Exact Matrix Elements in Supersymmetric Theories

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Abstract

The lowest representatives of the Form Factors relative to the trace operators of \( N = 1 \) Super Sinh-Gordon Model are exactly calculated. The novelty of their determination consists in solving a coupled set of unitarity and crossing equations. Analytic continuations of the Form Factors as functions of the coupling constant allows the study of interesting models in a uniform way, among these the latest model of the Roaming Series and the minimal supersymmetric models as investigated by Schoutens. A fermionic version of the \( c \)-theorem is also proved and the corresponding sum-rule derived.

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1 Introduction

The solution of a Quantum Field Theory (QFT) is provided by the complete set of correlation functions of its fields. Perturbation theory often proves to be an inadequate approach to this problem therefore more effective and powerful methods need to be developed. In this respect, one of the most promising methods is the Form Factor approach which is applicable to integrable models [1, 2]. This consists in computing exactly all matrix elements of the quantum fields and then using them to obtain the spectral representations of the correlators. In addition to the rich and interesting mathematical structure presented by the Form Factors themselves (which has been investigated in a series of papers, among which [2-13], the resulting spectral series usually show a remarkable convergent behaviour which allows approximation of the correlators (or quantities related to them) within any desired accuracy [11, 14, 15, 16, 17, 18, 19].

Particularly important examples of QFT are those which are invariant under a supersymmetry transformation which mixes the elementary bosonic and fermionic excitations. The subject of this paper is the investigation of the Form Factors of bidimensional supersymmetric theories, in particular of the Super Sinh-Gordon Theory (SShG) and of those models which can be obtained by its analytic continuations. In these models, the degeneracy of the spectrum dictated by the supersymmetry implies the existence of multi-channel scattering processes and the resulting $S$-matrix is necessarily non-diagonal. In this case, the complete determination of the matrix elements of the quantum fields for an arbitrary number of asymptotic particles becomes a mathematical problem of formidable complexity. Although this problem has not been solved in its full generality in this paper, however some progress has been nevertheless achieved in determining the lowest Form Factors of these non-diagonal scattering theories. In particular, a set of functions has been identified which might be used in future studies to achieve the determination of all Form Factors of the models. As we will see, the calculation of the lowest matrix elements of supersymmetric theories presents some novelties which makes their determination an interesting issue in itself. In fact, for the first time one has to solve a coupled set of unitarity and crossing equations which originate from the fermionic nature of the supersymmetric theories. It should be said that in the past the calculation of the lowest Form Factors of supersymmetric theories has been approached in a paper by Ahn [24] but it turns out that his results were incorrect, as briefly discussed in Appendix B.

The paper is organised as follows. The next section recalls the basic results of the scattering theories of bidimensional models with a $N = 1$ supersymmetry. Section 3 deals with the simplest super-symmetric scattering theory which will be finally identified with a particular point of the ordinary Sine-Gordon $S$-matrix. The Super Sinh-Gordon Theory (SShG) and the relevant features of the deformed superconformal models are the subject
of Section 4. The $S$-matrix of the SShG model and of others which can be obtained as analytic continuations thereof are analysed in Section 5. The determination of the lowest Form Factors of a particularly important class of operators – those of the trace of the super stress-energy tensor – is discussed in Section 6. In Section 7, a thorough check of their validity is obtained by means of the $c$-theorem sum rule as well as by comparing them with the limiting form obtained in two significant cases, the lastest representative of the Roaming Models and the first example of the Schoutens’s models. Section 8 is devoted to the fermionic formulation of the $c$-theorem which can be achieved in supersymmetric theories whereas Section 9 contains the summary of the results and conclusions. There are two appendices, the first relates to the properties of a function entering the Form Factor calculation, the second contains a brief discussion and criticism of previous results obtained by Ahn.

2 Generalities and Notation

Scattering theories of integrable super-symmetric theories have been discussed in detail in Schoutens’s paper [20] (see also [21]). However, for the sake of clarity, in this section we explicitly state all notations and conventions which will be used in this paper.

Let us consider a two-dimensional quantum field theory made of a bosonic and fermion particle both of mass $m$. The one particle-state of the bosonic and fermionic particles will be denoted by $|b(\beta)\rangle$ and $|f(\beta)\rangle$ respectively, where $\beta$ is the rapidity, i.e. the variable entering the dispersion relations $p_0 = m\cosh \beta$ and $p_1 = m\sinh \beta$ (in the following we will also consider the combinations $p_\pm = p_0 \pm p_1 = me^\pm \beta$).

Let us assume that such a theory is both integrable (i.e. there exists an infinite number of conserved charges) and invariant under a $\mathcal{N} = 1$ supersymmetry. This means that among the set of conserved quantities there are two fermionic charges $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ which satisfy the relations

$$\mathcal{Q}^2 = P_+ , \quad \overline{\mathcal{Q}}^2 = P_- , \quad \{\mathcal{Q}, \overline{\mathcal{Q}}\} = 0 ,$$

where $P_\pm$ are the right/left components $P_\pm = P_0 \pm P_1$ of the momentum operator. The action of these charges on the one-particle states can be represented as

$$\mathcal{Q} | b(\beta)\rangle = \omega \sqrt{m e^\beta/2} | f(\beta)\rangle ; \quad \overline{\mathcal{Q}} | f(\beta)\rangle = \delta \sqrt{m e^{\beta/2}} | b(\beta)\rangle ;$$
$$\mathcal{Q} | f(\beta)\rangle = \omega^{-1} \sqrt{m e^{-\beta/2}} | b(\beta)\rangle ; \quad \overline{\mathcal{Q}} | b(\beta)\rangle = \delta^{-1} \sqrt{m e^{-\beta/2}} | f(\beta)\rangle ,$$

$\mathcal{Q} | f(\beta)\rangle$ and $\overline{\mathcal{Q}} | b(\beta)\rangle$ for the in-state; $\mathcal{Q} | b(\beta)\rangle$ and $\overline{\mathcal{Q}} | f(\beta)\rangle$ for the out-state. The multi-particle states we have $|A_1(\beta_1)A_2(\beta_2)\cdots A_n(\beta_n)\rangle$, where each $A_i$ is either a $b$ or a $f$ particle and the rapidities are ordered in increasing order $\beta_1 > \beta_2 > \cdots > \beta_n$ for the in-state and in a decreasing order $\beta_1 < \beta_2 < \cdots < \beta_n$ for the out-state.
i.e. in terms of two matrices

\[ Q = \begin{pmatrix} 0 & \omega \\ \omega^{-1} & 0 \end{pmatrix}, \quad \overline{Q} = \begin{pmatrix} 0 & \delta \\ \delta^{-1} & 0 \end{pmatrix}, \tag{2.3} \]

satisfying \( Q^2 = \overline{Q}^2 = 1 \). The anti-commutativity of the operators \( Q \) and \( \overline{Q} \) gives the condition

\[ \omega = \pm i\delta, \tag{2.4} \]

with the actual values of \( \omega \) and \( \delta \) which will be fixed by later considerations.

The action of \( Q \) and \( \overline{Q} \) on a multi-particle states must take into account the fermionic nature of these operators and therefore involves brading relations. In the notation of ref.\[21\], we have

\[
Q | A_1(\beta_1)A_2(\beta_2) \ldots A_n(\beta_n) \rangle = \sqrt{m} \sum_{k=1}^{n} e^{\beta_k/2} | (Q_F A_1(\beta_1))(Q_F A_2(\beta_2)) \ldots (Q_F A_{n-1}(\beta_{n-1})(Q A_{n-1}(\beta_{n-1}))A_{n+1}(\beta_{n+1}) \ldots A_n(\beta_n) \rangle \tag{2.5}
\]

and

\[
\overline{Q} | A_1(\beta_1)A_2(\beta_2) \ldots A_n(\beta_n) \rangle = \sqrt{m} \sum_{k=1}^{n} e^{-\beta_k/2} | (Q_F A_1(\beta_1))(Q_F A_2(\beta_2)) \ldots (Q_F A_{n-1}(\beta_{n-1})(\overline{Q} A_{n-1}(\beta_{n-1}))A_{n+1}(\beta_{n+1}) \ldots A_n(\beta_n) \rangle \tag{2.6}
\]

where \( Q_F \) is the fermion parity operator, which on the basis \( | b \rangle \) and \( | f \rangle \) is represented by the diagonal matrix

\[ Q_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.7} \]

