GROUP APPROXIMATION IN CAYLEY TOPOLOGY AND COARSE GEOMETRY,
PART I: COARSE EMBEDDINGS OF AMENABLE GROUPS.

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Abstract. Objective of this series is to study metric geometric properties of coarse disjoint union of Cayley graphs. We employ the Cayley topology and observe connection between large scale structure of metric spaces and group properties of Cayley limit points. In this part I, we prove that a coarse disjoint union has property A of G. Yu if and only if all Cayley limit groups are amenable. As an application, we construct a coarse disjoint union of finite special linear groups which has property A but is of very poor compression into all uniformly convex Banach spaces.

1. Introduction

The concept of property A has been introduced by G. Yu [Yu00], plays the role of amenability in coarse geometry, and has been regarded as one of the most fundamental properties in coarse geometry (see Definition 4.6 in our setting). The concept of the coarse embedding has been defined by M. Gromov, and can be regarded as an injective morphism in coarse geometry (see Definition 4.9). Yu proved the following two celebrated properties in [Yu00]: for uniformly locally finite (see Definition 4.5) coarse space X,

- if X has property A, then X is coarsely embeddable into a Hilbert space;
- if X is coarsely embeddable into a Hilbert space, then the coarse Baum–Connes conjecture holds true for X,

We refer the readers to two books respectively by J. Roe [Roe03] and Nowak–Yu [NY12] for comprehensive treatments of these topics including the coarse Baum–Connes conjecture.

Box spaces have provided interesting examples of metric spaces. The construction goes as follows: choose an infinite and finitely generated group \((\Gamma, S)\) (S is a finite generating set) which is residually finite, and \(\{N_m\}_m\) is a decreasing sequence of normal subgroups of \(\Gamma\) of finite index with \(\bigcap_{m \in \mathbb{N}} N_m = \{e\}\). Then a coarse disjoint union of the sequence of Cayley graphs of the corresponding group quotients

\[
\Box \Gamma = \Box_{\{N_m\}} \Gamma = \bigsqcup_{m \in \mathbb{N}} \text{Cay}(\Gamma/N_m, S)
\]

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is called a box space of $\Gamma$ for $\{N_m\}_{m \in \mathbb{N}}$. E. Guentner observes the following relation between the coarse geometric properties of $\boxtimes \Gamma = \boxtimes \{N_m\}_{m \in \mathbb{N}}$ and group properties of the mother group $\Gamma$ (see Proposition 11.39 in [Roe03]) (for the definition of the $\alpha$-T-menability, see the paragraph above Theorem 5.7):

- $\boxtimes \Gamma$ has property A if and only if $\Gamma$ is amenable;
- $\boxtimes \Gamma$ admits a coarse embedding into a Hilbert space only if $\Gamma$ is a-T-manable.

There, however, exist various attractive examples of sequences $\{\text{Cay}(G^{(m)}, S_m)\}_{m \in \mathbb{N}}$ which is not in the framework. The following have been paid much attentions to.

- Example (i): $G^{(m)} := \mathfrak{S}_m$, the symmetric group on $\{1, \ldots, m\}$ is generated by $S_m := \{\sigma^{(m)}, \tau^{(m)}\}$. Here $\sigma^{(m)}$ is the cyclic permutation on the $m$-point set, and $\tau^{(m)}$ is the transposition on $\{1, 2\}$.
- Example (ii): Let $\{k_m\}_m$ be a sequence of integers at least 2. The group $G^{(m)}_{k_m} := \text{SL}(m, \mathbb{Z}/k_m \mathbb{Z})$ is generated by $\{\sigma^{(m)}, \tau^{(m)}, \upsilon^{(m)}\}$. Here $\sigma^{(m)}$ corresponds to the cyclic permutation on the coordinate (with some $(-1)$-entry if $m$ is even) and $\tau^{(m)} = e_{1, 2}(1)$ is an elementary matrix of size $m$ with 1 on $(1, 2)$-entry.
- Example (iii): The group $G^{(m)}_{k_m} := \text{SL}(m, \mathbb{Z}/k_m \mathbb{Z})$ is generated by $\{\sigma^{(m)}, \tau^{(m)}, \upsilon^{(m)}\}$. Here $\sigma^{(m)}, \tau^{(m)}$ are as above, and $\upsilon^{(m)} = e_{2, 1}(1)$ is an elementary matrix of size $m$ with 1 on $(2, 1)$-th entry.

These sequences are collections of groups “of unbounded rank”. They are not box spaces and the above theorems do not apply. To the best knowledge of the authors, it has not been known whether coarse disjoint unions of them have property A, or whether they admit coarse embeddings into a Hilbert space.

The goal of our sequel papers is to construct a general framework for this type of problems. We enlarge the setting and deal with general coarse disjoint unions of finite Cayley graphs with constant degree (see Definition 4.4 and Remark 5.4). The main problem is that we need to find an appropriate notion of “mother groups” in general setting. For the purpose, we employ the concept of the space of $k$-marked groups introduced by R. I. Grigorchuk [Gri83]. The space is equipped with a compact Hausdorff topology, which is called the Cayley topology. The space enables us to regard a group $G$ with fixed generating set $(s_1, s_2, \ldots, s_k)$ as a single point. For the precise definition, see Definition 2.3 and the book [dlH00, Section V.10] of de la Harpe. One interpretation of the convergence in the Cayley topology $(G^{(m)}, s_1^{(m)}, \ldots, s_k^{(m)}) \rightarrow (G^{(\infty)}, s_1^{(\infty)}, \ldots, s_k^{(\infty)})$ is the following condition: for every $r \in \mathbb{N}$, there exists $m = m_r \in \mathbb{N}$ such that for any $m \geq m_r$, $r$-balls in $\text{Cay}(G^{(m)}, s_1^{(m)}, \ldots, s_k^{(m)})$ is exactly as that in $\text{Cay}(G^{(\infty)}, s_1^{(\infty)}, \ldots, s_k^{(\infty)})$ with corresponding edges. This approximation property of metric balls is a key point of Guentner’s result above in the box space setting. Indeed, in the box space setting, the sequence $\{(\Gamma/N_m, S)\}_{m}$ (for fixed order in $S$) converges to the point $(\Gamma, S)$ in the space of $|S|$-marked groups.

One of the main results of this paper is the following:
Theorem A (Theorem 5.1, Theorem 5.7 and Corollary 5.6). Let \( k \geq 2 \) and \( \{G^{(m)}\}_{m \in \mathbb{N}} = \{(G^{(m)}, s^{(m)}_1, \ldots, s^{(m)}_k)\}_{m \in \mathbb{N}} \) be \( k \)-marked finite groups.

(i) A coarse disjoint union \( \bigsqcup_{m \in \mathbb{N}} \text{Cay}(G^{(m)}, s^{(m)}_1, \ldots, s^{(m)}_k) \) (as a metric space with an appropriate metric) has property A if and only if every element in \( \overline{\text{Cay}(G^{(m)}) \setminus \{G^{(m)}\}} \) is amenable.

(ii) A coarse disjoint union \( \bigsqcup_{m \in \mathbb{N}} \text{Cay}(G^{(m)}, s^{(m)}_1, \ldots, s^{(m)}_k) \) is coarsely embeddable into a Hilbert space only if every element in \( \overline{\text{Cay}(G^{(m)}) \setminus \{G^{(m)}\}} \) is \( \alpha \)-T-menable.

Here \( \overline{\text{Cay}(G^{(m)})} \) denotes the closure (in the Cayley topology) of the subset \( \{G^{(m)}\} \) of the space of \( k \)-marked groups \( G(k) \).

Theorem A relates coarse geometric properties of a sequence of finite groups with group properties of the Cayley limit points. In the statement, the Cayley limit groups play a role of “mother groups.” As a byproduct, we give the answers to the problems whether in Examples (i), (ii), and (iii) coarse disjoint unions have property A, and whether they admits coarse embeddings into a Hilbert space.

Theorem B (Theorem 6.3 and Theorem 6.6). We stick the notation to one in Examples (i)–(iii) above.

(i) A coarse disjoint union \( \bigsqcup_{m \in \mathbb{N} \geq 2} \text{Cay}(\mathfrak{S}_m, \sigma^{(m)}, \tau^{(m)}) \) has property A.

(ii) A coarse disjoint union \( \bigsqcup_{m \in \mathbb{N} \geq 2} \text{Cay}(\text{SL}(m, \mathbb{Z}/k_m \mathbb{Z}), \sigma^{(m)}, \tau^{(m)}) \) has property A for any \( \{k_m\}_m \).

(iii) A coarse disjoint union \( \bigsqcup_{m \in \mathbb{N} \geq 2} \text{Cay}(\text{SL}(m, \mathbb{Z}/k_m \mathbb{Z}), \sigma^{(m)}, \tau^{(m)}, \upsilon^{(m)}) \) has property A if and only if \( \sup_m k_m < \infty \). If \( \sup_m k_m = \infty \), then it does not admit coarse embeddings into a Hilbert space.

In items (ii) and (iii), we only deals with odd \( m \) (otherwise \( \sigma^{(m)} \) must have \((-1)\)-entry) in this paper, but we can recover the whole case without difficulty. We emphasize that in items (ii) and (iii), whereas these systems of generators \( \{\sigma^{(m)}, \tau^{(m)}\}_m \) and \( \{\sigma^{(m)}, \tau^{(m)}, \upsilon^{(m)}\}_m \) may look similar, the resulting two coarse disjoint unions possess considerably dissimilar coarse geometric properties.

Finally, we prove that a coarse disjoint union in item (ii) in Theorem B has very poor metric compression functions (see Definition 7.1 and Theorem 7.3). If we change the coefficient rings, then we extend the class of Banach space to one which contains all uniformly convex Banach spaces. For details on class of Banach spaces, see Definition 7.2.

Theorem C (Theorem 7.4). Let \( \sigma^{(m)}, \tau^{(m)} \in \text{SL}(m, \mathbb{F}_2[t]/(t^{k_m})) \) be the matrices defined as in Example (ii) and let \( \tau'^{(m)} = e_{1,2}(t) \) be the elementary matrix of size \( m \) whose \((1,2)\)-entry is \( t \). (The \( \mathbb{F}_2 \) denotes the field of order 2.) For every non-decreasing function \( \rho: \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) with \( \lim_{t \to +\infty} \rho(t) = +\infty \), there exists (explicit) \( \{k_m\}_m \) such that for

\[
X_{\mathbb{F}_2[t], (k_m)} := \bigsqcup_{m \in \mathbb{N} \geq 2} \text{Cay}(\text{SL}(m, \mathbb{F}_2[t]/(t^{k_m})), \sigma^{(m)}, \tau^{(m)}, \upsilon^{(m)}),
\]
\( \rho \) is not a compression function of \( X_{F_2[0],\{k_m\}} \) into \( Y \). Here \( Y \) is any element in the class of Banach spaces which is sphere equivalent to an element of the class of Banach space of type \( > 1 \). The class contains all uniformly convex Banach spaces.

The proof of Theorem C is based on breakthroughs of V. Lafforgue [Laf08, Laf09], together with spectral analysis. For the proof of Theorem 7.3 we employ a recent remarkable theorem of M. de la Salle [dlS13].

Note that this coarse disjoint union in Theorem C enjoys property A and hence that it coarsely embeds into \( \ell_2(N, Y) \) for any Banach space \( Y \). Theorem C states that nevertheless, we can choose parameters \( \{k_m\}_m \) and make compression functions as poor as we wish for a large class of Banach spaces. The first result in this direction is due to Arzhantseva–Druțu–Sapir [ADS09] and they have constructed a finitely generated group with property A (in fact with asymptotic dimension \( \leq 2 \)). Later Olshanskii–Osin [OO13] even have constructed such examples which are elementary amenable. Metric spaces in our theorem is not groups. However they are coarse disjoint unions of concrete Cayley graphs without lacunary. (We note that in the above two theorems they make lacunary in V. Lafforgue’s expander, and build groups form it.) Our Theorem C and Theorem 7.3 are not existence theorems, but the assertions that sequences of very concrete and in some sense handleable finite Cayley graphs, which have drawn heavy attention, enjoy considerable metric embedding properties.

**Organization of the paper:** In Section 2, we recall the definitions of the word metric on a group and the Cayley topology in the space of \( k \)-marked groups. Section 3 is for study of amenable groups in the space of marked groups, which shall play a key role in the proof of Theorem A. Section 4 is a preliminary part on coarse geometry. In Section 5, we prove Theorem A. In Section 6, we deal with examples to which Theorem A applies, and show Theorem B. Section 7 is for the proof of Theorem C and Theorem 7.3 which employs estimation of Banach spectral gaps. In Section 8, we shall announce our forthcoming results on fibered coarse embeddings in part II of this series of papers [MS].

### 2. Word metric and the Cayley topology

We recall the definition of the word metric on finitely generated groups. They allows us to regard finitely generated groups as geometric objects. In the rest of this paper, we in principle do not make distinction between \( G \) with word metric \( d_S \) and the Cayley graph \( \text{Cay}(G, S) \) in the definition below.

**Definition 2.1.** Let \( G \) be a group generated by a finite subset \( S \) of \( G \).

- The **word length** \( \ell_S(g) \) of an element \( g \in G \) is the smallest integer \( n \) for which there exists a sequence \( (s^{(1)}, s^{(2)}, \ldots, s^{(n)}) \) of \( S \cup S^{-1} \) such that \( g = s^{(1)} s^{(2)} \cdots s^{(n)} \). The word length of \( \text{id}_G \in G \) is defined to be 0.
- The **word metric** \( d_S \) is defined on \( G \) by \( d_S(g, h) = \ell_S(g h^{-1}) \).
- For a subset \( Y \) of \( G \) and a positive integer \( R \), let \( \text{nbhd}_R(Y) = \{ g \in G; \exists h \in Y, d_S(g, h) \leq R \} \) and let \( \partial_R(Y) \)
denote the boundary nbhd$_R(Y) \setminus Y$. For a singleton \{g\}, we simply write nbhd$_R(g)$ for nbhd$_R(\{g\})$.

- The Cayley graph \( \text{Cay}(G, S) \) of \((G, S)\) is an undirected graph whose vertex set is \(G\) and whose edge set consists of unoriented pairs \((g, sg)\) for any \(g \in G\) and \(s \in S\). The metric space structure (with the path metric) is exactly same as \(G\) with the word metric above, and the right multiplication of \(G\) is a graph automorphism action on \(\text{Cay}(G, S)\).

We note that the left multiplication \(G \ni h \mapsto gh \in G\) by \(g\) satisfies the equality \(d_S(gh, h) = \ell_S(g)\). There are two ways for the definition. We make use of a metric which is invariant under the multiplication from the right. Left invariant metrics are also widely used (see, for example, Gromov [Gro84], [Gro93]). Let \(F_k\) denote the free group generated by symbols \(a_1, a_2, \cdots, a_k\). Denote by \(d\) the word metric on \(F_k\) with respect to the generators \(a_1, a_2, \cdots, a_k\).

**Lemma 2.2.** Let \(N\) be a normal subgroup of \(F_k\). Denote by \(\pi\) the quotient map \(F_k \to G\). Let \(d_S\) be the word metric on the quotient group \(F_k/N\) with respect to the generators \(S = \{\pi(a_1), \pi(a_2), \cdots, \pi(a_k)\}\). Then the quotient map \(\pi\) is contractive with respect to the metrics \(d\) and \(d_S\). For \(b, c \in F_k\), \(d_S(\pi(b), \pi(c))\) is equal to \(\min\{d(bc^{-1}, x); x \in N\}\).

**Proof.** For the first assertion, it suffices to show that the word length of \(\pi(b)\) is at most \(\ell(b)\), for every \(b \in F_k\). The element \(b\) is expressed as a reduced word \(b = b^{(1)}b^{(2)} \cdots b^{(\ell(b))}\). The element \(\pi(b)\) of \(F_k/N\) is equal to \(\pi(b^{(1)})b^{(2)} \cdots b^{(\ell(b))}\). Since \(\pi(b^{(j)})\) is an element of \(S \cup S^{-1}\), the word length of \(\pi(b)\) is at most \(\ell(b)\).

For \(b, c \in F_k\) and \(x \in N\), \(d(bc^{-1}, x)\) is at least \(d_S(\pi(bc^{-1}), \text{id}_{F_k/N}) = d_S(\pi(b), \pi(c))\). It follows that \(d_S(\pi(b), \pi(c)) \leq \min\{d(bc^{-1}, x); x \in N\}\).

For \(b, c \in F_k\), define a natural number \(n\) by \(d_S(\pi(b), \pi(c))\). This is equal to the word length of \(\pi(bc^{-1})\). There exist elements \(a^{(1)}, a^{(2)}, \cdots, a^{(n)}\) of \(\{a_1, a_2, \cdots, a_k\} \cup \{a_1^{-1}, a_2^{-1}, \cdots, a_k^{-1}\}\) such that

\[\pi(bc^{-1}) = \pi(a^{(1)}a^{(2)} \cdots a^{(n)}).\]

It follows that there exists \(x \in N\) such that \(bc^{-1} = a^{(1)}a^{(2)} \cdots a^{(n)}x\). Therefore we have \(n \geq \min\{d(bc^{-1}, x); x \in N\}\). \(\square\)

We recall the Cayley topology, which allows us to regard a group as a point in a topological space. The space which we are going to use is homeomorphic to \(G_k\) introduced by Grigorchuk [Gri83]. We refer to the book [dH00, Section V.10] of de la Harpe for the definition. There are three ways to describe the set of discrete groups with \(k\) generators.

