COBORDISMS OF SUTURED MANIFOLDS

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Abstract. We introduce a natural notion of cobordism between sutured manifolds. Then we construct a map on sutured Floer homology, induced by cobordisms between balanced sutured manifolds. This map is a common generalization of the hat version of the cobordism map in Heegaard-Floer theory, and the contact gluing map recently defined by Honda, Kazez, and Matić. We show that \( SFH \), together with the above cobordism maps, form a type of TQFT in the sense of Atiyah. As a special case, our theory gives rise to a map on link Floer homology, induced by decorated link cobordisms.

1. Introduction

Sutured manifolds, introduced by Gabai in \cite{Gabai}, have been of great use in 3-manifold topology, and especially in knot theory. A parallel between convex surface theory and sutured manifolds was drawn in \cite{Honda}, making apparent their usefulness in contact topology. In \cite{Honda}, we defined an invariant called sutured Floer homology, in short \( SFH \), for balanced sutured manifolds. \( SFH \) can be viewed as a common generalization of the hat version of Heegaard-Floer homology, and link Floer homology, both defined by Ozsváth and Szabó in \cite{OzsvathSzabo2001} and \cite{OzsvathSzabo2005}. In \cite{Honda}, we showed that \( SFH \) behaves nicely under sutured manifold decompositions, which has several important consequences.

In the present paper, we define a notion of cobordism between sutured manifolds \((M_0, \gamma_0)\) and \((M_1, \gamma_1)\). It consists of a triple \(W = (W, Z, \xi)\), where \(W\) is a four-manifold with boundary and corners. The horizontal part of \(\partial W\) is \(-M_0 \cup M_1\). The vertical part \(Z\) is a cobordism between \(\partial M_0\) and \(-\partial M_1\), and carries a contact structure \(\xi\) such that both \(\partial M_0\) and \(\partial M_1\) are convex surfaces, with dividing set \(\gamma_0\) and \(\gamma_1\), respectively. Balanced sutured manifolds, together with equivalence classes of cobordisms between them, form a category. We extend \( SFH \) to a functor from this category to finitely generated Abelian groups, giving a type of TQFT.

There might seem to be an alternative definition for cobordisms between sutured manifolds. One could consider triples \((W, Z, F)\), where \(W\) is a four-manifold with corners, and has horizontal boundary \(-M_0 \cup M_1\). Furthermore, \(Z\) is a cobordism between \(\partial M_0\) and \(-\partial M_1\), and \(F \subset Z\) is a cobordism between \(\gamma_0\) and \(-\gamma_1\). A Morse-theoretic approach to define cobordism maps for such objects would require that every non-singular level set is a balanced sutured manifold. For this, the pair \((Z, F)\) has to be built up from pairs of 3-dimensional and 2-dimensional handles that are cut into two equal halves by the 2-dimensional handle, or equivalently, contact
handles. A contact handle decomposition of \( Z \) gives rise to a contact structure \( \xi \), and we arrive at the previous definition.

The construction of the map \( F_W \), assigned to a cobordism \( W \), goes as follows. The sutured manifold \((-M_0, -\gamma_0)\) is a sutured submanifold of \((-N, -\gamma_1)\) in the sense of \([9]\), where \( N = M_0 \cup -Z \). So the contact structure \( \xi \) on \( Z \) induces a gluing map

\[
\Phi_\xi : SFH(M_0, \gamma_0) \to SFH(N, \gamma_1),
\]

as described in \([9]\). Then one can view \( W \) as a cobordism \( W_1 = (W, Z_1, \xi_1) \) from \((N, \gamma_1)\) to \((M_1, \gamma_1)\) such that \( Z_1 = \partial M_1 \times I \), and \( \xi \) is the \( I \)-invariant contact structure such that \( \partial M_1 \times \{ t \} \) is a convex surface with dividing set \( \gamma_1 \times \{ t \} \) for every \( t \in I \). We call such a cobordism special, and it can be described using one-, two-, and three-handle attachments along the interior of \( N \). Generalizing the hat version of the cobordism maps on Heegaard-Floer homology, one gets a map \( F\hat{W}_1 \) induced by a special cobordisms \( W_1 \). Finally, we set \( F_W = F\hat{W}_1 \circ \Phi_\xi \). This map is functorial; i.e., \( F_{W \circ W'} = F_W \circ F_{W'} \). It is important to note that the map \( F_W \) is only well-defined up to sign, and this can not be improved as the sign of \( \Phi_\xi \) is ambiguous by subsection 7.3 of \([9]\).

As a special case, we get maps on link Floer homology, induced by decorated link cobordisms. More precisely, we consider decorated links \((Y, L, P)\), where \( P \) consists of a positive even number of alternating positive and negative marked points on each component of the oriented link \( L \). This gives rise to a decomposition of \( L \) into compact one-dimensional submanifolds \( R_+(P) \) and \( R_-(P) \), such that \( R_+(P) \cap R_-(P) = P \), and if we orient the arcs of \( R_+(P) \) consistently with \( L \), then they point from negative to positive marked points. A cobordism from the decorated link \((Y_0, L_0, P_0)\) to \((Y_1, L_1, P_1)\) is a triple \((X, F, \sigma)\), where \((F, \sigma)\) is a surface with divides such that \( \sigma \) consists of embedded curves connecting the marked points, and divides \( F \) into compact subsurfaces \( R_+(\sigma) \) and \( R_-(\sigma) \). We require that \( R_+(\sigma) \cap L_i = R_-(P_i) \) for \( i = 0, 1 \). If we give a framing to \( F \), then \( \sigma \) defines a contact structure \( \xi \) on \( \partial N(F) \), making \((X \setminus N(F), \partial N(F), \xi)\) into a cobordism \( W \) between the sutured manifolds \((Y_i \setminus N(L_i), P_i \times S^1)\) for \( i = 0, 1 \), complementary to the decorated links. If both \( H^1(X) \) and \( H^2(X) \) vanish, then the framing of \( F \) can be chosen in a canonical way. Note that

\[
SFH(Y_i \setminus N(L_i), P_i \times S^1) \cong \widehat{HF}(Y, L) \otimes V^{d_i},
\]

where \( V \cong \mathbb{Z}^2 \), and \( d_i \) depends on the distribution of the marked points on \( L_i \). Hence, the cobordism map \( F_W \) maps between certain link Floer homology groups.

Finally, we extend the notion of Weinstein cobordisms to cobordisms between contact manifolds with convex boundary. If \( W \) is a Weinstein cobordism from \((M_0, \gamma_0, \zeta_0)\) to \((M_1, \gamma_1, \zeta_1)\), then we can view \( W \) as a cobordism \( \overline{W} \) from \((-M_1, -\gamma_1)\) to \((-M_0, -\gamma_0)\). We will prove that \( F_{\overline{W}}(EH(M_1, \gamma_1, \zeta_1)) = EH(M_0, \gamma_0, \zeta_0) \). Here \( EH \) is the contact element in sutured Floer homology introduced in \([12]\).

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Sutured manifolds were originally introduced by Gabai in [5]. The following definition is slightly less general, in that it excludes toroidal sutures.

**Definition 2.1.** A sutured manifold $(M, \gamma)$ is a compact oriented 3-manifold $M$ with boundary, together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli. Furthermore, the interior of each component of $\gamma$ contains a suture; i.e., a homologically nontrivial oriented simple closed curve. We denote the union of the sutures by $s(\gamma)$.

Finally, every component of $R(\gamma) = \partial M \setminus \text{Int}(\gamma)$ is oriented. Define $R_+ (\gamma)$ (or $R_- (\gamma)$) to be those components of $\partial M \setminus \text{Int}(\gamma)$ whose normal vectors point out of (into) $M$. The orientation on $R(\gamma)$ must be coherent with respect to $s(\gamma)$; i.e., if $\delta$ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then $\delta$ must represent the same homology class in $H_1(\gamma)$ as some suture.

**Remark 2.2.** In this paper, we are not going to make a distinction between $\gamma$ and $s(\gamma)$, as it is usually clear from the context which one we mean.

**Definition 2.3.** Let $(M_0, \gamma_0)$ and $(M_1, \gamma_1)$ be sutured manifolds. A cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$ is a triple $\mathcal{W} = (W, Z, \xi)$, where

1. $W$ is a compact, oriented 4-manifold with boundary,
2. $Z$ is a compact, codimension-0 submanifold with boundary of $\partial W$, and $\partial W \setminus \text{Int}(Z) = -M_0 \sqcup M_1$,
3. $\xi$ is a positive contact structure on $Z$ such that $\partial Z$ is a convex surface with dividing set $\gamma_i$ on $\partial M_i$ for $i = 0, 1$.

**Remark 2.4.** We think of $W$ as a 4-manifold with corners along $\partial Z$. Furthermore, we say that $-M_0 \sqcup M_1$ is the horizontal and $Z$ is the vertical part of the boundary $\partial W$.

For orienting the boundary of a manifold, we always use the “outward normal first” convention. Note that if $S$ is a convex surface in a contact manifold with dividing set $\Gamma$, then the dividing set of $-S$ is $-\Gamma$. Hence, the dividing set of $\partial (-M_0)$ is $-\gamma_0$, and the dividing set of $\partial Z$ is $\gamma_0 \cup (-\gamma_1)$.

**Lemma 2.5.** If the sutured manifolds $(M_0, \gamma_0)$ and $(M_1, \gamma_1)$ are cobordant, then

$$\chi(R_+(\gamma_0)) - \chi(R_-(\gamma_0)) = \chi(R_+(\gamma_1)) - \chi(R_-(\gamma_1)).$$

Furthermore, the map $\pi_0(\gamma_i) \to \pi_0(\partial M_i)$ is surjective for $i = 0, 1$.

**Proof.** Recall that $(Z, \xi)$ is a contact manifold with convex boundary $\partial M_0 \sqcup -\partial M_1$ and dividing set $\gamma = \gamma_0 \cup (-\gamma_1)$. Thus

$$\chi(R_+(\gamma)) - \chi(R_-(\gamma)) = \langle e(\xi), [\partial Z] \rangle = 0.$$

Moreover, $\chi(R_+(\gamma)) = \chi(R_+(\gamma_0)) + \chi(R_-(\gamma_1))$ and $\chi(R_-(\gamma)) = \chi(R_-(\gamma_0)) + \chi(R_+(\gamma_1))$.

The second claim follows from the fact that the dividing set on a closed convex surface is never empty, see [6] p.230].

**Definition 2.6.** The cobordisms $\mathcal{W} = (W,Z,\xi)$ and $\mathcal{W}' = (W',Z',\xi')$ from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$ are called strongly equivalent if there is an orientation preserving diffeomorphism $d: W \to W'$ such that $d(Z) = Z'$ and $d_*(\xi) = \xi'$; furthermore, $d(x) = x$ for every $x \in M_0 \cup M_1$. Such a map $d$ is called a strong equivalence.
If \( \mathcal{W} = (W, Z, \xi) \) is a cobordism from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\) and \( \mathcal{W}' = (W', Z', \xi') \) is a cobordism from \((M'_0, \gamma'_0)\) to \((M'_1, \gamma'_1)\), then \( \mathcal{W} \) and \( \mathcal{W}' \) are called weakly equivalent if there is an orientation preserving diffeomorphism \( d: W \to W' \) such that \( d(Z) = Z' \), \( d(\gamma_0) = \gamma'_0 \), \( d(\gamma_1) = \gamma'_1 \), and \( d_*(\xi) = \xi' \). Such a map \( d \) is called a weak equivalence.

**Definition 2.7.** Let \((M, \gamma)\) be a sutured manifold such that the map \( \pi_0(\gamma) \to \pi_0(\partial M) \) is surjective. The trivial cobordism from \((M, \gamma)\) to \((M, \gamma)\) is the triple \( \mathcal{W} = (W, Z, \xi) \), where

1. \( W = M \times I \),
2. \( Z = \partial M \times I \),
3. \( \xi \) is the \( I \)-invariant contact structure on \( Z \) such that each \( \partial M \times \{t\} \) is a convex surface with dividing set \( \gamma \times \{t\} \) for every \( t \in I \).

**Remark 2.8.** To be absolutely precise, just as in [15], one should define a cobordism from \((N_0, \nu_0)\) to \((N_1, \nu_1)\) as a tuple \( \mathcal{W} = ((W, Z, \xi), (M_0, \gamma_0), (M_1, \gamma_1), h_0, h_1), \)

where \((W, Z, \xi)\), \((M_0, \gamma_0)\), and \((M_1, \gamma_1)\) are as in Definition 2.8 and for \( i = 0, 1 \) the map \( h_i: M_i \to N_i \) is an orientation preserving diffeomorphism such that \( h_i(\gamma_i) = \nu_i \). If we have two such cobordism \( \mathcal{W} \) and \( \mathcal{W}' \) from \((N_0, \nu_0)\) to \((N_1, \nu_1)\), then an equivalence between them is a diffeomorphism \( g: W \to W' \) such that \( g|M_i = (h'_i)^{-1} \circ h_i \) for \( i = 0, 1 \).

If \((N_0, \nu_0)\) and \((N_1, \nu_1)\) are disjoint, then we can safely restrict ourselves to cobordisms between them where \( M_i = N_i \) and \( h_i = \text{Id}_{N_i} \) for \( i = 0, 1 \), in which case equivalence becomes our strong equivalence. However, to define the identity morphism from \((N, \nu)\) to itself, one does need the above more precise approach. To keep the notation simple, we will be sloppy and use our previous not so precise terminology, which should not cause much confusion.

**Definition 2.9.** Suppose that \( \mathcal{W}_0 = (W_0, Z_0, \xi_0) \) is a cobordism from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\) and \( W_1 = (W_1, Z_1, \xi_1) \) is a cobordism from \((M_1, \gamma_1)\) to \((M_2, \gamma_2)\). Since \( \partial M_1 \) is a convex surface with dividing set \( \gamma_1 \) in both \((Z_0, \xi_0)\) and \((Z_1, \xi_1)\), we can glue the contact structures \( \xi_0 \) and \( \xi_1 \) together along \( \partial M_1 \) to obtain a well-defined contact structure \( \xi_1 \cup \xi_2 \) on \( Z_0 \cup \partial M_1 \times Z_1 \). Then the composition \( W_1 \circ \mathcal{W}_0 \) is the cobordism from \((M_0, \gamma_0)\) to \((M_2, \gamma_2)\) given by the triple \((W_0 \cup M, W_1, Z_0 \cup \partial M_1 \times Z_1, \xi_0 \cup \xi_1)\).

**Definition 2.10.** The cobordism category of sutured manifolds, \( \text{Sut} \), is given as follows. Its objects are sutured manifolds \((M, \gamma)\) such that the map \( \pi_0(\gamma) \to \pi_0(\partial M) \) is surjective. The set of morphisms from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\) is the set of strong equivalence classes of cobordisms from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\). Composition is given by Definition 2.9. The identity morphism from \((M, \gamma)\) to itself is the equivalence class of the trivial cobordism.

For a given \( k \in \mathbb{Z} \), those sutured manifolds that satisfy \( \chi(R_+(\gamma)) - \chi(R_-(\gamma)) = k \) form a full subcategory of \( \text{Sut} \) called \( \text{Sut}_k \). By Lemma 2.5 the sum category of \( \{ \text{Sut}_k : k \in \mathbb{Z} \} \) is exactly \( \text{Sut} \).

The following definition was introduced in [13].

**Definition 2.11.** A sutured manifold \((M, \gamma)\) is balanced if

1. \( \chi(R_+(\gamma)) = \chi(R_-(\gamma)) \),
2. the map \( \pi_0(\gamma) \to \pi_0(\partial M) \) is surjective,
Remark 2.12. The objects of $\textbf{Sut}_0$ are precisely those sutured manifolds that can be written as finite disjoint unions of balanced sutured manifolds and closed 3-manifolds.

It is also worth noting that if $(W, Z, \xi)$ is a cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$, then $(-W, -Z, \xi)$ is not a cobordism from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0)$, since $\xi$ is a negative contact structure on $-Z$. But we can view $(W, Z, \xi)$ as a cobordism from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0)$ by writing $\partial W = -(-M_1) \cup Z \cup -M_0$. Loosely speaking, this is turning the cobordism $W$ upside down.

Definition 3.1. Suppose that $\gamma$ is a balanced sutured manifold. Then $\gamma$ is a balanced cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$ if both $(M_0, \gamma_0, \xi_0)$ and $(M_1, \gamma_1, \xi_1)$ are balanced cobordisms from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$ and if $\xi_0$ and $\xi_1$ are homologous. Hence, a vector field that gives the positive co-orientation of $\xi_1 \cap \partial M_1$ can be taken to be the $v_0$ in [13] Notation 4.1.

Next, we will explore the dependence of the space $\text{Spin}^c(W)$ on the boundary condition; i.e., on the almost complex structure $J_0$. Analogously to Definition 3.1 for any almost complex structure $J_0$ on $TW|Z$, one can define the space of relative
Spin$^c$ structures Spin$^c(W, Z, J_0)$. The space $J(\xi)$ is contractible. So if we only consider $J_0$ and $J'_0$ in $J(\xi)$, then we can canonically identify the spaces Spin$^c(W, Z, J_0)$ and Spin$^c(W, Z, J'_0)$, that is why we can talk about Spin$^c(W)$. However, without fixing $\xi$, we run into difficulties. We start with a very simple lemma.

**Lemma 3.3.** Let $V$ be a 4-dimensional real vector space, together with an endomorphism $J$ such that $J^2 = -I$. Then every 3-dimensional subspace $U < V$ contains a unique $J$-invariant plane.

*Proof.* Think of $(V, J)$ as a complex vector space. Since two different complex lines span $V$ over $\mathbb{R}$, they cannot both lie in $U$. Thus $U \cap J(U)$ is the unique $J$-invariant 2-plane in $U$. \hfill \square

On the other hand, for a given oriented 2-plane $T < U$, the space

$$\{ J \in \text{End}(V) : J^2 = -I, J(T) = T \}$$

is contractible. Together with Lemma 3.3, this implies that the space of almost-complex structures on $V$ is homotopy equivalent to the oriented Grassmanian $\tilde{\text{Gr}}_2(\mathbb{R}^3) = S^2$.

Compare this with the fact that

$$\tilde{\text{Gr}}_2(\mathbb{R}^4) = S^2 \times S^2 = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}.$$  

Here one $S^2$ factor is given by the oriented 2-planes lying in $\mathbb{R}^3$, and the other factor by the oriented 2-planes containing the coordinate vector $e_4$. Thus $\pi_2(\tilde{\text{Gr}}_2(\mathbb{R}^4)) \cong \mathbb{Z} \oplus \mathbb{Z}$, and the first obstruction to homotoping two oriented 2-plane fields on a 4-manifold $X$ lies in $H^2(X; \mathbb{Z} \oplus \mathbb{Z})$. But $\pi_2(S^2) \cong \mathbb{Z}$, so the first obstruction to homotoping two almost-complex structures on $X$ lies in $H^2(X; \mathbb{Z})$.

**Definition 3.4.** Suppose that $(W, Z, \xi)$ is a cobordism of sutured manifolds. Given an almost-complex structure $J_0$ on $TW|Z$, let $\xi(J_0)$ denote the $J_0$-invariant 2-plane field inside $TZ$, given by Lemma 3.3. Furthermore, let $J(Z)$ be the space of almost-complex structures $J_0$ on $TW|Z$ such that $\xi(J_0)|\partial Z = \xi|\partial Z$.

**Proposition 3.5.** Suppose that $J_0$ and $J'_0$ are almost-complex structures in $J(Z)$ such that $\xi(J'_0) z \neq -\xi(J_0) z$ for every $z \in Z$. Then there is a homotopically unique path of almost-complex structures in $J(Z)$ connecting $J_0$ to $J'_0$, giving a canonical isomorphism between Spin$^c(W, Z, J_0)$ and Spin$^c(W, Z, J'_0)$.

*Proof.* If we fix a trivialization of $TZ$, then a 2-plane field on $Z$ corresponds to a section of the trivial $S^2$-bundle $Z \times S^2$ over $Z$. Now $(Z \times S^2) \setminus (-\xi(J_0))$ is a bundle with contractible fibres, and $\xi(J'_0)$ is a section of it. The space of such sections is contractible, hence the result follows. \hfill \square

**Definition 3.6.** Given a cobordism $(W, Z, \xi)$, let $J(\xi)'$ be the set of those $J'_0 \in J(Z)$ such that $\xi(J'_0) z \neq -\xi z$ for every $z \in Z$.

Proposition 3.5 implies that for every $J'_0 \in J(\xi)'$ we can canonically identify Spin$^c(W)$ with any Spin$^c(W, Z, J'_0)$.

**Remark 3.7.** If $J_0$ and $J'_0$ are arbitrary elements of $J(Z)$, then there is no homotopically unique path connecting them. Indeed, after fixing a trivialization of $TZ$, we can identify $J(Z)$ with the space of maps from $Z$ to $S^2$ that are fixed over $\partial Z$. The homotopy class of such a map $f : Z \to S^2$ over the 2-skeleton of $Z$ is described by
an element of $H^2(Z, \partial Z)$. Using the Pontrjagin-Thom construction, the dual of this element is given by the homology class of a link $f^{-1}(p)$, where $p$ is a regular value of $f$. Moreover, the whole homotopy class is given by the framed cobordism class of $f^{-1}(p)$ with framing $f^{-1}(q)$, where $q$ is a regular value close to $p$. Given maps $f_0$ and $f_1$ from $Z$ to $S^2$, and common regular values $p$ and $q$ of both, a homotopy between them is given by a framed cobordism $(F, \nu) \subset Z \times I$ between $(f_0^{-1}(p), f_0^{-1}(q))$ and $(f_1^{-1}(p), f_1^{-1}(q))$. Given two such cobordisms $(F, \nu)$ and $(F', \nu')$, the two-chain $\pi_Z(F - F')$ represents an element of $H_2(Z)$, where $\pi_Z : Z \times I \to Z$ is the projection. Hence one can associate to two paths connecting $f_0$ and $f_1$ a difference in $H_2(Z) \cong H^1(Z, \partial Z)$, which has to vanish if the two paths are relatively homotopic. But there is no canonical relative homotopy class of paths connecting $f_0$ and $f_1$ in general. Compare this with the long exact sequence of the triple $(W, Z, \partial Z)$:

$$H^1(Z, \partial Z) \to H^2(W, Z) \to H^2(W, \partial Z) \to H^2(Z, \partial Z).$$

4. Link cobordisms

**Definition 4.1.** For $i = 0, 1$, let $Y_i$ be a connected, oriented 3-manifold, and let $L_i$ be a non-empty oriented link in $Y_i$. Then a link cobordism from $(Y_0, L_0)$ to $(Y_1, L_1)$ is a pair $(X, F)$, where

1. $X$ is a connected, oriented cobordism from $Y_0$ to $Y_1$,
2. $F$ is a properly embedded, compact, oriented surface in $X$,
3. $F$ is an oriented cobordism from $L_0$ to $L_1$.

If the normal bundle $NF$ is trivial and $\nu$ is a trivialization of $NF$, then we call $\nu$ a framing of $F$.

**Definition 4.2.** Let $(X, F)$ be a link cobordism from $(Y_0, L_0)$ to $(Y_1, L_1)$. Then a Seifert cobordism bounded by $F$ is a compact oriented 3-manifold with corners $V$ embedded in $X$, whose codimension-1 faces are $F$ and the Seifert surfaces $V \cap Y_i$ of $L_i$ for $i = 0, 1$.

**Lemma 4.3.** Let $(X, F)$ be a link cobordism, and suppose that $H_i(X, \partial X) = 0$ for $i = 2, 3$. Then $F$ bounds a Seifert cobordism. Moreover, if $V$ and $V'$ are both Seifert cobordisms bounded by $F$, then they define the same trivialization $\nu_0$ of $NF$. We call $\nu_0$ the Seifert framing of $F$.

**Proof.** Since $H_2(X, \partial X) = 0$, we have $[F, \partial F] = 0$. A standard argument using the fact that $K(Z, 1) = S^3$ gives that $F$ bounds a Seifert cobordism $V$. If $V'$ is another Seifert cobordism, then $V - V'$ represents a 3-cycle in $C_3(X, \partial X)$. But $H_3(X, \partial X) = 0$, hence $[V - V'] = 0$. Identify a small tubular neighborhood of $F$ with $F \times D^2$, and let $V_0 = V \cap (F \times S^1)$ and $V'_0 = V' \cap (F \times S^1)$. Since $[V - V'] = 0$, we see that $[V_0 - V'_0] = 0$ in $H_2(F \times S^1, \partial F \times S^1) \cong H^1(F \times S^1) \cong H^0(F) \oplus H^1(F)$.

Let $s$ and $s'$ be the sections of the projection $p : F \times S^1 \to F$ corresponding to $V_0$ and $V'_0$, respectively. The obstruction to homotoping $s$ to $s'$ is an element $o \in H^1(F)$. By the K"unneth formula, the map $p^* : H^1(F) \to H^1(F \times S^1)$ is injective. Since $p^*(o) = PD[V_0 - V'_0] = 0$, the obstruction $o$ vanishes, and $s$ and $s'$ are homotopic. \qed
We say that the triple \( \mathcal{X} = (X, F, \sigma) \) is a decorated link cobordism \((W, Z, \xi)\). However, to define the contact structure \( \xi \), we need more structure, namely a set of dividing curves on \( F \). For this, let us recall [11, Definition 4.1].

**Definition 4.5.** A surface with divides \((\gamma, R_+ (\sigma), R_- (\sigma))\) is a compact oriented surface \( \gamma \), possibly with boundary, together with a properly embedded, oriented 1-manifold \( R_+ (\sigma) \) and a decomposition into two compact subsurfaces \( S = R_+ (\sigma) \cup R_- (\sigma) \) such that \( R_+ (\sigma) \cap R_- (\sigma) = \sigma \). The orientation on \( R_+ (\sigma) \) is the orientation induced from \( S \), while \( R_- (\sigma) \) has the opposite orientation, and \( \sigma \) is oriented as \( \partial R_+ (\sigma) \).

