A Reidemeister-Schreier theorem for finitely
$L$-presented groups

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Abstract

We prove a variant of the well-known Reidemeister-Schreier theorem for
finitely $L$-presented groups. More precisely, we prove that each finite index
subgroup of a finitely $L$-presented group is itself finitely $L$-presented. Our
proof is constructive and it yields a finite $L$-presentation for the subgroup.
We further study conditions on a finite index subgroup of an invariantly
finitely $L$-presented group to be invariantly $L$-presented itself.

Keywords: Reidemeister-Schreier theorem; infinite presentations; recursive
presentations; self-similar groups; Basilica group; Grigorchuk group; finite
index subgroups;

1 Introduction

Group presentations play an important role in computational group theory. In
particular finite group presentations have been subject to extensive research
in computational group theory dating back to the early days of computer-
algebra-systems [23]. Group presentations, on the one hand, provide an effective
description of the group. On the other hand, a description of a group by its
generators and relations leads to various decision problems which are known to
be unsolvable in general. For instance, the word problem of a finitely presented
group is unsolvable [27] [7]: see also [21]. Though various total and partial
algorithms for finitely presented groups are known [31]. For instance, the coset-
enumeration process introduced by Todd and Coxeter [32] enumerates the cosets
of a subgroup in a finitely presented group. If the subgroup has finite index,
coset-enumeration terminates and it computes a permutation representation
for the group’s action on the cosets. Coset-enumeration is a partial algorithm
as the process will not terminate if the subgroup has infinite index. However,
finite presentations often allow total algorithms that will compute factor groups
with a special type (including abelian quotients, nilpotent quotients [26] and,
in general, solvable quotients [20]).

Beside quotient and subgroup methods, the well-known theorem by Reide-
meister [29] and Schreier [30] allows to compute a presentation for a subgroup.
The Reidemeister-Schreier theorem explicitly shows that a finite index subgroup
of a finitely presented group is itself finitely presented. A similar result can be
shown for finite index ideals in finitely presented semi-groups [8]. In practice, the permutation representation for the group’s action on the cosets allows to compute the Schreier generators of the subgroup and the Reidemeister rewriting. The Reidemeister rewriting allows us to rewrite the relations of the group to relations of the subgroup [10, 31, 21]. Note that a method to compute a finite presentation for a finite index subgroup can be applied in the investigation of the structure of a group by its finite index subgroups; see [17].

Even though finitely presented groups have been studied for a long time, most groups are not finitely presented as there are uncountably many two-generator groups [24] but only countably many finite presentations [1]. A generalization of finite presentations are finite $L$-presentations which were introduced in [1]; however, there are still only countably many finite $L$-presentations. It is known that various examples of self-similar or branch groups (including the Grigorchuk group [10] and its twisted twin [4]) are finitely $L$-presented but not finitely presented [1]. Finite $L$-presentations are possibly infinite presentations with finitely many generators and whose relations (up to finitely many exceptions) are obtained by iteratively applying finitely many substitutions to a finite set of relations; see [1] or Section 2 below. A finite $L$-presentation is invariant if the substitutions which generate the relations induce endomorphisms of the group; see also Section 2. In fact, invariant finite $L$-presentations are finite presentations in the universe of groups with operators defined in [19, 26] in the sense that the operator domain of the group generates the infinitely many relations out of a finite set of relations.

Finite $L$-presentations allow computer algorithms to be applied in the investigation of the groups they define. For instance, they allow to compute the lower central series quotients [2], the Dwyer quotients of the group’s Schur multiplier [13], and even a coset-enumeration process exists for finitely $L$-presented groups [13]. It is the aim of this paper to prove the following variant of Reidemeister-Schreier’s theorem:

**Theorem 1.1** Each finite index subgroup of a finitely $L$-presented group is finitely $L$-presented.

If the finite index subgroup in Theorem 1.1 is normal and invariant under the substitutions (i.e., a normal and admissible subgroup in the notion of Krull & Noether [19, 26]), an easy argument gives a finite $L$-presentation for the subgroup; furthermore, if the group is invariantly finitely $L$-presented, then so is the subgroup. However, more work is needed if the subgroup is not invariant under the substitutions. Under either of two extra conditions (the subgroup is leaf-invariant, see Definition 5.8 or it is normal and weakly leaf-invariant, see Definition 7.2), we show that the subgroup is invariantly finitely $L$-presented as soon as the group is. We have not been able to get rid of these extra assumptions. In particular, it is not clear whether a finite index subgroup of an invariantly finitely $L$-presented group is always invariantly finitely $L$-presented. We show that the methods presented in this paper will (in general) fail to compute invariant $L$-presentations for the subgroup even if the group is invariantly $L$-presented. However, we are not aware of a method to prove that a given subgroup does not admit an invariant finite $L$-presentation at all.
Our proof of Theorem 1.1 is constructive and it yields a finite $L$-presentation for the subgroup. These finite $L$-presentations can be applied in the investigation of the underlying groups as the methods in [17] suggest for finitely presented groups. Notice that Theorem 1.1 was already posed in Proposition 2.9 of [1]. The proof we explain in this paper follows the sketch given in [1], but fixes a gap as the $L$-presentation of the group in Theorem 1.1 is possibly non-invariant. Even if the $L$-presentation is assumed to be invariant, the considered subgroup cannot be assumed to be invariant under the substitutions.

This paper is organized as follows: In Section 2 we recall the notion of a finite $L$-presentation and we recall basic group theoretic constructions which preserve the property of being finitely $L$-presented. Then, in Section 3 we recall the well-known Reidemeister-Schreier process. Before we prove Theorem 1.1 in Section 4 we construct in Section 3 a counter-example to the original proof of Theorem 1.1 in [1, Proposition 2.9]. Then, in Section 5 we introduce the stabilizing subgroups which are the main tools in our proof of Theorem 1.1. In Section 6 we study conditions on the finite index subgroup of an invariantly $L$-presented group to be invariantly $L$-presented itself. We conclude this paper by considering two examples of subgroup $L$-presentations in Section 7 including the normal closure of a generator of the Grigorchuk group considered in [3, 9]. We fix a mistake in the generating set of the normal closure $\langle d \rangle^\Phi$ using our Reidemeister-Schreier theorem for finitely $L$-presented groups. Therefore we show, in the style of [17], how these computational methods can be applied in the investigation of self-similar groups.

2 Preliminaries

In the following, we briefly recall the notion of a finite $L$-presentation and the notion a finitely $L$-presented group as introduced in [1]. Moreover, we recall some basic constructions for finite $L$-presentations.

A finite $L$-presentation is a group presentation of the form

$$\langle \mathcal{X} \mid Q \cup \bigcup_{\sigma \in \Phi^*} R^\sigma \rangle,$$

where $\mathcal{X}$ is a finite alphabet, $Q$ and $R$ are finite subsets of the free group $F$ over $\mathcal{X}$, and $\Phi^* \subseteq \text{End}(F)$ denotes the free monoid of endomorphisms which is finitely generated by $\Phi$. We also write $\langle \mathcal{X} \mid Q \mid \Phi \mid R \rangle$ for the finite $L$-presentation in Eq. (1) and $G = \langle \mathcal{X} \mid Q \mid \Phi \mid R \rangle$ for the finitely $L$-presented group.

A group which admits a finite $L$-presentation is finitely $L$-presented. An $L$-presentation of the form $\langle \mathcal{X} \mid \emptyset \mid \Phi \mid R \rangle$ is an ascending $L$-presentation and an $L$-presentation $\langle \mathcal{X} \mid Q \mid \Phi \mid R \rangle$ is invariant (and the group it presents is invariantly $L$-presented), if each endomorphism $\varphi \in \Phi$ induces an endomorphism of the group $G$; that is, if the normal subgroup $\langle Q \cup \bigcup_{\varphi \in \Phi^*} R^\varphi \rangle^F$ is $\varphi$-invariant. Each ascending $L$-presentation is invariant and each invariant $L$-presentation
Proposition 2.1 There are finite $L$-presentations that are not invariant.

Proof. The group $B = \langle \{a, b, t\} \mid \{a^i a^{-4}, b^{-1} b^{-2}, [a, b'] \mid i \in \mathbb{Z}\} \rangle$ is a metabelian, infinitely related group with trivial Schur multiplier $[6]$. By introducing a stable letter $u$, this group admits the finite $L$-presentation

$$\langle \{a, b, t, u\} \mid \{ub^{-1}\} \mid \{\sigma, \delta\} \mid \{a^i a^{-4}, b^{-1} b^{-2}, [a, u]\}\rangle,$$

where $\sigma$ is the free group homomorphism induced by the map $\sigma: a \mapsto a$, $b \mapsto b$, $t \mapsto t$, and $u \mapsto u^t$, while $\delta$ is the free group homomorphism induced by the map $\delta: a \mapsto a$, $b \mapsto b$, $t \mapsto t$, and $u \mapsto u^{-1}$. This finite $L$-presentation is not invariant $[14]$. □

The class of finitely $L$-presented groups contains all finitely presented groups:

Proposition 2.2 Each finitely presented group $\langle X \mid R \rangle$ is finitely $L$-presented by the invariant (or ascending) finite $L$-presentation $\langle X \mid \emptyset \mid \emptyset \mid R \rangle$.

Therefore, (invariant or ascending) finite $L$-presentations generalize the concept of finite presentations. Examples of finitely $L$-presented, but not finitely presented groups, are various self-similar or branch groups $[1]$ including the Grigorchuk group $[10, 22, 11]$ and its twisted twin $[4]$. However, the concept of a finite $L$-presentation is quite general so that other examples of infinitely presented groups are finitely $L$-presented as well. For instance, the groups in $[6] [18] [28]$ are all finitely $L$-presented.

Various group theoretic constructions that preserve the property of being finitely $L$-presented have been studied in $[1]$. For completeness, we recall some of these constructions in the remainder of this section.

Proposition 2.3 ([1] Proposition 2.7]) Let $G = \langle X \mid Q \mid \Phi \mid R \rangle$ be a finitely $L$-presented group and let $H = \langle Y \mid S \rangle$ be finitely presented. The group $K$ which satisfies the short exact sequence $1 \to G \to K \to H \to 1$ is finitely $L$-presented.

Proof. We recall the constructions from $[1]$ in the following: Let $\delta: H \to K$ be a section of $H$ to $K$ and identify $G$ with its image in $K$. Each relation $r \in S$ of the finitely presented group $H$ lifts, through the section $\delta$, to an element $g_r \in G$. As the group $G$ is normal in $K$, each generator $t \in Y$ of the finitely presented group $H$ acts, via $\delta$, on the subgroup $G$. Thus we have $x^{g(t)} = g_{x,t} \in G$ for each $x \in X$ and $t \in Y$. If $X \cup Y = \emptyset$, we may consider the following finite $L$-presentation

$$\langle X \cup Y \mid Q \cup \{r g_r^{-1} \mid r \in S\} \cup \{x^i g_{x,t}^{-1} \mid x \in X, t \in Y\} \mid \hat{\Phi} \mid R \rangle, \quad (2)$$

where the endomorphisms $\Phi$ of $G$’s finite $L$-presentation $\langle X \mid Q \mid \Phi \mid R \rangle$ are extended to endomorphisms $\hat{\Phi} = \{\hat{\sigma} \mid \sigma \in \Phi\}$ of the free group $F(X \cup Y)$ by

$$\hat{\sigma}: F(X \cup Y) \to F(X \cup Y), \quad \left\{ \begin{array}{ll}
  x & \mapsto x^\sigma, \quad \text{for each } x \in X \\
  y & \mapsto y, \quad \text{for each } y \in Y.
\end{array} \right.$$
As a finite group is finitely presented, Proposition 2.3 yields the immediate

Corollary 2.4 Each finite extension of a finitely L-presented group is finitely L-presented.

Note that the constructions in the proof of Proposition 2.3 above give a finite L-presentation for $K$ which is not ascending – even if the group $G$ is given by an ascending L-presentation. We therefore ask the following

Question 1 Is every finite extension of an invariantly (finitely) L-presented group invariantly (finitely) L-presented?

We do not have an answer to this question in general; though we suspect its answer is negative, see Remark 7.8. Given endomorphisms $\Phi$ of the normal subgroup $G$ in Proposition 2.3, one problem is to construct endomorphisms of the finite extension $K$ which restrict to $\Phi$. This does not seem to be possible in general.

A finite L-presentation for a free product of two finitely L-presented groups is given by the following improved version of [1, Proposition 2.6].

Proposition 2.5 The free product of two finitely L-presented groups is finitely L-presented. If both finitely L-presented groups are invariantly L-presented, then so is their free product.

Proof. Although a proof of the first claim can be found in [1], we summarize its construction for our proof of the second claim. Let $G = \langle X \mid Q \mid \Phi \mid R \rangle$ and $H = \langle Y \mid S \mid \Psi \mid T \rangle$ be finitely L-presented groups. Suppose that $X \cap Y = \emptyset$ holds. Then $G * H$ is finitely L-presented by $\langle X \cup Y \mid Q \cup S \mid \Phi \cup \Psi \mid R \cup T \rangle$ (see [1]), where the endomorphisms in $\Phi$ and in $\Psi$ are extended to endomorphisms $\hat{\Phi}$ and $\hat{\Psi}$ of the free group $F(X \cup Y)$ over $X \cup Y$ as follows: for each $\sigma \in \Phi$, we let

$$\hat{\sigma}: F(X \cup Y) \to F(X \cup Y), \quad \left\{\begin{array}{ll}
x \mapsto x^\sigma, & \text{for each } x \in X \\
y \mapsto y, & \text{for each } y \in Y;
\end{array}\right.$$  

and, accordingly, for each $\delta \in \Psi$. As an invariant L-presentation $\langle X \mid Q \mid \Phi \mid R \rangle$ can be considered as an ascending L-presentation $\langle X \mid \emptyset \mid \Phi \mid Q \cup R \rangle$, we can consider $Q$ and $S$ to be empty. Then the latter construction from [1] shows that the free product $G * H$ is ascendingly finitely L-presented and thus it is invariantly finitely L-presented.

We further have the following improved version of [1, Proposition 2.9]:

Proposition 2.6 Let $N \trianglelefteq G$ be a normal subgroup of a finitely L-presented group $G = \langle X \mid Q \mid \Phi \mid R \rangle$. If $N$ is finitely generated as a normal subgroup, the factor group $G/N$ is finitely L-presented. If, furthermore, $G$ is invariantly L-presented and the normal subgroup $N$ is invariant under the induced endomorphisms $\Phi$, then $G/N$ is invariantly L-presented.
Proof. Let $N = \langle g_1, \ldots, g_n \rangle^G$ be a finite normal generating set of the normal subgroup $N$. We consider the generators $g_1, \ldots, g_n$ as elements of the free group $F$ over $\mathcal{X}$. Then the finite $L$-presentation $\langle \mathcal{X} \mid Q \cup \{g_1, \ldots, g_n\} \mid \Phi \mid R \rangle$ is a finite $L$-presentation for the factor group $G/N$; see [1]. Suppose that $G$ is given by an invariant $L$-presentation $\langle \mathcal{X} \mid Q \mid \Phi \mid R \rangle$. Then $G = \langle \mathcal{X} \mid \emptyset \mid \Phi \mid Q \cup R \rangle$. As $N^\sigma \subseteq N$ holds, each $\sigma \in \Phi^*$ induces an endomorphism of the $L$-presented factor group $G/N$. Thus the images $g_1^\sigma, \ldots, g_n^\sigma$ are consequences of the relations of $G/N$’s finite $L$-presentation. Therefore $G/N \cong \langle \mathcal{X} \mid Q \mid \Phi \mid R \cup \{g_1, \ldots, g_n\} \rangle = \langle \mathcal{X} \mid \emptyset \mid \Phi \mid Q \cup R \cup \{g_1, \ldots, g_n\} \rangle$. \hfill \Box

Note that, if $G$ is invariantly $L$-presented and $N$ is a normal $\Phi$-invariant subgroup, then, in the notion of Krull & Noether [19] [20], the group $G$ is a group with operator domain $\Phi$ and the normal subgroup $N$ is an admissible subgroup. Proposition 2.3 and Proposition 2.6 yield the following straightforward

Corollary 2.7 Let $G$ and $H$ be finitely $L$-presented groups and let $F$ be a finitely generated group with isomorphisms $\psi: F \to G$ and $\phi: F \to H$. Then the amalgamated free product $G *_F H$ is finitely $L$-presented.

