Comments on the Chern–Simons photon term in the planar reduced QED description of graphene

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We revisit the Coleman–Hill theorem in the context of reduced planar QED, which can be supplemented with background electromagnetic fields. Using the global U(1) Ward identity for this non-local but still gauge invariant theory, we can confirm that the topological piece of the photon self-energy at zero momentum does not receive further quantum corrections apart from the potential one-loop contribution. This is of relevance to probe possible time parity odd dynamics in a planar sheet of graphene which has an effective description in terms of (2 + 1)-dimensional planar reduced QED.

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I. CONTEXT AND MOTIVATION

Quantum Electrodynamics in \((2 + 1)\) dimensions (QED\(_3\)) has been widely used as a toy model for Quantum Chromodynamics (QCD). This is due to the fact that although being Abelian, QED\(_3\) exhibits similar features as non-Abelian gauge theories, making it possible, for instance, to map and investigate chiral symmetry breaking and confinement into it \([1–5]\). The similarity is reinforced by the fact that a non-Abelian gauge theory at high temperature suffers a dimensional reduction and, if coupled to \(N_f\) fermion families, the non-Abelian interactions are suppressed by a factor of \(N_f^{-1}\), so that in the large \(N_f\) limit the theory can be considered approximately Abelian.

Recently, the emergence of the so-called Dirac and Weyl planar materials \([6]\), converted QED\(_3\) into a playground in which a potential link between high energy physics and condensed matter can emerge \([7, 8]\). Those are materials in which, due to the specific structure of their underlying lattice, the charge carriers present a relativistic-like behavior, being correctly described by a Dirac-like equation in some particular regimes. Particularly, the physical realization of graphene and other materials in two space dimensions, that are proved to contain a priori massless Dirac spinors, offer a direct connection to QCD since the continuum limit of the tight-binding theory, usually applied to describe their conduction electrons, naturally yields QED\(_3\) \([9]\).

Nevertheless, even though in these systems the fermions are constrained to remain in-plane and therefore are correctly described by a theory in \((2 + 1)\) dimensions, the gauge fields responsible for the interaction between these electrons are not subject to the same constraint. One of the most remarkable consequences of this fact is that the interaction between electrons remains the familiar \(\sim 1/r\) potential rather than the logarithmic one that would take place if the gauge fields were also restricted to the plane. Therefore, it is convenient and necessary to modify QED\(_3\) in order to merge the desired features of the two sectors of the theory, starting with a general \((3 + 1)\) theory and dimensionally reducing it to a non-local effective \((2 + 1)\) theory. This procedure was followed within similar approaches in \([10]\), and posteriorly in \([11]\), and received generically the name of reduced QED (RQED).

In the context of pure QED\(_3\), the most general structure of the action allows for a term in the gauge sector that breaks time reversal (T), namely the Chern–Simons (CS) term. Its presence gives a mass to the photon and, for this reason, it is also known as topological mass term. This term is important in several contexts in condensed matter, for instance it leads naturally to the transverse conductivity observed from the Hall effect and it is crucial to model high \(T_c\) superconductivity \([12]\). It was shown that radiative corrections coming from interaction terms can give a contribution for the topological photon mass up to one-loop. Remarkably, a theorem by Coleman and

\footnote{Actually, in the Abelian case there is no real topology involved and the term “topological” is used for historical reasons based on its non-Abelian counterpart.}
Hill [13] demonstrates that, apart from one-loop, all corrections to the topological mass term vanish identically to all orders. This was done in general grounds, considering the photon interacting with any massive scalar, spinor or vector field with arbitrary gauge invariant interactions. The massive nature of the field excitations interacting with the photon is crucial here, to avoid the typical infrared subtleties in lower-dimensional field theories. In particular does the Coleman–Hill theorem not hold in presence of massless degrees of freedom, as explicitly illustrated in e.g. [14]. Indeed, infrared singularities, typical for lower-dimensional field theories can disturb the argument.

