Fuzzy linear systems and core-EP inverses

Yuefeng Gao ‡ Jing Li †
College of Science, University of Shanghai for Science and Technology
Shanghai 200093, China

Abstract: The main purpose of this paper is to provide a solution of the consistent fuzzy linear system and a generalized solution of the inconsistent fuzzy linear system involving the core-EP inverse of an associated matrix. Before this can be achieved, it is necessary to study the block structure of the core-EP inverse. Finally, results are illustrated with some numerical examples.

Keywords: core-EP inverse; fuzzy linear system; block structure

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1 Introduction

The field of fuzzy linear systems have been developed rapidly since the appearance of fuzzy numbers in [3]. In many applications, the system’s parameters are represented by fuzzy numbers rather than crisp numbers, so it is important to solve fuzzy linear systems. There are numerous papers devoted to the investigation of solutions of fuzzy linear systems [1, 3, 6].

Block structures of generalized inverses such as the Moore-Penrose inverse, group inverse, W-weighted Drazin inverse and core inverse contributed to solving fuzzy linear systems, see [6–8] for example. More specifically, the method of using the Moore-Penrose inverse to solve fuzzy linear systems is firstly presented in [8]. Next, B. Mihailović et al. [7] got general solutions of fuzzy linear systems using the block structures of group inverses. M. Nikuie and M.Z. Ahmad [9] explained the effect of W-weighted Drazin inverses in solving singular fuzzy linear systems. Recently, H. Jiang et al. [6] gave a method for solving fuzzy linear systems using the block structures of core inverses. It is well known that the core-EP inverse introduced by K.M. Prasad and K.S. Mohana in [11] is a generalization of the core

‡Corresponding author. E-mail: yfgao91@163.com
†E-mail: jingli0204@163.com
inverse and it can be calculated by core-EP decomposition in [12]. It makes sense that solving fuzzy linear systems can be broadened via using core-EP inverses.

The paper is organized as follows: in Section 2, the definitions of fuzzy linear systems and several types of generalized inverses have arisen. In addition, we present an algorithm for computing the core-EP inverse. In Section 3, we shall focus our attention on the block structure of the core-EP inverse of an associated matrix, which affords a better insight into getting a solution of the fuzzy linear system. In Section 4, We divide the problems into consistent and inconsistent fuzzy linear systems. Based on the obtained theoretical results, methods for obtaining a solution of the fuzzy linear system are proposed. In Section 5, numerical examples about dealing with consistent and inconsistent fuzzy linear systems are implemented.

2 Preliminaries

In this section, we shall review some notations, definitions and results which play important roles in the rest sections.

2.1 Fuzzy number and fuzzy linear system

Brief definitions and theorems related to fuzzy numbers and fuzzy linear systems are given in this part. The notation $X^T$ means the transpose of a vector $X$ in the following content.

Definition 2.1. [13] We represent an arbitrary fuzzy number $\tilde{z}(r)$, in parametric form, by an ordered pair of functions $\tilde{z}(r) = (z(r), \overline{z}(r))$, $r \in [0, 1]$, which satisfies the following requirements:

1. $z(r)$ is a bounded left continuous nondecreasing function over $[0, 1]$.
2. $\overline{z}(r)$ is a bounded left continuous nonincreasing function over $[0, 1]$.
3. $z(r) \leq \overline{z}(r)$.

For each real number $\lambda$, the scalar multiplication and the addition of fuzzy numbers can be described as follows:

1. $\tilde{z}(r) + \tilde{w}(r) = (z(r) + w(r), \overline{z}(r) + \overline{w}(r))$,
2. $\lambda \tilde{z}(r) = \left\{ \begin{array}{ll}
(\lambda z(r), \lambda \overline{z}(r)), & \lambda \geq 0, \\
(\lambda \overline{z}(r), \lambda z(r)), & \lambda < 0,
\end{array} \right.$
3. $\tilde{z}(r) = \tilde{w}(r)$ if and only if $z(r) = w(r)$ and $\overline{z}(r) = \overline{w}(r)$. 
Definition 2.2. [4] The fuzzy linear matrix system $A\tilde{X} = \tilde{Y}$

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\vdots \\
\tilde{x}_n
\end{bmatrix}
= \begin{bmatrix}
\tilde{y}_1 \\
\tilde{y}_2 \\
\vdots \\
\tilde{y}_n
\end{bmatrix},
\]

(2.1)

where the matrix $A = [a_{ij}]$ is a real matrix and $\tilde{x}_i, \tilde{y}_i$, $i = 1, \ldots, n$, are fuzzy numbers, is called a fuzzy linear system (FLS).

