Nonparametric Double EWMA Control Charts based on Mood Statistic for Process Variability

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Abstract. This research aims to propose a nonparametric double exponentially weighted moving average (NDEWMA) control chart for detecting shifts in a process using the Mood statistic. A Monte Carlo simulation study was used. The data had a normal distribution and a non-normal distribution where the magnitude of shift size \( \delta \) was \( \delta = 1, 1.05, 1.10 - 2.00 \) and 3.00, respectively. A performance comparison of the control charts was evaluated by using the Average Run Length (ARL). In the case of an in-control process, a large ARL value is desired, while a small ARL value is desired when the process is out-of-control. From the results of the simulation study, it was observed that the proposed NDEWMA control chart is effective in detecting small shifts in the process and gives better performance compared to the nonparametric exponentially weighted moving average control (NEWMA) control chart based on Mood statistics.

1. Introduction
The main intention of Statistical Process Control (SPC) is a proven technique for cultivating productivity. Control charts are one of the efficient tools of SPC for detecting changes in means or variations of the process. Control charts are applied successfully in engineering, public health, economics, and finance as well as other areas of application. SPC charts such as the Shewhart control chart, which was first introduced by Walter A. Shewhart in 1924 [1], are widely used and powerful tools for monitoring and detecting variation in the process. It is sensitive to detecting relatively large shifts in the process mean (\( \geq 1.5\sigma \)). The control chart, namely the Exponentially Weighted Moving Average (EWMA) control chart, is primarily used to detect smaller shifts. Roberts [2] originally developed the EWMA control chart. The EWMA control chart is used in statistical process control to monitor the output of manufacturing processes by tracking the moving average of performance over the lifetime of the process. It has been used in various industries, especially the chemical industry.

Many researchers have been attempting to introduce various control charts that can detect small to moderate changes in a manufacturing process. Shama and Shama [3] proposed a double exponentially weighted moving average (DEWMA) control chart. Further, the DEWMA control charts are more sensitive for detecting small to moderate shifts in the process compared to the Shewhart control chart. In 2013, Alkahtani [4] compared the robustness and performance of the EWMA and DEWMA control charts in the case of normal and non-normal distributions. The results found that the DEWMA control chart was more effective for detecting shifts in the process than the EWMA control chart. Generally, control charts are designed to detect processes that have normal distribution or known distribution. In practice, however, the data have non-normal distribution or unknown distribution.
The classification of control charts including the Shewhart, the CUSUM and the EWMA control charts in the case of distribution-free are presented by Bakir [5]. The EWMA control charts are robust for detecting mean and variation in non-normal distribution [6]. Nandini [7] compared the performance of a control chart for monitoring changes in the location parameters of the process of unknown distribution. Three nonparametric estimators were compared, as follows: the Hodges-Lehmann (HL), the sign and the Mann-Whitney (MW) statistics. The results showed that all three methods worked well with symmetrically distributed data. Hidetoshi and Takashi [8] studied a nonparametric control chart based on the Mood statistics for detecting the variation of the process. In 2014, Zombade and Ghute [9] proposed the nonparametric CUSUM control chart for detecting process changes using Mood and Sukhatme statistics. They studied the cases of data with normal distribution, double exponential distribution and uniform distribution. The study found that the Mood statistic on the CUSUM control chart (NPCSM-M) and the Sukhatme statistics on CUSUM control chart (NPCSM-S) were more effective at detecting process change than the Mood statistics on the Shewhart control chart (NP-M) and Sukhatme statistics (NP-S) on the Shewhart control chart. Additionally, the Mood and Sukhatme statistics on the CUSUM control chart were found to be effective when the data was light tailed. Recently, Petcharar [10] proposed the EWMA control chart using Mood and Sukhatme statistics for monitoring process variation in logistic and uniform distribution. The results found that the EWMA control chart based on Mood statistics was more effective for detecting small changes in process variation, while the CUSUM control chart based on Mood statistics was more effective for detecting moderate and large changes in process variation.