Regarding the importance of RQED in the description of planar Dirac systems in condensed matter, precisely for those systems that allow for a direct analogy with QCD, it is important to verify if the Coleman–Hill theorem also holds for this theory. In this work we demonstrate that higher order radiative corrections are exactly vanishing in RQED, in the same way as for QED\textsuperscript{3}, meaning that the topological photon parameter\textsuperscript{2} arises at one-loop, or does not arise at all. In section II we discuss briefly how the tight-binding model yields QED\textsuperscript{3} in the continuum limit and present the general features of RQED, including its gauge invariance and freedom of gauge choice, before and after the reduction. In particular we discuss possible mass terms for the fermions that are important if we want to apply our theory directly to graphene. The role of electromagnetic background fields in the radiative corrections, important in manipulations to study transport phenomena in materials, is also briefly highlighted, with explicit computations relegated to a future longer paper. In section III we motivate our choice of mass terms and finally we prove that corrections of order higher than one are null, and we summarize the explicit one-loop computation in the absence of background fields, both from the (crucially different) two- as well as four-component spinor viewpoint. In section IV we present our final remarks.

II. PLANAR SYSTEMS AND RQED

Graphene is constituted by a single sheet of carbon atoms tightly packed into a two-dimensional honeycomb lattice [15]. The honeycomb array can be regarded in terms of two periodic sublattices \( L_A \) and \( L_B \). Here, we follow the convention of [15] and define the primitive two-dimensional vectors \( \vec{a}_i \) for sublattice \( L_A \) and \( \vec{b}_i \) for the reciprocal sublattice, as \( \vec{a}_1 = a(1/2, \sqrt{3}/2), \vec{a}_2 = a(1/2, -\sqrt{3}/2) \) and \( \vec{b}_1 = 2\pi a(1/2, \sqrt{3}/2), \vec{b}_2 = 2\pi a(1/2, -\sqrt{3}/2) \), where \( a \) is the sublattice spacing. It is also convenient to introduce the three near-neighbor vectors \( \vec{s}_i \),

\[
\vec{s}_1 = a(0, 1/\sqrt{3}), \quad \vec{s}_2 = a(1/2, -\sqrt{3}/6), \quad \vec{s}_3 = a(-1/2, -\sqrt{3}/6),
\]

where \( \ell = a/\sqrt{3} \) is the minimal lattice length.

\footnote{Not mass, see further in this paper.}
\footnote{Another usual convention can be found in [16].}
The inner orbitals are strongly bonded to their respective carbon atom while the π orbitals present a weak overlap. Following the usual tight-binding approach, only the interaction of each charge carrier with the nearest neighbors is considered. The Hamiltonian is written as

$$H = -t \sum_{\vec{r} \in L_A} \sum_{i=1}^{3} \left( a^\dagger(\vec{r}) b(\vec{r} + \vec{s}_i) + b^\dagger(\vec{r} + \vec{s}_i) a(\vec{r}) \right),$$

where the first sum is only along sublattice $L_A$, $t$ is the nearest-neighbor hopping energy and $a, a^\dagger(b, b^\dagger)$ are the anticommuting ladder operators in the sublattice $L_A(L_B)$. Applying a Fourier transformation it is straightforward to compute the energy-momentum dispersion relation $[9, 15]$:

$$E(k_x, k_y) = \pm t \sqrt{3 + 2\cos(\sqrt{3} k_y a) + 4\cos\left(\frac{\sqrt{3}}{2} k_y a\right)\cos\left(\frac{3}{2} k_x a\right)}.$$

The valence and conduction band, generated by the opposite signs in the dispersion relation, touch in six points (Dirac points), of which only two are inequivalent. Here we choose them to be $\vec{K}_\pm = \pm 2\pi/a(2/3, 0)$. Expanding the expression above around these zero energy points one can verify that the dispersion relation for each one of them is linear, $E \pm (\vec{p}) = \pm \hbar v_F |\vec{p}|$. Here, the Fermi velocity is determined by $v_F = \frac{3}{2} t \ell = \frac{\sqrt{3}}{2} a t \approx 0.5$. It was shown $[17]$ that the annihilation operators $a$ and $b$ can be accommodated in a spinor field when we expand around the above Dirac points and, therefore, it can be seen as relativistic-like fermion that obeys a Dirac-like equation. In resume, the continuum limit of the nearest neighbors approach in a tight-binding model applied to a pure hexagonal sublattice with two intertwined triangular sublattices yields a massless version of the fermion sector of QED$_3$.

