Zero-point energy of a free Brownian particle

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Abstract. We analyze an averaged energy $E$ of a free quantum Brownian particle coupled to an environment of absolute zero temperature (quantum vacuum) and study its dependence on the coupling strength $c$ between the particle and its surroundings. Impact of selected dissipation mechanisms is considered. In the weak coupling limit the energy tends to zero as $E \sim c \ln(1/c)$. In the strong coupling limit it diverges to infinity as $E \sim \sqrt{c}$. We pose a question whether for a fixed $c$ and an arbitrary dissipation mechanism there exists a non-zero lowest bound for $E$. The answer is: no.
1. Introduction

Processes and phenomena taking place at high temperature are well described by classical physics. In contrast, at low temperature quantum effects start to play a role in which fluctuations are an inherent part. The origin of quantum fluctuations is twofold: (i) the Heisenberg uncertainty principle and (ii) an environment of temperature $T$ being a source of quantum thermal noise. However, even at absolute zero temperature $T = 0$, there are still zero-point fluctuations of the environment which may cause observable effects. The existence of vacuum fluctuations is one of the most important predictions of modern quantum field theory. One can quote two celebrated examples for evidence of it: the Lamb shift [1, 2] and the Casimir effects [3, 4, 5]. Recent examples of experiments concerning zero-point fluctuations are e.g. in Refs. [6, 7, 8, 9, 10, 11, 12, 13, 14]. The related phenomenon is the zero-point energy being the lowest possible energy that a quantum mechanical system may have. A well-known example is a quantum harmonic oscillator of frequency $\omega_0$. If it is considered to be a closed system then its ground state energy is $(1/2)\hbar\omega_0$. If the oscillator is not perfectly isolated and interacts with its environment of temperature $T = 0$ then its zero-point energy is also $(1/2)\hbar\omega_0$. However, it is true only in the weak coupling limit. If the oscillator-environment coupling is not weak then its zero-point energy is greater than $(1/2)\hbar\omega_0$. The additional portion of energy comes from vacuum fluctuations. It is interesting to consider in this context a free quantum particle. Its energy is not quantized and its allowed values are the same as those of a classical counterpart. If it interacts with a heat bath of temperature $T$, then according to the classical statistical mechanics, the averaged energy is $(1/2)k_B T$ ($k_B$ is the Boltzmann constant) and it tends to zero when $T \to 0$. In the quantum regime, its average energy is non-zero even if $T \to 0$. In this paper we revisit this problem and study the impact of the coupling strength between the particle and its environment on the zero-point energy of the particle as well as the role of different dissipation mechanisms.

2. Model of a free quantum Brownian particle

For the paper to be self-contained, in this section we recall the most popular model of a free quantum Brownian particle of mass $M$ coupled to a heat bath which is defined by the Caldeira-Leggett Hamiltonian, see e.g. [15, 16, 17, 18, 19, 20, 21, 22],

$$H = \frac{p^2}{2M} + \sum_i \left[ \frac{p_i^2}{2m_i} + \frac{m_i\omega_i^2}{2} \left( q_i - \frac{c_i}{m_i\omega_i^2} x \right)^2 \right],$$

(1)

where the heat bath is modelled as a set of non-interacting harmonic oscillators. The operators $\{x, p\}$ are the coordinate and momentum operators of the Brownian particle and $\{q_i, p_i\}$ refer to the coordinate and momentum operators of the $i$-th thermostat oscillator of mass $m_i$ and the eigenfrequency $\omega_i$. The parameter $c_i$ characterizes the coupling between the particle and the $i$-th oscillator. All coordinate and momentum operators obey canonical equal-time commutation relations.
From the Heisenberg equations of motion for all coordinate and momentum operators $\{x, p, q_i, p_i\}$ one can obtain an effective equation of motion only for the particle coordinate $x(t)$ and momentum $p(t)$. It is called a generalized quantum Langevin equation and for the momentum operator of the Brownian particle it reads \[23\]

$$\dot{p}(t) + \frac{1}{M} \int_0^t \gamma(t-s)p(s)ds = -\gamma(t)x(0) + \eta(t), \quad t > 0,$$

where the dot denotes the derivative with respect to time and $\gamma(t)$ is the dissipation function (damping or memory kernel) given by

