Local wellposedness of the modified KP-I equations in periodic setting with small initial data

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Abstract

We prove local well-posedness of partially periodic and periodic modified KP-I equations, namely for \( \partial_t u + (-1)^{l+1} \partial_x^l u - \partial_x^{-1} \partial_y^2 u + u^2 \partial_x u = 0 \) in the anisotropic Sobolev space \( H^{s,s}(\mathbb{R} \times \mathbb{T}) \) if \( l = 3 \) and \( s > \frac{2}{3} \), in \( H^{s,s}(\mathbb{T} \times \mathbb{T}) \) if \( l = 3 \) and \( s > \frac{19}{8} \), and in \( H^{s,s}(\mathbb{R} \times \mathbb{T}) \) if \( l = 5 \) and \( s > \frac{5}{2} \). All three results require the initial data to be small.

1 Introduction

Let \( \mathbb{T} = \mathbb{R}/(2\pi \mathbb{Z}) \). This paper is dedicated to the study of the partially periodic and periodic equations of the hierarchy of modified Kadomtsev-Petviashvili I equations (mKP-I)

\[
\begin{align*}
\partial_t u + (-1)^{l+1} \partial_x^l u - \partial_x^{-1} \partial_y^2 u + u^2 \partial_x u &= 0, \\
u(0, x, y) &= u_0(x, y)
\end{align*}
\]

in the anisotropic Sobolev spaces \( H^{s_1,s_2}(M \times \mathbb{T}) \), where, if \( l = 3 \), \( M \) is either \( \mathbb{R} \) or \( \mathbb{T} \) and if \( l = 5 \), \( M = \mathbb{R} \).

The third order \( (l = 3) \) mKP-I equation (1) is the modified version of the third order KP-I equation

\[
\partial_t u + \partial_x^3 u - \partial_x^{-1} \partial_y^2 u + u \partial_x u = 0
\]

which appears when modeling certain long dispersive waves with weak transverse effects, as we see in [1, 16]. The modified KP equation appear in [7] which describe the evolution of sound waves in antiferromagnetics. These KP-I equations, as well as KP-II equations in which the sign of the term \( \partial_x^{-1} \partial_y^2 u \) in (2) is + instead of − appear in several physical contexts. Here the operator \( \partial_x^{-1} \) is defined via Fourier transform, \( \hat{\partial_x^{-1} f}(\xi, \eta) = \frac{1}{i\xi} \hat{f}(\xi, \eta) \).

The third order KP equations are well studied. The KP-II equations are much more well understood for the third order, mainly to the \( X^s_b \) method of Bourgain [4]. The third order KP-II initial value problem is globally wellposed in \( L^2 \) on both \( \mathbb{R} \times \mathbb{R} \) and \( \mathbb{T} \times \mathbb{T} \), see [4]. On \( \mathbb{R}^2 \), Takaoka and Tzvetkov [33] and Isaza and Mejía [15] pushed the low regularity local well-posedness theory down to the anistropic Sobolev space \( s_1 > -\frac{1}{3} \), \( s_2 \geq 0 \). Hardac [10] and Hardac, Herr and Koch in [10] and [11] reached the threshold \( s_1 \geq -\frac{1}{2} \), \( s_2 \geq 0 \) which
is the scaling critical regularity for the KP-II equation. As for the initial value problem on \( \mathbb{R} \times \mathbb{T} \), in order to study the stability of the KdV soliton under the flow of the KP-II equation, Molinet, Saut and Tzvetkov [24] proved global well-posedness on \( L^2(\mathbb{R} \times \mathbb{T}) \).

In the case of the KP-I initial value problem, as the Picard iterative methods in the standard Sobolev spaces, since the flow map fails to be \( C^2 \) at the origin in these spaces, as Molinet, Saut and Tzvetkov showed in [24]. Due to this fact, the wellposedness theory is more limited. For the third order KP-I equation, Kenig in [19] showed global well-posedness in the second energy space \( Z^2_{(3)} = \{ \phi \in L^2(\mathbb{R}^2) : \| (1 + \xi^2 + \frac{n^2}{\xi^2})\hat{\phi}(\xi, n)\|_{L^2_{\xi,n}} < \infty \} \) and later, in [13], Ionescu, Kenig and Tataru showed global well-posedness in the first energy space \( Z^1_{(3)} = \{ \phi \in L^2(\mathbb{R}^2) : \| (1 + \xi + \frac{n}{\xi})\hat{\phi}(\xi, n)\|_{L^2_{\xi,n}} < \infty \} \). Guo, Peng and Wang in [9] showed local well-posedness in \( H^{1,0}(\mathbb{R} \times \mathbb{T}) \). For the third order modified KP-I and KP-II equation, Saut [29] showed that the generalized KP-I/KP-II equation

\[
\begin{align*}
\partial_t u + \partial_x^3 u + \epsilon\partial_x^{-1}\partial_x^2 u + u^p\partial_x u &= 0, \\
u(0, x, y) &= u_0(x, y)
\end{align*}
\]

(\( \epsilon = \pm 1 \)) is locally well-posed in \( C([-T, T]; H^s(\mathbb{R}^2)) \cap C^1([-T, T]; H^{s-3}(\mathbb{R}^2)) \) for \( s \geq 3 \) with the momentum \( V(u)(t) = \int_{\mathbb{R}^2} u^2(t)dx dy \) and energy

\[
E(u)(t) = \int_{\mathbb{R}^2} \frac{(\partial_x u)^2}{2} - \epsilon (\partial_x^{-1}\partial_x u)^2 - \frac{u^{p+2}}{(p+1)(p+2)} dx dy
\]

being conserved quantities. Several blow-up result were found as well for the mKP-I equations. In [29], if \( p \geq 4 \), the corresponding solution \( u \) in (3) blows up in finite time, i.e. there exists \( \infty > T > 0 \) such that \( \lim_{t \to T^-} \| \partial_y u(\cdot, y) \|_{L^2_y} = +\infty \). Liu [23] improved the blow-up result for \( \frac{4}{3} \leq p < 4 \), also by showing that \( \lim_{t \to T^-} \| \partial_y u(\cdot, y) \|_{L^2_y} = +\infty \). Both proofs are based on some virial-type identities.

On \( \mathbb{R} \times \mathbb{T} \), Ionescu and Kenig [12] showed global well-posedness in the second energy space, i.e. \( Z^2_{(3)} = \{ \phi \in L^2 : \| (1 + \xi^2 + \frac{n^2}{\xi^2})\hat{\phi}(\xi, n)\|_{L^2_{\xi,n}} < \infty \} \) and Robert [27] proved global well-posedness in the first energy space \( Z^1_{(3)} = \{ \phi \in L^2 : \| (1 + \xi + \frac{n}{\xi})\hat{\phi}(\xi, n)\|_{L^2_{\xi,n}} < \infty \} \).

For our case of the third order partially periodic modified KP-I we prove the following theorem

**Theorem 1.1.** Assume \( \phi \in H^{s,\ast}(\mathbb{R} \times \mathbb{T}) \) with \( s > 2 \). Then the initial value problem

\[
\begin{align*}
\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u^2\partial_x u &= 0, \\
u(0, x, y) &= \phi(x, y)
\end{align*}
\]

admits a unique solution in \( C([-T, T] ; H^{s,\ast}(\mathbb{R} \times \mathbb{T})) \) with \( T = T(\| \phi \|_{H^{s,\ast}}) \) with \( u, \partial_x u, \partial_y u \in L^2_T L^\infty_{\xi,y} \) if \( \| \phi \|_{H^{s,\ast}} \) is sufficiently small. Moreover, the mapping \( \phi \to u \) is continuous from \( H^{s,\ast}(\mathbb{R} \times \mathbb{T}) \) to \( C([-T, T] ; H^{s,\ast}(\mathbb{R} \times \mathbb{T})) \).
For the $\mathbb{T} \times \mathbb{T}$, Ionescu and Kenig showed in [12] showed global well-posedness in the second energy space $Z_{2}(\mathbb{T} \times \mathbb{T})$ as defined above. Zhang showed in [35] that the third-order periodic KP-I equation is locally well-posed in a Besov type space, namely $B_{2,1}^{s}(\mathbb{T}^{2}) = \{ \phi : \mathbb{T}^{2} \to \mathbb{R} : \hat{\phi}(0, n) = 0 \text{ for all } n \in \mathbb{Z} \setminus \{0\} \text{ and } \|\phi\|_{B_{2,1}^{s}} = \sum_{n=0}^{\infty} 2^{s} \|1_{[2^{k-1},2^{k+1})}(m)\hat{\phi}(1 + \frac{|n|}{|m|^{1+|m|}})\|_{l_{n,m}} \}$.

Later, Robert [28] proved global well-posedness for the fifth-order KP-I in $Z_{2}(\mathbb{T} \times \mathbb{T})$, the natural energy space in this case.

For our case of the third order periodic modified KP-I we prove the following theorem

**Theorem 1.2.** Assume $\phi \in H^{s,s}(\mathbb{T} \times \mathbb{T})$ with $s > \frac{10}{8}$. Then the initial value problem

$$\begin{cases}
\partial_{t}u + \partial_{x}^{2}u - \partial_{x}^{-1}\partial_{y}^{2}u + u^{2}\partial_{x}u = 0, \\
u(0, x, y) = \phi(x, y)
\end{cases}$$

(5)

admits a unique solution in $C([-T, T] : H^{s,s}(\mathbb{T} \times \mathbb{T})$ with $T = T(\|\phi\|_{H^{s,s}})$ with $u, \partial_{x}u, \partial_{y}u \in L_{x}^{2}L_{y}^{\infty}$ if $\|\phi\|_{H^{s,s}}$ is sufficiently small. Moreover, the mapping $\phi \to u$ is continuous from $H^{s,s}(\mathbb{T} \times \mathbb{T})$ to $C([-T, T]; H^{s,s}(\mathbb{T} \times \mathbb{T}))$.

The fifth order ($l = 5$) mKP-I equation (1) is the modified version of the fifth order KP equation

$$\partial_{t}u - \partial_{x}^{5}u - \partial_{x}^{-1}\partial_{y}^{2}u + u\partial_{x}u = 0$$

(6)

which appears when modeling certain long dispersive waves with weak transverse effects, as we see in [2], [17]. The fifth order equation is part of a higher hierarchy of the third order KP equation (2). By the work of Saut and Tzvetkov in [32], we know that the fifth order KP-II initial value problem is globally wellposed in $L^{2}$ on both $\mathbb{R} \times \mathbb{R}$ and $\mathbb{T} \times \mathbb{T}$. In [20], local wellposedness in $H^{s,0}(\mathbb{R} \times \mathbb{T})$ for $s > -\frac{3}{4}$ and global wellposedness in $L^{2}$.

The fifth order KP-I initial value problem is known to be globally well-posed in the energy spaces $Z_{1}^{l}(\mathbb{R} \times \mathbb{R}) = \{ \phi \in L^{2} : \|(1 + x^{2} + \frac{\eta}{\xi})\hat{\phi}(\xi, \eta)\|_{L_{\xi}^{2}} < \infty \}$ on both $\mathbb{R} \times \mathbb{R}$ and $\mathbb{T} \times \mathbb{T}$ from the work of Saut and Tzvetkov in [30] and [31], using Picard iterative methods (see also [5]). Using the Fourier restriction norm method and sufficiently exploiting the geometric structure of the resonant set of [6] to deal with the high-high frequency interaction, Li and Xiao established in [21] the global well-posedness in $L^{2}(\mathbb{R}^{2})$. Guo et al. [8] established the local-wellposedness of the Cauchy problem in $H^{s,0}(\mathbb{R} \times \mathbb{R})$ for $s \geq -\frac{3}{4}$, Yan et al. [34] showed global well-posedness in $H^{s,0}(\mathbb{R} \times \mathbb{R})$ for $s > -\frac{6}{25}$ and finally Li et al. [22] proved global-wellposenedness in $H^{s,0}(\mathbb{R} \times \mathbb{R})$ for $s > -\frac{4}{7}$ and local well-posedness for $s > -\frac{9}{8}$. We conclude with the result from [12] which proves global-wellposedness on $\mathbb{R} \times \mathbb{T}$ in the first energy space $Z_{1}^{l}(\mathbb{T} \times \mathbb{T}) = \{ \phi \in L^{2} : \|(1 + x^{2} + \frac{\eta}{\xi})\hat{\phi}(\xi, n)\|_{L_{\xi}^{2}} < \infty \}$. For the fifth order modified KP-I, Esfahani [6] showed that the generalized KP-I equation

$$\begin{cases}
\partial_{t}u - \partial_{x}^{2}u - \partial_{x}^{-1}\partial_{y}^{2}u + u^{2}\partial_{x}u = 0, \\
u(0, x, y) = u_{0}(x, y)
\end{cases}$$

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is locally well-posed in $C([-T, T]; H^s(\mathbb{R}^2)) \cap C^1([-T, T]; H^{s-5}(\mathbb{R}))$ for $s \geq 5$. In the same paper, if $p \geq 4$, the corresponding solution $u$ in (7) blows up in finite time, i.e. there exists $\infty > T > 0$ such that $\lim_{t \to T^-} \|\partial_y u(t, y)\|_{L^2} = +\infty$. For our case of the fifth order partially periodic modified KP-I we prove the following theorem

**Theorem 1.3.** Assume $\phi \in H^{s,s}(\mathbb{R} \times \mathbb{T})$ with $s > \frac{5}{2}$. Then the initial value problem

$$
\begin{aligned}
\partial_t u - \partial_x^5 u - \partial_x^{-1} \partial_y^2 u + u^2 \partial_x u &= 0, \\
u(0, x, y) &= \phi(x, y)
\end{aligned}
$$

admits a unique solution in $C([-T, T] : H^{s,s}(\mathbb{R} \times \mathbb{T})$ with $T = T(\|\phi\|_{H^{s,s}})$ with $u, \partial_x u, \partial_y u \in L_T^2 L_x^\infty$ if $\|\phi\|_{H^{s,s}}$ is sufficiently small. Moreover, the mapping $\phi \to u$ is continuous from $H^{s,s}(\mathbb{R} \times \mathbb{T})$ to $C([-T, T]; H^{s,s}(\mathbb{R} \times \mathbb{T}))$.

For all our proofs, we will use the same method as used in Ionescu and Kenig [12]. We are bounding $\|u\|_{L_T^2 L_x^\infty investigations for $R \times \mathbb{T}$, which will help us to show the linear estimates in Section 4. Finally, in Section 6 we begin to prove Theorems 1.1, 1.2, 1.3, where we start with the existence result and unicity and finish by using a Bona-Smith argument as in [3] to prove continuity in the space and continuity of the flow map.

2 Notation and Preliminaries

We start by defining, for $g \in L^2(\mathbb{R} \times \mathbb{T})$, $\hat{g}(\xi, n)$ denote its Fourier transform in both $x$ and $y$. We define the Sobolev spaces which we will consider from now on: for $s_1, s_2 \geq 0$

$$
H^{s_1, s_2}(\mathbb{R} \times \mathbb{T}) = \{g \in L^2(\mathbb{R} \times \mathbb{T}) : \|g\|_{H^{s_1, s_2}} = \|\hat{g}(\xi, n)[(1 + \xi^2)^{\frac{s_1}{2}} + (1 + n^2)^{\frac{s_2}{2}}]\|_{L^2(\mathbb{R} \times \mathbb{Z})} < \infty \}
$$

and for $s \geq 0$

$$
H^s(\mathbb{R} \times \mathbb{T}) = \{g \in L^2(\mathbb{R} \times \mathbb{T}) : \|g\|_{H^s} = \|\hat{g}(\xi, n)[(1 + \xi^2 + n^2)^{\frac{s}{2}}]\|_{L^2(\mathbb{R} \times \mathbb{Z})} < \infty \}
$$

and so

$$
H^\infty(\mathbb{R} \times \mathbb{T}) = \cap_{k=0}^{\infty} H^k(\mathbb{R} \times \mathbb{T}).
$$

For $s \in \mathbb{R}$ we define the operators $J_x^s, J_y^s$ by

$$
\overline{J_x^s g}(\xi, n) = (1 + \xi^2)^{\frac{s}{2}} \hat{g}(\xi, n);
$$

and

$$
\overline{J_y^s g}(\xi, n) = (1 + n^2)^{\frac{s}{2}} \hat{g}(\xi, n).
$$
on $S'(\mathbb{R} \times \mathbb{T})$.

For $g \in L^2(\mathbb{T} \times \mathbb{T})$, $\hat{g}(m,n)$ denote its Fourier transform in both $x$ and $y$. In this case, we define similarly the Sobolev spaces which we will consider from now on: for $s_1, s_2 \geq 0$

$$H^{s_1,s_2}(\mathbb{T} \times \mathbb{T}) = \{ g \in L^2(\mathbb{T} \times \mathbb{T}) : \| g \|_{H^{s_1,s_2}} = \| \hat{g}(m,n)[(1 + m^2)^{\frac{s_1}{2}} + (1 + n^2)^{\frac{s_2}{2}}] \|_{L^2(\mathbb{Z} \times \mathbb{Z})} < \infty \}$$

and for $s \geq 0$

$$H^s(\mathbb{T} \times \mathbb{T}) = \{ g \in L^2(\mathbb{T} \times \mathbb{T}) : \| g \|_{H^s} = \| \hat{g}(m,n)[(1 + m^2 + n^2)^{\frac{s}{2}}] \|_{L^2(\mathbb{Z} \times \mathbb{Z})} < \infty \}$$

and so

$$H^\infty(\mathbb{T} \times \mathbb{T}) = \cap_{k=0}^\infty H^k(\mathbb{T} \times \mathbb{T}).$$

By slight abuse of notation, for $s \in \mathbb{R}$ we define the operators $J^s_x, J^s_y$ by

$$J^s_x g(m,n) = (1 + m^2)^{\frac{s}{2}} \hat{g}(m,n);$$

$$J^s_y g(m,n) = (1 + n^2)^{\frac{s}{2}} \hat{g}(m,n)$$

on $S'(\mathbb{T} \times \mathbb{T})$.

