How Time Works in Quantum Systems:
Overview of time ordering and time correlation in weakly perturbed
atomic collisions and in strongly perturbed qubits

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Time ordering may be defined by first defining the limit of no time ordering (NTO) in terms of a
time average of an external interaction, V(t). Previously, time correlation was defined in terms of
a similar limit called the independent time approximation (ITA). Experimental evidence for time
correlation has not yet been distinguished from experimental evidence for time ordering.
I. INTRODUCTION

A. Space and time

Both space and time are parameters used to mathematically describe observable properties of physical objects or systems of objects. In some ways space and time are similar, both intuitively and mathematically, since they are the most basic coordinates used to describe the dynamics of physical systems. It is often convenient and conventional to use $t$ and $\vec{r}$ to specify when and where an object is, even in quantum systems where the precision (or locality) of these coordinates is limited by the uncertainty principle or obscured by entanglement. Nevertheless, space and time differ fundamentally (even neglecting the second law of thermodynamics, which defines a direction in time but no preferred direction or arrow in space). Because there is no counterpart of temporal causality in a space-like context, an object can repeatedly return to any spatial location, but it cannot return or jump ahead temporarily in time. Much has been written about spatial properties of multi-particle systems, but less about time, even though three-dimensional space would seem much harder to deal with than one-dimensional time. Time, unlike space, seems somehow enigmatic. Newton refers to time as “like a river flowing.” Einstein calls time “that which a clock measures.” And Feynman refers to time as the space between events, i.e. that which keeps EVERYthing FROMhappenigALLatONCE.

One purpose of this paper is to see what happens when we try to carry an idea formulated in a spatial context into a time context. That idea is correlation (associated with non-randomness, or quantum entanglement). Time correlation \cite{1,2} has been directly related to time ordering, a causality condition implied by the time-dependent Schrödinger equation that constrains sequential interactions to occur in order of increasing time.

B. N-body problem

It is well known that the N-body problem, namely determining the quantum evolution of $N$ correlated or interacting bodies, is exponentially difficult. Kohn \cite{3} specifically estimates that the amount of computer storage capacity needed to solve an N-body problem scales as $e^{3N}$, where $3$ is simply a coefficient based on experience. Combining Kohn’s estimate with Moore’s law, which estimates, again based on experience, a doubling of computer capacity every 1.5 years, one can easily show that 6.5 years are required for computer capacity to grow enough to accommodate one additional body. Thus, increasing capacity from, say helium ($N = 3$ ignoring nuclear and subnuclear structure) to carbon ($N = 7$) is estimated to require about 25 years, progressing to water ($N = 11$ ignoring the inner shell) requires about 50 years, and DNA thousands of times the age of the universe. The N-body problem is difficult. On the other hand, since much of physics, chemistry, biology, materials science, nanotechnology, and quantum computing involves multiple particles, the problem is significant. In quantum computing, for example, much has been understood about individual qubits (regarded here as single “bodies”), but much less is understood about the coupled networks of qubits needed to build a quantum computer. Evidently, there is a need for sensible, well-defined approximations to the N-body problem. Perhaps the most widely used of these is the independent particle approximation (IPA), where the N-body problem is dramatically reduced in complexity to a problem of N independent bodies (or particles, quasi-particles, qubits, . . . ).

C. Correlation

It is sometimes argued that a theory is not well defined until a reliable method is developed to calculate corrections to the theory. In the case of the IPA, the corrections are called correlations and represent interconnections among the positions of the various particles. Correlation implies complexity: a system of independent (i.e. uncorrelated) particles is less complex than a system in which each particle’s behavior depends on the behavior of every other particle. From one viewpoint, correlation is a key to understanding how to make complex systems from simple ones. From the opposite point of view, in a very complicated system correlation can be a pathway to see order through a landscape of chaos.

Our approach is to focus on the first viewpoint – building complex systems from simple ones. If the correlations are sufficiently small, then the IPA is reasonably accurate. Correlation, one method to approach the N-body problem, is defined as the difference between a full solution and the IPA. Thus, the IPA limit of no correlation is used to define correlation itself. While this approach appears a bit awkward at first, it is conventional in the study of non-random processes and quantum entanglement as well. For example, in the limit of no entanglement the N-body wave function may be written as a simple product of independent particle solutions, and entanglement may be defined as the deviation from this limit.
In most applications, correlations arise from spatial interactions between particles. In N-electron atoms, for example, electron correlation is generated by the $1/r_{ij}$ Coulomb interactions between electrons. In this context, the IPA is also known as the independent electron approximation (IEA) and is defined by replacing these complicating correlation interactions by a sum of single-electron mean-field interactions. The resulting Hamiltonian is a sum of single-electron terms, even in the case of dynamic atomic systems, and the resulting wave function is a product of single-electron wave functions. One common method for analyzing deviations from the IEA is many-body perturbation theory (MBPT), used here and in other fields as well to describe correlation at various orders in a given perturbation.

One of the primary questions motivating the work described here is, “what happens if one tries to define an independent time approximation (ITA)”, where one eliminates correlations in time rather than the conventional correlations in space?” In such an approximation, multiple times may be used for multiple particles. This is simpler than using a single time for all particles, just as the use of N independent particle positions is simpler that trying to solve a single, but complicated, N-body problem. Time correlation may then be defined as a deviation from the ITA in analogy to the definition of spatial correlation by reference to the IPA.

D. Correlation in time and time ordering

In this paper we first review early efforts to define the ITA, in which a key step is to remove time ordering between fields acting on different particles. This is closely related to the approximation of no time ordering (NTO), where all time ordering is removed for fields acting either on different particles or on the same particle. These early efforts to define and study the ITA were based on a perturbation expansion of the external interaction $V(t)$. Since time correlation or time ordering first appear at second order in $V(t)$, the observable effects are small and the calculations difficult. Next we review more recent investigations of time ordering and the NTO approximation for qubits in non-perturbative external fields. Therefore, the organization of this paper is unusual. The harder problem of time correlation is addressed first, and the related but more tractable problem of time ordering is discussed second. We follow this unusual order because it reflects what has happened in recent years and also because the newer work on time ordering raises some intriguing questions and challenges. In particular we emphasize that experimental evidence for time correlation has not yet been distinguished from evidence for time ordering. That is, no difference between the ITA and the NTO approximations has yet been observed in any experiment, although recent experiments have been progressing in this direction, as we illustrate below.

