ON PROPERTIES OF SOLUTIONS OF THE p-HARMONIC EQUATION

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Abstract. A $2p$-times continuously differentiable complex-valued function $f = u + iv$ in a simply connected domain $\Omega \subseteq \mathbb{C}$ is $p$-harmonic if $f$ satisfies the $p$-harmonic equation $\Delta^p f = 0$. In this paper, we investigate the properties of $p$-harmonic mappings in the unit disk $|z| < 1$. First, we discuss the convexity, the starlikeness and the region of variability of some classes of $p$-harmonic mappings. Then we prove the existence of Landau constant for the class of functions of the form $Df = zf_z - \overline{f}_z$, where $f$ is $p$-harmonic in $|z| < 1$. Also, we discuss the region of variability for certain $p$-harmonic mappings. At the end, as a consequence of the earlier results of the authors, we present explicit upper estimates for Bloch norm for bi- and tri-harmonic mappings.

1. Introduction and Preliminaries

A complex-valued function $f = u + iv$ in a simply connected domain $\Omega \subseteq \mathbb{C}$ is called $p$-harmonic if $u$ and $v$ are $p$-harmonic in $\Omega$, i.e. $f$ satisfies the $p$-harmonic equation $\Delta^p f = 0$, where

$$\Delta^p f = \Delta \cdots \Delta f,$$

where $p$ is a positive integer and $\Delta$ represents the Laplacian operator

$$\Delta := 4 \frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Throughout this paper we consider $p$-harmonic mappings of the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. Obviously, when $p = 1$ (resp. $p = 2$), $f$ is harmonic (resp. biharmonic). The properties of harmonic [11, 15] and biharmonic [1, 2, 3, 18, 19] mappings have been investigated by many authors. Concerning $p$-harmonic mappings, we easily have the following characterization.

Proposition 1. A mapping $f$ is $p$-harmonic in $D$ if and only if $f$ has the following representation:

$$f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z),$$

where $G_{p-k+1}$ is harmonic for each $k \in \{1, \ldots, p\}$.

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Proof. We only need to prove the necessity since the proof for the sufficiency part is obvious. Again, as the cases \( p = 1, 2 \) are well-known, it suffices to prove the result for \( p \geq 3 \). We shall prove the proposition by the method of induction. So, we assume that the proposition is true for \( p = n (\geq 3) \).

Let \( F \) be an \((n + 1)\)-harmonic mapping in \( \mathbb{D} \). By assumption, \( \Delta F \) is \( n \)-harmonic and so can be represented as

\[
\Delta F(z) = \sum_{k=1}^{n} |z|^{2(k-1)} G_{n-k+1}(z),
\]

where \( G_{n-k+1} \) \((1 \leq k \leq n)\) are harmonic functions with

\[
G_{n-k+1}(z) = a_{0,n-k+1} + \sum_{j=1}^{\infty} a_{j,n-k+1} z^j + \sum_{j=1}^{\infty} b_{j,n-k+1} \overline{z}^j \quad \text{for} \quad k \in \{1, \ldots, n\}.
\]

Then

\[
\int_{0}^{z} \int_{0}^{\overline{z}} \Delta F \, d\overline{z} \, dz = \sum_{k=1}^{n} |z|^{2k} T_{p-k+1}(z) + g(z),
\]

where

\[
T_{p-k+1}(z) = \sum_{k=1}^{n} \left( \frac{a_{0,n-k+1}}{k^2} + \frac{\sum_{j=1}^{\infty} a_{j,n-k+1} z^j}{k(k+j)} + \frac{\sum_{j=1}^{\infty} b_{j,n-k+1} \overline{z}^j}{k(k+j)} \right)
\]

and \( g \) is a harmonic function in \( \mathbb{D} \). A rearrangement of the series in the sum shows that (1.1) holds for \( p = n + 1 \). \( \square \)

We remark that the representation (1.1) continues to hold even if \( f \) is \( p \)-harmonic in a simply connected domain \( \Omega \).

For a sense-preserving \( C^1 \)-mapping (i.e. continuously differentiable), we let

\[
\lambda_f = |f_z| - |f_{\overline{z}}| \quad \text{and} \quad \Lambda_f = |f_z| + |f_{\overline{z}}|
\]

so that the Jacobian \( J_f \) of \( f \) takes the form

\[
J_f = \lambda_f \Lambda_f = |f_z|^2 - |f_{\overline{z}}|^2 > 0.
\]

In [4], the authors obtained sufficient conditions for the univalence of \( C^1 \)-functions. Now we introduce the concepts of starlikeness and convexity of \( C^1 \)-functions.

Definition 1. A \( C^1 \)-mapping \( f \) with \( f(0) = 0 \) is called starlike if \( f \) maps \( \mathbb{D} \) univalently onto a domain \( \Omega \) that is starlike with respect to the origin, i.e. for every \( w \in \Omega \) the line segment \([0, w]\) joining 0 and \( w \) is contained in \( \Omega \). It is known that \( f \) is starlike if it is sense-preserving, \( f(0) = 0 \), \( f(z) \neq 0 \) for all \( z \in \mathbb{D} \setminus \{0\} \) and

\[
\frac{\partial}{\partial t} (\arg f(re^{it})) := \text{Re} \left( \frac{D f(z)}{f(z)} \right) > 0 \quad \text{for all} \quad z = re^{it} \in \mathbb{D} \setminus \{0\},
\]

where \( Df = zf_z - \overline{z} f_{\overline{z}} \) (cf. [23, Theorem 1]).
Definition 2. Let $f$ and $Df$ belong to $C^1(D)$. Then we say that $f$ is convex in $D$ if it is sense-preserving, $f(0) = 0$, $f(z) \cdot Df(z) \neq 0$ for all $z \in D \setminus \{0\}$ and

$$\text{Re} \left( \frac{D^2 f(z)}{Df(z)} \right) > 0 \quad \text{for all } z \in D \setminus \{0\}.$$ 

As $\text{arg } Df(re^{it})$ represents the argument of the outer normal to the curve $C_r = \{ f(re^{i\theta}) : 0 \leq \theta < 2\pi \}$ at the point $f(re^{it})$, the last condition gives that

$$\frac{\partial}{\partial t} \left( \text{arg } Df(re^{it}) \right) = \text{Re} \left( \frac{D^2 f(z)}{Df(z)} \right) > 0 \quad \text{for all } z = re^{it} \in D \setminus \{0\},$$

showing that the curve $C_r$ is convex for each $r \in (0, 1)$ (see [23, Theorem 2]). Non-analytic starlike and convex functions were studied by Mocanu in [23]. Harmonic starlike and harmonic convex functions were systematically studied by Clunie and Sheil-Small [11], and these two classes of functions have been studied extensively by many authors. See for instance, the book by Duren [15] and the references therein.

