Kinetic theory with angle-action variables

Pierre-Henri Chavanis

Laboratoire de Physique Théorique (UMR 5152 du CNRS),
Université Paul Sabatier
118 route de Narbonne,
31062 Toulouse Cedex 4, France
(chavanis@irsamc.ups-tlse.fr)

Abstract

We present a kinetic theory for one-dimensional inhomogeneous systems with weak long-range interactions. Starting from the Klimontovich equation and using a quasilinear theory valid at order $1/N$ in a proper thermodynamic limit $N \to +\infty$, we obtain a closed kinetic equation describing the relaxation of the distribution function of the system as a whole due to resonances between different orbits. This equation is written in angle-action variables. It conserves mass and energy and increases the Boltzmann entropy (H-theorem). Using a thermal bath approximation, we derive a Fokker-Planck equation describing the relaxation of a test particle towards the Boltzmann distribution under the combined effects of diffusion and friction. We mention some analogies with the kinetic theory of point vortices in two-dimensional hydrodynamics. We also stress the limitations of our approach and the connection with recent works.

Keywords: kinetic theory; long-range interactions; Fokker-Planck equations

Corresponding author: P.H. Chavanis; e-mail: chavanis@irsamc.ups-tlse.fr; Tel: +33-5-61558231; Fax: +33-5-61556065
I. INTRODUCTION

Systems with long-range interactions have been the object of considerable interest in recent years [1]. Their dynamics is very rich and presents many interesting features [2]. Therefore, the construction of a kinetic theory adapted to such systems is valuable. Several kinetic theories have been developed in the past. The first kinetic theory was constructed by Boltzmann [3] in his theory of gases. In that case, the particles do not interact except during strong collisions. The extension of this kinetic theory to the case of particles in Coulombian or Newtonian interaction was considered by Landau [4] in the case of plasmas and by Chandrasekhar [5] in the case of stellar systems. Developments and improvements of the kinetic theory of Coulombian plasmas were made by Lenard [6] and Balescu [7] using more formal approaches allowing to take into account collective effects. They showed in particular how collective effects can regularize the logarithmic divergence of the diffusion coefficient at the Debye length. On the other hand, a kinetic theory was developed by Dubin & O’Neil [8, 9] and Chavanis [10, 11] to describe the “collisional” evolution of charged rods in a magnetized plasma or point vortices in two-dimensional hydrodynamics. In that case, the particles interact via a logarithmic potential in two dimensions. A specificity of this model is that the particles have no inertia so that the collision term acts in position space. More recently, Bouchet, Dauxois and Chavanis [12-16] have considered the kinetic theory of the Hamiltonian Mean Field (HMF) model, a toy model of systems with long-range interactions which possesses a lot of interesting properties and which can be studied in great detail. Finally, in very recent works, Chavanis [2, 17] and Benedetti et al. [18] have developed the kinetic theory of a 2D Coulombian plasma and Valageas [19] has developed the kinetic theory of a 1D gravitational system in a cosmological context.

Kinetic theories are important to describe the relaxation of a Hamiltonian system of particles in interaction towards statistical equilibrium. They are also necessary to determine the timescale of the collisional relaxation and to prove whether the system will truly relax towards statistical equilibrium. Indeed, the convergence towards statistical equilibrium is not firmly established for complex systems with long-range interactions. A first reason is that systems with long-range interactions may exhibit non-markovian effects and spatial delocalization so that the Boltzmann $H$-theorem is difficult to prove and could even be wrong in a strict sense. Such problems have been discussed for example by Kandrup [20-22] in the case of self-gravitating systems and by Chavanis for point vortices [11] and for the HMF model [16]. These authors derive exact kinetic equations for these systems which apparently do not satisfy an $H$-theorem. The $H$-theorem is recovered only when additional assumptions (markovian approximation, local approximation, neglect of three body correlations,...) are implemented [34]. On the other hand, the relaxation of the system is due to a condition of resonance between different orbits and it may happen that the relaxation stops because there is no resonance anymore. This is the case for point vortices in two-dimensional hydrodynamics when the profile of angular velocity becomes monotonic [8, 11]. This is also the case for spatially homogeneous 1D systems for which the Landau-Lenard-Balescu collision term cancels out [2, 13, 14]. In these cases, the system remains frozen in a long-lived “metastable” state (a stationary solution of the Vlasov equation) which is not the statistical equilibrium state. This means either that the system will not reach statistical equilibrium or that the kinetic theory is incomplete. Indeed, the collision term calculated in the above-mentioned kinetic theories corresponds to a term of order $1/N$ in a systematic expansion carried out in a proper thermodynamic limit $N \to +\infty$ (one can rescale the parameters so that the
coupling constant scales like $k \sim 1/N$ while $E \sim N$, $T \sim 1$ and $V \sim 1$ [2]). Therefore, they describe the dynamics of the system on a timescale of order $N t_D$ where $t_D$ is the dynamical time. If the collision term cancels out because of the absence of resonances, the relaxation towards statistical equilibrium (if any) will take place on a timescale larger than $N t_D$ so that next order terms in the $1/N$ expansion must be taken into account. The calculation of these higher order terms has not been realized for any of the kinetic theories mentioned previously. However, for the HMF model, it is shown numerically that the relaxation towards statistical equilibrium takes place on a timescale of order $N^{1.7} t_D$ [23].

In all these works (except for point vortices), the system is assumed to be spatially homogeneous [17]. For the HMF model, a homogeneous phase exists if the energy is larger than a critical energy $E_c$. For Coulombian plasmas, the interaction is shielded on a distance corresponding to the Debye length so that the system is spatially homogeneous. Stellar systems, on the other hand, are clearly inhomogeneous but most kinetic theories assume that the encounters can be treated with a local approximation [24] as if the system were spatially homogeneous. This results in a logarithmic divergence of the diffusion coefficient which is cured by introducing an appropriate cut-off at the Jeans scale. It is expected that this divergence would be removed if effects of spatial inhomogeneity were taken into account. Therefore, the description of the dynamics of inhomogeneous systems with long-range interactions, like the clustered phase of the HMF model below the critical energy $E_c$, remains a challenging problem. In a previous paper [16], we have given some elements of kinetic theory for the inhomogeneous HMF model in a very special situation of low energy (close to the ground state) where the particles are strongly concentrated around a cluster at $\theta = 0$ so that they all perform a harmonic motion with the same frequency. In that “pathological” case, some curious behaviors have been predicted. However, in more general situations, the particles will have different frequencies and we may expect that the relaxation will be due to some sort of resonances as in the case of 2D point vortices [8, 11].

