POINTWISE ESTIMATES FOR BIPOLAR COMPRESSIBLE NAVIER-STOKES-POISSON SYSTEM IN DIMENSION THREE

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Abstract: The Cauchy problem of the bipolar Navier-Stokes-Poisson system (1.1) in dimension three is considered. We obtain the pointwise estimates of the time-asymptotic shape of the solution, which exhibit generalized Huygens’ principle as the Navier-Stokes system. This phenomenon is the most important difference from the unipolar Navier-Stokes-Poisson system. Due to non-conservative structure of the system, the difference from the unipolar Navier-Stokes-Poisson system. Due to non-conservative structure of the system, generalized Huygens’ principle as the Navier-Stokes system. This phenomenon is the most important considered. We obtain the pointwise estimates of the time-asymptotic shape of the solution, which exhibit with initial data under the influence of electrostatic force governed by the self-consistent Poisson equation, cf. [19]. In this paper, we mainly consider the three dimensions case.

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Key Words: Green’s function; Bipolar Navier-Stokes-Poisson system; Huygens’ principle.

1. Introduction.

The bipolar Navier-Stokes-Poisson system has been used to simulate the transport of charged particles under the influence of electrostatic force governed by the self-consistent Poisson equation, cf. [1]. In this paper, we are mainly concerned with the Cauchy problem of the bipolar Navier-Stokes-Poisson system (BNSP) in n dimensions:

\[
\begin{align*}
\partial_t \rho_1 + \text{div} J_1 &= 0, \\
\partial_t J_1 + \text{div} (\frac{\rho_1}{\mu_1} J_1) + \nabla P_1(\rho_1) &= \mu_1 \Delta (\frac{\Delta \phi}{\rho_1}) + \mu_2 \nabla \text{div} (\frac{\Delta \phi}{\rho_2}) + \rho_1 \nabla \phi, \\
\partial_t \rho_2 + \text{div} J_2 &= 0, \\
\partial_t J_2 + \text{div} (\frac{\rho_2}{\mu_2} J_2) + \nabla P_2(\rho_2) &= \mu_1 \Delta (\frac{\Delta \phi}{\rho_2}) + \mu_2 \nabla \text{div} (\frac{\Delta \phi}{\rho_2}) - \rho_2 \nabla \phi,
\end{align*}
\]

(1.1)

with initial data

\[
(\rho_1, \rho_2, J_1, J_2, \nabla \phi)(x, 0) = (\rho_{1,0}, \rho_{2,0}, J_{1,0}, J_{2,0}, \nabla \phi_0)(x), x \in \mathbb{R}^n.
\]

(1.2)

Here \(\rho_i(x, t), J_i(x, t) = \rho_i u_i, \phi(x, t)\) and \(P_i = P_i(\rho_i)\) represent the fluid density, momentum, self-consistent electric potential and pressure. The viscosity coefficients satisfy the usual physical conditions \(\mu_1 > 0, \mu_2 > 0\). \(\frac{\Delta \phi}{\rho_1} > 0\). \(\rho_0 > 0\) denotes the prescribed density of positive charged background ions, and in this paper is taken as a positive constant. In the present paper, we mainly consider the three dimensions case.

Now we mainly review some previous works on the Cauchy problem for some related models. For the compressible Navier-Stokes system, a lot of works have been done on the existence, stability and \(L^p\)-decay rates with \(p \geq 2\) for either isentropic or non-isentropic (heat-conductive) cases, cf. [2, 3, 7, 8, 9, 11, 12, 13, 10, 17, 20, 28, 29, 30] in various settings by using (weighted) energy method together with spectrum analysis. On the other hand, Liu and Zeng [26] first studied the pointwise estimates of solution to general hyperbolic-parabolic systems in one space dimension by using the method of Green’s function. Hoff and Zumbrun [5] investigated the \(L^p(\mathbb{R}^n)\) \((p \geq 1)\) estimates for the isentropic Navier-Stokes system in multi-dimensions. Then they in [6] deduced a detailed, pointwise description of the Green’s function for the related linearized Navier-Stokes system with artificial viscosity. Later, Liu and Wang [19] considered the pointwise estimates of the solution in odd dimensions and showed the generalized Huygens’ principle by using real analysis method.

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which was reconsidered in [18] by using the complex analysis method for the Green’s function recently. In particular, they in [18, 19] showed the solution behaves as

\[(1 + t)^{-\frac{3}{2}} \left(1 + \frac{|x|^2}{1+t} \right)^{-\frac{3}{4}} + (1 + t)^{-\frac{3n-1}{2}} \left(1 + \frac{(|x| - ct)^2}{1+t} \right)^{-\frac{3}{4}},\]

where the first profile is called as the diffusion wave(D-wave) and the second profile is called as the generalized Huygens’ wave(H-wave). Wang et al. [33, 38] also discussed the pointwise estimates of the solution for the damped Euler system when initial data is a small perturbation of the constant state and showed the solution behaves as the diffusion wave since the long wave of the Green’s function does not contain the wave operator. All of the results above showed the different asymptotic profiles in the pointwise sense for different systems. Indeed, the pointwise estimates of the solution play an important role in the description of the partial differential equations, since it can give explicit expressions of the time-asymptotic behavior of the solution. Moreover, one can get the global existence and optimal \(L^p\)-estimates of the solution directly from the pointwise estimates of the solution. Here, we give some comments on the diffusion wave(D-wave) and the generalized Huygens’ wave(H-wave) in (1.3) for the convenience of readers. As we know, for the \(L^p(\mathbb{R}^n)\)-estimate of these two waves, when \(p < 2\) the H-wave is the dominated part and when \(p > 2\) the D-wave is the dominated part, and \(p = 2\) is the critical case and the \(L^2\)-decay rate of these two waves is the same as the heat kernel. In other words, the usual \(L^2\)-estimates for some hyperbolic-parabolic coupled systems indeed conceal the hyperbolic characteristic of the solution. Besides, there are also other results in [21, 22, 23, 24, 25, 27, 39, 40] on the related models for the \(L^1\)-estimate, wave propagation pattern around shock wave by using the method of Green’s function.

Nowadays, for unipolar Navier-Stokes-Poisson system(NSP), there are also some results on the decay rate of the Cauchy problem when the initial data \((\rho_0, u_0)\) is a small perturbation of the constant state \((\bar{\rho}, 0)\). Li et al. [14] obtained the global existence and large time behavior of classical solution by spectrum method and energy method. They mainly found that the density of NSP system converges to its equilibrium state as the Navier-Stokes system, but the momentum of NSP system decays at \(L^2\)-decay rate \((1 + t)^{-\frac{1}{2}}\) as the Navier-Stokes system, and then they extended the similar result to the non-isentropic case [42]. On the other hand, in [32, 36], we investigated the pointwise estimates of the solution and showed the pointwise profile of the solution contains the D-wave but does not contain the H-wave, which is different from (1.3) for the Navier-Stokes system in [18, 19]. In fact, the NSP system is a hyperbolic-parabolic system with a non-local term arising from the electric field \(\nabla \phi\). The symbol of this nonlocal term is singular in the long wave of the Green’s function. It destroys the acoustic wave propagation, at the same time, this non-local term also destroys the time-decay rate and the regularity (integrability) on the space variable \(x\) for the momentum (or the velocity \(u\)). Recently, Wang [35] improved the \(L^2\)-decay rate and Wu et al. [37] improved the pointwise estimate under an additional assumption that the initial data of the electric potential term \(\nabla \phi\) is in \(L^1(\mathbb{R}^3)\), respectively. Li et al. [16] considered the similar problem in the negative Besov space to improve the \(L^2\)-decay rate.

Compared with NSP system, there are few results for the Cauchy problem of BNSP system (1.1)-(1.2) due to the interplay of two carriers which counteracts the influence of electric field. Li et al. [15] first gave the global existence and the \(L^2\)-decay rate of the classical solution as in [14] when the initial data is small in \(H^4 \cap L^1(\mathbb{R}^3)\) by changing the system (1.1) to (2.4). Later, Zou [43] improved the \(L^2\)-decay rate under the assumption on the initial data in the negative Besov space. Recently, Wang et al. [34] directly considered the system (1.1) and also obtained the \(L^2\)-decay rate of the solution as the Navier-Stokes system by using long wave and short wave decomposition method together with energy method when the initial data is small in \(H^4 \cap L^1(\mathbb{R}^3)\) together with an additional assumption of the antiderivative of two densities. Both the assumptions on the initial data in the negative Besov space and the antiderivative of two densities in [34, 43] are used to lift the \(L^2\)-decay rate of two momentums to \((1 + t)^{-\frac{1}{2}}\) as that of the Navier-Stokes system.

The motivation of this paper is as follows. Firstly, from the analysis on the Green’s function of BNSP system (1.1), we find that there exists the wave operator in the long wave of the Green’s function, therefore we hope that the pointwise estimates of the solution to the Cauchy problem (1.1)-(1.2) exhibit the generalized Huygens’ principle as in [18, 19], which is an absolutely different phenomenon from the unipolar case in
contains a variable \( n \) in the momentum equation. Based on the mechanism of linearized NSP system, we hope the solution from using the variable \( n \) can rate both on the time and the space for pointwise estimate of \( \rho \) because of the term \( \nabla n \) is conservative, which is called as a Navier-Stokes system on the total density \( \rho \). However, the system (1.1) is conservative, which is called as a Navier-Stokes system on the density and momentum, which allows to “borrow” the first order derivative from the nonlinear terms to deduce a H-wave above when estimating the convolution between the Green’s function and nonlinear terms. Lastly, with the pointwise estimates in hand, we can immediately obtain the \( L^p(p > 1) \)-estimate of the solution, which is a generalization for the usual \( L^p(p \geq 2) \)-decay rate of the solution given in [14, 34, 43].

We make a brief interpretation for the main idea of the proof. Due to the slower \( L^p \)-decay rate of the H-wave when \( p < 2 \), the authors in [18, 19] rely heavily on the conservative structure of the Navier-Stokes system on the density and momentum, which allows to “borrow” the first order derivative from the nonlinear terms to deduce a H-wave above when estimating the convolution between the Green’s function and nonlinear terms. However, the system (1.1) has not the conservative structure because of the presence of the terms \( \rho_1 \nabla \phi \) and \( \rho_2 \nabla \phi \) in the two momentum equations. Fortunately, by rewriting (1.1) into a new system (2.4) with the new variables \( (\rho_1 + \rho_2, J_1 + J_2, \rho_1 - \rho_2, J_1 - J_2) \), and from the Poisson equation \( \Delta \phi = \rho_1 - \rho_2 =: n_2 \), we find that the nonlinear term \( n_2 \nabla \phi = \Delta \phi \nabla \phi = \text{div}(\nabla \phi \otimes \nabla \phi - \frac{i}{4} |\nabla \phi|^2 I_{3 	imes 3}) \). Then the system (2.4)_{1,2} is conservative, which is called as a Navier-Stokes system on the total density \( n_1 = \rho_1 + \rho_2 \) and the total momentum \( w_1 = J_1 + J_2 \). Hence, it is possible to help us to obtain the H-wave as in [18, 19]. Nevertheless, to utilize this conservative structure in (2.4)_{1,2}, we need the pointwise estimate of the electronic potential \( \nabla \phi \), since there exists the term \( n_2 \nabla \phi \) in the equation (2.4)_2.

By the relation \( \nabla \phi = \sum_n n_2 \), we know that the pointwise estimate of \( \nabla \phi \) can only be derived from the pointwise estimate of \( n_2 \) in the system (2.4)_{3,4,5} with the variables \( n_2 = \rho_1 - \rho_2 \) and \( w_2 = J_1 - J_2 \). Thus, in the first step we should consider the NSP system (2.4)_{3,4,5}, which has still not the conservative structure because of the term \( (\rho_1 + \rho_2) \nabla \phi \) in (2.4)_4. As showed in [32, 37], because of the presence of the electric field \( \nabla \phi \), the long wave of the Green’s function of NSP only contains the D-wave and does not contain the H-wave, then one can deduce the pointwise estimates as the D-wave without the help from the conservative structure of the system. Based on the mechanism of linearized NSP system, we hope the solution \( (n_2, w_2) \) of the system (2.4)_{3,4,5} also behaves like the D-wave, though there exists both the D-wave and the H-wave arising from the highly coupled unknown variables of the nonlinear terms \( F_2(n_1, w_1, n_2, w_2) \) in (2.4)_4. Otherwise, once the pointwise estimate of \( n_2 = \rho_1 - \rho_2 \) contains a H-wave, it will generate a new H-wave with “worse” decay rate for the pointwise estimate of \( \nabla \phi \) from the relation between \( n_2 \) and \( \nabla \phi \). Thus, it is crucial for us to prevent the H-wave from happening in the pointwise estimate of \( n_2 \). On the other hand, when using the system (2.4)_{3,4,5} to get the pointwise estimate for \( n_2 \) and \( \nabla \phi \), one will encounter the nonlinear term \( n_1 \nabla \phi \) in the momentum equation (2.4)_4. Then it need the pointwise estimate of \( \nabla \phi \) in turn when deducing the pointwise estimate of \( n_2 \). What’s worse, there is a factor \( \varepsilon \) in Proposition 5.4 for the pointwise estimate of \( \nabla \phi \), that is, the nonlocal operator \( \frac{\nabla}{\varepsilon} \) acting on a D-wave for \( n_2 \) will lead to a D-wave with “bad” decay rate both on the time and the space for \( \nabla \phi \). Due to this mutual mechanism between \( n_2 \) and \( \nabla \phi \), and the presence of the factor \( \varepsilon \), one will go into an endless loop when making the nonlinear estimates and ultimately cannot close the ansatz (5.23). Hence, it is likewise crucial for us to avoid the presence of the term \( \nabla \phi \) in the nonlinear terms of the system on the variable \( n_2 \) as far as possible.

To overcome the difficulties from the system (2.4)_{3,4,5}, we mainly rely on the good form of nonlinear term \( F_2(n_1, w_1, n_2, w_2) \). In fact, after a careful computation we find that each nonlinear term in \( F_2(n_1, w_1, n_2, w_2) \) contains a variable \( n_2 \) or \( w_2 \), that is, there is not the interaction between the H-wave and the H-wave in \( F_2(n_1, w_1, n_2, w_2) \). Then, after a careful computation, we can prove the pointwise asymptotic shape of \( (n_2, w_2) \) only contains the D-wave. On the other hand, to avoid dealing with the nonlinear term \( n_1 \nabla \phi \) in the momentum equation (2.4)_4, we consider a new system on the variables \( \rho_1 - \rho_2 = n_2 \) and \( u_1 - u_2 =: w_2 \) in (5.21), where there is actually not the variable \( \nabla \phi \) in the nonlinear term \( F_4(n_1, v_1, n_2, v_2) \). That is, when deducing the pointwise estimate of \( n_2 \), we need not the pointwise estimate of \( \nabla \phi \), which is the key difference from using the variable \( (n_2, w_2) \). On the other side, we have to overcome the difficulty from the nonlocal operators in the Green’s function of the system (2.4)_{3,4,5}. Noticing the following proposition in Fourier space
of long wave part for the Green’s function for linearized NSP system (2.4)_{3,4,5}:

$$\hat{G}(\xi,t) = \begin{pmatrix} \hat{G}_{11} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{22} \end{pmatrix} \sim \begin{pmatrix} 1 & \xi \\ \xi & |\xi|^2 \end{pmatrix} e^{-C_1|\xi|^2t},$$

i.e., there are nonlocal operators with symbol $\frac{\xi}{|\xi|^2}$ and $\frac{\xi^T}{|\xi|^2}$ in Fourier space. Under the assumption on $\nabla \phi_0$ in (1.6), we can regard the nonlocal operator with the symbol $\frac{\xi}{|\xi|^2}$ in the Green’s function of (2.4)_{3,4,5} as the nonlocal operator with the symbol $\frac{\xi^T}{|\xi|^2}$ because of the Poisson equation (2.4)_{5}. In addition, for the nonlinear estimates, the operator $\frac{\xi}{|\xi|^2}$ in the long wave of the Green’s function just corresponds to the nonlinear term $F_3(n_1,v_1,n_2,v_2) = -\text{div}(\frac{n_1v_2+n_2v_1}{2})$ in (5.21), which is conservative due to the mass conservation on the variable $\rho_1 - \rho_2$. Then, one can still “borrow” the first order derivative from $F_3(n_1,v_1,n_2,v_2)$ to the Green’s function $\mathbb{G}_{21}$. From these facts above, we can obtain a refined pointwise estimate of the solution to unipolar Navier-Stokes-Poisson system as in (1.10). Then, all of the pointwise estimates for the other variables are based on (1.10).

Lastly, the presence of the factor $\varepsilon$ in the pointwise estimate for $\nabla \phi$ in Proposition 5.4 will bring us new difficulty when dealing with the nonlinear problem, since it decays slower than all of the variables in the Navier-Stokes system [18, 19]. In fact, owing to the presence of several different wave profiles: Riesz wave, Huygens’ wave, diffusion wave in long wave of the Green’s function of the Navier-Stokes system, to close the ansatz (5.23) on the variables $(n_1,v_1,n_2,v_2)$, all of which have the different (unsymmetrical) asymptotic profiles in the pointwise estimates, we have to make some subtle estimations for the convolutions between each term in the Green’s functions and the coupled nonlinear terms. Specifically, sometimes we will exchange time decay rate with the space decay rate to avoid producing the factor $\varepsilon$ in the time decay rate, and sometimes we will exchange the D-wave with the H-wave though it will waste the time decay rate to close the ansatz on the different pointwise profiles between $(\rho_1 - \rho_2, J_1 - J_2)$ and $D_x^\alpha (\rho_1 - \rho_2, J_1 - J_2)$ with $1 \leq |\alpha| \leq 2$.

Throughout this paper, $C$, $C_1$ and $c_0$ denote general positive constants which may vary in different estimates. We use $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$, where $W^{s,p}(\mathbb{R}^n)$ is the usual Sobolev space with its norm

$$\|f\|_{W^{s,p}(\mathbb{R}^n)} = \sum_{k=0}^s \|\partial_x^k f\|_{L^p(\mathbb{R}^n)}.$$ 

The following existence result is from [15, 34, 43], the unique difference is the regularity of the initial data in $H^6(\mathbb{R}^3)$ here.

**Theorem 1.1.** (Existence) Let $P_1'(\rho_1) > 0$ and $P_2'(\rho_2) > 0$ for $\rho_1 > 0$ and $\rho_2 > 0$ respectively, and $\bar{\rho} > 0$ be a constant. Assume that $(\rho_i - \bar{\rho}, J_i, \nabla \phi_0) \in H^6(\mathbb{R}^3)$ for $i = 1, 2$, with $\varepsilon_0 =: \|\rho_i - \bar{\rho}, J_i, \nabla \phi_0\|_{H^6(\mathbb{R}^3)}$ small. Then there is a unique global classical solution $(\rho_i, J_i, \nabla \phi)$ of the Cauchy problem (1.1)-(1.2) satisfying

$$\|\rho_i - \bar{\rho}, J_i, \nabla \phi\|_{H^6(\mathbb{R}^3)} \leq C\varepsilon_0. \quad (1.4)$$

Main results of this paper are stated as the following theorem.

**Theorem 1.2.** (Pointwise estimates) Under the assumptions of Theorem 1.1. If further, for $|\alpha| \leq 2$

$$|D_x^\alpha (\rho_{1,0} + \rho_{2,0} - 2\bar{\rho}, J_{1,0} + J_{2,0})| \leq C\varepsilon_0 (1 + |x|^2)^{-r_1}, \quad r_1 \geq \frac{21}{10}, \quad (1.5)$$

$$|D_x^\alpha (\rho_{1,0} - \rho_{2,0}, J_{1,0} - J_{2,0}), \nabla \phi_0| \leq C\varepsilon_0 (1 + |x|^2)^{-r}, \quad r > \frac{3}{2}, \quad (1.6)$$
then for the base sound speed $c := \sqrt{P_0'(\bar{\rho})}$ and a constant $\varepsilon$ satisfying $0 < \varepsilon \ll 1$, it holds that

$$\begin{align*}
|\rho_1 + \rho_2 - 2\bar{\rho}| &\leq C\varepsilon_0(1+t)^{-2}\left\{\left(1 + \frac{|x - ct|^2}{1 + t}\right)^{-\frac{3}{2}} - \varepsilon \right\}, \quad (1.7) \\
|J_1 + J_2| &\leq C\varepsilon_0(1+t)^{-2}\left(1 + \frac{|x - ct|^2}{1 + t}\right)^{-\frac{3}{2}} + C\varepsilon_0(1+t)^{-\frac{3}{2}} \left(1 + \frac{|x|^2}{1 + t}\right)^{-\frac{3}{2}}, \quad (1.8) \\
|\rho_1 - \rho_2| &\leq C\varepsilon_0(1+t)^{-2}\left(1 + \frac{|x|^2}{1 + t}\right)^{-\frac{3}{2}}, \quad (1.9) \\
|J_1 - J_2| &\leq C\varepsilon_0(1+t)^{-\frac{3}{2}} \left(1 + \frac{|x|^2}{1 + t}\right)^{-\frac{3}{2}}, \quad (1.10)
\end{align*}$$

which imply that

$$\begin{align*}
|\langle \rho_1 - \rho, \rho_2 - \bar{\rho} \rangle| &\leq C\varepsilon_0(1+t)^{-2}\left\{\left(1 + \frac{|x - ct|^2}{1 + t}\right)^{-\frac{3}{2}} - \varepsilon \right\}, \quad (1.11) \\
|\langle J_1, J_2 \rangle| &\leq C\varepsilon_0(1+t)^{-2}\left(1 + \frac{|x - ct|^2}{1 + t}\right)^{-\frac{3}{2}} + C\varepsilon_0(1+t)^{-\frac{3}{2}} \left(1 + \frac{|x|^2}{1 + t}\right)^{-\frac{3}{2}}. \quad (1.12)
\end{align*}$$

**Remark 1.1.** We find that the pointwise estimate of the densities is better than those of the momentums, which is resulted from two facts. On the one hand, the first row of the Green’s function of the Navier-Stokes system corresponding to the variable $\rho_1 + \rho_2$ only contains the Riesz’ wave and the rest rows of the Green’s function corresponding to the variable $J_1 + J_2$ contain both the Riesz wave and the Huygens’ wave. On the other hand, the assumption (1.6) on $\nabla \phi$ can enhance the decay rate of the variable $\rho_1 - \rho_2$ in the Navier-Stokes-Poisson system in (2.4)$_{3,4,5}$.

