Our universe hosts various large-scale structures from voids to galaxy clusters, so it would be interesting to find some simple and reasonable measure to describe the inhomogeneities in the universe. We explore two different methods for this purpose: the relative information entropy and the Weyl curvature tensor. These two quantities monotonically increase in the process of structure formation, characterizing the deviation of the actual distribution of matter from the unperturbed background. We calculate these two measures in the spherically symmetric Lemaître–Tolman–Bondi model in the dust universe. Both exact and perturbative calculations are presented, and we find that these two measures are correlated via a kinematical backreaction term.

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I. INTRODUCTION

The standard model of cosmology is usually based on two preconditions: (1) the dynamics of cosmological evolution is governed by Einstein’s general relativity; (2) the universe is spatially homogeneous and isotropic, as described mathematically by the Friedmann–Robertson–Walker (FRW) metric. The first precondition has stood many astronomical tests through the past century. However, the second one, always named as “cosmological principle”, is not so well-established. Data from various cosmological experiments, e.g. the detections of the anisotropies in the cosmic microwave background, have already confirmed with high precision that the universe is indeed very homogeneous at early times and large scales. Whereas, at late times (matter-dominated era) or small scales (10^2 Mpc), due to gravitational instability, regions that are slightly overdense will attract matter from the surroundings, in the process becoming even more overdense and vice versa. As a result, today the universe has a well-developed nonlinear structure that cannot at all scales be described by the FRW model. Consequently, the cosmological principle still deserves serious considerations.

The irreversibility of structure formation in the universe reminds us of the process of entropy increasing in thermodynamics. Their resemblance naturally leads us to attempt to introduce some kind of “entropy” to characterize the cosmological structure formation. This issue aroused attention of many people in recent years. Motivated by the Penrose conjecture and thermodynamical considerations on gravitational field, Clifton, Ellis, and Tavakol proposed a measure of gravitational entropy based on the Bel–Robinson tensor, which is the unique totally symmetric traceless tensor constructed from the Weyl tensor. It was shown that this measure is applicable to many models under certain conditions, ranging from the exact Schwarzschild black hole solution to the perturbed FRW model. Moreover, Sussman introduced a quasi-local scalar weighted average for the study of the Lemaître–Tolman–Bondi (LTB) dust model. Considering the asymptotic limits in this framework, Sussman and Larena pointed out that the proposal in Ref. is directly related to a negative correlation of the fluctuations of the energy density and the Hubble rate, for entropy production, time evolution and radial scaling. This result again convinces us that there may be some latent relationship between structure formation and gravitational entropy yet to be specified.

The aim of this paper is to explore some quantity that can measure the structure formation in the inhomogeneous universe, i.e. to investigate some way to describe the deviation of the actual distribution of matter from the FRW background. This problem has been discussed in Ref. where two different measures were studied: the Kullback–Leibler (KL) relative information entropy \( S_D \) and the contraction of the Weyl tensor \( C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho} \).

It was found that during cosmological evolution, \( S_D \) increases monotonically, and can be further linked to the averaged Weyl scalar in the cosmological perturbation theory (up to second order). Hence, they both serve...
as the reasonable measures for structure formation, and their only difference is a kinematical backreaction term $Q_D$,\[
\frac{S_D}{V_D} = \frac{9}{32\pi G} \left( \frac{v^2}{8} C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}_D + Q_D \right). \tag{1}
\]

The present paper is a successive research of Ref. [4]. Here, we work in a specific model — the spherically symmetric LTB metric — an exact solution to Einstein’s equations describing an inhomogeneous but still isotropic space-time. We will prove that Eq. (1) is valid only for the growing mode of the scalar cosmological perturbations in the context of the LTB model, and will extend this relation to more generic cases.

II. RELATIVE INFORMATION ENTROPY, WEYL CURVATURE, AND KINEMATICAL BACKREACTION

In this section, we explain the meanings of the KL relative entropy in information theory, the Weyl tensor in differential geometry, and the kinematical backreaction in averaging problem of the inhomogeneous universe, respectively.

A. KL Relative information entropy

The relative information entropy in cosmology is a direct analogue of the KL divergence widely used in statistics, probability theory, and information theory [5],\[
S\{p||q\} := \sum_i p_i \ln \frac{p_i}{q_i},
\]
which measures the difference between two probability distributions $\{p_i\}$ and $\{q_i\}$. Typically, $\{p_i\}$ denotes an actual distribution of data, while $\{q_i\}$ represents the presumed one or the theoretical description of $\{p_i\}$.

The KL divergence possesses several advantageous properties: (1) it is always nonnegative, $S\{p||q\} \geq 0$, with $S\{p||q\} = 0$ iff $p_i = q_i$ everywhere; (2) it is invariant under parameter transformations; (3) it is additive for independent distributions; (4) it remains well-defined for continuous distributions.

