Noise limits in matter-wave interferometry using degenerate quantum gases.

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We analyze the phase resolution limit of a Mach-Zehnder atom interferometer whose input consists of degenerate quantum gases of either bosons or fermions. For degenerate gases, the number of atoms within one de Broglie wavelength is larger than unity, so that atom-atom interactions and quantum statistics are no longer negligible. We show that for equal atom numbers, the phase resolution achievable with fermions is noticeably better than for interacting bosons.

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Atom interferometers offer a potential sensitivity that exceeds that of their optical counterparts by as much as $Mc^2/\hbar \omega \sim 10^{10}$. Besides being useful for fundamental tests of quantum mechanics and precision measurements of fundamental constants, this property offers great promise for inertial navigation as well as for geophysical applications, tests of general relativity, etc.

In particular, recent years have witnessed significant experimental progress towards the development of atomic rotation sensors with sensitivities now surpassing those of optical gyroscopes. So far, experiments with atom interferometers in the spatial domain have always been carried out in the nondegenerate regime where quantum statistics and many-body interactions are negligible and the interference pattern is the sum of single atom interference patterns. This is true even in those situations where Bose-Einstein condensates (BEC’s) have been used as sources: in such cases, the interference experiments were performed after the atoms have been released from their trapping potentials and allowed to undergo a ballistic expansion in order to eliminate any mean-field effects, see for example. Likewise, the theoretical analysis of the sensitivity of atom interferometers has also been confined to this regime. Refs. analyzed the quantum noise limit of a matter-wave Mach-Zehnder interferometer and showed that for uncorrelated inputs the phase resolution is $1/\sqrt{N}$, for bosons as well as for fermions. Here, $N$ is the total number of particles entering the interferometer. The phase resolution limit results from the intrinsic quantum fluctuations associated with the sorting of the particles by the beam splitters in the interferometer. The reason that bosons and fermions yield the same sensitivity in these treatments is a consequence of the assumption that the fermions are quasi-monochromatic, i.e. the fermions all occupy the same state. This can only be justified when the atomic wave functions do not overlap.

Very much like the invention of the laser revolutionized optics and interferometry, it would seem desirable to extend the operation of matter-wave interferometer to truly quantum-degenerate atom sources. It is therefore important to extend these theoretical results to the case of highly degenerate beams of bosons or fermions for which $\rho \lambda_d \gtrsim 1$, where $\rho$ is the atomic density and $\lambda_d$ is the de Broglie wavelength. In this limit, the effects of quantum statistics and of atom-atom interactions are no longer negligible, and it is expected that the sensitivity of the interferometer will decrease. For bosonic atoms, the interference pattern and phase sensitivity is degraded because of the atom-atom interactions, which give rise to a nonlinear phase shift as the atoms propagate along the arms of the interferometer. In contrast, for spin-polarized fermions — where s-wave scattering is absent for $T \sim 0$ — the phase sensitivity is degraded because the Pauli exclusion principle forces each of the atoms in the interferometer to occupy a different momentum state. Our central result is that contrary to what one might expect, the sensitivity of a Mach-Zehnder interferometer using a degenerate beam of fermions can be noticeably better than what can be achieved using a BEC.

Figure 1 shows the atomic Mach-Zehnder interferometer that we consider, with input ports $A$ and $B$. For convenience we assume that it can be treated as an effective one-dimensional system in which the propagation of the atoms transverse to the interferometer arms can be ignored. That is, we assume that the atoms are tightly confined in the directions transverse to the interferometer paths, with the same wave function $\phi_0(r_{\perp})$. This holds when $\mu < \hbar \omega_{\perp}$ where $\mu$ is the chemical potential for the bosons or fermions and $\hbar \omega_{\perp}$ is the spacing between the energy levels of the transverse wave functions, assumed to be harmonic. We introduce the annihilation (creation) operators $\hat{a}_{k} (\hat{a}_{k}^\dagger)$ and $\hat{b}_{k} (\hat{b}_{k}^\dagger)$ for atoms entering the input ports $A$ and $B$, respectively, with momentum $\hbar k$ directed along either the upper or lower interferometer path. They obey the commutation relations $\hat{a}_{k}\hat{a}_{k}^\dagger + \hat{a}_{k}^\dagger\hat{a}_{k} = \delta_{k,k'}$ and $\hat{b}_{k}\hat{b}_{k}^\dagger + \hat{b}_{k}^\dagger\hat{b}_{k} = \delta_{k,k'}$, where the minus sign is for bosons and the plus sign is for fermions.

