QUANTUM GEOMETRODYNAMICS
IN EXTENDED PHASE SPACE – II.

THE BIANCHI IX MODEL

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Abstract

The mathematically correct approach to constructing quantum geometrodynamics of a closed universe formulated in Part I is realized on the cosmological Bianchi-IX model with scalar fields. The physical adequacy of the obtained gauge-noninvariant theory to existing concepts about possible cosmological scenarios is shown. It is demonstrated that the Wheeler-DeWitt quantum geometrodynamics based on general quantum theoretical principles with probability interpretation of a closed universe wave function does not exist. The problem of the creation of the Universe is considered as a computational problem of a quantum reduction of the singular state “Nothing” to one of possible initial physical quantum states.

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In Part I of this work a mathematically correct gauge-noninvariant approach to quantum geometrodynamics (QGD) of a closed universe has been developed. Operationally it aimed at describing an integrated system including a physical object and observation means (OM). Presented below Part II contains detailed mathematical proofs of the possibility to apply this approach to simple but representative Bianchi-IX cosmology as a model.

The model, as well as the chosen class of gauge conditions and parametrizations are discussed in Sec. 9.

The Lagrangian dynamics is considered in Sec. 10 and in the same section the Hamiltonian dynamics in extended phase space (EPS) is constructed that is completely equivalent to the Lagrangian one. The possibility to construct the Hamiltonian dynamics in this approach is ensured by a special choice of a gauge condition in the differential form.

In Sec. 11 we construct the Hamiltonian formulation along the Batalin–Fradkin–Vilkovisky (BFV) line and compare the two approaches. We point out to the main mathematical reason why the two approaches turn out not to be equivalent: the reason is that the group of gauge transformations and the group of transformations generated by constraints are different. It results in a different structure of ghost sectors in the Lagrangian and Hamiltonian formulations. At the same time, the two approaches are in agreement in a gauge-invariant sector which can be singled out by asymptotic boundary conditions. In the case of a close universe one cannot appeal to the assumption about asymptotic states and the two formulations of the Hamiltonian dynamics in EPS lead to different results. Our preference is given to the approach presented in Sec. 10.

In Sec. 12, 13 a probable role of the gravitational vacuum condensate (GVC) in cosmological evolution is discussed and illustrated for the exact solution of conditionally-classical dynamical equations.

In Sec. 14 the Schrödinger equation for a wave function of the Universe is derived from a path integral by a standard method originated from Feynman; the Hamiltonian operator in the Schrödinger equation corresponds to the Hamiltonian constructed in Sec. 10. The structure of the general solution to the Schrödinger equation is analyzed in Sec. 15.

In Sec. 16 we discuss the possibility to go over from the formulation of QGD in extended phase space to the Wheeler–DeWitt theory. A special attention is paid to the demand of BRST-invariance of the

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wave function. It is shown that in the approach presented here this demand leads to the Wheeler–DeWitt equation only if the assumption about asymptotic states is made. At the same time, the known Wheeler–DeWitt theory is demonstrated to be implicitly gauge-noninvariant and undeducible correctly, as a physical theory of a closed universe, from general quantum theoretical principles.

The exact solution to the Schrödinger equation corresponding to the conditionally-classical exact solution (Sec. 13) is found in Sec. 17.

The verisimilitude of the physical content of the exact solution from the standpoint of existing concepts about cosmological scenarios justifies the attempt undertaken in Sec. 18 to extrapolate the developed above methodology to the problem of the creation of the Universe.

The two parts of the work have a through section numbering.

9. The Bianchi IX model

The interval in the cosmological Bianchi IX model looks like 

\[ ds^2 = N^2(t) dt^2 - \eta_{ab}(t)e^a_i e^b_k dx^i dx^k; \]  

\[ \eta_{ab}(t) = \text{diag} (a^2(t), b^2(t), c^2(t)), \]  

\[ e^1_i = (\sin x^3, -\cos x^3 \sin x^1, 0), \]  

\[ e^2_i = (\cos x^3, \sin x^3 \sin x^1, 0), \]  

\[ e^3_i = (0, \cos x^1, 1). \]

We shall also assume that the model includes an arbitrary number \( K \) of real scalar fields described by the Lagrangian

\[ L_{(scal)} = \frac{1}{2\pi^2} \sqrt{-g} \left[ \frac{1}{2} \sum_{i=1}^{K} \phi_i^\mu \phi_i^\nu - U_s(\phi_1, \ldots, \phi_K) \right], \]  

and use the following parametrization:

\[ a = \frac{1}{2} r \exp \left[ \frac{1}{2} \left( \sqrt{3} \varphi + \chi \right) \right]; \]  

\[ b = \frac{1}{2} r \exp \left[ \frac{1}{2} \left( -\sqrt{3} \varphi + \chi \right) \right]; \]  

\[ c = \frac{1}{2} r \exp (-\chi). \]

Writing out the Einstein equations in the given parametrization it is easy to notice [3] that the Bianchi IX model can be considered as a model of a Friedman-Robertson-Walker closed universe with \( r(t) \) being a scale factor, on which a transversal nonlinear gravitational wave \( \varphi(t), \chi(t) \) is superposed.

The action for a system of the scalar fields and gravitation can be presented in the simple form

\[ S_0 = \int dt \left[ \frac{1}{2} \mu_0^\gamma_{ab} \hat{Q}^a \hat{Q}^b - \frac{1}{\mu_0} U(Q) \right], \]  

where

\[ U(Q) = e^{2q} U_g(\varphi, \chi) + e^{3q} U_s(\phi), \]  

\[ U_g(\varphi, \chi) = \frac{2}{3} \left\{ \exp \left[ 2 \left( \sqrt{3} \varphi + \chi \right) \right] + \exp \left[ 2 \left( -\sqrt{3} \varphi + \chi \right) \right] + \exp(-4\chi) - \right. \]  

\[ -2 \exp \left[ -\left( \sqrt{3} \varphi + \chi \right) \right] - 2 \exp \left( \sqrt{3} \varphi - \chi \right) - 2 \exp(2\chi) \right\}, \]
\[ \mu_0 = \frac{r^3}{N}, \]  
\[ r = \exp \left( \frac{q}{2} \right), \]
\[ Q^a = (q, \varphi, \chi, \phi, \ldots); \]
indices \( a, b, \ldots \) are raised and lowered with the "metric"
\[ \gamma_{ab} = \text{diag}(-1, 1, 1, 1, \ldots); \]

A gauge variable should not inevitably be determined by the relation (4). Hereinafter we shall adhere to the parametrization
\[ \mu_0 = v(\mu, Q), \]

In the present work we shall confine our attention to the special class of gauges not depending on time,
\[ \mu = f(Q) + k; \quad k = \text{const}, \]
or, in a differential form,
\[ \dot{\mu} = f, a \dot{Q}^a, \]

an index after a comma here and further denoting differentiation with respect to generalized coordinates: \( f, a = \partial f / \partial Q^a \). Eq. (7) is a model form of the general gauge (10), Sec. 6, introducing the missing velocity \( \dot{\mu} \) to the Lagrangian and thus enabling to go over to Hamiltonian dynamics in EPS \( \mathbb{R} \). Practically, any gauge can be represented by Eq. (7) using an appropriate parametrization (5).

The latter reflects an obvious fact that the choice of a gauge variable parametrization and the choice of a gauge condition have an inseparable interpretation: they are both determined by a clock’s construction which time counting depends on, the influence of the physical subsystem with coordinates \( Q^a \) on the clock being taken into account. In particular, without loss of generality any gauge condition can be turned to \( \mu = \text{const} \) by choosing the function \( v \). In other words, splitting the definition procedure of a gauge variable into choosing of parametrization and imposing a gauge condition is quite conventional, and the familiar reparametrization noninvariance of the Wheeler – DeWitt equation \( \mathbb{R} \) (see, for example, \( \mathbb{R} \)) is essentially ill-hidden gauge noninvariance (see also Sec. [10]).

10. The Lagrangian and Hamiltonian dynamics

The ghost action corresponding to a gauge from the class (7) reads
\[ S_{\text{ghost}} = - \int dt i \bar{\theta} d \frac{dt}{dt} \left[ v(\mu, Q) \bar{\theta} - \left( \dot{\mu} - f, a \dot{Q}^a \right) \theta \right]. \]

Here \( w(\mu, Q) \equiv v(\mu, Q) / v, \mu; \ v, \mu = \partial v(\mu, Q) / \partial \mu; \ \theta, \ \bar{\theta} \) are the Faddeev-Popov ghosts after replacement \( \bar{\theta} \rightarrow -i \bar{\theta} \). The convenience of the latter is that the Lagrangian is real (Hermitian) when the variables \( \theta, \bar{\theta} \) are real, on account of the known complex conjugation rule for Grassmannian variables: \( \left( \bar{\theta} \theta \right)^* = \theta^* \bar{\theta}^* \).

It is convenient to write the ghost action in the dynamically-equivalent form to avoid the appearance of second time derivatives in the Lagrangian. Then, redefining the Lagrange multiplier \( \lambda \) of a gauge-fixing term we write the effective action in the form
\[ S_{\text{eff}} = \int dt \left\{ \frac{1}{2} v(\mu, Q) \gamma_{ab} \dot{Q}^a \dot{Q}^b - \frac{1}{v(\mu, Q)} U(Q) + \pi \left( \dot{\mu} - f, a \dot{Q}^a \right) + i w(\mu, Q) \dot{\bar{\theta}} \right\}, \]

where \( \pi \equiv \lambda + \dot{\bar{\theta}} \).