**Remark 1.2.** Comparing with the pointwise estimates for the Navier-Stokes system, the solution is also in $L^p(\mathbb{R}^3)$ with $p > 1$ in spite of an extra factor $\varepsilon$ in Theorem 1.2. That is, the factor $\varepsilon$ does not matter the range of $p$ and the pointwise estimates with optimal decay rate above for $(\rho_1 - \rho, \rho_2 - \bar{\rho}, J_1, J_2)$ are almost the same as in [18, 19]. Here the factor $\varepsilon$ is caused by the pointwise estimate for $\nabla \phi$, which does not exist in the Navier-Stokes system [18, 19].

**Remark 1.3.** The result in Theorem 1.2 also holds for other odd dimensions $n \geq 5$ without any new difficulty. On the other hand, it is more difficult to deduce the pointwise estimates for the problem (1.1)-(1.2) in even dimensions, especially in two dimensions, which will be considered in future.

From Theorem 1.2 and Remark 1.3, we have the following $L^p$-decay rates in odd dimensions $n \geq 3$.

**Corollary 1.3.** Under the assumptions in Theorem 1.2, we have the following optimal $L^p(\mathbb{R}^n)$ estimates of the solution in odd dimensions $n \geq 3$

$$\begin{align*}
\|\rho_1 - \rho, \rho_2 - \bar{\rho}, \rho_1 + \rho_2 - 2\bar{\rho}\|_{L^p(\mathbb{R}^n)} &\leq C\varepsilon_0(1+t)^{-\frac{3}{2} + \frac{1}{p} + \frac{3}{2}} \left(1 + \frac{|x - ct|^2}{1 + t}\right)^{-\frac{3}{2} - \varepsilon}, \quad 1 < p \leq \infty, \\
\|\rho_1 + \rho_2\|_{L^p(\mathbb{R}^n)} &\leq C\varepsilon_0(1+t)^{-\frac{3}{2} + \frac{1}{p}}, \quad 1 < p \leq \infty, \\
\|\langle J_1, J_2 \rangle\|_{L^p(\mathbb{R}^n)} &\leq \begin{cases} C\varepsilon_0(1+t)^{-\frac{3}{2} + \frac{1}{p} + \frac{3}{2}} \left(1 + \frac{|x - ct|^2}{1 + t}\right)^{-\frac{3}{2} - \varepsilon}, & 1 < p \leq 2, \\
C\varepsilon_0(1+t)^{-\frac{3}{2} + \frac{1}{p}}, & 2 \leq p \leq \infty, \end{cases} \\
\|\langle J_1 - J_2 \rangle\|_{L^p(\mathbb{R}^n)} &\leq C\varepsilon_0(1+t)^{-\frac{3}{2} + \frac{1}{p}}, \quad 1 < p \leq \infty.
\end{align*}$$

**Remark 1.4.** The usual $L^2$-estimates conceal the hyperbolic feature for the hyperbolic-parabolic coupled system. Our $L^p$-estimates above imply that the dominant part of $(\rho_1, \rho_2)$ is always the H-wave for all of $p > 1$, the dominant part of $(\rho_1 - \rho_2, J_1 - J_2)$ is always the D-wave for all of $p > 1$, and the dominant part of $(J_1, J_2)$ is the H-wave when $p < 2$ and the D-wave when $p \geq 2$. In fact, the $L^p$-decay rate of $(\rho_1, \rho_2)$ is completely new, which can hardly be deduced by using $L^2$-estimates together with Sobolev inequalities.

The rest of the paper is arranged as follows. In Section 2, we reformulate the system (1.1) into a new system that we will considered in the next sections. In Section 3 and Section 4, we give the pointwise
estimates for the Green’s functions of the Navier-Stokes system and the Navier-Stokes-Poisson system. The pointwise estimates of the solution to the nonlinear Cauchy problem (1.1)-(1.2) will be deduced in Section 5. Lastly, some useful lemmas on the nonlinear estimates are given in Appendix.

2. Reformulation of original problem.

Assume \( \tilde{\rho} = 1 \) and \( \sqrt{P_i(\tilde{\rho})} = c \) without loss of generality. Then the Cauchy problem (1.1)-(1.2) is reformulated as

\[
\begin{aligned}
\partial_t \rho_1 + \text{div} J_1 &= 0, \\
\partial_t J_1 - \mu_1 \Delta J_1 - \mu_2 \nabla \text{div} J_1 + c^2 \nabla \rho_1 - \nabla \phi &= -\text{div}\left( \frac{J_1 \otimes J_1}{\rho_1} \right) - \nabla (P_1(\rho_1) - c^2 \rho_1) + (\rho_1 - 1) \nabla \phi, \\
\partial_t \rho_2 + \text{div} J_2 &= 0, \\
\partial_t J_2 - \mu_1 \Delta J_2 - \mu_2 \nabla \text{div} J_2 + c^2 \nabla \rho_2 + \nabla \phi &= -\text{div}\left( \frac{J_2 \otimes J_2}{\rho_2} \right) - \nabla (P_2(\rho_2) - c^2 \rho_2) - (\rho_2 - 1) \nabla \phi, \\
\Delta \phi &= \rho_1 - \rho_2, \\
(\rho_1, J_1, \rho_2, J_2, \nabla \phi)(x, 0) &= (\rho_{1,0}, J_{1,0}, \rho_{2,0}, J_{2,0}, \nabla \phi_0)(x).
\end{aligned}
\]

(2.1)

Next, set

\[
n_1 = \rho_1 + \rho_2 - 2, \quad n_2 = \rho_1 - \rho_2, \quad w_1 = J_1 + J_2, \quad w_2 = J_1 - J_2,
\]

(2.2)

which give equivalently

\[
\rho_1 = \frac{n_1 + n_2}{2} + 1, \quad \rho_2 = \frac{n_1 - n_2}{2} + 1, \quad J_1 = \frac{w_1 + w_2}{2}, \quad J_2 = \frac{w_1 - w_2}{2}.
\]

(2.3)

From (2.2) and (2.3), it follows that the Cauchy problem (2.1) can be reformulated into the following Cauchy problem for the unknown \( (n_1, w_1, n_2, w_2, \nabla \phi) \)

\[
\begin{aligned}
\partial_t n_1 + \text{div} w_1 &= 0, \\
\partial_t w_1 - \mu_1 \Delta w_1 - \mu_2 \nabla \text{div} w_1 + c^2 \nabla n_1 &= F_1(n_1, w_1, n_2, w_2), \\
\partial_t n_2 + \text{div} w_2 &= 0, \\
\partial_t w_2 - \mu_2 \Delta w_2 - \mu_2 \nabla \text{div} w_2 + c^2 \nabla n_2 - 2 \nabla \phi &= F_2(n_1, w_1, n_2, w_2), \\
\Delta \phi &= n_2, \\
(n_1, w_1, n_2, w_2, \nabla \phi)(x, 0) &= (n_{1,0}, w_{1,0}, n_{2,0}, w_{2,0}, \nabla \phi_0)(x),
\end{aligned}
\]

(2.4)

where \((n_{1,0}, w_{1,0}, n_{2,0}, w_{2,0}) := (\rho_{1,0} + \rho_{2,0} - 2, J_{1,0} + J_{2,0}, \rho_{1,0} - \rho_{2,0}, J_{1,0} - J_{2,0})\), and

\[
F_1 = -\text{div}\left( \frac{(n_1 + n_2)(w_1 + w_2)}{2(n_1 + n_2) + 4} \right) + \frac{(n_1 - n_2)(w_1 - w_2)}{2(n_1 - n_2) + 4},
\]

\[
\nabla \left( P_1\left( \frac{n_1 + n_2}{2} + 1 \right) - c^2 \frac{n_1 + n_2}{2} + P_2\left( \frac{n_1 - n_2}{2} + 1 \right) - c^2 \frac{n_1 - n_2}{2} \right) + n_2 \nabla \phi - \mu_1 \Delta \frac{(n_1 - n_2)(w_1 - w_2)}{2(n_1 - n_2) + 4} + \frac{(n_1 + n_2)(w_1 + w_2)}{2(n_1 + n_2) + 4},
\]

\[
F_2 = -\text{div}\left( \frac{(n_1 + n_2)(w_1 + w_2)}{2(n_1 + n_2) + 4} \right) - \frac{(n_1 - n_2)(w_1 - w_2)}{2(n_1 - n_2) + 4},
\]

\[
\nabla \left( P_1\left( \frac{n_1 + n_2}{2} + 1 \right) - c^2 \frac{n_1 + n_2}{2} - P_2\left( \frac{n_1 - n_2}{2} + 1 \right) + c^2 \frac{n_1 - n_2}{2} \right) + n_1 \nabla \phi - \mu_2 \Delta \frac{(n_1 + n_2)(w_1 + w_2)}{2(n_1 + n_2) + 4} - \frac{(n_1 - n_2)(w_1 - w_2)}{2(n_1 - n_2) + 4}.
\]

(2.5)

3. Green’s function of linearized Navier-Stokes system.

In this section, we shall give an analysis on the Green’s function for the Navier-Stokes system by using complex method, which have been studied in [19] by using real method. In fact, recently Liu et al. [18] reconsidered it by using complex method. For completeness, we would like to restate it in another form with some slight differences.
First, the domain is divided into two parts: the finite Mach number region \{ |x| \leq 4c t \} and the outside finite Mach number region \{ |x| > 3c t \}. In the outside finite Mach number region, we will use the weighted energy estimate method to obtain the pointwise estimate. Inside the finite Mach number region, we will use the long-wave short-wave decomposition method. Such weighted method was introduced in [31] to study the boundary layer problem, and was used by other authors [21, 41].

Let

\[ \chi_1(\xi) = \begin{cases} 1, & |\xi| < \varepsilon_1, \\ 0, & |\xi| > 2\varepsilon_1, \end{cases} \]

and

\[ \chi_3(\xi) = \begin{cases} 1, & |\xi| > K + 1, \\ 0, & |\xi| < K, \end{cases} \]

be the smooth cut-off functions with \(2\varepsilon_1 < K\), and \(\chi_2 = 1 - \chi_1 - \chi_3\). Then we can divide the Green’s function into three parts: the long wave when \(|\xi|\) small, the short wave when \(|\xi|\) large and the middle part when \(|\xi|\) is in the middle:

\[ D_x^\alpha G(x, t) = \frac{1}{(2\pi)^n} \left( \int_{|\xi| \leq \varepsilon_1} + \int_{\varepsilon_1 \leq |\xi| \leq K} + \int_{|\xi| \geq K} \right) (i\xi)^\alpha \hat{G}(\xi, t)e^{ix \cdot \xi}d\xi \]

\[ := D_x^\alpha (\chi_1(D))G(x, t) + D_x^\alpha (\chi_2(D))G(x, t) + D_x^\alpha (\chi_3(D))G(x, t). \] (3.1)

Recall the linearized system on \((n_1, w_1)\) in (2.4)

\[ \begin{cases} \partial_t n_1 + \text{div} w_1 = 0, \\ \partial_t w_1 - \mu_1 \Delta w_1 - \mu_2 \nabla \text{div} w_1 + c^2 \nabla n_1 = 0. \end{cases} \] (3.2)

The symbol of the operator in (3.2) with \(\mu = \mu_1 + \mu_2\) is

\[ \tau^2 + \mu |\xi|^2 \tau + c^2 |\xi|^2 = 0. \] (3.3)

The eigenvalues of (3.3) for \(\lambda_{\pm}\) are

\[ \lambda_{\pm}(\xi) = -\mu|\xi|^2 \pm \sqrt{\mu^2|\xi|^4 - 4c^2|\xi|^2}. \] (3.4)

Then we consider the Green’s function for (3.2), i.e.,

\[ \begin{cases} (\frac{\partial}{\partial t} + A(D_x))G(x, t) = 0, \\ G(x, 0) = \delta(x), \end{cases} \] (3.5)

where \(\delta(x)\) is the Dirac function, the symbols of operator \(A(D_x)\) are

\[ A(\xi) = \begin{pmatrix} 0 & \sqrt{-1} \xi^\tau \\ \sqrt{-1} \xi \xi^\tau & \mu_1 |\xi|^2 I + \mu_2 \xi \xi^\tau \end{pmatrix}. \]

Thus by direct calculation, we get

\[ \hat{G}(\xi, t) = \begin{pmatrix} \hat{G}_{11} & \hat{G}_{12} \\ \hat{G}_{21} & \hat{G}_{22} \end{pmatrix}, \] (3.6)

where

\[ \hat{G}_{11} = \frac{\lambda_+ e^{\lambda_{-} t} - \lambda_- e^{\lambda_{+} t}}{\lambda_+ - \lambda_-}, \quad \hat{G}_{12} = -\sqrt{-1}c^2 \frac{e^{\lambda_{-} t} - e^{\lambda_{+} t}}{\lambda_+ - \lambda_-} \xi^\tau, \]

\[ \hat{G}_{21} = -\sqrt{-1}c^2 \frac{e^{\lambda_{+} t} - e^{\lambda_{-} t}}{\lambda_+ - \lambda_-} \xi, \quad \hat{G}_{22} = \frac{\xi^\tau \lambda_+ e^{\lambda_{+} t} - \lambda_- e^{\lambda_{-} t}}{|\xi|^2} \frac{\lambda_+ - \lambda_-}{\lambda_+ - \lambda_-} - e^{-\mu_2 |\xi|^2 t} \left( I - \frac{\xi \xi^\tau}{|\xi|^2} \right). \]
Sometime, we also rewrite the Fourier transform of the Green’s function as follows

\[
\hat{G}^+ = e^{\lambda_+ t} L_1 = \begin{pmatrix}
-\eta_- & -\sqrt{-1} \eta_0 \xi \\
-\sqrt{-1} \eta_0 \xi & -\eta_+
\end{pmatrix} e^{\lambda_+ t},
\]

\[
\hat{G}^- = e^{\lambda_- t} L_2 = \begin{pmatrix}
\eta_+ & \sqrt{-1} \eta_0 \xi \\
\sqrt{-1} \eta_0 \xi & \eta_-
\end{pmatrix} e^{\lambda_- t},
\]

\[
\hat{G}^0 = e^{-\mu_1 |\xi|^2 t} L_3 = \begin{pmatrix} 0 & 0 \\ 0 & I - \xi^2_0 |\xi|^2 \end{pmatrix} e^{-\mu_1 |\xi|^2 t},
\]

where \( \eta_0(\xi) = (\lambda_+(\xi) - \lambda_-(\xi))^{-1}, \eta_\pm(\xi) = \lambda_\pm(\xi) \eta_0(\xi) \).

**Lemma 3.1.** \([19]\) For a sufficiently small number \( \varepsilon_1 > 0 \) and a sufficiently large number \( K > 0 \), we have the following:

(i) when \( |\xi| < \varepsilon_1 \), \( \lambda_\pm \) are complex conjugates and have the following expansion:

\[
\lambda_+ = -\frac{\mu}{2} |\xi|^2 + \sqrt{-1} c |\xi| \left( 1 + \sum_{j=1}^{\infty} d_j |\xi|^{2j} \right),
\]

\[
\lambda_- = -\frac{\mu}{2} |\xi|^2 - \sqrt{-1} c |\xi| \left( 1 + \sum_{j=1}^{\infty} d_j |\xi|^{2j} \right);
\]

(ii) when \( \varepsilon_1 \leq |\xi| \leq K \), \( \lambda_\pm \) have the following spectrum gap property:

\[
\text{Re}(\lambda_\pm) \leq -C, \text{ for some constant } C > 0;
\]

(iii) when \( |\xi| > K \gg 1 \), \( \lambda_\pm \) are real and has the following expansion:

\[
\lambda_+ = -\frac{c^2}{\mu} + \frac{\mu}{2} \sum_{j=1}^{\infty} e_j |\xi|^{-2j},
\]

\[
\lambda_- = -\mu |\xi|^2 + \frac{c^2}{\mu} - \sum_{j=1}^{\infty} e_j |\xi|^{-2j}.
\]

Here \( c = \sqrt{\frac{P_1(\mu)}{\rho}} \) is defined as the equilibrium sound speed, and all \( d_j, e_j \) are real constants.

### 3.1. Pointwise estimate in the finite Mach number region.

#### 3.1.1. Long wave estimate. We first give the Kirchhoff method through the Fourier transform method, see also in \([18, 19, 23, 41]\):

**Lemma 3.2.** Let \( w(x, t) \) be a function given by its Fourier transformation in \( \mathbb{R}^3 \):

\[
\hat{w} = \frac{\sin(c|\xi|t)}{c|\xi|}, \quad \hat{w}_t = \frac{\cos(c|\xi|t)}{c|\xi|}.
\]

Then, for any function \( g(x) \) one has that

\[
w \ast g(x) = \frac{1}{4\pi} \int_{|y| = 1} g(x + cy) dy_S,
\]

\[
w_t \ast g(x) = \frac{1}{4\pi} \int_{|y| = 1} g(x + cy) dy_S + \frac{ct}{4\pi} \int_{|y| = 1} \nabla g(x + cy) \cdot y dy_S.
\]

To study the dissipation of different waves in the Navier-Stokes system, we need the following lemma with a slight modification.
Lemma 3.3. [25, 41] Let \( w(x, t) \) be the inverse Fourier transformation of \( \sin(c|\xi|t) \) given in Lemma 3.2. Then one has that
\[
\begin{align*}
|w* \frac{e^{-\frac{|x|^2}{2t(1+t)}}}{(4\pi (1+t))^{3/2}}| &\leq O(1)W_3(x, 1 + t, 2D), \\
|w_t* \frac{e^{-\frac{|x|^2}{2t(1+t)}}}{(4\pi (1+t))^{3/2}}| &\leq \frac{O(1)}{1 + t}W_3(x, 1 + t, 2D),
\end{align*}
\]
where
\[
W_3(x, 1 + t, D) = \begin{cases} \\
\frac{1}{(1+t)^{3/2}\sqrt{D}}, & |x| - ct \leq \sqrt{D(1+t)}, \\
\frac{-(1+t)^2}{(1+t)^{3/2}\sqrt{D}}, & |x| - ct \geq \sqrt{D(1+t)}.
\end{cases}
\]

The following lemma reveals the process of the coupling of isotropic and shear wave within a cone. We modify the relevant results from [23, 24], and we omit its proof for simplicity.

