MINIMAL SEESAW MECHANISM

D. Falcone

*Dipartimento di Scienze Fisiche, Università di Napoli, Via Cintia, Napoli, Italy*

In the framework of the seesaw mechanism, and adopting a typical form for the Dirac neutrino mass matrix, we discuss the impact of minimal forms of the Majorana neutrino mass matrix. These matrices contain four or three texture zeros and only two parameters, a scale factor and a hierarchy parameter. Some forms are not compatible with large lepton mixing and are ruled out. Moreover, a normal mass hierarchy for neutrinos is predicted.
I. INTRODUCTION

There is now a strong evidence for neutrino oscillations, especially through the SuperKamiokande, K2K, and SNO, KamLAND experiments [1]. Neutrino oscillations are naturally accounted for if neutrinos have small masses, so that leptons can mix in a similar way as quarks do [2]. Moreover, small neutrino masses can be achieved by means of the seesaw mechanism [3]. In this framework, the effective (Majorana) mass matrix of neutrinos $M_L$ is related to the Dirac neutrino mass matrix $M_\nu$ and the heavy Majorana neutrino mass matrix $M_R$ by the relation

$$M_L \simeq M_\nu M_R^{-1} M_\nu^T. \quad (1)$$

As a matter of fact, the seesaw formula (1) is valid at the high $M_R$ scale, and therefore one should determine both $M_\nu$ and $M_L$ at that scale, in order to find a consistent model. The effective matrix $M_L$ is partially described at the low scale through the analysis of several experiments [4]. On the other hand, the Dirac matrix $M_\nu$ is based on theoretical hints. Both have to be renormalized to the $M_R$ scale. Then the problem is to find models for $M_\nu$ and $M_R$ which reproduce the phenomenological forms of $M_L$ according to the master relation (1). Such a problem has been addressed in many papers (see, for instance, the review [5]). In the present article we consider a structure for $M_\nu$ which is usually adopted for charged fermion mass matrices, and minimal models for $M_R$. We select minimal forms which are compatible with phenomenology.

In section II we discuss the effective neutrino mass matrix. In section III we describe the Dirac and Majorana mass matrices of our minimal framework. Then, in section IV, the seesaw formula is applied and the resulting neutrino mass matrix is compared to phenomenology. A brief discussion is finally proposed.

II. THE EFFECTIVE NEUTRINO MASS MATRIX

Experimental informations on neutrino oscillations imply that the lepton mixing matrix is given by

$$U \simeq \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \epsilon e^{-i\delta} \\ -\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} e^{i\delta} & -\frac{1}{\sqrt{6}} e^{i\delta} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} e^{i\delta} & -\frac{1}{\sqrt{6}} e^{i\delta} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} e^{i\varphi_1/2} & e^{i\varphi_2/2} & 1 \end{pmatrix} \quad (2)$$

in the standard parametrization, where $\epsilon < 0.16$, $0 < \delta < 2\pi$, $0 < \varphi_1, \varphi_2 < 2\pi$ (see, for example, Ref.[6]). Moreover, neutrino oscillations determine the following square mass
differences,
\[ \Delta m_{32}^2 = m_3^2 - m_2^2 \simeq 3 \cdot 10^{-3} \text{eV}^2, \]  
\[ \Delta m_{21}^2 = m_2^2 - m_1^2 \simeq 7 \cdot 10^{-5} \text{eV}^2, \]  
where \( m_1, m_2, m_3 \) are the effective neutrino masses. Then, in the basis where the charged lepton mass matrix is diagonal, \( M_L \) is obtained by means of the transformation
\[ M_L = U^* D_L U^\dagger, \]  
with \( D_L = \text{diag}(m_1, m_2, m_3) \). Neglecting \( \epsilon \) in \( U \), except for \( U_{e3} \), the calculation leads to
\[
\begin{align*}
M_{ee} &\simeq \epsilon^2 m_3 + \frac{m_2}{3} + 2 \frac{m_1}{3}, \\
M_{e\mu} &\simeq \epsilon \frac{m_3}{\sqrt{2}} + \frac{m_2}{3} - \frac{m_1}{3}, \\
M_{e\tau} &\simeq \epsilon \frac{m_3}{\sqrt{2}} - \frac{m_2}{3} + \frac{m_1}{3}, \\
M_{\mu\mu} &\simeq \frac{m_3}{2} + \frac{m_2}{3} + \frac{m_1}{6}, \\
M_{\mu\tau} &\simeq \frac{m_3}{2} - \frac{m_2}{3} - \frac{m_1}{6}
\end{align*}
\]
not writing phases, which can be inserted by \( \epsilon \rightarrow \epsilon e^{i\delta}, \ m_1 \rightarrow m_1 e^{i\varphi_1}, \ m_2 \rightarrow m_2 e^{i\varphi_2} \).

