SET THEORY IS INTERPRETABLE IN THE
AUTOMORPHISM GROUP OF AN INFINITELY
GENERATED FREE GROUP

VLADIMIR TOLSTYKH

Introduction

In 1976 S. Shelah [8] showed that in the endomorphism semi-group of an
infinitely generated algebra which is free in a variety one can interpret some
set theory. It follows from his results that, for an algebra $F_\kappa$ which is free of
infinite rank $\kappa$ in a variety of algebras in a language $L$, if $\kappa > |L|$, then the first-
order theory of the endomorphism semi-group of $F_\kappa$, $\text{Th(End}(F_\kappa))$ syntactically
interprets $\text{Th}(\kappa, L_2)$, the second-order theory of the cardinal $\kappa$. This means that
for any second-order sentence $\chi$ of empty language there exists $\chi^*$, a first-order
sentence of semi-group language, such that for any infinite cardinal $\kappa$

$$\chi \in \text{Th}(\kappa, L_2) \iff \chi^* \in \text{Th(End}(F_\kappa)).$$

In his paper Shelah notes that it is natural to study the similar problem
for automorphism groups instead of endomorphism semi-groups; a priori the
expressive power of the first-order logic for automorphism groups is less than
the one for endomorphism semi-groups. For instance, according to Shelah’s
results of 1973 [4, 5] on permutation groups, one cannot interpret set theory
by means of first-order logic in the permutation group of an infinite set, the
automorphism group of an algebra in empty language. On the other hand, one
can do this in the endomorphism semi-group of such an algebra.

In [10] the author found a solution for the case of the variety of vector spaces
over a fixed division rings. If $V$ is a vector space over $D$ of an infinite dimension
$\kappa$, then theory $\text{Th}(\kappa, L_2)$ is interpretable in the first-order theory of $\text{GL}(V)$,
the automorphism group of $V$. When a division ring $D$ is a countable and defin-
able up to isomorphism by a second-order sentence, then theories $\text{Th}(\text{GL}(V))$
and $\text{Th}(\kappa, L_2)$ are mutually syntactically interpretable. In general case, the
formulation is a bit more complicated.

The main result of this paper, Theorem 7.1, states that a similar result holds
for the variety of all groups:

**Theorem 7.1.** Let $F$ be an infinitely generated free group of the rank $\kappa$.
Then the second-order theory of the set $\kappa$ and the elementary theory of $\text{Aut}(F)$
are mutually interpretable, uniformly in $F$.  

*Date*: March 30, 2022.

Supported by Russian Foundation of Fundamental Research Grant 96-01-00456.
As a corollary we have

**Theorem 7.2.** Let $F$ and $F'$ be infinitely generated free groups of ranks $\kappa$ and $\kappa'$, respectively. Then their automorphism groups are elementarily equivalent if and only if the cardinals $\kappa$ and $\kappa'$ are equivalent in the second-order logic as sets:

$$\text{Aut}(F) \equiv \text{Aut}(F') \iff \kappa \equiv_{L_2} \kappa'.$$

Let $F$ denote an infinitely generated free group. In Sections 1-4 we prepare ‘building materials’ for the first-order interpretation of infinitely generated free groups in their automorphism groups. We prove that the subgroup $\text{Inn}(F)$ of all conjugations (inner automorphisms of $F$) is $\emptyset$-definable in the group $\text{Aut}(F)$ (Theorem 4.1). Our key technical result states the set of all conjugations by powers of primitive elements is $\emptyset$-definable in $\text{Aut}(F)$. (Proposition 4.3).

We use in the proof of Theorem 1.1 a characterization of involutions in automorphism groups of free groups – known due to results of J. Dyer and G. P. Scott [3]. The Theorem enables us to prove the completeness of the automorphism groups of arbitrary non-abelian free groups. This generalizes the result of J. Dyer and E. Formanek of 1975 [2], who proved the completeness of automorphism groups of finitely generated non-abelian free groups.

In Section 5 we reconstruct (without parameters) in the group $\text{Aut}(F)$ the three-sorted structure $\langle \text{Aut}(F), F, S \rangle$, where $S$ denotes the set of all free factors of $F$. The basic relations of the latter structure are those of $\text{Aut}(F)$ and $F$, the actions of $\text{Aut}(F)$ on $F$ and $S$, the membership relation on $F \cup S$, and the relation $A = B \ast C$ on $S$.

The main result of the Section 6 is a recovering of a basis of $F$ in the structure $\langle \text{Aut}(F), F, S \rangle$. We interpret with definable parameters in the structure $\langle \text{Aut}(F), F, S \rangle$ the structure $\langle \text{Aut}(F), F, B \rangle$ (with natural relations), where $B$ is a free basis of $F$.

In the final section, using quite standard techniques, we prove that the elementary theory of the structure $\langle \text{Aut}(F), F, B \rangle$ and the second-order theory of the set $\kappa$, where $\kappa = \text{rank} F$, are mutually syntactically interpretable, uniformly in $\kappa$.

The author is grateful to his colleagues in Kemerovo State University Oleg Belegradek, Valery Mishkin, and Peter Biryukov for reading of a very draft of this paper and helpful comments. Some results of the paper were announced in the abstract [11].

1. Involutions

In what follows $F$ (unless otherwise stated) stands for an infinitely generated free group. Let $\overline{F} = F/[F,F]$. Clearly, $\overline{F}$ is a free abelian group of the same rank. The natural homomorphism $w \mapsto \overline{w}$ from the group $F$ to $\overline{F}$ provides the homomorphism $\text{Aut}(F) \to \text{Aut}(\overline{F})$. To denote this homomorphism we shall be
using the same symbol $\gamma$. We shall also say that an automorphism $\varphi \in \text{Aut}(F)$ induces the automorphism $\overline{\varphi} \in \text{Aut}(\overline{F})$.

In [3] J. Dyer and G. P. Scott obtained a description of automorphisms of $F$ of prime order. For involutions that description yields the following

**Theorem 1.1.** [3, p. 199] For every involution $\varphi$ in the group $\text{Aut}(F)$ there is a basis $\mathcal{B}$ of $F$ of the form

$$
\{ u : u \in U \} \cup \{ z, z' : z \in Z \} \cup \{ x, y : x \in X, y \in Y_x \}
$$
on which $\varphi$ acts as follows

$$
\begin{align*}
\varphi u &= u, & u &\in U, \\
\varphi z &= z', & z &\in Z, \\
\varphi z' &= z, \\
\varphi x &= x^{-1}, & x &\in X, \\
\varphi y &= xyx^{-1}, & y &\in Y_x.
\end{align*}
$$

Specifically, the fixed point subgroup of $\varphi$, $\text{Fix}(\varphi)$, is the subgroup $\langle u : u \in U \rangle$, and hence is a free factor of $F$.

We shall call a basis of $F$ on which $\varphi$ acts similar to (1.1) a **canonical** basis for $\varphi$. In view of (1.1) one can partition every canonical basis $\mathcal{B}$ for $\varphi$ as follows

$$
\mathcal{B} = U(\mathcal{B}) \cup Z(\mathcal{B}) \cup \{ z' : z \in Z(\mathcal{B}) \} \cup X(\mathcal{B}) \cup \bigcup_{x \in X(\mathcal{B})} Y_x(\mathcal{B}).
$$

We shall also call any set of the form $\{ x \} \cup Y_x$, where $x \in X(\mathcal{B})$ a **block** of $\mathcal{B}$, and the cardinal $|Y_x| + 1$ the **size** of a block. The subgroup generated by the set $Y_x$ will be denoted by $C_x$ ($\varphi$ operates on this subgroup as conjugation by $x$), and the subgroup generated by the block $\{ x \} \cup Y_x$ will be denoted by $H_x$. Sometimes we shall be using more ‘accurate’ notation like $C_x^\varphi$ or $H_x^\varphi$. The set $U(\mathcal{B})$ will be called the **fixed part** of $\mathcal{B}$.

Clearly, if $\mathcal{B}$ and $\mathcal{C}$ are some canonical bases for involutions $\varphi, \psi$, respectively, and the action of $\varphi$ on $\mathcal{B}$ is isomorphic to the action of $\psi$ on $\mathcal{C}$ (that is the corresponding parts of their canonical bases given by (1.2) are equipotent), symbolically $\varphi|\mathcal{B} \cong \psi|\mathcal{C}$, then $\varphi$ and $\psi$ are conjugate.

For the sake of simplicity we prove the converse (in fact a stronger result) only for involutions we essentially use: for involutions with $Z(\mathcal{B}) = \emptyset$ in all canonical bases $\mathcal{B}$’s. We shall call these involutions **soft** involutions. It is useful that involutions in $\text{Aut}(\overline{F})$ induced by them have a sum of eigen $\pm$-subgroups equal to $\overline{F}$, like involutions in general linear groups over division rings of characteristic $\neq 2$.

We shall say that involutions $\varphi, \psi \in \text{Aut}(F)$ have the same canonical form, if for all canonical bases $\mathcal{B}, \mathcal{C}$ of $\varphi$ and $\psi$, respectively, $\varphi|\mathcal{B} \cong \psi|\mathcal{C}$. Note that a priori we cannot even claim that the relation we introduce is reflexive.
Proposition 1.2. Let \( \varphi \in \text{Aut}(F) \) be a soft involution. An involution \( \psi \in \text{Aut}(F) \) is conjugate to \( \varphi \) if and only if \( \psi \) is soft and \( \varphi, \psi \) have the same canonical form.