- A group \((G, s_1, s_2, \cdots, s_k)\) with fixed and ordered \(k\) generators is called a marked group. We identify two marked groups \((G, s_1, s_2, \cdots, s_k)\) and \((G', s'_1, s'_2, \cdots, s'_k)\), if there exists an isomorphism \(\varphi: G \to G'\) satisfying \(\varphi(s_j) = s'_j\). We denote by \(G(k)\) the set of all the isomorphism classes of marked groups.

- Let \(G\) and \(G'\) be groups for which surjective homomorphisms \(\pi: F_k \to G\) and \(\pi': F_k \to G'\) exist. We say that \(\pi\) and \(\pi'\) are isomorphic, if there exists
a group isomorphism $\varphi: G \to G'$ such that $\varphi \circ \pi = \pi'$. We denote by $S\mathcal{H}(k)$ the set of all the isomorphism classes of surjective homomorphisms from $F_k$.

- Let $\mathcal{N}(k)$ denote the set of all the normal subgroups of $F_k$.

For a marked group $G$ with ordered generators $S = (s_1, s_2, \cdots, s_k)$, there exists a unique surjective homomorphism $\pi_{(G,S)}$ from $F_k$ such that $\pi_{(G,S)}(s_j) = s_j$. Let $\ker_{(G,S)}$ denote the kernel of the homomorphism $\pi_{(G,S)}$. We identify $G(k), S\mathcal{H}(k)$ and $\mathcal{N}(k)$ by the correspondences $(G, S) \leftrightarrow \pi_{(G,S)} \leftrightarrow \ker_{(G,S)}$. We often omit mentioning the bijections among $G(k), S\mathcal{H}(k)$ and $\mathcal{N}(k)$.

**Definition 2.3.** For $N \in \mathcal{N}(k)$ and a positive real number $R$, define a subset by $\mathcal{N}(k, N, R) = \{ N' \in \mathcal{N}(k); N' \cap \text{nbhd}_R(1_{F_k}) = N \cap \text{nbhd}_R(1_{F_k}) \}$ of $\mathcal{N}(k)$. The topology generated by $\{ \mathcal{N}(k, N, R); N \in \mathcal{N}(k), R > 0 \}$ is called the **Cayley topology**.

We also introduce a topology on $G(k)$ by the identification with $\mathcal{N}(k)$.

**Example 2.4.** Let $(G, s_1, \cdots, s_k)$ be a marked group. Let $H_1 \supset H_2 \supset \cdots$ be a decreasing sequence of normal subgroups whose intersection is $\{1_G\}$. Let $\pi_{(G,S)}: F_k \to G$ denote the canonical surjective homomorphism. The sequence $\{ \pi_{(G,S)}^{-1}(H_m) \}_{m=1}^{\infty} \subset \mathcal{N}(k)$ is decreasing and its intersection is $\ker(\pi_{(G,S)})$. It follows that the sequence of marked groups $\{ G/H_m \}_{m=1}^{\infty}$ converges to $G$.

For a subset $N$ of $F_k$, let $\chi_N \in \{0, 1\}^{F_k}$ denote the characteristic function of $N$. All the algebraic requirements for a normal subgroup correspond to closed subsets of the Cantor set $\{0, 1\}^{F_k}$. More concretely, $N$ is an element of $\mathcal{N}(k)$ if and only if for arbitrary elements $g, h \in F_k$, the following conditions hold:

- If $\chi_N(g) = 1$ and $\chi_N(h) = 1$, then $\chi_N(gh) = 1$. Or equivalently,
  $$\chi_N \in \{ \chi; \chi(g) = 0 \} \cup \{ \chi; \chi(h) = 0 \} \cup \{ \chi; \chi(gh) = 1 \} \subset \{ 0, 1 \}^{F_k}.$$

- If $\chi_N(g) = 1$, then $\chi_N(g^{-1}) = 1$. Or equivalently,
  $$\chi_N \in \{ \chi; \chi(g) = 0 \} \cup \{ \chi; \chi(g^{-1}) = 1 \} \subset \{ 0, 1 \}^{F_k}.$$

- If $\chi_N(g) = 1$, then $\chi_N(hgh^{-1}) = 1$. Or equivalently,
  $$\chi_N \in \{ \chi; \chi(g) = 0 \} \cup \{ \chi; \chi(hgh^{-1}) = 1 \} \subset \{ 0, 1 \}^{F_k}.$$

Therefore $\{ \chi_N; N \in \mathcal{N}(k) \}$ is a closed subset of the Cantor set.

**Proposition 2.5.** The space $\mathcal{N}(k)$ with the Cayley topology is homeomorphic to a closed subset of the Cantor set.

**Proof.** By definition, the Cayley topology on $\mathcal{N}(k)$ is homeomorphic to the pointwise-convergence topology on $\{ \chi_N; N \in \mathcal{N}(k) \}$. The latter topology coincides with the direct product topology. \(\square\)

As a consequence, the Cayley topology is a compact Hausdorff topology.
3. A TOPOLOGICAL PROPERTY OF THE SET OF AMENABLE GROUPS

Amenability is a fundamental property of discrete groups, which connects functional analytic features to geometric aspects. The original form of the definition was given by the existence of an invariant mean. Among a number of characterizations, we regard the following as a definition of amenability.

**Theorem 3.1.** Let $G$ be a group. Let $S$ be a finite subset which generates $G$. Introduce a metric on $G$ by the word length with respect to $S$. The group $G$ is amenable if and only if for every positive number $\epsilon$, there exists a finite subset $Y$ such that $\sharp(\partial_1(Y)) < \epsilon \sharp(Y)$.

The latter condition is called the Følner property of $G$ and the subset $Y$ is often called a Følner set. This property does not depend on the choice of $S$. Let $A(k)$ be the set of all the marked groups which are amenable. We study its topological feature (Theorem 3.5). We make use of several continuous functions on the space $N(k)$.

For every $g \in F_k$, we define a function $\text{Elm}(\cdot; g): N(k) \to \{0, 1\}$ by

$$\text{Elm}(N; g) = \begin{cases} 
0, & g \notin N, \\
1, & g \in N.
\end{cases}$$

Since $\text{Elm}(N; g)$ is the coordinate of $\chi_N \in \{0, 1\}^{F_k}$ at $g$, the function $\text{Elm}(\cdot; g)$ is continuous. For a non-empty finite subset $Z$ of $F_k$, we define a function $\text{Vol}(\cdot; Z)$ on $N(k)$ by

$$\text{Vol}(N; Z) = \sharp\{tN \subset F_k; t \in Z\}.$$

When we denote by $\pi$ the quotient map $F_k \twoheadrightarrow F_k/N$, the cardinality of $\pi(Z)$ is equal to $\text{Vol}(N; Z)$.

**Lemma 3.2.** The function $\text{Vol}(\cdot; Z)$ is continuous with respect to the Cayley topology.

**Proof.** We prove by induction on the cardinality of $Z$. For $Z = \{t_1, t_2\} \subset F_k$, we have

$$2 - \text{Elm}(N; t_2^{-1}t_1) = \begin{cases} 
2, & t_1N \neq t_2N, \\
1, & t_1N = t_2N.
\end{cases}$$

It follows that $\text{Vol}(\cdot; \{t_1, t_2\})$ coincides with the continuous function $2 - \text{Elm}(\cdot; t_2^{-1}t_1)$.

Let $Z$ be a non-empty finite subset of $F_k$. Suppose that $\text{Vol}(\cdot; Z)$ is continuous. For $f \in F_k$, we have

$$1 - \max_{t \in Z} \text{Elm}(N; t^{-1}f) = \begin{cases} 
1, & (\forall t \in Z, \ fN \neq tN), \\
0, & (\exists t \in Z, \ fN = tN).
\end{cases}$$

It follows that $\text{Vol}(\cdot; Z \cup \{f\})$ coincides with the function

$$\text{Vol}(\cdot; Z) + 1 - \max_{t \in Z} \text{Elm}(N; t^{-1}f).$$

The latter function is continuous. \qed
For a natural number $R$, define a function $\text{Rel}(\cdot, R)$ on marked groups by
\[
\text{Rel}(G, R) = \min \left\{ \frac{\sharp(\partial_1(Y))}{\sharp(Y)} ; \emptyset \neq Y \subset \text{nbhd}_R(1_G) \right\}.
\]
We use the same notation for the corresponding mapping $\mathcal{N}(k) \to \mathbb{R}$.

**Lemma 3.3.** The function $\text{Rel}(\cdot, R)$ is continuous with respect to the Cayley topology.

**Proof.** Let $N$ be an arbitrary element of $\mathcal{N}(k)$. Denote by $\pi_G : F_k \to F_k/N$ the quotient map. The marked group $G = (F_k/N, a_1N, \ldots, a_kN)$ corresponds to $N$. The family of finite subsets $\{F ; \emptyset \neq F \subset \text{nbhd}_R(1_G)\}$ is equal to
\[
\{\pi_G(Z) ; \emptyset \neq Z \subset \text{nbhd}_R(1_{F_k})\}.
\]
For every finite subset $Z$ of $F_k$, we have $\pi_G(\text{nbhd}_1(Z)) = \text{nbhd}_1(\pi_G(Z))$. It follows that
\[
\text{Rel}(G, R) = \min \left\{ \frac{\sharp(\text{nbhd}_1(Z))}{\sharp(Z)} - 1 ; \emptyset \neq Z \subset \text{nbhd}_R(1_{F_k}) \right\}
\]
\[
= \min \left\{ \frac{\sharp(\pi_G(\text{nbhd}_1(Z)))}{\sharp(\pi_G(Z))} - 1 ; \emptyset \neq Z \subset \text{nbhd}_R(1_{F_k}) \right\}
\]
\[
= \min \left\{ \frac{\text{Vol}(N; \text{nbhd}_1(Z))}{\text{Vol}(N; Z)} - 1 ; \emptyset \neq Z \subset \text{nbhd}_R(1_{F_k}) \right\}.
\]
This equation means that $\text{Rel}(\cdot, R)$ is a minimum of finitely many continuous functions. Hence the function $\text{Rel}(\cdot, R)$ is continuous.

**Proposition 3.4.** For every marked group $G$, the sequence $\{\text{Rel}(G; R)\}_{R=1}^\infty$ satisfies $\text{Rel}(G; R) \geq \text{Rel}(G; R+1)$. The group $G$ is amenable if and only if $\inf_R \text{Rel}(G; R) = 0$.

**Proof.** The first assertion is due to the inclusion
\[
\{Y ; \emptyset \neq Y \subset \text{nbhd}_R(1_G)\} \subset \{Y ; \emptyset \neq Y \subset \text{nbhd}_{R+1}(1_G)\}.
\]
Suppose that $G$ is amenable. Then for every positive number $\epsilon$, there exists a finite subset $Y \subset G$ such that $\sharp(\partial_1(Y)) < \epsilon \sharp(Y)$. For a natural number $R$ such that $\text{nbhd}_R(1_G)$ includes $Y$, we have $\text{Rel}(G; R) < \epsilon$. It follows that $\inf_R \text{Rel}(G; R) = 0$. Conversely, suppose that $\inf_R \text{Rel}(G; R) = 0$. Then for every positive number $\epsilon$, there exists a natural number $R$ such that $\text{Rel}(G; R) < \epsilon$. Hence there exists a subset $Y$ of $\text{nbhd}_R(1_G) \subset G$ satisfying the inequality $\sharp(\partial_1(Y)) < \epsilon \sharp(Y)$. 

**Theorem 3.5.** The set of amenable marked groups $\mathcal{A}(k)$ is an intersection of countably many open subsets of $\mathcal{G}(k)$. The relative topology on $\mathcal{A}(k)$ with respect to the Cayley topology is metrizable by a complete metric.

**Proof.** For a natural number $n$, define an open subset $\mathcal{F}_n$ of $\mathcal{G}(k)$ by
\[
\mathcal{F}_n = \bigcup_{R=1}^\infty \{G \in \mathcal{G}(k) ; \text{Rel}(G; R) < 1/n\}.
\]
The condition $\inf R \text{Rel}(G; R) = 0$ is equivalent to the following condition: $G \in \bigcap_{n=1}^{\infty} \mathcal{F}_n$. It follows that $\mathcal{A}(k)$ is identical to the intersection $\bigcap_{n=1}^{\infty} \mathcal{F}_n$. The second assertion follows from a general theorem for Polish spaces (see, for instance, [Kec95, Theorem 3.11]).

To construct an example in Section 4, we exploit the following lemma.

**Lemma 3.6.** Let $\{1, 2, \cdots \} \cup \{\infty\}$ be the one point compactification of $\{1, 2, \cdots \}$. For a marked group $(G, s_1, \cdots, s_k)$, define $\text{Ord}(G, s_1, \cdots, s_k) \in \{1, 2, \cdots \} \cup \{\infty\}$ by the order of $s_k \in G$. Then the function $\text{Ord} : \mathcal{G}(k) \to \{1, 2, \cdots \} \cup \{\infty\}$ is continuous.

**Proof.** For every natural number $n$, the subset

$$O_{\geq n} = \{ N \in \mathcal{N}(k); \text{the order of } a_k N \in F_k / N \text{ is at least } n \}$$

is equal to $\{ N \in \mathcal{N}(k); a_k \notin N, \cdots, a_k^{n-1} \notin N \}$. Its characteristic function is the continuous function $\max_{1 \leq i \leq n-1} (1 - \text{Elm} (\cdots ; a_k^i))$. It follows that $O_{\geq n}$ is closed and open. Considering the corresponding subset in $\mathcal{G}(k)$, we obtain the lemma. \qed

4. Preliminary on large scale geometry

4.1. Coarse space. In the framework of coarse geometry, we discuss large scale structures. A coarse structure of a set $X$ is formally a family of subsets of $X^2$. Details are described in Roe’s lecture note [Roe03, Chapter 2].

Let $X$ be a set. For subsets $T, T_1, T_2 \subset X^2$, define the inverse $T^{-1}$ and the product $T_1 \circ T_2$ by:

$$T^{-1} = \{(x, y) \in X^2 \ ; \ (y, x) \in T\},$$

$$T_1 \circ T_2 = \{(x, y) \in X^2 \ ; \ \exists z \in X, (x, z) \in T_1, (z, y) \in T_2\}.$$

For a point $x \in X$, we write $T[x]$ for the set $\{y \in X \ ; \ (y, x) \in T\}$.

**Definition 4.1** (Definition 2.3 in [Roe03]). Let $X$ be a set. A family $\mathcal{C}$ of subsets of $X^2$ is called a coarse structure on $X$ if $\mathcal{C}$ contains the diagonal subset $\Delta_X$ of $X^2$ and is closed under inverses, products, unions and subsets. The pair $(X, \mathcal{C})$ is called a coarse space. An element of $\mathcal{C}$ is called a controlled set.

A subset $Y$ of a coarse space $(X, \mathcal{C})$ naturally has a coarse structure defined by

$$\mathcal{C}|_Y = \{ T \cap (Y \times Y) \ ; \ T \in \mathcal{C} \}.$$

A metric space $(X, d)$ provides a typical example of a coarse space. A coarse structure $\mathcal{C}_d$ is defined on the set $X$ by $\mathcal{C}_d = \{ T \subset X^2 \ ; \ d \text{ is bounded on } T \}$.

**Example 4.2.** Let $X$ be a set on which a discrete group $G$ acts. For a finite subset $K \subset G$, let $T_K$ be the orbit of $K$. That is, $T_K = \{(kx, x) \in X^2; k \in K, x \in X\}$. The coarse structure $\mathcal{C}_G$ on $X$ is the collection $\{ T; \exists \text{ finite } K \subset G, T \subset T_K \}$. The group $G$ itself naturally has a coarse structure $\mathcal{C}_G$, which is defined by the left transformation action. Then the subset $T_K \subset G \times G$ is invariant under the right multiplication action by $K$. If $G$ is finitely generated, $\mathcal{C}_G$ is identical to the coarse structure $\mathcal{C}_d$ with respect to a word metric $d$. Therefore the coarse structure $\mathcal{C}_d$ does not depend on the choice of the generating set.
**Example 4.3.** (Coarse disjoint union)

For the rest of this paper, we study the disjoint union $X = \bigsqcup_{m=1}^{\infty} G^{(m)}$ in the following sense.

**Definition 4.4.** Let \( \{G^{(m)} = \left( G^{(m)}_1, s_1^{(m)}, s_2^{(m)}, \ldots, s_k^{(m)} \right) \}_{m=1}^{\infty} \) be a sequence of groups with fixed \( k \)-generators. Define surjective homomorphisms \( \pi^{(m)} : F_k \to G^{(m)} \) from the free group \( F_k = \langle a_1, \ldots, a_k \rangle \) by \( \pi^{(m)}(a_j) = s_j^{(m)} \). The set \( X = \bigsqcup_{m=1}^{\infty} G^{(m)} \) is equipped with an \( F_k \)-action defined by \( f \cdot g = \pi^{(m)}(f)g, g \in G^{(m)} \). This action gives a coarse structure \( \mathcal{C} \) on the disjoint union \( X \). Its restriction \( \mathcal{C}|_{G^{(m)}} \) to each component is identical to the coarse structure of the discrete group \( G^{(m)} \). This coarse space is called the coarse disjoint union of \( \{G^{(m)}\}_m \).

Denote by \( d_m \) the word metric on \( G^{(m)} \). The coarse structure \( \mathcal{C} \) of \( X \) is identical to

\[
\mathcal{C} = \left\{ T \subset \bigsqcup_{m=1}^{\infty} G^{(m)} \times G^{(m)} ; \exists R, \forall m, d_m \leq R \text{ on } T \cap G^{(m)} \times G^{(m)} \right\} ,
\]

The coarse disjoint union in the definition above is uniformly locally finite, that means, \( \bigsqcup_{m=1}^{\infty} G^{(m)} \) satisfies the following condition:

**Definition 4.5.** A coarse space \( (X, \mathcal{C}) \) is said to be uniformly locally finite if every controlled set \( T \in \mathcal{C} \) satisfies the inequality \( \sup_{x \in X} \| T[x] \| < \infty \).

