Consensus in Self-similar Hierarchical Graphs and Sierpiński Graphs: Convergence Speed, Delay Robustness, and Coherence

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Abstract—The hierarchical graphs and Sierpiński graphs are constructed iteratively, which have the same number of vertices and edges at any iteration, but exhibit quite different structural properties: the hierarchical graphs are non-fractal and small-world, while the Sierpiński graphs are fractal and “large-world”. Both graphs have found broad applications. In this paper, we study consensus problems in hierarchical graphs and Sierpiński graphs, focusing on three important quantities of consensus problems, that is, convergence speed, delay robustness, and coherence for first-order (and second-order) dynamics, which are, respectively, determined by algebraic connectivity, maximum eigenvalue, and sum of reciprocal (and square of reciprocal) of each nonzero eigenvalue of Laplacian matrix. For both graphs, based on the explicit recursive relation of eigenvalues at two successive iterations, we evaluate the second smallest eigenvalue, as well as the largest eigenvalue, and obtain the closed-form solutions to the sum of reciprocals (and square of reciprocals) of all nonzero eigenvalues. We also compare our obtained results for consensus problems on both graphs and show that they differ in all quantities concerned, which is due to the marked difference of their topological structures.

Index Terms—Distributed average consensus, multi-agent systems, hierarchical graph, Sierpiński graph, self-similar networks, graph Laplacians, convergence speed, delay robustness, network coherence.

I. INTRODUCTION

A s a fundamental research object with a long history [1], consensus problems cut across diverse areas of science and engineering. Typical examples include distributed computing [2], [3], load balancing [4]–[6], sensor networks [7]–[9], flocking [10], rendezvous [11], vehicle formation [12] and platooning [13]–[15], control system technique [16]–[18], and synchronization of coupled oscillators [19]. In the settings of networks (graphs) of agents, consensus means that agents represented by nodes (vertices) reach agreement on a certain issue, such as pace, load, or direction and velocity. Due to their wide applications, consensus problems have received a tremendous amount of attention and made great progress in past years [20], [21].

In this paper, we study consensus algorithms on graphs, with emphasis on some primary aspects: convergence rate [22], delay robustness [23], [24], robustness to noise [25]–[27], which are the theme of many previous work. All these three issues have a close relation to the eigenvalues of Laplacian matrix for the graph on which consensus algorithms are defined. Convergence speed measures the time of convergence of a consensus algorithm, which is closely related to the second smallest eigenvalue of Laplacian matrix [14], [33]. Delay robustness refers to the ability of consensus schemes resistant to communication delay between agents, with the allowable maximum delay determined by the largest eigenvalue [14]. [33]. Finally, robustness to noise can be gauged by the derivation of each vertex’s state from the global average of all current states, which is governed by all non-zero eigenvalues [27], [31].

As shown above, the three relevant issues for consensus algorithms are determined by the eigenvalues of Laplacian matrix for the underlying graphs. It is well established that the eigenvalues of Laplacian matrix of a graph depend on its topological structure. Thus, it is of theoretical and practical interest to unveil the profound effects of structural properties of networks on their Laplacian spectrum, such as small-world feature [35] that is ubiquitous in real-life networks [36]. However, it is very hard and even impossible to characterize the Laplacian spectrum of a generic graph. Most previous work about consensus problems either study part of the three problems or use the technique of numerical simulations. Then, it makes sense to study particular graphs with ideal structure, for which the properties or behaviors of Laplacian spectrum can be determined accurately.

In this paper, we study consensus problems on two families of self-similar determinstic graphs: hierarchical graphs [37] and Sierpiński graphs [38]. The major reasons to choose these two graphs as our research objects are as follows. First, they have a large variety of applications and are extensively studied. Our research is helpful for better understanding the two important graph families. Moreover, both graphs are constructed in an iterative way [39], have the same number of nodes and the same number edges at any iteration, but exhibit strongly different structural properties. Our work is instrumental to uncover the influence of topological properties on several consensus schemas. Finally, the full Laplacian spectrum for both graphs can be determined iteratively, which allows to study analytically the behaviors of various consensus algorithms that are dependent on the eigenvalues of Laplacian matrix. For both graphs, we provide recursive relations for the second smallest eigenvalues, the largest eigenvalues, the
sum of reciprocals (and reciprocals of square) of all non-zero eigenvalues, based on which we further obtain and compare the asymptotic behaviors for the studied consensus algorithms, and show that the behavior difference lies on the structure distinction of the graphs considered.

II. CONSENSUS PROBLEMS IN A GRAPH

Let \( G = (\mathcal{V}, \mathcal{E}) \) be an undirected connected network (graph) with \( N = |\mathcal{V}| \) nodes (vertices) and \( M = |\mathcal{E}| \) edges, where \( \mathcal{V} \) is the node set and \( \mathcal{E} \) is edge set. In this section, we give a brief introduction to consensus problems \([14, 13]\) in graph \( G \).

A. Matrix theory

The connectivity of a graph \( G \) is encoded in its adjacency matrix \( A \), the entry at row \( i \) (\( i = 1, 2, \ldots, N \)) and column \( j \) (\( j = 1, 2, \ldots, N \)) is \( a_{ij} = 1 \) (or 0) if nodes \( i \) and \( j \) are (not) connected by an edge. Let \( N_i(\mathcal{G}) \) denote the set of neighbors of node \( i \). Then, the degree of node \( i \) is \( d_i = \sum_{j=1}^{N} a_{ij} = \sum_{j \in N_i(\mathcal{G})} a_{ij} \), and the diagonal degree matrix of \( G \), denoted by \( D \), is defined as: the \( i \)th diagonal element is \( d_i \), while all the off-diagonal elements are zeros. Thus, the Laplacian matrix \( L \) of \( G \) is given as \( L = D - A \). Let \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N \) be the \( N \) eigenvalues of matrix \( L \) rearranged in an increasing order, that is \( \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N \). Since \( G \) is connected, its Laplacian matrix \( L \) always has a zero eigenvalue with a single degeneracy, i.e., \( \lambda_1 = 0 \), and all other eigenvalues are larger than zero, among which \( \lambda_2 \) is named the algebraic connectivity of the graph by Fiedler \([40]\), thus also called Fiedler eigenvalue.

B. Consensus problems

We next introduce several linear consensus problems under different assumptions.

1) Consensus without delay and noise: Let us consider graph \( G \) as a multi-agent system, where an agent and available information flow between two agents are, respectively, looked upon as a node and an edge in \( G \). We assume that the system of dynamic agents is described by \( \dot{x}_i(t) = u_i(t) \), where \( x_i(t) \in \mathbb{R} \) and \( u_i(t) \in \mathbb{R} \) denote, respectively, the state of agent \( i \) and the associated control input. It was shown by Olfati-Saber and Murray \([14, 13]\) that the following linear dynamic system

\[
\dot{x}_i(t) = \sum_{j \in N_i(\mathcal{G})} [x_j(t) - x_i(t)]
\]

with a collective dynamics

\[
\dot{x}(t) = -Lx(t)
\]

solves a consensus problem. In other words, the state of all agents in system \( \mathcal{G} \) asymptotically converges to the average value \( \bar{x} = (1/N) \sum_{i=1}^{N} x_i(0) \), where \( x_i(0) \) is the initial state of node \( i \).

The convergence speed of system \( \mathcal{G} \) can be measured by the algebraic connectivity \( \lambda_2 \): the larger the value of \( \lambda_2 \), the faster the convergence speed.

