Stability of compressed rods when their stiffness changes according to the law of the fourth power

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Abstract. Based on the previously obtained representations for the state parameters of a rod with arbitrary continuous bending stiffness, the stability problem for a family of rods whose stiffness varies according to the law of the fourth power is solved. The spectrum of critical forces is determined and formulas for curved forms of equilibrium are derived. We propose to consider the stability coefficient as a function of the variable $\alpha$, and having a set of values of this function corresponding to the values of the independent variable $0 < \alpha \leq 1$, approximate the function $K$ by a polynomial. The result is the expression $K = 0.0001\alpha + 0.7626\alpha^2 + 0.0002\alpha^3$. The power of the polynomial was chosen from the condition that the coefficient of determination should not be less than 0.9999. The directions for further research are presented: introduction of the results of this work into the practice of calculations for stability of real objects; application of general formulas for state parameters to the study of stability of rods with other laws of change in transverse stiffness encountered in practice.

1. Introduction

In many structures, variable stiffness rods are used to reduce material consumption. These are power line supports, wind turbine poles, antenna supports, crane arms, drilling towers, trestle supports, chimneys, structures in aircraft construction, bridge construction, machine building, etc. The problem of determining critical loads and forms of stability loss for such structures has always been one of the most important ones in mechanics. In recent years, this problem has become even more topical due to the use of new materials for lightweight structures, for which the problem of stability loss is of major importance. Nevertheless, most works on this subject are based on the use of approximate methods. Analytical solutions are limited, which is explained by the difficulties encountered in integrating differential equations with variable coefficients, to which the mathematical side of the problem is reduced.

2. Literature review

Fundamental studies of the problem of the stability of compressed rods have been performed by many famous scientists, among whom we should mention the works of L. Euler, F. S. Yasinsky, A. N. Dinnik, S. P. Timoshenko, A. S. Volmir, A. R. Rzhanitsyn, and others.

The attempt to find the optimal form of a rod compressed by a longitudinal force belongs to J. L. Lagrange [1]. To find the maximum compressive critical force of a rod of circular cross-section at its smallest volume, he introduced a value called efficiency. This problem has not yet been solved in general form [2, 3].

The stability of solid rods of variable cross-section with a power law of stiffness change has been studied most completely by A. N. Dinnik [4], and numerical results of calculations for such rods are given in work [5]. A review of the most frequent laws of stiffness change for different variants of rod fastening under short- and long-term load action is made in [6, 7]. As for the development of
calculation methods, it should be noted the works [8 - 14], in which many interesting results were obtained, which served as a basis for further research.

In the works of recent years, the use of numerical methods prevails, and, more often, it is the method of finite elements with implementation in popular software packages. Thus, in [15], a finite-element analysis of a beam with a variable cross-section in the COSMOS/M program is performed. Stability and dynamics of a thin-walled composite rod are considered in [16]; the problem is solved by the Galerkin method. The same issues with respect to metal boring mandrels are investigated in [17] on the basis of a modified Adomian decomposition method with subsequent verification in ANSYS.

Stability studies based on exact solutions of the differential equation of the curved rod axis are extremely rare in the literature [18].

3. The purpose of this work is to determine the spectrum of critical forces and formulas for curved forms of equilibrium of an elastic homogeneous centrally compressed rod, rigidly clamped at the ends, when the moment of inertia of the cross-section changes according to the law of the fourth degree.

4. Materials and Methods
The methods of structural mechanics and the theory of stability of compressed rods are used. The mathematical side of the problem is provided by applying the theory of differential equations and the theory of series.

5. Results and Discussion
At Euler loss of stability, the critical force is determined from the differential equation of the bent axis of the rod [10]

\[(E(x)I(x)y''(x))'' + Ny''(x) = 0 ,\]  

where \(E(x)I(x)\) – переменная изгибная жесткость стержня в точке \(x\);
\(E(x)\) – модуль упругости материала стержня;
\(I(x)\) – момент инерции сечения;
\(N\) – постоянная сжимающая сила;
\(y(x)\) – неизвестная функция, представляющая поперечное смещение (деформацию) сечения стержня в точке \(x\) (рисунок 1).

