Finite determinacy of matrices and ideals in arbitrary characteristic

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Abstract

Let $M_{m,n}$ be the ring of $m \times n$ matrices $A$ with entries in $R = K[[x_1,\ldots,x_s]]$, the ring of formal power series over the field $K$ of arbitrary characteristic with maximal ideal $m$. We call $A$ finitely determined if any matrix $B$, with $A - B \in m^kM_{m,n}$ for some finite integer $k$, is left–right equivalent to $A$, i.e. $B$ is contained in the $G$–orbit of $A$, where $G$ is the group of automorphisms of $R$ acting by coordinate change, combined with the multiplication of invertible matrices acting from the left and from the right on $M_{m,n}$. Finite determinacy is an important property, which implies that $A$ is $G$–equivalent to a matrix with polynomial entries but is in general much stronger. It has been intensively studied for one power series ($m=n=1$) over the complex and real numbers in connection with the classification of singularities. In positive characteristic the problem for matrices has been addressed in our previous paper [GP16], where it was shown that finite codimension of the tangent image, i.e. the image of the tangent space under the orbit map of $G$, is sufficient for finite determinacy. We showed also that the tangent image may differ from the tangent space to the orbit, in contrary to characteristic zero. The question whether finite codimension of the tangent image is necessary for finite determinacy remains open in general in positive characteristic.

In this paper we answer this question positively for 1-column matrices. For this we prove that the Fitting ideals of a finitely determined matrix (of arbitrary size) have maximal height. 1-column matrices are of particular interest, since $G$-equivalence of matrices corresponds to contact equivalence of the ideals in $R$ generated by their entries. Here two ideals $I$ and $J$ are contact equivalent if the $K$–algebras $R/I$ and $R/J$ are isomorphic. Our result on matrices implies that a positive dimensional ideal is finitely contact-determined if and only if it is an isolated complete intersection singularity. In addition we give explicitly computable and semicontinuous determinacy bounds. We discuss also several open problems which are of independent interest.
1 Introduction

Throughout this paper let $K$ denote a field of arbitrary characteristic and 

$$R := K[[x]] = K[[x_1, \ldots, x_s]]$$

the formal power series ring over $K$ in $s$ variables with maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_s \rangle$.

We denote by 

$$M_{m,n} := \text{Mat}(m,n,R)$$

the set of all $m \times n$ matrices with entries in $R$. Two matrices $A, B \in M_{m,n}$ are called $G_{lr}$-equivalent or left–right equivalent, denoted $A \overset{G_{lr}}{\sim} B$, if there are invertible matrices $U \in GL(m,R)$, $V \in GL(n,R)$, and an automorphism $\phi \in \text{Aut}(R)$ such that $B = U \cdot \phi(A) \cdot V$, where $A = [a_{ij}(x)]$ and $\phi(A) := [a_{ij}(\phi(x))]$ with $\phi(x) := (\phi(x_1), \ldots, \phi(x_s))$. A matrix $A \in M_{m,n}$ is said to be $G_{lr}$-k-determined if for each matrix $B \in M_{m,n}$ with $B - A \in \mathfrak{m}^k + \mathfrak{m}^{k+1} \cdot M_{m,n}$, we have $B \overset{G_{lr}}{\sim} A$, i.e. if $A$ is $G_{lr}$-equivalent to every matrix which coincides with $A$ up to and including degree $k$. $A$ is called finitely $G_{lr}$-determined if there exists a positive integer $k$ such that it is $G_{lr}$-k-determined.

The $G_{lr}$-equivalence classes are the orbits under the action of the semi-direct product

$$G := G_{lr} := (GL(m,R) \times GL(n,R)) \rtimes R,$$

$R := \text{Aut}(R)$, on $M_{m,n}$ defined by

$$(U,V,\phi,A) \mapsto U \cdot \phi(A) \cdot V.$$ 

The group $G_{lr}$ is denoted by $G_l$ if $V = \text{id}$ in the above definition and matrices in the same $G_l$-orbit are called left equivalent. In the following, we write $G$ instead of $G_{lr}$, for simplicity.

Finite determinacy for map-germs over the complex numbers with respect to various equivalence relations has been intensively studied by e.g. [Tou68], [Mat68], [Wal81], [Bri67], [Gal79], [dP80], [Dam81], [BdPW87], and many more. In [BK16], the authors study finite determinacy for matrices with entries in the convergent power series ring $\mathbb{C}\{x\}$, without giving determinacy bounds. In [Pha16] and [GP16] the authors started the study of finite determinacy for matrices of power series over fields of arbitrary characteristic. The case of one power series, i.e. $m = n = 1$, is classical over the complex and real numbers. It was treated over a field of arbitrary characteristic in [GK90] for contact equivalence and in [BGM12] for right and contact equivalence.

Let us give an overview of the main results together with some open problems of the present paper.
A fundamental observation in [GP16] was that, in positive characteristic, the tangent space to the orbit $GA$ does in general not coincide with the image of the tangent map of the orbit map. Instead of the tangent space to $GA$ we have to consider the image of the tangent map to the orbit map $G \to GA$, which we call the tangent image. The tangent image for several group actions has been determined in [GP16].

For $A \in m \cdot M_{m,n}$ we define the following submodules of $M_{m,n}$,

$$\tilde{T}_A(GA) := \langle E_{m,pq} \cdot A \rangle + \langle A \cdot E_{n,hl} \rangle + m \cdot \left< \frac{\partial A}{\partial x_\nu} \right> \text{ resp.}$$

$$\tilde{T}_A(GA) := \langle E_{m,pq} \cdot A \rangle + \langle A \cdot E_{n,hl} \rangle + \left< \frac{\partial A}{\partial x_\nu} \right>,$$

and call $\tilde{T}_A(GA)$ resp. $\tilde{T}_A(GA)$ the tangent image resp. extended tangent image at $A$ to the orbit $GA$. Here $\langle E_{m,pq} \cdot A \rangle$ is the $R$-submodule generated by $E_{m,pq} \cdot A$, $p,q = 1, \ldots, m$, with $E_{m,pq}$ the $(p,q)$-th canonical matrix of $\text{Mat}(m,m,R)$ (1 at place $(p,q)$ and 0 else) and $\left< \frac{\partial A}{\partial x_\nu} \right>$ is the $R$-submodule generated by the matrices $\frac{\partial A}{\partial x_\nu} = \left[ \frac{\partial a_{ij}}{\partial x_\nu}(x) \right], \nu = 1, \ldots, s$.

It is proved in [GP16, Corollary 4.4] that $\tilde{T}_A(GA)$ is the image of the tangent map of the orbit map $G \to GA$. Moreover, the following is proved in the same paper ([GP16, Theorem 3.2]), with $\text{ord}(A)$ the minimum of the orders of the entries of $A$.

**Theorem 1.1** ([GP16]).

1. Let $A = [a_{ij}] \in m \cdot M_{m,n}$. If there is an integer $k \geq 0$ such that

$$m^{k+2} \cdot M_{m,n} \subset m \cdot \tilde{T}_A(GA),$$

then $A$ is $G(2k-\text{ord}(A)+2)$-determined. Moreover, (1.1) holds iff $\tilde{T}_A(GA) \subset M_{m,n}$ (equivalently $\tilde{T}_A(GA)$) is of finite codimension over $K$.

2. If $\text{char}(K) = 0$ then the condition (1.1) for some $k$ is equivalent to $A$ being finitely $G$-determined.

**Remark 1.2.** The theorem gives a sufficient criterion for finite determinacy of $A$ in any characteristic, which is necessary and sufficient in characteristic 0. If $\text{char}(K) = 0$, the tangent image $\tilde{T}_A(GA)$ coincides with the tangent space $T_A(GA)$ to $GA$. In general we have $\tilde{T}_A(GA) \subset T_A(GA)$, where the inclusion can be strict in positive characteristic (see Example 2.9 in [GP16]).

**Problem 1.3.** It is not difficult to see that the finite codimension of $T_A(GA) \subset M_{m,n}$ is necessary for finite determinacy of $A$ (cf. [GP16]), but we do not know, for arbitrary
m and n, whether the finite codimension of $\tilde{T}_A(GA)$ in $M_{m,n}$ is necessary in positive characteristic. In general we do not even have a preference, it might be possible that there are counterexamples.

One of the main aims of this paper is to show that the finite codimension of $\tilde{T}_A(GA)$ in $M_{m,n}$ is in fact necessary and sufficient for finite determinacy of matrices with one column i.e. matrices in $M_{m,1}$. In this case left–right equivalence coincides with left equivalence and we prove in Theorem 3.8:

**Theorem 1.4.** For $A \in m \cdot \text{Mat}(m,1,R)$ with $K$ infinite, the following are equivalent:

1. $A$ is finitely left determined.
2. $\dim_K \left( M_{m,1}/\tilde{T}_A(GA) \right) =: d_e < \infty$.

Moreover, if condition 2. is satisfied then $A$ is $(2d_e - \text{ord}(A) + 2)$ left determined. In particular, finite determinacy is preserved under deformation of the entries of $A$.

Note that the assumption that $K$ is infinite is only needed for the implication 1. $\Rightarrow$ 2. and for $m < s$.

Consider now the ideals $I$ resp. $J$ generated by the entries of $A$ resp. $B \in M_{m,1}$. Then $A$ is left–equivalent to $B$ iff $I$ and $J$ are contact equivalent i.e. the analytic algebras $R/I$ and $R/J$ are isomorphic (c.f. Remark 4.5). Theorem 1.4 implies the following characterization of finite contact determinacy of ideals (Definition 4.4) in the power series ring $R = K[[x_1, \ldots, x_s]]$ (cf. Corollary 4.6).

**Corollary 1.5.** Let $I = \langle f_1, \ldots, f_m \rangle$ be an ideal in $m$ with $m$ the minimal number of generators of $I$ and $I_m(\left[ \frac{\partial f_i}{\partial x_j} \right])$ the ideal generated by the $m \times m$ minors of the $(m \times s)$ Jacobian matrix of the generators of $I$. For 2. assume that $K$ is infinite.

1. If $m \geq s$ then $I$ is finitely contact determined iff there is a $k$ such that $m^k \subset I$.
2. If $m \leq s$ then $I$ is finitely contact determined iff there is a $k$ such that $m^k \subset I + I_m(\left[ \frac{\partial f_i}{\partial x_j} \right])$.