2) Consensus under communication time-delay: As in many real-world situations, the communication between pairs of agents is often not instantaneous \([41]\). Instead, an agent reacts to the information or signal received from its neighbors with some finite time lag. Suppose that agent \( i \) receives a message sent by one of its neighbors \( j \) after a time delay \( \tau_{ij} \). Then, the dynamics of the system is governed by \([14, 13]\).

\[
\dot{x}_i(t) = \sum_{j \in N_i(\mathcal{G})} [x_j(t - \tau_{ij}) - x_i(t - \tau_{ij})] .
\]

Throughout this paper, we focus on the case of uniform delay, where the time delay \( \tau_{ij} \) for all pairs of nodes \( i \) and \( j \) is fixed to \( \tau \). Then, system \( \mathcal{G} \) becomes

\[
\dot{x}_i(t) = \sum_{j \in N_i(\mathcal{G})} [x_j(t - \tau) - x_i(t - \tau)]
\]

with a collective dynamics

\[
\dot{x}(t) = -Lx(t - \tau) .
\]

Olfati-Saber and Murray \([14]\) with an undirected and connected network topology, all nodes globally asymptotically reach an average consensus if and only if the following condition is satisfied:

\[
0 < \tau < \tau_{\text{max}} = \frac{\pi}{2\lambda_2}.
\]

Equation \( \mathcal{G} \) shows that the analysis of consensus problem in a connected undirected network with an equal time-delay in all links is reduced to spectral analysis of Laplacian matrix of the network. Specifically, the largest eigenvalue \( \lambda_n \) is a measure of delay robustness for achieving a agreement in a network: the smaller the largest eigenvalue \( \lambda_n \), the bigger the maximum delay \( \tau_{\text{max}} \), and vice versa. On the other hand, similarly to system \( \mathcal{G} \), the convergence speed of system \( \mathcal{G} \) is also determined by \( \lambda_2 \), which converges fast with the increasing of \( \lambda_2 \).

3) Consensus with white noise: Since autonomous systems must operate in uncertain environments without direct supervision, it is important that such systems should be robust with respect to environment uncertainty and communication uncertainty. Thus, it is of great interest to consider how robust distributed consensus algorithms are to external disturbances. For both first-order and second-order systems, robustness to uncertainty and noise can be quantified using the quantity called network coherence in terms of \( \mathcal{H}_2 \) norm \([27, 29, 31]\).

First-order noisy consensus. In the first-order consensus problem, each node has a single state subject to stochastic disturbances (noise). For simplicity, we assume that every agent is independently affected by white noise of the same intensity. The resulting system is an extension of system \( \mathcal{G} \) given by

\[
\dot{x}(t) = -Lx(t) + w(t),
\]

where \( w(t) \in \mathbb{R}^N \) is a random signal with zero-mean and unit variance. In contrast to standard consensus problem without noise, instead of converging to the average of the initial state values, the sequence of node states \( x(t) \) becomes a stochastic process and fluctuates around the average of the current node.
states. The variance of these fluctuations can be captured by network coherence. Without loss of generality, we consider the case $\sum_{i=1}^{N} x_i(0) = 0$.

Definition 2.1: For a graph $\mathcal{G}$, the first-order network coherence $H_1(\mathcal{G})$ is defined as the mean, steady-state variance of the deviation from the average of all node values,

$$H_1(\mathcal{G}) := \frac{1}{N} \sum_{i=1}^{N} \lim_{t \to \infty} \mathbb{E} \left\{ x_i(t) - \frac{1}{N} \sum_{j=1}^{N} x_j(t) \right\}.$$

It has been established [27]–[29], [31], [42] that $H_1(\mathcal{G})$ is fully determined by the $N - 1$ non-zero eigenvalues for Laplacian matrix $L$. Specifically, the network coherence of the first-order system is given by

$$H_1(\mathcal{G}) = \frac{1}{2N} \sum_{i=2}^{N} \frac{1}{\lambda_i}.$$  

(8)

Lower $H_1(\mathcal{G})$ implies good robustness of the system irrespective of the presence of noise, that is, nodes remain closer to consensus at the average of their current states.

In addition to the coherence of first-order noisy consensus, the sum $\Lambda_{\text{sum}} = \sum_{i=2}^{N} \frac{1}{\lambda_i}$ of the $N - 1$ non-zero eigenvalues for Laplacian matrix $L$ of a graph $\mathcal{G}$ also plays a key role in determining many other interesting properties of the graph. For example, as a measure of overall connectedness and robustness [43] of a graph, the Kirchhoff index [44], [45] defined as the sum of resistance distances between all the $N(N - 1)/2$ pairs of vertices, is equal to $N\Lambda_{\text{sum}}$ [46], [47]. Again for instance, as a metric of diffusion velocity or mean cost of search in a graph, the average of hitting times for random walks over all the $N(N - 1)$ pairs of vertices equals $2M_{\text{sum}}/\Lambda_{\text{sum}}$, which can thus be expressed in terms of $H_1(\mathcal{G})$ as $4M_{\text{sum}}/N(N - 1)H_1(\mathcal{G})$.

Second-order noisy consensus. In the second-order consensus problem, each node $i$ has two states, position $x_i(t)$ and velocity $v_i(t)$. For example, in the vehicular formation problem, each vehicle has a position and a velocity. The objective is for each vehicle to move with constant heading velocity while keeping a fixed, pre-specified distance between itself and all of its neighbors. In a second-order system, the node states consist of a position vector $x(t)$ and a velocity vector $v(t)$. The states are measured relative to the heading velocity $\bar{v}$ and position $\bar{x}(t)$. The equation governing the system dynamics is given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -L & -L \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I_n \end{bmatrix} w(t),$$

(9)

where $w(t) \in \mathbb{R}^N$ is a mutually uncorrelated white noise process. Notice that in system (9) stochastic perturbations enter only in the velocity terms.

The network coherence of the second-order system is defined in terms of the node positions only, and it captures how closely the vehicle formation follows the desired heading trajectory in steady-state.

Definition 2.2: For a graph $\mathcal{G}$, the second-order network coherence $H_2(\mathcal{G})$ is the mean, steady-state variance of the deviation of each vehicle’s position error from the average of all vehicle position errors.

As in the case of first-order dynamics, the variance can be related to the $H_2$ norm of the system defined by (9), and its value also depends on the nonzero eigenvalues of the Laplacian matrix $L$, given by [28], [31], [42]

$$H_2(\mathcal{G}) = \frac{1}{2N} \sum_{i=2}^{N} \frac{1}{\lambda_i^2}.$$  

(10)

A small $H_2(\mathcal{G})$ corresponds to good robust second-order system of agents subject to stochastic disturbances.

Thus, the interesting quantities for consensus problems such as convergence speed, delay robustness, coherence for both first-order and second-order dynamics, are all dependent on the eigenvalues of the Laplacian matrix.

C. Related work

We first review some related work on convergence rate and robustness of communication delay for consensus algorithms. Olfati-Saber demonstrated that it is possible to dramatically increase the second largest eigenvalue of the Laplacian matrix of a regular graph by rewiring some edges, without significantly decreasing the largest eigenvalue [22]. Olshevsky and Tsitsiklis provided lower bounds on the worst-case convergence time for many different types of linear, time-invariant, distributed consensus algorithms [23]. Müñz et al. studied the robustness of consensus schemes with different feedback delays and proposed a scalable delay-dependent design algorithm for consensus controllers for a class of linear multi-agent systems [20].

