End extending models of set theory via power admissible covers

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Abstract

Motivated by problems involving end extensions of models of set theory, we develop the rudiments of the power admissible cover construction (over ill-founded models of set theory), an extension of the machinery of admissible covers invented by Barwise as a versatile tool for generalising model-theoretic results about countable well-founded models of set theory to countable ill-founded ones. Our development of the power admissible machinery allows us to obtain new results concerning powerset-preserving end extensions and rank extensions of countable models of subsystems of ZFC. The canonical extension $\mathbf{KP}^P$ of Kripke-Platek set theory $\mathbf{KP}$ plays a key role in our work; one of our results refines a theorem of Rathjen by showing that $\Sigma_1^P$-Foundation is provable in $\mathbf{KP}^P$ (without invoking the axiom of choice).

1 Introduction

The admissible cover machinery was introduced by Barwise in the Appendix of his venerable book [Bar75] on admissible set theory. Admissible covers allow one to extend the range of infinitary compactness arguments from the domain of countable well-founded models of $\mathbf{KP}$ (Kripke-Platek set theory) to countable ill-founded models of $\mathbf{KP}$. For example, Barwise uses admissible covers in his book to prove a striking result: Every countable model of $\mathbf{ZF}$ has an extension to a model of $\mathbf{ZF} + V = L$.\textsuperscript{1} Admissible covers also appear in the work of Ressayre [Res], who showed that the results presented in the Appendix of [Bar75] pertaining to $\mathbf{KP}$ do not depend on the availability of the full scheme of foundation among the axioms of $\mathbf{KP}$; more specifically, they only require the scheme of foundation for $\Sigma_1 \cup \Pi_1$-formulae.\textsuperscript{2} Admissible covers were used more recently by Williams [Wil18], to show that certain class theories (including Kelley-Morse class theory) fail to have minimum transitive models (this result of Williams also appears in

\textsuperscript{1}This end extension result, together with certain elaborations of it, first appeared in an earlier paper of Barwise [Bar71]. It is also noteworthy that, as shown recently by Hamkins [Ham18], Barwise’s end extension theorem can also be proved using more classical techniques (without appealing to methods of admissible set theory).

\textsuperscript{2}Note that the full scheme of foundation is included in the axioms of $\mathbf{KP}$ in Barwise’s treatment [Bar75]. However, we follow the convention proposed by Mathias to only include $\Pi_1$-Foundation in the axiomatisation of $\mathbf{KP}$; this is informed by the fact, demonstrated by Mathias [Mat01], that many (but not all) results about Barwise’s $\mathbf{KP}$ can be carried out within Mathias’ $\mathbf{KP}$. 

1
their paper [Wi19], but with a different proof). In this paper we explore the variant power admissible cover of the notion of admissible cover in order to obtain new results in the model theory of set theory. The main inspiration for our results on end extensions arose from our joint work with Kaufmann [EKM] on automorphisms of models of set theory (see Theorem 5.10).

The canonical extension KP of Kripke-Platek set theory KP plays a key role in our work. KP is intimately related to Friedman’s so-called power admissible system \text{PAdm}, whose well-founded models are the the so-called power admissible sets [Fried2]. These two systems can accommodate constructions by \(\Sigma_1\)-recursions relative to the power set operation. The system KP has been closely studied by Mathias [Mat01] and Rathjen [Rat14], [Rat20]. In the latter paper Rathjen proves that \(\Sigma_1^P\)-Foundation is provable in KP + AC (where AC is the axiom of choice).

The highlights of the paper are as follows. In Corollary 3.3 we refine Rathjen’s aforementioned result by showing that \(\Sigma_1^P\)-Foundation is provable outright in KP. The rudiments of power admissible covers are developed in Section 4. In Section 5 the machinery of power admissible covers is put together with results of earlier sections to establish new results about powerset-preserving end extensions and rank extensions of models of set theory. For example in Theorem 5.7 we show that every countable model of \(\mathcal{M} \models \text{KP}^P\) has a topless rank extension, i.e., \(\mathcal{M}\) has a proper rank extension \(\mathcal{N} \models \text{KP}^P\) such that \(\text{Ord}^\mathcal{V} \setminus \text{Ord}^\mathcal{M}\) has no least element. This result generalises a classical theorem of Friedman that shows that every countable well-founded model of KP has a topless rank extension.

2 Background

We use \(\mathcal{L}\) throughout the paper to denote the language \(\{\in, =\}\) of set theory. We will make reference to generalisations of the Lévy hierarchy of formulae in languages extending \(\mathcal{L}\) that possibly contain constant and function symbols.

Let \(\mathcal{L}'\) be a language extending \(\mathcal{L}\). We use \(\Delta_0(\mathcal{L}')\) to denote the smallest class of \(\mathcal{L}'\)-formulae that is closed under the connectives of propositional logic and quantification in the form \(\exists x \in t \text{ and } \forall x \in t\), where \(t\) is a term of \(\mathcal{L}'\) and \(x\) is a variable that does not appear in \(t\). The classes \(\Sigma_1(\mathcal{L}'), \Pi_1(\mathcal{L}'), \Sigma_2(\mathcal{L}'), \ldots\) are defined inductively from \(\Delta_0(\mathcal{L}')\) in the usual way. We will write \(\Delta_0, \Sigma_1, \Pi_1, \ldots\) instead of \(\Delta_0(\mathcal{L}), \Sigma_1(\mathcal{L}), \Pi_1(\mathcal{L}), \ldots\), and we will use \(\Pi_\infty\) and \(\Pi_\infty(\mathcal{L}')\) to denote the class of all \(\mathcal{L}\)-formulae and \(\mathcal{L}'\)-formulae respectively. An \(\mathcal{L}'\)-formula is \(\Delta_n(\mathcal{L}')\), for \(n > 0\), if it is equivalent to both a \(\Sigma_n(\mathcal{L}')\)-formula and a \(\Pi_n(\mathcal{L}')\)-formula.

The class \(\Delta^0_n\) is the smallest class of \(\mathcal{L}\)-formulae that is closed under the connectives of propositional logic and quantification in the form \(Qx \subseteq y\) and \(Qx \in y\) where \(Q\) is \(\exists\) or \(\forall\), and \(x\) and \(y\) are distinct variables. The Takahashi classes \(\Delta^0_P, \Sigma^0_P, \Pi^0_n, \ldots\) are defined from \(\Delta^0_P\) in the same way as the classes \(\Delta_1, \Sigma_1, \Pi_1, \ldots\) are defined from \(\Delta_0\). If \(\Gamma\) is a collection of \(\mathcal{L}'\)-formulae and \(T\) is an \(\mathcal{L}'\)-theory, then we write \(\Gamma^T\) for the class of \(\mathcal{L}'\)-formulae that are provably in \(T\) equivalent to a formula in \(\Gamma\).

We will use capital calligraphic font letters \((\mathcal{M}, \mathcal{N}, \ldots)\) to denote \(\mathcal{L}\)-structures. If \(\mathcal{M}\) is an \(\mathcal{L}\)-structure, then, unless we explicitly state otherwise, \(\mathcal{M}\) will be used to denote the underlying set of \(\mathcal{M}\) and \(\mathcal{E}^\mathcal{M}\) will be used to denote the interpretation of \(\in\) in \(\mathcal{M}\).

\[\text{The precise relationship between Friedman's system and KP}^P\text{ is worked out in Section 6.19 of [Mat01].}\]
Let \( L' \) be a language extending \( L \) and let \( M \) be an \( L' \)-structure with underlying set \( M \). If \( a \in M \), then \( a^* \) is defined as follows:

\[
a^* := \{ x \in M \mid M \models (x \in a) \},
\]
as long as the structure \( M \) is clear from the context. Let \( \Gamma \) be a class of formulae. We say that \( A \subseteq M \) is \( \Gamma \)-definable over \( M \) if there exists a \( \Gamma \)-formula \( \phi(x, \bar{z}) \) and \( \bar{a} \in M \) such that \( A = \{ x \in M \mid M \models \phi(x, \bar{a}) \} \).

Let \( M \) and \( N \) be \( L' \)-structures. We will partake in the common abuse of notation and write \( M \subseteq N \) if \( M \) is a substructure of \( N \).

- We say that \( N \) is an end extension of \( M \), and write \( M \subseteq_e N \), if \( M \subseteq N \) and for all \( x, y \in N \), if \( y \in M \) and \( N \models (x \in y) \), then \( x \in M \).
- We say that \( N \) is a powerset-preserving end extension of \( M \), and write \( M \subseteq^P N \), if \( M \subseteq_e N \) and for all \( x, y \in N \), if \( y \in M \) and \( N \models (x \subseteq y) \), then \( x \in M \).
- We say that \( N \) is a topless powerset-preserving end extension of \( M \), and write \( M \subseteq_{\topless}^P N \), if \( M \subseteq^P N \), \( M \neq N \) and for all \( x \in N \), if \( c^x \subseteq M \), then \( c \in M \).
- We say that \( N \) is a blunt powerset-preserving end extension of \( M \), and write \( M \subseteq_{\blunt}^P N \), if \( M \subseteq^P N \), \( M \neq N \) and \( N \) is not a topless powerset-preserving end extension of \( M \).

Let \( \Gamma \) be a class of \( L \)-formulae. The following define the restriction of the ZF-provable schemes \textit{Separation}, \textit{Collection}, and \textit{Foundation} to formulae in the class \( \Gamma \):

\[
\text{(\( \Gamma \)-Separation)} \quad \text{For all } \phi(x, \bar{z}) \in \Gamma,
\forall \exists \forall w \exists y \forall x \in y \iff (x \in w) \land \phi(x, \bar{z}).
\]

\[
\text{(\( \Gamma \)-Collection)} \quad \text{For all } \phi(x, y, \bar{z}) \in \Gamma,
\forall \forall \exists w ((\forall x \in w) \exists y \phi(x, y, \bar{z}) \Rightarrow \exists C(\forall x \in w)(\exists y \in C)\phi(x, y, \bar{z})).
\]

\[
\text{(\( \Gamma \)-Foundation)} \quad \text{For all } \phi(x, \bar{z}) \in \Gamma,
\forall \exists (\exists x \phi(x, \bar{z}) \Rightarrow \exists y (\phi(y, \bar{z}) \land (\forall x \in y) \neg \phi(x, \bar{z}))).
\]

If \( \Gamma = \{ x \in z \} \) then we will refer to \( \Gamma \)-Foundation as Set-foundation.

We will also make reference to the following fragments of \textit{Separation} and \textit{Foundation} for formulae that are \( \Delta_n \) with parameters:

\[
\text{(\( \Delta_n \)-Separation)} \quad \text{For all } \Sigma_n \text{-formulae}, \phi(x, \bar{z}), \text{ and for all } \Pi_n \text{-formulae}, \psi(x, \bar{z}),
\forall \exists (\forall x (\phi(x, \bar{z}) \iff \psi(x, \bar{z})) \Rightarrow \forall w \exists y \forall x (x \in y \iff (x \in w) \land \phi(x, \bar{z}))).
\]

\[
\text{(\( \Delta_n \)-Foundation)} \quad \text{For all } \Sigma_n \text{-formulae}, \phi(x, \bar{z}), \text{ and for all } \Pi_n \text{-formulae}, \psi(x, \bar{z}),
\forall \exists (\forall x (\phi(x, \bar{z}) \iff \psi(x, \bar{z})) \Rightarrow (\exists x \phi(x, \bar{z}) \Rightarrow \exists y (\phi(y, \bar{z}) \land (\forall x \in y) \neg \phi(x, \bar{z}))).
\]
Similar definitions can also be used to express $\Delta^P_n$-Separation and $\Delta^P_n$-Foundation.

We use $\text{TCo}$ to denote the axiom that asserts that every set is contained in a transitive set.