For further group theoretic constructions which preserve the property of being finitely $L$-presented were refer to [1].

3 The Reidemeister-Schreier process

In the following, we briefly recall the Reidemeister-Schreier process for finite index subgroups as, for instance, outlined in [21] [31]. For this purpose, let $G$ be a group given by a group presentation $\langle \mathcal{X} \mid K \rangle$ where $\mathcal{X}$ is a (finite) alphabet which defines the free group $F$ and $K \subseteq F$ is a (possibly infinite) set of relations. Denote the normal closure of $K$ in $F$ by $K = \langle K \rangle^F$. Then $G = F/K$.

Let $\mathcal{U} \leq G$ be a finite index subgroup of $G$ given by its generators $g_1, \ldots, g_n$. Let $T \subseteq F$ be a Schreier transversal for $\mathcal{U}$ in $G$ (i.e., a transversal for $\mathcal{U}$ in $G$ so that every initial segment of an element of $T$ itself belongs to $T$, see [21]; note that we always acts by multiplication from the right). We consider the generators of $\mathcal{U}$ as words over the alphabet $\mathcal{X}$ and thus as elements of the free group $F$. Then the subgroup $U = \langle g_1, \ldots, g_n \rangle$ satisfies that $\mathcal{U} \cong UK/K$. In the style of [21], we define the Schreier map $\gamma: T \times \mathcal{X} \to F$ by $\gamma(t, x) = tx (\overline{tx})^{-1}$ where $\overline{tx}$ denotes the unique element $s \in T$ from the Schreier transversal so that $UKs = UKtx$ holds. The Schreier theorem (as, for instance, in [21] Proposition I.3.7]) shows that the subgroup $UK \leq F$ is freely generated by the Schreier generating set

$$\mathcal{Y} = \{ \gamma(t, x) \neq 1 \mid t \in T, x \in \mathcal{X} \}.$$ 

In particular, the Schreier theorem yields that a finite index subgroup of a finitely generated group is itself finitely generated. We consider the set $\mathcal{Y}$ as an alphabet and we denote by $F(\mathcal{Y})$ the free group over $\mathcal{Y}$. The Reidemeister rewriting $\tau$ is a map $\tau: F \to F(\mathcal{Y})$ given by

$$\tau(y_1 \cdots y_n) = \gamma(1, y_1) \cdot \gamma(y_1, y_2) \cdots \gamma(y_1 \cdots y_{n-1}, y_n)$$
where each \( y_i \in \mathcal{X} \cup \mathcal{X}^\prime \). In general, the Reidemeister rewriting \( \tau \) is not a group homomorphism; though, we have the following

**Lemma 3.1** For \( H \leq \text{UK} \), the restriction \( \tau: H \to F(\mathcal{Y}) \) is a homomorphism.

**Proof.** Let \( g, h \in H \) be given. Write \( g = g_1 \cdots g_n \) and \( h = h_1 \cdots h_m \) with each \( h_i, g_j \in \mathcal{X} \cup \mathcal{X}^\prime \). Then, as \( \overline{g_1 \cdots g_n} = \gamma = 1 \) holds, we obtain that

\[
\tau(gh) = \gamma(1,g_1) \cdots \gamma(\overline{g_1 \cdots g_n-1}, g_n) \cdot \gamma(1,h_1) \cdots \gamma(h_1 \cdots h_{m-1}, h_m) = \tau(g) \tau(h)
\]

while we already have \( \tau(1) = 1 \) by definition. \( \square \)

By Schreier’s theorem, the Reidemeister rewriting \( \tau: \text{UK} \to F(\mathcal{Y}) \) gives an isomorphism of free groups. A group presentation for the subgroup \( \mathcal{U} \equiv \text{UK}/K \) is given by the following well-known theorem; cf. [21, Section II.4].

**Theorem 3.2 (Reidemeister-Schreier Theorem)** If \( \tau \) denotes the Reidemeister-Schreier rewriting, \( T \) denotes a Schreier transversal for \( \mathcal{U} \) in \( G \), and if \( \langle \mathcal{X} \mid K \rangle \) is a presentation for \( G \), the subgroup \( \mathcal{U} \) is presented by

\[
\mathcal{U} \equiv \langle \mathcal{Y} \mid \{ \tau(tut^{-1}) \mid r \in \mathcal{K}, t \in T \} \rangle.
\]

**Proof.** We recall the proof for completeness: Notice that \( \mathcal{U} \equiv \text{UK}/K \equiv \tau(\text{UK})/\tau(K) \) holds. By Schreier’s theorem, we have \( \tau(\text{UK}) = F(\mathcal{Y}) \). It therefore suffices to determine a normal generating set for \( \tau(K) \). As \( \mathcal{K} \) is a normal generating set for \( \mathcal{K} \leq F \), a generating set for the image \( \tau(K) \) is given by

\[
\tau(K) = \langle \{ \tau(grg^{-1}) \mid r \in \mathcal{K}, g \in F \} \rangle.
\]

Since \( T \) is a transversal for \( \text{UK} \) in \( F \), each \( g \in F \) can be written as \( g = ut \) with \( t \in T \) and \( u \in \text{UK} \). This yields \( \tau(K) = \langle \{ \tau(utut^{-1}u^{-1}) \mid r \in \mathcal{K}, g = ut \in F \} \rangle \). For each relation \( r \in \mathcal{K} \), we have that \( trt^{-1} \in \text{UK} \) and \( u \in \text{UK} \). By Lemma 3.1, we obtain that

\[
\tau(utut^{-1}u^{-1}) = \tau(u) \tau(trt^{-1}) \tau(u)^{-1}.
\]

Therefore \( \tau(utut^{-1}u^{-1}) \) is a consequence of \( \tau(trt^{-1}) \) and hence, it can be omitted. Thus a normal generating set for \( \tau(K) \) is given by \( \tau(K) = \langle \{ \tau(trt^{-1}) \mid r \in \mathcal{K}, t \in T \} \rangle^{F(\mathcal{Y})} \). \( \square \)

In particular, if \( \mathcal{U} \) is a finite index subgroup of a finitely presented group \( G \), there exist a finite set of relations \( \mathcal{K} \) and a finite Schreier transversal \( T \) so that the subgroup \( \mathcal{U} \) is finitely presented by Theorem 3.2. This latter result for finitely presented groups is well-known and it is often simply referred to the Reidemeister-Schreier theorem for finitely presented groups. In this paper, we prove a variant of the Reidemeister-Schreier theorem for finitely \( L \)-presented groups.

### 4 A typical example of a subgroup \( L \)-presentation

Before proving Theorem 1.1 we first consider an example of a finite \( L \)-presentation for a finite index subgroup of a finitely \( L \)-presented group. For this purpose we consider a subgroup of the Basilica group [12]. The Basilica group satisfies the following
Proposition 4.1 (Bartholdi & Virág, [5]) The Basilica group $G$ is invariantly finitely $L$-presented by $G \cong \langle \{a, b\} \mid \emptyset \mid \{\sigma\} \mid \{[a, a^b]\} \rangle$ where $\sigma$ is the free group homomorphism induced by the map $a \mapsto b^2$ and $b \mapsto a$.

Since the Basilica group $G$ is invariantly $L$-presented, the substitution $\sigma$ induces an endomorphism of $G$. The group $G$ will often provide an exclusive (counter-)example throughout this paper.

Consider the subgroup $\mathcal{U} = \langle a, bab^{-1}, b^3 \rangle$ of the Basilica group. Then coset enumeration for finitely $L$-presented groups [13] shows that $\mathcal{U}$ is a normal subgroup of $G$ with index 3. A Schreier generating set for the subgroup $\mathcal{U}$ is given by $\{a, bab^{-1}, b^2ab^{-2}, b^3\}$. Write $x_1 = a$, $x_2 = bab^{-1}$, $x_3 = b^2ab^{-2}$, and $x_4 = b^3$.

Denote the free group over $\{a, b\}$ by $F$ and let $F$ denote the free group over $\{x_1, x_2, x_3, x_4\}$. For each $n \in \mathbb{N}_0$, we define $a_n = (2^n + 2)/3$ and $b_n = (2^n + 1)/3$.

Then the $\sigma$-images of the iterated relation $r = [a, a^b]$ can be rewritten with the Reidemeister rewriting $\tau: F \to F$. Their images have the form

$$\tau(r^{\sigma^{2n}}) = \begin{cases} [x_1^{2n}, x_4^{-a_n} x_3^{2n} x_2^{a_n}] & \text{if } n \text{ is even}, \\ [x_1^{2n}, x_4^{-a_n} x_2^{2n} x_3^{a_n}] & \text{if } n \text{ is odd}, \end{cases}$$

and

$$\tau(r^{\sigma^{2n+1}}) = \begin{cases} x_4^{-b_{n+1}} x_2^{-2n} x_4^{-a_{n+1}} x_3^{2n} x_4^{-b_{n+1}} x_2^{-b_{n+1}} x_4^{2n} x_1^{-1} & \text{if } n \text{ is even}, \\ x_4^{-a_{n+1}} x_3^{-2n} x_4^{-a_{n+1}} x_3^{-b_{n+1}} x_2^{2n} x_4^{a_{n+1}} x_3^{-2n} x_4^{a_{n+1}} x_1^{2n} & \text{if } n \text{ is odd}. \end{cases}$$

Note that $\tau(r^{\sigma^{2n}}) \in [F, F]$ though $\tau(r^{\sigma^{2n+1}}) \not\in [F, F]$. Therefore, the images $\tau(r^{\sigma^n})$ split into two classes which are recursive images of the endomorphism

$$\tilde{\delta}: F \to F, \begin{cases} x_1 \mapsto x_1^2, \\ x_2 \mapsto x_3^2, \\ x_3 \mapsto x_4 x_2 x_4^{-1}, \\ x_4 \mapsto x_4^2; \end{cases}$$

in the sense that $\tilde{\delta}$ satisfies

$$\tau(r^{\sigma^{2n}}) = [x_1, x_4^{-1} x_3 x_4]^{\tilde{\delta}^{2n}}$$

and

$$\tau(r^{\sigma^{2n+1}}) = (x_4^{-1} x_2^{-1} x_4^{-1} x_3 x_4 x_2^{-1} x_4 x_1)^{\tilde{\delta}^{2n}},$$

for each $n \in \mathbb{N}_0$. In Section [13] we will show that a finite $L$-presentation for the subgroup $\mathcal{U}$ is given by

$$\mathcal{U} \cong \langle \{x_1, \ldots, x_4\} \mid \emptyset \mid \{\tilde{\sigma}, \delta\} \mid \{[x_1, x_4^{-1} x_3 x_4], x_4^{-1} x_2^{-1} x_4^{-1} x_3 x_4 x_2^{-1} x_4 x_1]\rangle$$

where the endomorphism $\delta$ is induced by the mapping

$$\delta: F \to F, \begin{cases} x_1 \mapsto x_2, \\ x_2 \mapsto x_3, \\ x_3 \mapsto x_4 x_1 x_4^{-1}, \\ x_4 \mapsto x_4. \end{cases}$$

These subgroup $L$-presentations are typical for finite index subgroups of a finitely $L$-presented group. Besides, the subgroup $\mathcal{U}$ and its subgroup $L$-presentation provide a counter-example to the original proof of Theorem [13] in [13] as
there is no endomorphism \( \varepsilon \) of the free group \( \mathcal{F} \) such that \( \tau(\sigma^{n+1}) = \varepsilon(\sigma^n) \) for each \( n \in \mathbb{N}_0 \). A reason for the failure of the proof in [1] is that the subgroup \( \mathcal{U} \) is not \( \sigma \)-invariant but \( \sigma^2 \)-invariant. Therefore, the method suggested in the proof of [1, Proposition 2.9] will fail to compute a finite \( L \)-presentation for \( \mathcal{U} \).

5 Stabilizing subgroups

In this section, we introduce the stabilizing subgroups which will be central to what follows.

Let \( G = \langle X \mid Q \mid \Phi \mid R \rangle \) be a finitely \( L \)-presented group and let \( \mathcal{U} \) be a finite index subgroup of \( G \) which is generated by \( g_1, \ldots, g_n \), say. Denote the free group over \( X \) by \( F \) and let \( K = (Q \cup \cup_{\sigma \in \Phi^*} R^\sigma)^F \). We consider the generators \( g_1, \ldots, g_n \) of the subgroup \( \mathcal{U} \) as words over the alphabet \( X \). Thus the subgroup \( U = \langle g_1, \ldots, g_n \rangle \) of the free group \( F \) satisfies \( \mathcal{U} \cong UK/K \). The group \( F \) acts on the right-cosets \( UK \setminus F \) by multiplication from the right. Let \( \varphi: F \to \text{Sym}(UK \setminus F) \) be a permutation representation for the group’s action on \( UK \setminus F \). Note that this permutation representation can be computed with the coset-enumeration methods in [13]. We obtain the following

**Lemma 5.1** The kernel \( \ker(\varphi) \) is the normal core, \( \text{Core}_F(UK) \), of \( UK \) in \( F \).

**Proof.** As each \( g \in \ker(\varphi) \) stabilizes the right-coset \( UK \setminus 1 \), the kernel \( \ker(\varphi) \trianglelefteq F \) is contained in the subgroup \( UK = \varphi^{-1}(\text{Stab}_{\text{Sym}(UK \setminus F)}(UK \setminus 1)) \). Hence \( \ker(\varphi) \leq \text{Core}_F(UK) \). Recall that \( \text{Core}_F(UK) = \bigcap_{x \in F} xUKx^{-1} \). We show that each \( g \in \text{Core}_F(UK) \) acts trivially on the right-cosets \( UK \setminus F \). Let \( t \in T \) be given. Then, as \( F \) acts transitively on the cosets \( UK \setminus F \), there exists \( h \in F \) so that \( th = v \in UK \). Then \( h^{-1} = v^{-1}t \). As \( g \in \bigcap_{x \in F} xUKx^{-1} \), there exists \( u \in UK \) with \( g = huh^{-1} \). Hence \( UK t \cdot g = UK t h u h^{-1} = UK u h^{-1} = UK h^{-1} = UK v^{-1} \cdot t = UK t \) and so \( g \) acts trivially on the right-cosets \( UK \setminus F \). Thus we have \( g \in \ker(\varphi) \) and \( \text{Core}_F(UK) \leq \ker(\varphi) \). \( \square \)

In the following we define the stabilizing subgroups. These subgroups will be central to our proof of Theorem [1, Theorem 5.1] in Section 6.