II. THE N-BODY PROBLEM

In this section we consider solutions to the N-body problem described by a Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V}_S(t),$$

where there may be many terms in both the unperturbed $\hat{H}_0$, which we assume to be solvable, and in $\hat{V}_S(t)$, which we regard as an external time-dependent interaction (written in the Schrödinger representation). Where appropriate, $\hat{V}_S(t)$ may be treated as a classical external field. In the case of atomic collisions, this corresponds to the widely used semiclassical approximation (SCA) where the projectile is regarded as a classical particle if it is a proton or electron and a classical wave if it is a photon.

The N-body problem may be solved in various representations, depending on how $\hat{H}$ is separated into $\hat{H}_0$ and $\hat{V}_S(t)$, ranging from the Schrödinger representation where effectively $\hat{V}_S(t) = \hat{H}$ and $\hat{H}_0 = 0$ to the Heisenberg representation where $\hat{H}_0 = \hat{H}$ and $\hat{V}_S(t) = 0$. In the SCA, singularities can occur in the first order amplitudes when the Schrödinger representation is used. Except where otherwise specified we shall use an appropriate intermediate representation that takes maximum advantage of known solutions for $\hat{H}_0$, and in which $\hat{V}(t) = e^{i\hat{H}_0 t} \hat{V}_S(t) e^{-i\hat{H}_0 t}$ causes transitions between eigenstates of $\hat{H}_0$. We work in atomic units, in which $\hbar = m_e = 1$.

A. Formulation of the N-body problem in the time domain

We seek solutions $\Psi(r_1, \ldots, r_N; t)$ to the N-body problem described by Eq. (1). It is both conventional and sensible to separate the influence of the external interaction $\hat{V}(t)$ from the initial state at some time $t_0$ before the external fields are applied to the atomic system, namely,

$$\Psi(r_1, \ldots, r_N; t) = \tilde{U}(t, t_0) \Psi(r_1, \ldots, r_N; t_0).$$

(2)
Here all dynamics are contained in the time evolution operator (or Green’s function) $\hat{U}(t,t_0)$.

It is easily shown that in the intermediate representation $\hat{U}(t,t_0)$ satisfies

$$i\frac{d}{dt}\hat{U}(t,t_0) = \hat{V}(t)\hat{U}(t,t_0).$$

(3)

The solution for $\hat{U}(t,t_0)$, which may be verified by insertion into Eq. (3), is

$$\hat{U}(t,t_0) = 1 - i \int_{t_0}^{t} \hat{V}(t_1) dt_1 + (-i)^2 \int_{t_0}^{t} \hat{V}(t_1) dt_1 \int_{t_0}^{t} \hat{V}(t_2) dt_2$$
$$+ \cdots + (-i)^n \int_{t_0}^{t} \hat{V}(t_1) dt_1 \int_{t_0}^{t} \hat{V}(t_2) dt_2 \cdots \int_{t_0}^{t} \hat{V}(t_n) dt_n + \cdots$$
$$\equiv T \sum_n \frac{(-i)^n}{n!} \int_{t_0}^{t} \hat{V}(t') dt' = T e^{-i\int_{t_0}^{t} \hat{V}(t') dt'}.$$  

(4)

Here $T$ is the Dyson time-ordering operator, which arranges the interactions $\hat{V}(t')$ in order of increasing time, similar to the requirements of causality. The time ordering operator is central to the discussion in this paper, as it relates both to observable time ordering effects and to time correlation. The key idea is that $T\hat{V}(t_1)\hat{V}(t_2) = \hat{V}(t_2)\hat{V}(t_1)$ if $t_2 > t_1$ and $\hat{V}(t_1)\hat{V}(t_2)$ otherwise. The no time ordering (NTO) approximation is the approximation in which $T \to 1$, and the constraint of time ordering is not enforced. In this limit all time sequences of the $\hat{V}(t')$ are equally weighted, so that

$$\hat{U}(t,t_0) \to \hat{U}_{\text{NTO}}(t,t_0) = e^{-i\int_{t_0}^{t} \hat{V}(t') dt'}$$

(5)

in Eq. (4).

B. Independent particle approximation (IPA)

As noted above, the N-body problem is notoriously difficult to solve. A particularly useful approximation is the independent particle approximation (IPA). For an atom with $N$ electrons, $H_0 = \sum_j (-\nabla_j^2/2 - Z/r_j + \sum_{i<j} 1/r_{ij})$, where $Z$ is the nuclear charge and $r_j$ are the electron positions in a coordinate system centered on the nucleus, and $\hat{V}_S(t) = -\sum_j Z_p/|\vec{R}(t) - \vec{r}_j|$, where $Z_p$ and $\vec{R}(t)$ are the charge and position of the projectile. If $\sum_{i<j} 1/r_{ij}$ is approximated by a mean field $\sum_j \bar{v}(r'_j)$, then $\hat{H} = H_0 + \hat{V}_S(t)$ is reduced to a sum of single-particle terms that may be solved using separation of variables. One obtains the independent particle approximation (IPA) to the exact N-particle wave function, namely,

$$\Psi(r_1, \ldots, r_N; t) \approx \prod_j \psi_j(r_j, t) = \prod_j \hat{U}_j(t,t_0) \psi_j(r_j, t_0) \quad \text{(IPA)},$$

(6)

where each $\psi_j(r_j, t)$ is a single-particle wave function. We note here that in the IPA approximation, different times may be used for different particles, if so desired, since the particles are independent in both space and time. Whatever happens to one of the particles does not influence what happens to any of the others, although time ordering is retained within the evolution of each independent particle.

C. Independent time approximation (ITA)

In order to address the question, “what happens if one tries to define an independent time approximation (ITA), where one explores correlations in time rather than the conventional correlations in space?”, a list of comparisons between the ITA and IPA was developed [2]. An updated comparison is summarized in Table I.

There are similarities between the temporal independent time approximation and the spatial independent particle approximation, as seen in Table I. Time and space correlation can each be defined as a deviation from an uncorrelated limit, where the uncorrelated limit is given by a product form. Electron identity, which has been ignored here for simplicity of presentation, may be restored by antisymmetrizing the uncorrelated single-electron wave functions. The uncorrelated limit may also be described by an average of the appropriate correlation operator, as indicated in Table I.
Correlation may then be defined in terms of fluctuations away from the average, as is done in statistical mechanics. In both the spatial and temporal cases, the average term may form the basis for useful approximate calculations.

