Fine gradings of complex simple Lie algebras and Finite Root Systems

Gang Han
Department of Mathematics
Zhejiang University
China

Kang Lu
Department of Mathematics
Zhejiang University
China

Jun Yu
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139, USA

Abstract.
A $G$-grading on a complex semisimple Lie algebra $L$, where $G$ is a finite abelian group, is called quasi-good if each homogeneous component is 1-dimensional and 0 is not in the support of the grading.

Analogous to classical root systems, we define a finite root system $R$ to be some subset of a finite symplectic abelian group satisfying certain axioms. There always corresponds to $R$ a semisimple Lie algebra $L(R)$ together with a quasi-good grading on it. Thus one can construct nice basis of $L(R)$ by means of finite root systems.

We classify finite maximal abelian subgroups $T$ in Aut($L$) for complex simple Lie algebras $L$ such that the grading induced by the action of $T$ on $L$ is quasi-good, and show that the set of roots of $T$ in $L$ is always a finite root system. There are five series of such finite maximal abelian subgroups, which occur only if $L$ is a classical simple Lie algebra.

Contents

1 Introduction 2
2 Twisted group algebras and symplectic abelian groups 6

*Corresponding author. Work is supported by Zhejiang Province Science Foundation, grant No. LY14A010018.
1 Introduction

1.1. We will first briefly review gradings on a (not necessarily associative) algebra by abelian groups and mainly focus on gradings on a Lie algebra. All the algebras in the paper are assumed to be finite dimensional and over the field \( \mathbb{C} \) of complex numbers, although all of our main results can be generalized to any algebraically closed field of characteristic 0.

Let \( A \) be a finite dimensional algebra and \( G \) be a finitely generated additive abelian group. A \( G \)-grading \( \Gamma \) on \( A \) is the decomposition of \( A \) into a direct sum of subspaces

\[
\Gamma: A = \bigoplus_{g \in G} A_g
\]

such that

\[
A_g \cdot A_h \subset A_{g+h}, \quad \forall \ g, h \in G.
\]

We also say that \( A \) is \( G \)-graded. If \( A \) is a Lie algebra then the multiplication is understood to be the Lie bracket. The subset \( R = \{ g \in G | A_g \neq 0 \} \) of \( G \) is called the support of this grading and is denoted \( \text{Supp} \ \Gamma \). For any \( g \in R \), \( A_g \) is called the homogeneous component of degree \( g \) and each element in \( A_g \) is a homogeneous element. We will always assume that \( G \) is generated by \( R \), otherwise it could be replaced by its subgroup generated by \( R \). So \( G \) is always a finitely generated abelian group. Denote this grading by \( (\Gamma, G) \), or simply by \( \Gamma \). All the gradings mentioned in the paper will be some \( G \)-grading with \( G \) a (finitely generated) additive abelian group.

Let \( (\Gamma_i, G_i) \) be two gradings on \( A \) for \( i = 1, 2 \). Assume \( \phi \in \text{Aut}(A) \). If there is a group isomorphism \( \psi: G_1 \to G_2 \) such that \( \psi(\text{Supp} \ \Gamma_1) = \text{Supp} \ \Gamma_2 \) and for any \( a \in \text{Supp} \ \Gamma_1 \), \( \phi(A_a) = A_{\psi(a)} \), then \( \phi \) is called a grading homomorphism from \( (\Gamma_1, G_1) \) to \( (\Gamma_2, G_2) \) and denoted \( \phi: (\Gamma_1, G_1) \to (\Gamma_2, G_2) \). The grading homomorphism \( \phi: (\Gamma_1, G_1) \to (\Gamma_2, G_2) \) is an isomorphism of
gradings (which is called group equivalence in [K]) if $\phi^{-1}$ is also a grading homomorphism from $(\Gamma_2, G_2)$ to $(\Gamma_1, G_1)$. In this case the corresponding group homomorphism is an isomorphism.

Now let $A$ be a Lie algebra and denote it by $L$. Let $\text{Aut}(L)$ (resp. $\text{Int}(L)$) be respectively the group of automorphisms (resp. inner automorphisms) of $L$. Let the group $K$ be either $\text{Aut}(L)$ or $\text{Int}(L)$. Let $\Gamma$ be a $G$-grading on $L$. The group $\text{Aut}_K(\Gamma)$ consists of all the automorphisms in $K$ that permute the homogeneous components of $\Gamma$. The group $\text{Stab}_K(\Gamma)$ consists of all the automorphisms in $K$ that stabilize each homogeneous component of $\Gamma$. Clearly $\text{Stab}_K(\Gamma)$ is a normal subgroup of $\text{Aut}_K(\Gamma)$, and let

$$W_K(\Gamma) = \frac{\text{Aut}_K(\Gamma)}{\text{Stab}_K(\Gamma)}$$

be the Weyl group of the grading $\Gamma$ with respect to $K$, which describes the symmetry of $\Gamma$. Note that the Weyl group of $\Gamma$ defined in [EK] is with respect to $K = \text{Aut}(L)$.

People have done a lot of work in the area of abelian group gradings on Lie algebras (and on other types of algebras) and find many important applications for them. A good survey for it can be found in [K]. The classical Cartan decomposition of a semisimple Lie algebra $L$ is in fact a grading on $L$ by a free abelian group of the same rank as that of $L$. Alekseevskii found a class of interesting gradings on simple Lie algebras and computed the corresponding Weyl groups in [A]. Patera and his collaborators began to study abelian group gradings on Lie algebras systematically in [PZ] and [HPP], and they classified most of the fine gradings on classical simple Lie algebras. Later in [BSZ] Bahturin and his collaborators described all the abelian group gradings on classical simple Lie algebras except the simple Lie algebra of type $D_4$ (and on a class of simple Jordan algebras as well). In [BK] Bahturin and Kochetov classified the isomorphism classes of abelian group gradings on all the simple Lie algebras except the simple Lie algebra of type $D_4$ in terms of numerical and group-theoretical invariants. Draper and his collaborators classified all the fine gradings on the simple Lie algebra of type $D_4$ in [DV], classified all the gradings on the simple Lie algebra of type $G_2$ in [DM1] and classified all the fine gradings on the simple Lie algebra of type $F_4$ in [DM2]. In [EK] Elduque and Kochetov computed the Weyl groups for all the fine gradings on all the classical simple Lie algebras except $D_4$.

1.2. Now we recall the duality between abelian group gradings and abelian group actions.
An abelian algebraic group consisting of semisimple elements is called a quasitorus. Assume \( T \) is a complex quasitorus. Let \( \hat{T} \) be the set of algebraic group homomorphisms from \( T \) to \( \mathbb{C}^\times \). Then \( \hat{T} \) is an abelian group, called the character group or dual group of \( T \) with addition defined by

\[
(\alpha + \beta)(a) = \alpha(a) \cdot \beta(a), \text{ for any } \alpha, \beta \in \hat{T}; \text{ for any } a \in T.
\]

The group \( \text{Aut}(A) \) is a linear algebraic group. Assume \( T \subset \text{Aut}(A) \) is a quasitorus. Then the action of \( T \) on \( A \) induces a \( \hat{T} \)-grading on \( A \):

\[
\Gamma : A = \bigoplus_{g \in \hat{T}} A_g. 
\tag{1.1}
\]

Each element \( g \) in the support of this grading is called a root of \( T \) in \( A \), and \( A_g \) is called the root space of \( g \). If \( T' \) is another quasitorus in \( \text{Aut}(A) \), then \( T \) and \( T' \) induces isomorphic group gradings on \( A \) if and only if \( T \) and \( T' \) are conjugate in \( \text{Aut}(A) \).

Conversely given a grading \( A = \bigoplus_{g \in G} A_g \) with \( G \) a finitely generated abelian group, let \( \hat{G} \) be the dual group of \( G \), which is a (multiplicative) abelian group consisting of homomorphisms from \( G \) to \( \mathbb{C}^\times \). Then the grading induces a \( \hat{G} \)-action on \( A \):

\[
\sigma \cdot X = \sigma(g)X, \text{ for all } \sigma \in G, X \in A_g. 
\tag{1.2}
\]

This action gives a homomorphism \( \hat{G} \to \text{Aut}(A) \), which is injective as the support of the grading generates \( G \). So the homomorphism embeds \( \hat{G} \) as a quasitorus in \( \text{Aut}(A) \) and we will identify \( \hat{G} \) with its image in \( \text{Aut}(A) \).

Assume that \( A \) is a semisimple Lie algebra and denote it by \( L \). Assume \( T \subset \text{Aut}(L) \) is a quasitorus. Then the action of \( T \) on \( L \) induces a \( \hat{T} \)-grading \( \Gamma \) on \( L \). If \( T \) is a maximal quasitorus in \( \text{Aut}(L) \) (resp. in \( \text{Int}(L) \)) then \( \Gamma \) is called a fine grading (resp. fine inner grading). A grading \( \Gamma \) is quasi-good if 0 is not in \( \text{Supp } \Gamma \) and each homogeneous component is 1-dimensional. A quasitorus \( T \subset \text{Aut}(L) \) induces a quasi-good grading on \( L \) if and only if \( T \) is a finite quasitorus in \( \text{Aut}(L) \) such that its centralizer in \( \text{Aut}(L) \) is also finite. If \( T \) is a finite maximal quasitorus in \( \text{Aut}(L) \) (resp. in \( \text{Int}(L) \)) such that the induced grading \( \Gamma \) on \( L \) is quasi-good, then \( \Gamma \) is called a good grading (resp. good inner grading) and \( T \) is called a good finite maximal quasitorus in \( \text{Aut}(L) \) (resp. in \( \text{Int}(L) \)).

A good grading must be quasi-good, but not conversely. See Lemma 3.3 for details.

Let \( L \) be a simple Lie algebra and the group \( K \) be either \( \text{Aut}(L) \) or \( \text{Int}(L) \). Let \( T \subset K \) be a maximal quasitorus, then it induces a \( G \)-grading \( \Gamma \).
on $L$ with $G = \hat{T}$. Define the Weyl group of $T$ with respect to $K$ to be

$$W_K(T) = N_K(T)/Z_K(T).$$

By Proposition 2.4 of [H1], one has $N_K(T) = \text{Aut}_K(\Gamma)$, $Z_K(T) = \text{Stab}_K(\Gamma)$ and thus

$$W_K(T) = W_K(\Gamma). \quad (1.3)$$

1.3. Given a finite maximal abelian group $T$ of a compact Lie group $K$, Vogan defined the corresponding finite root datum analogous to the classical root datum in [HV], and conjectured that the corresponding Weyl group of $T$ with respect to the identity component group of $K$ is generated by the set of root transvections in it. The conjecture remains open by now. For any compact simple Lie algebra $L_0$, Yu in [Y] classified all the abelian subgroups $T$ of the compact Lie group $\text{Aut}(L_0)$ such that $T$ has the same dimension as that of its centralizer in $\text{Aut}(L_0)$, which include all the maximal abelian subgroups of $\text{Aut}(L_0)$, and also calculated all the Weyl groups.

For a complex simple Lie algebra $L$, the group $\text{Aut}(L)$ is a simple algebraic group. One knows that there is a one-to-one correspondence between complex reductive algebraic groups and compact Lie groups. A $G$-grading $\Gamma$ on a simple Lie algebra $L$ is a fine grading (resp. a fine inner grading) if and only if the dual group $\hat{G}$ embeds as a maximal quasitorus in $\text{Aut}(L)$ (resp. in $\text{Int}(L)$). Thus the works in [HV] and [Y] are closely related to the works mentioned in 1.1. Furthermore, by (1.3) the Weyl group of the fine grading $(\Gamma, G)$ and the Weyl group of the maximal quasitorus $\hat{G}$ are the same.

1.4. A root system $R$ is a finite set of elements in a Euclidean space $V$ satisfying certain axioms. There is a one-to-one correspondence between reduced and irreducible root systems and simple complex Lie algebras (up to isomorphism).

In [A], for a Jordan subgroup $T$ of $\text{Aut}(L)$ with $L$ a complex simple Lie algebra, an alternating bicharacter on $T$ is defined to study the Weyl group of $T$. In [HY], similar alternating bicharacters are also defined on elementary abelian 2-groups of compact simple Lie groups. An abelian group equipped with an alternating bicharacter is called a symplectic abelian group. Motivated by these works, in Section 3 we define a finite root system $(G, \beta, R)$ to be a finite set $R$ of elements in some finite symplectic abelian group $(G, \beta)$ satisfying certain axioms (Definition 3.4). Given any finite root system $(G, \beta, R)$, one always obtains a semisimple Lie algebra $L(R)$ with a standard (quasi-good) grading on it as well as an embedding $\hat{G} \hookrightarrow \text{Aut}(L(R))$. 

5
(See Proposition 3.7). The finite root system \((G, \beta, R)\) is called \textit{good} if \(\hat{G}\) is a finite maximal quasitorus in \(\text{Aut}(L(R))\) or in \(\text{Int}(L(R))\). Based on the main results in [Y], in Theorem 6.1 we classify good finite maximal quasitorus \(T\) in \(\text{Aut}(L)\) for any complex simple Lie algebra \(L\). In Section 5 we show case by case that for any such pair \((L, T)\) one can always find some unique irreducible finite root system \((G, \beta, R)\), where \(G = \hat{T}\) and \(R\) is the set of roots of \(T\) in \(L\), such that \(L \cong L(R)\) and the grading on \(L\) induced by \(T\) is isomorphic to the standard grading on \(L(R)\). As a corollary, we classify all the reduced and irreducible finite root systems \(R\) whose corresponding Lie algebra \(L(R)\) is simple and the standard grading on it is a good inner grading.

It is known that Vogan’s conjecture holds in these cases.

1.5. The paper is structured as follows. In Section 2 some results on twisted group algebras and symplectic abelian groups are collected, to prepare for Section 3, where we define finite root systems, corresponding quasi-good standings and Weyl groups. In Section 4 we will recall and prove some results about the gradings on simple Lie algebras. In Section 5 we give some typical examples of finite root systems \((G, \beta, R)\) and associated gradings, also give the embedding of \(\hat{G}\) in \(\text{Aut}(L(R))\). In Section 6 we classify good finite maximal quasitorus in \(\text{Aut}(L)\) for any complex simple Lie algebra \(L\), and prove our main result Theorem 6.1. The proofs in the last section heavily rely on [Y].

We use \(\mathbb{Z}_n\) to denote \(\mathbb{Z}/n\mathbb{Z}\) for any positive integer \(n\). The other notations in the paper are standard.

2 Twisted group algebras and symplectic abelian groups

Let \(G\) be a finite abelian group. Let

\[ A = \oplus_{a \in G} \mathbb{C} u_a \]

be a vector space with basis \(u_a\), and assume that there is a map \(\xi : G \times G \to \mathbb{C}^\times\) such that

\[ u_a \cdot u_b = \xi(a, b) u_{a+b}, \text{ for all } a, b \in G. \tag{2.1} \]

The following result can be found in Section 1, Chapter 2 of [Ka2].

**Lemma 2.1.** \((A, \cdot)\) is a \(G\)-graded associative algebra if and only if

\[ \xi(a, b) \xi(a + b, c) = \xi(a, b + c) \xi(b, c), \text{ for all } a, b, c \in G, \]
i.e., $\xi \in Z^2(G, C^\times)$. In this case $A$ is called a twisted group algebra of $G$ over $C$.

