Truncated Euler-Maruyama method for a class of non-autonomous stochastic differential equations

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Abstract

This paper studies the numerical approximation to a class of non-autonomous stochastic differential equations with the Hölder continuity in the temporal variable and the super-linear growth in the state variable. The truncated Euler-Maruyama method is proved to be convergent to this type of stochastic differential equations. The convergence rate is given, which is related the Hölder continuity.

Keywords: Truncated Euler-Maruyama method, non-autonomous stochastic differential equations, strong convergence, super-linear coefficients

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1. Introduction

Stochastic differential equations (SDEs) have broad applications in many areas such as finance, physics, chemistry and biology \cite{1,2}. However, most SDEs do not have the explicit expressions of the true solutions. The numerical methods and the rigorous numerical analysis of those methods become extremely important \cite{3,4}.

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In this paper, we investigate the numerical approximation to a class of non-autonomous stochastic differential equations of the Itô type

\[
\begin{align*}
\frac{dx(t)}{dt} &= \mu(t, x(t)) dt + \sum_{r=1}^{m} \sigma^r(t, x(t)) \, dW^r(t), \quad t \in [t_0, T], \\
x(t_0) &= x_0,
\end{align*}
\]

where the coefficients obey the Hölder continuity in the temporal variable, and super-linear growth condition in the state variable. The detailed mathematical descriptions are in Section 2.

For non-autonomous SDEs with the non-differential temporal variable in the coefficients, the randomized techniques are used to construct the Euler type method [5] and the Milstein type method [6]. However, most papers that investigated non-autonomous SDE only consider the global Lipschitz condition for the state variable.

The classic Euler-Maruyama (EM) method has been proved divergent for SDEs with super-linearly growing state variable [7]. While bearing in mind the idea that explicit methods have their advantages in simple algorithm structure and relatively lower computational cost in the simulations of a large number of sample paths [8], the tamed Euler method [9] and the truncated Euler-Maruyama method [10] are developed to approximate SDEs with super-linearly growing state variable. Some other interesting works on explicit methods for SDEs with the super-linear state variable are [11, 12, 13, 14, 15, 16, 17, 18, 19, 20], we just mention some of them and refer the readers to the references therein. However, it seems that those explicit methods proposed to tackle the super-linearity in the state variable do not take the non-autonomous coefficients into consideration.

To our best knowledge, there is few works on SDEs with both the Hölder continuity in the temporal variable and the super-linearity in the state variable. To fill up this gap, in this paper we will investigate the strong convergence of the newly developed truncated Euler-Maruyama method for non-autonomous SDEs with the non-differential temporal variable and the super-linear state variable.

The paper is constructed as follows. Section 2 briefly introduce the truncated
Euler-Maruyama method and useful lemmas. The main result of the strong convergence with the rate is presented and proved in Section 3. A numerical example is given in Section 4 to demonstrate and validate the theoretical result.

2. Mathematical preliminaries

This section is divided into three parts. In Section 2.1 the notations and assumptions are introduced. To keep the paper self-contained, the truncated Euler Maruyama method is briefed in Section 2.2. Some useful lemmas are presented in Section 2.3.

2.1. Notations and assumptions

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ denote the probability expectation with respect to $\mathbb{P}$. If $x \in \mathbb{R}^d$, then $|x|$ is the Euclidean norm. Let $x^T$ denotes the transposition of $x$. Moreover, for two real numbers $a$ and $b$, we use $a \lor b = \max(a, b)$ and $a \land b = \min(a, b)$.

For $d, m \in \mathbb{N}$, let $W : [t_0, T] \times \Omega \to \mathbb{R}^m$ be a standard $\{\mathcal{F}_t\}_{t \in [t_0, T]}$-Wiener process. Moreover, let $x : [t_0, T] \times \Omega \to \mathbb{R}^d$ be an $\{\mathcal{F}_t\}_{t \in [t_0, T]}$-adapted stochastic process that is a solution to Itô-type stochastic differential equation

$$
\begin{align*}
\text{dx}(t) &= \mu(t, x(t)) \, dt + \sum_{r=1}^{m} \sigma^r(t, x(t)) \, dW^r(t), \quad t \in [t_0, T], \\
x(t_0) &= x_0,
\end{align*}
$$

where the drift coefficient function $\mu : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and the diffusion coefficient function $\sigma^r : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ for $r \in \{1, 2, \cdots, m\}$.

We impose the following assumptions on the drift and diffusion coefficients.

**Assumption 2.1.** Assume that the coefficient $\mu$ and $\sigma^r$ satisfy the local Lipschitz condition that for any real number $R > 0$, there exists a $K_R > 0$ such that

$$
|\mu(t, x) - \mu(t, y)| \lor |\sigma^r(t, x) - \sigma^r(t, y)| \leq K_R|x - y|,
$$

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for all \( t \in [t_0, T] \), \( r \in \{1, 2, \cdots, m\} \), and \(|x| \lor |y| \leq R\).

