Abstract

We evaluate the matrix elements for the processes $\pi^0 K^0 \to \pi^0 K^0$ and $\pi^- K^+ \to \pi^0 K^0$ in the presence of isospin breaking terms at leading and next-to-leading order. As a direct application the relevant combination of the S-wave scattering lengths involved in the pion-kaon atom lifetime is determined. We discuss the sensitivity of the results with respect to the input parameters.

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1 Introduction

Chiral perturbation theory [1] has become one of the most used tools in exploring QCD low-energy dynamics. It applies in the non-perturbative regime of QCD. The initial QCD lagrangean, $L_0$, is then replaced by an effective one which contains the same symmetries as the fundamental $L_0$, composed by a string of higher and higher dimension operators involving derivatives and quark masses. Technically, the new lagrangean is not renormalizable (in the Wilsonian sense) but fortunately at a given order in the momenta (quark mass) expansion the number of needed counter-terms is finite. Thus assuming that the chiral series converges [2] one can truncate it at a given order and deal with a finite number of unknown constants. Restricting the analysis to the next-to-leading order in the mesonic sector (see below) one can obtain the unknown constants from the existing data and large-$N_c$ arguments. To this respect one makes use of the experimental knowledge on the pseudoscalar masses and decay constants, pion vector form-factor and $K_{l4}$ form-factors. Therefore none of those data informations can be used to claim any theoretical predictability. To exhibit the consistency of the theory one has to use other processes where the low-energy constants are given as mere inputs and confront the theoretical results with the experimental data. To this aim the $\pi - \pi$ scattering lengths have deserved a careful examination [3] but unfortunately they only bring information about the SU(2) sector. In line with the previous general argument, $\pi - K$ scattering stems for the simplest meson-meson scattering process that involves strangeness and can be used as an independent source of information on the validity of the extra assumptions that common wisdom assesses to hold in chiral perturbation theory, as for instance large-$N_c$ arguments. This will,
hopefully, bring some insight on the role of the strange quark mass inside the chiral expansion. Recently it has been noticed that some observables depend strongly on the number of light sea quarks \[4\]. This fact can cast doubts on the validity of the chiral expansion in the SU(3) sector where \(m_s\) is treated as a small parameter. Before any judgment is taken it is necessary to make more accurate experiments and precision calculations on testing processes. In that sense the next proposal for the measurement of the lifetime and lamb-shift in \(\pi^– K^+\) at CERN constitutes a major step from the experimental side \[6\]. Experiments on this system will constitute one of the most stringent test on chiral symmetry breaking existing up to the moment. Even though we want to stress that our treatment will not allow to deal with bound states, and a more refined analysis in the line of the one performed for the \(\pi– \pi\) atom is needed \[7, 8\]. In a very brief way the lifetime (\(\tau\)) of the \(A_{\pi K}\) atom, is given in terms of \[9\]

\[
\frac{1}{\tau_{n,0}} \propto (a_0^{1/2} - a_0^{3/2})^2.
\]

Thus any theoretical insight on the shift for the scattering lengths due to the isospin breaking terms constitutes a key role for a real estimate of \(A_{\pi K}\) atom lifetime.

In this paper our aim is to incorporate some of the theoretical effects that were not taken into account in a previous analysis \[5\]. We shall deal first with the more academic \(\pi^0 K^0 \rightarrow \pi^0 K^0\) process where there is no presence of an explicit virtual soft-photon, but electromagnetic effects will appear in the expressions of the scattering lengths as differences of charged and neutral pseudoscalar masses. We proceed the analysis considering the \(\pi^- K^+ \rightarrow \pi^0 K^0\) transition where in addition explicit exchange of virtual photons should be considered.

The paper is organized as follows: in sec. 2 we review briefly the inclusion of electromagnetic corrections inside the framework of effective lagrangeans, emphasizing the role of the low-energy constants. We continue with the isospin decomposition for the \(\pi – K\) scattering amplitudes, analyzing the scattering lengths at leading order in the isospin limit and comparing them with the isospin breaking corrections in sec. 3. Next, in sec. 4 we proceed with the analysis of the \(\pi^0 K^0 \rightarrow \pi^0 K^0\) process at next-to-leading order, reviewing first the kinematics and carefully explaining how to deal with isospin breaking effects, strong and electromagnetic ones, in order to have an expression consistent with the chiral power counting. In sec. 5 we turn to the evaluation of the experimental mode \(\pi^- K^+ \rightarrow \pi^0 K^0\) emphasizing the soft-photon contribution, the extraction of the Coulomb pole at threshold and the proper definition of observables once isospin breaking corrections are taken into account. In a more technical section, sec. 6 we explain how to perform the threshold expansion of the non-Coulomb part of the scattering amplitude. In sec. 7 we review the experimental and theoretical status of the \(\pi – K\) S-wave scattering lengths and we present our results discussing them in terms of all input parameters. We make use of our findings to determine the lifetime of \(A_{\pi K}\) in sec. 8. Sec. 9 summarizes our results. Finally, for not interrupting the discussion we have collected in the appendices all relevant expressions.

2 The effective lagrangean to lowest order

This section covers briefly the inclusion of electromagnetic corrections in a systematic way in the low-energy theory describing hadron interactions \[1\] \[2\]. Due to the smallness of the electromagnetic constant, \(\alpha_{em}\), these effects have been theoretically neglected so far in the \(\pi K\) scattering process, but is well known that near threshold isospin breaking effects can enhance considerably some observables.

In presence of electromagnetism it is convenient to split the lowest order effective lagrangean in three terms

\[
\mathcal{L}_2 = \mathcal{L}^\gamma + \mathcal{L}^{(2)}_{QCD} + \mathcal{L}^C.
\] (2.1)

The foregoing lagrangean possesses the same symmetry restrictions as the one in the strong sector. Additionally one has to impose an extra symmetry, charge conjugation, affecting only the photon fields and the spurions (see below). The first term in eq. (2.1) corresponds to the usual Maxwell lagrangean.
containing the classical photon field, \( A_\mu \), and the field strength tensor, \( F_{\mu\nu} \)

\[
\mathcal{L}_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A_\nu)^2, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.2}
\]

The second term formally describes the dynamics of the strong interaction sector \([4]\) and is given at lowest order by

\[
\mathcal{L}^{(2)}_{\text{QCD}} = \frac{F_2^2}{4} (d^\mu U d_\mu U + \chi^\mu U + \chi U^\dagger). \tag{2.3}
\]

As usual brackets, \( \langle \ldots \rangle \), stand for trace over flavour. The field \( u(x) \), parametrizes the dynamics of the low-energy modes in terms of elements of the Cartan subalgebra \([3]\). The covariant derivative is slightly modified with respect to the pure QCD interaction expression to accommodate the electromagnetic field

\[
d_\mu U = \partial_\mu U - i (v_\mu + Q_R A_\mu + a_\mu) U + iU (v_\mu + Q_L A_\mu - a_\mu), \tag{2.4}
\]

\( Q_R \) and \( Q_L \) are the aforementioned spurions fields, containing the sources for the electromagnetic operators \( A_\mu \bar{q}_L \left( \frac{\gamma^\mu}{2} \right) \gamma^\nu q_L \) and \( A_\mu \bar{q}_R \left( \frac{\lambda^a_\mu}{2} \right) \gamma^\nu q_R \). Furthermore, from now on we set them to their constant value

\[
Q_L(x) = Q_R(x) = Q, \quad Q = \frac{e}{3} \times \text{diag}(2, -1, -1). \tag{2.5}
\]

While, as usual, \( a_\mu \) and \( v_\mu \) stand for the axial and vector sources respectively. The scalar and pseudoscalar sources are contained inside the SU(3) matrix \( \chi \) as

\[
\chi = s + ip = 2B_0 \mathcal{M} + \ldots, \quad \mathcal{M} = \text{diag}(m_u, m_d, m_s). \tag{2.6}
\]

At lowest order the last term in eq. (2.1), \( \mathcal{L}^C \), determines the masses of the mesons in the chiral limit, which are of purely electromagnetic origin. This means that even at tree level the pole of the two-point Green function is shifted from its QCD value modifying therefore the kinematics of the low-energy region

\[
\mathcal{L}^C = C(QUQU^\dagger) = -\frac{2e^2 C}{F_0^2} (\pi^- \pi^+ + K^- K^+) + \ldots. \tag{2.7}
\]

Obviously the coupling is taken to be universal, thus we expect the same contribution to the masses for charged pions and kaons. Furthermore, for simplification, the results will be presented in terms of

\[
Z = \frac{C}{F_0^2} = \frac{M_{\pi^+}^2 - M_{\pi^0}^2}{2e^2F_0^2}. \tag{2.8}
\]

The inclusion of isospin breaking terms can be seen in a very naive way as coming through two different sources in eq. (2.1). First a pure strong isospin breaking, i.e. \( m_u \neq m_d \). And second an electromagnetic interaction, \( e \neq 0 \), eq. (2.7). In view of the seemingly different role of both contributions one has to relate them in a consistent way. For instance, the lagrangean eq. (2.1) involves an expansion in several parameters: \( p, m, \varepsilon \) and \( \delta \). Where \( p \) refers to the external momenta, \( m \) to the quark-masses, \( \varepsilon \) to the electric charge and finally \( \delta = m_d - m_u \). For being consistent, eq. (2.1) should contain operators with the same chiral order in the series expansion, thus a possible solution can be the choice \( m \sim \varepsilon^2 \sim O(p^2) \). At the next-to-leading order, \( O(p^4) \), none of the following terms are impeded to appear by chiral power counting: \( p^4, m^2, p^2 m, p^2, \varepsilon^2, m \varepsilon^2, p^2 \delta, m \delta, \varepsilon^4 \) and \( \delta^2 \), although there is quantitative support to the assumption, often used in phenomenological discussions, that the \( \delta^4 \) and \( \delta^2 \) contributions are tiny and can therefore be safely disregarded in front of the rest.

Hitherto we have listed all possible electromagnetic, lowest order operators. Once quantum fluctuations are considered using vertices from the functional (2.1) results are ultraviolet divergent. Those divergences depend of the regularisation method employed in the loops diagrams. As is customary we shall adopt the modified \( \overline{\text{MS}} \) subtraction scheme. In order to remove these ultraviolet divergences, higher
order operators, modulated by simple constants, should be incorporated into the theory with the guidance of the previously mentioned symmetry requirements. This allows to deal with a theory which is ultraviolet finite order by order in the parameter expansion and hopefully it adequately describes several observables.

These modulated constants are order parameters of the effective theory (low-energy constants) and in the case at hand are determined by the underlying low-energy dynamics of QCD and QED. For instance at lowest order the order parameters are given by $F_0$ (eq. (2.3)), $B_0$ (eq. (2.4)) and $C$ (eq. (2.7)). Describing the lowest pseudoscalar decay constant, the vacuum condensate parameter and the electromagnetic pion mass in the chiral limit respectively. For the strong SU(3) sector, at next chiral order, there are ten new low-energy constants $L_1, L_1, . . . , L_{10}$ and two high-energy constants. At this level of accuracy the ten low-energy constants may be extracted almost independently one from each other by matching some observables with the corresponding experimental determination. In the electromagnetic SU(3) sector there is also need of higher order operators, up to 16 modulated via $K_1, . . . , K_{16}$, to render any observable free of ultraviolet divergences. To gain some information on these low-energy constants one has to resort to models $[14]$, to sum-rules $[17]$ or to a simple crude order estimates. All in all, one can see that the inclusion of electromagnetic corrections to hadronic processes increases enormously the number of low-energy constants thus washing out any predictability.

### 3 Isospin breaking corrections at leading order

Under strong interactions symmetry pions are assigned to a triplet of states, “isotriplet states”, whereas kaons can be collected into doublets. This means that the pion and the kaon are isospin eigenstates with eigenvalues 1 and $1/2$ respectively. Therefore, the amplitudes for the $\pi - K$ scattering processes are solely described in terms of two independent isospin-eigenstates amplitudes $T^{1/2}$ and $T^{3/2}$

\[ \mathcal{M}(\pi^0 K^0 \to \pi^0 K^0) = \frac{1}{3} T^{1/2} + \frac{2}{3} T^{3/2}, \]
\[ \mathcal{M}(\pi^- K^+ \to \pi^0 K^0) = -\frac{\sqrt{2}}{3} T^{1/2} + \frac{\sqrt{2}}{3} T^{3/2}, \]
\[ \mathcal{M}(\pi^- K^+ \to \pi^- K^0) = \frac{2}{3} T^{1/2} + \frac{1}{3} T^{3/2}, \]
\[ \mathcal{M}(\pi^+ K^- \to \pi^+ K^0) = T^{3/2}. \]

It is obvious that under $s \leftrightarrow u$ crossing the last two matrix elements are related. In particular one finds

\[ T^{1/2}(s, t, u) = \frac{3}{2} T^{3/2}(u, t, s) - \frac{1}{2} T^{3/2}(s, t, u), \]

thus in the isospin limit it is sufficient to compute one of the processes. It is convenient to use, instead of the invariant amplitudes $T^l$, the partial wave amplitudes $t^l_f$ defined in the $s$-channel by

\[ T^l(s, \cos \theta) = 32\pi \sum_l (2l + 1) P_l(\cos \theta) t^l_f(s), \]

or by the inverse expression

\[ t^l_f(s) = \frac{1}{64\pi} \int_{-1}^{+1} d(\cos \theta) P_l(\cos \theta) T^l(s, \cos \theta), \]

where $l$ is the total angular momentum, $\theta$ the scattering angle in the center of mass frame and $P_l$ are the Legendre polynomials with $P_0(\cos \theta) = 1$. Near threshold the partial wave amplitudes can be parametrized in terms of the scattering lengths, $a^l_f$ and slope parameters, $b^l_f$. In the normalization (3.3) the real part of the partial wave amplitude reads

\[ \text{Re } t^l_f(s) = q^{2l} \{ a^l_f + b^l_f q^2 + \mathcal{O}(q^4) \}. \]
with \( q \) being the center of mass three-momentum.

Let us estimate the scattering lengths in the isospin limit at tree level. Using for instance the first two processes in (3.1) one can disentangle the values of each scattering length separately. In order to match the prescription of the scattering lengths given in [18] we define them in terms of the charged pion and kaon masses. They read

\[
a_{0}^{1/2} = \frac{M_{\pi} + M_{K}}{16\pi F_{0}^2} = 0.157 \ (0.129),
\]

\[
a_{0}^{3/2} = -\frac{M_{\pi} + M_{K}}{32\pi F_{0}^2} = -0.079 \ (-0.064),
\]

(3.6)

where the first quoted number refers to the choice \( F_{0}^{2} = F_{\pi}^{2} \) while the second is for \( F_{0}^{2} = F_{\pi} F_{K} \).