There are also notable differences between temporal and spacial correlation, as detailed in Table I. While correlation in space arises in the asymptotic target Hamiltonian \( \hat{H}_0 \), and affects both the asymptotic initial wave function \( \Psi(t_0) \) and the evolution operator \( \hat{U}(t, t_0) \), correlation in time occurs only in the time evolution operator \( \hat{U}(t, t_0) \). Correlation in space comes from \( 1/r_{ij} \) inter-electron interactions within the target. In the IPA, phase coherence and time correlation between electrons are both lost, as seen, for example, by noting that matrix elements of \( \hat{V} \) in Eq. 4 may be complex and that the time order is significant except when all the \( \hat{V} \) go to a common \( \bar{V} \). Time correlation arises from time ordering of the external interaction \( \hat{V}(t) \) acting on different particles. The ITA is intimately related to the NTO since \( T \to T_{nv} = 1 \) in both cases. However, in the ITA \( T \to 1 \) is applied only to the cross terms affecting different particles, while time ordering for each individual particle is retained. In the ITA, each particle evolves independently in time, although the initial state may be spatially correlated. Thus, the initial state \( \sum_k c_k \prod_j \psi_j^{(k)}(r_j, t_0) \) evolves to \( \sum_k c_k \prod_j \psi_j^{(k)}(r_j, t_j) \), where the state of each particle may be evolved using its own independent time \( t_j \). Removing some or all of the time ordering terms is straightforward in practice since the \( T_{cor} \) terms are easily identified, at least at second order in perturbation theory.

| Cause: | Spatial correlation | Temporal correlation |
|-------|---------------------|---------------------|
| \( \hat{v}_{ij} = 1/r_{ij} \) | \( T \) and \( \hat{V}(t) \) | \( T \) and \( \hat{V}(t) \) |
| spatially varying internal | time ordering | time ordering |
| Coulomb interactions | external interactions | external interactions |

### III. ATOMIC COLLISIONS

In this section we review time ordering and time correlation using perturbation theory for the interaction of charged particles with helium. From Eq. 3 we see that the leading effect due to time ordering arises at second order in the \( \hat{V}(t) \) expansion. The system has an infinite number of states but the external interaction \( \hat{V}(t) \) only acts on the system twice. In Sec. IV we consider a strongly perturbed qubit, a two-state system that interacts with the external field \( \hat{V}(t) \) an infinite number of times.

Through second order in \( \hat{V}(t) \), the time evolution operator \( \hat{U}(t, t_0) \) is given by

\[
\hat{U}(t, t_0) = Te^{-i \int_{t_0}^t \hat{V}(t') dt'} \simeq 1 - i \int_{t_0}^t \hat{V}(t_1) dt_1 - \int_{t_0}^t \hat{V}(t_1) dt_1 \int_{t_0}^{t_1} \hat{V}(t_2) dt_2
\]

\[
\equiv 1 - i \int_{t_0}^t \hat{V}(t_1) dt_1 - T \frac{1}{2!} \int_{t_0}^t \hat{V}(t_1) dt_1 \int_{t_0}^{t_1} \hat{V}(t_2) dt_2.
\]

This may also be obtained by integrating Eq. 3 through second order. Note that as \( T \to 1 \) the order in which the interactions act may be interchanged. It is the difference between the limit as \( T \to 1 \) and the full result that defines the effects of time ordering. In the case of weak correlation considered here, the NTO limit of \( T \to 1 \) will coincidentally yield the ITA as well. The trick now is to separate \( T \) from \( T_{nv} = 1 \).
A. Primary results

1. Various pathways to ITA

To separate the time ordering effects from the non-time ordering (NTO) effects, it is useful to write,

\[ T = T_{av} + (T - T_{av}) \]

where \( T_{av} \) yields the NTO approximation and \( T - T_{av} \) yields the effects of time ordering. This decomposition of \( T \) into an average part plus fluctuations is central for this paper. Note that \( T_{av} = 1 \) is required to satisfy the initial condition \( \hat{U}(t_0, t_0) = 1 \).

One conceptual pathway to the ITA proceeds by analogy with the NTO limit. In the ITA, time ordering is enforced among all potentials acting on an individual particle (e.g. for \( \hat{V}_i(t_1)\hat{V}_j(t_2) \) at second order), but ignored for different particles in cross terms such as \( \hat{V}_i(t_1)\hat{V}_j(t_2) \) when \( i \neq j \). This distinguishes the ITA from the NTO approximation where time ordering is removed for all terms.

Another clever conceptual pathway to time ordering and the ITA, noticed by Godunov \( \text{[2]} \), is via use of commutator relations. Consider the identity, which we call the Godunov identity \( \text{[8]} \),

\[ \hat{V}(t_1)\hat{V}(t_2) = \frac{1}{2} \left( \hat{V}(t_1)\hat{V}(t_2) + \hat{V}(t_2)\hat{V}(t_1) \right) + \frac{1}{2}[\hat{V}(t_1),\hat{V}(t_2)]. \]

In the case of atomic collisions with helium, the projectile interacts with both electrons so that \( \hat{V}(t') = \hat{V}_1(t') + \hat{V}_2(t') \).

Now \([\hat{V}(t_1),\hat{V}(t_2)]\) contains both \([\hat{V}_1(t_1),\hat{V}_1(t_2)]\) and \([\hat{V}_1(t_1),\hat{V}_2(t_2)]\) \((i \neq j)\) terms. In the NTO approximation, all commutator terms are eliminated, i.e. \([\hat{V}(t_1),\hat{V}(t_2)] \rightarrow 0\) and \( \hat{V}(t_1)\hat{V}(t_2) \rightarrow \frac{1}{2} \left( \hat{V}(t_1)\hat{V}(t_2) + \hat{V}(t_2)\hat{V}(t_1) \right) \) so that the time evolution operator at second order is given by the average value of both possible time sequences. In the ITA, only cross commutator terms between fields acting on different particles are neglected, i.e. \([\hat{V}_i(t_1),\hat{V}_j(t_2)] \rightarrow 0\) and \( \hat{V}_i(t_1)\hat{V}_j(t_2) \rightarrow \frac{1}{2} \left( \hat{V}_i(t_1)\hat{V}_j(t_2) + \hat{V}_j(t_2)\hat{V}_i(t_1) \right) \) only for \( i \neq j \).

