A tractable class of binary VCSPs via M-convex intersection

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Abstract

A binary VCSP is a general framework for the minimization problem of a function represented as the sum of unary and binary cost functions. An important line of VCSP research is to investigate what functions can be solved in polynomial time. Cooper–Živný classified the tractability of binary VCSP instances according to the concept of “triangle,” and showed that the only interesting tractable case is the one induced by the joint winner property (JWP). Recently, Iwamasa–Murota–Živný made a link between VCSP and discrete convex analysis, showing that a function satisfying the JWP can be transformed into a function represented as the sum of two M-convex functions, which can be minimized in polynomial time via an M-convex intersection algorithm if the value oracle of each M-convex function is given.

In this paper, we give an algorithmic answer to a natural question: What binary VCSP instances can be solved in polynomial time via an M-convex intersection algorithm? Under a natural condition, we solve this problem by devising a polynomial-time algorithm for obtaining a concrete form of the representation in the representable case. Our result presents a larger tractable class of binary VCSPs, which properly contains the JWP class. We also show the co-NP-hardness of testing the representability of a given binary VCSP instance as the sum of two M-convex functions.

Keywords: valued constraint satisfaction problems, discrete convex analysis, M-convexity

1 Introduction

The \textit{valued constraint satisfaction problem} (VCSP) provides a general framework for discrete optimization (see [36] for details). Informally, the VCSP framework deals with the minimization problem of a function represented as the sum of “small” arity functions, which are called \textit{cost functions}. It is known that various kinds of combinatorial optimization problems can be formulated in the VCSP framework. In general, the VCSP is NP-hard. An important line of research is to investigate what restrictions on classes of VCSP instances ensure polynomial time solvability. Two main types of VCSPs with restrictions are \textit{structure-based VCSPs} and \textit{language-based VCSPs} (see e.g., [6, 36]). Structure-based VCSPs deal with restrictions on the hypergraph structure representing the appearance of variables in a given instance. For example, Gottlob–Greco–Scarcello [12] showed that, if the hypergraph corresponding to a VCSP instance has a...
bounded hypertree-width, then the instance can be solved in polynomial time. Language-based VCSPs deal with restrictions on cost functions that appear in a VCSP instance. Kolmogorov–Thapper–Živný [20] gave a precise characterization of tractable valued constraint languages via the basic LP relaxation. Kolmogorov–Krokhin–Rollínek [19] gave a dichotomy for all language-based VCSPs (see also [2, 35] for a dichotomy for all language-based CSPs).

Hybrid VCSPs, which deal with a combination of structure-based and language-based restrictions, have emerged recently [6]. Among many kinds of hybrid restrictions, a binary VCSP, VCSP with only unary and binary cost functions, is a representative hybrid restriction that includes numerous fundamental optimization problems. Cooper–ˇZivn´y [4] showed that if a given binary VCSP with only unary and binary cost functions, is a representative hybrid restriction that induces numerous fundamental optimization problems. Cooper–ˇZivn´y [4] showed that if a given binary VCSP instance satisfies the joint winner property (JWP), then it can be minimized in polynomial time. The same authors classified in [5] the tractability of binary VCSP instances according to the concept of “triangle,” and showed that the only interesting tractable case is the one induced by the JWP (see also [6]). Furthermore, they introduced cross-free convexity as a generalization of JWP, and devised a polynomial-time minimization algorithm for cross-free convex instances $F$, provided a “cross-free representation” of $F$ is given (see section 2.3 for detail).

In this paper, we introduce a novel tractability principle going beyond triangle and cross-free representation for binary VCSPs. A binary VCSP is formulated as follows, where $D_1, D_2, \ldots, D_r$ ($r \geq 2$) are finite sets.

**Given:** Unary cost functions $F_p : D_p \to \mathbb{R}$ for $p \in \{1, 2, \ldots, r\}$ and binary cost functions $F_{pq} : D_p \times D_q \to \mathbb{R} \cup \{+\infty\}$ for $1 \leq p < q \leq r$.

**Problem:** Find a minimizer of $F : D_1 \times D_2 \times \cdots \times D_r \to \mathbb{R} \cup \{+\infty\}$ defined by

$$F(X_1, X_2, \ldots, X_r) := \sum_{1 \leq p \leq r} F_p(X_p) + \sum_{1 \leq p < q \leq r} F_{pq}(X_p, X_q).$$  \hfill (1)

Our tractability principle is built on discrete convex analysis (DCA) [25, 27], which is a theory of convex functions on discrete structures. In DCA, L-convexity and M-convexity play primary roles; the former is a generalization of submodularity, and the latter is a generalization of matroids. A variety of polynomially solvable problems in discrete optimization can be understood within the framework of L-convexity/M-convexity (see e.g., [25, 27]) recently, it has also turned out that discrete convexity is deeply linked to tractable classes of VCSPs. L-convexity is closely related to the tractability of language-based VCSPs. Various kinds of submodularity induce tractable classes of language-based VCSP instances [20], and a larger class of such submodularity can be understood as L-convexity on certain graph structures [14]. On the other hand, Iwamasa–Murota–Živný [18] have pointed out that M-convexity plays a role in hybrid VCSPs. They revealed the reason for the tractability of a VCSP instance satisfying the JWP from a viewpoint of M-convexity. We here continue this line of research, and explore further applications of M-convexity in hybrid VCSPs.

A function $f : \{0, 1\}^n \to \mathbb{R} \cup \{+\infty\}$ is called M-convex [22] if it satisfies the following generalization of the matroid exchange axiom: for $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \text{dom } f$, and $i \in \{1, 2, \ldots, n\}$ with $x_i > y_i$, there exists $j \in \{1, 2, \ldots, n\}$ with $y_j > x_j$ such that

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j),$$

where, for a function $f : \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$, the effective domain is denoted as $\text{dom } f := \{x \in \mathcal{D} \mid f(x) < +\infty\}$, and $\chi_i$ is the $i$th unit vector. An M-convex function can be minimized

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1Although M-convex functions are defined on $\mathbb{Z}^n$ in general, we only need functions on $\{0, 1\}^n$ here. M-convex functions on $\{0, 1\}^n$ are equivalent to the negative of valuated matroids introduced by Dress–Wenzel [4, 10].
in a greedy fashion similarly to the greedy algorithm for matroids. Furthermore, a function \( f : \{0, 1\}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) that is representable as the sum of two M-convex functions is called \( M_2 \)-convex. As a generalization of matroid intersection, the problem of minimizing an \( M_2 \)-convex function, called the \( M \)-convex intersection problem, can also be solved in polynomial time if the value oracle of each constituent M-convex function is given [23, 24]; see also [26, Section 5.2]. Our proposed tractable class of VCSPs is based on this result.

Let us return to binary VCSPs. The starting observation for relating VCSP to DCA is that the objective function \( F \) on \( D_1 \times D_2 \times \cdots \times D_r \) can be regarded as a function \( f \) on \( \{0, 1\}^n \) by the following correspondence between the domains:

\[
 D_p := \{1, 2, \ldots, n_p\} \ni i \leftrightarrow (0, \ldots, 0, 1, 0, \ldots, 0) \quad \text{(with \( i \) in the \( n_p \)th position)} \quad (p \in \{1, 2, \ldots, r\}).
\]

With this correspondence, the minimization of \( F \) can be transformed to that of \( f \). A binary VCSP instance \( F \) is said to be \( M_2 \)-representable if the function \( f \) obtained from \( F \) via the correspondence (2) is \( M_2 \)-convex.

It is shown in [18] that a binary VCSP instance satisfying the JWP can be transformed to an \( M_2 \)-representable instance\(^\dagger\) and two \( M \)-convex summands can be obtained in polynomial time. Here the following natural question arises: What binary VCSP instances are \( M_2 \)-representable? In this paper, we give an algorithmic answer to this question by considering the following problem:

**Testing \( M_2 \)-Representability**

**Given:** A binary VCSP instance \( F \).

**Problem:** Determine whether \( F \) is \( M_2 \)-representable or not. If \( F \) is \( M_2 \)-representable, obtain a decomposition \( f = f_1 + f_2 \) of the function \( f \) into two \( M \)-convex functions \( f_1 \) and \( f_2 \), where \( f \) is the function transformed from \( F \) via \( (2) \).

We assume the following condition for an instance \( F \).

\((\star)\): For all \( p \in \{1, 2, \ldots, r\} \) and \( d \in D_p \), there is \( X = (X_1, X_2, \ldots, X_r) \in \text{dom} \, F \) with \( X_p = d \).

This assumption is fairly reasonable, though checking whether a binary VCSP instance satisfies \((\star)\) is NP-hard in general (since it is equivalent to solving a general binary CSP). If all binary cost functions \( F_{pq} \) appearing in a given instance only take finite values, then the instance obviously satisfies \((\star)\). Furthermore, every binary VCSP instance can be reduced to an instance satisfying \((\star)\) by redefining \( D_p \leftarrow D_p \setminus \{d\} \) for \( p \in [r] \) and \( d \in D_p \) such that \( X_p \neq d \) for all \( X \in \text{dom} \, F \).

Our main result is the following:

**Theorem 1.1.** For binary VCSP instances satisfying \((\star)\), Testing \( M_2 \)-Representability can be solved in \( O(n^6) \) time.

An \( M_2 \)-convex function \( f \) can be minimized in polynomial time if such a decomposition can be obtained in polynomial time. Thus we obtain the following corollary of Theorem 1.1.

**Corollary 1.2.** An \( M_2 \)-representable binary VCSP instance satisfying \((\star)\) can be minimized in polynomial time.

\(^\dagger\)In [18], a binary VCSP instance satisfying the JWP was transformed into the sum of two \( M^\flat \)-convex functions. It can be easily seen that this function can also be transformed into the sum of two \( M \)-convex functions.
Our result provides us with cross-free representations, and presents a new tractable class of binary VCSPs that goes beyond JWP. A nice feature of our contribution is that the tractability based on M2-representability is independent of a particular representation of a given instance, while the tractability based on JWP or cross-free convexity depends on a representation; see section 2.3.

We also show the following theorem, which implies the NP-hardness of checking (*) when combined with Theorem 1.1.

**Theorem 1.3.** Testing M2-Representability is co-NP-hard.

Our approach to a polynomial-time algorithm for Testing M2-Representability is outlined as follows:

- We establish a unique representation theorem of M2-convex functions arising from binary VCSP instances (Theorem 2.2).
- With this result, our problem can be separated into two subproblems named Decomposition and Laminarization. The former is the problem of obtaining the unique representation of a given M2-convex function, and the latter is the problem of making a laminar family from a given family of subsets by means of certain transformations.
- We devise a polynomial-time algorithm for each problem, Decomposition and Laminarization (Theorems 3.7 and 4.11).

A unique representation theorem (Theorem 2.2), a polynomial-time algorithm for Decomposition (Theorem 3.7), and that for Laminarization (Theorem 4.11) are the major results of this paper. In particular, Laminarization seems to be an interesting problem of combinatorial nature in its own right.

**Organization.** In Section 2, we introduce the representation theorem (Theorem 2.2) of quadratic M2-convex functions arising from VCSP instances as well as the subproblems, Decomposition and Laminarization. We also prove Theorem 1.3. In Sections 3 and 4, we present polynomial-time algorithms for Decomposition and Laminarization, respectively.

**Notation.** Let \( \mathbb{Z} \), \( \mathbb{R} \), \( \mathbb{R}_+ \), and \( \mathbb{R}_{++} \) denote the sets of integers, reals, nonnegative reals, and positive reals, respectively. In this paper, functions can take the infinite value \( +\infty \), where \( a < +\infty \), \( a + \infty = +\infty \) for \( a \in \mathbb{R} \), and \( 0 \cdot (+\infty) = 0 \). Let \( \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \). For a positive integer \( k \), we define \([k] := \{1, 2, \ldots, k\} \).

**Remark 1.4.** This is a full version of an extended abstract [15], which did not include all proofs and only dealt with finite-valued CSPs, with cost functions taking only finite real values. In this paper, we deal with general-valued CSPs, where the cost functions can take both finite real values and the infinite value \( (+\infty) \).

## 2 Towards testing M2-representability

### 2.1 Representation theorem

We introduce a class of quadratic functions on \( \{0, 1\}^n \) that has a bijective correspondence to binary VCSP instances. Let \( \mathcal{A} := \{A_1, A_2, \ldots, A_r\} \) be a partition of \([n]\) with \( |A_p| \geq 2 \) for \( p \in [r] \).
We say that \( f : \{0,1\}^n \rightarrow \mathbb{R} \) is a VCSP-quadratic function of type \( \mathcal{A} \) if \( f \) is represented as

\[
 f(x_1, x_2, \ldots, x_n) := \begin{cases} 
 \sum_{i \in [n]} a_i x_i + \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j & \text{if } \sum_i x_i = r, \\
 +\infty & \text{otherwise,}
\end{cases}
\]  

(3)

where \( a_i \in \mathbb{R} \) and \( a_{ij} \in \mathbb{R} \) with \( a_{ij} := +\infty \) for all \( i, j \in A_p \) for some \( p \in [r] \). We assume \( a_{ij} = a_{ji} \) for distinct \( i, j \in [n] \).

Suppose that a binary VCSP instance \( F \) of the form (1) is given, where we assume \( F_{pq} = F_{qp} \) for distinct \( p, q \in [r] \). The transformation of \( F \) to \( f \) based on (2) in Section 1 is formalized as follows. Choose a partition \( \mathcal{A} := \{ A_1, A_2, \ldots, A_r \} \) of \([n]\) with \( |A_p| = n_p(= |D_p|) \) and a bijective correspondence \( A_p \rightarrow D_p \). Define \( a_i := F_p(d) \) if \( i \in A_p \) corresponds to \( d \in D_p \), \( a_{ij} := F_{pq}(d, e) \) if \( i \in A_p \) and \( j \in A_q \) correspond to \( d \in D_p \) and \( e \in D_q \), respectively, and \( a_{ij} := +\infty \) otherwise. Then the function \( f \) in (3) is a VCSP-quadratic function of type \( \mathcal{A} \). Note that condition (\star) for \( F \) is equivalent to the following condition for \( f \):

(\star\star): For all \( i \in [n] \), there is \( x = (x_1, x_2, \ldots, x_n) \in \text{dom } f \) with \( x_i = 1 \).

The class of \( M^2 \)-convex VCSP-quadratic functions admits a decomposition into simpler functions \( \ell_X \). For \( X \subseteq [n] \), let \( \ell_X : \{0,1\}^n \rightarrow \mathbb{R} \) be defined by

\[
 \ell_X(x) := \sum_{k_-(X) < k < k_+(X)} \left| k - \sum_{i \in X} x_i \right|,
\]

where

\[
 k_-(X) := \text{ the number of indices } p \in [r] \text{ with } X \supseteq A_p,
\]

\[
 k_+(X) := \text{ the number of indices } p \in [r] \text{ with } X \cap A_p \neq \emptyset.
\]

That is, \( \ell_X(x) \) is the sum of the distances from \( x \in \{0,1\}^n \) to hyperplanes \( \{ x \in \mathbb{R}^n \mid \sum_{i \in X} x_i = k \} \) for \( k_-(X) < k < k_+(X) \). In the following, we consider subsets \( X \) with \( k_-(X) + 2 \leq k_+(X) \), and denote the family of such subsets \( X \) by

\[
 \Pi = \Pi_\mathcal{A} := \{ X \subseteq [n] \mid k_-(X) + 2 \leq k_+(X) \}.
\]

In other words, \( X \in \Pi \) if and only if \( \emptyset \neq X \cap A_p \neq A_p \) for more than one \( p \in [r] \).

A family \( \mathcal{F} \subseteq \Pi \) is said to be laminar if \( X \subseteq Y \), \( X \supseteq Y \), or \( X \cap Y = \emptyset \) holds for all \( X, Y \in \mathcal{F} \). For a subpartition \( \mathcal{B} \) of \([n]\) (which often corresponds to quadratic coefficients with the infinite value), a family \( \mathcal{F} \subseteq \Pi \) is said to be \( \mathcal{B} \)-laminar if \( \mathcal{F} \cup \mathcal{B} \) is laminar and all elements in \( \mathcal{B} \) are minimal in \( \mathcal{F} \cup \mathcal{B} \). Define \( \delta_\mathcal{B} : \{0,1\}^n \rightarrow \mathbb{R} \) by \( \delta_\mathcal{B}(x) := 0 \) if \( \sum_{i \in [n]} x_i = r \) and \( \sum_{i \in X} x_i \leq 1 \) for each \( B \in \mathcal{B} \), and \( \delta_\mathcal{B}(x) := +\infty \) otherwise. Note that \( \delta_\mathcal{B} \) is the indicator function of \( \{ x \in \{0,1\}^n \mid \sum_{i \in [n]} x_i = r \} \). Then the following holds.

**Lemma 2.1.** For any \( \mathcal{B} \)-laminar family \( \mathcal{L} \subseteq \Pi \) and any positive weight \( c : \mathcal{L} \rightarrow \mathbb{R}_{++} \), the function \( \sum_{X \in \mathcal{L}} c(X) \ell_X + \delta_\mathcal{B} \) is \( M \)-convex.

**Proof.** We use the well-known fact that laminar convex functions are \( M^2 \)-convex (see [27 Section 6.3]). Let \( \delta_{\leq 1} : \mathbb{Z} \rightarrow \mathbb{R} \) the unary discrete convex function defined by \( \delta_{\leq 1}(t) := 0 \) for \( t \leq 1 \), and \( \delta_{\leq 1}(t) := +\infty \) for \( t \geq 2 \). Then \( \delta_{\leq 1} \) coincides with the function \( x \mapsto \sum_{B \in \mathcal{B}} \delta_{\leq 1}(\sum_{i \in B} x_i) \) on \( \{ x \in \{0,1\}^n \mid \sum_i x_i = r \} \). On the other hand, \( c(X) \ell_X \) can be regarded as \( x \mapsto g_X(\sum_{i \in X} x_i) \) for a unary discrete convex function \( g_X : \mathbb{Z} \rightarrow \mathbb{R} \), where \( X \in \mathcal{L} \). Since \( \mathcal{L} \cup \mathcal{B} \) is laminar, the function \( x \mapsto \sum_{X \in \mathcal{L}} c(X) \ell_X(x) + \sum_{B \in \mathcal{B}} \delta_{\leq 1}(\sum_{i \in B} x_i) \) on \( \{0,1\}^n \) is \( M^2 \)-convex. Furthermore it is known [30 Theorem 3.1] that the restriction of an \( M^2 \)-convex function \( f \) to \( \{ x \in \{0,1\}^n \mid \sum_i x_i = r \} \) is \( M \)-convex if the effective domain is nonempty after the restriction. Hence \( \sum_{X \in \mathcal{L}} c(X) \ell_X + \delta_\mathcal{B} \) is \( M \)-convex. \( \square \)
Our representation theorem (Theorem 2.2) says that an $M_2$-convex VCSP-quadratic function is always represented as the sum of $\sum_{X \in \mathcal{L}} c(X)\ell_X + \delta_B$ and a linear function on $\text{dom}\, \delta_A = \{x \in \{0,1\}^n | \sum_{i \in A_p} x_i = 1 \text{ for all } p \in [r]\}$. To state it precisely, there are substantial complications to be resolved. In our setting, we are given a VCSP-quadratic function $f$ of type $A$, which is defined only on $\text{dom}\, \delta_A$. It can happen that functions $\ell_X$ and $\ell_Y$ are identical on $\text{dom}\, \delta_A$ (i.e., $\ell_X + \delta_A = \ell_Y + \delta_A$) even when $X \neq Y$. Thus we have to make a judicious choice between them to demonstrate $M_2$-representability of $f$.

To cope with such complications, we define an equivalence relation $\sim$ by: $X \sim Y \iff \ell_X + \delta_A = \ell_Y + \delta_A$. For $\mathcal{F} \subseteq \Pi$, let $\mathcal{F}/\sim$ be the set of representatives (in $\Pi/\sim$) of all elements in $\mathcal{F}$. The equivalence relation is extended to subsets $\mathcal{F},\mathcal{G}$ of $\Pi$ by: $\mathcal{F} \sim \mathcal{G} \iff \mathcal{F}/\sim = \mathcal{G}/\sim$. A subset $\mathcal{P}$ of $\Pi/\sim$ is said to be $B$-laminar if $\mathcal{P} \cap (B/\sim) = \emptyset$ and there is a $B$-laminar family $\mathcal{L} \subseteq \Pi$ with $\mathcal{P} = \mathcal{L}/\sim$. A family $\mathcal{F} \subseteq \Pi$ is said to be $B$-laminarizable if $\mathcal{F}/\sim$ is $B$-laminar. For simplicity, the equivalence class of $X \in \Pi$ is also denoted by $X$, and a member of $\Pi/\sim$ is also denoted by its representative $X$.

Our first result is a representation theorem of $M_2$-convex functions. For a VCSP-quadratic function $f$ of type $A$, let $G_f^\infty$ be a multipartite graph on $[n]$ with partition $A$ such that edge $(i,j)$ $(i \in A_p, j \in A_q, p \neq q)$ exists if and only if $a_{ij} = +\infty$. Define $r(G_f^\infty)$ as the number of connected components of $G_f^\infty$. A connected component with at least one edge is said to be non-isolated.