If a metric space is uniformly locally finite as a coarse space, it is often called a metric space with bounded geometry.

### 4.2. Property A

Property A is an amenability-type condition for metric spaces and coarse spaces. It admits a number of characterizations. In this paper, we treat the following as a definition.

**Definition 4.6** (Definition 11.35 of [Roe03]). A uniformly locally finite coarse space \( (X, \mathcal{C}) \) has property A if the following condition holds: for every positive number \( \epsilon \) and every controlled set \( T \), there exists a map \( X \ni x \mapsto \eta_x \in \ell_2(X) \) assigning unit vectors such that \( \{(x, y) \in X^2 ; x \in \text{supp}(\eta_y)\} \) is controlled and that for every \( (x, y) \in T \), \( \| \eta_x - \eta_y \| < \epsilon \).

**Remark 4.7.** We may replace \( \ell_2(X) \) with any other Hilbert space \( \mathcal{H} \) (following an idea in [BNW07, Theorem 3]). As a consequence, property A passes to subsets.

The following are examples of coarse spaces with property A.

**Example 4.8.**

- Every subgroup of the general linear group over a field has property A (Guentner, Higson and Weinberger [GHW05]).
- If a group \( \Gamma \) is a hyperbolic relative to subgroups with property A, then \( \Gamma \) has property A (Ozawa [Oza06, Corollary 3]).
- Asymptotic dimension for metric spaces was introduced by Gromov [Gro93, Section 1E]. For a metric space \( X \) with bounded geometry, if the asymptotic dimension of \( X \) is finite, then \( X \) has property A (Higson and Roe [HR00, Lemma 4.3]).
Let \( G \) be a finitely generated residually finite group. Choose a decreasing sequence \( H_1 \supset H_2 \supset \cdots \) of finite index normal subgroups such that \( \bigcap H_m = \{1 \} \). The disjoint union \( \sqcup G = \bigsqcup_{m=1}^\infty G/H_m \) naturally has a coarse structure (see Example 4.3) and is often called a box space of \( G \). By Example 2.4 the closure of \( \{G/H_m\}_{m=1}^\infty \) with respect to the Cayley topology is equal to \( \{G/H_m\}_{m=1}^\infty \sqcup G \). It has been proved by Guentner that the box space \( \sqcup G \) has property A if and only if \( G \) is amenable (see, e.g., [Roe03 Proposition 11.39]). Theorem 5.1 is a generalization of this observation.

4.3. Coarse embedding of metric spaces. We introduce a kind of inclusion in the category of coarse spaces, which is called a coarse embedding.

**Definition 4.9.** Let \((X,C_X)\) and \((Y,C_Y)\) be coarse spaces. Let \( f : X \to Y \) be a map.

- **(Definition 2.21 of Roe03)** The map \( f \) is said to be bornologous, if for every controlled set \( T \in C_X \), the subset \( f^{\{2\}}(T) = \{(y_1,y_2) \in T, f(x_1) = y_1, f(x_2) = y_2\} \) of \( X \times Y \) is an element of \( C_Y \).
- **(Remark 3.16 of Roe03)** The map \( f \) is said to be effectively proper, if for every controlled set \( E \in C_Y \), the subset \( \{(x_1, x_2) \in X^2 ; (f(x_1), f(x_2)) \in E\} \) of \( X \times X \) is an element of \( C_X \).
- **(Remark 3.16 of Roe03)** The map \( f \) is called a coarse embedding, if \( f \) is bornologous and effectively proper.

In the case that two coarse structures \( C_X \) and \( C_Y \) are induced by metrics \( d_X \) and \( d_Y \), a map \( f : X \to Y \) is a coarse embedding if and only if there exist non-decreasing functions \( \rho_+, \rho_- : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \rho_- \leq \rho_+ \), \( \lim_{t \to +\infty} \rho_-(t) = +\infty \) and

\[
\rho_-(d_X(x_1, x_2)) \leq d_Y(f(x_1), f(x_2)) \leq \rho_+(d_X(x_1, x_2)).
\]

5. Property A for a sequence of marked amenable groups

The subject of this section is property A for a coarse disjoint union of marked amenable groups.

**Theorem 5.1.** Let \( \left\{ G^{(m)} = (G^{(m)}(s_1^{(m)}, \ldots, s_k^{(m)}) \right\}_{m=1}^\infty \) be a sequence of marked groups. Suppose that every group \( G^{(m)} \) is amenable. Let \( \overline{\{G^{(m)}\}}_{Cayley} \) denote the closure in the space \( \mathcal{G}(k) \). Define a coarse structure on \( \bigsqcup_{m=1}^\infty G^{(m)} \) as in Example 4.3. The coarse space \( \bigsqcup_{m=1}^\infty G^{(m)} \) has property A if and only if all the elements of \( \{G^{(m)}\}_{Cayley} \setminus \{G^{(m)}\} \) are amenable.

Before proving this theorem, we prepare two lemmata.

**Lemma 5.2.** Let \( \pi : F \to G \) be a homomorphism between groups. Let \( \phi \) be a positive definite function on \( G \). Then the function \( \psi = \phi \circ \pi \) on \( F \) is positive definite.

**Proof.** Let \((u, \mathcal{H}, \eta)\) be the GNS-construction for the positive definite function \( \phi \). Then the function \( \phi \) is expressed as \( \phi(g) = \langle u(g)\eta, \eta \rangle \). The function \( \psi = \phi \circ \pi \) is
on $F$ is expressed as $\psi(f) = \langle u(\pi(f))\eta, \eta \rangle$. For $n \in \mathbb{N}$, $f_1, f_2, \cdots, f_n \in F$ and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{C}$, we have
\[
\sum_{i,j=1}^{n} \alpha_j \alpha_i \psi(f_j^{-1} f_i) = \sum_{i,j=1}^{n} \alpha_j \alpha_i \langle u(\pi(f_i))\eta, u(\pi(f_j))\eta \rangle = \left\| \sum_{i=1}^{n} \alpha_i u(\pi(f_i))\eta \right\|^2 \geq 0.
\]
It follows that $\psi$ is positive definite.

**Lemma 5.3.** Let $\pi : F \to G$ be a surjective homomorphism between groups. Denote by $N$ the kernel of $\pi$. Let $\psi$ be a positive definite function on $F$. Suppose that $\psi(h) = 1$ for every $h \in N$. Then there exists a positive definite function $\phi$ on $G$ such that $\psi = \phi \circ \pi$.

**Proof.** Let $(v, \mathcal{H}, \eta)$ be the GNS-construction for the positive definite function $\psi$. Then the function $\psi$ is expressed as $\psi(f) = \langle v(f)\eta, \eta \rangle$. Since $1_{F_k} \in N$, we have $\|\eta\|^2 = \psi(1_{F_k}) = 1$. For every $h \in N$ and $f \in F$, we have
\[
\|v(h)v(f)\eta - v(f)\eta\|^2 = 2\|\eta\|^2 - 2\text{Re}(\langle v(hf)\eta, v(f)\eta \rangle) = 2 - 2\text{Re}(\psi(f^{-1}hf)) = 0.
\]
Hence $N$ is included in the kernel of the group homomorphism $v$. It follows that there exists a group homomorphism $u : G \to U(\mathcal{H})$ such that $v = u \circ \pi$. The function $\phi(g) = \langle u(g)\eta, \eta \rangle$ on $G$ satisfies the required conditions. \hfill \Box

**Proof of Theorem 5.1.** We note that by assumption we may replace $\{G^{(m)}\}_{m=1}^{\infty}$ with $\{G^{(m)}\}_{m=1}^{\infty}$ in the statement of theorem. First, suppose that every element $G$ of $\{G^{(m)}\}_{m=1}^{\infty}$ is amenable. By Proposition 3.4, the sequence $\{\text{Rel}(G; R)\}_{R=1}^{\infty}$ decreases and tends to 0. Since $\{G^{(m)}\}_{m=1}^{\infty}$ is compact, the sequence of the functions $\{\text{Rel}(\cdot; R)\}_{R=1}^{\infty}$ uniformly converges to 0 on $\{G^{(m)}\}_{m=1}^{\infty}$ by Dini’s theorem. Take an arbitrary controlled set $T$ of $\bigcup_{m=1}^{\infty} G^{(m)}$. There exists a natural number $C$ such that
\[
T \subset \bigcup_{m=1}^{\infty} \{(g, h) \in G^{(m)} \times G^{(m)} : d_m(g, h) \leq C\},
\]
where $d_m$ is the word metric on $G^{(m)}$. Since $\text{Rel}(\cdot, R)$ uniformly converges to 0, for every positive number $\epsilon$, there exists a natural number $R$ such that
\[
\text{Rel}(G^{(m)}; R) < \frac{\epsilon^2}{2C}, \quad m \in \mathbb{N}.
\]
For each $m$, choose a finite subset $Y^{(m)}$ of $G^{(m)}$ such that
\[
\sharp (\partial_1 (Y^{(m)})) < \frac{\epsilon^2}{2C} \sharp (Y^{(m)}), \quad Y^{(m)} \subset \text{nbhd}_R (1_{G^{(m)}}).
\]
Define a unit vector \( \eta^{(m)} \) in \( \ell_2(G^{(m)}) \subset \ell_2(\bigsqcup_{m=1}^{\infty} G^{(m)}) \) by
\[
\eta^{(m)} = \frac{1}{\sqrt{\#(Y^{(m)})}} \sum_{y \in (Y^{(m)})^{-1}} \delta_y.
\]

For \( g \in G^{(m)} \), we define a unit vector \( \eta_g^{(m)} \in \ell_2(G^{(m)}) \) by
\[
\eta_g^{(m)} = \eta^{(m)} \ast \delta_g = \frac{1}{\sqrt{\#(Y^{(m)})}} \sum_{y \in (Y^{(m)})^{-1}} \delta_{gy}.
\]

The support of \( \eta_g^{(m)} \) is included in \( \text{nbhd}_R(g) \subset G^{(m)} \). It follows that
\[
\left\{(g, h) \in \left( \bigsqcup_{m=1}^{\infty} G^{(m)} \right)^2; g \in \text{supp}(\eta_h^{(m)}) \right\}
\]
is a controlled set. For every pair \( (g, h) \in (G^{(m)})^2 \), we have
\[
\| \eta_g^{(m)} - \eta_h^{(m)} \|^2 = \frac{\#((Y^{(m)})^{-1}g \triangle (Y^{(m)})^{-1}h)}{\#(Y^{(m)})} = 2\frac{\#(gh^{-1}Y^{(m)} \setminus Y^{(m)})}{\#(Y^{(m)})}.
\]

If \( (g, h) \) is a member of the controlled set \( T \), then the word length of \( gh^{-1} \) is at most \( S \). It follows that
\[
\| \eta_g^{(m)} - \eta_h^{(m)} \|^2 \leq 2C \frac{\#(\partial_1(Y^{(m)}))}{\#(Y^{(m)})} < \epsilon^2, \quad (g, h) \in T \cap (G^{(m)} \times G^{(m)}).
\]

Because \( m \) is arbitrary and \( \epsilon \) does not depend on \( m \), the coarse space \( \bigsqcup_{m=1}^{\infty} G^{(m)} \) has property A.

For the converse, we exploit the limiting and averaging technique described in [Roe03, Proposition 11.39]. Suppose that \( \bigsqcup_{m=1}^{\infty} G^{(m)} \) has property A. We are going to prove that an arbitrary element \( G \) of \( \{G^{(m)}\} \) is amenable. Since the topological space \( G(k) \) is second countable, there exists a subsequence of \( \{G^{(m)}\}_{m=1}^{\infty} \) which converges to \( G \). Since property A passes to subsets, we may assume that \( \{G^{(m)}\}_{m=1}^{\infty} \) converges to \( G \). Let \( \epsilon \) be an arbitrary (small) positive number and let \( C \) be an arbitrary (large) natural number. Then there exist a natural number \( R \) and a map
\[
\eta: \bigsqcup_{m=1}^{\infty} G^{(m)} \to \ell_2 \left( \bigsqcup_{m=1}^{\infty} G^{(m)} \right)
\]
satisfying the following properties:

- If \( g, h \in G^{(m)} \) and \( d_m(g, h) \leq C \), then \( \| \eta_g - \eta_h \| < \epsilon \).
- If \( g \in G^{(m)} \), then \( \text{supp}(\eta_g) \subset \text{nbhd}_R(g) \subset G^{(m)} \).

We define a positive definite kernel \( \Phi \) on \( \bigsqcup_{m=1}^{\infty} G^{(m)} \) by
\[
\Phi(g, h) = \langle \eta_g, \eta_h \rangle, \quad (g, h) \in \bigsqcup_{m=1}^{\infty} G^{(m)} \times \bigsqcup_{m=1}^{\infty} G^{(m)}.
\]

The kernel \( \Phi \) satisfies the following:

- For \( g \in G^{(m)} \), \( \Phi(g, g) = 1 \).
• For \((g, h) \in G^{(m)} \times G^{(m)}\), if \(d_m(g, h) \leq C\), then \(|1 - \Phi(g, h)| < \epsilon\).
• For \((g, h) \in G^{(m)} \times G^{(m)}\), if \(d_m(g, h) > 2R\), then \(\Phi(g, h) = 0\).

Since the group \(G^{(m)}\) is amenable, there exists a left invariant mean \(\mu^{(m)}\) on \(G^{(m)}\). For \(g \in G^{(m)}\), define a bounded function \(\Phi_g\) on \(G^{(m)}\) by \(\Phi_g(h) = \Phi(g^{-1}h, h)\). Define a function \(\phi\) on \(G^{(m)}\) by

\[
\phi_m(g) = \mu^{(m)}(\Phi_g).
\]

The function \(\phi_m\) satisfies the following:

• If the word length of \(g\) is no more than \(C\), then \(|1 - \phi_m(g)| < \epsilon\).
• If the word length of \(g\) is greater than \(2R\), then \(\phi_m(g) = 0\).

We claim that \(\phi_m\) is a positive definite function. Let \(\lambda\) denote the left translation action on \(\ell_{\infty}(G^{(m)})\): \([\lambda_g(\zeta)](h) = \zeta(g^{-1}h)\). For \(n \in \mathbb{N}\), \(g_1, g_2, \ldots, g_n \in G^{(m)}\) and \(\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}\), the value of \(\sum_{i,j=1}^n \overline{\alpha_j} \alpha_i \lambda_{g_j} \left( \Phi_{g_j^{-1}g_i} \right)(h)\) is not negative, since

\[
\sum_{i,j=1}^n \overline{\alpha_j} \alpha_i \lambda_{g_j} \left( \Phi_{g_j^{-1}g_i} \right)(h) = \sum_{i,j=1}^n \overline{\alpha_j} \alpha_i \Phi(g_i^{-1}h, g_j^{-1}h).
\]

It follows that

\[
\sum_{i,j=1}^n \overline{\alpha_j} \alpha_i \phi_m(g_j^{-1}g_i) = \mu^{(m)} \left( \sum_{i,j=1}^n \overline{\alpha_j} \alpha_i \lambda_{g_j} \left( \Phi_{g_j^{-1}g_i} \right) \right) \geq 0.
\]

Thus we obtain the claim.

Let \(\psi_m\) denote the composition of the quotient map \(\pi^{(m)}: F_k \to G^{(m)}\) and the function \(\phi_m\). By Lemma 4.2, \(\psi_m\) is a positive definite function of \(F_k\). Denote by \(d\) the word metric of \(F_k\). The function \(\psi_m\) satisfies the following:

1. For \(f \in F_k\), if \(\ell(f) \leq C\), then \(|1 - \psi_m(f)| < \epsilon\).
2. If \(d(f, \ker(\pi^{(m)})) > 2R\), then \(\psi_m(f) = 0\).

The second assertion is due to Lemma 2.2. Since \(\psi_m(1_{F_k}) = 1\), we have \(|\psi_m(f)| \leq 1\) for every \(f \in F_k\). By the compactness of the disc \(\{z \in \mathbb{C}; |z| \leq 1\}\), there exists a subsequence \(\{G^{(m(l))}\}_{l=1}^\infty\) such that for every \(f \in F_k\), the sequence \(\{\psi_{m(l)}(f)\}_{l=1}^\infty\) converges. Its limit \(\psi_\infty(f)\) is also a positive definite function on \(F_k\). If \(f\) is in the kernel of the quotient map \(\pi: F_k \to G\), then \(f\) is in the kernel of \(\pi^{(m)}: F_k \to G^{(m)}\) for large enough \(m\). It follows that \(\psi_\infty\) is 1 on the normal subgroup \(\ker(\pi) \subset F_k\). Lemma 5.3 shows that \(\psi_\infty\) induces a positive definite function \(\phi\) on \(G\).

For \(g \in G\), if the word length of \(g\) is at most \(C\), then \(|1 - \phi(g)| < \epsilon\) by condition \([\Pi]\). By condition \([\Pi]\), for \(f \in F_k\), if \(d(f, \ker(\pi)) > 2R\), then \(\psi_\infty(f) = 0\). It follows from Lemma 2.2 that if the word length of \(g \in G\) is greater than \(2R\), then \(\phi(g) = 0\). We conclude that \(G\) is amenable.