The complex differential operator

$$D = z \frac{\partial}{\partial z} - \overline{z} \frac{\partial}{\partial \overline{z}}$$

defined by Mocanu [23] on the class of complex-valued $C^1$-functions satisfies the usual product rule:

$$D(af + bg) = aD(f) + bD(g) \quad \text{and} \quad D(fg) = fD(g) + gD(f),$$

where $a, b$ are complex constants, $f$ and $g$ are $C^1$-functions. The operator $D$ possesses a number of interesting properties. For instance, the operator $D$ preserves both harmonicity and biharmonicity (see also [3]). In the case of $p$-harmonic mappings, we also have the following property of the operator $D$.

Proposition 2. $D$ preserves $p$-harmonicity.

Proof. Let $f$ be a $p$-harmonic mapping with the form

$$f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z),$$

where each $G_{p-k+1}(z)$ is harmonic in $D$ for $k \in \{1, \ldots, p\}$. As $D(|z|^2) = 0$, the product rule shows that $D(|z|^{2(k-1)}) = 0$ for each $k \in \{1, \ldots, p\}$. In view of this and the fact that $D$ preserves harmonicity gives that

$$D(f(z)) = \sum_{k=1}^{p} \left[ |z|^{2(k-1)} D(G_{p-k+1}(z)) + D(|z|^{2(k-1)}) G_{p-k+1}(z) \right]$$

$$= \sum_{k=1}^{p} |z|^{2(k-1)} D(G_{p-k+1}(z)).$$

$\square$
One of the aims of this paper is to generalize the main results of Abdulhadi, et. al. \cite{3} to the case of $p$-harmonic mappings. The corresponding generalizations are Theorems 1 and 2.

The classical theorem of Landau for bounded analytic functions states that if $f$ is analytic in $D$ with $f(0) = f'(0) - 1 = 0$, and $|f(z)| < M$ for $z \in D$, then $f$ is univalent in the disk $D_{\rho} := \{z \in \mathbb{C} : |z| < \rho\}$ and in addition, the range $f(D_{\rho})$ contains a disk of radius $M\rho^2$ (cf. \cite{20}), where

$$\rho = \frac{1}{M + \sqrt{M^2 - 1}}.$$ 

Recently, many authors considered Landau’s theorem for planar harmonic mappings (see for example, \cite{6, 8, 9, 13, 16, 22, 28}) and biharmonic mappings (see \cite{1, 7, 8, 21}). In Section 4, we consider Landau’s theorem for $p$-harmonic mappings with the form $D(f)$ when $f$ belongs to certain classes of $p$-harmonic mappings. Our results are Theorems 3 and 4.

In a series of papers the second author with Yanagihara and Vasudevarao (see \cite{24, 25, 29, 30}) have discussed the regions of variability for certain classes of univalent analytic functions in $D$. In Section 5 (see Theorem 5), we solve a related problem for certain $p$-harmonic mappings. Finally, in Section 6, we present explicit upper estimates for Bloch norm for bi- and tri-harmonic mappings (see Corollaries 3 and 4).

2. **Lemmas**

For the proofs of our main results we require a number of lemmas. We begin to recall the following version of Schwarz lemma due to Heinz (\cite[Lemma]{17}) and Colonna \cite[Theorem 3]{12}, see also \cite{6, 8, 9}.

**Lemma A.** Let $f$ be a harmonic mapping of $D$ such that $f(0) = 0$ and $f(D) \subset D$. Then

$$|f(z)| \leq \frac{4}{\pi} \arctan |z| \leq \frac{4}{\pi} |z| \text{ for } z \in D$$

and

$$\Lambda_f(z) \leq \frac{4}{\pi} \frac{1}{(1 - |z|^2)} \text{ for } z \in D.$$ 

**Lemma B.** (\cite[Lemma 2.1]{22}) Suppose that $f(z) = h(z) + \bar{g}(z)$ is a harmonic mapping of $D$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ for $z \in D$. If $J_f(0) = 1$ and $|f(z)| < M$, then

$$|a_n|, |b_n| \leq \sqrt{M^2 - 1}, \ n = 2, 3, \ldots,$$

$$|a_n| + |b_n| \leq \sqrt{2M^2 - 2}, \ n = 2, 3, \ldots$$
and

\begin{equation}
\lambda_f(0) \geq \lambda_0(M) := \begin{cases} 
\sqrt{2} & \text{if } 1 \leq M \leq M_0, \\
\sqrt{M^2 - 1 + \sqrt{M^2 + 1}} & \text{if } M > M_0,
\end{cases}
\end{equation}

where \( M_0 = \frac{\pi}{2 \sqrt{2 \pi^2 - 16}} \approx 1.1296 \).

The following lemma concerning coefficient estimates for harmonic mappings is crucial in the proofs of Theorems 1 and 2. This lemma has been proved by the authors in [10] with an additional assumption that \( f(0) = 0 \). However, for the sake of clarity, we present a slightly different proof than that in [10].

