Leading Chiral Logarithms of $K_S \to \gamma\gamma$ and $K_S \to \gamma \ l^+l^-$ at two Loops

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Abstract

We obtain the leading divergences at two-loop order for the decays $K_S \to \gamma\gamma$ and $K_S \to \gamma \ l^+l^-$ using only one-loop diagrams. We then find the double chiral logarithmic corrections to the decay branching ratio of $K_S \to \gamma\gamma$ and to the decay rate for $K_S \to \gamma \ l^+l^-$. It turns out that these effects are numerically small and therefore make a very small enhancement on the branching ratio and decay rate. We also derive an expression for the corrections of type $\log \mu \times \text{LEC}$. Numerical analysis done for the process $K_S \to \gamma\gamma$ shows that these single logarithmic effects can be sizable but come with opposite signs with respect to the double chiral logarithms.

keywords: Kaon rare decay; Weak and Strong Chiral Lagrangian; Leading Chiral Logarithm

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1 Introduction

The non-leptonic kaon decay $K_S \to \gamma\gamma$ provides a decent testing bed for the effective chiral Lagrangian method. The reason hinges in the fact that to one-loop order, there is no short-distance effects due to the Furry’s theorem [1] and the decay amplitude at one-loop order in chiral perturbation theory (ChPT) is free from unknown low energy constants (LECs) [2, 3]. For a good recent review on the kaon physics within the Standard Model one may consult [4]. The decay rate and branching ratio at one-loop order in ChPT are evaluated in [2, 3]. It gives for the branching ratio $\text{Br}(K_S \to \gamma\gamma) = 2.1 \times 10^{-6}$. This finding is in good agreement with the experimental measurement of NA31 that obtained $\text{Br}(K_S \to \gamma\gamma) = (2.4 \pm 0.9) \times 10^{-6}$ [5] and with that of KLOE that measured $\text{Br}(K_S \to \gamma\gamma) = (2.26 \pm 0.12) \times 10^{-6}$ [6]. On the other hand, the most recent measurement from NA48, obtained $\text{Br}(K_S \to \gamma\gamma) = 2.71 \times 10^{-6}$ [7] with a total uncertainty of about 3%. The latter experiment opens up the possibility of a sizable correction from effects beyond one-loop order.

It is demonstrated within the dispersion relation approach in [8, 9] that in fact the $\pi\pi$ rescattering in the S-wave channel are important effects especially for processes without counterterm contribution at one-loop order. They used in [8] the Padé approximation for the Omnès function in the full unitarization procedure and found a significant enhancement in branching ratio, $\text{Br}(K_S \to \gamma\gamma) = 2.3 \times 10^{-6}$. This brings the theoretical prediction into better agreement with the present experimental world average, $\text{Br}(K_S \to \gamma\gamma) = (2.63 \pm 0.17) \times 10^{-6}$ [10].

We now turn to the main point which motivates the present research. What would be the size of the two-loop effects or order $p^6$ in chiral perturbation theory for the weak decay $K_S \to \gamma\gamma$? In fact, the presence of unknown low energy constants in the weak Lagrangian at next-to-leading order (NLO) have hindered our predictivity within ChPT at full two-loop order. However, of all different types of contributions to the full two-loop calculation, there is a part which is independent of the unknown constants and can be evaluated using leading two-loop divergences. These are the so-called Leading Logarithmic (LL) corrections. Since the chiral LL corrections (single logs) are absent at one-loop order for $K_S \to \gamma\gamma$, it is deemed interesting to investigate the importance of the LL contribution at two-loop order, i.e. chiral double log corrections. In the weak sector, the first study on the LL effects at two-loop order is done for the decay $K \to \pi\pi$ in [11]. In the decay $K \to \pi\pi$, pion loop integral and kaon loop integral do not decouple in the subclass of two-loop diagrams which are needed in order to find the double chiral log corrections. So, for the
process $K \to \pi\pi$, it was not possible to unambiguously define the double log corrections from only one-loop calculations.

Two comments are appropriate to mention. In the study of $\pi\pi$ scattering to two-loop accuracy within SU(2) ChPT, it was found that the bulk portion of the correction to the threshold parameters are due to the chiral logarithms \cite{12}. This is not commonly the case in the SU(3) ChPT calculations. For instance, the double chiral logs in the vector form factor, $f_+(0)$, related to the semi-leptonic kaon decay only make up about 25% of the NNLO correction \cite{13}. We are therefore curious to know what happens about the size of the leading logs for a process like $K_S \to \gamma\gamma$ with only two outgoing photons, reminding the fact that a combination of ChPT and dispersion relations can satisfactorily predict the experimental data.

Along the same line we study the decay $K_S \to \gamma l^+l^-$ with $l = e, \mu$. Although this decay is not observed experimentally yet, it is interesting from the vantage point of having the same low energy structure as the decay $K_S \to \gamma\gamma$ to be investigated within ChPT.

A related rare process is $K_S \to \gamma^*\gamma^*$. In view of the recent LHCb measurement on the rare decay $K_S \to \mu^+\mu^-$ \cite{14}, other possible rare decays at LHCb, namely, $K_S \to l^+l^-l^+l^-$ and $K_S \to l_1^+l_1^-l_2^+l_2^-$ \cite{15} which occur through the decay $K_S \to \gamma^*\gamma^*$ are studied in \cite{16} emphasizing on the vector meson dominance contribution at $\mathcal{O}(p^6)$. These studies make the conclusion that at the prospect of the experimental data on the relevant $K_S$ and $K_L$ decays at LHCb, our theoretical predictions can be verified. It would be also interesting that with the LHCb upgrade we may get experimental data for the rare decay $K_S \to \gamma l^+l^-$. The structure of the article is as follows. A brief introduction to the weak and strong chiral Lagrangian up to NLO is given in Section 2. In Section 3 the kinematics for the process $K_S \to \gamma\gamma$ and $K_S \to \gamma l^+l^-$ are discussed and one-loop result for $K_S \to \gamma\gamma$ decay is reviewed in Section 4. The procedure in which we can derive the leading log corrections and its link to the leading divergences are explained in Section 5. Our analytical result concerning the single and double log correction are provided by Section 6. Section 7 summarizes our numerical results. The divergent part of the integrals are given in Appendix A.

