Altruistic Contents of Quantum Prisoner’s Dilemma

Taksu Cheon
Laboratory of Physics, Kochi University of Technology, Tosa Yamada, Kochi 782-8502, Japan

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We examine the classical contents of quantum games. It is shown that a quantum strategy can be interpreted as a classical strategy with effective density-dependent game matrices composed of transposed matrix elements. In particular, successful quantum strategies in dilemma games are interpreted in terms of a symmetrized game matrix that corresponds to an altruistic game plan.

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Information processing with quantum mechanical bits, or qubits, has become a major area of research activities. Among them, the quantum game \[ \{ |0\rangle, |1\rangle \} \] occupies a somewhat peculiar position. This is because usual applications of game theory – sociology, economics, evolutionary biology \[ \{ |0\rangle, |1\rangle \} \] – are so far removed from the realm of quantum mechanics. It would be fair to say that quantum game theory has mostly been regarded as, and indeed pursued as, an object of pure intellectual curiosity. It is then somewhat puzzling to see quantum strategies “succeed” in some of the long-standing problems of classical game theory, whose solutions usually call for rather involved concepts and techniques. We cannot help wondering whether it is just a coincidence. Could it be that the quantum game theory amounts to an effective shorthand for advanced classical solutions?

In this article, we set out to give an answer to such questions by examining in detail the working of quantum strategies in a two-by-two game. When the quantum correlations are present, the classical separation of strategies of two players is in general not guaranteed. Also, the correlations result in purely quantum component in the payoff functions. However, it is pointed out that there is a particular choice of quantum strategies that ensures both classical separability and intactness of classical payoff functions. We show that this particular set of strategies, which we call pseudoclassical, can be reinterpreted as a classical game played with a mixture of altruistic game plans \[ \{ |0\rangle, |1\rangle \} \]. We further show, through numerical examples on the well-known case of the prisoner’s dilemma, that the altruism effectively incorporated in the pseudoclassical treatment is indeed at the core of the success of quantum strategies.

We consider a symmetric two strategy game, described by a Hermitian operator \( Q \) that is to be specified later, and played by two players with quantum strategies \( |\alpha\rangle \) and \( |\beta\rangle \), both of which are linear combinations of basis strategy vectors \( |0\rangle \) and \( |1\rangle \). We set

\[
|\alpha\beta\rangle = U_{\alpha\beta} |00\rangle
\]

The unitary rotation matrix \( U_{\alpha\beta} \) is given by \( U_{\alpha\beta} = U_\alpha \otimes U_\beta \) where \( U_\alpha \) and \( U_\beta \) act on the qubits representing the first and the second players respectively. We adopt the notations

\[
U_\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 \\ -\alpha_1^* & \alpha_0^* \end{pmatrix}, \quad U_\beta = \begin{pmatrix} \beta_0 & \beta_1 \\ -\beta_1^* & \beta_0^* \end{pmatrix},
\]

with complex numbers satisfying the conditions \( |\alpha_0|^2 + |\alpha_1|^2 = 1 \) and \( |\beta_0|^2 + |\beta_1|^2 = 1 \). The payoff of the game to the first player is given by

\[
\Pi(\alpha, \beta) = \langle \alpha\beta | Q | \alpha\beta \rangle,
\]

where the quantum game operator \( Q \) is set to be diagonal with respect to basis qubits of both players

\[
\langle i'j'|Q|ij\rangle = \delta_{i',i}\delta_{j',j} A_{ij},
\]

where \( A_{ij} \) is the classical game matrix. With the definitions \( \alpha_i = \sqrt{x_i} e^{i\xi_i} \) and \( \beta_i = \sqrt{y_i} e^{i\upsilon_i} \), the payoff \( \Pi \) depends only on the absolute values of \( \alpha_i \) and \( \beta_i \), and takes the form

\[
\Pi(x, y) = \sum_{i,j} x_i A_{ij} y_j
\]

which is identical to the payoff of the purely classical mixed strategies that are described by the strategy density vectors \( x = (x_0, x_1) \) and \( y = (y_0, y_1) \) with conditions \( x_0 + x_1 = 0 \) and \( y_0 + y_1 = 1 \). Because of the symmetry of the system, the payoff for the player 2 is given by the conjugate payoff

\[
\Pi^\dagger(x, y) = \Pi(y, x).
\]

Clearly, we have \( \Pi(x, x) = \Pi^\dagger(x, x) \), which simply means both players with same strategy earn the same payoff.

