Maximal f-vectors of Minkowski sums of large numbers of polytopes

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Abstract

It is known that in the Minkowski sum of \( r \) polytopes in dimension \( d \), with \( r < d \), the number of vertices of the sum can potentially be as high as the product of the number of vertices in each summand [2]. However, the number of vertices for sums of more polytopes was unknown so far.

In this paper, we study sums of polytopes in general orientations, and show a linear relation between the number of faces of a sum of \( r \) polytopes in dimension \( d \), with \( r \geq d \), and the number of faces in the sums of less than \( d \) of the summand polytopes. We deduce from this exact formula a tight bound on the maximum possible number of vertices of the Minkowski sum of any number of polytopes in any dimension. In particular, the linear relation implies that a sum of \( r \) polytopes in dimension \( d \) has a number of vertices in \( O(n^{d-1}) \) of the total number of vertices in the summands, even when \( r \geq d \). This bound is tight, in the sense that some sums do have that many vertices.

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1 Introduction

The Minkowski sum of two polytopes is defined as \( P_1 + P_2 = \{ x_1 + x_2 : x_1 \in P_1, x_2 \in P_2 \} \). Minkowski sums are of interest in various fields of theoretical and applied mathematics. While some applications only require sums of two polytopes in low dimensions (e.g. motion planning [7][8]), others use iterative sums of many polytopes in higher dimensions [9][12]. It is therefore desirable to study the complexity of such sums.

A trivial bound on the number of vertices of a sum is found as follows. Every vertex of a Minkowski sum decomposes into a sum of vertices of the summands. Therefore, there cannot be more vertices in the sum than there are possible decompositions. Thus, a trivial bound on the number of vertices in a Minkowski sum is the product of the number of vertices in the summands. That is, if \( P_1, \ldots, P_r \) are polytopes, and \( f_0(P) \) is the number of vertices of a polytope, then \( f_0(P_1 + \cdots + P_r) \leq f_0(P_1) \cdots f_0(P_r) \).

If we sum \( r \) polytopes in dimension \( d \), with \( r < d \), then the trivial bound is tight, that is, it is possible to choose summands with any number of vertices so that their sum has as many vertices as the trivial bound [2].

However, if we sum \( r \) polytopes in dimension \( d \) with \( r \geq d \), the trivial bound cannot be reached, except when summing \( d \) segments [10][11]. We assume here and in the rest of the article that all polytopes have at least two vertices, since a summand of only one vertex can be ignored without changing the properties of the sum.

The \( f \)-vector of a polytope encodes its number of faces of different dimensions. Maximal \( f \)-vectors are obtained for a particular class of Minkowski sums, called sums of polytopes in general orientations. We will therefore restrict our study to such sums.

We recently presented in [1] a result on sums of 3-dimensional polytopes in general orientations. We showed that the number of vertices in a sum of \( r \) summands can be deduced from the number of vertices in the summands and the number of vertices in sums of each of the \( \binom{r}{2} \) pairs of summands. Using this result, we found a tight upper bound on the number of vertices and facets in sums of 3-dimensional polytopes.

The basic reasoning of this previous result is to define a unique witness, called western-most corner, for all but two vertices of a polytope. These witnesses have the property that a western-most corner for a Minkowski sum of any number of summands is also necessarily a western-most corner for the sum of some pair of the summands. So the number of western-most corners in the total sum, and thus its number of vertices, can be found by examining sums of one or two summands only.

This prompted us to examine the possibility of extending the reasoning to higher dimensions and other faces, which resulted in this article. Our main result is presented in Theorem 1. It is a linear relation between the number of faces of a sum of \( r \) polytopes and the number of faces in the sums of less than \( d \) of the summand polytopes:

**Theorem 1** Let \( P_1, \ldots, P_r \) be \( d \)-dimensional polytopes in general orientations, \( r \geq d \), and each polytope full-dimensional. For any \( k \) in \( 0, \ldots, d-1 \),

\[
f_k(P_1 + \cdots + P_r) - \alpha = \sum_{j=1}^{d-1} (-1)^{d-1-j} \binom{r-1-j}{d-1-j} \sum_{S \in \mathcal{C}_j'} (f_k(P_S) - \alpha),
\]

where \( \mathcal{C}_j' \) is the family of subsets of \( \{1, \ldots, r\} \) of cardinality \( j \), \( P_S \) is the sum of polytopes \( \sum_{i \in S} P_i \); \( \alpha = 2 \) if \( k = 0 \) and \( d \) is odd, \( \alpha = 0 \) otherwise.
A slightly more general result also applies when summands are not full-dimensional. The intuitive explanation of the theorem is that for any face of the whole sum, we can find a witness of its existence by examining the faces of the same dimension in sums of \( d - 1 \) summands. However, if that witness exists in some sum of \( d - 2 \) summands, we will find it in many different sums of \( d - 1 \) summands. So we need to offset this by removing an appropriate number of times the witnesses in sums of \( d - 2 \) summands. But that in turn removes too many times witnesses that exist in some sum of \( d - 3 \) summands, so we need to add them back, etc. This implies that the total sum is smaller than the term for \( j = d - 1 \):

**Corollary 2**  Let \( P_1, \ldots, P_r \) be \( d \)-dimensional polytopes in general orientations, \( r \geq d \), and each polytope full-dimensional. For any \( k \) in \( 0, \ldots, d - 1 \),

\[
f_k(P_1 + \cdots + P_r) \leq \sum_{\mathcal{S} \subseteq \mathcal{P}_{r-1}} f_k(P_{\mathcal{S}}).
\]

From this result, we deduce bounds for the maximum possible number of vertices in a Minkowski sum of polytopes, for fixed number of vertices in the summands.

We find in particular that a sum of \( r \) polytopes in dimension \( d \), \( r \geq d \), where summands have \( n \) vertices in total, has less than \( \binom{n}{d-1} \) vertices, which is in \( O(n^{d-1}) \). In the case where each summand has at most \( n \) vertices, then the number of vertices of the sum is less than \( \binom{r}{d-1} n^{d-1} \), which is in \( O(r^{d-1} n^{d-1}) \). This is better than the previous known bound which was in \( O(r^{d-1} n^{2(d-1)}) \).