The action (8), strictly speaking, makes sense in the quantum theory only. In the path integral approach (8) yields the extremal equations
\[ \left( v(\mu, Q) \dot{Q}_a \right) - \frac{1}{2} v, a \dot{Q}^b \dot{Q}_b - \frac{1}{v^2(\mu, Q)} v, a U(Q) + \frac{1}{v(\mu, Q)} U, a - \dot{\pi} f, a + i w, a \dot{\bar{\theta}} = 0, \]
where $\alpha$ The Hamiltonian (14) gives the canonical equations

The corresponding Hamiltonian is

In the operator approach the same equations are considered as equations for operators with canonical commutation relations. (The introduction of the latter ones is ensured by phase space extension). The construction of the quantum theory of the Bianchi-IX model is described in Sec. [4] now we discuss the so-called “conditionally-classical” model, where the quantities appearing in (8) are conditionally treated as classical $c$-numerical and Grassmannian functions. The set (8) – (13) differs from classical equations of general relativity by the presence of ghosts and the inclusion of a gauge condition in the number of the equations of motion (as a result, the Lagrange multiplier $\pi$ becomes a dynamical variable).

An important property of the set of equations is its completeness enabling to formulate the Cauchy problem (see [5]) and to obtain unambiguous extremals of the action. The equations are degenerate on the subset of variables of the classical Einstein theory corresponding to a trivial solution for ghosts and the Lagrange multiplier $\pi$ turns one back to the gauge-invariant classical set of equations. But, being incomplete, the classical set of equations requires eliminating explicit substitution of trivial solutions for ghosts and the Lagrange multiplier.

We construct a Hamiltonian dynamics in extended phase space by introducing canonical momenta

The corresponding Hamiltonian is

The Hamiltonian (14) gives the canonical equations

\[
\hat{P}_a = \frac{1}{v^2(\mu, Q)} v_a \left[ \frac{1}{2} \left( P_a P^b + 2 \pi f_{ab} P^b + \pi^2 f_{ab} f^b \right) + U(Q) \right] \quad \text{and} \quad \hat{\theta} = \frac{1}{2} \left( v_a P^a + P^a \right) ;
\]

\[
\hat{Q}^a = \frac{1}{v(\mu, Q)} \left( P^a + \pi f^{a} \right) ;
\]

\[
\hat{\pi} = \frac{1}{\pi} \left( P_a P^a + 2 \pi f_{a} P^a + \pi^2 f_{a} f^a \right) + U(Q) \quad \text{and} \quad \hat{\bar{\theta}} = \frac{1}{\pi^2} \left( v \bar{v} \right) ;
\]

\[
\hat{\mu} = \frac{1}{v(\mu, Q)} \left( P^a + \pi f^{a} \right) ;
\]

\[
\hat{\rho} = 0 ;
\]

(10) \]

(11)

(12)

(13)

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Quantum geometrodynamics in extended phase space – II

\[ \dot{\theta} = -\frac{i}{w(\mu, Q)} \rho; \quad (20) \]
\[ \dot{\rho} = 0; \quad (21) \]
\[ \dot{\bar{\theta}} = -\frac{i}{w(\mu, Q)} \bar{\rho}. \quad (22) \]

It is easy to see that (15) – (22) are equivalent to the set of Eqs. (9) – (13); the constraint (10) and the gauge condition (11) acquiring the status of Hamiltonian equations (17), (18).

The primary (according to Dirac’s terminology) constraint \( \pi = 0 \) does not appear in this approach because of the special choice of a gauge condition (7); the secondary constraint is modified as a consequence of the extension of phase space. Once again, one can use trivial solution for \( \pi \) and ghosts to eliminate gauge-dependent terms. Another detail of the Dirac approach is that gauge variables serve as Lagrange multipliers of constraints. This makes one restrict the class of admissible parametrizations,

\[ v(\mu, Q) = \frac{u(Q)}{\mu}. \quad (23) \]

Making use of the trivial solutions for \( \pi \) and ghosts reduces Eq. (17) in the class of parametrization (23) to Dirac’s secondary constraint,

\[ T = \frac{1}{2u(Q)} P_a P^a + \frac{1}{u(Q)} U(Q) = 0. \quad (24) \]

The physical Hamiltonian in this class of parametrizations is proportional to the secondary constraint, \( H_0 = \mu T \). The constraint \( T \), in its turn, does not depend on a gauge variable.

It is generally accepted that the trivial solutions for \( \pi \) and ghosts can be singled out by asymptotic boundary conditions (if one ignores the problem of Gribov’s copies). From this viewpoint asymptotic boundary conditions lead to Dirac’s constraints \( \pi = 0, \ T = 0 \).

The investigation of the set of equations (13) – (22) shows that there exist a conserved quantity

\[ \Omega = w(Q, \mu) \pi \dot{\theta} - H \theta = -i \pi \rho - H \theta. \quad (25) \]

\( \Omega \) generates the following transformations of variables in extended phase space:

\[ \delta Q^a = \{Q^a, \Omega\} \bar{\epsilon} = -\frac{\partial H}{\partial P_a} \theta \epsilon = -\dot{Q}^a \theta \epsilon; \quad (26) \]
\[ \delta \mu = \{\mu, \Omega\} \bar{\epsilon} = -i \rho \bar{\epsilon} - \frac{\partial H}{\partial \pi} \theta \epsilon = w(\mu, Q) \dot{\theta} \epsilon - \dot{\mu} \theta \epsilon; \quad (27) \]
\[ \delta \theta = \{\theta, \Omega\} \bar{\epsilon} = 0; \quad (28) \]
\[ \delta \bar{\theta} = \{\bar{\theta}, \Omega\} \bar{\epsilon} = -i \pi \bar{\epsilon}; \quad (29) \]
\[ \delta P_a = \{P_a, \Omega\} \bar{\epsilon} = \frac{\partial H}{\partial Q^a} \theta \epsilon = -P_a \theta \epsilon; \quad (30) \]
\[ \delta \pi = \{\pi, \Omega\} \bar{\epsilon} = -i \frac{\partial H}{\partial \mu} \theta \epsilon = \dot{\pi} \theta \epsilon; \quad (31) \]
\[ \delta \bar{\rho} = \{\bar{\rho}, \Omega\} \bar{\epsilon} = H \bar{\epsilon}; \quad (32) \]
\[ \delta \rho = \{\rho, \Omega\} \bar{\epsilon} = 0. \quad (33) \]

The transformations (26) – (29) are BRST-transformations in the Lagrangian formalism. This enables one to associate \( \Omega \) with the BRST generator.

Dirac’s constraints generate transformations for \( Q^a, \mu \) that differ from (20), (24). The difference in groups of transformations results in the nonequivalence of the Lagrangian formulation and the Hamiltonian one based on the set of constraints \( \pi, T \).
11. Comparison with the BFV approach

It is interesting to compare these results with those obtained by a direct application of the BFV approach.

The BFV method inherits some features of the Dirac approach, in particular, one should restrict the class of admissible parametrizations (23). We have the full set of constraints, \( \mathcal{G}_\alpha = (\pi, T) \). The BRST-generator is

\[
\Omega_{BFV} = \eta^\alpha \mathcal{G}_\alpha = T \theta - i \rho. \tag{34}
\]

Taking a gauge-fixing function in the standard form,

\[
\bar{\psi} = i \bar{\theta} \chi(Q, P) + \bar{\rho},
\]

one obtains the effective action

\[
S_{BFV} = \int dt \left[ P_a \dot{Q}^a + \pi \dot{\pi} + \dot{\bar{\theta} \rho} + \{ \bar{\psi}, \Omega \} \right]
\]

\[
= \int dt \left[ P_a \dot{Q}^a + \pi \dot{\pi} + \dot{\bar{\theta} \rho} + \dot{\theta} \bar{\rho} + i \bar{\theta} \{ \chi, T \} \theta - i \rho - \mu T - \pi \chi(Q, P) \right]. \tag{35}
\]

It is easy to see that the effective action (35) leads to a set of equations which differs from (15) – (22).

The mathematical reason why the two formulations of Hamiltonian dynamics are different is that the transformations generated by the constraint (24) do not coincide with gauge transformations in Lagrangian formalism. Correspondingly, the BRST charge \( \Omega \) generates transformations which differ from those generated by (25). The form of transformations determines the structure of ghost Lagrangian, so different groups of transformations lead to nonequivalent formulations in extended phase space. The situation is typical for the theory of gravity.

The following may serve as a “circumstantial evidence” in favor of the method of constructing Hamiltonian dynamics in EPS presented in Sec. 10. The path integral approach, in contrast of canonical quantization, does not require to construct the Hamiltonian form of the theory at the classical level. Hamiltonian operator is obtained when deriving a Schrödinger equation from the path integral with the action in Lagrangian form. It is remarkable that the application of this procedure to the path integral with the effective action (8) yields the Hamiltonian operator corresponding to the Hamiltonian (14) (see Sec. 14).

The two formulations of Hamiltonian dynamics in extended phase space could enter into agreement in the class of parametrizations (23) in a gauge-invariant sector which supposedly can be singled out by asymptotic boundary conditions.

For a physical system possessing asymptotic states, neither of the two formulations seems to have any advantages. However, being applied to a system without asymptotic states, as a closed universe is, they may give different results. In particular, the demand of BRST-invariance of state vectors, \( \Omega \ket{\Psi} = 0 \), where \( \Omega \) is given by (23), may not lead to the Wheeler – DeWitt equation. We shall return to this point in Sec. 16 after constructing the quantum version of Hamiltonian dynamics in EPS.

12. The role of gravitational vacuum condensate

An important feature of the conditionally-classical model is the presence of a gravitational vacuum condensate (see Sec. 8), its quasi-energy-momentum tensor (quasi-EMT) being determined by a parametrization and a gauge condition. To simplify further calculations it is convenient to use an exponential parametrization

\[
v(\mu, Q) = \exp(\zeta(\mu, Q)). \tag{36}
\]

In a simple case, when the gauge function \( f(Q^a) \) depends only on \( Q^1 = q \) and \( v = \exp(\zeta(\mu, q)) \), the quasi-EMT is isotropic:

\[
T^\nu_{\mu(obs)} = \text{diag}(\varepsilon_{(obs)}, -p_{(obs)}, -p_{(obs)}, -p_{(obs)}), \quad \varepsilon_{(obs)} = -\frac{i}{2\pi^2(\zeta, \mu)^k} \exp(\zeta k - 3q),
\]
Quantum geometrodynamics in extended phase space – II

\[ p_{\text{obs}} = \left\{ 1 - \frac{2}{3} \left[ (\zeta_q)_k + (\zeta_{\mu})_k f_q \right] \right\} \epsilon, \]

where an index \( k \) denotes that the substitution \( \mu = f(Q^a) + k \) has been made; Eq. (22), Sec. 8, reads

\[ (\zeta_\mu)^{-1} \dot{\pi} = E_k. \]

So, the GVC is a continual medium with the equation of state essentially depending on a parametrization and a gauge, the latter two thus being cosmological evolution factors. In other words, one of the main tasks of theoretical cosmology appears to be finding out laws limiting a number of admissible parametrizations and gauges, i.e. the investigation of the GVC nature. Today we do not know how to approach this problem, but the results of the present work, to all appearance, inevitably lead to it as a constituent of the quantum measurement problem, mentioned in the report of Penrose \(^8\) in the sense of finding new approaches to the construction of quantum theory of gravity \(^1\).