In the case that all the components \((G^{(m)}, d_m)\) are finite, there are two ways to define coarse structures. In the coarse structure \(\mathcal{C}\) defined in Definition 4.4, two elements \(g, h \in \bigsqcup_{m=1}^\infty G^{(m)}\) belonging to different components are not connected. That is, there exists no element of \(\mathcal{C}\) which includes the pair \((x, y)\). This implies that there exists no metric on the disjoint union which is compatible with the structure \(\mathcal{C}\).
Remark 5.4. In many references, coarse embeddings of metric spaces have been attracted much attention. To make the disjoint union $X = \bigcup_{m=1}^{\infty} G^{(m)}$ a metric space, we often define a metric function $d$ on $X$ as follows:

$$d(g, h) = \begin{cases} d_m(g, h), & (g, h) \in G^{(m)} \times G^{(m)}, \\ \diam(X_m) + \diam(X_n) + m + n, & (g, h) \in G^{(m)} \times G^{(n)}, m \neq n, \end{cases}$$

where $d_m$ is the word metric on each component. The coarse structure $\mathcal{C}_d$ given by $d$ is described as

$$\left\{ T \subset X^2 : \exists m, \exists T_1, \exists T_2, \text{s.t. } T = T_1 \sqcup T_2, T_1 \subset \left( \bigcup_{n=1}^{m} G^{(n)} \right)^2, T_2 \in \mathcal{C} \right\}.$$

The coarse structure $\mathcal{C}_d$ is uniformly locally finite, or equivalently, $(X, d)$ is a metric space with bounded geometry.

Although $\mathcal{C}_d$ is larger than $\mathcal{C}$, we have the following.

**Proposition 5.5.** The coarse space $(X, \mathcal{C})$ has property A if and only if the metric space $(X, d)$ with bounded geometry also has property A.

**Proof.** Suppose that $(X, \mathcal{C})$ has property A. Let $\epsilon$ be an arbitrary positive number and let $T$ be an arbitrary element of $\mathcal{C}_d$. There exists a decomposition of $T$

$$T = T_1 \sqcup T^{(m)} \sqcup T^{(m+1)} \sqcup \cdots,$$

such that $T_1 \subset (G^{(1)} \cup \cdots \cup G^{(m-1)})^2$, $T^{(n)} = T \cap (G^{(n)})^2$ and that $\bigcup_{n=m}^{\infty} T^{(n)} \in \mathcal{C}$. There exists a mapping $\eta: \bigcup_{n=m}^{\infty} G^{(n)} \to \ell_2(\bigcup_{n=m}^{\infty} G^{(n)})$ assigning unit vectors such that

- for $g \in G^{(n)}$, $\eta_g \in \ell_2(G^{(n)})$,
- for every $(g, h) \in T^{(n)}$, $\|\eta_g - \eta_h\| < \epsilon$,
- $\bigcup_{n=m}^{\infty} \{(g, h) \in G^{(n)} \times G^{(n)} : \eta_h(g) \neq 0\} \in \mathcal{C}$.

Choose and fix some unit vector $\eta_0$ in $\ell_2(G^{(1)} \cup \cdots \cup G^{(m-1)})$. For $n = 1, \cdots, m-1$ and $g \in G^{(n)}$, define $\eta_g$ by $\eta_0$ Then the mapping $\eta: X \to \ell_2(X)$ satisfies the following:

- for every $(g, h) \in T$, $\|\eta_g - \eta_h\| < \epsilon$,
- $\{(g, h) \in X \times X : \eta_h(g) \neq 0\} \subset (G^{(1)} \cup \cdots \cup G^{(m-1)})^2 \sqcup \bigcup_{n=m}^{\infty} T^{(n)} \in \mathcal{C}_d$.

It follows that $(X, \mathcal{C}_d)$ has property A. The converse can be showed in a similar manner. 

The immediate corollary to Theorem 5.1 and Proposition 5.5 is the following.

**Corollary 5.6.** Let $\{G^{(m)}\}_{m=1}^{\infty}$ be marked finite groups with $k$ generators. The metric space $(\bigcup_{m=1}^{\infty} G^{(m)}, d)$ has property A if and only if all the elements of $\{G^{(m)}\}^{\text{Cayley}} \setminus \{G^{(m)}\}$ is amenable. In particular, it coarsely embeds into a Hilbert space if all the elements of $\{G^{(m)}\}^{\text{Cayley}} \setminus \{G^{(m)}\}$ is amenable.

\[1\] The specific form of $d$ is not important.
We shall proceed in the proof of item (ii) of Theorem 5.7. A countable discrete group $G$ is said to be $a$-$T$-menable if $G$ admits an isometric action on a Hilbert space which is metrically proper. Through Schoenberg’s argument, this condition is equivalent to the following: there exists a sequence of positive definite functions $\phi_m : G \to \mathbb{R}$ which satisfies $\phi_m(1_G) = 1$ and which is $C_0$ (namely, vanishes at infinity) such that $\phi_m$ pointwisely converges to the constant 1 function pointwisely. We note that amenable groups are $a$-$T$-maneble, and that $F_2$ for instance lies in the difference of these two classes. For details on $a$-$T$-menable groups, see a book of Cherix–Cowling–Jolissaint–Julg–Valette [CCJ+01].

**Theorem 5.7.** Let $\{G^{(m)}\}_{m=1}^{\infty}$ be marked finite groups with $k$ generators. If the metric space $\bigcup_{m=1}^{\infty} G^{(m)}$ coarsely embeds into a Hilbert space, then every element of $\{G^{(m)}\}^\text{Cayley}_G \setminus \{G^{(m)}\} \subset G(k)$ is $a$-$T$-menable.

**Proof.** Let $G$ be an arbitrary element of $\{G^{(m)}\}^\text{Cayley}_G$. Suppose that the metric space $X = \bigcup_{m=1}^{\infty} G^{(m)}$ coarsely embeds into a Hilbert space. By replacing with a subsequence, we may assume that the sequence $\{G^{(m)}\}_{m=1}^{\infty}$ converges to $G$.

By Theorem 5.2.8 of [NY12], for every (small) $\epsilon > 0$ and every (large) $C \in \mathbb{N}$, there exist a Hilbert space $\mathcal{H}$, a map $\eta : X \to \mathcal{H}$, and a non-decreasing function $\rho : \mathbb{N} \to [0, \sqrt{2})$ converging to $\sqrt{2}$ satisfying that

- for every $m$ and every $x \in G^{(m)}$, $\|\eta_x\| = 1$,
- for every $(x, y) \in (G^{(m)})^2$, $\|\eta_x - \eta_y\| \geq \rho(d_m(x, y))$,
- if $d_m(x, y) \leq C$, then $\|\eta_x - \eta_y\| < \epsilon$.

Define a positive definite kernel $\Phi_m$ on $(G^{(m)})^2$ by $\Phi_m(x, y) = \text{Re}\langle \eta_x, \eta_y \rangle$. Then $\Phi_m$ satisfies

- $\Phi_m(x, y) = (2 - \|\eta_x - \eta_y\|^2)/2 \leq (2 - \rho(d_m(x, y))^2)/2$,
- if $d_m(x, y) \leq C$, then $\Phi_m(x, y) = 1 - \text{Re}\langle \eta_x, \eta_x - \eta_y \rangle \geq 1 - \|\eta_x - \eta_y\| > 1 - \epsilon$.

Define a function $\phi_m$ on $G^{(m)}$ by

$$\phi_m(g) = \frac{1}{\sharp(G^{(m)})} \sum_{h \in G^{(m)}} \Phi_m(g^{-1}h, h).$$

For every $n \in \mathbb{N}$, $g_1, \ldots, g_n \in G^{(m)}$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, we have

$$\sum_{i,j=1}^{n} \overline{\alpha_j} \alpha_i \phi_m(g_j^{-1}g_i) = \frac{1}{\sharp(G^{(m)})} \sum_{i,j=1}^{n} \sum_{h \in G^{(m)}} \overline{\alpha_j} \alpha_i \Phi_m(g_i^{-1}g_jh, h)$$

$$= \frac{1}{\sharp(G^{(m)})} \sum_{h \in G^{(m)}} \sum_{i,j=1}^{n} \overline{\alpha_j} \alpha_i \Phi_m(g_i^{-1}h, g_j^{-1}h)$$

$$\geq 0.$$ 

It follows that $\phi_m$ is a positive definite function on $G^{(m)}$. Denote by $\ell_m$ the word length function on $G^{(m)}$. Since $d_m(g^{-1}h, h)$ is given by $\ell_m(g^{-1}) = \ell_m(g)$, we have

- $\Phi_m(g^{-1}h, h) \leq (2 - \rho(\ell_m(g))^2)/2$,
- if $\ell_m(g) \leq C$, then $\Phi_m(g^{-1}h, h) > 1 - \epsilon$. 

Therefore the positive definite function \( \phi_m \) satisfies the following conditions:

- \( \phi_m(1_G) = 1 \),
- \( \phi_m(g) \leq (2 - \rho(\ell_m(g))^2)/2 \),
- if \( \ell_m(g) \leq C \), then \( \phi_m(g) > 1 - \epsilon \).

Define positive definite functions \( \psi_m \) on \( F_k \) by \( \psi_m = \phi_m \circ \pi^{(m)} \), where \( \pi^{(m)} \) is the quotient map \( F_k \to G^{(m)} \). Replacing with a subsequence, we may further assume that \( \psi_m(f) \) converges for every \( f \in F_k \). Denote by \( \psi \) the limit of \( \{\psi_m\} \).

Let \( \pi_G \) denote the quotient map \( F_k \to G \). For \( f \in \text{Ker}(\pi_G) \), if \( m \) is large enough, then \( f \) is in \( \text{Ker}(\pi^{(m)}) \). Hence \( \psi(f) = \lim_{m \to \infty} \psi_m(f) = 1 \). By Lemma 5.3 there exists a positive definite function \( \phi \) on \( G \) such that \( \psi = \phi \circ \pi_G \). The function \( \phi \) satisfies

- \( \phi(1_G) = 1 \),
- \( \phi(g) = \lim_{m \to \infty} \phi_m(g) \leq \lim_{m \to \infty} (2 - \rho(\ell_m(g))^2)/2 \leq (2 - \rho(\ell(g))^2)/2 \),
- if \( \ell(g) \leq C \), then \( \phi(g) > 1 - \epsilon \),

where \( \ell \) stands for the word length of \( G \). By the second condition, the function \( \phi_\infty \) is an element of \( c_0(G) \). It follows that \( G \) is a-T-menable. \( \square \)

Remark 5.8. The converse of Theorem 5.7 does not hold true. The group \( \text{PSL}(2, \mathbb{Z}) \) is a-T-menable. For a prime number \( p \), we denote by \( \pi_p \) the quotient map \( \text{PSL}(2, \mathbb{Z}) \to \text{PSL}(2, \mathbb{Z}/p\mathbb{Z}) \). The elements \( s_1 = \left\{ \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\} \) and \( s_2 = \left\{ \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \) generate \( \text{PSL}(2, \mathbb{Z}) \). The sequence

\[
\{(\text{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \pi_p(s_1), \pi_p(s_2)); p \text{ prime}\}
\]

converges to \( (\text{PSL}(2, \mathbb{Z}), s_1, s_2) \). Selberg's theorem in \( \text{Sel65} \) implies that the sequence of the Cayley graphs

\[
\{\text{Cay}(\text{PSL}(2, \mathbb{Z}/p\mathbb{Z}), \pi_p(s_1), \pi_p(s_2)); p \text{ is prime}\}
\]

is a sequence of expanders (\( \text{[Lub94], Theorem 4.4.2}] \)). As a consequence, \( X \) does not coarsely embed into a Hilbert space. Note that in the opposite direction of this example, Arzhantseva–Guentner–Špakula \( \text{[AGS13]} \) have shown that there exists a box space \( \square_{\{N_m\}} F_2 \) of \( F_2 \) which \( \text{does} \) coarsely embed into a Hilbert space.

6. Examples

6.1. An example by a sequence of symmetric groups. We study the sequence \( \{S_m\}_{m=2}^\infty \) of symmetric groups. Kassabov's theorem \( \text{[Kas07]} \) states that this sequence is an expander sequence if the generating sets are appropriately chosen. We are going to show that the coarse disjoint union \( X = \bigsqcup_{m=2}^\infty S_m \) can also have property A. We explicitly describe the generating set of each component.

Identify \( S_m \) with the set of all the bijections on \( \mathbb{Z}/m\mathbb{Z} \). For \( j \in \mathbb{Z} \), we denote by \( [j] \) the coset \( j + m\mathbb{Z} \in \mathbb{Z}/m\mathbb{Z} \). Let \( \sigma^{(m)} \) and \( \tau^{(m)} \) be the elements of \( S_m \) defined by

\[
\sigma^{(m)}([j]) = [j + 1], \quad [j] \in \mathbb{Z}/m\mathbb{Z},
\tau^{(m)}([0]) = [1], \quad \tau^{(m)}([1]) = [0], \quad \tau^{(m)}([j]) = [j], \quad [j] \in (\mathbb{Z}/m\mathbb{Z}) \setminus \{[0], [1]\}.
\]
Let $\mathcal{G}$ be the group consisting of bijections on $\mathbb{Z}$ generated by 
\[ \sigma(j) = j + 1, \quad j \in \mathbb{Z}, \]
\[ \tau(0) = 1, \quad \tau(1) = 0, \quad \tau(j) = j, \quad j \in \mathbb{Z} \setminus \{0, 1\}. \]

**Lemma 6.1.** The group $\mathcal{G}$ is amenable.

**Proof.** Let $\mathcal{G}_\infty$ denote the infinite symmetric group 
\[ \{ \gamma : \mathbb{Z} \to \mathbb{Z}; \text{ bijection, } \gamma(j) = j \text{ except for finitely many } j \} \]

The group $\mathbb{Z}$ acts on $\mathcal{G}_\infty$ by the automorphism $\alpha : \gamma \mapsto \sigma \circ \gamma \circ \sigma^{-1}$. We claim that the marked group $\mathcal{G}$ is isomorphic to the semi-direct product $\mathbb{Z} \ltimes \mathcal{G}_\infty$ with respect to the automorphism $\alpha$. It is routine to prove that the map $\iota : \mathbb{Z} \ltimes \mathcal{G}_\infty \to \mathcal{G}$ defined by $\iota(\alpha, \gamma) = \sigma \circ \gamma \circ \sigma^{-1}$ is the surjective group homomorphism. To show that $\iota$ is injective, suppose that $n \in \mathbb{Z}$ and $g \in \mathcal{G}_\infty$ satisfies $\iota(\alpha^n, g) = \sigma^n g = \text{id}$. If an integer $j$ is large, then we have $\sigma^n g(j) = n + j$. Thus we have $n = 0$ and $g = \text{id}$.

It follows that $\mathcal{G}$ is isomorphic to $\mathbb{Z} \ltimes \mathcal{G}_\infty$. Since $\mathbb{Z}$ and $\mathcal{G}_\infty$ are amenable, $\mathcal{G}$ is also amenable. $\square$

**Lemma 6.2.** The sequence of marked groups $\{(\mathcal{G}_m, \sigma^{(m)}, \tau^{(m)})\}_{m=1}^{\infty}$ converges to $(\mathcal{G}, \sigma, \tau)$.

**Proof.** We use the symbols $s$ and $t$ for the standard generators of the free group $F_2$. Let $N_m$ denote the kernel of the surjective homomorphism $\pi^{(m)} : F_2 \to \mathcal{G}_m$ defined by $\pi^{(m)}(s) = \sigma^{(m)}$, $\pi^{(m)}(t) = \tau^{(m)}$. Let $N$ denote the kernel of the surjective homomorphism $\pi : F_2 \to \mathcal{G}$ defined by $\pi(s) = \sigma$, $\pi(t) = \tau$.

We are going to show that $\{N_m\} \subset \mathcal{N}(2)$ converges to $N \in \mathcal{N}(2)$ in the Cayley topology. Denote by $\ell$ the word metric of $F_2$ with respect to $s, t$. Let $f \in F_2$ be an arbitrary element. It suffices to show that $f$ is in $N$ if and only if $f$ is in $N_m$, for every $m$ greater than $2\ell(f) + 2$.

