Complete Supersymmetric Quantum Mechanics of Magnetic Monopoles in $N = 4$ SYM Theory

Dongsu Bak $^a$, Kimyeong Lee $^b$ and Piljin Yi $^b$

$^a$ Physics Department, University of Seoul, Seoul 130-743, Korea

$^b$ School of Physics, Korea Institute for Advanced Study
207-43, Cheongryangryi-Dong, Dongdaemun-Gu, Seoul 130-012, Korea

We find the most general low energy dynamics of 1/2 BPS monopoles in the $N = 4$ supersymmetric Yang-Mills (SYM) theories when all six adjoint Higgs expectation values are turned on. When only one Higgs is turned on, the Lagrangian is purely kinetic. When all the rest five are turned on a little bit, however, this moduli space dynamics is augmented by five independent potential terms, each in the form of half the squared norm of a Killing vector field on the moduli space. A generic stationary configuration of the monopoles can be interpreted as stable non BPS dyons, previously found as nonplanar string webs connecting D3-branes. The supersymmetric extension is also found explicitly, and gives the complete quantum mechanics of monopoles in $N = 4$ SYM theory.

1Electronic Mail: dsbak@mach.uos.ac.kr
2Electronic Mail: kimyeong@kias.re.kr
3Electronic Mail: piljin@kias.re.kr
1 Introduction

Recently new stable solitons in Yang-Mills theories were constructed, whose electric charges and magnetic charges are not proportional to each other. These new solitons exist only when more than one adjoint Higgs fields are involved, so are natural in Yang-Mills theories with extended supersymmetry. Their classical aspects have been studied in the context of $N = 4$ supersymmetric Yang-Mills theories \[1, 2, 3, 4, 5, 6\]. Of these dyons, some preserve 1/4 of $N = 4$ supersymmetry, and thus are known as 1/4 BPS dyons, while others do not preserve any supersymmetry at all.

It is also well known that the supersymmetric Yang-Mills theories arise as a low energy description of parallel D3 branes in the type IIB string theory \[7\]. In this context, dyons arise as string webs ending on D3 branes. For instance, more traditional 1/2 BPS monopoles and dyons are represented by straight \((p,q)\) string segments ending on a pair of D3 branes, while a 1/4 BPS dyon corresponds to a properly oriented, planar web of strings ending on more than two D3 branes\[8\]. The stable non-BPS states are realized as the most general form of string web, which is typically nonplanar. Some non-BPS dyons can be thought of as deformation of 1/4 BPS dyons which results as D3 positions themselves get moved and become nonplanar in the six transverse directions. (A numerical study of such non BPS dyons as a field theory configuration has been performed within spherically symmetric ansatz and the resulting brane configurations were found to agree with that of the type IIB string theory\[6\].)

The detail of the 1/4 BPS field configurations has been explored in Ref. \[1\]. The BPS equations satisfied by these field configurations consist of two pieces: one is the old magnetic BPS equation for some 1/2 BPS monopole configuration while the other is a covariant Laplace equation for the additional Higgs field in the 1/2 BPS monopole background. Because of this two-tier structure of the BPS equations, the parameter space of a 1/4 BPS dyon is identical to the moduli space of the first, magnetic BPS equations. The only subtlety here is that some of the monopole parameters transmute to classical electric charges, as we compare the two.

The main lesson we learn from this fact is that the nonrelativistic dynamics that incorporates 1/4 BPS dyonic states can be formulated as that of dynamics on the same old monopole moduli space but with new interactions. This is true, at least when the monopole rest mass is dominant over the electric part of the energy. The kinetic term of the modified effective Lagrangian is given
by the moduli space metric of the underlying 1/2 BPS monopoles, while the potential term comes from the additional Higgs field and is found to be a half of the squared norm of a Killing vector. The effective Lagrangian have a BPS bound of its own. Classically and quantum mechanically, the 1/4 BPS dyons arise as specific bound states that saturate such a low energy BPS bound. This new dynamics incorporates the 1/4 BPS dyons as well as more traditional 1/2 BPS monopoles and dyons.

There have been a couple of derivations of this effective low energy Lagrangian of monopoles that produces 1/4 BPS dyonic configurations. The first such derivation relied on the relation between the BPS energy of 1/4 BPS configurations and the conserved electric charges [9]. The trick here is to realize that the BPS energy can be estimated in two different ways. One from the field theory, and the other from realizing 1/4 BPS states as a bound state of monopoles in the low energy sense. By comparing the former, exact, formula to the latter, one can in fact identify the form of the potential term as one half of the electric part of the BPS energy, expressed in terms of Higgs expectation values and monopole moduli space geometry. The resulting potential is exact within the nonrelativistic approximation. The potential leads to the long range attraction or repulsion between dyons [10].

Shortly thereafter, there appeared an alternate derivation by two of the authors [11]. Here, the field theoretic Lagrangian is calculated for a given initial field data, which are made of 1/4 BPS configuration and its field velocity. The Lagrangian, after integrated over the space, turns out to give the sum of minus rest mass and the kinetic parts which consists of quadratic and also linear terms in velocity of moduli coordinates. This is somewhat similar to the consideration of the zero mode dynamics of a particle with nonzero momentum. After shifting the cyclic coordinates related to conserved electric charges, one gets the low energy Lagrangian including the potential energy.

In both of the previous derivations, the 1/4 BPS dyons were used as the convenient stepping stone that leads to the above low energy Lagrangian. However, the dynamics is really that of monopoles, which naturally produces 1/4 BPS dyons as bound states. The effective Lagrangian was successfully understood because the states therein were understood very well by other means. This funny state of affairs begs for a question whether there exists a more fundamental derivation of the dynamics based only on the properties of 1/2 BPS monopoles.

As we will see, there is indeed such a derivation. In particular, since the method does not rely
on BPS properties of monopole bound states, it is applicable to situations where bound states are typically non BPS. Such outcomes are generic when more than two Higgs take independent vacuum expectation values. In fact, we are going to find the exact low energy Lagrangian when all six Higgs are turned on. On the other hand, note that the low energy dynamics is meaningful only when the kinetic and the potential energies are much smaller than the rest mass of the monopoles [9]. Because of this, one combination of Higgs field must be chosen to be large, so that monopoles have rest masses much larger than the electric and the kinetic part of energy. This separates six Higgs into one with large expectations and five remaining ones with small and independent expectation values.