**Lemma C.** Let \( f = h + \overline{g} \) be a harmonic mapping of \( \mathbb{D} \) such that \( |f(z)| < M \) with \( h(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( g(z) = \sum_{n=1}^{\infty} b_n z^n \). Then \( |a_0| \leq M \) and for any \( n \geq 1 \)

\begin{equation}
|a_n| + |b_n| \leq \frac{4M}{\pi}.
\end{equation}

The estimate (2.2) is sharp. The extremal functions are \( f(z) \equiv M \) or \( f_n(z) = 2M\alpha \pi \arg \left( \frac{1 + \beta z^n}{1 - \beta z^n} \right) \), where \( |\alpha| = |\beta| = 1 \).

**Proof.** Without loss of generality, we assume that \( |f(z)| < 1 \). For \( \theta \in [0, 2\pi) \), let \( v_\theta(z) = \text{Im} \left( e^{i\theta} f(z) \right) \) and observe that

\[ v_\theta(z) = \text{Im} \left( e^{i\theta} h(z) + e^{-i\theta} g(z) \right) = \text{Im} \left( e^{i\theta} h(z) - e^{-i\theta} g(z) \right). \]

Because \( |v_\theta(z)| < 1 \), it follows that

\[ e^{i\theta} h(z) - e^{-i\theta} g(z) \prec K(z) = \lambda + \frac{2}{\pi} \log \left( \frac{1 + z\xi}{1 - z} \right), \]

where \( \xi = e^{-i\pi \text{Im}(\lambda)} \) and \( \lambda = e^{i\theta} h(0) - e^{-i\theta} g(0) \). The superordinate function \( K(z) \) maps \( \mathbb{D} \) onto a convex domain with \( K'(0) = \lambda \) and \( K'(0) = \frac{2}{\pi} (1 + \xi) \), and therefore, by a theorem of Rogosinski [26, Theorem 2.3] (see also [14, Theorem 6.4]), it follows that

\[ |a_n - e^{-2i\theta} b_n| \leq \frac{2}{\pi} |1 + \xi| \leq \frac{4}{\pi} \quad \text{for } n = 1, 2, \ldots \]

and the desired inequality (2.2), with \( M = 1 \), is a consequence of the arbitrariness of \( \theta \) in \( [0, 2\pi) \).

For the proof of sharpness part, consider the functions

\[ f_n(z) = \frac{2M\alpha}{\pi} \text{Im} \left( \log \frac{1 + \beta z^n}{1 - \beta z^n} \right), \quad |\alpha| = |\beta| = 1, \]
whose values are confined to a diametral segment of the disk $\mathbb{D}_M$. Also,  
$$f_n(z) = \frac{2M\alpha}{i\pi} \left( \sum_{k=1}^{\infty} \frac{1}{2k-1} (\beta z^n)^{2k-1} - \sum_{k=1}^{\infty} \frac{1}{2k-1} (\bar{\beta} z^n)^{2k-1} \right),$$
which gives
$$|a_n| + |b_n| = \frac{4M}{\pi}.$$ 

The proof of the lemma is complete. \(\Box\)

As an immediate consequence of Lemmas B and C, we have

Corollary 1. Let $f = h + g$ be a harmonic mapping of $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} b_n z^n$ and $|f(z)| \leq M$. If $J_f(0) = 1$ and $M \geq \frac{\pi}{\sqrt{\pi^2 - 8}}$, then for any $n \geq 2,$
$$|a_n| + |b_n| \leq \frac{4M}{\pi} \leq \sqrt{2M^2 - 2}.$$ 

3. The convexity and the starlikeness

The following simple result can be used to generate (harmonic) starlike and convex functions.

Theorem 1. Let $f$ be a univalent $p$-harmonic mapping with the form

$$f(z) = G(z) \sum_{k=1}^{p} \lambda_k |z|^{2(k-1)},$$

where $G$ is a locally univalent harmonic mapping and $\lambda_k$ ($k = 1, \ldots, p$) are complex constants. Then we have the following:

(a) $\frac{D(f)}{f} = \frac{D(G)}{G}$ and $\frac{D^2(f)}{G} = \frac{D^2(G)}{G}$.

(b) $f$ is convex (resp. starlike) if and only if $G$ is convex (resp. starlike).

Proof. (a) The two equalities are immediate consequences of the formula

$$D \left( G(z) \sum_{k=1}^{p} \lambda_k |z|^{2(k-1)} \right) = D(G(z)) \sum_{k=1}^{p} \lambda_k |z|^{2(k-1)}.$$ 

So, we omit the details.

(b) It suffices to prove the case of convexity since the proof for the starlikeness is similar.

Let $z = re^{it}$, where $0 < r < 1$ and $0 \leq t < 2\pi$. Then

$$f(z) = G(z) \sum_{k=1}^{p} \lambda_k |z|^{2(k-1)} = G(re^{i\theta}) \sum_{k=1}^{p} \lambda_k r^{2(k-1)},$$

so that

$$\frac{\partial f(re^{it})}{\partial t} = \frac{\partial G(re^{i\theta})}{\partial t} \sum_{k=1}^{p} \lambda_k r^{2(k-1)}.$$
On properties of solutions of the p-harmonic equation

and

\[ \frac{\partial^2 f(re^{it})}{\partial t^2} = \frac{\partial^2 G(re^{it})}{\partial t^2} \sum_{k=1}^{p} \lambda_k r^{2(k-1)}. \]

Therefore Part (a) yields

\[ \frac{\partial}{\partial t} \left( \arg \frac{\partial f(re^{it})}{\partial t} \right) = \text{Re} \left( \frac{D^2(f)}{D(f)} \right) = \text{Re} \left( \frac{D^2(G)}{D(G)} \right) = \frac{\partial}{\partial t} \left( \arg \frac{\partial G(re^{it})}{\partial t} \right), \]

from which the proof of Part (b) of this theorem follows.

As an immediate consequence of Theorem 1(a), we easily have the following.

**Corollary 2.** Let \( f \) be a univalent p-harmonic mapping defined as in Theorem 1. If \( f \) is convex and \( D(f) \) is univalent, then \( D(f) \) is starlike.

