Coarsening kinetics of a two-dimensional O(2) Ginzburg–Landau model: the effect of reversible mode coupling

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Received 24 November 2010
Accepted 14 February 2011
Published 14 March 2011

Online at stacks.iop.org/JSTAT/2011/P03013
doi:10.1088/1742-5468/2011/03/P03013

Abstract. We investigate, via numerical simulations, the phase ordering kinetics of a two-dimensional soft spin O(2) Ginzburg–Landau model when a reversible mode coupling is included via the conserved conjugate momentum of the spin order parameter (Model E). Coarsening of the system, when quenched from a disordered state to zero temperature, is observed to be enhanced by the existence of the mode coupling terms. The growth of the characteristic length scale $L(t)$ exhibits an effective superdiffusive growth exponent that can be interpreted as a positive logarithmic-like correction to a diffusive growth, i.e., $L(t) \sim (t \ln t)^{1/2}$. In order to understand this behavior, we introduced a simple phenomenological model of coarsening based on the annihilation dynamics of a vortex–antivortex pair, incorporating the effect of vortex inertia and logarithmically divergent mobility of the vortex. With a suitable choice of the parameters, numerical solutions of the simple model can fit the full simulation results very adequately. The effective growth exponent in the early time stage is larger due to the effect of the vortex inertia, which crosses over into the late time stage characterized by positive logarithmic correction to a diffusive growth. We also investigated the nonequilibrium autocorrelation function from which the so-called $\lambda$ exponent can be extracted. We get $\lambda \simeq 1.99(2)$ which is distinctly larger than the value of $\lambda \simeq 1.17$ for the purely dissipative Model A dynamics of non-conserved O(2) models.
Coarsening kinetics of a two-dimensional $O(2)$ Ginzburg–Landau model

Keywords: coarsening processes (theory)

ArXiv ePrint: 1011.5040

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1. Introduction

The dynamics of statistical systems quenched from high temperature disordered states to low temperature ordered states has been a subject of interest for several decades [1]–[7]. In typical situations, the average length scale $L(t)$ of the ordered domain grows in time as a power law $L(t) \sim t^{1/z}$, where the growth exponent $1/z$ depends on the dimension of the space and that of the relevant order parameters, in addition to the conserved or non-conserved nature of the latter in the relaxation dynamics [5]. Usually characteristic topological defects such as vortices or domain walls are generated in the initial disordered state, and the annihilation of these defects provides the main mechanism of coarsening and phase ordering in the system. The observed self-similarity of these coarsening systems at different time instants is usually represented by the so-called dynamic scaling hypothesis of the equal time spatial correlation function of the order parameter. The dynamic scaling hypothesis has been an important ingredient in deriving the properties of phase ordering in the late time stage. For example, in combination with the so-called energy scaling method [8], domain growth laws could be extracted for almost all model systems with purely dissipative dynamics.

Previous works on phase ordering focused mostly on the cases where the effect of dissipation dominates the coarsening and the motion of the topological defects. On the other hand, in reality, there exist various interesting systems exhibiting dynamic processes that cannot be described solely by dissipative dynamics. One example can be found in the case of magnetic spin systems, where the spins are influenced by neighboring spins via precession interaction terms that are energy conserving [9]. These elements in the dynamics are called reversible (non-dissipative) mode coupling. In fact, among the various model systems (Model A, B, C, E, F, G, H, J) classified by Hohenberg and Halperin [10] that are known to describe the dynamic critical phenomena, only Models A, B, and C are based on dissipative dynamics alone, with the remaining ones, Models E, F, G, H, and J, retaining reversible mode couplings. Hence, it is quite natural to extend studies on the phase ordering dynamics to these model systems.

doi:10.1088/1742-5468/2011/03/P03013
Indeed some works have been carried out along this direction. A most familiar example is the phase separation dynamics of a binary fluid, in which reversible coupling between the hydrodynamic flow and the relative concentration, advection of the order parameter field by the fluid flow, plays an important role [11,12]. Effects of hydrodynamic flow on the phase ordering kinetics and defect dynamics of nematic liquid crystals have been studied [13]–[16]. The influence of precession on the phase ordering of an isotropic Heisenberg magnet in three dimensions (Model J [9]) has also been studied [17].

A particularly interesting example is the phase ordering kinetics of the Bose gas, i.e., time evolution of the Bose–Einstein condensation, which was studied via the Gross–Pitaevskii (GP) equation in two and three dimensions, demonstrating the importance of the reversible Josephson precession term in the dynamics [18,19]. An alternative approach to this problem is to use the appropriate stochastic model known as Model F [20] in which the complex order parameter field $\psi$ is (both statically and dynamically) coupled to the conserved real field $m$.