When switching-on the isospin-breaking effects, we are not allowed anymore to refer to the scattering lengths in a given isospin-state and new terms in addition to the previous ones arise. Furthermore in principle relations as (3.2) do not hold anymore. This failure can be seen already at tree level in \( \mathcal{O}(\epsilon^{2}) \) and \( \mathcal{O}(\epsilon) \) terms. The modified scattering lengths in presence of isospin breaking can be split as

\[
a_{0}(00; 00) = \frac{1}{3} a_{0}^{1/2} + \frac{2}{3} a_{0}^{3/2} + \Delta a_{0}(00; 00),
\]

\[
a_{0}(+-; 00) = -\frac{\sqrt{2}}{3} a_{0}^{1/2} + \frac{\sqrt{2}}{3} a_{0}^{3/2} + \Delta a_{0}(+-; 00),
\]

\[
a_{0}(+-; -+) = \frac{2}{3} a_{0}^{1/2} + \frac{1}{3} a_{0}^{3/2} + \Delta a_{0}(+-; -+),
\]

\[
a_{0}(++; ++) = a_{0}^{3/2} + \Delta a_{0}(++; ++),
\]

where \( a_{0}^{1/2} \) and \( a_{0}^{3/2} \) denote the strong (isospin limit) S-wave scattering lengths and \( \Delta a_{0}(i, j; l, m) \) represents the leading correction to the corresponding combination of scattering lengths due to the isospin breaking effects. The evaluation of these corrections is straightforward and can be read from the scattering amplitude at leading order

\[
\Delta a_{0}(00; 00) = \frac{1}{32\pi F_{0}^2} \left[ \frac{1}{4} (\Delta_{\pi} - \Delta_{K}) + \frac{1}{\sqrt{3}} (M_{\pi}^2 + M_{K}^2) \right] = 0.0032 \ (0.0026),
\]

\[
\Delta a_{0}(+-; 00) = \frac{1}{32\sqrt{2} F_{0}^2} \left[ \frac{1}{\sqrt{3}} (M_{\pi}^2 + M_{K}^2) - \frac{3}{4} \frac{M_{\pi} + M_{K}}{M_{\pi}^2 - M_{K}^2} \Delta_{K} \right] = -0.0016 \ (-0.0013),
\]

\[
\Delta a_{0}(+-; -+) = \frac{\Delta_{K}}{32\sqrt{2} F_{0}^2} \left[ \frac{3}{4} \frac{M_{\pi} + M_{K}}{M_{\pi}^2 - M_{K}^2} \right] = 0.0014 \ (0.0012),
\]

\[
\Delta a_{0}(++; ++) = \frac{\Delta_{K}}{32\sqrt{2} F_{0}^2} \left[ \frac{3}{4} \right] = 0.0014 \ (0.0012),
\]

with \( \Delta_{m} = M_{n} - M_{n}^0 \). Notice that whereas in the isospin limit the combination for the scattering lengths cancels in the \( \pi^{0} K^0 \rightarrow \pi^{0} K^0 \) process this is no longer true when isospin breaking terms are considered. In the \( \pi^{-} K^{+} \rightarrow \pi^{0} K^{0} \) process the isospin breaking in the combination of the scattering lengths is roughly a couple of orders of magnitude smaller than the leading isospin limit quantity. Even though the future experimental bounds on the combination of scattering lengths for this process is quite restrictive being worthwhile to control higher order corrections. For the rest of processes the isospin effects are also rather small, roughly two order of magnitude less than the isospin limit counter parts.

4 \( \pi^{0} K^{0} \rightarrow \pi^{0} K^{0} \) process

Let us start considering the following process

\[
\pi^{0}(p_{\pi_{1}})K^{0}(p_{K_{1}}) \rightarrow \pi^{0}(p_{\pi_{2}})K^{0}(p_{K_{2}}).
\]

(4.1)

\(^{1}\)We use \( F_{0} = F_{\pi} = 93.4 \) MeV. See Sec. [1] for the rest of values.

\(^{2}\)In the sequel we shall use \( \frac{F_{K}}{F_{\pi}} = 1.22 \).
Our aim in this section will be to compute its amplitude taking into account all possible isospin breaking effects. Even if the process has not the same experimental interest as the $\pi^-K^+\rightarrow\pi^0K^0$ reaction it is worth to be considered because both processes share almost the same features and complications, exception of the one photon exchange contribution (see sec. 5.1).

4.1 Kinematics

The amplitude for the process, eq. (4.1), can be studied on general grounds in terms of the Mandelstam variables

$$s = (p_{\pi_1} + p_{K_1})^2, \quad t = (p_{\pi_1} - p_{\pi_2})^2, \quad u = (p_{\pi_1} - p_{K_2})^2.$$  \hspace{1cm} (4.2)

In the isospin limit and at lowest order of perturbation theory (corresponding to the PCAC results), see diagram (a) in fig. 1, the off-shell amplitude is given by \[18, 19\]

$$\mathcal{M}(s, t, u) = \frac{1}{6F_0^2} \left\{ M_K^2 + M_\pi^2 - \frac{u + s}{2} + t \right\}.$$  \hspace{1cm} (4.3)

It is worth to review briefly the kinematics of the process that will be needed subsequently. In the center of mass frame the Mandelstam variables are defined in terms of $q$ and $\theta$ by

$$s = (\sqrt{M_{\pi_0}^2 + q^2} + \sqrt{M_{K_0}^2 + q^2})^2,$$

$$t = -2q^2(1 - \cos \theta),$$

$$u = (\sqrt{M_{\pi_0}^2 + q^2} - \sqrt{M_{K_0}^2 + q^2})^2 - 2q^2(1 + \cos \theta).$$

4.2 General framework

Hitherto we have considered the process eq. (4.1) at leading order. In this section we shall sketch the role of isospin breaking at next-to-leading order. Indeed, as has been mentioned in the introduction, the isospin violating terms that are retained in the $\pi - K$ scattering differ from the ones in the $\pi - \pi$ reaction due to the inclusion of the s-quark. The later process can be described fully in terms of SU(2) quantities where, for instance, the difference between the charged and neutral pion masses are order $\delta^2$ and thus can be disregarded, whereas in the former there exist intermediate strangeness states like $\pi - K$ that give contributions of order $e$ and $\delta$.

To be consistent with the chiral power counting one has to take into account all possible scenarios. For instance, given a generic $2 \rightarrow 2$ reaction mediated via the diagram (b) in fig. 1 there are two different possibilities of incorporating isospin violating terms: (i) consider that one of the vertices breaks isospin through the $e^2$ terms or via the quark-mass difference, thus at the order we shall work the other vertex and the two propagators are taken in the isospin limit, or (ii) if both vertices are taken in the isospin limit that forces to consider the splitting between charged and neutral masses (in a given channel) in the same triplet for the pions or in the doublet for the kaons in the propagators. Thus within this prescription we shall consider in the chiral series terms up to including $\delta$ and $e^2$ corrections besides the usual $p^4$ at next-to-leading order.

As a matter of fact, the previous distinction, disentangling strong and electromagnetic contributions to the isospin breaking terms, is quite artificial as one can realize when the pseudoscalar masses are rewritten...
in terms of bare quantities. Even though it constitutes a great conceptual help because ultraviolet divergences involving \( e^2 \) and \( \delta \) terms do not mix at this order allowing to keep track of each term independently.

For the case we are interested in, involving only neutral particles, there is no direct contribution from virtual photon loops, therefore the amplitude is safe of infrared singularities and all \( e^2 \) dependence is due to the e.m. mass difference of mesons or by the integration of hard photon loops. Hence to obtain the amplitude including all \( O(\alpha_{em}) \) corrections one needs to restrict the evaluation to the one-particle-irreducible diagrams depicted in fig. 1 corresponding to the Born amplitude, (a), unitary contribution, (b), tadpole, (c) and finally the counter-term piece (d). For the precise expressions of this last contribution we refer the reader to the original literature [14, 11].

To ascertain the correctness of our expression we look at the scale independence of the result once all contributions of the one-particle-irreducible diagrams are added, the wave function renormalization for the external field are taken into account and the \( \pi - \eta \) mixing is treated correctly (see below). Furthermore, when restricting the expressions to the isospin limit we recover the results given in [5].

Once the amplitude is finite in terms of bare quantities we have to renormalize the coupling constant, \( F_0 \), and the masses appearing at lowest order. For the latter contribution we obtain agreement with the results quoted in [14] for the terms up to including \( \epsilon \) corrections and with [11] for the electromagnetic ones. While for the former we shall use two choices: the first one is to fully renormalize \( F_{\pi}^2 \) as \( F_{\pi}^2 \), and for comparison purposes as the combination \( F_{\pi} F_{K} \). To this end we use the isospin limit quantities [14]

\[
F_\pi = F_0 \left\{ 1 + 4 \frac{M_\pi^2 + 2M_K^2}{F_0^2} L_4^r + 4 \frac{M_\pi^2}{F_0^2} L_5^r - 2\mu_\pi - \frac{1}{2} \mu_K \right\},
\]

(4.4)

and

\[
F_K = F_0 \left\{ 1 + \frac{M_\pi^2 + 2M_K^2}{F_0^2} L_4^r + 4 \frac{M_\pi^2}{F_0^2} L_5^r - 3 \frac{1}{4} \mu_\pi - \frac{3}{2} \mu_K - \frac{3}{4} \mu_\eta \right\}.
\]

(4.5)

Being \( \mu_P \) the finite part of the well-known tadpole integral. We refrain to use \( F_{\pi^0} \) in the numerical estimates of \( F_\pi \) because experimentally is quite poorly known and instead we shall make use of the charged decay constant value.

The latest contribution enters through the \( \pi - \eta \) mixing. At lowest order the mixing angle is given by

\[
\epsilon = \epsilon^{(2)} = \frac{\sqrt{3}}{4} \frac{m_d - m_s}{m_s - m}.
\]

(4.6)

Notice that given the order of accuracy we are considering, \( \epsilon^{(2)} \) does not suffice and the next-to-leading order contribution, \( \epsilon^{(4)} \), to the mixing angle needs to be considered. We shall use the same approach as in [22] where we refer for a more detailed explanation. It consists essentially in diagonalizing the mixing matrix at the lowest order redefining in that way the \( \pi^0 \) and \( \eta \) fields. While higher order terms in the mixing are treated by direct computation of the S-matrix off-diagonal elements. This procedure is equivalent to the one outlined in [23].

Taking into account all mentioned contributions we obtain the renormalized S-matrix element, \( \mathcal{M}^{00,00} \), for the transition \( \pi^0 K^0 \to \pi^0 K^0 \) that is gathered in app. A where we refer the reader for a detailed exposition.

5  \( \pi^- K^+ \to \pi^0 K^0 \) process

In this section we shall consider the more relevant process

\[
\pi^-(p_{\pi^-}) K^+(p_{K^+}) \to \pi^0(p_{\pi^0}) K^0(p_{K^0}),
\]

(5.1)
with the following Mandelstam variables

\[ s = (p_{\pi^+} + p_K)^2, \quad t = (p_{\pi^-} - p_{\pi^0})^2, \quad u = (p_{\pi^-} - p_{K^0})^2. \]  

(5.2)

In the center of mass frame these variables read

\[ s = (E_1 + E_2)^2, \]
\[ t = M_{\pi^\pm}^2 + M_{\pi^0}^2 - 2E_1 \sqrt{M_{\pi^0}^2 + q'^2} + 2qq' \cos \theta, \]
\[ u = M_{\pi^\pm}^2 + M_{K^0}^2 - 2E_1 \sqrt{M_{K^0}^2 + q'^2} - 2qq' \cos \theta, \]  

(5.3)

with

\[ E_1^2 = M_{\pi^\pm}^2 + q'^2, \]
\[ E_2^2 = M_{K^0}^2 + q'^2, \]
\[ q'^2 = \frac{1}{4} \left[ E_1 + E_2 + \frac{\Delta_{\pi^0 K^0}}{\Delta_{\pi^\pm K^\pm}} (E_1 - E_2) \right]^2 - M_{\pi^0}^2, \]  

(5.4)

and being \( q(q') \) the three-momentum of the charged (neutral) particles.

As has been pointed out earlier, the relevance of this process is intimately related to the lifetime of the \( \pi - K \) system. Even though we want to stress that our formalism only allows us to deal with free, on-shell external particles, in clear contrast with the \( \pi - K \) atom where the states are bounded and off-shell [7, 8]. We shall not pursue here the more complete approach.

For the construction of most of the graphs (those equivalent to fig. 1) we shall use the same arguments presented in the preceding section. For the remaining ones we shall sketch their treatment in the next section.

5.1 Soft photon contribution

In the case of the \( \pi^- K^+ \rightarrow \pi^0 K^0 \) process, besides the corrections due to the mass difference of the up and down quarks and those generated by the integration of hard photons one has to consider corrections due to virtual photons. At order \( e^2p^2 \) these corrections arise from the wave function renormalization of the charged particles just as from the one-photon exchange diagrams depicted in fig. 2. The result of diagram (a) in this figure reduces at threshold to combinations of polynomials and logarithms. Whereas the second, (b), needs a closer consideration. It develops at threshold (see below) a singular behaviour.

This singularity is issued from the ultraviolet finite three-point function \( C \) defined by

\[
C(M_P^2, M_Q^2, m_1^2; p_1, p_2) = \frac{1}{i} \int \frac{d^d l}{(2\pi)^d} \frac{1}{[(l^2 - m_1^2) [(p_1 - l)^2 - M_P^2] [(p_2 - l)^2 - M_Q^2]} ,
\]  

(5.5)

with the on-shell conditions \( p_1^2 = M_P^2, p_2^2 = M_Q^2 \) and \( m_1^2 \) acting as an infrared cut-off for the photon mass. Using standard techniques, the integral (5.5) can be expressed in terms of logarithms and dilogarithms. For \( p^2 \equiv (p_1 - p_2)^2 > (M_P + M_Q)^2 \), and \( m_1^2 \to 0 \) it is given as follows

\[
32\pi^2 \lambda_{PQ}(p^2) C(M_P^2, M_Q^2, m_1^2; p_1, p_2) = \]
of the real part of the amplitude takes the following form

\[ -\left\{ \log\left(\frac{\lambda_{PQ}(p^2)}{p^2m_\gamma^2}\right) - \frac{1}{2} \log\left(\frac{[\Delta_{PQ} - \lambda_{PQ}(p^2)]^2 - p^4}{[\Delta_{PQ} + \lambda_{PQ}(p^2)]^2 - p^4}\right) - i\pi \right\} \log\left(\frac{[p^2 - \lambda_{PQ}(p^2)]^2 - \Delta_{PQ}^2}{[p^2 + \lambda_{PQ}(p^2)]^2 - \Delta_{PQ}^2} + 2i\pi\right) \]

\[ +2\text{Li}_2\left[\frac{p^2 + \Delta_{PQ} + \lambda_{PQ}^2(p^2)}{p^2 + \Delta_{PQ} - \lambda_{PQ}^2(p^2)}\right] - 2\text{Li}_2\left[\frac{p^2 - \Delta_{PQ} - \lambda_{PQ}^2(p^2)}{p^2 - \Delta_{PQ} + \lambda_{PQ}^2(p^2)}\right], \tag{5.6} \]

where

\[ \Delta_{PQ} = M_Z^2 - M_Q^2, \quad \lambda_{PQ}(p^2) = [p^2 - (M_P + M_Q)^2][p^2 - (M_P - M_Q)^2], \]

and finally the dilogarithm function is defined as

\[ \text{Li}_2 (z) = -\int_0^z dt \frac{\log(1-t)}{t}. \]

The contribution of the \( C \) function via the diagram (b) in fig. 2 to the amplitude is [c.f. eq. (5.6)]

\[ \frac{e^2}{\sqrt{2}F_0^2} (s-u)(s-\Sigma_{\pi K})C(M_\pi^2, M_K^2, m_\pi^2; p_{\pi-}, -p_{K+}). \]

Expanding the real part of the preceding function in the vicinity of the threshold by the use of eqs. (5.7) and (5.4) one obtains a Coulomb type behaviour, i.e. \( q^{-1} \). Then schematically the threshold expansion of the real part of the amplitude takes the following form

\[ \text{Re} \, M^{+,00}_t(s,t,u) = -\frac{M_{\pi+}+M_{K-}}{M_{\pi+}+M_{K-}} \frac{e^2 \mu_{\pi K}}{4q} + \text{Re} \, M^{+,00}_{t,\text{thr.}} + \mathcal{O}(q), \tag{5.7} \]

with

\[ \mu_{\pi K} = \frac{M_{\pi+}+M_{K-}}{M_{\pi+}+M_{K-}}, \tag{5.8} \]

being the reduced mass of the \( \pi - K \) system.