$$\gamma(t) = \sum_i \frac{c_i^2}{m_i\omega_i^2} \cos(\omega_i t) \equiv \int_0^\infty J(\omega) \cos(\omega t) d\omega$$

and

$$J(\omega) = \sum_i \frac{c_i^2}{m_i\omega_i^2} \delta(\omega - \omega_i)$$

is the spectral function of the thermostat which contains information on its modes and the Brownian particle-thermostat interaction. Remark: The definition \[4\] of the spectral density $J(\omega)$ differs from another frequently used form $\tilde{J}(\omega) = \omega J(\omega)$. We prefer the definition \[4\] because of a direct relation to the Fourier transform of the dissipation function \[3\], i.e. $\hat{\gamma}(\omega) = J(\omega)$. We note that here the Ohmic case corresponds to $J(\omega) = \text{const}$. The operator $\eta(t)$ can be interpreted as quantum thermal noise acting on the Brownian particle and has the form

$$\eta(t) = \sum_i c_i \left[ q_i(0) \cos(\omega_i t) + \frac{p_i(0)}{m_i\omega_i} \sin(\omega_i t) \right],$$

which depends on the thermostat operators $\{q_i(0), p_i(0)\}$ at the initial moment of time.

The solution of Eq. \[2\] reads

$$p(t) = R(t)p(0) - x(0) \int_0^t R(t-u)\gamma(u)du + \int_0^t R(t-u)\eta(u)du,$$

where $R(t)$ is a response function determined by its Laplace transform $\hat{R}_L(z)$, namely,

$$\hat{R}_L(z) = \frac{M}{Mz + \hat{\gamma}_L(z)}.$$

Here, $\hat{\gamma}_L(z)$ is a Laplace transform of the dissipation function $\gamma(t)$ defined in the standard way as

$$\hat{\gamma}_L(z) = \int_0^\infty e^{-zt}\gamma(t)dt.$$

In the thermodynamic limit of the infinitely extended heat bath and for $t \to \infty$ when a thermal equilibrium state is reached, the average kinetic energy $E$ of the Brownian particle can be presented in the form (for detailed derivation, see Ref. \[23\])

$$E = \lim_{t \to \infty} \frac{1}{2M} \langle p^2(t) \rangle = \int_0^\infty E(\omega)\mathcal{P}(\omega)d\omega,$$

where $E(\omega)$ is the spectral density of the kinetic energy and $\mathcal{P}(\omega)$ is the distribution function of the fluctuations.
where

\[ \mathcal{E}(\omega) = \frac{\hbar \omega}{4} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \]  

(10)

and

\[ \mathbb{P}(\omega) = \frac{1}{\pi} \left[ \hat{R}_L(i\omega) + \hat{R}_L(-i\omega) \right]. \]  

(11)

The notation \( \langle \cdot \rangle \) stands for averaging over the initial state of the composite system. The function \( \mathbb{P}(\omega) \) fulfils all conditions imposed on the probability density: (i) it is non-negative, i.e. \( \mathbb{P}(\omega) \geq 0 \), and (ii) normalized on the positive real half-line, i.e. \( \int_0^\infty d\omega \mathbb{P}(\omega) = 1 \). The corresponding proof is presented in Ref. [24]. Eqs. (9)-(11) constitute a quantum counterpart of the energy equipartition theorem well known for classical systems. It says that in quantum physics energy is not equally distributed among the degrees of freedom but it is allocated according to the corresponding probability density function \( \mathbb{P}(\omega) \), i.e. if the thermostat oscillator has eigenfrequency \( \omega \) then its contribution to \( \mathcal{E} \) is determined by \( \mathbb{P}(\omega) \). Because the model is exactly solvable the probability density \( \mathbb{P}(\omega) \) is exact and determined by the Laplace transform \( \hat{R}_L(z) \) of the response function \( R(t) \). In turn, it contains the Laplace transform \( \hat{\gamma}_L(z) \) of the memory function \( \gamma(t) \) which, via Eq. (4), depends on the spectral function \( J(\omega) \) which then, via Eq. (3), comprises all information on the oscillator-thermostat interaction and frequencies of the heat bath modes.

