HARMONIC DIFFERENTIAL FORMS FOR PSEUDO-REFLECTION GROUPS
I. SEMI-INVARIANTS

JOSHUA P. SWANSON AND NOLAN R. WALLACH

ABSTRACT. We give a type-independent construction of an explicit basis for the semi-invariant harmonic differential forms of an arbitrary pseudo-reflection group in characteristic zero. Our “top-down” approach uses the methods of Cartan’s exterior calculus and is in some sense dual to related work of Solomon [22], Orlik–Solomon [14], and Shepler [20, 21] describing (semi-)invariant differential forms. We apply our results to a recent conjecture of Zabrocki [29] which provides a representation theoretic-model for the Delta conjecture of Haglund–Remmel–Wilson [7] in terms of a certain non-commutative coinvariant algebra for the symmetric group. In particular, we verify the alternating component of a specialization of Zabrocki’s conjecture.

1. Introduction

Recently there has been a great deal of research activity in algebraic combinatorics studying diagonal actions of the symmetric group $S_n$ on $k$ sets of $n$ commuting indeterminants and $\ell$ sets of $n$ anti-commuting indeterminants. Orellana–Zabrocki [13] describe the $S_n$-invariants of these polynomial rings combinatorially and summarize some of their history. The $k = 2, \ell = 0$ case has received a very large amount of attention through the study of the diagonal coinvariants $\mathbb{Q}[x_n, y_n]/D_n$ where $D_n$ is the diagonal coinvariant ideal generated by all homogeneous $S_n$-invariants of positive degree and $x_n$ is shorthand for $x_1, \ldots, x_n$ [9, 10].

Zabrocki [29] has recently given a conjectured description of the tri-graded $S_n$-module isomorphism type of the coinvariant algebra when $m = 2, \ell = 1$:

\begin{equation}
\text{GrFrob}(\mathbb{Q}[x_n, y_n, \theta_n]/SD_n; q, t, z) = \sum_{i=1}^{n} z^{n-k} \Delta'_{e_k-1} e_n ,
\end{equation}

where GrFrob is the tri-graded Frobenius series, $SD_n$ is the super diagonal coinvariant ideal generated by homogeneous $S_n$-invariants of
positive degree, $e_n$ is an elementary symmetric function, and $\Delta'_f$ is a certain modified Macdonald eigenoperator. Equation (11) may be interpreted as a conjectural representation-theoretic model for the Delta conjecture of Haglund–Remmel–Wilson [7], and both conjectures remain open. See [7, 29] for details and further references.

The special case $t = 0$ of these conjectures involving one set of commuting and one set of anti-commuting variables has received special attention. Haglund–Rhoades–Shimozono [8] had earlier given a different representation-theoretic model for this specialization of the Delta conjecture. Motivated by Zabrocki’s conjecture, the second author [28] gave a conjectural description of the harmonics of $\mathbb{Q}[x_n, \theta_n]/J_n$ where $J_n$ is the super coinvariant ideal generated by homogeneous $S_n$-invariants of positive degree. This description further motivated Rhoades–Wilson [18] to recently construct another representation-theoretic model for the $t = 0$ specialization of the Delta conjecture arising from the leading terms of those harmonics. In this paper, we restrict our attention to natural pseudo-reflection group generalizations of the $t = 0$ case of Zabrocki’s coinvariant algebra model.

In [28], several of the implications of Zabrocki’s conjecture at $t = 0$ were proven, including a determination of the bi-graded Hilbert series of the alternants in the $S_n$-coinvariants and certain vanishing bounds. The key to the results in [28] is the fact that one is looking at differential forms with polynomial coefficients and thus one has at one’s disposal the full power of Cartan’s exterior calculus. The idea of adding anti-commuting variables to prove theorems about the commuting variables appeared in the work of Solomon [22] and, later, in the paper of Orlik–Solomon [14] in the case of a finite unitary subgroup $G \leq \text{GL}_n(\mathbb{C})$ generated by pseudo-reflections. Differential forms and derivations have also been exploited in this context more recently [16, 17]. The purpose of the present paper is to use the methods of differential forms to give a uniform generalization of the type $A$ description in [28] of the alternant polynomial differential forms and coinvariants for an arbitrary pseudo-reflection group.

Our main results are as follows. See the subsequent sections for missing definitions and Section 4 and Section 5 for proofs. Let $V$ be an $n$-dimensional vector space over an arbitrary field $F$ of characteristic 0, let $G \leq \text{GL}(V)$ be a pseudo-reflection group, let $M$ be an $r$-dimensional $G$-module, and let $\chi$ be a one-dimensional character of $G$.

We consider the semi-invariant differential forms

\begin{equation}
(S(V^*) \otimes \wedge^* M)^\chi
\end{equation}
where $S(V^*)$ is the symmetric algebra on $V^*$, $\wedge M^*$ is the exterior algebra on $M^*$, and $W^\chi := \{ w \in W : \sigma \cdot w = \chi(\sigma)w, \forall \sigma \in G \}$. In certain circumstances, such as when $M = V$, we give two explicit bases for (2). The following result may be thought of as an analogue of Solomon’s classic result [22] describing $(S(V^*) \otimes \wedge V^*)^G$ as a Grassmann algebra over $S(V^*)^G$. Here $\Delta_\chi$ is the minimal-degree element in the $\chi$-isotypic component of $S(V^*)$, the $d_i^*$ are certain differential operators depending on $M$ (see Definition 4.4), the $f_i$ are basic invariants of $G$, and $J_{M^*}$ is the Jacobian of $M^*$ (see Definition 3.3).

**Theorem 4.10.** Suppose $J_{M^*} | \Delta_\chi$. Then either of the sets

$$\{f_1^{a_1} \cdot \cdot \cdot f_n^{a_n} d_1^{i_1} \cdot \cdot \cdot d_k^{i_k} \Delta_\chi : 1 \leq i_1 < \cdots < i_k \leq r, a_j \in \mathbb{Z}_{\geq 0}\}$$

or

$$\{d_1^{i_1} \cdot \cdot \cdot d_k^{i_k} f_1^{a_1} \cdot \cdot \cdot f_n^{a_n} \Delta_\chi : 1 \leq i_1 < \cdots < i_k \leq r, a_j \in \mathbb{Z}_{\geq 0}\}$$

form bases for $(S(V^*) \otimes \wedge M^*)^\chi$.

We have the following enumerative corollary. Let $e_i^{M^*}$ denote the exponents of $M^*$ (see Definition 3.1) and let $d_i$ denote the degrees of $G$.

**Corollary 4.11.** Suppose $J_{M^*} | \Delta_\chi$. Then

$$\text{Hilb}((S(V^*) \otimes \wedge M^*)^\chi; q, z) = q^\deg \Delta_\chi \frac{\prod_{i=1}^r (1 + zq^{-e_i^{M^*}})}{\prod_{i=1}^n (1 - q^{d_i})}.$$ 

**Remark 1.1.** Orlik–Solomon [14] described $(S(V^*) \otimes \wedge M^*)^G$ when $J_M = \Delta_M$, up to a non-zero scalar, as an exterior algebra over $S(V^*)^G$, and Shepler [20, 21] gave an analogous result for $(S(V^*) \otimes \wedge V^*)^\chi$ using “$\chi$-wedging.” These exterior algebra structures may be considered “bottom-up” descriptions. Theorem 4.10 involves the operators $d_i^*$, which decrease $S(V^*)$-degree, applied to $\Delta_\chi$, so this and related work in [21, §6] can be considered “top-down.” The top-down description turns out to be more useful in our analysis of the coinvariant algebra.

Let $J_M^*$ denote the ideal in $S(V^*) \otimes \wedge M^*$ generated by all homogeneous $G$-invariants of positive degree. Our overarching motivation has been to describe the semi-invariant elements of the coinvariant algebra,

$$(S(V^*) \otimes \wedge M^*/J_M^*)^\chi.$$ 

We give the following explicit basis for (3) using the harmonics of $S(V^*) \otimes \wedge M^*$ (see Definition 4.1).
**Theorem 5.7.** Suppose $J_M \mid \Delta_\chi$ and $M^G = 0$. Then the $2^r$ elements
\[
\{d_{i_1}^* \cdots d_{i_k}^* \Delta_\chi \mid 1 \leq i_1 < \cdots < i_k \leq r\}
\]
form a basis of $\mathcal{H}(S(V^*) \otimes \wedge M^*)^\chi$, and their images descend to a basis of $(S(V^*) \otimes \wedge M^*/J^*_M)^\chi$.

**Corollary 5.8.** Suppose $J_M \mid \Delta_\chi$ and $M^G = 0$. Then

\[
(4) \quad \text{Hilb}((S(V^*) \otimes \wedge M^*/J^*_M)^\chi; q, z) = q^{\deg \Delta_\chi} \prod_{i=1}^{r}(1 + zq^{-\epsilon_i^M}).
\]

The hypotheses of Theorem 5.7 are satisfied whenever $\chi = \det M^*$, $M^G = 0$, and the pseudo-reflections of $G$ act by pseudo-reflections or the identity on $M$. In this case, the right-hand side of (4) is

\[
\prod_{i=1}^{r}(q^{\epsilon_i^M} + z).
\]

The alternating component of the $t = 0$ specialization of Zabrocki’s conjecture then follows from Corollary 5.8 when $G$ consists of $n \times n$ permutation matrices and $M$ is the $(n-1)$-dimensional standard representation.

The classical coinvariant algebra of $G$ is $S(V^*)/I^*$ where $I^*$ is the ideal generated by all non-constant homogeneous $G$-invariants. It is well-known that the top-degree component of $S(V^*)/I^*$ is the image of $S(V^*)^{\det V}$, which has motivated much of our work. We will explore which bidegrees of $S(V^*) \otimes \wedge M^*/J^*_M$ are non-zero in a future article [27].

The rest of the paper is organized as follows. In Section 2, we review background material on differential forms and pseudo-reflection groups. Section 3 concerns exponents and basic derivations of pseudo-reflection groups. Section 4 introduces $G$-harmonics for the polynomial algebra and constructs explicit bases for certain pieces of the differential algebras; see Theorem 4.5 and Theorem 4.10. In Section 5, we prove our main result, Theorem 5.7, which gives a basis for the harmonics and the coinvariants. In Section 6, we discuss the technical condition $\Delta_M = J_M$, up to a non-zero scalar, which appears in some of our results. A more explicit but less general version of many of these results that may be more palatable to algebraic combinatorialists can be found in [26].
2. Polynomial differential forms

We now describe several actions and pairings involving polynomial differential forms and related objects. All of the constructions in this section use standard ideas from differential geometry.

