The Zealot Voter Model

Ran Huo and Rick Durrett *
Dept. of Math, Duke U.,
P.O. Box 90320, Durham, NC 27708-0320

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Abstract

Inspired by the spread of discontent as in the 2016 presidential election, we consider a voter model in which 0’s are ordinary voters and 1’s are zealots. Thinking of a social network, but desiring the simplicity of an infinite object that can have a nontrivial stationary distribution, space is represented by a tree. The dynamics are a variant of the biased voter: if \( x \) has degree \( d(x) \) then at rate \( d(x)p_k \) the individual at \( x \) consults \( k \geq 1 \) neighbors. If at least one neighbor is 1, they adopt state 1, otherwise they become 0. In addition at rate \( p_0 \) individuals with opinion 1 change to 0. As in the contact process on trees, we are interested in determining when the zealots survive and when they will survive locally.

1 Introduction

In our process, space is represented by a tree \( T \) in which the degree of each vertex \( x \) satisfies \( 3 \leq d_{\text{min}} \leq d(x) \leq M \). Voters can be in state 0 (ordinary voter) or 1 (zealot). Given a probability distribution \( p_k \) on \( \{0,1,2,\ldots,d_{\text{min}}\} \), if \( k \geq 1 \) then at rate \( d(x)p_k \) the voter \( x \) picks \( k \) neighbors without replacement. The voter becomes 1 if at least one of the chosen neighbors is a 1, otherwise it becomes 0. In addition at rate \( p_0 \) voters change their opinion from 1 to 0.

Our process is additive in the sense of Harris [7] and hence can be constructed on a graphical representation with independent Poisson processes \( T_n^{x,i} \), \( n \geq 1, 0 \leq i \leq d_{\text{min}} \).

- The \( T_n^{x,0} \) have rate \( p_0 \). At these times we write a \( \delta \) at \( x \) that will kill a 1 at the site.
- The \( T_n^{x,i} \) have rate \( d(x)p_i \). At time \( T_n^{x,i} \) we write a \( \delta \) at \( x \) that will kill a 1 at the site. In addition we draw oriented arrows to \( x \) from \( i \) neighbors \( y_1, \ldots y_i \) chosen at random without replacement from the set of neighbors. If any of the \( y_i \) are in state 1, then \( x \) will be in state 1. Otherwise it will be in state 0.

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Intuitively, the process can be defined by introducing fluid at the sites $A = \{x : \xi_0(x) = 1\}$. The fluid flows up the graphical representation, being blocked by $\delta$'s and flowing across edges in the direction of their orientations. The state at time $t$, $\xi_t^A$ is the set of points that can be reached by fluid at time $t$ starting from some site in $A$ at time 0.

A nice feature of this construction is that it allows us to define a dual process in which fluid flows down the graphical representation, is blocked by $\delta$'s and flows across edges in a direction opposite their orientations. We let $\zeta_{B,t}^s$ be the points reachable at time $t-s$ starting from $B$ at time $t$. It is immediate from the construction that

$$\{\xi_t^A \cap B \neq \emptyset\} = \{A \cap \zeta_{t}^{B,t} \neq \emptyset\} \quad (1)$$

It should be clear from the construction that the distribution of $\zeta_{B,t}^s$ for $0 \leq s \leq t$ does not depend on $t$ and is a coalescing branching random walk (COBRA) with the following rules. A particle at $x$ dies at rate $p_0$ and at rate $d(x)p_k$ it dies after giving birth to offspring that occupy $k$ of the neighboring sites chosen at random without replacement. For more details see Griffeath [6].

In the case $p_0 = 0$ this pair of dual processes has been studied by Cooper, Radzik, and Rivera [2]. In their situation the zealot voter model is called a biased infection with a persistent source (BIPS). The phrase persistent source refers to the fact that the BIPS model has one individual that stays infected forever. Their main interest is in the cover time for COBRA, i.e., the time for the process to visit all of the sites. By duality this is related to time for the BIPS to reach all 1’s.

In this paper, when we say that a process survives we mean that with positive probability it avoids becoming $\emptyset$. We say a process survives locally if with positive probability the root 0 is occupied infinitely many times.

When $A = B = \{0\}$, (1) implies

$$P(0 \in \xi_t^0) = P(0 \in \zeta_t^0)$$

so local survival of one process implies local survival of the other. Taking one of the sets $= T$ and the other $= \{0\}$ we get

$$P(\xi_t^0 \neq \emptyset) = P(0 \in \zeta_t^T) \quad P(\zeta_t^0 \neq \emptyset) = P(0 \in \xi_t^T)$$

so survival of one process implies that the other has a nontrivial stationary distribution obtained by letting $t \to \infty$ in $\zeta_t^T$ or $\xi_t^T$. Our first result is very general.

**Theorem 1.** On any tree with degrees $3 \leq d(x) \leq M$, the zealot voter model survives if

$$\sum_{k \geq 2} (k - 1)p_k - p_0 > 0.$$ 

The result is proved by comparing the growth of the process at the “frontier” with a branching process. For the definition of frontier, see the text before Lemma 2.1. Note that the degree distribution does not appear in the condition. The next result shows that the condition is far from necessary.
1.1 Results for $d$–regular Trees

Let $\beta = 1 - (d-1)^{-2}$ be the probability that two independent random walks on the $d$-regular tree that start at distance two never hit. See Lemma 3.1 for a proof of this.

**Theorem 2.** On a $d$-regular tree the COBRA dies out if

$$d\beta \sum_{k \geq 2} (k-1)p_k - p_0 < 0. \tag{2}$$

When this holds the zealot voter model does not have a nontrivial stationary distribution.

To explain the condition, note that in the dual, a particle dies at rate $p_0$ and gives birth to $k$ particles at rate $dp_k$. To get an upper bound on the growth of the dual (i) we ignore coalescence between individuals that are not siblings, and (ii) if $k$ particles are born we number them $1, 2, \ldots, k$ and ignore coalescence between particles $i > 1$ and $j > 1$. This gives an upper bound on the dual COBRA.

**Theorem 3.** If (2) holds then the zealot voter model dies out on a $d$-regular tree.

**Proof.** Theorem 2 is proved by showing the expected number of particles in the COBRA, denoted as $E[\zeta_0^t]$, converges to 0 as $t \to \infty$. By symmetry,

$$E[\zeta_0^t] = \sum_x P(x \in \zeta_0^t) = \sum_x P(0 \in \zeta_x^t) = P(0 \in \zeta_1^t)$$

where $\zeta_1^t$ is the COBRA starting with all sites occupied. This implies that with the condition, COBRA has no stationary distribution so the zealot voter model dies out. \hfill \Box

To study the local survival of our voter model, we note that by duality

$$P(0 \in \zeta_0^t) = P(0 \in \zeta_0^t)$$

so we can instead study the local survival of the COBRA.

**Theorem 4.** Given a $d$-regular tree $T$, the zealot voter model dies out locally if

$$\mu < \frac{d(1-p_0) + p_0}{2\sqrt{d-1}}.$$

If $p_0 = 0$ this is $\mu < d/(2\sqrt{d-1})$.

This result is proved by comparing with a branching random walk. The second bound is sharp for the branching random walk with no death. That is, the corresponding branching random walk visits the root with positive probability if $\mu > d/(2\sqrt{d-1})$ and that the root is visited finitely many times if $\mu < d/(2\sqrt{d-1})$. This result can be found in Pemantle and Stacey [9]. There they studied the branching random walk on trees where each particle gives birth at a rate $\lambda$ and independently, it dies at rate 1. Since our branching process has simultaneous births and deaths we modify their proof to cover our situation and give the proof in Lemma 4.1.
To give sufficient conditions for local survival, we follow a tagged particle in the COBRA. If there is a particle produced on the site closer to the root, we follow this particle; otherwise we follow a new particle chosen uniformly at random from the offspring and ignore the rest. The recurrence of the tagged particle implies the local survival of COBRA. Using this idea leads to a simple proof of a condition for local survival, but the result is not very accurate.

