Criteria for the entanglement of composite systems of identical particles

GianCarlo Ghirardi†
Department of Theoretical Physics of the University of Trieste, and
International Centre for Theoretical Physics “Abdus Salam”, and
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Trieste, Italy
and

Luca Marinatto‡
Department of Theoretical Physics of the University of Trieste, and
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Trieste, Italy

Abstract

We identify a general criterion for detecting entanglement of pure bipartite quantum states describing a system of two identical particles. Such a criterion is based both on the consideration of the Slater-Schmidt number of the fermionic and bosonic analog of the Schmidt decomposition and on the evaluation of the von Neumann entropy of the one-particle reduced statistical operators.

1 Introduction

Entanglement is both one of the most peculiar and counterintuitive features of Quantum Mechanics and one of the most valuable resources for Quantum Information and Quantum Computation. In fact the possibility of implementing teleportation processes [1], of devising efficient quantum algorithms outperforming the classical ones in solving certain computational problems [2] and of exhibiting secure cryptographical protocols [3], rests on the striking physical properties of entangled states. Accordingly, having an exhaustive knowledge of this phenomenon is extremely relevant both from the theoretical and from the practical point of view. However, despite that almost all the physical implementations of the above-mentioned processes developed so far involve identical particles, the very notion of entanglement in systems composed of indistinguishable elementary constituents seems to be lacking both of a satisfactory theoretical formalization and of a clear physical understanding. In fact, while when dealing with bipartite composite systems composed of distinguishable particles and described by pure quantum states there exist several equivalent criteria for revealing the entanglement (non-factorizability of the states, determination of the Schmidt number, evaluation of the von Neumann entropy of the reduced single-particle statistical operators), an uncritical application of these criteria to quantum states...
describing a pair of identical particles seem to suggest (mistakenly) that non-entangled states of identical particles cannot exist. This stems from the symmetrization postulate which, forcing the states of identical particles to possess definite symmetry properties under any permutation of their labels, forbids the occurrence of factorized states. In order to clarify this subtle issue, where the unavoidable correlations arising from the truly indistinguishable nature of the particles are sometimes confused with the genuine correlations due to the entanglement, we have been led to exhibit two (equivalent and, in our opinion, satisfactory) criteria for deciding whether a given state is entangled or not. The first is based on the possibility of attributing a complete set of objective properties to each component particle of the composed quantum system. The second, whose analysis represents the main topic of this paper, is based both on the consideration of the Slater-Schmidt number of the fermionic and bosonic analog of the Schmidt decomposition and on the evaluation of the von Neumann entropy of the one-particle reduced statistical operator. We will present briefly and schematically both criteria, showing their complete equivalence and we address the interested reader to the original papers for a more exhaustive and detailed treatment.

2 Entanglement for distinguishable particles

Let us first briefly review the basic features of non-entangled (pure) state vectors associated to composite systems of two distinguishable particles. Given a bipartite state \(|\psi(1, 2)\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2\), the following three equivalent statements represent necessary and sufficient conditions in order that the state can be considered as non-entangled:

1. \(|\psi(1, 2)\rangle\) is factorized, i.e. there exist two single-particle states \(|\phi_1\rangle \in \mathcal{H}_1\) and \(|\chi_2\rangle \in \mathcal{H}_2\) such that \(|\psi(1, 2)\rangle = |\phi_1\rangle \otimes |\chi_2\rangle\). If this is the case a definite quantum state is assigned to each component subsystem so that, since such states are simultaneous eigenstates of a complete set of commuting observables, it is possible to predict with certainty the measurement outcomes of such a set of operators. These outcomes represent objective properties which we can legitimately consider as possessed by the physical constituents.

2. The Schmidt number of \(|\psi(1, 2)\rangle\), that is the number of non-zero coefficients appearing in the Schmidt decomposition of the state, equals 1.

3. If consideration is given to the reduced statistical operator \(\rho^{(i)}\) of one of the two subsystems \((i = 1, 2)\), its von Neumann entropy \(S(\rho^{(i)}) = -\text{Tr} [\rho^{(i)} \log \rho^{(i)}]\) equals 0. Since the von Neumann entropy measures the uncertainty about the quantum state to attribute to a physical system, its value being null reflects the fact that, in this situation, there is no uncertainty at all concerning the properties of each subsystem.

