Comparison of Microcanonical and Canonical Hadronization

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Abstract

Average multiplicities and transverse momenta of hadrons are calculated using a microcanonical hadronization description for a cluster of given total energy and volume. As a function of the total energy, we determine the critical volume above which the microcanonical description coincides with the canonical one, and compare the results with those obtained using one of standard canonical models. We show that the critical volume depends on the energy and the mass of the hadrons. For heavy particles, volumes above \(50 \text{ fm}^3\) are needed, even more than \(100 \text{ fm}^3\) if one considers transverse momenta. Thus the prediction of heavy hadron multiplicities in pp, Kp, and \(e^+e^-\) reactions requires a microcanonical approach, whereas for heavy ion reactions a canonical calculation is valid. We conclude by showing the importance of the feeding for the observed hadron multiplicities.

1 Introduction

The statistical description of proton-proton and heavy-ion reactions has already a long tradition. It was Hagedorn \([1, 2]\), who showed in a sequence of papers that many aspects of these reactions are close to that one expects assuming that the transition matrix element is constant and therefore the distribution of final state particles is completely determined by phase space.

More recently the statistical interpretation of nuclear reactions regained interest, after it had been demonstrated that in heavy ion reactions at

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CERN–SPS and RHIC energies the multiplicities of a multitude of non strange hadrons is in remarkable agreement with the assumption that all particles are created in thermal equilibrium at a temperature which is close to that expected from lattice calculations for the transition of a hadron gas towards a quark-gluon plasma [3-5]. Also strange hadrons fit into this picture if one includes a penalty factor for each strange quark which is contained in a hadron.

Later this approach has been successfully extended towards AGS and SIS energies [6], as well as towards pp and $e^+e^-$ reactions [7, 8]. Whereas the former are large systems with a large reaction volume, the latter ones yield only a small hadron multiplicity and require volumes of the order of 25 fm$^3$. For such small systems, it is not evident that a canonical or a grand canonical description is justified.

Therefore it is worthwhile to check whether for these small volumes and particle multiplicities a microcanonical description still coincides with the canonical one. It is the purpose of this article to investigate this question employing for the first time a numerical realization of the algorithm to calculate the microcanonical phase space which has been presented by us in ref. [9]. For the present investigation, we limit the number of hadrons to 54, which include pseudoscalar and vector mesons (octet and singlet) as well as the octet and decouplet of baryons and antibaryons. Strange particles are produced according to phase space, i.e. without applying any suppression factor. The results are compared to that of two canonical calculations using the approach of Becattini et al. [4, 7, 8], but without strangeness suppression: in the first calculation the number of hadrons which can be produced is limited to the same 54 species allowed in our microcanonical treatment, in the second calculation the standard set of hadrons [4, 7, 8] is included. In principle we can include more hadrons in the microcanonical ensemble. This makes a detailed comparison more difficult, because the less known decay channels of the additional hadrons have to agree.

2 Microcanonical Calculation

Following the general philosophy of statistical approaches to hadron production, we suppose that the result of a high energy collision can be considered as a distribution of “clusters”, “droplets”, or “fireballs”, which move relative to each other. Here, we are only interested in $4\pi$ particle yields and average transverse momenta, and therefore collective longitudinal motion needs not to be considered, and the distribution of clusters may be identified with one single “equivalent cluster”, being characterized by its volume $V$ (the sum of individual proper volumes), its energy $E$ (the sum of all the cluster masses) , and the net flavor content $Q = (N_u - N_{\bar{u}}, N_d - N_{\bar{d}}, N_s - N_{\bar{s}})$. 
The basic assumption is that a cluster, characterized by $V, E,$ and $Q$, decays “statistically” according to phase space. More precisely, the probability of a cluster to hadronize into a configuration $K = \{h_1, \ldots, h_n\}$ of hadrons $h_i$ is given by the microcanonical partition function $\Omega(K)$ of an ideal, relativistic gas of the $n$ hadrons $h_i$ [9].