A common characteristic used for comparing the performance of control charts is Average Run Length (ARL); the expected number of observations taken from an in-control process until the control chart falsely signals out-of-control is denoted by $\text{ARL}_0$. An ARL will be regarded as acceptable if it is large enough to keep the level of false alarms at an acceptable level. A second common characteristic is the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control, denoted by $\text{ARL}_1$.

The objective of this paper is to develop a nonparametric double exponentially weighted moving average (DEWMA) control chart for monitoring process dispersion using Mood statistics. The rest of the paper is organized as follows: the double exponentially weighted moving average control chart is presented in Section 2. The nonparametric Mood statistics used in the study are described in Section 3. Section 4 gives the design structure of the nonparametric exponentially weighted moving average (NEWMA) control chart. The nonparametric double exponentially weighted moving average (NDEWMA) control chart based on Mood statistics is proposed in Section 5. The performance evaluation of the proposed nonparametric double exponentially weighted moving average (NDEWMA) control charts for dispersion with respect to the average run length (ARL) values of different shift levels are discussed in Section 6. Finally, Section 5 provides the conclusion.

2. Double Exponentially Weighted Moving Average Control Chart

Shama and Shama [3] proposed a double exponentially weighted moving average control chart, namely DEWMA control chart, for monitoring the small to moderate changes in process dispersion. Suppose that, $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ at time $i$ from normal and non-normal distributions.

The DEWMA statistic at time $i$ denoted by $DEWMA_i$ is defined as follows:

$$EWM_A = \lambda X_i + (1 - \lambda)EWM_{i-1},$$

$$DEWMA_i = \lambda EWM_i + (1 - \lambda)DEWMA_{i-1},$$

where $\lambda$ is an exponential smoothing parameter, $0 < \lambda < 1$

$EWM_A$ is the initial value of EWMA statistic

$DEWMA_A$ is the initial value of DEWMA statistic.

The mean for double exponentially weighted moving average, $E(DEWMA)$ and the variance for double exponentially weighted moving average, $\text{Var}(DEWMA)$ are given as follows (Shama and Shama, [3])
\[ E(\text{DEWMA}) = \mu_0, \]  
\[ \text{Var}(\text{DEWMA}) = \lambda^4((1 + (1 - \lambda)^2 - i^2 + 2i + 1)(1 - \lambda)^2i + (2i^2 + 2i + 1)(1 - \lambda)^{2i+2} - i(1 - \lambda)^{2i+4}) / (1 - (1 - \lambda)^2)^3)\sigma_0^2. \]  

The Upper Control Limit of DEWMA control chart is defined by the following recursion:

\[ UCL_{\text{DEWMA}} = \mu_0 + L_1\sqrt{\text{Var}(\text{DEWMA})}. \]  

The Lower Control Limit of DEWMA control chart is defined by the following recursion:

\[ LCL_{\text{DEWMA}} = \mu_0 - L_1\sqrt{\text{Var}(\text{DEWMA})}. \]  

For the large sample cases, the control limits become:

\[ UCL_{\text{DEWMA}} = \mu_0 + L_1\sigma \sqrt{\lambda(2 - 2\lambda + \lambda^2) / (2 - \lambda^3)}, \]  
\[ LCL_{\text{DEWMA}} = \mu_0 - L_1\sigma \sqrt{\lambda(2 - 2\lambda + \lambda^2) / (2 - \lambda^3)}, \]

where \( \lambda \) is an exponential smoothing parameter, \( 0 < \lambda < 1 \)

\( \text{L}_1 \) is the width of the control limit of control chart.

3. Nonparametric Mood Statistic

Mood Statistics are utilized for charting. Suppose that \( X \) is an in-control process with sample size \( m \), denote by \( X = (X_1, ..., X_m) \). Suppose that \( Y = (Y_1, ..., Y_n) \) denotes an arbitrary test sample of size \( n \). Let \( R_1 < R_2 < \ldots < R_m \) be the combined-sample ranks of the \( X \)-value in increasing order of magnitude. Therefore, the Mood statistics are follows:

\[ M = \sum_{i=1}^{m} \left( R - \frac{N + 1}{2} \right)^2, \]  

where \( N = m + n \).