Assume that $\xi, \xi' \in Z^2(G, C^\times)$. Then $\xi$ and $\xi'$ define isomorphic $G$-graded associative algebras if and only if

$$\xi'(a,b)/\xi(a,b) = \eta(a)\eta(b)/\eta(a+b)$$

for some $\eta : G \to C^\times$, i.e., $\xi$ and $\xi'$ are cohomologous.

Thus the equivalence classes of twisted group algebras of $G$ over $C$ are in 1-1 correspondence with $H^2(G, C^\times)$.

A map $\beta : G \times G \to C^\times$ is called an alternating bicharacter on $G$ if it is multiplicative in each variable and has the property that $\beta(a,a) = 1$ for any $a \in G$, and is called a symmetric bicharacter on $G$ if it is multiplicative in each variable and has the property that $\beta(a,b) = \beta(b,a)$ for any $a,b \in G$. It is clear that all the alternating bicharacters (resp. all the symmetric bicharacters) on $G$ form an abelian group and will be denoted by $\bigwedge^2(G, C^\times)$ (resp. Sym$^2(G, C^\times)$). Note that $\bigwedge^2(G, C^\times)$ is denoted by $P_{as}(G)$ in Chapter 8 of [Ka3].

By Proposition 2.1 in Chapter 8 of [Ka3], there is a short exact sequence of abelian groups

$$1 \to \text{Ext}(G, C^\times) \to H^2(G, C^\times) \xrightarrow{\psi} \bigwedge^2(G, C^\times) \to 1,$$

where $\text{Ext}(G, C^\times) = \{\pi \in H^2(G, C^\times)|\alpha(a,b) = \alpha(b,a)\text{ for all }a,b \in G\}$ which is 0 as $C$ is algebraically closed and $\psi(\xi)(a,b) = \xi(a,b)\xi(b,a)^{-1}$. As $H^2(G, C^\times) \cong Z^2(G, C^\times)/B^2(G, C^\times)$, the isomorphism $\psi$ can be written as a short exact sequence of abelian groups

$$1 \to B^2(G, C^\times) \to Z^2(G, C^\times) \xrightarrow{\Psi} \bigwedge^2(G, C^\times) \to 1,$$

where the map $\Psi$ is

$$\Psi : Z^2(G, C^\times) \to \bigwedge^2(G, C^\times), \Psi(\xi)(a,b) = \xi(a,b)\xi(b,a)^{-1}. \quad (2.3)$$

As $G$ is abelian, elements in $\text{Hom}_Z(G \otimes G, C^\times)$ are bimultiplicative maps $G \times G \to C^\times$, which are clearly 2-cocycles in $Z^2(G, C^\times)$. Then (2.3) restricts to a surjective group homomorphism

$$\Psi : \text{Hom}_Z(G \otimes G, C^\times) \to \bigwedge^2(G, C^\times)$$

with the kernel consisting of all the symmetric bicharacters on $G$. Denote this kernel by $\text{Sym}^2(G, C^\times)$. Thus one has

$$1 \to \text{Sym}^2(G, C^\times) \to \text{Hom}_Z(G \otimes G, C^\times) \to \bigwedge^2(G, C^\times) \to 1. \quad (2.5)$$
If \( \beta : G \times G \to \mathbb{C}^\times \) is an alternating bicharacter on \( G \), then \( (G, \beta) \) is called a \textit{symplectic abelian group}. The radical of \( \beta \) is

\[ \text{Rad}(\beta) = \{ a \in G | \beta(a, b) = 1, \text{ for all } b \in G \}. \]

One says that \( \beta \) is \textit{nonsingular} if \( \text{Rad}(\beta) = 0 \).

By Lemma 2.7 and Corollary 2.10 in Chapter 8 of [Ka3], one gets

**Proposition 2.2.** One has \( C_\xi[G] \cong \bigoplus_{i=1}^k M(n, \mathbb{C}) \), \( k = |\text{Rad}(\beta)| \) and \( |G| = kn^2 \). \( C_\xi[G] \) is simple if and only if \( \beta \) is nonsingular.

As a corollary one has

**Corollary 2.3.** For any \( \beta \in \wedge^2(G, \mathbb{C}^\times) \), choose \( \xi \in H^2(G, \mathbb{C}^\times) \) with \( \Psi(\xi) = \beta \), then we get a \( G \)-graded Lie algebra \( L(\xi) \) obtained from the associative algebra \( C_\xi[G] \), i.e.,

\[ [u_a, u_b] = (\xi(a, b) - \xi(b, a))u_{a+b}. \]

The Lie algebra structure of \( L(\xi) \) depends only on \( \beta \) and not on the \( \xi \) chosen.

One has \( L \cong \mathfrak{gl}(n, \mathbb{C}) \oplus k \) with \( k = |\text{Rad}(\beta)| \) and \( |G| = kn^2 \). In particular Isomorphic alternating bicharacters on \( G \) corresponds to isomorphic \( L(\xi) \).

Let \( \xi \in H^2(G, \mathbb{C}^\times) \) and \( \beta = \Psi(\xi) \in \wedge^2(G, \mathbb{C}^\times) \), then for any \( a, b \in G \),

\[ \beta(a, b) = \xi(a, b)\xi(b, a)^{-1}. \]

Let \( L(\xi) \) be the reductive Lie algebra obtained from \( C_\xi[G] \). In \( L(\xi) \) one has

**Lemma 2.4.** (1) For any \( a, b \in G \), \( [u_a, u_b] = 0 \) if and only if \( \xi(a, b) = \xi(b, a) \) if and only if \( \beta(a, b) = 1 \). In particular, \( [u_{ma}, u_{ka}] = 0 \) for any \( a \in G \) and \( m, n \in \mathbb{Z} \).

(2) \( L \) is commutative if and only if \( \xi(a, b) = \xi(b, a) \) for all \( a, b \in G \) if and only if \( \beta(a, b) \equiv 1 \) for all \( a, b \in G \).

(3) The center of \( L \) is

\[ Z(L) = \bigoplus_{a \in \text{Rad}(\beta)} \mathbb{C}u_a. \] (2.6)

(4) If \( \text{gcd}(\text{ord}(a), \text{ord}(b)) = 1 \) for some \( a, b \in G \), then \( \beta(a, b) = 1 \), \( \xi(a, b) = \xi(b, a) \) and \( [u_a, u_b] = 0 \).

**Proof.** (1) is obvious and (2),(3) follows from (1).

Let us prove (4). Assume \( \text{ord}(a) = m, \text{ord}(b) = n \). Then \( \beta(a, b)^m = \beta(ma, b) = 1 \) and \( \beta(a, b)^n = \beta(a, nb) = 1 \). As \( \text{gcd}(m, n) = 1 \), \( \beta(a, b) = 1 \). Then \( \xi(a, b) = \xi(b, a) \) and \( [u_a, u_b] = 0 \) by definition. \( \square \)
Let $L' = [L, L]$ be the derived Lie algebra of $L = L(\xi)$.

**Proposition 2.5.** (1) $L' = \oplus_{a \not\in \text{Rad}(\beta)} \mathbb{C}u_a$ is the semisimple ideal of $L$ and $L = Z(L) \oplus L'$.

(2) If $R$ is a subset of $G$ satisfying
a) $R \subset G \setminus \text{Rad}(\beta)$ and $R$ generates $G$,
b) If $a \in R$ then $-a \in R$, and
c) If $\beta(a, b) \neq 1$ then $a + b \in R$,
then $L(R) = \oplus_{a \in R} \mathbb{C}u_a$ is a semisimple Lie subalgebra of $L$.

**Proof.** (1) Let $(,)$ be the Killing form on $L$. One knows by Corollary 2.3 that $L \cong gl(n, \mathbb{C})^\otimes k$ for some integer $n, k$, and is reductive.

If $a, b \not\in \text{Rad}(\beta)$ while $a + b \in \text{Rad}(\beta)$, $\beta(a, b) = \beta(a, a + b) = 1$ so $[u_a, u_b] = 0$. Thus $L_1 = \oplus_{a \not\in \text{Rad}(\beta)} \mathbb{C}u_a$ is a Lie subalgebra of $L$, and by Lemma 2.4.3,

$$L = Z(L) \oplus L_1.$$ 

Then $L_1$ is an ideal of $L$ as $[L, L_1] = [L_1, L_1] \subset L_1$. As $L' = [L, L] = [L_1, L_1] \subset L_1$, and $L = Z(L) \oplus L'$, one must have $L' = L_1$. So $L' = \oplus_{a \not\in \text{Rad}(\beta)} \mathbb{C}u_a$ is the semisimple ideal of $L$ and $L = Z(L) \oplus L'$.

(2) Let $(,)$ be the Killing form on $L(R)$. Assume $a, b \in R$. If $\beta(a, b) = 1$ then $[u_a, u_b] = 0$ by Lemma 2.4.1. If $\beta(a, b) \neq 1$ then $[u_a, u_b] = (\xi(a, b) - \xi(b, a))u_{a+b} \in L(R)$. So $L(R)$ is a Lie subalgebra of $L$.

It is clear that

$$(u_a, u_b) = 0, i f a + b \neq 0.$$ 

Choose $\xi \in \text{Hom}_{\mathbb{Z}}(G \otimes G, \mathbb{C}^\times)$ with $\Psi(\xi) = \beta$, i.e., $\beta(a, b) = \xi(a, b)\xi(b, a)^{-1}$. Note that $\xi(a, b)$ is always a root of unity. One has

$$
(u_a, u_{-a}) = tr(ad_{u_{-a}} \cdot ad_{u_a}) \\
= \sum_{b \in R} (\xi(a, b) - \xi(b, a)) (\xi(-a, a + b) - \xi(a + b, -a)) \\
= \sum_{b \in R} (2 - \xi(a, b)\xi(b, -a) - \xi(b, a)\xi(-a, b)).
$$

As $\xi(a, b)\xi(-a, b) = [\xi(a, b)\xi(b, -a)]^{-1}$ and $|\xi(b, a)\xi(-a, b)| = 1$, one has

$$
2 - (\xi(a, b)\xi(b, -a) + \xi(b, a)\xi(-a, b)) \geq 0,
$$

with equality holds if and only if $\xi(a, b)\xi(b, -a) = 1$, i.e., $\beta(a, b) = 1$.

So $(u_a, u_{-a}) = 0$ if and only if $\beta(a, b) = 1$ for any $b \in R$, which is equivalent to $a \in \text{Rad}(\beta)$ since $R$ generates $G$ by a). But $R \cap \text{Rad}(\beta) = \emptyset$,
so \((u_a, u_a) > 0\) for any \(a \in R\). Assume \(x = \sum_{a \in R} \lambda_a u_a \) \((\lambda_a \in \mathbb{C})\) is in the radical of the Killing form \((,\), then for any \(a \in R\), \((x, u_a) = \lambda_a (u_a, u_a) = 0\) thus \(\lambda_a = 0\). So \(x = 0\) and the radical of the Killing form \((,\) is 0. Therefore \((,\) is nondegenerate and \(L(R)\) is semisimple.

3 Finite root systems and corresponding quasi-good gradings on semisimple Lie algebras

If \(T\) is a complex quasitorus, then one knows that \(\hat{T}\) is a finitely generated abelian group, and the dual group of \(\hat{T}\) is naturally isomorphic to \(T\) by Pontryagin’s duality. An algebraic group homomorphism of quasitori \(\phi : T \to S\) will induce a group homomorphism of finitely generated abelian groups \(\hat{\phi} : \hat{S} \to \hat{T}\), the dual homomorphism of which is just the algebraic group homomorphism \(\phi : T \to S\) under the canonical identification of a quasitorus with its bidual. The following result is clear.

**Lemma 3.1.** An algebraic group homomorphism of quasitori \(T \to S\) is injective if and only if \(\hat{S} \to \hat{T}\) is surjective.

**Lemma 3.2.** Assume that \(L\) is a semisimple Lie algebra.

(1) An inclusion of quasitori \(T \to S\) of \(\text{Aut}(L)\) induces a surjective group homomorphism \(\hat{S} \to \hat{T}\) and a corresponding grading homomorphism.

(2) Conversely, a grading homomorphism on \(L\) induced by the identity map \(\text{id} : (\Gamma_1, G) \to (\Gamma_2, H)\) induces an inclusion of quasitori \(\hat{H} \to \hat{G}\) of \(\text{Aut}(L)\).

Proof. (1) The inclusion of quasitori \(T \to S\) of \(\text{Aut}(L)\) clearly induces a group homomorphism \(p : \hat{S} \to \hat{T}\), which is surjective by last lemma. Let

\[
\Gamma_1 : L = \bigoplus_{g \in \hat{T}} L_g
\]

and

\[
\Gamma_2 : L = \bigoplus_{g \in \hat{S}} L_g
\]

be the respective gradings by \(\hat{T}\) and by \(\hat{S}\). If \(g \in \hat{S}\) and \(X \in L_g\), then it is easy to see that \(X \in L_{p(g)}\) where \(L_{p(g)}\) is the homogeneous component of degree \(p(g)\) for \((\Gamma_2, \hat{T})\). Thus the identity map on \(L\) induces a grading homomorphism.

(2) Conversely, a grading homomorphism induced by the identity map \(\text{id} : (\Gamma_1, G) \to (\Gamma_2, H)\) induces a surjective group homomorphism \(G \to H\) by definition, which induces an injective algebraic group homomorphism of quasitori \(\hat{H} \to \hat{G}\) by last lemma. □
Lemma 3.3. For a quasi-good $G$-grading $\Gamma$ on $L$ induced by a finite quasitorus $T \subset \text{Aut}(L)$, there is a unique good $G_1$-grading $\Gamma_1$ on $L$ induced by a finite maximal quasitorus $T_1 \subset \text{Aut}(L)$ such that the identity map on $L$ induces a grading homomorphism $\text{id} : (\Gamma_1, G_1) \to (\Gamma, G)$, which is injective on the support of $\Gamma_1$. In particular, $T_1$ is the centralizer of $T$ in $\text{Aut}(L)$, and $T \subset T_1$.

Proof. Let $T_1$ be the centralizer of $T$ in $\text{Aut}(L)$. Then $T_1$ consists of semisimple automorphisms of $L$ preserving each 1-dimensional homogeneous component of $\Gamma$, thus is a maximal quasitorus of $\text{Aut}(L)$. Let $G_1 = \widehat{T_1}$. Then the $G_1$-grading $\Gamma_1$ is good and has the desired property.

Next we will prove uniqueness. If there is another good $G_2$-grading $\Gamma_2$ on $L$ induced by a finite quasitorus $T_2 \subset \text{Aut}(L)$ such that the identity map on $L$ induces a grading homomorphism $\text{id} : (\Gamma_2, G_2) \to (\Gamma, G)$, then by Lemma 3.2 (2) $T \to T_2$ is an inclusion of quasitorus. Then $T_2 \subset Z(T) = T_1$ and then $T_2 = T_1$ as $T_2$ is a maximal quasitorus in $\text{Aut}(L)$. The assertion $T \subset T_1$ is clear.

Now we will define finite root systems.

Let $G$ be a finite (additive) abelian group with an alternating bicharacter $\beta : G \times G \to \mathbb{C}^\times$. Then $(G, \beta)$ is a symplectic abelian group. Assume $\beta$ is nontrivial, i.e., $\text{Rad}(\beta) \neq G$.