$$\Omega(K) = \frac{V^n}{(2\pi\hbar)^3n} \prod_{i=1}^{n} g_i \prod_{\alpha \in S} \frac{1}{n_{\alpha}} \int \prod_{i=1}^{n} d^{3}p_i \delta(E - \Sigma \varepsilon_i) \delta(\Sigma \bar{p}_i) \delta_{Q,\Sigma q_i},$$

with $\varepsilon_i = \sqrt{m_i^2 + p_i^2}$ being the energy, and $\bar{p}_i$ the 3-momentum of particle $i$. The term $\delta_{Q,\Sigma q_i}$ ensures flavour conservation; $q_i$ is the flavour vector of hadron $i$. The symbol $S$ represents the set of hadron species considered: we take $S$ to contain the pseudoscalar and vector mesons ($\pi, K, \eta, \eta', \rho, K^*, \omega, \phi$) and the lowest spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ baryons ($N, \Lambda, \Sigma, \Xi, \Delta, \Sigma^*, \Xi^*, \Omega$) and the corresponding antibaryons. $n_{\alpha}$ is the number of hadrons of species $\alpha$, and $g_i$ is the degeneracy of particle $i$.

We are going to employ Monte Carlo techniques, so we have to generate randomly configurations $K$ according to the probability distribution $\Omega(K)$. We need a method in particular for intermediate size droplets, covering droplet masses from few GeV up to 100 or 1000 GeV. So the method should work for particle numbers $n = |K|$ between 2 and $10^3$, which means, we have to deal with a huge configuration space. Such problems are well known in statistical physics, and the method at hand is to construct a Markov process. So for a given cluster with mass $E$, volume $V$, and flavor $Q$, we start from some arbitrary initial configuration $K_0$, and generate a sequence $K_{I_{\text{eq}}}, K_{I_{\text{eq}}}, \ldots$, with $I_{\text{eq}}$ being sufficiently large to have reached equilibrium (which is defined to be the steady state of the Markov process). If we repeat this procedure many times, getting configurations $K_{I_{\text{eq}}}^{(1)}, K_{I_{\text{eq}}}^{(2)}, \ldots$, these configurations are distributed as $\Omega(K)$. We need a transition probability $p$ such that it leads to an equilibrium distribution $\Omega(K)$, with the initial transient $I_{\text{eq}}$ being as small as possible. Such an algorithm has been realized for the first time in [9].

The problem is solved in several steps. One first writes the phase space integral as

$$\phi(E,m_1,\ldots,m_n) = \int \prod_{i=1}^{n} d^{3}p_i \delta(E - \Sigma \varepsilon_i) \delta(\Sigma \bar{p}_i)$$

$$= (4\pi)^n \int \prod_{i=1}^{n} dp_i \prod_{i=1}^{n} p_i^2 \delta(E - \sum_{i=1}^{n} \varepsilon_i) W(p_1,\ldots,p_n),$$

with $p_i = |\bar{p}_i|$, and with the “random walk function” $W$ given as

$$W(p_1,\ldots,p_n) := \frac{1}{(4\pi)^n} \int \prod_{i=1}^{n} d\Omega_i \delta(\sum_{i=1}^{n} p_i \hat{e}_i),$$
with $\hat{e}_i = \vec{p}_i / |\vec{p}_i|$. The name “random walk function” is due to the fact that $W$ represents the probability to return back to the origin after $n$ “random walks” $p_i \hat{e}_i$ with given step sizes $p_i$. In [9], many details can be found about an efficient calculation of $W$ for any $n$ (big or small).

The next step amounts to getting rid of the energy delta function. A variable transformation gives

$$
\phi(E, m_1, \ldots, m_n)
= \int_0^1 dr_1 \cdots \int_0^1 dr_{n-1} \psi(E, m_1, \ldots, m_n; r_1, \ldots, r_{n-1}),
$$

with

$$
\psi(E, m_1, \ldots, m_n; r_1, \ldots, r_{n-1})
= (4\pi)^n T^{n-1} \prod_{i=1}^n p_i \varepsilon_i W(p_1, \ldots, p_n).
$$

The symbol $T$ denotes the total kinetic energy $E - \sum m_i$, and the absolute values of the momenta are expressed in terms of the $r_i$ as

$$
p_i = \sqrt{t_i(t_i + 2m_i)},
t_i = T(x_i - x_{i-1}), \quad x_0 = 0,
x_i = x_{i+1}\sqrt{r_i}, \quad x_n = 1.
$$

In principle one may use Monte Carlo techniques to calculate the integral, but this is very time consuming. A more elegant method amounts to generalizing the hadronic final state by considering not only hadron species, but also their momenta. So we use the generalized configurations

$$
G = \{h_1, \ldots, h_n; r_1, \ldots, r_{n-1}\},
$$

where the $r_i$ are related to the momenta $p_i$ via eq. (5). The weight of such a configuration is

$$
\Omega(G) = \frac{V^n}{(2\pi\hbar)^3n} \prod_{i=1}^n g_i \prod_{\alpha \in S} \frac{1}{n^\alpha} \psi(E, m_1, \ldots, m_n; r_1, \ldots, r_{n-1}).
$$

This expression is well suited to generate configurations $G$ according to $\Omega(G)$, by constructing Markov chains, for details see [9].