A mixed Nash equilibrium \( x^* \) is defined by the condition

\[
\partial_x \Pi(x, y)|_{x=y=x^*,} = 0,
\]

with the implied assumption that it gives local maximum, not minimum. This strategy \( x^* \) is the best response of a player against an opponent playing his/her best response. Since our game is symmetric, both players should play the same strategy \( x^* \) to obtain the expected payoff \( \Pi(x^*, x^*) \). There are, however, special cases where there is no solution to \( x \) within the valid range \( 0 \leq x_1 \leq 1 \),
in which case, the best response becomes the pure Nash equilibrium, \( x^* = (0, 1) \) or \((1, 0)\), depending on the sign of \( \partial_x \Pi(x, y) \).

The Nash equilibrium, obtained as the result of individual pursuit of optimality, is not always an ideal outcome for both players. In fact, if we consider \( \Pi(x, x) \) as a function of strategy density \( x \), and seek the value of \( x \) that maximizes this function by

\[
\partial_x \Pi(x, x) \bigg|_{x=x^*} = 0,
\]

the strategy vector \( x^* \) is Pareto efficient, or both players are best off, assuming this extremum is indeed a maximum. If a \( x^* \) is equal to \( x^0 \) the outcome of the game is described as Pareto efficient Nash equilibrium. Considering the relation

\[
\partial_x \Pi(x, x) = \partial_{x_i} \{ \Pi(x, y) + \Pi^\dagger(x, y) \} \bigg|_{x=y},
\]

we see that a game with the self-adjoint payoff has a Pareto efficient Nash equilibrium, namely,

\[
x^* = x^0 \quad \text{if} \quad \Pi^\dagger(x, y) = \Pi(x, y),
\]

because simultaneous constraints (7) and (8) hold for this case. When, on the other hand, the Nash equilibrium \( x^* \) is remote from the Pareto efficient value \( x^0 \), the outcome of the game is less than optimal for both players. This is the situation where the term dilemma is invoked, the representative of which is the well-known Prisoner’s dilemma, that has been the subject of much studies. In fact, we might even say that the search of the dynamics that brings Pareto efficient outcome to dilemma games constitute the bulk part of the recent works in evolutionary game theory.

Quantum strategies, which has been offered as an exotic alternative to the solution of dilemma games, deviate from the classical strategies with the introduction of quantum correlations. Following the scheme of Eisert, Wilkens, and Lewenstein, we define

\[
J_\gamma = e^{-i\gamma/2} \otimes \sigma_2
\]

(11)

to obtain a correlated state

\[
J_\gamma |00\rangle = \cos \frac{\gamma}{2} |00\rangle + i \sin \frac{\gamma}{2} |11\rangle,
\]

(12)

and construct a correlated quantum strategy vector

\[
|\Psi_{\alpha, \beta}(\gamma)\rangle = J_\gamma U_{\alpha, \beta} J^\dagger_\gamma |00\rangle.
\]

(13)

The payoff to the first player now becomes

\[
\Pi_\gamma(\alpha, \beta) = \langle \Psi_{\alpha, \beta}(\gamma) | Q | \Psi_{\alpha, \beta}(\gamma) \rangle.
\]

(14)

With the split of \( U_{\alpha, \beta} \) into real and imaginary components \( U_{\alpha, \beta} = R_{\alpha, \beta} + i I_{\alpha, \beta} \), we obtain

\[
\Pi_\gamma(\alpha, \beta) = \langle 00 | (R_{\alpha, \beta} Q R_{\alpha, \beta} + I_{\alpha, \beta} Q I_{\alpha, \beta}) | 00 \rangle
\]

\[
- \sin^2 \gamma \left\{ \langle 00 | I_{\alpha, \beta} Q I_{\alpha, \beta} | 00 \rangle - \langle 11 | I_{\alpha, \beta} Q I_{\alpha, \beta} | 11 \rangle \right\}
\]

\[
- 2 \sin \gamma \langle 11 | I_{\alpha, \beta} Q R_{\alpha, \beta} | 00 \rangle.
\]

(15)

We write the payoff for the first player in an analogous form to the classical case, using the strategy densities \( x \) and \( y \) of the first and the second players;

\[
\Pi_\gamma(\alpha, \beta) = \sum_{i,j} x_i B_{ij}(\gamma) y_j.
\]

(16)

The effective payoff matrix \( B_{ij}(\gamma) \) is written as

\[
B_{ij}(\gamma) = A_{ij} + B_{ij}^{exc}(\gamma) + B_{ij}^{cor}(\gamma),
\]

(17)

where the “exchange” contribution \( B_{ij}^{exc}(\gamma) \) comming from these second terms of (15) is given by

\[
B_{ij}^{exc}(\gamma) = - \sin^2 \gamma \sin^2 (\xi_i + \nu_j) (A_{ij} - A_{ij}),
\]

