Conic Nonholonomic Constraints on Surfaces and Control Systems

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Abstract
This paper addresses the equivalence problem of conic submanifolds in the tangent bundle of a smooth 2-dimensional manifold. Those are given by a quadratic relation between the velocities and are treated as nonholonomic constraints whose admissible curves are trajectories of the corresponding control systems, called quadratic systems. We deal with the problem of characterising and classifying conic submanifolds under the prism of feedback equivalence of control systems, both control-affine and fully nonlinear. The first main result of this work is a complete description of non-degenerate conic submanifolds via a characterisation under feedback transformations of the novel class of quadratic control-affine systems. This characterisation can explicitly be tested on structure functions defined for any control-affine system and gives a normal form of quadratisable systems and of conic submanifolds. Then, we consider the classification problem of regular conic submanifolds (ellipses, hyperbolas, and parabolas), which is treated via feedback classification of quadratic control-nonlinear systems. Our classification includes several normal forms of quadratic systems (in particular, normal forms not containing functional parameters as well as those containing neither functional nor real parameters) and, as a consequence, gives a classification of regular conic submanifolds.

Keywords Nonlinear control system · Feedback equivalence · Conic submanifolds · Nonholonomic constraint · Normal forms · Pseudo-Riemannian geometry

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1 Introduction

Let $\mathcal{X}$ be a smooth connected manifold of dimension $n = 2$ (a surface), equipped with local coordinates $x$. In the tangent bundle $T\mathcal{X}$ of $\mathcal{X}$, we consider a smooth 3-dimensional submanifold $\mathcal{S}$, a hypersurface, given by

$$\mathcal{S} = \{(x, \dot{x}) \in T\mathcal{X}, \ S(x, \dot{x}) = 0\},$$

where $S : T\mathcal{X} \to \mathbb{R}$ is a smooth scalar function satisfying $\text{rk} \frac{\partial S}{\partial x}(x, \dot{x}) = 1$ for all $(x, \dot{x}) \in \mathcal{S}$. Two submanifolds $\mathcal{S} \subset T\mathcal{X}$ and $\tilde{\mathcal{S}} \subset T\tilde{\mathcal{X}}$ are said to be equivalent if there exists a diffeomorphism $\phi : \mathcal{X} \to \tilde{\mathcal{X}}$ such that $\phi(S) = \tilde{\mathcal{S}}$, where the tangent map $\phi_* : T\mathcal{X} \to T\tilde{\mathcal{X}}$ is given by $\phi_* (x, \dot{x}) = (\phi(x), D\phi(x)\dot{x})$ and $D\phi$ is the derivative of $\phi$. If $S = \{S(x, \dot{x}) = 0\}$ and $\tilde{S} = \{\tilde{S}(\tilde{x}, \tilde{\dot{x}}) = 0\}$, then the above definition amounts to saying that there exists a smooth nonvanishing scalar function $\delta : T\mathcal{X} \to \mathbb{R}$ such that

$$\tilde{S}(\phi(x), D\phi(x)\dot{x}) = \delta(x, \dot{x}) S(x, \dot{x}).$$

The last equality implies that for all $(x, \dot{x}) \in \mathcal{S}$, we have $(\tilde{x}, \tilde{\dot{x}}) = (\phi(x), D\phi(x)\dot{x}) \in \tilde{\mathcal{S}}$; hence, the map $\phi_*$ sends the graph of $S^{-1}(0)$ into that of $\tilde{S}^{-1}(0)$. Equivalence of submanifolds $\mathcal{S} = \{S(x, \dot{x}) = 0\}$ and $\tilde{S} = \{\tilde{S}(\tilde{x}, \tilde{\dot{x}}) = 0\}$ means simply that the implicit underdetermined ordinary differential equations $S(x, \dot{x}) = 0$ and $\tilde{S}(\tilde{x}, \tilde{\dot{x}}) = 0$ are equivalent; see e.g. [4, Definition 2].

It is natural to ask how to characterise and classify submanifolds $\mathcal{S} \subset T\mathcal{X}$ (with $\mathcal{X}$ and $\mathcal{S}$ of arbitrary dimensions) of certain particular classes, for instance the class of linear submanifolds given by $S_{\text{lin}}(x, \dot{x}) = \omega(x)\dot{x} = 0$ or the class of affine submanifolds given by $S_{\text{aff}}(x, \dot{x}) = \omega(x)\dot{x} + h(x) = 0$, where $\omega$ is a smooth vector-valued differential 1-form on $\mathcal{X}$ and $h$ is a smooth vector-valued function on $\mathcal{X}$. Those questions have been widely studied under the prism of Pfaffian equations (linear and affine) and go back to Pfaff, Darboux, Cartan [8, 9, 20]. Although the problem of classification of Pfaffian equations is still open in its full generality, many important results have been obtained for various classes of linear Pfaffian equations (contact and quasi-contact case, Martinet case, singularities, see [11, 14, 18, 31, 32]) and of affine Pfaffian equations (dimension two [12], three [21, 22], and arbitrary [11, 33]).

We will call $S_q$ a quadratic, or a conic, submanifold of $T\mathcal{X}$ if it is given by $S_q = \{S_q(x, \dot{x}) = 0\}$, where $S_q : T\mathcal{X} \to \mathbb{R}$ is a quadratic map of the form

$$S_q(x, \dot{x}) = \dot{x}^T g(x) \dot{x} + 2 \omega(x) \dot{x} + h(x),$$

with all involved objects being smooth, i.e. for each point $x \in \mathcal{X}$, the set $S_q(x)$ forms a conic curve in $T_x \mathcal{X}$ (recall that we consider the case of $\mathcal{X}$ being a surface). If $S_q$ is a conic submanifold, then so is any submanifold equivalent to it. Indeed, if $S_q = \{S(x, \dot{x}) = 0\}$, then $\tilde{S} = \tilde{S}(\tilde{x}, \tilde{\dot{x}}) = 0$, where $\tilde{S}(\tilde{x}, \tilde{\dot{x}}) = S_q(\phi^{-1}(\tilde{x}), D\phi^{-1}(\tilde{x})\tilde{\dot{x}})$ is quadratic because $D\phi$ acts linearly on fibres. Moreover, we actually have $\tilde{S}(\tilde{x}, \tilde{\dot{x}}) = \tilde{\dot{x}}^T \tilde{g}(\tilde{x}) + 2 \tilde{\omega}(\tilde{x}) \tilde{\dot{x}} + \tilde{h}$ with $g = \phi^* \tilde{g}$, $\omega = \phi^* \tilde{\omega}$, and $h = \phi^* \tilde{h}$. On the other hand, a map $S$ defining a quadratic submanifold $S_q = \{S(x, \dot{x}) = 0\}$ need not be quadratic because $S(x, \dot{x})$ is of the form $\delta(x, \dot{x}) S_q(x, \dot{x})$ and, in general, the latter will not be quadratic (unless $\delta$ depends on $x$ only); see Example 2.1 below. The first purpose of this work is to provide a local characterisation of submanifolds $\mathcal{S}$ that are equivalent to quadratic submanifolds $S_q$. In particular, we will identify submanifolds that define, in each $T_x \mathcal{X}$, elliptic, hyperbolic, and parabolic conics, given respectively by

$$S_E = \{a^2(\dot{z} - c_0)^2 + b^2(\dot{y} - c_1)^2 = 1\}, \quad S_H = \{a^2(\dot{z} - c_0)^2 - b^2(\dot{y} - c_1)^2 = 1\},$$

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and \( S_P = \{ a\dot{y}^2 - \dot{z} + b\dot{y} + c = 0 \} \),

where \( a, b, c_0, c_1 \), and \( c \) are smooth functions of \( x = (z, y) \), satisfying \( a(\cdot) \neq 0 \) and \( b(\cdot) \neq 0 \) in the elliptic and hyperbolic cases and \( a(\cdot) \neq 0 \) in the parabolic case; we choose the order \((z, y)\) to be consistent with some normal forms existing in the literature (e.g. see [1], where quadratic submanifolds are used as canonical models which admit a parabolic Lie group of symmetries). We will also discuss the case of passing smoothly from one type to another. We call them elliptic, hyperbolic, and parabolic submanifolds, and to state and discuss general facts about them, we set \( S_Q = \{ S_E, S_H, S_P \} \).

The second goal of this work is to provide a classification of elliptic \( S_E \), hyperbolic \( S_H \), and parabolic \( S_P \) submanifolds. In the elliptic and hyperbolic cases, we will first describe the submanifolds with the functions \( a(x) = b(x) \) and call them conformally-flat; second, if, additionally, \( a = b = 1 \), then we call them flat elliptic or hyperbolic submanifolds. Finally, we will describe the forms for which we also have \( c_0, c_1 \in \mathbb{R} \) (in the same coordinate system as the one for which we have \( a = b = 1 \)), called a constant-form elliptic and hyperbolic submanifolds; the particular case of \( c_0 = c_1 = 0 \) is called a null-form to emphasise the absence of any (functional, continuous, or discrete) parameter. In the parabolic case, we will describe the submanifolds with \( a = 1 \), called weakly-flat, and characterise those with, additionally, \( b = 0 \) (called strongly-flat) and, moreover, with \( c \in \mathbb{R} \) (called constant-form); in the particular case where \( c = 0 \), we call the parabolic submanifold a null-form. Our classification is summarised by Table 1 below.

Our analysis of the equivalence problem of submanifolds will be based on attaching to a submanifold \( S = \{ S(x, \dot{x}) = 0 \} \subset T\mathcal{X} \) two control systems. First,

\[
\mathcal{E}_S : \dot{x} = F(x, w), \quad x \in \mathcal{X}, \quad w \in W \subset \mathbb{R},
\]

where \( \dot{x} - F(x, w) = 0 \) is a regular parametric representation of the submanifold \( S \), that is, for all values of the parameter \( w \in W \) (interpreted as a scalar control), we have \( S(x, F(x, w)) = 0 \) and \( \text{rk} \frac{\partial F}{\partial w}(x, w) = 1 \), and second,

\[
\Sigma_S : \begin{cases} 
\dot{x} = F(x, w), \\
\dot{w} = u
\end{cases}, \quad (x, w) \in \mathcal{X} \times W, \quad u \in \mathbb{R},
\]

called, respectively, a first and second extension of \( S \). Notice that for \( \mathcal{E}_S \), the control \( w \) enters in a nonlinear way, whereas for \( \Sigma_S \), the control \( u \) enters in an affine way, but for the price of augmenting the dimension of the state space. Observe that since \( S(x, \dot{x}) = 0 \), defining \( S \), relates the positions \( x \) with the velocities \( \dot{x} \), it describes a nonholonomic constraint. We say that a smooth curve \( x(t) \in \mathcal{X} \) satisfies the nonholonomic constraint given by \( S \) if we have \( (x(t), \dot{x}(t)) \in S \). Clearly, \( x(t) \) satisfies the nonholonomic constraint described by \( S \) (equivalently, satisfies the implicit differential equation \( S(x(t), \dot{x}(t)) = 0 \)) if and only if \( x(t) \) is a trajectory of \( \mathcal{E}_S \) for a certain smooth control \( w(t) \) or, equivalently, \( (x(t), w(t)) \) is

| Table 1 Classification of elliptic, hyperbolic, and parabolic submanifolds |
|---------------------------------------------------------------|
| Elliptic and hyperbolic submanifolds classification | Parabolic submanifolds classification |
|---------------------------------------------------------------|
| \( a = b \) | Conformally-flat | \( a = 1 \) | Weakly-flat |
| \( a = b = 1 \) | Flat | \( a = 1, b = 0 \) | Strongly-flat |
| \( a = b = 1, (c_0, c_1) \in \mathbb{R}^2 \) | Constant-form | \( a = 1, b = 0, c \in \mathbb{R} \) | Constant-form |
| \( a = b = 1, c_0 = c_1 = 0 \) | Null-form | \( a = 1, b = 0, \text{and } c = 0 \) | Null-form |
a trajectory of $\Sigma_S$ for a smooth control $u(t)$. A crucial observation that links studying submanifolds $S \subset TX$ and their extensions $\Xi_S$ and $\Sigma_S$ is that the equivalence of submanifolds corresponds to the equivalence of control systems $\Xi_S$ and $\Sigma_S$ via feedback transformations, general for $\Xi_S$ and control-affine for $\Sigma_S$, as ensured by Proposition 2.1.

Organisation of the paper. In the next section, we will recall some definitions of control theory and we will show that the problem of characterising and classifying submanifolds of $TX$ can be replaced by that of characterising and classifying their first and second extensions $\Xi_S$ and $\Sigma_S$ under feedback transformations (see Proposition 2.1). Moreover, we will give a first rough classification of non-degenerate conic submanifolds, introducing elliptic $S_E$, hyperbolic $S_H$, and parabolic $S_P$ subclasses (see Lemma 2.1). In Sect. 3, we will define a general second extension of a conic submanifold $S_q$, called a quadratic system $\Sigma_q$. Just as we do for conic submanifolds, we will identify elliptic $\Sigma_E$, hyperbolic $\Sigma_H$, and parabolic $\Sigma_P$ systems as particular cases of $\Sigma_q$, see Definition 3.1 and Proposition 3.1. In Theorem 3.1, we will fully characterise the class of quadratic systems by means of a checkable relation between well-defined structure functions attached to any control-affine system. The conditions obtained in that theorem allow to give a normal form for all quadratisable control-affine systems (i.e. systems $\Sigma$ that are feedback equivalent to $\Sigma_q$), see Theorem 3.2, which in turn leads to a normal form for all non-degenerate conic submanifolds $S_q$. We will also show how our characterisation and the normal form apply to the classes of elliptic, hyperbolic, and parabolic control-affine systems, which gives us a deeper insight into our conditions and in our normal form (see Corollary 3.1 and Corollary 3.2). Finally, in Sect. 4, we will be interested in the classification of elliptic, hyperbolic, and parabolic submanifolds as presented in Table 1. This problem is dealt with using the classification of their first extensions (treated as control-nonlinear systems) under feedback transformations. We first show that the classification of elliptic, hyperbolic, and parabolic submanifolds presented in Table 1 is reflected by properties of a triple of vector fields attached to their parametrisations; see Lemma 4.1 statements (i) to (iv) for the elliptic and hyperbolic cases and statements (v) to (viii) for the parabolic case. To every quadratic control-nonlinear system $\Xi_E$, $\Xi_H$, or $\Xi_P$ (first extensions of elliptic, hyperbolic, and parabolic submanifolds, respectively), we will attach a frame of the tangent bundle (see the paragraph before Proposition 4.1) and we give conditions for that frame to be commutative: it turns out that in the elliptic and hyperbolic cases, this requires that a certain pseudo-Riemannian metric is flat (see Proposition 4.3), whereas in the parabolic case, this problem can be solved without any extra assumptions (see Proposition 4.5). Then, we show how we can additionally normalise the systems while preserving the commutativity of that frame. Our classifications include several normal and canonical forms, given by Proposition 4.4 for elliptic and hyperbolic systems and by Theorem 4.1 for parabolic systems. We summarise the structure of the paper in Fig. 1.

Related works. A classification of quadratic control systems was initiated by Bonnard in [5]. His work differs from our as he considered homogeneous systems of degree 2 with respect to all state variables. Hence, his class of quadratic control systems is a subclass of our parabolic systems (where we require that only one variable enters quadratically), but he considers the general dimension $n$, while our results concern 3-dimensional systems only. In [16], Krener and Kang studied the problem of equivalence, via feedback, to polynomial systems of degree 2 modulo higher order terms. This work was continued in [15, 28] for any degree, but all those results are given for formal classification only. In [13], particularly in Example 2.2, Jakubczyk deals with a general elliptic system and studies its microlocal equivalence via symbols of its critical Hamiltonian. Examples of control systems subject to conic nonholonomic constraints appear in various domains of physics and engineering applications. In the next section, we will discuss Dubins’ car [10] which is a simple model
of a vehicle, as well as its hyperbolic counterpart [19]. In [23], the same elliptic model, as that of Dubins’ car, is studied to minimise the energy of its trajectories, which is the famous Euler’s elastica problem. We also mention [34], where the planar tilting manoeuvre problem is considered under small angle assumption, the studied control system is elliptic with respect to the states.

2 Preliminaries

Main notations.

\( \mathcal{X}, T \mathcal{X}, x = (z, y) \) A smooth 2d manifold, its tangent bundle, and its local coordinates.

\( \mathcal{M}, \xi = (x, w) = (z, y, w) \) A smooth 3d manifold and its local coordinates.

\( \phi, \phi_* \) A diffeomorphism and its tangent map.

\( S \) Smooth submanifold of \( T \mathcal{X} \) given by an equation of the form 
\[ S(x, \dot{x}) = 0, \text{ with } \text{rk} \frac{\partial S}{\partial \dot{x}}(x, \dot{x}) = 1. \]

\( S_q \) Quadratic (conic) submanifold described by 
\[ S_q(x, \dot{x}) = \dot{x}^T g(x) \dot{x} + 2\omega(x) \dot{x} + h(x) = 0. \]

\( S_Q = \{ S_E, S_H, S_P \} \) Set of elliptic, hyperbolic, and parabolic submanifolds.

\( \Sigma_S \) Regular parametrisation of a submanifold \( S \), called a first extension, seen as a control-nonlinear system.

\( \Sigma_S \) Extension of \( \Sigma_S \), called a second extension of a submanifold \( S \), seen as a control-affine system on a three dimensional manifold \( \mathcal{M} \).

In this section, we introduce all tools and concepts that we will need in this paper. First, we define in full generality the notions of submanifolds of the tangent bundle and of control-nonlinear systems. Second, we recall some notions on the equivalence of submanifolds (of the tangent bundle) and of control systems. Third, most importantly, we show that the problem of equivalence of submanifolds can be replaced by that of equivalence of their first and second extensions, see Proposition 2.1. Finally, we present quadratic submanifolds, and we discuss the subclasses of elliptic, hyperbolic, and parabolic submanifolds. In the paper, the word
smooth will always mean $C^\infty$-smooth, and throughout all systems, functions, manifolds and submanifolds are assumed to be smooth.

**Basic notions.** We introduce basic notions following the approach proposed in [27], see also [7]. Let $X$ be a smooth $n$-dimensional manifold. A *field of admissible velocities* or a *velocities constraint* on $X$ is a fibred manifold $\pi_S : S \to X$, where $S$ is an $(m+n)$-dimensional submanifold of $TX$, and thus, the following diagram is commutative

$$
\begin{array}{c}
S \\
\pi_S
\end{array}
\xymatrix{
& TX \\
\pi_T X & \ar_{\pi} X}
$$

In the diagram, the map $\pi_S$ is a surjective submersion, $\pi_{T,X}$ is the canonical projection, and $\iota : S \hookrightarrow TX$ is the inclusion map attaching to any $s \in S$ the same point $s$ considered as a point of $TX$. Denote by $S_x = \pi_S^{-1}(x) \subset T_x X$ the fibers of $S$. Any $S_x$ is an $m$-dimensional submanifold of $T_x X$ that consists of all velocities admissible at $x \in X$. If a field of admissible velocities $\pi_S : S \to X$ is a fiber bundle, with a typical fiber being a manifold $W$ of dimension $m$ (in particular, if $S = X \times W$), then the sets $S_x$ of admissible velocities at any $x \in X$ are diffeomorphic to each other and they all are diffeomorphic to $W$. Notice, however, that we do not assume $S$ to be a fiber bundle (just a fibred manifold only). In this case, the fibers $S_x$ need not be diffeomorphic meaning that the sets of admissible velocities $S_x$ may change completely when passing from one $x \in X$ to another. We will see this phenomenon to appear for quadratic constraints.

A *control system* $Ξ = (F, \pi_V)$ on $X$ consists of a fibred manifold $\pi_V : V \to X$, called the control bundle, where $\pi_V$ is a surjective submersion, and a map $F : V \to TX$ such that the following diagram is commutative:

$$
\begin{array}{c}
V \\
\pi_V
\end{array}
\xymatrix{
& TX \\
\pi_{T,X} & \ar_{\pi} X}
$$

where $\pi_{T,X}$ denotes the canonical projection. If the control bundle $\pi_V : V \to X$ is the Cartesian product $V = X \times W$ of the state space manifold $X$ and the control space manifold $W$ of dimension $m$, then the control system $Ξ$ takes the usual form

$$
Ξ : \dot{x} = F(x, w), \ x \in X, \ w \in W.
$$

If $V$ is a nontrivial fibred manifold, then $Ξ$ takes the form $\dot{x} = F(x, w)$ in local coordinates $(x, w)$ adapted to the fibred manifold structure of $V$.

There is a perfect correspondence between fields of admissible velocities and regular control systems. First, to any field of admissible velocities $\pi_S : S \to X$, we can associate a control system. Namely, we define the control bundle $\pi_V : V \to X$ by choosing $V = S$, $\pi_V = \pi_S$, and $F = \iota$. If the inclusion $\iota$ is an injective immersion (resp. an embedding), then $F = \iota$ is, obviously, an injective immersion (resp. embedding). Conversely, to any control system $(F, \pi_V)$, we can define point-wise the set of admissible velocities at $x \in X$ by $S_x = F(\pi_V^{-1}(x))$. If $\pi_V : V \to X$ is a fiber bundle, with a typical fiber being a manifold $W$ (in particular, if $V = X \times W$), then $S_x = F(x, W)$. If for the control system $Ξ$ the map $F$ is an injective immersion (resp. an embedding), then $S$, defined point-wise by $S_x$, is an immersed (resp. embedded) submanifold of $TX$ and $S_x$ are immersed (resp. embedded).
submanifolds of the corresponding \( T_x\mathcal{X} \). The reason is that the fibres \( \pi_\mathcal{V}^{-1}(x) \) are embedded submanifolds because \( \pi_\mathcal{V} : \mathcal{V} \to \mathcal{X} \) is a fibred manifold.

**Equivalence notions.** From now on, we will assume that \( \mathcal{X} \) is a surface, that is \( n = 2 \), and that the fibred submanifold \( S \), defining a velocities constraint, is a hypersurface in \( T\mathcal{X} \). Observe, however, that the notions introduced below, and in particular Proposition 2.1, are also valid in the general dimension setting. In this paper, we will be mostly working locally and, in all local statements, we will add the adjective “local” instead of using the notion of germs (which would be more proper but less frequently used in control theory). In local coordinates \((x, \dot{x})\), a submanifold \( S \) of \( T\mathcal{X} \) can be expressed as the zero level set \( S = \{ S(x, \dot{x}) = 0 \} \) of a local map \( S : T\mathcal{X} \to \mathbb{R} \) defined in a neighbourhood of \((x_0, \dot{x}_0)\). Since \( S \) is a fibred manifold, it follows that \( \text{rk} \frac{\partial S}{\partial x}(x, \dot{x}) = 1 \), which we thus assume throughout.