A remarkable feature of the threshold expansion of the scattering amplitude is that neither \( \text{Re} \, M^{+,00}_{t,\text{thr.}} \) nor the long-range force of the photon exchange is affected by the infrared singularity, which only contributes to \( \mathcal{O}(q^2) \) or to higher orders terms. In fact, the scattering amplitude contains two infrared divergent contributions. The first one comes from the wave function renormalization of the charged particles and can be read from the Born-type amplitude (B.3). The second infrared divergent piece is due to diagram (b) in fig. 2 and is contained in the \( C \) function (5.6). Adding both contributions one has the following infrared piece for the real part of the amplitude

\[ \text{Re} \, M^{+,00}_\pi = \left(\frac{s-u}{\sqrt{2}F_0^2}\right) \left[\frac{e^2}{16\pi^2} \log(m_\pi^2)\right] \left(1 + \frac{1}{2} \frac{s-\Sigma_{\pi K}}{M_{\pi+}+M_{K-}} \log\left(\frac{s-\lambda_{\pi K}^+(s)}{s+\lambda_{\pi K}^+(s)}\right) - \Delta_{\pi K}^2\right). \]

As is expected when evaluated at threshold, this expression vanishes rendering \( \text{Re} \, M^{+,00}_{t,\text{thr.}} \) as an infrared finite quantity. Although the scattering lengths defined in this way will be infrared finite the slope parameters will not. In order to define an infrared finite observable one should notice that the cancellation of infrared divergences takes place order by order in \( \alpha_{em} \). This requires that besides the virtual photon corrections, to take into account real soft photon emission from the external particles. Notice that the experimental result will include this bremsstrahlung effect. In our case is just sufficient to consider one single photon emission, which amplitude reads

\[ M^{-+,00}_\gamma = \frac{e}{2\sqrt{2}F_0^2} \epsilon_\mu(k_\gamma) [-2(p_{\pi\gamma} - p_{K+})]^\mu \]
we shall use the following substitutions for the masses at threshold and neutral masses to the corresponding charged ones. The procedure is rather cumbersome and because in order to match the prescription given in [18] for the scattering lengths we need to shift isospin scattering amplitude around the threshold values. Even though, this step is not quite straightforward let us explain how we obtain the scattering lengths from the scale invariant amplitudes eqs. (A.1) and (6.1).

\[
\sigma(s; \Delta E) = \sigma^{-+;00}(s) + \sigma^{+-+-00}(s; \Delta E),
\]

where \(\Delta E\) corresponds to the detector resolution. Once this is done, the corrected scattering length \(\tilde{a}_0(+-+-00)\) might be defined from the threshold expansion of the infrared finite cross-section, eq. (5.9), by subtracting the Coulomb pole term and excluding the corrections due to the mass squared differences in the phase-space in the following way

\[
\sigma(s; \Delta E) = \frac{1}{32\pi s} \left\{ \frac{\lambda_{\pi^0 K^0}}{\lambda_{\pi^+ K^+}}(s) \right\} \cdot \left( - \frac{M_{\pi^+} M_{K^+}}{\sqrt{2} F_0^2} \frac{e^2}{q} + \frac{32\pi a_0(+-+-00) + O(q)}{2} \right) .
\]

We have checked that the corrections due to the real, soft photon emission are negligible and thus we expect that the corrected scattering length \(\tilde{a}_0(+-+-00)\) will only differ beyond our accuracy from the one obtained using eq. (5.7), that is, from the infrared finite real part of the scattering amplitude at threshold.

Finally, in order to present our results we shall collect in a single, infrared finite expression (denoted by \(M_{\text{Soft photon}}\) in the tables) the contributions at threshold of eqs. (B.6, B.7, B.20).

6 Threshold expansion

Let us explain how we obtain the scattering lengths from the scale invariant amplitudes eqs. (A.1) and (B.1). Since we are only interested in the S-wave threshold parameters it is just sufficient to expand the scattering amplitude around the threshold values. Even though, this step is not quite straightforward because in order to match the prescription given in [18] for the scattering lengths we need to shift isospin limit and neutral masses to the corresponding charged ones. The procedure is rather cumbersome and we shall use the following substitutions for the masses

\[
M_{\pi^0}^2 \to M_{\pi^\pm}^2 - \Delta_\pi , \quad M_{K^0}^2 \to M_{K^\pm}^2 - \Delta_K ,
\]

where \(\Delta_i\) are small quantities that in the case of pions contain at leading order \(\epsilon^2\) pieces while for kaons contain both \(\epsilon^2\) and \(\epsilon\) terms. It is therefore sufficient to expand all quantities up to first order in \(\Delta_i\). For instance in the charged \(\to\) neutral transition we obtain for the kinematical variables

\[
\begin{align*}
    s &= (M_{\pi^\pm} + M_{K^\pm})^2 , \\
    t &= - \frac{M_{K^\pm}}{M_{\pi^\pm} + M_{K^\pm}} \Delta_\pi - \frac{M_{\pi^\pm} M_{K^\pm}}{M_{\pi^\pm} + M_{K^\pm}} \Delta_K , \\
    u &= (M_{\pi^\pm} - M_{K^\pm})^2 - \frac{M_{\pi^\pm} M_{K^\pm}}{M_{\pi^\pm} + M_{K^\pm}} \Delta_\pi - \frac{M_{K^\pm}^2}{M_{\pi^\pm} + M_{K^\pm}} \Delta_K .
\end{align*}
\]

Once this step is performed and in order to book-keep properly the power counting any charged mass multiplying \(\Delta_i\) or \(\epsilon\) pieces is settled to its isospin limit

\[
\frac{M_{\pi^\pm}}{M_{\pi^\pm} + M_{K^\pm}} \approx \frac{M_{\pi}}{M_{\pi} + M_{K}} \left( 1 + O(\epsilon) + O(\epsilon^2) \right) .
\]

Only for estimating higher order corrections we shall eventually keep charged masses in the ratios introduced in eq. (6.2).
Even if the outlined procedure for shifting the neutral masses is rather involved it allows to expand all one loop integrals in an analytic form with quite compact expressions. For instance in the s-channel for the neutral → neutral transition we obtain after performing the mentioned steps the following expansion

$$\text{Re} J(M_{\pi^0}, M_{K^0}; (M_{\pi^0} + M_{K^0})^2) = \frac{1}{16\pi^2} \left[ 1 + \frac{M_{\pi^0} M_{K^0}}{M_{K^0}^2 - M_{\pi^0}^2} \log \left( \frac{M_{K^0}^2}{M_{\pi^0}^2} \right) \right]$$

$$- \frac{1}{32\pi^2} \left( \frac{1}{M_{K^0}^2 - M_{\pi^0}^2} \right) \left( \frac{M_K}{M_{\pi^0}} \Delta x - \frac{M_{\pi^0}}{M_K} \Delta K \right) \left[ -2(M_{K^0}^2 - M_{\pi^0}^2) + (M_{\pi^0}^2 + M_K^2) \log \left( \frac{M_K^2}{M_{\pi^0}^2} \right) \right].$$

For loop functions involving the η mass the use of Gell-Mann–Okubo relation reduces its expression considerably.

In the neutral → neutral process, although the kinematics allows $t \propto q^2 \approx 0$ at threshold, there is no need to consider the expansion of the $J$ function in powers of $q$. This is the case because all channels contributions behave like polynomial $\times J$ without any inverse power of kinematical variables ($t$ in this case). This does not turn out to be the case in the charged → neutral transition. There, one deals with terms like $J/t$, where in the isospin limit $t \propto q^2 \rightarrow 0$. Expanding near threshold $J[m_1^2, m_2^2; t(q^2)] \approx bq^2 + \ldots$ we shall obtain contributions from the terms linear in $q^2$. Besides this last remark there is no more differences in the treatment of the two processes.

Due to the relevance of the process $\pi^- K^+ \rightarrow \pi^0 K^0$ we collect, besides the expression of the scattering amplitude, app. B.3 the expression of the S-wave scattering lengths in app. B.3.

## 7 Results and discussion

Let us first point out that due to the high threshold of production, $\sqrt{s_{th}} \sim 632$ MeV, it is not necessary true that a single one-loop calculation is enough to approach the physical values for the scattering lengths. Even though and due to the fuzzy existing $\pi^- K^+$ data we consider that this can only be answered once the size of the next-to-next- to-leading order is computed. Also there are opening of intermediate particle productions, for instance $K K$ in the t-channel, that presumably affects strongly the chiral series convergence.

### 7.1 Input parameters

Before presenting our results we want to stress the relevance of the $\pi^- K$ scattering process. Its importance goes beyond the determination of some threshold quantities, but is a touchstone in the knowledge of spontaneous symmetry breaking. Up to nowadays chiral perturbation theory has been used to parametrize the low-energy QCD phenomenology. Lacking of enough processes to determine all low-energy constants one has to resort in theoretical inputs (or prejudices). For instance the determination of the low-energy constants in the electromagnetic sector have in general a quite mild impact on the results and therefore have been relegated to a secondary place and only recently received some attention [6, 7] due to the increasing precision in the experiment. But very little is known about them exception of model estimates. In our treatment these constants can play an important role, and we include them as given in [6], where they are estimated by means of resonance saturation. (Hereafter all our results are given at the scale $\mu = M_p$).

\[
\begin{align*}
K_1^r &= -6.4 \cdot 10^{-3}, & K_2^r &= -3.1 \cdot 10^{-3}, & K_3^r &= 6.4 \cdot 10^{-3}, & K_4^r &= -6.2 \cdot 10^{-3}, \\
K_5^r &= 19.9 \cdot 10^{-3}, & K_6^r &= 8.6 \cdot 10^{-3}, & K_7^r \ldots K_{19}^r &= 0, & K_{11}^r &= 0.6 \cdot 10^{-3}, \\
K_{12}^r &= -9.2 \cdot 10^{-3}, & K_{13}^r &= 14.2 \cdot 10^{-3}, & K_{14}^r &= 2.4 \cdot 10^{-3}.
\end{align*}
\]

If instead we use a naïve dimensional analysis the value assigned to each of them would have been restricted to be inside the range

$$|K_1^r| \lesssim \frac{1}{16\pi^2}.$$
which is taken as a crude indication on the error. Notice that the central values quoted in [20, 17] lie inside this error band.

Contrary to the previous case the low-energy constants in the strong sector are better known. In a series of works [14, 27] most of the next-to-leading low-energy constants were pinned down. In addition to the experimental data a large-N_c arguments were used to settle the marginal relevance of some operators (those entering together with \(L_4\) and \(L_6\)). The use of \(\pi - K\) data in the \(T^{3/2}\) channel can disentangle (in principle) the value of \(L_4^\eta\) due to its product with \(M_{K^0}^2\) which enhances its sensitivity to the role of \(m_s\) [28].

In order to have a more complete control over our results we use two different set of constants [22]. The first one was obtained by fitting simultaneously the next-to-leading expressions of the meson masses, decay constants and the threshold values of the \(K_{\ell 4}\) form-factors to their experimental values. We shall refer to it as set I and is given by:

\[
\begin{align*}
10^3 \cdot L_1^\eta &= 0.46 \pm 0.23, & 10^3 \cdot L_2^\eta &= 1.49 \pm 0.23, & 10^3 \cdot L_3^\eta &= -3.18 \pm 0.85, \\
10^3 \cdot L_1^K &= 0 \pm 0.5^*, & 10^3 \cdot L_2^K &= 1.46 \pm 0.2, & 10^3 \cdot L_3^K &= 0 \pm 0.3^*, \\
10^3 \cdot L_1^\pi &= -0.49 \pm 0.15, & 10^3 \cdot L_2^\pi &= 1.00 \pm 0.20, 
\end{align*}
\]

The second set (set II) is obtained with the same inputs and under the same assumptions as the previous one, but this time the fitted expressions are next-to-next-to-leading quantities

\[
\begin{align*}
10^3 \cdot L_1^\eta &= 0.53 \pm 0.25, & 10^3 \cdot L_2^\eta &= 0.71 \pm 0.27, & 10^3 \cdot L_3^\eta &= -2.72 \pm 1.12, \\
10^3 \cdot L_1^K &= 0 \pm 0.5^*, & 10^3 \cdot L_2^K &= 0.91 \pm 0.15, & 10^3 \cdot L_3^K &= 0 \pm 0.3^*, \\
10^3 \cdot L_1^\pi &= -0.32 \pm 0.15, & 10^3 \cdot L_2^\pi &= 0.62 \pm 0.20, 
\end{align*}
\]

As one can see comparing both sets the central value of some of the low-energy constants are sizeable shifted from one to another. We stress at this point that the error in the determination of the scattering lengths is mainly associated with the errors on low-energy constants. The other quantities involved in the calculation are hadron masses and for those there are rather accurate determinations. For the latter we use [29]

\[
\begin{align*}
M_{\pi^\pm} &= 139.570 \text{ MeV}, & M_{\pi^0} &= 134.976 \text{ MeV}, \\
M_{K^\pm} &= 493.677 \text{ MeV}, & M_{K^0} &= 497.672 \text{ MeV}.
\end{align*}
\]

Notice that in principle there is no need of considering \(M_{\pi^0}^2\) as an additional input. It always appears through loop propagators and therefore is sufficient to consider it via the Gell-Mann–Okubo relation in the presence of isospin breaking

\[
\Delta_{GMO} \equiv M_{\eta^0}^2 + M_{\pi^0}^2 - \frac{2}{3} \left( M_{K^\pm}^2 + M_{K^0}^2 \right) - \frac{2}{3} M_{\pi^\pm}^2 = 0.
\]

Even though we shall also use the value \(M_{\eta^0} = 547.30 \text{ MeV}\) and consider the difference as an indication of higher order corrections.

Furthermore we make use of the isospin limit quantities \(M_{\pi}\) and \(M_{K}\) that can be defined through combinations of the physical masses

\[
M_{\pi^0}^2 = M_{\pi^\pm}^2, \quad M_{K^0}^2 = \frac{1}{2} \left( M_{K^\pm}^2 + M_{K^0}^2 + \gamma \left[ M_{\pi^0}^2 - M_{\pi^\pm}^2 \right] \right).
\]

The factor \(\gamma\) will take into account any deviation from Dashen’s theorem [30]. At lowest order (\(\gamma = 1\)) the e.m. relation between the pseudoscalar masses reads

\[
(M_{K^\pm}^2 - M_{K^0}^2)_{\text{e.m.}} = M_{\pi^\pm}^2 - M_{\pi^0}^2.
\]

\(^\dagger\) Notice that in the decay \(K \to \pi\pi\ell\nu\) the \(s\) quark is involved. Thus although if in principle it can be used to obtain the value of \(L_4^\eta\) the form-factors \(F\) and \(G\) turn out to be rather insensitive to its actual value [24].

\(^\ddagger\) Quantities with a star are theoretical inputs.
Next-to-leading contributions to the previous relation can be quite sizeable. As an indication we shall use

\[
(M_{K^+}^2 - M_{K^0}^2) \bigg|_{\text{e.m.}} = (1.84 \pm 0.25) (M_{\pi^+}^2 - M_{\pi^0}^2).
\]

(7.2)

The only remaining input is \( F_\pi \) and we shall discuss its value below.