**Theorem 5.** On a $d$-regular tree the zealot voter model survives locally if $p_0 = 0$ and $\mu > d/2$.

**Proof.** Note that if $i$ is the number of particles produced in a branching event and $q_i$ is the probability all of them going further from to the root then

$$q_i = \binom{d-1}{k} = \frac{(d-1)!}{k!(d-1-k)!} \cdot \frac{k!(d-k)!}{k!} = \frac{d-k}{d}$$

Thus if we follow the particle that gets closer to the root then it jumps by $-1$ with probability

$$\sum_k p_k \frac{k}{d} = \frac{\mu}{d}$$

and the tagged particle will be positive recurrent if $\mu > d/2$.

Our next Theorem, which uses some ideas from the proof of Lemma 4.57 in Liggett’s 1999 book [5], gives a more precise result.

**Theorem 6.** On a $d$-regular tree the zealot voter model survives locally if $p_0 = 0$ and

$$\mu > \frac{d}{\sqrt{d-1} + 1}.$$ 

Combining this with Theorem 4 we notice that when $p_0 = 0$ the phase transition of local survival $\mu_t$ satisfies

$$\mu_t \in \left[ \frac{d}{2\sqrt{d-1}}, \frac{d}{1 + \sqrt{d-1}} \right]$$

### 1.2 Results for Galton-Waston Trees

To prove an analogue of Theorem 2 we formulate our model as a voter model perturbation: let $\tilde{p}_i = \varepsilon p_i$ when $i \neq 1$ and choose $\tilde{p}_1$ to make the $\tilde{p}_i$ sum to 1. A random walk that jumps to each neighbor at rate 1 has a reversible stationary distribution that is uniform on the graph. Let $\pi_m$ be the fraction of vertices in the tree with degree $m$, and let $\mu_{m,k}$ be the expected number of surviving particles in the dual when we pick $k$ neighbors of a vertex of degree $m$ at random and run the coalescing random walk to time $\infty$.

**Theorem 7.** Let $\delta > 0$. If $\varepsilon$ is small then the COBRA dies out if

$$\sum_m \pi_m \sum_k k \mu_{m,k} - 1 - p_0 < \delta$$

and survives if the last quantity is $> \delta$. 

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(4)
This result can be easily proved using the techniques in [8]. The key idea is that when \( \varepsilon \) is small most of the steps in the dual are random walk steps, and the random walk is transient, so any coalescence occurs soon after branching, and the dual is essentially a coalescing branching random walk. These ideas go back to [3], where they were used on \( \mathbb{Z}^d \) with \( d \geq 3 \). In Section 4 we will provide more details.

Remark. The last result concerns the survival of the dual, which is the same as the existence of a nontrivial stationary distribution for the zealot voter model.

Our next result concerns local survival. Given any Galton-Waston tree \( T^{GW} \), let \( M \) denote its maximal degree. Further, let \( \mu_l(G) \) denote the threshold for local survival of the COBRA on graph \( G \). Note the expected number of new born particles at each time are the same on both trees. Since particles on tree \( T_M \) have more tendency to move further away from the root, a simple comparison leads to

\[
\mu_l (T^{GW}(\eta_t)) \leq \mu_l (T^M(\eta_t))
\]

where \( \eta_t \) is the BRW without coalescence. The comments under Theorem 4 says for \( p_0 = 0 \),

\[
\mu_l (T^M(\eta_t)) = M/(2\sqrt{M-1})
\]

It follows immediately that

**Theorem 8.** If \( p_0 = 0 \) and \( \mu < M/(2\sqrt{M-1}) \) then COBRA and the zealot voter model both die out locally.

Next we look for conditions implying local survival. On a tree we define the level \( \ell_x \) of a vertex \( x \) to be its distance to the root. As on \( d \)-regular trees, our strategy is to follow a tagged particle and seek conditions guaranteeing its recurrence. If \( \phi(X_t) \) is a harmonic function for the tagged particle \( X_t \), i.e. \( \phi(X_t) \) is a martingale, then it follows from the optional stopping theorem that If \( T_0 \) is the time to hit the root and \( T_N \) is the first time the walk hits a site at level \( N \)

\[
\phi(1) \geq \left( \min_{x: \ell_x = N} \phi(x) \right) P_1 (T_N < T_0)
\]

(3)

where the subscript 1 on \( P \) indicates that \( X_0 \) is at level 1. From (3) we see that if \( \phi(x) \) goes to \( \infty \) along all paths to \( \infty \) in the tree, then the tagged particles is recurrent. In order for \( \phi \) to be harmonic

\[
\phi(x + 1) - \phi(x) = \frac{p_x}{1 - p_x} [\phi(x) - \phi(x - 1)] = \frac{\mu}{d(x) - \mu} [\phi(x) - \phi(x - 1)]
\]

where \( p_x = \mu/d(x) \) is the probability the tagged particle moves closer to the root. Taking logarithms, then this is

\[
\log [\phi(x + 1) - \phi(x)] = \log [\phi(x) - \phi(x - 1)] + \log \left[ \frac{\mu}{d(x) - \mu} \right]
\]
As we will now explain, there is a natural mapping from the log-increments of the harmonic function to a branching process on $\mathbb{R}$. If we consider a particle at level $x$ to be at $\log [\phi(x) - \phi(x - 1)]$ on $\mathbb{R}$ then $d(x) - 1$ new particles will be dispersed to

$$\log [\phi(x) - \phi(x - 1)] + \log \left[ \frac{\mu}{d(x) - \mu} \right].$$

As a result along any genealogical path, the distance between two consecutive generations is i.i.d with law the same as $\log [\mu/(d(x) - \mu)]$.

This process just described is different from the usual branching random walk in which children are dispersed independently from their parent. However Biggins [1] has proved results for more general branching random walks that contain ours as a special case. Let $F(t) = E(\zeta(-\infty,t])$ be the expected number of children that lie in $(-\infty,t]$ and define the Laplace transform of the mean measure by

$$m(\theta) = \int e^{-\theta t} dF(t)$$

**Theorem 9.** If $\min_{\theta \geq 0} m(\theta) < 1$ then the leftmost particle in the branching random walk goes to $\infty$. This implies $\phi$ goes to $\infty$ along all paths to $\infty$ in the tree and we have local survival.

