DYNAMICAL NONCOMMUTATIVE GRAPHENE

Ilyas Haouam∗
Laboratoire de Physique Mathématique et de Physique Subatomique (LPMPS),
Université Frères Mentouri, Constantine 25000, Algeria

S Ali Alavi†
Department of Physics, Hakim Sabzevari University, P.O. Box 397, Sabzevar, Iran

We study graphene in a two-dimensional dynamical noncommutative space in the presence of a constant magnetic field. The model is solved using perturbation theory and to the second order of perturbation. The energy levels of the system are calculated and the corresponding eigenstates are obtained. For all cases, the energy shift depends on the dynamical noncommutative parameter $\tau$. Using the accuracy of energy measurement we put an upper bound on the noncommutativity parameter $\tau$. In addition, we investigate some of the thermodynamic quantities of the system at zero temperature limit and extreme relativistic case, which reveals interesting differences between commutative and dynamical noncommutative spaces.

Keywords: Noncommutative graphene ; Dirac equation ; dynamical noncommutative space ; zero temperature limit

∗ ilyashaouam@live.fr
† s.alavi@hsu.ac.ir
I. INTRODUCTION

Graphene is a crystalline two-dimensional (2D) material, which is an allotrope of carbon consisting of a single layer of atoms arranged in a 2D honeycomb lattice [1, 2]. Graphene is thought about to be the world’s thinnest, strongest and most conductive material of both electricity and heat. Its charge carriers exhibit giant intrinsic mobility, have zero effective mass, and can travel for micrometers without scattering at room temperature [3]. It is considered to have the potential to revolutionize whole industries in the fields of electricity, conductivity, energy generation, sensors and more. So there is strong motivation for researchers around the world to study the properties of graphene (see e.g. [4–8]). It is also well-known that graphene plays a substantial role in various branches of science. Its experimental realization has opened new horizons in material science and condensed-matter physics. Graphene is considered as one of the most famous materials in the history in electronics and material science for its outstanding several unique mechanical, optical, electrical, transport and thermodynamic properties [9–16].

Moreover, graphene physics is one of rich and vibrant fields of investigation, which has attracted a lot of attention from scientists since experimental observations revealed the existence of electrical charge carriers behaving like massless Dirac quasi-particles [1, 2, 17, 18]. It was observed that the low-energy electronic excitations at the corners of graphene Brillouin zone can be described by a 2D Dirac fermions with linear dispersion relation [9, 18]. Later, this effect led to testing many aspects of relativistic phenomena that usually requires large energy in experiments [19–21].

On the other side, the investigation of quantum systems in a noncommutative (NC) space has been a matter of much interest in the recent years. Many types of noncommutativity have been considered (see [22–27] for an overview). However, one type of noncommutativity of space, is particularly important to us, it is called dynamical noncommutativity (position-dependent noncommutativity), characterized by $\Theta$ being considered as a function of coordinates, i.e. $\Theta \rightarrow \Theta(X, Y)$.

Clearly, studying NC geometry is very important for understanding phenomena at short distances and has great impact in many areas of modern physics such as quantum physics, high energy, cosmology, gravity. For a review of NC quantum mechanics and NC field theories, see [28–38].

In this article, we attempt to examine the effects of dynamical noncommutative (DNC) space on the graphene in the presence of an external constant magnetic field. In the same context, Bastos et al [39] have studied graphene in the framework of NC quantum mechanics and determined the Hamiltonian and the corresponding energy spectrum. Likewise, Santos et al [40] employed the statistics theory to investigate the thermodynamical properties of graphene in a NC phase-space in the presence of a constant magnetic field. They exactly found the main thermodynamical properties of graphene in NC phase-space. In addition, Boumali [41] calculated the thermal properties of graphene under a magnetic field through the 2D Dirac oscillator (DO). He showed that using the approach of effective mass, the model of a 2D DO can be used to describe the thermal properties of graphene under an uniform magnetic field. The main thermodynamic quantities of graphene have been found by using an approach based on the zeta function. More recently, Khordad et al [42] considered a NC description of graphene and employed extensive and non-extensive entropies to study magnetic susceptibility of graphene in NC phase-space... etc.

This paper is outlined as follows. In Section. II, the DNC space is shortly reviewed. In Section. III, the 2D graphene is investigated, where we briefly review the graphene. In sub-section III A, we extend the problem to DNC space. Then, in sub-sections III B & III C we obtain the solution of Dirac equation and its spectrum in DNC space. In sub-section III D, based on the second-order correction on energy, an upper bound on the DNC parameter is found. In sub-section III E, the thermodynamics properties of graphene at zero temperature are investigated. We present our conclusion in Section. IV.

II. DYNAMICAL NONCOMMUTATIVIVE SPACE

Let us first review the substantial relations of the DNC space algebra. It is well-known that in NC spaces (at the tiny scale), the position coordinates do not commute with each other anymore. The NC coordinates satisfy the following deformed commutation relation:

$$[x^{nc}_\mu, x^{nc}_\nu] = i\Theta_{\mu\nu},$$

where $\Theta_{\mu\nu}$ is an anti-symmetric tensor. The simplest case is the situation where $\Theta$ is constant, which we call non-dynamical noncommutative space (NC space or $\Theta$-space).

In Ref.[23], an interesting generalization to DNC spaces was proposed so that the fundamental objects in this type of DNC spaces are string like therefore it is a good motivation to study physics in these spaces. In this reference, $\Theta_{\mu\nu}$ is chosen to be a function of coordinates as $\theta(X, Y) = \Theta \left(1 + \tau Y^2 \right)$. Certainly, there is a multitude of other possibilities, such as $\theta(X, Y) = \Theta/[1 + \Theta \alpha \left(1 + Y^2 \right)]$ (following Gomes et al in [43]).
The commutation relations for a 2D DNC space (or \( \tau \)-space) are as follows \([23]\):

\[
\begin{align*}
[X,Y] &= i\Theta (1 + \tau Y^2), \\
[X,P_x] &= i\hbar (1 + \tau Y^2), \\
[X,P_y] &= 2i\tau Y (\Theta P_y + \hbar X), \\
[Y,P_x] &= 0, \\
[Y,P_y] &= 0. \\
\end{align*}
\]

(2)

It is worth mentioning that \( \tau \) and \( \Theta \) have dimensions of \( L^{-1} \) and \( L^2 \), respectively.

