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The Kodaira dimension of the moduli space of Prym varieties

Received May 13, 2008 and in revised form September 15, 2008

Abstract. We study the enumerative geometry of the moduli space $R_g$ of Prym varieties of dimension $g - 1$. Our main result is that the compactification of $R_g$ is of general type as soon as $g > 13$ and $g$ is different from 15. We achieve this by computing the class of two types of cycles on $R_g$: one defined in terms of Koszul cohomology of Prym curves, the other defined in terms of Raynaud theta divisors associated to certain vector bundles on curves. We formulate a Prym–Green conjecture on syzygies of Prym-canonical curves. We also perform a detailed study of the singularities of the Prym moduli space, and show that for $g \geq 4$, pluricanonical forms extend to any desingularization of the moduli space.

Prym varieties provide a correspondence between the moduli spaces of curves and abelian varieties $M_g$ and $A_{g-1}$, via the Prym map $P_g : R_g \to A_{g-1}$ from the moduli space $R_g$ parameterizing pairs $[C, \eta]$, where $[C] \in M_g$ is a smooth curve and $\eta \in \text{Pic}^0(C)[2]$ is a torsion point of order 2. When $g \leq 6$ the Prym map is dominant and $R_g$ can be used directly to determine the birational type of $A_{g-1}$. It is known that $R_g$ is rational for $g = 2, 3, 4$ (see [Dol] and references therein and [Ca] for the case of genus 4) and unirational for $g = 5$ (cf. [IGS] and [V2]). The situation in genus 6 is strikingly beautiful because $P_6 : R_6 \to A_5$ is equidimensional (precisely $\dim(R_6) = \dim(A_5) = 15$). Donagi and Smith showed that $P_6$ is generically finite of degree 27 (cf. [DS]) and the monodromy group equals the Weyl group $WE_6$ describing the incidence correspondence of the 27 lines on a cubic surface (cf. [D1]). There are three different proofs that $R_6$ is unirational (cf. [DI], [MM], [Y1]). Verra has very recently announced a proof of the unirationality of $R_7$ (see also Theorem 0.8 for a weaker result). The Prym map $P_g$ is generically injective for $g \geq 7$ (cf. [FS]), although never injective. In this range, we may regard $R_g$ as a partial desingularization of the moduli space $P_g(R_g) \subset A_{g-1}$ of Prym varieties of dimension $g - 1$.

The scheme $R_g$ admits a suitable modular compactification $\overline{R}_g$, which is isomorphic to (1) the coarse moduli space of the stack $\overline{M}_g(\mathcal{B}Z_2)$ of Beauville admissible double covers (cf. [B], [ACV]) and (2) the coarse moduli space of the stack of Prym curves (cf. [BCF]). The forgetful map $\pi : R_g \to M_g$ extends to a finite map $\pi : \overline{R}_g \to \overline{M}_g$.

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The aim of this paper is to initiate a study of the enumerative and global geometry of \( \overline{M}_g \), in particular to determine its Kodaira dimension. The main result of the paper is the following:

**Theorem 0.1.** The moduli space of Prym varieties \( \overline{R}_g \) is of general type for \( g > 13 \) and \( g \neq 15 \). The Kodaira dimension of \( \overline{R}_{15} \) is at least 1.

We point out in Remark 2.9 that the existence of an effective divisor \( D \in \text{Eff}(\overline{M}_{15}) \) of slope \( s(D) < 6 + 12/(g+1) = 27/4 \) (that is, violating the Harris–Morrison Slope Conjecture on \( \overline{M}_{15} \)), would imply that \( \overline{R}_{15} \) is of general type. There are known examples of divisors \( D \in \text{Eff}(\overline{M}_g) \) satisfying \( s(D) < 6 + 12/(g+1) \) for every genus of the form \( g = s(2s + si + i + 1) \) with \( s \geq 2 \) and \( i \geq 0 \) (cf. [F1], [F2]). No such examples have been found yet on \( \overline{M}_{15} \), though they are certainly expected to exist.

The normal variety \( \overline{R}_g \) has finite quotient singularities and an important part of the proof is concerned with showing that pluricanonical forms defined on the smooth part \( \overline{R}_g^{\text{reg}} \subset \overline{R}_g \) can be lifted to any resolution of singularities \( \overline{R}_g \to \overline{R}_g \), that is, we have isomorphisms

\[
H^0(\overline{R}_g^{\text{reg}}, K_{\overline{R}_g}^{\otimes l}) \cong H^0(\overline{R}_g, K_{\overline{R}_g}^{\otimes l})
\]

for \( l \geq 0 \). This is achieved in the last section of the paper. The locus of non-canonical singularities in \( \overline{R}_g \) is also explicitly described: A Prym curve \( [X, \eta, \beta] \in \overline{R}_g \) is a non-canonical singularity if and only if \( X \) has an elliptic tail \( C \) with \( \text{Aut}(C) = \mathbb{Z}_6 \) such that the line bundle \( \eta_C \in \text{Pic}^0(C)[2] \) is trivial (cf. Theorem 6.7).

We outline the strategy to prove that \( \overline{R}_g \) is of general type for given \( g \). If \( \lambda = \pi^*(\lambda) \in \text{Pic}(\overline{R}_g) \) is the pull-back of the Hodge class and \( \delta_0', \delta_0'', \delta_{g-1}^{\text{ram}} \in \text{Pic}(\overline{R}_g) \) and \( \delta_i, \delta_{g-i}, \delta_{i:g-i} \in \text{Pic}(\overline{R}_g) \) for \( 1 \leq i \leq [g/2] \) are boundary divisor classes such that

\[
\pi^*(\delta_0) = \delta_0' + \delta_0'' + 2\delta_{g-1}^{\text{ram}} \quad \text{and} \quad \pi^*(\delta_i) = \delta_i + \delta_{g-i} + \delta_{i:g-i} \quad \text{for} \quad 1 \leq i \leq [g/2]
\]

(see Section 2 for a precise definition of these classes), then one has the formula

\[
K_{\overline{R}_g} \equiv 13\lambda - 2(\delta_0' + \delta_0'') - 3\delta_{g-1}^{\text{ram}} - 2 \sum_{i=1}^{[g/2]} (\delta_i + \delta_{g-i} + \delta_{i:g-i}) - (\delta_1 + \delta_{g-1} + \delta_{1:g-1}).
\]

We show that this class is big by explicitly constructing effective divisors \( D \) on \( \overline{R}_g \) such that one can write \( K_{\overline{R}_g} \equiv \alpha \cdot \lambda + \beta \cdot D + \text{[effective combination of boundary classes]} \) for certain \( \alpha, \beta \in \mathbb{Q}_{>0} \) (see (2) for the inequalities the coefficients of such \( D \) must satisfy).

We carry out an enumerative study of divisors on \( \overline{R}_g \) defined in terms of pairs \( [C, \eta] \) such that the 2-torsion point \( \eta \in \text{Pic}^0(C) \) is transversal with respect to the theta divisors associated to certain stable vector bundles on \( C \). We fix integers \( k \geq 2 \) and \( b \geq 0 \) and then define the integers

\[
i := kb + k - b - 2, \quad r := kb + k - 2, \quad g := ik + 1, \quad d := rk.
\]

The Brill–Noether number \( \rho(g, r, d) := g - (r + 1)(g - d + r) = 0 \) and a general \( [C] \in \mathcal{M}_g \) carries a finite number of line bundles \( L \in W^r_d(C) \). For each such line
bundle \( L \), if \( Q_L \) denotes the dual of the Lazarsfeld bundle defined by the exact sequence (see [3])

\[
0 \to Q_L^\vee \to H^0(C, L) \otimes \mathcal{O}_C \to L \to 0,
\]

we compute that \( \mu(Q_L) = d/r = k \) and then \( \mu(\bigwedge^i Q_L) = ik = g - 1 \). In these circumstances we define the Raynaud divisor (degeneration locus of virtual codimension 1)

\[
\Theta_{\bigwedge^i Q_L} := \{ \eta \in \text{Pic}^0(C) : H^0(C, \bigwedge^i Q_L \otimes \eta) \neq 0 \}.
\]

This is a virtual divisor inside \( \text{Pic}^0(C) \), in the sense that either \( \Theta_{\bigwedge^i Q_L} = \text{Pic}^0(C) \) or else \( \Theta_{\bigwedge^i Q_L} \) is a divisor on \( \text{Pic}^0(C) \) belonging to the linear system \( |\bigwedge^i \eta| \) (cf. [3]). We study the relative position of \( \eta \) with respect to \( \Theta_{\bigwedge^i Q_L} \) and introduce the following locus on \( \overline{R}_g \):

\[
D_{g,k} := \{ [C, \eta] \in R_g : \exists L \in \mathcal{W}_g^i(C) \text{ such that } \eta \in \Theta_{\bigwedge^i Q_L} \}.
\]

When \( k = 2, i = b \), then \( g = 2i + 1, d = 2g - 2 \) and \( D_{2i+1;2} \) has a new incarnation using the proof of the Minimal Resolution Conjecture [FMP]. In this case, \( L = K_C \) (a genus \( g \) curve has only one \( \mathcal{W}_{2g-2}^1 \)) and [FMP] gives an identification of cycles

\[
\Theta_{\bigwedge^i Q_K} = C_i - C_i \subset \text{Pic}^0(C),
\]

where the right-hand side stands for the \( i \)-th difference variety of \( C \).

We prove in Section 2 that \( D_{g,k} \) is an effective divisor on \( \overline{R}_g \). By specialization to the \( k \)-gonal locus \( \mathcal{M}_{g,k} \subset \mathcal{M}_g \), we show that for a generic \([C, \eta] \in \overline{R}_g \) the vanishing \( H^0(C, \bigwedge^i Q_L \otimes \eta) = 0 \) holds for every \( L \in \mathcal{W}_g^i(C) \) (Theorem 2.3). Then we extend the determinantal structure of \( D_{g,k} \) to a partial compactification of \( \overline{R}_g \), which enables us to compute the class of the compactification \( \overline{D}_{g,k} \). Precisely we construct two vector bundles \( E \) and \( F \) over a stack \( \overline{R}_g^0 \) which is a partial compactification of \( \overline{R}_g \), such that \( \text{rank}(E) = \text{rank}(F) \), together with a vector bundle homomorphism \( \phi : E \to F \) such that \( Z_1(\phi) \cap \overline{R}_g = D_{g,k} \). Then we explicitly determine the class \( c_1(F - E) \in \Lambda^1(\overline{R}_g^0) \) (Theorem 2.8). The cases of interest for determining the Kodaira dimension of \( \overline{R}_g \) are when \( k = 2, 3 \), for which we obtain the following results:

**Theorem 0.2.** The closure of the divisor \( D_{2i+1;2} = \{ [C, \eta] \in \mathcal{R}_{2i+1} : h^0(C, \bigwedge^i Q_{K_C} \otimes \eta) \geq 1 \} \) inside \( \overline{\mathcal{R}}_{2i+1} \) has class given by the following formula in \( \text{Pic}(\overline{\mathcal{R}}_{2i+1}) \):

\[
\overline{D}_{2i+1;2} = \frac{1}{2i-1} \left( \frac{2i}{i} \right) \left( (3i+1)\lambda - \frac{i}{2} (\delta_0^2 + \delta_i^2) - \frac{2i+1}{4} \delta_{0}^{\text{ram}} - (3i-1)\delta_{x-1} - i(\delta_{1,2} + \delta_{1}) \cdots \right).
\]

To illustrate Theorem 0.2 in the simplest case, \( i = 1 \) hence \( g = 3 \), we write \( D_{3;2} = \{ [C, \eta] \in \mathcal{R}_3 : \eta = \mathcal{O}_C(x - y), x, y \in C \} \). The analysis carried out in Section 5 shows that the vector bundle morphism \( \phi : E \to F \) is generically non-degenerate along the
boundary divisors $\Delta^\prime_0$, $\Delta^\text{ram}_0 \subset \overline{R}_3$ and degenerate (with multiplicity 1) along the divisor $\Delta^\prime_0 \subset \overline{R}_3$ of Wirtinger covers. Theorem 0.2 reads

\[ \overline{D}_{3:2} \equiv c_1(\mathcal{F} - \mathcal{E}) - \delta''_0 \equiv 8\lambda - \delta^\prime_0 - 2\delta''_0 - \frac{3}{2}\delta^\text{ram}_0 - 6\delta_1 - 4\delta_2 - 2\delta_{1:2} \in \text{Pic}(\overline{R}_3), \]

and then $\pi_*\overline{D}_{3:2} = 56(9\lambda - \delta_0 - 3\delta_1) = 56\cdot \overline{M}_{3,2}^1 \in \text{Pic}(\overline{M}_3)$ (see Theorem 5.1). Theorem 0.2 is consistent with the following elementary fact (see e.g. [HF]): If $[\tilde{C} \to C] \in \mathcal{R}_3$ is an étale double cover, then $[\tilde{C}] \in \mathcal{M}_3$ is hyperelliptic if and only if $[C] \in \mathcal{M}_3$ is hyperelliptic and $\eta = \mathcal{O}_C(x - y)$, with $x, y \in C$ being Weierstrass points.

**Theorem 0.3.** For $g \geq 1$ and $r = 3b + 1$ the class of the divisor $D_{6b+4:3}$ on $\mathcal{R}_{6b+4}$ is given by

\[ D_{6b+4:3} \equiv \frac{4}{r} \left( 6b + 3 \right) \times (3b + 2)(b + 2)\lambda - \frac{3b^2 + 7b + 3}{6}(\delta^\prime_0 + \delta''_0) - \frac{24b^2 + 47b + 21}{24}\delta^\text{ram}_0 - \cdots \].

Theorem 2.8, 0.2 and 0.3 specify precisely the $\lambda$, $\delta^\prime_0$, $\delta''_0$ and $\delta^\text{ram}_0$ coefficients in the expansion of $[\overline{D}_{rk}]$. Good lower bounds for the remaining boundary coefficients of $[\overline{D}_{rk}]$ can be obtained using Proposition 1.9. The information contained in Theorems 0.2 and 0.3 is sufficient to finish the proof of Theorem 0.1 for odd genus $g = 2i + 1 \geq 15$.

When $b = 0$, hence $i = r = k - 2$, Theorem 2.8 has the following interpretation:

**Theorem 0.4.** Fix integers $k \geq 3$, $r = k - 2$ and $g = (k - 1)^2$. The locus

\[ D_{rk} := \{ [C, \eta] \in \mathcal{R}_g : \exists L \in \text{Pic}_g(k-2)(C) \text{ such that } H^0(C, L \otimes \eta) \neq 0 \} \]

is a divisor on $\mathcal{R}_g$. The class of its compactification inside $\overline{R}_g$ is given by the formula

\[ \overline{D}_{rk} \equiv g! \frac{1! \cdots (k-2)!}{(k-1)! \cdots (2k-3)!} \left( \frac{(k^4 - 4k^3 + 11k^2 - 14k + 2)}{12} \right) (\delta^\prime_0 + \delta''_0) - \frac{(k^2 - 2k + 3)(2k^2 - 4k + 1)}{12} \delta^\text{ram}_0 - \cdots \]

in $\text{Pic}(\overline{R}_g)$.

When $k = 3$ and $g = 4$, the divisor $D_{4:3}$ consists of Prym curves $[C, \eta] \in \mathcal{R}_4$ for which the plane Prym-canonical model $\iota : \mathcal{C} \to [\mathcal{K}_C \otimes \eta] \to \mathbb{P}^2$ has a triple point. Note that for a general $[C, \eta] \in \mathcal{R}_4$, $\iota(C)$ is a 6-nodal sextic. We can then verify the formula

\[ \pi_*\overline{D}_{4:3} = 60(34\lambda - 4\delta_0 - 14\delta_1 - 18\delta_2) = 60 \cdot \mathcal{G} \mathcal{P}^1_{4:3} \in \text{Pic}(\overline{M}_4), \]

where $\mathcal{G} \mathcal{P}^1_{4:3} \subset \overline{M}_4$ is the divisor of curves with a vanishing theta-null. This is consistent with the set-theoretic equality $\pi(D_{4:3}) = \mathcal{G} \mathcal{P}^1_{4:3}$, which can be easily established (see Theorem 5.4).
Another case which has a simple interpretation is when $b = 1$, $i = r - 1$, and then $g = (2k - 1)(k - 1)$, $d = 2k(k - 1)$. Since rank($Q_L$) = $r$ and det($Q_L$) = $L$, by duality we have $\bigwedge^i Q_L = M_{L_k} \otimes L$, hence points $[C, \eta] \in D_{(2k - 1)(k - 1); k}$ can be described purely in terms of multiplication maps of sections of line bundles on $C$:

**Theorem 0.6.** Fix integers $k \geq 2$ and $g = (2k - 1)(k - 1)$. The locus

$$D_{g,k} = \{ [C, \eta] \in \mathcal{R}_g : \exists L \in W_{2k}^{2k - 2}(C)$$

with $H^0(L) \otimes H^0(\eta) \rightarrow H^0(L \otimes \eta)$ not bijective

is a divisor on $\mathcal{R}_g$. The class of its compactification inside $\overline{\mathcal{R}}_g$ equals

$$\overline{\mathcal{D}}_{g,k} \equiv g! \frac{1!2! \cdots (k - 1)!}{3(2k^2 - 3k - 1)(2k - 1)!(2k)!! \cdots (3k - 2)!} \times \left(6(8k^5 - 36k^4 + 78k^3 - 95k^2 + 49k - 6)\lambda - (8k^5 - 36k^4 + 70k^3 - 71k^2 + 29k - 2)\delta_0' \delta_0''

- \frac{1}{2} (32k^5 - 144k^4 + 262k^3 - 245k^2 + 107k - 14)\delta_{\text{ram}} - \cdots \right).$$

The second class of (virtual) divisors is provided by Koszul divisors on $\overline{\mathcal{R}}_g$. For a pair $(C, L)$ consisting of a curve $[C] \in \mathcal{M}_g$ and a line bundle $L \in \text{Pic}(C)$, we denote by $K_{i,j}(C, L)$ its $(i, j)$-th Koszul cohomology group (cf. [L]). For a point $[C, \eta] \in \mathcal{R}_g$ we set $L := K_C \otimes \eta$ and we stratify $\mathcal{R}_g$ using the syzygies of the Prym-canonical curve $C \xrightarrow{[L]} \mathbb{P}^{g-2}$. We define the stratum

$$U_{g,i} := \{ [C, \eta] \in \mathcal{R}_g : K_{i,2}(C, K_C \otimes \eta) \neq \emptyset \},$$

that is, $U_{g,i}$ consists of those Prym curves $[C, \eta] \in \mathcal{R}_g$ for which the Prym-canonical model $C \xrightarrow{[L]} \mathbb{P}^{g-2}$ fails to satisfy the Green–Lazarsfeld property ($N_i$) in the sense of [CL], [L].

**Theorem 0.6.** There exist two vector bundles $\mathcal{G}_{i,2}$ and $\mathcal{H}_{i,2}$ of the same rank defined over a partial compactification $\overline{\mathcal{R}}_{2i+6}$ of the stack $\mathcal{R}_{2i+6}$, together with a morphism $\phi : \mathcal{H}_{i,2} \rightarrow \mathcal{G}_{i,2}$ such that

$$U_{2i+6,i} := \{ [C, \eta] \in \overline{\mathcal{R}}_{2i+6} : K_{i,2}(C, K_C \otimes \eta) \neq 0 \}$$

is the degeneracy locus of the map $\phi$. The virtual class of $[U_{2i+6,i}]$ is given by the formula

$$[U_{2i+6,i}]_{\text{virt}} = c_1(\mathcal{G}_{i,2} - \mathcal{H}_{i,2}) = \binom{2i + 2}{i} \left( \frac{3(2i + 7)}{i + 3} \lambda - \frac{3}{2} \delta_{\text{ram}} - (\delta_0' + \alpha \delta_0'') \right) - \cdots ,$$

where the constant $\alpha$ satisfies $\alpha \geq 1$. 


The compactification $\tilde{R}_g$ has the property that if $\tilde{R}_g \subset R_g$ denotes its coarse moduli space, then $\text{codim}(\pi^{-1}(\mathcal{M}_g \cup \Delta_0) - \tilde{R}_g) \geq 2$. In particular Theorem 0.6 precisely determines the coefficients of $\lambda$, $\delta_0^\prime$, $\delta_0^\ast$ and $\delta_0^\text{tum}$ in the expansion of $[\tilde{U}_{2i+6, i}]^\text{var}$. We also show that if $i < 2i + 6$ then $K_{i, 2}(C, K_C \otimes \eta) \neq 0$ for any $[C, \eta] \in R_g$. By analogy with the case of canonical curves and the classical M. Green Conjecture on syzygies of canonical curves (see [V o]), we conjecture that the morphism of vector bundles $\phi : \mathcal{G}_{i, 2} \to \mathcal{H}_{i, 2}$ over $R_{2i+6}$ is generically non-degenerate:

**Conjecture 0.7** (Prym–Green Conjecture). For a generic point $[C, \eta] \in R_g$ and $g \geq 2i + 6$, we have $K_{i, 2}(C, K_C \otimes \eta) = 0$. Equivalently, the Prym-canonical curve $C \hookrightarrow \mathbb{P}^{g-2}$ satisfies the Green–Lazarsfeld property $(N_i)$ whenever $g \geq 2i + 6$. For $g = 2i + 6$, the locus $U_{2i+6, i}$ is an effective divisor on $R_{2i+6}$.

Proposition 3.1 shows that, if true, Conjecture 0.7 is sharp. In [F4] we verify the Prym–Green Conjecture for $g = 2i + 6$ with $0 \leq i \leq 4, i \neq 3$. In particular, this together with Theorem 0.6 proves that $\tilde{R}_g$ is of general type for $g = 14$.

The strata $U_{g, i}$ have been considered before for $i = 0, 1$, in connection with the Prym–Torelli problem. Unlike the classical Torelli problem, the Prym–Torelli problem is a subtle question: Donagi’s tetragonal construction shows that $P_g$ fails to be injective over points $[C, \eta] \in \pi^{-1}(\mathcal{M}_{g, 4})$ where the curve $C$ is tetragonal (cf. [D2]). The locus $U_{0, 0}$ consists of those points $[C, \eta] \in R_g$ where the differential

$$(dP_g)_{[C, \eta]} : H^0(C, K_C^{\otimes 2})^\vee \to (\text{Sym}^2 H^0(C, K_C \otimes \eta))^\vee$$

is not injective and thus the infinitesimal Prym–Torelli theorem fails. It is known that $(dP_g)_{[C, \eta]}$ is generically injective for $g \geq 6$ (cf. [B], or [De, Corollaire 2.3]), that is, $U_{0, 0}$ is a proper subvariety of $R_g$. In particular, for $g = 6$ the locus $U_{6, 0}$ is a divisor of $R_6$, which gives another proof of Conjecture 0.7 for $i = 0$.