To apply this result to our examples, we begin by noting that

$$m(\theta) = \sum_{j \geq 3} q_j (j - 1) \left( \frac{j - \mu}{\mu} \right)^\theta$$

It is not easy to use this formula with Theorem[9] to get explicit predictions, so we focus on Galton-Waston tree with degrees only 3 and 4. Let $\mu = 3q_3 + 4q_4$ and

$$\nu(0) = \min_{\theta \geq 0} m(\theta).$$

We have computed the threshold for various $\mu$ in Section 4.3. See also Figure 1.
2 Proof of Theorem 1

We begin by deriving a differential equation. Let \( d_k(x) = (d(x) - 1) \cdots (d(x) - (k - 1)) \). Note that \( d_k(x) \) is the number of ways of picking \( k \) of \( x \)'s neighbors without replacement, when the order of the choices is important.

\[
\frac{d}{dt} \sum_x P(\xi_t(x) = 1) = -p_0 \sum_x P(\xi_t(x) = 1) \\
+ \sum_{x, y_k \sim x} \frac{p_k}{d_k(x)} \sum_{x*_{(k-1)} \neq y_k} [P(\xi_t(x) = 1, \xi_t(y_k) = 0) - P(\xi_t(x) = 1, \text{all } \xi_t(y_i) = 0)] \quad (4) \\
+ \sum_{x, y_k \sim x} \frac{p_k}{d_k(x)} \sum_{x*_{(k-1)} \neq y_k} P(\xi_t(x) = \xi_t(y_k) = 0, \xi_t(y_i) = 1 \text{ for some } i < k)
\]

**Proof.** For simplicity, we use \( x \ast k \) to indicate that we sum over all choices of \( k \) different neighbors \( y_1, ..., y_k \) of \( x \) when order is important.

\[
\frac{d}{dt} P(\xi_t(x) = 1) = -p_0 P(\xi_t(x) = 1) \\
- p_1 \sum_{y \sim x} P(\xi_t(x) = 1, \xi_t(y) = 0) \\
+ p_1 \sum_{y \sim x} P(\xi_t(x) = 0, \xi_t(y) = 1) \\
- \sum_k \sum_x \frac{p_k}{d_k(x)} \sum_{x*k} P(\xi_t(x) = 1, \text{all } \xi_t(y_i) = 0) \\
+ \sum_k \sum_x \frac{p_k}{d_k(x)} \sum_{x*k} P(\xi_t(x) = 0, \text{some } \xi_t(y_i) = 1)
\]
If we sum over \( x \) then the second and third terms cancel. If we fix \( k \) the last two

\[
= - \sum_x \frac{p_k}{d_k(x)} \sum_{x+k} P(\xi_t(x) = 1, \text{ all } \xi_t(y_i) = 0) \\
+ \sum_x \frac{p_k}{d_k(x)} \sum_{x+k} P(\xi_t(x) = 0, \text{ some } \xi_t(y_i) = 1)
\]

(5)

Splitting “some” into

\[
P(\xi_t(x) = 0, \xi_t(y_k) = 1) + P(\xi_t(x) = \xi_t(y_k) = 0, \xi_t(y_i) = 1 \text{ for some } i < k)
\]

Summing over the first probability and writing \( x \ast (k-1) \neq y_k \) to indicate that we sum over the \( k-1 \) distinct neighbors \( y_1, \ldots, y_{k-1} \) that are different from \( y_k \) and order is important, this is

\[
\sum_{x,y_k \sim x} \frac{p_k}{d_k(x)} \sum_{x \ast (k-1) \neq y_k} P(\xi_t(x) = 0, \xi_t(y_k) = 1)
\]

\[
= \sum_{x,y_k \sim x} p_k P(\xi_t(x) = 0, \xi_t(y_k) = 1)
\]

\[
= \sum_{y_k \ast y_k} p_k P(\xi_t(x) = 0, \xi_t(y_k) = 1)
\]

\[
= \sum_{y_k \ast y_k} \frac{p_k}{d_k(y_k)} \sum_{y_k \ast (k-1) \neq x} P(\xi_t(x) = 0, \xi_t(y_k) = 1)
\]

Interchanging the role of \( x \) and \( y_k \), the above

\[
= \sum_{x,y_k \sim x} \frac{p_k}{d_k(x)} \sum_{x \ast (k-1) \neq y_k} P(\xi_t(x) = 1, \xi_t(y_k) = 0)
\]

Then (5) can be reformulated as

\[
= - \sum_k \sum_x \frac{p_k}{d_k(x)} \sum_{x+k} P(\xi_t(x) = 1, \text{ all } \xi_t(y_i) = 0) \\
+ \sum_{x,y_k \sim x} \frac{p_k}{d_k(x)} \sum_{x \ast (k-1) \neq y_k} P(\xi_t(x) = 1, \xi_t(y_k) = 0) \\
+ \sum_{x,y_k \sim x} \frac{p_k}{d_k(x)} \sum_{x \ast (k-1) \neq y_k} P(\xi_t(x) = \xi_t(y_k) = 0, \xi_t(y_i) = 1 \text{ for some } i < k)
\]

Combining the first two summations gives the desired result.

To proceed we need some more notation. Pick a vertex from the tree to be the root and call it \( x_0 \). Given a vertex \( x \) in the tree we say that \( x' \) is a child of \( x \) if it is further away from the root than \( x \) is. We define the subtree generated by \( x' \), \( S(x') \) to be all of the vertices that can be reached from \( x' \) without going through \( x \). For any finite set on the tree \( A \), define its frontier \( F(A) \) as the set of sites \( x \in A \) that have a child \( x' \) such that the subtree \( S(x') \cap A = \emptyset \) and define \( H(A) \) to be the set of all such children \( x' \). That is, \( x \in H(A) \) if and only if \( S(x') \cap A = \emptyset \) and \( x' \) has parent in \( F(A) \).
Lemma 2.1. $|H(A)| \geq |A|$ and $|F(A)| \geq |A|/(M-1)$.

Proof. We prove the first result by induction on the cardinality of $|A|$. If $|A| = 1$, the result is trivial as $|H(A)| \geq \text{deg}(x) - 1 \geq 2$. Suppose now that the result is true for all $B$ with $|B| \leq n - 1$ and let $|A| = n$. Let $x \in A$ be the point with the largest distance to the root and let $B = A \setminus \{x\}$. Then by induction $|H(B)| \geq n - 1$. Since none of the descendents of $x$ are in $A$, but $x$ might be in $H(B)$.

$$|H(A)| \geq |H(B)| - 1 + \text{deg}(x) - 1 \geq (n - 1) - 1 + 2 = n$$

The second result follows from the first since $|H(A)| \leq (M-1)|F(A)|$.

\[\]
Lemma 2.3.

Let $\overline{S}$ a neighbor $x$ any tree $T$ from $y$.

Proof. Once there exists $\xi_t(x) = \xi_t(y_k) = 0$, some $\xi_t(y_i) = 1$.

In the second last line, $d(x) - 1$ gives the choices for $y_k$. $k - 1$ is because we have $k - 1$ choices from $y_1, \ldots, y_{k-1}$ to be on the frontier. Suppose $y_1$ is chosen to be in the frontier, then the number of choices for $y_2, \ldots y_{k-1}$ is $\binom{d(x) - 2}{k-1}$. The final inequality comes from Lemma 2.1.

To prove Theorem 1, we will define a process $\overline{\xi}_t$ that gives a lower bound on $\xi_t$. Choose a neighbor $x_1$ of the root $x_0$. (See Figure 2 for a picture.) Set all the sites outside of $S(x_1) \cup \{x_0\}$ to be always equal to 0. Let $\overline{\xi}_t$ be the process restricted to $S_1 \equiv S(x_1) \cup \{x_0\}$. Let $\overline{A}_t = \{x : \overline{\xi}_t(x) = 1\}$. Let $H^*(\overline{A}_t) = H(\overline{A}_t) \cap S(x_1)$ and let $F^*(\overline{A}_t) = F(\overline{A}_t) \cap S(x_1)$.

Lemma 2.3.

$$\frac{d}{dt} E|\overline{A}_t| \geq \gamma E|A_t| - (\gamma + 1)(M - 1)$$

Proof. When we write the differential equation for $E|\overline{A}_t|$ there is a term

$$-p_1[d(x_0) - 1]P(\overline{\xi}_t(x_0) = 1)$$

that cannot be cancelled. Note that if $x_0 \in \overline{A}_t$, then $x_2, \ldots, x_{d(x_0)} \in H(\overline{A}_t)$ so

$$|H^*(\overline{A}_t)| \geq |H(\overline{A}_t)| - [d(x_0) - 1] \geq |\overline{A}_t| - (d(x_0) - 1)$$

where the last inequality follows from Lemma 2.1. Using $d(x_0) \leq M$ the desired result follows.