Recall that $R/I$ is called a complete intersection if $\dim(R/I) = s - \text{mng}(I)$ where $\text{mng}(I)$ is the minimal number of generators of $I \subset R$. The complete intersection $R/I$ is called an isolated complete intersection singularity (ICIS) if the ideal $I + I_m(\left[ \frac{\partial f_i}{\partial x_j} \right])$ contains a power of the maximal ideal.
Theorem 1.6. Let $K$ be infinite and $I \subset R$ be an ideal with $\dim(R/I) > 0$. Then $I$ is finitely contact determined if and only if $I$ is an ICIS.

In this case $I$ is contact $(2\tau(I) - \ord(I) + 2)$-determined, where $\tau(I)$ is the Tjurina number of $I$. Moreover, if $J$ is a deformation of $I$, then $J$ is contact $(2\tau(I) + 1)$-determined.

Problem 1.7. The assumption that $K$ is infinite is due to our method of proof. In most cases, in particular in Theorem 1.6, we do not know whether it is necessary. For hypersurfaces we can in fact show, that finite determinacy is equivalent to isolated singularity for arbitrary $K$ (cf. Theorem 4.13).

To prove the results we derive a necessary condition for finite $G$-determinacy in section 2, Theorem 2.7, by using Fitting ideals. We show in particular, that the Fitting ideals of a finitely $G$-determined matrix have maximal height. For this we use the specialization of ideals depending on parameters, which was introduced by W. Krull and then extended and systematically studied by D.V. Nhi and N.V. Trung for finitely generated modules over polynomial rings.

Problem 1.8. It would be interesting to develop a satisfactory theory for specialization of ideals in power series rings depending on parameters, and to see which properties continue to hold for a generic substitution of the parameters by field elements. We show in Example 2.1 (which is of independent interest) that a straightforward generalization of specialization from polynomial rings to power series rings does not work. In Remark 2.2 we propose an approach which seems to be reasonable for uncountable fields $K$ e.g. for $\mathbb{C}$. For a concrete open problem see Problem 2.6.

In section 3 we study $G$–equivalence for 1-column matrices and use the results of section 2 to prove Theorem 1.4. We need and prove a semicontinuity result for modules over a power series ring depending on parameters (Proposition 3.7) which should be well known, but for which we could not find a reference.

In section 4 we apply the results of section 3 to contact–equivalence for general ideals and for complete intersections and prove Corollary 1.5 (c.f Corollary 4.6) and Theorem 1.6 (c.f Theorem 4.10). Finally we prove in Theorem 4.13 that (for any $K$) a power series $f \in K[[x]]$ is finitely contact (resp. right) determined iff the Tjurina number (resp. the Milnor number) of $f$ is finite.

2 A necessary finite determinacy criterion by Fitting ideals

In this section, we establish a necessary condition for finite $G$-determinacy of matrices in $M_{m,n} = \text{Mat}(m,n,R)$, $R = K[[x]]$, $x = (x_1, \ldots, x_s)$. Without loss of generality, we assume that $n \leq m$. 
We need a result about the specialization of the ideal generated by minors of a matrix over the polynomial ring depending on parameters. For an ideal $I \subset K(u)[x]$, where $u = (u_1, \ldots, u_r)$ is a new set of indeterminates, and for $a \in K^r$ the specialization $I_a$ of $I$ is defined as $I_a := \{ f(a, x) \mid f(u, x) \in I \cap K[u][x] \}$, which is an ideal of $K[x]$. We say that a property holds for generic $a \in K^r$ if there exists a non-empty Zariski open set $U \subset K^r$ such that the considered property holds for all $a \in U$. The specialization of parameters was initiated by W. Krull for ideals and then extended and systematically studied by D.V. Nhi and N.V. Trung for finitely generated modules over $K(u)[x]$ in [NT99]. To get meaningful results about specializations we need that the field $K$ is infinite.

Notice that the straightforward definition of specialization $I_a := \{ f(a, x) \mid f(u, x) \in I \cap K[u][x] \}$ for an ideal $I$ in the power series ring $K(u)[[x]]$ does not work in general. $I_a$ may be 0 for all $a$ even if $I \neq 0$, as the following example shows. This example is due to Osgood and was also used by Gabrielov in his counterexample to the nested approximation theorem in the analytic case (cf. [Ro13]):

**Example 2.1.** Consider the morphism

$$\hat{\varphi} : \mathbb{C}[u, x_1, x_2] \to \mathbb{C}[y_1, y_2], \quad u \mapsto y_1, \ x_1 \mapsto y_1 y_2, \ x_2 \mapsto y_1 y_2 \cdot \exp(y_2),$$

and let $\varphi : \mathbb{C}(u)[x_1, x_2] \to \mathbb{C}(y_1, y_2)$ be given by the same assignment. It is shown in [Os16] that $\text{Ker}(\hat{\varphi}) = 0$. However, $I := \text{Ker}(\varphi) \neq 0$ since it contains $x_2 - x_1 \cdot \exp(x_1/u)$, while $I \cap \mathbb{C}(u)[[x_1, x_2]] = 0$.

**Remark 2.2.** The problem is, that elements in the ring $K(u)[[x]]$ may have infinitely many denominators. A reasonable definition for a specialization in this ring is

$$I_a := \{ f(a, x) \mid f(u, x) \in I \text{ s.t. no denominator of } f \text{ vanishes at } a \}.$$

It is very likely that with this definition (which can easily be extended to finitely generated submodules of $(K(u)[[x]])^P$) many properties of $I$ hold also for $I_a$ (e.g. the Hilbert-Samuel functions coincide) if $a$ is contained in the complement of countably many closed proper subvarieties of $K^r$. However, for this to be useful we do not only have to assume that $K$ is infinite but that $K$ is uncountable.

We do not pursue this here, since we need only the specialization for ideals $I \subset K(u)[x]$.

For a matrix $A \in Mat(m, n, P)$, $P$ a commutative Noetherian ring, and an integer $t$, $1 \leq t \leq n$ ($\leq m$), let $I_t(A)$ denote the ideal of $P$ generated by all $t \times t$ minors of $A$. For an ideal $J$ of $K[u, x]$, let $J^e = J \cdot K(u)[x]$ denote the extension of $J$. 

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Lemma 2.3. Let $M = [g_{ij}(u, x)] \in \text{Mat}(m, n, K[u, x])$ with $K$ infinite. For $a \in K^r$, let $M_a = [g_{ij}(a, x)] \in \text{Mat}(m, n, K[x])$. Then, with the notations from above, we have for generic $a \in K^r$

$$I_t(M_a) = (I_t(M)^e)_a$$

for all $t = 1, \ldots, n$. Moreover, for generic $a \in K^r$

$$\text{ht}(I_t(M_a)) = \text{ht}(I_t(M)^e)$$

for all $t = 1, \ldots, n$.

Proof. For $t = 1, \ldots, n$ and for all $a \in K^r$, $I_t(M_a)$ is the ideal of $K[x]$ generated by $d^{(t)}_1(a, x), \ldots, d^{(t)}_t(a, x)$, where $d^{(t)}_1(u, x), \ldots, d^{(t)}_t(u, x)$ are the $t \times t$ minors of $M$. On the other hand, for each generator $f^{(t)}(u, x)$ of the finitely generated ideal $I_t(M)^e \cap K[u, x]$ of $K[u, x]$, there is a polynomial $b^{(t)}(u) \in K[u] \setminus \{0\}$ such that $b^{(t)} \cdot f^{(t)} \in I_t(M)$. Therefore, for $a \in K^r$ which is not a zero of any of $b^{(t)}(u)$, we have that $(I_t(M)^e)_a$ is generated by $d^{(t)}_1(a, x), \ldots, d^{(t)}_t(a, x)$. Hence, the first assertion holds. The second statement follows from the first and [NT99, Theorem 3.4 ii) and Theorem 3.2].

The following theorem shows that we can modify a matrix with polynomial entries by adding polynomials of arbitrary high order such that the Fitting ideals of the modified matrix have maximal height. The proof was communicated to the authors by Ngô Viêt Trung in [Tru15] for $t = n$. As his arguments works for arbitrary $t$, we present it here in general.

Theorem 2.4. Let $A = [f_{ij}] \in \text{Mat}(m, n, K[x])$, $f_{ij} \in \langle x_1, \ldots, x_s \rangle \cdot K[x]$ with $K$ infinite. Then for any $N \geq 1$, there exists a matrix $B = [g_{ij}] \in \text{Mat}(m, n, K[x])$, where the entries have the form

$$g_{ij} = \sum_{k=1}^{s} a_{ijk} x_k^N,$$

with $a_{ijk} \in K$, such that for all $t \in \{1, \ldots, n\}$ we have

$$\text{ht}(I_t(A + B)) = \min\{s, (m - t + 1)(n - t + 1)\}.$$  

Proof. For $t \in \{1, \ldots, n\}$, set

$$m_t := (m - t + 1)(n - t + 1).$$

For all $i = 1, \ldots, m, j = 1, \ldots, n$, let

$$F_{ij} = \sum_{k=1}^{s} u_{ijk} x_k^N + f_{ij},$$

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where \( u = \{ u_{ijk} \mid i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, s \} \) is a set of new indeterminates. Set \( S = K[u, x] \) and let
\[
M = [F_{ij}] \in \text{Mat}(m, n, S).
\]
For \( t = 1, \ldots, n \), let \( I_t \) resp. \( I_t(M) \) be the ideal generated by the \( t \times t \) minors of \( M \) in \( S \) resp. in \( K(u)[x] \). The main work is done to prove the following

**Claim:** For all \( t = 1, \ldots, n \),
\[
\text{ht}(I_t(M)) = \min\{s, m_t\}.
\]
Indeed, fix \( t \in \{1, \ldots, n\} \). For every \( k = 1, \ldots, s \) and for all \( i, j \) we have that in \( S[\frac{1}{x_k}] \)
\[
\frac{1}{x_k}F_{ij} = u_{ijk} + \frac{1}{x_k} \left( \sum_{h \neq k} u_{ikh}x_h^N + f_{ij} \right).
\]
Therefore, the elements \( \frac{1}{x_k}F_{ij}, k = 1, \ldots, s, i = 1, \ldots, m, j = 1, \ldots, n \) are algebraically independent over \( K[u', x, \frac{1}{x_k}] \), where \( u' = u \setminus \{ u_{ijk}, i = 1, \ldots, m, j = 1, \ldots, n \} \). This means that \( \frac{1}{x_k} \cdot M \) is a generic matrix over the ring \( K[u', x, \frac{1}{x_k}] \) (generic in the sense that the entries are indeterminates, not to be confused with generic points). Note that
\[
K[u'][x][\frac{1}{x_k}][\frac{1}{x_k}F_{ij}] \mid i = 1, \ldots, m, j = 1, \ldots n = S[\frac{1}{x_k}].
\]
It is well known that the determinantal ideals of a generic matrix are prime and have maximal height (cf. e.g. [BV88, (2.13) and (5.18)]). Hence, \( I_t \left( \frac{1}{x_k} \cdot M \right) \), the ideal of \( S[\frac{1}{x_k}] \) generated by all \( t \times t \) minors of \( \frac{1}{x_k} \cdot M \), is a prime ideal of the height \( m_t \).