Definition 3.4. A subset $R$ of $G$ is a finite root system in $(G, \beta)$ if it satisfies

- **FRS0.** $R \subset G \setminus \text{Rad}(\beta)$ and $R$ generates $G$.
- **FRS1.** If $a \in R$, then $-a \in R$.
- **FRS2.** If $\beta(a, b) \neq 1$, then $a + b \in R$.

We also say that $(G, \beta, R)$ is a finite root system. Let $\text{Sp}(G, \beta)$ denote the group of isometries of $(G, \beta)$.

The finite root system $R$ is called reduced if the alternating bicharacter $\beta$ is nonsingular.

Two finite root system $(G_i, \beta_i, R_i)$ ($i = 1, 2$) are isomorphic, denoted $R_1 \cong R_2$ or $(G_1, \beta_1, R_1) \cong (G_2, \beta_2, R_2)$, if there is a group isomorphism $\varphi : G_1 \to G_2$ preserving the respective alternating bicharacters and $\varphi(R_1) = R_2$. 

11
For a given finite abelian group $G$, if it admits a nonsingular alternating bicharacter then $G \cong H \times H$ for some abelian group $H$, and if there are two nonsingular alternating bicharacters $\beta_i$ on $G$ for $i = 1, 2$, then $(G, \beta_1)$ and $(G, \beta_2)$ are isometric. See Lemma 1.6 and Theorem 1.8 of [Ka3] for details. If $(G, \beta)$ is a nonsingular symplectic abelian group, then $R = G \setminus \{0\}$ is a finite root system in $(G, \beta)$ and the following result is clear.

**Lemma 3.5.** If $(G_1, \beta_1, R_i)_{i=1,2}$ are reduced finite root systems with $R_i = G_i \setminus \{0\}$ and $G_1 \cong G_2$, then $(G_1, \beta_1)$ and $(G_2, \beta_2)$ are isometric. See Lemma 1.6 and Theorem 1.8 of [Ka3] for details.

Let $\varepsilon \in \mathbb{C}$ be a primitive $n$-th root of unity. Let $\mu_n = \{\varepsilon^i | i = 0, \ldots, n-1\}$ be the cyclic subgroup of $\mathbb{C}^\times$ of order $n$. For any subgroup $H$ of $G$ of order $n$, define an action of $\mu_n$ on $H$ as follows:

$$\varepsilon^i \cdot h = ih, \forall h \in H.$$ 

Given a finite root system $(G, \beta, R)$, assume $a \in R$ has order $n$. Fix a primitive $n$-th root of unity $\varepsilon$ and define

$$s_a : G \to G, b \mapsto b - \beta(a, b).a,$$

which is called a transvection on $G$. It is directly verified that $s_a^n = 1$ thus $s_a$ is invertible, and by (FRS2) one has

$$s_a(R) = R, \forall a \in R.$$

**Lemma 3.6.** For any $a, b, c \in G$, one has

$$\beta(s_a(b), s_a(c)) = \beta(b, c),$$

so $s_a \in \text{Sp}(G, \beta)$.

**Proof.** Assume $a$ has order $n$ and $\varepsilon$ is a fixed primitive $n$-th root of unity. Then $\beta(a, b) = \varepsilon^i$, $\beta(a, c) = \varepsilon^j$ for some $i, j \in \mathbb{Z}$.

$$\beta(s_a(b), s_a(c)) = \beta(b - \beta(a, b).a, c - \beta(a, c).a)$$

$$= \beta(b, c)\beta(a, c)^{-i} \beta(b, a)^{-j}$$

$$= \beta(b, c)(\varepsilon^j)^{-i}(\varepsilon^{-i})^{-j}$$

$$= \beta(b, c)$$

\[\square\]
The Weyl group $W(R)$ of $R$ is defined to be the subgroup of $Sp(G, \beta)$ generated by $\{s_a | a \in R\}$.

Let $L(R) = \bigoplus_{a \in R} C u_a$ be the Lie algebra with the Lie bracket defined by some $\xi \in \text{Hom}_\mathbb{Z}(G \otimes G, \mathbb{C}^\times)$ such that $\Psi(\xi) = \beta$. By Proposition 2.5 (2), $L(R)$ is a semisimple Lie algebra.

The grading

$$\Gamma : L(R) = \bigoplus_{a \in R} C u_a$$

is a $G$-grading on $L(R), 0 \notin R$, and each of its homogeneous component has dimension 1. Thus $\Gamma$ is a quasi-good grading on $L(R)$. We refer to $\Gamma$ as the standard $G$-grading on $L(R)$ or the standard $G$-grading of $R$. Together with Lemma 3.3 one has

**Proposition 3.7.** For any finite root system $(G, \beta, R)$, there corresponds to it a semisimple Lie algebra $L(R)$ with a standard quasi-good $G$-grading on it.

This grading is good if and only if $\hat{G}$ is a maximal quasitorus in $\text{Aut}(L(R))$, i.e., the centralizer of $\hat{G}$ in $\text{Aut}(L(R))$ equals $\hat{G}$.

A finite root system $(G, \beta, R)$ is called good if $L = L(R)$ is simple and $\hat{G}$ is a finite maximal quasitorus in $\text{Aut}(L)$ or in $\text{Int}(L)$. A finite root system $R$ in $(G, \beta)$ is called reducible if $G$ is an orthogonal product of two subgroups $G_1$ and $G_2$, $R$ is a disjoint union of two nonempty orthogonal subsets $R_1$ and $R_2$, and $R_i$ is a finite root system in $G_i$ for $i = 1, 2$. It is clear that in this case $L(R) = L(R_1) \oplus L(R_2)$ is a direct sum of semisimple ideals $L(R_1)$ and $L(R_2)$. A finite root system $R$ in $(G, \beta)$ is irreducible if it is not reducible. In particular, if $(G, \beta, R)$ is good then $L(R)$ is simple thus it must be irreducible.

**Lemma 3.8.** Assume that $(G, \beta, R)$ is a finite root system and $H$ is any subgroup of $\text{Rad}(\beta)$. Let $\overline{G} = G/H$, $\overline{R}$ be the image of $R$ in $\overline{G}$, and $\overline{\beta}$ be the alternating bicharacter on $\overline{G}$ induced by $\beta$, i.e., $\overline{\beta}(\overline{g}, \overline{h}) = \beta(g, h)$ for any $\overline{g}, \overline{h} \in \overline{G}$.

1. One has that $(\overline{G}, \overline{\beta}, \overline{R})$ is a finite root system. If $H = \text{Rad}(\beta)$ then $\overline{R}$ is a reduced finite root system, and is referred as the corresponding reduced finite root system of $R$.

2. There is a surjective Lie algebra homomorphism from $L(R)$ to $L(\overline{R})$ that maps $L(R)g$ to $L(\overline{R})\overline{g}$ for any $g \in R$. If $L(R)$ is simple then this is an isomorphism of Lie algebras, and in this case the homogeneous components of $L(R)$ and $L(\overline{R})$ are the same.
Proof. (1) is clear.

(2) Let \( \xi \in \text{Hom}_\mathbb{Z}(G \otimes \mathbb{C}, \mathbb{C}^\times) \) such that \( \Psi(\xi) = \beta \). Let \( \xi \in \text{Hom}_\mathbb{Z}(G \otimes G, \mathbb{C}^\times) \) be the pull-back of \( \xi \), i.e., for any \( g, h \in G \), \( \xi(g, h) = \xi(\bar{g}, \bar{h}) \). Then it is directly verified that \( \Psi(\xi) = \beta \).

Then \( \phi : L(R) \to L(\bar{R}), u_g \mapsto u_{\bar{g}} \) defines a surjective Lie algebra homomorphism, as

\[
[u_g, u_h] = (\xi(g, h) - \xi(h, g))u_{g+h}
\]

and

\[
[u_{\bar{g}}, u_{\bar{h}}] = (\bar{\xi}(\bar{g}, \bar{h}) - \bar{\xi}(\bar{h}, \bar{g}))u_{\bar{g}+\bar{h}} = (\xi(g, h) - \xi(h, g))u_{g+h}.
\]

The last assertion in (2) is clear. \( \square \)

If two finite root systems \((G_1, \beta_1, R_1)\) are isomorphic for \( i = 1, 2 \), then it is clear that \( G_1 \cong G_2 \) and the corresponding standard gradings are isomorphic. The converse may not hold in general, but it holds in a special case.

**Lemma 3.9.** Assume that the standard gradings of two finite root systems \((G_i, \beta_i, R_i)\) are isomorphic for \( i = 1, 2 \) and \( G_1 \cong G_2 \cong \mathbb{Z}_n^2 \) for some positive integer \( n \). Then \((G_1, \beta_1, R_1) \cong (G_2, \beta_2, R_2)\).

Proof. By assumption, there is a Lie algebra isomorphism \( \psi : L(R_1) \to L(R_2) \) and a group isomorphism \( \phi : G_1 \to G_2 \) such that \( \phi(R_1) = R_2 \) and for any \( a \in R_1 \), \( \psi(L(R_1)a) = L(R_2)\phi(a) \), i.e., \( \psi(u_a) = \lambda_a u_{\phi(a)} \) for some \( 0 \neq \lambda_a \in \mathbb{C} \). As \( \psi \) preserves Lie brackets, for any \( a, b \in R_1 \), \( [u_a, u_b] = 0 \) if and only if \( [u_{\phi(a)}, u_{\phi(b)}] = 0 \). Then by Lemma 2.1 (1), for any \( a, b \in R_1 \),

\[
\beta_1(a, b) = 1 \text{ if and only if } \beta_2(\phi(a), \phi(b)) = 1.
\]

As \( G_1 \cong G_2 \cong \mathbb{Z}_n^2 \), any nonzero element in \( G_1 \) and \( G_2 \) has order 2, thus \( \beta_1 \) and \( \beta_2 \) take values in \( \pm 1 \in \mathbb{C}^\times \). So (3.1) implies for any \( a, b \in R_1 \),

\[
\beta_1(a, b) = -1 \text{ if and only if } \beta_2(\phi(a), \phi(b)) = -1.
\]

As \( R_1 \) generates \( G_1 \), one has \( \beta_2(\phi(a), \phi(b)) = \beta_1(a, b) \) for any \( a, b \in G_1 \). Therefore \( \phi : G_1 \to G_2 \) preserves the respective alternating bicharacters and \( \phi(R_1) = R_2 \) thus \( \phi \) is an isomorphism of the two finite root systems \((G_i, \beta_i, R_i)\) for \( i = 1, 2 \). \( \square \)

Finally we will give two more examples of finite root systems.

Let \( q : \mathbb{F}_2^k \to \mathbb{F}_2 \) be a quadratic form and

\[
\alpha_1 : \mathbb{F}_2^k \times \mathbb{F}_2^k \to \mathbb{F}_2, \alpha_1(a, b) = q(a + b) - q(a) - q(b)
\]
be its polarization. Assume that $q$ is nonsingular, i.e. $\alpha_1$ is a nonsingular alternating bilinear form. One knows that there are two types of nonsingular quadratic forms on $\mathbb{F}_2^{2k}$ up to isomorphism. We will always identify the group $G = \mathbb{Z}_2^{2k}$ with the additive group of $\mathbb{F}_2^{2k}$.

**Lemma 3.10.** Assume that $\alpha_1$ is nonsingular. Let

$$\alpha : G \times G \to \mathbb{C}^\times, \alpha(a, b) = (-1)^{\alpha_1(a,b)}.$$ 

Assume that $R = \{a \in G \mid q(a) = 1\}$ generate $G$. Then $R$ is a finite root system in the symplectic abelian group $(G, \alpha)$.

**Proof.** FRS0 and FRS1 are obvious. Now we verify FRS2. Assume $a, b \in R$ and $\alpha(a, b) \neq 1$. Then $\alpha(a, b) = -1$, $\alpha_1(a, b) = 1$ and $q(a + b) = q(a) + q(b) + \alpha_1(a, b) = 1$, so $a + b \in R$. \[\square\]

Let

$$\beta_1 : \mathbb{F}_2^{2k} \times \mathbb{F}_2^{2k} \to \mathbb{F}_2, \beta_1(a, b) = \sum_{i=1}^{k} (a_{2i}b_{2i-1} - a_{2i-1}b_{2i})$$

be the unique nonsingular alternating bilinear form on $\mathbb{F}_2^{2k}$ up to isomorphism. Let

$$\beta : G \times G \to \mathbb{C}^\times, \beta(a, b) = (-1)^{\beta_1(a,b)}.$$ 

Then $(G, \beta)$ is a nonsingular symplectic abelian group. Up to isomorphism there are two quadratic forms on $\mathbb{F}_2^{2k}$ that polarize to $\beta_1$,

$$g : \mathbb{F}_2^{2k} \to \mathbb{F}_2, g(a) = \sum_{i=1}^{k} a_{2i-1}a_{2i}, \quad (3.2)$$

and

$$f : \mathbb{F}_2^{2k} \to \mathbb{F}_2, f(a) = a_1^2 + a_2^2 + \sum_{i=1}^{k} a_{2i-1}a_{2i}. \quad (3.3)$$

So we have the following two examples of finite root systems and corresponding standard gradings.

**Lemma 3.11.** Assume $G = \mathbb{Z}_2^{2k}$ with $k \geq 3$. Let $R = \{a \in G \mid g(a) = 1\}$. Then $R$ is a finite root system in $(G, \beta)$. 

15
Proof. The subset

\[ B = \{ e_{2i-1} + e_{2i},\ e_{2i-1} + e_{2i} + e_{2i+1} | i = 1, \ldots, k \} \ (e_{2k+1} = e_1) \]

of \( R \) generates \( G \), thus by last lemma \( R \) is a finite root system in \((G, \beta)\).

One will see that

\[ L(R) \cong so(2^k, \mathbb{C}) \]

by Lemma 4.1 (1), and the corresponding grading is (4.6).

**Lemma 3.12.** Assume \( G = \mathbb{Z}_{2^k} \) with \( k \geq 1 \). Let \( R = \{ a \in G | f(a) = 1 \} \). Then \( R \) is a finite root system in \((G, \beta)\).

**Proof.** It is directly verified that

\[
B = \begin{cases} 
\{e_1\} \cup \{e_1 + e_2\} & \text{if } k = 1; \\
\{e_1, e_{2i-1} + e_{2i} | i = 1, \ldots, k\} \cup \{e_{2i-1} + e_{2i} + e_{2i+1} | i = 1, \ldots, k - 1\} & \text{if } k > 1.
\end{cases}
\]

is contained in \( R \) and generates \( G \), thus \( R \) is a finite root system in \((G, \beta)\).

One will see that

\[ L(R) \cong sp(2^k, \mathbb{C}) \]

by Lemma 4.1 (2), and the corresponding grading is (4.7).

### 4 Some results about the gradings on simple Lie algebras

The gradings on classical simple Lie algebras are all described in [BSZ], which can be obtained by the gradings on matrix algebras.