Particularly important is the representation of the two super-charges on the two-particle states \( | b(\beta_1)b(\beta_2) \rangle, | f(\beta_1)f(\beta_2) \rangle, | f(\beta_1)b(\beta_2) \rangle, | b(\beta_1)f(\beta_2) \rangle \). The first two states belong to the \( F = 1 \) sector whereas the remaining two states to the \( F = -1 \) sector. By choosing for convenience \( \beta_1 = \beta/2 \) and \( \beta_2 = -\beta/2 \), the operator \( Q \) will be represented by the matrix

\[
Q(\beta) = \begin{pmatrix} 0 & 0 & \omega x & \omega x^{-1} \\ 0 & 0 & -\omega^{-1} x^{-1} & \omega^{-1} x \\ \omega^{-1} x & -\omega x^{-1} & 0 & 0 \\ \omega^{-1} x^{-1} & \omega x & 0 & 0 \end{pmatrix}, \tag{2.8}
\]

where \( x \equiv e^{\beta/4} \). For \( \overline{Q} \) we have analogously

\[
\overline{Q}(\beta) = \begin{pmatrix} 0 & 0 & \delta x^{-1} & \delta x \\ 0 & 0 & -\delta^{-1} x & \delta^{-1} x^{-1} \\ \delta^{-1} x^{-1} & -\delta x & 0 & 0 \\ \delta^{-1} x & \delta x^{-1} & 0 & 0 \end{pmatrix}. \tag{2.9}
\]
In the following we will also need the representation matrix of the operator $Q\overline{Q}$ on the above two particle states, given by

\[
(Q\overline{Q})(\beta) = 2 \begin{pmatrix}
\frac{\omega}{\delta} & -\omega \delta \sinh \frac{\beta}{2} & 0 & 0 \\
\frac{1}{\omega \delta} \sinh \frac{\beta}{2} & -\frac{\omega}{\delta} & 0 & 0 \\
0 & 0 & 0 & -\frac{\omega}{\delta} \cosh \frac{\beta}{2} \\
0 & 0 & -\frac{\omega}{\delta} \cosh \frac{\beta}{2} & 0
\end{pmatrix}.
\]  

(2.10)

Let us consider now the structure of the elastic two-body $S$-matrix of the theory. Since fermions can only created or destroyed in couples, we have the following scattering channels

\[
\begin{align*}
|b(\beta_1)b(\beta_2)| & = A(\beta_{12}) |b(\beta_2)b(\beta_1)| + B(\beta_{12}) |f(\beta_2)f(\beta_1)|; \\
|f(\beta_1)f(\beta_2)| & = C(\beta_{12}) |b(\beta_2)b(\beta_1)| + D(\beta_{12}) |f(\beta_2)f(\beta_1)|; \\
|f(\beta_1)b(\beta_2)| & = E(\beta_{12}) |f(\beta_2)b(\beta_1)| + F(\beta_{12}) |b(\beta_2)f(\beta_1)|; \\
|b(\beta_1)f(\beta_2)| & = G(\beta_{12}) |f(\beta_2)b(\beta_1)| + H(\beta_{12}) |b(\beta_2)f(\beta_1)|,
\end{align*}
\]  

(2.11)

where $\beta_{12} = \beta_1 - \beta_2$. Respecting the ordering of the 2-particle states as above, the $S$-matrix can be then represented by

\[
S(\beta) = \begin{pmatrix}
A & B & 0 & 0 \\
C & D & 0 & 0 \\
0 & 0 & E & F \\
0 & 0 & G & H
\end{pmatrix}.
\]  

(2.12)

The amplitudes $A(\beta)$ and $D(\beta)$ describe the transmission channels of two bosons and two fermions respectively whereas the amplitudes $B(\beta)$ and $C(\beta)$ are related to the annihilation-creation channels of two bosons into two fermions and vice versa. The other amplitudes $E(\beta)$ and $H(\beta)$ describe the reflection channels of the boson-fermion scattering whereas the remaining $F(\beta)$ and $G(\beta)$ describe the pure transmission processes in the $F = -1$ sector.

Let us now impose the invariance of the scattering processes under the action of the supersymmetric charges. First consider the invariance of the $S$-matrix with respect to the operator $Q\overline{Q}$: with the choice $\beta_1 = \beta/2$ and $\beta_2 = -\beta/2$, by taking into account the different ordering of the rapidities of the out-states, this implies the following constraint on the scattering amplitudes

\[
Q\overline{Q}(\beta)S(\beta) = S(\beta)(Q\overline{Q})(-\beta).
\]  

(2.13)

In the $F = -1$ sector, this equation leads to the conditions

\[
E(\beta) = H(\beta) \quad , \quad F(\beta) = G(\beta),
\]  

(2.14)
whereas in the $F = 1$ sector, eq. (2.13) implies

$$B(\beta) = (\omega \delta)^2 C(\beta) ;$$

$$2\omega^2 C(\beta) = [A(\beta) + D(\beta)] \sinh \frac{\beta}{2} ;$$

$$\frac{2}{\delta^2} B(\beta) = [A(\beta) + D(\beta)] \sinh \frac{\beta}{2} .$$

Hence, we can choose $B(\beta) = C(\beta)$ provided that the constants $\omega$ and $\delta$ satisfy the condition

$$(\omega \delta)^2 = 1 .$$

Since from the closure of the supersymmetric algebra $\omega = \pm i \delta$, it follows that $\delta$ has to satisfy the condition $\delta^4 = -1$ and therefore we have the following possibilities:

$$\delta = e^{i \frac{\pi}{4}} , \ \omega = \pm e^{-i \frac{\pi}{4}} .$$

or

$$\delta = e^{-i \frac{\pi}{4}} , \ \omega = \pm e^{i \frac{\pi}{4}} .$$

In the following we will adopt the values $\omega = e^{i \frac{\pi}{4}}$ and $\delta = e^{-i \frac{\pi}{2}}$. With this choice, we have then

$$B(\beta) = C(\beta) , \ 2B(\beta) = -i [A(\beta) + D(\beta)] \sinh \frac{\beta}{2} .$$

In light of all the above equations, the $S$ matrix can be written then as

$$S(\beta) = \begin{pmatrix}
A & B & 0 & 0 \\
B & D & 0 & 0 \\
0 & 0 & E & F \\
0 & 0 & F & E
\end{pmatrix} .$$

Further conditions on the above functions are provided by the invariance of the $S$-matrix under each supercharge separately (having already considered the invariance under $Q \overline{Q}$, one can consider only one of them, say $Q$) in the form

$$Q(\beta) S(\beta) = S(\beta) Q(-\beta) .$$

This equation is equivalent to the following conditions

$$A(\beta) - D(\beta) = 2F(\beta) ;$$

$$A(\beta) + D(\beta) = \frac{2}{\cosh \frac{\beta}{2}} E(\beta) ;$$

$$E(\beta) \tanh \frac{\beta}{2} = i B(\beta) .$$
Finally, crossing symmetry implies

\[ A(i\pi - \beta) = A(\beta) ; \]
\[ D(i\pi - \beta) = D(\beta) ; \tag{2.23} \]
\[ F(i\pi - \beta) = F(\beta) ; \]
\[ E(i\pi - \beta) = B(\beta) , \]

whereas the unitarity condition \( S(\beta)S(-\beta) = 1 \) gives

\[ A(\beta)A(-\beta) + B(\beta)B(-\beta) = 1 ; \]
\[ A(\beta)B(-\beta) + B(\beta)D(-\beta) = 0 ; \]
\[ B(\beta)A(-\beta) + D(\beta)B(-\beta) = 0 ; \]
\[ B(\beta)B(-\beta) + D(\beta)D(-\beta) = 1 ; \tag{2.24} \]
\[ E(\beta)E(-\beta) + F(\beta)F(-\beta) = 1 ; \]
\[ E(\beta)F(-\beta) + F(\beta)E(-\beta) = 0 . \]

In the next section we will discuss several examples of scattering theories which fulfill the above set of equations.

### 3 The simplest SUSY S-matrix

The simplest supersymmetric S-matrix can be obtained by noticing that the amplitude \( F(\beta) \) is invariant under crossing. This allows us to make the consistent choice \( F(\beta) = 0. \) In view of the equations (2.22), the S matrix can be written in this case as

\[
S(\beta) = \begin{pmatrix}
-1/\cosh \frac{\beta}{2} & i \tanh \frac{\beta}{2} & 0 & 0 \\
 i \tanh \frac{\beta}{2} & -1/\cosh \frac{\beta}{2} & 0 & 0 \\
 0 & 0 & -1 & 0 \\
 0 & 0 & 0 & -1
\end{pmatrix} R(\beta) , \tag{3.1}
\]

where the function \( R(\beta) \) has to satisfy the equations

\[ R(\beta)R(-\beta) = 1 ; \]
\[ R(i\pi - \beta) = -i \tanh \frac{\beta}{2} R(\beta) . \tag{3.2} \]

There is a geometrical method of solving the above coupled set of equations for a meromorphic function with only poles and zeros, none of which are in the physical strip \( 0 \leq \text{Im} \beta \leq \pi . \) First of all, the poles and zeros of \( R(\beta) \) are linked to each other by the first eq. of (3.2) because if the function \( R(\beta) \) has a pole (zero) in \( i\pi \eta \), it should necessary have a zero (pole) in \(-i\pi \eta \). On the other hand, changing the rapidity \( \beta \) according
to $\beta \to i\pi - \beta$ is a reflection with respect to the point $i\pi$. The function $-i \tanh \frac{\beta}{2}$ has zeros in $\beta = \pm 2k\pi i$ and poles in $\beta = \pm (k+1)\pi i$ ($k = 0,1,\ldots$) and its infinite product representation is given by

$$-i \tanh \frac{\beta}{2} = \prod_{k=0}^{\infty} \frac{[2k\pi - i\beta][(2k+1)\pi + i\beta]}{[(2k+1)\pi + i\beta][(2k+1)\pi - i\beta]}.$$  \hspace{1cm} (3.3)