These properties inspire people to apply this idea to cosmology. In Ref. [6], Hosoya, Buchert, and Morita defined the relative information entropy density $S_D/V_D$ as a functional of the actual and averaged distributions of mass densities, $\rho$ and $\langle \rho \rangle_D$, in the inhomogeneous universe,\[
\frac{S_D}{V_D} = \left\langle \rho \ln \frac{\rho}{\langle \rho \rangle_D} \right\rangle_D, \tag{2}
\]
where $V_D$ is the volume of a domain $D$. Furthermore, it proves that the increasing of $S_D$ encodes the noncommutivity of temporal evolution and spatial averaging of the mass density $\rho$,
\[
\frac{\dot{S}_D}{V_D} = \langle \dot{\rho} \rangle_D - \langle \rho \rangle_D^2. \tag{3}
\]

B. Weyl curvature and Penrose conjecture

As matter and the geometry of space-time are closely interrelated in general relativity, it is also possible to depict the inhomogeneous distribution of matter via geometrical concepts. This idea was suggested by Penrose that the Weyl curvature tensor could play the role of gravitational entropy [7].

In differential geometry, the Weyl tensor $C_{\mu\nu\lambda\rho}$ is a measure of the curvature of a pseudo-Riemannian manifold. In 4-dimensional space-time, the Weyl tensor is defined as
\[
C_{\mu\nu\lambda\rho} := R_{\mu\nu\lambda\rho} - g_{\mu[\lambda} R_{\rho]\nu] + g_{\nu[\lambda} R_{\rho]\mu} + \frac{1}{3} g_{\mu[\lambda} g_{\rho]\nu]} R, \]
where $R_{\mu\nu\lambda\rho}$ is the Riemann tensor, $R_{\mu\nu}$ is the Ricci tensor, and $R$ is the Ricci scalar. It is evident that the Weyl tensor has the same symmetries as the Riemann tensor; besides, there is an important extra symmetry: $C^\lambda_{\mu\nu\lambda} = 0$, i.e. the Weyl tensor is trace-free. Therefore, one can regard the Weyl tensor as a part of the Riemann tensor containing the components not captured by the Ricci tensor. In this way, it is locally independent of the energy–momentum tensor, so the Weyl tensor may be viewed as a purely geometrical description of an inhomogeneous space-time. For all these reasons, the full contraction of the Weyl tensor $C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}$ is the principal scalar that we can construct.

An important theorem on the Weyl tensor states that the necessary and sufficient condition for a metric to be conformally flat is that its Weyl tensor vanishes everywhere. A metric is said to be conformally flat, if it can be reduced to the Minkowski form by a differentiable transformation $g_{\mu\nu} = \Omega^2 \delta_{\mu\nu}$.

Let us apply this theorem to two different metrics. One is the homogeneous FRW metric, $ds^2 = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j$. By a conformal transformation $dt = a(t) d\eta$, we have $ds^2 = a^2(\eta)(-d\eta^2 + \delta_{ij} dx^i dx^j)$. Therefore, the FRW metric is conformally flat, and its Weyl tensor vanishes automatically. The other one is an inhomogeneous metric: the Schwarzschild metric of a black hole with mass $M$, $ds^2 = -(1 - \frac{2GM}{r}) dt^2 + (1 - \frac{2GM}{r})^{-1} dr^2 + r^2 d\Omega^2$. This metric is not conformally flat, and its Weyl scalar is nonzero consequently, $C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} = 48(GM)^2/\pi^6$. Meanwhile, the entropy $S$ of the Schwarzschild black hole is $S = \frac{A}{4G} \times 4\pi(2GM)^2$, with $4\pi(2GM)^2$ being the area of its event horizon. These observations led Penrose to conjecture that there could be some latent relationship between the thermodynamical entropy $S$ of the black hole and the geometrical Weyl scalar $C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}$.
We may further ponder upon this “Penrose conjecture” in the evolution of the universe. In the early universe, space-time is almost homogeneous and thus conformally flat, so its Weyl scalar vanishes. But at late times, the cosmic structures decouple from the global expansion and become gravitationally bound systems. In general, these systems will eventually end their evolutions towards an ensemble of randomly distributed black holes, and the corresponding Weyl scalars will appear gradually in every place in the inhomogeneous universe. As a result, the averaged Weyl scalar \( \langle C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \rangle_D \) will grow monotonically, and thus may behave as a measure for structure formation, or a kind of entropy in some sense.

### C. Kinematical backreaction

We have seen that the two seemingly uncorrelated quantities, \( S_D \) and \( \langle C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \rangle_D \), are both related to the spatial averaging problem in the inhomogeneous universe. How to average a physical observable in the perturbed space-time is a long-standing and very complicated issue \[8\]. Generally speaking, the cosmological information of a physical observable is encoded on the past light-cone, so the averaging problem in cosmology is essentially a problem of light propagation in the inhomogeneous space-time \[8\]. However, for the objects with redshifts \( \ll 1 \), spatial averaging on a constant-time hypersurface is already a good enough approximation.

In the following, we adopt the averaging formalism proposed by Buchert in Ref. \[10\], and focus only on the scalar observables in the dust universe during the matter-dominated era, when large-scale structures significantly form. The metric of the inhomogeneous universe can be written in the synchronous gauge as \( ds^2 = -dt^2 + g_{ij}(t, x) dx^i dx^j \), and Riemannian volume average of a scalar \( O(t, x) \) in a comoving domain \( D \) at time \( t \) is defined as

\[
\langle O \rangle_D := \frac{1}{V_D(t)} \int_D O(t, x) \sqrt{\det g_{ij}} \, d^3x,
\]

with \( V_D(t) := \int_D \sqrt{\det g_{ij}} \, d^3x \) being Riemannian volume of \( D \), and we are thus allowed to introduce an effective scale factor \( a_D(t)/a_D(t_0) := (V_D(t)/V_D(t_0))^{1/3} \). For the perturbative calculations in Section \[IV\] we further define Euclidean volume average as the integral on the FRW background without \( \sqrt{\det g_{ij}} \),

\[
\langle O \rangle := \frac{1}{\Omega(t)} \int D O(t, x) \, d^3x.
\]