**Definition 4.6.** A decorated link is a triple \((Y, L, P)\), where \( L \) is a non-empty oriented link in the connected oriented 3-manifold \( Y \), and \( P \subset L \) is a finite set of oriented points. We also require that for every component \( L_0 \) of \( L \) the number \(|L_0 \cap P|\) is positive and even. We have a decomposition into compact one-manifolds \( L = R_+ (P) \cup R_- (P) \) such that \( R_+ (P) \cap R_- (P) = P \), and if we orient the components of \( R_+ (P) \) and \( R_- (P) \) such that they point from the negative points to the positive points of \( P \), then the orientation of \( R_+ (P) \) agrees with the orientation of \( L \).

We can canonically assign a balanced sutured manifold \( W(Y, L, P) = (M, \gamma) \) to every decorated link \((Y, L, P)\), as follows. Identify \( \partial N(L) \) with \( L \times S^1 \) such that for every \( x \in L \) the curve \( \{x\} \times S^1 \) is a meridian of \( L \). Let \( M = Y \setminus N(L) \), and \( \gamma \subset \partial N(L) = L \times S^1 \) is \( P \times S^1 \) with the product orientation. Finally, \( R_{\pm} (\gamma) = R_{\pm} (P) \times S^1 \).

**Definition 4.7.** We say that the triple \( \mathcal{X} = (X, F, \sigma) \) is a decorated link cobordism from \((Y_0, L_0, P_0)\) to \((Y_1, L_1, P_1)\) if

- \((X, F)\) is a link cobordism from \((Y_0, L_0)\) to \((Y_1, L_1)\),
- \((F, \sigma)\) is a surface with divides such that \( \partial \sigma = -P_0 \cup P_1 \),
- \( R_+ (\sigma) \cap L_i = R_+ (P_i) \) and \( R_- (\sigma) \cap L_i = R_- (P_i) \) for \( i = 0, 1 \),
- if \( F_0 \) is a closed component of \( F \), then \( \sigma \cap F_0 \neq \emptyset \).

Two decorated link cobordisms \( \mathcal{X} = (X, F, \sigma) \) and \( \mathcal{X}' = (X', F', \sigma') \) from the decorated link \((Y_0, L_0, P_0)\) to \((Y_1, L_1, P_1)\) are said to be strongly equivalent if there is an orientation preserving diffeomorphism \( d: X \to X' \) such that \( d(F) = F' \) and \( d(\sigma) = \sigma' \) in the oriented sense; moreover, \( d(y) = y \) for every \( y \in Y_0 \cup Y_1 \).

If \( \mathcal{X} = (X, F, \sigma) \) is a cobordism from \((Y_0, L_0, P_0)\) to \((Y_1, L_1, P_1)\) and \( \mathcal{X}' = (X', F', \sigma') \) is a cobordism from \((Y_0', L_0', P_0')\) to \((Y_1', L_1', P_1')\), then we say that \( \mathcal{X} \) and \( \mathcal{X}' \) are weakly equivalent if there exists an orientation preserving diffeomorphism \( d: X \to X' \) such that \( d(F) = F' \) and \( d(\sigma) = \sigma' \) in the oriented sense.

Decorated links and strong equivalence classes of decorated link cobordisms form a category \( \text{DLink} \) with the obvious composition and identity morphisms.

Suppose that \( (X, F, \sigma) \) is a decorated link cobordism. By [8, Theorem 2.11] and [8, Section 4], there is an \( S^1 \)-invariant horizontal contact structure \( \xi_\sigma \) on \( F \times S^1 \) such that

1. \( \partial F \times S^1 \) is convex with dividing set isotopic to \( P \times S^1 \),
2. for every \( t \in S^1 \) the surface \( F \times \{t\} \) is convex with dividing set \( \sigma \times \{t\} \).
Furthermore, if $F$ has no $S^2$ or $T^2$ components, this correspondence is bijective between the isotopy classes of those $\sigma$’s that have no homotopically trivial components, and the isotopy classes of universally tight contact structures on $(F \times S^1, P \times S^1)$.

**Definition 4.8.** Let $(X, F, \sigma)$ be a decorated link cobordism from $(Y_0, L_0, P_0)$ to $(Y_1, L_1, P_1)$, together with a framing $\nu$ of $F$. Using $\nu$, identify a regular neighborhood of $F$ with $F \times D^2$. Then we define the cobordism $W = W(X, F, \sigma, \nu)$ to be the triple $(W, Z, \xi)$, where $W = X \setminus (F \times B^2)$ and $Z = F \times S^1$, finally $\xi = \xi_{\sigma}$. Note that $W$ is a cobordism from $W(Y_0, L_0, P_0)$ to $W(Y_1, L_1, P_1)$.

If $H_i(X, \partial X) = 0$ for $i = 2, 3$, then we let $W(X, F, \sigma) = W(X, F, \sigma, \nu_0)$, where $\nu_0$ is the Seifert framing of $F$.

Of course, $H_i(X, \partial X) = 0$ for $i = 2, 3$ if and only if $H^i(X) = 0$ for $i = 1, 2$.

**Proposition 4.9.** Let $\mathcal{X}_0: (Y_0, L_0) \to (Y_1, L_1)$ and $\mathcal{X}_1: (Y_1, L_1) \to (Y_2, L_2)$ be decorated link cobordisms. Suppose that $H^1(Y_1) = 0$, and assume $H^i(Y_j) = 0$ for $i = 1, 2$ and $j = 0, 1$. Then $H^i(X_0 \cup Y_1, X_1) = 0$ for $i = 1, 2$.

**Proof.** From the Mayer-Vietoris sequence

$$H^1(Y_1) \to H^2(X_0 \cup Y_1, X_1) \to H^2(X_0) \oplus H^2(X_1)$$

we conclude that $H^2(X_0 \cup Y_1, X_1) = 0$. The following segment is also exact:

$$\tilde{H}^0(Y_1) \to H^1(X_0 \cup Y_1, X_1) \to H^1(X_0) \oplus H^1(X_1).$$

Since $Y_1$ is connected, we see that $H^1(X_0 \cup Y_1, X_1) = 0$. \qed

**Definition 4.10.** The subcategory $\text{DLink}_0$ of $\text{DLink}$ is defined as follows. Its objects are decorated links $(Y, L, P)$ such that $H^1(Y) = 0$; i.e., $Y$ is a rational homology 3-sphere. The morphisms are strong equivalence classes of decorated link cobordisms $\mathcal{X} = (X, F, \sigma)$ that satisfy $H^i(X) = 0$ for $i = 1, 2$. By Proposition 4.9, the composition of such morphisms is well defined.

The following proposition is straightforward to verify using the definitions. Recall that for an object $(Y, L, P)$ of $\text{DLink}_0$, the sutured manifold $W(Y, L, P)$ was introduced in Definition 4.9, and for a morphism $\mathcal{X}$ in $\text{DLink}_0$, the cobordism $W(\mathcal{X})$ was defined in Definition 4.8.

**Proposition 4.11.** The map $W$ is a functor from $\text{DLink}_0$ to $\text{BSut}$. Furthermore, if $\mathcal{X}$ and $\mathcal{X}'$ are strongly/weakly equivalent, then $W(\mathcal{X})$ and $W(\mathcal{X}')$ are also strongly/weakly equivalent.

Hence, the composition $SFH \circ W$ gives a functor from $\text{DLink}_0$ to $\text{Ab}$, well defined up to sign. If $(Y, L, P)$ is an object of $\text{DLink}$, and the components of $L$ are $L_1, \ldots, L_k$, set

$$d = d(Y, L, P) = \sum_{i=1}^k (|L_i \cap P|/2 - 1).$$

Then

$$SFH(W(Y, L, P)) \cong \widetilde{HF}(Y, L) \otimes V^\otimes d,$$

where $V = \widetilde{HF}(S^1 \times S^2) \cong \mathbb{Z}^2$. 
5. Special cobordisms

Here, we extend the hat version of the cobordism map introduced in [19] to the class of sutured manifold cobordisms that are trivial along the boundary.

**Definition 5.1.** We say that a cobordism \( \mathcal{W} = (W, Z, \xi) \) from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\) is special if

1. \( W \) is balanced,
2. \( \partial M_0 = \partial M_1 \), and \( Z = \partial M_0 \times I \) is the trivial cobordism between them,
3. \( \xi \) is the \( I \)-invariant contact structure on \( Z \) such that each \( \partial M_0 \times \{ t \} \) is a convex surface with dividing set \( \gamma_0 \times \{ t \} \) for every \( t \in I \).

In particular, it follows from (3) that \( \gamma_0 = \gamma_1 \).

**Definition 5.2.** Special cobordisms \( \mathcal{W} = (W_1, Z, \xi) \) and \( \mathcal{W}' = (W', Z, \xi) \) from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\) are called strongly equivalent if there is an orientation preserving diffeomorphism \( d: W \to W' \) such that \( d(x) = x \) for every \( x \in \partial W \), and such a \( d \) is called a strong equivalence.

If \( \mathcal{W} \) is a special cobordism from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\), and \( \mathcal{W}' \) is a special cobordism from \((M'_0, \gamma'_0)\) to \((M'_1, \gamma'_1)\), then \( \mathcal{W} \) and \( \mathcal{W}' \) are called weakly equivalent if there is an orientation preserving diffeomorphism \( d: W \to W' \) such that \( d(\gamma_0) = \gamma'_0 \), and for every \((p, t) \in Z = \partial M_0 \times I \) we have \( d(p, t) = (d(p), t) \in Z' = \partial M'_0 \times I \). Such a \( d \) is called a weak equivalence.

**Remark 5.3.** Balanced sutured manifolds and strong equivalence classes of special cobordisms form a category \( \text{BSut} \).

Now, we make \( SFH \) into a functor from \( \text{BSut} \) to \( \text{Ab} \); i.e., we define the map \( \Phi_W \) if \( \mathcal{W} \) is a special cobordism. For this, we generalize [19]. Some of the necessary steps have already been done by Grigsby and Wehrli in [7], we review and extend their results first. In particular, we include the contact structure \( \xi \) on \( Z \) into the theory.

**Definition 5.4.** A balanced sutured multi-diagram is a tuple \( (\Sigma, \eta^0, \ldots, \eta^n) \), where \( \Sigma \) is a compact, oriented, open surface, and there is a non-negative integer \( d \) such that for every \( 0 \leq i \leq n \) the set \( \eta^i \) consists of \( d \) pairwise disjoint simple closed curves \( \eta^i_1, \ldots, \eta^i_d \subset \text{Int}(\Sigma) \) that are linearly independent in \( H_1(\Sigma) \).

**Remark 5.5.** By a slight abuse of notation, we will also write \( \eta^i \) for the 1-dimensional submanifold \( \bigcup \eta^i \) of \( \text{Int}(\Sigma) \).

Suppose that we are given a balanced sutured multi-diagram \( (\Sigma, \eta^0, \ldots, \eta^n) \). Then we associate to it a balanced cobordism

\[ \mathcal{W}_{\eta^0, \ldots, \eta^n} = (W_{\eta^0, \ldots, \eta^n}, Z_{\eta^0, \ldots, \eta^n}, \xi_{\eta^0, \ldots, \eta^n}). \]

For \( 0 \leq i \leq n \), let \( U_i \) be the compression body obtained from \( \Sigma \times I \) by attaching 2-handles along \( \eta^i \times \{ 0 \} \subset \Sigma \times \{ 0 \} \). Then

\[ \partial U_i = \Sigma^1 \cup (\partial \Sigma \times I) \cup \Sigma_{\eta^i}, \]

where \( \Sigma^1 = \Sigma \times \{ 1 \} \) and \( \Sigma_{\eta^i} \) is obtained from \( \Sigma \times \{ 0 \} \) by performing surgery along each component of \( \eta^i \times \{ 0 \} \).
Let $P_{n+1}$ denote a topological $(n+1)$-gon, with vertices $v_i$ for $i \in \mathbb{Z}_{n+1}$, labeled in a clockwise fashion. Denote the edge connecting $v_i$ to $v_{i+1}$ by $e_i$. Then let

$$W_{\eta^0, \ldots, \eta^n} = \left( \biguplus_{i=0}^n (e_i \times U_i) \right) \sim (e_i \times \Sigma),$$

where we round the corners along each $\{v_i\} \times \Sigma$ for $i \in \mathbb{Z}_{n+1}$.

Denote by $(M_{i,j}, \gamma_{i,j})$ the balanced sutured manifold defined by the diagram $(\Sigma, \eta^i, \eta^j)$. Then $M' = M_{0,1} \cup \cdots \cup M_{n-1,n} \cup -M_{0,n} \subset \partial W$, and we will write $Z_{\eta^0, \ldots, \eta^n}$ for $\partial W \setminus \text{Int}(M')$.

Finally, we define the contact structure $\xi = \xi_{\eta^0, \ldots, \eta^n}$ on the balanced sutured manifold $(Z, \gamma) = (Z_{\eta^0, \ldots, \eta^n}, \gamma_{0,1} \cup \cdots \cup \gamma_{n-1,n} \cup \gamma_{n,0})$ by giving a sutured manifold hierarchy of $(Z, \gamma)$. A sutured manifold hierarchy is a special case of a convex hierarchy by [11], hence it gives rise to a contact structure $\xi$. The space of contact structures compatible with a given convex hierarchy is contractible, which enables us to talk about canonically defined relative Spin$^c$ structures in Spin$^c(W, Z, \xi)$.

Note that $Z$ consists of three parts, $Z_{10} = P_{n+1} \times \partial \Sigma$ and $Z_{11} = \bigcup_{i=0}^n (e_i \times \Sigma \times I)$, finally $Z_2 = \bigcup_{i=0}^n (e_i \times \Sigma')$, see Figure 1. We put $Z_1 = Z_{10} \cup Z_{11}$; then

$$Z_1 = \left( P_{n+1} \cup \bigcup_{i=0}^n (e_i \times I) \right) \times \partial \Sigma = P_{2n+2} \times \partial \Sigma.$$
a union of the product sutured manifolds \((\Sigma_{n'} \times I, \partial \Sigma_{n'} \times I)\) for \(i \in \mathbb{Z}_{n+1}\), and \(S^1 \times D^2\) components with \((2n+2)\) longitudinal sutures on each. Every product piece has a canonical product disk decomposable contact structure. We further decompose each \(S^1 \times D^2\) along \(\{pt\} \times D^2\) to get a \(D^3\) with a unique suture. Our sequence of decompositions terminates in a product sutured manifold, hence gives the contact structure \(\xi\).

**Proposition 5.6.** Take \(W_{\eta_0,\ldots,\eta_n} = (W, Z, \xi)\). Let \(\xi' = \xi'_{\eta_0,\ldots,\eta_n}\) be the 2-plane field inside \(TW|Z\) such that on \(Z_{10}\) it is tangent to \(\Sigma\), on \(Z_2\) it is tangent to \(\Sigma_{n'}\), and on \(Z_{11}\) it interpolates between the two. Choose an arbitrary almost-complex structure \(J'_0\) compatible with \(\xi'\). Then \(J'_0 \in J(\xi')\), hence we can canonically identify \(\text{Spin}^c(W, Z, J'_0)\).

**Proof.** All we need to check is that the 2-plane field \(\xi(J'_0)\) in \(TZ\) never agrees with \(-\xi\). For an illustration of the following argument, see Figure \(\PageIndex{1}\). Since \(\xi'|Z_{10}\) is tangent to \(\Sigma\), we can choose \(J'_0\) such that the \(J'_0\)-invariant 2-plane contained in \(TZ_{10}\) is tangent to \(P_{n+1}\). On \(Z_2\), the planes \(\xi(J'_0)\) agree with \(\xi'\), which are tangent to \(\Sigma_{n'}\). In \(Z_{11}\), for every \(i \in \mathbb{Z}_{n+1}\), \(p \in e_i\), and \(q \in \partial \Sigma\), as we go along the arc \(\{p\} \times I \times \{q\}\) from 0 to 1, the planes \(\xi(J'_0)\) rotate \(\pi/2\) about the arc.

The contact structure \(\xi\) is a perturbation of the horizontal foliation on each \(\Sigma_{n'} \times I\), and behaves just like \(\xi(J'_0)\) on \(Z_{11}\). On \(Z_{10}\), it is a perturbation of the foliation by multi-saddles, so it is close to \(\xi(J'_0)\). In particular, \(\xi\) and \(\xi(J'_0)\) are never opposite. The second statement follows from Proposition \(\PageIndex{3.5}\). 

As usual, we denote by \(T_{\eta_i}\) the \(d\)-torus \(\eta^1 \times \cdots \times \eta^d\) inside \(\text{Sym}^d(\Sigma)\). For \(i \in \mathbb{Z}_{n+1}\), let \(x_{i+1} \in T_{\eta_i} \cap T_{\eta_i+1}\). Then we write \(\pi_2(x_0,\ldots, x_n)\) for the homotopy classes of Whitney \((n+1)\)-gons inside \(\text{Sym}^d(\Sigma)\) connecting \(x_0,\ldots, x_n\). Now we recall \[\PageIndex{7}\] Proposition 3.7, which relates Whitney \((n+1)\)-gons and \(\text{Spin}^c\) structures.

**Proposition 5.7.** Suppose that \((\Sigma, \eta^0,\ldots, \eta^n)\) is a sutured multi-diagram, and \(W = W_{\eta^0,\ldots, \eta^n}\) is the associated cobordism. Then there is a well-defined map \(s: \pi_2(x_0,\ldots, x_n) \to \text{Spin}^c(W)\) such that

\[s(\Psi)|M_{i+1} = s(x_i)\]

for every \(i \in \mathbb{Z}_{n+1}\).

**Proof.** The construction in \[\PageIndex{7}\] associates to any Whitney \((n+1)\)-gon a 2-plane field on \(W\) that agrees with \(\xi' = \xi'_{\eta^0,\ldots, \eta^n}\), along \(Z\). Let \(J'_0\) be an almost complex-structure on \(Z\) compatible with \(\xi'\). So we get an element \(s(\Psi)\) of \(\text{Spin}^c(W, Z, J'_0)\) that satisfies the property \(s(\Psi)|M_{i+1} = s(x_i)\). But Proposition \(\PageIndex{5.6}\) provides a canonical isomorphism between \(\text{Spin}^c(W, Z, J'_0)\) and \(\text{Spin}^c(W)\). 

**Definition 5.8.** Let \((\Sigma, \eta^0,\ldots, \eta^n)\) be a balanced sutured multi-diagram. Let \(D_1,\ldots, D_l\) denote the closures of the components of \(\Sigma \setminus (\eta^0 \cup \cdots \cup \eta^n)\) disjoint from \(\partial \Sigma\). Then the set of domains

\[D(\Sigma, \eta^0,\ldots, \eta^n) = \mathbb{Z}\langle D_1,\ldots, D_l \rangle.\]

For a domain \(D \in D(\Sigma, \eta^0,\ldots, \eta^n)\), we write \(D \geq 0\) if \(D \in \mathbb{Z}_{\geq 0}\langle D_1,\ldots, D_l \rangle\). As usual, if

\[(x_0,\ldots, x_n) \in (T_{\eta^n} \cap T_{\eta^0}) \times \cdots \times (T_{\eta^n} \cap T_{\eta^0}),\]

then \(D(x_0,\ldots, x_n)\) denotes the set of domains connecting \(x_0,\ldots, x_n\).
Finally, an \((n+1)\)-periodic domain is an element \(P \in D(\Sigma, \eta^0, \ldots, \eta^n)\) such that \(\partial P\) is a \(\mathbb{Z}\)-linear combination of curves in \(\eta^0, \ldots, \eta^n\).

The following proposition implies that any two Whitney \((n+1)\)-gons in the affine set \(\pi_2(x_0, \ldots, x_n)\) differ by an \((n+1)\)-periodic domain.

**Proposition 5.9.** If \(\pi_2(x_0, \ldots, x_n) \neq \emptyset\), then

\[
\pi_2(x_0, \ldots, x_n) \cong \ker \left( \bigoplus_{i=0}^{n} H_1(\eta^i) \to H_1(\Sigma) \right) \cong H_2(W_{\eta^0, \ldots, \eta^n}).
\]

Furthermore,

\[
coker \left( \bigoplus_{i=0}^{n} H_1(\eta^i) \to H_1(\Sigma) \right) \cong H_1(W_{\eta^0, \ldots, \eta^n}).
\]

**Proof.** This appeared as Proposition 3.3 and 3.4 in [7]. \(\Box\)

The correspondence in Proposition 5.9 can be made explicit by associating to each periodic domain \(P\) an element \(H(P)\) of \(H_2(W_{\eta^0, \ldots, \eta^n})\) as follows. Pick a point \(x \in P_{n+1}\), and connect \(x\) to every \(e_i\) by a straight arc \(a_i\). For each \(\eta_j\), let \(E_i \subset W_{\eta^0, \ldots, \eta^n}\) denote the union of the annulus \(a_i \times \eta_j \subset P \times \Sigma\), the annulus \((a_i \cap e_i) \times \eta_j \times I\), and the core disk of the 2-handle attached to \((a_i \cap e_i) \times \eta_j \times \{0\}\). Suppose that \(\partial P = \sum_{i,j} c_{ij}^i \eta_j^j\). Then

\[
H(P) = \{x\} \times P + \sum_{i,j} c_{ij}^j E_i^j \in H_2(W_{\eta^0, \ldots, \eta^n}).
\]

The following is a corrected version of [7, Proposition 3.9].

**Proposition 5.10.** Let \(\Psi, \Psi' \in \pi_2(x_0, \ldots, x_n)\). Then \(s(\Psi) = s(\Psi')\) if and only if \(\Psi - \Psi'\) can be written as a \(\mathbb{Z}\)-linear combination of doubly-periodic domains.

**Definition 5.11.** A balanced sutured multi-diagram is admissible if every non-trivial \((n+1)\)-periodic domain has both positive and negative coefficients.

The following statement is [7, Lemma 3.12].

**Lemma 5.12.** Every balanced sutured multi-diagram is isotopic to an admissible one.

For the following, see [7, Proposition 3.14].

**Proposition 5.13.** If \((\Sigma, \eta^0, \ldots, \eta^n)\) is admissible, then for every

\[
(x_0, \ldots, x_n) \in (T_{\eta^0} \cap T_{\eta^1}) \times \cdots \times (T_{\eta^{n-1}} \cap T_{\eta^n}),
\]

the set \(\{D \in D(x_0, \ldots, x_n) : D \geq 0\}\) is finite.

Let \((\Sigma, \eta^0, \ldots, \eta^n)\) be an admissible sutured multi-diagram, and for every \(1 \leq i \leq n\), let \(x_i \in T_{\eta^{i-1}} \cap T_{\eta^i}\) and \(y \in T_{\eta^{n}} \cap T_{\eta^0}\). Fix a complex structure on \(\Sigma\), and a 1-parameter variation of the induced almost-complex structure on \(\text{Sym}^d(\Sigma)\). As usual, we denote by \(M(\phi)\) the moduli-space of pseudo-holomorphic representatives of Whitney \((n+1)\)-gons lying in the homotopy class \(\phi \in \pi_2(x_1, \ldots, x_n, y)\). The Maslov index; i.e., the expected dimension, of \(M(\phi)\) is denoted by \(\mu(\phi)\). If \(n = 1\) and \(\mu(\phi) = 1\), then there is a natural \(\mathbb{R}\)-action on \(M(\phi)\), we let \(\hat{M}(\phi) = M(\phi) / \mathbb{R}\).

When \(n > 1\) and \(\mu(\phi) = 0\), the moduli space \(M(\phi)\) is compact. After choosing an
appropriate orientation system, we can talk about the algebraic count \( \#M(\phi) \). In the case \( n = 1 \) and \( \mu(\phi) = 1 \), the reduced moduli space \( \tilde{M}(\phi) \) will be compact.

For \( 0 \leq i < j \leq n \), we let

\[
CF(\Sigma, \eta^i, \eta^j) = \mathbb{Z}(T_{\eta^i} \cap T_{\eta^j}).
\]

This becomes a chain complexes when endowed with the differential that counts points in \( \tilde{M}(\phi) \), where \( \phi \) is a homotopy class of Whitney bigons with boundary on \( T_{\eta^i} \) and \( T_{\eta^j} \) and having \( \mu(\phi) = 1 \). Its homology is the sutured Floer homology group \( SFH(M_{i,j}, \gamma_{i,j}) \).

**Definition 5.14.** Let \( (\Sigma, \eta^0, \ldots, \eta^n) \) be an admissible sutured multi-diagram with \( n > 2 \), and fix a relative Spin\(^c\) structure \( s \in \text{Spin}^c(\mathcal{W}_{\eta^0, \ldots, \eta^n}) \). Then we have chain maps

\[
f_{\eta^0, \ldots, \eta^n} : \bigotimes_{i=1}^n CF(\Sigma, \eta^{i-1}, \eta^i) \to CF(\Sigma, \eta^0, \eta^n)
\]

and

\[
f_{\eta^0, \ldots, \eta^n}(\cdot, s) : \bigotimes_{i=1}^n CF(\Sigma, \eta^{i-1}, \eta^i, s|M_{i-1,i}) \to CF(\Sigma, \eta^0, \eta^n, s|M_{0,n}),
\]

defined by the formulas

\[
f_{\eta^0, \ldots, \eta^n}(x_1 \otimes \cdots \otimes x_n) = \sum_{y \in T_{\eta^0} \cap T_{\eta^n}} \sum_{\phi \in \pi_2(x_1, \ldots, x_n, y) : \mu(\phi) = 0} \#M(\phi) \cdot y
\]

and

\[
f_{\eta^0, \ldots, \eta^n}(x_1 \otimes \cdots \otimes x_n, s) = \sum_{y \in T_{\eta^0} \cap T_{\eta^n}} \sum_{\phi \in \pi_2(x_1, \ldots, x_n, y) : \mu(\phi) = 0, s(\phi) = s} \#M(\phi) \cdot y.
\]

We denote by \( F_{\eta^0, \ldots, \eta^n} \) and \( F_{\eta^0, \ldots, \eta^n}(\cdot, s) \) the maps induced on the homology.

