ABSTRACT

Graph neural networks generalize conventional neural networks for graph-structured data and have garnered widespread attention because of their impressive representation capability. Despite these significant achievements, the performance of Euclidean models in graph-related learning remains constrained by the representational capacity of Euclidean geometry, particularly for datasets exhibiting a highly non-Euclidean latent anatomy. Recently, hyperbolic spaces have grown in popularity for processing graph data with tree-like structures or power-law distributions, thanks to their exponential growth property. In this survey, we provide a comprehensive review of the current hyperbolic graph neural networks1, integrating them into a unified framework and detailing the variations of each component. Moreover, we highlight a series of pertinent applications across diverse fields. Most importantly, we pinpoint several challenges that could serve as directives for further enhancing the accomplishments of graph learning in hyperbolic spaces.

Keywords Hyperbolic geometry · Graph neural network · Graph representation learning

1 Introduction

Graphs are data structures that extensively exist in real-world complex systems, varying from social networks [15, 62], protein interaction networks [52], recommender systems [9, 65, 64], knowledge graphs [56], to the financial transaction systems [40]. They form the basis of innumerable systems owing to their widespread utilization, allowing relational knowledge about interacting entities to be stored and accessible rapidly. Consequently, graph-related learning tasks gain increasing attention in machine learning and network science research. Many researchers have applied Graph Neural Networks (GNNs) for a variety of tasks, including node classification [23, 53, 59], link prediction [22, 71], and graph classification [61, 11] by embedding nodes in low-dimensional vector spaces, encoding topological and semantic information simultaneously. Many GNNs are built in Euclidean space in that it feature a vectorial structure, closed-form distance and inner-product formulae and is a natural extension of our intuitively appealing visual three-dimensional space [14].

Despite the effectiveness of Euclidean space for graph-related learning tasks, its ability to encode complex patterns is intrinsically limited by its polynomially expanding capacity. Although nonlinear techniques [3] assist in mitigating this issue, complex graph patterns may still need an embedding dimensionality that is computationally intractable. As revealed by recent research [4] many complex data show non-Euclidean underlying anatomy. For example, the tree-like structure extensively exists in many real-world networks, such as the hypernym structure in natural languages, the subordinate structure of entities in the knowledge graph, the organizational structure for financial fraud, and the power-law distribution in recommender systems.2 In these situations, Euclidean space fails to make the most powerful or adequate geometrical representations.

1https://github.com/marlin-codes/HGNNs
2The power-law distribution can be traced back to the hierarchical structures [39].
Recently, hyperbolic space has gained increasing popularity in processing tree-like graph data. Figure 1 depicts three prevalent models for hyperbolic space, and they are isomorphic. The typical geometric property of hyperbolic space is that its volume increases exponentially in proportion to its radius, whereas Euclidean space grows polynomially. Such a geometric trait brings two benefits, enabling it to deal well with tree-like graph data. The first one is that hyperbolic space exhibits minimal distortion and fits the hierarchies particularly well since the space closely matches the growth rate of tree-like data while the Euclidean space cannot. The second is that even though with a low-embedding dimension space, hyperbolic models are surprisingly able to produce a high-quality representation, which makes it especially favorable in low-memory and low-storage scenarios.

Scope and Structure of the Survey. To the best of our knowledge, no surveys have been conducted mainly on the methods and applications of Hyperbolic Graph Neural Networks (HGNNs), and the most recent relevant study [37] mainly sketches the progress of hyperbolic neural networks without focusing on the latest methods and applications in graph field. In this work, we attempt to fill this gap by scoping the latest research efforts on HGNNs. The main contributions of this work are summarized below:

- We provide a detailed technique review over existing HGNN models, unifying them by a general framework and outlining the variants of each module. Additionally, we discuss recent studies on theoretical and empirical analysis of HGNN models.
- We systematically categorize the applications and divide them into numerous scenarios. For each case, we present several major applications and their corresponding methods.
- We summarize several challenges and opportunities for future research, providing insights for further flourishing the achievements of graph learning built with hyperbolic spaces.

2 Preliminaries and Notation

In this section, we briefly introduce a list of some of the most helpful Riemannian geometry concepts, definitions, and operations in hyperbolic geometry. For a more detailed introduction, please refer to [25] and [26].