**Theorem 2.2.** Let $f$ be a VCSP-quadratic function of type $A = \{A_1, A_2, \ldots, A_r\}$ satisfying condition $(\ast\ast)$, and $\mathcal{B}$ be the set of non-isolated connected components of $G_f^\infty$. Then $f$ is $M_2$-convex if and only if one of the following conditions (I) and (II) holds:

(I) $r(G_f^\infty) = r$ or $r + 1$, and $f = \delta_A + \delta_B + \text{ (linear function)}$.

(II) $r(G_f^\infty) \geq r + 2$, and there exist a $\mathcal{B}$-laminar family $\mathcal{P}_f \subseteq \Pi/\sim$ and a positive weight $c_f : \mathcal{P}_f \rightarrow \mathbb{R}^\ast$ such that

$$f = \left(\sum_{X \in \mathcal{P}_f} c_f(X)\ell_X + \delta_B\right) + \delta_A + \text{ (linear function)},$$

where “(linear function)” means a function $x \mapsto \sum_i p_i x_i + \alpha$ for some $(p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

In the case of (II), $\mathcal{P}_f$ and $c_f$ in (4) are uniquely determined.

By Theorem 2.2, an $M_2$-convex function $f$ has the summand $f_1 := \sum_{X \in \mathcal{L}} c_f(X)\ell_X + \delta_B$ with $\text{dom}\, f_1 = \text{dom}\, \delta_B$, where $\mathcal{L}$ is a $\mathcal{B}$-laminar family with $\mathcal{L}/\sim = \mathcal{P}_f$. Since $f$ satisfies condition $(\ast\ast)$, so does $f_1$. This implies $k_-(B) = 0$ and $k_+(B) \geq 2$ for every $B \in \mathcal{B}$. The proof of Theorem 2.2 is given in Sections 2.4 and 2.5.

### 2.2 Decomposition and Laminarization

To test for $M_2$-representability by Theorem 2.2 for the case of $r(G_f^\infty) \geq r + 2$, we first solve the following problem Decomposition, which detects non-$M_2$-convexity of $f$ or obtains decomposition $\mathcal{L}$.

**Decomposition**

**Given:** A VCSP-quadratic function $f$ of type $A$ satisfying condition $(\ast\ast)$. 


**Problem:** Either detect the non-$M_2$-convexity of $f$, or obtain some $P \subseteq \Pi/\sim$ and $c : P \to \mathbb{R}_{++}$ satisfying

$$f = \left( \sum_{X \in P} c(X) \ell_X + \delta_B \right) + \delta_A + \text{(linear function)}, \quad (5)$$

where $P$ is not required to be $B$-laminar in general, but in case of $M_2$-convex $f$, $P$ and $c$ should coincide, respectively, with $P_f$ and $c_f$ in $[1]$.

We emphasize that DECOMPOSITION may possibly output the decomposition $[5]$ even when the input $f$ is not $M_2$-convex, but if DECOMPOSITION detects the non-$M_2$-convexity then indeed the input $f$ is not $M_2$-convex.

Suppose that decomposition $[5]$ is obtained after solving DECOMPOSITION. In this case we have $P$ at hand. Then we have to check for the $B$-laminarizability of an arbitrarily chosen family $F \subseteq \Pi$ with $F/\sim = P$. This motivates us to consider the following problem.

**Laminarization**

**Given:** $F \subseteq \Pi$ and a subpartition $B \subseteq \Pi$ of $[n]$ satisfying $k_-(B) = 0$ for each $B \in B$.

**Problem:** Determine whether there exists a $B$-laminar family $L$ with $F \sim L$. If it exists, obtain a $B$-laminar family $L$ with $F \sim L$.

**Laminarization** is a purely combinatorial problem on a set system. Indeed, the equivalence relation $\sim$ can be rephrased in a combinatorial way as follows. For $X \in \Pi$, define

$$\langle X \rangle := \bigcup \{ A_p \in A \mid \emptyset \neq X \cap A_p \neq A_p \}$$

which is the union of $A_p$ contributing to $\ell_X + \delta_A$ nonlinearly. One can see the following.

**Lemma 2.3.** For $X,Y \in \Pi$, $X \sim Y$ if and only if $\{ \langle X \rangle \cap X, \langle X \rangle \setminus X \} = \{ \langle Y \rangle \cap Y, \langle Y \rangle \setminus Y \}$.

**Laminarization** can be regarded as the problem of transforming a given family $F$ to a laminar family by repeating the following operation: replace $X \in F$ with $[n] \setminus X$, $X \cup A_p$, or $X \setminus A_p$ with some $A_p$ satisfying $\langle X \rangle \cap A_p = \emptyset$. Figure $\square$ illustrates an example of the input (left) and an output (right) of Laminarization.

![Figure 1: The left figure illustrates the input $F$ of Laminarization and the right figure illustrates an output of Laminarization, where black nodes indicate elements of $[n]$, gray rectangles indicate $A_1, A_2, A_3$, and $A_4$, and solid curves indicate elements of $\Pi$.](image)
A decomposition \( f = f_1 + f_2 \) into two M-convex functions \( f_1 \) and \( f_2 \) can be constructed from \( c_f \) and \( \mathcal{L} \) found by DECOMPOSITION and LAMINARIZATION as \( f_1 := \sum_{X \in \mathcal{L}} c_f(X) \ell_X + \delta_B \) and \( f_2 := f - f_1 \). By Lemma 2.1, \( f_1 \) is an M-convex function, and \( f_2 \) is a linear function on \( \text{dom} \delta_A \).

For the case of \( r(G^\delta_f) = r \) or \( r + 1 \), we devise an \( O(n^6) \)-time algorithm for checking the linearity of \( f \) in Section 3.1. For the case of \( r(G^\delta_f) \geq r+2 \), we devise an \( O(n^5) \)-time algorithm for DECOMPOSITION in Section 3.2 and an \( O(n^4) \)-time algorithm for LAMINARIZATION in Section 4. Thus we obtain Theorem 1.1.

**Remark 2.4.** Our representation theorem (Theorem 2.2) and decomposition algorithm (in Section 3) are inspired by the polyhedral split decomposition due to Hirai [13]. This general decomposition principle decomposes, by means of polyhedral geometry, a function on a finite set \( \mathcal{D} \) of points of \( \mathbb{R}^n \) into a sum of simpler functions, called split functions, and a residue term. Actually, (5) can be viewed as a specialization of the polyhedral split decomposition, where \( \mathcal{D} = \text{dom}(\delta_A + \delta_B) \), and \( \ell_X + \delta_A + \delta_B \) is a sum of split functions. We refer the reader to [13] for details.

2.3 Relation to other problems

**Relation to JWP.** A binary VCSP instance \( F \) of the form [1] is said to satisfy the JWP (Joint Winner Property) [4] if

\[
F_{ij}(a,b) \geq \min\{F_{jk}(b,c),F_{ik}(a,c)\}
\]

for all distinct \( i,j,k \in [r] \) and all \( a \in D_i, b \in D_j, c \in D_k \). It is shown in [4] that if \( F \) satisfies the JWP, then \( F \) can be transformed, in polynomial time, into a function \( F' \) satisfying the JWP, argmin \( F' \subseteq \text{argmin} F \), and

\[
|\text{argmin}\{F'_{ij}(a,c),F'_{ij}(a,d),F'_{ij}(b,c),F'_{ij}(b,d)\}| \geq 2
\]

(7)

for any distinct \( i,j \in [r] \), distinct \( a,b \in D_i \), and distinct \( c,d \in D_j \). A function with the JWP satisfying (7) is said to be Z-free. Z-free functions can be minimized in polynomial time. Thus, if \( F \) satisfies the JWP, then \( F \) can be minimized in polynomial time. It is shown in [18] that Z-free instances are M_2-representable. The tractability based on M_2-representability depends solely on the function values, and is independent of how the function \( F \) is given. Indeed, an M_2-representable instance \( F \) can be characterized by the existence of a Z-free instance \( F' \) that satisfies \( F'(X) = F(X) \) for all \( X \).

This stands in sharp contrast with the tractability based on the JWP, which depends heavily on the representation of \( F \). For example, let \( F(X) = \sum F_{p}(X_p) + \sum F_{pq}(X_p,X_q) \) be a binary VCSP instance satisfying the JWP. By choosing a pair of distinct \( p,q \in [r] \), \( d \in D_p \), and \( \alpha \in \mathbb{R} \) arbitrarily, replace \( F_{p}(d) \) and \( F_{pq}(d,X_q) \) by \( F_{p}(d) + \alpha \) and \( F_{pq}(d,X_q) - \alpha \), respectively. Then \( F \) does not change but violates the JWP in general. Our result can explore such hidden M_2-convexity.

**Relation to cross-free convexity.** A pair \( X,Y \subseteq [n] \) is said to be crossing if \( X \cap Y, [n] \setminus (X \cup Y), X \setminus Y, \text{and } Y \setminus X \) are all nonempty. A family \( \mathcal{F} \subseteq 2^n \) is said to be cross-free if there is no crossing pair in \( \mathcal{F} \). A (not necessarily binary) VCSP instance \( F : D_1 \times D_2 \times \cdots \times D_r \rightarrow \mathbb{R} \) is said to be cross-free convex [5] if the function \( f : \{0,1\}^n \rightarrow \mathbb{R} \) obtained from \( F \) via the correspondence (2) can be represented as

\[
f(x) = \delta_A(x) + \sum_{X \in \mathcal{F}} g_X \left( \sum_{i \in X} x_i \right),
\]

(8)
where $\mathcal{A} := \{A_1, A_2, \ldots, A_r\}$ is a partition of $[n]$ with $|A_p| = |D_p|$, $\mathcal{F} \subseteq 2^{[n]}$ is cross-free, and $g_X$ is a unary discrete convex function for each $X \in \mathcal{F}$. Note that the polynomial-time minimization algorithm proposed in [6] for cross-free convex instances does not work unless the cross-free representation [8] is given.

Cross-free convexity is a special class of $M_2$-representability, where a (not necessarily binary) VCSP instance $F$ is $M_2$-representable if the function $f$ obtained from $F$ via the correspondence (2) is $M_2$-convex. Indeed, it is clear that $\delta_A$ is $M_2$-convex. Furthermore, by a similar argument to Lemma 2.1, the function $x \mapsto \sum_{X \in \mathcal{F}} g_X \left( \sum_{i \in X} x_i \right)$ is also $M_2$-convex on $\{x \in \{0,1\}^n \mid \sum_i x_i = r\}$. Thus $f$ is $M_2$-convex, and hence, $F$ is $M_2$-representable.

In case of binary VCSPs, the cross-free convexity and the $M_2$-representability are equivalent by Lemma 2.1 and Theorem 2.2. Hence our result provides, in binary VCSPs, a polynomial-time minimization algorithm for cross-free convex instances even when the expression (8) is not given.

**Application to quadratic pseudo-Boolean function minimization.** Consider a pseudo-Boolean function $f : \{0,1\}^n \to \mathbb{R}$ represented as $f(x_1, x_2, \ldots, x_n) = \sum_{i \in [n]} a_i x_i + \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j$.

For such $f$, we define $\hat{f} : \{0,1\}^n \times \{0,1\}^n \to \mathbb{R}$ by

$$\hat{f}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}) := \left\{ \begin{array}{ll} \sum_{i \in [n]} a_i x_i + \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j + \sum_{i \in [n]} \infty \cdot x_i x_{n+i} & \text{if } \sum_{i \in [2n]} x_i = n, \\
\infty & \text{otherwise.} \end{array} \right.$$ 

Then we have dom $\hat{f} = \{x \in \{0,1\}^{2n} \mid x_{n+i} = 1 - x_i \text{ for each } i \in [n]\}$ and $f(x_1, \ldots, x_n) = \hat{f}(x_1, \ldots, x_n, 1 - x_1, \ldots, 1 - x_n)$ for any $x \in \{0,1\}^n$. Hence minimizing $f$ is equivalent to minimizing $\hat{f}$.

Our result provides a new tractable class of quadratic pseudo-Boolean functions for minimization (see e.g., [1,7]). We can regard $\hat{f}$ as a function of the form (3) with the partition $\mathcal{A} := \{A_1, A_2, \ldots, A_n\}$ of $[2n]$ given by $A_i = \{i, n+i\}$ for $i \in [n]$. Therefore, if $\hat{f}$ is $M_2$-convex, then we can obtain two $M$-convex functions $\hat{f}_1$ and $\hat{f}_2$ satisfying $\hat{f} = \hat{f}_1 + \hat{f}_2$ by our proposed algorithm, and can minimize $\hat{f}$ (and hence $f$) in polynomial time.

We give an example of such a minimizable function. Define $f : \{0,1\}^4 \to \mathbb{R}$ by

$$f(x_1, x_2, x_3, x_4) := 4x_1 x_2 + x_1 x_3 + 3x_2 x_3 + 2x_2 x_4 + (\text{linear function}).$$

Then $\hat{f} : \{0,1\}^8 \to \mathbb{R}$ with $x = (x_1, x_2, \ldots, x_8) \in \{0,1\}^8$ is represented as

$$\hat{f}(x) = \left\{ \begin{array}{ll} 4x_1 x_2 + x_1 x_3 + 3x_2 x_3 + 2x_2 x_4 + \infty \cdot \sum_{i \in [4]} x_i x_{i+4} + \text{(linear function)} & \text{if } \sum_{i \in [8]} x_i = 4, \\
\infty & \text{otherwise.} \end{array} \right.$$ 

By solving DECOMPOSITION, we obtain $\mathcal{P} = \{123, 12, 23, 24\} \subseteq \Pi/\sim$ and $c(123) = 1/2$, $c(12) = 3/2$, $c(23) = 1$, and $c(24) = 1$, where we denote $\{i_1, i_2, \ldots, i_k\}$ by $i_1 i_2 \cdots i_k$ for distinct $i_1, i_2, \ldots, i_k$. By solving LAMINARIZATION, we obtain a laminar family $\mathcal{L} = \{123, 12, 1237, 123457\}$ (see also Figure 1). Note $12 \sim 1237$ and $24 \sim 123457$ by Lemma 2.3. Thus $\hat{f}$ is $M_2$-convex, and we obtain $\mathcal{P} = \mathcal{P}_f$ and $c = c_f$. The two $M$-convex functions for $\hat{f}$ are given by $\sum_{X \in \mathcal{L}} c_f(X) \ell_X + \delta_0$ and a linear function on dom $\delta_A$.

### 2.4 Proof of the characterization

In this subsection, we prove the characterization part of Theorem 2.2. That is, a VCSP-quadratic function $f$ of type $\mathcal{A}$ satisfying condition (***) is $M_2$-convex if and only if $f$ is a linear function
on \( \text{dom}(\delta_A + \delta_B) \) (the case of \( r(G_f^\infty) = r \) or \( r + 1 \)), or \( \square \) holds for some \( \mathcal{P}_f \) and \( c_f \) (the case of \( r(G_f^\infty) \geq r + 2 \)).

We first review fundamental facts about a general quadratic (not necessarily VCSP-quadratic) function \( g : \{0, 1\}^n \rightarrow \mathbb{R} \) represented as

\[ g(x_1, x_2, \ldots, x_n) = \left\{ \begin{array}{ll}
\sum_{i \in [n]} a_i x_i + \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j & \text{if } \sum_{i \in [n]} x_i = r, \\
+\infty & \text{otherwise},
\end{array} \right. \tag{9} \]

where \( r \in \mathbb{Z} \) with \( r \geq 2 \), \( a_i \in \mathbb{R} \), and \( a_{ij} = a_{ji} \in \mathbb{R} \). We assume condition (**) for \( g \). Let \( G_g^\infty \) be a graph on node set \([n]\) such that edge \( \{i, j\} \) (\( i \neq j \)) exists if and only if \( a_{ij} = +\infty \). Note that \( G_g^\infty \) is not a multipartite graph in general (unlike \( G_f^\infty \) for a VCSP-quadratic function \( f \)). Define \( r(G_g^\infty) \) as the number of connected components of \( G_g^\infty \). Let \( A_1, A_2, \ldots, A_m \) be the node sets of the non-isolated connected components of \( G_g^\infty \). Let \( \delta_g \) be the indicator function of \( \text{dom} \ g \), which is defined as \( \delta_g(x) := 0 \) for \( x \in \text{dom} \ g \) and \( \delta_g(x) := +\infty \) for \( x \notin \text{dom} \ g \). Then the M-convexity of \( g \) is characterized by the following lemma, which is a refinement of the result of \cite{16} and \cite{31}.

**Lemma 2.5** \((\cite{17}, \text{Theorem 3.1})\). A function \( g \) of the form (9) satisfying condition (**) is M-convex if and only if each connected component of \( G_g^\infty \) is a complete graph and one of the following conditions (I), (II), and (III) holds:

(I): \( r(G_g^\infty) \geq r + 2 \) and

\[ a_{ij} + a_{kl} \geq \min\{a_{ik} + a_{jl}, a_{il} + a_{jk}\} \tag{10} \]

holds for every distinct \( i, j, k, l \in [n] \).

(II): \( r(G_g^\infty) = r + 1 \) and

\[ a_{ij} + a_{kl} = a_{il} + a_{jk} \tag{11} \]

holds for every \( p \in [m], \) distinct \( i, k \in A_p, \) and distinct \( j, l \in [n] \setminus A_p \).

(III): \( r(G_g^\infty) = r \) and

\[ a_{ij} + a_{kl} = a_{il} + a_{jk} \tag{12} \]

holds for every distinct \( p, q \in [m], \) distinct \( i, k \in A_p, \) and distinct \( j, l \in A_q \).

In particular, (II) or (III) holds if and only if \( g \) is represented as \( g = \delta_g + \) (linear function). An M-convex function \( g \) of the form (9) is said to be non-trivial if \( r(G_g^\infty) \geq r + 2 \). We say that \((a_{ij})_{i,j \in [n]}\) satisfies the anti-tree metric property if (10) holds, and that \((a_{ij})_{i,j \in [n]}\) satisfies the anti-ultrametric property if

\[ a_{ij} \geq \min\{a_{ik}, a_{jk}\} \tag{13} \]

holds for all distinct \( i, j, k \in [n] \). It is known \cite{8} that the anti-ultrametric property is stronger than the tree-metric property \cite{10}. Anti-ultrametric property has a graphical interpretation, as follows.
Lemma 2.6 ([18, Lemma 8]). \((a_{ij})_{i,j \in [n]}\) satisfies the anti-ultrametric property if and only if there are a subpartition \(B\) of \([n]\), a \(B\)-laminar family \(L \subseteq 2^{[n]} \setminus \{n\}\), and a positive weight \(c : L \to \mathbb{R}_+\) such that
\[
a_{ij} = \begin{cases} 
+\infty & \text{if } i, j \in B \text{ for some } B \in \mathcal{B}, \\
\sum \{c(L) \mid L \in \mathcal{L} \text{ with } i, j \in L\} - \alpha^* & \text{otherwise},
\end{cases}
\]
where \(\alpha^* := \min_{i,j \in [n]} a_{ij}\).

Quadratic coefficients \((a_{ij})_{i,j \in [n]}\) are called \(\mathcal{B}\)-coefficients if \(a_{ij} = +\infty \iff i, j \in B \text{ for some } B \in \mathcal{B}\). The following is a variation of a well-known technique (Farris transform) in phylogenetics [33] to transform a tree metric to a ultrametric, and follows from the validity of Algorithm I in [17].

Lemma 2.7 ([17]). Suppose that \((a_{ij})_{i,j \in [n]}\) satisfies the anti-tree metric property. Let \(\alpha^* := \min_{i,j \in [n]} a_{ij}\) and \(b_k := \min_{i,j \in [n]} a_{kj} - \alpha^*\) for \(k \in [n]\). Then \((a_{ij} - b_i - b_j)_{i,j \in [n]}\) satisfies the anti-ultrametric property and \(\min_{i \in [n]} a_{ij} = \alpha^*\) holds for any \(i \in [n]\).

We now return to VCSP-quadratic functions. The following is a key proposition.

Proposition 2.8. Let \(f\) be a VCSP-quadratic function of type \(\mathcal{A}\) satisfying condition (\(\ast\ast\)), and \(\mathcal{B}\) be the set of non-isolated connected components (as a set of nodes) of \(G_f^\infty\). Then \(f\) is \(M_2\)-convex if and only if one of the following conditions (I) and (II) holds:

(I) \(r(G_f^\infty) = r \vee r + 1\), and \(f = \delta_a + \delta_b + (\text{linear function})\).

(II) \(r(G_f^\infty) \geq r + 2\), and \(f\) can be represented as
\[
f(x) = \sum_{i,j \in [n]} a_{ij} x_i x_j + \delta_a(x) + (\text{linear function}),
\]
where \((a_{ij})_{i,j \in [n]}\) are \(\mathcal{B}\)-coefficients satisfying the anti-ultrametric property.

Proof. Note that, for any subpartition \(C\) of \([n]\), \(\delta_C\) is an M-convex function that can be represented as \(\delta_C(x) = \sum_{C \in \mathcal{C}} \sum_{i,j \in C, i < j} \alpha \cdot x_i x_j\) on \(\{x \in \{0,1\}^n \mid \sum_i x_i = r\}\). Then the set of non-isolated connected components of \(G_f^\infty\) is equal to \(\mathcal{C}\).