Since \( \Delta m_{21}^2 \ll \Delta m_{32}^2 \), we have two main mass spectra for light neutrinos, the normal hierarchy \( m_1 < m_2 \ll m_3 \), and the inverse hierarchy \( m_1 \simeq m_2 \gg m_3 \). For the normal hierarchy the dominant elements are given by
\[ M_L \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} m_3, \]  
with \( m_3 \simeq \sqrt{\Delta m_{32}^2} \), and for the inverse hierarchy they are given by
\[ M_L \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} m_{1,2}, \]  
with \( m_{1,2} \simeq \sqrt{\Delta m_{32}^2} \). Both contain a democratic \( \mu\tau \) block, due to near maximal \( U_{\mu3} \).

The difference stands in the element \( ee \), which is suppressed in (6) but dominant in (7). Now, according to Ref. [7], the general structure of \( M_L \) is not changed by renormalization. Therefore, we can take matrices (6) and (7) as simple forms at the high scale, the zero elements meaning suppressed with respect to dominant elements. They correspond to distinct predictions for the double beta decay parameter \( M_{ee} \), since for the normal hierarchy we get \( M_{ee} \sim \sqrt{\Delta m_{21}^2} \), while for the inverse hierarchy we have \( M_{ee} \sim \sqrt{\Delta m_{32}^2} \).
III. DIRAC AND MAJORANA MASS MATRICES

In order to apply the seesaw formula, we need the expression of the Dirac and Majorana mass matrices. We take a typical form for the three mass matrices of charged fermions [8]:

\[ M_d \simeq \begin{pmatrix}
0 & \sqrt{m_dm_s} & 0 \\
\sqrt{m_dm_s} & m_s & \sqrt{m_dm_b} \\
0 & \sqrt{m_dm_b} & m_b,
\end{pmatrix} \quad (8) \]

\[ M_u \simeq \begin{pmatrix}
0 & \sqrt{m_um_c} & 0 \\
\sqrt{m_um_c} & m_c & \sqrt{m_um_t} \\
0 & \sqrt{m_um_t} & m_t
\end{pmatrix} \quad (9) \]

for down and up quarks, and

\[ M_e \simeq \begin{pmatrix}
0 & \sqrt{m_em_\mu} & 0 \\
\sqrt{m_em_\mu} & m_\mu & \sqrt{m_em_\tau} \\
0 & \sqrt{m_em_\tau} & m_\tau
\end{pmatrix} \quad (10) \]

for charged leptons. Then, since \( M_e \sim M_d \), a natural choice is also \( M_\nu \sim M_u \). In fact, we have \( m_d/m_s \sim m_s/m_b \sim \lambda^2 \), \( m_e/m_\mu \sim m_\mu/m_\tau \sim \lambda^2 \), and \( m_u/m_c \sim m_c/m_t \sim \lambda^4 \), where \( \lambda = 0.22 \) is the Cabibbo parameter. The renormalization of quark mass matrices does not affect their expression in terms of powers of \( \lambda \) [10]. Therefore, for the Dirac neutrino mass matrix we take

\[ M_\nu \simeq \begin{pmatrix}
0 & a & 0 \\
a & b & c \\
0 & c & 1
\end{pmatrix} m_t, \quad (11) \]

with \( a \ll b \sim c \ll 1 \). As order in \( \lambda \) we have \( a \sim \lambda^6 \), \( c \sim \lambda^4 \). Expressions (8) and (9) lead to small quark mixings, while lepton mixings \( U_{e2} \) and \( U_{\mu3} \) are large. The matrix \( M_R \) should produce, through the seesaw formula, large lepton mixings [11].