Lemma 3.4. There exists a constant \( C \), such that
When \( 0 < t \leq 1 \),
\[
\begin{align*}
\left| \int_0^t s \int_{|y|=1} \frac{e^{-\frac{|x+y|^2}{2t}}}{t^{5/2}} dS_y \frac{ds}{ds} \right| &= C e^{-\frac{|x|^2}{2t}} \left( t^{5/2} \right)^{1/2}, \quad \text{for } |x| \leq ct^{1/2}, \\
\left| \int_0^t s \int_{|y|=1} \frac{e^{-\frac{|x+y|^2}{2t}}}{t^{5/2}} dS_y \frac{ds}{ds} \right| &= C e^{-\frac{|x|^2}{2t}} \left( t^{5/2} \right)^{1/2}, \quad \text{for } |x| \geq ct^{1/2};
\end{align*}
\]

When \( t \geq 1 \),
\[
\begin{align*}
\left| \int_0^t s \int_{|y|=1} \frac{e^{-\frac{|x+y|^2}{2t}}}{t^{5/2}} dS_y \frac{ds}{ds} \right| &= C e^{-\frac{|x|^2}{2t}} \left( t^{5/2} \right)^{1/2}, \quad \text{for } |x| \leq ct^{1/2}, \\
\left| \int_0^t s \int_{|y|=1} \frac{e^{-\frac{|x+y|^2}{2t}}}{t^{5/2}} dS_y \frac{ds}{ds} \right| &= C \left( t^{5/2} \right)^{1/2} \left( 1 + \frac{|x|^2}{t} \right) + e^{-\frac{|x|^2}{2t}}, \quad \text{for } ct^{1/2} \leq |x| \leq ct, \\
\left| \int_0^t s \int_{|y|=1} \frac{e^{-\frac{|x+y|^2}{2t}}}{t^{5/2}} dS_y \frac{ds}{ds} \right| &= C e^{-\frac{|x|^2}{2t}} \left( t^{5/2} \right)^{1/2}, \quad \text{for } |x| \geq ct.
\end{align*}
\]

Now we invert each part of Green’s function separately. When \( |\xi| \ll 1 \), the Taylor expansion gives the following explicit form of Green’s function
\[
\chi_1(\xi) \hat{G}(\xi, t) = e^{\lambda_1 t}L_1 + e^{\lambda_2 t}L_2 + e^{-\mu_1|\xi|^2 t}L_3
\]
\[
= e^{-\frac{\mu|\xi|^2}{2}(1+O(1))|\xi|^2 t} \left( e^{\frac{\mu}{2}|\xi|^2 t + O(1)} |\xi|^2 t \right) \left( e^{-\frac{\mu}{2}|\xi|^2 t} \right) \left( e^{-\frac{\mu}{2}|\xi|^2 t} \right) (L_1 + L_2 + \sqrt{c} |\xi| |\beta| |\xi|^2 t) (L_1 - L_2))
\]
\[
= e^{-\mu|\xi|^2 t}L_3 + e^{-\frac{\mu|\xi|^2}{2} + O(1)} |\xi|^2 t \left( e^{-\frac{\mu|\xi|^2}{2} + O(1)} |\xi|^2 t \right) \left( e^{-\frac{\mu|\xi|^2}{2} + O(1)} |\xi|^2 t \right) (L_1 + L_2)
\]
\[
+ \frac{\sin(c|\xi| t)}{|c| |\xi|} \left( e^{-\frac{\mu|\xi|^2}{2} + O(1)} |\xi|^2 t \right) \left( e^{-\frac{\mu|\xi|^2}{2} + O(1)} |\xi|^2 t \right) (L_1 - L_2))
\]
\[
= e^{-\mu|\xi|^2 t}L_3 + \sin(c|\xi| t) e^{-\frac{\mu|\xi|^2}{2} + O(1)} |\xi|^2 t \left( \sin(c|\xi| t) |\beta| \sin(|\beta| |\xi|^2 t) + (L_1 + L_2) + \sqrt{c} |\xi| |\beta| |\xi|^2 t) (L_1 - L_2) \right)
\]
\[
+ \frac{\sin(c|\xi| t)}{|c| |\xi|} \left( e^{-\frac{\mu|\xi|^2}{2} + O(1)} |\xi|^2 t \right) \left( c|\xi| |\beta| |\xi|^2 t) (L_1 + L_2) + \sqrt{c} |\xi| |\beta| |\xi|^2 t) (L_1 - L_2) \right).
\]

Lemma 3.5. For a sufficiently small \( \xi \), we have the following estimates for \( L_1 \) and \( L_2 \):
\[
L_1 + L_2 = \begin{pmatrix} 1 & 0 \\
0 & \frac{\xi^T}{|\xi|^2} \end{pmatrix}, \quad L_1 - L_2 = \begin{pmatrix} iO(1) & O(|\xi|) \\
O(|\xi|) & iO(1) \frac{\xi^T}{|\xi|^2} \end{pmatrix}.
\]
Here $O(\xi)$ is an analytic function of $\xi$.

Then, we define

$$L_a = \begin{pmatrix} 0 & 0_{1 \times 3} \\ 0_{3 \times 1} & \frac{\xi \xi^T}{|\xi|^2} \end{pmatrix}, L_b = \begin{pmatrix} 0 & 0_{1 \times 3} \\ 0_{3 \times 1} & \frac{\xi \xi^T}{|\xi|^2} \end{pmatrix}, \quad L_c = \begin{pmatrix} 0 & 0_{1 \times 3} \\ 0_{3 \times 1} & I_{3 \times 3} \end{pmatrix},$$

$$L_{r_1} = \begin{pmatrix} 0 & 0_{1 \times 3} \\ 0_{3 \times 1} & \cos(c|\xi|t - 1) \frac{\xi \xi^T}{|\xi|^2} \end{pmatrix}, \quad L_{r_2} = \begin{pmatrix} 0 & 0_{1 \times 3} \\ 0_{3 \times 1} & \cos(c|\xi|t) \frac{\xi \xi^T}{|\xi|^2} \end{pmatrix}.$$  

First, we divide the third term in the right hand side of (3.7) into three parts

$$e^{-\mu_1|\xi|^2 t}L_3 = e^{-\mu_1|\xi|^2 t}L_c + e^{-\mu_1|\xi|^2 t}L_{r_1} - e^{-\mu_1|\xi|^2 t}L_{r_2},$$

where the first part is identified as the rotational pairing: $\hat{\mathcal{E}}(\xi, t) = e^{-\mu_1|\xi|^2 t}L_c$, and the second part is identified as the Riesz pairing-I: $\hat{\mathcal{R}}_1(\xi, t) = e^{-\mu_1|\xi|^2 t}L_{r_1}$. We also define the Riesz pairing-II as

$$\hat{\mathcal{R}}_2(\xi, t) = (e^{-\mu_1|\xi|^2 t}O(1)|\xi|^3 t - e^{-\mu_1|\xi|^2 t})L_{r_2}.$$  

The Huygens’ pair is defined as

$$\hat{\mathcal{H}}(\xi, t) = \cos(c|\xi|t)e^{2\mu_1|\xi|^2 t + O(1)|\xi|^3 t} \{\cos(|\xi|\beta(|\xi|^2)t)(L_1 + L_2 - L_a) + \sqrt{-1}\sin(|\xi|\beta(|\xi|^2)t)(L_1 - L_2 - L_b)\} + \frac{\sin(c|\xi|t)}{c|\xi|}e^{-\mu_1|\xi|^2 t + O(1)|\xi|^3 t}\{(-c|\xi|\sin(|\xi|\beta(|\xi|^2)t)(L_1 + L_2 - L_a) + \sqrt{-1}c|\xi|\cos(|\xi|\beta(|\xi|^2)t)(L_1 - L_2 - L_b)\}.$$  

Obviously,

$$\chi_1(\xi)\hat{\mathcal{G}}(\xi, t) = \hat{\mathcal{E}} + \hat{\mathcal{H}} + \hat{\mathcal{R}}_1 + \hat{\mathcal{R}}_2 + \hat{\mathcal{R}}_E,$$

where $\hat{\mathcal{R}}_E$ is the rest term with higher order of $|\xi|$. Next, we will give the estimate of each term above.

**Lemma 3.6.** The rotational wave (or called entropy wave) have the following estimate:

$$|D_x^2\mathcal{E}(x, t)| \leq C(1 + t)^{-\frac{|\alpha|}{4} - \frac{1}{8}} \frac{e^{-\frac{1}{4}\sqrt{-1}|\xi|^2}}{(1 + t)^{3/2}} + Ce^{-\sqrt{-1}|x|/t)}/C.$$  

**Proof.** The estimate of the rotational wave relies on the complex analysis. First, we have

$$D_x^2\mathcal{E}(x, t) := \frac{1}{(2\pi)^3} \int_{|\xi| \leq \varepsilon_1} (\sqrt{-1}\xi)^\alpha e^{\sqrt{-1}\xi x} e^{-\mu_1|\xi|^2 t + O(1)|\xi|^3 t}L_c d\xi.$$  

(3.10)  

We only consider the case $|\alpha| = 0$ since an additional factor $\xi$ will produce extra decay rate of $(1 + t)^{-1/2}$.

For each $x \in \mathbb{R}^3$, there is an orthogonal matrix $Q$ depending on $x$ such that $Qx = (|x|, 0, 0)^T$. Let $\eta = (\eta_1, \eta_2, \eta_3)^T = Q\xi$, then

$$|\mathcal{E}(x, t)| \leq C \frac{1}{(2\pi)^3} \int_{|\eta| \leq \varepsilon_1} |e^{\sqrt{-1}|\eta|^2} e^{-\mu_1|\eta|^2 t + O(1)|\eta|^3 t}L_c| d\eta,$$

since $\eta$ is bounded and the orthogonal matrix preserves inner products and norms.

Let $B := [-\frac{\varepsilon_1}{\sqrt{3}}, \frac{\varepsilon_1}{\sqrt{3}}] \times [-\frac{\varepsilon_1}{\sqrt{3}}, \frac{\varepsilon_1}{\sqrt{3}}] \times [-\frac{\varepsilon_1}{\sqrt{3}}, \frac{\varepsilon_1}{\sqrt{3}}], \quad \mathbb{B} \subset \{ |\eta| \leq \varepsilon_1 \}$. Then the integration is divided into two regions:

$$|\mathcal{E}(x, t)| \leq \frac{1}{(2\pi)^3} \int_B |e^{\sqrt{-1}|\eta|^2} e^{-\mu_1|\eta|^2 t + O(1)|\eta|^3 t}L_c| d\eta$$

$$+ \frac{1}{(2\pi)^3} \int_{|\eta| \leq \varepsilon_1 \cap \mathbb{B}^c} |e^{\sqrt{-1}|\eta|^2} e^{-\mu_1|\eta|^2 t + O(1)|\eta|^3 t}L_c| d\eta.$$  

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The integrand in the second term is $|e^{-\frac{1}{2}(|\eta|^2(1+\varepsilon)+O(1)|\eta|^3(1+\varepsilon))}L_c| = O(1)e^{-\frac{\varepsilon t}{C}}$ since $|\eta| \geq \frac{\varepsilon}{\sqrt{3}}$. Thus the second term is $O(1)e^{-\frac{\varepsilon t}{C}}$. The first term can be estimated as follows

$$|\mathcal{E}(x, t)| \leq \frac{1}{(2\pi)^3} \int \mathcal{E}(x, t) \leq \frac{1}{(2\pi)^3} \int \mathcal{E}(x, t) \leq O(1) \int \mathcal{E}(x, t) \leq O(1) \int \mathcal{E}(x, t).$$

Since $L_c$ is analytic in $\eta_1$ near the origin, we choose the contour as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$:

$$\Gamma_1 = \{ z : \text{Im}z = 0, \ |\text{Re}z| \leq \frac{1}{\sqrt{3}} \xi_1 \}, \quad \Gamma_2 = \{ z : \text{Re}z = -\frac{1}{\sqrt{3}} \xi_1, \ 0 \leq \text{Im}z \leq \sigma \},$$

$$\Gamma_3 = \{ z : \text{Im}z = \sigma, \ |\text{Re}z| \leq \frac{1}{\sqrt{3}} \xi_1 \}, \quad \Gamma_4 = \{ z : \text{Re}z = \frac{1}{\sqrt{3}} \xi_1, \ 0 \leq \text{Im}z \leq \sigma \}, \text{ with } \sigma = \frac{\xi_1}{\sqrt{3}} x_1.$$

One the path $\Gamma_3$,

$$\int_{\Gamma_3} e^{\frac{1}{2}(|\eta|^2(1+\varepsilon)+O(1)|\eta|^3(1+\varepsilon))} \leq O(1) \int_{\Gamma_3} e^{\frac{1}{2}(|\eta|^2(1+\varepsilon)+O(1)|\eta|^3(1+\varepsilon))} \leq C e^{-\frac{\varepsilon t}{O(1)}} \leq C e^{-\frac{\varepsilon t}{(1+t)^{3/2}}}.$$

Along the path $\Gamma_2$ and $\Gamma_4$, we can get the exponential decay rate

$$\int_{\Gamma_2, \Gamma_4} e^{\frac{1}{2}(|\eta|^2(1+\varepsilon)+O(1)|\eta|^3(1+\varepsilon))} \leq O(1) \int_{\Gamma_2, \Gamma_4} e^{\frac{1}{2}(|\eta|^2(1+\varepsilon)+O(1)|\eta|^3(1+\varepsilon))} \leq C e^{-\frac{\varepsilon t}{(1+t)^{3/2}}} \leq C e^{-\frac{\varepsilon t}{(1+t)^{3/2}}}.$$

This gives the estimate of the rotational wave.

**Lemma 3.7.** The Huygens’ wave $\mathcal{H}(x, t)$ has the following estimate:

$$|D_x^a \mathcal{H}(x, t)| \leq C(1 + t)^{-\frac{1}{2}} e^{-\frac{\xi^2}{2(1+t)^2}(1 + t)^2} + C e^{-\frac{\xi^2}{2(1+t)^3}}.$$

**Proof.** First, for convenience’s sake we rewrite (3.8) as

$$\mathcal{H}(x, t) = w_k(\cdot, t) * K_1(\cdot, t) + w_k(\cdot, t) * K_2(\cdot, t),$$

where $w(x, t)$ and $w_k(x, t)$ is defined in Lemma 3.2. The estimates of $K_1$ and $K_2$ are similar to that of $\mathcal{E}(x, t)$. Note that on the contour $\Gamma$, one also has the growth rate

$$|\cos(\xi \beta(\xi^2 t^2)|, |\sin(\xi \beta(\xi^2 t^2))| \leq C e^{O(1)|\xi^3| t}.$$

Thus, by the proof of Lemma 3.6, we have

$$|K_1(x, t)| \leq C e^{-\frac{\xi^2}{2(1+t)^{3/2}}} + C e^{-\frac{\xi^2}{2(1+t)^{3/2}}} + C e^{-\frac{\xi^2}{2(1+t)^{3/2}}}.$$

$$|\nabla K_1(x, t)| + |K_2(x, t)| \leq C e^{-\frac{\xi^2}{2(1+t)^{3/2}}} + C e^{-\frac{\xi^2}{2(1+t)^{3/2}}}.$$
where an additional decay rate of $(1 + t)^{-1/2}$ is gained due to the extra factor $|\xi|$ in $\nabla K_1$ and $K_2$. Using Lemma 3.3 and 3.4, we have
\[
|w * K_2(x, t)| = \left| \frac{t}{4\pi} \int_{|y|=1} K_3(x + cyt) dS_y \right| \leq \frac{t}{4\pi} \int_{|y|=1} |K_2(x + cyt)| dS_y
\]
\[
\leq \frac{t}{4\pi} \int_{|y|=1} e^{-\frac{|x + cyt|^2}{4(1 + t)^2}} dS_y \leq \frac{1}{\sqrt{1 + t}} W_3(x, 1 + t, 2D);
\]
\[
|w_t * K_1(x, t)| \leq \frac{1}{4\pi} \int_{|y|=1} |K_1(x + cyt)| dS_y + \frac{ct}{4\pi} \int_{|y|=1} |\nabla K_1(x + cyt) \cdot y| dS_y \leq C \frac{W_3(x, 1 + t, 2D)}{1 + t} + \frac{ct}{4\pi} C (1 + t)^{-3} e^{-\frac{|x - ct|^2}{4(1 + t)^2}} \leq C \frac{1}{1 + t} W_3(x, 1 + t, 2D). \tag{3.11}
\]
Here we have used the important estimate of
\[
\int_{|y|=1} e^{-\frac{|x + cyt|^2}{4(1 + t)^2}} dS_y \leq C(1 + t)^{-1} e^{-\frac{|y|^2}{C(1 + t)^2}}, \text{ for } \beta > 0,
\]
which was proved by Lemma 2.3 in [19]. Hence
\[
|D^\alpha_x \mathcal{H}(x, t)| = |D^\alpha_x w_t * K_1| + |D^\alpha_x w * K_2| \leq C(1 + t)^{-|\alpha|/2} \left\{ \chi_{|x| \leq ct} (1 + t)^{-\frac{|\alpha|}{2}} \left( 1 + \frac{x^2}{1 + t} \right)^{-\frac{|\alpha|}{2}} + e^{-\frac{|x - ct|^2}{2(1 + t)^2}} + e^{-\frac{|x|^2}{2(1 + t)^2}} \right\}.
\]

**Proof.** Recall that
\[
\mathcal{R}_1(x, t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{\sqrt{-1} \xi \cdot x} e^{-\mu_1 |\xi|^2 t} \, L_x \, d\xi.
\]
Since
\[
\int_{\mathbb{R}^3} e^{\sqrt{-1} \xi \cdot x} (\cos(c|\xi|t) - 1) \frac{\xi \xi^T}{|\xi|^2} e^{-\mu_1 |\xi|^2 t} d\xi = \frac{\partial^2}{\partial x^i \partial x_k} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{\sqrt{-1} \xi \cdot x} (\cos(c|\xi|t) - 1) \frac{1}{|\xi|^2} e^{-\mu_1 |\xi|^2 t} d\xi,
\]
\[
= -c^2 \frac{\partial^2}{\partial x^i \partial x_k} \int_0^t \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{\sqrt{-1} \xi \cdot x} \cos(c|\xi|s) \frac{1}{|\xi|} e^{-\mu_1 |\xi|^2 t} d\xi ds,
\]
\[
= C \int_0^t \int_{|y|=1} e^{-\frac{|x + cyt|^2}{4(1 + t)^2}} dS_y ds,
\]
then by using Lemma 3.4, we prove this lemma. □

**Lemma 3.9.** The Riesz wave-II has the following estimate:
\[
|D^\alpha_x \mathcal{R}_2(x, t)| \leq C(1 + t)^{-|\alpha|} e^{-\frac{(1 + t)^{3/2}}{(1 + t)^2}}.
\]
Proof. After a direct computation, we have

\[
\frac{1}{(2\pi)^3} \int_{B} \chi_{|\xi| \leq \varepsilon t} e^{\sqrt{T} \xi \cdot x} \left( e^{-\frac{\mu}{2} |\xi|^2t + \mathcal{O}(1) |\xi|^3t} - e^{-\mu_{11}|\xi|^2t} \right) L_{\xi_2} d\xi = \frac{1}{(2\pi)^3} \int_{B} \chi_{|\xi| \leq \varepsilon t} \left( e^{-\frac{\mu}{2} |\xi|^2t + \mathcal{O}(1) |\xi|^3t} - e^{-\mu_{11}|\xi|^2t} \right) \frac{\xi_{\xi_2}}{|\xi|^2} e^{\sqrt{T} \xi \cdot x} d\xi
\]

\[
= \frac{1}{(2\pi)^3} \int_{B} \chi_{|\xi| \leq \varepsilon t} \left( e^{\frac{\mu}{2} |\xi|^2t + \mathcal{O}(1) |\xi|^3t} - e^{-\mu_{11}|\xi|^2t} \right) \frac{\xi_{\xi_2}}{|\xi|^2} e^{\sqrt{T} \xi \cdot x} d\xi
\]

\[
= \omega \frac{1}{(2\pi)^3} \int_{B} \chi_{|\xi| \leq \varepsilon t} \left( e^{\frac{\mu}{2} |\xi|^2t + \mathcal{O}(1) |\xi|^3t} - e^{-\mu_{11}|\xi|^2t} \right) \frac{\xi_{\xi_2}}{|\xi|^2} e^{\sqrt{T} \xi \cdot x} d\xi
\]

\[
= \omega \left( \int_{B} \chi_{|\xi| \leq \varepsilon t} \left( e^{\frac{\mu}{2} |\xi|^2t + \mathcal{O}(1) |\xi|^3t} - e^{-\mu_{11}|\xi|^2t} \right) \frac{\xi_{\xi_2}}{|\xi|^2} e^{\sqrt{T} \xi \cdot x} d\xi \right). \]

The contour integral gives

\[
\left| \int_{|\xi| \leq \varepsilon t} e^{-|\xi|^2} e^{\sqrt{T} \xi \cdot x} d\xi \right| \leq C \frac{e^{-\frac{\beta^2}{T(s+1)^2}}}{(1+t)^{5/2}},
\]

which imply that

\[
|\mathcal{R}_2(x,t)| = \left| \frac{1}{(2\pi)^3} \int_{|\xi| \leq \varepsilon t} e^{\sqrt{T} \xi \cdot x} \left( e^{-\frac{\mu}{2} |\xi|^2t + \mathcal{O}(1) |\xi|^3t} - e^{-\mu_{11}|\xi|^2t} \right) L_{\xi_2} d\xi \right|
\]

\[
= \left| \frac{1}{(2\pi)^3} \int_{B} \chi_{|\xi| \leq \varepsilon t} \left( e^{-\frac{\mu}{2} |\xi|^2t + \mathcal{O}(1) |\xi|^3t} - e^{-\mu_{11}|\xi|^2t} \right) L_{\xi_2} d\xi \right| \leq C \frac{e^{-\frac{\beta^2}{T(s+1)^2}}}{(1+t)^{5/2}}.
\]

We complete the proof of this lemma. \( \square \)

The rest is

\[
\left| \mathcal{R}_\mathcal{E}(x,t) \right| = \left| \cos(|\xi| t) e^{-\frac{\mu}{2} |\xi|^2t + \mathcal{O}(1) |\xi|^3t} \sqrt{-1} \sin(|\xi| |\beta| |\xi|^2 t) L_b + \frac{\sin(|\xi| t)}{|\xi|} e^{-\frac{\mu}{2} |\xi|^2t + \mathcal{O}(1) |\xi|^3t} \left( -c|\xi| \sin(|\xi| |\beta| |\xi|^2 t) L_a + \sqrt{-1} e|\xi| \cos(|\xi| |\beta| |\xi|^2 t) L_b \right) \right|
\]

\[
= \left| \cos(|\xi| t) e^{-\frac{\mu}{2} |\xi|^2t + \mathcal{O}(1) |\xi|^3t} \sqrt{-1} \sin(|\xi| |\beta| |\xi|^2 t) \frac{\xi_{\xi_2}}{|\xi|^2} \right|
\]

\[
+ \frac{\sin(|\xi| t)}{|\xi|} e^{-\frac{\mu}{2} |\xi|^2t + \mathcal{O}(1) |\xi|^3t} \left( -c|\xi| \sin(|\xi| |\beta| |\xi|^2 t) \frac{\xi_{\xi_2}}{|\xi|^2} + \sqrt{-1} e|\xi| \cos(|\xi| |\beta| |\xi|^2 t) \frac{\xi_{\xi_2}}{|\xi|^2} \right) \right|.
\]

For any analytic function \( \beta(|\xi|^2) \), we have \( \sin(|\xi| |\beta| |\xi|^2 t) = |\xi| |\beta| |\xi|^2 t - \frac{1}{6} (|\xi| |\beta| |\xi|^2 t)^3 + \cdots \), then \( \mathcal{R}_\mathcal{E}(x,t) \) has the following estimate:

\[
|\mathcal{R}_\mathcal{E}(x,t)| \leq C \omega \frac{e^{-\frac{|\xi|^2}{2(1+t)^2}}}{(1+t)^{5/2}} + Cw \frac{e^{-\frac{|\xi|^2}{2(1+t)^2}}}{(1+t)^{5/2}} \leq C \frac{e^{-\frac{|\xi|^2}{2(1+t)^2}}}{(1+t)^{5/2}}.
\]

Now we conclude the estimate of the long wave:

PROPOSITION 3.10. The long wave component contains several waves

\[
\chi_1(D)G(x,t) = \mathcal{E}(x,t) + \mathcal{H}(x,t) + \mathcal{R}_1(x,t) + \mathcal{R}_2(x,t) + \mathcal{R}_\mathcal{E}(x,t),
\]

and has the following estimate

\[
|D_x \chi_1(D)(G_{11}, G_{12}, G_{21})| \leq C(1+t)^{-\frac{3}{4}|\xi|} \frac{e^{-\frac{3|\xi|^2}{C(1+t)}}}{(1+t)^{5/2}} + Ce^{-\frac{3|\xi|^2}{C(1+t)}},
\]

\[
|D_x \chi_1(D)G_{22}(x,t)| \leq C(1+t)^{-\frac{3}{4}|\xi|} \left( \chi_{|\xi| \leq \varepsilon}(1+t)^{-\frac{3}{2}} \frac{e^{-\frac{3|\xi|^2}{1+t}}}{(1+t)^{5/2}} + Ce^{-\frac{3|\xi|^2}{1+t}} \right) + Ce^{-\frac{3|\xi|^2}{1+t}}.
\]

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3.1.2. Short wave estimate.