The finiteness of the above sums is ensured by Proposition 5.13 since if \( \phi \) supports a pseudo-holomorphic representative, then its domain \( D(\phi) \geq 0 \).

**Proposition 5.15.** Let \( (\Sigma, \eta^0, \ldots, \eta^n) \) be an admissible multi-diagram, and set \( \mathcal{W} = \mathcal{W}_{\eta^0, \ldots, \eta^n} \). Then there are only finitely many \( s \in \text{Spin}^c(\mathcal{W}) \) for which the map \( f_{\eta^0, \ldots, \eta^n}(\cdot, s) \) is non-zero, and

\[
f_{\eta^0, \ldots, \eta^n} = \sum_{s \in \text{Spin}^c(\mathcal{W})} f_{\eta^0, \ldots, \eta^n}(\cdot, s).
\]

An analogous statement holds for \( F_{\eta^0, \ldots, \eta^n} \) and \( F_{\eta^0, \ldots, \eta^n}(\cdot, s) \).

**Proof.** Each \( T_{\eta^i} \cap T_{\eta^j} \) is finite, so there are only finitely many choices for \( x_1, \ldots, x_n \) and \( y \), and for each choice there are only finitely many \( \phi \in \pi_2(x_1, \ldots, x_n, y) \) such that \( M(\phi) \neq \emptyset \) by Proposition 5.13. Finally, such a \( \phi \) can only appear in the formula defining \( f_{\eta^0, \ldots, \eta^n}(\cdot, s(\phi)) \). The result follows. \( \square \)
5.1. **Naturality of sutured Floer homology.** First, we generalize the associativity theorem of the triangle maps [14, Theorem 8.16] to the sutured setting.

Fix a sutured quadruple diagram \((\Sigma, \eta^0, \ldots, \eta^3)\), and let \(\mathcal{W}_{\eta^0, \ldots, \eta^3}\) be the corresponding cobordism. Then we have a restriction map

\[
\text{Spin}^c(\mathcal{W}_{\eta^0, \ldots, \eta^3}) \to \text{Spin}^c(\mathcal{W}_{\eta^0, \eta^1, \eta^2, \eta^3}) \times \text{Spin}^c(\mathcal{W}_{\eta^0, \eta^2, \eta^3}),
\]

which corresponds to splitting the cobordism \(\mathcal{W}_{\eta^0, \ldots, \eta^3}\) along an embedded copy of \(M_{02}\). There is a subgroup

\[
\delta H^1(M_{02}, \partial M_{02}) < H^2(\mathcal{W}_{\eta^0, \ldots, \eta^3}, Z_{\eta^0, \ldots, \eta^3})
\]

whose orbits on \(\text{Spin}^c(\mathcal{W}_{\eta^0, \ldots, \eta^3})\) are the fibers of the restriction map, where \(\delta\) is the coboundary map in the corresponding relative Mayer-Vietoris sequence. Similarly, we have a restriction map

\[
\text{Spin}^c(\mathcal{W}_{\eta^0, \ldots, \eta^3}) \to \text{Spin}^c(\mathcal{W}_{\eta^0, \eta^1, \eta^3}) \times \text{Spin}^c(\mathcal{W}_{\eta^2, \eta^3}),
\]

which corresponds to splitting along \(M_{13}\), and a subgroup

\[
\delta H^1(M_{13}, \partial M_{13}) < H^2(\mathcal{W}_{\eta^0, \ldots, \eta^3}, Z_{\eta^0, \ldots, \eta^3}).
\]

**Theorem 5.16.** Let \((\Sigma, \eta^0, \eta^1, \eta^2, \eta^3)\) be an admissible sutured quadruple diagram, and fix a \(\delta H^1(M_{02}, \partial M_{02}) + \delta H^1(M_{13}, \partial M_{13})\) orbit \(\mathcal{G}\) in \(\text{Spin}^c(\mathcal{W}_{\eta^0, \ldots, \eta^3})\). For any \(s \in \mathcal{G}\) and \(i \in \{0, 1, 2\}\) the restriction \(s_{i,i+1} = s|M_{i,i+1}\) is independent of the choice of \(s\), pick an element \(x_{i,i+1} \in SFH(M_{i,i+1}, \gamma_{i,i+1}, s_{i,i+1})\). Furthermore, for \(0 \leq i < j < k \leq 4\), let \(F_{ijk} = F_{\eta^i, \eta^j, \eta^k}\) and \(s_{ijk} = s|\mathcal{W}_{\eta^0, \eta^i, \eta^k}\). Then

\[
\sum_{\sigma \in \mathcal{G}} F_{023}(F_{012}(x_{01} \otimes x_{12}, s_{012}) \otimes x_{23}, s_{023}) = \\
= \sum_{\sigma \in \mathcal{G}} F_{013}(x_{01} \otimes F_{123}(x_{12} \otimes x_{23}, s_{123}), s_{013}).
\]

**Proof.** Every subdiagram of an admissible sutured multi-diagram is also admissible. Hence the proof of [14, Theorem 8.16] works in this setting too, since the admissibility of \((\Sigma, \eta^0, \ldots, \eta^3)\) ensures the finiteness of all the counts of pseudo-holomorphic bigons, triangles, and rectangles that appear in the formula for the chain homotopy connecting the two sides. \(\square\)

In a similar manner, one can prove an associativity result without fixing an orbit \(\mathcal{G}\) of Spin\(^c\) structures on \(\mathcal{W}_{\eta^0, \ldots, \eta^3}\).

**Theorem 5.17.** Let \((\Sigma, \eta^0, \eta^1, \eta^2, \eta^3)\) be an admissible sutured quadruple diagram, and for every \(i \in \{0, 1, 2\}\) pick an element \(x_{i,i+1} \in SFH(M_{i,i+1}, \gamma_{i,i+1})\). Then

\[
F_{023}(F_{012}(x_{01} \otimes x_{12}) \otimes x_{23}) = F_{013}(x_{01} \otimes F_{123}(x_{12} \otimes x_{23})).
\]

Now we state a naturality theorem for sutured Floer homology, which shows that different choices of sutured diagrams for the same balanced sutured manifold give rise to canonically isomorphic Floer homology groups.

**Theorem 5.18.** Let \((M, \gamma)\) be a balanced sutured manifold, and let \((\Sigma, \alpha, \beta)\) and \((\Sigma', \alpha', \beta')\) be admissible sutured diagrams defining it. Furthermore, fix a Spin\(^c\) structure \(s \in \text{Spin}^c(M, \gamma)\). Then there is an isomorphism

\[
\Psi_s : SFH(\alpha, \beta, s) \to SFH(\alpha', \beta', s),
\]

uniquely determined up to an overall factor of \(\pm 1\).
such that Definition 5.22. A triple diagram and going to define maps $L_{\partial N}\to L_{\partial N}$. Definition 5.23. $A_1 \leq A$. Definition 5.21. in product sutured manifolds, in the context of a link surgery spectral sequence. Some of the following notions already appeared in Section 4 of [7] for framed links. The map associated to a framed link. Here, we generalize Section 4 of [19]. Furthermore, there is an obvious isomorphism $K$ pairwise disjoint, smoothly embedded circles meridian of $K$ cobordism $\Psi$. Let $(\Sigma, \alpha, \beta)$ and $(\Sigma', \alpha', \beta')$ be admissible sutured diagrams defining $(M, \gamma)$ and $(M', \gamma')$, respectively. Since $\partial \phi(\Sigma) = \phi(\gamma) = \gamma'$, both $(\phi(\Sigma), \phi(\alpha), \phi(\beta))$ and $(\Sigma', \alpha', \beta')$ define $(M', \gamma')$. So Theorem 5.18 gives an isomorphism $\Psi_s : SFH(\phi(\alpha), \phi(\beta), \phi_s(s)) \to SFH(\alpha', \beta', \phi_s(s))$. Furthermore, there is an obvious isomorphism $SFH(\alpha, \beta, s) \to SFH(\phi(\alpha), \phi(\beta), \phi_s(s))$. □

5.2. The map associated to a framed link. Here, we generalize Section 4 of [19]. Some of the following notions already appeared in Section 4 of [7] for framed links in product sutured manifolds, in the context of a link surgery spectral sequence.

Definition 5.21. A framed link $L$ in a sutured manifold $(M, \gamma)$ is a collection of $n$ pairwise disjoint, smoothly embedded circles $K_1, \ldots, K_n \subset \text{Int}(M)$, together with a choice of homology classes $\ell_i \in H_1(\partial N(K_i))$, with $m_i \cdot \ell_i = 1$, where $m_i$ is the meridian of $K_i$.

By attaching two-handles along the framed link $L$, we naturally obtain a special cobordism $W(L)$ from $(M, \gamma)$ to a sutured manifold $(M(L), \gamma)$. Note that $M(L)$ is obtained by surgery along $L$, and $\gamma$ is left unchanged.

For every framed link $L$ in $(M, \gamma)$ and Spin$^c$ structure $s \in \text{Spin}^c(W(L))$, we are going to define maps $F_L : SFH(M, \gamma) \to SFH(M(L), \gamma)$ and $F_{L,s} : SFH(M, \gamma, s|M) \to SFH(M(L), \gamma, s|M(L))$.

Definition 5.22. A bouquet $B(L)$ for the link $L$ is a 1-complex embedded in $M$ which is the union of $L$ with a collection of arcs $a_1, \ldots, a_n$, such that for every $1 \leq i \leq n$ the arc $a_i$ connects $K_i$ and $R_+(\gamma)$. We denote the punctured torus $\partial N(K_i) \setminus N(a_i)$ by $F_i$.

Definition 5.23. A sutured triple diagram subordinate to the bouquet $B(L)$ is a triple diagram $(\Sigma, \alpha, \beta, \delta) = (\Sigma, \{\alpha_1, \ldots, \alpha_d\}, \{\beta_1, \ldots, \beta_d\}, \{\delta_1, \ldots, \delta_d\})$, such that
Figure 2. A proper stabilization of a triple diagram is obtained by taking the connected sum with the above diagram at the point marked by $x$.

1. The diagram $(\Sigma, \{\alpha_1, \ldots, \alpha_d\}, \{\beta_{n+1}, \ldots, \beta_d\})$ defines the sutured manifold $(M', \gamma') = (M \setminus N(B(L)), \gamma)$.
2. The curves $\delta_{n+1}, \ldots, \delta_d$ are small isotopic translates of the $\beta_{n+1}, \ldots, \beta_d$.
3. For $i = 1, \ldots, n$, after surgering out $\beta_{n+1}, \ldots, \beta_d$, the induced curves $\beta_i$ and $\delta_i$ on $\mathbb{R}^+ (\gamma')$ lie in the punctured torus $F_i$.
4. For $i = 1, \ldots, n$, the curve $\beta_i$ represents a meridian of $K_i$ that is disjoint from all the $\delta_j$ for $i \neq j$, and meets $\delta_i$ in a single transverse intersection point.
5. For $i = 1, \ldots, n$, the homology class of $\delta_i$ corresponds to the framing $\ell_i$.

Definition 5.24. By a stabilization of a triple diagram $(\Sigma, \alpha, \beta, \delta)$ subordinate to some bouquet, we mean the following. Take the connected sum of $(\Sigma, \alpha, \beta, \delta)$ with a diagram $(E, \alpha_{d+1}, \beta_{d+1}, \delta_{d+1})$, where $E$ is a genus one surface, $|\alpha_{d+1} \cap \beta_{d+1}| = 1$, and $\delta_{d+1}$ is a small isotopic translate of $\beta_{d+1}$ such that $|\beta_{d+1} \cap \delta_{d+1}| = 2$, see Figure 2. Furthermore, we say that a stabilization is proper if the connected sum tube joins a component of $\Sigma \setminus (\alpha \cup \beta \cup \delta)$ that intersects $\partial \Sigma$ nontrivially with the component of $E \setminus (\alpha_{d+1} \cup \beta_{d+1} \cup \delta_{d+1})$ disjoint from the isotopy connecting $\beta_{d+1}$ and $\delta_{d+1}$.

The following lemma generalizes [19, Lemma 4.5].

Lemma 5.25. Let $(M, \gamma)$ be a balanced sutured manifold, together with a framed link $L \subset M$ and associated bouquet $B(L)$. Then there is a sutured triple diagram subordinate to $B(L)$, and any two such triple diagrams can be connected by a sequence of the following moves:

1. isotopies and handleslides amongst $\{\alpha_1, \ldots, \alpha_d\}$,
2. isotopies and handleslides amongst $\{\beta_{n+1}, \ldots, \beta_d\}$, while carrying along the curves $\delta_{n+1}, \ldots, \delta_d$, as well,
3. proper stabilizations,
4. for $1 \leq i \leq n$, an isotopy of $\beta_i$, or a handleslide of $\beta_i$ across a $\beta_j$ with $n+1 \leq j \leq d$,
5. for $1 \leq i \leq n$, an isotopy of $\delta_i$, or a handleslide of $\delta_i$ across a $\delta_j$ with $n+1 \leq j \leq d$.

Proof. By [19, Proposition 2.13], there exists a sutured diagram

$$(\Sigma, \{\alpha_1, \ldots, \alpha_d\}, \{\beta_{n+1}, \ldots, \beta_d\})$$
defining \((M', \gamma') = (M \setminus N(B(L)), \gamma)\). The curves \(\delta_{n+1}, \ldots, \delta_d\) are chosen to be small translates of \(\beta_{n+1}, \ldots, \beta_d\), respectively. The proof of [13 Proposition 2.15] does not use the assumption that the number of \(\alpha\) and \(\beta\) curves agree, hence any two sutured diagrams defining \((M', \gamma')\) can be connected by moves (1), (2), and stabilizations. To see that proper stabilizations suffice, note that we can obtain an arbitrary stabilization by performing a proper stabilization, followed by a sequence of handleslides. This is possible since every component of \(\Sigma \setminus \alpha\) and \(\Sigma \setminus \beta\) intersects \(\partial \Sigma\) nontrivially.

Since \(\Sigma\) surgered along \(\beta_{n+1}, \ldots, \beta_d\) is canonically diffeomorphic to \(R_+(\gamma')\), parts (3)-(5) of Definition 5.23 prescribe how to choose \(\beta_1, \ldots, \beta_n\) and \(\delta_1, \ldots, \delta_n\). For \(1 \leq i \leq n\), the link specifies the homology classes of \(\beta_i\) and \(\delta_i\) in \(F_i\). Different choices \(\beta_i\) and \(\delta_i\) can be connected by an isotopy in \(F_i\). It follows that in \(\Sigma\) they can be connected by a sequence of isotopies and handleslides across the \(\{\beta_{n+1}, \ldots, \beta_d\}\). The same argument works for \(\delta_1, \ldots, \delta_n\). These give rise to moves (4) and (5). \(\Box\)

The following proposition is a generalization of [19 Proposition 4.3].

**Proposition 5.26.** Let \((\Sigma, \alpha, \beta, \delta)\) be a triple diagram subordinate to the bouquet \(B(L)\) in \((M, \gamma)\).

1. \(W_{\alpha, \beta, \delta}\) is a cobordism from \((M, \gamma)\) to the disjoint union of \((M(L), \gamma)\) and \((M_\beta, \gamma_\beta, \delta) = (R_+ \times I, \partial R_+ \times I) \# (\#^{d-n} (S^2 \times S^1))\),

where \(R_+ = R_+(\gamma)\).

2. Let \(W_\emptyset\) be the cobordism from \((R_+ \times I, \partial R_+ \times I)\) to \(\emptyset\), corresponding to the sutured monodrome diagram \((R_+, \emptyset)\). Take the boundary connected sum of \(W_\emptyset\) and \(\#^{d-n} (D^3 \times S^1)\) along \(\text{Int}(R_+ \times I)\); we get a cobordism \(W'_\emptyset\) from \((M_\beta, \gamma_\beta, \delta)\) to \(\emptyset\). If we glue \(W_{\alpha, \beta, \delta}\) and \(W'_\emptyset\) along \((M_\beta, \gamma_\beta, \delta)\), then we obtain \(W(L)\).

**Proof.** Of course, \((\Sigma, \alpha, \beta)\) defines \((M, \gamma)\). By Definition 5.23, the diagram \((\Sigma, \alpha, \beta)\) defines \((M(L), \gamma)\), since we glue 2-handles to the complement \((M', \gamma')\) of the bouquet \(B(L)\) along curves specified by the framing of \(L\). Finally, let \(\Sigma_\beta\) be \(\Sigma\) surgered along \(\beta\). Then the diagram \((\Sigma, \beta, \delta)\) defines

\[
(\Sigma_\beta \times I, \partial \Sigma_\beta \times I) \# (\#^n S^3) \# (\#^{d-n} (S^2 \times S^1)),
\]

where \(\beta_i\) and \(\delta_i\) give the \(S^3\) components for \(1 \leq i \leq n\), and the \(S^2 \times S^1\) components for \(n+1 \leq i \leq d\). However, \(\Sigma_\beta = R_+(\gamma)\), which concludes the proof of (1).

Now we prove (2); i.e., that gluing \(W'_\emptyset\) to \(W_{\alpha, \beta, \delta}\) gives \(W(L)\). Let \(W_{\alpha, \beta, \delta} = (W, Z, \xi)\) and \(W'_\emptyset = (W', Z', \xi')\). As usual, \(P_3\) denotes a topological triangle, and we label its edges \(e_\alpha, e_\beta,\) and \(e_\delta\) in a clockwise fashion. Furthermore, \(U_\alpha, U_\beta,\) and \(U_\delta\) are the compression bodies corresponding to \(\alpha, \beta,\) and \(\delta\), respectively. Recall that \(W\) is obtained from \(P_3 \times \Sigma\) by gluing \(e_\alpha \times U_\alpha, e_\beta \times U_\beta,\) and \(e_\delta \times U_\delta\) along \(e_\alpha \times \Sigma, e_\beta \times \Sigma,\) and \(e_\delta \times \Sigma\), respectively.

Let \(P_1\) be a monogon with edge \(e\). Then \(W'\) is the boundary connected sum of

\[
W_\emptyset = (P_1 \times \Sigma_\beta) \cup (e \times (R_+ \times I))
\]

and

\[
V = (\#^n D^4) \# (\#^{d-n} (D^3 \times S^1)).
\]

We can also think of each \(D^3 \times S^1\) summand of \(V\) as a one-handle \(D^3 \times I\) attached to \(W_\emptyset\) along \(D^3 \times \partial I\).
Suppose that $n + 1 \leq i \leq d$. Let $B_i \subset U_\beta$ be the 2-handle attached to $\Sigma \times I$ along $\beta_i \times \{0\}$, together with a regular neighborhood of $\beta_i \times I \subset \Sigma \times I$. Similarly, $D_i \subset U_\delta$ corresponds to $\delta_i$. Since $\delta_i$ is a small isotopic translate of $\beta_i$, gluing the $i$-th $D^3 \times I$ summand of $V$ to $(e_\beta \times B_i) \cup (e_\delta \times D_i)$ along $\partial D^3 \times I$ gives $(e_\beta \cup e_\delta) \times B_i$.

Now fix some $1 \leq i \leq n$. Then $\beta_i$ and $\delta_i$ intersect in one point, forming an $S^3$ summand of $M_{\beta, \delta}$. This $S^3$ summand can be written as the union of two solid tori $B_i$ and $D_i$, where $B_i$ has meridian $\beta_i$ and $D_i$ has meridian $\delta_i$. Let $H_i$ be the union of the $i$-th $D^4$ summand of $V$ and $D_i \times e_\delta$. Then $H_i$ is a two-handle glued along $B_i \cup (\partial D_i \times e_\delta)$. Since the framing of $B_i$ is given by $\delta_i$, this shows that $H_i$ is the two-handle of $W(\mathbb{L})$ corresponding to $K_i$. The other way around, take $W(\mathbb{L})$ and glue $M(\mathbb{L}) \times I$ on top. Let $D^2 \times D^2 \subset W(\mathbb{L})$ be the two-handle glued to $N(K_i)$ along $\partial D^2 \times D^2$. Then

$$(W(L) \cup (M(\mathbb{L}) \times I)) \ \setminus \ (D^2 \times D^2) \cup (D^2 \times \partial D^2 \times I) = W(L \setminus K_i).$$

If we glue $V$ to $W$ by identifying $\partial V$ with the $\#^{d-n}(S^2 \times S^1)$ part of $M_{\beta, \delta}$, then we get an $R_+ \times I$ in $\partial(W \cup V)$. To this $R_+ \times I$ we glue $W_0$, which does not change the cobordism up to diffeomorphism.

Next, we will check that $(Z \cup Z', \xi \cup \xi')$ is contactomorphic to $(\partial M \times I, \zeta)$, where $\zeta$ is the $I$-invariant contact structure such that every $\partial M \times \{t\}$ is a convex surface with dividing set $\gamma$. This will conclude the proof of $W_{\alpha, \beta, \delta} \cup W_\emptyset = W(L)$.

Recall that $Z = Z_1 \cup Z_2$, where $Z_1 = P_0 \times \partial \Sigma$ and $Z_2 = (e_\alpha \times \Sigma_\alpha) \cup (e_\beta \times \Sigma_\beta) \cup (e_\delta \times \Sigma_\delta)$, see Figure 3. The contact structure $\xi$ is given by a hierarchy that starts with decomposing along a set of product annuli $A \subset Z_2$ parallel to $(e_\alpha \times \partial \Sigma_\alpha) \cup (e_\beta \times \partial \Sigma_\beta) \cup (e_\delta \times \partial \Sigma_\delta)$, then along surfaces $P_0 \times \{q\}$, for one $q$ in each component of $Z_1$.

Similarly, $Z' = Z'_1 \cup Z'_2$, where $Z'_1 = P_2 \times \partial R_+ = P_2 \times \partial \Sigma$ and $Z'_2 = e \times R_+$. Furthermore, $\xi'$ is defined by decomposing the sutured manifold $(Z', \gamma_{\beta, \delta})$ along product annuli $A'$ parallel to $e \times \partial R_+$, after which we get $Z'_1$, a union of tori with two longitudinal sutures on each, and $Z'_2$, a product sutured manifold. Hence $Z'$ is the manifold $R_+ \times I$ with $\xi'$ being the canonical $I$-invariant contact structure.
In particular, \((Z', \gamma_{\beta, \delta})\) is diffeomorphic to the product sutured manifold \((R_+ \times I, \partial R_+ \times I)\).

Let \(g_\alpha\) be the edge of \(P_0\) lying between \(e_\beta\) and \(e_\delta\), and let \(e_\beta \cap g_\alpha = v_\alpha^-\) and \(e_\delta \cap g_\alpha = v_\alpha^+\). Both \(\Sigma_\beta\) and \(\Sigma_\delta\) are naturally identified with \(R_+\), and \(Z'\) is glued to \(Z\) along
\[
(\{v_\alpha^-\} \times \Sigma_\beta) \cup (\{v_\alpha^+\} \times \Sigma_\delta) \cup (g_\alpha \times \partial \Sigma)
\]
using this identification. More precisely, \(R_-(Z', \gamma_{\beta, \delta}) = R_+ \times \{0\}\) is glued to \(\{v_\alpha^-\} \times \Sigma_\beta\) and \(R_+(Z', \gamma_{\beta, \delta}) = R_+ \times \{1\}\) is glued to \(\{v_\alpha^+\} \times \Sigma_\delta\), whereas \(\gamma_{\beta, \delta}\) is glued to \(g_\alpha \times \partial \Sigma\). It follows that \(Z \cup Z'\) is diffeomorphic to \((\Sigma_\alpha \cup \gamma_{\alpha, \beta} \Sigma_\beta) \times I = \partial M \times I\).

The dividing set of \(\xi \cup \xi'\) on \(\partial M \times \{i\}\) is \(s(\gamma_{\alpha, \beta}) \times \{i\}\) for both \(i = 0\) and \(i = 1\).

The product annuli \(A\) and \(A'\) for \((Z, \gamma)\) and \((Z', \gamma_{\beta, \delta})\) glue up to a set of product annuli \(A \cup A'\) inside \(\partial M \times I\). After decomposing \(\partial M \times I\) along \(A \cup A'\), we get a product sutured manifold diffeomorphic to
\[
(\Sigma_\alpha \times I, \partial \Sigma_\alpha \times I) \cup (\gamma_{\alpha, \beta} \times I, \partial \gamma_{\alpha, \beta} \times I) \cup (\Sigma_\beta \times I, \partial \Sigma_\beta \times I).
\]
The decomposing surfaces \(P_0 \times \{q\}\) and \(P_2 \times \{q\}\) glue together to give product disks inside \((\gamma_{\alpha, \beta} \times I, \partial \gamma_{\alpha, \beta} \times I)\). Hence \(\xi \cup \xi'\) is given by a hierarchy which starts with the product annuli \(A \cup A'\), and continue with decompositions along product disks, and is consequently \(I\)-invariant. Since the dividing set on \(\partial M \times \{0\}\) is \(s(\gamma_{\alpha, \beta}) \times \{0\}\), we have \(\xi \cup \xi' = \zeta\).

Alternatively, using the description of \(\xi\) in Proposition 5.20, we see that the 2-plane field \(\xi \cup \xi'\) is a perturbation of a 2-plane field that is tangent to the product foliation on \(\Sigma_\alpha \times I\) and \(\Sigma_\beta \times I\), and rotates \(\pi\) as we traverse \(\gamma_{\alpha, \beta} \times \{t\}\) from \(\Sigma_\alpha\) to \(\Sigma_\beta\). Hence the dividing set of \(\xi \cup \xi'\) on each \(\partial M \times \{t\}\) is close to \(s(\gamma_{\alpha, \beta}) \times \{t\}\).