For a nonsingular symmetric or skew-symmetric bilinear form \( \phi \) on \( \mathbb{C}^n \), one knows that the adjoint map \( * : \mathcal{M}(n, \mathbb{C}) \rightarrow \mathcal{M}(n, \mathbb{C}) \) defined by

\[
\phi(Xu, v) = \phi(u, X^*v)
\]

is an involution (i.e., involutive anti-automorphism) on \( \mathcal{M}(n, \mathbb{C}) \). If \( \Phi \) is the matrix of \( \phi \) with respect to the standard basis of \( \mathbb{C}^n \), then in matrix form one has

\[
X^* = \Phi^{-1}X^t\Phi,
\]
Conversely one can show that any involution on $M(n, \mathbb{C})$ can be defined in this way by some nonsingular symmetric or skew-symmetric bilinear form on $\mathbb{C}^n$. See Section 5 of [BSZ] for details. We call $(M(n, \mathbb{C}), \ast)$ an involutive matrix algebra. In the remaining of the section, $M$ will be assumed to be a matrix algebra.

Assume that $(M, \ast)$ is an involutive matrix algebra. Let

$$\text{Aut}(M, \ast) = \{ \sigma \in \text{Aut}(M) | \sigma \circ \ast = \ast \circ \sigma \}.$$ 

If $H$ is a subgroup of Aut$(M, \ast)$, then its action on $M$ is compatible with $\ast$, i.e., $h \circ \ast = \ast \circ h$ for any $h \in H$, and we say that $(M, \ast)$ has an $H$-action. If $M$ has a $G$-grading compatible with $\ast$, i.e., $M_g^\ast = M_g$ for any $g \in G$, then we say that $(M, \ast)$ has a $G$-grading. It is clear that $(M, \ast)$ has a $G$-grading if and only if $(M, \ast)$ has a $\hat{G}$-action.

Let

$$K(M, \ast) = \{ X \in M | X^\ast = -X \}$$

and

$$H(M, \ast) = \{ X \in M | X^\ast = X \}.$$

The subspace $K(M, \ast)$ is closed under the Lie bracket $[a, b] = ab - ba$ and is always regarded as a Lie algebra with this Lie bracket in the paper. A $G$-grading on $(M, \ast)$ clearly induces a $G$-grading on $K(M, \ast)$ (resp. on $H(M, \ast)$).

One knows that simple Lie algebras of type $B$, $C$ and $D$ can be realized as some $K(M, \ast)$ for suitable $(M, \ast)$, where $M = M(n, \mathbb{C})$ and $\ast$ corresponds to some nonsingular symmetric or skew-symmetric bilinear form $\phi$ on $\mathbb{C}^n$.

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2, M = M(2, \mathbb{C})$. Let

$$X_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } Y_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(4.1)

Let $Z_{(i,j)} = X_2^i Y_2^j$ for $(i, j) \in G$. The grading $M = \oplus_{a \in G} M_a$, where $M_a = \mathbb{C}Z_a$ for any $a \in G$, is referred as the standard $G$-grading on $M$. See Example 3.1.

Assume that $\ast$ is an involution on $M$ corresponding to the skew-symmetric form with the matrix $\Phi_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. By Lemma 3 (1) in [BSZ], $(M, \ast)$ has the $G$-grading with

$$K(M, \ast) = \{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \} = M_{(1,0)} \oplus M_{(0,1)} \oplus M_{(1,1)}$$

(4.2)
and
\[ H(M,*) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\} = M_{(0,0)}. \]

Assume that * is an involution on M corresponding to the symmetric form with the matrix \( I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

By Lemma 3 (3) of [BSZ], \((M,*)\) has the \(G\)-grading with
\[ K(M,*) = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \right\} = M_{(1,1)} \quad (4.3) \]
and
\[ H(M,*) = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right\} = M_{(0,0)} \oplus M_{(0,1)} \oplus M_{(1,0)}. \]

Next let us introduce the direct product gradings on associative algebras. Assume for \(i = 1, 2\), \(A_i\) is an associative algebra, and \((\Gamma_i, G_i)\) is the respective gradings on \(A_i\), then \((\Gamma, G_1 \times G_2)\) is the direct product grading on \(A = A_1 \otimes A_2\) with
\[ \Gamma : A_1 \otimes A_2 = \bigoplus_{(a,b) \in G_1 \times G_2} (A_1 \otimes A_2)_{(a,b)} \]
where
\[ (A_1 \otimes A_2)_{(a,b)} = (A_1)_a \otimes (A_2)_b, (a, b) \in G_1 \times G_2. \]

If \((M_i,*_i)\) is an involutive matrix algebra for \(i = 1, 2\), then their direct product \((M_1,*_1) \otimes (M_2,*_2)\) is defined to be the involutive matrix algebra \((M_1 \otimes M_2,*_1 \otimes *_2)\), where \((C \otimes B)^{*_1 \otimes *_2} = C^{*_1} \otimes B^{*_2}\) for any \(C \otimes B \in M_1 \otimes M_2\). If \((M_i,*_i)\) has a \(G_i\)-grading, then the \(G_1 \times G_2\)-grading on \(M_1 \otimes M_2\) is compatible with \(*_1 \otimes *_2\), thus \((M_1 \otimes M_2,*_1 \otimes *_2)\) has the naturally defined \(G_1 \times G_2\)-grading.

Assume that \((M_i,*_i)\) has a \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-grading for \(i = 1, \ldots, k\), where \(M_i = M(2, \mathbb{C})\) and the \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-grading on \(M_i\) is standard for all \(i\). Then the tensor product \((M,*_1)\) of \((M_i,*_i)\) for \(i = 1, \ldots, k\) has a \(G\)-grading with
\[ G = (\mathbb{Z}_2 \times \mathbb{Z}_2)^k = \mathbb{Z}_2^{2k}. \]
Thus \(K(M,*)\) also has a \(G\)-grading. Assume that \(*_i\) corresponds to a nonsingular bilinear form \(\phi_i\), then * corresponds to the nonsingular bilinear form \(\phi = \phi_1 \otimes \cdots \otimes \phi_k\). Assume the total number of skew-symmetric factors in \(\phi\) is \(m\). If \(m\) is even (resp. odd), then \(\phi\) is symmetric (resp. skew-symmetric).
Next we will compute the support \( R \) of the \( G \)-grading on \( K(M, \ast) \) in the two cases: \( m = 0 \) or \( m = 1 \). (It is not hard to show that the grading on \( K(M, \ast) \) is isomorphic to the first case if \( m \) is even, and is isomorphic to the second case if \( m \) is odd.)

Let

\[
\beta_1 : \mathbb{F}_2^{2k} \times \mathbb{F}_2^{2k} \to \mathbb{F}_2, \beta_1(a, b) = \sum_{i=1}^{k} (a_{2i}b_{2i-1} - a_{2i-1}b_{2i}) \tag{4.4}
\]

be a nonsingular bilinear form on \( \mathbb{F}_2^{2k} \).

Recall the nonsingular quadratic forms \( g \) and \( f \) on \( \mathbb{F}_2^{2k} \) defined in (3.2) and (3.3). One knows that they both polarize to \( \beta_1 \).

Let

\[
\beta : \mathbb{F}_2^{2k} \times \mathbb{F}_2^{2k} \to \mathbb{C}^X, \beta(a, b) = (-1)^{\beta_1(a, b)} = (-1)^{\sum_{i=1}^{k} (a_{2i}b_{2i-1} - a_{2i-1}b_{2i})} \tag{4.5}
\]

**Lemma 4.1.** Let \( k \geq 1 \) be an integer and \( G = \mathbb{Z}_2^{2k} \).

(1) Assume the total number of skew-symmetric factors \( \phi_i \) in \( \phi \) is 0 and the matrix for each \( \phi_i \) is \( I_2 \). Then \( K(M, \ast) \cong \mathfrak{so}(2^k, \mathbb{C}) \), \( R = \{ a \in G \setminus \{0\} \mid g(a) = 1 \} \). One has \( |R| = 2^{2k-1} - 2^{k-1} \), and this grading on \( \mathfrak{so}(2^k, \mathbb{C}) \) is

\[
\mathfrak{so}(2^k, \mathbb{C}) = \bigoplus_{a \in G, g(a) = 1} \mathbb{C}X_2^{a_1}Y_2^{a_2} \otimes X_2^{a_3}Y_2^{a_4} \otimes \cdots \otimes X_2^{a_{2k-1}}Y_2^{a_{2k}}. \tag{4.6}
\]

Conversely, let \( R \) be as above. Then \( R \) is a finite root system in \( (G, \beta) \) and the standard grading on \( L(R) \) is isomorphic to the \( G \)-grading on \( \mathfrak{so}(2^k, \mathbb{C}) \). Moreover, this grading is good if \( k \geq 3 \).

(2) Assume the total number of skew-symmetric factors \( \phi_i \) in \( \phi \) is 1, the matrix of \( \phi_1 \) is \( \Phi_1 \) and the matrix of \( \phi_i \) is \( I_2 \) for \( i > 1 \). Then \( K(M, \ast) \cong \mathfrak{sp}(2^k, \mathbb{C}) \), \( R = \{ a \in G \setminus \{0\} \mid f(a) = 1 \} \). One has \( |R| = 2^{2k-1} + 2^{k-1} \), and this grading on \( \mathfrak{sp}(2^k, \mathbb{C}) \) is

\[
\mathfrak{sp}(2^k, \mathbb{C}) = \bigoplus_{a \in G, f(a) = 1} \mathbb{C}X_2^{a_1}Y_2^{a_2} \otimes X_2^{a_3}Y_2^{a_4} \otimes \cdots \otimes X_2^{a_{2k-1}}Y_2^{a_{2k}}. \tag{4.7}
\]

Conversely, let \( R \) be as above. Then \( R \) is a finite root system in \( (G, \beta) \) and the standard grading on \( L(R) \) is isomorphic to the \( G \)-grading on \( \mathfrak{sp}(2^k, \mathbb{C}) \). Moreover, this grading is good.
\textit{Proof.} First let us consider the case \( k = 1 \). Then \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( M = M(2, \mathbb{C}) \). Then by (4.2) the support of \( \mathfrak{so}(2, \mathbb{C}) = K(M, \ast) \) is
\[
\{(1, 1)\} = \{(x_1, x_2) \in G | x_1 x_2 = 1\}. \tag{4.8}
\]
By (4.2) the support of \( \mathfrak{so}(2, \mathbb{C}) = K(M, \ast) \) is
\[
\{(0, 1), (1, 0), (1, 1)\} = \{(x_1, x_2) \in G | x_1^2 + x_2^2 + x_1 x_2 = 1\}. \tag{4.9}
\]
Next let us consider the general case. Then \( G = \mathbb{Z}_2^{2k} \). Let \( C = C_1 \otimes \cdots \otimes C_k \in M(2, \mathbb{C}) \otimes \cdots \otimes M(2, \mathbb{C}), C_i \) being one of \( \mathbb{Z}_a \) with \( a \in \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Let \( v_1 \otimes \cdots \otimes v_k, u_1 \otimes \cdots \otimes u_k \in (\mathbb{C}^2)^{\otimes k} \). Then
\[
\phi_i(C_i(v_i), u_i) = \begin{cases} 
-\phi_i(v_i, C_i(u_i)) & \text{if } C_i \in K(M, \ast_i); \\
\phi_i(v_i, C_i(u_i)) & \text{if } C_i \in H(M, \ast_i). \end{cases} \tag{4.10}
\]
One has
\[
\phi(C_1 \otimes \cdots \otimes C_k(v_1 \otimes \cdots \otimes v_k), u_1 \otimes \cdots \otimes u_k) = \phi_1(C_1(v_1), u_1) \cdots \phi_k(C_k(v_k), u_k),
\]
and
\[
\phi(v_1 \otimes \cdots \otimes v_k, C_1 \otimes \cdots \otimes C_k(u_1 \otimes \cdots \otimes u_k)) = \phi_1(v_1, C_1(u_1)) \cdots \phi_k(v_k, C_k(u_k)).
\]
Thus by (4.10)
\[
\phi(C_1 \otimes \cdots \otimes C_k(v_1 \otimes \cdots \otimes v_k), u_1 \otimes \cdots \otimes u_k) = -\phi(v_1 \otimes \cdots \otimes v_k, C_1 \otimes \cdots \otimes C_k(u_1 \otimes \cdots \otimes u_k))
\]
if and only if there are exactly odd number of \( C_i \in K(M, \ast_i) \).

(1) Assume the total number of skew-symmetric factors \( \phi_i \) in \( \phi \) is 0 and the matrix for each \( \phi_i \) is \( I_2 \). Recall
\[
g : \mathbb{Z}_2^{2k} \rightarrow \mathbb{Z}_2, g(x) = \sum_{i=1}^{k} x_{2i-1} x_{2i}.
\]
Let \( a = (a_1, a_2, \cdots, a_{2k-1}, a_{2k}) \in \mathbb{Z}_2^{2k} \). Then by (4.8) \( a \in R \) if and only if \( g(a) = 1 \), so the support \( R \) of \( K(M, \ast) \) is those \( a \in G \) with \( g(a) = 1 \). Similarly the support of \( H(M, \ast) \) is those \( a \in G \) with \( g(a) = 0 \). The fact \( |R| = 2^{2k-1} - 2^{k-1} \) is simple and we omit its proof. The grading of \( K(M, \ast) \cong \mathfrak{so}(2^k, \mathbb{C}) \) is clearly (4.6).

Recall the definition of \( \beta \) in (4.5). Then \( \beta \in \wedge^2(G, \mathbb{C}^\times) \) is a nonsingular alternating bicharacter on \( G \). By Lemma 3.11, \( R \) is a finite root system in \( (G, \beta) \). Let \( \xi : \mathbb{Z}_2^{2k} \times \mathbb{Z}_2^{2k} \rightarrow \mathbb{C}^\times, \xi(a, b) = (-1)^{\sum_{i=1}^{k} a_{2i-1} b_{2i-1}} \). Then \( \Psi(\xi) = \beta \).
Conversely, define \( \phi : L(R) \rightarrow \mathfrak{so}(2^k, \mathbb{C}) \),

\[
\phi(u_a) = Z_{(a_1, a_2)} \otimes Z_{(a_3, a_4)} \otimes \cdots \otimes Z_{(a_{2k-1}, a_{2k})}.
\]

where \( a = (a_1, a_2, \cdots, a_{2k-1}, a_{2k}) \). Then it is easy to verify that this is an isomorphism of \( G \)-gradings as follows.

\[
[\phi(u_a), \phi(u_b)] = [Z_{(a_1, a_2)} \otimes Z_{(a_3, a_4)} \otimes \cdots \otimes Z_{(a_{2k-1}, a_{2k})}, Z_{(b_1, b_2)} \otimes Z_{(b_3, b_4)} \otimes \cdots \otimes Z_{(b_{2k-1}, b_{2k})}]
\]

\[
= Z_{(a_1, a_2)}Z_{(b_1, b_2)} \otimes \cdots \otimes Z_{(a_{2k-1}, a_{2k})}Z_{(b_{2k-1}, b_{2k})} - Z_{(b_1, b_2)}Z_{(a_1, a_2)} \otimes \cdots \otimes Z_{(b_{2k-1}, b_{2k})}Z_{(a_{2k-1}, a_{2k})}
\]

\[
= [\xi(a, b) - \xi(b, a)]Z_{(a_1+b_1, a_2+b_2)} \otimes \cdots \otimes Z_{(a_{2k-1}+b_{2k-1}, a_{2k}+b_{2k})}
\]

\[
= [\xi(a, b) - \xi(b, a)]\phi(u_a+b)
\]

\[
= \phi([u_a, u_b]).
\]

(4.11)

Assume \( k \geq 3 \). As \( PO(2^k, \mathbb{C}) \subset PGL(2^k, \mathbb{C}) \), one knows that \( \tilde{G} \cong \mathbb{Z}_2^{2k} \) embeds as a maximal quasitorus in \( \text{Aut}(\mathfrak{so}(2^k, \mathbb{C})) \cong PO(2k, \mathbb{C}) \), so this grading is a good grading. (If \( k = 1 \) or \( 2 \) then \( L(R) \) is not simple.)

