A note on state space representations of locally stationary wavelet time series

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Abstract

In this note we show that the locally stationary wavelet process can be decomposed into a sum of signals, each of which following a moving average process with time-varying parameters. We then show that such moving average processes are equivalent to state space models with stochastic design components. Using a simple simulation step, we propose a heuristic method of estimating the above state space models and then we apply the methodology to foreign exchange rates data.

Some key words: wavelets, Haar, locally stationary process, time series, state space, Kalman filter.

1 Introduction

Nason et al. (2000) define a class of locally stationary time series making use of non-decimated wavelets. Let \{y_t\} be a scalar time series, which is assumed to be locally stationary, or stationary over certain intervals of time (regimes), but overall non-stationary. For more details on local stationarity the reader is referred to Dahlhaus (1997), Nason et al. (2000), Francq and Zakoan (2001), and Mercurio and Spokoiny (2004). For example, Figure 1 shows the nonstationary process considered in Nason et al. (2000), which is the concatenation of 4 stationary moving average processes, but each with different parameters. We can see that within each of the four regimes, the process is weakly stationary, but overall the process is non-stationary.

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The locally stationary wavelet (LSW) process is a doubly indexed stochastic process, defined by

\[ y_t = \sum_{j=-J}^{-1} \sum_{k=0}^{T-1} w_{jk} \psi_{j,t-k} \xi_{jk}, \]  

(1)

where \( \xi_{jk} \) is a random orthonormal increment sequence (below this will be iid Gaussian) and \( \{\psi_{jk}\}_{j,k} \) is a discrete non-decimated family of wavelets for \( j = -1, -2, \ldots, -J, \ k = 0, \ldots, T - 1 \), based on a mother wavelet \( \psi(t) \) of compact support. Denote with \( I_A(x) \) the indicator function, i.e. \( I_A(x) = 1 \), if \( x \in A \) and \( I_A = 0 \), otherwise. The simplest class of wavelets are the Haar wavelets, defined by

\[ \psi_{jk} = 2^{j/2} I_{\{0, \ldots, 2^{-j-1}-1\}}(k) - 2^{j/2} I_{\{2^{-j-1}, \ldots, 2^{-j}-1\}}(k), \]

for \( j \in \{-1, -2, \ldots, -J\} \) and \( k \in \{\ldots, -2, -1, 0, 1, 2, \ldots\} \), where \( j = -1 \) is the finest scale. It is also assumed that \( E(\xi_{jk}) = 0 \), for all \( j \) and \( k \) and so \( y_t \) has zero mean. The orthonormality assumption of \( \{\xi_{jk}\} \) implies that \( \text{Cov}(\xi_{jk}, \xi_{\ell m}) = \delta_{j\ell} \delta_{km} \), where \( \delta_{jk} \) denotes the Kronecker delta, i.e. \( \delta_{jj} = 1 \) and \( \delta_{jk} = 0 \), for \( j \neq k \).

Figure 1: Concatenation of four MA time series with different parameters. Overall the process is not stationary. The dotted vertical lines indicate the transition between one MA process and the next.
The parameters $w_{jk}$ are the amplitudes of the LSW process. The quantity $w_{jk}$ characterizes the amount of each oscillation, $\psi_{j,t-k}$ at each scale, $j$, and location, $k$ (modified by the random amplitude, $\xi_{jk}$). For example, a large value of $w_{jk}$ indicates that there is a chance (depending on $\xi_{jk}$) of an oscillation, $\psi_{j,t-k}$, at time $t$. Nason et al. (2000) control the evolution of the statistical characteristics of $y_t$ by coupling $w_{jk}$ to a function $W_j(z)$ for $z \in (0,1)$ by $w_{jk} = W_j(k/T) + O(T^{-1})$. Then, the smoothness properties of $W_j(z)$ control the possible rate of change of $w_{jk}$ as a function of $k$, which consequently controls the evolution of the statistical properties of $y_t$. The smoother $W_j(z)$ is, as a function of $z$, the slower that $y_t$ can evolve. Ultimately, if $W_j(z)$ is a constant function of $z$, then $y_t$ is weakly stationary.