Abdulhadi, et. al. [3, Theorem 1] discussed the univalence and the starlikeness of biharmonic mappings in \( \mathbb{D} \). A natural question is whether [3, Theorem 1] holds for p-harmonic mappings. The following result gives a partial answer to this problem.

**Theorem 1.** Let \( f \) be a p-harmonic mapping of \( \mathbb{D} \) satisfying

\[ f(z) = |z|^{2(p-1)}G(z), \]

where \( G \) is harmonic, orientation preserving and starlike. Then \( f \) is starlike univalent.

**Proof.** We see that the Jacobian \( J_f \) of \( f \) is

\[ J_f = |f_z|^2 - |f_{\overline{z}}|^2 \]

\[ = |z|^{4(p-1)}(|G_z|^2 - |G_{\overline{z}}|^2) + 2(p-1)|z|^{4p-6}|G|^2 \text{Re} \left( \frac{D(G)}{G} \right) \]

\[ \geq |z|^{4(p-1)}(|G_z|^2 - |G_{\overline{z}}|^2). \]

Hence \( J_f(z) > 0 \) when \( 0 < |z| < 1 \) and obviously, \( J_f(0) = 0 \). The univalence of \( f \) follows from a standard argument as in the proof of [3, Theorem 1]. Finally, Theorem 1 implies that \( f \) is starlike.

4. THE LANDAU THEOREM

We now discuss the existence of the Landau constant for two classes of p-harmonic mappings.

**Theorem 3.** Let \( f(z) = \sum_{k=1}^{p} |z|^{2(k-1)}G_{p-k+1}(z) \) be a p-harmonic mapping of \( \mathbb{D} \) satisfying \( \Delta G_{p-k+1}(z) = f(0) = G_p(0) = J_f(0) - 1 = 0 \) and for any \( z \in \mathbb{D} \), \( |G_{p-k+1}(z)| \leq M, \) where \( M \geq 1 \). Then there is a constant \( \rho (0 < \rho < 1) \) such that \( D(f) \) is univalent in \( \mathbb{D}_\rho \), where \( \rho \) satisfies the following equation:

\[ \lambda_0(M) - \frac{T(M)}{(1-\rho)^2} \sum_{k=2}^{p} (2k-1)\rho^{2(k-1)} - \sum_{k=1}^{p} \frac{2T(M)\rho^{2k-1}}{(1-\rho)^3} - \frac{16M}{\pi^2} s_0 \arctan \rho = 0 \]
with
\[ s_0 = \left(\frac{\sqrt{17} - 1}{\sqrt{17} - 3}\right) \sqrt{\frac{2}{5 - \sqrt{17}}} \approx 4.1996, \]

\[ T(M) = \begin{cases} 
\sqrt{2M^2 - 2} & \text{if } 1 \leq M \leq M_1 := \frac{\pi}{\sqrt{\pi^2 - 8}} \approx 2.2976 \\
\frac{4M}{\pi} & \text{if } M > M_1
\end{cases} \tag{4.1} \]

and \( \lambda_0(M) \) is given by (2.1). Moreover, the range \( D(f)\(\mathbb{D}_\rho) \) contains a univalent disk \( \mathbb{D}_R \), where

\[ R = \rho \left[ \lambda_0(M) - \sum_{k=2}^{p} \frac{T(M)\rho^{2(k-1)}}{(1 - \rho)^2} - \frac{16M}{\pi^2} s_0 \arctan \rho \right]. \]

**Proof.** For each \( k \in \{1, 2, \ldots, p\} \), let

\[ G_{p-k+1}(z) = a_{0,p-k+1} + \sum_{j=1}^{\infty} a_{j,p-k+1} z^j + \sum_{j=1}^{\infty} b_{j,p-k+1} \bar{z}^j, \]

where \( a_{0,p} = 0 \). We define the function \( H \) as

\[ H = D \left( \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1} \right) = \sum_{k=1}^{p} |z|^{2(k-1)} D(G_{p-k+1}). \]

Using Lemmas B, C and Corollary 1, we have

\[ |a_{n,p}| + |b_{n,p}| \leq T(M), \]

where \( T(M) \) is given by (4.1), and

\[ |a_{j,p-k+1}| + |b_{j,p-k+1}| \leq \frac{4M}{\pi} \]

for \( j \geq 1, n \geq 2 \) and \( 2 \leq k \leq p \).

We observe that

\[ J_f(0) = |(G_p)_z(0)|^2 - |(G_p)_\bar{z}(0)|^2 = J_{G_p}(0) = 1 \]

and hence by Lemmas A and B, we have

\[ \lambda_f(0) \geq \lambda_0(M), \]

where \( \lambda_0(M) \) is given by (2.1). Now, we define

\[ q(x) = \frac{2 - x^2}{(1 - x^2)x} \quad (0 < x < 1). \]

Then there is an \( r_0 = \sqrt{\frac{5 - \sqrt{17}}{2}} \approx 0.66 \) such that

\[ q(r_0) = \min_{0<x<1} q(x) = \left(\frac{\sqrt{17} - 1}{\sqrt{17} - 3}\right) \sqrt{\frac{2}{5 - \sqrt{17}}} = s_0. \]
For each $\theta \in [0, 2\pi)$, the function

$$G_\theta(z) = (G_p)_z(z) - (G_p)_z(0) + ((G_p)_\bar{z}(z) - (G_p)_\bar{z}(0))e^{i(\pi - 2\theta)}$$

is clearly a harmonic mapping of $\mathbb{D}$ and satisfies $G_\theta(0) = 0$. Moreover, it follows from Lemma A that

$$\Lambda G_p(z) \leq \frac{4M}{\pi} \frac{1}{1 - |z|^2} \quad \text{for} \quad z \in \mathbb{D}.$$ 

In particular, this observation yields that

$$|G_\theta(z)| \leq \Lambda G_p(z) + \Lambda G_p(0) \leq \frac{4M}{\pi} \left(1 + \frac{1}{1 - |z|^2}\right) = \frac{4M}{\pi} |z| q(|z|)$$

for all $z \in \mathbb{D}$.