Below we shall illustrate the role of the GVC in the cosmological evolution in a simple case of a parametrization and a gauge allowing to obtain an exact solution to the conditionally-classical set of Eqs. (9) – (13), which an appropriate exact quantum solution turns into in a semiclassical limit.

### 13. The conditionally-classical exact solution

Taking the parametrization and the gauge

\[ v = \exp(\mu); \quad \mu = k \]

\((k = \text{const})\), one can obtain an exact particular solution to Eqs. (9) – (13) with \( Q^2 = \varphi = 0 \). It is the same gauge that was used and discussed in Sec. 8. (see (12)). The existence of the particular solution \( \varphi = 0 \) gives the formal opportunity to consider the model without this degree of freedom.

Under the condition (37) the ghost variables vanish from Eqs. (9) – (11), and the latter form a closed set concerning the physical variables; the state equation of the GVC becomes extremely hard,

\[ p = \epsilon = -\frac{\dot{\pi}}{2\pi^2} \exp(k - 3q), \]

\[ \dot{\pi} = E. \]

Note that conditionality of the classical approach is shown here in the presence of ghosts in the integral of motion (39)

\[ \dot{\pi} = \dot{\lambda} - \dot{\theta} \dot{\theta}, \]

i.e. forms constructed on Grassmannian variables appear as parameters of the theory.

Let us turn to the case of a single massless linear scalar field and put

\[ U_s(\phi) = 0, \]

in (3). Now we have the simple equation for \( Q^4 = \phi \)

\[ \dot{\phi} = 0, \]

\(^1\)The complexity of the problem, the impossibility to solve it within the framework of the conceptions of modern theoretical physics, in our opinion, lies in the following. The introduction of a gauge condition even in the classical theory of gravity is an operation establishing the integrity of the system “a physical object + observation means”. The integrity consists in the fact that space-time dynamics of gravitational and matter fields looks differently in various reference systems. The possibility to extract a gauge-invariant information in the classical theory is operationally ensured, according to its well-known conception, by that interactions between an object and observation means can be made as weak as one would like. In quantum theory (QT) the integrity of a system is established by introducing commutation relations. The specificity of quantum theory of gravity is that we try to realize the same physical idea, the idea of integrity, by means of two independent mathematical operations which are parametrization-and-gauging and quantization. Apparently, in a future theory it is necessary to have a physical principle and a formalism that would unify these two operations.
The scalar field behaves as a medium with a positive energy density and with an extremely hard equation of state
\[ p_{(\text{scal})} = \varepsilon_{(\text{scal})} \propto \exp(-3q)\dot{\phi}^2 = C_s^2 \exp(-3q) \]
like that of the GVC \((38)\).

As one can see, in the present model the Universe is filled with the two-component medium described by the parameters \(E\) and \(C_s\). Below we will show that the relation between the two parameters essentially affects cosmological evolution at the quantum stage of the Universe existence as well as at the semiclassical one. Here is the difference between our consideration and the usual investigation of the Bianchi-IX model in general relativity.

The equations for \(q, \chi\) take the form:
\[
\ddot{q} - \frac{4}{3} \left[ \exp(2q - 4\chi) - 4 \exp(2q - \chi) \right] = 0,
\]
\[
\ddot{\chi} - \frac{4}{3} \left[ 2 \exp(2q - 4\chi) - 2 \exp(2q - \chi) \right] = 0.
\]
Integration is simplified with the substitution
\[
z_1 = 2q - 4\chi, \quad z_2 = 2q - \chi;
\]
after replacing
\[ t \rightarrow e^{-k}t \]
the solution is written in the form
\[
\exp \left( q - \frac{1}{2} \chi \right) = \frac{\alpha}{\cosh[2\alpha(t - t_0)]},
\]
\[
\exp \left( q - 2\chi \right) = \frac{\beta}{\cosh[2\beta(t - t_1)]},
\]
where \(\alpha, \beta, t_0, t_1\) are the integration constants. Without loss of generality by shifting the origin of time coordinate one can put \(t_1 = 0\). For the metric \((1) - (2)\) one finds:
\[
a^2 = b^2 = \frac{1}{4} \exp(q + \chi) = \frac{a^2 \cosh(2\beta t)}{4\beta \cosh^2[2\alpha(t - t_0)]};
\]
\[
c^2 = \frac{1}{4} \exp(q - 2\chi) = \frac{\beta}{4 \cosh[2\beta t]}.
\]
From the constraint equation \((10)\) with \(\phi = 0\) and \((40)\) it follows:
\[
\frac{1}{2} \left[ \dot{x}^2 - \dot{q}^2 \right] + \frac{2}{3} \left[ \exp(2q - 4\chi) - 4 \exp(2q - \chi) \right] = E_k - \frac{1}{2} C_s^2,
\]
where \(E_k = e^k E\); hence, in turn,
\[
\alpha^2 - \frac{1}{4} \beta^2 = \frac{3}{8} \left[ E_k - \frac{1}{2} C_s^2 \right].
\]
The dynamics of the model depends qualitatively on a relation between \(C_s\) and \(E_k\).

1. Empty space (there is neither scalar field, \(C_s = 0\), nor GVC, \(E_k = 0\); \(\alpha = \frac{\beta}{2}\).

In the limit \(t = \pm \infty\)
\[
a^2 = b^2 = \frac{\beta}{8}; \quad c^2 = 0,
\]

i.e. the metric (1) asymptotically takes the form

\[ ds^2 = (\beta c)^2 dt^2 - \frac{\beta}{8} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \]

\[ \vartheta = x^1, \quad \varphi = x^2. \]

When reaching singularity in one of the dimensions, two others form a stationary space of constant curvature. Here one deals with a regime of dynamical compactification, a space with simple topology being compactified.

2. Space is filled with the medium having a positive energy density: \( E_k < \frac{1}{2} c_s^2; \quad \alpha > \frac{1}{2} \beta. \)

For \( \alpha = \beta, \ t_0 = 0 \) the model is isotropic.

For \( \alpha = \beta, \ t_0 \neq 0 \) the model is anisotropic, but the singularity has an isotropic nature.

For \( \frac{1}{2} \beta < \alpha < \beta \) in a pre-singular state \( a^2 = b^2 \gg c^2 \), i.e. \( (2+1) \)-dimensional space-time arises where 2-space has a constant curvature.

For \( \alpha > \beta \) in a pre-singular state \( c^2 \gg a^2 = b^2 \), however, the model is not reduced to a space of less dimensions.

In all the cases for \( E_k < \frac{1}{2} c_s^2 \) space at singularity is contracted to a point.

3. Space is filled with the medium having a negative energy density: \( E_k > \frac{1}{2} c_s^2; \quad \alpha < \frac{1}{2} \beta. \)

At \( t = \pm \infty \) the third space dimension is compactified \( (c^2 \to 0) \), and the remaining two-dimensional space of constant curvature infinitely expands. In the special case \( \alpha = \frac{\beta}{4} \) the scale factor \( a = b \) increases exponentially in proper time.

So, the GVC affecting coupling between the constants \( \alpha \) and \( \beta \) through the controlling parameter \( E_k \) determines a cosmological scenario which may contain the following phenomena:

1) cosmological expansion and contraction of space;
2) cosmological singularity;
3) compactification of space dimensions;
4) asymptotically stationary space of less dimensions;
5) inflation of the Universe.

One can see that even such a simplified model reveals a number of effects which seem to be probable from the standpoint of modern cosmological ideas. The introduction of the GVC to the theory enlarges the number of possible cosmological scenarios, a concrete value of the parameter \( E_k \) being formed at the quantum stage of the Universe existence.

