}, \\ o|z-z_0|^{-1/2}, & z \in \Sigma^{(2)} \cap \mathcal{U}_{z_0}. \end{cases}
\]
Recall that \( z_0 = -\frac{1}{4}(x + \alpha) \). Due to \( |z - z_0| \geq \rho/2 \) outside \( \mathcal{U}_{z_0} \), if we omit the jump condition of \( M_{\text{RHP}} \), the estimate infers to that exponentially small error exists. Next, we establish a model RH problem and prove that its solution can be approximated by finite sum of soliton solutions.

Riemann-Hilbert Problem 6.1. Find a matrix value function \( M^{(\text{out})} \), admitting
\begin{itemize}
  \item \( M^{(\text{out})}(x, t; z) \) is analytical in \( \mathbb{C} \setminus (\Sigma^{(2)} \cup \mathbb{Z} \cup \mathbb{Z}^*) \);
  \item As \( z \to \infty \),
  \[
  M^{(\text{out})}(x, t; z) = I + o(z^{-1});
  \]
  \item \( M^{(\text{out})}(x, t; z) \) has simple poles at each point in \( \mathbb{Z} \cup \mathbb{Z}^* \) admitting the same residue condition in RHP 4.2 with \( M^{(\text{out})}(x, t; z) \) replacing \( M^{(2)}(x, t; z) \).
\end{itemize}

To obtain the solution of \( M^{(\text{out})}(x, t; z) \), we first study RHP 2.2 for the case of reflectionless. Under this condition, we know that \( M \) has no jump and get the following Riemann-Hilbert problem from RHP 2.2.
Riemann-Hilbert Problem 6.2. Find a matrix value function \( M(x, t; z|\sigma_d) \), admitting

- \( M(x, t; z|\sigma_d) \) is analytical in \( \mathbb{C} \setminus (\mathbb{Z} \cup \mathbb{Z}^*) \);
- \( M(x, t; z|\sigma_d) = I + O(z^{-1}), \quad z \to \infty \);
- \( M(x, t; z|\sigma_d) \) satisfies the following residue conditions at simple poles \( z_k \in \mathbb{Z} \) and \( z_k^* \in \mathbb{Z}^* \)

\[
\text{Res} M(x, t; z|\sigma_d) = \lim_{z \to z_k} M(x, t; z|\sigma_d)N_k, \\
\text{Res} M(x, t; z|\sigma_d) = \lim_{z \to z_k^*} M(x, t; z|\sigma_d)\sigma_2N_k^*\sigma_2,
\]

where \( \sigma_d = \{(z_k, c_k), z_k \in \mathbb{Z}\}_{k=1}^N \), which satisfies \( z_k \neq z_j \) for \( k \neq j \), are scattering data, and

\[
N_k = \begin{pmatrix} 0 & 0 \\ \gamma_k(x, t) & 0 \end{pmatrix}, \quad \gamma_k(x, t) = c_ke^{2im(z_k)}, \quad (6.5)
\]

\[
\theta(z_k) = 2z^2 + \alpha z - \gamma + \frac{x}{t} \epsilon. \quad (6.6)
\]

Then, according to the Liouville’s theorem, the uniqueness of the solution is obvious. Based on the symmetry that shown in Eq. (2.19), we obtain \( M(x, t; z|\sigma_d) = -\sigma M^\ast(x, t; z^\ast|\sigma_d)\sigma \), from which we can derive the following expansion, i.e.,

\[
M(x, t; z|\sigma_d) = I + \sum_{k=1}^N \left[ \frac{1}{z - z_k} \begin{pmatrix} \xi_k(x, t) & 0 \\ \eta_k(x, t) & 0 \end{pmatrix} + \frac{1}{z - z_k^*} \begin{pmatrix} 0 & -\eta_k^\ast(x, t) \\ 0 & \xi_k^\ast(x, t) \end{pmatrix} \right], \quad (6.7)
\]

where \( \xi_k(x, t) \) and \( \eta_k(x, t) \) are unknown coefficients to be determined. Next, using the similar way that has been shown in literature [23], we obtain the following proposition.

**Corollary 6.3.** For the given scattering data \( \sigma_d \), the RHP (6.2) has the unique solution.

So, for given scattering data \( \sigma_d \), the unique solution of RHP (6.2) can be shown that

\[
u_{sol}(x, t; \sigma_d) = 2i \lim_{z \to +\infty} (z M(z|\sigma_d))_{12} e^{2i \int_{-\infty}^{\infty} n \Lambda}, \quad (6.8)
\]

\[
u_{sol}(x, t; \sigma_d) = 2i \lim_{z \to -\infty} (z M(z|\sigma_d))_{21} e^{-2i \int_{-\infty}^{\infty} n \Lambda}. \quad (6.9)
\]

Now, we need to establish the relation between \( M(x, t; z|\sigma_d) \) and \( M^{(out)}(x, t; z) \). We first give the following proposition, and further explained it in the following two subsections.
Corollary 6.4. There exist unique solution \( M^{\text{out}} \) of RHP 6.1. Particularly,

\[
M^{\text{out}}(z) = M^{\delta_{0}}(z)\delta(z) = M^{\delta_{0}}(z)\sigma_{d}^{\text{out}}(z)
\]

where \( M^{\delta_{0}}(z) \) is the solution of RHP 6.5, with \( \Delta = \Delta_{0} \) and \( \sigma_{d}^{\text{out}} = \{(z_k, \tilde{c}_k(z_0))\}^N_{k=1} \) with

\[
\tilde{c}_k(z_0) = c_k e^{\frac{i}{2} \int_{z_0}^{z} \frac{\sigma_{d}(\zeta) d\zeta}{\zeta - z_k}}.
\]