Following this approach, the action of the system reads:

$$S_f = \int d^3x \left[ \bar{\psi} \left( D_0 - \vec{\gamma} \cdot \vec{\nabla} \right) \psi \right],$$

where here only the first two spatial gamma matrices $\vec{\gamma}$ enter. Interactions with external sources or alterations on the underlying lattice, for instance using a substrate or doping, could produce a gap between the bands. This can be represented at the level of the action by a specific Dirac mass term, $m\bar{\psi}\psi$, or interaction terms involving the matter current. Let us refer to $[9, 18, 19]$ for such possibilities and classification of the mass terms. Interaction terms that are bilinear in the fermion field will change the basic symmetries of the action, depending on their particular gamma matrices structure. In this paper we work in the chiral basis, where the gamma matrices and the fifth gamma matrix are given by:

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \quad i = 1, 2, 3,$$

$^4$ Our convention for the Minkowski metric is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1)$. 

with $I_2$ is the $2 \times 2$ identity and $\sigma^i$ are the standard Pauli matrices. Among the several possibilities of interaction, one can observe that certain terms are completely equivalent to the Dirac mass term as they correspond to a change in the variables in the path integral. Since there is no axial anomaly in $(2+1)$ dimensions the result must describe the same physics. This is the case for the (anti-Hermitian) mass terms $m\bar{\psi}\gamma^3\psi$ and $im\bar{\psi}\gamma^5\psi$, that can be reached from the standard Dirac mass term by performing the following unitary transformations in the fermion fields \[20\], respectively:

\[
\psi \to e^{i\beta\gamma^5}\psi; \quad \bar{\psi} \to \bar{\psi}e^{i\beta\gamma^5}, \tag{6a}
\]

\[
\psi \to e^{i\alpha\gamma^3}\psi; \quad \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma^3}, \tag{6b}
\]

with appropriate choices of the “angles” $\alpha$ and $\beta$. In case of massless fermions, (6a) and (6b) both constitute symmetries of the theory and are part of a larger U(2) invariance, see \[9\].

We remark that this is a feature of the continuum limit and discretization can bring differences between those terms. For example, the tight binding lattice models that would induce the three masses are different \[18, 19\], but they share their continuum limit. Notice also that all these masses correspond to a $T$-even sector \[9\].

Considering these variations of the Dirac mass in the continuum, it is particularly useful to go with $m\bar{\psi}\gamma^3\psi$ when working with a four-component representation of the fermion field, since in this way it is possible to decompose and rewrite the action in terms of two decoupled two-component spinors. This point will be discussed in more detail below in Section III B. The subtle differences between both formulations can also be appreciated from \[21\].

Besides the variants of the Dirac mass, one other specific mass term is particularly important, the Haldane mass $m_o\gamma^3\gamma^5$ \[23\]. This one is totally independent of the masses previously discussed, as it corresponds to a $T$-odd bilinear term. The special interest in it relies on the fact that in pure QED$_3$ it can be directly related to the CS term.

The gauge sector of pure QED$_3$ is described by

\[
\mathcal{L}_{\text{QED}_3} = \int d^3x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\xi} (\partial \cdot A)^2 - \frac{\theta}{2} g^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right],
\]

where the first is the usual Maxwell term, the second is a linear gauge fixing term and, the last one is the CS term. On one hand, the one-loop radiative corrections from a fermion with Haldane mass generates a $T$-odd piece in the photon polarization tensor \[24, 25\], which can be translated into the presence of the CS term in the gauge sector of the action. The Coleman–Hill theorem \[13\] guarantees that no higher order corrections are allowed.

\[5\] We refer to T-even or T-odd in the four-component spinor language. In the two-component description the symmetry behaviour of the fermion mass terms can be different, see \[21, 22\].
so the connection of the two terms is clearly pictured. On the other hand, the presence of a CS term generates dynamically a Haldane mass for the fermions \(^{26}\) already at one-loop as well.