2.1. Quantum environment noise

We assume the initial state of the composite system in a product form, i.e., \( \rho(0) = \rho_S \otimes \rho_B \), where \( \rho_S \) is an arbitrary state of the Brownian particle and \( \rho_B \) is the canonical Gibbs state of the heat bath of temperature \( T \), namely,

\[ \rho_B = \exp(-H_B/k_B T)/\mathrm{Tr}[\exp(-H_B/k_B T)], \]  

(12)

where

\[ H_B = \sum_i \left[ \frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 q_i^2 \right] \]  

(13)

is the Hamiltonian of the heat bath. The factorization of the composite system state means that there are no initial correlations between the particle and thermostat. The initial preparation of the heat bath turns the force \( \eta(t) \) into the operator-valued quantum thermal noise which in fact is a family of non-commuting operators whose commutators are \( c \)-numbers. This noise is unbiased and its mean value is zero

\[ \langle \eta(t) \rangle = \mathrm{Tr} [\eta(t) \rho_B] = 0. \]  

(14)

Its symmetrized correlation function depends only on the time difference,

\[ C(t-s) = \frac{1}{2} \langle \eta(t) \eta(s) + \eta(s) \eta(t) \rangle \]

\[ = \int_0^\infty \frac{\hbar \omega}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right) J(\omega) \cos[\omega(t-s)] d\omega. \]  

(15)
The higher order correlation functions are expressed by \( C(t_i - t_j) \) and have the same form as statistical characteristics for classical stationary Gaussian stochastic processes. Therefore \( \eta(t) \) defines a quantum stationary Gaussian process with time homogeneous correlations.

The assumption (12) allows to consider \( \mathcal{E}(\omega) \) defined in Eq. (10) as an equilibrium kinetic energy per one degree of freedom of the thermostat of temperature \( T \) at initial time \( t = 0 \) [26], namely,

\[
\mathcal{E}(\omega_i) = \text{Tr} \left[ \frac{p_i^2}{2m_i} \rho_B \right] = \frac{\hbar \omega_i}{4} \coth \left( \frac{\hbar \omega_i}{2k_B T} \right). \tag{16}
\]

It says that the mean kinetic energy \( E \) of the Brownian particle in Eq. (9) is equal to the mean kinetic energy per one degree of freedom of the thermostat free oscillators, i.e., \( E = \langle \mathcal{E} \rangle \). The averaging \( \langle \cdot \rangle \) is twofold [25]: (i) over the Gibbs state \( \rho_B \) for the thermostat free (non-interacting with the Brownian particle) oscillators resulting in \( \mathcal{E}(\omega) \) given by Eq. (16) and (ii) over frequencies \( \omega \) of those thermostat oscillators which contribute to \( E \) according to the probability distribution \( P(\omega) \). However, it should be explicitly stressed that \( \mathcal{E}(\omega) \) is not equal to the mean kinetic energy of a single thermostat oscillator at the equilibrium state of the composite system (i.e. for \( t \to \infty \)).

2.2. Consistency of modeling

Modelling by means of the generalized quantum Langevin equation should be based on the physically consistent specification of quantities describing the considered system. These are the dissipation kernel \( \gamma(t) \), the spectral density \( J(\omega) \) or the correlation function \( C(t) \). They are all related to each other, namely,

- The dissipation function
  \[
  \gamma(t) = \int_0^\infty J(\omega) \cos(\omega t) d\omega. \tag{17}
  \]

- The spectral density
  \[
  J(\omega) = \frac{2}{\pi} \int_0^\infty \gamma(t) \cos(\omega t) dt. \tag{18}
  \]

- The correlation function of quantum noise
  \[
  C(t) = \int_0^\infty \frac{\hbar \omega}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right) J(\omega) \cos(\omega t) d\omega. \tag{19}
  \]

The natural question which comes to mind is what constraints should be imposed on them in order to get the consistent description, especially without any divergences of directly measurable observables.