In the rest of the article, we shortly present the theory in Section 2. We first give an introduction to the concepts of west and western-most corner in three dimensions in Section 3, then extend them formally to higher dimensions in Section 4. We examine an introduction to the concepts of west and western-most corner in three dimensions in Section 3, then extend them formally to higher dimensions in Section 4. We examine the concepts of west and western-most corner in three dimensions in Section 3, then extend them formally to higher dimensions in Section 4.

### 2 Minkowski sums

Let \( P_1, \ldots, P_r \) be given polytopes. Their Minkowski sum is the polytope defined as

\[
P_1 + \cdots + P_r = \{ x_1 + \cdots + x_r : x_i \in P_i, \forall i \}.
\]

We assume in the following, and in the rest of the article, that every polytope is full-dimensional.

A nontrivial face of a polytope \( P \) in dimension \( d \) is the intersection of \( P \) with a support hyperplane of \( P \). Vertices, edges, facets, ridges are the faces of dimension 0, 1, \( d - 1 \) and \( d - 2 \) respectively. Thus, we can associate to each vector in the unit sphere \( S^{d-1} \) a face of the polytope, which is the intersection of the polytope with the support hyperplane to which the vector is outwardly normal: \( S(P; l) = \{ x \in P : \langle l, x \rangle \geq \langle l, y \rangle, \forall y \in P \} \).

Conversely, each face \( F \) of a \( d \)-dimensional polytope \( P \) can be associated with a region of the sphere \( S^{d-1} \), called the normal region, which is the set of unit vectors outwardly normal to some support hyperplane of \( P \) whose intersection with \( P \) is \( F \):

\[
\mathcal{N}(F; P) = \{ l \in S^{d-1} | F = S(P; l) \} = \{ l \in S^{d-1} | \langle l, x \rangle > \langle l, y \rangle, \forall x \in F, y \in P \setminus F \}.
\]

The normal region of a facet of \( P \) is thus a single point of \( S^{d-1} \), corresponding to the unit vector outwardly normal to the facet. The normal region of a face of dimension \( k \) is a relatively open subset of \( S^{d-1} \) of dimension \( d - 1 - k \).
We call a subset of the sphere $S^{d-1}$ **spherically convex** if for any two points in the subset, any shortest arc of great circle between the two points is inside the subset. If the polytope $P$ is full-dimensional, the normal regions of faces of $P$ are all disjoint, relatively open and spherically convex. They determine a subdivision of $S^{d-1}$ into a spherical cell complex which we call the *Gaussian map* of the polytope: $G(P) = \{N(F;P) : F \text{ face of } P\}$.

A property of Minkowski sums is that faces of the sum have a unique *decomposition* in faces of the summand. Let $F$ be a face of the Minkowski sum $P = P_1 + \cdots + P_r$, and $l$ be in $N(F;P)$. Then $F = F_1 + \cdots + F_l$, where $F_i = S(P_i;l)$ is a face of $P_i$. We can deduce that the normal region of a face of the sum is equal to the intersection of the normal regions of the faces in its decomposition: $N(F;P) = N(F_1;P_1) \cap \cdots \cap N(F_r;P_r)$. Thus the Gaussian map of the Minkowski sum is the *common refinement* of the Gaussian map of the summands:

$$G(P_1 + \cdots + P_r) = \{N(F_1;P_1) \cap \cdots \cap N(F_r;P_r) : F_i \text{ face of } P_i\}.$$ 

A polytope and its Gaussian map being dual structures, it is possible to study the number of faces of a polytope by studying the number of cells of its Gaussian map.

We say that a face of a Minkowski sum has an *exact decomposition* $F = F_1 + \cdots + F_r$ when its dimension is the sum of the dimension of the faces in its decomposition: $\dim(F) = \dim(F_1) + \cdots + \dim(F_r)$. That is, the decomposition is exact when there are no two parallel segments inside different faces in the decomposition. We say that polytopes are in general orientations when all faces of their Minkowski sum have an exact decomposition.

For fixed $f$-vectors of summands, the maximum number of faces of any dimension in the sum can always be reached by summands in general orientations. Therefore, we can assume summands are in general orientations when looking for sums with maximum number of faces.

Let $F$ be a face of the Minkowski sum $P = P_1 + \cdots + P_r$ of $d$-dimensional polytopes in general orientations. The face $F$ decomposes into a sum $F_1 + \cdots + F_r$ of faces of the summands, with $\dim(F) = \dim(F_1) + \cdots + \dim(F_r)$. Even if $r > d$, there are therefore at most $\dim(F)$ faces in the decomposition that have a dimension of more than 0. Let the *support* $I_F \subseteq \{1, \ldots, r\}$ of $F$ be the set of indices of these faces, with $|I_F| \leq \dim(F)$. Note that for any subface $G$ of $F$, $G$ decomposes into a sum $G_1 + \cdots + G_r$, where $G_i \subseteq F_i$ for all $i$; and so, $I_G \subseteq I_F$.

For any $S = \{i_1, \ldots, i_s\}$ subset of $\{1, \ldots, r\}$, let us define the *partial sum* $P_S = P_{i_1} + \cdots + P_{i_s}$.

### Lemma 3

Let $F$ be a facet of a Minkowski sum $P = P_1 + \cdots + P_r$ of $d$-dimensional polytopes in general orientations. Its normal region $N(F;P)$ is a node of $G(P)$. Then $N(F;P)$ is also a node of the Gaussian map $G(P_S)$ of a partial sum if and only if $I_F \subseteq S$.