In this paper, we present a kinetic theory for one-dimensional inhomogeneous systems with weak long-range interactions. Starting from the Klimontovich equation and using a quasilinear theory (Sec. II) valid at order $1/N$ in a proper thermodynamic limit $N \to +\infty$, we obtain a closed kinetic equation describing the relaxation of the distribution function of the system as a whole due to resonances between different orbits (Sec. IV). This equation is written in angle-action variables (Sec. III). It conserves mass and energy and increases the Boltzmann entropy so that a H-theorem can be proved (Sec. V). Using a thermal bath approximation (Sec. VII), we derive a Fokker-Planck equation that describes the relaxation of a test particle towards the Boltzmann distribution under the combined effects of diffusion and friction. Unfortunately, our derivation of the kinetic equation is based on a factorization hypothesis (28) that we cannot justify and that is probably not rigorously valid. However, we expect that more general approaches will be able to circumvent this difficulty. The important point is that the final kinetic equation (37) that we obtain is closed, without undetermined parameter, and that it possesses a lot of interesting properties (conservation laws and H-theorem). This gives some confidence in its relevance to describe the dynamics of the system even if the factorization hypothesis does not hold. In fact, in a very recent work, Valageas [19] has independently tackled a similar problem with a different method that avoids the factorization hypothesis. We shall discuss the connection between the two approaches in Sec. IX and show that our kinetic equation (37) can also be obtained from the approach of Valageas [19]. This provides therefore a rigorous justification for this kinetic equation without ad hoc assumption.
II. THE KLIMONTOVICH EQUATION

Let us consider a one-dimensional Hamiltonian system of particles in interaction described by the equations of motion

\[
\frac{dx_i}{dt} = \frac{\partial H}{\partial v_i}, \quad \frac{dv_i}{dt} = -\frac{\partial H}{\partial x_i},
\]

\[
H = \frac{1}{2} \sum_{i=1}^{N} v_i^2 + \sum_{i<j} u(x_i - x_j).
\]

(1)

We assume that the particles interact via a weak long-range binary potential \( u = u(|x_i - x_j|) \). This assumption is necessary to justify the quasilinear theory that we shall develop in the sequel. This potential of interaction can be, for example, the potential \( u = -\frac{k}{2\pi} \cos(\theta_i - \theta_j) \) of the HMF model [14]. We introduce the exact distribution function \( f_d(x,v,t) = \sum_{i=1}^{N} \delta(x - x_i(t))\delta(v - v_i(t)) \) and the corresponding potential \( \Phi_d(x,t) = \int u(x - x') f_d(x',v',t) dx' dv' \). The exact distribution function satisfies the Klimontovich equation

\[
\frac{\partial f_d}{\partial t} + [H_d,f_d] = 0,
\]

(2)

where \( H_d = v^2/2 + \Phi_d(x,t) \) is the exact energy of an individual particle and we have introduced the Poisson bracket

\[
[H,f] = \frac{\partial H}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial f}{\partial p},
\]

(3)

with \( H = H_d, f = f_d, q = x \) and \( p = v \). The Klimontovich equation (2) contains exactly the same information as the Hamilton equations (1).

We introduce a smooth distribution function \( f = \langle f_d \rangle \) which is the statistical average of the exact distribution function. We note \( \Phi = \langle \Phi_d \rangle \) the corresponding potential. Then, we write \( f_d = f + \delta f \) and \( \Phi_d = \Phi + \delta \Phi \). Inserting this decomposition in the Klimontovich equation (2), we obtain

\[
\frac{\partial f}{\partial t} + \frac{\partial \delta f}{\partial t} + [H,f] + [\delta H,f] + [\delta \Phi,f] + [\delta \Phi,\delta f] = 0,
\]

(4)

where we have used \( H_d = H + \delta H \) with \( H = v^2/2 + \Phi(x,t) \) and \( \delta H = \delta \Phi \). Taking the average of Eq. (4), we get

\[
\frac{\partial f}{\partial t} + [H,f] = -\langle [\delta \Phi,\delta f] \rangle.
\]

(5)

Subtracting this relation from Eq. (4) and neglecting nonlinear terms, we obtain an equation for the perturbation

\[
\frac{\partial \delta f}{\partial t} + [H,\delta f] + [\delta \Phi,f] = 0.
\]

(6)

This linearization is the essence of the quasilinear theory [25]. This approximation can be shown to be equivalent to keeping only terms of order \( 1/N \) in a proper thermodynamic limit \( N \to +\infty \) [2]. In the \( N \to +\infty \) limit, the term in the right hand side of Eq. (5) vanishes and we obtain the Vlasov equation

\[
\frac{\partial f}{\partial t} + [H,f] = 0.
\]

(7)
This equation admits an infinite number of stationary solutions of the form \( f = f(\epsilon) \) where \( \epsilon = v^2/2 + \Phi(x) \) is the individual energy. Such steady distributions can be reached through a process of violent relaxation involving only meanfield effects [26, 27]. If we keep terms of order \( 1/N \) or smaller, we shall obtain a kinetic equation taking into account the development of correlations between particles due to granularities (finite \( N \) effects). This is similar to the effect of “collisions” in the theory of gases. Thus, the right hand side of Eq. (5) is similar to the collision term in the Boltzmann equation. This term is expected to drive the system towards statistical equilibrium. Therefore, it is expected to select the Boltzmann distribution among all possible steady solutions of the Vlasov equation (see, however, the discussion of Sec. VA). Since this term is of order \( 1/N \) or smaller, the effect of “collisions” is a slow process that takes place on a timescale \( N t_D \) or larger. Therefore, there is a scale separation between the dynamical time \( t_D \) which is the timescale at which the system reaches a steady state of the Vlasov equation through violent relaxation due to inertial effects and the “collisional” time \( t_{\text{coll}} \) at which the system is expected to relax towards statistical equilibrium due to thermodynamical effects. Because of this scale separation, the distribution function will pass by a series of stationary solutions of the Vlasov equation depending only on energy, slowly evolving with time due to collisions (finite \( N \) effects). Hence, the distribution function can be approached by

\[
f(x, v, t) \simeq f(\epsilon, t).
\]

This means that the system is approximately in mechanical equilibrium [35] at each stage of the dynamics (due to violent relaxation) and that the “collisions” will drive it to thermodynamical equilibrium. The purpose of this paper is to obtain a closed expression for the collision term in energy space when the rapid dynamics has been averaged over the orbits. This is similar to the orbit-averaged-Fokker-Planck equation in astrophysics [24].

III. ANGLE-ACTION VARIABLES

The previous decomposition has separated the smooth meanfield distribution \( f(\epsilon, t) \) which evolves slowly with time from the fluctuations \( \delta f \) which are expected to have a decorrelation time on a much shorter timescale [36]. For inhomogeneous systems, it proves convenient to introduce angle-action variables [24] constructed from the slowly evolving distribution. Then, when we focus on the evolution of the perturbation, we can consider that the smooth distribution is “frozen”. This will allow us to determine the collision term. Then, we can take into account the time dependence of the smooth distribution function through the kinetic equation (5). We define the action

\[
J = \frac{1}{2\pi} \int v dx,
\]

where

\[
v(x) = \sqrt{2(\epsilon - \Phi(x))}.
\]

By definition, the action is a function \( J = J(\epsilon) \) of the energy so it is constant on the iso-energetic lines in phase space. As discussed previously, we assume in a first step that the potential \( \Phi(x, t) \) is “frozen” on the timescale on which the fluctuations have essential correlations so that \( J \) depends only on the energy (it is a new coordinate). We also introduce an angle variable

\[
\phi = \Omega \int_0^x \frac{dx}{v}.
\]
This angle parameterizes the iso-energetic line with energy $\epsilon$. Therefore, a particle with coordinates $(x, v)$ in phase-space can be described equivalently by the angle-action variables $(\phi, J)$. Noting that the period of an orbit with energy $\epsilon$ is

$$T(\epsilon) = \int dt = \int \frac{dx}{v} = \frac{2\pi}{\Omega(\epsilon)},$$

we find that $\Omega = \Omega(\epsilon) = \Omega(J)$ is the pulsation of the orbit with energy $\epsilon$. We also note for future reference that $J(\epsilon)$ represents the surface of phase space with energy smaller than $\epsilon$. Therefore, the surface of phase space with energy between $\epsilon$ and $\epsilon + d\epsilon$ is $g(\epsilon)d\epsilon$ with

$$g(\epsilon) = \frac{dJ}{d\epsilon}.$$  

This is similar to the density of states in statistical mechanics (but here in the $\mu$-space).