## 2 Chiral Lagrangians at $\mathcal{O}(p^2)$ and $\mathcal{O}(p^4)$

We employ chiral effective Lagrangians in order to study the low energy dynamics of the strong and weak interactions. The Lagrangians we use in the present
work are the leading order and next-to-leading order chiral Lagrangians. The expansion parameter is in terms of external momenta, \( p \), and quark masses, \( m_q \). Quark masses are counted of order \( p^2 \) due to the lowest order mass relation \( m_\pi^2 = B_0(m_u + m_d) \). Here we briefly discuss the leading order and next-to-leading order strong and weak chiral Lagrangian. The leading order Lagrangian which is of order \( p^2 \), has the form

\[
\mathcal{L}_2 = \mathcal{L}_S^2 + \mathcal{L}_W^2. \tag{1}
\]

The subscript in \( \mathcal{L}_2 \) indicates the chiral order. \( \mathcal{L}_S^2 \) refers to the strong sector with \( \Delta S = 0 \) and \( \mathcal{L}_W^2 \) stands for the effective weak interaction with \( \Delta S = \pm 1 \). For the strong part we use \[17\]

\[
\mathcal{L}_S^2 = \frac{F_0^2}{4} \langle u_\mu u^\mu + \chi \rangle, \tag{2}
\]

where \( F_0 \) is the pion decay constant at chiral limit and we define the matrices \( u_\mu \) and \( \chi_\pm \) as the following

\[
u_\mu = i u_\mu^\dagger D_\mu U u^\dagger = u_\mu^\dagger, \quad u^2 = U, \quad \chi_\pm = u_\mu^\dagger \chi u^\dagger \pm u \chi^\dagger u. \tag{3}\]

The matrix \( U \in SU(3) \) contains the octet of light pseudo-scalar mesons with its exponential representation given in terms of meson fields matrix as

\[
U(\phi) = \exp(i \sqrt{2} \phi / F_0), \tag{4}
\]

where

\[
\phi(x) = \begin{pmatrix}
\frac{\pi_3}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+
\\
\pi^- & -\frac{\pi_3}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0
\\
K^- & K^0 & -\frac{2 \eta_8}{\sqrt{6}}
\end{pmatrix}.
\tag{5}
\]

We use the method of external fields discussed in \[18\]. The external fields are then defined through the covariant derivatives as

\[
D_\mu U = \partial_\mu U - iv_\mu U + iU l_\mu. \tag{6}
\]
The right-handed and left-handed external fields are expressed by $r_\mu$ and $l_\mu$ respectively. In the present work we set

$$r_\mu = l_\mu = e A_\mu \left( \begin{array}{c} 2/3 \\ -1/3 \\ -1/3 \end{array} \right).$$  \hspace{1cm} (7)$$

The electron charge is denoted by $e$ and $A_\mu$ is the classical photon field. The Hermitian $3 \times 3$ matrix $\chi$ involves the scalar (s) and pseudo-scalar (p) external densities and is given by $\chi = 2B_0(s + ip)$. The constant $B_0$ is related to the pion decay constant and quark condensate. For our purpose it suffices to write

$$\chi = 2B_0 \left( \begin{array}{ccc} m_u & m_d & m_s \end{array} \right).$$  \hspace{1cm} (8)$$

The $\Delta S = \pm 1$ part of the weak effective Lagrangian contains both the $\Delta I = 1/2$ piece and the $\Delta I = 3/2$ transition and has the form \cite{9}

$$L_{W2} = F_0^4 \left[ G_8 \langle \Delta_{32} u_\mu u^\mu \rangle + G_8' \langle \Delta_{32} \chi_+ \rangle \\ + G_{27} t^{ij,kl} \langle \Delta_{ij} u_\mu \rangle \langle \Delta_{kl} u^\mu \rangle \right] + \text{h.c},$$  \hspace{1cm} (9)$$

where the low energy constants $G_8$ and $G_{27}$ are defined in terms of dimension-less couplings $g_8$ and $g_{27}$ as

$$G_{8,27} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* g_{8,27}. \hspace{1cm} (10)$$

The matrix $\Delta_{ij}$ is given by

$$\Delta_{ij} = u \lambda_{ij} u^\dagger, \quad (\lambda_{ij})_{ab} = \delta_{ia} \delta_{jb}. \hspace{1cm} (11)$$

The non-zero components of the tensor $t^{ij,kl}$ are

$$t^{21,13} = t^{13,21} = \frac{1}{3}, \quad t^{22,23} = t^{23,22} = -\frac{1}{6}, \quad t^{23,33} = t^{33,23} = -\frac{1}{6}, \quad t^{23,11} = t^{11,23} = \frac{1}{3}. \hspace{1cm} (12)$$

At order $p^4$, the chiral Lagrangian consists of two parts as

$$L_4 = L_{S4} + L_{W4}. \hspace{1cm} (13)$$
\begin{table}
\begin{tabular}{|l|c|c|c|c|}
\hline
\(N_i\) & \(n_i\) & \(L_i\) & \(l_i\) & \(H_i\) \\
\hline
\(N_5\) & \(3/2\) & \(L_1\) & \(3/32\) & \(H_1\) \\
\(N_7\) & \(-9/8\) & \(L_2\) & \(3/16\) & \(H_2\) \\
\(N_8\) & \(-1/2\) & \(L_3\) & \(0\) & \\
\(N_9\) & \(3/4\) & \(L_4\) & \(1/8\) & \\
\(N_{10}\) & \(2/3\) & \(L_5\) & \(3/8\) & \\
\(N_{11}\) & \(-13/18\) & \(L_6\) & \(11/144\) & \\
\(N_{12}\) & \(-5/12\) & \(L_7\) & \(0\) & \\
\(N_{14}\) & \(1/4\) & \(L_8\) & \(5/48\) & \\
\(N_{15}\) & \(1/2\) & \(L_9\) & \(1/4\) & \\
\(N_{16}\) & \(-1/4\) & \(L_{10}\) & \(-1/4\) & \\
\(N_{17}\) & \(0\) & & & \\
\(N_{18}\) & \(-1/8\) & & & \\
\(N_{37}\) & \(-1/8\) & & & \\
\hline
\end{tabular}
\end{table}

Table 1: The low energy constants of the strong and weak effective chiral Lagrangians at order \(p^4\) which contribute to decays \(K_S \to \gamma\gamma\) or \(K_S \to \gamma l^+l^-\) are shown along with the coefficients of the divergent part.

