The $SO(5)$ Landau model is the mathematical platform of the 4D quantum Hall effect and provide a rare opportunity for a physical realization of the fuzzy four-sphere. We present an integrated analysis of the $SO(5)$ Landau models and the associated matrix geometries through the Landau level projection. With the $SO(5)$ monopole harmonics, we explicitly derive matrix geometry of a four-sphere in an arbitrary Landau level: In the lowest Landau level the matrix coordinates are given by the generalized $SO(5)$ gamma matrices of the fuzzy four-sphere satisfying the quantum Nambu algebra, while in higher Landau level the matrix geometry becomes a nested fuzzy structure with no counterpart in classical geometry. The internal fuzzy geometry structure is discussed in the view of a $SO(4)$ Pauli-Schrödinger model and the $SO(4)$ Landau model, where we unveil a hidden singular gauge transformation between their background non-Abelian field configurations. Relativistic versions of the $SO(5)$ Landau model are also investigated and relation to the Berezin-Toeplitz quantization is clarified. We finally discuss the matrix geometry of the Landau models in even higher dimensions.
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\section{Introduction}

More than forty years ago, Yang proposed a $SU(2)$ generalization \cite{1} of the Dirac’s monopole \cite{2}. The set-up behind the Yang’s $SU(2)$ monopole stems from a beautiful mathematical concept of the 2nd Hopf map associated with the generalization of complex numbers to quaternions \cite{3, 4}. The Yang’s monopole field configuration on $S^4$ is conformally equivalent to the BPST instanton configuration on $\mathbb{R}^4$ \cite{5} and possesses the $SO(5)$ global rotational symmetry. Yang also succeeded to construct generalized monopole harmonics in the $SU(2)$ monopole background \cite{6}. This set-up was used in the context of the Zhang and Hu’s $SO(5)$ Landau model and 4D quantum Hall effect \cite{7} that realize natural higher dimensional counterparts of the Wu and Yang’s $SO(3)$ Landau model \cite{8} and the Haldane’s 2D quantum Hall effect on a two-sphere \cite{9, 10}.

The non-commutative geometry is the emergent geometry of the Landau models and governs the dynamics of the quantum Hall effect \cite{12, 13}. The Landau level projection truncates the whole quantum mechanical Hilbert space to a sub-space and provides a physical set-up where the non-commutative geometry naturally appears. Along this line, the fuzzy four-sphere geometry has been discussed in the context of the $SO(5)$ Landau model \cite{7, 14, 15}. It is known that the fuzzy four-sphere exhibits intriguing mathematical structure not observed in the fuzzy two-sphere: While the algebra of the fuzzy two-sphere is given by the $SU(2)$ algebra \cite{16, 17}, the five coordinates of the fuzzy four-sphere \cite{18, 19} are not closed by themselves within the Lie algebra but bring extra non-commutative coordinates constituting “internal” fuzzy structure \cite{20, 21, 22}. Such a peculiar structure makes the studies of higher dimensional non-commutative geometry more interesting and attractive. There are two ways to represent the fuzzy four-sphere algebraically: (i) Lie algebra \cite{20, 21, 22}: the enlarged algebra of the fuzzy four-sphere is the $SO(6) \simeq SU(4)$ giving rise to fuzzy fibre space (Fig.1):

\begin{equation}
[X_a, X_b] = i\alpha X_{ab}.
\end{equation}

(ii) Four-Lie bracket \cite{23, 24}: With the quantum Nambu bracket \cite{25, 26}, the fuzzy four-sphere coordinates are closed by themselves without introducing extra fuzzy coordinates. The internal structure is implicit, and the internal geometry reflects its existence in the degeneracy of (fuzzy) three-sphere latitudes (Fig.1):

\begin{equation}
[X_a, X_b, X_c, X_d] = (I + 2)\alpha^3 \epsilon_{abcde} X_e.
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fuzzy_s4.png}
\caption{Two geometric pictures of the fuzzy four-sphere. In the left figure, $X_{ab}$ \cite{1} span the fuzzy $S^2$-fibre on the original fuzzy manifold “$S^4_F$”. In the right figure, the internal geometric structure is accounted for by the degeneracy of the fuzzy three-sphere latitudes on $S^4_F$.}
\end{figure}

In the previous studies \cite{27, 28, 29}, we demonstrated that the quantum Nambu geometry actually appear in the higher dimensional Landau models, and is elegantly intertwined with exotic ideas of differential

\footnote{Interested readers may consult review articles \cite{10, 11} and references therein, for early developments of the higher dimensional Landau models and quantum Hall effects.}
topology, quantum anomaly, and string theory. However, the deduction of the non-commutative geometry from the Landau models has been rather heuristic and the obtained results are justified in the thermodynamic limit. A rigorous way to derive the non-commutative geometry is accomplished by the Landau level projection not resorting to any approximation, and the results will capture every detail of the emergent non-commutative geometry. The Landau level projection method can also be applied to an arbitrary Landau level (not limited to the lowest Landau level) whose non-commutative geometry has rarely been investigated, in contrast to the Berezin-Toeplitz quantization focused on zero-modes. The practical procedure of the Landau level projection is quite straightforward: We just sandwich coordinates of interest by Landau level basis states to derive their matrix-valued counterparts in a given Landau level. Since the total Hilbert space of the Landau model is mathematically well-defined, the truncated subspace of the Landau level necessarily provides a sound formulation of non-commutative geometry. Based on this observation, we derived matrix geometries of the \( SO(3) \) Landau models \[[31]\] and the \( SO(4) \) Landau models \[[32]\]. We extend this project to the \( SO(5) \) Landau models. Not just rendering the similar analysis, we integrate the previous results with new \( SO(5) \) results to present a comprehensive view of the emergent fuzzy geometry of the Landau models. We unveil hidden relations between the background topological field configurations of the Landau models, and also discuss the matrix geometry of the Landau models in an arbitrary dimension.

This paper is organized as follows. In Sec. 2 we review the \( SU(2) \) monopole and \( SO(5) \) Landau problem in a modern terminology. Using the \( SO(5) \) Landau level eigenstates, we derive the matrix geometry of the \( SO(5) \) Landau model in Sec. 3. Sec. 4 discusses the internal fuzzy three-sphere structure with emphasis on its relation to the \( SO(4) \) Landau model. We also clarify relations among the background topological field configurations in low dimensional Landau models. Relativistic version of the \( SO(5) \) Landau model and its associated zero-modes are analyzed in Sec. 5. In Sec. 6 we extend the matrix geometry analysis to even higher dimensions. Sec. 7 is devoted to summary and discussions.

2 Review of the Yang’s \( SU(2) \) monopole system

In this section, we review the Yang’s \( SU(2) \) monopole system \[[1, 6]\] and the Zhang and Hu’s \( SO(5) \) Landau model \[[7]\] adding some more informations.

2.1 \( SU(2) \) monopole and \( SO(5) \) angular momentum operators

With stereographic coordinates of \( S^3 \)-latitude on \( S^4 \), Yang gave an expression of the \( SU(2) \) monopole gauge field \[[1]\]. However, the original expression is rather cumbersome to handle and we then adopt the Zhang and Hu’s concise notation of the \( SU(2) \) (anti-)monopole gauge field \[[7]\]:

\[
A_m = -\frac{1}{r(r + x_5)} \bar{\eta}^i_{mn} x_n S_i \quad (m, n = 1, 2, 3, 4), \quad A_5 = 0, 
\]

where \( S_i \) \((i = 1, 2, 3)\) denote the \( SU(2) \) matrix of the spin \( I/2 \) representation:

\[
S_i S_i = \frac{I}{2} \left( \frac{I}{2} + 1 \right) 1_{I+1}.
\]

The field strength, \( F_{ab} = \partial_a A_b - \partial_b A_a + i [A_a, A_b] \), is given by

\[
F_{mn} = -\frac{1}{r^2} x_m A_n + \frac{1}{r^2} x_n A_m + \frac{1}{r^2} \bar{\eta}^i_{mn} S_i, \quad F_{m5} = -F_{5m} = \frac{1}{r^2} (r + x_5) A_m.
\]

The non-trivial homotopy for the \( SU(2) \) monopole field configuration on \( S^4 \) is guaranteed by

\[
\pi_3(SU(2)) \simeq \mathbb{Z},
\]

\[\text{See [30, 7, 10] also.}\]
and the second Chern number associated with (3) is evaluated as
\[ c_2 = \frac{1}{8\pi^2} \int_{S^4} \text{tr} \, F^2 = -\frac{1}{6} I(I + 1)(I + 2), \]  
where \( F = \frac{1}{2} F_{ab} dx_a \wedge dx_b \). We construct the covariant angular momentum operators, \( \Lambda_{ab} \), as
\[ \Lambda_{ab} = -ix_a D_b + ix_b D_a, \]
with
\[ D_a = \partial_a + iA_a, \]
and the total SO(5) angular momentum operators as
\[ L_{ab} = \Lambda_{ab} + r^2 F_{ab}. \]
In detail,
\[ L_{mn} = L_{mn}^{(0)} + \bar{\eta}_{mn}^i S_i, \quad L_{m5} = L_{m5}^{(0)} - \frac{1}{r + x_5} \eta_{mn}^i x_n S_i. \]

2.2 The SO(5) Casimir operator and SO(5) monopole harmonics

In usual textbook derivation of the spherical harmonics, the polar coordinates are adopted to represent the SO(3) Casimir. In a similar manner, we decompose the SO(5) Casimir operator to the SO(4) part and the remaining azimuthal angle part. We introduce the polar coordinates of a four-sphere (with unit radius) as
\[ x_1 = \sin \xi \sin \chi \sin \theta \cos \phi, \quad x_2 = \sin \xi \sin \chi \sin \theta \sin \phi, \quad x_3 = \sin \xi \sin \chi \cos \theta, \quad x_4 = \sin \xi \cos \chi, \quad x_5 = \cos \xi, \]
where
\[ 0 \leq \xi \leq \pi, \quad 0 \leq \chi \leq \pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi. \]
The SO(5) Casimir is expressed by the sum of the SU(2)_L \oplus SU(2)_R Casimir parts and \( x_5 \)-part
\[ \sum_{a < b = 1}^{5} L_{ab}^2 = -\frac{1}{1 - x_5^2} \frac{\partial}{\partial x_5} \left( (1 - x_5^2)^2 \frac{\partial}{\partial x_5} \right) + 2 \frac{1}{1 - x_5} J^2 + 2 \frac{1}{1 + x_5} K^2 + S_i^2 \]
\[ = -\frac{1}{\sin^2 \xi} \frac{\partial}{\partial \xi} \left( \sin^3 \xi \frac{\partial}{\partial \xi} \right) + 2 \frac{1}{1 - \cos \xi} J^2 + 2 \frac{1}{1 + \cos \xi} K^2 + S_i^2, \]
where \( J_i \) and \( K_i \) are the SU(2)_L and SU(2)_R operators given by
\[ J_i = \frac{1}{4} \eta_{mn}^i L_{mn} = \frac{1}{4} \eta_{mn}^i L_{mn}^{(0)} = J_i^{(0)}, \]
\[ K_i = \frac{1}{4} \bar{\eta}_{mn}^i L_{mn} = \frac{1}{4} \bar{\eta}_{mn}^i L_{mn}^{(0)} + S_i = K_i^{(0)} + S_i. \]
Here, \( \eta_{mn}^i \) and \( \bar{\eta}_{mn}^i \) denote the 't Hooft symbols:
\[ \eta_{mn}^i = \epsilon_{mn4} + \delta_{mi} \delta_{n4} - \delta_{m4} \delta_{ni}, \quad \bar{\eta}_{mn}^i = \epsilon_{mn4} - \delta_{mi} \delta_{n4} + \delta_{m4} \delta_{ni}. \]
Note that the SU(2) (anti-)monopole gauge field does not act to the SU(2)_L operators but acts to the SU(2)_R operators only, as if the right SU(2) angular momentum acquires additional SU(2) spin angular momentum.
2.2.1 The differential equation for the $S^3$-latitude

Let us first analyze the eigenvalue problem of the $SO(4)$ Casimir operator. The $SO(4)$ Casimir eigenstates that satisfy

$$J^2 Y_{j,m_j; k,m_k}(\Omega_3) = j(j+1)Y_{j,m_j; k,m_k}(\Omega_3), \quad K^2 Y_{j,m_j; k,m_k}(\Omega_3) = k(k+1)Y_{j,m_j; k,m_k}(\Omega_3), \quad (18a)$$

$$J_z Y_{j,m_j; k,m_k}(\Omega_3) = m_j Y_{j,m_j; k,m_k}(\Omega_3), \quad K_z Y_{j,m_j; k,m_k}(\Omega_3) = m_k Y_{j,m_j; k,m_k}(\Omega_3), \quad (18b)$$

with $\Omega_3 \equiv (\chi, \theta, \phi)$ are given by the $SO(4)$ spinor spherical harmonics:

$$Y_{j,m_j; k,m_k}(\Omega_3) = \sum_{m_R = -j}^{j} \frac{l/2}{i/2} C^{k,m_k}_{j,m_R; l/2,s_z} \Phi_{j,m_j; j,m_R}(\Omega_3) \otimes |I/2, s_z\rangle. \quad (19)$$

Here $C^{k,m_k}_{j,m_R; l/2,s_z}$ represent the Clebsch-Gordan coefficients, and $\Phi_{j, m_j; j,m_R}(\Omega_3)$ ($j = 0, 1/2, 1, 3/2, \cdots$) are the $SO(4)$ spherical harmonics $^{[32]}$

$$\Phi_{j, m_L; j,m_R}(\Omega_3) = \sum_{l=-j}^{j} \sum_{m=-l}^{l} C^{m}_{l,m_L; m_R} Y_{plm}(\Omega_3) \bigg|_{p=2j} \quad (20)$$

with the $SO(4)$ spherical harmonics $^{[3]}$

$$Y_{plm}(\Omega_3) = 2l! \sqrt{\frac{2(p+1)(p-l)!}{\pi(p+l+1)!}} \sin^l(\chi) C^{l+1}_{p-l}(\cos \chi) \cdot Y_{lm}(\theta, \phi) \quad (23)$$

$$(l = 0, 1, 2, \cdots, p \text{ and } m = -l, -l+1, \cdots, l).$$

Here, $C^{l+1}_{p-l}$ denote the Gegenbauer polynomials, and $Y_{lm}(\theta, \phi)$ stand for the $SO(3)$ spherical harmonics:

$$C^{\alpha}_{n}(x) \equiv \frac{(-2)^n}{n!} \frac{\Gamma(n+\alpha)\Gamma(n+2\alpha)}{\Gamma(\alpha)\Gamma(2n+2\alpha)} (1-x^2)^{-\alpha+1/2} \frac{d^n}{dx^n} [(1-x^2)^{n+\alpha-1/2}], \quad (24)$$

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}} \frac{(l-m)!}{(l+m)!} P_{l}^{[m]}(\cos \theta) e^{im\phi}. \quad (25)$$

Since the (anti-)monopole gauge field only contributes to the $SU(2)_{R}$ angular momentum operator, in $^{[20]}$ the original $SU(2)_{R}$ index $j$ of the $SO(4)$ spherical harmonics is contracted with the gauge spin index $I/2$ to form the $SU(2)_{R}$ composite spin $k$ just as the usual $SU(2)$ angular momentum composition rule. Therefore, $k$ takes

$$k = j + \frac{I}{2}, \quad j + \frac{I}{2} - 1, \cdots, \quad j - \frac{I}{2} \quad (26)$$

or

$$s \equiv j - k = \frac{I}{2}, \quad \frac{I}{2} - 1, \cdots, -\frac{I}{2} \quad (27)$$

$s$ signifies the difference between the left and right $SU(2)$ quantum numbers, and hence the name the chirality parameter $^{[28], [29]}$. Though $k$ and $j$ are two independent $SU(2)$ group indices, in the present

$^{3}$The $SO(4)$ spherical harmonics $^{[28]}$

$$J^{(0)}_{pl,m}(\Omega_3) = \frac{p(p+1)}{2} Y_{pl,m}(\Omega_3) = \frac{p(p+1)}{2} Y_{pl,m}(\Omega_3). \quad (21)$$

The dimension of the $SO(4)$ spherical harmonics is given by

$$\sum_{l=0}^{p} (2l+1) = (p+1)^2 = (2j+1)^2 \left| j = \frac{p-1}{2} \right. \quad (22)$$
The range of $k$ is not arbitrary but restricted as (26) with a given $j$. In $(I+1)$-component vector-like notation, the $SO(4)$ spinor spherical harmonics is expressed as

$$Y_{j,m_j; k,m_k}(\Omega_3) = \sum_{m_R=-j}^{j} \begin{pmatrix} C_{j,m_R; \frac{1}{2} - \frac{j}{2}}^{k,m_k} \Phi_{j,m_j; j,m_R}(\Omega_3) \\ C_{j,m_R; \frac{1}{2} + \frac{j}{2}}^{k,m_k} \Phi_{j,m_j; j,m_R}(\Omega_3) \\ \vdots \\ C_{j,m_R; -I/2}^{k,m_k} \Phi_{j,m_j; j,m_R}(\Omega_3) \end{pmatrix}. \quad (28)$$

From two indices $j$ and $k$, we introduce the $SO(4)$ Landau level index $n$:

$$n = j + k - I/2. \quad (29)$$

$n$ essentially denotes the sum of two $SU(2)$ quantum numbers. With $n$ and $s$, $j$ and $k$ are represented as

$$(j, k)_{SO(4)} = \left( \frac{n}{2} + I + s, \frac{n}{2} + I - s \right)_{SO(4)}. \quad (30)$$

Notice that while the (anti-)monopole only acts to the $SU(2)$ operator, with a given $j$ and $k$ are totally equivalent in the sense that either of $j$ and $k$ starts from $n/2$ and ends at $I/2 + n/2$ (see Fig[2] also).

In the view of the $SO(5)$ representation theory, the $SO(5)$ monopole harmonics are specified by two integers,

$$(p,q)_{SO(5)} = (I + N, N)_{SO(5)}. \quad (N = 0, 1, 2, \cdots) \quad (31)$$

The $SO(4)$ decomposition generally tells that, with a given $N$, $n$ takes

$$n = 0, 1, 2, \cdots, N, \quad (32)$$

and the $SO(4)$ decomposition can be expressed as

$$(I + N, N)_{SO(5)} = \sum_{n=0}^{N} \oplus \left( \sum_{s=-I/2}^{I/2} \oplus (j, k)_{SO(4)} \right), \quad (33)$$

where $j$ and $k$ are given by (30). The intervals of $n$ and $j$ are $\Delta n = 1$ and $\Delta j = 1/2$. Accompanied with the decomposition, the degeneracy of $SO(4)$ irreducible representation is counted as

$$d(I, n) = \sum_{s=-I/2}^{I/2} (2j + 1)(2k - 1) = \frac{1}{6}(I + 1)(I^2 + (6n + 5)I + 6(n + 1)^2), \quad (34)$$

and the $SO(5)$ Landau level degeneracy $D(I, N)$ is given by

$$\sum_{n=0}^{N} d(I, n) = \frac{1}{6} (N + 1)(I + 1)(I + N + 2)(I + 2N + 3). \quad (35)$$

$3\overline{3}$ implies that $N$th $SO(5)$ Landau level consists of the inner $SO(4)$ Landau levels with $n = 0, 1, 2, \cdots, N$:

$$\mathcal{H}_{SO(5)}^{(N)} = \sum_{n=0}^{N} \oplus \left( \sum_{s=-I/2}^{I/2} \mathcal{H}_{SO(4)}^{(n,s)} \right). \quad (36)$$

See Appendix A.1 for the $SO(5)$ representation theory and the $SO(4)$ decomposition.