### 7.2 Scattering lengths

The studies on the \( \pi - K \) scattering started quite earlier \[18, 19\] within a current algebra approach and followed latter by a series of works based on dispersion relations by means of unitarity and crossing-symmetry \[31\] (see also \[26\] for recent works using the same technique). With the advent of chiral perturbation theory the process was analyzed once more \[5\]. Nowadays its interest has been revival and new approaches like the inverse method amplitude \[34\]–\[36\] or like the treatment of kaons as heavy particles \[37\] have been considered. *Grosso modo* all mentioned techniques lead to a fairly unique prediction for the scattering lengths, inside the range \([0.16, 0.24]\) for \( a_0^{1/2} \) and \([-0.05, -0.07]\) for \( a_0^{3/2} \). The corrections to the current algebra values \[c.f. \text{eq. (3.4)}\] are roughly 20\% for \( a_0^{1/2} \) and 30\% for \( a_0^{3/2} \), thus being in the ball park of the usual shifts between the next-to-leading and leading order quantities in processes where chiral perturbation applies.

Even if it seems that from the theoretical point of view there is some general consensus the experimental results on the scattering lengths are more spread. In the earlier experiments most of the collected data on \( \pi - K \) were obtained via the scattering of kaons on proton or neutron targets. After the data were analyzed by determining the contribution of the one pion exchange. This technique does not allow to obtain the initial pion on-mass-shell and hence some extrapolation is needed. Even if one can perform this extrapolation the approach is model dependent. The obtained central values \[\text{eq. (3.4)}\] are roughly 20\% for \( a_0^{1/2} \) and 30\% for \( a_0^{3/2} \), thus being in the ball park of the usual shifts between the next-to-leading and leading order quantities in processes where chiral perturbation applies.

This will be used as a reference point for us.

Let us turn now to discuss our findings. We shall begin commenting briefly on the neutral \( \pi \rightarrow \pi \) case. Just for illustrative purposes we use the value \( F_\pi = 93.4 \text{ MeV} \) which contains partially some effects due to electromagnetic corrections. As we shall see none of our conclusions is affected by this choice. In table 1 we have collected the disentangled contributions to each of the terms given in app. \[\text{A}\] with \( F_\pi^2 \) as renormalization choice for the coupling constant. We show the isospin limit contribution (first column) and the different corrections to it. Notice that by the definitions \[\text{eq. (6.1)}\] of \( \Delta_i \) in terms of physical masses, \( \Delta_\pi \) is just an \( e^2 \) piece while \( \Delta_K \) contains \( Z e^2 \) and \( \epsilon \) terms. This is the reason why in most of the cases the \( \Delta_K \) contribution is enhanced with respect to the \( \Delta_\pi \) one. The first thing to remark is that isospin breaking corrections are roughly one order of magnitude smaller than the isospin limit quantities. It is worth to remind that the isospin limit correction comes purely from the next-to-leading terms. We have kept the physical eta mass inside the loop diagrams. In parentheses are quoted the contributions corresponding to set I instead of set II. The final result is given by

\[
3 a_0(00; 00) = a_0^{1/2} + 2 a_0^{3/2} + 3 \Delta a_0(00; 00) = 0.0679 + 0.0094 \pm 0.0511 (0.0885 + 0.0077 \pm 0.0462),
\]

(7.4)

renormalizing \( F_\pi^2 \) as \( F_\pi^2 \). The first quoted number corresponds to the isospin limit and the second to the isospin breaking corrections. Interesting enough are the assigned error that wash out any sensitivity with respect to the choice of the set or by any of the choices in the renormalization of the coupling constant.
to renormalize the coupling constant as $F$ with $14$

Quantities between brackets correspond to set I in quadrature. The dominant contributions come by far from the low-energy constants (see below). They come through the uncertainty in the low-energy constants and we have propagate them experimentally. The proposal claims an accuracy of $20\% - 30\%$ on the measurement of the lifetime, this roughly translate into $10\% - 15\%$ of accuracy for the determination scattering lengths. As before we have disentangled all contributions and explored all possible scenarios allowed by the input parameters. In our estimates we shall use the more appropriate value $F_\pi = 92.4$ MeV which does not contain electromag-

Table 1: Different chiral contributions to the combination of scattering lengths proportional to $a_0(00; 00)$. We have chosen to renormalize the coupling constant as $F_\pi^2$ with the value $F_\pi = 93.4$ MeV. In parentheses we show the values obtained with set I.

| $a_0^{1/2} + 2a_0^{3/2}$ | $\epsilon$ | $\Delta_\pi$ | $\Delta_K$ | $\epsilon^2$ |
|-------------------------|-------------|-------------|-------------|-------------|
| Tree                    | 0.0052      | 0.0010      | 0.0034      | -           |
| Born(a)                 | -0.0034 (0.0138) | -0.0020 (0.0025) | -0.0006 (0.0007) | -0.0019 (0.0011) |
| Born(b)                 | 0.0081 (0.0113) | 0.0019 (0.0012) | 0 (0.0012) | -0.0001 (0.0002) |
| Born(c)                 | -          | -           | -           | 0           |
| Mixing                  | -          | 0.0012      | -           | 0           |
| $F_\pi^2$               | -          | 0.0013 (0.0014) | 0.0002 (0.0002) | 0.0009 (0.0009) |
| Tad-pole                | -0.0447    | 0.0020      | 0           | -0.0007     |
| s-channel               | 0.0506     | -0.0008     | -0.0015     | 0.0036      |
| t-channel               | -0.0001    | -0.0008     | 0           | 0           |
| u-channel               | 0.0575     | -0.0013     | -0.0018     | 0           |
|                         | 0.0679 (0.0885) | 0.0067 (0.0056) | -0.0025 (0.0038) | 0.0052 (0.0059) | 0 |

Table 2: Different chiral contributions to the combination of scattering lengths proportional to $\text{Re} \mathcal{M}^{++}(00)$. We have choose to renormalize $F_\pi^2$ as $F_\pi^2$ with the value $F_\pi = 92.4$ MeV. We show the first four digits without any rounding. Quantities between brackets correspond to set I.

| $a_0^{1/2} - 3a_0^{3/2}$ | $\epsilon$ | $\Delta_\pi$ | $\Delta_K$ | $\epsilon^2$ |
|-------------------------|-------------|-------------|-------------|-------------|
| Tree                    | 0.2408      | 0.0026      | 0.0018      | -0.0009     |
| Born(a)                 | -0.1060 (0.1389) | -0.0010 (0.0014) | -0.0007 (0.0008) | -0.0001 (0.0003) |
| Born(b)                 | 0.0540 (0.0867) | 0.0006 (0.0012) | -0.0006 (0.0008) | 0 (0.0005) |
| Mixing                  | -          | 0.0010      | -           | 0           |
| $F_\pi^2$               | 0.0664 (0.0688) | 0.0007 (0.0007) | -0.0008 (0.0009) | -0.0003 (0.0003) |
| Tad-pole                | 0.0415     | 0.0011      | 0.0002      | -0.0005     |
| s-channel               | 0.0283     | 0.0013      | 0.0005      | 0           |
| t-channel               | -0.0312    | -0.0004     | 0.0001      | -0.0004     |
| u-channel               | -0.0265    | -0.0009     | 0           | 0.0001      |
| Soft photon             | -          | -           | -           | -0.0004     |
|                         | 0.2674 (0.2695) | 0.0047 (0.0053) | 0.0005 (0) | -0.0011 (0.0020) | -0.0004 |

(see below). They come through the uncertainty in the low-energy constants and we have propagate them in quadrature. The dominant contributions come by far from the low-energy constants $L_5$ and $L_4$ of the pure strong sector whereas the electromagnetic ones give an imperceptible contribution. Unfortunately this is not an experimental mode because this strong sensitivity would constitute a cross-check in the consistency of the values for $L_5$ and $L_4$. For sake of completeness we also show the final result if instead we renormalize $F_\pi^2$ as $F_\pi F_K$ and using set II

$$3a_0(00; 00) = a_0^{1/2} + 2a_0^{3/2} + 3 \Delta_0(00; 00) = 0.0456 + 0.0087 \pm 0.0343 .$$ (7.5)

While the isospin breaking effects are almost unaffected with respect to eq. (7.4) we want to remark the shift in the central value.

Let us turn now to discuss the most interesting mode, the charged $\to$ neutral transition. The experimental proposal claims an accuracy of $20\% - 30\%$ on the measurement of the lifetime, this roughly translate in $10\% - 15\%$ of accuracy for the determination scattering lengths. As before we have disentangled all contributions and explored all possible scenarios allowed by the input parameters. In our estimates we shall use the more appropriate value $F_\pi = 92.4$ MeV which does not contain electromag-
more the role of the low-energy constant breaking effects. It also seems that with this choice of renormalization of coupling constants we weight

\[ F_{\pi}F_K \]

and \[ \Delta_K \]

origin. Furthermore they are dominated clearly by isospin breaking terms. A closer look to the errors reveals that they have mainly an electromagnetic

\[ F_{\pi}F_K \]

netic corrections. Even if the most natural choice is \( F_{\pi}F_K \) we shall proceed as in the previous case and also show the results using \( F_{\pi}^2 \) as an indication of the sensitivity to this parameter.

Like this case is by far of greater interest we have refined slightly some considerations that we shall bear in mind in the remainder: (i) we have used the Gell-Mann–Okubo relation for the eta mass inside the loop functions. (ii) Only kept the first term in the r.h.s. of eq. (6.3). And (iii) we assume Dashen's theorem to hold at next-to-leading order. (We shall discuss below the uncertainties associated with these considerations.) Similarly to the previous table the numbers quoted in parentheses refer to set I.

We have collected the results for \( F_{\pi}^2 \) in table 3. As one can see the difference between the two sets is at most 2% in the final result. Also the isospin breaking effects are roughly two order of magnitude smaller than the isospin limit quantity. In this case it seems that the experimental setup sensitivity is not enough to detect the new effect we have incorporated. Adding all partial contributions we obtain

\[
-\frac{3}{\sqrt{2}}a_0(+-;00) = a_0^{1/2} - a_0^{3/2} - \frac{3}{\sqrt{2}}\Delta_0(+-;00) = 0.2674 + 0.0037 \pm 0.0022 (0.2695 + 0.0033 \pm 0.0023)
\]

(7.6)

stressing that the contents of the previous expression is only indicative because the way we have chosen to renormalize the coupling constant. Notice that the assigned errors are at the same footing than the isospin breaking terms. A closer look to the errors reveals that they have mainly an electromagnetic origin. Furthermore they are dominated clearly by \( K_{10}^r \) and \( K_{11}^r \) low-energy constants. In the strong sector the dominant errors come (by order of dominant contribution) from \( L_5^r \) and \( L_6^r \). Those low-energy constants are strongly correlated.

Our main results are collected in table 3. It corresponds to the choice of renormalization \( F_{\pi}F_K \) for the decay constant. Once more isospin breaking effects turn out to be two order of magnitude smaller than the isospin limit quantities. Furthermore, as in table 3 but in a more accentuated way, table 3 shows a strong cancellation between the isospin breaking contributions (see for instance the contribution of \( \Delta_\pi \) and \( \Delta_K \)). Adding all contributions from the table one obtains

\[
-\frac{3}{\sqrt{2}}a_0(+-;00) = 0.2412 + 0.0037 \pm 0.0022 (0.2520 + 0.0038 \pm 0.0043) .
\]

(7.7)

Notice that once more the theoretical errors are small but still competitive in size with the isospin breaking effects. It also seems that with this choice of renormalization of coupling constants we weight more the role of the low-energy constant \( L_6^r \), (this is reflected in the sizeable uncertainty if we use set

| \( a_0^{1/2} - a_0^{3/2} \) | \( \epsilon \) | \( \Delta_\pi \) | \( \Delta_K \) | \( e^2 \) |
|---|---|---|---|---|
| Tree | 0.1974 | 0.0021 | 0.0015 | -0.0008 | -
| Born(a) | -0.0712 (-0.0933) | -0.0007 (-0.0009) | -0.0004 (-0.0005) | 0 (0.0002) | -
| Born(b) | 0.0363 (0.0582) | 0.0004 (0.0008) | -0.0004 (-0.0005) | 0 (0) | -
| Mixing | - | 0.0007 | - | 0 | -
| \( F_{\pi}F_K \) | 0.0705 (0.0815) | 0.0007 (0.0008) | 0 (0) | -0.0001 (-0.0001) | -
| Tad-pole | 0.0279 | 0.0007 | 0.0001 | -0.0003 | -
| s-channel | 0.0192 | 0.0009 | 0.0003 | 0 | -
| t-channel | -0.0209 | -0.0003 | 0.0001 | 0.0002 | -
| u-channel | -0.0179 | -0.0006 | 0 | 0.0001 | -
| Soft photon | - | - | - | - | -0.0003

Table 3: Different chiral contributions to the combination of scattering lengths proportional to \( ReM^{+1/2} \). The entries in the table differ from table 2 just in the renormalization of the coupling constant. Here we have used the combination \( F_{\pi}F_K \) with the constrain \( F_{\pi}^2 = 1.22 \) and the value \( F_{\pi} = 92.4 \) MeV. We show the first four digits without any rounding. Quantities between brackets correspond to set I.
I). It is worth to stress at this point that the expression of the amplitude for the charged \( \to \) neutral transition does not depend on the low-energy constant \( L'_4 \) whereas \( L'_6 \) only comes inside isospin breaking terms. Due to the numerical irrelevance of those last terms and even lacking a complete knowledge on the role of large-\( N_c \) suppressed operators, our estimates (in that sense) are precise and unambiguous. The difference between the central values of the two sets are at most 5\%. Hence, without taking into account the error bars, and in the most optimistic case, 10\% of accuracy in the experimental result, any sensitivity to the set of low-energy constants is just borderline. Before concluding let us comment about the role of using the physical eta mass, the full expression in the l.h.s in eq. (6.3) and Dashen’s theorem, eq. (7.2). If instead of using the Gell-Mann–Okubo relation (7.1) one uses the physical eta mass, the sensitivity to the set of low-energy constants is just borderline. Before concluding let us comment about the role of using the physical eta mass, the full expression in the l.h.s in eq. (6.3) and Dashen’s theorem, eq. (7.2). If instead of using the Gell-Mann–Okubo relation (7.1) one uses the physical eta mass, the central value in (7.7) increases by \( \sim \) eq. (7.2). If instead of using the Gell-Mann–Okubo relation (7.1) one uses the physical eta mass, the central value in (7.7) increases by \( \sim 0.0003 \). Higher order terms as introduced by eq. (6.3) also increases (7.7) roughly by an amount \( \sim 0.003 \). And finally the shift allowed by the violation of Dashen’s theorem goes beyond the accuracy we quote. A consistent way of incorporating these effects in our estimates is to consider them as a crude guess of higher order corrections and we shall treat them as theoretical uncertainties, therefore we add all three in quadrature and to the previous error bar in eq. (7.7). This leads to the final estimate for the combination of scattering lengths

\[
-\frac{3}{\sqrt{2}}a_0(+-;00) = a_0^{1/2} - a_0^{3/2} - \frac{3}{\sqrt{2}}\Delta_0(+-;00) = \left\{ \begin{array}{ll}
\text{(SetI)} & 0.2520 \pm 0.0038 \pm 0.0073 \\
\text{(SetII)} & 0.2412 \pm 0.0037 \pm 0.0045.
\end{array} \right.
\]