In this work, we investigate the phase ordering process of systems governed by a simpler model, namely, Model E [20], in which there is no static coupling between $\psi$ and $m$. Specifically, we focus our investigation on the effect of the reversible spin precession term on the phase ordering process in the soft spin O(2) models in two dimensions. We especially try to compare the characteristics of the phase ordering dynamics of these model systems with those in the case where dissipative dynamics alone is considered. The linearized hard spin version of this model was employed by Nelson and Fisher [21] to describe the behavior of the spin wave in the two-dimensional anisotropic ferromagnet and the hydrodynamics of the propagation of third sound in a thin film of He4. In [18,19], the same set of linearized equations was used to demonstrate the crucial role played by the non-dissipative precession term in the phase ordering process of the defect free case in the two-dimensional $XY$ model.

In terms of phase ordering dynamics, O(2) models in two dimensions are of particular interest because the aforementioned energy scaling methods do not provide a definitive result on the domain growth law. In equilibrium, the two-dimensional ferromagnetic $XY$ model exhibits a Berezinskii–Kosterlitz–Thouless (BKT) transition at $T_{\text{BKT}}$ due to the unbinding of vortex–antivortex pairs [22]. Below $T_{\text{BKT}}$, the system has a quasi-ordered phase which is characterized by an algebraic decay of the order parameter correlation function for long distances. The critical exponent governing this power law decay decreases continuously down to zero temperature. That is, the system is critical at equilibrium for all non-vanishing temperatures below $T_{\text{BKT}}$. Therefore, the coarsening dynamics in this model at finite temperatures is expected to exhibit critical dynamic scaling instead of simple dynamic scaling.

Large number of works using the purely dissipative dynamics have been carried out on the coarsening dynamics of the two-dimensional $XY$/O(2) models [23]–[46]. After all these efforts, it is now agreed that, in the phase ordering dynamics of ordinary O(2) models in two dimensions without mode coupling terms and with non-conserved order parameter, the growing length scale exhibits a logarithmic correction to the diffusive growth as $L(t) \sim (t/\ln t)^{1/2}$. Here, the logarithmic correction can be attributed to a logarithmic divergence (with the system size) of the effective friction constant (or inverse mobility) of a moving vortex in the dissipative dynamics with non-conserved order parameter [47]–[51].

doi:10.1088/1742-5468/2011/03/P03013
In a related simulation work [42] on the coarsening dynamics of superconducting Josephson junction arrays based on the dynamics of a resistively shunted junction model [52], it was revealed that there is no logarithmic correction in the growth law, resulting in a purely diffusive growth law. This absence of logarithmic correction was understood in terms of finiteness of the effective friction constant (or inverse mobility) of a moving vortex in the limit of large system size, which is due to the particular type of dissipative coupling in Josephson junction arrays [42].

In addition, the coarsening dynamics of the two-dimensional \(XY\) model with a purely Hamiltonian dynamics [44, 45], [53]–[55] exhibited a growing length scale with an apparent late time power law growth as \(L(t) \sim t^{1/z}\) with the exponent \(1/z\) larger than the diffusive exponent \(1/2\). This kind of (dynamics-dependent) non-universal growth law motivated us to investigate further the phase ordering dynamics in related model systems.

Actually, in Model E, we find that the growing length scale \(L(t)\) exhibits an apparent superdiffusive growth as \(L(t) \sim t^{1/z}\) with the ‘effective’ dynamic exponent \(1/z\) a little larger than the diffusive exponent \(1/2\). This can be contrasted with the case where reversible mode couplings are absent where the apparent growth exponents with values a little smaller than \(1/2\) are obtained due to the (negative) logarithmic corrections mentioned above. The apparent superdiffusive exponent in the case of reversible mode coupling may suggest that the asymptotic behavior can be represented as a diffusive growth with a positive logarithmic corrections of the form \(L(t) \sim (t \ln t)^{1/2}\), as discussed in detail below.

One can extract another growing length scale from the excess energy relaxation. It is expected that the so-called energy scaling method [8] can be applied to the present case as well since the reversible contribution does not dissipate energy. This method gives the relation \(\Delta E(t) \sim L_E^{-2}(t) \ln L_E(t)\) (between the excess energy \(\Delta E(t)\) and a growing length \(L_E(t)\)). We find that the length scale \(L_E(t)\) derived from the above relation agrees with the growing length scale of the domains \(L(t)\) almost the entire time.

In order to understand the characteristics of the length scale growth in the present system, we noted that the coarsening and the resultant growth of the length scale are determined by the annihilation of vortex–anti-vortex pairs. Therefore, it is plausible to assume that the time \(t\) that it takes for the system to grow up to a length scale \(L(t)\) will correspond to the time that it takes for vortex–anti-vortex pairs of separation \(L(t)\) to annihilate. On the basis of this picture, we devised a simple phenomenological model of dynamics for the annihilation of a vortex–anti-vortex pair where we incorporate the effect of vortex inertia and logarithmically divergent mobility of the vortex [56]–[58]. We found that this model can describe closely the simulation results. We argue that the microscopic mechanism of these coarsening features, including the growth law, is closely related to the fact that there exists one conserved quantity in this model. That is, the \(m\) field component (see section 2), which is conjugate to the planar spin, is conserved. This corresponds to the third \((z)\) component of the spin in Heisenberg spin systems with easy plane anisotropy [21]. Due to the conservation of this component, there appear propagating spin wave modes at low temperatures below \(T_{KT}\) where the vortices and antivortices are neglected. In the case of coarsening dynamics where there exist a lot of vortices and antivortices in the initial stage, we do not expect the spin waves to be fully propagating. We expect rather that, due to the interaction of the vortices with the background vortices and antivortices, they propagate only within a short range and then scatter off the vortices (and antivortices)
in a complicated manner such that the mobility of an isolated vortex is enhanced with a logarithmic dependence on the size of the system.