Debarre proved that $U_{g, 1}$ is a proper subvariety of $R_g$ for $g \geq 9$ (cf. [De, Théorème 2.2]). This immediately implies that for $g \geq 9$ the Prym map $P_g$ is generically injective, hence the Prym–Torelli theorem holds generally. Debarre’s proof unfortunately does not cover the interesting case $g = 8$, when $U_{8, 1} \subset R_8$ is an effective divisor (cf. [F4]).

The proof of Theorem 0.1 is finished in Section 4, using in an essential way results from [F3]. We set $g' := 1 + \frac{g+1}{8} \left(\frac{2g}{g-1}\right)$ and then we consider the rational map which associates to a curve one of its Brill–Noether loci

$$\phi : \overline{\mathcal{M}}_{g+1} \dashrightarrow \overline{\mathcal{M}}_{g+1}^{1+\frac{g-1}{8} \left(\frac{2g}{g-1}\right)}, \quad \phi[Y] := W^{1}_{g+1}(Y),$$

where $W^{1}_{g+1}(Y) := \{L \in \text{Pic}^{g+1}(Y) : h^0(Y, L) \geq 2\}$. If $\chi : R_g \to \overline{\mathcal{M}}_{2g-1}$ is the map given by $\chi([C, \eta]) := [\hat{C}]$, where $f : \hat{C} \to C$ is the étale double cover with the property that $f_* \mathcal{O}_{\hat{C}} = \mathcal{O}_C \otimes \eta$, then using [F3] we compute the slope of manyrdis of effective divisors of type $\chi^* \phi^*(A)$, where $A \in \text{Ample}(\overline{\mathcal{M}}_g)$. This proves Theorem 0.1 for even genus $g = 2i + 6 \geq 18$. 

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We mention in passing, as an immediate application of Proposition 1.9, a different proof of the statement that $\overline{R}_g$ has good rationality properties for low $g$ (see again the introduction for the history of this problem). Our proof is quite simple and uses only numerical properties of Lefschetz pencils of curves on K3 surfaces:

**Theorem 0.8.** For all $g \leq 7$, the Kodaira dimension of $\overline{R}_g$ is $-\infty$.

We close by summarizing the structure of the paper. In Section 1 we introduce the stack $\overline{R}_g$ of Prym curves and determine the Chern classes of certain tautological vector bundles. In Section 2 we carry out the enumerative study of the divisors $D_{g,k}$ while in Section 3 we study Koszul divisors on $\overline{R}_g$ in connection with the Prym–Green Conjecture. The proof of Theorem 1.1 is completed in Section 4 while Section 5 is concerned with the enumerative geometry of $\overline{R}_g$ for $g \leq 5$. In Section 6 we describe the behaviour of singularities of pluricanonical forms of $\overline{R}_g$. There is a significant overlap between some of the results of this paper and those of [Be]. Among the things we use from [Be] we mention the description of the branch locus of $\pi$ and the fact that $\overline{R}_g$ is isomorphic to the coarse moduli space of $\overline{M}_g(B\mathbb{Z}_2)$ (see Section 1). However, some of the results in [Be] are not correct, in particular the statement in [Be, Chapter 3] on singularities of $\overline{R}_g$. Hence we carried out a detailed study of singularities of $\overline{R}_g$ in Section 6 of our paper.

1. The stack of Prym curves

In this section we review a few facts about compactifications of $\overline{R}_g$. As a matter of terminology, if $\mathcal{M}$ is a Deligne–Mumford stack, we denote by $\mathcal{M}$ its coarse moduli space (this is contrary to the convention set in [ACV] but it makes sense, at least from a historical point of view). All the Picard groups of stacks or schemes we are going to consider are with rational coefficients.

We recall that $\pi : \overline{R}_g \to \overline{M}_g$ is the $(2^{2g} - 1)$-sheeted cover which forgets the point of order 2 and we denote by $\overline{R}_g$ the normalization of $\overline{M}_g$ in the function field of $\overline{R}_g$. By definition, $\overline{R}_g$ is a normal variety and $\pi$ extends to a finite ramified covering $\pi : \overline{R}_g \to \overline{M}_g$. The local behaviour of this branched cover has been studied in the thesis of M. Bernstein [Be] as well as in the paper [BCF]. In particular, the scheme $\overline{R}_g$ has two distinct modular incarnations which we now recall. If $X$ is a nodal curve, a smooth rational component $E \subset X$ is said to be exceptional if $\#(E \cap (X - E)) = 2$. The curve $X$ is said to be quasi-stable if any two exceptional components of $X$ are disjoint. Thus a quasi-stable curve is obtained from a stable curve by blowing up each node at most once.

We denote by $\text{st}(X) \in \overline{M}_g$ the stable model of $X$. We have the following definition (cf. [BCF]):

**Definition 1.1.** A Prym curve of genus $g$ consists of a triple $(X, \eta, \beta)$, where $X$ is a genus $g$ quasi-stable curve, $\eta \in \text{Pic}^0(X)$ is a line bundle of degree 0 such that $\eta_E = O_E(1)$ for every exceptional component $E \subset X$, and $\beta : \eta \otimes \beta \to O_X$ is a sheaf homomorphism which is generically non-zero along each non-exceptional component of $X$. 
A family of Prym curves over a base scheme $S$ consists of a triple $(X \rightarrow S, \eta, \beta)$, where $f : X \rightarrow S$ is a flat family of quasi-stable curves, $\eta \in \text{Pic}(X)$ is a line bundle and $\beta : \eta \otimes \mathcal{O}_X \rightarrow \mathcal{O}_X$ is a sheaf homomorphism, such that for every point $s \in S$ the restriction $(X_s, \eta_{X_s}, \beta_{X_s} : \eta_{X_s} \otimes \mathcal{O}_{X_s} \rightarrow \mathcal{O}_{X_s})$ is a Prym curve.

We denote by $\overline{R}_g$ the non-singular Deligne–Mumford stack of Prym curves of genus $g$. The main result of [BCF] is that the coarse moduli space of $\overline{R}_g$ is isomorphic to the normalization of $M_g$ in the function field of $R_g$. On the other hand, it is proved in [Be] that $\overline{R}_g$ is also isomorphic to the coarse moduli space of the Deligne–Mumford stack $\overline{M}_g(B\mathbb{Z}_2)$ of $\mathbb{Z}_2$-admissible double covers introduced in [B] and later in [ACV]. For intersection theory calculations the language of Prym curves is better suited than that of admissible covers. In particular, the existence of a degree 0 line bundle $\eta$ over the universal Prym curve will be often used to compute the Chern classes of various tautological vector bundles defined over $\overline{R}_g$. Throughout this paper we use the isomorphism between rational Picard groups $\epsilon^* : \text{Pic}(\overline{R}_g) \rightarrow \text{Pic}(\overline{R}_g)$ induced by the map $\epsilon : \overline{R}_g \rightarrow \overline{R}_g$ from the stack to its coarse moduli space.

**Remark 1.2.** If $(X, \eta, \beta)$ is a Prym curve with exceptional components $E_1, \ldots, E_r$ and $(p_i, q_i) = E_i \cap X - E_i$ for $i = 1, \ldots, r$, then obviously $\beta_{E_i} = 0$. Moreover, if $\tilde{X} := X - \bigcup_{i=1}^r E_i$ (viewed as a subcurve of $X$), then we have an isomorphism of sheaves

$$\eta \otimes \mathcal{O}_X \sim \mathcal{O}_{\tilde{X}}(-p_1 - q_1 - \cdots - p_r - q_r).$$

(1)

It is straightforward to describe all Prym curves $[X, \eta, \beta] \in \overline{R}_g$ whose stable model has a prescribed topological type. We do this when $st(X)$ is a 1-nodal curve and we determine in the process the boundary components of $\overline{R}_g - R_g$.

**Example 1.3** (Curves of compact type). If $st(X) = C \cup D$ is a union of two smooth curves $C$ and $D$ of genus $i$ and $g - i$ respectively meeting transversally at a point, we use (1) to note that $X = C \cup D$ (that is, $X$ has no exceptional components). The line bundle $\eta$ on $X$ is determined by the choice of two line bundles $\eta_C \in \text{Pic}^0(C)$ and $\eta_D \in \text{Pic}^0(D)$ satisfying $\eta_C \otimes \mathcal{O}_C = \mathcal{O}_C$ and $\eta_D \otimes \mathcal{O}_D = \mathcal{O}_D$ respectively. This shows that for $1 \leq i \leq [g/2]$ the pull-back under $\pi$ of the boundary divisor $\Delta_i \subset \overline{M}_g$ splits into three irreducible components

$$\pi^*(\Delta_i) = \Delta_i + \Delta_{g-i} + \Delta_{i;g-i},$$

where the generic point of $\Delta_i \subset \overline{R}_g$ is of the form $[C \cup D, \eta_C \neq \mathcal{O}_C, \eta_D = \mathcal{O}_D]$, the generic point of $\Delta_{g-i}$ is of the form $[C \cup D, \eta_C = \mathcal{O}_C, \eta_D \neq \mathcal{O}_D]$, and finally $\Delta_{i;g-i}$ is the closure of the locus of points $[C \cup D, \eta_C \neq \mathcal{O}_C, \eta_D \neq \mathcal{O}_D]$ (see also [Be] p. 9).

**Example 1.4** (Irreducible one-nodal curves). If $st(X) = C_{qg} := C/(y \sim q)$, where $[C, y, q] \in \mathcal{M}_{g-1,2}$, then there are two possibilities, depending on whether $X$ has an exceptional component or not. Suppose first that $X = C$ and $\eta \in \text{Pic}^0(X)$. If $\nu : C \rightarrow X$ is the normalization map, then there is an exact sequence

$$1 \rightarrow C^* \rightarrow \text{Pic}^0(X) \rightarrow \text{Pic}^0(C) \rightarrow 0.$$
It is easy to establish a dictionary between Prym curves and Beauville admissible covers. Then by \( \eta \), there is a unique way to identify the fibres \( \eta_C(y) \) and \( \eta_C(q) \) such that \( \eta \not= O_X \), and this corresponds to the classical Wirtinger cover of \( X \). We denote by \( \Delta_0' = \Delta_0^{\text{vir}} \) the closure in \( \overline{R_g} \) of the locus of Wirtinger covers. If \( \eta_C \not= O_C \), then for each such choice of \( \eta_C \in \text{Pic}^0(C)[2] \) there are two ways to glue \( \eta_C(y) \) and \( \eta_C(q) \). This provides another \( 2 \times (2^{g-2} - 1) \) Prym curves having \( C' \) as their stable model. We set \( \Delta_0' \subset \overline{R_g} \) to be the closure of the locus of Prym curves with \( \eta_C \not= O_C \).

We now treat the case when \( X = C \cup \{y, q\} \) \( E \), with \( E \) being an exceptional component. Then \( \eta_E = O_E(1) \) and \( \eta_C^{\otimes 2} = O_C(-y - q) \). The analysis carried out in [BCF, Proposition 12] shows that \( \pi \) is simply ramified at each of these \( 2^{g-2} \) Prym curves in \( \pi^{-1}(\{C'\}) \). We denote by \( \Delta_0^{\text{ram}} \subset \overline{R_g} \) the closure of the locus of Prym curves \( [C \cup \{y, q\} \ E, \eta, \beta] \) and \( \Delta_0^{\text{ram}} \) is the ramification divisor of \( \pi \). Moreover one has the relation

\[
\pi^*(\Delta_0) = \Delta_0' + \Delta_0'' + 2\Delta_0^{\text{ram}}.
\]

It is easy to establish a dictionary between Prym curves and Beauville admissible covers. We explain how to do this in codimension 1 in \( \overline{R_g} \) (see also [D2, Example 1.9]). The general point of \( \Delta_0' \) corresponds to an étale double cover \( [\tilde{C} \overset{\tilde{f}}{\rightarrow} C] \in \overline{R}_{g-1} \) induced by \( \eta_C \). We denote by \( y, q, i = 1, 2 \) the points lying in \( f^{-1}(y) \) and \( f^{-1}(q) \) respectively. Then

\[
\overline{M}_{2g-1} \ni \begin{array}{ccc}
\tilde{C} & \overset{\tilde{f}}{\rightarrow} & C \\
y_1 \sim q_1, y_2 \sim q_2 & & y \sim q
\end{array} \in \overline{M}_g
\]

is an admissible double cover, defined up to a sign. This ambiguity is then resolved in the choice of an element in \( \text{Ker}(\nu^* : \text{Pic}^0(C) \otimes [2] \rightarrow \text{Pic}^0(C)[2]) \).

If \( [C/(y \sim q, \eta), \beta] \) is a general point of \( \Delta_0'' \), then we take identical copies \( [C_1, y_1, q_1] \) and \( [C_2, y_2, q_2] \) of \( [C, y, q] \in \overline{M}_{g-1} \). The Wirtinger cover is obtained by taking

\[
\overline{M}_{2g-1} \ni \begin{array}{cc}
C_1 \cup C_2 & \overset{\tilde{f}}{\rightarrow} C \\
y_1 \sim q_2, y_2 \sim q_1 & y \sim q
\end{array} \in \overline{M}_g.
\]

If \( [C \cup \{y, q\} \ E, \eta, \beta] \in \Delta_0^{\text{ram}} \), then \( \eta_C \in \sqrt{O_C(-y - q)} \) induces a \( 2 : 1 \) cover \( \tilde{C} \overset{\tilde{f}}{\rightarrow} C \) branched over \( y \) and \( q \). We set \( \{\tilde{y}\} := f^{-1}(y) \), \( \{\tilde{q}\} := f^{-1}(q) \). The Beauville cover is

\[
\overline{M}_{2g-1} \ni \begin{array}{cc}
\tilde{C} & \overset{\tilde{f}}{\rightarrow} C \\
\tilde{y} \sim \tilde{q} & y \sim q
\end{array} \in \overline{M}_g.
\]

As usual, one denotes by \( \delta_0', \delta_0'' \), \( \delta_0^{\text{ram}} \), \( \delta_1 \), \( \delta_{g-1} \), \( \delta_{1:g-1} \) \( \in \text{Pic} \overline{R}_g \) the stacky divisor classes corresponding to the boundary divisors of \( \overline{R}_g \). We also set \( \lambda := \pi^*(\lambda) \in \text{Pic} \overline{R}_g \).

Next we determine the canonical class \( K_{\overline{R}_g} \):

**Theorem 1.5.** One has the following formula in \( \text{Pic} \overline{R}_g \):

\[
K_{\overline{R}_g} = 13\lambda - 2(\delta_0' + \delta_0'') - 3\delta_0^{\text{ram}} - 2 \sum_{i=1}^{[g/2]} (\delta_i + \delta_{g-1} + \delta_{1:g-1}) - (\delta_1 + \delta_{g-1} + \delta_{1:g-1}).
\]
Proof. We use that \( K_{\overline{\mathcal{M}}_g} \equiv 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{g/2} \) (cf. \cite{HM}), together with
the Hurwitz formula for the cover \( \pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g \). We find that \( K_{\overline{\mathcal{R}}_g} = \pi^*(K_{\overline{\mathcal{M}}_g}) + \delta_{g}^{\text{ram}} \).

Using this formula as well as the results of Section 6, we conclude that in order to prove \( \overline{\mathcal{R}}_g \) is of general type for a certain \( g \), it suffices to exhibit a single effective divisor

\[
D = a\lambda - b_0\delta_0' - b_0''\delta_0'' - \sum_{i=1}^{[g/2]} (b_i\delta_i + b_{g-i}\delta_{g-i} + b_{i:g-i}\delta_{i:g-i}) \in \text{Eff}(\overline{\mathcal{R}}_g)
\]

satisfying the following inequalities:

\[
\begin{align*}
\max \left\{ \frac{a}{b_0'}, \frac{a}{b_0''} \right\} &< \frac{13}{2}, \\
\max \left\{ \frac{a}{b_1}, \frac{a}{b_{g-1}}, \frac{a}{b_{1:g-1}} \right\} &< \frac{13}{3}, \\
\max_{i \geq 1} \left\{ \frac{a}{b_i}, \frac{a}{b_{g-i}}, \frac{a}{b_{i:g-i}} \right\} &< \frac{13}{2}.
\end{align*}
\]  

(2)

1.1. The universal Prym curve

We start by introducing the partial compactification \( \overline{\mathcal{M}}_g := \mathcal{M}_g \cup \Delta_0 \) of \( \mathcal{M}_g \), obtained by adding to \( \mathcal{M}_g \) the locus \( \Delta_0 \subset \overline{\mathcal{M}}_g \) of one-nodal irreducible curves \( [C_{\eta q} := C/(y \sim q)] \), where \([C, [y, q]] \in \mathcal{M}_{g-1,2}\). Let \( p : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g \) denote the universal curve. We denote \( \overline{\mathcal{R}}_g := \pi^{-1}(\overline{\mathcal{M}}_g) \subset \overline{\mathcal{R}}_g \) and note that the boundary divisors \( \Delta_0 := \Delta_0 \cap \overline{\mathcal{R}}_g \), \( \Delta_0' := \Delta_0 \cap \overline{\mathcal{R}}_g \) and \( \Delta_0^{\text{ram}} := \Delta_0^{\text{ram}} \cap \overline{\mathcal{R}}_g \) become disjoint inside \( \overline{\mathcal{R}}_g \). Finally, we set \( \mathcal{Z} := \overline{\mathcal{R}}_g \times \overline{\mathcal{M}}_{g,1} \) and denote by \( p_1 : \mathcal{Z} \to \overline{\mathcal{R}}_g \) the projection.

To obtain the universal family of Prym curves over \( \overline{\mathcal{R}}_g \), we blow up the codimension 2 locus \( V \subset \mathcal{Z} \) corresponding to points

\[
v = ([C \cup_{y=q} E, \eta_C \in \sqrt{\mathcal{O}_C(-y-q)}], \eta_E = \mathcal{O}_E(1), v(y) = v(q)) \in \Delta_0^{\text{ram}} \times \overline{\mathcal{M}}_{g,1}
\]

(recall that \( v : C \to C_{y=q} \) denotes the normalization map). Suppose that \( (t_1, \ldots, t_{g-3}) \) are local coordinates in an étale neighbourhood of \([C \cup_{y=q} E, \eta_C, \eta_E] \in \overline{\mathcal{R}}_g \) such that the local equation of \( \Delta_0^{\text{ram}} \) is \((t_1 = 0)\). Then \( \mathcal{Z} \) around \( v \) admits local coordinates \((x, y, t_1, \ldots, t_{g-3})\) satisfying the equation \( xy = t_1^2 \). In particular, \( \mathcal{Z} \) is singular along \( V \).

We denote by \( \mathcal{X} := \text{Bl}_V(Z) \) and by \( f : \mathcal{X} \to \overline{\mathcal{R}}_g \) the induced family of Prym curves. Then for every \([X, \eta, \beta] \in \overline{\mathcal{R}}_g \), we have \( f^{-1}([X, \eta, \beta]) = X \).

On \( \mathcal{X} \) there exists a Prym line bundle \( P \in \text{Pic}(\mathcal{X}) \) as well as a morphism of \( \mathcal{O}_X \)-modules \( B : P^{\otimes 2} \to \mathcal{O}_X \) with the property that \( P_{\pi^{-1}(X, \eta, \beta)} = \eta \) and \( B_{\pi^{-1}(X, \eta, \beta)} = \beta : \eta^{\otimes 2} \to \mathcal{O}_X \) for all points \([X, \eta, \beta] \in \overline{\mathcal{R}}_g \) (see e.g. \cite{C}, the same argument carries over from the spin to the Prym moduli space).

We set \( E_0', E'' \) and \( E^{\text{ram}} \subset \mathcal{X} \) to be the proper transforms of the boundary divisors \( p_1^{-1}(\Delta_0'), p_1^{-1}(\Delta_0) \) and \( p_1^{-1}(\Delta_0^{\text{ram}}) \) respectively. Finally, we define \( E_0 \) to be the exceptional divisor of the blow-up map \( \mathcal{X} \to \mathcal{Z} \).
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We recall that $g : \mathcal{Y} \to S$ is a family of nodal curves and $L, M$ are line bundles on $\mathcal{Y}$; then $(L, M) \in \text{Pic}(S)$ denotes the bilinear Deligne pairing of $L$ and $M$.

**Proposition 1.6.** If $f : \mathcal{X} \to \tilde{R}_g$ is the universal Prym curve and $\mathcal{P} \in \text{Pic} (\mathcal{X})$ is the corresponding Prym bundle, then one has the following relations in $\text{Pic} (\tilde{R}_g)$:

(i) $\langle \omega_f, \mathcal{P} \rangle = 0$.
(ii) $\langle \mathcal{O}_\mathcal{X}(\mathcal{E}_0), \mathcal{O}_\mathcal{X}(\mathcal{E}_0) \rangle = -2\delta_{0}^{\text{ram}}$.
(iii) $\langle \mathcal{O}_\mathcal{X}(\mathcal{P}), \mathcal{O}_\mathcal{X}(\mathcal{P}) \rangle = -\delta_{0}^{\text{ram}}/2$.

**Proof.** The sheaf homomorphism $B : \mathcal{P}^{\otimes 2} \to \mathcal{O}_\mathcal{X}$ vanishes (with order 1) precisely along the exceptional divisor $\mathcal{E}_0$, hence $[\mathcal{E}_0] = -2c_1(\mathcal{P})$. Furthermore, we have the relations $f^*(\Delta_0^{\text{ram}}) = \mathcal{E}_0^{\text{ram}} + \mathcal{E}_0$ and $f_*([\mathcal{E}_0^{\text{ram}}] \cdot [\mathcal{E}_0]) = 2\delta_{0}^{\text{ram}}$ (in the fibre $f^{-1}([C \cup \{y, q\}, E, \eta_C])$ the divisors $\mathcal{E}_0$ and $\mathcal{E}_0^{\text{ram}}$ meet over two points, corresponding to whether the marked point equals $y$ or $q$). Now (ii) and (iii) follow simply from the push-pull formula. For (i), it is enough to show that $\omega_f|_{\mathcal{E}_0}$ is the trivial bundle. This follows because for any point $[X, \eta, \beta] \in \tilde{R}_g$ we have $\omega_X \otimes \mathcal{O}_E = 0$ for any exceptional component $E \subset X$. \hfill \Box

We now fix $i \geq 1$ and set $\mathcal{N}_i := f_* (\omega_f^{\otimes i} \otimes \mathcal{P}^{\otimes i})$. Since $R^1 f_* (\omega_f^{\otimes i} \otimes \mathcal{P}^{\otimes i}) = 0$, Grauert’s theorem implies that $\mathcal{N}_i$ is a vector bundle over $\tilde{R}_g$ of rank $(g - 1)(2i - 1)$.