For an integer $n$, denote by $t_n$ the conjugate $s^n t s^{-n}$. Using integers 
\[ p, n(1), \ldots, n(k) \in \{-\ell(f), -\ell(f) + 1, \ldots, \ell(f)\}, \]
we can uniquely express $f$ as 
\[ f = s^p t_{n(1)} t_{n(2)} \cdots t_{n(k)}. \]

Note that $\pi(t_{n(j)})$ is the transposition $\tau_{n(j)}$ acting on $\mathbb{Z}$ given by 
\[ \tau_{n(j)} : n(j) \mapsto n(j) + 1, \quad n(j) + 1 \mapsto n(j). \]
and that $\pi^{(m)}(t_{n(j)})$ is the transposition $\tau^{(m)}_{n(j)}$ acting on $\mathbb{Z}/m\mathbb{Z}$ defined by 
\[ \tau^{(m)}_{n(j)}([n(j)]) = [n(j) + 1], \quad \tau^{(m)}_{n(j)}([n(j) + 1]) = [n(j)]. \]

Suppose that $f \in N = \text{Ker} (\pi)$ and that $m > 2\ell(f) + 2$. Since $\pi(t_{n(j)})$ does not move $\ell(f) + 2$, we have 
\[ \ell(f) + 2 = [\pi(f)]([\ell(f) + 2]) = [\sigma^p][\ell(f) + 2] = p + \ell(f) + 2. \]
Thus we have $p = 0$. By the expression $f = t_{n(1)} t_{n(2)} \cdots t_{n(k)}$, we also have 
\[ \pi(t_{n(1)} t_{n(2)} \cdots t_{n(k)}) = \tau_{n(1)} \tau_{n(2)} \cdots \tau_{n(k)} = \text{id}. \]
Since \(|n(j)| \leq \ell(f)\) for every \(j\), This means that \(\tau^{(m)}_{n(j)}\) does not move elements of
\[
(Z/mZ) \setminus \{-\ell(f) + mZ, -\ell(f) + 1 + mZ, \ldots, \ell(f) + 1 + mZ\}.
\]
The inequality \(m > 2\ell(f) + 2\) implies that the above set is not empty. It follows that
\[
id = \tau^{(m)}_{t_{n(1)}}, \tau^{(m)}_{n(2)} \cdots \tau^{(m)}_{n(k)} = \pi^{(m)}(t_{n(1)}t_{n(2)} \cdots t_{n(k)}) = \pi^{(m)}(f).
\]

Conversely, suppose that \(f \in N_m = \text{Ker}(\pi^{(m)})\) and the inequality \(m > 2\ell(f) + 2\). Since \([\ell(f) + 2] \in Z/mZ\) is not an element of \([-\ell(f)], \ldots, [\ell(f) + 1]\), we have
\[
[\ell(f) + 2] = [\pi^{(m)}(f)][(\ell(f) + 2)] = [(\sigma^{(m)})^{\ell}(\tau^{(m)}_{n(1)}), \tau^{(m)}_{n(2)} \cdots \tau^{(m)}_{n(k)}]\ (\ell(f) + 2) = [(\sigma^{(m)})^{\ell}([\ell(f) + 2])] = [p + \ell(f) + 2].
\]
Since \(-\ell(f) \leq p \leq \ell(f)\), we have \(p = 0\). By the expression \(f = t_{n(1)}t_{n(2)} \cdots t_{n(k)}\), we also have
\[
\pi^{(m)}(t_{n(1)}t_{n(2)} \cdots t_{n(k)}) = \tau^{(m)}_{t_{n(1)}} \tau^{(m)}_{n(2)} \cdots \tau^{(m)}_{n(k)} = \text{id}.
\]
Recall that \(\tau_{n(j)}\) moves elements \(n(j), n(j) + 1\) of \([-\ell(f)], \ldots, \ell(f) + 1\). The inequality \(m < 2\ell(f) + 2\) implies that the quotient map \(Z \rightarrow Z/mZ\) is injective on the subset \([-\ell(f)], \ldots, \ell(f) + 1\). It follows that
\[
id = \tau_{n(1)} \tau_{n(2)} \cdots \tau_{n(k)} = \pi(t_{n(1)}t_{n(2)} \cdots t_{n(k)}) = \pi(f).
\]
We obtain the claim. \(\square\)

Denote by \(d_m\) the word metric on \(\mathcal{G}_m\) with respect to the generating set \(\{\sigma^{(m)}, \tau^{(m)}\}\).
Introduce a metric \(d\) on the disjoint union \(\bigsqcup_{m=1}^{\infty} \mathcal{G}_m\) as in the paragraph preceding Proposition \(\ref{5.5}\).

**Theorem 6.3.** The coarse disjoint union \(X = (\bigsqcup_{m=1}^{\infty} \mathcal{G}_m, d)\) has property A. In particular, \(X\) coarsely embeds into a Hilbert space.

**Proof.** By Lemma \(\ref{6.2}\) the closure of \(\{(\mathcal{G}_m, \sigma^{(m)}, \tau^{(m)}); m \in Z\}\) with respect to the Cayley topology is \(\{(\mathcal{G}_m, \sigma^{(m)}, \tau^{(m)}); m \in Z\} \cup \{(\mathcal{G}, \sigma, \tau)\}\). The closure consists of amenable groups. By Theorem \(\ref{5.1}\) the coarse space \(\bigsqcup_{m=1}^{\infty} \mathcal{G}_m\) defined in Example \(\ref{4.3}\) has property A. By Proposition \(\ref{5.5}\) the metric space \(\bigsqcup_{m=1}^{\infty} \mathcal{G}_m, d)\) also has property A. \(\square\)

### 6.2. A sequence of special linear groups.

In this subsection and the next subsection, \(m\) stands for odd integers at least 3. Let \(\{k_m\}_{m=3,5,\ldots}\) be a sequence of integers which are at least 2. Define \(G^{(m)}_{k_m} := G^{(m)}\) by \(\text{SL}(m, Z/k_mZ)\). The following
three elements generate $G^{(m)}$.

\[
\sigma^{(m)} = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1
\end{pmatrix}, \quad \tau^{(m)} = \begin{pmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1
\end{pmatrix},
\]
and

\[
v^{(m)} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
1 & 1 & \ddots & \vdots \\
0 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix}.
\]

We denote by $d_m$ the word metric on $\text{SL}(m, \mathbb{Z}/k_m\mathbb{Z})$ with respect to the generators $(\sigma^{(m)}, \tau^{(m)}, v^{(m)})$. Introduce a metric $d$ on $X := X_{\{k_m\}} = \bigcup_{m=1}^{\infty} \text{SL}(m, \mathbb{Z}/k_m\mathbb{Z})$ as in the paragraph preceding Proposition 5.5. We note that $X$ depends on the choices of sequences $\{k_m\}_m$. In Theorem 6.6 we answer the following question — Does $X$ coarsely embed into a Hilbert space?

We often index the columns and the rows by the finite set $\mathbb{Z}/m\mathbb{Z}$. The top-left entries are called $([0])$ matrix coefficients, and the bottom-right entries are called $([m-1], [m-1])$ matrix coefficients. The matrix coefficients are concretely described by the following:

\[
\sigma_{i,j}^{(m)} = \begin{cases}
1, & [i] = [j+1], \\
0, & [i] \neq [j+1],
\end{cases} \quad \tau_{i,j}^{(m)} = \begin{cases}
1, & [i] = [j], \\
1, & ([i], [j]) = ([0], [1]), \\
0, & \text{otherwise},
\end{cases}
\]

\[
v_{i,j}^{(m)} = \begin{cases}
1, & [i] = [j], \\
1, & ([i], [j]) = ([1], [0]), \\
0, & \text{otherwise}.
\end{cases}
\]

Recall that the function

\[\text{Ord}: \mathcal{G}(3) \ni (G, s_1, s_2, s_3) \mapsto (\text{the order of } s_3) \in \{1, 2, \ldots \} \cup \{\infty\}\]

is continuous (Lemma 3.6). Note that $\text{Ord}(\text{SL}(m, \mathbb{Z}/k_m\mathbb{Z}), \sigma^{(m)}, \tau^{(m)}, v^{(m)})$ is $k_m$. For every convergent subsequence of the marked groups $\text{SL}(m(l), \mathbb{Z}/k_{m(l)}\mathbb{Z})$, one of the following statements holds:

- $\lim k_{m(l)} = \infty$,
- there exists a natural number $k$ such that for large enough $l$, $k_{m(l)} = k$.

To study limit points of $\{\text{SL}(m, \mathbb{Z}/k_m\mathbb{Z}), \sigma^{(m)}, \tau^{(m)}, v^{(m)}\}$, we only have to consider the case that $\lim m_k = \infty$ and the case that $k_m = k$.

---

Since $m$ is odd, the determinant of $\sigma^{(m)}$ is 1.
Let $A$ be the unital associative and commutative ring. (We mainly treat the case where $A$ equals $\mathbb{Z}$, $\mathbb{F}_2[t]$, or a finite quotient ring of them.) We define a group $G(\infty, A)$ (as the abstract form) by the semidirect product $\mathbb{Z} \ltimes \text{SL}(\infty, A)$. Here $\text{SL}(\infty, A)$ is the multiplicative group of invertible matrices $g = (g_{i,j})_{i,j \in \mathbb{Z}}$, $g_{i,j} \in A$, with finite support, namely, for sufficiently large $n \in \mathbb{Z}$, $g_{i,j} = \delta_{i,j}$ if $|i|, |j| \geq n$ such that $\det(g) = 1$ (the determinant is well-defined by the finite support condition); and $\mathbb{Z}$ acts on $\text{SL}(\infty, A)$ by the bilateral coordinate shifts. Note that $\text{SL}(\infty, A)$ can be seen as the increasing union of $\text{SL}(2m + 1, A)$, $m \geq 1$ for some appropriate inductive system. In particular, if $A$ is a finite ring, then $\text{SL}(\infty, A)$ can be seen as the increasing union of some finite groups.

Here we shall give more explicit form of $G(\infty, A)$. An infinite matrix $\gamma = (\gamma_{i,j})_{i,j \in \mathbb{Z}}$ is said to be an element of $G(\infty, A)$, if $\gamma_{i,j} \in A$ and if there exist $n \in \mathbb{N}$ and $p \in \mathbb{Z}$ such that

- for $i \in \mathbb{Z}$, if $|i| > n$, then $\gamma_{i,p+i} = 1$,
- for $i, j \in \mathbb{Z}$, if $|i| > n$ and $i \neq j$, then $\gamma_{i,p+j} = 0$,
- for $i, j \in \mathbb{Z}$, if $|i| \leq n$ and $|j| > n$, then $\gamma_{i,p+j} = 0$,
- $\det(\gamma_{i,j})_{-n \leq i,j \leq n} = 1$.

The multiplication on $G(\infty, A)$ is the same as the following operation: $(\gamma \gamma')_{i,j} = \sum_{n \in \mathbb{Z}} \gamma_{i,n} \gamma'_{n,j}$. Note that the correspondence $p : \gamma \mapsto p$ is well-defined and a group homomorphism onto $\mathbb{Z}$, and the kernel is $\text{SL}(\infty, A)$.

In this explicit form, we define three elements $\sigma, \tau, \upsilon$ of $G(\infty, A)$ by

$$
\sigma_{i,j} = \begin{cases} 1, & i = j + 1, \\ 0, & i \neq j + 1 \end{cases}, \quad \tau_{i,j} = \begin{cases} 1, & i = j \text{ or } (i,j) = (0,1), \\ 0, & \text{otherwise} \end{cases}, \quad \upsilon_{i,j} = \begin{cases} 1, & i = j \text{ or } (i,j) = (1,0), \\ 0, & \text{otherwise} \end{cases}.
$$

Note that in the abstract form above, $\sigma$ corresponds to a generator of $\mathbb{Z}$ (in $\mathbb{Z} \ltimes \text{SL}(\infty, A)$), and $\tau, \upsilon$ lies in $\text{SL}(\infty, A)$. We furthermore mention that in the case where $A = \mathbb{Z}$ or $A = \mathbb{Z}/k\mathbb{Z}$, $\sigma, \tau,$ and $\upsilon$ generate the whole $G(\infty, A)$.

The following is straightforward from the abstract form of $G(\infty, A)$ and the fact that the amenability is stable under taking increasing unions and extensions.

**Lemma 6.4.** The group $G(\infty, \mathbb{Z})$ contains a subgroup which is isomorphic to $\text{SL}(3, \mathbb{Z})$. In particular, $G(\infty, \mathbb{Z})$ is not a-T-menable. The group $G(\infty, \mathbb{Z}/k\mathbb{Z})$ is amenable.

The following lemma indicates the Cayley limit groups in the example concerned.

**Lemma 6.5.** Let $\{k_m\}$ be a sequence of natural numbers whose limit is $\infty$. Then the sequence $\{\text{SL}(m, \mathbb{Z}/k_m\mathbb{Z}), \sigma^{(m)}, \tau^{(m)}, \upsilon^{(m)}\}_{m=3,5,\ldots}$ converges to $(G(\infty, \mathbb{Z}), \sigma, \tau, \upsilon)$.

For a natural number $k$, the sequence $\{\text{SL}(m, \mathbb{Z}/k\mathbb{Z}), \sigma^{(m)}, \tau^{(m)}, \upsilon^{(m)}\}_{m=3,5,\ldots}$ converges to $(G(\infty, \mathbb{Z}/k\mathbb{Z}), \sigma, \tau, \upsilon)$.

**Proof.** Consider the case that $\lim_{m \to \infty} k_m = \infty$. Let $s, t, u$ be symbols which generate the free group $F_3$. Denote by $\pi^{(m)}$ the surjective homomorphism from $F_3$ onto $\text{SL}(m, \mathbb{Z}/k_m\mathbb{Z})$ such that $\pi^{(m)}(s) = \sigma^{(m)}, \pi^{(m)}(t) = \tau^{(m)}, \pi^{(m)}(u) = \upsilon^{(m)}$. Denote by $\pi$ the surjective homomorphism from $F_3$ onto $G(\infty, \mathbb{Z})$ such that $\pi(s) = \sigma$, $\pi(t) = \tau$, $\pi(u) = \upsilon$. Then $\pi^{(m)}$ converges to $\pi$ in the $L^\infty$-norm, and hence $\{\text{SL}(m, \mathbb{Z}/k_m\mathbb{Z}), \sigma^{(m)}, \tau^{(m)}, \upsilon^{(m)}\}_{m=3,5,\ldots}$ converges to $(G(\infty, \mathbb{Z}), \sigma, \tau, \upsilon)$.
\[ \pi(t) = \tau, \pi(u) = v. \]
For an integer \( n \), define \( t_n, u_n \in F_3, \tau_n^{(m)}, v_n^{(m)} \in \text{SL}(m, \mathbb{Z}/k_m\mathbb{Z}) \) and \( \tau_n, v_n \in G(\infty, \mathbb{Z}) \) by
\[
\begin{align*}
t_n &= s^nts^{-n}, \quad u_n = s^nu^{-n}, \\
\tau_n^{(m)} &= (\sigma^{(m)})^n\tau^{(m)}(\sigma^{(m)})^{-n}, \quad v_n^{(m)} = (\sigma^{(m)})^n\tau^{(m)}(\sigma^{(m)})^{-n}, \\
\tau_n &= \sigma^n\tau\sigma^{-n}, \quad v_n = \sigma^n\tau\sigma^{-n}.
\end{align*}
\]

Take an arbitrary element \( f \) of \( F_3 \). Let \( \ell \) denote the word length of \( F_3 \). For the element \( f \), there exist integers \( l, p \) and elements \( w_1, \ldots, w_l \in \{ t_n; -\ell(f) \leq n \leq \ell(f) \} \cup \{ u_n; -\ell(f) \leq n \leq \ell(f) \} \) such that
\[
f = s^pw_1 \cdots w_l, \quad 0 \leq l \leq \ell(f), \quad |p| \leq \ell(f).
\]
Suppose that \( \pi(f) = \text{id} \). Take an arbitrary natural number \( m \) satisfying that \( m > 2\ell(f) + 2 \). Since \( \rho \circ \pi(w_j) = 0 \), we have
\[
0 = \rho \circ \pi(f) = \rho \circ \pi(s^p) = p.
\]
We also have
\[
id = \pi(f) = \pi(w_1) \cdots \pi(w_l).
\]
Since \( \pi(w_j) \) is an element of \( \{ \tau_n^{(m)}; -\ell(f) \leq n \leq \ell(f) \} \cup \{ v_n^{(m)}; -\ell(f) \leq n \leq \ell(f) \} \), we may regard \( \pi(w_j) \) as a matrix located on \( \{ -\ell(f), -\ell(f) + 1, \ldots, \ell(f), \ell(f) + 1 \} \). Rearrange the indices by
\[
(\{[0], [1], \ldots, [\ell(f) + 1]\} \cup \{[-\ell(f)], \ldots, [-1]\})^2 \subset (\mathbb{Z}/m\mathbb{Z})^2,
\]
via the quotient map \([\cdot]: \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}\). By the inequality \( m < 2\ell(f) + 2 \), the subsets \( \{[0], [1], \ldots, [\ell(f) + 1]\} \) and \( \{[-\ell(f)], \ldots, [-1]\} \) are disjoint. We may regard the elements of \( \text{SL}(m, \mathbb{Z}/k_m\mathbb{Z}) \) as matrices located on \( (\mathbb{Z}/m\mathbb{Z})^2 \). By taking quotient of the equation (1) with respect to the ideal \( k_m\mathbb{Z} \) of \( \mathbb{Z} \), we have
\[
id = \pi^{(m)}(w_1) \cdots \pi^{(m)}(w_l) = \pi^{(m)}(f).
\]
Conversely, suppose that there exists a natural number \( m \) such that \( \pi^{(m)}(f) = \text{id}, m > 2\ell(f) + 2, k_m > 2\ell(f) \). Note that \( \pi^{(m)}(f) \) is expressed by
\[
\pi^{(m)}(f) = (\sigma^{(m)})^p \pi^{(m)}(w_1) \cdots \pi^{(m)}(w_l).
\]
Since \( \pi^{(m)}(w_j) \) is an element of \( \{ \tau_n^{(m)}; -\ell(f) \leq n \leq \ell(f) \} \cup \{ v_n^{(m)}; -\ell(f) \leq n \leq \ell(f) \} \), we may regard \( \pi^{(m)}(w_j) \) as a matrix located on
\[
(\{[0], [1], \ldots, [\ell(f) + 1]\} \cup \{[-\ell(f)], \ldots, [-1]\})^2 \subset (\mathbb{Z}/m\mathbb{Z})^2
\]
Since \( m > 2\ell(f) + 2 \), the \((\ell(f) + 2)\)-nd column of \( \pi^{(m)}(f) \) is 1 at the \((p + \ell(f) + 2)\)-nd row. Since \( 0 \leq p \leq \ell(f) \), we have \( p = 0 \). It follows that
\[
id = \pi^{(m)}(w_1) \cdots \pi^{(m)}(w_l).
\]
Since \( 2 \ell(f) < k_m \), while we compute \( \pi(w_1) \cdots \pi(w_l) \), coefficients do not exceed the interval \( \{-k_m, \ldots, k_m\} \). Therefore we have
\[
id = \pi(w_1) \cdots \pi(w_l) = \pi(f).
\]
Therefore the marked groups \((\mathrm{SL}(m, \mathbb{Z}/k_m\mathbb{Z}), \sigma^{(m)}, \tau^{(m)}, v^{(m)})\) converges to \((G(\infty, \mathbb{Z}), \sigma, \tau, v)\). We can show the second statement by a similar technique.