Definition 2.3. [4] A fuzzy number vector $\tilde{X}(r) = [\tilde{x}_1(r), \tilde{x}_2(r), \ldots, \tilde{x}_n(r)]^T$, where

\[\tilde{x}_i(r) = (\underline{x}_i(r), \bar{x}_i(r)), \quad i = 1, \ldots, n, \quad r \in [0,1],\]

is a solution of FLS (2.1) if it satisfies

\[
\begin{align*}
\sum_{j=1}^n a_{ij} \underline{x}_j(r) &= \underline{y}_i(r) \\
\sum_{j=1}^n a_{ij} \bar{x}_j(r) &= \bar{y}_i(r)
\end{align*}
\]

for $i = 1, \ldots, n$.

An significant fact was noted in [4], in order to get a solution of the FLS $A\tilde{X} = \tilde{Y}$ (2.1), it is sufficient to solve the following crisp linear system:

\[
SX(r) = Y(r), \quad r \in [0,1],
\]

i.e.

\[
\begin{bmatrix}
\underline{s}_{11} & \underline{s}_{12} & \cdots & \underline{s}_{1,2n} \\
\underline{s}_{21} & \underline{s}_{22} & \cdots & \underline{s}_{2,2n} \\
\vdots & \vdots & \ddots & \vdots \\
\underline{s}_{2n,1} & \underline{s}_{2n,2} & \cdots & \underline{s}_{2n,2n}
\end{bmatrix}
\begin{bmatrix}
\underline{x}_1(r) \\
\vdots \\
\underline{x}_n(r)
\end{bmatrix}
= \begin{bmatrix}
\underline{y}_1(r) \\
\vdots \\
\underline{y}_n(r)
\end{bmatrix},
\]

where $s_{ij}$ are determined as follows:

\[
a_{ij} \geq 0 \Rightarrow s_{ij} = a_{ij}, \quad s_{i+n,j+n} = a_{ij},
\]

\[
a_{ij} < 0 \Rightarrow s_{i,j+n} = -a_{ij}, \quad s_{i+n,j} = -a_{ij},
\]

and all the remaining $s_{ij}$ are taken zero.

We use the next notation to represent the structure of $S$:

\[
S = \begin{bmatrix}
D & E \\
E & D
\end{bmatrix}.
\]
where \( D \) and \( E \) are \( n \times n \) matrices, \( D = \begin{bmatrix} a_{ij}^+ \end{bmatrix}, E = \begin{bmatrix} a_{ij}^- \end{bmatrix}, a_{ij}^+ = a_{ij} \lor 0 \) and \( a_{ij}^- = -a_{ij} \lor 0 \). In this case, \( S \) is called the associated matrix of \( A \). We observe that \( A = A^+ - A^- = D - E \) and \( |A| = A^+ + A^- = D + E \).

According to [5], we get the following conclusions:

1. \( \text{rank}(S) < \text{rank}(S|Y) \), \( SX = Y \) does not have any solution, \( \tilde{A}X = \tilde{Y} \) is called an inconsistent FLS.

2. \( \text{rank}(S) = \text{rank}(S|Y) \), \( SX = Y \) has a solution, \( \tilde{A}X = \tilde{Y} \) is called a consistent FLS.

Furthermore, \( \text{(i)} \ \text{rank}(S) = \text{rank}(S|Y) = 2n \), \( SX = Y \) has the unique solution; \n(\text{ii}) \ \text{rank}(S) = \text{rank}(S|Y) < 2n \), \( SX = Y \) has infinite solutions.

### 2.2 core-EP inverse

While the original core inverse is restricted to matrices of index one, the core-EP inverse exists for any square matrices. That is to say, it extends the core inverse of a matrix from index one to an arbitrary index. We begin with recalling some related definitions. As usual, \( A^* \) denotes the transpose of the matrix \( A \). The index of matrix \( A \in \mathbb{R}^{n \times n} \), denoted by \( \text{ind}(A) = k \), is the smallest nonnegative integer \( k \) such that \( \text{rank}(A^{k+1}) = \text{rank}(A^k) \).

Given \( A \in \mathbb{R}^{n \times n} \), the matrix \( X \) satisfying one or more of the following matrix equations has been studied extensively.

\[
\begin{align*}
(1) \ AXA &= A; & (1^k) \ AX^{k+1} &= A^k; \\
(2) \ XAX &= X; & (2') \ AX^2 &= X; \\
(3) \ (AX)^* &= AX; & (4) \ (XA)^* &= XA.
\end{align*}
\]

**Definition 2.4.** For a matrix \( A \in \mathbb{R}^{n \times n} \) with index \( k \).

(i) \( \text{(10)} \) \( X \) is the Moore-Penrose inverse of \( A \) if and only if \( X \) satisfies \( (1), (2), (3), (4) \), denoted by \( X = A^\dagger \).