Manifold and Tangent Space. Riemannian geometry is a sub-field of differential geometry in which a smooth manifold $\mathcal{M}$ is associated with a Riemannian metric $g^{\mathcal{M}}$. An $n$-dimensional manifold $(\mathcal{M}, g^{\mathcal{M}})$ is a topological space, a generalization of a 2-dimensional surface with high dimensions. For each point $x$ in $\mathcal{M}$, a tangent space $T_x\mathcal{M}$ is defined as the first-order approximation of $\mathcal{M}$ around $x$, which is an $n$-dimension vector space and isomorphic to $\mathbb{R}^n$. The Riemannian manifold metric $g^{\mathcal{M}}$ assigns a smoothly varying positive definite inner product $\langle \cdot, \cdot \rangle: T_x\mathcal{M} \times T_x\mathcal{M} \rightarrow \mathbb{R}$ on the tangent space, which allows us to define several geometric properties, such as geodesic distances, angles, and curvature.
Geodesics and Induced Distance Function. For a curve \( \gamma : [\alpha, \beta] \rightarrow M \), the shortest length of \( \gamma \), i.e., geodesics, is defined as \( L(\gamma) = \int_{\alpha}^{\beta} \|\gamma'(t)\|_{\gamma} \, dt \). Then the distance of \( u, v \in M \), \( d_M(u, v) = \inf L(\gamma) \) where \( \gamma \) is a curve that \( \gamma(\alpha) = u \) and \( \gamma(\beta) = v \).

Maps and Parallel Transport. The maps define the relationship between the hyperbolic space and the corresponding tangent space. For a point \( x \in M \) and vector \( v \in T_x M \), there exists a unique geodesic \( \gamma : [0, 1] \rightarrow M \) where \( \gamma(0) = x \) and \( \gamma'(0) = v \). The exponential map \( \exp_x : T_x M \rightarrow M \) is defined as \( \exp_x(v) = \gamma(1) \) and logarithmic map \( \log_x \) is the inverse of \( \exp_x \). The parallel transport \( P_{T_{x} \rightarrow y}(v) = v - \frac{c(x,y)}{1 + \frac{c(x,y)}{2}} (x + y) \) achieves the transportation from point \( x \) to \( y \) that preserves the metric tensors.

Hyperbolic Models. Hyperbolic geometry is a Riemannian manifold with a constant negative curvature. There exist multiple equivalent hyperbolic models, like the Poincaré ball model, Lorentz model, and Klein model, which show different characteristics but are mathematically equivalent. We here mainly introduce two widely studied and adopted hyperbolic models in HGNNs, i.e., the Poincaré ball model and the Lorentz model. Let \( ||.|| \) be the Euclidean norm and \( \langle ., . \rangle \) be the Minkowski inner product, respectively. The two models are given by Definition 2.1 and Definition 2.2, respectively. The related formulas and operations, e.g., distances, maps, and parallel transports, are further summarized in Table 1, where \( \oplus \), and \( \gamma y \) are M"obius addition [50] and gyration operator [50], respectively.

| Manifold          | Poincaré Ball Model                  | Lorentz Model                  |
|-------------------|--------------------------------------|---------------------------------|
| Metric            | \( \mathcal{B}^n \) = \{ x \in \mathbb{R}^n : \langle x, x \rangle_{\mathcal{B}} < -\frac{1}{c} \} \) | \( \mathcal{L}^n \) = \{ x \in \mathbb{R}^{n+1} : \langle x, x \rangle_{\mathcal{L}} = 1/c \} \) |
| Induced distance  | \( d_{\mathcal{B}}(x, y) = \frac{1}{\sqrt{c(1 - \frac{c}{2})}} \sqrt{1 \frac{c}{|x - y|^2}} \) | \( d_{\mathcal{L}}(x, y) = \frac{1}{\sqrt{c(1 - \frac{c}{2})}} \sqrt{1 \frac{c}{|x - y|^2}} \) |
| Exponential map   | \( \exp_{\mathcal{B}}(v) = x \oplus \left( \frac{\tan \frac{c}{2} |v|}{\sqrt{c}} \right) \frac{\bar{v}}{||\bar{v}||} \) | \( \exp_{\mathcal{L}}(v) = \cos \left( \frac{c}{2} |v| \right) x + \sinh \left( \frac{c}{2} |v| \right) \frac{\bar{v}}{||\bar{v}||} \) |
| Parallel transport| \( P_{T_{x} \rightarrow y}(v) = v - \frac{c(x,y)}{1 + \frac{c(x,y)}{2}} (x + y) \) | \( P_{T_{x} \rightarrow y}(v) = v - \frac{c(x,y)}{1 + \frac{c(x,y)}{2}} (x + y) \) |

In the following, we use \( \mathcal{H} \) to represent the case that is applicable to both \( \mathcal{B} \) and \( \mathcal{L} \). However, in certain situations, we may employ a more precise notation to describe a specific model.

3 HGNN Methodologies

Graph Neural Networks (GNNs) have recently demonstrated significant superiority in graph-related tasks and applications. They excel in explicitly encoding node attributes and their interactions while implicitly capturing high-order dependencies. Hyperbolic GNNs (HGNNs) have further achieved remarkable success in studying graph data, particularly with a tree-like structure, thanks to the unique properties of hyperbolic spaces. Implementing HGNNs involves three fundamental steps: feature transformation, neighborhood aggregation, and non-linear activation. Prior to that, it is necessary to initialize the features by projecting the input Euclidean feature onto the hyperbolic manifold. In the following, we will first introduce the hyperbolic initial layer in Section (3.1) and then present the details of the three steps in Section (3.2), Section (3.3), and Section (3.4), respectively.