Let \( F \) be a field of characteristic 0 and let \( V \) be an \( F \)-vector space of dimension \( n < \infty \). We identify \( V^{**} = V \). Let \( G \) be an arbitrary subgroup of \( \text{GL}(V) \) and let \( M \) be an \( F[G] \)-module of dimension \( r < \infty \).

2.1. Symmetric and exterior algebras.

**Definition 2.1.** Let \( S(V^*) := \text{Sym}(V^*) \) denote the symmetric algebra on \( V^* \) over \( F \), i.e. the algebra of polynomial functions on \( V \). Let \( \bigwedge M^* \) denote the exterior or Grassmann algebra on \( M^* \).

The tensor products \( S(V) \otimes \bigwedge M \) and \( S(V^*) \otimes \bigwedge M^* \) are algebras of differential forms with polynomial coefficients. Each of \( S(V) \), \( S(V^*) \), \( \bigwedge M \), and \( \bigwedge M^* \) is naturally graded, so we have four non-commutative, bigraded \( F \)-algebras

\[
S(V) \otimes \bigwedge M, \quad S(V^*) \otimes \bigwedge M^*, \quad S(V) \otimes \bigwedge M, \quad S(V^*) \otimes \bigwedge M^*.
\]

The \( G \)-action on \( V \) extends multiplicatively to yield natural \( G \)-actions on \( S(V) \) and \( \bigwedge M \). As usual, \( G \) acts on \( V^* \) contragrediently via \( g^* \lambda := \lambda \circ g^{-1} \). Thus \( S(V), S(V^*), \bigwedge M, \) and \( \bigwedge M^* \) are all naturally graded \( G \)-modules, so the algebras in (5) are bigraded \( G \)-modules via the diagonal actions of

\[
g \otimes g, \quad g \otimes g^*, \quad g^* \otimes g, \quad g^* \otimes g^*.
\]

2.2. Pairings and differential operators on \( S(V^*) \otimes \bigwedge M^* \). Our next goal is to describe natural actions of \( S(V) \otimes \bigwedge M \) and \( S(V^*) \otimes \bigwedge M^* \) on \( S(V^*) \otimes \bigwedge M^* \). These actions will be fundamental in later sections.

The following observation is essentially trivial.

**Lemma 2.2.** The natural pairing \( M^* \times M \to F \) given by \( \langle \lambda, v \rangle := \lambda(v) \) is \( G \)-invariant and perfect.

By “perfect”, we mean that \( \langle \lambda, v \rangle = 0 \) for all \( v \in V \) implies \( \lambda = 0 \), and \( \langle \lambda, v \rangle = 0 \) for all \( \lambda \in V^* \) implies \( v = 0 \). We may naturally extend such pairings to symmetric, exterior, and tensor products as follows.

**Lemma 2.3.** Suppose \( W, W_1, W_2 \) are finite-dimensional \( G \)-modules with \( G \)-invariant perfect pairings

\[
\langle -, - \rangle : W^* \times W \to F
\]

and

\[
\langle -, - \rangle_i : W_i^* \times W_i \to F.
\]
(i) The pairing
\[ W_1^* \otimes W_2^* \times W_1 \otimes W_2 \rightarrow F \]
\[ \langle \omega_1 \otimes \omega_2, w_1 \otimes w_2 \rangle := \langle \omega_1, w_1 \rangle_1 \langle \omega_2, w_2 \rangle_2 \]
is $G$-invariant and perfect.

(ii) The pairing
\[ \wedge W^* \times \wedge W \rightarrow F \]
\[ \langle \omega_1 \wedge \cdots \wedge \omega_k, w_1 \wedge \cdots \wedge w_\ell \rangle := \delta_{k\ell} \det(\langle \omega_i, w_j \rangle)_{i,j=1}^k \]
\[ := \delta_{k\ell} \sum_{\sigma \in S_k} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^k \langle \omega_i, w_{\sigma(i)} \rangle \]
is $G$-invariant and perfect.

(iii) The pairing
\[ S(W^*) \times S(W) \rightarrow F \]
\[ \langle \omega_1 \cdots \omega_k, w_1 \cdots w_\ell \rangle := \delta_{k\ell} \text{perm}(\langle \omega_i, w_j \rangle)_{i,j=1}^k \]
\[ := \delta_{k\ell} \sum_{\sigma \in S_k} \prod_{i=1}^k \langle \omega_i, w_{\sigma(i)} \rangle \]
is $G$-invariant and perfect, where \text{perm} denotes the matrix permanent.

Proof. In each case, $G$-invariance is immediate and non-degeneracy can be checked quickly using dual bases. \qed

Corollary 2.4. We have a natural $G$-invariant perfect pairing
\[ \langle -, - \rangle : S(V^*) \otimes \wedge^* \times S(V) \otimes \wedge^* \rightarrow F. \]

Furthermore, we have natural $G$-equivariant identifications of $S(V^*)$ with $S(V)^*$ and of $\wedge^*$ with $(\wedge^*)^*$.

Definition 2.5. For $s \in S(V)$ and $f \in S(V^*)$, we have multiplication operators defined by
\[ m_s : S(V) \rightarrow S(V) \quad \text{and} \quad m_f : S(V^*) \rightarrow S(V^*) \]
\[ m_s(u) := su \quad \text{and} \quad m_f(h) := fh. \]

For $m \in \wedge M$ and $\mu \in \wedge M^*$, we have multiplication operators
\[ \epsilon_m : \wedge M \rightarrow \wedge M \quad \text{and} \quad \epsilon_\mu : \wedge M^* \rightarrow \wedge M^* \]
\[ \epsilon_m(\ell) := m \wedge \ell \quad \text{and} \quad \epsilon_\mu(\nu) := \mu \wedge \nu. \]
Definition 2.6. For \( s \in S(V) \) and \( f \in S(V^*) \), we have adjoint operators defined by

\[
\partial_s : S(V^*) \to S(V^*) \quad \text{and} \quad \partial_f : S(V) \to S(V)
\]

\[
\langle \partial_s(h), u \rangle = \langle h, m_s(u) \rangle \quad \text{and} \quad \langle h, \partial_f(u) \rangle = \langle m_f(h), u \rangle
\]

for all \( u \in S(V) \), \( h \in S(V^*) \).

For \( m \in \wedge M \) and \( \mu \in \wedge M^* \), we have adjoint operators defined by

\[
\iota_m : \wedge M^* \to \wedge M^* \quad \text{and} \quad \iota_\mu : \wedge M \to \wedge M
\]

\[
\langle \iota_m(\nu), \ell \rangle = \langle \nu, \epsilon_m(\ell) \rangle \quad \text{and} \quad \langle \nu, \iota_\mu(\ell) \rangle = \langle \epsilon_\mu(\nu), \ell \rangle
\]

for all \( \ell \in \wedge M \), \( \nu \in \wedge M^* \).

Combining these operators yields actions of \( S(V) \otimes \wedge M \) and \( S(V) \otimes \wedge M^* \) on \( S(V^*) \otimes \wedge M^* \) as follows. These actions will be used prominently in later sections.

Lemma 2.7. The maps

\[
S(V) \otimes \wedge M \to \text{End}_F(S(V^*) \otimes \wedge M^*)
\]

\[
s \otimes m \mapsto \partial_s \otimes \iota_m
\]

and

\[
S(V) \otimes \wedge M^* \to \text{End}_F(S(V^*) \otimes \wedge M^*)
\]

\[
s \otimes \mu \mapsto \partial_s \otimes \epsilon_\mu
\]

are \( G \)-equivariant \( F \)-algebra morphisms.

Proof. The multiplication operators taken together yield \( F \)-algebra morphisms, and the same is true of their adjoints. By definition, \( g \in G \) acts on \( \psi \in \text{End}_F(S(V^*) \otimes \wedge M^*) \) by \( (g\psi)(\omega) := g\psi(g^{-1}\omega) \). For \( G \)-equivariance, we see immediately that

\[
m_{gs}(gu) = gm_s(u) \quad \text{and} \quad m_{gf}(gh) = gm_f(h)
\]

\[
\epsilon_{gm}(gu) = g\epsilon_m(u) \quad \text{and} \quad \epsilon_{g\mu}(g\nu) = g\epsilon_\mu(\nu).
\]

It follows from this and the \( G \)-invariance of \( \langle -,- \rangle \) that

\[
\partial_{gs}(gf) = g\partial_s(f) \quad \text{and} \quad \partial_{gf}(gu) = g\partial_f(u)
\]

\[
\iota_{gm}(g\nu) = g\iota_m(\nu) \quad \text{and} \quad \iota_{g\mu}(g\ell) = g\iota_\mu(\ell).
\]

The claimed \( G \)-equivariance follows.

Example 2.8. A special case of the preceding construction gives a distinguished and familiar \( G \)-invariant endomorphism of \( S(V^*) \otimes \wedge V^* \).
Let \( v_1, \ldots, v_n \in V \) be a basis and let \( \lambda_1, \ldots, \lambda_n \in V^* \) be its dual basis. Then

\[
\sum_{j=1}^n v_j \otimes \lambda_j \in S(V) \otimes \wedge V^*
\]
is independent of the choice of basis and is hence \( G \)-invariant. The action of this element is the exterior derivative, namely

\[
(7) \quad d := \sum_{j=1}^n \partial v_j \otimes \epsilon \lambda_j \in \text{End}_F(S(V^*) \otimes \wedge V^*).
\]

It satisfies \( g(d\omega) = d(g\omega) \) for all \( g \in G, \omega \in S(V^*) \otimes \wedge V^* \).

The operators above acting on \( S(V^*) \otimes \wedge M^* \) generate the following super Weyl algebra. All of the operators that we will be using in this paper come from the action of this algebra on \( S(V^*) \otimes \wedge M^* \).

**Proposition 2.9.** The subalgebra of \( \text{End}_F(S(V^*) \otimes \wedge M^*) \) generated by \( m_\mu \otimes \text{id}, \partial_v \otimes \text{id}, \text{id} \otimes \epsilon \mu, \text{id} \otimes \iota_v \) for \( v \in V, \mu \in M^* \) is isomorphic to the tensor product of simple algebras

\[
\mathbb{D}(V) \otimes \text{Cliff}(M \oplus M^*)
\]

where \( \mathbb{D}(V) \) is the algebra generated by \( \partial_v, m_\lambda \) which is the Weyl algebra on \( \text{dim}(V) \) variables, and \( \text{Cliff}(M \oplus M^*) \) is the Clifford algebra of the split form \( \langle m + \mu, \ell + \nu \rangle_{\text{cl}} := \mu(\ell) + \nu(m) \) which is generated by \( \iota_m, \epsilon_\mu \).