Now, from Eq. (9), we have

$$\frac{dJ}{d\epsilon} = \frac{1}{2\pi} \int \frac{dx}{\sqrt{2(\epsilon - \Phi)}} = \frac{1}{2\pi} \int \frac{dx}{v} = \frac{1}{\Omega(\epsilon)},$$

so that $g(\epsilon) = 1/\Omega(\epsilon)$. Accordingly, the density of particles with energy $\epsilon$ is

$$N(\epsilon) = f(\epsilon)g(\epsilon) = \frac{f(\epsilon)}{\Omega(\epsilon)}.$$  

Since only the meanfield Hamiltonian appears in the advective term of Eqs. (5)-(6), it only involves the unperturbed motion of the particles. The smooth Hamiltonian equations for the conjugate variables $(q, p)$ can be written

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$  

If we use the variables $(x, v)$, we find that the dynamics is relatively complicated because the potential explicitly appears in the second equation. Therefore, this equation $dv/dt = -\Phi(x)$ cannot be easily integrated except if $\Phi = 0$, i.e. for a homogeneous system. In that case, the unperturbed equations of motion reduce to $x = vt + x_0$, i.e. a motion at constant velocity. Now, the angle-action variables are constructed so that the Hamiltonian does not depend on the angles $\phi$. Therefore, the Hamilton equations for the conjugate variables $(\phi, J)$ are

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial J} = \Omega(J), \quad \frac{dJ}{dt} = -\frac{\partial H}{\partial \phi} = 0.$$  

From these equations, we find that $J = \text{Cst.}$ and $\phi = \Omega(J)t + \phi_0$ so that the equations of motion are very simple in such variables. They extend naturally the trajectories with constant velocity for homogeneous systems. This is why this choice of variables is relevant to develop the kinetic theory. Now, in terms of the angle-action variables, using the relations

$$[H, \delta f] = \frac{\partial H}{\partial J} \frac{\partial \delta f}{\partial \phi} - \frac{\partial H}{\partial \phi} \frac{\partial \delta f}{\partial J} = \Omega(J) \frac{\partial \delta f}{\partial \phi},$$

$$[\delta \Phi, f] = \frac{\partial \delta \Phi}{\partial J} \frac{\partial f}{\partial \phi} - \frac{\partial \delta \Phi}{\partial \phi} \frac{\partial f}{\partial J} = -\frac{\partial \delta \Phi}{\partial \phi} \frac{\partial f}{\partial J},$$

we have
we find that Eqs. (5)-(6) take the form
\begin{equation}
\frac{\partial f}{\partial t} = \frac{\partial}{\partial J} \langle \delta f \frac{\partial \delta \Phi}{\partial \phi} \rangle, \tag{20}
\end{equation}
and
\begin{equation}
\frac{\partial \delta f}{\partial t} + \Omega(J) \frac{\partial \delta f}{\partial \phi} - \frac{\partial \delta \Phi}{\partial \phi} \frac{\partial f}{\partial J} = 0. \tag{21}
\end{equation}

IV. THE QUASILINEAR THEORY

The equations (20)-(21) form the starting point of the quasilinear theory. The objective is to solve Eq. (21) for the fluctuation \(\delta f\) and substitute the result back into Eq. (20). Here, we follow the approach of [30, 31]. We introduce the Fourier transforms [37]:
\begin{equation}
\delta f(\phi,J,t) = \sum_m \int \delta \hat{f}_m(J,\omega) e^{im\phi} e^{-i\omega t} d\omega, \tag{22}
\end{equation}
\begin{equation}
\delta \Phi(\phi,J,t) = \sum_m \int \delta \hat{\Phi}_m(J,\omega) e^{im\phi} e^{-i\omega t} d\omega. \tag{23}
\end{equation}
Substituting these expressions in Eq. (21), we obtain
\begin{equation}
[\omega - m \Omega(J)] \delta \hat{f}_m = -m \frac{\partial f}{\partial J} \delta \hat{\Phi}_m. \tag{24}
\end{equation}
Therefore, the perturbed distribution function is given by
\begin{equation}
\delta \hat{f}_m(J,\omega) = \frac{-m \frac{\partial f}{\partial J}(J)}{\omega - m \Omega(J)} \delta \hat{\Phi}_m(J,\omega) + \hat{g}_m(J,\omega), \tag{25}
\end{equation}
where \(\hat{g}_m(J,\omega)\) is solution to the equation
\begin{equation}
[\omega - m \Omega(J)] \hat{g}_m(J,\omega) = 0, \tag{26}
\end{equation}
and is related to the initial conditions [25, 30, 31]. Here, \(\hat{g}_m(J,\omega)\) arises because of the discrete nature of the system, i.e. the fact that the exact distribution \(f_d\) is a sum of \(\delta\)-functions.

The Fourier transforms of the potential and of the distribution function are related to each other by an equation of the form (Appendix A)
\begin{equation}
\delta \hat{\Phi}_m(J,\omega) = \sum_{m'} \int A_{mm'}(J,J') \delta \hat{f}_{m'}(J',\omega) dJ', \tag{27}
\end{equation}
where \(A_{mm'}(J,J')\) are the Fourier components of the potential of interaction written in angle-action variables. At that stage, we shall make a rough approximation and assume that the function \(A_{mm'}(J,J')\) can be factorized [38] according to
\begin{equation}
A_{mm'}(J,J') = A_m(J) A_{m'}(J'). \tag{28}
\end{equation}
Although this approximation is clearly a weakness of our approach, we can give the following arguments to motivate our study: 1. Without a decomposition of this type, it appears difficult to go any further in our approach. 2. There may exist some potentials of interaction for which this approximation holds exactly or approximately. 3. Such factorization could be obtained rigorously by using another system of coordinates or another basis on which the potential of interaction could be developed. 4. This approximation does not seem to be crucial: in particular the final kinetic equation (37) can be expressed in terms of $A_{mn'}(J, J')$ alone provided that we neglect collective effects. Therefore, the approximation (28) may lead to an inaccurate description of collective effects while preserving the correct structure of the kinetic equation. Assuming that Eq. (28) holds, we obtain

$$\frac{\delta \hat{\Phi}_m(J, \omega)}{A_m(J)} = \sum_{m'} \int A_{m'}(J') \delta \hat{f}_{m'}(J', \omega) dJ'.$$  

(29)

This ratio is independent on $m$ and $J$. Substituting Eq. (25) in Eq. (29) we find

$$\frac{\delta \hat{\Phi}_m(J, \omega)}{A_m(J)} = - \sum_{m'} \int A_{m'}(J')^2 \frac{m' \Omega'(J')}{\omega - m' \Omega(J')} \delta \hat{\Phi}_m(J', \omega) dJ' + \sum_{m'} \int A_{m'}(J') \hat{g}_{m'}(J', \omega) dJ'.$$  

(30)

Using the remark following Eq. (29), this can be rewritten

$$\frac{\delta \hat{\Phi}_m(J, \omega)}{A_m(J)} \left\{ 1 + \sum_{m'} \int A_{m'}(J')^2 \frac{m' \Omega'(J')}{\omega - m' \Omega(J')} dJ' \right\} = \sum_{m'} \int A_{m'}(J') \hat{g}_{m'}(J', \omega) dJ'.$$  

(31)

We introduce the dielectric function

$$\epsilon(\omega) = 1 + \sum_{m'} \int A_{m'}(J')^2 \frac{m' \Omega'(J')}{\omega - m' \Omega(J')} dJ',$$  

(32)

so that

$$\delta \hat{\Phi}_m(J, \omega) = \frac{A_m(J)}{\epsilon(\omega)} \sum_{m'} \int A_{m'}(J') \hat{g}_{m'}(J', \omega) dJ'.$$  

(33)

At this stage, we must require that $\epsilon(\omega) \neq 0$ for any real $\omega$. This is the case if the distribution function $f(\epsilon)$ is stable with respect to the Vlasov equation (see Appendix B). In fact, the distribution function $f(\epsilon)$ which depends only on the energy results from a dynamical relaxation process as described at the end of Sec. II. This dynamical process (violent relaxation) necessarily puts the system on a stable stationary solution of the Vlasov equation. Therefore, $f(\epsilon, t)$ is a stable steady solution of the Vlasov equation at each stage of the dynamics which slowly evolves with time due to finite $N$ effects (collisions).