3.1 Hyperbolic Initialization Layer

Consider a graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) with a vertex set \( \mathcal{V} \) and an edge set \( \mathcal{E} \). Let \( (x^E_i) \in \mathcal{V} \) represent the \( n \)-dimensional input node features, where \( x^E \in \mathbb{R}^n \) and the superscript \( E \) indicates the Euclidean space. For each vertex \( i \in \mathcal{V} \), its \( n \)-dimensional input node feature is denoted as \( x^E_i \), residing in the Euclidean space. To refer to the tangent space at \( x \) and the hyperbolic state (in either the Poincaré ball model or the Lorentz model), we will use the superscripts
and \( \mathcal{H}(\mathcal{B}/\mathcal{L}) \), respectively. In the existing literature, the Euclidean node features \( x^E \) is typically derived from four different sources: (i) pre-trained embedding [48, 7], (ii) directly from node attributes [5, 29, 73], (iii) randomly sampling [33, 34, 42, 65, 64, 43] or (iv) one-hot projection [38, 66, 63].

**Initialization layer in Poincaré ball model.** To project the feature \( x^E \) onto the Poincaré ball model \( \mathcal{B} \), the exponential map is commonly used. This can be expressed as:

\[
x^B = \exp_o^c(x^E),
\]

where the \( x^E \) is interpreted as a node situated in the tangent space at the origin, that is \( x^E \in T_o\mathcal{B}^n \).

**Initialization layer in Lorentz model.** In the Lorentz model, a similar approach is adopted. Notably, when considering \( x^E \) as a point in the tangent space at the origin, it becomes necessary to prepend a zero value to \( x^E \) [5]. Thus, we have \( x^{T_o} = (0, x^E) \), satisfying the condition \( \langle 0, (0, x^E) \rangle_\mathcal{L} = 0 \). This point is then projected into the Lorentz model using the exponential map:

\[
x^L = \exp_o^c(x^{T_o}) = \exp_o^c((0, x^E)).
\]

Besides, Gulcehre et al. introduced a projection using pseudo polar coordinates, which has the same final expression as the Equation (2).

Once we have obtained the initialization state of the nodes, we proceed with feature transformation and neighborhood aggregation, followed by non-linear activation. The current methods primarily utilize two approaches for these transformations: the tangent space approach and the fully hyperbolic transformation.

- In the tangent space method, nodes are initially mapped to the tangent space using the logarithmic map. As the tangent space is isomorphic to the Euclidean space, the transformation takes place there. Subsequently, the nodes are mapped back to the hyperbolic space using the exponential map.
- The fully hyperbolic transformation is conducted entirely within the hyperbolic space without any involvement of the tangent space.

### 3.2 Hyperbolic Feature Transformation

Hyperbolic feature transformation mainly consists of matrix-vector multiplication and bias addition. In the subsequent sections, we will provide a detailed review of these operations.

**Matrix-vector Multiplication.** To implement matrix-vector multiplication, the tangential method is applicable for both the Poincaré ball model and the Lorentz model. Let \( x^B \in \mathcal{B}^n, W \in \mathbb{R}^{d \times n} \), then in Poincaré ball model, the matrix-vector multiplication [14, 29] is defined by

\[
W \odot^c x^B := \exp_o^c(W \log_o^c(x^B)).
\]

The node features \( x^B \) are projected from the Poincaré ball model to the tangent space, which is isometric to Euclidean space. After performing the matrix-vector multiplication operation in the tangent space, they are mapped back to the Poincaré ball.

In the context of the Lorentz model, one method to achieve matrix-vector multiplication is similar to that of the Poincaré ball model. Chami et al. directly employed Equation (3) on the Lorentz model. However, this approach breaks the constraints inherent to the Lorentz manifold. To rectify this, it is imperative that the node features remain within the tangent space at the origin after the matrix \( M \) multiplication\(^3\). As suggested by [75, 66], this can be accomplished by modifying the values of the last \( n \) coordinates. The formula is defined by

\[
W \odot^c x^L := \exp_o^c((0, W \log_o^c(x^L))[1:n]),
\]

where \( x^L \in \mathcal{L}^n, W \in \mathbb{R}^{d \times n} \). This method ensures the first coordinate is consistently zero, signifying that the resultant transformation is invariably within the tangent space at \( o \).