Applying the averaging procedures on the energy constraint, Raychaudhuri equation, and continuity equation, we arrive at the generalized Friedmann equations for the irrotational dust universe \[10\],

\[
\left( \frac{\dot{a}_D}{a_D} \right)^2 = \frac{8\pi G}{3} \langle \rho \rangle_D - \frac{\langle R \rangle_D + Q_D}{6},
\]

\[
\frac{\ddot{a}_D}{a_D} = -\frac{4\pi G}{3} \langle \rho \rangle_D + \frac{Q_D}{3},
\]

\[
\langle \rho \rangle_D + 3 \frac{\dot{a}_D}{a_D} \langle \rho \rangle_D = 0.
\]

From these effective equations, we see that besides the ordinary entries in the Friedmann equations for the FRW model, two extra terms influence the evolution of the perturbed universe: the averaged 3-dimensional spatial curvature \( \langle R \rangle_D \) and the so-called “kinematical backreaction”

\[
Q_D := \frac{2}{3} (\langle \theta^2 \rangle_D - \langle \theta \rangle_D^2) - 2 \langle \sigma^2 \rangle_D. \tag{4}
\]

\( Q_D \) bears this name because (1) it consists of kinematical quantities: the volume expansion scalar \( \theta \) and the shear scalar squared \( \sigma^2 := \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu} \); (2) from Eq. \(4\), if \( Q_D > 0 \), it plays the role of effective dark energy, and thus influences the evolution of the background universe.

### III. LTB MODEL

In this section, we first introduce the frequently used inhomogeneous LTB model, and then calculate \( S_D \), \( \langle C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \rangle_D \), and \( Q_D \) in this specific model, respectively.

#### A. LTB model and its solutions

The LTB metric \[11\] is an exact spherically symmetric (isotropic) solution to Einstein’s equations, which reads

\[
ds^2 = -dt^2 + \frac{R'(t, r)^2}{1 + f(r)} \, dr^2 + R(t, r)^2 \, d\Omega^2, \tag{5}
\]

where \( R(t, r) \) is a function of the cosmic time \( t \) and the comoving radius \( r \), and \( f(r) > -1 \) is an arbitrary function of \( r \), with \( f(r)/2 \) being the energy per unit mass of the dust at the comoving radius \( r \). In the following, we denote the partial derivative with respect to \( t \) by \( \dot{R}(t, r) \) and that to \( r \) by \( R'(t, r) \). It is obvious that if we further demand spatial homogeneity in this model, \( R(t, r) = a(t)r \) and \( f(r) = -kr^2 \), the LTB metric reduces to the FRW model naturally.

Substitution of the LTB metric into the Einstein equations yields the dynamical equations for the isotropic dust universe,

\[
\frac{F'}{R^2 R'} = 8\pi G \rho, \quad f = \ddot{R} + 2R \ddot{R}, \tag{6}
\]

where

\[
F(r) = -2R^2 \ddot{R} = \dddot{R} - f \dot{R}. \tag{7}
\]
is the second arbitrary function of \( r \), with \( F(r)/2 \) denoting the gravitational mass within the sphere at the comoving radius \( r \).

The solutions to \( R(t, r) \) can be categorized into three classes — the parabolic, hyperbolic, and elliptic evolutions:

1. for \( f = 0 \),
   \[ R = \left( \frac{9F}{4} \right)^{1/3} (t - T)^{2/3}, \tag{8} \]
2. for \( f > 0 \),
   \[ R = \frac{F}{2f} (\cosh \eta - 1), \quad t - T = \frac{F}{2f^{3/2}} (\sinh \eta - \eta), \tag{9} \]
3. for \( f < 0 \),
   \[ R = \frac{F}{-2f} (1 - \cos \eta), \quad t - T = \frac{F}{2(-f)^{3/2}} (\eta - \sin \eta), \tag{10} \]

where \( T = T(r) \) is the third arbitrary function of \( r \), describing the time of big bang at the comoving radius \( r \).

Furthermore, in any of the three cases above, the volume expansion scalar and the shear scalar squared are given as

\[ \theta = \frac{2 \dot{R}}{R} + \frac{\dot{R}'}{R'}, \quad \sigma^2 = \left. \frac{1}{3} \left( \frac{\dot{R}}{R} - \frac{\dot{R}'}{R'} \right)^2 \right|_{D}. \tag{11} \]

These results will be used in the following.

**B. Exact calculations in the LTB model**

Now we exactly calculate \( S_D, \langle C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \rangle_D \), and \( Q_D \) in the LTB model. However, the direct calculation of \( S_D \) is impossible, due to the logarithmic function in its definition in Eq. (2), so we turn to the time derivative of \( S_D \), with the help of Eq. (3).