Substituting Eq. (33) in Eq. (25), we obtain

$$\delta \hat{f}_m(J, \omega) = - \frac{m \partial f(J)}{\omega - \Omega(J)} \frac{A_m(J)}{\epsilon(\omega)} \sum_{m'} \int A_{m'}(J') \hat{g}_{m'}(J', \omega) dJ' + \hat{g}_m(J, \omega).$$  

(34)

We can now compute the diffusion current using

$$\langle \delta f \frac{\partial \Phi}{\partial \phi} \rangle = i \sum_{m, m'} m' \int d\omega d\omega' \langle \delta \hat{f}_m(J, \omega) \delta \hat{\Phi}_{m'}(J, \omega') \rangle e^{i(m+m')\phi} e^{-i(\omega+\omega')t},$$  

(35)
and the expressions (33) and (34). From that point, the calculations are similar to those carried out for the usual quasilinear theory [25, 30, 31] and they will not be repeated (see, e.g., Appendix B of [2]). We finally obtain the kinetic equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial}{\partial J} \sum_{m,m'} \int \frac{1}{|\epsilon(m'\Omega(J'))|^2} mA_m(J)^2A_{m'}(J')^2\delta(m\Omega(J) - m'\Omega(J'))$$

$$\times \left\{ f(J')m\frac{\partial f}{\partial J} - f(J)m'\frac{\partial f}{\partial J'} \right\} dJ'.$$

(36)

Using Eq. (28), the product \(A_m(J)A_{m'}(J')\) can be replaced by \(A_{mm'}(J,J')\). Then, we note that the function \(A_m(J)\) occurs only in the dielectric function. From now on, we shall neglect collective effects and take \(|\epsilon(m'\Omega(J'))| = 1\). It is likely that our approach does not describe collective effects accurately. Therefore, the kinetic equation that we shall consider is

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial}{\partial J} \sum_{m,m'} \int mA_{mm'}(J,J')^2\delta(m\Omega(J) - m'\Omega(J')) \left\{ f(J')m\frac{\partial f}{\partial J} - f(J)m'\frac{\partial f}{\partial J'} \right\} dJ'.$$

(37)

This equation depends only on \(A_{mm'}(J,J')\) which can be calculated from Eq. (A5) for any potential. It does not depend on the individual terms \(A_m(J)\) and \(A_{m'}(J')\) which would be undetermined if Eq. (28) were not valid. Therefore, the kinetic equation (37) is well-posed for any potential. This may be an indication that its form does not crucially depend on the approximations that we have made in the course of the derivation. Furthermore, we shall see in the next section that this kinetic equation respects all the conservation laws of the Hamiltonian system (conservation of mass and energy) and increases the Boltzmann entropy. This again suggests that this equation provides a good description of the out-of-equilibrium dynamics even if our manner to derive it is not entirely satisfactory. This equation is the main result of the paper. It can be viewed as the counterpart of the Landau-Lenard-Balescu equation of plasma physics for spatially inhomogenous one-dimensional systems with weak long-range interactions. We note, in particular, that the relaxation is due to a condition of resonance \(m\Omega(J) = m'\Omega(J')\) between different orbits. This is similar to the kinetic equation obtained for point vortices in two-dimensional hydrodynamics [11]:

$$\frac{\partial P}{\partial t} = -N\gamma^2 \frac{\partial}{\partial r} \int_{r_1}^{+\infty} r_1dr_1\delta(\Omega(r) - \Omega(r_1)) \ln \left[ 1 - \left( \frac{r_<}{r_>} \right)^2 \right] \left( \frac{1}{r} P_1 \frac{\partial P}{\partial r} - \frac{1}{r_1} P \frac{\partial P_1}{\partial r_1} \right),$$

(38)

where \(\Omega(r)\) is the angular velocity, \(\gamma\) is the circulation of a point vortex and \(r_<\) (resp. \(r_>\)) denotes the smallest (resp. largest) value of \(r\) and \(r_1\). Here again, the relaxation is due to a condition of resonance \(\Omega(r) = \Omega(r_1)\) when the profile of angular velocity is non-monotonic. Equation (38) also neglects collective effects but the more accurate treatment of Dubin & O’Neil [8] allows to take them into account.

V. PROPERTIES OF THE KINETIC EQUATION

Let us derive some general properties of the kinetic equation (37). This equation can be written in the form of a conservative equation

$$\frac{\partial f}{\partial t} = -\frac{\partial Q}{\partial J}.$$  

(39)
with a current

\[ Q = -\frac{1}{2} \sum_{m,m'} \int mA_{mm'}(J, J') \delta(m\Omega(J) - m'\Omega(J')) \left\{ f(J')m \frac{\partial f}{\partial J} - f(J)m' \frac{\partial f}{\partial J'} \right\} dJ'. \]  

(40)

We impose the boundary conditions \( Q = 0 \) at \( J = J_{min} \) and \( J = J_{max} \) so as to satisfy the conservation of mass.

## A. Equilibrium state

The Boltzmann distribution

\[ f_{eq} = Ae^{-\beta\epsilon(J)}, \]  

(41)

is a stationary solution of the kinetic equation (37). Indeed, using Eq. (14), we have

\[ \frac{\partial f_{eq}}{\partial J} = -\beta f_{eq}\epsilon'(J) = -\beta f_{eq}\Omega(J). \]  

(42)

Introducing this relation in Eq. (40), we see that the integrand is proportional to

\[ (m\Omega(J) - m'\Omega(J'))\delta(m\Omega(J) - m'\Omega(J')) = 0. \]  

(43)

Therefore, the current vanishes and

\[ \frac{\partial f_{eq}}{\partial t} = 0. \]  

(44)

An interesting question is whether the Boltzmann distribution is the only steady solution of the kinetic equation (37). Indeed, the current also vanishes if there is no resonance, i.e. if \( m\Omega(J) \neq m'\Omega(J') \) for all orbits \((m, J) \neq (m', J')\). In the case of point vortices described by a kinetic equation of the form (38), the Boltzmann distribution is a steady state but the relaxation stops as soon as the profile of angular velocity becomes monotonic, so that no resonance is possible, even if the system is not at statistical equilibrium. This is shown numerically in [33]. The question is whether a similar situation occurs in the more complex case of Eq. (37). Note, parenthetically, that we considered in [16] a case where there is no resonance. In that case, other terms must be taken into account in the kinetic theory and it is found that the system has a curious behavior.