**Definition 5.2** Let \( G = \langle X \mid Q \mid \Phi \mid R \rangle \) be a finitely \( L \)-presented group and let \( \mathcal{U} \leq G \) be a finite index subgroup which admits the permutation representation \( \varphi: F \to \text{Sym}(UK \setminus F) \). The stabilizing subgroup of \( \mathcal{U} \) is

\[
\tilde{\mathcal{L}} = \bigcap_{\sigma \in \Phi^*} (\sigma \varphi)^{-1}(\text{Stab}_{\text{Sym}(UK \setminus F)}(UK \setminus 1)). 
\]

The stabilizing core of \( \mathcal{U} \) is

\[
\mathcal{L} = \bigcap_{\sigma \in \Phi^*} \ker(\sigma \varphi). 
\]

For each \( \sigma \in \Phi^* \), we denote by \( ||\sigma|| \) the usual word-length in the generating set \( \Phi \). The free monoid \( \Phi^* \) has the structure of a \( |\Phi| \)-regular tree with its root being the identity map \( \text{id}: F \to F \). We can further endow the monoid \( \Phi^* \) with a
length-plus-(from the right)-lexicographic ordering $<$ by choosing an arbitrary ordering on the (finite) generating set $\Phi$. We then define $\sigma \prec \delta$ if $\|\sigma\| < \|\delta\|$ or, otherwise, if $\sigma = \sigma_1 \cdots \sigma_n$ and $\delta = \delta_1 \cdots \delta_n$, with each $\sigma_i, \delta_j \in \Phi$, and there exists a positive integer $1 \leq k \leq n$ such that $\sigma_i = \delta_i$ for each $k < i \leq n$, and $\sigma_n \prec \delta_k$. Since $\Phi$ is finite, the constructed ordering $<$ is a well-ordering on the monoid $\Phi^*$; see [31]. Thus, there is no infinite descending sequences $\sigma_1 \succeq \sigma_2 \succeq \cdots$ in $\Phi^*$.

We consider Algorithm 1 below. If $\varphi: F \to \text{Sym}(UK\backslash F)$ denotes a permutation representation as in Definition 5.2, the algorithm yields the existence of $\sigma \in \Phi$ with each $\varphi$.

**Algorithm 1:** Computing a finite set of endomorphisms $V \subseteq \Phi^*$; see also [13]

IteratingEndomorphisms($\mathcal{X}, \mathcal{Q}, \Phi, R, \mathcal{U}, \varphi$)

Initialize $S := \Phi$ and $V := \{\text{id}: F \to F\}$.

Choose an ordering on $\Phi = \{\phi_1, \ldots, \phi_n\}$ with $\phi_i \prec \phi_{i+1}$.

while $S \neq \emptyset$

Remove the first entry $\delta$ from $S$.

if not ($\exists \sigma \in V: \delta \varphi = \sigma \varphi$) then

Append $\phi_1 \delta, \ldots, \phi_n \delta$ to $S$.

Add $\delta$ to $V$.

return($V$)

**Lemma 5.3** The algorithm IteratingEndomorphisms terminates and it returns a finite set of endomorphisms $V \subseteq \Phi^*$ satisfying the following property:

For each $\sigma_1 \in \Phi^*$ there exists a unique $\sigma_n \in V$ so that $\sigma_1 \varphi = \sigma_n \varphi$. The element $\sigma_n$ is minimal with respect to the total ordering $\prec$ constructed above.

**Proof.** Let $\mathcal{X}$ be a basis of the free group $F$. Then a homomorphism $\psi: F \to \text{Sym}(UK\backslash F)$ is uniquely defined by the image of this basis. Since $UK\backslash F$ is finite, the symmetric group $\text{Sym}(UK\backslash F)$ is finite. Moreover, as $F$ is finitely generated, the set of homomorphisms $\text{Hom}(F, \text{Sym}(UK\backslash F))$ is finite. Therefore the algorithm IteratingEndomorphisms can add only finitely many elements to $V$. Thus the stack $S$ will eventually be reduced and the algorithm terminates.

The ordering $\prec$ on $\Phi$ can be extended to a total and well-ordering on the free monoid $\Phi^*$ as described above. The elements in the stack $S$ are always ordered with respect to the total and well-ordering $\prec$. They further always succeed those elements in $V$. In particular, the elements in $V$ are minimal. Let $\sigma_1 \in \Phi^*$ be given. Then there exists $w \in \Phi^*$ maximal subject to the existence of $\delta \in V$ so that $\sigma_1 = w\delta$. If $\|w\| = 0$ holds, then $\sigma_1 \in V$ and the claim is proved. Otherwise, there exists $\psi \in \Phi$ so that $\sigma_1 = v\psi\delta$ for some $v \in \Phi^*$ and $\psi\delta \not\in V$. The algorithm yields the existence of $\varepsilon \in V$ so that $\varepsilon \prec \psi\delta$ and $\psi\delta \varphi = \varepsilon \varphi$. We also have that $\sigma_2 = v\varepsilon \prec v\psi\delta = \sigma_1$. This rewriting process yields a descending sequence $\sigma_1 \succeq \sigma_2 \succeq \cdots$ of endomorphisms. As $\prec$ is a well-ordering there exists $\sigma_n \in V$ so that $\sigma_1 \succeq \sigma_2 \succeq \cdots \succeq \sigma_n$ and $\sigma_1 \varphi = \sigma_n \varphi$. Clearly, the element $\sigma_n$ is unique. $\square$
If \( \varphi : F \to \text{Sym}(UK \setminus F) \) is a permutation representation for an infinite index subgroup \( UK \leq F \), we cannot ensure finiteness of the set \( V \) above.

For finite \( L \)-presentations \( \langle X \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle \) with \( \Phi = \{ \sigma \} \), Algorithm \ref{alg} and Lemma \ref{lem:psi} yield the following

**Corollary 5.4** If \( \Phi = \{ \sigma \} \), there exist integers \( 0 \leq i < j \) with \( \sigma^i \varphi = \sigma^j \varphi \).

The set \( V \subseteq \Phi^* \) returned by Algorithm \ref{alg} satisfies the following

**Lemma 5.5** The set \( V \) can be considered as a subtree of \( \Phi^* \). The image of the finite set \( V \) and the image of the monoid \( \Phi^* \) in \( \text{Hom}(F, \text{Sym}(UK \setminus F)) \) coincide.

**Proof.** The identity mapping \( \text{id} : F \to F \) is contained in the set \( V \) and it represents the root of \( V \). Let \( \sigma \in V \) be given. Then either \( \sigma \in \Phi \) or there exists \( \psi \in \Phi \) and \( \delta \in \Phi^* \) so that \( \sigma = \psi \delta \). In the first case, the identity mapping \( \text{id} : F \to F \) is a unique parent of \( \sigma \in \Phi \). Suppose that \( \sigma = \psi \delta \) holds. We need to show that \( \delta \in V \). The algorithm \textsc{IteratingEndomorphisms} only adds elements from the stack \( S \) to \( V \). Thus at some stage of the algorithm we had \( \sigma = \psi \delta \in S \); however, this element is added to the stack \( S \) as a child of the element \( \delta \). The uniqueness follows from the freeness of \( \Phi^* \). The second argument follows immediately from Algorithm \ref{alg} and Lemma \ref{lem:psi}.

We define a binary relation \( \sim \) on the free monoid \( \Phi^* \) by defining \( \sigma \sim \delta \) if and only if the unique element \( \sigma_n \in \Phi^* \) in Lemma \ref{lem:psi} coincides for both \( \sigma \) and \( \delta \). Thus \( \sigma \sim \delta \) holds if and only if \( \sigma \varphi = \delta \varphi \). This definition yields the immediate

**Lemma 5.6** The relation \( \sigma \sim \delta \) is an equivalence relation. Each equivalence class is represented by a unique element in \( V \) which is minimal with respect to the total and well-ordering \( \prec \).

Recall that \( \varphi : F \to \text{Sym}(UK \setminus F) \) is a permutation representation for the group’s action on the right-cosets \( UK \setminus F \). Therefore, if \( T \) denotes a transversal for \( UK \) in \( F \), then \( \sigma \sim \delta \) implies that \( UK t \cdot g^\sigma = UK t \cdot g^\delta \), for each \( t \in T \) and \( g \in F \).

We therefore obtain the following

**Lemma 5.7** If \( \sigma \in \Phi^* \) satisfies \( \sigma \varphi = \varphi \), then the subgroup \( UK \) is \( \sigma \)-invariant.

There are \( \sigma \)-invariant subgroups \( UK \) that do not satisfy \( \sigma \varphi = \varphi \).

**Proof.** As \( \sigma \varphi = \varphi \) holds, we have \( UK t \cdot g^\sigma = UK t g \) for each \( t \in T \) and \( g \in F \). Let \( g \in UK \) be given. Then \( UK 1 \cdot g^\sigma = UK 1 \cdot g = UK 1 \) and so \( g^\sigma \in UK \). The index-2 subgroup \( U = \langle a, b^2, bab^{-1} \rangle \) of the Basilica group satisfies \( (UK)^\sigma \subseteq UK \) and \( \sigma \varphi \neq \varphi \). This (and similar results in the remainder of this paper) can be easily verified with a computer-algebra-system such as GAP. \( \Box \)

The latter observation motivates the following

**Definition 5.8** Let \( G = \langle X \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle \) be a finitely \( L \)-presented group and let \( \mathcal{U} \leq G \) be a finite index subgroup with permutation representation \( \varphi \). Then the \( \varphi \)-leaves \( \Psi \subseteq \Phi^* \setminus V \) of \( V \subseteq \Phi^* \) are defined by

\[
\Psi = \{ \psi \delta \mid \psi \in \Phi, \delta \in V, \psi \delta \notin V, \psi \delta \varphi = \varphi \}. \quad (6)
\]

The subgroup \( \mathcal{U} \) is leaf-invariant if \( \Psi = \{ \psi \delta \mid \psi \in \Phi, \delta \in V, \psi \delta \notin V \} \) holds.
As \( \Phi \subseteq \text{End}(F) \) is finite and the equivalence \( \sim \) yields finitely many equivalence classes, the set of \( \varphi \)-leaves \( \Psi \) of \( V \subseteq \Phi^* \) is finite. We obtain the following

**Lemma 5.9** If \( \mathcal{U} \) is a leaf-invariant subgroup of \( G \), then each \( \varphi \)-leaf \( \psi \delta \in \Psi \) induces an endomorphism of \( \text{UK} \). Moreover, each \( \sigma_1 \in \Phi^* \) can be written as \( \sigma_1 = v \sigma \) with \( v \in V \) and \( \sigma \in \Psi^* \subseteq \text{End}(\text{UK}) \).

**Proof.** We again follow the ideas of Algorithm 1. By Lemma 5.7, the condition \( \psi \sigma \varphi = \varphi \) implies \( \psi \sigma \) -invariance of \( \text{UK} \) and hence \( \Psi^* \subseteq \text{End}(\text{UK}) \). Write \( W = \{ \psi \delta \mid \psi \in \Phi, \delta \in V, \psi \delta \not\in V \} \) and let \( \sigma_1 \in \Phi^* \) be given. There exists \( w \in \Phi^* \) maximal subject to the existence of \( \delta \in V \) so that \( \sigma_1 = w \delta \). If \( \|w\| = 0 \), then \( \sigma_1 = \delta \cdot \text{id} \) with \( \delta \in V \) and \( \text{id} \in \Psi^* \). Otherwise, there exists \( \psi \in \Phi \) and \( \sigma_2 \in \Phi^* \) so that \( \sigma_1 = \sigma_2 \psi \delta \) and \( \psi \delta \not\in V \). Note that \( \psi \delta \in W \). Since \( \mathcal{U} \) is leaf-invariant, we have \( W = \Psi \) and hence \( \psi \delta \in \Psi \). Therefore \( \psi \delta \) induces an endomorphism of \( \text{UK} \) and we can continue with the prefix \( \sigma_2 \) of \( \sigma_1 \). Clearly \( \sigma_2 < \sigma_1 \). Rewriting the prefix \( \sigma_2 \) yields a descending sequence \( \sigma_1 \succ \sigma_2 \ldots \in \Phi^* \).

As \( \succ \) is a well-ordering, we eventually have \( \sigma_1 \succ \sigma_2 \succ \ldots \succ \sigma_n \) with \( \sigma_n \in V \).

If the finite \( L \)-presentation \( \langle X \mid Q \mid F \mid R \rangle \) satisfies \( \Phi = \{ \sigma \} \) and if there exists a minimal positive integer \( 0 < j \) so that \( \sigma^j \varphi = \varphi \) holds, then the set \( W = \{ \psi \delta \mid \psi \in \Phi, \delta \in V, \psi \delta \not\in V \} \) in the proof of Lemma 5.9 above becomes \( W = \{ \sigma^j \} \). Note the following

**Remark 5.10** The condition \( \sigma^j \varphi = \sigma^0 \varphi \) is essential for the \( \sigma^j \)-invariance of the subgroup. For instance, the subgroup \( \mathcal{U} = \langle a, b^{-1}, b^{-1}a^2b, b^3, b^2ab^{-2} \rangle \) of the Basilica group satisfies \( \sigma^4 \varphi = \sigma^3 \varphi \) but it is not \( \sigma \)-invariant.

The stabilizing subgroup \( \tilde{\mathcal{L}} \) introduced in Definition 5.2 satisfies the following

**Proposition 5.11** Let \( V \subseteq \Phi^* \) be the finite set returned by Algorithm 1. The stabilizing subgroup \( \tilde{\mathcal{L}} \) satisfies that

\[
\tilde{\mathcal{L}} = \bigcap_{\sigma \in V} (\sigma \varphi)^{-1}(\text{Stab}_{\text{Sym}(\text{UK}\setminus F)}(\text{UK}1)).
\]

The stabilizing subgroup \( \tilde{\mathcal{L}} \) is \( \Phi \)-invariant (i.e., we have \( \tilde{\mathcal{L}}^\psi \subseteq \tilde{\mathcal{L}} \) for each \( \psi \in \Phi \)). It is contained in the subgroup \( \text{UK} \) and it has finite index in \( F \). The stabilizing subgroup \( \tilde{\mathcal{L}} \) is the largest \( \Phi^* \)-invariant subgroup of \( \text{UK} \). It is not necessarily normal in \( F \).

**Proof.** Write \( \mathcal{K} = \bigcap_{\sigma \in V} (\sigma \varphi)^{-1}(\text{Stab}_{\text{Sym}(\text{UK}\setminus F)}(\text{UK}1)) \). Clearly \( \tilde{\mathcal{L}} \subseteq \mathcal{K} \), holds.