**Proposition 1.7.** For each integer $i \geq 1$ the following formula holds in $\text{Pic}(\tilde{R}_g)$:

$$c_1(\mathcal{N}_i) = \left( \frac{i}{2} \right) (12\lambda - 3\delta' - 3\delta'' - 2\delta_{0}^{\text{ram}}) + \lambda - \frac{i^2}{4}\delta_{0}^{\text{ram}}.$$

**Proof.** We apply Grothendieck–Riemann–Roch to the universal Prym curve $f : \mathcal{X} \to \tilde{R}_g$:

$$c_1(\mathcal{N}_i) = f_* \left[ \left( 1 + i c_1(\omega_f \otimes \mathcal{P}) + \frac{i^2 c_1^2(\omega_f \otimes \mathcal{P})}{2} \right) \left( 1 - \frac{c_1(\omega_f)}{2} + \frac{c_1^2(\omega_f) + \vert \text{Sing}(f) \vert}{12} \right) \right],$$

and then use Proposition 1.6 and Mumford’s formula $(\kappa_1)_{\tilde{R}_g} = 12\lambda - 3\delta' - 3\delta'' - 2\delta_{0}^{\text{ram}}$. \hfill \Box

### 1.2. Inequalities between coefficients of divisors on $\tilde{R}_g$

We use pencils of curves on $K3$ surfaces to establish certain inequalities between the coefficients of effective divisors on $\tilde{R}_g$. Using $K3$ surfaces we construct pencils that fill up the boundary divisors $\Delta_1$, $\Delta_{g-1}$ and $\Delta_{i:g-1}$ for $1 \leq i \leq \lfloor g/2 \rfloor$ when $g \leq 23$. The use of such pencils in the context of $\overline{M}_g$ has already been demonstrated in [FP].

We start with a Lefschetz pencil $B \subset \overline{M}_i$ of curves of genus $i$ lying on a fixed $K3$ surface $S$. The pencil $B$ is induced by a family $f : B_{1;i}(S) \to \mathbb{P}^1$ which has $i^2$ sections corresponding to the base points and we choose one such section $\sigma$. Using $B$, for each $g \geq i + 1$ we create a genus $g$ pencil $B_i \subset \overline{M}_g$ of stable curves, by gluing a fixed curve $[C_2, p] \in \mathcal{M}_{g-i, 1}$ along the section $\sigma$ to each member of the pencil $B$. Then we have the
following formulas on $\mathcal{M}_g$ (cf. [FP] Lemma 2.4]):

$$B_i \cdot \lambda = i + 1, \quad B_i \cdot \delta_0 = 6i + 18, \quad B_i \cdot \delta_i = -1, \quad B_i \cdot \delta_j = 0 \text{ for } j \neq i.$$ 

We fix $1 \leq i \leq [g/2]$ and lift $B_i$ in three different ways to pencils in $\mathcal{R}_g$. First we choose a non-trivial line bundle $\eta_2 \in \text{Pic}^0(C_2)[2]$. Let us denote by $A_{g-i} \subset \Delta_{g-i} \subset \mathcal{R}_g$ the pencil of Prym curves $[C_2 \cup_{\eta_2} f^{-1}(\lambda), \eta_{C_2} = \eta_2, \eta_{f^{-1}(\lambda)} = \mathcal{O}_{f^{-1}(\lambda)}]$, with $\lambda \in \mathbb{P}^1$.

Next, we denote by $A_i \subset \Delta_i \subset \mathcal{R}_g$ the pencil consisting of Prym curves

$$[C_2 \cup_{\eta_2} f^{-1}(\lambda), \eta_{C_2} = \mathcal{O}_{C_2}, \eta_{f^{-1}(\lambda)} = \mathcal{O}_{f^{-1}(\lambda)}[2]],$$

where $\lambda \in \mathbb{P}^1$.

Clearly $\pi(A_i) = B_i$ and $\deg(A_i/B_i) = 2^{2i} - 1$. Finally, $A_{i:g-i} \subset \Delta_{i:g-i} \subset \mathcal{R}_g$ denotes the pencil of Prym curves $[C_2 \cup f^{-1}(\lambda), \eta_{C_2} = \eta_2, \eta_{f^{-1}(\lambda)} \in \text{Pic}^0(f^{-1}(\lambda))[2]]$. Again, we have $\deg(A_{i:g-i}/B_i) = 2^{2i} - 1$.

**Lemma 1.8.** If $A_i$, $A_{g-i}$ and $A_{i:g-i}$ are pencils defined above, we have the following relations:

- $A_{g-i} \cdot \lambda = i + 1, A_{g-i} \cdot \delta_0' = 6i + 18, A_{g-i} \cdot \delta_i = A_{g-i} \cdot \delta_{g-i} = 0$, and $A_{g-i} \cdot \delta_{g-i} = -1$.
- $A_i \cdot \lambda = (i + 1)(2^{2i} - 1), A_i \cdot \delta_i' = (2^{2i-1} - 2)(6i + 18), A_i \cdot \delta_i'' = 6i + 18$.
- $A_{i,g-i} \cdot \lambda = (i + 1)(2^{2i} - 1), A_{i,g-i} \cdot \delta_0'' = (2^{2i-1} - 1)(6i + 18)$.
- $A_{i,g-i} \cdot \delta_0' = 2^{2i-2}(6i + 18), A_{i,g-i} \cdot \delta_0'' = 0$ and $A_{i,g-i} \cdot \delta_{g-i} = -(2^{2i} - 1)$.

Note that all these intersections are computed on $\mathcal{R}_g$. The intersection numbers of $A_i$, $A_{g-i}$ and $A_{i:g-i}$ with the generators of $\text{Pic}(\mathcal{R}_g)$ not explicitly mentioned in Lemma 1.8 are all equal to 0.

**Proof.** We treat in detail only the case of $A_i$, the other cases being similar. Using [FP] we find that $(A_i \cdot \lambda)_{\mathcal{M}_g} = (\pi(A_i) \cdot \lambda)_{\mathcal{M}_g} = (2^{2i} - 1)(B_i \cdot \lambda)_{\mathcal{M}_g}$. Furthermore, since $A_i \cap \Delta_{g-i} = A_i \cap \Delta_{i:g-i} = \emptyset$, we can write the formulas

$$(A_i \cdot \delta_0)_{\mathcal{M}_g} = (A_i \cdot \pi^*(\delta_i))_{\mathcal{M}_g} = (2^{2i} - 1)(B_i \cdot \delta_i)_{\mathcal{M}_g}.$$

Clearly $(A_i \cdot \delta_0)_{\mathcal{M}_g} = (B_i \cdot \delta_0)_{\mathcal{M}_g} = 6i + 18$, whereas the intersection $A_i \cdot \delta_0'$ corresponds to choosing an element in $\text{Pic}^0(f^{-1}(\lambda))[2]$, where $f^{-1}(\lambda)$ is a singular member of $B$. There are $2(2^{2i-2} - 1)(6i + 18)$ such choices.

**Proposition 1.9.** Let $D \equiv a \lambda - b_{i,g-i} \delta_0 - b_{i,g-i} \delta_i - b_{i,g-i} \delta_{g-i} + b_{i,g-i} \delta_{g-i} \in \text{Pic}(\mathcal{R}_g)$ be the closure in $\mathcal{R}_g$ of an effective divisor in $\mathcal{R}_g$. Then if $1 \leq i \leq [g/2]$, 1), we have the following inequalities:

1. $a(i + 1) - b_{i,g-i} (6i + 18) + b_{i,g-i} \geq 0$.
2. $a(i + 1) - b_{i,g-i} (6i + 18) \frac{2^{2i-2} - 2^{2i-1}-1}{2^{2i-1}} - b_{i,g-i} (6i + 18) \frac{2^{2i-1}-1}{2^{2i-1}} + b_{i,g-i} \geq 0$.
3. $a(i + 1) - b_{i,g-i} (6i + 18) \frac{2^{2i-2}}{2^{2i-1}} - b_{i,g-i} (6i + 18) \frac{2^{2i-1}-2}{2^{2i-1}} - b_{i,g-i} (6i + 18) \frac{1}{2^{2i-1}} + b_i \geq 0$. 


Either \( \Theta \), \( R \), or \( \eta \) is a divisor on \( \mathcal{R}_g \) and \( \eta \) is a linear series of \( \mathcal{R}_g \). For a fixed point \([C, \eta] \in \mathcal{R}_g\) we shall study the relative position of \( \eta \in \text{Pic}^0(C)[2] \) with respect to certain pluri-theta divisors on \( \text{Pic}^0(C) \).

We start by fixing a smooth curve \( C \). If \( E \in U_C(r, d) \) is a semistable vector bundle on \( C \) of integer slope \( \mu(E) := d/r \in \mathbb{Z} \), then following Raynaud [R], we introduce the determinantal cycle

\[
\Theta_E := \{ \eta \in \text{Pic}^{g-\mu-1}(C) : H^0(C, E \otimes \eta) \neq 0 \}.
\]

Either \( \Theta_E = \text{Pic}^{g-\mu-1}(C) \), or else, \( \Theta_E \) is a divisor on \( \text{Pic}^{g-\mu-1}(C) \) and then \( \Theta_E \equiv r \cdot \theta \). In the latter case we say that \( \Theta_E \) is the theta divisor of the vector bundle \( E \). Clearly, \( \Theta_E \) is a divisor if and only if \( H^0(C, E \otimes \eta) = 0 \) for a general bundle \( \eta \in \text{Pic}^{g-\mu-1}(C) \).

Let us now fix a globally generated line bundle \( L \in \text{Pic}^d(C) \) such that \( h^0(C, L) = r + 1 \). The Lazarsfeld vector bundle \( M_L \) of \( L \) is defined using the exact sequence on \( C \)

\[
0 \rightarrow M_L \rightarrow H^0(C, L) \otimes \mathcal{O}_C \rightarrow L \rightarrow 0
\]

(see also [GL], [L], [Vo], [F1], [FMP] for many applications of these bundles). It is customary to denote \( Q_L := M_L^* \), hence \( \mu(Q_L) = d/r \). When \( L = K_C \), one writes \( Q_C := Q_{K_C} \). The vector bundles \( Q_L \) (and all its exterior powers) are semistable under mild genericity assumptions on \( C \) (see [L] or [F1] Proposition 2.1). In the case \( \mu(Q_L) = g - 1 \), when we expect \( \Theta \otimes Q_L \) to be a divisor on \( \text{Pic}^d(C) \), we may ask whether for a given point \([C, \eta] \in \mathcal{R}_g\) the condition \( \eta \in \Theta \otimes Q_L \) is satisfied or not.

Throughout this section we denote by \( G_d^f \) the Deligne–Mumford stack parameterizing pairs \([C, l]\), where \([C] \in \mathcal{M}_g \) and \( l = (L, V) \in G_d^f(C) \) is a linear series of type \( g_d \).
We fix integers $k \geq 2$ and $b \geq 0$. We set integers
\[ i := kb + k - b - 2, \quad r := kb + k - 2, \]
\[ g := k(kb + k - b - 2) + 1 = ik + 1, \quad d := k(kb + k - 2). \]

Since $\rho(g, r, d) = 0$, a general curve $[C] \in \mathcal{M}_g$ carries a finite number of (obviously complete) linear series $l \in G_d'(C)$. We denote this number by
\[ N := g! \frac{1! \cdots r!}{(k - 1)! \cdots (k - 1 + r)!} = \deg(\Theta_{\mathcal{O}/\mathcal{M}_g}). \]

We also note that we can write $g = (r + 1)(k - 1)$ and $d = rk$, and moreover each line bundle $L \in W_d'(C)$ satisfies $h^1(C, L) = k - 1$. Furthermore, we compute $\mu(\bigwedge^i Q_L) = ik = g - 1$ and then we introduce the following virtual divisor on $\mathcal{R}_g$:
\[ D_{g,k} := \{ [C, \eta] \in \mathcal{R}_g : \exists L \in W_d'(C) \text{ such that } h^0(C, \bigwedge^i Q_L \otimes \eta) \geq 1 \}. \]

From the definition it follows that $D_{g,k}$ is either pure of codimension 1 in $\mathcal{R}_g$, or else $D_{g,k} = \mathcal{R}_g$. We shall prove that the second possibility does not occur.

For $[C, \eta] \in \mathcal{R}_g$ and $L \in W_d'(C)$ one has the following exact sequence on $C$:
\[ 0 \rightarrow \bigwedge^i M_L \otimes K_C \otimes \eta \rightarrow \bigwedge^i H^0(C, L) \otimes K_C \otimes \eta \rightarrow \bigwedge^{i-1} M_L \otimes L \otimes K_C \otimes \eta \rightarrow 0, \]
from which, using Serre duality, one derives the following equivalences:
\[ [C, \eta] \in D_{g,k} \iff h^1(C, \bigwedge^i M_L \otimes K_C \otimes \eta) \geq 1 \]
\[ \iff \bigwedge^i H^0(C, L) \otimes H^0(C, K_C \otimes \eta) \rightarrow H^0(C, \bigwedge^{i-1} M_L \otimes L \otimes K_C \otimes \eta) \]
\[ \text{is not an isomorphism.} \quad (3) \]

Note that obviously $\text{rank}(\bigwedge^i H^0(C, L) \otimes H^0(C, K_C \otimes \eta)) = \binom{\ell}{i+1}(g - 1)$, while
\[ h^0(C, \bigwedge^{i-1} M_L \otimes L \otimes K_C \otimes \eta) = \chi(C, \bigwedge^{i-1} M_L \otimes L \otimes K_C \otimes \eta) \]
\[ = \binom{r}{i-1} (-k(i - 1) + d + g - 1) = \binom{r + 1}{i} (g - 1) \]
(\text{use that } M_L \text{ is a semistable vector bundle and that } \mu(\bigwedge^{i-1} M_L \otimes L \otimes K_C \otimes \eta) > 2g - 1). \]

\textbf{Remark 2.1.} As pointed out in the introduction, an important particular case is $k = 2$, when $i = b, g = 2i + 1, r = 2i, d = 4i = 2g - 2$. Since $W^2_{g-2}(C) = \{K_C\}$, it follows that $[C, \eta] \in D_{2i+1,2} \iff \eta \in \Theta \bigwedge^i Q_C$. The main result from [FMP] states that for any $[C] \in \mathcal{M}_g$ the Raynaud locus $\Theta \bigwedge^i Q_C$ is a divisor in $\text{Pic}^0(C)$ (that is, $\bigwedge^i Q_C$ has a theta divisor) and we have an equality of cycles
\[ \Theta \bigwedge^i Q_C = C_i - C_i \subset \text{Pic}^0(C), \]
where the right-hand side denotes the $i$-th difference variety of $C$, that is, the image of the difference map

$$
\phi : C_i \times C_i \to \text{Pic}^0(C), \quad \phi(D, E) := \mathcal{O}_C(D - E).
$$

Using Lazarsfeld’s filtration argument [L] Lemma 1.4.1, one finds that for a generic choice of distinct points $x_1, \ldots, x_{g-2} \in C$, there is an exact sequence

$$
0 \to \bigoplus_{i=1}^{g-2} \mathcal{O}_C(x_i) \to Q_C \to K_C \otimes \mathcal{O}_C(-x_1 - \cdots - x_{g-2}) \to 0,
$$

which implies the inclusion $C_i - C_i \subset \Theta_{\wedge^i} Q_C$. The importance of (4) is that it shows that $\Theta_{\wedge^i} Q_C$ is a divisor on $\text{Pic}^0(C)$, that is, $H^0(C, \wedge^i Q_C \otimes \eta) = 0$ for a generic $\eta \in \text{Pic}^0(C)$.

**Theorem 2.2.** For every genus $g = 2i + 1$ we have the following identification of cycles on $\mathcal{R}_g$:

$$
D_{2i+1,2} := \{[C, \eta] \in \mathcal{R}_g : \eta \in C_i - C_i\}.
$$

Next we prove that $D_{g,k}$ is an actual divisor on $\mathcal{R}_g$ for any $k \geq 2$ and we achieve this by specialization to the $k$-gonal locus $\mathcal{M}_{g,k}$ in $\mathcal{M}_g$.

**Theorem 2.3.** Fix $k \geq 2$, $b \geq 1$ and $g, r, d, i$ defined as above. Then $D_{g,k}$ is a divisor on $\mathcal{R}_g$. Precisely, for a generic $[C, \eta] \in \mathcal{R}_g$ we have $H^0(C, \wedge^i Q_L \otimes \eta) = 0$ for every $L \in W_d^r(C)$.

**Proof.** Since there is a unique irreducible component of $\mathcal{G}_{g,k}(\mathcal{R}_g/\mathcal{M}_g) := \mathcal{G}_{g,k} \times_{\mathcal{M}_g} \mathcal{R}_g$ mapping dominantly onto $\mathcal{R}_g$, in order to prove that $D_{g,k}$ is a divisor it suffices to exhibit a single triple $[C, L, \eta] \in \mathcal{G}_{g,k}(\mathcal{R}_g/\mathcal{M}_g)$ such that (1) the Petri map

$$
\mu_0(C, L) : H^0(C, L) \otimes H^0(C, K_C \otimes L^\vee) \to H^0(C, K_C)
$$

is an isomorphism and (2) the torsion point $\eta \in \text{Pic}^0(C)[2]$ is such that $\eta \notin \Theta_{\wedge^i} Q_L$.

Proposition 2.1.1 from [CM] ensures that for a generic $k$-gonal curve $[C, A] \in \mathcal{G}_{g,k}$ of genus $g = (r+1)(k-1)$ one has $h^0(C, A^{\otimes j}) = j + 1$ for $1 \leq j \leq r + 1$. In particular there is an isomorphism $\text{Sym}^j H^0(C, A) \cong H^0(C, A^{\otimes j})$. Using this and Riemann–Roch, we obtain

$$
h^0(C, K_C \otimes A^{\otimes (j-i)}) = (k-1)(r+1-j)\text{ for } 0 \leq j \leq r + 1.
$$

Thus there is a generically injective rational map $\mathcal{G}_{g,k} \dashrightarrow \mathcal{G}_{g,k}$ given by $[C, A] \mapsto [C, A^{\otimes r}]$. (The use of such a map has been first pointed out to us in a different context by S. Keel.) We claim that $\mathcal{G}_{g,k}$ maps into the “main component” of $\mathcal{G}_{g,k}$ which maps dominantly onto $\mathcal{M}_g$. To prove this it suffices to check that the Petri map

$$
\mu_0(C, A^{\otimes r}) : H^0(C, A^{\otimes r}) \otimes H^0(C, K_C \otimes A^{\otimes (r-j)}) \to H^0(C, K_C)
$$

is an isomorphism (remember that $H^0(C, A^{\otimes r}) \cong \text{Sym}^r H^0(C, A)$). We use the base point free pencil trick to write down the exact sequence

$$
0 \to H^0(K_C \otimes A^{\otimes (j+1)}) \to H^0(A) \otimes H^0(K_C \otimes A^{\otimes (j-1)}) \xrightarrow{\mu_j(A)} H^0(K_C \otimes A^{\otimes (j-1)}).
$$
One can now easily check that \( \mu_j(A) \) is surjective for \( 1 \leq j \leq r \) by using the formulas
\[
h^0(C, K_C \otimes A^{(-j)}) = (k - 1)(r + 1 - j) \text{ valid for } 0 \leq j \leq r + 1.
\]
This turns implies that \( \mu_0(C, A^{\otimes r}) \) is surjective, hence an isomorphism.

We now check condition (2) and note that for \([C, L = A^{\otimes r}] \in \mathcal{G}'_d\), the Lazarsfeld bundle splits as \( Q_L \equiv A^{\otimes r} \). In particular, \( \Lambda^i Q_L \equiv \otimes^{r+i} A^{\otimes i} \), hence the condition \( H^0(C, A^{\otimes r} \otimes \eta) \neq 0 \) is equivalent to \( H^0(C, A^{\otimes i} \otimes \eta) \neq 0 \), that is, the translate of the theta divisor \( W_{g-1}(C) - A^{\otimes \eta} \subset \text{Pic}^0(C) \) cannot contain all points of order 2 on \( \text{Pic}^0(C) \).

We assume by contradiction that for any \([C, A] \in \mathcal{G}'_1\) and any \( \eta \in \text{Pic}^0(C)[2] \), we have \( H^0(C, A^{\otimes \eta} \otimes \eta) \geq 1 \). We use that \( \mathcal{G}'_1 \) is irreducible and specialize \( C \) to a hyperelliptic curve and choose \( A = g^1_1 \otimes \mathcal{O}_C(x_1 + \cdots + x_{k-2}) \), with \( x_1, \ldots, x_{k-2} \in C \) being general points. Finally we take \( \eta := \mathcal{O}_C(p_1 + \cdots + p_i + 1 - q_1 - \cdots - q_i) \in \text{Pic}^0(C)[2] \), with \( p_1, \ldots, p_i, q_1, \ldots, q_i \) being distinct ramification points of the hyperelliptic \( g^1_1 \). It is now straightforward to check that \( H^0(C, A^{\otimes \eta} \otimes \eta) = 0 \).