**Proof.** Let $F_1 + \cdots + F_r$ be the decomposition of $F$, with $\dim(F_i) > 0$ if and only if $i \in I_F$. Since the summands are in general orientations, the decomposition

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1. There exist different definitions of convexity on a sphere. Note that according to this one, the only convex set containing antipodal points is the whole sphere.
is exact and \( \dim(F) = d - 1 = \dim(F_1) + \cdots + \dim(F_r) = \sum_{i \in I_F} \dim(F_i) \). The normal region \( \mathcal{N}(F; P) \) contains a single unit vector \( l \); and \( \mathcal{N}(F; P) \) is a node of \( \mathcal{G}(P_S) \) if and only if \( \dim(S(P_S; l)) = d - 1 \). Again, the decomposition is exact and \( \dim(S(P_S; l)) = \sum_{i \in S} \dim(S(P_i; l)) = \sum_{i \in S} \dim(F_i) \). Since \( \dim(F_i) > 0 \) if and only if \( i \in I_F \), and \( \sum_{i \in I_F} \dim(F_i) = d - 1 \), the result is obvious.

\section{Sums of polytopes in dimension 3}

To facilitate the comprehension of the proof of Theorem 1, we present informally in this section the argument for three dimensions, where it is more readily understood. We present the full proof for general dimensions in Section 4. The result for three dimensions has already been published, though only for vertices [1].

In dimension 3, the Gaussian map of a polytope is a spherical cell complex of \( S^2 \), which can be described as a planar graph embedded in \( S^2 \). The normal regions of facets, edges and vertices of the polytope are nodes, edges and faces of the graph respectively. Note that the normal regions of edges, edges of the graph, are arcs of great circles of \( S^2 \). The Gaussian map of a Minkowski sum is the overlay of the Gaussian maps of the summands.

Let \( P = P_1 + \cdots + P_r \) be a sum of 3-dimensional polytopes in general orientations. We choose on \( S^2 \) two antipodal points in generic position, so that no edge of \( \mathcal{G}(P) \) is aligned with them. In particular, the points are inside two distinct faces of \( \mathcal{G}(P) \). We call these two points \emph{north pole} and \emph{south pole}. We define \emph{west} in the usual way with respect to the poles, as a direction turning around the poles, clockwise from the north pole.

For any spherically convex subset \( C \) of \( S^2 \) that does not contain either pole, we define the \emph{western-most point} of \( C \) as the point in the closure of \( C \) that is to the west of all points in \( C \). We also define as \emph{western-most corner} of \( C \) the subset of \( C \) at distance less than \( \varepsilon \) of the western-most point, for a small \( \varepsilon > 0 \). Note that if \( C \) is a cell of \( \mathcal{G}(P) \), the western-most point is a node of \( \mathcal{G}(P) \) incident to \( C \), which is unique because no edge of \( \mathcal{G}(P) \) is aligned with the poles.

The normal region \( \mathcal{N}(F; P) \) of a facet \( F \) of the Minkowski sum \( P \) is a node of \( \mathcal{G}(P) \). Because the summands are in general orientations, there are no three edges of different \( \mathcal{G}(P_i) \) intersecting in a single point. Thus, a node in \( \mathcal{G}(P) \) is either a node in some \( \mathcal{G}(P_i) \), in which case \( I_F = \{i\} \), or it is the intersection of two edges in some \( \mathcal{G}(P_i + P_j) \), in which case \( I_F = \{i, j\} \) (See Figure 1). So a western-most corner in \( \mathcal{G}(P) \) is always a western-most corner in some \( \mathcal{G}(P_i) \), or a western-most corner in some \( \mathcal{G}(P_i + P_j) \) whose western-most point is the intersection of two edges.

So we can find the number of western-most corners in \( \mathcal{G}(P) \) by counting those in all \( \mathcal{G}(P_i) \) and those in all \( \mathcal{G}(P_i + P_j) \) whose western-most point is an intersection of edges. But then, the western-most corners in \( \mathcal{G}(P_i + P_j) \) also include those whose western-most point is a node of \( \mathcal{G}(P_i) \) or \( \mathcal{G}(P_j) \). Denoting as \( w_k(.) \) the number of western-most corners of \( k \)-dimensional cells in a Gaussian map, this means that
\[ w_k(\mathcal{G}(P)) = \sum_{i=1}^{r} w_k(\mathcal{G}(P_i)) + \sum_{1 \leq i < j \leq r} (w_k(\mathcal{G}(P_i + P_j)) - w_k(\mathcal{G}(P_i)) - w_k(\mathcal{G}(P_j))) \]

\[ = \sum_{1 \leq i < j \leq r} w_k(\mathcal{G}(P_i + P_j)) - (r - 2) \sum_{i=1}^{r} w_k(\mathcal{G}(P_i)), \quad k = 0, 1, 2. \]

Intuitively, we sum the number of western-most corners in different \( \mathcal{G}(P_i + P_j) \), and subtract \((r - 2)\) times the western-most corners in the \( \mathcal{G}(P_i) \), since they are each counted \((r - 1)\) times in the first sum.

But since there is one distinct western-most corner for every cell of a Gaussian map except the two 2-dimensional cells that contain a pole, and the cells of a Gaussian map correspond to faces of the underlying polytope, we have that for any polytope \( \mathcal{P} \),

\[ w_0(\mathcal{G}(P)) = f_2(\mathcal{P}), \quad w_1(\mathcal{G}(P)) = f_1(\mathcal{P}) \quad \text{and} \quad w_2(\mathcal{G}(P)) = f_0(\mathcal{P}) - 2. \]

Replacing \( w_k \) in the above equation by these, we get Theorem 1 for \( d = 3 \).

Here is a subtle detail of the argument. Let us say that a point \( p \) in the closure of a subset \( C \) of the sphere \( S^2 \) is a local optimum of \( C \) if \( p \) is the western-most point for the intersection of \( C \) with some open set containing \( p \). The reason we use the direction west is that the level curves for west, the meridians of geography, are arcs of great circles, i.e. geodesics; they intersect only once any other geodesic inside a spherically convex set. This guarantees that all local optima are also western-most points. This would not be the case had we used the direction south, because the level curves for south, the parallels, are not geodesics.