The spin autocorrelation function $A(t)$ was also calculated, which is expected to be related to the growing length scale $L(t)$ through a new exponent $\lambda$ as $A(t) \sim L^{-\lambda}(t)$. We could extract the value of $\lambda$ in the long time limit as $\lambda \approx 1.99(2)$. This result may be interpreted as $\lambda = d = 2$ for the present case. We note that this value is quite distinct from the corresponding value of $\lambda \approx 1.17$ for the case of no reversible mode coupling, which was calculated theoretically as well as numerically [26, 30, 33, 59, 60].

2. Model E

We consider the ordering kinetics of the soft spin model known as Model E whose Hamiltonian is given by

$$H[\vec{\phi}, \sigma] = \int d^2r \left[ \frac{1}{2}(\nabla \vec{\phi})^2 + \frac{1}{4}(\vec{\phi}^2 - 1)^2 + \frac{K}{2}m^2 \right]$$

where $\vec{\phi}$ is the two-component vector order parameter $\vec{\phi} = (\phi_1, \phi_2)$ and $m$ is the third component conjugate to spin $\vec{\phi}$ (which corresponds to the $z$ component of the three-dimensional magnetization in Heisenberg spin systems with easy plane anisotropy). When $K$ is set to zero, the above Hamiltonian becomes equivalent to that of the O(2) Ginzburg–Landau model. Therefore, the order parameter has O(2) rotational symmetry on $(\phi_1, \phi_2)$-space and the equilibrium average of $m$ is zero. The Model E Hamiltonian, (1), does not contain static couplings between the order parameter $\vec{\phi}$ and the $m$ field. (On the other hand, Model F does contain such a static coupling.) However, as can be seen below, dynamic couplings between these two fields arise from reversible mode coupling terms.

The two fields $\vec{\phi}$ and $m$ satisfy the following Poisson bracket relations [10]:

$$\{\phi_1(r), m(r')\} = -g\phi_2(r)\delta(r - r'), \quad \{\phi_2(r), m(r')\} = g\phi_1(r)\delta(r - r'), \quad \{\phi_1(r), \phi_2(r')\} = 0$$

where $g$ denotes the strength of the mode coupling that is an analog of the gyromagnetic ratio of the spins. These Poisson bracket relations generate the reversible mode couplings in the equations of motion, which cause spin precession. They play a key role in the critical dynamics [10]. Note that the order parameter $\vec{\phi}$ is not a conserved quantity. Instead, the conserved quantity in this system is the $m$ component which causes the precession of the order parameter $\vec{\phi}$ in the $x$–$y$ plane of the spin space [10].

The dynamics of the system described by the fields $\psi_i$ ($=\phi_1, \phi_2, m$) can be written as

$$\frac{\partial \psi_i}{\partial t} = \sum_j \left[ \{\psi_i, \psi_j\} \frac{\delta H}{\delta \psi_j} - \Gamma_{ij} \frac{\delta H}{\delta \psi_j} \right] + \eta_i(\vec{r}, t)$$

where $\Gamma_{ij}$ denotes generalized kinetic coefficients ($\Gamma$ and $D$ below) and $\xi_i$ denotes the thermal noises ($\eta_1, \eta_2$ and $\zeta$ below). $\{A, B\}$ is the Poisson bracket of two generic dynamic variables $A$ and $B$ as defined above (2). With (2) and the above relation, the equations...
of motion become
\[
\begin{align*}
\frac{\partial \phi_1}{\partial t} &= -g\phi_2 \frac{\delta H}{\delta m} - \Gamma \frac{\delta H}{\delta \phi_1} + \eta_1(\vec{r}, t), \\
\frac{\partial \phi_2}{\partial t} &= g\phi_1 \frac{\delta H}{\delta m} - \Gamma \frac{\delta H}{\delta \phi_2} + \eta_2(\vec{r}, t), \\
\frac{\partial m}{\partial t} &= g \left( -\phi_1 \frac{\delta H}{\delta \phi_2} + \phi_2 \frac{\delta H}{\delta \phi_1} \right) + D \nabla^2 \frac{\delta H}{\delta m} + \zeta(\vec{r}, t),
\end{align*}
\]