$4$In the original literature of Yang [6], (37) is referred to as the $SU(2)$ monopole harmonics, but in the present paper we call them the $SO(5)$ monopole harmonics to emphasize their $SO(5)$ covariance.
2.2.2 Azimuthal part eigenvalue problem

The \(SO(5)\) Casimir operator was decomposed to the azimuthal part \(\xi\) and the hyper-latitude \(S^3\) part \((15)\). To solve the differential equation of the \(SO(5)\) Casimir operator, Yang adopted the method of separation of variables \([6]\):

\[
\Psi_{j,k}(\xi, \Omega_3) = G(\xi) \cdot Y_{j,k}(\Omega_3),
\]

(37)

Here, \(Y_{j,k}(\Omega_3)\) denote the \(SO(4)\) monopole harmonics \((19)\) with the constraint

\[
j + k = n + \frac{I}{2}.
\]

(38)

The \(SO(5)\) eigenvalue problem

\[
\sum_{a<b=1}^{5} L_{ab}^2 \Psi(\xi, \Omega_3) = \lambda \Psi(\xi, \Omega_3),
\]

(39)

gives the eigenvalue equation for \(G(\xi)\):

\[
\left[ -\frac{1}{\sin^3 \xi} \frac{d}{d\xi} (\sin^3 \xi \frac{d}{d\xi}) + 2 \frac{1}{1 - \cos \xi} j(j + 1) + 2 \frac{1}{1 + \cos \xi} k(k + 1) + \frac{I}{2} (\frac{I}{2} + 1) \right] G(\xi) = \lambda G(\xi).
\]

(40)

Yang showed that the difference of two Casimir indices is exactly equal to the \(SU(2)\) monopole index \(I\) \([6]\):

\[
p - q = I.
\]

(41)

Therefore, when we identify \(q\) with the \(SO(5)\) Landau level index \(N\) \((= 0, 1, 2, \cdots)\), the \(SO(5)\) monopole harmonics correspond to the irreducible representation with the indices \((31)\). The \(SO(5)\) Casimir eigenvalues are readily obtained as

\[
\lambda_N = \frac{1}{2} p^2 + \frac{1}{2} q^2 + 2p + q = N^2 + N(I + 3) + \frac{1}{2} I(I + 4),
\]

(42)

and the corresponding degeneracy is

\[
D(I, N) = \frac{1}{6} (p + 2)(q + 1)(p + q + 3)(p - q + 1) = \frac{1}{6} (N + 1)(I + 1)(I + N + 2)(I + 2N + 3),
\]

(43)
which is equal to (55). The normalized $SO(5)$ monopole harmonics are derived as

$$\Psi_{N;j,m;j,m}(\Omega_4) = G_{N,j,k}(\xi) \cdot Y_{j,m;j,k,m}(\Omega_3), \quad (\Omega_4 = (\xi, \chi, \theta, \phi))$$

(44)

where [14]

$$G_{N,j,k}(\xi) = \sqrt{N + \frac{l}{2} + \frac{3}{2}} \cdot \frac{1}{\sin \xi} \cdot d_{N+l+1,j-k,-j-k-1}(\xi)$$

$$= \sqrt{N + \frac{l}{2} + \frac{3}{2}} \cdot (-1)^{2j+1} \cdot \frac{(N + \frac{l}{2} + j + k + 1)!}{(N + \frac{l}{2} + j + k)!} \cdot \frac{(N + \frac{l}{2} + j - k)!}{(N + \frac{l}{2} + j + k)!} \cdot \frac{1}{\sin \xi} \cdot \left(\sin \frac{\xi}{2}\right)^{2j+1} \cdot \left(\cos \frac{\xi}{2}\right)^{-2k-1} \cdot \frac{1}{\sin \xi} \cdot \left(\cos \xi\right).$$

(45)

Here $d_{l,m,g}(\xi)$ denotes the Wigner’s small $d$-function and the three indices are identified with $(l, m, g) = (N + \frac{l}{2} + 1, s, -n + \frac{l}{2} - 1)$ in [14]. In the small $d$-function $d_{l,m,g}(\xi)$, its two magnetic indices, $m$ and $g$, generally take (half-)integer values between $-l$ and $l$, while in the present case $m = s$ and the range of $s$ is restricted to $|s| \leq \frac{l}{2}$ which is smaller than $l = N + \frac{l}{2} + 1$ (except for $N = 0$). We find that a subset of $d$-function is utilized in [14].

The orthonormal relation for (44) is given by

$$\int d\Omega_4 \Psi_{N;j,m;j,m}(\Omega_4)^{\dagger} \Psi_{N';j',m';j',m'}(\Omega_4) = \delta_{NN'} \delta_{jj'} \delta_{mm'}$$

$$= \int_0^\pi d\xi \sin^3 \xi \cdot G_{N,j,k}(\xi)^* \cdot G_{N',j',k'}(\xi) \cdot \int_{S^3} d\Omega_3 \cdot Y_{j,m;j,k,m}(\Omega_3)^{\dagger} \cdot Y_{j',m';j',k',m'}(\Omega_3) = \delta_{NN'} \delta_{jj'} \delta_{mm'} \delta_{mm'}. \quad (48)$$

For instance, the $SO(5)$ spinor representation $(N, I) = (0, 1)$ is obtained as

$$\Psi_{0,1/2,1/2,0,0}(\Omega_4) = -\sqrt{\frac{3}{4\pi^2}} \sin \frac{\xi}{2} \cdot \left(\cos \chi - i \sin \chi \sin \theta \cos \phi \right) \propto \psi_1 \equiv \frac{1}{\sqrt{2(1 + x_5)}} \left(\begin{array}{c} x_4 - i x_3 \\ -i x_1 + x_2 \end{array}\right),$$

$$\Psi_{0,1/2,-1/2,0,0}(\Omega_4) = -\sqrt{\frac{3}{4\pi^2}} \sin \frac{\xi}{2} \cdot \left(-i \sin \chi \sin \theta \cos \phi \right) \propto \psi_2 \equiv \frac{1}{\sqrt{2(1 + x_5)}} \left(\begin{array}{c} -i x_1 - x_2 \\ x_4 + i x_3 \end{array}\right), \quad (49a)$$

$$\Psi_{0,0,1/2,1/2}(\Omega_4) = -\sqrt{\frac{3}{4\pi^2}} \cos \frac{\xi}{2} \cdot \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \propto \psi_3 \equiv \sqrt{\frac{1 + x_5}{2}} \left(\begin{array}{c} 1 \\ 0 \end{array}\right),$$

$$\Psi_{0,0,1/2,-1/2}(\Omega_4) = -\sqrt{\frac{3}{4\pi^2}} \cos \frac{\xi}{2} \cdot \left(\begin{array}{c} 0 \\ 1 \end{array}\right) \propto \psi_4 \equiv \sqrt{\frac{1 + x_5}{2}} \left(\begin{array}{c} 0 \\ 1 \end{array}\right). \quad (49b)$$

From

$$G_{N,j,k}(x_5) = (1 - x_5)^j (1 + x_5)^{-k-1} \cdot \frac{P^{2j+1, -(2k+1)}_{N + \frac{l}{2} + 1 - j + k}}{N + \frac{l}{2} + 1 - j + k} \cdot \sim x_5^{N + \frac{l}{2}} \quad (50)$$

and

$$Y_{j,k}(\Omega_3) \sim y_\mu^{2j} \sim \frac{1}{(1 - x_5^2)^{j}} x_\mu^{2j}, \quad (51)$$

The small $d$-function can also be expressed as

$$d_{l,m,g}(\xi) = (e^{-i\xi S_y^{(l)}})_{m,g} \quad (46)$$

where $S_y^{(l)}$ denotes $y$-component of the $SU(2)$ spin matrix with spin magnitude $l$:

$$S^{(l)} \cdot S^{(l)} = l(l + 1) \cdot 1_{2l+1}. \quad (47)$$
the behavior of $\Psi_{N;j,k}$ can be read off as

$$\Psi_{N;j,k} \sim x_a^{N+\frac{j}{2}}.$$ (52)

At $I = 0$, (44) is reduced to the $SO(5)$ spherical harmonics as expected (see Appendix C).

2.3 The $SO(5)$ Landau model

The $SO(5)$ Landau model [7] is a Landau model on a four-sphere in the $SU(2)$ monopole background. With the covariant derivatives $D_a$, the $SO(5)$ Landau Hamiltonian is given by

$$H = \frac{1}{2M} \sum_{a=1}^{5} D_a^2, \quad r=1$$ (53)

which can be rewritten as

$$H = \frac{1}{2M} \sum_{a<b} (L_{ab}^2 - F_{ab}^2), \quad (54)$$

where we used $\Lambda_{ab}F_{ab} = F_{ab}\Lambda_{ab} = 0$. From (5), we can readily derive

$$\sum_{a<b} F_{ab}^2 = \sum_{m<n} (\bar{\eta}_{mn}S_i)^2 = 2S_i^2 = \frac{1}{2}I(I + 2),$$ (55)

and the $SO(5)$ Landau Hamiltonian is diagonalized as

$$E_N^a = \frac{1}{2M}(N^2 + N(I + 3) + I),$$ (56)

with the Landau level degeneracy (43). In particular, the lowest Landau level degeneracy is given by

$$D_{LLL}(I) \equiv D(I, N = 0) = \frac{1}{6}(I + 1)(I + 2)(I + 3).$$ (57)

The Landau level eigenstates are given by the $SO(5)$ monopole harmonics (44).

The lowest Landau level degeneracy (57) is simply understood as the number of the fully symmetric representation [7],

$$\frac{1}{\sqrt{m_1! \cdot m_2! \cdot m_3! \cdot m_4!}} \psi_1^{m_1} \psi_2^{m_2} \psi_3^{m_3} \psi_4^{m_4}$$ (58)

where $m_1, m_2, m_3, m_4$ are non-negative integers subject to

$$m_1 + m_2 + m_3 + m_4 = I$$ (59)

and $\psi$s denote the components of the 2nd Hopf spinor [8].

We can see equivalence between the fully symmetric representation (58) and the $SO(5)$ monopole expression (44) as follows. With the higher spin basis

$$e_A^{(I/2)} = \frac{1}{\sqrt{(\frac{I}{2} + A)!((\frac{I}{2} - A)!}} \phi_1^{\frac{I}{2} + A} \phi_2^{\frac{I}{2} - A}, \quad (A = I/2, I/2 - 1, \cdots, -I/2)$$ (61)

We will discuss the 2nd Hopf map in Sec. 3.2.
can be expanded as
\[ \frac{1}{\sqrt{m_1m_2m_3m_4}} \psi_1^{m_1} \psi_2^{m_2} \psi_3^{m_3} \psi_4^{m_4} = \sum_{A=-1/2}^{1/2} \psi_A^{(m_1,m_2,m_3,m_4)} e_A^{(I/2)}. \]

From the expansion coefficients on the right-hand side of (62), we can construct the \((I + 1)\)-component “vector” as
\[ \Psi_{N=0;j,m,j,m} = -\frac{\sqrt{(I + 2)(I + 3)}}{4\pi} \begin{pmatrix} \psi_{I/2}^{(m_1,m_2,m_3,m_4)} \\ \psi_{I/2-1}^{(m_1,m_2,m_3,m_4)} \\ \vdots \\ \psi_{-I/2}^{(m_1,m_2,m_3,m_4)} \end{pmatrix}, \]
which is exactly equal to the \(N = 0\) \(SO(5)\) monopole harmonics (44) under the identification
\[ j = \frac{1}{2}(m_1 + m_2), \quad m_j = \frac{1}{2}(m_1 - m_2), \quad k = \frac{1}{2}(m_3 + m_4), \quad m_k = \frac{1}{2}(m_3 - m_4). \]

3. Four-sphere matrix geometry

In this section, we investigate the matrix geometry of the \(SO(5)\) Landau model. First, we discuss a general structure of the matrix geometry deduced from the \(SO(5)\) irreducible decomposition rule. Next, we discuss the lowest Landau level matrix geometry at the quantum limit \(I = 1\) and at the classical limit \(I \gg 1\). Finally, through the Landau level projection, we explicitly derive the matrix geometry in an arbitrary Landau level.

3.1 General form of matrix coordinates

In this section, we utilize the notation \([N, I]\) to specify the \(SO(5)\) irreducible representation instead of \((p, q)_{SO(5)} = (N + I, N)\). First, let us see a general structure of the matrix elements of the four-sphere coordinates:
\[ \langle N', I_2 | x_a | N, I_2 \rangle. \]

Here \(|N, I_2\rangle\) is the abbreviation of the \(N\)th Landau level eigenstates (44), and the \(SO(5)\) vector \(x_a\) carries the \(SO(5)\) index \([1, 0]\), and hence the \(SO(5)\) index of \(x_a | N, I_2 \rangle\) is given by
\[ [1, 0] \otimes [N, I], \]
which is irreducibly decomposed as (83)
\[ [1, 0] \otimes [N, I] = [[N + 1, I]] \oplus [[N - 1, I]] \oplus [[N, I]] \oplus [[N + 1, I - 2]] \oplus [[N - 1, I + 2]], \]
where

\[
\begin{align*}
[[N + 1, -2]] &\equiv -[[N, 0]], \\
[[N, -1]] &\equiv [[-1, I]] \equiv \phi \text{ (empty set).}
\end{align*}
\]

(72a)

(72b)

See Appendix A.2 for several examples of (67). The corresponding dimension-counting is given by

\[
5 \otimes D[[N,I]] = D[[N,I]] \oplus D[[N+1,I]] \oplus D[[N+1,I-2]] \oplus D[[N-I+2]] \oplus D[[N-1,I]].
\]

(73)

With a \(SU(2)\) monopole background fixed \(I\), (67) implies that the Landau level transition, if occurred, only takes place between the adjacent Landau levels:

\[
|N, I/2\rangle \rightarrow |N + 1, I/2\rangle, \quad |N - 1, I/2\rangle \rightarrow |N, I/2\rangle.
\]

(74)

Consequently, the matrix elements only have finite values between the adjacent inter Landau levels and intra Landau levels:

\[
\langle N', I/2| x_a |N, I/2\rangle \neq 0 \quad \text{only for } \Delta N \equiv N - N' = 0, \pm 1,
\]

(75)

as depicted in Fig 3.

Figure 3: The matrix elements of four-sphere coordinates. The shaded regions stand for non-zero blocks: The red shaded squares denote the matrix elements in intra Landau levels, while the blue shaded rectangles represent the matrix elements between inter Landau levels.

\[\text{\textsuperscript{12}}\text{ is a special case of more general formula}
\]

\[
[[N + I + 1, -I - 2]] = -[[N, I]]
\]

(68)

or

\[
D[[N+I+1,-I-2]] = -D[[N,I]].
\]

(69)

\[\text{\textsuperscript{68}}\text{ is verified by the fact that the } [[N,I]] \text{ irreducible representation is specified by the polynomial } \text{[55]}
\]

\[
\xi(x,y)_{[[N,I]]} = x^{N+I+2}y^{N+1} - x^{N+1}y^{N+I+2} + y^{N+I+2} + \frac{1}{x^{N+I+2}} - \frac{1}{x^{N+I+2}y^{N+1}} + \frac{1}{y^{N+I+2}} - \frac{x^{N+I+2}}{y^{N+1}}.
\]

(70)

which has the property

\[
\xi(x,y)_{[[N+I+1,-I-2]]} = -\xi(x,y)_{[[N,I]]}.
\]

(71)
3.2 The 2nd Hopf map and Bloch four-sphere (quantum limit: \( I = 1 \))

The Yang’s \( SU(2) \) monopole is closely related to the 2nd Hopf map \([3, 4, 7]\). Using quaternions \( q_m (m = 1, 2, 3, 4) \), the 2nd Hopf map, \( S^7 \rightarrow S^4 \), is realized as

\[
\psi \rightarrow \psi^\dagger \gamma_a \psi = x_a, \quad (a = 1, 2, 3, 4, 5)
\]

where \( \gamma_a \) are

\[
\gamma_m = \begin{pmatrix} 0 & \bar{q}_m \\ \bar{q}_m & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\]

with \( \bar{q}_m = \{ -q_i, q_4 \} \). \( \psi \) which we refer to as the 2nd Hopf spinor is a two-component quaternionic spinor

\[
\psi = (\psi_1 \psi_2)^t
\]

subject to

\[
\psi^\dagger \psi = 1
\]

and signifies the total manifold \( S^7 \). \( x_a \) satisfy the normalization condition \( \sum_{a=1}^{5} x_a^2 = (\psi^\dagger \psi)^2 = 1 \) and are regarded as the coordinates on the base-manifold \( S^4 \). The \( S^3 \)-fibre part of \( S^7 \) is projected out in the map \( \psi \rightarrow \psi^\dagger \gamma_a \psi = x_a \). The four-sphere associated with the 2nd Hopf map can be considered as a 4D version of the Bloch sphere \([38]\).

Due to the algebras of the quaternions, it is shown that \( \gamma_a \) satisfy

\[
\{ \gamma_a, \gamma_b \} = 2 \delta_{ab},
\]

and act as the \( SO(5) \) gamma matrices. This will be more transparent when we introduce a matrix realization of the quaternions:

\[
q_m = \{-i\sigma_{i=1,2,3,12}\}, \quad \bar{q}_m = \{i\sigma_{i=1,2,3,12}\}.
\]

Substituting \( q_m \) to \( \gamma_m \), \( \gamma_a \) now become the familiar \( SO(5) \) \( 4 \times 4 \) gamma matrices, and the corresponding \( SO(5) \) generators are obtained as

\[
\sigma_{ab} = -i \frac{1}{4} [\gamma_a, \gamma_b],
\]

where

\[
\sigma_{mn} = \frac{1}{2} \left( \eta_{mn} \sigma_i \begin{pmatrix} 0 & 0 \\ 0 & \bar{\eta}_{mn} \sigma_i \end{pmatrix} \right), \quad \sigma_{m5} = \frac{i}{2} \begin{pmatrix} 0 & 0 \\ q_m & 0 \end{pmatrix}.
\]

The 2nd Hopf spinor \( \psi \) is also promoted to a \( 4 \times 2 \) matrix \( \Psi \) subject to

\[
\Psi^\dagger \Psi = 1_2.
\]

The \( S^3 \)-fibre part represents the \( SU(2) \) gauge degrees of freedom that acts to \( \Psi \) as

\[
\Psi \rightarrow \Psi \cdot g, \quad (g \in SU(2))
\]

A possible choice of \( \Psi \) is

\[
\Psi(\Omega_4) = \frac{1}{\sqrt{2(1+x_5)}} \left( \begin{array}{c} x_m \bar{q}_m \\ 1 + x_5 \end{array} \right) = \frac{1}{\sqrt{2(1+x_5)}} \left( \begin{array}{c} x_4 1_2 + ix_i \sigma_i \\ (1+x_5) 1_2 \end{array} \right).
\]

Interestingly, \( \Psi(\Omega_4) \) consists of the \( N = 0 \) \( SO(5) \) spinor multiplet for \( I = 1 \) \([49]\):

\[
\Psi(\Omega_4)^\dagger = (\psi_1 \psi_2 \psi_3 \psi_4).
\]

This implies that the 2nd Hopf map encodes informations of the lowest Landau level of the minimum \( SU(2) \) monopole index \( I = 1 \), which we call the quantum limit. For the \( SO(5) \) spinors, the \( SU(2) \) gauge transformation \( \Psi(\Omega_4) \) acts as

\[
\psi_\alpha \rightarrow g^\dagger \cdot \psi_\alpha, \quad (\alpha = 1, 2, 3, 4)
\]
and the gauge field is given by

$$A = -i \Psi^\dagger d\Psi = -i \sum_{\alpha=1}^{4} \psi_{\alpha}^\dagger d\psi_{\alpha} = -\frac{1}{2(1 + x^5)} \bar{\eta}_{mn} x_{m} \sigma_{l} dx_{m},$$

which is exactly equal to the Yang’s monopole gauge field \(3\) for \(I = 1\). Under the gauge transformation \(84\), the gauge field is transformed as expected:

$$A = -i \psi_{\alpha}^\dagger \rightarrow -i \sum_{\alpha=1}^{4} g^\dagger \psi_{\alpha}^\dagger (d\psi_{\alpha}^\dagger \cdot g + \psi_{\alpha}^\dagger \cdot dg) = g^\dagger Ag - ig^\dagger dg. \quad (89)$$

Including the \(SU(2)\) gauge degrees of freedom, the 2nd Hopf spinor is generally given by

$$\psi(\Omega_4, G) = \begin{pmatrix} \psi_1(\Omega_4, G) \\ \psi_2(\Omega_4, G) \\ \psi_3(\Omega_4, G) \\ \psi_4(\Omega_4, G) \end{pmatrix} = \Psi(\Omega_4) \begin{pmatrix} \phi_1(G) \\ \phi_2(G) \end{pmatrix} \quad (90)$$

or

$$\psi_{\alpha}(\Omega_4, G) = \sum_{i=1}^{2} \Psi_{\alpha i}(\Omega_4) \phi_i(G). \quad (\alpha = 1, 2, 3, 4) \quad (91)$$

Here \(\Psi(\Omega_4)\) signifies the base-manifold \(S^4\) and \((\phi_1 \phi_2)^t\) denotes a normalized \(SU(2)\) spinor taking its value on the \(S^3\)-fibre. With some appropriate inner product, we orthonormalize \(\phi_i\) \((i = 1, 2)\) as

$$\langle \phi_i | \phi_j \rangle = \delta_{ij}, \quad (92)$$

and the normalization condition of \(\psi\) is restated as

$$\Psi(\Omega_4)^\dagger \Psi(\Omega_4) = \sum_{\alpha=1}^{4} \psi_{\alpha}^\dagger \psi_{\alpha} = 1_2. \quad (93)$$

With this simple set-up, we discuss the \(SU(2)\) gauge invariance and the \(SO(5)\) covariance of the matrix geometry. The \(SU(2)\) gauge transformation \(84\) can be reinterpreted as the transformation of the \(\phi\)-part:

$$\phi_i \rightarrow g_{ij} \phi_j, \quad (94)$$

while the \(SO(5)\) global transformation acts to \(\Psi\)-part as\(^8\)

$$\Psi \rightarrow U^\dagger \Psi, \quad (97)$$

where

$$U \equiv e^{i \sum_{a<br} \omega_{ab} \sigma_{ab}}. \quad (98)$$

---

\(^8\) \(\psi_{\alpha=1,2,3,4}\) constitute the lowest Landau level eigenstates

$$L_{ab} \psi_{\alpha} = (\sigma_{ab})_{\beta \alpha} \psi_{\beta}, \quad (95)$$

and under the \(SO(5)\) transformation generalized by \(L_{ab}\), they behave as

$$\psi_{\alpha} \rightarrow \psi_{\beta} U_{\beta \alpha}, \quad (96)$$

or \(97\).
We define the matrix elements of observable \(O(\Omega_4)\) as

\[
\langle \psi_\alpha | O | \psi_\beta \rangle \equiv \frac{2}{A(S^4)} \int d\Omega_4 \Psi_{\alpha i} O \Psi^*_{\beta i} = \frac{2}{A(S^4)} \sum_{i=1}^{2} \int d\Omega_4 \Psi_{\alpha i} O \Psi^*_{\beta i} = \frac{2}{A(S^4)} \int d\Omega_4 \psi^\dagger_{\alpha} O \psi_{\beta}.
\]

The final expression implies that the matrix elements are evaluated through the integration of the operator sandwiched by the \(SO(5)\) spinors. In particular, the matrix elements of \(x_a\) are given by

\[
(X_a)_{\alpha\beta} = \frac{2}{A(S^4)} \sum_{i=1}^{2} \int d\Omega_4 x_a \Psi_{\alpha i} \Psi^*_{\beta i} = \frac{2}{A(S^4)} \int d\Omega_4 x_a \psi^\dagger_{\alpha} \psi_{\beta},
\]

or

\[
X_a = \frac{2}{A(S^4)} \int d\Omega_4 x_a P,
\]

where \(P\) denotes a \(4 \times 4\) projection matrix

\[
P = \Psi \Psi^\dagger = \frac{1}{2} (1 + \sum_{a=1}^{5} x_a \gamma_a).
\]

From (93) \(P^2 = P\), and \(P\) is invariant under the \(SU(2)\) gauge transformation (84). Therefore, \(X_a\) are obviously gauge invariant quantities. From the 2nd Hopf map \(x_a = \Psi^\dagger \gamma_a \Psi\), \(X_a = \int d\Omega_4 x_a \Psi^\dagger\) can also be represented as

\[
X_a = \frac{2}{A(S^4)} \int d\Omega_4 P \gamma_a P.
\]

With the following formulas

\[
P \gamma_a P = \frac{1}{2} x_a (1 + x_b \gamma_b), \quad \int_{S^4} d\Omega_4 x_a = 0, \quad \int_{S^4} d\Omega_4 x_a x_b = \frac{1}{5} A(S^4) \delta_{ab},
\]

we can easily evaluate (104) as

\[
X_a = \frac{1}{5} \gamma_a.
\]

Thus in the quantum limit, the lowest Landau level matrix geometry is given by the \(SO(5)\) gamma matrices (77) up to a proportional factor. Under the \(SO(5)\) global transformation (97), \(X_a\) are transformed as

\[
X_a = \frac{2}{A(S^4)} \int d\Omega_4 x_a \Psi \Psi^\dagger \rightarrow \frac{2}{A(S^4)} U^\dagger \int d\Omega_4 x_a \Psi \Psi^\dagger \cdot U = U^\dagger X_a U = R_{ab} X_b,
\]

where we used the \(SO(5)\) covariance of the gamma matrices

\[
U^\dagger \gamma_a U = R_{ab} \gamma_b \quad (R \equiv e^{i \sum_{a < b} \omega_{ab} \Sigma^A_{ab}}, \quad \Sigma^A_{ab} \equiv -i \delta_{ac} \delta_{bd} + i \delta_{ad} \delta_{bc}).
\]

(107) indicates that the matrix coordinates transform as a \(SO(5)\) vector as expected.