In this new derivation, the 1/2 BPS monopole configurations are primary and of order one. As we take the expectation values of the additional five Higgs fields to be small, we take them to be of order $\eta << 1$ quantities, where $\eta$ characterizes the ratio of the additional Higgs expectation values to the first Higgs which shows up in magnetic BPS equation. We then solve the field equation for the five additional Higgs fields classically to the leading order in $\eta$, given a static monopole background. The problem again reduces to a second order Laplace equation for each of five Higgs. We put the result back into the field theory Lagrangian to obtain the potential term as a function of Higgs expectations. Of course, independently of this, we also consider slow motions of monopoles and derive the kinetic term as well. The resulting action is accurate to the order, $\eta^2$ and $v^2$ where $v$ is the typical monopole velocity.

The modified effective Lagrangian is again based on the monopole moduli space, since, to the leading order, the above two computations of kinetic and potential terms, do not interfere with each other. In particular, the kinetic term is given by the same moduli space metric. The potential is now half the sum of squared norms of five Killing vectors. These five Killing vector fields are picked out by the five additional and small Higgs expectation values. For generic but small vev's of these five Higgs, even the lowest energy configuration with generic electric charges is non BPS.

The view we take here is similar to the method of obtaining the Coulomb potential between two massive point particles. In that case, we consider the limit the electric coupling $e$ is small and the velocity $v$ of charged particles is small. Then, the leading solution of the Maxwell equation is Coulombic. To the order $v^2 + e^2$, the Lagrangian is the standard kinetic energy with the Coulomb potential. The retarded or relativistic effects would be of order $v^2e^2$, $v^4$ and, thus, negligible.
Recall that the planar string web for the 1/4 BPS dyonic configurations are made of the webs of fundamental strings and D strings. The key cause for the web is the attractive force between fundamental and D strings. For the 1/4 BPS supersymmetry, the orientation of the each string vertex should be consistent. In dyonic picture, the 1/4 BPS configurations correspond to dyons in finite separation with the delicate balance between Higgs force and electromagnetic force. They appear naturally as the BPS configuration for the low energy dynamics. Thus they are the lowest energy configurations for a given set of the electric charge, which characterizes the BPS energy. In the non BPS case, again the non planar string webs are formed from the fundamental strings and D strings. In the low energy Lagrangian, they would correspond to the lowest energy configuration for the given set of electric charges, which could not saturate the BPS energy bound.

The supersymmetric completion of the low energy dynamics should have eight real supercharges. For the cases with one Higgs expectation and two Higgs expectations, the supersymmetric low energy dynamics are known [12, 9, 13]. The former is completely determined by the monopole moduli space metric, which happens to be hyperKähler, while the latter also involves additional potential determined by a single linear combination of triholomorphic Killing vector field on the moduli space. When all six Higgs fields are involved, the low energy dynamics involve up to five linearly independent combinations of such Killing vectors. The final goal of this paper is to write down this supersymmetric low energy Lagrangian explicitly, which completes the low energy interaction of monopoles in $\mathcal{N} = 4$ Yang-Mills theory, to the extent that the nonrelativistic approximation makes sense.

The plan of the paper is as follows. In Sec. 2, we review the 1/2 BPS monopoles and the 1/4 BPS equations. In Sec. 3, we derive the effective Lagrangian for the non BPS configurations. In Sec. 4, we explore this Lagrangian. In Sec. 5, we find the supersymmetric completion of the Lagrangian. In Sec. 6, we conclude with some remarks.

---

4 The supersymmetric quantum mechanics can be obtained as the dimensional reduction of the six dimensional (0,8) supersymmetric sigma model [16] to a one dimensional quantum mechanics, by the Scherk-Schwarz mechanism [17]. This suggests that the form of the supersymmetric Lagrangian is the most general nonlinear sigma model with eight real supercharges.
2 BPS Bound and Primary BPS equation

We begin with the $N = 4$ supersymmetric Yang-Mills theory. We choose the compact semi-simple group $G$ of the rank $r$. We divide the six Higgs fields into $b$ and $a_I$ with $I = 1, \ldots, 5$. The bosonic part of the Lagrangian is given by

$$L = \frac{1}{2} \int d^3x \left\{ E^2 + (D_0 b)^2 + (D_0 a_I)^2 \right\} - \frac{1}{2} \int d^3x \left\{ B^2 + (D b)^2 + (D a_I)^2 + (-i[a_I, b])^2 + \sum_{i<j} (-i[a_I, a_J])^2 \right\},$$

(1)

where $D_0 = \partial_0 - iA_0$, $D = \nabla - iA$, and $E = \partial_0 A - DA_0$. The four vector potential $(A_0, A) = (A_0^a T^a, A^a T^a)$ and the group generators $T^a$ are traceless hermitian matrices such that $\text{tr} T^a T^b = \delta^{ab}$.

As shown in Ref. [1], there is a BPS bound on the energy functional, which is saturated only when configurations satisfy

$$B = D b,$$

(2)

$$E = c_I D a_I,$$

(3)

$$D_0 b - i[c_I a_I, b] = 0,$$

(4)

$$D_0 c_I a_I = 0,$$

(5)

together with the Gauss law,

$$D \cdot E - i[b, D_0 b] - i[c_I a_I, D_0 c_I a_I] = 0.$$  

(6)

where $c_I$ is a unit vector in five dimensions. In addition, the rest of the Higgs field should be trivial on this configuration, or

$$D_0(a_I - c_I c_J a_J) = 0$$

(7)

$$D(a_I - c_I c_J a_J) = 0$$

(8)

$$[b, a_I - c_I c_J a_J] = 0.$$  

(9)

This condition implies the 1/4 BPS configuration should be planar. The BPS energy is then

$$Z = b \cdot g + c_I a_I \cdot q,$$

(10)

where $b$ and $a_I$ are vacuum expectation values of the Higgs fields, while $g$ and $q$ are magnetic and electric charges respectively.
Equation (2) is the old BPS equation for 1/2 BPS monopoles and is called the primary BPS equation. The BPS bound is saturated if the additional equations are satisfied. For 1/4 BPS configurations, the additional equations are from the energy bound and the Gauss law, which are put into a single equation,

\[ D^2 c_I a_I - [b, [b, c_I a_I]] = 0, \]  

which is called the secondary BPS equation. In addition, the \( a_I - c_I c_I a_I \) which is orthogonal to \( c_I \) vector should commute with all other fields and constant in space time. One last step necessary to solve for the 1/4 BPS dyon is to put \( A_0 = -c_I a_I \).

