Let \((N, g)\) be an \(n\)-dimensional complete Riemannian manifold with nonempty boundary \(\partial N\). Assume that the Ricci curvature of \(N\) has a negative lower bound \(\text{Ric} \geq -(n-1)c^2\) for some \(c > 0\), and the mean curvature of the boundary \(\partial N\) satisfies \(H \geq (n-1)c_0 > (n-1)c\) for some \(c_0 > c > 0\). Then a known result (see [12]) says that 
\[
\sup_{x \in N} d(x, \partial N) \leq \frac{1}{c_0} \coth^{-1} \frac{1}{c}. 
\]
In this paper, we prove that if the boundary \(\partial N\) is compact, then the equality holds if and only if \(N\) is isometric to the geodesic ball of radius \(\frac{1}{c_0} \coth^{-1} \frac{1}{c}\) in an \(n\)-dimensional hyperbolic space \(\mathbb{H}^n(-c^2)\) of constant sectional curvature \(-c^2\). Moreover, we also prove an analogous result for manifold with nonempty boundary and with \(m\)-Bakry-Émery Ricci curvature bounded below by a negative constant.

1. Introduction

Let \((N, g)\) be an \(n\)-dimensional complete Riemannian manifold. The classical Bonnet and Myers’ theorem says that if the Ricci curvature of \((N, g)\) has a positive lower bound \(\text{Ric} \geq (n-1)c^2 > 0\), then the diameter of \(N\) is at most \(\pi/c\). Cheng [4] proved that if the diameter is equal to \(\pi/c\), then \(N\) is isometric to the \(n\)-sphere of constant sectional curvature \(c^2\).

Recently, M. M. Li [10] considered \(n\)-dimensional complete Riemannian manifold \((N, g)\) with nonnegative Ricci curvature and with mean convex boundary \(\partial N\). M. M. Li proved that if the mean curvature of the boundary \(\partial N\) satisfies \(H \geq (n-1)c_0 > 0\) for some constant \(c_0 > 0\), then
\[
\sup_{x \in N} d(x, \partial N) \leq \frac{1}{c_0}, \tag{1.1}
\]
where \(d\) denotes the distance function on \(N\). Moreover, if \(\partial N\) is compact, then \(N\) is also compact and equality holds in (1.1) if and only if \(N\) is isometric to an \(n\)-dimensional Euclidean ball of radius \(1/c_0\). Here the mean curvature \(H\) of \(\partial N\) is defined as the trace of the second fundamental form of \(\partial N\) in \(N\), that is, \(H = \sum_{i=1}^{n-1} \langle \nabla e_i \nu, e_i \rangle\) for any orthonormal basis \(e_1, \cdots, e_{n-1}\) of tangent bundle \(T\partial N\), with respect to the outward unit normal \(\nu\) of \(\partial N\).

Note that by a similar argument as in the proof of the inequality (1.1), if the Ricci curvature of \(N\) has a negative lower bound \(\text{Ric} \geq -(n-1)c^2\) for
$c > 0$, and the mean curvature of the boundary \( \partial N \) satisfies \( H \geq (n-1)c_0 > (n-1)c \) for some constant \( c_0 > c > 0 \), then one can prove that

\[
\sup_{x \in N} d(x, \partial N) \leq \frac{1}{c} \coth^{-1} \frac{c_0}{c}.
\]  

(1.2)

In fact the distance bound (1.1) and (1.2) can be viewed as a Riemannian version of Hawking’s singularity theorem (see e.g. [6]). The proof of the distance bound (1.1), (1.2) is a standard argument by using the second variation formula of arc-length and can be found in Yanyan Li and Luc Nguyen’s paper [12, section 2]. In the first part of this paper, we study the equality case of (1.2). We have

**Theorem 1.1.** Let \( (N^n, g) \) be an \( n \)-dimensional complete Riemannian manifold with nonempty boundary \( \partial N \). Assume that the Ricci curvature of \( N \) has a lower bound \( \text{Ric} \geq -(n-1)c^2 \) for some \( c > 0 \), and the mean curvature of the boundary \( \partial N \) satisfies \( H \geq (n-1)c_0 > (n-1)c > 0 \) for some constant \( c_0 > c > 0 \). Then we know that the diameter estimate (1.2) holds in \( N \).

If \( \partial N \) is compact, then (1.2) implies that \( N \) is also compact. We show if the equality holds in (1.2), then \( N \) is isometric to a geodesic ball of radius \( \frac{1}{c} \coth^{-1} \frac{c_0}{c} \) in an \( n \)-dimensional hyperbolic space \( \mathbb{H}^n(-c^2) \) of constant sectional curvature \( -c^2 \).

The proof of Theorem 1.1 will be given in section 2. We first include the proof of the distance bound (1.2) for convenience of readers (see [12]). Then we consider the rigidity result when the equality occurs in (1.2). By rescaling of metric, it suffices to consider the case \( c = 1 \). The proof can be roughly divided into three parts. Firstly, by a Frankel type argument (see [8],[10]), we show that under the curvature assumption of Theorem 1.1, the boundary \( \partial N \) is connected. Secondly, by using a similar argument as in M. M. Li’s paper [10], we show that if the equality occurs in (1.2), then \( N \) is a geodesic ball of radius \( \coth^{-1} c_0 \) centered at some point \( x_0 \). Finally, by showing that the Laplacian comparison (2.6) assumes equality everywhere in \( N \), we obtain that \( N \) has constant sectional curvature \(-1\) and is isometric to the hyperbolic ball. In the last step of the proof, a Heintze-Karcher’s argument [7] will be used and is a key ingredient to the proof.