---

9 The factor in front of the integration is introduced for the normalization

\[
\frac{2}{A(S^4)} \int_{S^4} d\Omega_4 \Psi \Psi^\dagger = 14.
\]

10 Alternatively, we can obtain (106) by performing the integration (101) with (49).
3.3 Heuristic derivation of the fuzzy geometry (classical limit: $I \gg 1$)

Next, we consider the opposite limit $I \gg 1$, which we refer to as the classical limit by the analogy of quantum spin model $S \gg 1$. Refining the heuristic discussions of [27], we will show how the non-commutative geometry takes place in this limit.

At $I \gg 1$, the field strength term becomes dominant in the angular momentum $L_{ab}$ [10]:

$$L_{ab} \rightarrow r^2 F_{ab}.$$  \hspace{1cm} (109)

The coordinates $x_a$ can be extracted from the $SU(2)$ field strength [5] as [27]

$$\frac{1}{r^3} x_a = \frac{2}{4! c_2(I)} \epsilon_{abcde} \text{tr}(F_{bc} F_{de}). \hspace{1cm} (110)$$

Here $c_2(I)$ denotes the 2nd Chern number [4]:

$$c_2(I) = -\frac{1}{6} I(I+1)(I+2) = -D(I-1,0). \hspace{1cm} (111)$$

Replacing $F_{ab}$ with $L_{ab}$ in (110), we have

$$X_a \sim \frac{2r}{4! c_2(I)} \epsilon_{abcde} L_{bc} L_{de} \text{tr}(1_{\text{internal space}}). \hspace{1cm} (112)$$

Since $L_{ab}$ are the $SO(5)$ operators, the coordinates now become the operators given by (112). $\text{tr}$ in (110) is taken in the “internal” fuzzy space $S_F^2$ with dimension $(I+1)$ [27,15], and so

$$\text{tr}(1_{\text{internal space}}) = I + 1. \hspace{1cm} (113)$$

In the lowest Landau level, we may replace the $SO(5)$ operators $L_{ab}$ with the $SO(5)$ matrices $\Sigma_{ab}$ of the fully symmetric irreducible representation:

$$L_{ab} \rightarrow \Sigma_{ab}, \hspace{1cm} (114)$$

and (112) turns into

$$X_a = \frac{2}{4! c_2(I)} \epsilon_{abcde} \Sigma_{bc} \Sigma_{de} = -\frac{1}{4I(I+2)} \epsilon_{abcde} \Sigma_{bc} \Sigma_{de}. \hspace{1cm} (115)$$

Since in the fully symmetric representation $\Sigma_{ab}$ satisfy [11]

$$\epsilon_{abcde} \Sigma_{bc} \Sigma_{de} = -2(I+2) \Gamma_a, \hspace{1cm} (118)$$

(115) is greatly simplified as

$$X_a = \frac{1}{I} \Gamma_a. \hspace{1cm} (119)$$

[11] The gamma matrices in the fully symmetric representation are constructed as

$$\Gamma^{(I)}_a \equiv (\gamma_0 \otimes 1 \otimes 1 \cdots 1 + 1 \otimes \gamma_0 \otimes 1 \cdots 1 + \cdots + 1 \otimes 1 \cdots 1 \otimes \gamma_0)_{\text{sym.}}, \hspace{1cm} (116)$$

which satisfy

$$\sum_{a=1}^5 \Gamma_a^{(I)} \Gamma_a^{(I)} = \frac{I(I+4)}{8} \cdot \gamma_8^{(I+1)(I+2)(I+3)}, \hspace{1cm} (117a)$$

$$[\Gamma_a^{(I)}, \Gamma_b^{(I)}, \Gamma_c^{(I)}] = 8(I+2) \epsilon_{abcde} \Gamma_e^{(I)}. \hspace{1cm} (117b)$$

In this paper, we will drop $(I)$ on the shoulder of $\Gamma_a^{(I)}$ for brevity otherwise stated.
Therefore in the classical limit, the lowest Landau level matrix coordinates are given by the \( SO(5) \) gamma matrices in the fully symmetric representation.

From [11], we have

\[
[X_a, X_b] = i\frac{2}{\mathcal{T}}\Sigma_{ab},
\]

(120)

and around the north-pole \( X_5 = \frac{1}{2}\Gamma_5 \sim -1_{I+1} \) is reduced to

\[
[X_n, X_n] = i\frac{2}{\mathcal{T}}^2 \Sigma_{nn} \sim i\frac{2}{\mathcal{T}}^2 \eta_{nn} S_i,
\]

(121)

which realizes the non-commutative algebra of Zhang and Hu [7].

### 3.4 Landau level projection and matrix geometry (arbitrary \( I \) and \( N \))

We have obtained the matrix geometry at the quantum limit and the classical limit. Here, we apply the Landau level projection to derive more general results. The explicit form of the \( SO(5) \) monopole harmonics is crucial in the present analysis.

#### 3.4.1 Landau level matrix elements

We perform integrations in the azimuthal part and the \( S^3 \)-latitude part separately. The \( S^4 \)-coordinates are decomposed to the azimuthal part and the \( S^3 \)-latitude part:

\[
x_m = \sin \xi \, y_m, \quad x_5 = \cos \xi,
\]

(122)

where \( x_m \) are expressed by the product of the radius of \( S^3 \)-latitude and the (normalized) \( S^3 \)-coordinates:

\[
y_1 = \sin \chi \sin \theta \cos \phi, \quad y_2 = \sin \chi \sin \theta \sin \phi, \quad y_3 = \sin \chi \cos \theta, \quad y_4 = \cos \chi.
\]

(123)

The area element of \( S^4 \) is expressed as

\[
d\Omega_4 = d\xi \sin^3 \xi \, d\Omega_3,
\]

(124)

with the \( S^3 \) area element

\[
d\Omega_3 = \sin^2 \chi \sin \theta \, d\chi \, d\theta \, d\phi.
\]

(125)

For instance, an integration on \( S^4 \) is carried out as

\[
\langle \Psi_{N,j},m';k',m_k|\Psi_{N,j,m;j,k,m_k}\rangle = \langle G_{N,j,k'}|G_{N,j,k}\rangle \cdot \langle Y_{j',m';k',m_k}|Y_{j,m;j,k,m_k}\rangle
\]

\[
= \int_0^\pi d\xi \sin^3 \xi \, G_{N,j,k'}(\xi)^* G_{N,j,k}(\xi) \cdot \int d\Omega_3 \, Y_{j',m';k',m_k}(\Omega_3)^\dag Y_{j,m;j,k,m_k}(\Omega_3).
\]

(126)

As discussed in Sec 2.2, the \( N \)th \( SO(5) \) Landau level consists of inner \( SO(4) \) Landau levels with \( n = 0,1,2,\ldots,N \). In the \( SO(4) \) language, \( x_m \) acts as a vector with the \( SO(4) \cong SU(2)_L \otimes SU(2)_R \) index \((j, k) = (1/2, 1/2)\) and \( x_5 \) acts as a scalar with \((j, k) = (0, 0)\). For the \( SO(4) \) Landau level index \( n \) [56] and the chirality parameter \( s \) [27], differences are given by \( \Delta n = \Delta j + \Delta k \) and \( \Delta s = \Delta j - \Delta k \). The \( SO(4) \) selection rule tells that the matrix coordinates have non-zero values only for the cases

\[
\langle x_m \rangle : (\Delta n, \Delta s) = (\pm 1, 0), \quad (0, \pm 1),
\]

(127a)

\[
\langle x_5 \rangle : (\Delta n, \Delta s) = (0,0).
\]

(127b)

Regions of the non-zero matrix elements are depicted in Fig 4 that expresses fine internal structures of Fig 3. With this in mind, we shall evaluate the matrix elements of \( x_5 \) and \( x_m \).

---

12\( \Gamma_5 \) of is given by a \( D(I, 0) \times D(I, 0) \) block diagonal matrix whose most upper-left/lower-right block is given by \( \mp I \cdot 1_{I+1} \).
Figure 4: Matrix coordinates for $I = 3$. There are non-zero matrix elements in the shaded color regions. The blue, green, purple shaded regions are specified by $\Delta N = \pm 1$, $(\Delta n, \Delta N) = (\pm 1, 0)$ and $(\Delta s, \Delta n, \Delta N) = (\pm 1, 0, 0)$, respectively. The red shaded regions correspond to $\Delta N = \Delta n = \Delta s = 0$. The red-framed squares (with inner red and purple squares) denote the $SO(4)$ Landau level subspaces. Obviously, the matrix geometry exhibits a nesting structure.

- Matrix coordinates for $x_5$

The matrix elements of $x_5$ are diagonalized as

$$
\langle x_a \rangle = \langle \Psi_{N,j',m',k'} | x_5 | \Psi_{N,j,m} \rangle = \langle G_{N,j',k'} | x_5 | G_{N,j,k} \rangle \cdot \langle Y_{j',m',k'} | Y_{j,m} \rangle = \langle G_{N,j,k} | x_5 | G_{N,j,k} \rangle \cdot \delta_{j,j'} \delta_{k,k'} \delta_{m,m'} \delta_{m',m},
$$

with

$$
\langle G_{N,j,k} | x_5 | G_{N,j,k} \rangle = (N + \frac{I}{2} + \frac{3}{2}) \int_0^\pi d\xi \sin \xi d_{N+\frac{1}{2}+1,s,-n-\frac{1}{2}-1}(\xi) \cos \xi d_{N+\frac{1}{2}+1,s,-n-\frac{1}{2}-1}(\xi)
$$

$$
= -\frac{2n+I+2}{(2N+I+2)(2N+I+4)} \cdot 2s,
$$

where we used (13) and a formula for the small $d$-function. The matrix coordinate (129) takes equally spaced discrete values specified by the chiral parameter $s = I/2, I/2 - 1, \ldots, -I/2$, which are regarded as latitudes of a fuzzy four-sphere. Such a structure is very similar to that of the fuzzy two-sphere, but while the latitudes of fuzzy two-sphere are not degenerate, the latitudes of fuzzy four-sphere are degenerate giving rise to the internal structure.

- Matrix coordinates for $x_{m=1,2,3,4}$

As (127) indicates, there are two cases in which $\langle x_m \rangle$ take finite values. The first case is $(\Delta n, \Delta s) = (\pm 1, 0)$ representing transition between two adjacent $SO(4)$ Landau levels (two adjacent $SO(4)$ lines in Fig 2) corresponding to the green shaded regions in Fig 4, while the second case

$$
(\Delta n, \Delta s) = (0, \pm 1)
$$

$$
\int_0^\pi d\theta \sin \theta d_{l, m, \phi}(\theta) \cos \theta d_{l, m, \phi}(\theta) = \frac{2q}{l(l+1)(2l+1)} m.
$$
represents transition between the two adjacent sub-bands specified by \( s \) inside a \( SU(4) \) Landau level (two adjacent dots on an identical \( SO(4) \) line in Fig 2) corresponding to the small purple shaded regions in Fig 4. In the following, we focus on the second case, which in the language of the \( SU(2)_L \otimes SU(2)_R \) corresponds to

\[
 j' = j + \frac{\sigma}{2}, \quad k' = k - \frac{\sigma}{2} \quad (\sigma = +, -)
\]  

(132)

Under the condition (132), we have

\[
 \langle \Psi_{N;j',m';k'}|x_m|\Psi_{N;j,m;k}\rangle
 = \sum_{\sigma=+,-} \langle G_{N,j+\frac{\sigma}{2},k-\frac{\sigma}{2}}|\sin\xi|G_{N,j,k}\rangle \cdot Y^{(\sigma,-\sigma)}_{m}(j,k)(m',m'; m, m_k) \cdot \delta_{j',j+\frac{\sigma}{2}} \cdot \delta_{k',k-\frac{\sigma}{2}}.
\]  

(133)

where

\[
 Y^{(\sigma,-\sigma)}_{m}(j,k)(m',m'; m, m_k) \equiv \langle Y_{j+\frac{\sigma}{2},m';m-\frac{\sigma}{2},m}|Y_{j,m;k,m_k}\rangle.
\]  

(134)

\( Y^{(\sigma,-\sigma)}_{m}(j,k) \) are regarded as \((2j+\sigma+1)(2k-\sigma+1) \times (2j+1)(2k+1)\) rectangular matrices with magnetic indices \((m',m';m,m_k)\), and \( Y^{(\sigma,-\sigma)}_{m}(j,k) \) and \( Y^{(-,\sigma)}_{m}(j+\frac{\sigma}{2},k-\frac{\sigma}{2}) \) are in the relation of Hermitian conjugate. We can evaluate the \( S^3 \)-radius part of (133) as

\[
 \langle G_{N,j+\frac{\sigma}{2},k-\frac{\sigma}{2}}|\sin\xi|G_{N,j,k}\rangle = (N + \frac{I}{2} + \frac{3}{2}) \int_0^\pi d\xi \sin^2\frac{\xi}{2} \int_{N+\frac{1}{2},1,s'-n-\frac{1}{2}}^{N+\frac{1}{2},1,s-n-\frac{1}{2}} (\xi) \right|_{s'=s+\sigma} = -\frac{2n + I + 2}{(2N + I + 2)(2N + I + 4)} \cdot 2\sqrt{(N + \frac{I}{2} - \sigma s + 1)(N + \frac{I}{2} + \sigma s + 2)}.
\]  

(135)

In the last equation, we used another formula of the small \( d \)-function. Next, we turn to the unit-\( S^3 \) part (134). Notice that \( y_m \) can be expanded by the \( SO(4) \) spherical harmonics [14):

\[
 y_1 = -i\frac{\pi}{2}(\Phi_{\frac{1}{2},\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}} - \Phi_{\frac{1}{2},\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}}), \quad y_2 = -\frac{\pi}{2}(\Phi_{\frac{1}{2},\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}} + \Phi_{\frac{1}{2},\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}}),
\]

\[
 y_3 = i\frac{\pi}{2}(\Phi_{\frac{1}{2},\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}} + \Phi_{\frac{1}{2},\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}}), \quad y_4 = \frac{\pi}{2}(\Phi_{\frac{1}{2},\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}} - \Phi_{\frac{1}{2},\frac{1}{2}} \Phi_{\frac{1}{2},\frac{1}{2}}).
\]  

(137)

With an integration formula for the \( SO(4) \) spherical harmonics, a bit of calculation (see Appendix D.1)

\[
 \int_0^\pi d\theta \sin\theta d_{l,m',g}(\theta) \sin\theta d_{l,m,g}(\theta)|_{m'=-m+1} = \frac{2g}{l(l+1)(2l+1)} \sqrt{(l + m)(l + m + 1)}.
\]  

(136)
shows\textsuperscript{13}
\begin{align*}
Y_{m=1,2}^{(+)}(j, k) &= (-i)^m \frac{1}{2} (-1)^{n+I} \left\{ j + \frac{1}{2} \ k - \frac{1}{2} \right\} \\
&\times (\delta_{m', m+j+\frac{1}{2}, m+k+\frac{1}{2}} \sqrt{(j + m_j + 1)(k - m_k)} - (-1)^m \delta_{m', m+j-\frac{1}{2}, m+k+\frac{1}{2}} \sqrt{(j - m_j + 1)(k + m_k)}), \\
Y_{m=3,4}^{(+)}(j, k) &= -(-i)^m \frac{1}{2} (-1)^{n+I} \left\{ j + \frac{1}{2} \ k - \frac{1}{2} \right\} \\
&\times (\delta_{m', m+j+\frac{1}{2}, m+k+\frac{1}{2}} \sqrt{(j + m_j + 1)(k + m_k)} - (-1)^m \delta_{m', m+j-\frac{1}{2}, m+k+\frac{1}{2}} \sqrt{(j - m_j + 1)(k - m_k)}).
\end{align*}

\begin{equation}
Y_{m}^{(\sigma, -\sigma)}(j, k) \text{ do not depend on the } SO(5) \text{ Landau level } N \text{ and denote the matrix coordinates of three-sphere as we shall discuss in Sec\textsuperscript{4}. The matrix coordinates \textsuperscript{13,5} are thus completely determined as \textsuperscript{13,5} and \textsuperscript{13,9} in an arbitrary Landau level.}
\end{equation}

### 3.4.2 Fuzzy four-sphere in the lowest Landau level

With the general results above, it is easy to derive the lowest Landau level \((N = n = 0)\) matrix coordinates\textsuperscript{16}
\begin{align*}
\langle \Psi_{N=0;j', m_j; k', m_k} | x_m | \Psi_{N=0;j, m_j; k, m_k} \rangle &= \frac{-2}{I + 4} \times \\
\left( \sqrt{\left( \frac{I}{2} + s \right)} \left( \frac{I}{2} - s \right) \right) Y_{m}^{(+,-)}(j, k) \delta_{j', j} \delta_{k', k} + \sqrt{\left( \frac{I}{2} + s \right)} \left( \frac{I}{2} + s \right) Y_{m}^{(-,+)}(j, k) \delta_{j', j} \delta_{k', k}, \tag{141a}
\end{align*}
\begin{align*}
\langle \Psi_{N=0;j', m_j; k', m_k} | x_5 | \Psi_{N=0;j, m_j; k, m_k} \rangle &= \frac{-2}{I + 4} s \delta_{j, j'} \delta_{k, k'} \delta_{m_j, m'_j} \delta_{m_k, m'_k}. \tag{141b}
\end{align*}

Since the matrix geometry of \(x_5\) is given by the diagonal matrix with eigenvalues of equal spacing \textsuperscript{11,12}, the present geometry can be regarded as a stacking of the matrix-valued three-spheres along \(x_5\)-axis with equal spacing [see Fig\textsuperscript{1}]. The right-hand side of (141) are identical to the \(SO(5)\) gamma matrices in the fully symmetric representation \((p, q) = (I, 0)\), so we have
\begin{equation}
X_a \equiv \langle x_a \rangle_{(n=0,N=0)} = \frac{1}{I + 4} \Gamma_a, \tag{142}
\end{equation}

\textsuperscript{15} Similarly,
\begin{align*}
Y_{m=1,2}^{(-)}(j, k) &= (-i)^m \frac{1}{2} (-1)^{n+I} \left\{ j - \frac{1}{2} \ k + \frac{1}{2} \right\} \\
&\times (\delta_{m', m+j+\frac{1}{2}, m+k+\frac{1}{2}} \sqrt{(j + m_j + 1)(k + m_k + 1)} - (-1)^m \delta_{m', m+j-\frac{1}{2}, m+k+\frac{1}{2}} \sqrt{(j - m_j + 1)(k - m_k + 1)}), \\
Y_{m=3,4}^{(-)}(j, k) &= -(-i)^m \frac{1}{2} (-1)^{n+I} \left\{ j - \frac{1}{2} \ k + \frac{1}{2} \right\} \\
&\times (\delta_{m', m+j+\frac{1}{2}, m+k+\frac{1}{2}} \sqrt{(j + m_j + 1)(k + m_k + 1)} - (-1)^m \delta_{m', m+j-\frac{1}{2}, m+k+\frac{1}{2}} \sqrt{(j - m_j + 1)(k - m_k + 1)}). \tag{138}
\end{align*}

\textsuperscript{16} In the special case \(s = \frac{1}{2} \sigma\), (141a) becomes
\begin{equation}
\langle \Psi_{N=0;j', j} + \frac{5}{2}, m_j; k' - \frac{5}{2}, m_k} | x_m | \Psi_{N=0;j, m_j; k, m_k} \rangle = -\frac{1}{I + 4} (I + 3) Y_{m}^{(\sigma, -\sigma)}(j, k). \tag{140}
\end{equation}

\(Y_{m}^{(\sigma, -\sigma)}(j, k)\) realizes the matrix for the fuzzy three-sphere \textsuperscript{32}. 