The SU(3) strong Lagrangian at next to leading order contains 10+2 independent operators with corresponding low energy constants (LECs) \([20]\):

\[
\mathcal{L}_{S4} = L_1 \langle u_\mu u^\mu \rangle^2 + L_2 \langle u_\mu u^\nu \rangle \langle u^\mu u_\nu \rangle + L_3 \langle u_\mu u^\mu u_\nu u^\nu \rangle + L_4 \langle u_\mu u^\nu \rangle \langle \chi_+ \rangle \\
+ L_5 \langle u_\mu u^\nu \chi_+ \rangle + L_6 \langle \chi_+ \rangle^2 + L_7 \langle \chi_- \rangle^2 + \frac{1}{2} L_8 \langle \chi_+^2 + \chi_-^2 \rangle \\
- i L_9 \langle f_+^{\mu\nu} u_\mu u_\nu \rangle + \frac{1}{4} L_{10} \langle f_{\mu\nu} f_+^{\mu\nu} - f_{-\mu\nu} f_-^{\mu\nu} \rangle \\
+ \frac{1}{2} H_1 \langle f_{\mu\nu} f_+^{\mu\nu} + f_{-\mu\nu} f_-^{\mu\nu} \rangle + \frac{1}{4} H_2 \langle \chi_+^2 - \chi_-^2 \rangle.
\]

Terms with \(H_1\) and \(H_2\) are only needed for renormalization and do not appear in physical processes. The field strength tensor is defined as

\[
f_\pm^{\mu\nu} = u F_\pm^{\mu\nu} u^\dagger \pm u^\dagger F_\pm^{\mu\nu} u, \\
F_\pm^{\mu\nu} = \partial_\mu \partial_\nu - \partial_\nu \partial_\mu - i [l_\mu, l_\nu], \\
F_\pm^{\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - i [r_\mu, r_\nu].
\]

In order to absorb the infinities arising from the loop integrals, the low energy constants need to be renormalized in an appropriate way. This is done in \([20]\).
by splitting the constants into a finite renormalized part and an infinite piece as

\[ L_i = L_i^r + \frac{l_i \mu^{d-4}}{16\pi^2} \left( \frac{1}{d-4} + c \right), \]

\[ H_i = H_i^r + \frac{h_i \mu^{d-4}}{16\pi^2} \left( \frac{1}{d-4} + c \right), \]

(16)

where \( d = 4 - \epsilon \) is the space-time dimension in dimensional regularization. The constant \( c \) depends on the regularization scheme used and for ChPT in the minimal subtraction scheme we have \( c = -\frac{1}{2} (\log 4\pi + \gamma_E + 1) \). The coefficients \( l_i \) and \( h_i \) are listed in Table. 1.

The non-leptonic weak octet Lagrangian at NLO is discussed in full detail in [21]. The Lagrangian with all the terms relevant for the decays \( K^0 \rightarrow \gamma \gamma \) or \( \bar{K}^0 \rightarrow \gamma l^+ l^- \) is

\[ \mathcal{L}_{W4} = F_0^2 G_8 \left[ N_5 \mathcal{O}_5^8 + N_7 \mathcal{O}_7^8 + N_8 \mathcal{O}_8^8 + N_9 \mathcal{O}_9^8 + N_{10} \mathcal{O}_{10}^8 \right. \\
+ N_{11} \mathcal{O}_{11}^8 + N_{12} \mathcal{O}_{12}^8 + N_{14} \mathcal{O}_{14}^8 + N_{15} \mathcal{O}_{15}^8 + N_{16} \mathcal{O}_{16}^8 \\
\left. + N_{17} \mathcal{O}_{17}^8 + N_{18} \mathcal{O}_{18}^8 + N_{37} \mathcal{O}_{37}^8 \right]. \]

(17)

We use the basis for the operators in the Lagrangian above as set in [22]. In the same fashion as we do in the strong Lagrangian, the weak LECs in the weak Lagrangian have to be renormalized in a proper way. The renormalization procedure is performed by evaluating the one-loop divergences in [21] with

\[ N_i = N_i^r + \frac{n_i \mu^{d-4}}{16\pi^2} \left( \frac{1}{d-4} + c \right). \]

(18)

The constants \( n_i \) are quoted in Table. 1.

3 Kinematics for the decays \( K^0 \rightarrow \gamma \gamma \) and \( \bar{K}^0 \rightarrow \gamma l^+ l^- \)

The decay amplitude of \( K^0 \rightarrow \gamma \gamma \) with the following momentum assignment

\[ K^0(p) \rightarrow \gamma(q_1)\gamma(q_2), \]

(19)

has the form

\[ A(K^0 \rightarrow \gamma \gamma) = M_{\mu\nu}(q_1, q_2) \epsilon_1^\mu(q_1) \epsilon_2^\nu(q_2), \]

(20)
where $\epsilon_1^\mu$ and $\epsilon_2^\nu$ are the polarization four-vectors of the outgoing photons carrying momenta $q_1$ and $q_2$ respectively. Due to the gauge invariance, Lorentz symmetry and Bose symmetry, $M_{\mu\nu}(q_1, q_2)$ takes on the specific form

$$M_{\mu\nu}(q_1, q_2) = F(p^2) \left( q_{1\nu}q_{2\mu} - q_1.q_2 g_{\mu\nu} \right),$$

where $p = q_1 + q_2$ and $q_1^2 = q_2^2 = 0$ for photons with on-shell masses. The decay width for a decay with two particles in the final state reads

$$\Gamma(K_S \to \gamma\gamma) = \frac{M_K^2}{384\pi} |F(p^2 = m_K^2)|^2.$$

The decay $K_S \to \gamma l^+l^-$ takes place via the decay $K_S \to \gamma\gamma^*$ with one photon being off-shell decaying into a lepton pair $e^+e^-$ or $\mu^+\mu^-$. The decay amplitude is parameterized as

$$A(K_S \to \gamma l^+l^-) = \frac{1}{q_1^2} M_{\mu\nu}(q_1, q_2) \epsilon_2^\nu(q_2) \bar{u}(k) \gamma^\mu v(k'),$$

where, Lorentz gauge symmetry restricts $M_{\mu\nu}(q_1, q_2)$ to have the following form

$$M_{\mu\nu}(q_1, q_2) = G(q_1^2) \left( q_{1\nu}q_{2\mu} - q_1.q_2 g_{\mu\nu} \right).$$

The partial decay width for the process $K_S \to \gamma l^+l^-$ normalized to the decay width of $K_S \to \gamma\gamma$ is

$$\frac{1}{\Gamma_{K_S \to \gamma\gamma}} \frac{d\Gamma}{dz} = \frac{2}{z(1-z)^3} \frac{|G(z)|^2}{|G(0)|^2} \frac{1}{\pi} \Im \Pi(z),$$

in which the electromagnetic spectral function related to the lepton pair is expressed by

$$\frac{1}{\pi} \Im \Pi(z) = \frac{\alpha}{3\pi} \left( 1 + 2 \frac{r_l^2}{z} \right) \sqrt{1 - 4z^2/\Theta(z - 4r_l^2)},$$

where $r_l = m_l/m_K$ and $z = q_1^2/m_K^2$.