For the first-order noisy consensus problem, using the measure in (8) Young et al. derived analytical expressions for the coherence in path, cycle, and star graphs [29]. Patterson et al. provided a series of theoretical study on the coherence in fractal trees (T-fractal and Vicsek fractal) [32], [42], and tori and lattices of different dimensions [31], with the mean square performance and robust yet fragile nature of torus also being considered by Ma and Elia [48]. Very recently, we presented analytical solutions for network coherence in the small-world Farey graph [49], [50], as well as the scale-free small-world Koch network [51]. Summers et al. addressed the graph topology design problem [52]. For a given graph, choose a fixed number of edges added to it, with an object to minimize the coherence of the resultant graph.

In contrast to the first-order dynamics, related work about second-order consensus problem is relatively less. Patterson and Bamieh studied the network coherence for two fractal trees—T-fractal and Vicsek fractal [52], [42], and also considered the coherence for tori and lattices with their collaborators [31]. We addressed the same problem in Koch network and compared the result with that in regular ring lattice with identical average degree 3 [51].

Previous work showed that the interested quantities (convergence rate, maximum communication delay, and the coherence of both first-order and second-order noisy consensus problems) are closely related to the structural properties of underlying networks, and unveiled partial influences of some particular features on the behavior of consensus on networks, such as small-world phenomenon [22], [49], fractal dimension [32].
However, the diversity of realistic network leads to the existence of diverse structural properties. For example, some real networks (e.g. power grid [53]) have an exponentially-decaying degree distribution [54], although many real-life networks have a power-law degree distribution [55]. To the best our knowledge, extensive analytical research about various consensus algorithms in small-world graphs with an exponential degree distribution is much less. The consensus problems in such graphs are not well characterized.

Motivated by previous work, in the sequel, we will study consensus problems in two much studied self-similar networks: hierarchical graphs [37] and Sierpiński graphs [38]. Both graphs have the same number of nodes and edges but exhibit different structures. For example, hierarchical graphs have a degree distribution of an exponential form, while the degree of all vertices in Sierpiński graphs is identical except some special vertices. Moreover, the eigenvalues of Laplacian matrices for both graphs can be determined analytically, which permits us to determine explicitly relevant quantities for consensus problems, to show how they scale with the system size, and further to explore the impacts of network architectures on the consensus problems.

III. Network construction and properties

In this section, we introduce two families of self-similar graphs that are generated iteratively by different assembly mechanisms [37]–[39]. The first class of graphs is hierarchical graphs [37] generated by a branching iteration; and the second type of graphs is Sierpiński graphs [38] created by a nested iteration. Both networks have the same numbers of nodes and edges, but display quite different structural properties. Thus, they are good candidate networks for studying consensus problems, in order to unveil the effects of topologies on consensus performance.

A. Hierarchical graphs

The hierarchical graphs are created by the hierarchical product of graphs introduced first by Godsil and McKay [56].

Definition 3.1: [56] Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be two graphs, with a root vertex labeled by 1. Then, the hierarchical product $G_2 \square G_1$ of $G_1$ and $G_2$ is a graph with vertices $x_2x_1$, $x_1 \in V_1$ ($i = 1, 2$), and edges $(x_2x_1, y_2y_1)$, where either $y_2 = x_2$ and $y_1$ or $y_1 = x_1 = 1$ and $y_2$ and $x_2$ are adjacent in $G_2$.

Note that the hierarchical product is associative [37], we can iteratively apply the operation of hierarchical product to any graph. The hierarchical graphs are those graphs generated by iteratively using the operation of hierarchical product to the complete graph.

Definition 3.2: [37] Let $K_k$ denote the complete graph with $k$ ($k \geq 3$) vertices, labelled by 1, 2, ..., $k$. Then the hierarchical graphs $H(n, k)$, $n \geq 1$, are the hierarchical product of $n$ replicas of $K_k$, with $H(1, k)$ isomorphic to $K_k$. That is, $H(n, k) = H(n - 1, k) \square K_k = K_k \square \cdots \square K_k$.

Fig. 1 illustrates a hierarchical graph $H(3, 3)$.

The hierarchical graphs $H(n, k)$ can be alternatively constructed in the following way [57].

### Definition 3.3:
For $n = 1$, $H(1, k)$ is the complete graph $K_k$. For $n > 1$, $H(n, k)$ is obtained from $H(n - 1, k)$; for each vertex in $H(n - 1, k)$, we create a complete graph $K_{k-1}$ and connect all its $k - 1$ vertices to their mother vertex in $H(n - 1, k)$.

Informally, a graph is self-similar, if it displays identical structure at every scale. For a formal definition of self-similar graph, we refer the reader to [38]. It has been shown that many real complex networks are self-similar [39]. The hierarchical graphs have a self-similar structure. In $H(n, k)$, there are $k$ vertices with highest degree $n(k - 1)$. We called them hub vertices. Given $H(n - 1, k)$, $H(n, k)$ can be obtained by performing the following operations. First, we create a complete graph $K_{k-1}$ and generate $k$ copies of $H(n - 1, k)$. Then, we identify the $k$ vertices of the complete graph $K_{k-1}$ and $k$ hub vertices of $k$ different replicas of $H(n - 1, k)$.

Let $N_n$ and $E_n$ denote the order (number of vertices) and size (number of edges) of hierarchical graphs $H(n, k)$. According to the construction algorithms, we have

$$N_n = k^n,$$

and

$$E_n = \frac{k^{n+1} - k}{2},$$

In the hierarchical graphs $H(n, k)$, the degree spectrum is discrete: the number $N(\delta, n)$ of vertices of degree $n(k - 1)$, $(n - 1)(k - 1)$, $(n - 2)(k - 1)$, ..., $2(k - 1)$, $k - 1$, is $k$, $(k - 1)k$, $(k - 1)k^2$, ..., $(k - 1)k^{n-2}$, $(k - 1)k^{n-1}$, respectively. Thus, the degree of vertices in the hierarchical graphs $H(n, k)$ follows an exponential distribution, with its cumulative degree distribution

$$P_{\text{cum}}(\delta) = \sum_{\delta' \geq \delta} N(\delta', n)/N_n \sim k^1 - \delta \text{ decaying exponentially with the degree } \delta \text{ but independent of } n.$$ Such a degree distribution has been previously observed in some real technology networks [53], [54]. Besides, the hierarchical graphs $H(n, k)$ exhibit the typical small-world characteristics of real-life networks [55]. They have high average clustering coefficient, and both their diameter and average distance grow logarithmically with the network order [57].

The hierarchical graphs have been used to mimic real networks, such as biological networks [39] and polymer networks [57].
Sierpiński graphs

The Sierpiński graphs $S(n, k) = (\mathcal{V}(S(n, k)), \mathcal{E}(S(n, k)))$ \((n \geq 1 \text{ and } k \geq 3)\) were introduced by Klavžar and Milutinović [38] as a two-parametric generalization of the Tower of Hanoi graph [60], [61]. They are defined on the vertex set comprising of all \(n\)-tuples of integers \(1, 2, \cdots, k\), that is, $\mathcal{V}(S(n, k)) = \{1, 2, \cdots, k\}^n$. All vertices in $S(n, k)$ can be labelled in the form $u_1 u_2 \cdots u_n$, where $u_i \in \{1, 2, \cdots, k\}$ for all $i=1, 2, \cdots, n$. Two vertices $p = p_1 p_2 \cdots p_n$ and $q = q_1 q_2 \cdots q_n$ are connected to each other by an edge if and only if there exists an integer $h$ (\(1 \leq h \leq n\)) such that

(a) $p_i = q_i$ for $1 \leq i \leq h-1$;
(b) $p_h \neq q_h$;
(c) $p_i = q_i$ and $p_{i+1} = q_{i+1}$ for $h+1 \leq i \leq n$.