**Lemma 3.11.** When $|\xi| \geq K$, $L_i$ has the following estimates:

\[ L_1 = \begin{pmatrix} 1 + \mathcal{O}(|\xi|^{-2}) & \sqrt{-\xi^T} \mathcal{O}(|\xi|^{-2}) \\ \sqrt{-\xi} \mathcal{O}(|\xi|^{-2}) & \xi^T \mathcal{O}(|\xi|^{-4}) \end{pmatrix}, \]

\[ L_2 = \begin{pmatrix} \mathcal{O}(|\xi|^{-2}) & \sqrt{-\xi^T} \mathcal{O}(|\xi|^{-2}) \\ \sqrt{-\xi} \mathcal{O}(|\xi|^{-2}) & \mathcal{O}(|\xi|^{-2}) + \mathcal{O}(|\xi|^{-4}) \xi^T \end{pmatrix}, \]

\[ L_3 = \begin{pmatrix} \mathcal{O}(|\xi|^{-2}) & \sqrt{-\xi^T} \mathcal{O}(|\xi|^{-2}) \\ \sqrt{-\xi} \mathcal{O}(|\xi|^{-2}) & \xi^T \mathcal{O}(|\xi|^{-4}) \end{pmatrix}. \]

**Proposition 3.12.** There exists a constant $C$ such that the short wave part has the following estimates:

\[ |D^a_x(\chi_3(D) G(x, t) - G_{S_2}(x, t))| \leq C e^{-\frac{bt}{(3+|\xi|)^2}} + C e^{-(|x|+t)/C}. \]

Here $G_{S_2}$ is defined in the following Lemma 3.13, which contains a Dirac function and a Dirac-like function with exponential decay rate on the time $t$. And it is from the terms in short wave of Green’s function corresponding to the eigenvalue $\lambda_+ \sim -\frac{e^t}{t}$ for the short wave. For convenience, we denote the term $e^{-\frac{bt}{(3+|\xi|)^2}}$ as $G_{S_1}(x, t)$, which is from the terms in short wave of Green’s function corresponding to the eigenvalue $\lambda_- \sim -\mu|\xi|^2$.

**Lemma 3.13.** [6, 18] The singular term $G_{S_2}$ in short wave satisfies

\[ G_{S_2}(x, t) = e^{-\frac{bt}{|\xi|^2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta(x) + f(x), \]

\[ |f(x)| \leq e^{-bt} \tilde{f}(x), \quad \tilde{f}(x) \in L^1, \quad \tilde{f}(x) = \begin{cases} C|x|^{-2}, & |x| \leq R, \\ C|x|^{-N}, & |x| > R, \end{cases} \]

for some positive constant $b$ and any given positive integer $N$.

3.1.3. Middle part estimate.

The following results is standard for the middle part of the Green’s function as in [33, 18], we omit the details for simplicity.

**Lemma 3.14.** When $\{\varepsilon_1 \leq |\xi| \leq K\}$, $L_1 + L_2 + L_3$ is bounded and analytic. The only possible pole for $L_1 + L_2 + L_3$ is $\xi = 0$ and has been excluded here.

**Proposition 3.15.** There exists a constant $C$ such that the middle part of the Green’s function has the following estimate:

\[ |D^a_x \chi_2(D) G(x, t)| \leq C e^{-(|x|+t)/C}. \]

3.2. Energy estimate outside the finite Mach number region.

Consider the initial value problem for the linear part of the system (2.4) outside the finite Mach number region $\{|x| > 3c t\}$:

\[
\begin{align*}
\partial_t n_1 + \text{div} w_1 &= 0, \\
\partial_t w_1 + c^2 \nabla n_1 &= \mu_1 \Delta w_1 + \mu_2 \nabla \text{div} w_1, \\
n_1(x, 0) &= \rho_{1,0}(x), \quad w_1(x, 0) = w_{1,0}(x).
\end{align*}
\]

Multiplying each side of the two equations in (3.14) by $ce^{\alpha(|x|-Mct)} n_1$ and $e^{\alpha(|x|-Mct)} w_1$ respectively and integrating with respect to $x$ on $\mathbb{R}^3$, we have the $L^2$ estimate of $n_1$, $w_1$. Similarly, one can derive the $L^2$
estimate of $\nabla_x^\alpha n_1$ and $\nabla_x^\alpha w_1$ when $|\alpha| \geq 1$. Hence, with the help of Sobolev embedding theorem, we can prove the following Proposition.

**Proposition 3.16.** For $|x| > 4ct$, there exists a positive constant $C$ independent of $|x|$ and $t$ such that

$$|D_x^\alpha G(x,t) * n_{1,0}(x)| + |D_x^\alpha G(x,t) * w_{1,0}(x)| \leq C e^{-(|x|+t)/C}.$$

Up to now, from Proposition 3.10-3.16, we get the following pointwise estimates for each term in the Green's function $G(x,t)$, which contains the H-wave and D-wave mentioned in §1.

**Proposition 3.17.** For any $|\alpha| \geq 0$, we have for some constant $C > 0$

$$|D_x^\alpha (G_{11} - G_{S_2})| + |D_x^\alpha (G_{12} - G_{S_2})| + |D_x^\alpha (G_{21} - G_{S_2})| \leq C(1 + t)^{-\frac{4 + |\alpha|}{t} e^{-\frac{(|x|+ct)^2}{t}}} + e^{-\frac{|\alpha|^2}{2t(3+|\alpha|)/2}},$$

$$|D_x^\alpha (G_{22} - G_{S_2})| \leq C(1 + t)^{-\frac{3 + |\alpha|}{2}} \left\{ (1 + t)^{-\frac{1}{2}} e^{-\frac{(|x|+ct)^2}{t}} + \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3 + |\alpha|}{2}} + e^{-\frac{|\alpha|^2}{2t(3+|\alpha|)/2}} \right\},$$

where $G_{S_2}$ is defined in Lemma 3.13.

### 4. Green's function of linearized Navier-Stokes-Poisson system

In this section, we shall review the pointwise estimates for the Green’s function for the linearized Navier-Stokes-Poisson equations, which has been considered in Wang and Wu [32] by using real analysis method. Here we do not reconsider this term by using the complex method as in [18] for the Navier-Stokes system for simplicity.

Recall the linearized system on $(n_2, w_2)$ in (2.4) with $c = P^1_x(\rho_1) = P^2_x(\rho_2)$

$$\begin{cases}
\partial_t n_2 + \text{div} w_2 = 0, \\
\partial_t w_2 + c^2 \nabla n_2 - \mu_1 \Delta w_2 - \mu_2 \nabla \text{div} w_2 + 2 \nabla \phi = 0,
\end{cases}$$

(4.1)

The symbol of the operator in (4.1) is

$$\lambda^2 + (\mu_1 + \mu_2)|\xi|^2 \lambda + (c^2|\xi|^2 + 2) = 0.$$ (4.2)

Here, $\lambda$ and $\xi^\tau = (\xi_1, \xi_2, \cdots, \xi_n)$ correspond to $\frac{\partial^2}{\partial x^2}$ and $(D_{x_1}, \cdots, D_{x_n})$ respectively, where $D_{x_j} = \sqrt{-1/\partial_{x_j}}$ with $j = 1, 2, \cdots, n$. It is easy to find that the eigenvalues of (4.2) for $\lambda$ are

$$\lambda = \lambda_\pm(\xi) = -\mu|\xi|^2 + \sqrt{\mu^2|\xi|^4 - 4(c^2|\xi|^2 + 2)},$$ (4.3)

where $\mu = \mu_1 + \mu_2$.

Now we consider the Green’s function for (4.1) defined as

$$\begin{cases}
(\frac{\partial}{\partial t} + A(D_x) ) \mathcal{G}(x,t) = 0, \\
\mathcal{G}(x,0) = \delta(x),
\end{cases}$$

(4.4)

where $\delta(x)$ is the Dirac function, the symbols of operator $A(D_x)$ are

$$A(\xi) = \begin{pmatrix}
0 & \sqrt{-1} \xi^\tau \\
\sqrt{-1} (c^2 + \frac{1}{|\xi|^2}) & \mu_1 |\xi|^2 + \mu_2 \xi^\tau
\end{pmatrix}.$$ \(\text{(4.5)}\)

Applying the Fourier transform with respect to the variable $x$ to (4.4), we get by a direct calculation

$$\hat{\mathcal{G}}(\xi, t) = \begin{pmatrix}
\hat{G}_{11}(\xi, t) & \hat{G}_{12}(\xi, t) \\
\hat{G}_{21}(\xi, t) & \hat{G}_{22}(\xi, t)
\end{pmatrix} = \begin{pmatrix}
\frac{\lambda_{+} e^{\lambda_{+} t} - \lambda_{-} e^{-\lambda_{+} t} - \xi^\tau}{\lambda_{+} - \lambda_{-}} e^{\mu_1|\xi|^2 t} I + \frac{-\sqrt{\lambda_{+} e^{\lambda_{+} t} - \lambda_{-} e^{-\lambda_{+} t} - \xi^\tau}}{\lambda_{+} - \lambda_{-}} e^{-\mu_1|\xi|^2 t} I + \frac{\xi^\tau}{|\xi|^2}
\\
\frac{\eta_{+} e^{\lambda_{+} t} - \eta_{-} e^{-\lambda_{+} t}}{\lambda_{+} - \lambda_{-}} - \sqrt{-\frac{1}{|\xi|^2}} \epsilon_{\eta_{+} e^{\lambda_{+} t} - \eta_{-} e^{-\lambda_{+} t}}
\\
\frac{-\sqrt{-\frac{1}{|\xi|^2}} \epsilon_{\eta_{+} e^{\lambda_{+} t} - \eta_{-} e^{-\lambda_{+} t}}}{\lambda_{+} e^{\lambda_{+} t} - \lambda_{-} e^{-\lambda_{+} t} - e^{-\mu_1|\xi|^2 t} (I - \frac{\epsilon_{\xi^\tau}}{|\xi|^2}) + \frac{\epsilon_{\xi^\tau}}{|\xi|^2} (\eta_{+} e^{\lambda_{+} t} - \eta_{-} e^{-\lambda_{+} t})}
\end{pmatrix}.$$
where $\eta_0(\xi) = (\lambda_+ + \lambda_-)(\xi - \lambda_-(\xi))^{-1}$, $\eta_\pm(\xi) = \lambda_\pm(\xi)\eta_0(\xi)$.

Sometimes, we also rewrite the Fourier transform of the Green’s function as follows

$$\widehat{\chi}^0 = \begin{pmatrix} 0 & 0 \\ 0 & I - \xi \xi^\top \end{pmatrix} e^{-\mu_1|\xi|^2t},$$

Lemma 4.1. [32] For a sufficiently small number $\varepsilon_1 > 0$ and a sufficiently large number $K > 0$, we have the following:

(i) when $|\xi| < \varepsilon_1$, $\lambda_\pm$ has the following expansion:

$$\lambda_+ = \frac{\mu}{2} |\xi|^2 + \sum_{j=2}^{\infty} a_j |\xi|^{2j} + \sqrt{-1} \left(\sqrt{2} + \sum_{j=1}^{\infty} b_j |\xi|^{2j}\right),$$

$$\lambda_- = \frac{\mu}{2} |\xi|^2 + \sum_{j=2}^{\infty} a_j |\xi|^{2j} - \sqrt{-1} \left(\sqrt{2} + \sum_{j=1}^{\infty} b_j |\xi|^{2j}\right);$$

(ii) when $\varepsilon_1 \leq |\xi| \leq K$, $\lambda_\pm$ has the following spectrum gap property:

$$\text{Re}(\lambda_\pm) \leq -C,$$

for some constant $C > 0$;

(iii) when $|\xi| > K$, $\lambda_\pm$ has the following expansion:

$$\lambda_+ = -\frac{c^2}{\mu} + \sum_{j=1}^{\infty} c_j |\xi|^{-2j},$$

$$\lambda_- = -\frac{\mu}{2} |\xi|^2 + \frac{c^2}{\mu} - \sum_{j=1}^{\infty} c_j |\xi|^{-2j}.$$
By differentiating both sides of \( \chi_1(\xi) \hat{G}_{11}^+ \) and \( \chi_1(\xi) \hat{G}_{12}^- \) with respect to \( \xi \), because of the smoothness of \( \chi_1(\xi) \hat{G}_{11}^+ \) and \( \chi_1(\xi) \hat{G}_{12}^- \) near \( |\xi| = 0 \), we can immediately obtain the following lemma.

**Lemma 4.2.** If \( |\xi| \) is sufficiently small, then there exists a constant \( b > 0 \), such that

\[
|D_x^{\alpha} (\xi^\alpha \chi_1(\xi) \hat{G}_{11}^+(\xi,t))| + |D_x^{\alpha} (\xi^\alpha \chi_1(\xi) \hat{G}_{12}^-(\xi,t))| \\
\leq C(|\xi|^{(\alpha+1)l}|\xi|^{\beta};(1+|t|\xi^2)|^{(\alpha+1)l}e^{-b|\xi|^2t},
\]

\[
|D_x^{\alpha} (\xi^\alpha \chi_1(\xi) \hat{G}_{12}^-(\xi,t))| + |D_x^{\alpha} (\xi^\alpha \chi_1(\xi) \hat{G}_{12}^-(\xi,t))| \\
\leq C(|\xi|^{(\alpha+1)l}|\xi|^{\beta};(1+|t|\xi^2)|^{(\alpha+1)l}e^{-b|\xi|^2t}.
\]

Then, we will use the following key estimates for the Green’s function like a Heat kernel.

**Lemma 4.3.** If \( \hat{f}(\xi,t) \) has compact support on \( \xi \) in \( \mathbb{R}^n \), and satisfies for a constant \( b > 0 \) that

\[ |D_x^{\alpha} \hat{f}(\xi,t)| \leq C(|\xi|^{(\alpha+1)l}|\xi|^{\beta};(1+|t|\xi^2)^m \exp(-b|\xi|^2t), \]

for any multiindexes \( \alpha, \beta \) with \( |\beta| \leq 2N \) (the integer \( N \) could be arbitrary large), then

\[ |D_x^\alpha f(x,t)| \leq C_N(1+t)^{-n\alpha/k-1/2} \left(1 + \frac{|x|^2}{1+t}\right)^{-N}, \]

where \( k \) and \( m \) are any fixed integers and \( (\alpha) = \max(0,\alpha) \).

**Proof.** If \( |\beta| < k + |\alpha| \), then we have by direct calculation that

\[
|x^\beta D_x^\alpha f(x,t)| = C \left| \int e^{-\xi^2} D_x^\beta \xi^\alpha \hat{f}(\xi,t) d\xi \right| \\
\leq C \left| \int \xi^{(\alpha+l)|\xi|^{\beta};(1+|t|\xi^2)^m e^{-b|\xi|^2t} d\xi \right| \\
\leq C(1+t)^{-n\alpha/k-1/2} \left(1 + \frac{|x|^2}{1+t}\right)^{-N}.
\]

If \( |\beta| \geq k + |\alpha| \), one also can find that

\[
|x^\beta D_x^\alpha f(x,t)| = C \left| \int e^{-\xi^2} D_x^\beta \xi^\alpha \hat{f}(\xi,t) d\xi \right| \\
\leq C \left| \int \xi^{(\alpha+l)|\xi|^{\beta};(1+|t|\xi^2)^m e^{-b|\xi|^2t} d\xi \right| \\
\leq C(1+t)^{-n\alpha/k-1/2} \left(1 + \frac{|x|^2}{1+t}\right)^{-N}.
\]

Let \( |\beta| = 0 \) when \( |x|^2 \leq 1 + t \), and \( |\beta| = 2N \) when \( |x|^2 > 1 + t \), we get

\[ |D_x^\alpha f(x,t)| \leq C(1+t)^{-n\alpha/k-1/2} \min \left\{ 1, \left( \frac{1+t}{|x|^2} \right)^N \right\}. \]

Since

\[ 1 + \frac{|x|^2}{1+t} \leq 2 \left\{ \begin{array}{ll} 1, & |x|^2 \leq 1 + t, \\ \frac{1}{1+t}, & |x|^2 > 1 + t, \end{array} \right. \]

we have

\[ \min \left\{ 1, \left( \frac{1+t}{|x|^2} \right)^N \right\} \leq \frac{2^N}{(1+\frac{1}{1+t})^N} = 2^N \left( 1 + \frac{|x|^2}{1+t} \right)^{-N}. \]

This completes the proof. \( \square \)
Notice that when \( \hat{f}(\xi, t) \) has not compact support in \( \xi \), the result in Lemma 4.3 should be

\[
|D_x^\alpha f(x, t)| \leq C_N t^{-\frac{n+|\alpha|+k}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-N}.
\]

From Lemma 4.2 and 4.3, we can immediately give the following estimates for \( \chi_1(D)\hat{G}_{11} \) and \( \chi_1(D)\hat{G}_{12} \).

**Remark 4.1.** This estimate is crucial to help us deduce the asymptotic profile \((1 + \frac{|x|^2}{1 + t})^{-\frac{N}{2}}\) in the pointwise estimates, which is an improvement of those in [32] for the Navier-Stokes-Poisson system in \( \mathbb{R}^n \), where the asymptotic profile for the velocity (or the momentum) is \((1 + t)^{-\frac{n+1}{4}}(1 + \frac{|x|^2}{1 + t})^{-\frac{n+1}{2}}\).

Hence, we have the following pointwise estimates for \( \chi_1(D)\hat{G}_{21} \) and \( \chi_1(D)\hat{G}_{22} \).

**Proposition 4.4.** When \( |\xi| \) is sufficiently small, we have for any \( |\alpha| \geq 0 \) and any integer \( N > 0 \),

\[
|D_x^\alpha (\chi_1(D)\hat{G}_{11}(x, t))| \leq C(1 + t)^{-\frac{(n+|\alpha|)/2}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-N},
\]

\[
|D_x^\alpha (\chi_1(D)\hat{G}_{12}(x, t))| \leq C(1 + t)^{-\frac{(n+1+|\alpha|)/2}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-N}.
\]

Two remain terms \( \chi_1(D)\hat{G}_{21}(\xi, t) \) and \( \chi_1(D)\hat{G}_{22}(\xi, t) \) containing Calderon-Zygmund operators \( R_i \) and \( R_{ij} \) with the symbols \( \frac{\xi_i}{|\xi|^2} \) and \( \frac{\xi_i \xi_j}{|\xi|^4} \) respectively can be estimated by the following two lemmas.

**Lemma 4.5.** [6] When \( |\xi| \) is sufficiently small in \( \mathbb{R}^n \), it holds that \( C > 0 \) that

\[
|D_x^\alpha (R_i * \chi_1(D)H(x, t))| \leq C(\alpha)(1 + t)^{-\frac{\alpha |\xi|}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{\alpha |\xi|}{2}},
\]

where \( H(x, t) \) is the heat kernel.

Next, for the term \( R_i(x, t) \) in \( \chi_1(D)\hat{G}_{21} \), by using the same method in Lemma 4.4, we have the following.