We say that two velocities constraints \( \pi_S : S \to \mathcal{X} \) and \( \pi_{\tilde{S}} : \tilde{S} \to \tilde{\mathcal{X}} \) are (locally) equivalent if there exists a (local) diffeomorphism \( \phi : \mathcal{X} \to \tilde{\mathcal{X}} \) whose tangent map \( \phi_* : T\mathcal{X} \to T\tilde{\mathcal{X}} \) sends \( S \) onto \( \tilde{S} \), that is \( \phi_*(S) = \tilde{S} \). We say that two maps \( S : T\mathcal{X} \to \mathbb{R} \) and \( \tilde{S} : T\tilde{\mathcal{X}} \to \mathbb{R} \) are locally \( V \)-equivalent at \((x_0, \dot{x}_0)\) and \((\tilde{x}_0, \dot{\tilde{x}}_0)\), respectively, if there exists a local diffeomorphism \( \phi : \mathcal{X} \to \tilde{\mathcal{X}} \), satisfying \( \phi_*(x_0, \dot{x}_0) = (\tilde{x}_0, \dot{\tilde{x}}_0) \), and a non-vanishing function \( \delta : T\mathcal{X} \to \mathbb{R}^* \) such that

\[
\tilde{S}(\phi(x)), \quad D\phi(x)\dot{x} = \delta(x, \dot{x})S(x, \dot{x})
\]

holds locally around \((x_0, \dot{x}_0)\). This notion is a natural adaptation (to the case of maps defined on \( T\mathcal{X} \)) of the notion of \( V \)-equivalence (\( V \) referring to a variety) of germs of maps \( (\mathbb{R}^q, 0) \to (\mathbb{R}^p, 0) \) as defined in [2]. The following three examples illustrate the notion of \( V \)-equivalence of maps and its correspondence with the equivalence of submanifolds (that correspondence will be proven with full generality in Proposition 2.1 below).

**Example 2.1** The submanifolds \( S \) and \( \tilde{S} \) given by \( S(x, \dot{x}) = \dot{z} - (1 + \sqrt{1 + \dot{y}})^2 = 0 \), around \((x_0, \dot{x}_0) = (0, 0)\), and \( \tilde{S}(\tilde{x}, \dot{\tilde{x}}) = \dot{\tilde{z}} - (\frac{\dot{\tilde{y}}}{\tilde{y}})^2 = 0 \), around \((\tilde{x}_0, \dot{\tilde{x}}_0) = (0, 0)\), respectively, are locally equivalent by the diffeomorphism \( \phi(x) = (z, y - z) \) because the maps \( S \) and \( \tilde{S} \) are locally \( V \)-equivalent via

\[
(\tilde{z}, \tilde{y}) = \phi(x) = (z, y - z) \quad \text{and} \quad \delta(x, \dot{x}) = -\frac{1}{4} \left( \dot{z} - \dot{\tilde{y}} - 2 - 2\sqrt{1 + \dot{y}} \right).
\]

**Example 2.2** Since on a 2-dimensional manifold all metrics are locally conformally flat (see [26, Addendum 1 of Chapter 9] for the Riemannian case and [25, Theorem 7.2] for the Lorentzian one), it follows that the submanifold given by \( S(x, \dot{x}) = a(x)\dot{z}^2 + 2b(x)\dot{z}\dot{y} + c(x)y^2 - 1 = 0 \) (where \( ac - b^2 \neq 0 \)) is locally equivalent to that given by \( \tilde{S}(\tilde{x}, \dot{\tilde{x}}) = \left( \dot{\tilde{z}}^2 \pm \frac{c(x)}{a(x)}y^2 \right) - r(\tilde{x})^2 = 0 \).

**Example 2.3** Since \( \text{rk} \frac{\partial S}{\partial x}(x, \dot{x}) = 1 \), assume that \( \frac{\partial S}{\partial z}(x_0, \dot{x}_0) \neq 0 \); recall that \( x = (z, y) \). By the implicit function theorem, we can locally write \( S(x, \dot{x}) = \delta(x, \dot{x}) (\dot{z} - s(x, \dot{y})) \), where \( \delta(x_0, \dot{x}_0) \neq 0 \). So, a submanifold \( S \) is always locally equivalent to the one given by \( \dot{z} - s(x, \dot{y}) = 0 \).

Two control systems \( \Xi = (F, \pi_\mathcal{V}) \) and \( \tilde{\Xi} = (\tilde{F}, \pi_{\tilde{\mathcal{V}}} \) are feedback equivalent if there exists a diffeomorphism \( \phi : \mathcal{X} \to \tilde{\mathcal{X}} \) and a fiber preserving lift \( \Phi : \mathcal{V} \to \tilde{\mathcal{V}} \) of \( \phi \), i.e. \( \phi_*(\pi_\mathcal{V}) = \pi_{\tilde{\mathcal{V}}} \Phi \), such that

\[
D\phi(x)F(v) = \tilde{F}(\Phi(v)),
\]
for any \( v \in \mathcal{V} \). Namely, the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\pi_{\mathcal{V}}} & \mathcal{V} \\
\downarrow \phi & & \downarrow \phi \\
\mathcal{X} & \xrightarrow{\pi_{\tilde{\mathcal{V}}}} & \tilde{\mathcal{V}}
\end{array}
\quad
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{F} & T\mathcal{X} \\
\uparrow \phi & & \uparrow \phi \\
\tilde{\mathcal{V}} & \xrightarrow{\tilde{F}} & T\tilde{\mathcal{X}}
\end{array}
\]

Locally, in adapted coordinates \((x, w)\) of \(\mathcal{V}\), the fiber preserving lift \(\phi\) takes the form \(\Phi(x, w) = (\phi(x), \psi(x, w))\) and (1) becomes the usual feedback equivalence

\[D\phi(x)F(x, w) = \tilde{F}(\phi(x), \psi(x, w)).\]

The map \(\phi\) plays the role of a coordinates change in the state space \(\mathcal{X}\), and the map \(\psi\) is called a feedback transformation as it changes the parametrisation by control \(w\) in a way that depends on the state \(x\). If the diffeomorphism \(\Phi\) is defined in a neighbourhood of \((x_0, w_0)\) only, and \(\Phi(x_0, w_0) = (\tilde{x}_0, \tilde{w}_0)\), then we say that \(\mathcal{X}_{/X_1}\) satisfies \(\text{rk} \partial w (x, \tilde{w}_0) = 0\), and \(\mathcal{X}_{/X_1}\) is \(\text{locally feedback equivalent at } (x_0, w_0)\) (and \(\tilde{x}_0, \tilde{w}_0\), respectively). If \(\pi_{\mathcal{V}} : \mathcal{V} \to \mathcal{X}\) is a fiber bundle, with a typical fiber \(\mathcal{W}\), and the diffeomorphism \(\Phi\) is defined on the product \(\mathcal{X}_0 \times \mathcal{W}\), that is, it is global with respect to \(w\), where \(\mathcal{X}_0\) is a neighbourhood of a state \(x_0\), and \(\phi(x_0) = \tilde{x}_0\), then we say that \(\mathcal{X}\) and \(\tilde{\mathcal{X}}\) are locally feedback equivalent at \(x_0\) and \(\tilde{x}_0\), respectively. The latter local feedback equivalence will be especially useful for the class of control-affine systems

\[\Sigma : \dot{\xi} = f(\xi) + g(\xi)u, \quad u \in \mathbb{R},\]

where \(\xi \in \mathcal{M}\), \(f\) and \(g\) are smooth vector fields on \(\mathcal{M}\); below the state space \(\mathcal{M}\) will be of dimension 3 so \(\mathcal{M}\) should not be confused with \(\mathcal{X}\), which is of dimension 2. For control-affine systems, we will restrict the feedback transformations to the control-affine transformations

\[\psi(\xi, u) = \alpha(\xi) + \beta(\xi)u,\]

where \(\alpha(\xi)\) and \(\beta(\xi)\) are smooth functions satisfying \(\beta(\cdot) \neq 0\). In that case, we denote the feedback transformation by the triple \((\phi, \alpha, \beta)\), and if \(\phi = \text{id}\), then this action is called a pure feedback transformation and is denoted \((\alpha, \beta)\). Observe that a general control system of the form \(\mathcal{X} : \dot{x} = F(x, w)\), where \((x, w) \in \mathcal{X} \times \mathcal{W}\), can be extended to a control-affine system \(\mathcal{X}^e\) by augmenting the state space with the control \(w\) and introducing the new control \(u = \dot{w}\), which gives

\[\mathcal{X}^e : \begin{cases} 
\dot{x} = F(x, w), \\
\dot{w} = u,
\end{cases} \quad u \in \mathbb{R}.
\]

Notice that \(\mathcal{X}^e\) lives on the manifold \(\mathcal{M} = \mathcal{X} \times \mathcal{W}\) of dimension \(n = 3\).

Recall that \(\mathcal{S} = \{S(x, \dot{x}) = 0\}\) is a fibre hypersurface of \(T\mathcal{X}\), and that \(\mathcal{S} : T\mathcal{X} \rightarrow \mathbb{R}\) satisfies \(\text{ rk } \frac{\partial S}{\partial x}(x, \dot{x}) = 1\) for all \((x, \dot{x}) \in S\). Therefore, as already discussed in the Introduction, we can locally attach to \(\mathcal{S}\) its regular parametrisation \(\dot{x} - F(x, w) = 0\) satisfying \(S(x, F(x, w)) = 0\), thus defining a control-nonlinear system \(\mathcal{Z}_S : \dot{x} = F(x, w)\), for which \(\text{rk } \frac{\partial F}{\partial w}(x, w) = 1\). Since constructing \(\mathcal{Z}_S\) requires to introduce an extra variable \(w\) (a control), we will also call it a first extension of \(\mathcal{S}\). Then, we can attach to \(\mathcal{Z}_S\) its extension \(\mathcal{Z}_S^e\), called also a prolongation, denoted \(\Sigma_S^e\) and given by

\[\Sigma_S^e : \begin{cases} 
\dot{x} = F(x, w), \\
\dot{w} = u,
\end{cases} \quad u \in \mathbb{R},
\]

and called a second extension of \(\mathcal{S}\). To distinguish different control systems attached to the submanifolds \(\mathcal{S}\) and \(\tilde{\mathcal{S}}\), we will denote \(\mathcal{Z}_S\) (resp. \(\Sigma_S^e\)) by \(\mathcal{Z}_S^e\) (resp. by \(\Sigma_S^e\)). The following
Indeed, if \( \partial \) extension, \( \text{rk} \), equivalent via a diffeomorphism \( F \) will prove that and thus \( \partial \) \( \{ \) argument will strongly rely on the regularity of the submanifold, \( \text{rk} \) (of the state space) to analyse both first extensions in the same coordinate system. Then, equivalence under feedback of their first extensions, we will first use a diffeomorphism \( S \) \( \Rightarrow \) \( \{ \) equivalence under feedback transformations of their corresponding first and second extensions. Proof The proof of (i) \( \Leftrightarrow \) (ii) follows immediately from [2, Section 6.5]. (ii) \( \Rightarrow \) (iii). To show that the \( V \)-equivalence of maps (defining submanifolds) implies the equivalence under feedback of their first extensions, we will first use a diffeomorphism (of the state space) to analyse both first extensions in the same coordinate system. Then, we will show that the parameters (controls) of those two first extensions are related by a pure feedback transformation. To obtain invertibility of the feedback transformation, our argument will strongly rely on the regularity of the submanifold, \( \text{rk} \) \( \frac{\partial S}{\partial x} = 1 \), and of its first extension, \( \text{rk} \) \( \frac{\partial F}{\partial w} = 1 \). Assume that the maps \( S : T X \rightarrow \mathbb{R} \) and \( \tilde{S} : T \tilde{X} \rightarrow \mathbb{R} \) are \( V \)-equivalent via a diffeomorphism \( \tilde{x} = \phi(x) \) and a nonvanishing function \( \delta(x, \dot{x}) \), that is, \( \tilde{S}(\phi(x), D\phi(x)\dot{x}) = \delta(x, \dot{x})S(x, \dot{x}) \). Consider \( \Xi_S : \dot{x} = F(x, w) \) and \( \tilde{\Xi}_S : \dot{\tilde{x}} = \tilde{F}(\tilde{x}, \tilde{w}) \), two regular parametrisations of the corresponding submanifolds \( S \) and \( \tilde{S} \). Then, using \( 0 = \tilde{S}(\tilde{x}, \dot{\tilde{x}}) = \tilde{S}(\tilde{x}, \tilde{F}(\tilde{x}, \tilde{w})) \), we have

\[
\tilde{S}(\phi(x), \tilde{F}(\phi(x), \tilde{w})) = \delta(x, (D\phi(x))^{-1} \tilde{F}(\phi(x), \tilde{w}))S(x, (D\phi(x))^{-1} \tilde{F}(\phi(x), \tilde{w})), \quad \forall \tilde{w} \in \tilde{W},
\]

implying \( S(x, \tilde{F}(x, \tilde{w})) = 0 \), where \( \tilde{F}(x, \tilde{w}) = (D\phi(x))^{-1} \tilde{F}(\phi(x), \tilde{w}) \). Therefore, \( \dot{x} = F(x, w) \) and \( \dot{\tilde{x}} = \tilde{F}(\tilde{x}, \tilde{w}) \) are two regular parametrisations of the same submanifold \( S \). We will prove that \( F(x, w) \) and \( \tilde{F}(x, \tilde{w}) \) are related by an invertible (pure) feedback transformation of the form \( \tilde{w} = \psi(x, w) \).

Since \( \text{rk} \) \( \frac{\partial S}{\partial x}(x_0, \dot{x}_0) = 1 \), we may assume that \( \frac{\partial S}{\partial x}(x_0, \dot{x}_0) \neq 0 \), where \( x = (z, y) \) and, by the implicit function theorem, we have \( S(x, \dot{x}) = \delta(x, \dot{x})(\dot{z} - s(x, \dot{y})) = 0 \). Let \( \dot{z} = F_1(x, w) \), \( \dot{y} = F_2(x, w) \) be a regular parametrisation of \( S \), it follows that \( \frac{\partial F_2}{\partial w}(x_0, w_0) \neq 0 \). Indeed, if \( \frac{\partial F_2}{\partial w}(x_0, w_0) = 0 \), then \( \dot{z} - s(x, \dot{y}) = 0 \) (recall that \( \delta \neq 0 \)), hence \( F_1 - s(x, F_2) = 0 \) and thus \( \frac{\partial F_1}{\partial w} - \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial w} = 0 \) implying that \( \frac{\partial F_1}{\partial w}(x_0, w_0) = 0 \); contradiction.

Hence, the two regular parametrisations of \( S = \{ S(x, \dot{x}) = 0 \} \) given by

\[
\Xi_S : \begin{cases} 
\dot{z} = F_1(x, w) \\
\dot{y} = F_2(x, w)
\end{cases} \quad \text{and} \quad \tilde{\Xi}_S : \begin{cases} 
\dot{\tilde{z}} = \tilde{F}_1(x, \tilde{w}) \\
\dot{\tilde{y}} = \tilde{F}_2(x, \tilde{w})
\end{cases}
\]
satisfy \( \frac{\partial F_2}{\partial w}(x_0, w_0) \neq 0 \) and \( \frac{\partial \hat{F}_2}{\partial w}(x_0, \hat{w}_0) \neq 0 \). Therefore, \( \hat{w} = \hat{F}_2^{-1}(x, \hat{y}) = \tilde{F}_2^{-1}(x, F_2(x, w)) \), where \( \tilde{F}_2^{-1} \) is the inverse with respect to the second argument. And using \( \hat{z} = s(x, \hat{y}) \), we obtain \( \hat{z} = \hat{F}_1(x, w) = s(x, \hat{y}) = \hat{F}_1(x, \hat{w}) \). Thus, \( \Xi_S \) and \( \tilde{\Xi}_S \) are feedback equivalent via \( \hat{w} = \psi(x, w) \) with \( \psi(x, w) = \hat{F}_2^{-1}(x, F_2(x, w)) \), and the systems \( \Xi_S \) and \( \tilde{\Xi}_S \) are feedback equivalent since \( \tilde{\Xi}_S \) is the system \( \Xi_S \) mapped via the diffeomorphism \( \tilde{x} = \phi(x) \).

\[(iii) \Rightarrow (ii)\] Assume that the two regular parametrisations \( \Xi_S : \dot{x} = F(x, w) \) and \( \tilde{\Xi}_S : \dot{\tilde{x}} = \tilde{F}(\tilde{x}, \tilde{w}) \) of \( S \) and \( \tilde{S} \), respectively, are feedback equivalent via \( \tilde{x} = \phi(x) \) and \( \tilde{w} = \psi(x, w) \). Denote \( \tilde{\phi} = \phi^{-1} \) and apply the diffeomorphism \( x = \tilde{\phi}(\tilde{x}) \) to \( \tilde{\Xi}_S \) to obtain a new vector field, parametrised by \( \tilde{w}, \tilde{F}(x, w) = D\tilde{\phi}(\tilde{x}))\tilde{F}(\tilde{x}, \tilde{w}) \) related to \( F(x, w) \) by the pure feedback transformation \( \tilde{w} = \psi(x, w) \). Denote \( F = (F_1, F_2)' \) and \( \tilde{F} = (\hat{F}_1, \hat{F}_2)' \) in the \( x = (z, y) \) coordinates. Without loss of generality, we can assume that \( \frac{\partial \hat{F}_2}{\partial w}(x_0, w_0) \neq 0 \) and \( \frac{\partial \hat{F}_2}{\partial w}(x_0, w_0) \neq 0 \).

Now, apply to \( \tilde{S} = \{ \tilde{S}(\tilde{x}, \tilde{w}) = 0 \} \) the same diffeomorphism \( x = \tilde{\phi}(\tilde{x}) \), whose inverse is denoted by \( \tilde{x} = \phi(x) \), and set \( \tilde{S}(\tilde{x}, \tilde{w}) = \tilde{S}(\phi(x), D\phi(x)\tilde{x}) \). Since by definition of \( \tilde{\Xi}_S \) we have \( \tilde{S}(\tilde{x}, \tilde{F}(\tilde{x}, \tilde{w})) = 0 \), we conclude that \( \tilde{S}(x, \tilde{F}(x, \tilde{w})) = 0 \). Then, we claim that \( \frac{\partial \tilde{S}}{\partial \tilde{x}}(x_0, \tilde{x}_0) \neq 0 \); indeed, if \( \frac{\partial \tilde{S}}{\partial \tilde{x}}(x_0, \tilde{x}_0) = 0 \), then \( \frac{\partial \tilde{S}}{\partial \tilde{x}}(x, F(x, w)) = 0 \) yields \( \frac{\partial \tilde{S}}{\partial \tilde{x}} \frac{\partial F_1}{\partial \tilde{w}} = 0 \) and thus \( \frac{\partial \tilde{S}}{\partial \tilde{x}}(x_0, \tilde{x}_0) = 0 \) giving a contradiction. The same observation holds for \( \frac{\partial \tilde{S}}{\partial \tilde{x}}(x_0, \tilde{x}_0) \neq 0 \). Thus, by the implicit function theorem, we have \( S(x, \tilde{x}) = \delta(x, \hat{x})(\tilde{z} - s(x, \tilde{y})) \) and \( \tilde{S}(\tilde{x}, \tilde{w}) = \hat{S}(\phi(x), D\phi(x)\tilde{x}) \).

Using \( \tilde{z} = s(x, \tilde{y}) \) and \( \hat{z} = \hat{s}(x, \hat{y}) \), we obtain \( s(x, y) = F_1(x, w) = \hat{F}_1(x, \hat{w}) = \hat{s}(x, \hat{y}) \); hence, we have
\[
S(x, \tilde{x}) = \delta(x, \hat{x})(\tilde{z} - s(x, \tilde{y})) = \delta(x, \hat{x})(\hat{z} - \hat{s}(x, \hat{y})) = \frac{\delta}{\delta} \tilde{s}(\phi(x), D\phi(x)\tilde{x})
\]
establishing the \( V \)-equivalence between \( S \) and \( \tilde{\Sigma} \).

\[(iii) \Rightarrow (iv)\] If \( \Xi_S \) and \( \tilde{\Xi}_S \) are feedback equivalent, then
\[
D\phi(x)F(x, w) = \tilde{F}(\phi(x), \psi(x, w)).
\]

Thus, the diffeomorphism \( (\phi(x), \psi(x, w)) \), of the augmented state space \( (x, w) \), together with the control-affine feedback
\[
\tilde{\tilde{u}} = \frac{\partial \psi}{\partial x}F(x, w) + \frac{\partial \psi}{\partial w}u,
\]
transform \( \Sigma_S \) into \( \tilde{\Xi}_S \).

\[(iv) \Rightarrow (iii)\] Assume that \( \Sigma_S \) and \( \tilde{\Xi}_S \) are feedback equivalent via \( (\tilde{x}, \tilde{w}) = \varphi(x, w) \) and \( \tilde{\tilde{u}} = \alpha(x, w) + \beta(x, w)u \). Since the distribution span \( \{ \frac{\partial \varphi}{\partial w} \} \) is sent by \( \varphi_* \) into span \( \{ \frac{\partial \psi}{\partial w} \} \), it follows that \( \varphi \) has the triangular form \( \varphi(x, w) = (\phi(x), \psi(x, w)) \). Therefore, feedback equivalence of the systems \( \Xi_S \) and \( \tilde{\Xi}_S \) is established via the diffeomorphism \( \tilde{x} = \phi(x) \) and the reparametrisation \( \tilde{w} = \psi(x, w) \).

\[\square\]

**Remark 2.1** The use of extensions of control-nonlinear systems was introduced in [29] and used to study controllability and observability of nonlinear systems, and then to analyse linearisability and decoupling [30]. Moreover, notice that the same proof as that of \((iii) \Leftrightarrow (iv)\)
shows that any two control-nonlinear systems $\Xi$ and $\tilde{\Xi}$ (which need not be regular parametrisations of submanifolds) are feedback equivalent if and only if their extensions $\Xi^e$ and $\tilde{\Xi}^e$ are equivalent via control-affine feedback; see [12].

**Remark 2.2** The equivalence (ii)$\iff$(iii) does not hold, in general, if the parametrisation $\Xi_S : \tilde{\xi} = F(x, w)$ does not satisfy the regularity condition $\frac{\partial F}{\partial w}(x, w) \neq 0$ that we assume. To see that, consider the submanifold $S$ given by $\tilde{\xi} = F(x, w)$ and $\tilde{\Xi}_S : \tilde{\xi} = \tilde{\xi}_0$. The parametrisations of $S$

$$\Xi_S : \begin{cases} \tilde{\xi} = u^2 \\
\tilde{\eta} = w \end{cases} \text{ and } \tilde{\Xi}_S : \begin{cases} \tilde{\xi} = \tilde{w}^6 \\
\tilde{\eta} = \tilde{w}^3 \end{cases}$$

are not feedback equivalent around $w_0 = 0$, and the reason is that $\tilde{\Xi}_S$ fails to satisfy $\frac{\partial F}{\partial w}(\tilde{w}_0) \neq 0$ at $\tilde{w}_0 = 0$.