Fig. 2 illustrates the Sierpiński graph $S(3, 3)$. The order and size of the Sierpiński graphs $S(n, k)$ are identical to those corresponding to hierarchical graphs $\mathcal{H}(n, k)$. In other words, $N_n = k^n$ and $E_n = \frac{k^{n+1}-k}{k-1}$.

Analogously to hierarchical graphs $\mathcal{H}(n, k)$, Sierpiński graphs $S(n, k)$ are also self-similar. However, Sierpiński graphs $S(n, k)$ have different structural properties from those of hierarchical graphs $\mathcal{H}(n, k)$. First, Sierpiński graphs are more homogeneous. In $S(n, k)$, there are $k$ vertices of label form $i_1 \cdots i_k$ (\(1 \leq i_1 \leq k\)), which are called extreme vertices. Each extreme vertex has a degree $k-1$, while the degree of any other vertex is $k$. Second, Sierpiński graphs are not small-world. The diameter [38] of $S(n, k)$ is $2^n - 1$ increasing exponentially with $n$, and thus growing as a power-law function of network order $N_n$.

Definition 3.4: [62] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected undirected graph. For any vertex $v \in \mathcal{V}$, define $B(r)$ as the radius $r$, centered at $v$, i.e., $B(r) = \{u \in \mathcal{V} : d(u, v) \leq r\}$, where $d(u, v)$ denotes the shortest-path length between the two vertices $u$ and $v$ in $\mathcal{G}$. The fractal dimension of $\mathcal{G}$ is

$$d_f(\mathcal{G}) := \limsup_{r \to \infty} \frac{\log B(r)}{\log r}.$$ 

Sierpiński graphs $S(n, k)$ are fractal [63], [64], with their fractal dimension being $d_f(S(n, k)) = \frac{\log k}{\log 2}$.

Except for the intrinsic theoretical interest [65], [66], Sierpiński graphs play an important role in topology, mathematics, and computer science. For example, adding an “open edge” to each of the $k$ extreme vertices in Sierpiński graphs yields the WK-recursive networks. These resultant networks have many favorable structure properties, including a high degree of regularity, symmetry and scalability, which can be applied as a model of interconnection networks widely used in implementing large-scale distributed systems [67], [68]. Recently, great efforts have been dedicated to investigating various issues of the WK-Recursive networks, such as topological properties [69], broadcasting algorithms [70], and fault tolerance [71].

IV. Consensus Algorithms in Hierarchical Graphs

In this section, we study analytically several quantities related to consensus algorithms in hierarchical graphs $\mathcal{H}(n, k)$, concentrating on convergence speed of first-order consensus without delay and noise, delay robustness of first-order consensus, as well as coherence of first-order and second-order noisy consensus algorithms. For all these consensus algorithms, we derive explicit or recursive expressions for those quantities concerned, as well as their asymptotic scalings.

A. Spectra of Laplacian matrix

Let $L_n$ denote the Laplacian matrix of hierarchical graphs $\mathcal{H}(n, k)$. And let $\Psi_n$ represent the set of eigenvalues of $L_n$ defined as

$$\Psi_n = \{\lambda_1(n), \lambda_2(n), \cdots, \lambda_{\Omega(n)}(n)\},$$

satisfying $0 = \lambda_1(n) \leq \lambda_2(n) \leq \cdots \leq \lambda_{\Omega(n)}(n)$. Note that $\Psi_n$ is a multiset. That is, the distinctness of elements in $\Psi_n$ is neglected. It has been shown [57] that all eigenvalues of $L_n$ can be determined recursively.

Lemma 4.1: All the nonzero eigenvalues in the set $\Psi_n$ of matrix $L_n$ can be classified into two subsets $\Psi^{(1)}_n \cup \Psi^{(2)}_n \cup \{0\}$, where $\Psi^{(1)}_n$ contains only eigenvalue $k$ with multiplicity $(k-2)k^{n-1}+1$, $\Psi^{(2)}_n$ includes $2k^{n-1}-2$ elements $\lambda_i(n)$ and $\tilde{\lambda}_i(n)$ generated by $\lambda_i(n-1)$, $i = 2, 3, \cdots, k^{n-1}$, through the following relations

$$\tilde{\lambda}_i(n) = \frac{1}{2} \sqrt{k^2 + 2k \lambda_i(n-1) + \left(\lambda_i(n-1)\right)^2 - 4 \lambda_i(n-1)},$$
$$\lambda_i(n) = \frac{1}{2} \sqrt{k^2 + 2k \lambda_i(n-1) + \left(\lambda_i(n-1)\right)^2 - 4 \lambda_i(n-1)} + k + \lambda_i(n-1),$$

which obey relation $\tilde{\lambda}_2(n) \leq \tilde{\lambda}_3(n) \leq \cdots \leq \tilde{\lambda}_{k^{n-1}}(n) < k-2 < \lambda_{2'}(n) \leq \lambda_{3'}(n) \leq \cdots \leq \lambda_{k^{n-1}-1}(n)$.

For $\mathcal{H}(1, k)$, the set of eigenvalues of its Laplacian matrix is

$$\Psi_1 = \{0, k, k, \cdots, k\}.$$

![Fig. 2. The Sierpiński graph $S(3, 3)$ and its vertex labeling.](image)
Then, by iteratively applying Lemma 4.1, we can obtain all the $k^n$ eigenvalues of the Laplacian matrix for the hierarchical graphs $H(n, k)$ for all $n \geq 1$.

**B. Convergence speed and delay robustness**

Let $\epsilon_n$ and $\zeta_n$ be, respectively, the second smallest eigenvalue and the largest eigenvalue for Laplacian matrix of hierarchical graphs $H(n, k)$. As shown above, $\epsilon_n$ and $\zeta_n$ measure the convergence speed and delay robustness of first-order consensus algorithms on hierarchical graphs $H(n, k)$. In this subsection, we characterize the asymptotic behaviors of the two critical eigenvalues $\epsilon_n$ and $\zeta_n$.

**Theorem 4.2:** For hierarchical graphs $H(n, k)$, when $n$ is sufficient large, the asymptotic behaviors of the second smallest eigenvalue $\epsilon_n$ and the largest eigenvalue $\zeta_n$, are

$$\epsilon_n \approx k^{2-n} \quad (17)$$

and

$$\zeta_n \approx (k - 1)n, \quad (18)$$

respectively.

**Proof:** By (14), the second smallest eigenvalue $\epsilon_n$ of the Laplacian matrix for hierarchical graphs $H(n, k)$ obeys recursion relation

$$\epsilon_n = \frac{1}{2} \left( k + \epsilon_{n-1} - \sqrt{k^2 + 2k\epsilon_{n-1} + \epsilon_{n-1}^2 - 4\epsilon_{n-1}} \right)$$

$$= \frac{1}{2} \left( k + \epsilon_{n-1} - k \sqrt{1 + \frac{2\epsilon_{n-1}}{k} + \frac{\epsilon_{n-1}^2 - 4\epsilon_{n-1}}{k^2}} \right). \quad (19)$$

Analogously, the largest eigenvalue $\zeta_n$ satisfies

$$\zeta_n = \frac{1}{2} \left( \zeta_{n-1} \sqrt{1 + \frac{k^2}{\zeta_{n-1}^2} + \frac{2k - 4}{\zeta_{n-1}} + k + \zeta_{n-1}} \right). \quad (20)$$

Applying the following relation

$$\sqrt{1-x} = 1 - \frac{1}{2}x + o(x), \quad (x \to 0),$$

(21)

and ignoring the higher order infinitesimal in the radical terms lead to

$$\epsilon_n \approx \frac{\epsilon_{n-1}}{k} \quad (22)$$

and

$$\zeta_n \approx \zeta_{n-1} + k - 1. \quad (23)$$

Considering the fact that $\epsilon_1 = \zeta_1 = k$, we obtain the asymptotic behaviors of $\epsilon_n$ and $\zeta_n$ given in (17) and (18).