In [19, 20] we obtained the exact solution of equation (1) in the case of arbitrary continuous transverse stiffness. Here we will assume that the moment of inertia changes according to the law of the fourth power

\[I(x) = I_0 \left(1 - (1 - \alpha) \frac{x}{l}\right)^4 ,\]

where \(I_0\) – момент инерции сечения стержня в точке \(x = 0\);
\(\alpha\) – a constant that satisfies the condition \(0 < \alpha \leq 1\).
The stress-strain state of the rod is fully characterized by the parameters: displacement $y(x)$, angle of rotation $\varphi(x)$, bending moment $M(x)$ and shear force $Q(x)$.

For the parameters we will use the representations from [19-21]. Before writing out these representations, note that due to the homogeneity of the rod $E(x) = E = \text{const}$. Also we will introduce a notation of $u = 1 - (1 - \alpha)x/l$. Then, as applied to this case, the above formulas in dimensionless format take the form:

$$y(x) = y(0) + \varphi(0)lx_x(x) - M(0)\frac{x}{N}(1 - X_x(x)) - Q(0)\frac{l}{N}\left(\frac{x}{l} - X_x(x)\right);$$  \hspace{1cm} (2)

$$\varphi(x) = \varphi(0)X_x(x) + M(0)\frac{1}{Nl}X_x(x) - Q(0)\frac{1}{N}\left(1 - X_x(x)\right);$$  \hspace{1cm} (3)

$$M(x) = \varphi(0)lX_x(x) + M(0)X_x(x) + Q(0)lX_x(x);$$  \hspace{1cm} (4)

$$Q(x) = Q(0);$$  \hspace{1cm} (5)

$$X_n(x) = \alpha_{n,0}(x) - K\alpha_{n,1}(x) + K^2\alpha_{n,2}(x) - K^3\alpha_{n,3}(x) + \ldots (n = 1, 2);$$  \hspace{1cm} (6)

$$\alpha_{n,0}(x) = \left(\frac{x}{l}\right)^{n-1}, \quad \alpha_{n,k}(x) = \frac{1}{l^2}\int_0^l \int_0^x \alpha_{n,k-1}(x)dxdu \quad (k = 1, 2, 3, \ldots);$$  \hspace{1cm} (7)

$$N = \frac{K}{E\frac{l^2}{l^2}}.$$  \hspace{1cm} (8)

Here $X_n(x), \alpha_{n,0}(x) = l\alpha_{n,0}(x) \quad (n = 1, 2)$ – dimensionless functions and $K$ – dimensionless oscillation factor to be determined.

Using a specially derived formula

$$\int \int_0^l \left(\frac{x}{u}\right)^i dxdu = \frac{1}{(i + 1)(i + 2)}u\left(\frac{x}{u}\right)^{i+2} \quad (i = 1, 2, 3, \ldots),$$  \hspace{1cm} (9)

integrals (7) can be calculated explicitly. The proof of this formula is beyond the scope of this paper. Let us point out only one of the possible variants of such a proof. By differentiating both parts of formula (9) twice, we arrive at an identity. This means that the left part of the formula is equal to the right part up to an arbitrary polynomial of the first degree. However, considering the fact that both these parts together with their first derivatives turn to zero when $x = 0$, we conclude that the above polynomial is an identical zero.

First, by a direct twofold integration we obtain

$$\alpha_{n,1}(x) = \frac{1}{l^2}\int_0^l \int_0^x \alpha_{n,0}(x)dxdu = u\left(\frac{1}{2!}\left(\frac{x}{lu}\right)^2 + \frac{1 - \alpha}{3!}\left(\frac{x}{lu}\right)^3\right).$$

Then, using formula (9), we successively find:

$$\alpha_{n,k}(x) = \frac{1}{l^2}\int_0^l \int_0^x \alpha_{n,k-1}(x)dxdu = u\left(\frac{1}{(2k)!}\left(\frac{x}{lu}\right)^{2k} + \frac{1 - \alpha}{(2k + 1)!}\left(\frac{x}{lu}\right)^{2k+1}\right) \quad (k = 2, 3, 4, \ldots);$$

$$\alpha_{n,k}(x) = \frac{1}{l^2}\int_0^l \int_0^x \alpha_{n,2,k-1}(x)dxdu = \frac{1}{(2k + 1)!}u\left(\frac{x}{lu}\right)^{2k+1} \quad (k = 1, 2, 3, \ldots).$$