According to the previous proposition, in order to deal with the problem of equivalence of submanifolds of the tangent bundle $T\mathcal{X}$, it is interesting, first, to study the problem of equivalence of general control-nonlinear systems $\Xi$ and, second, the problem of equivalence of general control-affine systems $\Sigma$.

**Conic submanifolds.** A map $S : T\mathcal{X} \rightarrow \mathbb{R}$ will be called quadratic if it is of the form

$$S_q(x, \tilde{\xi}) = \tilde{\xi}^t g(x) \tilde{\xi} + 2 \omega(x) \tilde{\xi} + h(x),$$

that is, $S_q$ is a smooth polynomial of degree two in $\tilde{\xi}$. A velocities constraint $S$ is called quadratic (or conic) if there exists a quadratic map $S_q$ such that $S = \{S_q(x, \tilde{\xi}) = 0\}$. As we already observed in the Introduction, if $S$ is quadratic, then any $\tilde{S}$, equivalent to $S$, is also quadratic. On the other hand, if a map $S_q$ is quadratic, then a map $\tilde{S}$ that is $V$-equivalent to $S_q$ need not be quadratic (although they define equivalent quadratic velocities constraints).

The map $S_q$ can be represented by the triple $S_q = (g, \omega, h)$, where $g$ is a symmetric $(0, 2)$-tensor (possibly degenerated), $\omega$ is a one-form, and $h$ is a function. Clearly, two conics $S_q$ of $T\mathcal{X}$ and $\tilde{S}_q$ of $T\mathcal{X}$, given by $(g, \omega, h)$ and $(\tilde{g}, \tilde{\omega}, \tilde{h})$, respectively, are equivalent if and only if there exists a diffeomorphism $\tilde{\xi} = \phi(x)$ and a non-vanishing function $\delta = \delta(x)$ on $\mathcal{X}$ such that $\delta g = \phi^* \tilde{g}$, $\delta \omega = \phi^* \tilde{\omega}$, and $\delta h = \phi^* \tilde{h}$. In particular, observe that the tensors $g$ and $\tilde{g}$, which are (pseudo-)Riemannian metrics (possibly degenerated), are conformally equivalent.

It is well-known in affine geometry that such conic equations can be classified by the signature of the matrix $M_q(x) = \begin{pmatrix} g(x) & \omega(x) \\ \omega(x) & h(x) \end{pmatrix}$ and that of $g(x)$. We will use the following two determinants

$$\Delta_1(x) = \det(M_q(x)) \quad \text{and} \quad \Delta_2(x) = \det(g(x)).$$

Of course, $\Delta_1$ and $\Delta_2$ depend on the choice of coordinates; however, since a diffeomorphism $\tilde{\xi} = \phi(x)$ transforms them as $\Delta_i = \theta^2 \phi^*(\Delta_i)$, for $i = 1, 2$, where $\theta(x) = \det D\phi(x)$, their zero-level sets $\{\Delta_i(x) = 0\}$ are invariantly related to the submanifold $S_q$. In this work, we will characterise non-degenerate conics, that is, non-empty and satisfying $\Delta_1 \neq 0$; notice that non-empty conics at points of degeneration $\Delta_1(x) = 0$ form in fact (pairs of) linear subspaces of $T_x\mathcal{X}$. Excluding empty $S_q$ is needed when considering elliptic submanifolds (see lemma below), and it implies that $M_q$ is indefinite. The non-degeneracy assumption $\Delta_1(x) \neq 0$ implies that $\frac{\partial S_q}{\partial \tilde{\xi}}(x, \tilde{\xi}) \neq 0$ (but the converse does not hold in general) and $\text{rk} g(x) \geq 1$.

If $\Delta_2(x_0) \neq 0$ or if $\Delta_2 \equiv 0$ in a neighbourhood of $x_0$, then we can describe three particular types of conics given by the classification lemma below. Notice, however, that this lemma
Lemma 2.1 (Classification of non-degenerate conics) Consider a nonempty conic \( S_q \), given by \((g, \omega, h)\), and assume \( \Delta_1(x_0) \neq 0 \). Then, locally around \( x_0 \), we have

(i) If \( \Delta_2(x_0) > 0 \), then \( S_q \) is equivalent to \( S_E = \{a^2(\dot{z} - c_0)^2 + b^2(\dot{y} - c_1)^2 = 1\} \),

(ii) If \( \Delta_2(x_0) < 0 \), then \( S_q \) is equivalent to \( S_H = \{a^2(\dot{z} - c_0)^2 - b^2(\dot{y} - c_1)^2 = 1\} \),

(iii) If \( \Delta_2 \equiv 0 \), then \( S_q \) is equivalent to \( S_P = \{a\dot{y}^2 - \dot{z} + b\dot{y} + c = 0\} \),

where \( a, b, c_0, c_1, \) and \( c \) are smooth functions satisfying \( a \neq 0 \) and \( b \neq 0 \) in the elliptic and hyperbolic cases, and \( a \neq 0 \) in the parabolic case.

We call \( S_E \), resp. \( S_H \), resp. \( S_P \), an elliptic, resp. a hyperbolic, resp. a parabolic, submanifold, and we will use the notation \( S_q \) to denote the set \( \{S_E, S_H, S_P\} \) of those three particular forms. Observe that for the parabolic form \( S_P \), the non-degeneracy assumption \( \Delta_1 \neq 0 \) implies \( \text{rk } g = 1 \) and the existence of a nonvanishing one-form \( \omega = -dz + b dy \) satisfying \( \omega \neq \text{ann (ker } g) \), whereas in the elliptic and hyperbolic cases, this one-form, given by \( \omega = -(c_0 dz \pm c_1 dy) \), can vanish at some points. Those three classes of submanifolds are related to the signature of the metric \( g \); indeed if \( \text{sgn } (g) \) is constant in a neighbourhood of \( x_0 \), then \( S_E \), resp. \( S_H \), resp. \( S_P \), corresponds to \( \text{sgn } (g) = (+, +) \), resp. \( \text{sgn } (g) = (+, -) \), resp. \( \text{sgn } (g) = (+, 0) \); notice that we can always assume that there is at least one positive eigenvalue, otherwise take the equivalent submanifold defined by \( \bar{S} = -S \).

**Proof** Consider a submanifold \( S_q \) given in local coordinates by \( S_q(x, \dot{x}) = 0 \), with

\[
S_q(x, \dot{x}) = \dot{x}^t \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \dot{x} + 2(\omega_1, \omega_2) \dot{x} + h,
\]

where all functions \( g_{ij}, \omega_1, \omega_2, \) and \( h \) depend smoothly on \( x \in \mathcal{X} \).

(i) – (ii) We deal with \( \Delta_2(x_0) \neq 0 \), that is, the elliptic and hyperbolic cases together. In those cases, \( g \) is a non-degenerate symmetric \((0, 2)\)-tensor; therefore, it can be interpreted as a pseudo-Riemannian metric. Since on 2-dimensional manifolds all metrics are smoothly diagonalisable (actually, all metrics are conformally flat; see [26, Addendum 1 of Chapter 9] for the Riemannian case and [25, Theorem 7.2] for the Lorentzian one), introduce coordinates \( \tilde{x} = \phi(x) = (z, y) \) such that in those coordinates, \( S_q \) can be written (we drop the tildes for more readability) as

\[
S_q = \lambda_1^2(z)^2 \pm \lambda_2^2(y)^2 + 2\omega(x) \dot{x} + h(x)
\]

or, equivalently (since \( \lambda_1(\cdot) \neq 0 \) and \( \lambda_2(\cdot) \neq 0 \), as

\[
S_q = \lambda_1^2 \left( \dot{z} + \frac{\omega_1}{\lambda_1} \right)^2 \pm \lambda_2^2 \left( \dot{y} + \frac{\omega_2}{\lambda_2} \right)^2 + h - \lambda_1^2 \left( \frac{\omega_1}{\lambda_1} \right)^2 \mp \lambda_2^2 \left( \frac{\omega_2}{\lambda_2} \right)^2.
\]

Notice that for this form, we have \( \Delta_1 = \pm \lambda_1^2 \lambda_2^2 \left[ h - \frac{(\omega_1)^2}{\lambda_1^2} \mp \frac{(\omega_2)^2}{\lambda_2^2} \right] \), which, by our assumption, does not vanish. Denote \( c_0 = -\frac{\omega_1}{\lambda_1}, c_1 = \mp \frac{\omega_2}{\lambda_2} \), and divide by \( h = -\lambda_1^2 \left( \frac{\omega_1}{\lambda_1} \right)^2 \pm \lambda_2^2 \left( \frac{\omega_2}{\lambda_2} \right)^2 \), observe that \( \tilde{h} \neq 0 \) as \( \tilde{h} = \mp \frac{1}{\lambda_1^2 \lambda_2^2} \Delta_1 \), to obtain

\[
S_q = \frac{\lambda_1 \lambda_2}{\pm \Delta_1} (\dot{z} - c_0)^2 \pm \frac{\lambda_2^2 \lambda_4}{\pm \Delta_1} (\dot{y} - c_1)^2 - 1.
\]
where the upper, resp. lower, sign corresponds to the elliptic, resp. hyperbolic, case. In the elliptic case, if $\Delta_1 > 0$, the conic is empty which is excluded by assumption; therefore, $\Delta_1 < 0$ and set $a^2 = \frac{\lambda_1^2}{-\Delta_1}$ and $b^2 = \frac{\lambda_2^2}{-\Delta_1}$ to obtain $S_E$. In the hyperbolic case, if $\Delta_1 < 0$, then permute the variables $(z, y)$, and we can thus always obtain a conic defined by $S_q$ in the above form with $\Delta_1 > 0$. Then, set $a^2 = \frac{\lambda_1^2}{\Delta_2}$ and $b^2 = \frac{\lambda_2^2}{\Delta_2}$ to obtain $S_H$.

(iii) Assume $\Delta_2 \equiv 0$. Since $\Delta_1 \neq 0$, we have $\text{rk } g(x) = 1$ in a neighbourhood of $x_0$, which implies that $g_{11}(x_0)g_{22}(x_0) \neq 0$, and thus, without loss of generality, we can assume that $g_{22}(x_0) \neq 0$. Then, by $\text{rk } g(x) = 1$, the distribution $\text{ker } g = \text{span } \left\{ g_{22}(x) \frac{\partial}{\partial z} - g_{12}(x) \frac{\partial}{\partial y} \right\}$ is locally of constant rank, and thus, we introduce coordinates $\tilde{x} = (\tilde{z}, \tilde{y})$ such that $g = \text{span } \left\{ \frac{\partial}{\partial \tilde{z}} \right\}$ in which we have

$$\tilde{S}_q = \tilde{a}(\tilde{x}) \tilde{y}^2 + 2\tilde{\omega}(\tilde{x}) \tilde{x} + \tilde{h}(\tilde{x}),$$

whose determinant $\Delta_1 = -\tilde{a}(\tilde{\omega}_1)^2 \neq 0$ implies that $\tilde{\omega}_1 \neq 0$. Dividing $\tilde{S}_q$ by $-2\tilde{\omega}_1$, we obtain the desired form $S_P$ with $a = \frac{\tilde{a}}{-2\tilde{\omega}_1}, b = \frac{\tilde{\omega}_2}{\tilde{\omega}_1}$, and $c = \frac{\tilde{h}}{2\tilde{\omega}_1}$. $\square$

**Example 2.4** There is a well known example of a control-nonlinear system subject to an elliptic constraint $S_E$, namely Dubins’ car [10]. The state of the system is the centre of mass of the vehicle $(z, y) \in \mathbb{R}^2$, and we control the orientation of the vehicle (with respect to the $z$-axis) via $w \in S^1$. Then, the dynamics of Dubins’ car reads

$$\begin{cases}
\dot{z} = r \cos(w) \\
\dot{y} = r \sin(w),
\end{cases} \quad r \in \mathbb{R}^*,
$$

which clearly is a first extension, i.e. a regular parametrisation, of the elliptic submanifold $z^2 + \dot{y}^2 = r^2$.

We have established all notions necessary for our characterisation and classification of conic submanifolds $S_q$ by studying the feedback equivalence of their first and second extensions $\Sigma_{S_q}$ and $\Sigma_{S_q}$.

### 3 Quadraticable Control-Affine Systems

In this section, we introduce the novel class of quadratic control-affine systems $\Sigma_q$ that describes second extensions of quadratic submanifolds $S_q$ given by

$$S_q(x, \dot{x}) = \dot{x} \cdot g(x) \dot{x} + 2\omega(x) \dot{x} + h(x) = 0$$

and satisfying $\Delta_1 \neq 0$. Next, we address the equivalence problem of a control-affine system $\Sigma$ to a quadratic control-affine system $\Sigma_q$, and in that way, due to Proposition 2.1, we provide a characterisation of quadratic submanifolds $S_q$. As a corollary, we will give a characterisation of the elliptic, hyperbolic, and parabolic submanifolds via a characterisation of corresponding subclasses of quadratic control-affine systems. Moreover, by studying our conditions, we will give a normal form of control-affine systems that are feedback equivalent to a quadratic one, and as a consequence, this will give us a normal form of quadratic submanifolds that smoothly passes from the elliptic to the hyperbolic classes.
On a 3-dimensional manifold $\mathcal{M}$, equipped with local coordinates $\xi$, we consider the control-affine system

$$\Sigma: \dot{\xi} = f(\xi) + g(\xi)u,$$

with a scalar control $u \in \mathbb{R}$ and smooth vector fields $f$ and $g$. A control-affine system $\Sigma$ is denoted by the pair $\Sigma = (f, g)$, and we set $\mathcal{G} = \text{span}\{g\}$ the distribution spanned by the vector field $g$. Moreover, we will use the following notations: given two vector fields $g$ and $f$ on $\mathcal{M}$, by $[g, f]$ we denote the Lie bracket of $g$ and $f$, in coordinates we have $[g, f] = \frac{\partial g}{\partial \xi} f - \frac{\partial f}{\partial \xi} g$, and $\text{ad}^k g f = [g, \text{ad}^{k-1} g f]$ stands for the iterated Lie bracket, with the convention $\text{ad}^0 g f = f$.

**Definition 3.1 (Quadratic and quadratisable systems)** We say that a control-affine system $\Sigma = (f, g)$ is \textit{quadratisable} if it is feedback equivalent to

$$\Sigma_q : \begin{cases} \dot{x} = f_q(x, w) \\
\dot{w} = u \end{cases},$$

where $x \in \mathcal{X}$, a 2-dimensional manifold, $w \in \mathcal{W} \subset \mathbb{R}$, and $f_q(x, w)$ satisfies

$$\left(\frac{\partial^2 f_q}{\partial \xi^2} - \frac{\partial f_q}{\partial \xi w} \right) (x, w) \neq 0,$$

and

$$\frac{\partial^3 f_q}{\partial w^3} = \tau_q(x) \frac{\partial f_q}{\partial w},$$

with $\tau_q$ a smooth function of the indicated variable $x \in \mathcal{X}$. A system $\Sigma_q$ of the above form is called quadratic.

For a quadratic system $\Sigma_q = (f_q, g_q)$, where $f_q = f_q \frac{\partial}{\partial x}$ and $g_q = \frac{\partial}{\partial w}$, we show by applying Appendix A with $\zeta = f_1$ and $\eta = f_2$, that any smooth $f_q$, satisfying relation (3) locally around $(x_0, w_0)$, can be written as

$$f_q(x, w) = A(x) \sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k+1}}{(2k + 1)!} \tau_q^k(x) + B(x) \sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k+1}}{(2k + 1)!} \tau_q^k(x) + C(x), \quad (4)$$

where $A, B, C$ are smooth. Since $\mathcal{G} = \text{span}\{g_q\} = \text{span}\{\frac{\partial}{\partial w}\}$ is involutive and of constant rank one, it is integrable and its integral curves can be identified with the points $x$ of $\mathcal{X}$ and

\[ \text{Springer} \]
The following proposition shows that $\Sigma_q$ is a second extension of a quadratic submanifold $S_q$, thus justifies to call $\Sigma_q$ a quadratic system, and identifies the elliptic, hyperbolic, and parabolic subclasses by describing three normal forms of $f_q = f_q \frac{\partial}{\partial \tau_q}$ given for $\tau_q \neq 0$ and $\tau_q \equiv 0$.

**Proposition 3.1** Locally around $\xi_0$ belonging to $\mathcal{M}$, the following statements hold.

(i) $\Sigma_q$ is a second extension of a conic submanifold $S_q$, and, conversely, any second extension $\Sigma_{S_q}$ of a conic submanifold $S_q$ is feedback equivalent to a system of the form $\Sigma_q$.

(ii) If $\tau_q(\xi_0) < 0$, resp. $\tau_q(\xi_0) > 0$, resp. $\tau_q \equiv 0$, then $\Sigma_q$ is locally feedback equivalent to $\Sigma_E$, resp. $\Sigma_H$, resp. $\Sigma_P$, given by $f_q$ of, respectively, the form

\[
\begin{align*}
 f_E &= A(x) \cos(\hat{\theta}) + B(x) \sin(\hat{\theta}) + C(x), \\
 f_H &= A(x) \cosh(\hat{\theta}) + B(x) \sinh(\hat{\theta}) + C(x),
\end{align*}
\]

\[
\begin{align*}
 f_P &= A(x)w^2 + B(x)w + C(x),
\end{align*}
\]

where in all three cases $A \land B \neq 0$.

(iii) $\Sigma_E$, resp. $\Sigma_H$, resp. $\Sigma_P$, is a second extension of a conic submanifold $S_q$ satisfying $\Delta_2 > 0$, resp. $\Delta_2 < 0$, resp. $\Delta_2 \equiv 0$.

The conic submanifold $S_q$ of item (iii) is, by Lemma 2.1, equivalent to $S_E$ (if $\Delta_2 > 0$), resp. $S_H$ (if $\Delta_2 < 0$), resp. $S_P$ (if $\Delta_2 \equiv 0$). So it is natural to call $\Sigma_E$ an elliptic system, $\Sigma_H$ a hyperbolic system, and $\Sigma_P$ a parabolic system. We will denote by $Q$ the set $\{E, H, P\}$ and, consequently, $f_Q = \{f_E, f_H, f_P\}, g_Q = \{g_E, g_H, g_P\}, f_Q = \{f_E, f_H, f_P\}, \text{and} \Sigma_Q = \{\Sigma_E, \Sigma_H, \Sigma_P\}$.

**Proof** (i) Consider $f_q$ given by (4), for simplicity of the notations, we assume $w_0 = 0$. Notice that the two rank 1 distributions $A = \text{span} \{A\}$ and $B = \text{span} \{B\}$ satisfy $A(x_0) \oplus B(x_0) = T_{x_0} \mathbb{R}^2$. Thus, we can locally choose two independent smooth functions $\phi$ and $\psi$ satisfying $d\phi(x_0) \neq 0$, $d\psi(x_0) \neq 0$, $d\phi \in \text{ann}(A)$, and $d\psi \in \text{ann}(A).$ In the $(z, y) = (\phi, \psi)$-coordinates, we have $A = \text{span} \{ \frac{\partial}{\partial \phi} \}$ and $B = \text{span} \{ \frac{\partial}{\partial \psi} \}$ and, consequently, $A = a \frac{\partial}{\partial \tau} \text{and} B = b \frac{\partial}{\partial \tau}$, where $a$ and $b$ are smooth functions satisfying $a(x_0)b(x_0) \neq 0$. We set $C = c_0 \frac{\partial}{\partial \tau} + c_1 \frac{\partial}{\partial \tau}$, where $c_0$ and $c_1$ are smooth functions of $x$.

Using Cauchy products, we compute

\[
\begin{align*}
\left( \frac{\dot{z} - c_0}{a} \right)^2 &= \left( \sum_{k=0}^{+\infty} \frac{w^{2k+2}}{(2k+2)!} \tau_q^k \right)^2 = \sum_{k=0}^{+\infty} \frac{(8 \cdot 4k - 2)w^{2k+4} \tau_q^k}{(2k+4)!}, \\
\left( \frac{\dot{y} - c_1}{b} \right)^2 &= \left( \sum_{k=0}^{+\infty} \frac{w^{2k+1}}{(2k+1)!} \tau_q^k \right)^2 = \sum_{k=0}^{+\infty} \frac{2 \cdot 4k w^{2k+2} \tau_q^k}{(2k+2)!}.
\end{align*}
\]

\[
\begin{align*}
&= \frac{w^2 + \sum_{k=0}^{+\infty} 2 \cdot 4k w^{2k+2} \tau_q^k}{(2k+2)!} \\
&= \frac{w^2 + \sum_{k=0}^{+\infty} 2 \cdot 4k+1 w^{2k+4} \tau_q^{k+1}}{(2k+4)!} \\
&= \frac{w^2 + \tau_q \sum_{k=0}^{+\infty} 8 \cdot 4k w^{2k+4} \tau_q^k}{(2k+4)!}.
\end{align*}
\]
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To prove that any second extension of a conic submanifold $S_q$ to a system of the form

$$\Sigma_1$$

is feedback equivalent to a system of the form given by (5). To this end, consider

$$S_q = x^1 g x + 2o x + h = a z^2 + 2b z \dot{y} + c y^2 + 2d z + 2e \dot{y} + h = 0,$$

where all functions $a, b, c, d, e, h$ depend smoothly on $x = (z, y)$. By $\Delta_1(x_0) \neq 0$, we have $\text{rk} g(x_0) \geq 1$, where $g = \left( \begin{smallmatrix} b & c \\ -c & a \end{smallmatrix} \right)$. If $\text{rk} g(x_0) = 2$, then we can apply the results of Lemma 2.1 to get $S_q$ in the form of (5). If $\text{rk} g(x_0) = 1$, then we may assume that $c(x_0) \neq 0$. Indeed, if $a(x_0) = c(x_0) = 0$, then $b(x_0) \neq 0$ implying that $\text{rk} g(x_0) = 2$ so either $a(x_0) \neq 0$ or $c(x_0) \neq 0$ and we can always suppose $c(x_0) \neq 0$ by permuting $y$ and $z$, if necessary.

