RICCI EXPANDERS AND TYPE III RICCI FLOW

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Abstract. In this paper, we study how to get the Ricci expanders from $W_+\pm$-functional through the heat kernel estimate of the conjugate heat equation to the type III singularity of Ricci flow. The Gaussian upper and lower bounds are established for the related heat kernel in accordance to the interesting work of Cao-Zhang for the type I Ricci flow.

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1. Introduction

In this paper, we study the question how to get the Ricci expanders from $W_+\pm$-functional through the heat kernel estimate of the conjugate heat equation to the type III singularity of Ricci flow ([6]). As we shall see, the gradient estimate is an important step in solving the Hamilton conjecture. Given a complete non-compact Riemannian manifold $(M, g)$ of dimension $n$. Recall that a family $(g(t))$ of Riemannian metrics on $M^n$ is called a Ricci flow if $g(t)$ satisfies the following Ricci flow equation

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(g(t)), \text{ on } M,$$

with $g(0) = g$. We shall assume the Ricci flow $(M, g(t))$ a type III singularity in the sense that the flow is $k$-non-collapsed on all scales for some constant $k > 0$ (see the work of G.Perelman [11]) and is defined in $(0, \infty)$ with the curvature bound

$$|Rm(g(t))| \leq \frac{A}{A + t}$$

for some uniform constant $A > 0$. Then the conjugate heat equation associated to the Ricci flow $(g(t))$ is

$$u_t = -\Delta u + Ru$$

where $\Delta$ is the Laplacian-Beltrami operator of the evolving metric $g(t)$. Note that the adjoint equation to the conjugate heat equation associated to Ricci flow is

$$u_t = \Delta u.$$
We remark that the maximum principle is true for this heat equation associated to the Ricci flow \((M, g(t))\) (see [13]). Since we are studying the heat equation on complete non-compact Riemannian manifold, the heat kernel is restricted to the minimal fundamental solution \(G = G(z,l;x,t), l < t\), to (3) (or \(u(x,t) = G(z,l;x,t)\) to (4)). Note here that we have used an important observation that the fundamental solution to the heat equation (4) can also be regards as the fundamental solution to the conjugate heat equation (3). As R.Hamilton [6] expected, the type III singularity of Ricci flow gives an expanding soliton. For the \(W_+\)-functional

\[
W_+(g, G, \sigma) = \int_M [\sigma(|\nabla u|^2 + R) - f + n]udv_g, \quad \sigma = t - T > 0,
\]

with

\[
G = e^{-f/(4\pi\sigma)}^{n/2},
\]

introduced by M.Freldman, T.Ilmanen, L.Ni [5] (following the W-functional of G.Perelman), we have

\[
\frac{d}{dt}W_+(g(t), G(t), t - T) = \int_M 2(t - T)|Rc + D^2f + \frac{g}{2(t - T)}|^2dv_g \geq 0.
\]

Fix \(T = 0\). Hence one may get an Ricci expander by studying \(W_+\) along the heat kernel and the limit of some normalization of \(g(t)\) as \(t \to \infty\).

Our main result is below.

**Theorem 1.** Let \((M, g(t)), t \in [0, \infty)\), be a non-flat, type III k-non-collapsed Ricci flow with non-negative Ricci curvature and for some \(A > 0\),

\[
|Rm(g(t))| \leq \frac{A}{A + t}, \quad t > 0.
\]

Then at any point \(x_0 \in M\), a sequence of times \(\tau_k \to \infty\), and a sequence of re-scaled metrics

\[
g_k(x, s) = \tau_k^{-1}g(x, s\tau_k)
\]

such that the pointed Ricci flow sequence \((M, x_0, g_k)\) converges to a non-flat gradient expanding Ricci soliton in the sense of Cheeger-Gromov sense.

This result is used in [8], where part of Hamilton’s conjecture has been proved and a full study about the singularities of Ricci flow with positive Ricci pinching condition is presented. Related work by us using Yamabe flow is [9].

