INTERACTIVE VERSUS NON-INTERACTIVE LOCALLY DIFFERENTIALLY PRIVATE ESTIMATION: TWO ELBOWS FOR THE QUADRATIC FUNCTIONAL

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Local differential privacy has recently received increasing attention from the statistics community as a valuable tool to protect the privacy of individual data owners without the need of a trusted third party. Similar to the classical notion of randomized response, the idea is that data owners randomize their true information locally and only release the perturbed data. Many different protocols for such local perturbation procedures can be designed. In most estimation problems studied in the literature so far, however, no significant difference in terms of minimax risk between purely non-interactive protocols and protocols that allow for some amount of interaction between individual data providers could be observed. In this paper we show that for estimating the integrated square of a density, sequentially interactive procedures improve substantially over the best possible non-interactive procedure in terms of minimax rate of estimation.

In particular, in the non-interactive scenario we identify an elbow in the minimax rate at $s = \frac{3}{4}$, whereas in the sequentially interactive scenario the elbow is at $s = \frac{1}{2}$. This is markedly different from both, the case of direct observations, where the elbow is well known to be at $s = \frac{1}{4}$, as well as from the case where Laplace noise is added to the original data, where an elbow at $s = \frac{3}{4}$ is obtained.

We also provide adaptive estimators that achieve the optimal rate up to log-factors, we draw connections to non-parametric goodness-of-fit testing and estimation of more general integral functionals and conduct a series of numerical experiments. The fact that a particular locally differentially private, but interactive, mechanism improves over the simple non-interactive one is also of great importance for practical implementations of local differential privacy.

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1. Introduction. In the modern information-age an increasing amount of private and sensitive data about each and every one of us (such as medical information, smartphone user behavior, etc.) is perpetually being collected, electronically stored, processed and analyzed. This trend is opposed by an increasing desire for data privacy protection and stricter regulations as expressed, for instance, by the EU General Data Protection Regulation\footnote{https://gdpr-info.eu} which is in effect since May 2018. On the technological side, a particularly fruitful approach to data privacy protection, that is considered insusceptible to privacy breaches, is ‘differential privacy’, formally introduced by Dwork et al. (2006). However, the design and development of optimal statistical estimation procedures under differential privacy is still at its beginnings. A few first contributions in that direction are Butucea et al. (2020); Cai, Wang and Zhang (2019); Duchi, Jordan and Wainwright (2013a,b, 2014); Rohde and Steinberger (2020); Smith (2008, 2011); Wasserman and Zhou (2010); Ye and Barg (2017).

In this paper we focus on the concept of $\alpha$-local differential privacy (LDP) to protect the information of individual data providers. The general notion of $\alpha$-differential privacy, as introduced by Dwork et al. (2006), denotes a private data release mechanism that produces an output $Z$ based on original and confidential data $X_1,\ldots,X_n$, such that the conditional distribution of $Z$ given $X = (X_1,\ldots,X_n)$ satisfies

\begin{equation}
\sup_A \sup_{x,x'} \frac{\Pr(Z \in A | X = x)}{\Pr(Z \in A | X = x')} \leq e^\alpha,
\end{equation}

where the first supremum runs over all measurable sets and $d_0(x, x') := |\{i : x_i \neq x'_i\}|$ denotes the Hamming distance between $x$ and $x'$. Clearly, a smaller $\alpha$ implies a stronger privacy protection. Throughout this paper, we restrict to the case $\alpha \leq 1$, that is, the privacy protection is not allowed to deteriorate as the sample size increases. The ‘local’ paradigm within differential privacy describes a situation where no trusted third party is available that can do data collection and processing, but the original data $X_i$ have to be ‘sanitized’ already on the data providers ‘local machine’ (cf. Evfimievski, Gehrke and Srikant, 2003). This is also closely related to the classical idea of randomized response (Warner, 1965). In such local privacy protocols, even though the data providers trust nobody with their original data, some amount of interaction may be allowed between individuals. Here we consider two popular protocols for locally private estimation. First, we study the non-interactive protocol, where individual $i$ generates a private view $Z_i$ of its original data.
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\( X_i \) independently of all the other individuals. Furthermore, we also consider the sequentially interactive protocol where the \( i \)-th individual also has access to the previously sanitized data \( Z_1, \ldots, Z_{i-1} \) of other individuals in order to generate its own \( Z_i \). Of course, sequentially interactive protocols are more flexible than non-interactive ones and have the potential to retain more information about the original unobserved data sample.

Our goal is to provide a complete picture of the minimax theory of estimating the integrated square \( D(f) = \int f^2(x)dx \) of the density \( f \) of the original i.i.d. data \( X_1, \ldots, X_n \), under local differential privacy. The quadratic functional plays an important role in statistics, for instance, in goodness-of-fit testing. Very recently, Lam-Weil, Laurent and Loubes (2020) have investigated goodness-of-fit testing based on the quadratic functional for the non-interactive protocol of differential privacy, where each individual uses the same channel to produce a sanitized observation \( Z_i \). They succeeded in deriving the non-interactive minimax rate of testing over the particular Besov classes \( B_s^{2\infty} \). Here, we study quadratic functional estimation over the scale of general Besov classes \( B_s^{pq} \), \( p \geq 2 \), for both, non-interactive and sequentially interactive locally differentially private mechanisms. Contrary to most existing results on locally differentially private estimation, we find that for estimating the quadratic functional, using a sequentially interactive protocol considerably improves over the non-interactive one, even in terms of minimax rate of convergence. This phenomenon, that sequentially interactive procedures improve substantially over non-interactive ones, can not be observed for many other private estimation problems such as density estimation (Butucea et al., 2020), high-dimensional regression and mean estimation (Duchi, Jordan and Wainwright, 2018) and estimation of general linear functionals of the true data generating distribution (Rohde and Steinberger, 2020). However, Kasiviswanathan et al. (2011) showed that so-called masked-parity functions can only be learned with interactive procedures but not with purely non-interactive ones. Furthermore, for certain pointer-chasing games that are usually not studied in the statistics literature, Joseph, Mao and Roth (2020); Joseph et al. (2019) showed that going through multiple rounds of interaction in local differential privacy can reduce the sample complexity by a polynomial factor in the problem dimension. Only very recently Acharya et al. (2022) showed that sequential interaction can somewhat improve the rate of \( s \)-sparse mean estimation from \( \frac{sd}{na^2} \log \frac{ed}{s} \) for non-interactive protocols to \( \frac{sd}{na^2} \). Moreover, following up on the present paper, the separation between non-interactive and sequentially interactive privacy mechanisms was further examined in Berrett and Butucea (2020) and Butucea and Issartel (2021) for testing and non-linear functional estima-
tion with discrete data. Sequentially interactive locally differentially private mechanisms for statistical estimation have also been proposed, e.g., in Duchi, Jordan and Wainwright (2018, Section 3.2.2) and Duchi and Ruan (2020), but without rigorously establishing the superiority over non-interactive procedures.

Our main contributions are the following:

• In the non-interactive case we construct an \( \alpha \)-differentially private data release mechanism and estimator for \( \int f^2(x)dx \) based on U-statistics and sanitized empirical wavelet coefficients. Our procedure is related to the one of Butucea et al. (2020) and is shown to achieve the minimax rate (up to log factors) within the class of \( \alpha \)-non-interactive differentially private procedures over Besov classes \( B^p_{s} \) with \( p \geq 2, q \geq 1 \). In this case, for \( \alpha \in (0, 1] \), the optimal convergence rate is given by

\[
(n\alpha^2)^{-\frac{4s}{4s+3}} \lor (n\alpha^2)^{-1/2}.
\]

Notice the elbow at \( s = 3/4 \), where the nonparametric rate transitions into the rate of parametric \( \alpha \)-private estimation of \( \sqrt{n\alpha^2} \). Also observe that the minimax rate of testing in Lam-Weil, Laurent and Loubes (2020) corresponds to the square-root of the nonparametric part of our rate with respect to \( n \), but is suboptimal with respect to \( \alpha \) when it tends to zero.

• The crucial point is that we improve the classical U-statistics approach by considering a two-step procedure that requires sequential (but still locally differentially private) interaction between data owners. The first part \( X^{(1)} = (X_1, \ldots, X_{n/2}) \) of the sample is used to locally construct sanitized data \( Z^{(1)} = (Z_1, \ldots, Z_{n/2}) \) and an estimate \( \hat{f}^{(1)} \) of the density \( f \), using the method of Butucea et al. (2020). Then, conditional on \( Z^{(1)} \), we estimate the linear functional \( f \mapsto \int \hat{f}^{(1)}(x)f(x)dx \) by the method of Rohde and Steinberger (2020) in a locally private way. Since \( \hat{f}^{(1)} \) has to be provided to the owners of the second half of the data \( X^{(2)} = (X_{n/2+1}, \ldots, X_n) \) in order for them to generate sanitized data \( Z^{(2)} = (Z_{n/2+1}, \ldots, Z_n) \), the two-step procedure is inherently sequentially interactive. We establish its optimality within the class of all sequentially interactive procedures (up to log factors) by proving lower bounds on the corresponding minimax risk using a private version of the generalized Le Cam method (see also Duchi and Ruan, 2018, Section 5). The achieved rate is given by

\[
(n\alpha^2)^{-\frac{4s}{4s+2}} \lor (n\alpha^2)^{-1/2}.
\]
Notice that the elbow is now at $s = 1/2$. The fact that sequentially interactive methods may improve substantially over non-interactive ones is also an important lesson for implementations of local differential privacy.

- We discuss two practically important applications for estimation of the quadratic functional: estimating more general integral functionals and goodness-of-fit testing.
- We provide a non-interactive as well as a sequentially interactive $\alpha$-locally differentially private estimator of the quadratic functional, both of which do not depend on the smoothness $s$ of the density $f$, and we prove that they attain the respective minimax lower bounds, up to logarithmic factors in $n\alpha^2$.
- Several numerical experiments are conducted which show that the sequential procedure can also be superior to the non-interactive one in smaller samples.

1.1. Background on estimating quadratic functionals. One particularly interesting non-linear functional is the quadratic functional. Bickel and Ritov (1988) were the first to discover the so-called elbow phenomenon arising for estimating the integrated square of a density based on independent, identically distributed (i.i.d.) observations: While a $\sqrt{n}$-efficient estimator exists for Hölder smoothness to the exponent $s > 1/4$, the minimax rate of convergence over Hölder balls is $n^{-4s/(4s+1)}$ whenever $s \leq 1/4$ although the standard information bound is strictly positive and finite, see Ritov and Bickel (1990). Within the Gaussian sequence space model and minimax estimation of the squared $\ell_2$-norm of the sequential parameter, Donoho and Nussbaum (1990) found a corresponding phenomenon over $\ell_2$-ellipsoids. A fully data-driven procedure for quadratic functionals, based on model selection, with the functional class being some $\ell_p$ or Besov body for $0 < p < 2$, is developed in Laurent and Massart (2000). Estimation via quadratic rules of the quadratic functional over parameter spaces which are not quadratically convex is studied in Cai and Low (2005). It is shown that the near minimaxity of optimal quadratic rules typically does not hold when the parameter space is not quadratically convex. The maximum risk of quadratic procedures over any parameter space is established to be equal to the maximum risk over the quadratic convex hull. It also follows from the results that for Besov balls and $\ell_p$ balls with $0 < p < 2$, quadratic rules can be minimax rate optimal only if the minimax quadratic risk is of order $n^{-1}$. The minimax quadratic risk also exhibits the well-known elbow phenomenon as mentioned above for Hölder balls. More precisely, with $B^2_{pq}(M)$ denoting the centered
ball of radius $M$ in the Besov class $B^{pq}_{s}$,

$$\inf_D \sup_{f \in B^{pq}_{s}(M)} \mathbb{E}_f \left[ (\hat{D} - D(f))^2 \right] \asymp \begin{cases} \frac{M^2}{n}, & \frac{1}{2} - \frac{1}{2p} \leq s' \\ \frac{1}{n^{2 - \frac{s}{1+2\sigma}}}, & s' < \frac{1}{2} - \frac{1}{2p}, \end{cases}$$

where $0 < p < 2$ and

$$s' := s - \frac{1}{p} + \frac{1}{2} > 0.$$

In the same setting of sparse $\ell_p$ and Besov bodies, Cai and Low (2006) construct an adaptive minimax-optimal estimator selecting among a collection of penalized nonquadratic estimators. A detailed comparison to the results of Laurent and Massart (2000) is given in their Section 3.3. Klemelä (2006) studies estimation of quadratic functionals for $\ell_p$ bodies with $2 < p < \infty$. Butucea (2007) treats the problem of quadratic functional estimation on Sobolev classes in the convolution model, where the noise distribution is known and its characteristic function decays either polynomially or exponentially asymptotically. Particularly under polynomial decay at exponent $-\sigma$, the elbow between parametric and nonparametric rate is present again but shifted from Sobolev smoothness $1/4$ to $1/4 + \sigma$. Collier, Comminges and Tsybakov (2017) realize minimax estimation of linear and quadratic functionals over sparsity classes.

1.2. Organization of the paper. The paper is organized as follows. In Section 2 we discuss some preliminaries on differential privacy and Besov spaces and introduce the formal notation. Section 3 contains our main results on the non-interactive case, including minimax lower bounds and a minimax rate optimal non-interactive estimation procedure. In Section 4 we present a sequentially-interactive estimation procedure that improves on the rate of the non-interactive method from Section 3. We also establish its optimality within the larger class of sequentially interactive procedures by proving matching lower bounds (up to log factors). In Section 5 we discuss consequences and applications of our work for locally private estimation of more general integral functionals and for goodness-of-fit testing. In Section 6 we present adaptive estimators for both the non-interactive as well as the sequentially interactive case. Finally, in Section 7 we summarize the results of extensive numerical experiments to compare and evaluate the performance of our procedures in small samples. All the proofs are collected in the supplementary material (Butucea, Rohde and Steinberger, 2022).

2. Preliminaries and notation. We consider the situation where our $n$ data providers hold confidential data $X_1, \ldots, X_n$ assumed to be i.i.d. on
[0, 1] with common probability density function (pdf) \( f : [0, 1] \to \mathbb{R}_+ \), \( f \in L^2[0, 1] \). We want to estimate the quadratic functional \( D(f) = \int_0^1 f^2(x) \, dx \). However, we do not observe the original data \( X_1, \ldots, X_n \), but only the sanitized data \( Z_1, \ldots, Z_n \) on the measurable space \((\mathcal{Z}, \mathcal{G}) := (\prod_{i=1}^n Z_i, \bigotimes_{i=1}^n \mathcal{G}_i)\). The conditional distribution of the observations \( Z = (Z_1, \ldots, Z_n) \) given the original sample \( X = (X_1, \ldots, X_n) \) is described by the \emph{channel distribution} \( Q \). That is, \( Q \) is a Markov probability kernel from \(([0, 1]^n, \mathcal{B}([0, 1])^\otimes n)\) to \((\mathcal{Z}, \mathcal{G})\), where \( \mathcal{B}([0, 1]) \) denotes the Borel sets of \([0, 1]\) and \( \otimes n \) denotes the \( n \)-fold product sigma field. For ease of notation we suppress the \( n \)-dependence of \( Q \). Hence, the joint distribution of the observation vector \( Z = (Z_1, \ldots, Z_n) \) on \( \prod_{i=1}^n Z_i \) is given by \( Q_f := Q_{\mathbb{P}_f^n} \), i.e., the measure \( A \mapsto \int_{[0,1]^n} Q(A|x_1, \ldots, x_n) \prod_{i=1}^n f(x_i) \, dx, \ A \in \mathcal{G} \), where \( \mathbb{P}_f(B) = \int_B f(x_1) \, dx_1, \ B \in \mathcal{B}([0, 1]) \). Finally, whenever \( f \) and \( Q \) are fixed and clear from the context, we write \((\Omega, \mathcal{F}, \mathbb{P})\) for the underlying probability space on which random vectors like \( X \) and \( Z \) are defined, and we denote by \( \mathbb{E} \) and \( \text{Var} \) the corresponding expectation and variance operators.

2.1. Preliminaries on Besov spaces. For the necessary background on Besov spaces we mainly follow Härdle et al. (1998) and Giné and Nickl (2016, Section 4.3). For any \( h > 0 \), let \( \Delta_h \) denote the \( h \)-shift difference operator, acting pointwise on any real-valued function \( g \) on \([0, 1]\) as

\[
\Delta_h g(t) = \begin{cases} 
  g(t + h) - g(t) & \text{if } 0 \leq t \leq 1 - h \\
  0 & \text{otherwise.}
\end{cases}
\]

For any \( 2 \leq r \in \mathbb{N} \), \( \Delta_h^r = \Delta_h \circ \Delta_h^{r-1} \) inductively defines its \( r \)-fold composition and if \( |g|^p \) is Lebesgue integrable, \( p \geq 1 \),

\[
\omega_r(g, t, p) = \sup_{h \in [0, t]} \| \Delta_h^r g \|_{L_p}
\]

denotes the \( r \)-th modulus of smoothness in the Lebesgue space \( L_p \). For any \( s > 0 \) and \( 1 \leq q \leq \infty \), the Besov space \( B_s^q \) is given as

\[
B_s^q = \{ f \in L_p([0, 1]) : \| f \|_{B_s^q} < \infty \}, \quad \text{for } 1 \leq p < \infty,
\]

and with \( C[0, 1] \) denoting the real-valued continuous functions on the unit interval,

\[
B_s^{\infty q} = \{ f \in C([0, 1]) : \| f \|_{B_s^{\infty q}} < \infty \}.
\]

Here,

\begin{equation}
(2.1) \quad \| f \|_{B_s^q} = \begin{cases} 
\| f \|_{L_p} + \left( \sum_{j=0}^{\infty} \left[ 2^{js} \omega_r(f, 2^{-j}, p) \right]^q \right)^{1/q} & \text{if } 1 \leq q < \infty, \\
\| f \|_{L_p} + \sup_{j \geq 0} \left[ 2^{js} \omega_r(f, 2^{-j}, p) \right] & \text{if } q = \infty,
\end{cases}
\end{equation}

defines the Besov norm, where \( r = \lceil s \rceil \) is the smallest integer strictly larger than \( s \). Note that by classical Besov space embeddings (cf. Giné and Nickl (2016, Prop. 4.3.9)) for \( p \leq 2 \) and Jensen’s inequality for \( p > 2 \), the relation \( s > (1/p - 1/2)_+ \) reveals that \( B^{pq}_s \subset L_2 \).

For the scaling function \( \phi = \psi_{-10} = \mathbb{1}_{(0,1]} \) with wavelet \( \psi = \mathbb{1}_{(0,1/2]} - \mathbb{1}_{(1/2,1]} \), define \( \psi_{jk} = 2^{j/2}\psi(2^j \cdot -k) \) for \( j \in \mathbb{N} \cup \{0\}, k \in \{0, 1, \ldots, 2^j - 1\} \). The corresponding family

\[
\{ \psi_{-10}, \psi_{jk} : j \in \mathbb{N} \cup \{0\}, k \in \{0, 1, \ldots, 2^j - 1\} \}
\]
defines the orthonormal Haar wavelet basis of the Hilbert space \( L_2 \). Throughout, we will describe the regularity of the Lebesgue density \( f \) by its membership in an appropriate Besov ball. For \( L > 0 \),

\[
\mathcal{P}^{pq}_s(L) = \left\{ f : [0,1] \to \mathbb{R} : f \geq 0, \int_0^1 f(x) \, dx = 1, \|f\|_{B^{pq}_s} \leq L \right\}
\]
denotes the subset of Lebesgue probability densities on the unit interval within the centered Besov ball of radius \( L \). For any \( f \in \mathcal{P}^{pq}_s(L) \) with \( s > (1/p - 1/2)_+ \), an application of Parseval’s identity reveals the representation

\[
\int_0^1 f(x)^2 \, dx = \sum_{j \geq -1} \sum_{k=0}^{(1\vee 2^j)-1} \langle f, \psi_{jk} \rangle^2 = \sum_{j \geq -1} \sum_{k=0}^{(1\vee 2^j)-1} \beta_{jk}^2
\]

with the wavelet coefficients \( \beta_{jk} = \beta_{jk}(f) = \langle f, \psi_{jk} \rangle_{L_2} \). Note that for general parameter constellations \( p, q, s \), the Besov spaces cannot be defined equivalently in terms of Haar wavelet coefficient norms. Nevertheless, the sequences \( (\beta_{jk})_{k=0,\ldots,2^j-1} \) of above introduced coefficients satisfy the following relation with respect to the modulus of smoothness. For any \( 1 \leq p \leq \infty \), there exists some constant \( C_p > 0 \), such that for any \( f \in B^{pq}_s \) with \( s < 1 \),

\[
2^{j(1/2-1/p)} \|\beta_j(f)\|_{\ell_p} \leq C_p \omega_1(f, 2^{-j}, p)
\]

for \( j \geq 0 \), see Devore, Jawerth and Popov (1992).

2.2. Interactive and non-interactive differential privacy. Recall that for \( \alpha \in (0,1] \), a channel distribution \( Q \) is called \( \alpha \)-differentially private, if

\[
\sup_{A \in \mathbb{R}^n} \sup_{x, x' \in [0,1]^n} \frac{Q(A|x)}{Q(A|x')} \leq e^\alpha.
\]
where \( d_0(x, x') := |\{i : x_i \neq x'_i\}| \) is the Hamming distance between \( x \) and \( x' \).

Note that for this definition to make sense, the probability measures \( Q(\cdot|x) \), for different \( x \in [0, 1]^n \), have to be equivalent and we interpret \( 0_0 \) as equal to 1.

Next, we introduce two specific classes of locally differentially private channels. A channel distribution \( Q : (\otimes_{i=1}^n G_i) \times [0, 1]^n \rightarrow [0, 1] \) is said to be \( \alpha \)-sequentially interactive (or provides \( \alpha \)-sequentially interactive differential privacy) if the following two conditions are satisfied. First, we have for all \( A \in \otimes_{i=1}^n G_i \) and \( x_1, \ldots, x_n \in [0, 1] \),

\[
Q(A | x_1, \ldots, x_n) = \int_{Z_1} \cdots \int_{Z_n} Q_n(A_{z_{1:n-1}} | x_n, z_{1:n-1}) Q_{n-1}(d_{z_{n-1}} | x_{n-1}, z_{1:n-2}) \cdots Q_1(d_{z_1} | x_1),
\]

where, for each \( i = 1, \ldots, n \), \( Q_i \) is a channel from \([0, 1] \times \otimes_{j=1}^{i-1} G_j \) to \( Z_i \).

Here, \( z_{1:n} = (z_1, \ldots, z_n)^T \) and \( A_{z_{1:n-1}} = \{ z \in Z_n : (z_1, \ldots, z_{n-1}, z)^T \in A \} \) is the \( z_{1:n-1} \)-section of \( A \). Second, we require that the conditional distributions \( Q_i \) satisfy

\[
\sup_{A \in \otimes_{i=1}^n G_i} \sup_{x_i, x'_i, z_1, \ldots, z_{i-1}} Q_i(A | x_i, z_1, \ldots, z_{i-1}) \leq e^\alpha \quad \forall i = 1, \ldots, n.
\]

By the usual approximation of integrands by simple functions, it is easy to see that (2.5) and (2.6) imply (2.4). This notion coincides with the definition of sequentially interactive channels in Duchi, Jordan and Wainwright (2018) and Rohde and Steinberger (2020). We note that (2.6) only makes sense if for all \( x_i, x'_i, z_1, \ldots, z_{i-1}, \) the probability measure \( Q_i(\cdot|x_i, z_{1:i-1}) \) is absolutely continuous with respect to \( Q_i(\cdot|x'_i, z_{1:i-1}) \). Here, the idea is that individual \( i \) can only use \( X_i \) and previous \( Z_j, j < i \), in its local privacy mechanism, thus leading to the sequential structure in the above definition. In the rest of the paper we only consider \( \alpha \)-sequentially interactive channels, which we sometimes simply call \( \alpha \)-private channels.

An important subclass of sequentially interactive channels are the so called non-interactive channels \( Q \) that are of product form

\[
Q(A_1 \times \cdots \times A_n | x_1, \ldots, x_n) = \prod_{i=1}^n Q_i(A_i | x_i), \quad \forall A_i \in G_i, x_i \in [0, 1].
\]
In that case it is also called $\alpha$-non-interactive. Both, $\alpha$-non-interactive and $\alpha$-sequentially interactive channels satisfy the $\alpha$-local differential privacy constraint as defined in the introduction. Of course, every $\alpha$-non-interactive channel is also $\alpha$-sequentially interactive.

2.3. Locally, differentially private minimax risk. For a fixed channel distribution $Q$ from $([0,1]^n, \mathcal{B}([0,1]^n))$ to $(Z, \mathcal{G})$, the minimax risk of the above estimation problem is given by

$$\mathcal{M}_n(Q, \mathcal{P}_s^{pq}) = \inf_{\hat{D}_n} \sup_{f \in \mathcal{P}_s^{pq}} \mathbb{E}_{Q \in \mathcal{P}_s^{pq}} \left[ (\hat{D}_n - D(f))^2 \right],$$

where the infimum runs over all estimators $\hat{D}_n : Z \to \mathbb{R}$. Next, define the set of $\alpha$-non-interactive channels

$$Q^{(NI)}_\alpha := \bigcup_{(Z, \mathcal{G})} \{ Q : Q \text{ is } \alpha\text{-non-interactive from } [0,1]^n \text{ to } Z \},$$

where the union runs over all $n$-fold product spaces, and the set of $\alpha$-sequentially interactive channels

$$Q^{(SI)}_\alpha := \bigcup_{(Z, \mathcal{G})} \{ Q : Q \text{ is } \alpha\text{-sequentially interactive from } [0,1]^n \text{ to } Z \}.$$

Clearly, $Q^{(NI)}_\alpha \subseteq Q^{(SI)}_\alpha$. Therefore, we distinguish the $\alpha$-private minimax risks

$$\mathcal{M}^{(NI)}_{n,\alpha}(\mathcal{P}_s^{pq}) = \inf_{Q \in Q^{(NI)}_\alpha} \mathcal{M}_n(Q, \mathcal{P}_s^{pq})$$

and

$$\mathcal{M}^{(SI)}_{n,\alpha}(\mathcal{P}_s^{pq}) = \inf_{Q \in Q^{(SI)}_\alpha} \mathcal{M}_n(Q, \mathcal{P}_s^{pq}).$$

Note that the above infima include all possible product spaces $(Z, \mathcal{G})$.

In the sequel we will derive upper and lower bounds on both of these minimax risks (for appropriate subsets of $\mathcal{P}_s^{pq}$). In each case, we will also present an explicit construction of a locally private estimation procedure that attains the lower bound (up to logarithmic factors).
2.4. Further notation. We write $a \lor b = \max(a, b)$ and $a \land b = \min(a, b)$. Throughout, $C, C_0, c$ are positive finite constants that do neither depend on sample size $n$ nor on an unknown parameter $f$, but might depend on $s, p, q, L$ or other constants used to describe the parameter space for $f$, and might change from one occurrence to another. We sometimes write $a \lesssim b$ to mean $a \leq C \cdot b$, for a finite constant $C > 0$ that does not depend on $n$, $f$ and $\alpha$. Finally, $a \asymp b$ means that $a \lesssim b$ and $b \lesssim a$.