(ii) \( \text{(2)} \) \( X \) is the core inverse of \( A \) if and only if \( X \) satisfies \( (1), (2)' \), \( (3) \), denoted by \( X = A^\# \).

(iii) \( \text{(11)} \) \( X \) is the core-EP inverse of \( A \) if and only if \( X \) satisfies \( (1^k), (2)' \), \( (3) \), denoted by \( X = A^{\otimes} \).

Here, we describe a method for computing the core-EP inverse by applying the core-EP decomposition \([12]\). Let \( A \in \mathbb{R}^{n \times n}, \text{ind}(A) = k \), then there exists the unitary matrix \( U \) such that

\[
A = U \begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^*,
\]
where $T$ is nonsingular and $N^k = 0$. The core-EP inverse of $A$ has the following form:

$$A^{\ominus} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$  \hfill (2.3)

We present an algorithm for computing the core-EP inverse of $A$, when $\text{ind}(A) = k$, as follows:

| Table 1: Computation of the core-EP inverse |
|--------------------------------------------|
| **Algorithm:** Computation of $A^{\ominus} : \text{ind}(A) = k$ |
| 1. Input: $A$ is a $n \times n$ matrix; |
| 2. Calculate sub-matrices $U$, $T$, $S$ and $N$; |
| 3. Calculate matrix $T^{-1}$; |
| 4. Determine the core-EP inverse using (2.3). |

It is worth mentioning that the core-EP inverse $A^{\ominus}$ can also be expressed as (see [11]):

$$A^{\ominus} = A^k[(A^*)^kA^{k+1}]^\dagger(A^*)^k.$$  \hfill (2.4)

3 Block structure of core-EP inverse of the associated matrix $S$

The block structure of $S^{\ominus}$, which we shall consider in this part, is a powerful tool in solving FLS. Provided that the structure of the associated matrix $S$ is $\begin{bmatrix} D & E \\ E & D \end{bmatrix}$, where $D, E \in \mathbb{R}^{n \times n}$.

**Lemma 3.1.** Let $S$ be an arbitrary $2n \times 2n$ matrix with the form

$$S = \begin{bmatrix} D & E \\ E & D \end{bmatrix},$$

where both $D$ and $E$ are $n \times n$ matrices, then

$$S^n = \begin{bmatrix} \frac{1}{4}(D+E)^n + \frac{1}{4}(D-E)^n & \frac{1}{4}(D+E)^n - \frac{1}{4}(D-E)^n \\ \frac{1}{4}(D+E)^n - \frac{1}{4}(D-E)^n & \frac{1}{4}(D+E)^n + \frac{1}{4}(D-E)^n \end{bmatrix}.$$  \hfill (2.5)

**Proof:** This result can be proved by mathematical induction.

Firstly,
\[ S^2 = \begin{bmatrix} D^2 + E^2 & DE + ED \\ DE + ED & D^2 + E^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(D + E)^2 + \frac{1}{2}(D - E)^2 & \frac{1}{2}(D + E)^2 - \frac{1}{2}(D - E)^2 \\ \frac{1}{2}(D + E)^2 - \frac{1}{2}(D - E)^2 & \frac{1}{2}(D + E)^2 + \frac{1}{2}(D - E)^2 \end{bmatrix}. \]

If the situation of \( n = k \) is true, then we have
\[
S^k = \begin{bmatrix} \frac{1}{2}(D + E)^k + \frac{1}{2}(D - E)^k & \frac{1}{2}(D + E)^k - \frac{1}{2}(D - E)^k \\ \frac{1}{2}(D + E)^k - \frac{1}{2}(D - E)^k & \frac{1}{2}(D + E)^k + \frac{1}{2}(D - E)^k \end{bmatrix}.
\]

When \( n = k + 1 \),
\[
S^{k+1} = \begin{bmatrix} D & E \\ E & D \end{bmatrix} \begin{bmatrix} \frac{1}{2}(D + E)^k + \frac{1}{2}(D - E)^k & \frac{1}{2}(D + E)^k - \frac{1}{2}(D - E)^k \\ \frac{1}{2}(D + E)^k - \frac{1}{2}(D - E)^k & \frac{1}{2}(D + E)^k + \frac{1}{2}(D - E)^k \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(D + E)^{k+1} + \frac{1}{2}(D - E)^{k+1} & \frac{1}{2}(D + E)^{k+1} - \frac{1}{2}(D - E)^{k+1} \\ \frac{1}{2}(D + E)^{k+1} - \frac{1}{2}(D - E)^{k+1} & \frac{1}{2}(D + E)^{k+1} + \frac{1}{2}(D - E)^{k+1} \end{bmatrix}.
\]

It completes the proof. \( \square \)

We deal with the block structure of the core-EP inverse in the next statement.