Let \( g \in \mathcal{K} \) and \( \sigma \in \Phi^* \) be given. By Lemma 5.9 there exists a unique \( \delta \in V \) so that \( \sigma \varphi = \delta \varphi \). This yields that \( \text{UK}1 \cdot g^\delta = \text{UK}1 \cdot g^\delta = \text{UK}1 \). Thus

\[
g \in (\sigma \varphi)^{-1}(\text{Stab}_{\text{Sym}(\text{UK}\setminus F)}(\text{UK}1)) \text{ and so } \mathcal{K} \subseteq \tilde{\mathcal{L}}.
\]

Let \( \psi \in \Phi \) and \( \sigma \in V \) be given. Then either \( \psi \sigma \in V \) or there exists \( \delta \in V \) so that \( \psi \sigma \varphi = \delta \varphi \). In the first case, the image \( g^{\psi \sigma} \) of the element \( g \in \tilde{\mathcal{L}} \subseteq (\sigma \varphi)^{-1}(\text{Stab}_{\text{Sym}(\text{UK}\setminus F)}(\text{UK}1)) \) stabilizes the right-coset \( \text{UK}1 \) and thus \( g^{\psi \sigma} \in \text{UK} \). In the second case, there exists \( \delta \in V \) so that \( \psi \sigma \varphi = \delta \varphi \) and \( g^{\psi \sigma} = g^\delta \varphi \). As \( g \in \tilde{\mathcal{L}} \subseteq (\delta \varphi)^{-1}(\text{Stab}_{\text{Sym}(\text{UK}\setminus F)}(\text{UK}1)) \), the image \( g^\delta \) stabilizes the right-coset \( \text{UK}1 \). Hence \( \text{UK}1 \cdot g^{\psi \sigma} = \text{UK}1 \cdot g^\delta = \text{UK}1 \) and thus
$g^\psi\sigma \in UK$. Therefore, in both cases considered above, we have that $g^\psi\sigma \in UK$ and so $g^\psi \in (\sigma\varphi)^{-1}(\text{Stab}_{\text{Sym}(UK,F)}(UK))$. As $\sigma$ was arbitrarily chosen, we have $g^\psi \in \tilde{\mathcal{L}}$ which proves the $\psi$-invariance of the stabilizing subgroup $\tilde{\mathcal{L}}$.

Let $g \in \tilde{\mathcal{L}}$ be given. As id $\in V$ holds, $g \in \varphi^{-1}(\text{Stab}_{\text{Sym}(UK,F)}(UK))$. Hence the element $g \in \tilde{\mathcal{L}}$ stabilizes the right-coset $UK_1$. Thus $UK_1 \cdot g = UK_1$ and $g \in UK$. Since $\tilde{\mathcal{L}}$ is the intersection of finitely many finite index subgroups of $F$, the stabilizing subgroup has finite index in $F$.

Let $N$ be a $\Phi^*$-invariant subgroup satisfying $\tilde{\mathcal{L}} \leq N \leq UK \leq F$. Then, for each $\sigma \in \Phi^*$, we have $N^\sigma \subseteq N$. Let $g \in N$ be given. Then $g^\sigma \in N \leq UK$, as $N$ is $\sigma$-invariant. Thus $UK_1 \cdot g^\sigma = UK_1$ and $g \in (\sigma\varphi)^{-1}(\text{Stab}_{\text{Sym}(UK,F)}(UK))$.

Since $\sigma \in \Phi^*$ was arbitrary, we have $g \in \tilde{\mathcal{L}}$ and thus $\tilde{\mathcal{L}} = N$.

The stabilizing subgroup $\tilde{\mathcal{L}} = \langle a,bab^{-1},b^{-1}a^2b,b^2ab^{-2},b^3a^{-1}b,b^{-1}ab^3 \rangle$ of the subgroup $\mathcal{U} = \langle a,bab^{-1},b^{-1}a^2b,b^2ab^{-2},b^3a^{-1}b,b^{-1}ab^3 \rangle$ of the Basilica group is not normal in $F$. □

The stabilizing subgroup $\tilde{\mathcal{L}}$ always satisfies that $\tilde{\mathcal{L}} \subseteq UK$. Conditions for equality are given by the following

**Lemma 5.12** The stabilizing subgroup satisfies $\tilde{\mathcal{L}} = UK$ if and only if $(UK)^\psi \subseteq UK$ for all $\psi \in V$. Moreover, we have $(UK)^\psi \subseteq UK$ for all $\psi \in V$ if and only if $(UK)^\delta \subseteq UK$ for all $\delta \in \Phi^*$.

**Proof.** We already proved that $\tilde{\mathcal{L}} \subseteq UK$ holds. Let $g \in UK$ and $\psi \in V$ be given. If $(UK)^\psi \subseteq UK$ holds, then $UK_1 \cdot g^\psi = UK_1$ and thus $g^\psi \in \varphi^{-1}(\text{Stab}_{\text{Sym}(UK,F)}(UK))$. As $\psi$ was arbitrary, we have $g \in \tilde{\mathcal{L}}$ and therefore $UK \leq \tilde{\mathcal{L}}$. On the other hand, suppose that $UK \leq \tilde{\mathcal{L}}$ holds. Then, as $\tilde{\mathcal{L}}$ is $\psi$-invariant, for each $\psi \in \Phi^*$, we have that $(UK)^\psi \subseteq \tilde{\mathcal{L}} \subseteq UK$ which proves the $V$-invariance of $UK$.

Clearly, as $V \subseteq \Phi^*$ holds, the $\Phi^*$-invariance of $UK$ yields the $V$-invariance of $UK$. On the other hand, suppose that $UK$ is $V$-invariant. Let $\delta \in \Phi^*$ be given. Then there exists $\sigma \in V$ so that $\sigma\varphi = \delta\varphi$. If $g \in UK$, then the image $g^\sigma$ stabilizes the right-coset $UK_1$ as $UK$ is $V$-invariant. We therefore obtain $UK_1 \cdot g^\sigma = UK_1 \cdot g^\delta = UK_1$ and hence $g^\delta \in UK$ which proves the $\Phi^*$-invariance of $UK$. □

In the style of [13], we define a binary relation $\sim_{\varphi}$ on the free monoid $\Phi^*$ as follows: For $\sigma, \delta \in \Phi^*$ we define $\sigma \sim_{\varphi} \delta$ if and only if there exists a homomorphism $\pi: \text{im}(\delta\varphi) \rightarrow \text{im}(\sigma\varphi)$ so that $\sigma\varphi = \delta\varphi\pi$ holds. It is known [13] that it is decidable whether or not $\sigma \sim_{\varphi} \delta$ holds. This yields that

**Lemma 5.13** Let $V \subseteq \Phi^*$ be the finite set returned by Algorithm [1]. Then there exists a subset $\tilde{V} \subseteq V$ with the following property: For each $\sigma \in \Phi^*$ there exists a unique element $\delta \in W$ so that $\sigma \sim_{\varphi} \delta$ and $\delta$ is minimal with respect to the ordering $\prec$ in Lemma 5.3.

**Proof.** This is straightforward as the set $V$ returned by Algorithm [1] is an upper bound on $\tilde{V}$ because $\sigma \prec \delta$ implies both $\sigma \sim_{\varphi} \delta$ or $\delta \sim_{\varphi} \sigma$. □
Again, the set $\tilde{V}$ in Lemma 5.13 can be considered a subtree of $\Phi^*$ or even as a subtree of $V$. The binary relation $\sim_\varphi$ is reflexive and transitive but not necessarily symmetric. The equivalence relation $\sim$ and the relation $\sim_\varphi$ are related by the following

**Lemma 5.14** Let $G = \langle X \mid Q \mid \Phi \mid R \rangle$ be a finitely L-presented group and let $\varphi: F \to \text{Sym}(\mathcal{U}K\backslash F)$ be a permutation representation. For $\sigma, \delta \in \Phi^*$, we have

(i) We have $\sigma \sim_\varphi \delta$ and $\delta \sim_\varphi \sigma$ if and only if the homomorphism $\pi: \text{im}(\delta \varphi) \to \text{im}(\sigma \varphi)$ with $\sigma \varphi = \delta \varphi \pi$ is bijective.

(ii) If $\sigma \sim_\varphi \delta$, then $\sigma \sim_\varphi \delta$ and $\delta \sim_\varphi \sigma$. The converse is not necessarily true.

(iii) If $k > 0$ is minimal so that $\sigma^k \sim_\varphi \text{id}$, there exists a minimal positive integer $\ell$ so that $\ell \mid k$ and $\sigma^\ell \sim_\varphi \text{id}$. If $\Phi = \{\sigma\}$, then the set $\tilde{V}$ from Lemma 5.13 becomes $V = \{\text{id}, \sigma, \ldots, \sigma^{k-1}\}$.

(iv) If $\ell$ is a minimal positive integer so that both $\sigma^\ell \sim_\varphi \text{id}$ and $\text{id} \sim_\varphi \sigma^\ell$ hold, there exists $k \geq \ell$ so that $\sigma^k \sim_\varphi \text{id}$. If $\Phi = \{\sigma\}$, then the set $V$ returned by Algorithm 1 becomes $V = \{\text{id}, \sigma, \ldots, \sigma^{k-1}\}$.

(v) The subgroup $\mathcal{U} = \langle a, b^2, bab^{-1} \rangle$ of the Basilica group satisfies $\sigma \sim_\varphi \text{id}$ but there is positive integer $\ell > 0$ so that $\sigma^\ell \sim_\varphi \text{id}$ holds.

**Proof.** If the homomorphism $\pi: \text{im}(\delta \varphi) \to \text{im}(\sigma \varphi)$ with $\sigma \varphi = \delta \varphi \pi$ is bijective, then we obtain $\sigma \varphi \pi^{-1} = \delta \varphi$ and thus $\delta \sim_\varphi \sigma$. On the other hand, suppose that both $\sigma \sim_\varphi \delta$ and $\delta \sim_\varphi \sigma$ hold. Then there are homomorphisms $\pi: \text{im}(\sigma \varphi) \to \text{im}(\delta \varphi)$ and $\tau: \text{im}(\delta \varphi) \to \text{im}(\sigma \varphi)$ so that $\delta \varphi = \sigma \varphi \pi$ and $\sigma \varphi = \delta \varphi \tau$. This yields $\delta \varphi = \sigma \varphi \pi = \delta \varphi \tau \pi$ and $\sigma \varphi = \delta \varphi \tau = \sigma \varphi \pi \tau$. Hence $\pi$ and $\tau$ are isomorphisms.

Since $\sigma \sim_\varphi \delta$ implies $\delta \varphi = \sigma \varphi$, we immediately obtain both $\sigma \sim_\varphi \delta$ and $\delta \sim_\varphi \sigma$. The subgroup $\mathcal{U} = \langle a, bab^{-1}, b^3 \rangle$ of the Basilica group admits the permutation representation $\varphi: a \mapsto (\cdot), b \mapsto (1,2,3)$. We have $\sigma^2 \varphi: a \mapsto (\cdot), b \mapsto (1,3,2)$ and therefore $\sigma^2 \sim_\varphi \text{id}$ and $\text{id} \sim_\varphi \sigma^2$. Though $\sigma^2 \varphi \neq \varphi$.

Suppose that $\sigma^k \sim_\varphi \text{id}$ or $\sigma^k \varphi = \varphi$ holds. Then $\text{im}(\varphi) \supseteq \text{im}(\sigma \varphi) \supseteq \ldots \supseteq \text{im}(\sigma^k \varphi) = \text{im}(\varphi)$. Clearly, there exists a positive integer $0 < j \leq k$ minimal subject to the existence of $0 \leq i < j$ so that $\sigma^i \sim_\varphi \sigma^j$. Hence, there exists a homomorphism $\pi: \text{im}(\sigma^i \varphi) \to \text{im}(\sigma^j \varphi)$ so that $\sigma^i \varphi = \sigma^j \varphi \pi$. Note that $\pi$ is surjective. As $k - i > 0$, we have $\sigma^{k-i} \sigma^i \varphi = \sigma^{k-i} \sigma^i \varphi \pi = \sigma^k \varphi \pi = \varphi \pi$. On the other hand, we have $\sigma^{k-i} \sigma^i \varphi = \sigma^{k-i} \sigma^i \varphi = \sigma^{k-i} \varphi$. Hence $\sigma^{k-i} \varphi = \varphi \pi$. If $i > 0$, the latter contradicts the minimality of $j$. Thus $i = 0$ and we have $\sigma^i \varphi = \varphi \pi$ for a homomorphism $\pi: \text{im}(\varphi) \to \text{im}(\sigma^i \varphi)$. Since $\text{im}(\varphi) \supseteq \text{im}(\sigma \varphi) \supseteq \ldots \supseteq \text{im}(\sigma^k \varphi) = \text{im}(\varphi)$, the homomorphism $\pi$ is an automorphism of the finite group $\text{im}(\varphi)$. As $\text{im}(\varphi)$ is finite, the automorphism $\pi$ has finite order $n$, say. Suppose that $nj < k$ holds. Then we can write $k = s \cdot nj + t$ with $0 \leq t < nj$ and $s \in \mathbb{N}$. This yields $\varphi = \sigma^k \varphi = \sigma^t \sigma^{s \cdot nj} \varphi = \sigma^t \varphi (\pi^n)^s = \sigma^t \varphi$ and $\sigma^t \sim_\varphi \text{id}$. By the minimality of $j$, we have $t \geq j$. Therefore, we can write $t = m \cdot j + \ell$ with $0 \leq \ell < j$ and $m \in \mathbb{N}$. This yields that $\varphi = \sigma^t \varphi = \sigma^t \varphi \pi^m$ and thus $\sigma^t \sim_\varphi \text{id}$. If $\ell > 0$, then $\sigma^t \sim_\varphi \text{id}$ contradicts the minimality of $j$. Thus $t = mj$ and $j \mid k$ because $k = (sn + mj)$. This yields that $\varphi = \sigma^k \varphi = \sigma^{(sn + mj)} \varphi = \varphi \pi^{sn + mj}$ and, as $n$ is the order of the automorphism $\pi$, we obtain $n \mid sn + mj$ and $nj \mid k$. If, on
the other hand, \( nj > k \) holds, then \( j \leq k < nj \) and we can write \( k = mj + \ell \) with \( 0 \leq \ell < j \) and \( m \in \mathbb{N} \). Then \( \varphi = \sigma^k \varphi = \sigma^{mj + \ell} \varphi = \sigma^\ell \varphi^m \) and so \( \sigma^\ell \sim \varphi \) id.

The minimality of \( j \) yields \( \ell = 0 \) as above and hence \( k = mj \). Moreover, we have \( \varphi = \sigma^k \varphi = \sigma^{mj} \varphi = \varphi^m \) and thus the order \( n \) of the automorphism \( \pi \) divides the integer \( m \); in particular, we obtain \( nj \mid mj = k \) which contradicts the assumption \( k < nj \). Write \( \ell = nj \). If \( \Phi = \{ \sigma \} \), then the set \( \{ \text{id}, \sigma, \ldots, \sigma^{\ell - 1} \} \) is an upper bound on the set \( \tilde{V} \) from Lemma 5.13 because \( \sigma^\ell \sim \varphi \) id holds. By the minimal choice of \( \ell \), we obtain that \( \tilde{V} = \{ \text{id}, \sigma, \ldots, \sigma^{\ell - 1} \} \).