For any set $A$ let $1_A$ denote its the characteristic function. Given a Banach space $X$, a measurable function $u : \mathbb{R} \to X$, and an exponent $p \in [1, \infty]$, we define

$$\| u \|_{L^p X} = \left[ \int_\mathbb{R} (\| u(t) \|_X^p) dt \right]^{\frac{1}{p}}$$

if $p \in [1, \infty)$ and

$$\| u \|_{L^\infty X} = \text{esssup}_{t \in \mathbb{R}} \| u(t) \|_X$$

Also, if $I \subseteq \mathbb{R}$ is a measurable set, and $u : I \to X$ is a measurable function, we define

$$\| u \|_{L^p_I X} = \| 1_I(t) u \|_{L^p X}.$$ 

For $T \geq 0$, we define $\| u \|_{L^p_{-T,T} X} = \| u \|_{L^p_{-T,T} X}$

We also introduce the Kato-Ponce commutator estimates (as in Lemma XI from [18] and Appendix 9.A from [12]):

**Lemma 1.** (a) Let $m \geq 0$ and $f, g \in H^m(\mathbb{R})$. If $s \geq 1$ then

$$\| J^s_R (fg) - f J^s_R g \|_{L^2} \leq C_s [ \| J^s_R f \|_{L^2} \| g \|_{L^\infty} + (\| f \|_{L^\infty} + \| \partial f \|_{L^\infty}) \| J^{s-1}_R g \|_{L^2} ]$$

and if $s \in (0, 1)$ then

$$\| J^s_R (fg) - f J^s_R g \|_{L^2} \leq C_s \| J^s_R f \|_{L^2} \| g \|_{L^\infty}.$$
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(b) Let \( m > 0 \) and \( f, g \in H^m(\mathbb{T}) \). If \( s \geq 1 \), then
\[
\| J^s_T(fg) - f J^s_T g \|_{L^2} \leq C_s \| J^s_T f \|_{L^2} \| g \|_{L^\infty} + (\| f \|_{L^\infty} + \| \partial f \|_{L^\infty}) \| J^{s-1}_T g \|_{L^2}
\]
and if \( s \in (0, 1) \) then
\[
\| J^s_T(fg) - f J^s_T g \|_{L^2} \leq C_s \| J^s_T f \|_{L^2} \| g \|_{L^\infty}
\]

We have the following corollary which we will use later.

**Corollary.** Let \( M \) be either \( \mathbb{R} \) or \( \mathbb{T} \).

(a) If \( s \geq 1 \), then we have \( \| J^s_M(u^3) \| \lesssim \| u \|_{L^\infty}^2 \| J^s_M u \|_{L^2} + \| u \|_{L^\infty} \| \partial u \|_{L^\infty} \| J^{s-1}_M u \|_{L^2} \).

(b) If \( s \in (0, 1) \), then we have \( \| J^s_M(u^3) \| \lesssim \| u \|_{L^\infty} \| J^s_M u \|_{L^2} \).

## 3 Dispersive Estimates

For integers \( k = 0, 1, \ldots \) we define the operators \( Q^k_x, Q^k_y, \tilde{Q}^k_x, \tilde{Q}^k_y \) on \( H^\infty(\mathbb{R} \times \mathbb{T}) \) by
\[
\tilde{Q}^k_x g(\xi, n) = 1_{[2^{k-1}, 2^k]}(|\xi|) \quad \text{if} \quad k \geq 1
\]
with
\[
\tilde{Q}^0_x g(\xi, n) = 1_{[0, 1]}(|\xi|)
\]
and
\[
\tilde{Q}^k_y g(\xi, n) = 1_{[2^{k-1}, 2^k]}(|n|) \quad \text{if} \quad k \geq 1
\]
with
\[
\tilde{Q}^0_y g(\xi, n) = 1_{[0, 1]}(|n|).
\]
Also, \( \tilde{Q}^k_x = \sum_{k'=0}^k Q^k_{x}, \tilde{Q}^k_y = \sum_{k'=0}^k Q^k_{y} \), \( k \geq 1 \).

By slight abuse of notation, we define the operators \( Q^k_x, Q^k_y, \tilde{Q}^k_x, \tilde{Q}^k_y \) on \( H^\infty(\mathbb{T} \times \mathbb{T}) \) by
\[
\tilde{Q}^k_x g(m, n) = 1_{[2^{k-1}, 2^k]}(|m|) \quad \text{if} \quad k \geq 1
\]
with
\[
\tilde{Q}^0_x g(m, n) = 1_{[0, 1]}(|m|)
\]
and
\[
\tilde{Q}^k_y g(m, n) = 1_{[2^{k-1}, 2^k]}(|n|) \quad \text{if} \quad k \geq 1
\]
with
\[
\tilde{Q}^0_y g(m, n) = 1_{[2^{k-1}, 2^k]}(|n|).
\]
Also, \( \tilde{Q}^k_x = \sum_{k'=0}^k Q^k_{x}, \tilde{Q}^k_y = \sum_{k'=0}^k Q^k_{y} \), \( k \geq 1 \).

We are stating the dispersion estimates for the partially periodic and fully periodic cases that appear in Kenig and Ionescu [12].
Theorem 3.1. For $t \in \mathbb{R}$ let $W_3(t)$ denote the operator on $H^\infty(\mathbb{R} \times \mathbb{T})$ defined by the Fourier multiplier $(\xi, n) \mapsto e^{i(\xi \cdot x + \frac{\xi^2}{2} t)}$. Assume $\phi \in H^\infty(\mathbb{R} \times \mathbb{T})$. Then for any $\epsilon > 0$, we have
\[\|W_3(t) \tilde{Q}_y^{2j} Q_x^j \phi\|_{L^2_{-j} L^\infty_y} \leq C \epsilon 2^{\epsilon j} \|\tilde{Q}_y^{2j} Q_x^j \phi\|_{L^2_y} \quad (9)\]
and
\[\|W_3(t) Q_y^{2j+k} Q_x^j \phi\|_{L^2_{-j-k} L^\infty_y} \leq C \epsilon 2^{\epsilon j} \|Q_y^{2j+k} Q_x^j \phi\|_{L^2_y} \quad (10)\]
for any integers $j \geq 0$ and $k \geq 1$.

Theorem 3.2. For $t \in \mathbb{R}$ let $\tilde{W}_3(t)$ denote the operator on $H^\infty(\mathbb{T} \times \mathbb{T})$ defined by the Fourier multiplier $(m, n) \mapsto e^{i(m \cdot y + \frac{m^2}{2} t)}$ Assume $\phi \in H^\infty(\mathbb{T} \times \mathbb{T})$. Then for any $\epsilon > 0$, we have
\[\|\tilde{W}_3(t) \tilde{Q}_y^{2j} Q_x^j \phi\|_{L^2_{-j} L^\infty_y} \leq C \epsilon 2^{\epsilon j} \|\tilde{Q}_y^{2j} Q_x^j \phi\|_{L^2_y} \quad (11)\]
and
\[\|\tilde{W}_3(t) Q_y^{2j+k} Q_x^j \phi\|_{L^2_{-j-k} L^\infty_y} \leq C \epsilon 2^{\epsilon j} \|Q_y^{2j+k} Q_x^j \phi\|_{L^2_y} \quad (12)\]
for any integers $j \geq 0$ and $k \geq 1$.

Theorem 3.3. For $t \in \mathbb{R}$ let $W_5(t)$ denote the operator on $H^\infty(\mathbb{R} \times \mathbb{T})$ defined by the Fourier multiplier $(\xi, n) \mapsto e^{i(\xi \cdot x + \frac{\xi^2}{2} t)}$. Assume $\phi \in H^\infty(\mathbb{R} \times \mathbb{T})$. Then, for any $\epsilon > 0$,
\[\|W_5(t) \tilde{Q}_y^{3j} Q_x^j \phi\|_{L^2_{-j} L^\infty_y} \leq C \epsilon 2^{\epsilon j} \|\tilde{Q}_y^{3j} Q_x^j \phi\|_{L^2_y} \quad (13)\]
and
\[\|W_5(t) Q_y^{3j+k} Q_x^j \phi\|_{L^2_{-j-k} L^\infty_y} \leq C \epsilon 2^{\epsilon j} \|Q_y^{3j+k} Q_x^j \phi\|_{L^2_y} \quad (14)\]
for any integers $j \geq 0$ and $k \geq 1$.

4 Linear Estimate

We continue by adapting the argument in [12] to get the linear estimates.

Proposition 1. Assume $N \geq 4$, $u \in C^1([-T, T] : H^{-N-1}(\mathbb{R} \times \mathbb{T}))$, $f \in C([-T, T] : H^{-N}(\mathbb{R} \times \mathbb{T})$ with $T \in [0, \frac{1}{2}]$ and
\[\partial_t \partial_x^3 - \partial_x^{-1} \partial_y^3 u = \partial_x f \text{ on } \mathbb{R} \times \mathbb{T} \times [-T, T].\]

Then for any $\epsilon > 0$, we have
\[\|u\|_{L^2_t L^\infty_x} \leq C \epsilon \left(\|J_x^1 u\|_{L^2_x L^\infty_y} + \|J_y^1 J_x^1 u\|_{L^2_x L^\infty_y} + \|J_y^1 J_x^1 f\|_{L^1_t L^2_y}\right). \]
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**Proof.** Without loss of generality, we may assume that $u \in C^1([-T, T] : H^\infty(\mathbb{R} \times \mathbb{T}))$ and $f \in C([-T, T] : H^\infty(\mathbb{R} \times \mathbb{T}))$. It suffices to prove that if, for $\epsilon > 0$,

$$
\| \widetilde{Q}_y^{2j} Q_x^j u \|_{L^2_t L_x^\infty} \leq C_\epsilon 2^{-\frac{j+1}{2}} \left[ \| J_x^{1+\epsilon} u \|_{L^\infty_t L_x^2} + \| J_x^{1+\epsilon} f \|_{L^1_t L_x^2} \right] 
$$

(15)

and

$$
\| Q_y^{2j+k} Q_x^j u \|_{L^2_t L_x^\infty} \leq C_\epsilon 2^{-\frac{j+1}{2}} \left[ \| J_x^{-1} J_y^{1+\epsilon} u \|_{L^\infty_t L_x^2} + \| J_x^{1+\epsilon} f \|_{L^1_t L_x^2} \right]
$$

(16)

for any integers $j \leq 0$ and $k \leq 1$. For (15), we partition the interval $[-T, T]$ into $2^j$ equal subintervals of length $2T2^{-j}$, denoted by $[a_{j,l}, a_{j,l+1}), l = 1, \ldots, 2^j$. By Duhamel’s formula, for $t \in [a_{j,l}, a_{j,l+1})$,

$$
u(t) = W_3(t - a_{j,l})[u(a_{j,l})] + \int_{a_{j,l}}^t W_3(t - s) [\partial_x f(s)] ds.
$$

It follows from the dispersive estimate (9) that

$$
\| 1_{[a_{j,l}, a_{j,l+1})}(t) \widetilde{Q}_y^{2j} Q_x^j u \|_{L^2_x L_x^\infty}
\leq C_\epsilon \| 1_{[a_{j,l}, a_{j,l+1})}(t) W_3(t - a_{j,l}) \widetilde{Q}_y^{2j} Q_x^j u(a_{j,l}) \|_{L^2_x L_x^\infty}
+ C_\epsilon \| 1_{[a_{j,l}, a_{j,l+1})}(t) \int_{a_{j,l}}^t W_3(s) \widetilde{Q}_y^{2j} Q_x^j \partial_x f(s) ds \|_{L^2_x L_x^\infty}
\leq C_\epsilon 2^{\frac{j}{2}} \| \widetilde{Q}_y^{2j} Q_x^j u(a_{j,l}) \|_{L^2_x}
+ C_\epsilon 2^{\frac{j}{2}} 2^{j} \| 1_{[a_{j,l}, a_{j,l+1})}(t) \widetilde{Q}_y^{2j} Q_x^j f \|_{L^1_t L^2_x}. 
$$

(17)

For the first term of the right-hand side of (17), we have

$$
\sum_{l=1}^{2^j} 2^{\frac{j}{2}} \| 1_{[a_{j,l}, a_{j,l+1})}(t) \widetilde{Q}_y^{2j} Q_x^j u(a_{j,l}) \|_{L^2_x}
\leq \sum_{l=1}^{2^j} 2^{\frac{j}{2}} 2^{-(1+\epsilon)j} \| 1_{[a_{j,l}, a_{j,l+1})}(t) \widetilde{Q}_y^{2j} Q_x^j J_x^{1+\epsilon} u(a_{j,l}) \|_{L^2_x}.
$$

(18)
For the second term of the right-hand side of (17) we have

\[
\sum_{l=1}^{2^j} 2^{\frac{3s}{2}} \left\| 1_{[a_j,t,a_j,t+1]}(t) \tilde{Q}_y^{2j} Q_x^j f \right\|_{L_t^1 L_y^2}
\]

\[
\lesssim \sum_{l=1}^{2^j} 2^{\frac{3s}{2}} 2^{-1(1+\varepsilon)j} \left\| 1_{[a_j,t,a_j,t+1]}(t) \tilde{Q}_y^{2j} Q_x^j J_x^{1+\varepsilon} f \right\|_{L_t^1 L_y^2}
\]

\[
\lesssim 2^{-\frac{s}{4}} \sum_{l=1}^{2^j} \left\| 1_{[a_j,t,a_j,t+1]}(t) \tilde{Q}_y^{2j} Q_x^j J_x^{1+\varepsilon} f \right\|_{L_t^1 L_y^2}
\]

\[
\lesssim 2^{-\frac{s}{4}} \left\| \tilde{Q}_y^{2j} Q_x^j J_x^{1+\varepsilon} f \right\|_{L_t^1 L_y^2}
\]

\[
\lesssim 2^{-\frac{s}{4}} \left\| J_x^{1+\varepsilon} f \right\|_{L_t^1 L_y^2}.
\]

Therefore, (18) and (19) give (15).

For (16), we partition the interval \([-T, T]\) into \(2^{j+k}\) equal subintervals of length \(2T 2^{-j-k}\), denoted by \([b_{j,l}, b_{j,l+1})\), \(l = 1, \ldots, 2^{j+k}\). By Duhamel’s formula, for \(t \in [b_{j,l}, b_{j,l+1}]\),

\[
u(t) = W_3(t - b_{j,l})[u(b_{j,l})] + \int_{b_{j,l}}^t W_3(t - s)[\partial_x f(s)] ds.
\]

It follows from the dispersive estimate (10) that

\[
\left\| 1_{[b_{j,l}, b_{j,l+1}]}(t) Q_y^{2j+k} Q_x^j u \right\|_{L_t^1 L_y^2}
\]

\[
\lesssim C \left\| 1_{[b_{j,l}, b_{j,l+1}]}(t) W_3(t - b_{j,l}) Q_y^{2j+k} Q_x^j u(b_{j,l}) \right\|_{L_t^1 L_y^2}
\]

\[
+ \left\| 1_{[b_{j,l}, b_{j,l+1}]}(t) \int_{b_{j,l}}^t W_3(s) Q_y^{2j+k} Q_x^j \partial_x f(s) ds \right\|_{L_t^1 L_y^2}
\]

\[
\lesssim C 2^{\frac{3s}{4}} \left\| 1_{[b_{j,l}, b_{j,l+1}]}(t) Q_y^{2j+k} Q_x^j u(b_{j,l}) \right\|_{L_y^2}
\]

\[
+ C 2^{\frac{s}{4}} 2^{j} \left\| 1_{[b_{j,l}, b_{j,l+1}]}(t) Q_y^{2j+k} Q_x^j f \right\|_{L_t^1 L_y^2}.
\]

For the first term of the right-hand side of (20), we have

\[
\sum_{l=1}^{2^{j+k}} 2^{\frac{3s}{4}} \left\| 1_{[b_{j,l}, b_{j,l+1}]}(t) Q_y^{2j+k} Q_x^j u(b_{j,l}) \right\|_{L_y^2}
\]

\[
\lesssim \sum_{l=1}^{2^{j+k}} 2^{\frac{3s}{4}} 2^{-1(1+\varepsilon)(2j+k)} \left\| 1_{[b_{j,l}, b_{j,l+1}]}(t) Q_y^{2j+k} Q_x^j J_x^{1+\varepsilon} J_y^{1+\varepsilon} u(b_{j,l}) \right\|_{L_y^2}
\]

\[
\lesssim 2^{j+k} 2^{\frac{3s}{4}} 2^{-1(1+\varepsilon)(2j+k)} \left\| Q_y^{2j+k} Q_x^j J_x^{1+\varepsilon} J_y^{1+\varepsilon} u \right\|_{L_y^\infty L_y^2}
\]

\[
\lesssim 2^{-\frac{(j+k)}{2}} \left\| J_x^{1+\varepsilon} J_y^{1+\varepsilon} u \right\|_{L_y^\infty L_y^2}.
\]
For the second term of the right-hand side of (20) we have

\[
\sum_{l=1}^{2^{j+k}} 2^{j/2} 2^j |1_{[b_{j,l}, b_{j,l+1})}(t)| Q_{2j}^{j+k} Q_x^j f \|_{L^1_T L^2_y} 
\]  

\[\lesssim \sum_{l=1}^{2^{j+k}} 2^{j/2} 2^j 2^{-j/2} 2^{-(2j+k)} |1_{[b_{j,l}, b_{j,l+1})}(t)| Q_{2j}^{j+k} Q_x^j J_x^1 J_y^2 f \|_{L^1_T L^2_y} \]  

\[\lesssim 2^{-\frac{\epsilon(j+k)}{2}} \sum_{l=1}^{2^{j+k}} |1_{[b_{j,l}, b_{j,l+1})}(t)| Q_{2j}^{j+k} Q_x^j J_x^1 J_y^2 f \|_{L^1_T L^2_y} \]

\[\lesssim 2^{-\frac{\epsilon(j+k)}{2}} \|Q_{2j}^{j+k} Q_x^j J_x^1 J_y^2 f \|_{L^1_T L^2_y} \]

\[\lesssim 2^{-\frac{\epsilon(j+k)}{2}} \|J_x^1 J_y^2 f \|_{L^1_T L^2_y} \] \hspace{1cm} \text{ (22)}

Therefore, (21) and (22) give (16). \hfill \Box

**Proposition 2.** Assume \( N \geq 4, u \in C^1([-T, T] : H^{-N-1}(\mathbb{T} \times \mathbb{T})), f \in C([-T, T] : H^{-N}(\mathbb{T} \times \mathbb{T})) \) with \( T \in [0, \frac{1}{2}] \) and

\[\{ \partial_t + \partial_x^3 - \partial_x^{-1} \partial_y^2 \} u = \partial_x f \text{ on } \mathbb{T} \times \mathbb{T} \times [-T, T].\]

Then for any \( \epsilon > 0 \), we have

\[\| u \|_{L^2_T L^\infty_y} \leq C \| |J_x^{1/2}+\epsilon| u \|_{L^\infty_T L^2_y} + \| J_x^{-1/2} J_y^1 J_y^1+\epsilon u \|_{L^\infty_T L^2_y} + \| J_x^{1/2}+\epsilon J_y^1 f \|_{L^1_T L^2_y}.\]