First, using the local and averaged continuity equations, we may rewrite Eq. (3) as

\[ \frac{\dot{S}_D}{V_D} = \langle \dot{\rho} \rangle_D - \langle \rho \dot{\rho} \rangle_D = \langle \rho \dot{\theta} \rangle_D - \langle \theta \dot{\rho} \rangle_D. \tag{12} \]

Using Eqs. (4), (7), and (11), Eq. (12) can be reexpressed in terms of \( \dot{R} \) and its partial derivatives,

\[ \frac{\dot{S}_D}{V_D} = \frac{1}{4\pi G} \left[ \left( \frac{2 \dot{R}}{R} + \frac{\dot{R}'}{R'} \right) \left( \frac{2 \dot{R}'}{R'} + \frac{\dot{R}''}{R''} \right) \right]_D \tag{13} \]

Second, the calculation of the averaged Weyl scalar is straightforward, although a little bit tedious. From the LTB metric in Eq. (5), we have

\[ C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} = \frac{1}{3} \left[ \left( \frac{2 \dot{R}'}{R} + \frac{\dot{R}''}{R'} \right) \left( \frac{2 \dot{R}''}{R'} + \frac{\dot{R}'''}{R''} \right) \right]_D. \tag{14} \]

This result is expressed in terms of \( f \) and \( R \) in the LTB metric. While, we may further utilize Eq. (6) and simplify it to a more compact form,

\[ C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} = \frac{16}{3} \left( \frac{\dot{R}}{R} - \frac{\dot{R}'}{R'} \right)^2, \tag{15} \]

so

\[ \langle C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \rangle_D = \frac{16}{3} \left\langle \left( \frac{\dot{R}}{R} - \frac{\dot{R}'}{R'} \right)^2 \right\rangle_D. \tag{16} \]

Last, from Eq. (11), we have the kinematical backreaction,

\[ \frac{\dot{Q}_D}{V_D} = \frac{2}{3} \left( \left( \frac{2 \dot{R}}{R} + \frac{\dot{R}'}{R'} \right)^2 \right)_D - \left( \frac{2 \dot{R}}{R} + \frac{\dot{R}'}{R'} \right)_D \tag{17} \]

The results in Eqs. (13–15) are the exact expressions for \( \frac{\dot{S}_D}{V_D}, \langle C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \rangle_D \), and \( Q_D \) in the LTB model. In general, it is a highly nontrivial work to figure out some concise relation between them, like that in Eq. (11), which is obtained in second order cosmological perturbation theory.

In Ref. [12], numerical calculations were performed in a toy model to illustrate the evolutionary behaviour of \( S_D \). Nevertheless, in order to gain quantitative relation of these three terms, we still have to appeal to the perturbative approach. This will be the task in the next section.

**IV. PERTURBATIVE CALCULATIONS IN THE LTB MODEL**

In this section, we regard the LTB model as a spatially flat FRW model plus linear (first order) spherical perturbations. In this way, the three arbitrary functions \( f(r), F(r), \) and \( T(r) \) in the LTB metric are solved as

\[ f(r) = \frac{20}{9} \Psi'(r)r, \tag{16} \]
\[ F(r) = \frac{4}{9} \left[ 1 + \frac{10}{3} \Psi(r) \right], \tag{17} \]
\[ T(r) = \frac{3}{2} \frac{\Psi'(r)}{r}. \tag{18} \]
where $\Psi$ and $\Phi$ are the linear spherical scalar perturbations.

Below, we calculate $\mathcal{S}_D/V_D$, $\langle C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}\rangle_D$, and $\mathcal{Q}_D$ in cosmological perturbation theory up to second order, but in fact only need to consult the first order perturbative results. This trick lies on the fact that all these three quantities are already of second order. We pick $\mathcal{S}_D/V_D$ for an example. If we expand the mass density to second order, $\rho = \rho^{(0)} + \rho^{(1)} + \rho^{(2)}$, we have

$$\frac{\mathcal{S}_D}{V_D} = \left\langle \rho \ln \frac{\rho}{(\rho)_{D}} \right\rangle_D = \frac{\left(\rho^{(1)}\right)^2}{2\rho^{(0)}} + \cdots. \quad (19)$$

We see that the leading term in Eq. (19) is the variance of the mass density, and is thus of second order. Therefore, we are entitled to use Euclidean average $\langle \cdots \rangle$ to replace Riemannian average $\langle \cdots \rangle_D$, as their difference is at even higher orders. Similarly, this argument holds for $\langle C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}\rangle_D$ and $\mathcal{Q}_D$.

For the three solutions for $f$, we start from the simplest $f = 0$ case, where there is only the decaying mode of the cosmological perturbations. Next, we proceed to the growing mode in the $f \neq 0$ case, and finally to the general case with both the decaying and growing modes taken into account.

### A. Decaying mode

In the $f = 0$ case, from Eq. (16), $\Psi = 0$ and $\Phi$ is a constant. Using Eqs. (8) and (17), we expand $R$ to first order,

$$R(t, r) = r t^{2/3} \left( 1 + \frac{10}{9} \Psi + \frac{\Phi'}{rt} \right). \quad (20)$$

We see from Eq. (20) that the first two terms $1 + 10\Psi/9$ are constant in time, and the third one $\Phi'/rt$ represents a decaying mode in $R$. But this term should not be simply disregarded at present, because the constant perturbation $10\Psi/9$ can be viewed as a fraction of the background metric and thus does not contribute to the perturbative results. This will be seen in the following Eqs. (23)–(27).