We will consider extensions of the following subsystems of ZFC:

- $S_1$ is the $\mathcal{L}$-theory with axioms: Extensionality, Emptyset, Pair, Union, Set difference, and Powerset.
- $M$ is obtained from $S_1$ by adding $\text{TCo}$, Infinity, $\Delta_0$-Separation, and Set-foundation.
- $\text{Mac}$ is obtained from $M$ by adding $\text{AC}$ (the axiom of choice).
- $M^-$ is obtained from $M$ by removing Powerset.
- $\text{KP}$ is the $\mathcal{L}$-theory with axioms: Extensionality, Pair, Union, $\Delta_0$-Separation, $\Delta_0$-Collection and $\Pi_1$-Foundation.
- $\text{KP}^-$ is obtained from $\text{KP}$ by removing $\Pi_1$-Foundation.
- $\text{KPI}$ is obtained $\text{KP}$ by adding Infinity.
- $\text{KP}_{\Pi}$ is obtained from $\text{M}$ by adding $\Delta^P_0$-Collection and $\Pi^P_1$-Foundation.
- MOST is obtained from $\text{M}$ by adding $\Sigma_1$-Separation and $\text{AC}$.

In subsystems of ZFC that include Infinity we can also consider the following restriction of $\Gamma$-Foundation:

\[(\Gamma\text{-Foundation on } \omega) \text{ For all } \phi(x, \vec{z}) \in \Gamma, \]
\[\forall \vec{z}(\exists x \in \omega) \phi(x, \vec{z}) \Rightarrow (\exists y \in \omega)(\phi(y, \vec{z}) \land (\forall x \in y) \neg \phi(x, \vec{z}))).\]

The second family of theories that we will be concerned with are extensions of the variant of Kripke-Platek Set Theory with urelements that is introduced in [Bar75, Appendix].

Let $\mathcal{L}^*$ be obtained from $\mathcal{L}$ by adding a second binary relation $E$, a unary predicate $U$, and a unary function symbol $F$. The intended interpretation of $U$ is to distinguish urelements from sets. The binary relation $E$ is intended to be a membership relation that holds between urelements, and $\in$ is intended to be a membership relation that can hold between sets or urelements and sets.

Let $\mathcal{L}^*_P$ be obtained from $\mathcal{L}^*$ by adding a new unary function symbol $P$. An $\mathcal{L}^*_P$-structure is a structure $\mathfrak{A}_M = \langle M; A, \in^A, F^A, P^A \rangle$, where $M = \langle M, E^M \rangle$, $M$ is the extension of $U$, $A$ is the extension of $\neg U$, $\in^A$ is the interpretation of $\in$, $E^A$ is the interpretation of $E$, $F^A$ is the interpretation of $F$, and $P^A$ is the interpretation of $P$.

$\mathcal{L}^*$-structures will be presented in the same format, but without an interpretation of $P$. The $\mathcal{L}^*$- and $\mathcal{L}^*_P$-theories presented below will ensure that $E^A \subseteq M \times M$, and $\in^A \subseteq (M \cup A) \times A$.

Following [Bar75], we simplify the presentation of $\mathcal{L}^*$- and $\mathcal{L}^*_P$-formulae by treating these languages as two-sorted rather than one-sorted.

When writing $\mathcal{L}^*$- and $\mathcal{L}^*_P$-formulae we will use the convention below of Barwise [Bar75].
The variables $p, q, p_1, \ldots$ range over elements of the domain that satisfy $U$ (urelements).

- the variables $a, b, c, d, f, \ldots$ range over elements of the domain that satisfy $\neg U$ (sets); and

- the variables $x, y, z, w, \ldots$ range over all elements of the domain.

Therefore, $\forall p(\cdots)$ is an abbreviation of $\forall x(U(x) \Rightarrow \cdots)$, $\exists a(\cdots)$ is an abbreviation of $\exists x(\neg U(x) \land \cdots)$, etc.

In section 4 we will see that certain $\mathcal{L}$-structures can interpret $\mathcal{L}^*$- and $\mathcal{L}^*_p$-structures in which the urelements are isomorphic to the original $\mathcal{L}$-structure. It is this interaction that motivates our unorthodox convention of using $E^M, E^N, \ldots$ to denote the interpretation of $\in$ in the $\mathcal{L}$-structures $M, N, \ldots$. It should be noted that this convention differs from Barwise [Bar75] where $E$ is consistently used to denote the interpretation of $\in$ in $\mathcal{L}$-structures.

The following are analogues of axioms, fragments of axiom schemes and fragments of theorem schemes of ZFC in the languages $\mathcal{L}^*$ and $\mathcal{L}^*_p$:

- (Extensionality for sets) $\forall a \forall b(a = b \iff \forall x(x \in a \iff x \in b))$.

- (Pair) $\forall x \forall y \exists a \forall z(z \in a \iff z = x \lor z = y)$.

- (Union) $\forall a \exists b(\forall y \in a)(\forall x \in y)(x \in b)$.

Let $\Gamma$ be a class of $\mathcal{L}^*_p$-formulae.

- ($\Gamma$-Separation) For all $\phi(x, \bar{z}) \in \Gamma$,
  $\forall \exists \forall a \exists b \forall x(x \in b \iff (x \in a) \land \phi(x, \bar{z}))$.

- ($\Gamma$-Collection) For all $\phi(x, y, \bar{z}) \in \Gamma$,
  $\forall \exists \forall a((\forall x \in a)\exists y \phi(x, y, \bar{z}) \Rightarrow \exists b(\forall x \in a)(\exists y \in b)\phi(x, y, \bar{z}))$.

- ($\Gamma$-Foundation) For all $\phi(x, \bar{z}) \in \Gamma$,
  $\forall \exists (\exists x \phi(x, \bar{z}) \Rightarrow \exists y(\phi(y, \bar{z}) \land (\forall w \in y) \neg \phi(w, \bar{z})))$.

The following axiom in the language $\mathcal{L}^*$ describes the desired behaviour of the function symbol $F$:

- ($\dagger$) $\forall p \forall x(x \in F(p) \iff x \in F(p)) \land \forall a(F(a) = \emptyset)$.

The next axiom, in the language $\mathcal{L}^*_p$, says that the function symbol $P$ is the usual powerset function:

- (Powerset) $\forall a \forall b(b \in P(a) \iff b \subseteq a)$.

We will have cause to consider the following theories:

- $\text{KPU}_{\text{cov}}$ is the $\mathcal{L}^*$-theory with axioms: $\exists a(a = a)$, $\forall p \forall x(x \notin p)$, Extensionality for sets, Pair, Union, $\Delta_0(\mathcal{L}^*)$-Separation, $\Delta_0(\mathcal{L}^*)$-Collection, $\Pi_1(\mathcal{L}^*)$-Foundation and ($\dagger$).
• KPU$_{\text{cov}}^P$ is the $\mathcal{L}_P^*$-theory obtained from KPU$_{\text{cov}}$ by adding Powerset, $\Delta_0(\mathcal{L}_P^*)$-
Separation, $\Delta_0(\mathcal{L}_P^*)$-Collection and $\Pi_1(\mathcal{L}_P^*)$-Foundation.

Definition 2.1 Let $\mathcal{M} = \langle M, E^M \rangle$ be an $\mathcal{L}$-structure.

An admissible set covering $\mathcal{M}$ is an $\mathcal{L}^*$-structure

$$\mathfrak{A}_M = \langle M; A, e^A, F^A \rangle \models \text{KPU}_{\text{cov}},$$

such that $e^A$ is well-founded.

A power admissible set covering $\mathcal{M}$ is an $\mathcal{L}_P^*$-structure

$$\mathfrak{A}_M = \langle M; A, e^A, F^A, P^A \rangle \models \text{KPU}_{\text{cov}}^P$$

such that $e^A$ is well-founded.

We use $\text{Cov}_M = \langle M; A_M, e, F_M \rangle$ to denote the smallest admissible set covering $\mathcal{M}$
whose membership relation $\in$ coincides with the membership relation of the metatheory.

We use $\text{Cov}^P_M = \langle M; A_M, e, F_M, P_M \rangle$ to denote the smallest power admissible set
covering $\mathcal{M}$ whose membership relation coincides with the membership relation of the metatheory.

Note that if $\mathfrak{A}_M = \langle M; A, e^A, F^A, \ldots \rangle$ is an admissible set covering $\mathcal{M}$, then $\mathfrak{A}_M$
is isomorphic to a structure whose membership relation $\in$ is the membership relation of the metatheory.

Definition 2.2 Let $\mathcal{M} = \langle M, E^M \rangle$ be an $\mathcal{L}$-structure, and let

$$\mathfrak{A}_M = \langle M; A, e^A, F^A, P^A \rangle \models \text{KPU}_{\text{cov}}^P.$$

We use $\text{WF}(A)$ to denote the largest $B \subseteq_e A$ such that $\langle B, e^B \rangle$ is well-founded.

The well-founded part of $\mathfrak{A}_M$ is the $\mathcal{L}_P^*$-structure

$$\text{WF}(\mathfrak{A}_M) = \langle M; \text{WF}(A), e^A, F^A, P^A \rangle.$$

Note that $\text{WF}(\mathfrak{A}_M)$ is always isomorphic to an $\mathcal{L}_P^*$-structure whose membership relation $\in$
coincides with the membership relation of the metatheory.

As usual, in the theories $\mathcal{M}^-$, $\text{KP}^-$ and KPU$_{\text{cov}}$ the ordered pair $\langle x, y \rangle$ is coded by
the set $\{\{x\}, \{x, y\}\}$. This definition ensures that there is a $\Delta_0$-formula $\text{OP}(x)$ that says
that $x$ is an ordered pair, and functions

$$\text{fst}(\langle x, y \rangle) = x \text{ and } \text{snd}(\langle x, y \rangle) = y,$$

whose graphs are defined by $\Delta_0$-formulae. In KPU$_{\text{cov}}$ the rank function, $\rho$, and support
function, $\text{sp}$, are defined by recursion:

$$\rho(p) = 0 \text{ for all urelements } p, \text{ and } \rho(a) = \sup\{\rho(x) + 1 \mid x \in a\} \text{ for all sets } a;$$

$$\text{sp}(p) = \{p\} \text{ for all urelements } p, \text{ and } \text{sp}(a) = \bigcup_{x \in a} \text{sp}(x) \text{ for all sets } a.$$

The theory KPU$_{\text{cov}}$ proves that both of these are total and their graphs are $\Delta_1(\mathcal{L}^*)$. In
the theory KP, in which everything is a set, the rank function, $\rho$, is $\Delta_1$ and remains
provably total. We say that $x$ is a pure set if $\text{sp}(x) = \emptyset$. We say that $x$ is an ordinal if $x$ is a hereditarily transitive pure set; where:

$$\text{Transitive}(x) \iff \neg U(x) \land (\forall y \in x)(\forall z \in y)(z \in x),$$

and

$$\text{Ord}(x) \iff (\text{Transitive}(x) \land (\forall y \in x)(\text{Transitive}(y))).$$

Therefore, both ‘$x$ is transitive’ and ‘$x$ is an ordinal’ can be expressed using $\Delta_0(L^*)$-formulae. In the theories $M^-$ and $\text{KP}$, we can omit the reference to the predicate $U$ in the definition of ‘$x$ is transitive’, thus making both the property of being transitive and the property of being an ordinal into $\Delta_0$ properties.

The rank function allows us to strengthen the notion of powerset-preserving end extensions for models of $\text{KP}$. Let $L'$ be a language extending $L$. Let $M$ and $N$ be $L'$-structures that satisfy $\text{KP}$.

- We say that $N$ is a rank extension of $M$, and write $M \subseteq_e^r N$, if $M \subseteq_e^P N$ and for all $x, y \in N$, if $y \in M$ and $N \models (\rho(x) \leq \rho(y))$, then $x \in M$.

- We say that $N$ is a topless rank extension of $M$, and write $M \subseteq_e^r N$, if $M \subseteq_e^P N$, $M \neq N$ and for all $c \in N$, if $c^* \subseteq M$, then $c \in M$.

- We say that $N$ is a blunt rank extension of $M$, and write $M \subseteq_e^r N$, if $M \subseteq_e^P N$, $M \neq N$ and $N$ is not a topless rank extension of $M$.

Note that $\text{KP}^-$ is a subtheory of $M^- + \Delta_0$-Collection. We will make use of the following results:

- A consequence of [Mat01, Theorem Scheme 6.9(i)] is that $M$ proves $\Delta_0^P$-Separation.