### 4 ChPT result at $O(p^4)$ for $K_S \to \gamma\gamma$ decay

The decay amplitude gets no tree-level contribution of order $p^2$ and $p^4$. This is because all the particles involved here are neutral particles and on top of that due to the chiral symmetry. Thus, the leading non-zero part of the amplitude
originates from loop diagrams constructed out of strong and weak Lagrangians of order $p^2$. The relevant Feynman diagrams for this decay is depicted in Fig. 1. Since tree diagrams are absent here, we therefore expect that the sum of all the Feynman diagrams ends up finite, i.e. all infinities from loop integrals vanish.

We show our result in a form that full agreement with the earlier formula given in [2, 3] can be simply understood. The following analytical result is achieved

$$F^{(4)}(p^2) = \frac{2}{\pi} (G_8 + \frac{2}{3} G_{27}) \alpha_{em} F_0 \left( \frac{p^2 - m_\pi^2}{p^2} \right) \left[ 1 + \frac{m_\pi^2}{p^2} \log \left( \frac{\beta - 1}{\beta + 1} \right) \right] - (m_\pi^2 \to m_K^2),$$

(27)

where $\beta = \sqrt{1 - 4m_\pi^2/p^2}$. $m_\pi$ and $F_0$ are the bare pion mass and bare pion decay constant, respectively. One can see from the expression above that the pion loop contribution decouples from the kaon one. The bare parameters which appear in the decay amplitude makes the definition of the amplitude at this order numerically ambiguous. One convenient way of resolving the issue is to shift the bare quantities into their physical values but at the same time, one should keep track of all corrections which now go over to higher order. Hence, we define $F^{(4)}$ in terms of physical quantities such that $F^{(4)} = F^{(4)}_{phys} + F^{(6)}$. This can be done by correcting the bare parameters up to one-loop order as $m_\pi^2 = m_{\pi,phys}^2 - \delta m^2$ for pion mass and $F_0 = F_{\pi} - \delta F$ for the pion decay constant. The corrections $\delta m^2$ and $\delta F$ provided by [20] contain chiral logarithms and NLO low energy constants. We present here only part of the
correction \( F^{(6)} \) which entails chiral logarithms and LECs:

\[
F^{(6)} = -\frac{2}{\pi}(G_8 + \frac{2}{3}G_{27})\frac{\alpha_{\text{em}}}{F_0} \left\{ -(1 - \frac{m_\pi^2}{2p^2})L_\pi - \frac{1}{2}L_K + \frac{m_\pi^2}{6p^2}L_\eta \\
-4(1 + \frac{m_\pi^2}{p^2})\{m_\pi^2L_\pi^e + (m_\pi^2 + 2m_K^2)L_4^e\} \\
+16\frac{m_\pi^2}{p^2}(m_\pi^2 + 2m_K^2)L_6^e + 16\frac{m_\pi^4}{p^2}L_8^e \right\},
\]

(28)

where \( L_i = \frac{m_i^2}{16\pi^2}\log(\frac{\mu^2}{m_i^2}) \). This type of contribution is necessary to be regarded when one considers the amplitude at full NNLO. The decay amplitude for related process \( K_S \to \gamma l^+l^- \) at NLO is discussed in detail in [23].

5 Leading Logarithms in ChPT

Chiral perturbation theory is a non-renormalizable field theory, in the sense that the cancellation of the infinities arising from loop integrals at a given order, requires local operators with higher derivatives with respect to the lowest order Lagrangian. In the strong sector for instance, there are only two operators in the leading Lagrangian, 10+2 operators in the next-to-leading Lagrangian and 90+4 operators in the next-to-next-to-leading Lagrangian. The number of operators, thus, grows fast in going to higher orders. It was pointed out by Weinberg for the first time [24] that we can obtain information about the structure of the leading divergences at two-loop in a non-renormalizable field theory like ChPT by performing only one-loop calculations. In addition, this means that we can get the leading logarithmic corrections at two-loops from one-loop computations in ChPT. The generalization of this idea to any higher order is carried out in [25]. They derived in [25] general relations that allows one to determine the leading and subleading poles at any order in terms of one-loop diagrams. In the following we recapitulate some results obtained in [26, 27] emphasizing on the relation which connect double chiral logarithmic corrections at two-loops to one-loop diagrams. Besides, we find that logarithmic corrections of type \( \log\mu \times (L_i \text{ or } N_i) \) can be obtained by determining singularities like \( (L_i \text{ or } N_i)/\epsilon \).

In general we can expand the bare Lagrangian with increasing power of \( \hbar \) as

\[
\mathcal{L} = \mathcal{L}_{(0)} + \hbar \mathcal{L}_{(1)} + \hbar^2 \mathcal{L}_{(2)} + \ldots ,
\]

(29)
where \( \mathcal{L}_n \) itself consists of a series of operators as
\[
\mathcal{L}_n = \mu^{-n\epsilon} \sum_i c^n_i \mathcal{O}^n_i. \tag{30}
\]

The energy scale \( \mu \) is defined such that the renormalized Lagrangian at all \( \hbar \)-order has space-time dimension \( d \) where \( d = 4 - \epsilon \). The renormalization at a given order is achievable by subtracting the needed infinities from the low energy constants in order to absorb the infinities coming from loop integrals and in the end to find finite result for a quantity at hand. Thus, it is necessary to write out the bare low energy constants \( c^n_i \) as
\[
c^n_i = c^n_{i,0} + \frac{1}{\epsilon} c^n_{i,1} + \ldots + \frac{1}{\epsilon^n} c^n_{i,n}. \tag{31}
\]

We call \( c^n_{i,0} \), the renormalized low energy constant to be determined from phenomenology. Let’s call \( L^n_l \) loop diagrams of order \( n \) with \( l \) as the number of loops in the diagrams. A loop integral can be expanded in powers of poles in \( \epsilon \),
\[
L^n_l = L^n_{l,0} + \frac{1}{\epsilon} L^n_{l,1} + \ldots + \frac{1}{\epsilon^l} L^n_{l,l}. \tag{32}
\]