Suppose that both \( \sigma^\ell \sim \varphi \) id and \( \text{id} \sim \sigma^\ell \) hold. Then, as we already proved above, there exists an isomorphism \( \pi : \text{im}(\varphi) \to \text{im}(\sigma^\ell \varphi) \) with \( \sigma^\ell \varphi = \varphi \pi \). Since \( \text{im}(\sigma^\ell \varphi) \subseteq \text{im}(\varphi) \) and \( \pi \) is bijective, \( \pi \) is an automorphism of \( \text{im}(\varphi) \). Then automorphism \( \pi \) of the finite group \( \text{im}(\varphi) \), has finite order \( n \), say. Write \( k = n\ell \).

Then \( \sigma^k \varphi = \sigma^{n\ell} \varphi = \varphi^n = \varphi \) and so \( \sigma^k \sim \text{id} \). Suppose that \( \Phi = \{ \sigma \} \) and that the integer \( \ell > 0 \) above is minimal. Then, by our minimal choice of \( k \), we obtain \( V = \{ \text{id}, \sigma, \ldots, \sigma^{k - 1} \} \) for the set \( V \) returned by Algorithm \textbf{1}.

The permutation representation \( \varphi : F \to \text{Sym}(UK \setminus \text{F}) \) of the subgroup \( U = \langle a, b^2, bab^{-1} \rangle \) is induced by the map \( a \mapsto (\cdot) \) and \( b \mapsto (1, 2) \). Therefore, \( U \) satisfies that \( \varphi \sim \sigma^\ell \) id and \( |\text{im}(\varphi)| = 2 \) though \( |\text{im}(\sigma^\ell \varphi)| = 1 \). In particular, for each \( \ell \geq 1 \), we have \( |\text{im}(\sigma^\ell \varphi)| = 1 \) and thus there is no integer \( \ell \) so that \( \sigma^\ell \sim \text{id} \) holds. However, we have \( \sigma^2 \varphi = \sigma \varphi \) so that the set \( V = \{ \text{id}, \sigma, \sigma^2 \} \) returned by Algorithm \textbf{1} is finite.

The stabilizing core \( \mathcal{L} \) introduced in Definition 5.2 satisfies the following

**Proposition 5.15** Let \( V \subseteq \Phi^* \) be the finite set returned by Algorithm \textbf{2}. The stabilizing core \( \mathcal{L} \) satisfies that

\[
\mathcal{L} = \bigcap_{\sigma \in V} \ker(\sigma \varphi).
\]

Moreover, \( \mathcal{L} \) is the largest \( \Phi \)-invariant subgroup of \( UK \) which is normal in \( F \) and thus \( \mathcal{L} = \text{Core}_F(\mathcal{L}) \). It is finitely generated, it has finite index in \( F \), and it contains all iterated relations \( R \) of the \( L \)-presentation \( \langle X \mid Q \mid \Phi \mid R \rangle \) of \( G \).

We have \( \mathcal{L} \subseteq \mathcal{L} \subseteq UK \subseteq F \) and \( \mathcal{L} \subseteq \text{Core}_F(UK) \subseteq UK \subseteq F \).

**Proof.** Write \( \mathcal{K} = \bigcap_{\sigma \in V} \ker(\sigma \varphi) \). Clearly \( \mathcal{L} \subseteq \mathcal{K} \). Let \( g \in \mathcal{K} \) be given. Then, for all \( t \in T \), we have \( UK t \cdot g^\delta = UK t \) for each \( \sigma \in V \). Let \( \delta \in \Phi^* \) be given. By Lemma 5.3 there exists \( \sigma \in V \) with \( \delta \varphi = \sigma \varphi \). Thus \( UK t \cdot g^\delta = UK t \cdot g^\sigma = UK t \) for each \( t \in T \). Hence \( g^\delta \) stabilizes all right-cosets \( UK t \) and thus \( g \in \ker(\delta \varphi) \).

As \( \delta \in \Phi^* \) was arbitrarily chosen, we have \( \mathcal{L} = \mathcal{K} \).

The stabilizing core \( \mathcal{L} \) is normal in \( F \) because it is the intersection of normal subgroups. Since \( \mathcal{L} \subseteq \ker(\varphi) = \text{Core}_F(UK) \) holds, the stabilizing core \( \mathcal{L} \) is contained in \( UK \). As \( \mathcal{L} = \bigcap_{\sigma \in \Phi^*} \ker(\sigma \varphi) \) holds, the subgroup \( \mathcal{L} \) is \( \Phi \)-invariant.

Let \( N \) be a \( \Phi \)-invariant subgroup which is normal in \( F \) and which satisfies \( \mathcal{L} \leq N \leq UK \). Let \( g \in N \), \( t \in T \), and \( \sigma \in V \) be given. Since \( N \) is \( \Phi \)-invariant, we have \( g^\sigma \in N \). As \( N \subseteq F \) we also have \( tg^\sigma t^{-1} \in N \) or \( tg^\sigma = vt \) for some \( v \in N \subseteq UK \). Thus \( UK t \cdot g^\sigma = UK vt = UK t \) and so \( g \in \ker(\sigma \varphi) \). As \( \sigma \in V \) was arbitrarily chosen, we have \( g \in \mathcal{L} \). This yields that \( N \subseteq \mathcal{L} \) and hence \( N = \mathcal{L} \).
The stabilizing core \( L \) has finite index in \( F \) because it is the intersection of finitely many finite index subgroups \( \ker(\sigma \varphi) \) with \( \sigma \in V \). Moreover, \( L \) is finitely generated as a finite index subgroup of a finitely generated free group \( F \). Let \( r \in \mathcal{R} \) be an iterated relator of the \( L \)-presentation \( \langle X \mid Q \mid \Phi \mid \mathcal{R} \rangle \) of \( G \). Then, for each \( \sigma \in V \), the image \( r^\sigma \) is a relator of \( G \) as well and thus we have \( r \in \ker(\sigma \varphi) \) and so \( r \in \mathcal{L} \).

As \( L \) is \( \Phi \)-invariant, we have \( L \subseteq \tilde{L} \). Since \( L \) is normal in \( F \) and a subgroup of \( UK \), we have \( L \subseteq \text{Core}_F(UK) \).\( \square \)

Because the stabilizing core \( L \) contains the iterated relations \( \mathcal{R} \) of the \( L \)-presentation, the normal closure \( (\bigcap_{\sigma \in \Phi} \mathcal{R}^\sigma)^F \) is contained in \( L \) as well. This yields the immediate

**Corollary 5.16** If \( G = \langle X \mid Q \mid \Phi \mid \mathcal{R} \rangle = \langle X \mid \emptyset \mid \Phi \mid Q \cup \mathcal{R} \rangle \) is invariantly \( L \)-presented so that \( G = F/K \), we have \( K \subseteq L \subseteq \tilde{L} \subseteq UK \subseteq F \). Hence, the subgroup \( \mathcal{U} \cong UK/K \subseteq F/K = G \) contains the \( \Phi \)-invariant normal subgroup \( L/K \). The index \( [UK/K : L/K] = [UK : L] \) is finite.

Whence the subgroup \( \mathcal{U} \) in Corollary 6.16 is a finite extension of \( L/K \). Since \( L \) is the largest \( \Phi \)-invariant subgroup which is normal in \( F \), the stabilizing subgroup \( \tilde{L} \) is normal in \( F \) if and only if \( L = \tilde{L} \) holds. Moreover, we have the following

**Lemma 5.17** We have \( \tilde{L} = L \) if and only if \( \tilde{L} \subseteq \text{Core}_F(UK) \) holds.

**Proof.** We have \( L \subseteq \tilde{L} \) and \( \tilde{L} \subseteq \text{Core}_F(UK) \). On the other hand, suppose that \( \tilde{L} \subseteq \text{Core}_F(UK) \) holds. Let \( g \in \tilde{L} \) and \( \sigma \in V \) be given. Then \( g^\sigma \in \tilde{L} \), as \( \tilde{L} \) is \( \sigma \)-invariant. Since \( g^\sigma \in \tilde{L} \subseteq \text{Core}_F(UK) \) holds, we have \( tg^\sigma t^{-1} \in \text{Core}_F(UK) \subseteq UK \), for each \( t \in T \). This yields that \( UK t \cdot g^\sigma = UK t \) and thus \( g^\sigma \) acts trivially on the right-cosets \( UK \backslash F \). In particular, we have \( g^\sigma \in \ker(\varphi) \) and \( g \in \ker(\sigma \varphi) \). As \( \sigma \in V \) was arbitrarily chosen, we have \( g \in L = \bigcap_{\sigma \in V} \ker(\sigma \varphi) \).\( \square \)

If \( UK \subseteq F \) is a normal subgroup, then \( \tilde{L} \subseteq UK = \text{Core}_F(UK) \) holds and hence, we obtain the immediate

**Corollary 5.18** If \( UK \subseteq F \), then \( L = \tilde{L} \).

Note the following

**Remark 5.19** There are subgroups that satisfy \( \text{Core}_F(UK) \subseteq \tilde{L} \). For instance, the subgroup \( \mathcal{U} = \langle a, b^2, bab^{-1}, bab^{-2}a^{-1}b^{-1} \rangle \) of the Basilica group is \( \Phi \)-invariant (and hence \( \tilde{L} = UK \)) by Lemma 5.12 but not normal in \( G \).

There are subgroups that satisfy \( \tilde{L} \subseteq \text{Core}_F(UK) \). For instance, the subgroup \( \mathcal{U} = \langle a^2, b, aba^{-1} \rangle \) of the Basilica group has index 2 in \( G \) (and thus it is normal in \( G \)); though the subgroup \( \mathcal{U} \) is not \( \sigma \)-invariant.

There are subgroups that neither satisfy \( \tilde{L} \subseteq \text{Core}_F(UK) \) nor \( \text{Core}_F(UK) \subseteq \tilde{L} \). For instance, the subgroup \( \mathcal{U} = \langle a, bab^{-1}, b^{-1}a^2b, b^3ab^2, b^5a^{-1}b \rangle \) of the Basilica group satisfies \( [F : \tilde{L}] = [F : \text{Core}_F(UK)] \) and \( \tilde{L} \neq \text{Core}_F(UK) \).
6 The Reidemeister-Schreier theorem

In this section, we finally prove our variant of the Reidemeister-Schreier theorem in Theorem 1. For this purpose, let $G = \langle \mathcal{X} \mid Q \mid \Phi \mid R \rangle$ be a finitely $L$-presented group and let $U \leq G$ be a finite index subgroup given by its generators $g_1, \ldots, g_n$, say. We consider the generators $g_1, \ldots, g_n$ as elements of the free group $F$ over $\mathcal{X}$. Denote the normal closure of the relations of $G$ by $K = \langle Q \cup \bigcup_{\sigma \in \Phi} R^\sigma \rangle^F$ and let $U = \langle g_1, \ldots, g_n \rangle \leq F$. Then $U \cong UK/K$. If $T \subseteq F$ denotes a Schreier transversal for $UK$ in $F$, the Reidemeister-Schreier Theorem in Section 3 shows that the subgroup $U$ admits the group presentation

$$ U \cong \langle \mathcal{Y} \mid \{ \tau(tqt^{-1}) \mid t \in T, q \in Q \} \cup \bigcup_{\sigma \in \Phi^*} \{ \tau(t^\sigma t^{-1}) \mid t \in T, r \in R \} \rangle, $$

where $\tau$ is the Reidemeister rewriting. We will construct a finite $L$-presentation from the group presentation in Eq. (7). First, we note the following

**Theorem 6.1** Let $G = \langle \mathcal{X} \mid Q \mid \Phi \mid R \rangle$ be invariantly finitely $L$-presented. Each $\Phi$-invariant normal subgroup with finite index in $G$ is invariantly $L$-presented.

**Proof.** Let $G = \langle \mathcal{X} \mid Q \mid \Phi \mid R \rangle$ be an invariantly finitely $L$-presented group and let $U \leq G$ be a $\Phi$-invariant normal subgroup with finite index in $G$. Every invariantly $L$-presented group can be considered as an ascendingly $L$-presented group by Proposition 222. Therefore, we may consider $Q = \emptyset$ in the following. Consider the notation introduced above. As $G$ is invariantly $L$-presented, we have $K^\sigma \subseteq K$ for each $\sigma \in \Phi^*$. Since the subgroup $U$ is $\Phi$-invariant, we also have $U^\sigma \subseteq U$ and therefore $(UK)^\sigma \subseteq UK$ for each $\sigma \in \Phi^*$. By Lemma 5,12 we have $\hat{\mathcal{L}} = UK$. Furthermore, as $UK \subseteq F$ holds, we have $\hat{\mathcal{L}} = \mathcal{L}$ and thus $UK = \hat{\mathcal{L}} = \hat{\mathcal{L}}$.

Let $t \in T$ be given. As $U \leq G$ holds, the mapping $\delta_t : UK \to UK, g \mapsto tqt^{-1}$ defines an automorphism of $UK$. The Reidemeister rewriting $\tau : UK \to F(\mathcal{Y})$ is an isomorphism of free groups and therefore the endomorphisms $\hat{\Phi} \cup \{ \delta_t \mid t \in T \}$ of $UK$ translate to endomorphisms $\tilde{\Phi} \cup \{ \tilde{\delta}_t \mid t \in T \}$ of the free group $F(\mathcal{Y})$.

Consider the invariant finite $L$-presentation

$$ \langle \mathcal{Y} \mid \emptyset \mid \tilde{\Phi} \cup \{ \tilde{\delta}_t \mid t \in T \} \mid \{ \tau(r) \mid r \in R \} \rangle. \tag{8} $$

In order to prove that the finite $L$-presentation in Eq. (8) defines the subgroup $U$, it suffices to prove that each relation of the presentation in Eq. (7) is a consequence of the relations of the $L$-presentation in Eq. (8) and vice versa. For $t \in T$, $r \in R$, and $\sigma \in \Phi^*$, we consider the relation $\tau(t^\sigma t^{-1})$ of the group presentation in Eq. (7). Clearly, this relation is contained in the finite $L$-presentation in Eq. (8) as there exists $\tilde{\sigma} \in \tilde{\Phi}^*$ so that $\tau(r)^{\tilde{\sigma}} = \tau(r^\sigma)$. Then $(\tau(r))^{\tilde{\delta}_t} = \tau(t^\sigma t^{-1})$. On the other hand, consider the relation $\tau(r)^{\delta}$ of the finite $L$-presentation in Eq. (8) where $r \in R$ and $\tilde{\sigma} \in \tilde{\Phi} \cup \{ \tilde{\delta}_t \mid t \in T \}^\ast$.

Write $\Psi = \tilde{\Phi} \cup \{ \tilde{\delta}_t \mid t \in T \}$. Since $1 \in T$ and $id \in \Phi^*$, we can write each image of an element $\tilde{\delta} \in \Psi$ as $\tau(g)^{\tilde{\delta}} = \tau(tg^{-1} t^{-1})$ for some $t \in T$ and $\delta \in \Phi^*$. Since $\tilde{\sigma} \in \Psi^*$, we can write $\tilde{\sigma} = \tilde{\sigma}_1 \cdots \tilde{\sigma}_n$ with each $\tilde{\sigma}_i \in \Psi$. Then the image $\tau(r)^{\tilde{\sigma}}$ has the form

$$ \tau(r)^{\tilde{\sigma}} = \tau(t_1 \cdots t^n), $$

where $t_1 \cdots t^n$ is a word in the generators $t_1, \ldots, t_n$ of the free group over $\mathcal{X}$. This completes the proof of Theorem 6.1.
Since \( T \) is a transversal for \( UK \) in \( F \), we can write \( t_n \cdots t_1^a b^2 a^{-1} b^{-1} b a^{-1} b^{-2} a b^{-1} = u t \) where \( t \in T \) and \( u \in UK \). This yields that \( \tau(\sigma)^{\hat{\sigma}} = \tau(ut r^a s t^{-1} u^{-1}) = \tau(u) t r^a s t^{-1} = \tau(u) t r^a s t^{-1}) \tau(u)^{-1} \), which is a consequence of \( \tau(t r^a s t^{-1} u^{-1}) \). The latter relation \( \tau(t r^a s t^{-1}) \) is a relation of the group presentation in Eq. (7).