Using part (2) of Proposition 5.26, we get a restriction map
\[
\gamma : \text{Spin}^c(W(L)) \to \text{Spin}^c(X_{\alpha, \beta, \delta}).
\]
This will enable us to define maps on \(SFH\) induced by cobordisms equipped with \(\text{Spin}^c\) structures. By the connected sum formula \[13,\ Proposition 9.15\], we have
\[
SFH((R_+ \times I, \partial R_+ \times I)\#(\#^{d-n}(S^2 \times S^1))) \cong \Lambda^* H^1(\#^{d-n}(S^1 \times S^2)).
\]
This is supported in the \(\text{Spin}^c\) structure \(s_0\) that is characterized by

1. \(s_0(R_+ \times I, \partial R_+ \times I)\) is homologous to the 2-plane field tangent to the horizontal foliation,
2. \(s_0(\#^{d-n}(S^1 \times S^2))\) extends to \(\#^{d-n}(S^1 \times D^3)\), or equivalently, its first Chern class vanishes.

We introduce the shorthand
\[
M(R_+, d-n) = (R_+ \times I, \partial R_+ \times I)\#(\#^{d-n}(S^2 \times S^1))\).
\]
The “top-dimensional” homology group of \(SFH(M(R_+, d-n))\) is
\[
\Lambda^{d-n} H^1(\#^{d-n}(S^1 \times S^2)) \cong \mathbb{Z},
\]
where by dimension we mean the relative Maslov grading. We will denote the generator of this group by \(\Theta\), which is well-defined up to sign.

**Lemma 5.27.** Let \(s \in \text{Spin}^c(W(L))\) be an arbitrary \(\text{Spin}^c\) structure. Then
\[
(r(s)|M(R_+, d-n) = s_0.
\]
Proof. To see that \( r(5) | M(R_+, d - n) \) satisfies (1) characterizing \( s_0 \), notice that the 
\((R_+ \times I, \partial R_+ \times I)\) component of \( M(R_+, d - n) \) is parallel to the 
\((Z', \gamma_{3, \delta}) \subset \partial W(L)\) in the proof of Proposition 5.26, which carries the “horizontal” Spin' structure. 
Property (2) is satisfied because \( s \) is an extension of \( r(5) \) to \( Z^{d-n}(S^1 \times D^3) \subset \mathcal{W} \).

**Definition 5.28.** Let \( L \) be a framed link in \( (M, \gamma) \), and fix an \( s \in \text{Spin}'(W(L)) \). 
Then we define maps

\[
F_L: SFH(M, \gamma) \to SFH(M(L), \gamma)
\]

and

\[
F_{L, s}: SFH(M, \gamma, s|M) \to SFH(M(L), \gamma, s|M(L)),
\]

as follows. Pick a bouquet \( B(L) \) for \( L \), and an admissible triple diagram \((\Sigma, \alpha, \beta, \delta)\) subordinate to this bouquet. As above, \( \Theta \) is the generator of the top-dimensional homology of

\[
SFH(M_{3, \delta}, \gamma_{3, \delta}, s_0) = SFH(M(R_+, d - n), s_0),
\]

well defined up to sign. Then we let \( F_L(x) = F(x \otimes \Theta) \) and \( F_{L, s}(x, s) = F(x \otimes \Theta, r(s)) \), 
which makes sense by Lemma 5.27.

The following theorem ensures that the above definition is independent of the 
choice of bouquet and subordinate triple diagram.

**Theorem 5.29.** Let \( (M, \gamma) \) be a balanced sutured manifold equipped with a framed 
link \( L \). Suppose that \((\Sigma_1, \alpha_1, \beta_1, \delta_1)\) and \((\Sigma_2, \alpha_2, \beta_2, \delta_2)\) are admissible triple 
diagrams subordinate to bouquets \( B_1 \) and \( B_2 \) for \( L \), respectively. Then we have a 
commutative diagram

\[
\begin{array}{ccc}
SFH(\alpha_1, \beta_1, s|M) & \xrightarrow{F_{L, s}} & SFH(\alpha_1, \delta_1, s|M(L)) \\
\downarrow \Psi_1 & & \downarrow \Psi_2 \\
SFH(\alpha_2, \beta_2, s|M) & \xrightarrow{F_{L, s}} & SFH(\alpha_2, \delta_2, s|M(L)),
\end{array}
\]

where \( \Psi_1 \) and \( \Psi_2 \) are isomorphisms induced by equivalences between the sutured 
diagrams (cf. Theorem 5.13). An analogous statement holds for \( F_L \).

**Proof.** We closely follow the proof of [19] Theorem 4.4. As many of the details are 
completely analogous, we only elaborate on the parts where a new idea is necessary.

First, assume that \( B_1 = B_2 \). Then \((\Sigma_1, \alpha_1, \beta_1, \delta_1)\) and \((\Sigma_2, \alpha_2, \beta_2, \delta_2)\) can be 
connected by a sequence of moves (1)-(5) of Lemma 5.25. Just as in [19] Lemma 
4.7, we obtain that the maps \( F_{L, s} \) commute with the isomorphisms induced by 
proper stabilizations. The argument is somewhat simpler in our case, since the fact 
that we are stabilizing near \( \partial \Sigma \), and that holomorphic discs avoid the boundary, 
make neck-stretching unnecessary. From here, the claim in the case \( B_1 = B_2 \) follows 
in a way completely analogous to the proof of [19] Proposition 4.6.

Now we show that \( F_{L, s} \) is independent of the bouquet, in the spirit of [19] Lemma 
4.8. Suppose that \( B \) and \( B' \) are a pair of bouquets that differ in the choice of one 
path \( \alpha_1 \) and \( \alpha_1' \). We construct two triples \((\Sigma, \alpha, \beta, \delta)\) and \((\Sigma, \alpha, \beta', \delta')\) such that \( \beta' \) 
is obtained from \( \beta \) by a sequence of isotopies and handleslides, and \( \delta' \) is obtained 
from \( \delta \) by a sequence of isotopies and handleslides.

To this end, consider \((M'', \gamma'') = (M \setminus N(B \cup B'), \gamma)\). Note that \( \chi(R_-(\gamma'')) - \chi(R_+(\gamma'')) = 2(n + 1) \). By [13] Proposition 2.13, there exists a sutured diagram

\[
(\Sigma, \{\alpha_1, \ldots, \alpha_d\}, \{\beta_{n+2}, \ldots, \beta_d\})
\]
defining \((M'', \gamma'')\). Let \(\beta_1, \ldots, \beta_n\) be meridians of the components \(K_1, \ldots, K_n\) of \(L\), respectively. Furthermore, \(\beta_{n+1}\) is a meridian of \(a_1\). Similarly, \(\beta'_{n+1}\) is a meridian of \(a'_1\), and we set \(\beta'_i = \beta_i\) if \(1 \leq i \leq d\) and \(i \neq n + 1\). The curves \(\delta_1, \ldots, \delta_n\) correspond to the framing of \(L\), and for \(i = n + 1, \ldots, d\) the curve \(\delta_i\) is a small isotopic translate of \(\beta_i\). Finally, \(\delta'_{n+1}\) is a small isotopic translate of \(\beta'_{n+1}\), and we set \(\delta'_i = \delta_i\) if \(1 \leq i \leq d\) and \(i \neq n + 1\). Then \((\Sigma, \alpha, \beta, \delta)\) is subordinate to \(B\), while \((\Sigma, \alpha, \beta', \delta')\) is subordinate to \(B'\).

If we surger \(\Sigma\) along \(\beta_1, \ldots, \beta_n, \beta_{n+2}, \ldots, \beta_d\), then we obtain \(R_+(\gamma)\# T^2\), with two disjoint, embedded, homologically non-trivial curves lying in the \(T^2\) component, induced by \(\beta_{n+1}\) and \(\beta'_{n+1}\). These curves must then be isotopic in \(T^2\), thus \(\beta'_{n+1}\) can be obtained by handle sliding \(\beta_{n+1}\) over some collection of the \(\beta_1, \ldots, \beta_n, \beta_{n+2}, \ldots, \beta_d\). We can obtain \(\delta'_{n+1}\) from \(\delta_{n+1}\) in an analogous manner. Consequently, \(\beta'\) is obtained from \(\beta\) by a sequence of isotopies and handleslides, and \(\delta'\) is obtained from \(\delta\) by a sequence of isotopies and handleslides.

From here, the result follows as described in the last paragraph of the proof of [19, Lemma 4.8]. So the map \(F_{L,s}\) is independent of both the bouquet and the subordinate triple diagram. \(\square\)

We have the following analogue of [19, Proposition 4.9].

**Proposition 5.30.** Let \(L\) be a framed link in \((M, \gamma)\). Suppose that we are given a partition \(L_1 \cup L_2\) of \(L\). Then we have cobordisms

\[ W(L_1) : (M, \gamma) \to (M(L_1), \gamma), \]

and if we view \(L_2\) as a link in \(M(L_1)\),

\[ W(L_2) : (M(L_1), \gamma) \to (M(L), \gamma). \]

**Proof.** The proof is completely analogous to the proof of [19, Proposition 4.9]. Note that one can disregard the last paragraph there, as one can always achieve admissibility for the associativity theorem (Theorem 5.16) in our case. \(\square\)

### 5.3. One- and three-handles

As in [19], let \(U = (U, Z, \xi)\) be the cobordism from \((M, \gamma)\) to \((M', \gamma')\) obtained by attaching a single one-handle \(H\) along \(\text{Int}(M)\). This is also a special cobordism, with \(Z = \partial M \times I\), and \(\xi\) being the \(I\)-invariant contact structure such that for every \(t \in I\) the surface \(\partial M \times \{t\}\) is convex with dividing set \(\gamma \times \{t\}\). There are two possibilities for \((M', \gamma'):\)

1. If the feet of \(H\) lie in the same component \((M_0, \gamma_0)\) of \((M, \gamma)\), then \((M', \gamma') = (M \# (S^1 \times S^2), \gamma)\), where the connected sum is taken between \(M_0\) and \(S^1 \times S^2\).

2. Otherwise, there are distinct components \((M_1, \gamma_1)\) and \((M_2, \gamma_2)\) of \((M, \gamma)\) such that \((M', \gamma')\) has \((M_1, \gamma_1)\) and \((M_2, \gamma_2)\) replaced by \((M_1, \gamma_1)\#(M_2, \gamma_2)\). In both cases, we have \(SFH(M', \gamma') \cong SFH(M, \gamma) \otimes \mathbb{Z}^2\) by [13, Proposition 9.15].

As the restriction map \(H^2(U, Z) \to H^2(M, \partial M)\) is an isomorphism, \(\text{Spin}^c(U) \cong \text{Spin}^c(M, \gamma)\). In case (1), a \(\text{Spin}^c\) structures \(s'\) on \((M', \gamma')\) extends over \(U\) if and only if \(s'|(S^1 \times S^2) = s_0\), where \(s_0\) is characterized by \(c_1(s_0) = 0\). In case (2), every
Spin$^c$ structure extends from $(M', \gamma')$ over $U$. Given a Spin$^c$ structure $s \in \text{Spin}^c(U)$, we also write $s$ for the corresponding element of Spin$^c(M, \gamma)$ by a slight abuse of notation, and let $s' = s((M', \gamma'))$. Note that in case (1), we have $s' = s\#s_0$. We define the map for one-handle addition as follows.

**Definition 5.31.** Let $(A, \alpha, \beta)$ be a sutured diagram, where $A$ is an annulus, and $\alpha$ and $\beta$ are homologically non-trivial, transverse simple closed curves such that $|\alpha \cap \beta| = 2$. Let $\theta \in \alpha \cap \beta$ be the intersection point with higher relative grading.

Suppose that $(\Sigma, \alpha, \beta)$ is a balanced diagram for $(M, \gamma)$. Remove two open disks $D_1$ and $D_2$ from the interior of two components of $\Sigma \setminus (\alpha \cup \beta)$ that intersect $\partial \Sigma$ non-trivially, as follows. In case (1), we require that $D_1$ and $D_2$ lie in the same component of $\Sigma_0 \setminus (\alpha \cup \beta)$, where $\Sigma_0$ is the component of $\Sigma$ corresponding to $M_0$. In case (2), for $i = 1, 2$, the disk $D_i$ lies in the component $\Sigma_i$ of $\Sigma$ that corresponds to $M_i$. In both cases, let $\Sigma^0 = \Sigma \setminus (D_1 \cup D_2)$, and write $\Sigma'$ for the surface obtained from $\Sigma^0$ by gluing $A$ along $\partial D_1 \cup \partial D_2$. Then

$$(\Sigma', \alpha', \beta') = (\Sigma^0 \cup A, \alpha \cup \{\alpha\}, \beta \cup \{\beta\})$$

is a balanced diagram defining $(M', \gamma')$.

For $s \in \text{Spin}^c(U)$, we define the map

$$g_{U,s}: CF(\Sigma, \alpha, \beta, s) \rightarrow CF(\Sigma', \alpha', \beta', s')$$

by the formula $g_{U,s}(x) = x \times \{\theta\}$. This makes sense since $s(x \times \{\theta\}) = s'$. The feet of $A$ are glued near $\partial \Sigma$, hence $g_{U,s}$ is a chain map. The induced map on homology

$$G_{U,s}: \text{SFH}(M, \gamma, s) \rightarrow \text{SFH}(M', \gamma', s')$$

is the map associated to the one-handle $H$. Similarly, we define the map

$$G_U: \text{SFH}(M, \gamma) \rightarrow \text{SFH}(M', \gamma')$$

to be the one induced by $g_{U}(x) = x \times \{\theta\}$.

The following theorem is an analogue of [19, Theorem 4.10].

**Theorem 5.32.** The map $G_{U,s}$ depends only on the cobordism $U$ and the Spin$^c$ structure $s \in \text{Spin}^c(U)$ in the following sense. If $(\Sigma_1, \alpha_1, \beta_1)$ and $(\Sigma_2, \alpha_2, \beta_2)$ are equivalent balanced diagrams for $(M, \gamma)$, then the corresponding balanced diagrams $(\Sigma'_1, \alpha'_1, \beta'_1)$ and $(\Sigma'_2, \alpha'_2, \beta'_2)$ are equivalent, and we have a commutative diagram

$$
\begin{array}{ccc}
\text{SFH}(\alpha_1, \beta_1, s) & \xrightarrow{\psi_1} & \text{SFH}(\alpha'_1, \beta'_1, s') \\
\downarrow & & \downarrow \\
\text{SFH}(\alpha_2, \beta_2, s) & \xrightarrow{\psi_2} & \text{SFH}(\alpha'_2, \beta'_2, s')
\end{array}
$$

where the vertical maps are the isomorphisms induced by the equivalences of the sutured diagrams (cf. Theorem 5.18). An analogous statement holds for $G_U$.

**Proof.** Since $A$ is glued near $\partial \Sigma_1$, and all Heegaard moves avoid $\partial \Sigma_1$, the equivalence from $(\Sigma_1, \alpha_1, \beta_1)$ to $(\Sigma_2, \alpha_2, \beta_2)$ induces an equivalence from $(\Sigma'_1, \alpha'_1, \beta'_1)$ to a diagram $(\Sigma'_2, \alpha'_2, \beta'_2)$. Then $(\Sigma'_2, \alpha'_2, \beta'_2)$ and $(\Sigma'_2, \alpha'_2, \beta'_2)$ are both obtained from $(\Sigma_2, \alpha_2, \beta_2)$ by gluing $(A, \alpha, \beta)$, but along different disks. By a sequence of handleslides across $\alpha$ and $\beta$, supported in $A$, we get an equivalence from $(\Sigma'_2, \alpha'_2, \beta'_2)$ to $(\Sigma'_2, \alpha'_2, \beta'_2)$. From here, the result follows in a way analogous to the last paragraph of the proof of [19, Theorem 4.10]. \qed
Dually, if $V$ is the special cobordism from $(M', \gamma')$ to $(M, \gamma)$ obtained by adding a single three-handle along a two-sphere $S \subset M'$, then $(M', \gamma')$ is related with $(M, \gamma)$ as in case (1) if $S$ is non-separating, or as in case (2) if $S$ is separating. In the latter case, we require that if we desum $(M', \gamma')$ along $S$, then our sutured manifold stays balanced. There is a special kind of compatible balanced diagram induced by the embedded sphere.

**Lemma 5.33.** Let $S \subset M'$ be an embedded sphere in the balanced sutured manifold $(M', \gamma')$. Then there is a balanced diagram $(\Sigma', \mathbf{\alpha}', \mathbf{\beta}')$ for $(M', \gamma')$ of the form

$$(\Sigma', \mathbf{\alpha}', \mathbf{\beta}') = (\Sigma^0 \cup A, \mathbf{\alpha} \cup \{\alpha\}, \mathbf{\beta} \cup \{\beta\}),$$

where $(\mathbf{A}, \mathbf{\alpha}, \mathbf{\beta})$ is as in Definition 5.31. Moreover, if we have two such diagrams that are equivalent, $(\Sigma_1^0 \cup A, \mathbf{\alpha}_1 \cup \{\alpha\}, \mathbf{\beta}_1 \cup \{\beta\})$ and $(\Sigma_2^0 \cup A, \mathbf{\alpha}_2 \cup \{\alpha\}, \mathbf{\beta}_2 \cup \{\beta\})$, then $(\Sigma_1, \mathbf{\alpha}_1, \mathbf{\beta}_1)$ and $(\Sigma_2, \mathbf{\alpha}_2, \mathbf{\beta}_2)$ are equivalent balanced diagrams.

**Proof.** Pick a regular neighborhood $N(S)$ of $S$, and let $\partial N(S) = S_\pm$. Since $S$ desums $(M', \gamma')$ into a balanced $(M, \gamma)$, one can choose properly embedded arcs $c_\pm$ inside $M' \setminus N(S)$ that connect a point of $S_\pm$ with $\gamma'$. Then $D_\pm = S_\pm \setminus N(\gamma_\pm)$ are product disks inside $(M' \setminus (N(c_+)) \cup N(c_-)), \gamma''$, where $\gamma''$ is obtained by closing $\gamma' \setminus (N(c_+) \cup N(c_-))$ all the way around $N(\gamma_+) \cup N(\gamma_-)$. See the proof of Proposition 9.15. Of course, $(M' \setminus (N(c_+) \cup N(c_-)), \gamma'')$ is isomorphic with $(M', \gamma')$. If we decompose along both $D_\pm$ and $D_\mp$, then we get the disjoint union of $(M, \gamma)$ and the twice punctured sphere $S^3(2)$. Let $(\Sigma, \mathbf{\alpha}, \mathbf{\beta})$ be an arbitrary balanced diagram for $(M, \gamma)$, and note that $(\mathbf{A}, \mathbf{\alpha}, \mathbf{\beta})$ defines $S^3(2)$. Hence we obtain a diagram for $(M', \gamma')$ of the required form by joining the feet of $A$ to the appropriate components of $\partial \Sigma$ by two strips, as in taking boundary sums. Any two such sutured diagrams which arise in this manner are equivalent, through an equivalence that leaves $\alpha$ and $\beta$ unchanged. Indeed, if $c_\pm$ end on different components of $\gamma$, then we can connect the corresponding diagrams by a sequence of handleslides over the curves $\alpha$ and $\beta$.

To prove the second statement, observe that handleslides across $\alpha$ and $\beta$ correspond to isotopies in the surface obtained by surgering the Heegaard surface along $\alpha$. Hence the sequence of Heegaard moves that leave $\alpha$ and $\beta$ unchanged, descend to an equivalence from $(\Sigma_1, \mathbf{\alpha}_1, \mathbf{\beta}_1)$ to $(\Sigma_2, \mathbf{\alpha}_2, \mathbf{\beta}_2)$. □

**Definition 5.34.** Let $S \subset M'$ be an embedded sphere in the balanced sutured manifold $(M', \gamma')$, and take a split balanced diagram

$$(\Sigma', \mathbf{\alpha}', \mathbf{\beta}') = (\Sigma^0 \cup A, \mathbf{\alpha} \cup \{\alpha\}, \mathbf{\beta} \cup \{\beta\})$$

defining it, as in Lemma 5.33. For $s \in \text{Spin}^c(V)$, we define

$$e_{\Sigma, s} : CF(\mathbf{\alpha}', \mathbf{\beta}', s|M') \to CF(\mathbf{\alpha}, \mathbf{\beta}, s|M)$$

such that for $y \in \alpha \cap \beta$, we have $e_{\Sigma, s}(x \times \{y\}) = 0$ if $y = \emptyset$, and $e_{\Sigma, s}(x \times \{y\}) = x$ if $y$ is the intersection point of smaller relative grading. Then $e_{\Sigma, s}$ is a chain map, since both components of $\partial A$ lie close to $\partial S'$. It induces the required map on homology

$$E_{\Sigma, s} : SFH(M', \gamma', s|M') \to SFH(M, \gamma, s|M).$$

Similarly, we also have a map

$$E_{\Sigma} : SFH(M', \gamma') \to SFH(M, \gamma).$$
Theorem 5.35. Let $S \subset M'$ be an embedded sphere in the balanced sutured manifold $(M', \gamma')$, and let $$ (\Sigma'_1, \alpha'_1, \beta'_1) = (\Sigma^0_1 \cup A, \alpha_1 \cup \{\alpha\}, \beta_1 \cup \{\beta\}) $$ and $$ (\Sigma'_2, \alpha'_2, \beta'_2) = (\Sigma^0_2 \cup A, \alpha_2 \cup \{\alpha\}, \beta_2 \cup \{\beta\}) $$ be a pair of equivalent split balanced diagrams. If $W$ is the three-handle cobordism corresponding to $S$ and $s \in \text{Spin}^c(V)$, then the following square commutes: $$ \begin{array}{ccc} SFH(\alpha'_1, \beta'_1, s') & \xrightarrow{E^1_{V,s}} & SFH(\alpha_1, \beta_1, s) \\ \downarrow \psi_1 & & \downarrow \psi_2 \\ SFH(\alpha'_2, \beta'_2, s') & \xrightarrow{E^2_{V,s}} & SFH(\alpha_2, \beta_2, s), \end{array} $$ where the vertical maps are the isomorphisms induced by the equivalences. An analogous result holds for $E_V$.

Proof. This is analogous to [19 Theorem 4.12].

5.4. The map associated to a special cobordism.

Definition 5.36. Let $W$ be a special cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$. We decompose $W$ as $W = W_3 \circ W_2 \circ W_1$, where $W_1: (M_0, \gamma_0) \to (M'_0, \gamma'_0)$ consist of one-handle additions, $W_2: (M'_0, \gamma'_0) \to (M'_1, \gamma'_1)$ consists of two-handle additions, and $W_3: (M'_1, \gamma'_1) \to (M_1, \gamma_1)$ consist of three-handle additions. Concretely, $W_2$ can be represented as a framed link $L \subset M'_1 = M_1 \# (\#^\ell(S^1 \times S^2))$. Fix an $s \in \text{Spin}^c(W)$, and let $s_i = s|W_i$ for $i = 1, 2, 3$. Then we define $$ F_{W,s} = E_{W_3,s_3} \circ F_{L,s_2} \circ G_{W_1,s_1}, $$ where $E_{W_3,s_3}$ and $G_{W_1,s_1}$ are the composites of the maps $E$ and $G$ induced by the various one- and three-handles. Similarly, we let $$ F_W = E_{W_3} \circ F_L \circ G_{W_1}. $$

Theorem 5.37. Let $W$ be a special cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$. Fix a Spin$^c$ structure $s \in \text{Spin}^c(W)$, and let $t_i = s|M_i$ for $i = 0, 1$. Then the map $$ F_{W,s}: SFH(M_0, \gamma_0, t_0) \to SFH(M_1, \gamma_1, t_1) $$ is an invariant of $W$, uniquely defined up to sign. More precisely, if $\overline{W}$ from $(\overline{M}_0, \overline{\gamma}_0)$ to $(\overline{M}_1, \overline{\gamma}_1)$ is a special cobordism and $\phi: W \to \overline{W}$ is a weak equivalence, then we have the commutative diagram $$ \begin{array}{ccc} SFH(M_0, \gamma_0, t_0) & \xrightarrow{F_{W,s}} & SFH(M_1, \gamma_1, t_1) \\ \downarrow (\phi|_{M_0}) & & \downarrow (\phi|_{M_1}), \\ SFH(\overline{M}_0, \overline{\gamma}_0, \overline{t}_0) & \xrightarrow{F_{W,s}} & SFH(\overline{M}_1, \overline{\gamma}_1, \overline{t}_1), \end{array} $$ where $\overline{s} = \phi_*(s)$ and $\overline{t}_i = \overline{s}|\overline{M}_i$ for $i = 0, 1$. An analogous statement holds for $F_W$.

Proof. We follow the structure of the proof of [19 Theorem 3.1].

Lemma 5.38. The maps $G_{W_1,s}$ are invariant under the ordering of the one-handles, and handleslides among them.
Lemma 5.39. Fix a framed link \( L \) in \((M, \gamma)\), and let \( L' \) be a framed link obtained by handleslides amongst the components of \( L \). Then the maps \( F_{L,s} \) and \( F_{L',s} \) are equal.

**Proof.** We generalize the proof of [19, Lemma 4.14]. Suppose that \( K'_1 \) is obtained from \( K_1 \) by a handleslide over \( K_2 \), and \( K'_i = K_i \) for \( i = 2, \ldots, n \). To perform this handleslide, one needs a path \( \sigma \subset \text{Int}(M) \) joining \( K_1 \) to \( K_2 \). After the handleslide, there is a natural path \( \sigma' \) joining \( K'_1 \) and \( K_2 \).