We now make some remarks about the proof. At the first step, we should make sure that the \(W_+\)-functional is well-defined. This will be achieved by obtaining gradient estimate of the positive solution \(u\) and the upper and lower bounds of \(u\) in space-time. According [14] (see also the work of Cao-Hamilton [11], both following the idea of R.Hamilton), for any positive solution \(u\) to (4) on \(M \times [0,T]\), we have

\[
|\nabla \log u(x,t)| \leq \sqrt{\frac{1}{t}} \sqrt{\log \frac{M}{u(x,t)}}
\]
for $M = \sup_{M \times [0,T]} u$ and $(x,t) \in M \times [0,T]$ and moreover, for any $\delta > 0$, $t_0 < t \leq T$, and $x, y \in M$:
\[(7) \quad u(y,t) \leq C_1 u(x,t)^{1/(1+\delta)} M^{\delta/(1+\delta)} e^{C_2 d(x,y,t)^2/t}\]
where $C_1$ and $C_2$ are positive constants depending only on $\delta$, $d(x,y,t)$ is the distance between $x$ and $y$ in the metric $g(t)$. We shall follow the recent interesting works of Zhang [14], Cao-Zhang [2] to derive the desired heat kernel estimates and the result is presented in section 2. Our main heat kernel estimate for the Ricci flow in Theorem 1 is the same result as Theorem 3.1 in [2]. In section 3 we give the argument of Theorem 1.

2. Heat kernel estimate along the Ricci flow

Let us see some analytic parts of the Type III singularity of the Ricci flow $(M,g(t))$.

1. $(M,g(t))$ has a space-time doubling property. Namely, the distances of two points $x, y \in M$ at two different times $t > s > 0$ are comparable in terms of $s/t$ (which is comparable in the sense that it is bigger than a small constant). In fact, let $\gamma(\tau; g(t))$ be the minimizing geodesic connecting $x$ and $y$ in the metric $g(t)$. Then $d(x,y,t) = L(\gamma; g(t))$. Then by using $0 \leq Rc(g(t)) \leq A + t$, we get
\[0 \geq \frac{d}{dt} d(x,y,t) = -\int Rc(\gamma', \gamma')d\tau \geq -\frac{A}{A+t} d(x,y,t),\]
where the derivative is in Lipschitz sense. The latter implies that $(s/t)^A \leq \frac{d(x,y,s)}{d(x,y,t)} \leq 1$.

2. Similarly, local volume comparable property is also true. In fact, we have
\[0 \geq \frac{d}{dt} \int_{B(x,\sqrt{3}t_4)} dv g(t) = -\int_{B(x,\sqrt{3}t_4)} Rdv g(t) \geq -\frac{A}{A+t} \int_{B(x,\sqrt{3}t_4)} dv g(t).\]

Upon integration we know that the volumes of the balls $(x, \sqrt{t_3}; t_4)$ in terms of the metric $g(t_5)$ are comparable for $t_3, t_4, t_5 \in [s,t]$.

3. According to the result of E.Hebey we know that the following Sobolev inequality holds for the $k$-non-collapsed $(M, g(t))$ with $Ric(g(t)) \geq 0$. Namely for all $v \in H^1(B(x,r;t))$, we have
\[(\int |v|^{2n/(n-2)}(n-2)/n \leq \frac{c_n r^2}{|B(x,r;t)|^{2/n}} \int \|
abla v\|^2 + r^{-2} v^2 dv g(t).\]

We shall choose $r = c \sqrt{t}$ for $c \in (0,1)$. Then by assumption $|Ric| \leq \frac{A}{A+t}$ and the k-non-collapsing property, we know that $|B(x,r;t)| \geq k A^{-n} t^{n}/2$. 
Hence, we obtain the following uniform Sobolev inequality along the Ricci flow. For all \( v \in H^1(B(x, \sqrt{t}; t)) \), we have

\[
\left( \int |v|^{2n/(n-2)}(n-2)/n \right) \leq \frac{c_n A^2}{k^{2/n}} \int [\|\nabla v\|^2 + t^{-1}v^2]dv(t).
\]