Since \( I_t \cdot S[\frac{1}{x_k}] = I_t \left( \frac{1}{x_k} \cdot M \right) \), we have that \( I_t \cdot S \left[ \frac{1}{x_k} \right] \) is a prime ideal and
\[
\text{ht} \left( I_t \cdot S \left[ \frac{1}{x_k} \right] \right) = m_t.
\]
Since \( I_t \cdot S \left[ \frac{1}{x_k} \right] \) is a prime ideal, \( I_t \) has a prime component, say \( P_k^{(t)} \), which does not contain \( x_k \), and all other associated primes must contain \( x_k \).

Now, let \( k' \in \{1, \ldots, s\} \) and \( k' \neq k \). By a similar argument, \( I_t \left( \frac{1}{x_k} \cdot M \right) \cdot S \left[ \frac{1}{x_k \cdot x_{k'}} \right] \) is also a prime ideal of \( S \left[ \frac{1}{x_k \cdot x_{k'}} \right] \). Moreover, since
\[
I_t \left( \frac{1}{x_k} \cdot M \right) \cdot S \left[ \frac{1}{x_k \cdot x_{k'}} \right] = I_t \cdot S \left[ \frac{1}{x_k \cdot x_{k'}} \right],
\]
so is $I_t \cdot S \left[ \frac{1}{x_k} \right]$. Therefore, $I_t$ has a prime component $P_{k,k'}^{(t)}$, which does not contain $x_k$ and $x_{k'}$, and all other associated primes must contain the product $x_k \cdot x_{k'}$. Assume that $P_k^{(t)} \neq P_{k,k'}^{(t)}$. Then $P_{k,k'}^{(t)} \in \{P_k^{(t)}, P_{k'}^{(t)}\}$. If $P_{k,k'}^{(t)} = P_k^{(t)}$, it would contain $x_{k'}$ since $P_{k,k'}^{(t)}$ is an associated prime which differs from $P_{k,k'}^{(t)}$, a contradiction.

So, $P_k^{(t)} = P_{k,k'}^{(t)}$ for all $k \neq k'$. Let $P_t$ denote this prime component. Then $P_t$ does not contain any $x_k$, $k = 1, \ldots, s$, and all other associated primes of $I_t$ must contain all $x_1, \ldots, x_s$. Let $Q_t$ be the intersection of all primary components of $I_t$ whose associated primes contain $x_1, \ldots, x_s$. Then

$$I_t = P_t \cap Q_t.$$ 

Moreover, since $P_t$ is the only associated primes of $I_t$ which does not contain any $x_k$, $I_t \cdot S \left[ \frac{1}{x_k} \right]$ is the extension of $P_t$ in $S \left[ \frac{1}{x_k} \right]$, and

$$\text{ht}(P_t) = \text{ht} \left( I_t \cdot S \left[ \frac{1}{x_k} \right] \right) = m_t.$$ 

We have

$$I_t(M) = (I_t)^e = (P_t)^e \cap (Q_t)^e.$$ 

Let first $t$ be such that $s \leq m_t$.

We consider two cases:

Case 1: Suppose that $(P_t)^e$ is strictly contained in $K(u)[x]$. Then

$$\text{ht} \left( (P_t)^e \right) = \text{ht}(P_t) = m_t \geq s,$$

so that $\text{ht} \left( (P_t)^e \right) = s$. On the other hand, since all other associated primes of $I_t$ contain $x_1, \ldots, x_s$, we have

$$\langle x_1, \ldots, x_s \rangle \cdot K(u)[x] \subset \left( \sqrt{Q_t} \right)^e \subset \sqrt{(Q_t)^e}.$$ 

This implies that all associated primes of $(Q_t)^e$ contain $\langle x_1, \ldots, x_s \rangle \cdot K(u)[x]$, and thus they have the height $s$. Hence, in this case $\text{ht}(I_t(M)) = s$.

Case 2: Suppose that $(P_t)^e = K(u)[x]$. Then

$$I_t(M) = (Q_t)^e.$$ 

By an argument as in case 1, all associated primes of $(Q_t)^e$ have the same height $s$ so that $\text{ht}(I_t(M)) = s$.

Let now $t$ be such that $s > m_t$. 

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Since \( \langle x_1, \ldots, x_s \rangle \cdot S \subset \sqrt{Q_t} \), we have
\[
\text{ht}(P_t) = m_t < s = \text{ht}(\langle x_1, \ldots, x_s \rangle \cdot S) \leq \text{ht}(\sqrt{Q_t}),
\]
i.e. \( P_t \) has the least height among the associated primes of \( I_t \). Hence, \( \text{ht}(I_t) = \text{ht}(P_t) = m_t \).

By the generic perfection [HE71], \( I_t \) is a perfect ideal of \( S \), and \( S/I_t \) is a Cohen-Macaulay ring. Hence, all associated primes of \( I_t \) have the same height [HE71] and \( Q_t \) does not exist, showing that \( I_t = P_t \). So, in this case,
\[
I_t(M) = (I_t)^e = (P_t)^e
\]
and
\[
\text{ht}(I_t(M)) = \text{ht}(P_t) = m_t.
\]
This finishes the claim.

Now let \( r = mns \), i.e. \( r \) is the number of the new indeterminates \( u = u_{ijk} \). By Lemma 2.3, for generic \( a \in K^r \), the specialization satisfies

\[
\text{ht}(I_t(M_a)) = \text{ht}(I_t(M)) = \min\{s, m_t\}
\]
for all \( t = 1, \ldots, n \). This finishes the proof. \( \square \)

**Remark 2.5.** (1) The above proof shows that the conclusion holds for all \( B \) with entries \( a_{ijk} \) in an open dense subset of \( K^{mns} \).

(2) The proof shows also that \( I_t(M) \) is a Cohen-Macaulay prime ideal if \( m_t < s \).

**Problem 2.6.** Does an analogous statement as in Theorem 2.4 holds for matrices with entries in \( K[[x]] \) instead of \( K[x] \), e.g. if we use the specialization defined in Remark 2.2 and assume that \( K \) is uncountable?

The following proposition provides a necessary condition for finite determinacy for matrices with entries in \( R = K[[x_1, \ldots, x_s]] \) with respect to the group \( G \).

**Theorem 2.7.** Let \( A = [a_{ij}] \in \mathfrak{m} \cdot \text{Mat}(m, n, R) \) be finitely G-determined, with \( K \) infinite. Then the following holds:

1. For all \( t \in \{1, \ldots, n\} \), we have
   \[
   \text{ht}(I_t(A)) = \min\{s, (m - t + 1)(n - t + 1)\}.
   \]

2. (i) If \( s \geq mn \) then \( \text{ht}(I_1(A)) = mn \), i.e. \( \{a_{ij}\} \) is an \( R \)-sequence.

(ii) If \( s \leq mn \) then \( I_1(A) \supset \mathfrak{m}^k \) for some positive integer \( k \), i.e. the entries of \( A \) generate an \( \mathfrak{m} \)-primary ideal in \( R \). This holds for arbitrary \( K \).
Proof. For $t = 1, \ldots, n$, set $m_t := (m - t + 1)(n - t + 1)$.

1. Assume that $A$ is $G$-determined. Let $A_0 = jet_t(A)$. Then $A \sim A_0$, i.e. there exist matrices $U \in GL(m, R)$ and $V \in GL(n, R)$ and an automorphism $\phi \in Aut(R)$ such that

$$A = U \cdot \phi(A_0) \cdot V.$$ 

This yields for all $t = 1, \ldots, n$

$$\text{ht}(I_t(A)) = \text{ht} (\phi (I_t(A_0))) = \text{ht} (I_t(A_0)).$$

Applying Theorem 2.4, there is a matrix $B = [g_{ij}] \in Mat(m, n, K[x])$ with entries of the form

$$g_{ij} = \sum_{k=1}^{s} c_{ijk} x^{l+1}_k, \ c_{ijk} \in K,$$

such that for all $t \in \{1, \ldots, n\}$ we have

$$\text{ht} I_t(A_0 + B) = \min\{s, m_t\}.$$ 

Now for all $t = 1, \ldots, n$, let $I_t(A_0 + B)^e$ be the ideal of $R$ generated by all $t \times t$ minors of $A_0 + B$. Since the homomorphism

$$i : K[x] \hookrightarrow K[[x]] = R.$$

is flat, we have

$$\min\{s, m_t\} = \text{ht} I_t(A_0 + B) \leq \text{ht} (I_t(A_0 + B)^e) \leq \min\{s, m_t\},$$

where the last inequality follows by [Mat89, Theorem 13.10]. Thus,

$$\text{ht} (I_t(A_0 + B)^e) = \min\{s, m_t\}.$$ 

On the other hand, since $A_0 \subset A_0 + B$, the heights of the ideals of $R$ generated by $t \times t$ minors of $A_0$ and of $A_0 + B$ are equal. Therefore, $\text{ht}(I_t(A)) = \min\{s, m_t\}.$

2. If $mn \leq s$ and $A$ is finitely $G$-determined then by the previous part,

$$\text{ht}(I_1(A)) = mn.$$ 

This implies that $\{a_{ij}\}$ is an $R$-sequence.

If $s \leq mn$ and $A$ is finitely $G$-determined we do not need Theorem 2.4 (where $K$ is infinite was used) to show that

$$\text{ht}(I_1(A)) = s.$$ 

In fact, we may choose the first $s$ entries of the matrix $[g_{i,j}]$ to be $x_1^N, \ldots, x_s^N$, with $N$ sufficiently big. Then $\text{ht} (I_1(A_0 + B)) = s$, which can be seen for arbitrary $K$ by choosing a global degree ordering on the variables and applying [GP08, Theorem 5.2.6]. This implies $\dim (R/I_1(A)) = 0$ and hence $I_1(A) \supset m^k$ for some $k$. \qed
Remark 2.8. The above necessary condition is of course not sufficient. For example, \( f = x^k \in K[[x, y]] \) is not finitely contact determined by Theorem 4.11 in any characteristic. However, \( \text{ht} (\langle f \rangle) = 1 \).