We show first how renormalization procedure works at one-loop order. It is worth mentioning that it was proven in [25] that the physical amplitude can be made finite with only taking into account the one-particle irreducible diagrams at each order. At one-loop order, the loop diagrams are made out of vertices taken from lowest order Lagrangian, \( \mathcal{L}(0) \) and there is a contribution from counterterms taken from \( \mathcal{L}(1) \) which all together add up to
\[
\{ c^0_i \} L^1_l + \mu^{-\epsilon} \{ c^1_i \} L^1_0 = \{ c^0_{i,0} \} L^1_{l,0} + \frac{1}{\epsilon} \{ c^0_{i,0} \} L^1_{l,1} + \mu^{-\epsilon} \{ c^1_{i,0} \} L^1_{0,0} + \frac{1}{\epsilon} \mu^{-\epsilon} \{ c^1_{i,1} \} L^1_{0,0}, \tag{33}
\]
where \( \{ \ldots \} \) indicates the combination of all relevant low energy constants. Taking into account the expansion \( \mu^{-\epsilon} = 1 - \epsilon \log \mu + \ldots \), the cancellation of the infinities at one-loop order results in
\[
\{ c^1_{i,1} \} L^1_{0,0} = - \{ c^0_{i,0} \} L^1_{l,1}. \tag{34}
\]

Applying the relation above, the \( \log \mu \) dependent portion of the one-loop amplitude comes out
\[
- \{ c^1_{i,1} \} L^1_{0,0} \log \mu = \{ c^0_{i,0} \} L^1_{l,1} \log \mu. \tag{35}
\]
At two-loop order the full expression contains both local and non-local divergences. The former contribution comes from a tree diagram derived from $L_{(2)}$ and one-loop diagrams with vertices from $L_{(0)}$ and $L_{(1)}$ as well as two-loop diagrams with vertices from $L_{(0)}$. The latter contribution originates from two-loop diagrams with vertices from Lagrangians $L_{(0)}$ and from one-loop diagrams which involve vertices from both $L_{(0)}$ and $L_{(1)}$. We therefore can express the full result followed with expansion in $\epsilon$-poles as

$$
\mu^{-2\epsilon}\{c^2_i\} L^2_0 + \mu^{-\epsilon}\{c^1_i\} L^2_1 + \{c^0_i\} L^2_2 = \mu^{-2\epsilon}\left[\frac{1}{\epsilon}\{c^2_i,0\} + \frac{1}{\epsilon^2}\{c^2_i,1\}\right] L^2_{0,0}
+ \mu^{-\epsilon}\left[\{c^1_i,0\} + \frac{1}{\epsilon}\{c^1_i,1\}\right]\left[L^2_{1,0} + \frac{1}{\epsilon}L^2_{1,1}\right]
+ \{c^0_i\}\left[L^2_{2,0} + \frac{1}{\epsilon}L^2_{2,1} + \frac{1}{\epsilon^2}L^2_{2,2}\right].
$$

(36)

We substitute the expansion $\mu^{-2\epsilon} = 1 - 2\epsilon \log \mu + 2\epsilon^2 \log^2 \mu + \ldots$ into the above relation and ask for the cancellation of infinities of type $1/\epsilon^2$ and $\log \mu/\epsilon$, it turns out

$$
\{c^2_i,2\} L^2_{0,0} + \{c^1_i,1\} L^2_{1,1} + \{c^0_i,0\} L^2_{2,2} = 0,
2\{c^2_i,2\} L^2_{0,0} + \{c^1_i,1\} L^2_{1,1} = 0.
$$

(37)

The solution of the relations above reads

$$
\{c^0_i,0\} L^2_{2,2} = -\frac{1}{2}\{c^1_i,1\} L^2_{1,1},
$$

(38)

and

$$
\{c^2_i,2\} L^2_{0,0} = \{c^0_i,0\} L^2_{2,2}.
$$

(39)

We can now obtain the $\log^2 \mu$ dependent part of the full result at two-loop order by only collecting the relevant terms in Eq. (36) while we send $\epsilon \to 0$ and using Eq. (38) and Eq. (39) it gives

$$
2 \{c^2_i,2\} L^2_{0,0} \log^2 \mu + \frac{1}{2} \{c^1_i,1\} L^2_{1,1} \log^2 \mu = -\frac{1}{2} \{c^1_i,1\} L^2_{1,1} \log^2 \mu.
$$

(40)

The final result found in Eq. (40) is important because it tells us that the coefficient of the leading logarithmic correction at two-loop order can be achieved by finding the double pole coefficient stemming from one-loop diagrams at next-to-next-to leading order.
We are also interested to find corrections with single logarithms multiplied by the low energy constants. To this end, we turn back to Eq. (36) and restrict our attention to terms with divergences as $1/\epsilon$. The cancellation of these infinities requires the relation

$$\{c_{1,1}^2\}L_{0,0}^2 + \{c_{1,0}\}L_{1,1}^2 + \{c_{1,1}^1\}L_{1,0}^2 + \{c_{1,0}^0\}L_{2,1}^2 = 0. \tag{41}$$

Now we pick out terms proportional to $\log \mu$ in Eq. (36) and set $\epsilon \to 0$, the result is

$$-2\{c_{1,1}^2\}L_{0,0}^2 \log \mu - \{c_{1,1}^1\}L_{1,0}^2 \log \mu - \{c_{1,0}^1\}L_{1,1}^2 \log \mu = \{c_{1,1}^1\}L_{1,0}^2 \log \mu + \{c_{1,0}^1\}L_{1,1}^2 \log \mu + 2\{c_{1,0}^0\}L_{2,1}^2 \log \mu, \tag{42}$$

where to get the equality we have used Eq. (41) in which the term $\{c_{1,1}^2\}L_{0,0}^2$ is removed in favour of the rest. Notice that the term $\{c_{1,0}^1\}L_{1,1}^2 \log \mu$ in the second line is considered to be the correction of type $L_1 \log \mu$ or $N_1 \log \mu$, in which $\{c_{1,0}^1\}L_{1,1}^2$ is the coefficient of the single $\epsilon$-pole. It should be noted that one may compute the single log corrections directly using the one-loop diagrams but it sounds the easiest to follow the strategy discussed above.