**Proof.** Without loss of generality, we may assume that \( u \in C^1([-T, T] : H^{\infty}(\mathbb{T} \times \mathbb{T})) \) and \( f \in C([-T, T] : H^{\infty}(\mathbb{T} \times \mathbb{T})) \). It suffices to prove that if, for \( \epsilon > 0 \),

\[\|Q_y^{2j} Q_x^j u \|_{L^2_T L^\infty_y} \leq C \epsilon 2^{-\frac{\epsilon}{4}} \left[ \|J_x^{1/4}+\epsilon u \|_{L^\infty_T L^2_y} + \|J_x^{1/4}+\epsilon f \|_{L^1_T L^2_y} \right] \] \hspace{1cm} \text{ (23)}

and

\[\|Q_y^{2j+k} Q_x^j u \|_{L^2_T L^\infty_y} \leq C \epsilon 2^{-\frac{\epsilon(j+k)}{2}} \left[ \|J_x^{-1/2} J_y^1+\epsilon u \|_{L^\infty_T L^2_y} + \|J_x^{1/2} J_y^1 f \|_{L^1_T L^2_y} \right] \] \hspace{1cm} \text{ (24)}

for any integers \( j \leq 0 \) and \( k \leq 1 \). For (23), we partition the interval \([-T, T]\) into \(2^j\) equal subintervals of length \(2T2^{-j}\), denoted by \([a_{j,l}, a_{j,l+1}), l = 1, \ldots, 2^j\). By Duhamel’s formula, for \( t \in [a_{j,l}, a_{j,l+1})\),

\[u(t) = \tilde{W}_{(3)}(t - a_{j,l})[u(a_{j,l})] + \int_{a_{j,l}}^t \tilde{W}_{(3)}(t - s)[\partial_x f(s)]ds.\]
It follows from the dispersive estimate \( (11) \) that
\[
\| 1_{[a_{ij}, a_{ij+1})}(t) \widetilde{Q}_y^{2j} Q_x^j u_x \|_{L_y^2 L_x^\infty} \\
\leq C \| 1_{[a_{ij}, a_{ij+1})}(t) \widetilde{W}_y(3)(t-a_{ij}) \widetilde{Q}_y^{2j} Q_x^j u_x(a_{ij}) \|_{L_y^2 L_x^\infty} \\
+C \| 1_{[a_{ij}, a_{ij+1})}(t) \int_{a_{ij}}^t \widetilde{W}_y(s) \widetilde{Q}_y^{2j} Q_x^j f_x(s) ds \|_{L_y^2 L_x^\infty} \\
\lesssim C 2^{(\frac{3}{4} + \frac{j}{2})} \| \widetilde{Q}_y^{2j} Q_x^j u_x(a_{ij}) \|_{L_y^2 L_x^\infty} \\
+C 2^{(\frac{3}{4} + \frac{j}{2})} 2^{j} \| 1_{[a_{ij}, a_{ij+1})}(t) \widetilde{Q}_y^{2j} Q_x^j f_x \|_{L_y^1 L_x^\infty} \tag{25}
\]

For the first term of the right-hand side of (25), we have
\[
\sum_{l=1}^{2^j} 2^{(\frac{3}{8} + \frac{j}{2})} \| 1_{[a_{ij}, a_{ij+1})}(t) \widetilde{Q}_y^{2j} Q_x^j u_x(a_{ij}) \|_{L_y^2 L_x^\infty} \\
\lesssim \sum_{l=1}^{2^j} 2^{(\frac{3}{8} + \frac{j}{2})} 2^{(\frac{11}{8} + \epsilon)j} \| 1_{[a_{ij}, a_{ij+1})}(t) \widetilde{Q}_y^{2j} Q_x^j f_x^{\frac{11}{8} + \epsilon} \|_{L_y^2 L_x^\infty} \\
\lesssim 2^{\frac{3}{4} + \frac{j}{2}} \| J_x^{\frac{11}{8} + \epsilon} u_x \|_{L_y^\infty L_x^2}. \tag{26}
\]

For the second term of the right-hand side of (25) we have
\[
\sum_{l=1}^{2^j} 2^{(\frac{3}{8} + \frac{j}{2})} \| 1_{[a_{ij}, a_{ij+1})}(t) \widetilde{Q}_y^{2j} Q_x^j f_x \|_{L_y^1 L_x^2} \\
\lesssim \sum_{l=1}^{2^j} 2^{(\frac{3}{8} + \frac{j}{2})} 2^{(\frac{11}{8} + \epsilon)j} \| 1_{[a_{ij}, a_{ij+1})}(t) \widetilde{Q}_y^{2j} Q_x^j f_x^{\frac{11}{8} + \epsilon} \|_{L_y^1 L_x^2} \\
\lesssim 2^{-\frac{3}{4} + \epsilon} \sum_{l=1}^{2^j} \| 1_{[a_{ij}, a_{ij+1})}(t) \widetilde{Q}_y^{2j} Q_x^j f_x^{\frac{11}{8} + \epsilon} \|_{L_y^1 L_x^2} \tag{27}
\]

Therefore, (26) and (27) give (23).

For (24), we partition the interval \([-T, T]\) into \(2^{j+k}\) equal subintervals of length \(2T 2^{-j-k}\), denoted by \([b_{jl}, b_{jl+1})\), \(l = 1, \ldots, 2^{j+k}\). By Duhamel’s formula, for \(t \in [b_{jl}, b_{jl+1})\),
\[
u(t) = \widetilde{W}_y(3)(t-b_{jl})[u(b_{jl})] + \int_{b_{jl}}^t \widetilde{W}_y(3)(t-s)[\partial_x f(s)] ds.
\]
Proposition 3. Assume $N \geq 4$, $u \in C^1([-T, T] : H^{-N-1}(\mathbb{R} \times \mathbb{T}))$, $f \in C([-T, T] : H^{-N}(\mathbb{R} \times \mathbb{T}))$, $T \in [0, \frac{1}{2}]$ and

$$[\partial_t - \partial_x^5 - \partial_x^{-1}\partial_y^2] u = \partial_x f \text{ on } \mathbb{R} \times \mathbb{T} \times [-T, T].$$
Then, for any $1 \leq p \leq 2$, we have
\[ \|u\|_{L^p_y L^\infty_x} \lesssim C_p T^{2 - \frac{3p}{2p}} \left( \|J_x \frac{3p-4}{2p} + \epsilon\right) u_{L^\infty_y L^2_x} + \|J_x \frac{2p-4}{2p} + \epsilon\right) u_{L^\infty_y L^2_x} + \|J_y \frac{p-2}{p} + \epsilon\right) u_{L^\infty_y L^2_x} \]
where $p' = \max\left(\frac{3p-4}{2p} + \epsilon, \frac{1}{p} - \epsilon\right)$.

**Proof.** Without loss of generality, we may assume that $u \in C^1([-T, T] : H^\infty(\mathbb{R} \times \mathbb{T}))$ and $f \in C([-T, T] : H^\infty(\mathbb{R} \times \mathbb{T}))$. It suffices to prove that if, for $\epsilon > 0$,
\[ \|\tilde{Q}_y^j \tilde{Q}_x^j u\|_{L^p_y L^\infty_x} \leq C \epsilon^{2 - \frac{2j}{2}} T \frac{2p-4}{2p} \|J_x \frac{3p-4}{2p} + \epsilon\right) u_{L^\infty_y L^2_x} + \|J_y \frac{p-2}{p} + \epsilon\right) u_{L^\infty_y L^2_x} \] (31)
and
\[ \|Q_y^{3j+k} Q_x^j u\|_{L^p_y L^\infty_x} \leq C \epsilon^{2 - \frac{2j}{2}} T \frac{2p-4}{2p} \|J_x \frac{3p-4}{2p} + \epsilon\right) u_{L^\infty_y L^2_x} + \|J_y \frac{p-2}{p} + \epsilon\right) u_{L^\infty_y L^2_x} \] (32)
for any integers $j \leq 0$ and $k \leq 1$. For (31), we partition the interval $[-T, T]$ into $2^{2j}$ equal subintervals of length $2T 2^{-2j}$, denoted by $[a_{j,l}, a_{j,l+1})$, $l = 1, \ldots, 2^{2j}$. The term in the left-hand side of (31) using Hölder’s inequality, is dominated by
\[ \sum_{l=1}^{2^{2j}} \|1_{[a_{j,l}, a_{j,l+1})}(t) \tilde{Q}_y^j \tilde{Q}_x^j u\|_{L^p_y L^\infty_x} \]
\[ \leq C \|1_{[a_{j,l}, a_{j,l+1})}(t)\|_{L^p_x L^\infty_y} \sum_{l=1}^{2^{2j}} \|1_{[a_{j,l}, a_{j,l+1})}(t) \tilde{Q}_y^j \tilde{Q}_x^j u\|_{L^p_y L^\infty_x} \] (33)
\[ \leq C \epsilon^{2 - \frac{2j}{2}} T \frac{2p-4}{2p} \sum_{l=1}^{2^{2j}} \|1_{[a_{j,l}, a_{j,l+1})}(t) \tilde{Q}_y^j \tilde{Q}_x^j u\|_{L^p_y L^\infty_x}. \]

By Duhamel’s formula, for $t \in [a_{j,l}, a_{j,l+1}]$,
\[ u(t) = W_5(t - a_{j,l})[u(a_{j,l})] + \int_{a_{j,l}}^{t} W_5(t - s)[\partial_x f(s)]ds. \]
It follows from the dispersive estimate [13] that
\[ \|1_{[a_{j,l}, a_{j,l+1})}(t) \tilde{Q}_y^j \tilde{Q}_x^j u\|_{L^p_y L^\infty_x} \]
\[ \leq C \epsilon \|1_{[a_{j,l}, a_{j,l+1})}(t) W_5(t - a_{j,l}) \tilde{Q}_y^j \tilde{Q}_x^j u(a_{j,l})\|_{L^p_y L^\infty_x} \]
\[ + \|1_{[a_{j,l}, a_{j,l+1})}(t) \int_{a_{j,l}}^{t} W_5(s) \tilde{Q}_y^j \tilde{Q}_x^j \partial_x f(s)ds\|_{L^p_y L^\infty_x} \]
\[ \lesssim C \epsilon \|1_{[a_{j,l}, a_{j,l+1})}(t) \tilde{Q}_y^j \tilde{Q}_x^j u(a_{j,l})\|_{L^p_y L^\infty_x} \]
\[ + C \epsilon \|1_{[a_{j,l}, a_{j,l+1})}(t) \tilde{Q}_y^j \tilde{Q}_x^j f\|_{L^p_y L^\infty_x}. \]
Local wellposedness of the mKP-I equations in periodic setting

For the first term of the right-hand side of (34), we have

\[
\sum_{l=1}^{2^{2j}} 2 \frac{2^{2j}}{2p} 2^{2j}(-\frac{1}{2} + \frac{j}{2}) j \|1_{[a_j, a_j, l+1]} (t) \tilde{Q}^{3j} Q^j_x u(a_j) \|_{L^2_y}^2 \\
\lesssim \sum_{l=1}^{2^{2j}} 2 \frac{2^{2j}}{2p} 2^{2j}(-\frac{1}{2} + \frac{j}{2}) j \|1_{[a_j, a_j, l+1]} (t) \tilde{Q}^{3j} Q^j_x J_{x}^{5p-4} u(a_j) \|_{L^2_y}^2 (35)
\]

\[
\lesssim 2^{2j} \frac{2^{2j}}{2p} 2^{2j}(-\frac{1}{2} + \frac{j}{2}) j \|1_{[a_j, a_j, l+1]} (t) \tilde{Q}^{3j} Q^j_x J_{x}^{5p-4} u \|_{L^\infty_t L^2_y} \\
\lesssim 2^{-\frac{j}{2}} \|J_x^{5p-4} u\|_{L^\infty_t L^2_y}.
\]

For the second term of the right-hand side of (34) we have

\[
\sum_{l=1}^{2^{2j}} 2 \frac{2^{2j}}{2p} 2^{2j}(-\frac{1}{2} + \frac{j}{2}) j \|1_{[a_j, a_j, l+1]} (t) \tilde{Q}^{3j} Q^j_x f \|_{L^1_t L^2_y}^2 \\
\lesssim \sum_{l=1}^{2^{2j}} 2 \frac{2^{2j}}{2p} 2^{2j}(-\frac{1}{2} + \frac{j}{2}) j \|1_{[a_j, a_j, l+1]} (t) \tilde{Q}^{3j} Q^j_x J_{x}^{5p-4} f \|_{L^1_t L^2_y}^2 (36)
\]

\[
\lesssim 2^{-\frac{j}{2}} \|\tilde{Q}^{3j} Q^j_x J_{x}^{5p-4} f\|_{L^1_t L^2_y} \\
\lesssim 2^{-\frac{j}{2}} \|J_x^{5p-4} f\|_{L^1_t L^2_y}.
\]

Therefore, (35) and (36) give (31).

For (32), we partition the interval \([-T, T]\) into \(2^{2j+k}\) equal subintervals of length \(2T 2^{-2j-k}\), denoted by \([b_j, l, b_{j+l+1}]\), \(l = 1, \ldots, 2^{2j+k}\). The term in the left-hand side of (32), using Hölder’s inequality, is dominated by

\[
\sum_{l=1}^{2^{2j+k}} \|1_{[b_j, l, b_{j+l+1}]} (t) Q_y^{3j+k} Q_x^j u \|_{L^p_y L^q_y} \\
\leq C \|1_{[b_j, l, b_{j+l+1}]} (t) \|_{L^q_y L^\infty_y} \sum_{l=1}^{2^{2j+k}} \|1_{[b_j, l, b_{j+l+1}]} (t) Q_y^{3j+k} Q_x^j u \|_{L^p_y L^q_y} \\
\leq C 2^{\frac{2j+k}{2p}} (2^{2j+k}) T^2 \sum_{l=1}^{2^{2j+k}} \|1_{[a_j, a_j, l+1]} (t) Q_y^{3j+k} Q_x^j u \|_{L^p_y L^q_y}.
\]

By Duhamel’s formula, for \(t \in [b_j, l, b_{j+l+1}]\),

\[
u(t) = W(5)(t - b_j, l) [u(b_j, l)] + \int_{b_j, l}^t W(5)(t - s) [\partial_x f(s)] ds.
\]
It follows from the dispersive estimate (14) that

\[
\|1_{[b_j,t,b_{j+1}]}(t)Q^{3j+k}Q_x^j u\|_{L^2_x L^\infty_{t,y}}^2 \\
\leq C_\epsilon \|1_{[b_j,t,b_{j+1}]}(t)W(5)(t-b_j,t)Q^{3j+k}Q_x^j u(b_j,t,\cdot)\|_{L^2_x L^\infty_{y}}^2 \\
+ \|1_{[b_j,t,b_{j+1}]}(t)\int_{b_j,t}^t W(5)(s)Q^{3j+k}Q_x^j \partial_x f_s ds\|_{L^2_x L^\infty_{y}}^2
\]

(37)

\[
\lesssim C_\epsilon 2\left(-\frac{1}{2}+\frac{1}{2}\right)\frac{2}{2p-1}\|Q^{3j+k}Q_x^j u(b_j,t,\cdot)\|_{L^2_{t,x}} \\
+ C_\epsilon 2\left(-\frac{1}{2}+\frac{1}{2}\right)\frac{2}{2p-1}\|1_{[b_j,t,b_{j+1}]}(t)Q^{3j+k}Q_x^j f\|_{L^1_{t,x}}
\]

For the first term of the right-hand side of (37), we have

\[
\sum_{j=1}^{2^j+k} 2^{-\frac{2p}{2p-1}(2j+k)} 2^{2\frac{1}{2p-1}\epsilon(2j+k)} 2^{\frac{2p-1}{p}(3j+k)} 2^{\frac{2p-1}{p}(3j+k)} \|1_{[b_j,t,b_{j+1}]}(t)Q^{3j+k}Q_x^j x_j \|_{L^2_{t,x}} \\
\lesssim \sum_{j=1}^{2^j+k} 2^{-\frac{2p}{2p-1}(2j+k)} 2^{\frac{2p-1}{p}(3j+k)} 2^{-\frac{2p-1}{p}(3j+k)} \|Q^{3j+k}Q_x^j x_j \|_{L^2_{t,x}} \\
\lesssim 2^{\frac{1}{2}(3j+k)} 2^{-\frac{2p}{2p-1}(2j+k)} 2^{\frac{2p-1}{p}(3j+k)} 2^{-\frac{2p-1}{p}(3j+k)} \|Q^{3j+k}Q_x^j x_j \|_{L^2_{t,x}} \\
\lesssim 2^{-\frac{1}{2}(3j+k)} \|Q^{3j+k}Q_x^j x_j \|_{L^2_{t,x}}
\]

(38)

For the second term of the right-hand side of (37) we have

\[
\sum_{j=1}^{2^j+k} 2^{-\frac{2p}{2p-1}(2j+k)} 2^{\frac{1}{2p-1}\epsilon(2j+k)} 2^{\frac{2p-1}{p}(3j+k)} 2^{\frac{2p-1}{p}(3j+k)} \|1_{[b_j,t,b_{j+1}]}(t)Q^{3j+k}Q_x^j x_j \|_{L^2_{t,x}} \\
\lesssim \sum_{j=1}^{2^j+k} 2^{-\frac{2p}{2p-1}(2j+k)} 2^{\frac{2p-1}{p}(3j+k)} 2^{-\frac{2p-1}{p}(3j+k)} \|Q^{3j+k}Q_x^j x_j \|_{L^2_{t,x}} \\
\lesssim 2^{-\frac{1}{2}(3j+k)} 2^{-\frac{2p}{2p-1}(2j+k)} 2^{\frac{2p-1}{p}(3j+k)} 2^{-\frac{2p-1}{p}(3j+k)} \|Q^{3j+k}Q_x^j x_j \|_{L^2_{t,x}} \\
\lesssim 2^{-\frac{1}{2}(3j+k)} \|Q^{3j+k}Q_x^j x_j \|_{L^2_{t,x}}
\]

(39)

Therefore, (38) and (39) give (32).

Remark. For the fifth order case, we will use the linear estimate

\[
\|u\|_{L^2_x L^\infty_{t,y}} \lesssim \|J_x^{\frac{1}{2}+\delta} u\|_{L^p_{t,x}} + \|J_y^{\frac{1}{2}+\delta} f\|_{L^2_{t,x}} + \|J_x^{\frac{1}{2}+\delta} J_y^{\frac{1}{2}+\delta} f\|_{L^2_{t,x}}.
\]
5 A Priori Estimates

We are going to bound \( f_u(T) = \| u \|_{L^2_T L^\infty_y} + \| \partial_x u \|_{L^2_T L^\infty_y} + \| \partial_y u \|_{L^2_T L^\infty_y} \).