As discussed before, in order to correctly describe electrons confined to a plane but whose interaction is the usual Coulomb interaction, it is necessary to consider the gauge fields living in the three-dimensional spatial bulk rather than in the two-dimensional spatial plane. To obtain a consistent theory combining the suitable conditions for fermions and gauge fields, the authors in \(^{10, 11}\) start with the gauge theory in four dimensions and integrate out the gauge field. Being deliberately brief, we consider standard QED, written as

\[
S_{\text{QED}} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2\kappa} (\partial \cdot A)^2 + j_\mu A^\mu \right]. \tag{7}
\]

The Dirac matter currents are supposed to be

\[
j_\mu = \begin{cases} 
  i \bar{\psi} \gamma^\mu \psi \delta(x_3) & \text{for } \mu = 0, 1, 2, \\
  0 & \text{for } \mu = 3,
\end{cases} \tag{8}
\]

with the fermion fields only dependent on \((x_0, x_1, x_2)\). This formally expresses the fact that the fermion dynamics is restricted to happen in the \((x_1, x_2)\)-plane, i.e. the planar graphene sheet. The current is conserved, \(\partial_\mu j^\mu = 0\). The easiest way to proceed is to Wick rotate to Euclidean space and to Fourier transform in order to integrate out the four-dimensional gauge field, leading to

\[
S_{\text{eff}} = \int d^4p \left[ \hat{j}_\mu \hat{D}^{\mu\nu}_{\text{Tr}}(\vec{p}) \hat{j}_\nu \right], \tag{9}
\]

where \(\vec{p} = (p_0, p_1, p_2)\), \(\hat{D}^{\mu\nu}_{\text{Tr}} = \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{(\vec{p}^2 + p_3^2)} \right) \frac{1}{(\vec{p}^2 + p_3^2)}\) is the (gauge independent) transverse projection of the free photon propagator, which appears due to the conserved fermion current. As the Fourier-transformed currents will not depend on \(p_3\), we can integrate out the latter, leading to

\[
S_{\text{eff}} = \int d^3p \left[ \hat{j}_\mu \hat{D}^{\mu\nu}_{\text{Tr}}(\vec{p}) \hat{j}_\nu \right]. \tag{10}
\]

The indices \(\mu, \nu\) are from now on restricted to \(x_0, x_1, x_2\) and we can forget about the \(\delta(x_3)\) in the definition of the current \(j_\mu\). Furthermore, we set

\[
\hat{D}^{\mu\nu}_{\text{Tr}}(\vec{p}) = \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{1}{2p}, \quad p = \sqrt{\vec{p}^2}. \tag{11}
\]

\(^6\) Without topological term that is.

\(^7\) Denoted by the \(\hat{\cdot}\)-notation throughout the remainder of the text.
It is worth underlining that in passing from (9) to (10), an irrelevant longitudinal term appearing in \( \hat{j}_\mu \cdots \hat{j}_\nu \) has been dropped from (10). It is then easily recognized that effective action (10) can be equivalently reformulated in terms of an Euclidean non-local gauge invariant three-dimensional theory, with gauge fixed action

\[
S_{\text{RQED}_3} = \int \! d^3x \left[ \frac{1}{2} F_{\mu\nu} - \frac{1}{\sqrt{-\partial^2}} F_{\mu\nu} + \bar{\psi}(i\partial + m)\psi + \frac{1}{2\xi}(\partial \cdot A)^2 \right],
\]

after the introduction of a new and now three-dimensional Abelian gauge field that, with a slight abuse of notation, we have again called \( A_\mu \). The gauge fixing term before or after the reduction does not need to be the same, so we have opted here for a simple linear gauge fixing rather than the involved reduced non-local gauge fixing term kept in \([10, 11]\). The gauge parameter \( \zeta \) here also carries a dimension, unlike \( \xi \) in (7). The renormalization properties of \( \text{RQED} \equiv \text{RQED}_3 \) were discussed in \([27, 28]\). It should be noted that (12) generates already at tree level a branch cut in the complex momentum plane in the photon propagator, with branch point at \( p^2 = 0 \). It is exactly the presence of the \( 1/\sqrt{-\partial^2} \) in the kinetic gauge term that also allows to keep the electromagnetic coupling constant \( e \) to remain dimensionless, even in a (reduced) three-dimensional space-time. Indeed, the new gauge field \( A_\mu \) still has mass dimension 1, while for standard QED\(_3 \) that mass dimension would amount to \( 1/2 \). The non-local operator \( \sqrt{-\partial^2} \) is to be understood via its three-dimensional Fourier space representation \([10]\)

\[
\frac{1}{\sqrt{-\partial^2}}(\vec{x} - \vec{x}') = \int \! d^3k \frac{e^{i\vec{k}(\vec{x} - \vec{x}')}}{2\pi^3 k}, \quad k = \sqrt{k^2}.
\]