In the type IIB string realization of \( U(N) \) Yang-Mills theories, the above BPS equations imply that the corresponding 1/4 BPS string web lies on a plane. However, the D3 branes which are not connected to the web do not need to lie on the plane. Even when \( D3 \) branes lie on a single plane, one can find planar string webs which is not BPS as the orientations of the string junctions are not uniform. In this paper, we consider the special class of non BPS configurations which correspond the non planar web, which would have been 1/4 BPS configurations when we put D3 branes to a single plane by small deformations of their positions. In addition, we consider the string web is almost linear. For this case, we do not need to solve the full quadratic field equations. We consider an approximation by considering the quantity

\[ \eta \sim \frac{|a_I|}{|b|}, \]  

(12)

to be much smaller than unity, throughout this paper.

Within such approximation, we may solve the field equation in two steps. First one solve the first-order, magnetic BPS equation. Once this is done, the solution of this primary BPS equation describes the collection of 1/2 BPS monopoles, and the Higgs field \( b \) takes the form

\[ b \simeq b \cdot H - \frac{g \cdot H}{4\pi r}, \]  

(13)

asymptotically, where \( H \) is the Cartan subalgebra. We are interested in the case where the expectation value \( b \) breaks the gauge group \( G \) maximally to Abelian subgroups \( U(1)^r \). Then, there exists a unique set of simple roots \( \beta_1, \beta_2, \ldots, \beta_r \) such that \( \beta_A \cdot b > 0 \) [16]. The magnetic and electric charges are given by

\[ g = 4\pi \sum_{A=1}^{r} n_A \beta_A, \]  

(14)
where integer $n_A \geq 0$. For each simple root $\beta_A$, there exist a fundamental monopole of magnetic charge $4\pi \beta_A/e$, which comes with four bosonic zero modes: The integer $n_A$ can be thought of as the number of the $\beta_A$ fundamental monopoles. The moduli space of such 1/2 BPS configurations has the dimension of $4 \sum_A n_A$. We will consider the case where all $n_A$ are positive so that the monopoles do not separate into mutually noninteracting subgroups.

Let us denote the moduli space coordinates by $z^m$. If we parameterize BPS monopole solutions by the moduli coordinate $z$'s, $A_\mu(x, z^m) = (A(x, z^m), b(x, z^m))$ with $\mu = 1, 2, 3, 4$, the zero modes are in general of the form,

$$\delta_m A_\mu = \frac{\partial A_\mu}{\partial z^m} + D_\mu \epsilon_m,$$

(15)

where $D_\mu \epsilon_m = \partial_\mu \epsilon_m - i [A_\mu, \epsilon_m]$ with understanding $\partial_4 = 0$. The zero modes around the 1/2 BPS configurations are determined by perturbed primary BPS equation plus a gauge fixing condition,

$$D \times \delta_m A = \nabla \delta_m b - i [\delta_m A, b],$$

(16)

$$D_\mu \delta_m A_\mu = 0,$$

(17)

which forces the actual zero modes to be a sum of two terms. Given this definition of zero modes, one can define a natural metric on the moduli space spanned by the collective coordinate $z$'s

$$g_{mn}(z) = \int d^3 x \, \text{tr} \, \delta_m A_\mu \delta_m A_\mu.$$

(18)

With such a metric, the Lagrangian (11) for the monopoles of the primary BPS equation can be expanded for small velocities as

$$\bar{\mathcal{L}} = -g \cdot b + \mathcal{L} + \cdots,$$

(19)

where the first term is the rest mass of the monopoles,

$$g \cdot b = \frac{1}{2} \int d^3 x \, \text{tr} \left\{ B^2 + (D b)^2 \right\}.$$

(20)

Ignoring the other five Higgs fields, the low energy dynamics that actually dictates the motion of these 1/2 BPS configurations would be given by the purely kinetic, nonrelativistic Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{mn}(z) \dot{z}^m \dot{z}^n.$$

(21)

As there are $r$ unbroken global $U(1)$ symmetries, the corresponding electric charges should be conserved. In other words, $\mathcal{L}$ should have $r$ cyclic coordinates corresponding to these gauge
transformations. In particular, we can choose a basis such that a cyclic coordinate is denoted by \( \xi^A \) \((A = 1, ..., r)\) corresponds to the center of mass phase of monopoles of \( \beta_A \) root. In geometrical terms, the cyclic coordinates \( \xi^A \)’s generate Killing vectors,

\[
K_A = \frac{\partial}{\partial \xi^A}.
\]  

(22)

Finally, let us divide the moduli coordinates \( z^m \) to \( \xi^A \) and the rest \( y^i \), upon which the Lagrangian (21) can be rewritten as

\[
L = \frac{1}{2} h_{ij}(y) \dot{y}^i \dot{y}^j + \frac{1}{2} L_{AB}(y)(\xi^A + w^A_i(y) \dot{y}^i)(\xi^B + w^B_j(y) \dot{y}^j),
\]  

(23)

which defines the quantities \( h, L, \) and \( w \)’s. In particular,

\[
L_{AB} = g_{mn} K_A^m K_B^n.
\]  

(24)

Notice that all metric components are independent of \( \xi^A \).

3 Additional Higgs and Monopole Dynamics

Let us now explore the low energy dynamics of monopoles when additional Higgs fields, \( a_I \), are turned on. When expectation values \( a_I \) are turned on, the monopole solutions of the primary BPS equation are not, in general, solutions to the full field equations. Monopoles exert static forces on other monopoles. For sufficiently small \( a_I \), these forces arise from the extra potential energy due to nontrivial \( a_I \) fields; The combined effect of \( a_I \) and of the monopole background induce some nontrivial behavior to \( a_I \), which “dresses” the monopoles and contributes to the energy of the system.

To find this potential, we imagine a static configuration of monopoles, which are held fixed by some external force. Let us try to dress it with a time-independent \( a_I \) field with the smallest possible cost of energy. The energy functional for such \( a_I \) fields is

\[
\Delta E = \frac{1}{2} \int d^3 x \text{ tr } \left\{ (D a_I)^2 + (-i [a_I, b])^2 \right\},
\]  

(25)

to the leading order where we ignore terms of higher power in \( \eta \), such as \([a_I, a_J]^2\). We can find the minimal “dressing” field \( a_I \) by solving the second order equation,

\[
D^2 a_I - [b, [b, a_I]] = 0.
\]  

(26)
Solving this for $a_I$ and inserting them back into the energy functional above, we should find the minimal cost of energy for the static monopole configuration.

The same type of the second order equation appeared in construction of 1/4 BPS dyons, where the projected Higgs field $c_I a_I$ obey such an equation. However, we must emphasize that we are performing a very different task here. Specifically, in the construction of 1/4 BPS dyons, BPS equations force $-c_I a_I$ to be identified with the time-component gauge field, $A_0$, which determines electric charges. Here we are simply solving for the reaction of the scalar fields $a_I$ to the given monopole configuration.