The beam splitters 1 and 2 are identical 50-50 beam splitters, which in the case of the first beam splitter has the action

$$\begin{pmatrix} \hat{a}_{u,k} \\ \hat{a}_{l,k} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} \begin{pmatrix} \hat{a}_{k} \\ \hat{b}_{k} \end{pmatrix}$$

where $\hat{a}_{u,k}(\hat{a}_{l,k})$ are the operators for the upper (lower) paths of the interferometer. The mirrors $M_1$ and $M_2$...
where \( \hat{a}_k \) and \( \hat{c}_k \) are the annihilation operators for the output ports \( D \) and \( C \), respectively.

In what follows, we consider only uncorrelated inputs where the initial quantum state can be written as the factorized product of Fock states \( |\Psi_A\rangle|\Psi_B\rangle \). In this case, we recall that for noninteracting atoms occupying the same state, the expectation value of \( \hat{N} \) is \( \langle \hat{N} \rangle = (N_A - N_B) \cos \theta \) where \( \theta \) is the phase difference between the two interferometer paths and \( N_A \) and \( N_B \) are the number of atoms entering via the two input ports. Hence, \( \hat{N} \) is a direct measure of \( \theta \), with fluctuations

\[
\Delta \theta = \frac{\Delta N}{|\partial \langle \hat{N} \rangle / \partial \theta|}
\]

and \( \Delta N \equiv [\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2]^{1/2} \). We now compare \( \Delta \theta \) for the cases of a single mode BEC and of a spin-polarized Fermi gas.

**Bose-Einstein Condensate:** Here we consider \( N_A \) bosons entering port \( A \) and \( N_B \) entering port \( B \) in a time \( \tau \), each atom having velocity \( v_0 = \hbar k_0 / m \). As such, \( \langle \hat{N} \rangle \) represents the number of atoms counted in the interval \( \tau \). The quantum state describing this incident beam is

\[
|\Psi(0)\rangle = (\hat{a}^\dagger_{k_0})^{N_A} (\hat{b}^\dagger_{k_0})^{N_B} |0\rangle / \sqrt{N_A!N_B!}.
\]

We evaluate the moments of \( \hat{N} \) by using an angular momentum representation for the bosonic operators \( 2 \hat{J}_z = \hat{a}^\dagger_{k_0} \hat{a}_{k_0} - \hat{b}^\dagger_{k_0} \hat{b}_{k_0} \), \( 2 \hat{J}_y = \hat{a}^\dagger_{k_0} \hat{b}_{k_0} + \hat{b}^\dagger_{k_0} \hat{a}_{k_0} \), and \( \hat{J}^2 = \hat{N}/2(\hat{N}+1) \) where \( \hat{N} = \hat{a}^\dagger_{k_0} \hat{a}_{k_0} + \hat{b}^\dagger_{k_0} \hat{b}_{k_0} \). In this representation, the incident beam state is \( |\Psi(0)\rangle = |j,m\rangle \), where

\[
j = (N_A + N_B)/2, \quad m = (N_A - N_B)/2.
\]

After some algebra one finds

\[
\langle \hat{N} \rangle = 2m d_{m,m}(\beta) \cos \theta + \left( \sqrt{(j-m)(j+m+1)} d_{m+1,m}^{(j)}(\beta) + \sqrt{(j+m)(j-m+1)} d_{m-1,m}^{(j)}(\beta) \right) \sin \theta,
\]

\[
\langle \hat{N}^2 \rangle = j(j+1) + m^2 + f_{j,m}(2\beta) \cos(2\theta + \beta) + g_{j,m}(2\beta) \sin(2\theta + \beta),
\]

where

\[
f_{j,m}(2\beta) = (3m^2 - j(j+1)) d_{m,m}^{(j)}(2\beta) - \frac{1}{2} \sqrt{(j-m+2)(j+m-1)(j-m+1)(j+m+1)} \left[ d_{m+2,m}^{(j)}(2\beta) + d_{m-2,m}^{(j)}(2\beta) \right],
\]

\[
g_{j,m}(2\beta) = (2m-1) \sqrt{(j-m+1)(j-m+1)} d_{m-1,m}^{(j)}(2\beta) + (2m+1) \sqrt{(j+m+1)(j-m+1)} d_{m+1,m}^{(j)}(2\beta),
\]

and \( d_{m,m}^{(j)}(\beta) = |j,m| \exp(-i\beta \hat{J}_y) |j,m\rangle \) are the matrix elements of a rotation about the \( y \)-axis by an angle \( \beta = \ldots \)
\[ \chi(t_l + t_u) \]. In these expression, \( \theta \) is again the phase difference between the two interferometer paths,

\[ \theta = (\omega_{k_0} + \omega_\perp + \chi(N_A + N_B - 1)/2)(t_u - t_l) + \beta/2, \] (7)

where \( t_l = 2\ell_l/v_0 \) (\( t_u = 2\ell_u/v_0 \)) is the transit time for a point of constant atomic phase through the lower (upper) arm of the interferometer. Note that when \( \chi \equiv 0 \), \( d_{m,m}^{(0)}(0) = \delta_{m',m} \) and our results agree with prior calculations\[12\].