Let \( (N^n, g) \) be an \( n \)-dimensional complete smooth Riemannian manifold and \( f \) be a smooth function on \( N \). We denote \( \nabla, \Delta \) and \( \nabla^2 \) the gradient, Laplacian and Hessian operator on \( N \) with respect to \( g \), respectively. Given \( m \in [n, \infty) \), the \( m \)-Bakry-Émery Ricci curvature of \( (N, g) \) (see [2]) is defined by

\[
\text{Ric}_f^m = \text{Ric} + \nabla^2 f - \frac{1}{m-n} \nabla f \otimes \nabla f, \quad (m > n).
\]

(1.3)

When \( m = \infty \), the last term of (1.3) is interpreted as 0 and this gives the Bakry-Émery Ricci curvature \( \text{Ric}_f = \text{Ric} + \nabla^2 f \). When \( m = n \), this term only makes sense if \( f \) is constant and in this case \( \text{Ric}_f^m := \text{Ric} \).
Recently, the study of manifold with $m$-Bakry-Émery Ricci curvatures $\text{Ric}_f^m$ attracts many interests. Analogous to the Ricci curvature case, if one assumes that the $m$-Bakry-Émery Ricci curvature of $(N, g)$ satisfies $\text{Ric}_f^m \geq (m - 1)c^2 > 0$, Qian [18] proved that $\text{diam}(N) \leq \pi/c$ and then Ruan [19] proved that equality holds if and only if $N$ is isometric to the $n$-sphere of constant sectional curvature $c^2$. Recently, the authors [9] proved an analogous result of M. M. Li’s theorem [10] in the $m$-Bakry-Émery Ricci curvature case. Let $(N^n, g)$ be an $n$-dimensional complete Riemannian manifold with nonempty boundary $\partial N$ and $f$ be a smooth function on $N$. The $f$-mean curvature $H_f$ of $\partial N$ in $N$ is given by

$$H_f = H - \langle \bar{\nabla} f, \nu \rangle,$$

where $H$ is the mean curvature of $\partial N$ in $N$ and $\nu$ is the outward unit normal of $\partial N$. When $f$ is constant, $H_f$ is just the mean curvature $H$.

Assume that the $m$-Bakry-Émery Ricci curvature is nonnegative on $N$, and the $f$-mean curvature of the boundary $\partial N$ satisfies $H_f \geq (m - 1)c_0 > 0$ for some constant $c_0 > 0$. Then the authors [9] proved that

$$\sup_{x \in N} d(x, \partial N) \leq \frac{1}{c_0}.$$  \hspace{1cm} (1.4)

Moreover, if we assume that $\partial N$ is compact, then $N$ is also compact and equality holds in (1.4) if and only if $N$ is isometric to an $n$-dimensional Euclidean ball of radius $1/c_0$.

In the second part of this paper, we consider the manifold with nonempty boundary and with $m$-Bakry-Émery Ricci curvature bounded below by a negative constant.

**Theorem 1.2.** Let $(N^n, g)$ be an $n$-dimensional complete Riemannian manifold with nonempty boundary and $f$ be a smooth function on $N$. If the $m$-Bakry-Émery Ricci curvature of $N$ has a negative lower bound, i.e., $\text{Ric}_f^m \geq -(m - 1)c^2$ for some constant $c > 0$ on $N$, and the $f$-mean curvature of the boundary $\partial N$ satisfies $H_f \geq (m - 1)c_0 > (m - 1)c$ for some constant $c_0 > c > 0$, then we have the distance bound:

$$\sup_{x \in N} d(x, \partial N) \leq \frac{1}{c} \operatorname{coth}^{-1} \frac{c_0}{c}.$$  \hspace{1cm} (1.5)

If $\partial N$ is compact, then (1.5) shows that $N$ is compact. Moreover, if the equality holds in (1.5), we have $m = n$, and $N$ is isometric to a geodesic ball of radius $\frac{1}{c} \operatorname{coth}^{-1} \frac{c_0}{c}$ in an $n$-dimensional hyperbolic space $\mathbb{H}^n(-c^2)$ of constant sectional curvature $-c^2$.

The proof of Theorem 1.2 is similar with the proof of Theorem 1.1 with some adjustment. Without loss of generality, we assume that $c = 1$. When the equality occurs in (1.5), arguing as the proof of Theorem 1.1 we show that $N$ is equal to a geodesic ball of radius $\operatorname{coth}^{-1} c_0$ centered at some point $x_0$ and, the $f$-Laplacian comparison (3.7) assumes equality everywhere in $N$. There the generalized Heintze-Karcher theorem due to V. Bayle [1] (see also [14,15]) plays an important role. Finally, by using the Reilly formula for
the boundary \( \partial N \)

a lower bound \( \text{Ric} \) \( \partial N \) satisfies

Since above inequality implies

d \( \partial N \) fold with nonempty boundary

Choose \( 0 < c \leq \) each \( 1 \)

(0) = 0 and \( \varphi \) \( \gamma \): \([0, T]\) (see \([12]\)). For any point \( x \in N \), since \( N \) is complete, there exists a geodesic \( \gamma: [0, d] \to N \) parametrized by arc length with \( \gamma(0) = x \), \( \gamma(d) \in \partial N \) and \( d = d(x, \partial N) \). Choose an orthonormal basis \( e_1, \ldots, e_{n-1} \) for \( T_{\gamma(d)} \partial N \) and let \( e_i(s) \) be the parallel transport of \( e_i \) along \( \gamma \). Let \( V_i(s) = \varphi(s)e_i(s) \) with \( \varphi(0) = 0 \) and \( \varphi(d) = 1 \). From the first variation formula, we have that for each \( 1 \leq i \leq n-1 \)

\[
0 = \delta \gamma(V_i) = \langle \gamma'(d), V_i(d) \rangle - \langle \gamma'(0), V_i(0) \rangle - \int_0^d \langle \gamma''(s), V_i(s) \rangle ds
\]

which implies that \( \gamma'(d) \) is orthogonal to \( \partial N \) at \( \gamma(d) \). The second variation formula gives that

\[
0 \leq \sum_{i=1}^{n-1} \delta^2 \gamma(V_i, V_i) = \int_0^d \left( (n - 1)\varphi'(s)^2 - \varphi(s)^2 \text{Ric} \langle \gamma'(s), \gamma'(s) \rangle \right) ds
\]

\[
+ \langle \nabla V_i(d), \gamma'(d) \rangle - \langle \nabla V_i(0), \gamma'(0) \rangle
\]

\[
= \int_0^d \left( (n - 1)\varphi'(s)^2 - \varphi(s)^2 \text{Ric} \langle \gamma'(s), \gamma'(s) \rangle \right) ds - H(\gamma(d)).
\]

Since \( \text{Ric} \geq -(n - 1)c^2 \) in \( N \) and \( H \geq (n - 1)c_0 > (n - 1)c > 0 \) on \( \partial N \), the above inequality implies

\[
0 \leq \int_0^d (\varphi'(s)^2 + c^2 \varphi^2(s)) ds - c_0.
\]

Choose

\[
\varphi(s) = \frac{\sinh(cs)}{\sinh(cd)} \quad 0 \leq s \leq d,
\]

which satisfies \( \varphi(0) = 0 \) and \( \varphi(d) = 1 \). By substituting the above chosen \( \varphi(s) \) into (2.1), we have

\[
c_0 \leq c \coth(cd).
\]

Therefore we have \( d \leq \frac{1}{c} \coth^{-1} \frac{c}{c_0} \) and this is the distance bound (1.2).