Since the second equation can be written as

$$R(i\pi - \beta)R(-\beta) = -i \tanh \frac{\beta}{2},$$  \hspace{1cm} (3.4)

to solve this equation we therefore need to find a function whose superposition of poles and zeros coming from the combination of the two reflections $\beta \to i\pi - \beta$ and $\beta \to -\beta$ match those of (3.3). It is simple to see that such a function should have zeros and poles of increasing multiplicities and its infinite product representation is given by

$$R(\beta) = \prod_{k=0}^{\infty} \frac{[(2k+1)\pi + i\beta][(2k+3)\pi + i\beta][2(k+1)\pi - i\beta]^2}{[(2k+1)\pi - i\beta][(2k+3)\pi - i\beta][2(k+1)\pi + i\beta]^2}.$$  \hspace{1cm} (3.5)

The above function can also be written as

$$R(\beta) = \prod_{k=0}^{\infty} \frac{\Gamma \left( k + \frac{1}{2} - i\frac{\beta}{2\pi} \right) \Gamma \left( k + \frac{3}{2} - i\frac{\beta}{2\pi} \right) \Gamma^2 \left( k + 1 + i\frac{\beta}{2\pi} \right)}{\Gamma \left( k + \frac{1}{2} + i\frac{\beta}{2\pi} \right) \Gamma \left( k + \frac{3}{2} + i\frac{\beta}{2\pi} \right) \Gamma^2 \left( k + 1 - i\frac{\beta}{2\pi} \right)},$$  \hspace{1cm} (3.6)

or as

$$R(\beta) = \exp \left[ \frac{i}{2} \int_{0}^{\infty} dt \frac{\sin \frac{\beta t}{2}}{\cosh \frac{t}{2} \sinh \frac{\beta t}{2}} \right].$$  \hspace{1cm} (3.7)

It is now interesting to note that the simplest supersymmetric $S$-matrix as above coincides with the $S$-matrix coming from another integrable model. Consider, in fact, the $S$-matrix in the solitonic sector of the ordinary Sine-Gordon model, given in full generality by\footnote{The two-particle states are ordered as $|SS\rangle, |S\bar{S}\rangle, |\bar{S}S\rangle, |\bar{S}\bar{S}\rangle$, where $S$ denotes the soliton whereas $\bar{S}$ the anti-soliton.} 2

$$S(\beta) = \hat{R}(\beta) \begin{pmatrix} \frac{i \sin \frac{\beta}{2}}{\sinh \frac{\pi(\beta-\pi)}{2}} & \frac{\sinh \frac{\pi\beta}{2}}{\sinh \frac{\pi(\beta-\pi)}{2}} & 0 & 0 \\ \frac{\sin \frac{\pi\beta}{2}}{\sin \frac{\pi(\beta-\pi)}{2}} & \frac{i \sin \frac{\pi\beta}{2}}{\sin \frac{\pi(\beta-\pi)}{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$  \hspace{1cm} (3.8)

where

$$\hat{R}(\beta) = \exp \left[ -i \int_{0}^{\infty} dt \frac{\sinh \frac{t}{2}}{\cosh \frac{t}{2} \sinh \frac{\beta t}{2}} \sin \frac{\beta t}{\pi} \right].$$  \hspace{1cm} (3.9)
The parameter $\xi$ is related to the coupling constant $g$ of the Lagrangian of the model

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 + \frac{m^2}{g^2} \cos(g \varphi) ,$$

(3.10)

by $\xi = \frac{\pi g^2}{8 \pi - g^2}$.

By comparing the $S$-matrix (3.1) with the one of the Sine-Gordon model, it is now easy to see that the two coincide for the particular value $\xi = 2 \pi$, i.e. $g^2 = 16 \pi/3$. This is a repulsive point of the Sine-Gordon model, no additional bound states are present at this value and therefore the full $S$-matrix of the SG model is reduced to the one in (3.8). Since the ordinary Sine-Gordon model can be regarded as a massive integrable deformation of the gaussian conformal model with an action

$$A_0 = \frac{1}{2} \int d^2 x (\partial_\mu \varphi)^2 ,$$

(3.11)

(of central charge $C = 1$), the deforming operator $(e^{ig \varphi} + e^{-ig \varphi})$ which leads to the Sine-Gordon model has at the point $\xi = 2 \pi$ the conformal dimension $\Delta = 2/3$. The relevance of these observations as well as the consequences of the identity between the $S$-matrices will be topics of discussion later in this paper.

### 4 The SShG and Superconformal Models

The aim of this section is to illustrate several properties of the Super Sinh-Gordon model as well as some features of the superconformal models and their deformations.

#### 4.1 Lagrangian of the SShG model

In the euclidean space, the Super Sinh-Gordon model can be defined in terms of its action

$$\mathcal{A} = \int d^2 z d^2 \theta \left[ \frac{1}{2} D \Phi \overline{D} \Phi + i \frac{m}{\lambda^2} \cosh \lambda \Phi \right] ,$$

(4.1)

where the covariant derivatives are defined as

$$D = \partial_\theta - \theta \partial_z ;$$

$$\overline{D} = \partial_\bar{\theta} - \bar{\theta} \partial_{\bar{z}} ,$$

(4.2)

and the superfield $\Phi(z, \bar{z}, \theta, \bar{\theta})$ has an expansion as

$$\Phi(z, \bar{z}, \theta, \bar{\theta}) = \varphi(z, \bar{z}) + \theta \psi(z, \bar{z}) + \bar{\theta} \overline{\psi}(z, \bar{z}) + \theta \bar{\theta} \mathcal{F}(z, \bar{z}) .$$

(4.3)

The integration on the $\theta$ variables as well as the elimination of the auxiliary field $\mathcal{F}(z, \bar{z})$ by means of its algebraic equation of motion leads to the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_z \varphi \partial_{\bar{z}} \varphi + \overline{\psi} \partial_z \overline{\psi} + \psi \partial_{\bar{z}} \overline{\psi}) + \frac{m^2}{2 \lambda^2} \sinh^2 \lambda \varphi + im \overline{\psi} \psi \cosh \lambda \varphi .$$

(4.4)
Of all the different ways of looking at the SSHG model, one of the most convenient is to consider it as a deformation of the superconformal model described by the action

\[ \mathcal{A}_r = \frac{1}{2} \int (\partial_z \varphi \partial_{\bar{z}} \varphi + \overline{\psi} \partial_{\bar{z}} \psi + \psi \partial_z \psi) . \]  

(4.5)

This superconformal model has central charge \( C = 3/2 \). At this point, it is useful to briefly remind some properties of the superconformal models and their deformations.

### 4.2 Superconformal models and their deformation

For a generic superconformal model, the supersymmetric charges can be represented by the differential operators

\[ Q = \partial_\theta + \theta \partial_z ; \quad \overline{Q} = \partial_{\bar{\theta}} + \bar{\theta} \partial_{\bar{z}} . \]  

(4.6)

The analytic part of the stress-energy tensor \( T(z) \) and the current \( G(z) \) which generates the supersymmetry combine themselves into the analytic superfield

\[ W(z, \theta) = G(z) + \theta T(z) , \]  

(4.7)

which is called the super stress-energy tensor. For the anti-analytic sector we have correspondingly \( \overline{W}(\bar{z}) = \overline{G}(\bar{z}) + \bar{\theta} \overline{T}(\bar{z}) \). These fields are mapped one into the other by means of the super-charges

\[ T(z) = \{G(z), Q\} , \quad \partial_z G = [T(z), Q] ; \]
\[ \overline{T}(\bar{z}) = \{\overline{G}(\bar{z}), \overline{Q}\} , \quad \partial_{\bar{z}} \overline{G} = [\overline{T}(\bar{z}), \overline{Q}] , \]  

(4.8)

and their Operator Product Expansion reads

\[ T(z)T(w) = \frac{C}{2(z-w)^4} + 2 \frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \]
\[ T(z)G(w) = \frac{3}{2} \frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} + \cdots \]  

(4.9)
\[ G(z)G(w) = \frac{2C}{3(z-w)^3} + 2 \frac{T(w)}{(z-w)^2} + \cdots \]

with analogous relations for the anti-analytic fields. As it is well known \[25\], reducible unitary representations of the \( N = 1 \) superconformal symmetry occurs for the discrete values of the central charge

\[ C = \frac{3}{2} - \frac{12}{m(m+2)} . \]  

(4.10)

At these values, realizations of the \( N = 1 \) superconformal algebra are given in terms of a finite number of superfields in the Neveu-Schwartz sector and a finite number of ordinary
conformal primary fields in the Ramond sector. Their conformal dimensions are given by
\[
\Delta_{p,q} = \frac{[(m+2)p-mq]^2 - 4}{8m(m+2)} + \frac{1}{32}[1 - (-1)^{p-q}],
\]
where \(p - q\) even corresponds to the primary Neveu-Schwartz superfields \(N_{p,q}^{(m)}(z,\theta)\) and \(p - q\) odd to the primary Ramond fields \(R_{p,q}^{(m)}(z)\). These fields enter the so-called superconformal minimal models \(SM_m\).