14. Quantum dynamics

It is essentially important to note that in the EPS formalism a dynamical Schrödinger equation is a direct and unambiguous consequence of canonical quantization procedure by no means depending on our concepts about gauge invariance or noninvariance of the theory. Really, a Schrödinger equation can be derived from the quantum canonical equations

\[ \dot{X} = i \{H, X\}, \quad (44) \]

written in the matrix form,

\[ \langle \Psi_1 | \dot{X} | \Psi_2 \rangle \equiv \frac{\partial}{\partial t} \langle \Psi_1 | X | \Psi_2 \rangle = \left( \frac{\partial \Psi_1}{\partial t} \right) X | \Psi_2 \rangle + \langle \Psi_1 | X | \frac{\partial \Psi_2}{\partial t} \rangle, \]

\[ \langle \Psi_1 | [H, X] | \Psi_2 \rangle = (\langle \Psi_1 | H \rangle (X | \Psi_2 \rangle) - (\langle \Psi_1 | X \rangle (H | \Psi_2 \rangle), \]
where $|\Psi_1\rangle$, $|\Psi_2\rangle$ are arbitrary state vectors. Joining the latter two formulae into the equation corresponding to \((44)\) and taking into account the arbitrariness of the state vectors $|\Psi_1\rangle$, $|\Psi_2\rangle$ and $X|\Psi_1\rangle$, $X|\Psi_2\rangle$, one comes to mutually conjugate dynamical Schrödinger equations with the Hamiltonian operator \((14)\) defined in the EPS:

$$i \frac{\partial |\Psi\rangle}{\partial t} = H|\Psi\rangle, \quad -i \frac{\partial |\Psi\rangle}{\partial t} = \langle \Psi | H.$$

A dynamical Schrödinger equation, surely, can also be obtained in the path integral formalism having certain advantages over the operator one. In the latter, as is generally known, the operator ordering problem is not resolvable. When deriving a Schrödinger equation from a path integral, ordering turns to be bound up with a way of a final definition of the path integral as the limit of a multiple integral and with a choice of a gauge variable parametrization. The parametrization choice determines a path integral measure as well, the latter being identical with the measure of a normalizing integral – probability measure. Let us consider a path integral for a wave function in the coordinate representation. Such a wave function, according to the stated above, is defined on the extended configurational space with the coordinates $Q^a$, $\mu$, $\theta$, $\bar{\theta}$:

$$\Psi(Q^a, \mu, \theta, \bar{\theta}; t) = \int \langle Q^a, \mu, \theta, \bar{\theta}; t \mid Q^a_{(0)}, \mu_{(0)}, \theta_{(0)}, \bar{\theta}_{(0)}; t_0 \rangle \Psi(Q^a_{(0)}, \mu_{(0)}, \theta_{(0)}, \bar{\theta}_{(0)}; t_0) \times M \left( Q^a_{(0)}, \mu_{(0)} \right) d\theta_{(0)} d\bar{\theta}_{(0)} d\mu_{(0)} \prod_b dQ^b_{(0)}. \quad (45)$$

The transition amplitude, appearing in \((15)\),

$$\langle Q^a, \mu, \theta, \bar{\theta}; t \mid Q^a_{(0)}, \mu_{(0)}, \theta_{(0)}, \bar{\theta}_{(0)}; t_0 \rangle =
$$

$$= C \int \exp \left[ iS(t, t_0) \right] \prod_{t_0 < \tau < t} M \left( Q^a_{(\tau)}, \mu_{(\tau)} \right) d\mu_{(\tau)} d\theta_{(\tau)} d\bar{\theta}_{(\tau)} \prod_b dQ^b_{(\tau)} d\pi_{(\tau)} \approx 0$$

is given by the gauged action \((8)\).

As is well known, a path integral is not defined in internal terms. Proceeding from the standard treatment we shall consider it as the limit at $\epsilon_i \to 0$ of the following integral:

$$\Psi^{(N)}(Q^a, \mu, \theta, \bar{\theta}) = C \int \exp \left\{ i \sum_{i=1}^{N} S(t_i, t_{i-1}) \right\} \Psi^{(0)}(Q^a, \mu, \theta, \bar{\theta}) \times
$$

$$\times \prod_{i=0}^{N-1} M \left( Q^a_{(i)}, \mu_{(i)} \right) d\mu_{(i)} d\theta_{(i)} d\bar{\theta}_{(i)} \prod_b dQ^b_{(i)} d\pi_{(i+1)},$$

where $t_i - t_{i-1} = \epsilon_i$,

$$S(t_i, t_{i-1}) \approx \epsilon_i \left\{ \frac{1}{2} \exp(\zeta_{(i)}) \dot{Q}^a_{(i)} \dot{Q}^a_{(i)} - \exp(-\zeta_{(i)}) U(Q^a_{(i)}) + \pi_{(i)} | \hat{\mu}_{(i)} - \hat{f}(Q^a_{(i)}) \right\} + \frac{i}{\zeta_{(i)} \mu_{(i)}} \hat{\theta}_{(i)} \dot{\bar{\theta}}_{(i)} \right\}.$$

The further process of the path integral definition consists in choosing an approximation for the paths between $t_{i-1}$ and $t_i$. The standard procedure, which we still do not see any reason to deviate from, prescribes approximating by classical paths. In our case it means applying the equations of motion \((3) - (3)\). To calculate the path integral it is necessary to express all the velocities in terms of the values of coordinates at the end points of a time step $\epsilon_i$ by means of the equations of motion \((3) - (3)\), i.e., to solve the Cauchy problem that could be put only on the basis of the complete set of equations \((3) - (3)\). When solving the problem with the required accuracy the expansions in series of $\epsilon_i$ powers are used,

$$\dot{Q}^a_{(i)} = \frac{1}{\epsilon_i} \left( Q^a_{(i)} - Q^a_{(i-1)} \right) + \frac{1}{2} \epsilon_i \ddot{Q}^a_{(i)} - \frac{1}{6} \epsilon_i^2 \dddot{Q}^a_{(i)} + \cdots$$

It should be emphasized that in the case of the lack of asymptotic states there exist no other way compatible with the principles of QT. The known attempts (see, for example, \((3)\)) to derive gauge-invariant equations make use of the boundary conditions of the type

$$\theta(t_0) = \theta(t) = 0.$$
and the analogues ones for \( \mu \) and ghosts. Then the higher derivatives with respect to time are consequently expressed in terms of the first derivatives through the equations of motion. As a result, all the velocities appear in the form of power series of the differences

\[
q^i_{(i)} = Q^a_{(i)} - Q^a_{(i-1)},
\]

\[
m^i_{(i)} = \mu_{(i)} - \mu_{(i-1)},
\]

\[
\eta_{(i)} = \theta_{(i)} - \theta_{(i-1)},
\]

\[
\tilde{\eta}_{(i)} = \tilde{\theta}_{(i)} - \tilde{\theta}_{(i-1)}.
\]

In particular, the gauge condition is approximated by the following:

\[
\dot{\mu}_{(i)} - \dot{f}_{(i)} = \frac{1}{\epsilon_i} \left( \mu_{(i)} - \mu_{(i-1)} - \dot{f}_{(i)} + f_{(i-1)} \right) = \frac{1}{\epsilon_i} \left( m_{(i)} - f^{(i)} q^a_{(i)} + \frac{1}{2} f^{(i)} q^a_{(i)} q^b_{(i)} + \cdots \right).
\]

When deriving the Schrödinger equation it is sufficient to consider a one-step time interval \( t - t_0 = \epsilon \). The further procedure consist of substituting the expansions for the velocities into the exponent \( \exp(iS) \), developing it as series in \( \epsilon \), developing the measure \( M (Q^a(t), \mu(t)) = M (Q^a(t) - q^a, \mu(t) - m) \) as series in \( q^a, m \), and developing the wave function \( \Psi^{(0)} \) as series in \( q^a, m, \eta, \tilde{\eta} \). After performing simple integrations over \( \pi \) and \( m \), the path integral is reduced to Gaussian quadratures over the ghost and physical variables. Then by the standard Feynman method in first order one obtains the Schrödinger equation

\[
i \frac{\partial \Psi(Q^a, \mu, \theta, \bar{\theta}; t)}{\partial t} = H \Psi(Q^a, \mu, \theta, \bar{\theta}; t)
\]

the zero-order terms will give the constraints between the measure \( M \), step \( \epsilon \) and parametrization function \( \zeta \):

\[
\frac{1}{\epsilon \zeta_{, \mu}} (\epsilon e^{-\zeta}) \frac{K + 3}{2} M = \text{const.}
\]

The requirement for the Hamiltonian to be Hermitian gives raise to another constraint between the measure and the parametrization,

\[
M = \text{const} \cdot \zeta_{, \mu} \exp \left( \frac{K + 3}{2} \zeta \right).
\]

The independence of the parameter \( \epsilon \) on the variables \( Q^a, \mu \) follows from (47), (48).

The Hamiltonian in the Schrödinger equation obtained by the path integral method can be presented in the form

\[
H = -i \zeta_{, \mu} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2M} \frac{\partial}{\partial Q^a} \tilde{G}^{\alpha \beta} \frac{\partial}{\partial Q^a} + e^{-\zeta} (U - V),
\]

where \( M \) is defined by the formula (48), \( \tilde{G}^{\alpha \beta} = MG^{\alpha \beta} \),

\[
V = -\frac{3}{12} \left( \frac{\zeta_{, \mu}}{\zeta_{, \mu}} \right)^2 + \frac{\zeta_{, \mu}}{3\zeta_{, \mu}} + \frac{K + 1}{6\zeta_{, \mu}} + \frac{1}{24} \left( K^2 + 3K + 14 \right) \zeta_{a} \zeta^{a} + \frac{K + 2}{6} \zeta_{a},
\]

\[
\zeta_{a} = \frac{\partial \zeta}{\partial Q^{a}} + f_{, a} \frac{\partial \zeta}{\partial \mu},
\]

\[
\theta(t_0) = \theta(t) = 0,
\]

\[
\pi(t_0) = \pi(t) = 0.
\]

The application of these conditions in the path integral at \( t - t_0 = \epsilon \to 0 \) when deriving the Schrödinger equation and the Wheeler–DeWitt equation eliminates from the path integral all the structures associated with a gauge and restore the initial divergent integral with the gauge-invariant action. Appealing to these conditions does not make sense unless one points out the way to eliminate the contribution of the coordinate effects in the path integral owing to the degeneracy of the Cauchy problem for the action extremals. Such boundary conditions are not applied practically even as asymptotic conditions in S-matrix problems where the vacuum of ghosts and of 3-vector and 3-tensor gravitons is given but not fixed values of ghost fields and Lagrange multipliers.
Let us compare the operator formalism and the path integral one. Taking into account the form of a self-conjugate momentum operator in the presence of a nontrivial measure,

\[ P_\alpha = -i \left( \frac{\partial}{\partial Q_\alpha} + \frac{M_{\alpha\alpha}}{2M} \right), \]  

it is easy to see that in the operator formalism Eq. (49) corresponds to the Hamiltonian (14) ordered according to

\[ H = \frac{1}{2} \sum_{i=1}^{n} a_i \xi_i P_\alpha \xi_i^{-2} G^{\alpha\beta} \xi_i P_\beta + e^{-\zeta} U - i \zeta, \rho \bar{\rho}, \]  

with \( \sum_{i=1}^{n} a_i = 1 \), the comparison of the Hamiltonians (52) and (49) gives 5 equations for the 5 independent parameters \( r, s, a_1 \).