4 Extension to higher dimensions

We extend in this section the argument of Section 3 to higher dimensions. First, we extend the definition of west and western-most corners. We prove that the extension has the same property that (about) every cell of a Gaussian map has a single western-most corner. We also prove that the western-most corner of some cell in the Gaussian map of a Minkowski sum is also a western-most corner of some cell in any Gaussian map of a partial sum where its western-most point is a node of the map. Finally, we present the formula that allows us to count the number of western-most corners in the Gaussian map of the Minkowski sum.

Here is a brief summary of the proof. Let \( P = P_1 + \cdots + P_r \) be a Minkowski sum of \( d \)-dimensional polytopes in general orientations. Because the summands are in general orientations, a node of \( \mathcal{G}(P) \) is also a node of \( \mathcal{G}(P_S) \) if and only if the support of its underlying facet is contained in \( S \). This implies that in all \( \mathcal{G}(P_S) \) where a node exists, the local geometry of the map around the node is the same as in \( \mathcal{G}(P) \). Therefore, if the node is a local optimum in \( \mathcal{G}(P) \), it is also a local optimum in any \( \mathcal{G}(P_S) \) of which it is a node. This implies that a western-most corner in \( \mathcal{G}(P) \) is also a western-most corner in \( \mathcal{G}(P_S) \) if and only if its western-most point in \( \mathcal{G}(P) \) exists in \( \mathcal{G}(P_S) \), i.e. if and only if \( S \) contains the support of the underlying facet of the node. This is what allows us to use a counting argument for deducing the number of western-most corners in \( \mathcal{G}(P) \) from the number of western-most corners in the Gaussian map of partial sums.
Lemma 4  In a $d$-dimensional space, for any finite family $\{U_1, \ldots, U_n\}$ of $2$-dimensional linear subspaces, it is possible to find a linear subspace $U$ of dimension $d-2$ such that $U \cap U_i = \{0\}$ for any $i$ in $1, \ldots, n$.

Proof.  The orthogonal complements $U_i^\perp$ of the subspaces $U_i$ in the family are of dimension $d-2$. If we choose a vector $u$ that is not in these orthogonal complements, then for any $i$, $\text{span}(\{u\} \cup U_i^\perp)$ is of dimension $d-1$. If we choose now a vector $v$ that is not any of these subspaces of dimension $d-1$, then for any $i$, $\text{span}(\{u, v\} \cup U_i^\perp)$ is the $d$-dimensional space. Define $U = \text{span}(\{u, v\})^\perp$. If a vector is in $U \cap U_i$, it is orthogonal to $\text{span}(\{u, v\} \cup U_i^\perp)$, which is the whole space, and so it is the origin 0. 

This is a simple extension of the fact that in three dimensions, for any number of planes going through the origin, we can choose a vector that is in none of the planes.

Note that the intersection of $U$ with $S^{d-1}$ is equivalent to the sphere $S^{d-3}$. If $d = 3$, the intersection is the two antipodal points on $S^2$ that we named north and south poles in Section 3, and if $d = 2$, it is the empty set because $U = \{0\}$.

We choose an orthonormal basis $e_1, \ldots, e_d$ of the $d$-dimensional space such that $U = \text{span}\{e_3, \ldots, e_d\}$. We then define successive parametrizations of the spheres $S^n$, $1 \leq n \leq d-1$ as follows:

$S^1 = \{\sin(\theta_1)e_1 + \cos(\theta_1)e_2 : \theta_1 \in [0, 2\pi]\},$

$S^n = \{\sin(\theta_n)S^{n-1} + \cos(\theta_n)e_{n+1} : \theta_n \in [0, \pi]\}, \quad n = 2, \ldots, d-1.$

Note that a point of $S^{d-1}$ is in $U$ if and only if $\sin(\theta_j) = 0$ for some $j = 2, \ldots, d-1$. For any point of $S^{d-1}$ not in $U$, we define the direction west as $\hat{\theta}_1$, the direction of augmentation of $\theta_1$. Note that for $d = 3$, it is equivalent to the definition of Section 3, and for $d = 2$, it is a direction running around $S^1$. In $S^{d-1}$, west is not defined on the subspace $U$ because $\hat{\theta}_1 = 0$, so that the intersection of $U$ with $S^{d-1}$ is a sphere of dimension $d-3$ that plays the same role as poles in three dimensions.

Recall that in our parametrization of $S^{d-1}$, $\theta_1$ is in $[0, 2\pi)$. Formally, for any points $p$ and $q$ of $S^{d-1}$ that are not in $U$, we say that $p$ is to the west of $q$ if $\theta_1(p) \in [\theta_1(q), \theta_1(q) + \pi]$ and $\theta_1(q) < \pi$, or if $\theta_1(p) \in [\theta_1(q), 2\pi) \cup [0, \theta_1(q) - \pi]$ and $\theta_1(q) \geq \pi$.

For any spherically convex subset $C$ of $S^{d-1}$ that does not intersect $U$, we define the western-most point of $C$ as the point in the closure of $C$ that is to the west of all points in $C$. The next lemma, proved in Appendix A.1, shows that the western-most point exists.
Lemma 5 If a spherically convex subset \( C \) of \( S^{d-1} \) does not intersect \( U \), it is in a hemisphere defined by \( \theta_1 \in [\alpha, \alpha + \pi] \) or \( \theta_1 \in [0, \alpha] \cup [\alpha + \pi, 2\pi) \) for some \( \alpha \in [0, \pi) \).

We also define as western-most corner of \( C \) the subset of \( C \) at distance less than \( \varepsilon \) of the western-most point, where \( \varepsilon > 0 \) is smaller than the distance between any two non-incident cells in \( G(P) \). Note that the western-most point of \( C \) is also the western-most point of the western-most corner of \( C \).

Recall that the Gaussian map of a polytope is a subdivision of \( S^{d-1} \) into a spherical cell complex. For any cell \( C \) of \( G(P) \) that does not intersect \( U \), the western-most point of \( C \) is a node incident to \( C \), which is unique because otherwise there would be a great circle containing an edge of \( G(P) \) and intersecting \( U \), which contradicts the way we chose \( U \). As a consequence, there is one unique western-most corner for each cell of a Gaussian map that does not intersect \( U \).