**Assumption 2.2.** Assume that there exist positive constants \( \beta \) and \( M \) such that

\[
|\mu(t, x) - \mu(t, y)| \lor |\sigma^r(t, x) - \sigma^r(t, y)| \leq M(1 + |x|^\beta + |y|^\beta)|x - y|,
\]

for all \( x, y \in \mathbb{R}^d \) and \( r \in \{1, 2, \cdots, m\} \).

It can be observed from Assumption 2.2 that all \( t \in [t_0, T] \), \( r \in \{1, 2, \cdots, m\} \) and \( x \in \mathbb{R}^d \)

\[
|\mu(t, x)| \lor |\sigma^r(t, x)| \leq K|x|^\beta + 1, \tag{2}
\]

where

\[
K = 2M + \sup_{t_0 \leq t \leq T} (|\mu(t, 0)| + \max_{1 \leq r \leq m} |\sigma^r(t, 0)|).
\]

**Assumption 2.3.** Assume that there exists a pair of constants \( q > 2 \) and \( L_1 > 0 \) such that

\[
(x - y)^T(\mu(t, x) - \mu(t, y)) + \frac{q - 1}{2} \sum_{r=1}^{m} |\sigma^r(t, x) - \sigma^r(t, y)|^2 \leq L_1|x - y|^2,
\]

for any \( t \in [t_0, T] \) and \( x, y \in \mathbb{R}^d \).

**Assumption 2.4.** Assume that there exists a pair of constants \( p > 2 \) and \( L_2 > 0 \) such that

\[
x^T\mu(t, x) + \frac{p - 1}{2} \sum_{r=1}^{m} |\sigma^r(t, x)|^2 \leq L_2(1 + |x|^2), \tag{3}
\]

for any \( t \in [t_0, T] \) and \( x \in \mathbb{R}^d \).

**Remark 2.5.** It is clear that Assumption 2.4 may be derived from Assumption 2.3 but with more complicated coefficient in front of \(|\sigma^r(t, x)|^2\). To keep the notation simple, we state Assumption 2.4 as a new assumption, which would not shrink the range of SDEs covered by this paper at all.
Assumption 2.6. Assume that there exist constants $\gamma \in (0, 1]$, $\alpha \in (0, 1]$, $K_1 > 0$ and $K_2 > 0$ such that

$$|\mu(t_1, x) - \mu(t_2, x)| \leq K_1 (1 + |x|^{\beta+1}) |t_1 - t_2|^\gamma,$$

$$|\sigma^r(t_1, x) - \sigma^r(t_2, x)| \leq K_2 (1 + |x|^{\beta+1}) |t_1 - t_2|^\alpha,$$

for any $t \in [t_0, T]$, all $x \in \mathbb{R}^d$ and $r \in \{1, 2, \ldots, m\}$, where the $\beta$ is the same as that in Assumption 2.2.

2.2. The truncated Euler-Maruyama method for non-autonomous SDEs

This part is to recall the truncated EM numerical scheme. To define the truncated EM numerical solutions with time $t$, we first choose a strictly increasing continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(u) \to \infty$ as $u \to \infty$ and

$$\sup_{t_0 \leq t \leq T} \sup_{|x| \leq u} (|\mu(t, x)| \vee |\sigma(t, x)|) \leq f(u), \quad \forall u \geq 1.$$

Denote by $f^{-1}$ the inverse function of $f$. It is clear that $f^{-1}$ is a strictly increasing continuous function from $[f(u), \infty)$ to $\mathbb{R}_+$. We also choose a constant $\hat{h} \geq 1 \wedge f(1)$ and a strictly decreasing function $\kappa : (0, 1] \to [f(1), \infty)$ such that

$$\lim_{\Delta \to 0} \kappa(\Delta) = \infty, \quad \Delta^{\frac{1}{2}} \kappa(\Delta) \leq \hat{h}, \quad \forall \Delta \in (0, 1].$$

For a given step size $\Delta \in (0, 1]$ let us define the truncated mapping $\pi_\Delta : \mathbb{R}^d \to \{x \in \mathbb{R}^d : |x| \leq f^{-1}(\kappa(\Delta))\}$ by

$$\pi_\Delta(x) = \left(|x| \wedge f^{-1}(\kappa(\Delta))\right) \frac{x}{|x|},$$

where we set $x/|x| = 0$ when $x = 0$.