But, on account of the mentioned above definition ambiguity of the path integral, such an ordering should not be treated as the only possible one. The ordering problem as well as the problem of the choice of a gauge variable has no solution in the QGD framework. The existence of the two problems indicates the incompleteness of the theory. They have the common origin – the lack of understanding what is a measurement process in quantum gravity. Indeed, the choice of parametrization and gauge in total determine an instrument tuning (a time counting scale). The ordering problem arises from the operator noncommutativeness which, in turn, is the consequence of unremovable instrument affection on a physical system.

When dealing with the quantum physics domains available to experimental investigation, the experiment itself indicates solutions to problems. But in quantum gravity it is necessary to achieve a new level of understanding for that, in other words, the question is about the creation of a quantum measurement theory (see [8, 12]). In our opinion, the proposed modification of QGD giving possibility to describe jointly a physical object and OM may be considered as a first step on that way.

15. The structure of the general solution

The dynamical Schrödinger equation (46) with the Hamiltonian (13) has the general solution which should be considered as the quantum analogue of the solution to the Cauchy problem for conditionally-classical equations (9) – (13). Now we will turn to the analysis of the general solution structure that, as it is obvious in advance, is not gauge-invariant.

The general solution to the Schrödinger equation in the coordinate representation is a wave function

\[ \Psi = \Psi(Q^a, Q^0, \theta, \bar{\theta}, t), \]  

depending on time \( t \), physical variables \( Q^a \), the gauge variable \( Q^0 = \mu \), and ghost variables \( \theta, \bar{\theta} \). Let us show that making use of the Hamiltonian structure only one can establish the dependence of the wave function on the variables \( Q^0, \theta, \bar{\theta} \).

To begin with, note that in the class of gauges (10) not depending on time explicitly the general solution to the Schrödinger equation (41) is expandable in stationary state eigenfunctions satisfying the stationary Schrödinger equation

\[ H \Psi_n(Q^a, Q^0, \theta, \bar{\theta}) = E_n \Psi_n(Q^a, Q^0, \theta, \bar{\theta}). \]  

One of the canonical equations, the gauge equation

\[ [H, Q^0 - f(Q^a)] = 0 \]  

means the commutativeness of the Hamiltonian with the operator \( Q^0 - f(Q^a) \) and, consequently, an arbitrary solution to Eq. (53) can be presented in the form of a superposition of this operator eigenstates \( |k\rangle \),

\[ \{Q^0 - f(Q^a)\}|k\rangle = k|k\rangle. \]
so, finally, all the three versions to the one, 

\[ |k| = \delta \left( Q^0 - f(Q^a) - k \right), \]

so the general solution to the Schrödinger equation has the structure

\[ \Psi(Q^a, Q^0, \theta, \bar{\theta}; t) = \int \Phi_k(Q^a, \theta, \bar{\theta}; t) \delta \left( Q^0 - f(Q^a) - k \right) \, dk. \]

Since there is no other (independent) integral of motion for the \( Q^0 \) variable, the functions \((54)\) make the only basis depending on the gauge variable \( Q^0 \), i.e. the general solution to the dynamical Schrödinger equation inevitably has the structure \((55)\). One may come to the same conclusion by investigating the structure of the wave function in the path integral formalism.

The Schrödinger equation for \( \Phi_k(Q^a, \theta, \bar{\theta}; t) \) reads

\[ i \frac{\partial \Phi_k(Q^a, \theta, \bar{\theta}; t)}{\partial t} = H_k \Phi_k(Q^a, \theta, \bar{\theta}; t), \]

\[ H_k = -i (\zeta, \mu)_k \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} \frac{1}{2} \exp(-\zeta_k) \left( \frac{\partial^2}{\partial Q^a \partial Q^a} + Z^a_k \frac{\partial}{\partial Q^a} \right) + \exp(-\zeta_k) (U - V), \]

\[ Z^a_k = \frac{(\zeta, \mu)_k}{(\zeta, \mu)_k} + \frac{K + 1}{2} \zeta^a_k, \]

\( V \) being defined by Eq. \((\text{34})\), \((\zeta^a)_k = \zeta^a_k \equiv \partial \zeta_k / \partial Q^a \).

So, the wave function dependence on \( \mu \) is determined by Eq. \((55)\). As will be shown below, such a structure of the wave function under a certain restriction on the \( \Phi_k \) dependence on \( k \) makes the normalizing integral over the variable \( \mu \) transformed into an integral over \( k \) to be convergent. As for the dependence on the ghosts, it is strictly enough fixed by the Schrödinger equation in combination with the usual demand of norm positivity. Indeed, in the general case the wave function can be presented by the series in Grassmannian variables,

\[ \Phi_k(Q^a, \theta, \bar{\theta}; t) = \Psi^0_k(Q^a, t) + \Psi^1_k(Q^a, t) \theta + \Psi^1_k(Q^a, t) \bar{\theta} + \Psi^2_k(Q^a, t) \bar{\theta} \theta. \]

After substituting into \((54)\) one obtains the equations for the components

\[ i \frac{\partial \Psi^0_k}{\partial t} = H^0_k \Psi^0_k - i (\zeta, \mu)_k \Psi^2_k, \]

\[ i \frac{\partial \Psi^i_k}{\partial t} = H^i_k \Psi^i_k, \quad i = 1, 1, 2, \]

\[ H^0_k = H_k + i (\zeta, \mu)_k \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}}, \]

and the normalization condition imposes the constraints on these components: from the norm positivity demand it follows

\[ -i \int \left( \Psi^0_k * \Psi^2_k - \Psi^2_k * \Psi^0_k + \Psi^1_k * \Psi^1_k - \Psi^1_k * \Psi^1_k \right) \bar{\theta} \theta \, d\theta \, d\bar{\theta} > 0. \]

One of the consequences of the nonequality is \( \Psi^2_k = 0 \), or \( \Psi^0_k = 0 \), or \( \Psi^2_k = i \Psi^0_k \), and Eqs. \((57)\), \((58)\) reduce all the three versions to the one,

\[ \Psi^0_k = \Psi^2_k = 0, \]

\[ \Psi^1_k = i \Psi^1_k, \]

so, finally,

\[ \Phi_k(Q^a, \theta, \bar{\theta}; t) = \Psi_k(Q^a, t)(\bar{\theta} + i \theta), \]
where $\Psi_k(Q^a, t)$ is a solution to Eq. (58):

$$i \frac{\partial \Psi_k(Q^a, t)}{\partial t} = -\frac{1}{2M_k} \frac{\partial}{\partial Q^a} M_k \exp(-\zeta_k) \gamma^{ab} \frac{\partial \Psi_k(Q^a, t)}{\partial Q^b} + \exp(-\zeta_k) (U - V_k) \Psi_k(Q^a, t),$$

$$M_k = (\zeta, \mu)_k \exp \left( \frac{K}{2} + \frac{3}{2} \zeta_k \right).$$

The unitarity property of the wave function of a physical state takes the form

$$\int \Psi^*_k(Q^a, t) \delta(\mu - f(Q^a) - k') \Psi_k(Q^a, t) \delta(\mu - f(Q^a) - k) \, dk \, dk' \, M(Q^a, \mu) \, d\mu \prod_{a=1}^{K+3} dQ^a =$$

$$= \int \Psi^*_k(Q^a, t) \Psi_k(Q^a, t) \, dk \prod_{a} dQ^a. \quad (60)$$

Thus, the general solution (53), (54) to Eq. (46) under the condition the $\Psi_k(Q^a, t)$ to be a sufficiently narrow packet over $k$ is normalizable with respect to the gauge variable as well as to the ghosts and the physical variables.

The peculiarity of the amplitude (59) lies in the fact that the theory does not control its dependence on the free parameter $k$. From the standpoint of classical dynamic equations the parameter $k$ sets an initial condition for the variable $\mu$ and by the same determines an initial clock setting. In QT, however, there exist no physical state with a fixed $k$ value. Really, the unitarity requirement (see (60)) allows the existence of a physical state represented by a packet over $k$ narrow enough to fit a certain classical $k$ value, but not by a $\delta$-shaped packet. So, in the theory an additional degree of freedom appears that could be named an observer’s degree of freedom. Unlike the quantum uncertainties associated with operator noncommutativeness, QGD do not control even a width of a $k$-packet.

The general solution structure

$$\Psi(Q^a, Q^0, \theta, \bar{\theta}, t) = \int \Psi(E_k; Q^a) \exp(-iE_k t)(\bar{\theta} + i\theta) \delta(\mu - f(Q^a) - k) \, dE_k \, dk, \quad (61)$$

where $\Psi(E_k; Q^a)$ is a solution to the stationary equation

$$H^0_k \Psi_k(Q^a) = E_k \Psi_k(Q^a), \quad (62)$$

proves mathematically all the statements of Sec. 5: the wave function carries the information on 1) a physical object, 2) OM, 3) correlations between a physical object and OM. OM are represented by the factored part of the wave function – by the $\delta$-function of a gauge and by the ghosts; a physical object is described by the function $\Psi_k$; the correlations are present in the effective potential $V_k$ which the solution depends on and also in the wave function time dependence; after going over to the stationary states, they are present in the effective potential and the spectrum $E_k$.

It is of principal significance to emphasize that the procedure of constructing the wave function (61) is the only strict mathematical method to do it, by no way corresponding to the Wheeler – DeWitt QGD. The question arises: do Eq. (46) and its solutions have any relation to the Wheeler – DeWitt theory? In other words, whether it is possible to extract such a physical part of the (51) that would satisfy the Wheeler – DeWitt equation and could be reasonably interpreted?