A third pathway to the ITA is through elimination of off-energy-shell effects. The effect of the Dyson time ordering operator at second order may be expressed as \( T\hat{V}(t_1)\hat{V}(t_2) = \Theta(t_1 - t_2)\hat{V}(t_1)\hat{V}(t_2) + (1 \leftrightarrow 2) \), and thus

\[ \frac{1}{2!} T \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{V}(t_1)\hat{V}(t_2) dt_1 dt_2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Theta(t_1 - t_2)\hat{V}(t_1)\hat{V}(t_2) dt_1 dt_2. \]

Taking \( T \rightarrow 1 \) is equivalent to replacing \( \Theta(t_1 - t_2) \) by the constant \( 1/2 \). The Fourier transform of the Heavyside theta function is well known, namely

\[ \int_{-\infty}^{+\infty} e^{iE(t_1-t_2)} \Theta(t_1 - t_2) d(t_1 - t_2) = \pi \delta(E) + iP_e \frac{1}{E}. \]

The principal value term accesses off-energy-shell states, which violate energy conservation during the short collision time in a manner consistent with the uncertainty relation. Since taking \( T \rightarrow 1 \) gives the \( \pi \delta(E) \) term in Eq. 23, it is the principal value term that carries the effects of time ordering. Ignoring the off-shell contribution (which is conveniently \( \pi/2 \) out of phase with the time-averaged term) whenever the fields \( \hat{V}(t_1) \) and \( \hat{V}(t_2) \) act on different particles yields the ITA.

There is yet a fourth pathway to the NTO or ITA. For potentials that are time-independent in the intermediate representation, \([\hat{V}(t_1),\hat{V}(t_2)] \rightarrow 0\) and the NTO approximation is exact. In general, the NTO approximation may be obtained by replacing the true external potential \( \hat{V}(t) \) with its time average over the duration of the interaction, i.e. \( \hat{V}(t') \rightarrow \overline{V} = \frac{1}{t - t_0} \int_{t_0}^{t} \hat{V}(t') dt' \), so that the exponent \( \int \hat{V}(t') dt' \) in the time-ordered exponential of Eq. 2 is replaced by \( \overline{V} \cdot (t - t_0) \). The replacement of the external time-dependent interaction by its time average is analogous to the replacement of the true inter-particle interaction by its mean-field value in the IPA, which involves averaging over the positions of all but one of the particles.

We note that the limit of constant potential \( \hat{V}(t) \) in which the NTO approximation becomes exact is distinct from the adiabatic limit, in which the potential merely changes slowly with time. Furthermore, the content of the NTO or ITA approximation depends on the representation used, as we will see explicitly in Sec. 4.

Thus, for a given decomposition \( \hat{H} = \hat{H}_0 + \hat{V}_S(t) \), it is the interaction-representation potential \( \hat{V}(t) = e^{i\hat{H}_0 t}\hat{V}_S(t)e^{-i\hat{H}_0 t} \) that must be constant for the NTO or ITA approximation to be exact. In the Heisenberg representation, \( \hat{V}(t) = 0 \) by construction, and the NTO or ITA is always trivially exact. To summarize, pathways to the ITA approximation in this section include:
1. \( T \rightarrow 1 \),
2. \([\hat{V}_i(t_1), \hat{V}_j(t_2)] \rightarrow 0\),
3. \( P_{\text{off}} \rightarrow 0 \),
4. \( \hat{V}(t) \rightarrow \overline{V} \).

The difference between the ITA and NTO approximations is that in the second item above, the commutator disappears for all terms in the NTO but only for the cross terms in the ITA.

2. Economy of NTO

A variety of second-order calculations with and without time ordering have been done by Godunov and collaborators in recent years \([2][10]\). Most of these calculations have studied two-electron transitions in atoms caused by high-velocity collisions with protons or electrons, where second Born methods are applicable. The advantage of looking at two-electron transitions is that second order terms are often dominant, since both elastic scattering and single-electron transitions, corresponding to the first two terms in Eq. (7), are experimentally eliminated. A disadvantage is that the resulting cross sections are quite small. The Godunov code is remarkable in that the off-shell terms can be computed exactly at second order in perturbation theory, in contrast with most other existing calculations, which use closure approximations to avoid this relatively difficult calculation. Dropping the difficult but interesting off-shell terms yields a result without any time ordering. Comparing results with and without inclusion of off-shell terms then yields the time ordering or time correlation effect. The algebra required for the off-shell terms is relatively tedious, as reflected by the off-shell calculation typically requiring several hundred times more computer time. Thus the NTO (and possibly the ITA) represents a substantial reduction of computer time and algebraic effort. When valid, this approximation can therefore be used to attack problems harder than those that require the full off-shell terms. Unfortunately, at this point no simple, physically transparent criterion is known to us that determine when the NTO or ITA approximation is valid.

B. Experimental evidence

There have been ten or so experiments that show some evidence for time ordering and time correlation effects for two-electron transitions in high velocity collisions. Perhaps the most dramatic evidence is the factor of two difference found \([11]\) in the double ionization of helium in collisions with protons and antiprotons at several MeV. There seems to be agreement that this difference is due to time ordering and time correlation, but no definitive theoretical studies have yet been done. Some studies \([12]\) of Auger profiles also support the need for time correlation and time ordering. The clearest direct comparison of experimental data with time-ordered and non-time-ordered theoretical calculations is for polarization of light emitted after excitation-ionization of helium by proton impact \([13]\).

In this study, time ordering or time correlation has a 20% effect on the predicted polarization. The experiment, with 5% errors, is in excellent agreement with Godunov’s calculations, except at the lowest energy where perturbation theory is expected to break down, and clearly shows the importance of time ordering or time correlation effects. There is a need for more calculations by different authors to confirm these effects of time correlation and time ordering.

In all of these two-electron transition experiments, one may do a simultaneous expansion in the external interaction \( \hat{V}(t) \) and the correlation interaction \( \hat{v} \). Time correlation effects are associated with commutators of the form \( [\hat{V}_i(t_1), \hat{V}_j(t_2)] \), where \( i \) and \( j \) label two different electrons, and are second order in \( \hat{V} \). In addition, there are time ordering effects not related to time correlation: these are associated with commutators \( [\hat{V}_i(t_1), \hat{V}_i(t_2)] \) and enter at second order in \( \hat{V} \) and first order in \( \hat{v} \) (since the commutator vanishes for \( \hat{v} = 0 \)). Thus the difference between time ordering and time correlation is a third order effect, smaller by at least a factor of 10 than either the time ordering effect or the time correlation effect taken individually. The experiments show direct evidence for time correlation but do not distinguish between time correlation and time ordering.

There is also one clear and definitive study by the group of Thomas \([14]\) that shows direct evidence for time ordering in atoms interacting with a time-varying magnetic field.
FIG. 1: Calculations with and without time ordering between electrons are compared to experimental data. Here polarized light is emitted from helium following $1s \rightarrow 2p$ excitation of one electron accompanied by ionization of the second electron [13]. The polarization fraction is plotted as a function of the velocity of the incident proton. The first-order calculation (Born 1) has no time ordering. The second-order calculation is shown both with time ordering (Born 2 full) and in the no time ordering (NTO) approximation (Born 2 unc). As explained in the text, the NTO approximation is the same as the independent time approximation (ITA) at this order since the interaction is weak.