Using Tong’s trick[19], we notice that $D a_I$ and $-i[b,a_I]$ can be thought of as global gauge zero modes, $D_\mu a_I$, which satisfy the gauge fixing condition, $D_\mu D_\mu a_I = 0$. Thus, $D_\mu a_I$ can be regarded as a linear combination of gauge zero modes, and subsequently each $a_I$ picks out a linear combination of $U(1)$ Killing vector fields on the moduli space, which are

$$K^m_A \frac{\partial}{\partial z_m} = \frac{\partial}{\partial \xi_A}. \quad (27)$$

More precisely, each $K_A$ corresponds to a gauge zero mode,

$$K^m_A \delta_m A_\mu, \quad (28)$$

and each $D_\mu a_I$ is a linear combination of them,

$$D_\mu a_I = a^A_I K^m_A \delta_m A_\mu, \quad (29)$$

when we expand the asymptotic value $a_I = \sum_A a^A a^A$, where $\lambda_A$’s are the fundamental weights such that $\lambda_A \cdot \beta_B = \delta_{AB}$.

We then express the potential energy $V$, obtained by minimizing the functional $\Delta E$ in Eq. (25) in the monopole background, in terms of the monopole moduli parameters as

$$V = \frac{1}{2} \int d^3x \text{tr} \left\{ (a^A_I K^m_A \delta_m A_\mu)(a^B_I K^n_B \delta_n A_\mu) \right\} = \frac{1}{2} g_{mn} a^A_I K^m_A a^B_I K^n_B. \quad (30)$$

The value of this potential depends on the monopole configuration we started with, which induces the static force on monopoles. The low energy effective Lagrangian was purely kinetic when $a_I$ were absent. In the presence of $a_I$’s and of their expectation values $a_I$, however, the Lagrangian picks up a potential term,

$$\mathcal{L} = \frac{1}{2} g_{mn} \dot{z}^m \dot{z}^n - V \quad (31).$$
which can be written more explicitly as,

\[
\mathcal{L} = \frac{1}{2} g_{mn}(z) \dot{z}^m \dot{z}^n - \frac{1}{2} g_{mn}(z) a_I^A k^m_A a_I^B k^n_B \\
= \frac{1}{2} h_{ij}(y) \dot{y}^i \dot{y}^j + \frac{1}{2} L_{AB}(y)(\ddot{\xi}^A + \omega^A_i(y) \dot{y}^i)(\ddot{\xi}^B + \omega^B_j(y) \dot{y}^j) - \frac{1}{2} L_{AB}(y) a_I^A a_I^B.
\]

where the index \( I \) runs from 1 to 5, and labels the five potential terms.

The procedure we employed here should be a very familiar one. When we talk about, say, Coulombic interaction between charged particles, we also fix the charge distribution by hand, and then estimate the potential energy it costs. Of course, there is a possibility of more interaction terms involving velocities of moduli as well as \( a_I \) fields, but in the low energy approximation here, the only relevant terms of such kind would be of order \( v \eta \). However, it is clear that neither backreaction of \( a_I \) to the magnetic background nor the time-dependence of \( a_I \)'s can produce such a term. Thus, to the leading quadratic order in \( v \) and \( \eta \), the above Lagrangian captures all bosonic interactions among monopoles in the presence of \( a_I \)'s.

### 4 1/4 BPS and Non BPS Configurations

The total energy of the field configuration within this nonrelativistic approximation is then

\[
E = b \cdot g + \mathcal{E},
\]

where the nonrelativistic energy is derived from \( \mathcal{L} \), and can be written as

\[
\mathcal{E} = \frac{1}{2} g_{mn}(z) \left( \dot{z}^m \dot{z}^n + a_I^A k^m_A a_I^B k^n_B \right).
\]

The energy \( \mathcal{E} \) has a BPS bound of its own. With an arbitrary five dimensional unit vector \( c_I \), we can rewrite the energy as

\[
\mathcal{E} = \frac{1}{2} g_{mn}(z) (\dot{z}^m - c_I a_I^A k^m_A)(\dot{z}^n - c_J a_J^B k^n_B) + \frac{1}{2} g_{mn} a_{IJ}^A K^m_A k^n_B + c_I g_{mn} \dot{z}^m a_I^A k^n_A.
\]

\textsuperscript{5}While the low energy dynamics turns out to be quite simple, there is a subtlety in reconstructing the actual field configuration for a given low energy motion on the moduli space. For the magnetic part of the configuration, \( A_\mu \), the trajectory on moduli space can be represented reliably by allowing time-dependence of the moduli parameters. Namely, the time-dependent field configuration would be \( A_\mu = \tilde{A}_\mu(x; z_m(t)) \) where \( \tilde{A}_\mu(x; z) \) is the solution of the primary BPS equation. For the additional Higgs fields, however, the naive ansatz \( a_I = \tilde{a}_I(x; z_m(t)) \) does not work, where \( \tilde{a}_I(x; z_m) \) solves the static second order equation (32) in the background of \( \tilde{A}_\mu(x; z) \). Such an ansatz would involve fluctuations of nonnormalizable modes, as \( \tilde{a}_I \) has a \( z_m \)-dependent \( 1/r \) tail. Rather, the time-dependence of \( a_I \) field has much nicer large \( r \) behavior, and this can be seen easily by solving the full field equation for \( a_I \) order by order in \( v \).
where \( a_{\perp}^A = a_I^A - c_I^I a_J^J \) is the part of \( a_I^A \) orthogonal to \( c_I^I \). Since there is \( r \) \( U(1) \) symmetries with Killing vectors \( K^m_A \), there are \( r \) conserved charges

\[
q_A = K^m_A \frac{\partial \mathcal{L}}{\partial \dot{z}^m} = g_{mn} K^m_A \dot{z}^n . \tag{36}
\]

As the metric \( g_{mn} \) are positive definite, there is a bound on the energy,

\[
E \geq |c_I a_I^A q_A| . \tag{37}
\]

This bound is saturated when

\[
\dot{z}^m - c_I a_I^A K^m_A = 0 \tag{38}
\]

\[
a_{\perp}^A = a_I^A - c_I c_J a_J^J = 0 . \tag{39}
\]

The second equation is satisfied if, for instance, only one additional Higgs fields are relevant, while the first equation implies that the conserved charges are

\[
q_A = g_{mn} K^m_A c_I a_B^B K^n_B . \tag{40}
\]

Quantum counterpart of such BPS configurations have been explored in Ref. [13]. In field theory terms, these BPS states of low energy dynamics preserve 1/4 of field theory supersymmetries.