In the following we try to get the Gaussian upper and lower bounds for the heat kernel to the heat equation. The general program for this work is in three steps below. 1). We derive a weaker on-diagonal upper bound

\[
 u(x, t) \leq \text{const.} \frac{f(t)}{f(t)}
\]

for some increasing function in \( t \). Here \( u(x, t) = G(x_0, 0; x, t) \). The method is the Moser iteration and in this step, the uniform Sobolev inequality plays a important role. 2). We derive the Gaussian upper bound by the exponential weight method due to E.B.Davies. 3). We derive a on-diagonal lower bound at some point and then we obtain the full Gaussian lower bound by using the gradient estimate obtained by Zhang \[14] and Cao-Hamilton \[1], both papers follow the idea of the work of R.Hamilton for standard heat equation.

However, such a machinery can be go through in our setting. So we shall prove the Gaussian upper and lower bound of the heat kernel in a new way (which is also observed by X.Cao and Zhang \[2\]).

Once we have the Gaussian upper and lower bound of the heat kernel, we immediately see the W^+_-functional is well-defined along the Ricci flow.

Let us now go to the detail. Note that if \( G = G(z, l; x, t) \), \( l < t \), is the fundamental solution of

\[
 u_l = -\Delta u + Ru
\]

along the Ricci flow, then as a function of \((x, t)\), \( p(x, t; x_0, l) := G(x_0, l; x, t) \), \( l < t \), is the fundamental solution to the heat equation

\[
 u_t = \Delta u.
\]

Let \( u = u(x, t) \) be a positive solution to the heat equation \[11\] in the region

\[
 Q_{\sigma r} := \{(y, s) \in M \times [\tau - (\sigma r)^2, \tau, d(y, x; s) \leq \sigma r]\}
\]

Here \( r = \sqrt{t}/8 > 0 \), \( \sigma \in [1, 2] \). Then for any \( p \geq 1 \) we have

\[
 \partial_t u^p \leq \Delta u^p.
\]

Choose a non-negative smooth function \( \phi : [0, \infty) \to [0, 1] \) such that

\[
 |\phi'| \leq \frac{2}{(\sigma - 1)r}, \quad \phi' \leq 0,
\]

and \( \phi(\rho) = 1 \) for \( 0 \leq \rho \leq r \) and \( \phi(\rho) = 0 \) for \( \rho \geq \sigma r \). Then we choose a smooth non-negative function \( \eta \) such that

\[
 |\eta'| \leq \frac{2}{(\sigma - 1)^2 r^2}, \quad \eta' \geq 0,
\]

and \( \eta(\rho) = 1 \) for \( \tau - r^2 \leq s \leq \tau \) and \( \phi(s) = 0 \) for \( s \geq \tau - (\sigma r)^2 \).