### 2.3. Derivations and anti-derivations.

The differential operators \( \partial_s \) and \( \partial_f \) are the usual polynomial differential operators acting on the polynomial ring or its dual. The operators \( \epsilon_m \) and \( \epsilon_\mu \) are called exterior products and the operators \( \iota_m \) and \( \iota_\mu \) are called interior products. For completeness and concreteness, we briefly summarize some of their well-known properties.

If \( v \in V \), then \( \partial_v \in \text{End}_F(S(V^*)) \) is given as usual by

\[
(8) \quad \partial_v f(x) = \left. \frac{df(x + tv)}{dt} \right|_{t=0}.
\]

The operators \( \partial_v \) satisfy the classical Leibniz rule

\[
(9) \quad \partial_v f h = (\partial_v f) h + f(\partial_v h)
\]

for all \( v \in V \) and \( f, h \in S(V^*) \), and hence are derivations. If \( s \in S(V) \), we have

\[
(10) \quad (\partial_s f)(0) = \langle f, s \rangle = (\partial_f s)(0).
\]
Combining these last two observations, for all $v \in V$ and $\lambda \in V^*$, we have

$$\partial_v m_\lambda - m_\lambda \partial_v = \lambda(v) \text{id}_{S(V^*)}.$$  

It follows from the perm description in Lemma 2.3 that if $v_1, \ldots, v_n \in V$ is a basis with dual basis $\lambda_1, \ldots, \lambda_n \in V^*$, then

$$\partial_v \lambda_\beta = \delta_{\alpha \leq \beta} \frac{\beta!}{(\beta - \alpha)!} \lambda_\beta - \alpha,$$

where we have used multi-index notation and all operations are component-wise.

Analogously, one may check that if $m_1, \ldots, m_r \in M$ is a basis with dual basis $\mu_1, \ldots, \mu_r \in M^*$, then

$$\iota_m \mu^J = \pm \delta_{I \subset J} \mu^{J-I},$$

where we have used the natural analogue of multi-index notation in this setting, e.g. $\mu^J := \mu_{j_1} \wedge \cdots \wedge \mu_{j_\ell}$ if $J = \{j_1 < \cdots < j_\ell\}$. The sign may be determined explicitly by iterating the well-known identity

$$\iota_m (\mu_1 \wedge \cdots \wedge \mu_\ell) = \sum_{j=1}^\ell (-1)^{j-1} \mu_j(m) \mu_1 \wedge \cdots \wedge \hat{\mu}_j \wedge \cdots \wedge \mu_\ell.$$

More generally, the operators $\iota_m$ for $m \in M$ are anti-derivations in the sense that

$$\iota_m (\mu \wedge \nu) = (\iota_m \mu) \wedge \nu + (-1)^k \mu \wedge (\iota_m \nu)$$

for all $\mu \in \wedge^k M^*$ and $\nu \in \wedge M^*$. In particular, for all $m \in M$ and $\xi \in M^*$, we have

$$\iota_m \epsilon_\xi + \epsilon_\xi \iota_m = \xi(m) \text{id}_{\wedge M^*}.$$

See for instance [12, pp. 356-359] for further details.

2.4. Hermitian forms. While our key definitions and statements all involve the natural perfect pairings from Section 2.2, some of our proofs require replacing them with Hermitian forms. For later use, we now recall some elementary properties of Hermitian forms and relate the pairing $S(V^*) \otimes \wedge M^* \times S(V) \otimes \wedge M \to F$ to a Hermitian form on $S(V^*) \otimes \wedge M^*$. In this subsection, let $F$ be a subfield of $\mathbb{C}$ closed under complex conjugation and let $G \leq \text{GL}(V)$ be an arbitrary finite subgroup.

A Hermitian form on an $F$-vector space $W$ is a map $$(-, -) : W \times W \to F$$
such that if \( w \in W \) is fixed, the map \( v \mapsto (v, w) \) is \( F \)-linear, and \( (v, w) = (w, v) \). That is, a Hermitian form is linear in the first argument and conjugate-linear in the second argument. Note that \( (w, w) = (w, w) \in F \cap \mathbb{R} \). A Hermitian form is said to be positive-definite if \( (w, w) > 0 \) for all \( 0 \neq w \in W \).

**Lemma 2.10.** Let \( W = \bigoplus_{j=0}^{\infty} W_j \) be a graded \( F \)-vector space with \( \dim W_j < \infty \) for all \( j \). Suppose that \( (−, −) \) is a positive-definite Hermitian form on \( W \) for which \( (W_i, W_j) = 0 \) if \( i \neq j \). If \( X \) is a graded subspace of \( W \) and

\[ X^\perp := \{ w \in W : (x, w) = 0, \forall x \in W \}, \]

then \( W = X \oplus X^\perp \).

**Proof.** It is enough to consider the case when \( \dim W < \infty \). Note that if \( x \in X \cap X^\perp \), then \( (x, x) = 0 \), so \( x = 0 \). We must show \( X + X^\perp \) spans \( W \). Let \( x_1, \ldots, x_k \) be a basis for \( X \). Then \( X^\perp \) is the kernel of the \( F \)-linear map \( W \to F^k \) given by \( w \mapsto ((w, x_1), \ldots, (w, x_k)) \). By the rank-nullity theorem, \( \dim X + \dim X^\perp \geq \dim W \), and the result follows. \( \square \)

Since \( F \) is closed under complex conjugation, every finite-dimensional \( F \)-vector space \( W \) has a positive-definite Hermitian form given by choosing a basis \( \omega_1, \ldots, \omega_k \) of \( W^* \) and using

\[ (u, v) := \sum_{i=1}^{k} \omega_i(u)\overline{\omega_i(v)} \]

for all \( u, v \in W \). We may strengthen this construction and relate it to the canonical pairing

\[ \langle −, − \rangle : S(V^*) \otimes \wedge M^* \times S(V) \otimes \wedge M \to F \]

from Section 2.2 as follows.

**Lemma 2.11.** There is a positive-definite \( G \)-invariant Hermitian form

\[ (−, −) : S(V^*) \otimes \wedge M^* \times S(V^*) \otimes \wedge M^* \to F \]

and a conjugate-linear \( G \)-equivariant ring isomorphism

\[ \tau : S(V) \otimes \wedge M \to S(V^*) \otimes \wedge M^* \]

such that, for all \( \eta \in S(V^*) \otimes \wedge M^* \) and \( \omega \in S(V) \otimes \wedge M \),

\[ \langle \eta, \omega \rangle = \langle \eta, \tau(\omega) \rangle. \]
Proof. Let $W$ be a finite-dimensional $G$-module. Since $F$ is assumed closed under complex conjugation, there is a positive-definite Hermitian form on $W$. Since $G$ is assumed finite, the Hermitian form may be taken to be $G$-invariant by Weyl’s unitarian trick. We may extend $G$-invariant positive-definite Hermitian forms in three ways analogous to Lemma 2.3.

(1) Suppose $W_1, W_2$ have $G$-invariant positive-definite Hermitian forms $(\langle -,- \rangle_1, W_1, W_2) := (w_1, w'_1)(w_2, w'_2)$ extended bilinearly is a well-defined $G$-invariant Hermitian form. Moreover, it remains positive-definite as can be checked on orthogonal bases of $W_1, W_2$.

(2) A $G$-invariant positive-definite Hermitian form on $W$ induces a $G$-invariant positive-definite Hermitian form on the $k$th exterior power $\wedge^k W$ by $(w_1 \wedge \cdots \wedge w_k, w'_1 \wedge \cdots \wedge w'_k) := \det((w_i, w'_j))_{i,j=1}^k$, where perm denotes the matrix permanent.

(3) A $G$-invariant positive-definite Hermitian form on $W$ induces a $G$-invariant positive-definite Hermitian form on the $k$th symmetric power $S^k(W)$ by $(w_1 \cdots w_k, w'_1 \cdots w'_k) := \text{perm}((w_i, w'_j))_{i,j=1}^k$, where perm denotes the matrix permanent.

Combining these constructions yields a $G$-invariant positive-definite Hermitian form $(-,-)$ on $S(V^*) \otimes \wedge M^*$. We may thus define $\tau$ by (18).

The conjugate-linearity and $G$-equivariance of $\tau$ follow quickly from (18) and the corresponding properties of $(-,-)$ and $(-,-)$. It remains to show that $\tau$ is multiplicative. On $\wedge M^*$, we have

$(w_1 \wedge \cdots \wedge w_k, \tau(w'_1 \wedge \cdots \wedge w'_k)) = \langle w_1 \wedge \cdots \wedge w_k, w'_1 \wedge \cdots \wedge w'_k \rangle$

$= \det(\langle w_i, w'_j \rangle) = \det((w_i, \tau(w'_j)))$

$= \langle w_1 \wedge \cdots \wedge w_k, \tau(w'_1) \wedge \cdots \wedge \tau(w'_k) \rangle$.

The full calculation is exactly analogous. □

2.5. Pseudo-reflection groups and coinvariants. We now recall some basic facts concerning pseudo-reflection groups and establish some further notation. We again let $F$ be an arbitrary field of characteristic 0.

Definition 2.12. A pseudo-reflection is an element $g \in \text{GL}(V)$ such that

$$\dim \ker(g - I) = n - 1.$$ 

A pseudo-reflection of order two is called a reflection. A (pseudo) reflection group is a finite subgroup of $\text{GL}(V)$ generated by (pseudo) reflections.
For the rest of this subsection, let $G$ denote a pseudo-reflection group.

Shephard–Todd [19] and Chevalley [4] showed that the $G$-invariants $S(V^*)^G$ are generated by $n$ algebraically independent, homogeneous elements $f_1, \ldots, f_n \in S(V^*)^G$ of positive degrees $d_1, \ldots, d_n$. The $f_i$ are called basic invariants of $G$ and the $d_i$ are called the degrees of $G$.

Recall that the Hilbert series of a graded vector space $M = \oplus_{i=0}^{\infty} M_i$ with $\dim M_i < \infty$ is the formal power series
\[
\text{Hilb}(M; q) := \sum_{i=0}^{\infty} q^i \dim M_i.
\]
When $M = \oplus_{i,j=0}^{\infty} M_{i,j}$ is bigraded, we use a bivariate power series
\[
\text{Hilb}(M; q, z) := \sum_{i,j=0}^{\infty} q^i z^j \dim M_{i,j}.
\]

If $g \in \text{GL}(V)$ is a (pseudo) reflection, then $g^* \in \text{GL}(V^*)$ is as well, so $G^* \leq \text{GL}(V^*)$ is a pseudo-reflection group, and so $S(V)^G$ is similarly generated by algebraically independent, homogeneous elements $z_1, \ldots, z_n \in S(V)^G$. Since $G$ acts completely reducibly, $\dim M^G = \dim (M^*)^G$ in general, so $\text{Hilb}(S(V)^G; q) = \text{Hilb}(S(V^*)^G; q)$, and we may take $\deg f_i = \deg z_i$.