These 1/4 BPS configurations describe static dyons spreading out in space such that the electromagnetic force and the Higgs force are in delicate balance. They are the BPS configurations of the low energy effective action when, in effect, only one linear combination of the Killing vector fields, \( c_I a_I^A K^A \), is relevant. For more general cases, when \( a_{\perp}^A \) cannot be taken to be zero, the BPS bound are not saturated. Nevertheless, there must exist the lowest energy state with any given charge, which would correspond to stable non BPS states. (These non BPS configurations correspond to the string web which is not planar.) Such a stable dyonic configuration can be found classically considering the nonrelativistic energy functional.

The energy functional for a given set of electric charges

\[
q_A = \frac{\partial \mathcal{L}}{\partial \dot{\xi}^A} \tag{41}
\]

is

\[
\mathcal{E} = \frac{1}{2} h_{ij} \dot{y}^i \dot{y}^j + U_{\text{eff}}(y) . \tag{42}
\]
where the effective potential is

\[ U_{\text{eff}} = \frac{1}{2} L_{AB}(y) q_A q_B + \frac{1}{2} L_{AB} a_A^I a_B^I \]  

(43)

with the inverse of \( L_{AB} \) is denoted by \( L^{AB} \). The minimum of the energy is achieved by the configurations which are static in \( y^i \) and satisfy

\[ \frac{\partial}{\partial y^i} U_{\text{eff}}(y) = 0 \]  

(44)

In general the family of stable solutions \( y^i \) for a given \( q_A \), if they exist, will form a submanifold of the moduli space. However, it is not clear whether there will be always \( q_A \) satisfying Eq. (44) for some \( y^i \). In fact, it is known that for some case with too large values of \( q_A \) there is no solution to such equations\(^1\).

The general analysis of Eq. (44) will be complicated. One case where it can be solved explicitly is when the magnetic background contains only one fundamental monopole of each kind; That is, suppose that, for each simple root \( \beta_A, A = 1, \ldots, r \), we have one fundamental monopole at \( x_A \) and with the \( U(1) \) phase \( \xi_A \). Denote the relative position vectors between adjacent (in the Lie algebra sense) monopoles by \( r_A = x_{A+1} - x_A \) for \( A = 1, \ldots, r - 1 \) and also define the corresponding relative phases by \( \zeta_A \). For the phases, the redefinition is such that the charges \( \tilde{q}_A \), associated with \( \xi_A \)'s is related to \( q_A \)'s by \( \tilde{q}_A = q_{A+1} - q_A \). The metric is then decomposed into two decoupled pieces\(^2\):

\[ ds^2 = \left( \sum m_A \right)^{-1} \left( d\left( \sum_{A=1}^{r} m_A x_A \right)^2 + \frac{16 \pi^2}{e^2} d\left( \sum_{A=1}^{r} \xi_A \right)^2 \right) \]

\[ + \sum_{A=1}^{r-1} \sum_{B=1}^{r-1} \left( C^{AB} d\mathbf{r}_A \cdot d\mathbf{r}_B + C_{AB} (d\zeta^A + \mathbf{w}(r_A) \cdot d\mathbf{r}_A)(d\zeta^B + \mathbf{w}(r_B) \cdot d\mathbf{r}_B) \right) \]  

(45)

where \( m_A \) are the masses of the \( r \) fundamental monopoles. The \((r-1) \times (r-1)\) matrices \( C^{AB} \) and \( C_{AB} \) are inverses of each other,

\[ \sum_{B=1}^{r-1} C^{AB} C_{BC} = \delta^A_C \]  

(46)

and are explicitly known

\[ C^{AB} = \mu^{AB} + \delta^{AB} \frac{\lambda_A}{|r_A|} \]  

(47)

with the reduced mass matrix \( \mu_{AB}, A, B = 1, \ldots, r - 1 \), and some coupling constants \( \lambda_A \). The vector potential \( \mathbf{w}(\mathbf{r}) \) is the Dirac potential;

\[ \nabla \frac{1}{r} = \nabla \times \mathbf{w}(\mathbf{r}). \]  

(48)
In the new coordinate, the potential also decomposes into two parts, one of which is independent of moduli coordinates,

\[ U_{\text{eff}}(r^A) = \frac{1}{2} \sum_{A,B=1}^r L^{AB}(y) q_A q_B + \frac{1}{2} \sum_{A,B=1}^r L_{ABA}^I a_I^A a_I^B \]

\[ = \text{constant} + \frac{1}{2} \sum_{A,B=1}^{r-1} L^{AB} \tilde{q}_A \tilde{q}_B + \frac{1}{2} \sum_{A,B=1}^{r-1} C_{AB} \tilde{a}_I^A \tilde{a}_I^B \]  

(49)

The vacuum expectation values in the new basis, \( \tilde{a}_I^A, A = 1, \ldots, r-1 \), are found from \( a_I^A, A = 1, \ldots, r \), using the relationship,

\[ \sum_{A=1}^r a_I^A q_A = \sum_{A=1}^{r-1} \tilde{a}_I^A \tilde{q}_A + \tilde{a}_I^0 \sum_{A=1}^r m_A \]  

(50)

with \( \tilde{a}_I^0 \) to be determined from this as well.

The minimum of the potential is found by looking for the critical point,

\[ 0 = \frac{\partial}{\partial r_C} U_{\text{eff}} = \sum_{A,B=1}^{r-1} \frac{\partial C^{AB}}{2 r_C} \left( \tilde{q}_A \tilde{q}_B - \sum_{I=1}^5 \sum_{A',B'=1}^{r-1} C_{AA'} C_{BB'} \tilde{a}_I^{A'} \tilde{a}_I^{B'} \right) \]  

(51)

As \( \partial C^{AB}/\partial r_C = -\delta_{AB} \delta_{BC} r_C/(r_C)^3 \), the condition reduces to

\[ 0 = \frac{\partial}{\partial r_C} U_{\text{eff}} = -\frac{\lambda_C r_C}{r_C^3} \left( (\tilde{q}_C)^2 - \sum_{I=1}^5 \sum_{A,B=1}^{r-1} C_{CA} C_{CB} \tilde{a}_I^A \tilde{a}_I^B \right) \]  

(52)

and we find that the critical points are such that the charges are given as functions of \( \tilde{r}_A \) as follows,

\[ |\tilde{q}_C| = \sqrt{\sum_{I=1}^5 \sum_{A,B=1}^{r-1} C_{CA}(\tilde{r}) C_{CB}(\tilde{r}) \tilde{a}_I^A \tilde{a}_I^B} \]  

(53)

Once this is satisfied, \( \dot{\tilde{r}}_A = 0 \) solves the equations of motion, so the solution describes static configurations of many distinct monopoles, each dressed by the electric charges. They correspond to stable non BPS dyons in the field theoretic description.