The next lemma says that under the assumption of Theorem (1.1) the boundary \( \partial N \) is connected.

**Lemma 2.1.** Let \((N^n, g)\) be an \( n \)-dimensional complete Riemannian manifold with nonempty boundary \( \partial N \). Assume that the Ricci curvature of \( N \) has a lower bound \( \text{Ric} \geq -(n - 1)c^2 \), and the mean curvature of the boundary \( \partial N \) satisfies \( H \geq (n - 1)c_0 > (n - 1)c \) for some constant \( c_0 > c > 0 \). Then the boundary \( \partial N \) is connected.
Proof. We use the similar argument as in [8, 10]. Suppose \( \partial N \) is not connected, let \( \Sigma \) be one of its components. Then \( \Sigma \) and \( \partial N \setminus \Sigma \) have a positive distance apart, i.e., \( d(\Sigma, \partial N \setminus \Sigma) = l > 0 \). Since \( \Sigma \) and \( \partial N \setminus \Sigma \) are compact, there exists a minimizing geodesic \( \gamma : [0, l] \to N \) parametrized by arc-length which realize the distance between \( \Sigma \) and \( \partial N \setminus \Sigma \). Note that \( \gamma(0) \in \Sigma \), \( \gamma(l) \in \partial N \setminus \Sigma \) and \( \gamma(s) \) lies in the interior of \( N \) for all \( s \in (0, l) \). Moreover, \( \gamma'(0) \perp T_{\gamma(0)} \partial N \) and \( \gamma'(l) \perp T_{\gamma(l)} \partial N \), i.e., \( \gamma \) is a free boundary geodesic. Choose an orthonormal basis \( e_1, \ldots, e_{n-1} \) for \( T_{\gamma(0)} \partial N \) and let \( e_i(s) \) be the parallel transport of \( e_i \) along \( \gamma \). Let \( V_i(s) = \varphi(s)e_i(s) \) with \( \varphi(0) = \varphi(l) = 1 \). Then the second variation formula of arc length gives that

\[
0 \leq \sum_{i=1}^{n-1} \delta^2 \gamma(V_i, V_i) = \int_0^l \left( (n-1)\varphi'(s)^2 - \varphi(s)^2 \text{Ric}(\gamma'(s), \gamma'(s)) \right) \, ds \\
+ \langle \nabla V_i(t) \gamma(l), \gamma'(l) \rangle - \langle \nabla V_i(0), \gamma'(0) \rangle \\
= \int_0^l \left( (n-1)\varphi'(s)^2 - \varphi(s)^2 \text{Ric}(\gamma'(s), \gamma'(s)) \right) \, ds \\
- H(\gamma(l)) - H(\gamma(0)). \tag{2.3}
\]

Now we choose

\[
\varphi(s) = \frac{\cosh c(s - \frac{l}{2})}{\cosh(c/2)}, \quad 0 \leq s \leq l.
\]

Then \( \varphi(0) = \varphi(l) = 1 \). Substituting the above chosen \( \varphi(s) \) into (2.3), and using the assumption \( \text{Ric} \geq -(n-1)c^2 \) in \( N \) and \( H \geq (n-1)c_0 > (n-1)c > 0 \) on \( \partial N \), we obtain

\[
0 \leq 2(n-1)c \tanh\left(\frac{cl}{2}\right) - 2(n-1)c_0. \tag{2.4}
\]

Since \( l \) is the distance of \( \Sigma \) and \( \partial N \setminus \Sigma \) and \( \gamma(t) \) realizes this distance, we have \( l \leq 2 \sup_{x \in N} d(x, \partial N) \leq \frac{2}{c} \coth^{-1} \frac{c_0}{c} \) by (1.2). Therefore the right-hand side of (2.4) is bounded from above by

\[
2(n-1)c \tanh(\coth^{-1} \frac{c_0}{c}) - 2(n-1)c_0 \\
= \frac{2(n-1)}{c_0} (c^2 - \frac{c_0^2}{c^2}) < 0,
\]

which is a contradiction. We conclude that the boundary \( \partial N \) is connected. \( \square \)

Now we continue the proof of Theorem 1.1. Assume that \( \partial N \) is compact, then (1.2) implies \( N \) is also compact. Suppose equality holds in (1.2). By rescaling the metric of \( N \), we can assume that \( c = 1 \). Since \( N \) is compact, there exists one point \( x_0 \in N \) such that

\[
d(x_0, \partial N) = \coth^{-1} c_0. \tag{2.5}
\]

We denote \( \rho_0 = \coth^{-1} c_0 \) for simplicity. We first show that
**Lemma 2.2.** Under the assumption of Theorem 1.1, if the equality holds in (1.2), then $N$ is equal to the geodesic ball of radius $\rho_0$ centered at $x_0$.