We now prove that a western-most corner of some cell of \( G(P) \) is also a western-most corner of a cell of the Gaussian map of any partial sum \( G(P_S) \) if and only if its western-most point is a node of \( G(P_S) \). We call a point \( p \) in the closure of a subset \( C \) of \( S^{d-1} \) a local optimum of \( C \) if \( p \) is the western-most point of the intersection of \( C \) with some open subset of \( S^{d-1} \) containing \( p \).

Lemma 6 A point \( p \) is a local optimum of a cell \( C \) of a Gaussian map \( G \) if and only if it is a western-most point of \( C \).

The proof is in Appendix A.1. For the next lemma, recall that \( \varepsilon > 0 \) is smaller than the distance between any two non-incident cells in \( G(P) \).

Lemma 7 Let \( F \) be a facet of \( P \), with its normal region \( \mathcal{N}(F; P) \) a node of \( G(P) \). Let \( p \) be a point of \( S^{d-1} \) at distance less than \( \varepsilon \) of \( \mathcal{N}(F; P) \). For any partial sum \( P_S \) with \( I_F \subseteq S \), the dimensions of the cells containing \( p \) in \( G(P) \) and \( G(P_S) \) are the same.

Proof. In the Gaussian map \( G(P) \), the subset of \( S^{d-1} \) at a distance less than \( \varepsilon \) of \( \mathcal{N}(F; P) \) intersects only the normal regions of subfaces of \( F \). Therefore, for any point \( p \) in that subset, \( S(P; p) \) is a subface of \( F \). Recall that for any subface \( G \) of a facet \( F \), \( I_G \subseteq I_F \). So for any partial sum \( P_S \) such that \( I_F \subseteq S \), \( I_G \subseteq S \), which means that not only \( P_S \) has a facet with the same normal region as \( F \), but \( S(P_S; p) \) is a subface of that facet with the same dimension as \( G \), and \( p \) is in a cell of the same dimension in \( G(P_S) \) as in \( G(P) \). \( \blacksquare \)

We finally have the tools to prove:

Lemma 8 Let \( W \) be a western-most corner of a cell \( C \) in \( G(P) \), with \( \mathcal{N}(F; P) \) the western-most point of \( C \). Then \( W \) is a western-most corner of some cell of the same dimension in the Gaussian map of a partial sum \( G(P_S) \) if and only if \( I_F \subseteq S \).

Proof. First, if \( I_F \not\subseteq S \), then \( \mathcal{N}(F; P) \) is not a node of \( G(P_S) \), and so \( W \) cannot be a western-most corner. Suppose \( I_F \subseteq S \); then \( \mathcal{N}(F; P) \) is a node of \( G(P_S) \). Furthermore, by Lemma 7, the points in \( W \) are in a cell of the same dimension in \( G(P_S) \) as in \( G(P) \), and the points in the closure of \( W \) are in a cell of the same dimension in \( G(P_S) \) as in \( G(P) \). As a consequence, since \( \mathcal{N}(F; P) \) is the western-most point of \( W \) in \( G(P) \), it is also the western-most point of \( W \) in \( G(P_S) \). But \( W \) is the intersection, of the cell it is in, with an open subset, and so \( \mathcal{N}(F; P) \) is a local optimum of the cell that contains
Figure 2: Representation of a map in $S^3$ by stereographic projection in Euclidean space. West defined in $S^3 \setminus U$ is turning around the intersection of $S^3$ with a 2-dimensional subspace $U$. Cells which intersect the subspace have their western-most corner defined by a different direction west defined on the intersection, which is equivalent to $S^1$.

This is the most important lemma. It is the ultimate goal of the definitions in this section, which is to have a witness of the existence of a cell, a witness whose presence in the Gaussian maps of partial sums depends on a simple rule.

However, according to the definitions so far, cells intersecting $U$ do not have a western-most corner. In any cell that intersects $U$, it is possible to turn around $U$, always going west, much like the way it is possible to turn around a pole on $S^2$. To deal with this problem, we consider the restriction of the Gaussian map to $U$. Let us denote as $S_U$ the intersection of $S^{d-1}$ with $U$. $S_U$ is a sphere equivalent to $S^{d-3}$, and the restriction of a spherical cell complex on $S^{d-1}$ to $S_U$ also defines a spherical cell complex on $S_U$. In fact, the restriction to $S_U$ of the Gaussian map on $S^{d-1}$ of a $d$-dimensional polytope is the Gaussian map on $S_U$ of the orthogonal projection of the polytope onto $U$.

Since $S_U$ is a sphere equivalent to $S^{d-3}$, we can define west on $S_U$ as we have done for $S^{d-1}$ (See Figure 2). For any cell of $\mathcal{G}(P)$ that intersects $S_U$, we define its western-most corner as the western-most corner of its intersection with $S_U$ in the restriction of $\mathcal{G}(P)$ to $S_U$. If $d > 5$, this again does not define a westernmost point for every cell, because west is not defined on the intersection of $S_U$ with a subspace of dimension $d-4$; so we restrict the Gaussian map to that subspace, and start again recursively.

We present now the complete construction. We have chosen a subspace $U$ of dimension $d-2$ such that its intersection with any two-dimensional plane containing an edge of $\mathcal{G}(P)$ is just the origin. Let us write $U^{d-2} = U$, and denote as $G^{d-2}$ the restriction of the Gaussian map $\mathcal{G}(P)$ to $U^{d-2}$. $G^{d-2}$ is a spherical cell complex on $S^{d-3}$. Then, for any $i$ larger than 2, we define from $U^i$ and $G^i$ a subspace $U^{i-2}$, which is a subspace of $U^i$, such that the intersection of $U^{i-2}$ with any two-dimensional plane containing an edge of $G^i$ is just the origin. We then define $G^{i-2}$ as the restriction of $G^i$ to $U^{i-2}$, which is a spherical cell complex on $S^{i-3}$. This defines a sequence of subspaces $U^{d-2} \supset U^{d-4} \supset \cdots$ and a sequence of spherical cell complexes $G^{d-2} \supset G^{d-4} \supset \cdots$. If
$d$ is even, the sequences end with $G^2$, which is a spherical cell complex on $S^1$, and $U^0$ is just the origin and does not intersect $S^1$. If $d$ is odd, they end with $G^3$, a spherical cell complex on $S^2$, and $U^1$ is a one-dimensional subspace whose intersection with $S^2$ defines two antipodal points that we called north and south pole in Section 3.