3 A finite determinacy criterion for column matrices

Theorem 2.7 shows that finite \( G \)-determinacy of matrices in \( M_{m,n} \) is rather restrictive. A criterion which is at the same time necessary and sufficient for finite \( G \)-determinacy for arbitrary \( m,n \) in positive characteristic is unknown to us.

In this section we prove such a criterion for 1-column matrices. The main result of this section is Theorem 3.8, where we prove that finite \( G \)-determinacy for a 1-column matrix \( A \) is equivalent to finite codimension of the extended tangent image \( \tilde{T}_A(GA) \) (as well as of the tangent image \( T_A(GA) \)) at \( A \) to the orbits \( GA \).

From the results in section 1 and 2, the following proposition is easy to obtain.

Proposition 3.1. Let \( A = [a_1 \ a_2 \ \ldots \ a_m]^T \in \text{Mat}(m, 1, m) \) be such that \( m \geq s \), where \( s \) is the number of variables. \( A \) is finitely \( G \)-determined if and only if \( m \cdot \langle a_1, \ldots, a_m \rangle \supset m^{k+2} \) for some \( k \in \mathbb{N} \setminus \{0\} \); i.e. \( I_1(A) \) is \( m \)-primary. Furthermore, \( A \) is then \( G \)\((2k - \text{ord}(A) + 2)\)-determined.

Proof. If \( A \) is finitely \( G \)-determined then by Theorem 2.7.(ii), we get the claim. Conversely, if \( m \cdot \langle a_1, \ldots, a_m \rangle \supset m^{k+2} \) for some \( k \in \mathbb{N} \setminus \{0\} \) then
\[
m^{k+2} \cdot M_{m,1} \subset m \cdot I_1(A) \cdot M_{m,1} + m^2 \cdot \left( \frac{\partial A}{\partial x_1}, \ldots, \frac{\partial A}{\partial x_s} \right) = m \cdot \tilde{T}_A(GA).
\]
By Theorem 1.1, \( A \) is \( G \)\((2k - \text{ord}(A) + 2)\)-determined. \( \square \)

For a matrix \( A = [a_1 \ a_2 \ \ldots \ a_m]^T \in M_{m,1} \), we denote by \( \text{Jac}(A) := \left[ \frac{\partial a_i}{\partial x_j} \right] \in M_{m,s} \) the Jacobian matrix of the vector \( (a_1, \ldots, a_m) \in R^m \) and call it the Jacobian matrix of \( A \).

Lemma 3.2. Let \( A = [a_1 \ a_2 \ \ldots \ a_m]^T \in \text{Mat}(m, 1, m) \) be such that \( m \leq s \), where \( s \) is the number of variables. Let \( \Theta_{(G,A)} \) be a presentation matrix of \( M_{m,1}/\tilde{T}_A^e(GA) \), where \( \tilde{T}_A^e(GA) \) is the extended tangent image at \( A \) to the orbit \( GA \). Then
\[
\sqrt{I_1(A) + I_m(\text{Jac}(A))} = \sqrt{I_m(\Theta_{(G,A)})}.
\]

Proof. A presentation matrix of \( M_{m,1}/\tilde{T}_A^e(GA) \) is the following two-block matrix of size...
\[ m \times (m^2 + s) \]

\[
\Theta_{(G,A)} = \begin{bmatrix}
\frac{\partial \bar{a}_1}{\partial x_1} & \frac{\partial \bar{a}_1}{\partial x_2} & \cdots & \frac{\partial \bar{a}_1}{\partial x_{m}} \\
\frac{\partial \bar{a}_2}{\partial x_1} & \frac{\partial \bar{a}_2}{\partial x_2} & \cdots & \frac{\partial \bar{a}_2}{\partial x_{m}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \bar{a}_m}{\partial x_1} & \frac{\partial \bar{a}_m}{\partial x_2} & \cdots & \frac{\partial \bar{a}_m}{\partial x_{m}} \\
\bar{a} & \bar{0} & \bar{0} & \cdots & \bar{0} \\
\bar{0} & \bar{a} & \bar{0} & \cdots & \bar{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\bar{0} & \bar{0} & \bar{0} & \cdots & \bar{a}
\end{bmatrix},
\]

where \( \bar{a} := (a_1, \ldots, a_m) \) and \( \bar{0} := (0, \ldots, 0) \in K^m \). We first claim

\[ \sqrt{\langle a_1, \ldots, a_m \rangle + I_m(Jac(A))} \subset \sqrt{I_m(\Theta_{(G,A)})}, \]

Indeed, obviously we have

\[ \sqrt{\langle a_1, \ldots, a_m \rangle + I_m(Jac(A))} \subset \sqrt{\langle a_1, \ldots, a_m \rangle} + \sqrt{I_m(Jac(A))}. \]

Moreover,

\[ I_m(Jac(A)) \subset I_m(\Theta_{(G,A)}) \]

since \( Jac(A) \) is a block of \( \Theta_{(G,A)} \) and

\[ \langle a_1, \ldots, a_m \rangle^m \subset I_m(\Theta_{(G,A)}) \]

since \( \langle a_1, \ldots, a_m \rangle^m \) is the ideal generated by \( m \times m \) minors of the right-hand block of \( \Theta_{(G,A)} \). By taking the radicals we have

\[ \sqrt{\langle a_1, \ldots, a_m \rangle} + \sqrt{I_m(Jac(A))} \subset \sqrt{I_m(\Theta_{(G,A)})}, \]

and by taking the radical of both sides of this inclusion, we get the claim.

We now claim that

\[ I_m(\Theta_{(G,A)}) \subset \langle a_1, \ldots, a_m \rangle + I_m(Jac(A)). \]

Since \( m \leq s \), each \( m \times m \) minor \( T \) of \( \Theta_{(G,A)} \) either comes only from \( Jac(A) \) or contains at least a column of the right-hand block. In the first case, clearly \( T \in I_m(Jac(A)) \). In the second case, using Laplace’s expansion in a column of \( \Theta_{(G,A)} \), we get \( T \in \langle a_1, \ldots, a_m \rangle \). Hence, the claim follows. By taking radicals we prove the lemma.

**Proposition 3.3.** Let \( A = [a_1 \ a_2 \ \ldots \ a_m]^T \in Mat(m,1,m) \) be such that \( m \leq s \), where \( s \) is the number of variables. If there is some integer \( k \geq 0 \) such that

\[ \langle a_1, \ldots, a_m \rangle + I_m(Jac(A)) \supset m^k \quad (3.1) \]

then \( A \) is finitely \( G \)-determined.
Proof. This follows directly from Lemma 3.2 and [GP16, Proposition 4.2.5]. \qed

We show now that there exist finitely $G$-determined matrices in $Mat(m, 1, R)$ with entries of arbitrary high order.

**Proposition 3.4.** Let $R = K[[x_1, x_2, \ldots, x_s]]$ and $M_{m, 1} = Mat(m, 1, R)$ with $m \leq s$. Let $\text{char}(K) = p \geq 0$, $N \geq 2$ an integer and if $p > 0$ assume that $p \nmid N$. Choose $c_{ij} \in K, i = 1, \ldots, m, j = 1, \ldots, s$, such that no maximal minor $m_1, \ldots, m_r$ of the $m \times s$ matrix $[c_{ij}]_{i=1,\ldots,m, j=1,\ldots,s}$ vanishes (which is always possible if $K$ is infinite). Set

$$A := \begin{bmatrix} f_1 & f_2 & \ldots & f_m \end{bmatrix}^T \in M_{m, 1},$$

$$f_i := c_{i1}x_1^N + \ldots + c_{is}x_s^N, \quad i = 1, \ldots, m.$$ Then $A$ is finitely $G$-determined.

**Proof.** By assumption $K^{ms} \setminus V(m_1 \ldots m_r) \neq \emptyset$, which is obviously true if $K$ is infinite. Since $p \nmid N$, $I_m(Jac(A))$ is the ideal of $R$ generated by the products $m_j \cdot x_{j1}^{N-1} \cdot \ldots \cdot x_{jm}^{N-1}$, where $j_i \in \{1, 2, \ldots, s\}$ for all $i = 1, \ldots, m$ and $j_i \neq j_k$ for all $i \neq k$ and where $m_j$ is the $m \times m$-minor of $[c_{ij}]$ corresponding to the columns $j = (j_1, \ldots, j_m)$.

Let $J$ be the ideal of $K[x]$ generated by $f_1, f_2, \ldots, f_m$ and all $m \times m$ minors of $Jac(A)$.

We claim that $V(J) = \{0\}$ in $\overline{K}$, with $\overline{K}$ an algebraic closure of $K$. Indeed, let $a = (a_1, a_2, \ldots, a_s) \in V(J)$. Then at least $s - m + 1$ components of $a$ must be zero, since $m_j \neq 0$ and hence $a$ is a zero of all products $x_{j1}^{N-1} \cdot x_{j2}^{N-1} \cdot \ldots \cdot x_{jm}^{N-1}$, where $j_i \in \{1, 2, \ldots, s\}$ for all $i = 1, \ldots, m$ and $j_i \neq j_k$ for all $i \neq k$, and $m \leq s$. Without loss of generality we assume that the last $s - k \geq s - m + 1$ components of $a$ are zero. Then $f_i(a) = c_{i1}a_1^N + c_{i2}a_2^N + \ldots + c_{ik}a_k^N$.

Consider the homogeneous system $(H)$ of $m$ linear equations in $k$ variables $y_1, \ldots, y_k$

$$(H): \begin{cases} c_{11}y_1 + c_{12}y_2 + \ldots + c_{1k}y_k = 0 \\ c_{21}y_1 + c_{22}y_2 + \ldots + c_{2k}y_k = 0 \\ \vdots \\ c_{m1}y_1 + c_{m2}y_2 + \ldots + c_{mk}y_k = 0. \end{cases}$$

Since $a$ is also a zero of $f_1, \ldots, f_m$, it follows that $y_j = a_j^N$ is a solution of $(H)$ for $j = 1, \ldots, k$ and $k \leq m - 1$. By the choice of $c_{ij}$ all maximal minors of the $m \times s$ matrix $[c_{ij}]_{i=1,\ldots,m, j=1,\ldots,s}$ are non-zero. Hence, there must be a non-zero $k \times k$ sub-determinant of the coefficient matrix of $(H)$. This implies that $(H)$ has only the trivial solution and thus the first $k$ components of $a$ are zero. Since the last $s - k$ components of $a$ are zero, $a = 0$ and the claim follows. As a consequence, $\text{dim}(K[x]/J) = 0$, and hence $\text{ht}(J) = s$. 