We construct a bouquet \( B(\mathbb{L}) \) for \( \mathbb{L} \) as follows. Isotope \( \sigma \) fixing \( \partial \sigma \) until \( \sigma \cap R_+ (\gamma) = \{ p \} \). Then choose the arcs \( a_1 \) and \( a_2 \) such that they run close to \( \sigma \) and they both end near \( p \). We pick \( a_3, \ldots, a_n \) in an arbitrary way. Let \((\Sigma, \alpha, \beta, \delta)\) be a triple diagram subordinate to \( B(\mathbb{L}) \), such that \( \beta_1 \) is dual to \( K_1 \) and \( \beta_2 \) is dual to \( K_2 \). Let \((\Sigma, \alpha', \beta', \delta')\) be the triple diagram where \( \beta'_1 \) is obtained as a handleslide of \( \beta_2 \) over \( \beta_1 \) along \( \sigma, \) and \( \gamma'_1 \) is obtained as a handleslide of \( \gamma_1 \) over \( \gamma_2 \) along \( \sigma, \) all the other \( \beta'_i = \beta_i \) and \( \gamma'_i = \gamma_i \). Then \((\Sigma, \alpha, \beta', \delta')\) is subordinate to a bouquet \( B(\mathbb{L}') \) for \( \mathbb{L}', \) constructed using \( \sigma' \).

We then have the commutative diagram

\[
\begin{array}{ccc}
SFH(M, \gamma, s|M) & \xrightarrow{\Theta_{\beta, \delta}} & SFH(M|L), \gamma, s|M(L')) \\
\downarrow_{\Theta_{\beta, \delta'}} & & \downarrow_{\Theta_{\beta', \delta'}} \\
SFH(M, \gamma, s|M) & \xrightarrow{\Theta_{\beta', \delta'}} & SFH(M|L'), \gamma, s|M(L')).
\end{array}
\]

Commutativity follows from associativity, and the observation that

\[
F_{\beta, \delta'}(\Theta_{\beta, \delta'} \otimes \Theta_{\delta', \delta'}, s_0) = \pm \Theta_{\beta, \delta'} = \pm F_{\beta, \delta'}(\Theta_{\beta, \delta} \otimes \Theta_{\delta, \delta'}, s_0),
\]

according to the handleslide invariance of the homology groups. \( \square \)

Lemma 5.40. The maps \( E_{V_1, s_1} \) are invariant under the ordering of the three-handles, and handleslides amongst them.

**Proof.** We verify independence of the ordering of the handles; i.e.,

\[
E_{V_2, t_2} \circ E_{V_1, t_1} = E_{V_1, t_1} \circ E_{V_2, t_2},
\]

where \( V_1 \) and \( V_2 \) are three-handle cobordisms associated to disjoint spheres \( S_1 \) and \( S_2 \) in \((M', \gamma')\), respectively. We can find a balanced diagram for \((M', \gamma')\) that contains two disjoint copies of the diagram \((A, \alpha, \beta)\), in both copies \( \partial A \) glued near \( \partial \Sigma \). It is clear from the definitions that the two composite maps agree. Handleslide invariance follows from this. \( \square \)

Lemma 5.41. Let \( W_1 \) be the cobordism from \((M, \gamma)\) to \((M', \gamma')\) obtained by adding a one-handle \( H \) to \((M, \gamma)\). Furthermore, let \( W_2 \) be the cobordism arising from a two-handle attached along any framed knot \( K \) in \( M' \) that cancels \( H \). Pick \( \text{Spin}^c \) structures \( s_i \) on \( W_i \) for \( i = 1, 2 \). Then the induced map

\[
F_{K, s_2} \circ G_{W_1, s_1} : SFH(M, \gamma, s_1) \to SFH(M, \gamma, s_1)
\]

corresponds to the identity map.
Proof. Since \( H \) is canceled by a two-handle, both of its feet have to lie in the same component of \( (M, \gamma) \), hence \( (M', \gamma') = (M, \gamma) \# (S^1 \times S^2) \). Then one can construct a triple diagram of the form
\[
(\Sigma', \alpha', \beta', \delta') = (\Sigma^0 \# A, \alpha \cup \{ \alpha \}, \beta \cup \{ \beta \}, \delta \cup \{ \delta \}),
\]
where
- \((\Sigma, \alpha, \beta)\) is a balanced diagram defining \((M, \gamma)\),
- there is a component \( c \) of \( \partial \Sigma \), and disks \( D_1 \) and \( D_2 \) lying in the component of \( \Sigma \setminus (\alpha \cup \beta) \) containing \( c \), such that \( \Sigma^0 = \Sigma \setminus (D_1 \cup D_2) \),
- \( A \) is an annulus attached to \( \Sigma^0 \) along \( \partial D_1 \cup \partial D_2 \),
- the curve \( \alpha \) is a homologically non-trivial simple closed curve in \( A \), and \( \beta \) is a small Hamiltonian translate of \( \alpha \),
- \( \delta \) is a small Hamiltonian translate of \( \beta \), and \( \delta \) intersects both \( \alpha \) and \( \beta \) transversally in a single point,
- the triple diagram \((\Sigma', \alpha', \beta', \delta')\) is subordinate to some bouquet for \( K \), in such a way that \( \beta \) is a meridian of \( K \), and \( \delta \) represents the framing of \( K \).

For an illustration, see Figure 4. Since \( D_1 \) and \( D_2 \) lie in the same component of \( \Sigma \setminus (\alpha \cup \beta) \) as \( c \), there is an arc \( a \) that connects a point \( p \) of \( \delta \) with \( c \), and whose interior lies in \( \Sigma^0 \setminus (\alpha' \cup \beta' \cup \delta') \). Indeed, connect \( \partial D_1 \) to \( c \) with an arc \( a' \) inside \( \Sigma \setminus (\alpha \cup \beta) \) that intersects \( \delta \) transversally, and take \( a \) to be the closure of the last component of \( a' \setminus \delta \). Since \( a \cap \beta = \emptyset \) and \( \delta \) is a small Hamiltonian translate of \( \beta \), we can also achieve that \( a \cap \delta = \emptyset \).

We can also assume that \( \delta \cap \beta = \emptyset \), since \( \delta \) is a small Hamiltonian translate of \( \beta \). Next, we achieve that \( \delta \cap \alpha = \emptyset \). Just push the curves in \( \alpha \) that intersect \( \delta \) simultaneously using a finger move along the arcs \( \delta \setminus (A \cup p) \) towards \( \partial D_1 \cup \partial D_2 \), then handleslide them over \( \alpha \). This process can be done away from the arc \( a \). Furthermore, since \( \delta \cap \beta = \emptyset \), the new \((\Sigma, \alpha, \beta)\) differs from the old one by isotoping the \( \alpha \) curves inside \( \Sigma \setminus \beta \). So we can suppose that we started with a triple diagram where \( \delta \cap \alpha = \emptyset \).
We use the diagram \((\Sigma^0 \cup A, \alpha \cup \{\alpha\}, \beta \cup \{\beta\})\) to define the map \(G_{W_1,s_1}\), and we compute \(F_{K,z_2}\) using the triple \((\Sigma', \alpha', \beta', \delta')\). The fact that the arc \(a\) connects \(\delta\) and \(\partial \Sigma\) and its interior avoids \(\alpha' \cup \beta' \cup \delta'\) ensures that every domain in the triple diagram \((\Sigma', \alpha', \beta', \delta')\) has multiplicity zero on one side of \(\delta \setminus A\). Here, we can talk about the two sides of \(\delta \setminus A\) since \(\delta \cap (\alpha \cup \beta) = \emptyset\). Note that the only component of \(\alpha' \cup \beta' \cup \delta'\) that intersects \(\partial D_1 \cup \partial D_2\) is \(\delta\). Moreover, both \(\partial D_1 \setminus \delta\) and \(\partial D_2 \setminus \delta\) are connected, so every domain in the triple diagram \((\Sigma', \alpha', \beta', \delta')\) is the disjoint union of a domain supported in \(A\) and a domain supported in \(\Sigma^0\); i.e., it has zero multiplicity along \(\partial D_1 \cup \partial D_2\). Hence the composite map induces the same map on homology as the map
\[
CF(\Sigma, \alpha, \beta) \to CF(\Sigma', \alpha', \delta')
\]
given by \(x \mapsto x' \times \{d\}\), where \(\alpha \cap \delta = \{d\}\), and \(x' \in T_\alpha \cap T_\delta\) is the intersection point closest to \(x \in T_\alpha \cap T_\beta\). So this is equivalent to the map induced by stabilization. \(\square\)

**Lemma 5.42.** Let \(W_1\) be the cobordism obtained by attaching a two-handle to \((M, \gamma)\) along a framed knot \(K\), and let \(W_2\) be a three-handle attached along a two-sphere that cancels the knot. Pick Spin\(^c\) structures \(s_i\) on \(W_i\) for \(i = 1, 2\). Then, the composite
\[
E_{W_2,s_2} \circ F_{K,s_1} : SFH(M, \gamma, s_2) \to SFH(M, \gamma, s_2)
\]
corresponds to the identity map.

**Proof.** The proof follows from “turning around” the proof of Lemma 5.41. \(\square\)

Now, we are ready to prove Theorem 5.37. Let \(W = (W, Z, \xi)\) be our special cobordism. One can find a Morse function on \(W\) that has no zero- and four-handles, and which is the projection map \(\partial M \times I \to I\) on \(Z\). One can also achieve that the index \(i\) critical points map to \(i/4\), respectively. By Kirby calculus, any two such Morse functions can be connected through a sequence of pair creations and cancelations, and a sequence of handleslides, without introducing any zero- or four-handles. The above lemmas ensure that the map \(F_{W_2,s}\) is invariant under all Kirby moves, and hence, it is a four-manifold invariant. \(\square\)

### 5.5. Properties of the special cobordism maps.

**Proposition 5.43.** Let \(W\) be a special cobordism. Then there are only finitely many \(s \in \text{Spin}^c(W)\) for which \(F_{W,s} \neq 0\), and
\[
F_W = \sum_{s \in \text{Spin}^c(W)} F_{W,s}.
\]

**Proof.** Note that if we write \(W = W_3 \circ W_2 \circ W_1\), and \(s, s' \in \text{Spin}^c(W)\) satisfy \(s|W_2 = s'|W_2\), then \(s = s'\). So the claim follows from Proposition 5.15. \(\square\)

**Proposition 5.44.** Let \((M, \gamma)\) be a balanced sutured manifold. If \(W = (W, Z, \xi)\) is the trivial cobordism from \((M, \gamma)\) to \((M, \gamma)\), then \(F_W\) is the identity of \(SFH(M, \gamma)\). Furthermore, the restriction map from \(\text{Spin}^c(W)\) to \(\text{Spin}^c(M, \gamma)\) is an isomorphism, and for every \(s \in \text{Spin}^c(W)\), the map \(F_{W,s}\) is the identity of \(SFH(M, \gamma, s|M)\).

**Proof.** Since \(W = M \times I, Z = \partial M \times I\), and \(\xi\) is \(I\)-invariant, \(\text{Spin}^c(W)\) and \(\text{Spin}^c(M, \gamma)\) are obviously isomorphic. The rest follows from the fact that there is a relative handle decomposition of \(W\) with no handles at all. If \(L = \emptyset\), then both \(F_L\) and \(F_{L,s}\) are identity maps, as they are defined via a triple diagram \((\Sigma, \alpha, \beta, \delta)\), where \(\delta_i\) is a small Hamiltonian translate of \(\beta_i\) for \(i = 1, \ldots, d\). \(\square\)
We have the following analogue of [19, Theorem 3.4].

**Theorem 5.45.** Let $W_1$ be a special cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$, and $W_2$ a special cobordism from $(M_1, \gamma_1)$ to $(M_2, \gamma_2)$, and set $W = W_2 \circ W_1$. Fix Spin\(^c\) structures $\mathfrak{s}_i \in \text{Spin}^c(W_i)$ for $i = 1, 2$ such that $\mathfrak{s}_1|_{M_1} = \mathfrak{s}_2|_{M_1}$. Then

$$FW_2, s_2 \circ FW_1, s_1 = \sum_{\{s \in \text{Spin}^c(W) : s|_{W_1} = s_1, s|_{W_2} = s_2\}} \pm FW, s.$$ 

Moreover, $FW = FW_2 \circ FW_1$.

**Proof.** Just as in the proof of [19, Theorem 3.4], this follows from the corresponding result for links, Proposition 5.46 together with the fact that we can commute two-handle additions past one-handle additions; and similarly, three-handle additions past two-handle additions. We only state these two results in our case, their proofs are essentially the same as the proofs of Proposition 4.18 and 4.19 in [19].

**Proposition 5.46.** Let $(M, \gamma)$ be a balanced sutured manifold, and suppose that $(M', \gamma')$ is obtained by attaching one-handles to $(M, \gamma)$. Furthermore, let $L$ be a framed link in $(M, \gamma)$, and let $L'$ be the corresponding framed link in $(M', \gamma')$. Then we get a one-handle cobordism $V$ from $(M, \gamma)$ to $(M', \gamma')$, and a one-handle cobordism $V(L)$ from $(M(L), \gamma)$ to $(M'(L'), \gamma')$. Then the following diagram commutes:

$$\begin{array}{ccc}
SFH(M, \gamma) & \xrightarrow{F_L} & SFH(M(L), \gamma) \\
\downarrow G_V & & \downarrow G_{V(L)} \\
SFH(M', \gamma') & \xrightarrow{F_{L'}} & SFH(M'(L'), \gamma')
\end{array}$$

and there is an analogous diagram that also includes relative Spin\(^c\) structures.

**Proposition 5.47.** Let $L'$ be a framed link in $(M', \gamma')$, which is disjoint from a two-sphere $S \subset M$. Then we have a three-handle cobordism $U$ from $(M', \gamma')$ to $(M, \gamma)$, let $L$ be the corresponding framed link in $(M, \gamma)$. We also have a three-handle cobordism $U(L)$ from $(M'(L'), \gamma')$ to $(M(L), \gamma)$. Then the following diagram commutes:

$$\begin{array}{ccc}
SFH(M', \gamma') & \xrightarrow{F_U} & SFH(M'(L'), \gamma') \\
\downarrow E_{U} & & \downarrow E_{U(L)} \\
SFH(M, \gamma) & \xrightarrow{F_L} & SFH(M(L), \gamma)
\end{array}$$

and there is an analogous diagram that also includes relative Spin\(^c\) structures.

\[\square\]

### 6. The contact invariant $EH$ and the gluing map $\Phi_\xi$

Here, we review the necessary definitions and result from [12, 16, and 9], and enrich the theory with Spin\(^c\) structures.

#### 6.1. The contact invariant $EH$

Suppose that $(M, \xi)$ is a contact 3-manifold with convex boundary and dividing set $\gamma$ on $\partial M$. We denote such a contact manifold by $(M, \gamma, \xi)$. In [12], Honda, Kazez, and Matić defined an invariant of $(M, \gamma, \xi)$ which is an element $EH(M, \gamma, \xi)$ of $SFH(-M, -\gamma)$, also see [16]. Note that over $\mathbb{Z}$, this element is well-defined only up to sign. We briefly review the construction of this contact invariant.
Definition 6.1. A partial open book decomposition is a pair $(S, h : P \to S)$, where

- $S$ is a compact oriented surface with $\partial S \neq \emptyset$, called the page,
- $P \subset S$ is a compact subsurface, such that each component of $\partial P$ is polygonal with consecutive sides $r_1, \ldots, r_{2n}$, and $r_i \subset \partial S$ for $i$ even,
- $h : P \to S$ is a diffeomorphism such that $h|_{P \cap \partial S} = \text{Id}$.

The partial open book $(S, h)$ defines a contact manifold $(M, \gamma, \xi)$ as follows. Let $M = S \times I / \sim_h$, where $\sim_h$ is the equivalence relation such that $(x, t) \sim_h (x, t')$ for all $x \in \partial S$ and $t \in I$, and $(x, 1) \sim_h (h(x), 0)$ for all $x \in P$. Furthermore, $R_+ (\gamma) = \text{Int}(S \setminus P) \times \{1\}$ and $R_- (\gamma) = \text{Int}(S \setminus h(P)) \times \{0\}$, whereas $\gamma = \partial R_+ (\gamma) = \partial R_- (\gamma)$.

To define $\xi$, notice that one can obtain $(M, \gamma)$ by gluing together the product sutured manifolds $(S \times I, \partial S \times I)$ and $(P \times I, -\partial P \times I)$. Each of them carries a unique product disc decomposable contact structure, $\xi$ is obtained by gluing these together.

Given a contact manifold $(M, \gamma, \xi)$, there exists a compatible partial open book decomposition $(S, h : P \to S)$ by [12, Theorem 1.3]. The closure of the surface $S \setminus P$ is naturally identified with $R_+ (\gamma)$. A set $\{b_1, \ldots, b_d\}$ of properly embedded, pairwise disjoint arcs in $P$ is called a basis for $(S, R_+ (\gamma))$ if $S \setminus \bigcup_{i=1}^d b_i$ deformation retracts onto $R_+ (\gamma)$. In fact, $\{b_1, \ldots, b_d\}$ is a basis for $H_1 (P; \partial S)$. Fix such a basis.

For $i = 1, \ldots, d$, let $a_i$ be an arc that is isotopic to $b_i$ by a small isotopy such that the following hold:

- The endpoints of $b_i$ are isotoped along $\partial S$, in the direction given by the boundary orientation of $S$.
- The arcs $a_i$ and $b_i$ intersect transversely in one point in $\text{Int}(S)$.

Then we obtain a balanced diagram $(\Sigma, \alpha, \beta)$ defining $(M, \gamma)$ by setting

$$\Sigma = (S \times \{0\}) \cup -(P \times \{1/2\}),$$

and taking $\beta_i = \partial (b_i \times [0, 1/2])$ and $\alpha_i = (a_i \times \{1/2\}) \cup (h(a_i) \times \{0\})$ for $i = 1, \ldots, d$, see Figure 5. To see that the orientations match up, observe that $\gamma = \partial \Sigma$, and $\Sigma$ is oriented as the boundary of the $\alpha$ compression body.
Let \( x_i = (a_i \cap b_i) \times \{1/2\} \) for \( i = 1, \ldots, d \). Then \( x = (x_1, \ldots, x_d) \) is a cycle in \( CF(-\Sigma, \alpha, \beta) \). Indeed, if \( D \) is the closure of a component of \(-\Sigma \setminus (\alpha \cup \beta)\) such that \( x_i \in \partial D \) and \( \partial D \cap a_i \) points out of \( x_i \), then \( D \cap \partial \Sigma \neq \emptyset \). So every domain emanating from \( x \) is forced to be zero. Note that \((-\Sigma, \alpha, \beta)\) defines \((-M, -\gamma)\).

Hence \( x \) defines a class in \( SFH(-M, -\gamma) \), which is shown to be an invariant of the contact manifold \((M, \gamma, \xi)\) in \([12\text{, Theorem 3.1}]\). This is the contact invariant \( EH(M, \gamma, \xi) \).

**Remark 6.2.** Note that we changed the notation of \([12]\), where they labeled our \( \alpha \) curves by \( \beta \), our \( \beta \) curves by \( \alpha \), and our \(-\Sigma \) by \( \Sigma \). We did this to make the underlying orientation conventions more transparent. In fact, with the notations of \([12]\), one has \( x \in CF(\Sigma, \beta, \alpha) \), but their \((\Sigma, \alpha, \beta)\) defines \((M, -\gamma)\) instead of \((M, \gamma)\).

If \((\Sigma, \alpha, \beta)\) defined \((M, \gamma)\), then \((\Sigma, \beta, \alpha)\) would define \((-M, \gamma)\). However, this is not a real issue as \( SFH(-M, \gamma) \) and \( SFH(-M, -\gamma) \) are canonically isomorphic by \([4\text{, Proposition 2.12}]\).

**Example 6.3.** To illustrate the above construction, let us review \([16\text{, Example 2}]\). For an illustration, see Figure 5. In this example, the page \( S \) of our partial open book decomposition is an annulus, and \( P \) is a radial segment of \( S \). The diffeomorphism \( h: P \to S \) is the restriction of a left-handed Dehn twist to \( P \). This partial open book decomposition \((S, h: P \to S)\) defines a contact manifold \((M, \gamma, \xi)\). In the resulting chain complex \( CF(-\Sigma, \alpha, \beta) \), there are three generators; we denote them by \( x, y, \) and \( z \), as in Figure 5. Observe that \( \partial x = 0 \), while \( \partial y = x \) and \( \partial z = x \). Thus \( SFH(-M, -\gamma) \cong \mathbb{Z} \) and \( EH(M, \gamma, \xi) = [x] = 0 \). The sutured manifold \((M, \gamma)\) is the once punctured sphere \( S^3(1) \), and \( \xi \) is an overtwisted contact structure on it.

As described above \([4\text{, Proposition 2.12}]\), if \( v \) represents a \( \text{Spin}^c \) structure on \((M, \gamma)\), then the same vector field also represents a \( \text{Spin}^c \) structure on \((-M, -\gamma)\). Hence there is a canonical identification between \( \text{Spin}^c(M, \gamma) \) and \( \text{Spin}^c(-M, -\gamma) \), and we will not distinguish between the two.

**Proposition 6.4.** Suppose that \((M, \gamma, \xi)\) is a contact manifold, and let \( s_\xi \in \text{Spin}^c(M, \gamma) \) be the homology class of the vector field \( \xi^\perp \). Then

\[
EH(M, \gamma, \xi) \in SFH(-M, -\gamma, s_\xi).
\]

**Proof.** The partial open book used to define the contact class \( EH(M, \gamma, \xi) \) is constructed using a relative contact handle decomposition of \((M, \gamma, \xi)\). One takes \( R_-(\gamma) \times I \), and attaches \( d \) contact one-handles, then \( d \) contact two-handles. Suppose that \( H \) is a contact one- or two-handle. This means that \( \xi|H \) is contactomorphic to the unique tight contact structure on \( D^3 \) with convex boundary and connected dividing set. Denote by \( \nu \) the dividing set on \( \partial H \).

The handle decomposition gives a balanced diagram \((-\Sigma, \alpha, \beta)\) for \((-M, -\gamma)\), which agrees with the one arising from the partial open book decomposition. The Heegaard surface \( \Sigma \) is obtained from \( R_-(\gamma) \times \{1\} \) by performing the one-handle surgeries. If \( H \) is a one-handle, then its belt circle is an \( \alpha \)-curve that intersects \( \nu \) in exactly two points. If \( H \) is a two-handle, then its attaching circle is a \( \beta \)-curve that also intersects \( \nu \) in two points. The contact element is represented by an intersection point \( x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \) such that \( x \cap H \) lies on \( R_-(\nu) \) if \( H \) is a one-handle, and on \( R_+(\nu) \) if \( H \) is a two-handle.

The construction of the \( \text{Spin}^c \) structure \( s(x) \) associated to an \( x \) coming from a relative handle decomposition is explained in subsection 3.4 of \([4]\). A vector field \( v \)
Figure 6. A contact one-handle above a contact two-handle, together with the vector field $v_0$ used for constructing the relative $\text{Spin}^c$ structures.

representing $s(x)$ is obtained as follows. One takes the vector field $v_0$ on $M$ that agrees with $\partial/\partial t$ on $M \times I$. On a one-handle $H = D^1 \times D^2$, we take

$$v_0 = -x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y} + z \cdot \frac{\partial}{\partial z},$$

while on a two-handle $H = D^2 \times D^1$ one considers

$$v_0 = -x \cdot \frac{\partial}{\partial x} - y \cdot \frac{\partial}{\partial y} + z \cdot \frac{\partial}{\partial z}.$$

We also choose a smooth one-chain $\theta$ in $M$ that connects the centers of the one- and two-handles, and passes through $x$. Then one extends $v_0(M \setminus N(\theta))$ to a nowhere zero vector field $v$ on $M$. Suppose that $\theta_1, \ldots, \theta_d$ are the components of $\theta$, and $\theta_i$ connects the centers of the handles $H_{\alpha_i}$ and $H_{\beta_i}$ corresponding to $\alpha_i$ and $\beta_i$, respectively, for $i = 1, \ldots, d$. Along $\theta_i$, the vectors $v_0$ point from the center of $H_{\alpha_i}$ towards the center of $H_{\beta_i}$, whereas $v$ points in the opposite direction.

Now we show that $v$ and $\xi^\perp$ are homotopic over the two-skeleton of $M$; i.e., they are homologous. This will follow if we prove that for every one- or two-handle $H$, and for every $p \in \partial H$, we have

$$v_p \neq - (\xi^\perp)_p.$$

Indeed, on $R_-(\gamma) \times I$ both $v$ and $\xi^\perp$ point up. If $v$ is generic, the set where $v$ and $\xi^\perp$ are opposite represent the difference of the corresponding $\text{Spin}^c$ structures. If each component of this difference cycle lies in a handle, then it is obviously null-homologous.
First, observe that $\xi^\perp$ points into $H$ along $R_-(\nu)$, and points out of $H$ along $R_+(\nu)$. Hence, one has 
\[(v_0)_p = -(\xi^\perp)_p\]
for exactly one point $p \in \partial H$, which we can assume to be $x \cap H$. Indeed, if $H$ is a one-handle $H_{x_1}$, then along its belt circle $\alpha_1$ the field $v_0$ points in, and $\alpha_1 \cap R_-(\nu)$ consists of an arc, whose “midpoint” $x_i$ is where $v_0$ and $\xi^\perp$ are opposite. Similarly, if $H$ is a two-handle $H_{x_2}$ with attaching circle $\beta$, then the midpoint $x_1$ of the arc $\beta \cap R_+(\gamma)$ is where $v_0$ and $\xi^\perp$ are opposite. However, as described above, $v$ is opposite to $v_0$ along $\theta$. In particular, $v$ and $\xi^\perp$ point in the same direction at $x_i$, and are never opposite elsewhere along the two-skeleton.

\[
6.2. \text{The gluing map } \Phi_\xi.
\]

**Definition 6.5.** We say that $(M', \gamma')$ is a sutured submanifold of the sutured manifold $(M, \gamma)$ if $M'$ is a submanifold with boundary of $M$, and $M' \subset \text{Int}(M)$. A connected component $C$ of $M \setminus \text{Int}(M')$ is called isolated if $C \cap \partial M = \emptyset$.