## B. Conservation of mass and energy

The conservation of mass

\[ M = \int f(J, t)dJ, \]  

(45)

is clear from the conservative form of Eq. (39) and the boundary conditions. Let us show that the kinetic equation (37) conserves the energy

\[ E = \int f(J, t)\epsilon(J)dJ. \]  

(46)
The time derivative of the energy can be written
\[ \dot{E} = \int \frac{\partial f}{\partial t}(J,t)\epsilon(J)dJ = \int \Omega(J)Q\,dJ, \tag{47} \]
where we have integrated by parts and used \( \epsilon'(J) = \Omega(J) \). Then, introducing Eq. (40) in Eq. (47), we obtain
\[ \dot{E} = -\frac{1}{2} \sum_{m,m'} \int dJdJ'\Omega m A_{mm'}(J,J')^2 \delta(m\Omega - m'\Omega') \left\{ f'm \frac{\partial f}{\partial J} - f'm' \frac{\partial f}{\partial J'} \right\}. \tag{48} \]
Interchanging the dummy variables \( m, m' \) and \( J, J' \) we find that
\[ \dot{E} = \frac{1}{2} \sum_{m,m'} \int dJdJ' m' A_{mm'}(J,J')^2 \delta(m\Omega - m'\Omega') \left\{ f'm \frac{\partial f}{\partial J} - f'm' \frac{\partial f}{\partial J'} \right\}, \tag{49} \]
where we have used that \( A_{mm'}(J,J') = A_{m'm}(J',J) \). Summing these two expressions, we obtain
\[ \dot{E} = -\frac{1}{4} \sum_{m,m'} \int dJdJ' A_{mm'}(J,J')^2 (m\Omega - m'\Omega') \delta(m\Omega - m'\Omega') \left\{ f'm \frac{\partial f}{\partial J} - f'm' \frac{\partial f}{\partial J'} \right\}, \tag{50} \]
which is equal to zero according to Eq. (43). Therefore \( \dot{M} = \dot{E} = 0 \).

C. H-theorem

Let us show that the entropy
\[ S = -\int f(J,t) \ln f(J,t)dJ, \tag{51} \]
increases. The time derivative of the entropy can be written
\[ \dot{S} = -\int \frac{1}{f} \frac{\partial f}{\partial J} QdJ. \tag{52} \]
Introducing Eq. (40) in Eq. (52), we obtain
\[ \dot{S} = \frac{1}{2} \int dJdJ' \frac{1}{f} \frac{\partial f}{\partial J} \sum_{m,m'} m A_{mm'}(J,J')^2 \delta(m\Omega - m'\Omega') \left\{ f'm \frac{\partial f}{\partial J} - f'm' \frac{\partial f}{\partial J'} \right\}. \tag{53} \]
Interchanging the dummy variables \( m, m' \) and \( J, J' \) we find that
\[ \dot{S} = -\frac{1}{2} \int dJdJ' \frac{1}{f} \frac{\partial f}{\partial J} \sum_{m,m'} m' A_{mm'}(J,J')^2 \delta(m\Omega - m'\Omega') \left\{ f'm \frac{\partial f}{\partial J} - f'm' \frac{\partial f}{\partial J'} \right\}. \tag{54} \]
Summing these two expressions, we obtain
\[ \dot{S} = \frac{1}{4} \int dJdJ' \frac{1}{ff'} \sum_{m,m'} A_{mm'}(J,J')^2 \delta(m\Omega - m'\Omega') \left( f'm \frac{\partial f}{\partial J} - f'm' \frac{\partial f}{\partial J'} \right)^2, \tag{55} \]
which is positive. Therefore \( \dot{S} \geq 0 \). Note that the above properties persist for the kinetic equation (36).
VI. A MORE PRECISE KINETIC EQUATION

In the preceding derivation, we have considered that the meanfield potential was “frozen”. In fact, this field evolves on a slow timescale and this evolution must be taken into account in the advective term of the kinetic equation (5). Let us write the kinetic equation in the form

\[
\frac{\partial f}{\partial t} + [H, f] = C(f),
\]

where \( C(f) \) is the “collision” term. Since the distribution function (8) only depends on the energy (approximately), we have \([H, f] \simeq 0\). On the other hand,

\[
\frac{\partial}{\partial t} f(x, v, t) \simeq \frac{\partial f}{\partial t} + \frac{\partial \Phi}{\partial t} \frac{\partial f}{\partial \epsilon},
\]

where we have used \( \epsilon/\partial t = \Phi/\partial t \) since \( \epsilon = v^2/2 + \Phi(x, t) \). Therefore, the kinetic equation (56) becomes

\[
\frac{\partial f}{\partial t} + \frac{\partial \Phi}{\partial t} \frac{\partial f}{\partial \epsilon} \simeq C(f).
\]

The idea now is to average over the orbits in order to eliminate the dependence on \((x,v)\), using

\[
\langle X \rangle(\epsilon, t) = \frac{\oint X(x,v,t) \frac{dx}{v}}{\oint \frac{dx}{v}},
\]

where we recall that \( \oint dx/v = 2\pi g(\epsilon) \) is the density of states with energy \( \epsilon \). The orbit-averaged-kinetic equation (58) then becomes

\[
\frac{1}{2\pi} \oint \frac{dx}{v} \left[ \frac{\partial f}{\partial t} + \frac{\partial \Phi}{\partial t} \frac{\partial f}{\partial \epsilon} - C(f) \right] = 0.
\]

The first term is simply

\[
\frac{1}{2\pi} \oint \frac{dx}{v} \frac{\partial f}{\partial t} = \frac{\partial J}{\partial \epsilon} \frac{\partial f}{\partial t}.
\]

Noting that

\[
\frac{\partial J}{\partial t} = -\frac{1}{2\pi} \oint \frac{1}{v} \frac{\partial \Phi}{\partial t} dx,
\]

the second term can be written

\[
\frac{1}{2\pi} \oint \frac{dx}{v} \frac{\partial \Phi}{\partial t} \frac{\partial f}{\partial \epsilon} = -\frac{\partial J}{\partial \epsilon} \frac{\partial f}{\partial t}.
\]

Finally, the last term is equal to

\[
\frac{1}{2\pi} \oint \frac{dx}{v} C(f) = \frac{\partial J}{\partial \epsilon} \frac{\partial Q}{\partial J} = \frac{\partial Q}{\partial \epsilon}.
\]

Therefore, the final form of the kinetic equation is

\[
\frac{\partial J}{\partial \epsilon} \frac{\partial f}{\partial t} - \frac{\partial J}{\partial \epsilon} \frac{\partial f}{\partial \epsilon} = \frac{1}{2\epsilon} \sum_{m,m'} \oint mA_{mm'}(J, J')^2 \delta(m\Omega(J) - m'\Omega(J'))
\]

\[
\times \left\{ \int f(J')m \frac{\partial f}{\partial J} - f(J)m' \frac{\partial f}{\partial J'} \right\} dJ',
\]

where, now, \( J = J(\epsilon, t) \).
VII. THE THERMAL BATH APPROXIMATION

The kinetic equation (37) describes the evolution of the system “as a whole”. We have seen that it respects the microcanonical structure of the initial Hamiltonian system as it conserves mass and energy. Furthermore, due to the development of correlations between particles, the Boltzmann entropy increases. We shall now use this kinetic theory to describe the relaxation of a test particle in a thermal bath of field particles. Then, we interpret $f(J, t) = P(J, t)$ in the kinetic equation (37) as the distribution function of the test particle and $f(J', t) = f_0(J')$ as the static distribution function of the field particles with which it interacts. This procedure transforms the integrodifferential equation (37) into a differential equation of the Fokker-Planck type [39]:

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial}{\partial J} \sum_{m,m'} \int m A_{mm'}(J, J')^2 \delta(m \Omega(J) - m' \Omega(J')) \left\{ f_0(J') m \frac{\partial P}{\partial J} - P(J) m' \frac{df_0}{dJ'} \right\} dJ'.$$

(66)

If we consider that the field particles are in thermal equilibrium, one has

$$f_0(J') = Ae^{-\beta \epsilon(J')}.$$

(67)