3. Non-interactive privacy protocols. In this section we present an $\alpha$-non-interactive privacy mechanism and subsequent estimator for the quadratic functional $D(f) = \int_0^1 f^2(x) \, dx$ and establish its minimax optimality within the class $Q^{(NI)}_\alpha$ of all $\alpha$-non-interactive procedures.

3.1. Upper bounds. We first propose a non-interactive privacy mechanism, related to the one of Butucea et al. (2020), that is based on adding Laplace noise to empirical wavelet coefficients. The subsequent estimator is a standard U-statistic of order 2.

Let us define the following privacy mechanism using the Haar basis generated by $(\phi, \psi)$, with $\phi(x) = \mathbb{1}_{(0,1]}(x)$ and $\psi(x) = \mathbb{1}_{(0,1]}(x) - \mathbb{1}_{(1/2,1]}(x), x \in \mathbb{R}$. Fix $\alpha > 0$, $a > 1$ and $J \in \mathbb{N}$. Given its original data $X_i$, individual $i$ generates a random array $Z_i$ with $(j, k)$-th component

$$Z_{ijk} = \psi_{jk}(X_i) + \sigma_j \cdot \frac{\sigma}{\alpha} \cdot W_{ijk}, \quad j = -1, \ldots, J - 1, k = 0, \ldots, \lfloor 2^j - 1 \rfloor,$$

where $\sigma_{-1} = 1, \sigma_j = (1 \lor j)^a 2^j/2$ for $j \geq 0$, and $\sigma = 4 + 2 \sum_{j=1}^\infty \frac{1}{2^j}$. Moreover, $W_{ijk}$ are i.i.d. Laplace distributed with density $f_W(x) = \frac{1}{2} \exp(-|x|)$. Note that $W_{ijk}$ are all centered, with variance 2. We write $Q^{(NI)}$ for the conditional distribution (Markov kernel, channel distribution) of $(Z_1, \ldots, Z_n)$ given $(X_1, \ldots, X_n)$. In particular, the channel $Q^{(NI)}$ is non-interactive. The following result establishes that $Q^{(NI)} \in Q^{(NI)}_\alpha$. Its proof is deferred to Section A.1 in the supplement (Butucea, Rohde and Steinberger, 2022).

**Proposition 3.1.** For any $J \in \mathbb{N}$ and $a > 0$, $\alpha > 0$, the privacy mechanism $Q^{(NI)}$ defined in (3.1) is $\alpha$-non-interactive.

We shall use the notation

$$\hat{\beta}_{jk} = \frac{1}{n} \sum_{i=1}^n Z_{ijk}, \quad j = -1, \ldots, J - 1, k = 0, \ldots, (1 \lor 2^j) - 1,$$
with \( \hat{\beta}_{-1,0} \) also called \( \hat{\alpha}_{00} \). Since \( W_{ijk} \) are i.i.d. centered, with variance 2, we get for \( j = -1, \ldots, J - 1, k = 0, \ldots, (1 \lor 2^j) - 1 \):

\[
\mathbb{E}(\hat{\beta}_{jk}) = \beta_{jk} \quad \text{and} \quad \text{Var}(\hat{\beta}_{jk}) = \frac{1}{n} \left( \text{Var}(\psi_{jk}(X_1)) + 2\sigma_f^2 \cdot \frac{\sigma^2}{\alpha^2} \right).
\]

Finally, let us define the private estimator \( \hat{D}_n \) of \( D = D(f) \), by

\[
(3.3) \quad \hat{D}_n = \frac{1}{n(n-1)} \sum_{i\neq h} \sum_{j=-1}^{J-1} \sum_{k=0} \left( \land 2^j \right) - 1 Z_{ijk} \cdot Z_{hjk}.
\]

We are now in the position to formulate our first main result on the risk of \( \hat{D}_n \). Its proof is deferred to Section A.2 of the supplementary material (Butucea, Rohde and Steinberger, 2022).

**Theorem 3.2.** For finite constants \( L, M_2, M_3 > 0, 1 \leq p, q \leq \infty \) and \( s > \left( \frac{1}{2} - \frac{1}{2^j} \right)_+ \), consider \( \mathcal{P}_p^q(L, M_2, M_3) = \mathcal{P}_p^q(L) \cap \{ f \in L_3([0,1]) : \| f \|_{L_2} \leq M_2, \| f \|_{L_3} \leq M_3 \} \). Put \( s' = s - \left( \frac{1}{p} - \frac{1}{2} \right)_+ \). Then, for every \( n \in \mathbb{N} \) and \( \alpha \in (0,1] \), with \( n\alpha^2 > 1 \), the estimator \( \hat{D}_n \) with \( J = J_n \) given by

\[
2^J_n = \begin{cases} \left( \frac{\alpha^2}{(\log(n\alpha^2))^{4a+1}} \right)^{\frac{1}{3}}, & s' > \frac{3}{4}, \\ \left( n\alpha^2 \right)^{\frac{2}{4a+3}}, & 0 < s' \leq \frac{3}{4}, \end{cases}
\]

verifies

\[
\sup_{f \in \mathcal{P}_p^q(L, M_2, M_3)} \mathbb{E}_{\mathcal{Q}_f} \left[ \left| \hat{D}_n - D(f) \right|^2 \right] \lesssim \tau_n^{(NI)}(\alpha, a, s'),
\]

where

\[
\tau_n^{(NI)}(\alpha, a, s') = \begin{cases} \frac{1}{n\alpha^2}, & s' > \frac{3}{4}, \\ \left( \log(n\alpha^2) \right)^{4a+1}(n\alpha^2)^{-\frac{s'}{4a+3}}, & 0 < s' \leq \frac{3}{4}. \end{cases}
\]

### 3.2. Lower bounds.

We now show that the rate of the non-interactive U-statistics approach introduced in the previous subsection is indeed optimal for estimating the quadratic functional within the class of all \( \alpha \)-non-interactive procedures. See Section A.3 in the supplementary material (Butucea, Rohde and Steinberger, 2022) for the proof of the following theorem.
Theorem 3.3. Fix \( n \in \mathbb{N}, \alpha \in (0, \infty), s \in (0, 1), p \geq 2, q \geq 1, L > 1, M \geq 2 \) and consider the class \( \mathcal{P}_{s}^{pq}(L, M) := \{ f \in \mathcal{P}_{s}^{pq}(L) : \| f \|_{\infty} \leq M \} \). Define \( z_{\alpha} := e^{2\alpha} - e^{-2\alpha} \). If \( nz_{\alpha}^{2} \geq 2 \), then there exists a constant \( c > 0 \), not depending on \( n \) and \( \alpha \), such that

\[
\inf_{Q \in Q_{\alpha}^{(NI)}} \inf_{D_{n} \in \mathcal{P}_{s}^{pq}(L, M)} \sup_{f \in \mathcal{P}_{s}^{pq}(L)} \mathbb{E}_{Q_{n}} \left[ \left| \hat{D}_{n} - D(f) \right|^{2} \right] \geq \frac{c}{\left( \log(nz_{\alpha}^{2}) \right)^{2}} \left( nz_{\alpha}^{2} \right)^{-\frac{8s}{4s+3}}.
\]

Here, the set \( Q_{\alpha}^{(NI)} \) contains all \( \alpha \)-non-interactive channels.

Remark 3.4. Since \( \frac{1}{2}(e^{2\alpha} - e^{-2\alpha}) \leq e^{2\alpha} - 1 \), we immediately get the slightly smaller lower bound

\[
c' \left[ n(e^{2\alpha} - 1)^{2} \right]^{-\frac{8s}{4s+3}} \left( \log \left[ n(e^{2\alpha} - 1)^{2} \right] \right)^{-2},
\]

which, for bounded \( \alpha \), reduces to an expression in terms of the more familiar quantity \( na^{2} \), i.e.,

\[
c'' \left( na^{2} \right)^{-\frac{8s}{4s+3}} \left( \log(na^{2}) \right)^{-2}.
\]

Theorem 3.3 shows that the rate obtained in Theorem 3.2 is indeed optimal (up to logarithmic factors), at least in the case \( p \geq 2 \), that is, \( s' = s \).

Finally, we note that one can easily deduce a lower bound of the form \( c(na^{2})^{-1} \), even for the larger class \( Q_{\alpha}^{(SI)} \supseteq Q_{\alpha}^{(NI)} \) and over general Besov classes \( \mathcal{P}_{s}^{pq}(L, M) \), \( s > 0, 1 \leq p, q \leq \infty \), using Corollary 3.1 of Rohde and Steinberger (2020). To that end, we only need to lower bound the modulus of continuity of the quadratic functional w.r.t. the total variation distance, that is,

\[
\omega_{TV}(\varepsilon) := \sup \left\{ \| D(f_{0}) - D(f_{1}) \| : \frac{1}{2} \int | f_{0} - f_{1} | \leq \varepsilon, f_{0}, f_{1} \in \mathcal{P}_{s}^{pq}(L, M) \right\},
\]

by an expression of order \( \varepsilon \), because a minimax lower bound is of the form \( c_{0} [\omega_{TV}(c_{1}(na^{2})^{-1/2})]^{2} \). But this can easily be done for \( \varepsilon \in (0, 1] \), by choosing \( f_{0} \equiv 1 \) and \( f_{1}(x) = f_{0}(x) + \delta g(x/\varepsilon) \), for some non-trivial \( g \in B_{s}^{pq} \) with \( \int_{0}^{1} g(x)dx = 0 \) and \( \| g \|_{\infty} < \infty \), and for \( 0 < \delta \leq \sqrt{(L-1)/\| g \|_{\infty}} \| g \|_{B_{s}^{pq}} \). This choice implies that \( f_{1}(x) \geq 0, \| f_{1} \|_{B_{s}^{pq}} \leq 1 + \varepsilon \delta \| g \|_{B_{s}^{pq}} \leq L, \| f_{1} \|_{\infty} \leq 1 + \delta \| g \|_{\infty} \leq 2 \leq M, \| D(f_{0}) - D(f_{1}) \| = \varepsilon \delta^{2} \| g \|_{2}^{2} \) and \( \int | f_{0} - f_{1} | = \varepsilon \delta \| g \|_{1} \). Thus, \( \omega_{TV}(\varepsilon \delta \| g \|_{1}/2) \geq \varepsilon \delta^{2} \| g \|_{2}^{2} \).
4. Sequentially interactive privacy protocols. In Section 3 we have presented an $\alpha$-non-interactive procedure for estimating the quadratic functional $D = \int_0^1 f^2(x)dx$ and established its minimax optimality within the class of all $\alpha$-non-interactive procedures. If we leave this class, however, and also allow for sequential interaction between data owners, then we can improve substantially over the rate of the best non-interactive procedure. In the present section we pursue such improvements and prove their optimality.

4.1. Upper bounds. We first provide a concrete example of a locally private estimation procedure which relies on some sequential communication between individual data providers and which achieves a faster convergence rate than that of Section 3.

For convenience, we assume that the sample size is $2n$ and we split the data providing individuals into two groups of size $n$, such that the first group holds data $X^{(1)} = (X_1^{(1)}, \ldots, X_n^{(1)})$ and the second group holds the data $X^{(2)} = (X_1^{(2)}, \ldots, X_n^{(2)})$. Now, the individuals owning the data $X^{(1)}$ use the non-interactive privacy mechanism (3.1), which is based on the Haar wavelets, to generate arrays $Z_i = Z_i^{(1)}$ based on their private information $X_i^{(1)}$. We write $Z^{(1)} = (Z^{(1)}_1, \ldots, Z^{(1)}_n)$. These sanitized data are now used to estimate the unknown data generating density $f \in P_{pq}^s(L)$ at a point $x \in [0,1]$, by (cf. Butucea et al., 2020)

\begin{equation}
\hat{J}^{(1)}_J(x) := \sum_{j=-1}^{J-1} \sum_{k=0}^{(1\vee 2^j)-1} \hat{\beta}_{jk} \psi_{jk}(x),
\end{equation}

with $\hat{\beta}_{jk}$ as in (3.2), i.e.,

$$\hat{\beta}_{jk} = \frac{1}{n} \sum_{i=1}^{n} Z_{ijk}, \quad j = -1, \ldots, J - 1, \quad k = 0, \ldots, (1 \vee 2^j) - 1.$$

Now, in order to privately estimate the quadratic functional $D = D(f) = \int_0^1 f^2(x)dx$, we instead privately estimate the (random) linear functional

$$f \mapsto \int_0^1 \hat{J}^{(1)}_J(x)f(x)dx.$$

This second step is carried out using the rate optimal mechanism of Rohde and Steinberger (2020), that is, for some tuning parameter $\tau > 0$ and for
given $Z^{(1)}$ (or $j^{(1)}_J$), each individual from the second group independently generates $Z^{(2)}_i$ by

$$Z^{(2)}_i = \begin{cases} \tau^{\alpha_i+1}e^{-\alpha_i}, & \text{with probability } \frac{1}{2} \left( 1 + \frac{\Pi_\tau f^{(1)}_J(X^{(2)}_i)}{\tau^{\alpha_i+1}e^{-\alpha_i}} \right), \\ \tau^{\alpha_i+1}e^{-\alpha_i}, & \text{with probability } \frac{1}{2} \left( 1 - \frac{\Pi_\tau f^{(1)}_J(X^{(2)}_i)}{\tau^{\alpha_i+1}e^{-\alpha_i}} \right), \end{cases}$$

where $\Pi_\tau[y] = (\tau \wedge y) \vee (-\tau)$. Write $Z^{(2)} = (Z^{(2)}_1, \ldots, Z^{(2)}_n)$. Note that the projection of $f^{(1)}_J(X^{(2)}_i)$ onto $[-\tau, \tau]$ ensures that the probabilities belong to $[0, 1]$. Moreover, notice that we have $E[Z^{(2)}_i | Z^{(1)}_i, X^{(2)}_i] = \Pi_\tau f^{(1)}_J(X^{(2)}_i)$ and $E[Z^{(2)}_i | Z^{(1)}_i] = \int_0^1 \Pi_\tau f^{(1)}(x) f(x) dx \to \int_0^1 f^{(1)}(x) f(x) dx$ as $\tau \to \infty$. Our final estimator is then given by

$$\tilde{D}_n = \tilde{D}_{n,\tau} = \frac{1}{n} \sum_{i=1}^n Z^{(2)}_i.$$

We denote the above mechanism that outputs $(Z^{(1)}, Z^{(2)})$, given original data $(X^{(1)}, X^{(2)})$, by $Q^{(SI)}$. It clearly has a sequential structure because each $Z^{(2)}_i$ in the second group depends on the sanitized data $Z^{(1)}$ from the first group through $f^{(1)}_J$, but on none of the other $Z^{(2)}_j, j \neq i$. It is also easy to see that it satisfies (2.6) and hence, it is $\alpha$-sequentially interactive, i.e., $Q^{(SI)} \in Q^{(SI)}_\alpha$. The following theorem presents an upper bound on the risk of the estimation method proposed in (4.3). Its proof is deferred to Section B.2 of the supplement.

**Theorem 4.1.** Fix $M, L > 0$, $1 \leq p, q \leq \infty$ and $s > \left( \frac{1}{p} - \frac{1}{2} \right)_+$ and consider the Besov class $\mathcal{F}_{s}^{pq}(L,M) := \{ f \in \mathcal{F}_{s}^{pq}(L) : \|f\|_{\infty} \leq M \}$. Define $s' = s - \left( \frac{1}{p} - \frac{1}{2} \right)_+$. For $n \in \mathbb{N}$, $\alpha \in (0, 1]$, consider the estimator $\tilde{D}_n$ defined in (4.3) based on the private wavelet estimator $j^{(1)}_J$ in (4.1), with cut-off $\tau^{2} = [K^2 M^2 (1 + J^{2a+1} 2^{J(1-2(s'+1/2))})] \lor 1$, for a sufficiently large constant $K \geq 2$ (that can be chosen independently of $n$ and $\alpha$) and for $J = J_n$ such that $2^{J_n} = (n\alpha^2)^{\frac{1}{2(p'+1)}}$, where $a > 1$ is the constant from the privacy mechanism (3.1). Then,

$$\sup_{f \in \mathcal{F}_{s}^{pq}(L,M)} \mathbb{E}_{Q_{f}^{(SI)}} \left[ \left| \tilde{D}_n - D(f) \right|^2 \right] \lesssim \alpha^{(SI)}_n(\alpha, a, s').$$
with
\[
\tau_n^{(SI)}(\alpha, s') = \begin{cases} 
\frac{1}{n\alpha^2}, & s' > \frac{1}{2} \\
(\log(n\alpha^2))2\alpha+1(n\alpha^2)^{-\frac{s'}{2\alpha+1}}, & s' \leq \frac{1}{2}, 
\end{cases}
\]
provided that \(n\alpha^2 > c_0\), for a finite constant \(c_0 > 0\) that does not depend on \(n\) and \(\alpha\).

Theorem 4.1 shows that faster rates than those of Section 3 can be attained using a sequentially interactive privacy mechanism. Indeed, the elbow effect occurs at the value \(s' = \frac{1}{2}\) instead of \(s' = \frac{3}{4}\) in Theorem 3.2, and in case \(s' \leq \frac{1}{2}\) we have that
\[
(n\alpha^2)^{-\frac{4s'}{2\alpha+1}}/\log(n\alpha^2)^{\frac{s'}{2\alpha+1}} \to 0, \quad \text{as } n\alpha^2 \to \infty.
\]
Intuitively, a sequentially interactive privacy mechanism increases the information that the sanitized sample contains about the unknown parameter of interest. However, that this additional information can be exploited to obtain faster rates than those of non-interactive procedures cannot be observed for the problem of density estimation in \(L_r\) or of estimating linear functionals of the density (cf. Butucea et al., 2020; Rohde and Steinberger, 2020).

4.2. Lower bounds. In this subsection we show that the rate of the sequentially interactive procedure introduced in Subsection 4.1 is indeed optimal. See Section B.3 in the supplement (Butucea, Rohde and Steinberger, 2022) for the proof of the following theorem.

**Theorem 4.2.** Fix \(n \in \mathbb{N}, \alpha \in (0, \infty), s \in (0, 1), p, q \in [1, \infty], L > 1, M \geq 2\) and let the class \(\overline{P}_{p,q}^s(L, M)\) be defined as in Theorem 4.1. Define \(z_\alpha := e^{2\alpha} - e^{-2\alpha}\). Then, if \(nz_\alpha^2 \geq 1\), there exists a constant \(c > 0\) not depending on \(n\) and \(\alpha\), such that
\[
\inf_{Q \in \mathcal{Q}_{n}^{(SI)}} \inf_{\hat{D}_n, f \in \overline{P}_{p,q}^s(L, M)} \sup \mathbb{E}_{Q \in \mathcal{Q}_{n}^{(SI)}} \left[ (\hat{D}_n - D(f))^2 \right] \geq c \left[ nz_\alpha^2 \right]^{-\frac{4s'}{2\alpha+1}}.
\]
In view of Remark 3.4, the lower bound can further be bounded from below by \(c' [n\alpha^2]^{-\frac{4s'}{2\alpha+1}}\), provided that \(\alpha\) is bounded. Theorem 4.2 shows that the rate in Theorem 4.1 is optimal, at least in the regime where \(s = s'\), that is, \(p \geq 2\), and up to log factors. Recall that in the argument following Theorem 3.3 we have already established the parametric lower bound of order \((n\alpha^2)^{-1}\).
5. Applications. Next, we discuss two common applications where estimation of the quadratic functional plays an important role: estimating more general integral functionals and goodness-of-fit testing.

5.1. Integral functionals of the density. Suppose we want to estimate other integral functionals $T(f) = \int \phi(f(x)) dx$ of the bounded density $f$, such as, for example, the entropy $\int f(x) \log(f(x)) dx$. If $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is three times continuously differentiable, we can follow ideas of Birgé and Massart (1995) (see also Giné and Nickl, 2016, Section 5.3.1), and perform a Taylor expansion of $\phi$ at a suitable preliminary estimator followed by successive estimation of the resulting linear and quadratic functionals. More specifically, let $\hat{f}_n$ be a preliminary estimator of $f$, based on a subset $X^{(1)}$ of the whole sample and corresponding sanitized data $Z^{(1)}$, and write

$$\int_0^1 \phi(f) = \int_0^1 \left[ \phi(\hat{f}_n) + \phi'(\hat{f}_n)(f - \hat{f}_n) + \frac{1}{2} \phi''(\hat{f}_n)(f - \hat{f}_n)^2 \right] + G_n$$

$$= \int_0^1 \left[ \phi(\hat{f}_n) - \phi'(\hat{f}_n)\hat{f}_n + \frac{1}{2} \phi''(\hat{f}_n)\hat{f}_n^2 \right]$$

$$+ \int_0^1 f \cdot \left[ \phi'(\hat{f}_n) - \phi''(\hat{f}_n)\hat{f}_n \right] + \frac{1}{2} \int_0^1 f^2 \cdot \phi''(\hat{f}_n) + G_n,$$

where $|G_n| \leq \frac{1}{\|\phi''\|_\infty} \int |f - \hat{f}_n|^3$. Now, it remains to plug in optimal estimators of the linear and quadratic integral functionals $f \mapsto \int_0^1 f \cdot \psi_1$ and $f \mapsto \int_0^1 f^2 \cdot \psi_2$, for known functions $\psi_1$ and $\psi_2$, constructed with the remaining data sample $X^{(2)}$.

First note that according to Rohde and Steinberger (2020), the rate for $\alpha$-privately estimating the linear functional $f \mapsto \int_0^1 f \cdot \psi_1$ over a convex parameter space, is $(n\alpha^2)^{-1/2}$, provided that the function $x \mapsto \psi_1(x) := \phi'(\hat{f}_n(x)) - \phi''(\hat{f}_n(x))\hat{f}_n(x)$ is bounded on $(0, 1)$. Hence, estimating the linear term in the expansion (5.1) will never dominate the rate.\(^2\)

Next, for the preliminary estimator $\hat{f}_n$ based on sanitized data $Z^{(1)}$, let us consider the minimax adaptive estimator in Butucea et al. (2020), which has the property that, for privacy level $\alpha \in (0, 1]$ and $r \geq 1$,

$$\sup_{f \in \mathcal{P}_n^r} \mathbb{E}_{Q^* \mathbb{P}_n^1} \left[ \int_0^1 |f - \hat{f}_n|^r \right] \lesssim (\log n)^C \left\{ \begin{array}{ll}
(n\alpha^2)^{-\frac{r}{2}+\frac{1}{p}}, & s > \frac{r}{p} - 1, \\
\left(\frac{n\alpha^2}{\log(n\alpha^2)}\right)^{-\frac{r(s-1/p+1/s)}{2(s-1/p+1/s)}}, & \frac{1}{p} < s \leq \frac{r}{p} - 1.
\end{array} \right.$$

\(^2\)In case of a functional like the entropy, $\phi(f) = f \log(f)$, where $\phi'$ is unbounded on $(0, 1)$, one usually assumes that both $f$ and $\hat{f}_n$ are bounded from below by some positive constant.
where $Q^*$ is the optimal adaptive non-interactive channel of Butucea et al. (2020) that generates $Z^{(1)}$ from original data $X^{(1)}$ and does not depend on knowledge of $s$. This non-interactive procedure is actually shown to be rate optimal even among all sequentially interactive privacy mechanisms. For simplicity, here we ignore logarithmic terms in all the rates and only consider the case $p \geq 2$, which implies that only the first of the two regimes above occurs and that $s' = s$.

Had we done only a first order expansion instead of (5.1), then the remainder term would dominate and the resulting private estimator would converge at a rate of $(n\alpha^2)^{-\frac{1}{2}} \lor (n\alpha^2)^{-\frac{4s}{4s+3}}$. In view of our results in Section 4.1, however, the quadratic functional can be estimated at a rate of $(n\alpha^2)^{-\frac{1}{2}} \lor (n\alpha^2)^{-\frac{4s}{4s+3}}$ and the remainder term $G_n$ in (5.1) converges at the rate $(n\alpha^2)^{-\frac{3}{2s+1}}$, both of which are always faster than the rate of the first order expansion. Thus, the expansion (5.1) improves over the first order expansion. Furthermore, if $s \geq 1/2$, then both, $G_n$ and the quadratic functional estimate converge at the parametric rate and a higher order expansion would not improve the overall rate any further. If, on the other hand, $s < 1/2$, then we might be able to improve the rate further by considering a third order expansion.

However, if we restrict to non-interactive privacy mechanisms, then the quadratic functional can only be estimated at the rate $(n\alpha^2)^{-\frac{1}{2}} \lor (n\alpha^2)^{-\frac{4s}{4s+3}}$ (cf. Section 3) and this is always worse than the rate of the remainder term $G_n$. Thus, further expansion of $\phi$ to fourth or higher order can not improve the rate in the non-interactive case.

Hence, in some cases, our rates for estimating the quadratic functional already determine the rates for the estimation of much more general integral functionals $T(f) = \int \phi(f(x)) dx$ with three times continuously differentiable $\phi$. This is in contrast with the direct case when $X_1, ..., X_n$ are observed, where both, the quadratic and the cubic functional can be estimated at the rate $(n\alpha^2)^{-\frac{1}{2}} \lor (n\alpha^2)^{-\frac{4s}{4s+3}}$ and the remainder term $G_n$ converges at the rate $(n\alpha^2)^{-\frac{3s}{4s+3}}$. Thus, the second order expansion is always dominated by the remainder term (for $s < 1/4$ the remainder term converges strictly slower) and a third order expansion will be more efficient in terms of rate. Due to the inverse problem that local differential privacy introduces, the cubic term is not always necessary in the private setting.

5.2. **Goodness-of-fit tests.** The most frequent application of our results is goodness-of-fit testing for the underlying density $f$. Due to the regularizing
properties of the $L_2$ norm, testing rates are usually faster than estimation rates of $f$ (with pointwise or integrated risks). The nonparametric test problem writes $H_0 : f \equiv f_0$ for fixed, given $f_0$ in $\mathcal{P}_s^{pq}$, against the alternative

$$H_1(f_0, C\varphi_n) : f \in \mathcal{P}_s^{pq}, \quad \|f - f_0\|_2 \geq C\varphi_n,$$

for some constant $C > 0$ and sequence $\varphi_n$ of real numbers decreasing to 0. In the context of local differential privacy, test procedures $\Delta_n$ will be defined as measurable functions of the sanitized sample $Z_1, \ldots, Z_n$, which is generated from the privacy mechanism $Q \in Q_\alpha \subseteq Q_\alpha^{(SI)}$. The risk measure of a test procedure for a given privacy mechanism is defined by

$$\mathcal{T}_n(Q, \Delta_n, C\varphi_n) := Q_{P_{f_0}}(\Delta_n = 1) + \sup_{f \in H_1(f_0, C\varphi_n)} Q_{P_f}(\Delta_n = 0).$$

Let $\gamma$ belong to $(0,1)$. We say that a test procedure $\Delta_n$ associated to a privacy mechanism $Q$ attains the testing rate $\varphi_n$ if, for a constant $C > 0$,

$$\limsup_{n \to \infty} \mathcal{T}_n(Q, \Delta_n, C\varphi_n) \leq \gamma.$$

This rate is the minimax rate of testing among all $\alpha$-sequentially interactive procedures if, for some $0 < C^* < C$,

$$\liminf_{n \to \infty} \inf_{Q \in Q_\alpha} \inf_{\Delta_n} \mathcal{T}_n(Q, \Delta_n, C\varphi_n) \geq \gamma > 0.$$

We distinguish the cases of non-interactive privacy mechanisms $Q_\alpha = Q_\alpha^{(NI)}$ and of sequentially interactive privacy mechanisms $Q_\alpha = Q_\alpha^{(SI)}$.