Dividing $S_q = 0$ by $c$, we obtain (we keep the same names for all remaining functions):

$$S_q = a z^2 + 2b z \dot{y} + y^2 + 2d z + 2e \dot{y} + h = 0.$$ 

Choose local coordinates $(\tilde{z}, \tilde{y}) = (z, \psi(z, y))$, with $\frac{\partial \psi}{\partial y} \neq 0$, that rectify the line-distribution $\{ \partial \tilde{z} - b \partial \tilde{y} \}$, i.e. $\frac{\partial \psi}{\partial z} - b \frac{\partial \psi}{\partial y} = 0$. Therefore, $\dot{\tilde{z}} = \tilde{z}$ and $\dot{\tilde{y}} = \frac{\partial \psi}{\partial \tilde{z}} \tilde{z} + \frac{\partial \psi}{\partial \tilde{y}} \tilde{y} = \frac{\partial \psi}{\partial y} (b \tilde{z} + \tilde{y})$. Thus $\tilde{S}_q$, which is $S_q$ in $(\tilde{z}, \tilde{y})$-coordinates, reads

$$\tilde{S}_q = \frac{1}{b^2} \tilde{y}^2 + \tilde{a} \tilde{z}^2 + 2 \tilde{d} \tilde{z} + 2 e \tilde{y} + \tilde{h} = 0,$$

where $\tilde{b} = \frac{\partial \psi}{\partial y}$ and $\tilde{a} = \tilde{a}(\tilde{x}_0) = 0$ because $\text{rk} \tilde{g}(\tilde{x}_0) = 1$. By $\Delta_1(\tilde{x}_0) \neq 0$ and $\tilde{a}(\tilde{x}_0) = 0$, we conclude that $\tilde{a}(\tilde{x}_0) \neq 0$, and thus dividing by $\tilde{d}$ (and removing the "tildes" from the coordinates), we get

$$\tilde{S}_q = \frac{\tilde{S}_q}{\tilde{d}} = \frac{1}{\tilde{b}^2} \tilde{y}^2 + \tilde{A} \tilde{z}^2 + 2 \tilde{z} + 2 e \tilde{y} + \tilde{h} = \left( \frac{\dot{\tilde{z}} - \tilde{c}_1}{\tilde{b}} \right)^2 + \tilde{A} \tilde{z}^2 + 2 \tilde{z} + \tilde{H} = 0,$$

with $\tilde{b} = \tilde{b}/\tilde{d}$ (if $\tilde{d} < 0$, then we take $\tilde{z} = -\tilde{z}$ to get $\tilde{d} > 0$), $\tilde{A} = \tilde{a}/\tilde{d}$ (satisfying $\tilde{A}(x_0) = 0$), $\tilde{c}_1 = -\tilde{c} \tilde{b}^2$, and $\tilde{H} = \tilde{h} - \tilde{e}^2 \tilde{b}^2$ (see also [6] for another proof for the smooth diagonalisation of a symmetric $(0, 2)$-tensor in the case of rank deficiency). To get the form (5), we will now prove that there exists, locally around $x_0$, functions $\tilde{\tau}_q$, $\tilde{a} \neq 0$, and $\tilde{c}_0$ such that $\tilde{A} \tilde{z}^2 + 2 \tilde{z} + \tilde{H} = -\tilde{\tau}_q \left( \frac{\dot{\tilde{z}} - \tilde{c}_0}{\tilde{a}} \right)^2 - 2 \left( \frac{\tilde{z} - \tilde{c}_0}{\tilde{a}} \right).$ We obtain

$$\frac{-\tilde{\tau}_q}{\tilde{a}^2} = \tilde{A}, \quad \frac{\tilde{\tau}_q \tilde{c}_0}{\tilde{a}^2} - \frac{1}{\tilde{a}} = 1, \quad \text{and} \quad \frac{-\tilde{\tau}_q (\tilde{c}_0)^2}{\tilde{a}^2} + 2 \frac{\tilde{c}_0}{\tilde{a}} = \tilde{H}.$$
Hence, from the second equation, \( \frac{\ddot{c}_0}{a} = -\ddot{A}(c_0)^2 - \ddot{c}_0 \) implying that \( A(c_0)^2 + 2\ddot{c}_0 + \dddot{H} = 0 \). The last equation possesses a smooth local solution \( c_0(z, y) = \frac{-\dddot{H}}{1 + \sqrt{1 - AH}} \); recall that \( A(x_0) = 0 \). So

\[
\bar{S}_q = \left( \frac{\dot{y} - \ddot{c}_1}{b} \right)^2 - \bar{\tau}_q \left( \frac{\dot{z} - \ddot{c}_0}{a} \right)^2 - 2 \left( \frac{\dot{z} - \ddot{c}_0}{a} \right) = 0,
\]

where \( \bar{a} = \frac{1}{1 + A c_0} \), which is well defined since \( A(x_0) = 0 \), and \( \tau_q = -\bar{A}\bar{a}^2 \), proving that \( \bar{S}_q \) is of the desired form (5). From (5), we go to \( \Sigma_q \) following the passage already presented in the first part of the proof.

(ii) If \( \tau_q < 0 \), then \( \frac{\partial^2 f_q}{\partial w^2} = \tau_q \frac{\partial^2 A}{\partial w^2} \) implies that \( f_q = A \cos(\sqrt{-\tau_q} w) + B \sin(\sqrt{-\tau_q} w) + C \), where \( A, B, \) and \( C \) depend on \( x \) and, clearly, \( \frac{\partial^2 f_q}{\partial w^2} \neq 0 \) implies \( A \neq 0 \) (for this case, as well as for the next two cases). Via the change of coordinate \( \tilde{w} = \sqrt{-\tau_q} w \), we get \( f_q = f_E \). If \( \tau_q > 0 \), then \( \frac{\partial^2 f_q}{\partial w^2} = \tau_q \frac{\partial^2 A}{\partial w^2} \) implies that \( f_q = A \cosh(\sqrt{\tau_q} w) + B \sinh(\sqrt{\tau_q} w) + C \) which, via the change of coordinate \( \tilde{w} = \sqrt{\tau_q} w \), gives \( f_q = f_H \).

Finally, if \( \tau_q \equiv 0 \), then \( \frac{\partial^2 f_q}{\partial w^3} \equiv 0 \) implies that \( f_q = f_P = Aw^2 + Bw + C \).

(iii) By item (i), \( \Sigma_q \) is a second extension of a conic submanifold \( S_q \) for which we have \( \Delta_2 = -\frac{\tau_q}{a^2 b^2} \). Thus, \( \Delta_2 > 0 \), resp. \( \Delta_2 < 0 \), resp. \( \Delta_2 \equiv 0 \) if and only if \( \tau_q < 0 \), resp. \( \tau_q > 0 \), resp. \( \tau_q \equiv 0 \), which, by item (ii), correspond to \( \Sigma_E \), resp. \( \Sigma_H \), resp. \( \Sigma_P \).

\( \Box \)

Notice that \( \tau_q \) plays for \( \Sigma_q \) an analogous role to that played by \( \Delta_2 \) for \( S_q \); indeed, we have \( \sgn(\Delta_2(x)) = -\sgn(\tau_q(x)) \). In particular, the sign of \( \tau_q \) identifies the subclasses of elliptic, hyperbolic, and parabolic control-affine systems. Moreover, statement (i) shows that every second extension of a quadratic submanifold \( S_q \) is feedback equivalent to a quadratic system \( \Sigma_q \) and the other way around; therefore to obtain a characterisation of quadratic submanifolds, it is crucial to characterise the class of quadratisable systems.

The remaining part of this section is organised as follows. First, we will state our main theorem giving necessary and sufficient conditions characterising the class of quadratic control-affine systems \( \Sigma_q \). Second, by carefully studying the conditions of that theorem, we will give a normal form of all quadratisable systems.

### 3.1 Characterisation of Quadratisable Control-Affine Systems

We now focus on the feedback equivalence of a general control-affine system \( \Sigma : \dot{x} = f(\xi) + g(\xi)u \) with a quadratic control-affine system of the form \( \Sigma_q \). The theorem below gives checkable necessary and sufficient conditions in terms of the vector fields \( f \) and \( g \) of \( \Sigma \) for the existence of a smooth feedback transformation \( (\varphi, \alpha, \beta) \) that locally brings \( \Sigma \) into a quadratic system \( \Sigma_q \). Equivalence to particular subcases of \( \Sigma_q \), namely elliptic \( \Sigma_E \), hyperbolic \( \Sigma_H \), and parabolic \( \Sigma_P \), is provided by Corollary 3.1 below.

**Theorem 3.1 (Feedback quadratisation)** Let \( \Sigma = (f, g) \) be a control-affine system on a 3-dimensional smooth manifold with a scalar control. The system \( \Sigma \) is, locally around \( \xi_0 \in \mathcal{M} \), feedback equivalent to a quadratic system \( \Sigma_q \) if and only if

\( (C1) \ g \wedge \text{ad}_g f \wedge \text{ad}_g^2 f (\xi_0) \neq 0, \)
The structure functions $\rho$ and $\tau$ in the decomposition $\text{ad}^3_{gq}f = \rho \text{ad}^2_{g}f + \tau \text{ad}_{g}f \pmod{G}$ satisfy, locally around $\xi_0$,

$$L_g(\chi) - \frac{2}{3}\rho\chi = 0,$$

where $\chi = 3L_g(\rho) - 2\rho^2 - 9\tau$.

Condition (C1) is a regularity condition; it ensures that the vector fields $g$, $\text{ad}_g f$, and $\text{ad}_{g}^2 f$ are locally linearly independent and thus that they form a local frame; hence, the structure functions $(\rho, \tau)$ of (C2) are well defined. The main idea behind this theorem is to observe that if for $\Sigma$ we have $\text{ad}^3_{g}f = \tau(x) \text{ad}_g f$, modulo $G = \text{span} \{g\}$, i.e. the third Lie derivative of $f$ along $g$ is proportional to the first Lie derivative of $f$ along $g$, modulo $G$, then with the help of a diffeomorphism, we can obtain the form $\Sigma_q$, see the sufficiency part of the proof for details. Thus, condition (C2) shows how that relation changes when we allow for feedback transformations $(\alpha, \beta)$.

**Proof** Necessity. Consider the affine control system $\Sigma$ given by two smooth vector fields $f$ and $g$ and recall that $G$ is the distribution $G = \text{span} \{g\}$. Let $(\alpha, \beta)$ form a control-affine feedback and let $\phi$ be a diffeomorphism such that $\Sigma$ is, locally, transformed into $\Sigma_q$ via $\phi$ and $(\alpha, \beta)$. In coordinates $\tilde{\xi} = \phi(\xi) = (\tilde{x}, \tilde{u}) = (\tilde{z}, \tilde{y}, \tilde{w})$, we denote $\tilde{f}_q = \tilde{f}_q \frac{\partial}{\partial \tilde{x}}$ and $\tilde{g}_q = \frac{\partial}{\partial \tilde{w}}$ the vector fields of $\Sigma_q$, $\tilde{G}$ the distribution spanned by $\tilde{g}_q$, and $(\tilde{\rho}, \tilde{\tau})$ the structure functions of $\Sigma_q$, defined as in (C2). By definition of feedback equivalence the following relations between $(f, g)$ and $(\tilde{f}_q, \tilde{g}_q)$ hold: $\tilde{f}_q = \phi_*(f + \alpha g)$ and $\tilde{g}_q = \phi_*(g \beta)$.

The system $\Sigma_q$ is quadratic, so by Definition 3.1, we have $\frac{\partial^3 f_q}{\partial w^3} \wedge \frac{\partial^2 f_q}{\partial w^2}(\tilde{x}_0, \tilde{w}_0) \neq 0$, which implies that (C1) holds for $\Sigma_q$, and we also have $\frac{\partial^3 f_q}{\partial w^3} = \tilde{f}_q \frac{\partial^2 f_q}{\partial w^2}$; thus, we get $\tilde{\rho} = 0$ and $\tilde{\tau} = \tilde{\tau}(\tilde{z}, \tilde{y})$. Therefore for $\Sigma_q$, we have $\tilde{\chi}(\tilde{z}, \tilde{y}) = -9\tilde{\tau}(\tilde{z}, \tilde{y})$ implying $L_{\tilde{g}_q} \tilde{\chi} - \frac{2}{3}(3L_{\tilde{g}_q} \tilde{\chi}) = 0$. Hence, $\Sigma_q$ satisfies (C1) and (C2), and we will now prove that those conditions are invariant under diffeomorphisms $\phi$ and feedback transformations $(\alpha, \beta)$.

Clearly, (C2) is invariant under diffeomorphisms $\phi$. We have checked that (C2) holds for $\Sigma_q = (\tilde{f}_q, \tilde{g}_q)$ and, clearly, (C2) is invariant under feedbacks since they conjugate structure functions. Moreover, (C2) is invariant under the transformation $\tilde{f}_q \mapsto \tilde{f}_q + \alpha \tilde{g}_q$, since the expression of $\text{ad}^3_{\tilde{g}_q} \tilde{f}_q$ is considered modulo the distribution $\tilde{G}$. Finally, under the action of $\beta$ the brackets, with $\tilde{g}_q = g \beta$, are transformed by

$$\text{ad}_{\tilde{g}_q} f_q = \beta \text{ad}_{\tilde{g}_q} f \pmod{\tilde{G}},$$

$$\text{ad}^2_{\tilde{g}_q} f_q = \beta^2 \text{ad}^2_{\tilde{g}_q} f + \beta L_g(\beta) \text{ad}_{\tilde{g}_q} f \pmod{\tilde{G}},$$

$$\text{ad}^3_{\tilde{g}_q} f_q = (\beta^3 \rho + 3\beta^2 L_g(\beta)) \text{ad}^2_{\tilde{g}_q} f + (\beta^2 \tau + \beta L_g(\beta) L_g(\beta)) \text{ad}_{\tilde{g}_q} f \pmod{\tilde{G}},$$

$$= (\rho \beta + 3L_g(\beta)) \text{ad}^2_{\tilde{g}_q} f + \left(\tau \beta^2 + L_g(\beta) L_g(\beta) - \rho \beta L_g(\beta) \right) \pmod{\tilde{G}}.$$
Since for $\Sigma_q$ the structure function $\tilde{\rho} = 0$, we have the relation $L_g(\beta) = -\frac{\beta \rho}{\tau}$ and thus $\tilde{\chi} = -9\tilde{\tau}$, which is equal to

$$
\tilde{\chi} = -9\left(\tau \beta^2 + L_g(\beta L_g(\beta))\right) = -9\left(\tau \beta^2 + L_g\left(-\frac{\beta^2 \rho}{3}\right)\right),
$$

$$
= -9\left(\tau \beta^2 - \frac{1}{3}(\rho L_g(\beta^2) + \beta^2 L_g(\rho))\right) = -9\beta^2\left(\tau - \frac{1}{3} L_g(\rho) + \frac{2}{9} \rho^2\right) = \beta^2 \chi.
$$

And finally,

$$
L_{\tilde{g}_q}(\tilde{\chi}) = \beta L_g(\beta^2 \chi) = \beta^3 L_g(\chi) + 2\beta^2 \chi L_g(\beta) = \beta^3 L_g(\chi) - \frac{2}{3} \beta^3 \chi \rho = 0,
$$

showing the necessity of relation (6) and concludes the necessity part of the proof.

Sufficiency. There are two steps in the sufficiency part. The first one consists in building a vector field $\tilde{g}$ such that $\text{ad}^2 f = \tau \text{ad} \tilde{\xi} f$ mod $G$ with $L_{\tilde{g}}(\tau) = 0$. Then, we will construct a diffeomorphism $\phi$ that brings $\Sigma$ into the form $\Sigma_q$.

Consider the system $\Sigma : \tilde{\xi} = \tilde{f} + g u$, for which we assume $g \wedge \text{ad}^2 f \wedge \text{ad} \tilde{\xi} f(\xi_0) \neq 0$ and suppose that relation (6) holds for the structure functions $\rho$ and $\tau$ of $\Sigma$. Choose a function $\beta \neq 0$ satisfying $L_g(\beta) = -\frac{\beta \rho}{\tau}$, which exists locally since $g \neq 0$ by condition (C1); to guarantee that $\beta \neq 0$, we actually may solve the equation $L_g(\ln \beta) = -\frac{\rho}{\tau}$. Define the system $\tilde{\Sigma} : \tilde{\xi} = \tilde{f} + g \tilde{u}$, where $\tilde{g} = g \beta$ and $\tilde{f} = f$, then by (7) the structure function $\tilde{\rho}$ of $\tilde{\Sigma}$ vanishes. Therefore, we have $\tilde{\chi} = -9\tilde{\tau}$ and thus relation (6) implies that $L_{\tilde{g}}(\tau) = 0$.

Since $\tilde{g} \neq 0$, we apply a diffeomorphism $(z, y, w) = \phi(\xi)$ such that $\phi_* \tilde{g} = g_q = \frac{a}{\pi w}$ and denote $f_q = \phi_* \tilde{f}$, and $\tau_q \circ \phi = \tilde{\tau}$. Therefore, the decomposition $\text{ad}^3 f_q = \tau_q \text{ad} g_q f_q$ mod $G$ implies that $f_q = f_q^1 \frac{\partial}{\partial z} + f_q^2 \frac{\partial}{\partial y} + f_q^3 \frac{\partial}{\partial w}$ satisfies

$$
\frac{\partial^3 f_q^i}{\partial w^3} = \tau(z, y) \frac{\partial f_q^i}{\partial w},
$$

for $i = 1, 2$. Applying the feedback $u = f_q^3(z, y, w) + \tilde{u}$, we obtain the form $\Sigma_q$ with $f_q = f_q^1 \frac{\partial}{\partial z} + f_q^2 \frac{\partial}{\partial y}$ and $g_q = \frac{a}{\pi w}$. The condition $\frac{\partial^2 f_q^i}{\partial w^2} \wedge \frac{\partial f_q^i}{\partial w} \neq 0$ follows from (C1) and feedback invariance of the latter. □

The following corollary shows that we can test on the structure functions of $\Sigma$ if the equivalent quadratic system $\Sigma_q$ will be of elliptic, hyperbolic, or parabolic type.

**Corollary 3.1** Under conditions (C1) and (C2) of the previous theorem we have, locally around $\xi_0$,

(i) $\Sigma$ is feedback equivalent to $\Sigma_E$ if and only if $\chi(\xi_0) > 0$,

(ii) $\Sigma$ is feedback equivalent to $\Sigma_H$ if and only if $\chi(\xi_0) < 0$,

(iii) $\Sigma$ is feedback equivalent to $\Sigma_P$ if and only if $\chi \equiv 0$ in a neighbourhood of $\xi_0$,

where $\chi = 3L_g(\rho) - 2\rho^2 - 9\tau$.

Notice that $\Sigma$ is locally feedback equivalent to $\Sigma_P$ if and only if it satisfies (C1) and $\chi \equiv 0$, condition (C2) being satisfied automatically.

**Proof** From the necessity part of the proof of Theorem 3.1 we know that for $\Sigma_q$, with structure functions $\rho = 0$ and $\tau = \tau_q$, we have $\chi = -9\tau_q$, and we observed that under pure feedback transformations $(\alpha, \beta)$, we have $\tilde{\chi} = \beta^2 \chi$; thus, the sign of $\chi$ is invariant as well as the locus
where it vanishes. Moreover, by statement (ii) of Proposition 3.1, $\Sigma_q$ is elliptic if $\tau_q > 0$, equivalently $\chi < 0$, $\Sigma_q$ is hyperbolic if $\tau_q > 0$, equivalently $\chi < 0$, and $\Sigma_q$ is parabolic if $\tau_q \equiv 0$, equivalently $\chi \equiv 0$. Hence, the necessity of the stated conditions is established.

Conversely, in the sufficiency part of the proof of Theorem 3.1, we obtained $\Sigma$ with structure functions $(\check{\beta}, \check{\tau}) = (0, \check{\tau}(z, y))$ via a suitable feedback transformations. Since $\check{\chi} = \beta^2 \check{\chi}$, we have $-9\check{\tau} = \beta^2 \check{\chi}$, and thus, we get $\text{sgn} (\check{\tau}) = -\text{sgn} (\chi)$ and the conclusion follows by statement (ii) of Proposition 3.1.

3.2 Normal Form of Quadratisable Control-Affine Systems

Any control-affine system $\Sigma$, under the regularity assumption $g \land \text{ad}_g f (\xi_0) \neq 0$, can be written (after applying a suitable feedback transformation) locally around $0 \in \mathbb{R}^3$ as

$$\Sigma_h : \begin{cases} \dot{z} = h(z, y, w) \\
\dot{y} = w + \varepsilon \\
\dot{w} = u
\end{cases},$$

with $h$ a smooth function and $\varepsilon = 0$ or 1. The parameter $\varepsilon$, which is invariant under local feedback transformations, is $\varepsilon = 0$ if either $(f \land g \land \text{ad}_g f) (\xi_0) \neq 0$ or $(f \land g) (\xi_0) = 0$ and $\varepsilon = 1$ otherwise, i.e. $(f \land g) (\xi_0) \neq 0$ but $(f \land g \land \text{ad}_g f) (\xi_0) = 0$. By applying Theorem 3.1, we will give in this subsection a normal form of all smooth functions $h(z, y, w)$ that describe quadratisable systems, that is, control-affine systems feedback equivalent to $\Sigma_q$. In what follows, we assume to work locally around $0 \in \mathbb{R}^3$, and all derivatives are taken with respect to $w$ and denoted by prime, double prime, etc. Whenever we apply $\ln(a)$, we assume that $a > 0$ (if not, we take the absolute value).

**Theorem 3.2** (Normal form of quadratisable control-affine systems) The following statements are equivalent, locally around $0 \in \mathbb{R}^3$:

(i) $\Sigma_h$ is feedback equivalent to a quadratic system $\Sigma_q$;

(ii) The function $h$ satisfies $h''(0) \neq 0$ and, in a neighbourhood, it holds

$$9h^{(5)} (h'')^2 - 45h^{(4)} h^{(3)} h'' + 40 \left(h^{(3)}\right)^3 = 0,$$

(recall that the derivatives are taken with respect to $w$);

(iii) The second derivative of $h$ is of the following form

$$h''(x, w) = a(dw^2 + ew + 1)^{-3/2},$$

where $a = a(x)$, $d = d(x)$, and $e = e(x)$ are smooth functions satisfying $a(0) \neq 0$;

(iv) The function $h$ is given by

$$h(x, w) = 2a \left( \frac{w^2}{(\sqrt{dw^2 + ew + 1} + 1)^2 - dw^2} \right) + bw + c,$$

where $a$, $b$, $c$, $d$, $e$ are any smooth functions of $x$ such that $a(0) \neq 0$.