(4)

where \( \Gamma \) and \( D \) are the kinetic coefficients, and the Gaussian thermal noises \( \eta \) and \( \zeta \) at temperature \( T \) satisfy
\[
\begin{align*}
\langle \eta(\vec{r}, t) \eta(\vec{r}', t') \rangle &= 2\Gamma T \delta(t-t'), \\
\langle \zeta(\vec{r}, t) \zeta(\vec{r}', t') \rangle &= -2DT \nabla^2 \delta(\vec{r}-\vec{r}') \delta(t-t'), \\
\langle \eta(\vec{r}, t) \zeta(\vec{r}', t') \rangle &= 0.
\end{align*}
\]

(5)

The equation of motion (3) can be explicitly written as
\[
\begin{align*}
\frac{\partial \phi_1}{\partial t} &= -gK \phi_2 m + \Gamma [\nabla^2 \phi_1 + (1 - \phi_1^2 - \phi_2^2) \phi_1] + \eta_1(\vec{r}, t), \\
\frac{\partial \phi_2}{\partial t} &= gK \phi_1 m + \Gamma [\nabla^2 \phi_2 + (1 - \phi_1^2 - \phi_2^2) \phi_2] + \eta_2(\vec{r}, t), \\
\frac{\partial m}{\partial t} &= g \vec{\nabla} \cdot (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) + DK \nabla^2 m + \zeta(\vec{r}, t).
\end{align*}
\]

(6)

We can see again that, in the limit of \( g = 0 \), the order parameter and \( m \) are decoupled, resulting in an equation of motion for the order parameter equivalent to that of the time-dependent Ginzburg–Landau equation for the \( \text{O}(2) \) model. The equation for \( m \) in (6) can be written in the form of a continuity equation
\[
\frac{\partial m}{\partial t} = -\nabla \cdot \vec{J}_m, \quad \vec{J}_m \equiv g(\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2) - DK \nabla m + \vec{\zeta}(\vec{r}, t)
\]

(7)

where \( J_m \) is the corresponding current density. We see from (7) that \( m \) is a conserved quantity. In (7), the new thermal noise \( \xi \) satisfies
\[
\zeta(\vec{r}, t) \equiv -\nabla \cdot \vec{\xi}, \quad \langle \xi(\vec{r}, t) \xi(\vec{r}', t') \rangle = 2DT \delta(t-t') \delta(\vec{r}-\vec{r}').
\]

(8)

Note that the equations of motion (6) are invariant under the transformation \( m \to -m \), \( \phi_1 \to \phi_2 \), and \( \phi_2 \to \phi_1 \).

An important physical element brought in by the reversible mode coupling term is the existence of the propagating spin wave, especially when we can neglect the existence of vortices and antivortices. In [18], in the quench dynamics of the Gross–Pitaevskii equation, a linear growth of \( z = 1 \) was found, which was conjectured to be due to the existence of propagating spin waves. Here in the case of Model E, however, we find only a minor enhancement with logarithmic correction to a diffusive growth. This implies that the spin wave here is not fully propagating, but interacting in some complicated manner with the vortices or antivortices, resulting in suppression of the propagation due to scattering with the vortices and antivortices. See also the discussions in section 4 on related questions.
3. Simulation results and discussion

In order to investigate the phase ordering kinetics of Model E, we integrated the spatially discretized form of the equations of motion (5) at zero temperature $T = 0$, i.e., in the absence of thermal noise, starting from a random initial configuration (corresponding to an equilibrium at $T = \infty$). As for the values of the parameters, we put $\Gamma = D = K = 1$. In this work, we fix the intensity of the precession $g = 1$. Simulations are performed on systems with spatial discretization of square lattice type of dimensions up to $3000 \times 3000$ with periodic boundary conditions. For the integration time interval of Euler method, we chose $\delta t = 0.05$. The simulation results were the same for smaller time intervals.

Two main quantities of interest in phase ordering kinetics of this system are the equal time spatial correlation of the order parameter defined by
\[ C(r, t) = \frac{1}{N} \left\langle \sum_i \vec{\phi}_i(t) \cdot \vec{\phi}_{i+r}(t) \right\rangle, \tag{9} \]
and the spin autocorrelation function $A(t)$ defined as
\[ A(t) = \frac{1}{N} \left\langle \sum_i \vec{\phi}_i(0) \cdot \vec{\phi}_i(t) \right\rangle. \tag{10} \]

The equal time spatial spin correlation function satisfies the simple dynamic scaling (figure 1)
\[ C(r, t) = G(r/L(t)) \tag{11} \]
with the correlation length $L(t)$ defined from $C(r, t)|_{r=L(t)} = C_0$. Here $C_0$ was conveniently taken as $C_0 = 0.4$. The correlation length $L(t)$ is found to grow in time as an apparent power law
\[ L(t) \sim t^\phi, \quad \phi \simeq 0.54(1). \tag{12} \]
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Figure 2. (a) Growing length $L(t)$ versus $t$ (open circles) manifesting an apparent power law growth $L(t) \sim t^\phi$ with the effective growth exponent of $\phi \simeq 0.54(1)$ (dotted line). (b) $L(t)^2/t$ versus $t$ from the full simulation (open circles) is displayed here together with the result of a numerical solution (solid line) of a simple dynamic model of vortex–antivortex pair annihilation with a suitably chosen set of parameters that agrees reasonably well with the simulation results. Note that the result from the simple model is scaled by a constant factor to fit the full simulation results.