In the limit \( \tau \to 0 \), we recover the non-dynamical \((\Theta\text{-noncommutative}) \) commutation relations:

\[
\begin{align*}
[x^{nc}, y^{nc}] &= i\Theta, \\
[x^{nc}, p^{nc}_x] &= i\hbar, \\
[x^{nc}, p^{nc}_y] &= 0, \\
[y^{nc}, p^{nc}_x] &= 0, \\
[y^{nc}, p^{nc}_y] &= 0.
\end{align*}
\]

(3)

The coordinate \( X \) and the momentum \( P_y \) are not Hermitian, which makes the Hamiltonian that depends on these variables be non-Hermitian. We may represent algebra (2) in terms of the standard Hermitian NC variables operators \( x^{nc}, y^{nc}, p^{nc}_x, p^{nc}_y \) as

\[
X = \left( 1 + \tau (y^{nc})^2 \right) x^{nc}, \quad Y = y^{nc}, \quad P_y = \left( 1 + \tau (y^{nc})^2 \right) p^{nc}_y, \quad P_x = p^{nc}_x.
\]

(4)

From this representation, we can see that some of the operators involved above are no longer Hermitian. However, Fring et al \([23]\) has fixed this problem by converting the non-Hermitian variables into a Hermitian one. Using a Dyson map \( \eta \Omega \eta^{-1} = o = O^\dagger \) (with \( \eta = (1 + \tau Y^2)^{-1/2} \)), so the new Hermitian variables \( x, y, p_x \) and \( p_y \) in terms of NC variables are expressed as follows

\[
\begin{align*}
x &= \eta X \eta^{-1} = (1 + \tau Y^2)^{-1/2} \left( 1 + \tau (y^{nc})^2 \right)^{1/2} X (1 + \tau Y^2)^{1/2}, \\
y &= \eta Y \eta^{-1} = (1 + \tau (y^{nc})^2)^{-1/2} y^{nc} (1 + \tau (y^{nc})^2)^{1/2} = y^{nc}, \\
p_x &= \eta P_x \eta^{-1} = (1 + \tau (y^{nc})^2)^{-1/2} p^{nc}_x (1 + \tau (y^{nc})^2)^{1/2} = p^{nc}_x, \\
p_y &= \eta P_y \eta^{-1} = (1 + \tau (y^{nc})^2)^{-1/2} p^{nc}_y (1 + \tau (y^{nc})^2)^{1/2} = p^{nc}_y.
\end{align*}
\]

(5)

The new DNC variables satisfy the following commutation relations

\[
\begin{align*}
[x, y] &= i\Theta (1 + \tau y^2), \\
[x, p_x] &= i\hbar (1 + \tau y^2), \\
[x, p_y] &= 2i\tau (\Theta P_y + \hbar X), \\
[y, p_x] &= 0, \\
[y, p_y] &= 0.
\end{align*}
\]

(6)

Now, through Bopp-shift transformation, one can express the NC variables in terms of the standard commutative variables \([31]\):

\[
\begin{align*}
x^{nc} &= x^s - \frac{\hbar}{\Theta^2} p^y, \\
y^{nc} &= y^s + \frac{\hbar}{\Theta^2} p^y, \\
p^{nc}_x &= p^s_x, \\
p^{nc}_y &= p^s_y.
\end{align*}
\]

(7)

where the index \( s \) refers to the standard commutative space. Noting that in the DNC space, there is a minimum length for \( X \) in a simultaneous \( X, Y \) measurement \([23]\):

\[
\Delta X_{\text{min}} = \Theta \sqrt{\tau} \sqrt{1 + \frac{\langle Y \rangle^2}{\rho}},
\]

(8)

with no minimal length in \( Y \). As well, in a simultaneous \( Y, P_y \) measurement, we have a minimal momentum as

\[
\Delta (P_y)_{\text{min}} = \hbar \sqrt{\tau} \sqrt{1 + \frac{\langle Y \rangle^2}{\rho}}.
\]

(9)

As mentioned before, the motivation behind dynamical noncommutativity is that objects in 2D space are string-like \([23]\).
### III. DYNAMICAL NONCOMMUTATIVE GRAPHENE

Graphene is a 2D stable allotrope of carbon. It is used for constructing other nanoscale carbons, so that it is the basic structural element of the other carbon allotropes such as 3D Graphite, 1D carbon nanotube (CNT), 0D fullerene, i.e. C60, C50, C6 and 3D Diamond.

The time-independent Dirac equation simply reads

\[ H_D \psi(r) = E \psi(r), \tag{10} \]

the Dirac Hamiltonian is given by

\[ H_D = c \vec{\alpha} \cdot \vec{p} + \beta mc^2, \tag{11} \]

\(\alpha, \beta\) are the Dirac matrices. In the case of graphene, we have massless particles move through the honeycomb lattice where \( c \rightarrow v_F \simeq 10^6 m.s^{-1} \) (Fermi velocity) \(^6\), which gives

\[ H_D = v_F \vec{\sigma} \cdot \vec{p}. \tag{12} \]

From electronics properties point of view, graphene is a zero-gap semiconductor, in which low-energy quasiparticles within each valley can formally be described by the Dirac Hamiltonian. The wave function \( \psi(r) \) gives the electron states around the points \( K_i \) and the Dirac block diagonalized Hamiltonian is given by

\[
H_D|_{K_i} = \begin{pmatrix}
H_{k_1} & 0 & 0 & \cdots \\
0 & H_{k_2} & 0 & 0 \\
0 & 0 & H_{k_3} & 0 \\
\vdots & 0 & 0 & \ddots
\end{pmatrix}.	ag{13}
\]

The most important blocks amid the blocks of the Hamiltonian (13) are the blocks of the two specific wavenumbers, namely the Dirac points \( K \) and \( K' \) respectively, specified by \(^44\)

\[ K = \frac{2\pi}{3} \left( \frac{1}{\sqrt{3}} \right), \quad K' = \frac{2\pi}{3} \left( \frac{-1}{\sqrt{3}} \right). \tag{14} \]

These two Dirac points control the elementary excitation of graphene. The action of electrons around the Dirac points \( K \) and \( K' \) at the corners of Brillouin zone is described by the Dirac Hamiltonian where in the case of \( K' \), we have \( \vec{\sigma}^* = (\sigma_x, -\sigma_y, \sigma_z) \). In this manner, we can write

\[ H_D|_{K,K'} = \begin{pmatrix}
H_K & 0 \\
0 & H_{K'}
\end{pmatrix} = v_F \begin{pmatrix}
\vec{\sigma} \cdot \vec{p} & 0 \\
0 & \vec{\sigma}^* \cdot \vec{p}
\end{pmatrix}, \tag{15}
\]

consequently \( \psi_{K,K'}(r) \) is the two-component wavefunction, which gives the electron states around the the Dirac points \( K \) and \( K' \).