The non-stationarity in the above studies is better understood as local-stationarity so that the $w_{jk}$’s are close to each other. To elaborate on this, if $w_{jk} = w_j$ (time invariant), then $y_t$ would be weakly stationary. The attractiveness of the LSW process, is its ability to consider time-changing $w_{jk}$’s.

Nason et al. (2000) define the evolutionary wavelet spectrum (EWS) to be $S_j(z) = |W_j(z)|^2$ and discuss methods of estimation. Fryzlewicz et al. (2003) and Fryzlewicz (2005) modify the LSW process to forecast log-returns of non-stationary time series. These authors analyze daily FTSE 100 time series using the LSW toolbox. Fryzlewicz and Nason (2006) estimate the EWS by using a fast Haar-Fisz algorithm. Van Bellegem and von Sachs (2008) consider adaptive estimation for the EWS and permit jump discontinuities in the spectrum.

In this paper we show that the process $y_t$ can be decomposed into a sum of signals, each of which follows a moving average process with time-varying parameters. We deploy a heuristic approach for the estimation of the above moving average process and an example, consisting of foreign exchange rates, illustrates the proposed methodology.

## 2 Decomposition at scale $j$

The LSW process (1) can be written as

$$y_t = \sum_{j=-J}^{-1} x_{jt},$$

(2)

where

$$x_{jt} = \sum_{k=0}^{T-1} w_{jk} \psi_{j,t-k} \xi_{jk}.$$  

(3)

For computational simplicity and without loss in generality, we omit the minus sign of the scales ($-J, \ldots, -1$) so that the summation in equation (2) is done from $j = 1$ (scale $-1$) until $j = J$ (scale $-J$).
Using Haar wavelets, we can see that at scale 1, we have from (3) that

\[ x_{1t} = \psi_{1,0} w_{1t} \xi_{1t} + \psi_{1,-1} w_{1,t-1} \xi_{1,t-1}, \]

since there are only 2 non-zero wavelet coefficients. Then we can re-write (3) as

\[ x_{1t} = \alpha^{(0)}_{1t} \xi_{1t} + \alpha^{(1)}_{1t} \xi_{1,t-1}, \]

which is a moving average process of order one, with time-varying parameters \( \alpha^{(0)}_{1t} \) and \( \alpha^{(1)}_{1t} \). This process can be referred to as TVMA(1) process.

In a similar way, for any scale \( j = 1, \ldots, J \), we can write

\[ x_{jt} = \psi_{j,0} w_{jt} \xi_{jt} + \psi_{j,-1} w_{j,t-1} \xi_{j,t-1} + \cdots + \psi_{j,-2^{j-1}} w_{j,t-2^{j-1}+1} \xi_{j,t-2^{j-1}+1} \]

so that we obtain the TVMA\((2^j - 1)\) process

\[ x_{jt} = \alpha^{(0)}_{jt} \xi_{jt} + \alpha^{(1)}_{jt} \xi_{j,t-1} + \cdots + \alpha^{(2^{j-1}-1)}_{jt} \xi_{j,t-2^{j-1}+1}, \]

(4)

where \( \alpha^{(0)}_{jt} = \psi_{j,-\ell} w_{jt-\ell} \), for all \( \ell = 0, 1, \ldots, 2^j - 1 \) and \( j = 1, \ldots, J \). Thus the process \( y_t \) is the sum of \( J \) TVMA processes. However, we note that not all time-varying parameters \( \alpha^{(\ell)}_{jt} \) \( (\ell = 0, 1, \ldots, 2^j - 1) \) are independent, since, for a fixed \( j \), they are all functions of the \( \{ w_{jt} \} \) series.

We advocate that \( w_{jt} \) is a signal and as such we treat it as an unobserved stochastic process. Indeed, from the slow evolution of \( w_{jt} \), we can postulate that \( w_{jt} - w_{j,t-1} \approx 0 \), which motivates a random walk evolution for \( w_{jt} \) or \( w_{jt} = w_{j,t-1} + \zeta_{jt} \), where \( \zeta_{jt} \) is a Gaussian white noise, i.e. \( \zeta_{jt} \sim N(0, \sigma_j^2) \), for \( \sigma_j^2 \) a known variance, and \( \zeta_{jt} \) is independent of \( \zeta_{kt} \), for all \( j \neq k \).