Furthermore, Dai et al. defined a new matrix for the Lorentz feature transformation as:

\[
W \otimes x^L := W x^L
\]

\[
s.t. \ W = \begin{bmatrix} 1 & 0^\top \\ 0 & \tilde{M} \end{bmatrix}, \tilde{M}^\top \tilde{M} = I,
\]

\(^3\)Supposing we use Lorentz origin as the reference point.
where $W \in \mathbb{R}^{n+1}$, $\tilde{M}$ is a transformation sub-matrix, acting as a rotation matrix. $0$ is a column vector of zeros, and $I$ is an identity matrix. Compared to Equation (4), this method employs $\tilde{M}$ to define $W$. This transformation respects the Lorentz constraints, let $y$ be the result, then we have the following:

$$
(y, y) = -x_0^2 + \left(\tilde{M}x_{1:n}\right)^T \tilde{M}x_{1:n}
$$

$$
= -x_0^2 + x_{1:n}^T \tilde{M}^T \tilde{M} x_{1:n}
$$

$$
= -x_0^2 + x_{1:n}^T x_{1:n}
$$

$$
= -1.
$$

Then, it is derived that the transformed result $y$ lies in the Lorentz model. Recent study [8] shows that the above linear transformation only considers a special rotation but no boost. They then derived a general transformation method, which is equipped with both rotation and boost operations as well bias addition, normalization, etc., i.e.,

$$
W \odot \xi = \left(\begin{bmatrix} v^T & M \end{bmatrix}\right) \odot \xi = \begin{bmatrix} \sqrt{\|\phi(Mx, v)\|^2 - 1/K} \phi(Mx, v) \end{bmatrix},
$$

where $\phi(Mx, v) = \frac{\sigma(v^T x + b')}{\|Mh(x) + b\|}(Mh(x) + b)$, where $\sigma$ is the sigmoid function, $b$ and $b'$ are bias terms, $\lambda > 0$ controls the scaling range, and $h$ is the activation function.

In general, implementing feature transformation in HGNN is mainly composed of two different methods: tangent space method, fully in the hyperbolic manifold.

**Bias Addition.** To implement bias addition, tangent space at origin is still a useful medium, and the formula in $B$ and $L$ can be uniformly expressed as:

$$
x^H \odot b^H = \exp_{x^H}^e \left( P_{\alpha \rightarrow x^H}^c \left( \log_{x^H}^{c} (b^H) \right) \right),
$$

where $b^H$ is the bias in $H^n$, and the equations of parallel transport $P_{x^H \rightarrow y^H}(\cdot)$ in $B$ and $L$ are summarized in Table 1. In [8], they incorporated the bias addition in Equation (7).

### 3.3 Hyperbolic Neighborhood Aggregation

Hyperbolic neighborhood aggregation is a technique that leverages the hyperbolic space to aggregate information from neighboring nodes in a graph. This process can be broadly divided into two main steps: (1) computation of neighborhood weights and (2) computation of mean aggregation.

**Computation of neighborhood weights.** The determination of neighborhood weights is crucial in aggregating information effectively. Various strategies can be employed to calculate these weights, including utilizing structure information, feature information, hyperbolic distance, or a combination of these methods, which are summarized in Table 2. Below, we delve into each of these strategies. Consider a central node $i$, and we assume that $N(i)$ represents the immediate one-hop neighbor of node $i$, with $j$ being an element of $N(i)^4$.

1. **Structure information:** The weight $\alpha_{ij}$ can be defined based on node degree [29]:

$$
\alpha_{ij} = 1/\sqrt{d_id_j},
$$

| Formula | Source | Reference | Description |
|---------|--------|-----------|-------------|
| $\alpha_{ij} = 1/\sqrt{d_id_j}$ | SI | [29] | Simplified spectral convolution based on node degree $d_i, d_j$ |
| $\alpha_{ij} = \text{softmax}_{i \in N(i)}(\text{MLP}(\alpha_{ij}))$ | SI | [67] | Curvature-based weight $\kappa$ sensitive to local graph structure |
| $\alpha_{ij} = \text{softmax}_{i \in N(i)}(\text{LeakReLU}(W^T [x_i^T, x_j^T]))$ | SI | [5] | Weighting adaptive to node feature |
| $\alpha_{ij} = \text{softmax}_{i \in N(i)}(-d_i(x_i^H, x_j^H))$ | HD | [74] | Based on negative hyperbolic distance $d_i$ between nodes |
| $\alpha_{ij} = \text{softmax}_{i \in N(i)}(-d_i(x_i^H, x_j^H))$ | HD | [75, 8] | Weighting with squared hyperbolic distance, emphasizing geometry |
| $\alpha_{ij} = f(-\beta d_i(x_i^H, x_j^H) - \gamma)$ | HD | [17] | Flexible weighting incorporating squared hyperbolic distance $d_i^2$ |
| $\alpha_{ij} = \text{softmax}_{i \in N(i)}(\text{LeakReLU}(W^T [x_i, x_j]) \times d_i(x_i^H, x_j^H))$ | FLHD | [77] | Combines hyperbolic distance $d_i$ with node feature $x$ |
| $\alpha_{ij} = \text{softmax}_{i \in N(i)}(\text{LeakReLU}(W^T [x_i, x_j]) \times d_i(x_i^H, x_j^H))$ | FLSI | [5] | Integrates node feature $x$ with node degree information $d_i, d_j$ |
| $\alpha_{ij} = \frac{w_{ij}}{d_i + w_{ij}}$ | FLSI | [67] | Unifies node curvature information $w_{ij}$ with node feature $w_{ij}$ |