It is interesting to find that the apparent domain growth exponent in the late time regime ($\phi \simeq 0.54$) is somewhat larger than the diffusive value of $1/2$ (figure 2(a)). This can be contrasted with the conventional purely diffusive case ($g = 0$) where the effective growth exponents obtained numerically are invariably smaller than $1/2$. Now, in order to understand this behavior, we construct a simple phenomenological model of coarsening of the system as follows. To begin with, we note that the coarsening of the system is dominated by the annihilation of vortex–antivortex pairs. It is easy to see that, in order for the system to grow up to a length scale $L$, vortex pairs of sizes on the order of $L$ must be already annihilated. Even though this process of annihilation of vortex pairs is a very complicated many-body process involving many vortex pairs, we assume that we can simplify the whole coarsening process (corresponding to the growth of the length scale up to $L$) through the annihilation of a single vortex–antivortex pair of size $L$ with a suitably defined interaction potential and other dynamical parameters.

From previous work on the dynamics of a vortex in the related model systems of anisotropic Heisenberg spin systems (which is based on a collective variable approach) [56]–[58], we might assume that the vortex acts like a small particle with finite ‘mass’ moving under the influence of an external force with some (length-scale-dependent) mobility. Then the distance $R$ between a vortex and an antivortex would be described by the following equation of motion:

$$M(R) \frac{d^2 R}{dt^2} + \frac{1}{\mu(R)} \frac{dR}{dt} = F(R) = -\frac{k}{R}. \quad (13)$$

Here we denote the length-dependent effective mass of a vortex as $M(R)$ and the length-dependent mobility of a vortex as $\mu(R)$. We also assume naturally that the vortex–
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antivortex pair is interacting via a Coulombic force $F(R) = -k/R$ in two dimensions with a proportionality constant $k$. We will set $k = 1$.

One important remaining question is how we set the functional form of $M(R)$ and $\mu(R)$. From the apparent superdiffusive behavior of the growing length scale obtained from the simple power law fit with an exponent a little larger than 1/2, we guessed that the growing length scale might be represented, at least in the late time regime, as $L(t) \sim (t \ln t)^{1/2}$. If we neglect the mass term, this kind of positive logarithmic correction can be obtained by assuming a logarithmically divergent vortex mobility $\mu(R)$. A similar phenomenon of logarithmically divergent mobility was also found in the quasi-two-dimensional diffusion of colloids [61] or diffusion of protein molecules on the membranes [62] under the hydrodynamic effect, and also in the vortex diffusion in the anisotropic Heisenberg system in two dimensions with spin precession [56]–[58]. As for the form of $M(R)$, we turned to the anisotropic Heisenberg model where the effective mass of a vortex with logarithmic dependence on the length scale was derived [56]–[58].

Motivated by these phenomena in analogous systems, we assumed that the effective mass and the mobility are logarithmically dependent on $R$ as

$$M(R) = m_0 + m_1 \ln(R/r_0), \quad \mu(R) = \mu_0 + \mu_1 \ln(R/r_0) \quad (14)$$

where $m_0, m_1, \mu_0, \mu_1$ are constants and $r_0$ denotes a shortest cutoff length scale in the system (corresponding to the vortex core size). Note that $m_0$ corresponds to the effective mass at the shortest cutoff length scale and similarly for $\mu_0$ for the vortex mobility.

The way we obtain the growth law from this simple model of vortex–antivortex pair dynamics is as follows. We first begin with a finite value of the size of vortex–antivortex pair $R = R^*$. For convenience, we also set the initial velocity of the vortex to be zero. Then we numerically solve the model equations of vortex–antivortex pair dynamics, (13) and (14), to obtain the time $\tau$ when the size of the vortex–antivortex pair $R$ becomes equal to $r_0$ the lower cutoff length scale. By plotting the resulting relation $R^*$ and $\tau$ we get the growth law of the coarsening dynamics.

It would be ideal if we could derive the parameters $m_0, m_1, \mu_0, \mu_1$ from the full dynamic equations (6) in the absence of thermal noise. But we do not know how to proceed from this equation to calculate these parameters. We simply tried to fit our simulation results (especially the dependence of the growing length scale on time) with the numerical solutions of our simple dynamic model with our tuned values of the parameters such that the two results agree most favorably.