According to formula (6) we will now have:

$$X_n(x) = u\sum_{k=0}^\infty (-1)^k K^k \left(\frac{x}{lu}\right)^{2k} + \frac{1 - \alpha}{(2k + 1)!}\left(\frac{x}{lu}\right)^{2k+1}\right) = u\left(\cos K\frac{x}{lu} + \frac{1 - \alpha}{\sqrt{K}}\sin K\frac{x}{lu}\right);$$  \hspace{1cm} (10)
\[ X_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} K^k \left( \frac{x}{lu} \right)^{2k+1} = \frac{u}{\sqrt{K}} \sin \sqrt{K} \frac{x}{lu}. \]  

(11)

Using the known functions \( X_n(x) \) \((n=1,2)\), we find:

\[ \hat{X}_1(x) = (1-\alpha)^2 \frac{x}{lu} \cos \sqrt{K} \frac{x}{lu} - \left( \frac{1-\alpha}{\sqrt{K}} + \frac{1}{u} \right) \sin \sqrt{K} \frac{x}{lu}; \]

(12)

\[ \hat{X}_2(x,K) = \frac{1}{u} \cos \sqrt{K} \frac{x}{lu} - \frac{1-\alpha}{\sqrt{K}} \sin \sqrt{K} \frac{x}{lu}. \]

(13)

To the case of both fixed ends corresponds the boundary conditions: \( y(0) = 0; \ \varphi(0) = 0; \ y(l) = 0; \ \varphi(l) = 0. \) Realization of these conditions with formulas (2), (3) leads to the system of equations

\[
\begin{align*}
(1-X_1(l))M(0)+l(1-X_2(l))Q(0) &= 0; \\
\frac{X_1(l)}{l}M(0) - (1-\hat{X}_2(l))Q(0) &= 0.
\end{align*}
\]

(14)

For nontrivial coincidence of the system, its definition should be equal to zero. Hence, given equations (10)-(13), we arrive at the characteristic equation

\[ 2 - 2 \cos \frac{\sqrt{K}}{\alpha} - \frac{\sqrt{K}}{\alpha} \sin \frac{\sqrt{K}}{\alpha} = 0. \]

After elementary trigonometric transformations, this equation is equivalent to the set of two equations:

\[
\begin{align*}
\sin \frac{\sqrt{K}}{2\alpha} &= 0; \\
\cos \frac{\sqrt{K}}{2\alpha} - \frac{\sqrt{K}}{2\alpha} \cos \frac{\sqrt{K}}{2\alpha} &= 0.
\end{align*}
\]

(15) (16)

We obtain the set of stability coefficients by combining the roots of each of these equations into a single set and arranging them in ascending order. As the calculations show, in the specified ordered set the roots of equation (15) are located at odd numbers, and the roots of equation (16) are located at even numbers.

Since equation (15) is elementary, the stability coefficients with odd numbers are determined exactly: \( K_1 = (2\alpha\pi)^2, \ K_3 = (4\alpha\pi)^2, \ldots \) Consequently, by formula (8) the critical forces with odd indices are also determined exactly

\[ N_1 = (2\alpha\pi)^2 \frac{EI_0}{l^4}, \ N_3 = (4\alpha\pi)^2 \frac{EI_0}{l^4}, \ldots \]

The transcendental equation (16) is used to find stability coefficients with even numbers \( K_2, K_4, \ldots \). Finding the roots of such equation is not difficult. For this purpose there are many different software possibilities. Let us restrict ourselves here to the root of \( K_2 \). Its values corresponding to each value \( K_2 \) in increments of 0.1 are given in Table 1.

### Table 1. Values of stability coefficients.

| \( \alpha \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |  \\
|---|---|---|---|---|---|---|---|---|---|---|
| \( K_2 \) | 0.8076 | 3.2305 | 7.2687 | 12.922 | 20.191 | 29.075 | 39.574 | 51.688 | 65.418 | 80.763 |
It is quite clear that for any other values $\alpha$ it is possible to find stability coefficients. In order to do this without solving the characteristic equation, the idea arises to express the found table dependence between the parameters $K$ and $\alpha$ analytically as well.