20
which reproduces the quantum limit result (106) at $I = 1$ and the classical limit result (119) at $I \gg 1$.

The matrix geometry (142) realizes the quantum Nambu geometry of the fuzzy four-sphere [23, 24]:

$$\sum_{a=1}^{5} X_a X_a = \frac{I}{I+4} \cdot \frac{1}{6} (I+1)(I+2)(I+3),$$

(143a)

$$[X_a, X_b, X_c, X_d] = (I+2) \left( \frac{2}{I+4} \right)^3 \epsilon_{abcd} X_e,$$

(143b)

where $[\cdots]$ of (143b) signifies the quantum Nambu bracket [25, 26].

In the thermodynamic limit $I \to \infty$, (143b) is reduced to the condition of a four-sphere with unit radius. As discussed above, the stacking of the matrix-valued three-sphere latitudes along $x_5$-axis constitutes the fuzzy four-sphere geometry. One may wonder if the stacking along the $x_5$-axis breaks the $SO(5)$ symmetry of the four-sphere. However, this is not the case. Recall that we have adopted $x_5$ as a special axis. If we had chosen $x_1$ as a special axis, we would have had the stack along the $x_1$-axis. Therefore, the picture of the stack along $x_5$-axis is a kind of “gauge-artifact” by choosing $x_a$ as a special axis in $\mathbb{R}^5$, and the fuzzy four-sphere certainly respects the $SO(5)$ symmetry.

### 3.4.3 Nested matrix geometry in higher Landau levels

Let us consider the matrix geometry in higher $SO(5)$ Landau levels. With a given $SO(5)$ Landau level $N$, there are $N + 1$ inner $SO(4)$ Landau levels indexed by $n = 0, 1, 2, \cdots, N$, and further in each of the $SO(4)$ Landau levels there are $I + 1$ sub-bands indexed by the chiral parameter $s$. Each sub-band $s$ realizes the matrix-valued $S^3$-latitude, and a stack of such $(I + 1)$ matrix-valued $S^3$-latitudes along the $x_5$-axis constitute a quasi-fuzzy four-sphere geometry in the $SO(4)$ Landau levels. Therefore inside the $N$th $SO(5)$ Landau level, there are $N + 1$ quasi-fuzzy spheres that form a nested structure as a whole [Fig.6]. Recall that the range of the chiral parameter $s$ is restricted to $|s| = \frac{I}{2}$ and does not cover the whole range of the matrix size specified by $j + k = n + \frac{I}{2}$ (except for $n = 0$). This implies that the corresponding matrix geometry is not a complete fuzzy four-sphere but a fuzzy four-sphere like geometry with removed north and south “caps” due to the uncovered parameter regions of $s$. We referred to this geometry as quasi-fuzzy four-sphere. Each $SO(4)$ Landau level accommodates a quasi-fuzzy four-sphere geometry, and so $N$th $SO(5)$ Landau level realizes $N + 1$ quasi-fuzzy four-spheres with different matrix size depending on the $SO(4)$ index $n$. In this way, $N + 1$ quasi-fuzzy four-spheres exhibit a concentric nested structure in the
Nth $SO(5)$ Landau level as depicted in Fig.6. The lowest Landau level ($N = 0$) is exceptional, because the nested structure no longer exists and only a fuzzy four-sphere geometry remains.

As the quantum states of the nested fuzzy geometry are given by a $SO(5)$ irreducible representation (or the $SO(5)$ monopole harmonics), the nested fuzzy geometry has the $SO(5)$ covariance. Meanwhile each quasi-fuzzy four sphere does not possess the $SO(5)$ covariance, since it is solely constructed by the $SO(4)$ irreducible representations. There exist non-vanishing off-diagonal matrix elements between the adjacent $SO(4)$ Landau levels (as represented by the green shaded rectangular blocks in Fig.4). Borrowing the string theory interpretation that the off-diagonal parts signify interactions between the fuzzy objects represented by the diagonal block matrices, one may say that the quasi-fuzzy four-spheres of the adjacent $SO(4)$ Landau levels interact and conspire to maintain the $SO(5)$ covariance of the nested fuzzy geometry. Furthermore, the nested fuzzy geometry has the $SO(5)$ symmetry. Apparently as a classical geometry the nested structure [Fig.6] does not have the $SO(5)$ symmetry, but it does in a quantum mechanical sense. The reason is essentially same as of the discussion below Eq.(144). We had chosen $x_5$ as a special axis, and we obtained the $x_5$-axis preferred picture like Fig.6 but if we had chosen the $x_1$ axis, we would have had a similar nested structure along the $x_1$-axis. Actually we can adopt any axis in $\mathbb{R}^5$, and then the nested structure has to have the $SO(5)$ symmetry. Therefore, the nested fuzzy geometry is a $SO(5)$ symmetric quantum geometry that does not have its counterpart in classical geometry.

4 Internal fuzzy structure and the $SO(4)$ Landau models

We discuss a physical model that realizes the matrix-valued three-sphere geometry inside the $SO(5)$ Landau model. We also clarify relations among Landau models in different dimensions.

4.1 $SU(2)$ meron gauge field and $SO(4)$ Pauli-Schrödinger Hamiltonian

We first construct a physical model whose eigenstates are given by the $SO(4)$ spinor spherical harmonics. The expression of the $SO(4)$ part of the $SO(5)$ free angular momentum operators are exactly equal to the $SO(4)$ free angular momentum operators (see Appendix C):

$$L_{mn}^{(0)} = -ix_m \frac{\partial}{\partial x_n} + ix_n \frac{\partial}{\partial x_m} = -iy_m \frac{\partial}{\partial y_n} + iy_n \frac{\partial}{\partial y_m}.$$  (145)
The SO(4) angular momentum $L_{mn}$ can also be represented only in terms of the $S^3$-coordinates:

$$L_{mn} = -iy_m \frac{\partial}{\partial y_n} + iy_n \frac{\partial}{\partial y_m} + \bar{\eta}_{mn} S_i.$$  

(146)

Therefore, the SO(4) analysis in Sec. 2.2.1 can be rewritten entirely in the language of $S^3$ without resorting to any information of the original manifold $S^4$. We then explore the SO(4) problem as an independent problem defined on $S^3$, and just utilize the $S^3$-coordinates. Interestingly, (146) can be realized as the SO(4) angular momentum operators in the meron gauge field introduced by Alfaro, Fubini and Furlan as a solution of pure Yang-Mills field equation [39, 40]:

$$A^{\text{AFF}} = \frac{1}{2r^2} \bar{\eta}_{mn} y_n \sigma_i \, dy_m.$$  

(147)

where $r = \sqrt{y_m y_m}$. The meron gauge field is simply obtained by the dimensional reduction of the Yang’s $SU(2)$ monopole gauge field:

$$A = -\frac{1}{2r(r+x_5)} \bar{\eta}_{mn} x_n \sigma_i dx_m \xrightarrow{x_5 \rightarrow 0} A^{\text{AFF}} = - \frac{1}{2r^2} \bar{\eta}_{mn} y_n \sigma_i dy_m.$$  

(148)

Notice that the Yang’s monopole has the string-like singularity, while the meron only has the point-like singularity at the origin. The corresponding field strength is given by:

$$F^{\text{AFF}}_{mn} = \frac{1}{r^2} y_m A^{\text{AFF}}_n + \frac{1}{r^2} y_n A^{\text{AFF}}_m + \frac{1}{2} \bar{\eta}_{mn} \sigma_i,$$  

(151)

and the total angular momentum operator is

$$L_{mn} = \Lambda_{mn} + r^2 F^{\text{AFF}}_{mn} = -iy_m \partial_n + iy_n \partial_m + y_m A^{\text{AFF}}_n - y_n A^{\text{AFF}}_m + r^2 F^{\text{AFF}}_{mn}$$

$$= -iy_m \partial_n + iy_n \partial_m + \frac{1}{2} \bar{\eta}_{mn} \sigma_i,$$  

(152)

where

$$\Lambda_{mn} = -iy_m (\frac{\partial}{\partial y_n} + iA^{\text{AFF}}_n) + iy_n (\frac{\partial}{\partial y_m} + iA^{\text{AFF}}_m).$$  

(153)

With the replacement of $\frac{1}{2} \sigma_i$ with higher $SU(2)$ spin matrix $S_i$, turns to the SO(4) angular momentum (146). The SO(4) Casimir is given by

$$L_{mn}^2 = 4(\mathbf{J}^2 + \mathbf{K}^2),$$  

(154)

where $\mathbf{J}$ and $\mathbf{K}$ are the following $SU(2)_L$ and $SU(2)_R$ operators

$$J_i = \frac{1}{4} \bar{\eta}_{mn} L_{mn} = J_i^{(0)} = -\frac{1}{2} \bar{\eta}_{mn} y_m \frac{\partial}{\partial y_n},$$  

(155a)

$$K_i = \frac{1}{4} \bar{\eta}_{mn} L_{mn} = K_i^{(0)} + S_i = -i \frac{1}{2} \bar{\eta}_{mn} y_m \frac{\partial}{\partial y_n} + S_i,$$  

(155b)

The associated 2nd Chern number is evaluated

$$Q = \frac{1}{32\pi^2} \int_{S^4} d^4x \epsilon_{mnpq} tr(F^{\text{AFF}}_{mn} F^{\text{AFF}}_{pq}) = -\frac{1}{2},$$  

(149)

leading to the name “meron”. For the meron field configuration with general spin $S_i^{(1/2)}$, the 2nd Chern number is evaluated as

$$Q = \frac{1}{12} l(l+1)(l+2).$$  

(150)
and the $SU(2)_L$ and $SU(2)_R$ Casimir eigenvalues are given by

$$J^2 = j(j + 1), \quad K^2 = k(k + 1),$$

(156)

with

$$j + k = n + \frac{I}{2} \quad (n = 0, 1, 2, \cdots), \quad s = j - k = \frac{I}{2} - 1, \cdots, -\frac{I}{2},$$

(157)

or

$$j = \frac{n}{2} + \frac{s}{2}, \quad k = \frac{n}{2} - \frac{s}{2},$$

(158)

Their simultaneous eigenstates are given by the $SO(4)$ spinor spherical harmonics (28).

In the meron field background, we introduce a $SO(4)$ Landau-like Hamiltonian

$$H_{PS} = \frac{1}{2M} \sum_{m<n=1}^4 \Lambda_{mn}^2.$$  

(159)

As usual, (159) can be rewritten as

$$H_{PS} = \frac{1}{2M} \sum_{m<n} (L_{mn}^2 - F_{mn}^2) = \frac{1}{2M} (2J^2 + 2K^2 - S^2),$$  

(160)

where we used $\sum_{m<n} \Lambda_{mn} F_{mn} = \sum_{m<n} F_{mn} \Lambda_{mn} = 0$ and $\sum_{m<n} F_{mn}^2 = S^2$. (159) can also be expressed as

$$H_{PS} = \frac{1}{2M} (2J^{(0)}^2 + 2K^{(0)}^2 + 4K^{(0)} \cdot S + \frac{I}{2} \frac{I}{2} + 1),$$  

(161)

which realizes a $SO(4)$ generalization of the original Pauli-Schrödinger Hamiltonian [41] with spin-orbit coupling. For this reason, we refer to (159) as the $SO(4)$ Pauli-Schrödinger Hamiltonian in this paper. From (169), the eigenvalues of the Pauli-Schrödinger Hamiltonian are readily obtained as

$$E_n(s) = \frac{1}{2M} (n(n+2) + \frac{I}{2} (2n+1) + s^2),$$  

(162)

where $n$ denotes the $SO(4)$ Landau levels and $s$ denotes the sub-bands in the $SO(4)$ Landau levels [Fig. 7]. The $SO(4)$ Landau level eigenstates are actually the $SO(4)$ spinor spherical harmonics $Y_{j,mj; k,ms}$ with (157), and so the previous three-sphere matrix geometry (134) is considered to be realized in the $SO(4)$ Landau level. In this way, we can reformulate the $SO(4)$ part of the $SO(5)$ Landau model with the $SO(4)$ Pauli-Schrödinger model. In other words, the $SO(5)$ Landau model accommodates the $SO(4)$ Pauli-Schrödinger model as its internal model.

### 4.2 Singular gauge transformation and $SO(4)$ matrix geometry

Curiously, the energy levels (162) are exactly equal to the Landau levels of the $SO(4)$ Landau Hamiltonian proposed by Nair and Daemi [42]. This coincidence implies a hidden relation between the $SO(4)$ Pauli-Schrödinger model and the $SO(4)$ Landau model. In the following, we adopt the notation of [28, 32]. The $SO(4)$ Landau Hamiltonian is given by

$$H = \frac{1}{2M} \sum_{m<n=1}^4 \Lambda_{mn}^2,$$  

(163)

where

$$\Lambda_{mn} = -i y_m (\frac{\partial}{\partial y_n} + i A_{m}^{ND}) + i y_n (\frac{\partial}{\partial y_m} + i A_{n}^{ND}),$$  

(164)
with the Nair-Daemi $SU(2)$ gauge field\(^{18}\)

\[
A^\text{ND} = -\frac{1}{r(r+y_4)}\epsilon_{ijk}y_jS_k\,dy_i. \tag{165}
\]

Obviously, the Nair-Daemi $SU(2)$ gauge field has a Dirac string-like singularity. The corresponding field strength is derived as

\[
F^\text{ND}_{ij} = -y_iA^\text{ND}_j + y_jA^\text{ND}_i + \epsilon_{ijk}S_k, \quad F^\text{ND}_{i4} = (1 + y_4)A^\text{ND}_i. \tag{166}
\]

The eigenvalues of the $SO(4)$ Landau Hamiltonian\(^{163}\) are given by \(^{162}\) and the corresponding eigenstates, i.e., the $SO(4)$ monopole harmonics (in the Dirac gauge), are given by \(^{32\,17\,19}\)

\[
\Phi_{j,m_j;k,m_k}(\chi,\theta,\phi) = \tilde{g}(\theta,\phi) \left( \begin{array}{c}
\Phi_{j,m_j;k,m_k}(\chi,\theta,\phi)_{1/2} \\
\Phi_{j,m_j;k,m_k}(\chi,\theta,\phi)_{1/2-1} \\
\vdots \\
\Phi_{j,m_j;k,m_k}(\chi,\theta,\phi)_{-1/2}
\end{array} \right), \tag{168}
\]

where\(^{20}\)

\[
\tilde{g}(\theta,\phi) \equiv D^\langle(1/2)\rangle(\phi,\theta,0) = e^{-i\phi S_z}e^{-i\theta S_y}. \tag{169}
\]

\(^{18}\)The Nair-Daemi $SU(2)$ monopole gauge field is equivalent to the spin connection of $S^3$.

\(^{19}\)\(\tilde{g}\) constitutes an orthonormal set:

\[
\langle \Phi_{j,m_j;k,m_k}, \Phi_{j',m_j',k',m_k'} \rangle \equiv \int_{S^3} d\Omega_3 \Phi_{j,m_j;k,m_k}(\Omega_3)^\dagger \Phi_{j',m_j',k',m_k'}(\Omega_3) = \delta_{j,j'}\delta_{k,k'}\delta_{m_j,m_j'}\delta_{m_k,m_k'}. \tag{167}
\]

\(^{20}\)\(\tilde{g}(\theta,\phi)\) is a gauge function that relates the $SO(4)$ monopole harmonics in the Dirac gauge and the Schwinger gauge \(^{32}\).
and
\[ \Phi_{j,m;j,k,m_k}(\Omega_3) = \sqrt{\frac{(2j + 1)(2k + 1)}{2\pi^2(1 + 1)}} \times \sum_{m_j' = -j}^{j} \sum_{m_k' = -k}^{k} \langle I/2, A|j, m_j'; k, m_k' \rangle D^{(l)}(\chi, \theta, \phi)_{m_j',m_j} D^{(l')}(-\chi, \theta, \phi)_{m_k',m_k}, \] (170)

with the Wigner’s $D$-function
\[ D^{(l)}(\chi, \theta, \phi) = e^{-i\chi S^{(l)}_z} e^{-i\theta S^{(l)}_y} e^{-i\phi S^{(l)}_z}. \] (171)

With the preparation, we now discuss a relation between the $SO(4)$ Pauli-Schrödinger model and the $SO(4)$ Landau model.

We have seen that the meron gauge field has the point-like singularity, while the Nair-Daemi’s $SU(2)$ monopole has the string-like singularity. It is well known a similar situation occurs in a lower dimension. In this sense the meron is a 4D generalization of the Wu-Yang monopole configuration [44, 45]. Let us first recall the singular transformation that relates the singular transformation
\[ \Phi = e^{-ixS^{(l)}_z} e^{-i\theta S^{(l)}_y} e^{-i\phi S^{(l)}_z}. \]

\[ A^{\text{WY}} = \frac{1}{2} A^{\text{D}} g - ig^\dagger dg, \] (174)

where $g(\theta, \phi) = e^{-i\theta S^{\dagger}_z - 1 S^{\dagger}_u} = e^{-i\phi S^{\dagger}_y} e^{i\theta S^{\dagger}_z},$ (176)

with $S^1$-latitude coordinates $\hat{z}_1 = \cos \phi$, $\hat{z}_2 = \sin \phi$. A bit of consideration tells that the $SU(2)$ monopole and the meron gauge fields are also related by the following $SU(2)$ singular transformation
\[ A^{\text{ND}} = g^\dagger A^{\text{AFF}} g - ig^\dagger dg, \] (177)

where
\[ g(\chi, \theta, \phi) = e^{-i\chi \sum_{i=1}^{3} \hat{y}_i S_i} = g(\theta, \phi) e^{-i\chi S^{\dagger}_z} \hat{y}(\theta, \phi)^\dagger. \] (178)

Here $\hat{y}(\theta, \phi)$ is given by [169], and $\hat{y}_i$ are the coordinates on $S^2$-latitude parameterized as
\[ \hat{y}_{i=1,2,3} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \] (179)

$^{21}$For $S_i = \frac{1}{2} \sigma_i$, (176) becomes
\[ g(\theta, \phi) = e^{-i\frac{1}{2} \hat{y}(\hat{z}_2 S^{\dagger}_z - 1 S^{\dagger}_u)} = e^{-i\hat{y} S^{\dagger}_z} e^{i\theta S^{\dagger}_y} e^{i\phi S^{\dagger}_z} \left( \begin{array}{cc} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} e^{-i\phi} \\ -\sin \frac{\theta}{2} e^{i\phi} & \cos \frac{\theta}{2} \end{array} \right), \] (175)

whose columns are the 1st Hopf spinors (see [19] for instance).
The SO(4) Pauli-Schrödinger model is transformed to the SO(4) Landau model by the singular gauge transformation \(\mathcal{T}_{\text{Landau}}\). Indeed, one can also confirm that the SO(4) monopole harmonics and the SO(4) spinor spherical harmonics are related as

\[
\Phi_{j,m; k,m}(\chi, \theta, \phi) = (-1)^{j+s} g(\chi, \theta, \phi)^{j} Y_{j,m;j,k,m} \chi, \theta, \phi.
\]

Consequently, the matrix elements are related as

\[
\langle \Phi_{j',m'; k',m'} | y_m | \Phi_{j,m;j,k,m} \rangle = -\langle Y_{j',m'; k',m'} | y_m | Y_{j,m;j,k,m} \rangle,
\]

where we used \((-1)^{j+s' + s} = -(-1)^{j+s} = -1\). In Appendix D, we rigorously evaluate both sides of (181) and explicitly check its validity. Therefore, the matrix geometry of the SO(4) Pauli-Schrödinger model is exactly equal to the matrix geometry of the SO(4) Landau model, and hence the SO(4) Landau model describes the internal fuzzy geometry of the SO(5) Landau model. This demonstrates the idea of the dimensional hierarchy \([29, 15]\) relating the Landau physics in different dimensions. In Fig.8, we summarize the relations among the Landau models in various dimensions. For a better understanding of this section, we also elucidate the case \((j, k) = (1/2, 0) \oplus (0, 1/2)\) in Appendix D.3.