In order to compare the numerical solutions of the simple dynamic model with the simulation results, we plot in figure 2(b) $L^2(t)/t$ versus $t$ from which we can check the behavior of the (multiplicative) correction to the diffusive growth. It shows that there exists a smooth crossover from early time ($t < 300.0$) behavior with larger slope to later time behavior with smaller (slower) slope. The solid line in figure 2(b) represent the results of numerical solutions from the simple model of vortex annihilation where we chose the parameters $m_0 = m_1 = 1.0, \mu_0 = 1.0,$ and $\mu_1 = 0.225$. We see that the numerical solution exhibits a reasonable agreement with the full simulation results (open circles). From our numerical solutions to the simple model we see that the early time behavior comes from the effect of the inertial term. If the mobility of the vortex is assumed to be a constant independent of the size of the vortex–antivortex pair, then the inertial effect ceases to be effective in the longer time regime and the correction term would approach a constant
plateau. However, the real simulation of the coarsening exhibits a steady increase in the correction in the late time regime. That is why we incorporated a length-dependent mobility of the vortex with a logarithmic dependence on the size of the vortex–antivortex pair. We found that, as the coefficient $\mu_1$ of the logarithmic term increases, the later time correction with logarithmic behavior gets stronger (data not shown here).

In the long time limit with large separation between vortices and antivortices, we can expect the inertial effect to be negligible. Therefore, due to the logarithmic divergence of the vortex mobility, we can analytically get the dominant asymptotic growth law as

$$L(t) \sim (t \ln t)^{1/2}.$$ 

The numerical solution of (13) in the long time region, shown in figure 2(b), confirms this growth law.

In addition, shown in figure 3 are the relaxations of the vortex number density $\rho(t)$ and the excess energy density $\Delta E(t)$ in a log–log plot. From the logarithmic slopes, we get apparent power law relaxation in time $\rho(t) \sim t^{-1.03(1)}$ and $\Delta E(t) \sim t^{-0.95(1)}$. In order to understand the relationship between these relaxation behaviors and also this apparent growth exponent (12), we note that there exists the so-called energy scaling relation between the excess energy $\Delta E(t)$ and the length scale $L(t)$ of the domain growth for the case of O(2) models in two dimensions, i.e., $\Delta E(t) \sim L^{-2}(t) \ln(L(t)/a_0)$ with $a_0$ denoting a short distance cutoff corresponding approximately to the size of a vortex core. This was first derived by Bray and Rutenberg [8]. Since the reversible mode coupling terms satisfy the energy conservation, this relation should be valid in the presence of mode coupling as well. From the excess energy relaxation (obtained numerically) we can extract a length scale $L_E(t)$ through the relation $\Delta E(t) \sim L_E^{-2}(t) \ln(L_E(t)/a_0)$. We can also extract another length scale $L_V(t)$ from the defect density relaxation that corresponds to the average separation between neighboring defects in such a way that $\rho(t) \sim L_V^{-2}(t)$, which

\[ \text{Figure 3. Relaxation of the vortex number density } \rho(t) \text{ (solid line) and the excess energy } \Delta E(t) \text{ (dashed line) exhibiting } \rho(t) \sim t^{-1.03(1)} \text{ and } \Delta E(t) \sim t^{-0.95(1)} \text{ respectively.} \]
is based on the assumption that the vortices are uniformly (and randomly) distributed in two-dimensional space. These two length scales as well as the numerical solution to the aforementioned simple model of vortex–antivortex annihilation, together with $L(t)$, are displayed in figure 4(a).

This figure shows that the length scale $L_E(t)$ agrees with the growing length scale of the domains $L(t)$ in almost entire times, which implies that the above Bray–Rutenberg relation is valid in the zero-temperature coarsening of Model E. The Bray–Rutenberg relation, together with the growth law (15), leads to the asymptotic relaxation behavior for $\Delta E(t)$ as follows:

$$\Delta E(t) \sim t^{-1} \left( 1 + \frac{\ln(\ln t)}{\ln t} \right).$$

(16)

Figure 4(a) also shows that there is considerable difference between the length scale $L_V(t) \sim \rho^{-1/2}(t)$ and the length scale $L(t)$. Although this discrepancy appears to become smaller for the longer times, their agreement in the long time limit does not seem to be guaranteed. This feature seems to indicate that the assumption of vortices being uniformly distributed is not valid. This is also shown in figure 4(b), depicting the vortex number density $\rho(t)$ versus $L(t)$, which can be fitted for a wide range of time with $\rho(t) \sim L(t)^{-1.92(1)}$.

The nonequilibrium spin autocorrelation function $A(t)$ is expected to be related to the growing length scale $L(t)$ through a new nonequilibrium exponent $\lambda$ as

$$A(t) \sim L^{-\lambda}(t).$$

(17)
We could extract the value of $\lambda$ by plotting $A(t)$ versus $L(t)$ as shown in figure 5, where we can see that, in the long time limit, the value of $\lambda$ approaches $\lambda \simeq 1.99(2)$. This value is much larger than the value of $\lambda \simeq 1.17$ for the case of a non-conserved O(2) model with no reversible mode coupling [26, 30, 33, 59, 60]. It appears that the higher mobility of the vortices in the present model causes a faster decay of the autocorrelation and hence a larger value of the $\lambda$ exponent ensues compared to the case of an O(2) model without mode coupling terms. It would be interesting to prove the conjecture $\lambda = d = 2$ analytically.