Two useful intermediate steps are listed below, before giving the final results,

$$\dot{R} = \frac{2}{3} t r^{1/3} \left( 1 + \frac{10}{9} \Psi - \frac{\Phi'}{2rt} \right), \quad (21)$$

$$R' = t^{2/3} \left( 1 + \frac{10}{9} \Psi + \frac{\Phi''}{t} \right). \quad (22)$$

Substituting Eqs. (17), (20), and (22) into Eq. (5), we obtain

$$\rho = \frac{1}{6\pi G t^2} \left( 1 - \frac{2\Phi'}{rt} - \frac{\Phi''}{t} \right).$$

Thus, we have the mass density at the background and first order,

$$\rho^{(0)}(t) = \frac{1}{6\pi G t^2}, \quad \rho^{(1)}(t, r) = -\frac{1}{6\pi G t^2} \left( \frac{2\Phi'}{r} + \Phi'' \right).$$

Substituting these results into Eq. (19), we attain the relative information entropy in the LTB model (up to second order),

$$\frac{\mathcal{S}_D}{V_D} = \frac{1}{12\pi G t^4} \left[ \left( \frac{2\Phi'}{r} + \Phi'' \right)^2 - \left( \frac{2\Phi'}{r} + \Phi'' \right)^2 \right]. \quad (23)$$

Above, we change $\langle \cdots \rangle_D$ to $\langle \cdots \rangle$, as we have already explained.

Immediately, the time derivative and convexity of the relative information entropy read

$$\dot{\frac{\mathcal{S}_D}{V_D}} = \frac{1}{9\pi G t^6} \left[ \left( \frac{2\Phi'}{r} + \Phi'' \right)^2 - \left( \frac{2\Phi'}{r} + \Phi'' \right)^2 \right], \quad (24)$$

$$\ddot{\frac{\mathcal{S}_D}{V_D}} = \frac{1}{27\pi G t^8} \left[ \left( \frac{2\Phi'}{r} + \Phi'' \right)^2 - \left( \frac{2\Phi'}{r} + \Phi'' \right)^2 \right]. \quad (25)$$

Obviously, $\mathcal{S}_D/V_D$, $\dot{\mathcal{S}_D}/V_D$, and $\ddot{\mathcal{S}_D}/V_D$ are all positive definite. This indicates that the relative information entropy in the LTB model not only increases monotonically, but even in an accelerated way. Here, we should mention that if we start from Eq. (13), we will arrive at the same result as that in Eq. (24).

In like manner, substituting Eq. (20) into Eqs. (14) and (15), we get the averaged Weyl scalar,

$$\langle C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho} \rangle_D = \frac{64}{27t^6} \left( \frac{\Phi'}{r} - \Phi'' \right)^2, \quad (26)$$

and the kinematical backreaction,

$$\mathcal{Q}_D = \frac{2}{3t^4} \left[ \left( \frac{2\Phi'}{r} + \Phi'' \right)^2 - \left( \frac{2\Phi'}{r} + \Phi'' \right)^2 - \left( \frac{\Phi'}{r} - \Phi'' \right)^2 \right]. \quad (27)$$

Above, we do not combine the first and third terms in $\mathcal{Q}_D$, as we notice that the third term in Eq. (27) exactly cancels the averaged Weyl scalar in Eq. (26) (up to a coefficient).

From Eqs. (23), (24), and (27), we eventually find

$$\frac{\mathcal{S}_D}{V_D} = \frac{9}{32\pi G} \left( \frac{t^2}{8} \langle C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho} \rangle_D + \frac{3}{4} \mathcal{Q}_D \right). \quad (28)$$

This final relation looks rather like that in Eq. (11), but with a small difference: the coefficient $4/9$ in front of $\mathcal{Q}_D$. However, this difference does not indicate that one of the results in Eqs. (11) and (28) is incorrect; actually, there is no contradiction. In Ref. (1), merely the growing mode of the linear perturbations were considered, but here for the $f = 0$ case, we see from Eq. (20) that only
the decaying mode exists. Therefore, Eq. (28) is not only consistent with Eq. (1), but also supplements Eq. (1) with the decaying mode in cosmological perturbation theory, which was not extensively discussed in Ref. 4.

Similarly, we obtain

\[
\frac{\mathcal{S}_D}{V_D} = \frac{3}{8\pi G} \left( \frac{t}{8} (C_{\mu\nu\lambda\rho}^{\mu\nu\lambda\rho})_D + \frac{4}{9} \frac{Q_D}{t} \right),
\]

(29)

\[
\frac{\mathcal{S}_D}{V_D} = \frac{1}{8\pi G} \left( \frac{1}{8} (C_{\mu\nu\lambda\rho}^{\mu\nu\lambda\rho})_D + \frac{4}{9} \frac{Q_D}{t^2} \right).
\]

(30)

The coefficient 4/9 also remains in these two relations for the same reason.

**B. Growing mode**

For the case with a non-vanishing \( f \), we first Taylor expand \( t - T \) in Eqs. (9) and (10), solve the parameter \( \eta \), and then substitute it into the corresponding \( R \). For both the cases \( f > 0 \) and \( f < 0 \), after some algebra, we arrive at the same result,

\[
R(t, r) = rt^{2/3} \left( 1 + \frac{10}{9} \Psi + \frac{\Psi'}{rt} + \frac{\Psi'^t}{r} \right).
\]

(31)

This result is the same as that in Eq. (29), but with an additional term \( \Psi' t^{2/3} / r \), because \( \Psi' \) is now nonzero in the case \( f \neq 0 \). This term is the growing mode in the perturbative expansion of \( R \), and will dominate in \( R \) as \( t \) increases. For this reason, we may first neglect the decaying mode \( \Psi' / rt \) in Eq. (31), and only keep the growing and constant ones,

\[
R(t, r) = rt^{2/3} \left( 1 + \frac{10}{9} \Psi + \frac{\Psi'^t}{r} \right).
\]

(32)

Now, \( R \) is the function of \( \Psi \) only.