**Definition 2.13.** The coinvariant ideal of $S(V)$ is the ideal $\mathcal{I}$ generated by all homogeneous $G$-invariants of positive degree. We write $\mathcal{I}^*$ for the coinvariant ideal of $S(V^*)$. The coinvariant algebra of $S(V)$ or $S(V^*)$ is $S(V)/\mathcal{I}$ or $S(V^*)/\mathcal{I}^*$, respectively.

**Remark 2.14.** Chevalley [4] showed that $S(V^*)/\mathcal{I}^*$, a graded algebra, carries the regular representation of $G$ and that
\[
(19) \quad S(V^*)^G \otimes S(V^*)/\mathcal{I}^* \cong S(V^*)
\]
as a graded $S(V^*)^G$-module.

### 2.6. Fields of definition.

In order to use the machinery of Hermitian forms, we require representations defined over subfields of the complex numbers which are closed under complex conjugation. In practice, pseudo-reflection groups are typically constructed in terms of explicit unitary matrices over cyclotomic fields, in which case these properties are trivial. More care is required to handle the general case and avoid artificial assumptions.

Suppose $W$ is an $n$-dimensional vector space over a field $K$ of characteristic 0. The character field of a representation $\rho: G \to \text{GL}(W)$ is the subfield $\mathbb{Q}(\rho) \subset K$ generated by the set $\{\text{Tr} \rho(g) | g \in G\}$.
Lemma 2.15. If $|G| < \infty$, then $\mathbb{Q}(\rho)$ is isomorphic with a subfield of $\mathbb{C}$ closed under complex conjugation.

Proof. Choosing some basis for $W$, the matrix coefficients of $\rho(g)$ are $c^g_{ij} \in K$. Clearly $\mathbb{Q}(\rho) \subset \mathbb{Q}(c^g_{ii})$. Since $\{c^g_{ii}\}$ is finite, we may identify $\mathbb{Q}(c^g_{ii})$ with a subfield of $\mathbb{C}$, so $\mathbb{Q}(\rho)$ may be identified with a subfield of $\mathbb{C}$. Diagonalizing now shows that $\text{Tr} \rho(g)$ is of the form $\sum_i \zeta^{\alpha_i}$ for $\zeta = \exp(2\pi i / |G|) \in \mathbb{C}$. Thus

$$\text{Tr} \rho(g^{-1}) = \sum_i \zeta^{-\alpha_i} = \sum_i \zeta^{\alpha_i} = \overline{\text{Tr} \rho(g)}$$

so $\mathbb{Q}(\rho) \subset \mathbb{C}$ is closed under complex conjugation. □

We say $\rho: G \to \text{GL}(W)$ is defined over a subfield $K'$ of $K$ if there is some basis of $W$ for which $\rho(G) \subset \text{GL}_n(K')$. If $\rho$ is defined over $K'$, then clearly $K' \supset \mathbb{Q}(\rho)$. In favorable circumstances, the representation is actually defined over its character field.

Theorem 2.16 (Clark–Ewing [5]; see [11, Appendix B, p. 359]). If $\rho: G \to \text{GL}(W)$ is an irreducible representation of a finite group $G$ over a field $K$ of characteristic 0 and $\rho(G)$ contains a pseudo-reflection, then $\rho$ is defined over $\mathbb{Q}(\rho)$.

Benard gave a similar result for all representations of pseudo-reflection groups in characteristic 0. The only known proofs are case-by-case using the Shephard–Todd classification.

Theorem 2.17 (Benard [1, Thm. 1]). Let $G$ be a pseudo-reflection group over a field of characteristic 0. Then the character field $K$ of $G$ is a splitting field of $G$. That is, every representation of $G$ over a field containing $K$ is defined over $K$.

Corollary 2.18. If $G$ is a pseudo-reflection group over a field $K$ of characteristic 0 and $\rho: G \to \text{GL}(W)$ is a representation of $G$ over $K$, then $\rho$ is defined over a subfield of $K$ which is isomorphic to a subfield of $\mathbb{C}$ closed under complex conjugation.

3. Exponents, Jacobians, and Vandermondiens

We continue to let $V$ be an $n$-dimensional vector space over a field $F$ of characteristic 0. Let $G \leq \text{GL}(V)$ be a pseudo-reflection group and suppose $M$ is an $r$-dimensional $G$-module. In this section we recall the $M$-exponents, the basic derivations for $M$, and related notions.
3.1. **Exponents.** Chevalley’s results in Remark 2.14 (cf. [15] Lemma 6.45) imply that there are homogeneous elements \( \omega_1^M, \ldots, \omega_r^M \in (S(V^*) \otimes M^*)^G \) of bi-degrees \((e_1^M, 1), \ldots, (e_r^M, 1)\) such that

\[
(S(V^*) \otimes M^*)^G = S(V^*)^{G \omega_1^M} \oplus \cdots \oplus S(V^*)^{G \omega_r^M}.
\]

**Definition 3.1.** The \( \omega_i^M \) are called basic derivations for \( M \) over \( S(V^*) \) (cf. [15] Def. 6.50]). The \( e_i^M \) are the \( M \)-exponents. The exponents of \( G \) are the \( V \)-exponents \( e_1, \ldots, e_n := e_1^V, \ldots, e_n^V \), and the coexponents of \( G \) are the \( V^* \)-exponents \( e_1^*, \ldots, e_n^* := e_1^{V^*}, \ldots, e_n^{V^*} \).

The basic derivations are not unique, but the \( M \)-exponents are uniquely determined by \( M \) up to rearrangement. If \( M \) is absolutely irreducible, the \( M \)-exponents are the degrees in which \( M \) appears in \( S(V^*)/\mathcal{I}^* \).

Similarly, the basic derivations of a \( G \)-module \( M \) of dimension \( r \) over \( S(V) \) (rather than \( S(V^*) \)) are homogeneous elements \( \tilde{\omega}_1^M, \ldots, \tilde{\omega}_r^M \in (S(V) \otimes M)^G \) such that

\[
(S(V) \otimes M)^G = S(V)^G \tilde{\omega}_1^M \oplus \cdots \oplus S(V)^G \tilde{\omega}_r^M.
\]

After rearrangement, bi-deg \( \tilde{\omega}_i^M = \text{bideg} \omega_i^M \) since \( \text{Hilb}((S(V) \otimes M)^G; q) = \text{Hilb}((S(V^*) \otimes M^*)^G; q) \).

**Example 3.2.** Since basic invariants \( f_1, \ldots, f_n \in S(V^*)^G \) are algebraically independent and char \( F = 0 \), the classical Jacobian criterion gives \( \det(\partial_{v_j} f_i) \neq 0 \) for any basis \( v_1, \ldots, v_n \) of \( V \). This observation and the well-known fact that \( e_i = d_i - 1 \) imply \( df_1, \ldots, df_n \in (S(V^*) \otimes V^*)^G \) form a set of basic derivations for \( V \). That is, we may take

\[
\omega_i^V = df_i = \sum_{j=1}^n \partial_{v_j} f_i \otimes \lambda_j,
\]

where \( \lambda_1, \ldots, \lambda_n \) is the dual basis of \( v_1, \ldots, v_n \).

Likewise, the basic invariants \( z_1, \ldots, z_n \in S(V)^G \) yield basic derivations \( \tilde{\omega}_i^V = dz_i = \sum_{j=1}^n \partial_{v_j} z_i \otimes v_j \in (S(V) \otimes V)^G \).

3.2. **Jacobians and Vandermondians.** The preceding example motivates the following general notion. Suppose \( M^* \) has basis \( \mu_1, \ldots, \mu_r \). We may expand the basic derivations for \( M \) over \( S(V^*) \) as

\[
\sum_{j=1}^r J_{ij}^M \otimes \mu_j := \omega_i^M \in (S(V^*) \otimes M^*)^G
\]

for some \( J_{ij}^M \in S(V^*) \).

**Definition 3.3.** The Jacobian of \( M \) is

\[
J_M := \det(J_{ij}^M)^*_{i,j=1} \in S(V^*).
\]
The Jacobian is non-zero and is uniquely determined up to a non-zero constant \([14\text{, Prop. 2.5}].\) Note that \(\deg J_M = e_1^M + \cdots + e_r^M.\)

**Example 3.4.** We have

\[
\omega_1^V \wedge \cdots \wedge \omega_n^V = df_1 \wedge \cdots \wedge df_n = J_V \eta \in S(V^*) \otimes \wedge^n V^*
\]

for some \(0 \neq \eta \in \wedge^n V^*.\) Since this expression is \(G\)-invariant, it follows that \(J_V\) belongs to \(S(V^*)^{\det_V}\), the \(\det_V\)-isotypic component of \(S(V^*).\) Steinberg \([24]\) proved that

\[
(22) \quad S(V^*)^{\det_V} = S(V^*)^G J_V.
\]

Indeed, \(e_i = d_i - 1\) follows from \((22)\). For general \(M\), we similarly have some \(0 \neq \xi \in \wedge^r M\) such that

\[
(23) \quad \omega_1^M \wedge \cdots \wedge \omega_r^M = J_M \xi \quad \text{and} \quad J_M \in S(V^*)^{\det_M}.
\]

Consequently,

\[
(24) \quad S(V^*)^{\det_M} \supset S(V^*)^G J_M,
\]

though the containment may be strict, which motivates the following. By \((20)\), there exists an element

\[
\Delta_M \in S(V^*)^{\det_M}
\]

such that

\[
S(V^*)^{\det_M} = S(V^*)^G \Delta_M.
\]

**Definition 3.5.** We call \(\Delta_M\) the **Vandermondiand of \(M\) over \(S(V^*).\)**

The Vandermondiand of \(M\) is non-zero and is uniquely determined up to a non-zero scalar. When \(G\) consists of permutation matrices, \(\Delta_V\) is the classical Vandermondiand determinant. Steinberg’s result \((22)\) implies

\[
\Delta_V = J_V
\]

up to a non-zero scalar. In general, only

\[
\Delta_M \mid J_M
\]

is true. We have \(\Delta_M = J_{\det M} = \Delta_{\det M}\) up to a non-zero scalar, as can be seen from \((20)\). In particular, when \(M\) is one-dimensional, we may take \(\Delta_M = J_M.\)

Interchanging \(V\) and \(V^*\), we have the Jacobian and Vandermondiand with respect to \(S(V)\) instead of \(S(V^*).\)
Definition 3.6. The Jacobian of $M$ with respect to $S(V)$ is defined by
\[ \tilde{J}_M \eta = \tilde{\omega}_1^M \wedge \cdots \wedge \tilde{\omega}_r^M \in (S(V) \otimes \wedge M)^G \]
for $0 \neq \eta \in \wedge M^r$. The Vandermondian of $M$ with respect to $S(V)$ is defined by
\[ S(V)^{\det M^*} = S(V)^G \tilde{\Delta}_M . \]

As before, $\tilde{J}_M$ and $\tilde{\Delta}_M$ are each determined by $M$ up to a non-zero scalar. Since $\text{bideg } \tilde{\omega}_i^M = \text{bideg } \omega_i^M$, we have $\deg \tilde{J}_M = \deg J_M$ and $\deg \tilde{\Delta}_M = \deg \Delta_M$.