**Lemma 4.6.** When \( |\xi| \) is sufficiently small and \( x \in \mathbb{R}^n \), we have for some \( C > 0 \) that

\[
|D_x^\alpha (R_i * \chi_1(D)H(x, t))| \leq C(1 + t)^{-\frac{n+1+|\alpha|}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{n+1+|\alpha|}{2}}.
\]

**Remark 4.1.** This estimate is crucial to help us deduce the asymptotic profile \((1 + \frac{|x|^2}{1 + t})^{-\frac{N}{2}}\) in the pointwise estimates, which is an improvement of those in [32] for the Navier-Stokes-Poisson system in \( \mathbb{R}^n \), where the asymptotic profile for the velocity (or the momentum) is \((1 + t)^{-\frac{n+1}{4}}(1 + \frac{|x|^2}{1 + t})^{-\frac{n+1}{2}}\).

Hence, we have the following pointwise estimates for \( \chi_1(D)\hat{G}_{21} \) and \( \chi_1(D)\hat{G}_{22} \).

**Proposition 4.7.** When \( |\xi| \) is sufficiently small in \( \mathbb{R}^n \), we have for any \( |\alpha| \geq 0 \) that

\[
|D_x^\alpha (\chi_1(D)\hat{G}_{21}(x, t))| \leq C(1 + t)^{-\frac{n+1+|\alpha|}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{n+1+|\alpha|}{2}},
\]

\[
|D_x^\alpha (\chi_1(D)\hat{G}_{22}(x, t))| \leq C(1 + t)^{-\frac{n+|\alpha|}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{n+1}{2}}.
\]

The following proposition gives the pointwise estimates of the Green’s function in the middle part, which is a standard result for the pointwise estimates as in [32, 37].

**Proposition 4.8.** For fixed \( \varepsilon \) and \( R \) defined in the cut-off functions, there exists a positive constant \( c_0 \) and \( C \) such that

\[
|D_x^\alpha (\chi_2(D)\hat{G}(x, t))| \leq Ce^{-c_0t} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-N},
\]

for any integer \( N > 0 \).

Next, it is absolutely the same as in [18, 19] for the short wave of the Green’s function for the Navier-Stokes system, one can immediately get the following result.

**Proposition 4.9.** For \( |\xi| \) being sufficiently large, there exists distribution \( \mathcal{G}_{S_2} \) such that for some \( c_0 > 0 \)

\[
|D_x^\alpha (\chi_3(D)(\hat{G}(x, t) - \mathcal{G}_{S_2}(x, t)))| \leq Ce^{-c_0t} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-N}.
\]
In summary, we have the following pointwise results for the Green’s function $G(x,t)$.

**Proposition 4.10.** For any $|\alpha| \geq 0$ and $x \in \mathbb{R}^3$, we have

$$|D_x^\alpha (G_{11}(x,t) - G_{S_2}(x,t))| \leq C(1 + t)^{-\frac{3 + |\alpha|}{2}} \left(1 + \frac{|x|^2}{1 + t} \right)^{-N} + e^{-\frac{4|\alpha|}{1 + t}} \left(1 + \frac{|x|^2}{1 + t} \right)^{-N}, \quad (4.13)$$

$$|D_x^\alpha (G_{12}(x,t) - G_{S_2}(x,t))| \leq C(1 + t)^{-\frac{3 + |\alpha|}{2}} \left(1 + \frac{|x|^2}{1 + t} \right)^{-N} + e^{-\frac{4|\alpha|}{1 + t}} \left(1 + \frac{|x|^2}{1 + t} \right)^{-N}, \quad (4.14)$$

$$|D_x^\alpha (G_{21}(x,t) - G_{S_2}(x,t))| \leq C(1 + t)^{-\frac{3 + |\alpha|}{2}} \left(1 + \frac{|x|^2}{1 + t} \right)^{-N} + e^{-\frac{4|\alpha|}{1 + t}} \left(1 + \frac{|x|^2}{1 + t} \right)^{-N}, \quad (4.15)$$

$$|D_x^\alpha (G_{22}(x,t) - G_{S_2}(x,t))| \leq C(1 + t)^{-\frac{3 + |\alpha|}{2}} \left(1 + \frac{|x|^2}{1 + t} \right)^{-N} + e^{-\frac{4|\alpha|}{1 + t}} \left(1 + \frac{|x|^2}{1 + t} \right)^{-N}, \quad (4.16)$$

where $G_{S_2}$ is defined in Proposition 4.9 and the integer $N$ could be arbitrary large. Similar to Proposition 3.12, we also denote $G_{S_1} = e^{-\frac{4|\alpha|}{1 + t}}(1 + \frac{|x|^2}{1 + t})^{-N}$.

5. Pointwise estimates for nonlinear system.

This section devotes to the pointwise estimates of the solution to the nonlinear system. The following lemma will be used to estimate the convolutions between the Green’s function and the initial data.

**Lemma 5.1.** There exists a constant $C > 0$, such that

$$I_1 := \int_{\mathbb{R}^3} e^{-\frac{(x-y)^2}{4(1 + t)}}(1 + |y|^2)^{-r_1} dy \leq C \left(1 + \frac{x^2}{1 + t} \right)^{-r_1}, \quad \text{for } r_1 > \frac{3}{2}.$$  

$$I_2 := \int_{\mathbb{R}^3} \left(1 + \frac{|x-y|^2}{1 + t} \right)^{-\frac{q}{2}}(1 + |y|^2)^{-r_1} dy \leq C \left(1 + \frac{x^2}{1 + t} \right)^{-\frac{q}{2}}, \quad \text{for } r_1 > \frac{3}{2}$$  

$$I_3 := \int_{\mathbb{R}^3} e^{-\frac{(x-y-ct)^2}{4(1 + t)}}(1 + |y|^2)^{-r_1} dy \leq C \left(1 + \frac{|x|-ct}{1 + t} \right)^{-\frac{q}{2} - \epsilon}, \quad \text{for } r_1 \geq \frac{21}{10}.$$  

**Proof.** We only prove $I_3$. When $(|x| - ct)^2 \leq 4(1 + t)$,

$$I_3 \leq C \leq C \left(1 + \frac{|x|-ct}{1 + t} \right)^{-\frac{q}{2} - \epsilon}.$$  

When $(|x| - ct)^2 > 4(1 + t)$, we break integration into two parts. If $|y| \geq \frac{|x|-ct}{2}$,

$$I_4 \leq C(1 + (|x|-ct)^2)^{-\frac{q}{2} - \epsilon} \int_{\mathbb{R}^3} e^{-\frac{(x-y-ct)^2}{4(1 + t)}}(1 + |y|^2)^{-r_1} dy \leq C(1 + (|x|-ct)^2)^{-\frac{q}{2} - \epsilon} (1 + t)^{\frac{q}{2} - \epsilon} \left(\int_{\mathbb{R}^3} \left(1 + |y|^2\right)^{-\frac{q}{2} - \epsilon} dy \right)^{\frac{q}{2} - \epsilon} \leq C \left(1 + \frac{|x|-ct}{1 + t} \right)^{-\frac{q}{2} - \epsilon}, \quad \text{since } r_1 \geq \frac{21}{10}.$$  

Here we have used Young’s inequality with $p = \frac{q}{2}$ and the fact

$$\int_{\mathbb{R}^3} \left(1 + \frac{|y|-a|^2}{b} \right)^{-\frac{q}{2}} dy \leq C(b^{\frac{q}{2}} + b^{\frac{q}{2} + \epsilon})^N, \quad \text{for } N > \frac{q}{2}.$$  

If $|y| < \frac{|x|-ct}{2}$, then $|x-y| - ct \geq \frac{|x|-ct}{2}$. Thus, it holds that

$$I_3 \leq Ce^{-\frac{(x-y-ct)^2}{4(1 + t)}^{\frac{q}{2} - \epsilon}} \int_{\mathbb{R}^3} e^{-\frac{(x-y-ct)^2}{4(1 + t)}}(1 + |y|^2)^{-r_1} dy \leq C \left(1 + \frac{|x|-ct}{1 + t} \right)^{-\frac{q}{2} - \epsilon}.$$  

This proves the third estimate. \square
We begin to study the pointwise estimates of the solution \((n_1, w_1, n_2, w_2)\) for (2.4). By Duhamel’s principle, the solution \((n_1, w_1)\) can be expressed as

\[
D_x^n \left( \begin{array}{c} n_1 \\ w_1 \end{array} \right) = D_x^n \left( \begin{array}{cc} G_{11} & G_{12} \\ G_{21} & G_{22} \end{array} \right) * \left( \begin{array}{c} n_{1,0} \\ w_{1,0} \end{array} \right) + \int_0^t D_x^n \left( \begin{array}{cc} G_{11} & G_{12} \\ G_{21} & G_{22} \end{array} \right) (\cdot, t-s) * \left( \begin{array}{c} 0 \\ F_1(n_1, w_1, n_2, w_2) \end{array} \right) (\cdot, s) ds
\]

(5.2)

and \((n_2, w_2)\) can be expressed as

\[
D_x^n \left( \begin{array}{c} n_2 \\ w_2 \end{array} \right) = D_x^n \left( \begin{array}{cc} G_{11} & G_{12} \\ G_{21} & G_{22} \end{array} \right) * \left( \begin{array}{c} n_{2,0} \\ w_{2,0} \end{array} \right) + \int_0^t D_x^n \left( \begin{array}{cc} G_{11} & G_{12} \\ G_{21} & G_{22} \end{array} \right) (\cdot, t-s) * \left( \begin{array}{c} 0 \\ F_2(n_1, w_1, n_2, w_2) \end{array} \right) (\cdot, s) ds.
\]

(5.3)

Without loss of generality, we assume that for \(|\alpha| \leq 2\)

\[
|D_x^n(n_{1,0}, w_{1,0})| \leq C\varepsilon_0(1 + |x|^2)^{-r_1}, \quad r_1 \geq \frac{21}{10}, \\
|D_x^n(n_{2,0}, w_{2,0})| + |\nabla \phi_0| \leq C\varepsilon_0(1 + |x|^2)^{-r_2}, \quad r_2 \geq \frac{3}{2}.
\]

(5.4)

Here \(\varepsilon_0\) is sufficiently small.

Now we first study the pointwise estimates for the solution \((\hat{w}_1, \hat{w}_1)\) to the linearized system of (2.4)_{1,2}. From (3.6), we have

\[
D_x^n \hat{w}_1 = D_x^n G_{1,1}(t) * n_{1,0} + D_x^n G_{1,2} * w_{1,0} := R_1 + R_2.
\]

(5.5)

We rewrite \(R_1\) as

\[
R_1 = D_x^n (G_{11} - G_{S_1} - G_{S_2}) * n_{1,0} + D_x^n G_{S_1} * n_{1,0} + D_x^n G_{S_2} * n_{1,0} := R_1^1 + R_1^2 + R_1^3,
\]

(5.6)

where the Delta-like function \(G_{S_2}\) is generated from the higher frequency part in Green’s function defined in Lemma 3.13, and \(G_{S_1}\) has exponential decay rate and is only singular as \(t \to 0\).

From Lemma 5.1, the assumption (5.4) and Proposition 3.17, we have for \(R_1^1\) that

\[
R_1^1 \leq C(1 + t)^{-\frac{4+\alpha}{1+\alpha}} \left( 1 + \frac{|x| - ct}{1 + t} \right)^{-\frac{4}{N} - \frac{\alpha}{N} - \varepsilon}.
\]

(5.7)

For \(R_1^2\), when \(|x|^2 \leq 1 + t\), from Lemma 3.13, we have

\[
\left| \int G_{S_2}(x-y)n_{1,0}(y)dy \right| \leq e^{-\frac{2\alpha}{\pi} n_{1,0}(x)} + \int_{|x-y| \leq R} \frac{C e^{-bt}}{|x-y|^2} (1 + |y|^2)^{-\frac{4+\alpha}{N}} dy \\
+ \int_{|x-y| > R} \frac{C(N) e^{-bt}}{|x-y|^N} (1 + |y|^2)^{-\frac{4+\alpha}{N}} dy \\
\leq C e^{-C_1 t} \leq C e^{-C_1 t} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{4}{N}}.
\]

(5.8)

When \(|x|^2 > 1 + t\), we have

\[
\left| \int G_{S_2}(x-y)n_{1,0}(y)dy \right| \leq e^{-\frac{2\alpha}{\pi} n_{1,0}(x)} + \int_{|x-y| \leq |x|^2} \frac{C e^{-bt}}{|x-y|^2} (1 + |y|^2)^{-\frac{4+\alpha}{N}} dy \\
+ \int_{|x-y| > |x|^2} \frac{C(N) e^{-bt}}{|x-y|^N} (1 + |y|^2)^{-\frac{4+\alpha}{N}} dy \\
\leq C e^{-\frac{2\alpha}{\pi} (1 + |x|^2)^{-\frac{4+\alpha}{N}}} + C e^{-bt} \left( 1 + \frac{|x|^2}{4} \right)^{-\frac{4}{N}} + C(N) e^{-bt} (1 + |x|)^{-N} \\
\leq C e^{-C_t} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{4}{N}}
\]

(5.9)
For $R_1^2$, we divide the time interval into $0 \leq t \leq 1$ and $t > 1$. When $t > 1$, there is not singularity in $R_1^2$, so one can deal with this term as for $R_1^1$. When $0 \leq t \leq 1$ and $|x| \leq 1 + t$, by using $L^2$-energy estimate for the existence result and Sobolev inequality, we have

$$|R_1^2| = |G_{S_2} \ast D^n_x n_{1,0}| \leq C \leq Ce^{-|x|-t} \leq Ce^{-C_1t} \left(1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}}.$$  

(5.10)

When $t > 1$ and $|x| > 1 + t$,

$$|R_1^2| = |G_{s_1} \ast D^n_x n_{1,0}| \leq e^{-\frac{|x|}{R_1} t^{-3/2}} \int_{R^3} e^{-\frac{|x-y|^2}{4R^2}} (1 + |y|^2)^{-\frac{3}{2}} dy$$

$$= e^{-\frac{|x|}{R_1} t^{-3/2}} \left\{ \int_{|y| \geq \frac{|x|}{R_1}} e^{-\frac{|x-y|^2}{4R^2}} (1 + |y|^2)^{-\frac{3}{2}} dy + \int_{|y| < \frac{|x|}{R_1}} e^{-\frac{|x-y|^2}{4R^2}} (1 + |y|^2)^{-\frac{3}{2}} dy \right\}$$

$$\leq Ce^{-\frac{|x|}{R_1} t^{-3/2}} + Ce^{-\frac{|x|}{R_1} t^{-3/2}} e^{-\frac{|x|^2}{4R^2}}$$

$$\leq Ce^{-C_1t} \left(1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}}.$$  

(5.11)

Then, we have completed the estimate of $R_1$ as

$$|R_1| = |D^n_x G_{1,1}(t) \ast n_{1,0}| \leq C(1 + t) |\frac{|x|}{t} - c|^{-\frac{3}{2} - \varepsilon} \left(1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}} + (1 + t) \left(1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}}.$$  

(5.12)

synchronously, we have the same estimate for $R_2$.

As a result, we have

$$|D^n_x w_1| \leq C \varepsilon_0 (1 + t) |\frac{|x|}{t} - c|^{-\frac{3}{2} - \varepsilon} \left(1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}} + (1 + t) \left(1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}}.$$  

(5.13)

Similarly, we also have

$$|D^n_x w_2| \leq C \varepsilon_0 \left(1 + t + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2} - \varepsilon} \left(1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}} + (1 + t) \left(1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}}.$$  

(5.14)

Let us emphasize here that the method above on dealing with the convolution between the short wave component of the Green’s function and the nonlinear terms is standard, see [32, 36] and the references therein. Thus, in the following we mainly deal with the convolution between nonlinear terms and the leading part of the Green’s function $G(x, t) - G_{S_1} - G_{S_2}$.

Next, we consider the $(\overline{w_2}, \overline{w_2})$. Firstly, we have

$$D^n_x w_2 = D^n_x G_{11}(t) \ast n_{2,0} + D^n_x G_{12} \ast w_{2,0} := R_3 + R_4,$$

(5.15)

$$D^n_x w_2 = D^n_x G_{21}(t) \ast n_{2,0} + D^n_x G_{22} \ast w_{2,0} := R_5 + R_6.$$  

(5.16)

When $|\alpha| = 0$, from the assumption

$$|\nabla \phi_0| \leq C \varepsilon_0 \left(1 + |x|^2 \right)^{-r} \text{ with } r \geq \frac{3}{2},$$

and the fact $n_{2,0} = \text{div} \nabla \phi_0$, we find that for $|\alpha| \geq 0$

$$|D^n_x G_{11} \ast n_{2,0}| = |D^n_x (G_{11} - G_{S_1} - G_{S_2}) \ast n_{2,0}| + |(G_{S_1} + G_{S_2}) \ast D^n_x n_{2,0}|$$

$$\leq |D^n_x (G_{11} - G_{S_1} - G_{S_2}) \ast \phi_0| + C \varepsilon_0 e^{-C_1 t} \left(1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}}$$

$$\leq C \varepsilon_0 (1 + t)^{-\frac{3}{2}} \left(1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}} + C \varepsilon_0 e^{-C_1 t} \left(1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}}$$

$$\leq C \varepsilon_0 (1 + t)^{-\frac{3}{2}} \left(1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}} \text{, where } |\beta| = |\alpha| + 1.$$  

(5.17)
Similarly, by using Proposition 4.10, we have

\[ |R_4| = |D^o_xG_{22} \ast w_{2,0}| \leq C \varepsilon_0 (1 + t)^{-\frac{3+|\alpha|}{2}} \left(1 + \frac{|x|^2}{1+t}\right)^{-\frac{1}{2}}, \quad (5.18) \]

\[ |R_5| + |R_6| \leq C \varepsilon_0 (1 + t)^{-\frac{3+|\alpha|}{2}} \left(1 + \frac{|x|^2}{1+t}\right)^{-\frac{1}{2}}. \quad (5.19) \]

Up to now, we have obtained the following pointwise estimates for the linearized system.

**Proposition 5.2.** For \(0 \leq |\alpha| \leq 2\), we have

\[
|D^o_xw_1| \leq C \varepsilon_0 (1 + t)^{-\frac{3+|\alpha|}{2}} \left\{ 1 + \frac{1}{1+t} \right\} (1 + \frac{|x|^2}{1+t})^{-\frac{1}{2}},
\]

\[
|D^o_xw_2| \leq C \varepsilon_0 (1 + t)^{-\frac{3+|\alpha|}{2}} \left\{ 1 + \frac{1}{1+t} \right\} (1 + \frac{|x|^2}{1+t})^{-\frac{1}{2}}.
\]

As mentioned in §1, to deduce the pointwise estimates for the solution \((n_1, v_1)\) of the nonlinear problem, we have to first obtain the pointwise estimate of the electric field \(\nabla \phi\). To this end, we shall consider the following new system on the variable \(n_2\) and \(v_2 = u_1 - u_2\), to avoid dealing with the term \(\nabla \phi\) in the nonlinear term of this system. In fact, we can rewrite the system (2.4) as

\[
\begin{aligned}
\partial_t n_2 + \text{div} v_2 &= -\text{div} \left( \frac{n_1 n_2 + n_2 v_1}{2} \right) := F_3(n_1, v_1, n_2, v_2) \\
\partial_t v_2 + h'(1) n_2 - \mu_1 \Delta v_2 - \mu_2 \text{div} v_2 + 2 \nabla \phi &= \nabla \{ h'(1) n_2 - h(1 + \frac{n_1 + n_2}{2}) + h(1 + \frac{n_1 - n_2}{2}) \} + (v_1 \cdot \nabla) v_2 \\
&\quad + (v_2 \cdot \nabla) v_1 + \left( \frac{n_1 + n_2}{2} \right) (\mu_1 \Delta v_2 + \mu_2 \text{div} v_2) + \left( \frac{n_1 + n_2}{2} \right) (\mu_1 V \frac{n_1 + n_2}{2} + \mu_2 \text{div} \frac{n_1 + n_2}{2}) \\
&:= F_4(n_1, v_1, n_2, v_2),
\end{aligned}
\]

where \(h'(\rho) = \frac{\mu'(\rho)}{\rho}\) and \(v_1 = u_1 + u_2\).

Furthermore, we find that

\[ F_4(n_1, v_1, n_2, v_2) = O(1) (D(n_1 n_2) + v_1 Dv_2 + v_2 Dv_1 + n_1 D^2 v_2 + n_2 D^2 v_2 + n_2 D^3 v_1), \quad (5.22) \]

that is, there is a factor \(n_2\) or \(v_2\) in each term of \(F_3(n_1, v_1, n_2, v_2)\) and \(F_4(n_1, v_1, n_2, v_2)\). This fact is key for us to deduce the pointwise estimate of \(n_2\) and \(v_2\).