In addition,

\[
\begin{align*}
\upsilon_{\text{sol}}(x, t; \sigma_{d}^{\text{out}}) &= \lim_{\varepsilon \to 0} 2i\varepsilon M_{12}^{\text{out}}(x, t; z)e^{2i\int_{z_{\min}}^{z} \Lambda}, \\
\upsilon_{\text{sol}}(x, t; \sigma_{d}^{\text{out}}) &= \lim_{\varepsilon \to 0} 2i\varepsilon M_{21}^{\text{out}}(x, t; z)e^{-2i\int_{z_{\min}}^{z} \Lambda},
\end{align*}
\]

where \( \upsilon_{\text{sol}}(x, t; \sigma_{d}^{\text{out}}) \) and \( v_{\text{sol}}(x, t; \sigma_{d}^{\text{out}}) \) are the N-soliton solution of Eq. (1.1) corresponding the scattering data \( \sigma_{d}^{\text{out}} \).

6.1.1. Renormalization of the Riemann-Hilbert problem for reflectionless case

For the case of reflectionless, recall that

\[
M(x, t; z|\sigma_{d}) = \left( \frac{\mu_{-1}(x, t; z)}{s_{11}(z)}, \mu_{+2}(x, t; z) \right), \quad z \in \mathbb{C}^{+}, \quad s_{11}(z) = \prod_{k=1}^{N} \left( \frac{z - z_k}{z - \zeta_k} \right).
\]

Let \( \Delta \subseteq \{1, 2, \cdots, N\}, \quad \forall \subseteq \{1, 2, \cdots, N\} \setminus \Delta \), and define

\[
\begin{align*}
s_{11,\Delta} &= \prod_{k \in \Delta} \frac{z - z_k}{z - \zeta_k}, \quad s_{11,\forall} = \prod_{k \in \forall} \frac{z - z_k}{z - \zeta_k}.
\end{align*}
\]

Then, we introduce the normalization transformation

\[
M^{\Delta}(x, t; z|\sigma_{d}) = M_{\Delta}(x, t; z|\sigma_{d})s_{11,\Delta}(z)\gamma_{1},
\]

which splits the poles between the columns of \( M(x, t; z|\sigma_{d}) \) by selecting different \( \Delta \).

Then, we can get the modified Riemann-Hilbert problem.

**Riemann-Hilbert Problem 6.5.** Given scattering data \( \sigma_{d} = \{(z_k, \tilde{c}_k)\}^N_{k=1} \) and \( \Delta \subseteq \{1, 2, \cdots, N\} \). Find a matrix value function \( M^{\Delta} \), admitting

- \( M^{\Delta}(x, t; z|\sigma_{d}) \) is analytical in \( \mathbb{C} \setminus (\mathbb{Z} \cup \mathbb{Z}^{*}) \);
- \( M^{\Delta}(x, t; z|\sigma_{d}) = I + O(z^{-1}), \quad z \to \infty; \)

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• $M^\sigma(x, t; z|\sigma_d)$ satisfies the following residue conditions at simple poles $z_k \in \mathbb{Z}$ and $z_k^* \in \mathbb{Z}^*$

$$
\begin{align*}
\text{Res } M^\sigma(x, t; z|\sigma_d) &= \lim_{z \to z_k} M^\sigma(x, t; z|\sigma_d) N_k^\Delta, \\
\text{Res } M^\sigma(x, t; z|\sigma_d) &= \lim_{z \to z_k^*} M^\sigma(x, t; z|\sigma_d) \sigma_2(N_k^\Delta)^* \sigma_2,
\end{align*}
$$

where

$$
N_k^\Delta = \begin{cases}
0 & k \notin \Delta, \\
\gamma_k^\Delta & k \in \Delta,
\end{cases}
\gamma_k^\Delta = \begin{cases}
\pm 1 & k \notin \Delta, \\
\frac{e^{-i(s_{11}\delta(z_k))} - e^{-2i\pi(z_k)}}{c_k(s_{11}\delta(z_k))} & k \in \Delta,
\end{cases}
$$

$$
\theta(z_k) = 2z_k^2 + \alpha z - \gamma + \frac{\pi}{t} z.
$$

Since $M^\sigma(x, t; z|\sigma_d)$ is directly transformed from $M(x, t; z|\sigma_d)$, it is obvious that RHP6.5 has unique solution.

6.1.2. Long-time behavior of soliton solutions

When $N = 1$, the scattering data $\sigma_d = \{(z_1 = \xi + i\eta, c_1)\}$, we derive that

$$
\begin{align*}
u(x, t) &= 2\eta \text{sech}(2\eta \Omega(x, t)) e^{-2i(2(\xi-\eta)^2 - \alpha(x) + \gamma(x)) t + \xi(x) + \frac{\pi}{4} + \arg(c_1)} e^{2i \int_{\xi(x)}^{\alpha(x)} \Delta}, \\
v(x, t) &= 2\eta \text{sech}(2\eta \Omega(x, t)) e^{2i(2(\xi-\eta)^2 - \alpha(x) + \gamma(x)) t + \xi(x) + \frac{\pi}{4} + \arg(c_1)} e^{-2i \int_{\xi(x)}^{\alpha(x)} \Delta},
\end{align*}
$$

where $\Omega(x, t) = 4\xi t + \alpha t - x - \frac{1}{2\eta} \log\left(\frac{e^{2ix}}{e^{2\eta}}\right)$. In fact, after the elastic collisions, the $N$-soliton asymptotically separate into $N$ single-soliton solutions as $t \to \infty$, of course, except the non-generic case which we do not discuss here such as two points of scattering data lie on a vertical line.