In summary, each relation of the group presentation in Eq. (7) is a consequence of the finite \( L \)-presentation in Eq. (5) and vice versa. \( \square \)

In order to prove our Reidemeister-Schreier theorem for finitely \( L \)-presented groups, we need to consider finite index subgroups that are not normal. For this purpose, we need to construct the relations \( \tau(tr a t^{-1}) \), with \( t \in T \), \( r \in \mathcal{R} \), and \( \sigma \in \Phi \). The overall strategy in this paper is to construct the relations as iterated images of the form \( \tau(s r^{-1})^{\hat{\sigma}} \) for \( s \in T \) and some \( \hat{\sigma} \in \Phi^* \). If the subgroup \( U \) is normal as in Proposition 6.3, the conjugation action \( \delta_i : UK \to UK \) enables us to first construct the image \( \tau(r^\sigma) = \tau(r)^{\hat{\sigma}} \) and then to consider the conjugates \( \tau(r^\sigma)^{\hat{\sigma}} = \tau(tr^\sigma t^{-1}) \). However, in general, it is not sufficient to take as iterated relations those \( \tau(tr t^{-1})^\sigma = \tau(t^r r^\sigma t^{-\sigma}) \), with \( t \in T \) and \( r \in \mathcal{R} \), as \( \sigma \) may not be invertible over \( \{t^\sigma \mid t \in T\} \). More precisely, we have the following.

**Remark 6.2** Let \( \mathcal{U} = \{a, b^2, ba b^{-1}, bab^{-2} a^{-1} b^{-1}, ba^{-1} b^{-2} a b^{-1}\} \) be a subgroup of the Basilica group \( G \). The subgroup \( \mathcal{U} \) is \( \sigma \)-invariant and thus we can consider the iterated images \( \{\tau(r^\sigma)^{\hat{\sigma}} \mid r \in \mathcal{R}, \sigma \in \Phi^*\} \). A Schreier transversal \( T \) for \( \mathcal{U} \) in \( G \) is given by \( T = \{1, b, ba, ba^2, bab, ba^2 b\} \). We have \( T^\sigma = \{1, a, ab^2, ba^2, ab^2 a, ab^2 b\} \). Note that \( T^\sigma \subseteq UK \). Thus we cannot ensure that the iterated images \( \{\tau(tr t^{-1})^\sigma \mid r \in \mathcal{R}, t \in T, \sigma \in \Phi^*\} \) contain all relations in Eq. (7). As the subgroup \( \mathcal{U} \) is not normal in \( G \), we cannot consider the conjugate action as well. However, an invariant finite \( L \)-presentation for the subgroup \( \mathcal{U} \) can be computed with Theorem 7.1 as the subgroup \( \mathcal{U} \) is leaf-invariant (see Section 7 below).

In the following, we use Theorem 6.1 to prove our variant of the Reidemeister-Schreier Theorem for invariantly finitely \( L \)-presented groups first.

**Proposition 6.3** Every finite index subgroup of an invariantly finitely \( L \)-presented group is finitely \( L \)-presented.

**Proof.** Let \( \mathcal{U} \) be a finite index subgroup of an invariantly finitely \( L \)-presented group \( G = F/K \). By Corollary 5.10 the subgroup \( \mathcal{U} \cong UK/K \) contains a normal subgroup \( L/K \) with finite index in \( G \) and which is \( \Phi \)-invariant. By Theorem 6.1 the subgroup \( L/K \leq F/K \) is finitely \( L \)-presented. The subgroup \( \mathcal{U} \) is a finite extension of a finitely \( L \)-presented group and thus, by Corollary 2.24, the subgroup \( \mathcal{U} \) is finitely \( L \)-presented itself. \( \square \)

Recall that we do not have a method to construct an invariant \( L \)-presentation for a finite extension of an invariantly \( L \)-presented group. Therefore, we cannot ensure invariance of the finite \( L \)-presentation obtained from Corollary 5.10. We will study in Section 7 conditions on a subgroup of an invariantly \( L \)-presented group that ensure the invariance of the subgroup \( L \)-presentation. First, we complete our proof of Theorem 1.1.
7 Invariant subgroup \( L \)-presentations

The algorithms in \([2, 15]\) are much more efficient on invariant \( L \)-presentations. Therefore, we will study conditions on the subgroup \( \mathcal{U} \) of an invariantly \( L \)-presented group \( G \) to be invariantly \( L \)-presented itself. By Theorem 6.1, each \( \Phi \)-invariant normal subgroup \( \mathcal{U} \) of an invariantly \( L \)-presented group \( G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle \) is invariantly \( L \)-presented as soon as \( |G : \mathcal{U}| \) is finite.

Let \( \varphi : F \to \text{Sym}(UK\setminus F) \) be a permutation representation as usual. Recall that the subgroup \( \mathcal{U} \) is leaf-invariant, if the \( \varphi \)-leafs

\[
\Psi = \{ \psi \delta \mid \psi \in \Phi, \delta \in V, \psi \delta \not\in V, \psi \delta \varphi = \varphi \},
\]

of \( V \) satisfy \( \Psi = \{ \psi \delta \mid \psi \in \Phi, \delta \in V, \psi \delta \not\in V \} \); cf. Definition 6.8. This definition yields the following

**Theorem 7.1** Each leaf-invariant, finite index subgroup of an invariantly finitely \( L \)-presented group is invariantly finitely \( L \)-presented.

**Proof.** Let \( G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle \) be invariantly finitely \( L \)-presented and let \( \mathcal{U} \leq G \) be a leaf-invariant finite index subgroup of \( G \). Clearly, we can consider \( \mathcal{Q} = \emptyset \) in the following. The \( \varphi \)-leafs \( \Psi \) satisfy \( \Psi = \{ \psi \delta \mid \psi \in \Phi, \delta \in V, \psi \delta \not\in V \} \).

By Lemma 5.9 each \( \varphi \)-leaf \( \psi \delta \in \Psi \subseteq \Phi^* \) defines an endomorphism of the subgroup \( UK \). Moreover, Lemma 5.9 shows that each \( \sigma \in \Phi^* \) can be written as \( \sigma = \vartheta \delta \) with \( \vartheta \in V \) and \( \delta \in \Psi^* \). Consider the finite \( L \)-presentation

\[
\langle \mathcal{Y} \mid \emptyset \rangle \{ \psi \delta \mid \psi \delta \in \Psi \} \{ \tau(\vartheta) t^{-1} \mid \vartheta \in V, r \in \mathcal{R}, t \in T \},
\]

where \( \mathcal{Y} \) denotes the Schreier generators of \( UK \) and \( \widehat{\psi} \sigma \) denotes the endomorphism of the free group \( F(\mathcal{Y}) \) induced by the endomorphisms \( \psi \sigma \) of \( UK \). For \( t \in T, \sigma \in \Phi^* \), and \( r \in \mathcal{R} \), the relation \( \tau(t \sigma t^{-1} \tau^{-1}) \) of the group presentation in Eq. (7) can be obtain from the above \( L \)-presentation as follows: Since each \( \sigma \in \Phi^* \) can be written as \( \sigma = \vartheta \delta \) with \( \vartheta \in V \) and \( \delta \in \Psi^* \), we claim that the relation \( \tau(t \sigma t^{-1}) \) is a consequence of the image \( \tau(t \vartheta t^{-1}) \delta \). The latter image
satisfies that \( \tau(t^\delta r^\delta t^{-1}) = \tau(t^\delta r^\delta t^{-\delta} t^\delta) = \tau(t^\delta r^\delta t^{-\delta}). \) As \( \delta \in \Psi^*, \) we can write
\( \delta = \delta_1 \cdots \delta_n \) with each \( \delta_i \in \Psi. \) Recall that \( \delta_i \varphi = \varphi \) holds. Thus the right-coset
\( UK \) satisfies that \( UK 1 \cdot t^\delta \varphi = UK 1 \cdot t = UK t \) and therefore \( UK t^\delta \varphi = UK t. \) Hence, there exists \( u \in UK \) so that \( t^\delta = ut \) and we obtain
\[ \tau(t^\delta r^\delta t^{-1}) = \tau(t^\delta r^\delta t^{-\delta}) = \tau(ut r^\delta t^{-1} u^{-1}) = \tau(u t r^\delta t^{-1}) \tau(u)^{-1} \]
which is a consequence of \( \tau(t r^\delta t^{-1}) \) and vice versa. Similarly, every relation
of the \( L \)-presentation in Eq. \( \text{(9)} \) is a consequence of the relations in Eq. \( \text{(7)} \.
Therefore, the invariant finite \( L \)-presentation in Eq. \( \text{(9)} \) defines the leaf-invariant
finite index subgroup \( \mathcal{U}. \)

For finite \( L \)-presentations \( \langle \mathcal{X} | \mathcal{Q} | \Phi | \mathcal{R} \rangle \) with \( \Phi = \{\sigma\}, \) the leaf-invariance of
the subgroup \( \mathcal{U} \) yields the existence of a positive integer \( j \) so that \( \sigma^j \varphi = \varphi \) holds.
If we assume the positive integer \( j \) to be minimal, then \( V = \{\text{id}, \sigma, \ldots, \sigma^{j-1}\} \)
and \( \Psi = \{\sigma^j\}. \) In this case, the invariant finite \( L \)-presentation in Eq. \( \text{(9)} \)
becomes
\[ \mathcal{U} \cong \langle \mathcal{V} | \emptyset | \{\sigma^j\} | \{ \tau(tr^{\sigma^j} t^{-1}) | t \in \mathcal{T}, r \in \mathcal{R}, 0 \leq i < j \} \rangle. \]
Note that the subgroup \( \mathcal{U} \) in Theorem \( \text{7.1} \) is not necessarily normal in \( G. \) However,
leaf-invariance of a subgroup is a restrictive condition on the subgroup.
We try to weaken this condition with the following

**Definition 7.2** Let \( G = \langle \mathcal{X} | \mathcal{Q} | \Phi | \mathcal{R} \rangle \) be a finitely \( L \)-presented group and
let \( \mathcal{U} \leq G \) be a finite index subgroup with permutation representation \( \varphi. \) Then
the subgroup \( \mathcal{U} \) is weakly leaf-invariant, if
\[ \Psi = \{\psi \delta | \psi \in \Phi, \delta \in V, \psi \delta \notin V, \psi \delta \sim_{\varphi} \text{id} \} \]
satisfies \( \Psi = \{\psi \delta | \psi \in \Phi, \delta \in V, \psi \delta \notin V \}. \)

The notion of a weakly leaf-invariant subgroup is less restrictive than leaf-invariance,
as low-index subgroups of the Basilica groups suggest: Among the
4956 low-index subgroups of the Basilica group with index at most 20 there
are 2539 weakly leaf-invariant subgroups; only 156 of these subgroups are leaf-invariant.
More precisely, Table \( \text{I} \) shows the number of subgroups (\( \leq \)) that are
normal (\( \leq \)), maximal (\( \text{max} \)), leaf-invariant (\( \text{l.i.} \)), weakly leaf-invariant (\( \text{w.l.i.} \)),
and the number of subgroups that are weakly leaf-invariant and normal (\( \leq + \))
and \( \text{w.l.i.} \). For finite \( L \)-presentations \( \langle \mathcal{X} | \mathcal{Q} | \Phi | \mathcal{R} \rangle \) with \( \Phi = \{\sigma\}, \) each leaf-invariant
subgroup is weakly leaf-invariant by Lemma \( \text{5.14} \) (iii). On the other hand, a weakly leaf-invariant subgroup with \( \Phi = \{\sigma\} \) such that both \( \sigma^\ell \sim_{\varphi} \text{id} \)
and \( \text{id} \sim_{\varphi} \sigma^\ell \) hold, is leaf-invariant by Lemma \( \text{5.14} \) (iv). There are subgroups of
a finitely \( L \)-presented group that are weakly leaf-invariant but not leaf-invariant;
see Lemma \( \text{5.14} \) (v). If \( \Phi \) contains more than one generator, then we may ask
the following

**Question 2** Is every leaf-invariant subgroup weakly leaf-invariant?
Table 1: Subgroups of the Basilica group with index at most 20.

| index | ≤ | ≤ | max l.i. | w.l.i | ≤ + w.l.i |
|-------|---|---|--------|------|---------|
| 1     | 1 | 1 | 1      | 1    | 1       |
| 2     | 3 | 3 | 3      | 0    | 3       |
| 3     | 7 | 4 | 7      | 4    | 4       |
| 4     | 19| 7 | 0      | 19   | 7       |
| 5     | 11| 6 | 11     | 6    | 6       |
| 6     | 39| 13| 0      | 14   | 12      |
| 7     | 15| 8 | 15     | 8    | 8       |
| 8     | 163| 19| 0     | 139  | 19      |
| 9     | 115| 13| 9     | 49   | 13      |
| 10    | 83 | 19| 0      | 22   | 18      |
| 11    | 23 | 12| 23     | 12   | 12      |
| 12    | 355| 31| 0      | 98   | 28      |
| 13    | 27 | 14| 27     | 14   | 14      |
| 14    | 115| 25| 0      | 30   | 24      |
| 15    | 77 | 24| 0      | 24   | 24      |
| 16    | 1843| 47| 0     | 1531 | 43      |
| 17    | 35 | 18| 35     | 18   | 18      |
| 18    | 1047| 44| 0     | 366  | 40      |
| 19    | 39 | 20| 39     | 20   | 20      |
| 20    | 939| 45| 0      | 158  | 42      |

The problem is that Definitions 5.8 and 7.2 depend on the minimal sets \( V \) and \( \tilde{V} \) which satisfy \( \tilde{V} \subseteq V \) but which may differ in general. We do not have an answer to this question.

Moreover the sets \( V \) and \( \tilde{V} \) in the Definitions 5.8 and 7.2 may also depend on the ordering \( \prec \) chosen in our Algorithm 1. Though we have the following

**Lemma 7.3** The conditions leaf-invariance and weak leaf-invariance do not depend on the choice of the ordering \( \prec \) in Algorithm 1.

**Proof.** We show the claim for the weaker condition of weak leaf-invariance and we show this by proving that the set \( \tilde{V} \) in Lemma 5.13 does not depend on the ordering. Suppose that a subgroup \( \mathcal{U} \) is weakly leaf-invariant with respect to the ordering \( \prec \). Let \( V_\prec \) and \( \tilde{V}_\prec \) be the sets with respect to the orderings \( \prec \) and \( \prec_\prec \), respectively. We first show that \( V_\prec \subseteq V_\prec \) holds. Let \( \sigma \in V_\prec \) be a \( \prec \)-minimal counter-example with \( \sigma \notin V_\prec \). As \( \text{id} \in V_\prec \), we have \( \sigma \neq \text{id} \) and therefore we can write \( \sigma = \psi \delta \) with \( \psi \in \Phi \) and \( \delta \in \Phi^* \). Now, \( V_\prec \) can be considered as a subtree and hence, we have \( \delta \in V_\prec \) and \( \delta \prec \sigma \). By the minimality of \( \sigma \), we have \( \delta \in V_\prec \).