5 Relativistic SO(5) Landau models

We explore relativistic version of the SO(5) Landau model and clarify relation to the matrix geometry of the Berezin-Toeplitz quantization \([46]\).

5.1 Geometric quantities of \(S^4\)

In the parameterization \([13]\), the metric of \(S^4\) is given by

\[
ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2
= d\xi^2 + \sin^2\xi d\chi^2 + \sin^2\xi \sin^2\chi d\theta^2 + \sin^2\xi \chi \sin^2\theta d\phi^2.
\]
We adopt the vierbein in the Schwinger gauge as
\[ e^4 = d\xi, \quad e^1 = \sin \xi d\chi, \quad e^2 = \sin \xi \sin \chi d\theta, \quad e^3 = \sin \xi \sin \chi \sin \theta d\phi. \] (184)
With \( \omega_{mn} \) satisfying the Cartan structure equation, \( de^m + \omega_{mn}e_n = 0 \), the spin connections
\[ \omega_L = \frac{1}{2} \eta_{mn} \omega^{mn}, \quad \omega_R = \frac{1}{2} \tilde{\eta}_{mn} \omega^{mn}, \] (185)
are given by (see Appendix B for details)
\[ \omega_L = \omega_R = 0, \quad \omega_L = \omega_R = -\cos \xi \frac{1}{2} q_x, \quad \omega_R = \omega_R = \cos \xi \sin \chi \frac{1}{2} q_y - \cos \chi \frac{1}{2} q_z, \]
\[ \omega_L = \omega_R = \cos \xi \sin \chi \sin \theta \frac{1}{2} q_z + \cos \chi \sin \theta \frac{1}{2} q_y + \cos \theta \frac{1}{2} q_x. \] (186)
The \( SO(4) \) matrix-valued spin connection is constructed as
\[ \omega = \begin{pmatrix} \omega_L & 0 \\ 0 & \omega_R \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \omega^i_L q_i & 0 \\ 0 & \omega^i_R q_i \end{pmatrix} = \frac{1}{4} \omega^{mn} \begin{pmatrix} \eta_{mn} q_i & 0 \\ 0 & \tilde{\eta}_{mn} q_i \end{pmatrix}, \] (187)
which carries the \( SU(2)_L \otimes SU(2)_R \) index:
\[ (1/2, 0) \oplus (0, 1/2). \] (188)

5.2 Spinor \( SO(5) \) Landau model

We consider a relativistic spinor particle on \( S^4 \), which feels the connection of the base-manifold \( S^4 \) as well as the external \( SU(2) \) monopole gauge field. In other words, the relativistic particle interacts with the synthetic gauge field of the \( SO(4) \) connection \([187]\) and the \( SU(2) \) monopole field \([8, 23]\)
\[ A_a = \omega_a \otimes 1_{I+1} + 1_4 \otimes A_a. \] (189)
For the \( SO(4) \cong SU(2)_L \otimes SU(2)_R \) gauge group, the synthetic gauge field is irreducibly decomposed as
\[ ((1/2, 0) \oplus (0, 1/2)) \oplus (I/2) = (0, I/2 + 1/2) \oplus (0, I/2 - 1/2) \oplus (1/2, I/2), \] (190)
and their corresponding dimensions are
\[ (2 \oplus 2) \otimes (I + 1) = (I + 2) \oplus I \oplus (2I + 2). \] (191)
The field strength is given by
\[ F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b] = f_{ab} \otimes 1_{I+1} + 1_4 \otimes F_{ab}, \] (192)
with \( f_{ab} = \partial_a \omega_b - \partial_b \omega_a + i[\omega_a, \omega_b] = e_a \wedge e_b \) (Appendix B.1). The \( SO(5) \) angular momentum in the synthetic gauge field is given by
\[ L_{ab} = A_{ab} + F_{ab}, \] (193)

---

22 We choose the numbering of the vierbein as \([181]\) so that the \( SO(5) \) Dirac-Landau operator is reduced to the \( SO(4) \) Dirac-Landau operator of \([22]\) at \( \xi = \frac{\pi}{2} \) (see \([212]\)). The area of \( S^4 \) is calculated as
\[ A(S^4) = \int e^4 \wedge e^1 \wedge e^2 \wedge e^3 = \int_0^\pi d\xi \sin^3 \xi \int_0^\pi d\chi \sin^2 \chi \int_0^{2\pi} d\theta \sin \theta \int_0^{2\pi} d\phi = \frac{8\pi^2}{3}. \] (183)

23 \([180]\) is represented in the Schwinger gauge, while \([3]\) in the Dirac gauge (see Appendix B for details), so it will be convenient to adopt one gauge in constructing the synthetic gauge field \([189]\).
with $A_{ab}$ being the covariant angular momentum operator

\[ A_{ab} \equiv -ix_a(\partial_b + iA_b) + ix_b(\partial_a + iA_a). \]  

(194)

We introduce the spinor $SO(5)$ Landau Hamiltonian as

\[ H = \frac{1}{2M} \sum_{a < b = 1}^5 A_{ab}^2 = \frac{1}{2M}(\sum_{a < b = 1}^5 \mathcal{L}_{ab}^2 - \sum_{m < n = 1}^4 \Sigma_{mn}^2). \]  

(195)

The decomposition (190) implies that, with some appropriate unitary transformation, the spinor $SO(5)$ Landau Hamiltonian is transformed as

\[ H \to \begin{pmatrix} H^{(0,1/2)} & 0 & 0 \\ 0 & H^{(0,-1/2)} & 0 \\ 0 & 0 & H^{(1/2)} \end{pmatrix}. \]  

(196)

Here $H^{(I,I/2)}$ denotes a $SO(5)$ Landau Hamiltonian in the $SO(4)$ monopole background with $SO(4)$ matrices

\[ \Sigma_{mn}^{(I,I/2)} = \eta^i_{mn} S_i^{(I,I/2)} \otimes 1_{I+1} + 1_{I+1} \otimes \bar{\eta}^i_{mn} S_i^{(I-I/2)}. \]  

(197)

In particular, $H^{(0,-1/2)}$ in (196) is just a non-relativistic Landau Hamiltonian with the $SU(2)$ (anti-)monopole index $(I-1)/2$.  

5.3 $SO(5)$ Dirac-Landau operator and zero-modes

The Dirac-Landau operator on $S^4$, which we call the $SO(5)$ Dirac-Landau operator, is constructed as

\[ -i\mathcal{D}_{S^4} = -ie_\mu \gamma^m (\partial_\mu + i\omega_\mu \otimes 1_{I+1} + i\mathbf{1}_4 \otimes A_\mu) \]

\[ = -ie_\mu \gamma^m \otimes \tilde{D}_\mu, \]  

(198)

where $\tilde{D}_\mu$ ($\mu = \xi, \chi, \theta, \phi$) are newly introduced covariant derivatives including the contribution of the spin connection:

\[ \tilde{D}_\xi = D_\xi + \frac{3}{2} \cot \xi 1_{I+1}, \quad \tilde{D}_\chi = D_\chi + \cot \chi 1_{I+1}, \quad \tilde{D}_\theta = D_\theta + \frac{1}{2} \cot \theta 1_{I+1}, \quad \tilde{D}_\phi = D_\phi \]  

(199)

with

\[ D_\mu = \partial_\mu + iA_\mu. \]  

(200)

The second terms on the right-hand sides of (199) are attributed to the spin connections $\omega_\mu$. We adopt the $SU(2)$ gauge field in (200) as

\[ A_\mu = \omega_1^{R_\mu} S_2^{(1/2)} + \omega_2^{R_\mu} S_3^{(1/2)} + \omega_3^{R_\mu} S_y^{(1/2)}, \]  

(201)

or more explicitly,

\[ A_\xi = 0, \quad A_\chi = -\cos \xi S_2^{(1/2)}, \quad A_\theta = -\cos \xi \sin \chi S_2^{(1/2)} - \cos \chi S_y^{(1/2)}, \]

\[ A_\phi = -\cos \xi \sin \chi \sin \theta S_y^{(1/2)} + \cos \chi \sin \theta S_z^{(1/2)} - \cos \theta S_x^{(1/2)}, \]  

(202)

which denotes the Yang’s (anti-)monopole in the Schwinger gauge (see Appendix B). From (198), one can find that the Dirac-Landau operator respects the chiral “symmetry”:

\[ \{ -i\mathcal{D}_{S^4}, \gamma^5 \otimes 1_{I+1} \} = 0. \]  

(203)
Therefore, the positive and negative relativistic Landau levels of the \(SO(5)\) Dirac-Landau operator are symmetric with respect to the zero. The \(SO(5)\) Dirac-Landau operator does not have any \(SO(5)\) indices and is invariant under the \(SO(5)\) rotations:

\[
[-i \mathcal{P}_{s^4}, L_{ab}] = 0.
\]  

(204)

Here \(L_{ab}\) and \(-i \mathcal{P}_{s^4}\) are respectively given by (193) and (198), and one may in principle verify (204) by using the explicit forms of the operators. The Dirac-Landau operator eigenstates are degenerate with respect to the \(SO(5)\) rotational symmetry and can be expanded by the eigenstates of the spinor \(SO(5)\) Landau model. In particular, the \(SO(5)\) Dirac-Landau operator zero-modes are exactly equal to the lowest Landau level eigenstates of the non-relativistic Landau Hamiltonian \(H^{(0,\frac{2}{3} - \frac{1}{3})}\) of (196) as we shall see in Sec. 5.3.2.

5.3.1 Dimensional reduction to the \(SO(4)\) Dirac-Landau operator

On the equator \(\xi = \pi/2\), the \(SU(2)\) gauge field (202) is reduced to the \(SU(2)\) gauge field of the \(SO(4)\) Landau model (32):

\[
(A_\chi, A_\theta, A_\phi) \to (0, - \cos \chi S_y^{(1/2)}, \cos \chi \sin \theta S_x^{(1/2)} - \cos \theta S_z^{(1/2)}).
\]  

(205)

(198) can be decomposed as

\[
- i \mathcal{P}_{s^4} = -i \gamma^4 \otimes \hat{D}_\xi - i \frac{1}{\sin \xi} \left( \gamma^1 \otimes \hat{D}_\chi + \frac{1}{\sin \chi} \gamma^2 \otimes \hat{D}_\theta + \frac{1}{\sin \chi \sin \theta} \gamma^3 \otimes \hat{D}_\phi \right),
\]  

(206)

where

\[
\hat{D}_\xi \equiv \partial_\xi + i A_\xi + \frac{3}{2} \cot \xi \mathbf{1}_{I+1} = \partial_\xi + \frac{3}{2} \cot \xi \mathbf{1}_{I+1},
\]

\[
\hat{D}_\chi \equiv \partial_\chi + i A_\chi + \cot \chi \mathbf{1}_{I+1} = \partial_\chi - i A_\chi + \cot \chi \mathbf{1}_{I+1},
\]

\[
\hat{D}_\theta \equiv \partial_\theta + i A_\theta + \frac{1}{2} \cot \theta \mathbf{1}_{I+1} = \partial_\theta + i A_\theta + \cot \theta \mathbf{1}_{I+1},
\]

\[
\hat{D}_\phi \equiv \partial_\phi + i A_\phi = \partial_\phi - i \cos \xi \sin \theta S_y^{(1/2)} + i \cos \chi \sin \theta S_x^{(1/2)} - \cos \theta S_z^{(1/2)}.
\]  

(207)

When we take the gamma matrices (77) as

\[
\gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \sigma_z \\ -i \sigma_z & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & i \sigma_x \\ -i \sigma_x & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & i \sigma_y \\ -i \sigma_y & 0 \end{pmatrix},
\]  

(208)

(203) becomes

\[
-i \mathcal{P}_{s^4} = -i (\partial_\xi + \frac{3}{2} \cot \xi) \begin{pmatrix} 0_{2(I+1)} & 1 \\ 1 & 0_{2(I+1)} \end{pmatrix} + \frac{1}{\sin \xi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbb{P}_{s^3},
\]  

(209)

where \(-i \mathbb{P}_{s^3}\) is given by

\[
-i \mathbb{P}_{s^3} = \cos \xi \sum_{i=1}^3 \sigma_i \otimes S_i^{(1/2)}.
\]  

(210)

---

\(^{24}\)The choice (202) is different from the previous one (77). We adopt (202) so that the \(SO(4)\) Dirac-Landau operator (211) coincides with the expression of (32).

\(^{25}\)One may readily check that in the absence of the \(SU(2)\) monopole gauge field, (209) is reduced the free Dirac operator (77).
Hamiltonian. We will focus on the case Hamiltonian (195), the eigenvalue problem of (213) is essentially equivalent to that of the spinor Landau SO square of the Dirac-Landau operator shares the same from the chiral symmetry, and only have the degeneracy of the SO rotational symmetry and the chiral symmetry. The zero-modes, however, do not have the degeneracy

\[ -i\mathcal{P}_{S^3} = -i\sigma^3 \otimes (\partial_\chi + \cot \chi \, 1_{I+1}) - i \frac{1}{\sin \chi} \sigma^1 \otimes (\partial_\theta - i \cos \chi \, S_y^{(1/2)} + \frac{1}{2} \cot \theta \, 1_{I+1}) - i \frac{1}{\sin \chi \sin \theta} \sigma^2 \otimes (\partial_\phi + i \cos \chi \sin \theta \, S_x^{(1/2)} - i \cos \theta \, S_z^{(1/2)}). \]  

(211)

On the equator of \( S^3 \, (\xi = \frac{\pi}{2}) \), \(-i\mathcal{P}_{S^3}\) is reduced to \(-i\mathcal{P}_{S^3}\), and so is the SO(5) Dirac-Landau operator:

\[ -i\mathcal{P}_{S^3}|_{\xi = \frac{\pi}{2}} = -i\mathcal{P}_{S^3}. \]  

(212)

The relativistic SO(5) Landau model thus embeds the relativistic SO(4) Landau model on the equator as the non-relativistic Landau model does. The fuzzy three-sphere geometry is realized in the SO(4) relativistic Landau model [32], and then the SO(5) relativistic Landau model accommodates a fuzzy three-sphere geometry as its sub-geometry on the equator, which suggests existence of the fuzzy four-sphere as a whole geometry.

5.3.2 Zero-modes and the matrix geometry

The square of the Dirac-Landau operator [198] and the SO(5) Casimir [193] are related as [48] [27]

\[ (-i\mathcal{P}_{S^4})^2 = \sum_{a<b=1}^5 \mathcal{L}_{ab}^2 - \sum_{a<b=1}^5 F_{ab}^2 + \frac{1}{8} \mathcal{R}_{S^4} = \sum_{a<b=1}^5 \mathcal{L}_{ab}^2 - 2 \cdot \frac{I}{2} (\frac{I}{2} + 1) + \frac{3}{2}. \]  

(213)

Here, we used \( \sum_{a<b=1}^5 F_{ab}^2 = \sum_{\mu<n} (\hat{\eta}^I_{mn} S_i^{(1/2)})^2 = 2 \mathbf{S}^{(1/2)} S_i^{(1/2)} = 2 \cdot \frac{d}{2} (\frac{d}{2} + 1) \) and \( R_{S^4} = d(d-1)|_{d=4} = 12 \) [31]. The square of the Dirac-Landau operator respects the SO(5) rotational symmetry and the chiral symmetry as well:

\[ [(-i\mathcal{P}_{S^4})^2, \mathcal{L}_{ab}] = [(-i\mathcal{P}_{S^4})^2, \gamma_5 \otimes 1_{I+1}] = 0. \]  

(214)

Consequently, the eigenvalues of \((-i\mathcal{P}_{S^4})^2\) generally have two kinds of degeneracies coming from the SO(5) rotational symmetry and the chiral symmetry. The zero-modes, however, do not have the degeneracy from the chiral symmetry, and only have the degeneracy of the SO(5) rotational symmetry. Since the square of the Dirac-Landau operator shares the same SO(5) Casimir \( \sum_{a<b=1}^5 \mathcal{L}_{ab}^2 \) with the spinor Landau Hamiltonian [195], the eigenvalue problem of (213) is essentially equivalent to that of the spinor Landau Hamiltonian. We will focus on the case

\[ (p, q)_{SO(5)} = (N + 2J, N) \bigg|_{J = \frac{I}{2} - \frac{1}{2}}, \]  

(215)

which corresponds to \( H^{(0,\frac{1}{2}-\frac{1}{2})} \) in [196]. In this case, the SO(5) Casimir eigenvalues are given by

\[ \sum_{a<b=1}^5 \mathcal{L}_{ab}^2 = N^2 + N(2J + 3) + 2J(J + 2) \bigg|_{J = \frac{I}{2} - \frac{1}{2}} = N^2 + N(I + 2) + \frac{1}{2}(I - 1)(I + 3), \]  

(216)

and then

\[ (-i\mathcal{P}_{S^4})^2 = N^2 + N(I + 2). \]  

(217)

Obviously, the zero eigenvalue is realized at \( N = 0 \). Since the eigenvalues of the Dirac-Landau operator are real values, the zero-modes of the square of the Dirac-Landau operator are equal to the zero-modes of the Dirac-Landau operator. Notice that \( N \) acts as the Landau level index in the non-relativistic Hamiltonian.
\[ H^{(0, \frac{1}{2} - \frac{1}{2})} \], and so the zero-modes are identical to the lowest Landau level eigenstates of \( H^{(0, \frac{1}{2} - \frac{1}{2})} \). Then, the zero-mode degeneracy is readily obtained as
\[ D(I - 1, N)|_{N=0} = \frac{1}{6} I(I + 1)(I + 2). \] (218)

The Atiyah-Singer index theorem also gives the same result about the number of zero-modes, \(-c_2(I) = D(I - 1, N)|_{N=0} = \frac{1}{6} I(I + 1)(I + 2)\). In [46], the fuzzy four-sphere geometry was derived in the Berezin-Toeplitz method by taking matrix elements sandwiched by the Dirac-Landau operator zero-modes. Since the zero-modes are identical to the non-relativistic lowest Landau level eigenstates, the matrix geometry obtained in the non-relativistic analysis (142) exactly coincides with that of the Berezin-Toeplitz quantization.

6 Even higher dimensional Landau model and matrix geometry

We extend the discussions of Sec.3 to even higher dimensions and investigate the matrix geometry in the \( SO(2k + 1) \) Landau model.