(If part). If (I) holds, then \(f\) is a linear function on the domain of the sum of two indicator functions of partition matroids. Hence \(f\) is \(M_2\)-convex. If (II) holds, the function \(\sum a_{ij} x_i x_j\) on \(\{x \in \{0,1\}^n \mid \sum_i x_i = r\}\) is a non-trivial M-convex function by Lemma 2.5. Hence \(f\) is \(M_2\)-convex.

(Only-if part). Let \(f_1, f_2 : \{0,1\}^n \to \mathbb{R}\) be M-convex functions with \(f = f_1 + f_2\). Since \(f\) satisfies condition (\(\ast\ast\)), \(f_1\) and \(f_2\) also satisfy condition (\(\ast\ast\)) by dom \(f = \text{dom } f_1 \cap \text{dom } f_2\). Let \(\mathcal{C}_1\) and \(\mathcal{C}_2\) be the sets of non-isolated connected components of \(G_f^\infty\) and \(G_f^\infty\), respectively. Since \(f_1\) and \(f_2\) are M-convex, each member in \(\mathcal{C}_1 \cup \mathcal{C}_2\) is a complete graph by Lemma 2.5. Hence dom \(f_1\) = dom \(\delta_{\mathcal{C}_1}\) and dom \(f_2\) = dom \(\delta_{\mathcal{C}_2}\) hold.

First we prove dom \(f = \text{dom}(\delta_a + \delta_b)\). It is clear that dom \(f \supseteq \text{dom}(\delta_a + \delta_b)\). If dom \(f \supseteq \text{dom}(\delta_a + \delta_b)\), then there is \(B \in \mathcal{B}\) such that the subgraph \(G_f^\infty\) of \(G_f^\infty\) induced by \(B\) is not complete. Therefore we need at least two cliques to cover \(G_f^\infty\). This means that we need at least three subpartitions \(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_k\) \((k \geq 3)\) to satisfy dom \(f = \text{dom}(\sum_{i=1}^k \delta_{\mathcal{C}_i})\). On the other hand, we have dom \(f = \text{dom}(f_1 + f_2) = \text{dom}(\delta_{\mathcal{C}_1} + \delta_{\mathcal{C}_2})\), a contradiction.

Next we show that the above \(\mathcal{C}_1, \mathcal{C}_2\) can be taken as \(\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset\) and \(\mathcal{C}_1 \cup \mathcal{C}_2 = \mathcal{A} \cup \mathcal{B}\). By dom \(\delta_a + \delta_b) = \text{dom}(\delta_{\mathcal{C}_1} + \delta_{\mathcal{C}_2})\), it must hold that \(\mathcal{C}_1 \cup \mathcal{C}_2 \supseteq \mathcal{A} \cup \mathcal{B}\) and, for each \(C \in \mathcal{C}_1 \cup \mathcal{C}_2\), there is \(A \in \mathcal{A} \cup \mathcal{B}\) such that \(C \subseteq A\). Hence it suffices to show that if \(\mathcal{C}_1 \subseteq \mathcal{C}_2\) for some
$C_1 \in C_1$ and $C_2 \in C_2$, then we can modify $f_1$ so that $f_1$ is M-convex with $f = f_1 + f_2$ and dom $f_1 = \text{dom } \delta_{C_1 \setminus \{C_1\}}$.

If $f_1$ is a linear function on dom $f_1$, then $f_1 + f_2 = \delta_{C_1} + (\text{linear function}) + f_2 = \delta_{C_1 \setminus \{C_1\}} + (\text{linear function}) + f_2$, since dom$(\delta_{C_1} + f_2) = \text{dom } f_1 \cup \text{dom } f_2$ by $C_1 \subseteq C_2$. Hence we can modify $f_1$ as $\delta_{C_1 \setminus \{C_1\}} + (\text{linear function})$, as required. If $f_1$ is non-trivial, then, by Lemma 2.7, the quadratic coefficients of $f_1$ is represented as $(a_{ij}^1 + b_i + b_j)_{i,j \in [n]}$, where $b_i \in \mathbb{R}$ and $(a_{ij}^1)_{i,j}$ satisfies the anti-ultrametric property. By redefining $a_{ij}^1 \leftarrow M$ if $i, j \in C_1$ for a sufficiently large $M$, we have dom $f = \text{dom } \delta_{C_1 \setminus \{C_1\}}$ and the value of $f_1 + f_2$ does not change. Furthermore $(a_{ij}^1)_{i,j}$ still satisfies the anti-ultrametric property. Hence $f_1$ is M-convex, as required.

Let $f_1$ and $f_2$ be M-convex functions with $f = f_1 + f_2$, dom $f_1 = \text{dom } \delta_{C_1}$, and dom $f_2 = \text{dom } \delta_{C_2}$, where $C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 = \mathcal{A} \cup \mathcal{B}$. We consider two cases (i) both $f_1$ and $f_2$ are linear functions and (ii) at least one of $f_1$ and $f_2$ is a non-trivial M-convex function.

(i). Since $f_1 = \delta_{C_1} + (\text{linear function})$ and $f_2 = \delta_{C_2} + (\text{linear function})$, we have $f = f_1 + f_2 = \delta_{C_1} + \delta_{C_2} + (\text{linear function}) = \delta_{A} + \delta_{B} + (\text{linear function})$. Hence we can modify $f_1$ and $f_2$ as $f_1 \leftarrow \delta_A$ and $f_2 \leftarrow \delta_B + (\text{linear function})$, as required (note that the quadratic coefficients of $\delta_A$ satisfy the anti-ultrametric property).

(ii). Suppose that $f_1$ is a non-trivial M-convex function represented as $f_1(x) = \sum a_{ij} x_i x_j$ on $\{x \in \{0,1\}^n \mid \sum x_i = r\}$. We can assume that $(a_{ij})_{i,j}$ satisfies the anti-ultrametric property. Indeed, by Lemma 2.7, $(a_{ij} - b_i - b_j)_{i,j \in [n]}$ satisfies the anti-ultrametric property for some $b_i$ ($i \in [n]$). Hence $f_1(x) = \sum (a_{ij} - b_i - b_j) x_i x_j + \sum_i \sum_j b_i x_i = \sum (a_{ij} - b_i - b_j) x_i x_j + (\text{linear function})$ on dom $\delta_A$. Thus we can redefine $a_{ij} \leftarrow a_{ij} - b_i - b_j$ for distinct $i, j \in [n]$.

Suppose $C_1 = \mathcal{B}$. Then we have $C_2 = \mathcal{A}$ by $C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 = \mathcal{A} \cup \mathcal{B}$. Hence the number of connected components of $G_2^\infty$ is equal to $|\mathcal{A}| = r$. By the M-convexity of $f_2$ and Lemma 2.5, it holds that $f_2 = \delta_{A} + (\text{linear function})$; we are done.

Suppose $C_1 \neq \mathcal{B}$. Since $(a_{ij})_{i,j}$ satisfies the anti-ultrametric property, by Lemma 2.6, there are $C_1$-laminar family $\mathcal{L}$ and a positive weight $c_1 : \mathcal{L} \to \mathbb{R}_{++}$ representing $(a_{ij})_{i,j}$. Hence $f_1(x) = \sum_{L \in \mathcal{L}} c_1(L) \sum_{i,j \in L} x_i x_j + \delta_{C_1}(x)$. Let $\mathcal{A}_1 := \mathcal{A} \cap \mathcal{C}_1$, $\mathcal{A}_1 := \bigcup_{A \in \mathcal{A}} A$, $\mathcal{L}_1 := \{L \setminus \mathcal{A}_1 \mid L \in \mathcal{L}\}$, and $c_1'(L) := \sum \{c_1(L') \mid L' \in \mathcal{L}_1 \}$. Note $\emptyset \neq \mathcal{A}_1 \neq [n]$. Since, for every $L \in \mathcal{C}_1$ such that $\{L\}$ is $\mathcal{A}_1$-laminar, we have $\sum_{i,j \in L} x_i x_j + \delta_A = \sum_{i,j \in L \setminus \mathcal{A}_1} x_i x_j + \delta_A + (\text{linear function})$, one can see $f_1(x) + \delta_A(x) = \sum_{L \in \mathcal{L}_1} c_1'(L) \sum_{i,j \in L} x_i x_j + \delta_A(x) + (\text{linear function})$. By the definition of $\mathcal{A}_1$, every $B \in \mathcal{B} \cap \mathcal{C}_2$ satisfies $B \subseteq \mathcal{A}_1$. Hence $\mathcal{L}_1$ is $\mathcal{B}$-laminar.

Suppose that $f_2$ is a linear function. Then

$$f_1(x) + f_2(x) = \sum_{L \in \mathcal{L}_1} c_1'(L) \sum_{i,j \in L} x_i x_j + \delta_A(x) + \delta_B(x) + (\text{linear function}).$$

Since $\mathcal{L}'$ is $\mathcal{B}$-laminar, we can modify $f_1(x) \leftarrow \sum_{L \in \mathcal{L}_1} c_1'(L) \sum_{i,j \in L} x_i x_j + \delta_B(x)$ and $f_2 \leftarrow \delta_A + (\text{linear function})$, as required.

Suppose that $f_2$ is a non-trivial M-convex function represented as $f_2(x) = \sum b_{ij} x_i x_j$ on $\{x \in \{0,1\}^n \mid \sum x_i = r\}$. We can assume that $(b_{ij})_{i,j}$ satisfies the anti-ultrametric property by a similar argument as $f_1$. By Lemma 2.6, there are $C_2$-laminar family $\mathcal{L}_2$ and a positive weight $c_2 : \mathcal{L}_2 \to \mathbb{R}_{++}$ representing $(b_{ij})_{i,j}$. Let $\mathcal{L}_2' := \{L \cap \mathcal{A}_1 \mid L \in \mathcal{L}_2\}$ and $c_2'(L) := \sum c_2(L') \mid L' \cap \mathcal{A}_1 = L$. Then it holds that $f_2(x) + \delta_A(x) = \sum_{L \in \mathcal{L}_2'} c_2'(L) \sum_{i,j \in L} x_i x_j + \delta_A(x) + (\text{linear function})$.

Hence we have

$$f_1(x) + f_2(x) = \sum_{L \in \mathcal{L}_1'} c_1'(L) \sum_{i,j \in L} x_i x_j + \sum_{L \in \mathcal{L}_2'} c_2'(L) \sum_{i,j \in L} x_i x_j + \delta_A(x) + \delta_B(x) + (\text{linear function}).$$

Furthermore $\mathcal{L}_2'$ is $\mathcal{B}$-laminar. Since, for all $L_1 \in \mathcal{L}_1'$ and $L_2 \in \mathcal{L}_2'$, we have $L_1 \subseteq [n] \setminus \mathcal{A}_1$ and $L_2 \subseteq \mathcal{A}_1$. Hence $\mathcal{L}_1' \cup \mathcal{L}_2'$ is also $\mathcal{B}$-laminar. Thus we can modify $f_1(x) \leftarrow \sum_{L \in \mathcal{L}_1'} c_1'(L) \sum_{i,j \in L} x_i x_j + \sum_{L \in \mathcal{L}_2'} c_2'(L) \sum_{i,j \in L} x_i x_j + \delta_A(x) + \delta_B(x) + (\text{linear function}).$
\[ \sum_{L \in \mathcal{L}} c(L) \sum_{i,j \in L} x_i x_j + \delta_B(x) \] and \( f_2 \leftarrow \delta_A + \text{(linear function)} \), as required. This completes the proof of Proposition 2.8.

By the proof of Proposition 2.8, we immediately obtain the following necessary condition for \( M_2 \)-convexity, which can be checked in \( O(n^2) \) time.

**Corollary 2.9.** Let \( f \) be a VCSP-quadratic function of type \( A \) satisfying condition \( (**) \). If \( f \) is \( M_2 \)-convex, each connected component of \( G_f^\infty \) is a complete multipartite graph with partition \( A \).

With the use of Proposition 2.8, we can prove the characterization part of Theorem 2.2 as follows. The first case (I) of Theorem 2.2, where \( r(G_f^\infty) = r + 1 \), follows immediately from (I) of Proposition 2.8. The second case (II) of Theorem 2.2, where \( r(G_f^\infty) \geq r + 2 \), can be proved as follows. By Lemma 2.6 and Proposition 2.8, we have

\[ f(x) = \left( \sum_{L \in \mathcal{L}} c(L) \sum_{i,j \in L} x_i x_j + \delta_B \right) + \delta_A + \text{(linear function)} \]

for some \( \mathcal{B} \)-laminar family \( \mathcal{L} \) and \( c : \mathcal{L} \to \mathbb{R}_{++} \). We can assume \( \mathcal{L} \subseteq \Pi \), that is, \( k_-(L) + 2 \leq k_+(L) \) holds for all \( L \in \mathcal{L} \). Indeed, for \( \mathcal{L} \subseteq \Pi \) with \( k_+(L) < k_-(L) + 1 \), the function \( x \mapsto \sum_{i,j \in L} x_i x_j \) is linear on \( \text{dom} \ \delta_A \).

**Lemma 2.10.** \( \sum_{i,j \in L} x_i x_j + \delta_A = (1/2) \ell_L + \delta_A + \text{(linear function)} \).

**Proof.** With reference to \( L \), define \( (\hat{a}_{ij})_{i,j \in [n]} \) by: \( \hat{a}_{ij} := 1/2 \) if \( i, j \in (L) \cap L \) or \( i, j \not\in (L) \setminus L \), and \( \hat{a}_{ij} := 0 \) otherwise. Let \( \hat{f}_1(x) := \sum \hat{a}_{ij} x_i x_j \). We can see that \( \hat{f}_1 + \delta_A = (1/2) \ell_L + \delta_A \). Indeed, we can verify by straightforward calculations that, for \( x \in \text{dom} f \) with \( \sum_{i \in L} x_i = s \), both are equal to

\[ \frac{1}{4} ((s - k_-(L))(s - k_-(L) - 1) + (k_+(L) - s)(k_+(L) - s - 1)). \]

Define \( (a_{ij})_{i,j \in [n]} \) by: \( a_{ij} := +\infty \) if \( i, j \in A_p \) for some \( p \in [r] \), \( a_{ij} := 1 \) else if \( i, j \in L \), and \( a_{ij} := 0 \) otherwise. Then it holds that \( \sum_{i,j \in L} x_i x_j + \delta_A(x) = \sum a_{ij} x_i x_j \) if \( \sum x_i = r \), and \( = +\infty \) otherwise.

In the following, we show that \( g := \left( \sum_{i,j \in L} x_i x_j + \delta_A \right) - \hat{f}_1 \) is actually equal to \( \delta_A + \text{(linear function)} \). The quadratic coefficients of \( g \) are \( (a_{ij} - \hat{a}_{ij}) \). The set of connected components of \( G_f^\infty \) is \( \{A_1, A_2, \ldots, A_r\} \). Furthermore, we can verify that \( (a_{ij} - \hat{a}_{ij}) + (a_{kl} - \hat{a}_{kl}) \) holds for distinct \( i, k \in A_p \) and distinct \( j, l \in A_q \) for every distinct \( p, q \in [r] \). Hence, by Lemma 2.5, \( g = \delta_A + \text{(linear function)} \).

The above argument shows that \( \sum_{i,j \in L} x_i x_j + \delta_A = \hat{f}_1 + \delta_A + \text{(linear function)} = (1/2) \ell_L + \delta_A + \text{(linear function)} \).

By Proposition 2.8 and Lemma 2.10, \( f \) with \( r(G_f^\infty) \geq r + 2 \) is \( M_2 \)-convex if and only if there exist a \( \mathcal{B} \)-laminar family \( \mathcal{L} \subseteq \Pi \) and \( c : \mathcal{L} \to \mathbb{R}_{++} \) that represent \( f \) as

\[ f = \left( \sum_{L \in \mathcal{L}} c(L) \frac{\ell_L + \delta_B}{2} \right) + \delta_A + \text{(linear function)} \]

By defining

\[ c_f(X) := \sum_{L \in \mathcal{L} \sim X} c(L)/2 \quad (14) \]

for \( X \in \mathcal{L} \sim \) and \( \mathcal{P}_f := \mathcal{L} \sim \), we obtain the representation (14). Here, \( \mathcal{P}_f \) is \( \mathcal{B} \)-laminar as a result of the \( \mathcal{B} \)-laminarity of \( \mathcal{L} \).
2.5 Proof of the uniqueness

In this subsection, we prove the uniqueness of \( P_f \) and \( c_f \) in Theorem 2.2 for the case of \( r(G_f^\infty) \geq r + 2 \). Let \( f \) be a VCSP-quadratic function of type \( A \) and \( B \) be the set of non-isolated connected components of \( G_f^\infty \). The convex closure \( \overline{f} : \text{conv}(\text{dom} \ f) \to \mathbb{R} \) of \( f \) is the maximum convex function satisfying \( \overline{f}(x) = f(x) \) for \( x \in \text{dom} \ f \), which is given by

\[
\overline{f}(x) := \sup \left\{ \sum_{i \in [n]} c_i x_i + b \mid c \in \mathbb{R}^n, \ b \in \mathbb{R}, \ f(y) \geq \sum_{i \in [n]} c_i y_i + b \ (y \in \text{dom} \ f) \right\}.
\]

Note \( \text{conv}(\text{dom} \ f) = \{x \in [0,1]^n \mid \sum_{i \in A_p} x_i = 1 \text{ for all } p \in \{r\} \text{ and } \sum_{i \in B} x_i \leq 1 \text{ for all } B \in B\} \) and \( \overline{f}(x) = f(x) \) for all \( x \in \text{dom} \ f \).

Suppose that \( f \) is an \( M_2 \)-convex function. By the characterization part of Theorem 2.2 \( f \) is represented as

\[
f = \left( \sum_{L \in \mathcal{L}} c_f(L) \ell_L + \delta_{B} \right) + \delta_A + \text{(linear function)},
\]

where \( \mathcal{L} \) is a \( B \)-laminar family with \( \mathcal{L}/\sim = P_f \) and \( |\mathcal{L}| = |P_f| \). Then \( \overline{f} \) is explicitly written as follows.

\[
\overline{f}(x) = \sum_{L \in \mathcal{L}} c_f(L) \sum_{k = \ell(L) < k < k_+} |k - \sum_{i \in L} x_i| + \text{(linear function)} \quad (x \in \text{conv}(\text{dom} \ f)).
\] \hfill (15)

**Proof.** We denote by \( \hat{f} \) the right-hand side of (15). It is clear that \( f(x) = \hat{f}(x) \) for \( x \in \text{dom} \ f \). Since \( \hat{f} \) is piecewise linear and convex, \( \hat{f}(z) \leq \overline{f}(z) \) for \( z \in \text{conv}(\text{dom} \ f) \) by the definition of \( \overline{f} \). Thus it suffices to show \( \hat{f}(z) \geq \overline{f}(z) \) for \( z \in \text{conv}(\text{dom} \ f) \).

Take any \( z \in \text{conv}(\text{dom} \ f) \). Then \( z \) satisfies the following system of inequalities and equations for some integers \( k_L \) for all \( L \in \mathcal{L} \):

\[
0 \leq z_i \leq 1 \hspace{1cm} (i \in [n]),
\]

\[
\sum_{i \in A_p} z_i = 1 \hspace{1cm} (p \in \{r\}),
\]

\[
\sum_{i \in B} z_i \leq 1 \hspace{1cm} (B \in \mathcal{B}),
\]

\[
k_L - 1 \leq \sum_{i \in L} z_i \leq k_L \hspace{1cm} (L \in \mathcal{L}).
\] \hfill (19)

The coefficient matrix \( M \) of the system (16–19) is totally unimodular. Indeed, let \( M' \) be the matrix whose columns are the characteristic vectors of the members of \( \mathcal{L} \cup \mathcal{B} \) and \( \{A_1, A_2, \ldots, A_r\} \). \( M \) is represented as \( M = (I - I M' - M')^T \), where \( I \) is the \( n \times n \) identity matrix. Since \( \mathcal{L} \cup \mathcal{B} \) and \( \{A_1, A_2, \ldots, A_r\} \) are laminar, \( M' \) is totally unimodular [11]; see also [32, Theorem 41.11]. Thus \( M \) is also totally unimodular.