### 6 Calculation of the Leading Logarithm

As we saw, at one-loop order there is no chiral logarithmic correction to the kaon decay to di-photon. Therefore, one expects the LL corrections to show up at two-loop order. We explained in the previous section that to obtain the double log corrections we only need to know the double poles from one-loop diagrams. All the necessary subdiagrams of order $p^6$ for the decays $K_S \to \gamma\gamma$ or $K_S \to \gamma l^+l^-$ are displayed in Fig. 2. We only need the divergent part of the Feynman integrals so as to find the double pole contribution of the full amplitude. In Appendix. A we give the divergent part of the resulting integrals. We start with the process $K_S \to \gamma\gamma$. Let us parameterize what we obtain here as $A^{(6)} = \{c_{1,1}^1\}L_{1,1}^2$, being the coefficient of the double pole divergences. Including both pion and kaon loops the result explicitly is

$$A^{(6)} = -\frac{4\pi\alpha_{em}}{(16\pi^2)^2} \frac{G_S}{F_0} \frac{1}{6}(m_s - \hat{m}) (q_1 \cdot q_2 - q_1 \cdot q_2 g_{\mu\nu}). \tag{43}$$

This result to be gauge invariant is regarded as a non-trivial check on our analytical calculations. One additional way to verify the result is to note that Eq. (38) restricts $A^{(6)}$ to obey the relation $A^{(6)} \propto A^{(6)}_{\text{tree}}$, where $A^{(6)}_{\text{tree}}$ is the
Figure 2: A subset of Feynman diagrams of order $p^6$ which contribute to the double logarithms at two-loop order. The new vertices are: $\Box$ is a $p^4$ weak vertex from $\mathcal{L}_{W4}$ and $\times$ is a $p^4$ strong vertex generated by $\mathcal{L}_{S4}$. 
amplitude of the tree diagram of order $p^6$. The octet weak Lagrangian of order $p^6$ contains many operators but there is only one operator which can make the transition $K_S \to \gamma\gamma$ possible, see discussions in [28]. The relevant Lagrangian is parameterized as

$$L_{W6} = -i \frac{4\pi \alpha_{em}}{(16\pi^2)^2} G_S c_1 F^{\mu\nu} F_{\mu\nu} \langle \Delta \chi_+ \rangle,$$

where $c_1$ is an unknown low energy constant. It is then a straightforward task to find the decay amplitude as

$$A_{\text{tree}}^{(6)} = -\frac{4\pi \alpha_{em}}{(16\pi^2)^2} G_S 8 F_0 9 c_1 (m_s - \hat{m}) (q_{1\nu} q_{2\mu} - q_{1\nu} q_{2\mu}),$$

(45)

We are therefore convinced that the relation $A^{(6)} \propto A_{\text{tree}}^{(6)}$ holds and $A^{(6)}$ has the correct structure. One important observation which turns out from our direct computation is that the pion loop integrals can be disentangled from the kaon loop integrals in our expression$^2$. Our result with only pion integrals reads

$$A^{(6)}_\pi = \frac{4\pi \alpha_{em}}{(16\pi^2)^2} G_S 1 F_0 3 \hat{m} (q_{1\nu} q_{2\mu} - q_{1\nu} q_{2\mu}) - \frac{4\pi \alpha_{em}}{(16\pi^2)^2} G_S 2 F_0 9 (q_{1\nu} q_{2\mu}) q_{1\mu} q_{2\nu},$$

(46)

and with only kaon integrals leads to

$$A^{(6)}_K = -\frac{4\pi \alpha_{em}}{(16\pi^2)^2} G_S 1 F_0 6 (m_s + \hat{m}) (q_{1\nu} q_{2\mu} - q_{1\nu} q_{2\mu}) + \frac{4\pi \alpha_{em}}{(16\pi^2)^2} G_S 2 F_0 9 (q_{1\nu} q_{2\mu}) q_{1\mu} q_{2\nu}.$$

(47)

In the two relations above, terms proportional to $q_{1\mu} q_{2\nu}$ do not contribute to the physical amplitude since photons in the process here are on the mass-shell and consequently $q_{1\alpha}, q_{2\alpha} = 0$. With the formula provided by Eq. (40) we are now able to write down our formula for the leading log correction

$$F_{\text{LL}}^{(6)} = \frac{4\pi \alpha_{em}}{(16\pi^2)^2} G_S 1 F_0 24 \left( m_\pi^2 \log^2 \left( \frac{m_\pi^2}{\mu^2} \right) - m_K^2 \log^2 \left( \frac{m_K^2}{\mu^2} \right) \right),$$

(48)

where employed are the leading order mass relations, $m_\pi^2 = 2B_0 \hat{m}$ and $m_K^2 = B_0 (m_s + \hat{m})$.

$^2$This is not the case however, when we look at the full set of Feynman diagrams at two-loop order.
In addition, it is of interest to find analytically logarithmic corrections of type \( \log \mu \times L_i^r \) and \( \log \mu \times N_i^r \). It is explained in section 5 that we only need to sequester divergent terms as \( L_i^r / \epsilon \) and \( N_i^r / \epsilon \) and then with the application of Eq. (42) we find

\[
F^{(6)}_{\log \times \text{LEC}} = -\frac{4\pi \alpha_{em}}{32\pi^2} G_8 \frac{1}{F_0} \left\{ \frac{16}{3} m_\pi^2 (N_{17}^r + 2N_{18}^r + N_{15}^r - N_{14}^r) \right. \\
+ (L_9^r + L_{10}^r)(16 p^2 - 32 m_\pi^2) \left\} \log\left(\frac{m_\pi^2}{\mu^2}\right) - \left( m_\pi^2 \rightarrow m_K^2 \right). \tag{49}\]

We redo our calculations for the process \( K_S \rightarrow \gamma l^+ l^- \). In this case we should keep in mind that \( q_2^2 = 0 \) but \( q_1^2 \neq 0 \). The coefficient of the double pole divergences including both pion and kaon integrals gives rise to

\[
B^{(6)} = -\frac{4\pi \alpha_{em}}{(16\pi^2)^2} G_8 \frac{1}{F_0} \left( m_s - \hat{m} \right) (q_1 q_2 g_{\mu \nu} - \frac{1}{9} q_1 q_2) g_{\mu \nu} \right). \tag{50}\]

This is identical to \( A^{(6)} \), the amplitude of \( K_S \rightarrow \gamma \gamma \). It is also possible for this process to separate the contribution of the pion integrals and kaon integrals. Taking only pion integrals into account we obtain for the amplitude

\[
B^{(6)}_\pi = \frac{4\pi \alpha_{em}}{(16\pi^2)^2} G_8 \frac{1}{F_0} \left( \frac{1}{3} \hat{m} (q_1 q_2 g_{\mu \nu} - \frac{1}{9} q_1 q_2) q_1 q_2 - \frac{2}{9} (q_1 q_2) q_1 q_2 \right), \tag{51}\]

and taking only kaon integrals we find

\[
B^{(6)}_K = -\frac{4\pi \alpha_{em}}{(16\pi^2)^2} G_8 \frac{1}{F_0} \left( \frac{1}{6} (m_s + \hat{m}) (q_1 q_2 g_{\mu \nu} - \frac{1}{9} q_1 q_2) g_{\mu \nu} \right) + \frac{2}{9} (q_1 q_2) q_1 q_2 - \frac{1}{9} q_1 q_2 \right). \tag{52}\]