(2) Assume the total number of skew-symmetric factors \( \phi_i \) in \( \phi \) is 1, the matrix of \( \phi_1 \) is \( \Phi_1 \) and the matrix of \( \phi_i \) is \( I_2 \) for \( i > 1 \). Recall

\[
f : \mathbb{Z}_2^{2k} \rightarrow \mathbb{Z}_2, f(x) = (x_1^2 + x_2^2 + x_1x_2) + \sum_{i=2}^{k} x_{2i-1}x_{2i} = x_1^2 + x_2^2 + \sum_{i=1}^{k} x_{2i-1}x_{2i}.
\]

Then by (4.8) and (4.9) \( a = (a_1, a_2, \cdots, a_{2k-1}, a_{2k}) \in R \) if and only if \( f(a) = 1 \), so the support \( R \) of \( K(M, \ast) \) is those \( a \in G \) with \( f(a) = 1 \). Similarly the support of \( H(M, \ast) \) is those \( a \in G \) with \( f(a) = 0 \). The fact \( |R| = 2^{2k-1} + 2^{k-1} \) is simple and we omit its proof. The grading of \( K(M, \ast) \cong \mathfrak{so}(2^k, \mathbb{C}) \) is clearly (4.7).

Let \( \beta \) be as above. By Lemma 3.12 \( R \) is a finite root system in \( (G, \beta) \). Let

\[
\xi' : G \times G \rightarrow \mathbb{C}^\times, \xi'(a, b) = (-1)^{a_1b_1+a_2b_2+\sum_{i=1}^{k} a_{2i}b_{2i-1}}
\]

where \( a = (a_1, \cdots, a_{2k}), b = (b_1, \cdots, b_{2k}) \in G \). Then \( \Psi(\xi') = \beta \). By Lemma 2.24 and 2.27, there is some \( \eta : G \rightarrow \mathbb{C}^\times \) satisfying \( \xi(a, b)/\xi(b, a) = \eta(a)\eta(b)/\eta(a+b) \) for any \( a, b \in G \).

Conversely, define \( \phi : L(R) \rightarrow \mathfrak{sp}(2^k, \mathbb{C}) \),

\[
\phi(u_a) = \eta(a)Z_{(a_1, a_2)} \otimes Z_{(a_3, a_4)} \otimes \cdots \otimes Z_{(a_{2k-1}, a_{2k})}.
\]

where \( a = (a_1, a_2, \cdots, a_{2k-1}, a_{2k}) \in R \). Then one can verify that this is an isomorphism of \( G \)-gradings as in (4.11).
5 Good finite maximal quasitorus in Aut(L) for simple Lie algebras L and corresponding finite root systems

Given a complex simple Lie algebra L, a finite maximal quasitorus T of Aut(L) or Int(L) is said to be good if the induced grading on L is quasi-good, i.e., \( \dim L_\alpha \leq 1 \) for every character \( \alpha \in \hat{T} \). In this case, the set of roots R of T in L is a subset in \( \hat{G} = \hat{\hat{T}} \), and we will find some alternating bicharacter \( \beta \) on G such that \((G, \beta, R)\) is a finite root system, \( L \cong L(R) \) and the grading on L induced by T is isomorphic to the standard grading on L(R). It will be seen that the finite root system is unique up to isomorphism. It is a pity that we do not find a canonical way to construct the alternating bicharacter.

Recall that a finite root system \((G, \beta, R)\) is good if \( L = L(R) \) is simple and \( \hat{G} \) is a finite maximal quasitorus in Aut(L) or in Int(L). In the following examples, we will construct good finite root systems \((G, \beta, R)\), and identify \( L = L(R) \) with the corresponding standard gradings on it. Then we will give the embedding of \( T = \hat{G} \) in Aut(L) or Int(L). We also give \( W(R) \) in each case. People have known that \( W_K(\Gamma) = W(R) \) always holds in these cases, where \( K = \text{Int}(L) \).

In Section 6 we will prove that these exhaust all the good finite root systems.

5.1. Reduced finite root systems \((G, \beta, R)\) with \( R = G \setminus \{0\}\)

**Example 5.1.** Let \( G = \mathbb{Z}_n \times \mathbb{Z}_n \). Let \( \varepsilon \) be a primitive \( n \)-th root of unity. For any \((i, j), (s, t) \in \mathbb{Z}_n \times \mathbb{Z}_n = G\), let \( \beta((i, j), (s, t)) = \varepsilon^{js-it} \). It is clear that \( \beta \) is a nonsingular alternating bicharacter on \( G \) and \( R = G \setminus \{0\} \) is a finite root system in \((G, \beta)\).

Let \( \xi \in Hom_{\mathbb{Z}}(G \otimes G, \mathbb{C}^\times) \) be defined by \( \xi((i, j), (s, t)) = \varepsilon^{js} \). Then \( \Psi(\xi) = \beta \).

Next we will show \( L(R) \cong \mathfrak{sl}(n, \mathbb{C}) \).
Let $M = M(n, \mathbb{C})$ and $G = \mathbb{Z}_n \times \mathbb{Z}_n$. Let

$$X = X_n = \begin{pmatrix} \varepsilon^{n-1} & 0 & \cdots & 0 \\ 0 & \varepsilon^{n-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad Y = Y_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (5.1)$$

Then $XY = \varepsilon YX$ and $X^n = Y^n = I_n$ where $I_n$ will always denote the $n \times n$ identity matrix. Then

$$\Gamma : M = \bigoplus_{(i,j) \in G} \mathbb{C}X^iY^{-j} \quad (5.2)$$

is a $G$-grading on $M$, where $M_{(i,j)} = \mathbb{C}X^iY^{-j}$, called the $\varepsilon$-grading on $M$ in [BSZ].

Then

$$\mathfrak{sl}(n, \mathbb{C}) = \bigoplus_{(i,j) \in G \setminus \{0\}} \mathbb{C}X^iY^{-j} \quad (5.3)$$

is a $G$-grading on $\mathfrak{sl}(n, \mathbb{C})$.

Define

$$\varphi : L(R) \to \mathfrak{sl}(n, \mathbb{C}), \ u_{i,j} \mapsto X^iY^{-j}.$$

$$[\varphi(u_{i,j}), \varphi(u_{s,t})] = X^iY^{-j}X^sY^{-t} - X^sY^{-t}X^iY^{-j}$$

$$= (\varepsilon^{js} - \varepsilon^{it})X^{i+s}Y^{-(j+t)}$$

$$= (\xi((i, j), (s, t)) - \xi((s, t), (i, j)))X^{i+s}Y^{-(j+t)}$$

$$= (\xi((i, j), (s, t)) - \xi((s, t), (i, j)))\varphi(u_{i+s,j+t})$$

$$= \varphi([u_{i,j}, u_{s,t}]).$$

Thus $\varphi : L(R) \to \mathfrak{sl}(n, \mathbb{C})$ is an isomorphism of $G$-graded Lie algebras.

Let $P_n$ be the subgroup of $GL(n, \mathbb{C})$ of order $n^3$ generated by $X_n$ and $Y_n$. Let $K = \text{Int}(\mathfrak{sl}(n, \mathbb{C})) = PGL(n, \mathbb{C})$ and $P_n$ be the image of $P_n$ in $\text{Int}(\mathfrak{sl}(n, \mathbb{C}))$, i.e., $P_n$ is generated by $Ad_X$ and $Ad_Y$. One knows that

$$\widehat{G} = P_n \cong \mathbb{Z}_n \times \mathbb{Z}_n.$$

It is proved that $\widehat{G}$ is a maximal quasitorus of $\text{Aut}(\mathfrak{sl}(n, \mathbb{C}))$ in [HPP]. So this is a good grading on $\mathfrak{sl}(n, \mathbb{C})$. By [A] one knows that

$$W_K(\widehat{G}) \cong W_K(\Gamma) = W(R) \cong SL(2, \mathbb{Z}_n).$$

23
Example 5.2. Assume \( n = n_1 \cdots n_k \) with each \( n_i > 1 \) and

\[
n_i | n_{i+1}, \quad i = 1, \ldots, k - 1.
\]

For \( t = 1, \ldots, k \), let \( G_t = \mathbb{Z}_{n_t} \times \mathbb{Z}_{n_t} \). Let \( \beta_t : G_t \times G_t \to \mathbb{C}^\times, \beta_t((i,j),(s,l)) = \varepsilon_t^{js-il} \) be the alternating bicharacter on \( G_t \) as in last example, where \( \varepsilon_t \) is a primitive \( n_t \)-th root of unity. Let \( (G,\beta) \) be the orthogonal direct product symplectic abelian group of \( (G_1,\beta_1), \ldots, (G_k,\beta_k) \), i.e.,

\[
G = G_1 \times \cdots \times G_k
\]

and

\[
\beta((a_1, \ldots, a_k),(b_1, \ldots, b_k)) = \beta_1(a_1,b_1) \cdots \beta_k(a_k,b_k)
\]

where \( a_i, b_i \in G_i \). Then \( \beta \) is a nonsingular alternating bicharacter on \( G \) and \( R = G \setminus \{0\} \) is a finite root system in \( (G,\beta) \). Let \( \xi \in \text{Hom}_\mathbb{Z}(G \otimes G, \mathbb{C}^\times) \) be defined by

\[
\begin{align*}
\xi(((i_1,j_1), \ldots, (i_k,j_k)),((s_1,t_1), \ldots, (s_k,t_k))) &= \prod_{i=1}^k \varepsilon_t^{j_is_i}. \\
\end{align*}
\]

Then \( \Psi(\xi) = \beta \).

Next we will show that \( L(R) \cong \mathfrak{sl}(n,\mathbb{C}) \).

Let \( M = M(n,\mathbb{C}) \). Then \( M \cong M(n_1,\mathbb{C}) \otimes \cdots \otimes M(n_k,\mathbb{C}) \) has a \( G \)-grading as follows. Assume \( M(n_i,\mathbb{C}) = \bigoplus_{a \in G_i} \mathbb{C}X_{n_1}^a Y_{-n_1}^{-a} \) is the standard \( \varepsilon_i \)-grading defined in last example. Then \( M \) has the tensor product grading

\[
\begin{align*}
M &= \bigoplus_{(i_1,j_1, \ldots, i_k,j_k) \in G_1 \times \cdots \times G_k} \mathbb{C} \cdot X_{n_1}^{i_1} Y_{-n_1}^{-j_1} \otimes \cdots \otimes X_{n_k}^{i_k} Y_{-n_k}^{-j_k}, \\
\end{align*}
\]

and \( \mathfrak{sl}(n,\mathbb{C}) \) has the grading

\[
\begin{align*}
\mathfrak{sl}(n,\mathbb{C}) &= \bigoplus_{(i_1,j_1, \ldots, i_k,j_k) \in G_1 \times \cdots \times G_k \setminus \{0\}} \mathbb{C} \cdot X_{n_1}^{i_1} Y_{-n_1}^{-j_1} \otimes \cdots \otimes X_{n_k}^{i_k} Y_{-n_k}^{-j_k}. \\
\end{align*}
\]

Define

\[
\varphi : L(R) \to \mathfrak{sl}(n,\mathbb{C}), \quad u((i_1,j_1, \ldots, (i_k,j_k))) \mapsto X_{n_1}^{i_1} Y_{-n_1}^{-j_1} \otimes \cdots \otimes X_{n_k}^{i_k} Y_{-n_k}^{-j_k}.
\]

As in last example one can verify that \( \varphi : L(R) \to \mathfrak{sl}(n,\mathbb{C}) \) is an isomorphism of \( G \)-graded Lie algebras.

We remark that for \( R \) to be a finite root system in \( (G,\beta) \) the condition (5.4) is unnecessary. But with this condition different finite root systems are not isomorphic.
As \( L(R) \) is simple, the finite root system \( R \) in this example is reduced and irreducible. We say that it is of type \( I(n_1, \cdots, n_k) \), with each \( n_i \geq 1 \) and \( n_i | n_{i+1} \) for \( i = 1, \cdots, k - 1 \). For example, the finite root system in last example is of type \( I(n) \).

Now we will describe the dual group \( \hat{G} \) in \( \text{Aut}(L(R)) \).

Let \( D_n \) be the subgroup of \( GL(n, \mathbb{C}) \) consisting of all the diagonal matrices and \( D_n \subset PGL(n, \mathbb{C}) \) be its quotient group in \( PGL(n, \mathbb{C}) \) under the canonical projection \( GL(n, \mathbb{C}) \to PGL(n, \mathbb{C}) \). Let \( P_n \cong \mathbb{Z}_n \times \mathbb{Z}_n \) be the subgroup of \( PGL(n, \mathbb{C}) \) as defined in Example 5.1. Assume \( n = tn_1 \cdots n_k , t \geq 1 , n_i > 1 \), then

\[
D_t \times P_{n_1} \times \cdots \times P_{n_k} \twoheadrightarrow PGL(n, \mathbb{C})
\]

by the adjoint action of \( D_t \otimes P_{n_1} \otimes \cdots \otimes P_{n_k} \) on \( M(t, \mathbb{C}) \otimes M(n_1, \mathbb{C}) \otimes \cdots \otimes M(n_k, \mathbb{C}) \cong M(n, \mathbb{C}) \). By Theorem 3.2 of [HPP], any maximal quasitorus of \( PGL(n, \mathbb{C}) \) is conjugate to one and only one of the \( D_t \times P_{n_1} \times \cdots \times P_{n_k} \) with \( n = tn_1 \cdots n_k , t \geq 1 , n_i > 1 \) and each \( n_i \) dividing \( n_{i+1} \). Thus any finite maximal quasitorus in \( PGL(n, \mathbb{C}) \) (one must have \( t = 1 \)) is conjugate to

\[
Q(n_1, \cdots, n_k) = P_{n_1} \times \cdots \times P_{n_k}
\]

(5.8)

where \( n = n_1 \cdots n_k , n_i > 1 \) and each \( n_i \) divides \( n_{i+1} \). The dual group \( \hat{G} \) of \( G = \prod_{i=1}^{k} \mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i} \) in this example is \( Q(n_1, \cdots, n_k) \) in (5.8), which is a maximal quasitorus in \( K = \text{Int}( \mathfrak{sl}(n, \mathbb{C}) ) = PGL(n, \mathbb{C}) \). The group \( Q(n_1, \cdots, n_k) \) is also a maximal quasitorus in \( \text{Aut}( \mathfrak{sl}(n, \mathbb{C}) ) \) except the case \( n_1 = \cdots = n_k = 2 \).

One knows from [H2] that

\[
W_K(\hat{G}) \cong W_K(\Gamma) = W(R) \cong Sp(G, \beta).
\]

5.2. Finite root systems \((G, \beta, R)\) such that \( W(R) \) is a symmetric group

Let \( G_1 = \mathbb{Z}_2^n \). For \( i = 1, \cdots, n \), let \( e_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in G_1 \) with 1 in the \( i \)-th position. Let \( R = \{ e_i + e_j | 1 \leq i < j \leq n \} \) and \( G \cong \mathbb{Z}_2^{n-1} \) be the subgroup of \( G_1 \) generated by \( R \).