The Witten index \(Tr(-1)^F\) of the superconformal models can be computed by initially defining them on a cylinder \([26]\): the Hamiltonian on the cylinder is given by \(H = Q^2 = L_0 - C/24\), where as usual \(C/24\) is the Casimir energy on the cylinder and \(L_0 = \frac{1}{2\pi i} \oint dz z T(z)\). Considering that for any conformal state \(|a\rangle\) with \(\Delta > C/24\) there is the companion state \(Q|a\rangle\) of opposite fermionic parity, their contributions cancel each other in \(Tr(-1)^F\) and therefore only the ground states with \(\Delta = C/24\) (which are not necessarily paired) enter the final expression of the Witten index. For the minimal models, there is a non-zero Witten index only for \(m\) even. Therefore the lowest superconformal minimal model with a non-zero Witten index is the one with \(m = 4\), which has a central charge \(C = 1\) and corresponds to the class of universality of the critical Ashkin-Teller model. The superconformal theory with \(C = 3/2\) made of free bosonic and fermionic fields also has a non-zero Witten index, because an unpaired Ramond field \(R(z)\) is explicitly given by the spin field \(\sigma(z)\) of Majorana fermion \(\psi(z)\) with conformal dimension \(\Delta = 1/16\), i.e. by the magnetization operator of the Ising model.

The above observations become important in the understanding of the off-critical dynamics relative to the deformation of the action of the superconformal minimal models \(SM_m\) by means of the relevant supersymmetric Neveu-Schwartz operator \(N_{1,3}^{(m)}(z,\theta)\). In fact, as shown in \([26]\), the massless Renormalization Group flow generated by such an operator preserves the Witten index. Therefore the long distance behaviour of the deformed \(SM_m\) minimal model – controlled by the action
\[
A_m + \gamma \int d^2z d^2\theta N_{1,3}^{(m)},
\]
(for a positive value of the coupling constant \(\gamma\), with the usual conformal normalization of the superfield) – is ruled by the fixed point of the minimal model \(SM_{m-2}\). Therefore the action \((4.12)\) describes the RG flow
\[
SM_m \rightarrow SM_{m-2},
\]
with a corresponding jump in steps of two of the central charge \(C\), i.e.
\[
\Delta C = C(m) - C(m - 2).
\]

\(^3\)This is in contrast of what happens in the ordinary conformal minimal models, where the deformation of the conformal action by means of the operator \(\phi_{1,3}\) induces a massless flow between two next neighborod minimal models.
Therefore, a cascade of massless flows which start from $C = 3/2$ and progress by all $N^{(m)}_{1,3}$ deformations of $\mathcal{SM}_m$ met along the way must necessarily pass through the model with $C = 1$ in the second to last step rather than the lowest model with $C = 7/10$ (Figure 1). We will see that this is indeed the scenario which is described by a specific analytical continuation of the coupling constant of the Super-Sinh-Gordon model, the so-called Roaming Models.

For the time being, let us further discuss the deformation of the $C = 3/2$ super-conformal theory which leads to the SShG model. At the conformal point, the explicit realization of the component of the super stress-energy tensor are given by

$$
T(z) = -\frac{1}{2} [ (\partial_z \varphi)^2 - \psi \overline{\partial} \psi ] ;
G(z) = i \psi \partial_z \varphi ,
$$

and they satisfy the conservation laws $\partial_z T(z) = \partial_z G(z) = 0$. Once this superconformal model is deformed according to the Lagrangian (4.4), the new conservation laws are given by

$$
\partial_z T(z, \overline{z}) = \partial_z \Theta(z, \overline{z}) ;
\partial_z G(z, \overline{z}) = \partial_z \chi(z, \overline{z}) ,
$$

where

$$
\Theta(z, \overline{z}) = \frac{m^2}{2 \lambda^2} \sinh^2 \lambda \varphi + im \overline{\psi} \psi \cosh \lambda \varphi ,
\chi(z, \overline{z}) = \frac{m}{\lambda} \overline{\psi} \sinh \lambda \varphi .
$$

For the anti-analytic part of the super stress-energy tensor we have

$$
\partial_{\overline{z}} \overline{T}(z, \overline{z}) = \partial_{\overline{z}} \overline{\Theta}(z, \overline{z}) ;
\partial_{\overline{z}} \overline{G}(z, \overline{z}) = \partial_{\overline{z}} \overline{\chi}(z, \overline{z}) ,
$$

where $\Theta(z, \overline{z})$ is as before and the other fields are given by

$$
\overline{G}(z, \overline{z}) = -i \overline{\psi} \partial_{\overline{z}} \varphi ;
\overline{\chi}(z, \overline{z}) = \frac{m}{\lambda} \overline{\psi} \sinh \lambda \varphi .
$$

The operators $\Theta(z, \overline{z})$, $\chi(z, \overline{z})$ and $\overline{\chi}(z, \overline{z})$ belong to the trace of the supersymmetric stress-energy tensor and they are related each other by

$$
\Theta(z, \overline{z}) = \{ \chi(z, \overline{z}), \mathcal{Q} \} ,
\partial_{\overline{z}} \chi(z, \overline{z}) = [ \Theta(z, \overline{z}), \mathcal{Q} ] ;
\overline{\Theta}(z, \overline{z}) = \{ \overline{\chi}(z, \overline{z}), \mathcal{Q} \} ,
\partial_z \overline{\chi}(z, \overline{z}) = [ \Theta(z, \overline{z}), \mathcal{Q} ] ,
$$

\footnote{As well known, the lowest model with $C = 7/10$ corresponds to the class of universality of the Tricritical Ising Model.}
where the charges of supersymmetry are expressed by

\[
Q = \int G(z, \bar{z}) dz + \chi(z, \bar{z}) d\bar{z} ; \\
\bar{Q} = \int G'(z, \bar{z}) d\bar{z} + \bar{\chi}(z, \bar{z}) dz .
\]

(4.20)

In addition to the conservation laws (4.15) and (4.17), the SShG model possesses higher integrals of motion which were explicitly determined in [23]. Therefore its scattering processes are purely elastic and factorizable, and its two-body S-matrix is discussed in the next section.

5 The S-matrix of the SSHG

The S-matrix of the SSHG model has been determined in [24]. It is given by

\[
S(\beta) = Y(\beta) \begin{pmatrix}
1 - \frac{2i \sin \pi \alpha}{\sinh \frac{\beta}{2}} & -\frac{\sin \pi \alpha}{\cosh \frac{\beta}{2}} & 0 & 0 \\
-\frac{\sin \pi \alpha}{\cosh \frac{\beta}{2}} & -1 - \frac{2i \sin \pi \alpha}{\sinh \frac{\beta}{2}} & 0 & 0 \\
0 & 0 & -\frac{i \sin \pi \alpha}{\sinh \frac{\beta}{2}} & 1 \\
0 & 0 & 1 & -\frac{i \sin \pi \alpha}{\sinh \frac{\beta}{2}}
\end{pmatrix},
\]

(5.1)

where

\[
Y(\beta) = \frac{\sinh \frac{\beta}{2}}{\sinh \frac{\beta}{2} + i \sin \pi \alpha} U(\beta, \alpha) ,
\]

(5.2)

and the function \( U(\beta) \) is given by

\[
U(\beta) = \exp \left[ i \int_{0}^{\infty} dt \sinh \alpha t \, \sinh(1 - \alpha) t \, \frac{\sin \beta t}{\pi} \right] .
\]

(5.3)

The angle \( \alpha \) is a positive quantity, related to the coupling constant \( \lambda \) of the model by

\[
\alpha = \frac{1}{4\pi} \frac{\lambda^2}{1 + \frac{\lambda^2}{4\pi}}.
\]

(5.4)

This equation implies that the SShG is a quantum field theory invariant under the strong-weak duality

\[
\lambda \to \frac{4\pi}{\lambda}.
\]

(5.5)

It is easy to see that such S-matrix fulfills all constraints of section 2. The prefactor of the S-matrix admits the following infinite product representation

\[
Y(\beta) = \frac{1}{\Gamma \left( -i \frac{\beta}{2\pi} \right) \Gamma \left( -i \frac{\beta}{2\pi} + \frac{1}{2} \right) \Gamma \left( \frac{1}{2} + i \frac{\beta}{2\pi} \right) \Gamma \left( 1 + i \frac{\beta}{2\pi} \right)}
\times \prod_{k=0}^{\infty} \frac{\Gamma \left( k + \alpha + \frac{1}{2} + i \frac{\beta}{2\pi} \right) \Gamma \left( k + \frac{3}{2} - \alpha + i \frac{\beta}{2\pi} \right) \Gamma^2 \left( k + 1 + i \frac{\beta}{2\pi} \right)}{\Gamma \left( k + \alpha + \frac{1}{2} - i \frac{\beta}{2\pi} \right) \Gamma \left( k + \frac{3}{2} - \alpha - i \frac{\beta}{2\pi} \right) \Gamma^2 \left( k + 1 - i \frac{\beta}{2\pi} \right)}
\times \prod_{k=0}^{\infty} \frac{\Gamma \left( k + \alpha - i \frac{\beta}{2\pi} \right) \Gamma \left( k + 1 - \alpha - i \frac{\beta}{2\pi} \right) \Gamma^2 \left( k + \frac{1}{2} - i \frac{\beta}{2\pi} \right)}{\Gamma \left( k + \alpha + i \frac{\beta}{2\pi} \right) \Gamma \left( k + 2 - \alpha + i \frac{\beta}{2\pi} \right) \Gamma^2 \left( k + \frac{3}{2} + i \frac{\beta}{2\pi} \right)}.
\]

(5.6)
from which one can explicitly see that this S-matrix does not have poles into the physical strip. It is now interesting to analyse several analytic continuations of the above S-matrix in the parameter \( \alpha \).