- The availability of the collection scheme for the relevant class of formulae means that the class of formulae that are equivalent to a $\Sigma_1$-formula and the class of formulae that are equivalent to a $\Pi_1$-formula are closed under bounded quantification in the theory $\text{KP}^-; \text{ the class of formulae equivalent to a } \Sigma_1^P\text{-formula and the class of formulae that are equivalent to a } \Pi_1^P\text{-formula are closed under bounded quantification in the theory } M^- + \Delta_0^P\text{-Collection; the class of formulae equivalent to a } \Sigma_1(L^*)\text{-formula and the class of formulae that are equivalent to a } \Sigma_1^P(L^*)\text{-formula are closed under bounded quantification in the theory } KPU_{\text{Cov}}; \text{ and the class of formulae equivalent to a } \Sigma_1^P(L^*_p)\text{-formula are closed under bounded quantification in the theory } KPU_{\text{Cov}}^P.$

- The proof of [Bar75, I.4.4] shows:
  1. $\text{KP}^- \vdash \Sigma_1\text{-Collection}$;
  2. $M^- + \Delta_0^P\text{-Collection} \vdash \Sigma_1^P\text{-Collection}$;
  3. $KPU_{\text{Cov}} \vdash \Sigma_1(L^*)\text{-Collection}$; and
  4. $KPU_{\text{Cov}}^P \vdash \Sigma_1(L^*_p)\text{-Collection}$.

- The argument used in [Bar75, I.4.5] shows:
  1. $\text{KP}^- \vdash \Delta_1\text{-Separation}$;
2. $M^- + \Delta_0^P$-Collection $\vdash \Delta_1^P$-Separation;
3. $KPU_{\text{ Cov}} \vdash \Delta_1(L^\ast)$-Separation; and
4. $KPU_{\text{ Cov}}^P \vdash \Delta_1(L^\ast_P)$-Separation.

The following is Mathias’s calibration [Mat01, Proposition Scheme 6.12] of [Tak, Theorem 6].

**Theorem 2.3** The following inclusions hold between the indicated classes of formulae $(n \geq 1)$:

1. $\Sigma_1 \subseteq (\Delta_1^P)_{\text{ MOST}}$ and $\Delta_0^P \subseteq \Delta_2^S$.
2. $\Sigma_{n+1} \subseteq (\Sigma_n^P)_{\text{ MOST}}$.
3. $\Pi_{n+1} \subseteq (\Pi_n^P)_{\text{ MOST}}$.
4. $\Delta_{n+1} \subseteq (\Delta_n^P)_{\text{ MOST}}$.
5. $\Sigma_n^P \subseteq \Sigma_{n+1}^S$.
6. $\Pi_n^P \subseteq \Pi_{n+1}^S$.
7. $\Delta_n^P \subseteq \Delta_{n+1}^S$.

- As noted by Mathias in [Mat01, Corollary 6.15], MOST+$\Pi_1$-Collection and MOST+$\Delta_0^P$-Collection axiomatise the same theory. This fact follows from part 1 of Theorem 2.3 and the results mentioned above and will be repeatedly used throughout this paper.

The $\Sigma_1^P$-Recursion Theorem [Mat01, Theorem 6.26] shows that the theory $KP^P$ is capable of constructing the levels of the cumulative hierarchy:

$$V_0 = \emptyset \text{ and for all ordinals } \alpha, \quad V_{\alpha+1} = P(V_\alpha) \quad \text{and, if } \alpha \text{ is a limit ordinal, } V_\alpha = \bigcup_{\beta \in \alpha} V_\beta.$$  

More precisely, let $RK(\alpha, f)$ be the $L$-formula:

$$(f \text{ is a function}) \land (\alpha \text{ is an ordinal}) \land \text{dom}(f) = \alpha \land \left( (\forall \beta \in \alpha) \left( (\beta \text{ is a limit ordinal}) \Rightarrow f(\beta) = \bigcup_{\gamma \in \beta} f(\gamma) \right) \land (\exists \gamma \in (\beta = \gamma + 1) \Rightarrow ((\forall x \subseteq f(\gamma))(x \in f(\beta)) \land (\forall x \in f(\beta))(x \subseteq f(\gamma))) \right).$$

Note that $RK(f, \alpha)$ is a $\Delta_0^P$-formula.

**Lemma 2.4** The theory $KP^P$ proves

(I) for all ordinals $\alpha$, there exists $f$ such that $RK(\alpha, f)$;

(II) for all ordinals $\alpha$ and for all $f$, if $RK(f, \alpha + 1)$, then

$$f(\alpha) = \{ x \mid \rho(x) < \alpha \}.$$
Therefore, the theory $\text{KP}^P$ proves that the function $\alpha \mapsto V_\alpha$ is total and that the graph of this function is $\Delta^P_1$-definable.

In contrast, the theory $\text{MOST}$ does not prove that the function $\alpha \mapsto V_\alpha$ is total (see Example 2.7 below). Note that the availability of $\text{AC}$ in $\text{MOST}$ allows us to identify cardinals with initial ordinals. Consider the $\Delta^P_0$-formula $\text{BFEXT}(R, \kappa)$ defined by:

\begin{align*}
(R \text{ is an extensional relation on } X \text{ with a top element}) \land \\
(\forall S \subseteq X)(S \neq \emptyset \Rightarrow (\exists x \in S)(\forall y \in S)((y, x) \notin R)).
\end{align*}

The following lemma captures two important features of the theory $\text{MOST}$ that follow from [Mat01, Theorem 3.18].

**Lemma 2.5** The theory $\text{MOST}$ proves the following statements:

(I) for all $\langle X, R \rangle$ with $\text{BFEXT}(X, R)$, there exists a transitive set $T$ such that $\langle X, R \rangle \sim \langle T, \in \rangle$;

(II) there exist arbitrarily large initial ordinals;

(III) for all cardinals $\kappa$, the set $H_{\leq \kappa} = \{ x \mid |\text{TC}(x)| \leq \kappa \}$ exists.

In the theory $\text{MOST}$, the formula “$X = H_{\leq \kappa}$” is $\Delta^P_1$ with parameters $X$ and $\kappa$:

\begin{align*}
(\kappa \text{ is a cardinal}) \land \\
(\forall R \subseteq \kappa \times \kappa)(\text{BFEXT}(R, \kappa) \Rightarrow (\exists x, f, T \in X)(T = \text{TC} \{ x \} \land f : R \cong \in \restriction T)) \land \\
(\forall x \in X)(\exists T, f, T \in X)(T = \text{TC} \{ x \} \land (f : T \rightarrow \kappa \text{ is injective})).
\end{align*}

The next result is a special case of [Gor, Corollary 6.11]:

**Lemma 2.6** Let $\mathcal{M}$ and $\mathcal{N}$ be models of $\text{KP}^P$. If $\mathcal{M} \subseteq^P \mathcal{N}$, then $\mathcal{M} \subseteq^t \mathcal{N}$. □

The following examples show that neither of assumptions that $\mathcal{M}$ in Lemma 2.6 satisfies $\Delta^P_0$-Collection and $\Pi^P_1$-Foundation can be removed. The structure $\mathcal{M}$ defined in Example 2.7 satisfies all of the axioms of $\text{KP}^P$ except $\Delta^P_0$-Collection. The structure $\mathcal{M}$ defined in Example 2.8 satisfies all of the axioms of $\text{KP}^P$ except $\Pi^P_1$-Foundation.

**Example 2.7** Let $\mathcal{N} = \langle N, E^N \rangle \models \text{ZF} + \forall = L$, and

\[ \mathcal{M} = \langle (H^N_{\aleph_n})^*, E^N \rangle. \]

Then $\mathcal{M} \models \text{MOST} + \Pi^\infty_1$-Separation, and $\mathcal{M} \subseteq^P \mathcal{N}$, but $\mathcal{N}$ is not a rank extension of $\mathcal{M}$.

**Example 2.8** Let $\mathcal{N} = \langle N, E^N \rangle$ be an $\omega$-nonstandard model of $\text{ZF} + \forall = L$. Let $\mathcal{M} = \langle M, E^N \rangle$, where

\[ M = \bigcup_{n \in \omega} (H^N_{\aleph_n})^*. \]

Then $\mathcal{M} \models \text{MOST} + \Pi^1_1$-Collection, and $\mathcal{M} \subseteq^P \mathcal{N}$, but $\mathcal{N}$ is not a rank extension of $\mathcal{M}$.

---

4The abbreviation BFEXT has long been used by NF-theorists for well-founded extensional relations with a top, it is an abbreviation of Bien Fondée Extensionnelle, extensively employed by the French-speaking NF-ists in Belgium.
The following recursive definition can be carried out within $\mathsf{KPU}_{\text{cov}}$ thanks to the ability of $\mathsf{KPU}_{\text{cov}}$ to carry out $\Sigma_1(\mathcal{L}^*)$-recursions. The recursion defines an operation $\varphi^\gamma$ for coding the infinitary formulae of $\mathcal{L}_{\text{inf}}$, where $\mathcal{L}_{\text{inf}}$ be the language obtained from $\mathcal{L}$ by adding new constant symbols $\bar{a}$ for each urelement $a$ and a new constant symbol $c$.

- for all ordinals $\alpha$, $\varphi^\gamma(\alpha) = \langle 0, \alpha \rangle$,
- for all urelements $a$, $\varphi^\gamma(a) = \langle 1, a \rangle$,
- $\varphi^\gamma(\bar{c}) = \langle 2, 0 \rangle$,
- if $\phi$ is an $\mathcal{L}_{\text{inf}}$-formula and $x$ is a free variable of $\phi$, then $\varphi^\gamma(\exists x \phi) = \langle 3, \varphi^\gamma, \varphi^\gamma \rangle$,
- if $\phi$ is an $\mathcal{L}_{\text{inf}}$-formula and $x$ is a free variable of $\phi$, then $\varphi^\gamma(\forall x \phi) = \langle 4, \varphi^\gamma, \varphi^\gamma \rangle$,
- if $\Phi$ is a set of $\mathcal{L}_{\text{inf}}$-formulae such that only finitely many variables appear as a free variable of some formula in $\Phi$, then $\varphi^\gamma(\bigvee_{\phi \in \Phi} \phi) = \langle 5, \Phi^* \rangle$, where $\Phi^* = \{ \varphi^\gamma \mid \phi \in \Phi \}$,
- if $\Phi$ is a set of $\mathcal{L}_{\text{inf}}$-formulae such that only finitely many variables appear as a free variable of some formula in $\Phi$, then $\varphi^\gamma(\bigwedge_{\phi \in \Phi} \phi) = \langle 6, \Phi^* \rangle$, where $\Phi^* = \{ \varphi^\gamma \mid \phi \in \Phi \}$,
- if $\phi$ is an $\mathcal{L}_{\text{inf}}$-formula, then $\varphi^\gamma(\neg \phi) = \langle 7, \varphi^\gamma \rangle$,
- if $s$ and $t$ are terms of $\mathcal{L}_{\text{inf}}$, then $\varphi^\gamma(s = t) = \langle 8, \varphi^\gamma, \varphi^\gamma \rangle$,
- if $s$ and $t$ are terms of $\mathcal{L}_{\text{inf}}$, then $\varphi^\gamma(s \in t) = \langle 9, \varphi^\gamma, \varphi^\gamma \rangle$.

Let $M = \langle M, E^M \rangle$ be an $\mathcal{L}$-structure and let $\mathfrak{A}_M = \langle M; A, \in, F^\mathfrak{A}_M, P^\mathfrak{A}_M \rangle$ be a power admissible set covering $M$. We use $\mathcal{L}_{\text{inf}}^M$ to denote the fragment of $\mathcal{L}_{\text{inf}}$ that is coded in $\mathfrak{A}_M$. The $\mathcal{L}_{\text{inf}}^M$-formulae in the form $s = t$ or $s \in t$, where $s$ and $t$ are $\mathcal{L}_{\text{inf}}^M$-terms, are the \textbf{atomic formulae} of $\mathcal{L}_{\text{inf}}^M$. The formula that identifies the codes of the atomic formulae of $\mathcal{L}_{\text{inf}}^M$ is $\Delta_1(\mathcal{L}^*)$-definable over $\mathfrak{A}_M$. Similarly, other important properties of codes of $\mathcal{L}_{\text{inf}}^M$-constituents, such as being a \textbf{variable}, \textbf{constant}, \textbf{well-formed formula}, \textbf{sentence}, \ldots, are all $\Delta_1(\mathcal{L}^*)$-definable over $\mathfrak{A}_M$. We will often equate an $\mathcal{L}_{\text{inf}}^M$-theory $T$ with that subset of $\mathfrak{A}_M$ of codes of $\mathcal{L}_{\text{inf}}^M$-sentences in $T$.