In Fig. 3, we report the exact values and approximative results for the second smallest eigenvalue $\epsilon_n$ and the largest eigenvalue $\zeta_n$ corresponding to hierarchical graphs $H(n, k)$ with various $n$ and $k$. In the figure, solid symbols represent the exact results iteratively generated by (19) or (20), while the straight lines denote the approximative results given by (21) or (18). It can be seen that both the exact and approximate results agree well with each other. Moreover, as is expected, the approximate values of the largest eigenvalue (the second smallest eigenvalue) are slightly smaller (larger) than the corresponding exact ones, since we ignore the infinitesimal of higher order during the derivation of the approximate formulas.

**Theorem 4.2** shows that, as $n \to \infty$, the second smallest eigenvalue $\epsilon_n$ is inversely proportional to the order $N_n$, satisfying $\epsilon_n \approx k^2N_n^{-1}$, while the largest eigenvalue $\zeta_n$ increases logarithmically with $N_n$.

**C. First-order and second-order coherence**

We proceed to determine the explicit expressions and their leading behaviors for the coherence of the first-order and second-order noisy consensus algorithms in hierarchical graphs $H(n, k)$, which are denoted as $H_1(H(n, k))$ and $H_2(H(n, k))$, respectively.

1) First-order coherence: With the known results about the eigenvalues for Laplacian matrix of hierarchical graphs $H(n, k)$, we can obtain the first-order network coherence.

**Theorem 4.3:** For the hierarchical graphs $H(n, k)$ with $N_n$ vertices, the first-order coherence of the system with dynamics defined in (7) is

$$H_1(H(n, k)) = \frac{(2n-1)k^n - 2nk^n - 1}{2k^{n+1}}. \quad (24)$$

In the limit of large $n$, $H_1(H(n, k))$ grows with network order $N_n$ as

$$\lim_{n \to \infty} H_1(H(n, k)) = \frac{k - 1}{k^2} \log_k N_n. \quad (25)$$

**Proof:** Based on the previously established result, the first-order coherence of $H(n, k)$ is

$$H_1(H(n, k)) = \frac{1}{2N_n} \sum_{i=2}^{N_n} \frac{1}{\lambda_i^{(n)}}, \quad (26)$$

where $\lambda_i^{(n)}$, $2 \leq i \leq N_n$, are all the non-zero eigenvalues of the Laplacian matrix for $H(n, k)$. We now determine the sum on the right-hand side (rhs) of (26), denoted as $\Lambda_n$, which can be evaluated as

$$\Lambda_n = \sum_{i=2}^{N_n} \frac{1}{\lambda_i^{(n)}} = \sum_{\lambda_i^{(n)} \in \Psi_n^{(1)}} \frac{1}{\lambda_i^{(n)}} + \sum_{\lambda_i^{(n)} \in \Psi_n^{(2)}} \frac{1}{\lambda_i^{(n)}} \quad (27)$$

Let $\Lambda_n^{(1)}$ and $\Lambda_n^{(2)}$ stand for the two sum terms on the rhs of (27). For $\lambda_i^{(n)} \in \Psi_n^{(1)}$, we have

$$\Lambda_n^{(1)} = \sum_{\lambda_i^{(n)} \in \Psi_n^{(1)}} \frac{1}{\lambda_i^{(n)}} = (k - 2)k^{n-2} + \frac{1}{k}. \quad (28)$$
For \( \gamma(n) \in \Psi^{(2)}_n \), it can be evaluated as follows. From Lemma 4.1 each eigenvalue \( \lambda_i(n-1) \) in \( \Psi_{n-1} \) gives rise to two eigenvalues \( \gamma(n) \) and \( \gamma(n) \) in \( \Psi^{(2)}_n \), which obey relations \( \gamma(n) + \gamma(n) = k + \lambda_i(n-1) \) and \( \gamma(n) - \gamma(n) = \lambda_i(n-1) \). Then,

\[
\frac{1}{\gamma(n)} + \frac{1}{\gamma(n)} = \frac{\lambda_i(n) + \lambda_i(n)}{\lambda_i(n) - \lambda_i(n)} = \frac{k}{\lambda_i(n-1)} + 1. \tag{29}
\]

Consequently, we have

\[
\Lambda^{(2)}_n = \sum_{\gamma(n) \in \Psi^{(2)}_n} \frac{1}{\gamma(n)} = k \cdot \Lambda_{n-1} + N_{n-1} - 1. \tag{30}
\]

Combining (27), (28) and (30), we obtain the recursive relation for \( \Lambda_n \) as \( \Lambda_n = k \Lambda_{n-1} + 2(k-1)k^{n-2} - \frac{k}{k-1} \), which, together with the initial condition \( \Lambda_1 = \frac{k}{2} \), is solved to yield

\[
\Lambda_n = \frac{(2n-1)k^n - 2nk^{n-1} + 1}{k^n + 1}. \tag{31}
\]

Substituting (31) into (26), we obtain the explicit expression for the first-order coherence in the hierarchical graphs \( H(n, k) \):

\[
H_1(H(n, k)) = \frac{(2n-1)k^n - 2nk^{n-1} + 1}{2k^{n+1}}. \tag{32}
\]

For large \( n \), (32) implies

\[
\lim_{n \to \infty} H_1(H(n, k)) = \frac{k - 1}{k^2} n. \tag{33}
\]

Because \( n = \log_k N_n \), for \( n \to \infty \), \( H_1(H(n, k)) \) can be expressed in terms of \( N_n \) as

\[
\lim_{n \to \infty} H_1(H(n, k)) = \frac{k - 1}{k^2} \log_k N_n. \tag{34}
\]

This completes the proof.

Theorem 4.3 indicates that for large \( n \), the first-order coherence of hierarchical graphs \( H(n, k) \) grows as a logarithmic function of \( N_n \).

2) Second-order coherence: We now derive the second-order coherence in the hierarchical graphs \( H(n, k) \).

**Theorem 4.4:** For the hierarchical graphs \( H(n, k) \), the second-order coherence of the system with dynamics defined in (27) is

\[
H_2(H(n, k)) = \frac{k - k^2 + k^n(k^2 - 5k - 6) + k^{2n}(4k + 6)}{2k^{n+3}(1 + k)} + \frac{2n(1 - k)}{k^3}. \tag{35}
\]

In the limit of large \( n \), the leading term of \( H_2(H(n, k)) \) can be represented in terms of network order \( N_n \) as

\[
H_2(H(n, k)) \sim \frac{2k + 3}{k^3(k + 1)} N_n. \tag{36}
\]

**Proof:** By definition,

\[
H_2(H(n, k)) = \frac{1}{2N_n} \sum_{i=2}^{N_n} \frac{1}{\lambda_i(n)^2}. \tag{37}
\]

Let \( \Gamma_n \) denote the sum on the rhs of (37). Then, \( \Gamma_n \) can be evaluated as

\[
\Gamma_n = \sum_{\lambda_i(n) \in \Psi^{(2)}_n} \frac{1}{\lambda_i(n)^2} + \sum_{\lambda_i(n) \in \Psi^{(2)}_n} \frac{1}{\lambda_i(n)^2}. \tag{38}
\]