Let \( P \) be the polyhedron defined by the system (16–19). Then \( P \) is an integer polyhedron by the total unimodularity of \( M \). Hence all extreme points \( y_i \) of \( P \) belong to \( \text{dom} \ f \). By \( z \in P \), we have \( z = \sum_i \lambda_i y_i \) for some coefficients \( \lambda_i \) of a convex combination. Therefore \( \hat{f}(z) = \sum_i \lambda_i \hat{f}(y_i) = \sum_i \lambda_i f(y_i) \) holds, where the first equality follows from the linearity of \( \hat{f} \) on \( P \). Since \( f(y_i) = \overline{f}(y_i) \) and \( \overline{f} \) is convex, we obtain \( \sum_i \lambda_i f(y_i) = \sum_i \lambda_i \overline{f}(y_i) \geq \overline{f}(z) \), and hence \( \hat{f}(z) \geq \overline{f}(z) \). \( \square \)
We are ready to show the uniqueness part of Theorem 2.2. By Lemma 2.11, the set of nondifferentiable points of \( \overline{f} \) (with respect to the set of relative interior points of \( \text{conv}(\text{dom } f) \)) is given by

\[
\bigcup_{L \in \mathcal{L}, \ k_-(L) < k_+(L)} \left\{ x \in \text{conv}(\text{dom } f) \left| \sum_{i \in L} x_i = k \right. \right\} =: P(\mathcal{L}),
\]

where \( \mathcal{L} \) is a \( \mathcal{B} \)-laminar family with \( \mathcal{L}/\sim = \mathcal{P}_f \) and \( |\mathcal{L}| = |\mathcal{P}_f| \). Suppose, to the contrary, that there is another \((\mathcal{P}_f', c_f')\) which satisfies the conditions in Theorem 2.2 and let \( \mathcal{L}' \) be another \( \mathcal{B} \)-laminar family satisfying \( \mathcal{L}'/\sim = \mathcal{P}_f' \) and \( |\mathcal{L}'| = |\mathcal{P}_f'| \).

If \( \mathcal{P}_f \neq \mathcal{P}_f' \), i.e., \( \mathcal{L} \not\sim \mathcal{L}' \), then \( P(\mathcal{L}) \neq P(\mathcal{L}') \). This means that \( \overline{f} \) has two different sets of nondifferentiable points \( P(\mathcal{L}) \) and \( P(\mathcal{L}') \), a contradiction. Hence \( \mathcal{P}_f = \mathcal{P}_f' \) holds, and we can assume \( \mathcal{L} = \mathcal{L}' \). If \( c_f(L) > c_f'(L) \) for some \( L \in \mathcal{L} \), we can easily see that \( \overline{f} - c_f'(L) \) has two different sets of nondifferentiable points, a contradiction. Hence \( c_f(L) = c_f'(L) \) holds for all \( L \in \mathcal{L} \).

### 2.6 Co-NP-hardness of Testing M₂-Representability

In this subsection, we show the co-NP-hardness of Testing M₂-Representability in the general case without the assumption \((*)\) (Theorem 1.3). For quadratic coefficients \((a_{ij})_{i,j} \) with \( a_{ij} = 0 \) or \(+\infty\) for all distinct \( i, j \in [n] \) and \( r \geq 2 \), let \( g : \{0, 1\}^n \to \{0, +\infty\} \) be the quadratic function defined by

\[
g(x) := \begin{cases} 
\sum_{1 \leq i < j \leq n} a_{ij} x_i x_j & \text{if } \sum_{i \in [n]} x_i = r, \\
+\infty & \text{otherwise}.
\end{cases}
\]

The following lemma follows from the proof of Theorem 1.2 in [17].

**Lemma 2.12 ([17])**. Given a quadratic coefficient \((a_{ij})_{i,j} \) with \( a_{ij} = 0 \) or \(+\infty\) for each distinct \( i, j \in [n] \) and \( r \geq 2 \), the problem of checking the M-convexity of \( g \) is co-NP-hard.

Let \( \hat{A}_p := \{(1, p), (2, p), \ldots, (n, p)\} \) for \( p \in [r] \) and \( \hat{A} := \{\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_r\} \), which is a partition of \([n] \times [r]\). We construct from \((a_{ij})_{i,j} \) a VCSP-quadratic function \( \hat{g} : \{0, 1\}^{[n] \times [r]} \to \{0, +\infty\} \) of type \( \hat{A} \) as

\[
\hat{g}(x) = \begin{cases} 
\sum_{(i,p) \neq (j,q)} \hat{a}_{(i,p),(j,q)} x_{(i,p)} x_{(j,q)} & \text{if } \sum_{(i,p) \in [n] \times [r]} x_{(i,p)} = r, \\
+\infty & \text{otherwise},
\end{cases}
\]

where \( \hat{a}_{(i,p),(j,q)} := a_{ij} \) if \( i \neq j \) and \( p \neq q \), and \( \hat{a}_{(i,p),(j,q)} := +\infty \) otherwise. By Lemma 2.12 and the following proposition, we obtain Theorem 1.3.

**Proposition 2.13.** \( \hat{g} \) is M₂-convex if and only if \( g \) is M-convex.

**Proof.** Let \( X \) be the set of \( i \in [n] \) such that there is \( x \in \text{dom } g \) with \( x_i = 1 \). It is known [17] that \( g \) is M-convex if and only if each connected component of the subgraph \( \overline{G}_g[X] \) of \( \overline{G}_g \) induced by \( X \) is a complete graph.

On the other hand, let \( \hat{X} \) be the set of \( (i, p) \in [n] \times [r] \) such that there is \( x \in \text{dom } \hat{g} \) with \( x_{(i,p)} = 1 \). Let \( \hat{g}|_{\hat{X}} : \{0, 1\}^\hat{X} \to \{0, +\infty\} \) be the function \( \{0, 1\}^\hat{X} \ni x \mapsto \hat{g}(y) \), where \( y_{(i,p)} = x_{(i,p)} \) if \( (i,p) \in \hat{X} \) and \( y_{(i,p)} = 0 \) otherwise. By the definition of \( \hat{X} \), \( \hat{g} \) is M₂-convex.
if and only if \( \hat{g}|_{\hat{X}} \) is M_2-convex. Note that \( \hat{g}|_{\hat{X}} \) is a VCSP-quadratic function of type \( \hat{A}|_{\hat{X}} := \{ \hat{A}_1 \cap \hat{X}, \hat{A}_2 \cap \hat{X}, \ldots, \hat{A}_r \cap \hat{X} \} \) and satisfies condition (**) \( \hat{g}|_{\hat{X}} \) is M_2-convex if and only if each connected component of \( G_{\hat{g}}^\infty \) is a complete multipartite graph with partition \( \hat{A}|_{\hat{X}} \). Indeed, the if part is obvious, and the only-if part follows from Corollary 2.9. Hence \( \hat{g} \) is M_2-convex if and only if each connected component of the subgraph \( G_{\hat{g}}^\infty [\hat{X}] \) of \( G_{\hat{g}}^\infty \) induced by \( \hat{X} \) is a complete multipartite graph with partition \( \hat{A} \).

Finally we note that the completeness of connected components of \( G_{\hat{g}}^\infty [X] \) is equivalent to that of connected components of \( G_{\hat{g}}^\infty [\hat{X}] \).

\[ \square \]

3 Algorithm for Decomposition

Let \( f \) be a VCSP-quadratic function of type \( \mathcal{A} \) satisfying (**) and \( \mathcal{B} \) be the set of non-isolated connected components of \( G_f^\infty \). In this section, we devise an \( O(n^6) \)-time algorithm for Testing M_2-Representability for the case of \( r(G_f^\infty) = r \) or \( r + 1 \) (Section 3.1), and an \( O(n^5) \)-time algorithm for Decomposition for the case of \( r(G_f^\infty) \geq r + 2 \) (Section 3.2). By Corollary 2.9, we can assume that each connected component of \( G_f^\infty \) is a complete multipartite graph with partition \( \mathcal{A} \).

3.1 Case of \( r(G_f^\infty) = r \) or \( r + 1 \)

By Theorem 2.2, \( f \) is M_2-convex if and only if \( f \) admits a decomposition \( f = f_1 + f_2 \), where \( f_1 \) and \( f_2 \) are linear functions with dom \( f_1 = \text{dom} \delta_{\mathcal{A}} \) and dom \( f_2 = \text{dom} \delta_{\mathcal{B}} \). We assume that \( f_1 \) and \( f_2 \) are represented by \( (a_{ij})_{i,j} \) and \( (b_{ij})_{i,j} \), respectively, as in (9). Then, by Lemma 2.5, \( f \) is M_2-convex if and only if the following system of equations (20)–(22) has a solution:

\[
\begin{align*}
(a_{ij})_{i,j} & \text{ satisfies (12),} & (20) \\
(b_{ij})_{i,j} & \text{ satisfies (11) if } r(G_f^\infty) = r + 1 \text{ and (12) if } r(G_f^\infty) = r, & (21) \\
a_{ij} + b_{ij} & = F_{pq}(d,e) & (22)
\end{align*}
\]

for \( i \in A_p \) corresponding to \( d \in D_p \) and \( j \in A_q \) corresponding to \( e \in D_q \) such that \( F_{pq}(d,e) < +\infty \). Furthermore, for (20) and (21), the number of equations can be reduced to \( O(n^2) \) by the following observation.

Lemma 3.1 (see e.g., [17]). Suppose \( (a_{ij})_{i \in [N],j \in [M]} \), where \( N \) and \( M \) are positive integers with \( N,M \geq 2 \). Then \( a_{ij} + a_{kl} = a_{il} + a_{kj} \) holds for every distinct \( i,k \in [N] \) and distinct \( j,l \in [M] \) if and only if \( a_{ij} + a_{i+1,j+1} = a_{i+1,j} + a_{i,j+1} \) holds for every \( i \in [N-1] \) and \( j \in [M-1] \).

This observation enables us to check the solvability of (20)–(22) in \( O(n^6) \) time. Indeed, the number of variables \( (a_{ij})_{i,j} \) and \( (b_{ij})_{i,j} \) is \( O(n^2) \) and the number of equations is \( O(n^2) \). Therefore, the Gaussian elimination can be done in \( O((n^2)^3) = O(n^6) \) time (see e.g., [24], Section 4.3).

3.2 Case of \( r(G_f^\infty) \geq r + 2 \)

3.2.1 Outline

To describe our algorithm, we need the concept of restriction of a VCSP-quadratic function. Let \( f \) be a VCSP-quadratic function of type \( \mathcal{A} = \{ A_1, A_2, \ldots, A_r \} \). For \( Q \subseteq [r] \), let \( \mathcal{A}_Q := \{ A_p \}_{p \in Q} \)
and \( A_Q := \bigcup_{q \in Q} A_q \). For \( Q \subseteq [r] \), the restriction \( f_Q : \{0, 1\}^{A_Q} \to \overline{\mathbb{R}} \) of \( f \) to \( Q \) is a VCSP-quadratic function of type \( A_Q \) defined by
\[
f_Q(x) := \begin{cases} 
\sum_{i \in A_Q} a_i x_i + \sum_{i, j \in A_Q (i < j)} a_{ij} x_i x_j & \text{if } \sum_i x_i = |Q|, \\
+\infty & \text{otherwise}.
\end{cases}
\] (23)

**Lemma 3.2.** If \( f \) is \( M_2 \)-convex, so is the restriction \( f_Q \) for each \( Q \subseteq [r] \).

**Proof.** By Proposition 2.8, we can assume that \((a_{ij})_{i,j \in [n]}\) satisfies the anti-ultrametric property. Hence \((a_{ij})_{i,j \in A_Q}\) also has the anti-ultrametric property. Therefore (23) is naturally viewed as a decomposition of \( f_Q \) to an M-convex function and a linear function on \( \text{dom } \delta_{A_Q} \). \( \square \)

We abbreviate \( \Pi_A \) and \( \Pi_{A_Q} \) to \( \Pi \) and \( \Pi_Q \), respectively. Let \( B_Q := \{ B' \in \Pi_Q \mid \exists B \in \mathcal{B} \text{ with } B \cap A_Q = B' \} \). If \( f \) is \( M_2 \)-convex, then \( f_Q \) can also be represented in a form similar to (24), i.e.,
\[
f_Q = \left( \sum_{X \in \mathcal{P}_f} c_{f_Q}(X) \ell_X + \delta_{B_Q} \right) + \delta_{A_Q} + \text{(linear function)},
\] (24)

where \( \ell_X \), \( \delta_{A_Q} \), and \( \delta_{B_Q} \) are defined on \( \{0, 1\}^{A_Q} \). For nonempty \( Q \subseteq [r] \), we define \( \sim_Q \) for elements of \( \Pi/\sim \) by:
\[
X \sim_Q Y \Leftrightarrow \{((X) \cap A_Q) \cap X, (X) \cap A_Q) \setminus X\} = \{((Y) \cap A_Q) \cap Y, (Y) \cap A_Q) \setminus Y\};
\]
recall (6) for the notation \( \langle X \rangle \). Note that \( \{((X) \cap A_Q) \cap X, (X) \cap A_Q) \setminus X\} \) is well-defined for \( X \in \Pi/\sim \), and that every \( X \in \Pi_Q/\sim \) can also be regarded as an element in \( \Pi/\sim \).

The following lemma says that partial information about \( \mathcal{P}_f \) and \( c_f \) is obtained from \( \mathcal{P}_{f_Q} \) and \( c_{f_Q} \).

**Lemma 3.3.** Suppose that \( f \) is \( M_2 \)-convex. Then the coefficient \( c_{f_Q} \) in (24) is given by
\[
c_{f_Q}(Y) = \sum \{c_f(X) \mid X \in \mathcal{P}_f, X \sim_Q Y\} \quad (Y \in \mathcal{P}_{f_Q}).
\]

**Proof.** For \( X \in \Pi/\sim \) and \( Q \subseteq [r] \), let \( (\ell_X)_Q \) be the restriction of \( \ell_X \) to \( \{0, 1\}^{A_Q} \). Note that \( (\ell_X)_Q + \delta_{A_Q} \) is linear on \( \text{dom } \delta_{A_Q} \) if \( X \cap A_Q \not\in \Pi_Q \), and that \( (\ell_X)_Q + \delta_{A_Q} = (\ell_Y)_Q + \delta_{A_Q} \Leftrightarrow X \sim_Q Y \)
if \( X \cap A_Q, Y \cap A_Q \in \Pi_Q \). Therefore
\[
f_Q = \left( \sum_{X \in \mathcal{P}_f} c_f(X)(\ell_X)_Q + \delta_{B_Q} \right) + \delta_{A_Q} + \text{(linear function)}
\]
\[
= \left( \sum_{Y \in \mathcal{P}_f} \ell_Y \cdot \sum \{c_f(X) \mid X \in \mathcal{P}_f, X \sim_Q Y\} + \delta_{B_Q} \right) + \delta_{A_Q} + \text{(linear function)}.
\]

Hence it must hold that \( c_{f_Q}(Y) = \sum \{c_f(X) \mid X \in \mathcal{P}_f, X \sim_Q Y\} \) for \( Y \in \mathcal{P}_{f_Q} \). \( \square \)

Our algorithm to obtain decomposition (5) is outlined as follows, where we abbreviate \{\( \{p, q\}\) and \{\( p \)\} to \( pq \) and \( p \), respectively, and also \( \mathcal{P}_{f_{pq}} \) and \( c_{f_{pq}} \) to \( \mathcal{P}_{pq} \) and \( c_{pq} \), respectively:
We obtain a decomposition of the restriction $f_Q$ for $Q = \{1, 2\}, \{1, 2, 3\}, \ldots, \{1, 2, 3, \ldots, r\}$ in turn:

$$f_Q = \left( \sum_{X \in \mathcal{F}_Q} c_Q(X)x_X + \delta_{B_Q} \right) + \delta_{A_Q} + \text{(linear function)}. \quad (25)$$

In the initial case for $Q = \{1, 2\}$, we can obtain the decomposition [25] by Algorithm 1 (Section 3.2.2).

To construct the decomposition [25] for $Q = [r']$ from that for $Q = [r'-1]$, we first compute $(\mathcal{P}_{pr'}, c_{pr'})$ for all $p \in [r'-1]$ by Algorithm 1 and then, with this information, extend $(\mathcal{P}_{[r'-1]}, c_{[r'-1]})$ to $(\mathcal{P}_{[r']}, c_{[r']})$ by Algorithm 2 (Section 3.2.3). This procedure is justified by Lemmas 3.2 and 3.3.

We perform the above extension step for $r' = 3$ to $r' = r$, to arrive at the decomposition (5) of $f$. This is described in Algorithm 3.

### 3.2.2 Initial case ($r = 2$)

To compute $\mathcal{P}_{pq}$ and $c_{pq}$ for all distinct $p, q \in [r]$, we consider DECOMPOSITION algorithm for the case of $r = 2$. Namely $A = \{A_1, A_2\}$. Note that $\Pi = \Pi_{\{A_1, A_2\}} = \{X \subseteq [n] \mid \emptyset \neq X \cap A_p \neq A_p \text{ for } p = 1, 2\}$.

**Algorithm 1** (for DECOMPOSITION in the case of $r = 2$):

**Input:** A VCSP-quadratic function $f$ of type $\{A_1, A_2\}$.

**Step 0:** Define $\alpha^* := \min_{i,j \in [n]} a_{ij}$.

**Step 1:** For $i \in [n]$ with $b_i := \min_{j \in [n]} a_{ij} - \alpha^* > 0$, update $a_{ij} := a_{ij} - b_i$ for $j \in [n] \setminus \{i\}$ in turn.

**Step 2:** Let the distinct finite values of $a_{ij}$ ($i \in A_1, j \in A_2$) be given by $\alpha_1 > \alpha_2 > \cdots > \alpha_m = \alpha^*$. For $\alpha \in \mathbb{R}$, define a graph $G_{\alpha} := ([n], E_{\alpha})$ by $E_{\alpha} := \{(i, j) \mid i \in A_1, j \in A_2, \alpha \leq a_{ij}\}$. If, for some $\alpha \in \{\alpha_1, \alpha_2, \ldots, \alpha_{m-1}\}$, a (non-isolated) connected component of $G_{\alpha}$ is not a complete bipartite graph, then output "f is not $M_2$-convex" and stop.

**Step 3:** For $s \in [m-1]$, denote by $\mathcal{L}^{s}$ the set of non-isolated connected components $L$ of $G_{\alpha_s}$ satisfying $L \not\supseteq B$ for all $B \in \mathcal{B}$. For $L \in \mathcal{L}^{s} \setminus \mathcal{L}^{s-1}$ with $s \in [m-1]$, let $\alpha_L := \alpha_s$, where $\mathcal{L}^0 := \emptyset$. Define a laminar family $\mathcal{L}$ by $\mathcal{L} := \bigcup_{s=1}^{m-1} \mathcal{L}^{s}$. For $L \in \mathcal{L}$, define $c : \mathcal{L} \to \mathbb{R}_{++}$ by $c(L) := (\alpha_L - \alpha_{L^+})/2$, where $L^+$ is the minimal element in $\mathcal{L}$ properly containing $L$ if $L$ is not maximal, and $\alpha_{L^+} := \alpha^*$ if $L$ is maximal.

**Step 4:** Turn $c : \mathcal{L} \to \mathbb{R}_{++}$ to $\mathcal{L} \sim \to \mathbb{R}_{++}$ by defining the value $c$ on an equivalence class as the sum of $c(L)$ over (at most two) members $L$ in the equivalence class (cf. [14]). Output $\mathcal{P} := \mathcal{L} \sim$ and $c$.

Note that, for distinct $L, L' \in \Pi$, we have $L \sim L' \iff L = [n] \setminus L'$.

**Proposition 3.4.** Algorithm 1 solves DECOMPOSITION in $O(n^2)$ time.

For the proof of the validity of Algorithm 1, we need the following lemma.
Lemma 3.5 ([17] Lemma 4.2). Suppose that \((a_{ij})_{i,j\in[n]}\) satisfies the anti-tree metric property (10). If \(\min_{j\in[n]} a_{ij} = \alpha^*\) holds for all \(i \in [n]\), then \((a_{ij})_{i,j\in[n]}\) satisfies the anti-ultrametric property (13).

Proof of Proposition 3.4 (Validity). After the updates in Step 1, \(f\) still keeps the form \(f(x) = \sum_{i,j} a'_{ij} x_i x_j + \delta_A + (\text{linear function})\), since \(\sum_j b_i x_i x_j + \delta_A = rb_i x_i + \delta_A\) is linear. By the operation in Step 1, it holds that

\[
\min_{j'\in A_2} \min_{i'\in A_1} a_{i'j'} = \min_{i'\in A_1} a_{i'j} = \alpha^*
\]

for any \(i \in A_1\) and \(j \in A_2\).

Suppose that \(f\) is \(M_2\)-convex. By Proposition 2.8, we have \(f(x) = \sum a'_{ij} x_i x_j + \delta_A + (\text{linear function})\), where \((a'_{ij})_{i,j}\) are B-coefficients satisfying the anti-ultrametric property. Since \(\sum a_{ij} x_i x_j = \sum a'_{ij} x_i x_j + (\text{linear function})\), for some \(b'_i, b'_j \in \mathbb{R}\) we have \(a_{ij} = a'_{ij} = b'_i + b'_j\) for every \(i \in A_1, j \in A_2\). Let \(\overline{a}_{ij} := a'_i + b'_i + b'_j\) for distinct \(i, j \in [n]\). Then \(a_{ij} = \overline{a}_{ij}\) holds for any \(i \in A_1, j \in A_2\), and \((\overline{a}_{ij})_{i,j}\) are B-coefficients satisfying the anti-tree metric property (10).

We can redefine B-coefficients \((\overline{a}_{ij})\) so as to meet the anti-ultrametric property while maintaining \(a_{ij} = \overline{a}_{ij}\) for any \(i \in A_1, j \in A_2\), as follows. Let \(\beta := \alpha^* - \min \overline{a}_{ij}\). Note that \(\beta \geq 0\) holds by (26) and \(a_{ij} = \overline{a}_{ij}\) for \(i \in A_1, j \in A_2\). Suppose \(\beta = 0\). Then \(\min \overline{a}_{ij} = \alpha^*\) holds. Furthermore, we have \(\min \overline{a}_{ij} = \alpha^*\) for every \(i \in [n]\). Hence, by Lemma 3.5 \((\overline{a}_{ij})_{i,j}\) satisfies the anti-ultrametric property, as required.