Since $xq(x) - 1 = \frac{1}{1 - x^2}$ is an increasing function in the interval $(0, 1)$, the inequality (4.2) shows that for any $z \in \mathbb{D}_{r_0}$,

$$|G_\theta(z)| \leq \frac{4M}{\pi} m_0,$$

where $m_0 = (2 - r_0^2)/(1 - r_0^2)$. Next, we consider the mapping $F$ defined on $\mathbb{D}$ by

$$F(z) = \frac{\pi}{4Mm_0} G_\theta(r_0 z).$$

Applying Lemma A to the function $F(z)$ yields that for $z \in \mathbb{D}_{r_0}$,

$$|G_\theta(z)| \leq \frac{16M}{\pi^2} m_0 \arctan \left(\frac{|z|}{r_0}\right) \leq \frac{16M}{\pi^2} s_0 \arctan |z|,$$

where $s_0 = m_0/r_0$.

Now, we fix $\rho$ with $\rho \in (0, 1)$. To prove the univalency of $H$, we choose two distinct points $z_1, z_2$ in $\mathbb{D}_\rho$. Let $\gamma = \{(z_2 - z_1)t + z_1 : 0 \leq t \leq 1\}$ and $z_2 - z_1 = |z_1 - z_2|e^{i\theta}$. We find that
\[ |H(z_1) - H(z_2)| \]
\[ = \left| \int_{\gamma} H_z(z) \, dz + H_\overline{\gamma}(z) \, d\overline{z} \right| \]
\[ \geq \left| \int_{\gamma} (G_p)_z(0) \, dz - (G_p)_\overline{\gamma}(0) \, d\overline{z} \right| \]
\[ - \left| \int_{\gamma} \sum_{k=2}^{p} k |z|^{2(k-1)} (G_p)_{z^{2k}(z)} \, dz - (G_p)_{\overline{\gamma}^{2k}(z)} \, d\overline{z} \right| \]
\[ - \left| \int_{\gamma} \sum_{k=2}^{p} k |z|^{2(k-2)} (G_p)_{z^{2k}(z)} \, dz - (G_p)_{\overline{\gamma}^{2k}(z)} \, d\overline{z} \right| \]
\[ - \left| \int_{\gamma} (\rho z) (z) - (G_p)_z(0) \, dz - [(G_p)_\overline{\gamma}(z) - (G_p)_\overline{\gamma}(0)] \, d\overline{z} \right| \]
\[ \geq |z_1 - z_2| \left\{ \lambda_0(0) - |G_\varphi(\rho)| \right\} \]
\[ - \sum_{k=1}^{p} \rho^{2(k-1)} \sum_{n=2}^{\infty} n(n - 1)(|a_{n,p-k+1}| + |b_{n,p-k+1}|) \rho^{n-1} \]
\[ - \sum_{k=2}^{p} (2k - 1) \rho^{2(k-2)} \sum_{n=2}^{\infty} n(|a_{n,p-k+1}| + |b_{n,p-k+1}|) \rho^{n+1} \}
\[ > |z_1 - z_2| \left[ \lambda_0(M) - \frac{T(M)}{(1 - \rho)^2} \sum_{k=2}^{p} (2k - 1) \rho^{2(k-1)} \right. \]
\[ - \left. \sum_{k=1}^{p} \frac{2T(M) \rho^{2k-1}}{(1 - \rho)^3} - \frac{16M}{\pi^2} s_0 \arctan \rho \right]. \]

Let
\[ P(\rho) = \lambda_0(M) - \frac{T(M)}{(1 - \rho)^2} \sum_{k=2}^{p} (2k - 1) \rho^{2(k-1)} - \sum_{k=1}^{p} \frac{2T(M) \rho^{2k-1}}{(1 - \rho)^3} - \frac{16M}{\pi^2} s_0 \arctan \rho. \]

Then it is easy to verify that \( P(\rho) \) is a decreasing function on the interval \((0, 1)\),
\[ \lim_{\rho \to 0^+} P(\rho) = \lambda_0(M) \quad \text{and} \quad \lim_{\rho \to 1^-} P(\rho) = -\infty. \]

Hence there exists a unique \( \rho_0 \) in \((0, 1)\) satisfying \( P(\rho_0) = 0 \). This observation shows that \( |H(z_1) - H(z_2)| > 0 \) for arbitrary two distinct points \( z_1, z_2 \) in \(|z| < \rho_0\) which proves the univalency of \( D(F) \) in \( \mathbb{D}_{\rho_0} \).
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For any $z$ with $|z| = \rho_0$, we have

$$|H(z)| = \left| \sum_{k=1}^{p} |z|^{2(k-1)} \left( z(G_{p-k+1})z(z) - \bar{z}(G_{p-k+1})\bar{z}(z) \right) \right|$$

$$\geq \left| z(G_p)z(0) - \bar{z}(G_p)\bar{z}(0) \right|$$

$$- \left| z[(G_p)z(z) - (G_p)z(0)] - \bar{z}[(G_p)\bar{z}(z) - (G_p)\bar{z}(0)] \right|$$

$$- \sum_{k=2}^{p} |z|^{2(k-1)} \left( z(G_{p-k+1})z(z) - \bar{z}(G_{p-k+1})\bar{z}(z) \right)$$

$$\geq \rho_0 \left[ \lambda_0(M) - \sum_{k=2}^{p} \frac{T(M)\rho_0^{2(k-1)}}{(1 - \rho_0)^2} \right] - \frac{16M}{\pi^2} s_0 \arctan \rho_0$$

$$= R$$

and the proof of the theorem is complete.

From Table 1, we see that Theorem 3 improves Theorem 1.1 of [7] for the case $p = 2$, and the results for the rest of the values of $p$ are new. In Table 1, third and fourth columns refer to values obtained from Theorem 3 for cases $p = 2, 3, 4$ for certain choices of $M$, while the right two columns correspond to the values obtained from [7, Theorem 1.1] for the case $p = 2$.