**Theorem 6.6.** For the coarse disjoint union \(X = X_{\{k_m\}} = \bigsqcup_{m=3,5,\ldots} \mathrm{SL}(m, \mathbb{Z}/k_m\mathbb{Z})\), the following statements hold:

- \(X\) has property A if and only if \(\sup_m k_m < \infty\).
- \(X\) does not coarsely embed into a Hilbert space if and only if \(\sup_m k_m = \infty\).

**Proof.** Consider the case that the sequence \(\{k_m\}\) does not have an upper bound. Take a subsequence \(m(l)\) such that \(\lim_l m(l) = \infty\). By the first statement of Lemma 6.5, the limit of
\[
\{(\mathrm{SL}(m, \mathbb{Z}/k_{m(l)}\mathbb{Z}), \sigma^{(m)}, \tau^{(m)}, v^{(m)})\}_{m=l}^{\infty}
\]
is \((G(\infty, \mathbb{Z}), \sigma, \tau, v)\). By the first statement of Lemma 6.4, this group is not \(\alpha\)-T-menable. By Theorem 5.7, the coarse disjoint union \(X\) does not embed into a Hilbert space.

Consider the case that the sequence \(\{k_m\}\) has an upper bound \(K\). By the second assertion of Lemma 6.9, an accumulation point of
\[
\{(\mathrm{SL}(m, \mathbb{Z}/k_m\mathbb{Z}), \sigma^{(m)}, \tau^{(m)}, v^{(m)})\}_{m=1}^{\infty}
\]
is of the form
\[
(G(\infty, \mathbb{Z}/k\mathbb{Z}), \sigma, \tau, v), \quad 0 \leq k \leq K.
\]
By the second assertion of Lemma 6.4, this is an amenable group. By Theorem 5.1, the coarse disjoint union \(X\) has property A. By Proposition 5.3, the metric space \((\bigsqcup_{m=1}^{\infty} \mathrm{SL}(m, \mathbb{Z}/k_m\mathbb{Z}), d)\) also has property A. \(\square\)

6.3. Second example by special linear groups. For an odd natural number \(m\), \(\mathrm{SL}(m, \mathbb{Z}/k_m\mathbb{Z})\) is generated by \(\sigma^{(m)}\) and \(\tau^{(m)}\). The sequence of the marked group \(\{(\mathrm{SL}(m, \mathbb{Z}/k_m\mathbb{Z}), \sigma^{(m)}, \tau^{(m)})\}_{m=3,5,\ldots}\) always has property A.

**Theorem 6.7.** For every sequence of integers \(\{k_m \geq 2\}_{m=3,5,\ldots}\), the coarse disjoint union \(\bigsqcup_{m=3,5,\ldots} \mathrm{SL}(m, \mathbb{Z}/k_m\mathbb{Z})\) with respect to the generators \(\{\sigma^{(m)}, \tau^{(m)}\}_{m=3,5,\ldots}\) has property A.

As in the previous subsection, we exploit Theorem 5.1. Notice that the order of \(\tau^{(m)}\) is \(k_m\). Recall that the function \(\text{Ord}\) in Lemma 3.6 is continuous. In order to observe limits with respect to the Cayley topology, we only have to consider the case that \(\lim_m k_m = \infty\) and the case that there exists an integer \(k\) such that \(k_m = k\) for every \(m\).

Let \(A\) be the unital associative and commutative ring. We define a subgroup \(H(\infty, A)\) of \(G(\infty, A)\) as follows. In an abstract form, \(H(\infty, A) := \mathbb{Z} \rtimes UT(\infty, A)\), where \(UT(\infty, A)\) denotes the subgroup of \(\mathrm{SL}(\infty, A)\) consisting of upper triangular matrices (with finite support) with 1 on all diagonal entries. The \(\mathbb{Z}\)-action preserves \(U(\infty, A)\).

An explicit form of \(H(\infty, A)\) is the following: An infinite matrix \(\gamma = (\gamma_{i,j})_{i,j \in \mathbb{Z}}\) is said to be an element of \(H(\infty, A)\), if \(\gamma_{i,j} \in A\) and if there exist \(n \in \mathbb{N}\) and \(p \in \mathbb{Z}\) such that
• for $i \in \mathbb{Z}$, $\gamma_{i,p+i} = 1$,
• for $i, j \in \mathbb{Z}$, if $i > j$, then $\gamma_{i,p+j} = 0$,
• for $i, j \in \mathbb{Z}$, if $\gamma_{i,p+j} \neq 0$ and $i \neq j$, then $|i| \leq n$ and $|j| \leq n$.

Note that the correspondence $\rho : \gamma \mapsto p$ is well-defined and a group homomorphism onto $\mathbb{Z}$, and that the kernel of that is $UT(\infty, A)$. The elements $\sigma$ and $\tau$ defined in the previous subsection are elements of $H(\infty, A)$.

The first part of the following lemma is straightforward, and the second part follows from the observation that $UT(\infty, A)$ is the increasing union of some nilpotent subgroups and is hence amenable.

**Lemma 6.8.** For $A = \mathbb{Z}$ or $A = \mathbb{Z}/k\mathbb{Z}$, the group $H(\infty, A)$ is generated by $\sigma$ and $\tau$. The group $H(\infty, A)$ is amenable.

Let $F_2$ denote the free group with two generators $s$, $t$. Denote by $\varpi^{(m)}$ the surjective homomorphism from $F_2$ onto $SL(m, \mathbb{Z}/k_m\mathbb{Z})$ such that $\varpi^{(m)}(s) = \sigma^{(m)}$, $\varpi^{(m)}(t) = \tau^{(m)}$. Denote by $\varpi$ the surjective homomorphism from $F_2$ onto $H(\infty, \mathbb{Z})$ such that $\varpi(s) = \sigma$, $\varpi(t) = \tau$.

**Lemma 6.9.** Let $\{k_m\}$ be a sequence of natural numbers whose limit is $\infty$. Then the sequence $\{(SL(m, \mathbb{Z}/k_m\mathbb{Z}), \sigma^{(m)}, \tau^{(m)})\}_{m=3,5,\ldots}$ converges to $(H(\infty, \mathbb{Z}), \sigma, \tau)$.

For a natural number $k$, the sequence $\{(SL(m, \mathbb{Z}/k\mathbb{Z}), \sigma^{(m)}, \tau^{(m)})\}_{m=3,5,\ldots}$ converges to $(H(\infty, \mathbb{Z}/k\mathbb{Z}), \sigma, \tau)$.

**Proof.** The proof is almost identical to that of Lemma 6.5. \hfill \Box

It follows that all the accumulation points of $\{(SL(m, \mathbb{Z}/k_m\mathbb{Z}), \sigma^{(m)}, \tau^{(m)})\}_{m=3,5,\ldots}$ consist of amenable groups. Theorem 5.1 implies Theorem 6.7. \hfill \Box

### 6.4. More examples of special linear groups.

Here we consider more examples of sequences of linear groups, whose coefficient rings are not quotient rings of $\mathbb{Z}$, but of $\mathbb{F}_p[t]$, for $p$ prime. For simplicity, we let $p = 2$. For an odd natural number $m$, $SL(m, \mathbb{F}_2[t]/(t^{k_m}))$ is generated by $\sigma^{(m)}$, $\tau^{(m)}$, $\tau'^{(m)}$, $\upsilon^{(m)}$, and $\upsilon'^{(m)}$. Here $\sigma^{(m)}$, $\tau^{(m)}$, and $\upsilon^{(m)}$ are defined in the same way as in Subsection 6.2, and the other two are defined by

$$
\tau^{(m)} = \begin{pmatrix}
1 & t & 0 & \cdots & 0 \\
0 & 1 & 0 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}, \quad \upsilon^{(m)} = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
t & 1 & \ddots & \ddots & \\
0 & 0 & \ddots & \ddots & \\
\vdots & \ddots & \ddots & 0 & \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}
$$

Observe that $\sigma^{(m)}$, $\tau^{(m)}$, $\tau'^{(m)}$ generate $SL(m, \mathbb{F}_2[t]/(t^{k_m}))$. In this way we consider two sequences of finite marked groups, parameterized by $\{k_m\}$, $\{(SL(m, \mathbb{F}_2[t]/(t^{k_m})); \sigma^{(m)}, \tau^{(m)}, \tau^{(m)}, \upsilon^{(m)}, \upsilon'^{(m)})\}_{m}$ and $\{(SL(m, \mathbb{F}_2[t]/(t^{k_m})); \sigma^{(m)}, \tau^{(m)}, \tau^{(m)})\}_{m}$.

Similar arguments to these in Subsections 6.2 and 6.3 show the following theorem.

**Theorem 6.10.** The following hold true.
For the coarse disjoint union
\[ \bigsqcup_{m=3,5,\ldots} (\text{SL}(m, \mathbb{F}_2[t]/(t^{k_m})); \sigma^{(m)}, \tau^{(m)}, \upsilon^{(m)}, \nu^{(m)}), \]
the following statements hold:

- It has property A if and only if \( \sup_m k_m < \infty \).
- It does not coarsely embed into a Hilbert space if and only if \( \sup_m k_m = \infty \).

For every sequence of integers \( \{k_m \geq 2\}_{m=3,5,\ldots} \), the coarse disjoint union
\[ \bigsqcup_{m=3,5,\ldots} (\text{SL}(m, \mathbb{F}_2[t]/(t^{k_m})); \sigma^{(m)}, \tau^{(m)}) \]
has property A.

7. COARSE DISJOINT UNION WITH PROPERTY A WITH VERY SMALL COMPRESSION FUNCTION

In this section, we use freely the symbol \( a \preceq b \) for two nonnegative functions form the same parameter set \( \mathcal{P} \) if there exists positive multiplicative constant \( C > 0 \) independent of \( p \in \mathcal{P} \), \( a(p) \leq C b(p) \) holds true. We use the symbol \( a \asymp b \) if \( a \preceq b \) and \( a \succeq b \) hold. We use the symbol \( a \preceq_Y b \) if the parameter set \( \mathcal{P} \) has the variable \( Y \) and the positive multiplicative constant \( C = C_Y \) may depend on the choice of \( Y \). We write \( a \preceq b \) if \( a \preceq b \) holds but \( a \asymp b \) fails to be true.

7.1. Preliminaries and results. In the previous section we observe that behaviors of coarse disjoint unions of \( \{\text{SL}(m, \mathbb{Z}/k_m \mathbb{Z})\}_m \) (or \( \{\text{SL}(m, \mathbb{F}_2[t]/(t^{k_m}))\}_m \)) heavily depend on the choices of system of generators. In this section, we study compression functions of them when they have property A, in terms of the choices of the parameters \( \{k_m\} \). We give the definition of compression function between metric spaces.

**Definition 7.1.** Let \((X, d_X)\) and \((Y, d_Y)\) be (infinite) metric spaces. Let \( \rho: \mathbb{R}_+ \to \mathbb{R}_+ \) be a map with \( \lim_{t \to +\infty} \rho(t) = +\infty \). We say that \( \rho \) is a compression function of \( X \) into \( Y \) if there exists a coarse embedding \( f: X \to Y \) such that
\[ \rho(d_X(x_1, x_2)) \preceq d_Y(f(x_1), f(x_2)), \quad x_1, x_2 \in X \]
holds true.

We denote by \( \mathcal{R}_Y(X) \) the class of all compression functions of \( X \) into \( Y \).

We note that \( \mathcal{R}_Y(X) \neq \emptyset \) if and only if \( X \) is coarsely embeddable into \( Y \) (see Definition 4.9). Also note that because we allow positive constant multiplication in the inequality above (recall the definition of \( \preceq \)), compression functions only tell growth orders of lower bounds of coarse embeddings.

In this paper, we consider the case where \( X \) is a coarse disjoint union of finite Cayley graphs and \( Y \) is a Banach space. We need certain classes of Banach spaces to state our result.

**Definition 7.2.** Let \( Y \) be a Banach space. We denote by \( S(Y) \) the unit sphere of \( Y \).
We refer the readers to [BL00] for comprehensive treatment for the uniform convexity, the sphere equivalence, types and cotypes on Banach spaces. We note that every Banach space \( Y \) has type 1 and cotype \( \infty \), and that if \( Y \) has type \( p \) and cotype \( q \), then it has type \( p' \in [1, p] \) and cotype \( q' \in [q, \infty) \). For basics on complex interpolation, we refer to [BL76] and [BL00]. We mention that \( l_p \)-spaces, \( L_p \)-spaces, and noncommutative \( L_p \)-spaces are sphere equivalent to a Hilbert space for any
$p \in [1, \infty)$. Unless $p = 1$, they are superreflexive and also $\theta$-Hilbertian ($\theta$ depends on $p$). We also note that any superreflexive Banach space with unconditional basis (that means, convergences are unconditional with respect to this basis) and its quotient are also sphere equivalent to a Hilbert space. It is known (see [Pis10]) that any subspace of a quotient of an ultrapower of a $\theta$-Hilbertian space is uniformly curved. All uniformly curved Banach spaces are superreflexive. All superreflexive Banach spaces have type $> 1$. Finally we note that class $\mathcal{E}_4$ and the class of Banach spaces of type $> 1$ are respectively closed under taking $2$-Bochner integrable spaces, namely, the operation of constructing $L_2([0,1], Y)$ from $Y$. In particular, these two classes are respectively closed under taking $\ell_2$-direct sum.

The following assertions are our main results in this section. The reason why for $Y \in \mathcal{E}_4$ the theorem applies is due to de la Salle’s result [dlS13].

**Theorem 7.3.** Let $X_Z := X_{Z_{\{k_m\}}} := \bigsqcup_{m=3,5,\ldots} (\text{SL}(m, \mathbb{Z}/k_m\mathbb{Z}); \sigma^{(m)}, \tau^{(m)}))$. Let $\rho: \mathbb{R}_+ \nearrow \mathbb{R}_+$ be a map which satisfies $\lim_{t \to +\infty} \rho(t) = +\infty$. Then there exists a sequence $\{k_m\}$ of positive integers, may be explicitly written in terms of $\rho$, such that for any Banach space $Y$ in the class indicated below,

$$\rho \not\in \mathcal{R}_Y (X_Z) := \mathcal{R}_Y (X_{Z_{\{k_m\}}})$$

holds true. Here the class of $Y$ is the class consisting of all Banach spaces which are sphere equivalent to some uniformly curved Banach spaces or some elements in $\mathcal{E}_4$.

In particular, the class of $Y$ to which this theorem applies contains any noncommutative $L_p$-space for $p \in [1, \infty)$, a quotient space of an ultrapower of a $\theta$-Hilbertian space, a quotient space of a superreflexive Banach space with unconditional basis, and a Banach space sphere equivalent to a Banach space of type $p$ and cotype $q$ with $1/p - 1/q < 1/4$.

If we consider group quotients of $\{\text{SL}(m, \mathbb{F}_2[t])\}_m$, then we gain more. This result employs V. Lafforgue’s results [Laf08], [Laf09].

**Theorem 7.4.** Let $X_{\mathbb{F}_2[t]} := X_{\mathbb{F}_2[t]_{\{k_m\}}} := \bigsqcup_{m=3,5,\ldots} (\text{SL}(m, \mathbb{F}_2[t]/(t^{k_m})); \sigma^{(m)} \tau^{(m)}, \tau^{(m)})).$ Let $\rho: \mathbb{R}_+ \nearrow \mathbb{R}_+$ be a map which satisfies $\lim_{t \to +\infty} \rho(t) = +\infty$. Then there exists a sequence $\{k_m\}$ of positive integers, may be explicitly written in terms of $\rho$, such that for any Banach space $Y$ in the class indicated below,

$$\rho \not\in \mathcal{R}_Y (X_{\mathbb{F}_2[t]}) := \mathcal{R}_Y (X_{\mathbb{F}_2[t]_{\{k_m\}}})$$

holds true. Here the class of $Y$ is the class consisting of all Banach spaces which are sphere equivalent to some Banach spaces of type $> 1$.

In particular, the class of $Y$ to which this theorem applies contains Banach spaces sphere equivalent to uniformly convex Banach spaces.

We note that by Theorems 6.7 and 6.10, these coarse disjoint unions enjoy property $A$, and hence admit coarse embeddings into a Hilbert space and into $\ell_2(\mathbb{N}, Z)$ for any Banach space $Z$. We emphasize that in Theorems 7.3 and 7.4, $\{k_m\}$ is fixed before we let $Y$ move over the respectively indicated class.
7.2. Decay of Banach spectral gaps and distortions. For the proof of Theorems 7.3 and 7.4, we shall employ concepts of Banach spectral gaps and distortions of finite graphs.

Definition 7.5. Let $G = (V, E)$ be a finite connected (undirected) graphs, where $V$ is the vertex set and $E$ is the oriented edge set. We denote by $d(G)$ the maximal degree of $G$. We view $G$ as a metric space equipped with the path metric and denote by $\text{diam}(G)$ the diameter of $G$. Let $Y$ be a Banach space.

1. (See Definition 1.1 in [Mim13].) The Banach 2-spectral gap (or, the Banach spectral gap) of $G$ with target in $Y$, denoted by $\lambda_1(G; Y) := \lambda_1(G; Y, 2)$, is defined by the following formula:

$$\lambda_1(G; Y) := \lambda_1(G; Y, 2) := \frac{1}{2} \inf_{f : V \to Y} \frac{\sum_{v \in V} \sum_{e \in E : e = (v, w)} \|f(w) - f(v)\|^2}{\sum_{v \in V} \|f(v) - m(f)\|^2}.$$ 

Here $f$ moves among all nonconstant maps $V \to Y$, and $m(f)$ denotes the mean $\sum_{v \in V} f(v)/|V|$ of $f$.

2. The distortion of $G$ with target in $Y$, denoted by $c_Y(G)$, is defined by the following formula:

$$c_Y(G) := \inf_{f : V \to X, \text{biLipschitz}} \|f\|_{\text{Lip}}\|f^{-1}\|_{\text{Lip}}.$$ 

Here $\|\cdot\|_{\text{Lip}}$ means the Lipschitz constant.