IV. QUBITS

A qubit is a very simple two-state (e.g. on and off) quantum system whose state population may be changed by an external potential $\hat{V}(t)$, in analogy with the way in which an atomic state may be changed by the Coulomb potential $\hat{V}(t)$ of the projectile in an atomic collision. Qubits are building blocks for the complex interconnected N-qubit systems that can be used for quantum computation and quantum information. The extension from a single qubit to a system of N interconnected qubits is analogous to the extension from a one-electron atom to a correlated N-electron atom. The interaction of qubits with each other and with their environment still have to be dealt with before a quantum computer becomes a reality.

In this section we consider time ordering in a single qubit, analogous to time ordering in scattering from atomic hydrogen. To better understand time ordering effects, we work in the time domain rather than the more common energy (or frequency) domain. The advantages of working with qubits include the possibility of easily handling non-perturbative external potentials and the existence in some cases of analytic solutions, so that numerical calculations can be avoided. This yields new ways to think about time ordering mathematically and physically. Specific effects due to time ordering in a simply pulsed qubit are shown, for example, in Fig. 3 below. The idea is to extend the above analysis of time ordering for weakly perturbed atomic collisions to the case of strongly perturbed qubits. In order to make use of transparent analytic expressions, we spend some effort discussing the time evolution of qubits pulsed sharply in time, commonly referred to as “kicked” qubits. With the exception of the last brief subsection, there is little emphasis here on time correlation between qubits, simply because little work has been done on this problem to our knowledge. So our emphasis in this section is almost entirely on time ordering. One new feature here will be to explore what happens to time ordering under change of representation – specifically when we change from the Schrödinger to the intermediate representation.

A. Overview of single qubits

The qubit wave function $\psi(t)$ is a linear superposition of the “on” and “off” states, namely
\[
\Psi(t) = a_1(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2(t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\] (12)

Population can be transferred from one state to the other by applying an external potential \( \hat{V}(t) \), which can have the form of a single pulse characterized by a time duration, \( \tau \). The full Hamiltonian can be written in terms of the Pauli spin matrices,

\[
\hat{H}(t) = \hat{H}_0 + \hat{V}_S(t)
\]
\[
= \begin{bmatrix} -\Delta E/2 & 0 \\ 0 & +\Delta E/2 \end{bmatrix} + \begin{bmatrix} 0 & V(t) \\ V(t) & 0 \end{bmatrix}
\]
\[
= -\frac{\Delta E}{2}\sigma_z + V(t)\sigma_x.
\] (13)

The time dependence of the system can be described by the time evolution operator \( \hat{U}(t,t_0) \). Allowing \( t_0 = 0 \) to simplify the notation, we can write \( \psi(t) = \hat{U}(t)\psi(0) \), where

\[
\hat{U}(t) = \begin{bmatrix} U_{11}(t) & U_{12}(t) \\ U_{21}(t) & U_{22}(t) \end{bmatrix} = Te^{-i\int_0^t \hat{V}(t')dt'}
\] (14)

Here \( \hat{V}(t') = e^{i\hat{H}_0 t}' V_S(t')e^{-i\hat{H}_0 t}' \) is the interaction potential in the intermediate representation, and \( \hat{U}(t) \) is the formal solution to the differential equation \( \hat{U}(t) \). In the Schrödinger representation, \( \hat{V}(t') \) is replaced by \( \hat{H}(t') = \hat{H}_0 + \hat{V}_S(t') \).

The time evolution of a qubit depends on the energy splitting \( \Delta E \) and the time dependence \( V(t) \) of the external potential. Depending on the complexity of \( V(t) \), \( \hat{U}(t) \) may or may not have a simple analytic form. In the latter case, numerical solutions of the coupled differential equations (13) may obscure information about these quantum systems. Analytic solutions, when available, are more convenient and easy to analyze.

B. Simply pulsed qubits

Here we consider simply pulsed qubits – that is, qubits subject to an external field \( V(t) \) that has a finite duration in time and a sensibly simple shape, such as a simple rectangular or Gaussian pulse. Such pulses are convenient for studying qubits in the time domain. In some cases they also lead to convenient analytic solutions, even for strong fields.

1. Qubit map

A qubit map, such as that shown in Fig. 2 is a tool enabling one to visualize how the possible behavior of simply pulsed qubits depends on the variables \( \Delta E \) and \( V(t) \). In order to make the qubit map a helpful visual tool, the map coordinates are taken to be a dimensionless level splitting \( \Delta E \tau/2 \) (where \( \tau \) is the time duration of the pulse \( V(t) \)) and a dimensionless pulse strength \( \int_0^\infty V(t')dt' \). These two coordinates determine the effect of the unperturbed Hamiltonian \( \hat{H}_0 \) and of the external potential \( \hat{V}(t) \) on the evolution operator \( \hat{U}(t) \). It is useful to think of these two variables as independent phase angles or action-like integrals.

2. RWA solutions

The kicked qubits we focus on below provide an alternative to the well-established rotating wave approximation (RWA) method \( [15, 16, 17] \) based on a single resonant transition frequency, that may be detuned. Our emphasis here is on kicked (i.e. strongly time-localized) potentials because: i) the physics is then naturally analyzed in the time domain, and ii) kicks have been less widely explored than the RWA approach, which works well for sharp pulses in the reciprocal frequency space. While the RWA method is useful for two-state systems perturbed by an external interaction of narrow bandwidth, it fails to describe simply kicked two-state systems where the pulse bandwidth may be very broad.
is small, ∆E is the energy difference between the two states of the qubit, and V(t) is the external potential. When the total phase associated with the external potential is small, i.e. \( \int_0^\infty V(t') dt' \ll 2\pi \), then the expression for the time evolution operator may be expanded in powers of V using \( e^{-iV(t') dt'} \approx 1 - iV(t') dt' \) and only the first few terms retained. This corresponds to standard perturbation theory for either quickly perturbed qubits where \( \tau \) is small or slowly perturbed qubits where \( \tau \) is large. Similarly, if the phase associated with the energy splitting is small, \( \Delta E \tau/2 \ll 2\pi \), then we can treat the qubit as degenerate, and the solution can be expanded in powers of \( \Delta E \tau \). If both phases are large, then the adiabatic approximation generally applies.