Define \( \xi = \phi(d(x, y; s))\eta(s) \).
Set $w = u^p$ and using $w\xi^2$ as a testing function to the above differential inequality we deduce that
\[
\int \nabla w \cdot \nabla (w\xi^2) dv_g(s) ds \leq - \int \partial_s w w\xi^2 dv_g(s) ds.
\]
Note that the left hand side is
\[
\int |\nabla (w\xi)|^2 dv_g(s) ds - \int |\xi|^2 w^2 dv_g(s) ds
\]
and the right hand side is
\[
\int w^2 \xi^2 \partial_s dv_g(y,s) ds - \frac{1}{2} \int (w\xi)^2 R dv_g(s) ds - \frac{1}{2} \int (w\xi)^2 dv_g(y,\tau),
\]
Using $R \geq 0$ and our choice of $\phi$ and $\eta$ we know that the latter is bounded by
\[
\frac{c}{(\sigma - 1)^2 r^2} \int w^2 dv_g(s) ds - \frac{1}{2} \int (w\xi)^2 dv_g(y,\tau).
\]
Re-arranging the above relations we get
\[
\int |\nabla (w\xi)|^2 dv_g(s) ds + \frac{1}{2} \int (w\xi)^2 dv_g(y,\tau) \leq \frac{c}{(\sigma - 1)^2 r^2} \int w^2 dv_g(s) ds.
\]
Note that the Sobolev inequality gives us
\[
\left( \int |w\xi|^{2n/(n-2)}(n-2)/n \right)^{1/n} \leq \frac{c_n A^2}{k^{2/n}} \int \|\nabla (w\xi)\|^2 + r^{-2} (w\xi)^2 dv_g(t).
\]
by Holder inequality
\[
\int (w\xi)^{2(1+2/n)} dv_g(s) \leq \int (w\xi)^{2n/(n-2)}(n-2)/n \left( \int (w\xi)^2 dv_g(s) \right)^{2/n}.
\]
Then using the trick as in [4] we get that
\[
\int_{Q_{r/2}(x,\tau)} w^{2\theta} \leq c(k, A) \left( \frac{1}{(\sigma - 1)^2 r^2} \right) \int_{Q_{\sigma r}(x,\tau)} w^{2\theta}
\]
with $\theta = 1 + 2/n$. Choose
\[
\sigma_0 = 2, \sigma_i = 2 - \sum_{i=1}^i 2 - j, \quad p = \theta^i,
\]
and we find that
\[
\sup_{Q_{r/2}(x,\tau)} u^2 \leq \frac{c(k, A)}{r^{n+2}} \int_{Q_r(x,\tau)} u^2 dv_g(s) ds.
\]
Using the general trick of Li-Schoen [10], we then find the $L^1$ mean value inequality in the form below:
\[
\sup_{Q_{r/2}(x,\tau)} u \leq \frac{c(k, A)}{r^{n+2}} \int_{Q_r(x,\tau)} udv_g(s) ds.
\]
Using \( u(x, t) = G(x_0, 0; x, t) \), \( r = \sqrt{t} \) and the fact \( \int_M u dv_g(s) = 1 \), we obtain
\[
G(x_0, 0; x, t) \leq \frac{c(k, A)}{t^{n/2}}.
\]

Next we prove the lower bound by the trick of Perelman [11]. This argument is borrowed from [2]. Let \( u = u(x, t) = G(x, t; x_0, t_0) \) for \( t < t_0 \). Claim that for some uniform constant \( C > 0 \), we have, for \( t < t_0 \),
\[
G(x_0, t; x_0, t_0) \geq C \frac{1}{\tau^{n/2}} e^{-\frac{1}{2} \int_{t_0}^{t} \sqrt{t_0 - s} R(x_0, s) ds}
\]
where \( \tau := t_0 - t \).

Following G.Perelman, we set
\[
u = (4\pi\tau)^{-n/2} e^{-f}.
\]
Using Perelman’s differential Harnack inequality for the fundamental solution we have that for gamma \( \gamma(t) = x_0 \), we have
\[
- \partial_t f(x_0, t) \leq \frac{1}{2} R(x_0, t) - \frac{1}{2\tau} f(x_0, t).
\]
Then for any \( t_2 < t < t_0 \) we can integrate the above inequality to obtain
\[
f(x_0, t_0) \sqrt{t_0 - t_2} \leq f(x_0, t_1) \sqrt{t_0 - t_1} + \frac{1}{2} \int_{t_2}^{t_1} \sqrt{t_0 - s} R(x_0, s) ds.
\]
We remark that by the asymptotic formula for \( G \) we know that for \( t_1 \) approaches to \( t_0 \), \( f(x_0, t_1) \) stays bounded since \( G(x_0, t_1; x_0, 0)(t_0 - t_1)^{n/2} \) is bounded between two positive constants. Then for any \( t \leq t_0 \),
\[
f(x_0, t) \leq \frac{1}{2} \int_{t}^{t_0} \sqrt{t_0 - s} R(x_0, s) ds.
\]
Hence,
\[
G(x_0, t; x_0, t_0) \geq c(4\pi\tau)^{-n/2} e^{-\frac{1}{2\tau} \int_{t_0}^{t} \sqrt{t_0 - s} R(x_0, s) ds}.
\]
Using the assumption \( |R(x, s)| \leq A/(t_0 - s) \), we then know that
\[
G(x_0, t; x_0, t_0) \geq c(4\pi\tau)^{-n/2}.
\]
In summarize, we have obtained the below.