For each $G^i$, spherical cell complex of $S^{i-1}$, we can define a direction west for every point of $S^{i-1}$ that is not on $U^{i-2}$, as we have done for $G(P)$ on $S^{d-1}$. Then for any cell $C$ of $G(P)$, let $i$ be the smallest number such that the intersection of $C$ with $U^i$ is nonempty. We then define the western-most point of $C$ to be the western-most point of $C \cap U^i$, cell of $G^i$ on the sphere $S^{i-1}$. The western-most corner of $C$ is also the western-most corner of $C \cap U^i$.

Note that if a cell $C$ is of dimension $d - k - 1$, i.e. it is the normal region in $G(P)$ of a $k$-dimensional face, it does not intersect $U^i$ for any $i \geq k$, because $U^i$ was chosen so as not to intersect edges and nodes of $G^{i+2}$, which are restrictions to $G^{i+2}$ of cells of dimension $d - i - 1$ and $d - i - 2$ in $G(P)$. For instance, if $d$ is odd, only normal region of vertices may intersect with $U^1$. Since $U^1$ only intersects $S^{d-1}$ in two antipodal points, there are exactly two cells of dimension $d - 1$ in any Gaussian map that intersect $U^1$. These are the only two cells that do not have a western-most corner. When $d$ is even, west is defined on every point of $G^2$, spherical cell complex on $S^3$, and so every cell of a Gaussian map has a western-most corner.

We have now defined a western-most corner for every cell of Gaussian maps, with the exception, if $d$ is odd, of the two cells that contain a pole. As before for cells that do not intersect $U$, the western-most corner of a cell of $G(P)$ is also a western-most corner of a cell of the same dimension in the Gaussian map of a partial sum $G(P S)$ if and only if $S$ contains the support of its western-most point, or rather the support of the cell whose restriction is its western-most point. The cardinality of the support is always less than $d$.

Now that we have a complete definition of western-most corners, all that remains is to count them. The support of any face of $P$ has cardinality less than $d$, so all western-most corners of cells of $G(P)$ can be found in the Gaussian map of partial sums of at most $d - 1$ summands. It is not difficult to see that for any $j \geq |I_F|$, there are $\binom{r - |I_F|}{j - |I_F|}$ subsets of $\{1, \ldots, r\}$ of cardinality $j$ that contain $I_F$.

The formula of the main theorem was found by observing low-dimensional cases. It is based on the following combinatorial equivalence:

**Lemma 9** For any $1 \leq s < d \leq r$,

$$
\sum_{j=1}^{d-1} (-1)^{d-1-j} \binom{r-1-j}{d-1-j} \binom{r-s}{j-s} = 1.
$$

The proof is in Appendix A.2

By this Lemma, if we count all the western-most corners in partial sums of $j$ polytopes, multiply by $(-1)^{d-1-j} \binom{r-1-j}{d-1-j}$, and sum over $j$, we end up counting exactly once each western-most corner, no matter what is the cardinality of the relevant support. Therefore, if $w_k(.)$ is the number of western-most corners of $k$-dimensional cells in a
where \( \mathcal{C}_j \) is the family of subsets of \( \{1, \ldots, r\} \) of cardinality \( j \). Since there is one western-most corner of a \( k \)-dimensional cell for each \( d - 1 - k \) face of the underlying polytope, this proves Theorem 4 for any \( d \) and \( k \). The only exception is that if \( d \) is odd, any Gaussian map has two regions of dimension \( d - 1 \) that contain the poles, and that have no western-most corner, and so in that case, \( w_{d-1}(G(P)) = f_0(P) - 2 \). This gives the special case of the theorem for \( d \) odd and \( k = 0 \).

In order to prove Corollary 2, it is enough to point out that each western-most corner of cells of \( G(P) \) can be found at least once (and often a lot more) in the Gaussian map of partial sums of \( d - 1 \) summands. And so, the last term of the sum in Theorem 1 is an upper bound on the number of faces.

5 Maximum number of vertices

Using Corollary 2, we show bounds on the number of vertices of Minkowski sums. The trivial bound tells us that if \( r < d \), then \( f_0(P_1 + \cdots + P_r) \leq \prod_{i=1}^r f_0(P_i) \). Consequently, if \( r \geq d \), we get by Corollary 2 that

\[
f_0(P_1 + \cdots + P_r) \leq \sum_{S \in \mathcal{C}_{d-1}} \prod_{i \in S} f_0(P_i).
\]

This can be seen as enumerating all possible combinations of \( d - 1 \) vertices chosen each from a different summand. This is necessarily lower than all possible combinations of \( d - 1 \) vertices from the summands. If the summands have \( n \) vertices in total, this upper bound is \( \binom{n}{d-1} \), which is in \( O(n^{d-1}) \). If each summands has \( n \) vertices, then we have:

\[
f_0(P_1 + \cdots + P_r) \leq \sum_{S \in \mathcal{C}_{d-1}} n^{d-1} = \binom{r}{d-1} n^{d-1},
\]

which is in \( O(r^{d-1} n^{d-1}) \). The previous known bound was in \( O(r^{d-1} n^{2(d-1)}) \) [5].