The following is the Barwise Compactness Theorem \cite[III.5.6]{Bar75} tailored for countable admissible $\mathcal{L}_P$-structures.

\textbf{Theorem 2.9} \textit{(Barwise Compactness Theorem)} Let $\mathfrak{A}_M = \langle M; A, \in, F^\mathfrak{A}_M, P^\mathfrak{A}_M \rangle$ be a power admissible set covering $M$. Let $T$ be an $\mathcal{L}_{\text{inf}}^M$-theory that is $\Sigma_1(\mathcal{L}_P^*)$-definable over $\mathfrak{A}_M$ and such that for all $T_0 \subseteq T$, if $T_0 \in A$, then $T_0$ has a model. Then $T$ has a model.
3 The scheme of $\Sigma^P_1$-Foundation

Motivated by the apparent reliance of the constructions presented in the next section on $\Sigma^P_0$-Foundation, this section investigates the status of this scheme in the theories MOST + $\Pi_1$-Collection and $\text{KP}^P$. We begin by showing that $\text{KP}^P$ proves $\Sigma^P_1$-Foundation. In contrast, $\Sigma^P_1$-Foundation is not provable in MOST + $\Pi_1$-Collection but does hold in every $\omega$-standard model of this theory.

In [Rat20, Lemma 4.4] it is shown that $\text{KP}^P + \text{AC}$ proves $\Sigma^P_1$-Foundation\footnote{Rathjen proves the scheme that asserts that set induction holds for all $\Pi^P_1$-formulae, which, in the theory $\text{KP}^P$, is equivalent to $\Sigma^P_1$-Foundation.}. Here we use a modification of a choiceless scheme of dependant choices introduced in [Rat92] to show that $\Sigma^P_1$-Foundation can be proved in $\text{KP}^P$. The following is [Rat92, Definition 3.1]:

**Definition 3.1** Let $\phi(x,y,z)$ be an $\mathcal{L}$-formula. Define $\delta^\phi(a,b,f,z)$ to be the formula:

\[
\begin{align*}
\text{a is an ordinal } \Rightarrow & \quad \left( f \text{ is a function } \land \text{ dom}(f) = a + 1 \land f(0) = \{b\} \land \\
& \quad \left( \forall u \in a \left( \left( \forall x \in \text{ dom}(f) \right) \left( \exists y \in f(u + 1) \phi(x,y,z) \right) \land \\
& \quad \left( \forall y \in f(u + 1) \left( \forall x \in f(u) \phi(x,y,z) \right) \right) \right) \right). \\
\end{align*}
\]

By considering the variables $z$ to be parameters, $\phi(x,y,z)$ defines a directed graph. The formula $\delta^\phi(a,b,f,z)$ says that for all $0 \leq i \leq a$, $f(i)$ is a collection of vertices lying at a stage $i$ on a directed path of length $a$ starting at $b$ in this graph. In the next definition we introduce a formula that, given $b$, $f$ and $z$, says that $f$ is a function with domain $\omega$ and for all $n \in \omega$, $\delta^\phi(n+1,b,f \upharpoonright (n+1),z)$.

**Definition 3.2** Let $\phi(x,y,z)$ be an $\mathcal{L}$-formula. Define $\delta^\phi_\omega(b,f,z)$ to be the formula:

\[
\begin{align*}
& \quad f \text{ is a function } \land \text{ dom}(f) = \omega \land f(0) = \{b\} \land \\
& \quad \left( \forall u \in \omega \left( \left( \forall x \in \text{ dom}(f) \right) \left( \exists y \in f(u + 1) \phi(x,y,z) \right) \land \\
& \quad \left( \forall y \in f(u + 1) \left( \forall x \in f(u) \phi(x,y,z) \right) \right) \right) \right). \\
\end{align*}
\]

Note that if $\phi(x,y,z)$ is a $\Delta^P_0$-formula ($\Delta_0$-formula) then both $\delta^\phi(a,b,f,z)$ and $\delta^\phi_\omega(b,f,z)$ are both $\Delta^P_0$-formulae (respectively $\Delta_0$-formulae). The following is a modification of Rathjen’s $\Delta_0$-weak dependant choices scheme ($\Delta_0$-WDC) from [Rat92]:

($\Delta^P_0$-WDC$ \omega$) For all $\Delta^P_0$-formulae, $\phi(x,y,z)$,

\[
\forall z (\forall x \exists y \phi(x,y,z) \Rightarrow \forall w \exists f \delta^\phi_\omega(w,f,z)).
\]

The next result is based on the proof of [Rat92, Proposition 3.2]:

**Theorem 3.3** The theory $\text{M} + \Delta^P_0$-WDC$ \omega$ proves $\Sigma^P_1$-Foundation.

**Proof** Work in the theory $\text{M} + \Delta^P_0$-WDC$ \omega$. Suppose, for a contradiction, that there is an instance of $\Sigma^P_1$-Foundation that fails. Let $\phi(x,y,z)$ be a $\Delta^P_0$-formula and let $\bar{a}$ be a finite sequence of sets such that the class $C = \{ x \mid \exists y \phi(x,y,\bar{a}) \}$ is nonempty and has no $\in$-least element. Let $b$ and $d$ be such that $\phi(b,d,\bar{a})$ holds. Now, since $C$ has no $\in$-least element,

\[
\forall x \forall u \exists v (\phi(x,u,\bar{a}) \Rightarrow (y \in x) \land \phi(y,v,\bar{a})).
\]
Therefore, we have have $\forall x \exists y \theta(x, y, \vec{a})$ where $\theta(x, y, \vec{a})$ is

$$x = \langle x_0, x_1 \rangle \land y = \langle y_0, y_1 \rangle \land (\phi(x_0, x_1, \vec{a}) \Rightarrow (y_0 \in x_0) \land \phi(y_0, y_1, \vec{a})).$$

Note that $\theta(x, y, \vec{a})$ is a $\Delta_0^P$-formula. Therefore, using $\Delta_0^P$-WDC$_\omega$, let $f$ be such that $\delta^0_* ((b, d), f, \vec{a})$. Now, $\Delta_0^P$-Separation facilitates induction for $\Delta_0^P$-formulae and proves that for all $n \in \omega$, 

$$f(n) \neq \emptyset \land (\forall x \in f(n))(x = \langle x_0, x_1 \rangle \land \phi(x_0, x_1, \vec{a})) \land (\forall x \in f(n))(\exists y \in f(n+1))(x = \langle x_0, x_1 \rangle \land y = \langle y_0, y_1 \rangle \land y_0 \in x_0) \land .$$

Let $B = \text{TC} \{b\}$. Induction for $\Delta_0$-formulae suffices to prove that for all $n \in \omega$,

$$(\forall x \in f(n))(x = \langle x_0, x_1 \rangle \land x_0 \in B).$$

Consider

$$A = \left\{ x \in B \mid (\exists n \in \omega)(\exists z \in f(n)) \left( \exists y \in \bigcup \left( z = \langle x, y \rangle \right) \right) \right\},$$

which is a set by $\Delta_0$-Separation. Now, let $x \in A$. Let $y$ and $n \in \omega$ be such that $\langle x, y \rangle \in f(n)$. Therefore, there exists $w \in f(n+1)$ such that $w = \langle u, v \rangle$ and $u \in x$. So $u \in A$ and $u \in x$, which shows that $A$ has no $\in$-least element. This contradicts Set-Foundation in $M$ and proves the theorem. $\square$

The fact that $\text{KP}^P$ proves $\Sigma_1^P$-Foundation follows from the fact that $\text{KP}^P$ proves $\Delta_0^P$-WDC$_\omega$. The proof of Theorem 3.5 is inspired by the argument used in the proof of [FLM Theorem 4.15]. The stratification of the universe into ranks allows us to select sets of paths through a relation defined by a $\Delta_0^P$-formula $\phi(x, y, \vec{z})$ with parameters $\vec{z}$.

**Definition 3.4** Let $\phi(x, y, \vec{z})$ be an $\mathcal{L}$-formula. Define $\eta^\phi(a, b, f, \vec{z})$ by

$$\forall u \in a \exists \alpha \exists X \left( \delta^\phi(a, b, f, \vec{z}) \land (\alpha \text{ is an ordinal}) \land (X = V_\alpha) \land (\forall x \in f(u+1))(x \in X) \land (\forall y \in X)(\forall x \in f(u))(\phi(x, y, \vec{z}) \Rightarrow y \in f(u+1)) \land Y = V_\beta \Rightarrow (\forall \beta \in \alpha)(\forall Y \in X) \left( (\exists x \in f(u))(\forall y \in Y) \neg \phi(x, y, \vec{z}) \right) \right).$$

The formula $\eta^\phi(a, b, f, \vec{z})$ asserts that $f$ is a function with domain $a + 1$ such that $f(0) = \{b\}$ and for all $u \in a$, $f(u+1)$ is the set of $y$ of rank $\alpha$ such that there exists $x \in f(u)$ with $\phi(x, y, \vec{z})$ and $\alpha$ is the minimal ordinal such that for all $x \in f(u)$, there exists $y$ of rank $\alpha$ such that $\phi(x, y, \vec{z})$. Recall that, in the theory $\text{KP}^P$, the formula $`X = V_\alpha`$ is $\Delta_0^P$ with parameters $X$ and $\alpha$. Therefore, if $\phi(x, y, \vec{z})$ is a $\Delta_0^P$-formula, then $\eta^\phi(a, b, f, \vec{z})$ is equivalent to a $\Sigma_1^P$-formula in the theory $\text{KP}^P$.

**Theorem 3.5** The theory $\text{KP}^P$ proves $\Delta_0^P$-WDC$_\omega$.

**Proof** Work in the theory $\text{KP}^P$. Let $\phi(x, y, \vec{z})$ be a $\Delta_0^P$-formula. Let $\vec{a}$ be sets such that $\forall x \exists y \phi(x, y, \vec{a})$ holds. Let $b$ be a set. We begin by claiming that for all $n \in$
\( \omega, \exists f \eta^\delta(n, b, f, \vec{a}) \). Suppose, for a contradiction, that this does not hold. Using \( \Pi^1_1 \)-Foundation, there exists a least \( m \in \omega \) such that \( \neg \exists f \eta^\delta(m, b, f, \vec{a}) \). It is straightforward to see that \( m \neq 0 \). Therefore, there exists a function \( g \) with \( \text{dom}(g) = m \) such that \( \eta^\delta(m - 1, b, g, \vec{a}) \) holds. Consider the class

\[
A = \{ \alpha \in \text{Ord} \mid \forall X (X = V_\alpha \Rightarrow (\forall x \in g(m - 1))(\exists y \in X)\phi(x, y, \vec{a})) \}.
\]

Applying \( \Delta_0^P \)-Collection to the formula \( \phi(x, y, \vec{a}) \) shows that \( A \) is nonempty. Therefore, by \( \Pi^1_1 \)-Foundation, there exists a least element \( \beta \in A \). Let

\[
C = \{ y \in V_\beta \mid (\exists x \in g(m - 1))\phi(x, y, \vec{a}) \},
\]

which is a set by \( \Delta_0^P \)-Separation. Now, let \( f = g \cup \{ \langle m, C \rangle \} \). So, \( \eta^\delta(m, b, f, \vec{a}) \), which is a contradiction. Therefore, for all \( n \in \omega \), \( \exists f \eta^\delta(n, b, f, \vec{a}) \). Note that for all \( n \in \omega \) and for all \( f \) and \( g \), if \( \eta^\delta(n, b, f, \vec{a}) \) and \( \eta^\delta(n, b, g, \vec{a}) \), then \( f = g \). Now, using \( \Sigma^P_1 \)-Collection, we can find a set \( D \) such that \( (\forall n \in \omega)(\exists f \in D)\eta^\delta(n, b, f, \vec{a}) \). Let

\[
f = \{ \langle n, X \rangle \in \omega \times \text{TC}(D) \mid (\exists g \in D)(\eta^\delta(n, b, g, \vec{a}) \land g(n) = X) \}
\]

\[
= \{ \langle n, X \rangle \in \omega \times \text{TC}(D) \mid (\forall g \in D)(\eta^\delta(n, b, g, \vec{a}) \Rightarrow g(n) = X) \}.
\]

