Bounds on Variation of Spectral Subspaces under $J$-Self-adjoint Perturbations

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Abstract. Let $A$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Assume that the spectrum of $A$ consists of two disjoint components $\sigma_0$ and $\sigma_1$. Let $V$ be a bounded operator on $\mathcal{H}$, off-diagonal and $J$-self-adjoint with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ where $\mathcal{H}_0$ and $\mathcal{H}_1$ are the spectral subspaces of $A$ associated with the spectral sets $\sigma_0$ and $\sigma_1$, respectively. We find (optimal) conditions on $V$ guaranteeing that the perturbed operator $L = A + V$ is similar to a self-adjoint operator. Moreover, we prove a number of (sharp) norm bounds on the variation of the spectral subspaces of $A$ under the perturbation $V$. Some of the results obtained are reformulated in terms of the Krein space theory. As an example, the quantum harmonic oscillator under a $\mathcal{PT}$-symmetric perturbation is discussed.

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1. Introduction

Let $A$ be a (possibly unbounded) self-adjoint operator on a Hilbert space $\mathcal{H}$. Assume that $V$ is a bounded operator on $\mathcal{H}$. It is well known that in such a case the spectrum of the perturbed operator $L = A + V$ lies in the closed $\|V\|$-neighborhood of the spectrum of $A$ even if $V$ is non-self-adjoint. Thus, if the spectrum of $A$ consists of two disjoint components $\sigma_0$ and $\sigma_1$, that is, if

$$\text{spec}(A) = \sigma_0 \cup \sigma_1 \quad \text{and} \quad \text{dist}(\sigma_0, \sigma_1) = d > 0,$$

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then the perturbation $V$ with a sufficiently small norm does not close the gaps between $\sigma_0$ and $\sigma_1$ in $\mathbb{C}$. This allows one to think of the corresponding disjoint spectral components $\sigma_0'$ and $\sigma_1'$ of the perturbed operator $L = A + V$ as a result of the perturbation of the spectral sets $\sigma_0$ and $\sigma_1$, respectively.

Assuming (1.1), by $E_A(\sigma_0)$ and $E_A(\sigma_1)$ we denote the spectral projections of $A$ associated with the disjoint Borel sets $\sigma_0$ and $\sigma_1$, and by $\mathcal{H}_0$ and $\mathcal{H}_1$ the respective spectral subspaces, $\mathcal{H}_0 = \text{Ran } E_A(\sigma_0)$ and $\mathcal{H}_1 = \text{Ran } E_A(\sigma_1)$. If there is a possibility to associate with the disjoint spectral sets $\sigma_0'$ and $\sigma_1'$ the corresponding spectral subspaces of the perturbed (non-self-adjoint) operator $L = A + V$, we denote them by $\mathcal{H}_0'$ and $\mathcal{H}_1'$. In particular, if one of the sets $\sigma_0'$ and $\sigma_1'$ is bounded, this can easily be done by using the Riesz projections (see, e.g. [24, Sec. III.4]).

In the present note we are mainly concerned with bounded perturbations $V$ that possess the property

$$V^* = JVJ,$$

(1.2)

where $J$ is a self-adjoint involution on $\mathcal{H}$ given by

$$J = E_A(\sigma_0) - E_A(\sigma_1).$$

(1.3)

Operators $V$ with the property (1.2) are called $J$-self-adjoint.

A bounded perturbation $V$ is called diagonal with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ if it commutes with the involution $J$, $VJ = JV$. If $V$ anticommutes with $J$, i.e. $VJ = -JV$, then $V$ is said to be off-diagonal. Clearly, any bounded $V$ can be represented as the sum $V = V_{\text{diag}} + V_{\text{off}}$ of the diagonal, $V_{\text{diag}}$, and off-diagonal, $V_{\text{off}}$, terms. The spectral subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ remain invariant under $A + V_{\text{diag}}$ while adding a non-zero $V_{\text{off}}$ does break the invariance of $\mathcal{H}_0$ and $\mathcal{H}_1$. Thus, the core of the perturbation theory for spectral subspaces is in the study of their variation under off-diagonal perturbations (cf. [25]). This is the reason why we add to the hypothesis (1.2) another basic assumption, namely that all the perturbations $V$ involved are off-diagonal with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$.

We recall that if an off-diagonal perturbation $V$ is self-adjoint in the usual sense, that is, $V^* = V$, then the condition

$$\|V\| < \frac{d}{2}$$

(1.4)

ensuring the existence of gaps between the perturbed spectral sets $\sigma_0'$ and $\sigma_1'$ may be essentially relaxed. Generically, if no assumptions on the mutual position of the initial spectral sets $\sigma_0$ and $\sigma_1$ are made except (1.1), the sets $\sigma_0'$ and $\sigma_1'$ remain disjoint for any off-diagonal self-adjoint $V$ satisfying the bound $\|V\| < \sqrt{d}$ (see [27, Theorem 1 (ii)] and [49, Theorem 5.7 (ii)]). If, in addition to (1.1), it is known that one of the sets $\sigma_0$ and $\sigma_1$ lies in a finite gap of the other set then this bound may be relaxed further: for the perturbed sets $\sigma_0'$ and $\sigma_1'$ to be disjoint it only suffices to require that $\|V\| < \sqrt{d}$ (see [26, Theorem 3.2 and Remark 3.3]; cf. [27, Theorem 2 (i)] and [49, Theorem 5.7 (iii)]). Finally, if the sets $\sigma_0$ and $\sigma_1$ are subordinated, say $\sup \sigma_0 < \inf \sigma_1$, then no requirements on $\|V\|$ are needed.