Let

\[
\beta : G_1 \times G_1 \to \mathbb{C}^\times , \beta(a, b) = (-1)^{\sum_{i=1}^{n} a_i (\sum_{i=1}^{n} b_i) - \sum_{i=1}^{n} a_i b_i}.
\]

Let

\[
\xi : G_1 \times G_1 \to \mathbb{C}^\times , \xi(a, b) = (-1)^{\sum_{1 \leq j < i \leq n} (a_i b_j)}.
\]

Then \( \Psi(\xi) = \beta \). We will show that \( R \) is a finite root system in \((G, \beta)\). Now we distinguish the cases \( n \) is odd or even.
Example 5.3. Assume $n = 2k + 1$ is odd with $k \geq 1$.

Let $G_1 = \mathbb{Z}_2^{2k+1}$. It is directly verified that $\text{Rad}(\beta) = \{0, (1, \cdots, 1)\}$.

Now $R = \{e_i + e_j|1 \leq i < j \leq n = 2k + 1\}$ and $G \cong \mathbb{Z}_2^{2k}$ is the subgroup generated by $R$. It is easy to verify that $\beta|G$ is nonsingular, and that $R \subset G$ satisfies FRS0, FRS1 and FRS2, thus is a reduced finite root system in $(G, \beta)$. Then we will show that $L(R) \cong \mathfrak{so}(2k + 1, \mathbb{C})$.

For any $e_i + e_j \in R$ we will always assume $i < j$. Define

$$\varphi : L(R) \to \mathfrak{so}(2k + 1, \mathbb{C}), \varphi(u_{e_i+e_j}) = 2(E_{ij} - E_{ji}).$$

Then for any $i, j, s, t \in \{1, \cdots, 2k + 1\}$ with $i < j, s < t$, by simple computations one has

$$\xi(e_i + e_j, e_s + e_t) - \xi(e_s + e_t, e_i + e_j) = 0 \text{ if } \{i, j\} = \{s, t\} \text{ or } \{i, j\} \cap \{s, t\} = \text{O},$$

$$\xi(e_i + e_j, e_i + e_t) - \xi(e_i + e_t, e_i + e_j) = -2 \text{ if } j < t,$$

and

$$\xi(e_i + e_j, e_j + e_t) - \xi(e_j + e_t, e_i + e_j) = 2 \text{ if } j < t.$$

Then

$$[u_{e_i+e_j}, u_{e_s+e_t}] = \begin{cases} 0 & \text{if } \{i, j\} = \{s, t\} \text{ or } \{i, j\} \cap \{s, t\} = \text{O}; \\ -2u_{e_j+e_t} & \text{if } i = s, j < t; \\ 2u_{e_i+e_t} & \text{if } s = j < t. \end{cases}$$

Also one has

$$[2(E_{ij} - E_{ji}), 2(E_{st} - E_{it})] = \begin{cases} 0 & \text{if } \{i, j\} = \{s, t\} \text{ or } \{i, j\} \cap \{s, t\} = \text{O}; \\ -2 \cdot 2(E_{jt} - E_{ij}) & \text{if } i = s, j < t; \\ 2 \cdot 2(E_{it} - E_{it}) & \text{if } s = j < t. \end{cases}$$

So $\varphi : L(R) \cong \mathfrak{so}(2k + 1, \mathbb{C})$ is an isomorphism of Lie algebras and the standard grading on $L(R)$ is in fact the $\mathbb{Z}_2^{2k}$-grading $\Gamma$ on $\mathfrak{so}(2k + 1, \mathbb{C})$:

$$\Gamma : L = \bigoplus_{1 \leq i < j \leq 2k+1} L_{e_i+e_j}$$

with $L = \mathfrak{so}(2k + 1, \mathbb{C})$ and $L_{e_i+e_j} = C(E_{ij} - E_{ji})$.

As $L(R)$ is simple, the finite root system $R$ in this case is reduced and irreducible. We say that it is of type $II(k)$ with $k \geq 1$. As $\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C})$, it is easy to see that

$$II(1) \cong I(2).$$
In this paper let
\[ SO(n, \mathbb{C}) = \{ A \in GL(n, \mathbb{C}) | A^t A = I_n \}. \]

Let \( K = \text{Int}(so(2k+1, \mathbb{C})) = SO(2k+1, \mathbb{C}) \). One knows that \( \hat{G} \cong \mathbb{Z}_2^{2k} \)
embeds as
\[ SO(2k+1, \mathbb{C}) \cap \{ \text{diag}(\pm 1, \cdots, \pm 1) \}, \quad (5.9) \]
thus is a maximal quasitorus in \( K \). So this is a good grading on \( so(2k+1, \mathbb{C}) \).

By [A] one has
\[ W_K(\hat{G}) \cong W_K(\Gamma) = W(R) \cong S_{2k+1} = S_n. \]

Example 5.4. Assume \( n = 2k \) is even with \( k \geq 3 \).

(1) Let \( G_1 = \mathbb{Z}_2^{2k} \). Then \( \text{Rad}(\beta) = \{ 0 \} \). Now \( R = \{ e_i + e_j | 1 \leq i < j \leq 2k \} \) and \( G \cong \mathbb{Z}_2^{2k-1} \) is the subgroup generated by \( R \). Then \( \text{Rad}(\beta|G) = \{ 0, (1, \cdots, 1) \} \). As in last example, \( R \) is a (nonreduced) finite root system in \((G, \beta)\).

And
\[ \varphi : L(R) \rightarrow so(2k, \mathbb{C}), \varphi(u_{e_i+e_j}) = 2(E_{ij} - E_{ji}) \]
defines an isomorphism of \( L(R) \) and \( so(2k, \mathbb{C}) \), and the standard grading on \( L(R) \) is in fact a \( \mathbb{Z}_2^{2k-1} \)-grading on \( so(2k, \mathbb{C}) \).

As \( so(2k, \mathbb{C}) \) is simple for \( k \geq 3 \) and \( \text{Rad}(\beta|G) \neq 0 \), the finite root system \( R \) is nonreduced and irreducible. We say that it is of type \( IV'(k) \) with \( k \geq 3 \).

For \( k \neq 4 \), \( \text{Aut}(so(2k, \mathbb{C})) \cong PO(2k, \mathbb{C}) = O(2k, \mathbb{C})/\{ \pm I_{2k} \} \). If \( k = 4 \) then \( PO(2k, \mathbb{C}) \) is a subgroup of \( \text{Aut}(so(2k, \mathbb{C})) \) of index 3. One knows that \( \hat{G} \) embeds as
\[ (O(2k, \mathbb{C}) \cap \{ \text{diag}(\pm 1, \cdots, \pm 1) \})/\{ \pm I_{2k} \} \quad (5.10) \]
in \( PO(2k, \mathbb{C}) \) which is a maximal quasitorus in \( \text{Aut}(so(2k, \mathbb{C})) \) (including the case \( k = 4 \)). So this is a good grading on \( so(2k, \mathbb{C}) \).

Let \( K = \text{Int}(so(2k, \mathbb{C})) \). Then by simple computation one has
\[ W_K(\hat{G}) \cong W_K(\Gamma) = W(R) \cong S_{2k} = S_n. \]

(2) (Continued) Let \( H = \text{Rad}(\beta|G) \cong \mathbb{Z}_2 \) and \( \overline{G} = G/H \cong \mathbb{Z}_2^{2k-2} \).
Then as \( so(2k, \mathbb{C}) \) is simple for \( k \geq 3 \), by Lemma 3.8 \( R \) is a reduced and irreducible finite root system in \( \overline{G} \). We say that it is of type \( IV(k) \) with \( k \geq 3 \). As \( so(6, \mathbb{C}) \cong sl(4, \mathbb{C}) \), it is not hard to check that
\[ IV(3) \cong I(2, 2). \]
Let $T$ be the dual group of $\mathcal{G}$, which embeds as

$$\SO(2k, \mathbb{C}) \cap \{\text{diag}(\pm 1, \ldots, \pm 1)\}/\{\pm I_{2k}\}$$

in $\text{Aut}(\mathfrak{so}(2k, \mathbb{C})) = O(2k, \mathbb{C})/\{\pm I_{2k}\}$, which is a maximal quasitorus in $\text{Int}(\mathfrak{so}(2k, \mathbb{C})) = \SO(2k, \mathbb{C})/\{\pm I_{2k}\}$. So this is a good inner grading on $\mathfrak{so}(2k, \mathbb{C})$.

Let $K = \text{Int}(\mathfrak{so}(2k, \mathbb{C}))$. By [A] one also has

$$W_K(T) \cong W_K(\Gamma) = W(R) \cong S_{2k} = S_n.$$  

5.3. Finite root systems constructed from nonsingular quadratic forms over $\mathbb{F}_2$  

Let $q : \mathbb{F}_2^{2k+1} \rightarrow \mathbb{F}_2$ be a quadratic form and $\alpha_1 : \mathbb{F}_2^{2k+1} \times \mathbb{F}_2^{2k+1} \rightarrow \mathbb{F}_2$ be its polarization, i.e., $\alpha_1(a, b) = q(a + b) - q(a) - q(b)$. Assume that $q$ is nonsingular, i.e., $\alpha_1$ is an alternating bilinear form whose radical is 1-dimensional and $\text{Rad}(q) = 0$, where $\text{Rad}(q) = \{a \in \text{Rad}(\alpha_1)|q(a) = 0\}$. One knows that there is only one type of nonsingular quadratic form on $\mathbb{F}_2^{2k+1}$ up to isomorphism, and

$$h : \mathbb{F}_2^{2k+1} \rightarrow \mathbb{F}_2, h(a) = \sum_{i=1}^{k} a_{2i-1}a_{2i} + a_{2k+1}^2$$

is a such nonsingular quadratic form. One has

$$\alpha_1(a, b) = h(a + b) - h(a) - h(b) = \sum_{i=1}^{k} (a_{2i}b_{2i-1} - a_{2i-1}b_{2i})$$

Identify the group $G = \mathbb{Z}_2^{2k+1}$ with the additive group of $\mathbb{F}_2^{2k+1}$. Let

$$\alpha : G \times G \rightarrow \mathbb{C}^\times, \alpha(a, b) = (-1)^{\alpha_1(a, b)}.$$  

Lemma 5.5. Assume $k \geq 2$. Then the subset

$$R = \{a \in G \setminus \text{Rad}(\alpha)|h(a) = 1\}$$

is a finite root system in the symplectic abelian group $(G, \alpha)$.

Proof. FRS1 is clear. Now we prove FRS2. Note that $\text{Rad}(\alpha) = \text{Rad}(\alpha_1) = \{0, e_{2k+1}\}$. Assume that $a, b \in R$ and $\alpha(a, b) \neq 1$, i.e., $\alpha_1(a, b) \neq 0$. Then $h(a + b) = h(a) + h(b) + \alpha_1(a, b) = 1$. If $a + b = 0$ then $a = b$ which contradicts
to $\alpha(a, b) \neq 1$. If $a + b = e_{2k+1}$ then $h(a) \neq h(b)$ which contradicts to $a, b \in R$. Thus $a + b \in R$.

Finally we prove FRS0. It is easy to see that the subset

$$B = \{e_{2i-1} + e_{2i}, e_{2i} + e_{2k+1} | i = 1, \ldots, k\} \cup \{e_j \}_{j=2k-3}^{2k+1}$$

of $R$ generates $G$.

This finite root system $(\mathbb{Z}_2^{2k+1}, \alpha, R)$ ($k \geq 2$) is nonreduced with $\text{Rad}(\alpha) = \{0, e_{2k+1}\}$, and is said to be of type $I'(2, \ldots, 2)$, where the number of 2’s is $k$.

**Lemma 5.6.** Assume $k \geq 2$. The corresponding reduced finite root system of $(\mathbb{Z}_2^{2k+1}, \alpha, R)$ is of type $I(2, \ldots, 2)$. One has $(\mathbb{Z}_2^{2k+1}, \alpha, R)$ is nonreduced and irreducible, and $L(R) = L(\overline{R}) = \mathfrak{sl}(2k, \mathbb{C})$.

**Proof.** Let $G = \mathbb{Z}_2^{2k+1}$. As $\text{Rad}(\alpha) = \{0, e_{2k+1}\}$, $\overline{G} = G/\text{Rad}(\alpha) = \mathbb{Z}_2^{2k}$, and the quotient group homomorphism is $p : \mathbb{Z}_2^{2k+1} \rightarrow \mathbb{Z}_2^{2k}, (a_1, \ldots, a_{2k+1}) \mapsto (a_1, \ldots, a_{2k})$. Thus

$$\overline{\alpha_1}(a, b) = \sum_{i=1}^{k} (a_{2i}b_{2i-1} - a_{2i-1}b_{2i})$$

and

$$\overline{\alpha} : \overline{G} \times \overline{G} \rightarrow \mathbb{C}^\times, \overline{\alpha}(a, b) = (-1)^{\overline{\alpha_1}(a, b)}$$

is nonsingular.

Next we will show that $\overline{R} = \mathbb{Z}_2^{2k} \setminus \{0\}$. It is clear that $\overline{R} \subseteq \mathbb{Z}_2^{2k} \setminus \{0\}$. For any $a \in \mathbb{Z}_2^{2k} \setminus \{0\}$, if $h(a, 0) = 1$ then $(a, 0) \in R$ and $p(a, 0) = a$; if $h(a, 0) = 0$ then $(a, 1) \in R$ and $p(a, 1) = a$. Thus the corresponding reduced finite root system of $(\mathbb{Z}_2^{2k+1}, \alpha, R)$ is $(\mathbb{Z}_2^{2k}, \overline{\alpha}, \mathbb{Z}_2^{2k} \setminus \{0\})$, which is of type $I(2, \ldots, 2)$. As $L(R) \cong \mathfrak{sl}(2k, \mathbb{C})$ is simple, by Lemma 3.8, $R$ is irreducible and $L(R) = L(\overline{R}) = \mathfrak{sl}(2k, \mathbb{C})$.

Recall that $\text{Aut}(\mathfrak{sl}(2k, \mathbb{C})) = \langle PGL(2k, \mathbb{C}), \tau \rangle$ where $\tau : \mathfrak{sl}(2k, \mathbb{C}) \rightarrow \mathfrak{sl}(2k, \mathbb{C}), A \mapsto -A^t$ is an involutive outer automorphism of $\mathfrak{sl}(2k, \mathbb{C})$ for $k > 1$. The dual group $\widetilde{G}$ is embedded as the maximal quasitorus

$$< Q(2, \ldots, 2), \tau > \cong \mathbb{Z}_2^{2k+1} \tag{5.13}$$

in $\text{Aut}(\mathfrak{sl}(2k, \mathbb{C}))$. 

29
Similar as in Example 5.2, one has
\[ W_K(\hat{G}) \cong W_K(\Gamma) = W(R) \cong Sp(2k, \mathbb{F}_2). \]

**Example 5.7.** Assume \( G = \mathbb{Z}_{2k}^2 \) with \( k \geq 3 \). Let \( R = \{ a \in G \mid g(a) = 1 \} \). By Lemma 3.11, \( R \) is a finite root system in \((G, \beta)\). One knows that \( L(R) \cong \mathfrak{so}(2k, \mathbb{C}) \) by Lemma 4.1 (1), and the corresponding grading is (4.6). So the finite root system \( R \) is reduced and irreducible.