The converse of the previous statements are easily proven. Accordingly, a bipartite quantum system is described by an entangled state \(|\psi(1, 2)\rangle\) if and only if one of the three following equivalent conditions holds true: i) the state is not factorizable; ii) the Schmidt number of the state is strictly greater than one; iii) the von Neumann entropy of both reduced statistical operators is strictly positive.

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1. The only exception is represented by a system of bosons described by the same state vector.
2. For our convenience, the log function is intended to be in base 2 rather than in the natural base e.
In this situation it is no more possible to attribute a definite quantum state to each constituent and therefore no objective property whatsoever can be claimed to be possessed by them; accordingly, a positive value of the von Neumann entropy reflects this uncertainty concerning the state to attribute to the particles.

3 Two identical particles

When passing to the case of composite quantum systems consisting of identical particles, it has first of all to be noticed that the symmetrization postulate poses severe constraints on the mathematical form of their associated state vectors. In fact, a couple of identical fermions or bosons must be described by antisymmetric and symmetric states respectively, states which are clearly not factorized in almost all cases. Consequently their Schmidt decomposition involves more than one term and the von Neumann entropy of the reduced statistical operators is strictly positive. It is therefore clear that, if one would resort to the same criteria used for distinguishable particles in order to detect entanglement, one would be led to conclude that non-entangled states of identical particles cannot exist (with the only exception represented by two bosons in the same state). To tackle the problem in the correct way, one has to stick to the idea that the physically most interesting and fundamental feature of non-entangled states is that both constituents possess a complete set of objective properties. In Refs. [4, 5] we have taken precisely this attitude, and we have given the following definitions:

Definition 3.1 The identical constituents $S_1$ and $S_2$ of a composite quantum system $S = S_1 + S_2$ are non-entangled when both constituents possess a complete set of properties.

Definition 3.2 Given a composite quantum system $S = S_1 + S_2$ of two identical particles described by the normalized state vector $|\psi(1, 2)\rangle$, we will say that one of the constituents possesses a complete set of properties iff there exists a one-dimensional projection operator $P$, defined on the single particle Hilbert space $\mathcal{H}$, such that:

$$\langle \psi(1, 2) | \mathcal{E}_P(1, 2) | \psi(1, 2) \rangle = 1$$

(3.1)

where

$$\mathcal{E}_P(1, 2) = P^{(1)} \otimes [ I^{(2)} - P^{(2)} ] + [ I^{(1)} - P^{(1)} ] \otimes P^{(2)} + P^{(1)} \otimes P^{(2)}.$$  

(3.2)

The second Definition is useful to make precise the meaning of the statement "both constituents possess a complete set of properties" in the considered peculiar situation where it is not possible, both conceptually and practically, to distinguish the two particles. In fact condition of Eq. (3.1) gives the probability of finding at least one of the two identical particles (without saying which one) in the state associated to the one-dimensional projection operator $P$. Since any state vector is a simultaneous eigenvector of a complete set of commuting observables, condition of Eq. (3.1) allows to attribute to at least one of the particles the complete set of properties (eigenvalues) associated to the considered set of observables.

Starting from the above Definitions, we have proven the following Theorems

Theorem 3.1 The identical fermions $S_1$ and $S_2$ of a composite quantum system $S = S_1 + S_2$ described by the normalized state $|\psi(1, 2)\rangle$ are non-entangled iff $|\psi(1, 2)\rangle$ is obtained by antisymmetrizing a factorized state.
Theorem 3.2 The identical bosons of a composite quantum system $S = S_1 + S_2$ described by the normalized state $|\psi(1,2)\rangle$ are non-entangled iff either the state is obtained by symmetrizing a factorized product of two orthogonal states or it is the product of the same state for the two particles.

These two Theorems clearly show why non-factorized state vectors describing two identical particles have not to be considered as necessarily entangled, since there exist cases in which property attribution is still possible in spite of their non-factorizable form. In the situation described by Theorems 3.1 and 3.2, the states do not exhibit the peculiar non-local correlations between measurement outcomes which are typical of the entangled states and, consequently, it is not possible to use them to violate any Bell’s inequality or to perform any teleportation process [4, 5].