Due to Proposition 5.2, we first introduce the following ansatz for \(|\alpha| \leq 2\):

\[
M(T) = \sup_{0 \leq t \leq T} \left\{ ||D^o_x n_1(\cdot, t)\psi^{-1}_1(\cdot, t)||_{L^\infty} + ||D^o_x w_1(\cdot, t)\psi^{-1}_2(\cdot, t)||_{L^\infty} + ||n_2(\cdot, t)\psi^{-1}_3(\cdot, t)||_{L^\infty} + ||w_2(\cdot, t)\psi^{-1}_4(\cdot, t)||_{L^\infty} + ||D^o_x n_2(\cdot, t)\psi^{-1}_5(\cdot, t)||_{L^\infty} + ||D^o_x w_2(\cdot, t)\psi^{-1}_6(\cdot, t)||_{L^\infty} + (1 + t)^3 (||D^3(\cdot, w_1, n_2, w_2)||_{L^\infty} + ||D^4(\cdot, n_1, w_1, n_2, w_2)||_{L^\infty}) \right\},
\]

(5.23)
where for any $0 < \varepsilon \ll 1$ that

$$
\psi_1(x, t) = (1 + t)^{-\frac{3}{2}} \left\{ \left( 1 + \frac{x^2}{1 + t} \right)^{-\frac{3}{2} + \varepsilon} + \left( 1 + \frac{(|x| - ct)^2}{1 + t} \right)^{-\frac{3}{2} + \varepsilon} \right\} ,
$$

$$
\psi_2(x, t) = (1 + t)^{-\frac{3}{2}} \left\{ \left( 1 + \frac{x^2}{1 + t} \right)^{-\frac{3}{2} - \varepsilon} + (1 + t)^{-\frac{3}{2}} \left( 1 + \frac{(|x| - ct)^2}{1 + t} \right)^{-\frac{3}{2} - \varepsilon} \right\} ,
$$

$$
\psi_3(x, t) = (1 + t)^{-2} \left( 1 + \frac{x^2}{1 + t} \right)^{-\frac{3}{2}} , \quad \psi_4(x, t) = (1 + t)^{-\frac{3}{2}} \left( 1 + \frac{x^2}{1 + t} \right)^{-\frac{3}{2}} ,
$$

$$
\psi_5(x, t) = (1 + t)^{-2} \left( 1 + \frac{x^2}{1 + t} \right)^{-\frac{3}{2} + \varepsilon} , \quad \psi_6(x, t) = (1 + t)^{-\frac{3}{2}} \left( 1 + \frac{x^2}{1 + t} \right)^{-\frac{3}{2} - \varepsilon} .
$$

In what follows, we mainly prove that $M(T) \leq C$.

In fact, in [34] the authors have deduced the following decay rate of the solution $(\rho_1, u_1, \rho_2, u_2)$ when the initial data is in $H^l(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ with $l \geq 4$ that

$$
\|D_x^a (\rho_1 - 1, u_1, \rho_2 - 1, u_2)\|_{L^2(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{3}{2} - \frac{|a|}{2}}, \quad |a| \leq l .
$$

(5.24)

After a similar procedure as in [34], when the initial data is in $H^6(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$, one can immediately get the similar results for the higher order derivative of the solution. In addition, by using Sobolev inequality in $\mathbb{R}^3$

$$
\|f\|_{L^\infty} \leq C\|Df\|_{L^2} \|D^2f\|_{L^2} ,
$$

we know the ansatz on $D_2^a (n_1, w_1, n_2, w_2)$ with $3 \leq |\beta| \leq 4$ is reasonable in the present paper. Thus, in the following we mainly focus on the pointwise estimates for the $D_x^a (n_1, w_1, n_2, w_2)$ when $|a| \leq 2$.

From the ansatz (5.23) and the relation $w_1 = (\rho_1 + 1)u_1 + (\rho_2 + 1)u_2$ and $w_2 = (\rho_1 + 1)u_1 - (\rho_2 + 1)u_2$, we have

$$
|v_1| \leq C(|w_1| + |n_2|), \quad |v_2| \leq C(|w_2| + |n_2|), \quad |w_1| \leq C(|v_1| + |n_2|), \quad |w_2| \leq C(|v_2| + |n_2|) .
$$

(5.25)

In addition, for $1 \leq |a| \leq 2$ we get

$$
\begin{align*}
|D_x^a v_2(x, t)| & \leq CM(T)(\psi_3(x, t) + \psi_4(x, t)), \\
|D_x^a v_1(x, t)| & \leq CM(T)(\psi_2(x, t) + \psi_3(x, t)).
\end{align*}
$$

(5.26)

Now, we begin to derive the pointwise estimates for $n_2$ and $v_2$. by Duhamel’s principle, the nonlinear part of $|D_x^a n_2|$ is as follows

$$
\begin{align*}
& \int_0^t D_x^a G_{11}(\cdot, t - s) * F_3(\cdot, s) ds + \int_0^t D_x^a G_{12}(\cdot, t - s) * F_4(\cdot, s) ds \\
= & \int_0^t D_x^a (G_{11} - G_{S_1} - G_{S_2})(\cdot, t - s)) * F_3(\cdot, s) ds + \int_0^t D_x^a (G_{S_1} + G_{S_2})(\cdot, t - s) * F_3(\cdot, s) ds \\
& + \int_0^t D_x^a (G_{12} - G_{S_1} - G_{S_2})(\cdot, t - s)) * F_4(\cdot, s) ds + \int_0^t D_x^a (G_{S_1} + G_{S_2})(\cdot, t - s) * F_4(\cdot, s) ds \\
:= & R_7 + R_8 + R_9 + R_{10}.
\end{align*}
$$

(5.27)

We first consider the case $|a| = 0$ to obtain the estimate for $n_2$. We divide the time interval into two parts: short time $[0, \frac{t}{2}]$ and long time $(\frac{t}{2}, t)$. For $R_7$, since the nonlinear term $F_3$ has the conservative
structure, and from (5.25), we have for the short time
\[
\left| \int_0^{\gamma} (G_{11} - G_{S_1} - G_{S_2})(\cdot, t - s) * F_3(\cdot, s) ds \right|
= \left| \int_0^{\gamma} D_2(G_{11} - G_{S_1} - G_{S_2})(\cdot, t - s) * (n_1 v_2 + n_2 v_1)(\cdot, s) ds \right|
\leq \int_0^{\gamma} |D_2(G_{11} - G_{S_1} - G_{S_2})(\cdot, t - s)| |n_1 v_2 + n_2 v_1|(\cdot, s) ds
\leq CM^2(T) \int_0^{\gamma} \int_{\mathbb{R}^3} (1 + t - s)^{-2} \left( 1 + \frac{|x - y|^2}{1 + t - s} \right)^{-N} (1 + s)^{-\frac{\gamma}{2}} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-\frac{\gamma}{2}} dyds
\cdot \left\{ \left( 1 + \frac{|y|^2}{1 + s} \right)^{-(\frac{\gamma}{2} - \epsilon)} + \left( 1 + \frac{(|y| - cs)^2}{1 + s} \right)^{-(\frac{\gamma}{2} - \epsilon)} \right\}
\]
(5.28)

When $|x|^2 \leq 1 + t$, by using (5.2) and Young’s inequality, we have
\[
R_1^2 + R_2^2 \leq CM^2(T)(1 + t)^{-2} \left\{ \int_0^{\gamma} \left( (1 + s)^{-\frac{\gamma}{2}} (1 + \frac{|y|^2}{1 + s}) dyds \right) + \int_0^{\gamma} \left( (1 + s)^{-\frac{\gamma}{2}} (1 + \frac{(|y| - cs)^2}{1 + s}) dyds \right) \right\}
\leq CM^2(T)(1 + t)^{-2} \left\{ 1 + \int_0^{\gamma} \int_{\mathbb{R}^3} (1 + \frac{|x - y|^2}{1 + t - s}) (1 + s)^{-\frac{\gamma}{2}} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-\frac{\gamma}{2}} \left( 1 + \frac{(|y| - cs)^2}{1 + s} \right)^{-\frac{\gamma}{2}} dyds \right\}
\leq CM^2(T)(1 + t)^{-2} \left\{ 1 + \int_0^{\gamma} \int_{\mathbb{R}^3} (1 + s)^{-\frac{\gamma}{2}} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-\frac{\gamma}{2}} \left( 1 + \frac{(|y| - cs)^2}{1 + s} \right)^{-\frac{\gamma}{2}} dyds \right\}
\leq CM^2(T)(1 + t)^{-2} \leq C(1 + t)^{-2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{\gamma}{2}},
\]
where we have used the estimate (5.34).

When $|x|^2 > 1 + t$ and $|y| > \frac{|x|^2}{1 + t}$, by using (5.2), Young’s inequality and Lemma 6.1, we have
\[
R_1^2 = CM^2(T) \int_0^{\gamma} \int_{\mathbb{R}^3} (1 + t - s)^{-2} \left( 1 + \frac{|x - y|^2}{1 + t - s} \right)^{-N} (1 + s)^{-\frac{\gamma}{2}} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-\frac{\gamma}{2}} dyds
\leq CM^2(T)(1 + t)^{-2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{\gamma}{2}} (1 + s)^{-\frac{\gamma}{2}} \int_0^{\gamma} (1 + t - s)^{-2} (1 + s)^{-\frac{\gamma}{2}} (1 + \frac{(|y| - cs)^2}{1 + s})^{-\frac{\gamma}{2}} dyds
\leq CM^2(T)(1 + t)^{-2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{\gamma}{2}},
\]
and
\[
R_2^2 = CM^2(T) \int_0^{\gamma} \int_{\mathbb{R}^3} (1 + t - s)^{-2} \left( 1 + \frac{|x - y|^2}{1 + t - s} \right)^{-N} (1 + s)^{-\frac{\gamma}{2}} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-\frac{\gamma}{2}} \left( 1 + \frac{(|y| - cs)^2}{1 + s} \right)^{-\frac{\gamma}{2}} dyds
\leq CM^2(T)(1 + t)^{-2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{\gamma}{2}} (1 + s)^{-\frac{\gamma}{2}} \int_0^{\gamma} (1 + t - s)^{-2} (1 + s)^{-\frac{\gamma}{2}} (1 + \frac{(|y| - cs)^2}{1 + s})^{-\frac{\gamma}{2}} dyds
\leq CM^2(T)(1 + t)^{-\frac{\gamma}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{\gamma}{2}}.
\]
Similarly, when $|x|^2 > 1 + t$ and $|y| \leq \frac{|x|^2}{1 + t}$, it also holds that
\[
|R_7| \leq CM^2(T)(1 + t)^{-2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{\gamma}{2}}.
\]

\[\text{\Hb}\]
Next, we consider the long time part of the nonlinear term. We have
\[
\left| \int_{\frac{3}{4}}^{t} (G_{11} - G_{S_1} - G_{S_2})(\cdot, t-s) * F_3(\cdot, s) ds \right| = \int_{\frac{3}{4}}^{t} D_x (G_{11} - G_{S_1} - G_{S_2})(\cdot, t-s) * (n_1 v_2 + n_2 v_1)(\cdot, s) ds \\
\leq \int_{\frac{3}{4}}^{t} |D_x (G_{11} - G_{S_1} - G_{S_2})(\cdot, t-s) * (|n_1 v_2| + |n_2 v_1|)(\cdot, s) ds \\
\leq CM^2(T) \int_{\frac{3}{4}}^{t} \int_{\mathbb{R}^3} (1 + t-s)^{-2} \left( 1 + \frac{|x-y|^2}{1+t-s} \right)^{-N} (1 + s)^{-\frac{2}{N}} \left( 1 + \frac{|y|^2}{1+s} \right)^{-\frac{4}{N}} dy ds \\
\cdot \left\{ (1 + \frac{|y|^2}{1+s})^{-(\frac{4}{N} - \varepsilon)} + (1 + \frac{(|y| - cs)^2}{1+s})^{-(\frac{4}{N} - \varepsilon)} \right\} dy ds := R_7^3 + R_7^4. 
\tag{5.29}
\]
When $|x|^2 \leq 1 + t$, we have
\[
R_7^3 + R_7^4 \leq CM^2(T)(1 + t)^{-\frac{2}{N}} \int_{\frac{3}{4}}^{t} (1 + t-s)^{-2} (1 + t-s)^{\frac{2}{N}} ds \\
\leq CM^2(T)(1 + t)^{-3} \leq CM^2(T)(1 + t)^{-3} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{2}{N}}.
\]
When $|x|^2 > 1 + t$ and $|y| > \frac{|x|}{2}$,
\[
R_7^3 = CM^2(T) \int_{\frac{3}{4}}^{t} \int_{\mathbb{R}^3} (1 + t-s)^{-2} \left( 1 + \frac{|x-y|^2}{1+t-s} \right)^{-N} (1 + s)^{-\frac{2}{N}} \left( 1 + \frac{|y|^2}{1+s} \right)^{-(3-\varepsilon)} dy ds \\
\leq CM^2(T)(1 + t)^{-\frac{2}{N}} \left( 1 + \frac{|x|^2}{1+t} \right)^{-(3-\varepsilon)} \int_{\frac{3}{4}}^{t} (1 + t-s)^{-2} (1 + t-s)^{-\frac{2}{N}} ds \\
\leq CM^2(T)(1 + t)^{-3} \left( 1 + \frac{|x|^2}{1+t} \right)^{-(3-2\varepsilon)}.
\]
\[
R_7^4 = CM^2(T) \int_{\frac{3}{4}}^{t} \int_{\mathbb{R}^3} (1 + t-s)^{-2} \left( 1 + \frac{|x-y|^2}{1+t-s} \right)^{-N} (1 + s)^{-\frac{2}{N}} \left( 1 + \frac{|y|^2}{1+s} \right)^{-\frac{2}{N}} \left( 1 + \frac{(|y| - cs)^2}{1+s} \right)^{-(\frac{4}{N} - \varepsilon)} dy ds \\
\leq CM^2(T)(1 + t)^{-\frac{2}{N}} \left( 1 + \frac{|x|^2}{1+t} \right)^{\frac{2}{N}} \int_{\frac{3}{4}}^{t} (1 + t-s)^{-2} (1 + t-s)^{-\frac{2}{N}} ds \\
\leq CM^2(T)(1 + t)^{-3} \left( 1 + \frac{|x|^2}{1+t} \right)^{-\frac{2}{N}}.
\]
Similarly, one can get for $|x|^2 > 1 + t$ and $|y| \leq \frac{|x|}{2}$ that
\[
R_7^3 + R_7^4 \leq CM^2(T)(1 + t)^{-2} \left( 1 + \frac{|x|^2}{1+t} \right)^{-\frac{2}{N}}.
\]
Notice that there exists an additional derivative in $G_{12}$ though the nonlinear term $F_4$ has not a divergence form. Then, after a similar calculus as for $R_7$, and by using the ansatz (5.23), one can immediately obtain
\[
R_0 \leq CM^2(T)(1 + t)^{-2} \left( 1 + \frac{|x|^2}{1+t} \right)^{-\frac{2}{N}}.
\]
Next we consider $R_8$ and $R_{10}$. In spite of singularity of $G_{S_1}$ as $t \to 0$, the convolution between $G_{S_1}$ and the nonlinear terms has not this singularity when $|x| \leq 1$. Thus, one can similarly deduce the estimate
for this convolution as \( R_7 \) and \( R_9 \). Then, we consider the Dirac-like function \( G_{S_2} \). We need the following standard nonlinear estimates as in [19, 18].

**Lemma 5.3.** There exists a constant \( C > 0 \) such that

\[
\int_0^t \int_{\mathbb{R}^3} e^{-b(t-s)} \hat{f}(x-y)(1+s) -\frac{\alpha}{2} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-\alpha} dyds \leq C(1+t) -\frac{\alpha}{2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\alpha},
\]

\[
\int_0^t \int_{\mathbb{R}^3} e^{-b(t-s)} \hat{f}(x-y)(1+s) -\frac{\alpha}{2} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-\alpha} dyds \leq C(1+t) -\frac{\alpha}{2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\alpha},
\]

where \( b > 0 \) and \( \hat{f}(x) \) is defined in Proposition 4.9, which is a Dirac-like function.

This lemma implies that the convolution between \( G_{S_2} \) and the nonlinear terms is almost the same as the convolution between \( e^{bt}\delta(x) \) and the nonlinear terms. Thus, in what follows, we regard \( G_{S_2} \) as \( e^{bt}\delta(x) \). Notice the difference between \( \psi_3, \psi_4 \) and \( \psi_5, \psi_6 \) in the ansatz (5.23). We have to carefully estimate the convolutions between \( G_{S_2} \) and \( F_3, F_4 \) corresponding to the variables \( n_2, w_2 \). In fact, from the relation (5.26) and the equations (5.21), we can rewrite \( F_3 \) and \( F_4 \) as

\[
|F_3| + |F_4| = \mathcal{O}(1) \{ Dn_1(w_2 + n_2) + Dn_2w_1 + n_2Dv_1 + Dn_2n_1 + w_1Dw_2 \\
+ w_1Dn_2 + n_1D^2w_2 + n_1D^2n_2 + n_1D^2w_2 + n_1D^2n_2 + \cdots \},
\]

where “\( \cdots \)” denotes the terms containing \( n_2 \) or \( w_2 \). Hence, for the term “\( \cdots \)” above one can use the ansatz (5.23) and Lemma 5.3 to get the pointwise profile with the factor \( a = \frac{3}{2} \) in Lemma 5.3. For the other terms above, from the ansatz (5.23), it is not so obvious to obtain the same result. We just take the term \( w_1Dw_2 \) in (5.32) for example. In fact, we have

\[
|w_1Dw_2|(x,t) \leq CM^2(T) \left\{ (1+t)^{-\frac{3}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\left(\frac{3}{2} - \epsilon\right)} + (1+t)^{-2} \left( 1 + \frac{(|x| - ct)^2}{1 + t} \right)^{-\left(\frac{3}{2} - \epsilon\right)} \right\}
\]

\[
\cdot (1+t)^{-\frac{3}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\left(\frac{3}{2} - \epsilon\right)}
\]

\[
\leq CM^2(T)(1+t)^{-3} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\left(3 - 2\epsilon\right)} + CM^2(T)(1+t)^{-\frac{3}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\left(\frac{3}{2} - \epsilon\right)} \left( 1 + \frac{(|x| - ct)^2}{1 + t} \right)^{-\left(\frac{3}{2} - \epsilon\right)}
\]

\[
\leq CM^2(T)(1+t)^{-3} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\left(3 - 2\epsilon\right)} + CM^2(T)(1+t)^{-\frac{3}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\left(\frac{3}{2} - \epsilon\right)} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\epsilon}
\]

\[
\leq CM^2(T)(1+t)^{-3} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\left(3 - 2\epsilon\right)} + (1+t)^{-\left(\frac{3}{2} - 2\epsilon\right)} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}}
\]

\[
\leq CM^2(T)(1+t)^{-3} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}}.
\]

Here we have used the following estimate for \( a_1 > 0 \) that

\[
\left( 1 + \frac{(|x| - ct)^2}{1 + t} \right)^{-a_1} \leq C(1+t)^{2a_1} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-a_1},
\]

which can be proved by dividing the domain into two parts: \(|x| < 2ct\) and \(|x| \geq 2ct\). For simplicity, we omit the details. Lastly, the other terms in (5.32) could be treated similarly.

In a conclusion, we have the following estimate for \( n_2 \)

\[
|n_2(x,t)| \leq C(\varepsilon_0 + M^2(T))(1+t)^{-\frac{3}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}}.
\]

Synchronously, one can have the estimate of \( v_2 \) that

\[
|v_2(x,t)| \leq C(\varepsilon_0 + M^2(T))(1+t)^{-\frac{3}{2}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{2}}.
\]
which together with the relation (5.25) yields

\[ |w_2(x,t)| \leq C(\varepsilon_0 + M^2(T))(1 + t)^{-\frac{3}{2}} \left(1 + \frac{|x|^2}{1 + t}\right)^{-\frac{3}{4}}. \]  

(5.37)

That is, we have completed the estimate for \((n_2, w_2)\) as

\[ |n_2(x,t)| \leq C(\varepsilon_0 + M^2(T)) \psi_3, \]

\[ |w_2(x,t)| \leq C(\varepsilon_0 + M^2(T)) \psi_4. \]  

(5.38)

Now, from the relation \(\nabla \phi = \nabla_\n \cdot n_2\), we can immediately give the following proposition on \(\nabla \phi\), which is key for us to deduce the optimal pointwise estimate for \(n_1\) and \(w_1\).