#### A. EXTENSION TO DYNAMICAL NONCOMMUTATIVE SPACE

Let us consider a layer of graphene in an external constant magnetic field along the \( z \) axis. We introduce \( B\hat{z} \) through the minimal coupling to the vector potential, so we define the canonical momentum as

\[ \vec{p}^s \rightarrow \vec{p}^s - \frac{e}{c} \vec{A}^s, \tag{16} \]

where \( \vec{A}^s \) is the electromagnetic vector potential and in the symmetric gauge is given by

\[ \vec{A}^s = \frac{B}{2} (-y^s, x^s, 0). \tag{17} \]

The 2d time-independent Dirac equation around two Dirac points \( K \) and \( K' \) reads

\[ H_D \psi_{K,K'}(r^s) = v_F \begin{pmatrix}
\vec{\sigma} \cdot \left( \vec{p}^s - \frac{e}{c} \vec{A}^s \right) & 0 \\
0 & \vec{\sigma}^* \cdot \left( \vec{p}^s - \frac{e}{c} \vec{A}^s \right)
\end{pmatrix} \psi_{K,K'}(r^s) = E_{K,K'} \psi_{K,K'}(r^s). \tag{18} \]
The Dirac equation around Dirac point $K$ is given by

$$H_K \psi^K(r^s) = v_F \vec{\sigma} \cdot \left( \vec{p}^s - \frac{e}{c} \vec{A}^s \right) \psi^K(r^s) = E_K \psi^K(r^s),$$

(19)

where $\psi^K = ( \phi^A \phi^B )^t$ is two dimensional eigenstate, i.e. an eigenvector that describes the probability of an electron state to be on sub-lattice $A$ in the upper component, or on the sub-lattice $B$ in the lower component of the eigenstate, and $t$ denotes transpose. A similar equation can also be obtained for the Dirac point $K'$ with $\psi^{K'} = ( \phi^{A'} \phi^{B'} )^t$.

$\vec{\sigma}$ are the three $2 \times 2$ Pauli matrices, which are given by

$$\sigma_x = \alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \alpha_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_z = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

(20)

Therefore, for the Dirac point $K$, one has the following Hamiltonian

$$H_K (x_i^s, p_i^s) = v_F \left\{ \alpha_1 p_x^s + \alpha_2 p_y^s + \frac{\hbar}{2 l_B^2} (\alpha_1 y^s - \alpha_2 x^s) \right\},$$

(21)

where we have used $l_B = \sqrt{\frac{\hbar}{c B}}$, which is the magnetic length.

The above Hamiltonian in DNC space turns to

$$H_K (x, p) = v_F \left\{ \alpha_1 p_x + \alpha_2 p_y + \frac{\hbar}{2 l_B} (\alpha_1 y - \alpha_2 x) \right\}.$$ 

(22)

Now, using equation (5), we express the Hamiltonian above in terms of NC variables

$$H_K (x_i^{nc}, p_i^{nc}) = v_F \alpha_1 p_x^{nc} + v_F \alpha_2 \left( 1 + \tau (y^{nc})^2 \right)^{\frac{1}{2}} p_y^{nc} \left( 1 + \tau (y^{nc})^2 \right)^{\frac{1}{2}} + v_F \frac{\hbar}{2 l_B^2} \left\{ \alpha_1 y^{nc} - \alpha_2 \left( 1 + \tau (y^{nc})^2 \right)^{\frac{1}{2}} x^{nc} \left( 1 + \tau (y^{nc})^2 \right)^{\frac{1}{2}} \right\}.$$ 

(23)

Since $\tau$ is very small, the parentheses can be expanded to the first order using

$$\left( 1 + \tau (y^{nc})^2 \right)^{\frac{1}{2}} = 1 + \frac{1}{2} \tau (y^{nc})^2,$$

(24)

thus, equation (23) becomes

$$H_K (x_i^{nc}, p_i^{nc}) = v_F \left\{ \alpha_1 p_x^{nc} + \alpha_2 \left\{ p_y^{nc} + \frac{\hbar}{2 l_B} \tau (y^{nc})^2 p_y^{nc} + \frac{\hbar^2}{2 l_B^2} \left( y^{nc} \right)^2 \right\} \right\} + v_F \frac{\hbar}{2 l_B^2} \left\{ \alpha_1 y^{nc} - \alpha_2 \left\{ x^{nc} + \frac{\hbar}{2 l_B} \tau (y^{nc})^2 x^{nc} + \frac{\hbar^2}{2 l_B^2} \left( y^{nc} \right)^2 \right\} \right\}.$$ 

(25)

Now using the Bopp-shift transformation (7), the Hamiltonian (25) can be expressed in terms of the standard commutative variables

$$H_K (x_i^s, p_i^s) = v_F \alpha_1 p_x^s + v_F \alpha_2 p_y^s + v_F \alpha_2 \left\{ \frac{1}{2} \tau \left( y^s + \frac{\hbar}{2 l_B} p_y^s \right)^2 p_y^s + \frac{1}{2} \tau \left( y^s + \frac{\hbar}{2 l_B} p_y^s \right)^2 \right\}$$

$$+ v_F \frac{\hbar}{2 l_B^2} \left\{ \alpha_1 \left( y^s + \frac{\hbar}{2 l_B} p_y^s \right) - \alpha_2 \left\{ x^s - \frac{\hbar}{2 l_B} \tau p_y^s \left( y^s + \frac{\hbar}{2 l_B} p_y^s \right)^2 \right\} \right\}.$$ 

(26)