4 A new criterion to detect entanglement

In the previous Section we have described a physically meaningful and unambiguous criterion for deciding whether a given state describing two identical particles is entangled or not. The criterion is based on the possibility of attributing a definite quantum state to each component particle, without of course being able to specify which particle has which property owing to their indistinguishability. Recently other criteria for detecting entanglement have appeared in the literature [7, 8, 9, 10]. While some of them [7, 8] correctly deal with the case of identical fermions, an inappropriate treatment of the (subtle) boson case is presented in Ref. [8] while in Ref. [9] the use of the entropy criterion is misleading. Most of them [8, 9, 10] seem to agree that the use of such criteria presents some obscure aspects. The purpose of this Section is to exhibit a unified and unambiguous criterion, based both on the consideration of the Slater-Schmidt number of the fermionic and bosonic analog of the Schmidt decomposition and on the evaluation of the von Neumann entropy of the one-particle reduced statistical operators, to identify whether a state is entangled or not.

As we will show, such a criterion turns out to be in complete agreement with the previous criterion based on the property attribution and it clarifies all the obscure aspects which have been pointed out by the authors of Refs. [8, 9, 10].

For the sake of simplicity we will deal separately with the cases of two identical fermions and two identical bosons.

4.1 The fermion case

The notion of entanglement for systems composed of two identical fermions has been discussed in Ref. [7] where a “fermionic analog of the Schmidt decomposition” has been exhibited. Such a decomposition results from a nice extension to the set of the antisymmetric complex matrices of a well-known theorem holding for antisymmetric real matrices and it states that:

Theorem 4.1.1 Any state vector $|\psi(1,2)\rangle$ describing two identical fermions of spin $s$ and, consequently, belonging to the antisymmetric manifold $A(C^{2s+1} \otimes C^{2s+1})$, can be written as:

$$|\psi(1,2)\rangle = \sum_{i=1}^{(2s+1)/2} a_i \frac{1}{\sqrt{2}} \left[ |2i-1\rangle_1 \otimes |2i\rangle_2 - |2i\rangle_1 \otimes |2i-1\rangle_2 \right],$$

(4.1)
where the states \{ |2i - 1\rangle, |2i\rangle \} with \( i = 1 \ldots (2s+1)/2 \) constitute an orthonormal basis of \( \mathbb{C}^{2s+1} \), and the complex coefficients \( a_i \) (some of which may vanish) satisfy the normalization condition \( \sum_i |a_i|^2 = 1 \).

The \textit{Slater number} of \( |\psi(1,2)\rangle \) is then defined as the number of non-zero coefficients \( a_i \) appearing in the decomposition of Eq. (4.1). Two possible cases can occur:

\textbf{Slater Number} = 1. In this situation the state \( |\psi(1,2)\rangle \) has the form of a single Slater determinant:

\[
|\psi(1,2)\rangle = \frac{1}{\sqrt{2}} \left[ |1\rangle_{1} \otimes |2\rangle_{2} - |2\rangle_{1} \otimes |1\rangle_{2} \right] \tag{4.2}
\]

Since the state has been obtained by antisymmetrizing the product of two orthogonal states, \( |1\rangle \) and \( |2\rangle \), it must be considered as non-entangled according to our previous criterion of Theorem 3.1. The reduced single-particle statistical operators of each particle (it does not really matter which one we consider since, due to symmetry considerations, they are equal) and their von Neumann entropy (expressed in base 2) are:

\[
\rho^{(1 \text{ or } 2)} = \frac{1}{2} \left[ |1\rangle \langle 1| + |2\rangle \langle 2| \right] \tag{4.3}
\]

\[
S(\rho^{(1 \text{ or } 2)}) \equiv -\text{Tr} \left[ \rho^{(1 \text{ or } 2)} \log \rho^{(1 \text{ or } 2)} \right] = 1 \tag{4.4}
\]

It should be obvious that we cannot pretend that the operator \( \rho^{(1 \text{ or } 2)} \) of Eq. (4.3) describes the properties of \textit{precisely} the first or of the second particle of the system, due to their indistinguishability. Accordingly, in this case, the value \( S(\rho^{(1 \text{ or } 2)}) = 1 \) correctly measures only the unavoidable uncertainty concerning the quantum state to attribute to each of the two identical physical subsystems and it has nothing to do with any uncertainty arising from any actual form of entanglement.