It is known in the direct case (when $X_1, \ldots, X_n$ are observed) that the optimal test is based on the optimal estimator of the quadratic functional $\|f - f_0\|_2^2$. Instead of a plug-in procedure, the test statistic is based on optimal estimators of $D(f) = \|f\|_2^2$ and of the linear functional $L = \int_0^1 f_0 f$. We already mentioned that for bounded $f_0$ in $\mathcal{P}_s^{pq}$, the linear functional $L$ can be estimated by $\hat{L}_n$ at rate $(n\alpha^2)^{-1/2}$ via an estimator based on a non-interactive privacy mechanism (see Rohde and Steinberger, 2020), therefore the rates will be driven by the estimator of the quadratic functional $D(f)$. The test procedure $\Delta_n^{(NI)} = 1$, iff $\hat{D}_n^{(NI)} - 2\hat{L}_n + \|f_0\|_2^2 > C\varphi_n^{(NI)}$, where $\hat{D}_n^{(NI)}$ is the procedure in (3.3) with $2^I = (n\alpha^2)^{-2/(4a'+1)}$, attains the rate $\varphi_n^{(NI)}$, where

$$t_n^{(NI)} = \varphi_n^{(NI)} = (n\alpha^2)^{-2\delta'/4a'+3} \cdot \log^{a+1}(n\alpha^2), \quad a > 1.$$
The test procedure $\Delta_n^{(SI)} = 1$, iff $\hat{D}_n^{(SI)} - 2\hat{L}_n + \|f_0\|_2^2 > C\ell_n^{(SI)}$, where $\hat{D}_n^{(SI)}$ is the procedure in (4.3), attains the rate $\varphi_n^{(SI)}$, where

$$t_n^{(SI)} = \varphi_n^{(SI)} = (\alpha a^2)^{-\frac{2}{\pi^2} + \frac{3}{2}} \cdot \log\frac{2}{3} = (\alpha a^2), \quad a > 1.$$  

The upper bounds are simple consequences of the upper bounds for estimating the quadratic functional $D(f)$. It is also easy to deduce the corresponding lower bounds (without the log factors) from the proofs of the lower bounds on estimation. Indeed, in these proofs, the estimation risk is first reduced to the risk for testing and this is further bounded from below.

Lam-Weil, Laurent and Loubes (2020) have recently derived similar results for goodness-of-fit testing over spaces $B_2^{s\infty}$ in the special case of non-interactive privacy with identical privacy mechanisms on each sample $X_i, Q \times n$. Their innovative method for establishing lower bounds is generalized here in order to take into account general non-interactive privacy mechanisms $\prod_{i=1}^n Q_i$, in order to achieve optimality over Besov $B_{s,p}^q$, $s > 0, p \geq 2, q \geq 1$, smoothness classes and optimality with respect to the privacy level $\alpha$ when it tends to 0.

The testing approach above has been followed by Berrett and Butucea (2020) for goodness-of-fit testing of discrete distributions with separation defined by the $L_2$ and $L_1$ norms. In the case of discrete distributions, it has also been noticed that the analogous interactive privacy mechanism introduced here allows for faster rates of testing than any non-interactive privacy mechanism.

6. Adaptation to the smoothness. Notice that the estimators considered so far use the smoothness $s$ of the unknown underlying probability density in order to determine the optimal resolution level $J$, and this resolution level plays a role in both, the construction of the sanitized data and in the estimation procedure of the quadratic functional. In addition, our sequentially interactive procedure relied on an optimal truncation parameter $\tau$ that also depended on $s$. In this section we show how to aggregate procedures for different values of the resolution level $J$ and select the optimal $\hat{J}$ and the associated estimator $\hat{D}_J$ in a data driven way by minimizing a penalized criterion.

6.1. Non-interactive setup. Let us consider the sanitized samples $Z_{ijk}, j = -1, \ldots, J_{max} - 1, k = 0, \ldots, 2^j - 1$ in (3.1). Recall that we use a fixed arbitrary constant $a > 1$ and $\sigma = 4 + 2\sum_{j=1}^{\infty} \frac{1}{j^2}$ in the construction of $Z'$s. We
use the estimator $\hat{D}_n$ as defined in (3.3), but to emphasize the dependence on $J$ in the definition, we now write $\hat{D}_J$ instead of $\hat{D}_n$, for some $J \leq J_{\text{max}}$, and we suppress the dependence on $n$.

For some constant $C > 0$, define
\[
\text{pen}^{(NI)}(J) = C \frac{J^{2a}2^{3J/2}}{n^{\alpha^2}} \log(2^{4J+1}),
\]
where $J \in \mathcal{J} = \{1, 2, \ldots, J_{\text{max}}\}$ such that the largest value $J_{\text{max}}$ in $\mathcal{J}$ satisfies, for some $\kappa > 4(a + 1)$,
\[
\frac{2^{3J_{\text{max}}}}{n^{2\alpha^4}} \leq \log^{-\kappa}(n^{\alpha^2}).
\]
If $C > 0$ is chosen sufficiently large, the penalty allows us to define the final purely data-driven estimator as follows:
\[
\hat{D}_n^{(NI)} := \max \left\{ \hat{D}_J - \text{pen}^{(NI)}(J) : J \in \mathcal{J} \right\}.
\]
The proof of the following theorem is deferred to Section C.1 of the supplement (Butucea, Rohde and Steinberger, 2022).

**Theorem 6.1.** Under the assumptions and notations of Theorem 3.2, the adaptive estimator $\hat{D}_n^{(NI)}$ associated to the penalty $\text{pen}^{(NI)}$ above is such that
\[
\sup_{f \in \mathcal{P}_2^q(L,M_2,M_3)} \mathbb{E}_{Q_f^{(NI)}} \left[ \left| \hat{D}_n^{(NI)} - D(f) \right|^2 \right] \lesssim_{\log} \mathfrak{r}_n^{(NI)}(\alpha, a, s'),
\]
for every $n \in \mathbb{N}$ and $\alpha \in (0, 1]$, with $n^{\alpha^2} > 1$. Here, $\lesssim_{\log}$ indicates that polynomial factors in $\log(n^{\alpha^2})$ have been omitted.

We proceed similarly to Laurent (2005) in order to build the adaptive procedure in the non-interactive case. However, the $U$-statistic that we build is based on randomized versions $Z_{ijk}$ of $\psi_{jk}(X_i)$ and therefore we can decompose the estimator $\hat{D}_J$ of $D$ at each resolution level $J$ into terms that already appeared in Laurent (2005) but also two additional terms due to the Laplace random variables $W_{ijk}$. Our proof is mainly dedicated to dealing with these additional terms. For example, standard concentration inequalities for $U$-statistics of order 2 in Houdré and Reynaud-Bouret (2003) do not apply to unbounded random variables and we tailor a proof using the coupling inequality in de la Peña and Montgomery-Smith (1995).
6.2. Interactive setup. Like the non-adaptive one, our interactive adaptive procedure also proceeds in two steps. First, half of the data providers generate sanitized samples in a non-interactive way as before and these are used to build preliminary estimators of the wavelet coefficients and of the underlying probability density. Then, the second part of the data providers use this information to generate sanitized samples that are subsequently used both for estimating the quadratic functional at an arbitrary resolution level \( J \), as well as for estimating the appropriate penalty term. Indeed, the variance of our interactive non-adaptive procedure depends on the smoothness of the unknown density through its wavelet coefficients and we need to estimate this quantity in order to build a purely data-dependent penalty.

Let \( J_{\text{max}} = J_{\text{max}}(na^2, B) \) be defined such that

\[
\frac{2^{2J_{\text{max}}}}{na^2} \times \frac{1}{\log(B(na^2))},
\]

for some large enough \( B > 0 \). Our interactive adaptive procedure is defined as follows. First, the sample is divided into two equally sized parts, where we assume for simplicity that \( n \in 2\mathbb{N} \). Based on the first sample, we then generate for each \( j \in \{-1, 0, 1, \ldots, J_{\text{max}} - 1\} \) and \( k \in \{0, 1, \ldots, (1 \vee 2^j) - 1\} \) the random variables \( Z^{(1)}_{ijk} \) as given in (3.1) and then build \( \hat{\beta}_{jk} \) as in (3.2).

Now, in the second step, the set \( \{n/2+1, \ldots, n\} \) is decomposed into \( J_{\text{max}} + 1 \) (approximately) equally sized parts \( \mathcal{N}_j, j \in \{-1, 0, 1, \ldots, J_{\text{max}} - 1\} \). For each \( j \) and each individual \( X_i \) with \( i \in \mathcal{N}_j \), we generate

\[
Z^{(2,j)}_i := \pm e^{\alpha} + 1 \quad \text{with probability } \frac{1}{2} \left( 1 \pm \frac{\Pi [\sum_{k=0}^{(1 \vee 2^j) - 1} \hat{\beta}_{jk} \psi_{jk}(X^{(2)}_i)]}{\tau e^{\alpha + 1} e^{-1}} \right),
\]

where \( \tau = \log^K(na^2) \) for some large enough \( K = K(a, B) > 0 \). Define the estimator at resolution level \( J \) as

\[
\hat{D}_J := \sum_{j=-1}^{J-1} \frac{1}{|\mathcal{N}_j|} \sum_{i \in \mathcal{N}_j} Z^{(2,j)}_i.
\]

Next, define

\[
\hat{\text{pen}}^2(J) := \frac{1}{na^2} \sum_{j=-1}^{J-1} \sigma_j^2 \left( \frac{1}{|\mathcal{N}_j|} \sum_{i \in \mathcal{N}_j} Z^{(2,j)}_i \right),
\]

where \( \sigma_j \) is as in (3.1). Then

\[
\hat{J} := \arg\max_{J \in \{1, \ldots, J_{\text{max}}\}} \left( \hat{D}_J - \hat{\text{pen}}(J) \right).
\]
The final estimator is then $\hat{D}_n^{(SI)} = \hat{D}_j$. The following theorem is proved in Section C.2 of the supplement (Butucea, Rohde and Steinberger, 2022).

**Theorem 6.2.** Under the assumptions and notations of Theorem 4.1, the adaptive estimator $\hat{D}_n^{(SI)}$ associated to the data-driven $\hat{J}$ in (6.1) satisfies

$$\sup_{f \in P_{\alpha,a}(L,M)} \mathbb{E}_{Q(f)} \left[ \left| \hat{D}_n^{(SI)} - D(f) \right|^2 \right] \lesssim r_n^{(SI)}(\alpha, a, s'),$$

up to logarithmic factors.

7. Numerical results. In this section we complement our theoretical findings about the private minimax convergence rates by an extensive simulation study to further investigate potential strengths and weaknesses of the non-interactive and sequentially interactive locally private estimation procedures suggested above.

In our first round of numerical experiments we consider Hölder smooth data generating densities $f_s$. More specifically, let

$$f_s(x) := (s + 1)x^s, \quad x \in (0, 1], s \in (0, 1).$$

Then $f_s$ belongs to the Hölder space

$$C^s((0, 1]) := \left\{ f \in C((0, 1]) : \|f\|_\infty + \sup_{x \neq y, x, y \in [0, 1]} \frac{|f(x) - f(y)|}{|x - y|^s} < \infty \right\}$$

which is itself identical to $B^{s,\infty}$ with equivalent norms (cf. Giné and Nickl, 2016, Proposition 4.3.23). Figure 1 shows examples of $f_s$ for different smoothness parameters $s$. We implement the non-interactive procedure in (3.3) and the private wavelet density estimator in (4.1) both with $\sigma_{-1} = 0$, $\sigma_j = (1 \lor 2^{j/2})(2^j + 1)$, $j \geq 0$, and $\sigma = 1$. For the second step in the sequentially interactive procedure, we compute a more practical version that is, however, harder to analyze theoretically, by choosing $\tau = \|\tilde{f}_j^{(1)}\|_\infty$. It is easy to see that with these modifications both procedures still satisfy the privacy constraint. In particular, we have $\psi_{-1,0}(x) = \phi(x) = 1$, for all $x \in (0, 1]$ and hence $Z_{i,-1,0} = 1$ and $\hat{\beta}_{i,-1,0} = 1$, irrespective of the sensitive information $X_i$. In Figures 2 to 4 we plot mean squared errors based on 100 Montecarlo iterations of samples of size $n = 1000$ each ($2n = 1000$ in case of the two-step procedure), for different values of the tuning parameter $J$. The parameter $J$ determines the number of resolution levels to be included in the wavelet estimator, so a large value of $J$ results in an estimator that
includes more details but also more Laplace noise, whereas a small $J$ leads to a larger bias. Notice that for $J = 0$ both procedures disregard the original data $X_1, \ldots, X_n$, the non-interactive procedure simply returns $\hat{D}_n = 1$ and the sequentially interactive one essentially does the same but adds a little bit of random noise in the second step.

We see that for a smaller privacy level of $\alpha = 1$ (Figure 2), the sequentially interactive procedure (SI) consistently outperforms the non-interactive (NI) one. However, for both methods, adding higher resolution levels to the estimator does not really pay off. Although we have a sample size of $n = 1000$, still we are basically unable to pick up any signal under all the noise of the privacy mechanism, and simply assuming a uniform density (i.e., $J = 0$ and $\hat{D}_n = 1$) and suffering the resulting bias is acceptable compared to the higher variance that results from increasing $J$. There are two options for how to alleviate this problem. Either we further increase the sample size $n$, which gets challenging in terms of simulation runtime, or we can increase the privacy level $\alpha$. For $\alpha = 10$ (Figure 3), we observe a clear benefit of increasing the resolution $J$, except for the very non-smooth case $s = 1/8$, which corresponds to a density that is close to uniform. However, at the same time the superiority of the SI mechanism is reduced and is lost altogether in case $\alpha = 100$ (Figure 4). Of course, this should not surprise us, because for large $\alpha$ the NI differentially private procedure is almost equivalent to the conventional U-statistics estimator based on direct observation of $X_i$ (the Laplace...
Fig 2. MSE of non-interactive (NI) and sequentially interactive (SI) procedure with privacy level $\alpha = 1$, and true data generating distributions as in Figure 1.

Fig 3. MSE of non-interactive (NI) and sequentially interactive (SI) procedure with privacy level $\alpha = 10$, and true data generating distributions as in Figure 1.
noise in (3.1) vanishes as $\alpha \to \infty$), in which case a sample splitting approach is clearly inferior. Moreover, notice that the noise added in the second step (4.2) of the sequentially interactive mechanism does not disappear even as $\alpha \to \infty$.

From these observations we conclude that it is hard to see the superiority of the sequential mechanism in terms of convergence rate as $n \to \infty$ in a simulation, because in order to pick up enough local structure of the true density underneath all the added differentially private noise we need very large sample sizes. However, we also see that the sequential mechanism has an advantage when dealing with certain global features of the true density as in Figure 1 ($s = 7/8$). Therefore, we conduct a second round of simulations with very smooth but more structured beta-densities (Figure 5).

For a privacy parameter of $\alpha = 1$, in Figure 6 we see that both the NI as well as the SI mechanism can pick up some signal beneath the differentially private noise. Moreover, the SI procedure is never worse than the NI mechanism and achieves a smaller optimum at a value of $J$ that may also be different from the optimal $J$ of the NI procedure.

Finally, we numerically investigate the role of the splitting ratio in the sequentially-interactive two-step procedure. By $n_1 \leq n$ we denote the number of observations that are used in the first step to compute the private wavelet density estimator (4.1). In Figure 7 we plot MSEs of the SI proce-
Fig 5. Beta($b_1, b_2$) densities.

Fig 6. MSE of non-interactive (NI) and sequentially interactive (SI) procedure with privacy level $\alpha = 1$, and true data generating distributions as in Figure 5.
dure as a function of $J$ and $\frac{n_1}{n}$. For reference, the black dashed line denotes the MSE of the NI procedure with optimally chosen $J$. Original data $X_i$ were generated from the Beta$(20, 2)$-distribution. The nearly constant MSE curves for small values of $J$ can be explained by the fact that the bias of the SI procedure does not depend on the splitting ratio $\frac{n_1}{n}$ while at the same time the bias dominates the MSE for small $J$. However, the fraction of data points $\frac{n_1}{n}$ used in the first step of the SI procedure strongly influences the variance which dominates the MSE for larger values of $J$. We also see in Figure 7 that in order to beat the NI mechanism with the SI procedure, the choice of $J$ is not as critical ($J = 2, 3, 4$ would do), provided that about two thirds of the data go into the first wavelet density estimation part of SI.

We conclude this numerical section by pointing out again that although the precise quantitative behavior of the convergence rates of the two local differentially private procedures (investigated in Sections 3 and 4) can only be observed in very large samples, we still see clear benefits of the sequentially-interactive procedure over the simple non-interactive one also for smaller samples and when the privacy protection is strong and $\alpha$ is small.

SUPPLEMENTARY MATERIAL

Supplement to “Interactive versus non-interactive locally differentially private estimation: Two elbows for the quadratic functional” (supplement.pdf). The supplementary material contains all the proofs.

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Fig 7. MSE of NI with optimal tuning $J = 3$ (black dashed horizontal line) and of SI in dependence on $J$ and fraction $n_1/n$ of data points used in step one. Original private data were generated from the Beta(20, 2)-distribution.
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APPENDIX A: PROOFS OF SECTION 3 (NON-INTERACTIVE PROTOCOLS)

A.1. Proof of Proposition 3.1. It suffices to show that the one dimensional marginal channel satisfies (2.8). Note that the conditional density of $Z_1 | X_1 = x$ is given by

$$q(z|x) = \prod_{j=-1}^{J-1} \prod_{k=0}^{(1/2)^{-1}} \frac{\alpha}{2\sigma \cdot \sigma_j} \exp \left(-\frac{\alpha}{\sigma \cdot \sigma_j} |z_{jk} - \psi_{jk}(x)| \right), \quad z \in \mathbb{R}^M,$$
where $M = \sum_{j=-1}^{J-1} (1 \vee 2^j)$. Therefore,

$$
\frac{q(z|x)}{q(z|x')} = \prod_{j=-1}^{J-1} \prod_{k=0}^{(1/2^{j})-1} \exp \left( -\frac{\alpha}{\sigma \cdot \sigma_j} (|z_{jk} - \psi_{jk}(x)| - |z_{jk} - \psi_{jk}(x')|) \right)
$$

$$
\leq \exp \left( \frac{\alpha}{\sigma} |\phi(x) - \phi(x')| + \frac{\alpha}{\sigma} \sum_{j=0}^{J-1} \sum_{k=0}^{2j/2} (1/2^{j})-1 \sum_{k=0}^{(1/2^{j})-1} |\psi(2^j x - k) - \psi(2^j x' - k)| \right)
$$

For any fixed $x \neq x' \in [0,1]$, $|\phi(x) - \phi(x')| \leq 1$ and, for $j \geq 0$,

$$
\sum_{k=0}^{(1/2^{j})-1} |\psi(2^j x - k) - \psi(2^j x' - k)| \leq 2.
$$

Thus,

$$
\frac{q(z|x)}{q(z|x')} \leq \exp \left( \frac{\alpha}{\sigma} \left[ 2 + \sum_{j=1}^{J-1} \frac{1}{j^2} \right] \right) \leq \exp(\alpha),
$$

for $\sigma = 4 + 2 \sum_{j=1}^{\infty} \frac{1}{j^2}$. \hfill \Box

**A.2. Proof of Theorem 3.2.** Fix $f \in \tilde{P}_s^{pq}(L, M_2, M_3)$. It first follows from Lemma A.3 that the bias of the estimator $\hat{D}_n$ is bounded as follows:

$$
D - E_{Q^{(N)}}[\hat{D}_n] = \sum_{j \geq J} \|\beta_j\|^2 \leq \begin{cases} 
C2^{-2js'}, & 0 \leq s' < 1 \\
C2^{-j^3/2}, & s' \geq 1,
\end{cases}
$$

with $C > 0$.

We remark that, in order to use only the Haar basis in our construction, a rough control of the bias is performed for $s' \geq 1$. We have to embed our Besov body into a larger one with smoothness parameter smaller than 1 and we chose a somehow arbitrary value $5/6$ that is larger than $3/4$. This is enough to get the parametric rate in the regime where $s' > \frac{3}{4}$.

Next, we study the variance of the private estimator $\hat{D}_n$ with $J \geq 2$. Note that

$$
\hat{D}_n - D = \frac{1}{n(n-1)} \sum_{i \neq h} \sum_{j=-1}^{J-1} \sum_{k=0}^{(1/2^{j})-1} (Z_{ijk} - \beta_{jk})(Z_{hjk} - \beta_{jk})
$$

$$
+ \frac{2}{n} \sum_{i=1}^{n} \sum_{j=-1}^{J-1} \sum_{k=0}^{(1/2^{j})-1} (Z_{ijk} - \beta_{jk})\beta_{jk} = T_1 + T_2.
$$
Now,

\[
\text{Var}(T_1) = \frac{4}{n^2(n-1)^2} \text{Var} \left( \sum_{i<h} \sum_j \sum_k (Z_{ijk} - \beta_{jk})(Z_{hjk} - \beta_{jk}) \right)
\]

\[
= \frac{2}{n(n-1)} \text{Var} \left( \sum_j \sum_k (Z_{1jk} - \beta_{jk})(Z_{2jk} - \beta_{jk}) \right).
\]

We can decompose the previous sum into uncorrelated terms as follows:

\[
\sum_j \sum_k (Z_{1jk} - \beta_{jk})(Z_{2jk} - \beta_{jk}) = \sum_j \sum_k (\psi_{jk}(X_1) - \beta_{jk})(\psi_{jk}(X_2) - \beta_{jk})
\]

\[
+ \sum_j \sum_k \sigma_j \psi_{jk} (X_1) W_{1jk} + \sum_j \sum_k \sigma_j \psi_{jk} (X_2) W_{2jk}.
\]

Therefore, using independence, the inequality \((a-b)^2 \leq 2a^2 + 2b^2\) and \(\mathbb{E}(W_{1jk}^2) = 2\), we get

\[
\text{Var}(T_1) = \frac{2}{n(n-1)} \left\{ \mathbb{E} \left[ \left( \sum_j \sum_k (\psi_{jk}(X_1) - \beta_{jk})(\psi_{jk}(X_2) - \beta_{jk}) \right)^2 \right] \right.
\]

\[
+ 2 \cdot \frac{\sigma_j^2}{\alpha^2} \mathbb{E} \left[ \left( \sum_j \sum_k \sigma_j (\psi_{jk}(X_1) - \beta_{jk}) W_{1jk} \right)^2 \right]
\]

\[
+ \frac{\sigma_j^4}{\alpha^4} \mathbb{E}(W_{1jk}^2) \right\} 2^j \sum_j \sigma_j^4 \cdot 2^j \right) \right\}
\]

\[
\leq \frac{2}{n(n-1)} \left\{ 2 \mathbb{E} \left[ \left( \sum_j \sum_k \psi_{jk}(X_1) \psi_{jk}(X_2) \right)^2 \right] + 2 \left( \sum_j \sum_k \beta_{jk}^2 \right)^2
\]

\[
+ 8 \frac{\sigma_j^2}{\alpha^2} \mathbb{E} \left[ \left( \sum_j \sum_k \sigma_j \psi_{jk}(X_1) \right)^2 \right] + 8 \frac{\sigma_j^2}{\alpha^2} \sum_j \sum_k \sigma_j^2 \beta_{jk}^2
\]

\[
+ 4J^{4a+123J} \frac{\sigma_j^4}{\alpha^4} \right\}.
\]
Now, we easily see that
\[
\left( \sum_j \sum_k \beta_{jk}^2 \right)^2 = D^2 \leq M_2^4,
\]
and that
\[
\sum_j \sum_k \sigma_j^2 \beta_{jk}^2 = \sum_{j=-1}^{J-1} j^{2a+2j} \| \beta_j \|_2^2 \leq J^{2a+1} 2^J.
\]
Furthermore, by Jensen’s inequality and the fact that for \( k_1 \neq k_2 \), the basis functions \( \psi_{jk_1} \) and \( \psi_{jk_2} \) have disjoint support, we get
\[
E \left[ \left( \sum_j \sum_k \psi_{jk}(X_1) \psi_{jk}(X_2) \right)^2 \right] \leq (J + 1) \cdot E \left[ \left( \sum_j \sum_k \psi_{jk}(X_1) \psi_{jk}(X_2) \right)^2 \right] \\
\leq (J + 1) \sum_j \sum_k (E \left[ \psi_{jk}^2(X_1) \right])^2 \\
= (J + 1) \sum_j \sum_k \left( E \left[ (1 \vee 2^j) \mathbb{1}_{[0,1]}((1 \vee 2^j)X_1 - k) \right] \right)^2 \\
= (J + 1) \sum_j \left( (1 \vee 2^j) E \left[ \mathbb{1}_{[0,1]}((1 \vee 2^j)X_1) \right] \right)^2 \\
\leq (J + 1) \sum_{j=-1}^{J-1} (1 \vee 2^j)^2 \leq (J + 1) \cdot 2^{2J}.
\]
Moreover, similar but simpler considerations yield
\[
E \left[ \left( \sum_j \sigma_j \sum_k \psi_{jk}(X_1) \right)^2 \right] \leq (J + 1) \cdot \sum_j \sigma_j^2 \sum_k \left( \sum_j \psi_{jk}(X_1) \right)^2 \\
\leq (J + 1) \cdot \sum_{j=-1}^{J-1} \sigma_j^2 (1 \vee 2^j) \leq (J + 1) \left( 2 + \sum_{j=1}^{J-1} j^{2a+2j} \right) \\
\leq (J + 1)^{2a+2} 2^{2J}.
\]
In conclusion, there exists some constant \( C > 0 \), not depending on \( f \), \( n \) or \( \alpha \leq 1 \), such that
\[
\text{Var}(T_1) \leq C \left( \frac{J \cdot 2^J}{n^2} + \frac{j^{2a+2} 2^J}{n^2 \alpha^2} + \frac{j^{4a+1} 2^J}{n^2 \alpha^4} \right) \leq 3C \frac{j^{4a+1} 2^{3J}}{n^2 \alpha^4}.
\]
Next,

\[
\text{Var}(T_2) = \frac{4}{n} \text{Var} \left( \sum_j \sum_k (Z_{1jk} - \beta_{jk})\beta_{jk} \right)
\]

\[
= \frac{4}{n} \text{Var} \left( \sum_j \sum_k (\psi_{jk}(X_1) - \beta_{jk})\beta_{jk} \right) + \frac{4}{n} \text{Var} \left( \sum_j \sum_k \sigma_j \frac{\sigma}{\alpha} W_{1jk} \beta_{jk} \right)
\]

\[
= \frac{4}{n} \text{Var} \left( \sum_j \sum_k \psi_{jk}(X_1)\beta_{jk} \right) + \frac{4}{n} \sigma^2 \sum_j \sum_k 2\beta^2_{jk}
\]

\[
= \frac{4}{n} \left( \int_0^1 (P_Jf)^2 \cdot f - \left( \int_0^1 (P_Jf) \cdot f \right)^2 \right) + \frac{8\sigma^2}{n\alpha^2} \sum_{J=-1}^{J-1} \sum_{m=0}^{2J-1} \sigma^2 \parallel \beta_j \parallel^2,
\]

where \( P_Jf = \sum_{J=-1}^{J-1} \sum_k \beta_{jk} \psi_{jk} = \sum_{m=0}^{2J-1} \alpha_{Jm} \phi_{Jm} \) is the projection of \( f \) onto the linear space \( V_J = \text{span}\{ \phi_{Jm} : m = 0, \ldots, 2J - 1 \} \), \( \phi_{Jm} = 2^{J/2} \phi(2^J x - m) \) and \( \alpha_{Jm} = \langle f, \phi_{Jm} \rangle \). Because of the special structure of the Haar basis functions \( \phi_{Jm} \) we have \( (P_Jf)^2 \in V_J \) and therefore \( (P_Jf)^2 \perp (f - P_Jf) \in V_J^\perp \), so that we obtain

\[
\int_0^1 (P_Jf)^2 \cdot f - \left( \int_0^1 (P_Jf) \cdot f \right)^2 \leq \int_0^1 (P_Jf)^3 = \int_0^1 \left( \sum_k \alpha_{Jk} \phi_{Jk}(x) \right)^3 dx
\]

\[
= \int_0^1 \sum_k \alpha^3_{Jk} \phi_{Jk}^3(x) dx = \sum_k 2^{J/2} \alpha^3_{Jk}
\]

\[
= \sum_k 2^{J/2} \left( \int_0^1 f \cdot \phi_{Jk} \right)^3.
\]