**Theorem 3.2.** Let \( A \) be the coefficient matrix of FLS and \( S \) be its associated matrix. The core-EP inverse \( S^{(S)} \) of \( S \) is

\[
S^{(S)} = \begin{bmatrix} H & Z \\ Z & H \end{bmatrix}
\]

if and only if
\[
H = \frac{1}{2}[(D + E)^{\odot} + (D - E)^{\odot}], \quad Z = \frac{1}{2}[(D + E)^{\odot} - (D - E)^{\odot}].
\]

**Proof.** \( \Rightarrow \) Firstly, since, as the concept of the core-EP inverse stated, \( S^{(S)}S^{k+1} = S^k \), then,
\[
\begin{bmatrix} H & Z \\ Z & H \end{bmatrix} \begin{bmatrix} D & E \\ E & D \end{bmatrix}^{k+1} = \begin{bmatrix} D & E \\ E & D \end{bmatrix}^{k}.
\]
So,
\[
\begin{bmatrix} HD + ZE & ZD + HE \\ ZD + HE & HD + ZE \end{bmatrix} \begin{bmatrix} D & E \\ E & D \end{bmatrix}^{k} = \begin{bmatrix} D & E \\ E & D \end{bmatrix}^{k}.
\]

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In view of Lemma 3.1, we claim that

\[
\begin{bmatrix}
  HD + ZE & ZD + HE \\
  ZD + HE & HD + ZE \\
\end{bmatrix}
\begin{bmatrix}
  \frac{1}{4}(D + E)^k + \frac{1}{4}(D - E)^k \\
  \frac{1}{4}(D + E)^k - \frac{1}{4}(D - E)^k \\
\end{bmatrix}
\begin{bmatrix}
  \frac{1}{2}(D + E)^k + \frac{1}{2}(D - E)^k \\
  \frac{1}{2}(D + E)^k - \frac{1}{2}(D - E)^k \\
\end{bmatrix}
= \begin{bmatrix}
  \frac{1}{4}(D + E)^k + \frac{1}{4}(D - E)^k \\
  \frac{1}{4}(D + E)^k - \frac{1}{4}(D - E)^k \\
\end{bmatrix}
\]

which guarantees,

\[
(ZD + HE)[(D + E)^k + (D - E)^k] + (HD + ZE)[(D + E)^k - (D - E)^k]
= (D + E)^k - (D - E)^k,
\]

\[
(HD + ZE)[(D + E)^k + (D - E)^k] + (ZD + HE)[(D + E)^k - (D - E)^k]
= (D + E)^k + (D - E)^k.
\]

This amounts to the two formulas,

\[
(H + Z)(D + E)^{k+1} + (H - Z)(D - E)^{k+1} = (D + E)^k + (D - E)^k,
\]

\[
(H + Z)(D + E)^{k+1} - (H - Z)(D - E)^{k+1} = (D + E)^k - (D - E)^k.
\]

Clearly these mean,

\[
(H + Z)(D + E)^{k+1} = (D + E)^k, \quad (H - Z)(D - E)^{k+1} = (D - E)^k,
\]

i.e.

\[
H + Z = (D + E)^{(1^k)}, \quad H - Z = (D - E)^{(1^k)}.
\]

Hence,

\[
H = \frac{1}{2} \left[ (D + E)^{(1^k)} + (D - E)^{(1^k)} \right],
Z = \frac{1}{2} \left[ (D + E)^{(1^k)} - (D - E)^{(1^k)} \right].
\]

Secondly, from \(S(S^\oplus)^2 = S^\oplus\), it follows that

\[
\begin{bmatrix}
  D & E \\
  E & D \\
\end{bmatrix}
\begin{bmatrix}
  H & Z \\
  Z & H \\
\end{bmatrix}
= \begin{bmatrix}
  H & Z \\
  Z & H \\
\end{bmatrix}.
\]

It gives

\[
\begin{bmatrix}
  (DH + EZ)H + (DZ + EH)Z & (DH + EZ)Z + (DZ + EH)H \\
  (DH + EZ)Z + (DZ + EH)H & (DH + EZ)H + (DZ + EH)Z \\
\end{bmatrix}
= \begin{bmatrix}
  H & Z \\
  Z & H \\
\end{bmatrix}.
\]