\textbf{Slater number} > 1. In this case the state \( |\psi(1,2)\rangle \) is written in term of more than one single Slater determinant and, consequently, it must be considered as a truly entangled state. In fact, according to our criterion of Theorem 3.1, there is no way to obtain \( |\psi(1,2)\rangle \) by antisymmetrizing the tensor product of two orthogonal states. Moreover the reduced single-particle statistical operators and their associated von Neumann entropy are:

\[
\rho^{(1 \text{ or } 2)} = \sum_{i=1}^{(2s+1)/2} \frac{|a_i|^2}{2} \left[ |2i - 1\rangle \langle 2i - 1| + |2i\rangle \langle 2i| \right] \tag{4.5}
\]

\[
S(\rho^{(1 \text{ or } 2)}) = -\sum_{i=1}^{(2s+1)/2} |a_i|^2 \log \frac{|a_i|^2}{2} = 1 - \sum_{i=1}^{(2s+1)/2} |a_i|^2 \log |a_i|^2 > 1 \tag{4.6}
\]

In this case the fact that the von Neumann entropy is strictly greater than one correctly measures both the uncertainty deriving from the indistinguishability of the particles and the one connected to the genuine entanglement of the state.

The previous two cases are summarized in the following Theorem:
Theorem 4.1.2 A state vector $|\psi(1,2)\rangle$ describing two identical fermions is non-entangled iff its Slater number is equal to 1 or, equivalently, iff the von Neumann entropy of the one-particle reduced density operator $S(\rho^{(1\,or\,2)})$ is equal to 1.

4.2 The boson case

The case of bipartite systems composed of two identical bosons is slightly more articulated and subtle than the fermionic case. Let us start, as before, by considering the bosonic Schmidt decomposition of an arbitrary state vector $|\psi(1,2)\rangle$ belonging to the symmetric manifold $S(C^{2s+1} \otimes C^{2s+1})$ and describing two identical bosons:

Theorem 4.2.1 Any state vector describing two identical $s$-spin boson particles $|\psi(1,2)\rangle$ and, consequently, belonging to the symmetric manifold $S(C^{2s+1} \otimes C^{2s+1})$ can be written as

$$|\psi(1,2)\rangle = \sum_{i=1}^{2s+1} b_i |i\rangle_1 \otimes |i\rangle_2,$$  \hspace{1cm} (4.7)

where the states $\{|i\rangle\}$, with $i = 1, \ldots, 2s+1$, constitute an orthonormal basis for $C^{2s+1}$, and the real nonnegative coefficients $b_i$ satisfy the normalization condition $\sum_i b_i^2 = 1$.

The Schmidt number of the state of Eq. (4.7) is defined, as usual, as the number of non-zero coefficients $b_i$ appearing in the decomposition and the following cases must be analyzed:

Schmidt number = 1. In this case the state is factorized, i.e., $|\psi(1,2)\rangle = |i^\star\rangle \otimes |i^\star\rangle$, and it describes two identical bosons in the same state $|i^\star\rangle$. It is obvious that such a state must be considered as non-entangled since one knows precisely the properties of both constituents and, consequently, no uncertainty remains about which particle has which property. This fact perfectly agrees with the von Neumann entropy of the single-particle reduced statistical operators $S(\rho^{(1\,or\,2)})$ being null.

Schmidt number = 2. According to Eq. (4.7), the most general state with Schmidt number equal to 2 has the following form:

$$|\psi(1,2)\rangle = b_1 |1\rangle_1 \otimes |1\rangle_2 + b_2 |2\rangle_1 \otimes |2\rangle_2,$$  \hspace{1cm} (4.8)

where $b_1^2 + b_2^2 = 1$.