One has to perform the thermodynamic limit for the heat bath by assuming that the number of oscillators forming thermostat tends to infinity. Otherwise all dynamical quantities are (quasi)periodic functions of time and there is no dissipation and a thermodynamic equilibrium state. Under such assumption the dissipation kernel \( \gamma(t) \) decays to zero as \( t \to \infty \) and the singular spectral density \( J(\omega) \) defined in Eq. (4) tends
to a (piecewise) continuous function. Moreover, by inspecting the counter-term in the Hamiltonian given by Eq. (1) (for the discussion of the counter-term see e.g. Ref. [22])

$$\frac{1}{2} \sum_i c_i^2 m_i \omega_i^2 x_i^2 = \frac{1}{2} k_0 x^2$$

we find that the force constant $k_0$ should be finite, i.e.,

$$k_0 = \sum_i \frac{c_i^2}{m_i \omega_i^2} = \int_0^\infty J(\omega) d\omega = \gamma(0) < \infty.$$  (21)

It follows that the spectral density $J(\omega)$ is integrable on the positive half-line and the integral is related to the dissipation function $\gamma(t)$ at the initial moment of time $t = 0$. Therefore $\gamma(t)$ as a decaying function of time is finite. The open question is whether the correlation function $C(t)$ should be finite for all instants of time, in particular $C(0)$ which is related to the second moment of quantum noise $\langle \eta^2(t) \rangle$. In classical statistical mechanics the latter is often modeled as Gaussian white noise whose the second moment does not exist.

3. Zero-point energy of the Brownian particle

At non-zero thermostat temperature $T > 0$, the averaged energy of the free quantum Brownian particle given by Eq. (9) is always greater than at zero temperature $T = 0$. When $T \to 0$ then $\coth(\hbar \omega/2 k_B T) \to 1$ and Eq. (9) reduces to the form

$$E = \frac{\hbar}{4} \int_0^\infty \omega \mathbb{P}(\omega) d\omega,$$  (22)

which is proportional to the first statistical moment of the probability density $\mathbb{P}(\omega)$. It can be interpreted as an averaged kinetic energy $\hbar \omega/4$ per one degree of freedom of thermostat oscillators which contribute to $E$ according to the probability distribution $\mathbb{P}(\omega)$. The latter quantity, c.f. Eq. (11), is defined solely by the dissipation function $\gamma(t)$. The choice of $\gamma(t)$ is arbitrary although in principle it should be determined by properties of the environment. As outlined above to guarantee the consistent description $\gamma(t)$ needs to be a bounded decaying function of time. In the following we consider several examples of $\gamma(t)$ in order to test how $E$ depends on $\gamma(t)$ and whether there is universal behaviour of $E$ which is robust against changes of the dissipation mechanism determined by $\gamma(t)$.

3.1. Analytically tractable case: Drude model

The so-called Drude model is defined by the exponentially decaying damping function [22]

$$\gamma_D(t) = \gamma_0 e^{-t/\tau_c},$$  (23)

where $\gamma_0 > 0$ is the particle-thermostat coupling strength and $\tau_c > 0$ is the memory time which characterizes the degree of non-Markovianity of the Brownian particle dynamics.
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Its inverse \( \omega_c = 1/\tau_c \) is the Drude cutoff frequency. The corresponding spectral density takes the form

\[
J_D(\omega) = \frac{2}{\pi} \frac{\gamma_0 \tau_c}{1 + \tau_c^2 \omega^2}.
\] (24)

The probability distribution is found to be [23]

\[
P_D(\omega) = \frac{2 M \gamma_0}{\pi \tau_c} \frac{1}{(M \omega^2 - \gamma_0)^2 + (M \omega/\tau_c)^2}
\] (25)

and the averaged energy of the Brownian particle is given by the formula

\[
E = \frac{1}{2\pi} \frac{M \gamma_0}{\tau_c} \int_0^\infty \frac{\hbar \omega}{(M \omega^2 - \gamma_0)^2 + (M \omega/\tau_c)^2} d\omega.
\] (26)

We note that there are three parameters of the system \( \{M, \gamma_0, \tau_c\} \). The dimensionless quantities can be introduced as follows

\[
\mathcal{E} = \frac{\tau_c E}{\hbar} = \frac{E}{\hbar \omega_c}, \quad x = \tau_c \omega = \frac{\omega}{\omega_c}, \quad c = \frac{\gamma_0 \tau_c^2}{M} = \frac{\gamma_0}{M \omega_c^2},
\] (27)

which transform the relation (26) to the form

\[
\mathcal{E} = \mathcal{E}(c) = \frac{1}{2\pi} \int_0^\infty \frac{cx}{(x^2 - c)^2 + c^2} dx.
\] (28)