16. Gauge-invariant QGD and the requirement of BRST-invariance of physical states

To begin with, let us discuss a possibility to construct a wave function, corresponding to the Wheeler – DeWitt QGD, by going over from the general solution to some particular solution. As it is known, the
transition to the Wheeler–DeWitt QGD means the separation of physical variable subspace from EPS. For this purpose it is not enough to separate the physical part $\Psi_k(Q^a)$ from the general solution: one also has to banish correlations between the properties of the physical object and those of OM. The latter are given, first of all, by the GVC parameter. Hence, to eliminate correlations, firstly, we put $E_k = 0$. Secondly, making use of the noted in Sec. 9 possibility to go over to any given gauge function $f(Q^a)$ by means of transformation of a parametrization function $v(\mu, Q^a)$ it is necessary to pass to the gauge

$$\mu = k.$$  

And, thirdly, the measure should be factored: $M = M_1(\mu) M_2(Q^a)$. In view of Eq. (18) the measure factorization requires, in turn, factoring the parametrization function that means

$$\zeta(\mu, Q^a) = \zeta_1(\mu) + \zeta_2(Q^a),$$

the conservation of additivity of the function (64) when going over to the gauge (63) imposing one more restriction on the parametrization function: $\zeta_1$ should be linear in $\mu$,

$$\zeta_1(\mu) = A + B\mu.$$  

Under these conditions one obtains the Wheeler–DeWitt equation for the physical part of the wave function

$$H_{ph}\Psi(Q^a) = 0,$$  

$$H_{ph} = -\frac{1}{2M_2} \frac{\partial}{\partial Q^a} M_2 \exp(-\zeta_2) \gamma^{ab} \frac{\partial}{\partial Q^b} + \exp(-\zeta_2) U + \frac{1}{6} R,$$  

$$R = -\exp(-\zeta_2) \left[ \frac{1}{4} \left( K^2 + 3K + 14 \right) \zeta_2 \zeta_2 + (K + 2) \zeta_2 \right].$$  

$R$ being the scalar curvature constructed on the metric $G^{ab} = \exp(-\zeta_2)\gamma^{ab}$.

Eq. (66) possesses all formal properties of the Wheeler–DeWitt equation including parametrization noninvariance and the lack of any visible vestige of a gauge. However, the described above method of deriving Eq. (66) makes it apparent that in the gauge class (6), any change of the parametrization function $\zeta(\mu, Q^a)$ is mathematically equivalent to a new gauge. Hence, the generally known parametrization noninvariance of the Wheeler–DeWitt theory, as a matter of fact, is the ill-hidden gauge noninvariance. This circumstance has to be taken into account when estimating the status of the Wheeler–DeWitt theory.

On the other hand, by origin, the only Hamiltonian eigenvalue $E_k = 0$ fixed by Eq. (66) is by construction a line in a continuous spectrum of the Hamiltonian (67), hence the solution is unnormalizable. In other words, on the way of formal singling out the particular solution to Eq. (46) one would fail to obtain a wave function having a physical meaning generally adopted in QT$^3$.

Another approach to the problem of the existence of gauge invariant states is based on singling out BRST-invariant solutions. In this case the procedure of derivation of the Wheeler–DeWitt equation does not consist in fixing an eigenvalue in a Hamiltonian spectrum, but in reducing the spectrum itself to a single value by means of the BRST-invariance requirement.

It is worth noticing that this requirement should be treated as an independent postulate: the BRST invariance of the action cannot be a foundation for introducing them. It is known that the invariance of an action under some global transformations does not mean the invariance of state vectors; the latter ones have to be just covariant, subjected to the appropriate unitary transformations$^4$.

In the BFV scheme a wave function of a physical state should obey the superselection rules

$$\Omega_{BFV} |\Psi\rangle = 0,$$  

$^3$A positive solution to this problem needs, apart from a discrete $H_{ph}$ spectrum, the presence of the line $E = 0$ in it.

$^4$For instance, in the standard QT of fields the Lorentz invariance of an action leads to the Lorentz covariance of state vectors. The only Lorentz-invariant vector is a vacuum vector which is a particular solution obtained from the general one.
where $\Omega_{BFV}$ is given by (34). The Wheeler–DeWitt equation
\[
\mathcal{T} |\Psi\rangle = 0 \tag{70}
\]
follows immediately from (69) as a consequence of the arbitrariness of BFV ghosts $\{\eta^a\}$. This result is quite natural, because the theory is constructed in such a way for operator Dirac’s constraints to be inevitably satisfied.

In the approach presented in this paper the sense of the requirement
\[
\Omega |\Psi\rangle = 0, \tag{71}
\]
where $\Omega$ is given by (25), is more questionable. The generator $\Omega$ in this case cannot be presented as a combination of constraints with infinitesimal parameters replaced by ghosts.

One may suppose that the condition (71) together with the quantum version of the primary constraint,
\[
\pi |\Psi\rangle = 0, \tag{72}
\]
lead to the Wheeler–DeWitt equation. (At least, for a system with asymptotic states one may expect that the primary constraint $\pi = 0$ is valid.) However, for any operator ordering, the additional condition (72) does not reduce (71) to the Wheeler–DeWitt equation (70), because of the noncommutativity of operators. The more consistent way to avoid the problems arising from the noncommutativity of operators is to impose the constraints $\pi = 0$ at the classical level before quantization. As was pointed out in Sec. 10, if the constraint $\pi = 0$ holds, one comes from (17) to the secondary constraint $\mathcal{T} = 0$. It is exactly what is implied in the BFV approach. Indeed, originally the BFV approach was developed for constructing the S-matrix of an arbitrary constrained system, i.e. the existence of asymptotic states was supposed, therefore, the constraints $\pi = 0$, $\mathcal{T} = 0$ have to hold.

There is still the third way to obtain the Wheeler–DeWitt equation (66)–(68) mentioned in the footnote 2, p. 11. Without adducing appropriate calculations let us note that on this way two incorrect mathematical structures are necessarily used: 1) a gauge-invariant path integral approximated on partially degenerate action extremals without pointing out a procedure of removing coordinate effects; 2) the definition of a wave function through a divergent integral over gauge variables.

The derivation of the Wheeler–DeWitt equation from a path integral was investigated by Halliwell [17]. There were many references to the paper [17], so it seems to be relevant to comment it briefly. The feature of the consideration presented in [17] is that the assumption about asymptotic states was made, so the Fradkin–Vilkovisky theorem was presumed to be valid for the case of a closed universe as well. As a consequence of independence of the path integral on a gauge condition, a particular gauge, $\dot{N} = 0$, was chosen. At the same time, as it has been demonstrated in the present paper, if the same calculations had been made for an arbitrary gauge condition, it would have led to an equation containing information about the chosen gauge.

The described methods of deriving the Wheeler–DeWitt equation reveal the following.

1. The wave function not containing information about correlations between the physical object and OM is the same in all the approaches.

2. In accordance with Sec. 5, if the state vector allows to compute average values of observable quantities, then information about the correlations has to be contained in it inevitably. In other words, there is no physical (normalizable) quantum state without a GVC.

It is to be stated that there is no QGD as a gauge-invariant theory of physical states in a closed universe based on the general principles of QT. The illusion of the existence of such a theory arises if one forgets about the necessity of singling out gauge orbit representatives and tries to come to an agreement about some special quantization rules. As to the correct path integral method, it shows unambiguously that being applied to a closed universe, the ordinary QT of gravity is gauge-noninvariant. And this feature of the theory is the evidence of its adequacy to the phenomena in question: it answers to the conditions of observations in a closed universe, where there is no possibility to remove an instrument for infinite distance from the
quantum geometrodynamics in extended phase space – II

object and thus to avoid influence of inertial fields, locally indistinguishable from a gravitational field, on the instrument. As it was mentioned in Sec. 6, a transition to another reference system (RS) implies a physical operation in the whole Universe scale, and one hardly can think that such an operation could take place without physical consequences in view of nonvanishing correlations between properties of the object and those of OM.

Some notions about the physical content of the QGD based on Eqs. (46), (49), (50) one can get from the presented in the next section exact solution to this equation corresponding to the considered in Sec. 13 conditionally-classical solution.

17. The exact solution to the Schrödinger equation

The task of constructing the wave function (61) is reduced to searching for a solution to the stationary equation (62) for the physical part of the wave function under the parametrization-and-gauge condition (37). This equation reads

\[ -\frac{1}{2} \frac{\partial^2 \Psi_k}{\partial Q^a \partial Q^a} + U(Q^a)\Psi_k(Q^a) = E_k \Psi_k(Q^a), \] (73)

\( Q^a = (q, \phi, \chi) \). Substitution (41) enables to separate the variables in the equation, and it can be written in the following manner

\[ \left( 6\hat{L}_1 - \frac{3}{2} \hat{L}_2 + \frac{1}{2} \hat{L}_3 - kE \right) \Psi_k(z_1, z_2, \phi) = 0, \]

where

\[ \hat{L}_1 = -\frac{\partial^2}{\partial z_1^2} + \frac{1}{9} \exp(z_1), \]

\[ \hat{L}_2 = -\frac{\partial^2}{\partial z_2^2} + \frac{16}{9} \exp(z_2), \]

\[ \hat{L}_3 = -\frac{\partial^2}{\partial \phi^2}. \]

The eigenfunctions of the operators \( \hat{L}_1, \hat{L}_2 \) appropriate to the positive eigenvalues \( \frac{\nu_1^2}{4}, \frac{\nu_2^2}{4} \) are modified Bessel functions with an imaginary index,

\[ \psi_\nu(z) = \frac{1}{\sqrt{2\pi \Gamma(i\nu)}} K_{\nu} \left[ A \exp\left(\frac{z}{2}\right) \right]; \quad (A_1; A_2) = \left( \frac{2}{3}; \frac{8}{3} \right). \]

So far as \( \exp\left(\frac{z_1}{2}\right) = 4c^2; \quad \exp\left(\frac{z_2}{2}\right) = 4ac, \) the quantum number \( \nu_1 \) determines probability distribution for the scale \( c \), and so does the quantum number \( \nu_2 \) for the scale \( a = b \) at a given \( c \) value. Note, that for any \( \nu \) there exist a quasiclassical solution to the problem,

\[ \psi_\nu(z) = \frac{1}{\sqrt{2\pi \Gamma(i\nu)}} \exp\left(\frac{\pi\nu}{2}\right) \left( \frac{\nu^2}{4} - \frac{A^2}{4} \exp(z) \right)^{\frac{1}{2}} \times \]

\[ \times \cos \left[ 2 \left( \sqrt{\frac{\nu^2}{4} - \frac{A^2}{4} \exp(z)} - \frac{\nu}{2} \right. \right. \left. \sqrt{1 - \frac{A^2}{\nu^2} \exp(z)} \right] + \pi, \quad z < z_\nu; \]

\[ \psi_\nu(z) = \frac{1}{2\sqrt{2}\Gamma(i\nu)} \exp\left(\frac{\pi\nu}{2}\right) \left( \frac{A^2}{4} \exp(z) - \frac{\nu^2}{4} \right)^{\frac{1}{2}} \times \]
\[
\times \exp \left[ -2 \left( \sqrt{\frac{A^2}{4}} \exp(z) - \frac{\nu^2}{4} - \frac{\nu}{2} \arctan \sqrt{\frac{A^2}{\nu^2} \exp(z) - 1} \right) \right], \quad z > z_\nu,
\]

\[z_\nu = \ln \left( \frac{\nu^2}{A^2} \right)\]

being a classical turning-point.