**Proof.** It is clear that the geodesic ball $B_{\rho_0}(x_0)$ of radius $\rho_0$ centered at $x_0$ is contained in $N$. We show that $N$ is just equal to the geodesic ball $B_{\rho_0}(x_0)$. Let $\rho = d(x_0, \cdot)$ be the distance function from $x_0$. Since the Ricci curvature of $N$ satisfies $Ric \geq -(n-1)$, the Laplacian of $\rho$ satisfies (see [17,20])

$$\bar{\Delta} \rho \leq (n-1) \coth \rho,$$

in the sense of distribution. Let $\Sigma = \{q \in \partial N : \rho(q) = \rho_0\}$, which is clearly a closed set in $\partial N$ by the continuity of $\rho$. Since $\partial N$ is connected, to show $\Sigma = \partial N$, it suffices to show that $\Sigma$ is also open in $\partial N$, that is for any $q \in \Sigma$, there is an open neighborhood $U$ of $q$ in $\partial N$ such that $\rho \equiv \rho_0$ on $U$. If $q$ is not a conjugate point to $x_0$ in $N$, then the geodesic sphere $\partial B_{\rho_0}(x_0)$ is a smooth hypersurface near $q$ in $N$. Note that $\bar{\Delta} \rho$ is the mean curvature $H$ of the geodesic sphere and $\rho = \rho_0$ on the geodesic sphere, by the Laplacian comparison inequality (2.6), we have that the mean curvature $H$ of the geodesic sphere is at most $(n-1)\rho_0$. However, by the assumption of Theorem 1.1, the mean curvature of $\partial N$ is at least $(n-1)\rho_0$. Then from the maximum principle (see [3]), we have that $\partial N$ and $\partial B_{\rho_0}(x_0)$ coincides in a neighborhood of $q$. This implies that $\Sigma$ is open near any $q$ which is not a conjugate point. A similar process in [10] (see also Calabi [3]) makes us to work through the argument to conclude that $\rho$ is constant near $q$ in $\partial N$, when $q$ is a conjugate point of $x_0$. This proves that $\partial N$ is just the geodesic sphere $\partial B_{\rho_0}(x_0)$ and $N$ is the geodesic ball $B_{\rho_0}(x_0)$ of radius $\rho_0$ centered at $x_0$.

Since any $q \in \partial N$ can be joined by a minimizing geodesic $\gamma$ parameterized by arc-length from $x_0$ to $q$, and $\gamma$ is orthogonal to $\partial N = \partial B_{\rho_0}(x_0)$ at $q$, the geodesic $\gamma$ is uniquely determined by $q$ and $q$ is not in the cut locus of $x_0$. So that $\rho = d(x_0, \cdot)$ is smooth up to the boundary $\partial N$. From the proof of lemma 2.2 the mean curvature $H$ of the boundary $\partial N$ satisfies $H = (n-1)\coth \rho_0$. Since $\partial N$ is the geodesic sphere $\partial B_{\rho_0}(x_0)$, the Laplacian of $\rho$ is equal to the mean curvature $H$ on the boundary $\partial N$. Therefore

$$\bar{\Delta} \rho = (n-1) \coth \rho,$$

holds on the boundary $\partial N$. We next show that (2.7) holds everywhere in $N$.

On the one hand, since $Ric \geq -(n-1)$ on $N$, the Laplacian comparison (2.6) for $\rho$ holds in the classical sense. Note that $|\nabla \rho| = 1$ in $N$, then we have

$$\bar{\Delta} \cosh \rho = \bar{\Delta} \rho \sinh \rho + \cosh \rho |\nabla \rho|^2 \leq n \cosh \rho.$$

Integrating the above inequality over $N$ and by divergence theorem, we have

$$\int_{\partial N} \sinh \rho (\nabla \rho, \nu) d\mu \leq n \int_N \cosh \rho dv,$$
where $\nu$ is the outward unit normal of $\partial N$ in $N$ and $d\mu$, $dv$ are volume elements on $\partial N$ and $N$ respectively. Note that $\rho = \rho_0$ and $(\nabla \rho, \nu) = 1$ on $\partial N$, the above inequality gives that

$$\int_{\partial N} \sinh \rho_0 d\mu \leq n \int_N \cosh \rho(x) dv. \quad (2.8)$$

On the other hand, we prove the reversed inequality also holds in (2.8).

**Lemma 2.3.** We have

$$\int_{\partial N} \sinh \rho_0 d\mu \geq n \int_N \cosh \rho(x) dv. \quad (2.9)$$

**Proof.** The proof is inspired by Heintze-Karcher’s paper [7]. Note that any $q \in \partial N$ can be joined by a minimizing geodesic $\gamma$ parameterized by arc-length from $x_0$ to $q$, and $\gamma$ is orthogonal to $\partial N = \partial B_{\rho_0}(x_0)$ at $q$, the geodesic $\gamma$ is uniquely determined by $q$ and $q$ is not in the cut locus of $x_0$. The exponential map of the normal bundle $T^\perp \partial N$ of $\partial N$ in $N$ is surjective. For any $q \in \partial N$, the curve $\gamma(t) = \exp_q(-t\nu) \ (0 \leq t \leq \rho_0)$ is the geodesic connecting $q$ and $x_0$, i.e., $\gamma(\rho_0) = x_0$. We have

$$\int_N \cosh \rho(x) dv \leq \int_{\partial N} \int_0^{\rho_0} \cosh(\exp_q(-t\nu(q))) |\det(d\exp_q)_{t\nu}| dt d\mu_q,$$

Since $\text{Ric} \geq -(n-1)$ in $N$, Corollary 3.3.2 of [7] gives that

$$|\det(d\exp_q)_{t\nu}| \leq (\cosh t - \frac{H(q)}{n-1} \sinh t)^{n-1}. \quad (2.10)$$

Note that $\rho(\exp_q(-t\nu(q))) = \rho_0 - t$, and

$$\cosh(\rho_0 - t) = \cosh \rho_0 \cosh t - \sinh \rho_0 \sinh t = \sinh \rho_0(\coth \rho_0 \cosh t - \sinh t).$$

From the proof of Lemma 2.2, we have $H(q) = (n-1) \coth \rho_0$ on $\partial N$. So we have

$$\int_N \cosh \rho(x) dv \leq -\int_{\partial N} \int_0^{\rho_0} \frac{1}{n} \sinh \rho_0 \frac{d}{dt}(\cosh t - \coth \rho_0 \sinh t)^n dt d\mu_q$$

$$= \frac{1}{n} \int_{\partial N} \sinh \rho_0 d\mu$$

which gives the inequality (2.9). \qed

Combining (2.8) and (2.9), we have that (2.7) holds everywhere in $N$. Thus we conclude that $N$ has constant sectional curvature $-1$ and is isometric to the hyperbolic ball (see [17]). The proof of Theorem 1.1 is completed.
3. Proof of Theorem 1.2