**Proof** (i)$\Rightarrow$(ii). It is a straightforward application of the conditions of Theorem 3.1 with the structure functions of $\Sigma_h$ given by $\rho = h^{(3)}$ and $\tau = 0$ yielding $\chi = 3\rho' - 2\rho^2$. By (C1), we have $h''(0) \neq 0$ and then condition (C2) reads

$$\chi' - \frac{2}{3} \rho \chi = 3\rho'' - 6\rho\rho' + \frac{4}{3}\rho^3 = 0,$$

$\square$ Springer
which, by plugging \( p = \frac{h^{(3)}}{h''} \) into the last equation, gives (9).

(ii)⇒(iii). Assume that \( h \) satisfies \( h''(0) \neq 0 \) and (9). Set \( \rho = \frac{h^{(3)}}{h''} \), then \( \rho = \rho(x, w) \) fulfills
\[ 3\rho'' - 6\rho\rho' + \frac{4}{3}\rho^3 = 0, \]
namely, the second equation of (12). By a change of variable, it is easy to obtain that the solutions of (12) are of the following form (see Appendix B.1 for the proof)
\[ \rho(x, w) = -\frac{3}{2} \frac{2d(x)w + e(x)}{d(x)w^2 + e(x)w + 1}. \]  
(13)

This form can be integrated, using \( \rho = \frac{h^{(3)}}{h''} = \ln(h'') \), into \( h''(x, w) = a(x) \left( d(x)w^2 + e(x)w + 1 \right)^{-3/2} \) with \( a, d \), and \( e \) any smooth functions such that \( a(0) \neq 0 \).

(iii)⇒(iv). To show (11), we integrate the second derivative of \( h \) given by (10). Denote \( p = p(x, w) = d(x)w^2 + e(x)w + 1 \) and \( \Delta = \Delta(x) = e(x)^2 - 4d(x) \). First, we obtain (see Appendix B.2 for details)
\[ h'(x, w) = \frac{2aw(\sqrt{p} + 1)}{\sqrt{p}(ew + 2 + 2\sqrt{p})} + b, \]
with \( b \) an arbitrary smooth function of \( x \). Integrate once more to get
\[ h(x, w) = \frac{2a}{\Delta \sqrt{p}} \left( ew\sqrt{p} - 2p \right) + \frac{4a}{\Delta} bw + c = \frac{2a}{\Delta} \left( ew + 2 - 2\sqrt{p} \right) + bw + c \]
\[ = \frac{2aw^2}{ew + 2 + 2\sqrt{p}} + bw + c = \frac{2aw^2}{(\sqrt{p} + 1)^2 - dw^2} + bw + c. \]

(iv)⇒(i). Given \( \Sigma_h \) with \( h \) defined by (11), we will construct a feedback transformation that brings the system into \( \Sigma_q \). First, we introduce coordinates, centred at 0 \( \in \mathbb{R}^2 \), \((\tilde{z}, \tilde{y}) = \phi(z, y)\), where \( \tilde{y} = y \), such that \( \phi_* \left( b \frac{\partial}{\partial z} + \frac{\partial}{\partial y} \right) = \frac{\partial}{\partial \tilde{y}} \). Those coordinates transform the system \( \Sigma_h \) into
\[ \left\{ \begin{array}{l}
\tilde{z} = 2\tilde{a} \frac{w^2}{(\sqrt{p} + 1)^2 - dw^2} + \tilde{c} \\
\tilde{y} = w + \varepsilon \\
\tilde{w} = u
\end{array} \right., \]
where \( \tilde{a}, \tilde{c}, \tilde{d}, \) and \( \tilde{p} \) are new functions satisfying \( \tilde{a}(0) \neq 0 \) and \( d = \tilde{d} \circ \phi \) and \( p = \tilde{p} \circ \phi \).

Second, we set \( \tilde{w}^2 = \frac{w^2}{(\sqrt{p} + 1)^2 - dw^2} \) or, equivalently, \( w = \tilde{w} \left( \tilde{e}\tilde{w} \pm 2\sqrt{\tilde{d}\tilde{w}^2 + 1} \right) \), which brings the above system into (after applying a suitable feedback along the last component)
\[ \tilde{\Sigma}_h : \left\{ \begin{array}{l}
\tilde{z} = 2\tilde{a}\tilde{w}^2 + \tilde{c} \\
\tilde{y} = \tilde{e}\tilde{w}^2 \pm 2\tilde{w}\sqrt{\tilde{d}\tilde{w}^2 + 1} + \varepsilon \\
\tilde{\omega} = \tilde{u}
\end{array} \right.. \]  
(14)

The structure functions of \( \tilde{\Sigma}_h \) are given by \( \tilde{\phi} = -\frac{3\tilde{d}\tilde{w}}{\tilde{d}\tilde{w}^2 + 1} \) and \( \tilde{\tau} = \frac{3\tilde{d}}{\tilde{d}\tilde{w}^2 + 1} \), then apply the feedback \( \tilde{u} = \beta\tilde{u} \), where \( \beta(\tilde{x}, \tilde{w}) = \sqrt{\tilde{d}\tilde{w}^2 + 1} \) (it is a solution of the equation \( \frac{\partial}{\partial \tilde{w}} = -\frac{\tilde{p}}{\tilde{y}} \)), to obtain a new vector field \( \tilde{g} = \tilde{p} \frac{\partial}{\partial \tilde{w}} \) and structure functions of \( (\tilde{f}, \tilde{g}) \), where \( \tilde{f} \) is the drift of \( \tilde{\Sigma}_h \), are given by \( \tilde{\rho} = 0 \) and \( \tilde{\tau} = 4\tilde{d}(x) \). To complete the form, it remains to find a new \( \tilde{w} = \psi(\tilde{x}, \tilde{w}) \) such that the diffeomorphism \( (\tilde{x}, \tilde{w}) = \Psi(\tilde{x}, \tilde{w}) = (\tilde{x}, \psi(\tilde{x}, \tilde{w})) \) satisfies

\[ \begin{array}{l}
\tilde{\phi} = -\frac{3\tilde{d}\tilde{w}}{\tilde{d}\tilde{w}^2 + 1} \\
\tilde{\tau} = \frac{3\tilde{d}}{\tilde{d}\tilde{w}^2 + 1}
\end{array} \]
This theorem provides a normal form of submanifolds. Indeed, observe that the differential 1-form \( \omega \) of the transformation; in the following corollary, we show that this function is the key of the expression of to which \( a(\tilde{c} - c) = 2a(\tilde{y} - \varepsilon) \) plays an important role for the shape of the transformation; in the following corollary, we show that this function is the key of the normal form of quadratisable control systems.

**Corollary 3.2** Assume that \( \Sigma_h \) is given by \( h \) of the form (11). Then, \( \Sigma \) is feedback equivalent to \( \Sigma_P \), resp. \( \Sigma_E \), resp. \( \Sigma_H \) if and only if \( d \equiv 0 \), resp. \( d < 0 \), resp. \( d > 0 \). Moreover, the normalising feedback transformation is given by

\[
\tilde{w} = \frac{w}{1 + \sqrt{ew + 1}} \quad \text{for} \quad \Sigma_P,
\]

\[
\text{resp.} \quad \sin^2(\sqrt{-d}\tilde{w}) = \frac{-dw^2}{ew + 2 + 2\sqrt{p}} \quad \text{for} \quad \Sigma_E,
\]

\[
\text{resp.} \quad \sinh^2(\sqrt{d}\tilde{w}) = \frac{dw^2}{ew + 2 + 2\sqrt{p}} \quad \text{for} \quad \Sigma_H,
\]

where \( p = p(x, w) = d(x)w^2 + e(x)w + 1 \).

**Proof** First, we show that \( \Sigma_h \) is feedback equivalent to \( \Sigma_P \), resp. \( \Sigma_E \), resp. \( \Sigma_H \), if and only if \( d \equiv 0 \), resp. \( d < 0 \), resp. \( d > 0 \). From Corollary 3.1, we know that we have to compute the sign of \( \chi \), which is given by \( \chi = -9d \) for \( \Sigma_h \) (this can easily be deduced from the expression of \( \rho \) given by (13)). Since \( p(0) = 1 > 0 \), we have \( sgn(\chi) = -sgn(d) \) and thus the conclusion follows.

We now show how to explicitly transform \( \Sigma_h \) into \( \Sigma_P \), resp. \( \Sigma_E \), resp. \( \Sigma_H \). From the last part of the proof of the previous theorem, we know that a suitable parametrisation \( \tilde{w} \) is given by the following two steps

\[
\tilde{w}^2 = \frac{w^2}{ew + 2 + 2\sqrt{p}}, \quad \text{and} \quad \tilde{w} = \int_0^{\tilde{w}} \frac{1}{\sqrt{d\tilde{w}^2 + 1}} d\tilde{w}.
\]
Assume $d = 0$, then the procedure reduces to the first step only, and thus $\tilde{w}^2 = \tilde{w}^2 = \frac{w^2}{ew+2+2\sqrt{ew+1}} = \left(\frac{w}{1+\sqrt{ew+1}}\right)^2$, and we choose $\tilde{w} = \frac{w}{1+\sqrt{ew+1}}$. Assume $d < 0$, then the second step of the procedure leads to $\tilde{w} = \frac{1}{\sqrt{-d}}\arcsin(\sqrt{-d}\tilde{w})$. Hence, a reparametrisation is given by $\sin^2(\sqrt{-d}\tilde{w}) = \frac{-dw^2}{ew+2+2\sqrt{p}}$. Assume $d > 0$, then from the second step of the procedure we have $\tilde{w} = \frac{1}{\sqrt{d}}\arcsinh(\sqrt{d}\tilde{w})$. Hence, a reparametrisation is given by $\sinh^2(\sqrt{d}\tilde{w}) = \frac{dw^2}{ew+2+2\sqrt{p}}$.

**Remark 3.2 (Interpretation of parametrisating functions)** In the normal form (11), there are 5 parametrisating functions. However, only $d = d(x)$ and $e = e(x)$ play a significant role in the shape of the submanifold $S_q$. Indeed, $a$ is a scaling of the submanifold, $c$ is the value of $h$ at $w = 0$, and by an appropriate choice of coordinates (as in (14)), we can always assume that $b \equiv 0$. From the above corollary, the role of $d$ is clear: its sign around $x_0 = 0 \in \mathbb{R}^2$ determines the nature of the submanifold, that is, whether the submanifold is elliptic, hyperbolic, or parabolic.

The role of the function $e$ is, however, more subtle. Clearly, $h$ is well defined whenever $p > 0$ and, for a given $d$, the function $e$ determines the region in which $p > 0$ (in particular, whether $h$ is defined globally with respect to $w$ or not). If $d \equiv 0$, then $p > 0$ holds everywhere ($h$ is defined globally) if and only if $e \equiv 0$ that is, $h$ is explicitly given by $h = 2aw^2 + bw + c$. If $d < 0$, then we have $p > 0$ only between its roots and the parametrisation is never global. Finally, if $d > 0$, then the parametrisation is global if and only if $\Delta < 0$ (where $\Delta$ is the discriminant of $p = 0$), that is $|e| < 2\sqrt{d}$.

### 4 Classification of Quadratic Systems

Notations.

\[
\mathcal{E}_Q = \{\mathcal{E}_{EH}, \mathcal{E}_P\}, \quad \mathcal{E}_{EH} = \{\mathcal{E}_E, \mathcal{E}_H\}
\]

First extension of elliptic, hyperbolic, parabolic submanifolds, seen as a control-nonlinear system on $X$.

\[
\mathcal{E}_Q = (A, B, C)
\]

Triple of vector fields attached to a first extension of a quadratic submanifold.

\[(\alpha, \beta)\]

Reparametrisation (feedback) acting on quadratic nonlinear systems, given by Proposition 4.1.

\[(\mu_0, \mu_1), \gamma = (\gamma_0, \gamma_1)\]

Structure functions attached to $(A, B, C)$ by $[A, B] = \mu_0 A + \mu_1 B$ and $C = \gamma_0 A + \gamma_1 B$; see (18).

$\lambda$

Notation for $\lambda = (\gamma_1, \mp \gamma_0)$, used in the elliptic and hyperbolic cases.

$\Gamma_E, \Gamma_H, \Gamma_P$

Functions given by $\Gamma_E = (\gamma_0)^2 + (\gamma_1)^2$, $\Gamma_H = (\gamma_0)^2 - (\gamma_1)^2$, and $\Gamma_P = \gamma_0^2 + (\gamma_1)^2$.

$g_{\pm}, k_{\pm}$

A (pseudo-)Riemannian metric defined by $g_{\pm}(A, A) = 1$, $g_{\pm}(B, B) = \pm 1$, and $g_{\pm}(A, B) = 0$, and $k_{\pm}$ its Gaussian curvature; see (21).

From Theorem 3.1, we know how to characterise control-affine system equivalent to the quadratic form $\Sigma_q$ and, in particular, we know how to characterise the subclasses of elliptic, hyperbolic, and parabolic systems (see Corollary 3.1). We are now interested in classifying, under feedback transformations, the systems inside those three subclasses. Indeed, due to

1 On [https://www.geogebra.org/m/tyb4ygpb](https://www.geogebra.org/m/tyb4ygpb) the reader can play with those parameters (the functions $a, b, c, d, e$ become real numbers when fixing $x \in X$).
Proposition 2.1, the proposed classification of those classes provides an equivalent classification of the elliptic, hyperbolic, and parabolic submanifolds (see Lemma 4.1 below). To this end, we consider the quadratic nonlinear system

$$\mathcal{E}_Q : \dot{x} = f_Q(x, w),$$

where \( x \in \mathcal{X} \) is the 2-dimensional state, \( w \in \mathbb{R} \) plays the role of a control that enters in a nonlinear way, and \( f_Q \) is a \( w \)-parameterised vector field on \( \mathcal{X} \) given by either

- \( f_E = A(x) \cos(w) + B(x) \sin(w) + C(x) \), defining \( \mathcal{E}_E \), or
- \( f_H = A(x) \cosh(w) + B(x) \sinh(w) + C(x) \), defining \( \mathcal{E}_H \), or
- \( f_P = A(x)w^2 + B(x)w + C(x) \), defining \( \mathcal{E}_P \).

In each of the three cases, \( A, B, \) and \( C \) are smooth vector fields on \( \mathcal{X} \) satisfying \((A \wedge B)(x_0) \neq 0\). We call \( \mathcal{E}_E \) an elliptic system, \( \mathcal{E}_H \) a hyperbolic system, and \( \mathcal{E}_P \) a parabolic system, because in each fiber \( T_x \mathcal{X} \), the system \( \mathcal{E}_E, \mathcal{E}_H, \mathcal{E}_P \) parametrises an ellipse, resp. a hyperbola, resp. a parabola. A quadratic nonlinear system \( \mathcal{E}_Q \) is then represented by the triple \((A, B, C)\) of three smooth vector fields satisfying \( A \wedge B \neq 0 \). We call the pair \((A, B)\) a \( Q \)-frame, and if additionally \([A, B] = 0\), then we call \((A, B)\) a commutative \( Q \)-frame. We will denote by the index \( EH \) objects attached to either the elliptic or the hyperbolic case, as those two are treated in a similar manner.

For quadratic submanifolds, elliptic and hyperbolic, of the form \( \mathcal{S}_{EH} = \{a^2(\dot{z} - c_0)^2 \pm b^2(\dot{y} - c)^2 = 1\} \) and parabolic of the form \( \mathcal{S}_P = \{a\dot{y}^2 - \dot{z} + b\dot{y} + c = 0\} \), we distinguished specific classes (conformally-flat, flat, constant and null forms for elliptic and hyperbolic submanifolds; weakly and strongly flat, constant and null forms for parabolic submanifolds) that are presented in Table 1 of the Introduction. Our goal is to characterise those types of quadratic submanifolds, and we show in the next lemma that the classification of elliptic, hyperbolic, and parabolic submanifolds \( \mathcal{S}_Q \) presented in Table 1 is reflected in properties of the control system \( \mathcal{E}_Q = \langle A, B, C \rangle \).

**Lemma 4.1** Consider a quadratic submanifold \( \mathcal{S}_Q \) together with its regular parametrisation \( \mathcal{E}_Q = \langle A, B, C \rangle \).

(i) \( \mathcal{S}_{EH} \) is locally equivalent to a conformally-flat elliptic/hyperbolic submanifold if and only if \( \mathcal{E}_{EH} \) is locally feedback equivalent to \( \mathcal{E}_{EH} \), whose \( EH \)-frame \((A, B)\) is given by \( A = r(x) \frac{\partial}{\partial z} \) and \( B = r(x) \frac{\partial}{\partial y} \) for some nonvanishing function \( r(x) \).

(ii) \( \mathcal{S}_{EH} \) is locally equivalent to a flat elliptic/hyperbolic submanifold if and only if \( \mathcal{E}_{EH} \) is locally feedback equivalent to \( \mathcal{E}_{EH} \), whose frame \((A, B)\) is commutative.

(iii) \( \mathcal{S}_{EH} \) is locally equivalent to a constant-form elliptic/hyperbolic submanifold if and only if \( \mathcal{E}_{EH} \) is locally feedback equivalent to \( \mathcal{E}_{EH} \), whose \( EH \)-frame \((A, B)\) is commutative and, additionally, \([A, C] = [B, C] = 0\).

(iv) \( \mathcal{S}_{EH} \) is locally equivalent to a null-form elliptic/hyperbolic submanifold if and only if \( \mathcal{E}_{EH} \) is locally feedback equivalent to \( \mathcal{E}_{EH} \), whose \( EH \)-frame \((A, B)\) is commutative and, additionally, \( C = 0 \).

(v) \( \mathcal{S}_P \) is locally equivalent to a weakly-flat parabolic submanifold if and only if \( \mathcal{E}_P \) is locally feedback equivalent to \( \mathcal{E}_P \), whose \( P \)-frame \((A, B)\) is commutative.

(vi) \( \mathcal{S}_P \) is locally equivalent to a strongly-flat parabolic submanifold if and only if \( \mathcal{E}_P \) is locally feedback equivalent to \( \mathcal{E}_P \), whose \( P \)-frame \((A, B)\) is commutative and, additionally, \( A \wedge C = 0 \).

(vii) \( \mathcal{S}_P \) is locally equivalent to a constant-form parabolic submanifold if and only if \( \mathcal{E}_P \) is locally feedback equivalent to \( \mathcal{E}_P \), whose \( P \)-frame \((A, B)\) is commutative and, additionally, \([A, C] = [B, C] = 0\).
(viii) $S_P$ is locally equivalent to a null-form parabolic submanifold if and only if $\Xi_P$ is locally feedback equivalent to $\Xi_P$, whose $P$-frame $(A, B)$ is commutative and, additionally, $C = 0$.

Recall that a general conic submanifold $S_q$ is equivalent to an elliptic $S_E$, resp. a hyperbolic $S_H$, resp. a parabolic $S_P$, submanifold if and only the determinant $\Delta_2$ satisfies $\Delta_2 > 0$, resp. $\Delta_2 < 0$, resp. $\Delta_2 \equiv 0$; see Lemma 2.1. Therefore, the above lemma allows to check equivalence of $S_q$ to a submanifold of any of the subclasses listed in Table 1.

**Proof** It is a straightforward computation to check that for the submanifolds $S_Q$ of the indicated forms, their first extensions $\Xi_Q = (A, B, C)$ have the triple $(A, B, C)$ or the $Q$-frame $(A, B)$ satisfying the stated conditions. Conversely, in local coordinates $x = (z, y)$ in which either $A = r(x) \frac{\partial}{\partial z}$ and $B = r(x) \frac{\partial}{\partial y}$, for (i), or $A = \frac{\partial}{\partial z}$ and $B = \frac{\partial}{\partial y}$, for (ii) to (viii), the system $\Xi_Q$ is a regular parametrisation of $S_Q$ with the desired properties. Now, all items (i) to (viii) follow from Proposition 2.1.

The above lemma asserts that to achieve the classification of elliptic, hyperbolic, and parabolic submanifolds presented in Table 1, it is crucial to classify, under feedback transformations, quadratic control systems $\Xi_Q = (A, B, C)$ with the properties presented in Table 2, which will be the goal of the remaining part of this section. In particular, we will show that the characterisation of item (i), resp. item (v), is always satisfied by any $\Xi_{EH}$ and thus by the corresponding $S_{EH}$, resp. by any $\Xi_P$ and thus by the corresponding $S_P$, while the characterisations of the remaining classes of systems, and thus of the corresponding submanifolds, require non-trivial conditions.

Although the systems of the form $\Xi_Q$ are nonlinear with respect to the control $w$, the feedback transformations that preserve this class are not as general as possible. Indeed, feedback transformations which preserve the class of quadratic system $\Xi_Q$ are affine (and even of Brockett type, in the case of elliptic and hyperbolic systems) with respect to the control $w$, as ensured by the next proposition, which also shows how feedback acts on the triple $(A, B, C)$.

**Proposition 4.1 (Reparametrisation of quadratic nonlinear systems)** Consider two quadratic systems $\Xi_Q$ and $\tilde{\Xi}_Q$ around $(x_0, w_0)$ and $(\tilde{x}_0, \tilde{w}_0)$, respectively.