The eigenfunctions of the operator \( \hat{L}_3 \) are plane waves

\[\psi_p(\phi) = \frac{1}{\sqrt{2\pi}} \exp(ip\phi)\]

that is in agreement with the classical solution \([10]\).

The general solution to Eq. (73) for a given value of the parameter \( E_k \), describing the GVC state, is a superposition

\[\Psi_{Ek}(z_1, z_2, \phi) = \int_{-\infty}^{\infty} dp \exp (z_1 z_2) \int dp_1 dp_2 c_1(\nu_1, \nu_2, p) \psi_{\nu_1}(z_1) \psi_{\nu_2}(z_2) \psi_p(\phi) \delta \left( \frac{3}{2} \nu_1^2 - \frac{3}{8} \nu_2^2 + \frac{1}{2} p^2 - E_k \right). \quad (74)\]

However, the stationary states which the wave functions \([74]\) correspond to are not physical being un-normalizable since \( E_k \) has a continuous spectrum. A physical state is described by a time-dependent wave packet:

\[\Psi_k(z_1, z_2, \phi, t) = \int_{-\infty}^{\infty} dE_k c_2(E_k) \Psi_{Ek}(z_1, z_2, \phi) \exp \left[ -iE_k(t - t_0) \right]. \quad (75)\]

Let us note, that in expressions \([74]\), \([73]\) the quantity \( E_k \) appears to be a controlling parameter providing, through the \( \delta \)-function, the correlation of the quantum numbers \( \nu_1, \nu_2, p \), and, by that, a probability distribution of space scales at the quantum stage of the Universe evolution.

One can obtain the classical evolution law by computing the mean values of the operators \( \exp \left( \frac{z_1}{2} \right) \), \( \exp \left( \frac{z_2}{2} \right) \) over the packet \([73]\). To do it the matrix elements will be required,

\[\int_{-\infty}^{\infty} dz \exp \left( \frac{z}{2} \right) \psi^*_\nu(z) \psi_\nu(z) = \left[ 2\pi \Gamma(-i\mu) \Gamma(i\nu) \right]^{-1} \int_{-\infty}^{\infty} dz \exp \left( \frac{z}{2} \right) K_{-i\mu} \left[ A \exp \left( \frac{z}{2} \right) \right] K_{i\nu} \left[ A \exp \left( \frac{z}{2} \right) \right] = \]

\[\pi \left[ 4A \Gamma(-i\mu) \Gamma(i\nu) \right]^{-1} \left\{ \cosh \left[ \frac{\pi}{2} (\mu + \nu) \right] \cosh \left[ \frac{\pi}{2} (\mu - \nu) \right] \right\}^{-1}. \quad (76)\]

For the packet \([74] - [75] \) to describe really a classically evolving Universe, it needs to be sufficiently narrow, i.e. \( c_1(\nu_1, \nu_2, p) \) and \( c_2(E_k) \) do not have to deviate from zero values beyond a small vicinity of their arguments near \( (\bar{\nu}_1, \bar{\nu}_2, \bar{\nu}) \) and \( \bar{E}_k \). Therefore,

\[\mu + \nu \approx 2\bar{\nu}; \quad \mu - \nu = \frac{A\omega}{2\bar{\nu}}, \quad (77)\]

where \( \omega = A^{-1}(\mu^2 - \nu^2) \) is the difference between two values of the parameter \( E_k \) corresponding to the quantum numbers \( \mu \) and \( \nu \). Note that the matrix element \([74] \) depends weakly on \( \nu \) and decreases quickly when \( |\mu - \nu| \) increasing. So, making use of the approximations \([74] \) one obtains for the average of the exponents \( \exp \left( \frac{z}{2} \right) \),

\[\exp \left( \frac{z}{2} \right) = \frac{1}{4} \tanh(\pi\bar{\nu}) \int_{-\infty}^{\infty} d\omega \frac{\exp \left[ -i\omega(t - t_0) \right]}{\cosh \left( \frac{\pi A\omega}{4\bar{\nu}} \right)} = \frac{\bar{\nu} \tanh(\pi\bar{\nu})}{A \cosh \left[ (2A^{-1} \bar{\nu}(t - t_0)) \right]. \quad (78)\]
In the classical limit $\tilde{\nu}$ is large, hence $\tanh(\pi \tilde{\nu}) \approx 1$, and, comparing (78) with the classical expression (42), one concludes that

$$\alpha = \frac{\tilde{\nu}_2}{A_2} = \frac{3}{8} \tilde{\nu}_2; \quad \beta = \frac{\tilde{\nu}_1}{A_1} = \frac{3}{2} \tilde{\nu}_1.$$

From (74) the equation for the mean values follows ($k = 0$):

$$\frac{3}{2} \tilde{\nu}_1^2 - \frac{3}{8} \nu_2^2 + \frac{1}{2} \tilde{p}^2 = \tilde{E},$$

(79)

because $\nu_1^2 \approx \tilde{\nu}_1^2$ and so on. The comparison of (43) and (79) gives

$$\bar{p} = C_s.$$

Evidently, to every classical cosmological evolution scenario some configuration of the wave packet (74) – (75) must correspond, though not all solutions to the Schrödinger equation describe classical universes, and, in addition, even those wave packets, for which the transition to the quasiclassical regime is possible, may turn to be unstable. Therefore in our approach the known problem of initial conditions for classical evolution is formulated as the problem of choice of quantum state of the Universe. The quantum state in the Bianchi-IX model is determined by a concrete kind of the function

$$\tilde{C}(\nu_1, \nu_2, p, E_k) = c_2(E_k) c_1(\nu_1, \nu_2, p),$$

describing a wave packet structure.

We do not know how the choice of the quantum state is made. Perhaps, it is made according to statistical laws in the process of the creation of the Universe from “Nothing”. In Sec. 15 we will discuss the hypothesis according to which the act of the creation is thought as a quantum transition taking place out of time from the special singular state to one of the physical states of the Universe which the wave packets (74) – (75) correspond to.

A quasiclassical wave packet may also be written in the form

$$\Psi_k(z_1, z_2, \phi, t) = \int \int dE_k dp d\nu_1 d\nu_2 \tilde{C}(\nu_1, \nu_2, p, E_k) \exp \left[ -iE_k(t - t_0) + i\sigma_1(\nu_1) + i\sigma_2(\nu_2) \right] \times$$

\[ \times \psi_p(\phi) \delta \left( \frac{3}{2} \nu_1^2 - \frac{3}{8} \nu_2^2 + \frac{1}{2} p^2 - E_k \right). \]

Here the sum $\sigma_1(\nu_1) + \sigma_2(\nu_2)$ is the part of the classical action $S(z_1, z_2, \phi, t)$, determining its dependence on $z_1$ and $z_2$ (the scalar field is treated as essentially quantum). The functions $\sigma(z)$ satisfy the equations

$$\frac{\partial \sigma}{\partial z} = \sqrt{\frac{\nu^2}{4} - \frac{A^2}{4}} \exp(z).$$

(80)

If the dependence of the classical action on the variables $z$ is given one can reconstruct the evolution law (12) with the help of the standard procedure. But the two mentioned methods of going over to the classical limit are applicable owing to the explicit dependence of the general solution on time. And this, in turn, is due to the indication of the concrete choice of a RS where the time $t$ is measured, available in the theory.

In the classical limit a classical subsystem of the physical object itself can be considered as a RS. Such a subsystem cannot fill the whole space; it is admissible that it occupies a limited region of space. So, we will refer to such RS as to a local one. The appearance of the time $\tau$ introduced as a parameter along a classical path is associated with this very RS.

A derivative with respect to path length can be defined by the following way:

$$\frac{d}{dt} = u(\tau) \nabla S \nabla,$$
where \( u(\tau) \) is an arbitrary function, \( \nabla S \) is a tangent vector to the path:

\[
\nabla = \left( 2\sqrt{3} \frac{\partial}{\partial z_1}, \ i\sqrt{3} \frac{\partial}{\partial z_2} \right).
\]

On the other hand,

\[
\frac{d}{d\tau} = \frac{dz_1}{d\tau} \frac{\partial}{\partial z_1} + \frac{dz_2}{d\tau} \frac{\partial}{\partial z_2},
\]

whence

\[
\frac{dz_1}{d\tau} = 12u(\tau) \frac{d\sigma_1}{dz_1}; \quad \frac{dz_2}{d\tau} = -3u(\tau) \frac{d\sigma_2}{dz_2}.
\]

From (80) one obtains

\[
\exp\left(\frac{z_1}{2}\right) = \beta \cosh^{-1} (2\beta [\hat{u}(\tau) - \tau_0]); \quad \exp\left(\frac{z_2}{2}\right) = \alpha \cosh^{-1} (2\alpha [\hat{u}(\tau) - \tau_0]); \quad (81)
\]

The time \( \tau \) of a local observer emerges irrespectively of the existence of the time \( t \), but both the times may correlate between each other. To bring in correspondence the expressions (42) and (81) it is sufficient to put \( u(\tau) = 1 \).