By inserting (53) to the effective potential (49), the energy of the configuration is determined as a function of the monopole positions. The latter \( U(1) \) charge is not determined by moduli parameters. It should be remarked that these states become 1/4 BPS, when only one Higgs vacuum expectation value out of five is nonvanishing or all the directions of them are parallel with each other.
5 Supersymmetric Extension

So far we have concentrated on the bosonic part of the low energy effective Lagrangian in the moduli space approximation. The supersymmetric extension of the bosonic action can be achieved in two different routes. A direct approach is to follow the same strategy of the bosonic part. Namely, identify first the moduli fluctuations and their coordinates of the fermionic part, and integrate out all the other fluctuation except the monopole moduli variables using the original field theoretic Lagrangian.

While such a derivation would be more desirable, the symmetry of the system seems to offer us a shortcut and allow us to fix the SUSY completion of the effective Lagrangian uniquely. We shall follow the latter approach here.

Since the background configurations of monopoles preserves half of the 16 supersymmetries of the original $N=4$ SYM theory, the quantum mechanics should have four complex, or eight real supercharges. Furthermore, the low energy effective theory should be consistent with the $SO(6)$ R-symmetry of the four dimensional $N=4$ SYM theory. Out of six Higgs fields, we picked out one, $b$, associated with construction of monopoles, so only $SO(5)$ subgroup of $SO(6)$ may show up. For instance, when we consider the extreme case of $a_I = 0$, the SUSY quantum mechanics must have full $SO(5)$ R-symmetry.

The additional Higgs expectations $a_I$ break the remaining $SO(5)$ rotational symmetry of the field theory, as well, spontaneously. On the other hand, in the low energy dynamics of monopoles, $a_I$ are small parameters, so the breaking of $SO(5)$ is explicit and soft. Thus, $SO(5)$ is not a symmetry of the low energy dynamics. Nevertheless, because all $a_I$ are on an equal footing (unlike $b$), the low energy dynamics must remain invariant when we rotate the $a_I$ in addition to rotating dynamical degrees of freedom. Thus, although this $SO(5)$ is not a symmetry of the low energy theory in the conventional sense, this provides us with an interesting consistency checkpoint. Later we will find and write down this $SO(5)$ transformation explicitly.

Existence of the four complex supercharges is already quite restrictive. The supersymmetry requires the geometry to be hyperKähler, to begin with, equipped with three complex structures.
that satisfy
\[ \mathcal{I}^{(s)} \mathcal{I}^{(t)} = -\delta^{st} + \epsilon^{stu} \mathcal{I}^{(u)}, \]  \[ D_m \mathcal{I}^{(s)} n_p = 0. \]  \[ (54) \]

In the absence of \( a_I \)'s, the dynamics would be a sigma model onto the hyperKähler moduli space of monopoles. The bosonic potential introduced by \( a_I \)'s can be rewritten in terms of five triholomorphic Killing vector fields, \( G_I \equiv a_I \cdot K \), as
\[ \frac{1}{2} \sum_{I=1}^{5} G_I^m G_I^m g_{mn}. \]  \[ (55) \]

Alvarez-Gaume and Freedman \[21\] discussed how such Killing vector fields can be incorporated into supersymmetric Lagrangian while maintaining four complex supercharges in the two-dimensional context. In this two-dimensional setting, they showed that up to four triholomorphic Killing vectors can be accommodated. This result presumably has something to do with the fact that the supersymmetric Lagrangian can also be obtained via the Scherk-Schwarz dimensional reduction \([15]\) from the six dimensional (0,8) nonlinear sigma model action presented in Ref. \([14]\).

Since we are considering quantum mechanics instead of two-dimensional field theory, this suggests to us that one should be able to incorporate up to five such Killing vectors to the effective Lagrangian. Thus, generalizing their result to quantum mechanics, we obtain the following unique supersymmetric completion of the low energy dynamics,
\[ \mathcal{L} = \frac{1}{2} \left( g_{mn} \dot{z}^m \dot{z}^n + i g_{mn} \bar{\psi}^m \gamma^0 D_t \psi^n + \frac{1}{6} R_{mnpq} \bar{\psi}^m \psi^p \psi^q \psi^r - g_{mn} G_I^m G_I^n - i D_m G_I n \bar{\psi}^m (\Omega^I \psi)^n \right) \]  \[ (56) \]

where \( \psi^m \) is a two component Majorana spinor, \( \gamma^0 = \sigma_2, \gamma^1 = \sigma_1, \gamma^2 = -i \sigma_3, \bar{\psi} = \psi^T \gamma^0 \). The operator \( \Omega_I \)'s are defined respectively by \( \Omega_4 = \delta^m_n \gamma_{\alpha \beta}, \Omega_5 = \delta^m_n \gamma_{\alpha \beta}^2 \) and \( \Omega_s = i \mathcal{I}^{(s)} m n \delta_{\alpha \beta} \) for \( s = 1, 2, 3 \).

The supersymmetry algebra by itself requires some properties of \( G_I \)'s, in addition to hyperKähler properties of \( g_{mn} \). \( G_I \) must satisfy
\[ D_m G_I n + D_n G_I m = 0, \]  \[ (57) \]

or equivalently \( \mathcal{L}_{G_I} g = 0 \), that is, \( G_I \) must be a Killing vector. In addition, the rotated version \((\mathcal{I}^{(s)} G_I)_m \)'s must also satisfy
\[ D_m (\mathcal{I}^{(s)} G_I)_n - D_n (\mathcal{I}^{(s)} G_I)_m = 0 \]  \[ (58) \]
Taken together with the Killing properties of $G_I$ and the closedness of the Kähler forms, this also implies that $G_I$ are triholomorphic,

$$\mathcal{L}_{G_I} T^{(s)} = 0$$

(60)

Of course, for the specific case of monopole dynamics, these two conditions are satisfied because each $K_A$ is a triholomorphic Killing vector field on the moduli space. One last requirement on $G_I$’s from SUSY algebra is,

$$G_I^m T_{mn}^{(s)} G_J^m = 0.$$ 

(61)

for $s = 1, 2, 3$. This condition is met for triholomorphic $G_I$’s, provided that the commutators vanish, $[G_I, G_J] = 0$. Since $K_A$’s all commute among themselves, this last condition is also satisfied in the above low energy dynamics of monopoles.