It is known that $1 \leq c_Y(G) \leq \text{diam}(G)$ for $\tilde{Y} = \ell_2(N, Y)$. (The latter inequalities is due to J. Bourgain.) The following two results enables us to bound the distortion from below by the Banach spectral gap, and to bound compression function of a coarse disjoint from above by the order of distortions of a sequence of finite graphs.

Theorem 7.6. (Special case of the generalized Jolissaint–Valette inequality, see [JV11] and Theorem 2.6 in [Mim13])

Let $G = (V, E)$ be a finite connected Cayley graph. Then for any Banach space $Y$, we have that

$$c_Y(G) \geq \text{diam}(G) \sqrt{\frac{\lambda_1(G; Y)}{2d(G)}}.$$ 

In particular, $c_Y(G) \geq d(G) \lambda_1(G; Y)^{\frac{1}{2}} \cdot \text{diam}(G)$.

A generalized version of the following lemma is used by T. Austin [Aus11] to find an amenable group with the Hilbert compression exponent 0. (The compression exponent is the supremum of $\alpha \in [0, 1]$ such that $\rho(t) = t^\alpha$ is a compression function.)

Lemma 7.7. Let $\{G(m)\}_m$ be a sequence of finite connected graphs with $\text{diam}(G(m)) \not\to \infty$. Let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be a map with $\lim_{t \to +\infty} \rho(t) = +\infty$ which satisfies that $\rho(t)/t$ is nonincreasing on $t$ large enough. Let $Y$ be a Banach space. If for $m$ large enough

$$\frac{\text{diam}(G(m))}{\rho(\text{diam}(G(m)))} \leq c_Y(G(m))$$

then $\lim_{m \to \infty} \rho(\text{diam}(G(m))) = +\infty$. Otherwise, $\rho(\text{diam}(G(m))) \not\to +\infty$.
holds, then we have that
\[ \rho \not\in \mathcal{R}_Y \left( \bigsqcup_m G^{(m)} \right). \]

For the convenience of the readers, we let the proof of Lemma 7.7 be included.

**Proof of Lemma 7.7.** Suppose in contrary that there exists a coarse embedding
\[ f : \bigsqcup_m G^{(m)} \rightarrow Y \]
such that
\[ \rho(d(v, w)) \preceq \| f(v) - f(w) \|, \quad v, w \in \bigsqcup_m G^{(m)} \]
holds. Set \( f_m := f \mid_{G^{(m)}} : V_m \rightarrow Y \). We may assume that \( f_m \) is a 1-Lipschitz map and that each \( f_m \) is biLipschitz. Then for \( m \) sufficiently large, we have the following order inequalities.

\[ \frac{\text{diam}(G^{(m)})}{\rho(\text{diam}(G^{(m)}))} \preceq c_Y(G^{(m)}) \leq \| f_m^{-1} \|_{\text{Lip}} \leq \max_{v \neq w \in V_m} \frac{d(v, w)}{\| f_m(v) - f_m(w) \|} \]

\[ \preceq \max_{v \neq w \in V_m} \frac{d(v, w)}{\rho(d(v, w))} \leq \frac{\text{diam}(G^{(m)})}{\rho(\text{diam}(G^{(m)}))}. \]

This is a contradiction. \( \square \)

### 7.3. Proofs of the results modulo analysis of Banach spectral gaps

The final keys to the proofs of Theorems 7.3 and 7.4 are the following spectral order inequalities.

**Theorem 7.8.** Let \( \{k_m\}_m \) be a sequence of positive integers. Let \( Y \) be a Banach space which is sphere equivalent to a uniformly curved Banach space. Then we have

\[ \lambda_1(G^{(m)}_Z, Y) \succsim_Y m^{-4}. \]

Here \( G^{(m)}_Z := G^{(m)}_{Z,k_m} := \text{Cay}(\text{SL}(m, \mathbb{Z}/k_m \mathbb{Z}); \sigma^{(m)}, \tau^{(m)})). \)

(This asserts that the positive multiplicative constant in the symbol “\( \succsim \)” may depend on \( Y \), but does not depend on \( \{k_m\}_m \).)

**Theorem 7.9.** We stick the notation in Theorem 7.8

(i) Let \( Y \) be a Banach space sphere equivalent to a member in \( \mathcal{E}_A \). Then we have that

\[ \lambda_1(G^{(m)}_Z, Y) := \lambda_1(G^{(m)}_{Z,k_m}, Y) \succsim_Y m^{-6}. \]

(ii) Let \( Y \) be a Banach space sphere equivalent to a Banach space of type \( > 1 \). Then we have that

\[ \lambda_1(G^{(m)}_{F_2[t]}, Y) := \lambda_1(G^{(m)}_{F_2[t],k_m}, Y) \succsim_Y m^{-6}. \]

Here \( G^{(m)}_{F_2[t]} := G^{(m)}_{F_2[t],k_m} := \text{Cay}(\text{SL}(m, \mathbb{F}_2[t]/(t^k_m)); \sigma^{(m)}, \tau^{(m)})). \)

We shall postpone the proofs of Theorems 7.8 and 7.9 to the next two subsections.
Proof. (Theorems 7.3 and 7.4 modulo Theorems 7.8 and 7.9)

If necessary by replacing $\rho$ with smaller function, we may assume that $\rho(t)/t$ is nonincreasing on sufficiently large $t$. Then Lemma 7.7 together with the following diameter estimate ends our proof: for instance, for $G_{Z,km}^{(m)}$, by the at most exponential growth it is straightforward to show that

$$\text{diam}(G_{Z,km}^{(m)}) \gtrsim \log |G_{Z,km}^{(m)}| \asymp m^2 \log km.$$  

(In fact, Kassabov–Riley [KR07] show that $\text{diam}(G_{Z,km}^{(m)}) \asymp m^2 \log km$.) □

For instance, if we take $\rho(t) = \log \log t \lor 0$, then

$$k_m \gtrsim e^{e^{Cm^3}}$$

for some $C > 0$ does the job for $X_Z$.  

7.4. Proof of Theorem 7.8. First, we relate the Banach spectral gap to some representation theoretic quantity.

Definition 7.10. Let $Y$ be a Banach space. Take the $\ell_2$-stabilization of $Y$, namely, $\tilde{Y} = \ell_2(\mathbb{N}, Y)$.

(i) Let $G$ be a finite group and $S$ be a symmetric generating set.

Let $\pi_Y = \pi_{Y,G}$ be the left regular representation of $G$ on $\ell_2(G, \tilde{Y})$, namely, we define for $\xi \in \ell_2(G, \tilde{Y})$ and $g \in G$,

$$\pi_Y(g)\xi(x) := \xi(g^{-1}x).$$

Then the representation space $\ell_2(G, \tilde{Y})$ decomposes into the following two $G$-representation subspaces

$$\ell_2(G, \tilde{Y}) = \ell_2(G, \tilde{Y})^{\pi_Y(G)} \oplus \ell_2(G, \tilde{Y})_0.$$

Here $\ell_2(G, \tilde{Y})^{\pi_Y(G)}$ is the space of all $\pi_Y(G)$-invariant vectors (in other words, the space of constant functions on $G$ to $\tilde{Y}$), and $\ell_2(G, \tilde{Y})_0$ is the canonical complement of it, more precisely, the space of functions with zero sum, defined as follows:

$$\ell_2(G, \tilde{Y})_0 := \left\{ \xi \in \ell_2(G, \tilde{Y}); \sum_{x \in G} \xi(x) = 0 \right\}.$$

We use the same symbol $\pi_Y$ for the restriction of $\pi_Y$ on $\ell_2(G, \tilde{Y})_0$.

(ii) Let $\Gamma$ be an infinite and finitely generated group (with nontrivial finite group quotient) and $S$ be a symmetric finite generating set.

(1) (Definition 3.2 in [Mim13]) The 2-(\tau)-type constant for $(\Gamma, S)$ on $Y$, denoted by $\kappa_Y^{(\tau)}(\Gamma, S)$, is defined as follows.

$$\kappa_Y^{(\tau)}(\Gamma, S) := \inf_G \kappa_Y(G, S).$$
Here $G$ moves among all nontrivial finite group quotients of $\Gamma$.

(2) The Kazhdan constant for $(\Gamma, S)$, denoted by $\mathcal{K}(\Gamma, S)$, is defined as follows.

$$\mathcal{K}(\Gamma, S) := \inf_{(\pi, \mathcal{H}) \neq \{\pi\}_{\mathcal{H}(\tau)}} \inf_{\xi \in \mathcal{H}(\tau)} \sup_{s \in S} \frac{\|\xi - \pi(s)\xi\|}{\|\xi\|}.$$ 

Here $(\pi, \mathcal{H})$ runs over all nontrivial unitary representations of $\Gamma$.

In what follows, we simply call $\kappa_Y^{(\tau)}(\Gamma, S)$ the $(\tau)$-type constant for $(\Gamma, S)$ on $Y$. If $Y$ is an infinite dimensional Hilbert space $\ell_2$, then $\kappa_Y^{(\tau)}(\Gamma, S) \geq \mathcal{K}(\Gamma, S)$ holds.

The following lemma follows from the definition.

**Lemma 7.11.** Let $G$ be a finite group and $S_1$ and $S_2$ be two symmetric finite generating sets. Let $Y$ be a Banach space. For $q \in \mathbb{N}$ such that $(S_1)^q \supset S_2$, we have that

$$\kappa_Y(G, S_1) \geq \frac{1}{q} \kappa_Y(G, S_2).$$

If $\Gamma = G$ is an infinite group in the setting above, then we have that

$$\kappa_Y^{(\tau)}(\Gamma, S_1) \geq \frac{1}{q} \kappa_Y^{(\tau)}(\Gamma, S_2).$$

Although $\kappa_Y^{(\tau)}$ heavily depends on the choice of symmetric finite generating set $S$, the lemma above says that the property of having strictly positive $(\tau)$-type constant on $Y$ is independent of the choice.

We also need relative version of these notions in the next subsection.

**Definition 7.12.** (i) Let $G$ be a finite group, $S$ be a symmetric generating set, and $L \lhd G$ be a normal subgroup. Then the $G$-representation space $\ell_2(G, \bar{Y})$ for $\pi_Y(G)$ decomposes as a $G$-representation space into

$$\ell_2(G, \bar{Y}) = \ell_2(G, \bar{Y})^{\pi_Y(L)} \oplus \ell_2(G, \bar{Y})_{L,0},$$

Here $\ell_2(G, \bar{Y})^{\pi_Y(L)}$ is the space of $\pi_Y(L)$-invariant vectors and $\ell_2(G, \bar{Y})_{L,0}$ is the canonical complement of it, more precisely, the space of all functions that have zero-sum on each coset of $L$. We define the relative displacement constant $\kappa_Y(G, L, S)$ for $\pi_Y$ of $(G, L, S)$ as follows.

$$\kappa_Y(G, L, S) := \inf_{0 \neq \xi \in \ell_2(G, \bar{Y})_{L,0}} \sup_{s \in S} \frac{\|\xi - \pi_Y(s)\xi\|}{\|\xi\|}.$$ 

(ii) Let $\Gamma$ be an infinite and finitely generated group (with nontrivial finite group quotient), $S$ be a symmetric finite generating set, and $\Lambda \lhd \Gamma$ be a normal subgroup of $\Gamma$. The relative $2-$$(\tau)$-type constant for $(\Gamma, \Lambda, S)$ on $Y$, denoted by $\kappa_Y^{(\tau)}(\Gamma, \Lambda, S)$, is defined as follows.

$$\kappa_Y^{(\tau)}(\Gamma, \Lambda, S) := \inf_{G} \kappa_Y(G, G(\Lambda), S).$$

Here $G$ moves among all nontrivial finite group quotients of $\Gamma$, and $G(\Lambda) \lhd G$ denote the relative quotient of $\Lambda \lhd \Gamma$. 
In what follows, we simply call \( \kappa_Y^{(r)}(\Gamma, \Lambda, S) \) the relative \((\tau)\)-type constant for \((\Gamma, \Lambda, S)\) on \(Y\).

Note that in the setting above as a \(G\)-representation space, we have
\[
\ell_2(G, \tilde{Y}) \cong \ell_2(G/L, \ell_2(L, \tilde{Y}))
\]
and that \(\ell_2(L, \tilde{Y})\) is isomorphic to \(\tilde{Y}\) because we take the \(\ell_2\)-stabilization \(\tilde{Y}\) of \(Y\). Also we mention that by definition \(\kappa_Y^{(r)}(\Gamma, \Gamma, S) = \kappa_Y^{(r)}(\Gamma, S)\).

We observe the following norm estimate.

**Lemma 7.13.** We stick the notation in item (i) of Definition 7.12.

1. If we write \(\xi \in \ell_2(G, \tilde{Y})\) as \(\xi = \xi_1 + \xi_0\) according to the decomposition\n\[
\ell_2(G, \tilde{Y}) = \ell_2(G, \tilde{Y})^{\pi_Y(L)} \oplus \ell_2(G, \tilde{Y})_{L,0},
\]
then we have \(\|\xi_1\| \leq \|\xi\|\) and \(\|\xi_0\| \leq 2\|\xi\|\).

2. For any \(0 \neq \eta \in \ell_2(G, \tilde{Y})_0 = \ell_2(G, \tilde{Y})_{G,0}\), we have that\n\[
\sup_{\gamma \in G} \|\eta - \pi_Y(\gamma)\eta\| > \|\eta\|.
\]

Indeed, the former inequality in item (1) follows from the fact that the norm of the mean is at most the mean of the norms, and this implies the latter one. Item (2) can be showed by observing that otherwise the mean of \(\{\pi_Y(\gamma)\eta\}_{\gamma \in G}\) would be a non-zero \(\pi_Y(G)\)-invariant vector in \(\ell_2(G, \tilde{Y})_0\), which is a contradiction.

The following lemma is showed in a standard argument. For the proof, we refer to Lemma 3.3 in [Mim13].

**Lemma 7.14.** Let \(G\) be a finite group and \(S\) be a symmetric generating set. Then for any Banach space \(Y\), we have that
\[
\lambda_1(Cay(G, S); Y) = \lambda_1(Cay(G, S); \tilde{Y}) \asymp_{|S|} \kappa_Y(G, S)^2.
\]

In particular, for an infinite group and its symmetric finite generating set \((\Gamma, S)\), we have that
\[
\inf_G \lambda_1(Cay(G, S); Y) \asymp_{|S|} \kappa_Y^{(r)}(\Gamma, S)^2,
\]
where \(G\) moves over all nontrivial finite group quotients of \(\Gamma\).

We also need the following two facts. On the former one we refer to Section 3 of [Pis10]. For a Banach space \(Y\), we call a sequence of finite graphs \(\{G^{(m)}\}\) a family of \((Y, 2)\)-anders if they have uniformly bounded degree and \(\inf_m \lambda_1(G^{(m)}; Y) > 0\) holds true. (The “2” comes from that \(\lambda_1(G; Y) = \lambda_1(G; Y, 2)\) is defined in terms of square sums.) The notion of expanders in usual sense is identical to being \((\ell_2, 2)\)-anders.

**Proposition 7.15** (Pisier [Pis10]). Let \(Z\) be a uniformly curved Banach space. Then for any finite graph \(G\), there exists \(\epsilon > 0\) such that
\[
\lambda_1(G; Z) \geq \epsilon = \epsilon(\lambda_1(G; \ell_2), d(G), Z).
\]
Here \(\epsilon\) only depends on \(\lambda_1(G; \ell_2), d(G)\) (the maximal degree of \(G\)), and \(\delta_Z\) (see item (5) of Definition 7.2). The \(\epsilon\) increases on \(\lambda_1(G; \ell_2)\).

In particular, if a sequence \(\{G^{(m)}\}_m\) of finite graphs is a family of expanders, then it is a family of \((Z, 2)\)-anders for any uniformly curved Banach space \(Z\).
**Proposition 7.16** (Proposition 4.3 in [Mim13]). Let $\Gamma$ be a finitely generated infinite group, (with nontrivial finite group quotient), $S$ be a symmetric finite generating set, and $\Lambda \triangleleft \Gamma$ be a normal subgroup of $\Gamma$. Let $Z$ be a Banach space. Then for any Banach space $Y$ which is sphere equivalent to $Z$, there exists $\epsilon \geq 0$ such that

$$\kappa_Y^{(r)}(\Gamma, \Lambda, S) \geq \epsilon = \epsilon(\kappa_Z^{(r)}(\Gamma, \Lambda, S), Y).$$

Here $\epsilon$ only depends on $\kappa_Z^{(r)}(\Gamma, \Lambda, S)$, and moduli of continuity of $\phi$ and $\phi^{-1}$, where $\phi: S(Y) \to S(Z)$ is a uniform homeomorphism in the definition of sphere equivalence, and does not depend on $(\Gamma, \Lambda)$. The $\epsilon$ increases on $\kappa_Z^{(r)}(\Gamma, \Lambda, S)$, and $\epsilon > 0$ if $\kappa_Z^{(r)}(\Gamma, \Lambda, S) > 0$.

In particular, $\kappa_Y^{(r)}(\Gamma, \Lambda, S) > 0$ if and only if $\kappa_Z^{(r)}(\Gamma, \Lambda, S) > 0$.

In the proposition above by setting $\Lambda = \Gamma$, we obtain that $\kappa_Y^{(r)}(\Gamma, S) > 0$ if and only if $\kappa_Z^{(r)}(\Gamma, S) > 0$.