Proof. Let \( F_2 = F/2F \) that is the quotient group of \( F \) by the subgroup of even elements. Natural homomorphisms \( F \to F \) and \( F \to F_2 \), gives us a homomorphism \( \mu : \text{Aut}(F) \to \text{Aut}(F_2) \). Clearly, the family of all involutions in \( \ker \mu \) coincides with the family of all soft involutions. Therefore an involution which is conjugate to a soft involution is soft, too.

Lemma 1.3. Let \( \varphi \) be a soft involution with a canonical basis \( \mathcal{B} \).

(i) Suppose \( a \) is an element in \( F \) such that \( \varphi a = a^{-1} \). Then \( a = \varphi(w)w^{-1} \) or \( a = \varphi(w)xw^{-1} \) for some \( x \in X = X(\mathcal{B}) \) and \( w \in F \).

(ii) Suppose \( C \) is a maximal subgroup of \( F \) on which \( \varphi \) acts as conjugation by \( x \in X(\mathcal{B}) \):

\[
C = \{ c \in F : \varphi(c) = xcx^{-1} \}.
\]

Then \( C = C_x^\varphi = \langle Y_x \rangle \).

Proof. (i) By \ref{1.1} we have

\[
F = \text{Fix}(\varphi) \ast \prod_{x \in X} H_x,
\]

where each factor is \( \varphi \)-invariant. Then \( a = a_1 \ldots a_n \), where every \( a_i, i = 1, \ldots, n \) is an element of a free factor in expansion \ref{1.3}, \( a_i \) and \( a_{i+1} \) lie in different factors for every \( i = 1, \ldots, n-1 \) (that is the sequence \( a_1, a_2, \ldots, a_n \) is reduced). Hence if \( \varphi(a) = a^{-1}, \) or \( \varphi(a_1) \ldots \varphi(a_n)a_1 \ldots a_n = 1 \) then

\[
\varphi(a_n)a_1 = 1, \quad \varphi(a_{n-1})a_2 = 1, \quad \ldots
\]

Therefore \( a = \varphi(w_0)w_0^{-1} \) or \( a = \varphi(w_0)v_0w_0^{-1} \), where \( v \in H_x \) for some \( x \in X \) and \( \varphi(v) = v^{-1} \).

So let us prove, using induction on length of a word \( v \) in the basis \( \mathcal{B} \), that \( v = \varphi(w_1)w_1^{-1} \) or \( v = \varphi(w_1)xw_1^{-1} \). The only words \( v \) of length one in \( H_x \) with \( \varphi v = v^{-1} \) are \( x \) and \( x^{-1} = \varphi(x)xx^{-1} \).

An arbitrary element \( v \in H_x \) can be written in the form

\[
v = x^{k_1}y_1x^{k_2}y_2\ldots x^{k_m}y_m,
\]

where \( y_i \in C_x \), the elements \( x^{k_1} \) and \( y_m \) could be equal to 1, but any other element is non-trivial. Since \( \varphi \) acts on \( C_x \) as conjugation by \( x \) we have

\[
\varphi(v) = x^{-k_1+1}y_1x^{-k_2}y_2\ldots x^{-k_m}y_mx^{-1}.
\]

Suppose that \( \varphi(v)v = 1 \). We then have

\[
x^{-k_1+1}y_1x^{-k_2}y_2\ldots x^{-k_m}y_mx^{-1}x^{k_1}y_1x^{k_2}y_2\ldots x^{k_m}y_m = 1.
\]

Let first \( y_m \neq 1 \). Then \( k_1 = 1 \) and \( y_m = y_1^{-1} \). Hence

\[
v = xy_1x^{-1}(x^{k_2+1}y_2\ldots x^{k_m})y_1^{-1} = \varphi(y_1)y_1^{-1}.
\]
It is easy to see that \( \varphi(t) = t^{-1} \) and length of \( t \) is less than length of \( v \). In the case when \( y_m = 1 \) we have \( k_m \neq 0 \) and \( k_1 = k_m + 1 \). Therefore,

\[
v = x^{k_m}(xy_1 \ldots y_{m-1}) = \varphi(x^{-k_m})tx^{k_m},
\]

and we again have that \( \varphi(t) = t^{-1} \) and \( |t| < |v| \).

(ii) Let \( \varphi = xcx^{-1} \) and \( c = c_1c_2 \ldots c_n \), where \( c_i \) are elements in free factors from (1.3) and the sequence \( c_1, c_2, \ldots , c_n \) is reduced. Suppose that \( n \geq 2 \). Due to the \( \varphi \)-invariance of our free factors, the sequence \( \varphi(c_1), \varphi(c_2), \ldots , \varphi(c_n) \) is also reduced and must represent the same element as the sequence \( x, c_1, c_2, \ldots , c_n, x^{-1} \).

It is easy to see that it is possible if both \( c_1, c_n \) lie in \( H_x \). In particular, \( n \geq 3 \). It implies that \( \varphi(c_1) = xc_1 \) and \( \varphi(c_n) = c_nx^{-1} \). It easily follows from (1.4) and (1.3) there is no \( v \in H_x \) with \( \varphi(v) = xv \); it of course means that for every \( v \in H_x \) the equality \( \varphi(v) = vx^{-1} \) is also impossible, since it is equivalent to \( \varphi(v^{-1}) = xv^{-1} \).

Thus, if \( \varphi(c) = xcx^{-1} \), then \( c \in H_x \). By applying formulae (1.4) and (1.3), one can readily conclude that \( c \) must be in \( C_x \). \( \square \)

REMARKS. (a) Note that \( a = \varphi(w)w^{-1} \) cannot be a primitive element (i.e. a member of a basis of \( F \)) since \( \overline{\varphi} \) is an even element of \( \overline{F} \). Indeed, it follows from (1.3) that

\[
\overline{w} = w(U) + \overline{w(X)} + \overline{w(Y)}.
\]

where \( w(U) \in \text{Fix}(\varphi) \), \( w(X) \) is an element in the subgroup generated by \( X \), and \( w(Y) \) is an element in the subgroup generated by the set \( Y = \bigcup_{x \in X} Y_x \). Hence

\[
\overline{a} = \overline{\varphi(w)w^{-1}} = \overline{\varphi(w)} - \overline{w}
\]

\[
= (\overline{w(U)} - \overline{w(X)} + \overline{w(Y)}) - (\overline{w(U)} + \overline{w(X)} + \overline{w(Y)})
\]

\[
= -2\overline{w(X)}.
\]

(b) Using a similar argument we see that if \( \varphi(w_1)x_1w_1^{-1} = \varphi(w_2)x_2w_2^{-1} \), where \( x_1, x_2 \in X \), then \( x_1 = x_2 \) (the element \( \overline{x_1} - \overline{x_2} \) is even if and only if \( x_1 = x_2 \)).

Suppose now that soft involution \( \psi \) is a conjugate of \( \varphi \): \( \psi = \sigma^{-1} \varphi \sigma \). Let

\[
B' = U' \cup X' \cup \bigcup_{x' \in X'} Y'_{x'x'}
\]

be a canonical basis for \( \psi \). Fixed point subgroups of \( \varphi \) and \( \psi \) are clearly isomorphic. Thus, \( |U'| = |U| \).

If \( \psi x' = x'^{-1} \), where \( x' \in X' \), then \( \varphi(\sigma x') = (\sigma x')^{-1} \). By (1.3) (i) and the above remarks there is a unique \( x \in X \) such that

\[
\sigma x' = \varphi(w)xw^{-1}
\]

The mapping \( x' \mapsto x \) determined in such a way is injective, because otherwise we can find two distinct elements in a basis of a free abelian group group whose difference is even.
Hence, $|X'| \leq |X|$ and by symmetry $|X'| = |X|$.

We claim now that
\[ \sigma C'_x = wC_x w^{-1}. \]
It will complete the proof, because in this case $|Y'_x| = |Y_x|$.

Let $y' \in C'_x$, and $b = \sigma y'$. Since $\psi y' = x'y'x'^{-1}$, then $\varphi b = (\sigma x')b(\sigma x')^{-1}$.

Therefore we have
\[ \varphi(w^{-1}bw) = \varphi(w^{-1})\varphi(b)\varphi(w) = x(w^{-1}bw)x^{-1}. \]

Hence by (ii) $w^{-1}bw \in C_x$.