6.1 Quantum limit

First, we analyze the quantum limit, \( I = 1 \). We introduce the following map analogous to the Hopf maps:
\[ \Psi \rightarrow x_a 1_{2k-1} = \Psi^\dagger \gamma_a \Psi. \quad (a = 1, 2, \cdots, 2k + 1) \] (219)

Here, \( \gamma_a \) are \( SO(2k + 1) \) gamma matrices
\[ \gamma_i = \begin{pmatrix} 0 & i\gamma'_i \\ -i\gamma'_i & 0 \end{pmatrix}, \quad \gamma_{2k + 1} = \begin{pmatrix} 0 & 1_{2k-1} \\ 1_{2k-1} & 0 \end{pmatrix}, \quad \gamma_{2k + 1} = \begin{pmatrix} -1_{2k-1} & 0 \\ 0 & 1_{2k-1} \end{pmatrix}, \] (220)

with \( \gamma'_i (i = 1, 2, \cdots, 2k - 1) \) being the \( SO(2k - 1) \) gamma matrices, and \( \Psi \) denotes a \( 2^k \times 2^{k - 1} \) complex spinor given by
\[ \Psi = \frac{1}{\sqrt{2(1 + x_{2k + 1})}} \begin{pmatrix} x_{2k} 1_{2k-1} + i \sum_{i=1}^{2k-1} \gamma'_i x_i \\ (1 + x_{2k + 1}) 1_{2k-1} \end{pmatrix} \cdot g, \] (221)

where \( g \) denotes \( SO(2k) \) gauge group element, and \( \Psi \) satisfies \( \Psi^\dagger \Psi = 1_{2k-1} \). The corresponding connection is obtained as
\[ A = -i \Psi^\dagger d\Psi = -i \frac{1}{1 + x_{2k + 1}} \sigma_{mn} x_n dx_m, \] (222)

with \( SO(2k) \) matrix generators
\[ \sigma_{mn} = -i \frac{1}{4} [\gamma'_m, \gamma'_n]. \quad (\gamma'_m = \{ \gamma'_m, 1_{2k-1} \}) \] (223)

(222) signifies a \( SO(2k) \) non-Abelian monopole gauge field [15]. We can construct the \( SO(2k + 1) \) Landau model in a similar manner to Sec.2.3 [15]. For instance, the \( SO(2k + 1) \) angular momentum operators are given by
\[ L_{ab} = -ix_a (\partial_b + iA_b) + ix_b (\partial_a + iA_a) + r^2 F_{ab}. \] (224)
When we represent $\Psi$ as
\[
\Psi = \begin{pmatrix}
\psi_1^\dagger \\
\psi_2^\dagger \\
\vdots \\
\psi_{2k}^\dagger
\end{pmatrix},
\]
(225)
the rows $\psi_{\alpha}$ ($\alpha = 1, 2, \cdots, 2^k$) denote a set of $SO(2k + 1)$ $2^{k-1}$-component spinors that transform as a multiplet under the $SO(2k + 1)$ transformation. They are the lowest Landau level eigenstates for $I = 1$. $\Psi$ yields the projection matrix
\[
P \equiv \Psi \Psi^\dagger = \frac{1}{2} (1_{2^k} + \sum_{a=1}^{2k+1} x_a \gamma_a),
\]
(228)
which is a gauge invariant quantity and simply expressed by the $S^{2k}$-coordinates. The matrix coordinates are expressed as
\[
(X_a)_{\alpha\beta} = \frac{2}{A(S^{2k})} \int d\Omega_{2k} \psi_{\alpha}^\dagger x_a \psi_{\beta},
\]
(230)
or
\[
X_a = \frac{2}{A(S^{2k})} \int d\Omega_{2k} \Psi x_a \Psi^\dagger = \frac{2}{A(S^{2k})} \int d\Omega_{2k} P \gamma_a P,
\]
(231)
where $A(S^{2k}) = \frac{2^{k+1} \pi^k}{(2k-1)!!}$ denotes the area of $S^{2k}$. With the formulas
\[
P \gamma_a P = \frac{1}{2} x_a (1 + x_b \gamma_b), \quad \int_{S^{2k}} d\Omega_{2k} x_a = 0, \quad \int_{S^{2k}} d\Omega_{2k} x_a x_b = \frac{1}{2k+1} A(S^{2k}) \delta_{ab},
\]
(232)
we can easily evaluate (231) as
\[
X_a = \frac{1}{2k+1} \gamma_a.
\]
(233)

6.2 Classical limit

Next, we consider the classical limit $I >> 1$, in which $L_{ab}$ (224) is reduced to
\[
L_{ab} \rightarrow r^2 F_{ab}.
\]
(234)
The coordinates $x_a$ can be extracted from the field strength as
\[
\frac{1}{r^{2k+1}} x_a = \frac{2}{(2k)! c_k(I)} \epsilon_{a a_1 a_2 \cdots a_{2k}} \text{tr}(F_{a_1 a_2} F_{a_3 a_4} \cdots F_{a_{2k-1} a_{2k}}),
\]
(235)
where $c_k(I)$ denotes the $k$th Chern number for the $SO(2k)$ gauge field:
\[
c_k(I) = \frac{1}{(2\pi)^k k!} \int_{S^{2k}} \text{tr}(F^k).
\]
(236)

27With (224) of $I = 1$, we can show,
\[
L_{ab} \psi_{\alpha} = (\sigma_{ab})_{\beta\alpha} \psi_{\beta},
\]
(226)
where
\[
\sigma_{ab} \equiv -\frac{i}{4} [\gamma_a, \gamma_b].
\]
(227)

28 The coefficient in front of the integration of (230) is added to be accounted for by the normalization of $\Psi$:
\[
\frac{2}{A(S^{2k})} \int_{S^{2k}} d\Omega_{2k} \Psi \Psi^\dagger = 1_{2^k},
\]
(229)
Substituting (234) to (235), we have
\[ X_a \sim \frac{2r}{(2k)! c_k(I)} \epsilon_{a_1 a_2 \cdots a_{2k}} L_{a_1 a_2} L_{a_3 a_4} \cdots L_{a_{2k-1} a_{2k}} \text{ tr}(1_{\text{internal space}}). \]  

(237)

Since \( L_{ab} \) are the \( SO(2k+1) \) operators, \( X_a \) also become operators. \( \text{tr} \) in (235) (and (236)) is taken for the “internal fuzzy space” \( S_F^{2k-2} \) with dimension \( [27, 15] \)

\[
D_{k-1}(I) = \prod_{l=1}^{k-1} \prod_{i=1}^{l} \frac{I + l + i - 1}{l + i - 1},
\]

(238)

and then
\[
\text{tr}(1_{\text{internal space}}) = D_{k-1}(I).
\]

(239)

In the lowest Landau level, the \( SO(2k+1) \) operators can be replaced with the \( SO(2k+1) \) matrix generators:
\[ L_{ab} \rightarrow \Sigma_{ab}, \]

(240)

and so (237) becomes
\[
X_a = \frac{2}{(2k)! c_k(I)} \frac{D_{k-1}(I)}{I} \epsilon_{a_1 a_2 \cdots a_{2k}} \Sigma_{a_1 a_2} \Sigma_{a_3 a_4} \cdots \Sigma_{a_{2k-1} a_{2k}}
\]

\[
= \frac{2}{I (2k)! D_k(I - 1)} \epsilon_{a_1 a_2 \cdots a_{2k}} \Sigma_{a_1 a_2} \Sigma_{a_3 a_4} \cdots \Sigma_{a_{2k-1} a_{2k}}
\]

\[
= \frac{1}{I!} \frac{I!!}{(I + 2k - 2)!!} \epsilon_{a_1 a_2 \cdots a_{2k}} \Sigma_{a_1 a_2} \Sigma_{a_3 a_4} \cdots \Sigma_{a_{2k-1} a_{2k}},
\]

(241)

where in the second equation the Atiyah-Singer index theorem was used \( [27, 48] \)

\[ c_k(I) = -D_k(I - 1). \]

(242)

Since the fully symmetric representation \( SO(2k+1) \) matrices satisfy
\[
\epsilon_{a_1 a_2 \cdots a_{2k}} \Sigma_{a_1 a_2} \Sigma_{a_3 a_4} \cdots \Sigma_{a_{2k-1} a_{2k}} = -\frac{k!(I + 2k - 2)!!}{I!!} \Gamma_a,
\]

(243)

(241) finally takes a concise form
\[ X_a = \frac{1}{I} \Gamma_a. \]

(244)

### 6.3 Even higher dimensional matrix geometry

The results in the two limits, (233) and (244), suggest that the matrix coordinates for general \( I \) take the form
\[ X_a = \frac{1}{I + 2k} \Gamma_a. \]

(245)

Since the \( SO(2k + 1) \) gamma matrices in the fully symmetric representation satisfy
\[
\sum_{a=1}^{2k+1} \Gamma_a \Gamma_a = I(I + 2k)1_{D_k(I)},
\]

\[
[\Gamma_{a_1}, \Gamma_{a_2}, \cdots, \Gamma_{a_{2k}}] = -\frac{k(2k)!! (I + 2k - 2)!!}{I!!} \epsilon_{a_1 a_2 \cdots a_{2k+1}} \Gamma_{a_{2k+1}},
\]

(246)
\( X_a \) satisfy the quantum Nambu geometry of the fuzzy 2k-sphere\(^4\) [23] [24]:
\[
\sum_{a=1}^{2k+1} X_a X_a = \frac{I}{I + 2k} 1_{D_k(I)},
\]
\[
[X_{a_1}, X_{a_2}, \cdots, X_{a_{2k}}] = -i^k C(k, I) \left( \frac{2}{I + 2k} \right)^{2k-1} \epsilon_{a_1a_2\cdots a_{2k+1}} X_{a_{2k+1}},
\]
with
\[
C(k, I) \equiv \frac{(2k)!!}{2^{2k-1} I!!}.\]
The quantum Nambu geometry thus naturally emerges as the lowest Landau level matrix geometry of the \( SO(2k + 1) \) Landau model. The matrix geometry\(^{246}\) will also be obtained by the Berezin-Toeplitz quantization, since the zero-modes of the Dirac-Landau operator are equal to the lowest Landau level eigenstates\(^{27}\) and the Atiyah-Singer theorem also hold in arbitrary even dimension.

Further, when we take into account the low dimensional results including odd dimensions\(^{31} 32 29\):
\[
S^2_F \text{ of } SO(3) \text{ Landau model: } \langle x_i \rangle_{\text{LLL}} = \frac{1}{I + 2} 2S_i,
\]
\[
S^2_F \text{ of } SO(4) \text{ Landau model: } \langle x_m \rangle_{\text{LLL}} = \frac{1}{I + 3} \Gamma_m,
\]
\(^{247}\) may be generalized to
\[
X_a = \frac{1}{I + d} \Gamma_a, \quad (a = 1, 2, \cdots, d + 1)
\]
for the \( SO(d + 1) \) Landau model.

### 7 Summary

In this work, we performed a comprehensive study of the \( SO(5) \) Landau models and their matrix geometries. With \( SO(5) \) monopole harmonics in a full form, we completely derived the matrix coordinates of four-sphere in an arbitrary Landau level. In the lowest Landau level, the matrix geometry is given by the generalized \( SO(5) \) gamma matrices realizing the quantum Nambu geometry. We showed that the matrix geometry obtained by the Landau level projection actually interpolates the matrix geometries of the quantum limit and the classical limit. In higher Landau level, the matrix geometry exhibits a nested fuzzy structure. The \( N \)th \( SO(5) \) Landau level accommodates \( N + 1 \) inner \( SO(4) \) Landau levels each of which realizes quasi-fuzzy four-sphere geometry. As a whole, there are \( N + 1 \) quasi-fuzzy four-spheres constituting a \( N + 1 \) concentric nested structure with \( SO(5) \) symmetry. The nested matrix geometry denotes a pure quantum geometry having no counterpart in classical geometry. We introduced a \( SO(4) \) Pauli-Schrödinger model with meron gauge field background that realizes the inner \( SO(4) \) part of the \( SO(5) \) Landau model. We established a singular gauge transformation between the \( SO(4) \) Pauli-Schrödinger model and the \( SO(4) \) Landau model, and demonstrated the internal fuzzy geometry is identical to the \( SO(4) \) Landau model matrix geometry. Explicit relations among other low dimensional Landau models with fuzzy geometries were summarized too. We also analyzed the relativistic \( SO(5) \) Landau models and clarified relation to the matrix geometry of the Berezin-Toeplitz quantization. Finally, we investigated even higher dimensional Landau model and discussed the associated quantum Nambu geometry in an arbitrary dimension.

The \( SO(5) \) Landau model and four-dimensional quantum Hall effect have opened a window to a research field of topological phases in higher dimension. This is not just rendered to be a theoretical issue. Recent technologies of quantum photonics in ultra cold atom have made experimental explorations possible with the idea of synthetic dimension\(^{49}\). The present analysis will be useful not only for the non-commutative geometry but also for the practical analysis of higher dimensional topological phases such as quantum Hall effect and Weyl semi-metal\(^{50} 51\).
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A Representation theory of the $SO(5)$ group

A.1 $SO(5)$ irreducible representation and the $SO(4)$ decomposition

The $SO(5)$ Casimir operator is constructed as

$$\sum_{a<b=1}^{5} L_{ab}^2.$$ \hfill (251)

From the representation theory, the $SO(5)$ irreducible representation is specified by two integers $(p, q)_{SO(5)}$ with the dimension

$$D = \frac{1}{6}(p+2)(q+1)(p+q+3)(p-q+1),$$ \hfill (252)

and the $SO(5)$ Casimir eigenvalue of (251) is given by \footnote{For representation theory of $SO(5)$ (and other classical Lie groups), readers may consult textbooks such as [33] and [34]. In the notation of [33], $[[a_1, a_2]] = [[q, p-q]]$ or $[l_1, l_2] = \left[\frac{1}{2}(p+q), \frac{1}{2}(p-q)\right]$, and in the notation of [34], $[\lambda_1, \lambda_2] = \left[\frac{1}{2}(p+q), \frac{1}{2}(p-q)\right]$.}

$$\frac{1}{2}(p^2 + q^2) + 2p + q.$$ \hfill (253)

According to the decomposition $SO(4) \simeq SU(2)_L \otimes SU(2)_R$, we can introduce the $SU(2)_L$ and $SU(2)_R$ angular momentum operators as

$$J_i = \frac{1}{4} \eta^i_{mn} L_{mn}, \quad K_i = \frac{1}{4} \bar{\eta}^i_{mn} L_{mn}.$$ \hfill (254)

Here, $\eta^i_{mn}$ and $\bar{\eta}^i_{mn}$ denote the 't Hooft symbols:

$$\eta^i_{mn} = \epsilon_{mni4} + \delta_{mi}\delta_{n4} - \delta_{ni}\delta_{m4}, \quad \bar{\eta}^i_{mn} = \epsilon_{mni4} - \delta_{mi}\delta_{n4} + \delta_{ni}\delta_{m4}.$$ \hfill (255)

$J_i$ and $K_i$ are mutually commutative

$$[J_i, K_j] = 0,$$ \hfill (256)

and satisfy

$$J_i^2 = j(j+1), \quad K_i^2 = k(k+1),$$ \hfill (257)

where each of $j$ and $k$ takes an integer or half-integer value. The dimension of the $SO(4)$ irreducible representation with $(j, k)$ is given by

$$(2j+1)(2k+1),$$ \hfill (258)

and the $SO(4)$ Casimir is expressed as

$$L_{mn}^2 = 4(J^2 + K^2) = 4(j(j+1) + k(k+1)).$$ \hfill (259)

The $SO(5)$ irreducible representation is decomposed to the $SO(4)$ irreducible representation as

$$(p, q)_{SO(5)} = \sum_{n=0}^{q} \oplus \left( \sum_{s=-\frac{p-q}{2}}^{\frac{p-q}{2}} (j, k)_{SO(4)} \right),$$ \hfill (260)

where

$$j = \frac{n}{2} + \frac{p-q}{4} + \frac{s}{2}, \quad k = \frac{n}{2} + \frac{p-q}{4} - \frac{s}{2}.$$ \hfill (261)
A.2 $SO(5)$ irreducible decomposition of direct products

From (67), we have

\[ N = 0 : \begin{bmatrix} 1, & 0 \end{bmatrix} \otimes \begin{bmatrix} 0, & I \end{bmatrix} = \begin{bmatrix} 0, & I \end{bmatrix} \oplus \begin{bmatrix} 1, & I \end{bmatrix} \oplus \begin{bmatrix} 1, & I - 2 \end{bmatrix}, \]

\[ I = 1 : \begin{bmatrix} 1, & 0 \end{bmatrix} \otimes \begin{bmatrix} N, & 1 \end{bmatrix} = \begin{bmatrix} N, & 1 \end{bmatrix} \oplus \begin{bmatrix} N + 1, & 1 \end{bmatrix} \oplus \begin{bmatrix} N - 1, & 3 \end{bmatrix} \oplus \begin{bmatrix} N - 1, & 1 \end{bmatrix}, \]

\[ I = 0 : \begin{bmatrix} 1, & 0 \end{bmatrix} \otimes \begin{bmatrix} N, & 0 \end{bmatrix} = \begin{bmatrix} N + 1, & 0 \end{bmatrix} \oplus \begin{bmatrix} N - 1, & 2 \end{bmatrix} \oplus \begin{bmatrix} N - 1, & 0 \end{bmatrix}. \] \hspace{1cm} (262)

In particular,

\[ \begin{bmatrix} 1, & 0 \end{bmatrix} \otimes \begin{bmatrix} 0, & 1 \end{bmatrix} = \begin{bmatrix} 1, & I \end{bmatrix} \oplus \begin{bmatrix} 1, & I - 2 \end{bmatrix}, \]

\[ \begin{bmatrix} 1, & 0 \end{bmatrix} \otimes \begin{bmatrix} N, & 1 \end{bmatrix} = \begin{bmatrix} N, & 1 \end{bmatrix} \oplus \begin{bmatrix} N + 1, & 1 \end{bmatrix} \oplus \begin{bmatrix} N - 1, & 2 \end{bmatrix}, \]

or

\[ 5 \otimes 4 = 4 \oplus 16, \]

\[ 5 \otimes 5 = 5 \oplus 14 \oplus (-5) \oplus 10 \oplus 1 = 14 \oplus 10 \oplus 1, \]

\[ 5 \otimes 10 = 10 \oplus 35 \oplus 5. \] \hspace{1cm} (264)

For other examples of the irreducible decomposition of the tensor product of $SO(5) \simeq USp(4)$, one may consult [37] for instance.

B The Dirac gauge and the Schwinger gauge for $S^4$

We introduce the Dirac gauge and the Schwinger gauge for $S^4$ and derive a gauge transformation between them.

B.1 Dirac gauge

As $S^4$ being a coset space

\[ S^4 \simeq SO(5)/SO(4), \] \hspace{1cm} (265)

the non-linear realization is given by [32]

\[ \Psi = e^{i\xi} y_m \sigma_m^5 \frac{1}{\sqrt{2(1 + x_5)}} \begin{pmatrix} 1 + x_5 & x_m q_m \\ -x_m q_m & 1 + x_5 \end{pmatrix}, \] \hspace{1cm} (266)

where $y_m$ are the $S^3$-coordinates [123], $q_m$ are quaternions [80], and

\[ \sigma_m^5 = i \frac{1}{2} \begin{pmatrix} 0 & -q_m \\ q_m & 0 \end{pmatrix}. \] \hspace{1cm} (267)

With the polar coordinates [123], (266) is expressed as

\[ \Psi = \begin{pmatrix} \cos \frac{\xi}{2} & 0 & \sin \frac{\xi}{2} (\cos \phi + i \sin \phi \cos \theta) & i \sin \frac{\xi}{2} \sin \phi \sin \theta e^{-i\phi} \\
- \sin \frac{\xi}{2} (\cos \phi - i \sin \phi \cos \theta) & \cos \frac{\xi}{2} & i \sin \frac{\xi}{2} \sin \phi \sin \theta e^{-i\phi} & 0 \\
i \sin \frac{\xi}{2} \sin \phi \sin \theta e^{i\phi} & -i \sin \frac{\xi}{2} \sin \phi \sin \theta e^{-i\phi} & \cos \frac{\xi}{2} & \sin \frac{\xi}{2} (\cos \phi + i \sin \phi \cos \theta) \\
\cos \frac{\xi}{2} & 0 & - \sin \frac{\xi}{2} (\cos \phi - i \sin \phi \cos \theta) & \cos \frac{\xi}{2} \end{pmatrix}. \] \hspace{1cm} (268)
We decompose $\Psi$ into two $4 \times 2$ rectangular matrices:

$$
\Psi = \left( \Psi_L, \Psi_R \right),
$$

where

$$
\Psi_L = \frac{1}{\sqrt{2(1 + x_5)}} \begin{pmatrix} 1 + x_5 \\ -x_m q_m \end{pmatrix}, \quad \Psi_R = \frac{1}{\sqrt{2(1 + x_5)}} \begin{pmatrix} x_m q_m \\ 1 + x_5 \end{pmatrix}.
$$

The 2nd Hopf map (76) can be expressed as

$$
x_m 1_2 = -\bar{\Psi}_L^\dagger \gamma_m \Psi_L = \bar{\Psi}_R^\dagger \gamma_m \Psi_R, \quad x_5 1_2 = \bar{\Psi}_L^\dagger \gamma_5 \Psi_L = -\bar{\Psi}_R^\dagger \gamma_5 \Psi_R,
$$

and the associated connections are derived as

$$
\omega^D_L = -i \Psi_L^\dagger d\Psi_L = -i \frac{1}{2(1 + x_5)} \eta^i_{mn} x_n q_i dx_m, \quad \omega^D_R = -i \Psi_R^\dagger d\Psi_R = -i \frac{1}{2(1 + x_5)} \eta^i_{mn} x_n q_i dx_m.
$$

Here $D$ of $\omega^D$ is for the Dirac gauge. More comprehensively,

$$
A^D \equiv -i \Psi^\dagger d\Psi^D = \begin{pmatrix} -i \Psi_L^\dagger d\Psi_L & -i \Psi_L^\dagger d\Psi_R \\ -i \Psi_R^\dagger d\Psi_L & -i \Psi_R^\dagger d\Psi_R \end{pmatrix} = \begin{pmatrix} \omega^D_L & -i \Psi_L^\dagger d\Psi_R \\ -i \Psi_R^\dagger d\Psi_L & \omega^D_R \end{pmatrix}
$$

$$
= -i \frac{1}{2(1 + x_5)} \begin{pmatrix} \eta^i_{mn} x_n q_i dx_m & -x_m q_m dx_5 + (1 + x_5) q_m dx_m \\ x_m q_m dx_5 - (1 + x_5) q_m dx_m & \eta^i_{mn} x_n q_i dx_m \end{pmatrix}.
$$

$\omega^D_L$ is equal to the $SU(2)$ (anti-)monopole gauge field (3) for $I = 1$:

$$
\omega^D_L = \frac{1}{4} \omega^D_{mn} \eta^i_{mn} \sigma_i.
$$

In the Cartesian coordinates, $\omega^D_{mn}$ are given by

$$
\omega^D_{mn} = \frac{1}{1 + x_5} (x_m dx_n - x_n dx_m),
$$

while in the polar coordinates,

$$
\omega^D_{12} = 2 \sin^2 \frac{\xi}{2} \sin^2 \chi \sin^2 \theta d\phi,
\omega^D_{13} = -2 \sin^2 \frac{\xi}{2} \sin^2 \chi \cos \phi d\theta + 2 \sin^2 \frac{\xi}{2} \sin^2 \chi \sin \theta \cos \sin \phi d\phi,
\omega^D_{14} = -2 \sin^2 \frac{\xi}{2} \sin \theta \cos \phi d\chi - 2 \sin^2 \frac{\xi}{2} \sin \chi \cos \chi \cos \phi d\theta + 2 \sin^2 \frac{\xi}{2} \sin \chi \cos \chi \sin \theta \sin \phi d\phi,
\omega^D_{23} = -2 \sin^2 \frac{\xi}{2} \sin \phi d\theta - 2 \sin^2 \frac{\xi}{2} \sin^2 \chi \sin \theta \cos \phi d\phi,
\omega^D_{24} = -2 \sin^2 \frac{\xi}{2} \sin \theta \sin \phi d\chi - 2 \sin^2 \frac{\xi}{2} \sin \chi \cos \chi \sin \phi d\theta - 2 \sin^2 \frac{\xi}{2} \sin \chi \cos \chi \cos \phi d\phi,
\omega^D_{34} = -2 \sin^2 \frac{\xi}{2} \cos \theta d\chi + 2 \sin^2 \frac{\xi}{2} \sin \chi \cos \chi \sin \phi d\theta.
$$