Then $|F^*(\overline{A}_t)| \geq |F(\overline{A}_t)| - 1$ and this is an equality if the root $x_0$ is in state 1.

Lemma 2.4. There exists $t_0 > 0$ such that

$$E|F^*(\overline{A}_{t_0})| > 1$$

(6)

for all trees with $3 \leq d_{\min} \leq d(x) \leq M$.

Proof. Once $E|\overline{A}_t| \geq 2(\gamma + 1)(M - 1)/\gamma$ it grows exponentially. We have shown that given any tree $T$, $E|\overline{A}_t|$ satisfies the equation

$$\frac{d}{dt} E|\overline{A}_t| \geq pE|\overline{A}_t| - (M - 1)$$

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Hence once $E|\tilde{A}_t| \geq L \gg 1$ for some $L$, $E|\tilde{A}_t|$ grows exponentially afterwards. Recall that $\tilde{A}_t^* = A_t' \cap S(x_1)$ and it satisfies $|\tilde{A}_t^*| \geq |\tilde{A}_t| - 1$. Note that $|\tilde{A}_t|$ jumps to $|\tilde{A}_t| - 1$ at rate $\leq M|\tilde{A}_t|$; and jumps to $|\tilde{A}_t| + 1$ at rate $\geq 1$ (when $|\tilde{A}_t| = 1$) or $\geq |\tilde{A}_t^*|\mu \geq \frac{\mu}{2}|\tilde{A}_t|$, where $\mu = 1 - p + 2p$ is the mean. Hence the process dominates a branching process that is independent of the tree structure and we can find a constant $C_1 > 0$ such that

$$P(|\tilde{A}_1| \geq L) > C_1 \quad \forall \text{Tree}$$

This implies that

$$E|\tilde{A}_t| \geq C_1 \exp\left[\frac{p}{2}(t - 1)\right] \quad \forall \text{Tree} \quad (7)$$

Note that by Lemma 2.1

$$|F^*(\tilde{A}_t)| \geq |F(\tilde{A}_t)| - 1$$

$$\geq \frac{1}{M - 1}|\tilde{A}_t| - 1$$

$$\geq \frac{1}{M - 1}|\tilde{A}_t| - 1$$

Hence a uniform $t_0$ is guaranteed for all trees. \hfill \Box

Now define a branching process $Z_n$ on any given tree $T$. Let $Z_0 = \{x_0\}$, where $x_0$ is the root. Then $Z_1 = F^*(\tilde{A}_{t_0}) = F(\tilde{A}_{t_0}) \cap S(x_1)$. Clearly, $x_0 \notin Z_1$. Inductively, given $Z_n$, note that for any $x \in Z_n$, $x$ has a child $x'$ such that $S(x') \cap \tilde{A}_{nt_0} = \emptyset$. That is, $S(x')$ is completely vacant. Using the arrows falling in $S(x') \cup \{x \text{ and its neighbors}\}$ for the time $[nt_0, \ (n + 1)t_0]$. Let $F^*(\tilde{A}_{t_0}^x,0)$ be the children of $x$. the superscript 0 means that to obtain $\tilde{A}_{t_0}^x,0$, we enforce 0-boundary condition $T^0$ on $x$. Hence all the neighbors of $x$ except for $x'$ are in state 0 during $[nt_0, \ (n + 1)t_0]$. Therefore all $\tilde{A}_{t_0}^x,0 \forall x \in Z_n$ are independent and $E|F^*(\tilde{A}_{t_0}^x,0)| > 1 \forall x \in Z_n$ by Lemma 2.4. Define the $n + 1$ th generation by

$$Z_{n+1} = \bigcup_{x \in Z_n} F^*(\tilde{A}_{t_0}^x,0)$$

**Lemma 2.5.** Let $\epsilon > 0$ such that $1 + \epsilon \leq E|F^*(\tilde{A}_{t_0})|$ where $t_0$ is given in Lemma 2.4. Then there exists $C > 0$ such that for all trees

$$E[Z_{n+1}|Z_n] \geq (1 + \epsilon)Z_n \quad (8)$$

$$\text{Var}(Z_{n+1}|Z_n) \leq CZ_n \quad (9)$$

**Proof.** We drop the 0 in the superscript for easy notation. Given any tree $T$, note that $Z_{n+1} = \bigcup_{x \in Z_n} F^*(\tilde{A}_{t_0}^x)$ and $F^*(\tilde{A}_{t_0}^x) \cap F^*(\tilde{A}_{t_0}^y) = \emptyset$ if $x \neq y$. Then

$$E[Z_{n+1}|Z_n, T] = \sum_{x \in Z_n} E\left[|F^*(\tilde{A}_{t_0}^x)||Z_n, T\right] > (1 + \epsilon)Z_n \quad (10)$$
To prove (9), let \( \eta \) be a branching process where \( \eta_0 = 1 \) and every particle gives a birth at rate \( M \) without death. Then given any tree, \( |\bar{A}_t| \) is stochastically bounded by \( |\eta_t| \). So now, let \( T^x \) denote the subtree rooted at \( x \) and by independence

\[
Var(Z_{n+1}\mid Z_n, T) = \sum_{x \in Z_n} Var(|\bar{A}_{t_0}^x| \mid T^x)
\]

\[
\leq \sum_{x \in Z_n} E[|\bar{A}_{t_0}^x|^2 \mid T]
\]

\[
\leq \sum_{x \in Z_n} E|\eta_{t_0}|^2
\]

\[= CZ_n
\]

Since \( T \) is arbitrary, we have completed the proof.

The following lemma shows that the process survives with positive probability.

**Lemma 2.6.** Suppose \( \epsilon \) is given by Lemma 2.5, then

\[
\liminf_{n \to \infty} \frac{Z_n}{(1 + \frac{\epsilon}{2})^n} \quad w.p.p.
\]

*(11)*

**Proof.** First by Lemma 2.5 and Chebyshev’s Inequality,

\[
P\left(Z_{n+1} < \left(1 + \frac{\epsilon}{2}\right)^{n+1} \mid Z_n \geq \left(1 + \frac{\epsilon}{2}\right)^n\right)
\]

\[
\leq P\left(|Z_{n+1} - E[Z_{n+1}\mid Z_n]| > \frac{\epsilon}{2}Z_n \mid Z_n \geq \left(1 + \frac{\epsilon}{2}\right)^n\right)
\]

\[
\leq E\left[\frac{CZ_n}{(\epsilon Z_n/2)^2}\mid Z_n \geq \left(1 + \frac{\epsilon}{2}\right)^n\right]
\]

\[
\leq \frac{4C}{\epsilon^2 (1 + \frac{\epsilon}{2})^n} = \frac{C'}{(1 + \frac{\epsilon}{2})^n} =: \delta_n
\]

Pick \( n_0 \gg 1 \) fixed such that \( \delta_{n_0} < 1 \). Since \( \delta_n \) is decreasing, we have

\[
P\left(\liminf_{n \to \infty} \frac{Z_n}{(1 + \frac{\epsilon}{2})^n} \geq 1 \mid Z_{n_0} \geq \left(1 + \frac{\epsilon}{2}\right)^{n_0}\right)
\]

\[
\geq \prod_{n=n_0}^{\infty} P\left(Z_{n+1} \geq \left(1 + \frac{\epsilon}{2}\right)^{n+1} \mid Z_{n} \geq \left(1 + \frac{\epsilon}{2}\right)^n\right)
\]

\[
\geq \prod_{n=n_0}^{\infty} (1 - \delta_n)
\]