In this scaling the parameter \( c \) is the dimensionless particle-thermostat coupling strength. It is impressive that now the system is completely characterized only by one parameter \( c \). The above integral (28) can be explicitly calculated yielding

\[
\mathcal{E}(c) = \frac{c}{4\pi \sqrt{1 - 4c}} \ln \frac{1 - 2c + \sqrt{1 - 4c}}{1 - 2c - \sqrt{1 - 4c}}, \quad c < 1/4,
\] (29)

\[
= \frac{c}{2\pi \sqrt{4c - 1}} \left[ \frac{\pi}{2} + \arctan \frac{2c - 1}{\sqrt{4c - 1}} \right], \quad c > 1/4,
\] (30)

\[
= \frac{1}{4\pi}, \quad c = \frac{1}{4}.
\] (31)
The dependence of $E(c)$ upon the coupling constant $c$ is depicted in Fig. 1. It is a monotonically increasing function of the latter parameter. If $c \to 0$ then $E(c) \to 0$ and $E(c) \to \infty$ when $c \to \infty$. In the weak coupling regime $c \ll 1$, the first two leading contributions to the energy have the form

$$E(c) = E_1(c) + E_2(c), \quad E_1(c) = \frac{c}{2\pi} \ln(1/c), \quad E_2(c) = \frac{c^2}{\pi} [\ln(1/c) - 1] \quad (32)$$

Their graphical representation is also depicted in Fig. 1. $E_1(c)$ is already known in the literature [22]. It is worth to note that the leading contribution to the Lamb shift is also logarithmic and reads $\alpha^5 \ln(1/\alpha)$, where $\alpha$ is a fine-structure constant. The correction $E_2(c)$ is the next leading contribution to $E(c)$ for small $c$. The term $(-c^2/\pi)$ is included to minimize the deviation from the exact value of the zero-point particle energy. We now return to the dimensional variables and the first leading term to the dimensional energy is

$$E_1 = \hbar \omega_c E_1 = \hbar \frac{\gamma_0 \tau_c}{2\pi M} \ln \left( \frac{M}{\gamma_0 \tau_c} \right). \quad (33)$$

It is a purely quantum contribution to the energy proportional to $\hbar$ and tends to zero when the coupling constant $\gamma_0 \to 0$ or the memory time $\tau_c \to 0$, or the particle mass $M \to \infty$. The asymptotics of $E(c)$ can be evaluated also for the limit of strong coupling. By inspecting (30) we find

$$E(c) \sim \sqrt{c}, \quad c \gg 1. \quad (34)$$

### 3.2. Selected examples of the dissipation mechanism

We now want to investigate whether the zero-point energy of the quantum Brownian particle is bounded from below by the function $E(c)$ given by Eqs. (29)-(31). For this purpose we consider different dissipation mechanisms $\gamma(t)$ and check the interrelations between the corresponding zero-point energies.

#### 3.2.1. The Debye-type model: algebraically decaying oscillations

As the first example we pick the following dissipation function

$$\gamma_S(t) = \gamma_0 \frac{\sin \left( \frac{t}{\tau_c} \right)}{t/\tau_c}. \quad (35)$$

The spectral density is of the Debye-type [27]

$$J_S(\omega) = \frac{\gamma_0}{\omega_c} \theta(\omega_c - \omega), \quad (36)$$

where $\omega_c = 1/\tau_c$ is the cut-off frequency and $\theta(x)$ denotes the Heaviside step function. This spectral density is constant on the compact support $[0, \omega_c]$ determined by the memory time $\tau_c$. Under this assumption the probability density can be presented as

$$P_S(\omega) = \frac{4M}{\gamma_0} \frac{\omega_c \theta(\omega_c - \omega)}{\pi^2 + [\ln(\omega_c + \omega) - \ln(\omega_c - \omega) - 2M\omega_c \omega/\gamma_0]^2} \quad (37)$$

and has the same compact support $[0, \omega_c]$ as the spectral function $J_S(\omega)$. 
3.2.2. Lorentzian decay. The Lorentz-type damping function reads

$$\gamma_L(t) = \gamma_0 \frac{1}{1 + (t/\tau_c)^2}$$

(38)
and the corresponding spectral density is given by

$$J_L(\omega) = \gamma_0 \tau_c e^{-\tau_c \omega}.$$

(39)