The magnitude of the differences between \( w_{j,t-1} \) and \( w_{jt} \) can be controlled by \( \sigma_j^2 \) and this controls on the degree of evolution of \( w_{jt} \) as a function of \( t \) and hence on \( y_t \) through (2).

At scale 1 we can write \( x_{1t} \) as

\[ x_{1t} = \psi_{1,0} w_{1t} \xi_{1t} + \psi_{1,-1} w_{1,t-1} \xi_{1,t-1} = (\psi_{1,0} \xi_{1t} + \psi_{1,-1} \xi_{1,t-1}) w_{1,t-1} + \psi_{1,0} \xi_{1t} \xi_{1t}, \]

where we have used \( w_{1t} = w_{1,t-1} + \zeta_{1t} \). Likewise at scale 2 we have

\[
\begin{align*}
x_{2t} &= \psi_{2,0} w_{2t} \xi_{2t} + \psi_{2,-1} w_{2,t-1} \xi_{2,t-1} + \psi_{2,-2} w_{2,t-2} \xi_{2,t-2} + \psi_{2,-3} w_{2,t-3} \xi_{2,t-3} \\
&= (\psi_{2,0} \xi_{2t} + \psi_{2,-1} \xi_{2,t-1} + \psi_{2,-2} \xi_{2,t-2} + \psi_{2,-3} \xi_{2,t-3}) w_{2,t-3} \\
&\quad + \psi_{2,0} \xi_{2,t-2} \xi_{2t} + \psi_{2,0} \xi_{2,t-1} \xi_{2t} + \psi_{2,0} \xi_{2t} \xi_{2t} \\
&\quad + \psi_{2,-1} \xi_{2,t-2} \xi_{2,t-1} + \psi_{2,-1} \xi_{2,t-1} \xi_{2,t-1} \\
&\quad + \psi_{2,-2} \xi_{2,t-1} \xi_{2,t-2},
\end{align*}
\]

where we have used \( w_{2,t-2} = w_{2,t-3} + \zeta_{2,t-2}, w_{2,t-1} = w_{2,t-3} + \zeta_{2,t-2} + \zeta_{2,t-1} \) and \( w_{2t} = w_{2,t-3} + \zeta_{2,t-2} + \zeta_{2,t-1} + \zeta_{2t} \).

In general we observe that at any scale \( j = 1, \ldots, J \) we can write

\[ x_{jt} = \sum_{k=0}^{2^j-1} \psi_{j,-k} \xi_{jt-k} w_{jt-2^j+1} + \sum_{k=0}^{2^j-2} \sum_{m=k}^{2^j-2} \psi_{j,-k} \xi_{jt-k} \zeta_{jt,t-m}, \quad t = 2^j, 2^j + 1, \ldots, \]

(5)
where the \( w_{jt} \)'s follow the random walk
\[
w_{jt,2i+1} = w_{jt,2i} + \zeta_{jt,2i+1}, \quad \zeta_{jt,2i+1} \sim N(0, \sigma_j^2).
\] (6)

### 3 A state space representation

For estimation purposes one could use a time-varying moving average model in order to estimate \( \{w_{jk}\} \) in (4). Moving average processes with time-varying parameters are useful models for locally stationary time series data, but their estimation is more difficult than that of time-varying autoregressive processes (Hallin, 1986; Dahlhaus, 1997). The reason for this is that the time-dependence of the moving average coefficients may result in identifiability problems. The consensus is that some restrictions of the parameter space of the time-varying coefficients should be applied; for more details the reader is referred to the above references as well as to Triantafyllopoulos and Nason (2007).