3. Kicked qubits

A useful approximation for simply pulsed qubits is the fast, narrow pulse or “kick” limit in which the width \( \tau \) of the pulse goes to zero, while the integrated strength or area under the external potential curve

\[
\alpha = \int_0^\tau V(t') dt'
\]  

remains fixed. Formally, the shape of a very narrow pulse of finite total strength \( \alpha \) may be expressed by a delta function: \( V(t') = \alpha \delta(t' - t_k) \), where \( t_k \) is the time at which the pulse is centered. The kicked region corresponds to the lower half of the qubit map in Fig. 2. Here the duration of the pulse is so short that \( \Delta E \tau/2 \ll 2\pi \), i.e. there is not enough time for the splitting \( \Delta E \) to have a significant effect while the pulse is active. The integrated strength of the pulse, \( \alpha \), may be either large or small in this region. If \( \alpha \) is large, we are in the lower right quadrant of the map, where the kicked region overlaps with the adiabatic region. If \( \alpha \) is small, we are in the lower left quadrant, where the kicked region overlaps with the perturbative region.

**Single kick**

For a two-state system subjected to a single kick at time \( t_k \), corresponding to \( V(t') = \alpha \delta(t' - t_k) \), the integration
over time is trivial and the time evolution operator in Eq. (13) becomes
\[ \hat{U}^k(t) = T \exp \left[ -i \int_0^t e^{-i\Delta E \sigma_z t'/2} \alpha \delta(t' - t_k) \sigma_x e^{i\Delta E \sigma_z t'/2} dt' \right] \]
\[ = \exp \left[ -i \alpha \left( e^{-i\Delta E t_k/2} 0 0 0 \right) \left( 0 1 1 0 \right) \left( e^{i\Delta E t_k/2} 0 0 e^{-i\Delta E t_k/2} \right) \right] \]
\[ = \exp \left[ -i \alpha \left( 0 e^{-i\Delta E t_k} 0 \right) \right] \]
\[ = \left( \begin{array}{cc} \cos \alpha & -ie^{-i\Delta E t_k} \sin \alpha \\ -ie^{i\Delta E t_k} \sin \alpha & \cos \alpha \end{array} \right) \]  \tag{16}
for \( t > t_k \). The last line of Eq. (16) was obtained by expanding the fourth line in powers of \( \alpha \) using the identity \( \left( e^{\Delta E t_k} \right)^{2n} = I \). Equivalently, one can take advantage of the useful identity (which we also used in the third line) \( e^{i\phi \tilde{\sigma} \cdot \hat{u}} = \cos \phi \hat{I} + i \sin \phi \hat{\sigma} \cdot \hat{u} \), where \( \hat{u} \) is an arbitrary unit vector. Note that \( \hat{U}^k(t) \) is independent of the final time \( t \) in the intermediate representation.

The occupation probabilities for a kicked qubit initially in state 1 are
\[ P_1(t) = |a_1(t)|^2 = |U^k_{11}(t)|^2 = \cos^2 \alpha \]
\[ P_2(t) = |a_2(t)|^2 = |U^k_{21}(t)|^2 = \sin^2 \alpha . \] \tag{17}
This simple example may be extended to a series of kicks. It is one of the few cases in which analytic solutions may be obtained for qubits controlled by external potentials.

**Double kicks**

The simplest example of a series of arbitrary kicks is a sequence of two kicks of strengths \( \alpha_1 \) and \( \alpha_2 \), applied at times \( t_1 \) and \( t_2 \) respectively, i.e. \( \hat{V}_S(t) = (\alpha_1 \delta(t - t_1) + \alpha_2 \delta(t - t_2)) \sigma_x \). Eq. (14) is then easily solved in the interaction representation, namely,
\[ \hat{U}^{k_2,k_1} = \hat{U}^{k_2} \times \hat{U}^{k_1} \]
\[ = \left( \begin{array}{cc} \cos \alpha_2 & -ie^{-i\Delta E t_2} \sin \alpha_2 \\ -ie^{i\Delta E t_2} \sin \alpha_2 & \cos \alpha_2 \end{array} \right) \times \left( \begin{array}{cc} \cos \alpha_1 & -ie^{-i\Delta E t_1} \sin \alpha_1 \\ -ie^{i\Delta E t_1} \sin \alpha_1 & \cos \alpha_1 \end{array} \right) \]
\[ = \left( \begin{array}{cc} U_{11} & U_{12} \\ U_{21} & U_{22} \end{array} \right), \]  \tag{18}
where
\[ U_{11} = \cos \alpha_1 \cos \alpha_2 - \sin \alpha_1 \sin \alpha_2 e^{-i\Delta E t_2} , \]
\[ U_{21} = -ie^{i\Delta E t_2/2} (\cos \alpha_1 \sin \alpha_2 e^{i\Delta E t_2/2} + \sin \alpha_1 \cos \alpha_2 e^{-i\Delta E t_2/2}) . \] \tag{19}

Here \( t_- = t_2 - t_1 \), and \( t_+ = t_1 + t_2 \). In the limit \( t_2 \rightarrow t_1 \), Eq. (18) reduces to Eq. (16) with \( \alpha \rightarrow \alpha_1 + \alpha_2 \). Note that \( [\hat{U}^{k_2}, \hat{U}^{k_1}] \neq 0 \) so that the time ordering of the interactions is important.

The algebra for a combination of two arbitrary kicks [13], one proportional to \( \sigma_y \) and the other proportional to \( \sigma_x \), is very similar to the above. Triple kicks are also straightforward to solve analytically.

4. Time ordering in a doubly kicked qubit

In this subsection we use our analytic expressions to examine the effect of the Dyson time ordering operator \( T \) in a kicked two-state system. Time ordering has been considered previously in the context of atomic collisions with charged particles [4, 6, 11, 14, 20] and differs somewhat from the order in which external pulses are applied, as illustrated below. As is intuitively evident, there is no time ordering in a singly kicked qubit since there is only one kick. The simplest kicked two-state system that shows an effect due to time ordering is the qubit kicked by two equal and
opposite pulses labeled $k$ and $-k$ separated by a time interval $t_1 = t_2 - t_1$. The evolution matrix for this system is [10],

$$
\hat{U}^{-k,k} = \left( \begin{array}{cc}
e^{-iE_{t-}/2} \left( \cos \frac{\Delta E t_-}{2} + i \cos 2\alpha \sin \frac{\Delta E t_-}{2} \right) \\
\quad -e^{iE_{t+}} \sin 2\alpha \sin \frac{\Delta E t_-}{2}
\end{array} \right) \left( \begin{array}{cc}
e^{-iE_{t+}/2} \sin 2\alpha \sin \frac{\Delta E t_-}{2} \\
\quad e^{iE_{t-}/2} \left( \cos \frac{\Delta E t_-}{2} - i \cos 2\alpha \sin \frac{\Delta E t_-}{2} \right) \end{array} \right).
$$

The time evolution operator $\hat{U}^{(0)}$ in the limit of no time ordering, i.e. in the approximation $T \to 1$, may in principle be generally obtained [21] by replacing $\int_0^t V(t') dt'$ with $\bar{V}$, where $\bar{V}$ is an average (constant) value of the interaction. In our case, it is then straightforward to show

$$
\hat{U}^{(0)-k,k} = e^{-i\bar{V}t} = \left( \begin{array}{cc}
\cos(2\alpha \sin \frac{\Delta E t_-}{2}) \\
\cos(2\alpha \sin \frac{\Delta E t_-}{2})
\end{array} \right)
$$

In this example we now have analytic expressions for the matrix elements of both the evolution operator $\hat{U}^{-k,k}$ that contains time ordering and the evolution operator $\hat{U}^{(0)-k,k}$ without time ordering.