Next, we recall [9, Theorem 1.1].

**Theorem 6.6.** Let $(M', \gamma')$ be a sutured submanifold of $(M, \gamma)$, and let $\xi$ be a contact structure on $M \setminus \text{Int}(M')$ with convex boundary, and dividing set $\gamma$ on $\partial M$ and $\gamma'$ on $\partial M'$. If $M \setminus \text{Int}(M')$ has $m$ isolated components, then $\xi$ induces a natural map 

\[
\Phi_\xi : SFH(-M', -\gamma') \to SFH(-M, -\gamma) \otimes V^\otimes m
\]

that is well-defined only up to an overall $\pm$ sign. Here $V = HF(S^1 \times S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ is a $\mathbb{Z}$-graded vector space, where the two summands have gradings that differ by one, say 0 and 1.

Moreover, if $\xi'$ is any contact structure on $M'$ with dividing set $\gamma'$ on $\partial M'$, then 

\[
\Phi_\xi(EH(M', \gamma', \xi')) = EH(M, \gamma, \xi' \cup \xi) \otimes (x \otimes \cdots \otimes x),
\]

where $x$ is the contact class of the standard contact structure on $S^1 \times S^2$.

Honda, Kazez, and Matić call $\Phi_\xi$ the “gluing map”. Its construction is rather involved, we only describe it briefly and highlight its properties that we will need later.

Suppose for now that $m = 0$. First, we choose an arbitrary balanced diagram $(-\Sigma_0', \alpha_0', \beta_0')$ for $(-M', -\gamma')$. Let $T = \partial M'$, and denote by $\zeta$ the $I$-invariant contact structure on $T \times I$ such that for every $t \in I$ the surface $T \times \{t\}$ is convex with dividing set $\gamma' \times \{t\}$. Then we construct two special partial open book decompositions for $(T \times I, \zeta)$, and denote by $(-\Sigma_1, \alpha_1, \beta_1)$ and $(-\Sigma_2, \alpha_2, \beta_2)$ the corresponding balanced diagrams. We set $\Sigma' = \Sigma_0' \cup \Sigma_1 \cup \Sigma_2'$, and extend $\alpha_0' \cup \alpha_1 \cup \alpha_2'$ and $\beta_0' \cup \beta_1 \cup \beta_2'$ to full sets of attaching circles $\alpha'$ and $\beta'$, respectively. We end up with a special kind of balanced diagram $(-\Sigma', \alpha', \beta')$ for $(-M', -\gamma')$, that [9] calls contact compatible with $\zeta$ near $\partial M'$. In the second step, using the contact structure $\xi$, one extends the balanced diagram $(-\Sigma', \alpha', \beta')$ to a balanced diagram $(-\Sigma, \alpha, \beta)$ for $(-M, -\gamma)$ such that $\Sigma' \subset \Sigma$, while $\alpha = \alpha' \cup \alpha''$ and $\beta = \beta' \cup \beta''$. The curves $\alpha''$ and $\beta''$ come from a partial open book decomposition, hence there is a distinguished intersection point $x'' = (x''_1, \ldots, x''_n) \in T_{\alpha''} \cap T_{\beta''}$, with the properties described above Remark 6.2.

More precisely, if $D$ is the closure of a component of $-\Sigma \setminus (\alpha \cup \beta)$ such that $x''_i \in \partial D$ and $\partial D \cap \alpha_i$ points out of $x''_i$, then $D \cap \partial \Sigma \neq \emptyset$.
We define the map 

\[ \phi_\xi: CF(-\Sigma', \alpha', \beta') \to CF(-\Sigma, \alpha, \beta) \]

on each generator \( y \in T_{\alpha'} \cap T_{\beta'} \) of \( CF(-\Sigma', \alpha', \beta') \) by the formula \( \phi_\xi(y) = (y, x'') \), and extend it to \( CF(-\Sigma', \alpha', \beta') \) linearly. This is a chain map, since every domain emanating from \((y, x'')\) has zero multiplicities around \( x'' \), hence must consist of a domain in \((\Sigma', \alpha', \beta')\) emanating from \( y \), plus the trivial domain from \( x'' \) to itself. The induced map on the homology is

\[ \Phi_\xi: SFH(-M', -\gamma') \to SFH(-M, -\gamma). \]

We point out that the definition of \( \Phi_\xi \) also works when \((M', \gamma') = \emptyset \), so \((M, \gamma, \xi)\) is a contact manifold. Then \( SFH(M', \gamma') = \mathbb{Z} \), and the map \( \Phi_\xi \) is given by

\[ \Phi_\xi(1) = EH(M, \gamma, \xi). \]

The naturality of the map \( \Phi_\xi \) means the following. Suppose that \((\Sigma', \alpha', \beta')\) and \((-\Sigma', \alpha', \beta')\) are balanced diagrams for \((-M', -\gamma')\) that are contact compatible with \( \zeta \) near \( \partial M' \), and let \((-\Sigma, \alpha, \beta)\) and \((-\Sigma, \alpha, \beta)\) be their extensions to \((-M, -\gamma)\), respectively. Then the following diagram is commutative:

\[
\begin{array}{ccc}
SFH(-\Sigma', \alpha', \beta') & \xrightarrow{\Phi_1} & SFH(-\Sigma, \alpha, \beta) \\
\downarrow \Psi_1 & & \downarrow \Psi_2 \\
SFH(-\Sigma', \alpha, \beta') & \xrightarrow{\Phi_2} & SFH(-\Sigma, \alpha, \beta)
\end{array}
\]

where \( \Psi_1 \) and \( \Psi_2 \) are isomorphisms induced by equivalences between the sutured diagrams (cf. Theorem 5.18). This implies that if \((M', \gamma')\) is a sutured submanifold of \((M, \gamma)\) with contact structure \( \xi \) on \( Z = M \setminus \text{Int}(M') \), and \((\overline{M'}, \overline{\gamma})\) is a sutured submanifold of \((\overline{M}, \overline{\gamma})\) with contact structure \( \zeta \) on \( \overline{Z} = \overline{M} \setminus \text{Int}(\overline{M'}) \), then an orientation preserving diffeomorphism \( d: M \to \overline{M} \) such that \( d(Z) = \overline{Z} \), \( d(\gamma') = \overline{\gamma} \), \( d^*(\xi) = \overline{\zeta} \) gives rise to a commutative diagram

\[
\begin{array}{ccc}
SFH(-M', -\gamma') & \xrightarrow{\Phi_\xi} & SFH(-M, -\gamma) \\
\downarrow (d|M') & & \downarrow d \circ \\
SFH(-\overline{M'}, -\overline{\gamma'}) & \xrightarrow{\Phi_\overline{\gamma}} & SFH(-\overline{M}, -\overline{\gamma}).
\end{array}
\]

For the case \( m > 0 \), see the discussion on page 7 of [9]. We now list some basic properties of this gluing map. The first is [9, Theorem 6.1].

**Theorem 6.7.** Let \((M, \gamma)\) be a balanced sutured manifold, and let \( \xi \) be an I-invariant contact structure on \( \partial M \times I \) with dividing set \( \gamma \times \{t\} \) on \( \partial M \times \{t\} \). Then the gluing map

\[ \Phi_\xi: SFH(-M, -\gamma) \to SFH(-M, -\gamma), \]

obtained by attaching \((\partial M \times I, \xi)\) to \((M, \gamma)\) along \( \partial M \times \{0\} \), is the identity map (up to an overall \( \pm \) sign, if over \( \mathbb{Z} \)).

The following statement is [9, Proposition 6.2].
Proposition 6.8. Consider the inclusions \((M_0, \gamma_0) \subset (M_1, \gamma_1) \subset (M_2, \gamma_2)\) of sutured submanifolds. For \(i = 0, 1\), let \(\xi_i\) be a contact structure on \(M_{i+1} \setminus \text{Int}(M_i)\) that has convex boundary and dividing set \(\gamma_j\) on \(\partial M_j\) for \(j = i, i + 1\). Then
\[
\Phi_{\xi_i} \circ \Phi_{\xi_0} = \Phi_{\xi_0 \cup \xi_1} : SFH(-M_0, -\gamma_0) \rightarrow SFH(-M_2, -\gamma_2),
\]
up to an overall \(\pm\) sign if over \(\mathbb{Z}\).

Definition 6.9. Let \((M', \gamma')\) be a sutured submanifold of \((M, \gamma)\), and let \(\xi\) be a contact structure on \(M \setminus \text{Int}(M')\) with convex boundary, and dividing set \(\gamma\) on \(\partial M\) and \(\gamma'\) on \(\partial M'\). If \(s' \in \text{Spin}^c(M', \gamma')\), and \(v'\) is a vector field representing \(s'\), then the homology class of the vector field \(v = v' \cup \xi\) is obviously independent of the choice of \(v'\). This defines a map
\[
f_{\xi} : \text{Spin}^c(M', \gamma') \rightarrow \text{Spin}^c(M, \gamma),
\]
by setting \(f_{\xi}(s')\) to be the homology class of \(v\).

Given \(s_1, s_2 \in \text{Spin}^c(M, \gamma)\), their difference \(s_1 - s_2\) is an element of \(H^2(M, \partial M)\), which we identify with \(H_1(M)\) using Poincaré duality.

Lemma 6.10. If \(s'_1, s'_2 \in \text{Spin}^c(M', \gamma')\), then
\[
f_{\xi}(s'_1) - f_{\xi}(s'_2) = e_*(s'_1 - s'_2),
\]
where \(e_* : H_1(M') \rightarrow H_1(M)\) is the map induced by the embedding \(e : M' \hookrightarrow M\).

Proof. Let \(v'_i\) be a vector field representing \(s'_i\) for \(i = 1, 2\). After fixing a trivialization of \(TM\), we can view \(v_1 = v'_1 \cup \xi\) and \(v_2 = v'_2 \cup \xi\) as maps from \(M\) to \(S^2\). If \(p\) is a common regular value of \(v_1\) and \(v_2\), then
\[
f_{\xi}(s'_1) - f_{\xi}(s'_2) = [v_1^{-1}(p) - v_2^{-1}(p)].
\]
But \(v_1\) and \(v_2\) agree outside \(M'\); moreover,
\[
v_1^{-1}(p) - v_2^{-1}(p) = (v'_1)^{-1}(p) - (v'_2)^{-1}(p).
\]
The right hand side, thought of as a one-cycle in \(M\), represents \(e_*(s'_1 - s'_2)\), which proves the lemma.

Proposition 6.11. Let \((M', \gamma')\) be a sutured submanifold of \((M, \gamma)\), and let \(\xi\) be a contact structure on \(M \setminus \text{Int}(M')\) with convex boundary, and dividing set \(\gamma\) on \(\partial M\) and \(\gamma'\) on \(\partial M'\). Pick a \(\text{Spin}^c\) structure \(s' \in \text{Spin}^c(M', \gamma')\), and choose an element \(x' \in SFH(-M', -\gamma', s')\). Then
\[
\Phi_{\xi}(x') \in SFH(-M, -\gamma, f_{\xi}(s')).
\]

Proof. Fix an arbitrary contact structure \(\xi'\) on \((M', \gamma')\) such that \(\partial M'\) is a convex surface with dividing set \(\gamma'\). Choose a partial open book decomposition defining \((M', \gamma', \xi')\), and let \((-\Sigma', \alpha', \beta')\) be the corresponding balanced diagram of \((-M', -\gamma')\). Extend \((-\Sigma, \alpha, \beta)\) it to a diagram \((-\Sigma, \alpha, \beta)\) defining \((-M, -\gamma)\), as in the definition of the map \(\Phi_{\xi}\). More precisely, we have \(\alpha = \alpha' \cup \alpha''\) and \(\beta = \beta' \cup \beta''\), and there is a distinguished intersection point \(x'' \in T_{\alpha''} \cap T_{\beta''}\) such that \(\phi_{\xi}(y) = (y, x'')\) for every \(y \in T_{\alpha'} \cap T_{\beta'}\). Let \(y_0 \in T_{\alpha'} \cap T_{\beta'}\) be the distinguished intersection point representing the contact class \(EH(M', \gamma', \xi')\).

Again, let \(e : M' \hookrightarrow M\) be the embedding. Then it follows from Lemma 6.10 and the above description of \(\phi_{\xi}\) that for any \(y \in T_{\alpha'} \cap T_{\beta'}\), we have
\[
f_{\xi}(\phi_{\xi}(y)) - f_{\xi}(\phi_{\xi}(y_0)) = e_*(\phi_{\xi}(y) - \phi_{\xi}(y_0)) = \]
Definition 7.1. Let $s$ that represents open book decomposition. This would replace the diagrams for $(W, Z, \xi)$.

First, we define $F_{\xi, s}$ which might simplify concrete computations of $\Phi_{\xi, s}$, this implies the claim of the proposition.

Remark 6.12. It is not true in general that for every $x' \in SFH(-M', -\gamma')$ there exists a contact structure $\xi'$ on $(M', \gamma')$ such that $x' = EH(M', \gamma', \xi')$.

Corollary 6.13. For any $s' \in Spin^c(M', \gamma')$, let

$$\Phi_{\xi, s'} = \Phi_{\xi}|SFH(-M', -\gamma', s').$$

Then

$$\Phi_{\xi, s'}: SFH(-M', -\gamma', s') \to SFH(-M, -\gamma, f_\xi(s')), \tag{59a}$$

and

$$\Phi_{\xi} = \bigoplus_{s' \in Spin^c(M', \gamma')} \Phi_{\xi, s'}.$$ 

Remark 6.14. In the remark on page 11 of [9], a construction is outlined for a map from $SFH(-M', -\gamma')$ to $SFH(-M, -\gamma)$ that depends on a $Spin^c$ structure $s' \in Spin^c(M', \gamma')$. Specifically, one would use a contact structure $\xi'$ on $(M', \gamma')$ that represents $s'$ to construct the balanced diagram for $(-M', -\gamma')$ via a partial open book decomposition. This would replace the diagrams for $(-M', -\gamma')$ that are contact compatible near the boundary. It is natural to conjecture that the restriction of this map to $SFH(-M', -\gamma', s')$ is precisely $\Phi_{\xi, s'}$, which might simplify concrete computations of $\Phi_{\xi, s'}$.

7. Construction of the cobordism map $F_{\mathcal{W}}$

Now we can give the construction of the map $F_{\mathcal{W}}$ for an arbitrary balanced cobordism $\mathcal{W} = (W, Z, \xi)$. It is a composition of the gluing map $\Phi_{\xi}$ induced by the contact structure $\xi$ and the cobordism map induced by a special cobordism.

Definition 7.1. Let $\mathcal{W} = (W, Z, \xi)$ be a balanced cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$. Following [9], we say that a component $Z_0$ of $Z$ is isolated if $Z_0 \cap M_1 = \emptyset$. First, we define $F_{\mathcal{W}}$ if $Z$ has no isolated components, see Figure 7

Observing the orientation conventions in Remark 2.1, note that $(-M_0, -\gamma_0)$ is a sutured submanifold of $(-N, -\gamma_1) = (-M_0 \cup Z, -\gamma_1)$. Thus Theorem 6.16 provides us with a map

$$\Phi_{\xi}: SFH(M_0, \gamma_0) \to SFH(N, \gamma_1),$$

well-defined up to an overall $\pm$ sign.

The sutured manifolds $(N, \gamma_1)$ and $(M_1, \gamma_1)$ have the same boundary and same sutures, and $\partial W = -N \cup M_1$. We can think of this as a special cobordism from $(N, \gamma_1)$ to $(M_1, \gamma_1)$, call it $W_1$. For each such cobordism we constructed a map

$$F_{\mathcal{W}_1}: SFH(N, \gamma_1) \to SFH(M_1, \gamma_1)$$

in section 5. Finally, we set

$$F_{\mathcal{W}} = F_{\mathcal{W}_1} \circ \Phi_{\xi}.$$
Figure 7. A balanced cobordism \( W \) having no isolated components on the left, and the corresponding special cobordism \( W_1 \) on the right.

In the general case, for each isolated component \( Z_0 \) of \( Z \), choose a small standard contact ball \( B_0 \subset \text{Int}(Z_0) \) with convex boundary, and connected dividing set \( \delta_0 \). Let \( B \) be the union of all such balls, and \( \delta \) the dividing set on \( \partial B \). Then we replace the cobordism \( W \) with \( W' = (W, Z', \xi') \), where \( Z' = Z \setminus \text{Int}(B) \) and \( \xi' = \xi|Z' \). The morphism \( W' \) has source \( (M_0, \gamma_0) \) and range \( (M'_1, \gamma'_1) = (M_1, \gamma_1) \sqcup (B, \delta) \). Since \( SFH(M'_1, \gamma'_1) \cong SFH(M_1, \gamma_1) \), and now \( Z' \) has no isolated components, we can define \( F_W = F_{W'} \).

The map \( F_W \) has a refinement along relative Spin\(^c\) structure on \( W \). We describe this next.

**Definition 7.2.** Let \( W = (W, Z, \xi) \) be a balanced cobordism from \( (M_0, \gamma_0) \) to \( (M_1, \gamma_1) \). Choose a relative Spin\(^c\) structure \( s \in \text{Spin}^c(W) \), and for \( i = 0, 1 \) set
\[
t_i = s|_{M_i} \in \text{Spin}^c(M_i, \gamma_i).
\]

First, suppose that \( Z \) has no isolated components. Then there is a natural restriction map from \( \text{Spin}^c(W) \) to \( \text{Spin}^c(W_1) \), we denote the image of \( s \) by \( s_1 = s|W_1 \). Note that
\[
s_1|N = f_\xi(t_0) \in \text{Spin}^c(N, \gamma_1).
\]

Hence the special cobordism \( W_1 \), endowed with the Spin\(^c\) structure \( s_1 \), induces a map
\[
F_{W_1, s_1} : SFH(N, \gamma_1, f_\xi(t_0)) \to SFH(M_1, \gamma_1, t_1).
\]

By Corollary 6.13, the sutured submanifold \((-M_0, -\gamma_0)\) of \((-N, -\gamma_1)\), together with the Spin\(^c\) structure \( t_0 \), give rise to a map
\[
\Phi_{\xi, t_0} : SFH(M_0, \gamma_0, t_0) \to SFH(N, \gamma_1, f_\xi(t_0)).
\]

So we can define
\[
F_{W, s} : SFH(M_0, \gamma_0, t_0) \to SFH(M_1, \gamma_1, t_1)
\]
by the formula \( F_{W, s} = F_{W_1, s_1} \circ \Phi_{\xi, t_0} \).
When $Z$ does have isolated components, then as before, we take the cobordism $\mathcal{W}'$ from $(M_0, \gamma_0)$ to $(M'_1, \gamma'_1) = (M_1, \gamma_1) \cup (B, \delta)$. Then set $s' = s|\mathcal{W}'$, and notice that $t'_1 = s'((M'_1, \gamma'_1))$ agrees with $t_1$ on $(M_1, \gamma_1)$, and is the vertical Spin$^c$ structure $t_B$ on the product $(B, \delta)$. Hence

$$SFH(M'_1, \gamma'_1, t'_1) = SFH(M_1, \gamma_1, t_1) \otimes SFH(B, \delta, t_B) \cong SFH(M_1, \gamma_1, t_1) \otimes \mathbb{Z},$$

and we can set $F_{\mathcal{W}, s} = F_{\mathcal{W}', s'}$.

The above construction motivates the following definition.

**Definition 7.3.** A cobordism $\mathcal{W} = (W, Z, \xi)$ from $(M_0, \gamma_0)$ to $(N, \gamma_1)$ is called a boundary cobordism if $\mathcal{W}$ is balanced, $N$ is parallel to $M_0 \cup (-Z)$, and we are also given a retraction $r: W \to M_0 \cup (-Z)$ such that $r|N$ is an orientation preserving diffeomorphism from $N$ to $M_0 \cup (-Z)$.

Given a boundary cobordism, we can view $(-M_0, -\gamma_0)$ as a sutured submanifold of $(-N, -\gamma_1)$, and $\xi$ is a contact structure such that $\partial M_0 \cup \partial N$ is a convex surface with dividing set $\gamma_0 \cup \gamma_1$. Hence a boundary cobordism gives rise to a map $\Phi: SFH(M_0, \gamma_0) \to SFH(N, \gamma_1)$.

**Definition 7.4.** Let $W = (W, Z, \xi)$ and $W' = (W', Z', \xi')$ be boundary cobordisms from $(M_0, \gamma_0)$ to $(N, \gamma_1)$, together with retractions $r$ and $r'$, respectively. Then we say that $W$ and $W'$ are strongly equivalent if there is a strong equivalence $d: W \to W'$ in the sense of Definition 7.3 that also respects the retractions; i.e., $d \circ r = r' \circ d$. Such a $d$ is called a strong equivalence.

If $W$ is a boundary cobordism from $(M_0, \gamma_0)$ to $(N, \gamma_1)$, and $W'$ is a boundary cobordism from $(M'_0, \gamma'_0)$ to $(N', \gamma'_1)$, then $W$ and $W'$ are said to be weakly equivalent if there is a weak equivalence $d: W \to W'$ in the sense of Definition 7.3 that also satisfies $d \circ r = r' \circ d$. We call such a $d$ a weak equivalence.

The naturality of $\Phi$ implies the following statement.

**Proposition 7.5.** Let $d$ be a weak equivalence between the boundary cobordisms $\mathcal{W} = (W, Z, \xi)$ from $(M_0, \gamma_0)$ to $(N, \gamma_1)$ and $\mathcal{W}' = (W', Z', \xi')$ from $(M'_0, \gamma'_0)$ to $(N', \gamma'_1)$. Furthermore, let $t \in \text{Spin}^c(M_0, \gamma_0)$ and $t' = d_*(t)$. Then there is a commutative diagram

\[
\begin{array}{ccc}
SFH(M_0, \gamma_0, t) & \xrightarrow{\Phi_{*, t}} & SFH(N, \gamma_1, f_*(t)) \\
\downarrow{(d|M_0)_*} & & \downarrow{(d|N)_*} \\
SFH(M'_0, \gamma'_0, t') & \xrightarrow{\Phi_{*, t'}} & SFH(N', \gamma'_1, f_*(t')),
\end{array}
\]

and there is an analogous diagram for $\Phi_{\xi}$ and $\Phi_{\xi'}$ that does not include Spin$^c$ structures.

**Lemma 7.6.** Suppose that $\mathcal{W} = (W, Z, \xi)$ is a balanced cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$ such that $Z$ has no isolated components. Then $\mathcal{W}$ can be written as a composition $\mathcal{W}_1 \circ \mathcal{W}_0$, where $\mathcal{W}_0$ is a boundary cobordism from $(M_0, \gamma_0)$ to some $(N, \gamma_1)$, and $\mathcal{W}_1$ is a special cobordism from $(N, \gamma_1)$ to $(M_1, \gamma_1)$.

This decomposition is unique in the following sense. Suppose that $d$ is a weak equivalence between $\mathcal{W}$ and another cobordism $\mathcal{W}'$ from $(M'_0, \gamma'_0)$ to $(M'_1, \gamma'_1)$, and we have a splitting $\mathcal{W}' = \mathcal{W}'_1 \circ \mathcal{W}'_0$ along $(N', \gamma'_1)$. Then there is a weak equivalence $d'$ such that $d'(N, \gamma_1) = (N', \gamma'_1)$ and $d(x) = d'(x)$ for every $x \in M_0 \cup M_1$. Furthermore,
$d'|W_0$ is a weak equivalence between the boundary cobordisms $W_0$ and $W_0'$, and $d'|W_1$ is a weak equivalence between the special cobordisms $W_1$ and $W_1'$. Finally, $d_*(s) = d'_*(s)$ for every $s \in \text{Spin}^c(W)$.

Proof. First, choose a collar neighborhood $Z_1 = \partial M_1 \times I$ of $\partial M_1$ in $Z$ such that $\partial M_1 = \partial M_1 \times \{0\}$, and for every $t \in I$ the surface $\partial M_1 \times \{t\}$ is convex with dividing set $\gamma_1 \times \{t\}$ in $\xi_1 = \xi|Z_1$. Set $Z_0$ to be the closure of $Z \setminus Z_1$. Then choose a properly embedded 3-manifold $N \subset W$ such that $\partial N = \partial M_0 \times \{1\}$, and $N$ is parallel to $M_0 \cup (-Z_0)$. Cutting $W$ along $N$ gives the required decomposition.

The uniqueness follows from the uniqueness of the “product” collar neighborhood ($Z_1, \xi_1$), see [12] Theorem 2.5.23]. So one can isotope $d$ relative to $M_0 \cup M_1$ through weak equivalences until $d'(N, \gamma_1) = (N', \gamma_1')$. Then a further isotopy in $\text{Int}(W')$ ensures that $d \circ r = r' \circ d$, where $r$ and $r'$ are the retractions for the boundary cobordisms $W_0$ and $W_0'$, respectively. Hence $r|W_0$ is an equivalence of boundary cobordisms. We can achieve that the map $d$ respects the product structures on $Z_1$ and $Z_2$; i.e., $d(x,t) = (d(x),t)$ for $(x,t) \in Z_1 = \partial M_1 \times I$, showing that $d|W_1$ is an equivalence of special cobordisms. The last statement follows from the fact that we got $d'$ from $d$ by isotoping it through weak equivalences.

Let $W = (W, Z, \xi)$ be a balanced cobordism such that $Z$ has no isolated components. An alternative way of thinking about the map $F_W$ is to write $W = W_1 \circ W_0$ as in Lemma 7.8, i.e., $W_0 = (W_0, Z_0, \xi_0)$ is a boundary cobordism from $(M_0, \gamma_0)$ to $(N, \gamma_1)$, and $W_1$ is a special cobordism from $(N, \gamma_1)$ to $(M_1, \gamma_1)$. Then set $F_W = F_{W_1} \circ F_{W_0}$, where $F_{W_1}$ is the map defined in section 5 and $F_{W_0} = \Phi_{\xi_0}$.