Therefore, to the first order in $\Theta$ and $\tau$, we have (the terms containing $\Theta \tau$ are also neglected)

$$H_K (x_i^s, p_i^s) = v_F \left\{ \alpha_1 p_x^s + \alpha_2 \left\{ p_y^s + \frac{1}{2} \tau (y^s)^2 p_y^s + \frac{1}{2} \tau (y^s)^2 \right\} \right\} + v_F \frac{\hbar}{2 l_B^2} \left\{ \alpha_1 \left( y^s + \frac{\hbar}{2 l_B} p_y^s \right) - \alpha_2 \left\{ x^s - \frac{\hbar}{2 l_B} \tau p_y^s \left( y^s \right)^2 \right\} \right\},$$

(27)

which may be written as

$$H_K = H_K^0 + H_K^{(\Theta)} + H_K^{(\tau)},$$

(28)
where

\[ H_K^{(0)} = v_F \left\{ \alpha_1 p_x^s + \alpha_2 p_y^s + \frac{\hbar}{2l_B^2} (\alpha_1 y^s - \alpha_2 x^s) \right\}, \tag{29} \]

\[ H_K^{(\Theta)} = v_F \frac{\Theta}{4l_B^2} (\alpha_1 p_x^s + \alpha_2 p_y^s), \tag{30} \]

\[ H_K^{(\tau)} = v_F \frac{\tau}{2} \alpha_2 \left\{ (y^s)^2 p_y^s + p_y^s (y^s)^2 - \frac{\hbar}{l_B^2} x^s (y^s)^2 \right\}. \tag{31} \]

A similar set of equations can also be obtained for the Dirac point \( K' \).

### B. Unperturbed System

Using equation (20), the Dirac Hamiltonian in commutative space (29) becomes

\[
H_K^{(0)} = v_F \left\{ \sigma_x p_x^s + \sigma_y p_y^s - e\sigma_x A_x^s - e\sigma_y A_y^s \right\} \\
= v_F \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x^s + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y^s + \frac{\hbar}{2l_B} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} y^s - \frac{\hbar}{2l_B} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} x^s \right\},
\]

or in more compact form

\[
H_K^{(0)} = v_F \begin{pmatrix} 0 & 0 \\ p_x^s - ip_y^s + \frac{\hbar}{2l_B} (y^s + ix^s) & 0 \end{pmatrix}.
\]

By introducing \( \kappa = \frac{\hbar}{2l_B} \), we have

\[
H_K^{(0)} = v_F \begin{pmatrix} 0 & 0 \\ p_x^s + ip_y^s + \kappa (y^s - ix^s) & 0 \end{pmatrix} = v_F \begin{pmatrix} 0 & h_{12} \\ h_{21} & 0 \end{pmatrix}.
\]

Let us now introduce the following creation and annihilation operators

\[
a_x = \frac{1}{\sqrt{2\kappa h}} (\kappa x^s + ip_x^s), \quad a_x^\dagger = \frac{1}{\sqrt{2\kappa h}} (\kappa x^s - ip_x^s), \tag{35} \]

\[
a_y = \frac{1}{\sqrt{2\kappa h}} (\alpha y^s + ip_y^s), \quad a_y^\dagger = \frac{1}{\sqrt{2\kappa h}} (\alpha y^s - ip_y^s), \tag{36} \]

which satisfy the following commutations relations

\[
[a_i, a_i^\dagger] = 1, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad i = x, y. \tag{37} \]

So different components of the Hamiltonian could be written in terms of the creation and annihilation operators as follows

\[
h_{12} = \sqrt{2\kappa h} (a_y^\dagger + ia_x^\dagger), \tag{38} \]

\[
h_{21} = h_{12}^\dagger = \sqrt{2\kappa h} (a_y - ia_x). \tag{39} \]

Thus, the Hamiltonian (34) takes the following form

\[
H_K^{(0)} = v_F \begin{pmatrix} 0 & 2\sqrt{\frac{\kappa h}{2}} (a_y^\dagger + ia_x^\dagger) \\ 2\sqrt{\frac{\kappa h}{2}} (a_y - ia_x) & 0 \end{pmatrix} = v_F \begin{pmatrix} 0 & \frac{\kappa}{\sqrt{2}} (a_y^\dagger + ia_x^\dagger) \\ \frac{\kappa}{\sqrt{2}} (a_y - ia_x) & 0 \end{pmatrix}, \tag{40} \]
in which the parameter \( g = 2\sqrt{\kappa \hbar} \) describes the coupling between different states in commutative space.

Let us return to the Dirac equation

\[
H_K^{(0)} \left| \psi_K^{(0)} \right\rangle = E_K^{(0)} \left| \psi_K^{(0)} \right\rangle,
\]

where \( E_K^{(0)}, \left| \psi_K^{(0)} \right\rangle \) are the eigenvalues and eigenkets of the Dirac Hamiltonian in commutative space, respectively. In two dimensions, \( \left| \psi_K^{(0)} \right\rangle \) is written as

\[
\left| \psi_K^{(0)} \right\rangle = \left( \left| \psi_A \right\rangle \left| \psi_B \right\rangle \right).
\]

By setting \( C = \frac{1}{\sqrt{2}} (a_y - ia_x) \), and inserting equation (40) in equation (41), we obtain the following system of equations

\[
v_F \begin{pmatrix} 0 & gC^\dagger \\ gC & 0 \end{pmatrix} \left( \left| \psi_A \right\rangle \left| \psi_B \right\rangle \right) = E_K^{(0)} \left( \left| \psi_A \right\rangle \left| \psi_B \right\rangle \right),
\]

\[E_K^{(0)} \left| \psi_A \right\rangle + v_F g C^\dagger \left| \psi_B \right\rangle = 0,
\]

\[v_F g C \left| \psi_A \right\rangle - E_K^{(0)} \left| \psi_B \right\rangle = 0.
\]

Equations (44) and (45) give

\[\left| \psi_B \right\rangle = \frac{v_F g C}{E_K^{(0)}} \left| \psi_A \right\rangle,
\]

\[- \left( E_K^{(0)} \right)^2 \left| \psi_A \right\rangle + v_F g^2 C^\dagger C \left| \psi_B \right\rangle = 0,
\]

so

\[v_F^2 g^2 C^\dagger C - \left( E_K^{(0)} \right)^2 \left| \psi_A \right\rangle = 0.
\]