Thus the element \( \sigma = \psi \delta \) satisfies \( \psi \in \Phi \), \( \delta \in V_\prec \), and \( \psi \delta \notin V_\prec \). Since the subgroup \( \mathcal{U} \) is weakly leaf-invariant with respect to \( \prec \), we have \( \psi \delta \sim_\prec \text{id} \) which contradicts the assumption that \( \sigma = \psi \delta \in V_\prec \). On the other hand, let \( \sigma \in V_\prec \) be \( \prec \)-minimal so that \( \sigma \notin V_\prec \). As \( \text{id} \in V_\prec \), we have \( \sigma \neq \text{id} \) and hence, we can...
write \( \sigma = \psi \delta \) with \( \psi \in \Phi \) and \( \delta \in \Phi^* \). Since \( V_\prec \) is a subtree of \( \Phi^* \), we also have \( \delta \in V_\succ \) and \( \delta \prec \sigma \). The minimality of \( \sigma \) yields that \( \delta \in V_\succ \). Since \( \psi \delta \notin V_\prec \), there exists \( \gamma \in V_\prec \) so that \( \sigma = \psi \delta \sim_\varphi \gamma \). Note that \( \gamma \in V_\prec \subseteq V_\succeq \) which contradicts that \( \sigma = \psi \delta \in V_\prec \) as there would exists \( \gamma \in V_\prec \) so that \( \sigma \sim_\varphi \gamma \) which is impossible. \( \square \)

It can be shown that the subgroup \( V = \langle x_1, x_2, x_3, x_4 x_1 x_4^{-1}, x_3^2 \rangle \) of the subgroup \( U \) in Section 4 is weakly leaf-invariant but it is not leaf-invariant. The notion of a weakly leaf-invariant subgroup yields the following

**Lemma 7.4** A normal subgroup \( UK \trianglelefteq F \) is \( \sigma \)-invariant if and only if \( \sigma \sim_\varphi \text{id} \).

**Proof.** Suppose that \( \sigma \sim_\varphi \text{id} \) holds. Then there exists a homomorphism \( \pi: \text{im}(\varphi) \to \text{im}(\sigma \varphi) \) so that \( \sigma \varphi = \varphi \pi \). Let \( g \in UK = \text{Core}_F(UK) = \ker(\varphi) \) be given. Then \( 1 = (g^2)\pi = g^2 \varphi \pi = (g^2)^\varphi \) and so \( g^2 \in \ker(\varphi) \subseteq UK \). In particular, the subgroup \( UK \) is \( \sigma \)-invariant. On the other hand, suppose that the normal subgroup \( UK \trianglelefteq F \) is \( \sigma \)-invariant. For \( g \in F \), we define the map \( \delta_g: UK \setminus F \to UK \setminus F, UK t \mapsto UK t \cdot g \). Note that, for \( g, h \in F \), we have that \( \delta_g \delta_h: UK \setminus F \to UK \setminus F, UK t \mapsto UK t \cdot gh \) and so \( \delta_g \delta_h = \delta_{gh} \). Then \( \delta_g \in \text{im}(\varphi) \).

We define a map \( \pi: \text{im}(\varphi) \to \text{Sym}(UK \setminus F), \delta_g \mapsto \delta_g \varphi \). Let \( g, h \in F \) be given. Then \( (\delta_g \delta_h)^\pi = (\delta_{gh})^\pi = (\delta_{gh}) = \delta_{gh} \varphi = \delta_g \varphi \delta_h \varphi = (\delta_g)^\pi (\delta_h)^\pi \). Suppose that, for \( g \in F \), the map \( \delta_g \) acts trivially on \( UK \setminus F \). Then, for each \( t \in T \), we have \( UK t \cdot g = UK t \) or \( tgt^{-1} \in UK \). Since \( UK \trianglelefteq F \), the latter yields that \( g \in UK \) and, as \( UK \) is \( \sigma \)-invariant, we also have that \( g^2 \in UK \). Thus \( tg^2t^{-1} \in UK \). Consider the image \( (\delta_g)^\pi = \delta_{g^2} \). Then, as \( tg^2t^{-1} \in UK \), the map \( \delta_g \) fixes \( UK t \).

Because \( t \in T \) was arbitrarily chosen, we have \( \delta_{g^2} = 1 \in \text{Sym}(UK \setminus F) \). Thus the map \( \pi \) defines a homomorphism that satisfies \( \sigma \varphi = \varphi \pi \). Thus \( \sigma \sim_\varphi \text{id} \). \( \square \)

Lemma 7.4 yields that a \( \Phi \)-invariant normal subgroup is weakly leaf-invariant. However, there exist subgroups which are weakly leaf-invariant but not \( \Phi \)-invariant (e.g. the subgroup \( U = \langle a, bab^{-1}, b^3 \rangle \) of the Basilica group in Section 4) satisfies \( \sigma^2 \sim_\varphi \text{id} \) but not \( \sigma \sim_\varphi \text{id} \); thus, it is weakly leaf-invariant but not \( \Phi \)-invariant). The condition \( UK \trianglelefteq F \) in Lemma 7.4 is necessary, as we have the following

**Remark 7.5** The condition \( UK \trianglelefteq F \) in Lemma 7.4 is necessary, as the subgroup \( U = \langle a, b^2, ba^3b^{-1}, bab^{-2}a^{-1}b^{-1}, ba^{-1}b^{-2}ab^{-1} \rangle \) of the Basilica group \( G \) is not normal in \( G \), it satisfies \( \langle UK \rangle^\sigma \subseteq UK \) but it does not satisfy \( \sigma \sim_\varphi \text{id} \).

On the other hand, the subgroup \( U = \langle a, bab, ba^{-1}b, b^3 \rangle \) of the Basilica group \( G \) satisfies \( \sigma \sim_\varphi \text{id} \) but it does not satisfy \( \langle UK \rangle^\sigma \subseteq UK \) as \( [F : \text{Core}_F(UK)] = [F : \mathcal{L}] = 8 \neq 4 = [F : UK] \).

A weakly leaf-invariant subgroup allows the following variant of our Reidemeister-Schreier theorem:

**Theorem 7.6** A weakly leaf-invariant normal subgroup which has finite index in an invariantly finitely \( L \)-presented group is invariantly finitely \( L \)-presented.

**Proof.** Let \( G = \langle \mathcal{X} \mid Q \mid \Phi \mid \mathcal{R} \rangle \) be invariantly finitely \( L \)-presented and let \( U \cong UK/K \) be a finite index normal subgroup of \( G \). As usual, we may
consider \( Q = \emptyset \) as \( G \) is invariably \( L \)-presented. Let \( \hat{V} \) be the subset \( \hat{V} \subseteq V \) given by Lemma \ref{lem:leaf-invariance}. Since \( \mathcal{U} \) is weakly leaf-invariant, the weak-leaves \( \Psi \) in Definition \ref{def:weak_leaf} satisfy \( \Psi = \{ \psi \delta \mid \psi \in \Phi, \delta \in \hat{V}, \psi \delta \notin \hat{V} \} \). By Lemma \ref{lem:finite_index}, each \( \psi \delta \in \Psi \) induces an endomorphism of the normal subgroup \( UK \leq F \). Let \( T \) be a Schreier transversal for \( UK \) in \( F \) and let \( \mathcal{Y} \) denote the Schreier generators of the subgroup \( UK \). Then each endomorphism \( \psi \delta \in \Psi \) of \( UK \) translates to an endomorphism \( \hat{\psi} \delta \) of the free group \( F(\mathcal{Y}) \). Consider the invariant and finite \( L \)-presentation

\[
\langle \mathcal{Y} \mid \emptyset \mid \{ \psi \delta \mid \psi \delta \in \Psi \} \cup \{ \hat{\delta}_t \mid t \in T \} \mid \{ \tau(r^\sigma) \mid r \in \mathcal{R}, \sigma \in \hat{V} \} \rangle, \tag{10}
\]

where \( \hat{\delta}_t \) denotes the endomorphism of \( UK \) which is induced by conjugation by \( t \in T \). The finite \( L \)-presentation in Eq. \ref{eq:finite_pres} defines the normal subgroup \( \mathcal{U} \). This assertion follows with the same techniques as above; in particular, it follows from rewriting the presentation in Eq. \ref{eq:finite_pres}.

The subgroup in Section \ref{sec:finite_pres} is a normal subgroup satisfying \( \sigma^2 \sim_{\varphi} \text{id} \) and hence, Theorem \ref{thm:finite_pres} shows that this subgroup is invariantly finitely \( L \)-presented.

Even non-invariant \( L \)-presentations may give rise to invariant subgroup \( L \)-presentations as the following remark shows:

\begin{remark} \label{rem:non-invariant_pres}
There are non-invariant \( L \)-presentation \( G = \langle X \mid Q \mid \Phi \mid \mathcal{R} \rangle \) and finite index subgroups \( \mathcal{U} \leq G \) that satisfy \( UK^\sigma \leq UK \) for each \( \sigma \in \Phi^* \). For instance, the finite \( L \)-presentation of Baumslag’s group \( G \) in \cite{Bass98} is non-invariant (cf. Proposition \ref{prop:Bass_pres}) while its index-3 subgroup \( \mathcal{U} = \langle a^3, b, t \rangle \) satisfies \( (UK)^\sigma \leq UK \) for each \( \sigma \in \Phi \). The subgroup \( \mathcal{U} \) even admits an invariant \( L \)-presentation over the generators \( x = a^3 \) and \( y = a^2ta^{-2} \) given by

\[
\langle \{ x, y \} \mid \emptyset \mid \{ \delta_1, \delta_2 \} \mid \{ y^{-1}xyx^{-4} \} \rangle
\]

where \( \delta_1 \) is induced by the mapping \( x \mapsto x \) and \( y \mapsto xyx^{-3} \) and \( \delta_2 \) is induced by the mapping \( x \mapsto x \) and \( y \mapsto xyx^{-2} \).
\end{remark}

The finite \( L \)-presentations for a finite index subgroup constructed in Proposition \ref{prop:finite_index_pres} and Theorem \ref{thm:finite_pres} are derived from the group’s \( L \)-presentation \( \langle X \mid Q \mid \Phi \mid \mathcal{R} \rangle \) by restricting to those endomorphisms in \( \Phi^* \) which restrict to the subgroup. However, there are subgroups of an invariantly \( L \)-presented group that do not admit endomorphisms in \( \Phi^* \) which restrict to the subgroup. In this case the finite \( L \)-presentation for the finite index subgroup needs to be constructed as a finite extension of the finitely \( L \)-presented stabilizing core \( \mathcal{L} \) as in the proof of Theorem \ref{thm:finite_pres}. The following remark gives an example of a subgroup of the invariantly finitely \( L \)-presented Basilica group which does not admit endomorphisms in \( \Phi^* \) that also restrict to the subgroup:

\begin{remark} \label{rem:Basilica}
Let \( \mathcal{U} = \langle b^2, a^3, ab^2a^{-1}, a^{-1}b^2a, bab^{-1}a \rangle \) denote a subgroup of the Basilica group \( G \). Then \( \mathcal{U} \) is a normal subgroup with index 6 in \( G \). We are not able to find an invariant finite \( L \)-presentation for \( \mathcal{U} \).
\end{remark}
The subgroup $\mathcal{U}$ admits the permutation representations $\varphi : F \rightarrow \text{Sym}(\mathcal{U}K\backslash F)$ and the $\sigma$-iterates

$$\varphi : \begin{cases} a & \mapsto (1,2,3)(4,6,5) \\ b & \mapsto (1,4)(2,5)(3,6) \end{cases}$$

and

$$\sigma \varphi : \begin{cases} a & \mapsto (1,2,3)(4,6,5) \\ b & \mapsto (1,2,3)(4,6,5) \end{cases}$$

as well as

$$\sigma^2 \varphi : \begin{cases} a & \mapsto (1,3,2)(4,5,6) \\ b & \mapsto (1,3,2)(4,5,6). \end{cases}$$

Clearly, we have $\sigma^3 \sim_\varphi \sigma$ but for each $0 < \ell < 3$ we do not have $\sigma^\ell \sim_\varphi \text{id}$. Note that the homomorphism $\pi : \text{im}(\sigma \varphi) \rightarrow \text{im}(\sigma^3 \varphi)$ with $\sigma^3 \varphi = \sigma \varphi \pi$ is bijective.

Suppose there existed $\sigma^n \in \Phi^\ast$ so that the subgroup $\mathcal{U}K$ is $\sigma^n$-invariant. By Lemma [7,4] the normal subgroup $\mathcal{U}K$ is $\sigma^n$-invariant if and only if $\sigma^n \sim_\varphi \text{id}$ holds. Clearly $n > 3$. Since $\sigma^n \sim_\varphi \text{id}$ holds, there exists a homomorphism $\psi : \text{im}(\varphi) \rightarrow \text{im}(\sigma^n \varphi)$ so that $\sigma^n \varphi = \varphi \psi$. We obtain $\varphi \psi = \sigma^n \varphi = \sigma^n \sigma^3 \varphi = \sigma^n - 3 \sigma \varphi \pi = \sigma^n - 2 \sigma^3 \varphi$. Iterating this rewriting process eventually yields a positive integer $0 < \ell < 3$ so that $\varphi \psi = \sigma^n \varphi = \sigma^\ell \varphi \pi^\ell$. As $\pi$ is bijective, this yields that $\sigma^\ell \varphi = \varphi (\pi^{-1})^\ell$ and hence $\sigma^\ell \sim_\varphi \text{id}$ which is a contradiction. Thus there is no positive integer $n \in \mathbb{N}$ so that $\sigma^n \sim_\varphi \text{id}$ and hence, no substitution in $\Phi^\ast$ restricts to the subgroup $\mathcal{U}K$.

Our method to compute a finite $L$-presentation for the subgroup $\mathcal{U}$ in Remark [6,8] is therefore given by our explicit proof of Theorem [11.1]. If the subgroup $\mathcal{U}$ in Remark [7,5] admits an invariant finite $L$-presentation, then the substitutions may not be related to the substitutions $\Phi$ of the finite $L$-presentation $\langle X \mid Q \mid \Phi \mid R \rangle$ of the Basilica group in Proposition [4,1]. It is neither clear to us whether $\mathcal{U}$ admits an invariant finite $L$-presentation nor do we know how to possibly prove that $\mathcal{U}$ does not admit such invariant $L$-presentation.

8 Examples of subgroup $L$-presentations

In this section, we again consider the subgroup $\mathcal{U} = \langle a, bab^{-1}, b^3 \rangle$ of the Basilica group $G$ as in Section [4]. We demonstrate how our methods apply to this subgroup and, in particular, how to compute the $L$-presentation in Section [4].