Suppose \(\beta > 0\). By \(\overline{a}_{ij} \geq \alpha^*\) for any \(i \in A_1, j \in A_2\), if \(\overline{a}_{i^*,j^*} = \alpha^* - \beta\), then \(i^*, j^* \in A_1\) or \(i^*, j^* \in A_2\) holds. Without loss of generality, we assume \(i^*, j^* \in A_1\). Since \((\overline{a}_{ij})_{i,j}\) satisfies (10), it holds that \(\overline{a}_{i^*,j^*} = \alpha^* + \beta\). Let \(b_i := \beta/2\) if \(i \in A_1\) and \(b_i := -\beta/2\) if \(i \in A_2\). We redefine \(\overline{a}_{ij}\) as \(\overline{a}_{ij} := \overline{a}_{ij} + b_i + b_j\). Then it is easy to see that \(a_{ij} = \overline{a}_{ij}\) holds for any \(i \in A_1, j \in A_2\), and that \((\overline{a}_{ij})_{i,j}\) are B-coefficients satisfying (10). Furthermore \(\alpha^* - \min \overline{a}_{ij} = 0\) holds. Hence, by Lemma 3.5 \((\overline{a}_{ij})_{i,j}\) satisfies the anti-ultrametric property, as required.

Thus, by Lemma 2.6 there is a L-bilinear family \(L\) and a positive weight \(\gamma : \mathcal{L} \to \mathbb{R}_+\) associated with \((\overline{a}_{ij})_{i,j}\), and we can assume \(L' \subseteq \Pi\). Since \(a_{ij} = \overline{a}_{ij}\) holds for any \(i \in A_1, j \in A_2\), each non-isolated connected component of \(G_\alpha\) is a complete bipartite graph for any \(\alpha \in \{\alpha_1, \alpha_2, \ldots, \alpha_{m-1}\}\). If there is \(L \in \mathcal{L}\) with \(L \sim B\) for some \(B \in \mathcal{B}\), we can redefine \(L', \gamma', (\overline{a}_{ij})_{i,j}\) as \(L' := L \setminus \{L\}, \gamma' := \gamma|_L\), and \((\overline{a}_{ij})_{i,j} := \text{the quadratic coefficients represented by } (L', \gamma)\), respectively, since \(\sum_{i,j \in L} x_i x_j + \delta_A + \delta_B = (\text{linear function})\). Thus we obtain \(L = \mathcal{L}\) and \(c = \gamma|_2\).

Let \(\mathcal{P} := \mathcal{L}/\sim\). Note that \(\mathcal{P}\) is L-bilinear. Since \(L \sim L' \iff L = L' = [n] \setminus L'\) holds for \(L, L' \in \Pi = \Pi[A_1, A_2]\), we can regard \(\mathcal{P}\) as a subset of \(\mathcal{L}\). By Lemma 2.10 \((\sum_{L \in \mathcal{P}} c(L) f_L(x) + \delta_B(x)) + \delta_A(x) = \sum a_{ij} x_i x_j + (\text{linear function})\). Thus, it holds that \(f = (\sum_{L \in \mathcal{P}} c(L) f_L + \delta_B) + \delta_A + (\text{linear function})\). By Theorem 2.2 we obtain \(\mathcal{P} = \mathcal{P}_L\) and \(c = c_f\).

By the above argument, if Algorithm 1 reaches Step 3, then it is revealed that \(f\) admits a decomposition \(f(x) = \sum a_{ij} x_i x_j + \delta_A(x) + \delta_B(x) + (\text{linear function})\) for B-coefficients \(a_{ij}\) with the anti-ultrametric property. This guarantees that \(f\) is \(M_2\)-convex by Theorem 2.2.

(Complexity). It is clear that Step 0 and Step 1 can be done in \(O(n^2)\) time, and that Step 4 of Algorithm 1 can be done in \(O(|\mathcal{L}|) = O(n)\) time.

We show that Steps 2 and 3 can be done in \(O(n^2)\) time, though they can be naively done in \(O(n^3)\) time. Our approach is based on the idea used in [17] (see also [33, 34]). Suppose that \(f\) is \(M_2\)-convex, and that we are given some \(L \in \mathcal{L}\). We can compute in \(O(|\mathcal{L}|^2)\) time the (disjoint) set \(\mathcal{L}'\) of all maximal members in \(\mathcal{L}\) properly contained in \(L\), sketched as follows. Let \(L_1 := A_1 \cap L\) and \(L_2 := A_2 \cap L\). Observe that \(a_L = \min_{j' \in L_2} a_{ij'} = \min_{j' \in L_1} a_{ij'}\) holds for each \(i \in L_1\) and \(j \in L_2\). Choose arbitrary \(i \in L_1\), and compute \(\text{argmin}_{j' \in L_2} a_{ij'}\). If \(L_2 = \text{argmin}_{j' \in L_2} a_{ij'}\), then there is no member of \(\mathcal{L}'\) containing \(i\). Otherwise, choose \(j' \in L_2 \setminus \{j' \in L_2: a_{ij'}\} \), and compute
argmin'\in\mathcal{L}, a_{ij}'. Then one can see that the (unique) member \(L'\) in \(\mathcal{L}\) containing \(i,j\) is equal to the union of \(L_1\setminus \text{argmin}'\in\mathcal{L}, a_{ij}'\) and \(L_2\setminus \text{argmin}'\in\mathcal{L}, a_{ij}'\). By repeating this procedure, we obtain \(L'\) in \(O(|\mathcal{L}|^2)\) time. Thus, starting from \(L = [n]\), we recursively apply this procedure to the \(L's\) so far obtained, and finally get \(L\) (as well as \(c : \mathcal{L} \rightarrow \mathbb{R}_{++}\)) in total \(O(n^2)\)-time. Even when \(f\) is not \(M_2\)-convex, we can apply this procedure and detect the non-\(M_2\)-convexity. Indeed, define \(a_{ij}' = a_L\) for the final \(L\) containing \(i,j\) in the above procedure. Then \(a_{ij}' = a_{ij}\) holds for any \(i,j\) if and only if \((a_{ij})_{i,j}\) satisfies the anti-ultrametric property, i.e., \(f\) is \(M_2\)-convex. \(\square\)

### 3.2.3 General case \((r \geq 3)\)

To obtain the decomposition \([25]\) of the restriction \(f_Q\) for \(Q = \{1, 2\}, \{1, 2, 3\}, \ldots, \{1, 2, 3, \ldots, r\}\) in turn, we need to extend \((\mathcal{P}_{[r-1]}, c_{[r-1]})\) to \((\mathcal{P}_{[r]}, c_{[r]})\) with the use of \((\mathcal{P}_{pr}, c_{pr})\) \((p \in [r'-1])\) for \(r' = 3, \ldots, r\). Algorithm 2 corresponds to this extension step. It is noted that, if \(\mathcal{P}\) is laminar, then \(|\mathcal{P}|\) is at most \(2n = 2A_{[r]}\) (see e.g., [32] Theorem 3.5).

**Algorithm 2 (for extending \(f'\) to \(f\)):**

**Input:** A VCSP-quadratic function \(f\) of type \(\mathcal{A}\) and restriction \(f' := f_{[r-1]}\) given as

\[
f'(X) = \left( \sum_{X \in \mathcal{P}'} c'(X)\ell_X + \delta_{\mathcal{B}_{[r-1]}} \right) + \delta_{\mathcal{A}_{[r-1]}} \quad \text{(linear function)}
\]

for a family \(\mathcal{P}' \subseteq \Pi_{[r-1]}/\sim\) with \(|\mathcal{P}'| \leq 2A_{[r-1]}\) and a positive weight \(c'\) on \(\mathcal{P}'\).

**Output:** Either detect the non-\(M_2\)-convexity of \(f\), or obtain expression

\[
f = \left( \sum_{X \in \mathcal{P}} c(X)\ell_X + \delta_{\mathcal{B}} \right) + \delta_{\mathcal{A}} \quad \text{(linear function)}
\]

with \(\mathcal{P} \subseteq \Pi/\sim\) satisfying \(|\mathcal{P}| \leq 2n = 2A_{[r]}\) and a positive weight \(c\) on \(\mathcal{P}\).

**Step 1:** For each \(p \in [r-1]\), execute Algorithm 1 for \(f_{pr}\). If Algorithm 1 returns “\(f_{pr}\) is not \(M_2\)-convex” for some \(p \in [r-1]\), then output “\(f\) is not \(M_2\)-convex” and stop. Otherwise, obtain \(\mathcal{P}_{pr}\) and \(c_{pr}\) for all \(p \in [r-1]\). Let \(\mathcal{P} := \emptyset\).

**Step 2:** If \(\mathcal{P}' = \emptyset\), go to Step 3. Otherwise, do the following: Let \(X_0\) be an element of \(\mathcal{P}' \cup \mathcal{B}_{[r-1]}\) such that \(\langle X_0 \rangle\) is maximal. Let \(\{p_1, p_2, \ldots, p_k\}\) be the set of indices \(p \in [r-1]\) with \(\langle X_0 \rangle = A_{\{p_1, p_2, \ldots, p_k\}}\). If there exist \(X \in \Pi/\sim\) and \(X_i \in \mathcal{P}_{pr}\) \((i = 1, 2, \ldots, k)\) such that \(X \sim_{[r-1]} X_0\) and \(X \sim_{pr} X_i\) for each \(i \in [k]\), then go to Step 2-1. Otherwise, go to Step 2-2.

**2-1:** Update as

\[
\mathcal{P} \leftarrow \mathcal{P} \cup \{X\},
\]

\[
c(X) \leftarrow \min\{c'(X_0), c_{pr}(X_1), c_{pr}(X_2), \ldots, c_{pr}(X_k)\},
\]

\[
c'(X_0) \leftarrow c'(X_0) - c(X),
\]

\[
c_{pr}(X_i) \leftarrow c_{pr}(X_i) - c(X) \quad (i \in [k]),
\]

\[
\mathcal{P}' \leftarrow \mathcal{P}' \setminus \{X_0\} \quad \text{if} \quad c'(X_0) = 0,
\]

\[
\mathcal{P}_{pr} \leftarrow \mathcal{P}_{pr} \setminus \{X_i\} \quad \text{if} \quad c_{pr}(X_i) = 0 \quad (i \in [k]),
\]

and go to Step 2, where \(c' : \mathcal{P}' \rightarrow \mathbb{R}_{++}\) is regarded as \(\mathcal{P}' \cup \mathcal{B}_{[r-1]} \rightarrow \mathbb{R}_{++}\) by \(c'(B) := +\infty\) for \(B \in \mathcal{B}_{[r-1]}\).
2-2: If $X_0 \in \mathcal{P}'$, update as $\mathcal{P} \leftarrow \mathcal{P} \cup \{X_0\}$, $\mathcal{P}' \leftarrow \mathcal{P}' \setminus \{X_0\}$, and $c(X_0) \leftarrow c'(X_0)$. Go to Step 2.

**Step 3:** Update as $\mathcal{P} \leftarrow \mathcal{P} \cup \bigcup_{i \in [k]} \mathcal{P}_{pr}$, and $c(X) \leftarrow c_{pr}(X)$ for $i \in [k]$ and $X \in \mathcal{P}_{pr}$. If $|\mathcal{P}| \leq 2n$, then output $\mathcal{P}$ and $c$. Otherwise, output “$f$ is not $M_2$-convex.”

The following proposition shows that Algorithm 2 works as expected.

**Proposition 3.6.** If $f$ is $M_2$-convex, and $\mathcal{P}' = \mathcal{P}_f$, and $c' = c_f$ hold, then $\mathcal{P} = \mathcal{P}_f$ and $c = c_f$ hold. Furthermore, Algorithm 2 runs in $O(n^4)$ time.

**Proof.** (Validity). First we show that the output $\mathcal{P}$ and $c$ satisfy $f = (\sum_{X \in \mathcal{P}} c(X)\ell_X + \delta_B) + \delta_A + (\text{linear function})$. Let $x|_Q := (x_i)_{i \in A_Q} \in \{0,1\}^{A_Q}$ for $x = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$ and $Q \subseteq [r]$. Then we have

$$f(x) = \begin{cases} \sum_{i,j \in A_{r-1}} a_{ij}x_ix_j + \sum_{p \in [r-1]} \sum_{i,j \in A_{pr}} a_{ij}x_ix_j + (\text{linear function}) & \text{if } \sum_i x_i = r, \\ \infty & \text{otherwise} \end{cases}$$

for $x \in \{0,1\}^n$. Hence we obtain

$$f(x) = \sum_{X \in \mathcal{P}'} c'(X)(\ell_X + \delta_{f'}) (x|_{[r-1]}) + \sum_{p \in [r-1]} \sum_{X \in \mathcal{P}_{pr}} c_{pr}(X)(\ell_X + \delta_{f_{pr}}) (x|_{[r]}) + (\text{linear function}).$$

We regard $c$ and $c_f$ as functions on $\Pi/\sim$ by defining $c(X) := 0$ for $X \in (\Pi/\sim) \setminus \mathcal{P}$ and $c_f(X) := 0$ for $X \in (\Pi/\sim) \setminus \mathcal{P}_f$, respectively. Since $f = (\sum_{X \in \mathcal{P}} c(X)\ell_X + \delta_B) + \delta_A + (\text{linear function})$ holds by the above argument, it suffices to prove $(i)$ $c(X) = c_f(X)$ for $X$ obtained in Step 2-1, $(ii)$ $c(X_0) = c_f(X_0)$ for $X_0$ obtained in Step 2-2 if $X_0 \in \mathcal{P}'$, and $(iii)$ $c(X) = c_f(X)$ for $X$ obtained in Step 3.

(i). Let $\hat{c} := \min \{c'(X_0), c_{pr}(X_1), c_{pr}(X_2), \ldots, c_{pr}(X_k)\}$. We prove $c_f(X) \leq \hat{c}$ holds. Indeed, by Lemma 3.3, we have $c'(X_0) \geq c_f(X)$ and $c_{pr}(X_i) \geq c_f(X)$ for $i \in [k]$.

Suppose, to the contrary, that $c_f(X) < \hat{c}$ holds. We show that all $Y_0 \in \mathcal{P}_f \cup B$ with $Y_0 \sim_{[r-1]} X$ satisfy $Y_0 \sim X$. This contradicts $c_f(X) < c'(X_0)$ and Lemma 3.3. Indeed, if $c'(X_0) = +\infty$, there is $B \in B$ with $B \sim_{[r-1]} X$. By $X_i \in \mathcal{P}_{pr}$ and $X_i \not\sim_{pr} B$ for all $i \in [k]$, we have $B \not\sim X$. If $c'(X_0) < +\infty$, there is $Y_0 \in B$ satisfying $Y_0 \sim_{[r-1]} X$ and $Y_0 \not\sim X$ by Lemma 3.3.

Assume $c'(X_0) < +\infty$. Take any $Y_0 \in \mathcal{P}_f$ with $Y_0 \sim_{[r-1]} X$. (By replacing $Y_0$ with $B$ satisfying $B \sim_{[r-1]} X$, we can treat the other case $c'(X_0) = +\infty$.) By $c_{pr}(X_i) > c_f(X)$ ($i \in [k]$) and Lemma 3.3 for every $i \in [k]$ there is $Y \in \mathcal{P}_f$ with $Y \sim_{[r-1]} X_i$ and $Y \not\sim X$. Take $Y \in \mathcal{P}_f$ satisfying $Y \not\sim X$ with $\{i \in [k] \mid Y \sim_{pr} X_i\}$ maximal among elements $Y' \in \mathcal{P}_f$ satisfying


\[ Y' \not\sim X. \] Let \( I := \{ i \in [k] \mid Y_i \sim_{p,r} X_i \} (\neq \emptyset). \) By the maximality of \( \langle X_0 \rangle \) and \( Y' \not\sim X \), we have \( [k] \setminus I \not\sim \emptyset; \) otherwise \( \langle Y \rangle \cap A_{[r-1]} \supseteq \langle X_0 \rangle \), contradicting the maximality of \( \langle X_0 \rangle \).

Choose arbitrary \( j \in [k] \setminus I \). Then there is \( Y_j \in P_j \) with \( Y_j \sim_{p,r} X_j \) and \( Y_j \not\sim X \). Furthermore, by the maximality of \( I \), there is \( i \in I \) such that \( Y_j \not\sim_{p,r} X_i \). Hence \( Y_j \not\sim_{p,r} Y' \not\sim_{p,r} X \) holds. In the following, we denote \( Y \) by \( Y_i \).

Since \( Y_i, Y_j, Y_0 \in P_j \), now the set of representatives \( \{ Y_i, Y_j, Y_0 \} \subseteq \Pi \) forms a \( \mathcal{B} \)-laminarizable family, and hence can be assumed to be laminar. We can also assume \( Y_i \cap A_{p_i} = Y_0 \cap A_{p_0}(\neq \emptyset) \) and \( Y_j \cap A_{p_j} = Y_0 \cap A_{p_0}(\neq \emptyset) \). Indeed, \( Y_i \cap A_{p_i} \neq Y_0 \cap A_{p_0}, \) means \( \langle Y \rangle \cap A_{p_i} = Y_0 \cap A_{p_0}. \) By the laminarity of \( Y_i \) and \( Y_0 \), we have \( Y_i \cap Y_0 = \emptyset \). Hence \( \langle Y \rangle \cap Y_0 \) is also laminar. Furthermore, note \( Y_i \cap A_r = Y_j \cap A_r(\neq \emptyset) \) by \( Y_i \sim_{p,r} Y_i \) and \( Y_j \sim_{p,r} X_j \).

By \( Y_0 \not\sim_{p,r} X \) \( \not\sim_{p,r} Y_i \) and laminarity, it holds that \( Y_0 \cap A_{p_0} \supseteq Y_i \cap A_{p_i} \) or \( Y_0 \cap A_{p_0} \subseteq Y_i \cap A_{p_i} \). Assume \( Y_0 \cap A_{p_0} = Y_i \cap A_{p_i} \subseteq Y_i \cap A_{p_i} \) (the argument for the other case is similar). Hence, by \( Y_0 \cap Y_i \neq \emptyset \) and \( Y_j \cap Y_i \neq \emptyset \), we have \( Y_0 \not\subseteq Y_i \not\subseteq Y_j \). By \( Y_j \not\sim_{p,r} Y_i \not\sim_{p,r} X \) and \( Y_i \not\sim Y_j \), we have \( Y_0 \cap A_{p_0} = Y_i \cap A_{p_i} \). Hence \( Y_0 \not\subseteq Y_i \not\subseteq Y_j \) holds. By \( Y_i \cap A_r = Y_j \cap A_r \), it holds that \( Y_i \cap A_r = Y_j \cap A_r = Y_0 \cap A_r \). This means \( Y_0 \sim X \), contradicting \( c_f(X) < c_f(X_0) \) and Lemma 3.3 (ii). By Lemma 3.3, it holds that

\[
c'(X_0) = \sum \{ c_f(Y) \mid Y \subseteq \Pi \sim, \ Y \sim_{[r-1]} X_0 \} = c_f(X_0) + \sum \{ c_f(Y) \mid Y \subseteq \Pi \sim, \ \langle Y \rangle \supseteq A_r, \ X_0 = Y \cap A_{[r-1]} \}. \]

Here the second term must be zero. Otherwise, by Lemma 3.3, we would have found \( X_1, X_2, \ldots, X_k \) in Step 2. Therefore \( c'(X_0) = c_f(X_0) \) holds. Thus we obtain \( c(X) = c_{p,k}(X) = c_f(X) \) for any \( i \in [k] \) and \( X \in P_{p,k} \) by a similar argument as for (ii).

(Complexity). Note \( |\mathcal{P}'| = O(|A_{[r-1]}|) \) and \( |\mathcal{P}_{pr}| = O(|A_{pr}|) \) for any \( p \in [r-1] \). Step 1 can be done in \( O(n^2) \) time by Proposition 3.4. In Step 2, we first need to sort the elements in \( \mathcal{P}' \) with respect to set-inclusion ordering in \( O(|A_{[r-1]}| \log |A_{[r-1]}|) = O(n \log n) \) time (this is done only once). For each iteration, we search for \( \{ X_1, X_2, \ldots, X_k \} \) satisfying the conditions described in Step 2. This can be done in \( O(|\bigcup_p \mathcal{P}_{pr}|) = O(n + r |A_r|) \) time by using the structure of \( \mathcal{P}_{pr} \) as follows. Consider \( X_0 \) and \( X_i \in \mathcal{P}_{pr} (i \in [k]) \) as elements in \( \Pi \). Then construct \( \mathcal{F}_i := \{ X_i \cap A_r \mid X_i \in \mathcal{P}_{pr}, \ X_r \cap A_{p_r} = X_0 \cap A_{p_0} \} \) for each \( i \in [k] \) (in \( O(|\bigcup_i \mathcal{P}_{pr}|) \) time). Note that there exists \( \{ X_1, X_2, \ldots, X_k \} \) satisfying the conditions in Step 2 if and only if \( \bigcap_i \mathcal{F}_i \neq \emptyset \).