When we put these results back into Eq. (23) terms proportional to \( q_{2\mu} \) vanish because for the on-shell photon we have \( q_{2\epsilon} = 0 \). We make use of the formula in Eq. (40) and obtain the leading log effects for the decay \( K_S \rightarrow \gamma l^+ l^- \) as

\[
G^{(6)}_{\text{LL}} = -\frac{4\pi \alpha_{em}}{(16\pi^2)^2} G_8 \frac{1}{F_0} \left( m_\pi^2 \log^2\left(\frac{m_\pi^2}{\mu^2}\right) - m_K^2 \log^2\left(\frac{m_K^2}{\mu^2}\right) \right). \tag{53}\]

The bottom line here is the observation that the leading log corrections for the decays \( K_S \rightarrow \gamma \gamma \) and \( K_S \rightarrow \gamma l^+ l^- \) are identical. We finish this subsection by presenting our calculations on the contributions of type \( \log \mu \times (L_i^r \text{ or } N_i^r) \) for
\[ Br(K_S \rightarrow \gamma \gamma) \times 10^6 \]

| \( \mu = 0.50 \text{ GeV} \) | 2.0399 | 2.0407 | 1.827 |
| \( \mu = 0.77 \text{ GeV} \) | 2.0399 | 2.0401 | 1.877 |
| \( \mu = 1.00 \text{ GeV} \) | 2.0399 | 2.0387 | 1.912 |

Table 2: The branching ratios for the decay \( K_S \rightarrow \gamma \gamma \) at three different values of the renormalization scale are compared including the double chiral log corrections.

\[
G_{\text{log} \times \text{LEC}}^{(6)} = -\frac{4\pi \alpha_{\text{em}}}{32\pi^2} G_8 \left\{ \frac{16}{3} m_\pi^2 (N_{37}^r + 2N_{18}^r + N_{15}^r - N_{14}^r) \right. \\
- \left. \frac{2}{3} q_1^2 (N_{17}^r + N_{16}^r + N_{15}^r + N_{14}^r) \right. \\
+ (L_9^r + L_{10}^r) (32 q_1 q_2 - 32 m_\pi^2 + 16 q_1^2) \log \left( \frac{m_\pi^2}{\mu^2} \right) \\
- (m^2_\pi \rightarrow m_K^2) \right. \}
\]

(54)

7 Numerical results

We are now ready to evaluate numerically the leading log contribution to the decay branching ratio. Let us begin with the NLO amplitude for the \( K_S \rightarrow \gamma \gamma \) decay given in Eq. (27). As input we use for the masses \( m_\pi = 0.136 \text{ GeV} \) and \( m_K = 0.497 \text{ GeV} \) and for the pion decay constant \( F_\pi = 0.0924 \text{ GeV} \). There is an ambiguity in knowing which values for the weak coupling \( g_8 \) and \( g_{27} \) should be used at this level of calculations. It is found \( g_8 = 4.99 \) and \( g_{27} = 0.297 \) from a fit to the decay \( K \rightarrow \pi\pi \) at leading order, see discussions in [4]. At NLO fit, \( g_8 \)
receives a rather significant reduction such that $g_8 = 3.62$ and $g_{27} = 0.286$ [4]. Since in this research, it is the matter of comparing the size of the LL effects with NLO result, it may suffice to use the leading order values of the weak couplings, namely, $g_8 = 4.99$ and $g_{27} = 0.297$. We also use the same values as introduced above when we compute the LL effects. In Table 2 we compare the decay branching ratio of $K_S \rightarrow \gamma \gamma$ both at one-loop order and with the inclusion of the LL effects at three different values of the renormalization scale, namely, $\mu = 1 \text{ GeV}$, 0.77 GeV and 0.5 GeV. As it is evident, the double log correction is the largest at $\mu = 0.5 \text{ GeV}$ and changes very little with varying $\mu$. But at any rate, the size of the correction is meager even though it goes in the right direction.

Moreover, we estimate the size of the single log effects. The values for the LECs used in the numerical calculations are listed in Table 3. Our numerical results shown in Table 2 indicate that these effects are significantly larger in magnitude than the LL corrections as expected, but they go in the opposite direction with respect to the LL effects.

Finally we calculate numerically the decay width of $K_S \rightarrow \gamma \ l^+ l^-$ normalized to the decay width of $K_S \rightarrow \gamma \gamma$ both for electron pair and muon pair in the final state. Input parameters are the same as those we used for the decay $K_S \rightarrow \gamma \gamma$. Our calculations summarized in Table 4 compare the NLO results and the NLO+LL effects at different renormalization scales. It is seen that the LL contribution has a very small impact on the decay width, though having the largest contribution at $\mu = 0.5 \text{ GeV}$.

| $\frac{\Gamma(K_S \rightarrow \gamma l^+ l^-)}{\Gamma(K_S \rightarrow \gamma \gamma)}$ | NLO | NLO+LL | NLO | NLO+LL |
|----------------|------|--------|------|--------|
| $l = e$  | $1.5967 \times 10^{-2}$ | $1.5969 \times 10^{-2}$ | $3.6924 \times 10^{-4}$ | $3.6930 \times 10^{-4}$ |
| $l = e$  | $1.5967 \times 10^{-2}$ | $1.5967 \times 10^{-2}$ | $3.6924 \times 10^{-4}$ | $3.6926 \times 10^{-4}$ |
| $l = \mu$ | $1.5967 \times 10^{-2}$ | $1.5964 \times 10^{-2}$ | $3.6924 \times 10^{-4}$ | $3.6915 \times 10^{-4}$ |

Table 4: The decay width of the process $K_S \rightarrow \gamma l^+ l^-$ normalized to the decay width of $K_S \rightarrow \gamma \gamma$ at three different values of the renormalization scale are compared including the double chiral log corrections.
8 Conclusions

The calculation of the leading logarithmic corrections at two-loop order has been the main aim behind the present work. These effects are the only part of the NNLO result that can be obtained from one-loop calculations and are free from unknown constants. We knew already from earlier works that LL effects are the sub-dominant part of the NNLO. In the case of \( K \to \gamma\gamma \), earlier findings based on dispersion relation technique suggests that the LL correction might be even smaller than those found in other studied processes. We have shown numerically that the size of the leading log corrections is very small indeed. Relying on earlier experiences and our finding here, we can confirm that the full NNLO correction cannot enhance the branching ratio significantly. In addition, we found analytically the single log corrections of type \( \log \mu \times \text{LEC} \) as part of the higher order effects. It turned out that these corrections are numerically large but we know that these will go through a cancellation among different contributions in the full NNLO.