**Proposition 5.4.** We have for a constant \(\varepsilon\) satisfying \(0 < \varepsilon \ll 1\) that

\[ |\nabla \phi(x,t)| \leq C(\varepsilon_0 + M^2(T)(1 + t)^{-\frac{3}{2}} \left(1 + \frac{|y|^2}{1 + t}\right)^{-\frac{3}{4}}. \]  

(5.39)

**Proof.** The relation \(\nabla \phi = \nabla_\n \cdot n_2\) yields that

\[ |\nabla \phi(x,t)| = (1 + t)^{-2} \int_{\mathbb{R}^3} |x - y|^{-2} \left(1 + \frac{|y|^2}{1 + t}\right)^{-\frac{3}{4}} dy. \]  

(5.40)

When \(|x|^2 \leq 1 + t\), if \(|x - y| \leq 1\), we know

\[ |\nabla \phi(x,t)| \leq C(1 + t)^{-2} \leq C(1 + t)^{-2} \left(1 + \frac{|y|^2}{1 + t}\right)^{-\frac{3}{4}}, \]  

(5.41)

and if \(|x - y| > 1\), by using Young’s inequality we have

\[ |\nabla \phi(x,t)| \leq C(1 + t)^{-\frac{3}{2}} \leq C(1 + t)^{-\frac{3}{2}} \left(1 + \frac{|y|^2}{1 + t}\right)^{-\frac{3}{4}}. \]  

(5.42)

When \(|x|^2 > 1 + t\), if \(|y| < \frac{|x|}{2}\), we know \(|x - y| \geq \frac{|x|}{2}\), which implies that

\[ |\nabla \phi(x,t)| \leq C(1 + t)^{-\frac{3}{4}} \left(1 + \frac{|y|^2}{1 + t}\right)^{-\frac{3}{4}} \int_{\mathbb{R}^3} |x - y|^{-2} \left(1 + \frac{|y|^2}{1 + t}\right)^{-\frac{3}{4}} dy \]

\[ \leq C(1 + t)^{-\frac{3}{4}} \left(1 + \frac{|y|^2}{1 + t}\right)^{-\frac{3}{4}}. \]  

(5.43)

When \(|y| \geq \frac{|x|}{2}\), we also obtain

\[ |\nabla \phi(x,t)| \leq C(1 + t)^{-\frac{3}{2}} \left(1 + \frac{|x|^2}{1 + t}\right)^{-\frac{3}{4}}. \]  

(5.44)

This proves Proposition 5.4. \(\square\)

Next, we shall go back to consider the conservation system on \(n_1\) and \(w_1\). We rewrite

\[ \int_0^t D_x^a(G_{12}(\cdot, t - s) \ast F_1(\cdot, s)) \, ds \]

\[ = \int_0^t D_x^a(G_{12} - G_{S_1} - G_{S_2})(\cdot, t - s) \ast F_1(\cdot, s) \, ds + \int_0^t D_x^a(G_{S_1} + G_{S_2})(\cdot, t - s) \ast F_1(\cdot, s) \, ds \]

\[ := R_{15} + R_{16}. \]  

(5.45)
For $R_{15}$, we mainly focus on the “worst” nonlinear term $n_2 \nabla \phi$ in $F_1$ of (2.4). In fact, we have
\[
R_{15} := \int_0^t D_x^\alpha (G_{12}\cdot t - s) * (n_2 \nabla \phi)(\cdot, s)ds = - \int_0^t D_x^\beta G_{12}(\cdot, t - s) * (|\nabla \phi|^2)(\cdot, s)ds, \tag{5.46}
\]
where $|\beta| = |\alpha| + 1$.

Since the Green’s function of the Navier-Stokes system contains the diffusion wave, the Riesz wave and the Hyngens’ wave, to estimate the interaction of these different waves, we should divide both the time $t$ and the space $x$ into several parts. In particular, as in [18] we define
\[
D_1 = \{x^2 \leq 1 + t\}, \quad D_2 = \{|x - ct|^2 \leq 1 + t\}, \quad D_3 = \{|x| \geq ct + \sqrt{1 + t}\},
\]
\[
D_4 = \{\sqrt{1 + t} \leq |x| \leq \frac{ct}{2}\}, \quad D_5 = \{|x| \leq ct - \sqrt{1 + t}\}.
\]

In fact, we have the following estimate for the nonlinear term $n_2 \nabla \phi$ in $F_1(n_1, w_1, n_2, w_2)$.

**Lemma 5.5.** We have
\[
N_1 := \left| \int_0^t H(\cdot, t - s) * (n_2 \nabla \phi)(\cdot, s)ds \right|
\leq CM^2(T)(1 + t)^{-2} \left\{ \left( 1 + \frac{|x - ct|^2}{1 + t} \right)^{-\left(\frac{5}{4} - \varepsilon\right)} + \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\left(\frac{5}{4} - \varepsilon\right)} \right\}, \tag{5.47}
\]
where $H(x, t)$ is the H-wave defined in Proposition 3.17.

**Proof.** When $(x, t) \in D_1 \cup D_2$, we divide the interval $[0, t]$ into two parts. In fact, by using Young’s inequality and the ansatz (5.23), we have
\[
\left| \int_0^t H(\cdot, t - s) * (n_2 \nabla \phi)(\cdot, s)ds \right| = \left| - \int_0^t (1 + t - s)^{-\frac{3}{2}} e^{-\frac{|x - (x - ct)(t - s)|^2}{2(1 + t - s)}} * (|\nabla \phi|^2)(\cdot, s)ds \right|
\leq CM^2(T) \int_0^t \int_{\mathbb{R}^3} (1 + t - s)^{-\frac{3}{2}} e^{-\frac{|x - (x - ct)(t - s)|^2}{2(1 + s)}} (1 + s)^{-\left(\frac{3}{4} - \varepsilon\right)} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-\left(\frac{5}{4} - \varepsilon\right)} dyds
\leq CM^2(T)(1 + t)^{-\frac{3}{2}} \int_0^t (1 + s)^{-\left(\frac{3}{4} - \varepsilon\right)} (1 + s)^{\frac{3}{4}} ds
\leq CM^2(T)(1 + t)^{-\frac{3}{2}} \leq C
\begin{cases}
(1 + t)^{-\frac{3}{4}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{3}{4}}, & (x, t) \in D_1; \\
(1 + t)^{-\frac{3}{4}} \left( 1 + \frac{|x - ct|^2}{1 + t} \right)^{-\frac{3}{4}}, & (x, t) \in D_2,
\end{cases}
\tag{5.48}
\]
and
\[
\left| \int_0^t H(\cdot, t - s) * (n_2 \nabla \phi)(\cdot, s)ds \right| = \left| \int_0^t (1 + t - s)^{-2} e^{-\frac{|x - (x - ct)(t - s)|^2}{2(1 + t - s)}} * (n_2 \nabla \phi)(\cdot, s)ds \right|
\leq CM^2(T) \int_0^t \int_{\mathbb{R}^3} (1 + t - s)^{-2} e^{-\frac{|x - (x - ct)(t - s)|^2}{2(1 + s)}} (1 + s)^{-\left(\frac{5}{4} - \varepsilon\right)} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-\left(\frac{5}{4} - \varepsilon\right)} dyds
\leq CM^2(T)(1 + t)^{-\left(\frac{5}{4} - \varepsilon\right)} \cdot (1 + t) \leq C(1 + t)^{-\left(\frac{5}{4} - \varepsilon\right)}
\begin{cases}
CM^2(T)(1 + t)^{-\frac{5}{4}} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-\frac{5}{4}}, & (x, t) \in D_1; \\
CM^2(T)(1 + t)^{-\frac{5}{4}} \left( 1 + \frac{|x - ct|^2}{1 + t} \right)^{-\frac{5}{4}}, & (x, t) \in D_2.
\end{cases}
\tag{5.49}
\]
When \((x, t) \in D_3\), and \(|y| \geq \frac{|x - ct|}{2}\), by using Lemma 6.1 and the ansatz (5.23), we can get

\[
\left| \int_0^t H(\cdot, t - s) * (n_2 \nabla \phi)(\cdot, s) ds \right|
\leq CM^2(T) \int_0^t \int_{\mathbb{R}^3} (1 + t - s)^{- \frac{7}{4}} e^{\frac{|(x - y) - (c(t-s))|^2}{c(t-s)}} (1 + s)^{-(3-\varepsilon)} \left(1 + \frac{|y|^2}{1 + s}\right)^{- (2-\varepsilon)} dy ds
\leq CM^2(T)(1 + t)^{- \frac{7}{4}} \left(1 + \frac{|x - ct|^2}{1 + t}\right)^{-(\frac{7}{4} - \varepsilon)} (1 + t)^{- (3-\varepsilon)} \left(1 + \frac{|y|^2}{1 + s}\right)^{- (2-\varepsilon)} ds
\]

and similarly

\[
\left| \int_0^t (1 + t - s)^{-2} e^{\frac{|(x - y) - (c(t-s))|^2}{c(t-s)}} (1 + s)^{-(\frac{7}{4} - \varepsilon)} \left(1 + \frac{|y|^2}{1 + s}\right)^{- (2-\varepsilon)} dy ds \right|
\leq CM^2(T) \int_0^t \int_{\mathbb{R}^3} (1 + t - s)^{-2} e^{\frac{|(x - y) - (c(t-s))|^2}{c(t-s)}} (1 + s)^{-(\frac{7}{4} - \varepsilon)} \left(1 + \frac{|y|^2}{1 + s}\right)^{- (2-\varepsilon)} dy ds
\leq CM^2(T)(1 + t)^{- (\frac{7}{4} - \varepsilon)} \left(1 + \frac{|x - ct|^2}{1 + t}\right)^{-(\frac{7}{4} - \varepsilon)} (1 + t)^{- (3-\varepsilon)} \left(1 + \frac{|y|^2}{1 + s}\right)^{- (2-\varepsilon)} ds
\leq CM^2(T)(1 + t)^{- (\frac{7}{4} - \varepsilon)} \left(1 + \frac{|x - ct|^2}{1 + t}\right)^{-(\frac{7}{4} - \varepsilon)}, \text{ where } p = \frac{4}{3}, q = 4. \tag{5.50}
\]

When \((x, t) \in D_3\) and \(|y| < \frac{|x - ct|}{2}\), by using the same method above, one can immediately obtain

\[
\left| \int_0^t (1 + t - s)^{-2} e^{\frac{|(x - y) - (c(t-s))|^2}{c(t-s)}} (n_2 \nabla \phi)(\cdot, s) ds \right| \leq CM^2(T)(1 + t)^{- (\frac{7}{4} - \varepsilon)} \left(1 + \frac{|x - ct|^2}{1 + t}\right)^{- \frac{3}{2}}. \tag{5.52}
\]

When \((x, t) \in D_4 : \sqrt{1 + \frac{1}{t}} \leq |x| \leq \frac{ct}{2}\), we have to divide the interval into several parts.

Case 1: \(0 \leq s \leq \frac{t}{2} - \frac{|x|}{2c}\). When \(|y| \leq \frac{c(t-s)}{4}\), it holds that \(c(t-s) - |x - y| \geq \frac{c(t-s)}{4}\). Then,

\[
\int_0^\frac{t}{2} \int_{|y| \leq \frac{c(t-s)}{4}} (1 + t - s)^{-\frac{7}{4}} e^{\frac{|(x - y) - (c(t-s))|^2}{c(t-s)}} (1 + s)^{-(3-\varepsilon)} \left(1 + \frac{|y|^2}{1 + s}\right)^{- (2-\varepsilon)} dy ds
\leq C(1 + t)^{-\frac{7}{4}} \left(1 + \frac{|x - ct|^2}{1 + t}\right)^{-\frac{7}{4}} \int_0^\frac{t}{2} \frac{t}{2} \int_{|y| \leq \frac{c(t-s)}{4}} (1 + s)^{-(3-\varepsilon)} (1 + s)^{\frac{3}{2}} ds
\leq C(1 + t)^{-\frac{7}{4}} \left(1 + \frac{|x - ct|^2}{1 + t}\right)^{-\frac{7}{4}}. \tag{5.53}
\]
When $|y| > \frac{ct - |x|}{4} > \frac{ct}{8} > \sqrt{1 + t}$, we have

$$
\int_0^{\frac{t}{2} - \frac{|x|}{4c}} \int_{|y| \le \frac{ct - |x|}{4c}} (1 + t - s)^{-\frac{5}{2}} e^{-\frac{(|y| - ct - s)^2}{(t - s)^2}} (1 + s)^{-(3-\varepsilon)} \left(1 + \frac{|y|^2}{1 + s}\right)^{-(2-\varepsilon)} dy ds
\le C(1 + t)^{-\frac{5}{2}} \left(1 + \frac{|x| - ct}{1 + t}\right)^{-(3-\varepsilon)} (1 + t)^{-(\varepsilon-\varepsilon)}
\times \int_0^{\frac{t}{2} - \frac{|x|}{4c}} \int_{|y| \le \frac{ct - |x|}{4c}} e^{-\frac{(|y| - ct - s)^2}{(t - s)^2}} (1 + s)^{-\frac{3}{2}} \left(1 + \frac{|y|^2}{1 + s}\right)^{-\frac{5}{2}} dy ds
\le C(1 + t)^{-(4-\varepsilon)} \left(1 + \frac{|x| - ct}{1 + t}\right)^{-(\varepsilon-\varepsilon)} \int_0^{\frac{t}{2} - \frac{|x|}{4c}} (1 + s)^{-\frac{3}{2}} (1 + t - s)^{\frac{5}{2}} (1 + s)^{\frac{3}{2}} ds
\le C(1 + t)^{-(\varepsilon-\varepsilon)} \left(1 + \frac{|x| - ct}{1 + t}\right)^{-(\varepsilon-\varepsilon)}.
$$

(5.54)

Case 2: $\frac{t}{2} - \frac{|x|}{4c} \le s \le \frac{t}{4}$. Since $|x| \le \frac{ct}{2}$, we know $s \ge \frac{t}{2} - \frac{|x|}{4c} > \frac{t}{4}$ and $\frac{t}{4} \le t - s \le t$. Thus,

$$
\int_{\frac{t}{4} - \frac{|x|}{4c}}^{\frac{t}{2}} \int_{\mathbb{R}^3} (1 + t - s)^{-\frac{5}{2}} e^{-\frac{(|y| - ct - s)^2}{(t - s)^2}} (1 + s)^{-(3-\varepsilon)} \left(1 + \frac{|y|^2}{1 + s}\right)^{-(2-\varepsilon)} dy ds
\le C(1 + t)^{-\frac{5}{2}} \left(1 + \frac{|x| - ct}{2c}\right)^{-(3-\varepsilon)} (1 + t)^2
\le CM^2(T)(1 + t)^{-(\varepsilon+\varepsilon)} \left(1 + \frac{|x| - ct}{4c}\right)^{-(3-2\varepsilon)} (1 + t)^2
\le C(1 + t)^{-\frac{5}{2}} \left(1 + \frac{|x| - ct}{1 + t}\right)^{-(\varepsilon+\varepsilon)} (1 + t)^{-(\varepsilon-\varepsilon)}
\le C(1 + t)^{-(\varepsilon-\varepsilon)} \left(1 + \frac{|x| - ct}{1 + t}\right)^{-(\varepsilon-\varepsilon)},
$$

(5.55)

where we have used the following estimate

$$
\int_0^t \int_{\mathbb{R}^3} e^{-\frac{(y - ct + s)^2}{(t - s)^2}} \left(1 + \frac{|y|^2}{1 + s}\right)^{-\alpha_2} dy ds \le C(1 + t)^2, \text{ for } \alpha_2 > \frac{3}{2}.
$$

Case 3: $\frac{t}{4} \le s \le t - \frac{|x|}{4c}$. Since $|x| \le \frac{ct}{2}$, we have

$$
\int_{\frac{t}{4}}^{\frac{t}{4} - \frac{|x|}{4c}} \int_{\mathbb{R}^3} (1 + t - s)^{-\frac{5}{2}} e^{-\frac{(|y| - ct - s)^2}{(t - s)^2}} (1 + s)^{-(3-\varepsilon)} \left(1 + \frac{|y|^2}{1 + s}\right)^{-(2-\varepsilon)} dy ds
\le C(1 + t)^{-\frac{5}{2}} \left(1 + \frac{|x|}{4c}\right)^{-(3-\varepsilon)} (1 + t)^2 \le CM^2(T)(1 + t)^{-(\varepsilon+\varepsilon)} (1 + t)^{-(1-2\varepsilon)} (1 + |x|)^{-2}
\le C(1 + t)^{-\frac{5}{2}} (1 + |x|)^{-(3-2\varepsilon)} \le C(1 + t)^{-\frac{5}{2}} \left(1 + \frac{|x|^2}{1 + t}\right)^{-\frac{5}{2}} (1 + t)^{-\frac{3}{2}}
\le C(1 + t)^{-\frac{5}{2}} \left(1 + \frac{|x|^2}{1 + t}\right)^{-(\varepsilon-\varepsilon)}.
$$

(5.56)
Case 4: $t - \frac{|x|}{2} \leq s \leq t$. When $|x - y| \geq \frac{|x|}{2}$, we have $|x - y| - c(t - s) \geq |x - y| - \frac{|x|}{2} \geq \frac{|x|}{4}$, then

$$
\int_t^{t - \frac{|x|}{4}} \int_{\mathbb{R}^3} (1 + t - s)^{-2} e^{-\frac{(x - y - c(t - s))^2}{4(t - s)}} (1 + s)^{-\frac{3}{2} - \epsilon} \left(1 + \frac{|y|^2}{1 + s}\right)^{-\frac{3}{2} - \epsilon} dy\, ds
\leq C(1 + t)^{-\frac{3}{2} - \epsilon} \left(1 + \frac{|x|^2}{1 + t}\right)^{-3} \int_t^{t - \frac{|x|}{4}} (1 + t - s)^{-2} (1 + s)^{-\frac{5}{4} + \epsilon} ds
\leq C(1 + t)^{-\frac{3}{2} - \epsilon} \left(1 + \frac{|x|^2}{1 + t}\right)^{-3} (1 + t)^{\frac{3}{4}} \left(1 + \frac{|x|^2}{1 + t}\right)^{\frac{3}{4}}
\leq C(1 + t)^{-\frac{3}{2} - \epsilon} \left(1 + \frac{|x|^2}{1 + t}\right)^{-\frac{3}{2} - \epsilon}.
$$

(5.57)

When $|x - y| < \frac{|x|}{2}$, we have $|y| \geq \frac{|x|}{2}$, which yields that

$$
\int_t^{t - \frac{|x|}{4}} \int_{\mathbb{R}^3} (1 + t - s)^{-2} e^{-\frac{(x - y - c(t - s))^2}{4(t - s)}} (1 + s)^{-\frac{3}{2} - \epsilon} \left(1 + \frac{|y|^2}{1 + s}\right)^{-\frac{3}{2} - \epsilon} dy\, ds
\leq C(1 + t)^{-\frac{3}{2} - \epsilon} \left(1 + \frac{|x|^2}{1 + t}\right)^{-3} \int_t^{t - \frac{|x|}{4}} (1 + t - s)^{-2} (1 + s)^{-\frac{5}{4} + \epsilon} ds
\leq C(1 + t)^{-\frac{3}{2} - \epsilon} \left(1 + \frac{|x|^2}{1 + t}\right)^{-3} (1 + t)^{\frac{3}{4}} \left(1 + \frac{|x|^2}{1 + t}\right)^{\frac{3}{4}}
\leq C(1 + t)^{-\frac{3}{2} - \epsilon} \left(1 + \frac{|x|^2}{1 + t}\right)^{-\frac{3}{2} - \epsilon}.
$$

(5.58)

Lastly, when $(x, t) \in D_5 : \frac{ct}{2} \leq |x| \leq ct - \sqrt{1 + t}$, we also divide the domain into four parts. The Case 1: $0 \leq s \leq \frac{ct}{2}$ and Case 2: $\frac{ct}{2} - \frac{|x|}{2c} \leq s \leq \frac{ct}{4}$ can be treated as those in $D_4$, we omit the details. For the Case 3: $\frac{ct}{4} \leq s \leq t - \frac{ct - |x|}{4c}$, we have

$$
\int_{\frac{ct}{4}}^{t - \frac{ct - |x|}{4c}} \int_{\mathbb{R}^3} (1 + t - s)^{-2} e^{-\frac{(x - y - c(t - s))^2}{4(t - s)}} (1 + s)^{-\frac{3}{2} - \epsilon} \left(1 + \frac{|y|^2}{1 + s}\right)^{-\frac{3}{2} - \epsilon} dy\, ds
\leq C(1 + ct - |x|)^{-2} (1 + t)^{-\frac{3}{2} - \epsilon} (1 + t)^2
\leq C(1 + t)^{-\frac{3}{2} - \epsilon} (1 + t)^{-\frac{3}{2} - \epsilon} (1 + ct - |x|)^{-2}
\leq C(1 + t)^{-\frac{3}{2} - \epsilon} (1 + ct - |x|)^{-\frac{3}{2} - \epsilon}
\leq C(1 + t)^{-\frac{3}{2} - \epsilon} \left(1 + \frac{(ct - |x|)^2}{1 + t}\right)^{-\frac{3}{2} - \epsilon} (1 + t)^{-\frac{3}{2} - \epsilon}
\leq C(1 + t)^{-\frac{3}{2} - \epsilon} \left(1 + \frac{|x|^2}{1 + t}\right)^{-\frac{3}{2} - \epsilon}.
$$

(5.59)
For the Case 4: \( t - \frac{ct - |x|}{4c} \leq s \leq t \), when \( |x - y| \geq \frac{ct - |x|}{2} \), we have \( |x - y| - c(t - s) \geq \frac{ct - |x|}{4} \). Then

\[
\int_{t - \frac{ct - |x|}{4c}}^{t} \int_{\mathbb{R}^3} (1 + t - s)^{-2} e^{-\frac{(|x - y| - c(t - s))^2}{c(t - s)}} (1 + s)^{-(\frac{3}{2} - \varepsilon)} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-(2 - \varepsilon)} \, dy \, ds \\
\leq C(1 + t)^{-\left(\frac{3}{2} - \varepsilon\right)} \left( 1 + \frac{(ct - |x|)^2}{1 + t} \right)^{-3} \int_{t - \frac{ct - |x|}{4c}}^{t} (1 + t - s)^{-2} (1 + t - s)^{\frac{3}{2}} ds \\
\leq C(1 + t)^{-\left(\frac{3}{2} - \varepsilon\right)} \left( 1 + \frac{(ct - |x|)^2}{1 + t} \right)^{-3} (1 + t)^{\frac{3}{2}} \left( 1 + \frac{(ct - |x|)^2}{1 + t} \right)^{\frac{3}{2}} (1 + t)^{\frac{3}{2}} \\
\leq C(1 + t)^{-\left(\frac{3}{2} - \varepsilon\right)} \left( 1 + \frac{(ct - |x|)^2}{1 + t} \right)^{-\frac{3}{2}}. \tag{5.60}
\]

When \( |x - y| < \frac{ct - |x|}{2} \), we have \( |y| \geq \frac{|x - ct|}{2} \), which yields that

\[
\int_{t - \frac{ct - |x|}{4c}}^{t} \int_{\mathbb{R}^3} (1 + t - s)^{-2} e^{-\frac{(|x - y| - c(t - s))^2}{c(t - s)}} (1 + s)^{-(\frac{3}{2} - \varepsilon)} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-(2 - \varepsilon)} \, dy \, ds \\
\leq C(1 + t)^{-\left(\frac{3}{2} - \varepsilon\right)} \left( 1 + \frac{(ct - |x|)^2}{1 + t} \right)^{-2} \int_{t - \frac{ct - |x|}{4c}}^{t} (1 + t - s)^{-2} (1 + t - s)^{\frac{3}{2}} ds \\
\leq C(1 + t)^{-\left(\frac{3}{2} - \varepsilon\right)} \left( 1 + \frac{(ct - |x|)^2}{1 + t} \right)^{-2} (1 + t) \left( 1 + \frac{|x| - ct)^2}{1 + t} \right)^{\frac{3}{2}} \\
\leq C(1 + t)^{-\left(\frac{3}{2} - \varepsilon\right)} \left( 1 + \frac{(ct - |x|)^2}{1 + t} \right)^{-\frac{3}{2}}. \tag{5.61}
\]

Hence, we complete the proof of this lemma. \( \square \)

Next, we consider the convolution between Riesz wave and the worst nonlinear term \( n_2 \nabla \phi \). We have the following lemma.