Define a distance

$$
\mu(I) = \min_{z_k \in \mathbb{Z}: \text{Im}(z_k) \leq \text{Im}(I)} \{\text{Im}(z_k) \text{dist}(\text{Re}z_k, I)\},
$$

and a space-time cone

$$
S(x_1, x_2, v_1, v_2) = \{(x, t), x = x_0 + vt \text{ with } x_0 \in [x_1, x_2] \text{, } v \in [v_1, v_2]\}.
$$

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Proposition 6.6. For given scattering data $\sigma_d^{\Delta_0} = \{(z_k, \hat{c}_k)\}$, for any fixed $x_1, x_2, v_1, v_2 \in \mathbb{R}$ and $x_1 < x_2$, $v_1 < v_2$. Let $I = \left[-\frac{v_2}{2}, -\frac{v_1}{2}\right]$, then $t \to \infty$ and $(x, t) \in S(x_1, x_2, v_1, v_2)$, we have

$$M^{\Delta_0}(z|\sigma_d^{\Delta_0}) = (I + O(e^{-\mu t}))M^{\Delta_0}(z|\hat{\sigma}_d(I)),$$

(6.23)

where $M^{\Delta_0}(z|\hat{\sigma}_d(I))$ is $N(I) = |Z(I)|$-soliton solutions corresponding to scattering data

$$\hat{\sigma}_d(I) = \{(z_k, c_k(I)), z_k \in Z(I)\}, \quad c_k(I) = c_k \prod_{z_j \in Z\setminus Z(I)} \frac{(z_k - z_j)^2}{(z_k - z_j)^2}.$$

(6.24)
From RHP (6.3) if we chose \( \Delta = \Delta_k \), it is easy to check that
\[
||N_k^{\ast}|M_k|| = \begin{cases} o(1) & k \in \mathcal{Z} \cap \mathcal{I}, \\
o(e^{-8t(l^I)}) & k \in \mathcal{Z} \setminus \mathcal{I}, \end{cases} \quad t \to \pm \infty, \quad (6.25)
\]
which infers to the residues with \( z_k \in \mathcal{Z} \setminus \mathcal{I} \) have little contribution to the solution \( M^{\ast}_k \).

Next, for each discrete spectrum point \( z_k \in \mathcal{Z} \setminus \mathcal{I} \), we make a small disk \( D_k \) whose radius is smaller than \( \mu(l) \). Denoting that \( \partial D_k \) is the boundary of \( D_k \). Then, we introduce that
\[
\phi(z) = \begin{cases} I - \frac{N_k^{\ast}(I)}{z - z_k} & z \in D_k, \\
\frac{\sigma(N_k^{\ast}(I)z)}{z - z_k} & z \in D_k^*, \\
I, & \text{elsewhere}, \end{cases} \quad (6.26)
\]
where
\[
\Delta^- (I) = \{ k : \Re(z_k) < -\frac{V_2}{2} \}, \quad \Delta^+ (I) = \{ k : \Re(z_k) > \frac{V_1}{2} \}.
\]
Making the transformation \( \hat{M}_k = M_k \phi(z) \), we can derive that \( \hat{M}_k \) has new jump in each boundary of the disk which can be denoted as \( \hat{V} \), i.e.,
\[
\hat{M}_k = M_k \hat{V}, \quad z \in \partial D_k \cup D_k^*.
\]
By using the estimate (6.25), we have
\[
||\hat{V} - I|| = o(e^{-8t(l^I)}), \quad z \in \partial D_k \cup D_k^*, \quad t \to \pm \infty. \quad (6.28)
\]
It is obvious that \( \hat{M}_k (z|\sigma_d) \) and \( M_k (z|\hat{\sigma}_d) \) the same poles and residue conditions. Thus,
\[
e(z) = \hat{M}_k (z|\sigma_d)[M_k (z|\hat{\sigma}_d)]^{-1} \quad (6.29)
\]
has no poles. However, there exists jump for \( z \in \cup_{z \in \mathcal{Z} \cap \mathcal{I}}(\partial D_k \cup \partial D_k^*) \),
\[
e_+ (z) = e_- (z) V_e,
\]
where \( V_e = M_k \hat{V} (\hat{\sigma}_d) \) which implies that
\[
||V_e - I|| = o(e^{-8t(l^I)}), \quad t \to \pm \infty. \quad (6.30)
\]
Then, using the theory of small-norm Riemann-Hilbert problems, one can easily derive that
\[
e(z) = I + o(e^{-8t(l^I)}), \quad t \to \pm \infty,
\]
which together with \( \hat{M}_k (z) = \hat{M}_k (z) \phi(z) \) gives the formula (6.23).
6.2. Local solvable model near phase point \( z = z_0 = -\frac{1}{2}(\rho + \alpha) \)

From (6.2), we can easily find that \( V^{(2)} - I \) does not have a uniform estimate for large time. Therefore, we use a local model \( M^{(\text{out})}(z)M^{(\text{pc})}(z_0, r_0) \) to match the jumps of \( M_{\text{RHP}} \) on \( \Sigma^{(2)} \cap \mathcal{U}_{z_0} \) to make the jump uniformly for the function \( E(z) \). Recall the definition of \( \theta(z) \) (3.1) and introduce the transformation

\[
\lambda = \lambda(z) = 2\sqrt{2t}(z - z_0). \tag{6.31}
\]