(i) Two elliptic systems $\Xi_E$ and $\tilde{\Xi}_E$ are locally feedback equivalent if and only if there exists a local diffeomorphism $\tilde{x} = \phi(x)$ and a reparametrisation (feedback) $w = \psi(x, \tilde{w})$, given by $\psi = \pm \tilde{w} + \alpha(x)$, satisfying

$$\tilde{A} = \phi_*(A \cos \alpha + B \sin \alpha), \quad \tilde{B} = \pm \phi_*( -A \sin \alpha + B \cos \alpha), \quad \tilde{C} = \phi_*(C).$$

| Elliptic and hyperbolic classification | Parabolic classification |
|---------------------------------------|-------------------------|
| Conformally-flat $A = r(x) \frac{\partial}{\partial z}$ and $B = r(x) \frac{\partial}{\partial y}$ | Weakly-flat $[A, B] = 0$ |
| Flat $[A, B] = 0$ | Strongly-flat $[A, B] = 0$ and $A \wedge C = 0$ |
| Constant-form $[A, B] = [A, C] = [B, C] = 0$ | Constant-form $[A, B] = [A, C] = [B, C] = 0$ |
| Null-form $[A, B] = 0$ and $C = 0$ | Null-form $[A, B] = 0$ and $C = 0$ |
(ii) Two hyperbolic systems $\Xi_H$ and $\tilde{\Xi}_H$ are locally feedback equivalent if and only if there exists a local diffeomorphism $\tilde{x} = \phi(x)$ and a reparametrisation (feedback) $w = \psi(x, \tilde{w})$, given by $\psi = \pm \tilde{w} + \alpha(x)$, satisfying

$$\tilde{A} = \phi_*(A \cosh \alpha + B \sinh \alpha), \quad \tilde{B} = \pm \phi_*(A \sinh \alpha + B \cosh \alpha), \quad \tilde{C} = \phi_*(C).$$  

(16)

(iii) Two parabolic systems $\Xi_P$ and $\tilde{\Xi}_P$ are locally feedback equivalent if and only if there exists a local diffeomorphism $\tilde{x} = \phi(x)$ and an invertible reparametrisation (feedback) $w = \psi(x, \tilde{w})$, given by $\psi = \alpha(x) + \beta(x)\tilde{w}$ and $\beta(\cdot) \neq 0$, satisfying

$$\tilde{A} = \phi_*(A \beta^2), \quad \tilde{B} = \phi_*(2A\alpha\beta + B\beta), \quad \tilde{C} = \phi_*(C + A\alpha^2 + B\alpha).$$  

(17)

**Proof** We show the necessity of each statement as the converse implications are immediate.

(i) Assume that $\Xi_E$ and $\tilde{\Xi}_E$ are locally equivalent via a diffeomorphism $\tilde{x} = \phi(x)$ and a reparametrisation $w = \psi(x, \tilde{w})$. Then, we have the following relation $\phi_* f_E(x, \psi(x, \tilde{w})) = \tilde{f}_E(\tilde{x}, \tilde{w})$, which we differentiate 3 times with respect to $\tilde{w}$ and using $\frac{\partial^3 \tilde{f}_E}{\partial \tilde{w}^3} = -\frac{\partial f_E}{\partial w}$, we conclude the relation $\phi_* \frac{\partial^3}{\partial \tilde{w}^3} f_E = -\phi_* \frac{\partial}{\partial w} f_E$, which translates into

$$A \left(-\psi''''(\sin(\psi)) + (\psi')^3 \sin(\psi) - 3\psi' \psi'' \cos(\psi)\right) + B \left(\psi''' \cos(\psi) - (\psi')^3 \cos(\psi) - 3\psi' \psi'' \sin(\psi)\right) = \alpha \psi'(\sin(\psi) - B \psi' \cos(\psi),$$

where the derivatives are taken with respect to $\tilde{w}$. Since the functions cos and sin are linearly independent, we obtain $\psi'' = 0$ and $(\psi')^2 = 1$. Thus, $\psi(x, \tilde{w}) = \pm \tilde{w} + \alpha(x)$. Applying this reparametrisation to $\Xi_E = (A, B, C)$, we obtain the relations of (15).

(ii) Exactly the same reasoning, using $f_H$ and the fact $\phi_* \frac{\partial^3}{\partial \tilde{w}^3} f_H = \phi_* \frac{\partial}{\partial w} f_H$, implies that $\psi(x, \tilde{w}) = \pm \tilde{w} + \alpha(x)$. Applying $\phi(x)$ and $w = \psi(x, \tilde{w}) = \pm \tilde{w} + \alpha$ to $\Xi_H = (A, B, C)$, we obtain the relations of (16).

(iii) We repeat again the same reasoning to $f_p$ with the property $\phi_* \frac{\partial^3}{\partial \tilde{w}^3} f_p = 0$. However, this time, we obtain the conditions $\psi''' = 0$ and $\psi'''' + 3\psi' \psi''' = 0$ on the reparametrisation $\psi$, which implies $\psi'''' = 0$, that is, $\psi(x, \tilde{w}) = \beta(x)\tilde{w} + \alpha(x)$, with $\beta$ satisfying $\beta(\cdot) \neq 0$. Applying this reparametrisation together with a diffeomorphism $\phi$ yields the relations of (17). $\square$

**Remark 4.1** (Local character of the results) Initially, $\Xi_Q$ was considered locally around a point $x_0$ and a control $w_0$, however, since $\Xi_Q$ is defined globally with respect to $w$ and, moreover, by the last proposition, the transformations $w = \psi(x, \tilde{w})$ are global with respect to $\tilde{w}$, so we will consider the systems $\Xi_Q$ and their equivalence locally in $x$ and globally with respect to $w$. All results below are stated assuming this structure.

We will develop relations involving structure functions attached to any fixed triple $(A, B, C)$ in a unique way and thus change accordingly with diffeomorphisms $\tilde{x} = \phi(x)$. So we will act on $(A, B, C)$ by $(\alpha, \beta)$ only ($\beta$ is $\pm 1$ in the elliptic and hyperbolic cases) and we will denote by $(\tilde{A}, \tilde{B}, \tilde{C})$ the result of that action (given by (15), or (16), or (17), with $\phi = \text{id}$), called a reparametrisation.

Observe that the reparametrisations of $\Xi_P$ depend on two smooth functions $\alpha$ and $\beta$ while those of $\Xi_E$ and $\Xi_H$ depend on one smooth function $\alpha$ only. Therefore, we expect the classification of parabolic systems to be less rich (less parametrising functions) than the classification of elliptic and hyperbolic systems. In the following subsections, we will first classify elliptic and hyperbolic systems as the procedures are similar, and then we will classify parabolic systems under reparametrisation actions.
4.1 Classification of Elliptic and Hyperbolic Systems

In this subsection, we classify elliptic and hyperbolic systems under the action of reparametrisations. Recall that our aim is to classify elliptic and hyperbolic nonholonomic constraints $S_E$ and $S_H$ (submanifolds of $T\mathcal{X}$), which are parameterised by systems of the form $\Xi_E$ and $\Xi_H$, respectively. The classification of submanifolds given in Table 1 of the Introduction is reflected in special properties of the vector fields $(A, B, C)$, attached to the control system $\Xi_E$ and $\Xi_H$, that we list in Table 2 above and summarise in Lemma 4.1. Firstly, we will give a normal form for both types of systems $\Xi_E$ and $\Xi_H$ showing that they actually depend on three smooth functions, that normal form corresponds to conformally-flat elliptic and hyperbolic submanifolds. Secondly, we will further develop their classification, in particular we will give conditions for the existence of commutative frames (corresponding to flat elliptic and hyperbolic submanifolds) and a complete characterisation of forms without functional parameters, corresponding to constant-form (and, in particular, null-form) elliptic and hyperbolic submanifolds.

Notations. In order to simplify and unify notations, in the following formulae, the upper sign always corresponds to the elliptic case and the lower sign to the hyperbolic case, e.g. we will use the symbol $\pm$ to design similar objects attached to the elliptic ($+$ case) and to the hyperbolic ($-$ case) systems and in the case of a $\mp$ symbol we have $-$ for elliptic systems and $+$ for hyperbolic ones. We denote $\Xi_{EH}$ elliptic and hyperbolic systems, and an EH-frame stands for an E-frame or an H-frame. To avoid unnecessary computations, we assume that EH-frames $(A, B)$ and $(\tilde{A}, \tilde{B})$ of two equivalent systems have the same orientation (we will come back to this simplification in Proposition 4.4); therefore we restrict reparametrisations of the control to $w = \tilde{w} + \alpha(x)$ thus resulting in the "+" sign in (15) and (16). Denoting by $\tilde{R}_{EH}(\alpha)$ the (trigonometric or hyperbolic) rotation matrix given by

$$\tilde{R}_{E}(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad \text{and} \quad \tilde{R}_{H}(\alpha) = \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{pmatrix},$$

respectively, we see from (15) and (16) that EH-frames are transformed by $(\tilde{A}, \tilde{B}) = (A, B)\tilde{R}_{EH}(\pm \alpha)$ under reparametrisations of the form $w = \tilde{w} + \alpha$. Introduce structure functions $(\mu_0, \mu_1)$ and $(\gamma_0, \gamma_1)$ uniquely defined by

$$[A, B] = \mu_0 A + \mu_1 B \quad \text{and} \quad C = \gamma_0 A + \gamma_1 B, \quad (18)$$

respectively. We denote $\gamma = (\gamma_0, \gamma_1)$, and $\mu = (\mu_1, \mp \mu_0)$, and set $\Gamma_{EH} = (\gamma_0)^2 \pm (\gamma_1)^2$.

We begin by a technical lemma showing how structure functions behave under reparametrisations of the control $w$.

Lemma 4.2 (Transformation of structure functions) Consider an elliptic/hyperbolic system $\Xi_{EH}$ with structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$. Then under the reparametrisation $w = \tilde{w} + \alpha(x)$, we have

$$(\tilde{\mu}_0, \tilde{\mu}_1) = (\mu_0 \mp L_A(\alpha), \mu_1 - L_B(\alpha)) \tilde{R}_{EH}(\alpha), \quad \text{and} \quad \tilde{\gamma} = \gamma \tilde{R}_{EH}(\alpha). \quad (19)$$

Proof Details of the computations can be found in Appendix C. \qed

Clearly, from (19), $\Gamma_{EH} = (\gamma_0)^2 \pm (\gamma_1)^2$ is invariant under reparametrisations, i.e. $\tilde{\Gamma}_{EH} = \Gamma_{EH}$. 
Proposition 4.2 (Conformal form of elliptic and hyperbolic systems)

(i) Any elliptic system $\mathcal{E}_E$, resp. hyperbolic system $\mathcal{E}_H$, always admits under a reparametrisation of the following conformal form, locally around $x_0$,

$$
\mathcal{E}_E^c : \dot{x} = r(x) \left( \frac{\cos(w)}{\sin(w)} \right) + \left( c_0(x) c_1(x) \right), \quad \text{resp.} \quad \mathcal{E}_H^c : \dot{x} = r(x) \left( \cosh(w) \right) + \left( c_0(x) c_1(x) \right),
$$

with $r$ a smooth function satisfying $r > 0$. 

(ii) Two conformal forms $\mathcal{E}_E^c$ and $\mathcal{E}_E^c$, resp. $\mathcal{E}_H^c$ and $\mathcal{E}_H^c$, are locally feedback equivalent if and only if there exists a local diffeomorphism $\tilde{x} = \phi(x) = (\phi_1(x), \phi_2(x))$, where $x = (z, y)$, satisfying

$$
\frac{\partial \phi_1}{\partial z} + \frac{\partial \phi_2}{\partial y} = \pm \frac{\partial \phi_1}{\partial y} \quad \text{and} \quad \left( \frac{\partial \phi_1}{\partial y} \right)^2 + \left( \frac{\partial \phi_1}{\partial z} \right)^2 = \left( \frac{\tilde{r}}{r} \right)^2, \quad \text{where} \quad \tilde{r} \text{is expressed in} \ x\text{-coordinates, i.e. we set} \ \tilde{r} = \tilde{r}(\phi(x)).
$$

We call $\mathcal{E}_{E,H}$ a conformal-form because the systems of that class parametrise elliptic and hyperbolic submanifolds for which the quadratic term, interpreted as a (pseudo-)Riemannian metric, is conformally flat.

Proof (i) For the system $\mathcal{E}_{E,H} = (A, B, C)$, define a (pseudo-)Riemannian metric $g_\pm$ on $\mathcal{X}$ by $g_\pm(A, A) = 1$, $g_\pm(B, B) = \pm 1$, and $g_\pm(A, B) = 0$. It is known that any non-degenerate metric on a manifold of dimension two is conformally flat (see [3, pp 15-35] or [26, Addendum 1 of chapter 9] for the elliptic case and [25, theorem 7.2] for the hyperbolic case). Therefore, there exists a local diffeomorphism $(\tilde{z}, \tilde{y}) = \tilde{x} = \phi(x)$ such that $g_\pm = g^* \tilde{g}_\pm$, where $\tilde{g}_\pm = \rho(\tilde{x}) (\tilde{d}z^2 \pm \tilde{d}y^2)$, $\rho > 0$. The vector fields $\tilde{A} = \phi_* A$ and $\tilde{B} = \phi_* B$ satisfy $\tilde{g}_\pm(A, A) = 1$, $\tilde{g}_\pm(\tilde{B}, \tilde{B}) = \pm 1$, and $\tilde{g}_\pm(\tilde{A}, \tilde{B}) = 0$ which implies that $(\tilde{A}, \tilde{B})$ is a (pseudo-)orthonormal frame. Finally, using the feedback $w \mapsto w - \alpha$ we can smoothly rotate $(\tilde{A}, \tilde{B})$ into $\left( r \frac{\partial}{\partial z}, r \frac{\partial}{\partial y} \right)$ with $r = \frac{1}{\sqrt{\rho}}$, which gives the desired form $\mathcal{E}_{E,H} = (\tilde{A}, \tilde{B}, \tilde{C})$, with $\tilde{C} = \phi_* C$. 

(ii) By relations (15) and (16), the reparametrisations do not act on $C$ and thus the relation $\tilde{C} = \phi_* C$ is necessary for the equivalence of conformal forms. Consider two elliptic conformal systems $\mathcal{E}_E^c$ and $\mathcal{E}_E^c$, resp. two hyperbolic conformal systems $\mathcal{E}_H^c$ and $\mathcal{E}_H^c$, with frames $(A, B)$ and $(\tilde{A}, \tilde{B})$ and related by a feedback $w = \tilde{w} + \alpha$ and a diffeomorphism $\phi$. In all computations below, $\tilde{r}$ is expressed in $x$-coordinates, that is, we set $\tilde{r}(\phi(x))$. Thus, using relation (15), resp. (16), we obtain

$$
\tilde{r} \cos(\alpha) = \frac{\partial \phi_1}{\partial y}, \quad \tilde{r} \sin(\alpha) = -\frac{\partial \phi_2}{\partial z},
$$

resp.

$$
\tilde{r} \cos(\alpha) = \frac{\partial \phi_1}{\partial y}, \quad \tilde{r} \sin(\alpha) = -\frac{\partial \phi_2}{\partial z},
$$

from which we deduce condition (20). Conversely, applying the diffeomorphism $\phi$ given by (20), together with the feedback $w = \tilde{w} + \alpha$, with $\alpha$ being a solution of

$$
\cos(\alpha) = \frac{r}{\tilde{r}} \frac{\partial \phi_1}{\partial z}, \quad \sin(\alpha) = -\frac{r}{\tilde{r}} \frac{\partial \phi_1}{\partial y},
$$

resp.

$$
\cos(\alpha) = \frac{r}{\tilde{r}} \frac{\partial \phi_1}{\partial z}, \quad \sin(\alpha) = -\frac{r}{\tilde{r}} \frac{\partial \phi_1}{\partial y},
$$
we transform $\Sigma_E^c$ into $\tilde{\Sigma}_E^c$, resp. $\Sigma_H^c$ into $\tilde{\Sigma}_H^c$.

**Remark 4.2** In the above proof, we used the metric $g_{\pm}$ on $\mathcal{X}$ defined by

$$g_{\pm}(A, A) = 1, \quad g_{\pm}(B, B) = \pm 1, \quad g_{\pm}(A, B) = 0. \quad (21)$$

This object will play a special role in the interpretation of the conditions describing the existence of a commutative EH-frame.

The above proposition shows that elliptic and hyperbolic systems $\Sigma_{EH}$ are parametrised by three smooth functions of two variables (and not by 6 functions defining the triple $(A, B, C)$). Now we will pass to the problem of commutative frames, and the following proposition gives equivalent algebraic and geometric conditions for the existence of a commutative EH-frame.

**Proposition 4.3** (Existence of a commutative EH-frame) Consider an elliptic/hyperbolic system $\Sigma_{EH} = (A, B, C)$ with structure functions $(\mu_0, \mu_1)$ of the EH-frame $(A, B)$. The following statements are equivalent locally around $x_0$:

(i) There exists a commutative EH-frame.

(ii) The structure functions $(\mu_0, \mu_1)$ attached to the EH-frame $(A, B)$ satisfy

$$-(\mu_0)^2 \mp (\mu_1)^2 \pm L_A(\mu_1) - L_B(\mu_0) = 0. \quad (22)$$

(iii) The Gaussian curvature $\kappa_\pm$ of the metric $g_{\pm}$ vanishes.

Notice that item (i) describes the following normal forms,

$$\Sigma'_E : \begin{cases} \dot{z} = \cos(w) + c_0(x) \\ \dot{y} = \sin(w) + c_1(x) \end{cases}, \quad \text{and} \quad \Sigma'_H : \begin{cases} \dot{z} = \cosh(w) + c_0(x) \\ \dot{y} = \sinh(w) + c_1(x) \end{cases},$$

whose structure functions are $\mu_0 = \mu_1 = 0$, $\gamma_0 = c_0$, and $\gamma_1 = c_1$. We call $\Sigma'_E$ a flat elliptic system and $\Sigma'_H$ a flat hyperbolic system.

**Proof** The equivalence between (ii) and (iii) is immediate since the left hand side of (22) is the Gaussian curvature $\kappa_\pm$ of $g_{\pm}$ (details of the computations are in Appendix D). We show that (i) is equivalent to (ii). If the EH-frame $(A, B)$ is equivalent via $w = \tilde{w} + \alpha(x)$ to a commutative EH-frame $(\tilde{A}, \tilde{B})$, then by (19), we immediately have $L_A(\alpha) = \pm \mu_0$ and $L_B(\alpha) = \mu_1$; the integrability condition of this system of first order partial differential equations gives (22). Conversely, consider the system $\tilde{\Sigma}_{EH} = (A, B, C)$ and construct $\alpha$ as a solution of the system $L_A(\alpha) = \pm \mu_0$ and $L_B(\alpha) = \mu_1$, whose solvability is guaranteed by the integrability condition given by (22). Then by (19), we see that the resulting EH-frame $(\tilde{A}, \tilde{B})$, of the system $\tilde{\Sigma}_{EH}$ obtained by the reparametrisation $w = \tilde{w} + \alpha$, is commutative. □

Notice that when proving Proposition 4.3, we have shown that the Gaussian curvature $\kappa_\pm$ of the metric $g_{\pm}$ is given by the left hand side of (22). Moreover, relation (19) implies that $\kappa_\pm$ is invariant under reparametrisations $w = \tilde{w} + \alpha$ and is therefore an equivariant of the feedback transformations of the system $\Sigma_{EH}$.

In the following proposition, we give first a classification of flat elliptic/hyperbolic systems; second, we characterise those without functional parameters, i.e. constant-forms; and third, we provide a canonical form for the latter. Recall that $\mathcal{A} = (\gamma_1, \mp \gamma_0)$ and that for flat elliptic/hyperbolic systems $\Sigma_{EH}$, we have $(\gamma_0, \gamma_1) = (c_0, c_1)$ so all statements of the proposition below are actually expressed in terms of structure functions. From now on, we will consider the group of feedback transformations consisting of $\tilde{x} = \phi(x)$ and $w = \pm \tilde{w} + \alpha(x)$. 

\[ Springer \]
The additional transformation \( w = -\tilde{w} + \alpha \) implies \((\tilde{A}, \tilde{B}) = (A, B) \tilde{R}_{EH}(\pm \alpha)\), where \( \tilde{R}_E(\alpha) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) - \alpha \end{pmatrix} \) and \( \tilde{R}_H(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) - \alpha \end{pmatrix} \), and the corresponding structure functions change by, compare (19),

\[
(\tilde{\mu}_0, \tilde{\mu}_1) = - (\mu_0 + L_A(\alpha), \mu_1 - L_B(\alpha)) \tilde{R}_{EH}(\alpha) \quad \text{and} \quad \tilde{\gamma} = \gamma \tilde{R}_{EH}(\alpha). \tag{23}
\]

**Proposition 4.4** (Characterisation and classification of flat elliptic/hyperbolic systems)

(i) Two flat elliptic systems \( \Xi'_E \) and \( \Xi''_E \), resp. two flat hyperbolic systems \( \Xi'_H \) and \( \Xi''_H \), are locally feedback equivalent around \( x_0 = 0 \in \mathbb{R}^2 \) if and only if there exists a constant \( \alpha \in \mathbb{R} \) satisfying

\[
R_{EH}^{-1}(\pm \alpha)C(x) = \tilde{C} \left( R_{EH}^{-1}(\pm \alpha)x \right), \tag{24}
\]

where \( R_{EH} \) stands for either \( \tilde{R}_{EH} \) or \( \tilde{R}_{EH} \).

(ii) An elliptic/hyperbolic system \( \Xi_{EH} \) is locally feedback equivalent to a constant-form, i.e. \( \Xi'_{EH} \) with \((c_0, c_1) \in \mathbb{R}^2\), if and only if one of the equivalent conditions of Proposition 4.3 holds and, additionally,

\[
L_A(\gamma) + _A\mu_0 = 0 \quad \text{and} \quad L_B(\gamma) \pm _A\mu_1 = 0. \tag{25}
\]

(iii) A constant-form elliptic system is always feedback equivalent, locally around \( x_0 = 0 \in \mathbb{R}^2 \), to the canonical form

\[
\Xi_E^{\Gamma} \colon \begin{cases} \dot{z} = \cos(w) + \sqrt{\Gamma_E} \\ \dot{y} = \sin(w) \end{cases}
\]

where \( \Gamma_E = (c_0)^2 + (c_1)^2 \in \mathbb{R} \) is a complete invariant of constant-form elliptic systems.

(iv) A constant-form hyperbolic system is always feedback equivalent, locally around \( x_0 = 0 \in \mathbb{R}^2 \), to one of the following canonical forms

\[
\Xi_H^{\Gamma_1} \colon \begin{cases} \dot{z} = \cosh(w) + \varepsilon \sqrt{\Gamma_H} \\ \dot{y} = \sinh(w) \end{cases}, \quad \text{or} \quad \Xi_H^{-\Gamma_1} \colon \begin{cases} \dot{z} = \cosh(w) \\ \dot{y} = \sinh(w) + \sqrt{-\Gamma_H} \end{cases},
\]

\[
\Xi_H^{0,0} \colon \begin{cases} \dot{z} = \cosh(w) \varepsilon \\ \dot{y} = \sinh(w) + 1 \end{cases}, \quad \text{or} \quad \Xi_H^{0,0} \colon \begin{cases} \dot{z} = \cosh(w) \\ \dot{y} = \sinh(w) \end{cases},
\]

where \( \Gamma_H = (c_0)^2 - (c_1)^2 \in \mathbb{R} \) and satisfies \( \Gamma_H > 0 \) for the first form, \( \Gamma_H < 0 \) for the second form, and \( \Gamma_H = 0 \) for the third and fourth ones, where \( \varepsilon = \text{sgn}(c_0) = \pm 1 \). Moreover \( \Gamma_H, \varepsilon \) is a complete invariant of constant-form hyperbolic systems.

Observe that if in items (i), (iii), and (iv) the considered systems are defined globally, then their feedback equivalence is also global and, in particular, the proposed canonical forms are also global.

**Remark 4.3** In item (iv), notice that there are two orbits of the local action of feedback transformations group for any \( \Gamma_H > 0 \), corresponding to \( \text{sgn}(c_0) = \varepsilon = \pm 1 \), one orbit for any \( \Gamma_H < 0 \), and three orbits for \( \Gamma_H = 0 \) corresponding, respectively, to \( \text{sgn}(c_0) = \varepsilon = \pm 1 \) and to \((c_0, c_1) = (0, 0)\). The invariant \( \varepsilon = \pm 1 \) corresponds to the parametrisation of one of two branches of the hyperbola \((\dot{z} - \sqrt{\Gamma_H})^2 - \dot{y}^2 = 1\).