On \( \mathbb{C}^2 \) there is a standard nonsingular orthogonal bilinear form \( \varphi_0 \) with matrix \( I_2 \). It is clear that \( P_2 \subset O(2, \mathbb{C}) \). Equip \((\mathbb{C}^2)^{\otimes k} \cong \mathbb{C}^{2k} \) with the nonsingular symmetric bilinear form \( \varphi_0^{\otimes k} \). Then \( P_2 \otimes \cdots \otimes P_2 \subset SO(2k, \mathbb{C}) \) for \( k \geq 2 \). Then \( Q(2, \cdots, 2) = P_2 \times \cdots \times P_2 (\cong \mathbb{Z}_{22}^k) \subset PSO(2k, \mathbb{C}) = \text{Int}(\mathfrak{so}(2k, \mathbb{C})) \). It is directly verified that in this case the dual group \( G \) of \( G = \mathbb{Z}_{22}^k \) embeds as the finite maximal quasitorus
\[ Q(2, \cdots, 2) = P_2 \times \cdots \times P_2 \cong \mathbb{Z}_{22}^k \] (5.14) in \( \text{Aut}(\mathfrak{so}(2k, \mathbb{C})) \). We say that it is of type \( V(k) \) with \( k \geq 3 \).

By \([A]\) one knows that
\[ W_K(\hat{G}) \cong W_K(\Gamma) = W(R) \cong O(G, g), \]
where \( G = \mathbb{Z}_{22}^k \) and \( O(G, g) \) denotes the group of linear isomorphisms of \( G \) preserving \( g \), usually denoted by \( O_{22}^k(2) \).

Note that the abelian groups in \( IV(4) \) and \( V(3) \) are both \( \mathbb{Z}_2^6 \) and the respective \( L(R) \) are both isomorphic to \( \mathfrak{so}(8, \mathbb{C}) \), but

**Lemma 5.8.**
\[ IV(4) \not\cong V(3). \]

**Proof.** Assume that \((G_1, \beta_1, R_1)\) is of the type \( IV(4) \) and \((G_2, \beta_2, R_2)\) is of the type \( V(3) \). Then \( \hat{G}_1 \) embeds in \( \text{Aut}(\mathfrak{so}(8, \mathbb{C})) = PO(8, \mathbb{C}) \) as in (5.11), which is not maximal abelian in \( PO(8, \mathbb{C}) \) as its centralizer in \( PO(8, \mathbb{C}) \) is as in (5.10). Since \( \hat{G}_2 \) is a maximal abelian subgroup in \( \text{Aut}(\mathfrak{so}(8, \mathbb{C})) \), \( \hat{G}_1 \) and \( \hat{G}_2 \) cannot be conjugate in \( \text{Aut}(\mathfrak{so}(8, \mathbb{C})) \) and \( IV(4) \not\cong V(3) \). \[\square\]
Example 5.9. Assume $G = \mathbb{Z}_{2}^{2k}$ with $k \geq 1$. Let $R = \{a \in G | f(a) = 1\}$. By Lemma 3.12, $R$ is a finite root system in $(G, \beta)$. One knows that

$$L(R) \cong \mathfrak{sp}(2^k, \mathbb{C})$$

by Lemma 4.1 (2), and the corresponding grading is (4.7).

So the finite root system $R$ is reduced and irreducible. We say that it is of type $III(k)$ with $k \geq 1$. As $\mathfrak{sp}(2^k, \mathbb{C}) \cong \mathfrak{sl}(2^k, \mathbb{C})$ and $\mathfrak{sp}(4, \mathbb{C}) \cong \mathfrak{so}(5, \mathbb{C})$, one can verify that $III(1) \cong I(2)$ and $III(2) \cong II(2)$.

On $\mathbb{C}^2$ there is a standard nonsingular skew-symmetric bilinear form $\varphi_1$ with matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Equip $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \cong \mathbb{C}^{2k}$ with the nonsingular skew-symmetric bilinear form $\varphi_1 \otimes \varphi_0 \otimes \cdots \otimes \varphi_0$. As $P_2 \subset \mathbb{C}^\times \mathfrak{Sp}(2, \mathbb{C})$ and $P_2 \subset O(2, \mathbb{C})$, $P_2 \otimes \cdots \otimes P_2$ acts on $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \cong \mathbb{C}^{2k}$ and $P_2 \otimes \cdots \otimes P_2 \subset \mathbb{C}^\times \mathfrak{Sp}(2^k, \mathbb{C})$. Then $Q(2, \cdots , 2) = P_2 \times \cdots \times P_2 \cong \mathbb{Z}_2^{2k} \subset P\mathfrak{Sp}(2^k, \mathbb{C}) = \text{Aut}(\mathfrak{sp}(2^k, \mathbb{C}))$. In this case the dual group $\widehat{G}$ of $G = \mathbb{Z}_{2}^{2k}$ embeds as the finite maximal quasitorus

$$Q(2, \cdots , 2) = P_2 \times \cdots \times P_2 \cong \mathbb{Z}_2^{2k} \quad \text{(5.15)}$$

in $\text{Aut}(\mathfrak{sp}(2^k, \mathbb{C}))$.

By [A], one knows that

$$W_K(\widehat{G}) \cong W_K(\Gamma) = W(R) \cong O(G, f),$$

where $G = \mathbb{Z}_{2}^{2k}$ and $O(G, f)$ denotes the group of linear isomorphisms of $G$ preserving $f$, usually denoted by $O^{2k}(2)$.

Proposition 5.10. Assume that $L$ is a complex simple Lie algebra, $T$ is a good finite maximal quasitorus in $\text{Aut}(L)$ or in $\text{Int}(L)$. If $(L, T)$ is as in above examples of this section, then there corresponds to $(L, T)$ a unique finite root system $(G, \beta, R)$ up to isomorphism such that $L = L(R)$ and the standard grading on $L(R)$ is isomorphic to the grading on $L$ induced by $T$.

Proof. In the above examples, we constructed finite root systems $(G, \beta, R)$ and identified $L = L(R)$ with the corresponding standard gradings on it, then gave the embedding of $T = \widehat{G}$ in $\text{Aut}(L)$ or $\text{Int}(L)$ and showed that $T$ is a finite maximal quasitorus in $\text{Aut}(L)$ or $\text{Int}(L)$. So $(G, \beta, R)$ is good, and is clearly a finite root system corresponding to $(L, T)$. We only need to
prove that it is the unique finite root system corresponding to \((L,T)\) up to isomorphism.

If \((L,T)\) is as in Example 5.2 (Example 5.1 is a special case of Example 5.2), then \(L = \mathfrak{sl}(n,\mathbb{C})\), \(T\) is as in 5.8 and the grading is as in 5.7. Thus if \((G,\beta,R)\) is a finite root system corresponding to \((L,T)\), then \(G = \hat{T}\) and \(R = G \setminus \{0\}\). Since \(R \subseteq G \setminus \text{Rad}(\beta)\), one has \(\text{Rad}(\beta) = \{0\}\) and thus \(\beta\) is nonsingular. By Theorem 1.8 of [Ka3], if there are two nonsingular alternating bicharacters \(\beta_i\) on \(G\) for \(i = 1,2\), then \((G,\beta_1)\) and \((G,\beta_2)\) are isometric. So the finite root system \((G,\beta,R)\) corresponds to \((L,T)\) is unique up to isomorphism.

If \((L,T)\) is as in other examples, then \(G = \hat{T}\) is always an elementary abelian 2-groups. So the finite root system corresponding to \((L,T)\) is unique up to isomorphism by Lemma 3.9.

\[
6 \quad \text{Classification of good finite maximal quasitorus in } \text{Aut}(L) \text{ with } L \text{ a complex simple Lie algebra and main results}
\]

Now we state our main result.

**Theorem 6.1.** Assume that \(L\) is a complex simple Lie algebra, \(T\) is a good finite maximal quasitorus in \(\text{Aut}(L)\) or in \(\text{Int}(L)\).

(1) There corresponds to \((L,T)\) a unique irreducible finite root system \((G,\beta,R)\) up to isomorphism.

(2) The pair \((L,T)\) and corresponding finite root system \(R\), where \(T\) is up to conjugation in \(\text{Aut}(L)\), is one and only one of the following.

1. \(L = \mathfrak{sl}(n,\mathbb{C}), n \geq 2\).
   
   (a) Assume \(n = n_1 \cdots n_k\) with \(n_i|n_{i+1}\) for \(i = 1,\ldots,k-1\) and \(n_i > 1\). Then \(T \cong \mathbb{Z}_{n_1}^2 \times \cdots \times \mathbb{Z}_{n_k}^2\) which embeds in \(\text{Int}(L)\) as in (5.8) is always a good finite maximal quasitorus in \(\text{Int}(L)\), which is also a good finite maximal quasitorus in \(\text{Aut}(L)\) except \(k \geq 2\) and \(n_1 = \cdots = n_k = 2\). The corresponding finite root system \(R\) is of type \(I(n_1,\ldots,n_k)\).
   
   (b) Assume \(n = 2^k, k \geq 2\). Then \(T \cong \mathbb{Z}_{2^{k+1}}^2\) which embeds in \(\text{Aut}(L)\) (and \(T \nsubseteq \text{Int}(L)\)) as in (5.13) is a good finite maximal quasitorus in \(\text{Aut}(L)\). The corresponding finite root system \(R\) is of type \(I'(2,\cdots,2)\).

2. \(L = \mathfrak{so}(2k+1,\mathbb{C}), k \geq 2\). Then \(T \cong \mathbb{Z}_2^{2k}\) which embeds in \(\text{Aut}(L)\) as in (5.9) is a good finite maximal quasitorus in \(\text{Aut}(L)\). The finite root
system $R$ is of type $II(k)$. In this case $\text{Aut}(L) = \text{Int}(L)$ and $T$ is also a good finite maximal quasitorus in $\text{Int}(L)$.

3. $L = \mathfrak{sp}(2k,\mathbb{C}), k \geq 3$, Then $T \cong \mathbb{Z}_2^{2k}$ which embeds in $\text{Aut}(L)$ as in (5.12) is a good finite maximal quasitorus in $\text{Aut}(L)$. The finite root system $R$ is of type $III(k)$. In this case $\text{Aut}(L) = \text{Int}(L)$ and $T$ is also a good finite maximal quasitorus in $\text{Int}(L)$.

4. $L = \mathfrak{so}(2k,\mathbb{C}), k \geq 4$. Then $T \cong \mathbb{Z}_2^{2k-1}$ which embeds in $\text{Aut}(L)$ (and $T \not\subseteq \text{Int}(L)$) as in (5.14) is a good finite maximal quasitorus in $\text{Aut}(L)$. The finite root system $R$ is of type $IV'(k)$. In this case $T \cap \text{Int}(L) \cong \mathbb{Z}_2^{2k-2}$ is a good finite maximal quasitorus in $\text{Int}(L)$, and the finite root system $R$ is of type $IV(k)$.

5. $L = \mathfrak{so}(2k,\mathbb{C}), k \geq 3$. Then $L \cong \mathbb{Z}_2^{2k}$ which embeds in $\text{Int}(L)$ as in (5.14) is a good finite maximal quasitorus in $\text{Aut}(L)$ as well as in $\text{Int}(L)$, and the finite root system $R$ is of type $V(k)$.

In particular there is no good finite maximal quasitorus in $\text{Aut}(L)$ for exceptional simple Lie algebras $L$.

We will prove it later.

Remark 6.2. If $T$ is a good finite maximal quasitorus in $\text{Aut}(L)$ for $L$ a simple Lie algebra, then either $T \subseteq \text{Int}(L)$ or $T \not\subseteq \text{Int}(L)$, in the latter case $T \cap \text{Int}(L)$ is a good finite maximal quasitorus in $\text{Int}(L)$. The corresponding irreducible finite root system of $(L,T)$ is reduced if $T$ is a good finite maximal quasitorus in $\text{Int}(L)$, and is nonreduced if $T$ is a good finite maximal quasitorus in $\text{Aut}(L)$ but $T \not\subseteq \text{Int}(L)$.

Remark 6.3. Let $K = \text{Int}(L)$ and $T = \hat{G}$. One knows from [A] and [H2] that in all the cases in the list, the Weyl group $W_K(T)$ of $T$ with respect to $K$ is isomorphic to the Weyl group $W(R)$, i.e., $W_K(T)$ is generated by root transvections. So Vogan’s conjecture holds in these cases. The Weyl group in Case (a) is $Sp(G,\beta)$ by [H2]. And the Weyl group in Case (b), (c), (d) and (e) are respectively $S_{2k+1}$, $O(G,f)$, $S_{2k}$ and $O(G,g)$ by [A], where $O(G,f)$ and $O(G,g)$ are the respective group of linear isomorphisms of $G$ preserving $f$ and $g$. $O(G,f)$ and $O(G,g)$ are usually denoted by $O^{2k}(2)$ and $O^{2k}_+(2)$ respectively. One also notes in [A] that in Case (b), (c), (d) and (e), $G$ is a subgroup of $SL(n,\mathbb{F}_2)$ acting irreducible on $\mathbb{F}_2^n$.

Corollary 6.4. Assume that $(G,\beta,R)$ is a finite root system such that $L(R)$ is a simple Lie algebra and the standard grading on $L(R)$ is a good inner grading, then $(G,\beta,R)$ is reduced and irreducible, and the type of $(G,\beta,R)$ is one and only one of the following:

(1) $I(n_1,\cdots,n_k)$ with $k \geq 1$, each $n_i > 1$ and $n_i | n_{i+1}$ for $i = 1,\cdots,k-1$;
(2) $II(k)$ with $k \geq 2$;
(3) $III(k)$ with $k \geq 3$;
(4) $IV(k)$ with $k \geq 4$;
(5) $V(k)$ with $k \geq 3$.

Proof. By Theorem 6.1 and Remark 6.2, one only need to verify that there is no redundancy in the list. Assume that two finite root systems $(G_i, \beta_i, R_i)$ $(i = 1, 2)$ in the list are isomorphic. Then $L(R_1) \cong L(R_2)$, which is possible only if they are of types $IV(m)$ and $V(k)$ respectively, where $m \geq 4, k \geq 3$ and

$$2m = 2^k. \quad (6.1)$$

Assume that $(G_1, \beta_1, R_1)$ is of type $IV(m)$ and $(G_2, \beta_2, R_2)$ is of type $V(k)$. Then $\hat{G}_1 \cong \mathbb{Z}_{2^{m-2}}^2$ and $\hat{G}_2 \cong \mathbb{Z}_{2^k}^2$ are isomorphic. So

$$2m - 2 = 2k. \quad (6.2)$$

Solving $(6.1)$ and $(6.2)$ one has $m = 4, k = 3$. But $IV(4) \not\cong V(3)$ by Lemma 5.8.

Now we will prove Theorem 6.1. Recall that for a complex simple Lie algebra $L$, a finite maximal quasitorus $T$ of $\text{Aut}(L)$ or $\text{Int}(L)$ is said to be good if $\dim L_\alpha \leq 1$ for every character $\alpha \in \hat{T}$. The main part of the proof is to classify good finite maximal quasitorus $T$ of $\text{Aut}(L)$ (or $\text{Int}(L)$). If the classification is done, then for each $(L, T)$ the finite root system $(G, \beta, R)$ is constructed in Section 5. Theorem 6.1 (1) follows from Proposition 5.10.