4. Summary and outlook

In this work, we studied the zero-temperature coarsening kinetics of Model E in which the order parameter field $\psi$ is dynamically coupled with an additional conserved field $m$. We found that the phase ordering kinetics of the two-dimensional O(2) model is modified considerably due to these reversible mode coupling terms. We can summarize the simulation results as follows. The growth of the typical length scale in Model E is found to exhibit an apparently superdiffusive behavior. We introduced a simple phenomenological model for the annihilation dynamics of a vortex-antivortex pair incorporating vortex inertia and logarithmically divergent mobility of the vortex. This model was shown to describe closely the positive logarithmic corrections of the simulation data, where the inertial effect dominates in the early time stage, while the positive logarithmic effect of the vortex mobility dominates in the late time stage.

We also investigated the autocorrelation function of the order parameter field. The numerical result for the value of $\lambda$ exponent for the present case of Model E is approximately $\lambda \simeq 1.99(2)$ which is quite distinct from the value $\lambda \simeq 1.17$ for the case of purely dissipative dynamics of an O(2) model without reversible mode couplings. Even though we do not have an analytic proof, it also leads us to conjecture that $\lambda = d = 2$.
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for Model E, which seems to be closely related to the higher mobility of the vortices as compared to the case of no mode coupling.

We think that the interaction of propagating spin waves (generated by the mode coupling) with the vortices (and antivortices) influences the motion of vortices (and antivortices) in such a way that the vortex mobility increases logarithmically in the sizes of vortex–antivortex pairs, thereby facilitating the annihilation of the vortex–antivortex pairs in the late time stage of the coarsening process with a logarithmic correction to a diffusive growth. Further investigation would be necessary to understand in more detail the interaction between the vortex defects as well as that between the vortices and the propagating spin waves.

In this work, we only considered the phase ordering kinetics of Model E in two dimensions, in which static coupling between the order parameter field and the conserved field \( m \) is ignored. In view of the recent surge of interest in the phase ordering kinetics in the ultra-cold atomic gases, it would be worthwhile to extend our study to the coarsening of Model F which possess a static coupling between the order parameter field and the conserved third component. In [18], the superfluid ordering kinetics of the Bose gas followed by an instantaneous quench from the high temperature normal state was studied using the Gross–Pitaevskii equation for the order parameter field in both two and three dimensions. The order parameter correlation function was shown to obey a critical dynamic scaling for \( d = 2 \) and a simple scaling for \( d = 3 \) with the growing length scale exhibiting a linear growth in time, i.e., \( L(t) \sim t^{1/2} \) with the dynamic exponent \( z \approx 1.0 \) due to the presence of propagating spin waves in both dimensions. Therefore it would be interesting to carry out a detailed study on the coarsening kinetics of the corresponding stochastic model, Model F, in order to see whether the observed linear growth in the Gross–Pitaevskii equation is preserved in Model F or not.

Recently, the Bose–Einstein condensate with nonzero spin, i.e., the spinor condensate, has been experimentally realized for alkali atoms such as \(^{23}\text{Na} \) [63] and \(^{87}\text{Rb} \) [64]. Interplay between superfluid and magnetic ordering in these spinor condensates leads to unusual phases and the corresponding topological defects\(^3\). One remarkable example is the existence of the nematic superfluid phase and the associated half-vortices in the so-called polar phase (observed in sodium atoms), which are usually observed in the nematic phase of liquid crystals [66,67]. Quite recently, the Model F dynamics was generalized and applied to the magnetic domain growth in the ferromagnetic phase of spinor condensates (observed in rubidium atoms), with and without the constraint of total magnetization conservation [68,69]. Extending our study towards these fascinating developments in the nonequilibrium dynamics of ultra-cold atoms would be quite rewarding.

Acknowledgments

This work was supported by a National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST; No. 2009-0090085). We also thank the Korea Institute for Advanced Study for providing computing resources on the Linux Cluster System (KIAS Center for Advanced Computation) for this work.