Using

$$\frac{df_0}{dJ'} = -\beta f_0(J') \Omega(J'),$$

(68)

we obtain the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial}{\partial J} \sum_{m,m'} \int A_{mm'}(J, J')^2 \delta(m \Omega - m' \Omega') f_0(J') \left\{ m \frac{\partial P}{\partial J} + \beta P m' \Omega' \right\} dJ'.$$

(69)

Using the $\delta$-function, this can be rewritten equivalently

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial}{\partial J} \sum_{m,m'} \int A_{mm'}(J, J')^2 \delta(m \Omega - m' \Omega') m^2 f_0(J') \left( \frac{\partial P}{\partial J} + \beta P \Omega \right) dJ'.$$

(70)

Introducing the diffusion coefficient

$$D(J) = \frac{1}{2} \sum_{m,m'} \int A_{mm'}(J, J')^2 \delta(m \Omega - m' \Omega') m^2 f_0(J') dJ',$$

(71)

we obtain the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial J} \left[ D(J) \left( \frac{\partial P}{\partial J} + \beta P \Omega(J) \right) \right].$$

(72)

This equation relaxes towards the Boltzmann distribution

$$P_\epsilon(J) \propto e^{-\beta \epsilon(J)}.$$

(73)

In terms of the energy, the Fokker-Planck equation (72) can be rewritten

$$\frac{\partial P}{\partial t} = \Omega(\epsilon) \frac{\partial}{\partial \epsilon} \left[ D(\epsilon) \Omega(\epsilon) \left( \frac{\partial P}{\partial \epsilon} + \beta P \right) \right].$$

(74)
Introducing $N(\epsilon, t) = P(\epsilon, t)/\Omega(\epsilon)$, the density of particles with energy $\epsilon$, we obtain

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial \epsilon} \left[ D_*(\epsilon) \left( \frac{\partial N}{\partial \epsilon} + \beta N \frac{dV}{d\epsilon} \right) \right],$$

(75)

where

$$D_*(\epsilon) = D(\epsilon)\Omega(\epsilon)^2 = \frac{1}{2} \sum_{m,m'} \int A_{mm'}(J, J')^2 \delta(m\Omega - m'\Omega') m^2 \Omega^2 f_0(J')dJ',$$

(76)

is the diffusion coefficient in energy space and

$$V(\epsilon) = \epsilon + \frac{1}{\beta} \ln \Omega(\epsilon),$$

(77)

is an effective potential. The stationary solution of Eq. (75) is

$$N_e(\epsilon) \propto \frac{1}{\Omega(\epsilon)} e^{-\beta \epsilon} \propto g(\epsilon) e^{-\beta \epsilon},$$

(78)

corresponding to the Boltzmann distribution. If we now consider a distribution of the field particles $f_0(J)$ such that $m\Omega(J) \neq m'\Omega(J')$ for any $(m, J) \neq (m', J')$ so that this distribution is stable on a timescale $Nt_D$ (see Sec. VA), the Fokker-Planck equation (66) becomes

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial J} \left[ D(J) \left( \frac{\partial P}{\partial J} - P \frac{d\ln f_0}{dJ} \right) \right],$$

(79)

with a diffusion coefficient

$$D(J) = \frac{1}{2} f_0(J) \sum_m \left| m \right| \frac{A_{mm}(J, J)^2}{|\Omega'(J)|}.$$  

(80)

To arrive at this expression, we have used the fact that the only contribution to the collision term comes from $m = m'$ and $J = J'$ and we have simplified the $\delta$-function according to $\delta(m(\Omega(J) - \Omega(J'))) = \delta(J - J')/|m\Omega'(J)|$.

Note, for comparison, that if we consider the relaxation of a test vortex in a bath of field vortices with static vorticity distribution $\omega_0(r)$, we obtain from Eq. (38) a Fokker-Planck equation of the form [10, 11, 33]:

$$\frac{\partial P}{\partial t} = -\frac{\gamma}{4r} \frac{\partial}{\partial r} \int_0^{+\infty} r' dr' \delta(\Omega(r) - \Omega(r')) \ln \left[ 1 - \left( \frac{r_<}{r_>} \right)^2 \right] \left( \frac{1}{r} \omega_0 \frac{\partial P}{\partial r} - \frac{1}{r^2} P \frac{d\omega_0}{dr} \right).$$

(81)

If we assume that the field vortices are at statistical equilibrium $\omega_0 = N\gamma P_0 = Ae^{-\beta\gamma \psi}$ (thermal bath), the Fokker-Planck equation becomes:

$$\frac{\partial P}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ r D(r) \left( \frac{\partial P}{\partial r} + \beta \gamma P \frac{d\psi}{dr} \right) \right],$$

(82)

with a diffusion coefficient

$$D(r) = -\frac{\gamma}{4r^2} \int_0^{+\infty} r' dr' \omega_0(r') \delta(\Omega - \Omega') \ln \left[ 1 - \left( \frac{r_<}{r_>} \right)^2 \right].$$

(83)
To arrive at these expressions, we have used the $\delta$-function and the fact that $\Omega(r) = -(1/r)d\psi/dr$. Note that these expressions are valid even if the profile of angular velocity is non-monotonic since the Boltzmann distribution is always a stationary solution of the kinetic equation (38), and it is expected not to evolve at all (statistical equilibrium state). If we now consider a distribution of the field vortices $\omega_0(r)$ associated with a monotonic profile of angular velocity $\Omega(r)$, so that it does not evolve under the effect of collisions on a timescale $Nt_D$, we find that the relaxation of a test vortex is governed by a Fokker-Planck equation of the form:

$$\frac{\partial P}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ rD(r) \left( \frac{\partial P}{\partial r} - P \frac{d \ln \omega_0}{dr} \right) \right],$$

with diffusion coefficient

$$D(r) = \frac{\gamma}{8} \frac{1}{|\Sigma(r)|} \ln N \omega_0(r),$$

where $\Sigma(r) = r\Omega'(r)$ is the local shear created by the field particles (the angular velocity is related to the vorticity by $\omega_0 = (1/r)(r^2\Omega)'$). To arrive at this expression, we have used the fact that the only contribution to the collision term comes from $r = r'$ and we have simplified the $\delta$-function according to $\delta(\Omega(r) - \Omega(r')) = \delta(r - r')/|\Omega'(r)|$. The analogy with the previous equations can be noted.

**VIII. THE HOMOGENEOUS CASE**

In the homogeneous case where $\Phi = 0$ and $\epsilon = v^2/2$, we have

$$\phi = x, \quad J = v, \quad \Omega(J) = v.$$  \hfill (86)

On the other hand,

$$A_{mm'} = 2\pi \delta_{mm'} \hat{u}_m,$$  \hfill (87)

where $\hat{u}_m$ is the Fourier transform of the potential. Therefore, the kinetic equation (37) becomes

$$\frac{\partial f}{\partial t} = 2\pi^2 \frac{\partial}{\partial J} \sum_m \int |m| \hat{u}_m^2 \delta(v - v') \left( f' \frac{\partial f}{\partial v} - f \frac{\partial f}{\partial v'} \right) dv' = 0.$$  \hfill (88)

Thus, for a one-dimensional homogeneous system, the collision term vanishes [2, 13, 14] (this result is well-known in plasma physics; see last paragraph of [30]). As a result, the relaxation time is larger than $Nt_D$. This has been shown numerically for the homogeneous HMF model where it is found that the system relaxes towards statistical equilibrium on a timescale $N^{1.7}t_D$ [23]. By contrast, for inhomogeneous systems, the collision term in Eq. (37) does not necessarily vanish at order $1/N$ because there are new resonances. Therefore, we may expect that the system will relax towards statistical equilibrium on a timescale of order $Nt_D$ provided that there are sufficient resonances to drive the relaxation to completion (see the remarks in Sec. V.A).