We have also studied the LL effects in the decay width of the processes \( K_S \to \gamma e^+e^- \) and \( K_S \to \gamma \mu^+\mu^- \). It turned out that double log corrections are also very small in these decay channels.

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10 Appendix A: One-loop integrals

As we mentioned in the text, for our purpose we only need the divergent piece of the resulting one-loop integrals. For integrals with more than one propagator we need to make use of the Feynman parameter formula and then we should redefine the momentum variable to get an integral with powers of one propagator. We only present here our final result.

\[
\frac{1}{i} \int \frac{d^dq}{(2\pi)^d} \frac{1}{q^2 - m^2} = \frac{m^2}{16\pi^2} \frac{2}{4-d} + \text{finite},
\]

(55)
\[
\frac{1}{i} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - m^2)^2} = \frac{1}{16\pi^2} \frac{2}{4 - d} + \text{finite}, \tag{56}
\]

\[
\frac{1}{i} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 - m^2)((q + p)^2 - m^2)} = \frac{1}{16\pi^2} \frac{2}{4 - d} + \text{finite}, \tag{57}
\]

\[
\frac{1}{i} \int \frac{d^d q}{(2\pi)^d} \frac{q^\mu}{(q^2 - m^2)((q + p)^2 - m^2)} = \frac{p^\mu}{16\pi^2} \frac{1}{4 - d} + \text{finite}, \tag{58}
\]

\[
\frac{1}{i} \int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu}{(q^2 - m^2)((q + p)^2 - m^2)} = \frac{2}{16\pi^2 (4 - d)} \left( \frac{p^\mu p^\nu}{3} + g^{\mu\nu} \right) + \text{finite}, \tag{59}
\]

\[
\frac{1}{i} \int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu}{(q^2 - m^2)((q + p)^2 - m^2)((q + p + s)^2 - m^2)} \frac{q^\alpha q^\beta}{(q^2 - m^2)} = \frac{2}{16\pi^2 (4 - d)} g^{\mu\nu} + \text{finite}, \tag{60}
\]

\[
\frac{1}{i} \int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu q^\alpha}{(q^2 - m^2)((q + p)^2 - m^2)((q + p + s)^2 - m^2)} \left( p^\mu g^{\nu\alpha} + p^\nu g^{\mu\alpha} + p^\alpha g^{\mu\nu} \right) = \frac{2}{16\pi^2 (4 - d)} \left( \frac{s^\mu g^{\nu\alpha} + s^\nu g^{\mu\alpha} + s^\alpha g^{\mu\nu}}{6} \right) + \text{finite}, \tag{61}
\]

where \( d = 4 - \epsilon \). We do not show integrals which are finite.

References

[1] M. K. Gaillard and B. W. Lee, Phys. Rev. D 10 (1974) 897.
[2] G. D’Ambrosio and D. Espriu, Phys. Lett. B 175 (1986) 237.
[3] J. L. Goity, Z. Phys. C 34 (1987) 341.
[4] V. Cirigliano, G. Ecker, H. Neufeld, A. Pich and J. Portoles, Rev. Mod. Phys. 84 (2012) 399 [arXiv:1107.6001 [hep-ph]].

[5] G. D. Barr et al. [NA31 Collaboration], Phys. Lett. B 351 (1995) 579.

[6] F. Ambrosino et al. [KLOE Collaboration], JHEP 0805 (2008) 051 [arXiv:0712.1744 [hep-ex]].

[7] A. Lai et al., Phys. Lett. B 551 (2003) 7 [arXiv:hep-ex/0210053].

[8] J. Kambor and B. R. Holstein, Phys. Rev. D 49 (1994) 2346 [arXiv:hep-ph/9310324].

[9] T. N. Truong, Phys. Lett. B 313 (1993) 221.

[10] K Nakamura et al., (Particle Data Group) 2010 J. Phys. G: Nucl. Part. Phys. 37 075021

[11] M. Buchler, Phys. Lett. B 633 (2006) 497 [hep-ph/0511087].

[12] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser and M. E. Sainio, Phys. Lett. B 374 (1996) 210 [hep-ph/9511397].

[13] J. Bijnens, G. Colangelo and G. Ecker, Phys. Lett. B 441 (1998) 437 [hep-ph/9808421].

[14] R. Aaij et al. [LHCb Collaboration], JHEP 1301 (2013) 090 [arXiv:1209.4029 [hep-ex]].

[15] RAaij et al. [LHCb Collaboration], Eur. Phys. J. C 73 (2013) 2373 [arXiv:1208.3355 [hep-ex]].

[16] G. D’Ambrosio, D. Greynat and G. g. Vulvert, Eur. Phys. J. C 73 (2013) 2678 [arXiv:1309.5736 [hep-ph]].

[17] S. Weinberg, Phys. Rev. 166 (1968) 1568.

[18] J. Gasser and H. Leutwyler, Annals Phys. 158 (1984) 142.

[19] J. A. Cronin, Phys. Rev. 161 (1967) 1483.

[20] J. Gasser and H. Leutwyler, Nucl. Phys. B 250 (1985) 465.

[21] J. Kambor, J. H. Missimer and D. Wyler, Nucl. Phys. B 346 (1990) 17.
[22] G. Ecker, J. Kambor and D. Wyler, Nucl. Phys. B 394 (1993) 101.
[23] G. Ecker, A. Pich and E. de Rafael, Nucl. Phys. B 303 (1988) 665.
[24] S. Weinberg, Physica A 96 (1979) 327.
[25] M. Buchler and G. Colangelo, Eur. Phys. J. C 32 (2003) 427 [hep-ph/0309049].
[26] J. Bijnens and L. Carloni, Nucl. Phys. B 827 (2010) 237 [arXiv:0909.5086 [hep-ph]].
[27] J. Bijnens and L. Carloni, Nucl. Phys. B 843 (2011) 55 [arXiv:1008.3499 [hep-ph]].
[28] G. Buchalla, G. D’Ambrosio and G. Isidori, Nucl. Phys. B 672 (2003) 387 [hep-ph/0308008].
[29] J. Bijnens and F. Borg, Eur. Phys. J. C 39 (2005) 347 [hep-ph/0410333].
[30] J. A. Vermaseren, arXiv:math-ph/0010025