By the laminarity of \( \mathcal{P}_{pr}, \mathcal{F}_i \) is nested, and can be represented as \( \mathcal{F}_i = \{ F^1_i, F^2_i, \ldots, F^q_i \} \) for \( i \in [k] \), where \( F^1_i \supseteq F^2_i \supseteq \cdots \supseteq F^q_i \) (this chain can be obtained while constructing \( \mathcal{P}_{pr} \) in Algorithm 1). If \( \bigcap_i \mathcal{F}_i \neq \emptyset \), we can obtain \( F \in \bigcap_i \mathcal{F}_i \) (and desired \( X_i = (X_0 \cap A_{p_0}) \cup F \in \mathcal{P}_{pr} \) for each \( i \in [k] \) in \( O(|\bigcap_i \mathcal{F}_i|) \) time. Indeed, take the maximal elements \( F^1_1, F^1_2, \ldots, F^1_k \) in \( \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k \), respectively. For each \( i \) with \( F^1_i \supseteq \bigcap j F^1_j \), update \( \mathcal{F}_i \leftarrow \mathcal{F}_i \setminus \{ F^1_i \} \), and do the same thing. By repeating this procedure, we can verify \( \bigcap_i \mathcal{F}_i \neq \emptyset \) or obtain \( F \in \bigcap_i \mathcal{F}_i \). Furthermore we need to calculate \( \min\{ c'(X_0), c_{p,r}(X_1), c_{p,r}(X_2), \ldots, c_{p,r}(X_k) \} \); this is done in \( O(k) \) time. Since \( |\mathcal{P}'| + |\bigcup_p \mathcal{P}_{pr}| \) decreases at least by one in each iteration in Step 2, the number of iterations in Step 2 is bounded by \( O(|\mathcal{P}'| + |\bigcup_{p \in [r-1]} \mathcal{P}_{pr}|) = O(n + r |A_r|) \). Hence the running-time of Algorithm 2 is bounded by \( O(n^4) \).

\[
\square
\]

Our proposed algorithm for \textsc{Decomposition} can be summarized as follows.

**Algorithm 3 (for \textsc{Decomposition}):**

**Step 1:** Execute Algorithm 1 for the restriction \( f_{12} \). If Algorithm 1 returns \( f_{12} \) is not \( M_2 \)-convex,” then output “\( f \) is not \( M_2 \)-convex” and stop. Otherwise, obtain \( P_{12} \) and \( c_{12} \).
Step 2: For \( r' = 3, \ldots, r \), execute Algorithm 2 for \( (f_{[r']}, P_{[r'-1]}, c_{[r'-1]}) \). If Algorithm 2 returns “\( f_{[r']} \) is not M2-convex,” output “\( f \) is not M2-convex” and stop. Otherwise, obtain \( c_{[r']} \) and \( P_{[r']} \).

Step 3: Output \( P := P_{[r]} \) and \( c := c_{[r]} \).

**Theorem 3.7.** Algorithm 3 solves DECOMPOSITION in \( O(n^5) \) time.

**Proof.** Since the running-time of Algorithm 2 for \( r' \) is bounded by \( O(|A_{[r'+1]}|^4) \) by Proposition 3.6, the running-time of Algorithm 3 is bounded by \( O(n^5) \).

The validity of Algorithm 3 can be proved as follows. Suppose that Algorithm 3 stops at Step 1 or Step 2. By Propositions 3.4 and 3.6, \( f \) is not M2-convex. Hence Algorithm 3 works correctly.

Suppose that Algorithm 3 reaches Step 3. Since \( f_{[2]} = \left( \sum_{X \in P_{[2]}} c_{[2]}(X) \ell_X + \delta_{B_{[2]}} \right) + \left( \sum_{X \in P_{[r']}} c_{[r']}(X) \ell_X + \delta_{B_{[r']}} \right) + \delta_A \) (linear function) by Proposition 3.4, we obtain \( f_{[r']}' = \left( \sum_{X \in P_{[r']}} c_{[r']}(X) \ell_X + \delta_{B_{[r']}} \right) + \delta_A \) (linear function) for all \( r' = 2, 3, \ldots, r \) by a similar argument for (29) in the proof of Proposition 3.6. Thus we have \( f = \left( \sum_{X \in P_{[r]}} c_{[r]}(X) \ell_X + \delta_{B} \right) + \delta_A \) (linear function) holds. Furthermore, if \( f \) is M2-convex, then \( P_{[r]} = P_f \) and \( c_{[r]} = c_f \) hold by Lemma 3.2 and Proposition 3.6. Hence Algorithm 3 works correctly.

**4 Algorithm for LAMINARIZATION**

For a VCSP-quadratic function \( f \) of type \( A \), suppose that we have obtained \( P \subseteq \Pi/\sim \) by solving DECOMPOSITION. The next step for solving TESTING M2-REPRESENTABILITY is to check for the B-laminarity of \( P \), where \( B \) is the set of non-isolated connected components of \( G_f^\infty \). Take \( F \subseteq \Pi \) with \( F/\sim = P \); such \( F \) can be constructed easily from \( P \). The input of LAMINARIZATION consists of \( F \) and \( B \).

**4.1 Outline**

For families \( G, H \subseteq \Pi \), we say that \( G \) is equivalent to \( H \) if \( G \sim H \). A family \( G \subseteq \Pi \) is said to be \( B \)-cross-free if there is no crossing pair in \( G \) and each pair of \( X \in G \) and \( B \subseteq \Pi \) satisfies \( X \not\subseteq B \) or \( X \cap B = \emptyset \). A B-laminar family can be constructed easily from a \( B \)-cross-free family \( G \) by switching \( X \mapsto [n] \setminus X \) for appropriate \( X \in G \) (see e.g., [21] Section 2.2); this can be done in \( O(|G|) \) time. Thus, by \( X \sim [n] \setminus X \), our goal is to construct a \( B \)-cross-free family equivalent to the input family.

In this section, we devise a polynomial-time algorithm for constructing a desired \( B \)-cross-free family. Our algorithm makes use of weaker notions of \( B \)-cross-freeness, called 2- and 3-local \( B \)-cross-freeness. The existence of a 2-local \( B \)-cross-free family is characterized by the existence of a 2-locally \( B \)-cross-free family (Section 4.2). The existence of such a 2-locally \( B \)-cross-free family can be checked easily by solving a 2-SAT problem. If a 2-locally \( B \)-cross-free family exists, a 3-locally \( B \)-cross-free family also exists, and can be constructed in polynomial time (Section 4.4). From a 3-locally \( B \)-cross-free family, we can construct a desired \( B \)-cross-free family in polynomial time by the uncrossing operations (Section 4.3). Thus we solve LAMINARIZATION.

Without loss of generality, we assume that \( X \subseteq \{X\} \) for every \( X \) in the input \( F \) and no distinct \( X, Y \) with \( X \sim Y \) are contained in \( F \), i.e., \( |F| = |F/\sim| \). For \( X \in F \), let \( X := \{X\} \setminus X \); note \( X \sim \overline{X} \). For \( X, Y, Z \in \Pi \), we define \( XY := \{X\} \cap \{Y\} \) and \( XYZ := \{X\} \cap \{Y\} \cap \{Z\} \). For \( X \in F \cup B \) and \( Q \subseteq [r] \) with \( A_Q \subseteq \{X\} \), the partition line of \( X \) on \( A_Q \) is a bipartition \( \{X \cap A_Q, \overline{X} \cap A_Q\} \) of \( A_Q \). We also assume that \( |F/\sim| \) is at most \( 2n \) and that \( X \subseteq Y, X \subseteq \overline{Y}, \)
$X \supseteq Y$, or $X \supseteq \overline{Y}$ holds on $\langle XY \rangle$ for distinct $X, Y \in F \cup B$ with $\langle XY \rangle \neq \emptyset$, since otherwise $F$ is not $B$-laminarizable.

We can also assume throughout that both $\langle X \rangle \setminus \langle Y \rangle$ and $\langle Y \rangle \setminus \langle X \rangle$ are nonempty for all distinct $X, Y \in F \cup B$. Indeed, for each $X \in F \cup B$, we add a new set $A_X$ with $|A_X| = 2$ to the ground set $[n]$ and to the partition $\mathcal{A}$ of $[n]$; the ground set will be $[n] \cup \bigcup_{X \in F \cup B} A_X$ and the partition will be $\mathcal{A} \cup \{A_X \mid X \in F \cup B\}$. Define $X_+ := X \cup \{x\}$, where $x$ is one of the two elements of $A_X$, $F_+ := \{X_+ \mid X \in F\}$, and $B_+ := \{B_+ \mid B \in B\}$. Note $\langle X_+ \rangle = \langle X \rangle \cup A_X$ and $\langle X_+ \rangle \setminus \langle Y \rangle \neq \emptyset$ for all $X_+, Y_+ \in F_+ \cup B_+$. Then it is easily seen that there exists a $B$-cross-free family $\mathcal{L}$ with $\mathcal{L} \sim \mathcal{F}$ if and only if there exists a $B_+$-cross-free family $\mathcal{L}_+$ with $\mathcal{L}_+ \sim \mathcal{F}_+$.

### 4.2 2-local $B$-cross-freeness

For $A \subseteq [n]$, a pair $X, Y \subseteq [n]$ is said to be crossing on $A$ if $(X \cap Y) \cap A, (X \cap Y) \cap A$, and $(Y \cap X) \cap A$ are all nonempty. A family $\mathcal{G} \subseteq \Pi$ is said to be crossing-free on $A$ if there is no crossing pair on $A$ in $\mathcal{G}$. A family $\mathcal{G} \subseteq \Pi$ is called 2-locally $B$-cross-free if no $X, Y \in \mathcal{G}$ is crossing on $(X) \cup (Y)$ and each pair of $X \in \mathcal{G}$ and $B \in \mathcal{B}$ satisfies $X \supseteq B$ or $X \cap B = \emptyset$. A $B$-cross-free family is 2-locally $B$-cross-free.

The $LC$-graph $G(F, B) = (V(F, B), E_f \cup E_b)$ of the input $F$ is an undirected graph defined by

\[
V(F, B) := \{XY \mid X, Y \in F \cup B, X \neq Y\},
\]

\[
E_f := \{\{XY, XZ\} \mid Y \neq Z, (\langle Y \rangle \setminus \langle X \rangle) \cap \langle Z \rangle \neq \emptyset\},
\]

\[
E_b := \{\{XY, YX\} \mid \langle XY \rangle \neq \emptyset\},
\]

where $XY$ is an abbreviation of ordered pair $(X, Y)$. LC stands for Local Cross-freeness.

Note that the structure of LC-graph depends only on $\{\{X \mid X \in F \cup B\}$. We call an edge $e \in E_f$ a forward edge and an edge $e \in E_b$ a backward edge. A backward edge $e = \{XY, YX\}$ is said to be flipping (resp. non-flipping) if $X \subseteq Y$ or $X \supseteq Y$ (resp. $X \subseteq \overline{Y}$ or $X \supseteq \overline{Y}$) holds on $\langle XY \rangle$;

**Example 4.1.** Let $\{1, 2, \ldots, 12\}$ be the ground set, and $\{\{2p - 1, 2p\} \mid p \in \{1, 2, \ldots, 6\}\}$ a partition of $\{1, 2, \ldots, 12\}$. With $X := \{1, 3, 5\}, Y := \{3, 5, 7\}, Z := \{5, 7, 9\}$, and $W := \{5, 11\}$, we define a family $\mathcal{F} := \{X, Y, Z, W\} \subseteq \Pi$. Then the edge sets $E_f$ and $E_b$ of the LC-graph $G(F, \emptyset)$ are given by

\[
E_f = \{\{XY, XZ\}, \{ZX, ZY\}, \{WX, WY\}, \{YW, WZ\}\},
\]

\[
E_b = \{\{XY, YX\}, \{XZ, ZX\}, \{YZ, ZY\}, \{XW, WX\}, \{YW, WY\}, \{ZW, WZ\}\}.
\]

For example, the forward edge $\{XY, XZ\}$ exists since $\langle \langle Y \rangle \setminus \langle X \rangle \rangle \cap \langle Z \rangle = \{7, 8\} \neq \emptyset$ by $\langle X \rangle = \{1, 2, 3, 4, 5, 6\}, \langle Y \rangle = \{3, 4, 5, 6, 7, 8\}$, and $\langle Z \rangle = \{5, 6, 7, 8, 9, 10\}$. The backward edge $\{XY, YX\}$ exists since $\langle XY \rangle = \langle X \rangle \setminus \langle Y \rangle = \{3, 4, 5, 6\} \neq \emptyset$. This backward edge is flipping since $X = Y$ on $\langle XY \rangle$.
An LC-labeling is a function $s : V(F, B) \to \{0, 1\}$ such that

$$s(XY) = \begin{cases} 
  s(XZ) & \text{if } \{XY, XZ\} \text{ is a forward edge,} \\
  s(YX) & \text{if } \{XY, YX\} \text{ is a non-flipping backward edge,} \\
  1 - s(YX) & \text{if } \{XY, YX\} \text{ is a flipping backward edge,}
\end{cases}$$

(30)

$$s(BX) = 0 \quad \text{if } B \in B \text{ and } X \in F \cup B,$$

(31)

$$s(BX) = 0 \quad \text{if } B \in B \text{ and } X \in F \cup B,$$

(32)

Note that (32) imposes no condition if the partition lines of $X$ and $Y$ on $\langle XY \rangle$ are the same. Node $XY \in V(F, B)$ is said to be fixed if the value of an LC-labeling $s$ for $XY$ is determined as (31) or (32), that is, if $X \in B$ and $Y \in F \cup B$, or $\langle XY \rangle \neq \emptyset$ and the partition lines of $X$ and $Y$ on $\langle XY \rangle$ are different.

An LC-labeling $s$ transforms the family $F$ to another family $F^s$ equivalent to $F$, which is given by

$$F^s := \{X^s \mid X \in F\}$$

with

$$X^s := X \cup \bigcup \{\langle Y \rangle \setminus \langle X \rangle \mid Y \in F \cup B \text{ with } s(YX) = 1\}.$$  

(33)

(34)

Thanks to condition (30) on forward edges, we have

$$X^s \cap (\langle Y \rangle \setminus \langle X \rangle) = \emptyset \text{ for } Y \in F \cup B \text{ with } s(YX) = 0.$$  

(35)

**Proposition 4.2.** There exists a 2-locally $B$-cross-free family equivalent to $F$ if and only if there exists an LC-labeling $s$ in $G(F, B)$. To be specific, $F^s$ is a 2-locally $B$-cross-free family equivalent to $F$.

**Proof.** Suppose that an LC-labeling $s$ exists. Then $F^s \sim F$ holds. We show that $F^s$ is 2-locally $B$-cross-free. For $X, Y \in F$ with $\langle XY \rangle = \emptyset$, $\{X^s, Y^s\}$ is 2-locally $B$-cross-free if and only if $X^s \supseteq \langle Y \rangle$ or $X^s \cap \langle Y \rangle = \emptyset$ holds and $Y^s \supseteq \langle X \rangle$ or $Y^s \cap \langle X \rangle = \emptyset$ holds. This follows from (34) and (35). For $X, Y \in F$ with $\langle XY \rangle \neq \emptyset$, we assume $X \subseteq Y$ on $\langle XY \rangle$ (the argument for the case of $X \subseteq Y$ is similar). If $X = Y$ on $\langle XY \rangle$, $\{X^s, Y^s\}$ is 2-locally $B$-cross-free if and only if $(s(XY), s(YX)) = (1, 0)$ or $(0, 1)$. If $X \subsetneq Y$ on $\langle XY \rangle$, $\{X^s, Y^s\}$ is 2-locally $B$-cross-free if and only if $(s(XY), s(YX)) = (0, 1)$. This follows from (30) on backward edges and (32). For $B \in B$ and $X \in F$, if $\langle BX \rangle \neq \emptyset$ and $B \subseteq X$ on $\langle BX \rangle$, $\{B, X^s\}$ is 2-locally $B$-cross-free if and only if $(s(BX), s(XB)) = (0, 1)$. If $\langle BX \rangle \neq \emptyset$ and $B \subseteq X$ on $\langle BX \rangle$, $\{B, X^s\}$ is 2-locally $B$-cross-free if and only if $(s(BX), s(XB)) = (0, 0)$. If $\langle BX \rangle = \emptyset$, $\{B, X^s\}$ is 2-locally $B$-cross-free if and only if $(s(BX), s(XB)) = (0, 0)$ or $(1, 1)$. This follows from (30) on backward edges and (31) and (32).

Suppose that there exists a 2-locally $B$-cross-free family $G$ with $G \sim F$ for $X \in F$, let $X'$ denote the (unique) member of $G$ with $X' \sim X$, where we can assume $X' \cap \langle X \rangle = X$ since otherwise we may replace $X'$ by $[n] \setminus X'$ (this replacement preserves the 2-local $B$-cross-freeness).

From $G$, we define a function $s : V(F, B) \to \{0, 1\}$ by

$$s(XY) := \begin{cases} 
  1 & \text{if } X' \supseteq \langle Y \rangle \setminus \langle X \rangle, \\
  0 & \text{if } X' \cap (\langle Y \rangle \setminus \langle X \rangle) = \emptyset.
\end{cases}$$
Then $s$ is well-defined by the 2-local $B$-cross-freeness of $G$. Also, by the 2-local $B$-cross-freeness, $s$ satisfies (30)–(32), where (31) follows from the fact that $B \subseteq X$ or $B \cap X = \emptyset$ must hold for all $B \in B$ and $X \in G$ with $\langle BX \rangle \neq \emptyset$. Hence $s$ is an LC-labeling. □

An LC-labeling is nothing but a feasible solution for the 2-SAT problem defined by the constraints (30)–(32). Therefore we can check the existence of an LC-labeling $s$ greedily in $O(|E_f \cup E_b|) = \Theta(n^4)$ time as follows, where $XY$ is called a defined node if the value of $s(XY)$ has been defined.

1. For each fixed node $XY$, define $s(XY)$ according to (31) or (32).
2. In each connected component of $G(F, B)$, execute a breadth-first search from a defined node $XY$, and define $s(ZW)$ for all reached nodes $ZW$ according to (30). If a conflict in value assignment to $s(ZW)$ is detected during this process, output “there is no LC-labeling.”
3. If there is an undefined node, choose any undefined node $XY$, and define $s(XY)$ as 0 or 1 arbitrarily. Then go to 2.

4.3 3-local $B$-cross-freeness

A family $G \subseteq \Pi$ is called 3-locally $B$-cross-free if $G$ is 2-locally $B$-cross-free and $\{X, Y, Z\}$ is cross-free on $\langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle$ for all $X, Y, Z \in G$ with $\langle XYZ \rangle \neq \emptyset$. A $B$-cross-free family is 3-locally $B$-cross-free, and a 3-locally $B$-cross-free family is 2-locally $B$-cross-free, whereas the converse is not true (see Example 4.5). We write $X \subseteq^* Y$ to mean $X \subseteq Y$ on $\langle X \rangle \cup \langle Y \rangle$.

Our objective of this subsection is to give an algorithm for constructing a desired $B$-cross-free family from a 3-locally $B$-cross-free family equivalent to the input $F$. The algorithm consists of repeated applications of an elementary operation that preserves 3-local $B$-cross-freeness. The operation is defined by (36) below, and is referred to as the uncrossing operation to $X, Y$. By the 2-local $B$-cross-freeness of $G$, the two cases in (36) exhaust all possibilities for $X, Y \in G$.

**Proposition 4.3.** Suppose that $G$ is 3-locally $B$-cross-free. For $X, Y \in G$, define

$$
G' := \begin{cases} 
G \setminus \{X, Y\} \cup \{X \cap Y, X \cup Y\} & \text{if } X \subseteq^* Y \text{ or } Y \subseteq^* X, \\
G \setminus \{X, Y\} \cup \{X \setminus Y, Y \setminus X\} & \text{if } X \subseteq^* [n] \setminus Y \text{ or } [n] \setminus Y \subseteq^* X.
\end{cases}
$$

(36)

Then $G'$ is a 3-locally $B$-cross-free family equivalent to $G$.

The proof of Proposition 4.3 is given at the end of this subsection.

**Algorithm 4 (for constructing a $B$-cross-free family):**

**Input:** A 3-locally $B$-cross-free family $G$.

**Step 1:** While there is a crossing pair $X, Y$ in $G$, apply the uncrossing operation to $X, Y$ and modify $G$ accordingly.

**Step 2:** Output $G$. □

Note that Algorithm 4 does not alter the family $B$, since $\{B, X\}$ cannot be a crossing pair for any $B \in B$ and $X \in G$.