Now, \( f \) is a set by \( \Delta_0^P \)-Separation and \( f \) is the function required by \( \Delta_0^P \)-WDC\( \omega \). \( \square \)

Combining Theorems 3.3 and 3.5 yields:

**Corollary 3.6** \( K^P + \Sigma^P_1 \)-Foundation. \( \square \)

We now turn to investigating \( \Sigma^P_1 \)-Foundation in the theory MOST + \( \Pi_1 \)-Collection. The following is an instance of [PK, Proposition 2] in the context of set theory:

**Theorem 3.7** Let \( \Sigma \) denote \( \Sigma^P_1 \)-Foundation on \( \omega \), and \( \Pi \) denote \( \Pi^1_1 \)-Foundation on \( \omega \). Then we have:

\[
M^- + \Delta_0^P \text{-Collection} + \Pi \vdash \Sigma, \quad \text{and} \quad M^- + \Sigma \vdash \Pi.
\]

**Proof** To see that \( \Pi \) implies \( \Sigma \), work in the theory \( M^- + \Delta_0^P \text{-Collection} \). We prove the contrapositive. Let \( \phi(x, \vec{z}) \) be a \( \Pi^P_0 \)-formula and let \( \vec{a} \) be sets such that the class \( \{ x \in \omega \mid \phi(x, \vec{a}) \} \) is nonempty and has no least element. Let \( p \in \omega \) be such that \( \phi(p, \vec{a}) \). Let

\[
C = \{ x \in \omega \mid \exists w (x + w = p \land (\forall y \in w)\neg \phi(y, \vec{a})) \}.
\]

Note that \( \Delta_0^P \)-Collection implies that \( C \) is a \( \Sigma^P_1 \)-definable subclass of \( \omega \). Moreover, \( p \in C \) and \( 0 \notin C \). Identical reasoning to that used above shows that \( C \) has no least element. Therefore \( \Sigma^P_1 \)-Foundation on \( \omega \) fails.

To see that \( \Sigma \) implies \( \Pi \), work in the theory \( M^- \). Again, we prove the contrapositive. Let \( \phi(x, \vec{z}) \) be a \( \Sigma^P_1 \)-formula and let \( \vec{a} \) be the sequence of set parameters such that the class \( \{ x \in \omega \mid \phi(x, \vec{a}) \} \) is nonempty and has no least element. Let \( p \in \omega \) be such that \( \phi(p, \vec{a}) \). Let

\[
C = \{ x \in \omega \mid \forall w (x + w = p \Rightarrow (\forall y \in w)\neg \phi(y, \vec{a})) \}.
\]

Note that \( C \) is a \( \Pi^P_0 \)-definable subclass of \( \omega \), \( p \in C \) and \( 0 \notin C \). Suppose that \( q \in C \) is a least element of \( C \). Let \( u \in \omega \) be such that \( q + u = p \). Now, \( \phi(u, \vec{a}) \), since \( q \) is the least of \( C \), and \( (\forall y \in u)\neg \phi(y, \vec{a}) \). But then \( u \) is a least element of \( \{ x \in \omega \mid \phi(x, \vec{a}) \} \), which is a contradiction. Therefore \( \Pi^1_1 \)-Foundation on \( \omega \) fails. \( \square \)
An examination of the proof of [Mat01, Proposition 9.22] yields:

**Theorem 3.8** The consistency of \( \text{Mac} \) is provable in \( M + \Pi^0_1 \)-Foundation on \( \omega \). □

The results of [Mat01] and [M] (see [M] Corollary 3.5) show that Mac and MOST + \( \Pi_1 \)-Collection have the same consistency strength. Therefore, Theorem 3.8 yields:

**Theorem 3.9** The consistency of MOST + \( \Pi_1 \)-Collection is provable in MOST + \( \Pi_1 \)-Collection + \( \Sigma^P_1 \)-Foundation. □

Therefore, \( \Sigma^P_1 \)-Foundation is not provable in MOST + \( \Pi_1 \)-Collection. However, we can show that \( \Sigma^P_1 \)-Foundation does hold in every model of MOST + \( \Pi_1 \)-Collection in which the natural numbers are standard.

In the context of the theory MOST + \( \Pi_1 \)-Collection, we can use the stratification of the universe into the sets \( H_{\leq \kappa} \) in the same way that we used the stratification of the universe into ranks in the proof of Theorem 3.5.

**Definition 3.10** Let \( \phi(x, y, z) \) be an \( L \)-formula. Define \( \chi^\phi(a, b, f, z) \) by

\[
\begin{align*}
(\forall u \in a) &\exists \exists X
\left( \\
&\begin{array}{l}
\delta^\phi(a, b, f, z)\land \\
(X = H_{<\kappa}) \land (\forall x \in f(u + 1))(x \in X)\land \\
(\forall y \in X)(\forall x \in f(u))(\phi(x, y, z) \Rightarrow y \in f(u + 1))\land \\
(\forall \lambda \in \kappa)(\exists x \in f(u))(\forall R \subseteq \lambda \times \lambda) \left( \exists T, f, y \\
\left( \begin{array}{l}
\text{BFE}^T(R, \lambda) \Rightarrow \\
T = \text{TC}(\{y\})\land \\
f : R \ni x \in T \land \neg \phi(x, y, z)
\end{array} \right) \right)
\end{array}
\right).
\end{align*}
\]

The formula \( \chi^\phi(a, b, f, z) \) asserts that \( f \) is a function with domain \( a + 1 \) such that \( f(0) = \{b\} \) and for all \( u \in a \), \( f(u + 1) \) is the set of all \( y \) in \( H_{\leq \kappa} \) such that there exists an \( x \in f(u) \) with \( \phi(x, y, z) \) and \( \kappa \) is the minimal cardinal such that for all \( x \in f(u) \), there exists \( y \in H_{\leq \kappa} \) with \( \phi(x, y, z) \). Recall that the formula expressing “\( X = H_{\leq \kappa} \)” is \( \Delta^P_1 \) with parameters \( X \) and \( \kappa \) in the theory MOST. Therefore, if \( \phi(x, y, z) \) is a \( \Delta^P_1 \)-formula, then \( \chi^\phi(a, b, f, z) \) is equivalent to a \( \Sigma^P_1 \)-formula in the theory MOST + \( \Pi_1 \)-Collection. In the proof of the next theorem the formula \( \chi^\phi(a, b, f, z) \) plays the role of \( \eta^\phi(a, b, f, z) \) in the proof of Theorem 3.5.

**Theorem 3.11** Let \( \mathcal{M} = \langle M, E^\mathcal{M} \rangle \) be an \( \omega \)-standard model of MOST + \( \Pi_1 \)-Collection. Then \( \mathcal{M} \models \Delta^P_0 \)-WDC\( \omega \).

**Proof** Let \( \phi(x, y, z) \) be a \( \Delta^P_0 \)-formula. Let \( \bar{a} \) be sets such that \( \mathcal{M} \models \forall x \exists y \phi(x, y, \bar{a}) \). Let \( b \) be a set. We begin by showing that

\[
\mathcal{M} \models (\forall n \in \omega) \exists f \chi^\phi(n, b, f, \bar{a}).
\]

Work inside \( \mathcal{M} \). Suppose, for a contradiction, that there exists \( n \in \omega \) such that \( \neg \exists f \chi^\phi(n, b, f, \bar{a}) \) holds. Therefore, since \( \mathcal{M} \) is \( \omega \)-standard, there is a least \( m \in \omega \) such that \( \neg \exists f \chi^\phi(m, b, f, \bar{a}) \). It is straightforward to see that \( m \neq 0 \). Therefore, there exists a function \( g \) with \( \text{dom}(g) = m \) such that \( \chi^\phi(m - 1, b, g, \bar{a}) \). Consider

\[
A = \{ \kappa \in \text{Ord}^\mathcal{M} \mid \mathcal{M} \models \forall X(X = H_{\leq \kappa} \Rightarrow (\forall x \in g(m - 1))(\exists y \in X)\phi(x, y, \bar{a})) \}
\]

\[
= \{ \kappa \in \text{Ord}^\mathcal{M} \mid \mathcal{M} \models \exists X(X = H_{\leq \kappa} \land (\forall x \in g(m - 1))(\exists y \in X)\phi(x, y, \bar{a})) \}
\].
Applying $\Delta^p_0$-Collection to $\phi(x, y, \vec{a})$ shows that $A$ is nonempty. Therefore, $\Delta^p_1$-Separation ensures that $A$ has an $\varepsilon$-least element $\lambda$. Let

$$C = \{ y \in H_{\leq \lambda} \mid (\exists x \in g(m - 1))\phi(x, y, \vec{a}) \},$$

which is a set by $\Delta^p_0$-Separation. Now, let $f = g \cup \{ (m, C) \}$. So, $\chi^\phi(m, b, f, \vec{a})$ which is a contradiction. This shows that $M$ starts with a structure which is a set by $\Delta^p_0$-Foundation. Therefore, $\delta_\omega(b, f, \vec{a})$ holds, which completes the proof that $\Delta^p_0$-WDC holds in $M$. □

Combining Theorems 3.3 and 3.11

**Corollary 3.12** If $M$ is an $\omega$-standard model of MOST $+$ $\Pi_1$-Collection, then $M \models \Sigma^p_1$-Foundation. □

### 4 Obtaining $\text{Cov}_M$ from $M$

[Bar75 Appendix] shows how the admissible cover, $\text{Cov}_M$, can be built from an $\mathcal{L}$-structure $M$ that satisfies KP $+$ $\Sigma_1$-Foundation. The construction proceeds in two stages. The first stage interprets a model of KPU inside $M$. The second stage takes the well-founded part of this interpreted model of KPU to obtain an admissible set covering $M$ that [Bar75 Appendix] shows is minimal. It should be noted that [Bar75 Appendix] starts with a structure $M$ that satisfies full $\Pi_\infty$-Foundation.

It is noted in [Res, Chapter 2] that all of the elements of Barwise’s construction of $\text{Cov}_M$ can be carried out when $M$ satisfies $\Pi_1 \cup \Sigma_1$-Foundation. The aim of this section is to review the construction of $\text{Cov}_M$ from $M$ and investigate the influence of the theory of $M$ on $\text{Cov}_M$. In particular, we will show that if $M$ is a model of KP $+$ Powerset $+$ $\Delta^p_0$-Collection $+$ $\Sigma^p_1$-Foundation, then $P$ can be interpreted in $\text{Cov}_M$ to make it a power admissible set.

Throughout this section we will work with a fixed $\mathcal{L}$-structure $M = \langle M, E^M \rangle$ that satisfies KP $+$ Powerset $+$ $\Delta^p_0$-Collection $+$ $\Sigma^p_1$-Foundation. We begin by expanding the interpretation of the theory KPU inside $M$ presented in [Bar75 Appendix Section 3] to obtain $\mathcal{L}'^M$-structure that satisfies Powerset. Working inside $M$, define the unary relations $\text{N}$ and $\text{Set}$, the binary relations $\mathcal{E}$ and $\mathcal{E}'$, and unary function symbols $F$ and $P$ by:

$$N(x) \iff \exists y (x = \langle 0, y \rangle);$$

$$x E' y \iff \exists w \exists z (x = \langle 0, w \rangle \land y = \langle 0, z \rangle \land w \in z);$$

15
∀x \, xN \equiv (\forall y (x = \langle 1, y \rangle \land (\forall z \in y)(N(z) \lor \text{Set}(z))))

x \in y \equiv \exists z (y = \langle 1, z \rangle \land x \in z);

\bar{F}(x) = \langle 1, X \rangle \text{ where } X = \{0, y\} \mid \exists w (x = \langle 0, w \rangle \land y \in w\};

\bar{P}(x) = \langle 1, X \rangle \text{ where } X = \{1, y\} \mid \exists w (x = \langle 1, w \rangle \land y \subseteq w\}.