Considering the stability coefficient $K$ as a function of the variable $\alpha$, and having a set of values of this function corresponding to the values of the independent variable $0 < \alpha \leq 1$, we approximate the function $K$ by a polynomial. As a result, we obtain $K_2 = 0.0001 \alpha + 80.762 \alpha^2 + 0.0002 \alpha^3$. The polynomial degree was chosen from the condition that the coefficient of determination should not be less than 0.9999. Therefore,

$$N_2 = (0.0001 \alpha + 80.762 \alpha^2 + 0.0002 \alpha^3) \frac{E I_0}{l^2}.$$ 

It remains to write out formulas for the curved forms of equilibrium of the rod. For this purpose let us transform the formula for displacements (2) to the form

$$y(x) = -\frac{M(0)}{N} \left[ 1 - u \left( \cos \frac{\sqrt{K}}{l} x + \frac{1 - \alpha}{\sqrt{K}} \sin \frac{\sqrt{K}}{l} x \right) - \eta \left( \frac{x}{l} - \frac{u}{\sqrt{K}} \sin \frac{\sqrt{K}}{l} x \right) \right], \quad (17)$$

where $\eta = -Q(0)/M(0)$ dimensionless parameter to be determined.

From the equations of the system (14) we find

$$\eta = \frac{1 - X_1(l)}{1 - X_2(l)} = \frac{1 - \alpha}{1 - \alpha \sin \frac{\sqrt{K}}{\alpha}},$$

After transformations we arrive at the formula

$$\eta = \frac{1 + \tan^2 \frac{\sqrt{K}}{2 \alpha} - \alpha \left( 1 + \tan^2 \frac{\sqrt{K}}{2 \alpha} + \frac{2(1 - \alpha)}{\sqrt{K}} \frac{\tan \frac{\sqrt{K}}{2 \alpha}}{2} \right)}{1 + \tan^2 \frac{\sqrt{K}}{2 \alpha} - \frac{2 \alpha}{\sqrt{K}} \frac{\tan \frac{\sqrt{K}}{2 \alpha}}{2}}. \quad (18)$$

If $K$ – the root of equation (15), then $\tan \left( \frac{\sqrt{K}}{2 \alpha} \right) = 0$, and if $K$ – the root of equation (16), then $\tan \left( \frac{\sqrt{K}}{2 \alpha} \right) = 0$. Substituting the found tangent values into formula (18), we find that for stability coefficients with odd indices $\eta = 1 - \alpha$, and for stability coefficients with even indices $\eta = 1 + \alpha$.

For the curved forms of equilibrium we take the representation

$$y_j(x) = C_j Y_j \left( \frac{x}{l} \right) \left( j = 1, 2, 3, \ldots \right),$$

where $C_j$ – constant size multiplier; $Y_j \left( \frac{x}{l} \right)$ – dimensionless function defining the law of the curved form of equilibrium.

Then, on the basis of formula (17) we obtain:

$$C_j = -\frac{M_j(0) l^2}{E I_0} \left( j = 1, 2, 3, \ldots \right).$$
\[ Y_j \left( \frac{x}{l} \right) = \begin{cases} \frac{u}{((j+1)\alpha)^2} \left( 1 - \cos(j+1)\alpha \pi \frac{x}{lu} \right) & (j = 1, 3, \ldots) \\ \frac{1}{K_j} \left[ 1 - (1 + \alpha) \frac{x}{l} - u \left( \cos \frac{K_j}{lu} \frac{x}{l} - \frac{2\alpha}{\sqrt{K_j}} \sin \frac{K_j}{lu} \frac{x}{l} \right) \right] & (j = 2, 4, \ldots). \end{cases} \]

If necessary, on the basis of representations (3) - (5) it is easy to write formulas for other parameters of the rod state corresponding to the considered case.

6. Conclusions

Based on the previously obtained representations for the state parameters of a rod with arbitrary continuous bending stiffness, the stability problem for a family of rods whose stiffness varies according to the law of the fourth degree is solved. The spectrum of critical forces is determined and formulas for curved forms of equilibrium are derived.

We propose to consider the stability coefficient as a function of the variable \( \alpha \), and having a set of values of this function corresponding to the values of the independent variable \( 0 < \alpha \leq 1 \), approximate the function \( K \) by a polynomial. As a result, we obtained the expression \( K_2 = 0.0001\alpha + 80.7626\alpha^2 + 0.0002\alpha^3 \). The degree of the polynomial was chosen from the condition that the coefficient of determination should not be less than 0.9999.

The following directions of activity seem to be promising: introduction of the results of this work into the practice of calculations for stability of real objects; application of general formulas for state parameters to the study of stability of rods with other laws of change in transverse stiffness encountered in practice.

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