B.2 Schwinger gauge

$\Psi$ (268) is factorized as

$$
\Psi(\xi, \chi, \theta, \phi) = H(\chi, \theta, \phi)^\dagger \cdot e^{i \xi \gamma_{45}} \cdot H(\chi, \theta, \phi),
$$
where
\[ e^{i\xi \sigma_{45}} = \begin{pmatrix} \cos \frac{\xi}{2} & 1_2 \\ -\sin \frac{\xi}{2} & 1_2 \end{pmatrix}, \]
and
\[ H(\chi, \theta, \phi) = e^{i\chi \sigma_{43}} e^{i\theta \sigma_{31}} e^{i\phi \sigma_{12}} = \begin{pmatrix} H_L(\chi, \theta, \phi) & 0 \\ 0 & H_R(\chi, \theta, \phi) \end{pmatrix}, \]
with the chiral Hopf spinor matrices \[32\]
\[ H_L(\chi, \theta, \phi) = e^{-i\frac{1}{4}\sigma_z} e^{i\frac{1}{2}\sigma_y} e^{i\frac{1}{2}\sigma_z} = \begin{pmatrix} \cos\left(\frac{\chi}{2}\right) & e^{-i\frac{1}{2}(\chi-\phi)} \\ -\sin\left(\frac{\chi}{2}\right) & e^{i\frac{1}{2}(\chi+\phi)} \end{pmatrix}, \]
\[ H_R(\chi, \theta, \phi) = H_L(-\chi, \theta, \phi) = e^{i\frac{3}{4}\sigma_z} e^{i\frac{1}{2}\sigma_y} e^{i\frac{1}{2}\sigma_z} = \begin{pmatrix} \cos\left(\frac{\chi}{2}\right) & e^{i\frac{1}{2}(\chi+\phi)} \\ -\sin\left(\frac{\chi}{2}\right) & e^{-i\frac{1}{2}(\chi-\phi)} \end{pmatrix}. \]

Though \( H \) is a 4\( \times \)4 matrix, \( H \) carries the \( SU(2) \) degrees of freedom. We introduce a new 4\( \times \)4 matrix \( \Phi \) as
\[ \Psi(\xi, \chi, \theta, \phi) = \Phi(\xi, \chi, \theta, \phi) \cdot H(\chi, \theta, \phi), \]
or
\[ \Phi(\xi, \chi, \theta, \phi) = H(\chi, \theta, \phi)^{\dagger} \cdot e^{i\xi \sigma_{45}}. \]

In the polar coordinates,
\[ \Phi = \begin{pmatrix} \cos \frac{\xi}{2} & \cos \frac{\chi}{2} e^{i\frac{1}{4}(\chi-\phi)} & -\cos \frac{\xi}{2} \sin \frac{\chi}{2} e^{-i\frac{1}{4}(\chi+\phi)} & -\cos \frac{\xi}{2} \sin \frac{\chi}{2} e^{-i\frac{1}{4}(\chi+\phi)} \\ \cos \frac{\xi}{2} \sin \frac{\chi}{2} e^{i\frac{1}{4}(\chi+\phi)} & \cos \frac{\chi}{2} & \sin \frac{\xi}{2} \sin \frac{\chi}{2} e^{-i\frac{1}{4}(\chi-\phi)} & \sin \frac{\xi}{2} \sin \frac{\chi}{2} e^{-i\frac{1}{4}(\chi-\phi)} \\ -\sin \frac{\xi}{2} \cos \frac{\chi}{2} e^{-i\frac{1}{4}(\chi+\phi)} & \sin \frac{\xi}{2} \sin \frac{\chi}{2} e^{i\frac{1}{4}(\chi-\phi)} & \cos \frac{\chi}{2} & \cos \frac{\xi}{2} \sin \frac{\chi}{2} e^{i\frac{1}{4}(\chi+\phi)} \\ -\sin \frac{\xi}{2} \sin \frac{\chi}{2} e^{-i\frac{1}{4}(\chi-\phi)} & -\sin \frac{\xi}{2} \cos \frac{\chi}{2} e^{i\frac{1}{4}(\chi+\phi)} & -\sin \frac{\xi}{2} \cos \frac{\chi}{2} e^{i\frac{1}{4}(\chi+\phi)} & \cos \frac{\xi}{2} \cos \frac{\chi}{2} e^{-i\frac{1}{4}(\chi-\phi)} \end{pmatrix}. \]

As in the case of \( \Psi \) \[269\], we decompose \( \Phi \) as
\[ \Phi = (\Phi_L, \Phi_R), \]
where
\[ \Phi_L = \begin{pmatrix} \cos \frac{\xi}{2} H^\dagger_L \\ -\sin \frac{\xi}{2} H^\dagger_R \end{pmatrix}, \quad \Phi_R = \begin{pmatrix} \sin \frac{\xi}{2} H^\dagger_L \\ \cos \frac{\xi}{2} H^\dagger_R \end{pmatrix}. \]

The corresponding connection is derived as
\[ \omega^S_L = -i\Phi_L^\dagger d\Phi_L = -\frac{i}{2}(H_L dH^\dagger_L + H_R dH^\dagger_R) - \frac{1}{2} \cos \xi \left( H_L dH^\dagger_L - H_R dH^\dagger_R \right), \]
where
\[ -\frac{1}{2}(H_L dH^\dagger_L + H_R dH^\dagger_R) = \cos \chi \sin \theta \, d\phi \frac{1}{2} \sigma_x - \cos \chi \sin \theta \, d\phi \frac{1}{2} \sigma_y - \cos \theta \, d\phi \frac{1}{2} \sigma_z, \]
\[ -\frac{1}{2}(H_L dH^\dagger_L - H_R dH^\dagger_R) = \sin \chi \, d\phi \frac{1}{2} \sigma_x + \sin \chi \, d\phi \frac{1}{2} \sigma_y + d\chi \frac{1}{2} \sigma_z. \]

\( S \) of \( \omega^S \) is for the Schwinger gauge \[30\]. \( \omega^S_L \) is explicitly given by
\[ \omega^S_L = \frac{i}{2} \omega^1_L y_i \]
where
\[ \omega^x_L = \cos \chi \sin \theta \ d\phi + \cos \xi \ \sin \chi \ d\theta, \quad \omega^y_L = -\cos \chi \ d\theta + \cos \xi \ \sin \chi \ d\phi, \quad \omega^z_L = -\cos \theta \ d\phi + \cos \xi \ d\chi. \]

With \( \omega^L_i \equiv \frac{1}{2} \eta^{i}_{mn} \omega^S_{mn} \), (287) can be rewritten as
\[
\omega^S_L = \frac{1}{4} \omega^S_{mn} \eta^i_{mn} q_i \ dx^\mu, \quad (dx^\mu = d\theta, d\phi, d\chi, d\xi)
\]
where
\[
\begin{align*}
\omega^S_{12} &= -\cos \theta d\phi, \\
\omega^S_{13} &= \cos \chi d\theta, \\
\omega^S_{14} &= \cos \xi \sin \chi \ d\phi, \\
\omega^S_{23} &= \cos \chi \sin \theta d\phi, \\
\omega^S_{24} &= \cos \xi \sin \chi \sin \theta d\phi, \\
\omega^S_{34} &= \cos \xi \ d\chi.
\end{align*}
\]

It is straightforward to check that (290) satisfy the Cartan structure equation:
\[
de^S_m + \omega^S_{mn} e^S_n = 0
\]
with the vierbein in the Schwinger gauge
\[
e^S_1 = \sin \xi \ \sin \chi \ d\theta, \quad e^S_2 = \sin \xi \ \sin \theta d\phi, \quad e^S_3 = \sin \xi \ d\chi, \quad e^S_4 = d\xi.
\]

Similarly, we have
\[
\omega^S_R = -i \Phi^1_R d\Phi^R = -i \frac{1}{2} (H_L dH_L^\dagger + H_R dH_R^\dagger) + i \frac{1}{2} \cos \xi \ (H_L dH_L^\dagger - H_R dH_R^\dagger) = i \frac{1}{2} \omega^i R q_i,
\]
with
\[
\omega^i R \equiv \frac{1}{2} \eta^i_{mn} \omega^S_{mn},
\]
and
\[
A^S = -i \Phi^1 d\Phi = \begin{pmatrix}
-i \Phi^1_L d\Phi_L & -i \Phi^1_L d\Phi_R \\
-i \Phi^1_R d\Phi_L & -i \Phi^1_R d\Phi_R
\end{pmatrix} = \begin{pmatrix}
\omega^S_L & -i \Phi^1_R d\Phi_R \\
-i \Phi^1_R d\Phi_L & \omega^S_R
\end{pmatrix}.
\]

### B.3 Gauge transformation and vierbein in the Dirac gauge

From the relation (280), we have
\[
\Psi_L = \Phi_L \cdot H_L, \quad \Psi_R = \Phi_R \cdot H_R,
\]
and so (293) and (295) are related as
\[
A^D = H^\dagger A^S H - i H^\dagger dH.
\]

(291) implies
\[
\omega^D_L = H_L^\dagger \omega^S_L H_L - i H_L^\dagger dH_L, \quad \omega^D_R = H_R^\dagger \omega^S_R H_R - i H_R^\dagger dH_R,
\]
or
\[
\omega^S_L = H_L \omega^D_L H_L^\dagger - i H_L dH_L^\dagger, \quad \omega^S_R = H_R \omega^D_R H_R^\dagger - i H_R dH_R^\dagger.
\]

We then find that the \( SO(4) \) matrix-valued spin connections
\[
\omega^{D/S} \equiv \begin{pmatrix}
\omega^D_L / \omega^{D/S} & 0 \\
0 & \omega^D_R / \omega^{D/S}
\end{pmatrix} = \frac{1}{4} \omega^{D/S} \left( \eta^i_{mn} \sigma_i, 0 \right) \left( \eta^i_{mn} \sigma_i \right),
\]

31 The numbering of the vierbein here is different from that of (184).
are also related by the gauge transformation
\[ \omega^D = H^\dagger \omega^S H - iH^\dagger dH. \] (301)

Under the SU(2) transformation \( H \), the gamma matrices are transformed as
\[ H^\dagger \gamma_m H = \gamma_n O_{nm} \] (302)
with
\[ O = (e^{i\theta t_{12}} e^{i\phi t_{34}})^t = e^{-i\theta t_{12}} e^{-i\phi t_{34}} \]
\[ = \begin{pmatrix}
\cos \theta \cos \phi & -\sin \phi & \cos \chi \sin \theta \cos \phi & \sin \chi \sin \theta \cos \phi \\
\cos \theta \sin \phi & \cos \phi & \cos \chi \sin \theta \sin \phi & \sin \chi \sin \theta \sin \phi \\
-\sin \theta & 0 & \cos \chi \cos \phi & \sin \chi \cos \phi \\
0 & 0 & -\sin \chi & \cos \chi
\end{pmatrix}. \] (303)

Here \( t_{mn} \) are the adjoint representation \( SO(4) \) generators:
\[ (t_{mn})_{pq} = -i\delta_{mp}\delta_{nq} + i\delta_{mq}\delta_{np}. \] (304)

Since the vierbein carries local coordinate indices, the vierbein transforms similarly to (302). Therefore, the vierbein in the Dirac gauge can be obtained from the vierbein in the Schwinger gauge:
\[ e^D_m = O_{mn} e^S_n. \] (305)

With the expression of \( e^S_m \), \( e^D_m \) are explicitly given by
\[ e^D_1 = \sin \chi \sin \theta \cos \phi d\xi + \sin \xi \cos \chi \sin \theta \cos \phi \ d\chi + \sin \xi \sin \chi \cos \theta \cos \phi \ d\theta - \sin \xi \sin \chi \sin \theta \sin \phi \ d\phi, \]
\[ e^D_2 = \sin \chi \sin \theta \sin \phi \ d\xi + \sin \xi \cos \chi \sin \theta \sin \phi \ d\chi + \sin \xi \sin \chi \cos \theta \sin \phi \ d\theta + \sin \xi \sin \chi \sin \theta \cos \phi \ d\phi, \]
\[ e^D_3 = \sin \chi \cos \theta d\xi + \sin \xi \cos \chi \cos \theta d\chi - \sin \xi \sin \chi \sin \theta d\theta \]
\[ e^D_4 = \cos \chi d\xi - \sin \xi \sin \chi d\chi. \] (306)

It is straightforward to show that (276) and (306) satisfy the Cartan structure equation:
\[ de_m^D + \omega^D_{mn} e_m^D = 0. \] (307)

We thus successfully obtained the vierbein in the Dirac gauge from the relation (305). On the other hand, it may be a formidable task to derive the vierbein in the Dirac gauge from the Cartan structure equation (307) with the spin connection (276).

**B.4 Curvature**

With (300), the curvature
\[ f^{D/S} = d\omega^{D/S} + i\omega^{D/S}^2 = \frac{1}{2} f^{D/S}_{mn} \sigma_{mn} \] (308)
is readily obtained as
\[ f^{D/S}_{mn} = e^D_m \wedge e^D_n. \] (309)

\( f^D \) and \( f^S \) are related by
\[ f^D = H^\dagger f^S H. \] (310)
The Riemann curvature can be read off from
\[ f^{D/S}_{mn} = \frac{1}{2} R^{m}_{npq} e^{D/S}_p \wedge e^{D/S}_q, \]  
(311)
as
\[ R_{1212} = R_{1313} = R_{1414} = R_{2323} = R_{2424} = R_{3434} = 1, \]  
(312)
and the Ricci scalar is obtained as
\[ R = R^m_{m mn} = 2 \times 6 = 12. \]  
(313)

C Reduction to the SO(5) spherical harmonics

C.1 SO(5) free angular momentum Casimir

In the polar coordinates, the SO(5) free angular momentum operators, \( L_{ab} = -i x_a \frac{\partial}{\partial x_b} + i x_b \frac{\partial}{\partial x_a} \), are expressed as
\[
\begin{align*}
L_{12} &= -i \partial_\phi, \quad L_{13} = i (\cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi), \quad L_{23} = i (\sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi), \\
L_{14} &= i (\sin \theta \cos \phi \partial_\chi + \cot \chi \cos \theta \cos \phi \partial_\phi), \\
L_{24} &= i (\sin \theta \sin \phi \partial_\chi + \cot \chi \cos \theta \sin \phi \partial_\phi), \quad L_{34} = i (\cos \theta \partial_\chi - \cot \chi \sin \theta \partial_\phi), \\
L_{15} &= i (\sin \chi \cos \phi \partial_\xi + \cot \xi \cos \chi \sin \phi \partial_\phi), \quad L_{16} = i (\sin \chi \sin \phi \partial_\xi), \\
L_{25} &= i (\sin \chi \sin \phi \partial_\xi + \cot \xi \cos \chi \sin \phi \partial_\phi), \quad L_{26} = i (\sin \chi \cos \phi \partial_\xi), \\
L_{35} &= i (\sin \chi \cos \phi \partial_\xi + \cot \xi \cos \chi \cos \phi \partial_\phi), \quad L_{36} = i (\sin \chi \cos \phi \partial_\xi), \\
L_{45} &= i (\cos \chi \partial_\xi - \cot \xi \sin \chi \partial_\phi).
\end{align*}
\]  
(314)
Notice that \( L_{mn} (m, n = 1, 2, 3, 4) \) do not depend on \( \xi \) and are equal to the polar coordinate expression of the SO(4) free angular momentum operators, \( L_{mn} = -i y_m \frac{\partial}{\partial y_n} + i y_n \frac{\partial}{\partial y_m} \). The Laplacian on \( S^4 \) is given by
\[
\Delta_{S^4} = \frac{1}{\sin^3 \xi} \partial_\xi (\sin^3 \xi \partial_\xi) + \frac{1}{\sin^2 \xi \sin^2 \chi} \partial_\chi (\sin^2 \chi \partial_\chi) + \frac{1}{\sin^2 \xi \sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \xi \sin^2 \theta} \partial_\phi^2
\]
\[ = - \sum_{a < b = 1}^5 L_{ab}^2, \]  
(315)
which is related to the Laplacian on \( S^3 \) as
\[
\Delta_{S^4} = \frac{1}{\sin^3 \xi} \partial_\xi (\sin^3 \xi \partial_\xi) + \frac{1}{\sin^2 \xi} \Delta_{S^3}.
\]  
(316)
The eigenvalues of the SO(5) free angular momentum Casimir is given by
\[ \sum_{a < b = 1}^5 L_{ab}^2 = N(N + 3), \]  
(317)
with degeneracy
\[ D = \frac{1}{6} (N + 1)(N + 2)(2N + 3). \]  
(318)
C.2 \textit{SO(5) spherical harmonics}

As found in [53], the \textit{SO(5)} spherical harmonics are usually given by

\[ \Phi_{Nnlm}(\Omega_4) = \Phi_{Nn}(\xi) \cdot Y_{nlm}(\Omega_3), \tag{319} \]

where

\[ \Phi_{Nn}(\xi) = \frac{2^{n+1}}{(N+1)!} \sqrt{\frac{(2N+3)(N-n)!}{2}} \cdot \frac{1}{\sin^{n+2}\xi} P_{N+n+2}^{-(n+1),-(n+1)}(\cos \xi) \]

\[ = \sqrt{\frac{2N+3}{2(N+n+2)!}} \cdot \frac{1}{\sin^2\xi} P_{N+1}^{n+1}(\cos \xi), \tag{320} \]

with the associated Legendre polynomials \( P_n^m(x) \). Meanwhile in the present paper, the \textit{SO(5)} monopole harmonics are given by [41] which should be reduced to the \textit{SO(5)} spherical harmonics at \( I = 0 \):

\[ \Psi_{N;\xi,m_L;m_R}(\Omega_4) = G_{N;\xi,m_L,m_R}(\Omega_4) \cdot Y_{m_L;\xi,m_R}(\Omega_3), \quad \left(-\frac{n}{2} \leq m_L, m_R \leq \frac{n}{2}\right) \tag{322} \]

where

\[ G_{N;\xi,m_L,m_R}(\xi) = \frac{N+3}{2N+1,0,-n-1}(\xi) \]

\[ = (-1)^{n+1} (N+1)! \sqrt{\frac{2N+3}{2(N+n+2)!}} \cdot \frac{1}{\sin^2\xi} \tan^{n+1}\left(\frac{\xi}{2}\right) P_{N+1}^{n+1,-(n+1)}(\cos \xi). \tag{323} \]

There are two distinct expressions for the \textit{SO(5)} spherical harmonics, [51] and [52]. From the property of the Jacobi polynomials

\[ (-1)^{n+1} \frac{(N+1)!}{(N-n)!} \tan^{n+1}\left(\frac{\xi}{2}\right) P_{N+1}^{n+1,-(n+1)}(\cos \xi) = P_{N+1}^{n+1}(\cos \xi), \tag{324} \]

we find that the azimuthal parts of the two expressions are identical:

\[ G_{N;\xi,m_L,m_R}(\xi) = \Phi_{Nn}(\xi), \tag{325} \]

and the previous \textit{SO(4)} analysis [52] has shown that the \textit{SO(4)} spherical harmonics parts are related as

\[ Y_{nlm}(\Omega_3) = i^l \sum_{m_L,m_R=-n/2}^{n/2} \langle l,m| n/2, m_L; n/2, m_R \rangle Y_{m_L;\xi,m_R}(\Omega_3). \tag{326} \]

Consequently, [51] and [52] are related by the linear combination

\[ \Phi_{Nnlm}(\Omega_4) = i^l \sum_{m_L,m_R=-n/2}^{n/2} \langle l,m| n/2, m_L; n/2, m_R \rangle \Psi_{N;\xi,m_L,m_R}(\Omega_4) \tag{327} \]

or

\[ \Psi_{N;\xi,m_L,m_R}(\Omega_4) = \sum_{m=-l}^{l} (-i)^l \langle n/2, m_L; n/2, m_R|l,m \rangle \Phi_{Nnlm}(\Omega_4), \tag{328} \]

which means that the two expression are actually equivalent.

\[ \text{32The associated Legendre polynomials } P_n^m(x) \text{ are related to the Jacobi polynomials } P_3^{(a,b)}(x) \text{ as} \]

\[ P_n^m(x) = 2^m \frac{(n+m)!}{n!} (1-x^2)^{m/2} P_{n+m}^{(-m,-m)}(x). \tag{321} \]
D  Matrix elements for three-sphere coordinates

Using the integration formula for three SO(4) monopole harmonics [see Sec.6.1 of [32]]

\[
\int d\Omega_3 \Phi^\dagger_{(l_L,m_L;\ell_R,n_R)} \Phi_{(l'_L,m'_L;\ell'_R,n'_R)} = \sqrt{\frac{(2l'_L+1)(2\ell'_R+1)(I+1)2}{\pi^2}} \left\{ \begin{array}{ccc} l_L & l_R & \ell' \ \\
\frac{1}{2} & \frac{1}{2} & 0 \end{array} \right\} C^{l_L,m_L}_{\frac{1}{2},\ell;\ell'_L,n_L} C^{l'_R,m'_R}_{\frac{1}{2},\ell';\ell'_R,n_R}
\]

\[
= \sqrt{\frac{(2l'_L+1)(2\ell'_R+1)(I+1)2}{\pi^2}} (-1)^{l_L+l'_L+\ell_R+\ell'_R+1} \left\{ \begin{array}{ccc} l_L & l_R & \ell' \ \\
\frac{1}{2} & \frac{1}{2} & 0 \end{array} \right\} C^{l_L,m_L}_{\frac{1}{2},\ell;\ell'_L,n_L} C^{l'_R,m'_R}_{\frac{1}{2},\ell';\ell'_R,n_R}
\]

\[
= \sqrt{\frac{(2l'_L+1)(2\ell'_R+1)}{\pi}} (-1)^{l_L+l'_L+2\ell_R+\ell'_R+\frac{1}{2}(I+1)} \left\{ \begin{array}{ccc} l_L & l_R & \ell' \ \\
\frac{1}{2} & \frac{1}{2} & 0 \end{array} \right\} C^{l_L,m_L}_{\frac{1}{2},\ell;\ell'_L,n_L} C^{l'_R,m'_R}_{\frac{1}{2},\ell';\ell'_R,n_R}
\]  \hspace{1cm} (329)

we will derive the three-sphere matrix coordinates.