We can now distinguish two subcases, depending on the values of the positive coefficients $b_1$ and $b_2$. If they are equal, that is $b_1 = b_2 = 1/\sqrt{2}$, the following Theorem holds:

Theorem 4.2.2 The condition $b_1 = b_2 = 1/\sqrt{2}$ is necessary and sufficient in order that the state $|\psi(1,2)\rangle = b_1 |1\rangle_1 \otimes |1\rangle_2 + b_2 |2\rangle_1 \otimes |2\rangle_2$ can be obtained by symmetrizing the factorized product of two orthogonal states.

In this situation, and in full accordance with our Theorem 3.2, one must consider this state as non-entangled since it is possible to attribute well definite state vectors to both particles.

\[\text{It is worth pointing out that the Schmidt decomposition of Eq. (4.7) is not always unique, as happens for the biorthonormal decomposition of states describing distinguishable particles.}\]
(of course, as usual, we cannot say which particle is associated to which state due to their indistinguishability). Moreover the von Neumann entropy of the reduced statistical operators $S(\rho^{(1\text{ or } 2)})$ is equal to 1 measuring, as happened in the fermion case with the state of Eq. (4.2), only the uncertainty arising from the indistinguishability of the particles.

On the contrary, when the two coefficients are different, that is $b_1 \neq b_2$, the following Theorem holds:

**Theorem 4.2.3** The condition $b_1 \neq b_2$ is necessary and sufficient in order that the state $|\psi(1, 2)\rangle = b_1|1\rangle_1 \otimes |1\rangle_2 + b_2|2\rangle_1 \otimes |2\rangle_2$ can be obtained by symmetrizing the factorized product of two non-orthogonal states.

According to our original criterion, this state must be considered as a truly entangled state since it is impossible to attribute to both particles definite properties. Moreover the von Neumann entropy of the reduced statistical operator $S(\rho^{(1\text{ or } 2)}) = -b_1^2 \log b_1^2 - b_2^2 \log b_2^2$ lies within the open interval $(0, 1)$. It correctly measures the uncertainty coming both from the indistinguishability of the particles and from the entanglement, and it is strictly less than one because, in a measurement process, there is a probability greater than $1/2$ to find both bosons in the same physical state ($|1\rangle$ or $|2\rangle$ depending whether $b_1 > b_2$ or vice versa).

**Schmidt number $\geq 3$.** In this situation the state is a genuine entangled one since it cannot be obtained by symmetrizing a factorized product of two orthogonal states and the von Neumann entropy of the reduced density operators is such that $S(\rho^{(1\text{ or } 2)}) \in (0, \log(2s + 1)]$.

In accordance with our analysis we can formulate the following Theorem which supplies us with a unified and unambiguous criterion for detecting the entanglement in the boson case:

**Theorem 4.2.4** A state vector $|\psi(1, 2)\rangle$ describing two identical bosons is non-entangled iff either its Schmidt number is equal to 1, or the Schmidt number is equal to 2 and the von Neumann entropy of the one-particle reduced density operator $S(\rho^{(1\text{ or } 2)})$ is equal to 1. Alternatively, one might say that the state is non-entangled iff either its von Neumann entropy is equal to 0, or it is equal to 1 and the Schmidt number is equal to 2.

It is clear from the previous analysis, differently from what happens in the case of two identical fermions, that the consideration of the Schmidt number alone (or of the von Neumann entropy) to detect entanglement for a pair of bosons is inappropriate. In fact there exist states with Schmidt number equal to 2, or with von Neumann entropy equal to 1, which can be non-entangled as well as entangled. Therefore, the only consistent way to overcome this problem turns out to be that of considering the two criteria together, as clearly stated in the Theorem.

## 5 Conclusions

In this paper we have reviewed in general the problem of deciding whether a state describing a system of two identical particles is entangled or not. Two equivalent criteria have been

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4In fact, if this would be true, the rank of the reduced density operator would be equal to two, in contradiction with the fact that a Schmidt number greater or equal to three implies a rank equal or greater to three.
exhibited: the first [4][5], in the spirit of the founder fathers of Quantum Mechanics, is based on the possibility of attributing a complete set of objective properties to both constituents (that is, a definite state vector) while the second [6] is based on the consideration of both the Slater-Schmidt number of the fermionic and bosonic analog of the Schmidt decompositions of the states describing the system and of the von Neumann entropy of the reduced statistical operators.

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