The nonlinearity has a two-fold effect on the interference pattern. This is seen by noting that the phase acquired by an atom propagating along the upper or lower path is \( \phi_s = (\omega_{k_0} + \omega_\perp + \chi \hat{\phi}_{s,k_0}) t_s \), and includes a self-phase modulation due to the other atoms traveling along the same path. However, the wave functions of the atoms entering the interferometer are coherently split between the two paths so that the effect of the nonlinearity is no longer a simple self-phase modulation when expressed in terms of the incident quantum fields,

\[ \hat{\phi}_u - \hat{\phi}_l = (\omega_{k_0} + \omega_\perp + \chi \hat{N}/2)(t_u - t_l) - \chi(t_u + t_l)\hat{J}_y. \]

The phase difference includes a self-phase modulation proportional to half the total number, a simple consequence of the fact that the atoms are equally split between the two paths. However \( \phi_s \) also includes a term proportional to \( \hat{J}_y \). This results from the fact that the wave functions of the atoms entering the interferometer are coherently split between the two paths at the first beam splitter. Hence, the \( \hat{J}_y \) contribution is a result of the indistinguishability of the bosons that enter via the two input ports.

We remark that \( \theta = \langle \hat{\phi}_u - \hat{\phi}_l \rangle + \beta/2 - \chi(t_u - t_l)/2 \) where the last two terms, \( \beta/2 - \chi(t_u - t_l)/2 \), can be interpreted as a "vacuum effect", resulting from the noncommutativity of \( \hat{\phi}_{s,k_0} \) and \( \hat{\phi}^\dagger_{s,k_0} \). Its main impact on \( \theta \) is to replace \( N \equiv N_A + N_B \) with \( N - 1 \), showing explicitly that the nonlinear phase shift vanishes if only one atom is present. The \( \beta/2 \) simply shifts the fringe pattern and is included in the definition of \( \theta \) for convenience.

It is clear from Eqs. \([3]\) and \([4]\) and the definition of \( d_{m,m}^{(j)}(\beta) \) that \( \beta j \) is the critical parameter that determines the effects of the atom-atom interactions on \( \Delta \theta \). Relevant values of \( \beta \) can be estimated by assuming a typical chemical potential of \( \mu/\hbar \sim 10^3 - 10^4 s^{-1} \) and that \( \phi_0(\omega_\perp) \) is the harmonic oscillator ground state for \( \omega_\perp \gtrsim \mu/\hbar \). Taking the perimeter of the interferometer to be \( \ell_l + \ell_u \sim 1 \) cm and typical recoil velocities, \( v_0 \sim 1-10 \) cm/s, gives \( \beta \gtrsim 10^{-3} \) for \( ^{87}\text{Rb} \) or \( ^{23}\text{Na} \), so that \( \beta j \gtrsim 1 \) typically.

The results of a numerical evaluation of Eqs. \([3]\) and \([4]\) are shown in Figs. 2 and 3. We find that \( \Delta \theta \) is minimum at \( \theta = (2n + 1)\pi/2 \) and exhibits sharp resonances at \( \theta = n\pi \) where \( n \) is an integer. These resonances disappear in the limit \( \beta \to 0 \).

These features agree qualitatively with a perturbation expansion for \( \Delta \theta \) to lowest order in \( \beta j \),

\[ \Delta \theta = \sqrt{[j(j+1) - m^2]/2m^2(1 + (\beta j)^2(a(m,j) + b(m,j) \cot^2 \theta))} \]

where \( a(m,j) \) and \( b(m,j) \) are positive definite and of order unity. Fig. 2 shows that for fixed \( N_A \) the normalized phase fluctuations, \( \Delta \theta(\beta)/\Delta \theta(\beta = 0) \), increase with increasing \( N_B \) and for fixed \( N \), the nonlinearity also results in an increase in the phase fluctuations with increasing \( N_B \). Figure 3 shows that for \( \beta \lesssim 10^{-3} \), \( \Delta \theta \) continues to scale as \( 1/\sqrt{N_A} \) when \( N_B = 0 \) while for \( \beta > 10^{-3} \), \( \Delta \theta \) can actually show an increase with increasing \( N_A \).

Fermi Gas: As with the bosons, we consider uniform beams of \( N_A \) and \( N_B \) fermions entering the interferometer via input ports A and B in a time \( \tau \) at average velocities \( v_0 \). Their densities are \( n_A = N_A/v_0 \tau \) and \( n_B = N_B/v_0 \tau \), and the quantum state for the incident
fermions is then given by
\[ |\Psi(0)\rangle = \prod_{|k-k_0| \leq k_{FA}} \hat{a}_k^\dagger \prod_{|k-k_0| \leq k_{FB}} \hat{b}_k^\dagger |0\rangle \]
where \( k_{FA} = \pi n_A \) and \( k_{FB} = \pi n_B \) are the Fermi wave numbers for the two beams. We note from the definitions of \( k_{FA} \) and \( k_{FB} \) that there exists a new fundamental length scale, \( L_F = v_0 \tau \) that is important for the description of the fermions but was unimportant for the bosons. It represents the length of the two incident beams and can be interpreted as the quantization volume for the fermions.