Flavor conservation is trivial to take into account, by considering only propositions in the Markov chain construction which conserve the total flavor $Q$.

Our algorithm provides a fast method to generate hadron configurations, characterized by the number of hadrons, their type, and their momenta. In this sense we have a real “event generator” of statistically generated hadrons.
3 Results

3.1 Multiplicities

In order to see best how the results of a microcanonical calculation approach those of a canonical one, we investigate the average particle density \( \rho_\alpha = n_\alpha / V \), where \( n_\alpha \) is the average particle multiplicity of hadron species \( \alpha \), and \( V \) is the volume. In the microcanonical approach, the average particle density is in general a function of the two variables \( E \) and \( V \). For large volumes, however, \( \rho \) should only depend on the ratio \( \varepsilon = E / V \), and this represents the limit where microcanonical and canonical description should coincide.

We therefore calculate the density \( \rho_\alpha \) as a function of the energy density \( \varepsilon \), for different volumes. We expect these curves to converge for large volumes.

Figs. 1-3 show the average particle density \( \rho_\alpha \) as a function of the energy density \( \varepsilon \) for three different volumes \( V = 12.5 \) (line), \( 25 \) (dashed line) and \( 50 \) (dotted) \( fm^3 \). The baryon density is fixed at 0.08 \( baryons/fm^3 \) and the total charge is 2 for reasons which will be explained later. The particle densities for \( V = 12.5 \) \( fm^3 \) and \( V = 25 \) \( fm^3 \) differ at low energy densities for heavy mesons as well as for many of the baryons. The deviation increases - as expected - either with increasing particle mass or with increasing strange quark content, because the strange quarks have to be compensated by other particles in order to obtain the quantum numbers of the two incoming protons.

We see as well that all mesons and almost all of the baryons have already arrived at the canonical limit for a volume as small as \( V = 25 \) \( fm^3 \). The only exceptions are the heavy triple strange barons \( \Omega \) and \( \bar{\Omega} \) at low energy densities, where the total energy of the disintegrating system is (at \( V = 25 fm^3 \)) only a couple of times larger than the mass of the \( \Omega, \bar{\Omega} \) pair.

From this observation we can conclude that the canonical limit is obtained if the total energy of the system is about 10 times the energy of the sum of the particles under consideration and of the particles necessary to compensate the deviation of the quantum numbers of the considered particle from that of the total cluster.

With the exception of the proton the density of all particles increases with increasing energy density. For the proton this is not the case because at very low energy densities the baryon cannot get rid of its charge, whereas at higher energies a produced meson can carry this charge leaving behind a neutral baryon.

In Figs. 1-3 we have marked as a dot the results of a canonical calculation with the parameters fitted to describe the particle yields observed in pp collisions at 27.4 GeV [4, 7]. The parameters used in this calculations are
Figure 1: Density of particles as a function of the energy density of a cluster for three different volumes: 12.5 \( fm^3 \) (full line), 25 \( fm^3 \) (dashed line) and 50 \( fm^3 \) (dotted line) for an initial baryon density of 0.08 baryons / \( fm^3 \) and a total charge of 2 using a microcanonical phase space calculation. The dots present the result of a canonical calculation provided by Becattini. In both cases the number of hadrons is limited to 54 and strange particles are not suppressed.
Figure 2: Same as previous figure, but for additional hadrons

$V = 25.5 \; fm^3$ and $T = 162 \; MeV$. In contradistinction to the calculation with these parameters which is presented in ref. [7], here the strange particles are not suppressed by a $\gamma_s$ factor and only the 54 hadron species mentioned above are produced. Therefore this canonical calculation can directly be compared with our microcanonical approach. The average energy of the clusters obtained in this calculation is 8.74 GeV, resulting in an average energy density of $\epsilon = 0.342 \; GeV/fm^3$. We observe that for all light hadrons
Figure 3: Same as previous figure, but for additional hadrons

the agreement is very nice, verifying that indeed at a volume of $V = 25 \, fm^3$
the multiplicity has arrived at its canonical limit. For the heavy particles, such as $\Omega$ and $\bar{\Omega}$, volumes above $50 \, fm^3$ are needed.