**Proposition 7.7.** Let $W$ be a cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$. If $W$ is a special cobordism, then the $F_W$ defined here agrees with the special cobordism map $F_W$ defined in section 4. If $W = (W, Z, \xi)$ is a boundary cobordism, then $F_W = \Phi_{\xi}$.

Proof. If $W$ is a special cobordism, then $W_0$ is the trivial cobordism from $(M_0, \gamma_0)$ to itself, so by Theorem 6.7 the map $\Phi_{W_0}$ is the identity, and $F_W = F_{W_1}$. Furthermore, $W = W_1$.

On the other hand, if $W$ is a boundary cobordism, then $W_1$ is the trivial cobordism from $(M_1, \gamma_1)$ to itself, so $W = W_0$. By Proposition 5.44 the map $F_{W_1}$ is the identity, hence $F_W = \Phi_{\xi_0}$.

Consequently, the maps $F_W$ generalize both special cobordism maps and gluing maps.

**Definition 7.8.** Suppose that $W = (W, Z, \xi)$ is a balanced cobordism such that $Z$ has no isolated components. Write $W$ as $W_1 \circ W_0$, where $W_0$ is a boundary cobordism and $W_1$ is a special cobordism. We say that the Spin$^c$ structures $s, s' \in \text{Spin}^c(W)$ are equivalent, in short $s \sim s'$, if $s|W_i = s'|W_i$ for $i = 0, 1$. We denote the set of equivalence classes by $\text{Spin}^c(W)/\sim$.

If $W$ does have isolated components, then we let $\text{Spin}^c(W)/\sim = \text{Spin}^c(W')/\sim$, where $W'$ is the cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1) \sqcup (B, \delta)$ introduced in Definition 7.1.

By construction, if $s \sim s'$, then $F_{W,s} = F_{W,s'}$. For $s \in \text{Spin}^c(W)/\sim$, let

$$F_{W,s} = F_{W,s},$$

where $s$ is an arbitrary representative of $s$. 
Remark 7.9. Suppose that \( \mathcal{W}_0 \) is a boundary cobordism from \((M_0, \gamma_0)\) to \((N, \gamma_1)\), and \( \mathcal{W}_1 \) is a special cobordism from \((N, \gamma_1)\) to \((M_1, \gamma_1)\). Let \( \mathcal{W} = \mathcal{W}_1 \circ \mathcal{W}_0 \). Using a relative Mayer-Vietoris sequence, we see that \( \text{Spin}^c(\mathcal{W})/\sim \) corresponds to the set of \( \delta H^1(N, dN) \) orbits in \( \text{Spin}^c(\mathcal{W}) \). Hence \( \text{Spin}^c(\mathcal{W}) = \text{Spin}^c(\mathcal{W})/\sim \) if \( H_2(N) = 0 \).

Also note that \( \text{Spin}^c(\mathcal{W}_0) \cong \text{Spin}^c(M_0, \gamma_0) \). For \( t_0 \in \text{Spin}^c(M_0, \gamma_0) \), let \( s_0 \) be the corresponding element of \( \text{Spin}^c(\mathcal{W}_0) \). Then \( s_0|N = f_\xi(t_0) \), and in general the map \( f_\xi \) is neither injective, nor surjective, cf. Lemma 6.11. Furthermore, \( s, s' \in \text{Spin}^c(\mathcal{W}) \) are equivalent if and only if \( s|M_0 = s'|M_0 \) and \( s|\mathcal{W}_1 = s'|\mathcal{W}_1 \).

**Proposition 7.10.** Given a balanced cobordism \( \mathcal{W} = (W, Z, \xi) \), we have

\[
F_W = \bigoplus_{s \in \text{Spin}^c(\mathcal{W})/\sim} F_{\mathcal{W}, s}.
\]

**Proof.** This follows from Proposition 6.13 and Corollary 6.13. \( \square \)

8. Properties of the cobordism map \( F_W \)

8.1. Naturality and functoriality. We start by proving a naturality result for the cobordism map \( F_W \).

**Theorem 8.1.** Suppose that \( \mathcal{W} = (W, Z, \xi) \) is a balanced cobordism from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\), and \( \mathcal{W}' = (W', Z', \xi') \) is a balanced cobordism from \((M_0', \gamma_0')\) to \((M_1', \gamma_1')\). Pick an \( s \in \text{Spin}^c(\mathcal{W}) \), and let \( t_i = s|M_i \) for \( i = 0, 1 \). If \( d \) is a weak equivalence from \( W \) to \( W' \), then we have a commutative diagram

\[
\begin{array}{ccc}
SFH(M_0, \gamma_0, t_0) & \xrightarrow{F_{\mathcal{W}, s}} & SFH(M_1, \gamma_1, t_1) \\
\downarrow{(d|\mathcal{M}_0)} & & \downarrow{(d|M_1)} \\
SFH(M_0', \gamma_0', t_0') & \xrightarrow{F_{\mathcal{W}', s'}} & SFH(M_1', \gamma_1', t_1'),
\end{array}
\]

where \( s' = d_* (s) \) and \( t'_i = s'|M_i \) for \( i = 0, 1 \). An analogous statement holds for \( F_{\mathcal{W}} \).

**Proof.** This follows from the corresponding naturality result for special cobordisms, Theorem 6.37 and the naturality of the gluing map \( \Phi_\xi \). More precisely, first assume that \( Z \) and \( Z' \) have no isolated components. Set \( N = M_0 \cup (-Z) \) and \( N' = M_0' \cup (-Z') \). Then \((M_0, \gamma_0)\) is a sutured submanifold of \((N, \gamma_1)\), and \((M_0', \gamma_0')\) is a sutured submanifold of \((N', \gamma_1')\). Furthermore, \( d|N \) is an orientation preserving diffeomorphism from \( N \) to \( N' \) such that \( d(Z) = Z' \), \( d(\gamma_0) = \gamma_0' \), \( d(\gamma_1) = \gamma_1' \), and \( d_* (\xi) = \xi' \). Hence the naturality of the gluing map gives the commutative diagram

\[
\begin{array}{ccc}
SFH(M_0, \gamma_0, t_0) & \xrightarrow{\Phi_\xi, t_0} & SFH(N, \gamma_1, f_\xi(t_0)) \\
\downarrow{(d|\mathcal{M}_0)} & & \downarrow{(d|N)} \\
SFH(M_0', \gamma_0', t_0') & \xrightarrow{\Phi_\xi', t_0'} & SFH(N', \gamma_1', f_\xi(t_0')).
\end{array}
\]

On the other hand, \( d \) gives rise to a weak equivalence between the special cobordism \( \mathcal{W}_1 \) from \((N, \gamma_1)\) to \((M_1, \gamma_1)\) and the special cobordism \( \mathcal{W}'_1 \) from \((N', \gamma_1')\) to \((M_1', \gamma_1')\).
Figure 8. The cobordism $W_1$ has one isolated component, while $W_2$ has two. In this example, $W''_2 \circ W'_1$ and $W'$ are different.

So Theorem 5.37 gives the commutative diagram

$$SFH(N, \gamma_1, f_\xi(t_0)) \xrightarrow{F_{W_1,s_1}} SFH(M_1, \gamma_1, t_1)$$

$$\downarrow (d|N)_* \downarrow (d|M)_*$$

$$SFH(N', \gamma'_1, f_{\xi'}(t'_0)) \xrightarrow{F_{W'_1,s'_1}} SFH(M'_1, \gamma'_1, t'_1).$$

Putting the above two commutative diagrams together gives the required diagram.

The case when $Z$ and $Z'$ have isolated components follows from this by deleting standard contact balls $(B, \delta)$ from $Z$, and then deleting the corresponding balls $B' = d(B)$ from $Z'$.  □

**Corollary 8.2.** If the balanced cobordisms $W$ and $W'$ from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$ are strongly equivalent, then $F_W = F_{W'}$. Furthermore, if $d$ is a strong equivalence, and $s \in \text{Spin}^c(W)$, then $F_{W,s} = F_{W',d(s)}$.

**Proof.** This follows from Theorem 8.1 by observing that for a strong equivalence $d$ and for $i = 0, 1$ the map $d|M_i$ is the identity of $M_i$, hence $(d|M_i)_*$ is the identity of $SFH(M_i, \gamma_i)$.  □

So $F_W$ can be defined on strong equivalence classes of cobordisms; i.e., on morphisms of the category of BSut. We now show that $F_W$ is in fact functorial.

**Theorem 8.3.** Let $W_1$ be a balanced cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$, and let $W_2$ be a balanced cobordism from $(M_1, \gamma_1)$ to $(M_2, \gamma_2)$. We write $W$ for the composition $W_2 \circ W_1$. Then

$$F_{W_2} \circ F_{W_1} = F_W.$$

Furthermore, if $s_i \in \text{Spin}^c(W_i)/\sim$ for $i = 1, 2$, then

$$F_{W_2,s_2} \circ F_{W_1,s_1} = \sum_{\{s \in \text{Spin}^c(W)/\sim : s|W_i = s_1, s|W_2 = s_2\}} F_{W,s}.$$
Proof. We show how to reduce to the case when neither \( W_1 \), nor \( W_2 \) has isolated components. Recall that in the general case, for \( i = 1, 2 \), the map \( F_{W_i} \), was defined to be \( F_{W_i} \), where \( W_i \) is a cobordism from \((M_i, 1, \gamma_i)\) to \\
\((M_i, 1, \gamma_i) \cup (B_i, \delta_i)\).

The cobordisms \( W_1' \) and \( W_2' \) do not compose if \( B_1 \neq \emptyset \), see Figure 8. To get around this problem, let \( B_1 \) be the trivial cobordism from \((B_1, \delta_i)\) to itself, and take \( W_2'' \) to be the disjoint union of \( W_2' \) and \( B_1 \). Then, assuming the result for no isolated components, we have \\
\( F_{W_2''} \circ F_{W_1} = F_{W_2''} \circ W_{1}' \).

Now \( F_{W_2''} = F_{W_2} \) by Proposition 5.44, so the left hand side is just \( F_{W_2} \circ F_{W_1} \).

To define the map \( F_W \), we use a cobordism \( \mathcal{W} \) with no isolated components. Notice that \( W_2'' \circ W_1' \) almost agrees with \( \mathcal{W} = (W, Z', \xi') \), except that some standard contact balls might be removed from \((Z', \xi')\) and added to \((M_2', \gamma_2')\). Such a situation is depicted on Figure 8. However, the following lemma will ensure that \( F_{W_2''} \circ W_{1}' = F_W \) still holds, and hence we still have \( F_{W_2} \circ F_{W_1} = F_W \) in the presence of isolated components.

Lemma 8.4. Suppose that \( \mathcal{W} = (W, Z, \xi) \) is a balanced cobordism from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\). Let \( B \subset \text{Int}(Z) \) be a standard contact ball in \((Z, \xi)\) with convex boundary and connected dividing set \( \delta \). Furthermore, let \( \mathcal{V} = (W, Z_0, \xi_0) \) be the cobordism from \((M_0, \gamma_0)\) to \((M_1, \gamma_1) \cup (B, \delta) \), where \( Z_0 = Z \setminus \text{Int}(B) \) and \( \xi_0 = \xi|_{Z_0} \). Then \( F_W = F_{\mathcal{V}_0} \), after we identify \( SFH(M_1, \gamma_1) \) with \( SFH((M_1, \gamma_1) \cup (B, \delta)) \).

Proof. First, assume that \( Z \) has no isolated components, then the same holds for \( Z_0 \). By definition, \( F_W = F_{W_1} \circ \Phi_\xi \) and \( F_{\mathcal{V}} = F_{\mathcal{V}_1} \circ \Phi_{\xi_0} \), where \( W_1 \) is a special cobordism from some sutured manifold \((N, \gamma_1)\) to \((M_1, \gamma_1)\). As described on page 7 of [2], the map \\
\( \Phi_{\xi_0} : SFH(M_0, \gamma_0) \to SFH(N, \gamma_1) \otimes V \)

agrees with \( \Phi_\xi \) composed with a map \\
\( SFH(N, \gamma_1) \to SFH(N, \gamma_1) \otimes V \),

given by connected summing with the suture manifold \( S^3(1) \). More precisely, take the same balanced diagram that gives \( \Phi_\xi \), remove an open ball \( D \) from the Heegaard surface, and add a circle \( \alpha \) and a circle \( \beta \) that are small Hamiltonian translates of each other, and are parallel to \( \partial D \), see Figure 9. If \( x \in \alpha \cap \beta \) is the intersection
agrees with $F_W$. So suppose that $W_1$ is a special cobordism. Then $W_1 = W_{11} \circ W_{10}$, where $W_{10}$ is a boundary cobordism and $W_{11}$ is a special cobordism. By definition, $F_{W_1} = F_{W_{11}} \circ F_{W_{10}}$. So

$$F_{W_1} \circ F_{W_{10}} = F_{W_{21}} \circ F_{W_{20}} \circ F_{W_{11}} \circ F_{W_{10}}.$$

We can uniquely write the cobordism $W_{20} \circ W_{11}$ in the form $\mathcal{V}_1 \circ \mathcal{V}_0$, where $\mathcal{V}_0$ is a boundary cobordism and $\mathcal{V}_1$ is a special cobordism. Then

$$\mathcal{V}_1 \circ \mathcal{V}_0 = W_{21} \circ \mathcal{V}_1 \circ \mathcal{V}_0 \circ W_{10},$$

and $W_{21} \circ \mathcal{V}_1$ is a special cobordism, whereas $\mathcal{V}_0 \circ W_{10}$ is a boundary cobordism. So, by the uniqueness part of Lemma 7.6,

$$F_{W_0} = F_{W_{21} \circ \mathcal{V}_1} \circ F_{\mathcal{V}_0} \circ W_{10}.$$

Using Theorem 5.45, we get

$$F_{W_{21} \circ \mathcal{V}_1} = F_{W_{21}} \circ F_{\mathcal{V}_1},$$

and Proposition 6.8 implies that

$$F_{\mathcal{V}_0 \circ W_{10}} = F_{\mathcal{V}_0} \circ F_{W_{10}}.$$

So we are done as soon as we show that

$$F_{\mathcal{V}_1} \circ F_{\mathcal{V}_0} = F_{W_{20}} \circ F_{W_{11}}.$$

This will follow from the next proposition.

**Proposition 8.5.** Suppose that $\mathcal{W} = \mathcal{W}_1 \circ \mathcal{W}_0$, where $\mathcal{W}_1$ is a boundary cobordism and $\mathcal{W}_0$ is a special cobordism. Let $\mathcal{V}_1 \circ \mathcal{V}_0$ be the unique decomposition of $\mathcal{W}$ into a boundary cobordism $\mathcal{V}_0$ and a special cobordism $\mathcal{V}_1$. Then

$$F_{\mathcal{W}_1} \circ F_{\mathcal{W}_0} = F_{\mathcal{V}_1} \circ F_{\mathcal{V}_0}.$$

**Proof.** Suppose that $(M', \gamma')$ is a sutured submanifold of $(M, \gamma)$, and assume $\xi$ is a contact structure on $M \setminus \text{Int}(M')$ which has no isolated components. The result follows if we show that the gluing map $\Phi_{\xi}$ commutes with one-, two-, and three-handle maps, corresponding to handle attachments to $(-M', -\gamma')$. I.e., we can assume that $\mathcal{W}_0$ and $\mathcal{V}_1$ are both $k$-handle cobordisms for $k = 1, 2, 3$. 

First, we consider the case $k = 2$. Let $L' \subset (-M', -\gamma')$ be the framed link along which we attach the two-handles, and we write $L$ when we consider this link in $(-M, -\gamma)$. Then construct a triple-diagram $(-\Sigma_0', \alpha_0', \beta_0', \delta_0')$ subordinate to some bouquet for $L'$. As explained after Theorem 6.6, we can extend $(-\Sigma_0', \alpha_0', \beta_0')$ to a diagram $(-\Sigma', \alpha', \beta')$ of $(-M', -\gamma')$ that is contact compatible near $\partial M'$, which we then further extend to a diagram $(-\Sigma, \alpha, \beta)$ defining $(-M, -\gamma)$ in a way compatible with $\xi$. Next, we extend $\delta_0'$ to a set of curves $\delta$ by taking small Hamiltonian translates of the curves in $\beta \setminus \beta_0'$, and denote by $\delta'$ the translates of the curves in $\beta' \setminus \beta_0'$, together with $\delta_0$. Then the two-handle map $F_{L'}$ is defined using the triple-diagram $(-\Sigma', \alpha', \beta', \delta')$, while $F_L$ can be defined using $(-\Sigma, \alpha, \beta, \delta).

Take intersection points $x \in T_\alpha \cap T_\beta$ and $y \in T_{\alpha'} \cap T_{\beta'}$. Furthermore, let $\Theta \in T_\beta \cap T_\delta$ and $\Theta' \in T_{\beta'} \cap T_{\delta'}$ be the distinguished intersection points. Then $\phi_\xi(y) = (y, x'')$ and $F_{L'}(y) = F(y \otimes \Theta')$, while $F_L(x) = F(x \otimes \Theta)$. Denote by $x''' \in T_{\alpha'} \cap T_{\beta'} \cap T_{\delta'}$ the point lying closest to $x''$, which is also $F(x'' \otimes (\Theta \setminus \Theta'))$, see Figure 10. So what we need to check is

$$(F(y \otimes \Theta'), x''') = F((y, x'') \otimes \Theta).$$

To see this, notice that the only domains in $(-\Sigma, \alpha, \beta, \delta)$ that contain both $\Theta$ and $x''$ as incoming corners consist of the small triangles next to $x''$, plus a domain in $(-\Sigma', \alpha', \beta')$, as all the $\alpha \setminus \alpha'$, $\beta \setminus \beta'$, and $\delta \setminus \delta'$ are used up in the small triangles. These small triangles have corners $x''$, $\Theta \setminus \Theta'$, and $x'''$, and appear in the count for $F((y, x'') \otimes \Theta)$. The above mentioned domains in $(-\Sigma', \alpha', \beta')$ are exactly the ones counted in $F(y \otimes \Theta')$. The argument is analogous to the proof of the invariance of the map $\Phi_\xi$ under handleslides in $(-\Sigma_0', \alpha_0', \beta_0')$, found in [9].

The cases $k = 1$ and $k = 3$ are analogous to the invariance of $\Phi_\xi$ under stabilization/destabilization in $(-\Sigma_0', \alpha_0', \beta_0')$. For example, if we attach a one-handle to $(-M', -\gamma')$, then we start out with an arbitrary diagram $(-\Sigma_0', \alpha_0', \beta_0')$ that defines $(-M', -\gamma')$, and extend it to $(-\Sigma', \alpha', \beta')$ that is contact compatible near $\partial M'$. Then we use the diagram $(-\Sigma')^0 \cup A, \alpha' \cup \{\alpha\}, \beta' \cup \{\beta\})$ to define the one-handle map by $g_{(\theta)}(y) = y \times \{\theta\}$, where $\theta \in \alpha \cap \beta$ is the intersection point with the bigger relative grading. Note that we glue $A$ to the subsurface $-\Sigma_0'$ of $-\Sigma'$. Then the result follows from

$$(y \times \{\theta\}, x'') = (y, x'') \times \{\theta\}.$$
The case $k = 3$ is very similar, we start with a diagram for $(-M', -\gamma')$ that represents the attaching sphere of the three-handle, given by Lemma 5.33, then “turn around” the argument for one-handles. This proves the proposition.

The second part, using the Spin$^c$ structures, follows from the first part and Proposition 7.10. This concludes the proof of Theorem 8.3.

**Proposition 8.6.** If $W = (W, Z, \xi)$ is the trivial cobordism from $(M, \gamma)$ to $(M', \gamma)$, then $F_W$ is the identity of $SFH(M, \gamma)$. Furthermore, the restriction map from $Spin^c(W)$ to $Spin^c(M, \gamma)$ is an isomorphism, and for every $s \in Spin^c(W)$, the map $F_{W,s}$ is the identity of $SFH(M, \gamma, s[M])$.

**Proof.** This follows from Proposition 5.44 and Proposition 7.7.

Together with Theorem 8.1, this implies the following.

**Corollary 8.7.** Let $(M, \gamma)$ be a balanced sutured manifold. If the diffeomorphisms $\phi_0, \phi_1 : (M, \gamma) \to (M', \gamma)$ are isotopic, then

$$(\phi_0)_*, (\phi_1)_* : SFH(M, \gamma) \to SFH(M', \gamma).$$

**Proof.** As explained on page 180 of [1], use Theorem 8.1 with both $W$ and $W'$ being the trivial cobordism from $(M, \gamma)$ to $(M', \gamma)$, and with $d : M \times I \to M \times I$ being an isotopy between $\phi_0$ and $\phi_1$. Then Proposition 8.6 gives $(\phi_0)_* = (\phi_1)_*$.

So the group $\Gamma(M, \gamma)$ of isotopy classes of orientation preserving diffeomorphisms of $(M, \gamma)$ acts on $SFH(M, \gamma)$. Furthermore, if $s \in Spin^c(M, \gamma)$ and $\phi_0$ and $\phi_1$ are isotopic, then $(\phi_0)_*(s) = (\phi_1)_*(s)$, call this Spin$^c$ structure $t$. Thus $\phi_0$ and $\phi_1$ induce the same map from $SFH(M, \gamma, s)$ to $SFH(M, \gamma, t)$.

**Remark 8.8.** If $(M', \gamma')$ is obtained from $(M, \gamma)$ using a convex decomposition, then we can view $(-M', -\gamma')$ as a sutured submanifold of $(-M, -\gamma)$, and there is a natural contact structure $\zeta$ on $M \setminus \text{Int}(M')$, see page 3 of [9]. Hence we can view this as a boundary cobordism $W$ from $(M', \gamma')$ to $(M, \gamma)$. For nice sutured manifold decompositions, the map $F_W = \Phi_\zeta$ is an embedding by [14].

### 8.2. Duality and blow-up.

It follows from [4, Proposition 2.12] that there is a natural bilinear pairing

$$CF(\Sigma, \alpha, \beta, s) \otimes CF(-\Sigma, \alpha, \beta, s) \to \mathbb{Z}.$$  

On the generators $x, y \in T_\alpha \cap T_\beta$ with $s(x) = s$ and $s(y) = \bar{s}$, it is given by $\langle x, y \rangle = 1$ if $x = y$, and $\langle x, y \rangle = 0$ otherwise. Just as in [19, Lemma 5.1], for every $a \in CF(\Sigma, \alpha, \beta, s)$ and $b \in CF(-\Sigma, \alpha, \beta, \bar{s})$, we have

$$\langle a, \partial_- b \rangle = \langle \partial_\Sigma a, b \rangle.$$  

So there is an induced pairing $\langle \ , \rangle$ on $SFH(M, \gamma, s) \otimes SFH(-M, -\gamma, s)$. Then $F_W$ satisfies the following duality result.

**Theorem 8.9.** Let $W$ be a special cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$. Then, the map induced by $W$, thought of as a cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$, is dual to the map induced by $W$, thought of as a cobordism from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0)$ (cf. Remark 2.12). That is, for every $s \in Spin^c(W)$ with $s|M_i = t_i$ for $i = 1, 2$, and each $x \in SFH(M_0, \gamma_0, t_0)$ and $y \in SFH(M_1, \gamma_1, t_1)$, we have

$$\langle F_{W,s}(x), y \rangle_1 = \langle x, F_{W,s}(y) \rangle_0.$$


Proof. This is completely analogous to [19, Theorem 3.5]. □

Question 8.10. Does Theorem 8.9 hold for arbitrary balanced cobordisms?

We will get back to this question shortly. Next, we generalize the hat version of the blow-up formula [19, Theorem 3.7] to our situation.

Theorem 8.11. Let $W_0$ be the trivial cobordism from the balanced sutured manifold $(M, \gamma)$ to itself. Consider the blowup $W = W_0 \# \mathbb{CP}^2$, where we take an internal connected sum, and write $E$ for the exceptional divisor. Furthermore, fix a Spin$^c$ structure $s \in \text{Spin}^c(W)$, and let $t = s(M \times \{0\})$. Then $s(M \times \{1\})$ is also $t$, and the map

$$F_w : SFH(M, \gamma, t) \to SFH(M, \gamma, t)$$

is the identity if $\langle c_1(s), E \rangle = \pm 1$, and is zero otherwise.

Proof. This follows from the same local computation as [19, Theorem 3.7]. □

8.3. Sutured Floer homology as a TQFT. An axiomatic description of topological quantum field theory, in short TQFT, was first given by Atiyah [1]. As it is explained in [2], an $(n+1)$-dimensional TQFT over a field $\mathbb{F}$ is a symmetric monoidal functor from the cobordism category of $n$-manifolds to the category of finite dimensional vector spaces over $\mathbb{F}$.

Theorem 8.12. The functor $SFH : BSut \to \text{Ab}$ is a $(3+1)$-dimensional TQFT in the sense of [1] and [2], except for the $\pm 1$ ambiguity when we are working with $\mathbb{Z}$ coefficients.

Proof. The axioms of [1] and [2] are equivalent. Since [2] takes the cobordism point of view, and we are working with cobordisms, we are going to check the axioms in [2].

In our case, we have a finitely generated $\mathbb{Z}$-module $SFH(M, \gamma)$ assigned to every balanced sutured manifold $(M, \gamma)$, and a homomorphism

$$F_W : SFH(M_0, \gamma_0) \to SFH(M_1, \gamma_1)$$

assigned to every balanced cobordism $W$ from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$.