In this section we use a heuristic approach for the estimation of the above models. First we recast model (5)-(6) into state space form. To end this we write
\[
x_{jt} = A_{jt}w_{jt,2i+1} + \nu_{jt},
\] (7)
where
\[
A_{jt} = \sum_{k=0}^{2j-1} \psi_{j,-k} \xi_{jt,-k} \quad \text{and} \quad \nu_{jt} = \sum_{k=0}^{2j-2} \sum_{m=k}^{2j-2} \psi_{j,-k} \xi_{jt,-k} \zeta_{jt,-m}, \text{ for } t = 2^j, 2^j+1, \ldots
\]

In addition we assume that \( \xi_{jt}^i \) is independent of \( \xi_{jt}^s \), for \( i = 1, 2 \) and for any \( t, s \), so that
\[
\nu_{jt} \sim N \left\{ 0, \sigma_j^2 \sum_{k=0}^{2j-1} \psi_{j,-k}^2 (2^j - k - 1) \right\}.
\] (8)

Equations (7), (6), (8) define a state space model for \( x_{jt} \) and by defining \( A_t = (A_{1t}, \ldots, A_{Jt})' \) and by noting that \( \nu_{jt} \) is independent of \( \nu_{kt} \), for any \( j \neq k \), we obtain by (2) a state space model for \( y_t \), which essentially is the superposition of \( J \) state space models of the form of (7), (6), (8), each being a state space model for each scale \( j = 1, \ldots, J \).

Given a set of data \( y^T = \{y_1, \ldots, y_T\} \), a heuristic way to estimate \( \{w_{jt}\} \), is to simulate independently all \( \xi_{jt} \) from \( N(0, 1) \), thus to obtain simulated values for \( A_{jt} \) and then, conditional on \( A_t \), to apply the Kalman filter to the state space model for \( y_t \). This procedure will give simulations from the posterior distributions of \( w_{jt} \) and also from the predictive distributions of \( y_{t+h}|y^t \). The estimator of \( w_{jt} \) and the forecast of \( y_{t+h} \) are conditional on the simulated values of \( \{\xi_{jt}\} \). For competing simulated sequences \( \{\xi_{jt}\} \) the performance of the above estimators/forecasts can be judged by comparing the respective likelihood functions (which are easily calculable by the Kalman filter) or by comparing the respective posterior and forecast densities (by using sequential Bayes factors). Another means of model performance may be the computation of the mean square forecast error.
Figure 2: Simulated values of posterior estimates of $\{ S_{jt} = w_{jt}^2 \}$, for $\{ y_{1t} \}$ (GBP rate). Shown are simulations of $\{ S_{1t} \}$ and $\{ S_{2t} \}$, corresponding to scales 1 and 2.

We illustrate this approach by considering foreign exchange rates data. The data are collected in daily frequency from 3 January 2006 to and including 31 December 2007 (considering trading days there are 501 observations). We consider two exchange rates: US dollar with British pound (GBP rate) and US dollar with Euro (EUR rate). After we transform the data to the log scale, we propose to use the LSW process in order to obtain estimates of the spectrum process $\{ S_{jt} = w_{jt}^2 \}$, for each scale $j$. We form the vector $y_t = (y_{1t}, y_{2t})'$, where $y_{1t}$ is the log-return value of GBP and $y_{2t}$ is the log-return value of EUR. For each series $\{ y_{1t} \}$ and $\{ y_{2t} \}$, respectively, Figures 2 and 3 show simulations of the posterior spectrum $\{ S_{jt} \}$, for scales 1 and 2. The smoothed estimates of these figures are achieved by first computing the smoothed estimates using the Kalman filter and then applying a standard Spline method (Green and Silverman, 1994). We note that, for the data set considered in this paper, the estimates of Figures 2 and 3 are less smooth than those produced by the method of Nason et al. (2000). However, a higher degree of smoothness in our estimates can be achieved by considering small values of the variance $\sigma_j^2$, which controls the smoothness of the shocks in the random walk of the $w$'s.
Figure 3: Simulated values of posterior estimates of \( \{S_{jt} = w_{jt}^2\} \), for \( \{y_{2t}\} \) (EUR rate). Shown are simulations of \( \{S_{1t}\} \) and \( \{S_{2t}\} \), corresponding to scales 1 and 2.

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