It suffices to show

\[
P\left(Z_{n_0} \geq \left(1 + \frac{\epsilon}{2}\right)^{n_0}\right) > 0
\]

*(12)*
Recall $|F^*(A_t)| \geq \frac{1}{M} |A_t| - 1$. For arbitrary tree $T$, $|A_t|$ stochastically dominates a branching process $\bar{\eta}_t$ where each particle dies at rate $M$ and gives birth at rate $\mu$. Hence

$$P \left( Z_{n_0} \geq \left( 1 + \frac{\epsilon}{2} \right)^{n_0} \right) \geq P \left( \bar{\eta}_{n_0} \geq M \left( 2 + \frac{\epsilon}{2} \right) |\bar{\eta}_0 = 1 \right) \left( 1 + \frac{\epsilon}{2} \right)^{n_0 - 1} > 0$$

\[3\] Results for $d$-regular trees

3.1 Extinction

**Lemma 3.1.** Let $h(x)$ be the probability two continuous time random walks separated by $x$ on a $d$-regular tree will hit.

$$h(x) = \left( \frac{1}{d-1} \right)^x$$

**Proof.** If the two particles are at distance $x > 0$ then the probability they are at distance $x + 1$ after the first jump is $(d-1)/d$, while they are at distance $x - 1$ with probability $1/d$.

$$
\left( \frac{1}{d-1} \right)^x = \frac{d-1}{d} \left( \frac{1}{d-1} \right)^{x+1} + \frac{1}{d} \left( \frac{1}{d-1} \right)^{x-1}
$$

$$= \frac{1}{d} \left( \frac{1}{d-1} \right)^x + \frac{d-1}{d} \left( \frac{1}{d-1} \right)^x = \left( \frac{1}{d-1} \right)^x$$

i.e., if $X_t$ is the distance between the two coalescing random walks $((1-p)/p)^{X_t}$ is a martingale. Since $h(0) = 1$, $h(x) \leq 1$ for $x \geq 0$ and $h(x) \to 0$ as $x \to \infty$ the desired result follows from the optional stopping theorem.

Let $\beta$ be the probability two newborn particles in the dual do not coalesce. Since two newborn particles are at distance two from each other

$$\beta = 1 - \frac{1}{(d-1)^2}.$$  

**Theorem 2** On a $d$-regular tree the COBRA dies out if

$$d\beta \sum_{k \geq 2} (k-1)p_k - p_0 < 0$$

**Proof.** In COBRA, a particle dies at rate $p_0$ and gives birth to $k$ particles at rate $dp_k$. To get an upper bound on the growth (i) we ignore coalescence between individuals that are not siblings, and (ii) if $k$ particles are born we number them $1, 2, \ldots, k$ and ignore coalescence between particles $i > 1$ and $j > 1$. Note that particles $2, \ldots, k$ each have probability $\beta$ of not coalescing with $1$. Since the parent particle dies but particle $1$ is added, the expected number of the remaining particles is $(k-1)\beta$. We use $\eta_0^0$ to denote the resulting system starting from a single particle then

$$\frac{d}{dt} E\eta_0^0 = \left[ -p_0 + d \sum_k p_k (k-1)\beta \right] E\eta_0^0$$

It is immediate that $f d\beta \sum_{k \geq 2} (k-1)p_k - p_0 < 0$ then $E|\zeta_t^0| \leq E|\eta_0^0| \to 0$.
3.2 Local Survival

Theorem 4 Given a $d$-regular tree $T$, the zealot voter model dies out locally if

$$\mu < \frac{d(1 - p_0) + p_0}{2\sqrt{d-1}}.$$ 

If $p_0 = 0$ this is $\mu < d/(2\sqrt{d-1})$.

Proof. Let 0 in the superscript denote the root in the following. We need to show

$$P(\xi_0^0 \cap \{0\} \neq \emptyset) \rightarrow 0 \text{ as } t \rightarrow \infty \quad (13)$$

By duality,

$$P(\xi_0^0 \cap \{0\} \neq \emptyset) = P(\zeta_0^0 \cap \{0\} \neq \emptyset) \quad (14)$$

Let $\eta_t^0 \supset \zeta_t^0$ be the BRW in which particles die at rate $p_0$ and at rate $dp_k$ die and give birth onto $k$ neighbors chosen without replacement. Particles in $\eta_t^0$ are independent BRW.

Lemma 3.2. Let $m(t, x) = E\eta_t^0(x)$ be the expected number of particles on site $x$ at time $t$. Then $m(t, x)$ satisfies the equation

$$\frac{d}{dt}m(t, x) = -\alpha m(t, x) + \sum_{y \sim x} m(t, y)\mu \quad \text{where } \alpha = d(1 - p_0) + p_0.$$

The solution is given by

$$m(t, x) = e^{(\mu - \alpha)dt}P(S_t^0 = x) \quad (15)$$

where $S_t^0$ is the random walk on tree $T$ that jumps at rate $d\mu$ to a neighbor chosen uniformly at random.

Proof. To check (15), note that using RHS for the right-hand side of the equation

$$\frac{d}{dt}RHS = (\mu - \alpha)d e^{(\mu - \alpha)dt}P(S_t^0 = x) + e^{(\mu - \alpha)dt} \left[-d\mu P(S_t^0 = x) + \sum_{y \sim x} d\mu \times \frac{1}{d}P(S_t^0 = y)\right]$$

$$= -\alpha d e^{(\mu - \alpha)dt}P(S_t^0 = x) + \sum_{y \sim x} \mu e^{(\mu - \alpha)dt}P(S_t^0 = y)$$

$$= -\alpha m(t, x) + \sum_{y \sim x} m(t, y)\mu$$

which gives the desired result. 

Let $X_t = |S_t^0|$. We couple $X_t$ to a simple random walk on $\mathbb{Z}$ called $\hat{X}_t$ which jumps to the left at rate $\mu$ and to the right at rate $(d - 1)\mu$ by using the following recipe: $\hat{X}_t$ follows the move of $X_t$ if $X_t \neq 0$; when $X_t$ jumps from 0 to 1, $\hat{X}_t$ jumps to the left with probability $1/d$. Clearly,

$$\hat{X}_t \leq X_t \quad \forall t \geq 0$$
and hence
\[ P(S_t^0 = 0) = P(X_t = 0) \leq P(\hat{X}_t \leq 0) \] (16)

Note that if \( \theta \leq 0 \) then
\[ P(\hat{X}_t \leq 0) \leq E e^{\theta \hat{X}_t} = \sum_{k=0}^{\infty} e^{-d\mu t} \cdot \frac{(d\mu)^k}{k!} \left( \frac{1}{d} e^{-\theta} + \frac{d-1}{d} e^{\theta} \right)^k \]
\[ = \exp \left\{ -d\mu t \left[ 1 - \left( \frac{1}{d} e^{-\theta} + \frac{d-1}{d} e^{\theta} \right) \right] \right\} \]

To optimize this bound we set
\[ 0 = \frac{d}{d\theta} \left[ 1 - \left( \frac{1}{d} e^{-\theta} + \frac{d-1}{d} e^{\theta} \right) \right] = \frac{1}{d} e^{-\theta} - \frac{d-1}{d} e^{\theta} \]

Solving we have \( e^{2\theta} = 1/(d-1) \) or \( e^{\theta} = 1/\sqrt{d-1} \), which leads to the bound
\[ P(\hat{X}_t \leq 0) \leq \exp \left\{ -(d-2\sqrt{d-1})\mu t \right\} \]

Using this with (15) and (16) we have
\[ m(t,0) = e^{(\mu-\alpha)dt} P(S_t^0 = 0) \]
\[ \leq \exp \left\{ \left[ (d - (d-2\sqrt{d-1}))\mu - d\alpha \right] t \right\} \]
\[ \leq \exp \left\{ \left( 2\sqrt{d-1}\mu - d\alpha \right) t \right\} \]

Since \( \alpha = p_0 + d(1-p_0) \) the exponent is negative. We have completed the proof. \( \square \)

**Theorem 6.** On a \( d \)-regular tree the zealot voter model survives locally if \( p_0 = 0 \) and \( \mu > \frac{d}{\sqrt{d-1} + 1} \).