But Jensen’s inequality yields

\[
\sum_k 2^{J/2} \left| 2^{-J/2} \int_0^1 f \cdot 2^{J/2} \phi_{Jk} \right|^3 \leq \sum_k 2^{J/2} 2^{-3J/2} \int_0^1 f^3 \cdot 2^{J/2} \phi_{Jk}
\]

\[
= \int_0^1 f^3(x) \cdot \sum_{k=0}^{2^J-1} \phi(2^J x - k) dx = \int_0^1 f^3 dx \leq M_3^3.
\]

On the other hand, recall that for the bias part we already showed that
\[ \| \beta_j \|_2 \lesssim 2^{-js'} \cdot \mathbf{1}_{\{0 < s' < 1\}} + 2^{-\frac{3}{2}j} \cdot \mathbf{1}_{\{s' \geq 1\}} \text{ where } s' = s - \left( \frac{1}{p} - \frac{1}{2} \right)_+. \] Thus
\[ \sum_{j=1}^{J-1} \sigma_j^2 \| \beta_j \|_2^2 \lesssim 2 + \sum_{j=1}^{J-1} 2^{2a} \left[ 2^{j(1-2s')} \cdot \mathbf{1}_{\{0 < s' < 1\}} + 2^{j(1-\frac{3}{2})} \cdot \mathbf{1}_{\{s' \geq 1\}} \right] \lesssim 1 \vee (J^{2a+1} \cdot 2^{J(1-2s')}). \]

Thus, for some constant \( C > 0 \),
\[ \text{Var}(T_2) \leq \frac{4M_3}{n} + C \cdot \left\{ 1 \vee \left( J^{2a+1} \cdot 2^{J(1-2s')} \right) \right\}. \]

Summing up the previous bounds, we get
\[ \mathbb{E}_{Q_{j}^{(NI)}}[(\hat{D}_n - D)^2] \lesssim 2^{-4Js'} \mathbf{1}_{\{0 < s' < 1\}} + 2^{-J\frac{10}{3}} \mathbf{1}_{\{s' \geq 1\}} + \frac{J^{4a+2}2^J}{n^2a^4} + \frac{1}{n} + \frac{1 \vee (J^{2a+1} \cdot 2^{J(1-2s')})}{na^2}. \]

With our choice of \( J \) the result of Theorem 3.2 follows, because
\[ \frac{1 \vee (J^{2a+1} \cdot 2^{J(1-2s')})}{na^2} = \frac{1 \vee \left( \left[ \frac{2}{4s' + 3} \log_2(na^2) \right]^{2a+1} \cdot (na^2)^{2(1-2s')} \right)}{na^2}, \]
\[ \leq \frac{1}{na^2} \vee \left( \left[ \frac{2}{4s' + 3} \log(na^2) \right]^4a + 1 \cdot (na^2)^{-\frac{8s'}{3}} \right). \]

\[ \square \]

**A.3. Proof of Theorem 3.3.** Fix a channel \( Q \in Q_{\alpha}^{(NI)} \) with marginal conditional densities \( q_i(z_i|x_i), i = 1, \ldots, n \) with respect to some reference probability measure \( \mu_i \) on \( Z_i \), as in Lemma B.3, that is, \( e^{-\alpha} \leq q_i(z_i|x) \leq e^{\alpha} \) and, in particular, \( q_i(z_i|x) \leq e^{2\alpha} q_i(z_i|x'), \) for all \( z_i \in Z_i \) and all \( x, x' \in [0, 1] \).

The lines of proof are similar to those of Lam-Weil, Laurent and Loubes (2020) that we generalize in order to: a) take into account possibly different mechanisms \( q_i \) for each \( i \), b) consider Besov smooth densities belonging to \( B^p_s \) with \( s > 0, p \geq 2, q \geq 1 \), instead of \( B^{\infty}_s \) and c) get the (nearly) optimal dependence with respect to \( \alpha \) when it tends to 0.

Let \( f_0 = \mathbf{1}_{[0,1]} \) and denote by \( g_{0,i}(z_i) = \int_0^1 q_i(z_i|x)dx \geq e^{-\alpha} \). For any \( i = 1, \ldots, n \), define the bounded linear operator \( K_i : L_2([0, 1]) \rightarrow L_2(Z_i, d\mu_i) \) by
\[ K_if = \int_0^1 q_i(\cdot|x)f(x) \frac{dx}{\sqrt{g_{0,i}(\cdot)}}, \quad f \in L_2([0, 1]). \]
Then with $K^*_i$ denoting its adjoint, the operator $K^*_iK_i$ is a symmetric integral operator with kernel $F_i(x, y) = \int q_i(z_i|x)q_i(z_i|y)/g_{0,i}(z_i)dz_i$:

$$K^*_iK_if(\cdot) = \int_{Z_i} q_i(z_i|x)\int_0^1 q_i(z_i|y)f(\cdot)dyd\mu_i(z_i) = \int_{Z_i} q_i(z_i|x)d\mu_i(z_i) = 1$$

by Fubini’s theorem. Next, let us note that $f_0$ is an eigenfunction of $K^*_iK_i$, associated to the eigenvalue $\lambda_{0,i} = 1$, for all $i$ from 1 to $n$:

$$K^*_iK_if_0(x) = \int_{Z_i} q_i(z_i|x)\int_0^1 q_i(z_i|y)f_0(\cdot)dyd\mu_i(z_i) = \int_{Z_i} q_i(z_i|x)d\mu_i(z_i) = 1$$

for all $x \in [0, 1]$. Now, define the operator

$$K := \frac{1}{n} \sum_{i=1}^n K^*_iK_i.$$ 

It is again symmetric and positive semidefinite and has the eigenfunction $w_0 = f_0$ associated to the eigenvalue $\lambda_0 = 1$. It is also an integral operator with kernel $F(x, y) = n^{-1}\sum_{i=1}^n F_i(x, y)$. Recall the Haar wavelet functions $(\psi_{jk})$ of Section 2.1 and define

$$W_m = \text{span}\{\psi_{mk} : k = 0, ..., 2^m - 1\}$$

as the linear subspace spanned by the orthonormal family consisting of $(\psi_{mk})_{k=0,...,2^m-1}$. Denote by $w_1, ..., w_{2^m}$ the eigenfunctions of $K$ as an operator on the linear $L_2([0, 1])$-subspace $W_m$, satisfying $\|w_k\|_{L_2} = 1$ and $\int_0^1 w_k(x)dx = 0$ since they are orthogonal to $f_0 = 1_{[0,1]}$. Moreover, we write $\lambda_1^2, ..., \lambda_{2^m}^2$ for the corresponding eigenvalues, respectively. Note that they are non-negative.

From now on, we denote by $z_\alpha = e^{2\alpha} - e^{-2\alpha} \leq e^2$ for $\alpha$ in $(0,1]$ and by

$$\lambda_{k,\alpha,m} = (\frac{\lambda_k}{z_\alpha}) \vee 2^{-m/2} \geq 2^{-m/2}.$$ 

Define the functions

$$f_\nu(x) = f_0(x) + 2^{-m(s+1)}\delta \sum_{j=1}^{2^m} \nu_j \frac{1}{\lambda_{j,\alpha,m}} \cdot w_j(x), \quad x \in [0, 1],$$

where $\nu_j \in \{-1, 1\}$, for $j = 1, ..., 2^m$ and $\delta = \delta_m > 0$ is to be specified later. By a slight abuse of notation, we identify $f_0$ with $f_\nu$ with $\nu = (0, ..., 0)$. 

Lemma A.1 shows that for the overwhelming part of possible vectors $\nu$, $f_\nu$ is a density, belongs to the right Besov space and the corresponding quadratic functional $D(f_\nu)$ is sufficiently far away from $D(f_0)$.

We choose the integer number $m$ such that:

$$nz_\alpha^2 \asymp 2^{2ms+3m/2}.$$  

Let us denote by $g_{\nu,i}$ the function $g_{\nu,i}(z_i) = \int_0^1 q_i(z_i|x)f_\nu(x)dx$, $z_i \in Z_i$, and see that

$$g_{\nu,i}(z_i) = g_{0,i}(z_i) + 2^{-m(s+\frac{1}{2})} \sum_{k=1}^{2^m} \frac{\nu_k}{\lambda_{k,\alpha,m}} \cdot \int_0^1 q_i(z_i|x)w_k(x)dx.$$  

Classical results allow us to reduce the lower bounds for estimating $f$ to testing between the probability measures

$$(A.1) \quad dQ_{0,n}(z_1, \ldots, z_n) := \prod_{i=1}^{n} g_{0,i}(z_i)d\mu_i(z_i)$$

(where, for $\mu := \bigotimes_{i=1}^{n} \mu_i$, $\prod_{i=1}^{n} g_{0,i}(z_i)$ is also the $\mu$-density of the product measure $Q_{Pf_0} = \bigotimes_{i=1}^{n} (Q_{if_0})$) and the averaged alternative

$$(A.2) \quad dQ_n(z_1, \ldots, z_n) := E_\nu \left[ \prod_{i=1}^{n} g_{\nu,i}(z_i)d\mu_i(z_i) \right],$$

where $E_\nu$ stands for expectation over i.i.d. Rademacher random variables $\nu_k$. Indeed, using Lemma A.1, we first reduce the maximal risk over $\mathcal{P}_s^{pq}$ to the maximal risk over the subfamily of pdf’s $\{f_\nu : \nu = 0 \text{ or } \nu \in A_\gamma\} \subseteq \mathcal{P}_s^{pq}$.
and then use the Markov inequality with $\Delta = \delta^2 \cdot 2^{-2ms}$ to get

$$\inf_{Q \in Q_{n}^{(N)}} \inf_{D_n} \sup_{f \in P_{n}^{(L,M)}} \mathbb{E}^{P_{n}} \left[ |\hat{D}_n - D(f)|^2 \right]$$

$$\geq \inf_{Q \in Q_{n}^{(N)}} \inf_{\nu \in A_{\gamma} \cup \{0\}} \sup_{D_n} \left( \frac{\Delta}{2} \right)^2 \mathbb{Q}^{P_{n}}_{f_{\nu}} \left( |\hat{D}_n - D(f_{\nu})| \geq \frac{\Delta}{2} \right)$$

$$\geq \left( \frac{\Delta}{2} \right)^2 \inf_{Q \in Q_{n}^{(N)}} \inf_{\nu \in A_{\gamma} \cup \{0\}} \max \left\{ \mathbb{Q}^{P_{n}}_{f_{\nu}} \left( |\hat{D}_n - D(f_{\nu})| \geq \frac{\Delta}{2} \right), \right. \right.$$

$$\left. \left. \mathbb{E}_{\nu} \left[ 1_{A_{\gamma}(\nu)} \mathbb{Q}^{P_{n}}_{f_{\nu}} \left( |\hat{D}_n - D(f_{\nu})| \geq \frac{\Delta}{2} \right) \right] \right\}$$

$$\geq \left( \frac{\Delta}{2} \right)^2 \inf_{Q \in Q_{n}^{(N)}} \inf_{\nu \in A_{\gamma} \cup \{0\}} \max \left\{ \mathbb{Q}^{P_{n}}_{f_{\nu}} (B), \right.$$

$$\left. \left. \mathbb{E}_{\nu} \left[ 1_{A_{\gamma}(\nu)} \mathbb{E}^{P_{n}}_{Q_{n}^{P_{n}}(B)} \cdot \mathbb{Q}^{P_{n}}_{f_{\nu}} \left( |\hat{D}_n - D(f_{\nu})| \geq \Delta \right) \right] \right\},$$

(A.3)

where we denote by $B$ the event $\{|\hat{D}_n - D(f_0)| \geq \frac{\Delta}{2}\}$. Note that, because of $|D(f_{\nu}) - D(f_0)| \geq \Delta$ (cf. Lemma A.1.(iii)), the complementary event of $B$, $\bar{B}$, implies that $|\hat{D}_n - D(f_{\nu})| \geq \frac{\Delta}{2}$. Thus, for any $\tau \in (0, 1)$ we can further bound from below the term in (A.3) by

$$\mathbb{E}^{P_{n}} \left[ \mathbb{E}_{\nu} \left[ 1_{A_{\gamma}(\nu)} \frac{d(Q^{P_{n}}_{f_{\nu}})}{d(Q^{P_{n}}_{f_{0}})} \cdot 1_{\bar{B}} \right] \right] = \mathbb{E}^{Q_{n,0}} \left[ \mathbb{E}_{\nu} \left[ 1_{A_{\gamma}(\nu)} \frac{d(Q^{P_{n}}_{f_{\nu}})}{d(Q^{P_{n}}_{f_{0}})} \cdot 1_{\bar{B}} \right] \right]$$

$$= \mathbb{E}^{Q_{n,0}} \left[ \frac{dQ_{n}}{dQ_{0,n}} - \mathbb{E}_{\nu} \left[ 1_{A_{\gamma}(\nu)} \frac{d(Q^{P_{n}}_{f_{\nu}})}{d(Q^{P_{n}}_{f_{0}})} \right] \cdot 1_{\bar{B}} \right]$$

$$\geq \mathbb{E}^{Q_{n,0}} \left[ \frac{dQ_{n}}{dQ_{0,n}} \cdot 1_{\bar{B}} \right] - \mathbb{E}_{\nu} \left[ 1_{A_{\gamma}(\nu)} \mathbb{E}^{Q_{n,0}} \left[ \frac{d(Q^{P_{n}}_{f_{\nu}})}{dQ_{0,n}} \right] \right]$$

$$\geq \mathbb{E}^{Q_{n,0}} \left[ \tau \cdot 1_{\frac{dQ_{n}}{dQ_{0,n}} \geq \tau} \cdot 1_{\bar{B}} \right] - \gamma$$

$$\geq \tau \cdot \left( Q_{n,0} \left( \frac{dQ_{n}}{dQ_{0,n}} \geq \tau \right) - Q_{n,0}(B) \right) - \gamma,$$

where we have used that $\mathbb{E}_{\nu}[1_{A_{\gamma}(\nu)}] \leq \gamma$, by Lemma A.1. If there exist $\epsilon, \tau \in (0, 1)$ and $n_0 \in \mathbb{N}$, all three not depending on $n$ and $\alpha$, such that, whenever $nz_{\alpha}^2 \geq n_0$, we have

(A.4) $Q_{n,0} \left( \frac{dQ_{n}}{dQ_{0,n}} \geq \tau \right) \geq 1 - \epsilon,$
then we conclude the proof by the following lower bound for the minimax risk

\[
\mathcal{M}_{n,\alpha}^{(N)}(\mathcal{P}_{\text{pq}}(L, M)) \geq \left( \frac{\Delta}{2} \right)^2 \left( \inf_{Q \in \mathcal{Q}_\alpha^{(N)}} \inf_{D_n} \max \left\{ Q_{0,n}(B), +\tau \cdot \left( Q_{0,n} \left( \frac{dQ_n}{dQ_{0,n}} \geq \tau \right) - Q_{0,n}(B) \right) \right\} - \gamma \right),
\]

\[
\geq \left( \frac{\Delta}{2} \right)^2 \left( \inf_{p \in (0,1)} \max \{ p, \tau(1 - \epsilon - p) \} - \gamma \right)
\]

\[
= \left( \frac{\Delta}{2} \right)^2 \cdot \left( \frac{\tau}{1 + \tau}(1 - \epsilon) - \gamma \right),
\]

provided that \( nz^2 \geq n_0 \). Thus, for an appropriate choice of \( \gamma \in (0,1) \) and with \( \delta = \delta_m \) as in Lemma A.1, \( \Delta^2 = 4^{-4ms} \times (nz^2)^{-8s/(4s+3)}|\log(nz^2)|^{-2} \) is the desired rate.

A sufficient condition for (A.4) is that for \( nz \alpha \geq n_0 \),

\[
\chi^2(Q_n, Q_{0,n}) := \int \left( \frac{dQ_n}{dQ_{0,n}} - 1 \right)^2 dQ_{0,n} \leq (1 - \tau)^4.
\]

Indeed, \( Q_{0,n}(\frac{dQ_n}{dQ_{0,n}} \geq \tau) \geq 1 - \frac{1}{(1 - \tau)^2} \int (\frac{dQ_n}{dQ_{0,n}} - 1)^2 dQ_{0,n} \geq 1 - (1 - \tau)^2 \) and this checks (A.4) with \( \epsilon = (1 - \tau)^2 \).

We have

\[
\chi^2(Q_n, Q_{0,n}) = -1 - \mathbb{E}_{Q_{0,n}} \left[ \left( \frac{dQ_n}{dQ_{0,n}} \right)^2 \right]
\]

\[
= -1 - \mathbb{E}_{Q_{0,n}} \left[ \mathbb{E}_\nu \prod_{i=1}^n \left( 1 + 2^{-m(s+1)} \delta \sum_{k=1}^{2^m} \frac{\nu_k}{\lambda_{k,\alpha,m}} \cdot \langle q_i(Z_i \cdot, w_k) \rangle \right) \right]^2
\]

\[
= -1 - \mathbb{E}_{Q_{0,n}} \left[ \mathbb{E}_{\nu_0} \prod_{i=1}^n \left( 1 + 2^{-m(s+1)} \delta \sum_{k=1}^{2^m} \frac{\nu_k}{\lambda_{k,\alpha,m}} \cdot \langle q_i(Z_i \cdot, w_k) \rangle \right) \right]^2
\]

\[
= -1 - \mathbb{E}_{Q_{0,n}} \prod_{i=1}^n \left( 1 + \mathbb{E}_{Q_{0,n}} 2^{-2m(s+1)} \delta^2 \sum_{k=1}^{2^m} \frac{\nu_{k_1}}{\lambda_{k_1,\alpha,m}} \cdot \langle q_i(Z_i \cdot, w_{k_1}) \rangle \right)
\]

\[
\cdot \frac{\nu_{k_2}}{\lambda_{k_2,\alpha,m}} \cdot \langle q_i(Z_i \cdot, w_{k_2}) \rangle,
\]

\[
\geq (1 - \tau)^2. \quad \epsilon = (1 - \tau)^2.
\]
where $\nu, \nu'$ are independent copies of vectors with i.i.d. Rademacher entries and we used that

$$\mathbb{E}_{Q_{0,n}} \left( \frac{\langle q_i(Z_i \cdot), w_k \rangle}{g_{0,i}(Z_i)} \right) = \int_{Z_i} \langle q_i(z_i \cdot), w_k \rangle d\mu_i(z_i) = \int_0^1 w_k(x) dx = 0.$$  

Note that

$$\sum_{k_1, k_2=1}^{2^m} \nu_{k_1} \nu_{k_2}' \frac{\nu_{k_1}}{\lambda_{k_1, \alpha, m}} \frac{\nu_{k_2}'}{\lambda_{k_2, \alpha, m}} \mathbb{E}_{Q_{0,n}} \left[ \frac{\langle q_i(Z_i \cdot), w_{k_1} \rangle}{g_{0,i}(Z_i)} \cdot \frac{\langle q_i(Z_i \cdot), w_{k_2} \rangle}{g_{0,i}(Z_i)} \right]$$

$$= \sum_{k_1, k_2=1}^{2^m} \nu_{k_1} \nu_{k_2}' \frac{1}{\lambda_{k_1, \alpha, m} \lambda_{k_2, \alpha, m}} \int_0^1 \int_0^1 F_i(x, y) w_{k_1}(x) w_{k_2}(y) dxdy.$$ 

Now, we use that $1 + u \leq \exp(u)$ for all real numbers $u$ and since $w_k$ are orthonormal eigenfunctions of $K = \frac{1}{n} \sum_{i=1}^n K_i^* K_i$ we also have that

$$\chi^2(Q_n, Q_{0,n}) \leq -1 + \mathbb{E}_{\nu, \nu'} \exp \left( 2^{-2m(s+1)} \delta^2 \sum_{k_1, k_2=1}^{2^m} \frac{\nu_{k_1} \nu_{k_2}'}{\lambda_{k_1, \alpha, m} \lambda_{k_2, \alpha, m}} \int_0^1 \int_0^1 \frac{1}{n} \sum_{i=1}^n F_i(x, y) w_{k_1}(x) w_{k_2}(y) dxdy \right).$$

Remember that

$$\int_0^1 \int_0^1 \frac{1}{n} \sum_{i=1}^n F_i(x, y) w_{k_1}(x) w_{k_2}(y) dxdy = \langle K w_{k_1}, w_{k_2} \rangle = \lambda_k^2 \cdot \langle w_{k_1}, w_{k_2} \rangle,$$

where $\lambda_k^2$ are eigenvalues of $K$.

We use that

$$\frac{\lambda_k^2}{\lambda_{k, \alpha, m}^2} = \frac{\lambda_k^2}{(z_\alpha^{-2} \cdot \lambda_k^2) \vee 2^{-m}} \leq z_\alpha^2,$$

and that $\nu_k \nu_k'$, $k = 1, \ldots, 2^m$, are Rademacher distributed and independent, to further obtain

$$\chi^2(Q_n, Q_{0,n}) \leq -1 + \mathbb{E}_{\nu} \left( \sum_{k=1}^{2^m} 2^{-2m(s+1)} \delta^2 \frac{\nu_k}{\lambda_{k, \alpha, m}^2} n \lambda_k^2 \right)$$

$$= -1 + \mathbb{E}_{\nu} \prod_{k=1}^{2^m} \exp \left( \nu_k 2^{-2m(s+1)} \delta^2 n \lambda_k^2 \right)$$

$$\leq -1 + \prod_{k=1}^{2^m} \cosh(2^{-2m(s+1)} \delta^2 n z_\alpha^2).$$
Let us further see that $\cosh(u) \leq \exp(u^2/2)$ for all real numbers $u$ and therefore

$$\chi^2(Q_n, Q_{0,n}) \leq -1 + \exp \left( \frac{1}{2} \sum_{k=1}^{2m} (2^{-2m(s+1)}\delta^2 n z_\alpha^2)^2 \right)$$

$$\leq -1 + \exp(2^{-m(4s+3)}\delta^4 \cdot n^2 z_\alpha^4),$$

which, for our choice of $m$ and $\delta = \delta_m$ (cf. Lemma A.1), tends to 0 as $n z_\alpha^2$ becomes large. This concludes the proof. $\Box$

### A.4. Auxiliary lemmas.

**Lemma A.1.** Let $P_\nu$ denote the uniform distribution on $\{-1, 1\}^{2m}$. For any $\gamma \in (0, 1)$, there exists $\delta = \delta_m = c/\sqrt{2 \log(2^{m+1}/\gamma)}$ in the definition of $f_\nu$ for some constant $c > 0$ independent of $m$ and a subset $A_\gamma \subseteq \{-1, 1\}^{2m}$ with $P_\nu(A_\gamma) \geq 1 - \gamma$, such that

(i) $f_\nu \geq 0$ and $\|f_\nu\|_\infty \leq M$, for all $\nu \in A_\gamma$,

(ii) $f_\nu \in P_{pq}^s(L)$ for all $\nu \in A_\gamma$, and

(iii) $|D(f_\nu) - D(f_0)| \geq \delta^2 \cdot 2^{-2ms}$ for all $\nu$.

**Proof.** Representing the orthonormal eigenvectors $w_1, \ldots, w_{2m}$ as linear combination

$$w_j = \sum_{k=0}^{2m-1} a_{kj} \psi_{mk},$$

of the basis functions $\psi_{mk}$, the $2m \times 2m$ matrix $(a_{kj})_{kj}$ of corresponding coefficients is orthogonal and

$$f_\nu(x) = f_0(x) + 2^{-m(s+1)} \delta \sum_{j=1}^{2m} \sum_{k=0}^{2m-1} \nu_j a_{kj} \frac{\lambda_{j,\alpha,m}}{\lambda_{j,\alpha,m}} \psi_{mk}(x).$$

For some $\gamma$ in $(0,1)$, define

$$A_\gamma = \left\{ \nu : \left| \sum_{j=1}^{2m} \nu_j \frac{a_{kj}}{\lambda_{j,\alpha,m}} \right| \leq 2^{m/2} \sqrt{2 \log \left( \frac{2^{m+1}}{\gamma} \right)} \text{ for all } 0 \leq k \leq 2m - 1 \right\}.$$
(i) Since the basis functions \( \psi_{mk}, k = 0, \ldots, 2^m - 1 \), have disjoint support and are bounded in absolute value by \( 2^m \), it has to be shown that

\[
\left| 2^{-m(s+1)} \delta \sum_{j=1}^{2^m} \nu_j \frac{a_{kj}}{\lambda_{j,\alpha,m}} \right| \leq 2^{-m/2}
\]

for all \( \nu \in A_\gamma \). But on this event we have

\[
\left| 2^{-m(s+1)} \delta \sum_{j=1}^{2^m} \nu_j \frac{a_{kj}}{\lambda_{j,\alpha,m}} \right| \leq 2^{m/2} \leq \delta \cdot 2^{-ms} \sqrt{2 \log(2m+1/\gamma)}.
\]

But by our choice of \( \delta \) and provided \( c \leq 1 \), it follows that

\[
\delta \cdot 2^{-ms} \sqrt{2 \log(2m+1/\gamma)} \leq 1.
\]

Note that this also proves that \( \|f_\nu\|_\infty \leq 2 \leq M \).