Now we classify good finite maximal quasitorus $T$ of $\text{Aut}(L)$ (or $\text{Int}(L)$). Since the finite maximal quasitorus $T$ of $\text{Aut}(L)$ will stabilize some compact real form $L_0$ of $L$, $T \subset \text{Aut}(L_0)$. Thus it is equivalent to the classification of finite maximal quasitorus of $\text{Aut}(L_0)$ (or $\text{Int}(L_0)$). To do this we use the classification of maximal abelian subgroups of compact simple Lie groups in [Y]. We will also show that there are no good gradings on exceptional simple Lie algebras.

When $L_0 = \mathfrak{su}(n)$ $(n \geq 2)$ and $T \subset \text{Int}(L_0)$ is a finite maximal abelian subgroup, there is no much to say, as every finite maximal abelian subgroup is of the form $(3.8)$ and satisfies the condition. We remark that, in this case $T$ is also a maximal abelian subgroup of $\text{Aut}(L_0)$ except when it is an elementary abelian 2-subgroup. In the latter case $T$ commutes with an outer involution $\tau$ in the conjugacy class of complex conjugation and $\langle \tau, T \rangle$ is a maximal abelian subgroup of $\text{Aut}(L_0)$. 

34
When \( L_0 = \mathfrak{so}(n) \ (n \geq 5) \) and \( T \subset \text{PO}(n) \) is a finite maximal abelian subgroup (this means only the case of \( L_0 = \mathfrak{so}(8) \) and \( T/T \cap \text{Int}(L_0) \cong C_3 \) needs a further investigation), let \( \pi : O(n) \to \text{PO}(n) \) be the natural projection. By [Y] Subsection 3.2], there is an antisymmetric and bimultiplicative function \( m : T \times T \to \{ \pm 1 \} \) and \( \text{ker} \ m \) is a diagonalizable subgroup (since \( T \) is assumed to a finite maximal abelian subgroup, \( \text{ker} \ m = B_T \) by [Y] Lemma 3.3], where \( B_T \) is a subgroup defined in [Y]). By [Y] Proposition 3.3], there exists integers \( s_0 \geq 1 \) and \( k \geq 0 \) such that \( n = s_0 \cdot 2^k \), and the centralizer of \( \pi^{-1}(T) \) in \( O(n) \) is

\[
C_{O(n)}(\pi^{-1}(T)) = \underbrace{O(2^k) \times \cdots \times O(2^k)}_{s_0},
\]

and \( \pi^{-1}(\text{ker} \ m) = Z(C_{O(n)}(\pi^{-1}(T)), \text{the center of} \ C_{O(n)}(\pi^{-1}(T)) \). We show that \( k = 0 \), or \( s_0 = 1 \), or \( (s_0, k) = (3, 1) \). Moreover, there is a unique conjugacy class of \( T \) while \( (s_0, k) = (3, 1) \). Let \( E_i = \pi(\text{diag}\{(i - 1) \cdot I_{2^k}, I_{2^k}, (s_0 - i) \cdot I_{2^k}\}) \). For any character \( \alpha \in \text{Hom}(T, S^1) \), if \( L_\alpha \neq 0 \) and \( \alpha|_{\text{ker} \ m} \neq 1 \), then there exists \( i \neq j \), \( \alpha(E_k) = -1 \) if \( k \neq i, j \) and \( \alpha(E_k) = 1 \) if \( k = i \) or \( j \). Given \( i \neq j \) and any character \( \alpha \in \text{Hom}(T, S^1) \) with \( \alpha(E_k) = -1 \) if \( k \neq i, j \) and \( \alpha(E_k) = 1 \) if \( k = i \) or \( j \), by calculation one shows that \( \dim L_\alpha = 1 \). Given a character \( \alpha \in \text{Hom}(T, S^1) \) with \( \alpha|_{\text{ker} \ m} = 1 \), the root space \( \dim L_\alpha \) is contained in the complexified Lie algebra of

\[
C_{O(n)}(\pi^{-1}(T)) = \underbrace{O(2^k) \times \cdots \times O(2^k)}_{s_0}.
\]

In [Y], we defined functions \( \mu_1, \ldots, \mu_{s_0} : T/\text{ker} \ m \to \{ \pm 1 \} \) by

\[
(A_1^2, \ldots, A_{s_0}^2) = (\mu_1(x)I_{2^k}, \ldots, \mu_{s_0}(x)I_{2^k})
\]

for any \( x \in T \) and \( \text{diag}\{A_1, \ldots, A_{s_0}\} \in \pi^{-1}(x) \). Let \( L_1 \) be the complexified Lie algebra of the \( i \)-th factor of \( C_{O(n)}(\pi^{-1}(T)) \). By calculation one shows that \( \dim(L_\alpha \cap L_i) \leq 1 \) and equality holds if and only if \( \mu_i(\text{ker}(m|_{\text{ker} \ m})) = -1 \). From these, we conclude that in the case of \( s_0 = 1 \) or \( k = 0 \), \( \dim L_\alpha \leq 1 \) for any character \( \alpha \); in the case of \( s_0 \geq 2 \) or \( k \geq 1 \), \( \dim L_\alpha \leq 1 \) for any character \( \alpha \) if and only if for any \( i \neq j \), there exists no \( x \in T \) with \( \mu_i(x) = \mu_j(x) = -1 \). Since \( \{x \in T|\mu_i(x) = \mu_j(x)\} \) is an index 2 subgroup, there exits no \( x \in T \) with \( \mu_i(x) = \mu_j(x) = -1 \) implies \( k = 1 \) and \( \mu_i \neq \mu_j \). Moreover, we have \( s_0 = 3 \) since \( n = s_0 \cdot 2^k \geq 5 \). Finally, as \( \mu_1, \mu_2, \mu_3 \) are non-equal to each other, by [Y] Proposition 3.4], we get a unique conjugacy class in the case of \( (s_0, k) = (3, 1) \). As \( \mathfrak{so}(6) \cong \mathfrak{su}(4) \), actually one shows that the above
subgroup $T$ of $\text{Aut}(\mathfrak{so}(6))$ corresponds to a subgroup of $\text{Int}(\mathfrak{su}(4))$ isomorphic to $(\mathbb{Z}/4\mathbb{Z})^2$ in the case $(s_0, k) = (3, 1)$. If $s_0 = 1$ then $n = 2^k$ and the finite maximal quasitorus $T \cong \mathbb{Z}_2^{2k}$ is of the form (5.14); if $k = 0$ then the finite maximal quasitorus $T \cong \mathbb{Z}_2^{n-1}$, which is of the form (5.9) if $n$ is odd and is of the form (5.10) if $n$ is even.

When $L_0 = \mathfrak{sp}(n)$ $(n \geq 2)$ and $T \subset \text{PSp}(n)$ is a finite maximal abelian subgroup, similarly there are integers $s_0 \geq 1$ and $k \geq 1$ with $n = s_0 \cdot 2^{k-1}$. An argument similar as the above for projective orthogonal groups shows $s_0 = 1$ if $\dim L_\alpha \leq 1$ for each $\alpha \in \text{Hom}(T, S^1)$. The case $k = 0$ can not happen since we have $k \geq 1$ automatically, and there is no exception like $(s_0, k) = (3, 1)$ in this case. In the case $s_0 = 1$ the finite maximal quasitorus $T \cong \mathbb{Z}_2^{2k}$ is of the form (5.15).

When $L_0 = \mathfrak{su}(n)$ $(n \geq 3)$, $T \subset \text{Aut}(\mathfrak{su}(n))$ is a finite maximal abelian subgroup, and $T \not\subset \text{Int}(\mathfrak{su}(n))$, similarly there are integers $s_0 \geq 1$ and $k \geq 0$ with $n = s_0 \cdot 2^k$ ([Y Proposition 3.10]). Similarly as for projective orthogonal groups, one shows that $s_0 = 1$ or $k = 0$ if $\dim L_\alpha \leq 1$ for each $\alpha \in \text{Hom}(T, S^1)$. When $s_0 = 1$, $T$ is conjugate to a subgroup of the form $\langle \tau_0, T' \rangle$, where $\tau_0$ is complex conjugation and $T'$ is an elementary abelian 2-subgroup of $\text{PO}(2^k)$ with rank $2k$ and with a nondegenerate skew-symmetric and bimultiplicative function $m : T' \times T' \to \{\pm 1\}$. This subgroup satisfies $\dim L_\alpha \leq 1$ for each weight $\alpha \in \text{Hom}(T, S^1)$. Note that in this case the root spaces of the group grading induced by $T$ and $T'$ coincides (that means if $g_\alpha, g_\beta \neq 0$, then $\alpha = \beta$ if and only if $\alpha|_{T'} = \beta|_{T'}$). The group $T'$ is a maximal elementary abelian 2-subgroup of $\text{Int}(\mathfrak{su}(n))$. The finite maximal quasitorus $T$ in the case $s_0 = 1$ is of the form (5.13). When $k = 0$, $T$ is conjugate to a subgroup of the form $\langle \tau_0, T' \rangle$, where $\tau_0$ is complex conjugation and $T'$ is an elementary abelian 2-subgroup consisting of all diagonal elements in $\text{PO}(n)$. For the character with $\alpha|_{T'} = 1$ and $\alpha(\tau_0) = -1$, $\dim L_\alpha = n - 1 > 1$. Hence, this subgroup is not good.

Now we turn to exceptional groups. If a finite abelian subgroup $T$ is good, one must have $|T| \geq \dim L + 1$. When $L_0 = \mathfrak{so}(8)$ and $T/T \cap \text{Int}(L_0) \cong C_3$, by [Y Proposition 5.1], there are two conjugacy classes of finite maximal abelian subgroups, with order 27 and 24 respectively. Both orders are smaller than $\dim \mathfrak{g} = 28$ and hence they are not good.

When $L_0$ is of type $G_2$ (or $F_4$), there is a unique conjugacy classes (or two conjugacy classes) of finite maximal abelian subgroups with order 8 (or 27, 32), they are not good as the order is less than $\dim \mathfrak{g}$.

When $L_0$ is of type $E_6$, any finite maximal abelian subgroup $T$ with $|T| > \dim L$ is conjugate to one of $F_3, F_{12}$ (cf. [Y Table 5 and Table 6]). The group $F_3 \cong (C_3)^4$ is not good since $\dim L_\alpha = 3$ if $\alpha|_{\{x \in F_3 | x \sim \theta_3\}} = 1$.
The $F_{12}$ is not good since $\dim L_\alpha = 6$ if $\alpha | F_{12} \cap \text{Int}(L_0) = 1$.

When $L_0$ is of type $\mathbf{E}_7$, any finite maximal abelian subgroup $T$ with $|T| > \dim L$ is conjugate to $F_7$ (see [Y, Table 7]). The group $F_7 \cong (C_2)^8$ is not good since $\dim L_\alpha = 7$ if $\alpha | F_7 \cap \text{Int}(L_0) = 1$ (see [Y, Paragraph after Proposition 8.2]).

When $L_0$ is of type $\mathbf{E}_8$, any finite maximal abelian subgroup $T$ with $|T| > \dim L$ is conjugate to one of $F_4$, $F_5$, $F_7$, $F_8$, $F_9$ (see [Y, Table 8]). The group $\text{Aut}(\mathfrak{e}_8)$ has a Klein four-subgroup $\Gamma_1$ with (see [HY, Table 6])

$$G^{\Gamma_1} \cong (E_6 \times U(1) \times U(1))/\langle (c, \omega, 1) \rangle \rtimes \langle z \rangle,$$

where $(\mathfrak{e}_6 \oplus i\mathbb{R} \oplus i\mathbb{R})^z = 1 \oplus 0 \oplus 0$. By [Y, Proposition 11.2], each of $F_4$, $F_5$, $F_7$, $F_8$ is conjugate to a subgroup of the form $T = T' \times \Gamma_1$, where $T' \subset E_6 \rtimes \langle z \rangle$. Since $T'$ is not good, $T$ is not as well. The group $F_9$ is not good since $\dim L_\alpha = 8$ if $\alpha | F_9 \cap \text{Int}(L_0) = 1$.

References

[A] A.V. Alekseevskii, *Finite commutative Jordan subgroups of complex simple Lie groups*, Funct. Anal. Appl., 8 (1974), no.4, 277-279.

[BSZ] Y.A. Bahturin, I.P. Shestakov, M.V. Zaicev, *Gradings on simple Jordan and Lie algebras*, J. Algebra, 283 (2005), no.2, 849-868.

[BK] Y. Bahturin, M. Kochetov *Classification of group gradings on simple Lie algebras of types A,B,C and D*, J. Algebra 324 (2010), no.11, 2971-2989.

[BZ] Y. Bahturin;M. Zaicev, *Gradings on Simple Lie Algebras of Type A*, J.Lie Theory 16 (2006), no. 4, 719-742.

[DM1] C. Draper, M. Cristina, *Gradings on G2*. Linear Algebra Appl. 418 (2006), no. 1, 85 C111

[DM2] C. Draper, M. Cristina, *Gradings on the Albert algebra and on $F_4$*. Rev. Mat. Iberoam. 25 (2009), no. 3, 841-908.

[EK] A. Elduque; M. Kochetov, *Weyl groups of fine gradings on simple Lie algebras of types A, B, C and D*. Serdica Math. J. 38 (2012), no. 1-3, 7-36.
[HV] G. Han and D. Vogan, *Finite maximal tori*, available at www.math.mit.edu/dav/finitetori.pdf, to appear in ’Symmetry: Representation Theory and Its Applications’.

[HY] J.-S. Huang and J. Yu, *Klein four subgroups of Lie algebra automorphisms*. Pacific J. Math. 262 (2013), no. 2, 397-420.

[DV] C. Draper, A. Viruel, *Group gradings on $\mathfrak{o}(8, \mathbb{C})$*. Rep. Math. Phys. 61 (2008), no. 2, 265-280.

[H1] G. Han, *The symmetries of the fine gradings of $\mathfrak{sl}(n^k, \mathbb{C})$ associated with direct product of Pauli groups*. J. Math. Phys. 51 (2010), no. 9, 092104, 15 pp.

[H2] G. Han, *The Weyl group of the fine grading of $\mathfrak{sl}(n, \mathbb{C})$ associated with tensor product of generalized Pauli matrices*. J. Math. Phys. 52 (2011), no. 4, 042109, 18 pp.

[HPP] M. Havlicek, J. Patera and E. Pelantova, *On Lie gradings II*, Linear Algebra Appl. 277, (1998),no. 1-3 , 97-125.

[Ka2] G. Karpilovsky, *Group representations Vol. 2*, North-Holland Mathematics Studies 180 (North-Holland Publishing Co., Amsterdam, 1994).

[Ka3] G. Karpilovsky, *Group representations Vol. 3*, North-Holland Mathematics Studies 180 (North-Holland Publishing Co., Amsterdam, 1994).

[K] M. Kochetov, *Gradings on finite-dimensional simple Lie algebras*, Acta. Appl. Math. 108 (2009), 101-127.

[PZ] Patera, J.; Zassenhaus, H. *On Lie gradings. I*. Linear Algebra Appl. 112 (1989), 87-159.

[Y] J. Yu, *Abelian subgroups of compact simple Lie groups*, preprint.

[Y2] J. Yu, *Elementary abelian 2-subgroups of compact Lie groups*. Geom. Dedicata 167 (2013), no. 1, 245-293.