The following perturbative calculations are totally parallel to those in Section IV A. First, we have

\[
\dot{R} = \frac{2r}{3t^{1/3}} \left( 1 + \frac{10}{9} \Psi + \frac{\Psi'^t}{r} \right),
\]

\[
R' = t^{2/3} \left( 1 + \frac{10}{9} \Psi + \frac{\Psi'}{t} + \Psi'' t^{2/3} \right),
\]

and

\[
\rho^{(1)} = -\frac{1}{6\pi G t^{4/3}} \left( \frac{2\Psi'}{r} + \Psi'' \right).
\]

Then, \( S_D/V_D \), \( (C_{\mu\nu\lambda\rho}^{\mu\nu\lambda\rho})_D \), and \( Q_D \) are obtained straightforwardly,

\[
\frac{S_D}{V_D} = \frac{1}{12\pi G t^{2/3}} \left[ \left( \frac{2\Psi'}{r} + \Psi'' \right)^2 - \left( \frac{2\Psi'}{r} + \Psi'' \right)^2 \right],
\]

(33)

and

\[
(C_{\mu\nu\lambda\rho}^{\mu\nu\lambda\rho})_D = \frac{64}{27t^{8/3}} \left( \Psi' r - \Psi'' \right)^2,
\]

(34)

and

\[
Q_D = \frac{8}{27t^{2/3}} \left[ \left( \frac{2\Psi'}{r} + \Psi'' \right)^2 - \left( \frac{2\Psi'}{r} + \Psi'' \right)^2 \right].
\]

(35)

From Eqs. (33)-(35), we recover the main results in Ref. 4,

\[
\frac{S_D}{V_D} = \frac{9}{32\pi G} \left( \frac{t^2}{8} (C_{\mu\nu\lambda\rho}^{\mu\nu\lambda\rho})_D + Q_D \right),
\]

(36)

\[
\frac{S_D}{V_D} = \frac{3}{8\pi G} \left( \frac{1}{8} (C_{\mu\nu\lambda\rho}^{\mu\nu\lambda\rho})_D + \frac{Q_D}{t} \right),
\]

(37)

\[
\frac{S_D}{V_D} = \frac{1}{8\pi G} \left( \frac{1}{8} (C_{\mu\nu\lambda\rho}^{\mu\nu\lambda\rho})_D + \frac{Q_D}{t^2} \right).
\]

(38)

Till now, we finally understand that this relation is valid only for the growing mode of the cosmological perturbations. As expected, the second order perturbative calculations in the LTB model for the \( f > 0 \) and \( f < 0 \) cases reconfirm this relation.

**C. General case**

With the preparations in Sects. IV A and IV B, we now present the general solutions for \( S_D/V_D \), \( (C_{\mu\nu\lambda\rho}^{\mu\nu\lambda\rho})_D \), and \( Q_D \), taking into account both the decaying and growing modes of the cosmological perturbations. We begin with

\[
R(t, r) = rt^{2/3} \left( 1 + \frac{10}{9} \Psi + \frac{\Psi'}{rt} + \frac{\Psi'^t}{r} \right),
\]

so we have

\[
\dot{R} = \frac{2r}{3t^{1/3}} \left( 1 + \frac{10}{9} \Psi - \frac{\Psi'}{2rt} + \frac{2\Psi'^t}{r} \right),
\]

\[
R' = t^{2/3} \left( 1 + \frac{10}{9} \Psi + \frac{\Psi'}{t} + \frac{10}{9} \Psi r + \Psi'' t^{2/3} \right),
\]

and

\[
\rho^{(1)} = -\frac{1}{6\pi G t^{4/3}} \left[ \left( \frac{2\Psi'}{r} + \Psi'' \right) + \left( \frac{2\Psi'}{r} + \Psi'' \right) t^{5/3} \right].
\]

From these general results, we eventually achieve the full expressions of the relative information entropy,

\[
\frac{S_D}{V_D} = \frac{1}{12\pi G t^{2/3}} \left[ \left( \frac{2\Psi'}{r} + \Psi'' + \frac{2\Psi'}{r} + \Psi'' t^{5/3} \right)^2 \right] - \left( \frac{2\Psi'}{r} + \Psi'' + \frac{2\Psi'}{r} + \Psi'' t^{5/3} \right)^2,
\]

(39)
the averaged Weyl scalar,

$$(\langle C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}\rangle_d) = 64 \frac{27}{27t^6} \left( \frac{\Phi'}{r} - \frac{\Phi''}{r} + \frac{\Psi' t^{5/3}}{r} - \frac{\Psi'' t^{5/3}}{r} \right)^2,$$

and the kinematical backreaction,

$$Q_D = \frac{8}{27t^4} \left[ \left( \frac{3\Phi'}{r} + \frac{\Phi''}{r} + \frac{2\Psi' t^{5/3}}{r} - \frac{\Psi'' t^{5/3}}{r} \right)^2 \right].$$

These are the final and complete results for $S_D/V_D$, $(C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho})_2$, and $Q_D$ that we hope to calculate in the present paper. In these expressions, both the decaying and growing modes are taken into account. In general, due to the crossing products of the two modes, the simple relations like Eqs. (23)–(30) or (30)–(33) can no longer be attained, because they only apply in certain special circumstances.