3.3. Gutkin’s formula. Stanley [23] expressed $\Delta_M$ and Gutkin [6] expressed $J_M$ as a product of linear forms vanishing on the reflecting hyperplanes of $G$, which we now summarize. See [16, §4] for historical discussion and [3, §4.5.2] for a proof using Molien’s theorem.

Let $A(G)$ be the set of reflecting hyperplanes of $G$, i.e. the fixed spaces of pseudo-reflections of $G$. For each $H \in A(G)$, fix some $\alpha_H \in V^*$ with $\ker \alpha_H = H$. Let $G_H$ denote the subgroup of $G$ fixing $H$ pointwise. Since $\sigma \in G_H$ acts trivially on a codimension 1 subspace, $\sigma \mapsto \det_{V^*} \sigma$ is a faithful representation $G_H \hookrightarrow F^\times$. Since finite subgroups of $F^\times$ are cyclic, $G_H$ is cyclic, $G_H$ is generated by a pseudo-reflection, and the irreducible $G_H$-representations are powers of $\det_{V^*}$. Consequently,
\[ M|_{\det_{V^*}} \cong \det_{V^*}^{m_{H,1}(M)} \oplus \cdots \oplus \det_{V^*}^{m_{H,r}(M)} \]
as $G_H$-modules, with $0 \leq m_{H,i}(M) < |G_H|$. Set \[ m_H(M) := m_{H,1}(M) + \cdots + m_{H,r}(M) . \]

Theorem 3.7 (Gutkin [6]; cf. [14, (2.11)] or [3, Theorem 4.38(1)]). Up to a non-zero scalar,
\[ J_M = \prod_{H \in A(G)} \alpha_H^{m_H(M)} . \]

Recall that $\Delta_M = J_{\det M}$ up to a non-zero scalar, so Theorem 3.7 gives product formulas for $\Delta_M$ as well. We have the following well-known special cases. See [3, Prop. 4.34(2)] and [14].

Corollary 3.8.
(a) $J_{V^*} = \prod_{H \in A(G)} \alpha_H .$
(b) $J_V = \prod_{H \in A(G)} \alpha_H^{|G_H|−1} .$
(c) $J_M$ equals $\Delta_M$ up to a non-zero scalar if and only if $m_H(M) < |G_H|$ for all $H \in A(G)$.
(d) If the pseudo-reflections of $G$ act on $M$ as pseudo-reflections or the identity, then $J_M$ equals $\Delta_M$ up to a non-zero scalar.

(e) If $M$ is a Galois conjugate of $V$, then $J_M$ equals $\Delta_M$ up to a non-zero scalar.

(f) If $\chi$ is one-dimensional, then $\Delta \chi | \Delta V$.

Proof. For (a), use $V^*|_{G_H} \cong \det V^* \oplus 1^{n-1}$. For (b), use $V|_{G_H} \cong \det V = \det V^* \oplus 1^{n-1}$. For (c), (25) implies

$$m_H(M) - m_H(\det M) \in \mathbb{Z}_{\geq 0}\lvert G_H\rvert,$$

which gives the result since $\Delta_M = J_{\det M}$. Now (d) follows from (c) since in this case (25) has at most one non-trivial summand, so $m_H(M) < |G_H|$, and (e) is a special case of (d). Finally, (f) follows from (b) and (d) since $m_H(\chi) = m_{H,1}(\chi) < |G_H|$ and $\Delta_V = J_V$. $\square$

4. Harmonics and semi-invariant bases

Our first goal in this section is to define the well-known $G$-harmonic polynomials using the constructions in Section 2. We prove their basic properties by constructing a $G$-invariant Hermitian form, and then use the harmonics to construct explicit bases for semi-invariant differential forms. The main results of this section are Theorem 4.5 and Theorem 4.10. $F$ remains a field of characteristic $0$, $V$ is an $n$-dimensional $F$-vector space, $G \leq \text{GL}(V)$ is a pseudo-reflection group, and $M$ is an $r$-dimensional $G$-module.

Definition 4.1. The spaces of $G$-harmonic elements of $S(V^*)$ and $S(V)$ are, respectively,

$$\mathcal{H}(V^*) := \{ f \in S(V^*) : \langle s, f \rangle = 0 \text{ for all } s \in \mathcal{I} \},$$

$$\mathcal{H}(V) := \{ s \in S(V) : \langle s, f \rangle = 0 \text{ for all } f \in \mathcal{I}^* \}.$$

That is, the harmonics are the orthogonal complements of the coinvariant ideals $\mathcal{I}$ and $\mathcal{I}^*$ with respect to the natural perfect pairing $\langle -, - \rangle$ from (6). The harmonics have the following basic properties.

Lemma 4.2. We have

(26) $\mathcal{H}(V^*) = \{ f \in S(V^*) : \partial_{x_j} f = 0, j = 1, \ldots, n \},$

(27) $S(V^*) = \mathcal{H}(V^*) \oplus \mathcal{I}^*,$

(28) $\mathcal{H}(V^*) \cong S(V^*)/\mathcal{I}^*$ as graded $G$-modules,

(29) $S(V^*) = S(V^*)^G \otimes \mathcal{H}(V^*),$

and similarly with $\mathcal{H}(V)$. 

Proof. (26) follows from the fact that $\mathcal{I}$ is the ideal generated by $z_1, \ldots, z_n$. (29) follows from (28) and Chevalley’s result in Remark 2.14, and (28) follows from (27).

As for (27), first suppose $F \subset \mathbb{C}$ is closed under complex conjugation. By Lemma 2.11, we have $\tau(S(V)^G) = S(V^*)^G$ and $\tau(\mathcal{I}) = \mathcal{I}^*$. Thus $\langle \mathcal{I}, f \rangle = \langle \tau(\mathcal{I}), f \rangle = \langle \mathcal{I}^*, f \rangle$, so $\mathcal{H}(V^*)$ is the orthogonal complement of $\mathcal{I}^*$ under a positive-definite Hermitian form. Now (27) follows from Lemma 2.10 in this case.

For the general case, by Corollary 2.18, $G$ is defined over a subfield $K$ of $F$ which may be identified with a subfield of $\mathbb{C}$ closed under complex conjugation. Let $v_1, \ldots, v_n$ be a basis for $V$ over $F$ such that the matrix of each $\sigma \in G$ is in $\text{GL}_n(K)$. Set $V_K := \bigoplus_{i=1}^n K v_i$. Then $S(V)^* = \text{Span}_F S(V_K)^* \cong F \otimes_K S(V_K)^*$ as $F$-algebras and graded $G$-modules. Let $R_G := \frac{1}{|G|} \sum_{\sigma \in G} \sigma$ be the projection onto $G$-invariants. Since $R_G$ is $F$-linear, $S(V^*)^G = R_G S(V^*) \cong R_G(F \otimes_K S(V_K)^*)$ $= F \otimes_K R_G S(V_K)^* = F \otimes_K S(V_K^*)^G$.

This implies $\mathcal{I}^* \cong F \otimes_K \mathcal{I}_K^*$, and hence $\mathcal{H}(V^*) \cong F \otimes_K \mathcal{H}(V_K^*)$. □

Remark 4.3. By Lemma 2.11 when $F \subset \mathbb{C}$ is closed under complex conjugation, we have $\tau((S(V) \otimes M)^G) = (S(V^*) \otimes M^*)^G$, so we may take $\tau(\tilde{\omega}_i M) = \omega_i^M$, $\tau(\tilde{J}_M) = J_M$, and $\tau(\tilde{\Delta}_M) = \Delta_M$.

Recall the $G$-equivariant $F$-algebra homomorphism $S(V) \otimes \wedge M^* \to \text{End}_F(S(V^*) \otimes \wedge M^*)$ $s \otimes \mu \mapsto \partial_s \otimes \epsilon_\mu$.

from Lemma 2.7.

Definition 4.4. For $i = 1, \ldots, r$, let $d_i^*: S(V^*) \otimes \wedge M^* \to S(V^*) \otimes \wedge M^*$ denote the action of the basic derivation $\tilde{\omega}_i M^* \in (S(V) \otimes \wedge M^*)^G$ from Section 3.

Theorem 4.5. Suppose $J_M \supset \Delta_\chi$. Then the $2^r$ elements

$$\{d_{i_1}^* \cdots d_{i_k}^* \Delta_\chi \mid 1 \leq i_1 < \cdots < i_k \leq r\}$$

form a basis of $(\mathcal{H}(V^*) \otimes \wedge M^*)^\chi$, and hence descend to a basis of $(S(V^*)/\mathcal{I}^* \otimes \wedge M^*)^\chi$. 
Proof. We suppose throughout that \( F \) is a subfield of \( \mathbb{C} \) closed under complex conjugation. The general case follows by arguing as in the proof of Lemma \ref{lem:complex-conjugation} using extension of scalars.

We have \( \Delta_\chi \in \mathcal{H}(V^*)^\chi \) since \( \Delta_\chi \) is the lowest-degree element of \( S(V^*)^\chi \) and so \( \partial_{\zeta_j} \Delta_\chi = 0 \) for \( j = 1, \ldots, n \). Since \( \tilde{\omega}_M^* \) is \( G \)-invariant, \( d_i^* \) preserves the \( \chi \)-isotypic component. Since \( \partial_s \) preserves \( \mathcal{H}(V^*) \), \( \partial_s \otimes \epsilon_\mu \) preserves \( \mathcal{H}(V^*) \otimes \wedge M^* \), so \( d_i^* \) preserves \( \mathcal{H}(V^*) \otimes \wedge M^* \) and \( d_1^* \cdots d_k^* \Delta_\chi \in (\mathcal{H}(V^*) \otimes \wedge M^*)^\chi \).