Appendix Section 3] notes that \( N, E', E \) and \( \bar{F} \) are defined by \( \Delta_0 \)-formulae in \( M \), and, using the Second Recursion Theorem ([Bar75, V.2.3.]), \( \text{Set} \) can be expressed using a \( \Sigma_1 \)-formula. [Res, Chapter 2] notes that the Second Recursion Theorem can be proved in \( KP + \Sigma_1 \)-Foundation. The function \( y = \bar{P}(x) \) is defined by a \( \Delta_0 \)-formula:

\[
y = \bar{P}(x) \text{ iff } \bigwedge \big(\forall z \subseteq \text{snd}(x)\big) \big((\forall z \subseteq \text{snd}(x))(\exists z \in \text{snd}(y)) \land (\forall w \in \text{snd}(y)) (\text{snd}(w) \subseteq \text{snd}(x)).\big)
\]

These definitions yield an interpretation, \( I \), of an \( L_\omega^\omega \)-structure that is summarised in Table 1 that extends the table in [Bar75, p. 373]:

| \( L_\omega^\omega \)-Symbol | \( L \) expression under \( I \) |
|--------------------------|---------------------------------|
| \( \forall x \)         | \( \forall x (N(x) \lor \text{Set}(x) \Rightarrow \cdots) \) |
| \( = \)                  | =                               |
| \( U(x) \)               | \( N(x) \)                      |
| \( x \in y \)            | \( xE'y \)                      |
| \( x \in y \)            | \( xEy \)                       |
| \( \bar{F}(x) \)         | \( \bar{F}(x) \)                |
| \( \bar{P}(x) \)         | \( \bar{P}(x) \)                |

In other words, \( \mathfrak{A}_N = \langle N, \text{Set}^M, E^M, \bar{E}^M, \bar{P}^M \rangle \), where \( N = \langle N^M, (E')^M \rangle \), is an \( L_\omega^\omega \)-structure. If \( \phi \) is an \( L_\omega^\omega \)-formula, then we write \( \phi^I \) for the translation of \( \phi \) into an \( L \)-formula of \( M \) described in Table 1. Note that the map \( x \mapsto \langle 0, x \rangle \) is an isomorphism between \( M \) and \( N \). The following is the refinement of [Bar75 Appendix Lemma 3.2] noted by [Res, Chapter 2]:

**Theorem 4.1** \( \mathfrak{A}_N \models KP_{\text{cov}} \). \( \square \)

We now turn to showing that axioms and axiom schemes transfer from \( M \) to \( \mathfrak{A}_N \).

**Lemma 4.2** \( \mathfrak{A}_N \models \text{Powerset} \).

**Proof** Let \( a \) be a set of \( \mathfrak{A}_N \). To see that \( \bar{P}(a) \) exists, note that \( a = \langle 1, a_0 \rangle \) and \( \bar{P}(a) = \langle 1, X \rangle \) where \( X = \{1\} \times \bar{P}(a_0) \). Therefore, the powerset axiom in \( M \) ensures that \( \bar{P} \) is total in \( \mathfrak{A}_N \). Now, let \( b \) be a set of \( \mathfrak{A}_N \). Work inside \( M \). Now, \( b = \langle 1, b_0 \rangle \). And,

\[
b \in \bar{P}(a) \text{ iff } b_0 \subseteq a_0,
\]

iff for all \( x \), if \( x \in b \), then \( x \in a \),

\[
\text{iff } (b \subseteq a)^I.
\]

Therefore, \( \mathfrak{A}_N \) satisfies Powerset. \( \square \)
Lemma 4.3 Let \( \phi(\bar{x}) \) be a \( \Delta_0(L^p_\ast) \)-formula. Then \( \phi^I(\bar{x}) \) is equivalent to a \( \Delta^P_1 \)-formula in \( \mathcal{M} \).

Proof We prove this lemma by induction on the complexity of \( \phi^I \). Note that, by the above observations, \( N(x) \) and \( xE'y \) can be written as \( \Delta_0 \)-formulae. Moreover, \( y = \bar{F}(x) \) is equivalent to a \( \Delta_0 \)-formula, and \( y = \bar{P}(x) \) is equivalent to a \( \Delta^P_0 \)-formula. Now, \( y \bar{E}(x) \) iff

\[
\text{fst}(y) = 0 \land \text{snd}(y) \in \text{snd}(x),
\]

which is \( \Delta_0 \). Similarly, \( y \bar{E}(t(x)) \) iff

\[
\text{fst}(y) = 1 \land \text{snd}(y) \subseteq \text{snd}(x),
\]

which is also \( \Delta_0 \). Now, suppose that \( t(x) \) is an \( L^p_\ast \)-term and both \( y = t(\bar{x}) \) and \( y \bar{E}t(\bar{x}) \) are \( \Delta^P_1 \) in \( \mathcal{M} \). Now, \( y = \bar{F}(t(x)) \)

\[
\text{if } \exists w(w = t(x) \land y = \bar{P}(w)),
\]

\[
\text{if } \forall w(w = t(x) \Rightarrow y = \bar{P}(w)).
\]

Similarly, \( y \bar{E}(t(x)) \)

\[
\text{if } \exists w(w = t(x) \land y \bar{E}p(w)),
\]

\[
\text{if } \forall w(w = t(x) \Rightarrow y \bar{E}p(w)).
\]

Therefore, both \( y = \bar{P}(t(x)) \) and \( y \bar{E}(t(x)) \) are \( \Delta^P_1 \) in \( \mathcal{M} \). Now, \( y = \bar{F}(t(x)) \)

\[
\text{if } \exists w(w = t(x) \land y = \bar{F}(w)),
\]

\[
\text{if } \forall w(w = t(x) \Rightarrow y = \bar{F}(w)).
\]

And, \( y \bar{E}(t(x)) \)

\[
\text{if } \exists w(w = t(x) \land y \bar{E}F(w)),
\]

\[
\text{if } \forall w(w = t(x) \Rightarrow y \bar{E}F(w)).
\]

Since \( \bar{F} \) and \( \bar{P} \) are both unary functions, this shows that for every \( L^p_\ast \)-term \( t(x) \), both \( y = t(x) \) and \( y \bar{E}t(x) \) are \( \Delta^P_1 \) in \( \mathcal{M} \). Finally, we need an induction step that allows us to deal with bounded quantification. Let \( \psi(x_0, \ldots, x_{n-1}) \) be an \( L^p_\ast \)-formula such that \( \psi^I(x_0, \ldots, x_{n-1}) \) is \( \Delta^P_1 \) in \( \mathcal{M} \). Now, \( (\exists x_0 \in \text{snd}(x_n))\psi^I(x_0, \ldots, x_{n-1}) \)

\[
\text{if } \exists x_0 \in \text{snd}(x_n)\psi^I(x_0, \ldots, x_{n-1}).
\]

Therefore, \( (\exists x_0 \in \text{snd}(x_n))\psi^I(x_0, \ldots, x_{n-1}) = ((\exists x_0 \in \text{snd}(w))\psi^I(x_0, \ldots, x_{n-1}))^I \) is \( \Delta^P_1 \) in \( \mathcal{M} \). Let \( t(x) \) be an \( L^p_\ast \)-term. Now, \( (\exists x_0 \in \text{snd}(x_n))\psi^I(x_0, \ldots, x_{n-1}) \)

\[
\text{if } \exists w(w = t(x_n) \land (\exists x_0 \in \text{snd}(w))\psi^I(x_0, \ldots, x_{n-1})),
\]

\[
\text{if } \forall w(w = t(x_n) \Rightarrow (\exists x_0 \in \text{snd}(w))\psi^I(x_0, \ldots, x_{n-1})).
\]

Therefore \( (\exists x_0 \in \text{snd}(x_n))\psi^I(x_0, \ldots, x_{n-1}) = ((\exists x_0 \in \text{snd}(x_n))\psi^I(x_0, \ldots, x_{n-1}))^I \) is \( \Delta^P_1 \) in \( \mathcal{M} \). The Lemma now follows by induction. \( \Box \)

Lemma 4.4 \( \mathcal{N} \models \Delta_0(L^p_\ast)\)-Separation.
**Proof** Let \( \phi(x, \bar{z}) \) be a \( \Delta_0(L^*_p) \)-formula, \( \bar{v} \) be sets and/or urelements of \( \mathfrak{A}_\mathcal{N} \) and \( a \) a set of \( \mathfrak{A}_\mathcal{N} \). Work inside \( \mathcal{M} \). Now, \( a = \langle 1, a_0 \rangle \). Let

\[
b_0 = \{x \in a_0 \mid \phi^I(x, \bar{v})\},
\]

which is a set by \( \Delta^P_1 \)-Separation. Let \( b = \langle 1, b_0 \rangle \). Therefore, for all \( x \) such that \( \Set(x) \),

\[
x \in \mathcal{E}b \iff x \in b_0 \land x \in \mathcal{E}a \land \phi^I(x, \bar{v}).
\]

Therefore, \( \mathfrak{A}_\mathcal{N} \) satisfies \( \Delta_0(L^*_p) \)-Separation. \( \Box \)

**Lemma 4.5** \( \mathfrak{A}_\mathcal{N} \models \Delta_0(L^*_p) \)-Collection.

**Proof** Let \( \phi(x, y, \bar{z}) \) be a \( \Delta_0(L^*_p) \)-formula. Let \( \bar{v} \) be a sequence of sets and/or urelements of \( \mathfrak{A}_\mathcal{N} \) and let \( a \) be a set of \( \mathfrak{A}_\mathcal{N} \) such that

\[
\mathfrak{A}_\mathcal{N} \models (\forall x \in a)(\exists y \phi(x, y, \bar{v})).
\]

Work inside \( \mathcal{M} \). Since \( a \) is a set of \( \mathfrak{A}_\mathcal{N} \), \( a = \langle 1, a_0 \rangle \). We have

\[
(\forall x \in a_0)(\exists y \in b)((N(y) \lor \Set(y)) \land \phi^I(x, y, \bar{v})).
\]

And,

\[
(\forall x \in a_0)(\exists y \in b)((N(y) \lor \Set(y)) \land \phi^I(x, y, \bar{v})).
\]

So, since \( (N(y) \lor \Set(y)) \land \phi^I(x, y, \bar{v}) \) is equivalent to a \( \Sigma^P_1 \)-formula, we can apply \( \Delta^P_0 \)-Collection to obtain \( b \) such that

\[
(\forall x \in a_0)(\exists y \in b)((N(y) \lor \Set(y)) \land \phi^I(x, y, \bar{v})).
\]

Let \( b_0 = \{y \in b \mid (N(y) \lor \Set(y))((b)\} \), which is a set by \( \Delta_0 \)-Separation. Let \( b_1 = \langle 1, b_0 \rangle \). Therefore \( \Set(b_1) \) and

\[
(\forall x \in a_0)(\exists y \in b_1) \phi^I(x, y, \bar{v}).
\]

So,

\[
\mathfrak{A}_\mathcal{N} \models (\forall x \in a)(\exists y \in b_1) \phi(x, y, \bar{v}).
\]

This shows that \( \mathfrak{A}_\mathcal{N} \) satisfies \( \Delta_0(L^*_p) \)-Collection. \( \Box \)

**Lemma 4.6** \( \mathfrak{A}_\mathcal{N} \models \Sigma_1(L^*_p) \)-Foundation.

**Proof** Let \( \phi(x, \bar{z}) \) be a \( \Sigma_1(L^*_p) \)-formula. Let \( \bar{v} \) be a sequence of sets and/or urelements be such that

\[
\{x \in \mathfrak{A}_\mathcal{N} \mid \mathfrak{A}_\mathcal{N} \models \phi(x, \bar{v})\}
\]

is nonempty.