**Lemma 5.6.** We have

\[
\mathcal{N}_2 := \left| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^3} (1 + t - s)^{-\frac{3}{2}} \left( 1 + \frac{|x - y|^2}{1 + t - s} \right)^{-2} (n_2 \nabla \phi)(y, s) dy ds \right| \\
+ \left| \int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^3} (1 + t - s)^{-2} \left( 1 + \frac{|x - y|^2}{1 + t - s} \right)^{-2} (|\phi|)^2(y, s) dy ds \right| \\
\leq CM^2(T)(1 + t)^{-2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-(2 - \varepsilon)} \tag{5.62}
\]

*Proof.* We divide the domain into several parts.

Case 1: \( 0 \leq s \leq \frac{t}{2} \). When \( |x|^2 \leq 1 + t \), we have

\[
\left| \int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^3} (1 + t - s)^{-2} \left( 1 + \frac{|x - y|^2}{1 + t - s} \right)^{-2} (1 + s)^{-(3 - \varepsilon)} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-(2 - \varepsilon)} \, dy \, ds \right| \\
\leq C(1 + t)^{-2} \int_{0}^{\frac{t}{2}} (1 + s)^{-(3 - \varepsilon)} (1 + s)^{\frac{3}{2}} ds \\
\leq C(1 + t)^{-2} \leq C(1 + t)^{-2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-2}. \tag{5.63}
\]
When \( |x|^2 > 1 + t \) and \( |y| \geq \frac{|x|}{2} \), we have
\[
\left| \int_0^t \int_{\mathbb{R}^3} (1 + t - s)^{-2} \left( 1 + \frac{|x - y|^2}{1 + t - s} \right)^{-2} (1 + s)^{-(3-\varepsilon)} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-(2-\varepsilon)} \right. dyds \right| \\
\leq C(1 + t)^{-2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-2} (1 + t)^{-2} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-(2-\varepsilon)} \\
\leq C(1 + t)^{-2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-(2-\varepsilon)}. \tag{5.64}
\]

When \( |x|^2 > 1 + t \) and \( |y| < \frac{|x|}{2} \), we have \( |x - y| \geq \frac{|y|}{2} \) and
\[
\left| \int_0^t \int_{\mathbb{R}^3} (1 + t - s)^{-2} \left( 1 + \frac{|x - y|^2}{1 + t - s} \right)^{-2} (1 + s)^{-(3-\varepsilon)} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-(2-\varepsilon)} \right. dyds \right| \\
\leq C(1 + t)^{-2} \left( 1 + \frac{|y|^2}{1 + t} \right)^{-2} \int_{\frac{t}{2}}^t (1 + s)^{-(3-\varepsilon)} (1 + s)^{\frac{3}{2}} ds \\
\leq C(1 + t)^{-2} \left( 1 + \frac{|y|^2}{1 + t} \right)^{-(2-\varepsilon)}. \tag{5.65}
\]

Case 2: \( \frac{t}{2} < s \leq t \). When \( |x|^2 \leq 1 + t \), we have
\[
\left| \int_0^t \int_{\mathbb{R}^3} (1 + t - s)^{-2} \left( 1 + \frac{|x - y|^2}{1 + t - s} \right)^{-2} (1 + s)^{-(3-\varepsilon)} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-(2-\varepsilon)} \right. dyds \right| \\
\leq C(1 + t)^{-(3-\varepsilon)} \int_{\frac{t}{2}}^t (1 + t - s)^{-2} (1 + t - s)^{\frac{3}{2}} ds \\
\leq C(1 + t)^{-(\frac{7}{2}-\varepsilon)} \leq C(1 + t)^{-(\frac{7}{2}-\varepsilon)} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-2}. \tag{5.66}
\]

When \( |x|^2 > 1 + t \) and \( |y| \geq \frac{|x|}{2} \), we have
\[
\left| \int_{\frac{t}{2}}^t \int_{\mathbb{R}^3} (1 + t - s)^{-2} \left( 1 + \frac{|x - y|^2}{1 + t - s} \right)^{-2} (1 + s)^{-(3-\varepsilon)} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-(2-\varepsilon)} \right. dyds \right| \\
\leq C(1 + t)^{-(3-\varepsilon)} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-(2-\varepsilon)} \int_{\frac{t}{2}}^t (1 + t - s)^{-2} (1 + t - s)^{\frac{3}{2}} ds \\
\leq C(1 + t)^{-(\frac{7}{2}-\varepsilon)} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-(2-\varepsilon)}. \tag{5.67}
\]

When \( |x|^2 > 1 + t \) and \( |y| < \frac{|x|}{2} \), we have \( |x - y| \geq \frac{|x|}{2} \) and
\[
\left| \int_{\frac{t}{2}}^t \int_{\mathbb{R}^3} (1 + t - s)^{-2} \left( 1 + \frac{|x - y|^2}{1 + t - s} \right)^{-2} (1 + s)^{-(3-\varepsilon)} \left( 1 + \frac{|y|^2}{1 + s} \right)^{-(2-\varepsilon)} \right. dyds \right| \\
\leq C(1 + t)^{-2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-2} \int_{\frac{t}{2}}^t (1 + s)^{-(3-\varepsilon)} (1 + s)^{\frac{3}{2}} ds \\
\leq C(1 + t)^{-2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-2}. \tag{5.68}
\]

Thus, we complete the proof of this lemma. \( \square \)

As in Lemma 5.5 and 5.6, one can deal with the convolutions \( \int_0^t (G_{12} - G_{S_1} - G_{S_2})(\cdot, t - s) \ast \tilde{F}_1(\cdot, s)ds \) and \( \int_0^t (G_{22} - G_{S_2} - G_{S_2})(\cdot, t - s) \ast \tilde{F}_1(\cdot, s)ds \), where \( \tilde{F}_1 \) is the other terms in \( F_1 \) except the worst nonlinear term \( u_2 \nabla \phi \) by using Lemma 6.2-6.4.
On the other hand, we consider the convolution between the singular part in short wave $G_{S_1}$ and $F_1(n_1, w_1, n_2, w_2)(x, t)$. Since $G_{S_1}$ is singular when $t \to 0$, this singularity no longer exists when taking the convolution. Then these terms can be estimated as those for the convolution between $G - G_{S_1} - G_{S_2}$ and the nonlinear terms above.

Next, for the convolution between the singular part $G_{S_2}$ and $F_1(n_1, w_1, n_2, w_2)(x, t)$, we have

$$
\left| \int_0^t G_{S_2}(\cdot, t - s) * F_1(n_1, w_1, n_2, w_2)(\cdot, s) \, ds \right|
$$

$$
\leq CM^2(T) \int_0^t e^{-C_1(t-s)}(|\delta(\cdot)| + \hat{f}(\cdot)) \cdot \{ (1 + t)^{-3}(\psi_1 + \psi_2 + \psi_3 + \psi_4) + \psi_1\psi_4 + \psi_2\psi_3 + \cdots \} \, ds
$$

$$
\leq CM^2(T)(1 + t)^{-3}\left( 1 + \frac{|x|^2}{1 + t} \right)^{-\left(\frac{2}{3} - \epsilon\right)} + \left( 1 + \frac{(|x| - ct)^2}{1 + t} \right)^{-\left(\frac{2}{3} - \epsilon\right)},
$$

(5.69)

where we have used the ansatz (5.23), Lemma 5.3 and the fact

$$
F_1 = \mathcal{O}(1)\{D(w_1^2 + w_2^2 + n_1^2 + n_2^2 + n_2w_1^2 + n_2w_2 + n_2w_2^2) + n_2\nabla \phi
+ D^2(n_1w_1 + n_2w_2 + n_1n_2w_1 + n_1n_2w_2 + w_1n_2^2 + n_2^2w_2) \}.
$$

(5.70)

Up to now, from Lemma 5.5, 5.6, 6.2, 6.3, 6.4, and Proposition 5.2, we can immediately obtain

$$
|n_1(x, t)| \leq C(\varepsilon_0 + M^2(T))\psi_1,
$$

$$
|w_1(x, t)| \leq C(\varepsilon_0 + M^2(T))\psi_2.
$$

(5.71)

For the higher order derivative of $D^2_F(n_1, w_1, n_2, w_2)$ with $|\alpha| = 1, 2$, one can easily obtain in the same way. In fact, for the short time part, one can put derivatives of any order on the Green’s function to get the same estimate as $|\alpha| = 0$. For the long time part, the unique difference from the case $|\alpha| = 0$ is in the convolution between $G_{S_1}$ (or $G_{S_2}$) due to the singularity as $t \to 0$, which only allows us to put first order derivative on the Green’s function. Thus, when $|\alpha| = 2$, taking the nonlinear term $D^2(n_1w_1)$ in $F_1$ for example, the term $D^2(n_1w_1) = D^3w_1 + D_2w_1Dw_1 + Dw_1D^2w_1 + n_1D^3w_1$ can easily be estimated from the ansatz (5.23), Lemma 6.2-6.4. The convolution between $D^2_{F_2}(G(x, t) - G_{S_1} - G_{S_2})$ (or $D^2_{F_2}(G(x, t) - G_{S_1} - G_{S_2})$) and the nonlinear terms $F_1$ and $F_2$ can be treated similarly.

Lastly, for the convolution between $G_{S_2}$ (or $G_{S_2}$) and the nonlinear terms $D^2_F F_1$ and $D^2_{F_2} F_2$ with $|\alpha| = 1, 2$, we have to face the third-order and the fourth-order derivatives of the solution $D^3(n_1, n_2, w_2)$ and $D^4(n_1, n_2, w_2)$, which is lack of the pointwise estimates for these terms due to the ansatz (5.23). Hence, we should estimate as follows. In fact,

$$
D^2 F_1 = \mathcal{O}(1)\{D^3w_1w_1 + D^3w_2w_2 + D^3n_1n_1 + D^3n_2n_2 + D^4n_1n_1 + D^4n_1n_1 + D^4n_2w_2 + D^4w_2n_2 + \cdots \},
$$

$$
D^2 F_2 = \mathcal{O}(1)\{D^3w_1w_2 + D^3w_2w_1 + D^3n_1n_2 + D^3n_2n_1 + D^4n_1n_2 + D^4n_1w_2 + D^4w_2n_1 + \cdots \},
$$

(5.72)

where “…” denotes the rest terms. We take the term above $D^3w_1w_1$ for example and have

$$
|D^3w_1w_1| \leq C(1 + t)^{-3}\left( 1 + \frac{|x|^2}{1 + t} \right)^{-\left(\frac{2}{3} - \epsilon\right)} + (1 + t)^{-2}\left( 1 + \frac{(|x| - ct)^2}{1 + t} \right)^{-\left(\frac{2}{3} - \epsilon\right)}
$$

$$
\leq C(1 + t)^{-\frac{2}{3}}\left( 1 + \frac{|x|^2}{1 + t} \right)^{-\left(\frac{2}{3} - \epsilon\right)} + C(1 + t)^{-5}\left( 1 + \frac{|x|^2}{1 + t} \right)^{-\left(\frac{2}{3} - \epsilon\right)}(1 + t)^{3-2\epsilon}
$$

$$
\leq C(1 + t)^{-2}\left( 1 + \frac{|x|^2}{1 + t} \right)^{-\left(\frac{2}{3} - \epsilon\right)},
$$

(5.73)
where we have used the inequality (5.34) again. Then, from Lemma 5.3, (5.72) and (5.73), we know when 1 ≤ |α| ≤ 2 that

\[
\int_0^t G_{S_2}(t') \ast D^a_x (F_1, F_2) \, ds + \int_0^t G_{S_2}(t') \ast D^a_x (F_1, F_2) \, ds \\
\leq C(1 + t)^{-2} \left(1 + \frac{|x|^2}{1 + t}\right)^{-(\frac{3}{2} - \varepsilon)} .
\]  

As a result, we have for 1 ≤ |α| ≤ 2 that

\[
|D^a_x n_1(x, t)| \leq C(\varepsilon_0 + M^2(T))\psi_1, \\
|D^a_x w_1(x, t)| \leq C(\varepsilon_0 + M^2(T))\psi_2, \\
|D^a_x n_2(x, t)| \leq C(\varepsilon_0 + M^2(T))\psi_3, \\
|D^a_x w_2(x, t)| \leq C(\varepsilon_0 + M^2(T))\psi_6.
\]  

In summary, from (5.38), (5.71) and (5.75), we can conclude that

\[ M(T) \leq C(\varepsilon_0 + M^2(T)), \]

which together with the smallness of \( \varepsilon_0 \) and the continuity of \( M(T) \) implies

\[ M(T) \leq C\varepsilon_0. \]

This closes the ansatz (5.23) and proves Theorem 1.2.

6. Appendix.

**Lemma 6.1.** [33] (1) When \( \tau \in [0, t] \) and \( a^2 \geq 1 + t \), for any positive constant \( l \), we have

\[
\left(1 + \frac{a^2}{1 + \tau}\right)^{-l} \leq 3^l \left(1 + \frac{a^2}{1 + t}\right)^{-l} .
\]

(2) When \( a^2 \leq 1 + t \), we have

\[
1 \leq 2^l \left(1 + \frac{a^2}{1 + t}\right)^{-l} .
\]

The following three lemmas are almost from [18, 19]. The unique difference in the present paper is the presence of the factor \( \varepsilon \) in these lemmas. In fact, the factor \( \varepsilon \) can be negligible in the proof. For simplicity, we omit the details. And \( t_0 \) below is defined as in [18]

\[ t_0 = \max \left\{ \frac{t}{2}, t - \frac{\sqrt{1 + t}}{4} \right\} \text{ in } D_1 \cup D_2 \cup D_3; \quad t_0 = t - \min \left\{ \frac{ct - |x|}{4c}, \frac{|x| - ct}{4c} \right\} \text{ in } D_4 \cup D_5. \]

**Lemma 6.2.** For any \( \gamma \geq 0 \), we have

\[
N_1^3 = \int_0^{t_0} \int_{\mathbb{R}^3} (t - s)^{-\frac{4 + \gamma}{2}} e^{-\frac{|x|^2}{\varepsilon(t - s)}} (1 + s)^{-3} \left(1 + \frac{|y|^2}{1 + s}\right)^{-(\frac{3}{2} - \varepsilon)} dyds \\
\leq C(1 + t)^{-\min\left\{\frac{4 + \gamma}{2}, \frac{3 + \gamma}{2}\right\}} \left(1 + \frac{x^2}{1 + t}\right)^{-(\frac{3}{2} - \varepsilon)} ,
\]

\[
N_2^3 = \int_0^{t_0} \int_{\mathbb{R}^3} (t - s)^{-\frac{4 + \gamma}{2}} (t - s)^{-\frac{1}{2}} e^{\frac{(l - \varepsilon s - \varepsilon^2 t^2)}{2(1 + s)}} (1 + s)^{-4} \left(1 + \frac{(|y| - cs)^2}{1 + s}\right)^{-(3 - 2\varepsilon)} dyds \\
\leq C(1 + t)^{-\min\left\{\frac{4 + \gamma}{2}, 3\right\}} \left(1 + \frac{x^2}{1 + t}\right)^{-(\frac{3}{2} - \varepsilon)} + \left(1 + \frac{|x|^2}{1 + t}\right)^{-(\frac{3}{2} - \varepsilon)},
\]

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\[ N_3 = \int_0^t \int_{\mathbb{R}^3} (t-s)^{-\frac{4+\gamma}{2}} e^{-\frac{(x-y-ct)(t-s)}{ct}} (1+s)^{-3} \left(1 + \frac{|y|^2}{1+s}\right)^{-(3-2\varepsilon)} dyds \]
\[ \leq C(1+t)^{-\frac{4+\gamma}{2}} \left(1 + \frac{x^2}{1+t}\right)^{-(\frac{3}{2}-\varepsilon)} + \left(1 + \frac{|x| - ct}{1+t}\right)^{-(\frac{3}{2}-\varepsilon)} \), \]

\[ N_4 = \int_0^t \int_{\mathbb{R}^3} (t-s)^{-\frac{4+\gamma}{2}} e^{-\frac{(x-y)^2}{ct}} (1+s)^{-4} \left(1 + \frac{|y-cs|^2}{1+s}\right)^{-(3-2\varepsilon)} dyds \]
\[ \leq C(1+t)^{-\frac{4+\gamma}{2}} \left(1 + \frac{x^2}{1+t}\right)^{-(\frac{3}{2}-\varepsilon)} + \left(1 + \frac{|x| - ct}{1+t}\right)^{-(\frac{3}{2}-\varepsilon)} \). \]

**Lemma 6.3.** For any \( \gamma \geq 0 \), we have

\[ N_5^3 = \int_{t_0}^t \int_{\mathbb{R}^3} (t-s)^{-2} e^{-\frac{(x-y-ct)(t-s)}{ct}} (1+s)^{-\frac{4+\gamma}{2}} \left(1 + \frac{|y|^2}{1+s}\right)^{-(\frac{3}{2}-\varepsilon)} dyds \]
\[ \leq C(1+t)^{-\frac{4+\gamma}{2}} \left(1 + \frac{x^2}{1+t}\right)^{-(\frac{3}{2}-\varepsilon)} \), \]

\[ N_6^3 = \int_{t_0}^t \int_{\mathbb{R}^3} (t-s)^{-2} e^{-\frac{(x-y-ct)(t-s)}{ct}} (1+s)^{-\frac{4+\gamma}{2}} \left(1 + \frac{|y-cs|^2}{1+s}\right)^{-(\frac{3}{2}-\varepsilon)} dyds \]
\[ \leq C\left((1+t)^{-\frac{4+\gamma}{2}} \left(1 + \frac{x^2}{1+t}\right)^{-(\frac{3}{2}-\varepsilon)} + (1+t)^{-\frac{4+\gamma}{2}} \left(1 + \frac{|x| - ct}{1+t}\right)^{-(\frac{3}{2}-\varepsilon)} \right), \]

\[ N_7^3 = \int_{t_0}^t \int_{\mathbb{R}^3} (t-s)^{-2} e^{-\frac{(x-y-ct)(t-s)}{ct}} (1+s)^{-\frac{4+\gamma}{2}} \left(1 + \frac{|y|^2}{1+s}\right)^{-(\frac{3}{2}-\varepsilon)} dyds \]
\[ \leq C \left((1+t)^{-\frac{4+\gamma}{2}} \left(1 + \frac{x^2}{1+t}\right)^{-(\frac{3}{2}-\varepsilon)} + (1+t)^{-\frac{4+\gamma}{2}} \left(1 + \frac{|x| - ct}{1+t}\right)^{-(\frac{3}{2}-\varepsilon)} \right), \]

**Lemma 6.4.** For any \( \gamma \geq 0 \), we have

\[ N_8^3 = \int_{t_0}^t \int_{\mathbb{R}^3} (t-s)^{-\frac{4+\gamma}{2}} \chi_{|x-y| \leq c(t-s)} \left(1 + \frac{|x-y|^2}{t-s}\right)^{-2} (1+s)^{-3} \left(1 + \frac{|y|^2}{1+s}\right)^{-(3-2\varepsilon)} dyds \]
\[ \leq C(1+t)^{-\frac{4+\gamma}{2}} \left(1 + \frac{x^2}{1+t}\right)^{-(\frac{3}{2}-\varepsilon)} \), \]

\[ N_9^3 = \int_{t_0}^t \int_{\mathbb{R}^3} (t-s)^{-\frac{4+\gamma}{2}} \chi_{|x-y| \leq c(t-s)} \left(1 + \frac{|x-y|^2}{t-s}\right)^{-2} (1+s)^{-\frac{4+\gamma}{2}} \left(1 + \frac{|y|^2}{1+s}\right)^{-(3-2\varepsilon)} dyds \]
\[ \leq C(1+t)^{-\frac{4+\gamma}{2}} \left(1 + \frac{x^2}{1+t}\right)^{-(\frac{3}{2}-\varepsilon)} \), \]
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