**Lemma 2.** Suppose \( u \in C([-T, T] : H^{s,a}(M \times \mathbb{T})) \) satisfies the initial value problems (4), (5), (8), with initial data \( \phi \in H^{s,a}(M \times \mathbb{T}) \) (here, \( M \) is either \( \mathbb{R} \) or \( \mathbb{T} \)). Then we have

\[
\| u \|_{L^\infty_T H^{s,a}} \lesssim \| \phi \|_{H^{s,a}} \exp(3 f_u(T)).
\]

**Proof.** First,

\[
\beta_f(t) = \| f(t) \|_{L^\infty_x} + \| \partial_x f(t) \|_{L^2_x}^2 + \| \partial_y f(t) \|_{L^2_x}^2
\]

for a function \( f \). If we apply to any of (4), (5), (8), the operator \( J_x^s \) and then we multiply by \( J_x^s u \), we get by integration by parts

\[
\frac{d}{dt} \| J_x^s u \|_{L^2_y}^2 = \int J_x^s u J_x^s (u^2 \partial_x u) = \int J_x^s u [J_x^s (u^2 \partial_x u) - u^2 J_x^s \partial_x u] + \int u^2 J_x^s u J_x^s \partial_x u
\]

\[
\lesssim \| J_x^s u \|_{L^2_y}^2 (\| u \|_{L^\infty_y}^2 + \| u \|_{L^\infty_y}^2 \| \partial_x u \|_{L^\infty_y}^2) \lesssim \| J_x^s u \|_{L^2_y}^2 \beta_u(t)
\]

therefore, by Grönwall’s inequality, we get that

\[
\| J_x^s u \|_{L^\infty_T L^2_y} \lesssim \| J_x^s \phi \|_{L^2_y} \exp(f_u(T)). \tag{40}
\]

Again, if we apply to any of (4), (5), (8), the operator \( J_y^s \) and then we multiply by \( J_y^s u \), we obtain integrating by parts,

\[
\frac{d}{dt} \| J_y^s u \|_{L^2_y}^2 = \int J_y^s u J_y^s (u^2 \partial_x u) = \int J_y^s u [J_y^s (u^2 \partial_x u) - u^2 J_y^s \partial_x u] + \int u^2 J_y^s u J_y^s \partial_x u
\]

and we denote \((I) = \int J_y^s u [J_y^s (u^2 \partial_x u) - u^2 J_y^s \partial_x u] \) and \((II) = \int u^2 J_y^s u J_y^s \partial_x u\). For the first term, by the Kato-Ponce commutator estimates we have

\[
(I) \lesssim \| J_y^s u \|_{L^2_y} \| J_x^s (u^2 \partial_x u) - u^2 J_x^s \partial_x u \|_{L^2_y}
\]

\[
\lesssim \| J_y^s u \|_{L^2_y} \| \partial_x u \|_{L^2_y} \| J_x^s u \|_{L^2_y} \| u \|_{L^\infty_y}^2 + \| u \|_{L^\infty_y} \| \partial_y u \|_{L^\infty_y} \| J_x^s u \|_{L^2_y} \partial_x u \|_{L^2_y}
\]

\[
\lesssim \| J_y^s u \|_{L^2_y} \| \partial_x u \|_{L^2_y} \| u \|_{L^\infty_y}^2 + \| u \|_{L^\infty_y} \| \partial_y u \|_{L^\infty_y}
\]

\[
+ \| J_y^s u \|_{L^2_y} \| J_x^s u \|_{L^2_y} \| u \|_{L^\infty_y}^2 + \| u \|_{L^\infty_y} \| \partial_y u \|_{L^\infty_y}
\]

\[
\lesssim \| J_y^s u \|_{L^2_y} \| \partial_x u \|_{L^\infty_y} \| J_x^s u \|_{L^2_y} \beta_u(t)
\]

\[
\lesssim \| J_y^s u \|_{L^2_y} \| \partial_x u \|_{L^\infty_y} \| J_x^s u \|_{L^2_y} \beta_u(t)
\]

By integration by parts, we get that

\[
(II) \lesssim \| J_y^s u \|_{L^2_y} \| \partial_x u \|_{L^\infty_y} \beta_u(t)
\].
Therefore, we get
\[ \frac{d}{dt} \|J_y^s u\|_{L^2_y}^2 \lesssim \|J_y^s u\|_{L^2_y}^2 \beta_u(t) + \|J_y^s u\|_{L^2_y} \|\phi\|_{H^{s+\varepsilon}} \exp(2f_u(T)) \beta_u(t) \]
so
\[ \frac{d}{dt} \|J_y^s u\|_{L^2_y} \lesssim \left( \|J_y^s u\|_{L^2_y} + \|\phi\|_{H^{s+\varepsilon}} \exp(2f_u(T)) \right) \beta_u(t) \]
hence, by Grönwall’s inequality, we get
\[ \|J_y^s u\|_{L^2_y} \lesssim \left( \|J_y^s \phi\|_{L^2_y} + \|\phi\|_{H^{s+\varepsilon}} \exp(2f_u(T)) \right) \exp(\int_0^t \|f_u(T)\| dt) \]
which yields that \( \|u\|_{L^\infty_t H^{s+\varepsilon}} \lesssim \|\phi\|_{H^{s+\varepsilon}} \exp(3f_u(T)) \).

**Proposition 4.** Let \( s > 2 \) and \( u_0 \in H^{s}(\mathbb{R} \times \mathbb{T}) \). Suppose \( u \in C([-T, T] : H^{s}(\mathbb{R} \times \mathbb{T})) \) satisfies the IVP (\[4\]). Then \( u, \partial_x u, \partial_y u \in L^2([-T, T]; L^\infty(\mathbb{R} \times \mathbb{T})) \). Moreover,
\[ f_u(T) = \|u\|_{L^2_t L^\infty_y} + \|\partial_x u\|_{L^2_t L^\infty_y} + \|\partial_y u\|_{L^2_t L^\infty_y} \leq C_T \]
for a suitable small \( T \), if \( \|u_0\|_{H^{s+\varepsilon}} \) is small enough.

**Proof.** From now on \( s > 2 + 2\delta \). By the linear estimate in Proposition 1, we have
\[ \|u\|_{L^2_t L^\infty_y} \lesssim \|J_x^{1+\delta} u\|_{L^\infty_t L^2_y} + \|J_y^{-1} J_x^{1+\delta} u\|_{L^\infty_t L^2_y} + \|J_x^{1+\delta} J_y^s (u^3)\|_{L^1_t L^2_y} \]
and
\[ \|\partial_x u\|_{L^2_t L^\infty_y} \lesssim \|J_x^{2+\delta} u\|_{L^\infty_t L^2_y} + \|J_y^{1+\delta} u\|_{L^\infty_t L^2_y} + \|J_x^{2+\delta} J_y^s (u^3)\|_{L^1_t L^2_y} \]...

so by the corollary of the Kato-Ponce commutator estimates
\[ \|J_x^s (u^3)\|_{L^2_y} \lesssim \|J_x^s u\|_{L^2_y} \|u\|_{L^2_y}^2 + \|J_y^s u\|_{L^2_y} \|u\|_{L^\infty_y} \|\partial_x u\|_{L^\infty_y} \leq \beta_u(t) \|J_x^s u\|_{L^2_y} \]
and
\[ \|J_y^s (u^3)\|_{L^2_y} \lesssim \|J_y^s u\|_{L^2_y} \|u\|_{L^2_y}^2 + \|J_y^s u\|_{L^2_y} \|u\|_{L^\infty_y} \|\partial_y u\|_{L^\infty_y} \leq \beta_u(t) \|J_y^s u\|_{L^2_y} \]
so therefore
\[ \|J_x^{2+\delta} J_y^s (u^3)\|_{L^1_t L^2_y} \lesssim f_u(T)^2 \|u\|_{L^\infty_t H^{s+\varepsilon}}. \]
Also,
\[ \|\partial_y u\|_{L^3_t L^2_{xy}} \lesssim \|J_x^{1+\delta} \partial_y u\|_{L^\infty_t L^2_{xy}} + \|J_x^{-1} J_y^{2+\delta} u\|_{L^\infty_t L^2_{xy}} + \|J_x^{1+\delta} J_y \partial_y (u^3)\|_{L^1_t L^2_{xy}}. \]

By the arithmetic-geometric inequality, we also have that
\[ \|J_x^{1+\delta} \partial_y u\|_{L^\infty_t L^2_{xy}} \lesssim \|J_x^{1+\delta} J_x^{1+\delta} (u^3)\|_{L^2_{xy}} \lesssim \|J_x^{1+\delta} J_y \partial_y (u^3)\|_{L^\infty_t L^2_{xy}} + \|J_y^{2+2\delta} (u^3)\|_{L^\infty_t L^2_{xy}} \]

the second inequality being true as \( s - 1 - \delta > 1 \). We also have
\[ \|J_x^{-1} J_y^{2+\delta} u\|_{L^\infty_t L^2_{xy}} \lesssim \|J_y^{2+\delta} (u^3)\|_{L^\infty_t L^2_{xy}}. \]

Since
\[ \|J_x^{1+\delta} J_y \partial_y (u^3)\|_{L^2_{xy}} \lesssim \|J_x^{1+\delta} J_y \partial_y (u^3)\|_{L^2_{xy}} \lesssim \|J_x^{1+\delta} J_y \partial_y (u^3)\|_{L^\infty_t L^2_{xy}} \]

the corollary of the Kato-Ponce commutator estimates gives
\[ \|J_x^{1+\delta} J_y \partial_y (u^3)\|_{L^2_{xy}} \lesssim \|J_x^{1+\delta} J_y \partial_y (u^3)\|_{L^2_{xy}} \lesssim \|J_x^{1+\delta} J_y \partial_y (u^3)\|_{L^\infty_t L^2_{xy}} \]

So we get from the previous estimates \( \|J_x^{1+\delta} \partial_y u\|_{L^1_t L^2_{xy}} \lesssim f_u(T)^2 \|\| u \|_{L^\infty_t H^{s,s}} \).

Hence,
\[ f_u(T) = \|u\|_{L^2_t L^\infty_{xy}} + \|\partial_x u\|_{L^2_t L^\infty_{xy}} + \|\partial_y u\|_{L^2_t L^\infty_{xy}} \lesssim \|u\|_{L^\infty_t H^{s,s}} (1 + f_u(T)^2). \]

Together by the previous lemma,
\[ f_u(T) \lesssim \|\phi\|_{H^{s,s}} (1 + f_u(T)^2) \exp(3f_u(T)) \]

and therefore, if \( \|\phi\|_{H^{s,s}} \) is small enough, by a continuity argument, we get \( f_u(T) \lesssim C \) for \( T \) sufficiently small. \( \square \)

**Proposition 5.** Let \( s > \frac{10}{8} \) and \( u_0 \in H^{s,s} (\mathbb{R} \times \mathbb{T}) \). Suppose \( u \in C([-T, T]; H^{s,s} (\mathbb{R} \times \mathbb{T})) \) satisfies the IVP (4). Then \( u, \partial_x u, \partial_y u \in L^2([-T, T]; L^\infty (\mathbb{R} \times \mathbb{T})). \) Moreover,
\[ f_u(T) = \|u\|_{L^2_t L^\infty_{xy}} + \|\partial_x u\|_{L^2_t L^\infty_{xy}} + \|\partial_y u\|_{L^2_t L^\infty_{xy}} \leq C_T \]

for a suitable small \( T \), if \( \|u_0\|_{H^{s,s}} \) is small enough.

**Proof.** From now on \( s > \frac{10}{8} + 2\delta \).

By the linear estimate in Proposition 2, we have
\[ \|u\|_{L^2_t L^\infty_{xy}} \lesssim \|J_x^{1+\delta} u\|_{L^\infty_t L^2_{xy}} + \|J_x^{-1} J_x^{1+\delta} u\|_{L^\infty_t L^2_{xy}} + \|J_x^{1+\delta} J_y \partial_y (u^3)\|_{L^1_t L^2_{xy}} \]
and
\[ \|\partial_x u\|_{L^2_t L^\infty_{xy}} \lesssim \|J_x^{1+\delta} u\|_{L^\infty_t L^2_{xy}} + \|J_x^{1+\delta} J_x^{1+\delta} u\|_{L^\infty_t L^2_{xy}} + \|J_x^{1+\delta} J_y \partial_y (u^3)\|_{L^1_t L^2_{xy}}. \]
By the estimates,

\[ \| J_x^{1/2 + \delta} u \|_{L^2_x L^2_y} \lesssim \| J_x^s u \|_{L^2_x L^2_y} \lesssim \| u \|_{L^\infty \dot{H}^{s,x}_y}, \]

\[ \| J_x^{1/2 + \delta} u \|_{L^2_x L^2_y} \lesssim \| J_x^{1/2 + \delta} u \|_{L^2_x L^2_y} + \| J_x^{1/2 + \delta} u \|_{L^2_x L^2_y} \lesssim \| u \|_{L^\infty \dot{H}^{s,x}_y}, \]

and

\[ \| J_x^{1/2 + \delta} J_y^s (u^3) \|_{L^2_x L^2_y} \lesssim \| J_x^{1/2 + \delta} (u^3) \|_{L^2_x L^2_y} + \| J_y^{1/2 + \delta} (u^3) \|_{L^2_x L^2_y} \lesssim \| J_x^{1/2 + \delta} (u^3) \|_{L^2_x L^2_y} + \| J_y^{1/2 + \delta} (u^3) \|_{L^2_x L^2_y} \]

so by the corollary of the Kato-Ponce commutator estimates

\[ \| J_x^{1/2 + \delta} J_y^s (u^3) \|_{L^2_x L^2_y} \lesssim \| J_x^s u \|_{L^2_x L^2_y} \| u \|_{L^\infty \dot{H}^{s,x}_y}, \]

and

\[ \| J_x^{1/2 + \delta} (u^3) \|_{L^2_x L^2_y} \lesssim \| J_x^s u \|_{L^2_x L^2_y} \| u \|_{L^\infty \dot{H}^{s,x}_y}, \]

so therefore

\[ \| J_x^{1/2 + \delta} J_y^s (u^3) \|_{L^2_x L^2_y} \lesssim f_u(T)^2 \| u \|_{L^\infty \dot{H}^{s,x}_y}. \]

Lastly,

\[ \| \partial_y u \|_{L^2_x L^\infty_y} \lesssim \| J_x^{1/2 + \delta} \partial_y u \|_{L^\infty_x L^2_y} + \| J_x^{1/2 + \delta} J_y^2 \partial_y u \|_{L^\infty_x L^2_y} + \| J_x^{1/2 + \delta} J_y^s u \|_{L^\infty_x L^2_y}, \]

the second inequality being true as \( s - \frac{11}{8} - \delta > 1 \). We also have

\[ \| J_x^{1/2 + \delta} J_y^s u \|_{L^2_x L^2_y} \lesssim \| J_y^s u \|_{L^2_x L^2_y} \| u \|_{L^\infty \dot{H}^{s,x}_y}. \]

Since

\[ \| J_x^{1/2 + \delta} J_y^s \partial_y (u^3) \|_{L^2_x L^2_y} \lesssim \| J_x^{1/2 + \delta} J_y^s (u^3) \|_{L^2_x L^2_y} \lesssim \| J_x^{1/2 + \delta} (u^3) \|_{L^2_x L^2_y} + \| J_y^{1/2 + \delta} (u^3) \|_{L^2_x L^2_y} \]

the corollary of the Kato-Ponce commutator estimates gives

\[ \| J_x^{1/2 + \delta} J_y^s \partial_y (u^3) \|_{L^2_x L^2_y} \lesssim \beta_u(t) \| J_x^{s} u \|_{L^2_y} + \| J_x^{1/2 + \delta} J_y^s u \|_{L^2_x L^2_y}. \]

So we get from the previous estimates \( \| J_x^{1/2 + \delta} \partial_y (u^3) \|_{L^2_x L^2_y} \lesssim f_u(T)^2 \| u \|_{L^\infty \dot{H}^{s,x}_y}. \)

Hence,

\[ f_u(T) = \| u \|_{L^2_x L^\infty_y} + \| \partial_x u \|_{L^2_x L^\infty_y} + \| \partial_y u \|_{L^2_x L^\infty_y} \lesssim \| u \|_{L^\infty \dot{H}^{s,x}_y} (1 + f_u(T)^2). \]

Together with the previous lemma

\[ f_u(T) \lesssim \| \phi \|_{H^{s,x}} (1 + f_u(T)^2) \exp(3 f_u(T)) \]

and therefore, if \( \| \phi \|_{H^{s,x}} \) is small enough, by a continuity argument, we get \( f_u(T) \lesssim C \) for \( T \) sufficiently small.
Proposition 6. Suppose $u$ satisfies the IVP \( u_t = f(u) \) with initial data \( u_0 \) and let \( s > \frac{5}{2} \).
Then, for \( t \in [-T, T] \), \( u \in L^2([-T, T]; L^\infty(\mathbb{R} \times \mathbb{T})) \). Moreover,
\[
f_u(T) = \|u\|_{L^2_T L^\infty_y} + \|\partial_x u\|_{L^2_T L^\infty_y} + \|\partial_y u\|_{L^2_T L^\infty_y} \leq C_T
\]
for a suitable small enough \( T \), if \( \|u_0\|_{H^{s,s}} \) is small enough.