![Graph](image.png)

FIG. 3: Difference in population transfer probability, $P_2 - P_2^{(0)}$ vs. $\epsilon = \sin(\Delta E t_- / 2)$ and $\phi = 2\alpha$. Here $t_- = t_2 - t_1$ is the time between the pulses and $\alpha = \int V(t') dt'$ is a measure of the interaction strength. The two-state system is kicked by a sharp pulse of strength $\alpha$ at time $t_1$ and by an equal and opposite pulse at time $t_2$. The difference, $P_2 - P_2^{(0)}$, is due to time ordering in this qubit.

It may be shown analytically [10] that for two kicks both proportional to $\sigma_x$, the order of the kicks does not change the final transfer probability $P_2$. However, interestingly, this does not mean that there is no effect due to time ordering in this case. As we show next, there is an effect due to time ordering in this case, even though interchanging the order of the kicks has no effect. The effect of time ordering on the occupation probabilities may be examined by considering the probability of population transfer from the on state to the off state with and without time ordering, namely from Eqs. (20) and (21).

$$
P_2 = |U_{21}|^2 = |\sin 2\alpha \sin \frac{\Delta E}{2} t_-|^2 = |\epsilon \sin \phi|^2,
$$

$$
P_2^{(0)} = |U_{21}^{(0)}|^2 = |\sin(2\alpha \sin \frac{\Delta E}{2} t_-)|^2 = |\sin \epsilon \phi|^2,
$$

where $\epsilon = \sin \frac{\Delta E}{2} t_-$ and $\phi = 2\alpha$.

The effect of time ordering is shown in Fig. 3 where $P_2 - P_2^{(0)}$ is plotted as a function of $\phi = 2\alpha$, corresponding to the strength of the kicks, and $\epsilon = \sin \frac{\Delta E}{2} t_-$, which varies with the time separation of the two kicks. The effect of time
ordering disappears in our example in the limit that either the interaction strength or the time separation between the pulses goes to zero. For small, but finite, values of both the interaction strength and the time separation between the pulses, the effect of time ordering is to reduce the transition probability from the initially occupied state to an initially unoccupied state. That is, in this regime time ordering reduces the maximum transfer of population from one state to another. As either of these two parameters gets sufficiently large, the effect of time ordering oscillates with increasing values of the interaction strength or the inter-pulse separation time. Time ordering effects are present even though $U^{-k,k} = U^{k,-k}$.

5. Time ordering vs. time reversal

Let us now pause to examine the difference between time ordering and time reversal in this simple, illustrative example. Reversal of time ordering means that, since kick strengths $\alpha_k$ and kick times $t_k$ are both interchanged, $t_\pm \to -t_\pm$ and $\alpha \to -\alpha$. In this case one sees from Eqs. (20) and (21) that $U^{-k,k}$ experiences a change of phase, while $U^{(0)-k,k}$ is invariant under change of time ordering. For time reversal $\hat{U} \to \hat{U}^\dagger$, $t_\pm \to -t_\pm$ and, since the initial and final states are also interchanged, $\hat{U} \to \hat{U}^\dagger$. Inspection of the same equations as above shows that $U^{-k,k}$ and $U^{(0)-k,k}$ are both invariant under time reversal, as expected. When the symmetry between the kicks $\alpha_2 \neq \pm \alpha_1$, the difference between $\hat{U}^{k_2,k_1} = \hat{U}^{k_2} \hat{U}^{k_1}$ and $\hat{U}^{k_1,k_2} = \hat{U}^{k_1} \hat{U}^{k_2}$ can be observed [15].

6. Numerical calculations of time ordering

As an illustrative specific example, we present the results of numerical calculations for $2s \to 2p$ transitions in atomic hydrogen caused by a Gaussian pulse of finite width $\tau$. The occupation probabilities of the $2s$ and $2p$ states are evaluated by integrating the two-state equations using a standard fourth order Runge-Kutta method. This enables us to verify the validity of our analytic solutions for kicked qubits in the limit $\tau \to 0$ and also to consider the effects of finite pulse width. In this system, the unperturbed level splitting is the $2s - 2p$ shift, $\Delta E = E_{2p} - E_{2s} = 4.37 \times 10^{-6}$ eV. The corresponding time scale is the Rabi time, $T_{\Delta E} = 2\pi/\Delta E = 972$ ps, which gives the period of oscillation between the states. In our numerical calculations we use for convenience a Gaussian pulse of the form $V(t) = (\alpha/\sqrt{\pi \tau}) e^{-(t-t_k)^2/\tau^2}$. The two-state coupled equations for the amplitudes $a_1$ and $a_2$ of Eq. (12) take the form implied by Eq. (23),

$$i \dot{a}_1 = -\frac{1}{2} \Delta E a_1 + \frac{\alpha}{\sqrt{\pi \tau}} e^{-(t-t_k)^2/\tau^2} a_2$$

$$i \dot{a}_2 = \frac{1}{2} \Delta E a_2 + \frac{\alpha}{\sqrt{\pi \tau}} e^{-(t-t_k)^2/\tau^2} a_1 .$$

Here the pulse is applied at $t_k = 150$ ps and we have chosen $\alpha = \pi/2$ so that in the limit of a perfect kick all of the population will be transferred from the $2s$ to the $2p$ state after $t = t_k$. In Fig. 4 the resulting transition probability is shown as a function of pulse width $\tau$ and as a function of observation time $T_f$ for $T_f > t_k$.