**Proof** (i) Consider, locally around \( 0 \in \mathbb{R}^2 \), two equivalent flat elliptic/hyperbolic systems \( \Xi'_{EH} \) and \( \Xi''_{EH} \) given by structure functions \((\mu_0, \mu_1, \gamma_0, \gamma_1) = (0, 0, c_0, c_1)\) and
Consider a flat elliptic system $(\bar{\mu}_0, \bar{\mu}_1, \tilde{\gamma}_0, \tilde{\gamma}_1) = (0, 0, \tilde{c}_0, \tilde{c}_1)$, respectively. Since they both have a commutative EH-frame, by (19) and (23), they differ by a reparametrisation $w = \pm \tilde{w} + \alpha$ satisfying $L_A(\alpha) = L_B(\alpha) = 0$ and thus $\alpha \in \mathbb{R}$. Applying this reparametrisation together with a diffeomorphism $\phi$ satisfying $\phi_* = \mathcal{R}_{EH}^{-1}(\pm \alpha)$, that is, $\tilde{x} = \phi(x) = \mathcal{R}_{EH}^{-1}(\pm \alpha)x$, transforms $\mathcal{Z}_{EH}$ into $\mathcal{Z}_{EH}'$ if and only if

$$
\begin{pmatrix}
\tilde{c}_0(\tilde{x}) \\
\tilde{c}_1(\tilde{x})
\end{pmatrix} = \mathcal{R}_{EH}^{-1}(\pm \alpha)
\begin{pmatrix}
c_0(x) \\
c_1(x)
\end{pmatrix},
$$

which is (24).

(ii) Assume that $\mathcal{Z}_{EH}$, given by structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$, is equivalent via $\tilde{x} = \phi(x)$ and $w = \pm \tilde{w} + \alpha$ to $\mathcal{Z}_{EH}'$ with structure functions $(\bar{\mu}_0, \bar{\mu}_1, \bar{\gamma}_0, \bar{\gamma}_1) = (0, 0, c_0, c_1)$, where $(c_0, c_1) \in \mathbb{R}^2$. Necessity of one (and thus any) of the conditions of Proposition 4.3 is clear, and by (19) and (23), we have first, $L_A(\alpha) = \pm \mu_0$ and $L_B(\alpha) = \mu_1$ and second, $\gamma \mathcal{R}_{EH}(\alpha) = \bar{\gamma} = (c_0, c_1)$; recall that $\mathcal{R}_{EH}$ stands for either $\mathcal{R}_{EH}$ or $\bar{\mathcal{R}}_{EH}$. By differentiating this last relation along $A$ and $B$ we obtain

$$
\begin{align*}
0 &= L_A(\gamma) \mathcal{R}_{EH}(\alpha) + \gamma L_A(\mathcal{R}_{EH}(\alpha)) \\
&= L_A(\gamma) \mathcal{R}_{EH}(\alpha) + \gamma \left( \pm L_A(\alpha) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \mathcal{R}_{EH}(\alpha),
\end{align*}
$$

$$
\begin{align*}
0 &= L_B(\gamma) \mathcal{R}_{EH}(\alpha) + \gamma L_B(\mathcal{R}_{EH}(\alpha)) \\
&= L_B(\gamma) \mathcal{R}_{EH}(\alpha) + \gamma \left( \pm L_B(\alpha) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \mathcal{R}_{EH}(\alpha),
\end{align*}

$$

thus proving that (25) holds. Conversely, assume that (22) and (25) hold for $\mathcal{Z}_{EH}$. By Proposition 4.3, $\mathcal{Z}_{EH}$ is equivalent to $\mathcal{Z}_{EH}'$ with a commutative EH-frame $(A, B)$, and applying (25) to the latter, we get $L_A(\gamma) = L_B(\gamma) = 0$ and therefore, we have $(c_0, c_1) \in \mathbb{R}^2$.

(iii) Consider a flat elliptic system $\mathcal{Z}_{EH}'$ with $(c_0, c_1) \in \mathbb{R}^2$, then relation (24) reads

$$
\begin{pmatrix}
\tilde{c}_0 \\
\tilde{c}_1
\end{pmatrix} = \begin{pmatrix}
\cos(\alpha) & \sin(\alpha) \\
-\sin(\alpha) & \pm \cos(\alpha)
\end{pmatrix} \begin{pmatrix}
c_0 \\
c_1
\end{pmatrix}.
$$

Take $\alpha$ as a solution of $-\sin(\alpha)c_0 + \cos(\alpha)c_1 = 0$, then we have $\tilde{c}_1 = 0$ and $\tilde{c}_0 = \pm \sqrt{\Gamma_E}$, with $\Gamma_E = (c_0)^2 + (c_1)^2$. If necessary, apply $\alpha = \pi$ to send $(\tilde{c}_0, \tilde{c}_1) = (-\sqrt{\Gamma_E}, 0)$ into $(\sqrt{\Gamma_E}, 0)$. The proof that $\mathcal{Z}_{EH}'$ is equivalent to $\mathcal{Z}_{EH}^{\Gamma_E}$ if and only if $\Gamma_E = \Gamma_E$ is immediate from (24).

(iv) Consider a flat hyperbolic system $\mathcal{Z}_{EH}'$ with $(c_0, c_1) \in \mathbb{R}^2$ and denote $\Gamma_H = (c_0)^2 - (c_1)^2$, then relation (24) reads

$$
\begin{pmatrix}
\tilde{c}_0 \\
\tilde{c}_1
\end{pmatrix} = \begin{pmatrix}
\cosh(\alpha) & -\sinh(\alpha) \\
-\sinh(\alpha) & \pm \cosh(\alpha)
\end{pmatrix} \begin{pmatrix}
c_0 \\
c_1
\end{pmatrix}.
$$

We consider four cases. First, assume that $\Gamma_H > 0$, that is $c_0 \neq 0$ and $-1 < \frac{c_0}{c_1} < 1$, and take $\alpha$ as the solution of $\tanh(\alpha) = \frac{c_0}{c_1}$ yielding $\tilde{c}_1 = 0$ and $\tilde{c}_0 = \text{sgn}(c_0) \sqrt{\Gamma_H}$, which gives the canonical form $\mathcal{Z}_{EH}^{\Gamma_H,1}$. Second, assume that $\Gamma_H < 0$, that is $c_1 \neq 0$ and $-1 < \frac{c_0}{c_1} < 1$, and take $\alpha$ as the solution of $\tanh(\alpha) = \frac{c_0}{c_1}$ yielding $\tilde{c}_0 = 0$ and $\tilde{c}_1 = \text{sgn}(c_1) \sqrt{-\Gamma_H}$. If $\text{sgn}(c_1) = -1$, then by applying (24") with $\alpha = 0$ and the
bottom row, we can always normalise $\text{sgn} (c_1)$ to $+1$ yielding the canonical form $\Sigma_{H}^{\Gamma_H}$. Third, assume that $\Gamma_H = 0$ and $c_0 = 0$ thus $c_1 = 0$ and therefore we immediately have the canonical form $\Sigma_{H}^{0,0}$. Fourth, and finally, assume that $\Gamma_H = 0$ and $c_0 \neq 0$, thus $c_1 = \varepsilon c_0$ with $\varepsilon = \pm 1$. If necessary, apply (24") with $\alpha = 0$ and the bottom sign to obtain $c_1 > 0$. Take $\alpha = \varepsilon \ln c_1$ and apply (24") with the upper sign to obtain $\bar{c}_1 = 1$ and $\bar{c}_0 = \varepsilon$. To show that $(\Gamma_H, \varepsilon)$ is a complete invariant is trivial by applying (24") to the canonical forms $\Sigma_{H}^{\Gamma_H,\varepsilon}$, $\Sigma_{H}^{-\Gamma_H}$, $\Sigma_{H}^{0,\varepsilon}$, and $\Sigma_{H}^{0,0}$. $\square$

We summarise the results of this subsection: we started from a general elliptic, resp. hyperbolic, system $\Sigma_E$, resp. $\Sigma_H$, which parametrises an elliptic, resp. a hyperbolic, submanifold $S_E$, resp. $S_H$. We showed that reparametrisations (pure feedback transformations) acting on the class of elliptic/hyperbolic systems $\Sigma_{EH}$ are affine (actually they are of the Brockett type $w = \pm \tilde{w} + \alpha$) with respect to the control and thus depend on one function $\alpha$ only. By transforming into the conformal form, we showed that elliptic/hyperbolic systems $\Sigma_{EH}$ are given by three arbitrary smooth functions. Next, we showed that the vanishing of the Gaussian curvature of a (pseudo-)Riemannian metric, associated with the EH-frame of $\Sigma_{EH}$, characterises the flat elliptic/hyperbolic systems $\Sigma_{EH}'$, which depend on two arbitrary smooth functions only. Finally, we gave conditions characterising the constant-form systems, i.e. elliptic/hyperbolic systems without functional parameters. In the elliptic case, equivalent constant-form systems correspond to the circles $\Gamma_E = (c_0)^2 + (c_1)^2 = \text{const.}$, and their canonical forms are parametrised by a closed half-line of real constants. On the other hand, in the hyperbolic case the structure is richer because equivalent systems correspond to connected branches of the hyperbolas $\Gamma_H = (c_0)^2 - (c_1)^2 = \text{const.}$; two connected components for $\Gamma_H > 0$, one for $\Gamma_H < 0$, and three for $\Gamma_H = 0$. Thus canonical forms of hyperbolic systems are parametrised by a real line of constants (the value of $\Gamma_H$) and by a discrete invariant $\varepsilon = \pm 1$ (if either $\Gamma_H > 0$ or $\Gamma_H = 0$ and $c_0 \neq 0$). Our characterisation of elliptic/hyperbolic systems gives an equivalent classification for elliptic/hyperbolic submanifolds that is summarised in Lemma 4.1 and explicitly given in [24].

4.2 Classification of Parabolic Systems

We now turn to the classification of parabolic systems $\Sigma_P = (A, B, C)$, of the form

$$\Sigma_P : \dot{x} = A(x)w^2 + B(x)w + C(x),$$

which is expected to be different from that of elliptic and hyperbolic systems because the allowed reparametrisations of the control $w$ depend on 2 smooth functions $(\alpha, \beta)$, see Proposition 4.1. Our aim is to get a classification of parabolic submanifolds $S_P$, as the one presented in Table 1 of the Introduction, via the properties of the triple $(A, B, C)$ presented in Table 2 at the beginning of Sect. 4. In particular, the existence of a commutative P-frame $(A, B)$ corresponds to weakly-flat parabolic submanifolds; existence of a commutative P-frame, which additionally satisfies $A \wedge C = 0$, corresponds to strongly-flat submanifolds; finally, existence of a commutative P-frame, which additionally satisfies $C$ constant (in the coordinates, where $(A, B)$ is rectified), corresponds to constant-form parabolic submanifolds; in particular, the case of $C = 0$ is called null-form. As in the elliptic/hyperbolic cases, we introduce the structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$ uniquely defined for any triple $(A, B, C)$ by

$$[A, B] = \mu_0 A + \mu_1 B \quad \text{and} \quad C = \gamma_0 A + \gamma_1 B.$$
By a direct computation, we obtain that under reparametrisations of the form \( w = \beta \tilde{w} + \alpha \) the structure functions \( (\mu_0, \mu_1, \gamma_0, \gamma_1) \) of \( (A, B, C) \) are transformed into the structure functions \( (\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\gamma}_0, \tilde{\gamma}_1) \) of \( (\tilde{A}, \tilde{B}, \tilde{C}) \) via the following relations:

\[
\begin{align*}
\tilde{\gamma}_0 &= \frac{1}{\beta^2} (\gamma_0 - 2\alpha \gamma_1 - \alpha^2), \\
\tilde{\gamma}_1 &= \frac{1}{\beta} (\gamma_1 + \alpha), \\
\tilde{\mu}_0 &= \beta \mu_0 - 2\alpha L_A(\beta) + 2\beta L_A(\alpha) - 2L_B(\beta) - 2\alpha (L_A(\beta) + \beta \mu_1), \\
\tilde{\mu}_1 &= \beta^2 \mu_1 + \beta L_A(\beta).
\end{align*}
\] (26) (27)

There are two main questions that we will answer. First, when does a commutative P-frame \( (\tilde{A}, \tilde{B}) \) exist, i.e. \( \tilde{\mu}_0 = \tilde{\mu}_1 = 0 \)? Second, provided that a commutative P-frame \( (A, B) \) has been normalised, how can we additionally simplify \( C \)? Contrary to the elliptic and hyperbolic cases the answer to the first question is always positive without any additional assumption, as ensured by the next result.

**Proposition 4.5 (Existence of a commutative P-frame)**

(i) For any P-frame \( (A, B) \) there exists, locally around \( x_0 \), a reparametrisation \( (\alpha, \beta) \) such that \( (\tilde{A}, \tilde{B}) \) is a commutative P-frame.

(ii) If \( (A, B) \) is a commutative P-frame, then \( (\tilde{A}, \tilde{B}) \) is also a commutative P-frame if and only if the reparametrisation \( (\alpha, \beta) \) satisfies

\[
L_A(\beta) = 0 \text{ and } \frac{1}{\beta} L_B(\beta) = L_A(\alpha).
\] (28)

**Proof** (i) Consider a P-frame \( (A, B) \) whose structure functions are \( (\mu_0, \mu_1) \). Apply a reparametrisation \( (\alpha, \beta), \beta \neq 0 \), given by a solution of the following system of equations, in which we solve the first equation for \( \beta \), and plug in \( \beta \) into the second equation to solve it for \( \alpha \),

\[
\begin{align*}
L_A(\beta) &= -\beta \mu_1 \\
L_A(\alpha) + \alpha \mu_1 &= \frac{1}{\beta^2} (2L_B(\beta) - \beta \mu_0).
\end{align*}
\]

Then formula (27) implies \( \tilde{\mu}_0 = \tilde{\mu}_1 = 0 \), i.e. \( \tilde{[\tilde{A}, \tilde{B}]} = 0 \). Notice that to ensure \( \beta \neq 0 \), we may actually solve \( L_A(\ln \beta) = -\mu_1 \).

(ii) Using relation (27) with \( \mu_i = \tilde{\mu}_i = 0 \), for \( i = 0, 1 \), we see that all reparametrisations \( (\alpha, \beta) \) have to satisfy relations (28). Conversely, if \( (A, B) \) is a commutative P-frame \( (\mu_0 = \mu_1 = 0) \) and \( (\alpha, \beta) \) is any solution of (28), with \( \beta \neq 0 \), then by (27) we obtain that \( \tilde{\mu}_0 = \tilde{\mu}_1 = 0 \), i.e. \( (\tilde{A}, \tilde{B}) \) is also a commutative P-frame. \( \Box \)

Immediately, item (i) of Proposition 4.5 gives the following prenormal forms of parabolic systems \( \mathcal{P} \). Recall that systems and equivalence are considered locally with respect to the state \( x \) and globally with respect to the control \( w \in \mathbb{R} \).

**Corollary 4.1 (Prenormal forms of \( \mathcal{P} \))** The parabolic system \( \mathcal{P} \) is always feedback equivalent to the following prenormal forms, locally around \( x_0 \):

\[
\begin{align*}
\mathcal{P}' : \quad &\begin{cases} 
\dot{z} = w^2 + c_0(x) \\
\dot{y} = w + c_1(x)
\end{cases}, & \quad \mathcal{P}'' : \quad &\begin{cases} 
\dot{z} = w^2 + b(x)w + \Gamma_P(x) \\
\dot{y} = w
\end{cases},
\end{align*}
\]

whose structure functions are \( (\mu'_0, \mu'_1, \gamma'_0, \gamma'_1) = (0, 0, c_0, c_1) \), and \( (\mu''_0, \mu''_1, \gamma''_0, \gamma''_1) = (\frac{\partial b}{\partial z}, 0, \Gamma_P, 0) \), respectively.
A parabolic system of the form $\mathcal{E}'_P$ is called weakly-flat, this terminology is justified by
generalisations of the previous result to higher dimensions that we will present in a future
paper.

**Remark 4.4** Since any parabolic system can be brought into $\mathcal{E}'_P$ (and into $\mathcal{E}''_P$), it follows
that all parabolic systems are locally parametrised by two functions of two variables ($c_0$
and $c_1$, or, equivalently, $b$ and $\Gamma_P$). This is in contrast with elliptic/hyperbolic systems $\mathcal{E}_{EH}$
parametrised by three functions of two variables (compare Proposition 4.2).

**Proof** Apply to $\mathcal{E}_P$ a reparametrisation $(\alpha, \beta)$ transforming its P-frame into a commutative
P-frame $(\tilde{A}, \tilde{B})$ and let $\phi$ be a local diffeomorphism introducing coordinates $x = (z, y)$ such
that $\phi_* \tilde{A} = \frac{\partial}{\partial z}$ and $\phi_* \tilde{B} = \frac{\partial}{\partial y}$. In this system of coordinates, $\Xi_P$ takes the form $\Xi'_{P}$. Then
apply to $\Xi'_{P}$ the reparametrisation $\tilde{w} = w + c_1(x)$ to obtain the form $\Xi''_{P}$ with $b = -2c_1$ and
$\Gamma_P = c_0 + (c_1)^2$. The computation of the structure functions is straightforward. □

Notice that the normal forms $\Xi'_{P}$ and $\Xi''_{P}$ are related by the reparametrisation $\tilde{w} = w + c_1(x)$. The function $\Gamma_P$ (appearing in $\Xi''_{P}$) will be of special importance in the remaining
part of this section, and in any P-frame $(A, B)$, commutative or not, we define it by setting

$$\Gamma_P = \gamma_0 + (\gamma_1)^2.$$ 

Recall that $(\gamma_0, \gamma_1)$ are defined by $C = \gamma_0 A + \gamma_1 B$. Clearly, diffeomorphisms act on $\Gamma_P$ by
conjugation and reparametrisations $(\alpha, \beta)$ act by $\beta^2 \Gamma_P = \Gamma_P$, as it can be computed from
formula (26).

The remaining part of this section shows how to additionally normalise $\Xi'_{P}$ and $\Xi''_{P}$ while
preserving the commutativity of the P-frame $(A, B)$. Although for a parabolic system $\Xi_P$, there always exists a commutative P-frame $(A, B)$, its explicit construction can be difficult or even impossible, as it requires to solve a system of first order PDEs. For this reason, we
will state our results for a general, not necessarily commutative, P-frame $(A, B)$.

**Theorem 4.1 (Normalisation of parabolic systems)** Let $\Xi_P = (A, B, C)$ be a parabolic
system with structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$. Then the following statements hold, locally
around $x_0$:

(i) $\Xi_P$ is strongly-flat, i.e. feedback equivalent to $\Xi'_{P}$ with $c_1 \equiv 0$, we will denote that
form $\Xi''_{P}$, if and only if

$$L_A^2(\gamma_1) + \gamma_1 (L_A(\mu_1) - (\mu_1)^2) = \frac{\mu_0 \mu_1}{2} + \frac{1}{2} L_A(\mu_0) + L_B(\mu_1).$$  \hspace{1cm} (29)

(ii) $\Xi_P$ is a constant-form, i.e. feedback equivalent to $\Xi'_P$ with $(c_0, c_1) \in \mathbb{R}^2$, satisfying
$c_0 + (c_1)^2 \in \mathbb{R}^+$, if and only if $\Gamma_P \neq 0$ and

$$L_A(\Gamma_P) + 2\mu_1 \Gamma_P = 0,$$

$$L_B(\Gamma_P) + 2\Gamma_P L_A(\gamma_1) = \Gamma_P \mu_0 - 2 \Gamma_P \gamma_1 \mu_1.$$  \hspace{1cm} (30)

Moreover, in this case we can always normalise $c_0 = \pm 1$ and $c_1 = 0$ and we will
denote that form $\Xi_{P}^\pm$.

(iii) $\Xi_P$ is a null-form, i.e. feedback equivalent to $\Xi'_P$ with $c_0 \equiv c_1 \equiv 0$, which we will
denote $\Xi_{P}^0$, if and only if (29) holds and, additionally, $\Gamma_P \equiv 0$. 

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A parabolic system of the form $\Xi''_P$ is called strongly-flat (the terminology is justified by generalisations of the previous result to higher dimensions). Equivalent statements can be formulated to obtain special structures of the second prenormal form. The constant-forms $\Xi''_P$ and $\Xi''_P$ are clearly special cases of $\Xi'''_P$, where $c_0$ is constant. Observe that contrary to the elliptic/hyperbolic case, where canonical forms of constant-form systems are parametrised by real-continuous parameters, in the parabolic case, there are only three distinct canonical forms of constant-form systems. The difference between the normal form $\Xi''_P$ and $\Xi'''_P$ lies in the existence or not of an equilibrium point: the control $w = 0$ defines an equilibrium of $\Xi'''_P$ while there are no equilibria for $\Xi''_P$.

**Proof** (i) Sufficiency: Consider a parabolic system $\Xi_P = (A, B, C)$, with structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$, and assume that relation (29) holds. Introduce the reparametrisation $\alpha = -\gamma_1$ and $\beta$ given as a solution of the following system of equations

$$\begin{cases}
\frac{1}{\beta} L_A(\beta) &= -\mu_1 \\
\frac{1}{\beta} L_B(\beta) &= \frac{1}{2} (\mu_0 - 2\gamma_1\mu_1 - 2L_A(\gamma_1))
\end{cases}$$

(31)

This system, rewritten for $\ln(\beta)$, admits solutions since the integrability condition

$$L_A(L_B(\ln(\beta))) - L_B(L_A(\ln(\beta))) = L_{[A,B]}(\ln(\beta)) = \mu_0 L_A(\ln(\beta)) + \mu_1 L_B(\ln(\beta))$$

(31’)

takes the form

$$\frac{1}{2} L_A(\mu_0) - \gamma_1 L_A(\mu_1) - L_A^2(\gamma_1) - \mu_1 L_A(\gamma_1) + L_B(\mu_1)$$

$$= -\frac{1}{2} \mu_0 \mu_1 - \gamma_1 (\mu_1)^2 - \mu_1 L_A(\gamma_1)$$

(31’’)

and is guaranteed by condition (29). Consider the system $\tilde{\Xi}_P = (\tilde{A}, \tilde{B}, \tilde{C})$ obtained by the above defined reparametrisation $(\alpha, \beta)$, and using relations (26) and (27) we deduce that the structure functions $(\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\gamma}_0, \tilde{\gamma}_1)$ of $\tilde{\Xi}_P$ satisfy $\tilde{\mu}_0 = \mu_1 = 0$ and $\tilde{\gamma}_1 = 0$. By choosing local coordinates $(z, y)$ such that $(\tilde{A}, \tilde{B}) = \left(\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right)$, we obtain the system $\Xi''_P$ with $c_1 = \tilde{\gamma}_1 = 0$, i.e. a strongly-flat parabolic system $\Xi'''_P$.