**Theorem 4.** Let $f(z) = |z|^{2(p-1)}G(z)$ be a $p$-harmonic mapping of $\mathbb{D}$ satisfying $G(0) = J_G(0) - 1 = 0$ and $|G(z)| \leq M$, where $M \geq 1$ and $G$ is harmonic. Then there is a constant $\rho$ ($0 < \rho < 1$) such that $D(f)$ is univalent in $\mathbb{D}_\rho$, where $\rho$ satisfies

\begin{table}
\begin{tabular}{|c|c|c|c|c|}
\hline
$M$ & $p$ & $\rho = \rho(M, p)$ & $R = R(M, \rho(M, p))$ & $\rho'$ & $R'$ \\
\hline
1.1296 & 2 & 0.0714741 & 0.0101601 & 0.0420157 & 0.00945379 \\
2 & 2 & 0.0206783 & 0.00227639 & 0.0139439 & 0.00164502 \\
2.2976 & 2 & 0.0155966 & 0.00151523 & 0.00108021 \\
3 & 2 & 0.00922255 & 0.00067425 & 0.000482413 \\
1.1296 & 3 & 0.071463 & 0.0101647 & - & - \\
2 & 3 & 0.0206782 & 0.00227641 & - & - \\
2.2976 & 3 & 0.0155966 & 0.00151523 & - & - \\
3 & 3 & 0.00922254 & 0.00067425 & - & - \\
1.1296 & 4 & 0.0714629 & 0.0101647 & - & - \\
2 & 4 & 0.0206782 & 0.00227641 & - & - \\
2.2976 & 4 & 0.0155966 & 0.00151523 & - & - \\
3 & 4 & 0.00922254 & 0.00067425 & - & - \\
\hline
\end{tabular}
\caption{Values of $\rho$ and $R$ for Theorem 3 for $p = 2$, and the corresponding values of $\rho'$ and $R'$ of [7, Theorem 1.1] (for $p = 2$)}
\end{table}
the following equation:

\[ \lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \frac{2T(M)\rho}{(1 - \rho)^2} = 0, \]

where the constants \( s_0, \lambda_0(M) \) and \( T(M) \) are the same as in Theorem 3. Moreover, the range \( D(f)(D_\rho) \) contains a univalent disk \( D_R \), where

\[ R = \rho^{2p-1} \left[ \frac{16M}{\pi^2} s_0 \arctan \rho \right]. \]

Especially, if \( M = 1 \), then \( G(z) = z \), i.e. \( f(z) = |z|^{2(p-1)}z \) which is univalent in \( \mathbb{D} \).

Proof. Let \( G(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}_n \). Using Lemmas B, C and Corollary 1, we have

\[ |a_n| + |b_n| \leq T(M) \quad \text{for} \ n \geq 2. \]

Note that

\[ J_G(0) = |a_1|^2 - |b_1|^2 = 1 \]

and hence, by Lemmas A and B, we have

\[ \lambda_G(0) \geq \lambda_0(M). \]

Next, we set \( H = D(f) = |z|^{2(p-1)}D(G) \) and fix \( \rho \) with \( \rho \in (0,1) \). To prove the univalency of \( f \), we choose two distinct points \( z_1, z_2 \) in \( D_\rho \). Let \( \gamma = \{(z_2 - z_1)t + z_1 :}
0 ≤ t ≤ 1} and \( z_2 - z_1 = |z_1 - z_2| e^{i\theta} \). Then
\[
|H(z_1) - H(z_2)| = \left| \int_{[z_1, z_2]} \left[ H(z) dz + H_\omega(z) d\omega \right] \right|
\]
\[
= \left| \int_{[z_1, z_2]} \rho |z|^{2(p-1)} (G_z(z) dz - G_\omega(z) d\omega) + |z|^{2(p-1)} (zG_z^2(z) dz - \bar{z}G_\omega(z) d\omega) + (p-1) |z|^{2(p-2)} (z^2G_z(z) dz - \bar{z}^2G_\omega(z) d\omega) \right|
\]
\[
\geq |z_1 - z_2| \left( \int_0^1 |z|^{2(p-1)} dt \right) \left\{ \lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \sum_{n=2}^{\infty} n(n-1)(|a_n| + |b_n|) \rho^{n-1} \right\}
\]
\[
> |z_1 - z_2| \left( \int_0^1 |z|^{2(p-1)} dt \right) \left[ \lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \frac{2T(M)\rho}{(1 - \rho)^3} \right].
\]
Since there exists a unique \( \rho \) in \( (0, 1) \) which satisfies the following equation:
\[
\lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \frac{2T(M)\rho}{(1 - \rho)^3} = 0,
\]
we see that \( H(z_1) \neq H(z_2) \) and so, \( H(z) \) is univalent for \( |z| < \rho_0 \).

Furthermore, we observe that for any \( z \) with \( |z| = \rho_0 \),
\[
|H(z)| = \rho_0^{2(p-1)} |zG_z(0) - \bar{z}G_\omega(0) + z(G_z(z) - G_\omega(0)) - \bar{z}(G_\omega(z) - G_\omega(0))|
\]
\[
\geq \rho_0^{2p-1} \left[ \lambda_0(M) - \frac{16M}{\pi^2} s_0 \arctan \rho_0 \right]
\]
\[
= R.
\]
The proof of the theorem is complete.
\[ M \quad p \quad \rho = \rho(M, p) \quad R = R(M, \rho(M, p)) \quad \rho' \quad R' \]

| $M$ | $p$ | $\rho = \rho(M, p)$ | $R = R(M, \rho(M, p))$ | $\rho'$ | $R'$ |
|-----|-----|----------------|----------------|-------|------|
| 1.1296 | 2 | 0.0281673 | 0.0000106985 | 0.0194864 | 3.54498 $\times 10^{-6}$ |
| 2 | 2 | 0.00856025 | 1.73218 $\times 10^{-7}$ | 0.00623202 | 6.5415 $\times 10^{-8}$ |
| 2.2976 | 2 | 0.00646284 | 6.4986 $\times 10^{-8}$ | 0.0047235 | 2.47902 $\times 10^{-8}$ |
| 3 | 2 | 0.0037942 | 1.2693 $\times 10^{-11}$ | – | – |
| 1.1296 | 3 | 0.0281673 | 8.48819 $\times 10^{-9}$ | – | – |
| 2 | 3 | 0.00856025 | 1.00669 $\times 10^{-13}$ | – | – |
| 2.2976 | 3 | 0.00646284 | 2.71435 $\times 10^{-12}$ | – | – |
| 3 | 3 | 0.0037942 | 2.47902 $\times 10^{-8}$ | – | – |