The mean for Mood statistic, \( E(M) \) and the variance for Mood statistic, \( \text{Var}(M) \) are given as follows (Hidetoshi and Takashi, [8]):

\[ E(M) = \frac{m(N^2 + 1)}{12}, \]  
\[ \text{Var}(M) = \frac{mn(N + 1)(N^2 - 4)}{180}. \]  

For the large sample sizes, the \( M \) statistic performs on the standard normal distribution, defined as follows:

\[ W = \frac{M - E(M)}{\sqrt{\text{Var}(M)}}. \]

4. The Nonparametric Exponential Weighted Moving Average Control Chart

This section proposes the design structure for the nonparametric EWMA control chart based on the Mood statistic for monitoring process variation. The Mood statistic \( W_t \) according to (9) is obtained from two independent random samples \( X = X_1, X_2, ..., X_m \) and \( Y = Y_1, Y_2, ..., Y_n \).
Hence, the nonparametric EWMA control chart based on Mood statistics is given by:

\[
NEWMA_t = \lambda W_i + (1 - \lambda) NEWMA_{t-1}, \quad t = 1, 2, \ldots
\]

(13)

where \( W_i \) is \( i \)th observation based on the Mood statistic

\[ \lambda \]

is an exponential smoothing parameter, \( 0 < \lambda < 1 \).

Consequently, the nonparametric EWMA control chart based on the Mood statistic is denoted by NEWMA. The control limits of NEWMA control chart as follows:

\[
UCL_{NEWMA} = E(NEWMA) + L_2\sqrt{Var(NEWMA)}
\]

(14)

\[
LCL_{NEWMA} = E(NEWMA) - L_2\sqrt{Var(NEWMA)}
\]

(15)

where \( L_2 \) is the width of control limit of control chart.

The statistics \( NEWMA \) plotted on the NEWMA control chart with the appropriate Upper Control Limit \( (UCL) \) and Lower Control Limit \( (LCL) \).

5. The Proposed Nonparametric Double Exponentially Weighted Moving Average Control Chart

In this section, the nonparametric double exponentially weighted moving average (NDEWMA) control chart based on the Mood statistics is proposed for monitoring small to moderate changes in process dispersion more efficiently. Let \( NDEWMA_1, NDEWMA_2, \ldots \) be the nonparametric of the double exponentially weighted moving average from a series of subgroups obtained from normal and exponential distributions. The nonparametric double exponentially weighted moving average based on Mood statistics at time \( i \) denoted by \( NDEWMA_i \) is defined as follows:

\[
NEWMA_t = \lambda W_i + (1 - \lambda) NEWMA_{t-1},
\]

(16)

\[
NDEWMA_t = \lambda NEWMA_i + (1 - \lambda) DNEWMA_{i-1},
\]

(17)

where \( W_i \) is the \( i \)th Mood statistic

\[ \lambda \]

is an exponential smoothing parameter, \( 0 < \lambda < 1 \)

\( NEWMA_i \) is the initial value of NEWMA statistic

\( NDEWMA_i \) is the initial value of NDEWMA statistic.

The Upper Control Limit of the NDEWMA control chart is defined by the following recursion:

\[
UCL_{NDEWMA} = E(NDEWMA) + L_4\sqrt{Var(NDEWMA)}
\]

(18)

The Lower Control Limit of the NDEWMA control chart is defined by the following recursion:

\[
LCL_{NDEWMA} = E(NDEWMA) - L_4\sqrt{Var(NDEWMA)}
\]

(19)

where \( L_4 \) is the width of control limit of the control chart.
6. Comparison of Performance of Control Chart

In this section, we provide a comparative discussion of the results obtained in Sections 4 and 5. Moreover, the efficiency of the NDEWMA control chart is also compared with the NEWMA control chart. We first consider the case in which observations are normal and exponential distributions. In a situation where the process is in-control state, mean and variance are mean $\mu = \mu_0$ and $\sigma = \sigma_0$. When the process is a shift in variation, let $\sigma_1 = \delta \sigma_0$, $(0 < \delta \neq 1)$ be the standard deviation for out of control state, where $\delta$ is the magnitude of shift size; $\delta = 1.05, 1.10 - 2.00$ and $3.00$, respectively.