**Proposition 4.4.** Algorithm 4 runs in $O(n^2)$ time, and the output $G$ is $B$-cross-free.
Example 4.5. Let $A_1 := \{1, 2\}, A_2 := \{3, 4\}, A_3 := \{5, 6\}, A_4 := \{7, 8\}$, and $B := \emptyset$. For $X := \{1, 3, 5, 6\}, Y := \{1, 3, 4, 7\}$, and $Z := \{1, 5, 7, 8\}$, we have $\langle X \rangle = \{1, 2, 3, 4\}, \langle Y \rangle = \{1, 2, 7, 8\}$, and $\langle Z \rangle = \{1, 2, 5, 6\}$. As is easily verified, $\{X, Y, Z\}$ is 2-locally B-cross-free. However, it is not 3-locally B-cross-free. Indeed, $\langle XYZ \rangle = A_1 \neq \emptyset$ and $\{X, Y\}$ is crossing on $\langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle = \{1, 2, \ldots, 8\}$. □

Remark 4.6. It is worth mentioning that the uncrossing operation does not preserve 2-local B-cross-freeness. In Example 4.5, for instance, the uncrossing operation to $X$ results in $X \cap Y = \{1, 3\}$ and $X \cup Y = \{1, 3, 4, 5, 6, 7\}$. Then $X \cap Y = \{1, 3\}$ and $Z = \{1, 5, 7, 8\}$ is crossing on $\langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle = \emptyset$. □

The rest of this subsection is devoted to the proof of Proposition 4.3. We first note the following facts, which are also used in the proof of Proposition 4.10 in Section 4.4.

Lemma 4.7. Let $\mathcal{G}$ be a 2-locally B-cross-free family. A triple $\{X, Y, Z\} \subseteq \mathcal{G}$ is cross-free on $\langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle$ if one of the following conditions holds:

1. $\langle XY \rangle \neq \emptyset$, and $\{X, Y\}$ is cross-free on $\langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle$.
2. $\langle XY \rangle \not\subseteq \langle Z \rangle$, and $\langle XZ \rangle$ or $\langle YZ \rangle$ is nonempty.
3. The partition lines of $X, Y, Z$ on $\langle XYZ \rangle$ are not the same.
4. $\langle XY \rangle = \emptyset$, and there is a path $\langle XY, XY_1, \ldots, XY_k \rangle$ in $G(\mathcal{G}, B)$ such that $\{X, Y_k, Z\}$ is cross-free on $\langle X \rangle \cup \langle Y_k \rangle \cup \langle Z \rangle$.

Proof. Let $S := \langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle$. Note that $\{X, Y\}$ is 2-locally B-cross-free if and only if so is $\{n \setminus X, Y\}$. Hence, by appropriate replacement $X \mapsto [n] \setminus X$ and/or $Y \mapsto [n] \setminus Y$, we can assume $X \subseteq^* Y$; we often use such replacement in this proof.

1. By symmetry, it suffices to show that $\{X, Z\}$ is cross-free on $S$. We assume $X \subseteq^* Z$ (the argument for the case of $Z \subseteq^* X$ is similar). There are two cases: (i) $\langle XY \rangle \not\subseteq \langle Z \rangle$ and (ii) $\langle XY \rangle \not\subseteq \langle Z \rangle$. Note that $X \subseteq^* Z$ implies $Z \supseteq \langle X \rangle \setminus \langle Z \rangle$ and $X \cap ((Z) \setminus \langle X \rangle) = \emptyset$.

   (i) By the 2-local B-cross-freeness of $\{Y, Z\}$ and $(XY) \setminus \langle Z \rangle \neq \emptyset$, $Z \supseteq \langle Y \rangle \setminus \langle Z \rangle$ holds. Thus $X \subseteq Z$ holds on $S$.

   (ii) We can assume $Y \subseteq X$ or $X \subseteq Y$ on $S$. Then, by $\langle XY \rangle = \langle XYZ \rangle$ and the 2-local cross-freeness of $\{Y, Z\}$, we have $Y \subseteq^* Z$ or $Z \subseteq^* Y$. If $Y \subseteq^* Z$, then $Z \supseteq \langle Y \rangle \setminus \langle Z \rangle$ holds on $S$. Hence $X \subseteq Z$ holds on $S$. If $Z \subseteq^* Y$, then $X \subseteq Y$ must hold on $S$ by $X \subseteq^* Z$. This means $X \subseteq^* Y$, i.e., $X \cap ((Y) \setminus \langle X \rangle) = \emptyset$. Hence $X \subseteq Z$ holds on $S$.

2. We can assume $X \subseteq^* Y$ and $\langle XYZ \rangle \neq \emptyset$. By $\langle XY \rangle \not\subseteq \langle Z \rangle$ and $\langle XYZ \rangle \neq \emptyset$, there are two cases: (i) $\langle XZ \rangle \not\subseteq \langle Y \rangle$ or (ii) $\langle XZ \rangle \not\subseteq \langle Y \rangle \setminus \langle XY \rangle$.

   (i) $X \subseteq^* Y$ implies $Y \supseteq \langle X \rangle \setminus \langle Y \rangle$. By $\langle XZ \rangle \not\subseteq \langle Y \rangle$, we have $Y \cap \langle (Z) \setminus \langle Y \rangle \rangle \neq \emptyset$. Hence, by the 2-local B-cross-freeness of $\{Y, Z\}$, $Y$ must contain $\langle Z \rangle \setminus \langle Y \rangle$. Therefore, it holds that $X \subseteq Y$ on $S$; then we use (1) (note $\langle XY \rangle \neq \emptyset$).
(ii). We assume $X \subseteq^* Z$ by the 2-local $\mathcal{B}$-cross-freeness of $\{X, Z\}$ (the argument for the case of $Z \subseteq^* X$ is similar). This implies $Z \supseteq (X) \setminus (Z)$. By $\emptyset \neq \langle XZ \rangle \subseteq \langle XY \rangle$, we have $Z \cap (\langle Y \rangle \setminus (Z)) \neq \emptyset$. Hence, by the 2-local $\mathcal{B}$-cross-freeness of $\{Y, Z\}$, $Z$ must contain $\langle Y \rangle \setminus (Z)$. Therefore, it holds that $Z \subseteq X$ on $S$; then we use (1).

(3). Note that $\langle XY \rangle$, $\langle YZ \rangle$, and $\langle ZX \rangle$ are all nonempty. We can assume that both $X$ and $Y$ properly contain $Z$ in $\langle XY \rangle \cap \langle YZ \rangle$. Necessarily $Z$ is disjoint from $(\langle X \rangle \cup (Y)) \setminus (Z) = (\langle X \rangle \cup \langle Y \rangle) \setminus \langle XYZ \rangle$ by the 2-local $\mathcal{B}$-cross-freeness of $\{X, Z\}$ and $\{Y, Z\}$. Hence $\{X, Z\}$ (or $\{Y, Z\}$) is cross-free on $S$; then we use (1).

(4). We can assume $X \subseteq^* Y$ by the 2-local $\mathcal{B}$-cross-freeness of $\{X, Y\}$. Then we can also assume $X \subseteq^* Z$ or $Z \subseteq^* X$. If $X \subseteq^* Z$, then $X$ does not meet $\langle Y \rangle \cup (Z) \setminus (X)$, and $\{X, Y\}$ is cross-free on $S$; then we use (1). Hence suppose $Z \subseteq^* X$. By $X \subseteq^* Y$ and the 2-local $\mathcal{B}$-cross-freeness of $\{X, Y_i\}$ for $i \in [k]$, it must hold that $X \subseteq^* Y_i$ for $i = 1, 2, \ldots, k$. Since $\{X, Y_k, Z\}$ is cross-free on $\langle X \rangle \cup \langle Y_k \rangle \cup \langle Z \rangle$, it holds that $Z \subseteq X \subseteq Y_k$ on $\langle X \rangle \cup \langle Y_k \rangle \cup \langle Z \rangle$. Here $\langle Z \rangle$ cannot meet $\langle Y_i \rangle$, $i \in [k]$. Otherwise sequence $X, Y_1, \ldots, Y_k, Z$ also forms a path in $G(\mathcal{G}, \mathcal{B})$, and $X \subseteq^* Z$, a contradiction to $Z \subseteq^* X$. By this fact with $\langle XY \rangle = \langle YZ \rangle$, sequence $Z, Y, Z_1, \ldots, Y_k$ also forms a path in $G(\mathcal{G}, \mathcal{B})$. By $Z \subseteq^* Y_k$ and the 2-local $\mathcal{B}$-cross-freeness of $\{Z, Y_i\}$ for $i \in [k]$, we have $Z \subseteq^* Y$. Now $Z \subseteq^* X$ and $Z \subseteq^* Y$ hold. This means that $Z$ does not meet $\langle X \rangle \cup \langle Y \rangle \setminus \langle Z \rangle$, which implies that $\{Y, Z\}$ is cross-free on $S$; then we use (1). \qed

We are now ready to give the proof of Proposition 4.3.

Proof of Proposition 4.3. We only prove that if $X \subseteq^* Y$, then $G' := \mathcal{G} \setminus \{X, Y\} \cup \{X \cap Y, X \cup Y\}$ is 3-locally $\mathcal{B}$-cross-free with $G' \sim \mathcal{G}$; the other case is similar.

First we prove $G' \sim \mathcal{G}$, that is, we show $X \sim X \cap Y$ and $Y \sim X \cup Y$. By $X \subseteq^* Y$, $X$ (resp. $Y$) coincides with $X \cap Y$ (resp. $X \cup Y$). Hence the partition line of $X$ (resp. $Y$) on $\langle X \rangle \cup \langle Y \rangle$ is equal to that of $X \cap Y$ (resp. $X \cup Y$) on $\langle X \rangle \cup \langle Y \rangle$. Furthermore, for any $p \in [r]$ with $A_p \cap \langle X \rangle = \emptyset$, and $X \cap Y \supseteq A_p$ or $X \cup Y \supseteq A_p$ holds and $X \cap Y \supseteq A_p$ or $X \cup Y \supseteq A_p$ holds. This means $X \sim X \cap Y$ and $Y \sim X \cup Y$; then $\langle X \rangle = \langle X \cap Y \rangle$ and $\langle Y \rangle = \langle X \cup Y \rangle$ hold.

Next we show that $G'$ is 2-locally $\mathcal{B}$-cross-free. Since the partition lines of $X$ and $Y$ are the same as those of $X \cap Y$ and $X \cup Y$, $\{X \cap Y, X \cup Y\}$ is also cross-free on $\langle X \rangle \cup \langle Y \rangle$. Hence $\{X \cap Y, X \cup Y\}$ is 2-locally $\mathcal{B}$-cross-free. In the following, we prove that $\{X \cap Y, X \cup Y\}$ is 2-locally $\mathcal{B}$-cross-free for each $Z \in \mathcal{G} \setminus \{X, Y\}$.

If $\{X, Y\}$ is cross-free on $\langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle$, then the partition lines of $X$ and $Y$ on $\langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle$ are the same as those of $X \cap Y$ and $X \cup Y$. Hence, by the 2-local $\mathcal{B}$-cross-freeness of $\mathcal{G}$, we obtain that $\{X \cap Y, X \cup Y, Z\}$ is also 2-locally $\mathcal{B}$-cross-free. Therefore, it suffices to deal with the cases of (i) $\langle XZ \rangle = \langle YZ \rangle = \emptyset$, (ii) $\langle XZ \rangle \neq \emptyset$ and $\langle XY \rangle = \langle YZ \rangle = \emptyset$, (iii) $\langle YZ \rangle \neq \emptyset$ and $\langle XY \rangle = \langle XZ \rangle = \emptyset$, and (iv) $\langle XY \rangle = \langle YZ \rangle = \langle ZX \rangle = \emptyset$. Indeed, for other cases, $\{X, Y, Z\}$ is cross-free on $\langle X \rangle \cup \langle Y \rangle$ by (2) of Lemma 4.7 reducing to the cross-free case above.

(i). By the 2-local $\mathcal{B}$-cross-freeness of $\mathcal{G}$, we have both $\langle X \rangle \subseteq (Z)$ or $\langle X \cap Z \rangle = \emptyset$ and $\langle Y \rangle \subseteq (Z)$ or $\langle Y \cap Z \rangle = \emptyset$. Hence both $\langle X \cap Y \rangle \subseteq (Z)$ or $\langle X \cap Y \rangle \subseteq (Z)$ or $\langle X \cup Y \rangle \subseteq (Z)$ or $\langle X \cup Y \rangle \subseteq (Z) = \emptyset$. Therefore $\{X \cap Y, X \cup Y, Z\}$ is 2-locally $\mathcal{B}$-cross-free.

(ii) and (iii). By symmetry, we show (ii) only. By $X \subseteq^* Y$, we have $\langle X \rangle \subseteq (Y) \setminus (Y)$. By $\langle XZ \rangle \neq \emptyset$ and $\langle XY \rangle = \langle YZ \rangle = \emptyset$, it holds that $\langle Y \rangle \cap (\langle Z \rangle \setminus (\langle Y \rangle)) \neq \emptyset$. By the 2-local $\mathcal{B}$-cross-freeness of $\{Y, Z\}$, $Y$ must contain $\langle Z \rangle \setminus (\langle Y \rangle)$. Therefore $X \subseteq Y$ holds on $\langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle$, reducing to the cross-free case.

(iv). $\langle XY \rangle = \langle YZ \rangle = \langle ZX \rangle \neq \emptyset$ implies $\langle XY \rangle \cup \langle YZ \rangle \cup \langle ZX \rangle \neq \emptyset$. Hence, by the 3-local $\mathcal{B}$-cross-freeness of $\mathcal{G}$, $\{X, Y, Z\}$ is cross-free on $\langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle$, reducing to the cross-free case.

Finally, we show that $G'$ is 3-locally $\mathcal{B}$-cross-free. Take distinct $S, T, U \in \mathcal{G}$ with $\langle STU \rangle \neq \emptyset$. If $\{S, T, U\} \cap \{X \cap Y, X \cup Y\} = \emptyset$, then $\{S, T, U\}$ does not change in the construction of $G'$. 28
Hence \( \{S, T, U\} \) is cross-free on \( \langle S \rangle \cup \langle T \rangle \cup \langle U \rangle \). If \( |\{S, T, U\} \cap \{X \cap Y, X \cup Y\}| = 1 \), then \( \{S, T, U\} \setminus \{X \cap Y, X \cup Y\} \) is cross-free on \( \langle S \rangle \cup \langle T \rangle \cup \langle U \rangle \). By the 2-local \( B \)-cross-freeness of \( G' \) shown above and (1) of Lemma 4.7, \( \{S, T, U\} \) is also cross-free on \( \langle S \rangle \cup \langle T \rangle \cup \langle U \rangle \). If \( |\{S, T, U\} \cap \{X \cap Y, X \cup Y\}| = 2 \) (assume \( S = X \cap Y \) and \( T = X \cup Y \)), then the partition lines of \( X \cap Y \) and \( X \cup Y \) on \( \langle X \rangle \cup \langle Y \rangle \cup \langle U \rangle \) do not change in the construction of \( G' \), since \( \{X, Y, U\} \) is cross-free on \( \langle X \rangle \cup \langle Y \rangle \cup \langle U \rangle \). Thus \( \{X \cap Y, X \cup Y\} \) is cross-free on \( \langle X \rangle \cup \langle Y \rangle \cup \langle U \rangle \). This completes the proof of Proposition 4.3.

4.4 Constructing 3-locally \( B \)-cross-free family

Our final task is to show that, for an input \( F \) equivalent to a 2-locally \( B \)-cross-free family, we can always construct a 3-locally \( B \)-cross-free family in polynomial time (see Figure 1). Specifically, we use the LC-graph \( G(F, B) \) introduced in Section 4.2 and construct an LC-labeling \( s \) with the desired property that the family \( F^s \) in (33) transformed from \( F \) by \( s \) is 3-locally \( B \)-cross-free. While the existence of an LC-labeling is guaranteed by the assumed equivalence of \( F \) to a 2-locally \( B \)-cross-free family (Proposition 4.2), we need to exploit a certain intriguing structure inherent in an LC-graph before we can construct such a special LC-labeling.

Lemma 4.7 indicates that, more often than not, a triple \( X, Y, Z \) in any 2-locally \( B \)-cross-free family is cross-free on \( \langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle \). To construct a 3-locally \( B \)-cross-free family, a particular care is needed for those triples \( X, Y, Z \) with \( \langle XY \rangle = \langle YZ \rangle = \langle ZX \rangle \neq \emptyset \) for which there exists no path \( (XY, XY_1, \ldots, XY_k) \) satisfying \( \langle XY \rangle \neq \langle XY_k \rangle \neq \emptyset \). This motivates the notion of special nodes and special connected components in the LC-graph \( G(F, B) \). For distinct \( X, Y \in F \), define

\[
R(XY) := \{Z \in F \mid \text{There is a path } (XY, XY_1, \ldots, XZ) \text{ using only forward edges}\},
\]

\[
R^*(XY) := \{Z \in R(XY) \mid \langle XZ \rangle \neq \emptyset\}.
\]

We say that a node \( XY \) (or an ordered pair of \( X \) and \( Y \)) with \( \langle XY \rangle \neq \emptyset \) is special if \( \langle XZ \rangle = \langle XY \rangle \) holds for all \( Z \in R^*(XY) \). For \( X, Y \in F \) with \( XY \) and \( YX \) both being special, let \( v(XY) \) denote the connected component (as a set of nodes) containing \( XY \) or \( YX \) in \( G(F, B) \). We call such a component special. Let \( v^*(XY) \) denote the set of nodes \( ZW \) in \( v(XY) \) with \( \langle ZW \rangle \neq \emptyset \).

A special component has an intriguing structure; the proof is given at the end of this section.

Proposition 4.8. If both \( XY \) and \( YX \) are special, then the following hold.

(i) \( v(XY) = (R^*(XY) \times R(YX)) \cup (R^*(YX) \times R(XY)) \).

(ii) \( v^*(XY) = (R^*(XY) \times R(YX)) \cup (R^*(YX) \times R^*(XY)) \).

(iii) If \( ZW \in v^*(XY) \), then \( ZW \) is special and \( \langle ZW \rangle = \langle XY \rangle \).

For a special component \( v = v(XY) \), we call \( \langle XY \rangle \) the center of \( v \); this is well-defined by (iii) of Proposition 4.8. For \( Q \subseteq [r] \), the set of all special components whose center coincides with \( A_Q \) is called the \( Q \)-flower if its size is at least two. The following proposition gives a concrete representation of the \( Q \)-flower; the proof is given at the end of this section.

Proposition 4.9. The \( Q \)-flower is given as

\[
\{v(X_i, X_j) \mid 1 \leq i < j \leq p\}
\]

for some \( p \geq 3 \) and distinct \( X_1, X_2, \ldots, X_p \in F \) such that \( R(X_i, X_j) = R(X_i, X_j) \) for all \( i, i' < j \), and \( R(X_i, X_j) \cap R(X_i', X_j') = \emptyset \) for all distinct \( j, j' \in [p], i < j \), and \( i' < j' \).
The above $X_1, X_2, \ldots, X_p$ are called the representatives of the $Q$-flower. Figure 2 illustrates an example of $Q$-flower.

A component $v$ is said to be fixed if $v$ contains a fixed node, and said to be free otherwise. A special component $v(XY)$ in the $Q$-flower is free if and only if the partition lines of $X'$ and $Y'$ on $A_Q$ are the same for all $X' \in R^*(YX)$ and $Y' \in R^*(XY)$. A free $Q$-flower is a maximal set of free components in the $Q$-flower such that the partition lines on $A_Q$ are the same. Now the set of free components of the $Q$-flower is partitioned to free $Q$-flowers each of which is represented as

$$\{v(X_sX_t) \mid 1 \leq s < t \leq q\}$$

with a subset $\{X_{i_1}, X_{i_2}, \ldots, X_{i_q}\}$ of the representatives. A free $Q$-flower (for some $Q \subseteq [n]$) is also called a free flower.

We now provide a polynomial-time algorithm to construct a 3-locally $B$-cross-free family $F^*$ by defining an appropriate LC-labeling $s$.

**Algorithm 5 (for constructing a 3-locally $B$-cross-free family):**

**Step 0:** Determine whether there exists a 2-locally $B$-cross-free family equivalent to $F$. If not, then output “$F$ is not $B$-laminarizable” and stop.

**Step 1:** For all fixed nodes $XY$, define $s(XY)$ according to (31) and (32). By a breath-first search, define $s$ on all other nodes in fixed components appropriately.

**Step 2:** For each component $v$ which is free and not special, take any node $XY$ in $v$. Define $s(XY)$ as 0 or 1 arbitrarily, and define $s(ZW)$ appropriately for all nodes $ZW$ in $v$. Then all the remaining (undefined) components are special and free.
Step 3: For each free flower, which is assumed to be represented as \( \{ v(X_i, X_j) \mid 1 \leq i < j \leq q \} \), do the following:

3-1: Define the value of \( s(X_i, X_j) \) for distinct \( i, j \in [q] \) so that \( \{ X_i^*, X_2^*, \ldots, X_q^* \} \) is cross-free on \( \bigcup_{i \in [q]} (X_i) \); such a labeling is given, for example, as

\[
s(X_i, X_j) := \begin{cases} 
1 & \text{if } i > j \text{ and } X_i = X_1 \text{ on } A_Q, \text{ or } i < j \text{ and } X_i = X_1 \text{ on } A_Q, \\
0 & \text{if } i < j \text{ and } X_i = X_1 \text{ on } A_Q, \text{ or } i > j \text{ and } X_i = X_1 \text{ on } A_Q, 
\end{cases}
\]

(37)

where \( A_Q \) is the center of the free flower.