### 5.1 The analytic continuation \( \alpha \to -\alpha \) and Schoutens’s model

Under this analytic continuation, the SSHG model goes into the Super Sine-Gordon model and the S-matrix (5.1) describes in this case the scattering of the lowest breather states of the latter model. The explicit expression is given by

\[
S(\beta) = \hat{Y}(\beta) \begin{pmatrix}
1 + \frac{2i \sin \pi \alpha}{\sinh \beta} & \frac{\sin \pi \alpha}{\cosh \frac{\beta}{2}} & 0 & 0 \\
\frac{\sin \pi \alpha}{\cosh \frac{\beta}{2}} & -1 + \frac{2i \sin \pi \alpha}{\sinh \beta} & 0 & 0 \\
0 & 0 & i \frac{\sin \pi \alpha}{\sinh \frac{\beta}{2}} & 1 \\
0 & 0 & 1 & i \frac{\sin \pi \alpha}{\sinh \frac{\beta}{2}}
\end{pmatrix}
\]  

(5.7)

where

\[
\hat{Y}(\beta) = \frac{\sinh \frac{\beta}{2}}{\sinh \frac{\beta}{2} - i \sin \pi \alpha} \hat{U}(\beta)
\]  

(5.8)

and

\[
\hat{U}(\beta) = U(\beta) \frac{\sinh \frac{\beta}{2} - i \sin \pi \alpha}{\sinh \frac{\beta}{2} + i \sin \pi \alpha} \left( \frac{\sinh \beta + i \sin(2\pi \alpha)}{\sinh \beta - i \sin(2\pi \alpha)} \right).
\]  

(5.9)

Notice that at the particular value \( \alpha = \pi/3 \) we can consistently truncate the theory at the Super Sine-Gordon breather sector only\[.\] At this value, the pole at \( \beta = 2\pi i/3 \) of the S-matrix can be regarded as due to the bosonic and fermionic one-particle states \( |b(\beta)\rangle \) and \( |f(\beta)\rangle \). These particles are therefore bound states of themselves, in the channels

\[
bb \to b \to bb; \quad ff \to b \to ff,
\]

for the bosonic particle \( b \), and in the channels

\[
bf \to f \to fb; \quad fb \to f \to bf,
\]

for the fermionic particle \( f \). There is of course a price to pay for this truncation: this means that the residues of the S-matrix at these poles will be purely imaginary. Such a

---

\[5\text{The general class of models considered by Schoutens is obtained by taking } \alpha = \pi/(2N + 1), N = 1, 2, \ldots \text{ and they correspond to the supersymmetric deformation of the non-unitary minimal superconformal models with central charge } C = -3N(4N + 3)/(2N + 2). \text{ For simplicity only the first is examined, the detailed discussion of the others requires the application of the bootstrap equations to their S-matrix.} \]
model therefore would be the supersymmetric analogous of the Yang-Lee model for the ordinary Sine-Gordon model [22]. It was considered originally by Schoutens [20] and it has been identified with the off-critical supersymmetric deformation of the non-unitary superconformal minimal model with central charge $C = -21/4$. The residues of the $S$-matrix are given by (Figure 2)

$$\Gamma_{bb}^b = i \sqrt{3\kappa};$$
$$\Gamma_{ff}^b = i \sqrt{3\kappa};$$
$$\Gamma_{fb}^f = i \sqrt{3\kappa};$$
$$\Gamma_{fb}^f = i \sqrt{3\kappa},$$

where

$$\kappa = \exp \left[ -\frac{1}{2} \int_0^\infty \frac{dt \sinh \frac{t}{2} \sinh \frac{2t}{3}}{t \cosh \frac{t}{2} \cosh \frac{3t}{2}} \right] = 0.7941001...$$

We will come back to this model for the discussion of its Form Factors.

5.2 The Roaming Models

The $S$ matrix (5.1) of the SShG model has zeros in the physical strip located at $\alpha_1 = i\pi\alpha$ and $\alpha_2 = i\pi(1 - \alpha)$. By varying the coupling constant $\lambda$, they move along the imaginary axis and they finally meet at the point $i\pi/2$, at the self-dual value of the coupling constant $\lambda^2 = 4\pi$. If we further increase the value of the coupling constant, they simply swap positions. But there is a more interesting possibility: as first proposed by Al. Zamolodchikov for the analogous case of the ordinary Sinh-Gordon model [27], once the two zeros meet at $i\pi$, they can enter the physical strip by taking complex values of the coupling constant (Figure 3). In this way, the location of the two zeros are given by

$$\alpha_{\pm} = \frac{1}{2} \pm i\alpha_0.$$

From the analytic $S$-matrix theory, the existence of complex zeros in the physical strip implies the presence of resonances in the system. By analysing the finite-size behaviour of the theory by means of the Thermodynamic Bethe Ansatz [28], the interesting result is that the net effect of these resonances consists in an infinite cascade of massless Renormalization Group flows generated by the Neveu-Schwartz fields $N^{m}_{1,3}$ and passing through all minimal superconformal model $SM_m$ with non-zero Witten index (see Figure 1). As discussed in the previous section, the ending point of this infinite-nested RG flow should describe the $N_{1,3}$ deformation of the superconformal model $SM_4$. Is this really the case? By taking the limit $\alpha \to i\infty$ into the $S$-matrix (5.1), it is easy to see that it reduces to the simplest supersymmetric $S$-matrix analysed in Section 3, which in turn coincides with
the one of the Sine-Gordon model at $\xi = 2\pi$. Since this $S$-matrix describes a massive deformation of the $C = 1$ model, in order to confirm the above roaming trajectory scenario the only thing that remains to check is the comparison of the anomalous dimension of deforming field. In the Sine-Gordon model at $\xi = 2\pi$, the anomalous dimension of the deforming field was determined to be $\Delta = \frac{2}{3}$, which is indeed the conformal dimension of the top component of the superfield $N_{1,3}$ in the model $\mathcal{SM}_4$. In the light of this result, it is now clear why in the roaming limit the value which is actually selected is $\xi = 2\pi$ among all possible values of the coupling constant of Sine-Gordon model.

In the next section we will see that the identity between the $S$-matrix of the two models also implies an identity between the Form Factors of the two theories.

6 Form Factors of the Trace Operators of the SShG Model

For integrable quantum field theories, the knowledge of the $S$-matrix is very often the starting point for a complete solution of quantum field dynamics in terms of an explicit construction of the correlation functions of all fields of the theory. This result can be obtained by computing first the matrix elements of the operators on the asymptotic states (the so-called Form Factors) [1, 2] and then inserting them into the spectral representation of the correlators. For instance, in the case of the two-point correlation function of a generic operator $\mathcal{O}(z, \bar{z})$ we have

$$G(z, \bar{z}) = \langle 0 | \mathcal{O}(z, \bar{z}) \mathcal{O}(0, 0) | 0 \rangle = \int_0^\infty da^2 \rho(a^2) K_0(a) ,$$

where $K_0(x)$ is the usual Bessel function. The spectral density $\rho(a^2)$ is given in this case by

$$\rho(a^2) = \sum_{n=0}^{\infty} \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_n}{2\pi} \delta(a - \sum_i m \cosh \beta_i) \delta(\sum_i \sinh \beta_i) \times$$

$$\langle 0 | \mathcal{O}(0, 0) | A_1(\beta_1) \cdots A_n(\beta_n) \rangle |^2 .$$

The Form-Factor approach has proved to be extremely successful for theories with scalar $S$-matrix, leading to an explicit solution of models of statistical mechanics interest such as the Ising model [1, 3, 14], the Yang-Lee model [15] or quantum field theories defined by a lagrangian, like the Sinh-Gordon model [1, 3, 13]. On the contrary, for theories with a non-scalar $S$-matrix the functional equations satisfied by the Form Factors are generally quite difficult to tackle and a part from the Sine-Gordon model or theories which can be brought back to it [1, 2, 4], there is presently no mathematical technique available for solving them in their full generality. Also in this case, however, the situation is not as impracticable as
it might seem at first sight. The reason consists in the fast convergent behaviour of the spectral representation series which approximates the correlation functions with a high level of accuracy even if truncated at the first available matrix elements [11, 14, 15, 16, 17, 18, 19]. In the light of this fact, in this section we will compute the lowest matrix elements of two of the most important operators of the theory, namely the trace operators $\Theta(z, \bar{z})$, $\chi(z, \bar{z})$ and $\bar{\chi}(z, \bar{z})$ of the supersymmetric stress-energy tensor of the SSshG model. For the operator $\Theta(0, 0)$, they are given by

$$
F_{bb}^{\Theta}(\beta) = \langle 0 | \Theta(0,0) | b(\beta_1)b(\beta_2) \rangle ;
$$

$$
F_{ff}^{\Theta}(\beta) = \langle 0 | \Theta(0) | f(\beta_1)f(\beta_2) \rangle ,
$$

(6.3)

whereas for the operators $\chi(0,0)$ we have instead

$$
F_{bf}^{\chi}(\beta_1, \beta_2) = \langle 0 | \chi(0,0) | b(\beta_1)f(\beta_2) \rangle ;
$$

$$
F_{fb}^{\chi}(\beta_1, \beta_2) = \langle 0 | \chi(0,0) | f(\beta_1)b(\beta_2) \rangle ,
$$

(6.4)