The equation (1.6) can be rewritten as follows
\[ \sigma^{-1}x = \psi(\sigma^{-1}w^{-1})x'\sigma^{-1}w. \]

Therefore
\[ \sigma^{-1}C_x \subseteq \sigma^{-1}w^{-1}C'_x\sigma^{-1}w, \]
or
\[ C_x \subseteq w^{-1}\sigma C'_x w, \]
and the result follows.

\[ \square \]

**Proposition 1.4.** The set of all soft involutions is $\varnothing$-definable in Aut$(F)$.

**Proof.** Let us now cite the above mentioned Dyer-Scott theorem in the full form.

**Theorem** (Dyer-Scott). Let $F$ be free, and $\alpha$ an automorphism of $F$ of prime order $p$. Then

\[ F = \text{Fix}(\alpha) \ast \prod_{i \in I}^* F_i \ast \prod_{\lambda \in \Lambda}^* F_{\lambda}, \]

where each factor is $\alpha$-invariant. Moreover,

(i) For each $i \in I$, $F_i$ has a basis $x_{i,1}, \ldots, x_{i,p}$ such that
\[ \alpha(x_{i,r}) = x_{i,r+1}\text{mod }p. \]

(ii) For each $\lambda \in \Lambda$, $F_\lambda$ has a basis
\[ x_{\lambda,1}, \ldots, x_{\lambda,p-1}, \quad \{y_j : j \in J_\lambda\} \]
such that
\[ \alpha(x_{\lambda,r}) = x_{\lambda,r+1}, \quad r = 1, \ldots, p - 2, \]
\[ \alpha(x_{\lambda,p-1}) = (x_{\lambda,1}, \ldots, x_{\lambda,p-1})^{-1}, \]
\[ \alpha(y_j) = x_{\lambda,1}^{-1}y_jx_{\lambda,1}, \quad j \in J_\lambda. \]

Let $\sigma \in \text{Aut}(F)$ be an element of prime order $p > 2$. Consider the natural homomorphism from Aut$(F)$ to Aut$(F_2)$. As an easy corollary of the Dyer-Scott Theorem, we have that the image of $\sigma$ under this homomorphism is non-trivial.
Two involutions $\varphi, \varphi_0$ in the automorphism group of two-generator free group $G = \langle a_1, a_2 \rangle$ such that

\[
\begin{aligned}
\varphi(a_1) &= a_2^{-1}, & \varphi_0(a_1) &= a_1 a_2, \\
\varphi(a_2) &= a_1^{-1}, & \varphi_0(a_2) &= a_2^{-1},
\end{aligned}
\]

are non-soft ($\{a_1, a_2 a_1\}$ is a canonical basis for latter, and $\{a_1, a_2^{-1}\}$ for former). Their product $\sigma = \varphi_0 \varphi$ is of the order three in $\text{Aut}(G)$:

\[
\begin{aligned}
\sigma(a_1) &= a_2, \\
\sigma(a_2) &= (a_1 a_2)^{-1}.
\end{aligned}
\]

Generalizing this example, we can easily observe that for every non-soft involution $\varphi \in \text{Aut}(F)$ there is a conjugate $\varphi_0 \in \text{Aut}(F)$ of $\varphi$ such that $\varphi_0 \varphi$ has order three.

The latter (first-order) property is evidently false for every soft involution $\varphi \in \text{Aut}(F)$, because for every conjugate $\varphi_0$ of $\varphi$, the product $\varphi_0 \varphi$ cannot have order three (all elements of order three in $\text{Aut}(F)$ have non-trivial images under the natural homomorphism from $\text{Aut}(F)$ in $\text{Aut}(F_2)$, but $\varphi_0 \varphi$ has not).

2. Anti-commutative conjugacy classes

The key roles in the group-theoretic characterization of conjugations in $\text{Aut}(F)$ will be played by two conjugacy classes of involutions. First class consists of involutions with a canonical form

\[
\begin{aligned}
\varphi x &= x^{-1}, \\
\varphi y &= xyx^{-1}, & y &\in Y
\end{aligned}
\]

that is $B = \{x\} \cup Y$ is a basis of $F$, canonical for $\varphi$, $U(B) = Z(B) = \emptyset, X(B)$ is a singleton set $\{x\}, Y_x = Y$ (any canonical form reproduced below is interpreted in a similar way). We shall call these involutions quasi-conjugations.

An arbitrary element $\varphi \in \text{Aut}(F)$ in the second class has the following canonical form:

\[
\varphi x = x^{-1}, & x \in X,
\]

that is there is a basis of $F$ such that $\varphi$ inverts all its elements. We shall use for these involutions the term symmetries.

Every symmetry induces in $\text{Aut}(F)$ the automorphism $-\text{id}_F$ and hence the product of two symmetries induces $\text{id}_F$. Therefore by [1.1] two symmetries commute if and only if they are coincide. Thus, the conjugacy class of all symmetries is, say, anti-commutative, since its elements are pairwise non-commuting. In the next section we shall prove that the class of all quasi-conjugation is also anti-commutative.

In order to characterize conjugations we shall use anti-commutative conjugacy classes of involutions, but we need not an exact determination of all such
conjugacy classes: it suffices to know that they lie in some ‘easy-to-define’ family. In the following proposition we formulate and prove a necessary condition of being an anti-commutative conjugacy class, but do not prove its sufficiency.

**Proposition 2.1.** Let \( \varphi \) be an involution in an anti-commutative conjugacy class. Then either \( \varphi \) has a canonical form such that

\[
\begin{align*}
\varphi u &= u, & u &\in U, \\
\varphi x &= x^{-1}, & x &\in X, \\
\varphi y &= xyx^{-1}, & y &\in Y,
\end{align*}
\]

where \(|X| \geq 2\), all the sets \( Y_x, x \in X \) have the same finite power \( n \) and \(|U| < n + 1\), or \( \varphi \) has a canonical form such that

\[
\begin{align*}
\varphi u &= u, & u &\in U, \\
\varphi x &= x^{-1}, \\
\varphi y &= xyx^{-1}, & y &\in Y,
\end{align*}
\]

where the cardinal \(|U|\) is finite and less than \(|Y| + 1\).

Furthermore, all involutions of the form \((2.1)\) are squares in \( \text{Aut}(F) \) and involutions of the form \((2.1)\) are not.

In other words, in the terminology introduced in Section 1, a canonical basis \( B \) for an involution in an anti-commutative conjugacy class either contains exactly one block and the power of the fixed part of \( B \) is less than the size of this block, or all blocks of \( B \) have the same finite size and the power of the fixed part is less than the size of any block. Clearly, symmetries have the form \((2.1)\) (all blocks of their canonical bases have the size one), and quasi-conjugations have the form \((2.1)\) (the size of the unique block is equal to rank \( F \)).

**Proof.** Show first that every anti-commutative class of involutions consists only of soft involutions. Indeed, let involution \( \varphi \) be an involution whose canonical basis \( B \) has non-empty ‘permutational’ part \( Z(B) \). Suppose \( \varphi \) takes \( z \in Z(B) \) to \( z' \). Consider an involution \( \psi \) which acts on \( B \setminus \{z, z'\} \) exactly as \( \varphi \) does, but taking \( z \) to \( z'^{-1} \). Clearly, \( \psi \) is conjugate to \( \varphi \) and commutes with \( \varphi \).

The following example demonstrates why the size of the fixed part of a canonical basis must necessarily be less than the size of each block:

\[
\begin{align*}
\varphi x &= x^{-1}, & \psi x &= x, \\
\varphi y &= xyx^{-1}, & \psi y &= y, \\
\varphi x_1 &= x_1, & \psi x_1 &= x_1^{-1}, \\
\varphi y_1 &= y_1, & \psi y_1 &= x_1 y_1 x_1^{-1}, \\
\varphi u &= u & \psi u &= u.
\end{align*}
\]
Let us make a technical remark. Involutions
\[
\begin{align*}
\varphi_x &= x^{-1}, \\
\varphi_y &= x y x^{-1}, \quad y \in Y
\end{align*} \quad \text{and} \quad \begin{align*}
\psi_x &= x^{-1}, \\
\psi_y &= x^{-1} y x^{-1}, \quad y \in Y
\end{align*}
\]
are conjugate: the second one acts ‘canonically’ on the set \( \{x, x^{-1} y : y \in Y\} \):
\( \varphi(x^{-1} y) = x(x^{-1} y)x^{-1} \).

Let now \( \varphi \) be a soft involution with a canonical basis \( B \) such that for some distinct \( x_1, x_2 \in X(B) \) either \( |Y_{x_1}| < |Y_{x_2}| \) or \( |Y_{x_1}| = |Y_{x_2}| \) and the cardinal \( |Y_{x_1}| \) is infinite (thus, there must be no neither a pair of blocks of different size nor a pair of infinite blocks). The fact that \( \varphi \) commutes with at least two its conjugates is a consequence of (2.1) and the following

**Claim 2.2.** Let \( G \) be a free group with a basis
\[
\{x, a\} \cup B \cup \{c\} \cup D \cup E,
\]
where \( |B| = |D| \). Then

(i) Involutions
\[
\begin{align*}
\varphi x &= x^{-1}, \\
\varphi a &= x^{-1} a x^{-1}, \\
\varphi b &= x^{-1} b x^{-1}, \\
\varphi c &= c^{-1}, \\
\varphi d &= c^{-1} d c^{-1}, \\
\varphi e &= x^{-1} e x^{-1}
\end{align*} \quad \text{and} \quad \begin{align*}
\psi x &= x^{-1}, \\
\psi a &= a^{-1}, \\
\psi b &= a^{-1} b a^{-1}, \quad b \in B, \\
\psi c &= x^{-1} c x^{-1}, \\
\psi d &= x^{-1} d x^{-1}, \quad d \in D, \\
\psi e &= x^{-1} e x^{-1}, \quad e \in E
\end{align*}
\]
are conjugate and commute.