V. CONCLUSIONS AND DISCUSSIONS

In recent years, the study of the inhomogeneous cosmological models and the corresponding problems, e.g. the averaging procedure, backreaction mechanism, and light propagation in perturbed space-time, has attracted much attention (see Refs. [14, 15] and the references therein). One relevant and important issue is to seek some simple and reasonable measure for the large-scale structure formation during cosmological evolution. In Ref. [3], two such measures were investigated: the relative information entropy $S_D$ and the averaged Weyl scalar $(C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho})_2$, and their relation is shown in Eq. (1).

In the present paper, in the specific LTB model, we verify this basic result, and simultaneously point out its range of application: only for the growing scalar mode. In general, due to the superposition of different perturbative modes, Eq. (1) will not be valid, and the fully nonlinear exact relation is still under finding. A next possible step should be to look for other quantities that vanish in the perturbative treatment, but are there in the full LTB solution.

At last, we give some general discussions.

1. From Eqs. (11) and (14), we find $C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \propto \sigma^2$. This is not just a coincidence, and we may have deeper insight from this proportion. The Weyl curvature may be irreducibly decomposed into the electric part $E_{\mu\nu} := C_{\mu\lambda\nu\rho} u^\lambda u^\rho$ and the magnetic part $H_{\mu\nu} := \frac{1}{2} C_{\mu\lambda\alpha\beta} C^{\alpha\beta\nu\rho} u^\lambda u^\rho$. In the LTB model (both in exact and perturbative approaches), the magnetic part vanishes, so $C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho} \propto E_{\mu\nu} E^{\mu\nu}$. At the same time, the shear tensor is proportional to the electric part, so $\sigma^2 \propto E_{\mu\nu} E^{\mu\nu}$ also. These facts explain the similar results in Eqs. (11) and (14).

2. To our knowledge, the Penrose conjecture has not yet been well formulated in a rigorous mathematical way. Hence, to construct possible scalars from the Weyl tensor should be the first step in this direction. According to the Petrov classification, in addition to $C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}$, there are other independent full contractions, e.g. $\epsilon_{\mu\nu\lambda\rho} C_{\lambda\rho\sigma\tau} C_{\sigma\tau}^{\mu\nu}$, $C_{\mu\nu\lambda\rho} C_{\lambda\rho\sigma\tau} C_{\sigma\tau}^{\mu\nu}$, or $\epsilon_{\mu\nu\lambda\rho} C_{\lambda\rho\sigma\tau} C_{\sigma\tau}^{\mu\nu}$ vanishes. While, for the last two possibilities, since there are three Weyl tensors inside, their results are beyond the first and second order perturbation theories. Moreover, in Ref. [17], it was shown that the trivial contraction $C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho}$ diverges and thus fails to be monotonic near the isotropic singularities. Therefore, some other candidates have been considered, e.g. $(C_{\mu\nu\lambda\rho} C^{\mu\nu\lambda\rho})/(R_{\mu\nu} R^{\mu\nu})$, which may help to evade this limitation, and this direction deserves further exploration.

3. From a mathematical point of view, the curvatures of space-time are measured by the Riemann tensor, consisting of the Ricci tensor and Weyl tensor, namely, Riemann = Ricci + Weyl. However, Einstein’s equations only associate the Ricci sector with energy–momentum tensor. We may naturally ask why the information stored in the Weyl sector is absent in general relativity? A possible answer is that the Weyl tensor is linked not to the dynamical, but to the thermodynamical aspect of gravitational fields. The evolution of our universe is doubtlessly irreversible, but on the contrary, a process governed by Einstein’s equations possesses the invariance of time reversal. Thus, the time asymmetry of cosmological evolution is not shown explicitly in Einstein’s equations. Is this information encoded in the Weyl tensor? Penrose proposed that some scalar invariant of the Weyl tensor could be identified with the gravitational entropy of the universe. This idea was further modified and developed by several authors [17]. Our present work helps to confirm this idea, and indicates that the Weyl tensor can be further related to the relative information entropy. These facts lead us to wonder if there exist equations that are parallel to Einstein’s equations and quantify the thermodynamical relationship between space-time and matter. These equations are expected to link the Weyl tensor with the concepts such as temperature or entropy. This question will be the topic of research in future.