For this Majorana mass matrix we consider minimal forms. These include matrices with four texture zeros:

\[ M_R = \begin{pmatrix}
0 & A & 0 \\
A & 0 & 0 \\
0 & 0 & B
\end{pmatrix} m_R, \quad (12) \]
and matrices with three texture zeros, that is the diagonal form

\[ M_R = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} m_R, \]

(15)

and the Zee-like form [12]

\[ M_R = \begin{pmatrix} 0 & A & B \\ A & 0 & C \\ B & C & 0 \end{pmatrix} m_R. \]

(16)

Parameters \( A, B, C \) can take values 1 and \( r < 1 \). Therefore, in such matrices there is a scale factor, related to \( m_R \), and possibly one hierarchy parameter \( r \).

IV. SEESAW MECHANISM

In this section we calculate the effective neutrino mass matrix by means of the seesaw formula (1) on mass matrices discussed in the previous section. Then we look for structure of the kind (6) and (7). We exclude possible cancellations during our analysis.

A. Four texture zeros

Matrix (12) leads to

\[
M_L \approx \begin{pmatrix} 0 & \frac{a^2}{A} & 0 \\ \frac{a^2}{A} + \frac{2ab}{A} + \frac{c^2}{B} & \frac{ac}{A} + \frac{c}{B} \\ 0 & \frac{ac}{A} + \frac{c}{B} & \frac{1}{B} \end{pmatrix} \left( \frac{m_i^2}{m_R} \right).
\]

(17)
Condition $M_{\mu\tau} \sim M_{\tau\tau}$ gives $A/B \sim ac$. Then $A = r \sim ac$ and $B = 1$. Condition $M_{\mu\mu} \sim M_{\tau\tau}$ is satisfied as a consequence. See also Ref. [13] for a discussion on this structure.

Matrix (13) leads to

$$M_L \simeq \begin{pmatrix} 0 & \frac{ac}{B} & \frac{a}{B} \\ \frac{ac}{B} & \frac{a^2}{A} + \frac{2bc}{B} & \frac{b}{B} + \frac{c^2}{B} \\ \frac{a}{B} & \frac{b}{B} + \frac{c^2}{B} & \frac{2c}{B} \end{pmatrix} \frac{m_t^2}{m_R}. \tag{18}$$

Here condition $M_{\mu\tau} \sim M_{\tau\tau}$ is satisfied while $M_{\mu\mu} \sim M_{\tau\tau}$ requires $A/B \sim a^2/c$. Hence $A = r \sim a^2/c$ and $B = 1$.

Matrix (14) leads to

$$M_L \simeq \begin{pmatrix} \frac{a^2}{B} & \frac{ab}{B} & \frac{ac}{B} \\ \frac{ab}{B} & \frac{b^2}{A} + \frac{2ac}{B} & \frac{bc}{B} + \frac{a}{A} \\ \frac{ac}{B} & \frac{bc}{B} + \frac{a}{A} & \frac{c^2}{B} + \frac{1}{C} \end{pmatrix} \frac{m_t^2}{m_R}. \tag{19}$$

so that $M_{\mu\tau} \sim M_{\tau\tau}$ gives $B/A \lesssim c^2/a$. Then $A = 1$ and $B = r \lesssim c^2/a$. The condition $M_{\mu\mu} \sim M_{\tau\tau}$ is valid as a consequence.

The normal hierarchy is achieved in all cases. However, note the three different values for the scale $m_R$, that is $m_t^2/m_3, cm_t^2/m_3, am_t^2/m_3$, respectively.