Thus

\[
H = (DH + EZ)H + (DZ + EH)Z, \quad Z = (DH + EZ)Z + (DZ + EH)H.
\]
This shows that, \( H + Z = (D + E)(H + Z)^2 \), \( H - Z = (D - E)(H - Z)^2 \), i.e.
\[
H + Z = (D + E)(^2), \quad H - Z = (D - E)(^2).
\]
We conclude that
\[
H = \frac{1}{2} \left[ (D + E)^2 + (D - E)^2 \right], \quad Z = \frac{1}{2} \left[ (D + E)^2 - (D - E)^2 \right].
\]
Lastly, according to \( SS^\dagger \ast = SS^\dagger \), we obtain
\[
\begin{bmatrix} H & Z \\ Z & H \end{bmatrix} \ast \begin{bmatrix} D & E \\ E & D \end{bmatrix} \ast = \begin{bmatrix} D & E \\ E & D \end{bmatrix} \ast \begin{bmatrix} H & Z \\ Z & H \end{bmatrix}.
\]
It is not difficult to prove,
\[
(DH + EZ)^\ast = DH + EZ, \quad (DZ + EH)^\ast = DZ + EH.
\]
So we have
\[
[(D + E)(H + Z)]^\ast = (D + E)(H + Z), \quad [(D - E)(H - Z)]^\ast = (D - E)(H - Z),
\]
i.e.
\[
H + Z = (D + E)^3, \quad H - Z = (D - E)^3.
\]
Clearly,
\[
H = \frac{1}{2} \left[ (D + E)^3 + (D - E)^3 \right], \quad Z = \frac{1}{2} \left[ (D + E)^3 - (D - E)^3 \right].
\]
To sum up,
\[
H = \frac{1}{2} \left[ (D + E)^\circ + (D - E)^\circ \right], \quad Z = \frac{1}{2} \left[ (D + E)^\circ - (D - E)^\circ \right].
\]
(\( \Rightarrow \)) Given
\[
H = \frac{1}{2} \left[ (D + E)^\circ + (D - E)^\circ \right], \quad Z = \frac{1}{2} \left[ (D + E)^\circ - (D - E)^\circ \right],
\]
it is immediate that
\[
(D + E)(H + Z)^2 = (H + Z), \quad (D - E)(H - Z)^2 = (H - Z),
\]
\[
[(D + E)(H + Z)]^\ast = (D + E)(H + Z), \quad [(D - E)(H - Z)]^\ast = (D - E)(H - Z),
\]
\[
(H + Z)(D + E)^k+1 = (D + E)^k, \quad (H - Z)(D - E)^k+1 = (D - E)^k.
\]

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First of all, from

\[ H = \frac{1}{2} (H + Z) + \frac{1}{2} (H - Z) \]
\[ = \frac{1}{2} (D + E)(H + Z)^2 + \frac{1}{2} (D - E)(H - Z)^2 \]
\[ = (DH + EZ)H + (DZ + EH)Z, \]

\[ Z = \frac{1}{2} (H + Z) - \frac{1}{2} (H - Z) \]
\[ = \frac{1}{2} (D + E)(H + Z)^2 - \frac{1}{2} (D - E)(H - Z)^2 \]
\[ = (DH + EZ)Z + (DZ + EH)H, \]

it follows that

\[
\begin{bmatrix}
D & E \\
E & D \\
\end{bmatrix}
\begin{bmatrix}
H & Z \\
Z & H \\
\end{bmatrix}
= 
\begin{bmatrix}
H & Z \\
Z & H \\
\end{bmatrix}.
\]

Secondly, note that

\[ DH + EZ = \frac{1}{2} (D + E)(H + Z) + \frac{1}{2} (D - E)(H - Z) \]
\[ = \frac{1}{2} (H + Z)^* (D + E)^* + \frac{1}{2} (H - Z)^* (D - E)^* \]
\[ = (DH + EZ)^*, \]

\[ EH + DZ = \frac{1}{2} (D + E)(H + Z) - \frac{1}{2} (D - E)(H - Z) \]
\[ = \frac{1}{2} (H + Z)^* (D + E)^* - \frac{1}{2} (H - Z)^* (D - E)^* \]
\[ = (EH + DZ)^*. \]

So we derive that

\[
\begin{bmatrix}
H & Z \\
Z & H \\
\end{bmatrix}

\begin{bmatrix}
D & E \\
E & D \\
\end{bmatrix}
^* = 
\begin{bmatrix}
D & E \\
E & D \\
\end{bmatrix}
\begin{bmatrix}
H & Z \\
Z & H \\
\end{bmatrix}.
\]

At last, we find that

\[ (H + Z)(D + E)^{k+1} = (D + E)^k, \]
\[ (H - Z)(D - E)^{k+1} = (D - E)^k. \]

So,

\[
(H + Z)(D + E)^{k+1} - (H - Z)(D - E)^{k+1} = (D + E)^k - (D - E)^k, \]
\[ (H + Z)(D + E)^{k+1} + (H - Z)(D - E)^{k+1} = (D + E)^k + (D - E)^k, \]

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i.e.
\[
(ZD + HE)[(D + E)^k + (D - E)^k] + (ZE + HD)[(D + E)^k - (D - E)^k]
\]
\[
= (D + E)^k - (D - E)^k,
\]
\[
(HD + ZE)[(D + E)^k + (D - E)^k] + (HE + ZD)[(D + E)^k - (D - E)^k]
\]
\[
= (D + E)^k + (D - E)^k.
\]