(ii) Furthermore,
\[
\begin{align*}
\text{rank } C^\varphi_x &= |\{a\} \cup B \cup E| = |B| + |E| + 1, \\
\text{rank } C^\varphi_c &= |D|.
\end{align*}
\]
Thus, \( \text{rank } C^\varphi_c \leq \text{rank } C^\varphi_x \), where equality holds only if both ranks are infinite.

**Proof.** Easy. \( \square \)

To complete the proof of the first statement in the Proposition, we should cut off involutions with a canonical form such that
\[
\begin{align*}
\varphi u &= u, \quad u \in U, \\
\varphi x &= x^{-1}, \\
\varphi y &= x y x^{-1}, \quad y \in Y,
\end{align*}
\]
where $|U|$ is infinite. To do this let us consider involutions $\varphi, \psi$ which act on a basis $U \cup \{x\} \cup Y$ of $F$ as follows

\[
\begin{align*}
\varphi u_0 &= u_0, & \psi u &= u_0^{-1}, \\
\varphi u &= u, & \psi u &= u, & u \in U \setminus \{u_0\}, \\
\varphi x &= x^{-1}, & \psi x &= u_0xu_0^{-1}, \\
\varphi y &= xyx^{-1}, & \psi y &= u_0yu_0^{-1}, & y \in Y,
\end{align*}
\]

where $u_0$ is a fixed element in $U$. It is readily seen that $\varphi$ and $\psi$ are conjugate and commute.

One can readily check that

\[
\begin{align*}
\sigma x &= a^{-1}, & \sigma^2 x &= x^{-1}, \\
\sigma y &= aba^{-1}, & \sigma^2 y &= xyx^{-1}, \\
\sigma a &= x, & \sigma^2 a &= a^{-1}, \\
\sigma b &= y & \sigma^2 b &= aba^{-1}.
\end{align*}
\]

It demonstrates that every automorphism of $F$ of the form (2.1) is a square in $\text{Aut}(F)$.

On the other hand, every automorphism of the form (2.1) cannot be a square. Indeed, every $\varphi \in \text{Aut}(F)$ of the form (2.1) induces in $\text{Aut}(F)$ an extremal involution, that is an involution such that one of its ±-eigen subgroups, $\langle \varphi \rangle$ in our case, has rank one. Suppose that $\varphi = \sigma^2$ in $\text{Aut}(F)$. We then have

\[
\varphi(\sigma x) = \sigma(\varphi x) = -\sigma x.
\]

Therefore $\sigma x = m \varphi$ for $m \in \mathbb{Z}$. Since $-1$ is not a square in $\mathbb{Z}$, the equation $\varphi = \sigma^2$ is impossible.

3. Quasi-conjugations

In this section we obtain a first-order characterization of quasi-conjugations in $\text{Aut}(F)$. First we prove that the conjugacy class of all quasi-conjugations is anti-commutative. Then we distinguish this class from other anti-commutative conjugacy classes of involutions: it is trivial in the case when rank $F = 2$ and more technical in the case when rank $F > 2$.

**Proposition 3.1.** The class of all quasi-conjugation is an anti-commutative conjugacy class.

**Proof.** Let $\varphi$ be a quasi-conjugation with a canonical basis $B = \{x\} \cup Y$:

\[
\begin{align*}
\varphi x &= x^{-1}, \\
\varphi y &= xyx^{-1}, & y \in Y
\end{align*}
\]

Every $\sigma \in \text{Aut}(F)$ which commutes with $\varphi$ takes $x$ to a primitive element of the form $\varphi(w)xw^{-1}$, $w \in F$ (Lemma [3]). It turns out that the only primitive elements of the form $\varphi(w)xw^{-1}$ are $x$ and $x^{-1}$. This fact is a consequence of the following result, which we shall use once more later.
Lemma 3.2. Let $\alpha$ be an involution with a canonical form such that
\begin{align*}
\alpha x &= x^{-1}, \\
\alpha y &= xyx^{-1}, \ y \in Y = Y_x, \\
\alpha u &= u, \ u \in U,
\end{align*}
and $a \in F$ a primitive element which $\alpha$ sends to its inverse. Then $a = vx^\pm 1v^{-1}$, where $v \in \langle U \rangle = \text{Fix}(\alpha)$.

Proof. As we observed earlier $a = \alpha(w)xw^{-1}$. It is easy to see that $a$ lies in the normal closure of $x$. One can use induction on length of a reduced word $w$ in the basis $\{x\} \cup Y \cup U$. We have
\begin{align*}
\alpha(xw_1)xw_1^{-1}x^{-1} &= x^{-1}(\alpha(w_1)xw_1^{-1})x^{-1}, \\
\alpha(uw_1)xw_1^{-1}u^{-1} &= u(\alpha(w_1)xw_1^{-1})u^{-1}, \\
\alpha(yw_1)xw_1^{-1}y^{-1} &= x \cdot y(x^{-1}\alpha(w_1)xw_1^{-1})y^{-1}.
\end{align*}

Proposition . ([5, II.5.15]) Let $F$ be a free group and the normal closure of $q \in F$ consists of a primitive element $p$. Then $q$ is conjugate to $p$ or $p^{-1}$.

Hence $a = bx^\varepsilon b^{-1}$, where $\varepsilon = \pm 1$. Since $\alpha(a) = a^{-1}$, we have
$$\alpha(b)x^{-\varepsilon}\alpha(b^{-1}) = bx^{-\varepsilon}b^{-1}.$$ It follows that $x^{-\varepsilon}$ and $b^{-1}\alpha(b)$ commute. Therefore both these elements lie in a cyclic subgroup of $F$ ([I, I.2.17]). It must be the subgroup $\langle x \rangle$, because $x$ is primitive. Hence $\alpha(b) = bx^k$. If $k$ is even, say $k = 2m$, then $\alpha(bx^m) = bx^m$ and $v = bx^m \in \langle U \rangle$. In the case when $k$ is odd, we have there is $z \in F$ such that $\alpha(z) = zx$. One can easily check that this is impossible (e.g. using ‘abelian’ arguments as in Section [O]).

Proposition 3.3. The centralizer of a quasi-conjugation $\varphi$ with a canonical basis $\{x\} \cup Y$ consists of automorphisms $\sigma$ of $F$ of the form
\begin{align*}
\sigma x &= x, \\
\sigma y &= \theta(y), \ y \in Y,
\end{align*}
and of the form
\begin{align*}
\sigma x &= x^{-1}, \\
\sigma y &= x\theta(y)x^{-1}, \ y \in Y,
\end{align*}
where $\theta \in \text{Aut}(C_x)$.

Proof. The Proposition easily follows from Lemma 3.2 and one more

Lemma 3.4. [1, p. 101] Let $F = G \ast H$ be a free factorization of a free group $F$. Suppose that $\alpha \in \text{Aut}(F)$ and $\alpha|_G$ is an endomorphism of $G$. Then $\alpha|_G \in \text{Aut}(G)$ if either rank $G$ or rank $H$ is finite.
By \[1.1.2\] if \(\sigma \in \text{Cen}(\varphi)\), then either \(\sigma x = x\) or \(\sigma x = x^{-1}\). Assume that \(\sigma\) fixes \(x\). Then, for each \(y \in C_x\)
\[
\varphi \sigma y = \sigma x \sigma y \sigma x^{-1} = x \sigma y x^{-1}.
\]
Therefore \(\sigma y \in C_x\).

If \(\sigma x = x^{-1}\), then \(\varphi \sigma x = x\). It follows that \(\varphi \sigma\) has the form (3.1), and hence \(\sigma\) must have the form (3.1). \(\square\)

We can prove now that \(\varphi\) is the unique quasi-conjugation in its centralizer, or, in other words, the conjugacy class of \(\varphi\) is anti-commutative. Indeed, there are no quasi-conjugations in the family of automorphisms of the form (3.1), because every quasi-conjugation has trivial fixed-point subgroup. Let \(\sigma\) be a quasi-conjugation of the form (3.1). Hence \(\theta^2 = \text{id}\), and \(\theta\) is either the identity automorphism of \(C_x\) or a soft involution. Assume that \(\theta\) is an involution. By Theorem [1.1] there is a canonical basis \(C\) of a free group \(C_x\) for \(\theta\) such that
\[
\theta u = u, \quad u \in U(C),
\quad \theta c = c^{-1}, \quad c \in X(C),
\quad \theta d = cdc^{-1}, \quad d \in Y_c(C).
\]
Therefore
\[
\sigma x = x^{-1},
\sigma u = xux^{-1}, \quad u \in U(C),
\sigma xc = (xc)^{-1}, \quad c \in X(C),
\sigma d = (xc)d xc^{-1}, \quad d \in Y_c(C).
\]
We obtain a canonical basis for \(\sigma\). This basis contains at least two blocks, and hence by Proposition [1.2] \(\sigma\) cannot be a quasi-conjugation. So \(\theta\) must be the identity, or, equivalently, \(\sigma = \varphi\) as desired. This completes the proof. \(\square\)

The next is

**Theorem 3.5.** The class of all quasi-conjugations is the unique anti-commutative conjugacy class \(K\) of involutions such that its elements are not squares and for every anti-commutative conjugacy class \(K'\) of involutions, whose elements are squares, all involutions in \(KK'\) are conjugate.