Firstly, the proof of diameter estimate (1.5) is similar with the proof of (1.2), which is also by using the second variation formula of arc-length. For any point \( x \in N \), since \( N \) is complete, there exists a geodesic \( \gamma : [0, d] \rightarrow N \) parametrized by arc-length with \( \gamma(0) = x, \gamma(d) \in \partial N \) and \( d = d(x, \partial N) \). Choose an orthonormal basis \( e_1, \ldots, e_{n-1} \) for \( T_{\gamma(d)} \partial N \) and let \( e_i(s) \) be the parallel transport of \( e_i \) along \( \gamma \). Let \( V_i(s) = \varphi(s)e_i(s) \) with \( \varphi(0) = 0 \) and \( \varphi(d) = 1 \). The first variation formula implies that \( \gamma'(d) \) is orthogonal to \( \partial N \) at \( \gamma(d) \). The second variation formula gives that

\[
0 \leq \sum_{i=1}^{n-1} \delta^2 \gamma(V_i, V_i) = \int_0^d \left( (n-1)\varphi'(s)^2 - \varphi(s)^2 \text{Ric}(\gamma'(s), \gamma'(s)) \right) ds - H(\gamma(d)).
\]

By the definition of \( m \)-Bakry-Émery Ricci curvature, we have

\[
0 \leq \int_0^d \left( (n-1)\varphi'(s)^2 - \varphi(s)^2 \text{Ric}_f^m(\gamma'(s), \gamma'(s)) \right) ds - H(\gamma(d)) \\
+ \int_0^d \varphi(s)^2 \left( \nabla^2 f(\gamma(s), \gamma'(s)) - \frac{1}{m-n} \langle \nabla f(\gamma(s)), \gamma'(s) \rangle^2 \right) ds
\]

\[
= \int_0^d \left( (n-1)\varphi'(s)^2 - \varphi(s)^2 \text{Ric}_f^m(\gamma'(s), \gamma'(s)) \right) ds - H(\gamma(d)) \\
+ \int_0^d \varphi(s)^2 \left( \frac{d^2}{ds^2} f(\gamma(s)) - \frac{1}{m-n} \left( \frac{d}{ds} f(\gamma(s)) \right)^2 \right) ds
\]

(3.1)

where we used the facts

\[
\frac{d}{ds} f(\gamma(s)) = \langle \nabla f(\gamma(s)), \gamma'(s) \rangle
\]

and

\[
\frac{d^2}{ds^2} f(\gamma(s)) = \nabla^2 f(\gamma'(s), \gamma'(s)).
\]

By integration by parts, we deduce from (3.1) that

\[
0 \leq \int_0^d \left( (n-1)\varphi'(s)^2 - \varphi(s)^2 \text{Ric}_f^m(\gamma'(s), \gamma'(s)) - 2\varphi(s)\varphi'(s) \frac{d}{ds} f(\gamma(s)) \right) \\
- \frac{1}{m-n} \varphi(s)^2 \left( \frac{d}{ds} f(\gamma(s)) \right)^2 ds + \varphi(d)^2 \langle \nabla f(\gamma(d)), \gamma'(d) \rangle \\
- \varphi(0)^2 \langle \nabla f(\gamma(0)), \gamma'(0) \rangle - H(\gamma(d)).
\]

(3.2)

Note that \( \varphi(0) = 0, \varphi(d) = 1 \) and \( \gamma'(d) \) is equal to the outward unit normal vector \( \nu \) at \( \gamma(d) \in \partial N \). The \( f \)-mean curvature \( H_f \) at \( \gamma(d) \) is

\[
H_f(\gamma(d)) = H(\gamma(d)) - \langle \nabla f(\gamma(d)), \nu(\gamma(d)) \rangle.
\]

Moreover, the Cauchy-Schwartz inequality implies

\[
-2\varphi(s)\varphi'(s) \frac{d}{ds} f(\gamma(s)) \leq (m-n)\varphi'(s)^2 + \frac{1}{m-n} \varphi(s)^2 \left( \frac{d}{ds} f(\gamma(s)) \right)^2.
\]
Thus from (3.2), we have
\[ 0 \leq \int_0^d \left( (m - 1)\varphi'(s)^2 - \varphi(s)^2 Ric_f^m(\gamma'(s), \gamma'(s)) \right) ds - H_f(\gamma(d)). \tag{3.3} \]
Choose
\[ \varphi(s) = \frac{\sinh(cs)}{\sinh(cd)}, \quad 0 \leq s \leq d, \]
which satisfies \( \varphi(0) = 0 \) and \( \varphi(d) = 1 \). Since \( Ric_f^m \geq -(m - 1)c^2 \) in \( N \) and \( H_f \geq (m - 1)c_0 > (m - 1)c \) on \( \partial N \), by substituting the above chosen \( \varphi(s) \) into (3.3), we have
\[ c_0 \leq c \coth(cd). \tag{3.4} \]
Therefore we have \( d \leq \frac{1}{2} \coth^{-1} \frac{m}{2} \) and this is the distance bound (1.5).

If the boundary \( \partial N \) is compact, then (1.5) implies \( N \) is also compact. Next we prove the rigidity result when the equality occurs in (1.5). As in the proof of Theorem 1.1, the curvature assumption of Theorem 1.2 implies the boundary \( \partial N \) is connected.

Lemma 3.1. Under the curvature assumption of Theorem 1.2, the boundary \( \partial N \) is connected.