Note that a construction from [2] allows us to choose \( d - 1 \) polytopes of \( n \) vertices each such that the sum has \( n^{d-1} \) vertices. It is easy to adapt this construction to choose \( r \) polytopes such that any partial sum of \( d - 1 \) summands has this many vertices, which by Theorem 1 means that the total sum has exactly (for \( d \) even) \( \sum_{j=1}^{d-1} (-1)^{d-1-j} \binom{r-1-j}{d-1-j} \binom{r}{d-1-j} n^j \) vertices.

Unfortunately, except in three dimensions, the maximum number of facets of a Minkowski sum of polytopes remains open, even for two summands in four dimensions, so we cannot write an upper bound for facets. We can however tell that if the number of facets in the sum of \( d - 1 \) polytopes is in \( O(p(n)) \), their number in the sum of \( r \geq d \) polytopes is in \( O(r^{d-1} p(n)) \). Finding \( p(n) \) should be the object of further research.
6 Summary

We have extended the intuitive concept of west from three dimensions to higher dimensions. Thanks to the properties of the concept, we were able to prove a relation on the number of vertices in sums of many polytopes, and show that this number has a comparatively low order of complexity. For faces of higher dimensions, the result also shows that the complexity of Minkowski sums of many polytopes is not much more complex than that of $d - 1$ summands.

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A Appendix

A.1 Proof of Lemma 5 and 6

We prove in this appendix that the subspace $U$ plays the same role as poles in 3 dimensions, and that our definition of west has the property that if we “optimize” in direction west over a spherically convex subset of $S^{d-1}$, a local optimum of the subset is also a global optimum. We start with a few lemmas:

Lemma 10 Let $p$ and $p'$ be distinct antipodal points of $S^{d-1}$. Suppose $p$ and $p'$ are on a same subspace. Then all points on the great circle of $S^{d-1}$ containing $p$ and $p'$ are on that subspace.

Proof. A great circle of $S^{d-1}$ is the intersection of $S^{d-1}$ with a 2-dimensional space. Suppose a great circle contains $p$, $p'$ and $q$, with $p$ and $p'$ on a subspace $L$, but $q \notin L$. Then the intersection of the 2-dimensional space containing the great circle with $L$ is 1-dimensional, and so it is a line going through the origin. Since $p$ and $p'$ are both on that line and in $S^{d-1}$, they are either the same or antipodal. \[\blacksquare\]

For any $\theta$, let us denote as $L(\theta)$ the subspace orthogonal to $\sin(\theta)e_1 + \cos(\theta)e_2$, i.e. the set \{ $p$ : $\langle p, \sin(\theta)e_1 + \cos(\theta)e_2 \rangle = 0$ \}.

Lemma 11 A point $p$ of $S^{d-1}$ is in $L(\theta) \cap S^{d-1}$ if and only if $\cos(\theta_1(p) - \theta)) = 0$ or $p \in U$.

Proof. Any $p$ in $S^{d-1}$ is written in our parametrization as $\rho s + u$, with $\rho \geq 0$, $s \in S^1$ and $u \in U$. We can write $s = \sin(\theta_1(p))e_1 + \cos(\theta_1(p))e_2$, and $\rho = 0$ if and only if $p \in U$. Therefore, $\langle p, \sin(\theta)e_1 + \cos(\theta)e_2 \rangle = \rho(\sin(\theta_1(p)) + \cos(\theta))\cos(\theta_1(p)) = \rho\cos(\theta_1(p) - \theta)$. So $p$ is in $L$ if and only if $\rho\cos(\theta_1(p) - \theta) = 0$, which is if and only if $\rho = 0$ or $\cos(\theta_1(p) - \theta) = 0$. The result follows. \[\blacksquare\]

Lemma 12 Let $K$ be great circle of $S^{d-1}$. Then either $K$ is inside $U$; or $K$ intersects $U$, $K \setminus U$ has two connected components $K_1$ and $K_2$ such that for any two points $p \in K_1$, $p' \in K_2$, $\theta_1(p) + \pi = \theta_1(p')$; or $K$ does not intersect $U$, and for any two distinct points $p$, $p'$ in $K$, $\theta_1(p) \neq \theta_1(p')$, and $\theta_1(p) + \pi = \theta_1(p')$ if and only if $p$ and $p'$ are antipodal.

Proof. Suppose $p$ and $p'$ distinct in $K$ such that $\theta_1(p) = \theta_1(p')$; or suppose $p$ and $p'$ non-antipodal in $K$ such that $\theta_1(p) + \pi = \theta_1(p')$; or suppose $p'$ is in $K \cap U$ and any $p$ in $K$. In all three cases, by Lemma 11 $p$ and $p'$ are both in the subspace $L(\theta_1(p) + \pi/2)$. By Lemma 11 all points on $K$ are in $L(\theta_1(p) + \pi/2)$. Then by Lemma 11 again, for any $q$ on the arc of great circle, $q \in U$ or $\cos(\theta_1(q) - (\theta_1(p) + \pi/2)) = 0$. Suppose $K \cap U$ contains more than two points. Then some of them are distinct and non-antipodal, and by Lemma 11 $K$ in inside $U$. Otherwise, for any $q$ and $q'$ antipodal on $K \setminus U$, $\theta_1(q') = \theta_1(q) \pm \pi$. So there must be two antipodal points of $K$ inside $U$ separating $q$ and $q'$, and so $K \setminus U$ has two connected components.
implies that there are two points $p$ and $p'$ in $K$ such that $\theta_1(p) + \pi = \theta_1(p')$, $p$ and $p'$ must be antipodal; and $K \cap U$ is empty.

Note that if a great circle of $S^{d-1}$ does not intersect $U$, then $\theta_1$ is different in any two points of the great circle. Since the parametrization is smooth on $S^{d-1} \setminus U$, $\theta_1$ augments monotonically and continuously in one direction around the great circle, except in one point when it drops from $2\pi$ to 0.

Let us recall Lemma 5 before proving it:

**Lemma 5** If a spherically convex subset $C$ of $S^{d-1}$ does not intersect $U$, it is in a hemisphere defined by $\theta_1 \in [\alpha, \alpha + \pi]$ or $\theta_1 \in [0, \alpha] \cup [\alpha + \pi, 2\pi]$ for some $\alpha \in [0, 2\pi)$.