In the basis of \( C^\dagger C = N \), we have

\[v_F^2 g^2 N - \left( E_K^{(0)} \right)^2 \left| \psi_A \right\rangle = 0, \text{ with } N \left| \psi_A \right\rangle = n \left| \psi_A \right\rangle.
\]

Thus, the energy spectrum is given by

\[E_{n \pm}^{(0)} \left| K \right\rangle = \pm \sqrt{v_F^2 g^2 n},
\]

which can be rewritten as

\[E_{n \pm}^{(0)} \left| K \right\rangle = \pm v_F \frac{\hbar}{l_B} \sqrt{2n}, \text{ for } n = 0, 1, 2, ...
\]

The result is in good agreement with that of ordinary quantum mechanics [39, 40]. Now, if we consider equations (44, 45), and substitute the equation (51), we get the corresponding wave function for the Dirac point \( K \)

\[\left| \psi_k^{(0)} \right\rangle = \left( \left| \psi_A \right\rangle \pm i \left| \psi_B \right\rangle \right)^t.
\]

Using the same method, the eigenvalues and eigenstates of the Dirac point \( K' \) can be obtained.
C. Perturbed System

If the dynamical noncommutativity parameter $\tau$ is non-zero, should be very small compared to the energy scales of the system, one can always treat the DNC effects as some perturbations of the commutative analogue, so we use time-independent perturbation theory to study the system in $\tau$-space. If it is non-zero, should be very small.

From Equations (35) and (36) we have

$$
x^s = \frac{1}{2\Gamma} \left( a_d + a_d^\dagger + a_g + a_g^\dagger \right), \quad y^s = \frac{1}{2\Gamma} \left( a_d - a_d^\dagger - a_g + a_g^\dagger \right),
$$

$$
p_x^s = \frac{1}{2\Gamma} \left( -a_d + a_d^\dagger - a_g + a_g^\dagger \right), \quad p_y^s = \frac{i}{2\Gamma} \left( a_d + a_d^\dagger - a_g - a_g^\dagger \right),
$$

with

$$
\Gamma = \frac{1}{\sqrt{2}\Gamma_B}.
$$

By introducing

$$
n = n_d + n_g \quad \text{and} \quad m = n_d - n_g,
$$

the eigenkets of the Hamiltonian could be represented as follows

$$
\left| n_d = \frac{n + m}{2}, \quad n_g = \frac{n - m}{2} \right>,
$$

we distinguish the following states:

| Ground state     | n=0, m=0          | (0,0) |
|------------------|-------------------|-------|
| First excited state | n=-1, m=1         | (1,0) |
|                   | n=-1, m=-1        | (0,1) |
| Second excited state | n=-2, m=2         | (2,0) |
|                   | n=-2, m=0         | (1,1) |
|                   | n=-2, m=-2        | (0,2) |
| Third excited state | n=-3, m=3         | (3,0) |
|                   | n=-3, m=1         | (2,1) |
|                   | n=-3, m=-1        | (1,2) |
|                   | n=-3, m=-3        | (0,3) |

Now, let move to calculate the different elements in our perturbed Hamiltonian. We start with $x^s y^s$, thus using equation (53) we have

$$
x^s y^s = \frac{1}{8\Gamma^2} \left[ a_d^3 - a_d^2 a_d - a_d^2 a_g - a_d a_d^\dagger a_g - a_d a_d^\dagger a_g^\dagger - a_g a_d^\dagger a_d + a_g a_d^\dagger a_d^\dagger + a_g a_g^\dagger a_g + a_g a_g^\dagger a_g^\dagger - a_d a_d^\dagger a_d^\dagger + a_g a_g^\dagger a_g^\dagger - a_d a_d^\dagger a_d^\dagger + a_g a_g^\dagger a_g^\dagger - a_d a_d^\dagger a_d^\dagger + a_g a_g^\dagger a_g^\dagger - a_d a_d^\dagger a_d^\dagger + a_g a_g^\dagger a_g^\dagger - a_d a_d^\dagger a_d^\dagger + a_g a_g^\dagger a_g^\dagger - a_d a_d^\dagger a_d^\dagger + a_g a_g^\dagger a_g^\dagger - a_d a_d^\dagger a_d^\dagger + a_g a_g^\dagger a_g^\dagger - a_d a_d^\dagger a_d^\dagger + a_g a_g^\dagger a_g^\dagger \right].
$$

(57)

Corrections due to $x^s y^s$ on energy spectrum to the first order is given by

$$
\langle n_d, n_g | x^s y^s | n_d, n_g \rangle,
$$

(58)

where for the ground state, we have $(0,0) x^s y^s (0,0)$, and using equation (57), one can check that

$$
\langle 0,0 | x^s y^s | 0,0 \rangle = 0.
$$

(59)
The first excited state is two fold degenerate \( |1,0 \rangle \) and \( |0,1 \rangle \), so we have the following perturbed matrix

\[
\begin{pmatrix}
0,1 | x^+y'^2 | 0,1 \\
0,1 | x^+y'^2 | 1,0 \\
1,0 | x^+y'^2 | 0,1 \\
1,0 | x^+y'^2 | 1,0
\end{pmatrix}.
\]

(60)

By doing necessary calculations we can show that the matrix element \( \langle 0,1 | x^+y'^2 | 0,1 \rangle \) vanishes

\[
\langle 0,1 | x^+y'^2 | 0,1 \rangle = 0,
\]

(61)

with the same calculations we can also check that the contributions of the rest of the matrix elements in equation (60) vanish.

The second excited state is three fold degenerate \( |2,0 \rangle, |1,1 \rangle, |0,2 \rangle \) and it is given by \( 3 \times 3 \) matrix. One can check again that its elements vanish.