Coset-enumeration for finitely presented groups [13] allows us to compute the permutation representation $\varphi : F \rightarrow \text{Sym}(\mathcal{U}K\backslash F)$ for the group’s action on the right-cosets. A Schreier transversal for $\mathcal{U}$ in $G$ is given by $T = \{ 1, b, b^2 \}$ and we have

$$\varphi : F \rightarrow \mathcal{S}_n, \begin{cases} a & \mapsto (1) \\ b & \mapsto (1,2,3). \end{cases}$$

Moreover, $\mathcal{U}$ is a normal subgroup with index 3 in $G$ and it satisfies $\sigma^2 \sim_\varphi \text{id}$. By Lemma [5,14] there exists an integer $k \geq 2$ so that $\sigma^k \sim \text{id}$; we can verify that $\sigma^4 \varphi = \varphi$ holds and thus we have $\sigma^4 \sim \text{id}$. In particular, the subgroup $\mathcal{U}$ is (weakly) leaf-invariant and normal. Therefore the following techniques apply to this subgroup:

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• As the subgroup $\mathcal{U}$ is a finite index subgroup of an invariantly $L$-presented group $G$, the general methods of Proposition 6.3 and Theorem 6.4 apply.

• As the subgroup $\mathcal{U}$ is leaf-invariant, the methods in Theorem 7.1 apply.

• As the subgroup $\mathcal{U}$ is weakly leaf-invariant and normal, the methods in Theorem 7.6 apply.

We demonstrate these different techniques for the subgroup $\mathcal{U}$. First, we consider the general method from Proposition 6.3. For this purpose, we first note that the stabilizing subgroup $L$ and stabilizing core $\hat{L}$ coincide by Corollary 5.18. The stabilizing subgroups $L = \hat{L}$ have index 9 in $F$ and a Schreier generating set for $L = \hat{L}$ is given by

$$
x_1 = a^3, \quad x_4 = abab^{-1}a^{-2}, \quad x_7 = a^3bab^{-1}, \quad x_{10} = b^2a^2ba^{-2}.
$$

$$
x_2 = bab^{-1}a^{-1}, \quad x_5 = ab^2a^{-1}b^{-2}, \quad x_8 = a^2b^2a^{-2}b^{-2}, \quad x_9 = b^2a^2b^{-2}.
$$

Let $F$ denote the free group over $\{a, b\}$ and let $\mathcal{F}$ denote the free group over $\{x_1, \ldots, x_{10}\}$. The Reidemeister rewriting $\tau: F \to \mathcal{F}$ allows us to rewrite the iterated relation $r = [a, a^3]$. We obtain $\tau(r) = x_1^{-1}x_{10}^{-1}x_6x_{10}^{-1}x_9x_3$. Furthermore, the rewriting $\tau$ allows us to translate the substitution $\sigma$ of the Basilica group to an endomorphism of the free group $\mathcal{F}$. For instance, we obtain a free group homomorphisms $\hat{\sigma}: \mathcal{F} \to \mathcal{F}$ which is induced by the map

$$
x_1 \mapsto x_3^2, \quad x_6 \mapsto x_8x_9,
$$

$$
x_2 \mapsto x_5, \quad x_7 \mapsto x_3x_2x_5x_6,
$$

$$
x_3 \mapsto x_1, \quad x_8 \mapsto x_3x_2x_4x_{10}^{-1}x_8^{-1},
$$

$$
x_4 \mapsto x_6x_2^{-1}x_3^{-1}, \quad x_9 \mapsto x_8x_{10}x_8x_{10}^{-1},
$$

$$
x_5 \mapsto x_8^{-1}, \quad x_{10} \mapsto x_8x_{10}x_7x_3x_3^{-1}.
$$

Similarly, the conjugation actions $\delta_a$ and $\delta_b$ which are induced by $a$ and $b$, respectively, translate to endomorphisms $\hat{\delta}_a$ and $\hat{\delta}_b$ of the free group $\mathcal{F}$. By Proposition 6.3, the stabilizing subgroups $L = \hat{L}$ are finitely $L$-presented by

$$L = L/K \cong \langle \{x_1, \ldots, x_{10}\} \mid 0 \mid \{\hat{\sigma}, \hat{\delta}_a, \hat{\delta}_b\} \mid \{x_1^{-1}x_{10}^{-1}x_6x_{10}^{-1}x_9x_3\} \rangle.
$$

The subgroup $\mathcal{U}$ satisfies the short exact sequence $1 \to L \to \mathcal{U} \to C_3 \to 1$ with a cyclic group $C_3 = \langle \alpha \mid \alpha^3 = 1 \rangle$ of order 3. Corollary 2.4 yields the following finite $L$-presentation for the subgroup $\mathcal{U}$:

$$\langle \{\alpha, x_1, \ldots, x_{10}\} \mid \{\alpha^3x_1^{-1}\} \cup \{(x_{10}^{-1})^\alpha x_1^{-1}\}_{1 \leq i \leq 10} | \hat{\Psi} \mid \{x_1^{-1}x_{10}^{-1}x_6x_{10}^{-1}x_9x_3\} \rangle.$$

where the substitutions $\hat{\Psi} = \{\hat{\sigma}, \hat{\delta}_a, \hat{\delta}_b\}$ of $L$’s finite $L$-presentation are dilated to endomorphisms $\hat{\Psi} = \{\hat{\sigma}, \hat{\delta}_a, \hat{\delta}_b\}$ of the free group over $\{\alpha, x_1, \ldots, x_{10}\}$ as in the proof of Proposition 2.3.

Secondly, the subgroup $\mathcal{U}$ is (weakly) leaf-invariant and normal and therefore, the methods in Section 7 apply. First, we consider the construction in Theorem 7.4 for leaf-invariant subgroups. A Schreier generating set for the subgroup $UK$ is given by $x_1 = a$, $x_2 = bab^{-1}$, $x_3 = b^2ab^{-2}$, and $x_4 = b^3$. Since
σ^4φ = φ holds, the subgroup U is σ^4-invariant and its suffices to rewrite the
relation r = [a, b] and its images. The images τ(tr^σt^{-1}) have the form:

| i | t = 1 | t = b |
|---|---|---|
| 0 | x_1^{-1}x_3^{-1}x_4x_1x_4^{-1}x_3x_4 | x_2^{-1}x_4^{-1}x_2x_1 | x_3^{-1}x_2^{-1}x_3x_2 |
| 1 | x_1^{-1}x_3^{-1}x_4x_1x_4^{-1}x_3x_4 | x_2^{-1}x_4^{-1}x_2x_1 | x_3^{-1}x_2^{-1}x_3x_2 |
| 2 | x_1^{-1}x_3^{-1}x_4x_1x_4^{-1}x_3x_4 | x_2^{-1}x_4^{-1}x_2x_1 | x_3^{-1}x_2^{-1}x_3x_2 |
| 3 | x_1^{-1}x_3^{-1}x_4x_1x_4^{-1}x_3x_4 | x_2^{-1}x_4^{-1}x_2x_1 | x_3^{-1}x_2^{-1}x_3x_2 |

Let R denote the set of relations above. The endomorphism σ^4 translates, via
τ, to an endomorphism of the free group over \{x_1, \ldots, x_4\} which is induced by

\[ \sigma^4 : \begin{cases} x_1 & \mapsto x_1^4 \\ x_2 & \mapsto x_2x_4^2x_4^{-1} \\ x_3 & \mapsto x_3^2x_3x_4^{-2} \\ x_4 & \mapsto x_4^2 \end{cases} \]

By Theorem 7.1 \( L \)-presentation for the subgroup U is given by

\[ U \cong \langle \{x_1, \ldots, x_4\} \mid \emptyset \rangle \langle \sigma^4, \mid \{R\} \rangle \]

Finally, the subgroup U is weakly leaf-invariant and normal and therefore, the
methods in Theorem 7.3 apply. As σ^2 ∼ id holds, it suffices to consider the
relations τ(r) and τ(r^σ) and their images under the substitutions σ^2 and δ_b (as
a Schreier transversal is given by \( T = \{1, b, b^2\} \)) which are induced by

\[ \sigma^2 : \begin{cases} x_1 & \mapsto x_1^2 \\ x_2 & \mapsto x_2 \\ x_3 & \mapsto x_3^2x_4^{-1} \\ x_4 & \mapsto x_4^2 \end{cases} \]

and

\[ \delta_b : \begin{cases} x_1 & \mapsto x_2 \\ x_2 & \mapsto x_3 \\ x_3 & \mapsto x_4x_1x_4^{-1} \\ x_4 & \mapsto x_4 \end{cases} \]

Theorem 7.6 yields the following finite \( L \)-presentation for the subgroup U:

\[ U \cong \langle \{x_1, \ldots, x_4\} \mid \emptyset \rangle \langle \sigma^2, \delta_b, \mid \{r, r^σ\} \rangle \]

### 8.1 An application to the Grigorchuk group

As a finite \( L \)-presentation of a group allows the application of computer algorithms,
we may use our constructive proof of Theorem 14 allows us to investigate
the structure of a self-similar group by its finite index subgroups as in [17].
As an application, we consider the Grigorchuk group, see [10], \( \mathcal{G} = \langle a, b, c, d \rangle \)
and its normal subgroup \( \langle d \rangle^G \). We show that the subgroup \( \langle d \rangle^G \) has a minimal
generating set with 8 elements and thereby we correct a mistake in [8, 9].

The Grigorchuk group \( \mathcal{G} \) satisfies the following well-known

**Proposition 8.1 (Lysěňok, 22)** The Grigorchuk group \( \mathcal{G} \) is invariantly
\( L \)-presented by \( \mathcal{G} \cong \langle \{a, b, c, d\} \mid \{a^2, b^2, c^2, d^2, bcd\} \mid \{σ\} \mid \{ad^3, (adacac)^4\} \rangle \),
where σ is the endomorphism of the free group over \{a, b, c, d\} induced by the
mapping \( a \mapsto ac, b \mapsto d, c \mapsto b, \) and \( d \mapsto c \).
It was claimed in [3, Section 4.2] and in [9, Section 6] that the normal closure \( \langle d \rangle^\oplus \) is 4-generated by \( \{ d, d^2, d^{ac}, d^{aca} \} \). In the following, we show that the Reidemeister Schreier Theorem can be used to prove that a generating set for \( \langle d \rangle^\oplus \) contains at least 8 elements. Our coset-enumeration for finitely presented groups [13] and our solution to the subgroup membership problem for finite index subgroups in [13] show that the subgroup

\[
\mathcal{D} = \langle d, d^2, d^{ac}, d^{aca}, d^{acac}, d^{acaca}, d^{acacac}, d^{acacaca} \rangle
\]

has index 16 in \( \mathcal{G} \) and it is a normal subgroup of \( \mathcal{G} \) so that \( \mathcal{G}/\mathcal{D} \) is a dihedral group of order 16. In particular, the subgroup \( \mathcal{D} \) and the normal closure \( \langle d \rangle^\oplus \) coincide. A permutation representation \( \varphi: F \to S_{16} \) for the group’s action on the right-cosets \( UK \setminus F \) is given by

\[
\varphi: F \to S_{16}, \quad \begin{cases} 
  a &\mapsto (1, 2)(3, 5)(4, 6)(7, 9)(8, 10)(11, 13)(12, 14)(15, 16) \\
  b &\mapsto (1, 3)(2, 4)(5, 7)(6, 8)(9, 11)(10, 12)(13, 15)(14, 16) \\
  c &\mapsto (1, 3)(2, 4)(5, 7)(6, 8)(9, 11)(10, 12)(13, 15)(14, 16) \\
  d &\mapsto (1).
\end{cases}
\]

Our variant of the Reidemeister-Schreier Theorem and the techniques introduced in Section [7] enable us to compute a subgroup \( L \)-presentation for \( \mathcal{D} \). For this purpose, we first note that \( \sigma^3 \sim_{\varphi} \text{id} \) holds and hence, the normal core \( \mathcal{D} = \text{Core}_F(\mathcal{UK}) = \ker(\varphi) \) is \( \sigma^3 \)-invariant. The normal core \( \text{Core}_F(\mathcal{UK}) \) has rank 49 and a Schreier transversal for \( \mathcal{D} \) in \( \mathcal{G} \) is given by

\[
1, a, b, ab, ba, aba, bab, (ab)^2, (ba)^2, a(ba)^2, b(ab)^2, (ab)^3, (ba)^3, a(ba)^3, b(ab)^3, (ab)^4.
\]

A finite \( L \)-presentation with generators \( d_0 = d, d_1 = d^2, d_2 = d^{ac}, d_3 = d^{aca}, d_4 = d^{acac}, d_5 = d^{acaca}, d_6 = d^{acacac}, d_7 = d^{acacaca} \) is given by

\[
\mathcal{D} \cong \langle \{ d_0, \ldots, d_7 \} \ | \ \{ \sigma, \delta_a, \delta_b \} \ | \ \mathcal{R} \rangle,
\]

where the iterated relations are

\[
\mathcal{R} = \{ d_0^3, [d_1, d_0], [d_1, d_4], [d_7, d_3, d_4]^4, [d_7 d_0, d_3 d_4]^4, (d_3 d_7 d_4 d_0)^2, (d_7 d_4 d_0 d_3^4)^2 \}
\]

and the endomorphisms are induced by the maps

\[
\delta_a: \begin{cases} 
  d_0 &\mapsto d_1 \\
  d_1 &\mapsto d_0 \\
  d_2 &\mapsto d_3 \\
  d_3 &\mapsto d_2 \\
  d_4 &\mapsto d_5 \\
  d_5 &\mapsto d_4 \\
  d_6 &\mapsto d_7 \\
  d_7 &\mapsto d_6
\end{cases}, \quad \delta_b: \begin{cases} 
  d_0 &\mapsto d_0 \\
  d_1 &\mapsto d_2 \\
  d_2 &\mapsto d_1 \\
  d_3 &\mapsto d_0, d_4 \\
  d_4 &\mapsto d_3, d_0 \\
  d_5 &\mapsto d_6 \\
  d_6 &\mapsto d_5 \\
  d_7 &\mapsto d_7
\end{cases}, \quad \text{and} \quad \sigma: \begin{cases} 
  d_0 &\mapsto d_0 \\
  d_1 &\mapsto d_0 d_3 \\
  d_2 &\mapsto d_0 d_7 \\
  d_3 &\mapsto d_0 d_4 d_3 \\
  d_4 &\mapsto d_0 d_7 d_4 d_3 \\
  d_5 &\mapsto d_0 d_7 d_4 d_3 \\
  d_6 &\mapsto d_0 d_7 d_4 d_3 \\
  d_7 &\mapsto d_0 d_7 d_4 d_3
\end{cases}.
\]
The \( L \)-presentation of \( D \) allows us to compute the abelianization \( D/[D, D] \) with the methods from \cite{2}. We obtain that \( D/[D, D] \cong (\mathbb{Z}/2\mathbb{Z})^8 \) is 2-elementary abelian of rank 8. Hence, the normal subgroup \( D \) has a minimal generating set of length 8 because a generating set with 8 generators was already given in Eq. (11) above. In particular, this shows that \( \langle d \rangle^G \neq \langle d, d^a, d^{ac}, d^{aca} \rangle \). Note that the mistake could have been detected by computing the abelianization of the image of \( \langle d \rangle^G \) in a finite quotient of \( G \) (e.g. the quotient \( G/\text{Stab}(n) \) for \( n \geq 4 \)), by hand or using a computer-algebra-system such as GAP.

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