Axiom (1) is naturality. It was shown in Corollary 5.20 that every orientation preserving diffeomorphism $\phi : (M_0, \gamma_0) \to (M_1, \gamma_1)$ induces an isomorphism $\phi_* : SFH(M_0, \gamma_0) \to SFH(M_1, \gamma_1)$, well-defined up to sign. Furthermore, this assignment is functorial; i.e., $(\psi \circ \phi)_* = \psi_* \circ \phi_*$. The naturality of the cobordism maps $F_W$ was stated in Theorem 8.1.

Axiom (2) is functoriality; i.e.,

$$F_{W_2 \circ W_1} = F_{W_2} \circ F_{W_1}.$$  

This was shown in Theorem 8.3.

Axiom (3), normalization, states that if $W$ is the trivial cobordism from $(M, \gamma)$ to $(M, \gamma)$, then

$$F_W : SFH(M, \gamma) \to SFH(M, \gamma)$$

is the identity. This was shown in Proposition 8.3.

Axiom (4) is multiplicativity. If we are working over a field, it states that there are functorial isomorphisms

$$SFH((M_1, \gamma_1) \sqcup (M_2, \gamma_2)) \cong SFH(M_1, \gamma_1) \otimes SFH(M_2, \gamma_2),$$
which easily follows from the definitions. If we are working over \( \mathbb{Z} \), a Künneth-type formula is satisfied, although the base ring \( \mathbb{Z} \) can essentially be replaced by \( \mathbb{Q} \) and \( \mathbb{Z}_p \). Furthermore, we set \( \text{SFH}(\emptyset) = \mathbb{Z} \), where here \( \emptyset \) is the empty balanced sutured manifold. It is straightforward to check that these isomorphisms fit into the commutative diagrams in \([2]\) that describe associativity and the unit. Finally, using the above identifications, we also have \( F_{W_1 \sqcup W_2} = F_{W_1} \otimes F_{W_2} \).

Axiom (5), symmetry, states that the isomorphism
\[
\text{SFH}((M_1, \gamma_1) \sqcup (M_2, \gamma_2)) \cong \text{SFH}((M_2, \gamma_2) \sqcup (M_1, \gamma_1))
\]
induced by the obvious diffeomorphism corresponds to the standard isomorphism of vector spaces
\[
\text{SFH}(M_1, \gamma_1) \otimes \text{SFH}(M_2, \gamma_2) \cong \text{SFH}(M_2, \gamma_2) \otimes \text{SFH}(M_1, \gamma_1).
\]
This is also straightforward. \( \square \)

**Remark 8.13.** Similar axioms hold if we are dealing with balanced sutured manifolds endowed with \( \text{Spin}^c \) structures, and morphisms being balanced cobordisms endowed with relative \( \text{Spin}^c \) structures. These do not form a proper category, as the composition of \((W_1, s_1)\) and \((W_2, s_2)\) does not have a well defined relative \( \text{Spin}^c \) structure on it. But if we replace functoriality with the formula in Theorem \([5,\text{Theorem 2.1.1}]\), we also get a type of TQFT.

Recall that \([1, \text{Axiom 2}]\) states that \( \text{SFH}(M, \gamma) \) and \( \text{SFH}(-M, -\gamma) \) are dual vector spaces if we are working over a field, and are related like integral homology and cohomology when working over \( \mathbb{Z} \). This was proved in \([4, \text{Proposition 2.12}]\). Furthermore, there has to be a functorial isomorphism between \( \text{SFH}(-M, -\gamma) \) and \( \text{SFH}(M, \gamma)^* \). This follows from the set of axioms in \([2]\). Indeed, if we view the trivial cobordism \( W \) from \((M, \gamma)\) to \((M, \gamma)\) as a cobordism from \((M, \gamma)\sqcup(-M, -\gamma)\) to \( \emptyset \), then we get a pairing
\[
\langle \cdot, \cdot \rangle' : \text{SFH}(M, \gamma) \otimes \text{SFH}(-M, -\gamma) \to \mathbb{Z},
\]
which is non-degenerate by \([22, \text{Theorem 2.1.1}]\), see also the right hand side of Figure \([11]\).

**Conjecture 8.14.** The pairing \( \langle \cdot, \cdot \rangle' \) agrees with the pairing \( \langle \cdot, \cdot \rangle \) appearing in Theorem \([8,\text{Theorem 8.1.1}]\).

**Corollary 8.15.** The above conjecture would give an affirmative answer to Question \([8,\text{Question 8.7.10}]\).

**Proof.** Indeed, note that Theorem \([8,\text{Theorem 8.1.1}]\) is true for arbitrary balanced cobordisms if we replace \( \langle \cdot, \cdot \rangle \) with \( \langle \cdot, \cdot \rangle' \). To see this, note that a cobordism \( W \) from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\) can be decomposed as follows. Let \( \overline{W} \) be \( W \) viewed as a cobordism from \((-M_1, -\gamma_1)\) to \((-M_0, -\gamma_0)\), and for \( i = 0, 1 \) let \( \mathcal{I}_i \) be the trivial cobordism from \((M_i, \gamma_i)\) to itself. Furthermore, let \( \mathcal{I}_i' \) be \( \mathcal{I}_i \) viewed as a cobordisms from \( \emptyset \) to \((-M_i, -\gamma_i)\sqcup(M_i, \gamma_i)\) and \( \mathcal{I}_i'' \) is \( \mathcal{I}_i \) viewed as a cobordism from \((M_i, \gamma_i)\sqcup(-M_i, -\gamma_i)\) to \( \emptyset \). Then
\[
W = (\mathcal{I}_0'' \sqcup \mathcal{I}_1) \circ (\mathcal{I}_0 \sqcup \overline{W} \sqcup \mathcal{I}_1) \circ (\mathcal{I}_0 \sqcup \mathcal{I}_1'),
\]
see the left hand side of Figure \([11]\). Note that we can compute \( F_{\mathcal{I}_1} \) using the decomposition
\[
\mathcal{I}_1 = (\mathcal{I}_1 \sqcup \mathcal{I}_1') \circ (\mathcal{I}_1'' \sqcup \mathcal{I}_1),
\]
see the right hand side of Figure \([11]\). \( \square \)
Using the same trick, one can rearrange the ingoing and outgoing ends of any balanced cobordism \( W \), and relate the induced map to \( F_W \) using \( \langle \cdot, \cdot \rangle' \).

If we were to follow the axioms in [1], instead of maps induced by balanced cobordisms, we would need to talk about elements \( EH(W') \in SFH(M, \gamma) \) associated to balanced cobordisms \( W' \) from \( \emptyset \) to \( (M, \gamma) \). It was explained in [1] how to translate between the two approaches. If we are given cobordism maps, the for a balanced cobordisms \( W' \) from \( \emptyset \) to \( (M, \gamma) \), we can simply take \( EH(W') = F_W(1) \), where \( 1 \in SFH(\emptyset) \cong \mathbb{Z} \).

In the other direction, if \( W \) is a balanced cobordism from \( (M_0, \gamma_0) \) to \( (M_1, \gamma_1) \), then we can also view \( W \) as a cobordism \( W' \) from \( \emptyset \) to \( (-M_0, -\gamma_0) \cup (M_1, \gamma_1) \). Hence

\[
EH(W') \in SFH((-M_0, -\gamma_0) \cup (M_1, \gamma_1)) \cong SFH(M_0, \gamma_0) \ast SFH(M_1, \gamma_1).
\]

So we do get a homomorphism \( F_W \) from \( SFH(M_0, \gamma_0) \) to \( SFH(M_1, \gamma_1) \) induced by the balanced cobordism \( W \). It is important to note that if we decide to follow [1], we need to use the pairing \( \langle \cdot, \cdot \rangle' \) and not \( \langle \cdot, \cdot \rangle \) to identify \( SFH(-M, -\gamma) \) with \( SFH(M, \gamma)^* \).

**Remark 8.16.** Another consequence of Conjecture [8.14] would be a new definition of the maps \( F_W \), only using the \( EH \) class in \( SFH \) and special cobordism maps. Indeed, let \( W = (W, Z, \xi) \) be a balanced cobordism from \( (M_0, \gamma_0) \) to \( (M_1, \gamma_1) \). Let \( \gamma = -\gamma_0 \cup \gamma_1 \). If we view \( W \) as a cobordism \( W' \) from \( \emptyset \) to \( (-M_0, -\gamma_0) \cup (M_1, \gamma_1) \), then, by definition,

\[
EH(W') = F_W(1) = F_{W'}(EH(Z, \gamma, \xi)),
\]

where \( W' \) is the obvious special cobordism from \( (Z, \gamma) \) to \( (-M_0, -\gamma_0) \cup (M_1, \gamma_1) \). But we saw above that using \( \langle \cdot, \cdot \rangle' \), we can view \( EH(W') \) as a homomorphism from \( SFH(M_0, \gamma_0) \) to \( SFH(M_1, \gamma_1) \), which agrees with \( F_W \). A priori, to compute \( \langle \cdot, \cdot \rangle' \), one already needs to use some gluing map, but Conjecture [8.14] would eliminate this problem. In particular, if \( W \) is a boundary cobordism, then we would get a new definition for the gluing map \( F_W = \Phi_\xi \).
However, if one starts out with this definition of $F_{W}$, then Axioms 2 and 3 seem to be very difficult to prove without using the theory of gluing maps [9].

Remark 8.17. A cobordism $W$ from the empty sutured manifold to itself is a smooth oriented 4-manifold $W$ with contact boundary $(Z, \xi)$. Here $\xi$ is a positive contact structure when $Z = \partial W$ is given the boundary orientation. Then, for every relative Spin$^c$ structure $s \in \text{Spin}^c(W)$, we have a map $F_{W,s} : Z \to Z$. This can be computed using just the contact class and the cobordism map in Heegaard-Floer homology. More precisely, let $W_0$ be the cobordism from $Y$ to $S^3$ obtained by removing an open ball from the interior of $W$, and set $s_0 = s|W$. Then

$$F_{W,s}(1) = \pm F_{W_0,s_0}(c(Y, \xi)) \in \widehat{HF}(S^3) \cong \mathbb{Z},$$

where $c(Y, \xi) \in \widehat{HF}(Y, s(\xi))$ is the contact invariant defined in [18], and agrees with $EH(Y, \xi)$ by [10].

Recall that in section [3] we defined a functor $W : \text{DLink}_0 \to \text{BSut}$, and for $(Y, L, P) \in \text{DLink}_0$ we have

$$SFH(W(Y, L, P)) \cong \widehat{HF}(Y, L) \otimes V^\otimes d.$$  

Then Theorem 8.12 and Proposition 4.11 give the following.

Corollary 8.18. The functor

$$SFH \circ W : \text{DLink}_0 \to \text{Ab}/\pm 1$$

is also a TQFT in the sense of [1] and [2], making link Floer homology functorial.

8.4. Weinstein cobordisms. We conclude with defining Weinstein cobordisms, and showing that they preserve the $EH$ class. The following definition extends the notions of [3] and [21] to sutured manifolds.

Definition 8.19. Suppose that $(M_0, \gamma_0, \zeta_0)$ and $(M_1, \gamma_1, \zeta_1)$ are contact manifolds. A Liouville cobordism from $(M_0, \gamma_0, \zeta_0)$ to $(M_1, \gamma_1, \zeta_1)$ is a pair $(W, \theta)$, where $W = (W, Z, \xi)$ is a balanced cobordism from $(M_0, \gamma_0)$ to $(M_1, \gamma_1)$, and $\theta$ is a one-form on $W$ with the following properties:

1. $\omega = d\theta$ is symplectic,
2. the Liouville vector field $X$ defined by $\iota_X \omega = \theta$ is transverse to every face of $\partial W$, enters $W$ through $M_0$, and exits $W$ through $Z \cup M_1$,
3. $\xi = \ker(\theta|Z)$, $\zeta_0 = \ker(\theta|M_0)$, and $\zeta_1 = \ker(\theta|M_1)$.

The Liouville cobordisms $(W, \theta)$ and $(W', \theta')$ are weakly (strongly) equivalent if there is a weak (strong) equivalence $d$ between $W$ and $W'$ such that $d^*(\theta') = \theta$.

Example 8.20. An important instance of a Liouville cobordism is the symplectization of a contact manifold $(M, \gamma, \zeta)$. Let $W = (W, Z, \xi)$ be the trivial cobordism from $(M, \gamma)$ to itself. Since $\partial M$ is a convex surface, there is a contact vector field $v$ on $M$ which is transverse to $\partial M$, points out of $M$, and $v \in \zeta$ exactly along $\gamma$. Let $\alpha$ be a contact one-form such that $\zeta = \ker(\alpha)$. Then $L_\alpha \alpha = \mu \alpha$ for some function $\mu : M \to \mathbb{R}$. By the Cartan formula, this is equivalent to $\iota_v d\alpha + d\iota_v \alpha = \mu \alpha$. The dividing set $\gamma$ coincides with the zero set of $\alpha(v)$ along $\partial M$. If we multiply $v$ by a sufficiently small positive scalar, then we can assume that $\mu < 1/2$. Take

$$\theta = e^t \alpha - d(e^t \alpha(v)).$$
Then
\[ \omega = d\theta = e^t dt \wedge \alpha + e^t d\alpha \]
is symplectic since \( \omega \wedge \omega = 2e^{2t} dt \wedge \alpha \wedge d\alpha \) is nowhere zero on \( W \).

I claim that
\[ X = (1 - \mu) \partial_t + v \]
is the Liouville vector field for \( \theta \). Indeed,
\[ i_X \omega = (1 - \mu) i_{\partial_t} \omega + i_v \omega. \]
Since \( i_{\partial_t} \omega = e^t \alpha \) and
\[ i_v \omega = -e^t \alpha(v) dt + e^t i_v d\alpha = -e^t \alpha(v) dt + e^t (\mu \alpha - d\alpha), \]
we can conclude that \( i_X \omega = \theta \). As \( 1 - \mu > 0 \), the vector field \( X \) points into \( W \) along \( M_0 \) and points out of \( W \) along \( M_1 \). Furthermore, since \( v \) points out of \( M \) along \( \partial M \), we see that \( X \) points out of \( W \) along \( \partial M \times I \).

For \( i = 0, 1 \), the contact structure
\[ \ker(\theta_i|M_i) = \ker(\alpha - d(\alpha(v))) \]
is isotopic to \( \zeta \) through contact structures. To see this, note that for every \( 0 \leq \varepsilon \leq 1 \), the same argument as above shows that
\[ \theta^\varepsilon = e^\varepsilon \alpha - d(e^\varepsilon \alpha(\varepsilon v)) \]
induces a contact structure \( \xi^\varepsilon = \ker(\theta^\varepsilon|M_i) \) on \( M_i \). For \( \varepsilon = 0 \), we get that \( \xi^0 = \ker(\alpha) = \zeta \), while \( \xi^1 = \ker(\theta(M_1)) \).

Finally, we show that \( \ker(\theta|\partial \theta) = \xi \). Since \( \mathcal{L}_{\partial_t} d = d\mathcal{L}_{\partial_t} \), we have \( \mathcal{L}_{\partial_t} \theta = \theta \). Hence \( \partial_t|\partial \theta \) is a contact vector field on \( (\partial \theta|\partial \theta) \). So for every \( 0 \leq t \leq 1 \) the surface \( \partial M \times \{t\} \) is convex. The dividing set on this surface is given by the equation \( \theta(\partial_t) = 0 \). However,
\[ \theta(\partial_t) = -d(e^t \alpha(v))(\partial_t) = -e^t \alpha(v), \]
and the function \( \alpha(v)|\partial M \) vanishes exactly along \( \gamma \).

**Definition 8.21.** Let \((W, \theta)\) from \((M_0, \gamma_0, \zeta_0)\) to \((M_1, \gamma_1, \zeta_1)\) be a Liouville cobordism, and let \( X \) be the corresponding Liouville vector field. A function \( H: W \to \mathbb{R} \) is a **Lyapunov function for** \( X \) if \( H \) is a smooth Morse function, and there exists \( \delta > 0 \) and a Riemannian metric on \( W \) such that \( dH(X) \geq \delta |X|^2 \).

**Example 8.22.** If \((W, \theta)\) is a symplectization, then \( H(x, t) = t \) is a Lyapunov function for \( X = (1 - \mu) \partial_t + v \).

**Definition 8.23.** We say that the Liouville cobordism \((W, \theta)\) from \((M_0, \gamma_0, \zeta_0)\) to \((M_1, \gamma_1, \zeta_1)\) is **Weinstein**, if there exists a Lyapunov function \( H \) for the Liouville vector field \( X \) such that

1. a collar neighborhood \( M_0 \times I \) of \( M_0 = M_0 \times \{0\} \) is a symplectization of \( (M_0, \gamma_0, \zeta_0) \) as in Example 8.20
2. \( H(x, t) = t \) for \((x, t) \in M_0 \times I \),
3. there is a collar neighborhood \( \partial M_0 \times [0, 2] \) of \( \partial M_0 \) in \( Z \) that extends \( \partial M_0 \times I \), where \( H(x, t) = t \),
4. \( H \equiv 2 \) on \((Z \cup M_1) \setminus (\partial M_0 \times [0, 2]) \),
5. \( H \) has no critical points on \( \partial W \).
Remark 8.24. Since $X$ points out of $W$ along $Z \cup M_1$, the negative gradient flow lines of $X$ can only exit $W$ along $M_0$. As explained in [3], all critical points of the Lyapunov function $H$ have Morse index at most two, and the stable manifolds intersected with regular levels $H^{-1}(c)$ are isotropic for the induced contact structure $\ker(\theta|H^{-1}(c))$. Using [23], we see that one can build $W$, viewed as a special cobordism from $M_0$ to $Z \cup M_1$, from the symplectization of $(M_0, \gamma_0, \zeta_0)$ by attaching Weinstein one- and two-handles. A Weinstein one-handle attachment changes the boundary by taking a connected sum with a standard contact $S^1 \times S^2$. Each Weinstein two-handle is attached along some Legendrian knot $K$ with framing $tb(K) - 1$.

Theorem 8.25. Let $(W, \theta)$ be a Weinstein cobordism from the contact manifold $(M_0, \gamma_0, \zeta_0)$ to $(M_1, \gamma_1, \zeta_1)$. Let $s \in \text{Spin}^c(W)$ be the Spin$^c$ structure associated with $\omega = d\theta$. As explained in Remark 8.24, we can view $W$ as a balanced cobordism $\overline{W}$ from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0)$. Then

$$F_{\overline{W}, s}(EH(M_1, \gamma_1, \zeta_1)) = EH(M_0, \gamma_0, \zeta_0).$$

Remark 8.26. Recall that for $i = 0, 1$ we have

$$EH(M_i, \gamma_i, \zeta_i) \in SFH(-M_i, -\gamma_i, s(\zeta_i)).$$

Furthermore, $s|M_i = s(\zeta_i)$, and

$$F_{\overline{W}_i, s}: SFH(-M_i, -\gamma_i, s(\zeta_i)) \rightarrow SFH(-M_0, -\gamma_0, s(\zeta_0)).$$

Proof. Consider the cobordism $\overline{W}_1 = (W, Z, \xi)$ from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0)$. First, suppose that $Z$ has no isolated components and set $N = M_1 \cup Z$. Then $F_{\overline{W}} = F_{\overline{W}_1} \circ \Phi_{\xi}$, where $\overline{W}_1$ is a special cobordism from $(-N, -\gamma_0)$ to $(-M_0, -\gamma_0)$.

By Theorem 6.6, we have

$$\Phi_{\xi}(EH(M_1, \gamma_1, \zeta_1)) = EH(N, \gamma_0, \zeta_0 \cup \xi).$$

We can view $\overline{W}_1$ as a special cobordism $W_1$ from $(M_0, \gamma_0)$ to $(N, \gamma_0)$ as explained in Remark 8.24, we can build $W_1$ from the symplectization of $(M_0, \gamma_0, \zeta_0)$ by first attaching Weinstein one-handles, then attaching Weinstein two-handles along a Legendrian link $L$ with framing $tb(L) - 1$. Hence we are done if we prove the result when $W$ is a Weinstein one- or two-handle cobordism.

Suppose that $W$ is a Weinstein one-handle cobordism, which connects $(M_0, \gamma_0, \zeta_0)$ with $(M_0 \# (S^1 \times S^2), \gamma_0, \zeta_1)$. The contact structure $\zeta_1$ agrees with $\zeta_0$ minus a standard contact ball on $M_0 \setminus B^3$, and is the unique tight contact structure minus a standard contact ball $\xi_{std}$ on $(S^1 \times S^2) \setminus B^3$. Fix a partial open book decomposition $(S_0, h_0: P_0 \rightarrow S_0)$ for $(M_0, \gamma_0, \zeta_0)$, and let $(\Sigma_0, \alpha_0, \beta_0)$ be the associated balanced diagram. Furthermore, let $(S, h: P \rightarrow S)$ be the partial open book decomposition of $((S^1 \times S^2)(1), \xi_{std})$ described in [16] Example 1. It agrees with the partial open book of Example 6.3 except that the map $h$ is the identity of $P$. We are going to denote by $(\Sigma, \alpha, \beta)$ the corresponding balanced diagram of $(S^1 \times S^2)(1)$. Then $\Sigma = T^2 \setminus B^2$, the curves $\alpha$ and $\beta$ intersect in exactly two points, and $\alpha$ is a small Hamiltonian translate of $\beta$. Let $y \in \alpha \cap \beta$ be the intersection point with the smaller relative grading. If we take the boundary connected sum of $S_0$ and $S$ along $\partial S_0 \setminus P_0$ and $\partial S \setminus P$, then we get a partial open book decomposition for $(M_0 \# (S^1 \times S^2), \gamma_0, \zeta_1)$, which induces the balanced diagram $(\Sigma_0 \# \Sigma, \alpha_0 \cup \{\alpha\}, \beta_0 \cup \{\beta\})$. Let $x \in \mathbb{T}_{\alpha_0} \cap \mathbb{T}_{\beta_0}$ be the distinguished generator representing $EH(M_0, \gamma_0, \zeta_0)$, and note that $y$ represents $EH((S^1 \times S^2)(1), \xi_{std})$. Then $x \times \{y\}$ represents $EH(M_0 \# (S^1 \times S^2), \gamma_0, \zeta_1)$. The curves $\alpha$ and $\beta$ bound a periodic domain which represents a sphere $\{p\} \times S^2$.
inside $S^1 \times S^2$. The cobordism $\overline{W}$ corresponds to a three-handle attached along \{p\} $\times S^2$, so by Definition 5.34 the map $F_{\overline{W}}$ takes $x \times \{y\}$ to $x$, which proves the claim for Weinstein one-handles.

Now assume that $W$ is a Weinstein two-handle cobordism corresponding to a Legendrian knot $K$ in $(M_0, \gamma_0, \zeta_0)$. Then the proofs of \cite{12} Proposition 4.4 and \cite{19} Theorem 3.5 imply that the EH class is preserved by $F_{\overline{W}}$. More concretely, in the proof of \cite{12} Proposition 4.4 they construct a partial open book decomposition $(S, h: P \to S)$ for $(M_0, \gamma_0, \zeta_0)$ which contains the Legendrian knot $K$ inside $P$. This gives rise to a triple diagram $(\Sigma, \alpha, \beta, \delta)$ which is subordinate to some bouquet for $K$. As described in the proof of \cite{19} Theorem 3.5, if $(\Sigma, \alpha, \beta, \delta)$ corresponds to the two-handle cobordism $W$, then $(-\Sigma, \alpha, \delta, \beta)$ corresponds to $\overline{W}$. We end up with the configuration depicted on Figure 10 where there is a distinguished triangle mapping the generator representing $EH(M_1, \gamma_1, \zeta_1)$ to the one representing $EH(M_0, \gamma_0, \zeta_0)$.

If $Z$ does have isolated components, then $F_{\overline{W}} = F_{\overline{W}}$, where $\overline{W}$ is the cobordism from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0) \sqcup (B, \delta)$ given by Definition 7.1. Note that $B \subset Z$ and $\ker(\theta | B) = \xi | B$ is a union of standard contact balls. Let $W'$ be $\overline{W}$ viewed as a cobordism from $(M_0, \gamma_0) \sqcup (-B, -\delta)$ to $(M_1, \gamma_1)$. Then $(W', \theta)$ is not a Liouville cobordism, since the Liouville vector field $X$ points out of $W$ along $B$. To fix this, attach a Weinstein one-handle to each component of $B$ along one of its free feet. Then $\theta$ plus the Liouville one-forms on the one-handles give a one-form $\theta'$ such that the new Liouville vector field $X'$ points in along the free feet of the one-handles. The contact structure $\xi$ on $Z$ is left unchanged, since $I \times S^2$ has a unique tight contact structure. The cobordism $(W', \theta')$ is also Weinstein, since a Lyapunov function $H$ on $W$ extends to the one-handles with a unique index one critical point in each. Let $\xi_0 = \ker(\theta' | (-B))$, then $(-B, \xi_0)$ is also a union of standard contact balls. If we apply the previous part to the Weinstein cobordism $(W', \theta')$, then we get that

$$F_{\overline{W}, s}(EH(M_1, \gamma_1, \zeta_1)) = EH(M_0, \gamma_0, \xi_0) \otimes EH(-B, -\delta, \xi_0),$$

where $s' = s|W'$. As $EH(-B, -\delta, \xi_0) = \pm 1 \in SFH(B, \delta)$, the right hand side maps to $EH(M_0, \gamma_0, \xi_0)$ under the isomorphism

$$SFH(-M_0, -\gamma_0) \otimes SFH(B, \delta) \to SFH(-M_0, -\gamma_0).$$

\[ \square \]

**Corollary 8.27.** Let $(W, \theta)$ from $(M_0, \gamma_0, \xi_0)$ to $(M_1, \gamma_1, \zeta_1)$ be a Weinstein cobordism. If $EH(M_0, \gamma_0, \zeta_0) \neq 0$, then $EH(M_1, \gamma_1, \zeta_1) \neq 0$.

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