In general one can show that to the first order the term \( x^+y'^2 \) in equation (31) has no correction to the energy of the system. Now, we calculate the second order contribution of \( x^+y'^2 \). For the ground state we should find the non-zero matrix elements \( \langle n_d, n_g | x^+y'^2 | 0,0 \rangle \), where the non-zero values are as follows:

\[
\{ \langle 3,0 | a_d^{13} | 0,0 \rangle, \langle 0,3 | a_g^{13} | 0,0 \rangle, -\langle 2,1 | a_d^{13} a_g^{13} | 0,0 \rangle, -\langle 1,2 | a_g^{13} a_d^{13} | 0,0 \rangle, -\langle 0,1 | a_d a_d a_g | 0,0 \rangle, -\langle 0,1 | a_d a_g a_d | 0,0 \rangle, -\langle 1,0 | a_g a_d a_d | 0,0 \rangle, -\langle 1,1 | a_g a_d a_d | 0,0 \rangle \}.
\]

Now, we consider the two other terms in equation (31) i.e., \( y'^2 p_g^s, p_g^s y'^2 \). Employing the relation

\[
[p_g^s, y'^2] = -2i\hbar y'^s,
\]

(62)

we rearrange the terms \( y'^2 p_g^s + p_g^s y'^2 \) as

\[
y'^2 p_g^s + p_g^s y'^2 = 2 p_g^s y'^s + 2i\hbar y'^s.
\]

(63)

Therefore, we calculate the contribution of \( p_g^s y'^s \) thus we have

\[
x^+y'^2 = -\hbar^2 [a_d^{13} - a_d a_d a_d - a_d a_g - a_d a_d a_g + a_d a_d a_d + a_d a_d a_g - a_d a_d a_d]
\]

(64)

One can check that the non-vanishing matrix elements of \( p_g^s y'^2 \) are the same as \( x^+y'^2 \), so we proceed to calculate these non-vanishing matrix elements. We use the following useful relation

\[
\left( a_d^{13} \right)^{n_d} \left( a_g^{13} \right)^{n_g} | 0,0 \rangle = \sqrt{n_d! n_g!} | n_d, n_g \rangle,
\]

(65)

thus we have

\[
\langle 3,0 | a_d^{13} | 0,0 \rangle = \sqrt{6} \langle 3,0 | 3,0 \rangle = \sqrt{6},
\]

(66)

\[
\langle 0,3 | a_g^{13} | 0,0 \rangle = \sqrt{6},
\]

(67)

\[
\langle 2,1 | a_d^{13} a_g^{13} | 0,0 \rangle = \sqrt{2} \langle 2,1 | 2,1 \rangle = \sqrt{2},
\]

(68)

\[
\langle 1,2 | a_g^{13} a_d^{13} | 0,0 \rangle = \sqrt{2}.
\]

(69)
\[ \langle 0, 1 | a_d a_d^\dagger a_g | 0, 0 \rangle = \langle 0, 1 | 0, 1 \rangle = 1, \]  
\tag{70}

with the same method we find

\[
\begin{align*}
\langle 0, 1 | a_d a_d^\dagger a_g^\dagger | 0, 0 \rangle &= \langle 0, 1 | a_d^\dagger a_d a_g | 0, 0 \\
&= \langle 1, 0 | a_d a_d^\dagger a_g | 0, 0 \\
&= \langle 1, 0 | a_d^\dagger a_g | 0, 0 \rangle = 1.
\end{align*}
\tag{71}
\]

Finally, we calculate the contribution of the last term in equation (31), i.e. 2i\hbar y^s. So the first order contribution of this term on the ground state is zero

\[ 2i\hbar \langle 0, 0 | y^s | 0, 0 \rangle = 0. \]  
\tag{72}

For the second order contribution, only \( a_d^\dagger \) and \( a_g^\dagger \) have non-zero contribution

\[ \langle 1, 0 | a_d^\dagger | 0, 0 \rangle = 1, \]  
\tag{73}

\[ \langle 0, 1 | a_g^\dagger | 0, 0 \rangle = 1. \]  
\tag{74}

One can check that the corrections due to \( 2i\hbar y^s \) on excited states (degenerate states) to the first order is zero. Thus we find that to the first order of perturbation, the corrections due to \( H^{(\tau)}_K \) on the excited degenerate states vanish.

**D. Upper bound on dynamical noncommutative parameter \( \tau \)**

In this subsection, we put an upper bound on the DNC parameter \( \tau \) using the accuracy of the energy measurement. The second order correction to the energy of the system is given by

\[ E_n^{(2)} = \sum_{k \neq n} \frac{\left| \langle \varphi_n | H^{(\tau)} | \varphi_k \rangle \right|^2}{E_n^{(0)} - E_k^{(0)}}, \]  
\tag{75}

but

\[ \left| \langle \varphi_n | H^{(\tau)} | \varphi_k \rangle \right|^2 \propto \frac{\hbar}{\Gamma_F^2 v_F^2 \tau^2}, \]  
\tag{76}

and

\[ E_n^{(0)} - E_k^{(0)} \propto \hbar \Gamma v_F, \]  
\tag{77}

where lower index 0 refers to the ground state.

\( E_n^{(2)} \) should be equal or less than the accuracy of the energy measurement. \( E_n^{(2)} \leq 10^{-3}\text{eV} \) \cite{39}, thus we have

\[ \frac{\hbar}{\Gamma_F^2 v_F^2 \tau^2} \leq 10^{-3}\text{eV}. \]  
\tag{78}

Using the following numerical values of relevant quantities: \( l_B \approx 2.5 \times 10^{-8} \text{ m} \) (for B=1 Tesla) ; \( v_F \approx 10^6 \text{m/s} \) and \( \hbar \approx 6 \times 10^{-15}\text{eVs} \), thus, the DNC parameter \( \tau \) satisfies

\[ \sqrt{\tau} \leq \frac{1}{10^{-7}} \text{ m}^{-1}. \]  
\tag{79}

Using the relation 1 Fermi\(^{-1} \approx 200 \text{ MeV} \), one can get

\[ \sqrt{\tau} \leq 10^{-6}\text{MeV} = 1\text{eV}. \]  
\tag{80}

Clearly, this bound is not a stringent bound. But if we take the accuracy of energy measurement \( 10^{-12}\text{eV} \) \cite{45, 46} (as in atomic physics), we obtain a much better stringent bound.