**Proof.** Let \(\varphi\) be a quasi-conjugation, \(K'\) an anti-commutative conjugacy class of involutions, and \(\varphi \not\in K'\). It suffices to prove that if \(\psi, \psi' \in K'\) both commute with \(\varphi\), then \(\varphi \psi\) and \(\varphi \psi'\) are conjugate.

Suppose first that fixed point subgroups of \(\psi\) and \(\psi'\) are trivial. Then both \(\psi\) and \(\psi'\) have the form (3.1):
\[
\psi x = x^{-1}, \quad \psi' x = x^{-1},
\quad \psi y = x\theta(y) x^{-1}, \quad \psi' y = x\theta'(y) x^{-1}, \quad y \in Y,
\]
where $\theta$ and $\theta'$ are in Aut$(C_x)$. It is easy to see that $\theta$ and $\theta'$ are soft involutions. Let $C$ be a canonical basis for $\theta$. As we observed above, the set

$$\{x\} \cup U(C) \cup \bigcup_{c \in X(C)} (\{xc\} \cup Y_c(C))$$

is a canonical basis for $\psi$, and (3.1) is a partition of this basis into blocks (there are at least two blocks, because $\psi$ must be a square). Since $\psi$ lies in an anti-commutative conjugacy class, all these blocks have the same size, and hence for all $c \in X(C)$

$$|U(C)| = |Y_c(C)|.$$

By applying a similar argument to $\psi'$, we see that $\theta$ and $\theta'$ are conjugate in Aut$(C_x)$. Hence $\varphi \psi$ and $\varphi \psi'$ are conjugate in Aut$(F)$: both these automorphisms fix $x$ and their restrictions on $C_x (\theta$ and $\theta')$ are conjugate.

Assume that $\psi$ and $\psi'$ have non-trivial fixed point subgroups. Both $\psi$ and $\psi'$ preserve the subgroup $C_x$ and fix the element $x$. It means (Prop. 1.1, Prop. 1.2) that restrictions of $\psi$ and $\psi'$ on $C_x$, say, $\theta$ and $\theta'$, respectively are conjugate in Aut$(C_x)$: $\theta' = \pi^{-1}\theta\pi$, where $\pi \in$ Aut$(C_x)$. The map $\pi$ can be extended to an element $\sigma \in$ Aut$(F)$ such that $\sigma x = x$. By 3.3 $\sigma$ commutes with $\varphi$, and hence

$$\sigma^{-1}(\varphi \psi)\sigma = \varphi \psi'.$$

Let us prove the converse. It suffices to prove that for each inv olution $\varphi$ of the form (2.1), which is not a quasi-conjugation, there are two symmetries $\psi$ and $\psi'$ such that $\varphi \psi$ and $\varphi \psi'$ are non-conjugate involutions.

(a) A natural way to define an action of a symmetry commuting with $\varphi$ on a block of a canonical basis for $\varphi$ is given by Proposition 3.3:

$$\varphi x = x^{-1}, \quad \psi x = x^{-1},$$
$$\varphi y = xyx^{-1}, \quad \psi y = xy^{-1}x^{-1}, \quad (\psi(xy) = (xy)^{-1}) \quad y \in Y_x.$$  

Clearly, the product of $\varphi$ and $\psi$ fixes $x$ and inverts each element in $Y_x$.

(b) A way to define an action of a symmetry on the fixed part of a canonical basis is obvious: a symmetry should invert each element.

Let now $B = U \cup \{x\} \cup Y$ be a canonical basis for $\varphi$:

$$\varphi u = u, \quad u \in U \quad (U \neq \emptyset),$$
$$\varphi x = x^{-1},$$
$$\varphi y = xyx^{-1}, \quad y \in Y.$$  

One can easily find a symmetry $\psi$ such that the product $\varphi \psi$ will be non-conjugate to each product of $\varphi$ with a symmetry obtained in a natural way,
using (a) and (b):

\[ \psi u_0 = u_0^{-1}, \]
\[ \psi u = u^{-1}, \quad u \in U \setminus \{u_0\}, \]
\[ \psi x = u_0 x^{-1} u_0^{-1}, \]
\[ \psi y = u_0 x y^{-1} x^{-1} u_0^{-1}, \quad y \in Y, \]

where \( u_0 \in U \). The reason is that any canonical basis for \( \varphi \psi \) contains a block of the size two (\( \{u_0\} \cup \{x\} \) in our example).

The proof of Theorem 3.5 is now complete. \( \square \)

All the hypotheses in the latter Theorem are surely first-order, and hence we have

**Theorem 3.6.** The set of all quasi-conjugations is first-order definable in \( \text{Aut}(F) \).

**Remark.** In fact, the set of all quasi-conjugations is definable in the automorphism group of an arbitrary non-abelian free group (whether finitely or infinitely generated). We do not prove this result here, but discuss key technical points. It is easy to see that the method of the proof of Proposition 3.1 works for arbitrary free groups of rank at least two. Then there are no further problems in a proof of definability of quasi-conjugations in automorphisms groups of two-generator free groups:

**Proposition.** Suppose that rank \( F = 2 \). Then the class of all quasi-conjugations is the unique anti-commutative conjugacy class \( K \) of involutions whose elements are not squares.

However, the property of being of a square does not always distinguish involutions of the form (2.1) from involutions of the form (2.1), and the generalization of Theorem 3.6 is formulated as follows:

**Theorem.** Let rank \( F > 2 \). Then the class of all quasi-conjugations is the unique anti-commutative conjugacy class \( K \) of involutions such that for every anti-commutative conjugacy class \( K' \) of involutions, all involutions in \( KK' \) are conjugate.

4. **Conjugations**

**Theorem 4.1.** The subgroup \( \text{Inn}(F) \) of all conjugations is \( \emptyset \)-definable subgroup of \( \text{Aut}(F) \).

**Proof.** We start with a quasi-conjugation \( \varphi \). Suppose \( \mathcal{B} = \{x\} \cup Y \) is a canonical basis for \( \varphi \). Let \( \Pi \) denote the set of all automorphisms of \( F \) of the form \( \pi = \sigma \sigma' \), where \( \sigma \) and \( \sigma' \) are in \( \text{Cen}(\varphi) \) and conjugate. By 3.3 conjugate \( \sigma, \sigma' \) in the centralizer of \( \varphi \) either both have the form (3.1) (when their fixed point
subgroups are non-trivial) or have the form (3.1). Therefore every \( \pi \in \Pi \) has the form
\[
\pi x = x, \quad \pi y = \theta(y), \quad y \in Y,
\]
where \( \theta \in \text{Aut}(C_x) \), that is \( \pi \) fixes \( x \) and preserves the subgroup \( C_x \).

In analogy to the term in the linear group theory we call any involution \( \psi \in \text{Aut}(F) \) with a canonical form such that
\[
\psi x = x^{-1}, \quad \psi y = y, \quad y \in Y
\]
extremal involution (compare with Section 3). Actually we need not extremal involutions in the proof of the Theorem, but we shall use them later.

**Lemma 4.2.** (i) All conjugations by powers of \( x \) are in the centralizer of the family \( \Pi \). Every member of \( \text{Cen}(\Pi) \) is either an involution or conjugation by a power of \( x \).

(ii) Every involution in \( \text{Cen}(\Pi) \) is either a quasi-conjugation or an extremal involution. Therefore the set of extremal involutions is \(\emptyset\)-definable in \( \text{Aut}(F) \).

**Proof.** Let
\[
\mathcal{C} = \{a, b\} \cup C
\]
be a basis of \( C_x \), and \( \tau \in \text{Aut}(F) \) an element of \( \text{Cen}(\Pi) \).

First we construct \( \pi \in \Pi \) such that the fixed point subgroup of \( \pi \) is the subgroup \( \langle x, a \rangle \). Since \( \tau \) must commute with \( \pi \) we shall have that
\[
\tau a = w_a(x, a),
\]
where \( w_a \) is a reduced word in letters \( x \) and \( a \).

To construct \( \pi \), let us consider conjugate \( \sigma, \sigma' \) in \( \text{Cen}(\varphi) \) which act on \( \mathcal{C} \) as follows
\[
\sigma x = x, \quad \sigma' x = x, \\
\sigma a = a^{-1}, \quad \sigma' a = a^{-1}, \\
\sigma b = b^{-1}, \quad \sigma' b = a^{-1}b^{-1}a, \\
\sigma c = c^{-1}, \quad \sigma' c = a^{-1}c^{-1}a, \quad c \in C.
\]
The restriction of \( \pi = \sigma \sigma' \) on \( C_x \) is conjugation by \( a \). Then it is easy to show that the fixed point subgroup of \( \pi \) is \( \langle x, a \rangle \). By Lemma 3.4 \( \tau \langle x, a \rangle = \langle x, a \rangle \).

A similar argument can be applied to an arbitrary primitive element in \( C_x \). Hence for every primitive \( d \in C_x \)
\[
\tau d = w_d(x, d),
\]
and \( \tau \) preserves the subgroup \( \langle x, d \rangle \). We then have
\[
\tau \langle x \rangle = \tau (\langle x, a \rangle \cap \langle x, b \rangle) = \langle x, a \rangle \cap \langle x, b \rangle = \langle x \rangle.
\]
Therefore $\tau x = x^{\pm 1}$. In particular, the word $w_a(x, a)$ must have explicit occurrences of $a$.