We now prove linear independence. Suppose
\[
\sum \zeta_{i_1, \ldots, i_k} d_{i_1}^* \cdots d_{i_k}^* \Delta_\chi = 0
\]
for scalars \( \zeta_{i_1, \ldots, i_k} \in F \). By homogeneity, we may suppose the subsets are each of size \( k \). For a fixed choice of \( \{i_1, \ldots, i_k\} \), let \( \{j_1, \ldots, j_{r-k}\} \) be its complement in \( [r] \). Since \( \tilde{\omega}_M^* \tilde{\omega}_M^* = -\tilde{\omega}_M^* \tilde{\omega}_M^* \), we have \( d_i^* d_j^* = -d_j^* d_i^* \). Applying \( d_{j_1}^* \cdots d_{j_{r-k}}^* \) to (30) gives
\[
\pm \zeta_{i_1, \ldots, i_k} d_{i_1}^* \cdots d_{i_k}^* \Delta_\chi = 0.
\]
We have \( \tilde{\omega}_M^* \cdots \tilde{\omega}_M^* = \tilde{\lambda}_M^* \eta \) where \( 0 \neq \eta \in \wedge^r M^* \), so \( d_1^* \cdots d_r^* = \partial_{\tilde{\lambda}_M^*} \otimes \epsilon_\eta \). Thus (31) becomes
\[
\zeta_{i_1, \ldots, i_k} \partial_{\tilde{\lambda}_M^*} \Delta_\chi \eta = 0.
\]
Since \( \Delta_\chi / J_{M^*} \in S(V^*) \) by assumption, \( \Delta_\chi / J_{M^*} = \tau^{-1}(\Delta_\chi / J_{M^*}) \in S(V) \). Applying \( \partial_{\Delta_\chi / J_{M^*}} \) to (32) and canceling \( \eta \) gives
\[
0 = \zeta_{i_1, \ldots, i_k} \partial_{\Delta_\chi} \Delta_\chi = \zeta_{i_1, \ldots, i_k} (\tilde{\Delta}_\chi, \Delta_\chi)
= \zeta_{i_1, \ldots, i_k} (\tau(\tilde{\Delta}_\chi), \Delta_\chi) = \zeta_{i_1, \ldots, i_k} (\Delta_\chi, \Delta_\chi).
\]
Since \( (\Delta_\chi, \Delta_\chi) > 0 \), we find \( \zeta_{i_1, \ldots, i_k} = 0 \).

From Chevalley’s result in Remark \[\ref{rem:chevalley}\] \( \mathcal{H}(V^*) \) carries the regular representation \( F[G] \) of \( G \). It is well-known that \( F[G] \otimes M \) is isomorphic to \( \dim M \) copies of \( F[G] \) as a \( G \)-module, since if \( M' \) is \( M \) with trivial \( G \)-action,

\[
F[G] \otimes M \rightarrow F[G] \otimes M'
\]
\[
g \otimes m \mapsto g \otimes g^{-1} m
\]
is a \( G \)-module isomorphism. Thus \( \dim(\mathcal{H}(V^*) \otimes \wedge M^*)^\chi = \dim \wedge M^* = 2^r \), so the linearly independent set is indeed a basis.

\[\square\]

Remark 4.6. The \( d_i^* \) essentially appear in \[\ref{ref:chevalley} \ §6, Prop. 18\] in slightly less generality. The preceding proof exploits their exterior algebra structure in a fundamental way. The condition \( J_{M^*} \mid \Delta_\chi \) is a generalization of the condition from \[\ref{ref:chevalley}\] that \( \chi \) is \textit{wholly-nontrivial}, meaning that \( \Delta_{\det V^*} \mid \Delta_\chi \).
Corollary 4.7. Suppose $J_M \mid \Delta_\chi$. Then

$$\text{Hilb}((\mathcal{H}(V^*) \otimes \wedge M^*)^\chi; q, z) = q^{\deg \Delta_\chi} \prod_{i=1}^{r}(1 + zq^{-e_i^M}).$$

Proof. The $d_i^*$ alter bidegree by $(-e_i^M, 1)$. \hfill \Box

We additionally have the following result. Alternatively, it is a straightforward consequence of Orlik–Solomon’s generalization [14, Thm. 3.1] of Solomon’s description [22] of $(S(V^*) \otimes \wedge V^*)^G$ as the exterior algebra over $S(V^*)^G$ generated by $d_f$. By Corollary 3.8(d), $J_M = \Delta_M$ up to a non-zero scalar whenever the pseudo-reflections of $G$ act on $M$ as pseudo-reflections or the identity.

Corollary 4.8. Suppose $J_M = \Delta_M$ up to a non-zero scalar. Then

$$(33) \quad \text{Hilb}(\text{Hom}_G(\wedge^k M, S(V^*)/I^*); q) = \sigma_k(q^{e_1^M}, \ldots, q^{e_r^M}),$$

where $\sigma_k$ denotes an elementary symmetric polynomial of degree $k$. In particular, if $\wedge^k M$ is absolutely irreducible, the right-hand side of (33) is the generating function for the degrees in which $\wedge^k M$ appears in the coinvariant algebra $S(V^*)/I^*$.

Proof. We have

$$(S(V^*)/I^* \otimes \wedge M^*)^G \cong \text{Hom}_G(\wedge^k M, S(V^*)/I^*),$$

so (33) is equivalent to

$$\text{Hilb}((S(V^*)/I^* \otimes \wedge M^*)^G; q, z) = \prod_{i=1}^{r}(1 + zq^{e_i^M}).$$

Since $\deg \Delta_M = \deg J_M = \sum_{i=1}^{r} e_i^M$, Corollary 4.7 gives

$$\text{Hilb}((S(V^*)/I^* \otimes \wedge M)^{\text{det} M}; q, z) = \prod_{i=1}^{r}(z + q^{e_i^M}) = z^r \prod_{i=1}^{r}(1 + z^{-1}q^{e_i^M}).$$

It thus suffices to show

$$\text{Hilb}((S(V^*)/I^* \otimes \wedge M)^{\text{det} M}; q, z) = z^r \text{Hilb}((S(V^*)/I^* \otimes \wedge M^*)^G; q, z^{-1}).$$

This follows from the well-known fact that for each $0 \leq k \leq r$, there is a (non-canonical) isomorphism of $\text{GL}(V)$-modules

$$\wedge^k M^* \cong \text{det}^{-1}_M \otimes \wedge^{r-k} M.$$  \hfill \Box
Remark 4.9. By Benard’s Theorem 2.17, every irreducible representation of a pseudo-reflection group \( G \) is absolutely irreducible over the character field of \( G \). Steinberg noted that if \( G \) is generated by \( n = \dim(V) \) pseudo-reflections and \( V \) is irreducible, then \( \wedge^k V \) is (absolutely) irreducible for all \( 0 \leq k \leq n \) (see [2, Ch. V, §2, Exercise 3(d)], [11, p. 250, Thm. A]).

We now restate and prove Theorem 4.10 and Corollary 4.11 from the Introduction.

**Theorem 4.10.** Suppose \( J_{M^*} \mid \Delta_\chi \). Then either of the sets

\[
\{ f_1^{a_1} \cdots f_n^{a_n} d_1^{i_1} \cdots d_k^{i_k} \Delta_\chi : 1 \leq i_1 < \cdots < i_k \leq r, a_j \in \mathbb{Z}_{\geq 0} \}
\]

or

\[
\{ d_1^{i_1} \cdots d_k^{i_k} f_1^{a_1} \cdots f_n^{a_n} \Delta_\chi : 1 \leq i_1 < \cdots < i_k \leq r, a_j \in \mathbb{Z}_{\geq 0} \}
\]

form bases for \( (S(V^*) \otimes \wedge M^*)^G \).

**Proof.** The fact that the first set is a basis follows from Theorem 4.5 and (29). For the second set, it suffices to verify linear independence. To do so, we sketch how to modify the proof of Theorem 4.5. Suppose \( F \) is a subfield of \( \mathbb{C} \) closed under complex conjugation, pick a homogeneous, orthogonal basis \( \{ \eta_\alpha \} \) of \( S(V^*)^G = S(V^*)^G \Delta_\chi \) with respect to the Hermitian form, and let \( g_\alpha \Delta_\chi = \eta_\alpha \) where \( \{ g_\alpha \} \) is a basis for \( S(V^*)^G \). Linear independence is unaffected if we replace \( \{ f_1^{a_1} \cdots f_n^{a_n} \} \) with \( \{ g_\alpha \} \). If

\[
0 = \sum_{\alpha, I} \zeta_{\alpha, I} d_1^{a_1} \cdots d_k^{a_k} g_\alpha \Delta_\chi,
\]

we may apply \( \partial_\Delta_\chi / J_M^* \partial_{\tau^{-1}(g_\beta)} d_1^{a_1} \cdots d_k^{a_k} \) to get

\[
0 = \sum_{\alpha} \pm \zeta_{\alpha, I} \partial_\Delta_\chi / J_M^* \partial_{\tau^{-1}(g_\beta)} g_\alpha \Delta_\chi \eta
\]

\[
= \sum_{\alpha} \pm \zeta_{\alpha, I} \langle \tau^{-1}(g_\beta \Delta_\chi), g_\alpha \Delta_\chi \rangle \eta
\]

\[
= \sum_{\alpha} \pm \zeta_{\alpha, I} \langle \overline{\eta_\beta}, g_\alpha \Delta_\chi \rangle \eta = (\overline{\eta_\beta}, \overline{\eta_\beta}) \zeta_{\beta, I} \eta.
\]

Now use \( (\overline{\eta_\beta}, \overline{\eta_\beta}) > 0 \).

**Corollary 4.11.** Suppose \( J_{M^*} \mid \Delta_\chi \). Then

\[
\mathrm{Hilb}((S(V^*) \otimes \wedge M^*)^G; q, z) = q^\deg \Delta_\chi \prod_{i=1}^r (1 + z q^{-e_i M^*}) \prod_{i=1}^n (1 - q^{d_i}).
\]
Remark 4.12. Corollary 4.11 gives Hilbert series for the invariants and alternants of the four algebras in (5). Table 1 gives a summary. The formulas may also be derived from Shepler’s results in [21]. Reiner–Shepler–Sommers [17] have recently given a related formula for \( \text{Hilb}(S(V^*) \otimes \wedge V^* \otimes \wedge V)^G; q, t, z) \) when \( G \) is a “coincidental” complex reflection group.

\[
\begin{array}{|c|c|c|c|}
\hline
& \text{Hilb}(\mathcal{A}^G; q, z) & \text{Hilb}(\mathcal{A}^{\text{det}V^*}; q, z) & \text{Hilb}(\mathcal{A}^{\text{det}V}; q, z) \\
\hline
S(V^*) \otimes \wedge V^* & \prod_{i=1}^n \frac{1+q^i z}{1-q^i} & \prod_{i=1}^n \frac{z+q^i t^i}{1-q^i} & q \sum_{i=1}^n e_i - e_i^* \prod_{i=1}^n \frac{z+q^i t^i}{1-q^i} \\
\hline
S(V) \otimes \wedge V & \prod_{i=1}^n \frac{1+q^i z}{1-q^i} & q \sum_{i=1}^n e_i - e_i^* \prod_{i=1}^n \frac{z+q^i t^i}{1-q^i} & \prod_{i=1}^n \frac{z+q^i t^i}{1-q^i} \\
\hline
S(V) \otimes \wedge V^* & \prod_{i=1}^n \frac{1+q^i z}{1-q^i} & \prod_{i=1}^n \frac{z+q^i t^i}{1-q^i} & - \\
\hline
S(V^*) \otimes \wedge V & \prod_{i=1}^n \frac{1+q^i z}{1-q^i} & - & \prod_{i=1}^n \frac{z+q^i t^i}{1-q^i} \\
\hline
\end{array}
\]

Table 1. Product formulas for invariants and alternants for the algebras \( \mathcal{A} \) listed in (5) when \( M = V \).