We denote the two sum terms on the rhs of (38) by \( \Gamma^{(1)}_n \) and \( \Gamma^{(2)}_n \), respectively. \( \Gamma^{(1)}_n \) can be expressed as

\[
\Gamma^{(1)}_n = \sum_{\lambda_i(n) \in \Psi^{(1)}_n} \frac{1}{\lambda_i(n)^2} = (k - 2)k^{n-3} + \frac{1}{k^2}. \tag{39}
\]

Considering

\[
\frac{1}{\lambda_i(n)^2} + \frac{1}{\lambda_i(n)^2} = \frac{(\lambda_i(n-1) + k)^2 - 2\lambda_i(n-1)}{(\lambda_i(n-1))^2}, \tag{40}
\]

\( \Gamma^{(2)}_n \) can be evaluated as

\[
\Gamma^{(2)}_n = \sum_{\lambda_i(n) \in \Psi_{n-1}} \left( 1 + \frac{k^2}{(\lambda_i(n-1))^2} + \frac{2k - 2}{(\lambda_i(n-1))^2} \right). \tag{41}
\]

Plugging (39) and (41) into (38) gives

\[
\Gamma_n = k^2 \Gamma_{n-1} + (2k - 2)\Gamma_{n-1} + k^{n-1} - 1, \tag{42}
\]

which under the initial condition \( \Gamma_1 = \frac{k^2 - 2k + 2}{k^2} \) is solved inductively to yield

\[
\Gamma_n = \frac{k - k^2 + k^n(k^2 - 5k - 6) + k^{2n}(4k + 6)}{k^3(k + 1)} + 4n \cdot k^{n-3}(1 - k). \tag{43}
\]

Inserting the expression for \( \Gamma_n \) into \( H_2(H(n, k)) = \frac{\Gamma_n}{2N_n} \) gives (35). When \( n \to \infty \), the dominating term of \( H_2(H(n, k)) \) increases with \( N_n \) as

\[
H_2(H(n, k)) \sim \frac{2k + 3}{k^3(k + 1)} N_n. \tag{44}
\]

This completes the proof.

Theorem 4.3 means that when the network order \( N_n \) is sufficiently large, the second-order coherence \( H_2(H(n, k)) \) behaves linearly with \( N_n \).

**V. Consensus Algorithms in Sierpiński Graphs**

In this section, we study consensus algorithms in Sierpiński graphs. We are concerned with the same quantities as those corresponding to the hierarchical graphs. We will show that the behaviors of related quantities are significantly different from those associated with the hierarchical graphs.

**A. Spectra of Laplacian matrix**

In the case without confusion, we use the same notations as those corresponding to the hierarchical graphs. Let \( L_n \) denote the Laplacian matrix of the Sierpiński graphs \( S(n, k) \), the eigenvalue set of which is denoted by \( \Phi_n \), given by

\[
\Phi_n = \left\{ \lambda_1(n), \lambda_2(n), \cdots, \lambda_{N_n}(n) \right\}, \tag{45}
\]

satisfying \( 0 = \lambda_1(n) = \lambda_2(n) = \cdots = \lambda_{N_{n-1}}(n) \).

The eigenvalues of the Laplacian matrix for Sierpiński graphs \( S(n, k) \) have been fully determined [63], which have been applied to numerous aspects, e.g., relaxation dynamics [72]. In [63], a recursive relation for the eigenvalues of
Laplacian matrix for Sierpiński graphs $S(n, k)$ was provided, as stated in the following lemma.

**Lemma 5.1:** For Sierpiński graphs $S(n, k)$, $n \geq 2$, the non-zero eigenvalues in set $\Phi_n$ can be classified into two subsets $\Phi_n^{(1)}$ and $\Phi_n^{(2)}$, satisfying $\Phi_n = \Phi_n^{(1)} \cup \Phi_n^{(2)} \cup \{0\}$, where $\Phi_n^{(1)}$ consists of two eigenvalues $k$ and $k+2$, the multiplicities of which are $\frac{k-1}{2} \left(k^{n-1} + \frac{k}{k-1}\right)$ and $\frac{k+2}{2} \left(k^{n-1} - 1\right)$, respectively; while $\Phi_n^{(2)}$ contains the rest $2k^{n-1} - 2$ nonzero eigenvalues. Moreover, each eigenvalue $\lambda_i^{(n-1)}$, $2 \leq i \leq k^{n-1}$, in set $\Phi_{n-1}$ generates two elements in set $\Phi_n$, both of which have the same degeneracy as that of $\lambda_i^{(n-1)}$ and are the roots of the following equation in $\lambda$:

$$\lambda^2 - (k+2)\lambda + \lambda_i^{(n-1)} = 0,$$  \hspace{1cm} (46)

Since $S(1, k)$ is isomorphic to $H(1, k)$, the set of eigenvalues for Laplacian matrix of $S(1, k)$ is

$$\Phi_1 = \Psi_1 = \{0, k, k, k, \ldots, k\}.$$  \hspace{1cm} (47)

Then, by recursively using Lemma 5.1 we obtain all the eigenvalues for $S(n, k)$ for any $n > 1$.

**B. Convergence speed and delay robustness**

For the Laplacian matrix of Sierpiński graphs $S(n, k)$, the second smallest eigenvalue $\epsilon_n$ and the largest eigenvalue $\zeta_n$ can be analytically determined as given in the following theorem.

**Theorem 5.2:** For Sierpiński graphs $S(n, k)$, the largest eigenvalue of its Laplacian matrix $\mathbf{L}_n$ is $\zeta_n = k + 2$ for all $n \geq 2$, while the second smallest eigenvalue $\epsilon_n$ of $\mathbf{L}_n$ satisfies

$$\epsilon_n \approx \frac{k}{(k+2)^{n-1}},$$  \hspace{1cm} (48)

as $n \to \infty$.

**Proof:** Let $f_1(x)$ and $f_2(x)$ be two functions of real number $x$ in interval $[0, k+2]$, defined by

$$f_1(x) = \frac{k+2 - \sqrt{(k+2)^2 - 4x}}{2},$$ \hspace{1cm} (49)

and

$$f_2(x) = \frac{k+2 + \sqrt{(k+2)^2 - 4x}}{2},$$ \hspace{1cm} (50)

respectively. It is easy to see that function $f_1(x)$ (or $f_2(x)$) is a monotonically increasing (or decreasing) function on its domain $x \in [0, k+2]$, satisfying $0 \leq f_1(x) \leq \frac{k+2}{2} \leq f_2(x) \leq k+2$.

According to Lemma 5.1 each eigenvalue $\lambda_i^{(n-1)}$ in set $\Phi_{n-1}$ gives rise to two eigenvalues $\tilde{\lambda}_i^{(n)}$ and $\tilde{\lambda}_i^{(n)}$ in set $\Phi_n^{(2)}$ through (46), with $\tilde{\lambda}_1^{(n)} = f_1(\lambda_1^{(n-1)})$ and $\tilde{\lambda}_2^{(n)} = f_2(\lambda_1^{(n-1)})$. For any eigenvalue $\lambda_i^{(n)} \in \Phi_n^{(2)}$, we have

$$\tilde{\lambda}_i^{(n)} = \frac{1}{2} \left(k+2 + \sqrt{(k+2)^2 - 4\lambda_i^{(n-1)}}\right).$$  \hspace{1cm} (51)

Since all the eigenvalues of a connected graph are greater than or equal to zero, (51) means that all the eigenvalues of Laplacian matrix for Sierpiński graphs $S(n, k)$ is less than or equals $k+2$. Thus, the largest eigenvalue of Laplacian matrix for $S(n, k)$ is always $\zeta_n = k + 2$ for $n \geq 2$.