Table 2. Values of $\rho$ and $R$ for Theorem 4 for $p = 2, 3$, and the corresponding values of $\rho'$ and $R'$ of [7, Theorem 1.2] (for $p = 2$)

We remark that Theorem 4 is an improved version of [7, Theorem 1.2] when $p = 2$. In order to be more explicit, we refer to Table 2 in which the third and fourth columns refer to values obtained from Theorem 4 for cases $p = 2, 3$ for certain choices of $M$, while the right two columns correspond to the values obtained from [7, Theorem 1.2] for the case $p = 2$.

5. The Region of Variability

Definition 3. Let $\mathcal{H}_p$ denote the set of all $p$-harmonic mappings of the unit disk $\mathbb{D}$ with the normalization $f_{z^{p-1}}(0) = (p - 1)!$ and $|f(z)| \leq 1$ for $|z| < 1$. Here we prescribe that $\mathcal{H}_0 = \emptyset$.

For a fixed point $z_0 \in \mathbb{D}$, let

\[ V_p(z_0) = \{ f(z_0) : f \in \mathcal{H}_p \setminus \mathcal{H}_{p-1} \}. \]

Now, we have

Theorem 5. (a) If $p = 1$, then $V_1(z_0) = \{1\}$;

(b) If $p \geq 2$, $V_p(z_0) = \mathbb{D}$.

Proof. We first prove (a). Let $f \in \mathcal{H}_1$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$. By Parseval’s identity and the hypotheses $|f(z)| \leq 1$ and $f(0) = 1$, we have

\[
\lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} (|h(re^{i\theta})|^2 + |g(re^{i\theta})|^2) \, d\theta
\]

\[ = |a_0|^2 + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) \leq 1. \]

This inequality implies that for any $n \geq 1$, $a_n = b_n = 0$ which gives that $f(z) \equiv 1$ for $z \in \mathbb{D}$. Thus, we have $V_1(z_0) = \{1\}$.

In order to prove (b), we consider the function

\[ \phi(z) = \frac{2^{p-1} - w}{1 - wz^{p-1}} = |z|^{2(p-1)} \sum_{n=1}^{\infty} w^{n(z^{n-1})^{(p-1)}} \left( z^{p-1} - w - \sum_{n=1}^{\infty} w^{n+1}(z^{(p-1)n}) \right). \]
where \( w \in \overline{D} \) and \( p \geq 2 \).

Then \( \phi_{p-1}(0) = (p-1)! \), \( \Delta^p \phi = 0 \) and therefore, \( \phi \in \mathcal{H}_p \setminus \mathcal{H}_{p-1} \). For each fixed \( a \in \overline{D} \), \( z \mapsto f_{a}(z) = (z^{p-1} - a)/(1 - a z^{p-1}) \) is a \( p \)-harmonic mapping and \( f_{a}(\overline{D}) \subset \overline{D} \).

Obviously, \( a \mapsto f_{a}(z_0) = \frac{z_0^{p-1} - a}{1 - a z_0} \) is a conformal automorphism of \( D \) and the image of \( \overline{D} \) under \( f_{a}(z_0) \) is \( D \) itself. By hypotheses, we obtain that for any \( g \in \mathcal{H}_p \setminus \mathcal{H}_{p-1} \), \( g(z_0) \in \overline{D} \). Hence \( V_0(z_0) \) coincides with \( \overline{D} \). The proof of this theorem is complete. \( \square \)

By the method of proof used in Theorem 5(a), we obtain the following generalization of Cartan’s uniqueness theorem (see [5] or [27, p. 23]) for harmonic mappings.

**Theorem 6.** Let \( f \) be a harmonic mapping in \( D \) with \( f(D) \subset D \) and \( f(0) = 1 \). Then \( f(z) = z \) in \( D \).

6. **Estimates for Bloch norm for bi- and tri-harmonic mappings**

In the case of \( p \)-harmonic Bloch mappings, the authors in [10] obtained the following result.

**Theorem 7.** Let \( f \) be a \( p \)-harmonic mapping in \( D \) of the form (1.1) satisfying \( B_f < \infty \), where

\[
B_f := \sup_{z,w \in D, \, z \neq w} \left| \frac{f(z) - f(w)}{\rho(z, w)} \right| < \infty \quad \text{with} \quad \rho(z, w) = \frac{1}{2} \log \left( 1 + \frac{|z - w|}{1 - |z - w|} \right).
\]

Then

\[
B_f := \sup_{z \in D} (1 - |z|^2) \left\{ \sum_{k=1}^{p} |z|^{2(k-1)} (G_{p-k+1})_z(z) + \sum_{k=1}^{p} (k - 1)|z|^{2(k-2)} G_{p-k+1}(z) \right\}
\]

\[
\geq \sup_{z \in D} (1 - |z|^2) \left| \sum_{k=1}^{p} |z|^{2(k-1)} (G_{p-k+1})_z(z) - \sum_{k=1}^{p} |z|^{2(k-1)} (G_{p-k+1})_\bar{z}(z) \right|
\]

(6.1)

and (6.1) is sharp. The equality sign in (6.1) occurs when \( f \) is analytic or anti-analytic.