In order to compute the class \([\mathcal{T}_{g,k}] \in \text{Pic}(\overline{M}_g)\) we extend the determinantal description of \( D_{g,k} \) to the boundary of \( \overline{M}_g \). We start by setting some notation. We denote by \( \mathcal{M}_g^0 \subset \mathcal{M}_g \) the open substack classifying curves \([C] \in \mathcal{M}_g\) such that \( W_{g-1}(C) = \emptyset \) and \( W_{g+1}(C) = \emptyset \). We know that \( \text{codim}(\mathcal{M}_g - \mathcal{M}_g^0, \mathcal{M}_g) \geq 2 \). We further denote by \( \Delta_g^0 \subset \Delta_g \subset \overline{M}_g \) the locus of curves \([C/(y \sim q)]\) where \([C] \in \mathcal{M}_{g-1}\) is a curve that satisfies the Brill–Noether theorem and where \( y, q \in C \) are arbitrary points. Note that every Brill–Noether general curve \([C] \in \mathcal{M}_{g-1}\) satisfies
\[
W'_{g-1}(C) = \emptyset, \quad W'_{g+1}(C) = \emptyset \quad \text{and} \quad \dim W'_g(C) = \rho(g - 1, r, d) = r.
\]
We set \( \overline{M}_g^0 := \mathcal{M}_g^0 \cup \Delta_g^0 \subset \overline{M}_g \). Then we consider the Deligne–Mumford stack
\[
\sigma_0 : \mathcal{G}'_d \rightarrow \overline{M}_g^0
\]
classifying pairs \([C, L] \) with \([C] \in \overline{M}_g^0\) and \( L \in G'_d(C) \) (cf. [EH], [F2], [Kh]; note that it is essential that \( \rho(g, r, d) = 0 \); at the moment there is no known extension of this stack over the entire \( \overline{M}_g \)). We remark that for any curve \([C] \in \overline{M}_g^0\) and \( L \in W'_d(C) \) we have \( h^0(C, L) = r + 1 \), that is, \( \mathfrak{G}'_d \) parameterizes only complete linear series. Indeed, for a smooth curve \([C] \in \mathcal{M}_g^0\) we have \( W'_{g+1}(C) = \emptyset \), so necessarily \( W'_d(C) = G'_d(C) \). For a point \([C_{yq} := C/(y \sim q)] \) \( \in \Delta_g^0 \) we have the identification
\[
\sigma_0^{-1}[C_{yq}] = \{ L \in W'_d(C) : h^0(C, L \otimes \mathcal{O}_C(-y - q)) = r \}.
\]
where we note that since the normalization \([C] \in \mathcal{M}_{g-1}\) is assumed to be Brill–Noether general, any sheaf \( L = \sigma_0^{-1}[C_{yq}] \) satisfies \( h^0(C, L \otimes \mathcal{O}_C(-y)) = h^0(C, L \otimes \mathcal{O}_C(-q)) = r \) and \( h^0(C, L) = r + 1 \). Furthermore, \( \sigma_0 : \mathfrak{G}'_d \rightarrow \overline{M}_g^0 \) is proper, which is to say that \( W'_d(C_{yq}) = W'_d(C_{yq}) \), where the left-hand side denotes the closure of \( W'_d(C_{yq}) \) in the variety \( \text{Pic}^0(C_{yq}) \) of torsion-free sheaves on \( C_{yq} \). This follows because a non-locally free
torsion-free sheaf in \( \mathcal{W}_d(C) - W_d(C) \) is of the form \( v_w(A) \), where \( A \in W'_d-1(C) \) and \( v : C \to C'q \) is the normalization map. But we know that \( W'_d-1(C) = \emptyset \), because \([C] \in \mathcal{M}_\Phi\) satisfies the Brill–Noether theorem. Since \( \rho(g, r, d) = 0 \), by general Brill–Noether theory, there exists a unique irreducible component of \( \mathcal{G} \) which maps onto \( \mathcal{M}_\Phi^0 \). It is certainly not the case that \( \mathcal{G} \) is irreducible, unless \( k \leq 3 \), when either \( \mathcal{G} = \mathcal{M}_k^0 \) (\( k = 2 \)), or \( \mathcal{G} \) is isomorphic to a Hurwitz stack (\( k = 3 \)). Let \( f_{r_d}^d : \mathcal{C}_{r_d}^d := \mathcal{M}_{g,1}^0 \times \mathcal{G}^d_r \to \mathcal{G}^d_r \) denote the pullback of the universal curve \( \mathcal{M}_{g,1}^0 \to \mathcal{M}_g^0 \) to \( \mathcal{G}^d_r \). Once we have chosen a Poincaré bundle \( \mathcal{L} \) on \( \mathcal{C}_{r_d}^d \) we can form the three codimension 1 tautological classes in \( A^1(\mathcal{G}^d_r) \):

\[
\begin{align*}
\alpha := (f_{r_d}^d)_*(c_1(\mathcal{L})^2), \\
\beta := (f_{r_d}^d)_*(c_1(\mathcal{L}) \cdot c_1(\omega_{\mathcal{G}^d_r})), \\
\gamma := (f_{r_d}^d)_*(c_1(\omega_{\mathcal{G}^d_r})^2) = (c_1)^0(1)_{\mathcal{M}_g^0}.
\end{align*}
\]

(5)

These classes depend on the choice of \( \mathcal{L} \) and behave functorially with respect to base change (see also Remark 2.7 for the precise statement regarding the choice of \( \mathcal{L} \)). We set \( \tilde{\mathcal{R}}^0 := \pi^{-1}(\mathcal{M}_g^0) \subset \mathcal{R} \) and introduce the stack of \( g^r_{d} \)'s on Prym curves:

\[
\sigma : \mathcal{G}^d_r(\tilde{\mathcal{R}}^0) := \mathcal{R} \times \mathcal{G}^d_r \to \mathcal{R}.
\]

By a slight abuse of notation we denote the boundary divisors by the same symbols, that is, \( \Delta^0, \Delta^0'' := \sigma^*(\Delta^0), \Delta^0''' := \sigma^*(\Delta^0'') \) and \( \Delta^0_{nm} := \sigma^*(\Delta^0_{nm}) \). Finally, we introduce the universal curve over the stack of \( g^r_{d} \)'s on Prym curves:

\[
f' : \mathcal{X}^r_d := \mathcal{X} \times \mathcal{G}^d_r(\tilde{\mathcal{R}}^0) \to \mathcal{G}^d_r(\tilde{\mathcal{R}}^0/\mathcal{M}_g^0).
\]

On \( \mathcal{X}^r_d \) there are two tautological line bundles, the universal Prym bundle \( \mathcal{P} \) which is the pull-back of \( \mathcal{P} \in \text{Pic}(\mathcal{X}) \) under the projection \( \mathcal{X}^r_d \to \mathcal{X} \), and a Poincaré bundle \( \mathcal{L} \in \text{Pic}(\mathcal{X}^r_d) \) characterized by the property \( \mathcal{L}|_{\mathcal{X} \times \mathcal{G}^d_r(\tilde{\mathcal{R}}^0/\mathcal{M}_g^0)} = L \in \mathcal{W}_d(C) \), for each point \([X, \eta, \beta, L] \in \mathcal{G}^d_r(\tilde{\mathcal{R}}^0/\mathcal{M}_g^0) \). Note that we also have the codimension 1 classes \( \alpha, \beta, \gamma \in A^1(\mathcal{G}^d_r(\tilde{\mathcal{R}}^0/\mathcal{M}_g^0)) \) defined by the formulas (3).

**Proposition 2.4.** Let \( C \) be a curve of genus \( g \) and let \( L \in \mathcal{W}_d(C) \) be a globally generated complete linear series. Then for any integer \( 0 \leq j \leq r \) and for any line bundle \( A \in \text{Pic}^a(C) \) such that \( a \geq 2g + d - r + j - 1 \), we have \( H^1(C, \bigwedge^j M_L \otimes A) = 0 \).

**Proof.** We use a filtration argument due to Lazarsfeld [1]. Having fixed \( L \) and \( A \), we choose general points \( x_1, \ldots, x_{r-1} \in \mathcal{C} \) such that \( h^0(C, L \otimes \mathcal{O}_C(-x_1 - \cdots - x_{r-1})) = 2 \) and then there is an exact sequence on \( C \):

\[
0 \to L^\vee(x_1 + \cdots + x_{r-1}) \to M_L \to \bigoplus_{i=1}^{r-1} \mathcal{O}_C(-x_i) \to 0.
\]

Taking the \( j \)-th exterior powers and tensoring the resulting exact sequence with \( A \), we find that in order to conclude that \( H^1(C, \bigwedge^i M_L \otimes A) = 0 \) for \( i \leq r \), it suffices to show that for \( 1 \leq i \leq r \) the following hold:
Proposition 2.5. For each point \([X, \eta, \beta, L] \in \mathfrak{S}_d^{[a]}(\mathbf{R}_g^0, \mathbf{M}_g^0)\) and \(0 \leq a \leq i - 1\), we have
\[
H^1(X, \bigwedge^a M_L \otimes L^{\otimes (i-a)} \otimes \omega_X \otimes \eta) = 0.
\]

Proof. If \(X\) is smooth, then the vanishing follows directly from Proposition 2.4. Assume now that \([X, \eta, \beta] \in \Delta^\text{ram}_0 \cup \Delta^\text{ram}_1\), that is, \(s_1(X) = X\) and \(\eta \in \text{Pic}^0(X)[2]\). As usual, we denote by \(v : C \to X\) the normalization map, and \(L_C := v^*(L) \in W_d(C)\) satisfies \(h^0(C, L_C \otimes \mathcal{O}_C(-y - q)) = r\), hence \(H^0(X, L) \cong H^0(C, L_C)\), which implies that \(v^*(M_L) = M_{L_C}\). Tensoring the usual exact sequence on \(X\),
\[
0 \to \mathcal{O}_X \to v_* \mathcal{O}_C \to v_* \mathcal{O}_C/\mathcal{O}_X \to 0,
\]
by the line bundle \(\bigwedge^a M_L \otimes L^{\otimes (i-a)} \otimes \omega_X \otimes \eta\), we find that a sufficient condition for the vanishing \(H^1(X, \bigwedge^a M_L \otimes L^{\otimes (i-a)} \otimes \omega_X \otimes \eta) = 0\) to hold is that
\[
H^1(C, \bigwedge^a M_{L_C} \otimes L_{C}^{\otimes (i-a)} \otimes K_C \otimes \eta_C)
= H^1(C, \bigwedge^a M_{L_C} \otimes L_{C}^{\otimes (i-a)} \otimes K_C(y + q) \otimes \eta_C) = 0.
\]

Since \(i < r\), this follows directly from Proposition 2.4.

We are left with the case when \([X, \eta, \beta] \in \Delta^\text{ram}_0\), when \(X := C \cup_{[y, y]} E\), with \(E\) being a smooth rational curve, \(L_C \in W_d(C), L_E = \mathcal{O}_E\) and \(\eta_C^2 = \mathcal{O}_C(-y - q)\). We also have \(M_{L_C} = M_{L_C} \otimes \mathcal{O}_C(-y - q)\). A standard argument involving the Mayer–Vietoris sequence on \(X\) shows that the vanishing of the group \(H^1(X, \bigwedge^a M_L \otimes L^{\otimes (i-a)} \otimes \omega_X \otimes \eta)\) is implied by the following vanishing conditions:
\[
H^1(C, \bigwedge^a M_{L_C} \otimes L_{C}^{\otimes (i-a)} \otimes K_C(y + q) \otimes \eta_C)
= H^1(C, \bigwedge^a M_{L_C} \otimes L_{C}^{\otimes (i-a)} \otimes K_C \otimes \eta_C) = 0.
\]

The conditions of Proposition 2.4 being satisfied \((i \leq r - 1)\), we finish the proof. □

Proposition 2.5 enables us to define a sequence of tautological vector bundles on \(\mathfrak{S}_d^{[a]}(\mathbf{R}_g^0, \mathbf{M}_g^0)\): First, we set \(\mathcal{H} := f'_*\mathcal{L}\). By Grauert’s theorem, \(\mathcal{H}\) is a vector bundle of rank \(r + 1\) with fibre \(\mathcal{H}(X, \eta, \beta, L) = H^0(X, L)\). For \(j \geq 0\) we set
\[
A_{0,j} := f'_*(L^{\otimes j} \otimes \omega_{f^*} \otimes \mathcal{P}_d^0),
\]
Proposition 2.6. For all $c$ sufficient to compute $R^1 f_*(\mathcal{N}^{\oplus j} \otimes \omega_f \otimes P_d') = 0$, we find that $A_0, j$ is a vector bundle over $\mathcal{S}'_d(R^0_g/M^0_g)$ of rank equal to $h^0(X, L^{\oplus j} \otimes \omega_X \otimes \eta) = jd + g - 1$. Next we introduce the global Lazarsfeld vector bundle $\mathcal{M}$ over $\mathcal{X}'_d$ by the exact sequence

$$0 \to \mathcal{A} \to f^*(\mathcal{H}) \to \mathcal{L} \to 0,$$

hence $\mathcal{M}_{i-1} = M_i$ for each $[X, \eta, \beta, L] \in \mathcal{S}'_d(R^0_g/M^0_g)$. Then for integers $a, j \geq 1$ we define the sheaf

$$A_{a,j} := f'_*(\bigwedge^a \mathcal{M} \otimes \mathcal{L}^{\oplus j} \otimes \omega_f' \otimes P_d').$$

For each $1 \leq a \leq i - 1$, we have proved that $R^1 f'_*(\bigwedge^a \mathcal{M} \otimes \mathcal{L}^{\oplus (i-a)} \otimes \omega_f' \otimes P_d') = 0$ (cf. Proposition 2.5), therefore $A_{a,i-a}$ is a vector bundle over $\mathcal{S}'_d(R^0_g/M^0_g)$ having rank

$$\text{rk}(A_{a,i-a}) = \chi(X, \bigwedge^a M_i \otimes L^{\oplus (i-a)} \otimes \omega_X \otimes \eta) = \left(\frac{i}{d}\right) k(i-a)(r+1).$$

Proposition 2.5 also shows that for all integers $1 \leq a \leq i - 1$, the vector bundles $A_{a,i-a}$ sit in exact sequences

$$0 \to A_{a,i-a} \to \bigwedge^a \mathcal{H} \otimes A_{0,i-a} \to A_{a-1,i-a+1} \to 0.$$  \hspace{1cm} (6)

We shall need the expression for the Chern numbers of $A_{a,i-a}$. Using (6) it will be sufficient to compute $c_1(A_{0,j})$ for all $j \geq 0$.

Proposition 2.6. For all $j \geq 0$ one has the following formula in $A^1(\mathcal{S}'_d(R^0_g/M^0_g))$:

$$c_1(A_{0,j}) = c_1(\bigwedge^j (\omega_f \otimes \mathcal{L}^{\oplus j} \otimes P_d'))$$

$$= f'_*(1 + c_1(\omega_f \otimes \mathcal{L}^{\oplus j} \otimes P_d') + \frac{c_2(\omega_f' \otimes \mathcal{L}^{\oplus j} \otimes P_d')}{2})$$

$$\cdot \left(1 - \frac{c_1(\omega_f')}{2} + \frac{c_1(\omega_f') + \text{Sing}(f'))}{12}\right) \bigg|_{\bigwedge^j}.$$

where $\text{Sing}(f') \subset \mathcal{X}'_d$ denotes the codimension 2 singular locus of the morphism $f'$, therefore $f'_*[\text{Sing}(f')] = \Delta_0 + \Delta_0'' + 2\Delta_{\text{ram}}$. We finish the proof using Mumford's formula $\kappa_1 = f'_*(c_1^2(\omega_f')) = 12\lambda - (\delta_0' + \delta_0'' + 2\delta_{\text{ram}})$ and noting that $f'_*(c_1(\mathcal{L}) \cdot c_1(P_d')) = 0$ (the restriction of $\mathcal{L}$ to the exceptional divisor of $f' : \mathcal{X}'_d \to \mathcal{S}'_d(R^0_g/M^0_g)$ is trivial) and $f'_*(c_1(\omega_f') \cdot c_1(P_d')) = 0$. Finally, according to Proposition 1.6 we have $f'_*(c_1^2(\omega_f')) = -\delta_{\text{ram}} / 2$. \hfill \square
Remark 2.7. While the construction of the vector bundles $\mathcal{A}_{a,i}$ depends on the choice of the Poincaré bundle $\mathcal{L}$ and that of the Prym bundle $\mathcal{P}^0_d$, it is easy to check that if we set the vector bundles $A := \bigwedge^1 \mathcal{H} \otimes \mathcal{A}_{0,0}$ and $B := \mathcal{A}_{-1,0}$, then the vector bundle $\Hom(A, B)$ on $\mathcal{G}_d(\mathcal{R}_g^0/\mathcal{M}_g^0)$, as well as the morphism

$$\phi \in H^0(\mathcal{G}_d(\mathcal{R}_g^0/\mathcal{M}_g^0), \Hom(A, B))$$

whose degeneracy locus is the virtual divisor $\overline{D}_{g,k}$, are independent of such choices.

More precisely, let us denote by $\Xi$ the collection of triples $\alpha := (\pi_\alpha, \mathcal{L}_\alpha, (\mathcal{P}^0_d)_\alpha)$, where $\pi_\alpha : \Sigma_\alpha \to \mathcal{G}_d(\mathcal{R}_g^0/\mathcal{M}_g^0)$ is an étale surjective morphism from a scheme $\Sigma_\alpha$, $(\mathcal{P}^0_d)_\alpha$ is a Prym bundle and $\mathcal{L}_\alpha$ is a Poincaré bundle on $p_{2,0} : X'_d \times \mathcal{G}_d(\mathcal{R}_g^0/\mathcal{M}_g^0) \to \Sigma_\alpha$. Recall that if $\Sigma \to \mathcal{G}_d(\mathcal{R}_g^0/\mathcal{M}_g^0)$ is an étale surjection from a scheme and $\mathcal{L}$ and $\mathcal{L}'$ are two Poincaré bundles on $p_2 : X'_d \times \mathcal{G}_d(\mathcal{R}_g^0/\mathcal{M}_g^0) \to \Sigma$, then the sheaf $N := p_{2*} \Hom(\mathcal{L}, \mathcal{L}')$ is invertible and there exists a canonical isomorphism $\mathcal{L} \otimes p_{2*} N \cong \mathcal{L}'$. For every $\alpha \in \Xi$ we construct the morphism between vector bundles of the same rank $\phi_\alpha : \mathcal{A}_{\alpha} \to \mathcal{B}_{\alpha}$ as above. Then since a straightforward cocycle condition is met, we find that there exists a vector bundle $\Hom(A, B)$ on $\mathcal{G}_d(\mathcal{R}_g^0/\mathcal{M}_g^0)$ together with a section $\phi \in H^0(\mathcal{G}_d(\mathcal{R}_g^0/\mathcal{M}_g^0), \Hom(A, B))$ such that for every $\alpha = (\pi_\alpha, \mathcal{L}_\alpha, (\mathcal{P}^0_d)_\alpha) \in \Xi$ we have

$$\pi^*_\alpha(\Hom(A, B)) = \Hom(A_{\alpha}, B_{\alpha}) \quad \text{and} \quad \pi^*_\alpha(\phi) = \phi_\alpha.$$ 

We are finally in a position to compute the class of the divisor $\overline{D}_{g,k}$.

Theorem 2.8. Fix integers $k \geq 2$, $b \geq 0$ and set $i := kb - b + k - 2$, $r := kb + k - 2$, $g := ik + 1$, $d := rk$ as above. Then there exists a morphism $\phi : \bigwedge^1 \mathcal{H} \otimes \mathcal{A}_{0,0} \to \mathcal{A}_{-1,1}$ between vector bundles of the same rank over $\mathcal{G}_d(\mathcal{R}_g^0/\mathcal{M}_g^0)$, such that the push-forward under $\alpha$ of the restriction to $\mathcal{G}_d(\mathcal{R}_g^0/\mathcal{M}_g^0)$ of the degeneration locus of $\phi$ is precisely the effective divisor $\overline{D}_{g,k}$. Moreover we have the following expression for its class in $A^1(\mathcal{R}_g)$:

$$\sigma_\alpha(e_1(\mathcal{A}_{-1,1} - \bigwedge^1 \mathcal{H} \otimes \mathcal{A}_{0,0})) = \left( \frac{r}{b} \frac{N}{(r + k)} (kr + k - r - 3) \left( \mathfrak{A} - \frac{\mathfrak{B}_0}{b} (\delta'^0_0 + \delta''_0) - \frac{\mathfrak{B}'_0}{12} \delta''_0 \right) \right),$$

where

$$\mathfrak{A} = (k^5 - 4k^4 + 5k^3 - 2k^2)b^3 + (3k^5 - 13k^4 + 24k^3 - 23k^2 + 9k)b^2$$

$$+ (3k^5 - 14k^4 + 43k^3 - 45k^2 + 24k - 4)b + k^5 - 5k^4 + 15k^3 - 25k^2 + 16k - 2,$$

$$\mathfrak{B}_0 = (k^5 - 4k^4 + 5k^3 - 2k^2)b^3 + (3k^5 - 13k^4 + 22k^3 - 17k^2 + 5k)b^2$$

$$+ (3k^5 - 14k^4 + 30k^3 - 33k^2 + 14k - 2)b + k^5 - 5k^4 + 13k^3 - 19k^2 + 10k$$

and

$$\mathfrak{B}'_0 = (4k^5 - 16k^4 + 20k^3 - 8k^2)b^3 + (12k^5 - 52k^4 + 85k^3 - 65k^2 + 20k)b^2$$

$$+ (12k^5 - 56k^4 + 111k^3 - 114k^2 + 53k - 8)b + 4k^5 - 20k^4 + 46k^3 - 58k^2 + 34k - 6.$$
Proof. To compute the class of the degeneracy locus of \( \phi \) we use the exact sequence (6) and Proposition 2.6. We write the following identities in \( A^1(\mathcal{O}_d(\mathbb{P}^0_{\mathfrak{g}}/\mathbb{M}^0_{\mathfrak{g}})) \):

\[
c_1(A_{i-1,1} - \bigwedge^i \mathcal{H} \otimes A_{0,0}) = \sum_{j=0}^i (-1)^{i-j} c_1(\bigwedge^{i-j} \mathcal{H} \otimes A_{0,i})
\]

\[
= \sum_{j=0}^i (-1)^{i-j} \left((ld + g - 1) \left(\binom{r}{i-j} c_1(\mathcal{H}) + \binom{r + 1}{i-j} c_1(A_{0,i})\right)\right)
\]

\[
= -k \left(\frac{kb + k - 4}{b - 1}\right) c_1(\mathcal{H}) + \frac{1}{2} \left(\frac{kb + k - 3}{b}\right) b
\]

\[
- \left(\frac{kb + k - 2}{b}\right) \lambda - \frac{kb + k - 2b}{b} - 3 \left(\frac{kb + k - 3}{b}\right) a + \frac{1}{4} \left(\frac{kb + k - 2}{b}\right) \delta_{\text{ram}}
\]

\[
= \left(\frac{r - 1}{b}\right) \left(-\frac{kb}{r - 1} c_1(\mathcal{H}) + \frac{1}{2} b - \frac{r - 2b - 1}{2(r - 1)} a - \frac{r - b}{4(r - b)} \delta_{\text{ram}}\right),
\]

where \( \delta_{\text{ram}} = \sigma^*(\delta^*_{\text{ram}}) \in A^1(\mathcal{O}_d(\mathbb{P}^0_{\mathfrak{g}}/\mathbb{M}^0_{\mathfrak{g}})) \). The classes \( a, b \in A^1(\mathcal{O}_d(\mathbb{P}^0_{\mathfrak{g}}/\mathbb{M}^0_{\mathfrak{g}})) \) and the line bundle \( \mathcal{H} \in \text{Pic}(\mathcal{O}_d(\mathbb{P}^0_{\mathfrak{g}}/\mathbb{M}^0_{\mathfrak{g}})) \) are defined in terms of a Poincaré bundle \( L \). If \( L' := L \otimes f^*(\mathcal{M}) \) is another Poincaré bundle with \( \mathcal{M} \in \text{Pic}(\mathcal{O}_d(\mathbb{P}^0_{\mathfrak{g}}/\mathbb{M}^0_{\mathfrak{g}})) \) and if \( a', b', \mathcal{H}' \) denote the classes defined in terms of \( L' \) using (5), then we have the formulas

\[
a' = a + 2dc_1(\mathcal{M}), \quad b' = b + (2g - 2)c_1(\mathcal{M}), \quad c_1(\mathcal{H}') = c_1(\mathcal{H}) + (r + 1)c_1(\mathcal{M}).
\]

A straightforward calculation shows that the class

\[
\Xi := -\frac{kb}{r - 1} c_1(\mathcal{H}) + \frac{1}{2} b - \frac{r - 2b - 1}{2(r - 1)} a \in A^1(\mathcal{O}_d(\mathbb{P}^0_{\mathfrak{g}}/\mathbb{M}^0_{\mathfrak{g}}))
\]

is independent of the choice of \( L \) and \( \sigma_a(\Xi) = \pi^*(\sigma_{a*}(\Xi_0)) \), where the \( \Xi_0 \in A^1(\mathcal{O}_d) \) is defined by the same formula (7) but inside \( \text{Pic}(\mathcal{O}_d) \). We outline below the computation of \( \pi^*(\sigma_{a*}(\Xi_0)) \), which uses (22) in an essential way.