(ii) We have already seen that for \( \nu \in A_\gamma \), \( f_\nu \) is a density. Note that \( \|f_\nu\|_{B^{p,q}_S} \leq 1 + \|f_\nu - f_0\|_{B^{p,q}_S} \). Since \( s < 1 \), the Haar wavelets can be used to characterize the Besov space and in view of Proposition 4.3.2 of Giné and Nickl (2016) and if \( c > 0 \) is sufficiently small, it remains to show that the wavelet coefficient norm of \( f_\nu - f_0 \) is bounded by \( L - 1 > 0 \). That is, we need to show that

\[
\sum_{k=0}^{2^m-1} \left| \langle f_\nu - f_0, \psi_{mk} \rangle \right|^p \leq (L - 1)^p \cdot 2^{-mp(s+1/2-1/p)}.
\]

Because of

\[
\langle f_\nu - f_0, \psi_{mk} \rangle = 2^{-m(s+1)} \delta \sum_{j=1}^{2^m} \nu_j \frac{a_{kj}}{\lambda_{j,\alpha,m}},
\]

for any \( 0 \leq k \leq 2^m - 1 \), this is the case if

\[
\sum_{k=0}^{2^m-1} \left| 2^{-m(s+1)} \delta \sum_{j=1}^{2^m} \nu_j \frac{a_{kj}}{\lambda_{j,\alpha,m}} \right|^p \leq (L - 1)^p \cdot 2^{-mp(s+1/2-1/p)}.
\]

But if \( \nu \in A_\gamma \), we have

\[
\sum_{k=0}^{2^m-1} \left| 2^{-m(s+1)} \delta \sum_{j=1}^{2^m} \nu_j \frac{a_{kj}}{\lambda_{j,\alpha,m}} \right|^p \leq 2^m \left( 2^{-m(s+1)} \delta 2^m \sqrt{2 \log \left( \frac{2m+1}{\gamma} \right)} \right)^p \leq 2^{-mp(s-1/p+1/2)}c^p.
\]
Thus, if \( c \leq L - 1 \) and \( \nu \in A_\gamma \), \( f_\nu \) belongs to \( \mathcal{P}_{s}^{pq}(L) \).

(iii) By orthonormality of \( f_0, w_1, \ldots, w_{2m} \), we have that

\[
\int_0^1 f_\nu^2(x)dx = \int_0^1 f_0^2(x)dx + 2^{-2m(s+1)} \frac{\delta^2}{s^2} \sum_{k=1}^{2m} \frac{1}{\lambda_k^2}. \]

We get

\[
|D(f_\nu) - D(f_0)| = 2^{-2m(s+1)} \delta^2 \sum_{k=1}^{2m} \left( 2^{m} \cdot \mathbb{1}_{z_\alpha^{-1} \lambda_k < 2^{-m/2}} + \frac{z_\alpha^2}{\lambda_k^2} \cdot \mathbb{1}_{z_\alpha^{-1} \lambda_k \geq 2^{-m/2}} \right).
\]

Denote by \( \kappa \) the number of values in the set \( K = \{ k : z_\alpha^{-1} \lambda_k \geq 2^{-m/2} \} \). We have

\[
|D(f_\nu) - D(f_0)| = 2^{-2ms} \delta^2 \left( 2^{-m} (2^m - \kappa) + 2^{-2m} \sum_{k \in K} \frac{z_\alpha^2}{\lambda_k^2} \right)
\]

\[
= 2^{-2ms} \delta^2 \left( 1 - 2^{-m} \kappa + 2^{-2m} \sum_{k \in K} \frac{1}{\lambda_k^2} \right)
\]

\[
\geq 2^{-2ms} \delta^2 \left( 1 - 2^{-m} \kappa + 2^{-2m} z_\alpha^2 \kappa^2 \left( \sum_{k \in K} \lambda_k^2 \right)^{-1} \right),
\]

where we used the inequality between harmonic and arithmetic mean. If we can prove

\[
(A.5) \quad \sum_{k=1}^{2m} \lambda_k^2 \leq z_\alpha^2,
\]

we conclude that

\[
|D(f_\nu) - D(f_0)| \geq 2^{-2ms} \delta^2 (1 - 2^{-m} \kappa + 2^{-2m} \kappa^2) \geq \frac{3}{4} \delta^2 2^{-2ms}.
\]

Thus, let us finish by the proof of \((A.5)\). It is easy to see that

\[
\sum_{k=1}^{2m} \lambda_k^2 = \sum_{k=1}^{2m} \int_0^1 \int_0^1 w_k(x)w_k(y) \frac{1}{n} \sum_{i=1}^{n} F_i(x, y) dx dy
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \int_{Z} \sum_{k=1}^{2m} \left( \int_0^1 \frac{g_i(z|x)w_k(x)dx} {g_{0,i}(z)} \right)^2 g_{0,i}(z)d\mu_i(z)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \int_{Z} \sum_{k=1}^{2m} \left( \int_0^1 \frac{g_i(z|x) - e^{-2\alpha}} {g_{0,i}(z)} w_k(x)dx \right)^2 g_{0,i}(z)d\mu_i(z)
\]

where we used the inequality between harmonic and arithmetic mean. If we can prove

\[
(A.5) \quad \sum_{k=1}^{2m} \lambda_k^2 \leq z_\alpha^2,
\]

we conclude that

\[
|D(f_\nu) - D(f_0)| \geq 2^{-2ms} \delta^2 (1 - 2^{-m} \kappa + 2^{-2m} \kappa^2) \geq \frac{3}{4} \delta^2 2^{-2ms}.
\]
as \( \int_0^1 w_k(x) \, dx = 0 \). By our choice of densities \( q_i \), we have \( 0 \leq f_{z,i}(x) := \frac{q_i(z|x)}{g_{0,i}(z)} - e^{-2\alpha} \leq z_\alpha \). Since the \( w_1, \ldots, w_{2^m} \) are orthonormal and \( \|f_{z,i}\|_{L_2} \leq z_\alpha \), it follows that

\[
\sum_{k=1}^{2^m} \left( \int_0^1 \left( \frac{q_i(z|x)}{g_{0,i}(z)} - e^{-2\alpha} \right) w_k(x) \, dx \right)^2 = \sum_{k=1}^{2^m} \langle f_{z,i}, w_k \rangle^2 = \left\| \sum_{k=1}^{2^m} \langle f_{z,i}, w_k \rangle w_k \right\|^2_{L_2} \leq z^2_\alpha.
\]

Moreover, \( \int_{\mathbb{R}} g_{0,i}(z) d\mu_i(z) = \int_0^1 \int_{\mathbb{R}} q_i(z|x) d\mu_i(z) dx = 1 \), and we arrive at \( \sum_{k=1}^{2^m} \lambda_k^2 \leq z^2_\alpha \), as desired.

**Lemma A.2.** If \( f \) belongs to the Besov ball \( B_{s,p,q}^p(\mathbb{R}) := \{ g \in B_{s,p,q}^p([0,1]) : \|g\|_{s,p,q} \leq R \} \) and \( s' := s - \left( \frac{1}{p} - \frac{1}{2} \right)_+ < 1 \), then the Haar coefficients of \( f \) satisfy

\[
\|\beta_j\|_2 \leq 2^{-j s'} \varepsilon_j, \quad j \geq 1,
\]

and \( \|\varepsilon\|_q \leq C_0 R \), for a constant \( C_0 \) that does not depend on \( f \).

**Proof of Lemma A.2.** We shall consider first the case \( p \geq 2 \). In that case, by the Hölder inequality with \( \frac{1}{2} = \frac{1}{p} + \left( \frac{1}{2} - \frac{1}{p} \right) \), we get

\[
\|\beta_j\|_2 \leq 2^{j \left( \frac{1}{2} - \frac{1}{p} \right)} \|\beta_j\|_p.
\]

Since \( s = s' < 1 \), by (2.3), we have that \( \|\beta_j\|_p = 2^{-j (s + \frac{1}{2} - \frac{1}{p})} \varepsilon_j \), for a sequence \( \{\varepsilon_j\} \in \ell_q \) with \( \|\varepsilon\|_q \leq \|f\|_{B_{s,p,q}^p} \leq C_0 R \), giving

\[
\|\beta_j\|_2 \leq 2^{-j s} \varepsilon_j.
\]

In the case \( 1 \leq p < 2 \), we use the continuous embedding

\[
B_{s,p,q}^p \subseteq B_{s-\frac{1}{p}+\frac{1}{2},2q}^{2q} = B_{s',2q}^{2q},
\]

which follows from the characterization of the Besov space in terms of wavelet coefficients. Again, by (2.3), we get

\[
\|\beta_j\|_2 = 2^{-j s'} \varepsilon_j,
\]

for a sequence \( \{\varepsilon_j\} \in \ell_q \) as desired. \( \square \)
Lemma A.3. Fix $J \geq 1$ and, for $j \geq J$ and $k = 0, \ldots, 2^j - 1$, let $\beta_{jk}$ be the Haar coefficients of $f \in P^p_q(L)$. Then, for $s' = s - \left(\frac{1}{p} - \frac{1}{2}\right)_+$, we have

$$\sum_{j \geq J} \|\beta_j\|_2^2 \lesssim 2^{-2Js'} I_{[0,1)}(s') + 2^{-J\frac{2}{3}} I_{[1,\infty)}(s')$$

Proof. We consider successively the cases where $1 \leq p < 2$ and where $p \geq 2$. If $1 \leq p < 2$, the continuous embedding $B^p_q \subseteq B^{2q}_{s - \frac{1}{p} + \frac{1}{2}}$ holds in view of the definition of the wavelet Besov norm. Now, in the case $s' = s - \frac{1}{p} + \frac{1}{2} < 1$ we get, by Lemma A.2, that

$$\sum_{j \geq J} \|\beta_j\|_2^2 \leq \sum_{j \geq J} 2^{-2js'} \varepsilon_j^2 \leq C \cdot 2^{-2Js'},$$

for some $C > 0$ that does not depend on $f$, by using that $\|\varepsilon\|_\infty \leq \|\varepsilon\|_q$. In case $s' \geq 1 > 5/6$, we use the further embedding $B^p_q \subseteq B^{2q}_{s'} \subseteq B^{2q}_{5/6}$. Thus, from Lemma A.2, we get that $\|\beta_j\|_2 \leq 2^{-j5/6} C$, which implies

$$\sum_{j \geq J} \|\beta_j\|_2^2 \leq \sum_{j \geq J} 2^{-j\frac{5}{6}} C^2 \leq C^2 \cdot 2^{-J\frac{5}{6}}.$$ 

If $p \geq 2$, and if $s' = s < 1$, then we apply directly Lemma A.2 to get

$$\sum_{j \geq J} \|\beta_j\|_2^2 \leq C 2^{-2Js},$$

for some constant $C > 0$.

If $s' = s \geq 1$, we use the embedding $B^p_q \subseteq B^{p_q}_{5/6}$ and conclude, again from Lemma A.2, that

$$\sum_{j \geq J} \|\beta_j\|_2^2 = \sum_{j \geq J} 2^{-j\frac{5}{6}} \varepsilon_j^2 \leq C \cdot 2^{-J\frac{5}{6}},$$

for some constant $C > 0$.

\[\square\]

Appendix B: Proofs of Section 4 (Sequentially Interactive Protocols)

B.1. Concentration of the sanitized density estimator.
Proposition B.1. Fix $M, L > 0$, $1 \leq p, q \leq \infty$ and $s > 0$, and let $\mathcal{P}^p_q(L, M)$ be defined as in Theorem 4.1. Then there exist constants $c_1, c_2 > 0$, such that for any $n \geq 1$, $\alpha \leq 1$ and $J \geq 2$, the estimator $\hat{f}_J^{(1)}$ in (4.1) satisfies

$$
\sup_{x \in [0, 1]} Q_{f_J}^{(SL)} \left( \left| \hat{f}_J^{(1)}(x) - \mathbb{E}_{f_J}^{(SL)} \left[ \hat{f}_J^{(1)}(x) \right] \right| \right) \geq \left[ c_1 \frac{J^{\alpha/2} J}{\alpha n \sqrt{u}} \wedge \left[ c_2 \frac{J^{\alpha/2} J}{n \alpha} u \right] \right] \leq 4e^{-u/2},
$$

for all $u > 0$ and all $f \in \mathcal{P}^p_q(L, M)$.

Proof. A centered random variable $Y$ is said to be sub-exponential with parameters $(\nu^2, b)$, $\nu^2 > 0$, $b > 0$, denoted by $SubExp(\nu^2, b)$, if

$$
\mathbb{E}[\exp(tY)] \leq \exp \left( \frac{\nu^2 t^2}{2} \right), \text{ for all } |t| < \frac{1}{b}.
$$

The Bernstein inequality states that if $Y$ is $SubExp(\nu^2, b)$, then

$$
\mathbb{P}(Y \geq t) \leq \exp \left( -\frac{1}{2} \left( \frac{t^2}{\nu^2} \wedge \frac{t}{b} \right) \right), \text{ for all } t > 0,
$$

or, equivalently,

$$
\mathbb{P}(Y \geq (\nu \sqrt{u}) \vee (bu)) \leq \exp \left( -\frac{u}{2} \right), \text{ for all } u > 0.
$$

We apply this to the two summands in the decomposition

$$
\hat{f}_J^{(1)}(x) - \mathbb{E}_f \left[ \hat{f}_J^{(1)}(x) \right] = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^{J-1} \sum_{k=0}^{(1/2)^j-1} [\psi_{jk}(X_i) - \beta_{jk}] \psi_{jk}(x)
$$

$$
+ \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^{J-1} \sum_{k=0}^{(1/2)^j-1} \left[ \sigma_j \frac{\sigma}{\alpha} W_{ijk} \psi_{jk}(x) \right]
$$

$$
= : T_1(x) + T_2(x).
$$

Let us start with $T_2$. A Laplace distribution has the following Laplace transform

$$
\mathbb{E} \left[ \exp \left( \frac{t}{n} \frac{\sigma}{\alpha} \cdot W_{ijk} \psi_{jk}(x) \right) \right] = \frac{1}{1 - \left( \frac{t}{n} \frac{\sigma}{\alpha} \psi_{jk}(x) \right)^2},
$$
for any $|t| < \frac{n_\alpha}{2\sigma_j(1 + 2^{j/2})}$.

By the Bernstein inequality we get for all $u > 0$,

$$(B.1) \quad \mathbb{P} \left( |T_2(x)| \geq \left( \frac{2\sigma J^a 2^j}{\alpha \sqrt{n} u} \right) \vee \left( \frac{2\sigma J^a 2^j}{\alpha n_\alpha u} \right) \right) \leq 2 e^{-u/2}.$$ 

For $T_1$, we use the fact that the Haar wavelets generate a multiresolution analysis of $L_2[0,1]$ with Projection operator $P_J$ onto $V_J = \text{span}\{\phi_{jm} : m = 0, \ldots, 2^j - 1\}$ to write

$$T_1(x) = \hat{f}_j^{(1)}(x) - [P_J f](x) = \frac{1}{n} \sum_{i=1}^{n} \kappa_J(x, X_i) - [P_J f](x)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{k=0}^{2^j - 1} (\phi_{jk}(X_i) - \alpha_{jk}) \phi_{jk}(x).$$
We have that
\[
\left| \sum_k [\phi_{j,k}(X_i) - \alpha_{j,k}]\phi_{j,k}(x) \right| \leq \sum_k [\phi_{j,k}(X_i) + \alpha_{j,k}]\phi_{j,k}(x) \\
\leq \sum_k \left( 2^{j/2} + \int_0^1 2^{j/2} \phi(2^j x - k)f(x)dx \right) \phi_{j,k}(x) \\
\leq 2 \cdot 2^{j/2} \sum_k 2^{j/2} \phi(2^j x - k) = 2^{j+1}.
\]

We write for \(|t| \leq n2^{-j-1}\) and \(|u| \leq \frac{|t|}{n} 2^{j+1} \leq 1\), that \(e^u \leq 1 + u + u^2e\).

Thus, we get
\[
\mathbb{E} \left[ \exp \left( \frac{t}{n} \sum_{i=1}^n \sum_k [\phi_{j,k}(X_i) - \alpha_{j,k}]\phi_{j,k}(x) \right) \right] \\
= \left( \mathbb{E} \exp \left( \frac{t}{n} \sum_k [\phi_{j,k}(X_1) - \alpha_{j,k}]\phi_{j,k}(x) \right) \right)^n \\
\leq \left( 1 + e \frac{t^2}{n^2} \mathbb{E} \left[ \left( \sum_k [\phi_{j,k}(X_1) - \alpha_{j,k}]\phi_{j,k}(x) \right)^2 \right] \right)^n \\
\leq \left( 1 + e \frac{t^2}{n^2} \text{Var} \left( \sum_k [\phi_{j,k}(X_1)]\phi_{j,k}(x) \right) \right)^n.
\]

The Haar basis is such that for \(F_j = 2^j \mathbb{1}_{[0,2^{-j}]}\), and for all \(x, y \in [0,1]\),
\[
\sum_k \phi_{j,k}(x)\phi_{j,k}(y) = \sum_k 2^j \mathbb{1}_{[k2^{-j},(k+1)2^{-j})^2}(x,y) \leq F_j(|x - y|),
\]
and it follows that
\[
\text{Var} \left( \sum_k \phi_{j,k}(X_1)\phi_{j,k}(x) \right) \leq \mathbb{E} \left[ \left( \sum_k \phi_{j,k}(X_1)\phi_{j,k}(x) \right)^2 \right] \\
\leq \mathbb{E} \left[ F_j(|X_1 - x|^2) \right] \leq 2^{j+1} \|f\|_{\infty}.
\]

Thus, for all \(t\) such that \(|t| \leq \frac{n}{2^{j+1}}\), we obtain
\[
\mathbb{E} \left[ \exp \left( \frac{t}{n} \sum_{i=1}^n \sum_k [\phi_{j,k}(X_i) - \alpha_{j,k}]\phi_{j,k}(x) \right) \right] \\
\leq \left( 1 + e \frac{t^2}{n^2} 2^{j+1} \|f\|_{\infty} \right)^n \\
\leq \exp \left( \frac{t^2}{2} e \frac{2^{j+2}}{n} \|f\|_{\infty} \right).
\]
Now, by the Bernstein inequality, we get for all \( u > 0 \),
\[
(B.2) \quad \mathbb{P}
\left(|T_1(x)| \geq \frac{\|f\|_\infty^{1/2} 2^{(J+2)/2} \sqrt{\mathbb{E}}}{\sqrt{n}} \sqrt{u} \right) \vee \left[\frac{2^J}{n^2} u\right] \leq 2e^{-u/2}.
\]

Putting together (B.2) and (B.1), we get the result. \( \square \)

**B.2. Proof of Theorem 4.1.** For \( f \in \mathcal{D}_{pq}^0(L) \), we have
\[
E_{Q_j(s)} \tilde{D}_n = E \left[ E \left[ \tilde{D}_n | Z^{(1)} \right] \right] = E \left[ E \left[ Z_1^{(2)} | X_1^{(2)}, Z^{(1)} \right] \right]
= E \left[ \int_0^1 \Pi_\tau \tilde{f}_J(1)(x) f(x) dx \right].
\]
Note that \( \Pi_\tau(v) = (\tau \wedge v) \vee (-\tau) = v - (v - \tau)_+ + (v - \tau)_- \). Thus, the bias can be written as
\[
D - E \left[ \tilde{D}_n \right] = D - E \left[ \int_0^1 \tilde{f}_J(1)(x) f(x) dx \right] + E \left[ \int_0^1 \left( \tilde{f}_J(1)(x) - \tau \right)_+ f(x) dx \right] \quad (B.3)
- E \left[ \int_0^1 \left( -\tilde{f}_J(1)(x) - \tau \right)_+ f(x) dx \right].
\]
First, compute \( E \left[ \int_0^1 \tilde{f}_J(1)(x) f(x) dx \right] = \int_0^1 \sum_{j=-1}^{J-1} \sum_{k=0}^{(1+2^{J-1})-1} \beta_{jk} \psi_{jk}(x) f(x) dx = \sum_{j=-1}^{J-1} \|\beta_{j}\|_2^2. \) Thus, by Lemma A.3, we get
\[
(B.4) \quad \left| D - E \left[ \int \tilde{f}_J(1)(x) f(x) dx \right] \right| = \sum_{j \geq J} \|\beta_{j}\|_2^2 \lesssim \begin{cases} 2^{-2J s'}, & 0 \leq s' < 1, \\ 2^{-J/2}, & s' \geq 1. \end{cases}
\]
Next, we treat the second term on the right-hand-side of (B.3), but we skip the third because a similar bound can be obtained by analogous arguments. We write
\[
E \left[ \int_0^1 \left( \tilde{f}_J(1)(x) - \tau \right)_+ f(x) dx \right]
= E \left[ \int_0^1 \left( \tilde{f}_J(1)(x) - E \left[ \tilde{f}_J(1)(x) \right] \right) + E \left[ \tilde{f}_J(1)(x) \right] - \tau \right)_+ f(x) dx \right]
\]
and we note that \( \tau \geq 2M \), where \( M \) is the uniform bound on \( \|f\|_\infty \). Then, using the fact that the Haar basis generates a multiresolution analysis of \( L_2([0, 1]) \) with projection operator \( P_J \) onto \( V_J = \text{span}\{\phi_{jm} : m = 0, \ldots, 2^J - 1\} \),
\[
\tau - E \left[ \tilde{f}_J(1)(x) \right] = \tau - P_J f = \tau - \frac{\tau - 2M}{2} > \frac{\tau - 2M}{2},
\]
and we observe that
\[
\tau - E \left[ \tilde{f}_J(1)(x) \right] \leq \frac{\tau - 2M}{2} = \frac{\tau - 2M}{2}.
\]

because $P_J f = \sum_{k=0}^{2^J-1} \alpha_{Jk} \phi_{Jk}(x) \geq 0$, $\alpha_{Jk} = \int_0^1 f(x) 2^J \phi(2^J x - k) dx \leq M 2^{-J/2}$ and,

\begin{align}
(B.5) \quad \sup_{x \in [0,1]} P_J f(x) & \leq \sup_{x \in [0,1]} M \sum_{k=0}^{2^J-1} \phi(2^J x - k) \leq M,
\end{align}

as the functions $\phi(2^J \cdot - k)$ have disjoint supports for different values of $k$. For $x \in [0,1]$, let us denote $Y_x = \hat{f}_J^{(1)}(x) - \mathbb{E}[\hat{f}_J^{(1)}(x)]$ and, using the previous consideration, write

\begin{align}
(B.6) \quad \mathbb{E} \left[ \int_0^1 (\hat{f}_J^{(1)}(x) - \tau)_+ f(x) \, dx \right] & \leq \mathbb{E} \left[ \int_0^1 (Y_x - \frac{\tau}{2})_+ f(x) \, dx \right] \\
& = \int_0^1 \int_0^\infty \mathbb{P}_f \left( Y_x - \frac{\tau}{2} \geq u \right) \, dw \, f(x) \, dx \\
& = \int_0^1 \int_{\tau/2}^\infty \mathbb{P}_f( Y_x \geq u ) \, du \, f(x) \, dx.
\end{align}

We apply Proposition B.1 to get, for some constant $c > 0$ not depending on $n, f, \alpha$ and $u$, and for all $u > 0$,

\begin{align}
(B.7) \quad \mathbb{P}(Y_x \geq u) & \leq 4 \exp \left( -\frac{c}{2} \left[ \left( \frac{n\alpha^2}{J^a 2^J u^2} \right) \wedge \left( \frac{n\alpha}{J^a 2^J u} \right) \right] \right),
\end{align}

which implies

\begin{align}
\frac{1}{4} \int_{\tau/2}^\infty \mathbb{P}(Y_x \geq u) \, du & \leq \int_{\tau/2}^{J^a 2^J / \alpha} \exp \left( -\frac{c}{2} \frac{n\alpha^2}{J^a 2^J u^2} \right) \, du + \int_{J^a 2^J / \alpha}^\infty \exp \left( -\frac{c}{2} \frac{n\alpha}{J^a 2^J u} \right) \, du
\end{align}

Next, we apply Lemma B.2 with $a_1 = 0$, $A_1 = \frac{c}{2} \frac{n\alpha^2}{J^a 2^J}$, $r_1 = 2$ and $v_1 = \frac{\tau}{2}$, and with $a_2 = 0$, $A_2 = \frac{c}{2} \frac{n\alpha}{J^a 2^J}$, $r_2 = 1$ and $v_2 = J^a 2^J / \alpha$. Note that in the
former case,

\[
A_1 r_1 v_1^r_1 = \frac{c}{2} \frac{na^2}{J^2a^2 \tau^2} \leq \frac{c(KM)^2}{2} \frac{na^2}{J^2a^2 \tau^2} J^{2a+1} 2^{-J(1-2(s' \wedge \frac{1}{2}))}
\]

\[
= \frac{c(KM)^2}{2} \frac{na^2}{J^2a^2 \tau^2} J^{2a+1} 2^{-J(1+2(s' \wedge \frac{1}{2}))}
\]

\[
= \frac{c(KM)^2}{2} \frac{na^2}{2(s' \wedge 1) + 1} \log(na^2) (na^2)^{-1-2(s' \wedge \frac{1}{2})}
\]

\[
\geq \frac{c(KM)^2}{6 \log 2} (na^2)^{-1-2(s' \wedge \frac{1}{2})} \log(na^2)
\]

\[
= \frac{c(KM)^2}{6 \log 2} (na^2)^{-2(s' \wedge 1) + 1} \log(na^2)
\]

\[
\geq \frac{c(KM)^2}{6 \log 2} \log(na^2) \geq 3,
\]

for \(na^2 \geq \exp(\frac{12 \log 2}{c(KM)^2})\). In the latter case \(A_2 r_2 v_2^r_2 = \frac{\tau}{2} n \geq 3\), if \(na^2 \geq 6/c\), because \(\alpha \leq 1\). Therefore, Lemma B.2 yields

\[
\int_{\tau/2}^{\infty} \exp \left( -\frac{c}{2} \frac{na^2}{J^2a^2 \tau^2} u^2 \right) du \leq \left( A_1 v_1 \right)^{-1} \exp \left( -A_1 r_1 v_1^r_1 \right)
\]

\[
= \frac{2v_1}{A_1 r_1 v_1^r_1} \exp \left( -\frac{1}{2} A_1 r_1 v_1^r_1 \right)
\]

\[
\leq \frac{\tau}{2} (na^2)^{\frac{c(KM)^2}{12 \log 2}},
\]

but for sufficiently large \(K\) this will always be smaller than the final rate. Similarly,

\[
\int_{J^{a^2} / \alpha}^{\infty} \exp \left( -\frac{c}{2} \frac{na^2}{J^2a^2 \tau^2} u^2 \right) du \leq 2A^{-1} \exp \left( -A_2 v_2^r_2 \right)
\]

\[
= \frac{J^{a^2}}{A_2 r_2 v_2} \exp \left( -A_2 r_2 v_2^r_2 \right)
\]

\[
= \frac{J^{a^2}}{\alpha} \frac{4}{en} e^{-n^2 \frac{c}{2}}
\]

\[
\leq \frac{J^{a^2}}{\alpha} \frac{4}{en^2} e^{-na^2 \frac{c}{2}},
\]

which is much smaller than the final rate. Thus, the only relevant contribution from the bias is the one of (B.4).