**Proof.** Let $T_1(C)$ be the set of values of $\theta_1$ over $C$ in our parametrization. Since $C$ does not intersect $U$, $T_1(C)$ is connected. Suppose $T_1(C)$ is $\{0; 2\pi\}$, then there are two points $p, p'$ in $C$ with $\theta_1(p) + \pi = \theta_1(p')$. Because $C$ is spherically convex, any shortest arc of great circle between $p$ and $p'$ is contained in $C$. If $p$ and $p'$ are antipodal, then $C$ is the whole sphere and intersects $U$. Otherwise, by Lemma 12 the arc of great circle again contains a point in $U$. This is a contradiction.

Otherwise, suppose without loss of generality that the supremum of $T_1(C)$ is $3\pi/2$. Then either $C$ is in the hemisphere defined by $\theta_1 \in [\pi/2, 3\pi/2]$, or there is a $\delta > 0$ such that there are two points $p, p'$ in $C$ with $\theta_1(p) + \pi = \theta_1(p') = 3/2 - \delta$. As above, this implies that $C$ contains a point in $U$, which is a contradiction.

Let us recall Lemma 6 before proving it:

**Lemma 6** A point $p$ is a local optimum of a cell $C$ of a Gaussian map $G$ if and only if it is a western-most point of $C$.

**Proof.** By definition, a western-most point is always a local optimum. Assume $p$ is a local optimum of $C$, and that some distinct $p'$ is the western-most point of $C$, and therefore also a local optimum. Then the shortest arc of great circle between $p$ and $p'$ is in $C$. Let $\alpha = \theta_1(p') - \theta_1(p)$. If $\cos(\alpha) \neq 0$, then by Lemma 12 the great circle defined by $p$ and $p'$ does not intersect $U$, and $\theta_1$ augments continuously from $p$ to $p'$ except possibly in one point when it jumps from $2\pi$ to $0$. So there is a $q$ in the intersection of the arc of great circle from $p$ to $p'$ with the open set that proves $p$ is a local optimum. For $q$ close enough, $\theta_1(q) > \theta_1(p)$, and so $p$ is not a local optimum, a contradiction.

Suppose now $\alpha = \pi$. If $p$ and $p'$ are antipodal, then any great circle containing $p$ and $p'$ is in $C$, and $C$ is the whole sphere, a contradiction. If $p$ and $p'$ are not antipodal, then by Lemma 12 there is a point in the arc of great circle from $p$ to $p'$ that is in $U$, and so $C$ intersects $U$. But this means $C$ has no western-most point, a contradiction.

Suppose now $\alpha = 0$. Then $p$ and $p'$ are incident to a cell where $\theta_1$ is fixed. But this means that the great circles containing edges of the cell intersect $U$, which contradicts the way we have chosen $U$.

Therefore, it is impossible to have a local optimum $p$ and a distinct western-most point $p'$ of a same cell.

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A.2 Proof of Lemma 9

We prove here the combinatorial equivalence used for the formulation of Theorem 1. The relation can also be derived from a protean family of equivalences of type

\[ \sum_{j=0}^{c} (-1)^{c-j} \binom{a+j}{b+c} \binom{c}{j} = \binom{a}{b} , \quad b < a , \quad c \geq 0. \]

See also [4, p. 169], [6, p. 149], [13, p. 285] on this subject.

Let us recall Lemma 9 before proving it:

**Lemma 9** For any \( 1 \leq s < d \leq r \),

\[ \sum_{j=1}^{d-1} (-1)^{d-1-j} \binom{r-1-j}{d-1-j} \binom{r-s}{j-s} = 1. \]

**Proof.** We prove the Lemma by induction over \( r \). We know that for any \( d \),

\[ \sum_{j=0}^{d} (-1)^{j} \binom{d}{j} = 0. \]

We can also write \( \sum_{j=0}^{d} (-1)^{d-s} \binom{d-s}{j-s} = 0 \), for any \( 1 \leq s < d \), and so we have \( \sum_{j=0}^{d} (-1)^{d-1-j} \binom{d-s}{j-s} = 1 \). We can also write

\[ \sum_{j=0}^{d-1} (-1)^{d-1-j} \binom{d-1-j}{d-1-j} \binom{d-s}{j-s} = 1. \]

This proves the relation for \( r = d \). Assume the relation is proved for \( r \). Then

\[ \sum_{j=0}^{d-1} (-1)^{d-1-j} \binom{r-1-j}{d-1-j} \binom{r-s}{j-s} = 1 , \]

\[ = \sum_{j=0}^{d-1} (-1)^{d-1-j} \binom{r-j}{d-1-j} \binom{r-s}{j-s} - \sum_{j=0}^{d-1} (-1)^{d-1-j} \binom{r-1-j}{d-2-j} \binom{r-s}{j-s} . \]

Replacing \( j \) in the second sum with \( j' - 1 \), we get

\[ \sum_{j=0}^{d-1} (-1)^{d-1-j} \binom{r-j}{d-1-j} \binom{r-s}{j-s} - \sum_{j=1}^{d} (-1)^{d-j'} \binom{r-j'}{d-1-j'} \binom{r-s}{j'-1-s} = 1. \]

In the second sum, the term \( j' = d \) gives zero, so we can remove it and add one for \( j' = 0 \), which also gives zero.

\[ \sum_{j=0}^{d-1} (-1)^{d-1-j} \binom{r-j}{d-1-j} \binom{r-s}{j-s} - \sum_{j=0}^{d-1} (-1)^{d-j'} \binom{r-j'}{d-1-j'} \binom{r-s}{j'-1-s} = 1. \]

Grouping the sums, we get

\[ \sum_{j=0}^{d-1} (-1)^{d-1-j} \binom{(r+1)-1-j}{d-1-j} \binom{(r+1)-s}{j-s} = 1, \]

and so the relation is true for \( r + 1 \), which proves Lemma 9 by induction.