$\phi_{ij}$ which is equipped with both rotation and boost operations as well bias addition, normalization, etc., i.e.,

$$
\phi_{ij} = \frac{\sigma(v^T x + b')}{\|Mh(x) + b\|}(Mh(x) + b),
$$

where $\sigma$ is the sigmoid function, $b$ and $b'$ are bias terms, $\lambda > 0$ controls the scaling range, and $h$ is the activation function.

Following the literature, we consider $i \in N(i)$.
where $\hat{d}_i = d_i + 1$ and $d_i$ is the degree of node $i$. This method, derived from the Euclidean graph convolutional network [23], provides a simplified approach to graph spectral convolution, primarily focusing on the topological information of the graph.

Additionally, the weight can also be defined using Ollivier Ricci curvature [35], which describes the deviation of a local pair of neighborhoods from a "flat" case,

$$\alpha_{ij} = \text{softmax}_{j \in N(i)}(\text{MLP}(\kappa_{ij})),$$  \hspace{1cm} (10)

where MLP represents Multi-layer Perception and curvature $\kappa$ reflects the local structure of the node $i$, which is computed by optimal transport [67].

(2) **Feature information** [5]: The weight $\alpha_{ij}$ using node feature is defined as follows:

$$\alpha_{ij} = \text{softmax}_{j \in N(i)}(\text{LeakReLU}(\mathbf{W}^T[x_i^{T_o}||x_j^{T_o}])), \hspace{1cm} (11)$$

This method computes the weight based on the hyperbolic features of nodes $i$ and $j$. The negative sign ensures that closer nodes have higher weights. Besides, Zhang et al. also introduced $K$-head attention, which is defined as:

$$\alpha_{ij} = \text{softmax}_{j \in N(i)}\left(-d_{ij}^H(x_i^H, x_j^H)\right), \hspace{1cm} (12)$$

Alternative approaches using hyperbolic distance include using squared distance [75]:

$$\alpha_{ij} = \text{softmax}_{j \in N(i)}\left(-d_{ij}^H(x_i^H, x_j^H)^2\right). \hspace{1cm} (13)$$

Similar to the previous method, the distance is squared, emphasizing the difference between close and distant nodes. Another more complex form [17] is given as:

$$\alpha_{ij} = f(-\beta d_{ij}^H(x_i^H, x_j^H) - \gamma), \hspace{1cm} (14)$$

where $\beta$ and $\gamma$ are parameters that can be set manually or learned along with the training process, and $f$ can be $\text{softmax}(\cdot)$. The hyperbolic distance is scaled by a factor $\beta$ and shifted by $\gamma$. Both $\beta$ and $\gamma$ can be learned or set manually. The function $f$ can be a softmax function, ensuring the weights are normalized.

(4) **Integration of feature information and hyperbolic distance** [76]: The $\alpha_{ij}$ is computed by

$$\alpha_{ij} = \text{softmax}\left(\text{LeakReLU}(\mathbf{W}^T[x_i^{T_o}||x_j^{T_o}]) \times d_{ij}^H(x_i^H, x_j^H)\right). \hspace{1cm} (15)$$

This approach combines feature attention and node distance. Compared with the pure feature method, it is modulated by the hyperbolic distance between the nodes, ensuring that both the node features and their relative positions in the hyperbolic space contribute to the weight.

(5) **Integration of feature and structure information**: The weight $\alpha_{ij}$ is computed by$^5$

$$\alpha_{ij} = \text{sigmoid}\left(\text{LeakReLU}(\mathbf{W}^T[x_i^{T_o}||x_j^{T_o}]) \times 1/\sqrt{d_i d_j}\right). \hspace{1cm} (16)$$

This method integrates both structural information and feature attention, ensuring that both the node features and their connectivity in the graph contribute to the weight. In [67], Yang et al. introduced another integration, which is formulated by:

$$\alpha_{ij} = \frac{w_s \tilde{k}_{ij} + w_f \tilde{f}_{ij}}{w_s + w_f},$$

where $\tilde{k}_{ij}$ and $\tilde{f}_{ij}$ is derived from Equation (10) and (11), respectively.