8 Perspectives

Using the previous estimates for the combination of scattering lengths, eq. (7.8), we can calculate partially the lifetime of the \( A_{\pi K} \) atoms. Inside a relativistic framework the probability of the transition of an \( A_{\pi K} \) atom

\[ A_{\pi K} \to \pi^0 + K^0 \]

can be casted as

\[ W_{n,0}(\pi^0 K^0) \propto \left( \text{Re} \ M_{\text{thr}}^{\pm,0} \right)^2. \]

Focusing on the isospin limit the previous relation is given by

\[ W_{n,0}(\pi^0 K^0) \approx \frac{8\pi}{9} \left( \frac{2\Delta m}{\mu_{\pi K}} \right)^{1/2} \left( \frac{a_0^{1/2} - a_0^{3/2})^2}{1 + \frac{2}{9}\mu_{\pi K}\Delta m(a_0^{1/2} + 2a_0^{3/2})^2} \right)^{1/2} \cdot \cdots \approx \frac{1}{\tau_{n,0}}. \]

where \( n \) is the principal quantum number,

\[ \Delta m = (M_{K^\pm} + M_{\pi^\pm}) - (M_{\pi^0} + M_{K^0}) \]

and \( \mu_{\pi K} \) is given in eq. (8.1). Finally \( \Psi_{n,0}(0) \) is the \( A_{\pi K} \) Coulomb wave function at the origin and it is given by

\[ |\Psi_{n,0}(0)|^2 = \frac{p_B^3}{\pi n^4}, \quad p_B = \frac{e^2}{4\pi \mu_{\pi K}}. \]

Furthermore the orbital angular momentum, \( l \), can be safely taken equal to zero. If one allows \( l > 0 \) the transition is suppressed by chiral power counting, i.e. turns to be of \( \mathcal{O}(e^4) \). Notice that in the previous relation \( \sqrt{\frac{2}{9}} \) enter precisely both combinations of scattering lengths we have found, \( a_0^{1/2} - a_0^{3/2} \) and \( a_0^{1/2} + 2a_0^{3/2} \). Even though the term added to unity in the denominator of (8.1) can be neglected, since it is of order \( 10^{-5} \), thus beyond the accuracy of our calculation or the experimental sensitivity and hence only the combination \( a_0^{1/2} - a_0^{3/2} \) turns to be of relevance. This simplify slightly the expression to

\[ \tau_{1,0} \approx \frac{0.274 \cdot 10^{-15}}{M_{\pi^\pm}^2} \left( \frac{2}{9} \right) a_0(+-;00)^{-2}. \]
Inserting eq. (7.8) in eq. (8.4) one finds the following value for the $A_{\pi K}$ lifetime in the ground state

$$\tau_{1,0} = 4.58 \cdot 10^{-15} \text{ s} \quad (4.20 \cdot 10^{-15} \text{ s}),$$

where the quantity quoted between brackets corresponds to set I.

Let us stress once more that in writing (8.1) we have not taken into account any source of isospin breaking and consequently a theoretical determination of the $\pi - K$ lifetime is just halfway. Hence the content of eq. (8.5) is just indicative. In fact eq. (8.1) is not suitable to handle bound state systems being by far the framework of a non-relativistic lagrangean [51] the most efficient way of treating bound states. Even though without any further control over the errors on the low-energy constants it seems not worth to pursue this analysis.

9 Summary

In this work we have estimated the role of isospin breaking effects in the transitions $\pi^0 K^0 \to \pi^0 K^0$ and $\pi^- K^+ \to \pi^0 K^0$. They turn out to be rather mild. Furthermore the former reaction is quite interesting because its sensitivity to the large-$N_c$ suppressed operator involving $L_4^\perp$. From a more practical point of view (essentially because the existence of experimental data) we have carefully evaluated the shift in the scattering lengths for the $\pi^- K^+ \to \pi^0 K^0$ reaction. While the error from the low-energy constants turns out to be compatible with the future experimental sensitivity, estimates of higher order corrections are at the same footing. Contrary to the previous case this reaction does not contain sizeable contribution from large-$N_c$ suppressed operators. Bearing in mind the results in eq. (7.8) and the expected experimental sensitivity we can conclude that in principle isospin breaking effects do not affect the determination of the lifetime. An accuracy in the determination of the $\pi - K$ S-wave scattering lengths at the same level as in pionium experiments will disentangle between the two sets of low-energy constants and thus will constitute a major step in understanding the basic structure of the effective lagrangean. As a direct application we have evaluated (partially) the expected lifetime of the $A_{\pi K}$ atom.

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Note added

When completing this work, ref. [52] appears. It contains partially our work. As has been shown the isospin breaking effects are quite mild and hidden by sizeable error bars. This reference essentially leads to the same numerical conclusions. Even though after some partial checks we notice that our expressions contain some log dependence from the three point function that are missing in the mentioned reference.

A $\pi^0 K^0 \to \pi^0 K^0$ scattering amplitude

In this appendix we collect all relevant formulae for the neutral→neutral amplitude. We are aware that the “digestion” of this kind of expressions is always hard, thus for sake of clarity we have not mixed (in a good extent) the contributions. This has probably enlarged slightly the expressions but we find this worth for any future comparison.

The amplitude is written as

$$\mathcal{M}(s, t, u) = \left\{ \mathcal{M}_{\text{tree}} + \sum_{i=a,b,c} \mathcal{M}_{(i)} \left|\begin{array}{c} \text{born} + \mathcal{M}_{\text{mixing}} + \mathcal{M}_{\text{tadpole}} + \mathcal{M}_{\text{t-channel}} \end{array}\right. \right\}$$
also that we explicitly use $F_0$ in at all orders. This stays for the non-renormalized decay constant.

The leading order contribution is given by

$$M \bigg|_{\text{tree}} = \frac{1}{4F_0^2} (t + \Delta_\pi - \Delta_K) + \frac{\epsilon}{2F_0^2 \sqrt{3}} (s + u - 2t). \quad (A.1)$$

The Born-type term containing the w.f.r. and bare masses renormalization contributions is given by

$$M_{(a)} \bigg|_{\text{born}} = \frac{1}{4F_0^2} (t + \Delta_\pi - \Delta_K) \left\{ \frac{1}{6} (\mu_{\pi^0} + 3\mu_{\eta} + 6\mu_{K^0} + 10\mu_{\pi^\pm} + 4\mu_{K^\pm}) - \frac{8}{F_0^2} \left[ 2(M_\pi^2 + 2M_K^2)L_4^r + \Sigma_{\pi K} L_5^r \right] + \frac{\epsilon}{\sqrt{3}} (\mu_{\eta} - \mu_\pi) - \frac{16}{F_0^2 \sqrt{3}} \left( \frac{\epsilon}{\sqrt{3}} \right) \Delta_{\pi K} (2L_6^r - L_5^r) \right\}$$

$$+ \left( \frac{2}{3} \frac{\mu_{\eta} - \mu_\pi}{\sqrt{3}} - \frac{16}{F_0^2 \sqrt{3}} \left( \frac{\epsilon}{\sqrt{3}} \right) \Delta_{\pi K} (2L_6^r - L_5^r) \right) \right\}$$

$$+ \frac{\Delta_\pi}{4F_0^2} \left[ \frac{2}{3} \mu_\eta + \frac{8}{F_0^2} \left\{ (M_\pi^2 + 2M_K^2)(2L_6^r - L_4^r) + 2M_K^2 (2L_6^r - L_5^r) \right\} \right]$$

$$+ \frac{\Delta_\pi}{4F_0^2} \left[ \frac{M_\pi^2}{16\pi^2 F_0^2} + 4\mu_\pi - 2\mu_K - \frac{2}{3} \mu_\eta \right]$$

$$+ \frac{\Delta_\pi}{4F_0^2} \left[ (M_\pi^2 + 2M_K^2)(2L_6^r - L_4^r) + 2M_K^2 (2L_6^r - L_5^r) - \Delta_{\pi K} L_5^r \right]$$

$$+ \frac{1}{2F_0^2} \left[ \left\{ \frac{\epsilon}{\sqrt{3}} \right\} (s + u - 2t) \left\{ \frac{1}{6} (11\mu_\pi + 3\mu_\eta + 10\mu_K) \right\} \right]$$

$$+ \frac{e^2 t}{18F_0^2} \left[ 24(K_4^r + K_5^r) - 18K_3^r + 9K_4^r + 14(K_5^r + K_6^r) \right]$$

$$+ \frac{e^2 M_K^2}{6F_0^2} \left[ \frac{3}{8\pi^2} - 9F_0^2 \mu_K \right]$$

$$+ \frac{2}{9} \left[ 12(K_4^r + K_5^r) - 5K_7^r - 5K_6^r - 5K_8^r - 4K_9^r - 4K_{10}^r + 50K_{11}^r + 54K_{12}^r \right]$$

$$+ \frac{e^2 M_K^2}{6F_0^2} \left[ - \frac{3}{8\pi^2} + 9F_0^2 \mu_\pi \right]$$

$$+ \frac{1}{9} \left[ 24(K_4^r + K_5^r - K_7^r - K_8^r) + 18K_3^r - 9K_4^r + 20(K_5^r + K_6^r) - 2K_9^r \right.$$
where as is customary

\[ \Sigma_m = M_{m^2}^2 + M_{\eta^0_n}^2, \quad \Sigma_{mn} = M_m^2 + M_n^2. \] (A.4)

The effect of the \( \pi - \eta \) mixing is taken into account by

\[
\mathcal{M} \left|_{\text{mixing}} = \frac{1}{144 F_0^2} \left[ 2(s + u - 2t) - \Delta_{\pi\eta} \right] \left\{ -3(\mu_K^\pm - \mu_K^0) \right. \\
- \frac{36e}{\sqrt{3}} (\mu_\pi - \mu_\eta) - \frac{24e}{\sqrt{3}} (\mu_\pi - \mu_K) + \frac{864e}{\sqrt{3} F_0^2} \Delta_{\pi\eta} (3L_7 + L_8) \right. \\
+ \frac{1}{144 F_0^2} \left[ 2(s + u - 2t) \right] - 1 \right\} \left\{ 12\Sigma_{\pi^0 K^0}(\mu_K^\pm - \mu_K^0) \right. \\
- \frac{96e}{\sqrt{3}} M_\pi^2 (\mu_\pi - \mu_K) + 8e^2 M_\pi^2 (3(2K_3^r - K_4^r) - 2(K_5^r + K_6^r)) + 2(K_5^r + K_1) \right\}. 
\] (A.5)

The Born-type contribution containing one insertion of the strong \( O(p^4) \) counter-terms is written as

\[
\mathcal{M}_{(b)} \bigg|_{\text{born}} = \frac{1}{t_0} \sum_{i=1}^{8} \mathcal{P}_i \mathcal{L}_i, 
\] (A.6)

with

\[
\begin{align*}
\mathcal{P}_1 &= 8(2M_{\pi^0}^2 - t)(2M_{K^0}^2 - t), \\
\mathcal{P}_2 &= 4 \left[ (\Sigma_{\pi^0 K^0} - s)^2 + (\Sigma_{\pi^0 K^0} - u)^2 \right], \\
\mathcal{P}_3 &= 2(1 - \frac{2e}{\sqrt{3}})(2M_{\pi^0}^2 - t)(2M_{K^0}^2 - t) + (1 + \frac{2e}{\sqrt{3}}) \left[ (\Sigma_{\pi^0 K^0} - s)^2 + (\Sigma_{\pi^0 K^0} - u)^2 \right], \\
\mathcal{P}_4 &= -\frac{2}{3} \left[ M_\pi^2 + 14M_K^2 - \frac{6e}{\sqrt{3}}(5M_\pi^2 - 2M_K^2) \right](2M_{\pi^0}^2 - t) \\
- \frac{2}{3} \left[ 13M_\pi^2 + 2M_K^2 - \frac{6e}{\sqrt{3}}(M_\pi^2 + 2M_K^2) \right](2M_{K^0}^2 - t) \\
+ \frac{2}{3} \left[ M_\pi^2 + 2M_K^2 - \frac{6e}{\sqrt{3}}(M_\pi^2 + 2M_K^2) \right], \\
\mathcal{P}_5 &= -\frac{2}{3} \left( M_\pi^2 + 3M_K^2 - \frac{12e}{\sqrt{3}} M_\pi^2 \right)(2M_{\pi^0}^2 - t) \\
- \frac{2}{3} \left[ 3M_\pi^2 + M_K^2 - \frac{4e}{\sqrt{3}}(5M_\pi^2 - 2M_K^2) \right](2M_{K^0}^2 - t) \\
+ \frac{2}{3} \left[ -\Sigma_{\pi K} + \frac{4e}{\sqrt{3}}(2M_\pi^2 - 5M_K^2) \right], \\
\mathcal{P}_6 &= \frac{8}{3} \left[ 2M_K^4 + 15M_\pi^2 M_K^2 + M_\pi^4 + \frac{16e}{\sqrt{3}}(M_K^4 + M_\pi^2 M_K^2 - 2M_\pi^4) \right], \\
\mathcal{P}_7 &= \frac{64e}{\sqrt{3}}(2M_K^4 - M_\pi^2 M_K^2 - M_\pi^4), \\
\mathcal{P}_8 &= \frac{8}{3} \left[ M_K^4 + 6M_\pi^2 M_K^2 + M_\pi^4 + \frac{2e}{\sqrt{3}}(19M_K^4 - 6M_\pi^2 M_K^2 - 13M_\pi^4) \right].
\end{align*}
\]
\[
\frac{4e^2}{2\pi F_0^2} (3K_7 + 3K_8 + K_9 + K_1^0) \Sigma_{\pi K} .
\] (A.7)

If mesons were really massless, any tadpole contribution type will vanish, as SU(3) symmetry is broken precisely by quark masses this does not turns out to be the case in nature. One thus obtains

\[
\mathcal{M}_{\text{tadpole}} = \left. \frac{\mu_{\pi^0}}{18F_0^2} \left[ t + 2M_{\pi^0}^2 - \frac{12\epsilon}{\sqrt{3}}(t + M_{\pi}^2 - 2M_{K}^2) \right] \right| _{t = 0}
\] (A.8)