When quantized, the spinors $\psi^E = e^E_m \psi^m$ commute with all the bosonic dynamical variables, especially with $p$’s that are canonical momenta of the coordinates $z$’s. The remaining fundamental commutation relations are

$$[z^m, p_n] = i \delta^m_n,$$

$$\{\psi^E, \psi^F\} = \delta^{EF} \delta_{\alpha \beta}.$$ 

(62)

(Consequently, the bosonic momenta $p$’s do not commute with $\psi^m$.) It is straightforward to show that the Lagrangian (57) is invariant under the N=4 supersymmetry transformations,

$$\delta(0) z^m = \bar{\epsilon} \psi^m,$$

$$\delta(0) \psi^m = -i z^m \gamma^0 \epsilon - \Gamma^m_{np} \bar{\epsilon} \psi^n \psi^p - i (G^I \Omega^I)^m \epsilon,$$

$$\delta(s) z^m = \epsilon(s) (T^{(s)} \psi)^m,$$

$$\delta(s) \psi^m = i (\bar{T}^{(s)} \bar{z})^m \gamma^0 \epsilon(s) + \bar{T}^{(s)} \Gamma^m_{np} \epsilon(s) (T^{(s)} \psi)^n (T^{(s)} \psi)^p - i (G^I \Omega^{I(s)})^m \epsilon(s),$$

(63)

(64)

where $\epsilon$ and $\epsilon(s)$ are spinor parameters and no summation convention is used for the index $s = 1, 2, 3$.

For the supercharges, let us first define supercovariant momenta by

$$\pi_m \equiv p_m - \frac{i}{2} \omega_{EF} m \bar{\psi}^E \gamma^0 \psi^F,$$

(65)

where $\omega_{EF} m$ is the spin connection. The corresponding N=4 SUSY generators in real spinors are then

$$Q_\alpha = \psi^m_\alpha \pi^m - (\gamma^0 \Omega^I \psi)^m G^I_m,$$

(66)

$$Q^{(s)}_\alpha = (T^{(s)} \psi)^m_\alpha \pi^m - (\gamma^0 T^{(s)} \Omega^I \psi)^m G^I_m.$$ 

(67)
These charges satisfy the N=4 superalgebra:

\[
\{Q_\alpha, Q_\beta\} = \{Q_\alpha^{(s)}, Q_\beta^{(s)}\} = 2\delta_{\alpha\beta} H - 2(\gamma^0 \gamma^1)_{\alpha\beta} Z_4 - 2(\gamma^0 \gamma^2)_{\alpha\beta} Z_5, \tag{68}
\]

\[
\{Q_\alpha, Q_\beta^{(s)}\} = 2\gamma^0_{\alpha\beta} Z_s, \quad \{Q_\alpha^{(1)}, Q_\beta^{(2)}\} = 2\gamma^0_{\alpha\beta} Z_3, \tag{69}
\]

\[
\{Q_\alpha^{(2)}, Q_\beta^{(3)}\} = 2\gamma^0_{\alpha\beta} Z_1, \quad \{Q_\alpha^{(3)}, Q_\beta^{(1)}\} = 2\gamma^0_{\alpha\beta} Z_2, \tag{70}
\]

where the Hamiltonian \( H \) and the central charges \( Z_I \) read

\[
H = \frac{1}{2} \left( \frac{1}{\sqrt{g}} \pi_m \sqrt{g} g^{mn} \pi_n + g^{mn} G^I_m G^I_n - \frac{1}{4} R_{mnpq} \tilde{\psi}^m \gamma^0 \psi^n \tilde{\psi}^p \gamma^0 \psi^q + i D_\mu G^I_\nu \tilde{\psi}^\mu \Omega_I \psi^n \right), \tag{71}
\]

\[
Z_I = G^m_I \pi_m - \frac{i}{2} D_m G^I_n \tilde{\psi}^m \gamma^0 \psi^n. \tag{72}
\]

It is straightforward to see that the SO(5) rotation is realized by the transformation,

\[
\psi \rightarrow e^{\frac{1}{2} \theta_{KL} J_{KL}} \psi, \tag{73}
\]

where \( \theta_{KL} \) is antisymmetric in its indices and the corresponding generators, \( J_{KL} (= -J_{LK}) \), denote

\[
J_{ab} = \epsilon_{abc} \mathcal{I}^{(c)}, \quad J_{45} = i\sigma_2, \quad J_{4a} = \sigma_1 \mathcal{I}^{(a)}, \quad J_{5a} = -\sigma_3 \mathcal{I}^{(a)} \tag{74}
\]

with \( a, b, c = 1, 2, 3 \). For example, the transformation reads explicitly

\[
\psi^m_\alpha \rightarrow \cos \theta \psi^m_\alpha + \sin \theta (\sigma_1 \mathcal{I}^{(1)} \psi)_\alpha, \tag{75}
\]

when only \( \theta_{41} = -\theta_{14} = \theta \) is nonvanishing.

Performing such SO(5) rotations, we obtain a theory with the vacuum expectation values \( a'_I = R_{IJ} a_J \) where \( R_{IJ} \) is the corresponding SO(5) rotation matrix satisfying \( R^T R = I \). More specifically, the induced transformation of \( G_I \) that is linear in the vacuum expectation value \( a^I \), is

\[
G_I \rightarrow \left( e^{\theta_{KL} J_{KL}} \right)_{IJ} G_J \tag{76}
\]

where \( (J_{KL})_{IJ} = \frac{1}{2} (\delta_{KI} \delta_{LJ} - \delta_{KJ} \delta_{LI}) \). When all \( a^I \) are parallel with each other, one may make only one Higgs expectation value nonvanishing by an appropriate SO(5) rotations; the result corresponds to 1/4 BPS effective Lagrangian in Ref. [1].