We next recall basics on elementary groups over a unital associative ring. Let $R$ be a unital associative ring (possibly noncommutative) and $n \geq 1$. Denote by $M_n(R)$ the matrix ring over $R$ of size $n$ by $n$. An *elementary matrix* of size $n$ is a matrix of the form $e_{i,j}(a) \in M_n(R)$, where $1 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j$, and $a \in R$. Here $e_{i,j}(a) \in M_n(R)$ is the matrix whose entries are $1$ on diagonal, the $(i,j)$-th entry is $a$, and all other entries are $0$. The *elementary group* of size $n$, denoted by $E(n, R)$, is the multiplicative group generated by all elementary matrices of size $n$ over $R$. If $R$ is commutative, then $E(n, R)$ is a subgroup of the special linear group $\text{SL}(n, R)$. This inclusion is in usual proper, but if $R$ is moreover a Euclidean ring, then by Gaussian elimination, we have $E(n, R) = \text{SL}(n, R)$. (It is the case where $R = \mathbb{Z}$ or $R = \mathbb{F}_2[t]$.)

We note that if $R$ is finitely generated as a ring, then $E(n, R)$ is finitely generated as a group provided $n \geq 3$. Indeed, let $1, a_1, \ldots, a_t$ be a finite generating set of $R$ as a ring and $n \geq 3$, then a commutator relation

$$[e_{i,j}(a), e_{j,k}(b)] = e_{i,k}(ab), \quad i \neq j \neq k \neq i$$

tells us that the following finite set

$$S_n = \{e_{i,j}(a); \; 1 \leq i \leq n, 1 \leq j \leq n, i \neq j, a \in \{\pm 1, \pm a_1, \ldots, \pm a_t\}\}$$

generates $E(n, R)$. We shall call this $S_n$ the *standard set of generators* of $E(n, R)$.

The following theorem by U. Hadad [Had07] plays a crucial role in the proof of Theorem 7.8 (See also a paper of Ershov and Jaikin-Zapirain [EJZ10] for more recent development.)

**Theorem 7.17** (Hadad [Had07]). Let $R$ be a unital associative ring generated by $l+1$ elements $1, a_1, \ldots, a_t$ with stable rank $r$. Assume that for some $d \geq \max\{r+1,3\}$, the elementary group $E(d, R)$ has Kazhdan’s property (T). More precisely, assume that for a standard set of generators $S_d$ of $E(d, R)$ (indicated above), there exists $\epsilon_0 > 0$ such that $K(E(d, R), S_d) > \epsilon_0$.

Then there exists $\epsilon = \epsilon(\epsilon_0, l)$ and $L = L(l)$ such that for any $m \geq d$, there exists an (explicitly written and different from $S_m$) symmetric finitely generating set $\tilde{S}_m$ of $E(m, R)$ which fulfills the following two properties:


• The set $\tilde{S}_m$ satisfies $|\tilde{S}_m| \leq L$.
• The Kazhdan constant satisfies $\mathcal{K}(E(m,R), \tilde{S}_m) > \epsilon$.

The point of this theorem is that $\epsilon$ and $L$ are independent of the choice of $m \geq d$. For the definition of stable range of a ring, we refer to [Had07]. What is needed here is that if $R = \mathbb{Z}$ or $R = \mathbb{F}_2[t]$, then $R$ is a Euclidean ring and its stable range is at most 2.

For the proof of Theorem 7.8 we need the explicit form of $\tilde{S}_m$ in Theorem 7.17. For simplicity, we consider the case where $m = 4n$ ($n \in \mathbb{N}$). Observe that $E(4n,R) = E(4,M_n(R))$ and that the ring $M_n(R)$ is generated by the following $l + 3$ elements as a ring:

$$1_{M_n(R)} = I_n, \begin{pmatrix} 0 & a_i & 0 \\ 0 & \ddots & 0 \\ \vdots & \ddots & 0 \\ 0 & \ddots & \ddots \\ 0 & \ddots & \ddots \end{pmatrix}, \quad 0 \leq i \leq l,$$

$$\text{and} \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}.$$ 

Here we set $a_0 := 1(\in R)$.

Then define $\tilde{S}_{4n}$ to be the standard set of generators of $E(4,M_n(R))$ associated with the ring generators above of $M_n(R)$. Then $\tilde{S}_m = \tilde{S}_{4n}$ consists of $24(l + 3)$ elements (this number is independent of $m$). In the case where $n$ is not divisible by 4, the explicit form of $\tilde{S}_m$ will be slightly complicated, but the idea of the construction is similar to one above.

Proof of Theorem 7.8. For simplicity, we demonstrate the proof for the case $m$ is divisible by 4. (We are considering odd $m$, but the argument below can be easily generalized to all $m \geq 3$.) Set $m = 4n$ and take the set of generators $\tilde{S}_{4n}$ in Theorem 7.17 as indicated in the previous paragraph. We again note that $|\tilde{S}_{4n}| = 24$ and it is independent of $n$ (in particular, uniformly bounded on $n$). Then Theorem 7.17 in particular implies that

$$\inf_n \kappa_{\ell^2}^{(r)}(\text{SL}(4n,\mathbb{Z}), \tilde{S}_{4n}) > 0.$$ 

In the view of Lemma 7.14 this implies that any sequence of finite group quotients of $\{(\text{SL}(4n,\mathbb{Z}), \tilde{S}_{4n})\}_{n}$ forms a family of expanders.

Let $Y$ be a Banach space which is sphere equivalent to a uniformly curved Banach space, and take a uniformly curved Banach space $Z$ such that $Y \sim^S Z$. By Lemma 7.14 and Proposition 7.15 we have that

$$\inf_n \kappa_{Z}^{(r)}(\text{SL}(4n,\mathbb{Z}), \tilde{S}_{4n}) > 0.$$ 

Next we apply Proposition 7.16 and obtain that there exists $\epsilon$ such that

$$\inf_n \kappa_{Y}^{(r)}(\text{SL}(4n,\mathbb{Z}), \tilde{S}_{4n}) \geq \epsilon = \epsilon(Y) > 0.$$ 

Here $\epsilon$ only depends on $Y$ (more precisely, $\delta_Z$ and moduli of continuity of $\phi: S(Y) \rightarrow S(Z)$ and $\phi^{-1}$), and does not depend on $n$. 

Recall that \( \{ (\sigma(4n))^\pm, (\tau(4n))^\pm \} \) is the set of generators of \( \text{SL}(4n, \mathbb{Z}) \) that we are interested in. (If \( m \) is odd, we need some change of the sign of one entry of \( \sigma \), but we omit the argument here.) Let \( T_{4n} \) be the set of generators indicated above. Then the argument in \cite{KR07} shows that for some \( q = q(n) \geq n^2 \), \( (T_{4n})^q \supset \tilde{S}_{4n} \). By Lemma 7.11 we therefore have that for \( m = 4n \),
\[
\kappa_Y^*(\text{SL}(m, \mathbb{Z}), T_m) \geq_Y m^{-2}.
\]
This argument, with slight modification, works for any \( m \geq 3 \). Finally, Lemma 7.14 ends our proof. \( \square \)

7.5. **Proof of Theorem 7.9.** In the proof of Theorem 7.9 we use certain kind of bounded generation for elementary groups. This method dates back to a work of Y. Shalom \cite{Sha99}. We furthermore shall appeal to the following deep results of de la Salle, and V. Lafforgue. Recall the definition of the relative \((\tau)\)-type constant from Definition 7.12.

**Theorem 7.18** (de la Salle \cite{dlS13}). The group \( \text{SL}(3, \mathbb{R}) \) has “strong property (\( T \))” with respect to \( \mathcal{E}_4 \), precisely indicated in \cite{dlS13}.

In particular, for any \( Z \in \mathcal{E}_4 \), there exists \( \delta = \delta(Z) > 0 \) such that
\[
\kappa_Z^*(\text{SL}(3, \mathbb{Z}), S_3) \geq \delta,
\]
where \( S_3 \) is a standard set of generators of \( \text{SL}(3, \mathbb{Z}) \).

**Theorem 7.19** (V. Lafforgue \cite{Laf08}, \cite{Laf09}). Let \( Z \) be a Banach space with type \( > 1 \). Let \( F \) be a nonarchimedean local field and \( G(F) \) be a semisimple algebraic group defined over \( F \) whose Lie algebra contains \( \mathfrak{sl}_3 \). Then \( G(F) \) has “strong property (\( T \))” with respect to the class of Banach spaces of type \( > 1 \), precisely indicated in \cite{Laf08}.

In particular, for any Banach space \( Z \) of type \( > 1 \), there exists \( \delta = \delta(Z) > 0 \) such that
\[
\kappa_Z^*(\text{SL}(3, \mathbb{F}_2[t]), S_3) \geq \delta,
\]
where \( S_3 \) is a standard set of generators of \( \text{SL}(3, \mathbb{F}_2[t]) \).

Here we note that \( \text{SL}(3, \mathbb{F}_2[t]) \) is a lattice of the algebraic group \( \text{SL}(3, \mathbb{F}_2((t^{-1}))) \), which satisfies the assumptions of the first-half of the results in Theorem 7.19. Recall from Subsection 7.1 that both in \( \mathcal{E}_4 \) and in the class of all Banach spaces of type \( > 1 \) are respectively stable under taking \( L_2\)-Bochner integrable spaces, in particular under taking \( \ell_2\)-direct sum.

Here we need the notion of relative property (\( \tau \)) on a Banach space \( Y \), which means, possessing strictly positive relative (\( \tau \))-type constant. Note that this notion is weaker than one of “relative property (\( T_Y \)).” For the precise definition of the latter, we refer to Section 2 of \cite{Mim11} and \cite{BFGM07}. Theorem 7.18 indicates that the pair \( \text{SL}(3, \mathbb{Z}) \ltimes \mathbb{Z}^3 \rhd \mathbb{Z}^3 \) has relative property (\( T_Z \)) for \( Z \in \mathcal{E}_4 \), which in particular implies that
\[
\kappa_Z^*(\text{SL}(3, \mathbb{Z}) \ltimes \mathbb{Z}^3, \mathbb{Z}^3, S) > 0
\]
for some (any) symmetric finite generating set \( S \) of \( \text{SL}(3, \mathbb{Z}) \ltimes \mathbb{Z}^3 \). In a similar way, we have from Theorem 7.19 that the pair \( \text{SL}(3, \mathbb{F}_2[t]) \ltimes (\mathbb{F}_2[t])^3 \rhd (\mathbb{F}_2[t])^3 \) has relative property (\( T_Z \)) for any Banach space \( Z \) of type \( > 1 \).
Therefore by combining with Proposition 7.16 we obtain that for any Banach space $Y$ sphere equivalent to some element in $\mathcal{E}_4$,

$$\kappa_Y^{(r)}(\text{SL}(3,\mathbb{Z}) \ltimes \mathbb{Z}^3, \mathbb{Z}^3, S) > 0,$$

where $S$ is any symmetric finite generating set of $\text{SL}(3,\mathbb{Z}) \ltimes \mathbb{Z}^3$. And that for any Banach space $Y$ sphere equivalent to a Banach space of type $> 1$,

$$\kappa_Y^{(r)}(\text{SL}(3,\mathbb{F}_2[t]) \ltimes (\mathbb{F}_2[t])^3, (\mathbb{F}_2[t])^3, S) > 0,$$

where $S$ is any symmetric finite generating set of $\text{SL}(3,\mathbb{F}_2[t]) \ltimes (\mathbb{F}_2[t])^3 \triangleright (\mathbb{F}_2[t])^3$.

For the proof of the following proposition, we employ a similar argument to the one in Lemma 2.10 in [Mim11]. Note that even when $Y$ is not superreflexive, the argument there works with the aid of item (1) of Lemma 7.13. What we need to observe is that for any $g \in U_m$, we can embed $E(3, R) \ltimes R^3$ into $E(m, R)$ such that the image of $R^3$ contains $g$. See also Corollary 3.5 in [Sha99].

**Proposition 7.20.** Let $m \geq 4$. For a finitely generated associative unital ring $R$, set $S_m$ to be the standard set of generators of $E(m, R)$. Denote by $U_m$ the set of all elementary matrices in $E(m, R)$, namely,

$$U_m := \{e_{i,j}(a); \ 1 \leq i \leq m, 1 \leq j \leq m, i \neq j, a \in R\}.$$

(Note that in contrary to $S_m$, $U_m$ is an infinite subset of $E(m, R)$ as soon as $R$ is infinite.) Let $Y$ be a Banach space for which $\kappa_Y^{(r)}(E(3, R) \ltimes R^3, R^3, S) > 0$. Here we embed $E(3, R) \ltimes R^3$ into $E(4, R)$ in a standard way and $S$ above is $S_4 \cap E(3, R) \ltimes R^3$.

Then the following holds true: for any finite group quotient $G$ of $E(m, R)$ and for any $\xi \in \ell_2(G, \hat{Y})$, we have that

$$\sup_{g \in U_m} \|\xi - \pi_Y(g)\xi\| \geq_Y \sup_{s \in S_m} \|\xi - \pi_Y(s)\xi\|.$$

(More precisely, the multiplicative constant only depends on $\kappa_Y^{(r)}(E(3, R) \ltimes R^3, R^3, S)^{-1}$.) Here we identify $G(U_m)$ (relative quotient of $U_m$ in $G$) with $U_m$.

**Proof of Theorem 7.39.** We first present the proof of item (i). Let $Y \in \mathcal{E}_4$. Let $m \geq 4$. Then Lemma 3.1 in [Had07] applies to $\text{SL}(m, \mathbb{Z})$. (Recall that the stable range of $\mathbb{Z}$ is 2.) By noting that every triangle matrix in $\text{SL}(m, \mathbb{Z})$ with 1 along diagonal is the product of at most $m(m-1)/2$ elementary matrices, we have that any element $\gamma$ in $\text{SL}(m, \mathbb{Z})$ can be written as

$$\gamma = g_1 \gamma_0 g_2,$$

where $g_1$ and $g_2$ are the products of at most $m(m-1)$ elementary matrices, and $\gamma_0$ lies in the copy of $\text{SL}(3, \mathbb{Z})$ in the left upper corner of $\text{SL}(m, \mathbb{Z})$. Then by Theorem 7.18 and Proposition 7.20 we conclude that

$$\kappa_Y^{(r)}(\text{SL}(m, \mathbb{Z}), S_m) \geq_Y m^{-2}$$

in a similar argument to one in [Sha99] or [Had07]. Here we employ item (2) of Lemma 7.13. The set of generators which we are interested in is $T_m := \{(\sigma^{(m)})^\pm, (\tau^{(m)})^\pm\}$. 
By [KR07], we observe that there exists \( q' = q'(m) \gtrsim m \) such that \((T_m)^{q'} \supset S_m\) holds. Therefore by Lemma 7.11 we obtain that
\[
\kappa_Y(\text{SL}(m, \mathbb{Z}), T_m) \gtrsim_Y m^{-3}.
\]
Lemma 7.13 ends our proof.

For item \((ii)\), we employ Theorem 7.19 and Proposition 7.20 for any Banach space \( Y \) sphere equivalent to a Banach space of type \( \gamma > 1 \). \( \square \)

We have thus completed the proofs of Theorems 7.3 and 7.4.

8. Further direction–fibered coarse embeddings

The notion of fibered coarse embeddability is introduced by Chen–Wang–Yu [CWY12] in their study of the maximal coarse Baum–Connes conjecture. For the precise definition, we refer to their paper. Chen–Wang–Wang [CWW12] show that a box space \( \boxtimes \Gamma \) admits fibered coarse embedding into a Hilbert space if and only if \( \Gamma \) is a-T-menable. Also, J. Deng observes that the class of compression functions of \( \boxtimes \Gamma \) for fibered coarse embeddings into a Hilbert space, which is defined in a similar manner to one in Definition 7.1 coincides with the class of equivariant compression functions of \( \Gamma \) into a Hilbert space.

In part II of our sequel papers [MS], we shall discuss relationship between fibered coarse embeddability of a coarse disjoint union of Cayley graphs of finite graphs and group properties of Cayley limit groups. We generalize these results mentioned above to the setting of generalized box spaces. As a byproduct, by combining with [dLS13], [Laf08], and [Laf09], we will obtain the following result, which we shall announce here.

**Theorem 8.1.** Let \( \{k_m\}_{m=3,5,...} \) be a sequence of positive integers with \( \sup_m k_m = \infty \).

1. Let \( W_Z := \bigsqcup_{m=3,5,...} (\text{SL}(m, \mathbb{Z}/k_m \mathbb{Z}); \sigma^{(m)}, \tau^{(m)}, \upsilon^{(m)})) \). Let \( Z \) be in the class of all Banach spaces of type \( p \) and cotype \( q \) with \( 1/p - 1/q < 1/4 \). Then for any Banach space \( Y \) indicated below, \( W_Z \) does not admit a fibered coarse embedding into \( Y \). Here the class of such \( Y \) is all Banach spaces which are isomorphic to \( [Z_0, Z_1]_\theta \) for some compatible couple \( (Z_0, Z_1) \) in complex interpolation with \( Z_0 \) isomorphic to some \( Z \) in the class above, and \( \theta \in (0, 1) \).

   In particular, the assertion applies to any \( \theta \)-Hilbertian space \( Y \).

2. Let \( W_{F_2[t]} := \bigsqcup_{m=3,5,...} (\text{SL}(m, F_2[t]/(t^{k_m})); \sigma^{(m)}, \tau^{(m)}, \tau'(m), \upsilon(m), \upsilon'(m))) \). Then \( W_{F_2[t]} \) does not admit fibered coarse embedding into any superreflexive Banach space \( Y \).

These theorems above, together with Theorems 6.7 and 6.10, shows that the choices of system of generators of finite quotients of \( \{\text{SL}(m, \mathbb{Z})\}_m \) or of \( \{\text{SL}(m, F_2[t])\}_m \) make extreme difference in large scale geometry of resulting generalized box spaces.

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