Hitherto we have shown the terms that are polynomials. When performing the one loop corrections one obtains an unitary piece. This is given in terms of non-analytical functions, i.e. essentially functions with a momenta dependence in the argument. In order to present them we have kept track on the internal propagators, thus the identification of each diagram is straight forward. They are given in the t-channel by

\[
\mathcal{M}_{\pi^0\pi^0} = \frac{M_{\pi^0}^2}{24F_0^2} \left\{ 4F_0^2 (1 - \frac{6\epsilon}{\sqrt{3}}) \mu_{\pi^0} + 3 \left[ t + \Delta_\pi - \Delta_K - \frac{2\epsilon}{\sqrt{3}}(3t - 2\Sigma_{\pi K}) \right] \mathcal{F}(M_{\pi^0}^2, M_{\pi^0}^2; t) \right\}.
\] (A.9)

\[
\mathcal{M}_{\eta\eta} = \frac{M_{\eta}^2}{72F_0^2} \left\{ 12F_0^2 (1 + \frac{2\epsilon}{\sqrt{3}}) \mu_{\eta} + 9t - 6M_{\eta}^2 - 2M_{\pi}^2 + 3(\Delta_K - \Delta_\pi) + \frac{6\epsilon}{\sqrt{3}}(3t - 2\Sigma_{\eta K}) \right\} \mathcal{F}(M_{\eta}^2, M_{\eta}^2; t) \right\}.
\] (A.10)

\[
\mathcal{M}_{\pi^0\eta} = \frac{1}{3F_0^2} \left( \frac{\epsilon}{\sqrt{3}} \right) \Delta_{\pi K} \left[ 2F_0^2 \mu_{\pi} + 2F_0^2 \mu_{\eta} + (3t - 4M_K^2) \mathcal{F}(M_{\pi}^2, M_{\eta}^2; t) \right].
\] (A.11)

\[
\mathcal{M}_{\pi^+\pi^-} = \frac{1}{36F_0^2} \left\{ 4F_0^2 \mu_{\pi^+} (5t - 3M_{\pi^0}^2) + 9t - M_{\pi^0}^2 \right\} \mathcal{F}(M_{\pi^+}^2, M_{\pi^0}^2; t) \right\}.
\] (A.12)

\[
\mathcal{M}_{K^0\bar{K}^0} = \frac{1}{72F_0^2} \left\{ -4F_0^2 \mu_{K^0} \left[ -5t + 3(\Delta_K - \Delta_\pi) + \frac{6\epsilon}{\sqrt{3}}(5t - 2\Sigma_{\pi K}) \right] + 9t \left[ t + \Delta_\pi - \Delta_K - \frac{2\epsilon}{\sqrt{3}}(3t - 2\Sigma_{\pi K}) \right] \mathcal{F}(M_{K^0}^2, M_{\bar{K}^0}^2; t) \right\}.
\] (A.13)
\[ M_{K+K^-} = \frac{1}{144 F_0^2} \left\{ 4F_0^2 \mu_{K^\pm} \left[ 5t + 3(\Delta K - \Delta \pi) + \frac{6e}{\sqrt{3}} (5t - 2\Sigma \pi K) \right] \right. \\
+ \left. 9t \left[ t + \Delta K - \Delta \pi + \frac{2e}{\sqrt{3}} (3t - 2\Sigma \pi K) \right] \frac{B}{B}(M_{K^\pm}^2, M_{K^\pm}^2; t) \right\}. \quad (A.14) \]

For the s(u)-channel the intermediate particles contribution is

\[ M \bigg|_{\text{s-channel}} = M_{\pi^+\pi^-} + M_{\eta^0K^0} + M_{\eta K^0}. \]

The above terms are given by

\[ M_{\pi^+\pi^-} = \frac{1}{8F_0^2} \left\{ 2F_0^2 \mu_{K^\pm} \left[ 3s - 3M_{\pi^\pm}^2 - M_{K^\pm}^2 + 4\Delta \pi + \frac{2e}{\sqrt{3}} (3s - 3M_{\pi^0}^2 - M_{K^0}^2) \right] \right. \\
+ \left. \left[ s - \Sigma_{\pi^\pm K^\pm} + 2\Delta \pi + \frac{e}{\sqrt{3}} (s - \Sigma_{\pi K}) \right]^2 \frac{B}{B}(M_{\pi^\pm}^2, M_{K^\pm}^2; s) \right. \\
+ \left. 2 \left[ s - \Sigma_{\pi^+ K^+} + 2\Delta \pi + \frac{e}{\sqrt{3}} (s - \Sigma_{\pi K}) \right] \right. \\
+ \left. \left[ s - \Delta_{\pi^0 K^0} + \frac{e}{\sqrt{3}} (s + 3\Delta \pi K) \right]^2 \frac{B_1}{B_1}(M_{\pi^0}^2, M_{K^0}^2; s) \right. \\
+ \left. 2s \left[ 2M_{K^0}^2 - t - \frac{2e}{\sqrt{3}} (2u - t - 2M_{\pi^0}^2) \right] \frac{B_2}{B_2}(M_{\pi^0}^2, M_{K^0}^2; s) \right\}. \quad (A.15) \]

\[ M_{\pi^0 K^0} = \frac{1}{144 F_0^2} \left\{ 6F_0^2 \mu_{K^0} \left[ s - M_{\pi^0}^2 - 3M_{K^0}^2 - \frac{4e}{\sqrt{3}} (3s - 3M_{\pi^0}^2 - 5M_{K^0}^2) \right] \right. \\
+ \left. \left[ 5M_{\pi^0}^2 + M_{K^0}^2 - s - \frac{6e}{\sqrt{3}} (5M_{\pi^0}^2 - 3M_{K^0}^2 - s) \right]^2 \frac{B}{B}(M_{\pi^0}^2, M_{K^0}^2; s) \right. \\
+ \left. 2(s + 3\Delta_{\pi^0 K^0}) \left[ 5M_{\pi^0}^2 + M_{K^0}^2 - s + \frac{12e}{\sqrt{3}} (s - 5M_{\pi^0}^2 + M_{K^0}^2) \right] \frac{B_1}{B_1}(M_{\pi^0}^2, M_{K^0}^2; s) \right. \\
+ \left. (1 - \frac{12e}{\sqrt{3}}) (s + 3\Delta_{\pi^0 K^0})^2 \frac{B_2}{B_2}(M_{\pi^0}^2, M_{K^0}^2; s) \right. \\
+ \left. 2s (1 - \frac{12e}{\sqrt{3}}) \left[ 4(u - t + \Delta_{\pi^0 K^0}) + 2M_{K^0}^2 - t \right] \frac{B_2}{B_2}(M_{\pi^0}^2, M_{K^0}^2; s) \right\}. \quad (A.16) \]

\[ M_{\eta K^0} = \frac{1}{432 F_0^2} \left\{ 18F_0^2 \mu_{K^0} \left[ 3s + M_{\eta^0}^2 - 5M_{K^0}^2 - \frac{2e}{\sqrt{3}} (6s + 6M_{\pi^0}^2 - 14M_{K^0}^2) \right] \right. \\
+ \left. \left[ 3s - 7M_{\eta^0}^2 + M_{K^0}^2 - \frac{2e}{\sqrt{3}} (3s + M_{\eta^0}^2 - 7M_{K^0}^2) \right]^2 \frac{B}{B}(M_{\eta^0}^2, M_{K^0}^2; s) \right. \\
+ \left. 6(s + 3\Delta_{\pi^0 K^0}) \left[ 3s - 7M_{\eta^0}^2 + M_{K^0}^2 - \frac{12e}{\sqrt{3}} (s - 5M_{\eta^0}^2 - M_{K^0}^2) \right] \frac{B_1}{B_1}(M_{\eta^0}^2, M_{K^0}^2; s) \right. \\
+ \left. 9(1 - \frac{4e}{\sqrt{3}}) (s + 3\Delta_{\pi^0 K^0})^2 \frac{B_2}{B_2}(M_{\eta^0}^2, M_{K^0}^2; s) \right. \\
+ \left. 18s (1 - \frac{4e}{\sqrt{3}}) \left[ 4(u - t + \Delta_{\pi^0 K^0}) + 2M_{K^0}^2 - t \right] \frac{B_2}{B_2}(M_{\eta^0}^2, M_{K^0}^2; s) \right\}. \quad (A.17) \]

The \( B_{ij} \) functions are defined in app. 3.
In this appendix we display the relevant formulae concerning the more interesting process. As before our philosophy has been not to mix too much the terms. One can verify that the following result is scale invariant.

### B.1 Scattering amplitude

The amplitude is written as

\[
\mathcal{M}(s, t, u) = \left\{ \mathcal{M}_{\text{tree}} + \sum_{i=a,b,c,d} \mathcal{M}_{(i)} \right\}_{\text{born}} + \mathcal{M}_{\text{mixing}} + \mathcal{M}_{\text{tadpole}} + \mathcal{M}_{\text{s-channel}} + \mathcal{M}_{\text{t-channel}} + \mathcal{M}_{\text{u-channel}} + \mathcal{M}_{\text{one-photon}} \right\}. \tag{B.1}
\]

In the above expression the splitting between the terms is essentially the same as in the pure neutral case, but in addition we have the last term that concerns the explicit photon exchange.

For the tree-level amplitude we have

\[
\mathcal{M}_{(a)} \bigg|_{\text{tree}} = \frac{u - s}{2\sqrt{2} F_0^2} \left\{ \frac{1}{6} \left(3\mu_{\pi\pi} + 3\mu_{\eta\eta} + 5\mu_{K\pi} + 8\mu_{\pi\eta} + 5\mu_{K\pi} \right) + \frac{8}{F_0^2} \left[ \frac{1}{6} \left( 3M_\pi^2 + 2M_K^2 \right) L_4^\pi + \Sigma_{\pi K} L_5^\pi \right] \right\} + \frac{M_\pi^2}{\sqrt{2} F_0^2} \left( \frac{\epsilon}{\sqrt{3}} \right) (\mu_\eta - \mu_\pi) + \frac{\Delta_\pi}{4\sqrt{2} F_0^2} \left\{ \frac{1}{6} \left( 11\mu_\pi - \mu_\eta + 10\mu_K \right) + \frac{8}{F_0^2} \left[ \frac{1}{6} \left( 3M_\pi^2 + 2M_K^2 \right) (L_4^\pi + 2L_6^\pi) - \Delta_{K\pi} L_5^\pi + 4M_K^2 L_8^\pi \right] \right\} + \frac{1}{2\sqrt{2} F_0^2} \left( \frac{\epsilon}{\sqrt{3}} \right) (s + u - 2t) \left\{ \frac{1}{6} \left( 11\mu_\pi + 3\mu_\eta + 10\mu_K \right) - \frac{8}{F_0^2} \left[ \frac{1}{6} \left( 3M_\pi^2 + 2M_K^2 \right) L_4^\pi + \Sigma_{\pi K} L_5^\pi \right] \right\}. \tag{B.3}
\]

As in the neutral to neutral transition, there is a term from the treatment of the \( \pi - \eta \) mixing

\[
\mathcal{M}_{\text{mixing}} = -\frac{1}{144\sqrt{2} F_0^2} \left\{ 2(s + u - 2t) - \Delta_{\pi\pi} + 2\Delta_\pi \right\} \left\{ -3(\mu_{K\pi} - \mu_{K\eta}) - \frac{36\epsilon}{\sqrt{3}} (\mu_\pi - \mu_\eta) - \frac{24\epsilon}{\sqrt{3}} (\mu_\eta - \mu_K) + \frac{864\epsilon}{\sqrt{3} F_0^2} \Delta_{\pi\eta} (3L_7 + L_8^\pi) \right\} - \frac{1}{144\sqrt{2} F_0^2} \left[ 2 \left( \frac{s + u - 2t + \Delta_\pi}{M_\pi^2 - M_\eta^2} \right) - 1 \right] \left\{ 12\Sigma_{\pi K\nu}(\mu_{K\pi} - \mu_{K\eta}) \right\}. \tag{B.3}
\]
The polynomials not displayed explicitly do not contribute to the process.

The equivalent contribution, but in the e.m. sector is casted as

\[ \mathcal{M}_{(b)} \bigg|_{\text{born}} = \frac{2}{\sqrt{2} F_0} \sum_{i=3}^{8} P_i L_i^r, \]  

being

\[ P_3 = \frac{2 \epsilon}{\sqrt{3}} (2 M_\pi^2 - t)(2 M_K^2 - t) \]
\[ + (1 - \frac{\epsilon}{\sqrt{3}})(\Sigma_{\pi^0 K^+} - u)(\Sigma_{\pi^+ K^0} - u) \]
\[ - (1 + \frac{\epsilon}{\sqrt{3}})(\Sigma_{\pi^0 K^0} - s)(\Sigma_{\pi^+ K^-} - s), \]
\[ P_4 = -2(M_\pi^2 + 2 M_K^2)(s - u) - \frac{2 \epsilon}{\sqrt{3}}(2 M_\pi^2 + 2 M_K^2)(2 \Sigma_{\pi K} - 3t), \]
\[ P_5 = -2 \Sigma_{\pi K}(s - u) + \frac{6 \epsilon}{\sqrt{3}} \Sigma_{\pi K} t + \frac{4 \epsilon}{\sqrt{3}}(M_K^4 - M_\pi^4 - 4 M_\pi^2 M_K^2), \]
\[ P_6 = -\frac{8 \epsilon}{\sqrt{3}}(2 M_K^4 - M_\pi^4 - 2 M_\pi^2 M_K^2), \]
\[ P_7 = -\frac{32 \epsilon}{\sqrt{3}}(2 M_K^4 - M_\pi^4 - 2 M_\pi^2 M_K^2), \]
\[ P_8 = -\frac{8 \epsilon}{3}(2 M_K^4 - M_\pi^4 - 10 M_\pi^2 M_K^2). \]

The polynomials not displayed explicitly do not contribute to the process.

The equivalent contribution, but in the e.m. sector is casted as

\[ \mathcal{M}_{(c)} \bigg|_{\text{born}} = \frac{2 \epsilon^2}{3 \sqrt{2} F_0^3} (2 K_1^r + 2 K_2^r - 4 K_3^r + 2 K_4^r + K_5^r + 4 K_6^r)((\Sigma_{\pi K} - s) \]
\[ - \frac{2 \epsilon^2}{9 \sqrt{2} F_0^3} (6 K_1^r + 6 K_2^r - 6 K_3^r + 3 K_4^r + 4 K_5^r + 4 K_6^r)((\Sigma_{\pi K} - u) \]
\[ + \frac{\epsilon^2}{9 \sqrt{2} F_0^3} (12 K_1^r - 6 K_2^r - 3 K_3^r - K_5^r - K_6^r)(2 M_\pi^2 - t) \]
\[ + \frac{\epsilon^2}{3 \sqrt{2} F_0^3} (K_1^r + K_6^r)(2 M_K^2 - t) \]
\[ - \frac{2 \epsilon^2}{9 \sqrt{2} F_0^3} \left[ 9(M_\pi^2 + 2 M_K^2)K_8^r - M_\pi^2 K_5^r + (17 M_\pi^2 + 18 M_K^2)K_{10}^r + 18 \Sigma_{\pi K} K_{11}^r \right] \]  

(B.6)

The \( \epsilon^2 \) contribution from the wave function renormalization reads

\[ \mathcal{M}_{(d)} \bigg|_{\text{born}} = \frac{\epsilon^2(u - s)}{2 \sqrt{2} F_0^3} \left[ -2 F_0^2 \bar{\mu}_\pi - 2 F_0^2 \bar{\mu}_K - \frac{1}{16 \pi^2} \left( 2 + \log \frac{m_\gamma^2}{M_\pi^2} + \log \frac{m_\gamma^2}{M_K^2} \right) \right] \]
\[ - \frac{1}{9} (48 K_1^r + 48 K_2^r - 18 K_3^r + 9 K_4^r + 34 K_5^r + 34 K_6^r) \]  