Then, we can derive that

\[
2t\theta = \frac{1}{2}t^2 - 4r_0^2 - 2yt, \tag{6.32}
\]

which means \( \mathcal{U}_{z_0} \) can be mapped into an expanding neighborhood of \( \lambda = 0 \). If we let

\[
r_0(z_0) = r(z_0)T_0(z_0)^{-2}e^{2i\nu(z_0)\log(2\sqrt{2t})}e^{-4r_0^2 - 2yt} \tag{6.33}
\]

and consider the fact that \( 1 - \chi z = 1 \) as \( z \in \mathcal{U}_{z_0} \), the jump of \( M_{\text{RHP}} \) in \( \mathcal{U}_{z_0} \) can be expressed as

\[
V^{(2)}(z) \big|_{z \in \mathcal{U}_{z_0}} = \begin{cases} 
\begin{pmatrix}
\lambda(z)^{\nu\rho z} e^{\frac{\nu z^2}{2}} \\
\lambda(z)^{\nu\rho z} e^{\frac{\nu z^2}{2}}
\end{pmatrix} & \begin{pmatrix}
1 & 0 \\
r_0(z_0) & 1
\end{pmatrix}, \quad z \in \Sigma_1, \\
\begin{pmatrix}
\lambda(z)^{\nu\rho z} e^{\frac{\nu z^2}{2}} \\
\lambda(z)^{\nu\rho z} e^{\frac{\nu z^2}{2}}
\end{pmatrix} & \begin{pmatrix}
1 & 0 \\
\frac{r_0^2(z_0)}{1 + \nu z_0^2} & 1
\end{pmatrix}, \quad z \in \Sigma_2, \\
\begin{pmatrix}
\lambda(z)^{\nu\rho z} e^{\frac{\nu z^2}{2}} \\
\lambda(z)^{\nu\rho z} e^{\frac{\nu z^2}{2}}
\end{pmatrix} & \begin{pmatrix}
1 & 0 \\
\frac{r_0^2(z_0)}{1 + \nu z_0^2} & 1
\end{pmatrix}, \quad z \in \Sigma_3, \\
\begin{pmatrix}
\lambda(z)^{\nu\rho z} e^{\frac{\nu z^2}{2}} \\
\lambda(z)^{\nu\rho z} e^{\frac{\nu z^2}{2}}
\end{pmatrix} & \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad z \in \Sigma_4.
\end{cases} \tag{6.34}
\]

Obviously, the above jump \( V^{(2)}(z) \big|_{z \in \mathcal{U}_{z_0}} \) is equivalent to the jump of the parabolic cylinder model problem (3.5) which we have obtain the solutions in appendix A. Then, since \( M^{(\text{out})}(z) \) an analytic and bounded function in the \( \mathcal{U}_{z_0}, M^{(\text{out})}(z)M^{(\text{pc})}(z_0, r_0) \) which has been defined in (6.1) admits the jump \( V^{(2)}(z) \) of \( M_{\text{RHP}}(z) \).

6.3. The small-norm RHP for \( E(z) \)

Recall the transform (6.1), we can derive that

\[
E(z) = \begin{cases} 
M_{\text{RHP}}(z)M^{(\text{out})}(z)^{-1}, \quad & z \in \mathbb{C} \setminus \mathcal{U}_{z_0}, \\
M_{\text{RHP}}(z)M^{(\text{pc})}(z_0, r_0)^{-1}M^{(\text{out})}(z)^{-1}, \quad & z \in \mathcal{U}_{z_0}.
\end{cases} \tag{6.35}
\]
which is analytic in $\mathbb{C} \setminus \Sigma^{(E)}$ where

$$\Sigma^{(E)} = \partial \mathcal{U}_0 \cup (\Sigma^{(2)} \setminus \mathcal{U}_0), \quad (6.36)$$

with clockwise direction for $\partial \mathcal{U}_0$. Then we can check that $E(z)$ satisfies the Riemann-Hilbert problem.

**Riemann-Hilbert Problem 6.7.** Find a matrix-valued function $E(z)$ such that

- $E$ is analytical in $\mathbb{C} \setminus \Sigma^{(E)}$;

- $E(z) = I + O(z^{-1})$, $z \to \infty$;

- $E_+(z) = E_-(z)V^{(E)(z)}$, $z \in \Sigma^{(E)}$, where

$$V^{(E)}(z) = \begin{cases} M^{(\text{out})}(z)V^{(2)}(z)M^{(\text{out})}(z)^{-1}, & z \in \Sigma^{(2)} \setminus \mathcal{U}_0, \\ M^{(\text{out})}(z)M^{(\text{pc})}(\xi, r_0)M^{(\text{out})}(z)^{-1}, & z \in \partial \mathcal{U}_0. \end{cases} \quad (6.37)$$

![Figure 6. Jump contour $\Sigma^{(E)} = \partial \mathcal{U}_0 \cup (\Sigma^{(2)} \setminus \mathcal{U}_0)$.](image)

Base on the estimate $V^{(2)} - I$ shown in (6.2) and the boundedness of $M^{(\text{out})}$, it is easy to check that

$$|V^{(E)}(z) - I| = \begin{cases} O(e^{-2|z_0|^2}), & z \in \Sigma^{(2)} \setminus \mathcal{U}_0, \\ O(r^{-1/2}), & z \in \partial \mathcal{U}_0. \end{cases} \quad (6.38)$$

then, we can derive that

$$\|(z - z_0)^k(V^{(E)} - I)\|_{L^p(\Sigma^{(E)})} = o(r^{-1/2}), \quad p \in [1, +\infty], \quad k \geq 0. \quad (6.39)$$
As a small-norm Riemann-Hilbert problem, the existence and uniqueness of RHP are guaranteed and we obtain that

$$E(z) = I + \frac{1}{2\pi i} \int_{\Sigma_E} \frac{\mu_E(s)(V^E(s) - I)}{s - z} ds$$  \hspace{1cm} (6.40)

where $\mu_E \in L^2(\Sigma(E))$ and satisfies

$$(1 - C_{\omega E})\mu_E = I,$$  \hspace{1cm} (6.41)

where $C_{\omega E}$ is an integral operator which is defined by

$$C_{\omega E} f = C_-(f(V^E - I)),$$

$$C_-(f(z) \overset{z \to \Sigma(E)}{\lim} \int_{\Sigma_E} \frac{f(s)}{s - z} ds,$$

where $C_-$ is the Cauchy projection operator. Then, according to the properties of the Cauchy projection operator $C_-$, and the estimate of $V^E - I$ shown in (6.38), we can derive that

$$\|C_{\omega E}\|_{L^2(\Sigma(E))} \leq \|C_-(\|L^2(\Sigma(E))\|V^E - I\|_{L^2(\Sigma(E))} \leq O(t^{-1/2}).$$  \hspace{1cm} (6.42)

which implies that $1 - C_{\omega E}$ is invertible and thus the existence and uniqueness of $\mu_E$ are guaranteed, and thus of $E(z)$. Now, it can be explained that the definition of $M_{RHP}$ is reasonable, and in turn we can solve (5.1) to the unknown $M^{(3)}$ which admits the pure $\bar{\partial}$-Problem (5.2).