(with an analogous result for the lowest Form Factors of the operator $\bar{\chi}(0,0)$). Since the operators $\Theta$, $\chi$ and $\bar{\chi}$ are related each other by supersymmetry, as consequence of eqs. (6.13) we have

$$
F_{bb}^{\Theta}(\beta) = \omega \left( e^{\beta_1/2} F_{fb}^{\chi} + e^{\beta_2/2} F_{bf}^{\chi} \right) ;
$$

$$
F_{ff}^{\Theta}(\beta) = -\omega \left( e^{\beta_1/2} F_{fb}^{\chi} - e^{\beta_2/2} F_{bf}^{\chi} \right) ;
$$

$$
F_{bb}^{\chi}(\beta) = \omega \left( e^{-\beta_1/2} F_{fb}^{\chi} + e^{-\beta_2/2} F_{bf}^{\chi} \right) ;
$$

$$
F_{ff}^{\chi}(\beta) = -\omega \left( e^{-\beta_1/2} F_{fb}^{\chi} - e^{-\beta_2/2} F_{bf}^{\chi} \right) .
$$

(6.5)

It is therefore sufficient to compute the two-particle Form Factors of the operator $\Theta(z, \bar{z})$ for determining those of $\chi(z, \bar{z})$ and $\bar{\chi}(z, \bar{z})$. Let us discuss the functional equations satisfied by $F_{bb}^{\Theta}(\beta)$ and $F_{ff}^{\Theta}(\beta)$.

The first set of equations (called the unitarity equations) rules the monodromy properties of the matrix elements as dictated by the $S$-matrix amplitudes

$$
F_{bb}^{\Theta}(\beta) = S_{bb}^{bb}(\beta) F_{bb}^{\Theta}(-\beta) + S_{bb}^{ff}(\beta) F_{ff}^{\Theta}(-\beta) ;
$$

$$
F_{ff}^{\Theta}(\beta) = S_{ff}^{bb}(\beta) F_{bb}^{\Theta}(-\beta) + S_{ff}^{ff}(\beta) F_{ff}^{\Theta}(-\beta) ,
$$

(6.6)

where $S_{bb}^{bb}(\beta)$ is the scattering amplitude of two bosons into two bosons and similarly for the other amplitudes.

The second set of equations (called crossing equations) express the locality of the operator $\Theta(z, \bar{z})$

$$
F_{bb}^{\Theta}(\beta + 2\pi i) = F_{bb}^{\Theta}(-\beta) ;
$$

$$
F_{ff}^{\Theta}(\beta + 2\pi i) = F_{ff}^{\Theta}(-\beta) .
$$

(6.7)

They depend on the difference of rapidities $\beta = \beta_1 - \beta_2$ since $\Theta(0,0)$ is a scalar operator.
One may be inclined to solve the coupled monodromy-crossing equations by initially diagonalising the $S$-matrix \[1\]. However, this method does not work in this case for a series of reasons, both of mathematical and physical origin. To simplify the notation, let us use in the following

\[ a \equiv \sin \pi \alpha \; ; \]
\[ s \equiv \sinh \frac{\beta}{2} \; ; \]
\[ c \equiv \cosh \frac{\beta}{2} \; . \]

Observe that the eigenvalues of the $S$-matrix of the SSHG model in the $F = 1$ sector are given by

\[ \zeta_{\pm} = - \frac{ia}{sc} \pm \sqrt{1 + \left( \frac{a}{c} \right)^2} \; , \tag{6.8} \]

with the corresponding eigenvectors

\[ |\zeta_{+}\rangle = N_+ \begin{bmatrix} \frac{a}{c} | b(\beta_1) b(\beta_2) \rangle + \left( 1 - \sqrt{1 + \left( \frac{a}{c} \right)^2} \right) | f(\beta_1) f(\beta_2) \rangle \end{bmatrix} \; ; \tag{6.9} \]
\[ |\zeta_{-}\rangle = N_- \begin{bmatrix} \frac{a}{c} | b(\beta_1) b(\beta_2) \rangle + \left( 1 + \sqrt{1 + \left( \frac{a}{c} \right)^2} \right) | f(\beta_1) f(\beta_2) \rangle \end{bmatrix} \; , \tag{6.10} \]

and the normalization constants given by

\[ N_{\pm} = \frac{1}{\sqrt{2 \left( 1 + \left( \frac{a}{c} \right)^2 \right) \mp \sqrt{1 + \left( \frac{a}{c} \right)^2}}} \; . \]

From a mathematical point of view, the branch cuts present in the eigenvalues (6.8) make it impossible to find their exponential integral representations – a step which is usually rather crucial in obtaining the corresponding Form Factor \[1\]. A more serious aspect, however, is the fact that the eigenvectors (6.9) do not have any satisfactory properties under the crossing transformation $\beta \rightarrow \beta + 2\pi i$. From a physical point of view, the origin of all these troubles is the different scattering property of the channel $bb \rightarrow bb$ with respect to the channel $ff \rightarrow ff$, which does not permit, in this case, to assign a reasonable physical meaning to the states which diagonalise the $S$-matrix.

The determination of the Form Factors of the operator $\Theta(z, \bar{z})$ must pass through a different route. The way that we will proceed is to introduce two auxiliary functions $F_{\pm}(\beta)$ by considering the scattering theory in the $F = -1$ sector and to use them as building blocks for constructing the matrix elements $F_{bb}^{\Theta}(\beta)$ and $F_{ff}^{\Theta}(\beta)$.
6.1 Auxiliary problem: two-particle FF in the $F = -1$ sector

Let us look for the eigenvalues of the $S$-matrix in the $F = -1$ sector. They are given by

$$
\lambda_+ = \frac{s - ia}{s + ia} U(\beta) ; \\
\lambda_- = -U(\beta) ,
$$

with the relative eigenvectors given by

$$
| \lambda_+ \rangle = \frac{1}{\sqrt{2}} \left( | b(\beta_1)f(\beta_2) \rangle + f(\beta_1)b(\beta_2) \right) ; \\
| \lambda_- \rangle = \frac{1}{\sqrt{2}} \left( | f(\beta_1)b(\beta_2) \rangle - b(\beta_1)f(\beta_2) \right) .
$$

The states in this sector are necessarily coupled to operators with a non-zero fermionic quantum number. For the purpose of obtaining the auxiliary functions $F_{\pm}(\beta)$ which will be used as building blocks to construct $F_{bb}^{\Theta}(\beta)$ and $F_{ff}^{\Theta}(\beta)$, it is sufficient to consider the coupling of the eigenstates (6.12) to a fictitious scalar operator $\Lambda(0)$ but with a non-zero fermionic quantum number. Let us denote the corresponding matrix elements as

$$
F_+(\beta) \equiv \langle 0 | \Lambda(0) | \lambda_+(\beta_1,\beta_2) \rangle ; \\
F_-^{\pm}(\beta) \equiv \langle 0 | \Lambda(0) | \lambda_-(\beta_1,\beta_2) \rangle .
$$

Under the condition of unitarity, these matrix elements satisfy the equations

$$
F_+(\beta) = \frac{s - ia}{s + ia} U(\beta) F_+(\beta) ; \\
F_-(-\beta) = -U(\beta) F_-(-\beta) .
$$

The crossing properties of the above matrix elements is more subtle. Let us consider, in fact, what happens to $F_+(\beta)$ and to $F_-^{\pm}(\beta)$ in the analytic continuation $\beta \rightarrow \beta + 2\pi i$, i.e. when the first particle in each of the two states entering the eigenvectors (6.12) goes around the operator $\Lambda(0)$. Since this operator has a fermionic quantum number, if the particle which goes around the operator is also a fermion we get an extra phase ($-1$), otherwise nothing, and therefore

$$
\langle 0 | \Lambda(0) (| b(\beta_1)f(\beta_2) \rangle + | f(\beta_1)b(\beta_2) \rangle) \rightarrow -\langle 0 | \Lambda(0) (| b(\beta_2)f(\beta_1) \rangle - | f(\beta_2)b(\beta_1) \rangle) \\
\langle 0 | \Lambda(0) (| f(\beta_1)b(\beta_2) \rangle - | b(\beta_1)f(\beta_2) \rangle) \rightarrow -\langle 0 | \Lambda(0) (| b(\beta_2)f(\beta_1) \rangle + | f(\beta_2)b(\beta_1) \rangle)
$$

i.e. under crossing the two matrix elements mix each other,

$$
F_+(\beta + 2\pi i) = -F_-^{\pm}(-\beta) ; \\
F_-^{\pm}(\beta + 2\pi i) = -F_+(-\beta) .
$$
In order to solve the coupled system of crossing-unitarity equations (6.14) and (6.16), it is convenient to separate the problem into two steps. The first step consists in finding a function $G(\beta)$ which solves the monodromy equation involving only the function $U(\beta)$, with the usual crossing property, i.e.

$$G(\beta) = U(\beta)G(-\beta) ;$$

$$G(\beta + 2\pi i) = G(-\beta) ,$$

(6.17)

The explicit expression of the function $G(\beta)$ can be found in Appendix A.