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[1] T. Clifton, G.F.R. Ellis, and R. Tavakol, Classical Quantum Gravity \textbf{30}, 125009 (2013).
[2] R.A. Sussman, Classical Quantum Gravity \textbf{30}, 065015 (2013); \textbf{30}, 065016 (2013).
[3] R.A. Sussman and J. Larena, Classical Quantum Gravity \textbf{31}, 075021 (2014).
[4] N. Li, \textit{et al.}, Phys. Rev. D \textbf{86}, 083539 (2012).
[5] S. Kullback and R.A. Leibler, Ann. Math. Stat. \textbf{22}, 79 (1951).
[6] A. Hosoya, T. Buchert, and M. Morita, Phys. Rev. Lett. \textbf{92}, 141302 (2004).
[7] R. Penrose, in \textit{General relativity, an Einstein centenary survey}, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979) p. 581.
[8] G.F.R. Ellis, in \textit{General relativity and gravitation}, edited by B. Bertotti, F. de Felice, and A. Pascolini (D. Reidel Publishing Company, Dordrecht, 1984) p. 215.
[9] V. Marra, E.W. Kolb, and S. Matarrese, Phys. Rev. D \textbf{77}, 023003 (2008); S. Räsänen, J. Cosmol. Astropart. Phys. 02 (2009) 011; G. Marozzi, J. Cosmol. Astropart. Phys. 01 (2011) 012; M. Gasperini, G. Marozzi, F. Nugier, and G. Veneziano, J. Cosmol. Astropart. Phys. 07 (2011) 008; I. Ben-Dayan, \textit{et al.}, J. Cosmol. Astropart. Phys. 04 (2012) 036; F. Nugier. \texttt{arXiv:1309.6512} [astro-ph.CO].
[10] T. Buchert, Gen. Relativ. Gravit. \textbf{32}, 105 (2000).
[11] A. Krasiński, \textit{Inhomogeneous cosmological models}, (Cambridge University Press, Cambridge, 1997), p. 100; G.F.R. Ellis, R. Maartens, and M.A.H. MacCallum, \textit{Relativistic cosmology}, (Cambridge University Press, Cambridge, 2012), p. 395.
[12] M. Morita, T. Buchert, A. Hosoya, and N. Li, AIP Conf. Proc. \textbf{1241}, 1074 (2010).
[13] M. Morita, K. Nakamura, and M. Kasai, Phys. Rev. D \textbf{57}, 6094 (1998).
[14] D.L. Wiltshire, Phys. Rev. Lett. \textbf{99}, 251101 (2007); J. Behrend, I.A. Brown, and G. Robbers, J. Cosmol. Astropart. Phys. 01 (2008) 013; N. Li and D.J. Schwarz, Phys. Rev. D \textbf{78}, 083531 (2008); A. Paranjape and T.P. Singh, Phys. Rev. Lett. \textbf{101}, 181101 (2008); T. Buchert and M. Carfora, Classical Quantum Gravity \textbf{25}, 195001 (2008); J. Larena, \textit{et al.}, Phys. Rev. D \textbf{79}, 083011 (2009); C. Clarkson, K. Ananda, and J. Larena, Phys. Rev. D \textbf{80}, 083525 (2009); K. Enqvist, M. Mattsson, and G. Rigopoulos, J. Cosmol. Astropart. Phys. 09 (2009) 022; A. Krasiński, C. Hellaby, K. Bolejko, and M.-N. Célerier, Gen. Relativ. Gravit. \textbf{42}, 2453 (2010); E. W. Kolb, V. Marra, and S. Matarrese, Gen. Relativ. Gravit. \textbf{42}, 1399 (2010); M. Mattsson and T. Mattsson, J. Cosmol. Astropart. Phys. 10 (2010) 021; V. Marra and M. Paakkonen, J. Cosmol. Astropart. Phys. 12 (2010) 021; V. Marra and A. Notari, Classical Quantum Gravity \textbf{28}, 164004 (2011); R.A. Sussman, Classical Quantum Gravity \textbf{28}, 235002 (2011); D.L. Wiltshire, Classical Quantum Gravity \textbf{28}, 164006 (2011); E.W. Kolb, Classical Quantum Gravity \textbf{28}, 164009 (2011); C. Clarkson, G. Ellis, J. Larena, and O. Umeh, Rept. Prog. Phys. \textbf{74}, 112001 (2011); M. Lavinto, S. Räsänen, and S.J. Szybka, J. Cosmol. Astropart. Phys. 12 (2013) 051; T. Buchert, C. Nayet, and A. Wiegand, Phys. Rev. D \textbf{87}, 123503 (2013).
[15] T. Buchert, Gen. Relativ. Gravit. \textbf{40}, 467 (2008); T. Buchert and S. Räsänen, Ann. Rev. Nucl. Part. Sci. \textbf{62}, 57 (2012).
[16] S.W. Goode, A.A. Coley, and J. Wainwright, Classical Quantum Gravity \textbf{9}, 445 (1992).
[17] S.W. Goode and J. Wainwright, Classical Quantum Gravity \textbf{2}, 99 (1984); W.B. Bonnor, Classical Quantum Gravity \textbf{3}, 495 (1986); S.W. Goode, Classical Quantum Gravity \textbf{8}, L1 (1991); O. Gru and S. Hervik, \texttt{arXiv:gr-qc/0205026}; R.P.A.C. Newman, Proc. Roy. Soc. Lond. A \textbf{443}, 493 (1993); W.C. Lim, H. van Elst, C. Uggla, and J. Wainwright, Phys. Rev. D \textbf{69}, 103507 (2004); L. Herrera, A. Di Prisco, and J. Ibanez, Phys. Rev. D \textbf{84}, 064036 (2011); N. Akerblom and G. Corneliussen, J. Math. Phys. (N.Y.) \textbf{53}, 012502 (2012); P. Mishra and T.P. Singh, Phys. Rev. D \textbf{89}, 123007 (2014).