The ten generators of SO(5) in (74) exhaust all the possible covariantly constant, antisymmetric structures present in the N=4 supersymmetric sigma model without potential, so the realization of the R-symmetry is rather unique.
The complex form of the supercharges are often useful. For this, we introduce \( \varphi^m \equiv \frac{1}{\sqrt{2}}(\psi_1^m - i\psi_2^m) \) and define \( Q \equiv \frac{1}{\sqrt{2}}(Q_1 - iQ_2) \). The supercharges in (66) can be rewritten as

\[
Q = \varphi^m \pi_m - \varphi^m (G_4^m - iG_5^m) - i \sum_{s=1}^{3} G_s^m (\mathcal{I}^{(s)} \varphi)^m \tag{77}
\]

\[
Q^\dagger = \varphi^m \pi_m - \varphi^m (G_4^m + iG_5^m) + i \sum_{s=1}^{3} G_s^m (\mathcal{I}^{(s)} \varphi^*)^m \tag{78}
\]

The supercharges \( Q^{(s)} \) and \( Q^{(s)\dagger} \) are found similarly from (67). Finally, \( \{Q, Q^\dagger\} = \{Q^{(s)}, Q^{(s)\dagger}\} = \mathcal{H} \), so the Hamiltonian is positive definite. All the central charges appear in other parts of the algebra.

In Ref. [13], the quantum 1/4 BPS wavefunctions have been constructed explicitly and the structure of the supermultiplet are identified. This construction has heavily relied upon the BPS nature of the states which first order equations. This kind of simplification does not occur in the case of the stable non BPS states, so we will leave the analysis of their wavefunctions for future works.

6 Conclusion

We have found the complete supersymmetric Lagrangian for the low energy dynamics of 1/2 monopoles when all six adjoint Higgs fields get expectation values. We consider the nonrelativistic dynamics of monopoles, which constrains the five additional Higgs to be small compared the first that gives mass to the monopoles. The bosonic part of the effective dynamics is found by a perturbative expansion of fields around purely magnetic monopole configurations, which agrees with previously found exact Lagrangian when only two Higgs fields, one large and another small, are involved. The supersymmetric extension is constructed and argued to be unique, given the four complex supercharges, and an would-be \( SO(5) \) R-symmetry that is softly broken by five Killing potential terms.

The dyonic state from the low energy dynamics would correspond to a web of strings ending on D3 branes, when realized as type IIB string theory configurations. This is possible for all classical gauge groups of the Yang-Mills theory. When the transverse positions of the D3 branes are planar, only one of the five Killing vectors become relevant, and the state saturate a BPS bound which is
linearly characterized by the values of electric charges. These are 1/4 BPS dyons in the field theory sense.

For a non planar distribution of D3 branes, on the other hand, at least two of the five Killing vectors are relevant. The resulting non planar web does not saturate a BPS bound, but exist as a stable dyonic state. The state will consist of several distinct monopoles, each dressed with some electric charges that are mostly determined by the inter-monopole separations. For a simple case, we gave a set of algebraic equations that can be used to determine the charge-position relationship.

We have not fully explored the low energy effective Lagrangian even classically. It is expected that there exists a clear correspondence between the energy of non planar string web and the minimum energy of the stable but non BPS states. The energy of the string web in an appropriate limit can be determined as sum of each length multiplied by tension. The detailed comparison will be of interest. It would be interesting to find out the field theoretic configuration for these dyonic non BPS configurations. The quantum mechanics of the supersymmetric Lagrangian is more involved than that of Ref. [13], which considered only one Killing potential. Nevertheless it is of some interest to find the ground state for the given electric charges.

Acknowledgments

D.B. is supported in part by Ministry of Education Grant 98-015-D00061. K.L. is supported in part by the SRC program of the SNU-CTP and the Basic Science and Research Program under BRSI-98-2418. D.B. and K.L. are also supported in part by KOSEF 1998 Interdisciplinary Research Grant 98-07-02-07-01-5.

References

[1] K. Lee and P. Yi, [hep-th/9804174]. Phys. Rev. D58 (1998) 066005.

[2] K. Hashimoto, H. Hata and N. Sasakura, [hep-th/9803127]. Phys. Lett. B431 (1998) 303; [hep-th/9804164]. Nucl. Phys. B535 (1998) 83; T. Kawano and K. Okuyama, [hep-th/9804139]. Phys. Lett. B432 (1998) 338.

[3] D. Bak, K. Hashimoto, B-H. Lee, H. Min and N. Sasakura, [hep-th/9901107]. Phys. Rev. D60 (1999) 046005.
[4] K. Lee, [hep-th/9903093], Phys. Lett. B458 (1999) 53; C. Houghton and K. Lee, Nahm Data and the Mass of 1/4-BPS States, [hep-th/9909218].

[5] T. Ioannidou, G. Papadopoulos, and P.M. Sutcliffe, NonBPS D-Branes from Sigma Model Solitons, [hep-th/9907156]. JHEP (1999) 9909:016.

[6] T. Ioannidou, and P.M. Sutcliffe, NonBPS String Junctions and Dyons in N=4 Super Yang-Mills, [hep-th/9907157].

[7] E. Witten, Nucl. Phys. B460 (1996) 335; A.A. Tseytlin, Nucl. Phys B469 (1996) 51; M.B. Green and Gutperle, Phys. Lett. B377 (1996) 28.

[8] O. Bergman, [hep-th/9712211]. Nucl. Phys. B525 (1998) 104; O. Bergman and B. Kol, [hep-th/9804160]. Nucl. Phys. B536 (1998) 149.

[9] D. Bak, C. Lee, K. Lee, and P. Yi, Low Energy Dynamics for 1/4 BPS Dyons, [hep-th/9906119].

[10] C. Fraser and T.J. Hollowood, Phys. Lett. B402 (1997) 106.

[11] D. Bak and K. Lee, Comments on the Moduli Dynamics of 1/4 BPS Dyons, [hep-th/9909035].

[12] J.P. Gauntlett, Nucl. Phys. B411 (1994) 443; J. Blum, Phys. Lett. B333 (1994) 92.

[13] D. Bak, K. Lee and P. Yi, Quantum 1/4 BPS Dyons, [hep-th/9907090].

[14] P. Deligne and D. Freed, Supersolutions, [hep-th/9901094].

[15] J. Scherk, and J.H. Schwarz, Nucl. Phys. B153 (1979) 61.

[16] E. Weinberg, Nucl. Phys. B167 (1980) 500.

[17] N.S. Manton, Phys. Lett. 110B (1982) 54.

[18] M.F. Atiyah and N.J. Hitchin, The Geometry and Dynamics of Magnetic Monopoles, (Prince ton University Press, Princeton, 1988).

[19] D. Tong, [hep-th/9902005], Phys. Lett. B460 (1999) 295.

[20] K. Lee, E.J. Weinberg, and P. Yi, [hep-th/9602167]. Phys. Rev. D54 (1996) 1633.

[21] L. Alvarez-Gaume and D. Freedman, Commun. Math. Phys. 91 (1983) 87.