The second step consists instead in finding two functions $f_+(\beta)$ and $f_-(-\beta)$ which solve the functional equations

$$f_+(\beta) = \frac{s - ia}{s + ia} f_+(-\beta) ;$$

$$f_-(-\beta) = -f_-(-\beta) ;$$

$$f_+(\beta + 2\pi i) = -f_-(\beta) ;$$

$$f_-(-\beta + 2\pi i) = -f_+(\beta) .$$

(6.18)

To this aim, let us write initially the integral representation of the eigenvalues $\lambda_+(\beta)$

$$\frac{s - ia}{s + ia} = \exp \left[ -4i \int_0^\infty \frac{dt \sinh \alpha t \sinh(1 - \alpha)t}{t} \frac{\sin \beta t}{\cosh t} \right] .$$

(6.19)

The minimal solutions of (6.18) are then given by

$$f_+(\beta) = \cosh \frac{\beta}{4} H_+(\beta) ;$$

$$f_-(-\beta) = i \sinh \frac{\beta}{4} H_-(-\beta) ,$$

(6.20)

where

$$H_+(\beta) = \exp \left[ 4 \int_0^\infty \frac{dt \sinh \alpha t \sinh(1 - \alpha)t}{t} \frac{\sin^2 \left( \frac{\beta - 2\pi i}{2\pi} \right)}{\cosh t \sinh 2t} \right] ;$$

$$H_-(-\beta) = \exp \left[ 4 \int_0^\infty \frac{dt \sinh \alpha t \sinh(1 - \alpha)t}{t} \frac{\sin^2 \left( \frac{\beta}{2\pi} \right)}{\cosh t \sinh 2t} \right] .$$

(6.21)

The function $H_-(-\beta)$ admits the equivalent representation

$$H_-(-\beta) = \prod_{k=0}^{N-1} \left| \frac{\Gamma \left( k + \frac{3}{2} + \frac{\beta}{4\pi} \right) \Gamma \left( k + 1 - \frac{\beta}{2} + \frac{i\beta}{4\pi} \right) \Gamma \left( k + \frac{1}{2} + \frac{\beta}{2} + \frac{i\beta}{4\pi} \right) \Gamma \left( k + 1 + \frac{\beta}{2} + \frac{i\beta}{4\pi} \right) \Gamma \left( k + \frac{3}{2} - \frac{\beta}{2} + \frac{i\beta}{4\pi} \right) \Gamma \left( k + 1 + \frac{\beta}{2} + \frac{i\beta}{4\pi} \right) }{\Gamma \left( k + \frac{1}{2} + \frac{\beta}{4\pi} \right) \Gamma \left( k + 1 - \frac{\beta}{2} + \frac{i\beta}{4\pi} \right) \Gamma \left( k + \frac{1}{2} + \frac{\beta}{2} + \frac{i\beta}{4\pi} \right) \Gamma \left( k + 1 + \frac{\beta}{2} + \frac{i\beta}{4\pi} \right) } \right|^2$$

(6.22)

or the mixed one, which is more useful for numerical calculation

$$H_-(-\beta) = \prod_{k=0}^{N-1} \left[ \left( 1 + \frac{\beta^2}{\pi(4k+2)} \right)^2 \left( 1 + \frac{\beta^2}{\pi(4k+4+2\alpha)} \right)^2 \left( 1 + \frac{\beta^2}{\pi(4k+6+2\alpha)} \right)^2 \right]^{k+1} \times$$

$$\times \exp \left[ 4 \int_0^\infty \frac{dt \sinh \alpha t \sinh(1 - \alpha)t}{t} \frac{\sin^2 \beta}{\cosh t \sinh 2t} \right] \left( N + 1 - N e^{-4\alpha t} \right) e^{-4\alpha t} \sin^2 \frac{\beta}{2\pi} t \right] .$$

(6.23)
Since the functions $H_\pm(\beta)$ are related each other by the equation $H_+(\beta + 2\pi i) = H_-(\beta)$, the corresponding infinite product or mixed representations of $H_+(\beta)$ follow easily. For large values of $\beta$, $H_+(\beta)$ and $H_-(\beta)$ go the same constant, given by

$$\lim_{\beta \to \infty} H_\pm(\beta) = \exp \left[ 2 \int_0^\infty \frac{dt \sinh \alpha t \sinh(1 - \alpha) t}{\cosh t \sinh 2t} \right].$$

(6.24)

So, summarising the results of this section, the final expressions of the functions $F_+(\beta)$ and $F_-(\beta)$ are given by

$$F_+(\beta) = G(\beta) f_+(\beta) ;$$

$$F_-(\beta) = G(\beta) f_-(\beta) ,$$

(6.25)

with $G(\beta)$ given in eq.(A.2) of Appendix A and $f_\pm(\beta)$ given in eq. (6.20).

### 6.2 Two-particle FF of the operator $\Theta$

Let us look for the two-particle Form Factors of the trace $\Theta$ of the stress-energy tensor as linear combination of the two functions $F_+(\beta)$ and $F_-(\beta)$ above determined, i.e.

$$F_{bb}(\beta) = A(\beta) F_+(\beta) + B(\beta) F_-(\beta) ;$$

$$F_{ff}(\beta) = C(\beta) F_+(\beta) + D(\beta) F_-(\beta) .$$

(6.26)

Let us now plug them into eq.(6.6). By using the monodromy equations satisfied by the $F_\pm(\beta)$ and by comparing the terms in front of each of these functions, we obtain the following equations

$$A(\beta) = \frac{s}{s - ia} \left[ \left( 1 - \frac{ia}{sc} \right) A(-\beta) - \frac{a}{c} C(-\beta) \right] ;$$

$$B(\beta) = -\frac{s}{s + ia} \left[ \left( 1 - \frac{ia}{sc} \right) B(-\beta) - \frac{a}{c} D(-\beta) \right] ;$$

$$C(\beta) = -\frac{s}{s - ia} \left[ \left( 1 + \frac{ia}{sc} \right) C(-\beta) + \frac{a}{c} A(-\beta) \right] ;$$

$$D(\beta) = \frac{s}{s + ia} \left[ \left( 1 + \frac{ia}{sc} \right) D(-\beta) + \frac{a}{c} B(-\beta) \right] .$$

(6.27)

By taking the free limit $\alpha \to 0$ in the above equations, it is easy to see that

$$A(\beta) = A(-\beta) ;$$

$$B(\beta) = -B(-\beta) ;$$

$$C(\beta) = -C(-\beta) ;$$

$$D(\beta) = D(-\beta) .$$

(6.28)

Let us assume that the above parity properties of the functions $A, \ldots, D$ are still valid for a non-zero value of the coupling constant $\alpha$ (after all, the coupling constant dependence
of the matrix elements $F_{bb}$ and $F_{ff}$ should be already included into the functions $F\pm(\beta)$. Under this hypothesis, eqs. (6.27) provide the relationships

\begin{align*}
\mathcal{C}(\beta) &= -i \frac{c-1}{s} A(\beta) ; \\
\mathcal{D}(\beta) &= i \frac{c+1}{s} B(\beta) .
\end{align*}

Hence, at this stage we have for the two-particle Form Factors of $\Theta(x)$

\begin{align*}
F_{bb}(\beta) &= A F_+ + B F_- , \\
F_{ff}(\beta) &= -i \left[ \frac{c-1}{s} A F_+ - \frac{c+1}{s} B F_- \right] .
\end{align*}

In order to determine the two remaining functions $A(\beta)$ and $B(\beta)$, let us consider once again the case $\alpha \to 0$ and let us impose the condition that in this limit the two-particle Form Factors reduce to their free limit

\begin{align*}
F_{bb}(\beta) &= 2 \pi m^2 ; \\
F_{ff}(\beta) &= -2 \pi i \sinh \frac{\beta}{2} .
\end{align*}

From this matching, the functions $A$ and $B$ are uniquely determined to be

\begin{align*}
A(\beta) &= 2 \pi \cosh \frac{\beta}{4} , \\
B(\beta) &= 2 \pi i \sinh \frac{\beta}{4} .
\end{align*}

For a generic value of the coupling constant, the correct expressions of the two-particle Form Factors of the operator $\Theta(0)$ can be obtained by imposing their normalization $F_{bb}(i\pi) = F_{ff}(i\pi) = 2\pi m^2$ and their final form are given by

\begin{align*}
F_{bb}^{\Theta}(\beta) &= 2\pi m^2 \frac{\tilde{F}_{bb}(\beta)}{F_{bb}(i\pi)} ; \\
F_{ff}^{\Theta}(\beta) &= 2\pi m^2 \frac{\tilde{F}_{ff}(\beta)}{F_{ff}(i\pi)} ,
\end{align*}

where

\begin{align*}
\tilde{F}_{bb}(\beta) &= \left[ \cosh^2 \frac{\beta}{4} H_+ (\beta) - \sinh^2 \frac{\beta}{4} H_- (\beta) \right] G(\beta) , \\
\tilde{F}_{ff}(\beta) &= \sinh \frac{\beta}{2} \left[ H_+ (\beta) + H_- (\beta) \right] G(\beta) .
\end{align*}

Notice that for large values of $\beta$, $F_{bb}^{\Theta}(\beta)$ tends to a constant whereas $F_{ff}^{\Theta}(\beta) \approx e^{\beta/2}$, both behaviour in agreement with Weinberg’s power counting theorem of the Feynman diagrams.
While we postpone non trivial checks of the validity of (6.33) to the next sections, let us use the Form Factors (6.33) to estimate the correlation function \( C(r) = \langle \Theta(r)\Theta(0) \rangle \) by means of formulas (6.1) and (6.2). In the free limit, the correlator is simply expressed in terms of Bessel functions,
\[
C(r) = m^4 \left( K_1^2(mr) + K_0^2(mr) \right).
\]

(6.35)

For a finite value of \( \alpha \), a numerical integration of (6.1) produces the graphs shown in Figure 4. As it was expected, in the ultraviolet limit the curve relative to a finite value of \( \alpha \) is steeper than the curve relative to the free case whereas it decreases slower at large values of \( mr \). This curve is expected to correctly capture the long distance behaviour of the correlator and to provide a reasonable estimate of their short distance singularity. However, for the exact estimation of the power law singularity at the origin one would of course need the knowledge of all higher particle Form Factors.