According to a specified in-control Average Run Length ($ARL_0$) equal to 370, we determine the parameter of control charts based on the Monte Carlo simulation technique. The Average Run Length was estimated by means of the number of simulation studies ($M$). We consider the situation of the sample sizes ($n$) are equal 5 and 10. The summarized information for these control charts are provided in Table 1 to Table IV.

The Run Length is given by

$$ RL_l = \begin{cases} 1; & LCL_l \leq C \leq UCL_l \\ 0; & \text{Otherwise} \end{cases} \quad (20) $$

where $l = 1, 2, \ldots, M$.

Therefore, the Average Run Length $= \frac{\sum RL_l}{M}$,

where $LCL_l$ is the Lower Control Limit of the chart

$UCL_l$ is the Upper Control Limit of the chart

$M$ is the number of the simulation study.

Table 1. Comparison of ARLs Values for $N(0,1)$ Between NDEWMA and NEWMA control charts

| Shift ($\delta$) | $\lambda = 0.05$ | | | $\lambda = 0.10$ | | |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                 | $n=5$           | $n=10$          | $n=5$           | $n=10$          |
|                 | NDEWMA         | NEWMA           | NDEWMA         | NEWMA           | NDEWMA         | NEWMA           | NDEWMA         | NEWMA           |
| $L_1=0.16295$   | $L_2=0.3437$   | $L_1=0.16295$   | $L_2=0.3437$   | $L_1=0.3007$   | $L_2=0.5502$   | $L_1=0.30135$  | $L_2=0.5512$   |
| 1.00            | 370.2033       | 370.6409        | 370.7577       | 370.6272        | 370.4638       | 370.2391       |
| 1.05            | **157.9211**   | 170.0728        | **109.7337**   | 119.082         | **173.2657**   | 192.0397       | **122.7031**   | 137.1468       |
| 1.10            | **90.5594**    | 97.5190         | **55.2363**    | 56.6904         | **99.6650**    | 113.1537       | **58.2735**    | 65.615         |
| 1.20            | **45.5586**    | 42.4015         | **28.0366**    | **24.6381**     | **46.5534**    | 51.8759        | **25.7216**    | 26.2598        |
| 1.30            | **31.3696**    | **28.5263**     | **20.4486**    | **15.7235**     | **29.6028**    | 31.1771        | **17.0488**    | 15.6696        |
| 1.40            | **24.9307**    | **20.9438**     | **16.9973**    | **15.7235**     | **29.6028**    | 31.1771        | **17.0488**    | **15.6696**    |
| 1.50            | **21.3770**    | **16.6885**     | **14.9741**    | **9.5606**      | **17.9185**    | **16.7505**    | **11.4533**    | **8.8041**     |
| 1.60            | **19.1122**    | **14.1179**     | **13.6389**    | **8.1610**      | **15.4879**    | **13.7739**    | **10.2024**    | **7.3852**     |
| 1.70            | **17.6026**    | **12.4013**     | **12.6557**    | **7.2115**      | **13.9457**    | **11.8289**    | **9.3761**     | **6.4634**     |
| 1.80            | **16.4385**    | **11.1082**     | **11.9437**    | **6.5346**      | **12.8173**    | **10.5097**    | **8.7512**     | **5.7803**     |
| 1.90            | **15.5555**    | **10.2052**     | **11.3734**    | **6.0302**      | **11.9780**    | **9.4958**     | **8.2800**     | **5.2967**     |
| 2.00            | **14.8645**    | **9.4413**      | **10.9152**    | **5.6295**      | **11.3106**    | **8.7116**     | **7.9162**     | **4.9238**     |
| 3.00            | **11.8481**    | **6.4533**      | **8.8608**     | **3.9666**      | **6.6640**     | **5.6868**     | **6.2974**     | **3.4040**     |
As shown in table 1, NDEWMA control charts are superior for detection $\delta \leq 1.10$ and value $\lambda = 0.05$, as well as NDEWMA control charts are superior for detection $\delta \leq 1.20$ and value $\lambda = 0.10$ in normal distribution process for all value of sample size $n = 5$ and 10, see figure 1 (a) and (b), respectively.