Proof. We take \( 0 < \delta < s - \frac{5}{2} \). By the linear estimate in Proposition 3, we have
\[
\|u\|_{L^2_T L^\infty_y} \lesssim \|J_x^{\frac{5}{2} + \delta} u\|_{L_T^\infty L^2_y} + \|J_y^{\frac{5}{2} + \delta} J_y^{1+\delta} u\|_{L_T^\infty L^2_y} + \|J_x^{\frac{1}{2} + \delta} J_y^{\delta}(u^3)\|_{L_T^1 L^2_y}
\]
with \( \|J_x^{\frac{5}{2} + \delta} u\|_{L_T^\infty L^2_y} \lesssim \|u\|_{L_T^\infty H^{s,s}_{xy}} \) and \( \|J_x^{\frac{5}{2} + \delta} J_y^{1+\delta} u\|_{L_T^\infty L^2_y} \lesssim \|u\|_{L_T^\infty H^{s,s}_{xy}} \). For the third term of the linear estimate we have \( \|J_x^{\frac{1}{2} + \delta} J_y^{\delta}(u^3)\|_{L_T^2 y} \leq \|J_x^{\frac{1}{2} + \delta} (u^3)\|_{L_T^2 y} + \|J_y^{\frac{1}{2} + \delta} (u^3)\|_{L_T^2 y} \) so by the Kato-Ponce commutator estimates,
\[
\|J_x^{\frac{1}{2} + \delta} (u^3)\|_{L_T^2 y} \lesssim \|J_x^{\frac{1}{2} + \delta} u\|_{L_T^\infty L^2_y} \|u\|_{L_T^\infty L^\infty_y} \lesssim \|J_y^{\frac{1}{2} + \delta} u\|_{L_T^\infty L^2_y} \|u\|_{L_T^\infty L^\infty_y}
\]
and so
\[
\|J_x^{\frac{1}{2} + \delta} (u^3)\|_{L_T^2 y} \lesssim \|J_y^{\frac{1}{2} + \delta} u\|_{L_T^\infty L^2_y} \|u\|_{L_T^\infty L^\infty_y} \lesssim f_u(T) \|u\|_{L_T^\infty H^{s,s}_{xy}}
\]
and finally we get
\[
\|u\|_{L^2_T L^\infty_y} \lesssim (1 + f_u(T)^2) \|u\|_{L_T^\infty H^{s,s}_{xy}} \lesssim \|\phi\|_{H^{s,s}_{xy}} \exp(f_u(T))(1 + f_u(T)^2).
\]

Now, we look at the second term of \( f_u(T) \). By the linear estimate applied to \( \partial_x u \), we have
\[
\|\partial_x u\|_{L^2_T L^\infty_y} \lesssim \|J_x^{\frac{5}{2} + \delta} u\|_{L_T^\infty L^2_y} + \|J_x^{\frac{1}{2} + \delta} J_y^{1+\delta} u\|_{L_T^\infty L^2_y} + \|J_x^{\frac{1}{2} + \delta} J_y^{\delta}(u^3)\|_{L_T^1 L^2_y}
\]
and so
\[
\|J_x^{\frac{1}{2} + \delta} (u^3)\|_{L_T^2 y} \lesssim \|J_x^{\frac{1}{2} + \delta} u\|_{L_T^\infty L^2_y} \|u\|_{L_T^\infty L^\infty_y} \lesssim \|J_y^{\frac{1}{2} + \delta} u\|_{L_T^\infty L^2_y} \|u\|_{L_T^\infty L^\infty_y}
\]
Finally we get
\[
\|\partial_x u\|_{L^2_T L^\infty_y} \lesssim (1 + f_u(T)^2) \|u\|_{L_T^\infty H^{s,s}_{xy}} \lesssim \|\phi\|_{H^{s,s}_{xy}} \exp(f_u(T))(1 + f_u(T)^2).
\]

Now, for the final term of \( f_u(T) \), by the linear estimate applied to \( \partial_y u \), we have
\[
\|\partial_y u\|_{L^2_T L^\infty_y} \lesssim \|J_x^{\frac{1}{2} + \delta} J_y^{1} u\|_{L_T^\infty L^2_y} + \|J_x^{\frac{1}{2} + \delta} J_y^{2 \delta} u\|_{L_T^\infty L^2_y} + \|J_x^{\frac{1}{2} + \delta} \partial_y (u^3)\|_{L_T^1 L^2_y}
\]
20
with
\[ \|J^{rac{1}{2}+\delta}_x J^1_y u\|_{L^\infty_T L^2_y} \lesssim \|J^{rac{3}{2}+\delta}_x u\|_{L^\infty_T L^2_y} + \|J^{rac{1}{2}+\delta}_y u\|_{L^\infty_T L^2_y} \lesssim \|u\|_{L^\infty_T H^{5/2}_y} \]
and \[ \|J^{rac{3}{2}+\delta}_x J^2_y u\|_{L^\infty_T L^2_y} \lesssim \|u\|_{L^\infty_T H^{5/2}_y}. \]
For the third term in the linear estimate, we have
\[ \|J^{rac{1}{2}+\delta}_x J^3_y (u^3)\|_{L^\infty_T L^2_y} \leq \|J^{rac{3}{2}+2\delta}_x (u^3)\|_{L^\infty_T L^2_y} + \|J^{rac{3}{2}+2\delta}_y (u^3)\|_{L^\infty_T L^2_y} \]
sO by the same reasoning, after applying the Kato-Ponce commutator estimates we get
\[ \|J^1_x J^5_y (u^2 \partial_y u)\|_{L^\infty_T L^2_y} \lesssim \|u\|_{L^\infty_T H^{5/2}_y} f_u(T)^2. \]
Finally, we get
\[ \|\partial_y u\|_{L^2_T L^\infty_y} \lesssim (1 + f_u(T)^2) \|u\|_{L^\infty_T H^{5/2}_y} \lesssim \|\phi\|_{H^{3/2}_{3y}} \exp(f_u(T))(1 + f_u(T)^2). \]
All in all, we have that \( f_u(T) \lesssim \|\phi\|_{H^{3/2}_{3y}} \exp(f_u(T))(1 + f_u(T)^2) \), and if \( \|\phi\|_{H^{3/2}_{3y}} \) is small, then by a continuity argument we get that \( f_u(T) \leq C \) if \( T \) is sufficiently small.

\[ \square \]

6 Existence and Uniqueness

We start by stating a well-known local-wellposedness result from Iorio and Nunes (see [14], Section 4):

**Lemma 3.** Assume \( \phi \in H^\infty(M \times \mathbb{T}) \), where \( M \) is either \( \mathbb{R} \) or \( \mathbb{T} \). Then there is \( T = T(\|\phi\|_{H^3}) > 0 \) and a solution \( u \in C([-T, T]: H^\infty(M \times \mathbb{T})) \) of the initial value problem
\[
\begin{aligned}
&\partial_t u + \partial_x^2 u - \partial_x^{-1} \partial_y^2 u + u^2 \partial_x u = 0, \\
u(0, x, y) = \phi(x, y).
\end{aligned}
\]
The proof for \( \mathbb{R} \times \mathbb{T} \) and \( \mathbb{T} \times \mathbb{T} \) is the same as the proof in [14] for \( \mathbb{R} \times \mathbb{R} \).

For the fifth order case we have the following result inspired also by Iorio and Nunes’ [14]

**Lemma 4.** Assume \( \phi \in H^\infty(\mathbb{R} \times \mathbb{T}) \). Then there is \( T = T(\|\phi\|_{H^3}) > 0 \) and a solution \( u \in C([-T, T]: H^\infty(\mathbb{R} \times \mathbb{T})) \) of the initial value problem
\[
\begin{aligned}
&\partial_t u - \partial_x^2 u - \partial_x^{-1} \partial_y^2 u + u^2 \partial_x u = 0, \\
u(0, x, y) = \phi(x, y).
\end{aligned}
\]

We proceed to prove the local well-posedness result.

**Theorem 6.1.** The initial value problem [4] is locally well-posed in \( H^{s,\alpha}(\mathbb{R} \times \mathbb{T}), s > 2 \). More precisely, given \( u_0 \in H^{s,\alpha}(\mathbb{R} \times \mathbb{T}), s > 2 \), there exists \( T = T(\|u_0\|_{H^{s,\alpha}}) \) and a unique solution \( u \) to the IVP such that \( u \in C([0, T]: H^{s,\alpha}(\mathbb{R} \times \mathbb{T})), u, \partial_x u, \partial_y u \in L^2_T L^\infty_y \). Moreover, the mapping \( u_0 \rightarrow u \) in \( C([0, T]: H^{s,\alpha}(\mathbb{R} \times \mathbb{T})) \) is continuous.
Theorem 6.2. The initial value problem (5) is locally well-posed in $H^{s,s}(\mathbb{T} \times \mathbb{T})$, $s > \frac{19}{8}$. More precisely, given $u_0 \in H^{s,s}(\mathbb{T} \times \mathbb{T})$, $s > \frac{19}{8}$, there exists $T = T(\|u_0\|_{H^{s,s}})$ and a unique solution $u$ to the IVP such that $u \in C([0, T] : H^{s,s}(\mathbb{T} \times \mathbb{T}))$, $u, \partial_x u, \partial_y u \in L^2_T L^\infty_x$. Moreover, the mapping $u_0 \rightarrow u$ in $C([0, T] : H^{s,s}(\mathbb{T} \times \mathbb{T}))$ is continuous.

Theorem 6.3. The initial value problem (5) is locally well-posed in $H^{s,s}(\mathbb{R} \times \mathbb{T})$, $s > \frac{5}{2}$. More precisely, given $u_0 \in H^{s,s}(\mathbb{R} \times \mathbb{T})$, $s > \frac{5}{2}$, there exists $T = T(\|u_0\|_{H^{s,s}})$ and a unique solution $u$ to the IVP such that $u \in C([0, T] : H^{s,s}(\mathbb{R} \times \mathbb{T}))$, $u, \partial_x u, \partial_y u \in L^2_T L^\infty_x$. Moreover, the mapping $u_0 \rightarrow u$ in $C([0, T] : H^{s,s}(\mathbb{R} \times \mathbb{T}))$ is continuous.

We present the proof for existence and uniqueness in the case of the third order mKP-I on $\mathbb{R} \times \mathbb{T}$, since the other two cases are similar.

Proof. Let $u_0 \in H^{s,s}(\mathbb{R} \times \mathbb{T})$ and fixed $u_{0,\epsilon} \in H^{s,s}(\mathbb{R} \times \mathbb{T}) \cap H^s_\infty(\mathbb{R} \times \mathbb{T})$ such that $\|u_0 - u_{0,\epsilon}\|_{H^{s,s}} \rightarrow 0$ and $\|u_{0,\epsilon}\|_{H^{s,s}} \leq 2\|u_0\|_{H^{s,s}}$.

We know by the Iorio-Nunes local well-posedness result that $u_{0,\epsilon}$ gives a unique solution $u_{\epsilon}$. We have by the a priori bound that $\|u_{\epsilon}\|_{L^2_T L^\infty_x} + \|\partial_x u_{\epsilon}\|_{L^2_T L^\infty_x} + \|\partial_y u_{\epsilon}\|_{L^2_T L^\infty_x} \leq C_T$ and by the previous result, $\sup_{0 < t < T} \|u_{\epsilon}\|_{H^{s,s}} \leq C_T$.

Henceforth,

$$\partial_t \|u_{\epsilon} - u_{\epsilon'}\|^2_{L^2} = \int (u_{\epsilon} - u_{\epsilon'})\partial_x \left(\frac{u_{\epsilon}^3}{3} - \frac{u_{\epsilon'}^3}{3}\right)$$
$$= \int \partial_x(u_{\epsilon} - u_{\epsilon'}) \cdot (u_{\epsilon} - u_{\epsilon'}) \frac{u_{\epsilon}^2 + u_{\epsilon} u_{\epsilon'} + u_{\epsilon'}^2}{3} =$$
$$= \int (u_{\epsilon} - u_{\epsilon'})^2 \partial_x \left[\frac{u_{\epsilon}^2 + u_{\epsilon} u_{\epsilon'} + u_{\epsilon'}^2}{3}\right]$$
$$\leq \|u_{\epsilon} - u_{\epsilon'}\|^2_{L^2} (\|u_{\epsilon}\|^2_{L^\infty} + \|\partial_x u_{\epsilon}\|^2_{L^\infty} + \|u_{\epsilon'}\|^2_{L^\infty} + \|\partial_x u_{\epsilon'}\|^2_{L^\infty})$$
$$\leq (\beta_{u_{\epsilon}}(t) + \beta_{u_{\epsilon'}}(t)) \|u_{\epsilon} - u_{\epsilon'}\|^2_{L^2}.$$

and by Grönwall’s inequality

$$\|u_{\epsilon} - u_{\epsilon'}\|^2_{L^\infty_T L^2_x} \lesssim_T \|u_{0,\epsilon} - u_{0,\epsilon'}\|^2_{L^2_y},$$

hence $\sup_{0 < t < T} \|u_{\epsilon} - u_{\epsilon'}\|_{L^2_y} \rightarrow 0$, hence we can find $u \in C([0, T] : H^{s',s'}(\mathbb{R} \times \mathbb{T})) \cap L^\infty([0, T] : H^{s,s}(\mathbb{R} \times \mathbb{T}))$ with $s' < s$. The fact that $u$ is a solution of the IVP is clear now. Uniqueness also comes from the previous Grönwall’s inequality.

7 Continuity with respect to time

We proceed by a standard Bona-Smith argument (3).
Definition 7.1. For $\phi \in H^{s}(\mathbb{R} \times \mathbb{T})$ with $s > 2$, let $\phi_k = P_k^{(3)}\phi$ where $\hat{P}_k^{(3)}g(\xi, n) = \hat{g}(\xi, n) \cdot 1_{[0,k]}(|\xi|) \cdot 1_{[0,k]}(|n|)$. Let

$$
\hat{h}_{\phi}^{(3)}(k) = \left[ \sum_{n \in \mathbb{Z}} \int_{|\xi|+|n| \leq k} |\hat{\phi}(\xi, n)|^2 [(1 + \xi^2)^s + (1 + n^2)^s] d\xi \right]^\frac{1}{2}.
$$

Clearly, $\hat{h}_{\phi}^{(3)}$ is nondecreasing in $k$ and $\lim_{k \to \infty} \hat{h}_{\phi}^{(3)}(k) = 0$. By Plancherel,

$$
\|\phi - \phi_k\|_{L^2_{x,y}} = \|\hat{\phi} - \hat{\phi}_k\|_{L^2_{x,y}} = \left[ \sum_{|n| \geq k} \sum_{|m| \geq k} |\hat{\phi}(m, n)|^2 \right]^\frac{1}{2}
\leq \left[ \sum_{m, n \in \mathbb{Z}, |m| + |n| \geq k} |\hat{\phi}(m, n)|^2 \left[ (1 + m^2)^s + (1 + n^2)^s \right]^\frac{1}{2} \right] \lesssim k^{-s} \hat{h}_{\phi}^{(3)}(k).
$$

Definition 7.2. For $\phi \in H^{s}(\mathbb{T} \times \mathbb{T})$ with $s > \frac{10}{3}$, let $\phi_k = P_k^{(3)}\phi$ where $\hat{P}_k^{(3)}g(m, n) = \hat{g}(m, n) \cdot 1_{[0,k]}(|m|) \cdot 1_{[0,k]}(|n|)$. Let $\hat{h}_{\phi}^{(3)}(k) = \left[ \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} |\hat{\phi}(m, n)|^2 \left[ (1 + m^2)^s + (1 + n^2)^s \right]^\frac{1}{2} \right]^\frac{1}{2}.

Clearly, $\hat{h}_{\phi}^{(3)}$ is nondecreasing in $k$ and $\lim_{k \to \infty} \hat{h}_{\phi}^{(3)}(k) = 0$. By Plancherel,

$$
\|\phi - \phi_k\|_{L^2_{x,y}} = \|\hat{\phi} - \hat{\phi}_k\|_{L^2_{x,y}} = \left[ \sum_{|n| \geq k} \sum_{|m| \geq k} |\hat{\phi}(m, n)|^2 \right]^\frac{1}{2}
\leq \left[ \sum_{m, n \in \mathbb{Z}, |m| + |n| \geq k} |\hat{\phi}(m, n)|^2 \left[ (1 + m^2)^s + (1 + n^2)^s \right]^\frac{1}{2} \right] \lesssim k^{-s} \hat{h}_{\phi}^{(3)}(k).
$$

Definition 7.3. For $\phi \in H^{s}(\mathbb{R} \times \mathbb{T})$ with $s > \frac{5}{2}$, let $\phi_k = P_k^{(5)}\phi$ where $\hat{P}_k^{(5)}g(\xi, n) = \hat{g}(\xi, n) \cdot 1_{[0,k]}(|\xi|) \cdot 1_{[0,k]}(|n|)$. Let $h_{\phi}^{(5)}(k) = \left[ \sum_{n \in \mathbb{Z}} \int_{|\xi|+|n| \geq k} |\hat{\phi}(\xi, n)|^2 \left[ (1 + \xi^2)^s + (1 + n^2)^s \right] d\xi \right]^\frac{1}{2}.

Clearly, $h_{\phi}^{(5)}$ is nondecreasing in $k$ and $\lim_{k \to \infty} h_{\phi}^{(5)}(k) = 0$. By Plancherel,

$$
\|\phi - \phi_k\|_{L^2_{x,y}} = \|\hat{\phi} - \hat{\phi}_k\|_{L^2_{x,y}} = \left[ \sum_{|n| \geq k} \int_{|\xi| \geq k} |\hat{\phi}(\xi, n)|^2 d\xi \right]^\frac{1}{2}
\leq \left[ \sum_{n \in \mathbb{Z}} \int_{|\xi| + |n| \geq k} |\hat{\phi}(\xi, n)|^2 \left[ (1 + \xi^2)^s + (1 + n^2)^s \right]^\frac{1}{2} \right] \lesssim k^{-s} h_{\phi}^{(5)}(k).
$$

In all the cases, from their respective definitions, we have that

$$
\|J_x^p \phi_k\|_{L^2_{x,y}} \lesssim C(T, \|\phi\|_{H^{s,s}})k^{p-s} \text{ and } \|J_y^p \phi_k\|_{L^2_{x,y}} \lesssim C(T, \|\phi\|_{H^{s,s}})k^{p-s}.
$$
Local wellposedness of the mKP-I equations in periodic setting

Since \( \phi_k \in H^\infty \), by local well-posedness result of Iorio and Nunes, they give rise to unique solutions \( u_k \) in \( H^\infty \). The above estimates together with (10) and (11), we also have

\[
\| J^p_x u_k \|_{L_x^\infty L_y^2} \leq C(T, \| \phi \|_{H^{s,s}}) k^{p-s}
\]

and

\[
\| J^p_y u_k \|_{L_x^\infty L_y^2} \leq C(T, \| \phi \|_{H^{s,s}}) k^{p-s}.
\]

Denote \( \omega = u_k - u_{k'} \) with \( k < k' \).

**Lemma 5.** We have the following estimates:

a) \[
\| J^s_x \omega \|_{L_x^\infty L_y^2} \leq \exp \left( \frac{1}{2} f(0) + \frac{1}{2} f(u_k(T)) \right) \left[ \| J^s_x \omega(0) \|_{L^2_y} + \| J^s_x u_k \|_{L_x^\infty L_y^2} (f(u_k(T)) + f(0)) \right]
\]

b) \[
\| J^s_y \omega \|_{L_x^\infty L_y^2} \leq \exp \left( \frac{1}{2} f(0) + \frac{1}{2} f(u_k(T)) \right) \left[ \| J^s_y \omega(0) \|_{L^2_y} + \| J^s_x u_k \|_{L_x^\infty L_y^2} (f(u_k(T)) + f(0)) \right]
\]

**Proof.**

\[
\partial_t \omega - \partial_x^2 \omega - \partial_x^{-1} \partial_y^2 \omega + \omega^2 \partial_x \omega + 3u_k^2 \partial_x \omega + 3u_k \omega \partial_x u_k - 3u_k \omega \partial_y \omega - 3 \omega^2 \partial_x u_k = 0.
\]

a) We apply \( J^s_x \omega \) to (44) and then we multiply by \( J^s_x \omega \), in order to get

\[
\frac{d}{dt} \| J^s_x \omega \|_{L^2}^2 = \int J^s_x (\omega^2 \partial_x \omega) J^s_x \omega + 3 \int J^s_x (u_k^2 \partial_x \omega) J^s_x \omega + 3 \int J^s_x (u_k \omega \partial_x u_k) J^s_x \omega
\]

and we will analyze each term in the sum.