Evidently,
\[
\frac{1}{2}(HD + ZE) \left[(D + E)^k - (D - E)^k\right] + \frac{1}{2}(HE + ZD) \left[(D + E)^k - (D - E)^k\right]
\]
\[
= \frac{1}{2} \left[(D + E)^k + (D - E)^k\right],
\]
\[
\frac{1}{2}(ZD + HE) \left[(D + E)^k - (D - E)^k\right] + \frac{1}{2}(ZE + HD) \left[(D + E)^k - (D - E)^k\right]
\]
\[
= \frac{1}{2} \left[(D + E)^k - (D - E)^k\right].
\]

With the application of Lemma 3.1,
\[
\begin{bmatrix}
  H & Z \\
  Z & H
\end{bmatrix}
\begin{bmatrix}
  D & E \\
  E & D
\end{bmatrix}^{k+1} = \begin{bmatrix}
  D & E \\
  E & D
\end{bmatrix}^k.
\]

In conclusion,
\[
\S\oplus = \begin{bmatrix}
  H & Z \\
  Z & H
\end{bmatrix}.
\]

\[\square\]

4 Methods for giving a solution of FLS

In this section, new methods involving core-EP inverses for giving a solution of FLS are proposed. For the consistent and inconsistent FLS, we will study them separately. We now state the main theorems of this paper. \(\mathcal{R}(X)\) denotes the column space of a matrix \(X\).

**Theorem 4.1.** \(\S\oplus Y\) is a solution of the crisp linear system \(SX = Y\) if and only if \(Y \in \mathcal{R}(S^k)\), where \(k = \text{ind}(S)\).

**Proof.** (\(\Leftarrow\)) Due to \(Y \in \mathcal{R}(S^k)\), we have \(Y = S^k T\), where \(T \in \mathbb{R}^{n \times 1}\).

According to the definition of core-EP inverse,
So,
\[
SS\oplus Y = SS\oplus S^k T = S^k T = Y.
\]
It turns out that $S^\ominus Y$ is a solution of $SX = Y$.

(\Rightarrow) Since $S^\ominus Y$ is a solution of $SX = Y$, that $SS^\ominus Y = Y$.

From $S^\ominus = S(S^\ominus)^2$, it follows that

$$Y = SS^\ominus Y = SS(S^\ominus)^2Y = S^2(S^\ominus)^2Y = \cdots = S^k(S^\ominus)^kY,$$

which leads to $Y \in R(S^k)$.

Proposed method 1:

(a) If $\text{ind}(S) = 0$, then the crisp linear system $SX = Y$ is consistent and $X = S^{-1}Y$ is clearly the unique solution of $SX = Y$.

(b) If $\text{ind}(S) \neq 0$ and $Y \in R(S^k)$, then the crisp linear system $SX = Y$ is consistent and $X = S^\ominus Y$ is a solution of $SX = Y$, see Theorem 4.1.

According to [5, Theorem 2.5 and Definition 2.5], if the crisp linear system $SX = Y$ (2.2) is inconsistent, then the associated FLS $A\tilde{X} = \tilde{Y}$ (2.1) is also inconsistent. Under this circumstances, we desire to find a generalized solution of the inconsistent FLS $A\tilde{X} = \tilde{Y}$ (2.1) by the following two approaches.

Proposed method 2:

(i) Through solving the consistent crisp linear system

$$SX = S^k(S^\ominus)^{(1,3)}Y.$$  \hfill (4.1)

By Theorem 4.1

$$X = S^\ominus S^k(S^\ominus)^{(1,3)}Y = S^\ominus Y$$ \hfill (4.2)

is a solution of the above crisp linear system.

(ii) Through solving the consistent crisp linear system

$$(S^k)^*SX = (S^k)^*Y.$$ \hfill (4.3)

By Theorem 4.1

$$X = [(S^k)^*S]^\ominus (S^k)^*Y = S^\ominus Y$$ \hfill (4.4)

is a solution of the above crisp linear system.

Definition 4.2. For the inconsistent FLS $A\tilde{X} = \tilde{Y}$ (2.1), $X = S^\ominus Y$ is a solution of the crisp linear system (4.7) (resp. (4.3)), then associated fuzzy number vector $\tilde{X}$ is called a generalized solution of the FLS $A\tilde{X} = \tilde{Y}$ (2.1).
5 Numerical examples

Our aim in this section is to use some examples about obtaining a solution of the consistent and inconsistent FLS to illustrate the methods presented in this paper. All the numerical tasks have been performed by using Matlab R2019b.