(18)

and the “correlation” contribution \( B_{ij}^{cor}(\gamma) \) coming from the last term is given by

\[
B_{ij}^{cor}(\gamma) = (-)^{i+j} 2 \sin \gamma \sin (\xi_i + \nu_j) \cos (\xi_i + \nu_j)
\]

\[
\times \sqrt{\frac{x_i y_j}{x_i y_j}} A_{ij}.
\]

(19)

Here the bars on top of the indices stand for the logical complementarity \( \bar{0} = 1 \) and \( \bar{1} = 0 \). The correlation term \( B_{ij}^{cor} \) has a singular strategy-density dependence. In this form, it is obvious that playing a given game specified by the matrix \( \{ A_{ij} \} \) with quantum strategy is formally equivalent to playing a related, but different game specified by the matrix \( \{ B_{ij}(\gamma) \} \) with purely classical mixed strategy.

If we take the “quantum strategy” at its face value, both amplitudes and phases of \( \alpha \) should be optimized to increase the payoff \( \Pi_\gamma(\alpha, \beta) \). Same apply for \( \beta \) with \( \Pi_\gamma(\beta, \alpha) \), and \( \alpha \) and \( \beta \) would settle down at the common value corresponding to the full quantum Nash equilibrium. However, such assumption would require a system consisting of quantum agents making choice of strategies either with intelligence, or under competitive evolutionary pressure. We might argue that, at this point, such approach is rather far fetched for real life ecosystems, apart from artificial experimental realizations with quantum computational circuits. In this article, with the purpose of analyzing the workings of quantum strategies, we regard only the amplitudes \( x_i \) and \( y_j \) as the optimizing variables and regard the angles \( \xi_i = \nu_i \) as external parameters, their equivalence being a natural reflection of the symmetry of players at the final outcome.

Of all possible quantum strategies, there are four subsets under which contributions from \( B_{ij}^{cor} \) disappears.

First is the trivial classical limit \( (\xi_0 = 0, \xi_1 = 0) \) or \( (\xi_0 = \frac{\pi}{2}, \xi_1 = \frac{\pi}{2}) \), at which we have \( B_{ij}(\gamma) = A_{ij} \).

The second, more interesting case is what we call pseudoclassical limit, \( (\xi_0 = 0, \xi_1 = \frac{\pi}{2}) \) or \( (\xi_0 = \frac{\pi}{2}, \xi_1 = 0) \), with which we have

\[
B_{ij}(\gamma) = \cos^2 \gamma A_{ij} + \sin^2 \gamma A_{ij}.
\]

(20)

For this case, the quantum payoff \( \Pi_\gamma(x, y) \) calculated from (20) is readily given by

\[
\Pi_\gamma(x, y) = \cos^2 \gamma \Pi(x, y) + \sin^2 \gamma \Pi^\dagger(x, y).
\]

(21)
A simple classical interpretation of the pseudoclassical case exist in terms of altruism; that is, a choice of \( \gamma \) gives a specific game plan, or a style in the selection of strategies of a player, that incorporate the interest of other player. For example, adopting \( \Pi_\varnothing(x, y) = \Pi(x, y) \) as one’s payoff amounts to exchanging the role of two players, thus identifying the payoff of the other party as one’s own, or playing with the totally altruistic game plan. On the other hand, adopting \( \Pi_0(x, y) = \Pi(x, y) \) means just stick to one’s own payoff as usual, or playing with selfish game plan. Anything in between, \( (0 < \gamma < \pi/2) \), gives a game plan which pursue varying mixture of selfish and altruistic payoff maximization.

A notable consequence of (24) is that, for a given common strategy for both players, the payoff for quantum game is identical to that of classical game:

\[
\Pi_\varnothing(x, x) = \Pi(x, x).
\]  

The Nash equilibrium of quantum game play \( x_\gamma^* \) is now given by

\[
\partial_x \Pi_\gamma(x, y)|_{x=y=x_\gamma^*} = 0,
\]  

and both players end up obtaining the payoff \( \Pi(x_\gamma^*, x_\gamma^*) \).

Among various game plans with different \( \gamma \) values, the one with \( \gamma = \pi/2 \) occupies a spacial place. Because of the equal mixture of “selfish” \( \Pi(x, y) \) and “altruistic” \( \Pi^\gamma(x, y) \), we have self-adjointness for the payoff, \( \Pi_\varnothing(x, y) = \Pi_{\varnothing}^\gamma(x, y) \). Then, from (11) we obtain a Pareto efficient Nash equilibrium for \( \gamma = \pi/4 \), namely

\[
x_\gamma^* = x^0,
\]  

which is the primary result of this article. Among the pseudoclassical game plans with a given \( \gamma \), therefore, \( \gamma = \pi/4 \) gives the optimal results for both players, and either \( \gamma = 0 \) or \( \gamma = \pi/2 \) gives the less favorable results.