(ii) Necessity: Assume that $\Xi_P = (A, B, C)$ is feedback equivalent to $\tilde{\Xi}_P$ of the form $\Xi''_P$ with $c_1 \equiv 0$, via $\phi$ and $(\alpha, \beta)$. Then for $\tilde{\Xi}_P$ we have $\tilde{\mu}_0 = \tilde{\mu}_1 = \tilde{\gamma}_1 = 0$. First, by (26) we obtain $\alpha = -\gamma_1$ and by (27) we obtain that $\beta$ satisfies the relations given by system (31) above. Therefore, by computing the integrability condition (31’), equivalently given by (31’’), we conclude relation (29).

(iii) Sufficiency: Consider a parabolic system $\Xi_P = (A, B, C)$, with structure functions $(\mu_0, \mu_1, \gamma_0, \gamma_1)$, and assume that $\Gamma_P = \gamma_0 + (\gamma_1)^2 \neq 0$ and that (30) holds. We follow the same reasoning as in the proof of sufficiency of item (i). Introduce a reparametrisation $(\alpha, \beta)$, where $\alpha = -\gamma_1$ and $\beta$ is a solution of the system (31). To assure the existence of $\beta$, we have to fulfil the integrability condition of (31), which is (31’), equivalently (31’’). To this end, we differentiate the second condition of (30) along $A$ and use $L_A(L_B((\cdot)) = \gamma_1$. Springer
Consider the system \( \tilde{\Sigma}_P = (\tilde{A}, \tilde{B}, \tilde{C}) \) obtained by the above defined reparametrisation \((\alpha, \beta)\) and using relations (26) and (27), we deduce that the structure functions \((\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\gamma}_0, \tilde{\gamma}_1)\) of \( \tilde{\Sigma}_P = (\tilde{A}, \tilde{B}, \tilde{C}) \) satisfy \( \tilde{\mu}_0 = \tilde{\mu}_1 = \tilde{\gamma}_1 = 0 \). Therefore \( \tilde{\Gamma}_P = \tilde{\gamma}_0 \), for which condition (30) yields \( \tilde{L}_A(\tilde{\Gamma}_P) = \tilde{L}_B(\tilde{\Gamma}_P) = 0 \), implying that \( \tilde{\Gamma}_P \) is constant (we still have \( \tilde{\Gamma}_P \neq 0 \) since \( \beta^2 \tilde{\Gamma}_P = \Gamma_P \)). Introduce coordinates \((\tilde{z}, \tilde{y})\) such that \( \tilde{A} = \frac{\partial}{\partial \tilde{z}} \) and \( \tilde{B} = \frac{\partial}{\partial \tilde{y}} \), in which the system takes the form (recall that \( \tilde{\gamma}_1 = 0 \))

\[
\begin{cases}
\frac{\dot{\tilde{z}}}{\gamma} = \tilde{w}^2 + c_0 \\
\frac{\dot{\tilde{y}}}{\gamma} = \tilde{w}
\end{cases}
\]

with \( c_0 \in \mathbb{R}^* \). Finally, defining new coordinates \((z, y)\) by \( z = \frac{\tilde{z}}{\sqrt{|c_0|}} \) and \( y = \frac{\tilde{y}}{\sqrt{|c_0|}} \), and reparametrising by \( w = \frac{\tilde{w}}{\sqrt{|c_0|}} \), yields the normal form \( \Sigma_P^{\pm} \).

**(ii) Necessity:** Assume that \( \Sigma_P \), whose structure functions are \((\mu_0, \mu_1, \gamma_0, \gamma_1)\) and \( \Gamma_P = \gamma_0 + \gamma_1^2 \), is feedback equivalent, via \( \phi \) and \((\alpha, \beta)\), to \( \tilde{\Sigma}_P \) of the form \( \Sigma_P' \) with \((c_0, c_1) \in \mathbb{R}^2 \) satisfying \( c_0 + c_1^2 \neq 0 \). For \( \tilde{\Sigma}_P \) we have \( \tilde{\mu}_0 = \tilde{\mu}_1 = 0 \) and \( \tilde{\gamma}_0 = 0, \tilde{\gamma}_1 = 1 \), hence \( \tilde{\Gamma}_P = c_0 + c_1^2 \neq 0 \) implying \( \Gamma_P \neq 0 \) since \( \beta^2 \tilde{\Gamma}_P = \Gamma_P \). By relation (27), we obtain that \( \beta \) satisfies the relations of system (31). Differentiating \( \Gamma_P = \beta^2 \tilde{\Gamma}_P \) along \( A \) we deduce

\[
L_A(\Gamma_P) = L_A(\beta^2 \tilde{\Gamma}_P) = 2 \tilde{\Gamma}_P \beta L_A(\beta) = -2 \tilde{\Gamma}_P \beta^2 \mu_1 = -2 \mu_1 \Gamma_P,
\]

giving the first relation of (30). A similar computation, by taking the derivative of \( \Gamma_P = \beta^2 \tilde{\Gamma}_P \) along \( B \), implies the second relation of (30).

**(iii) The proof of that statement is a special case of the proof of item (i) with the additional condition \( \Gamma_P \equiv 0 \).**

**Sufficiency:** Use the proof of the sufficiency of item (i) to bring the system \( \Sigma_P \) into \( \Sigma_P''' \). For the latter form we have \( \Gamma_P = c_0(x) \), hence \( c_0(x) \equiv 0 \) (due to \( \beta^2 \tilde{\Gamma}_P = \Gamma_P \) and assumption \( \Gamma_P \equiv 0 \)) and we obtain the normal form \( \Sigma_P^0 \).

**Necessity:** Assume that \( \Sigma_P \), whose structure functions are \((\mu_0, \mu_1, \gamma_0, \gamma_1)\) and \( \Gamma_P = \gamma_0 + \gamma_1^2 \), is feedback equivalent, via \( \phi \) and \((\alpha, \beta)\), to \( \tilde{\Sigma}_P \) of the form \( \Sigma_P^0 \) (which is, actually, \( \Sigma_P' \) with \( c_0 \equiv c_1 \equiv 0 \)). For that system we have \( \tilde{\mu}_0 = \tilde{\mu}_1 = 0 \) and \( \tilde{\Gamma}_P \equiv 0 \) and since \( \Gamma_P \) is transformed under \((\alpha, \beta)\) by \( \beta^2 \tilde{\Gamma}_P = \Gamma_P \), we obtain the necessity of \( \Gamma_P \equiv 0 \). The necessity of (29) is deduced from the necessity part of statement (i).

Observe that item (ii) of the above theorem does not explicitly require condition (29), while the normal form \( \Sigma_P^{\pm} \) satisfies \( c_1 \equiv 0 \) and hence that condition has to be hidden in (30). Indeed, this can be observed by differentiating \( \Gamma_P \) along \([A, B]\) and using the constraint (30), which after a short computation gives condition (29).

**Remark 4.5 (Interpretation of the conditions)** We now give a tangible interpretation of our conditions. To this end, consider the system \( \Sigma_P' \), for which we have \( \mu_0 = \mu_1 = 0, \gamma_0 = c_0(x), \gamma_1 = c_1(x) \) and thus \( \Gamma_P(x) = c_0(x) + (c_1(x))^2 \). First, condition (29) implies \( \frac{\partial^2 c_1}{\partial z^2} = 0 \), that is, \( c_1 \) is affine with respect to \( z \), namely, \( c_1(x) = c_1^0(y)z + c_1^1(y) \), and thus, \( \Gamma_P(x) \) is given by \( c_0(x) + (c_1^0(y)z + c_1^1(y))^2 \). This means that if a weakly-flat system \( \Sigma_P' \) is feedback equivalent to a strongly-flat system \( \Sigma_P'' \), then it is parametrised by 3 smooth functions, two of them being functions of \( y \) only, and it has the following form

\[
\begin{cases}
\dot{z} = w^2 + \Gamma_P(x) - (c_1^0(y)z + c_1^1(y))^2 \\
\dot{y} = w + c_1^0(y)z + c_1^1(y)
\end{cases}
\]
By additionally applying the first equation of (30), we obtain
\[ \frac{\partial c_0}{\partial z} + 2c_1^0(y)^2z + 2c_1^0(y)c_1^1(y) = 0 \]
and thus \( c_0(x) \) is a polynomial of degree 2 in \( z \), related to \( c_1(x) \) by \( c_0(x) = -(c_1(x))^2 + c_2(y) \), for an arbitrary smooth function \( c_2(y) \). We now have \( \Gamma_p = \Gamma_p(y) = c_2(y) \) and we use the second equation of (30). Thus, we get \( \Gamma_p(y) = G \exp \left( -2 \int c_1^1(y) \, dy \right) \), with \( G \in \mathbb{R} \). To summarise, any system \( \Xi'_p \) satisfying (29) and (30), i.e. equivalent to a constant-form parabolic system, is parametrised by two arbitrary smooth functions of \( y \) and a constant \( G \in \mathbb{R} \), and is expressed by the form

\[
\begin{cases}
\dot{z} = w^2 + \Gamma_p(y) - (c_1^0(y)z + c_1^1(y))^2, \\
\dot{y} = w + c_1^0(y)z + c_1^1(y)
\end{cases}
\]

where \( \Gamma_p(y) = G \exp \left( -2 \int c_1^1(y) \, dy \right) \). Finally, \( \Xi'_p \) (satisfying (29) and (30)) is feedback equivalent to \( \Xi^0_p \) if and only if \( G = 0 \), and is feedback equivalent to \( \Xi^+_p \), respectively to \( \Xi^-_p \), if \( G > 0 \), resp. \( G < 0 \). The distinction between the three normal forms comes from the sign of \( \Gamma_p \), which is thus a discrete invariant (and that sign is dictated by the value of the constant \( G \)).

5 Conclusions

In this paper, we studied the characterisation and the classification problem of 3-dimensional submanifolds of the tangent bundle of a smooth surface. We showed that the equivalence of submanifolds is reflected in the equivalence, under feedback transformations, of their first and second extensions treated as control-nonlinear and control-affine systems, respectively. We gave a complete characterisation of non-degenerate quadratic submanifolds and proposed a classification of the regular ones, namely the classes of elliptic, hyperbolic, and parabolic submanifolds. To achieve our characterisation results, we introduced the novel class of quadratic control-affine systems and we gave a characterisation of that class in terms of relations between structure functions. Using our characterisation, we identified the sub-classes of elliptic, hyperbolic, and parabolic control-affine systems (second extensions of elliptic, hyperbolic, and parabolic submanifolds). Moreover, we constructed a normal-form for all quadratisable systems and thus gave a normal form of quadratic submanifolds, that may smoothly pass from the elliptic to the hyperbolic classes. Finally, working within the class of control-nonlinear systems subject to a regular quadratic nonholonomic constraint, we exhibited several normal forms of elliptic, hyperbolic, and parabolic systems. In particular, we highlighted a connection between the Gaussian curvature of a well-defined metric and the existence of a commutative frame for elliptic and hyperbolic systems. Normal forms include systems without functional parameters. As a consequence of our classification of quadratic systems, we obtained a classification of elliptic, hyperbolic, and parabolic submanifolds.

The purpose of future works is two folds. First, we want to study the properties (such as controllability, stability, optimal trajectories...) of the class of control systems that we identified. Second, we plan to extend our results to higher dimensional quadratic nonholonomic constraints, in particular, we will generalise our results for parabolic systems to the case of control-nonlinear systems (with the state-space being an \( n \geq 3 \) dimensional manifold) subject to paraboloid nonholonomic constraints. Another interesting problem is to characterise control-nonlinear systems subject to any algebraic nonholonomic constraint. For instance, in order to generalise our results for parabolic systems, one can study polynomial systems, that is, systems subject to the nonholonomic constraint \( \dot{z} - \sum_{i=0}^d a_i(x) \dot{y}^i = 0, (z, y) \in \mathbb{R}^2 \).
Appendix A: Resolution of equation (3)

Consider \( \zeta \) an analytic function of \((z, y, w)\), we will abbreviate \((z, y)\) as \(x\) to shorten the notation. We show how to represent all analytic solutions around \((x_0, w_0)\) of the equation

\[
\frac{d^3 \zeta}{dw^3} = \tau(x) \frac{\partial \zeta}{\partial w}.
\]

First, we will find an expression for \( \frac{\partial \zeta}{\partial w} \) that will be then integrated to obtain the desired form. Consider the following linear system of PDEs:

\[
\begin{pmatrix}
\frac{\partial \zeta_1}{\partial w} \\
\frac{\partial \zeta_2}{\partial w}
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
\tau(x) & 0
\end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}
\]

given for the functions \( \zeta_1 = \frac{\partial \zeta}{\partial w} \) and \( \zeta_2 = \frac{\partial^2 \zeta}{\partial w^2} \). Solutions of this system (interpreted as a system of ODEs with \(x\) being a parameter) are expressed by the exponential of the matrix

\[
\begin{pmatrix}
0 & w \\
\tau(x) & 0
\end{pmatrix}
\]

given by the formula

\[
\exp \left( \begin{pmatrix}
0 & w \\
\tau(x) & 0
\end{pmatrix} \right) = \sum_{k=0}^{+\infty} \frac{w^{2k+1} \tau^k(x)}{(2k+1)!} \begin{pmatrix} 0 & 1 \\ \tau(x) & 0 \end{pmatrix} + \sum_{k=0}^{+\infty} \frac{w^{2k} \tau^k(x)}{(2k)!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

obtained by expressing the power series of the exponential and by regrouping the terms of odd and even degrees. Denote \( b(x) = \zeta_1(x, w_0) \) and \( a(x) = \zeta_2(x, w_0) \), thus we obtain

\[
\frac{\partial \zeta}{\partial w} = \zeta_1(x, w) = a(x) \sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k+1}}{(2k+1)!} \tau^k(x) + b(x) \sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k}}{(2k)!} \tau^k(x).
\]

Integration of this expression yields

\[
\zeta(x, w) = a(x) \sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k+2}}{(2k+2)!} \tau^k(x) + b(x) \sum_{k=0}^{+\infty} \frac{(w - w_0)^{2k+1}}{(2k+1)!} \tau^k(x) + c(x).
\]

Appendix B: Detailed computation of the proof of Theorem 3.2

We detail the computation needed in the proof of Theorem 3.2.

B.1 Resolution of equation (12)

We show how to, locally around \(0 \in \mathbb{R}^3\), solve the equation \( \rho'' - \frac{2}{3} \rho' + \frac{4}{9} \rho^3 = 0 \), where \( \rho = \rho(x, w) \) and the derivatives are taken with respect to \(w\). Introduce the new unknown function \( R(x, w) = \exp \left( -\frac{2}{3} \int_0^w \rho(x, t) \, dt \right) \) which satisfies \( R(x, 0) \neq 0 \) and

\[
R' = -\frac{2}{3} \rho R, \quad R'' = -\frac{2}{3} R \left( \rho' - \frac{2}{3} \rho^2 \right), \quad R''' = -\frac{2}{3} R \left( \rho'' - 2 \rho \rho' + \frac{4}{9} \rho^3 \right) = 0.
\]
Thus, $R(x, w) = a(x)w^2 + b(x)w + c(x)$ yielding $\rho = -\frac{3}{2} \frac{h'}{R} = -\frac{3}{2} \frac{2aw + b}{aw^2 + bw + c}$. Since $R(x, 0) = c(x) \neq 0$, thus taking $d(x) = \frac{a}{c}$ and $e(x) = \frac{b}{c}$ we obtain

\[
\rho(x, w) = -\frac{3}{2} \frac{2d(x)w + e(x)}{d(x)w^2 + e(x)w + 1}.
\]

B.2 Smooth form of $h'$

We integrate $h''(x, w) = a(x)(d(x)w^2 + e(x)w + 1)^{-3/2}$ to obtain a smooth expression of $h'(x, w)$ around 0. We can then derive a smooth closed form expression of $h'(x, w)$:

\[
h'(x, w) = a(x) \int \frac{1}{p(x, w)^{3/2}} dw = \frac{-2a(2dw + e)}{\Delta \sqrt{p}} + \bar{b}(x).
\]

Since $h'(x, 0) = b(x)$ is smooth, we have $\frac{-2ac}{\Delta} + \bar{b} = b(x)$, where $b(x)$ is a smooth function around 0. We can then derive a smooth closed form expression of $h'(x, w)$:

\[
h'(x, w) = \frac{-2a(2dw + e)}{\Delta \sqrt{p}} + \frac{2ae}{\Delta} + b = \frac{-2a}{\Delta \sqrt{p}} (2dw + e - e\sqrt{p}) + b
\]

\[
= \frac{-2a}{\Delta \sqrt{p}(ew + 2 + 2\sqrt{p})} (2dw + e - e\sqrt{p})(ew + 2 + 2\sqrt{p}) + b
\]

\[
= \frac{-2a}{\Delta \sqrt{p}(ew + 2 + 2\sqrt{p})} (2dw^2 + 4dw + 4w\sqrt{p} + e^2w + 2e
\]

\[
+ 2e\sqrt{p} - e^2\sqrt{p} - 2e\sqrt{p} - 2ep) + b
\]

\[
= \frac{-2a}{\Delta \sqrt{p}(ew + 2 + 2\sqrt{p})} (w\sqrt{p}(4d - e^2) + 4dw + 2dw^2 + 2e
\]

\[
+ e^2w - 2edw^2 - 2e^2w - 2e) + b
\]

\[
= \frac{-2a}{\Delta \sqrt{p}(ew + 2 + 2\sqrt{p})} (w\sqrt{p}(4d - e^2) + w(4d - e^2)) + b
\]

\[
= \frac{2aw}{\sqrt{p}(ew + 2 + 2\sqrt{p})} (\sqrt{p} + 1) + b.
\]

Appendix C: Details of the computations of Lemma 4.2

Denoting $c_{E}(x) = \cos(x)$, $c_{H}(w) = \cosh(w)$, $s_{E}(w) = \sin(w)$, and $s_{H}(w) = \sinh(w)$ and starting from the system

\[
\dot{x} = (A, B) \begin{pmatrix} c_{EH}(w) \\ s_{EH}(w) \end{pmatrix} + (A, B) \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix},
\]

we apply a reparametrisation $w = \tilde{w} + \alpha(x)$:

\[
\dot{x} = (A, B) \tilde{R}_{EH}(\pm \alpha) \begin{pmatrix} c_{EH}(\tilde{w}) \\ s_{EH}(\tilde{w}) \end{pmatrix} + (A, B) \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}
\]

\[
= (\tilde{A}, \tilde{B}) \begin{pmatrix} c_{EH}(\tilde{w}) \\ s_{EH}(\tilde{w}) \end{pmatrix} + (\tilde{A}, \tilde{B}) \tilde{R}_{EH}^{-1}(\pm \alpha) \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}.
\]
This yields $(\tilde{\gamma}_0, \tilde{\gamma}_1) = (\gamma_0, \gamma_1) \tilde{R}_{EH}^T(\pm \alpha)$, and by the definition of $\tilde{R}_{EH}(\alpha)$ we have
\[
\tilde{R}_E^T(\alpha) = \tilde{R}_E^T(-\alpha) = \tilde{R}_E(\alpha), \quad \text{and} \quad \tilde{R}_H^T(-\alpha) = \tilde{R}_H^{-1}(\alpha) = \tilde{R}_H(\alpha).
\]
Next, computing separately in the elliptic and hyperbolic cases, we have
\[
[A, B] = (\mu_0 \mp L_A(\alpha))A + (\mu_1 - L_B(\alpha))B,
\]
\[
(A, B)\tilde{R}_E(\pm \alpha) \left( \begin{array}{c} \tilde{\mu}_0 \\ \tilde{\mu}_1 \end{array} \right) = (A, B) \left( \begin{array}{c} \mu_0 \mp L_A(\alpha) \\ \mu_1 - L_B(\alpha) \end{array} \right).
\]
Thus,
\[
(\tilde{\mu}_0, \tilde{\mu}_1) = (\mu_0 \mp L_A(\alpha), \mu_1 - L_B(\alpha)) \tilde{R}_E^T(\pm \alpha)
\]
and relation (19) follows.

**Appendix D: Gaussian curvature for metric given in terms of vector fields**

Consider a 2-dimensional manifold $X$ and two smooth vector fields $A$ and $B$ satisfying $A \wedge B \neq 0$. Construct the (pseudo)-Riemannian metric $g_\pm$ defined by
\[
g_\pm(A, A) = 1, \quad g_\pm(B, B) = \pm 1, \quad \text{and} \quad g_\pm(A, B) = 0.
\]
We will give a formula for the Gaussian curvature of $g_\pm$ in terms of the structure functions $(\mu_0, \mu_1)$ uniquely defined by $[A, B] = \mu_0 A + \mu_1 B$. We will use the following formula for the covariant derivative
\[
\nabla_{E_i} E_j = \frac{1}{2} \sum_k (g_\pm([E_i, E_j], E_k) - g_\pm([E_i, E_k], E_j) - g_\pm([E_j, E_k], E_i)) E_k
\]
for $E_i, E_j, E_k \in \{A, B\}$, and the following formula for the Gaussian curvature of a 2-dimensional manifold, see [17, Proposition 1.11.3],
\[
\kappa_\pm = \frac{g_\pm((\nabla_B \nabla_A - \nabla_A \nabla_B + \nabla_{[A,B]})A, B)}{\det(g_\pm)},
\]
where $\det(g_\pm) = g_\pm(A, A)g_\pm(B, B) - g_\pm(A, B)^2 = \pm 1$. Computing, we have
\[
\nabla_A A = -\mu_0 B, \quad \nabla_A B = \mu_0 A, \quad \nabla_B A = \mp \mu_1 B, \quad \nabla_B B = \pm \mu_1 A.
\]
Then we can deduce
\[
\nabla_B \nabla_A A = -L_B(\mu_0)B \mp \mu_0 \mu_1 A, \quad \nabla_A \nabla_B A = \mp L_A(\mu_1)B \mp \mu_0 \mu_1 A,
\]
\[
\nabla_{[A,B]} A = \mu_0 \nabla_A A + \mu_1 \nabla_B A = -(\mu_0)^2 B \mp (\mu_1)^2 B.
\]
Thus
\[
\kappa_\pm = \pm g_\pm \left( -L_B(\mu_0) \pm L_A(\mu_1) - (\mu_0)^2 \mp (\mu_1)^2 \right) B, B,
\]
\[
= -L_B(\mu_0) \pm L_A(\mu_1) - (\mu_0)^2 \mp (\mu_1)^2.
\]
Therefore, we have obtained the expression of relation (22).
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