\(^3\) For an extensive recent review on spinor condensates, see [65].

doi:10.1088/1742-5468/2011/03/P03013
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References

[1] Gunton J D and Droz M, 1983 Introduction to the Theory of Metastable and Unstable States (Springer Lecture Notes in Physics) vol 183 (New York: Springer)

[2] Gunton J D, San Miguel M and Sahni P S, 1983 Phase Transitions and Critical Phenomena vol 8, ed C Domb and J L Lebowitz (New York: Academic)

[3] Furukawa H, 1985 Adv. Phys. 34 703

[4] Binder K, 1987 Rep. Prog. Theor. Phys. 50 783

[5] Bray A J, 1994 Adv. Phys. 43 357

[6] Onuki A, 2002 Phase Transition Dynamics (Cambridge: Cambridge University Press)

[7] Cugliandolo L F, 2010 Physica A 389 4360

[8] Bray A J and Rutenberg A D, 1994 Phys. Rev. E 49 R27

[9] Ma S and Mazenko G F, 1975 Phys. Rev. B 11 4077

[10] Hohenberg P C and Halperin B I, 1977 Rev. Mod. Phys. 49 5069

[11] Siggia E D, 1979 Phys. Rev. A 20 595

[12] Ahmad S, Das S K and Puri S, 2010 Phys. Rev. E 82 040107(R) and references therein

[13] Denniston C, Orlandini E and Yeomans J M, 2001 Phys. Rev. E 64 021701

[14] Tóth G, Denniston C and Yeomans J M, 2002 Phys. Rev. Lett. 88 105504

[15] Svensek D and Zumer S, 2002 Phys. Rev. E 66 021712

[16] Mondello M and Goldenfeld N, 1990 Phys. Rev. A 42 5865

[17] Briggs A J and Puri S, 1991 Phys. Rev. Lett. 67 2670

[18] Liu F and Mazenko G F, 1992 Phys. Rev. B 45 6989

[19] Blundell R E and Bray A J, 1994 Phys. Rev. E 50 5113

[20] Rojas F and Rutenberg A D, 2001 Phys. Rev. E 63 035101

[21] Qian H and Mazenko G F, 2003 Phys. Rev. E 68 021109

doi:10.1088/1742-5468/2011/03/P03013
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Qian H and Mazenko G F, 2004 Phys. Rev. E 70 031104
[44] Koo K, Baek W, Kim B and Lee S J, 2006 J. Korean Phys. Soc. 49 1977 [arXiv:cond-mat/0610590]
[45] Assel A and Zheng B, 2007 J. Phys. A: Math. Theor. 40 9957
[46] Lei X W and Zheng B, 2007 Phys. Rev. E 75 040104(R)
[47] Pleiner H, 1988 Phys. Rev. A 37 3986
[48] Ryskin G and Kremenetsky M, 1991 Phys. Rev. Lett. 67 1574
[49] Korshunov S E, 1994 Phys. Rev. B 50 13616
[50] Nogawa T and Nemoto K, 2009 J. Phys. Soc. Japan 78 064001
[51] Jelić A and Cugliandolo L F, 2011 J. Stat. Mech. P02032
[52] Kim B J, Minnhagen P and Olsson P, 1999 Phys. Rev. B 59 11506 and references therein
[53] Leocini X, Verga A D and Rufò S, 1998 Phys. Rev. E 57 6377
[54] Cerruti-Sola M, Clementi C and Pettini M, 2000 Phys. Rev. E 61 5171
[55] For the phase-ordering kinetics of a scalar $\phi^4$ model on a square lattice using the Hamiltonian dynamics, see Zheng B, 2000 Phys. Rev. E 61 153
Kockelkoren J and Chaté H, 2002 Phys. Rev. E 65 058101
Zheng B, 2002 Phys. Rev. E 65 058102
[56] Mertens F G and Bishop A R, 1999 Dynamics of Vortices in Two-Dimensional Magnets (Nonlinear Science at the Dawn of the 21st Century) ed P L Christiansen and M P Sørensen (New York: Springer)
[57] Kamppeter T, Mertens F G, Moro E, Sánchez A and Bishop A R, 1999 Phys. Rev. B 59 11349
[58] Kamppeter T, Mertens F G, Sánchez A, Bishop A R, Domínguez-Adame F and Grønbech-Jensen N, 1999 Eur. Phys. J. B 7 607
[59] Newman T J and Bray A J, 1990 J. Phys. A: Math. Gen. 23 L1279
Newman T J and Bray A J, 1990 J. Phys. A: Math. Gen. 23 4491
[60] Newman T J, Bray A J and Moore M A, 1990 Phys. Rev. B 42 4514
[61] Sané J, Padding J T and Louis A A, 2009 Phys. Rev. E 79 051402
[62] Saffman P G and Delbrück M, 1975 Proc. Nat. Acad. Sci. 72 3111
[63] Stenger J, Inouye S, Stamper-Kurn D M, Miesner H-J, Chikkatur A P and Ketterle W, 1998 Nature 396 345
[64] Sadler L E, Higbie J M, Leslie S R, Vengalattore M and Stamper-Kurn D M, 2006 Nature 443 312
[65] Ueda M and Kawaguchi Y, 2010 arXiv:1001.2072v2
[66] Mukerjee S, Xu C and Moore J E, 2006 Phys. Rev. Lett. 97 120406
[67] James A J A and Lamacraft A, 2010 arXiv:1009.0043v1
[68] Mukerjee S, Xu C and Moore J E, 2007 Phys. Rev. B 76 104519
[69] Lamacraft A, 2007 Phys. Rev. Lett. 98 160404

doi:10.1088/1742-5468/2011/03/P03013