We have (I) = \( \int J^s_x (\omega^2 \partial_x \omega) J^s_x \omega \), (II) = \( \int J^s_x (u_k^2 \partial_x \omega) J^s_x \omega \), (III) = \( \int J^s_x (u_k \omega \partial_x u_k) J^s_x \omega \), (IV) = \( \int J^s_x (u_k \omega \partial_x u_k) J^s_x \omega \), and (V) = \( \int J^s_x (\omega^2 \partial_x \omega) J^s_x \omega \).

For (I) = \( \int J^s_x (\omega^2 \partial_x \omega) J^s_x \omega = \int J^s_x (\omega^2 \partial_x \omega) J^s_x \omega \), and we will denote (I)\(_1 = \int J^s_x (\omega^2 \partial_x \omega) J^s_x \omega \) and (I)\(_2 = \int \omega^2 J^s_x \partial_x \omega J^s_x \omega \). For the first one, we have by the Kato-Ponce commutator estimate

\[
(I)\(_1 \leq \| J^s_x \omega \|_{L_x^\infty L_y^2} \| J^s_x (\omega^2 \partial_x \omega) \|_{L_y^2} \]

\[
\leq \| J^s_x \omega \|_{L_x^\infty L_y^2} (\| \partial_x \omega \|_{L_x^\infty} \| J^s_x \omega \|_{L_y^2} + \| \omega \|_{L_x^\infty} \| \partial_x \omega \|_{L_y^\infty}) \| J^s_x \omega \|_{L_x^\infty L_y^2} \]

\[
\leq \| J^s_x \omega \|_{L_x^\infty L_y^2} \beta_0(t)
\]
and

$$(I)_2 \leq \|J_x^s \omega\|^2_{L_{x y}^2} \|\omega\|_{L_{x y}^2} \|\partial_x ^k \omega\|_{L_{x y}^2} \leq \|J_x^s \omega\|^2_{L_{x y}^2} \beta_\omega(t)$$

so $(I) \lesssim \|J_x^s \omega\|^2_{L_{x y}^2} \beta_\omega(t)$.

Now, $(II) = \int J_x^s (u_k^s \partial_x \omega) J_x^s \omega = \int [J_x^s (u_k^s \partial_x \omega) - u_k^s J_x^s \partial_x^k \omega] J_x^s \omega + \int u_k^s J_x^s \partial_x^k \omega J_x^s \omega$ and we denote $(II)_1 = \int [J_x^s (u_k^s \partial_x \omega) - u_k^s J_x^s \partial_x^k \omega] J_x^s \omega$ and $(II)_2 = \int u_k^s J_x^s \partial_x^k \omega J_x^s \omega$. For the first term we have by the Kato-Ponce commutator estimate

$$(II)_1 \lesssim \|J_x^s \omega\|^2_{L_{x y}^2} \|J_x^s (u_k^s \partial_x \omega) - u_k^s J_x^s \partial_x^k \omega\|_{L_{x y}^2} \lesssim \|J_x^s \omega\|^2_{L_{x y}^2} \beta_{u_k}(t) + \|J_x^s \omega\|_{L_{x y}^2} \|J_x^s \partial_x u_k\|_{L_{x y}^2} \|\partial_x^k \omega\|_{L_{x y}^2} \|u_k\|_{L_{x y}^2} \|\partial_x \omega\|_{L_{x y}^2} \|\partial_x^k u_k\|_{L_{x y}^2}$$

Also, we have

$$(II)_2 \lesssim \|J_x^s \omega\|^2_{L_{x y}^2} \|u_k\|_{L_{x y}^2} \|\partial_x \omega\|_{L_{x y}^2} \lesssim \|J_x^s \omega\|^2_{L_{x y}^2} \beta_{u_k}(t)$$

Therefore,

$$(II) \lesssim \|J_x^s \omega\|^2_{L_{x y}^2} \beta_{u_k}(t) + \|J_x^s \omega\|_{L_{x y}^2} \|J_x^s \partial_x u_k\|_{L_{x y}^2} (\beta_{u_k}(t) + \beta_\omega(t))$$

Again, $(III) = \int J_x^s (u_k \omega \partial_x u_k) J_x^s \omega = \int [J_x^s (u_k \omega \partial_x u_k) - u_k \omega J_x^s \partial_x u_k] J_x^s \omega + \int u_k \omega J_x^s \partial_x u_k J_x^s \omega$ and we denote $(III)_1 = \int [J_x^s (u_k \omega \partial_x u_k) - u_k \omega J_x^s \partial_x u_k] J_x^s \omega$ and $(III)_2 = \int u_k \omega J_x^s \partial_x u_k J_x^s \omega$. We have by the Kato-Ponce commutator estimate

$$(III)_1 \lesssim \|J_x^s \omega\|^2_{L_{x y}^2} \|J_x^s (u_k \omega \partial_x u_k) - u_k \omega J_x^s \partial_x u_k\|_{L_{x y}^2} \lesssim \|J_x^s \omega\|^2_{L_{x y}^2} \beta_{u_k}(t) + \|J_x^s \omega\|_{L_{x y}^2} \|J_x^s \partial_x u_k\|_{L_{x y}^2} (\beta_{u_k}(t) + \beta_\omega(t)) + \|J_x^s \omega\|_{L_{x y}^2} \|u_k \omega \partial_x u_k\|_{L_{x y}^2} \|\partial_x \omega\|_{L_{x y}^2} \|u_k \omega \partial_x u_k\|_{L_{x y}^2}$$

Also, $(III)_2 \lesssim \|J_x^s \omega\|^2_{L_{x y}^2} \|J_x^{s+1} u_k\|_{L_{x y}^2} \|\omega\|_{L_{x y}^2} \|\omega\|_{L_{x y}^2} \|u_k\|_{L_{x y}^2} \|\omega\|_{L_{x y}^2}$ and so therefore

$$(III) \lesssim \|J_x^s \omega\|^2_{L_{x y}^2} \beta_{u_k}(t) + \|J_x^s \omega\|_{L_{x y}^2} \|J_x^s \partial_x u_k\| (\beta_{u_k}(t) + \beta_\omega(t)) + \|J_x^s \omega\|_{L_{x y}^2} \|J_x^{s+1} u_k\|_{L_{x y}^2} \|\omega\|_{L_{x y}^2} \|u_k\|_{L_{x y}^2} \|\omega\|_{L_{x y}^2}$$

Again, $(IV) = \int J_x^s (u_k \omega \partial_x u_k) J_x^s \omega = \int [J_x^s (u_k \omega \partial_x u_k) - u_k \omega J_x^s \partial_x \omega] J_x^s \omega + \int u_k \omega J_x^s \partial_x \omega J_x^s \omega$ and we denote $(IV)_1 = \int [J_x^s (u_k \omega \partial_x u_k) - u_k \omega J_x^s \partial_x \omega] J_x^s \omega$ and $(IV)_2 = \int u_k \omega J_x^s \partial_x \omega J_x^s \omega$. We have by the Kato-Ponce commutator estimate

$$(IV)_1 \lesssim \|J_x^s \omega\|^2_{L_{x y}^2} \|J_x^s (u_k \omega \partial_x u_k) - u_k \omega J_x^s \partial_x \omega\|_{L_{x y}^2} \lesssim \|J_x^s \omega\|^2_{L_{x y}^2} \beta_{u_k}(t) + \|J_x^s \omega\|_{L_{x y}^2} \|J_x^s \partial_x u_k\| (\beta_{u_k}(t) + \beta_\omega(t)) + \|J_x^s \omega\|_{L_{x y}^2} \|J_x^s \omega\|_{L_{x y}^2} \|u_k \omega \partial_x u_k\|_{L_{x y}^2} \|\partial_x \omega\|_{L_{x y}^2} \|u_k \omega \partial_x u_k\|_{L_{x y}^2} \|\partial_x \omega\|_{L_{x y}^2} \|\partial_x u_k\|_{L_{x y}^2} \|\omega\|_{L_{x y}^2} \|J_x^{s+1} \partial_x \omega\|_{L_{x y}^2}$$

$$(IV)_2 \lesssim \|J_x^s \omega\|^2_{L_{x y}^2} \beta_{u_k}(t) + \beta_\omega(t) + \|J_x^s \omega\|_{L_{x y}^2} \|J_x^s u_k\|_{L_{x y}^2} \beta_\omega(t).$$
Also, \((IV)_2 \lesssim \|J_x^s \omega\|_{L_y^2}^2 (\|\partial_x u_k\|_{L_y^2} \omega \|_{L_x^\infty} + \|\partial_x \omega\|_{L_y^2} \|u_k\|_{L_x^\infty}) \lesssim \|J_x^s \omega\|_{L_y^2}^2 (\beta_{uk}(t) + \beta_\omega(t))\) and so therefore
\[
(IV) \lesssim \|J_x^s \omega\|_{L_y^2}^2 (\beta_{uk}(t) + \beta_\omega(t)) + \|J_x^s \omega\|_{L_y^2} \|J_x u_k\|_{L_y^2} \beta_\omega(t).
\]

Again, \((V) = \int J_x^s (\omega^2 \partial_x u_k) J_x^s \omega = \int [J_x^s (\omega^2 \partial_x u_k) - \omega^2 J_x^s \partial_x u_k] J_x^s \omega + \int \omega^2 J_x^s \partial_x u_k J_x^s \omega\) and we denote \((V)_1 = \int [J_x^s (\omega^2 \partial_x u_k) - \omega^2 J_x^s \partial_x u_k] J_x^s \omega\) and \((V)_2 = \int \omega^2 J_x^s \partial_x u_k J_x^s \omega\). We have by the Kato-Ponce commutator estimate
\[
(V)_1 \lesssim \|J_x^s \omega\|_{L_y^2} \|J_x^s (\omega^2 \partial_x u_k)\|_{L_y^2} \omega^2 \|J_x^s \partial_x u_k\|_{L_y^2}
\lesssim \|J_x^s \omega\|_{L_y^2}^2 (\beta_{uk}(t) + \beta_\omega(t)) + \|J_x^s \omega\|_{L_y^2} \|J_x u_k\|_{L_y^2} \beta_\omega(t).
\]

Also, \((V)_2 \lesssim \|J_x^s \omega\|_{L_y^2} \|J_x^{s+1} u_k\|_{L_y^2} \|\omega\|_{L_y^2}^2\) and so therefore
\[
(V) \lesssim \|J_x^s \omega\|_{L_y^2}^2 (\beta_{uk}(t) + \beta_\omega(t)) + \|J_x^s \omega\|_{L_y^2} \|J_x u_k\|_{L_y^2} \beta_\omega(t) + \|J_x^s \omega\|_{L_y^2} \|J_x^{s+1} u_k\|_{L_y^2} \|\omega\|_{L_y^2}^2.
\]

Now, putting together all the terms we get that
\[
\frac{d}{dt} \|J_x^s \omega\|_{L_y^2}^2 \lesssim (\|J_x^s \omega\|_{L_y^2}^2 + \|J_x^s \omega\|_{L_y^2} \|J_x^s u_k\|_{L_y^2}) (\beta_\omega(t) + \beta_{uk}(t))
+ \|J_x^s \omega\|_{L_y^2} \|J_x^{s+1} u_k\|_{L_y^2} (\|\omega\|_{L_y^2}^2 + \|\omega\|_{L_y^2} \|u_k\|_{L_y^2}) 
\lesssim \|J_x^s \omega\|_{L_y^2}^2 (\beta_\omega(t) + \beta_{uk}(t)) + \|J_x^s u_k\|_{L_y^2} \|\omega\|_{L_y^2}^2 + \|u_k\|_{L_y^2}^2)
\]

We are using the following variant of Gronwall’s inequality:

**Lemma 6.** If \(\alpha(t), \beta(t)\) are two non-negative functions, and \(\frac{d}{dt} u(t) \leq u(t) \beta(t) + \alpha(t)\) for all \(t \in [0,T]\) then
\[
u(t) \leq e^{\int_0^t \beta(s)ds} \left( u(0) + \int_0^t \alpha(s)ds \right).
\]

By putting \(\beta(t) = \beta_\omega(t) + \beta_{uk}(t) \geq 0\) and
\[
\alpha(t) = \|J_x^s u_k\|_{L_y^2}^2 (\beta_\omega(t) + \beta_{uk}(t)) + \|J_x^{s+1} u_k\|_{L_y^2}^2 (\|\omega\|_{L_y^2}^2 + \|u_k\|_{L_y^2}^2) \geq 0
\]
by applying the lemma to \((45)\) we get
\[
\|J_x^s \omega\|_{L_y^2}^2 \lesssim \exp(f_\omega(T)^2 + f_{uk}(T)^2) \left[ \|J_x^s \omega(0)\|_{L_y^2}^2 + \|J_x^s u_k\|_{L_y^2}^2 \|\omega\|_{L_y^2}^2 + \|f_\omega(T)^2 + f_{uk}(T)^2) \right.
+ \|J_x^{s+1} u_k\|_{L_y^2}^2 \|\omega\|_{L_y^2}^2 + \|u_k\|_{L_y^2}^2 \right].
\]

therefore
\[
\|J_x^s u_k\|_{L_y^2}^2 \lesssim \exp \left[ \frac{1}{2} f_\omega(T) + \frac{1}{2} f_{uk}(T) \right] \left[ \|J_x^s \omega(0)\|_{L_y^2}^2 + \|J_x^s u_k\|_{L_y^2}^2 \|\omega\|_{L_y^2}^2 + \|u_k\|_{L_y^2}^2 \right].
\]
b) We apply \( J_y^s \) to (I) and then we multiply by \( J_y^s \omega \), in order to get

\[
\frac{d}{dt} \| J_y^s \omega \|_{L^2}^2 = \int J_y^s (\omega^2 \partial_x \omega) J_y^s \omega + 3 \int J_y^s (u_k^2 \partial_x \omega) J_y^s \omega + 3 \int J_y^s (u_k \omega \partial_x u_k) J_y^s \omega
\]

\[
- 3 \int J_y^s (u_k \omega \partial_x \omega) J_y^s \omega - 3 \int J_y^s (\omega^2 \partial_x u_k) J_y^s \omega
\]

and we will analyze each term in the sum.

We have \((I) = \int J_y^s (\omega^2 \partial_x \omega) J_y^s \omega\), \((II) = \int J_y^s (u_k^2 \partial_x \omega) J_y^s \omega\), \((III) = \int J_y^s (u_k \omega \partial_x u_k) J_y^s \omega\), \((IV) = \int J_y^s (\omega u_k \partial_x \omega) J_y^s \omega\) and \((V) = \int J_y^s (\omega^2 \partial_x u_k) J_y^s \omega\).

For \((I) = \int J_y^s (\omega^2 \partial_x \omega) J_y^s \omega = \int [J_y^s (\omega^2 \partial_x \omega) - \omega^2 J_y^s \partial_x \omega] J_y^s \omega + \int J_y^s (\omega \partial_x \omega) J_y^s \omega\) and \((I) = \int \omega^2 J_y^s \partial_x \omega J_y^s \omega\). For the first one, we have by the Kato-Ponce commutator estimate

\[
(I)_1 \leq \| J_y^s \omega \|_{L^2_y} \| J_y^s (\omega^2 \partial_x \omega) - \omega^2 J_y^s \partial_x \omega \|_{L^2_y}
\]

\[
\leq \| J_y^s \omega \|_{L^2_y} \| \partial_x \omega \|_{L^2_y} \cdot \| J_y^s \omega \|_{L^2_y} \cdot \omega \|_{L^2_y} + (\| \omega \|_{L^2_y}^2 + \| \omega \|_{L^\infty} \| \partial_y \omega \|_{L^\infty} \| J_y^s \omega \|_{L^2_y}) \| J_y^s \partial_x \omega \|_{L^2_y}
\]

\[
\leq (\| J_y^s \omega \|_{L^2_y}^2 + \| J_y^s \omega \|_{L^2_y} \| J_y^s \partial_x \omega \|_{L^2_y}) \cdot \beta (t)
\]

and

\[
(I)_2 \leq \| J_y^s \omega \|_{L^2_y}^2 \| \partial_x \omega \|_{L^\infty} \| J_y^s \omega \|_{L^2_y} \leq \| J_y^s \omega \|_{L^2_y} \| J_y^s \partial_x \omega \|_{L^2_y} \cdot \beta (t)
\]

so \((I) \leq (\| J_y^s \omega \|_{L^2_y}^2 + \| J_y^s \omega \|_{L^2_y} \| J_y^s \partial_x \omega \|_{L^2_y}) \cdot \beta (t)\).

Now, \((II) = \int J_y^s (u_k^2 \partial_x \omega) J_y^s \omega = \int [J_y^s (u_k^2 \partial_x \omega) - u_k^2 J_y^s \partial_x \omega] J_y^s \omega + \int u_k J_y^s \partial_x \omega J_y^s \omega\) and we denote \((II)_1 = \int [J_y^s (u_k^2 \partial_x \omega) - u_k^2 J_y^s \partial_x \omega] J_y^s \omega\) and \((II)_2 = \int u_k J_y^s \partial_x \omega J_y^s \omega\). For the first term we have by the Kato-Ponce commutator estimate

\[
(II)_1 \leq \| J_y^s \omega \|_{L^2_y} \cdot \| J_y^s (u_k^2 \partial_x \omega) - u_k^2 J_y^s \partial_x \omega \|_{L^2_y}
\]

\[
\leq \| J_y^s \omega \|_{L^2_y} \| \partial_x \omega \|_{L^\infty} \| J_y^s u_k \|_{L^2_y} \| u_k \|_{L^\infty} + (\| u_k \|_{L^\infty} + \| u_k \|_{L^\infty} \| \partial_y u_k \|_{L^\infty}) \| J_y^s \partial_x \omega \|_{L^2_y}
\]

\[
\leq \| J_y^s \omega \|_{L^2_y} \| \partial_x \omega \|_{L^\infty} \| u_k \|_{L^\infty} + \| u_k \|_{L^\infty} \| \partial_y u_k \|_{L^\infty} \| u_k \|_{L^\infty}
\]

Also, we have

\[
(II)_2 \leq \| J_y^s \omega \|_{L^2_y} \| u_k \|_{L^\infty} \| \partial_x u_k \|_{L^\infty} \leq \| J_y^s \omega \|_{L^2_y} \| \partial_x \omega \|_{L^\infty} \| u_k \|_{L^\infty} \| u_k \|_{L^\infty}
\]

Therefore,

\[
(II) \leq \| J_y^s \omega \|_{L^2_y} \| \partial_x \omega \|_{L^\infty} \| u_k \|_{L^\infty} \| \partial_x u_k \|_{L^\infty} \leq \| J_y^s \omega \|_{L^2_y} \| \partial_x \omega \|_{L^\infty} \| u_k \|_{L^\infty} \| u_k \|_{L^\infty}
\]
We have by the Kato-Ponce commutator estimate

\[ (III)_1 \lesssim \|y^*\omega\|_{\mathcal{L}_2} \|y^*u_k\|_{\mathcal{L}_2} + \|y^*\omega\|_{\mathcal{L}_2} \|\partial_x u_k\|_{\mathcal{L}_2} \]

and so therefore

\[ (III)_2 \lesssim \|y^*\omega\|_{\mathcal{L}_2} \|y^*u_k\|_{\mathcal{L}_2} \|\omega\|_{L^\infty} \|u_k\|_{L^\infty} + \|y^*\omega\|_{\mathcal{L}_2} \|y^*u_k\|_{\mathcal{L}_2} \|\omega\|_{L^\infty} \|u_k\|_{L^\infty} \]

and so

\[ (III) \lesssim \|y^*\omega\|_{\mathcal{L}_2} \|y^*u_k\|_{\mathcal{L}_2} \|\omega\|_{L^\infty} \|u_k\|_{L^\infty} + \|y^*\omega\|_{\mathcal{L}_2} \|y^*u_k\|_{\mathcal{L}_2} \|\omega\|_{L^\infty} \|u_k\|_{L^\infty} \]

Again, \((IV) = \int y^*(u_k\omega\partial_x u_k)J^*_y \omega = \int [y^*(u_k\omega\partial_x u_k) - u_k\omega J^*_y\partial_x u_k]J^*_y \omega + \int u_k\omega J^*_y\partial_x u_k J^*_y \omega\) and we denote \((IV)_1 = \int [y^*(u_k\omega\partial_x u_k) - u_k\omega J^*_y\partial_x u_k]J^*_y \omega\) and \((IV)_2 = \int u_k\omega J^*_y\partial_x u_k J^*_y \omega\).