Example 5.1. Consider the following $2 \times 2$ order consistent FLS:

\[
\begin{bmatrix}
-2 & 1 \\
4 & -2
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2
\end{bmatrix}
= \begin{bmatrix}
(-1 + 3r, 3 - r) \\
(-6 + 2r, 2 - 6r)
\end{bmatrix}.
\]

(5.1)

The extended $4 \times 4$ order crisp linear system $SX = Y$ is

\[
\begin{bmatrix}
0 & 1 & 2 & 0 \\
4 & 0 & 0 & 2 \\
2 & 0 & 0 & 1 \\
0 & 2 & 4 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
-\tilde{x}_1 \\
-\tilde{x}_2
\end{bmatrix}
= \begin{bmatrix}
-1 + 3r \\
-6 + 2r \\
-3 + r \\
-2 + 6r
\end{bmatrix}.
\]

(5.2)

Since $\text{rank}(S) = \text{rank}(S | Y) = 2$, then the crisp linear system $SX = Y$ is consistent.

Note that $\text{ind}(S) = 1$.

By Proposed method 1, a solution of (5.2) is given by

\[
X = S^\oplus Y = S^\oplus Y.
\]

The core inverse of $S$ is

\[
S^\ominus = \begin{bmatrix}
0.0000 & 0.1000 & 0.0500 & 0.0000 \\
0.1000 & 0.0000 & 0.0000 & 0.2000 \\
0.0500 & 0.0000 & 0.0000 & 0.1000 \\
0.0000 & 0.2000 & 0.1000 & 0.0000
\end{bmatrix}.
\]

Finally, a solution of the consistent linear system (5.2) is

\[
X = \begin{bmatrix}
\frac{\tilde{x}_1}{\tilde{x}_2} \\
-\frac{\tilde{x}_1}{\tilde{x}_2}
\end{bmatrix}
= \begin{bmatrix}
-0.7500 + 0.2500r \\
-0.5000 + 1.5000r \\
-0.2500 + 0.7500r \\
-1.5000 + 0.5000r
\end{bmatrix}.
\]

Then the associated fuzzy number vector

\[
\tilde{X} = \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2
\end{bmatrix}
= \begin{bmatrix}
(-0.7500 + 0.2500r, 0.2500 - 0.7500r) \\
(-0.5000 + 1.5000r, 1.5000 - 0.5000r)
\end{bmatrix}.
\]

is a solution of the FLS (5.1).
Example 5.2. Consider the following $3 \times 3$ order consistent FLS:

\[
\begin{bmatrix}
2 & 0 & 0 \\
-1 & 1 & 1 \\
-1 & -1 & -1
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3
\end{bmatrix}
= 
\begin{bmatrix}
(4r, -8) \\
(12 - r, -4 - 3r) \\
(8 + r, -8 - r)
\end{bmatrix}.
\tag{5.3}
\]

The extended $6 \times 6$ order crisp linear system $SX = Y$ is

\[
\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3 \\
\tilde{x}_4 \\
\tilde{x}_5 \\
\tilde{x}_6
\end{bmatrix}
= 
\begin{bmatrix}
4r \\
12 - r \\
8 + r \\
8 \\
4 + 3r \\
8 + r
\end{bmatrix}.
\tag{5.4}
\]

Note that $\text{ind}(S) = 2$. We can get

\[
S^2 = 
\begin{bmatrix}
4 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 4 & 1 & 1 \\
2 & 1 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 4 & 0 & 0 \\
4 & 1 & 1 & 0 & 1 & 1 \\
2 & 1 & 1 & 2 & 1 & 1
\end{bmatrix}.
\]

Then, $Y = S^2 \cdot \begin{bmatrix} r \\ 2 \\ -5 - r \\ 2 \\ 4 \\ 3 \end{bmatrix}$, $Y \in \mathcal{A}(S^2)$.

According to Theorem 4.1, $X = S^\odot Y$ is a solution of (5.4). So,

\[
S^\odot = 
\begin{bmatrix}
0.3750 & -0.1250 & 0.0000 & 0.1250 & 0.1250 & 0.0000 \\
-0.2500 & 0.2500 & 0.1250 & 0.0000 & 0.0000 & 0.1250 \\
-0.1250 & 0.1250 & 0.1250 & -0.1250 & 0.1250 & 0.1250 \\
0.1250 & 0.1250 & 0.0000 & 0.3750 & -0.1250 & 0.0000 \\
0.0000 & 0.0000 & 0.1250 & -0.2500 & 0.2500 & 0.1250 \\
-0.1250 & 0.1250 & 0.1250 & -0.1250 & 0.1250 & 0.1250
\end{bmatrix}.
\]
Therefore, a solution of (5.4) is

\[
X = \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3 \\
-x_1 \\
-x_2 \\
-x_3 \\
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
2r \\
5 - r \\
3 \\
4 \\
1 + r \\
3
\end{bmatrix},
\]

then the associated fuzzy number vector

\[
\tilde{X} = \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3
\end{bmatrix} = \begin{bmatrix}
(2r, -4) \\
(5 - r, 1 - r) \\
(3, -3)
\end{bmatrix}.
\]

is a solution of the FLS (5.3).