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Set $\hat{J} := I_1(A) + I_m(Jac(A)) := J \cdot K[[x]]$. Since the map $K[x] \mapsto K[[x]]$ is flat, we obtain by the going down theorem for flat morphism [Eis95, Lemma 10.11]

$$s = \text{ht}(J) \leq \text{ht}(\hat{J}) \leq s.$$  

This implies $\text{ht}(\hat{J}) = s$ so that $\dim(R/\hat{J}) = 0$. Therefore, $\hat{J}$ contains a power of the maximal ideal $m$. Applying Proposition 3.3, $A$ is finitely $G$-determined.

Example 3.5. $K^m \setminus V(m_1 \cdot \ldots \cdot m_r) = \emptyset$ may happen for finite $K$. Choose $K = \{0, 1\}$, $m = 2$ and $s = 4$. Then it is easy to see that at least one of the six 2-minors of $[c_{ij}]_{i=1,\ldots,2, j=1,\ldots,4}$ is 0 for any choice of $c_{ij} \in K$.

To prove that the sufficient criterion of Theorem 1.1 is also necessary for finite determinacy of matrices in $M_{m,1}$ we need the above proposition and the semi-continuity of the $K$-dimension of a 1-parameter family of modules over a power series ring.

Let $P = K[[t]][[x]]$, $K$ arbitrary, where $x = (x_1, \ldots, x_s)$, and $M$ a finitely generated $P$-module. For $t_0 \in K$, set

$$M(t_0) := M \otimes_{K[[t]]} (K[[t]]/(t - t_0)) \cong M/(t - t_0) \cdot M,$$

$$m_{t_0} := \langle x_1, \ldots, x_s, t - t_0 \rangle \subset P.$$

Lemma 3.6. For all $t_0 \in K$,

$$M_{m_{t_0}}/(t - t_0) \cdot M_{m_{t_0}} \cong M/(t - t_0) \cdot M$$

as $K[[t]]$-modules. In particular,

$$\dim_K M_{m_{t_0}}/(t - t_0) \cdot M_{m_{t_0}} = \dim_K M(t_0).$$

Proof. Since $P_{m_{t_0}}/(t - t_0) \cong P/(t - t_0)$ we get

$$M_{m_{t_0}}/(t - t_0)M_{m_{t_0}} \cong (M \otimes P_{m_{t_0}}) \otimes_{P_{m_{t_0}}/(t - t_0)} P_{m_{t_0}}/(t - t_0) \cong M \otimes P_{m_{t_0}}/(t - t_0) \cong M/(t - t_0)M.$$

Proposition 3.7. With the above notations, there is a nonempty open neighborhood $U$ of 0 in $\mathbb{A}^1$ such that for all $t_0 \in U$, we have

$$\dim_K M(t_0) \leq \dim_K M(0).$$
Proof. Without loss of generality we may assume that \( \dim_K M(0) < \infty \). Then \( M \) is quasi-finite but in general not finite over \( K[t] \). However, we show that the restriction of \( M \) to some open subset of \( \mathbb{A}^1 \) is in fact finitely generated over \( K[t] \) (Claim 5), so that we can apply the semicontinuity of the rank of the presentation matrix of \( M \) as \( K[t] \)-module.

The first steps in the proof are used to show that we can reduce to this case.

Consider a primary decomposition of \( \text{Ann}_P(M) \),

\[
I := \text{Ann}_P(M) = \bigcap_{i=1}^r Q_i \subset P.
\]

**Case 1:** \( V(\langle x_1, \ldots, x_s \rangle) \not\subset V(Q_i) \) for all \( i = 1, \ldots, r \) in \( \text{Spec}(P) \), i.e. \( Q_i \not\subset \langle x_1, \ldots, x_s \rangle \) for all \( i = 1, \ldots, r \). If \( Q_i \) denotes the image of \( Q \) under the morphism \( P \to P/\langle x_1, \ldots, x_s \rangle \cong K[t] \), then \( Q_i \neq 0 \) and \( \bar{Q}_i = \langle f_i \rangle \subset K[t] \) for some \( f_i \neq 0 \). Then

\[
U := \mathbb{A}^1 \setminus \bigcup_{i=1}^r V(f_i) \neq \emptyset
\]

and for all \( t_0 \in U \) we have \( Q_i \not\subset m_{t_0} \) for all \( i = 1, \ldots, r \). Since otherwise \( \bar{Q}_i \subset (t - t_0) \subset K[t] \) for some \( i \) which implies \( t_0 \in V(f_i) \), a contradiction. This implies that \( m_{t_0} \not\subset V(\text{Ann}_P M) = \text{Supp}(M) \) so that \( M_{m_{t_0}} = 0 \). Applying Lemma 3.6, we obtain

\[
\dim_K M(t_0) = \dim_K M_{m_{t_0}}/\langle t - t_0 \rangle \cdot M_{m_{t_0}} = 0 \leq \dim_K M(0).
\]

**Case 2:** \( V(\langle x_1, \ldots, x_s \rangle) \subset V(Q_i) \) for some \( i \in \{1, \ldots, r\} \), i.e. \( Q_i \subset \langle x_1, \ldots, x_s \rangle \).

**Claim 1:** For \( Q_i \subset \langle x_1, \ldots, x_s \rangle \) we have

\[
\sqrt{Q_i} = \langle x_1, \ldots, x_s \rangle,
\]

and \( Q_i \) is unique.

Indeed, by Lemma 3.6 we have \( \dim_K M_{m_0}/\langle t \rangle \cdot M_{m_0} = \dim_K M(0) < \infty \). Therefore \( M_{m_0} \) is quasi-finite and hence finite over the local ring \( P_{m_0} \). By Krull’s principal ideal theorem ([GLS07, Theorem B.2.1]) \( \dim(M_{m_0}) \leq 1 \).

Since \( Q_i \subset m_0 \), we have \( \text{Ann}_{P_{m_0}}(M_{m_0}) \subset (Q_i)_{m_0} \subset (\sqrt{Q_i})_{m_0} \) and hence

\[
\dim \left( P_{m_0}/(\sqrt{Q_i})_{m_0} \right) \leq \dim \left( P_{m_0}/\text{Ann}_{P_{m_0}}(M_{m_0}) \right) = \dim(M_{m_0}) \leq 1.
\]

On the other hand, since \( (\sqrt{Q_i})_{m_0} \subset \langle x_1, \ldots, x_s \rangle_{m_0} \), we have \( \dim \left( P_{m_0}/(\sqrt{Q_i})_{m_0} \right) \geq \dim \left( P_{m_0}/\langle x_1, \ldots, x_s \rangle_{m_0} \right) = 1 \) and therefore \( \dim \left( P_{m_0}/(\sqrt{Q_i})_{m_0} \right) = 1 \).

Since \( \sqrt{Q_i} \) and \( \langle x_1, \ldots, x_s \rangle \) are prime ideals of \( P \) contained in \( m_0 \) and \( \sqrt{Q_i} \subset \langle x_1, \ldots, x_s \rangle \), the equality of codimensions implies

\[
\sqrt{Q_i} = \langle x_1, \ldots, x_s \rangle
\]

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and that $\sqrt{Q_i}$ is a minimal associated prime of $\text{Ann}_P M$. Hence $Q_i$ is unique. 

Claim 2: Let, by Claim 1,

$$Q := Q_i$$

be the only primary component of $\text{Ann}_P M$ contained in $\langle x_1, \ldots, x_s \rangle$ and set

$$\tilde{M} := M/Q \cdot M,$$

$$\tilde{M}(t_0) := \tilde{M} \otimes K[t]/\langle t - t_0 \rangle$$

for $t_0 \in K$. Then $\tilde{M}(t_0) \cong M/\langle t - t_0 \rangle \cdot \tilde{M} \cong M/(Q + \langle t - t_0 \rangle) \cdot M$ and hence

$$\dim_K \tilde{M}(t_0) \leq \dim_K M(t_0). \quad (3.3)$$

Claim 3: Set

$$W := V(\tilde{Q}) \setminus \left( \bigcup_{Q_i \neq Q} V(\tilde{Q}_i) \right) = A^1 \setminus \left( \bigcup_{Q_i \neq Q} V(\tilde{Q}_i) \right) \neq \emptyset,$$

where $\tilde{Q}_i$ is the image of $Q_i$ under the projection $P \to P/\langle x_1, \ldots, x_s \rangle \cong K[t]$. 

We claim that for $t_0 \in W$ there is a $P$-module homomorphism

$$\varphi : \tilde{M}_{m_{t_0}} \cong M_{m_{t_0}}, \quad \varphi \left( \langle t - t_0 \rangle \cdot \tilde{M}_{m_{t_0}} \right) = \langle t - t_0 \rangle \cdot M_{m_{t_0}}. \quad (3.4)$$

In $\text{Spec}(P)$ we have

$$m_{t_0} \in V(Q) \setminus \left( \bigcup_{Q_i \neq Q} V(Q_i) \right),$$

implying that for all $i$ such that $Q_i \neq Q$ we have $(Q_i)_{m_{t_0}} = P_{m_{t_0}}$. Hence

$$Q_{m_{t_0}} = I_{m_{t_0}} \subset \text{Ann}_{P^{m_{t_0}}}(M_{m_{t_0}}),$$

yielding $(Q \cdot M)_{m_{t_0}} = 0$. As a result, we obtain a $P$-module homomorphism

$$\varphi : \tilde{M}_{m_{t_0}} \cong M_{m_{t_0}}/(Q \cdot M)_{m_{t_0}} = M_{m_{t_0}}$$

and the claim follows.