**Proof.** The proof is also by a Frankel type argument, see lemma 2.1. We include a proof here for exhibiting the adjustment. Suppose \( \partial N \) is not connected, let \( \Sigma \) be one of its components. Let \( \gamma(s) \) \( (0 \leq s \leq l) \) be the free boundary geodesic realizing the distance between \( \Sigma \) and \( \partial N \setminus \Sigma \). Choose an orthonormal basis \( e_1, \ldots, e_{n-1} \) for \( T_{\gamma(0)}\partial N \) and let \( e_i(s) \) be the parallel transport of \( e_i \) along \( \gamma \). Let \( V_i(s) = \varphi(s)e_i(s) \) with \( \varphi(0) = \varphi(l) = 1 \). Then the second variation formula of arc-length gives that
\[
0 \leq \sum_{i=1}^{n-1} \delta^2 \gamma(V_i, V_i) = \int_0^l \left( (n - 1)\varphi'(s)^2 - \varphi(s)^2 Ric(\gamma'(s), \gamma'(s)) \right) ds \\
\quad + \langle \bar{\nabla}_{V_i(l)} V_i(l), \gamma'(l) \rangle - \langle \bar{\nabla}_{V_i(0)} V_i(0), \gamma'(0) \rangle \\
= \int_0^l \left( (n - 1)\varphi'(s)^2 - \varphi(s)^2 Ric(\gamma'(s), \gamma'(s)) \right) ds \\
- H(\gamma(l)) - H(\gamma(0)).
\]
By the definition of \( m \)-Bakry-Émery Ricci curvature, and using the Cauchy-Schwartz inequality as in the proof of (1.5), we have
\[
0 \leq \int_0^l \left( (m - 1)\varphi'(s)^2 - \varphi(s)^2 Ric_f^m(\gamma'(s), \gamma'(s)) \right) ds \\
- H_f(\gamma(l)) - H_f(\gamma(0)). \tag{3.5}
\]
Since \( Ric_f^m \geq -(m - 1)c^2 \) in \( N \) and \( H_f \geq (m - 1)c_0 > (m - 1)c > 0 \) on \( \partial N \), we can argue as the proof of lemma 2.1 to get a contradiction by choosing
the function
\[ \varphi(s) = \frac{\cosh c(s - \frac{l}{2})}{\cosh(cl/2)}, \quad 0 \leq s \leq l. \]
in (3.3). Then we conclude that the boundary \( \partial N \) is connected. \( \square \)

Now assume that the equality occurs in (1.5). Without loss of generality, we assume that \( c = 1 \). By the compactness of \( N \), there exists one point \( x_0 \in N \) such that
\[ d(x_0, \partial N) = \coth^{-1} c_0. \]
(3.6)
We also denote \( \rho_0 = \coth^{-1} c_0 \) for simplicity.

**Lemma 3.2.** Under the assumption of Theorem 1.2, if the equality holds in (1.5), then \( N \) is equal to the geodesic ball of radius \( \rho_0 \) centered at \( x_0 \).

**Proof.** The proof is similar with the proof of lemma 2.2. The only difference is that we replace the Laplacian comparison (2.6) by the following \( f \)-Laplacian comparison. Since \( \text{Ric}_f^m \geq -(m - 1) \) in \( N \), the \( f \)-Laplacian comparison of the distance function \( \rho(x) = d(x_0, x) \) due to Qian [18] says that
\[ \tilde{\Delta}_f \cosh \rho(x) = \tilde{\Delta} \rho(x) - \tilde{\nabla} f \cdot \tilde{\nabla} \rho(x) \leq (m - 1) \coth \rho(x). \]
(3.7)
holds in the sense of distribution. \( \square \)

Next we show that the \( f \)-Laplacian comparison (3.7) assumes equality everywhere in \( N \). From lemma 3.2, \( N \) is the geodesic ball of radius \( \rho_0 \) centered at \( x_0 \). Any \( q \in \partial N \) can be joined by a minimizing geodesic \( \gamma \) parameterized by arc-length from \( x_0 \) to \( q \), and \( \gamma \) is orthogonal to \( \partial N = \partial B_{\rho_0}(x_0) \) at \( q \). The geodesic \( \gamma \) is uniquely determined by \( q \) and \( q \) is not in the cut locus of \( x_0 \). Then the distance function \( \rho(x) \) is smooth up to the boundary \( \partial N \). The \( f \)-Laplacian comparison (3.7) implies
\[ \tilde{\Delta}_f \cosh \rho(x) = \tilde{\Delta} \rho(x) \sinh \rho(x) + \cosh \rho(x) |\tilde{\nabla} \rho(x)|^2 \leq m \cosh \rho(x). \]
Integrating the above inequality over \( N \) with respect to the weighted volume element \( e^{-f} dv \) and by divergence theorem, we have
\[ \int_{\partial N} \sinh \rho(\tilde{\nabla} \rho, \nu) e^{-f} d\mu \leq m \int_N \cosh \rho(x) e^{-f} dv, \]
where \( \nu \) is the outward unit normal of \( \partial N \) in \( N \). Note that \( \rho = \rho_0 \) and \( \langle \tilde{\nabla} \rho, \nu \rangle = 1 \) on \( \partial N \), the above inequality gives that
\[ \int_{\partial N} \sinh \rho_0 e^{-f} d\mu \leq m \int_N \cosh \rho(x) e^{-f} dv. \]
(3.8)
On the other hand, we prove the reversed inequality also holds in (3.8).

**Lemma 3.3.** We have
\[ \int_{\partial N} \sinh \rho_0 e^{-f} d\mu \geq m \int_N \cosh \rho(x) e^{-f} dv. \]
(3.9)
Proof. To show (3.9), we need the generalized Heintze-Karcher theorem due to V. Bayle [1]. Note that any \( q \in \partial N \) can be joined by a minimizing geodesic \( \gamma \) parameterized by arc-length from \( x_0 \) to \( q \), and \( \gamma \) is orthogonal to \( \partial N = \partial B_{\rho_0}(x_0) \) at \( q \), the geodesic \( \gamma \) is uniquely determined by \( q \) and \( q \) is not in the cut locus of \( x_0 \). The exponential map of the normal bundle \( T_{\perp} \partial N \) of \( \partial N \) in \( N \) is surjective. For any \( q \in \partial N \), the curve \( \gamma(t) = \exp_q(-t
u) \) \( (0 \leq t \leq \rho_0) \) is the geodesic connecting \( q \) and \( x_0 \), i.e., \( \gamma(\rho_0) = x_0 \). Since \( \text{Ric}_m^f \geq - (m-1) \) in \( N \), the generalized Heintze-Karcher theorem in [1] implies

\[
\int_N \cosh \rho(x)e^{-f} \, dv \\
\leq \int_{\partial N} \int_0^{\rho_0} \cosh \rho(\exp_q(-t
u(q)))(\cosh t - \frac{H_f(q)}{m-1} \sinh t)^{m-1} dt e^{-f(q)} d\mu_q,
\]