\( \langle N \rangle \) and \( \Delta N^2 \) are now found to be
\[ \langle N \rangle = \cos \theta [\bar{n}_A S(\bar{n}_A \theta/2) - \bar{n}_B S(\bar{n}_B \theta/2)]/\alpha \]
\[ \Delta N^2 = (\bar{n}_A - \bar{n}_B - \cos 2\theta [\bar{n}_A S(\bar{n}_A \theta) - \bar{n}_B S(\bar{n}_B \theta)])/2 \alpha \]
where \( S(x) = \sin(x)/x \), \( \alpha = (2\pi/k_0)/L_F \), \( \theta = k_0 (\ell_2 - \ell_1) \) is the phase difference obtained for a single atom with momentum \( h k_0 \), and we have taken \( N_A > N_B \) without loss of generality. We have also introduced the dimensionless densities \( \bar{n}_A = 2\pi n_A/k_0 = 2k_{FA}/k_0 \) and \( \bar{n}_B = 2\pi n_B/k_0 = 2k_{FB}/k_0 \). They represent the number of atoms within the de Broglie wavelength \( \lambda_0 = (2\pi)/k_0 \), so that quantum degeneracy corresponds to \( \bar{n} \gtrsim 1 \). Note that \( \bar{n} \) is also a measure of the degree of monochromaticity of the incident Fermi beam. Earlier treatments of fermion interferometers dealt with quasi-monochromatic beams that correspond to \( \bar{n} \ll 1 \).

We first restrict ourselves to the case \( \bar{n}_B = 0 \). The phase uncertainty is then
\[ \Delta \theta = \frac{1}{2\sqrt{\bar{n}_A}} \times \frac{\sqrt{\theta^2(1 - \cos 2\theta S(\bar{n}_A \theta)/2)}}{|\cos \theta \cos(\theta \bar{n}_A/2) - S(\bar{n}_A \theta/2)/\theta \sin \theta + \cos \theta|} \]
which shows that \( \Delta \theta \sim 1/\sqrt{N_A} \). However, Eq. (8) also indicates that \( \Delta \theta \) is an increasing function of \( \theta \) and hence of the path length difference. This is easily understood by noting that the individual phases contributing to the total interference pattern are uniformly distributed over the interval \( [\theta - \delta \theta, \theta + \delta \theta] \), where \( \delta \theta = k_F |\ell_2 - \ell_1| = \bar{n}_A \theta/2 \).

Consequently, for degenerate fermion beams with \( \bar{n}_A \gtrsim 1 \), the interference pattern is undetectable for large path differences, \( |\theta| \gtrsim \pi \). This will continue to hold when \( \bar{n}_B \neq 0 \).

Figure 4 shows the minimum value of \( \Delta \theta \) for fixed \( \bar{n}_A \) and various \( \bar{n}_B \) plotted as a function of \( \sqrt{\bar{n}} \sim 1/\sqrt{N} \). It is clear that the phase fluctuations continue to scale as \( 1/\sqrt{N} \). Moreover, a comparison with Fig. 2 shows that increasing the number of atoms entering via input port B does not result in as dramatic an increase in \( \Delta \theta \) as with bosons. Whereas for bosons having only \( \sim 10\% \) of the atoms enter via port B could result in an increase in \( \Delta \theta \) by a factor of 5 or more for moderate values of \( \beta \), having more than a third of the atoms enter via B now results in an increase in \( \Delta \theta \) of less than a factor of 2.

Fig. 5 provides a direct comparison of \( \Delta \theta \) for the BEC and Fermi beams when \( N_B = 0 \). Remarkably, the phase resolution limit achievable with the Fermi gas is noticeably better than what can be achieved with even a non-interacting BEC. Moreover, increasing the fermions density enhances the phase resolution. Of course, this increase comes at the expense of a limited range of \( \theta \) over which the interference pattern is discernible. For BECs, by contrast, \( \Delta \theta \) is independent of \( \theta \) for \( \beta = 0 \) and even for \( \beta \neq 0, \Delta \theta \) is nearly constant except in the vicinity of the resonances.

In conclusion, we have examined the phase resolution limit for an atomic Mach-Zehnder using degenerate Bose and Fermi gases. Our results indicate that one can attain better phase resolution using Fermi gases at the expense of a limited range of path differences over which there is a discernible interference pattern.

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