Then, to reconstruct the solutions of $u(x, t)$ and $v(x, t)$, the asymptotic behavior of $E(z)$ as $z \to \infty$ is needed. According to the expression (6.40) of $E(z)$, we obtain that

$$E(z) = I + z^{-1}E_1 + O(z^{-2}), \quad z \to \infty,$$  \hspace{1cm} (6.43)

where

$$E_1(x, t) = -\frac{1}{2it} \int_{\Sigma_E} \mu_E(s)(V^E(s) - I) ds.$$  \hspace{1cm} (6.44)

Then, by applying (6.41) and the estimate (6.38), (6.39), we obtain that

$$E_1(x, t) = -\frac{1}{2it} \int_{\partial U_0} (V^E(s) - I) ds + o(t^{-1}).$$  \hspace{1cm} (6.45)

Finally, applying (6.37), (6.31) and (A.6) shown in Appendix A, we derive that

$$E_1(x, t) = \frac{1}{2t \sqrt{2t}} M^{(out)}(z_0) M^{(pc)}(z_0) M^{(out)}(z_0)^{-1} + O(t^{-1})$$

$$= \frac{1}{2t \sqrt{2t}} M^{(out)}(z_0) \begin{pmatrix} 0 & \beta_{12} \\ -\beta_{21} & 0 \end{pmatrix} M^{(out)}(z_0)^{-1} + O(t^{-1}),$$  \hspace{1cm} (6.46)
with
\[ \beta_{12}(z_0) = \beta_{21}(z_0) = \tau(z_0, +) e^{ix^2/(4t)+i\alpha^2/4-2\alpha x-i\nu(z_0) \log |8t|}. \]  
(6.47)

Here, \(|\tau(z_0, +)|^2 = |\nu(z_0)|^2\) and
\[ \arg \tau(z_0, +) = \frac{\pi}{4} + \arg \Gamma(i\nu(z_0)) - \arg r(z_0) - 2 \int_{-\infty}^{z_0} \ln |z-z_0|d\nu(s). \]

7. Pure \(\bar{\partial}\)-Problem

In this section, we analysis the remaining \(\bar{\partial}\)-Problem. The \(\bar{\partial}\)-RH problem \([5,2]\) for \(M^{(3)}(z)\) is equivalent to the following integral equation
\[ M^{(3)}(z) = I - \frac{1}{\pi} \int_{C} \frac{M^{(3)}W^{(3)}}{s-z} dA(s), \]  
(7.1)

where \(dA(s)\) is Lebesgue measure. Further, we write the equation \((7.1)\) in operator form
\[ (I - S)M^{(3)}(z) = 1, \]  
(7.2)

where \(S\) is Cauchy operator
\[ S[f](z) = -\frac{1}{\pi} \int_{C} \frac{f(s)W^{(3)}(s)}{s-z} dA(s). \]  
(7.3)

We need to prove that the inverse operator \((I - S)^{-1}\) exists, so that the solution \(M^{(3)}(z)\) exists.

**Lemma 7.1.** For large \(t\), there exists a constant \(c\) that makes the operator \((7.3)\) admits the following relation
\[ \|S\|_{L^\infty \to L^\infty} \leq ct^{-1/4}. \]  
(7.4)

**Proof.** We mainly prove the case that the matrix function supported in the region \(\Omega_1\), the other case can be proved similarly. Denoted that \(f \in L^p(\Omega_1)\) and \(s = p + iq\), then based on \((4.11)\) and \((5.2)\), we can obtain the following inequality
\[ |S[f](z)| \leq \frac{1}{\pi} \int_{\Omega_1} \frac{|f(s)M_{RHP}(s)\tilde{\partial}R_1(s)M_{RHP}^{-1}(s)|}{|s-z|} df(s) \]  
(7.5)
\[ \leq \|f\|_{L^p(\Omega_1)} \|M_{RHP}\|_{L^\infty(\hat{\Omega}_1)} \|M_{RHP}^{-1}\|_{L^\infty(\hat{\Omega}_1)} \frac{1}{\pi} \int_{\Omega_1} \frac{\|\tilde{\partial}R_1(s)\|e^{-8q(p-z_0)}}{|s-z|} df(s). \]  
(7.6)
where $\hat{\Omega}_1 \triangleq \Omega_1 \cap \text{supp}(1 - \chi_Z)$ is bounded far from the scattering data $z_k$ of $M_{RHP}$, and thus $\|M_{RHP}^{11}\|_{L^\infty(\hat{\Omega}_1)}$ are finite.