For the second smallest eigenvalue $\epsilon_n$, it obeys the following recursive relation:

$$\epsilon_n = \frac{k}{2} \left(2 - (k+2) \sqrt{1 - \frac{4\epsilon_{n-1}}{(k+2)^2}}\right).$$  \hspace{1cm} (52)

Using the approximate relation in (21) and applying a similar argument for the proof of Theorem 4.2 we obtain

$$\epsilon_n \approx \frac{\epsilon_{n-1}}{k+2},$$  \hspace{1cm} (53)

Considering $\epsilon_1 = k$, (52) leads to (47).

In Fig. 4 we report the comparison of approximate and exact results for the second smallest eigenvalue $\epsilon_n$ of the Laplacian matrix for Sierpiński graphs $S(n, k)$, which are generated by (47) and (51), respectively. In Fig. 4 solid symbols denote the accurate results, while the straight lines represent the approximate ones. Fig. 4 shows that the results yielded by (47) and (51) are consistent with each other, the difference between which is intangible.

Theorem 5.2 implies that as $n \to \infty$, the algebraic connectivity $\epsilon_n$ of Sierpiński graphs $S(n, k)$ is a power-law function of $N_n$ as $\epsilon_n \approx k(k+2)(N_n)^{-\frac{\log_2(k+2)}{2}}$, which is much smaller than the algebraic connectivity $k^2N_n^{-1}$ corresponding to hierarchical graphs $H(n, k)$. Thus, the speed of convergence of the consensus algorithm described by (5) in hierarchical graphs is considerably faster than in Sierpiński graphs.

On the other hand, the largest eigenvalue $\zeta_n$ for the Laplacian matrix of $S(n, k)$ is a constant $k + 2$, independent of $n$. While for hierarchical graphs $H(n, k)$, $\zeta_n$ grows linearly with $n$ as $\zeta_n \approx (k-1)n$, much larger than $k+2$. Therefore, the consensus algorithm described by (5) in Sierpiński graphs $S(n, k)$ is more robust to delay than in hierarchical graphs $H(n, k)$.

**C. First-order and second-order coherence**

Let $H_1(S(n, k))$ and $H_2(S(n, k))$ denote, respectively, the coherence of the first-order and second-order noisy consensus
algorithms in Sierpiński graphs $S(n, k)$. Below we determine their accurate expressions and leading scalings.

1) First-order coherence: Using the above recursive relations related to the eigenvalues for Laplacian matrix of Sierpiński graphs $S(n, k)$ at two successive iterations, we can deduce the first-order coherence.

**Theorem 5.3:** For the Sierpiński graphs $S(n, k)$ with $N_n$ vertices, the first-order coherence of the system with dynamics defined in (7) is

$$H_1(S(n, k)) = \frac{(k^2 + k + 2)(k - 1)(k + 2)^2 - 4k}{4k(k + 1)(k + 2)} - \frac{(k - 2)(k + 1)}{4k(k + 2)}. \tag{53}$$

In the limit of large $n$, $H_1(S(n, k))$ scales with the network order $N_n$ as

$$\lim_{n \to \infty} H_1(S(n, k)) = \frac{k^3 + k - 2}{4k(k + 1)(k + 2)} N_n^{\log_2(k+2) - 1}. \tag{54}$$

**Proof:** Let $\Theta_n$ denote the sum of reciprocals of all the $N_n - 1$ nonzero eigenvalues of Laplacian matrix for $S(n, k)$. Then, $H_1(S(n, k)) = \Theta_n^{(2)}$. By definition,

$$\Theta_n = \sum_{n=2}^{N_n} \frac{1}{\lambda_i^{(n)}} = \sum_{\lambda_i^{(n)} \in \Phi_n} \frac{1}{\lambda_i^{(n)}} + \sum_{\lambda_i^{(n-1)} \in \Phi_{n-1} \setminus \{0\}} \frac{1}{\lambda_i^{(n-1)}}. \tag{55}$$

Denote the two sum terms on the rhs of (55) as $\Theta_n^{(1)}$ and $\Theta_n^{(2)}$, respectively. By Lemma 5.1, the first sum is

$$\Theta_n^{(1)} = \frac{k - 2}{2} \left( k^{n-2} + \frac{1}{k - 2} + k^{n-1} - 1 \right). \tag{56}$$

We continue to compute the second sum term $\Theta_n^{(2)}$. According to Vieta’s formulas, the two roots $\lambda_{i,1}^{(n)}$ and $\lambda_{i,2}^{(n)}$ of (46) obey relations $\lambda_{i,1}^{(n)} + \lambda_{i,2}^{(n)} = k + 2$ and $\lambda_{i,1}^{(n)} \cdot \lambda_{i,2}^{(n)} = \lambda_i^{(n-1)}$, which indicate

$$\frac{1}{\lambda_{i,1}^{(n)}} + \frac{1}{\lambda_{i,2}^{(n)}} = \frac{\lambda_{i,1}^{(n)} + \lambda_{i,2}^{(n)}}{\lambda_{i,1}^{(n)} \cdot \lambda_{i,2}^{(n)}} = \frac{k + 2}{\lambda_i^{(n-1)}}. \tag{57}$$

Then the second sum term $\Theta_n^{(2)}$ in (55) can be evaluated as

$$\Theta_n^{(2)} = \sum_{\lambda_i^{(n)} \in \Phi_n} \frac{1}{\lambda_i^{(n)}} = \sum_{\lambda_i^{(n)} \in \Phi_{n-1} \setminus \{0\}} \frac{k + 2}{\lambda_i^{(n-1)}}. \tag{58}$$

Inserting (56) and (58) into (55) yields

$$\Theta_n = (k + 2)\Theta_{n-1} + \frac{k - 2}{2} \left( k^{n-2} + \frac{1}{k - 2} + k^{n-1} - 1 \right). \tag{59}$$

Using the initial condition $\Theta_1 = \frac{k + 1}{k}$, (59) is solved to obtain

$$\Theta_n = \frac{(k^2 + k + 2)(k - 1)(k + 2)^2 - (k - 2)(k + 1)^2k^n}{2k(k + 1)(k + 2)} - \frac{2}{(k + 1)(k + 2)}. \tag{60}$$

Substituting this result and $N_n = k^n$ into $H_1(S(n, k)) = \Theta_n^{(2)}$ yields

$$H_1(S(n, k)) = \frac{(k^2 + k + 2)(k - 1)(k + 2)^2 - 4k}{4k(k + 1)(k + 2)} - \frac{(k - 2)(k + 1)}{4k(k + 2)}. \tag{61}$$

In the large limit of $n$,

$$\lim_{n \to \infty} H_1(S(n, k)) = \frac{k^3 + k - 2}{4k(k + 1)(k + 2)} \left( \frac{k + 2}{k} \right)^n, \tag{62}$$

which can be expressed in terms of the network order $N_n$ as

$$\lim_{n \to \infty} H_1(S(n, k)) = \frac{k^3 + k - 2}{4k(k + 1)(k + 2)} N_n^{\log_2(k+2) - 1}. \tag{63}$$

This completes the proof.

Theorem 5.3 shows that for large Sierpiński graphs $S(n, k)$, the first-order coherence $H_1(S(n, k))$ grows sublinearly with $N_n$. This is contrast to its counterpart of hierarchical graphs $H_1(n, k)$, for which the first-order coherence increases logarithmically with $N_n$.