We claim now that the words $w_d$, where $d = a, b, ab$ have the same structure, that is any word $w_d(x, d)$ can be obtained from the word $w_a(x, a)$ by replacing occurrences of $a$ by $d$:

$$[w_a(x, a)]_d^a = w_d(x, d).$$

To prove this, it suffices to find in $\Pi$ an automorphism which takes $a$ to $b$ (a to $ab$).

Let $\sigma_1$ and $\sigma'_1$ be involutions in $\text{Cen}(\varphi)$ such that $\sigma_1$ and $\sigma'_1$ both fix the set $\{x\} \cup C$ pointwise and

$$\sigma_1 a = b^{-1}, \quad \sigma'_1 a = ab,$$
$$\sigma_1 b = a^{-1}, \quad \sigma'_1 b = b^{-1},$$

Clearly, $\sigma_1$ and $\sigma'_1$ are conjugate and $\pi_1 = \sigma_1 \sigma'_1$ sends $a$ to $b$. Since $\tau$ and $\pi_1$ commute, we have

$$\tau a = w_a(x, a) \Rightarrow \tau (\pi_1 a) = w_a(\pi_1 x, \pi_1 a)$$
$$\Rightarrow w_b(x, b) = w_a(x, b).$$

Thus, there is a reduced word $w$ in letters $x$ and, say, $t$ such that

$$[w(x, t)]_d^t = w_d(x, d),$$

where $d = a, b, ab$. We then have

$$\tau(ab) = w(x, ab) = \tau(a) \tau(b) = w(x, a)w(x, b),$$

and hence

$$w(x, ab) = w(x, a)w(x, b). \quad (4.1)$$

Now we show that the word $w(x, t)$ has the form $x^ktx^{-k}$, where $k \in \mathbb{Z}$. Assume that $w(x, t)$ has the (possibly non-reduced) form such that

$$x^{k_1} x^{k_2} x^{l_2} \cdots x^{k_m} x^{l_m},$$

where $k_1$ or $l_m$ could be equal to zero, whereas any other exponent is non-trivial. Then by (4.1)

$$x^{k_1} (ab)^{l_1} x^{k_2} (ab)^{l_2} \cdots x^{k_m} (ab)^{l_m} = x^{k_1} a^{l_1} x^{k_2} a^{l_2} \cdots x^{k_m} a^{l_m} x^{k_1} b^{l_1} x^{k_2} b^{l_2} \cdots x^{k_m} b^{l_m}. \quad (4.2)$$

The latter equality is evidently impossible when $m \geq 2$ and $l_m \neq 0$. Hence $l_m = 0$ and $k_m = -k_1$. Even after this reduction (4.2) fails, if $m \geq 3$. Therefore

$$x^{k_1} (ab)^{l_1} x^{-k_1} = x^{k_1} a^{l_1} b^{l_1} x^{-k_1},$$

and we have

$$(ab)^{l_1} = a^{l_1} b^{l_1}.$$ 

Since $a$ and $b$ are independent, $l_1 = 1$. 

Summing up, we see that \( \tau \) acts on \( B \) as follows
\[
\tau x = x^{\varepsilon},
\tau a = x^k a x^{-k},
\tau b = x^k b x^{-k},
\tau c = x^k c x^{-k}, \quad c \in C.
\]
where \( \varepsilon = \pm 1 \). In the case when \( \varepsilon = -1 \), \( \tau \) is an involution, otherwise \( \tau \) is conjugation by \( x^k \). Conversely, every automorphism of \( F \) of the form (4.3) is in \( \text{Cen}(\Pi) \).

Suppose \( \tau x = x^{-1} \). Then \( \tau a = x^{2m} a x^{-2m} \), where \( m \in \mathbb{Z} \) is equivalent to
\[
\tau(x^m a x^{-m}) = x^m a x^{-m}.
\]
It demonstrates that \( \tau \) of the form (4.3) is an extremal involution, if \( \varepsilon = -1 \) and \( k \) is even. Clearly, if \( k \) is odd, then \( \tau \) is a quasi-conjugation.

Thus, we have that

**Proposition 4.3.** The set of all conjugations by powers of primitive elements is \( \emptyset \)-definable in \( \text{Aut}(F) \).

It is easy to see now that the subgroup of all conjugations is definable in \( \text{Aut}(F) \). Indeed, let \( B \) is a basis of \( F \). Suppose that \( a = w(b_1, \ldots, b_n) \), where \( a \) is a non-trivial element in \( F \) and \( b_1, \ldots, b_n \in B \). Since \( B \) is infinite, there is \( b \in B \setminus \{b_1, \ldots, b_n\} \). Then
\[
b \text{ and } b^{-1} w(b_1, \ldots, b_n)
\]
are both primitive elements. Therefore \( a \) is the product of two primitive elements. It implies that an arbitrary conjugation in \( \text{Aut}(F) \) is the product of two conjugations by powers of primitive elements.

**Remark.** Modulo definability of quasi-conjugations, Proposition 4.3 remains true for arbitrary non-abelian free groups. A proof for free groups with more than two generators is the same as the proof just completed. We give a sketch of proof for two-generator free groups. Let \( \{x\} \cup \{y\} \) be a canonical basis for a quasi-conjugation \( \varphi \). The unique involution in the centralizer of \( \varphi \) commuting with at least two its conjugates is an involution \( \psi \) which fixes \( x \) and inverts \( y \) (Proposition 3). The centralizer of \( \psi \) consists of conjugations by the powers of \( x \), involutions, but some elements of infinite order. Fortunately, these elements can be distinguished from conjugations: they induces in \( \text{Aut}(F) \) automorphisms whose determinates equal to \(-1\), and hence one can not represent them as the product of conjugate elements from \( \text{Aut}(F) \). On the other hand, one can represent every conjugation by a power of \( x \) as the product of two conjugate involutions (like in the proof of Lemma 4).

It is known that the automorphism group of a centreless group \( G \) is complete if and only if the subgroup \( \text{Inn}(G) \) is a characteristic subgroup of \( \text{Aut}(G) \) (3).
Being definable in $\text{Aut}(F)$, the subgroup $\text{Inn}(F)$ is its characteristic subgroup. Therefore

**Theorem 4.4.** Let $F$ be an infinitely generated free group. Then the group $\text{Aut}(F)$ is complete.

As we noted in Introduction, the latter theorem generalizes the result of J. Dyer and E. Formanek, stating the completeness of automorphism groups of finitely generated non-abelian groups. One more purely algebraic result which is a consequence of Theorem 4.1 is the following

**Theorem 4.5.** The automorphism groups of infinitely generated free groups $F$ and $F'$ are isomorphic if and only if $F \cong F'$.

*Proof.* Any isomorphism from $\text{Aut}(F)$ to $\text{Aut}(F')$ preserves conjugations, and hence induces an isomorphism between $F$ and $F'$. \hfill \Box

5. *The group and its free factors*

Let us consider the structure $\mathfrak{F}$ such that the domain of $\mathfrak{F}$ consists elements of three sorts:

- the elements of the group $F$,
- the automorphisms of $F$,
- the free factors of $F$.

The basic relations of $\mathfrak{F}$ are

- those of standard relations on sorts,
- the actions of $\text{Aut}(F)$ on other sorts,
- the membership relation between elements of $F$ and the set of free factors;
- a ternary relation, say $R$, on the set free factors such that

$$R(A, B, C) \leftrightarrow A = B \ast C.$$ 

**Theorem 5.1.** The structure $\mathfrak{F}$ is interpretable without parameters in $\text{Aut}(F)$ by means of first order logic.

*Proof.* By Theorem 4.1 the subgroup of conjugations is $\emptyset$-definable in $\text{Aut}(F)$. Let $\tau_a$ denote conjugation by $a \in F$:

$$\tau_a(z) = aza^{-1}, \quad z \in F;$$

it will model the element $a \in F$. If $\sigma \in \text{Aut}(F)$, then

$$\sigma \tau_a \sigma^{-1} = \tau_{\sigma(a)}.$$ 

Thus, we have interpreted in $F$ the first sort of $\mathfrak{F}$.

Let us prove now that

**Lemma 5.2.** The set of all primitive elements of $F$ is $\emptyset$-definable in the reduct $\langle \text{Aut}(F), F \rangle$ of the structure $\mathfrak{F}$. 
Proof. Consider a quasi-conjugation \( \varphi \in \text{Aut}(F) \) with a canonical basis \( \{x\} \cup Y \), where \( C_x = \langle Y \rangle \). By Proposition 4.2 the set of extremal involutions \( \{\sigma\} \) of the form
\[
\sigma x = x^{-1}, \\
\sigma y = x^{2k}yx^{-2k}, \quad y \in Y
\]
is definable in \( \text{Aut}(F) \) with the parameter \( \varphi \). Take one such extremal involution \( \sigma \).
Clearly, \( F = \langle x \rangle \ast \text{Fix}(\sigma) \).