Claim 4: For $t_0 \in W$, we claim that

$$\dim_K \tilde{M}(t_0) = \dim_K M(t_0). \quad (3.5)$$

In fact, we have

$$\dim_K \tilde{M}(t_0) = \dim_K \tilde{M}/\langle t - t_0 \rangle \cdot \tilde{M} \quad \text{(by definition)}$$

$$= \dim_K \tilde{M}_{m_{t_0}}/\langle t - t_0 \rangle \cdot \tilde{M}_{m_{t_0}} \quad \text{(by an analogue of Lemma 3.6)}$$

$$= \dim_K M_{m_{t_0}}/\langle t - t_0 \rangle \cdot M_{m_{t_0}} \quad \text{(by (3.4))}$$

$$= \dim_K M(t_0). \quad \text{(by Lemma 3.6)}$$

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Claim 5: There is an open neighborhood \( V \) of 0 in \( \mathbb{A}^1 \) such that for all \( t_0 \in V \) we have

\[
\dim_K \bar{M}(t_0) \leq \dim_K \bar{M}(0).
\] (3.6)

Indeed, since \( \sqrt{Q} = \langle x_1, \ldots, x_s \rangle \) by (3.2), there is a positive integer \( n \) such that \( \langle x_1, \ldots, x_s \rangle^n \subset Q \). This implies that \( P/\langle x_1, \ldots, x_s \rangle^n \rightarrow P/Q \) is surjective. Since \( P/\langle x_1, \ldots, x_s \rangle^n \) is a finitely generated \( K[t] \)-module, so is \( P/Q \). Moreover, since \( M \) is a finitely generated \( P \)-module, \( M/Q \cdot M \) is a finitely generated \( P/Q \)-module. This implies that \( M/Q \cdot M \) is a finitely generated \( K[t] \)-module. Hence, there is a presentation

\[
K[t]^m \xrightarrow{\psi(t)} K[t]^n \rightarrow \bar{M} \rightarrow 0.
\]

For \( t_0 \in K \), by taking the tensor product with \( K[t]/(t - t_0) \) over \( K[t] \), we obtain the exact sequences

\[
K^m \xrightarrow{\psi(t_0)} K^n \rightarrow \bar{M}(t_0) \rightarrow 0.
\]

This yields

\[
\dim_K \bar{M}(t_0) = n - \text{rank} \psi(t_0).
\]

Since \( \text{rank} \psi(t) \) is lower semi-continuous on \( \mathbb{A}^1 \), there is an open neighborhood \( V \) of 0 in \( \mathbb{A}^1 \) such that for all \( t_0 \in V \) we get the desired inequality

\[
\dim_K \bar{M}(t_0) = n - \text{rank} \psi(t_0) \leq n - \text{rank} \psi(0) = \dim_K \bar{M}(0).
\]

Now let \( U := W \cap V \), which is a nonempty open subset of \( \mathbb{A}^1 \). Then for all \( t_0 \in U \), we have

\[
\dim_K M(t_0) = \dim_K \bar{M}(t_0) \tag{by (3.5)}
\]

\[
\leq \dim_K \bar{M}(0) \tag{by (3.6)}
\]

\[
\leq \dim_K M(0). \tag{by (3.3)}
\]

This proves the proposition. \( \Box \)

We now prove our main result of this section.

Theorem 3.8. Let \( A = [f_1, f_2, \ldots, f_m]^T \in \text{Mat}(m, 1, \mathfrak{m}) \). If \( K \) is infinite, the following are equivalent:

1. \( A \) is finitely \( G \)-determined.

2. \( \dim_K \left( M_{m,1}/\bar{T}_A^e(GA) \right) =: d_e < \infty \).

Moreover, if condition 2. is satisfied then \( A \) is \( G \left( 2d_e - \text{ord}(A) + 2 \right) \)-determined.
Proof. (2. ⇒ 1.) is a consequence of Theorem 1.1, where $K$ infinite is not needed.

(1. ⇒ 2.) Assume that $A$ is $G$ $k$-determined.

**Case 1:** $m < s$. By finite determinacy we may assume that $A = [f_1 \ f_2 \ \ldots \ f_m]^T$ is a matrix of polynomials. Let $\text{char}(K) = p > 0$ and $N \in \mathbb{N}$ be such that $N > k$ and $p \nmid N$. Let $B = [g_1 \ \ldots \ g_m]^T \in \text{Mat}(m, 1, m)$, where for all $i = 1, \ldots, m$

$$g_i = c_{i1}x_1^N + c_{i2}x_2^N + \ldots + c_{is}x_s^N$$

and $c_{ij} \in K$ are determined as in Proposition 3.4. Consider

$$B_t = B + tA = [g_1 + tf_1 \ \ldots \ g_m + tf_m]^T \in \text{Mat}(m, 1, K[[x]][t]).$$

For fixed $t \neq 0$, since $A$ is $G$ $k$-determined, so is $tA$, and we have

$$B_t \not\sim tA.$$

By the proof of Proposition 3.4, the ideal $Q_0$ of $K[[x]]$ generated by $g_1, \ldots, g_m$ and all $m \times m$ minors of $\text{Jac}(B)$ is zero dimensional, i.e.

$$\dim_K (K[[x]]/Q_0) < \infty.$$

Let $Q_t$ be the ideal of $K[[x]][t]$ generated by $g_1 + tf_1, \ldots, g_m + tf_m$ and all $m \times m$ minors of $\frac{\partial (g_i + tf_i)}{\partial x_j}$. Then by Proposition 3.7, there exists a nonempty open subset $U \subset K^1$ such that for all $t_0 \in U$,

$$\dim_K (K[[x]]/Q_{t_0}) \leq \dim_K (K[[x]]/Q_0) < \infty,$$

i.e. $Q_{t_0}$ contains a power of the maximal ideal $m$. Applying Lemma 3.2 and [GP16, Proposition 4.2], we obtain for all $t_0 \in U$

$$\dim_K \left( M_{m, 1}/\tilde{T}_{B_{t_0}}^e (GB_{t_0}) \right) < \infty.$$

Let $t_0 \in U$, $t_0 \neq 0$. Since $B_{t_0} \not\sim t_0 A$, we have

$$\dim_K \left( M_{m, 1}/\tilde{T}_{t_0 \cdot A}^e (G(t_0 \cdot A)) \right) < \infty.$$

Since $t_0 \neq 0$, the extended tangent image at $t_0 \cdot A$ to the orbit $G(t_0 \cdot A)$ equals the extended tangent image at $A$ to the orbit $GA$. Hence, the assertion follows.

**Case 2:** $m \geq s$ ($K$ infinite is not needed here). Apply Proposition 3.1, there is some positive integer $k$ such that $m^k \subset I_1 (A)$. This implies

$$m^k \cdot M_{m, 1} \subset I_1 (A) \cdot M_{m, 1} \subset \tilde{T}_{A}^e (GA)$$

and thus the claim follows. \qed
The following corollary shows that the sufficient condition (3.1) in Proposition 3.3 is also necessary for finite determinacy. In addition, a determinacy bound is provided.

**Corollary 3.9.** Let \( A = [a_1 \ a_2 \ \ldots \ a_m]^T \in \text{Mat}(m, 1, m), K \) infinite, be such that \( m \leq s \), where \( s \) is the number of variables. Then \( A \) is finitely \( G \)-determined if and only if there is some integer \( k \geq 0 \) such that

\[
\langle a_1, \ldots, a_m \rangle + I_m(\text{Jac}(A)) \supset \mathfrak{m}^k.
\]

Furthermore, \( A \) is then \( G(2km - \text{ord}(A) + 2) \)-determined.

**Proof.** By Proposition 3.3, it suffices to prove the necessity of the condition. Assume that \( A \) is finitely \( G \)-determined. By Theorem 3.8, we have

\[
\dim_K \left( M_{m,1}/\tilde{T}_A^e(GA) \right) < \infty.
\]

Then by [GP16, Proposition 4.2], the 0th Fitting ideal \( I_m(\Theta_{G,A}) \) contains a power of the maximal ideal \( \mathfrak{m} \), where \( \Theta_{G,A} \) is a presentation matrix of \( M_{m,1}/\tilde{T}_A^e(GA) \). Applying Lemma 3.2, we get the assertion.

Now we prove the determinacy bound. The inclusion \( \mathfrak{m}^k \subset \langle a_1, \ldots, a_m \rangle + I_m(\text{Jac}(A)) \) implies that

\[
\mathfrak{m}^{km} \subset \langle a_1, \ldots, a_m \rangle^m + I_m(\text{Jac}(A)).
\]

On the other hand, by the proof of Lemma 3.2, we have

\[
\langle a_1, \ldots, a_m \rangle^m + I_m(\text{Jac}(A)) \subset I_m(\Theta_{G,A}).
\]

Hence,

\[
\mathfrak{m}^{km} \subset I_m(\Theta_{G,A})
\]

and we obtain the determinacy bound by applying [GP16, Proposition 4.2]. \( \square \)

## 4 Contact equivalence of ideals and complete intersections

We will see below that \( G \)-equivalence for matrices in \( M_{m,1} \) is the same as contact equivalence for the ideals generated by the column entries. We use this to get a necessary and sufficient criterion for finite contact-determinacy of ideals in \( R = K[[x_1, \ldots, x_s]] = K[[x]] \). We then characterize finitely contact determined ideals as isolated complete intersection singularities. Recall that \( \text{mng}(I) \) denotes the minimal number of generators of the ideal \( I \subset R \).

**Definition 4.1.** Let \( I \) and \( J \) be proper ideals of \( R \). \( I \) is said to be contact equivalent to \( J \), \( I \overset{\cdot}{\sim} J \), if \( R/I \cong R/J \) as \( K \)-algebras.
Notice that \( I \sim J \) implies \( \text{mng}(I) = \text{mng}(J) \) by the lifting lemma [GLS07, Lemma 1.23]. The following lemma is proved in [Mat68, 2.3 Proposition] for map-germs, but the proof works also in our more general situation.

**Lemma 4.2.** Let \( Q \) be a unital Noetherian local ring with maximal ideal \( n \). Let \( I \) be an ideal of \( Q \). Assume that \( I \) can be generated by \( m \) elements, say \( \langle a_1, \ldots, a_m \rangle \) and \( \langle b_1, \ldots, b_m \rangle \). Let \( A = [a_1 \ a_2 \ \ldots \ a_m]^T \) and \( B = [b_1 \ b_2 \ \ldots \ b_m]^T \) be the column matrices corresponding to the two sets of generators of \( I \). Then there is an invertible matrix \( U \in \text{GL}(m, Q) \) such that \( B = U \cdot A \).