5. Coinvariant harmonics and semi-invariants

We next define the \( G \)-harmonic polynomial differential forms, which naturally involve operators \( \delta^*_i \) similar to the operators \( d^*_i \) of the preceding section. We prove the basic properties of the \( G \)-harmonics using Hermitian forms and we show that in fact the \( d^*_i \)’s preserve the \( G \)-harmonics. Finally, we prove our main result, Theorem 5.7, giving an explicit basis for the semi-invariant differential harmonics and coinvariants. We continue the notation of Section 4.

Definition 5.1. The coinvariant ideal of \( S(V^*) \otimes \wedge M^* \) is the ideal \( \mathcal{J}_M \) generated by homogeneous \( G \)-invariants of positive degree. The coinvariant ideal \( \mathcal{J}_M \) of \( S(V) \otimes \wedge M \) is defined analogously. When \( M = V \), we write \( \mathcal{J}^* := \mathcal{J}_V^* \) and \( \mathcal{J} := \mathcal{J}_V \) in analogy with the classical coinvariant ideals \( \mathcal{I}^* \) and \( \mathcal{I} \).

By Solomon’s theorem [22], \( \mathcal{J}^* = (f_1, \ldots, f_n, df_1, \ldots, df_n) \) is the ideal generated by \( \{f_1, \ldots, f_n, df_1, \ldots, df_n\} \) and \( \mathcal{J} = (z_1, \ldots, z_n, dz_1, \ldots, dz_n) \). More generally, Orlik-Solomon [14] show that, if \( J_M = \Delta_M \) up to a non-zero scalar, we have

\[
\begin{align*}
\mathcal{J}_M &= (f_1, \ldots, f_n, \omega_1^M, \ldots, \omega_r^M), \\
\mathcal{J}_M &= (z_1, \ldots, z_n, \tilde{\omega}_1^M, \ldots, \tilde{\omega}_r^M).
\end{align*}
\]
Definition 5.2. The spaces of $G$-harmonic elements of $S(V^*) \otimes \Lambda M^*$ and $S(V) \otimes \Lambda M$ are, respectively,

$\mathcal{H}(S(V^*) \otimes \Lambda M^*) := \{\omega \in S(V^*) \otimes \Lambda M^* : \langle \xi, \omega \rangle = 0 \text{ for all } \xi \in J_M\}$

$\mathcal{H}(S(V) \otimes \Lambda M) := \{\xi \in S(V) \otimes \Lambda M : \langle \xi, \omega \rangle = 0 \text{ for all } \omega \in J_M^*\}.$

That is, the harmonics are the orthogonal complements of the coinvariant ideals $J_M$ and $J_M^*$ with respect to the natural perfect pairing $\langle -, - \rangle$.

Recall the $G$-equivariant $F$-algebra homomorphism

$S(V) \otimes \Lambda M \to \text{End}_F(S(V^*) \otimes \Lambda M^*)$

$s \otimes m \mapsto \partial_s \otimes t_m.$

from Lemma 2.7.

Definition 5.3. For $i = 1, \ldots, r$, let

$\delta_i^*: S(V^*) \otimes \Lambda M^* \to S(V^*) \otimes \Lambda M^*$

denote the action of the basic derivation $\tilde{\omega}_i^M \in (S(V) \otimes \Lambda M)^G$.

Lemma 5.4. Suppose $J_M$ is generated by $j_1, \ldots, j_p$. Then

(34) $\mathcal{H}(S(V^*) \otimes \Lambda M^*) = \{\omega \in S(V^*) \otimes \Lambda M^* : j_i \cdot f = 0, i \in [p]\}$

(35) $S(V^*) \otimes \Lambda M^* = \mathcal{H}(S(V^*) \otimes \Lambda M^*) \oplus J_M^*$

(36) $\mathcal{H}(S(V^*) \otimes \Lambda M^*) \cong (S(V^*) \otimes \Lambda M^*) / J_M^*$ as bigraded $G$-modules

(37) $S(V^*) \otimes \Lambda M^* = (S(V^*) \otimes \Lambda M^*)^G \mathcal{H}(S(V^*) \otimes \Lambda M^*),$

and likewise with $\mathcal{H}(S(V) \otimes \Lambda M)$.

Proof. When $F$ is a subfield of $\mathbb{C}$ closed under complex conjugation, $\tau((S(V) \otimes \Lambda M)^G) = (S(V^*) \otimes \Lambda M^*)^G$ and $\tau(J_M) = J_M^*$. The first three thus follow exactly as in the proof of Lemma 4.2. Note that (37) merely asserts that the multiplication map

$$(S(V^*) \otimes \Lambda M^*)^G \times \mathcal{H}(S(V^*) \otimes \Lambda M^*) \to S(V^*) \otimes \Lambda M^*$$

is surjective; it is not in general injective. Surjectivity may be proven by induction using (35). \qed

Lemma 5.5. We have

$$d_i^* \delta_j^* + \delta_j^* d_i^* = \partial_{L_{i,j}},$$

with $L_{i,j} \in S(V)^G$. 

Proof. If $F$ is a subfield of $\mathbb{C}$ closed under complex conjugation, we may take
\[
\tilde{\omega}_i^M = \sum_j \tilde{J}_{ij}^M \otimes m_j \in (S(V) \otimes M)^G,
\]
\[
\delta_i^* = \sum_j \partial_{\tilde{j}_{ij}^M} \otimes \iota_{m_j},
\]
\[
\tilde{\omega}_i^{M^*} = \sum_j \tilde{J}_{ij}^{M^*} \otimes \tau(m_j) \in (S(V) \otimes M^*)^G,
\]
\[
d_i^* = \sum_j \partial_{\tilde{j}_{ij}^{M^*}} \otimes \epsilon_{\tau(m_j)}.
\]
We may transfer the Hermitian form on $M^*$ to a Hermitian form on $M$ defined by $(m_1, m_2) := (\tau(m_2), \tau(m_1))$. Note that $\tau(m_k)(m_\ell) = \langle m_\ell, \tau(m_k) \rangle$. Using (16), we calculate
\[
d_i^* \delta_j^* + \delta_j^* d_i^* = \sum_{k, \ell} (\partial_{\tilde{j}_{ik}^M} \partial_{\tilde{j}_{j\ell}^M}) (\epsilon_{\tau(m_k)} \epsilon_{m_\ell} + \epsilon_{m_\ell} \epsilon_{\tau(m_k)})
\]
\[
= \sum_{k, \ell} (\partial_{\tilde{j}_{ik}^M} \partial_{\tilde{j}_{j\ell}^M}) (m_k, m_\ell) = \partial_{L_{ij}}
\]
where
\[
L_{ij} := \sum_{k, \ell} (m_k, m_\ell) \tilde{J}_{ik}^{M^*} \tilde{J}_{j\ell}^M.
\]
Since $d_i^* \delta_j^* + \delta_j^* d_i^*$ is independent of the choice of basis $m_1, \ldots, m_r$, the same is true of $L_{ij}$. We have
\[
\tilde{\omega}_i^M = g\tilde{\omega}_i^M = \sum_j g \tilde{J}_{ij}^M \otimes gm_j
\]
and similarly for $\tilde{\omega}_i^{M^*}$, so
\[
L_{ij} = \sum_{k, \ell} (gm_k, gm_\ell) g \tilde{J}_{ik}^M * g \tilde{J}_{j\ell}^M
\]
\[
= g \sum_{k, \ell} (m_k, m_\ell) \tilde{J}_{ik}^{M^*} \tilde{J}_{j\ell}^M = gL_{ij}.
\]
The general case follows by extending scalars as in the proof of Lemma 4.2. \qed

**Corollary 5.6.** If $M^G = 0$, then $L_{ij}$ (as in Lemma 5.5) is homogeneous of positive degree and
\[
d_i^* \in \text{End}_F (\mathcal{H}(S(V^*) \otimes \wedge M^*)).
\]
Proof. Since $M^G = 0$, we have $(F \otimes M)^G = 0$, so $\deg_{S(V^*)} \omega^M_i \geq 1$. By (38), $\deg L_{ij} \geq 2$, so $L_{ij} \in \mathcal{I}$. Suppose $\omega \in \mathcal{H}(S(V^*) \otimes M^*)$, or equivalently $\delta_j^\ast \omega = 0$ and $\partial_{z_j} \omega = 0$. We must show $d_i^\ast \omega \in \mathcal{H}(S(V^*) \otimes M^*)$. Since $\partial_{z_j}$ commutes with $d_i^\ast$, we have $\partial_{z_j} (d_i^\ast \omega) = 0$. By Lemma 5.5, we have $\delta_j^\ast (d_i^\ast \omega) = -d_i^\ast (\delta_j^\ast \omega) + \partial L_{ij} \omega = 0 + 0 = 0$ since $L_{ij} \in \mathcal{I}$. □

We now restate and prove our main result from the introduction.

Theorem 5.7. Suppose $J \mid \Delta \chi$ and $M^G = 0$. Then the $2^r$ elements

$$\{d_{i_1}^\ast \cdots d_{i_k}^\ast \Delta \chi \mid 1 \leq i_1 < \cdots < i_k \leq r\}$$

form a basis of $\mathcal{H}(S(V^*) \otimes \wedge M^*)^\chi$, and their images descend to a basis of $(S(V^*) \otimes \wedge M^* / J^\ast M)^\chi$.