![Figure 1. Comparisons of ARL1 Values.](image)

Table 2. Comparison of ARLs Values for Exp(1) Between NDEWMA and NEWMA control charts

| Shift ($\delta$) | $\lambda = 0.05$ | $\lambda = 0.10$ |
|-----------------|-----------------|-----------------|
|                 | $n = 5$         | $n = 10$        | $n = 5$         | $n = 10$        |
| NDEWMA          | NEWMA           | NDEWMA          | NEWMA           | NDEWMA          | NEWMA           |
| $L_1 = 0.16175$ | $L_2 = 0.3416$  | $L_1 = 0.16310$ | $L_2 = 0.3435$  | $L_1 = 0.3001$  | $L_2 = 0.5484$  |
| 1.00            | 370.0123        | 370.0130        | 370.0874        | 370.3574        | 370.1691        | 370.3408        |
| 1.05            | 273.9897        | 284.7868        | 239.2840        | 250.6530        | 253.7215        | 265.5045        |
| 1.10            | 218.0966        | 231.7796        | 167.9299        | 179.9509        | 183.7099        | 198.4615        |
| 1.20            | 146.5364        | 159.8373        | 97.5413         | 106.2561        | 107.8606        | 123.3136        |
| 1.30            | 108.6423        | 120.3621        | 68.2558         | 73.4795         | 74.9983         | 87.2708         |
| 1.40            | 87.0849         | 96.0047         | 52.7027         | 55.4767         | 56.2781         | 65.5375         |
| 1.50            | 72.8064         | 80.1426         | 43.6007         | 44.3310         | 45.1594         | 52.0099         |
| 1.60            | 63.1776         | 69.0845         | 37.5486         | 37.9308         | 38.0251         | 43.3898         |
| 1.70            | 56.6473         | 61.4616         | 33.0123         | 34.1209         | 33.2738         | 37.6495         |
| 1.80            | 51.4833         | 55.8631         | 29.6667         | 31.3237         | 30.0045         | 33.5623         |
| 1.90            | 47.7386         | 51.4544         | 27.1753         | 29.1946         | 27.4687         | 30.3740         |
| 2.00            | 44.7039         | 48.1238         | 25.1732         | 27.6060         | 25.4817         | 28.1048         |
| 3.00            | 33.6303         | 35.844          | 18.3904         | 21.7820         | 18.6251         | 19.7027         |

As shown in table 2, NDEWMA control charts are superior to detect for all magnitude shift on exponential distribution process and for value of $\lambda = 0.05, 0.10$ and sample size $(n) = 5$ and 10, see figure 2 (a) and (b), respectively.
On comparing the performance between NDEWMA and NEWMA control charts can be summarized as follows:

- The NDEWMA and NEWMA control charts showed decreasing behavior of ARLs with increasing sample size ($n$).
- The NDEWMA control chart is performs better than the NEWMA control chart for detect small shifts ($\delta < 1.20$) on normal distribution than the NEWMA control chart, while the NEWMA control chart is good performs to detect moderate and large shifts ($\delta \geq 1.20$).
- The NDEWMA control chart performs better than the NEWMA control chart for detection dispersion in process on the exponential distribution of all magnitude shifts.

7. Conclusion
This paper proposed the NDEWAM control chart based on the Mood statistics for monitoring dispersion in processes. The performance of the proposed control chart indicates that the NDEWMA control chart is more effective in terms of detecting small shifts ($\delta < 1.20$) than the NEWMA control chart for normal distribution. Moreover, the NDEWMA control chart performs better for detecting all shifts in exponential distribution.

8. References
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