It is worth mentioning that the upper and lower bounds on the NC parameter \( \Theta \) were obtained in \cite{31, 47, 48}.
E. Thermodynamic properties- Dynamics at zero temperature

We pursue to determine the thermodynamic properties of the graphene under a magnetic field in DNC space at zero temperature. For many applications of interest to physicists the temperatures are “small”, so the zero temperature limit is correct and valid. At $T = 0$ the ground state of a system of $N$ fermions will have all single particle energy levels filled up to the Fermi energy $E_F$ and the remainder empty. In what follows we study the thermodynamic properties of graphene as a relativistic Fermi system in DNC space and at $T = 0$. In [49], the equations of state for the extreme relativistic case ($E = |\vec{p}|c$) at $T = 0$, are summarized as follows:

$$n(p_F) = \frac{g p_F^3}{6\pi^2 \hbar^3},$$  

and

$$u(p_F) = \frac{g c p_F^4}{8\pi^2 \hbar^3},$$

$$\mu = p_F c,$$

$$P = \frac{1}{3} u,$$

$$\frac{dP}{dn} = \frac{c}{3} p_F = \frac{1}{3} \mu,$$

$$\gamma = \frac{4}{3}.$$  

where $n$, $u$, $\mu$, $P$, $p_F$ and $g = 2s + 1$, are density of electrons, density of energy, chemical potential, pressure of the system, Fermi momentum and degeneracy factor (that counts the number of states available with the same momentum and position, where $s$ stands for spin), respectively. As we mentioned before, in the ground state of the N-fermion system, the particles occupy the lowest energy states available. Those states with energy below $E_F$ have unit probability to be occupied, those with energies above $E_F$ remain empty. On the other hand, it is well known fact in quantum mechanics that, for the ground state, the second-order perturbation energy correction term is negative. So the correction obtained in subsection “III D” for the ground state is negative, and therefore we have

$$E_F^{\tau \neq 0} < E_F^{\tau = 0}. $$  

By considering together equations (81-86) and (87), we find the interesting relations between thermodynamics properties of graphene in the DNC space and commutative one, which are summarized as follows

| $n$ $\neq 0$ $< n$ $= 0$  |
| $u$ $\neq 0$ $< u$ $= 0$  |
| $\mu$ $\neq 0$ $< \mu$ $= 0$  |
| $P$ $\neq 0$ $< P$ $= 0$  |
| $(dP/dn)$ $\neq 0$ $> (dP/dn)$ $= 0$  |
| $\gamma$ $\neq 0$ $= \gamma$ $= 0$.  |

It is worth mentioning that the dynamical noncommutative graphene is more compressible than the commutative one.

$$\left(\frac{dn}{dP}\right)^{\tau \neq 0} > \left(\frac{dn}{dP}\right)^{\tau = 0}.$$  

So, in summary, the behavior of graphene at zero temperature might play an important role in understanding more the dynamical noncommutativity and the structure of space-time.
IV. CONCLUSION

It is very interesting to investigate fundamental phenomena in DNC space. In this work, we have studied the effects of DNC space on the graphene in the presence of an external constant magnetic field. Graphene is described by Dirac equation so we investigate the Dirac equation in DCN space and derive the corrections due to dynamical noncommutativity on the Hamiltonian and energy spectrum. Moreover, using the accuracy of energy measurement we set an upper bound on DNC parameter $\sqrt{\tau} \leq 1$ eV, a bound that is not very stringent. Of course, that indicates that there is no contradiction between DNC effects and graphene’s physics. Then, we investigated some thermodynamic characteristics of the considered system at zero temperatures. We showed that, there are significant differences between thermodynamics properties of graphene at low temperatures in commutative and DNC spaces. Therefore high-precision very low temperature techniques can be used to determine whether there exists dynamical noncommutativity of space in nature.

[1] Geim, A., Novoselov, K. The rise of graphene. Nat. Mater. 6, 183 (2007). https://doi.org/10.1038/nmat1849
[2] Peres, N. M. R., & Ribeiro, R. M. Focus on graphene. New J. Phys. 11(9), 095002 (2009). https://doi.org/10.1088/1367-2630/11/9/095002
[3] Geim, A. K. Graphene: status and prospects. science, 324(5934), 1530 (2009). https://doi.org/10.1126/science.1158877
[4] Bonaccorso, F., Sun, Z., Hasan, T. et al. Graphene photonics and optoelectronics. Nat. Photon. 4, 611 (2010). https://doi.org/10.1038/nphoton.2010.186
[5] Mikhailov, S. A. & Ziegler, K. New electromagnetic mode in graphene. Phys. Rev. Lett. 99, 016803 (2007). https://doi.org/10.1103/PhysRevLett.99.016803
[6] Jiang, Z., et al. Infrared Spectroscopy of Landau Levels of Graphene. Phys. Rev. Lett. 98, 197403, (2007). https://doi.org/10.1103/PhysRevLett.98.197403
[7] Rastegar Sedehi, H. R., & Khordar, R. Investigation of specific heat in the monolayer graphene. Iran. J. Phys. Res. 20, 210415 (2009). https://doi.org/10.1088/1751-8113/42/21/210415
[8] Hwang, E. H. & Das Sarma, S. Dielectric function, screening, and plasmons in two-dimensional graphene. Phys. Rev. B 75, 205418 (2007). https://doi.org/10.1103/PhysRevB.75.205418
[9] Neto, A.H. Castro, et al. The electronic properties of graphene. Rev. Mod. Phys. 81, 109 (2009). https://doi.org/10.1103/RevModPhys.81.109
[10] Yokoyama, T. Controllable spin transport in ferromagnetic graphene junctions. Phys. Rev. B, 77(7), 073413 (2008). https://doi.org/10.1103/PhysRevB.77.073413
[11] Rusanov, A. I. Thermodynamics of graphene. Surf. Sci. Rep. 69(4), 296 (2014).https://doi.org/10.1016/j.surfrep.2014.08.003
[12] Wright, A. R., & al. Thermodynamic properties of graphene nanoribbons under zero and quantizing magnetic fields. Microelectron. J. 40(4), 716 (2009). https://doi.org/10.1016/j.microj.2008.11.004
[13] Pereira, J. M., Vasilopoulos, P., & Peeters, F. M. Graphene-based resonant-tunneling structures. Appl. Phys. Lett, 90(13), 132122(2007). https://doi.org/10.1063/1.2717092
[14] Rastegar Sedehi, H. R., & Khordar, R. Investigation of specific heat in the monolayer graphene. Iran. J. Phys. Res. 20(2), 355 (2020). https://doi.org/10.47176/ijpr.20.2.3829
[15] Hawamdeh, Mustafa M., et al. Thermodynamic properties of graphene using the static fluctuation approximation (SFA). Can. J. Phys. 95(3), 211 (2017). https://doi.org/10.1139/cjp-2016-0310
[16] Peres, N. M. R., & Ribeiro, R. M. Focus on graphene. New J. Phys. 11(9), 095002 (2009). https://doi.org/10.1088/1367-2630/11/9/095002
[17] Geim, A., Novoselov, K. The rise of graphene. Nat. Mater. 6, 183 (2007). https://doi.org/10.1038/nmat1849
[18] Fefferman, C.L., Weinstein, M.I. Wave Packets in Honeycomb Structures and Two-Dimensional Dirac Equations. Commun. Math. Phys. 326, 251 (2014). https://doi.org/10.1007/s00220-013-1847-2
[19] Katsnelson, M. I., & Novoselov, K. S. Graphene: New bridge between condensed matter physics and quantum electrodynamics. Solid State Commun. 143(1), 3 (2007). https://doi.org/10.1016/j.ssc.2007.02.043
[20] Katsnelson, M., Novoselov, K. & Geim, A. Chiral tunnelling and the Klein paradox in graphene. Nat. Phys. 2, 620 (2006). https://doi.org/10.1038/nphys384
[21] Katsnelson, M. I. Graphene: carbon in two dimensions. Materials Today, 10(1), 20 (2007). https://doi.org/10.1016/S1369-7021(06)71788-6
[22] Haoouam, I. Dirac Oscillator in Dynamical Noncommutative Space. Preprints , 2021020072 (2021). https://doi.org/10.20944/preprints202102.0072.v1
[23] Andreas Fring et al. Strings from position-dependent noncommutativity. J. Phys. A: Math. Theor. 43, 345401 (2010). . https://doi.org/10.1088/1751-8113/43/34/345401
[24] Alavi, S.A., Rezaei, N. Dirac equation, hydrogen atom spectrum and the Lamb shift in dynamical non-commutative spaces. Pramana. J. Phys 88, 77 (2017). https://doi.org/10.1007/s12043-017-1381-4
[25] Gomes M and Kupriyanov V G. Position-dependent noncommutativity in quantum mechanics. Phys. Rev. D79, 125011 (2009). https://doi.org/10.1103/PhysRevD.79.125011