Proof. Since $\mathcal{I} \subset \mathcal{J}_{M^*}$,

$$\mathcal{H}(V^*) \otimes \wedge M^* \supset \mathcal{H}(S(V^*) \otimes \wedge M^*)$$

We have $\Delta_{M^*} \in \mathcal{H}(S(V^*) \otimes \wedge M^*)$ since $\delta_i^\ast \Delta_{M^*} = 0$ trivially. By Corollary 5.6 the proposed elements belong to $\mathcal{H}(S(V^*) \otimes \wedge M^*)$. The result hence follows from Theorem 4.5. We may pass to the quotient by (35). □

Corollary 5.8. Suppose $J_{M^*} \mid \Delta \chi$ and $M^G = 0$. Then

$$\text{Hilb}((S(V^*) \otimes \wedge M^* / J_{M^*})^\chi; q, z) = q^{\deg \Delta \chi} \prod_{i=1}^r (1 + z q^{-e_i M^*}).$$

Remark 5.9. By (39), Theorem 5.7 gives the surprising result that

$$\mathcal{H}(V^*) \otimes \wedge M^*^\chi = \mathcal{H}(S(V^*) \otimes \wedge M^*)^\chi.$$

6. The condition $J_M = \Delta_M$

We retain the notation of the preceding sections and discuss the condition $J_M = \Delta_M$ appearing in some of our results. In type $A$, with precisely one exception, all irreducible examples are either one-dimensional or the defining representation.

Proposition 6.1. Suppose $G = S_n$ consists of $n \times n$ permutation matrices and let $M$ be an irreducible $S_n$-module. Then $J_M = \Delta_M$ up to a non-zero scalar if and only if $M$ is the trivial representation, the sign representation, the standard representation, or, when $n = 4$, the unique degree 2 representation.
Proof (Sketch). The irreducible \( \mathfrak{S}_n \)-representations are indexed by integer partitions \( \lambda \vdash n \). The number of exponents of the irreducible indexed by \( \lambda \) is the degree \( f^\lambda \). Since there are sub-exponentially many \( \lambda \vdash n \) and \( \sum_{\lambda \vdash n} (f^\lambda)^2 = n! \), one heuristically expects \( f^\lambda \) to grow super-exponentially in \( n \), making \( J_M = \Delta_M \) quite rare. One may for instance apply the arguments in [25, p. 15] to make this intuition rigorous, though we omit the details. \( \square \)

If \( G \) is a dihedral group, then every irreducible representation \( M \) takes reflections to reflections or the identity, so by Corollary 3.8(4), \( J_M = \Delta_M \) up to a non-zero scalar. We now give somewhat less trivial examples.

Example 6.2. Let \( \mathfrak{B}_n \subset O(n) \) be the Weyl group of type \( B_n \) realized as \( n \times n \) signed permutation matrices whose non-zero entries are \( \pm 1 \). Let \( \mathfrak{Z}_n \subset \mathfrak{B}_n \) be the group of diagonal matrices with diagonal entries \( \pm 1 \). Note that \( \mathfrak{S}_n = \mathfrak{B}_n/\mathfrak{Z}_n \). Thus we may consider the standard representation \( M \) of \( \mathfrak{S}_n \) as an irreducible \( \mathfrak{B}_n \)-representation. The exponents of \( M \) are the degrees in which \( M \) appears in the coinvariant algebra \( \mathbb{R}[\mathfrak{B}_n]/\mathcal{I} \), or equivalently in the harmonics \( \mathcal{H}(\mathfrak{B}_n) \). Since \( M \) is \( \mathfrak{Z}_n \)-invariant, these occur in \( \mathcal{H}(\mathfrak{B}_n)^{\mathfrak{Z}_n} \). By Chevalley’s result applied to \( \mathfrak{B}_n \) (see Remark 2.14), multiplication gives an isomorphism

\[
\mathbb{R}[x_1, \ldots, x_n]^{\mathfrak{B}_n} \otimes \mathcal{H}(\mathfrak{B}_n) \sim \mathbb{R}[x_1, \ldots, x_n].
\]

Taking \( \mathfrak{Z}_n \)-invariants and simplifying,

\[
\mathbb{R}[x_1^2, \ldots, x_n^2]^{\mathfrak{B}_n} \otimes \mathcal{H}(\mathfrak{B}_n)^{\mathfrak{Z}_n} \sim \mathbb{R}[x_1^2, \ldots, x_n^2].
\]

Again applying Chevalley’s result to \( \mathfrak{S}_n \), it follows that \( \mathcal{H}(\mathfrak{B}_n)^{\mathfrak{Z}_n} \cong \mathcal{H}(\mathfrak{S}_n) \) where \( x_i^2 \mapsto x_i \). In particular, \( J_M = \Delta_M \) up to a non-zero multiple, and both have degree \( 2 + 4 + \cdots + 2(n-1) = 2\binom{n}{2} = n(n-1) \).

Example 6.3. Let \( \mathfrak{D}_n \subset \mathfrak{B}_n \subset O(n) \) be the Weyl group of type \( D_n \) consisting of signed permutation matrices with evenly many negative entries. Let \( \mathfrak{Z}_n' = \mathfrak{D}_n \cap \mathfrak{Z}_n \). Note that \( \mathfrak{B}_n/\mathfrak{Z}_n' \cong \mathfrak{S}_n \). It is easily checked that

\[
\mathbb{R}[x_1, \ldots, x_n]^{\mathfrak{Z}_n'} = \mathbb{R}[x_1^2, \ldots, x_n^2] \oplus \mathbb{R}[x_1^2, \ldots, x_n^2] x_1 \cdots x_n
\]

and

\[
\mathbb{R}[x_1, \ldots, x_n]^{\mathfrak{D}_n} = \mathbb{R}[x_1^2, \ldots, x_n^2]^{\mathfrak{B}_n} \oplus \mathbb{R}[x_1^2, \ldots, x_n^2]^{\mathfrak{B}_n} x_1 \cdots x_n.
\]

Using the same argument as in Example 6.2, the standard representation \( M \) of \( \mathfrak{S}_n \) yields an irreducible representation of \( \mathfrak{D}_n \) with \( J_M = \Delta_M \) up to a non-zero multiple, and both have degree \( n(n-1) \).
The Weyl group $\mathfrak{F}_4$ of type $F_4$ is isomorphic with $\mathfrak{S}_3 \rtimes \mathfrak{D}_4$. One may then view the standard representation $M$ of $\mathfrak{S}_3$ as an $\mathfrak{F}_4$-module and check that the exponents of $M$ are 4, 8 and $J_M = \Delta_M$ up to a non-zero scalar. The hypotheses of Corollary 3.8(d) are satisfied in this case. Further examples exist. For example, Orlik–Solomon observe that Galois conjugates of $V$ and $V^*$ have this property. A complete classification is not known.

7. Acknowledgements

The first named author would like to thank John Mahacek, Vic Reiner, Brendon Rhoades, Bruce Sagan, Anne Shepler, and Mike Zabrocki for helpful discussions on this and related projects, and for generously sharing their preprints. The second named author thanks Adriano Gar- sia for pointing out that our result for the signum representation of $S_n$ can be derived from the Zabrocki conjecture.

References

[1] Benard, M. Schur indices and splitting fields of the unitary reflection groups. J. Algebra 38, 2 (1976), 318–342.
[2] Bourbaki, N. Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines. Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968.
[3] Broué, M. Introduction to complex reflection groups and their braid groups, vol. 1988 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2010.
[4] Chevalley, C. Invariants of finite groups generated by reflections. Amer. J. Math. 77 (1955), 778–782.
[5] Clark, A., and Ewing, J. The realization of polynomial algebras as cohomology rings. Pacific J. Math. 50 (1974), 425–434.
[6] Gutkin, E. A. Matrices that are connected with groups generated by reflections. Funkcional. Anal. i Priložen. 7, 2 (1973), 81–82.
[7] Haglund, J., Remmel, J. B., and Wilson, A. T. The Delta conjecture. Trans. Amer. Math. Soc. 370, 6 (2018), 4029–4057.
[8] Haglund, J., Rhoades, B., and Shimozono, M. Ordered set partitions, generalized coinvariant algebras, and the Delta conjecture. Adv. Math. 329 (2018), 851–915.
[9] Haiman, M. Vanishing theorems and character formulas for the Hilbert scheme of points in the plane. Invent. Math. 149, 2 (2002), 371–407.
[10] Haiman, M. D. Conjectures on the quotient ring by diagonal invariants. J. Algebraic Combin. 3, 1 (1994), 17–76.
[11] Kane, R. Reflection groups and invariant theory, vol. 5 of CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer-Verlag, New York, 2001.
[12] Lee, J. M. Introduction to smooth manifolds, vol. 218 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2003.
[13] Orellana, R., and Zabrocki, M. A combinatorial model for the decomposition of multivariate polynomials rings as an $S_n$-module, 2019. arXiv:1906.01125.

[14] Orlik, P., and Solomon, L. Unitary reflection groups and cohomology. *Invent. Math.* 59, 1 (1980), 77–94.

[15] Orlik, P., and Terao, H. *Arrangements of hyperplanes*, vol. 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[16] Reiner, V., and Shepler, A. V. Invariant derivations and differential forms for reflection groups. *Proc. Lond. Math. Soc. (3)* 119, 2 (2019), 329–357.

[17] Reiner, V., Shepler, A. V., and Sommers, E. Invariant theory for coincidental complex reflection groups, 2019. arXiv:1908.02663.

[18] Rhoades, B., and Wilson, A. T. Vandermondes in superspace, 2019. arXiv:1906.03315.

[19] Shephard, G. C., and Todd, J. A. Finite unitary reflection groups. *Canad. J. Math.* 6 (1954), 274–304.

[20] Shepler, A. V. Semi-invariants of finite reflection groups. *J. Algebra* 220, 1 (1999), 314–326.

[21] Shepler, A. V. Generalized exponents and forms. *J. Algebraic Combin.* 22, 1 (2005), 115–132.

[22] Solomon, L. Invariants of finite reflection groups. *Nagoya Math. J.* 22 (1963), 57–64.

[23] Stanley, R. P. Relative invariants of finite groups generated by pseudoreflections. *J. Algebra* 49, 1 (1977), 134–148.

[24] Steinberg, R. Invariants of finite reflection groups. *Canadian J. Math.* 12 (1960), 616–618.

[25] Swanson, J. P. On the existence of tableaux with given modular major index. *Algebr. Comb.* 1, 1 (2018), 3–21.

[26] Swanson, J. P. Alternating super-polynomials and super-coinvariants of finite reflection groups, 2019. arXiv:1908.00196.

[27] Swanson, J. P., and Wallach, N. R. Harmonic differential forms for pseudo-reflection groups II. Bi-degree bounds, 2019. In progress.

[28] Wallach, N. R. Some implications of a conjecture of Zabrocki to the action of $S_n$ on polynomial differential forms, 2019. arXiv:1906.11787.

[29] Zabrocki, M. A module for the Delta conjecture, 2019. arXiv:1902.08966.