Based on (4.6) and the estimates shown in Appendix B, from (7.5), we obtain that
\[
\|S\|_{L^\infty \to L^\infty} \leq c(I_1 + I_2 + I_3) \leq ct^{-1/4},
\] (7.7)
where
\[
I_1 = \int_{\hat{\Omega}_1} \frac{|\partial_x z(s)| |e^{-8q(p-z_0)}|}{|s-z|} df(s), \quad I_2 = \int_{\hat{\Omega}_1} \frac{|r'(p)| |e^{-8q(p-z_0)}|}{|s-z|} df(s),
\] (7.8)
and
\[
I_3 = \int_{\hat{\Omega}_1} \frac{|s-z_0|^{1/2} |e^{-8q(p-z_0)}|}{|s-z|} df(s).
\] (7.9)

Next, our purpose is to recover the large time asymptotic behaviors of $u(x,t)$ and $v(x,t)$. According to (2.25), we need determine the coefficient of the $z^{-1}$ term of $M^{(3)}$ in the Laurent expansion at infinity. Based on (7.1), we can derive that
\[
M^{(3)}(z) = 1 - \frac{1}{\pi} \int \int_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(3)}{s-z} dA(s) = 1 + \frac{M^{(1)}_1}{z} + o(z^{-2}),
\]
where
\[
M^{(1)}_1 = \frac{1}{\pi} \int \int_{\mathbb{C}} M^{(3)}(s)W^{(3)}(3) dA(s).
\]
In what follows, the the coefficient of the $z^{-1}$ term of $M^{(3)}$ in the Laurent expansion at infinity, i.e., $M^{(3)}_1$ satisfies the following property.

**Lemma 7.2.** For large $t$, there exists a constant $c$ that makes $M^{(3)}_1$ admits the following inequality
\[
M^{(3)}_1 \leq ct^{-3/4}.
\] (7.10)

The proof of this Lemma is similar to the process that shown in Appendix B.

**8. Soliton resolution for the cgNLS equation**

Now, we are going to construct the long time asymptotic of the cgNLS equation 1.1. Recall a series of transformation process including (3.8), (4.4), (5.1) and (6.1), i.e.,
\[
M(z) \iff M^{(1)}_1(z) \iff M^{(2)}_1(z) \iff M^{(3)}_1(z) \iff E(z),
\]
we then obtain

\[ M(z) = M^{(3)}(z)E(z)M^{(out)}(z)R^{(2)}(z)F^{\sigma_3}(z), \quad z \in \mathbb{C} \setminus \mathcal{U}_{\Omega}. \]

In order to recover the solution \( u(x, t) \) and \( v(x, t) \), we take \( z \to \infty \) vertically which implies \( z \in \Omega_2 \) or \( z \in \Omega_3 \), thus \( R^{(2)}(z) = I \). Then, we have

\[ M = \left( I + \frac{M^{(3)}}{z} + \cdots \right) \left( I + \frac{E_1}{z} + \cdots \right) \left( I + \frac{M^{(out)}}{z} + \cdots \right) \left( I + \frac{T_1\sigma_3}{z} + \cdots \right), \]

from which we can derive that

\[ M_1 = M^{(out)}_1 + E_1 + M^{(3)}_1 + T_1\sigma_3. \]

Then, according to the reconstruction formula (2.25) and the Lemma 7.2, we obtain that

\[
\begin{align*}
  u(x, t)e^{-2iJ^{(s)}_{k,\sigma_d}(z)} & = 2i(M^{(out)}_1)_{12} + 2i(E_1)_{12} + O(r^{-3/4}), \quad (8.1) \\
  v(x, t)e^{-2iJ^{(s)}_{k,\sigma_d}(z)} & = 2i(M^{(out)}_1)_{21} + 2i(E_1)_{21} + O(r^{-3/4}). \quad (8.2)
\end{align*}
\]

Based on the Corollary 6.4 and the formulae (6.46), (6.47), we obtain the soliton resolution

\[
\begin{align*}
  u(x, t)e^{-2iJ^{(s)}_{k,\sigma_d}(z)} & = u_{sol}(x, t; \sigma_d^{out})e^{-2iJ^{(s)}_{k,\sigma_d}(z)} + t^{-\frac{i}{2}f_1^+} + O(t^{-\frac{1}{2}}), \\
  v(x, t)e^{-2iJ^{(s)}_{k,\sigma_d}(z)} & = v_{sol}(x, t; \sigma_d^{out})e^{-2iJ^{(s)}_{k,\sigma_d}(z)} + t^{\frac{i}{2}f_2^+} + O(t^{\frac{1}{2}}),
\end{align*}
\]

where

\[
\begin{align*}
  f_1^+ & = \frac{1}{\sqrt{2}}m_{11}^2\tau(z_0, +)e^{\theta_1(x, t)} + \frac{1}{2\sqrt{2}}m_{12}^2\tau^*(z_0, +)e^{\theta_2(x, t)}, \\
  f_2^+ & = -\frac{1}{\sqrt{2}}m_{22}^2\tau^*(z_0, +)e^{\theta_1(x, t)} - \frac{1}{2\sqrt{2}}m_{21}^2\tau(z_0, +)e^{\theta_2(x, t)},
\end{align*}
\]

with \( z_0 = -\frac{i}{4}(\tau - \sigma) \),

\[
\begin{align*}
  \theta_1(x, t) & = \frac{ix^2}{4t} + \frac{\alpha^2 t}{4} - 2\alpha x - iv(z_0) \log |\text{det}l|, \\
  \theta_2(x, t) & = -\frac{ix^2}{4t} - \frac{\alpha^2 t}{4} - 2\alpha x + iv(z_0) \log |\text{det}l|
\end{align*}
\]

and \( m_{11}, m_{12}, m_{21} \) and \( m_{22} \) come from the elements of the matrix solution of RHP 6.2.

Via using Proposition 6.6, there exists exponential errors which are absorbed into the \( O(r^{-3/4}) \) term when we replace \( u_{sol}(x, t; \sigma_d^{out}) \) and \( v_{sol}(x, t; \sigma_d^{out}) \) with \( u_{sol}(x, t; \hat{\sigma}_d(I)) \) and
Finally, we can give the more general solution of $u(x,t)$ and $v(x,t)$ and summarise as the following theorem.