In ordinary quantum mechanics the time involving in the Schrödinger equation is also associated with an observer making measurements on a physical system. That time may fail to coincide with the time appearing in Heisenberg operator equations, as well as with the world time in which dynamics of the system can be described in case the system is quasiclassical. In quantum mechanics the hypothesis about the equivalence of the mentioned times is used, though this is nowhere specified. In the considered example we have manifested that the times used for describing the evolution of a quantum system are different in general. In QGD times associated with different observers can be brought to agreement with each other by choice of a gauge condition.

18. The problem of the creation of the Universe from “Nothing”

In the previous section an EPS scheme has been proposed that enables one to describe phenomenologically quantum evolution of the Universe as an integrated system including OM. But we have not touched upon the main question of quantum cosmology – the problem of the creation of the Universe from “Nothing” – and the associated problem of finding the initial state for cosmological evolution.

There are several approaches to the question about the quantum origin of the Universe, among the most significant works the papers by Hartle and Hawking [18] and Vilenkin [19, 20, 21] should be mentioned. These works are based on the Wheeler – DeWitt equation in the minisuperspace of two dimensions – the scale factor of the Friedman – Robertson – Walker space and a homogeneous scalar field. The creation of the Universe is thought to be a quantum tunneling from a classically forbidden region of the minisuperspace, and the state “Nothing” is described by the asymptotic of a wave function corresponding to the vicinity of scale factor values of Planckian order. The discussion was concentrated on the choice of boundary conditions for a wave function of the Universe, namely, the “no boundary” and tunneling prescription for the wave function.

The interest to the problem of creation of the Universe was aroused two years ago by the proposal of Hawking and Turok that the “no boundary” wave function may describe creation of an open inflationary universe [22]. The attempts were done to make cosmological predictions on the base of this proposal [23, 24].
At the same time, Hawking – Turok approach was seriously criticized by Linde [25, 26, 27] and Vilenkin [28] (see especially Linde’s review on the problem of quantum creation of the Universe, [25]).

In the other approach [29, 30] the creation of the Universe (more precisely, universes) is considered from the position of the second quantization (the third, regarding matter fields). The initial state is a vacuum of universes, and the creation act is analogous to particle creation in a nonstationary metric, the superspace metric playing its role. The 4-metric determinant (in the minisuperspace – the scale factor) acts as a time parameter causing nonstationarity of the metric. But the rightfulness of identification of this parameter with time beyond a quasiclassical region arouses doubt.

On the other hand, according to Grishchuk and Zeldovich [31], in the initial state, the state “Nothing”, there is neither space with its geometry nor time. The creation of the Universe is essentially a quantum-gravitational process in the result of which a classical space-time comes into existence. Hence, the process cannot be ordered by a geometric parameter, as it is done, for instance, in the approach based on the third quantization. There is no reason to employ some physical parameters to define the state “Nothing” as well.

Thus, the state “Nothing” belongs to a deeper level of reality beyond the bounds of the physical reality which quantum mechanics was formulated for. Strictly speaking, we have no grounds to describe such a state by a wave function obeying an Schrödinger equation.

But we cannot avoid making an attempt to apply quantum-mechanical concepts for describing the creation of the physical reality itself.

It may be supposed that the Universe appears as a result of spontaneous reduction of the state “Nothing”, potentially containing all the possible Universe states, to one of them. (Such a picture, as a matter of fact, is present in the Penrose’s report [3]). The wave function of the state “Nothing” is a superposition of all physical states but is not normalizable itself. From the classical viewpoint, the initial state of the Universe is a singular state without structure. From the QGD viewpoint, the Universe is localized in the region of small values of scale factor in minisuperspace. So far as this state cannot have any internal structure, there is no gravitation wave or matter field in it. Evidently, there exist no observer either. The wave function of the state must satisfy an equation describing an object with the only degree of freedom – the scale factor $r = \exp(q/2)$. This equation cannot be derived on the grounds of general theoretical principles since all of them anyway appeal to space-time notions. It can be only postulated on the basis of the above ideas.

We postulate the equation for the wave function of the state “Nothing” as follows:

$$\frac{d^2 \Psi_0}{dq^2} - 4 \exp(2q) \Psi_0 = 0. \tag{82}$$

It can be “obtained” from Eq. (73) by excluding all degrees of freedom but the scale factor. Its form, however, must not depend on a chosen concrete model. The similar equation for the wave function of the state “Nothing” was firstly suggested by Vilenkin [19].

The solution to Eq. (82) is the modified Bessel function of the zero order

$$\Psi_0(q) = C_0 K_0 \left[2 \exp(q)\right], \tag{83}$$

possessing all the mentioned properties; it describes a state with no matter, no observer and, consequently, no time, and with no space itself: the Universe is “locked” in the isotropic singularity $q = -\infty$ ($r = 0$).

Relative probabilities of transitions from the state “Nothing” to physical states, which the wave packets [74], (75) correspond to, at $t = t_0$ are given by the projections of the packets on the wave function [78]:

$$\int dq d\chi d\phi d\mu \Psi_0^* \int dk \Psi_k(q, \chi, \phi, t = t_0) \delta(\mu - k) = \int dE d\nu_1 d\nu_2 E^{-1} \sqrt{2\pi} \tilde{C}(\nu_1, \nu_2, 0, E) \xi(\nu_1, \nu_2),$$

$$\xi(\nu_1, \nu_2) = \int dq d\chi \Psi_0^*(q) \psi_{\nu_1}(q, \chi) \psi_{\nu_2}(q, \chi).$$

The time moment $t = t_0$ corresponds to the creation of the Universe, the act of the quantum transition from the state “Nothing” to one of the physical states itself taking place out of time. The act results in the

5Similarly, it is impossible to reject classical notions (among which we live) such as, for instance, particle coordinates and momenta, when interpreting quantum mechanics.
emergence of an observer (the GVC). From this moment we can consider the evolution of the Universe in
time, described in our model by changing in time of the wave packet (7 4), (7 5), with the supposed subsequent
going over to a quasiclassical regime.
Thus, the problem of the creation of the Universe from “Nothing” becomes a computational problem of
transition amplitudes to various physical states.

19. Conclusions

Let us briefly formulate the main results of the work.

1. It is shown that the standard Wheeler – DeWitt QGD fails to be constructed by correct mathemat-
ical methods of quantum theory based on the path integral formalism. The main physical cause for
the Wheeler – DeWitt QGD nonexistence in the frames of general QT is that there are no asymp-
totical quantum states providing non-interference of measurement means in interaction processes of
gravitational fields.

2. A mathematically correct QGD of the Bianchi-IX model is formulated in extended phase space (EPS);
the basic equation of the QGD in EPS is the dynamical Schrödinger equation describing time evolution
of a physical object (gravitational fields of a closed universe) and a classical gravitational vacuum con-
densate (GVC) breaking symmetry of the system after introducing continually distributed observation
means.

3. Gauge noninvariance is inherent in the proposed version of QGD in EPS in a fundamental manner; in
the frames of the Copenhagen operational interpretation of QT it is shown that this property adequately
reflects the conditions of observations in a closed universe in mental experiments operating with the
notions associated with the Landau-Lifshitz reference systems.

4. On the example of the exactly solvable quantum Bianchi-IX model with one gravitational-wave degree of
freedom it is shown that the GVC determines a cosmological scenario character through the controlling
parameter – its whole energy.

5. The new approach to the creation of the Universe as an objective reduction of the singular state
“Nothing” to one of alternative physical states existing in the gauge-noninvariant QGD in EPS is
proposed. It is shown that in this version there is the possibility to evaluate numerically relative
probabilities of various physical (normalizable) initial states (at the moment \( t = t_0 \)) of quantum
cosmological evolution.

The proposed QGD version we consider as a phenomenological extrapolation of the existing QT method-
ological principles and formalism on scales of a closed universe as a whole. Evidently, such an extrapolation,
even being mathematically correct, cannot be physically complete. The phenomenologeness of QT itself, the
lack of an answer to the question about nature of quantum integrity of a physical object and OM inside it
shows the necessity to make choice of mathematical procedures, finally defining the formalism of the theory,
without clear understanding the motives for this choice. In particular, the number of unsettled problems
includes:

1. The operator ordering problem, or the mathematically equivalent to it problem of the choice of path
integral approximation.

2. The problem of uncertainty about the choice of a parametrization fixing gauge variables, and of gauge
conditions realizing a choice of a reference system in the Landau-Lifshitz RS class; physically, this
problem consists in the lack of strict principles fixing the GVC properties.

The existence of these problems let the question about quantitative characteristics of quantum correlations
between the properties of a physical object and those of OM be open; the common origin of the problems
is insufficient understanding what the measuring process in quantum gravity is. To solve the problems a
self-consistent theory is required that primordially gives a description of the integrated system including 
OM, and contains a quantum theory of measurements as a constituent part.

We suppose that in the future theory a solution to the mentioned above problems will be obtained 
on the grounds of unification of the geometric and the quantum integrity conceptions. As to the existing 
phenomenological QT that is in principle gauge-noninvariant for a closed universe, we are forced to bind it to 
a concrete gauge class, and to bind the procedure of solving equations to a concrete gauge. In this situation 
for making gauge choice it is necessary to have some argumentation, ours consisting in the following:

1) the possibility of comparing with the canonical operator formalism;
2) the possibility to interpret directly the Landau-Lifshitz RS;
3) the availability of the wave properties of the GVC excitations.

The latter fixes a representative of a gauge class. From the future theory, likely, a deeper criteria for 
gauge choosing should be expected.

Let us note, finally, that after Penrose having discussed the necessity for a radical change of the 
QT content, we suppose that the new theory will require essentially new ideas and mathematical algorithms. 
According to its status, the theory should include a description of irreversible measurement processes; there-
fore, it should have one more substantial feature – time irreversibility. In other words, the presence of such a 
subsystem as a GVC in an integrated system must, likely, break Hermitianess of Hamiltonian of the system.

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