If we add an Euclidean CS term, \( i\theta \int \! d^3x \, \varepsilon_{\mu\nu\rho} A_\mu \partial_\nu A_\rho \) to the action in (12), we can deduce the tree level photon propagator for a reduced Maxwell-CS theory, namely

\[
\hat{D}_{\mu\nu}(\vec{p}) = \frac{1}{2p} \frac{1}{(1 + \theta^2)} \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) - \frac{1}{2p^2} \frac{\theta}{(1 + \theta^2)} \varepsilon_{\mu\nu\rho} p_\rho + \frac{\zeta}{p^2} \frac{p_\mu p_\nu}{p^2},
\]

From the CS term, we can infer that \( \theta \) here is actually a dimensionless parameter, so unlike in standard QED\(_3 \), it does not provide the theory with a "topological photon mass". On the other hand, \( \theta \neq 0 \) does influence the photon propagator, not only by the presence of a T-odd contribution, but also by a normalization of the photon propagator. Intuitively, this corresponds to a down-scaling of the strength of the photon propagator, an effect not unlike increasing the mass of the exchanged particle.

### III. ONE-LOOP EXACTNESS OF TOPOLOGICAL PHOTON TERM IN REDUCED PLANAR QED

Our aim is now to prove that there will be no T-odd contributions to the gauge sector, i.e. the CS term, coming from radiative corrections beyond one-loop, even in the presence of electromagnetic background fields. Indeed, we

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8 The physical content of the theory will anyhow be gauge invariant and thus independent of the chosen gauge.

9 It is consistent with the observation that the RQED at the classical level is scale invariant, at least when the fermions are massless.
notice that any background field, can be added to the action —be it to QED$_4$ or RQED$_3$— via the gauge principle of minimal coupling with the fermion fields. Background fields must be treated classically and, in the same way as the gauge sector, they must be defined in four dimensions. For possible interesting physics involving background fields see for instance [29, 30], including in-plane fields as also considered in [31]. For example, minimal coupling means we replace in (12) the covariant derivative as follows

$$i\partial \rightarrow i\partial + i\tilde{A}_0\gamma^0 + i\tilde{A}_3\gamma^3$$

(15)

where the barred gauge fields are classical in nature. $\tilde{A}_0$ can describe a potential (electric field $\vec{E}$) applied in or orthogonal to the graphene sheet, while $\tilde{A}_3$ can be used to couple an in-plane magnetic field $\vec{B} \parallel \vec{e}_1$. Taking the non-relativistic limit of the corresponding Dirac equation, the latter coupling will provide the necessary magnetic field-magnetic moment coupling relevant for the Zeeman term, considered in [31]. It is important to realize that although graphene is a sheet and the fermions will have no classical dynamics outside of the plane due to an in-plane magnetic field, there is still the option for further quantum effects in the plane. In general, we will from now on work with

$$S_{\text{RQED}_3} = \int d^3x \left[ \frac{1}{2} F_{\mu\nu} \frac{1}{\sqrt{-\text{det} g}} F^{\mu\nu} + \Psi (i\partial + iA_s\gamma^s + m\gamma^3 + m_o\gamma^3\gamma^5)\Psi + \frac{1}{2\xi} (\partial \cdot A)^2 \right].$$

(16)

with $s$ running from 0 to 3 to allow for the most generic electromagnetic background. We have also allowed for the Haldane mass as another source of T-odd physics.$^{10}$ As explained before, we opted for the $m\Psi\gamma^3\gamma\Psi$-representation of the Dirac mass, although the following argument does not depend on which fermion masses are present, the actual numbers can however.

A. All order proof based on Ward identity

First, we will use the power of the global Ward identity associated to charge conservation to prove that (16) will generate a CS mass term for the photon at one-loop order, or not at all. It is important that the fermions are massive of some sort to avoid spurious infrared singularities, so we can hereafter safely consider zero momentum expansions. Such approach was suggested in [32] for standard QED$_3$ whilst avoiding the combinatorial elements of the original proof of [13].