Work inside \( \mathcal{M} \). Consider \( \theta(\alpha, \bar{v}) \) defined by

\[
(\alpha \text{ is an ordinal}) \land \exists x((\Set(x) \lor N(x)) \land \rho(x) = \alpha \land \phi^I(x, \bar{v})).
\]

Note that \( \theta(\alpha, \bar{v}) \) is equivalent to a \( \Sigma^P_1 \)-formula. Therefore, using \( \Sigma^P_1 \)-Foundation, let \( \beta \) be an \( \varepsilon \)-least element of

\[
\{\alpha \in M \mid \mathcal{M} \models \theta(\alpha, \bar{v})\}.
\]

Let \( y \) be such that \( (N(y) \lor \Set(y)) \), \( \rho(y) = \beta \) and \( \phi^I(y, \bar{v}) \). Note that if \( x \in \mathcal{E}y \), then \( \rho(x) < \rho(y) \). Therefore \( y \) is an \( \varepsilon \)-least element of

\[
\{x \in \mathfrak{A}_\mathcal{N} \mid \mathfrak{A}_\mathcal{N} \models \phi(x, \bar{v})\}.
\]

\( \Box \)
The following combines [Bar75, II.8.4] with the characterisation of $\text{Cov}_\mathcal{M}$ proved in [Bar75, Appendix Section 3]:

**Theorem 4.7** The $\mathcal{L}^*$-reduct of $\text{WF}(\mathcal{A}_\mathcal{N})$, $\text{WF}^-(\mathcal{A}_\mathcal{N}) = \langle \mathcal{N}; \text{WF}(\text{Set}^\mathcal{M}), \mathcal{E}^\mathcal{M}, \bar{F}^\mathcal{M} \rangle$ is an admissible set covering $\mathcal{N}$ that is isomorphic to $\text{Cov}_\mathcal{M}$. $\square$

We now turn to extending this result to show that $\text{WF}(\mathcal{A}_\mathcal{N})$ is a power admissible set covering $\mathcal{N}$ and therefore the least power admissible set covering $\mathcal{N}$.

**Theorem 4.8** The structure $\text{WF}(\mathcal{A}_\mathcal{N}) = \langle \mathcal{N}; \text{WF}(\text{Set}^\mathcal{M}), \mathcal{E}^\mathcal{M}, \bar{F}^\mathcal{M}, \bar{P}^\mathcal{M} \rangle$ is a power admissible set covering $\mathcal{N}$. Moreover, $\text{WF}(\mathcal{A}_\mathcal{N})$ is isomorphic to $\text{Cov}_\mathcal{P}^\mathcal{M}$.

**Proof** Note that it follows immediately from Theorem 4.7 that $\text{WF}(\mathcal{A}_\mathcal{N}) = \langle \mathcal{N}; \text{WF}(\text{Set}^\mathcal{M}), \mathcal{E}^\mathcal{M}, \bar{F}^\mathcal{M}, \bar{P}^\mathcal{M} \rangle$ satisfies all of the axioms of $\text{KPU}_{\text{Cov}}$ plus full Foundation. The fact that $\text{WF}(\mathcal{A}_\mathcal{N}) \subseteq^P \mathcal{A}_\mathcal{N}$ implies that $\text{WF}(\mathcal{A}_\mathcal{N})$ satisfies Powerset and $\Delta_0(\mathcal{L}^\mathcal{P})$-Separation. To show that $\text{WF}(\mathcal{A}_\mathcal{N})$ satisfies $\Delta_0(\mathcal{L}^\mathcal{P})$-Collection, let $\phi(x, y, \bar{v})$ be a $\Delta_0(\mathcal{L}^\mathcal{P})$-formula. Let $\bar{v}$ be sets and/or urelements of $\text{WF}(\mathcal{A}_\mathcal{N})$ and let $a$ be a set of $\text{WF}(\mathcal{A}_\mathcal{N})$ such that

$$\text{WF}(\mathcal{A}_\mathcal{N}) \models (\forall x \in a)\exists y \phi(x, y, \bar{v}).$$

Consider the formula $\theta(\beta, \bar{z})$ defined by

$$(\beta \text{ is an ordinal}) \land (\forall x \in a)(\exists \alpha \in \beta)\exists y (\rho(y) = \alpha \land \phi(x, y, \bar{v})).$$

Since $\text{WF}(\mathcal{A}_\mathcal{N}) \subseteq^P \mathcal{A}_\mathcal{N}$, if $\beta$ is a nonstandard ordinal of $\mathcal{A}_\mathcal{N}$, then $\mathcal{A}_\mathcal{N} \models \theta(\beta, \bar{v})$. Using $\Delta_0(\mathcal{L}^\mathcal{P})$-Collection, $\theta(\beta, \bar{z})$ is equivalent to a $\Sigma_1(\mathcal{L}^\mathcal{P})$-formula in $\mathcal{A}_\mathcal{N}$. Therefore, by $\Sigma_1(\mathcal{L}^\mathcal{P})$-Foundation, $\{ \beta \mid \mathcal{A}_\mathcal{N} \models \theta(\beta, \bar{v}) \}$ has a least element $\gamma$. Note that $\gamma$ is an ordinal of $\text{WF}(\mathcal{A}_\mathcal{N})$. Now, consider the formula $\psi(x, y, \bar{z}, \gamma)$ defined by

$$\phi(x, y, \bar{z}) \land (\rho(y) < \gamma).$$

Note that

$$\mathcal{A}_\mathcal{N} \models (\forall x \in a)\exists y \psi(x, y, \bar{v}, \gamma).$$

Using $\Delta_0(\mathcal{L}^\mathcal{P})$-Collection in $\mathcal{A}_\mathcal{N}$, there exists a set $b$ of $\mathcal{A}_\mathcal{N}$ such that

$$\mathcal{A}_\mathcal{N} \models (\forall x \in a)(\exists y \in b)\psi(x, y, \bar{v}, \gamma).$$

Let $c = \{ x \in b \mid \rho(x) < \gamma \}$, which is a set in $\mathcal{A}_\mathcal{N}$ by $\Delta_1(\mathcal{L}^\mathcal{P})$-Separation. Now, $c$ is a set of $\text{WF}(\mathcal{A}_\mathcal{N})$ and

$$\text{WF}(\mathcal{A}_\mathcal{N}) \models (\forall x \in a)(\exists y \in c)\phi(x, y, \bar{v}).$$

Therefore, $\text{WF}(\mathcal{A}_\mathcal{N})$ satisfies $\Delta_0(\mathcal{L}^\mathcal{P})$-Collection. And so, $\text{WF}(\mathcal{A}_\mathcal{N})$ is a power admissible set covering $\mathcal{N}$. Finally, since the $\mathcal{L}^*$-reduct of $\text{WF}(\mathcal{A}_\mathcal{N})$ is isomorphic to $\text{Cov}_\mathcal{M}$, $\text{WF}(\mathcal{A}_\mathcal{N})$ is isomorphic to $\text{Cov}_\mathcal{P}^\mathcal{M}$. $\square$

The following theorem summarises the analysis undertaken in this section:

**Theorem 4.9** If $\mathcal{M} \models \text{KP} + \text{Powerset} + \Delta_0^\mathcal{P}$-Collection + $\Sigma_1^\mathcal{P}$-Foundation, then there is an interpretation of $\mathcal{P}$ in $\text{Cov}_\mathcal{M}$ that yields the power admissible set $\text{Cov}_\mathcal{P}^\mathcal{M}$. $\square$

This yields a version of [Bar75, Corollary 2.4] that will be useful for the compactness arguments in the next section.

**Theorem 4.10** Let $\mathcal{M} = \langle \mathcal{M}, \mathcal{E}^\mathcal{M} \rangle \models \text{KP} + \text{Powerset} + \Delta_0^\mathcal{P}$-Collection + $\Sigma_1^\mathcal{P}$-Foundation. For all $A \subseteq \mathcal{M}$, there exists $a \in M$ such that $a^* = A$ if and only if $A \in \text{Cov}_\mathcal{P}^\mathcal{M}$. $\square$
5 End extension results

In this section we use the Barwise Compactness Theorem for $L_{e}^{	ext{ee}}_{\text{Cov}_{M}}$ to show that every countable model of $\text{KP} + \text{powerset} + \Delta_0^P$-Collection + $\Sigma_1^P$-Foundation has a powerset-preserving end extension.

The following is an immediate consequence of Theorem 4.10.

**Lemma 5.1** Let $\mathcal{M} = \langle M, E^M \rangle \models \text{KP} + \text{Powerset} + \Delta_0^P$-Collection + $\Sigma_1^P$-Foundation, and let $T_0$ be an $L_{e}^{	ext{ee}}_{\text{Cov}_{M}}$-theory. If $T_0 \in \text{Cov}_{M}^P$, then there exists $b \in M$ such that

$$b^* = \{a \in M \mid \bar{a} \text{ is mentioned in } T_0\}.$$ 

□

The next result expands on comments made in [Bar75, p. 637] and connects definability in $\mathcal{M}$ to definability in $\text{Cov}_{M}^P$.

**Lemma 5.2** Let $\mathcal{M} = \langle M, E^M \rangle \models \text{KP} + \text{Powerset} + \Delta_0^P$-Collection + $\Sigma_1^P$-Foundation, and let $\phi(\bar{z})$ be a $\Delta_0^P$-formula. Then there exists a formula $\hat{\phi}(\bar{z})$ that is $\Delta_1(L_{P}^\star)$ in the theory $\text{KPU}_{\text{Cov}}^P_L$ such that for all $\bar{z} \in M$,

$$\mathcal{M} \models \phi(\bar{z}) \iff \text{Cov}_{M}^P \models \hat{\phi}(\bar{z}).$$

**Proof** Let $\phi(\bar{z})$ be a $\Delta_0$-formula. We prove the lemma by structural induction on the complexity of $\phi$. Without loss of generality we can assume that the only connectives of propositional logic appearing in $\phi$ are $\neg$ and $\lor$. If $\phi(z_1, z_2)$ is $z_1 \in z_2$, then let $\phi(z_1, z_2)$ be the $\Delta_0(L_{P}^\star)$-formula $z_1 \in z_2$. Therefore, for all $z_1, z_2 \in M$,

$$\mathcal{M} \models \phi(z_1, z_2) \iff \text{Cov}_{M}^P \models \hat{\phi}(z_1, z_2).$$

If $\phi(\bar{z})$ is $\neg \psi(\bar{z})$ and the lemma holds for $\psi(\bar{z})$, then let $\hat{\phi}(\bar{z}) = \neg \hat{\psi}(\bar{z})$. So, $\hat{\phi}(\bar{z})$ is $\Delta_1(L_{P}^\star)$ in the theory $\text{KPU}_{\text{Cov}}^P$ and for all $\bar{z} \in M$,

$$\mathcal{M} \models \phi(\bar{z}) \iff \text{Cov}_{M}^P \models \hat{\phi}(\bar{z}).$$

Suppose that $\phi(\bar{z})$ is $\psi_1(\bar{z}) \lor \psi_2(\bar{z})$ and the lemma holds for $\psi_1(\bar{z})$ and $\psi_2(\bar{z})$. Let $\hat{\phi}(\bar{z})$ be $\psi_1(\bar{z}) \lor \psi_2(\bar{z})$. Therefore, $\hat{\phi}(\bar{z})$ is $\Delta_1(L_{P}^\star)$ in the theory $\text{KPU}_{\text{Cov}}^P$ and for all $\bar{z} \in M$,

$$\mathcal{M} \models \phi(\bar{z}) \iff \text{Cov}_{M}^P \models \hat{\phi}(\bar{z}).$$

Suppose $\phi(y, \bar{z})$ is $(\exists x \in y)\psi(x, y, \bar{z})$, where $\exists \in \{\exists, \forall\}$, and the lemma holds for $\psi(x, y, \bar{z})$. Let $\hat{\phi}(y, \bar{z})$ be $(\exists x \in F(y))\hat{\psi}(x, y, \bar{z})$. So, $\hat{\phi}(\bar{z})$ is $\Delta_1(L_{P}^\star)$ in the theory $\text{KPU}_{\text{Cov}}^P$. Since $\text{Cov}_{M}^P$ satisfies $(\dagger)$, for all $y, \bar{z} \in M$,

$$\mathcal{M} \models \phi(y, \bar{z}) \iff \mathcal{M} \models (\exists x \in y)\psi(x, y, \bar{z})$$

iff

$$\text{Cov}_{M}^P \models (\exists x \in F(y))\hat{\psi}(x, y, \bar{z})$$

iff

$$\text{Cov}_{M}^P \models \hat{\phi}(y, \bar{z}).$$
Suppose that \( \phi(y, \vec{z}) \) is \( (Q \subseteq y) \psi(x, y, \vec{z}) \), where \( Q \in \{\exists, \forall\} \), and the lemma holds for \( \psi(x, y, \vec{z}) \). Let \( \hat{\phi}(y, \vec{z}) \) be \( (Qx \in P(F(y))) \exists p(F(p) = x \land \hat{\psi}(p, y, \vec{z})) \). Note that, in the theory \( \text{KPU}_{\text{Cov}} \), for all urelements \( \vec{z} \),

\[
(Qx \in P(F(y))) \exists p(F(p) = x \land \hat{\psi}(p, y, \vec{z})) \iff \text{Cov}^P_M \models \hat{\phi}(y, \vec{z}).
\]

Therefore, in the theory \( \text{KPU}_{\text{Cov}} \), \( \hat{\phi}(y, \vec{z}) \) is \( \Delta^1_1(L^*P) \). Moreover, by Theorem 4.10 and \((\dagger)\), for all \( \vec{z} \in M \),

\[
M \models \hat{\phi}(y, \vec{z}) \iff (Qx \in P(F(y))) \exists p(F(p) = x \land \hat{\psi}(p, y, \vec{z})) \iff \text{Cov}^P_M \models \hat{\phi}(y, \vec{z}).
\]

Therefore, the lemma follows by induction. \( \square \)

We are now able to use the machinery we have developed to establish the following result.