**Theorem 8.1.** With the initial value conditions that $u_0(x), v_0(x) \in H^{1,1}(\mathbb{R})$ and the assumption that the initial values $u_0(x), v_0(x)$ satisfy the Assumption (2.1), let $u(x,t)$ and $v(x,t)$ be the solution of cgNLS equations (1.1). The scattering data denoted as $\{r, \{z_k, c_k\}_{k=1}^{N}\}$ which generated from the initial values $u_0(x), v_0(x)$. For fixed $x_1, x_2, v_1, v_2 \in \mathbb{R}$ with $x_1 < x_2, v_1 < v_2$, and $I = \left[-\frac{v_2}{2}, -\frac{v_1}{2}\right]$, $z_0 = -\frac{1}{4}(\frac{t}{\alpha} + \alpha)$, then as $t \to \infty$ and $(x, t) \in S(x_1, x_2, v_1, v_2)$ which is defined in (6.22), the solution $u(x,t)$ and $v(x,t)$ can be expressed as

$$\begin{align*}
u(x,t) &= (\nu_{sol}(x,t; \hat{\sigma}_d(I)) + i^{-\frac{1}{2}} f_1^{\pm}) e^{2\beta \sum_{k=1}^{N} M(k,t) dt} + O(t^{-\frac{5}{2}}),
\end{align*}$$

$$\begin{align*}
u(x,t) &= (\nu_{sol}(x,t; \hat{\sigma}_d(I)) + i^{-\frac{1}{2}} f_2^{\pm}) e^{-2\beta \sum_{k=1}^{N} M(k,t) dt} + O(t^{-\frac{5}{2}}).
\end{align*}$$

$u(x,t) = (u_{sol}(x,t; \hat{\sigma}_d(I)) + i^{-\frac{1}{2}} f_1^{\pm}) e^{2\beta \sum_{k=1}^{N} M(k,t) dt} + O(t^{-\frac{5}{2}}),

$$\begin{align*}
\end{align*}$$

Finally, we can give the more general solution of $u(x,t)$ and $v(x,t)$ and summarise as the following theorem.
Remark 8.2. The results shown in Theorem 8.1 need the condition that the initial values $u_0(x), v_0(x) \in H^{1,1}(\mathbb{R})$, then the inverse scattering transform possesses well mapping properties, and it guarantees the condition that there exists no discrete spectrum on the real axis. It is noted that the asymptotic results only depend on the $H^1(\mathbb{R})$ norm of $r$, therefore, for any $u_0(x), v_0(x) \in H^{1,1}(\mathbb{R})$ admitting the Assumption (2.1), the process of the large-time analysis and calculations shown in this work is unchanged.

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Appendix A: The parabolic cylinder model problem

Here, we describe the solution of parabolic cylinder model problem [36, 37]. Define the contour $\Sigma^{pc} = \bigcup_{j=1}^{4} \Sigma_j^{pc}$ where

$$\Sigma_j^{pc} = \left\{ \lambda \in \mathbb{C} | \text{arg} \lambda = \frac{2j-1}{4} \pi \right\}.$$  \hfill (A.1)

For $r_0 \in \mathbb{C}$, let $v(r) = -\frac{1}{2\pi} \log(1+|r_0|^2)$, consider the following parabolic cylinder model Riemann-Hilbert problem.

Riemann-Hilbert Problem 8.3. Find a matrix-valued function $M^{(pc)}(\lambda)$ such that

- $M^{(pc)}(\lambda)$ is analytic in $\mathbb{C} \setminus \Sigma^{pc}$, \hfill (A.2)
- $M_+^{(pc)}(\lambda) = M_+^{(pc)}(\lambda) V^{(pc)}(\lambda)$, $\lambda \in \Sigma^{pc}$, \hfill (A.3)
- $M^{(pc)}(\lambda) = I + \frac{M_1}{\lambda} + O(\lambda^2)$, $\lambda \rightarrow \infty$. \hfill (A.4)
where

\[ V^{(pc)}(\lambda) = \begin{cases} 
A^{iv\sigma_3 e^{-\frac{\lambda^2}{4}}} \begin{pmatrix} 1 & 0 \\ r_0 & 1 \end{pmatrix}, & \lambda \in \Sigma_{1}^{pc}, \\
A^{iv\sigma_3 e^{-\frac{\lambda^2}{4}}} \begin{pmatrix} 1 & \frac{r_0^*}{1+|r_0|^2} \\ 0 & 1 \end{pmatrix}, & \lambda \in \Sigma_{2}^{pc}, \\
A^{iv\sigma_3 e^{-\frac{\lambda^2}{4}}} \begin{pmatrix} 1 & 0 \\ \frac{r_0}{1+|r_0|^2} & 1 \end{pmatrix}, & \lambda \in \Sigma_{3}^{pc}, \\
A^{iv\sigma_3 e^{-\frac{\lambda^2}{4}}} \begin{pmatrix} 1 & r_0^* \\ 0 & 1 \end{pmatrix}, & \lambda \in \Sigma_{4}^{pc}.
\end{cases} \] (A.5)

Figure 7. Jump matrix \( V^{(pc)} \).