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$^{10}$ Notice that a $\vec{E} \cdot \vec{B}$ would be another T-odd scalar quantity, if present.
If we decompose in Fourier space the three-dimensional photon $1PI$ propagator (self-energy) in its most general form in a linear covariant gauge that is compatible with all Ward (Slavnov-Taylor) identities,

$$\Pi_{\mu\nu}(\vec{p}) = \langle A_\mu(\vec{p})A_\nu(-\vec{p}) \rangle^{1PI} = \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \Pi(p^2) + \epsilon_{\mu\nu\rho\nu} p_\rho \Theta(p^2),$$

(17)

then $\lim_{p^2 \to 0} \Theta(p^2)$ is solely determined by one-loop corrections.

To show this explicitly, we start from the path integral,

$$I = \int [d\bar{\psi}] [d\psi] [dA_\mu] e^{-S_{\text{RQED3}}},$$

(18)

with $S_{\text{RQED3}}$ defined in (16). Then diagrammatically it is easily seen that at zero momentum, the graphs contributing to $\Pi_{\mu\nu}(p^2)$ are corresponding to those of the $1PI$ current-current correlator with zero momentum flow. We shall hence focus attention on $\langle j_\mu(x)j_\nu(y) \rangle^{1PI}$ and show that at zero momentum, it is fully determined at one-loop order.

Classically, we can couple the current $j_\mu(x)$ to the action via a local source $\eta_\mu(x)$ by considering

$$\Sigma = S_{\text{RQED3}} + \int d^3x \, \eta_\mu(x) j_\mu(x),$$

(19)

then

$$\partial_\mu \frac{\delta \Sigma}{\delta \eta_\mu(x)} = \bar{\psi}(x) \frac{\delta \Sigma}{\delta \bar{\psi}(x)} + \frac{\delta \Sigma}{\delta \psi(x)} \psi(x)$$

(20)

expresses that the current is conserved. This is nothing else than the Noether theorem in functional language. The classical identity becomes a Ward identity at the quantum level,

$$\partial_\mu \frac{\delta \Gamma}{\delta \eta_\mu} = \bar{\psi}(x) \bar{\psi}(x) + \frac{\delta \Gamma}{\delta \psi(x)} \psi(x),$$

(21)

Here, $\Gamma$ is the quantum effective action, viz. the generating functional for the $1PI$ correlation functions. We have also suppressed the space time variable $x$ to avoid notational clutter.

Let us now denote with $\mathcal{V}_0 \equiv -i \int d^3x \bar{\psi} A_\mu \gamma^\mu \psi$ the standard gauge-boson fermion vertex operator. Then we can infer from the Ward identity (21) that

$$\partial_\mu \langle j_\mu(x)j_\nu(y) \rangle^{1PI} = 0,$$

(22)
by taking another functional derivative of (21) w.r.t. \( \eta(y) \), followed by \( n \geq 0 \) derivatives w.r.t. the coupling constant \( e^{11} \) and setting all external sources and fields to zero at the end. Notice that each power of \( \mathcal{V}_0 \) is an integrated operator insertion, that is, one with zero momentum flow. Since (22) holds for any \( n \) and since any expectation value of operators evaluated with the path integral partition function (18) can be succinctly rewritten as

\[
\langle j_\mu(x) j_\nu(y) \rangle_{\text{PI}}^{\mathcal{S}_{\text{RQED}}} = \sum_{n \in \mathbb{N}} \langle j_\mu(x) j_\nu(y) e^n \mathcal{V}_0 \rangle_{\text{quad}}^{\text{PI}},
\]

where “quad” refers to the quadratic (free theory) approximation of \( \mathcal{S}_{\text{RQED}} \), we can equally well write

\[
\sum_{n \in \mathbb{N}} \partial_\mu \langle j_\mu(x) j_\nu(y) e^n \mathcal{V}_0 \rangle_{\text{quad}}^{\text{PI}} = 0,
\]

instead of (22).