**Theorem 5.3** Let \( S \) be a recursively enumerable \( L \)-theory such that

\[
S \vdash \text{KP + Powerset + } \Delta^P_0 \text{-Collection + } \Sigma^P_1 \text{-Foundation},
\]

and let \( M \) be a countable model of \( S \). Then there exists an \( L \)-structure \( N \) such that \( M \subseteq c \; N \models S \), and for some \( d \in N \), and for all \( x \in M \), \( N \models (x \in d) \).

**Proof** Let \( T \) be the \( L^\text{ee}_{\text{Cov}^P_M} \)-theory that contains:

- \( S \);
- for all \( a, b \in M \) with \( M \models (a \in b), \bar{a} \in \bar{b} \);
- for all \( a \in M \),
  \[
  \forall x (x \in \bar{a} \iff \bigvee_{b \in a} (x = \bar{b})) ;
  \]
- for all \( a \in M \),
  \[
  \forall x (x \subseteq \bar{a} \iff \bigvee_{b \subseteq a} (x = \bar{b})) ;
  \]
- for all \( a \in M \), \( \bar{a} \in c \).

Lemma 5.2 shows that \( T \subseteq \text{Cov}^P_M \) is \( \Sigma_1(L^*_P) \)-definable over \( \text{Cov}^P_M \). Let \( T_0 \subseteq T \) be such that \( T_0 \subseteq \text{Cov}^P_M \). Using Lemma 5.1 there exists \( c \in M \) such that

\[
c^* = \{a \in M \mid \bar{a} \text{ is mentioned in } T_0\}.
\]

Therefore, by interpreting each \( \bar{a} \) that is mentioned in \( T_0 \) by \( a \in M \) and interpreting \( c \) by \( c \), we can expand \( M \) to a model \( M' \) that satisfies \( T_0 \). Therefore, by the Barwise Compactness Theorem, there exists \( N \models T \). It is straightforward to see that the \( L \)-reduct of \( N \) is the desired extension of \( M \). \( \square \)
We first apply this result to show that countable models of KP have topless rank extensions that satisfy KP. This generalises [Frl Theorem 2.3], which shows that every countable transitive model of KP has a topless rank extension that satisfies KP.

**Theorem 5.4** (*Friedman*) Let $S$ be a recursively enumerable $\mathcal{L}$-theory such that $S \vdash KP$. If $M$ is a countable transitive model of $S$, then there exists $N \models S$ such that $M \subseteq^{\text{topless}} N$. 

It follows from [Gor. Theorem 4.8] that every countable nonstandard model of KP + $\Sigma^P_1$-Separation, $N$, is isomorphic to substructure $M$ of $N$ such that $M \subseteq^{\text{topless}} N$. We will make use of this result in the following form:

**Theorem 5.5** (*Gorbunov*) Let $M$ be a countable nonstandard model of KP + $\Sigma^P_1$-Separation. Then there exists $N \equiv M$ such that $M \subseteq^{\text{topless}} N$. 

We next note that if a model of KP has a blunt rank extension, then that model must satisfy the full scheme of separation.

**Lemma 5.6** Let $M = \langle M, E^M \rangle$ and $N = \langle N, E^N \rangle$ be models of KP. If $M \subseteq_{\text{blunt}} N$, then $M \models \Pi_{\infty}$-Separation.

**Proof** Assume that $M \subseteq N$, $E^M = E^N \upharpoonright M$ and $M \subseteq_{\text{blunt}} N$. Let $c \in N$ be such that $c^* \subseteq M$ and $c \notin M$. Working inside $N$, let $\alpha = \rho(c)$. Therefore, since $M \subseteq_{\text{blunt}} N$,

$$x \in (V^N_\alpha)^* \text{ if and only if } N \models (\rho(x) < \alpha)$$

$$\text{if and only if } N \models (\exists y \in c)(\rho(x) \leq \rho(y))$$

$$\text{if and only if } x \in M.$$ 

So, $M = (V^N_\alpha)^*$ and every instance of $\Pi_{\infty}$-Separation in $M$ can be reduced to an instance of $\Delta_0$-Separation in $N$ and, since $M \subseteq_{e} N$, the resulting set will be in $M$. Therefore, $M \models \Pi_{\infty}$-Separation. 

**Theorem 5.7** Let $S$ be a recursively enumerable $\mathcal{L}$-theory such that $S \vdash KP$. If $M$ is a countable model of $S$, then there exists $N \models S$ such that $M \subseteq_{\text{topless}}^{\text{topless}} N$.

**Proof** Let $M$ be a countable model of $S$. If $M$ is well-founded, then $M$ is isomorphic to a transitive model of KP and we can use Theorem 5.4 to find an $\mathcal{L}$-structure $N \models S$ such that $M \subseteq_{\text{topless}} N$. Therefore, assume that $M$ is nonstandard. By Corollary 5.6, $M$ satisfies KP + Powerset + $\Delta^P_0$-Collection + $\Sigma^P_1$-Foundation. Therefore, using Theorem 5.3 we can find an $\mathcal{L}$-structure $N \models S$ such that $N \models \Pi_{\infty}$-Separation. Therefore, since $M$ is nonstandard, we can apply Theorem 5.5 to obtain an $\mathcal{L}$-structure $N' \equiv M$ such that $M \subseteq_{\text{blunt}} N'$. 

We now turn to showing that every countable model of MOST + $\Pi_1$-Collection + $\Sigma^P_1$-Foundation has a topless powerset-preserving end extension that satisfies MOST + $\Pi_1$-Collection + $\Sigma^P_1$-Foundation. We first prove an analogue of Lemma 5.6 for models of MOST.
Lemma 5.8 Let $\mathcal{M} = \langle M, E^M \rangle$ and $\mathcal{N} = \langle N, E^N \rangle$ be models of MOST. If $\mathcal{M} \subseteq_{\text{blunt}} \mathcal{N}$, then $\mathcal{M} \models \Pi^1_{\infty}$-Separation.

Proof Assume that $M \subseteq N$, $E^M = E^N | M$ and $\mathcal{M} \subseteq_{\text{blunt}} \mathcal{N}$. Let $c \in N$ be such that $c^* \subseteq M$ and $c \notin M$. Work inside $\mathcal{N}$. Let $\kappa = |\text{TC}(c)|$ and note that $\kappa \notin M$. Consider

$$A = \{ \lambda \in \kappa \mid (\exists y \in c)(\lambda = |\text{TC}(y)|) \},$$

which is a set by $\Sigma_1$-Separation. Let $\mu = \sup A$ and note that $\mu$ is an initial ordinal. Work in the metatheory again. If $\mu \in M$, then so is $(\mu^+)\mathcal{N} = (\mu^+)\mathcal{M} \in M$. So, $H^N_{\mu^+} \in M$ and $\mathcal{N} \models (c \subseteq H_{\mu^+})$. And, $c \in M$, which is a contradiction. Therefore $\mu \notin M$. Now,

$$x \in (H^N_{\mu^+})^* \iff \mathcal{N} \models (|\text{TC}(x)| < \mu)$$

if and only if

$$\mathcal{N} \models (\exists y \in c)(|\text{TC}(x)| < |\text{TC}(y)|)$$

if and only if

$$x \in M.$$

So, $M = (H^N_{\mu^+})^*$ and every instance of $\Pi^1_{\infty}$-Separation in $\mathcal{M}$ can be reduced to an instance of $\Delta_0$-Separation in $\mathcal{N}$ and, since $\mathcal{M} \subseteq_{\text{p}} \mathcal{N}$, the resulting set will be in $\mathcal{M}$. Therefore, $\mathcal{M} \models \Pi^1_{\infty}$-Separation. \qed

Theorem 5.9 Let $S$ be an $\mathcal{L}$-theory such that $S \vdash$ MOST + $\Pi^1_1$-Collection + $\Sigma^P_1$-Foundation. If $\mathcal{M}$ is a countable model of $S$, then there exists a model $\mathcal{N}$ such that $\mathcal{M} \subseteq_{\text{p topless}} \mathcal{N} \models S$.

Proof This can be proved using an identical argument to the proof of Theorem 5.1 after observing that every transitive model of MOST + $\Pi^1_1$-Collection + $\Sigma^P_1$-Foundation is a model of KP and KP + $\Sigma^P_1$-Separation is a subtheory of MOST + $\Pi^1_1$-Collection + $\Pi^1_{\infty}$-Separation. \qed

The work [EKM] studies the class $\mathcal{C}$ of structures $\mathcal{I}_{\text{fix}(j)}$ where $j : \mathcal{M} \rightarrow \mathcal{M}$ is a nontrivial automorphism, $\mathcal{M}$ is an $\mathcal{L}$-structure that satisfies MOST, $j$ fixes every point in $(\omega^M)^*$ and $\mathcal{I}_{\text{fix}(j)}$ is the substructure of $\mathcal{M}$ that consists of elements $x$ of $\mathcal{M}$ such that $j$ fixes every point in $(\text{TC}^M(\{x\}))^*$. The results of [EKM, Section 3] show that every structure in $\mathcal{C}$ satisfies MOST + $\Pi^1_1$-Collection. Conversely, [EKM, Section 4] shows that a sufficient condition for a countable structure $\mathcal{M}$ that satisfies MOST + $\Pi^1_1$-Collection to be in $\mathcal{C}$ is that there exists $\mathcal{M} \subseteq_{\text{p topless}} \mathcal{N}$ such that $\mathcal{N}$ satisfies MOST + $\Pi^1_1$-Collection. Theorem 5.9 allows us to extend [EKM, Theorem B] by showing that $\mathcal{C}$ contains all countable models of MOST + $\Pi^1_1$-Collection + $\Sigma^P_1$-Foundation.

Theorem 5.10 Let $\mathcal{M} = \langle M, E^M \rangle$ be a countable model of MOST + $\Pi^1_1$-Collection + $\Sigma^P_1$-Foundation. Then there exists a model $\mathcal{N} = \langle N, E^N \rangle$ that satisfies MOST and a nontrivial automorphism $j : \mathcal{N} \rightarrow \mathcal{N}$ such that $\mathcal{M} \cong \mathcal{I}_{\text{fix}(j)}$, where $\mathcal{I}_{\text{fix}(j)}$ is the substructure of $\mathcal{N}$ with underlying set

$$\mathcal{I}_{\text{fix}(j)} = \{ x \in N \mid (\forall y \in \text{TC}^N(\{x\}))^*(j(y) = y) \}.$$

\qed

Combined with Corollary 5.12 and [EKM, Theorem 5.6] this shows that the class $\mathcal{C}$ contains every countable recursively saturated model of MOST + $\Pi^1_1$-Collection and every countable $\omega$-standard model of MOST + $\Pi^1_1$-Collection, providing a partial positive answer to Question 5.1 of [EKM]. A positive answer to the following question would positively answer Question 5.1 of [EKM]:
Question 5.11 Does every countable $\omega$-nonstandard model of MOST + $\Pi_1$-Collection have a topless powerset-preserving end extension that satisfies MOST + $\Pi_1$-Collection?

Note that [EKM Theorem 5.6] shows that this question has a positive answer when the countable model is recursively saturated.

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25