Note that \( \rho(\exp_q(-t
u(q))) = \rho_0 - t \) and

\[
\cosh(\rho_0 - t) = \cosh \rho_0 \cosh t - \sinh \rho_0 \sinh t = \sinh \rho_0(\coth \rho_0 \cosh t - \sinh t).
\]

From the proof of lemma 3.2 we have \( H_f(q) = (m-1) \coth \rho_0 \) on \( \partial N \). Therefore

\[
\int_N \cosh \rho(x)e^{-f} \, dv \\
\leq - \int_{\partial N} \int_0^{\rho_0} \frac{1}{m} \sinh \rho_0 \frac{d}{dt}(\cosh t - \coth \rho_0 \sinh t)^{m} dt e^{-f(q)} d\mu_q \\
= \frac{1}{m} \int_{\partial N} \sinh \rho_0 e^{-f} d\mu
\]

which gives the inequality (3.9). \( \square \)

Combining (3.8) and (3.9), the \( f \)-Laplacian comparison inequality (3.7) assumes equality everywhere in \( N \), i.e., we have that

\[
\Delta_{f\rho}(x) = (m-1) \coth \rho(x)
\]

holds in the classical sense everywhere in \( N \). Finally, we show that \( m = n \).

Lemma 3.4. We have \( m = n \).

Proof. Recall that for any function \( u \in C^3(N) \), Ma-Du [16] obtained the following Reilly formula for Bakry-Émery Ricci curvature \( \text{Ric}_f \):

\[
0 = \int_N (\text{Ric}_f(\nabla u, \nabla u) - |\Delta_f u|^2 + |\nabla^2 u|^2)e^{-f} \, dv \\
+ \int_{\partial N} \left( (\Delta_f u + H_f \frac{\partial u}{\partial v}) \frac{\partial u}{\partial v} - (\nabla u, \nabla \frac{\partial u}{\partial v}) + h(\nabla u, \nabla u) \right) e^{-f} \, d\mu.
\]  

Here, \( \Delta_f = \Delta - \nabla f \cdot \nabla \) and \( \nabla^2 \) are the \( f \)-Laplacian, gradient and Hessian on \( N \) respectively; \( \Delta_f = \Delta - \nabla f \cdot \nabla \) and \( \nabla \) are the \( f \)-Laplacian and gradient operators on \( \partial N \); \( \nu \) is the outward unit normal of \( \partial N \); \( H_f \) and \( h \) are the
$f$-mean curvature and second fundamental form of $\partial N$ in $N$ with respect to $\nu$ respectively.

Suppose on the contrary we have $m > n$, let $z = \frac{m}{n}$ and by the basic algebraic inequality $(a + b)^2 \geq z^2 - \frac{\nu^2}{z} - 1$ for $z > 1$, we have (see [9, 11])

\[
|\nabla^2 u|^2 \geq \frac{1}{n}(\Delta u)^2 = \frac{1}{n}(\bar{\Delta} f u + \nabla f \cdot \nabla u)^2 \\
\geq \frac{1}{n} \left( \frac{n}{m}(\bar{\Delta} f u)^2 - \frac{n}{m-n}(\nabla f \cdot \nabla u)^2 \right) \\
= \frac{1}{m}(\bar{\Delta} f u)^2 - \frac{1}{m-n}(\nabla f \cdot \nabla u)^2. \tag{3.12}
\]

Substituting this into (3.11) and using the definition (1.3) of $m$-Bakry-Émery Ricci curvature, we get

\[
0 \geq \int_N (\text{Ric}_m^p(\nabla u, \nabla u) - \frac{m-1}{m}|\bar{\Delta} f u|^2)e^{-f}dv - \int_{\partial N} H_f \frac{\partial u}{\partial \nu} \frac{\partial u}{\partial \nu} - (\nabla u, \nabla \frac{\partial u}{\partial \nu}) + h(\nabla u, \nabla u) e^{-f}d\mu \tag{3.13}
\]

Now since (3.10) holds everywhere in $N$, we have that $\bar{\Delta} f \cosh \rho(x) = \frac{m}{m-n} \cosh \rho(x)$ (3.14) also holds everywhere in $N$. Note that $\partial N$ is a geodesic ball of radius $\rho_0$ centered at $x_0$, $\text{Ric}_m^p \geq - (m-1)\cosh \rho_0$ in $N$ and $H_f = (m-1)\coth \rho_0$ on $\partial N$. Substituting $u(x) = \cosh \rho(x)$ into the Reilly inequality (3.13), using (3.14) and integrating by part, we obtain

\[
0 \geq \int_N (-(m-1)|\nabla u|^2 - \frac{m-1}{m}|\bar{\Delta} f u|^2)e^{-f}dv + \int_{\partial N} H_f \left( \frac{\partial u}{\partial \nu} \right)^2 e^{-f}d\mu \\
= \frac{m-1}{m} \int_N \bar{\Delta} f u (m u - \bar{\Delta} f u) e^{-f}dv - \int_{\partial N} (m-1) \frac{\partial u}{\partial \nu} e^{-f}d\mu \\
+ (m-1) \int_{\partial N} \coth \rho_0 \left( \frac{\partial u}{\partial \nu} \right)^2 e^{-f}d\mu \\
= - \int_{\partial N} (m-1) \sinh \rho_0 \cosh \rho_0 e^{-f}d\mu + (m-1) \int_{\partial N} \coth \rho_0 (\sinh \rho_0)^2 e^{-f}d\mu \\
= 0,
\]

where we used the facts $\rho = \rho_0$ and $\frac{\partial \rho}{\partial \nu} = 1$ on $\partial N$. Therefore, the algebraic inequality (3.12) assumes equality everywhere for $u(x) = \cosh \rho(x)$. Thus we have

\[
0 = \bar{\Delta} f \cosh \rho(x) + \frac{m}{m-n} \nabla f \cdot \nabla \cosh \rho(x) \\
= \bar{\Delta} \cosh \rho(x) + \frac{n}{m-n} \nabla f \cdot \nabla \cosh \rho(x)
\]
holds everywhere in $N$. Let $\omega(x) = \cosh \rho(x) - \cosh \rho_0$. Then

$$0 = \bar{\Delta} \omega(x) + \frac{n}{m-n} \nabla f \cdot \bar{\nabla} \omega(x)$$