Such a choice of the dissipation kernel leads to the following probability distribution

$$P_L(\omega) = \frac{4 \nu_0 e^{-\tau_c \omega}}{\pi^2 \nu_0^2 e^{-2 \tau_c \omega} + h^2(\omega)}, \quad \nu_0 = \frac{\gamma_0 \tau_c}{M},$$

(40)

where

$$h(\omega) = \nu_0 e^{-\tau_c \omega} \text{Ei}(\tau_c \omega) - \nu_0 e^{\tau_c \omega} \text{Ei}(-\tau_c \omega) - 2\omega$$

(41)

and Ei(z) is the exponential integral defined as

$$\text{Ei}(z) = \int_{-\infty}^{z} \frac{e^t}{t} dt.$$  

(42)

3.2.3. Family of algebraic decay. This class of dissipation functions is defined by the formula

$$\gamma_n(t) = \gamma_0 \frac{1}{(1 + t/\tau_c)^n},$$

(43)

where $n \in \mathbb{N}$ and $n \geq 2$. The corresponding spectral density reads

$$J_n(\omega) = \frac{\gamma_0 \tau_c}{\pi} \left[ e^{-i \tau_c \omega} E_n(-i \tau_c \omega) + e^{i \tau_c \omega} E_n(i \tau_c \omega) \right]$$

(44)

and $E_n(z)$ is the exponential integral,

$$E_n(z) = \int_1^\infty dt \frac{e^{-zt}}{t^n}.$$  

(45)

The probability distribution takes the form

$$P_n(\omega) = \frac{\nu_0}{\pi} \frac{e^{-i \tau_c \omega} E_n(-i \tau_c \omega) + e^{i \tau_c \omega} E_n(i \tau_c \omega)}{[\omega + i \nu_0 e^{-i \tau_c \omega} E_n(-i \tau_c \omega)][\omega - i \nu_0 e^{i \tau_c \omega} E_n(i \tau_c \omega)]},$$

(46)

where $\nu_0 = (\gamma_0 \tau_c)/M$.

3.3. Zero-point energy vs the dissipation mechanism

In Fig. 2 (e) we present dependence of the zero-point energy $E$ on the particle-thermostat coupling strength $c$ for different forms of the dissipation mechanism. To facilitate the analysis, we plot the damping kernel $\gamma(t)$, the spectral density $J(\omega)$ and the correlation function $C(t)$ in panels (a), (b) and (c), respectively. The reader can immediately note that the sequence (from the top to the bottom) of the zero-point energy curves $E(c)$ for different dissipation mechanisms is the same as the order of the damping kernels $\gamma(t)$ and the spectral densities $J(\omega)$ for small times $t$ and small frequencies $\omega$, respectively.

In contrast, it is rather difficult to reveal any universal pattern in the impact of the form of the corresponding probability densities $P_j(\omega), (j = D, S, L, n = 2, 4, 6)$ (not
Figure 2. (a): The dissipation kernel $\tilde{\gamma}(\tilde{t})$, (b): the spectral density $\tilde{J}(\tilde{\omega})$, (c): the correlation function $\tilde{C}(\tilde{t})$, in insert the magnified Debye and Lorentz ones, (d): the cumulative distribution function $\tilde{F}(\tilde{\omega})$ and (e): the zero-point energy of the quantum Brownian particle $E(c)$ all depicted for different dissipation mechanisms. The dimensionless quantities are: $\tilde{\gamma}(\tilde{t}) = \gamma(\omega_c t)/\gamma_0$, $\tilde{J}(\tilde{\omega}) = (\omega_c/\gamma_0)J(\omega_c \tilde{\omega})$, $\tilde{C}(\tilde{t}) = 2C(\omega_c t)/((\gamma_0 \hbar \omega_c))$ and $\tilde{F}(\tilde{\omega}) = F(\omega)$. The dimensionless variables are: $\tilde{t} = \omega_c t$ and $\tilde{\omega} = \omega/\omega_c$. In panel (d) $c = 0.25$.