We have by the Kato-Ponce commutator estimate

\[ (IV)_1 \lesssim \|y^*\omega\|_{\mathcal{L}_2} \|y^*u_k\|_{\mathcal{L}_2} \|\partial_x u_k\|_{\mathcal{L}_2} \]

and so

\[ (IV) \lesssim \|y^*\omega\|_{\mathcal{L}_2} \|y^*u_k\|_{\mathcal{L}_2} \|\omega\|_{L^\infty} \|u_k\|_{L^\infty} + \|y^*\omega\|_{\mathcal{L}_2} \|y^*u_k\|_{\mathcal{L}_2} \|\omega\|_{L^\infty} \|u_k\|_{L^\infty} \]

Also, \( (IV)_2 \lesssim \|y^*\omega\|_{\mathcal{L}_2} \|y^*u_k\|_{\mathcal{L}_2} \|\omega\|_{L^\infty} \|u_k\|_{L^\infty} \|\omega\|_{L^\infty} \|u_k\|_{L^\infty} \|\omega\|_{L^\infty} \|u_k\|_{L^\infty} \|\omega\|_{L^\infty} \|u_k\|_{L^\infty} \]

and so therefore

\[ (IV) \lesssim \|y^*\omega\|_{\mathcal{L}_2} \|y^*u_k\|_{\mathcal{L}_2} \|\omega\|_{L^\infty} \|u_k\|_{L^\infty} + \|y^*\omega\|_{\mathcal{L}_2} \|y^*u_k\|_{\mathcal{L}_2} \|\omega\|_{L^\infty} \|u_k\|_{L^\infty} \]

Also,
Again, \( (V) = \int J_y^s(\omega^2 \partial_x u_k) J_y^s \omega = \int [J_y^s(\omega^2 \partial_x u_k) - \omega^2 J_y^s \partial_x u_k] J_y^s \omega + \int \omega^2 J_y^s \partial_x u_k J_y^s \omega \) and we denote \((V)_1 = \int [J_y^s(\omega^2 \partial_x u_k) - \omega^2 J_y^s \partial_x u_k] J_y^s \omega \) and \((V)_2 = \int \omega^2 J_y^s \partial_x u_k J_y^s \omega \). We have by the Kato-Ponce commutator estimate

\[
(V)_1 \lesssim \|J_y^s \omega\|_{L^2_y} \|J_y^s(\omega^2 \partial_x u_k) - \omega^2 J_y^s \partial_x u_k\|_{L^2_y} \\
\lesssim \|J_y^s \omega\|_{L^2_y} \left[\|\partial_x u_k\|_{L^\infty_y} \|J_y^s \omega\|_{L^2_y} + \|\omega\|_{L^\infty_y} \|\partial_x \omega\|_{L^\infty_y} \right] \|J_y^s \partial_x u_k\|_{L^2_y} \\
\lesssim \|J_y^s \omega\|^2_{L^2_y} (\beta_u(t) + \beta_x(t)) + \|J_y^s \omega\|_{L^2_y} \|J_y^s u_k\|_{L^2_y} \beta_x(t) + \|J_y^s \omega\|_{L^2_y} \|J_y^s u_k\|_{L^2_y} \beta_x(t).
\]

Also,

\[
(V)_2 \lesssim \|J_y^s \omega\|_{L^2_y} \|J_x^{s+1} u_k\|_{L^2_y} \|\omega\|^2_{L^2_y} + \|J_y^s u_k\|_{L^2_y} \|J_x^s u_k\|_{L^2_y} \|\omega\|^2_{L^2_y}
\]

and so therefore

\[
(V) \lesssim \|J_y^s \omega\|^2_{L^2_y} (\beta_u(t) + \beta_x(t)) + \|J_y^s \omega\|_{L^2_y} \|J_y^s u_k\|_{L^2_y} \beta_x(t) + \|J_y^s \omega\|_{L^2_y} \|J_x^{s+1} u_k\|_{L^2_y} \|\omega\|^2_{L^2_y}.
\]

Now, putting together all the terms we get that

\[
\frac{d}{dt} \|J_y^s \omega\|^2_{L^2_y} \lesssim (\|J_y^s \omega\|^2_{L^2_y} + \|J_y^s \omega\|_{L^2_y} \|J_y^s u_k\|_{L^2_y})(\beta(t) + \beta_u(t)) \\
+ \left(\|J_y^s \omega\|_{L^2_y} \|J_x^s \omega\|_{L^2_y} \|J_x^s u_k\|_{L^2_y} \|J_y^s u_k\|_{L^2_y}\right)(\beta(t) + \beta_u(t)) \\
+ \|J_y^s \omega\|_{L^2_y} \|J_x^{s+1} u_k\|_{L^2_y} \|\omega\|^2_{L^2_y} + \|\omega\|_{L^\infty_y} \|u_k\|_{L^\infty_y} \\
\lesssim \|J_y^s \omega\|^2_{L^2_y} (\beta(t) + \beta_u(t)) \\
+ \left(\|J_y^s \omega\|^2_{L^2_y} + \|J_x^s u_k\|_{L^2_y} \|J_y^s u_k\|_{L^2_y}\right)(\beta(t) + \beta_u(t)) \\
+ \|J_y^s \omega\|_{L^2_y} \|J_x^{s+1} u_k\|_{L^2_y} \|\omega\|^2_{L^\infty_y} + \|u_k\|^2_{L^\infty_y} \] (46)
\]

Using the variant of Grönwall’s inequality from part a) and applying it to (46) with \( \beta(t) = \beta_x(t) + \beta_u(t) \geq 0 \) and

\[
\alpha(t) = \left(\|J_y^s \omega\|^2_{L^2_y} + \|J_x^s u_k\|_{L^2_y} \|J_y^s u_k\|_{L^2_y}\right)(\beta(t) + \beta_u(t)) + \|J_x^{s+1} u_k\|_{L^2_y} \|\omega\|^2_{L^2_y} + \|u_k\|^2_{L^\infty_y}
\]

\[
\|J_y^s \omega\|^2_{L^2_y} \lesssim \exp\left(f_u(T)^2 + f_u(T)^2\left[\|J_y^s \omega(0)\|^2_{L^2_y}
\right.\right. \\
+ \left.\left.\left(\|J_y^s \omega\|^2_{L^2_y} + \|J_x^s u_k\|^2_{L^2_y} + \|J_y^s u_k\|^2_{L^2_y}\right)(f_u(T)^2 + f_u(T)^2) + \|J_x^{s+1} u_k\|_{L^2_y} \|\omega\|^2_{L^\infty_y} + \|u_k\|^2_{L^\infty_y}\right) \right] \\
\]

therefore

\[
\|J_y^s \omega\|_{L^\infty T L^2_y} \lesssim \exp\left(\frac{1}{2} f_u(T) + \frac{1}{2} f_u(T)\right) \left[\|J_y^s \omega(0)\|_{L^2_y}
\right.\right. \\
+ \left.\left.\left(\|J_y^s \omega\|_{L^2_y} + \|J_x^s u_k\|^2_{L^2_y} + \|J_y^s u_k\|^2_{L^2_y}\right)(f_u(T) + f_u(T))
\right.\right. \\
+ \left.\left.\left.\|J_x^{s+1} u_k\|_{L^2_y} \|\omega\|^2_{L^\infty_y} + \|u_k\|^2_{L^\infty_y}\right) \right] \\
\]

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Lemma 7. Suppose \( u_k \) satisfies the IVP \( (*) \) with initial data \( \phi_k = P_{(3)}^6 \phi \). We have \( f_{u_k}(T) \lesssim k^{0-} \) and \( f_\omega(T) \lesssim k^{0-} \) as \( k \to \infty \).

**Proof.** Take \( \delta < \frac{s-2}{2} \). By the linear estimate in Proposition \[ \Box \]

\[
\| u_k \|_{L^2_t L^\infty_x} \lesssim \| J_x^{1+\delta} u_k \|_{L^\infty_t L^2_x} + \| J_y^{-1} J_x^{1+\delta} u_k \|_{L^\infty_t L^2_y} + \| J_x^{1+\delta} J_y^3 (u_k^3) \|_{L^1_x L^2_y},
\]

and since \( \| J_x^{1+\delta} u_k \|_{L^\infty_t L^2_x} \lesssim k^{1+\delta-s} \), \( \| J_x^{-1} J_y^{1+\delta} u_k \|_{L^\infty_t L^2_x} \lesssim k^{1+\delta-s} \), also

\[
\| J_x^{1+\delta} J_y^3 (u_k^3) \|_{L^1_x L^2_y} \lesssim \| J_x^{2+2\delta} (u_k^3) \|_{L^1_x L^2_y} + \| J_y^{2+2\delta} (u_k^3) \|_{L^1_y L^2_x} \lesssim C_T k^{1+\delta-s}
\]

by the corollary of the Kato-Ponce commutator estimate together with the a priori estimate

\[
\| J_x^{1+\delta} J_y^3 (u_k^3) \|_{L^1_x L^2_y} \lesssim \| J_x^{2+2\delta} (u_k^3) \|_{L^1_x L^2_y} + \| J_y^{2+2\delta} (u_k^3) \|_{L^1_y L^2_x} \lesssim f_{u_k}(T)^2 \| J_x^{2+2\delta} u_k \|_{L^1_x L^2_y} + \| J_y^{2+2\delta} u_k \|_{L^1_y L^2_x} \lesssim C_T k^{1+\delta-s}
\]

It yields \( \| u_k \|_{L^2_t L^\infty_x} \lesssim k^{1+\delta-s} \to 0 \) as \( k \to \infty \) since \( 1 + 2\delta < s \). We also have

\[
\| \partial_x u_k \|_{L^2_t L^\infty_x} \lesssim \| J_x^{2+\delta} u_k \|_{L^\infty_t L^2_x} + \| J_x^{1+\delta} u_k \|_{L^\infty_t L^2_x} + \| J_x^{2+\delta} J_y^3 (u_k^3) \|_{L^1_x L^2_y}
\]

and since \( \| J_x^{2+\delta} u_k \|_{L^\infty_t L^2_x} \lesssim k^{2+\delta-s} \), \( \| J_y^{1+\delta} u_k \|_{L^\infty_t L^2_y} \lesssim k^{1+\delta-s} \) and finally by the corollary of the Kato-Ponce commutator estimate together with the a priori estimate

\[
\| J_x^{2+\delta} J_y^3 (u_k^3) \|_{L^1_x L^2_y} \lesssim \| J_x^{2+2\delta} (u_k^3) \|_{L^1_x L^2_y} + \| J_y^{2+2\delta} (u_k^3) \|_{L^1_y L^2_x} \lesssim f_{u_k}(T)^2 \| J_x^{2+2\delta} u_k \|_{L^1_x L^2_y} + \| J_y^{2+2\delta} u_k \|_{L^1_y L^2_x} \lesssim C_T k^{2+2\delta-s}
\]

which gives us \( \| \partial_x u_k \|_{L^2_t L^\infty_x} \lesssim k^{2+2\delta-s} \to 0 \) as \( k \to 0 \) since \( 2 + 2\delta < s \).

Lastly,

\[
\| \partial_y u_k \|_{L^2_t L^\infty_x} \lesssim \| J_x^{1+\delta} \partial_y u_k \|_{L^\infty_t L^2_x} + \| J_x^{-1} J_y^{2+\delta} u_k \|_{L^\infty_t L^2_x} + \| J_x^{1+\delta} J_y^3 \partial_y u_k \|_{L^1_x L^2_y}
\]

and since

\[
\| J_x^{1+\delta} \partial_y u_k \|_{L^\infty_t L^2_x} \lesssim \| J_x^{2+\delta} u_k \|_{L^\infty_t L^2_x} + \| J_y^{2+\delta} u_k \|_{L^\infty_t L^2_y} \lesssim k^{2+\delta-s} + k^{2+\delta-s}
\]

also

\[
\| J_y^{-1} J_x^{2+\delta} u_k \|_{L^\infty_t L^2_x} \lesssim \| J_x^{2+\delta} u_k \|_{L^\infty_t L^2_x} \lesssim k^{2+\delta-s}
\]

and finally by the Kato-Ponce commutator estimate together with the a priori estimate

\[
\| J_x^{1+\delta} J_y^3 (u_k^3) \|_{L^1_x L^2_y} \lesssim \| J_x^{1+\delta} J_y^{1+\delta} u_k \|_{L^\infty_t L^2_y} \lesssim \| J_x^{2+2\delta} u_k \|_{L^\infty_t L^2_y} + \| J_y^{2+2\delta} u_k \|_{L^\infty_t L^2_y} \lesssim f_{u_k}(T)^2 \| J_x^{2+2\delta} u_k \|_{L^1_x L^2_y} + \| J_x^{2+2\delta} u_k \|_{L^1_x L^2_y} \lesssim C_T k^{2+2\delta-s}
\]

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This yields \( \| \partial_y u_k \|_{L^2_y L^\infty_y} \lesssim k^{2+2\delta-s} \to 0 \) as \( k \to \infty \) since \( 2 + 2\delta < s \). Therefore,

\[
f_{u_k}(T) \lesssim k^{2+2\delta-s}
\]

Since

\[
\| \omega \|_{L^2_y L^\infty_y} \lesssim \| u_k \|_{L^2_y L^\infty_y} + \| u_{\kappa'} \|_{L^2_y L_y^\infty} \lesssim k^{1+2\delta-s} + (k')^{1+2\delta-s} \lesssim k^{1+2\delta-s}
\]

and also \( f_\omega(T) \leq f_{u_k}(T) + f_{u_{\kappa'}}(T) \lesssim k^{2+2\delta-s} + k^{2+2\delta-s} \lesssim k^{2+2\delta-s} \), we get

\[
f_\omega(T) \lesssim k^{2+2\delta-s}.
\]

\[
\square
\]

**Lemma 8.** Suppose \( u_k \) satisfies the IVP \((\star)\) with initial data \( \phi_k = \tilde P_{(3)}^k \phi \). We have \( f_{u_k}(T) \lesssim k^{0-} \) and \( f_\omega(T) \lesssim k^{0-} \) as \( k \to \infty \).

**Proof.** Take \( \delta < \frac{s}{2} \). By the linear estimate in Proposition 2

\[
\| u_k \|_{L^2_y L^\infty_y} \lesssim \| J_x^{\frac{11}{8} + \delta} u_k \|_{L^\infty_y L^2_y} + \| J_y^{\frac{11}{8}} J_y^{1+\delta} u_k \|_{L^\infty_y L^2_y} + \| J_x^{\frac{11}{8} + \delta} J_y^3 (u_k^3) \|_{L^1_y L_y^2}
\]

and since \( \| J_x^{\frac{11}{8} + \delta} u_k \|_{L^\infty_y L^2_y} \lesssim k^{\frac{11}{8} + \delta-s} \), \( \| J_y^{\frac{11}{8}} J_y^{1+\delta} u_k \|_{L^\infty_y L^2_y} \lesssim k^{1+\delta-s} \), also by the Kato-Ponce commutator estimate together with the a priori estimate

\[
\| J_y^{\frac{11}{8} + \delta} J_y^3 (u_k^3) \|_{L^1_y L_y^2} \lesssim \| J_y^{\frac{11}{8} + 2\delta} (u_k^3) \|_{L^1_y L^2_y} + \| J_y^{\frac{11}{8} + 2\delta} (u_k^3) \|_{L^1_y L^2_y} \lesssim f_{u_k}(T)^2 (\| J_x^{\frac{11}{8} + 2\delta} u_k \|_{L^1_y L^2_y} + \| J_y^{\frac{11}{8} + \delta} u_k \|_{L^1_y L^2_y}) \lesssim C_T k^{\frac{11}{8} + \delta-s}.
\]

It yields \( \| u_k \|_{L^2_y L^\infty_y} \lesssim k^{\frac{11}{8} + 2\delta-s} \to 0 \) as \( k \to \infty \) since \( \frac{11}{8} + 2\delta < s \).

We also have

\[
\| \partial_x u_k \|_{L^2_y L^\infty_y} \lesssim \| J_x^{\frac{11}{8} + \delta} u_k \|_{L^\infty_y L^2_y} + \| J_y^{\frac{3}{8}} J_y^{1+\delta} u_k \|_{L^\infty_y L^2_y} + \| J_y^{1+\delta} (u_k^3) \|_{L^1_y L^2_y}
\]

and since \( \| J_x^{\frac{11}{8} + \delta} u_k \|_{L^\infty_y L^2_y} \lesssim k^{\frac{11}{8} + \delta-s} \), \( \| J_y^{\frac{3}{8}} J_y^{1+\delta} u_k \|_{L^\infty_y L^2_y} \lesssim k^{1+\delta-s} \), \( \| J_y^{1+\delta} (u_k^3) \|_{L^1_y L^2_y} \lesssim k^{1+\delta-s} \) and finally by the corollary of the Kato-Ponce commutator estimate together with the a priori estimate

\[
\| J_y^{\frac{11}{8} + \delta} J_y^3 (u_k^3) \|_{L^1_y L^2_y} \lesssim \| J_x^{\frac{11}{8} + 2\delta} (u_k^3) \|_{L^1_y L_y^2} + \| J_y^{\frac{11}{8} + 2\delta} (u_k^3) \|_{L^1_y L_y^2} \lesssim f_{u_k}(T)^2 (\| J_x^{\frac{11}{8} + 2\delta} u_k \|_{L^1_y L^2_y} + \| J_y^{\frac{3}{8} + \delta} u_k \|_{L^1_y L^2_y}) \lesssim C_T k^{\frac{11}{8} + 2\delta-s}
\]

which gives us \( \| \partial_x u_k \|_{L^2_y L^\infty_y} \lesssim k^{\frac{11}{8} + 2\delta-s} \to 0 \) as \( k \to 0 \) since \( 2 + 2\delta < s \).