We follow closely (F2) and denote by \( \mathbb{M}^0_{\mathfrak{g}} := \mathbb{M}^0_{\mathfrak{g}} \cup \Delta^0_{\mathfrak{g}} \cup \Delta^0_{\mathfrak{g}} \) the partial compactification of \( \mathbb{M}^0_{\mathfrak{g}} \) obtained from \( \mathbb{M}^0_{\mathfrak{g}} \) by adding the stack \( \Delta^0_{\mathfrak{g}} \subset \Delta_{\mathfrak{g}} \) consisting of curves \( [C, y] \), where \( [C, y] \in M_{k-1,1} \) is a Brill–Noether general pointed curve and \( [E, y] \in \mathbb{M}^0_{\mathfrak{g}} \). We extend \( \sigma_0 : \mathbb{O}_d \rightarrow \mathbb{M}_g \) to a proper map \( \sigma_1 : \mathbb{O}_d \rightarrow \mathbb{M}_g \) from the Deligne–Mumford stack of limit linear series \( \mathfrak{g}_d \), cf. (EH), (F2), (KH). Then for each \( n \geq 1 \) we consider the vector bundles \( \mathbb{O}_{0,n} \) over \( \mathbb{O}_d \) defined in (F2) Proposition 2.8 with the following description of their fibres:

- \( \mathbb{O}_{0,n}(C, L) = H^0(C, L^\otimes n) \) for each \( [C] \in M_{k-1,1} \) and \( L \in W_{k}^L(C) \).
- \( \mathbb{O}_{0,n}(C, L^\otimes n) = H^0(C, L^\otimes n) \oplus \mathbb{C} \cdot u^n \subset H^0(C, L^\otimes n) \), where \( t = (C, y, L \in W_{k}^L(C)) \in \sigma_{0}^{-1}([C, y]) \) with \( u \in H^0(C, L) \) being a section such that

\[
H^0(C, L) = H^0(C, L(-y - q)) \oplus \mathbb{C} \cdot u.
\]
We extend the classes \( \alpha, \beta \in A^1(\Omega_d^g) \) over the stack \( \Omega_d^g \) by choosing a Poincaré bundle over \( \Omega_g^1 \times \Omega_g^1 \) which restricts to line bundles of bidegree \((d,0)\) on curves \([C \cup E] \in \Delta_g^1\). Grothendieck–Riemann–Roch applied to the universal curve over \( \Omega_g^1 \) gives that
\[
c_1(\Omega_{0,n}) = \lambda - \frac{n}{2} b + \frac{n^2}{2} a \in A^1(\Omega_d^g) \quad \text{for all } n \geq 2, \tag{8}
\]
while obviously \( \sigma^* (\Omega_{0,1}) = \mathcal{H} \). We now fix a general pointed curve \([C, q] \in \mathcal{M}_{g-1}\) and an elliptic curve \([E, y] \in \mathcal{M}_{1,1}\) and consider the test curves (see also [F8, p. 7])
\[
C^0 := \{ [C/(y \sim q)]_y \in \mathcal{M}_g \} \subset \mathcal{M}_g^1 \quad \text{and} \quad C^1 := \{ [C \cup E]_y \in \mathcal{C}_g \} \subset \mathcal{M}_g^1.
\]
For \( n \geq 1 \), the intersection numbers \( C^0 \cdot (\sigma_0)_* (c_1(\Omega_{0,n})) \) and \( C^1 \cdot (\sigma_0)_* (c_1(\Omega_{0,n})) \) can be computed using [F8] Lemmas 2.6 and 2.13 and Proposition 2.12. Together with the relation (cf. [F8] p. 15) for details
\[
(\sigma_0)_* (c_1(\Omega_{0,n}))-12(\sigma_0)_* (c_1(\Omega_{0,n}))_b + (\sigma_0)_* (c_1(\Omega_{0,n}))_b = 0,
\]
this completely determines the classes \( (\sigma_0)_* (c_1(\Omega_{0,n})) \in A^1(\Omega_d^g) \). Then using \( \text{[8]} \) we find
\[
(\sigma_0)_* (a) = N \left( - \frac{r k (r^2 k^2 - 3 r^2 k + 3 r k^2 + 2 r^2 + 2 k^2 + 4 k - 7 r k - 4 r - 10)}{(r k - r + k + 3) (r k - r + k - 2)} \lambda + \frac{r k (r^2 k^2 - 3 r^2 k + 3 r k^2 - 8 r k + 2 r^2 + 2 k^2 + 4 k - 7 r k - 4 r - 10)}{6 (r k - r + k + 3) (r k - r + k - 2)} \delta_0 + \cdots \right),
\]
\[
(\sigma_0)_* (b) = N \left( - \frac{6 r k}{r k - r + k - 2} \lambda + \frac{r k}{r k - r + k - 2} \delta_0 + \cdots \right),
\]
is this completes the computation of the class \( (\sigma_0)_* (\mathcal{Z}) \) and finishes the proof. \( \square \)

The rather unwieldy expressions from Theorem 2.8 simplify nicely for \( k = 2, 3 \) when we obtain Theorems 0.2 and 0.3.

**Proof of Theorem 0.1 when \( g = 2i + 1 \).** We construct an effective divisor on \( \overline{\mathcal{M}_g} \) satisfying the inequalities (2) as follows: The pull-back to \( \overline{\mathcal{M}_g} \) of the Harris–Mumford divisor \( \mathcal{M}_{g, i+1}^1 \) of curves of genus \( 2i + 1 \) with a \( \mathcal{Z}_{i+1} \) is given by the formula
\[
\pi^*(\mathcal{M}_{g, i+1}^1) = \frac{(2i - 2)!}{(i + 1)(i - 1)!} \times \left( 6(i + 2) \lambda - (i + 1) (\delta_0' + \delta_0^{ram}) - \sum_{j=1}^{i} 3 j (g - j) (\delta_j + \delta_{g-j} + \delta_{j,g-j}) \right).
\]
We split $\overline{D}_{2i+1,2}$ into boundary components of compact type and their complement,

$$\overline{D}_{2i+1,2} \equiv E + \sum_{j=1}^{i} (a_j \delta_j + a_{g-j} \delta_{g-j} + a_{j+g-j} \delta_{j+g-j}),$$

where $a_j, a_{g-j}, a_{j+g-j} \geq 0$ and $\Delta_j, \Delta_{g-j}, \Delta_{j+g-j} \subseteq \text{supp}(E)$ for $1 \leq j \leq i$, and we consider the following positive linear combination on $R_g$:

$$A := \frac{i! (i-1)!}{(2i-1) (2i-3)!} \cdot \pi^* (\overline{M}_{2i+1,i+1}^{1}) + \frac{4(i!)^2}{(2i)!} \cdot E$$

$$\equiv \frac{4(i! + 5)}{i+1} \lambda - 2(\delta_0' + \delta_0'') - 3\delta_{\text{ram}} - \cdots,$$

where each of the coefficients of $\delta_j, \delta_{g-j}$ and $\delta_{j+g-j}$ in the expansion of $A$ is at least

$$\frac{6(i - 1) j (2i + 1 - j)}{(2i - 1)(i + 1)} \geq 2.$$

Since $\frac{4(i! + 5)}{i+1} < 13$ for $i \geq 8$, the conclusion now follows using (2). For $i = 7$ we find that $A \equiv 13\lambda - 2(\delta_0' + \delta_0'') - 3\delta_{\text{ram}} - \cdots$, hence $\kappa(\overline{R}_{15}) \geq 0$. To obtain $\kappa(\overline{R}_{15}) \geq 1$, we use the fact that on $\overline{M}_{15}$ there exists a Brill–Noether divisor other than $\overline{M}_{15,8}$, namely the divisor $\overline{M}_{15,14}$ of curves $[C] \in \overline{M}_{15}$ with a $g_{14}$. This divisor has the same slope $s(\overline{M}_{15,14}) = s(\overline{M}_{15,8}) = 27/4$, but $\text{supp}(\overline{M}_{15,14}) \neq \text{supp}(\overline{M}_{15,8})$. It follows that there exist constants $\alpha, \beta, \gamma, m \in \mathbb{Q}_{>0}$ such that

$$\alpha \cdot E + \beta \cdot \pi^* (\overline{M}_{15,8}) \equiv \bar{A} \cdot E + \gamma \cdot \pi^* (\overline{M}_{15,14}) \in |m K_{\overline{R}_{15}}|.$$

Thus we have found distinct multicanonical divisors on $\overline{M}_{15}$, that is, $\kappa(\overline{M}_{15}) \geq 1$. \qed

**Remark 2.9.** The same numerical argument shows that if one replaces $\overline{M}_{15,8}$ with any divisor $D \in \text{Eff}(\overline{M}_{15})$ with $s(D) < s(\overline{M}_{15,8}) = 27/4$, then $\overline{R}_{15}$ is of general type. Any counterexample to the Slope Conjecture on $\overline{M}_{15}$ makes $\overline{R}_{15}$ of general type.

### 3. Koszul cohomology of Prym canonical curves

We recall that for a curve $C$, a line bundle $L \in \text{Pic}^d(C)$ and integers $i, j \geq 0$, the Koszul cohomology group $K_{i,j}(C, L)$ is obtained from the complex

$$\bigwedge^{i+1} H^0(L) \otimes H^0(L^{(j-1)}) \xrightarrow{d_{i+1,i-1}} \bigwedge^i H^0(L) \otimes H^0(L^{(j)}) \xrightarrow{d_{i,j}} \bigwedge^{i-1} H^0(L) \otimes H^0(L^{(j+1)}),$$

where $d_{i,j}$ is the Koszul differential.
where the maps are the Koszul differentials (cf. [GL]). There is a well-known connection between Koszul cohomology groups and Lazarsfeld bundles. Assuming that $L$ is globally generated, a diagram chasing argument involving exact sequences of the type

$$0 \to \bigwedge^a M_L \otimes L^{\otimes b} \to \bigwedge^a H^0(L) \otimes L^{\otimes b} \to \bigwedge^{a-1} M_L \otimes L^{\otimes (b+1)} \to 0,$$

for various $a, b \geq 0$, yields the following identification (see also [GL, Lemma 1.10]):

$$K_{i,j}(C, L) = \frac{H^0(C, \bigwedge^i M_L \otimes L^{\otimes j})}{\text{Image} \{\bigwedge^{i+1} H^0(C, L) \otimes H^0(C, L^{\otimes (j-1)})\}}.$$  

(9)

We fix $[C, \eta] \in \mathcal{R}_g$, set $L := K_C \otimes \eta \in W^{g-2}_{2g-2}(C)$ and consider the Prym-canonical map $C \dashrightarrow \mathbb{P}^{g-2}$. We denote by $\mathcal{I}_C \subset \mathcal{O}_{\mathbb{P}^{g-2}}$ the ideal sheaf of the Prym-canonical curve.

By analogy with [F2] we study the Koszul stratification of $\mathcal{R}_g$ and define the strata

$$\mathcal{U}_{g,i} := \{[C, \eta] \in \mathcal{R}_g : K_{i,2}(C, K_C \otimes \eta) \neq \emptyset\}.$$

Using (9) we write the series of equivalences

$$[C, \eta] \in \mathcal{U}_{g,i} \iff H^1(C, \bigwedge^{i+1} M_L \otimes L) \neq \emptyset \iff h^0(C, \bigwedge^{i+1} M_L \otimes L) > \frac{(i+1)(2g-2) + (g-1)}{g-2}.$$

Next we write down the exact sequence

$$0 \to H^0(\bigwedge^{i+1} M_{\mathbb{P}^{g-2}}(1)) \overset{\partial}{\to} H^0(C, \bigwedge^{i+1} M_L \otimes L) \to H^1(\bigwedge^{i+1} M_{\mathbb{P}^{g-2}} \otimes \mathcal{I}_C(1)) \to 0,$$

and then also

$$\text{Coker}(\partial) = H^1(\mathbb{P}^{g-2}, \bigwedge^{i+1} M_{\mathbb{P}^{g-2}} \otimes \mathcal{I}_C(1)) = H^0(\mathbb{P}^{g-2}, \bigwedge^i M_{\mathbb{P}^{g-2}} \otimes \mathcal{I}_C(2)).$$

Using the well-known fact that $h^0(\mathbb{P}^{g-2}, \bigwedge^{i+1} M_{\mathbb{P}^{g-2}}(1)) = \binom{g-1}{i+2}$ (use for instance the Bott vanishing theorem), we end up with the following equivalence:

$$[C, \eta] \in \mathcal{U}_{g,i} \iff h^0(\mathbb{P}^{g-2}, \bigwedge^i M_{\mathbb{P}^{g-2}} \otimes \mathcal{I}_C(2)) > \frac{(g-3)}{i} \left(\frac{(g-1)(g-2i-6)}{i+2}\right) > \frac{g-3}{i} \left(\frac{(g-1)(g-2i-6)}{i+2}\right).$$  

(10)

**Proposition 3.1.**  
1. For $g < 2i + 6$, we have $K_{i,2}(C, K_C \otimes \eta) \neq \emptyset$ for any $[C, \eta] \in \mathcal{R}_g$, that is, the Prym-canonical curve $C \dashrightarrow \mathbb{P}^{g-2}$ does not satisfy property (N).  
2. For $g = 2i + 6$, the locus $\mathcal{U}_{g,i}$ is a virtual divisor on $\mathcal{R}_g$, that is, there exist vector bundles $G_{i,2}$ and $H_{i,2}$ over $\mathcal{R}_g$ such that $\text{rank}(G_{i,2}) = \text{rank}(H_{i,2})$, together with a bundle morphism $\phi : H_{i,2} \to G_{i,2}$ such that $\mathcal{U}_{g,i}$ is the degeneracy locus of $\phi$.  

Proof. Part (1) is an immediate consequence of (10), since we have the equivalence
\[ K_{t,2}(C, K_C \otimes \eta) = 0 \Leftrightarrow h^0(P_{g-2}, \bigwedge^i M_{P_{g-2}} \otimes I_C(2)) = \left( \frac{(g-3)(g-2i-6)}{i+2} \right). \]
For part (2) one constructs two vector bundles \( G_{t,2} \) and \( H_{t,2} \) over \( \bar{R}_g \) having fibres
\[ G_{t,2}[C, \eta] = H^0(C, \bigwedge^i M_{K_C \otimes \eta}(2)) \quad \text{and} \quad H_{t,2}[C, \eta] = H^0(P_{g-2}, \bigwedge^i M_{P_{g-2}}(2)). \]
There is a natural morphism \( \phi : H_{t,2} \rightarrow G_{t,2} \) given by restriction. We have
\[ \text{rank}(G_{t,2}) = \left( \frac{g-2}{i} \right) \left( \frac{-i(2g-2)}{g-2} + 3(g-1) \right) \quad \text{and} \quad \text{rank}(H_{t,2}) = (i+1) \left( \frac{g}{i+2} \right) \]
and the condition that \( \text{rank}(G_{t,2}) = \text{rank}(H_{t,2}) \) is equivalent to \( g = 2i + 6 \).

We describe a set-up that will be used to define certain tautological sheaves over \( \bar{R}_g \) and compute the class \([\bar{D}_{g,i}]^{vir}\). We use the notation from Subsection 1.1, in particular from Proposition 1.7 and recall that \( f : X' \rightarrow \bar{R}_g \) is the universal Prym curve, \( P \in \text{Pic}(X') \) denotes the universal Prym line bundle and \( N_i = f'_*(\omega^{\otimes i}_f \otimes P^{\otimes i}) \). We denote by \( T := E''_g \cap \text{Sing}(f) \) the codimension 2 subvariety corresponding to Wirtinger covers \([C_{yg}, \eta \in \text{Pic}(C_{yg})][2], v(y) = v(q) \in X \) (where \( v^*(\eta) = O_C \)), with the marked point being the node of the underlying curve \( C_{yg} \). Let us fix a point \( [X := C_{yg}, \eta, \beta] \in \bar{A}_{g}' \cup \bar{A}_{g}'' \)
where as usual \( v : C \rightarrow X \) is the normalization map. Then we have an identification
\[ \mathcal{N}_i[X, \eta, \beta] = \text{Ker}[H^0(C, \omega_C(y+q) \otimes \eta_C) \rightarrow (v_* \omega_C/O_C) \otimes \omega_X \otimes \eta \equiv C_{y,q}], \quad (11) \]
where the map is given by taking the difference of residues at \( y \) and \( q \). Note that when \( \eta_C = O_C \), that is, when\([X, \eta, \beta] \in \bar{A}_{g}'\), we have \( \mathcal{N}_i[X, \eta, \beta] = H^0(C, \omega_C) \).

For a point
\[ [X = C \cup [y, q], E, \eta_C \in \sqrt{\omega_C}(-y-q), \eta_E] \in \bar{A}_{g}'' \]
we have an identification
\[ \mathcal{N}_i[X, \eta, \beta] = \text{Ker}[H^0(C, \omega_C(y+q) \otimes \eta_C) \otimes H^0(E, \omega_E(1)) \rightarrow (\omega_X \otimes \eta) \equiv C_{y,q}], \quad (12) \]
We set
\[ \mathcal{M} := \text{Ker}[f^*(\mathcal{N}_1) \rightarrow \omega_f \otimes P]. \]

From the discussion above it is clear that the image of \( f^*(\mathcal{N}_1) \rightarrow \omega_f \otimes P \) is \( \omega_f \otimes P \otimes I_T \). Since \( T \subset X' \) is smooth of codimension 2 it follows that \( \mathcal{M} \) is locally free. For \( a, b \geq 0 \), we define the sheaf \( \mathcal{E}_{a,b} := f_* (\bigwedge^a \mathcal{M} \otimes \omega^{\otimes b}_f \otimes P^{\otimes b}) \) over \( \bar{R}_g \). Clearly \( \mathcal{E}_{a,b} \) is locally free. We have \( \mathcal{E}_{0,b} = \mathcal{N}_b \) for \( b \geq 0 \), and we always have left-exact sequences
\[ 0 \rightarrow \mathcal{E}_{a,b} \rightarrow \bigwedge^a \mathcal{E}_{0,1} \otimes \mathcal{E}_{0,b} \rightarrow \mathcal{E}_{a-1,b+1}, \quad (13) \]
which are right-exact off the divisor \( \bar{A}_{g}' \) (to be proved later). We then define inductively a sequence of vector bundles \([\mathcal{H}_{a,b}]_{a,b \geq 0} \) over \( \bar{R}_g \) in the following way: We set \( \mathcal{H}_{0,b} := \).
For Proposition 3.2. we prove that

\[ \text{Proof.} \]

For a point \([X, \eta, \beta] \in \tilde{\mathcal{R}}_g\), if we use the identification \(H^0(X, \omega_X \otimes \eta) = H^0(\mathbb{P}^{X^2}, \mathcal{O}_{\mathbb{P}^{X^2}}(1))\), we have a natural identification of the fibre

\[ H_{a,b}[X, \eta, \beta] = H^0(\mathbb{P}^{X^2}, \wedge^a M_{\mathbb{P}^{X^2}}(b)). \]

By induction on \(a \geq 0\), there exist vector bundle morphisms \(\phi_{a,b} : H_{a,b} \to E_{a,b}\).

**Proposition 3.2.** For \(b \geq 2\) and \(a \geq 0\) we have the vanishing of the higher direct images

\[ R^1 f_*(\wedge^a \mathcal{M} \otimes \omega_f^{\otimes b} \otimes \mathcal{D}^{\otimes b})|_{\mathcal{R}_a} = 0. \]

It follows that the sequences (13) are right-exact off the divisor \(\tilde{\Delta}_0^a\) of \(\tilde{\mathcal{R}}_g\).

**Proof.** Over the locus \(\mathcal{R}_E\) the vanishing is a consequence of Proposition 2.4. For simplicity we prove that \(R^1 f_*(\wedge^a \mathcal{M} \otimes \omega_f^{\otimes b} \otimes \mathcal{D}^{\otimes b}) \otimes \mathcal{O}_{\tilde{\Delta}_0^a} = 0\), the vanishing over \(\tilde{\Delta}_0^a\) being similar. We fix a point \([X = C \cup [y, q] \mathcal{E}, \eta_C, \eta_E] \in \tilde{\Delta}_0^{\mathcal{E}}\) with \(\eta_C^{\otimes 2} = \mathcal{O}_C(-y - q), \eta_E = \mathcal{O}_E(1)\) and set \(L := \omega_X \otimes \eta \in \text{Pic}^{\mathcal{E}^2}(X)\). We show that \(H^1(X, \wedge^a \mathcal{M} \otimes L^{\otimes b}) = 0\) for all \(a \geq 0\) and \(b \geq 2\). A Mayer–Vietoris argument shows that it suffices to prove that

\[ H^1(C, \wedge^a \mathcal{M}_L \otimes L^{\otimes b} \otimes \mathcal{O}_C) = 0, \quad H^1(E, \wedge^a \mathcal{M}_L \otimes L^{\otimes b} \otimes \mathcal{O}_E) = 0, \]

\[ H^1(C, \wedge^a \mathcal{M}_L \otimes L^{\otimes b} \otimes \mathcal{O}_C(-y - q)) = 0. \]

For \(L_C := L \otimes \mathcal{O}_C = K_C(y + q) \otimes \eta_C\) and \(L_E := L_E \otimes \mathcal{O}_E\), we write the exact sequences

\[ 0 \to H^0(C, L_C(-y - q)) \otimes \mathcal{O}_E \to M_L \otimes \mathcal{O}_E \to M_L \to 0, \]

\[ 0 \to H^0(E, L_E(-y - q)) \otimes \mathcal{O}_E \to M_L \otimes \mathcal{O}_E \to M_L \to 0, \]

and we find that \(M_L \otimes \mathcal{O}_C = M_L\) while obviously \(M_L_E = \mathcal{O}_E(-1)\). We conclude that the statements (15) and (16) for all \(a \geq 0\) and \(b \geq 2\) can be reduced to showing that

\[ H^1(C, \wedge^a M_{L_C} \otimes L_C^{\otimes b}) = H^1(C, \wedge^a M_{L_C} \otimes L_C^{\otimes b} \otimes \mathcal{O}_C(-y - q)) = 0 \quad \text{for all} \ a \geq 0, b \geq 2. \]

This is now a direct application of Proposition 2.4. \(\square\)

**Proof of Theorem 0.6.** We have constructed the vector bundle morphism \(\phi_{1,2} : H_{1,2} \to E_{1,2}\) over \(\mathcal{R}_g\). For \(g = 2l + 6\) we have \(\text{rank}(H_{1,2}) = \text{rank}(E_{1,2})\) and the virtual Koszul class \([\tilde{H}_{g,1}]^{\text{vir}}\) is given by \(e_1(E_{1,2} - H_{1,2})\). We recall that for a rank \(e\) vector bundle \(E\) over a
variety $X$ and for $i \geq 1$, we have the formulas $c_1(\wedge^i \mathcal{E}) = (c_{i-1}) c_1(\mathcal{E})$ and $c_1(\text{Sym}^i(\mathcal{E})) = (\epsilon^{i+1}) c_1(\mathcal{E})$. Using \cite{13}, we find that there exists a constant $\alpha \geq 0$ such that

$$c_1(\mathcal{E}_{i,2}) - \alpha \cdot \delta_0^i = \sum_{l=0}^{i} (-1)^l c_1(\wedge^{i-l} \mathcal{E}_{0,1} \otimes \mathcal{E}_{0,l+2}) = \sum_{l=0}^{i} (-1)^l \left( \binom{g+1}{i-1} \right) c_1(\mathcal{E}_{0,l+2})$$

while a repeated application of the exact sequence \cite{14} gives that

$$c_1(\mathcal{H}_{i,2}) = \sum_{l=0}^{i} (-1)^l c_1(\wedge^{i-l} \mathcal{H}_{0,1} \otimes \text{Sym}^{i+2}(\mathcal{H}_{0,1}))$$

$$= \sum_{l=0}^{i} (-1)^l \left( \binom{g+1}{i-1} \right) c_1(\text{Sym}^{i+2}(\mathcal{H}_{0,1})) + \binom{g+2}{i-1} c_1(\mathcal{H}_{0,1})$$

$$= \sum_{l=0}^{i} (-1)^l \left( \binom{g+1}{i-1} \binom{g+2}{i-1} \right) c_1(\mathcal{H}_{0,1}),$$

with $\mathcal{E}_{0,1} = \mathcal{H}_{0,1} = N_1$ and $\mathcal{E}_{0,l+2} = N_{l+2}$ for $l \geq 0$. Proposition \cite{17} finishes the proof.