**Computation of Mean Aggregation.** For mean aggregation, or weighted mean pooling, it cannot be computed by simply averaging the inputs, which may lead a deviation out of the hyperbolic manifold. Currently, there are three

\[\text{From https://github.com/HazyResearch/hgcn}\]
typical ways to implement mean aggregation: tangential method, Einstein midpoint, and Lorentzian centroid. 

1) Tangential method [29, 5] is defined by,

\[
AGG(x^H_i) := \exp^c_o \left( \sum_{j \in N_i} \alpha_{ij} \left( \log^c_o (x^H_j) \right) \right).
\]

(17)

2) Einstein midpoint method [17, 10] is formulated as,

\[
AGG(x^H_i) := \begin{cases} 
\mathbf{x}^K_i = p_{\mathcal{H} \rightarrow \mathcal{K}} (x^\ell_i), \\
\mathbf{m}^c_i = \sum_{j \in N_i} \gamma_j \mathbf{x}^K_j / \sum_{j \in N_i} \gamma_j, \\
\mathbf{x}^\ell_i = p_{\mathcal{K} \rightarrow \mathcal{H}} (\mathbf{m}^c_i)
\end{cases}
\]

(18)

where \(p_{\mathcal{M}_1 \rightarrow \mathcal{M}_2}\) denotes the projection from \(\mathcal{M}_1\) to \(\mathcal{M}_2\) and \(\gamma_j = (1 - \|x^K_j\|)^{-1/2}\) is the Lorentz factor.

3) Lorentzian centroid method [75, 8, 38] is expressed as,

\[
AGG(x^F_i) := \frac{\sum_{j \in N_i} \alpha_{ij} x_j^F}{\sqrt{c_i} \| \sum_{j \in N_i} \alpha_{ij} x_j^F \|_c}.
\]

(19)

The tangential mean computation is one of the most straightforward methods. It is applicable to the Poincaré ball and Lorentz models. However, directly executing the weighted mean in the tangent space needs extra caution to guarantee the results still live in the manifold. On the other hand, it lacks a differentiable Fréchet mean operation [30]. The Einstein midpoint is based on the Klein coordinates and applicable to Poincaré ball and Lorentz models by the isomorphic bijections. The Lorentzian centroid is designed for Lorentz model. Besides, there is another equivalent centroid for Poincaré ball model as shown in [41], i.e., Möbius gyromidpoint [51]. These three centroids can be characterized as a minimizer of the weighted sum of calibrated squared distance [41].

3.4 Non-linear Activation

The non-linear activation can be achieved with the same idea of matrix-vector multiplication [5], i.e.,

\[
\sigma^{\otimes c_{t-1} \otimes c} (x^H) = \exp^c_o \left( \sigma (\log^c_o (x^H)) \right),
\]

where \(\ell\) denotes the \(\ell - th\) layer. For the Lorentz model, [75] proposed a more accurate form,

\[
\sigma^{\otimes c_{t-1} \otimes c} (x^H) = \exp^c_o \left( 0, \sigma (\log^c_o (x^H_{[1:n_i]})) \right),
\]

(21)

which ensures that the result still lives in the Lorentz manifold. Besides, according to the manifold-preserving properties between the Lorentz model and Poincaré ball model, another way is to convert the Lorentzian feature to the Poincaré feature and implement the non-linearity in Poincaré ball model [10]

\[
\sigma^{\otimes c_{t-1} \otimes c} (x^H) = p_{\mathcal{B} \rightarrow \mathcal{H}} \left( \sigma (p_{\mathcal{H} \rightarrow \mathcal{B}} (x^H)) \right)
\]

Chen et al. incorporated the non-linear activation in Equation (7). Since the exponential map is a non-linear operation, which can produce the non-linearity, works by [42, 55] thus ignored it.

3.5 Overall View

In general, a unified hyperbolic graph convolutional layer can be formulated as,

\[
\begin{align*}
\mathbf{h}^{t-1, H}_i &= \left( \mathbf{W}^t \otimes^{c_{t-1}} x^{t-1, H}_i \right) \otimes^{c_{t-1}} \mathbf{b}^t, \\
y^{t, H}_i &= AGG^{c_{t-1}} \left( \mathbf{h}^{t, H}_i \right), \\
x^{t, H}_i &= \sigma^{\otimes c_{t-1} \otimes c} \left( y^{t, H}_i \right).
\end{align*}
\]

(22)

To reduce the maps between tangent space and hyperbolic manifold, some research works, for example, [29], achieve all three steps in the tangent space. Although this simplification can reduce some computation burden, its performance decreases as well, according to the experimental results [76].