Regarding the variance, we write

\[
\text{Var}[\tilde{D}_n] = \mathbb{E} \left[ \text{Var}(\tilde{D}_n | X^{(1)}) \right] + \text{Var} \left[ \mathbb{E}(\tilde{D}_n | X^{(1)}) \right]
\]
Now,
\[
\text{Var}(\tilde{D}_n|X^{(1)}) \leq \mathbb{E}(\tilde{D}_n^2|X^{(1)}) = \frac{1}{n}\mathbb{E}((Z_1^{(2)})^2|X^{(1)})
\]
\[
= \frac{\tau^2}{n} \left( \frac{e^\alpha + 1}{e^\alpha - 1} \right)^2 \leq (e + 1)^2 4(KM)^2 J^{2a+1} 2^{J(1-2(s'\wedge \frac{1}{2}))} \frac{1}{n\alpha^2},
\]
since \(\alpha \leq 1\) and \(\alpha \leq e^\alpha - 1\). Moreover,
\[
\text{Var}\left[\mathbb{E}(\tilde{D}_n|X^{(1)})\right] = \text{Var}\left[\int_0^1 \Pi x \left[\hat{f}^{(1)}(x)\right] f(x)dx\right]
\leq 3 \text{Var}\left[\int_0^1 (\hat{f}^{(1)}(x) - \tau)_+ f(x)dx\right] + 3 \text{Var}\left[\int_0^1 \hat{f}^{(1)}(x)f(x)dx\right]
\]
\[+ 3 \text{Var}\left[\int_0^1 (-\hat{f}^{(1)}(x) - \tau)_+ f(x)dx\right].
\]
Again, we only explicitly treat the first two terms, as the first and the third terms are handled analogously. We have
\[
\text{Var}\left[\int_0^1 (\hat{f}^{(1)}(x) - \tau)_+ f(x)dx\right] \leq \mathbb{E}\left[\left(\int_0^1 (\hat{f}^{(1)}(x) - \tau)_+ f(x)dx\right)^2\right]
\leq \int_0^1 \mathbb{E}\left[\left(\hat{f}^{(1)}(x) - \tau\right)_+^2\right] f(x)dx \leq \int_0^1 \mathbb{E}\left[\left(Y_x - \frac{\tau}{2}\right)_+^2\right] f(x)dx,
\]
as in (B.6). Thus, we may use (B.7) and Lemma B.2 with \(a_1 = 1\) and \(A_1, r_1\) and \(v_1\) as above, satisfying \(A_1 r_1 v_1^2 \geq 3\) and with \(a_2 = 1\) and \(A_2, r_2\) and \(v_2\) as above, satisfying \(A_2 r_2 v_2^2 \geq 3\), in order to get
\[
\mathbb{E}\left[\left(Y_x - \frac{\tau}{2}\right)_+^2\right] = 2 \int_0^\infty \int_0^\infty tP f\left(Y_x - \frac{\tau}{2} \geq t\right) dt
\leq 2 \int_\tau^\infty sP f(Y_x \geq s) ds
\leq 8 \int_{\tau/2}^{J_0^{a_2J}/\alpha} s \exp\left(-\frac{c}{2} \frac{n\alpha^2}{J^{2a_2J} s^2}\right) ds
+ 8 \int_{J_0^{a_2J}/\alpha}^\infty s \exp\left(-\frac{c}{2} \frac{n\alpha^2}{J^{2a_2J} s}\right) ds
\leq 8 \frac{3}{A_1 r_1} \exp(-A_1 v_1^2) + 8 \frac{3 v_2}{A_2 r_2} \exp(-A_2 v_2^2)
\]
\[
(\text{B.8}) \leq 12 \frac{J^{2a_2J}}{c} \frac{n\alpha^2}{n\alpha^2} \exp\left(-\frac{c}{2} \frac{n\alpha^2}{J^{2a_2J} \frac{\tau^2}{4}}\right) + 24 \frac{J^{2a_2J}}{c} \frac{n\alpha^2}{n\alpha^2} \exp\left(-\frac{n\alpha^2}{2}\right).
\]
Again, both terms in the last line of the previous display are smaller than the final rate of our estimator, provided that the constant $K$ is large enough. Finally, we consider the variance of the integrated estimator in (4.1), that is,

$$\text{Var} \left[ \int_0^1 \hat{f}_j^{(1)}(x)f(x) dx \right] = \text{Var} \left[ \sum_{j=-1}^{J-1} \sum_{k=0}^{(1\vee 2^j)-1} \hat{\beta}_{jk} \right]$$

$$\leq \text{Var} \left[ \sum_{j=-1}^{J-1} \sum_{k=0}^{(1\vee 2^j)-1} \frac{1}{n} \sum_{i=1}^{n} \psi_{jk}(X_i) \beta_{jk} \right]$$

$$+ \text{Var} \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{j=-1}^{J-1} \sum_{k=0}^{(1\vee 2^j)-1} \sigma_j \frac{\sigma}{\alpha} W_{ijk} \beta_{jk} \right]$$

$$= \frac{1}{n} \text{Var} \left[ \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \psi_{jk}(X_i) \beta_{jk} \right] + \frac{1}{n} \sum_{j=-1}^{J-1} \sum_{k=0}^{(1\vee 2^j)-1} \text{Var}(W_{j1k}) \beta_{jk}^2 \sigma_j^2$$

$$\leq \frac{1}{n} \int_0^1 \left( \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \psi_{jk}(x) \beta_{jk}^2 \right) f(x) dx + \frac{2}{n} \sum_{j=-1}^{J-1} \sum_{k=0}^{(1\vee 2^j)-1} \text{Var}(W_{j1k}) \beta_{jk}^2 \sigma_j^2$$

(B.9)

$$\leq \frac{1}{n} M^2 + C \frac{\sigma^2}{n \alpha^2} (1 \vee J^{2a+1} \cdot 2^{j(1-2s')}) ,$$

where the last inequality follows from (B.5) and Lemma A.3, the latter of which implies

$$\| \beta_j \|_2^2 \lesssim 2^{-2js'} \mathbb{1}_{[0,1)}(s') + 2^{-j} \mathbb{1}_{[1,\infty)}(s'),$$

and, in turn,

$$\sum_{j=0}^{J-1} j^{2a+2j} \| \beta_j \|_2^2 \lesssim 1 \vee J^{2a+1} \cdot 2^{j(1-2s')} .$$

By putting together (B.4) and (B.9) we get the stated result. \qed

**B.3. Proof of Theorem 4.2.** Fix a $Q \in \mathcal{Q}_\alpha^{(S')}$. To construct appropriate hypotheses, consider an $S$-regular orthonormal Daubechies wavelet basis

$$\mathcal{W} = \{ \phi_k = \phi(\cdot - r), \psi_{lk} = 2^{l/2} \psi \left( 2^l (\cdot - k) \right) : k \in \mathbb{Z}, l \in \mathbb{N}_0 \},$$
of $L^2(\mathbb{R})$ with $S > s$ (cf. Giné and Nickl, 2016, Theorem 4.2.10). This means, in particular, that $\text{supp} \phi \subseteq [0, 2S - 1]$, $\text{supp} \psi \subseteq [-S, S]$, $\int_{\mathbb{R}} \psi_{lk}(x) dx = 0$, $\left\| \sum_{k \in \mathbb{Z}} |\psi(\cdot - k)| \right\|_{\infty} < \infty$ and $\| \psi_{lk} \|_1 = 2^{-l/2} \| \psi \|_1$. Since $s < 1$, it suffices to take $S = 1$ and thus, for every fixed $l \geq 1$ and $k = 1, \ldots, 2^l$, the $\psi_{lk}$ are supported on $[0, 1]$. Clearly, $\psi_{lk} \in B^p_{s,p}(\mathbb{R})$, and therefore also $\psi_{lk} \in B^p_{s,p}([0, 1])$.

Now, for $m \geq 1$, $\delta > 0$ and $\nu \in \mathcal{V}_m := \{-1, 1\}^{2^m}$, define $f_{\nu}(x) := 1$ and

$$f_{\nu}(x) := 1 + 2^{-m(s+\frac{1}{2})} \delta \sum_{k=1}^{2^m} \nu_k \psi_{mk}(x), \quad x \in [0, 1].$$

Since $2^{-m/2} \left\| \sum_{k=1}^{2^m} \psi_{mk} \right\|_{\infty} = \left\| \sum_{k=1}^{2^m} (\psi(\cdot - k)) \right\|_{\infty} \leq \left\| \sum_{k \in \mathbb{Z}} (\psi(\cdot - k)) \right\|_{\infty} =: D_0 < \infty$, we see that $f_{\nu}$ is lower bounded by $\frac{1}{2}$ on $[0, 1]$ provided that $2^{-ms} \delta D_0 \leq \frac{1}{2}$, which holds (for all $m \geq 0$) if $\delta \leq (2D_0)^{-1}$. This shows that $f_{\nu}$ is a density on $[0, 1]$ with $\| f_{\nu} \|_{\infty} \leq 2$. Moreover, by Proposition 4.3.2 of Giné and Nickl (2016), we have

$$\| f_{\nu} \|_{B^p_{s,p}([0, 1])} \leq 1 + 2^{-m(s+\frac{1}{2})} \delta \left\| \sum_{k=1}^{2^m} \nu_k \psi_{mk} \right\|_{B^p_{s,p}([0, 1])} \leq 1 + 2^{-m(s+\frac{1}{2})} \delta \left\| \sum_{k=1}^{2^m} \nu_k \psi_{mk} \right\|_{B^p_{s,p}(\mathbb{R})} \leq 1 + c2^{-m(s+\frac{1}{2})} \delta \left\| \sum_{k=1}^{2^m} \nu_k \psi_{mk} \right\|_{B^p_{s,p}^{1,W}(\mathbb{R})} = 1 + c \delta \leq L,$n

provided that $\delta > 0$ is sufficiently small. Hence, $f_{\nu} \in \bar{B}^p_{s,p}(L, M)$ for every $\nu \in \mathcal{V}_m$. Moreover, by construction, $D(f_{\nu}) = D(f_0) + 2^{-2ms} \delta^2$. If $P_0$ and $P_{\nu}$ are the probability measures corresponding to $f_0$ and $f_{\nu}$, respectively, we write $Q^n_{\nu} := Q^n_{P_{\nu}}, Q^n_0 := Q^n_{P_0}$ and $Q^n := 2^{-2m} \sum_{\nu \in \mathcal{V}_m} Q^n_{\nu}$.

Set $\Delta := 2^{-2ms} \delta^2/2$. For a measurable function $\hat{D}_n : \mathcal{Z} \to \mathbb{R}$ and $f \in \bar{B}^p_{s,p}(L, M)$, define $S_f := \{ z \in \mathcal{Z} : |\hat{D}_n(z) - D(f)| \geq \Delta \}$, $S_{f,1} := \{ z \in \mathcal{Z} : \hat{D}_n(z) \geq 1 + \Delta, D(f) \leq 1 \}$ and $S_{f,2} := \{ z \in \mathcal{Z} : \hat{D}_n(z) < 1 + \Delta, D(f) \geq$
Next, for fixed \( z \) (B.11)

\[
1 + 2\Delta \}
\]

which obey the inclusions \( S_{j,i} \subseteq S_{j} \), for \( j = 1, 2 \). Now

\[
\sup_{f \in \mathcal{P}_n(L,M)} Q^n_{P^n_f} (S_f) \geq \sup_{f \in \mathcal{P}_n(L,M)} \max \left\{ Q^n_{P^n_{f_1}} (S_{f_1}), Q^n_{P^n_{f_2}} (S_{f_2}) \right\}
\]

\[
\geq \max \left\{ \max_{\nu \in \mathcal{V}_m} Q^n_{P^n_{f_1}} (S_{f_1,1}), \max_{\nu \in \mathcal{V}_m} Q^n_{P^n_{f_1}} (S_{f_1,2}) \right\}
\]

\[
= \max \left\{ Q^n_{P^n_0} \left( \hat{D}_n \geq 1 + \Delta \right), \max_{\nu \in \mathcal{V}_m} Q^n_{P^n_\nu} \left( \hat{D}_n < 1 + \Delta \right) \right\}
\]

\[
\geq \frac{1}{2} \left\{ Q^n_{P^n_0} \left( \hat{D}_n \geq 1 + \Delta \right) + 2^{-2m} \sum_{\nu \in \mathcal{V}_m} Q^n_{P^n_\nu} \left( \hat{D}_n < 1 + \Delta \right) \right\}
\]

\[
\geq \frac{1}{2} \inf \left\{ \mathbb{E}_{Q^n_0} [\phi] + \mathbb{E}_{Q^n} [1 - \phi] \right\} = \frac{1}{2} \left\{ 1 - \sup_{\phi \in \text{tests}} \mathbb{E}_{Q^n_0} [\phi] - \mathbb{E}_{Q^n} [\phi] \right\}
\]

(B.10)

\[
= \left( 1 - d_{TV} (Q^n_0, Q^n) \right) \geq \frac{1}{2} \left( 1 - \sqrt{D_{KL} (Q^n_0, Q^n) / 2} \right),
\]

where we have used Pinsker’s inequality in the last step.

We abbreviate the regular conditional distributions of \( Z_i \) given \( Z_1, \ldots, Z_{i-1} \) when \( X_i \) comes from \( \mathbb{P}_0 \) or \( \mathbb{P}_\nu \), by \( \mathcal{L}^{(0)}_{Z_i|Z_{1:(i-1)}} (dz_i) := \int_{[0,1]} Q_i(dz_i|x_i, z_{1:(i-1)})d\mathbb{P}_0(x_i) \)

and \( \mathcal{L}^{(\nu)}_{Z_i|Z_{1:(i-1)}} (dz_i) := \int_{[0,1]} Q_i(dz_i|x_i, z_{1:(i-1)})d\mathbb{P}_\nu(x_i) \), respectively, and we denote the joint distribution of \( Z_1, \ldots, Z_i \), when \( X_1, \ldots, X_i \) are i.i.d. from \( \mathbb{P}_0 \), by

\[
\mathcal{L}^{(0)}_{Z_1,\ldots,Z_i} (dz_1:i) := \mathcal{L}^{(0)}_{Z_{1:(i-1)}} (dz_1) \cdots \mathcal{L}^{(0)}_{Z_{2:1}} (dz_2) \mathcal{L}^{(0)}_{Z_1} (dz_1).
\]

Thus, by the convexity and tensorization property of the KL-divergence, we have

\[
D_{KL} (Q^n_0, \bar{Q}^n) \leq 2^{-2m} \sum_{\nu \in \mathcal{V}_m} D_{KL} (Q^n_0, Q^n_\nu)
\]

(B.11)

\[
= 2^{-2m} \sum_{\nu \in \mathcal{V}_m} \sum_{i=1}^n \int_{Z_{i-1}} D_{KL} \left( \mathcal{L}^{(0)}_{Z_{1:(i-1)}}, \mathcal{L}^{(\nu)}_{Z_{1:(i-1)}} \right) d\mathcal{L}^{(0)}_{Z_{1,\ldots,Z_{i-1}}}(dz_{1,\ldots,Z_{i-1}}).
\]

Next, for fixed \( z_{1:(i-1)} \in \mathcal{Z}^{i-1} \) (if \( i = 1 \) there is nothing to be fixed here), we bound the KL-divergence by the \( \chi^2 \)-divergence, as in Lemma 2.7 of Tsybakov (2009). Since \( Q \) is \( \alpha \)-sequentially interactive differentially private, Lemma B.3 establishes existence of a probability measure \( \mu_{z_{1:(i-1)}} \) and a
family of $\mu_{z_1; (i-1)}$-densities $z_i \mapsto q_i(z_i|x_i, z_{1; (i-1)})$ of $Q_i(\cdot|x_i, z_{1; (i-1)})$, $x_i \in X$, with

$$0 < q_i(z_i|x_i, z_{1; (i-1)}) \leq e^{2\alpha} q_i(z_i|x_i', z_{1; (i-1)}), \quad \forall z_i \in Z, \forall x_i, x_i' \in X.$$ 

Abbreviating $q_{z_1; (i-1)}^{(\nu)}(z_i) := \int_{[0,1]} q_i(z_i|x_i, z_{1; (i-1)})d\mathbb{P}_\nu(x_i)$, we see that

$$D_{KL}(\mathcal{L}_{Z_1|z_1; (i-1)}^{(0)}, \mathcal{L}_{Z_1|z_1; (i-1)}^{(\nu)})$$

\[\leq \int_Z \left[ \left( \int_{[0,1]} q_i(z_i|x_i, z_{1; (i-1)})d[\mathbb{P}_0 - \mathbb{P}_\nu](x_i) \right)^2 q_{z_1; (i-1)}^{(\nu)}(z_i) \right] d\mu_{z_1; (i-1)}(z_i) \]

(B.12)

where we choose $c_{\alpha,0} = \frac{1}{2}(e^{2\alpha} + e^{-2\alpha})$. But since

$$e^{-2\alpha} \leq \inf_{x_i' \in [0,1]} \frac{q_i(z_i|x_i, z_{1; (i-1)})}{q_i(z_i'|x_i', z_{1; (i-1)})} \leq \frac{q_i(z_i|x_i, z_{1; (i-1)})}{q_{z_1; (i-1)}^{(0)}(z_i)} \leq \sup_{x_i \in [0,1]} \frac{q_i(z_i|x_i, z_{1; (i-1)})}{q_i(z_i'|x_i', z_{1; (i-1)})} \leq e^{2\alpha},$$

and if we set $c_{\alpha,1} = \frac{1}{2}(e^{2\alpha} - e^{-2\alpha})$, we arrive at

$$g_i(x_i) := g_i(x_i|z_1, \ldots, z_i) := \frac{q_i(z_i|x_i, z_{1; (i-1)})}{q_{z_1; (i-1)}^{(0)}(z_i)} - c_{\alpha,0} \in [-c_{\alpha,1}, c_{\alpha,1}].$$

Next, we consider the average over $\nu$ of the inner squared integral in (B.12), i.e.,

$$2^{-m} \sum_{\nu \in \mathcal{V}_m} \left( \int_{[0,1]} g_i(x_i)[f_0(x_i) - f_\nu(x_i)]dx_i \right)^2$$

$$= \delta^2 2^{-m(s+\frac{1}{2})} 2^{-m} \sum_{\nu \in \mathcal{V}_m} \left( \sum_{k=1}^{2^m} v_k \int_{[0,1]} g_i(x_i)\psi_{mk}(x_i)dx_i \right)^2$$

$$= \delta^2 2^{-m(s+\frac{1}{2})} 2^m \sum_{k=1}^{2^m} \left( \int_{[0,1]} g_i(x_i)\psi_{mk}(x_i)dx_i \right)^2$$

$$\leq \delta^2 2^{-m(s+\frac{1}{2})} c_{\alpha,1}^2 \sum_{k=1}^{2^m} \|\psi_{mk}\|^2 \leq \delta^2 2^{-m(s+\frac{1}{2})} c_{\alpha,1}^2 \|\psi\|^2 2^m (2^{-m/2}\|\psi\|_1)^2$$

$$= \frac{2^{-m(2s+1)}}{4} (e^{2\alpha} - e^{-2\alpha})^2 (\delta\|\psi\|_1)^2.$$
Hence, using \( \frac{q_{z_1(z_1)}}{q_{z_1(z_1)}} \leq \|1/f\|_\infty \leq 2 \), we see that the average over \( \nu \) of \((B.12)\) is bounded by the same expression multiplied by 2, and \((B.11)\) yields
\[
D_{KL}(Q_0^n, \bar{Q}_n) \leq n^2 - m(2s + 1)^2 \left( e^{2\alpha} - e^{-2\alpha} \right)^2 c_0^2 = \frac{1}{2},
\]
where \( c_0 = \delta \|\psi\|_1 \) and \( m \) is chosen as \( m = \frac{\log(n(e^{2\alpha} - e^{-2\alpha})^2 c_0^2)}{2(s + 1)\log 2} \). In view of Markov’s inequality and \((B.10)\), this leads to
\[
\sup_{f \in \mathcal{P}^n(L,M)} E_{Q^n} \left[ \left| \hat{D}_n - D(f) \right|^2 \right] \geq \Delta^2 \sup_{f \in \mathcal{P}^n(L,M)} Q^n(f) \left\| \hat{D}_n - D(f) \right\|^2 > \Delta^2 \]
\[
\geq \Delta^2 \left( 1 - \sqrt{D_{KL}(Q_0^n, \bar{Q}_n)/2} \right)
\]
\[
\geq \frac{\Delta^2}{4} = \left[ n(e^{2\alpha} - e^{-2\alpha})^2 c_0^2 \right] - \frac{4\alpha}{2(s + 1)\log 2} \delta^4.
\]
which finishes the proof. \( \square \)

**B.4. Auxiliary lemmas for Section 4.** The following lemma is a non-asymptotic version of Lemma 2 in Butucea and Tsybakov (2008).

**Lemma B.2.** For arbitrary finite constants \( A, B, r, s > 0, a, b \geq 0 \) and \( v > 0 \), such that \( \frac{Ar^r}{a + 1} > 1 \), we have

\[
\int_v^\infty u^a e^{-Au} du \leq \frac{1}{Ar^r} v^{a+1-r} e^{-Ar^r} \left( 1 - \frac{a + 1}{Ar^r} \right)^{-1}, \quad \text{and} \quad
\int_0^v u^b e^{Bu} du \leq \frac{1}{Bs^s} v^{b+1-s} e^{Bs^s} \left( 1 + \frac{b + 1}{Bs^s} \right)^{-1}.
\]

**Proof.** To see \((B.13)\), simply integrate by parts to get
\[
\int_v^\infty u^a e^{-Au} du = \left[ \frac{u^{a+1}}{a + 1} e^{-Au} \right]_v^\infty - \int_v^\infty \frac{u^{a+1}}{a + 1} e^{-Au} (-Au^{r-1}) du
\]
\[
= -\frac{v^{a+1}}{a + 1} e^{-Ar^r} + \frac{Ar^r}{a + 1} \int_v^\infty u^{a+r} e^{-Au} du
\]
\[
\geq -\frac{v^{a+1}}{a + 1} e^{-Ar^r} + \frac{Ar^r}{a + 1} \int_v^\infty u^a e^{-Au} du.
\]
For $\frac{Arv^r}{a+1} > 1$, this is equivalent to
\[
\int_v^\infty u^a e^{-Au} du \leq \frac{u^{a+1}}{a+1} e^{-Arv^r} \left( \frac{Arv^r}{a+1} - 1 \right)^{-1},
\]
which implies the desired result. (B.14) follows analogously and without any further restrictions on the constants.

**Lemma B.3.** Let $\alpha \in (0, \infty)$, $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Z}, \mathcal{G})$ be measurable spaces and $Q$ a Markov kernel from $\mathcal{X}$ to $\mathcal{Z}$. If $Q(A|x) \leq e^\alpha Q(A|x')$, for all $A \in \mathcal{G}$ and all $x, x' \in \mathcal{X}$, then there exists a probability measure $\mu$ and a family of $\mu$-densities $(q_x)_{x \in \mathcal{X}}$, such that for every $x \in \mathcal{X}$, $dQ(\cdot|x) = q_x d\mu$ and $e^{-\alpha} \leq q_x(z) \leq e^\alpha$, for all $z \in \mathcal{Z}$.

**Proof.** Let $x_0 \in \mathcal{X}$ and $\mu := Q(\cdot|x_0)$. For a fixed $x \in \mathcal{X}$, we have $Q(\cdot|x) \ll \mu$, and we write $\tilde{q}_x$ for a corresponding density. Since $\int_A \tilde{q}_x d\mu = Q(A|x) \leq e^\alpha Q(A|x_0) = \int_A e^\alpha d\mu$ and $Q(A|x) \geq e^{-\alpha} Q(A|x_0) = \int_A e^{-\alpha} d\mu$, for all $A \in \mathcal{G}$, we have $e^{-\alpha} \leq \tilde{q}_x \leq e^\alpha$, $\mu$-almost surely. Let $N_x \in \mathcal{G}$ be the corresponding $\mu$-null set. Then define $q_x(z) = \tilde{q}_x(z)$, if $z \in N_x^c$, and set $q_x(z) = 1$, otherwise. Thus, $q_x$ is still a $\mu$-density of $Q(\cdot|x)$ with $e^{-\alpha} \leq q_x(z) \leq e^\alpha$.

**APPENDIX C: PROOFS OF SECTION 6 ON ADAPTIVE ESTIMATION**

**C.1. Proof of Theorem 6.1 (non-interactive protocol).** Let $\mathbb{P}_n$ denote the empirical measure with respect to $X_1, ..., X_n$. We decompose our estimator into the non-private version and new terms $A_J$ and $B_J$ due to privacy as follows:

\[
\hat{D}_J - 2 \left( \frac{1}{n} \sum_{i=1}^n f(X_i) - D \right) - D = U_n(H_J) - 2(\mathbb{P}_n - \mathbb{P})(f - f_J) - \| f - f_J \|^2 + A_J + B_J,
\]
where \( f_J(x) = \sum_{j=1}^{J-1} \sum_{k=0}^{(1/2^j)-1} \beta_{jk} \psi_{jk}(x) \) is the projection of \( f \) at the resolution level \( J \), \( U_n(H_J) \) is the \( U \)-statistic with kernel \( H_J \) given by

\[
H_J(x, y) = \sum_{j=1}^{J-1} \sum_{k=0}^{(1/2^j)-1} (\psi_{jk}(x) - \beta_{jk})(\psi_{jk}(y) - \beta_{jk}),
\]

\[
U_n(H_J) = \frac{1}{n(n-1)} \sum_{i,h=1}^{n} H_J(X_i, X_h)
\]

and, finally,

\[
A_J = \frac{2}{n(n-1)} \sum_{i \neq h} \sum_{j=1}^{J-1} \sum_{k=0}^{(1/2^j)-1} \sigma_j \frac{\sigma}{\alpha} W_{ijk} \psi_{jk}(X_h),
\]

\[
B_J = \frac{1}{n(n-1)} \sum_{i \neq h} \sum_{j=1}^{J-1} \sum_{k=0}^{(1/2^j)-1} \sigma_j^2 \frac{\sigma^2}{\alpha^2} W_{ijk} W_{hjk}.
\]

We decompose \( \text{pen}^{(NI)}(J) \) into the sum of \( \text{pen}(J), \text{pen}_A(J) \) and \( \text{pen}_B(J) \), for all \( J \) in \( J \), that we specify below. Let us denote

\[
V_J = U_n(H_J) - 2(\mathbb{P}_n - \mathbb{P})(f - f_J) - \|f - f_J\|^2 - \text{pen}(J)
\]

\[
+ A_J - \text{pen}_A(J)
\]

\[
+ B_J - \text{pen}_B(J).
\]

By the definition of \( \hat{D}^{(NI)} \) we get

\[
\hat{D}^{(NI)} - 2 \left( \frac{1}{n} \sum_{i=1}^{n} f(X_i) - D \right) - D = \sup_{J \in J} V_J.
\]

Using

\[
\left| \sup_{J \in J} V_J \right| = \sup_{J \in J}(V_J)_+ \lor \inf_{J \in J}(V_J)_-,\n\]

we have that

\[
\mathbb{E} \left[ (\sup_{J \in J} V_J)^2 \right] \leq \sum_{J \in J} \mathbb{E}[V_J^2] + \inf_{J \in J} \mathbb{E}(V_J)^2].
\]

We start with bounding the first term on the right-hand side in the former inequality. Following Laurent (2005), we similarly use the concentration inequality for \( U \)-statistics in Houdré and Reynaud-Bouret (2003) and get

\[
\mathbb{P} \left( |U_n(H_J)| \geq \frac{C}{n-1} \left[ \sqrt{2(J + 1)M2^Jn(n-1)\sqrt{t} + 8(J + 1)Mnt + 2(J + 1)2^Jt^2} \right] \right) \leq 5.6 \exp(-t),
\]
which, combined with the deviation of the empirical process part, provides the penalty
\[
pen(J) = \frac{\kappa_0}{n} \left( \sqrt{M(J + 1)2^J \log(2^J + 1)} + M(J + 1) \log(2^J + 1) + \frac{(J + 1)2^J \log^2(2^J + 1)}{n} \right).
\]
with \(\kappa_0\) as specified in Laurent (2005). We obtain
\[
\sum_{J \in J} \mathbb{E} \left[ \left( U_n(H_J) - 2(\mathbb{P}_n - \mathbb{P})(f - f_J) - \|f - f_J\|_2^2 - pen(J) \right)_+ \right] \lesssim \frac{1}{n}
\]
and
\[
\mathbb{E} \left[ \left( U_n(H_J) - 2(\mathbb{P}_n - \mathbb{P})(f - f_J) - \|f - f_J\|_2^2 - pen(J) \right)_- \right]^2 \leq C_0 (\|f - f_J\|_2^4 + pen^2(J))
\]
for some absolute constant \(C_0 > 0\).