Comparing these formulas with the canonical class of $\overline{\mathcal{R}}_g$, one finds that $\overline{\mathcal{R}}_g$ is of general type for $g > 12$.

4. Effective divisors on $\overline{\mathcal{R}}_g$

We now use in an essential way results from \cite{F3} to produce myriads of effective divisors on $\overline{\mathcal{R}}_g$. This construction, though less explicit than that of $\overline{\mathcal{U}}_{2g+6}$ and $\overline{\mathcal{D}}_{g,k}$, is still very effective and we use it to show $\overline{\mathcal{R}}_{18}$, $\overline{\mathcal{R}}_{20}$ and $\overline{\mathcal{R}}_{22}$ are of general type.

We consider the morphism $\chi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_{2g-1}$ given by $\chi ([C, \eta]) := [\tilde{C}]$, where $f : \tilde{C} \to C$ is the étale double cover determined by $\eta$. Thus one has

$$f_\ast \mathcal{O}_C = \mathcal{O}_C \oplus \eta$$

and

$$H^i(\tilde{C}, f^\ast L) = H^i(C, L) \oplus H^i(C, L \oplus \eta)$$

for any $L \in \text{Pic}(C)$, $i = 0, 1$.

The pullback map $\chi^\ast$ at the level of Picard groups has been determined by M. Bernstein in \cite{BG} Lemma 3.1.3. We record her results:

**Proposition 4.1.** The pullback map $\chi^\ast : \text{Pic}(\overline{\mathcal{R}}_g) \to \text{Pic}(\overline{\mathcal{M}}_{2g-1})$ is given as follows:

$$\chi^\ast(\lambda) = 2\lambda - \frac{1}{4}\delta_{\text{ram}}^0,$$

$$\chi^\ast(\delta_0) = \delta_{\text{ram}}^0 + 2 \left( \delta_0' + \delta_0'' + \sum_{i=1}^{[g/2]} \delta_{g-i} \right),$$

$$\chi^\ast(\delta_i) = 2\delta_{g-i} \quad \text{for } 1 \leq i \leq g - 1.$$
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**Proof.** The formula for \( \chi^*(\delta_i) \) when \( 1 \leq i \leq g - 1 \) is immediate. To determine \( \chi^*(\lambda) \) one notices that \( \chi^*(\kappa_1)_{\overline{M}_{g-1}} = 2(\kappa_1)_{\overline{R}_g} \) and the rest follows from Mumford’s formulas
\[
(\kappa_1)_{\overline{M}_{g-1}} = 12\lambda - \delta \in \text{Pic}(\overline{M}_{g-1}) \quad \text{and} \quad (\kappa_1)_{\overline{R}_g} = 12\lambda - \pi^*(\delta) \in \text{Pic}(\overline{R}_g).
\]

We set the integer \( g^2 = 1 + \frac{g - 1}{g} (2g) \). In [F3] we have studied the rational map
\[
\phi : \overline{M}_{g-1} \dashrightarrow \overline{M}_{1+g} \left( \frac{g}{g-1} \right), \quad \phi[Y] := W_{g+1}^1(Y),
\]
and determined the pullback map at the level of divisors \( \phi^* : \text{Pic}(\overline{M}_{g}) \rightarrow \text{Pic}(\overline{M}_{g-1}) \). In particular, we proved that if \( A \in \text{Pic}(\overline{M}_{g}) \) is a divisor of slope \( s(A) = s \), then the slope of the pullback \( \phi^*(A) \) is equal to (cf. [F3] Theorem 0.2)
\[
s(\phi^*(A)) = 6 + \frac{8g^3s - 32g^3 - 19g^2s + 66g^2 + 6gs - 16g - 3s + 6}{(g-1)(g+1)(g^2s - 2gs - 4g^2 + 7g + 3)}.
\]

To obtain effective divisors of small slope on \( \overline{R}_g \) we shall consider pullbacks \( (\phi^*)^*(\Lambda) \), where \( \Lambda \in \text{Ample}(\overline{M}_{g}) \). Of course, one can consider the cone \( \chi^*(\text{Ample}(\overline{M}_{g-1})) \), but a quick look at Proposition 4.1 shows that it is impossible to obtain in this way divisors on \( \overline{R}_g \) satisfying the inequalities (2). Pulling back merely effective divisors \( \overline{M}_{g-1} \), rather than ample ones, is problematic since \( \chi(\overline{R}_g) \) tends to be contained in many geometric divisors on \( \overline{M}_{g-1} \). In order for the pullbacks \( \chi^*(\Lambda) \) to be well-defined as effective divisors on \( \overline{R}_g \) we prove the following result:

**Proposition 4.2.** If \( \text{dom}(\phi) \subset \overline{M}_{g-1} \) is the domain of definition of the rational morphism \( \phi : \overline{M}_{g-1} \rightarrow \overline{M}_g \), then \( \chi(\overline{R}_g) \cap \text{dom}(\phi) = \emptyset \). It follows that for any ample divisor \( A \in \text{Ample}(\overline{M}_{g}) \), the pullback \( \chi^*(\Lambda) \in \text{Eff}(\overline{R}_g) \) is well-defined.

**Proof.** We take a general point \([C \cup E, \eta_C = O_C, \eta_E] \in \Delta_1 \subset \overline{R}_g \). The corresponding admissible double cover is then \( f : C_1 \cup y_1 \bar{E} \cup y_2 C_2 \rightarrow C \cup E \), where \([C_1, y_1] \) and \([C_2, y_2] \) are copies of \([C, y] \) mapping isomorphically to \([C, y] \), and \( f : \bar{E} \rightarrow E \) is the étale double cover induced by the torsion point \( \eta_E \in \text{Pic}^0(E)[2] \). We have \( C_i \cap \bar{E} = y_i \), where \( \bar{y}_i = y \). Thus \( \chi[C \cup E, O_C, \eta_E] := [C_1 \cup y_1 \bar{E} \cup y_2 C_2] \), where \( y_1, y_2 \in \bar{E} \) are such that \( O_{\bar{E}}(y_1 - y_2) = 2 \)-torsion point in \( \text{Pic}^0(\bar{E}) \).

Suppose now that \( X := C_1 \cup y_1 E \cup y_2 C_2 \) is a curve of compact type such that \([C_i, y_i] \in \overline{M}_{g-1,1} (i = 1, 2) \) and \([E, y_1, y_2] \in \overline{M}_{1,2} \) are all Brill–Noether general. In particular, the class \( y_1 - y_2 \in \text{Pic}^0(E) \) is not torsion. Then \( \phi([X]) := [\overline{W}_{g+1}^1(X)] \) is the variety of limit linear series \( g^1_{g+1} \) on \( X \). The general point of each irreducible component of \( \overline{W}_{g+1}^1(X) \) corresponds to a refined linear series \( l \) on \( X \) satisfying the compatibility conditions in terms of Brill–Noether numbers (see also [EH], [F3]):
\[
1 = \rho(l_{C_1}, y_1) + \rho(l_{C_2}, y_2) + \rho(l_E, y_1, y_2) = 1,
\]
\[
\rho(l_{C_1}, y_1), \rho(l_{C_2}, y_2), \rho(l_E, y_1, y_2) \geq 0.
\]

If \( \rho(l_{C_2}, y_2) = 1 \), we find two types of components of \( \overline{W}_{g+1}^1(X) \) which we describe: Since \( \rho(l_{C_1}, y_1) = 0 \), there exists an integer \( 0 \leq a \leq g/2 \) such that \( a^{l_{C_1}}(y_1) = (a, g + 2 - a) \).
On $E$ there are two choices for $l_E \in G^1_{g+1}(E)$ such that $a^E(y_1) = (a - 1, g + 1 - a)$. Either $a^E(y_2) = (a, g + 1 - a)$ (there is a unique such $l_E$), and then $l_{C_2}$ belongs to the connected curve $T_a := \{l_{C_2} \in G^1_{g+1}(C_2) : a^{C_2}(y_2) \geq (a, g + 1 - a)\}$, or else, $a^E(y_2) = (a - 1, g + 2 - a)$ (again, there is a unique such $l_E$), and then the $C_2$-aspect of $l$ belongs to the curve $T'_a := \{l_{C_2} \in G^1_{g+1}(C_2) : a^{C_2}(y_2) \geq (a - 1, g + 2 - a)\}$. Thus $\{l_{C_1}\} \times T_a$ and $\{l_{C_2}\} \times T'_a$ are irreducible components of $\overline{W}^1_{g+1}(X)$. If $\rho(l_E, y_1, y_2) = 1$, then there are three types of irreducible components of $\overline{W}^1_{g+1}(X)$ corresponding to the cases

\[
\begin{align*}
a^E(y_1) &= (a - 1, g + 1 - a), & a^E(y_2) &= (a - 1, g + 1 - a) \quad \text{for } 0 \leq a \leq g/2, \\
a^E(y_1) &= (a - 1, g + 1 - a), & a^E(y_2) &= (a, g - a) \quad \text{for } 1 \leq a \leq (g - 1)/2, \\
a^E(y_1) &= (a - 1, g + 1 - a), & a^E(y_2) &= (a - 2, g + 2 - a) \quad \text{for } 2 \leq a \leq (g - 1)/2.
\end{align*}
\]

Finally, the case $\rho(l_{C_1}, y_1) = 1$ is identical to the case $\rho(l_{C_2}, y_2) = 1$ by reversing the roles of the curves $C_1$ and $C_2$. The singular points of $\overline{W}^1_{g+1}(X)$ correspond to (necessarily) crude limit $g^1_{g+1}$’s satisfying $\rho(l_{C_1}, y_1) = \rho(l_{C_2}, y_2) = \rho(l_E, y_1, y_2) = 0$. For such $l$, there must exist two irreducible components of $X$, say $Y$ and $Z$, for which $Y \cap Z = \{x\}$ and such that $a^0_l(x) + a^l(x) = g + 2$ and $a^l(x) + a^0_l(x) = g + 1$. The point $l$ lies precisely on the two irreducible components of $\overline{W}^1_{g+1}(X)$: The one corresponding to refined limit $g^1_{g+1}$ with vanishing sequence on $Y$ equal to $(a^0_l(x) - 1, a^l(x))$, and the one with vanishing $(a^0_l(x), a^l(x) - 1)$ on $Z$. Thus $\overline{W}^1_{g+1}(X)$ is a stable curve of compact type, so $[X] \in \text{dom}(\phi)$. Using [13], this set-theoretic description applies to the image under $\phi$ of any point $[C_1 \cup_1, E \cup_{y_2} C_2]$, in particular to $[C_1 \cup_{y_1} E \cup_{y_2} C_2] = \chi(C \cup_{y} E)$. We have shown that $\chi(A) \cap \text{dom}(\phi) \neq \emptyset$.

**Proof of Theorem 4.1 for genus $g = 18, 20, 22$**. We construct an effective divisor on $\overline{\mathcal{M}}_g$ which satisfies the inequalities $\psi$ and which is of the form

$$\mu \pi^*(D) + \epsilon \pi^*(A) = \alpha \lambda - 2(\delta_0 + \delta_g) - 3\pi^*(\text{ram}) - \sum_{i=1}^{[g/2]} (b_i \delta_i + b_{g-i} \delta_{g-i} + b_{i;g-i} \delta_{i;g-i}),$$

where $A = s \lambda - \delta \in \text{Pic}(\overline{\mathcal{M}}_g)$ is an ample class (which happens precisely when $s > 11$, cf. [2]), $D \in \text{Eff}(\overline{\mathcal{M}}_g)$ and $\mu, \epsilon > 0$ and $\alpha < 13$. We solve this linear system using Proposition 4.1 and find that we must have

$$\epsilon = \frac{8}{12 - s(\phi^*(A))}, \quad \mu = \frac{16 - 2s(\phi^*(A))}{12 - s(\phi^*(A))}. $$

To conclude that $\overline{\mathcal{M}}_g$ is of general type, it suffices to check that the inequality

$$\alpha = \frac{8s(\phi^*(A))}{12 - s(\phi^*(A))} + \left(6 + \frac{12}{g + 1}\right) \frac{16 - 2s(\phi^*(A))}{12 - s(\phi^*(A))} < 13$$

has a solution $s = s(A) \geq 11$. Using [17], we find that this is the case for $g \geq 18$. □
5. The enumerative geometry of $\overline{M}_g$ in small genus

In this section we describe the divisors $D_{g,k}$ and $U_{g,i}$ for small $g$. We start with the case $g = 3$. This result has been first obtained by M. Bernstein [Be] Theorem 3.2.3] using test curves inside $\overline{M}_3$. Our method is more direct and uses the identification of cycles $C - C = \Theta_{QC} \subset \text{Pic}^0(C)$, valid for all curves $[C] \in M_3$.

**Theorem 5.1.** The divisor $D_{3,2} = \{[C, \eta] \in R_3 : \eta \in C - C\}$ is equal to the locus of étale double covers $[\tilde{C} \xrightarrow{f} C] \in R_3$ such that $[\tilde{C}] \in M_3$ is hyperelliptic. We have the equality of cycles $\overline{D}_{3,2} = 8\lambda - \delta_1' - 2\delta_0'' - \frac{1}{2} \lambda_{\text{ram}} - 6\delta_1 - 4\delta_2 - 2\delta_1' \subset \text{Pic}(\overline{M}_3)$. Moreover,

$$\pi_*(\overline{D}_{3,2}) = 56 \cdot \overline{M}_{3,2} = 56 \cdot (9\lambda - \delta_0 - 3\delta_1) \in \text{Pic}(\overline{M}_3).$$

This equality corresponds to the fact that for an étale double cover $f : \tilde{C} \to C$, the source $\tilde{C}$ is hyperelliptic if and only if $C$ is hyperelliptic and $\eta \in C - C \subset \text{Pic}^0(C)$.

**Proof.** We use the set-up from Theorem 2.8 and recall that there exists a vector bundle morphism $\phi \cdot H \otimes A_{0,0} \to A_{0,1}$ over $\overline{R}_3$ such that $Z(\phi) \cap R_3 = D_{3,2}$. Here $H = \pi^*(E)$, $A_{0,0}[X, \eta, \beta] = H^0(X, \omega_X \otimes \beta)$ and $A_{0,1}[X, \eta, \beta] = H^0(X, \omega_X^{\otimes 2} \otimes \beta)$, for each point $[X, \eta, \beta] \in \overline{R}_x$. Using (11) and (12) we check that both $\phi|_{\Delta_3^0}$ and $\phi|_{\Lambda_2'}$ are generically non-degenerate. Over a point $l = [C, \eta, \beta] \in \Delta_3^0$ corresponding to a Wirtinger covering (i.e. $\nu : C \to C_{3,y}$ with $[C] \in M_2$ and $\nu^*(\eta) = \hat{O}_C$), we have

$$\phi(t) : H^0(C, K_C) \otimes H^0(C, K_C \otimes \text{O}(y + q)) \to A_{0,1}(t) \subset H^0(C, \omega_C^{\otimes 2} \otimes \text{O}(2y + 2q)).$$

From the base point free pencil trick we find that $\text{Ker}(\phi(t)) = H^0(C, \text{O}(y + q))$, that is, $\phi|_{\Delta_3^0}$ is everywhere degenerate and the class $c_1(A_{0,1} - H \otimes A_{0,0}) - \delta_0'' \in \text{Pic}(\overline{R}_3)$ is effective. From the formulas $\pi_*(\lambda) = 63\lambda$, $\pi_*(\delta_0') = 30\delta_0$, $\pi_*(\delta_0'' = \delta_0$ and $\pi_*(\lambda_{\text{ram}}) = 16\delta_0$, we obtain

$$s(\pi_*(c_1(A_{0,1} - H \otimes A_{0,0}) - \delta_0'')) = 9.$$

The hyperelliptic locus $\overline{M}_{3,2}$ is the only divisor on $D \in \text{Eff}(\overline{M}_3)$ with $\Delta_i \not\subset \text{supp}(D)$ for $i = 0, 1$ and $s(D) \leq 9$, which leads to the formula $\pi_*(\overline{D}_{3,2}) = 56 \cdot \overline{M}_{3,2}$. □

**Theorem 5.2.** The divisor $\overline{D}_{5,2} := \{[C, \eta] \in R_5 : \eta \in C_2 - C_2\}$ equals the locus of étale double covers $[\tilde{C} \xrightarrow{f} C] \in R_5$ such that the genus 9 curve $\tilde{C}$ is tetragonal. We have the formula $\overline{D}_{5,2} = 14\lambda - 2(\delta_0' + \delta_0'') - 5 \lambda_{\text{ram}} - 10\delta_4 - 4\delta_4 = \cdots \in \text{Pic}(\overline{M}_5)$.

**Proof.** We start with an étale cover $f : \tilde{C} \xrightarrow{21} C$ corresponding to the torsion point $\eta = \text{O}_C(D - E)$ with $D, E \in C_2$. Then

$$H^0(\tilde{C}, \text{O}_C(f^*D)) = H^0(C, \text{O}_C(D)) \otimes H^0(C, \text{O}_C(E)),$$

that is, $f^*D \in G_4^2(\tilde{C})$ and $[\tilde{C}] \in \overline{M}_{9,4}$. Conversely, if $l \in G_4^2(\tilde{C})$, then $l$ must be invariant under the involution of $\tilde{C}$ and then $f_\eta(l) \in G_4^2(\tilde{C})$ contains two divisors of the type $2x + 2y = 2p + 2q$. Then we take $\eta = \text{O}_C(x + y - p - q)$, that is, $[C, \eta] \in \overline{D}_{5,2}$. □
**Remark 5.3.** Since \(\text{codim}(\overline{\mathcal{M}}_{0,4}, \mathcal{M}_{0}) = 3\) while \(D_{5,2}\) is a divisor in \(\mathcal{R}_{3}\), there seems to be a dimensional discrepancy in Theorem 5.2. This is explained by noting that for an étale double covering \(f : \tilde{C} \to C\) over a general curve \([C] \in \mathcal{M}_{2}\), the codimension 1 condition \(\text{gon}(\tilde{C}) \leq 5\) is equivalent to the seemingly stronger condition \(\text{gon}(\tilde{C}) \leq 4\). Indeed, if \(f \in G_{2}(\tilde{C})\) is base point free, then \(f\) is not invariant under the involution of \(\tilde{C}\) and \(\dim |f_*\mathcal{I}| \geq 2\), so \(G_{2}^{1}(C) \neq \emptyset\), a contradiction with the genericity assumption on \(C\).

**Theorem 5.4.** The divisor \(D_{4,3} = \{[C, \eta] \in \mathcal{R}_{4} : \exists A \in W_{3}^{1}(C) \text{ with } H^{0}(C, A \otimes \eta) \neq 0\}\) can be identified with the locus of Prym curves \([C, \eta] \in \mathcal{R}_{4}\) such that the Prym-canonical model \(\mathcal{C} : \frac{|K_{C} \otimes \eta|}{\mathbb{P}^{2}}\) is a plane sextic curve with a triple point. We also have the class formula

\[
\overline{\mathcal{D}}_{4,3} \equiv 8\lambda - \delta_{0} - 2\delta_{0}' - \frac{7}{4}\delta_{0}^\text{ram} - 4\delta_{3} - 7\delta_{1} - 3\delta_{1,3} - \cdots \in \text{Pic}(\overline{\mathcal{R}}_{4}),
\]

hence \(\pi_{4}(\overline{\mathcal{D}}_{4,3}) = 60 \cdot \mathcal{P}^{1}_{4,3} = 60(34\lambda - 4\delta_{0} - 14\delta_{1} - 18\delta_{2}) \in \text{Pic}(\overline{\mathcal{M}}_{4})\), where

\[
\mathcal{G} \mathcal{P}^{1}_{4,3} \subset \mathcal{M}_{4} : = \{[C] \in \mathcal{M}_{4} : \exists A \in W_{3}^{1}(C), A^\otimes = K_{C}\}
\]

is the Gieseker–Petri divisor of curves \([C] \in \mathcal{M}_{4}\) with a vanishing theta-null.

**Proof.** We start with a Prym curve \([C, \eta] \in \mathcal{R}_{4}\) such that there exists \(A \in W_{3}^{1}(C)\) with \(H^{0}(C, A \otimes \eta) \neq 0\). We claim that \(A^\otimes = K_{C}\), that is, \([C] \in \mathcal{G} \mathcal{P}^{1}_{4,3}\). Indeed, assuming the opposite, we find disjoint divisors \(D_{1}, D_{2} \in C_{3}\) such that \(D_{1} \in |A \otimes \eta|\) and \(D_{2} \in |K_{C} \otimes A^\vee \otimes \eta|\). In particular, the subspaces \(H^{0}(C, K_{C} \otimes \eta(-D_{i})) \subset H^{0}(C, K_{C})\) are both of dimension 2, hence they intersect non-trivially, that is, \(H^{0}(C, K_{C} \otimes \eta(-D_{1} - D_{2})) \neq 0\). Since \(D_{1} + D_{2} \equiv K_{C}\), this implies \(\eta = 0\), a contradiction.