For \( I = \langle a_1, \ldots, a_m \rangle \subset R \), we denote by \( A = [a_1 \ a_2 \ \ldots \ a_m]^T \in M_{m,1} \) the column matrix corresponding to the generators \( a_1, a_2, \ldots, a_m \) of \( I \).

**Proposition 4.3.** Let \( I \) and \( J \) be ideals of \( R \). Assume that \( I \) and \( J \) can be generated by \( m \) elements, say \( I = \langle a_1, \ldots, a_m \rangle \) and \( J = \langle b_1, \ldots, b_m \rangle \). Let \( A \) resp. \( B \) be the column matrices in \( M_{m,1} \) corresponding to the generators of \( I \) resp. \( J \). Then the following are equivalent:

1. \( I \sim J \).

2. There is an automorphism \( \phi \in \text{Aut}(R) \) such that \( \phi(I) = J \).

3. There are \( \phi \in \text{Aut}(R) \) and \( U \in \text{GL}(m, R) \) such that \( B = U \cdot \phi(A) \).

4. \( A^G \sim B \).

**Proof.** The equivalence of 1. and 2. follows from the lifting lemma [GLS07, Lemma 1.23] and the equivalence of 2. and 3. from Lemma 4.2. Since on \( M_{m,1} \) the orbits of the action of \( G = G_{lr} \) coincide with the orbits of the action of \( G_l \), condition 3. means that the matrices \( A \) and \( B \) are in the same \( G \)-orbit, showing the equivalence of 3. and 4. 

**Definition 4.4.** Let \( I \) be a proper ideal of \( R \) with \( \text{mng}(I) = m \) and \( a_1, \ldots, a_m \) a minimal set of generators of \( I \).

1. \( I \) is called **contact \( k \)-determined**, if for every ideal \( J \) of \( R \) which can be generated by \( m \) elements \( b_1, \ldots, b_m \) with \( b_i - a_i \in m^{k+1} \) for all \( i = 1, \ldots, m \), we have \( I \sim J \).

2. \( I \) is called **finitely contact determined** if there exists a \( k \) such that \( I \) is contact \( k \)-determined. The minimal such \( k \) is called the **contact determinacy** of \( I \).

**Remark 4.5.** 1. Let \( I \) be an ideal of \( R \) with \( \text{mng}(I) = m \) and \( a_1, \ldots, a_m \) a minimal set of generators of \( I \). Let \( A = [a_1 \ \ldots \ a_m]^T \in M_{m,1} \) be the column matrix corresponding to the generators of \( I \). It follows easily from Proposition 4.3 that \( I \) is contact \( k \)-determined if and only if the matrix \( A \) is \( G \) \( k \)-determined.
Indeed, assume that $I$ is contact $k$-determined. Let $B \in M_{m,1}$ be such that $B - A \in \mathfrak{m}^{k+1} \cdot M_{m,1}$. Then $I \sim J$, where $J$ is the ideal generated by the $m$ entries of $B$. By Proposition 4.3, $A \sim G B$. Hence, $A$ is $G$ $k$-determined. Conversely, suppose that $A$ is $G$ $k$-determined. Let $J = \langle b_1, \ldots, b_m \rangle$ be an ideal of $R$ such that $b_i - a_i \in \mathfrak{m}^{k+1}$ for all $i = 1, \ldots, m$ and $B$ its corresponding column matrix. Then $B - A \in \mathfrak{m}^{k+1} \cdot M_{m,1}$, and thus $B \sim G A$. Again by Proposition 4.3, we have $J \sim I$.

2. If $I \sim J$ then $I$ is contact $k$-determined if and only if $J$ is contact $k$-determined. Indeed, let $I = \langle a_1, \ldots, a_m \rangle$ and $J = \langle b_1, \ldots, b_m \rangle$ be minimal sets of generators of $I$ and $J$, respectively. Then by Proposition 4.3, the two column matrices $A$ and $B$ corresponding to these minimal sets of generators of $I$ and $J$, respectively, are $G$-equivalent. Hence, if $A$ is $G$ $k$-determined then so is $B$, and vice versa. Therefore, by 4.5.1, we get the claim. Taking $I = J$ we see that the above definition does not depend on minimal set of generators of $I$.

For an ideal $I$ of $R$, we denoted by $\text{ord}(I)$ the order of $I$, i.e. the maximal positive integer number $o$ such that $I \subset \mathfrak{m}^o$. The results on finite $G$-determinacy for one column matrices are transfered to ideals:

**Corollary 4.6.** Let $I = \langle f_1, \ldots, f_m \rangle$ be an ideal in $\mathfrak{m}$ with $m$ the minimal number of generators of $I$ and $I_m\left(\begin{bmatrix} \frac{\partial f_i}{\partial x_j} \end{bmatrix}\right)$ the ideal generated by the $m \times m$ minors of the $(m \times s)$ Jacobian matrix of the generators of $I$.

1. If $m \geq s$ then $I$ is finitely contact determined iff there is a $k$ such that $\mathfrak{m}^k \subset I$. Furthermore, if $\mathfrak{m}^k \subset I$ then $I$ is contact $(2k - \text{ord}(I) + 2)$-determined.

2. If $m \leq s$ and $K$ infinite, then $I$ is finitely contact determined iff there is a $k$ such that

$$\mathfrak{m}^k \subset I + I_m\left(\begin{bmatrix} \frac{\partial f_i}{\partial x_j} \end{bmatrix}\right).$$

Moreover, $I$ is then contact $(2km - \text{ord}(I) + 2)$-determined.

**Proof.** 1. The statement follows from Proposition 3.1 and Remark 4.5.1.

2. This assertion follows from Corollary 3.9 and Remark 4.5.1. \qed

**Definition 4.7.** Let $I$ be a proper ideal of $R$ with $\text{mng}(I) = m$ and $A \in M_{m,1}$ the column matrix corresponding to a minimal set of generators of $I$. We define

$$T_I := M_{m,1} / \left( I \cdot M_{m,1} + \left( \frac{\partial A}{\partial x_1}, \ldots, \frac{\partial A}{\partial x_s} \right) \right)$$

and

$$\tau(I) := \dim_K T_I.$$
Remark 4.8. 1. We have
\[ T_I = M_{m,1}/\tilde{T}_A^r(GA), \]
where \( \tilde{T}_A^r(GA) \) is the extended tangent image at \( A \) to the orbit \( GA \).

2. By Proposition 3.7 \( \tau(I) = \dim_K T_I \) is semicontinuous if we perturb the entries of \( A \).

3. If \( I = \langle a_1, \ldots, a_m \rangle \subset R \) set \( a := (a_1, \ldots, a_m) \). Then
\[ \frac{\partial a}{\partial x_i} = \left( \frac{\partial a_1}{\partial x_i}, \ldots, \frac{\partial a_m}{\partial x_i} \right) \in R^m, \]
and
\[ T_I \cong R^m/IR^m + \left( \frac{\partial a_i}{\partial x_j} \right) \cdot R^s, \]
where \( \left( \frac{\partial a_i}{\partial x_j} \right) : R^s \rightarrow R^m \) is the Jacobian matrix of \( a \) and \( \left( \frac{\partial a_i}{\partial x_j} \right) \cdot R^s \) is the \( R \)-submodule of \( R^m \) generated by the columns of the Jacobian matrix of \( a \).

Proposition 4.9. Let \( I \) be a proper ideal of \( R \) with \( \text{mng}(I) = m \). Let \( J \) be an ideal of \( R \) such that \( I \sim J \) and \( \phi \in \text{Aut}(R) \) such that \( \phi(I) = J \). Then

1. \( T_I \cong T_J \) as \( R \)-modules over \( \phi : R \cong R \).

2. \( \tau(I) = \tau(J) \).

Proof. 1. Since \( I \sim J \), we have \( \text{mng}(J) = m \). Let \( A, B \in M_{m,1} \) be the column matrices corresponding to a minimal set of generators of \( I \) and \( J \), respectively. Then by Proposition 4.3, \( A \sim B \) i.e. there are an invertible matrix \( U \in GL(m, R) \) and an automorphism \( \phi \in \text{Aut}(R) \) such that \( B = U \cdot \phi(A) \). Let \( \alpha := \gamma \circ \phi \), where \( \gamma \) is the \( R \)-module isomorphism
\[ \gamma : M_{m,1} \rightarrow M_{m,1} \\
C \mapsto U \cdot C \]
and \( \phi : M_{m,1} \rightarrow M_{m,1}, C \mapsto \phi(C) \). Then \( \alpha : M_{m,1} \rightarrow M_{m,1} \) is a module isomorphism over \( \phi : R \rightarrow R \), i.e. for all \( r \in R \) and \( C \in M_{m,1} \)
\[ \alpha(r \cdot C) = \phi(r) \cdot \alpha(C), \]
and by Lemma 3.1 of [GP16] we have
\[ \alpha \left( I \cdot M_{m,1} + \left\langle \frac{\partial A}{\partial x_1}, \ldots, \frac{\partial A}{\partial x_s} \right\rangle \right) = J \cdot M_{m,1} + \left\langle \frac{\partial B}{\partial x_1}, \ldots, \frac{\partial B}{\partial x_s} \right\rangle. \]
Hence, \( \alpha \) induces the module homomorphism over \( \phi : R \cong R \)
\[ \bar{\alpha} : T_I \rightarrow T_J, \; \bar{C} \mapsto \alpha(C), \]
since for all \( r \in R \) and \( \bar{C} \in T_I \), we have
\[
\bar{\alpha}(r \cdot \bar{C}) = \bar{\alpha}(r \cdot C) = \bar{\alpha}(r \cdot C) = \phi(r) \cdot \bar{\alpha}(C) = \phi(r) \cdot \bar{\alpha}(C).
\]

2. Follows from the first statement. \( \square \)

We now finish the paper by applying the previous results to ideals.

**Theorem 4.10.** Let \( I \) be a proper ideal of \( R \).

1. If \( \dim(R/I) = 0 \) then \( I \) is finitely contact determined.

2. If \( \dim(R/I) > 0 \) and \( K \) is infinite, then \( I \) is finitely contact determined if and only if \( R/I \) is an isolated complete intersection singularity.

**Proof.** Let \( A \) be the column matrix corresponding to a minimal set of generators \( a_1, \ldots, a_m \) of \( I \). Then \( I \) is finitely contact determined iff \( A \) is finitely \( G \)-determined by Remark 4.5.1.