For \( n \geq 0 \), each term in the expansion (23) can be expanded around zero momentum as

\[
\langle j_\mu(p) j_\nu(-p) \mathcal{V}_0 \rangle_{\text{quad}}^{\text{PI}} \rightarrow \lim_{k \to 0} \langle j_\mu(p + k/2) j_\nu(-p + k/2) \mathcal{V}_0 - 1 \mathcal{V}_k \rangle_{\text{quad}}^{\text{PI}},
\]

i.e. we let a small net momentum \( k \) flow through one of the vertices, keeping total momentum conservation in mind of course. This means we must exclude the \( n = 0 \) term as we need at least one vertex insertion. Due to the symmetry \( (\mu, p) \leftrightarrow (\nu, -p) \) present in expression (26), only the following expansion can hold at leading order in \((p, k)\),

\[
\langle j_\mu(p + k/2) j_\nu(-p + k/2) \mathcal{V}_0 - 1 \mathcal{V}_k \rangle_{\text{quad}}^{\text{PI}} = A_n \delta_{\mu\nu} + B_n \epsilon_{\mu\rho\nu} p_\rho \ldots
\]

Since \( k \) does not appear in the foregoing expression, we actually have \( B_n = b_n \) for \( n \geq 1 \) from the identification (26) together with the expansion (25).

\[\text{11} \] We assume that the \( e \) in front of the possible background fields has been absorbed into these fields.
The Fourier version of the constraint now reads

\[(p + k/2)_\mu \langle j_\mu(p + k/2) j_\nu(-p + k/2) \hat{\gamma}_0^{\mu-1} \hat{\gamma}_k \rangle^{\text{quad}}_{\text{1PI}} = 0\]  

(28)

Applying this to (27) leads, next to \(A_n = 0\), to \(b_n = 0\) for all \(n \geq 1\).

Putting everything back together, we have actually shown that

\[\langle j_\mu(p) j_\nu(-p) \rangle_{\text{SQED}}^{\text{1PI}} = b_0 \epsilon_{\mu\nu\rho} p_\rho + O(p^2),\]  

(29)

which is nothing else than the Coleman–Hill theorem for RQED, as the corresponding zeroth order diagram contributing to (29) is equivalent to the one-loop photon self-energy correction.

B. Four-component vs. two-component spinors

As we mentioned in Section II, there are several theoretical instances to create a mass gap in the Dirac regime of graphene \(\pi\)-electrons (see [9, 19, 33] for a detailed description of the different mass terms and their corresponding symmetry breaking). Here, we shall briefly survey how those mass terms reduce in the four- and two-component spinor description for these electrons. The spinors in \((2 + 1)\) dimensions are in a Lorentz \(SO(2, 1)\) reducible representation. We arrange the sublattice annihilation operators \((a\) and \(b)\) with their corresponding valley numbers (subscript + and −) as

\[\psi_+ = \begin{pmatrix} a_+ \\ b_+ \end{pmatrix}, \quad \psi_- = \begin{pmatrix} b_- \\ a_- \end{pmatrix},\]  

(30)

in two-component representation, and as

\[\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix},\]  

(31)

in the case of a four-component representation.

As in the four-component description we have at our disposal two matrices which anti-commute with respect to the rest \(\gamma^3\) and \(\gamma^5\), we have basically four kind of masses.\(^{14}\)

\(^{12}\) This condition also holds when the operators \(\hat{\gamma}_k\) would not be integrated, this can be shown by coupling the operator \(\bar{\psi} \gamma_0 \psi\) to the action \(\Sigma\) with another local source and by manipulating the corresponding Ward identity.

\(^{13}\) However, from the experimental point of view, is still very difficult to open a mass gap in a controllable way \(^{15}\).

\(^{14}\) We do not consider the internal spin 1/2 nature of the \(\pi\)-electrons. Considering it, the number of mass terms increases considerably \(^{19}\).
A standard mass term in four-component spinor language is of the form $m \overline{\psi} \psi = m \psi^\dagger \gamma^0 \psi$, which breaks both chiral symmetries (6a) and (6b), but it does not break time reversal symmetry in the four-dimensional matrix representation. This term mixes the flavours $+$ and $-$,

$$S_{\text{usual}} = - \int d^3 x m \overline{\psi} \psi = - \int d^3 x \left( \psi^\dagger_+ \psi_- + \psi^\dagger_- \psi_+ \right).$$