D.1 About the SO(4) spinor spherical harmonics

We evaluate the matrix elements of \( y_m \) sandwiched by the SO(4) spin spherical harmonics:

\[
\langle Y_{j'm'_j;k'm'_k} | y_m | Y_{jm_j;km_k} \rangle = \int d\Omega_3 Y^\dagger_{j'm'_j;k'm'_k} y_m Y_{jm_j;km_k}, \hspace{1cm} (330)
\]

With (28), (330) can be rewritten as

\[
\langle Y_{j'm'_j;k'm'_k} | y_m | Y_{jm_j;km_k} \rangle = \sum_{A=-\frac{1}{2}}^{\frac{1}{2}} \int d\Omega_3 (Y^\dagger_{j'm'_j;k'm'_k})^A y_m (Y_{jm_j;km_k})_A
\]

\[
= \sum_A \sum_{j'_j} \sum_{k'_k} C^{k'm'_k}_{j'm'_j} C^{km_k}_{jm_j} \int d\Omega_3 \Phi^\dagger_{j'm'_j;jm_j} y_m \Phi_{jm_j;km_k}, \hspace{1cm} (331)
\]

where we used that the Clebsch-Gordan coefficients are real. Since the SO(4) spherical harmonics are equal to the monopole harmonics for \( I = 0 \):

\[
\Phi_{j,m_j;k,m_k} \big|_{j=k=\frac{1}{2}} = \Phi_{j,m_j;\frac{1}{2}}, \hspace{1cm} (332)
\]

(324) gives

\[
\int d\Omega_3 \Phi^\dagger_{j'm'_j;jm_j} \Phi_{\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}} \Phi_{jm_j;km_k} = -\frac{1}{\pi} \sqrt{\frac{2j+1}{2j'+1}} (-1)^{2(j+j')} C^{j'm'_j}_{\frac{1}{2},\frac{1}{2};jm_j} C^{j'm'_j}_{\frac{1}{2},\frac{1}{2};jm_j}, \hspace{1cm} (333)
\]
where \( \begin{pmatrix} j' & j' & 0 \\ j & j & 1/2 \end{pmatrix} = -i(\ddot{\Phi} + \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) was used. Therefore, with respect to \( y_1 = -i\sum \Phi \) \( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \), (331) becomes

\[
\langle y_1 \rangle = -\frac{i\pi}{2} \sum_{A, m_R^j} \sum_{A, m_R^j} \sum_{j} C_{k m k}^{j m k} \int d\Omega_3 \Phi_{j m j}^j (\Phi_{j m j}^j - \Phi_{j m j}^j) \Phi_{j m j m_R}
\]

\[
= i(-1)^{2(j+j')} \frac{1}{2} \sqrt{\frac{2j + 1}{2j' + 1}} \sum_{A, m_R^j} \sum_{j} C_{k m k}^{j m k} \int d\Omega_3 \Phi_{j m j}^j (\Phi_{j m j}^j - \Phi_{j m j}^j) \Phi_{j m j m_R}
\]

\[
= i(-1)^{2(j+j')} \frac{1}{2} \sqrt{\frac{2j + 1}{2j' + 1}} \left( C_{j m j}^{j m j} \sum_{A, m_R^j} \sum_{A, m_R^j} C_{j m j}^{j m j} \frac{1}{2} A + C_{j m j}^{j m j} \sum_{A, m_R^j} \sum_{A, m_R^j} C_{j m j}^{j m j} \frac{1}{2} A \right)
\]

(334)

In (334) we need to calculate

\[
\sum_{A, m_R^j} \sum_{A, m_R^j} C_{j m j}^{k m k} C_{j m j}^{j m j} \frac{1}{2} A = (-1)^{j+j'} \sum_{A, m_R^j} \sum_{A, m_R^j} C_{j m j}^{k m k} C_{j m j}^{j m j} \frac{1}{2} A + C_{j m j}^{j m j} \frac{1}{2} A
\]

\[
= (-1)^{j+k+1} \sqrt{\frac{2j + 1}{2j' + 1}} \left( C_{k m k}^{k m k} \frac{1}{2} A \right) \frac{1}{2} A \left( \begin{array}{c} j \\ k \\ j' \end{array} \right)
\]

\[
= -i(-1)^{j+2k} \sqrt{\frac{2j + 1}{2j' + 1}} C_{k m k}^{k m k} \left( \begin{array}{c} j \\ k \\ j' \end{array} \right)
\]

(335)

where in the first and last equations we used

\[
C_{j m j}^{k m k} = (-1)^{j+j'-k} C_{j m j}^{k m k},
\]

and in the second equation, Eq.(12) in p.260 of [54]

\[
\sum_{a, b, \delta} C_{a c, b \delta}^{c e} C_{d b, \delta}^{d e} C_{a a, f \delta}^{d e} = (-1)^{b+c+d+f} \sqrt{\frac{2c + 1}{2d + 1}} C_{e f}^{c e} \left( \begin{array}{c} a \\ c \\ b \\ d \\ f \end{array} \right).
\]

Consequently,

\[
\langle Y_{j m j} | y_1 | Y_{j m j} \rangle = \sqrt{\frac{(2j + 1)(2k + 1)}{2}} (-1)^{j+2j+2k} \left( \begin{array}{c} j' \\ k' \\ j \end{array} \right) \left( C_{j m j}^{j m j} C_{k m k}^{k m k} - C_{j m j}^{j m j} C_{k m k}^{k m k} \right).
\]

(338)

We used the fact that \( j \) takes a half-integer or integer value and so \((-1)^{\delta j} = 1\), and the property of the 6j symbol, \( \left[ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \right] = \left[ \begin{array}{c} e \\ d \\ c \\ b \\ a \\ f \end{array} \right] \).

When \( j+k = j'+k' = n + \frac{1}{2} \), the signature part of (338) is simplified as

\[
(-1)^{n+j+2j'+2k} = (-1)^{n-k+k'},
\]

(339)
and the matrix elements of \( y_1 \) and other coordinates become

\[
\langle Y_{j',m';k',m_k'} \rangle_{y=m=1,2} | Y_{j,m;k,m_k} \rangle = (-i)^m \frac{\sqrt{2j + 1}(2k + 1)}{2} (-1)^{n+l-k+k'} \left\{ \begin{array}{ccc} j' & k' & \frac{l}{2} \\ j & k & \frac{1}{2} \end{array} \right\} (C_{\frac{1}{2}+\frac{1}{2};j,m_j} C_{\frac{1}{2}+\frac{1}{2};k,m_k} + (-1)^m C_{\frac{1}{2}+\frac{1}{2};j,m_j} C_{\frac{1}{2}+\frac{1}{2};k,m_k}),
\]

\[
\langle Y_{j',m';k',m_k'} \rangle_{y=m=3,4} | Y_{j,m;k,m_k} \rangle = (-i)^m \frac{\sqrt{2j + 1}(2k + 1)}{2} (-1)^{n+l-k+k'} \left\{ \begin{array}{ccc} j' & k' & \frac{l}{2} \\ j & k & \frac{1}{2} \end{array} \right\} (C_{\frac{1}{2}+\frac{1}{2};j,m_j} C_{\frac{1}{2}+\frac{1}{2};k,m_k} - (-1)^m C_{\frac{1}{2}+\frac{1}{2};j,m_j} C_{\frac{1}{2}+\frac{1}{2};k,m_k}).
\]

With the explicit form of Clebsch-Gordan coefficients

\[
C_{\frac{1}{2}+\frac{1}{2};j,m} = \delta_{j',j+\frac{1}{2}} \delta_{m',m+\frac{1}{2}} \sqrt{\frac{j + m + 1}{2j + 1} \pm \delta_{j',j-\frac{1}{2}} \delta_{m',m+\frac{1}{2}} \sqrt{\frac{j + m}{2j + 1},}
\]

(341) yields (139) and (138).

### D.2 About the \( SO(4) \) monopole harmonics

Next, we evaluate the matrix elements of \( y_m \) sandwiched by the \( SO(4) \) monopole harmonics. From (342) and (341), we immediately have

\[
\langle \Phi_{j',m';k',m_k'} | \Phi_{j,m;k,m_k} \rangle = (-i)^m \frac{\sqrt{2j + 1}(2k + 1)}{2} (-1)^{j+2j+k+\frac{l}{2}} \left\{ \begin{array}{ccc} j' & k' & \frac{l}{2} \\ j & k & \frac{1}{2} \end{array} \right\} C(j',m',j,m) C(k',m_k,k,m_k),
\]

(342a)

\[
\langle \Phi_{j',m';k',m_k'} | \Phi_{j,m;k,m_k} \rangle = (-i)^m \frac{\sqrt{2j + 1}(2k + 1)}{2} (-1)^{j+2j+k+\frac{l}{2}(I+1)} \left\{ \begin{array}{ccc} j' & k' & \frac{l}{2} \\ j & k & \frac{1}{2} \end{array} \right\} C(j',m',j,m) C(k',m_k,k,m_k).
\]

(342b)

When \( j + k = j' + k' = n + \frac{l}{2} \), the sign-part is simplified as

\[
(-1)^{j+2j+k+\frac{l}{2}(I+1)} = -i(-1)^{(j'+k'+j+k+k'+\frac{l}{2})} = -i(-1)^{(n+l+j+k+\frac{l}{2})} = -i(-1)^{j+k+\frac{l}{2}}
\]

(343)

and with (341) we have

\[
\langle \Phi_{j',m';k',m_k'} | \Phi_{j,m;k,m_k} \rangle = \sum_{\sigma=\pm} \lambda_{m,-\sigma}^{(\sigma,-\sigma)}(j,k) \delta_{j',j+\frac{l}{2}} \delta_{k',k-\frac{l}{2}},
\]

(344)
where $\mathcal{Y}^{(\sigma,-\sigma)}(j, k)(m'_j, m'_k; m_j, m_k) \equiv \langle \Phi_{j+m'_j} m'_k | - \frac{2}{\sqrt{2}} | \Phi_{j} m_j m_k \rangle$ are given by

$$
\mathcal{Y}^{(+-)}_{m=1,2}(j, k) = \left( -i \right)^{m_1} \frac{1}{2}(1)^n \left\{ \begin{array}{ccc} j & 1 & 1 \end{array} \right\}
$$

$$
\times \left( \delta_{m'_j, m_j + \frac{3}{2}} \delta_{m'_k, m_k + \frac{1}{2}} \right)
$$

$$
\sqrt{(j + m_j + 1)(k - m_k) - (1)^m \delta_{m'_j, m_j - \frac{1}{2}} \delta_{m'_k, m_k + \frac{1}{2}} \sqrt{(j - m_j + 1)(k + m_k)}},
$$

where $m_1 = 1, 3$. With a 4 × 2 matrix $\Phi_{n}(\Omega \pi \phi)$ can be represented by the $SO(4)$ spinor spherical harmonics $Y_{j, m_j; k, m_k}$ as

$$
\Phi_{AFF}(\Omega_{3}) = \frac{1}{\sqrt{2}} \left( y_m q_m \right).
$$

we can obtain the meron gauge field configuration [147]:

$$
A^{AFF} = -i \Phi^{AFF} \Phi^{AFF} \frac{1}{2} g_{mn} \chi \sigma_i \frac{d y_m}{d y_m},
$$

where $q_m q_m = \delta_{mm} - \tilde{n}_{mn} q_l$ was used. $\Phi^{AFF}(\Omega_{3})$ can be represented by the $SO(4)$ spinor spherical harmonics $Y_{j, m_j; k, m_k}$ as

$$
\Phi^{AFF}(\Omega_{3}) = \pi \left( | \Phi_1^{AFF} \rangle \right) | \Phi_2^{AFF} \rangle \left( | \Phi_3^{AFF} \rangle \right) | \Phi_4^{AFF} \rangle = \pi \left( - | Y_1 \rangle - | Y_2 \rangle | Y_3 \rangle | Y_4 \rangle \right)
$$

where

$$
| Y_1 \rangle = Y_{1/2, 1/2; 0, 0} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} -y_4 + i y_3 \\ iy_1 - y_2 \end{pmatrix} \left( \begin{array}{c} \cos \chi + i \sin \chi \cos \theta \\ i \sin \chi \sin \theta e^{i\phi} \end{array} \right),
$$

$$
| Y_2 \rangle = Y_{1/2, -1/2; 0, 0} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} iy_1 + y_2 \\ -y_1 - iy_3 \end{pmatrix} \left( \begin{array}{c} i \sin \chi \sin \theta e^{-i\phi} \\ -\cos \chi - i \sin \chi \cos \theta \end{array} \right),
$$

$$
| Y_3 \rangle = Y_{0, 0; 1/2, 1/2} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

$$
| Y_4 \rangle = Y_{0, 0; 1/2, -1/2} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

On may find that (345) is simply related to (139) and (138) as

$$
\mathcal{Y}^{(\sigma,-\sigma)}(j, k) = -\mathcal{Y}^{(\sigma,-\sigma)}_{m=1,2}(j, k).
$$

**D.3 Special case** $(j, k) = (1/2, 0) \oplus (0, 1/2)$

With a 4 × 2 matrix

$$
\Phi^{AFF}(\Omega_{3}) = \frac{1}{\sqrt{2}} \left( y_m q_m \right).
$$

we can obtain the meron gauge field configuration [147]:

$$
A^{AFF} = -i \Phi^{AFF} \Phi^{AFF} \frac{1}{2} g_{mn} \chi \sigma_i \frac{d y_m}{d y_m},
$$

where $q_m q_m = \delta_{mm} - \tilde{n}_{mn} q_l$ was used. $\Phi^{AFF}(\Omega_{3})$ can be represented by the $SO(4)$ spinor spherical harmonics $Y_{j, m_j; k, m_k}$ as

$$
\Phi^{AFF}(\Omega_{3}) = \pi \left( | \Phi_1^{AFF} \rangle \right) | \Phi_2^{AFF} \rangle \left( | \Phi_3^{AFF} \rangle \right) | \Phi_4^{AFF} \rangle = \pi \left( - | Y_1 \rangle - | Y_2 \rangle | Y_3 \rangle | Y_4 \rangle \right)
$$

where

$$
| Y_1 \rangle = Y_{1/2, 1/2; 0, 0} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} -y_4 + i y_3 \\ iy_1 - y_2 \end{pmatrix} \left( \begin{array}{c} \cos \chi + i \sin \chi \cos \theta \\ i \sin \chi \sin \theta e^{i\phi} \end{array} \right),
$$

$$
| Y_2 \rangle = Y_{1/2, -1/2; 0, 0} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} iy_1 + y_2 \\ -y_1 - iy_3 \end{pmatrix} \left( \begin{array}{c} i \sin \chi \sin \theta e^{-i\phi} \\ -\cos \chi - i \sin \chi \cos \theta \end{array} \right),
$$

$$
| Y_3 \rangle = Y_{0, 0; 1/2, 1/2} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

$$
| Y_4 \rangle = Y_{0, 0; 1/2, -1/2} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$

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It is not difficult to derive the $4 \times 4$ matrix elements of $y_m$ by performing the integration $\langle \Phi_\alpha^{\text{AFF}} | y_m | \Phi_\beta^{\text{AFF}} \rangle$ such as $\langle \Phi_1^{\text{AFF}} | y_1 | \Phi_4^{\text{AFF}} \rangle = \int d\Omega_3(-Y_{1/2,1/2}; 0,0) y_1 Y_{0,0,1/2,-1/2} = \frac{1}{4}i$: 

$$\langle y_m \rangle_{\Phi}^{\text{AFF}} = \frac{1}{4} \gamma_m, \quad (351)$$

and similarly

$$\langle y_m \rangle_{Y} = -\frac{1}{4} \gamma_m. \quad (352)$$

We introduce another $4 \times 2$ matrix

$$\Phi = \Phi^{\text{AFF}} \cdot g = \frac{1}{2 \sqrt{1 + y_4}} \begin{pmatrix} 1 + y_m q_m & 1 + y_m q_m \end{pmatrix}, \quad (353)$$

where $g$ is given by

$$g(\Omega_3) = e^{-i \frac{\pi}{4} \sum_{s=1}^{3} \hat{y}_s \sigma_s} = \hat{g}(\theta, \phi) e^{-i \frac{\pi}{4} \sigma_3} \hat{g}(\theta, \phi)^\dagger = \begin{pmatrix} \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \cos \theta & -i \sin \frac{\pi}{2} \sin \theta e^{-i \phi} \\ -i \sin \frac{\pi}{2} \sin \theta e^{i \phi} & \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \cos \theta \end{pmatrix}, \quad (354)$$

with

$$\hat{g}(\theta, \phi) = e^{-i \frac{\pi}{4} \sum_{s=1}^{3} \hat{y}_s \sigma_s} = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i \frac{\phi}{2}} & - \sin \frac{\theta}{2} e^{-i \frac{\phi}{2}} \\ \sin \frac{\theta}{2} e^{i \frac{\phi}{2}} & \cos \frac{\theta}{2} e^{i \frac{\phi}{2}} \end{pmatrix}. \quad (355)$$

$\Phi$ is related to the $SO(4)$ monopole harmonics (in the Dirac gauge) $\Phi_{j,m_j; k,m_k}$ as

$$\Phi(\Omega_3)^\dagger = \frac{1}{2 \sqrt{1 + y_4}} \begin{pmatrix} 1 + y_m q_m & 1 + y_m q_m \end{pmatrix} = \pi \begin{pmatrix} |\Phi_1\rangle & |\Phi_2\rangle & |\Phi_3\rangle & |\Phi_4\rangle \end{pmatrix}, \quad (356)$$

where

$$|\Phi_1\rangle \equiv \Phi_{1/2,1/2; \ 2,0,0} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \cos \phi \\ -i \sin \frac{\theta}{2} \sin \phi \end{pmatrix}, \quad |\Phi_2\rangle \equiv \Phi_{1/2,-1/2; \ 2,0,0} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} -i \sin \frac{\theta}{2} \sin \phi e^{-i \phi} \\ \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \cos \phi \end{pmatrix}, \quad (359a)$$

$$|\Phi_3\rangle \equiv \Phi_{0,0; \ 1/2,1/2} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \cos \phi \\ i \sin \frac{\theta}{2} \sin \phi e^{i \phi} \end{pmatrix}, \quad |\Phi_4\rangle \equiv \Phi_{0,0; \ 1/2,-1/2} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} i \sin \frac{\theta}{2} \sin \phi \cos \phi \\ \cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \cos \phi \end{pmatrix}. \quad (359b)$$

The corresponding matrix coordinates are derived as

$$\langle y_m \rangle_{\Phi} = \frac{1}{4} \gamma_m. \quad (360)$$

$\Phi^{S}_{j,m_j; k,m_k}$ is related to the Dirac gauge as

$$\Phi_{j,m_j; k,m_k} \equiv \tilde{g} \Phi^{S}_{j,m_j; k,m_k} \quad (358)$$

with $\tilde{g}$.
(351) implies that
\[ |\Phi_\alpha\rangle = g^\dagger |\Phi_{\alpha}^{\text{AFF}}\rangle, \]
and consequently
\[ \langle y_m |\Phi_{\alpha}^{\text{AFF}}\rangle = \langle y_m |\Phi_{\alpha}\rangle, \]
which is actually confirmed by comparing (351) with (360). Similarly, we have
\[ |\Phi_\alpha\rangle = -g^\dagger |Y_\alpha\rangle \quad (\alpha = 1,2), \quad |\Phi_\alpha\rangle = g^\dagger |Y_\alpha\rangle \quad (\alpha = 3,4), \]
which implies
\[ \langle \Phi_\alpha |y_m |\Phi_\beta\rangle = -\langle Y_\alpha |y_m |Y_\beta\rangle. \] 
This relation is also confirmed by comparing (360) with (352). (364) realizes the simplest version of (361).

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