If \( \dim(R/I) = 0 \) then \( m = mng(I) \geq s \) and \( I \) is finitely contact determined by Corollary 4.6.1.

Let \( \dim(R/I) > 0 \). If \( I \) is finitely contact determined then \( m < s \) by Corollary 4.6.1. By Theorem 2.7.1, we get \( \text{ht}(I_1(A)) = m \), i.e. \( \dim R/I = s - m \) and thus \( R/I \) is a complete intersection. By Corollary 4.6.2 \( I \) is an ICIS. \( \square \)

For a complete intersection we call the \( R/I \)-module \( T_I \) (see Definition 4.7), the **Tjurina module of** \( I \) and \( \tau(I) = \dim_K T_I \) the **Tjurina number of** \( I \) (see [GLS07, Theorem 1.16 and Definition 1.19] for the complex case).

We deduce that for a complete intersection \( R/I \), being an isolated singularity is equivalent to the finiteness of the Tjurina number in arbitrary characteristic.

**Theorem 4.11.** Let \( I \) be a proper ideal of \( R \) such that \( I \) defines a complete intersection. Then the following are equivalent for \( K \) infinite:

1. \( R/I \) is an isolated singularity.

2. The Tjurina number \( \tau(I) \) is finite.

3. \( I \) is finitely contact determined.

If any of these conditions is satisfied then \( I \) is contact \((2\tau(I) - \text{ord}(I) + 2)\)-determined. Moreover, if \( J \) an ideal such that \( R/J \) is a deformation of \( R/I \), then \( J \) is contact \( 2\tau(I) + 1 \)-determined, resp. \( 2\tau(I) \)-determined if \( \text{ord}(J) \geq 2 \).
Proof. Let $A$ be the column matrix corresponding to a minimal set of $m$ generators of $I$.

By definition $R/I$ is an isolated singularity if and only if there is a positive integer $h$ such that $I + I_h(Jac(A)) \supset m^h$. By Lemma 3.2, this is equivalent to the fact that $I_m(\Theta_{(G,A)}) \supset m^k$ for some positive integer $k$, where $\Theta_{(G,A)}$ is a presentation matrix of the $R$-module $M_m,1/T^*_A(GA)$. Applying [GP16, Proposition 4.2], this means that

$$\tau(I) = \dim_K M_{m,1}/T^*_A(GA) < \infty,$$

showing the equivalence of 1. and 2. The equivalence of 2. and 3. follows from Theorem 4.10.

Furthermore, applying [GP16, Proposition 4.2], $A$ is $G(2\tau(I) - \ord(I) + 2)$-determined so that $I$ is contact $(2\tau(I) - \ord(I) + 2)$-determined.

If $R/J$ is a deformation of $R/I$, $J$ defines an ICIS of the same dimension and hence $J$ is contact $(2\tau(J) + 1) \geq (2\tau(J) - \ord(J) + 2)$-determined. By semicontinuity of $\tau$ (Remark 4.8.2) $\tau(I) \geq \tau(J)$, implying that $J$ is $(2\tau(I) + 1)$-determined, resp. $2\tau(I)$-determined if $\ord(J) \geq 2$.

Remark 4.12. The determinacy of $I$, i.e. the minimum $k$ such that $I$ is contact $k$-determined, is in general not semicontinuous under deformations of $I$ (cf. [GLS07, Exercise 2.2.4]). It is therefore important, in particular in connection with classification and moduli problems, to have $2\tau(I) + 1$ as a semicontinuous bound for any deformation of $I$.

The fact that a complete intersection is finitely contact determined if it has an isolated singularity was already proved over the complex numbers by Brieskorn in the unpublished part of [Bri67, Theorem 1.3.1 and 1.3.5]. It is new and much more involved in positive characteristic. The determinacy bound is new in any characteristic.

We complete the section by considering in addition to contact equivalence also right equivalence for a hypersurface $f \in K[[x]]$, with $f$ being right equivalent to $g$ if $\phi(f) = g$ for some $\phi \in Aut(K[[x]])$.

Theorem 4.13. Let $K$ be any field and $f \in m \subset K[[x]]$.

1. $f$ is finitely contact determined iff the Tjurina number

$$\tau(f) = \dim_K K[[x]]/(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_s})$$

is finite. If this holds, $f$ is contact $(2\tau(f) - \ord(f) + 2)$-determined.

2. $f$ is finitely right determined iff the Milnor number

$$\mu(f) = \dim_K K[[x]]/(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_s})$$

is finite. If this holds, $f$ is right $(2\mu(f) - \ord(f) + 2)$-determined.

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**Proof.** Since the statement is obviously true for \( f = 0 \) we may assume that \( f \neq 0 \).

1. An argument, following the proof of Theorem 3.8.2 with \( m = 1 \), goes as follows. To see that we do not need any assumption about \( K \) we may choose in the proof \( g = g_1 = x_1^N + \cdots + x_s^N \) and get \( Q_t = (g + tf, Nx_1^{N-1} + t \partial f / \partial x_1, \cdots , N x_s^{N-1} + t \partial f / \partial x_s) \). Then

\[
\dim_K K[[x]]/Q_t \leq \dim_K K[x]/Q_t < \infty.
\]

The latter inequality follows e.g. by choosing a global degree ordering on \( K[x] \) such that the leading ideal \( Q_t \) is generated by \( x_1^{N-1}, \cdots , x_s^{N-1} \) if \( N \) is big enough and apply [GP08, Theorem 5.2.6]. The rest of the proof is unchanged.

2. If \( \mu(f) < \infty \), then \( f \) is finitely right determined by [GP16, Theorem 3.2] for any \( K \). The proof of the converse goes as in 1. with \( Q_t = (N x_1^{N-1} + t \partial f / \partial x_1, \cdots , N x_s^{N-1} + t \partial f / \partial x_s) \).

**Remark 4.14.** (1) Theorem 4.13 was already stated in [BGM12, Theorem 5]. However, the proof of the direction that finite determinacy implies the finiteness of \( \tau \) resp. \( \mu \) in [BGM12] contains a gap since it assumes the orbit map \( G^{(k)}(k) \to G^{(k)} \text{jet}_k(f) \) to be separable. Separability of the orbit map holds always in characteristic 0 and quite often in positive characteristic, but not always. (see [GP16, Example 2.9]).

(2) Note that (2) \( \Leftrightarrow \) (1) \( \Rightarrow \) (3) in Theorem 4.11 holds for any \( K \), as well as (3) \( \Rightarrow \) (1) for hypersurfaces by Theorem 4.13. However, we do not know whether (3) \( \Rightarrow \) (1) holds for finite \( K \) and a complete intersection which is not a hypersurface.

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**References**

[BdPW87] James W. Bruce, Andrew A. du Plessis, and Charles T. C. Wall, Determinacy and unipotency, Invent. Math. 88 (1987), no. 3, 521–554. Zbl 0596.58005

[BGM12] Yousra Boubakri, Gert-Martin Greuel, and Thomas Markwig, Invariants of hypersurface singularities in positive characteristic, Rev. Mat. Complut. 25 (2012), no. 1, 61–85. Zbl 1279.14004

[BK16] Genrich R. Belitskii, Dmitry Kerner, Finite determinacy of matrices over local rings.II. Tangent modules to the miniversal deformations for group-actions involving the ring automorphisms, ArXiv e-prints (2016).
[Bri67] Egbert Valentin Brieskorn, *Isolierte Singularitäten komplexer Räume*, Habilitationsschrift, Bonn, 1967.

[BV88] Winfried Bruns, Udo Vetter, *Determinantal rings*. Lecture Notes in Mathematics 1327, Berlin etc.: Springer-Verlag (1988). Zbl 0673.13006

[Dam81] James Damon, *Finite determinacy and topological triviality. I*, Invent. Math. 62 (1980/81), no. 2, 299–324. Zbl 0489.58003

[dP80] Andrew A. du Plessis, *On the determinacy of smooth map-germs*, Invent. Math. 58 (1980). Zbl 0446.58004

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. Zbl 0819.13001

[Gaf79] Terence Gaffney, *A note on the order of determination of a finitely determined germ*, Invent. Math. 52 (1979), no. 2, 127–130. Zbl 0419.58004

[GK90] Gert-Martin Greuel, Heike Kröning, *Simple singularities in positive characteristic*, Math. Z. 203 (1990), no. 2, 339?354. Zbl 0715.14001

[GLS07] Gert-Martin Greuel, Christoph Lossen, and Eugenii Shustin, *Introduction to Singularities and Deformations*, Springer Monographs in Mathematics, Springer, Berlin, 2007. Zbl 1125.32013

[GP08] Gert-Martin Greuel and Gerhard Pfister, *A Singular Introduction to Commutative Algebra*, 2nd Edition, Springer, Berlin, 2008, With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann. Zbl 1133.13001

[GP16] Gert-Martin Greuel and Thuy Huong Pham, *On finite determinacy for matrices of power series*, ArXiv e-prints (2016). To appear in Math.Zeitschrift.

[HE71] Melvin Hochster and John A. Eagon, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. 93 (1971), 1020–1058. Zbl 0244.13012

[Pha16] Thuy Huong Pham, *On finite determinacy of hypersurface singularities and matrices in arbitrary characteristic*, Ph.D. thesis, TU Kaiserslautern, 2016.

[Mat68] John N. Mather, *Stability of $C^\infty$ mappings. III. Finitely determined map-germs*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 35, 279–308. Zbl 0603.58028

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[Mat89] Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. Zbl 0666.13002

[NT99] Dam V. Nhi and Ngô V. Trung, *Specialization of modules*, Comm. Algebra 27 (1999), no. 6, 2959–2978. Zbl 0933.13009

[Os16] Osgood, William F., *On functions of several complex variables*, American M. S. Trans. 17, 1-8 (1916). JFM 46.0527.01

[Ro13] Rond, Guillaume, *Artin Approximation*, habilitation thesis, Luminy (2013).

[Tou68] Jean-Claude Tougeron, *Idéaux de fonctions différentiables. I*, Ann. Inst. Fourier (Grenoble) 18 (1968), no. fasc. 1, 177–240. Zbl 0251.58001

[Tru15] Ngô V. Trung, *Generic determinantal ideals in power series rings*, unpublished note, 2015.

[Wal81] Charles T. C. Wall, *Finite determinacy of smooth map-germs*, Bull. London Math. Soc. 13 (1981), no. 6, 481–539. Zbl 0451.58009

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