Let us deal now with the additional terms that occur due to privacy in our setup.

**First additional term:** Recall that
\[
A_J = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{J-1} \sum_{k=0}^{(1 \lor 2^j) - 1} W_{ijk} \sigma_j \xi_{ijk}, \quad \text{with } \xi_{ijk} = \frac{1}{n-1} \sum_{h \neq i} \psi_{jk}(X_h).
\]
The aim is to specify some potentially random and ideally small real numbers \(y_J, J \in J\), such that
\[
\sum_{J \in J} \mathbb{E}[(A_J - y_J)_+] \lesssim \frac{1}{n \alpha^2}.
\]
For any \(y = y((X_i)) > 0\),
\[
(\Delta_A) := \mathbb{E}[(A_J - y)_+] = \mathbb{E} \mathbb{E}[(A_J - y)_+ \mid (X_i)] = \mathbb{E} \int_0^\infty t \mathbb{P}([A_J - y]_+ > t(X_h)) dt.
\]
Due to the independence of \((X_i)\) and \((W_{ijk})\), we use Bernstein’s inequality for sums of independent Laplace random variables (cf. Boucheron, Lugosi and Massart (2013)) to get

\[
P(A_J > \eta \mid (X_i)) \leq \exp \left( - \frac{\eta^2}{2 \max_{i,j,k} \sigma_j^2 \sigma_{ijk}^2 \alpha^2 \sum_{i=1}^{n} \sum_{j,k} \sigma_{ijk}^2 + \eta n \sum_{i,j,k} \sigma_{ijk}^2 \alpha^2 \Xi_{ijk}^2} \right) =: C_1 + \eta \max_{i,j,k} \sigma_j \sigma_{ijk} |\Xi_{ijk}| =: C_2 \tag{C.1}
\]

where we drop the dependence on \(J\) in \(C_1 = C_{1,J}\) and \(C_2 = C_{2,J}\). Here and in what follows, the double sum over \(j,k\) is abbreviated by \(\sum_{j,k}\), dropping in particular its dependence on \(J\) if this is clear from the context.

Hence, using Lemma C.1,

\[
(\Delta_A) \leq \mathbb{E} \int_0^\infty t \mathbb{P}(A_J > t \mid (X_i)) dt \\
\leq \mathbb{E} \int_0^\infty t \exp \left( - \frac{t^2}{2 C_1 + tC_2} \right) dt \\
\leq \mathbb{E} \left\{ 2C_1 \exp \left( \frac{y^2}{4C_1} \right) + 4C_2 \left[ y \vee \frac{C_1}{C_2} \right] \exp \left( - \frac{y \vee C_1}{4C_2} \right) + 16C_2^2 \exp \left( - \frac{y \vee C_1}{4C_2} \right) \right\}.
\]

Now define

\[
y_J := 8 \sqrt{C_1} \log[2^{4J+1}]
\]

and insert it into (I), (II) and (III). Then

\[
(I) = 2C_1 \exp \left( - \frac{y_J^2}{4C_1} \right) \leq 2C_1 \exp \left( - 4 \log[2^{4J+1}] \right) = 2C_1 \frac{1}{2^{(4J+1)J}}.
\]

Concerning (II), note that if \(y_J > \frac{C_1}{C_2}\), then

\[
\frac{[y_J \vee \frac{C_1}{C_2}]}{4C_2^2} = 2 \sqrt{\frac{C_1}{C_2}} \log[2^{4J+1}] \geq 2 \log[2^{4J+1}].
\]
because $\sqrt{C_1} \geq C_2$. If $y_J \leq \frac{C_1}{C_2}$, then
\[ 8\sqrt{C_1} \log [2^{4J+1}] \leq \frac{C_1}{C_2} \iff \frac{\sqrt{C_1}}{4C_2} \geq 2 \log[2^{4J+1}] . \]

Consequently,
\[ (C.3) \quad \frac{[y_J \vee \frac{C_1}{C_2}]}{4C_2} \geq 2 \log [2^{4J+1}] . \]

Finally, $C_2 \sqrt{C_1} \leq C_1$ implies
\[ 4C_2 [y_J \vee \frac{C_1}{C_2}] \leq 8C_1 \log[2^{4J+1}] . \]

Summarizing,
\[ (II) = 4C_2 [y_J \vee \frac{C_1}{C_2}] \exp \left( - \frac{[y_J \vee \frac{C_1}{C_2}]}{4C_2} \right) \leq 8C_1 \log\left(2^{4J+1}\right) \exp \left( - 2 \log\left(2^{4J+1}\right) \right) . \]

Again by (C.3) and the inequality $C_2^2 \leq C_1$,
\[ (III) = 16C_2^2 \exp \left( - \frac{[y_J \vee \frac{C_1}{C_2}]}{4C_2} \right) \leq 16C_1 \exp \left( - 2 \log\left(2^{4J+1}\right) \right) . \]

This gives,
\[ (\Delta A) \leq E \int_{y_J}^{\infty} tP(\{X > t \mid (X_i)\}) dt \leq E \left[ 2C_1 \cdot \frac{1}{2(4J+1)^4} + \frac{32C_1 \log[2^{4J+1}]}{2^{2(4J+1)}} + \frac{16C_1}{2^{2(4J+1)}} \right] . \]

Now we are in the position to evaluate the expected value. Because of
\[ E \left[ \frac{1}{n-1} \sum_{h \neq i} \psi_{jk}(X_h) \right]^2 = \left( 1 - \frac{1}{n-1} \right) \beta_{jk}^2 + \frac{1}{n-1} \mathbb{E} \left[ \psi_{jk}(X_1)^2 \right] , \]
we get
\[ \mathbb{E} C_1 \leq \frac{2\alpha^2}{n\alpha^2} \sum_{j,k} 2^j (1 + j)^2 \alpha \left\{ \beta_{jk}^2 + \frac{\mathbb{E} \left[ \psi_{jk}(X_1)^2 \right]}{n-1} \right\} \]
\[ \leq \frac{1}{n\alpha^2} \left( j^{2a+1} 2^j + \frac{1}{n-1} 2^{2j} \right) . \]
Therefore, with $y_J$ in (C.2), $\mathbb{E}[(A_J - y_J)^2] \lesssim \frac{1}{n^{\alpha^2}}$. As a consequence,

$$\sum_{J \in \mathcal{J}} \mathbb{E}[(A_J - y_J)^2] \lesssim \frac{1}{n^{\alpha^2}}.$$ 

Finally, with $\bar{y}_J := 8\sqrt{\mathbb{E}C_1} \log[2^{4J+1}]$ and

$$\text{pen}_A(J) = 8 \left( \frac{J^{2a+1} \log(J)}{n^{\alpha^2}} + \frac{2^{2J}}{n^{\alpha^2}} \right)^{1/2} \log[2^{4J+1}],$$

such that $\bar{y}_J \leq \text{pen}_A(J)$, we arrive at

$$\sum_{J \in \mathcal{J}} \mathbb{E}[(A_J - \text{pen}_A(J))^2] \lesssim \frac{1}{n^{\alpha^2}} + \sum_{J \in \mathcal{J}} \log^2[2^{4J+1}] (\text{Var}(C_1))^{1/2}.$$ (C.4)

In order to bound $\text{Var}(C_1)$, observe that

$$C_1 = \frac{2}{n n} \sum_{i=1}^{n} \sum_{j,k} \sigma_j^2 \frac{\psi_j^2(X_h) \psi_j(X_{h'})}{n^{\alpha^2}} = \frac{2}{n n} \sum_{i=1}^{n} \sum_{j,k} \sigma_j^2 \frac{1}{(n-1)^2} \sum_{h,h' \neq i} \psi_j(X_h) \psi_j(X_{h'})$$

$$= \frac{2\sigma^2}{n^2(n-1)^2} \sum_{h=1}^{n} \sum_{j,k} \sigma_j^2 \psi_j(X_h) + \frac{2\sigma^2(n-2)}{n^2(n-1)^2} \sum_{h \neq h'} \sum_{j,k} \sigma_j^2 \psi_j(X_h) \psi_j(X_{h'})$$

$$=: C_{1,1} + C_{1,2}.$$ 

Note that $\text{Var}(C_1) \leq 2 \text{Var}(C_{1,1}) + 2 \text{Var}(C_{1,2})$. Therefore,

$$\text{Var}(C_{1,1}) \asymp \frac{1}{n^5 \alpha^4} \text{Var} \left( \sum_{j,k} \sigma_j^2 \psi_{j,k}^2(X_1) \right)$$

$$\lesssim \frac{1}{n^5 \alpha^4} \mathbb{E} \left( \sum_{j,j',k,k'} \sigma_j^2 \sigma_{j'}^2 \psi_{j,k}^2(X_1) \psi_{j',k'}^2(X_1) \right)^2$$

$$\lesssim \frac{1}{n^5 \alpha^4} \sum_{j \geq j',k} \sigma_j^2 \sigma_{j'}^2 \sum_{k'} \int \psi_{j,k}^2 \psi_{j',k'}^2 f$$

$$\lesssim \frac{J^{4a}}{n^5 \alpha^4} \sum_{j \geq j',k} 2^{2(j+j')} \sum_{k'} \int I_{j,k} I_{j',k'} \cdot M,$$
where we denote by $I_{j,k} = \left[ \frac{k}{2^j}, \frac{k+1}{2^j} \right]$. We use repeatedly that for any, $j$, $k$ and $j'$ such that $j \geq j'$, there are only $2^{j-j'}$ values of $k'$ such that $I_{j,k}I_{j',k'} \neq 0$. For these values of $k'$ the length of the interval $I_{j,k}I_{j',k'}$ is $2^{-j}$. Therefore $\sum_{k'} I_{j,k}I_{j',k'} \leq 2^{j-j'} \cdot 2^{-j}$. We obtain

$$
\text{Var}(C_{1,1}) \lesssim \frac{J^{4a}}{n^5 \alpha^4} \sum_{j \geq j',k} 2^{2j+j'} \lesssim \frac{J^{4a}}{n^5 \alpha^4} \sum_j 2^{4j} \approx \frac{J^{4a} \cdot 2^{4J}}{n^5 \alpha^4}.
$$

We plug this in (C.4) to get

$$
\sum_{J \in \mathcal{J}} \log^2 [2^{4J+1}] (\text{Var}(C_{1,1}))^{1/2} \lesssim \sum_{J \in \mathcal{J}} (4J + 1)^2 \frac{J^{2a} \cdot 2^{2J}}{n^5 \alpha^2} \lesssim \frac{1}{n} \cdot \frac{(J_{\max})^2 + 2a2^2J_{\max}}{n^5 \alpha^2} \lesssim \frac{1}{n},
$$

where we use that $2^{J_{\max}} \approx (n\alpha^2)^{2/3} \log^{-\kappa/3}(n\alpha^2)$ for some $\kappa > 4(a+1) > 0$.

Similarly, for the term appearing in $C_{1,2} - \mathbb{E}(C_{1,2})$,

$$
\mathbb{E} \left( \sum_{h \neq h'}^n \sum_{j,k} \sigma^2_j (\psi_{jk}(X_h)\psi_{jk}(X_{h'}) - \beta^2_{jk}) \right)^2 \\
= n(n-1) \mathbb{E} \left( \sum_{j,k} \sigma^2_j (\psi_{jk}(X_1)\psi_{jk}(X_2) - \beta^2_{jk}) \right)^2 \\
+ n(n-1)(n-2) \mathbb{E} \left( \sum_{j,k} \sigma^2_j (\psi_{jk}(X_1)\psi_{jk}(X_2) - \beta^2_{jk}) \right) \cdot \left( \sum_{j',k'} \sigma^2_j (\psi_{j'k'}(X_1)\psi_{j'k'}(X_3) - \beta^2_{j'k'}) \right) =: T_1 + T_2.
$$

We bound from above $\text{Var}(C_{1,2}) \lesssim \frac{1}{n^6 \alpha^4} (T_1 + |T_2|)$, with

$$
\frac{1}{n^6 \alpha^4} T_1 \lesssim \frac{1}{n^6 \alpha^4} \text{Var} \left( \sum_{h \neq h'}^n \sum_{j,k} \sigma^2_j \psi_{jk}(X_h)\psi_{jk}(X_{h'}) \right) \\
\lesssim \frac{1}{n^6 \alpha^4} \text{Var} \left( \sum_{j,k} \sigma^2_j \psi_{jk}(X_1)\psi_{jk}(X_2) \right)
$$
\[
\begin{align*}
\sum_{j,k} \left( \sum_{j',k'} \sigma_j^2 \psi_{jk}(X_1) \psi_{j'k'}(X_2) \right)^2 &\leq \frac{1}{n^4 \alpha^4} \mathbb{E} \left[ \left( \sum_{j,k} \sigma_j^2 \psi_{jk}(X_1) \psi_{j'k'}(X_1) \right)^2 \right] \\
&\leq \frac{1}{n^6 \alpha^4} \sum_{j,j',k,k'} \sigma_j^2 \sigma_{j'}^2 \mathbb{E}^2 \left[ \psi_{jk}(X_1) \psi_{j'k'}(X_1) \right] \\
&\leq \frac{1}{n^4 \alpha^4} \sum_{j,j',k} \sigma_j^2 \sigma_{j'}^2 M \left( \int I_{jk} I_{j'k'} \right)^2 M^2 \\
&\leq \frac{J^{4a}}{n^4 \alpha^4} \sum_{j \geq j'} 2^{(j+j')} \cdot 2^{j-j'} 2^{-2j} M^2 \\
&\leq \frac{J^{4a}}{n^4 \alpha^4} \sum_{j \geq j'} 2^j \cdot 2^{j+j'} \leq \frac{J^{4a}}{n^4 \alpha^4} 2^{3J}.
\end{align*}
\]

Therefore, for all \( J \) in \( \mathcal{J} \), we get

\[
(C.5) \quad \frac{1}{n^3 \alpha^2} \sqrt{T_1} \lesssim \frac{1}{n} \cdot \frac{J^{2a} \cdot 2^{3J/2}}{n \alpha^2}.
\]

Now,

\[
\begin{align*}
\frac{1}{n^6 \alpha^4} |T_2| &\leq \frac{J^{4a}}{n^4 \alpha^4} \left| \sum_{j,k,j',k'} 2^{j+j'} \left( \mathbb{E} \left[ \psi_{jk}(X_1) \psi_{j'k'}(X_1) \psi_{jk}(X_2) \psi_{j'k'}(X_3) \right] - \beta_{jk}^2 \beta_{j'k'}^2 \right) \right| \\
&= \frac{J^{4a}}{n^4 \alpha^4} \left| \sum_{j,k,j',k'} 2^{j+j'} \int \psi_{jk} \psi_{j'k'} f \cdot \beta_{jk} \beta_{j'k'} - \left( \sum_j 2^j \| \beta_j \|_2^2 \right) \right| \\
&\leq \frac{J^{4a}}{n^4 \alpha^4} \cdot \left( \sum_{j \geq j'} 2^{j+j'} \sum_{k'} 2^{\frac{1}{2} (j+j')} \int I_{jk} I_{j'k'} M \cdot | \beta_{jk} \beta_{j'k'} | + 2^{2j} M^2 \right) \\
&\leq \frac{J^{4a}}{n^4 \alpha^4} \cdot \left( \sum_{j \geq j'} 2^{\frac{3}{2} (j+j')} | \beta_{jk} | \left( \sum_{k'} \left[ \int I_{jk} I_{j'k'} \right]^2 \right)^{1/2} \| \beta_{j'k'} \|_2 + 2^{j} M^2 \right) \\
&\leq \frac{J^{4a}}{n^4 \alpha^4} \cdot \left( \sum_{j \geq j'} 2^{\frac{3}{2} (j+j')} | \beta_{jk} | 2^{-1/2 (j+j')} + 2^{j} M^2 \right) \\
&\leq \frac{J^{4a}}{n^4 \alpha^4} \cdot \left( \sum_{j \geq j'} 2^{j+j'} 2^{j/2} \| \beta_j \|_2 + 2^{j} M^2 \right) \leq \frac{2^3 J}{n^3 \alpha^4}.
\end{align*}
\]
Thus
\[
(C.6) \quad \frac{1}{n^3 \alpha^2} \sqrt{T_2} \lesssim \frac{1}{n} \cdot \frac{2^{3J/2}}{\sqrt{n \alpha^2}}.
\]
Combining (C.5) and (C.6), we get that for any \( \kappa \geq 4(a+1) \):
\[
\sum_{J \in \mathcal{J}} \log^2[2^{4J+1}](\text{Var}(C_{1,2}))^{1/2} \lesssim \frac{1}{n} \sum_{J \in \mathcal{J}} (4J+1)^2 \frac{2^{3J/2}}{\sqrt{n \alpha^2}} \lesssim \frac{1}{n} \cdot \frac{(J_{\max})^2 + 2a2^{3J_{\max}/2}}{\sqrt{n \alpha^2}} \lesssim \frac{1}{n}.
\]
Indeed, remember that \( 2^{3J_{\max}/(n \alpha^2)^2} \lesssim 1/\log^\kappa(n \alpha^2) \). Plugging the bounds for \( C_{1,1} \) and \( C_{1,2} \) into (C.4), we get that
\[
\sum_{J \in \mathcal{J}} \mathbb{E}[(A_J - \text{pen}_A(J))^2] \lesssim \frac{1}{n \alpha^2}.
\]
We conclude for the negative part that
\[
\mathbb{E}[(A_J - \text{pen}_A(J))^2] \leq \mathbb{E}A_J^2 + \text{pen}_A^2(J) \leq 2\text{pen}_A^2(J).
\]

**Second additional term:** Recall now
\[
B_J = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=-1}^{J-1} \frac{1}{(1 + 2)^{-1}} \sum_{k=0}^{J_{\max}} W_{ijk} \sigma_j^2 \left[ \frac{1}{n - 1} \sum_{h \neq i} W_{hjk} \right],
\]
The aim is to specify some ideally small real numbers \( y_J, J \in \mathcal{J} \), such that
\[
\sum_{J \in \mathcal{J}} \mathbb{E}[(B_J - y_J)_+] \lesssim \frac{1}{n^2 \alpha^4}.
\]
For any \( y > 0 \),
\[
(\Delta B) := \mathbb{E}[(B_J - y)_+] = \int_{0}^{\infty} t \mathbb{P}([B_J - y]_+ > t) dt \leq \int_{y}^{\infty} u \mathbb{P}(X > u) du.
\]
Let \( (W'_{ijk}) \) be an independent copy of \( (W_{ijk}) \) and
\[
B'_J = \frac{1}{n} \sum_{i=1}^{n} \sum_{j,k} W'_{ijk} \sigma_j^2 \left[ \frac{1}{n - 1} \sum_{h \neq i} W'_{hjk} \right].
\]
Then, with the notation 
\[ \zeta_{ijk} = \frac{1}{n^2} \sum_{h \neq i} W'_{hjk}, \]

\[ \mathbb{P}(B'_j > \eta \mid (W'_{hjk})_{hjk}) \leq \exp \left( -\frac{\eta^2}{2} \frac{1}{n^2} \frac{1}{n} \sum_{i=1}^{n} \sum_{j,k} \sigma_j^4 \sigma_k^4 \zeta_{ijk}^2 + \eta \frac{\max_{i,j,k} \sigma_j^2 \sigma_k^2 |\zeta_{ijk}|}{n \alpha^2} \right). \]

Here and in what follows, the double sum over \( j,k \) is abbreviated by \( \sum_{j,k} \), dropping in particular its dependence on \( J \) if this is clear from the context. Note that \( C_1 \) and \( C_2 \) are random and equally depend on \( J \).

By the decoupling inequality of de la Peña and Montgomery-Smith (1995), there exists some universal constant \( C > 0 \) with

\[ (\Delta_B) \leq C \int_{y}^{\infty} u \mathbb{P}\left( B'_j > \frac{u}{C} \right) du, \quad \frac{u}{C} = t \]

\[ = C^3 \int_{y/C}^{\infty} t \mathbb{P}(B'_j > t) dt \]

\[ = C^3 \int_{y/C}^{\infty} t \mathbb{E}\mathbb{P}(B'_j > t \mid W') dt \]

\[ = C^3 \mathbb{E} \int_{y/C}^{\infty} t \mathbb{P}(B'_j > t \mid W') dt \]

\[ \leq C^3 \mathbb{E} \int_{y/C=y'}^{\infty} t \exp \left( -\frac{t^2}{2} \frac{1}{C_1 + t C_2} \right) dt \]

\[ \leq C^3 \mathbb{E} \left\{ 2C_1 \exp \left( -\frac{y'^2}{4C_1} \right) + 4C_2 \left[ y' \vee \frac{C_1}{C_2} \right] \exp \left( -\frac{y' \vee \frac{C_1}{C_2}}{4C_2} \right) + \right. \]

\[ \left. + 16C_2^2 \exp \left( -\frac{y' \vee \frac{C_1}{C_2}}{4C_2} \right) \right\}, \]

where we have also used Lemma C.1.

As we have used Fubini’s theorem in the third equation, \( y \) is not allowed to depend on \( (W') \). Inspired by the choice for the previously treated first term, define

\[ y' := 8\sqrt{\mathbb{E}C_1} \log[2^{4J^2+1}]. \]

In order to bound (I), (II) and (III), we need concentration of \( C_1 \) around \( \mathbb{E}C_1 \) (cf. Lemma C.2).
We evaluate the expectation. Because of 
\[ E \left| \frac{1}{n-1} \sum_{h \neq i} W'_{hjk} \right|^2 = \frac{2}{n-1}, \]
we get 
\[ EC_1 = \frac{2\sigma^4}{n\alpha^4} \sum_{j,k} \sigma_j^2 \frac{2}{n-1} \approx \frac{1}{n^2\alpha^4} J^{4\alpha} 2^{3J}. \]

Now, we are in the position to continue with bounding (I), (II) and (III).

**Upper bound of \( E(I) \):** Let us denote by \( V = C_1/EC_1 \). Then 
\[ E(I) = 2EC_1 \cdot E \left[ V \exp \left( -\frac{16}{V} \cdot \log^2(2^{4J+1}) \right) \right]. \]

On the set \( \{ V \leq n^{-1/3} \} \), we get an easy upper bound 
\[ 2EC_1 \cdot n^{-1/3} \exp(-16n^{1/3} \log^2(2^{4J+1})) \leq 2EC_1 \cdot \frac{1}{2^{4J}}, \]
while on the set \( \{ V > n^{-1/3} \} \) we bound the exponential term by 1 and write 
\[ 2EC_1 \cdot \sum_{k \in \mathbb{N}} E \left[ V \exp \left( -\frac{16}{V} \cdot \log^2(2^{4J+1}) \right) \cdot 1 \left( \frac{1}{n^{1/3}} \vee k^5 < V \leq (k+1)^5 \right) \right] \]
\[ \lesssim EC_1 \cdot \sum_{k \geq 1} (k+1)^5 \left\{ 2^{J+2} \exp \left( -2^J k^5 \right) + \frac{1}{k^{10} n^{3/2} J} \right\} \]
\[ \lesssim EC_1 \cdot \left\{ \sum_{k \geq 1} \frac{(k+1)2^{J}k^5}{2^J k^3} \exp \left( -2^J k \right) + \frac{1}{n^{3/2} J} \right\}. \]

Finally, 
\[ E(I) \lesssim EC_1 \cdot \left\{ \frac{1}{2^{4J}} + \frac{1}{n^{3/2} J} \right\}. \]

**Upper bound of \( E(II) \):** If \( y' > C_1/C_2 \), then \( \sqrt{C_1} \geq C_2 \) implies 
\[ \frac{8\sqrt{EC_1} \log(2^{4J+1})}{4C_2} \geq 2 \frac{\sqrt{EC_1}}{\sqrt{C_1}} \log(2^{4J+1}). \]
If \( y' \leq C_1/C_2 \), we conclude that
\[
\frac{\sqrt{C_1}}{4C_2} \geq 2 \log(2^{4J+1}) \frac{\sqrt{EC_1}}{\sqrt{C_1}}.
\]

Put the two cases together to get
\[
(C.7) \quad \frac{[y' \lor C_1/C_2]}{4C_2} \geq 2 \left( \frac{\sqrt{EC_1}}{\sqrt{C_1}} + \frac{EC_1}{C_1} \right) \log [2^{4J+1}]
\]

Finally, \( C_2\sqrt{C_1} \leq C_1 \) implies
\[
4C_2 \left[ y' \lor \frac{C_1}{C_2} \right] \leq 8(C_1 + EC_1) \log [2^{4J+1}].
\]

Summarizing,
\[
(II) = 4C_2[y' \lor \frac{C_1}{C_2}] \exp \left( - \frac{[y' \lor \frac{C_1}{C_2}]^2}{4C_2} \right)
\]
\[
\leq 8(C_1 + EC_1) \log [2^{4J+1}] \exp \left( -2 \left( \frac{\sqrt{EC_1}}{\sqrt{C_1}} + \frac{EC_1}{C_1} \right) \log (2^{4J+1}) \right).
\]

Therefore, using again the notation \( V = C_1/EC_1 \) and proceeding as before, we get:
\[
\mathbb{E}(II) \lesssim EC_1 \log \left[ 2^{4J+1} \right] \mathbb{E} \left[ (V + 1) \cdot \exp \left( -2 \left( \frac{1}{\sqrt{V}} + \frac{1}{V} \right) \log [2^{4J+1}] \right) \right]
\]
\[
\lesssim EC_1 \cdot \log \left[ 2^{4J+1} \right] \cdot \left\{ \frac{1}{24J} + \frac{1}{n^{3/2}J} \right\}.
\]

Upper bound of \( \mathbb{E}(III) \): Again by (C.7) and the inequality \( C_2^2 \leq C_1 \),
\[
(III) = 16C_2^2 \exp \left( - \frac{[y' \lor \frac{C_1}{C_2}]^2}{4C_2} \right)
\]
\[
\leq 16C_1 \exp \left( -2 \left( \frac{\sqrt{EC_1}}{\sqrt{C_1}} + \frac{EC_1}{C_1} \right) \log (2^{4J+1}) \right).
\]

The same arguments as used for \( \mathbb{E}(II) \) reveal
\[
\mathbb{E}(III) \lesssim EC_1 \cdot \log \left[ 2^{4J+1} \right] \cdot \left\{ \frac{1}{24J} + \frac{1}{n^{3/2}J} \right\}.
\]
Finally, inserting \( y = Cy' = C8\sqrt{E}C_1 \log[2^{4J+1}] \) into \((\Delta_B)\), where \( C \) denotes the universal constant of the decoupling inequality of de la Peña and Montgomery-Smith (1995), we obtain

\[
(\Delta_B) \leq C^3E \int_{y'}^{\infty} t \mathbb{P}(X' > t \mid W')dt \end{equation}
\[
 \lesssim EC_1 \log \left[ 2^{4J+1} \right] \left\{ \frac{1}{2^{4J}} + \frac{1}{n^32^J} \right\}. \end{equation}
\]

Therefore, with \( y_J := C8\sqrt{E}C_1 \log[2^{4J+1}] \) we get the following upper bound

\[
\mathbb{E}[(B_J - y_J)^2] \lesssim \frac{1}{n^2\alpha^4} \sum_{J \in \mathcal{J}} \mathcal{J}_{[4a \alpha^3J]} \log[2^{4J+1}] \left\{ \frac{1}{2^{4J}} + \frac{1}{n^32^J} \right\} \end{equation}
\[
 \lesssim \frac{1}{n^2\alpha^4} \mathcal{J}_{\max} \left( 1 + \frac{2^{2\mathcal{J}_{\max}}}{n^3} \right). \end{equation}
\]

As a consequence, remembering also that \( 2^{3\mathcal{J}_{\max}} \lesssim n^2\alpha^4 \),

\[
\sum_{J \in \mathcal{J}} \mathbb{E}[(B_J - y_J)^2] \lesssim_{\text{log}} \frac{1}{n^2\alpha^4}, \end{equation}
\]

We conclude to the same inequality if we replace \( y_J \) with its bound from above \( \text{pen}_B(J) := 256\sigma^2(J + 1)^{2a+1}2^{3J/2}\sqrt{n\alpha^2} \) and for the negative part that

\[
\mathbb{E}[(B_J - \text{pen}_B(J))^2] \leq \mathbb{E}B_J^2 + \text{pen}_B^2(J) \leq 2\text{pen}_B^2(J). \end{equation}
\]

In order to conclude the proof of the theorem we see that

\[
\mathbb{E}\left( \left( \sup_J V_J^2 \right) \right) \lesssim \log \frac{1}{n\alpha^2} + \inf_{J \in \mathcal{J}} \left( \|f - f_J\|_2^2 + \text{pen}_A^2(J) + \text{pen}_B(J) \right) \end{equation}
\[
 \lesssim \log \frac{1}{n\alpha^2} + \inf_{J \in \mathcal{J}} \left( \|f - f_J\|_2^2 + \frac{2^J}{n\alpha^2} + \frac{2^{3J}}{n^2\alpha^4} \right) \end{equation}
\]

and the infimum is attained at the minimax rate, up to some logarithmic factors.