Define the truncated functions by

$$\mu_\Delta(t, x) = \mu(t, \pi_\Delta(x)), \quad \sigma_\Delta(t, x) = \sigma(t, \pi_\Delta(x)), \quad \text{(4)}$$

for $x \in \mathbb{R}^d$. It is easy to see that for any $t \in [t_0, T]$ and all $x \in \mathbb{R}^d$

$$|\mu_\Delta(t, x)| \vee |\sigma_\Delta(t, x)| \leq f(f^{-1}(\kappa(\Delta))) = \kappa(\Delta).$$
The discrete-time truncated EM numerical solutions \( x_\Delta(t_k) \), to approximate \( x(t_k) \) for \( t_k = k\Delta + t_0 \), are formed by setting \( x_\Delta(t_0) = x_0 \) and computing

\[
x_\Delta(t_{k+1}) = x_\Delta(t_k) + \mu_\Delta(t_k, x_\Delta(t_k))\Delta + \sum_{r=1}^{m} \sigma^r_\Delta(t_k, x_\Delta(t_k))\Delta W_r^k,
\]

for \( k = 0, 1, \cdots, N_\Delta \), where \( N_\Delta \) is the integer part of \( T/\Delta \) and we will set \( t_{N_\Delta + 1} = T \) while \( \Delta W_r^k = W_r^r(t_{k+1}) - W_r^r(t_k) \).

To form the continuous versions of truncated EM numerical schemes, we define

\[
\tau(t) = \sum_{k=0}^{N_\Delta} t_k I_{[t_k, t_{k+1})}(t), \quad t \in [t_0, T]
\]

There are two versions of the continuous-time truncated EM solutions. The first one is defined by

\[
x_\Delta(t) = x_0 + \int_{t_0}^{t} \mu_\Delta(\tau(s), x_\Delta(s))ds + \sum_{r=1}^{m} \int_{t_0}^{t} \sigma^r_\Delta(\tau(s), x_\Delta(s))dW^r(s),
\]

which is continuous for all \( t \in [t_0, T] \).

2.3. Some useful lemmas

In this subsection, some lemmas that will be essential for the proof of the main result in Section 3 are presented. The proofs of these lemmas are either straightforward or can be found in references. Therefore, to focus our attention on the proof of the main result, those lemmas are stated without proofs.

Lemma 2.7. Under Assumptions 2.1 and 2.4, the SDE (1) has a unique global solution \( x(t) \). Moreover,

\[
\sup_{t_0 \leq t \leq T} \mathbb{E}|x(t)|^p < \infty.
\]

The proof of the above lemma can be found in, for example, [21].
Lemma 2.8. For any $\Delta \in (0, 1]$ and any $\overline{p} > 0$, we have

$$\mathbb{E}|x_{\Delta}(t) - \tau_{\Delta}(t)|^{\overline{p}} \leq C_{\overline{p}}\Delta^{\overline{p}}(\kappa(\Delta))^{\overline{p}}, \quad \forall t \in [t_0, T],$$

where $C_{\overline{p}}$ is a positive constant dependent only on $\overline{p}$. Consequently

$$\lim_{\Delta \to 0} \mathbb{E}|x_{\Delta}(t) - \tau_{\Delta}(t)|^{\overline{p}} = 0, \quad \forall t \in [t_0, T].$$

Lemma 2.9. Let Assumptions 2.1 and 2.4 hold. Then

$$\sup_{0 < \Delta \leq 1} \sup_{t_0 \leq t \leq T} \mathbb{E}|x_{\Delta}(t)|^{p} \leq C,$$

where $C$ is a positive constant independent of $\Delta$.

From now on, the constants $C, C_1, C_2, C_3, C_31$ and $C_32$ stand for generic positive constants that are independent of $\Delta$ and their values may change between occurrences.

The proofs of Lemmas 2.8 and 2.9 follow straightforwardly from the proofs of Lemmas 3.1 and 3.2 in [10], by substituting $\mu_{\Delta}(t, \tau_{\Delta}(s))$ and $\sigma_{\Delta}(t, \tau_{\Delta}(s))$ for $\mu_{\Delta}(\tau_{\Delta}(s))$ and $\sigma_{\Delta}(\tau_{\Delta}(s))$, respectively.

Remark 2.10. From Lemmas 2.8 and 2.9 it is easily obtained that

$$\sup_{0 < \Delta < 1} \sup_{t_0 \leq t \leq T} \mathbb{E}|\tau_{\Delta}(t)|^{p} \leq C.$$

3. Main results

In this section the strong convergence of the truncated Euler-Maruyama method is proved to be of order $\min\{\alpha, \gamma, \frac{1}{2} - \varepsilon\}$ for non-autonomous SDEs with super-linear state variables, where $\varepsilon$ is an arbitrarily small positive constant. Our following main theorem describes this more precisely.

Theorem 3.1. Let Assumptions 2.1, 2.2, 2.3 and 2.6 hold. In addition, assume that (3) in Assumption 2.4 is true for any $p > 2$. Then for any $\overline{p} \in [2, p)$, $\Delta \in (0, 1]$ and any $\varepsilon \in (0, 1/4),$

$$\sup_{t_0 \leq t \leq T} \mathbb{E}|x(t) - x_{\Delta}(t)|^{\overline{p}} \leq C\Delta^{\overline{p}}(\gamma, \alpha, \frac{1}{2} - \varepsilon)^{\overline{p}},$$

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and
\[
\sup_{t_0 \leq t \leq T} \mathbb{E}|x(t) - \mathbf{F}_\Delta(t)|^\gamma \leq C \Delta^{\min(\gamma,\alpha,\frac{1}{\gamma}-\epsilon)} \gamma.
\]

**Remark 3.2.** To get Theorem 3.1 more precisely, to obtain the results hold for arbitrarily \( \epsilon \in (0, 1/4) \), Assumption 2.4 is strengthened by requiring (3) holding for any \( p > 2 \) instead of existing some \( p > 2 \).