Let \( z \) be a power of a primitive element with \( \varphi z = z^{-1} \). Suppose that for each \( v \in \text{Fix}(\sigma) \), the element \( zv \) is also a power of a primitive element. Then we claim that \( z = x^{\pm 1} \). It of course implies that primitive elements are definable.

Indeed, let \( z = t^k \), where \( t \) is primitive. Therefore \( (\varphi t)^k = (t^{-1})^k \), and \( \varphi t = t^{-1} \) \( \text{(1.2.17)} \). By 1.2 \( t = x \) or \( t = x^{-1} \). Thus, \( z = x^k \) or \( z = x^{-k} \) for some \( k \in \mathbb{N} \). Suppose \( |k| \geq 2 \). Let \( v = u^k \), where \( u \) is a primitive element in \( \text{Fix}(\sigma) \). Then \( zv \) cannot be a power of a primitive element due to the following result: the equality \( a^m b^p c^p = 1 \), where \( a, b, c \) are elements of a free group and \( m, n, p \geq 2 \) implies that all \( a, b, c \) lie in a cyclic subgroup of \( F \) \( \text{(sect. 6, ch. I)} \). Really, if \( zv = w^p \), where \( w \) is primitive and \( |p| > 1 \), then both \( x \) and \( u \) lie in a cyclic subgroup of \( F \). In the case when \( p = \pm 1 \) we have that \( \pm \overline{w} = \overline{w} \in kF \) and \( \overline{w} \) is non-primitive in \( F \). On the other hand, if \( z = x, x^{-1} \), then for each \( v \in \text{Fix}(\sigma) \) the element \( zv \) is primitive. \( \square \)

By the Dyer-Scott theorem (see Section 1), the fixed-point subgroup of an involution from \( \text{Aut}(F) \) is a free factor of \( F \), and conversely one can easily realize an arbitrary free factor of \( F \) as the fixed-point subgroup of some involution. If \( \varphi \) is an involution in \( \text{Aut}(F) \) then
\[
a \in \text{Fix}(\varphi) \iff \varphi \tau_a \varphi^{-1} = \tau_a.
\]

It also helps us to explain when involutions \( \varphi, \psi \) have the same fixed-point subgroups. To model the action of \( \text{Aut}(F) \) on the set of free factors of \( F \), we use the formula
\[
\text{Fix}(\sigma \varphi \sigma^{-1}) = \sigma \text{Fix}(\varphi).
\]

The next is an interpretation of the relation \( A = B \ast C \) on the set of free factors of \( F \). Let us interpret first the relation \( F = A \ast B \).

(i) Clearly, a free factor \( A \) of \( F \) has rank one if and only if \( A \) consists of only two distinct primitive elements. A free factorization
\[
F = A \ast B
\]
where rank \( A = 1 \) holds if and only if there is a quasi-conjugation with a canonical basis \( \{x\} \cup Y \), such that \( A = \langle x \rangle \) and \( B = \langle Y \rangle = C_x^z \). Therefore we should write that there is a quasi-conjugation \( \varphi \in \text{Aut}(F) \), a primitive element \( z \in F \) with \( \varphi z = z^{-1} \) such that \( z \in A \) and \( B = C_x^z \) (that is \( b \in B \) if and only if \( \varphi b = zbz^{-1} \)).
(ii) A soft non-extremal involution $\psi$ with non-trivial fixed-point subgroup (a member of a definable family by 1.4 and 4.2), which satisfies a definable condition

there is a primitive $x \in F$ with $\psi x = x^{-1}$ such that each primitive $a \in F$ with $\psi a = a^{-1}$ is equal to $\psi(w) x w^{-1}$ for some $w \in F$.

has by 1.2 and 1.3 the following canonical form

$$
\psi x = x^{-1}, \\
\psi y = x y x^{-1}, \quad y \in Y \neq \emptyset \\
\psi u = u, \quad u \in U \neq \emptyset.
$$

By 3.2 if $a$ is a primitive element, which $\psi$ sends to its inverse, then $a = v x^\varepsilon v^{-1}$, where $v \in \text{Fix}(\psi)$. We have

$$
\psi(b) = aba^{-1} \iff \psi(v^{-1} bv) = x^\varepsilon (v^{-1} bv)x^{-\varepsilon}.
$$

Hence $C_{a}^{\psi} = v C_{x^\varepsilon}^{\psi} v^{-1}$. Therefore

$$
F = \langle a \rangle * C_{a}^{\psi} * \text{Fix}(\psi).
$$

It follows from the above arguments that the factorization

$$
F = A * B
$$

where $A$ is of rank at least two, and $B$ is non-trivial holds if and only if there is an involution $\psi$ of the form (5.0) such that

- $B$ is equal to the fixed-point subgroup of $\psi$;
- there is a primitive $a \in F$ with $\psi a = a^{-1}$ such that $A$ is the least free factor of $F$ which contains $\langle a \rangle$ and $C_{a}^{\psi}$.

Let $A$ be a free proper factor of $F$ of rank $\geq 2$. Fix a free factor $D$ such that $F = A * D$. Consider the subgroup

$$
\Sigma = \{ \sigma \in \text{Aut}(F) : \sigma A = A, \sigma|_{D} = \text{id}_{D} \}.
$$

If $\sigma \in \Sigma$ has the form (5.1), then the restriction $\sigma$ on $A$, $\sigma|_{A}$ has an analogous canonical form in $\text{Aut}(A)$ (exactly one block in any canonical basis). Clearly, $\sigma|_{A}$ having such a canonical form is a quasi-conjugation if and only if $\text{Fix}(\sigma) = D$. Therefore $A = B * C$, where rank $B = 1$ holds if and only if there are $\sigma \in \Sigma$, a primitive $b \in B$ such that $\sigma|_{A}$ is a quasi-conjugation, $b$ and $b^{-1}$ are only primitive elements in $B$, and $C = C_{b}^{\sigma}$. Similarly, one can adapt the method in (ii) above to construct a definable with the parameter $D$ condition which is equivalent to $A = B * C$, where both $B$ and $C$ have rank at least two.

6. Getting a basis

Let $F$ be a free group of infinite rank. Consider free factors $A, B$ of $F$ with bases $\{a_{i} : i \in I\}$ and $\{b_{i} : i \in I\}$, respectively such that

$$
F = A * B.
$$
Suppose that \( \varphi \) is an involution with the canonical form
\[
\varphi a_i = a_i^{-1}, \quad i \in I,
\varphi b_i = a_i b_i a_i^{-1}.
\]
If \( \psi \) is an automorphism of \( F \) which fixes all the elements in the subgroup \( A \) and commutes with \( \psi \), then by (ii)
\[
\psi a_i = a_i, \quad i \in I,
\psi b_i = b_i^{\epsilon_i},
\]
where \( \epsilon_i = \pm 1 \). Let \( \Psi \) denote the set of all automorphisms of the form (6.0).

Then \( \Psi \) obviously satisfies the following properties:

(a) each \( \psi \in \Psi \) is an involution and the fixed-point subgroup of the involution \( \psi|_B \) either is trivial, or is a free factor of \( B \);
(b) for every element \( b \in B \) there is \( \psi \in \Psi \) such that \( \text{Fix}(\psi|_B) \) has finite rank and \( b \) is in \( \text{Fix}(\psi|_B) \);
(c) for each \( \psi \in \Psi \), if \( \text{rank Fix}(\psi|_B) > 1 \), then there are \( \psi_1, \psi_2 \in \Psi \) such that
\[
\text{Fix}(\psi|_B) = \text{Fix}(\psi_1|_B) \ast \text{Fix}(\psi_2|_B);
\]
(d) Let \( \Psi^1 \) denote the set of all elements in \( \Psi \) with rank \( \text{Fix}(\psi|_B) = 1 \); then for every \( \psi' \in \Psi^1 \) there is a free factor \( C \) of \( B \) such that
\[
B = \text{Fix}(\psi'|_B) \ast C,
\]
and \( C \) consists of all the subgroups \( \text{Fix}(\psi|_B) \), where \( \psi \in \Psi^1 \setminus \{ \psi' \} \).

The properties (a,b,c,d) imply that the set
\[
\mathcal{C} = \{ b : b \text{ is a primitive element in } \text{Fix}(\psi|_B), \text{ where } \psi \in \Psi^1 \}
\]
is just slightly greater than a ‘real’ basis of \( B \): there exists a basis \( \mathcal{B} \) of \( B \) such that \( \mathcal{C} = \mathcal{B}^\pm \). Indeed, let \( \mathcal{B} \) is any maximal system of representatives by an equivalence relation \( c \approx d \leftrightarrow (c = d^{-1}) \lor (c = d) \) on \( \mathcal{C} \). Then it follows from (b) and (c) that \( B \) is generated by \( \mathcal{B} \), because every subgroup of the form \( \text{Fix}(\psi|_B) \) of finite rank can be represented as a free product of fixed-point subgroups of the rank one of involutions in \( \Psi^1|_B \). The property (d) implies that \( \mathcal{B} \) is a free subset of \( B \).