4 Applications

Hyperbolic spaces have many successful applications in a variety of fields, including natural language processes, computer vision, etc., while the applications of HGNNs are mainly on recommendation systems, knowledge graphs, and drug molecules, where the dataset is graph-structured in natural and have the tree-like characteristic.
4.1 HGNNS for Recommender Systems

The recommender systems can be simplified as a bipartite graph, in which vertices represent users or items, and edges denote their interactions. Considering the prevalence of the power-law distribution in user-item networks, hyperbolic space has attracted considerable attention. In the following, we review the recommender systems built upon hyperbolic space from four aspects.

**Graph Neural Collaborative Filtering.** In the settings of graph neural collaborative filtering, the features of items and users are often not available. [42, 55, 65] built the feature via Gaussian sampling. Further, they incorporate multiple layers of HGNN to gather higher-order information for explicitly modeling user-item interactions.

**Social Network Enhanced Recommender System.** Apart from user-item interactions, users’ preferences are strongly tied to their social relationships (e.g., friends and followers). [54] designed a multi-aspect perceiving HGNN in hyperbolic space to capture multi-aspect interactions of users on item, via defining a learnable interactive relation for each specific user-item pair.

**Knowledge Graph Enhanced Recommender System.** Recommender systems incorporating the knowledge graph as side information can not only address the issues of data sparsity and cold start issues but also provide explanations for recommended items. [9] proposed a Lorentzian knowledge-enhanced graph convolutional network for recommendation, which extracts high-order interaction in user-item bipartite graphs and knowledge graph for recommendation.

**Session-based Recommender System.** Session-based recommendation learns the user preferences by analyzing the short-term and long-term patterns based on the user behavior. [18] proposed a contrastive learning manner for session-based recommendation and [27] presented a hyperbolic space-based hypergraph convolutional neural network to learn for session-based recommender systems.

4.2 HGNNS for Knowledge Graph

The knowledge graph is a graph-structured network representing real-world facts with triplets to store a large number of entity and relation information. Given that in large-scale knowledge graphs, the number of entities is scale-free and can be organized to the underlying hierarchical structure, hyperbolic geometry thus provides a powerful alternative to learning low-dimensional embedding while preserving the underlying hierarchy.

Knowledge graph completion and associations are two important applications. [57] proposed an attentive neural context aggregation to adaptively integrate the relational context for enhancing the ability to preserve the hierarchical relations. Moreover, [58] explored the mixed curvature for knowledge graph completion. For HGNN-based knowledge graph associations, [45] developed a HyperKA model that employs an HGNN for KG embedding and utilizes a hyperbolic transformation across embedding spaces to capture multi-granular knowledge associations.

Compared with the hyperbolic transnational model for knowledge graph, e.g., KyperKG [24] MuRP [1], AttH [6], and HERCULES [31], HGNN-based models show more competitive performance with further extracting and incorporating high-order relationships.

4.3 HGNNS for Molecular

Molecules are also naturally represented as graphs, with nodes representing atoms and edges representing chemical bonds. Recently, many studies used GNNs and some of their variations to predict chemical properties. For molecular applications, the research mainly focuses on molecular representation and generation. The basic motivation to apply hyperbolic spaces on molecules is to model the underlying hierarchical structure in it.

**Molecular Representation.** [60] developed a hyperbolic relational graph convolution network plus (HRGCN+) by combining molecular graphs and molecular descriptors for drug discovery. HRGCN+ allows medicinal chemists to understand models at both the atom and descriptor levels, which can also aid in the extraction of hidden information. [68] proposed to learn molecular embedding through the hyperbolic VAE framework to discover new side effects and re-position drugs.

**Molecular Generation.** Learning the implicit structure of molecules using hyperbolic space is an important method for extracting molecular latent hierarchical structure. Hyperbolic molecule generation is generally via graph generation models, including normalized flow [2], GAN [38], etc.
4.4 HGNNs for Other Applications

In this part, we will discuss more successful applications of HGNNs from different fields. For skeleton-based action recognition, [36] designed a hyperbolic spatial-temporal GCN that combines several dimensions on the manifold and provides an effective technique to explore the dimension for each ST-GCN layer. For quantitative trading and investment decision making, [40] modeled the inter-stock relations by HGNN and developed a stock model HyperStockGAT, which constructs the stock graph by the relation in Wikidata and their industry information. For medical ontology matching, [20] proposed MEDTO framework, which is built upon HGNN and a heterogeneous graph layer.

5 Challenges and Opportunities

Although we have witnessed the rapid development and achievements of HGNNs in recent years, there are still issues and challenges that need better solutions. [37] discussed several open problems which are also shared by HGNNs. In this section, we further summarize several challenges in the HGNN community, which also provides opportunities for future study.