Another interesting question is how the observed particle yield is influenced by feeding. It is addressed in figs. 4-6 which show for the 54 hadrons two densities: that of those particles which are produced directly and that present after electromagnetic and strong decays. Both microcanonical calculations are compared with the corresponding canonical results. We see
Figure 4: Density of particles as a function of the energy density of a cluster with a volume of $50\, fm^3$ in the microcanonical calculation (lines) in comparison with a canonical calculation at a energy density of $0.342\, GeV/fm^3$ with 54 hadrons (solid points) and 250 hadrons (open points), before (solid lines and squares) and after (full lines and circles) strong and electromagnetic decay.

that feeding is unimportant for the vector mesons, whereas for the pseudoscalar mesons feeding increases the yield by more than a factor of 2 for
the π’s and by more than 50% for the K’s. With the exceptions of protons, neutrons and Λ’s feeding is much less important in the baryonic sector. The strange (anti)baryons in the electromagnetic decay chain of Ω and ¯Ω show some feeding whereas for Δ’s, Ω’s, χ∗’s and Σ∗’s feeding is absent. We display in figs. 3-6 as well the results of the canonical calculation with the same parameters in which the standard set of hadrons is included. For the heavy hadrons both canonical results differ little, whereas the standard set yields more pions and kaons, the decay products of the hadrons not included in
Figure 6: Same as previous figure, but for additional hadrons

54 particle set. For the pions both canonical results differ by less than 50\% for all the other hadrons the differences are much less important.

### 3.2 Transverse momentum

In fig. 7, we plot the average transverse momentum $<p_T>$ as a function of the hadron mass $m$. We compare our microcanonical results (points) with
an energy density of $\epsilon = 0.342 \text{GeV/fm}^3$ to the canonical result, using

$$<p_T(m, T)> = \frac{(\pi m T/2)^{1/2} K_{5/2}(m/T)}{K_2(m/T)}$$

with a temperature of $T = 162 \text{ MeV}$ (solid lines). The $<p_T>$ values obtained in the canonical calculation of Becattini agree with that analytical formula.

On the left hand side, we display $<p_T>$ of hadrons before decay as a function of their mass, for a volume of $V = 25 \text{ fm}^3$, and a total baryon number $B = 0$ (stars) and $B = 2$ (squares). Whereas the $<p_T>$ of $\pi$'s and K's are close to the value one expects for an equivalent canonical system, for the heavier particles the deviation from the canonical value increases with increasing mass. The reason is easy to understand: If such a heavy particle is produced, the available energy is not sufficient to fill phase space up to the high momenta. We see as well that the $<p_T>$ of the $\Omega$ is $1.5$
time the value of $\pi$. Please note that relativity moderates the increase. In a nonrelativistic calculations we expect that $<p_T>$ increases as $\sqrt{m}$. We display on the left hand side as well the average transverse momenta after the strong electromagnetic interaction decay for the case of $B = 2$ (cycles). The decay modifies considerably the $<p_T>$ of the light mesons and baryons.

On the right hand side, we study the critical volume for a droplet with total baryon number $0$ (in order to keep the baryon density constant when changing the volume). We see that with increasing volume the $<p_T>$ of the heavy baryons approaches only very slowly the canonical value. However, even at volumes of $100 \, fm^3$ the asymptotic value is not reached. Thus kinematical observables are much more suited to observe the deviations between canonical and microcanonical systems.

### 4 Conclusions

We have presented the first numerical calculations of the multiplicity distribution of hadrons assuming that they are distributed according to microcanonical phase space. In this study we have limited ourselves to a set of 54 hadrons. The results have been compared with the results of a canonical system which has the same size and the same average energy, and agreement has been found. We observe that for the multiplicity the canonical limit is obtained, if the energy of the system is around ten times the mass of the observed particle plus that of those particles which are necessary to compensate its deviation from the quantum numbers of the cluster. These results raise doubts whether heavy hadrons in small systems like pp, Kp and $e^+e^-$ can be described in a canonical approach but shows as well that for a heavy collision a canonical treatment is valid. Both, canonical and microcanonical calculations, show that feeding is very important for the pseudoscalar mesons as well as for p,n and $\Lambda$ but that is of minor or no importance for the other particles. We observe, as predicted analytically, an increase of the average transverse momentum $<p_T>$ with the mass of the hadron. However, we also find that a very big volume is required to obtain the value of the equivalent canonical system.

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