To reduce the set \( \mathcal{C} \) to a basis of \( B \), let us add to the tuple of parameters \((A, B, \varphi)\) two involutions \( \pi_0, \pi_1 \) such that

(e) both \( \pi_0, \pi_1 \) fix \( A \) pointwise and preserve \( B \);
(f) \( \pi|_C \) is a bijection between the sets
\[
\mathcal{C}_0 = \{ c \in \mathcal{C} : \pi_0 c = c^{-1} \} \text{ and } \mathcal{C}_1 = \mathcal{C} \setminus \mathcal{C}_0,
\]
(g) for each \( c \in \mathcal{C}_1 \) there is \( c_0 \in \mathcal{C}_0 \) such that \( \pi_0 c = c_0 c_0^{-1} \).

Let \( \mathcal{B}_0 \) denote the set
\[
\{ c \in \mathcal{C}_0 : (\exists c' \in \mathcal{C}) \pi_0 c' = cc^{-1} \}.
\]
The inverse of each element \( c \in \mathcal{B}_0 \) is not in \( \mathcal{B}_0 \). Indeed, if \( \pi_0 c' = cc' c^{-1} \), then the only primitive elements \( z \) with \( \psi_0 z = c^{-1} z c \) in view of
\[
\pi_0 (cc' c^{-1}) = c^{-1} (cc' c^{-1}) c,
\]
appear \( cc' c^\pm 1 c^{-1} \), but they are not in \( \mathcal{C} \). Therefore the set
\[
\mathcal{B} = \mathcal{B}_0 \cup \pi_1 \mathcal{B}_0
\]
is a basis of \( \mathcal{B} \).

It is easy to see that the properties (a,b,c,d,e,f,g) are definable with parameters \((A,B,\varphi,\pi_0,\pi_1)\) in the structure \( \mathfrak{F} \) over \( F \): e.g. rank \( D \), where \( D \) is an arbitrary free factor of \( F \), is finite if and only if there is no automorphism in \( \text{Aut}(F) \) such that \( \varphi D \) is a proper subgroup of \( D \) (Lemma 3.4), we observed above that the condition rank \( D = 1 \) is a definable, etc.

To describe by means of first-order logic the tuple of parameters \((A,B,\varphi,\pi_0,\pi_1)\) we should explain that the first two parameters \( A \) and \( B \) are free factors of \( F \) such that \( F = A \ast B \) and there is an automorphism \( \rho \) of \( F \) which maps \( A \) onto \( B \): \( \rho A = B \); one can use as the third parameter \( \varphi \) any element from \( \text{Aut}(F) \), which satisfies the following definable with the parameters \( A,B \) condition: the set of all automorphisms of \( F \)
\[
\Psi = \{ \psi : \psi \varphi = \varphi \psi \text{ and } \psi|_A = \text{id}_A \}
\]
has the properties (a,b,c,d). There also are no problems in a description of \( \pi_0 \) and \( \pi_1 \).

Consider the group
\[
\{ \sigma : \sigma B = B \text{ and } \sigma|_A = \text{id}_A \}
\]
the group \( B \) of \( F \) as isomorphic copies of the group \( \text{Aut}(F) \) and \( F \), respectively. Thus, we have proved the following

**Theorem 6.1.** The structure \( \langle \text{Aut}(F), F, \mathcal{B} \rangle \) (with natural relations), where \( \mathcal{B} \) is a basis of \( F \) is interpretable in the structure \( \mathfrak{F} \) by means of first-order logic.

### 7. Interpretation of set theory

**Theorem 7.1.** Let \( F \) be an infinitely generated free group of rank \( \kappa \). Then the second-order theory of the set \( \kappa \) and the elementary theory of \( \text{Aut}(F) \) are mutually syntactically interpretable, uniformly in \( F \).

**Proof.** It is well-known that the second-order theory of a set \( X \) and first-order theory of the structure \( \langle X, X^X \rangle \), where \( X^X \) is the set of all functions from \( X \) to \( X \), are mutually syntactically interpretable.

In the previous section we have interpreted in \( \text{Aut}(F) \) the structure \( \langle \text{Aut}(F), F, \mathcal{B} \rangle \), where \( \mathcal{B} \) is a basis of \( F \). Let us partition \( \mathcal{B} \) into two equipotent subsets, say, \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \), by taking two automorphisms \( \pi_0, \pi_1 \in \text{Aut}(F) \) such that
\[
\begin{align*}
\bullet & \quad b \in \mathcal{B}_1 \iff \pi_0 b = b; \\
\bullet & \quad \pi_1 \mathcal{B}_1 = \mathcal{B}_2.
\end{align*}
\]
We shall interpret the set \( \kappa \) by the set \( B_1 \). Then we can interpret the set of functions from \( \kappa \) to \( \kappa \) using the set \( \Sigma = \{ \sigma \} \subseteq \text{Aut}(F) \) such that

\[
(\forall b \in B_1)(\sigma b = b) & (\forall b \in B_1)(\exists b' \in B_1)(\sigma(\pi^1 b) = \pi^1 b \cdot b').
\]

Clearly, every \( \sigma \in \Sigma \) determines a \((\pi^0, \pi^1)\)-definable function \( b \mapsto b' \) from \( B_1 \) to \( B_1 \), and, on the other hand, any function from \( B_1 \) to \( B_1 \) can be coded in such a way.

Thus, one can interpret (uniformly in \( F \)) in \( \text{Aut}(F) \) the structure \( \langle \kappa, \kappa \rangle \).

Let \( X \) be an infinite set. Let \( X_{II} \) denote the structure, with the domain \( \bigcup_{n \in \omega} R_n(X) \), where \( n \in \omega \), and for every \( n \) \( R_n(X) \) is the set of all \( n \)-placed relations on \( X \); the unique \( n \)-placed \((n \geq 1)\) basic relation on \( X_{II} \) says whether \( R(x_1, \ldots, x_{n-1}) \) is true or not for any tuple \( x_1, \ldots, x_{n-1} \in X \) and an arbitrary element \( R \in R_{n-1}(X) \). Clearly, the elementary theory of \( X_{II} \) and the second-order theory of \( X \) are mutually syntactically interpretable.

It is easy to see that one can interpret in \( X_{II} \) the automorphism group of a free group of rank \( |X| \). This will complete the proof of the Theorem.

Take a binary operation \( f_0 : X \times X \to X \) and a proper subset \( X_0 \subset X \) such that \( |X_0| = |X \setminus X_0| \) (they can be treated as suitable elements of \( X_{II} \)). Then we should explain that \( (X; f_0) \) is a group, and any map from \( X_0 \) to \( X \) can be extended to a homomorphism from the group \( (X; f_0) \) to a group \( (X; f) \), where \( f \) is an arbitrary group operation on \( X \). Thus, we can interpret in \( X_{II} \) the structure \( \langle F, F^F \rangle \), where \( F \) is a free group of rank \( |X| \). Clearly, the automorphism group of \( F \) is interpretable in the latter structure.

**Theorem 7.2.** Let \( F \) and \( F' \) be infinitely generated free groups of ranks \( \kappa \) and \( \kappa' \), respectively. Then their automorphism groups are elementarily equivalent if and only if the cardinals \( \kappa \) and \( \kappa' \) are equivalent in the second-order logic as sets:

\[
\text{Aut}(F) \equiv \text{Aut}(F') \iff \kappa \equiv_{L_2} \kappa'.
\]

**Proof.** By [7.1].

Theorem [7.1] implies also that

**Proposition 7.3.** The first-order theory of the automorphism group of an infinitely generated free group is undecidable and unstable.

**References**

[1] R. Cohen, ‘Classes of automorphisms of free group of infinite rank’, *Trans. AMS* 177 (1973) 99–119.

[2] J. Dyer, E. Formanek, The automorphism group of a free group is complete, *J. London Math. Soc.* 11 (1975) 181–190.

[3] J. Dyer, G. P. Scott, ‘Periodic automorphisms of free groups’, *Comm. Algebra* 3 (1975) 195–201.

[4] R. Lindon, P. Schupp, ‘Combinatorial group theory’ (Springer-Verlag, Berlin, etc., 1977).

[5] W. Magnus, Untersuchungen über einige unendliche discontinuierliche Gruppen, *Math. Ann.* 105 (1931) 52–74.
[6] S. Shelah, First-order theory of permutation groups, Israel. J. Math. 14 (1973) 149–162.
[7] S. Shelah, ‘Errata to: first-order theory of permutation groups’, Israel J. Math. 15 (1973) 437–441.
[8] S. Shelah, ‘Interpreting set theory in the endomorphism semi-group of a free algebra or in a category’, Annales Scientifiques de L’universite de Clermont fasc. 13 (1976) 1–29.
[9] W. Specht, Gruppentheorie (Berlin-Göttingen-Heidelberg, 1956).
[10] V. Tolstykh, Theories of infinite-dimensional linear groups (in Russian), C. Sci. Thesis (Kemerovo State University, Kemerovo, 1992).
[11] V. Tolstykh, ‘Puissance et pléritude: interprétation des groupes d’automorphismes des groupes libres’, in Quatrième Colloque Franco-Touranien de Théorie des Modèles, Résumés des Conférences, Marseille-Luminy, 26.05 au 30.05.97.

DEPARTMENT OF MATHEMATICS, KEMEROVO STATE UNIVERSITY, KRASNAJA, 6, 650043, KEMEROVO, RUSSIA

E-mail address: vlad@accord.kuzb-fin.ru