In the Schrödinger picture there are very large differences between the transition probabilities with and without time ordering, $P_2(T_f)$ and $P_2^{(0)}(T_f)$, even for an ideal kick. This occurs because the energy splitting $\Delta E$ is non-zero, and for $T_f > \alpha/\Delta E = \alpha T_{\Delta E}/2\pi$, the average potential $\langle V \rangle = \alpha/T_f$ becomes smaller than the energy splitting $\Delta E$. Thus, for a given pulse, the influence of the potential necessarily decreases at large $T_f$, and any transfer probability becomes exponentially small. In effect, the free propagation before and after the pulse diminishes the effect of the pulse itself in the Schrödinger picture, when time ordering is removed. This behavior contrasts with the intermediate picture result, where $P_2^{(0)}(T_f)$ depends on $\Delta E \tau/2$ but not on $T_f$, as seen on the left side of Fig. 4. The contrast is also evident on the right hand side of Fig. 4, where, after the pulse has died off, the value of $P_2^{(0)}(T_f)$ decays with increasing $T_f$, while $P_2^{(0)}(T_f)$ approaches a constant.

C. Networks of qubits

To our knowledge relatively little has been understood analytically for systems of coupled qubits. Understanding coupling between qubits is a well recognized challenge in the field of quantum computing [22]. In our view, developing a realistic analytic model for two coupled qubits could provide a useful and instructive example in the fields of quantum computing, quantum information and coherent control. Unitary evolution operators acting on a system of non-interacting qubits formally belong to the $SU(2) \times SU(2) \times \cdots \times SU(2)$ unitary group, which is simply the set
FIG. 4: Target state probability as a function of the pulse width \( \tau \) (on the left) and of observation time \( T_f \) (on the right). Here \( T_{\Delta E} = \frac{2\pi}{\Delta E} \) is the Rabi time for oscillations between the states, where \( \Delta E = E_{2p} - E_{2s} \). The heavy line denotes probability including time ordering, the dashed line denotes the probability in the Schrödinger picture without time ordering, and the dotted line represents the probability in the intermediate picture without time ordering. On the right, the lines begin at the midpoint of the pulse, \( T_f = t_k \). The Schrödinger results damp out for large \( T_f \) as explained in the text.

of all local qubit operations. This set is a subgroup of the full \( SU(2^N) \) dynamic group of \( N \) coupled qubits. The \( SU(4) \) dynamic group of two interacting qubits plays a fundamental role in the analysis of multi-qubit dynamics since any operator (i.e., quantum gate) from the full \( SU(2^N) \) group can be factorized as a product of \( SU(4) \) two-qubit gates. In connection with this property, it is useful to note that any 4-level quantum system can be used to encode a tensor-product four-dimensional Hilbert space of two qubits. The specific form of such encoding is completely determined by fixing one point on the orbit of the maximal \( SU(2) \times SU(2) \) subgroup of \( SU(4) \). Next, the 3-qubit system (e.g., a carrier space for GHZ multiparticle entangled states) can be similarly encoded in an arbitrary 8-level quantum system. This requires two steps: identifying the maximal \( SU(4) \times SU(2) \) subgroup of \( SU(8) \), and then adjusting the \( SU(2) \times SU(2) \) subgroup of the resulting \( SU(4) \) group.

It would be useful to have a well developed theory for time correlations between interacting qubits – one that clarifies how the time dependence of a field acting on one qubit impacts the time evolution of another qubit, for example in switching. However, defining the independent time approximation for \( N \)-qubit systems in a useful way is a challenge. Although working with \( n = 2^N \) degenerate states is doable in principle (setting aside the problem of solving an \( n \)-th order equation for \( n > 4 \)), this does not always give the NTO approximation, seemingly a prerequisite for the ITA approximation needed to define time correlations.

Furthermore, in many cases sequencing of external interactions can be problematic. For example, if \( \hat{V}(t') = \hat{V}_A(t') + \hat{V}_B(t') \) for two particles or qubits \( A \) and \( B \), and \( \hat{U} = Te^{-i \int \left( \hat{V}_A(t') + \hat{V}_B(t') \right) dt'} \), then the ITA is not \( \hat{U}_A \cdot \hat{U}_B \) in
general. The ITA is given by $\hat{U}_A \cdot \hat{U}_B = \hat{U}_B \cdot \hat{U}_A$ if $[\hat{V}_A(t), \hat{V}_B(t')] = 0$, but in that case all time correlations vanish and the ITA is exact. Also, it difficult for us to envision how one may satisfy $[\hat{U}_A(t), \hat{U}_B(t')] = 0$ with $[\hat{U}_J(t), \hat{U}_J(t')] \neq 0$ for $J = A, B$. That is, in what situations can one eliminate inter-particle time correlations while retaining time ordering for individual particles? If all commutators terms vanish, then time correlation effects and time ordering for individual particles are both absent. Similarly, time correlation effects disappear if all parts of a composite $\hat{V}(t)$ are replaced by the time averaged value $\overline{V}$, but this again is equivalent to eliminating all time ordering, even within single-particle evolution.

V. SUMMARY

In solutions of the time dependent Schrödinger equation there are only two sources of time dependence, namely the Dyson time ordering operator $T$ and the explicit time dependence of the interaction $V(t)$. This simplifies the study of how time dependence may influence the evolution of $N$-body quantum systems. The causal-like constraint of time ordering between fields acting on different particles can cause time correlation between the particles. That is, the time dependence of a field acting on one particle can influence the evolution of other particles. Correlation is traditionally studied by defining an uncorrelated limit. In the case of time correlation, we have called this the independent time approximation (ITA), and have pointed out similarities (e.g. in Table I) to the widely used and practical independent particle approximation (IPA) that eliminates spatial correlations between particles. Similarly, the limit of no time ordering (NTO) can be defined by eliminating all time ordering constraints, for fields acting on the same particle or on different particles. Thus, the ITA may be viewed as the NTO applied to cross terms only.

The ITA or NTO limit may be reached in several ways, but the most general seems to be to define a mean time-averaged coupling interaction (as is done in the IPA). Time ordering effects have been observed in weakly perturbed atomic collisions. One may also consider strongly perturbed systems of coupled qubits. However, relatively little has been done on this problem, which is a key problem in quantum computing. In this paper we considered the effect of time ordering on strongly perturbed single qubits. To do this, we focused on qubits subjected to fast strong external pulses called kicks, where useful analytic expressions for observable transition probabilities may be obtained. What we have found (but not discussed here) is that the NTO may be easier to implement than the ITA. This means that the ITA might be computationally awkward in general. We have also demonstrated that the NTO (and consequently the ITA) are dependent on the representation used. There is evidence that the intermediate representation is preferred. This suggests that one could find gauge dependence in specific NTO and ITA terms if MBPT is used, and raises a question [4][20] about the physical meaning of time ordering and time correlation.

In summary, the methods we have developed in this paper are intended to probe the nature of how time works in quantum N-body systems. The focal point in our approach is the influence of the constraint imposed by time ordering, which can lead to time correlation between different parts of the system.
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