The proof that the vector bundle morphism \(\phi : \mathcal{H} \otimes A_{0.0} \to A_{0.1}\) constructed in the proof of Theorem 2.8 is degenerate with order 1 along the divisor \(\Delta_{i}^\prime \subset \mathcal{R}_{4}\) follows from [11]. Thus \(c_{1}(A_{0.1} - \mathcal{H} \otimes A_{0.0}) - \delta_{0}'' \in \text{Pic}(\overline{\mathcal{R}}_{4})\) is an effective class and its push-forward to \(\overline{\mathcal{M}}_{4}\) has slope \(17/2\). The only divisor \(D \in \text{Eff}(\overline{\mathcal{M}}_{4})\) with \(\Delta_{i} \nsubseteq \text{supp}(D)\) for \(i = 0, 1, 2\) and \(s(D) \leq 17/2\) is the theta-null divisor \(\mathcal{G} \mathcal{P}^{1}_{4,3}\) (cf. [13], Theorem 5.1). \(\square\)

**Remark 5.5.** For a general point \([C, \eta] \in \mathcal{R}_{4}\), the Prym-canonical curve \(\iota : C : \frac{|K_{C} \otimes \eta|}{\mathbb{P}^{2}}\) is a plane sextic with 6 nodes which correspond to the preimages of \(\phi^{-1}(\eta)\) under the second difference map

\[
C_{2} \times C_{2} \to \text{Pic}^{0}(C), \quad (D_{1}, D_{2}) \mapsto O_{C}(D_{1} - D_{2}).
\]

Note that \(W_{2}(C) \cdot (W_{2}(C) + \eta) = 6\). For a general \([C, \eta] \in D_{4,3}\), the model \(i(C) \subset \mathbb{P}^{2}\) has a triple point. For a hyperelliptic curve \([C] \in \mathcal{M}_{4,2}\), out of the 255 = \(2^{8} - 1\) étale double covers of \(C\), there exist 210 for which \(C : \frac{|K_{C} \otimes \eta|}{\mathbb{P}^{2}}\) has an ordinary 4-fold point and no other singularity. The remaining 45 = \(2^{8}/2\) coverings correspond to the case \(\eta = O_{C}(x - y)\), with \(x, y \in C\) being Weierstrass points, when \(|K_{C} \otimes \eta|\) has two base points and \(\iota\) is a degree 2 map onto a conic.
The singularities of the moduli space of Prym curves

The moduli space $\overline{R}_g$ is a normal variety with finite quotient singularities. To determine its Kodaira dimension we consider a smooth model $\hat{R}_g$ of $R_g$ and then analyze the growth of the dimension of the spaces $H^0(\hat{R}_g, K_{\hat{R}_g}^\otimes l)$ of pluricanonical forms for all $l \geq 0$. In this section we show that in doing so one only needs to consider forms defined on $R_g$ itself.

**Theorem 6.1.** Fix $g \geq 4$ and let $\hat{R}_g \to \overline{R}_g$ be any desingularization. Then every pluricanonical form defined on the smooth locus $R_{\text{reg}}^g$ of $R_g$ extends holomorphically to $\hat{R}_g$, that is, for all integers $l \geq 0$ we have isomorphisms

$$H^0(\hat{R}_g, K_{\hat{R}_g}^\otimes l) \cong H^0(\overline{R}_g, K_{\overline{R}_g}^\otimes l).$$

A similar statement has been proved for the moduli space $M_g$ of curves (cf. [HM, Theorem 1]) and for the moduli space $S_g$ of spin curves (cf. [Lud, Theorem 4.1]). We start by explicitly describing the locus of non-canonical singularities in $R_g$, which has codimension 2. At a non-canonical singularity there exist local pluricanonical forms that do acquire poles on a desingularization. We show that this situation does not occur for forms defined on the smooth locus $R_{\text{reg}}^g$, and they extend holomorphically to $\hat{R}_g$.

**Definition 6.2.** An automorphism of a Prym curve $(X, \eta, \beta)$ is an automorphism $\sigma \in \text{Aut}(X)$ such that there exists an isomorphism of sheaves $\gamma : \sigma^*\eta \to \eta$ making the following diagram commutative:

$$
\begin{array}{ccc}
\sigma^*\eta \otimes^2 & \xrightarrow{\gamma \otimes^2} & \eta \otimes^2 \\
\sigma^*\beta \downarrow & & \downarrow \beta \\
\sigma^*\mathcal{O}_X & \cong & \mathcal{O}_X
\end{array}
$$

If $C := \text{st}(X)$ denotes the stable model of $X$ then there is a group homomorphism $\text{Aut}(X, \eta, \beta) \to \text{Aut}(C)$ given by $\sigma \mapsto \sigma_C$. The kernel $\text{Aut}_0(X, \eta, \beta)$ of this homomorphism is called the subgroup of inessential automorphisms of $(X, \eta, \beta)$.

**Remark 6.3.** The subgroup $\text{Aut}_0(X, \eta, \beta)$ is isomorphic to $\{\pm 1\}^{\text{CC}(\tilde{X})} / \{\pm 1\}$, where $\text{CC}(\tilde{X})$ is the set of connected components of the non-exceptional subcurve $\tilde{X}$ (compare [CCC Lemma 2.3.2] and [Lud Proposition 2.7]). Given $\gamma_j \in \{\pm 1\}$ for every connected component $\tilde{X}_j$ of $\tilde{X}$ consider the automorphism $\tilde{\gamma}$ of $\tilde{\eta} = \eta|_{\tilde{X}}$ which is multiplication by $\gamma_j$ in every fibre over $\tilde{X}_j$. Then there exists a unique inessential automorphism $\sigma$ such that $\tilde{\gamma}$ extends to an isomorphism $\gamma : \sigma^*\eta \to \eta$ compatible with the morphisms $\sigma^*\beta$ and $\beta$. Considering $(-\gamma_j)_j$ instead of $(\gamma_j)_j$ gives the same automorphism $\sigma$.

**Definition 6.4.** For a quasi-stable curve $X$, an irreducible component $C_j$ is called an **elliptic tail** if $p_a(C_j) = 1$ and $C_j \cap (\overline{X} - C_j) = \{p\}$. The node $p$ is then an elliptic tail node. A non-trivial automorphism $\sigma$ of $X$ is called an elliptic tail automorphism (with respect to $C_j$) if $\sigma|_{X - C_j}$ is the identity.
**Theorem 6.5.** Let $(X, \eta, \beta)$ be a Prym curve of genus $g \geq 4$. The point $[X, \eta, \beta] \in \overline{R}_g$ is smooth if and only if $\text{Aut}(X, \eta, \beta)$ is generated by elliptic tail involutions.

Throughout this section, $X$ denotes a quasi-stable curve of genus $g \geq 2$ and $C := st(X)$ is its stable model. We denote by $N \subset \text{Sing}(C)$ the set of exceptional nodes and $\Delta := \text{Sing}(C) - N$. Then $X$ is the support of a Prym curve if and only if $N$ considered as a subgraph of the dual graph $\Gamma(C)$ is eulerian, that is, every vertex of $\Gamma(C)$ is incident to an even number of edges in $N$ (cf. [BCF, Proposition 0.4]).

Locally at a point $[X, \eta, \beta]$, the moduli space $\overline{R}_g$ is isomorphic to the quotient of the versal deformation space $C_3^{g-3}$ of $(X, \eta, \beta)$ modulo the action of the automorphism group $\text{Aut}(X, \eta, \beta)$. If $C_3^{g-3} = \text{Ext}^1(\Omega_C^1, O_C)$ denotes the versal deformation space of $C$, then the map $C_3^{g-3} \to C_3^{g-3}$ is given by $t_i = t_i^2$ if $(t_i = 0) \subset C_3^{g-3}$ is the locus where the exceptional node $p_i \in N$ persists and $t_i = t_i$ otherwise. The morphism $\pi : \overline{R}_g \to \overline{M}_g$ is given locally by the map $C_3^{g-3}/\text{Aut}(X, \eta, \beta) \to C_3^{g-3}/\text{Aut}(C)$. One has the following decomposition of the versal deformation space of $(X, \eta, \beta)$:

$$C_3^{g-3} = \bigoplus_{i \in N} C_i \oplus \bigoplus_{i \in \Delta} C_i \oplus \bigoplus_{i \subset C} H^1(C_i^v, T_{C_i^v}(-D_j)),$$

where for a node $p_i \in N$ we denote by $(t_i = 0) \subset C_3^{g-3}$ the locus where the corresponding exceptional component $E_i$ persists, while for a node $p_i \in \Delta$ we denote by $(t_i = 0) \subset C_3^{g-3}$ the locus of those deformations in which $p_i$ persists. Finally, for a component $C_j \subset C$ with normalization $C_j^v$, if $D_j$ consists of the inverse images of the nodes of $C$ under the normalization map $C_j^v \to C_j$, the group $H^1(C_j^v, T_{C_j^v}(-D_j))$ parameterizes deformations of the pair $(C_j^v, D_j)$. This decomposition is compatible with the decomposition

$$C_3^{g-3} = \left( \bigoplus_{i \in \text{Sing}(C)} C_i \right) \oplus \left( \bigoplus_{i \in C} H^1(C_i^v, T_{C_i^v}(-D_j)) \right),$$

as well as with the actions of the automorphism groups on $C_3^{g-3}$ and $\overline{R}_g$ is smooth if and only if the action of $\text{Aut}(X, \eta, \beta)$ on $C_3^{g-3}$ is generated by quasi-reflections, that is, elements $\sigma \in \text{Aut}(X, \eta, \beta)$ having 1 as an eigenvalue of multiplicity precisely $3g - 4$. Theorem 6.5 follows from the following proposition.

**Proposition 6.6.** Let $\sigma \in \text{Aut}(X, \eta, \beta)$ be an automorphism of a Prym curve $(X, \eta, \beta)$ of genus $g \geq 4$. Then $\sigma$ acts on $C_3^{g-3}$ as a quasi-reflection if and only if $X$ has an elliptic tail $C_j$ such that $\sigma$ is the elliptic tail involution with respect to $C_j$.

**Proof.** Let $\sigma$ be an elliptic tail involution with respect to $C_j$. The induced automorphism $\sigma_C$ is an elliptic tail involution of $C$ and acts on the versal deformation space $C_3^{g-3}$ of $C$ as $t_1 \mapsto -t_1$ and $t_i \mapsto t_i$, $i \neq 1$. Here $t_1$ is the coordinate corresponding to the node $p_1 \in C_j \cap \overline{C} - C_j$. The node $p_1$ being non-exceptional, we have $t_1 = t_1$, hence...
exists a circuit of edges in $p_1$. For coordinates $t_i = t_i^1$, $\sigma$ is the identity in a neighbourhood of the corresponding exceptional component $E_i$, thus $\sigma \cdot t_i = t_i$.

Now let $\sigma \in \text{Aut}(X, \eta, \beta)$ act as a quasi-reflection with eigenvalues $\zeta$ and 1. As in the proof of [Lud, Proposition 2.15], there exists a node $p_1 \in C$ such that the action of $\sigma$ is given by $\sigma \cdot t_1 = \zeta t_1$ and $\sigma \cdot t_j = t_j$ for $j \neq 1$. When $p_1 \in N$, the induced automorphism $\sigma_C$ acts via $t_1 \mapsto \zeta^2 t_1$ and $\sigma_C \cdot t_j = t_j$ for $j \neq 1$. If $\zeta^2 \neq 1$, then $\sigma_C$ acts as a quasi-reflection and $p_1$ is an elliptic tail node, which contradicts the assumption $p_1 \in N$. Therefore $\sigma_C = \text{Id}_C$ and the exceptional component $E_1$ over $p_1$ is the only component on which $\sigma$ acts non-trivially. The graph $N \subset \Gamma(C)$ is eulerian and there exists a circuit of edges in $N$ containing $p_1$:

By Remark 6.3, $\sigma$ corresponds to an element $\pm(y_j) \in \{\pm 1\}^{CC(\widetilde{X})}/\pm 1$. Since $\sigma$ acts non-trivially on $E_1$ we find that $y_1 = -y_2$. In particular, there exists $i \neq 1$ such that $\sigma$ acts non-trivially on $E_i$. This is a contradiction which shows that the node $p_1$ is non-exceptional, $t_1 = t_i$ and $\sigma_C \cdot t_i = \zeta t_i$ and $\sigma_C \cdot t_j = t_j$ for $i \neq 1$. Thus $\sigma_C$ is an elliptic tail involution of $C$ with respect to an elliptic tail through the node $p_1$ and $\zeta = -1$. Since $\sigma$ fixes every coordinate corresponding to an exceptional component of $X$, it follows that $\sigma$ is an elliptic tail involution of $X$. $\square$

Theorem 6.7. Fix $g \geq 4$. A point $[X, \eta, \beta] \in \overline{\mathcal{R}_g}$ is a non-canonical singularity if and only if $X$ has an elliptic tail $C_j$ with $j$-invariant 0 and $\eta$ is trivial on $C_j$.

The proof is similar to that of the analogous statement for $\overline{\mathcal{R}_g}$ and we refer to [Lud, Theorem 3.1] for a detailed outline of the proof and background on quotient singularities. Locally at $[X, \eta, \beta]$, the space $\overline{\mathcal{R}_g}$ is isomorphic to a neighbourhood of the origin in $C_3^{3g-3}/\text{Aut}(X, \eta, \beta)$. We consider the normal subgroup $H$ of $\text{Aut}(X, \eta, \beta)$ generated by automorphisms acting as quasi-reflections on $C_3^{3g-3}$. The map $C_3^{3g-3} \rightarrow C_3^{3g-3}/H = C_3^{3g-3}$ is given by $v_i = t_i^2$ if $p_i$ is an elliptic tail node and $v_i = t_i$ otherwise. The automorphism group $\text{Aut}(X, \eta, \beta)$ acts on $C_3^{3g-3}$ and the quotient $C_3^{3g-3}/\text{Aut}(X, \eta, \beta)$ is isomorphic to $C_3^{3g-3}/\text{Aut}(X, \eta, \beta)$. Since $\text{Aut}(X, \eta, \beta)$ acts on $C_3^{3g-3}$ without quasi-reflections the Reid–Shepherd-Barron–Tai criterion applies to this new action.

We fix an automorphism $\sigma \in \text{Aut}(X, \eta, \beta)$ of order $n$ and a primitive $n$-th root of unity $\zeta_n$. If the action of $\sigma$ on $C_3^{3g-3}$ has eigenvalues $\zeta_n^{a_1}, \ldots, \zeta_n^{a_{3g-3}}$ with $0 \leq a_i < n$ for $i = 1, \ldots, 3g-3$, then following [Re2] we define the age of $\sigma$ by

$$\text{age}(\sigma; \zeta_n) := \frac{1}{n} \sum_{i=1}^{n} a_i.$$
We say that \( \sigma \) satisfies the **Reid–Shepherd-Barron–Tai inequality** if \( \text{age}(\sigma, \zeta_n) \geq 1 \). The **Reid–Shepherd-Barron–Tai criterion** states that \( \mathbb{C}^{3g-3}_{\sigma} / \text{Aut}(X, \eta, \beta) \) has canonical singularities if and only if every \( \sigma \in \text{Aut}(X, \eta, \beta) \) satisfies the Reid–Shepherd-Barron–Tai inequality (cf. [Re], [HM], [Lud]).

**Proof of the ‘if’ part of Theorem 6.7** Let \( (X, \eta, \beta) \) be a Prym curve, \( C = st(X) \) and \( C_j \subset X \) an elliptic tail with \( \text{Aut}(C_j) = \mathbb{Z}_6 \) and assume \( \eta_{C_j} = \mathcal{O}_{C_j} \). We fix an elliptic tail automorphism \( \sigma_C \) with respect to \( C_j \subset C \) such that \( \text{ord}(\sigma_C) = 6 \). Then \( \sigma_C \) acts on \( \mathbb{C}^{3g-3}_\sigma \) by \( t_1 \mapsto \zeta_6 t_1, t_2 \mapsto \zeta_6^2 t_2 \) for an appropriate sixth root of unity \( \zeta_6 \) and \( \sigma \cdot t_i = t_i \) for \( i \neq 1, 2 \). Here \( t_1, t_2 \in \text{Ext}^1(\Omega_C, \mathcal{O}_C) \) correspond to smoothing the node \( p_l \in C_j \cap \overline{C - C_j} \) and deforming the curve \( [C_j, p_l] \in \mathcal{M}_{1,1} \) respectively. Since \( \eta_{C_j} = \mathcal{O}_{C_j} \), the automorphism \( \sigma_C \) lifts to an automorphism \( \sigma \in \text{Aut}(X, \eta, \beta) \) such that \( \sigma \in \text{Aut}(X, \eta, \beta) \) is the identity. Then \( \sigma \) acts on \( \mathbb{C}^{3g-3}_{\sigma} \) as \( \sigma \cdot t_1 = \zeta_6 t_1, \sigma \cdot t_2 = \zeta_6^2 t_2 \) and \( \sigma \cdot t_i = t_i \) for \( i \neq 1, 2 \).

Since \( v_1 = t_1^2 \) and \( v_2 = t_2 \), the action of \( \sigma \) on \( \mathbb{C}^{3g-3}_{\sigma} \) is \( v_1 \mapsto \zeta_6^2 v_1, v_2 \mapsto \zeta_6^2 v_2 \) and \( v_i \mapsto v_i, i \neq 1, 2 \). We compute age(\( \sigma, \zeta_6^2 \)) = \( \frac{1}{3} + \frac{1}{3} + 0 + \cdots + 0 = \frac{2}{3} < 1 \), that is, \( [X, \eta, \beta] \in \overline{\mathcal{K}_g} \) is a non-canonical singularity. Similarly, an elliptic tail automorphism of order 3 with respect to \( C_j \) acts via \( t_1 \mapsto \zeta_3^2 t_1, t_2 \mapsto \zeta_3 t_2 \) and \( t_i \mapsto t_i, i \neq 1, 2 \), and then for the action on \( \mathbb{C}^{3g-3}_{\sigma} \) as \( v_1 \mapsto \zeta_3 v_1, v_2 \mapsto \zeta_3 v_2 \) and \( v_i \mapsto v_i, i \neq 1, 2 \). This gives a value of \( \frac{2}{3} \) for the age.

Suppose that \( [X, \eta, \beta] \in \overline{\mathcal{K}_g} \) is a non-canonical singularity. Then there exists an automorphism \( \sigma \in \text{Aut}(X, \eta, \beta) \) of order \( n \) which acts on \( \mathbb{C}^{3g-3}_{\sigma} \) such that \( \text{age}(\sigma, \zeta_n) < 1 \). Let \( p_{l_0}, p_{l_1} = \sigma \cdot (p_{l_0}), \ldots, p_{l_m-1} = \sigma^{m-1}(p_{l_0}) \) be distinct nodes of \( C \) which are cyclically permuted by the induced automorphism \( \sigma_C \) and \( p_{l_0} \) is not an elliptic tail node. The action of \( \sigma \) on the subspace \( \bigoplus_j \mathbb{C}_{C_j} \subset \mathbb{C}^{3g-3}_{\sigma} \) is given by the matrix

\[
B = \begin{pmatrix}
0 & c_1 & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \cdots & \cdots & \cdots & \vdots \\
c_m & 0 & \cdots & \cdots & 0 & 0
\end{pmatrix}
\]

for appropriate scalars \( c_j \neq 0 \). The pair \( ((X, \eta, \beta), \sigma) \) is said to be **singularity reduced** if for every such cycle we have \( \prod_{j=1}^m c_j \neq 1 \).

**Proposition 6.8 ([HM], [Lud] Proposition 3.6).** There exists a deformation \( (X', \eta', \beta') \) of \( (X, \eta, \beta) \) such that \( \sigma \) deforms to an automorphism \( \sigma' \in \text{Aut}(X', \eta', \beta') \) and the nodes of every cycle of nodes as above with \( \prod_{j=1}^m c_j = 1 \) are smoothed. The pair \( ((X', \eta', \beta'), \sigma') \) is then singularity reduced and the action of \( \sigma \) on \( \mathbb{C}^{3g-3}_{\sigma} \) and that of \( \sigma' \) on \( \mathbb{C}^{3g-3}_{\sigma'} \) have the same eigenvalues and hence the same age.

We fix a singularity reduced pair \( ((X, \eta, \beta), \sigma) \) with \( n := \text{ord}(\sigma) \geq 2 \) and assume that \( \text{age}(\sigma, \zeta_n) < 1 \). We denote this assumption by \((\bullet)\). Using [Lud] Proposition 3.7 we find that if \((\bullet)\) holds, the induced automorphism \( \sigma_C \) of \( C = st(X) \) fixes every node with the possible exception of two nodes which are interchanged.
Proposition 6.9. If (⋆) holds, then \( \sigma_C \) fixes all components of the stable model \( C \) of \( X \).

Proof. Let \( C_{i_0}, C_{i_1} = \sigma_C(C_{i_0}), \ldots, C_{i_{m-1}} = \sigma_C^{m-1}(C_{i_0}) \) be distinct components of \( C \), \( \sigma_C^n(C_{i_0}) = C_{i_0} \) and assume that \( m \geq 2 \). Most of the proof of Proposition 3.8. in \([Lud]\) applies to the case of Prym curves and implies that the normalization \( C_{i_0} \) is rational and there are exactly three preimages of nodes \( p_1^+, p_2^+, p_3^+ \in C_{i_0} \) mapping to different nodes of \( C \). By \([Lud] \) Proposition 3.7] at least one of \( p_1, p_2, p_3 \) is fixed by \( \sigma_C \). If either one or all three nodes are fixed, then \( g(C) = 2 \), impossible. Thus two nodes, say \( p_1 \) and \( p_2 \), are fixed by \( \sigma_C \) while \( p_3 \) is interchanged with a fourth node \( p_4 \). Interchanging \( p_3 \) and \( p_4 \) gives a contribution of \( \frac{1}{2} \) to \( g(\sigma, \zeta_n) \). Now consider the action of \( \sigma_C \) near \( p_1 \) and let \( xy = 0 \) be a local equation of \( C \) at \( p_1 \). We have \( \tau_1 = xy \mapsto yx = \tau_1 \) and \( \tau_1 \mapsto \pm \tau_1 \), where the minus sign is only possible if \( p_1 \in N \). Since \( p_1 \) is not an elliptic tail node and \( (X, \eta, \beta, \sigma) \) is singularity reduced, we have \( \tau_1 \mapsto -\tau_1 \), which gives an additional contribution of \( \frac{1}{2} \) to the age, that is, \( \sigma(\tau_n) > \frac{1}{2} + \frac{1}{2} = 1 \), contradicting (⋆). □

Proposition 6.10 ([HM pp. 28, 36], \([Lud]\) Proposition 3.9]). Assume that (⋆) holds and denote by \( \psi_j = \sigma_C^{r_j} \), the induced automorphism of the normalization \( C_j \) of the irreducible component \( C_j \) of \( C \). Then the pair \( (C_j, \psi_j) \) is one of the following types:

(i) \( \psi_j = \text{Id}_{C_j} \) and \( C_j \) arbitrary.

(ii) \( C_j \) is rational and \( \text{ord}(\psi_j) = 2, 4 \).

(iii) \( C_j \) is elliptic and \( \text{ord}(\psi_j) = 2, 4, 3, 6 \).

(iv) \( C_j \) is hyperelliptic of genus 2 and \( \psi_j \) is the hyperelliptic involution.

(v) \( C_j \) is hyperelliptic of genus 3 and \( \psi_j \) is the hyperelliptic involution.