**Proof.** Suppose \( l(x) \) is the distance from \( x \) to the root. Choose a self-avoiding path \( \{ e_n, -\infty < n < \infty \} \) in \( T^d \) such that \( e_0 = e \) is the root and \( |e_n - e_{n+1}| = 1 \). This gives an embedding of \( \mathbb{Z} \) into \( T^d \). Now define
\[ u(n) = P(e_n \in \zeta_t \text{ for some } t) \]
for \( n \geq 0 \). By the strong Markov property, for all \( n.m \geq 0 \)
\[ u(n + m) \geq u(n)u(m) \]
i.e., the sequence is supermultiplicative. Thus \( \beta(\mu) := \lim_{n \to \infty} [u(n)]^{1/n} \) exists.

Let \( S(e) \) denote the subtree starting from the root \( e_0 = e \) that does not include \( e_{-1} \). Consider a lower bound process \( \tilde{\zeta}_t \) where we ONLY use the particles in \( S(e) \). That is, once the particle at \( e \) moves towards \( e_{-1} \) it is ignored afterward. Our next step is to state a result from the contact process. This is Lemma 4.53 in [5] but the proof also works for our COBRA.
Let \( \lfloor \).

We will show particles from the subchain \( \{ \).

We now follow the proof of Proposition 4.57 in Liggett [5] to construct an embedded branching process. Let \( B_0 = \{ e \} \) and \( B_1 = \{ x \in \zeta_s : |x-e| = n \} \). For each \( x \in B_1 \), suppose \( e_{x-1,x} \) is the edge between \( x \) and its parent. We ignore all the Poisson arrows outside \( S(x) \cup \{ e_{x-1,x} \} \) and apply the same rules leading from \( B_0 \) to \( B_1 \). Hence once a descendant of \( x \in B_1 \) steps outside \( S(x) \), it is then ignored in the process of generating \( B_2 \) later on. Then we obtain a random subset \( B(x) \) of \( \{ y \in S(x) \cap \zeta_{2s} : |y-e| = 2n \} \). Let \( B_2 = \cup_{x \in B_1} B(x) \). We repeat the same rule to construct a branching process \( B_j \). Note \( B_j \subset \zeta_{js} \). Moreover \( B_j \) is supercritical since by (18) the offspring distribution has mean \( (d-1)^n a^n > 1 \). Then

\[
\lim_{j \to \infty} \frac{|B_j|}{((d-1)^n a^n)^j}
\]

exists and is positive with positive probability. As a result, we can find an \( \epsilon \) such that for all sufficiently large \( j \),

\[
P \left( |B_j| > \epsilon ((d-1)a)^nj \right) > \epsilon
\]

We will show particles from the subchain \( \{ B_{ji} \}_{i=0}^\infty \)'s are sufficient for local survival. Since it takes time \( ij \) to get \( B_{ji} \), let

\[
\kappa_i = P(e \in \zeta_{2ijs})
\]

Intuitively, a considerable number of particles in \( B_{ji} \) will trace back and the above probability should be bounded away from 0. Like to show the surviving probability of a supercritical branching process is positive, we will look for an iteration from \( r_i \) to \( r_{i+1} \). To do this, note if a particle \( x \in B_j \) does not trace back immediately, it might spend time \( 2ji \) traveling down in \( S(x) \) then come back. Such event has probability \( r_i \). If it then travels up to the root using time \( js \), we obtain \( e \in \zeta_{2(i+1)js} \), considering the time before \( x \) was born. It follows from the strong Markov property that

\[
r_{i+1} \geq P(x \in \zeta_{2(i+1)js} \text{ for some } x \in B_j)P(e_{nj} \in \zeta_{js})
\]

Let \( \lfloor y \rfloor \) be the largest integer \( \leq y \) and let \( N = \lfloor \epsilon((d-1)a)^nj \rfloor \). This is

\[
\geq P(|B_j| > N)[1 - (1 - r_i)^N]P(e_{nj} \in \zeta_{js})
\]

\[
\geq \epsilon[1 - (1 - r_i)^N]P(e_{nj} \in \zeta_{js})
\]
Using the strong Markov property on the last probability gives

\[ P(e_{nj} \in \tilde{s}_j) \geq \left( P(e_n \in \tilde{s}_s) \right)^j = a^{nj} \]

(22)

Let \( f(r) = \epsilon[1 - (1 - r)^N]a^{nj} \). Combining (21), (22) and (20) gives

\[ r_{i+1} \geq f(r_i) \]

Note \( f(r) \) is increasing over \([0, 1]\) with \( f(0) = 0 \). Moreover, \( f'(r) = \epsilon a^{nj}N(1 - r)^{N-1} \). So using the definition of \( N \),

\[ f'(0) = \epsilon a^{nj}N \]
\[ \geq \epsilon a^{nj}[(d-1)^{nj} - 1] \]
\[ = \epsilon^2[a^2(d-1)]^{nj} - \epsilon a^{nj} \]

Since \( a > 1/\sqrt{d-1} \) this is \( > 1 \) if \( j \) is chosen large enough. Thus \( f(r) \) has a fixed point \( r^* \in (0, 1] \). We will prove by induction that

\[ r_i \geq r^*, \forall i \]  

(23)

When \( i = 0 \), the inequality is trivial since \( r_0 = 1 \). Suppose \( r_i \geq r^* \). By the monotonicity of \( f(r) \), we have

\[ r_{i+1} \geq f(r_i) \geq f(r^*) = r^* \]

To generalize (23) to all time \( t \). Note particles die at rate \( d \). Precisely

\[ P(e \in \tilde{s}_t|e \in \tilde{s}_{2ijs}) \geq e^{-d(t-2ijs)} \]

In particular

\[ P(e \in \tilde{s}_t) \geq e^{-dj \epsilon r_i}, \text{ if } 2ijs < t < (i+1)js \]

We have completed the proof.