The reason is that these quantities depend on the magnitude of the coupling strength $c$ and therefore their interrelations generally vary for different values of the parameter $c$. However, it is instructive to analyze the cumulative distribution function $F_j(\omega)$, namely,

$$F_j(\omega) = \int_0^\omega \mathbb{P}_j(u)du, \quad j = D, S, L, n = 2, 4, 6.$$  \hspace{1cm} (47)
It is depicted in Fig. 2 (d). The correlation between $F_j(\omega)$ and $E(c)$ is evident: If the cumulative distribution function is greater then the zero-point energy $E(c)$ of the quantum Brownian particle is smaller. If for two probabilities $F_j(\omega) > F_k(\omega)$ then for the corresponding energies $E_j(c) < E_k(c)$. The above observations allow us to formulate the following conclusions:

(i) The rate of decay of the damping kernel $\gamma(t)$ crucially modify the energy $E$. If $\gamma(t)$ decreases rapidly then $E$ is small.

(ii) If the main contribution to the zero-point energy $E$ comes from the environment oscillators of small frequencies $\omega$ then $E$ is small.

(iii) There is no non-zero lower bound for the zero-point energy $E(c)$ of the free quantum Brownian particle. If for short times the decay of the memory function $\gamma(t)$ is faster and faster then the particle energy $E$ is smaller and smaller.

4. Critical discussion

4.1. Scaling of the memory kernel

In the literature the memory kernel $\gamma(t)$ is frequently defined in such a way that it tends to the Dirac delta when the memory time $\tau_c$ tends to zero. E.g. for the Drude model the most common form reads

$$\gamma(t) = \frac{\gamma}{\tau_c} e^{-t/\tau_c}. \quad (48)$$

Indeed, $\lim_{\tau_c \to 0} \gamma(t) = \delta(t)$ and for the integral part of Eq. (2) one gets

$$\frac{1}{M} \int_0^t \gamma(t-s)p(s)\,ds \to \frac{p(t)}{M}. \quad (49)$$

It is often called the white noise limit or Markovian approximation. Let us verify its consequences. Firstly, according to Eq. (21) in such a case $k_0 = \gamma(0) = \gamma/\tau_c$. When $\tau_c \to 0$ then $k_0 \to \infty$ and the counter-term in Eq. (20) diverges. As a consequence, the Hamiltonian is not correctly defined. Secondly, the zero-point energy of the Brownian particle tends to infinity, which is also non-physical. It is explicitly seen from Eq. (33) for $\gamma_0 = \gamma/\tau_c$. Indeed, $E_1 = \hbar \omega_c E_1 \propto \ln(1/\tau_c) \to \infty$. Moreover, if $\tau_c$ is varied as a control parameter then the force constant $k_0 = \gamma/\tau_c$ is modified and in consequence the Hamiltonian (1) is altered. In this way one compares e.g. the average energy $E$ for two different values of $\tau_c$, i.e. for two different Hamiltonians (two different systems), which is not a correct method.

4.2. The correlation function of vacuum noise

For classical systems the correlation function $C(t) = C_d(t)$ of thermal noise $\eta(t)$ in Eq. (5) is equal, up to a constant factor, to the the damping function $\gamma(t)$. Indeed, for high temperature

$$\coth\left(\frac{\hbar \omega}{2k_B T}\right) \approx \frac{2k_B T}{\hbar \omega}. \quad (50)$$
and from Eq. (19) it follows that

\[ C_{cl}(t) = k_B T \gamma(t). \] (51)

Properties of \( C_{cl}(t) \) can be deduced from Fig. 2 (a). At absolute zero temperature \( T = 0 \), its quantum counterpart \( C_0(t) \) is obtained from Eq. (19) and reads

\[ C_0(t) = \int_0^\infty \frac{\hbar \omega}{2} J(\omega) \cos(\omega t) d\omega. \] (52)