\( \square \)

C.2. Proof of Theorem 6.2 (sequentially interactive protocol).

Similarly to the non-interactive case, we write

\[
\mathbb{E} \left[ (\hat{D}_n^{(SI)} - D - \text{pen}(\hat{J}))^2 \right] = \mathbb{E} \left[ \sup_{J \leq \mathcal{J}_{\max}} (\hat{D}_J - D - \text{pen}(J))^2 \right] \end{equation}
\[
 \leq \sum_{J \leq \mathcal{J}_{\max}} \mathbb{E}[(V_J)^2] + \inf_{J \leq \mathcal{J}_{\max}} \mathbb{E}[(V_J)^2], \end{equation}
\]
where $V_J := \hat{D}_J - D - \hat{\text{pen}}(J)$.

We start with bounding \( \sum_{J \leq J_{\text{max}}} \mathbb{E}[(V_J)^2] \). To this aim, note first that

\[
\frac{1}{2} \mathbb{E}[(V_J)^2] \leq \mathbb{E}[(\hat{\text{pen}}(J) - \text{pen}(J))^2] + \mathbb{E}[(\hat{D}_J - D - \text{pen}(J))^2] := T_1 + T_2,
\]
say, with $\text{pen}^2(J) = \frac{1}{n \alpha^2} \sum_{j=-1}^{J-1} \sigma_j^2 \| \beta_j \|^2$. First

\[
T_1 := \mathbb{E}[(\hat{\text{pen}}(J) - \text{pen}(J))^2]
\]

\[
\leq \frac{1}{n \alpha^2} \sum_{j=-1}^{J-1} \sigma_j^2 \left| \mathbb{E} \left[ \left( \frac{1}{|N_j|} \sum_{i \in N_j} Z_i^{(2,j)} \right) \right] - \| \beta_j \|^2 \right|
\]

\[
\leq \frac{1}{n \alpha^2} \sum_{j=-1}^{J-1} \sigma_j^2 \left\{ \mathbb{E} \left[ \left| \mathbb{E} \left[ \left( \frac{1}{|N_j|} \sum_{i \in N_j} Z_i^{(2,j)} \right) - \int \left( \sum_k \hat{\beta}_{jk} \psi_{jk} \right) f \right] \right| \right] Z^{(1)} \right\}
\]

\[
+ \mathbb{E} \left[ \left\| \left( \sum_k \hat{\beta}_{jk} \psi_{jk} \right) f - \| \beta_j \|^2 \right\| \right]
\]

\[
\leq \frac{1}{n \alpha^2} \sum_{j=-1}^{J-1} \sigma_j^2 \left\{ \mathbb{E} \left[ \left\| \mathbb{E} \left[ \left( \frac{1}{|N_j|} \sum_{i \in N_j} Z_i^{(2,j)} \right) \right] Z^{(1)} \right\| \right] \right\}
\]

\[
+ \mathbb{E} \left[ \left\| \left( \sum_k \hat{\beta}_{jk} \psi_{jk} \right) f - \sum_k \hat{\beta}_{jk} \beta_{jk} \right\| \right] + \left\| \sum_k \hat{\beta}_{jk} \beta_{jk} - \| \beta_j \|^2 \right\|
\]

\[
= T_{1,1} + T_{1,2} + T_{1,3}, \text{ say.}
\]

We need to show that the sum $\sum_{J \leq J_{\text{max}}} T_1(J)$ stays bounded by a sequence not larger than the minimax rate, up to some logarithmic factors.

The first term in the sum above is bounded from above as follows

\[
T_{1,1} \leq \frac{1}{n \alpha^2} \sum_{j=-1}^{J-1} J^{2a_j} \frac{\tau}{\sqrt{n/(2J_{\text{max}})}} \leq \frac{J^{2a_j} \sqrt{J_{\text{max}}}}{(n \alpha^2)^{3/2}}.
\]

Summing this up over $J = 1, \ldots, J_{\text{max}}$, we see that this gives a rate of order $(n \alpha^2)^{-1}$ up to log-factors. By an argument analogous to (B.9), the third term in the sum can be bounded from above using that $\| \beta_j \|^2 \lesssim$
\[ 2^{-2j's'} I(s' < 1) + 2^{-J\frac{3}{2}} I(s' \geq 1}, \text{ (see Lemma A.2), by} \]

\[ T_{1,3} \leq \frac{1}{na^2} \sum_{j=-1}^{J-1} \sigma^2_j \text{Var}^{1/2} \left( \sum_k \hat{\beta}_{jk} \hat{\beta}_{jk} \right) \lessapprox \frac{1}{na^2} \sum_{j=-1}^{J-1} \sigma^2_j \frac{\sigma_j}{\sqrt{na^2}} \| \beta_j \|_2 \]

\[ \lessapprox \frac{1}{(na^2)^{3/2}} \mathbb{1}_{0<s'<1} + \frac{1}{(na^2)^{3/2}} \mathbb{1}_{s' \geq 1}. \]

Summing this up over \( J = 1, \ldots, J_{\max} \), we obtain the bound

\[ \frac{1}{na^2} \cdot \left( \frac{J_{\max}^3}{(na^2)^{1/2}} + \frac{J_{\max}^3}{(na^2)^{1/2}} \right). \]

Plugging in \( 2^{J_{\max}} \leq \sqrt{na^2/\log^{B/2}(na^2)} \), we see that in the regime where \( s' \in (0, \frac{1}{2}) \) this is bounded from above by \( (na^2)^{-\frac{4}{3}+\varepsilon} \), up to log-factors, while, for \( s' > \frac{1}{2} \), we end up with the parametric rate, again, up to log-factors. For the second term \( T_{1,2} \) in the sum, first recall that \( \Pi_\tau(v) - v = -(v - \tau)_+ + (v - \tau)_+ \) and thus

\[ T_{1,2} = \frac{1}{na^2} \sum_{j=-1}^{J-1} \sigma^2_j \mathbb{E} \left| \int_0^1 \Pi_\tau \left( \sum_k \hat{\beta}_{jk} \psi_{jk} \right) f - \sum_k \hat{\beta}_{jk} \psi_{jk} \right| \]

\[ \lessapprox \frac{1}{na^2} \sum_{j=-1}^{J-1} \sigma^2_j \mathbb{E} \left| \int_0^1 \left( \sum_k \hat{\beta}_{jk} \psi_{jk} - \tau \right)_+ f \right| + \frac{1}{na^2} \sum_{j=-1}^{J-1} \sigma^2_j \mathbb{E} \left| \int_0^1 \left( - \sum_k \hat{\beta}_{jk} \psi_{jk} - \tau \right)_+ f \right|. \]

In order to bound the two expected values within the sums above, we may replicate the arguments of the proof of Theorem 4.1 with \( \hat{f}_j^{(1)}(x) \) replaced by

\[ Y_j(x) := \hat{f}_{j+1}^{(1)}(x) - \hat{f}_j^{(1)}(x) = \sum_k \hat{\beta}_{jk} \psi_{jk}(x) \]

and \( \tau = \log^c(\sqrt{na^2}) \) in (B.6) and further on. Note that the truncation value \( \tau \) is free of \( s' \). Indeed, analogously to Proposition B.1, there exist constants \( c_1, c_2 > 0 \) such that

\[ |Y_j(x) - \mathbb{E} Y_j(x)| \leq \left[ c_1 \frac{(j + 1) a^2 u}{\sqrt{na^2}} \right] \vee \left[ c_2 \frac{(j + 1) a^2 u}{na^2} \right] \]

with probability larger than \( 1 - 4 \exp(-u/2) \) for all \( u > 0 \), such that the arguments for bounding (B.6) by means of Lemma B.2 with \( a_1 = 0, A_1 = \ldots \)
\( c_1 \frac{na^2}{(j+1)^2} \), \( r_1 = 2 \), \( v_1 = \frac{r}{2} \) and \( a_2 = 0 \), \( A_2 = \frac{c_2}{2} \frac{na^2}{(j+1)^2} \), \( r_2 = 1 \) and \( v_2 = (j+1)^2 \alpha \) are applicable. We get

\[
\mathbb{E} \int_0^1 \left( \pm \sum_k \hat{\beta}_{jk} \psi_{jk} - \tau \right) f \leq \mathbb{E} \int_0^1 \left( \pm (Y_j - \mathbb{E} Y_j) - \frac{\tau}{2} \right) + f
\]

\[
\lesssim \frac{4}{c_1} \frac{2(j+1)2^j}{\tau na^2} \exp \left( -\frac{c_1}{8} \frac{\tau^2 na^2}{(j+1)^2} \right) + \frac{4}{c_2} \frac{(j+1)^2}{na^2} \exp \left( -\frac{c_2}{2} \frac{\tau}{na^2} \right).
\]

This gives

\[
T_{1,2} \lesssim \frac{1}{(na^2)^2} \left\{ \frac{8}{c_1} \sum_j \frac{1}{\tau} (j+1)^2 \alpha \exp(-\frac{c_1}{2} \frac{na^2}{\tau^2 J_{max}^2}) \right\}
\]

By summing up over \( J \) we get the upper bound (up to constants), we get

\[
\frac{\int_{\tau}^{\frac{na^2}{2} J_{max}}}{(na^2)^2 \log \kappa(\tau/na^2)} \exp(-c \log^{2\kappa+2B-2a}(\kappa^2)) \lesssim \frac{1}{na^2},
\]

for some constant \( c > 0 \) and the last inequality holds for \( \kappa(a, B) > 0 \) chosen large enough.

Next, denote by \( D_J = \sum_{j=-1}^{J-1} \| \vec{\beta}_j \|^2 \) and let us decompose

\[
T_2 := \mathbb{E}[(\hat{D}_J - D - \text{pen}(J))^2]
\]

\[
\leq \mathbb{E} \left[ \left( \hat{D}_J - \sum_{j=-1}^{J-1} \int_0^1 \Pi \left( \sum_k \hat{\beta}_{jk} \psi_{jk} \right) f \right)^2 \right]
\]

\[
+ \mathbb{E} \left[ \left( \sum_{j=-1}^{J-1} \int_0^1 \Pi \left( \sum_k \hat{\beta}_{jk} \psi_{jk} \right) f - \sum_{j=-1}^{J-1} \sum_k \hat{\beta}_{jk} \beta_{jk} \right)^2 \right]
\]

\[
+ \mathbb{E} \left[ \left( \sum_{j=-1}^{J-1} \sum_k \hat{\beta}_{jk} \beta_{jk} - D_J - (D - D_J - \text{pen}(J))^2 \right) \right]
\]

\[= T_{2,1} + T_{2,2} + T_{2,3}, \text{ say.} \]
For the first term we write
\[ T_{2,1} = \mathbb{E} \left( \sum_j (Z_j^{(2,j)} - \mathbb{E}(Z_j^{(2,j)}|Z^{(1)})) \right)^2 \]
\[ \lesssim J \cdot \mathbb{E} \left( \frac{\tau^2}{|\mathcal{V}_j|\alpha^2} \right) \lesssim \log^2(n\alpha^2) \frac{J}{(n/J_{\text{max}})\alpha^2} \leq \log^2(n\alpha^2) \frac{J_{\text{max}}^2}{n\alpha^2}. \]

Next, we write \( \sum_k \hat{\beta}_{jk} \beta_{jk} = \int_0^1 Y_j f \) and get that
\[ T_{2,2} \leq J \cdot \sum_j \mathbb{E} \left( \left( \int_0^1 [\tau(Y_j) - Y_j] f \right)^2 \right) \]
\[ \leq J \cdot \sum_j \mathbb{E} \left( \left( \int_0^1 -[Y_j - \tau]_+ f + [-Y_j - f]_+ f \right)^2 \right) \]
\[ \leq J \cdot \sum_j 2 \cdot \max \mathbb{E} \left( \left( \int_0^1 (\pm(Y_j - \mathbb{E}Y_j) - \frac{\tau}{2})_+ f \right)^2 \right) \]
and we follow the lines of proof for the non-adaptive case with \( \hat{f}_j^{(1)} \) replaced by \( Y_j \) in Section B.2 and also analogously to the bound for the term \( T_{1,2} \) here above to get, for \( \kappa(a,B) > 0 \) large enough,
\[ T_{2,2} \lesssim \log \frac{1}{n\alpha^2}. \]

Finally,
\[ T_{2,3} \lesssim \int_0^\infty u \cdot \mathbb{P} \left( \sum_{j=1}^{J-1} \sum_k (\hat{\beta}_{jk} - \beta_{jk}) \beta_{jk} \geq u + D - D_J + \text{pen}_J \right) \, du \]
\[ \lesssim \int_{D - D_J + \text{pen}_J}^\infty u \cdot \mathbb{P} \left( \sum_{j=1}^{J-1} \sum_k (\hat{\beta}_{jk} - \beta_{jk}) \beta_{jk} \geq u \right) \, du. \]
\[ \text{(C.8)} \]
Here we use directly the concentration from Lemma C.1. We decompose
\[ \sum_{j=1}^{J-1} \sum_k (\hat{\beta}_{jk} - \beta_{jk}) \beta_{jk} = \mathbb{P}_{n/2} \sum_{j=1}^{J-1} \sum_k (\psi_{jk}(\cdot) - \beta_{jk}) \beta_{jk} \]
\[ + \frac{2}{n} \sum_{i=1}^{n/2} \sum_{j=1}^{J-1} \sum_k \sigma_j \sigma_k W_{ijk} \beta_{jk} \]
Analogously to the non-interactive setup, we use the Bernstein inequality in (C.1) with random $\Xi_{ijk}$ replaced by deterministic $\beta_{jk}$. The empirical process in the first term of the previous display possesses uniformly in $J$ an exponential tail bound which is smaller than the exponential tail bound for the second part and consequently covered by doubling the constant of the penalty. Indeed, the first term has variance of a smaller order uniformly in $J$ than the second term, and also an upper bound which is uniformly in $J$ smaller than $\sigma/(n\alpha) \max(\beta)$. We write

$$P\left(\sum_{j=1}^{J} \sum_{k} (\beta_{jk} - \beta_{jk}) \beta_{jk} \geq u \right) \leq \exp\left(-\frac{u^2}{2} \frac{\sigma^2}{n\alpha^2} \sum_j \sigma^2_j \|\beta_j\|^2 + u \frac{\sigma}{n\alpha} \max(\beta) \right).$$

Remark that $\max(\beta) \leq J^a$. We apply Lemma C.1 in order to bound the integral in (C.8) with $a_1 = \text{pen}(J)^2$, $a_2 = J^a/(n\alpha)$ and $y = D - D_J + \text{pen}(J)$ to get the upper bound

$$T_{2,3} \leq 2a_1 \exp\left(-\frac{y^2}{4a_1}\right) + 4a_2 \left[y \vee \frac{a_1}{a_2}\right] \exp\left(-\frac{y \vee \frac{a_1}{a_2}}{4a_2}\right)$$

$$+ 16a_2^2 \exp\left(-\frac{y \vee \frac{a_1}{a_2}}{4a_2}\right) \leq \text{pen}(J)^2 + a_2^2 \leq \text{pen}(J)^2.$$ 

By an argument analogous to (B.9), $\text{pen}(J)^2 = (n\alpha^2)^{-1} \sum_{j=-1}^{J-1} \sigma^2_j \|\beta_j\|^2$ can be bounded from above using that $\|\beta_j\|^2 \lesssim 2^{-2j} I(s' < 1) + 2^{-J^2} I(s' \geq 1)$, (see Lemma A.2). After summing up $\text{pen}(J)^2$ over $J$ we get the bound

$$J_{max} \frac{J_{max}^{2a} \cdot (1 \vee 2^{J_{max}(1 - 2s')})}{n\alpha^2} \lesssim J_{max}^{2a+1} (n\alpha^2)^{-\frac{1}{2} - s'} \cdot I(s' < \frac{1}{2}).$$

We see that, in case $s' < 1/2$, this bound is smaller than $(n\alpha^2)^{-2s'/(4s' + 1)}$ and that the bound is up to logarithmic terms smaller than $\nu_n^{(SI)}(\alpha, a, s')$.

We finish the proof by studying

$$\inf_{J \leq J_{max}} \mathbb{E}(V_J^2) \leq 4 \inf_{J \leq J_{max}} \left\{ \mathbb{E}((\hat{D}_J - D_J)^2) + (D - D_J)^2 + \text{pen}(J)^2 \right.$$ 

$$+ \mathbb{E}((\text{pen}(J) - \text{pen}(J))^2) \right\}$$

$$\lesssim \log \inf_{J \leq J_{max}} (D - D_J)^2 + \text{pen}(J)^2 + T_1,$$

where $T_1$ has been defined and bounded from above here above. We conclude that the risk of the adaptive estimator $\hat{D}^{(SI)}$ is up to some logarithmic factor bounded from above by $\nu_n^{(SI)}(\alpha, a, s')$. \qed
C.3. Auxiliary lemmas.

**Lemma C.1.** For any constants \( y, a_1, a_2 > 0 \),
\[
\int_{y}^\infty t \exp \left( -\frac{t^2}{2} \frac{1}{a_1 + a_2} + ta_2 \right) dt \\
\quad \leq 2a_1 \exp \left( -\frac{y^2}{4a_1} \right) + 4a_2 \left[ y \vee \frac{a_1}{a_2} \right] \exp \left( -\left[ \frac{y \vee \frac{a_1}{a_2}}{4a_2} \right]^2 \right) + \\
\quad + 16a_2^2 \exp \left( -\left[ \frac{y \vee \frac{a_1}{a_2}}{4a_2} \right]^2 \right).
\]

**Proof.** Because of
\[
\frac{t^2}{2} \frac{1}{a_1 + a_2} \geq \begin{cases} 
\frac{t^2}{4a_1} & \text{if } t \leq a_1/a_2 \\
\frac{t}{4a_2} & \text{if } t > a_1/a_2,
\end{cases}
\]
the left-hand side is upper bounded by
\[
\int_{y}^\infty t \exp \left( -\frac{t^2}{2} \frac{1}{a_1 + a_2} + ta_2 \right) dt \\
\leq \int_{y}^{(a_1/a_2) \vee y} t \exp \left( -\frac{t^2}{4a_1} \right) dt + \\
+ \int_{y \vee (a_1/a_2)}^\infty t \exp \left( -\frac{t}{4a_2} \right) dt.
\]
Both integrals can be evaluated explicitly:
\[
A_1 = -2a_1 \int_{y}^{(a_1/a_2) \vee y} t \frac{1}{2a_1} \exp \left( -\frac{t^2}{4a_1} \right) dt \\
= \left[ -2a_1 \exp \left( -\frac{t^2}{4a_1} \right) \right]_{y}^{(a_1/a_2) \vee y} \\
= 2a_1 \exp \left( -\frac{y^2}{4a_1} \right) - 2a_1 \exp \left( -\left[ \frac{a_1}{a_2} \vee y \right]^2 \right)
\]
and
\[
A_2 = -4a_2 \exp \left( -\frac{t}{4a_2} \right)_{y \vee (a_1/a_2)}^{\infty} + \int_{y \vee (a_1/a_2)}^\infty 4a_2 \exp \left( -\frac{t}{4a_2} \right) dt \\
= 4a_2 \left[ y \vee \frac{a_1}{a_2} \right] \exp \left( -\left[ \frac{y \vee \frac{a_1}{a_2}}{4a_2} \right] \right) + 16a_2^2 \exp \left( -\frac{y \vee \frac{a_1}{a_2}}{a_2} \right).
\]
Dropping the negative summand in the expression for \( A_1 \) reveals the bound. \( \square \)
LEMMA C.2 (Tail bound of \( C_1 \)). For any \( \gamma > n^{-1/3} \), \( n > 1 \),
\[
\mathbb{P}(C_1 > \gamma E \mathcal{C}_1) \lesssim 2^{J+2} \exp \left( -n \min \left\{ \frac{2^J \gamma}{n}, \frac{2^J/\sqrt{n}}{\sqrt{n}} \right\} \right) + \frac{1}{\gamma^2 n^2 2^J}.
\]

PROOF. We decompose \( \zeta_{ijk} = \frac{1}{n} \sum_{h=1}^{n} W_{hjk} - \frac{1}{n-1} W_{ijk} \) and use that \((a+b)^2 \leq 2a^2 + 2b^2\) together with \( n/(n-1) \leq 2 \). Due to the inequality
\[
\mathbb{P}(C_1 > \eta) \leq \mathbb{P} \left( \frac{16}{n} \sum_{j,k} \sigma_j^4 \sigma_k^4 \left[ \frac{1}{n} \sum_{i=1}^{n} W_{ijk} \right]^2 > \eta/2 \right)
+ \mathbb{P} \left( \frac{4}{n(n-1)^2} \sum_{j,k} \sigma_j^4 \sigma_k^4 W_{ijk}^2 > \eta/2 \right),
\]
it remains to bound both expressions on the right-hand side. By the union bound and the Bernstein exponential inequality for Laplace i.i.d. random variables derived, e.g., in Boucheron, Lugosi and Massart (2013), we get
\[
\mathbb{P} \left( \frac{1}{n} \sum_{j,k} \sigma_j^4 \sigma_k^4 \left[ \frac{1}{n} \sum_{i=1}^{n} W_{ijk} \right]^2 > \frac{\eta}{32} \right)
\leq 2^{J+1} \mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^{n} W_{i1} \right| > \frac{\sqrt{n} \alpha^2 \sqrt{\eta}}{8 \sigma^2 2^J 2^{2a}} \right)
\leq 2^{J+2} \exp \left( -n \min \left\{ \frac{\eta_n, J}{2}, \frac{1}{2 + \eta_n, J} \right\} \right)
\leq 2^{J+2} \exp \left( -n \min \left\{ \frac{\eta_n, J}{8}, \frac{\eta_n, J}{4} \right\} \right).
\]
Concerning the second expression, Chebychev’s inequality reveals
\[
\mathbb{P} \left( \frac{4}{n(n-1)^2} \sum_{j,k} \sigma_j^4 \sigma_k^4 W_{ijk}^2 > \eta/2 \right)
\leq \mathbb{P} \left( \frac{4}{n(n-1)^2} \sum_{j,k} \sigma_j^4 \sigma_k^4 (W_{ijk}^2 - 2) > \eta/4 \right)
+ \mathbb{P} \left\{ \frac{8 \sigma^4}{n(n-1)^2 \alpha^4} \sum_{j,k} \sigma_j^4 > \eta/4 \right\}
\leq \frac{16^2 \sigma^8}{\eta^2 n^3 (n-1)^4 \alpha^8} 2^5 J 8 J 8 \mathbb{E}(W_{ijk}^2 - 2)^2 + \mathbb{P} \left\{ \frac{8 \sigma^4}{n(n-1)^2 \alpha^4} 2^3 J + 1 2^4 a > \eta/4 \right\}.
\]
With \( \eta = \gamma \mathbb{E} C_1 = \gamma \frac{4\sigma^4}{n(n-1) \alpha^4} \sum_{j,k} \sigma_j^2 \), the indicator function above is equal to 0 for any \( \gamma > cn^{-1}J_{\max}^2 J_{\max}^{2a} \frac{2}{\sqrt{n}} \) for some constant \( c > 0 \). Moreover, we obtain \( \frac{2^{J/2}}{\sqrt{n}} \sqrt{\gamma} \lesssim \eta_{n,J} \). Therefore,

\[
\mathbb{P}(C_1 > \gamma \mathbb{E} C_1) \lesssim 2^{J+2} \exp \left( -n \min \left\{ \frac{2^J \gamma}{n}, \frac{2^{J/2} \sqrt{\gamma}}{\sqrt{n}} \right\} \right) + \frac{2^J \gamma^{8a}}{\gamma^2 (\mathbb{E} C_1)^2 n^7 \alpha^8} + \frac{1}{\gamma^2 n^{3/2} J}.
\]