On the other hand, for a one-dimensional homogeneous system, the velocity distribution of a test particle evolving in a bath of field particles with static distribution $f_0(v)$ is governed by the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = 2\pi^2 \frac{\partial}{\partial J} \sum_m \int |m| \hat{u}_m^2 \delta(v - v') \left( f_0' \frac{\partial P}{\partial v} - P \frac{df_0}{dv'} \right) dv'.$$  \hfill (89)
If the field particles are at statistical equilibrium with the Maxwellian distribution function \( f_0 \propto e^{-\beta v^2/2} \) (thermal bath), the Fokker-Planck equation becomes

\[
\frac{\partial P}{\partial t} = \frac{\partial}{\partial v} \left[ D(v) \left( \frac{\partial P}{\partial v} + \beta P v \right) \right],
\]

(90)

with a diffusion coefficient

\[
D(v) = 2\pi f_0(v) \sum_m \hat{u}_m^2 |m|,
\]

(91)

proportional to the distribution function of the thermal bath. In fact, since the kinetic equation vanishes at order 1/N, we can consider a bath with any stable distribution of the Vlasov equation; it will not change on a timescale \( Nt_D \). On the other hand, collective effects can be included in the kinetic theory through the dielectric function. These generalizations are discussed in [2, 13, 14]. They lead to a Fokker-Planck equation of the form

\[
\frac{\partial P}{\partial t} = \frac{\partial}{\partial v} \left[ D(v) \left( \frac{\partial P}{\partial v} - P \frac{d\ln f_0}{dv} \right) \right],
\]

(92)

with a diffusion coefficient

\[
D(v) = 2\pi^2 f_0(v) \sum_m \frac{\hat{u}_m^2 |m|}{\epsilon(m, mv)},
\]

(93)

where the dielectric function is

\[
\epsilon(m, \omega) = 1 + 2\pi \hat{u}_m \int \frac{f_0(v)}{\omega/m - v} dv.
\]

(94)

These kinetic equations are again very similar to the previous equations. For a Maxwellian distribution of the field particles \( f_0 = (\beta/2\pi)^{1/2} \rho \exp(-\beta v^2/2) \) (thermal bath), the dielectric function can be explicited [2]. The Fokker-Planck equation is then given by Eq. (90) with a diffusion coefficient

\[
D(v) = 2\pi^2 f_0(v) \sum_m \frac{\hat{u}_m^2 |m|}{1 + 2\pi \hat{u}_m \beta \rho B(\sqrt{\frac{2}{\beta}} v)^2 + 2\pi^3 \beta^2 \rho^2 \hat{u}_m^2 v^2 e^{-\beta v^2}},
\]

(95)

where \( B(x) = 1 - 2xe^{-x^2} \int_0^x e^t dt \). The result (91) is recovered if we neglect collective effects and take \( \epsilon(m, mv) = 1 \).

IX. CONNECTION WITH OTHER WORKS

While this paper was in course of redaction, we came accross the very interesting study of Valageas [19] for 1D self-gravitating systems. This author develops a kinetic theory for inhomogeneous systems by starting directly from the Hamiltonian equations of motion and using a perturbative expansion of the trajectories in powers of 1/N. He can therefore obtain the coefficients of diffusion and friction directly from the equations of motion. Although the methodology is different, the expressions of the diffusion and friction coefficients obtained by Valageas [19] are very similar to those obtained in our approach (we shall see, in fact,
that they coincide after simple transformations). Since this author does not make the factorization hypothesis (28) this shows that this assumption can be avoided by using a different approach. We note, however, that the approach of Valageas does not take into account collective processes encapsulated in the dielectric function (polarization cloud). This is precisely in order to obtain this function that the factorization (28) was necessary in our approach. Therefore, the agreement between the two approaches when collective effects are neglected is consistent.

There are, on the other hand, differences between the two approaches because Valageas [19] finally obtains a Fokker-Planck equation of the form

$$\frac{\partial N}{\partial t} = \frac{\partial^2}{\partial \epsilon^2}(D_\epsilon N) + \frac{\partial}{\partial \epsilon}(D_\epsilon \beta N),$$

(96)

leading to a stationary solution

$$N_\epsilon(\epsilon) \propto \frac{1}{D_\epsilon(\epsilon)} e^{-\beta \epsilon},$$

(97)

which differs from the expected Boltzmann distribution (78). In fact, in order to arrive at Eq. (96), Valageas makes some approximations that are, in fact, not necessary. Indeed, he obtains the following expressions for the diffusion and friction coefficients

$$\left\langle \frac{\Delta J}{\Delta t} \right\rangle = \frac{1}{2} \int dJ' f_0(J') \sum_{m,m'} \left( m \frac{\partial}{\partial J} - m' \frac{\partial}{\partial J'} \right) A_{mm'}(J, J') m \delta(m \Omega - m' \Omega'),$$

(98)

$$\left\langle \frac{(\Delta J)^2}{\Delta t} \right\rangle = \int dJ' f_0(J') \sum_{m,m'} A_{mm'}(J, J') m^2 \delta(m \Omega - m' \Omega'),$$

(99)

where we have used our notations to facilitate the comparison with the equations of the present paper. If we assume that the dynamics can be described by a Fokker-Planck equation, the evolution of the distribution function is given by

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial J^2} \left( \left\langle \frac{(\Delta J)^2}{\Delta t} \right\rangle P \right) - \frac{\partial}{\partial J} \left( \left\langle \frac{\Delta J}{\Delta t} \right\rangle P \right).$$

(100)

The Fokker-Planck equation (100) can be rewritten

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial J} \left( D \frac{\partial P}{\partial J} - P \eta \right),$$

(101)

where we have introduced the diffusion coefficient

$$D = \frac{1}{2} \left\langle \frac{(\Delta J)^2}{\Delta t} \right\rangle,$$

(102)

and the friction term

$$\eta = \left\langle \frac{\Delta J}{\Delta t} \right\rangle - \frac{\partial D}{\partial J}.$$  

(103)

Now, using Eqs. (98) and (99), we obtain

$$D = \frac{1}{2} \int dJ' f_0(J') \sum_{m,m'} A_{mm'}(J, J') m^2 \delta(m \Omega - m' \Omega'),$$

(104)
and

\[ \eta = -\frac{1}{2} \int dJ' f_0(J') \sum_{m,m'} m' \frac{\partial}{\partial J} A_{mm'}(J,J')^2 m \delta(m\Omega - m'\Omega'). \]  

(105)

Integrating by parts the second equation, we have equivalently

\[ \eta = \frac{1}{2} \int dJ' \frac{\partial f_0}{\partial J} \sum_{m,m'} A_{mm'}(J,J')^2 m m' \delta(m\Omega - m'\Omega'). \]  

(106)

Therefore, the Fokker-Planck equation (101) can be rewritten as

\[ \frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial}{\partial J} \sum_{m,m'} \int m A_{mm'}(J,J')^2 \delta(m\Omega(J) - m'\Omega(J')) \left\{ f_0(J') m \frac{\partial P}{\partial J} - P(J') m' \frac{df_0}{dJ'} \right\} dJ', \]

(107)

which coincides with our kinetic equation Eq. (66). Therefore, the two theories give the same results and the expressions of the diffusion and friction terms (98)-(99) can be obtained from our approach by starting from Eq. (66) and using the steps (98)-(107) in the other way round as done in [2]. When we perform a thermal bath approximation as in Sec. VII, we obtain from Eq. (106) that \( \eta = -D\delta\Omega(J) \) which can be interpreted as an appropriate Einstein relation. Substituting this relation in Eq. (101), we obtain Eqs. (72) and (75) without approximation. These Fokker-Planck equations relax towards the Boltzmann distribution (78), contrary to Eq. (96) obtained by Valageas [19]. This is more satisfactory on a physical point of view. Then, if we account for the fact that the distribution of the bath in Eq. (107) is not stationary but evolves slowly under the effect of "collisions" in a self-consistent way, we obtain the integrodifferential kinetic equation (37) or, more precisely, Eq. (65). Therefore, we can start from the approach of Valageas (which does not make the factorization hypothesis (28)) to rigorously justify the kinetic equation (37). This equation was not given by Valageas [19]. It describes the evolution of the system as a whole and conserves mass and energy. As we have seen, these conservation laws and the H-theorem directly result from the symmetrical form of the kinetic equation (37).