**Proof of Lemma 3.5.** Consider a simple random walk on \( \mathbb{Z} \) which takes steps

\[
\begin{cases}
+1, \text{ with probability } \frac{d-\mu}{d} \\
-1, \text{ with probability } \frac{\mu}{d}
\end{cases}
\]

Repeating the proof of Lemma 3.1 shows that \( \phi(x) = \left( \frac{\mu}{d-\mu} \right)^x \) is a martingale. The stopping time theorem of martingales shows

\[ P_n(T_0 < \infty) = \left( \frac{\mu}{d-\mu} \right)^n \]

Note

\[ P(e_n \in \tilde{s}_t \text{ for some } t > 0) \geq P_{e_n}(T_e < \infty) \geq \left( \frac{\mu}{d-\mu} \right)^n \]
where the second one is the probability that the COBRA initiated at \(e_n\) ever visits the root. Then

\[
[u(n)]^{1/n} \geq \frac{\mu}{d - \mu}
\]

Since \(\mu > \frac{d}{\sqrt{d-1}+1}\), by assumption we have

\[
\beta(\mu) = \lim_{n \to \infty} [\mu(n)]^{1/n}
\geq \frac{d}{d - \mu} - 1
\]

\[
> \frac{d}{d - d/(\sqrt{d-1}+1)} - 1
\]

\[
= \frac{1}{\sqrt{d-1}}
\]

\[\square\]

4 Results for Galton-Waston Trees

4.1 Survival of COBRA

We will now prove Theorem 7. If our process took place on \(\mathbb{Z}^d\) with \(d \geq 3\) then the next result would follow from Cox, Durrett, and Perkins [3]. Let \(\mu(\mathbb{Z}^d, k)\) be the mean number of particles that we have in the limit as \(t \to \infty\) when we start a coalescing random walk from \(k\) neighbors of 0 chosen without replacement.

**Theorem 10.** Let \(\delta > 0\). If \(\varepsilon > 0\) is small enough then the COBRA dies out if

\[
2d \sum_k p_k(\mu(\mathbb{Z}^d, k) - 1) - p_0 < -\delta
\]

and survives if the last quantity is \(> \delta\).

This result is proved by showing that if time and space are rescaled appropriately then the dual converges to a branching Brownian motion. See Chapter 2 of [3]. However if we only want to prove survival of the dual it is enough to prove that when time is run at rate \(1/\varepsilon\) the size of the dual converges to a branching process. The key to doing this is to let \(K(\varepsilon) \to \infty\) and not add the newly created particles to the dual until time \(K(\varepsilon)\) has elapsed (on the original time scale). Having done this we will not see any coalescence in the modified process. Its only effect is to reduce the number of particles produced.

The same approach works on the \(d\)-regular tree \(T^d\) to prove that if \(d \geq 3\)

**Theorem 11.** Let \(\delta > 0\). If \(\varepsilon > 0\) is small enough then the COBRA dies out if

\[
2d \sum_k p_k(\mu(T^d, k) - 1) - p_0 < -\delta
\]

and survives if the last quantity is \(> \delta\).

The Galton-Watson case is the same once we take into account the fact that when branchings occur the particle is a randomly chosen point of the tree.
4.2 Local Survival

4.2.1 Proof of Theorem 8

Since the branching random walk gives an upper bound for the COBRA, it suffices to show

Lemma 4.1. If $p_0 = 0$, then on a $d$-regular tree, the threshold for the local survival of $\eta^0_t$ satisfies

$$\mu_l(T^d) = \frac{d}{2\sqrt{d} - 1}$$

where $\eta^0_t$ is the branching random walk starting with 1 particle at the root.

To prove this result, define $M(v, n)$ to be the number of oriented loops of length $n$ starting from vertex $v$. It is well-known that the limit $L = \lim_{n \to \infty} M(v, 2n)^{1/2n} = \sup M(v, 2n)^{1/2n}$ exists for all graphs, independent of the choice of vertex. This follows from a simple supermultiplicativity argument. Furthermore, define an evolutionary walk from vertex $u$ to vertex $v$ to be a sequence $0 \leq T_{x_0} < T_{x_1} < \cdots < T_{x_m} < \infty$ with $x_0 = u, x_m = v$. Precisely, this corresponds to a path in the graphical representation such that the fluid can flow from $u$ to $v$. By definition, for a fixed path of length $n$ on the tree, the expected number of evolutionary walks on this path is $(\mu/d)^n$. This is because when a branching event occurs, the expected number of particles landing on a certain neighbor is $\mu/d$. We will show

Lemma 4.2. Suppose $L = \lim_{n \to \infty} M(v, 2n)^{1/2n}$. Then $\mu_l(T^d) = d/L$.

Proof. Let $X_n$ be the number of evolutionary walks of length $n$ starting and ending at the root $v_0$. Note $\{X_{2nk}\}$ dominates a branching process with offspring distribution given by $X_n$. In particular,

$$EX_n \geq (\mu/d)^n M(v_0, 2n)$$

So if $\mu > d/L$, this branching process is supercritical if $n$ is sufficiently large. Choose $n$ so that the above expectation is $> 1$. Note $\forall T > 0$, the expected number of evolutionary walks of length $n$ by time $T$ is

$$\leq \Gamma(d, n) = \frac{d^n}{(n-1)!} \int_0^T e^{-s} s^{n-1} ds \leq \frac{(dT)^n}{n!}$$

The $\Gamma(d, n)$ comes from a sum of $n$ exponential distributions with parameter $d$. Note this is summable with respect to $n$. By Borel-Cantelli Lemma, the maximal length of evolutionary walks within any finite time $T$ is bounded and thus the root has to be visited infinitely often. For the other direction, note the expected number of evolutionary walks traversing $v_0$ is bounded by

$$\sum_{n=1}^{\infty} (\mu/d)^n M(v_0, n) < \infty, \text{ if } \mu < d/L$$

The proof is complete. \qed
**Proof of Lemma 4.1.** It remains to compute $L$. This is given by Pemantle and Stacey [9]. To summarize, note for an oriented loop of length $2n$, $n$ steps are up (i.e. closer to $e_0$) $n$ steps are down. At each step, there are $d - 1$ choices to move farther away. Hence

$$M(0, 2n) = \left(\frac{2n}{n}\right)(d - 1)^n$$

Use Stirling formula the desired result follows. $\square$

### 4.2.2 Condition for Local Survival

We will now prove Theorem 9. In what follows, assume $p_0 = 0$. Let $p_x = \mu/d(x)$ be the probability that the particle at $x$ will be replaced by a child moving closer to the root. Define a harmonic function depending on the distance to the root by

$$\phi(x) = p_x\phi(x - 1) + (1 - p_x)\phi(x + 1) \quad (24)$$

Note (24) is equivalent to

$$\phi(x + 1) - \phi(x) = \frac{p_x}{1 - p_x} [\phi(x) - \phi(x - 1)]$$

$$= \frac{\mu}{d(x) - \mu} [\phi(x) - \phi(x - 1)]$$

Suppose $0 = x_0, x_1, \ldots, x_n = x$ is the path from the root to $x$. We have

$$\phi(x_n) - \phi(x_{n-1}) = \prod_{k=1}^{n-1} \frac{\mu}{d(x_k) - \mu} [\phi(x_1) - \phi(0)] \quad (25)$$

This recursion allows us to impose function $\phi(x)$ on each vertex $x \in G(V, E)$. By Theorem 6.4.8 in [4], if $\phi(x) \to \infty$ for all $l_x \to \infty$ then the dual survives locally. However, it is more convenient to pursue conditions such that the log increment $\log[\phi(x) - \phi(x - 1)] \to \infty$ instead as we will see later.