Thus, within pseudoclassical limit, thanks to the crucial identity (24), we are able to interpret the the quantum strategies as an effective way to incorporate the altruistic game plan, which can help improve the outcome of the game toward the optimal result.

There are two more cases for which the strategy density dependent term \( B_{ij}^{corr} \) drops out. One of them is the case of \( (\xi_0 = \frac{3\pi}{4}, \xi_1 = \frac{\pi}{4}) \) or \( (\xi_0 = \frac{\pi}{4}, \xi_1 = \frac{3\pi}{4}) \), with which we have

\[
\begin{align*}
B_{ii}(\gamma) &= \cos^2 \gamma A_{ii} + \sin^2 \gamma A_{aii}, \\
B_{ii}(\gamma) &= \cos^2 \gamma A_{aii} + \sin^2 \gamma A_{i}. 
\end{align*}
\]  

Another case is \( (\xi_0 = \frac{\pi}{4}, \xi_1 = \frac{3\pi}{4}) \) or \( (\xi_0 = \frac{3\pi}{4}, \xi_1 = \frac{\pi}{4}) \), with which we have

\[
\begin{align*}
B_{ii}(\gamma) &= \cos^2 \gamma A_{aii} + \sin^2 \gamma A_{i}, \\
B_{i}(\gamma) &= A_{i}. 
\end{align*}
\]

Although these appear analogous to the altruistic pseudoclassical case, there is no immediate interpretation in terms of the classical game, because of the existence of exchanged components in diagonal matrix elements. Moreover, there is no such relation as (22) in neither of these cases, and we have bona fide quantum contribution to the payoff \( \Pi_\gamma(x, x) - \Pi(x, x) \), which, by its nature, is classically non-interpretable.

As such, these cases can be thought of as belonging to the more general category of generic quantum cases that are given by arbitrary values for \( (\xi_0, \xi_1) \). We would then calculate Nash equilibrium for a given set of values of \( (\xi_0, \xi_1) \) from the full payoff \( \Pi_\gamma(\alpha, \beta) \), in a fashion analogous to (24). Other than possible numerical assessments, figuring out the meaning of quantum payoff for generic quantum cases is beyond the scope of the current work.

In fact, it is unlikely that it is readily interpretable, until we have sound scheme to place the game theory into the information theoretical framework, which we are still lacking. Conversely, the true role of the quantum game theory may be to lay foundation for the information theoretic approaches of the game theory.

We illustrate our arguments with a numerical example on the Prisoner’s dilemma. With positive real numbers \( a < b < c \), the classical game matrix is given by

\[
\{ A_{ij} \} = \begin{pmatrix} b & 0 \\ c & a \end{pmatrix}.
\]

The Nash equilibrium \( x^* = (1 - x_1^*, x_1^*) \) and its payoff
For separable, but fully quantum case \([25]\), the corresponding results are obtained by replacing both \(a\) and \(b\) by their average \((a + b)/2\) in the formulae. The results are rather similar to the pseudoclassical case because of the existence of altruistic exchange components in \([24]\). For another separable, but fully quantum case \([26]\), there is no improvement over the classical game play.

In FIG. 2, We show numerical example of generic quantum cases of \(\xi\) in the neighborhood of pseudoclassical case \(\xi_0 = 0, \xi_1 = \pi/2\). Both classical payoff \(\Pi\) and full quantum payoff \(\Pi_\gamma\) are shown. When we look at the classical payoffs, with generic “quantum” choice of strategies, \(\gamma\)–dependence is changed from the pseudoclassical case. However, the essential ingredient of the successful strategy at high value of \(\gamma\) – mixture of altruism – is still intact. The story is similar with full quantum payoff functions. Although the difference between the quantum payoff and the classical payoffs are non negligible, the overall feature does not change very much. While the results with only particular choices of angle parameters are shown here, we note that these are rather representative ones whose characteristics are shared by the results with other generic parameter values. We also add that in the fuller approach of “complete quantum strategy” implementation on the same Prisoner’s dilemma including the optimization of angles \([2]\), the stable quantum Nash equilibrium is numerically found to coincide precisely with the pseudoclassical limit in our terminology.

In conclusion, we have examined the source of the “success” of quantum strategies in dilemma games. We have identified altruism, which is expressed in the symmetrization of the classical game matrix, as the main cause. In the process, the pseudoclassical limit of quantum strategies with its intriguing characteristics is uncovered.

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