X. CONCLUSION

In this paper, we have introduced a kinetic equation (37) or (65) that describes the dynamics of spatially inhomogeneous one-dimensional systems with weak long-range interactions in angle-action variables. We have shown that this kinetic equation conserves mass and energy and increases the Boltzmann entropy (H-theorem). The evolution of the system is due to resonances between different orbits. The Boltzmann distribution is a stationary solution of this equation, but it is not necessarily the only one. The system can be frozen in another steady state of the Vlasov equation if there is no resonance, i.e. if \( m\Omega(J) \neq m'\Omega(J') \) for all orbits \((m,J) \neq (m',J')\). On the other hand, using a thermal bath approximation, we have obtained a Fokker-Planck equation (72) that describes the relaxation of a test particle towards statistical equilibrium. These nice properties suggest that these kinetic equations are relevant to describe the evolution of the system. Unfortunately, the derivation of the kinetic equation (37) relies on a factorization hypothesis (28) of the Fourier components of the potential in angle-action variables which is not rigorously justified. However, we have shown that the recent work of Valageas [19], which uses a different approach where this
assumption is not required, leads to the same results after some transformations. Therefore, we can start from this approach to rigorously derive Eqs. (37), (65) and (72) which were not given in [19]. Thus, these equations should provide a good description of the dynamics of inhomogeneous systems with long-range interactions. These results can be applied to the HMF model and to the dynamics of point vortices in two dimensions. This will be presented in a future work [33]. Other extensions are also possible.

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APPENDIX A: THE FUNCTION $A_{mm'}(J, J')$

The relation between the potential and the distribution function is given by

$$\Phi(x) = \int u(x - x')f(x', v')dx'dv'.$$  \hfill (A1)

Introducing angle-action variables such that $x = x(\phi, J)$ and $v = v(\phi, J)$ we obtain

$$\Phi(\phi, J) = \int u(x - x')f(\phi', J')d\phi'dJ',$$  \hfill (A2)

where we have used $dxdv = d\phi dJ$. Taking the inverse Fourier transform of Eq. (23) and using Eq. (A2), we find that

$$\hat{\Phi}_m(J) = \int u(x - x')f(\phi', J')e^{-i\phi d\phi' dJ'}.$$  \hfill (A3)

Inserting Eq. (22) in Eq. (A3) we get

$$\hat{\Phi}_m(J) = \sum_{m'} \int u(x - x')\hat{f}_{m'}(J')e^{i(m\phi - m\phi')}dJ.$$  \hfill (A4)

Defining

$$A_{mm'}(J, J') = \frac{1}{2\pi} \int u(x - x')e^{i(m\phi - m\phi')}d\phi d\phi',$$  \hfill (A5)

we finally obtain

$$\hat{\Phi}_m(J) = \sum_{m'} \int A_{mm'}(J, J')\hat{f}_{m'}(J')dJ.'$$  \hfill (A6)

APPENDIX B: DISPERSION RELATION FOR THE VLASOV EQUATION

Let us consider the linear dynamical stability of a stationary solution $f_0 = f_0(J)$ of the Vlasov equation

$$\frac{\partial f}{\partial t} + [H, f] = 0.$$  \hfill (B1)
If we substitute the decomposition $f = f_0 + \delta f$ in the Vlasov equation (B1) and neglect nonlinear terms, we find that the perturbation $\delta f$ obeys the linearized Vlasov equation

$$\frac{\partial \delta f}{\partial t} + \Omega(J) \frac{\partial \delta f}{\partial \phi} - \frac{\partial \delta \Phi}{\partial \phi} \frac{\partial f}{\partial J} = 0. \quad (B2)$$

The Vlasov and linearized Vlasov equations (B1)-(B2) are similar to the Klimontovich and linearized Klimontovich equations (2)-(21). It is important to note, however, that in the Vlasov equation the distribution functions $f$ and $\delta f$ are $\textit{smooth}$ distributions while in the Klimontovich equation they are expressed in terms of $\delta$-functions. Writing the perturbation in the form

$$\delta f(\phi, J, t) = \sum_m \delta \hat{f}_m(J, \omega) e^{im\phi} e^{-i\omega t}, \quad (B3)$$

we can proceed exactly like in Sec. IV except that $g = 0$ since the perturbation $\delta f$ of the smooth distribution $f$ does not involve any $\delta$-function. Therefore, we find that $\epsilon(\omega) = 0$, i.e.

$$\epsilon(\omega) = 1 + \sum_{m'} \int A(m', 0) \delta \hat{f}_m(J') \frac{m' \frac{\partial f_0}{\partial J}(J')}{\omega - m' \Omega(J')} dJ' = 0. \quad (B4)$$

This is the required dispersion relation from which we can study the linear dynamical stability of the steady state $f_0 = f_0(J)$ with respect to the Vlasov equation [32]. Note that the integration must be done by using the Landau contour. In general, $\omega = \omega_r + i\omega_i$ is a complex pulsation, except at the point of marginal stability $\omega_i = 0$ where it is real. In the kinetic theory of Sec. IV, we must require that the smooth meanfield distribution $f(\epsilon)$ is Vlasov stable, which is the case on the basis of physical grounds (see discussion in Sec. II). In that case, $\epsilon(\omega_r) \neq 0$ for any real $\omega_r$. 

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These assumptions can be justified in a proper thermodynamic limit with $N \rightarrow +\infty$. Therefore, the $H$-theorem holds in that limit. This is consistent with the results of equilibrium statistical mechanics which indicate that, for $N \rightarrow +\infty$, the distribution function $f(r, v)$ of statistical equilibrium maximizes the Boltzmann entropy $S_B[f]$ at fixed mass $M[f]$ and energy $E[f]$; see [2] for more details and references.

It is easy to show that if the distribution function $f = f(\epsilon, t)$ depends only on the energy, then the condition of hydrostatic equilibrium $\nabla p + \rho \nabla \Phi = 0$ holds at any time; see, e.g., Eq. (33) in [28].

This assumption is not completely obvious for systems with long-range interactions. For example, in the case of 3D stellar systems, the correlations of the force decrease algebraically, like $t^{-1}$ [29]. For the one-dimensional HMF model at $T = 0$, the force auto-correlation function has an oscillatory behaviour [16]. For larger values of $T$, one expects an exponential decay of the force auto-correlation function as in the homogeneous case $T > T_c$. However, the timescale of the exponential relaxation can be very large, especially close to the critical point $T_c$ [12, 14]. In any case, in the $N \rightarrow +\infty$ limit, there will be a time-scale separation between the decorrelation time of the fluctuations and the evolution of the smooth meanfield distribution which justifies the decomposition made in this paper.

The same approximation is made by Pichon [32] to investigate the linear dynamical stability of a steady solution of the Vlasov equation (7).