(B.7)
Like in the neutral case, we display the unitary contribution disentangling each term separately. We remind that the subscripts refer to the internal particles running in the propagators. For the $s$-channel one gets

\[
\mathcal{M}\bigg|_{s\text{-channel}} = -\frac{1}{12\sqrt{2} F_0^2} \left\{ \mathcal{M}_{\pi^- K^+} + \frac{1}{2} \mathcal{M}_{\pi^0 K^0} + \frac{1}{18} \mathcal{M}_{\eta K^0} \right\},
\]

where

\[
\mathcal{M}_{\pi^- K^+} = 2F_0^2 \left[ (s + \Delta_{\pi^- K^+} + 6\Delta_\pi) \right] \mu_{K^\pm} + (s + \Delta_{\pi^- K^+} + 6\Delta_\pi) \mu_{K^\mp} - \frac{4\epsilon}{\sqrt{3}} \left[ s - \Delta_{\pi^- K^+} + 6\Delta_\pi \right] \left[ \mathcal{B}(M_{\pi^\pm}, M_{K^\pm}; s) \right] + \left[ (s + \Delta_{\pi^- K^+} + 6\Delta_\pi) \right] \left[ s - \Delta_{\pi^0 K^0} + \frac{4\epsilon}{\sqrt{3}} \left( s - 5M_\pi^2 + 3M_K^2 \right) \right] \mathcal{B}_1(M_{\pi^\pm}, M_{K^\pm}; s) + \left[ (s + \Delta_{\pi^- K^+} + 6\Delta_\pi) \right] \left[ s - \Delta_{\pi^0 K^0} + \frac{4\epsilon}{\sqrt{3}} \left( s + 3\Delta_{\pi K} \right) \right] \mathcal{B}_{21}(M_{\pi^\pm}, M_{K^\pm}; s)
\]

\[
- 2s \left[ \Sigma_{\pi^- K^0} - u + 2(t - \Sigma_{K^+ K^0}) + \frac{4\epsilon}{\sqrt{3}} \left( \Sigma_{\pi K} - t \right) - \frac{5\epsilon}{\sqrt{3}} (\Sigma_{\pi K} - u) \right] \mathcal{B}_{22}(M_{\pi^\pm}, M_{K^\pm}; s).
\]

(B.9)
\[ -2F_0^2(1 + \frac{\epsilon}{\sqrt{3}}) \left[ 3s - 3M_{\pi^0}^2 - 5M_{K^0}^2 - \frac{6\epsilon}{\sqrt{3}}(3s - 3M_{\pi}^2 - M_K^2) \right] \mu_{K^0} \]
+ \left[ s - \Sigma_{\pi^0 K^0} + \frac{\epsilon}{\sqrt{3}}(s - 9M_{\pi}^2 + 7M_K^2) \right] \times \left[ 5M_{\pi^0}^2 + M_{K^0}^2 - s - \frac{6\epsilon}{\sqrt{3}}(5M_{\pi}^2 - 3M_K^2 - s) \right] \bar{B}(M_{\pi^0}^2, M_{K^0}^2; s) \\
+ \left\{ [s - \Delta_{\pi - K^+} + \frac{\epsilon}{\sqrt{3}}(s + 3\Delta_{\pi K})] \times \left[ 5M_{\pi^0}^2 + M_{K^0}^2 - s - \frac{6\epsilon}{\sqrt{3}}(5M_{\pi}^2 - 3M_K^2 - s) \right] \right. \\
- \left. \left( 1 - \frac{6\epsilon}{\sqrt{3}} \right) (s + 3s_{\pi^0 K^0}) \left[ s - \Sigma_{\pi^0 K^0} + \frac{\epsilon}{\sqrt{3}}(s - 9M_{\pi}^2 + 7M_K^2) \right] \right\} \bar{B}_1(M_{\pi^0}^2, M_{K^0}^2; s) \\
- \left( 1 - \frac{6\epsilon}{\sqrt{3}} \right) (s + 3s_{\pi^0 K^0}) \left[ s - \Delta_{\pi - K^+} + \frac{\epsilon}{\sqrt{3}}(s + 3s_{\pi K}) \right] \bar{B}_{21}(M_{\pi^0}^2, M_{K^0}^2; s) \\
- 2s(1 - \frac{6\epsilon}{\sqrt{3}}) \left[ 2(\Sigma_{\pi^0 K^+} - u) + t - \Sigma_{K^+ K^0} + \frac{4\epsilon}{\sqrt{3}}(2M_{\pi}^2 - t) \right] \\
+ \frac{\epsilon}{\sqrt{3}}(2M_{K}^2 - t) - \frac{4\epsilon}{\sqrt{3}}(\Sigma_{\pi K} - u) \right\} \bar{B}_{22}(M_{\pi^0}^2, M_{K^0}^2; s). \quad (B.10) \]

\[ M_{\eta K^0} = 18F_0^2 \left[ 3s + M_{\pi^0}^2 - 5M_{K^0}^2 - \frac{\epsilon}{\sqrt{3}}(15s + 13M_{\pi}^2 - 33M_K^2) \right] \mu_{K^0} \]
+ \left[ 3s - 7M_{\pi^0}^2 + M_{K^0}^2 - \frac{\epsilon}{\sqrt{3}}(9s - 5M_{\pi}^2 - 13M_K^2) \right] \times \left[ 3s - 7M_{\pi^0}^2 + M_{K^0}^2 - \frac{2\epsilon}{\sqrt{3}}(3s + M_{\pi}^2 - 7M_K^2) \right] \bar{B}(M_{\eta}^2, M_{K^0}^2; s) \\
+ \left\{ 3(1 - \frac{2\epsilon}{\sqrt{3}})(s + 3s_{\pi^0 K^0}) \left[ 3s - 7M_{\pi^0}^2 + M_{K^0}^2 - \frac{\epsilon}{\sqrt{3}}(9s - 5M_{\pi}^2 - 13M_K^2) \right] \right. \\
+ \left. 3 \left[ 3s - 7M_{\pi^0}^2 + M_{K^0}^2 - \frac{2\epsilon}{\sqrt{3}}(3s + M_{\pi}^2 - 7M_K^2) \right] \times \right\} \bar{B}_1(M_{\eta}^2, M_{K^0}^2; s) \\
+ 9(1 - \frac{2\epsilon}{\sqrt{3}})(s + 3s_{\pi^0 K^0}) \left[ s + 3s_{\pi^0 K^+} - \frac{3\epsilon}{\sqrt{3}}(s - \Delta_{\pi K}) \right] \bar{B}_{21}(M_{\eta}^2, M_{K^0}^2; s) \\
+ 18s(1 - \frac{2\epsilon}{\sqrt{3}}) \left[ 2(u - s - t) + 4(\Sigma_{\pi^0} - t) + \Sigma_{K^+} - t \right] \\
+ \frac{3\epsilon}{\sqrt{3}}(2M_{K}^2 - t) - \frac{6\epsilon}{\sqrt{3}}(\Sigma_{\pi K} - u) \right\} \bar{B}_{22}(M_{\eta}^2, M_{K^0}^2; s). \quad (B.11) \]

In the t-channel

\[ \mathcal{M} \bigg|_{\text{t-channel}} = \mathcal{M}_{\pi^0 \pi^0} + \mathcal{M}_{\pi^- \eta} + \mathcal{M}_{K^- K^0}, \quad (B.12) \]

with

\[ \mathcal{M}_{\pi^- \pi^0} = -\frac{1}{6\sqrt{2}F_0^2} \left( \frac{4\epsilon}{\sqrt{3}}(3t - M_{\pi}^2 - 4M_{K}^2) \right) F_0^2 \mu_{\pi} \]
Finally for the $u$-channel we get the following set of results.

\[ M_{\pi - \eta} = \frac{1}{3\sqrt{2} F_0^4} \left( \frac{\epsilon}{\sqrt{3}} \right) \left\{ -2(4M_K^2 - 3t)F_0^2 \mu_\eta + (\Sigma_{\pi\eta} - t)^2B(M_{\pi}^2, M_{\eta}^2; t) \right. \\
- \left. 2t(\Sigma_{\pi\eta} - t)B_1(M_{\pi}^2, M_{\eta}^2; t) + t^2B_{21}(M_{\pi}^2, M_{\eta}^2; t) + t^2B_{22}(M_{\pi}^2, M_{\eta}^2; t) \right\}. \]  

(B.13)

\[ M_{K - \bar{K}^0} = -\frac{1}{24\sqrt{2} F_0^4} \left( -\frac{4\epsilon}{\sqrt{3}}(3t - 2M_K^2)F_0^2 \mu_K \right) \\
+ \frac{1}{24\sqrt{2} F_0^4} \left( 5M_{\pi}^2 + 3M_{\eta}^2 - 3u + 3\Delta_x - \frac{\epsilon}{\sqrt{3}}(15u - 29M_{\pi}^2 + 13M_{\eta}^2) \right) \mu_{\pi^0} \\
+ (1 - \frac{\epsilon}{\sqrt{3}})(\Sigma_{K - \pi^+} - u + \Delta_x) \left( 5M_{K^+}^2 + M_{\pi}^2 - 6\sqrt{3}(\Sigma_{\pi\eta} - u) \right) \left( B(M_{K^+}^2, M_{\pi}^2; u) \right) \\
+ \left\{ \left[ 5M_{K^0}^2 + M_{\pi}^2 - 6\sqrt{3}(\Sigma_{\pi\eta} - u) \right] \left[ \Delta_{X + K^0} + u + \frac{\epsilon}{\sqrt{3}}(3\Delta_{X K^0} - u) \right] \right. \\
+ (1 - \frac{\epsilon}{\sqrt{3}})(1 + \frac{6\epsilon}{\sqrt{3}})(\Sigma_{K - \pi^+} - u + \Delta_x)(3\Delta_{X K^0} - u) \right\} B_1(M_{K^+}^2, M_{\pi}^2; u) \\
- \left. (1 + \frac{6\epsilon}{\sqrt{3}})(3\Delta_{X K^0} - u) \right\} B_2(M_{K^+}^2, M_{\pi}^2; u) \\
- \left. 2u(1 + \frac{6\epsilon}{\sqrt{3}}) \left[ \Sigma_x - t + 2(s - \Sigma_{X K^0}) \right] \right. \\
+ \left. \frac{\epsilon}{\sqrt{3}}(2M_{\pi}^2 - t) + \frac{4\epsilon}{\sqrt{3}}(2M_{K^0}^2 - t) - \frac{4\epsilon}{\sqrt{3}}(\Sigma_{\pi\eta} - s) \right\} B_{22}(M_{K^+}^2, M_{\pi}^2; u). \]  

(B.14)

Finally for the $u$-channel we get the following set of results.

\[ M_{u}\text{-channel} = M_{K - \pi^0} + M_{K - \eta} + M_{\pi - K^0}. \]  

(B.16)

For each term separately

\[ M_{K - \pi^0} = -\frac{1}{24\sqrt{2} F_0^4} \left( \right) \\
- 2F_0^2 \left[ 5M_{\pi}^2 + 3M_{\eta}^2 - 3u + 3\Delta_x - \frac{\epsilon}{\sqrt{3}}(15u - 29M_{\pi}^2 + 13M_{\eta}^2) \right] \mu_{\pi^0} \\
+ \left\{ \right. \\
0
\[ \mathcal{M}_{\pi - K_0} = -\frac{1}{12\sqrt{2}F_0^4}\left( -2F_0^2(1 - \frac{e}{\sqrt{3}})(3u - 3M_{\pi^0}^2 + M_{K_0}^2 - 3\Delta_\pi)\mu_{K_0} \right) 
+ (1 - \frac{e}{\sqrt{3}})(\Delta_{\pi K} - u) + 2\Delta_\pi)(3\Delta_{\pi K} - u)
+ \left\{ (\Delta_{\pi K} - u + 2\Delta_\pi)(3\Delta_{\pi K} - u) \right\} \overline{B}_1(M_{\pi^0}^2, M_{K_0}^2; u) 
+ (3\Delta_{\pi K} - u) \left( 3\Delta_{\pi K} - u - \frac{3e}{\sqrt{3}}(\Delta_{\pi K} + u) \right) \overline{B}_2(M_{\pi^0}^2, M_{K_0}^2; u) 
+ 2u \left[ 2(s - t - u) + 3\Sigma_{\pi K} - 2 \right] \frac{6e}{\sqrt{3}(\Sigma_{\pi K} - s)} - \frac{3e}{\sqrt{3}(2M_{\pi^0}^2 - t)} \right] \overline{B}_2(M_{\pi^0}^2, M_{K_0}^2; u). \] (B.18)

\[ \mathcal{M} \bigg|_{\text{one-photon}} = \frac{-e^2}{4\sqrt{2}F_0^2} \left( u + \Delta_{\pi K} \right) \left( -6F_0^2\mu_{\pi^0} + \frac{1}{4\pi^2} \right) 
- \frac{e^2}{4\sqrt{2}F_0^2} \left( u - \Delta_{\pi K} \right) \left( -6F_0^2\mu_{K} + \frac{1}{4\pi^2} \right) 
+ \frac{e^2}{2\sqrt{2}F_0^2} \left\{ -F_0^2\mu_{\pi^0} \left( \Sigma_{\pi K} + u - 2s \right) - F_0^2\mu_{K} \left( \Delta_{\pi K} + u - 2s \right) 
+ 2(u - s) \left( \frac{1}{16\pi^2} \right) + (\Sigma_{\pi K} - s)\overline{B}(M_{\pi^0}^2M_{K}^2; s) + (s - \Delta_{\pi K})\overline{B}_1(M_{\pi^0}^2M_{K}^2; s) 
+ 2(s - u)(s - \Sigma_{\pi K})C_{\pi K}(s) + 4(s - \Sigma_{\pi K}) \left[ uG_{\pi K}^{-}(s) + \Delta_{\pi K}G_{\pi K}^{+}(s) \right] \right\}, \] (B.20)

The only remaining piece is the soft photon exchange diagram. It contains an infrared divergence (included in the C function). Its result is given by the functions $G_{\pi K}^{-}$ and $G_{\pi K}^{+}$ are defined in the following appendix.
B.2 Scattering lengths

Due to the relevance of the process $\pi^- K^+ \to \pi^0 K^0$ we display the combination for the S-wave scattering lengths. After performing all the steps mentioned in sec. 3 we make the following substitutions

$$\Delta_K \to \Delta_\pi - \frac{4\epsilon}{\sqrt{3}}(M_K^2 - M_\pi^2), \quad \Delta_\pi \to 2e^2 Z F_0^2.$$  