In contrast, it is not proportional to \( \gamma(t) \) as in the classical case. Representative examples of \( C_0(t) \) are depicted in Fig. 2 (c). For the Drude model, the correlation function of quantum noise \( \eta(t) \) reads

\[ C_D(t) = -\frac{\gamma_0 \hbar \omega_c}{\pi} \left[ e^{-\omega_c t} \text{Ei}(\omega_c t) + e^{\omega_c t} \text{Ei}(-\omega_c t) \right], \] (53)

where \( \text{Ei}(x) \) is the exponential integral and \( \text{Ei}(x) \) denotes its principal value. When \( t \to 0 \) then \( C_D(t) \to \infty \) and the second moment of noise diverges, \( \langle \eta^2(t) \rangle \to \infty \). For the Debye-type model, it is bounded and has the simple form

\[ C_S(t) = \gamma_0 \frac{\hbar \omega_c}{2} \left[ \frac{\sin(\omega_c t)}{\omega_c t} + \frac{\cos(\omega_c t) - 1}{(\omega_c t)^2} \right], \quad \langle \eta^2(t) \rangle = C_S(0) = \gamma_0 \frac{\hbar \omega_c}{4}. \] (54)

For the Lorentzian decay it is also bounded and takes the form

\[ C_L(t) = \gamma_0 \frac{\hbar \omega_c}{2} \frac{1 - (\omega_c t)^2}{[1 + (\omega_c t)^2]^2}, \quad \langle \eta^2(t) \rangle = C_L(0) = \gamma_0 \frac{\hbar \omega_c}{2}. \] (55)

For the algebraic decay of \( \gamma(t) \) given by Eq. (43) there is no an analytical expression for \( C_0(t) \). Its numerical calculation is presented in Fig. 2 (c). For all members of the family of algebraic decay the second moment of noise does not exist, \( \langle \eta^2(t) \rangle = \infty \).

There are three crucial disparities: (i) In the classical case \( C_{cl}(t) \to 0 \) for \( T \to 0 \). In the quantum case \( C_0(t) \neq 0 \) at absolute zero temperature \( T = 0 \). (ii) \( C_0(0) \) can diverge for quantum systems while its classical counterpart \( C_{cl}(0) \) has to be finite, cf. Eq. (21). (iii) if \( C_{cl}(t) \) is positive \( C_0(t) \) may assume negative values. It means that quantum noise can exhibit negative correlations (anticorrelations) while its classical counterpart exhibits only positive ones. For tailored forms of the dissipation kernels classical noise may also be anticorrelated as it is the case for e.g. the Debye model.

4.3. Impact of temperature and potential energy

In order to complement the analysis, in Fig. 3 we show the influence of temperature and a potential on the average kinetic energy of the quantum Brownian particle. As an example we present the case of a harmonic oscillator for which the potential is \( U(x) = M \omega_0 x^2 / 2 \). It is an exactly solvable model [28]. As expected, if temperature of a thermal bath increases the average kinetic energy of the particle grows as well. It is obvious that the average potential energy becomes greater when the eigenfrequency \( \omega_0 \) increases. The same hold true for the total energy. What is in clear contrast to classical result is the dependence of the average kinetic energy on the eigenfrequency \( \omega_0 \). Here, the kinetic energy grows together with \( \omega_0 \) whereas classically it is independent
of the latter parameter and equal to \((1/2)k_B T\) as the equipartition theorem states. Solid lines represent the results for the free particle where the dashed ones correspond to the harmonic oscillator.

5. Summary

We revisited the paradigmatic model of a free quantum Brownian particle in contact with quantum vacuum, i.e. thermostat of absolute zero temperature, and studied the zero-point energy of the particle. We scrutinized the impact of various dissipation mechanisms and revealed universal behavior of the zero-point energy of the Brownian particle as a function of the rescaled coupling strength between the system and the environment. In particular, we pointed out an abuse of the common scaling of the damping kernel which tends to the Dirac delta when the memory time approaches zero.

We discussed the influence of the form of the dissipation kernel, the spectral density as well as the cumulative distribution function on the lowest possible energy of the system. For the Drude model we additionally obtained an exact analytical formula for the zero-point energy of the free Brownian particle. It allowed us to easily calculate the asymptotic forms the energy in the limit of weak and strong particle-environment coupling. Last but not least, we discussed influence of the harmonic potential on the zero-point energy of the particle.

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