5.1 Challenges I: Complex Structures

In graph representation learning, hyperbolic space is emerging as a potential alternative. The most noticeable advantages are credited to its exponential volume growth property, which makes this space to be much more embeddable than Euclidean space, especially for datasets with implicit tree-like layouts. However, the structure of a real-world network is always complex and complicated, in which some areas are tree-like, some are flat and some are circular. Directly embedding a graph with intricate layouts into the hyperbolic manifold inevitably leads to structural inductive biases and distortions.

Opportunities: To this point, there are some preliminary attempts, but there is still much potential for improvements. An intuitive approach is to combine Euclidean and/or spherical spaces to complement the hyperbolic space. For example, GIL [76] utilizes hyperbolic space and Euclidean space interactively and places different weights on two branches to cope with intricate complex graph structures. M²GNN [58] incorporates three different spaces, namely hyperbolic, Euclidean, and spherical spaces, for knowledge graph completion. SelfMGNN [44] resorts to a mixed-curvature space via the Cartesian product. On the other hand, ACE-HGNN [13] attempts to learn an optimal curvature to model the tree-like graph with different hyperbolicity via a multi-agent reinforcement learning framework. HGCL [28] enhances the modeling of HGNN by contrastive learning. These methods generally need to create multiple branches, which unavoidably increases the computational complexity to a certain extent. Besides, most of the above methods solve this issue in a global manner. A local identifier, like Ricci curvature [32, 49] can be more effective and efficient.

5.2 Challenge II: Geometry-aware Learning

Though HGNNs have made noticeable achievements, most of the efforts mainly focus on how to generalize GNNs into hyperbolic space by properly designing the transformation operations among the spaces. In many HGNNs, the optimization target (like cross-entropy in [5] and [75]) are generally similar to the Euclidean counterparts, which are geometrically irreverent with hyperbolic properties. On the other hand, hyperbolic space is curved, that is, locations closer to the origin are flatter and relatively narrow, whereas regions further from the origin are broader, and this property is seldom considered when designing HGNNs.

Opportunities: From the optimization target, how to integrate the hyperbolic geometry with the learning objective is the first valuable problem to explore. Recently, [65] presented a geometrically aware hyperbolic recommender system, paving the path for the community. The second opportunity is to make the learning process geometry-aware, keeping the awareness of node position rather than being equipped with a geometry-unconscious optimization target, like the constant margin in metric learning. Last, based on the hyperbolic geometric trait, the exponentially hyperbolic space provides spacious room for the samples to be well arranged, especially in the area far away from the origin. Intuitively, by encouraging the model to preserve the inherent data structure and pushing the overall embedding far away from the origin, the model representation ability could then be largely improved.

5.3 Challenges III: Trustworthy HGNNs

HGNNs have been proven to produce better representations of hierarchical graphs. However, there are still a few trustworthy issues to be addressed: (1) Where is the superiority of the hyperbolic space? For example, the better performance of the hyperbolic model originates from where? the better fitting of high-level nodes, tail nodes, or both?
It is unclear in what conditions the HGNNs are guaranteed to be better than their Euclidean counterparts. In other words, the generalization error and the robustness of HGNNs have not been well studied and analyzed.

**Opportunities:** [72] made an initial study empirically investigating the behavior of hyperbolic recommender systems. To figure out the characteristics of HGNNs, more in-depth analyses are needed. For the generalization ability, [46] pointed out that in hyperbolic ordinal embedding (HOE), the generalization error bound is at most exponential with respect to the embedding radius and HOE can represent a tree better than linear graph embedding [47]. The analysis facilitates the exploration of the generalization and robustness of HGNNs.

### 5.4 Challenges IV: Scalable HGNNs

Despite the tremendous success in the performance improvement, HGNNs also bring in more computational requirements compared with their Euclidean counterparts, e.g., the frequent exponential and logarithmic maps [69]. In real-world scenarios, graphs often on the scale of millions of nodes and edges, and computational resources are limited. The difficulty of extending HGNNs to large-scale graphs remains one of the main challenges.

**Opportunities:** A large body of research work has been proposed to improve the scalability of GNNs by decoupling the graph process, simplifying the feature transformation or aggregation operations [59, 12], or via a series sampling techniques [70, 19, 21], which could be further integrated with the property of HGNNs.

### 6 Conclusion

Hyperbolic space can be regarded as a continuous tree, making it well-suited for modeling datasets with latent hierarchy layouts. HGNNs extend GNNs into hyperbolic space and have achieved great success in graph data, especially with tree-like structures. In this survey, we present the technical details of HGNNs, including the methodologies, applications, challenges, opportunities, and current state-of-the-art models and algorithms of HGNN networks. More specifically, we unify them by a general framework and summarize the variants of each module. We also identified several challenges that need to be overcome. To some extent, these challenges serve as guidelines for flourishing the achievements of hyperbolic graph learning. Although many researchers are actively engaged in addressing these problems, we point to the numerous opportunities that still exist to contribute to the development of this important, ever-evolving field.

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