### B. Diagonal form

In this case the effective neutrino mass matrix is given by

$$M_L \simeq \begin{pmatrix} \frac{a^2}{B} & \frac{ab}{B} & \frac{ac}{B} \\ \frac{ab}{B} & \frac{b^2}{A} + \frac{c^2}{C} & \frac{bc}{B} + \frac{a}{C} \\ \frac{ac}{B} & \frac{bc}{B} + \frac{e}{C} & \frac{c^2}{B} + \frac{1}{C} \end{pmatrix} \frac{m_t^2}{m_R}. \tag{20}$$

Here condition $M_{\mu\tau} \sim M_{\tau\tau}$ gives $B/C \lesssim c^2$. Then $C = 1$ and $B = r \lesssim c^2$. Both $A = r$ and $A = 1$ are consistent with $M_{\mu\mu} \sim M_{\tau\tau}$. The normal hierarchy is obtained. The scale $m_R$ is given by $m_t^2/m_3$. Large lepton mixing can indeed be obtained even by means of small mixing in $M_\nu$ and zero mixing in $M_R$ (see Ref. [14]). In particular, for $A \simeq B \simeq c^2$ we get

$$M_L \simeq \begin{pmatrix} k^2 & k & k \\ k & 1 & 1 \\ k & 1 & 1 \end{pmatrix} \frac{m_t^2}{m_R}. \tag{21}$$
with \( k = a/c \). This form of \( M_L \) has already been proposed several times \[15\]. Moreover, the same form is realized in (19) for \( B \simeq c^2/a \), but with the scale \( m_R \) suppressed by the factor \( a \) with respect to (20).

C. Zee-like form

In this case, apart from an overall scale \( m^2_t/2m_R \), we get the following approximate effective matrix

\[
\begin{pmatrix}
- \frac{a^2 B}{AC} & \left[ - \frac{ab B}{AC} + \frac{a^2}{A} + \frac{ac}{C} \right] & \left[ - \frac{ac B}{AC} + \frac{a}{C} \right] \\
* & \left[ - \frac{b^2 B}{AB} - \frac{c^2 A}{BC} + \frac{2ab}{A} + \frac{2ac}{B} + \frac{2bc}{C} \right] & \left[ - \frac{bc B}{AC} - \frac{A}{BC} + \frac{2c}{C} \right] \\
* & * & \left[ - \frac{c^2 B}{AC} - \frac{A}{BC} + \frac{2c}{C} \right]
\end{pmatrix}
\]

In order to have a useful \( \mu\tau \) block, the leading terms must be those with \( AC \) in the denominator. Then the normal hierarchy is achieved for \( A = C = r \lesssim c \) and \( B = 1 \). Here the scale \( m_R \) is about \( m^2_t/m_R \).

V. DISCUSSION

We have studied the seesaw mechanism assuming simple forms of the fermion mass matrices and in particular minimal forms for the heavy neutrino mass matrix, which contain four or three texture zeros, a scale factor and a hierarchy parameter. Our minimal framework allows only the normal hierarchy for light neutrinos and not the inverse hierarchy. Large lepton mixing is achieved by tuning the hierarchy parameter \( r \) in the heavy neutrino mass matrix.

There is another possible mass spectrum for neutrinos, the degenerate spectrum, \( m_1 \simeq m_2 \simeq m_3 \), which gives the dominant elements \( M_L \sim \text{diag}(1,1,1)m_{1,2,3} \) or

\[
M_L \sim \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} m_{1,2,3}.
\]

One can easily check that such forms are not reproduced in our minimal framework. However, both the inverse hierarchy and the degenerate spectrum can be achieved in some nonminimal models \[16\]. Indeed, generally it is quite hard to yield degeneracy in \( M_L \) from hierarchy in \( M_\nu \) by means of \( M_R \) in the seesaw formula.
The present framework could also be embedded in a unified $SO(10)$ model. In fact, the relations $M_e \sim M_d$ and $M_\nu \sim M_u$ can be the result of a quark-lepton symmetry, and the high $M_R$ scale can as well be related to the unification or intermediate breaking scale of the supersymmetric or nonsupersymmetric model, respectively \cite{17}. Then, matrix (13) and especially matrix (14) correspond to the nonsupersymmetric model, while matrices (12), (15) and possibly (16) correspond to the supersymmetric model.

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