Taking log of the recursion formula (25) gives

$$\log[\phi(x_n) - \phi(x_{n-1})] = \log[\phi(x_1) - \phi(0)] + \sum_{k=1}^{n-1} \log \frac{\mu}{d(x_k) - \mu}$$

Suppose the Galton-Watson tree has degree distribution $\{q_j\}$. Now consider a branching random walk on $\mathbb{R}$ which has an initial particle at the origin. With probability $q_j$, it gives birth to $j - 1$ particles at $\log \frac{\mu}{j - \mu}$ and this forms a point process $Z$. The location of the first generation is denoted as $\{z_r^1\}$ where $r$ is the index of each individual. For each particle $x$ in the first generation, it generates new particles in a similar way. The location of its children has the same distribution as $\{z_r^1 + x\}$. We obtain the second generation by taking all the children of the first generation. Let $\{z_r^2\}$ be the locations of the second generation. The following generations are produced under the same manner. Denote $\{z_r^n\}$ as the location of
the $n$th generation individuals. Let $F(t) = E[Z(-\infty, t)]$ be the expected number of points in $Z$ to the left of $t$. Define

$$m(\theta) = \int_{-\infty}^{\infty} e^{-\theta t} dF(t) = \sum_{j \geq 3} q_j (j - \mu)^{\theta}$$

To avoid notational confusion, we use $\nu(a) = \inf\{e^{\theta a} m(\theta) : \theta \geq 0\}$. This is (2.1) defined in [1]. It follows from Corollary (3.4) in [1] that $\nu(0) < 1$ implies $\log[\phi(x) - \phi(x-1)] \to \infty$ for all $l_x \to \infty$. Hence $\nu(0) < 1$ is a sufficient condition for local survival.

Remark. On the $d$–regular tree,

$$m(\theta) = (d - 1) \left( \frac{d - \mu}{\mu} \right)^{\theta}$$

Then $\nu(0) < 1$ iff $\mu > d/2$, which gives another proof of Theorem 5.

4.3 Degree = 3 and 4

Our recursion is

$$\phi(x+) - \phi(x) = \frac{\mu}{d(x) - \mu} (\phi(x) - \phi(x-))$$

$x-$ is neighbor closer to root. $x^+$ is any neighbor further away

$$m(\theta) = 2q_3 \left( \frac{3 - \mu}{\mu} \right)^{\theta} + 3q_4 \left( \frac{4 - \mu}{\mu} \right)^{\theta}$$

$$m'(\theta) = 2q_3 \left( \frac{3 - \mu}{\mu} \right)^{\theta} \log \left( \frac{3 - \mu}{\mu} \right) + 3q_4 \left( \frac{4 - \mu}{\mu} \right)^{\theta} \log \left( \frac{4 - \mu}{\mu} \right)$$

Reacall $\nu(0) = \min\{m(\theta) : \theta \geq 0\}$.

4.3.1 $\mu > 2$

Since $\mu/(3 - \mu)$ and $\mu/(4 - \mu)$ are both $> 1$, $\phi(x_n) \to \infty$ along any path $x_n \to \infty$, so the process survives strongly.

4.3.2 $\mu \leq 3/2$

Since $\mu/(3 - \mu)$ and $\mu/(4 - \mu)$ are both $< 1$, $\phi(x_n) \to 0$ along any path $x_n \to \infty$. However this only tells us that the proof fails.

4.3.3 $3/2 < \mu \leq 2$

Case 1. Note that if $2q_3 > 1$ there is a path to $\infty$ (which may not start at the root) along which we take the products of $\mu/(3 - \mu)$ and hence $\phi(x_n) \to 0$, so the proof fails.
Figure 3: Local survival is possible only if $\mu'(0) < 0$. This is $q_3 > 0.85$ for $\mu = 1.6$; $q_3 > 0.65$ for $\mu = 1.7$; $q_3 > 0.45$ for $\mu = 1.8$ and $q_3 > 0.25$ for $\mu = 1.9$.

Case 2. $(4 - \mu)/\mu > (3 - \mu)/\mu$ so if

$$m'(0) = 2q_3 \log \left( \frac{3 - \mu}{\mu} \right) + 3q_4 \log \left( \frac{4 - \mu}{\mu} \right) > 0$$

then $m'(\theta) > 0$ for all $\theta > 0$ and the minimum occurs at 0. $m(0) = 2q_3 + 3q_4 \geq 2$, so again the proof fails. let $q_3 = p$ and $q_4 = 1 - p$. For fixed $\mu$, $m'(0)$ is linear in $p$ so the condition holds when

$$p < p_c = \frac{3 \log((4 - \mu)/\mu)}{3 \log((4 - \mu)/\mu) + 2 \log(\mu/(3 - \mu))}$$

Case 3. If $m'(0) < 0$ then a minimum at $\bar{\theta} > 0$ exists. Using (27) we want

$$2q_3 \left( \frac{3 - \mu}{\mu} \right)^{\theta} \log \left( \frac{\mu}{3 - \mu} \right) = 3q_4 \left( \frac{4 - \mu}{\mu} \right)^{\theta} \log \left( \frac{4 - \mu}{\mu} \right)$$

Cross multiplying

$$\left( \frac{4 - \mu}{3 - \mu} \right)^{\theta} = \frac{2q_3 \log(\mu/(3 - \mu))}{3q_4 \log((4 - \mu)/\mu)}$$

Let $A$ be the numerator and $B$ be the denominator of the fraction. $m'(0) < 0$ implies $A > B$. The LHS is 1 at $\theta = 0$ and increases $\to \infty$ as $\theta \to \infty$ so a solution exists. Taking logs

$$\theta \log((4 - \mu)/(3 - \mu)) = \log(A) - \log(B)$$
so we have

\[
\tilde{\theta} = \frac{\log(A) - \log(B)}{\log((4 - \mu)/(3 - \mu))}
\]

There does not seem to be a good formula for \(m(\tilde{\theta})\). To compute it numerically, we choose \(\mu = 1.6, 1.7, 1.8\) and 1.9 for

\[
\nu(0) = m(\tilde{\theta}) = 2q_3 \exp(\tilde{\theta} \log((3 - \mu)/\mu) + 3q_4 \exp(\tilde{\theta} \log((4 - \mu)/\mu))
\]

It shows from the table that the phase transition occurs at \(q_3 = 0.996, 0.97, 0.91\) and 0.82 respectively.

| \(q_3\) | \(\mu = 1.6\) | \(\mu = 1.7\) | \(\mu = 1.8\) | \(\mu = 1.9\) |
|---|---|---|---|---|
| 0.8 | 2.2 | 2.014149722 | 1.597069414 | 1.074539921 |
| 0.81 | 2.19 | 1.979137551 | 1.549560204 | **1.030929768** |
| 0.82 | 2.179999993 | 1.942042353 | 1.500601354 | **0.896331678** |
| 0.83 | 2.169035962 | 1.902724759 | 1.450116162 | 0.941977346 |
| 0.88 | 2.079229445 | 1.666138794 | 1.171184237 | 0.708137684 |
| 0.89 | 2.052259329 | 1.608953028 | 1.109102792 | 0.659079066 |
| 0.9 | 2.021286727 | 1.547575636 | **1.044453965** | 0.609119574 |
| 0.91 | 1.985646496 | 1.481425138 | **0.976943138** | 0.558174592 |
| 0.92 | 1.944464056 | 1.409753116 | 0.906197761 | 0.506138592 |
| 0.93 | 1.896552634 | 1.331568175 | 0.831734216 | 0.452876792 |
| 0.94 | 1.840236437 | 1.245508443 | 0.752903513 | 0.398211752 |
| 0.95 | 1.773023132 | 1.14961228 | 0.668796138 | 0.341900422 |
| 0.96 | 1.690934707 | **1.040865809** | 0.578059943 | 0.283591365 |
| 0.97 | 1.586938026 | **0.914185487** | 0.478505588 | 0.222735055 |
| 0.98 | 1.446391322 | 0.759629666 | 0.366073412 | 0.158358633 |
| 0.99 | 1.227510494 | 0.551306428 | 0.23102478 | 0.088284998 |
| 0.995 | **1.037752234** | | | |
| 0.996 | **0.98268267** | | | |
| 0.997 | 0.915774891 | | | |
| 0.998 | 0.828879261 | | | |
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