**Lemma 2.** Let \((M, g(t)), t \in [0, \infty)\), be a \( k \)-non-collapsed Ricci flow with bounded curvature \(|Rm| \leq \frac{A}{4\pi t} \), and non-negative Ricci curvature. Then there exist positive constant \( C_1 \) and \( C_2 \) which depends only on \( k, n, A \), such that for all \( x, x_0 \in M, t > 0 \), we have
\[
G(x_0, 0; x, t) \leq \frac{C_1}{t^{n/2}},
\]
and
\[
G(x_0, 0; x_0, t) \geq \frac{C_2}{t^{n/2}}.
\]
With this Lemma to replace Theorem 2.1 in [2], we can get the same result as Theorem 3.1 in [2], which is enough for our use of \( W_+ \)-functional.
3. Blow-down for the Type III Ricci flow

Choose suitable time sequence $\tau_k \to \infty$ and point sequence $x_k \in M$ for the blowing up metrics as in Hamilton [6]. Consider the pointed Ricci flow $(M, g_k, x_k)$ with

$$g_k(s) := \tau_k^{-1} g(\cdot, t_k + s\tau_k).$$

Let, for $s \in [1, 4]$ and for $x_0 = x_k$,

$$u_k = u_k(x, s) := \tau_k^{n/2} G(x, s\tau_k; x_0, 6\tau_k).$$

Then $u_k$ satisfies

$$\partial_s u_k = -\Delta_{g_k} u_k + R(g_k) u_k.$$

Recall that $f_k$ is defined by the relation

$$(4\pi s)^{-n/2} e^{-f_k} = u_k.$$

Using the upper bound for $u_k$, we know that

$$-f_k = \log u_k + \frac{n}{2} \log(4\pi s) \leq C_0$$

for all $k = 1, 2, \ldots$ and $s \in [1, 3]$. Since $\int_M u_k dg_k = 1$, by our uniform bounds for $u$, we know that there is a limit $u_\infty$ of $u_k$ as $k \to \infty$. Note also that

$$W^+(s) \leq C - n$$

for all $k = 1, 2, \ldots$ and $s \in [1, 3]$. By scaling we know that

$$W^+(s) = W^+(g, u, s\tau_k) \leq C - n,$$

where $u = u(x, t) = G(x, t; x_0, 6\tau_k)$, $0 \leq t \leq 4\tau_k$. Note that using the asymptotic behavior of $u$ (see [3]) we have

$$W^+(g, G(x, t; x_0, 6\tau_k), s\tau_k) \leq W^+(g, G(x, t; x_0, 6\tau_{k+1}), s\tau_k) + o_k(1),$$

and using the increasing property we have

$$W^+(g, G(x, t; x_0, 6\tau_{k+1}), s\tau_k) \leq W^+(g, G(x, t; x_0, 6\tau_{k+1}), s\tau_{k+1}).$$

Hence there exists the limit $W^*_\infty(s)$ for the sequence $W_{\tau_k}^+(s)$ as $k \to \infty$.

Then we can get an expanding Ricci soliton similar to Zhang did in the case for shrinking soliton [14]. Since the argument is almost the same, we omit the detailed proof. Thus we have completed the proof of Theorem 1.

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