To obtain Theorem 3.1 we first show Theorem 3.3 which gives the strong convergence with the format of the rate a little bit complicated. The proof of Theorem 3.1 is postponed after the proof of the following theorem.

**Theorem 3.3.** Let Assumptions 2.1, 2.2, 2.3, 2.4, and 2.6 hold and assume that \( p > (1 + \beta)q \). Then, for any \( \gamma \in [2, q) \) and \( \Delta \in (0, 1] \)
\[
\mathbb{E}|x(t) - x_\Delta(t)|^\gamma \leq C \left( (f^{-1}(\kappa(\Delta)))^{\left[1 + \beta\gamma - p\right]/p} + \Delta^{\gamma/2} (\kappa(\Delta))^\gamma + \Delta^{\gamma/2} + \Delta^{\gamma} \right),
\]
(5)

and
\[
\mathbb{E}|x(t) - \mathbf{F}_\Delta(t)|^\gamma \leq C \left( (f^{-1}(\kappa(\Delta)))^{\left[1 + \beta\gamma - p\right]/p} + \Delta^{\gamma/2} (\kappa(\Delta))^\gamma + \Delta^{\gamma/2} + \Delta^{\gamma} \right).
\]
(6)

**Proof.** Fix \( \gamma = [2, q) \) and \( \Delta \in (0, 1] \) arbitrarily. Let \( e_\Delta(t) = x(t) - x_\Delta(t) \) for \( t \in [t_0, T] \). By the Itô formula, we have for any \( t_0 \leq t \leq T \),
\[
\mathbb{E}|e_\Delta(t)|^\gamma \leq \mathbb{E} \int_{t_0}^t \gamma e_\Delta(s) |\gamma e_\Delta(s)|^{\gamma-2} \left( e_\Delta(s) |\mu(s, x(s)) - \mu(\tau(s), \mathbf{F}_\Delta(s))| + \frac{\gamma - 1}{2} \sum_{r=1}^m |\sigma_r(s, x(s)) - \sigma_r(\tau(s), \mathbf{F}_\Delta(s))|^2 \right) ds.
\]
(7)

By the Young inequality \( 2ab \leq \varepsilon a^2 + b^2/\varepsilon \) for any \( a, b \geq 0 \) and \( \varepsilon \) arbitrary, choosing \( \varepsilon = (q - \gamma)/(\gamma - 1) \) leads to
\[
\frac{\gamma - 1}{2} \sum_{r=1}^m |\sigma_r(s, x(s)) - \sigma_r(\tau(s), \mathbf{F}_\Delta(s))|^2
\]
\[
\leq \frac{\gamma - 1}{2} \sum_{r=1}^m \left( (1 + \frac{q - \gamma}{q - 1}) |\sigma_r(s, x(s)) - \sigma_r(s, x_\Delta(s))|^2 + (1 + \frac{\gamma - 1}{q - \gamma}) |\sigma_r(s, x_\Delta(s)) - \sigma_r(\tau(s), \mathbf{F}_\Delta(s))|^2 \right)
\]
\[
= \frac{\gamma - 1}{2} \sum_{r=1}^m |\sigma_r(s, x(s)) - \sigma_r(s, x_\Delta(s))|^2 + \frac{\gamma - 1}{2(q - \gamma)} \sum_{r=1}^m |\sigma_r(s, x_\Delta(s)) - \sigma_r(\tau(s), \mathbf{F}_\Delta(s))|^2.
\]
(8)
By Assumption 2.3, we have

\[ \mathbb{E}|e_\Delta(t)|^{\overline{7}} \leq \mathbb{E} \int_{t_0}^{t} \overline{7}|e_\Delta(s)|^{\overline{7}-2} \left( e_\Delta^T(s)[\mu(s, x(s)) - \mu(s, x_\Delta(s)) + \frac{D - 1}{2} \sum_{r=1}^{m} |\sigma^r(s, x(s)) - \sigma^r(s, x_\Delta(s))|^2 \right) ds \]

+ \mathbb{E} \int_{t_0}^{t} \overline{7}|e_\Delta(s)|^{\overline{7}-2} e_\Delta^T(t)[\mu(s, x_\Delta(s)) - \mu(s, x_\Delta(s))] ds

+ \mathbb{E} \int_{t_0}^{t} \overline{7}|e_\Delta(s)|^{\overline{7}-2} e_\Delta^T(t)[\mu(\tau(s), x_\Delta(s)) - \mu(\tau(s), x_\Delta(s))] ds

+ \mathbb{E} \int_{t_0}^{t} \overline{7}|e_\Delta(s)|^{\overline{7}-2} \overline{\overline{7}}(q - 1) \frac{m}{(q - 1)} \sum_{r=1}^{m} |\sigma^r(s, x_\Delta(s)) - \sigma^r(\tau(s), x_\Delta(s))|^2 ds

+ \mathbb{E} \int_{t_0}^{t} \overline{7}|e_\Delta(s)|^{\overline{7}-2} \overline{\overline{7}}(q - 1) \frac{m}{(q - 1)} \sum_{r=1}^{m} |\sigma^r(\tau(s), x_\Delta(s)) - \sigma^r(\tau(s), x_\Delta(s))|^2 ds.