Lastly,

\[
\| \partial_y u_k \|_{L^2_y L^\infty_y} \lesssim \| J_x^{\frac{11}{8} + \delta} \partial_y u_k \|_{L^\infty_y L^2_y} + \| J_y^{\frac{3}{8}} J_y^{1+\delta} u_k \|_{L^\infty_y L^2_y} + \| J_y^{\frac{11}{8} + \delta} J_y^3 \partial_y u_k \|_{L^1_y L^2_y}
\]

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and since
\[ \| J_\delta^{1/8 + \delta} \partial_y u_k \|_{L_y^\infty L_x^2} \lesssim \| J_x^{1/8 + 2\delta} u_k \|_{L_y^\infty L_x^2} + \| J_y^{1/8 + \delta} u_k \|_{L_y^\infty L_x^2} \lesssim k^{10/8 - \delta} + k^{11/8 + \delta - s} \]
also
\[ \| J_x^{-\delta/8} J_y^{2 + \delta} u_k \|_{L_y^\infty L_x^2} \lesssim \| J_y^{2 + \delta} \|_{L_y^\infty L_x^2} \lesssim k^{2 + \delta - s} \]
and finally by the Kato-Ponce commutator estimate together with the a priori estimate
\[ \| J_x^{1/8 + \delta} J_y^{1/8 + \delta} \partial_y (u_k^3) \|_{L_y^\infty L_x^2} \lesssim \| J_x^{1/8 + \delta} J_y^{1/8 + \delta} u_k \|_{L_y^\infty L_x^2} \lesssim \| J_x^{1/8 + \delta} u_k \|_{L_y^\infty L_x^2} + \| J_y^{1/8 + \delta} u_k \|_{L_y^\infty L_x^2} \lesssim f_u(T)^2 (\| J_x^{1/8 + \delta} u_k \|_{L_y^\infty L_x^2} + \| J_y^{1/8 + \delta} u_k \|_{L_y^\infty L_x^2}) \lesssim C_T k^{11/8 + \delta - s}. \]
This yields \( \| \partial_y u_k \|_{L_y^2 L_x^2} \lesssim k^{10/8 + \delta - s} \to 0 \) as \( k \to \infty \) since \( \frac{10}{8} + \delta < s \). Therefore,
\[ f_{u_k}(T) \lesssim k^{10/8 + \delta - s} \]

Since
\[ \| \dot{\omega} \|_{L_y^2 L_x^2} \lesssim \| u_k \|_{L_y^2 L_x^\infty} + \| u_k' \|_{L_y^2 L_x^\infty} \lesssim k^{11/8 + 2\delta - s} + (k')^{11/8 + 2\delta - s} \lesssim k^{11/8 + 2\delta - s} \]
and also \( f_{\omega}(T) \leq f_{u_k}(T) + f_{u_k'}(T) \lesssim k^{10/8 + 2\delta - s} + k^{11/8 + 2\delta - s} \lesssim k^{10/8 + 2\delta - s} \), we get
\[ f_{\omega}(T) \lesssim k^{10/8 + 2\delta - s} \]

Lemma 9. Suppose \( u_k \) satisfies the IVP (8) with initial data \( \phi_k = P^k(\delta) \phi \). We have \( f_{u_k}(T) \lesssim k^{0 -} \) and \( f_{\omega}(T) \lesssim k^{0 -} \) as \( k \to \infty \).

Proof. Take \( 0 < \delta < s - \frac{5}{2} \). By the linear estimate in Proposition 3
\[ \| u_k \|_{L_y^2 L_x^\infty} \lesssim \| J_x^{3/8 + \delta} u_k \|_{L_y^\infty L_x^2} + \| J_x^{-3/8 + \delta} J_y^{1/8 + \delta} u_k \|_{L_y^\infty L_x^2} + \| J_y^{3/8 + \delta} J_y^{1/8 + \delta} (u_k^3) \|_{L_y^1 L_x^2} \]
and since \( \| J_x^{3/8 + \delta} u_k \|_{L_y^2 L_x^2} \lesssim k^{\frac{3}{8} + \delta - s} \to 0 \) and
\[ \| J_x^{-3/8 + \delta} J_y^{1/8 + \delta} u_k \|_{L_y^\infty L_x^2} \lesssim \| J_y^{1/8 + \delta} u_k \|_{L_y^\infty L_x^2} \lesssim k^{11/8 - \delta - s} \]
and \( \| J_x^{1/8 + \delta} J_y^{1/8 + \delta} (u_k^3) \|_{L_y^2} \leq \| J_x^{1/8 + 2\delta} (u_k^3) \|_{L_y^2} + \| J_y^{1/8 + 2\delta} (u_k^3) \|_{L_y^2} \) so by the Kato-Ponce commutator estimates,
\[ \| J_x^{1/8 + 2\delta} (u_k^3) \|_{L_y^2} \leq \| J_x^{1/8 + 2\delta} u_k \|_{L_y^\infty} \| u_k \|_{L_y^2}^2 \] and \( \| J_y^{1/8 + 2\delta} (u_k^3) \|_{L_y^2} \lesssim \| J_y^{1/8 + \delta} u_k \|_{L_y^\infty} \| u_k \|_{L_y^2}^2 \)
and so
\[ \| J_x^{1/8 + \delta} J_y^{1/8 + \delta} (u_k^3) \|_{L_y^1 L_x^2} \lesssim C_T k^{\frac{3}{8} + \delta - s}. \]
It yields $\|u_k\|_{L^2_x L^\infty_y} \leq k^{\frac{5}{2} + \delta - s} \to 0$ as $k \to \infty$ since $\frac{5}{2} + \delta < s$.

We also have

$$\|\partial_x u_k\|_{L^2_t L^\infty_y} \leq \|J_x^{\frac{5}{2} + \delta} u_k\|_{L^\infty_t L^2_y} + \|J_x^{\frac{5}{2} + \delta} J_y^{1 + \delta} u_k\|_{L^\infty_t L^2_y} + \|J_x^{\frac{5}{2} + \delta} (u_k^2)\|_{L^1_t L^2_y}$$

and since $\|J_x^{\frac{5}{2} + \delta} u_k\|_{L^\infty_t L^2_y} \lesssim k^{\frac{5}{2} + \delta - s}$, $\|J_x^{\frac{5}{2} + \delta} J_y^{1 + \delta} u_k\|_{L^\infty_t L^2_y} \lesssim \|J_y^{1 + \delta} u_k\|_{L^\infty_t L^2_y} \lesssim k^{1 + \delta - s}$ and

$$\|J_x^{\frac{3}{2} + \delta} J_y^{3} (u_k^3)\|_{L^2_y} \leq \|J_x^{\frac{3}{2} + \delta} (u_k^3)\|_{L^2_y} + \|J_y^{\frac{3}{2} + \delta} (u_k^3)\|_{L^2_y}$$

by the Kato-Ponce commutator estimates,

$$\|J_x^{\frac{5}{2} + \delta} (u_k^3)\|_{L^2_y} \lesssim \|J_x^{\frac{5}{2} + \delta} u_k\|_{L^2_y} \leq \|J_x^{\frac{5}{2} + \delta} (u_k^3)\|_{L^2_y} \lesssim \|J_x^{\frac{5}{2} + \delta} u_k\|_{L^2_y} \|\omega\|_{L^1_x L^\infty_y}$$

and so

$$\|J_x^{\frac{5}{2} + \delta} J_y^{3} (u_k^3)\|_{L^1_t L^2_y} \lesssim C_T k^{\frac{5}{2} + \delta - s}.$$

This yields $\|\partial_x u_k\|_{L^1_t L^\infty_y} \lesssim k^{\frac{5}{2} + \delta - s} \to 0$ as $k \to \infty$ since $\frac{5}{2} + \delta < s$.

Lastly,

$$\|\partial_y u_k\|_{L^2_t L^\infty_y} \lesssim \|J_x^{\frac{5}{2} + \delta} \partial_y u_k\|_{L^\infty_t L^2_y} + \|J_x^{\frac{5}{2} + \delta} J_y^{2 + \delta} u_k\|_{L^\infty_t L^2_y} + \|J_x^{\frac{5}{2} + \delta} J_y^{3} \partial_y u_k\|_{L^1_t L^2_y}$$

and since

$$\|J_x^{\frac{5}{2} + \delta} \partial_y u_k\|_{L^2_t L^\infty_y} \lesssim \|J_x^{\frac{5}{2} + \delta} u_k\|_{L^2_y} \|\omega\|_{L^1_x L^\infty_y} + \|J_x^{\frac{5}{2} + \delta} (u_k^3)\|_{L^2_y} \leq k^{\frac{5}{2} + \delta - s} + k^{\frac{5}{2} + \delta - s}$$

and

$$\|J_x^{\frac{5}{2} + \delta} J_y^{2 + \delta} u_k\|_{L^2_y} \lesssim \|J_y^{2 + \delta} u_k\|_{L^2_y} \lesssim k^{2 + \delta - s}$$

and

$$\|J_x^{\frac{5}{2} + \delta} J_y^{3} \partial_y (u_k^3)\|_{L^2_y} \lesssim \|J_x^{\frac{5}{2} + \delta} J_y^{1 + \delta} (u_k^3)\|_{L^2_y} \leq \|J_x^{\frac{3}{2} + \delta} (u_k^3)\|_{L^2_y} + \|J_x^{\frac{3}{2} + \delta} (u_k^3)\|_{L^2_y}$$

so by the same reasoning, after applying the Kato-Ponce commutator estimates we get

$$\|J_x^{1 + \delta} J_y^{3} (u^2 \partial_y u_k)\|_{L^2_y} \lesssim C_T k^{\frac{5}{2} + \delta - s}.$$

This yields $\|\partial_y u_k\|_{L^2_t L^\infty_y} \lesssim k^{\frac{5}{2} + \delta - s} \to 0$ as $k \to \infty$ since $\frac{5}{2} + \delta < s$.

Therefore,

$$f_{u_k}(T) \lesssim k^{\frac{5}{2} + \delta - s}.$$

Since

$$\|\omega\|_{L^2_t L^\infty_y} \leq \|u_k\|_{L^2_t L^\infty_y} + \|u_{k'}\|_{L^2_t L^\infty_y} \lesssim k^{\frac{5}{2} + 2\delta - s} + (k')^{\frac{5}{2} + 2\delta - s} \lesssim k^{\frac{5}{2} + 2\delta - s}$$

and also $f_{\omega}(T) \leq f_{u_k}(T) + f_{u_{k'}}(T) \lesssim k^{\frac{5}{2} + \delta - s} + k^{\frac{5}{2} + \delta - s} \lesssim k^{\frac{5}{2} + \delta - s}$, we get

$$f_{\omega}(T) \lesssim k^{\frac{5}{2} + \delta - s}.$$
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**Corollary.** We have \( \|\omega\|_{H^{s,+}} \to 0 \) as \( k \to \infty \), where \( s > 2 \) for the initial value problem (7), \( s > \frac{19}{8} \) for the initial value problem (3) and \( s > \frac{5}{2} \) for initial value problem (8).

**Proof.** From (42) and Lemmas 7, 8 and 9 we get \( \|J_x^{s+1} u_k\|_{L_x^\infty L_y^2} \|\omega\|_{L_x^2 L_y^\infty} \lesssim k^1 \cdot k^{-1} = k^0 \)

and \( \|J_y^{s+1} u_k\|_{L_x^\infty L_y^2} \|\omega\|_{L_x^2 L_y^\infty} \lesssim k^1 \cdot k^{-1} = k^0 \)

and \( k^0 \rightarrow 0 \) as \( k \rightarrow \infty \). From the Lemmas 7, 8 and 9 we obtain

\[
\|J_x^s \omega\|_{L_x^\infty L_y^2} \lesssim \exp(k^0)(\|J_x^s \omega(0)\|_{L_x^\infty L_y^2} + Ck^0) \to 0
\]
as \( k \rightarrow \infty \).

From (43) and Lemmas 7, 8 and 9 we get \( \|J_y^{s+1} u_k\|_{L_x^\infty L_y^2} \|\omega\|_{L_x^2 L_y^\infty} \lesssim k^1 \cdot k^{-1} = k^0 \)

and \( \|J_y^{s+1} u_k\|_{L_x^\infty L_y^2} \|\omega\|_{L_x^2 L_y^\infty} \lesssim k^1 \cdot k^{-1} = k^0 \)

and \( k^0 \rightarrow 0 \) as \( k \rightarrow \infty \). From the Lemmas 7, 8 and 9 we obtain

\[
\|J_y^s \omega\|_{L_x^\infty L_y^2} \lesssim \exp(k^0)(\|J_y^s \omega(0)\|_{L_x^\infty L_y^2} + Ck^0) \to 0
\]
as \( k \rightarrow \infty \).

Therefore, as \( \|J_x^s \omega\|_{L_x^\infty L_y^2} + \|J_y^s \omega\|_{L_x^\infty L_y^2} \to 0 \) as \( k \rightarrow \infty \), it means that \( u \in C([0,T] : H^{s,+}) \).

## 8 Continuity of the flow map

We assume that \( T \in [0, \infty) \) and \( \phi^l \to \phi \) in \( H^{s,+}(M \times \mathbb{T}) \) as \( l \to \infty \). We are going to prove that \( u^l \to u \) in \( C([-T,T] : H^{s,+}(M \times \mathbb{T})) \) as \( l \to \infty \), where \( u^l \) and \( u \) are solutions of the initial value problem \( \partial_t u + (-1)^{d-1} \frac{d-1}{2} \partial_x^2 u - \partial_x^3 \partial_x^3 u + u^2 \partial_x u = 0 \) corresponding to initial data \( \phi^l \) and \( \phi \), for \( d = 3, M = \mathbb{R}, s > 2 \), for \( d = 3, M = \mathbb{T}, s > \frac{19}{8} \) and for \( d = 5, M = \mathbb{R}, s > \frac{5}{2} \).

For \( k \geq 1 \), let as before, \( \phi^l_k = P^k \phi^l \) and \( u^l_k \in C([-T,T] : H^\infty) \) the corresponding solutions. Denote by \( \omega_k = u_k - u \). We know from Lemma 5 that

\[
\|\omega_k\|_{H^{s,+}} \lesssim \exp(f_{\omega_k}(T) + f_{u_k}(T)) [\|\omega(0)\|_{H^{s,+}} + \|u_k\|_{L_x^\infty H^{s,+}} (f_{u_k}(T) + f_{\omega_k}(T))
\]

\[
+ (\|J_x^{s+1} u_k\|_{L_x^\infty L_y^2} + \|J_y^{s+1} u_k\|_{L_x^\infty L_y^2}) (\|\omega_k\|_{L_x^2 L_y^\infty} + \|u_k\|_{L_x^2 L_y^\infty})
\]

and since, by Lemmas 7, 8 and 9 we have \( f_{\omega_k}(T) + f_{u_k}(T) \lesssim k^0 \), \( \|\omega_k\|_{L_x^2 L_y^\infty} \lesssim k^1 \) so

\[
(\|J_x^{s+1} u_k\|_{L_x^\infty L_y^2} + \|J_y^{s+1} u_k\|_{L_x^\infty L_y^2}) (\|\omega_k\|_{L_x^2 L_y^\infty} + \|u_k\|_{L_x^2 L_y^\infty}) \lesssim k^0
\]

therefore

\[
\|u_k - u\|_{H^{s,+}} \lesssim \|\phi_k - \phi\|_{H^{s,+}} + C(T, \|\phi_k\|_{H^{s,+}}, \|\phi\|_{H^{s,+}}) k^0
\]

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By the same reasoning, we have that
\[ \|u_k^l - u^l\|_{H^{s,s}} \lesssim \|\phi_k^l - \phi^l\|_{H^{s,s}} + \mathcal{C}(T, \|\phi_k^l\|_{H^{s,s}}, \|\phi^l\|_{H^{s,s}})k^{0-}. \]

Now, denote \( \omega_k^l = u_k^l - u_k \). By the same estimates from Lemma 5 we have
\[ \|\omega_k^l\|_{H^{s,s}} \lesssim \exp(f_{u_k}(T) + f_{u_k}(T)) \left[ \|\omega_k^l(0)\|_{H^{s,s}} + \|u_k\|_{L_{T}^{x}H_{x,y}^{s,s}}(f_{u_k}(T) + f_{u_k}(T)) \right. \\
+ \left. (\|J_{x}^{s+1}u_k\|_{L_{T}^{x}L_{x,y}^{s}} + \|J_{y}^{s+1}u_k\|_{L_{T}^{y}L_{x,y}^{s}})(\|\omega_k^l\|_{L_{T}^{x}L_{x,y}^{s}} + \|u_k\|_{L_{T}^{x}L_{x,y}^{s}}) \right] \]
and again since, by Lemmas 7, 8 and 9 we have
\[ f_{u_k}(T) + f_{u_k}(T) \lesssim f_{u_k}(T) + f_{u_k}(T) \lesssim k^{0-}, \]
\[ \|\omega_k^l\|_{L_{T}^{x}L_{x,y}^{s}} \lesssim \|u_k\|_{L_{T}^{x}L_{x,y}^{s}} + \|u_k\|_{L_{T}^{x}L_{x,y}^{s}} \leq k \]
so
\[ (\|J_{x}^{s+1}u_k\|_{L_{T}^{x}L_{x,y}^{s}} + \|J_{y}^{s+1}u_k\|_{L_{T}^{y}L_{x,y}^{s}})(\|\omega_k^l\|_{L_{T}^{x}L_{x,y}^{s}} + \|u_k\|_{L_{T}^{x}L_{x,y}^{s}}) \leq k^{0-} \]
therefore
\[ \|u_k^l - u_k\|_{H^{s,s}} = \|\omega_k\|_{H^{s,s}} \lesssim \|\phi_k^l - \phi_k\|_{H^{s,s}} + \mathcal{C}(T, \|\phi_k^l\|_{H^{s,s}}, \|\phi_k\|_{H^{s,s}})k^{0-}. \]

By the triangle inequality, we get
\[ \|u^l - u\|_{H^{s,s}} \leq \|u_k - u\|_{H^{s,s}} + \|u_k^l - u_k\|_{H^{s,s}} + \|u_k^l - u_k\|_{H^{s,s}} \]

which, by letting \( k \to \infty \), we get \( \|u^l - u\|_{H^{s,s}} \lesssim \|\phi^l - \phi\|_{H^{s,s}} \) and proves the continuity of the flow map.

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