We know that the parabolic cylinder equation can be expressed as \[38\]

\[ \left( \frac{\partial^2}{\partial z^2} + \left( \frac{1}{2} - \frac{z^2}{2} + a \right) \right) D_z = 0. \]

As shown in the literature \[20, 39\], we obtain the explicit solution \( M^{(pc)}(\lambda, r_0) \):

\[ M^{(pc)}(\lambda, r_0) = \Phi(\lambda, r_0) P(\lambda, r_0) e^{iv\sigma_3 e^{-\frac{\lambda^2}{4}}} \lambda^{-iv\sigma_3}. \]
where

\[ \mathcal{P}(\lambda, r_0) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
-r_0 & 1 
\end{pmatrix}, & \lambda \in \Omega_1, \\
\begin{pmatrix} 1 & -r_0^* \\
0 & 1 
\end{pmatrix}, & \lambda \in \Omega_3, \\
\begin{pmatrix} 1 & 0 \\
-\frac{r_0}{1+|r_0|^2} & 1 
\end{pmatrix}, & \lambda \in \Omega_4, \\
\begin{pmatrix} 1 & r_0^* \\
0 & 1 
\end{pmatrix}, & \lambda \in \Omega_6, \\
I, & \lambda \in \Omega_2 \cup \Omega_5,
\end{cases} \]

and

\[ \Phi(\lambda, r_0) = \begin{cases} 
\begin{pmatrix} e^{-\frac{i\pi}{4}}D_{i\nu}(e^{-\frac{i\pi}{4}}\lambda) & -i\beta_{12}e^{-\frac{i\pi}{2}(\nu-1)}D_{-i\nu-1}(e^{-\frac{i\pi}{4}}\lambda) \\
\beta_{21}e^{-\frac{i\pi}{4}}D_{i\nu-1}(e^{-\frac{i\pi}{4}}\lambda) & e^{\frac{i\pi}{2}}D_{-i\nu}(e^{-\frac{i\pi}{4}}\lambda) 
\end{pmatrix}, & \lambda \in \mathbb{C}^+, \\
\begin{pmatrix} e^{\frac{i\pi}{4}}D_{i\nu}(e^{\frac{i\pi}{4}}\lambda) & -i\beta_{12}e^{-\frac{i\pi}{2}(\nu-1)}D_{-i\nu-1}(e^{\frac{i\pi}{4}}\lambda) \\
\beta_{21}e^{\frac{i\pi}{4}}D_{i\nu-1}(e^{\frac{i\pi}{4}}\lambda) & e^{-\frac{i\pi}{2}}D_{-i\nu}(e^{\frac{i\pi}{4}}\lambda) 
\end{pmatrix}, & \lambda \in \mathbb{C}^-,
\end{cases} \]

with

\[ \beta_{12} = \frac{\sqrt{2\pi e^{-i\pi/4}e^{-\pi/2}}}{r_0 \Gamma(-i\nu)}, \quad \beta_{21} = \frac{-\sqrt{2\pi e^{-i\pi/4}e^{-\pi/2}}}{r_0 \Gamma(i\nu)} = \frac{\nu}{\beta_{12}}. \]

Then, it is not hard to obtain the asymptotic behavior of the solution by using the well known asymptotic behavior of \( D_\nu(z) \),

\[ M^{(pc)}_1(r_0, \lambda) = I + \frac{M^{(pc)}_1}{\lambda} + O(\lambda^{-2}), \quad \text{(A.6)} \]

where

\[ M^{(pc)}_1 = \begin{pmatrix} 0 & \beta_{12} \\
-\beta_{21} & 0 
\end{pmatrix}. \]

**Appendix B: Detailed calculations for the pure \( \bar{\eta} \)-Problem**

**Proposition 8.4.** For \( t > 0 \), there exists constants \( c_j \) (\( j = 1, 2, 3 \)) such that \( I_j(j = 1, 2, 3) \) which defined in (7.8) and (7.9) possess the following estimate

\[ I_j \leq c_j t^{-4}, \quad j = 1, 2, 3. \quad \text{(B.1)} \]

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Proof. Let $s = p + iq$ and $z = \xi + i\eta$. Using the fact that

$$\left\| \frac{1}{s - z} \right\|_{L^2} (q + z_0) = \left( \int_{q+z_0}^{+\infty} \frac{1}{|s - z|^2} dp \right)^{1/2} \leq \frac{\pi}{q - \eta},$$

we can derive that

$$|I_1| \leq \int_0^{+\infty} \int_{q+z_0}^{+\infty} \frac{|\partial_X f(z(s))| e^{-8\eta q (p-z_0)}}{|s - z|} dp dq \leq \int_0^{+\infty} e^{-8\eta q} \left\| \partial_X f(z(s)) \right\|_{L^2} (q + z_0) dq \left\| \frac{1}{s - z} \right\|_{L^2} (q + z_0) dq \quad (B.2)$$

Similarly, considering that $r \in H^{1,1} (\mathbb{R})$, we obtain the estimate

$$|I_2| \leq \int_0^{+\infty} \int_{q+z_0}^{+\infty} \frac{|r'(p)| e^{-8\eta q}}{|s - z|} dp dq \leq c_1 r^{1/4}.$$

To obtain the estimate of $I_3$, we consider the following $L^2 (k > 2)$ norm

$$\left\| \frac{1}{\sqrt{|s - z_0|}} \right\|_{L^2} \leq \left( \int_{q+z_0}^{+\infty} \frac{1}{|p - z_0 + iq|^2} dp \right)^{1/2} \leq c q^{1/2} \quad (B.4)$$

Similarly, we can derive that

$$\left\| \frac{1}{|s - z|} \right\|_{L^2} \leq c |q - \eta|^{1/2}. \quad (B.5)$$

By applying (B.4) and (B.5), it is not hard to check that

$$|I_3| \leq \int_0^{+\infty} \int_{q}^{+\infty} \frac{|z - z_0|^{1/2} e^{-8\eta q (p-z_0)}}{|s - z|} dp dq \left\| \frac{1}{\sqrt{|s - z_0|}} \right\|_{L^2} \left\| \frac{1}{|s - z|} \right\|_{L^2} dq \leq c_3 r^{1/4}. \quad (B.6)$$

Now, we complete the estimates of $I_j (j = 1, 2, 3)$.

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