This implies

\[ \mathbb{E}|e_\Delta(t)|^{\overline{7}} \leq I_1 + I_2 + I_3, \]

where

\[ I_1 = \mathbb{E} \int_{t_0}^{t} \overline{7}|e_\Delta(s)|^{\overline{7}-2} \left( e_\Delta^T(s)[\mu(s, x(s)) - \mu(s, x_\Delta(s))] \right. \]

\[ + \frac{D - 1}{2} \sum_{r=1}^{m} |\sigma^r(s, x(s)) - \sigma^r(s, x_\Delta(s))|^2 \right) ds, \]

\[ I_2 = \mathbb{E} \int_{t_0}^{t} \overline{7}|e_\Delta(s)|^{\overline{7}-2} \left( e_\Delta^T(s)[\mu(s, x_\Delta(s)) - \mu(s, x_\Delta(s))] \right. \]

\[ + \overline{7}(q - 1) \sum_{r=1}^{m} |\sigma^r(s, x_\Delta(s)) - \sigma^r(\tau(s), x_\Delta(s))|^2 \right) ds, \]

and

\[ I_3 = \mathbb{E} \int_{t_0}^{t} \overline{7}|e_\Delta(s)|^{\overline{7}-2} \left( e_\Delta^T(s)[\mu(\tau(s), x_\Delta(s)) - \mu(\tau(s), x_\Delta(s))] \right. \]

\[ + \overline{7}(q - 1) \sum_{r=1}^{m} |\sigma^r(\tau(s), x_\Delta(s)) - \sigma^r(\tau(s), x_\Delta(s))|^2 \right) ds. \]

By Assumption 2.3 we have

\[ I_1 \leq C_1 \mathbb{E} \int_{t_0}^{t} |e_\Delta(s)|^{\overline{7}} ds, \]  

(8)
where $C_1 = K_2 \varpi$. Using the Young inequality and Assumption 2.6, we can derive

$$
I_2 \leq E \int_{t_0}^t |e_\Delta(s)|^{q-2} \left( \frac{1}{2} |e_\Delta(s)|^2 + \frac{1}{2} |\mu(s, x_\Delta(s)) - \mu(\tau(s), x_\Delta(s))|^2 \right) ds
+ \frac{(q-1)(q-1)}{(q-\theta)} \sum_{r=1}^m |\sigma^r(s, x_\Delta(s)) - \sigma^r(\tau(s), x_\Delta(s))|^2 ) ds
$$

$$
\leq C_2 \left( E \int_{t_0}^t |e_\Delta(s)|^q ds + E \int_{t_0}^t |\mu(s, x_\Delta(s)) - \mu(\tau(s), x_\Delta(s))|^q ds
+ \frac{2(q-1)(q-1)}{(q-\theta)} \sum_{r=1}^m E \int_{t_0}^t |\sigma^r(s, x_\Delta(s)) - \sigma^r(\tau(s), x_\Delta(s))|^q ds \right)
$$

Then by Lemma 2.7 we obtain

$$
I_2 \leq C_2 \left( E \int_{t_0}^t |e_\Delta(s)|^q ds + \Delta^\gamma \| + \Delta^\alpha \right). \tag{9}
$$

Rearranging $I_3$ gives

$$
I_3 \leq E \int_{t_0}^t \|e_\Delta(s)\|^{q-2} \left( e_\Delta^T(t)[\mu(\tau(s), x_\Delta(s)) - \mu(\tau(s), \varpi(s))]
+ \frac{2(q-1)(q-1)}{(q-\theta)} \sum_{r=1}^m |\sigma^r(\tau(s), x_\Delta(s)) - \sigma^r(\tau(s), \varpi(s))|^2 ) ds
+ E \int_{t_0}^t \|e_\Delta(s)\|^{q-2} \left( e_\Delta^T(t)[\mu(\tau(s), \varpi(s)) - \mu(\tau(s), \varpi(s))] + \frac{2(q-1)(q-1)}{(q-\theta)} \sum_{r=1}^m |\sigma^r(\tau(s), \varpi(s)) - \sigma^r(\tau(s), \varpi(s))|^2 ) ds
\right)
$$

$$
:= I_{31} + I_{32}. \tag{10}
$$
By using the Young inequality and Assumption 2.12 we can show that