The mass terms considered in Section III, i.e., $im \overline{\psi} \gamma^5 \psi$ and $m \overline{\psi} \gamma^3 \psi$, break one of the extended chiral symmetries, (6a) and (6b) respectively, but preserve time reversal symmetry in four-dimensional matrix representation. The first case is related to the Kekulé distortion [34], while we can see that the second one allow us to rewrite the action in a two-component spinor decomposition as

$$S_{\gamma^3} = - \int d^3 x m 0 \overline{\psi} \gamma^3 \gamma^5 \psi = - \int d^3 x m_0 \left( \psi^\dagger_+ \sigma^3 \psi_+ + \psi^\dagger_- \sigma^3 \psi_- \right).$$

(32)

We will call this term the “normal” mass, as is the usual mass for a two-component spinor in $(2+1)$ dimensions with two different decoupled flavours $+$ and $-$.

The last possibility is the Haldane mass term [23], which does not break the chiral symmetries (6a) and (6b), but does break time reversal symmetry [9]. This term also admits a decoupled two-component spinor decomposition,

$$S_{\text{Haldane}} = - \int d^3 x m_0 \overline{\psi} \gamma^3 \psi = - \int d^3 x m_0 \left( \psi^\dagger_+ \sigma^3 \psi_+ + \psi^\dagger_- \sigma^3 \psi_- \right).$$

(33)

We can see that the mass terms (32) and (33) have different relative sign for the two flavours $+$ and $-$.

The CS mass term can be generated by T-odd fermion one-loop corrections. These corrections at zero momentum are of the form [22, 25, 35, 36]

$$\Gamma_{\mu \nu} \sim \frac{m}{|m|} \varepsilon_{\rho \nu \lambda} A_\rho \partial_\lambda,$$

(34)

implying that the term (32) will give a net zero contribution for the CS photon mass, while (33) does contribute. More precisely, we will get at the level of the action a (exact) radiatively introduced T-photon term

$$S_{\text{CS}} = \int d^3 x \left( - \frac{e^2}{4\pi} \frac{m}{|m|} \varepsilon_{\nu \rho \lambda} \partial_\nu A_\rho \right)$$

(35)

when a Haldane term (33) is coupled to RQED. Here is a nice place to appreciate again the role of the dimensionless coupling in RQED. Indeed, in the case of QED$_3$ the $e^2$ in front of (35) is what “feeds” the dynamical topological photon mass $\theta$ thanks to $e^2$ having mass dimension 1, whereas now the dimensionless nature of $e^2$ gives a dimensionless parameter $\theta$ in front of the CS term.
IV. OUTLOOK

We have shown that, in the framework of reduced QED in $(2 + 1)$ dimensions, the topological piece of the photon self-energy at zero momentum only receive quantum corrections up to one-loop. Using fundamental arguments based on the U(1) Ward identity, we have proven that all the two- and higher-loop contributions are identically zero. In other words, besides holding for ordinary QED$_3$, the Coleman–Hill theorem thus also applies in the case we are dealing with a theory where the gauge fields are not constrained to the plane while the fermions are. For completeness, we have also derived the tree-level photon propagator for this theory, taking into account the CS term. Interestingly, for the RQED case, the parameter $\theta$ in front of the CS term is not a mass, as for QED$_3$, but somehow acts as a dimensionless suppressing factor in the photon propagator (see (14)). Moreover, we computed the exact value of $\theta$ in case the four-component Dirac fermions are massive for two different realizations of the mass term, both relevant for graphene studies, see (32) and (33).

Our observations pave the road to investigate deeper the interconnection between the CS photon term and Haldane fermion mass in the specific case of RQED. Any interaction term or fermion mass has a direct influence in the vector and axial current channels which, in the context of graphene physics, provide us with relevant observables for transport phenomena. A complete mapping between the two sectors of the theory would also allow us to quantitatively investigate how the presence of external electromagnetic fields effectively manifest itself in the fermion sector. An important piece of information will be encoded in the $\theta$-sector of the photon propagator, which we expect to be quite sensitive to such background fields. Numerical estimates for the influence of the CS term on the Haldane mass and/or $\gamma^3$-Dirac mass making use of Dyson–Schwinger equations, along the same lines as the QED$_3$ study of [37], are currently being prepared and will be reported in forthcoming work.

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