(3.15)

in $N$ and $\omega(x) = 0$ on $\partial N$. Multiplying (3.15) with $\omega(x)$ and integrating over $N$ with respect to $e^{m-nf} dv$, we get

$$0 = \int_N \omega \left( \bar{\Delta} \omega(x) + \frac{n}{m-n} \nabla f \cdot \bar{\nabla} \omega(x) \right) e^{m-nf} dv$$

$$= -\int_N |\bar{\nabla} \omega|^2 e^{m-nf} dv + \int_{\partial N} \omega \frac{\partial \omega}{\partial \nu} e^{m-nf} d\mu$$

$$= -\int_N |\bar{\nabla} \omega|^2 e^{m-nf} dv$$

where the third equality is due to the fact $\omega(x) = 0$ on $\partial N$. Therefore we have that $\omega(x) = \cosh \rho(x) - \cosh \rho_0$ is constant in $N$, which is a contradiction. Thus we conclude that $m = n$. □

Once we have $m = n$, the last statement of Theorem 1.2 follows from Theorem 1.1, and we complete the proof of Theorem 1.2.

Appendix A. Manifold with $\text{Ric}_f$ bounded below

In this appendix, we give a result on the diameter estimate for manifold $(N^n, g)$ with Bakry-Émery Ricci curvature $\text{Ric}_f$ bounded below. By assuming that the function $f$ is bounded, i.e., $|f| \leq k$, and $\text{Ric}_f \geq (n-1)c^2 > 0$ in $N$, Wei-Wylie [21] proved that the diameter of $N$ satisfies $\text{diam}(N) \leq (\pi + \frac{4k}{n-1})/c$. See a different upper bound $\text{diam}(N) \leq \sqrt{1 + \frac{2\pi k}{n-1}}/c$ obtained by Limoncu [13]. The following proposition deals with the manifold with $\text{Ric}_f \geq -(n-1)c^2$ for some $c \geq 0$ and with nonempty boundary. By assuming $|f| \leq k$ for some constant $k$, we have

**Proposition A.1.** Let $(N^n, g)$ be an $n$-dimensional complete Riemannian manifold with nonempty boundary and $f$ be a smooth bounded function $|f| \leq k$ on $N$. Assume that the Bakry-Émery Ricci curvature $\text{Ric}_f \geq -(n-1)c^2$ for some $c \geq 0$ on $N$, and the $f$-mean curvature of the boundary $\partial N$ satisfies $H_f \geq (n-1+4k)c_0 > (n-1+4k)c \geq 0$ for some constant $c_0 > c \geq 0$. Let $d$ denote the distance function on $N$. Then

$$\sup_{x \in N} d(x, \partial N) \leq \begin{cases} \frac{1}{c_0}, & \text{if } c = 0 \\ \frac{1}{c} \coth^{-1} \frac{c_0}{c}, & \text{if } c > 0 \end{cases}$$

(A.1)
Proof. As in the proof of diameter estimate (1.5), we have

\[ 0 \leq \int_0^d \left( (n-1) \varphi'(s)^2 - \varphi(s)^2 Ric_f(\gamma'(s), \gamma'(s)) \right) ds \]

\[ - 2 \int_0^d \varphi(s) \varphi'(s) \frac{d}{ds} f(\gamma(s)) ds - H_f(\gamma(d)) \]

\[ = \int_0^d \left( (n-1) \varphi'(s)^2 - \varphi(s)^2 Ric_f(\gamma'(s), \gamma'(s)) \right) ds - H_f(\gamma(d)) \]

\[ + 2 \int_0^d \frac{d}{ds} (\varphi(s) \varphi'(s)) f(\gamma(s)) ds - 2 \varphi(d) \varphi'(d) f(\gamma(d)), \quad (A.2) \]

where in the second equality we used the integration by parts and the fact \( \varphi(0) = 0 \).

If \( c = 0 \), by choosing \( \varphi(s) = s/d \) in (A.2). Using the assumption \( Ric_f \geq 0 \) in \( N \), \( |f| \leq k \) and \( H_f \geq (n - 1 + 4k)c_0 \) on \( \partial N \), we have

\[ 0 \leq \frac{n - 1}{d} + \frac{2}{d^2} \int_0^d f(\gamma(s)) ds - \frac{2}{d} f(\gamma(d)) - (n - 1 + 4k)c_0 \]

\[ \leq \frac{n - 1 + 4k}{d} - (n - 1 + 4k)c_0. \]

Therefore, we have

\[ d \leq \frac{1}{c_0}. \quad (A.3) \]

If \( c > 0 \), by choosing \( \varphi(s) = \sinh(cs)/\sinh(cd) \) for \( 0 \leq s \leq d \) in (A.2). Using the assumption \( Ric_f \geq -(n - 1)c^2 \) in \( N \), \( |f| \leq k \) and \( H_f \geq (n - 1 + 4k)c_0 > (n - 1 + 4k)c > 0 \) on \( \partial N \), we have

\[ 0 \leq (n - 1)c \coth(cd) + \frac{2c^2}{\sinh^2(cd)} \int_0^d \cosh(2cs) f(\gamma(s)) ds \]

\[ - 2c \coth(cd) f(\gamma(d)) - (n - 1 + 4k)c_0 \]

\[ \leq (n - 1 + 4k)c \coth(cd) - (n - 1 + 4k)c_0, \]

which is equivalent to

\[ d \leq \frac{1}{c} \coth^{-1} \frac{c_0}{c} \quad (A.4) \]

\[ \square \]

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