\[ I_{31} \leq E \int_{t_0}^t |\epsilon_\Delta(s)|^{\frac{\gamma-2}{\gamma}} \left( \frac{1}{2} |\epsilon_\Delta(t)|^2 + \frac{1}{2} |\mu(\tau(s), x_\Delta(s)) - \mu(\tau(s), \varphi_\Delta(s))|^2 \right) ds + \frac{2(q-1)(q-1)}{q} \sum_{r=1}^{m} |\sigma^r(\tau(s), x_\Delta(s)) - \sigma^r(\tau(s), \varphi_\Delta(s))|^2 \right) ds \\
\leq C_{31} \left( E \int_{t_0}^t |\epsilon_\Delta(s)|^{\gamma} ds + E \int_{t_0}^t |\mu(\tau(s), x_\Delta(s)) - \mu(\tau(s), \varphi_\Delta(s))|^{\gamma} ds \\
+ \sum_{r=1}^{m} |\sigma^r(\tau(s), x_\Delta(s)) - \sigma^r(\tau(s), \varphi_\Delta(s))|^{\gamma} ds \right) \\
\leq C_{31} \left( E \int_{t_0}^t |\epsilon_\Delta(s)|^{\gamma} ds + M E \int_{t_0}^t (1 + |x_\Delta(s)|^{\gamma} + |\varphi_\Delta(s)|^{\gamma}) |x_\Delta(s) - \varphi_\Delta(s)|^{\gamma} ds \right) \\
\]

Then, by the Hölder inequality, Lemma 2.7 and Lemma 2.8 we arrive at

\[ I_{31} \leq C_{31} \left( E \int_{t_0}^t |\epsilon_\Delta(s)|^{\gamma} ds + E \int_{t_0}^t (E|x_\Delta(s) - \varphi_\Delta(s)|^{\gamma})^{\frac{1}{\gamma}} ds \right) \\
\leq C_{31} \left( E \int_{t_0}^t |\epsilon_\Delta(s)|^{\gamma} ds + \Delta^{\frac{1}{\gamma}}(\kappa(\Delta))^{\gamma} ds \right) . \tag{11} \]

Similarly, we can show that

\[ I_{32} \leq C_{32} \left( E \int_{t_0}^t |\epsilon_\Delta(s)|^{\gamma} ds + E \int_{t_0}^t |\mu(\tau(s), \varphi_\Delta(s)) - \mu_{\Delta}(\tau(s), \varphi_\Delta(s))|^{\gamma} ds \\
+ \sum_{r=1}^{m} |\sigma^r(\tau(s), \varphi_\Delta(s)) - \sigma_{\Delta}^r(\tau(s), \varphi_\Delta(s))|^{\gamma} ds \right) . \]

Recalling the definition of truncated EM method [11] and Assumption 2.12 gives

\[ I_{32} \leq C_{32} \left( E \int_{t_0}^t |\epsilon_\Delta(s)|^{\gamma} ds + E \int_{t_0}^t |\mu(\tau(s), \varphi_\Delta(s)) - \mu(\tau(s), \pi_\Delta(\varphi_\Delta(s)))|^{\gamma} ds \\
+ \sum_{r=1}^{m} |\sigma^r(\tau(s), \varphi_\Delta(s)) - \sigma_{\Delta}^r(\tau(s), \pi_\Delta(\varphi_\Delta(s)))|^{\gamma} ds \right) \\
\leq C_{32} \left( E \int_{t_0}^t |\epsilon_\Delta(s)|^{\gamma} ds + M E \int_{t_0}^t (1 + |\varphi_\Delta(s)|^{\gamma} + |\pi_\Delta(\varphi_\Delta(s))|^{\gamma}) |\varphi_\Delta(s) - \pi_\Delta(\varphi_\Delta(s))|^{\gamma} ds \right) . \\
\]

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By the Hölder inequality, we then obtain
\[
I_{32} \leq C_{32} \left( E \int_{t_0}^{t} |e_{\Delta}(s)|^q \, ds + \int_{t_0}^{t} \left( E \left( 1 + |\mathcal{F}_{\Delta}(s)|^p + |\pi_{\Delta}(\mathcal{F}_{\Delta}(s))|^p \right) \right)^{\frac{q}{p}} \right)
\]
\[
\leq C_{32} \left( E \int_{t_0}^{t} |e_{\Delta}(s)|^{\frac{q}{\beta}} \, ds + \int_{t_0}^{t} \left( E \left( 1 + \pi_{\Delta}(s) \right) \right)^{\frac{q}{p}} \, ds \right)
\]
\[
\leq C_{32} \left( E \int_{t_0}^{t} |e_{\Delta}(s)|^{\frac{q}{\beta}} \, ds + \int_{t_0}^{t} \left( E \left( 1 + \pi_{\Delta}(s) \right) \right)^{\frac{q}{p}} \, ds \right)
\]
\[
\leq C_{32} \left( E \int_{t_0}^{t} \left( \mathbb{P} \left( |\mathcal{F}_{\Delta}(s)| > f^{-1}(\kappa(\Delta)) \right) \right)^{\frac{q}{p}} \, ds \right)
\]
\[
\leq C_{32} \left( E \int_{t_0}^{t} \left( \mathbb{P} \left( |\mathcal{F}_{\Delta}(s)| > f^{-1}(\kappa(\Delta)) \right) \right)^{\frac{q}{p}} \, ds \right)
\]
\[
\leq C_{32} \left( E \int_{t_0}^{t} |e_{\Delta}(s)|^{\frac{q}{\beta}} \, ds + \int_{t_0}^{t} \left( \mathbb{P} \left( |\mathcal{F}_{\Delta}(s)| > f^{-1}(\kappa(\Delta)) \right) \right)^{\frac{q}{p}} \, ds \right)
\]
(12)