This leads to the following expression

$$-\frac{3}{\sqrt{2}} a_0 (+; 00) = a_0^{1/2} - a_0^{1/2} - \frac{3}{\sqrt{2}} \Delta_0 (+; 00) =$$

$$\frac{3}{32\pi} \frac{M_\pi \mp M_K \mp}{F_0^2} + \frac{3}{32\pi} \frac{M_K - M_\pi}{256\pi F_0^2 M_K + M_\pi} \Delta_K + \frac{3}{32\pi} \frac{3M_K + 5M_\pi}{256\pi F_0^2 M_K + M_\pi} \Delta_\pi + \frac{\sqrt{3}}{64\pi F_0^2} (M_K^2 + M_\pi^2) \epsilon$$

$$+ \frac{3}{32\pi} \frac{M_\pi \mp M_K \mp}{F_0^2} \left\{ -8 (M_\pi^2 + 2M_K^2)^2 L_4^2 - \frac{1}{576\pi^2} \left( \frac{1}{M_K^2 - M_\pi^2} \right) \right\}$$

$$\left[ 27M_\pi^4 \log \left( \frac{M_\pi^2}{\mu^2} \right) - 2M_K^2 (18M_\pi^4 + 5M_\pi^2) \log \left( \frac{M_K^2}{\mu^2} \right) \right] + M_\pi^2 (28M_K^2 - 9M_\pi^2) \log \left( \frac{M_\pi^2}{\mu^2} \right)$$

$$- \frac{\sqrt{3}}{32\pi F_0^2} \left\{ - \frac{M_\pi}{432\pi^2 (M_K^2 - M_\pi^2) (4M_K^2 - M_\pi^2)} \times \right.$$  

$$(216M_K^4 - 604M_K^2 M_\pi + 862M_K^2 M_\pi^2 + 1331M_K^3 M_\pi^3 - 245M_K^4 M_\pi^4 - 283M_K M_\pi^5 + M_\pi^6)$$

$$+ 4 \left[ (M_K^4 - M_\pi^4)(L_5^2 - 2L_6^2) + 2M_K M_\pi (2L_5^2 (M_K - M_\pi) M_\pi + L_4^2 (6M_K^2 - 3M_\pi^2)) \right]$$

$$+ \frac{1}{36}(M_K^2 - M_\pi^2) (4M_K^2 - M_\pi^2) \times$$

$$\left[ (304M_K^4 - 60M_K^2 M_\pi + 472M_K^2 M_\pi^2 + 10M_K^2 M_\pi^3 + 99M_K M_\pi^4 + 8M_\pi^5) B_+$$

$$+ (-160M_K^4 + 24M_K^2 M_\pi + 310M_K M_\pi^2 + 26M_K^2 M_\pi^3 - 81M_K M_\pi^4 + 8M_\pi^5) B_- \right]$$

$$+ \frac{1}{\pi^2 (M_K^2 - M_\pi^2)^2 (M_K^2 + M_\pi^2)} \times$$

$$\left[ \frac{M_\pi^2}{192} (71M_K^2 - 79M_K^4 M_\pi - 174M_K^2 M_\pi^2 + 43M_K^3 M_\pi^3 - 44M_K M_\pi^4 + 3M_\pi^5) \log \left( \frac{M_\pi^2}{\mu^2} \right) \right]$$

$$+ \frac{M_K M_\pi}{864} (-162M_K^5 + 68M_K^3 M_\pi + 455M_K^3 M_\pi^2 + 645M_K^3 M_\pi^3 - 154M_K M_\pi^4 + 128M_\pi^5) \log \left( \frac{M_\pi^2}{\mu^2} \right)$$

$$+ \frac{1}{1728} (-36M_K^6 + 36M_\pi^6 M_\pi - 415M_K^6 M_\pi^2 + 251M_K^4 M_\pi^3$$

$$- 174M_K^4 M_\pi^3 - 277M_K^2 M_\pi^5 + 266M_K M_\pi^6 + 9M_\pi^7) \log \left( \frac{M_\pi^2}{\mu^2} \right) \right\}$$

$$- \frac{32e^2}{32\pi F_0^2} \left\{ \frac{1}{1728\pi^2 (M_K^2 - M_\pi^2)^2 (M_K + M_\pi)} \times \right.$$  

$$(4M_K^7 - 124M_K^5 M_\pi + 671M_K^5 M_\pi^2 +$$

$$+ 3967M_K^3 M_\pi^4 - 2276M_K^3 M_\pi^4 + 575M_K M_\pi^6 + 29M_\pi^7)$$

$$+ 2 \left[ (8L_4^2 + 3L_5^2) M_K^2 - 4(L_4 + 6L_6^2) M_K M_\pi + (4L_4^2 - L_5^2) M_\pi^2 \right]$$

$$M_K M_\pi \times$$

$$\left[ (-160M_K^4 + 48M_K^2 M_\pi + 274M_K^2 M_\pi^2 + 14M_K^2 M_\pi^3 - 45M_K M_\pi^4 - 20M_\pi^5) B_+$$

$$+$$
where for sake of clarity we have expressed the combination $a_0(+-;00)$ in terms of the bared coupling constant $F_0$. If one chooses to renormalize the decay constants as $F_\pi F_K$ the following term should be added to the previous expression

$$
\delta_{F_\pi F_K} = \frac{3M_\pi M_K^\pm}{256\pi} \left\{ -\frac{1}{16\pi^2 F_0^2} \left\{ -512\pi^2 [2(M_{\pi^\pm}^2 + 2M_{K^\pm}^2)L_5^3 + (M_{\pi^\pm}^2 + M_{K^\pm}^2)L_5^5] \\
+11M_{\pi^\pm}^2 \log \left( \frac{M_{\pi^\pm}^2}{\mu^2} \right) + 10M_{K^\pm}^2 \log \left( \frac{M_{K^\pm}^2}{\mu^2} \right) + (4M_{K^\pm}^2 - M_{\pi^\pm}^2) \log \left( \frac{M_{K^\pm}^2}{\mu^2} \right) \right\} \\
-\frac{3M_\pi M_K}{256\pi} \left( \frac{1}{16\pi^2 F_0^2} \right) \left( \frac{\epsilon}{\sqrt{3}} \right) \left\{ 20(M_{K^\pm}^2 - M_{\pi^\pm}^2) - 512\pi^2 [3M_{K^\pm}^2 (4L_4^4 + L_5^5) - M_{\pi^\pm}^2 (6L_4^4 + L_5^5)] \\
+11M_{\pi^\pm}^2 \log \left( \frac{M_{\pi^\pm}^2}{\mu^2} \right) + 10(3M_{K^\pm}^2 - 2M_{\pi^\pm}^2) \log \left( \frac{M_{K^\pm}^2}{\mu^2} \right) + 3(4M_{K^\pm}^2 - 3M_{\pi^\pm}^2) \log \left( \frac{M_{K^\pm}^2}{\mu^2} \right) \right\} \\
+\frac{3\epsilon^2}{256\pi} \left( \frac{1}{16\pi^2 F_0^2} \right) \left( \frac{\epsilon}{\sqrt{3}} \right) \left\{ 42M_{\pi^\pm} M_K + 512\pi^2 [M_{K^\pm}^2 (4L_4^4 + L_5^5) - 4M_{\pi^\pm} M_K (3L_4^4 + L_5^5) + M_{\pi^\pm}^2 (2L_4^4 + L_5^5)] \\
+11M_{\pi^\pm} (2M_K - M_\pi) \log \left( \frac{M_{\pi^\pm}^2}{\mu^2} \right) - 10M_K (M_K - 2M_\pi) \log \left( \frac{M_{K^\pm}^2}{\mu^2} \right) \right\} \\
+(-4M_{K^\pm}^2 + 6M_{\pi^\pm} M_K + M_{\pi^\pm}^2) \log \left( \frac{M_{\pi^\pm}^2}{\mu^2} \right) \right\} .
$$

(B.22)
Otherwise, if the desired renormalization is as \( F^2 \) the term to be added is

\[
\delta F^2 = \frac{3M_{\bar{p}}^2 M_{K^\pm}}{32\pi} \left( \frac{1}{16\pi^2F_0^4} \right) \times \frac{1}{16\pi^2F_0^4} \left( \frac{\epsilon}{3} \right) \left\{ 2(M_K^2 - M_{\pi}^2) - 128\pi^2[3(M_K^2 - M_{\pi}^2) L_4^1 + M_{\pi}^2 L_5^1] \right\} - 3M_{\pi} M_K \left( \frac{1}{16\pi^2 F_0^4} \right) \left( \frac{\epsilon}{\sqrt{3}} \right) \left\{ 2(M_K^2 - M_{\pi}^2) - 128\pi^2[3(M_K^2 - M_{\pi}^2) L_4^1 + M_{\pi}^2 L_5^1] \right\} + 2M_{\pi}^2 \log \left( \frac{M_{\pi}^2}{\mu^2} \right) + (3M_K^2 - 2M_{\pi}^2) \log \left( \frac{M_K^2}{\mu^2} \right) \right. 

\[
- \frac{3Z\pi^2}{32\pi} \left( \frac{1}{16\pi^2 F_0^4} \right) \left\{ -6M_{\pi} M_K - 128\pi^2[2M_{\pi}^2 L_4^1 + 2M_{\pi} M_K (3L_4^1 + L_5^1) + M_{\pi}^2 (L_4^1 + L_5^1)] \right\} + 2M_{\pi}(2M_K + M_{\pi}) \log \left( \frac{M_{\pi}^2}{\mu^2} \right) + M_K (M_K - 2M_{\pi}) \log \left( \frac{M_K^2}{\mu^2} \right) \right) \quad \text{(B.24)}
\]

\[
\left\{ 128\pi^2[3(M_K^2 - M_{\pi}^2) L_4^1 + M_{\pi}^2 L_5^1] - 2M_{\pi}^2 \log \left( \frac{M_{\pi}^2}{\mu^2} \right) - M_K^2 \log \left( \frac{M_K^2}{\mu^2} \right) \right\} \quad \text{(B.25)}
\]

\[
\frac{3Z\pi^2}{32\pi} \left( \frac{1}{16\pi^2 F_0^4} \right) \left\{ -6M_{\pi} M_K - 128\pi^2[2M_{\pi}^2 L_4^1 + 2M_{\pi} M_K (3L_4^1 + L_5^1) + M_{\pi}^2 (L_4^1 + L_5^1)] \right\} + 2M_{\pi}(2M_K + M_{\pi}) \log \left( \frac{M_{\pi}^2}{\mu^2} \right) + M_K (M_K - 2M_{\pi}) \log \left( \frac{M_K^2}{\mu^2} \right) \right) \quad \text{(B.26)}
\]

Obviously once the renormalization of the coupling constant is taken into account there are some cancellations that simplify slightly the expression for the combination of scattering lengths. In writing this expression we have applied extensively the Gell-Mann–Okubo relation to the polynomial terms and to the \( J \) functions but otherwise keep the eta mass inside the log functions. The functions \( B_+ \) and \( B_- \) are defined in the next appendix. Notice that this combination of scattering lengths is independent at leading order on the ratio \( \frac{m_{\pi}}{m_{\eta}} \), this implies that the uncertainty from this quantity is very small [10].

### C Loop Integrals

In this appendix we collect for completeness some familiar formulas. The \( \overline{B}_{ij} \) functions are the finite components of those defined in [23]. Using Lorenz decomposition and some simpler algebraic manipulation they can be reduced to the tadpole integral, \( \mu \), and the one-loop function, \( \overline{J} \), through the following set of finite relations

\[
\overline{B}(m_1^2, m_2^2; p^2) = \frac{\mu_{m_1} - \mu_{m_2}}{\Delta_{m_1 m_2}} + \overline{J}(m_1^2, m_2^2; p^2),
\]

\[
\overline{B}_1(m_1^2, m_2^2; p^2) = \frac{1}{2} \left( 1 + \frac{\Delta_{m_1 m_2}}{p^2} \right) \overline{J}(m_1^2, m_2^2; p^2) + \frac{\mu_{m_1} - \mu_{m_2}}{\Delta_{m_1 m_2}} \quad \text{(C.1)}
\]

\[
\overline{B}_2(m_1^2, m_2^2; p^2) = \frac{1}{3p^2} \left( \lambda_{m_1 m_2}(p^2) + 3p^2 m_1^2 \right) \overline{J}(m_1^2, m_2^2; p^2) + \frac{p^2 - m_1^2}{\Delta_{m_1 m_2}} \mu_{m_1} - \frac{p^2 - m_2^2}{\Delta_{m_1 m_2}} \mu_{m_2} + \left( \frac{1}{16\pi^2} \right) \left( 1 - \frac{3\Sigma_{m_1 m_2}}{p^2} \right) \quad \text{(C.1)}
\]

\[
\overline{B}_{22}(m_1^2, m_2^2; p^2) = \frac{1}{12p^4} \left( \lambda_{m_1 m_2}(p^2) \overline{J}(m_1^2, m_2^2; p^2) + \frac{2p^4}{3} \left( \frac{1}{16\pi^2} \right) \left( 1 - \frac{3\Sigma_{m_1 m_2}}{p^2} \right) \right) - \left( p^2 + \Delta_{m_1 m_2} \right) \mu_{m_1} - \left( p^2 - \Delta_{m_1 m_2} \right) \mu_{m_2} + \lambda_{m_1 m_2}(p^2) \frac{\mu_{m_1} - \mu_{m_2}}{\Delta_{m_1 m_2}} \quad \text{(C.1)}
\]
In the \( \mathcal{J} \) function we have to consider at least two branches

\[
32\pi^2 \mathcal{J}(m_1^2, m_2^2; p^2) = \begin{cases} 
\text{i) if } & p^2 \geq (m_1 + m_2)^2 \\
& 2 + \left( -\frac{\Delta_{m_1 m_2}}{p^2} + \frac{\Sigma}{\Delta m_{12}^2} \right) \log \left( \frac{m_1^2}{m_2^2} \right) - \frac{\lambda_{m_1 m_2}}{p^2} \log \left( \frac{p^2 + \lambda_{m_1 m_2}^2}{p^2 - \lambda_{m_1 m_2}^2} \right) \\
\text{ii) if } & (m_1 - m_2)^2 \leq p^2 \leq (m_1 + m_2)^2. \\
& 2 + \left( -\frac{\Delta_{m_1 m_2}}{p^2} + \frac{\Sigma}{\Delta m_{12}^2} \right) \log \left( \frac{m_1^2}{m_2^2} \right) \\
& -2 \sqrt{-\frac{\lambda_{m_1 m_2}^2}{p^2}} \left[ \arctg \left( \frac{p^2 + \lambda_{m_1 m_2}}{\sqrt{-\lambda_{m_1 m_2}^2}} \right) - \arctg \left( \frac{-p^2 + \lambda_{m_1 m_2}}{\sqrt{-\lambda_{m_1 m_2}^2}} \right) \right].
\end{cases}
\]

(C.2)

With the use of the Gell-Mann–Okubo relation the \( \mathcal{J} \) function can be reduced to simpler combinations of \( \log \) and \( \arctg \) functions. In the latter case we define following the functions

\[
B_{\pm} \equiv B(M_K, M_\pi) \pm B(M_K, -M_\pi),
\]

where

\[
B(x, y) = -\frac{\sqrt{(x-y)(2x+y)}}{12\pi^2(x+y)} \left[ \arctg \left( \frac{\sqrt{(x-y)(2x+y)}}{2(x-y)} \right) + \arctg \left( \frac{x+2y}{2\sqrt{(x-y)(2x+y)}} \right) \right].
\]

(C.3)

Besides the previous integrals in the calculation is needed the three-point function integrals. Those arise through the photon exchange diagram. The scalar function, the only one that is actually IR divergent, is given in eq. (5.5) while the remaining one is

\[
C^\mu(m_P^2, m_Q^2, m_k^2; p, k) = -i\mu^{\mu - D} \int \frac{d^D l}{(2\pi)^D} \frac{l^\mu}{(l^2 - m_l^2)(p - l)^2 - M_P^2 ((k - l)^2 - M_Q^2)} = (p - k)^\mu G_{PQ}^-(p, k) + (p + k)^\mu G_{PQ}^+(p, k).
\]

(C.4)

In the convention of [24] the previous decomposition reads

\[
G_{PQ}^+(p, k) \propto -\frac{C_{11}(p, k)}{2}, \quad G_{PQ}^-(p, k) \propto -\frac{C_{11}(p, k)}{2} + C_{12}(p, k).
\]

In terms of the basic functions we obtain

\[
G_{PQ}^+(q^2, p \cdot k) = \frac{1}{\mathcal{G}} \left\{ -\Delta_{PQ} \left[ J_{PQ}(q^2) - \frac{1}{16\pi^2} \right] - \frac{1}{32\pi^2} [(p + k)^2 - \Sigma_{PQ}] \log \left( \frac{M_P^2}{M_Q^2} \right) \right\},
\]

\[
G_{PQ}^-(q^2, p \cdot k) = \frac{1}{\mathcal{G}} \left\{ q^2 \left[ J_{PQ}(q^2) - \frac{1}{16\pi^2} \right] - \frac{1}{32\pi^2} \left( q^2 \frac{\Sigma_{PQ}}{\Delta_{PQ}} - \Delta_{PQ} \right) \log \left( \frac{M_P^2}{M_Q^2} \right) \right\},
\]

(C.5)

where

\[
\mathcal{G} = [q^2(p + k)^2 - \Delta_{PQ}^2] \quad \text{and} \quad q^2 = (p - k)^2.
\]

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