3-2: Define \( s(ZW) \) appropriately for all \( ZW \in v(X_i, X_j) \).

Step 4: Output \( \mathcal{F}^* \).

**Proposition 4.10.** The output \( \mathcal{F}^* \) is 3-locally \( B \)-cross-free, and Algorithm 5 runs in \( O(n^4) \) time.

**Proof.** We show the 3-local \( B \)-cross-freeness of \( \mathcal{F}^* \). Take any triple \( \{ X^*, Y^*, Z^* \} \) with \( \langle XY \rangle \notin \emptyset \). It suffices to deal with the case of \( \langle XY \rangle = \langle YZ \rangle = \langle ZX \rangle \neq \emptyset \) by (2) of Lemma 4.7. If \( \langle XY \rangle \) is not special, there is a path \( \langle XY, X_1Y_1, \ldots, X_kY_k \rangle \) in \( G(G, B) \) such that \( \emptyset \neq \langle XY \rangle \) and \( \langle YZ \rangle = \langle ZX \rangle \).

By (2) of Lemma 4.7, \( \{ X, Y, Z \} \) is cross-free on \( \langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle \). Hence, by (4) of Lemma 4.7, \( \{ X, Y, Z \} \) is cross-free on \( \langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle \). Therefore, we assume that \( XY, YX, YZ, ZY, ZX, ZX \) are special.

We can suppose that \( XY, YZ, ZX \) belong to special components of the \( Q \)-flower \( \{ v(X_i, X_j) \mid 1 \leq i < j \leq p \} \), i.e., \( \langle XY \rangle = \langle YZ \rangle = \langle ZX \rangle = A_Q \). By Proposition 4.9, we can assume \( X \in R^*(X_kX_k) \), \( Y \in R^*(X_kX_k) \), and \( Z \in R^*(X_kX_k) \) for distinct \( i, j, k \in [q] \) with \( i < j < k \).

Suppose that \( v(X_i, X_j), v(X_iX_k), \) or \( v(X_iX_k) \) is fixed. Then we can assume that there is \( \hat{X} \in R^*(X_kX_k) \) such that the partition lines of \( X, Y, Z \) are not the same. By (3) of Lemma 4.7, \( \{ X^*, Y^*, Z^* \} \) is cross-free on \( \langle \hat{X} \rangle \cup \langle Y \rangle \cup \langle Z \rangle \). Furthermore, since there is a path \( \langle XY = YX_0, YX_1, \ldots, YX_k = YX \rangle \), \( \{ X, Y, Z \} \) is cross-free on \( \langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle \) by (4) of Lemma 4.7.

Suppose that \( v(X_i, X_j), v(X_iX_k), \) and \( v(X_iX_k) \) are free. Then \( v(X_i, X_j), v(X_iX_k), v(X_iX_k) \) are contained in the same free \( Q \)-flower. By the definition of \( s \) (cf. (37)), \( \{ X_i^*, X_j^*, X_k^* \} \) is cross-free on \( \langle X_i \rangle \cup \langle X_j \rangle \cup \langle X_k \rangle \). By applying repeatedly (4) of Lemma 4.7, \( \{ X^*, Y^*, Z^* \} \) is cross-free on \( \langle X \rangle \cup \langle Y \rangle \cup \langle Z \rangle \).

Finally we see the running-time of Algorithm 5. By the argument at the end of Section 4.2, Step 0 can be done in \( O(n^4) \) time. We can also obtain an appropriate value of each \( s(XY) \) in Steps 1–3 in \( O(n^4) \) time. From \( s \), we can construct \( \mathcal{F}^* \) in \( O(|V(\mathcal{F}, B)|) = O(n^2) \) time. Thus the running-time of Algorithm 5 is bounded by \( O(n^4) \).

By Propositions 4.4 and 4.10, we obtain the following theorem.

**Theorem 4.11.** Algorithms 4 and 5 solve LAMINARIZATION in \( O(n^4) \) time.

The rest of this section is devoted to proving Propositions 4.8 and 4.9. First we show a key lemma about special nodes.

**Lemma 4.12.** If \( XY \) is special and \( \langle XY \rangle = \langle XY' \rangle \) for some \( X' \), then \( R(X'Y) \subseteq R(XY) \), and \( \langle X'Z \rangle \supseteq \langle XZ \rangle \) for any \( Z \in R(X'Y) \).

**Proof.** We prove \( Y_k \in R(XY) \) and \( \langle X'Y_k \rangle \supseteq \langle XY_k \rangle \) by induction on the length \( k \) of a path \( \langle XY = X'Y_0, X'Y_1, \ldots, X'Y_k \rangle \). In \( k = 0 \), we have \( Y_0 = Y \in R(XY) \) and \( \langle X'Y_0 \rangle \supseteq \langle XY_0 \rangle \). For induction step, suppose that \( Y_k \in R(XY) \) and \( \langle X'Y_k \rangle \supseteq \langle XY_k \rangle \) for \( k \geq 0 \). Since a forward edge \( \{ X'Y_k, X'Y_{k+1} \} \) exists, we have \( \langle Y_k, Y_{k+1} \rangle \setminus \langle X'Y_k \rangle \neq \emptyset \). Then \( Y_{k+1} \neq X \) holds. Indeed, if \( Y_{k+1} = X \), then \( \langle XY_k \rangle \setminus \langle X'Y_k \rangle \neq \emptyset \), a contradiction to \( \langle XY_k \rangle \supseteq \langle XY_k \rangle \). By \( \langle XY_k \rangle \supseteq \langle XY_k \rangle \),
we obtain \( \langle Y_k Y_{k+1} \rangle \setminus \langle XY_k \rangle \neq \emptyset \). Hence there is a forward edge \( \{XY_k, XY_{k+1}\} \). This means \( Y_{k+1} \in R(XY) \).

Suppose, to the contrary, that \( \langle X'Y_{k+1} \rangle \not\supseteq \langle XY_{k+1} \rangle \), i.e., \( \langle XY_{k+1} \rangle \setminus \langle X'Y_{k+1} \rangle \neq \emptyset \) holds. Note that \( \langle XY_{k+1} \rangle \setminus \langle X'Y_{k+1} \rangle = \langle XY_k \rangle \setminus \langle X'Y \rangle \) holds. Furthermore, by \( \langle XY \rangle = \langle X'Y \rangle \), we obtain \( \langle XY \rangle \supseteq \langle XY \rangle \). Hence we have \( \langle XY_{k+1} \rangle \setminus \langle XY \rangle \neq \emptyset \). However, by the speciality of \( XY \) and \( Y_{k+1} \in R( XY ) \), it must hold that \( \langle XY_{k+1} \rangle \equiv \langle XY \rangle \) or \( \langle XY_{k+1} \rangle = \emptyset \); this is a contradiction. \( \square \)

**Proof of Proposition** [4.8]. First we show the following three claims.

**Claim 1.** \( R( XY ) \cap R( YX ) = \emptyset \).

**Proof.** Suppose, to the contrary, that \( R( XY ) \cap R( YX ) \neq \emptyset \). For each \( Z \in R( XY ) \), we have \( \langle XZ \rangle \subseteq \langle YZ \rangle \) since \( \langle XZ \rangle = \langle XY \rangle \) or \( \langle XZ \rangle = \emptyset \).

Let \( Z \in R( XY ) \cap R( YX ) \) be an element such that the length \( k \) of a path \( \{ YX, \ldots, YZ \} \) in \( G( F, B ) \) is shortest. If \( k \geq 2 \), there is a forward edge \( \{ YZ_k, YZ_{k-1} \} \) and \( Z_{k-1} \neq X \). That is \( \langle Z_k Z_{k-1} \rangle \setminus \langle YZ_k \rangle \neq \emptyset \). By \( \langle XZ_k \rangle \subseteq \langle YZ_k \rangle \), we obtain \( \langle Z_k Z_{k-1} \rangle \setminus \langle XZ_k \rangle \neq \emptyset \). Hence a forward edge \( \{ XZ_k, XZ_{k-1} \} \) exists. This means \( Z_{k-1} \in R( XY ) \cap R( YX ) \), which contradicts the minimality of \( Z = Z_k \). Therefore a forward edge \( \{ YX, YZ \} \) exists for some \( Z \in R( XY ) \) for all \( Z \in R( YX ) \). That is, \( \langle XZ \rangle \setminus \langle XY \rangle \neq \emptyset \). Hence we obtain \( \emptyset \neq \langle XZ \rangle \neq \langle XY \rangle \). This contradicts the speciality of \( XY \).

**Claim 2.** For any \( Y' \in R^*( XY ) \), it holds that \( R( XY ) = R( Y' \) and \( \langle Y'Z \rangle = \langle YZ \rangle \) for any \( Z \in R( XY ) = R( Y'X ) \).

**Proof.** If \( R^*( XY ) = \{ Y \} \), the proof is trivial. Suppose \( R^*( XY ) \setminus \{ Y \} \neq \emptyset \). Take any \( Y' \in R^*( XY ) \setminus \{ Y \} \). Then there is a backward edge \( \{ Y'X, Y' \} \). Therefore, for all \( Z \in R( YX ) \), \( XY \) and \( Y'Z \) are connected. It holds that \( \langle YX \rangle = \langle Y'X \rangle \) by the speciality of \( XY \). Since \( XY \) is special and \( \langle YX \rangle = \langle Y'X \rangle \), by Lemma [4.12] we have \( R( Y'X ) \subseteq R( YX ) \) and \( \langle Y'Z \rangle \supseteq \langle YZ \rangle \) for all \( Z \in R( YX ) \).

In the following, we prove that, for each \( Z \in R( YX ) \), it holds that \( Z \in R( Y'X ) \) and \( \langle Y'Z \rangle \subseteq \langle YZ \rangle \), which imply \( R( Y'X ) = R( YX ) \) and \( \langle Y'Z \rangle = \langle YZ \rangle \). We show this by induction on the length of a path \( \{ YX = YX_0, YX_1, \ldots, YX_{k+1} = YZ \} \). For \( X_0 \), we have \( R( Y'X ) \ni X = X_0 \) and \( \langle YX_0 \rangle = \langle Y'X_0 \rangle \). Suppose \( R( Y'X ) \ni X_k \) and \( \langle YX_k \rangle \supseteq \langle Y'X_k \rangle \) by induction. Since a forward edge \( \{ X_k X_{k+1}, YX_{k+1} \} \) exists, we have \( \langle X_k X_{k+1} \rangle \setminus \langle YX_k \rangle \neq \emptyset \). By \( \langle YX_k \rangle \supseteq \langle Y'X_k \rangle \), we obtain \( \langle X_k X_{k+1} \rangle \setminus \langle Y'X_k \rangle \neq \emptyset \). Hence there is a forward edge \( \{ YX_k, YX_{k+1} \} \). This means \( R(Y'X) \ni X_{k+1} = Z \).

Suppose, to the contrary, that \( \langle YX_{k+1} \rangle \not\supseteq \langle Y'X_{k+1} \rangle \), i.e., \( \langle Y'X_{k+1} \rangle \setminus \langle YX_{k+1} \rangle \neq \emptyset \) holds. Then there is a forward edge \( \{ YX_{k+1}, YY' \} \). Hence we have \( R( YX ) \ni Y' \). However this contradicts \( R( YX ) \ni Y' \) by Claim [1] and \( R( XY ) \ni Y' \). Therefore we obtain \( \langle YX_{k+1} \rangle \supseteq \langle Y'X_{k+1} \rangle \). \( \square \)

**Claim 3.** For \( ZW \in v( XY ) \), there is a path from \( XY \) or \( YX \) to \( ZW \) containing at most one backward edge.

**Proof.** Suppose, to the contrary, that all paths from \( XY \) to \( ZW \) and from \( YX \) to \( ZW \) use at least two backward edges for some \( ZW \). Take such a path \( P \) with minimum backward edges. Denote the number of backward edges in \( P \) by \( k(\geq 2) \). Without loss of generality, we assume that \( P \) is a path from \( XY \) to \( ZW \). By \( k \geq 2 \), \( P \) has a subpath \( \{ XY = X_0 Y_0, \ldots, X_0 Y_1, Y_1 X_1, \ldots, Y_1 X_2, X_2 Y_2 \} \). Note that \( Y_1 \in R^*( XY ) \), and \( X_1 \in R^*( Y_1 X ) \). By Claim [2] \( \langle YX \rangle \ni Y \). Hence there is a path from \( XY \) to \( X_1 Y_1 \) using only one backward edge. Indeed, \( \{ YX = Y_0 X_0, \ldots, Y_0 X_1, X_1 Y_0, \ldots, X_1 Y_1 \} \) is such a path. This means that there is a path from \( XY \) to \( ZW \) with \( k - 1 \) backward edges, a contradiction to the minimality of \( P \). \( \square \)
First we show the statement (i) of Proposition 4.8. If \( Z \in R^*(XY) \) and \( W \in R(YX) \), then there is a path in \( G(\mathcal{F}, \mathcal{B}) \) such as \( \langle XY, \ldots, XZ, ZX, \ldots, ZW \rangle \) since \( R(YX) = R(ZX) \) by Claim 2, implying \( ZW \in v(XY) \). Conversely, if \( ZW \in v(XY) \), then there is a path from \( XY \) to \( YX \) with at most one backward edge by Claim 3. We may assume that there is such a path \( P \) from \( XY \) to \( ZW \). If \( P \) has no backward edge, then \( Z = X \in R^*(YX) \) and \( W \in R(YX) \) hold. If \( P \) has exactly one backward edge, then \( Z \in R^*(XY) \) and \( W \in R(ZX) = R(YX) \) by Claim 3. Thus we obtain (i).

Next we show (iii). If \( ZW \in v(XY) \), then there is a path from \( XY \) or \( YX \) to \( ZW \) with at most one backward edge by Claim 3. We may assume that there is such a path \( P \) from \( XY \) to \( ZW \). If \( P \) has no backward edge, then \( ZW = \langle XY \rangle \) or \( \langle ZW \rangle = \emptyset \) holds by the speciality of \( XY \). If \( P \) has one backward edge, then \( \langle ZW \rangle = \langle YW \rangle \) holds by Claim 2 and \( \langle YW \rangle = \langle XY \rangle \) or \( \langle YW \rangle = \emptyset \) holds by the speciality of \( YX \). Therefore, if \( ZW \in v^*(XY) \), then \( \langle ZW \rangle = \langle XY \rangle \), and the speciality of \( ZW \) is obvious. Thus we obtain (iii).

Finally we show (ii). For every \( Z \in R^*(XY) \) and \( W \in R^*(YX) \), we have \( \langle Z \rangle \supseteq \langle XY \rangle \subseteq \langle W \rangle \) by (iii). Hence \( \langle ZW \rangle \neq \emptyset \), implying \( ZW \in v^*(XY) \). Conversely, let \( ZW \in v^*(XY) \). By (i), we may assume \( Z \in R^*(XY) \) and \( W \in R(YX) \). Since \( W \) holds by (iii). Hence \( (W) \cap (Y) \supseteq \langle XY \rangle \neq \emptyset \). This means \( W \in R^*(YX) \).

**Proof of Proposition 4.8.** Let \( v(X_1X_2) \) be a special connected component with \( \langle v(Y_1Y_2) \rangle = A_Q \). Take any special connected component \( v(Y_1Y_2) \) with \( \langle v(Y_1Y_2) \rangle = A_Q \). It suffices to show that, (i) if \( R(X_1X_2) \cap R(Y_1Y_2) \neq \emptyset \), we have \( R(X_1X_2) = R(Y_1Y_2) \) (this implies \( v(Y_1Y_2) = v(Y_1X_2) \) by \( Y_2 \in R^*(X_1X_2) \)), and (ii) if \( R(X_1X_2) \cap R(Y_1Y_2) = \emptyset \), there exists a special connected component \( v(X_2Y_2) \) with \( \langle v(X_2Y_2) \rangle = A_Q \).

(i). If there exists \( Z \in R^*(X_1X_2) \cap R^*(Y_1Y_2) \), then it holds that \( X_1Z \) and \( Y_1Z \) are special and \( (X_1Z) = (Y_1Z) = A_Q \). Hence, by Lemma 4.12, we have \( R(X_1Z) \subseteq R(Y_1Z) \) and \( R(X_1Z) \supseteq R(Y_1Z) \), i.e., \( R(X_1Z) = R(Y_1Z) \). This implies \( R(X_1X_2) = R(X_1Z) = R(Y_1Z) = R(Y_1Y_2) \), as required. Thus, in the following, we show that there exists \( Z \in R^*(X_1X_2) \cap R^*(Y_1Y_2) \).

Suppose, to the contrary, \( R^*(X_1X_2) \cap R^*(Y_1Y_2) = \emptyset \). Note that \( R^*(X_1X_2) \cap R^*(Y_1Y_2) = \emptyset \) implies \( R^*(X_1X_2) \cap R(Y_1Y_2) = R(X_1X_2) \cap R^*(Y_1Y_2) = \emptyset \). Indeed, each \( Z \in R^*(X_1X_2) \cap R(Y_1Y_2) \) satisfies \( Z \supseteq A_Q \) by \( Z \in R^*(Y_1Y_2) \). Hence \( Z \in R^*(Y_1Y_2) \) holds by \( Z \in R(Y_1Y_2) \) and \( Y_1Z \neq \emptyset \).

Let \( Z \in R(X_1X_2) \cap R(Y_1Y_2) = (R(X_1X_2) \cap R(Y_1Y_2)) \setminus (R^*(X_1X_2) \cup R^*(Y_1Y_2)) \) be an element such that the length of a path \( X_1X_2 = X_1Z_{01}X_1Z_{1}, \ldots, X_1Z_k = X_1Z \) is shortest; by the assumption, \( k \geq 1 \). Since a forward edge \( \{Y_1Z_k, X_1Z_{k-1}\} \) exists, we have \( \langle Z_{k}Z_{k-1} \rangle \setminus \langle X_1Z_k \rangle \neq \emptyset \). Furthermore, by \( \langle X_1Z_k \rangle = \langle Y_1Z_k \rangle = \emptyset \), we obtain \( \langle Z_{k}Z_{k-1} \rangle \setminus \langle Y_1Z_k \rangle \neq \emptyset \). This means that a forward edge \( \{Y_1Z_k, Y_1Z_{k-1}\} \) exists and \( Z_{k-1} \in R(X_1X_2) \cap R(Y_1Y_2) \) holds, a contradiction to the minimality of \( k \).

(ii). First we show that \( \langle X_2Y_2 \rangle = A_Q \) or \( \langle X_2Y_2 \rangle = \emptyset \) holds for any \( X_2 \in R(X_1X_2) \) and \( Y_2 \in R(Y_1Y_2) \). Since, for any \( Z \in R(X_1X_2) \cup R(Y_1Y_2), \langle Z \rangle \supseteq A_Q \) or \( (Z) \cap A_Q = \emptyset \) holds by Proposition 4.8, we have \( (X_2Y_2) \supseteq A_Q \) or \( (X_2Y_2) \cap A_Q = \emptyset \) for each \( X_2 \in R(X_1X_2) \) and \( Y_2 \in R(Y_1Y_2) \) with \( \langle X_2Y_2 \rangle \neq \emptyset \). Suppose, to the contrary, that there exist \( X_2 \in R(X_1X_2) \) and \( Y_2 \in R(Y_1Y_2) \) with \( \langle X_2Y_2 \rangle \neq \emptyset \) and \( \langle X_2Y_2 \rangle \cap A_Q \neq \emptyset \). Then \( \langle X_2Y_2 \rangle \supseteq A_Q \) or \( \langle X_2Y_2 \rangle \cap A_Q = \emptyset \) holds. Hence we have \( \langle X_2Y_2 \rangle \setminus \langle X_1X_2 \rangle \neq \emptyset \). This means that there is a forward edge \( \{X_1X_2, X_1X_2\} \) and \( R(X_1X_2) \cap R(Y_1Y_2) \neq \emptyset \) holds, a contradiction.

By \( \langle X_2 \rangle \supseteq A_Q \subseteq \langle Y_2 \rangle \), we have \( \langle X_2Y_2 \rangle \neq \emptyset \). Hence, by the above argument, we obtain \( \langle X_2Y_2 \rangle = A_Q \). Furthermore \( Y_1Y_2 \) is special and \( (Y_1Y_2) = (X_2Y_2) \) holds. By Lemma 4.12, we obtain \( R(X_2Y_2) \subseteq R(Y_1Y_2) \). By \( \langle X_2Z \rangle = A_Q \) or \( \langle X_2Z \rangle = \emptyset \) for every \( Z \in R(X_2Y_2) \subseteq R(Y_1Y_2) \), it holds that \( X_2Y_2 \) is special. Furthermore, by Lemma 4.12 and the speciality of \( X_2Y_2 \), we also obtain \( R(X_2Y_2) \supseteq R(Y_1Y_2) \). Hence \( R(X_2Y_2) = R(Y_1Y_2) \) holds. By a similar argument, \( Y_2X_2 \) is special and \( R(Y_2X_2) = R(X_1X_2) \) holds. Thus, a special component \( v(X_2Y_2) \) with \( \langle v(X_2Y_2) \rangle = A_Q \) exists.

\( \square \)
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