[26] Haouam, I. The Non-Relativistic Limit of the DKP Equation in Non-Commutative Phase-Space. Symmetry, 11, 223 (2019). https://doi.org/10.3390/sym11020223

[27] Haouam, I. Analytical solution of (2+1) dimensional Dirac equation in time-dependent noncommutative phase-space. Acta. polytech. 60(2), 111(2020). https://doi.org/10.14311/AP.2020.60.0111

[28] Seiberg, N; Witten, E. String theory and noncommutative geometry. J. High Energy Phys. JHEP09, 032 (1999). https://doi.org/10.1088/1126-6708/1999/09/032

[29] Gingrich, D.M. Noncommutative geometry inspired blackholes in higher dimensions at the LHC. J. High Energ. Phys. 2010, 22 (2010). https://doi.org/10.1007/JHEP05(2010)022

[30] Gracia-Bondia, J. M. Notes on quantum gravity and noncommutative geometry: New Paths Towards Quantum Gravity. Springer, Berlin, Heidelberg, 2010. 3-58. https://doi.org/10.1007/978-3-642-11897-5_1

[31] Haouam, I. On the three-dimensional Pauli equation in noncommutative phase-space. Acta. polytech. 61(1), 230 (2021). https://doi.org/10.14311/AP.2021.61.0230

[32] Haouam, I. On the noncommutative geometry in quantum mechanics. J. Phys. Stud. 24(2), 2020 (2020). https://doi.org/10.30970/jps.24.2002

[33] Alavi, S. A. Berry’s Phase in Noncommutative Spaces. Phys. Scr.2003, 67, 366 (2003). https://doi.org/10.1238/Physica.Regular.067a00366

[34] Alavi, S. A. Lambda shift and stark effect in simultaneous space-space and momentum-momentum noncommutative quantum mechanics and $\theta$–deformed SU(2) algebra. Mod. Phys. Lett. A, 22 (05), 377 (2007). https://doi.org/10.1142/S0217732307018579

[35] Bastos, C., et al.. Noncommutative graphene. Int. J. Mod. Phys. A, 28(16), 1350064 (2013). https://doi.org/10.1142/S0217751X13500644

[36] Santos, V., Maluf, R. V., & Almeida, C. A. S. Thermodynamical properties of graphene in noncommutative phase-space. Annals of Physics, 349, 402 (2014). https://doi.org/10.1016/j.aop.2014.07.005

[37] Boumali A. Thermodynamic properties of the grapheme in a magnetic field via the two-dimensional Dirac oscillator. Phys. Scr. 90, 045702 (2015). https://doi.org/10.1088/0031-8949/90/4/045702

[38] Khordad, R., Rastegar Sedehi, H.R. Magnetic susceptibility of graphene in non-commutative phase-space: Extensive and non-extensive entropy. Eur. Phys. J. Plus 134, 133 (2019). https://doi.org/10.1140/epjp/12019-12558-5

[39] Gomes M and Kupriyanov V G. Position-dependent noncommutativity in quantum mechanics. Phys. Rev. D79, 125011 (2009). https://doi.org/10.1103/PhysRevD.79.125011

[40] Pefferman, C., & Weinstein, M. Honeycomb lattice potentials and Dirac points. J. Am. Math. Soc. 25(4), 1169 (2012). https://doi.org/10.1090/S0894-0347-2012-00745-0

[41] Essen, L., Donaldson, R., Bangham, M. et al. Frequency of the Hydrogen Maser. Nat. 229, 110 (1971). https://doi.org/10.1038/229110a0

[42] Alavi, S. A., Abbaspour, S. Dynamical noncommutative quantum mechanics. J. Phys. A: Math. Theor. 47 045303 (2014). https://doi.org/10.1088/1751-8113/47/4/045303

[43] Alavi, S. A. Hyperfine splitting in noncommutative spaces. Phys. Scr. 78, 015005 (2008). https://doi.org/10.1088/0031-8949/78/01/015005

[44] Alavi, S. A., Nodeh. S . Neutrino spin oscillations in gravitational fields in noncommutative spaces. Phys. Scr. 90, 035301 (2015). https://doi.org/10.1088/0031-8949/90/3/035301

[45] Jaffe. R. L, “DEGENERATE FERMION SYSTEMS”, Lecture Notes 8.322, Quantum Theory II 2006, Massachusetts Institute of Technology.