Substituting (11) and (12) into (10), we arrive at
\[
I_3 \leq C_{32} \left( E \int_{t_0}^{t} |e_{\Delta}(s)|^{\frac{q}{\beta}} \, ds + \int_{t_0}^{t} \left( \mathbb{P} \left( |\mathcal{F}_{\Delta}(s)| > f^{-1}(\kappa(\Delta)) \right) \right)^{\frac{q}{p}} \, ds \right)
\]
(13)

Then (8), (9) and (13) together imply that
\[
E |e_{\Delta}(t)|^{\frac{q}{\beta}} \leq C \left( E \int_{t_0}^{t} |e_{\Delta}(s)|^{\frac{q}{\beta}} \, ds + \int_{t_0}^{t} \left( \mathbb{P} \left( |\mathcal{F}_{\Delta}(s)| > f^{-1}(\kappa(\Delta)) \right) \right)^{\frac{q}{p}} \, ds \right)
\]
(14)

An application of the Gronwall inequality yields that
\[
E |e_{\Delta}(t)|^{\frac{q}{\beta}} \leq C \left( f^{-1}(\kappa(\Delta)) \right)^{\frac{q}{p}} \, ds + \int_{t_0}^{t} \left( \mathbb{P} \left( |\mathcal{F}_{\Delta}(s)| > f^{-1}(\kappa(\Delta)) \right) \right)^{\frac{q}{p}} \, ds
\]
(15)

which is the required assertion (5). The other assertion (6) follows from (5) and Lemma 2.8. The proof is therefore complete.

We are now ready to give the proof of Theorem 3.1.

**Proof of Theorem 3.1**

Recalling (2), we then define
\[
f(u) = Ku^{\beta+1}, \quad u \geq 1,
\]
which implies that
\[
f^{-1}(u) = \left( \frac{u}{K} \right)^{\frac{1}{\beta+1}}.
\]
Let

\[ \kappa(\Delta) = \Delta^{-\varepsilon} \] for some \( \varepsilon \in (0, \frac{1}{4}) \) and \( \hat{h} \geq 1 \).

Following Theorem 3.3, we obtain

\[ E|y(x(t)) - y(x(\Delta(t)))| \leq C \left( \Delta^{\frac{\varepsilon(p-\beta q - \beta)}{\beta + 1}} + \Delta^{\frac{(1-2\varepsilon)}{2}} + \Delta^{\gamma} + \Delta^{\alpha} \right), \] (14)

and

\[ E|y(x(t)) - y(x(\Delta(t)))| \leq C \left( \Delta^{\frac{\varepsilon(p-\beta q - \beta)}{\beta + 1}} + \Delta^{\frac{(1-2\varepsilon)}{2}} + \Delta^{\gamma} + \Delta^{\alpha} \right). \] (15)

Choosing \( p \) sufficiently large for

\[ \frac{\varepsilon(p - \beta q - \beta)}{\beta + 1} > \min(\gamma, \alpha, \frac{1}{2} - \varepsilon)q, \]

we can draw the assertions from (14) and (15) immediately. \( \square \)

4. Simulation

Two examples with the different theoretical convergence rates are presented in this section. Computer simulations are conducted to verify the theoretical results.

Example 4.1. Consider the scalar stochastic differential equation

\[
\begin{cases}
    dx(t) = \left(t(1-t)\frac{1}{4}x^2(t) - 2x^5(t)\right)dt + \left[t(1-t)\frac{1}{4}x^2(t)\right]dW(t), & t \in [t_0, T], \\
    x(t_0) = 2,
\end{cases}
\] (16)

where \( t_0 = 0 \) and \( T = 1 \). It is clearly that the drift and diffusion coefficients are locally Lipschitz continuous, i.e. Assumption 2.1 is satisfied.