**Example 5.3.** Consider the following 2\times2 order inconsistent FLS:

\[
\begin{bmatrix}
-1 & 1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2
\end{bmatrix} = \begin{bmatrix}
(3, 2 + r) \\
(4, 8r)
\end{bmatrix}.
\] (5.5)

The extended 4\times4 order crisp linear system \( SX = Y \) is

\[
\begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
-x_1 \\
-x_2
\end{bmatrix} = \begin{bmatrix}
3 \\
4 \\
-2 - r \\
-8r
\end{bmatrix}.
\] (5.6)

Since \( \text{rank}(S) = 2 \), \( \text{rank}(S \mid Y) = 4 \), \( \text{rank}(S) < \text{rank}(S \mid Y) \), then the crisp linear system \( SX = Y \) (5.6) is inconsistent and the FLS (5.3) is also inconsistent. Note that \( \text{ind}(S) = 2 \).

(i) We consider the following consistent crisp linear system:

\[
SX = S^2(S^2)^{(1, 3)}Y.
\] (5.7)

The core-EP decomposition of \( S \) is

\[
S = U \begin{bmatrix}
T & S \\
0 & N
\end{bmatrix} U^*,
\]

where

\[
U = \begin{bmatrix}
0.5000 & 0.5000 & 0.7071 & 0.0000 \\
0.5000 & 0.5000 & -0.7071 & 0.0000 \\
0.5000 & -0.5000 & 0.0000 & 0.7071 \\
0.5000 & -0.5000 & 0.0000 & -0.7071
\end{bmatrix},
\]

\[
T = \begin{bmatrix}
2.0000
\end{bmatrix}
\]
\( S = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \) and \( N = \begin{bmatrix} 0.0000 & -1.4142 & 1.4142 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \end{bmatrix} \).

By Algorithm,

\[
S\hat{=} = U \begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^* = \begin{bmatrix} 0.1250 & 0.1250 & 0.1250 & 0.1250 \\ 0.1250 & 0.1250 & 0.1250 & 0.1250 \\ 0.1250 & 0.1250 & 0.1250 & 0.1250 \\ 0.1250 & 0.1250 & 0.1250 & 0.1250 \end{bmatrix}.
\]

According to Proposed method 2, a solution of (5.7) is given by

\[
X = S\hat{=}Y.
\]

Therefore,

\[
X = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ -\bar{x}_1 \\ -\bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0.1250 & 0.1250 & 0.1250 & 0.1250 \\ 0.1250 & 0.1250 & 0.1250 & 0.1250 \\ 0.1250 & 0.1250 & 0.1250 & 0.1250 \\ -2 - r \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 4 \\ -8r \end{bmatrix} = \begin{bmatrix} 0.6250 - 1.1250r \\ 0.6250 - 1.1250r \\ 0.6250 - 1.1250r \\ 0.6250 - 1.1250r \end{bmatrix},
\]

then the associated fuzzy number vector

\[
\hat{X} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} (0.6250 - 1.1250r, -0.6250 + 1.1250r) \\ (0.6250 - 1.1250r, -0.6250 + 1.1250r) \end{bmatrix},
\]

is a generalized solution of the FLS (5.5).

(ii) We focus on the following consistent crisp linear system:

\[
(S^2)^*X = (S^2)^*Y. \tag{5.8}
\]

According to Proposed method 2,

\[
X = S\hat{=}Y = \begin{bmatrix} 0.1250 & 0.1250 & 0.1250 & 0.1250 \\ 0.1250 & 0.1250 & 0.1250 & 0.1250 \\ 0.1250 & 0.1250 & 0.1250 & 0.1250 \\ 0.1250 & 0.1250 & 0.1250 & 0.1250 \end{bmatrix}.
\]
is a solution of (5.8), then the associated fuzzy number vector
\[ \tilde{X} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} (0.6250 - 1.1250r, -0.6250 + 1.1250r) \\ (0.6250 - 1.1250r, -0.6250 + 1.1250r) \end{bmatrix}, \]
is a generalized solution of the FLS (5.5).

**Concluding remarks.** In the above context, we give a solution of the consistent FLS \( \tilde{A} \tilde{X} = \tilde{Y} \) and a generalized solution of the inconsistent FLS \( \tilde{A} \tilde{X} = \tilde{Y} \). It is natural to ask whether we can obtain the general solution of the FLS \( \tilde{A} \tilde{X} = \tilde{Y} \) and this question will be our future research topic.

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