(vi) \( C_j \) is bielliptic of genus 2 and \( \psi_j \) is the associated involution.

The possibility of \( \sigma_C \) interchanging two nodes does not appear (cf. \([Lud]\) Prop. 3.10]):

Proposition 6.11. Under the assumption (⋆), the automorphism \( \sigma_C \) fixes all the nodes of \( C \).

Proposition 6.12. Assume (⋆) holds. Let \( C_j \) be a component of \( C \) with normalization \( C_j \), \( D_j \) the divisor of the marked points on \( C_j \) and \( \psi_j = \sigma_C^{r_j} \). Then \( (C_j, D_j, \psi_j) \) is of one of the following types and the contribution to age(\( \sigma, \zeta_n \)) coming from \( H^1(C_j, T_{C_j}(\mathcal{O}(D_j))) \subset \mathbb{C}^{3g-3} \) is at least the following quantity \( w_j \):

(i) Identity component: \( \psi_j = \text{Id}_{C_j} \), arbitrary pair \( (C_j, D_j) \) and \( w_j = 0 \).

(ii) Elliptic tail: \( C_j \) is elliptic, \( D_j = p_1^+ \) and \( p_1^+ \) is fixed by \( \psi_j \).

order 2: \( \text{ord}(\psi_j) = 2 \) and \( w_j = 0 \)

order 4: \( C_j \) has j-invariant 1728, \( \text{ord}(\psi_j) = 4 \) and \( w_j = \frac{1}{2} \)

order 3, 6: \( C_j \) has j-invariant 0, \( \text{ord}(\psi_j) = 3 \) or 6 and \( w_j = \frac{1}{4} \)

(iii) Elliptic ladder: \( C_j \) is elliptic, \( D_j = p_1^+ + p_2^+ \), with \( p_1^+ \) and \( p_2^+ \) both fixed by \( \psi_j \).

order 2: \( \text{ord}(\psi_j) = 2 \) and \( w_j = \frac{1}{2} \)

order 4: \( C_j \) has j-invariant 1728, \( \text{ord}(\psi_j) = 4 \) and \( w_j = \frac{3}{4} \)

order 3: \( C_j \) has j-invariant 0, \( \text{ord}(\psi_j) = 3 \) and \( w_j = \frac{2}{3} \).
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(iv) Hyperelliptic tail: \( C_j^\nu \) has genus 2, \( \varphi_j \) is the hyperelliptic involution, \( D_j \) is of the form \( D_j = p_1^+ + p_1^- \) with \( p_1^+ \) fixed by \( \varphi_j \) and \( w_j = \frac{1}{i} \).

Proof. The proof is along the lines of the proof of Proposition 3.11 in [Lud]. The only difference occurs in the case of a singular elliptic tail on which \( \sigma \) acts with order 2.

Assume that \( C_j^\nu \) is rational, \( D_j = p_1^+ + p_1^- + p_2 \), with \( \text{ord}(\varphi_j) = 2 \) which fixes \( p_2 \) and interchanges \( p_1^+ \) and \( p_1^- \). If \( xy = 0 \) is an equation for \( C \) at \( p_1 \), then \( \sigma C \) acts via \( t_1 = xy \mapsto yx = t_1 \). Since \( p_1 \) is not an elliptic tail node and \( ((X, \eta, \beta), \sigma) \) is singularity reduced, the node \( p_1 \) must be exceptional and \( \sigma \cdot t_1 = -t_1 \).

A deformation of \((X, \eta, \beta)\) over the locus \((t_1 = 0)_{t \neq 1} \subset C^g_{3g-3} \) smooths \( p_1 \). Furthermore, \( \sigma \) deforms to an automorphism \( \sigma' \) of a general Prym curve \((X', \eta', \beta')\) over this locus, \( \varphi_j \) deforms to the involution \( \varphi'_j \) on the smooth elliptic tail \( C_j' \) such that it fixes the line bundle \( \eta'_{\nu, j} \), and the restrictions of \( \sigma \) and \( \sigma' \) to the complement of \( C_j \) resp. \( C_j' \) coincide. Over the non-exceptional subcurve \( \tilde{X} \subset X \) we have \((\tilde{\sigma}')^* \eta' \cong \eta'\). Thus \( \sigma \cdot t_1 = t_1 \), which is a contradiction. The case of a singular elliptic tail is thus excluded. \( \square \)

**Proposition 6.13.** Under the hypothesis (\( \bullet \)), the hyperelliptic tail case does not occur.

Proof. Let \( C_j \) be a genus 2 tail of \( C \) and \( C_j' \) the second component through \( p_1 \). The action of \( \sigma \) on \( H^1(C_j', TC_j'(-D_j)) \) contributes \( \frac{1}{2} \) to the age of \( \sigma \) and \( C_j \) has to be one of the cases of Proposition 6.12. If \( C_j' \) is elliptic, then \( g(C) = 3 \). If \( C_j' \) is a hyperelliptic tail or an elliptic ladder, the action on \( H^1(C_j', TC_j'(-D_j)) \) contributes at least \( \frac{1}{2} \). Therefore \( C_j' \) is an identity component. If \( xy = 0 \) is an equation for \( C \) at \( p_1 \), then \( \sigma C \) acts via \( t_1 = xy \mapsto -xy = -t_1 \). The node \( p_1 \) is disconnecting, hence non-exceptional, and it is not an elliptic tail node. Therefore, \( \nu_1 = \nu_1 = t_1 \) and \( \sigma \) acts as \( \sigma \cdot \nu_1 = -\nu_1 \). This gives an additional contribution of \( \frac{3}{2} \) to the age of \( \sigma \), finishing the proof. \( \square \)

**Proposition 6.14.** In situation (\( \bullet \)) the elliptic ladder cases do not occur.

Proof. Let \( C_j \) be an elliptic ladder of \( C \) of order \( n_j = \text{ord}(\varphi_j) \) and denote by \( C_j \) resp. \( C_j' \) the second component through the node \( p_1 \) resp. \( p_2 \). Since every elliptic ladder contributes at least \( \frac{1}{2} \) to the age, \( C_j \) and \( C_j' \) can only be elliptic tails or identity components.

If both are elliptic tails, then \( g(C) = 3 \), hence we may assume that \( C_j' \) is an identity component. If \( xy = 0 \) is an equation for \( C \) at \( p_1 \), then \( \sigma C \) acts as \( x \mapsto x, y \mapsto ay \) and \( t_1 \mapsto \alpha t_1 \), where \( \alpha \) is a primitive \( n_j \)-th root of 1. If \( p_1 \) is non-exceptional then \( \nu_1 = t_1 = t_1 \) and the space \( H^1(C_j', TC_j'(-D_j)) \oplus C \cdot \nu_1 \) contributes to the age at least

\[
1 = \begin{cases} 
\frac{1}{2} + \frac{1}{2} & \text{if } n_j = 2, \\
\frac{1}{2} + \frac{1}{2} & \text{if } n_j = 4, \\
\frac{1}{2} + \frac{1}{2} & \text{if } n_j = 3.
\end{cases}
\]

Therefore \( p_1 \in N \). Since \( N \subset \Gamma(C) \) is an eulerian subgraph, the node \( p_2 \) is also exceptional, both \( p_1 \) and \( p_2 \) are non-disconnecting and \( C_j' \) is an identity component as well. Moreover \( \sigma C \cdot t_i = \alpha t_i, i = 1, 2 \). Since \( \nu_i = t_i \) and \( t_i^2 = t_i \) for \( i = 1, 2 \), we find that

\[
1 = \begin{cases} 
\frac{1}{2} + \frac{1}{2} & \text{if } n_j = 2, \\
\frac{1}{2} + \frac{1}{2} & \text{if } n_j = 4, \\
\frac{1}{2} + \frac{1}{2} & \text{if } n_j = 3.
\end{cases}
\]
σ \cdot v_i = a_i v_i, \ i = 1, 2, \text{ where } a_i \text{ is a square root of } \sigma. \text{ Therefore, the contribution to the age of } \sigma \text{ coming from } H^1(C_j^\nu, T_{C_j^\nu}(-D_j)) \oplus \mathbb{C} \cdot v_1 \oplus \mathbb{C} \cdot v_2 \text{ is at least}

\begin{align*}
1 = \begin{cases} 
\frac{1}{3} + \frac{1}{4} + \frac{1}{5} & \text{if } n_j = 2, \\
\frac{3}{5} + \frac{1}{6} + \frac{1}{7} & \text{if } n_j = 4, \\
\frac{5}{7} + \frac{1}{6} + \frac{1}{8} & \text{if } n_j = 3,
\end{cases}
\end{align*}

and the case of elliptic ladders is excluded. \hfill \Box

**Proposition 6.15.** Under hypothesis \((\ast)\), the case of an elliptic tail of order 4 does not occur.

**Proof.** Let \(C_j\) be an elliptic tail of order 4, and \(C_j'\) another component of \(C\) through \(p_1\). Then \(\sigma_{C_j(C_j')} = \text{Id} \downharpoonright_{C_j'}\) and \(\sigma_C\) acts as \(t_1 = xy \mapsto \xi xy = \xi t_1\) for a suitable fourth root \(\xi_4\) of 1. Since \(p_1\) is an elliptic tail node, we have \(v_1 = t_1^2\) and \(\sigma \cdot v_1 = -v_1\). The action of \(\sigma\) on \(H^1(C_j', T_{C_j'}(-D_j)) \oplus \mathbb{C} \cdot v_1\) contributes \(\left\lfloor \frac{1}{2} + \frac{1}{2} \right\rfloor = 1\) to \(\text{age}(\sigma, \xi_4)\), excluding this case. \hfill \Box

**Proposition 6.16.** In situation \((\ast)\) there has to be at least one elliptic tail of order 3 or 6.

**Proof.** Assume to the contrary that every component of \(C\) is either an identity component or an elliptic tail of order 2. The action of \(\sigma\) on every space \(H^1(C_j^\nu, T_{C_j^\nu}(-D_j))\) is trivial. If \(p_1\) is the node of an elliptic tail of order 2, then \(\sigma_C \cdot t_1 = -t_1\) and we have \(v_1 = t_1^2 = t_1^2\) and \(\sigma \cdot v_1 = v_1\). In case \(p_1\) is non-exceptional but not an elliptic tail node, \(\sigma_C \cdot t_1 = t_1\). Since \(v_1 = t_1 = t_1\), we find that \(\sigma\) fixes \(v_1\). If \(p_1 \in \mathcal{N}\), then \(\sigma_C \cdot t_1 = t_1\) and \(v_1^2 = t_1^2 = t_1\) and \(\sigma\) acts as \(t_1 \mapsto \pm t_1\). Since \(\text{age}(\sigma, \xi_6) < 1\), there is exactly one node \(p_1\) such that \(\sigma \cdot v_1 = -v_1\), that is, \(\sigma\) acts as quasi-reflection on \(\mathbb{C}^{3g-3}_{\nu}\), a contradiction. \hfill \Box

**Proof of the "only if" part of Theorem 6.7.** We proved that if \(((X, \eta, \beta), \sigma)\) is a singularity reduced pair and \(\text{age}(\sigma, \xi_n) < 1\), where \(n = \text{ord}(\sigma)\), there exists an elliptic tail \(C_j \subset C\) with \(\text{Aut}(C_j) = \mathbb{Z}_n\) such that \(\text{ord}(\sigma_C) \in \{3, 6\}\). Since \(\sigma_{C_j'}(\eta_{C_j'}) \cong \eta_{C_j}\), we find that \(\eta_{C_j} = \partial C_j\). Let \(((X, \eta, \beta), \sigma)\) be a pair consisting of a Prym curve and an automorphism such that \(\text{age}(\sigma, \xi_n) < 1\). By Proposition 6.13, we may deform \(((X, \eta, \beta), \sigma)\) to a singularity reduced pair \(((X', \eta', \beta'), \sigma')\) such that the actions of \(\sigma\) on \(\mathbb{C}^{3g-3}_{\nu}\) and \(\sigma'\) on \(\mathbb{C}^{3g-3}_{\nu'}\) have the same ages. Therefore \(X'\) has an elliptic tail \(C_j'\) with \(\text{Aut}(C_j') = \mathbb{Z}_6\) such that \(\eta_{C_j'}' = \partial C_j\) is trivial and \(\sigma'\) acts on \(C_j'\) of order 3 or 6. In the deformation of \(((X, \eta, \beta), \sigma)\) to \(((X', \eta', \beta'), \sigma')\) elliptic tails are preserved, hence \(((X, \eta, \beta), \sigma)\) enjoys the same properties. \hfill \Box

**Remark 6.17.** If \(\sigma \in \text{Aut}(X, \eta, \beta)\) satisfies the inequality \(\text{age}(\sigma, \xi_n) < 1\) (with respect to the action on \(\mathbb{C}^{3g-3}_{\nu}\)), then \(\sigma\) is an elliptic tail automorphism and \(\text{ord}(\sigma) \in \{3, 6\}\). Indeed, we already know that \(\sigma_C \in \text{Aut}(C)\) acts with order 3 or 6 on an elliptic tail \(C_j\). The action of \(\sigma\) on \(H^1(C_j^\nu, T_{C_j^\nu}(-D_j))\) and the \(\nu\)-coordinate corresponding to the elliptic tail node on \(C_j\) contribute at least \(\frac{1}{2}\) to \(\text{age}(\sigma, \xi_n)\). Thus there is exactly one elliptic tail
of order 3 or 6 and \( \sigma_C \) is an elliptic tail automorphism of the same order. If \( \sigma \) is not an elliptic tail automorphism of \( X \), then there exists an exceptional component \( E_1 \subset X \) on which \( \sigma \) acts non-trivially. Since \( E_1 \) connects two non-exceptional components of \( X \) on which \( \sigma \) acts trivially, \( \sigma \cdot \nu_1 = -\nu_1 \), giving a contribution of \( \frac{1}{2} \) and an age \( \geq \frac{2}{3} + \frac{1}{2} \geq 1 \).

**Proof of Theorem 6.1.** We start with a pluricanonical form \( \omega \) on \( \overline{R}_{reg} \) and show that \( \omega \) lifts to a desingularization of a neighborhood of every point \([X, \eta, \beta] \in \overline{R}_{reg}\). We may assume that \([X, \eta, \beta] \) is a general non-canonical singularity of \( \overline{R}_{reg} \), hence \( X = C_1 \cup_p C_2 \), where \([C_1, p] \in \mathcal{M}_{g,-1,1} \) is general and \([C_2, p] \in \mathcal{M}_{1,1} \) has \( j \)-invariant 0. Furthermore \( \eta_{C_1} = O_{C_1} \) and \( \eta_{C_2} :\eta_{C_1} \to O_{C_1} \). We consider the pencil \( \phi : \mathcal{M}_{1,1} \to \overline{R}_{reg} \) given by \( \phi([C', p]) = [C' \cup_p C_1, \eta_{C'} = O_{C'}, \eta_{C_1} = \eta] \). Since \( \phi(\mathcal{M}_{1,1}) \cap \Delta_{0} = \emptyset \), we imitate [HM, pp. 41–44] and construct an explicit open neighbourhood \( \overline{R}_{reg} \supset S \supset \phi(\mathcal{M}_{1,1}) \) such that the restriction to \( S \) of \( \pi : \overline{R}_{reg} \to \mathcal{M}_{g} \) is an isomorphism and every form \( \omega \in H^0(\overline{R}_{reg}, K_{\overline{R}_{reg}}) \) extends to a resolution \( \hat{S} \) of \( S \). For an arbitrary non-canonical singularity we show that \( \omega \) extends locally to a desingularization along the lines of [Lud] Theorem 4.1. \( \square \)

**Acknowledgments.** Research of the first author was partially supported by an Alfred P. Sloan Fellowship, the NSF Grant DMS-0500747 and a Texas Research Assignment.

**References**

[ACGH] Arbarello, E., Cornalba, M., Griffiths, P., Harris, J.: Geometry of Algebraic Curves. Grundlehren Math. Wiss. 267, Springer (1985) [Zbl 0559.14017] MR 0770932

[ACV] Abramovich, D., Corti, A., Vistoli, A.: Twisted bundles and admissible coverings. Comm. Algebra 31, 3547–3618 (2003) [Zbl 1077.14034] MR 2007376

[BCF] Ballico, E., Casagrande, C., Fontanari, C.: Moduli of Prym curves. Documenta Math. 9, 265–281 (2004) [Zbl 1077.14034] MR 2007376

[B] Beauville, A.: Prym varieties and the Schottky problem. Invent. Math. 41, 149–196 (1977) [Zbl 0371.14010] MR 0572974

[Be] Bernstein, M.: Moduli of curves with level structures. Harvard Univ. Ph.D. Thesis (1999)

[CCC] Caporaso, L., Casagrande, C., Cornalba, M.: Moduli of roots of line bundles on curves. Trans. Amer. Math. Soc. 359, 3733–3768 (2007) [Zbl 1154.14022] MR 2302513

[Ca] Catanese, F.: On the rationality of certain moduli spaces related to curves of genus 4. In: Algebraic Geometry (Ann Arbor, MI), Lecture Notes in Math. 1008, Springer, 30–50 (1983) [Zbl 0557.14005] MR 0721300

[C] Cornalba, M.: A remark on the Picard group of spin moduli space. Rend. Lincei Mat. Appl. 2, 211–217 (1991) [Zbl 0768.14010] MR 1135424

[CH] Cornalba, M., Harris, J.: Divisor classes associated to families of stable varieties, with applications to the moduli space of curves. Ann. Sci. École Norm. Sup. 21, 455–475 (1988) [Zbl 0674.14006] MR 0974412

[CM] Coppens, M., Martens, G.: Linear series on a general \( k \)-gonal curve. Abh. Math. Sem. Univ. Hamburg 69, 347–371 (1999) [Zbl 0957.14018] MR 1729444

[De] Debarre, O.: Sur le problème de Torelli pour les variétés de Prym. Amer. J. Math. 111, 111–134 (1989) [Zbl 0699.14052] MR 0980302
Dolgachev, I.: Rationality of fields of invariants. In: Algebraic Geometry, Bowdoin, 1985, Proc. Sympos. Pure Math. 46, Part 2, Amer. Math. Soc., 3–16 (1987) Zbl 0659.14009 MR 0927970

Donagi, R.: The unirationality of $\mathcal{A}_5$. Ann. of Math. 119, 269–307 (1984) Zbl 0589.14003 MR 0740895

Donagi, R., The fibers of the Prym map. In: Curves, Jacobians, and Abelian Varieties (Amherst, MA, 1990), Contemp. Math. 136, Amer. Math. Soc. 55–125 (1992) Zbl 0783.14025 MR 1188194

Donagi, R., Smith, R.: The structure of the Prym map. Acta Math. 146, 25–102 (1981) Zbl 0538.14019 MR 0594627

Eisenbud, D., Harris, J.: Linear series: basic theory. Invent. Math. 85, 337–371 (1986) Zbl 0598.14003 MR 0846932

Farkas, G.: Syzygies of curves and the effective cone of $\overline{M}_g$. Duke Math. J. 135, 53–98 (2006) Zbl 1107.14019 MR 2259923

Farkas, G.: Koszul divisors on moduli spaces of curves. Amer. J. Math. 131, 819–867 (2009) Zbl pre0573661 MR 2530855

Farkas, G.: Rational maps between moduli spaces of curves and Gieseker–Petri divisors. J. Algebraic Geom. 19, 243–284 (2010)

Farkas, G.: The Prym–Green Conjecture. In preparation

Farkas, G., Mustaţă, M., Popa, M.: Divisors on $\overline{M}_{g,g+1}$ and the Minimal Resolution Conjecture for points on canonical curves. Ann. Sci. École Norm. Sup. 36, 553–581 (2003) Zbl 1063.14031 MR 2013926

Farkas, G., Popa, M.: Effective divisors on $\overline{M}_g$, curves on K3 surfaces and the Slope Conjecture. J. Algebraic Geom. 14, 241–267 (2005) Zbl 1081.14038 MR 2123229

Friedman, R., Smith, R.: The generic Torelli theorem for the Prym map. Invent. Math. 67, 473–490 (1982) Zbl 0506.14042 MR 0664116

Green, M., Lazarsfeld, R.: Some results on the syzygies of finite sets and algebraic curves. Compos. Math. 67, 301–314 (1988) Zbl 0671.14010 MR 0959214

Harris, J., Mumford, D.: On the Kodaira dimension of $\overline{M}_g$. Invent. Math. 67, 23–88 (1982) Zbl 0506.14016 MR 0664324

Eisenbud, D., Harris, J.: The Kodaira dimension of the moduli space of curves of genus $\geq 3$. Invent. Math. 90, 359–387 (1987) Zbl 0631.14023 MR 0910206

Izadi, E., Lo Giudice, M., Sankaran, G. K.: The moduli space of étale double covers of genus 5 curves is unirational. Pacific J. Math. 239, 39–52 (2009) Zbl pre05366395 MR 2449010

Khosla, D.: Tautological classes on moduli spaces of curves with linear series and a push-forward formula when $p=0$. arXiv:0704.1340

Lazarsfeld, R.: A sampling of vector bundle techniques in the study of linear systems. In: Lectures on Riemann Surfaces (Trieste, 1987), World Sci., 500–559 (1989) Zbl 0680.14003 MR 1082360

Ludwig, K.: On the geometry of the moduli space of spin curves. J. Algebraic Geom. 19, 133–171 (2010)

Mori, S., Mukai, S.: The uniruledness of the moduli space of curves of genus 11. In: Algebraic Geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math. 1016, Springer, 334–353 (1983) Zbl 0557.14015 MR 0726433

Raynaud, M.: Sections des fibrés vectoriels sur une courbe. Bull. Soc. Math. France 110, 103–125 (1982) Zbl 0503.14011 MR 0662131
The Kodaira dimension of the moduli space of Prym varieties

[Re] Reid, M.: Canonical 3-folds. In: Journées de Géométrie Algébrique d’Angers (Angers, 1979), Sijthoff & Noordhoff, Alphen aan de Rijn, 273–310 (1980) Zbl 0451.14014 MR 0605348

[Re2] Reid, M.: La correspondance de McKay. In: Séminaire Bourbaki, Vol. 1999/2000, Astérisque 276, 53–72 (2002) Zbl 0996.14006 MR 1886756

[T] Tai, Y.: On the Kodaira dimension of the moduli space of abelian varieties. Invent. Math. 68, 425–439 (1982) Zbl 0508.14038 MR 0669424

[V1] Verra, A.: A short proof of the unirationality of $\mathcal{A}_5$. Indag. Math. 46, 339–355 (1984) Zbl 0553.14010 MR 0763470

[V2] Verra, A.: On the universal principally polarized abelian variety of dimension 4. In: Curves and Abelian Varieties, Contemp. Math. 465, Amer. Math. Soc., 253–274 (2008) Zbl 1160.14032 MR 2457741

[Vo] Voisin, C.: Green’s generic syzygy conjecture for curves of even genus lying on a $K3$ surface. J. Eur. Math. Soc. 4, 363–404 (2002) Zbl 1080.14525 MR 1941089