Also, for any \( q > 2 \), such that

\[
(x - y)^T \left( \mu(t, x) - \mu(t, y) \right) + \frac{q - 1}{2} |\sigma^r(t, x) - \sigma^r(t, y)|^2 
\]

\[
\leq (x - y)^2 \left( [t(1-t)]\frac{1}{4} (x + y) - 2(x^4 + x^2y^2 + xy^3 + y^4) + \frac{q - 1}{2} [t(1-t)]\frac{1}{4} (x + y)^2 \right).
\]
But

\[-2(x^3y + xy^3) = -2xy(x^2 + y^2) \leq (x^2 + y^2)^2 = x^4 + y^4 + 2x^2y^2.\]

Therefore,

\[
(x - y)^T (\mu(t, x) - \mu(t, y)) + \frac{q - 1}{2} |\sigma^*(t, x) - \sigma^*(t, y)|^2 \\
\leq (x - y)^2 \left( |t(1 - t)\frac{d}{dt} (x + y) - x^{-4} - y^{-4} + (q - 1)|t(1 - t)\frac{d}{dt} (x^2 + y^2) \right) \\
\leq L_1 (x - y)^2,
\]

where the last inequality is due to the fact that polynomials with the negative coefficient for the highest order term can always be bounded from above. This indicates that Assumption 2.3 holds.

In addition, for any \(p > 2\), we have

\[
x^T \mu(t, x) + \frac{p - 1}{2} |\sigma(t, x)|^2 \leq x^3 - 2x^6 + \frac{p - 1}{2} |x|^4 \leq K_1 (1 + |x|^2),
\]

which means that Assumption 2.4 is satisfied.

Using the mean value theorem for the temporal variable, Assumptions 2.2 and 2.6 are satisfied with \(\alpha = \gamma = 1/4\) and \(\beta = 4\). According to Theorem 3.3, we know that

\[
E|x(t) - x_\Delta(t)|^q \leq C \left( (f^{-1}(\kappa(\Delta)))^{(5q-p)/p} + \Delta^{q/2} (\kappa(\Delta))^q + \Delta^{q/4} \right),
\]

and

\[
E|x(t) - x_\Delta(t)|^q \leq C \left( (f^{-1}(\kappa(\Delta)))^{(5q-p)/p} + \Delta^{q/2} (\kappa(\Delta))^q + \Delta^{q/4} \right).
\]

Due to that

\[
\sup_{0 \leq t \leq T} \sup_{|x| \leq u} (|\mu(t, x) \lor |\sigma(t, x)|) \leq 3u^5, \quad \forall u \geq 1,
\]

we choose \(f(u) = 3u^5\) and \(\kappa(\Delta) = \Delta^{-\varepsilon}\), for any \(\varepsilon \in (0, 1/4)\). As a result, \(f^{-1}(u) = (u/3)^{1/5}\) and \(f^{-1}(\kappa(\Delta)) = (\Delta^{-\varepsilon}/3)^{1/5}\). Choosing \(p\) sufficiently large, we can get from Theorem 3.1 that

\[
\sup_{0 \leq t \leq 1} E|x(t) - x_\Delta(t)|^q \leq C \Delta^{q/4},
\]
and
\[
\sup_{0 \leq t \leq 1} \mathbb{E}|x(t) - \mathbb{F}_\Delta(t)|^\nu \leq C\Delta^{\nu/4},
\]
which imply that the convergence rate of truncated EM method for the SDE (16) is 1/4.

Let us compute the approximation of the mean square error. We run \(M = 1000\) independent trajectories for every different step sizes, \(10^{-1}, 10^{-2}, 10^{-3}, 10^{-5}\). Because it is hard to find the true solution for the SDE, the numerical solution with the step size \(10^{-5}\) is regarded as the exact solution.

![Figure 1: The mean square errors between the exact solution and the numerical solutions for step sizes \(\Delta = 10^{-1}, 10^{-2}, 10^{-3}\).](image)

By the linear regression, also shown in the Figure 1(a), the slope of the errors against the step sizes is approximately 0.24629, which is quite close to the theoretical result.

**Example 4.2.** Consider the scalar stochastic differential equation
\[
\begin{align*}
\frac{dx}{dt}(t) &= \left( [(t - 1)(2 - t)]^{3/5}x^2(t) - 2x^3(t) \right) dt + \left( [(t - 1)(2 - t)]^{2/5}x^2(t) \right) dW(t), \quad t \in [t_0, T],
\end{align*}
\]
\[
\begin{align*}
x(t_0) &= 2,
\end{align*}
\]
where \(t_0 = 1\) and \(T = 2\). In the similar way as Example 4.1 we can verify that Assumptions 2.1, 2.3 and 2.4 hold.
Moreover, the mean value theorem is used to verify that Assumptions 2.2 and 2.6 are satisfied with $\alpha = 2/5$, $\gamma = 1/5$ and $\beta = 4$.

We can get from Theorem 3.1 that
\[
\sup_{1 \leq t \leq 2} \mathbb{E}|x(t) - x_{\Delta}(t)|^q \leq C\Delta^{7/5},
\]
and
\[
\sup_{1 \leq t \leq 2} \mathbb{E}|x(t) - x_{\Delta}(t)|^7 \leq C\Delta^{7/5},
\]
which implies that the convergence rate of truncated EM method for the SDE (17) is 1/5. Simulation is conducted using the same strategy as that in Example 4.1. Using the linear regression, also seen in the figure 1(b), the slope of the errors against the step sizes is approximately 0.20550, which coincides with the theoretical result.

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