2) Second-order coherence: We finally compute the second-order coherence in Sierpiński graphs $S(n, k)$.

**Theorem 5.4:** For the Sierpiński graphs $S(n, k)$ with $N_n$ vertices, the second-order coherence of the system with dynamics defined in (9) is

$$H_2(S(n, k)) = -\frac{7k^2 + 13k + 2}{2k^{n+1}(k + 1)^2(k + 2)^2(k + 3)} - \frac{(k - 2)(k^3 + 4k^2 + 4k + 2)}{2k^2(k + 2)^2(k + 3k + 4)} + \frac{(k - 1)(k + 2)^{n-2}(k + 2k + 2)}{2k^{n+1}(k + 1)^2(k + 3k + 4)} + \frac{k^5 + 7k^4 + 16k^3 + 28k^2 + 26k + 12}{2k^{n+2}(k + 1)^2(k + 3k^2 + 3k + 4)} - \frac{(k - 1)(k + 2)^{n+2}}{2k^{n+2}(k + 1)^2(k + 3k^2 + 3k + 4)}. \tag{64}$$

In large graphs ($n \to \infty$), the dominating term of $H_2(S(n, k))$ scales with network order $N_n$ as

$$H_2(S(n, k)) \sim h(k) N_n^{\log_2(k+2) - 1}, \tag{65}$$

where $h(k) = \frac{(k^5 + 7k^4 + 16k^3 + 28k^2 + 26k + 12)(k - 1)}{2k^2(k + 1)^2(k + 2)(k + 3k^2 + 3k + 4)}$.

**Proof:** Let $\Omega_n$ denote the sum of the reciprocals of square of all non-zero eigenvalues for Laplacian matrix of $S(n, k)$. By definition,

$$\Omega_n = \sum_{i=2}^{N_n} \frac{1}{\lambda_i^{(n)}} = \sum_{\lambda_i^{(n)} \in \Phi_n} \frac{1}{\lambda_i^{(n)}} + \sum_{\lambda_i^{(n-1)} \in \Phi_{n-1} \setminus \{0\}} \frac{1}{\lambda_i^{(n-1)}}. \tag{66}$$

The first sum term on the rhs of (66) can be expressed as

$$\sum_{\lambda_i^{(n)} \in \Phi_n} \frac{1}{\lambda_i^{(n)}} = \frac{k - 2}{2} \left( k^{n-3} + \frac{1}{k - 2} + k^{n-1} - 1 \right). \tag{67}$$
Using the following relation
\[ \frac{1}{\tilde{\lambda}_{i,1}^{(n)}} + \frac{1}{\tilde{\lambda}_{i,2}^{(n)}} = \frac{(k+2)^2 - 2\lambda_{i}^{(n-1)}}{\lambda_{i}^{(n-1)}}, \tag{68} \]
the second sum term in (66) can be evaluated as
\[ \sum_{\tilde{\lambda}^{(n)} \in \Phi^{(2)}_{0}} \frac{1}{\tilde{\lambda}_{i}^{(n-1)}} = \sum_{\lambda_{n-1} \in \Phi_{n-1}\setminus\{0\}} \frac{(k+2)^2 - 2\lambda_{i}^{(n-1)}}{\lambda_{i}^{(n-1)}} = (k+2)^2 \Omega_{n-1} - 2 \Theta_{n-1}. \tag{69} \]
Then, we have the following recursive relation for \( \Omega_n \):
\[ \Omega_n = (k+2)^2 \Omega_{n-1} - 2 \Theta_{n-1} + \frac{k-2}{2} \left( \frac{k-3}{2} + \frac{1}{k(k-2)} \right) + \frac{k^n-1}{(k+2)^2}. \tag{70} \]
Applying (69) and the initial value \( \Omega_1 = \frac{k-1}{k+2} \), (70) is solved to obtain the explicit expression for \( \Omega_n \). Plugging the expression for \( \Omega_n \) into relation \( H_2(S(n, k)) = N_n^{2h(k)} \) yields (64). For large \( n \), the leading term of \( H_2(S(n, k)) \) can be represented in terms of network order \( N_n \) as
\[ H_2(S(n, k)) \sim h(k)N_n^{2\log_{(k+2)}{n-1}}, \tag{71} \]
where the factor \( h(k) \) is a function of \( k \): \[ h(k) = \frac{(k^5 + 7k^4 + 16k^3 + 23k^2 + 26k + 12)(k-1)}{2k^2(k+1)^2(k+2)^2(k+3)(k^2 + 3k + 4)}. \tag{72} \]
This completes the proof.

Theorem 4.3 indicates that the asymptotic behavior for the second-order coherence in the Sierpiński graphs \( S(n, k) \) grows superlinearly with the network order \( N_n \), and is considerably larger than that associated with the hierarchical graphs \( H(n, k) \).

3) Analysis: In the above, we have demonstrated that the behaviors of related quantities for consensus algorithms in hierarchical graphs \( H(n, k) \) and Sierpiński graphs \( S(n, k) \) are strongly different from each other. The second smallest eigenvalue of \( H(n, k) \) is considerably larger than that of \( S(n, k) \), with their ratio being \( \left( \frac{k+2}{k-1} \right)^{n-1} \) and increasing exponentially with \( n \). The largest eigenvalue of \( H(n, k) \) is remarkably greater than that of \( S(n, k) \), with its ratio being \( \frac{k-1}{k+2} )^n \) and growing linearly with \( n \). In addition, the scalings of both the first-order and the second-order coherence in \( H(n, k) \) are smaller than those corresponding to \( S(n, k) \).

Because both hierarchical graphs \( H(n, k) \) and Sierpiński graphs \( S(n, k) \) are self-similar, and have the same number of vertices and edges, but differ in some structural aspects, such as degree distribution, average distance, and fractality, we argue that the difference for consensus algorithms in these two graphs lies in, at least partially, their structural discrepancy. For example, hierarchical graphs exhibit the small-world effect, while Sierpiński graph are “large-world”. The small-world structure can drastically reduce communication time between different vertices and speed up information diffusion in a network. Then, it is not difficult to understand that the speed of convergence of noiseless consensus algorithms in hierarchical graphs is faster than in Sierpiński graphs, and the coherence for both first-order and second-order noise consensus problems is lower in hierarchical graphs than in Sierpiński graphs.

VI. CONCLUSION

The self-similarity property is ubiquitous in real-world and man-made systems. In this paper, we studied consensus problems in two iteratively growing self-similar networks, the hierarchical graphs and the Sierpiński graphs. Both of these two networks have the same order and size at any iteration, but exhibit quite different topological properties. We studied in detail several important quantities of consensus problems in these two networks, including convergence speed, delay robustness, and coherence for first-order and second-order dynamics. We showed that the consensus problem can be solved faster in hierarchical graphs than in Sierpiński graphs. In contrast, the hierarchical graphs can tolerate smaller communication delay than Sierpiński graphs.

For the first-order and second-order noisy consensus algorithms, the asymptotic behaviors of network coherence also scale differently in the two graphs. For the first-order noisy consensus algorithm, the network coherence grows logarithmically with the number \( N \) of vertices in hierarchical graphs but sublinearly with network order \( N \) in Sierpiński graphs. For the second-order noisy consensus algorithm, the coherence grows linearly with \( N \) in hierarchical graphs, but superlinearly with \( N \) in Sierpiński graphs.

We demonstrated that the structure difference of the two self-similar networks is responsible for the observed distinct performance of the studied consensus algorithms defined on them.

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