Furthermore, if for each \( k \in \{1, 2, \ldots, p\} \), the harmonic functions \( G_{p-k+1} \) in (1.1) are such that \( |G_{p-k+1}(z)| \leq M \), then

\[
(6.2) \quad B_f \leq 2M \phi_p(y_0).
\]

Here \( y_0 \) is the unique root in \((0, 1)\) of the equation \( \phi'_p(y) = 0 \), where

\[
(6.3) \quad \phi_p(y) = \frac{2}{p} \sum_{k=1}^{p} y^{2(k-1)} + y(1 - y^2) \sum_{k=2}^{p} (k - 1)y^{2(k-2)}.
\]
The bound in (6.2) is sharp when \( p = 1 \), where \( M \) is a positive constant. The extremal functions are

\[
f(z) = \frac{2M\alpha}{\pi} \Im \left( \log \frac{1 + S(z)}{1 - S(z)} \right),
\]

where \( |\alpha| = 1 \) and \( S(z) \) is a conformal automorphism of \( \mathbb{D} \).

In order to emphasize the importance of this result, we recall that, when \( p = 1 \), (6.1) (resp. (6.2)) is a generalization of [12, Theorem 1] (resp. [12, Theorem 3]). In the case of \( p = 2 \) of Theorem 7, after some computation, one has the following simple formulation for biharmonic mappings.

**Corollary 3.** Let \( f = H + |z|^2G \) be a biharmonic mapping of \( \mathbb{D} \) such that \( B_f < \infty \). Then, we have

\[
B_f \geq \sup_{z \in \mathbb{D}} (1 - |z|^2) \left| |H_z + |z|^2G_z| - |H_\bar{z} + |z|^2G_{\bar{z}}| \right|
\]

and

\[
B_f \leq \frac{4M}{27\pi^3} \left( 8 + 36\pi^2 + (4 + 3\pi^2)^{3/2} \right) \approx 30.7682 M.
\]

**Proof.** According to our notation, (6.1) is equivalent to (6.4). In order to prove (6.5), we first observe that (6.2) is equivalent to

\[
B_f \leq 2M \sup_{0 < y < 1} \phi_2(y),
\]

where

\[
\phi_2(y) = \frac{2}{\pi} \left( 1 + y^2 \right) + y(1 - y^2).
\]

Now, to find \( \sup_{0 < y < 1} \phi_2(y) \), we compute the derivative

\[
\phi_2'(y) = 1 + \frac{4}{\pi} y - 3y^2 = -3(y - y_0) \left( y - \frac{2 - \sqrt{4 + 3\pi^2}}{3\pi} \right)
\]

so that \( \phi_2'(y) \geq 0 \) for \( 0 \leq y \leq y_0 \) and \( \phi_2'(y) \leq 0 \) for \( y_0 \leq y < 1 \). Hence

\[
y_0 = \frac{2 + \sqrt{4 + 3\pi^2}}{3\pi} \approx 0.82732
\]

is the critical point of \( \phi_2(y) \). Consequently, \( \phi_2(y) \leq \phi_2(y_0) \). A simple calculation shows that

\[
\phi_2(y_0) = \frac{2}{\pi} \left( 1 + y_0^2 \right) + y_0(1 - y_0^2)
\]

\[
= \frac{2}{\pi} \left( \frac{8 + 12\pi^2 + 4\sqrt{4 + 3\pi^2}}{9\pi^2} \right) + \left( \frac{2}{3\pi} + \frac{\sqrt{4 + 3\pi^2}}{3\pi} \right) \left( \frac{6\pi^2 - 8 - 4\sqrt{4 + 3\pi^2}}{9\pi^2} \right)
\]

\[
= \frac{2}{27\pi^3} \left( 16 + 42\pi^2 + 8\sqrt{4 + 3\pi^2} + \sqrt{4 + 3\pi^2} \left( 3\pi^2 - 4 - 2\sqrt{3\pi^2 + 4} \right) \right)
\]

\[
= \frac{2}{27\pi^3} \left( 8 + 36\pi^2 + (4 + 3\pi^2)^{3/2} \right) \approx 15.3841
\]
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and therefore, $B_f \leq 2M\phi_2(y_0)$ which is the desired inequality (6.5). The result follows. □

In the case of $p = 3$ of Theorem 7, we have

**Corollary 4.** Let $f = H + |z|^2G + |z|^4K$ be a triharmonic (i.e. 3-harmonic) mapping of the unit disk $\mathbb{D}$ such that $B_f<\infty$, where $H$, $G$ and $K$ are harmonic in $\mathbb{D}$. Then we have

$$B_f \geq \sup_{z\in\mathbb{D}}(1 - |z|^2) \left| |H_z + |z|^2G_z + |z|^4K_z| - |H_{z\tau} + |z|^2G_{z\tau} + |z|^4K_{z\tau}| \right|$$

and

$$B_f \leq 2M\phi_3(y_1) \approx 4.037006M,$$

where $\phi_3(y_1) = \sup_{0<y<1}\phi_3(y)$ and

$$\phi_3(y) = \frac{2}{\pi}(1 + y^2 + y^4) + y(1 + y^2 - 2y^4).$$

**Proof.** Set $p = 3$ in Theorem 7. Then, (6.6) is equivalent to (6.1) and therefore, it suffices to prove (6.7). The choice $p = 3$ in (6.2) shows that

$$B_f \leq 2M \sup_{0<y<1}\phi_3(y),$$

where $\phi_3(y)$ is obtained from (6.3).

We see that $\phi_3(y)$ has a unique positive root in $(0, 1)$. Also,

$$\phi'_3(y) = \frac{4}{\pi}(y + 2y^3) + 1 + 3y^2 - 10y^4.$$

Computations show that $\phi'_3(y) \geq 0$ for $0 \leq y \leq y_1$ and $\phi'_3(y) \leq 0$ for $y_1 \leq y < 1$. Hence

$$y_1 \approx 0.891951$$

is the only critical point of $\phi_3(y)$ in the interval $(0, 1)$. It follows that

$$\phi_3(y) \leq \phi_3(y_1) \approx 2.018503.$$

Thus, $B_f \leq 2M\phi_3(y_1)$ which is the desired inequality (6.7). □

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