7 \textit{C-theorem Sum Rule}

As mentioned in Section 4.1, the SSHG model can be seen as a massive deformation of the superconformal model with the central charge \( C = \frac{3}{2} \). While this fixed point rules the ultraviolet properties of the model, its large distance behaviour is controlled by a purely massive theory with \( C = 0 \). The variation of the central charge in this RG flow is dictated by the \textit{C-theorem} of Zamolodchikov [30], which we will discuss in more details in the next section in relation with its fermionic formulation. In this section, we are concerning with the integral version of the \text{C-theorem} in order to have non trivial checks of the validity of the Form Factors (6.33). In this formulation of the \textit{c-theorem}, the variation of the central charge \( \Delta C \) satisfies the sum rule
\[
\Delta C = \frac{3}{4\pi} \int d^2 x \ | x |^2 \ \langle 0 | \Theta(x)\Theta(0) | 0 \rangle_{\text{conn}} = \int_0^\infty d\mu \ c(\mu),
\]

(7.1)

where \( c(\mu) \) is given by
\[
c(\mu) = \frac{6}{\pi^2} \frac{1}{\mu^3} \Im G(p^2 = -\mu^2),
\]
\[
G(p^2) = \int d^2 x e^{-ipx} \langle 0 | \Theta(x)\Theta(0) | 0 \rangle_{\text{conn}}.
\]

(7.2)

Inserting a complete set of in-state into (7.2), the spectral function \( c(\mu) \) can be expressed as a sum on the FF’s
\[
c(\mu) = \frac{12}{\mu^3} \sum_{n=1}^\infty \int \frac{d\beta_1}{2\pi} \cdots \frac{d\beta_n}{2\pi} \delta(\mu - \sum_i m \cosh \beta_i) \delta(\sum_i m \sinh \beta_i) \times
\]
\[
| \langle 0 | \Theta(0,0) | A_1(\beta_1) \cdots A_n(\beta_n) \rangle |^2
\]

(7.3)
Since the term $|x|^2$ present in (7.1) suppresses the ultraviolet singularity of the two-point correlator of $\Theta$, the sum rule (7.1) is expected to be saturated by the first terms of the series (7.3). For the SShG model the first approximation to the sum rule (7.1) is given by the contributions of the two-particle states

$$\Delta C^{(2)} = \frac{3}{8\pi^2 m^4} \int_0^\infty \frac{d\beta}{\cosh^2 \beta} \left[ | F_{\Theta}^{bb}(2\beta) |^2 + | F_{\Theta}^{ff}(2\beta) |^2 \right] .$$

(7.4)

The numerical data relative to the above integral for different values of the coupling constant $\alpha$ is reported in Table 1. They are remarkably close to the theoretical value $\Delta C = \frac{3}{2}$, even for the largest possible value of the coupling constant, which is the self-dual point $\lambda = \sqrt{4\pi}$. In addition to this satisfactory check, a more interesting result is obtained by analysing the application of the $c$-theorem sum rule to models which are obtained as analytic continuation of the SShG.

### 7.1 C-theorem Sum Rule for the Roaming Model

As we have seen in Section 5.2, by taking the analytic continuation

$$\alpha \to \frac{1}{2} + i\alpha_0$$

(7.5)

and the limit $\alpha_0 \to \infty$, the $S$-matrix of the SShG model is mapped into the $S$-matrix of the ordinary Sine-Gordon model at $\xi = 2\pi$. This identity between the two $S$-matrices is expected to extend to other properties of the two theories as well. There are however some subtleties in this mapping.

One subtlety is that the right hand side of the sum rule (7.1) is a pure number ($\Delta C = \frac{3}{2}$ for the SShG model) – a quantity which therefore seems to be completely insensitive to the variation of the coupling constant of the model. On the other hand, if the SShG model is mapped onto the Sine-Gordon model in the limit $\alpha_0 \to \infty$ of the analytic continuation (7.5), the sum rule should instead jump discontinuously from the value $\Delta C = \frac{3}{2}$ to $\Delta C = 1$, simply because the latter model is a massive deformation of the conformal theory with $C = 1$. This is indeed what happens\footnote{We will not discuss further this statement since a detailed discussion of this aspect can be found in \cite{8}, where the analogous situation of the roaming limit mapping between the Sinh-Gordon model and the Ising model is analysed.}, the reason being the presence of an additional energy scale brought by the parameter $\alpha_0$ and the non-uniform convergence of the spectral series (7.3).

In the case of the SShG model, there is an additional point that however deserves to be carefully checked. Note that in the SShG model, the lowest approximation of the sum rule, eq. (7.4), is expressed as sum of the moduli square of both $F_{\Theta}^{bb}$ and $F_{\Theta}^{ff}$. In the
roaming limit, the original states $|bb⟩$ and $|ff⟩$ of the Sinh-Gordon model are mapped respectively onto the states $|SS⟩$ and $|SS⟩$ of the Sine-Gordon model. On the other hand, the trace $Θ$ of the stress-energy tensor of the Sine-Gordon model only couples to the symmetric combination of the soliton states

$$|+⟩ = \frac{1}{\sqrt{2}} \left( |SS⟩ + |SS⟩ \right) ,$$

and not to each of them individually! How this apparent discrepancy can be settled?

First let us consider how the two-particle Form Factor of the operator $Θ$ is computed within the Sine-Gordon theory for a generic value of $ξ$. The symmetry responsible for the coupling of the operator $Θ$ to the combination (7.6) is the invariance of $Θ$ under the charge conjugation. Hence, to compute the two-particle Form Factor

$$F_{SG}^{+}(β) = ⟨0 | Θ(0) | +⟩ ,$$

one needs to employ the corresponding eigenvalues of the $S$-matrix (3.8) given by

$$λ_{SG}^{±}(β) = \frac{\sinh \frac{π}{2ξ}(β + iπ)}{\sinh \frac{π}{2ξ}(β - iπ)} \hat{R}(β) ,$$

and solve the unitarity and crossing equations

$$F_{SG}^{+}(β) = λ_{+}(β) F_{SG}^{+}(-β) ;$$
$$F_{SG}^{+}(β + 2πi) = F_{SG}^{+}(-β) .$$

The solution of these equation gives us the two-particle Form Factor of $Θ$ of the Sine-Gordon model

$$F_{SG}^{+}(β) = -\sqrt{2}π^{2}m^{2} \xi \hat{G}(iπ,ξ) \left[ \frac{\sinh β}{\sinh \frac{π}{2ξ}(β - iπ)} \right] \hat{G}(β,ξ) ,$$

with

$$\hat{G}(β,ξ) = \exp \left[ \int_{0}^{∞} dt \frac{\sinh \frac{t}{2} \left( 1 - \frac{ξ}{π} \right)}{t \cosh \frac{t}{2} \sinh \frac{ξt}{2π} \sinh t} \sin^{2} \left( iπ - β \right) t \right] .$$

Observe that for $ξ = 2π$, the above expression (7.10) can be written equivalently as

$$F_{SG}^{+}(β) = -\sqrt{2}π^{2}m^{2} \sinh β \left[ \sinh \frac{β}{4} + i \cosh \frac{β}{4} \right] \hat{G}(β,2π) .$$

Once inserted into the sum rule (7.1), the result is $ΔC^{(2)} = 0.9924...$, i.e. a saturation within few percent of the exact value $ΔC = 1$ relative to this case.

Let us consider now the corresponding FF of the SSHG model in the roaming limit, eq. (6.25). For $α_{0} → ∞$, the function $G(β)$ reduces precisely to $\hat{G}(β,2π)$ whereas for
\( H_+ (\beta) \) we have

\[
H_+ (\beta) = \exp \left[ 2 \int_0^\infty \frac{dt \sin^2 \left( \frac{\beta - 2\pi i}{2\pi} t \right)}{t \sinh 2t} \right]; \quad (7.13)
\]

\[
H_- (\beta) = \exp \left[ 2 \int_0^\infty \frac{dt \sin^2 \beta t}{t \sinh 2t} \right].
\]

By using the formula

\[
\int_0^\infty \frac{dt}{t} e^{-pt} \sin^2 \frac{at}{2} = \frac{1}{4} \ln \left[ 1 + \left( \frac{a}{p} \right)^2 \right],
\]

and the infinite product representation

\[
\cosh x = \prod_{k=0}^{\infty} \left( 1 + \frac{4x^2}{(2k+1)^2\pi^2} \right),
\]

it is easy to see that they can be simply expressed as

\[
H_+ (\beta) = -i \sinh \frac{\beta}{4}, \quad H_- (\beta) = \cosh \frac{\beta}{4}. \quad (7.14)
\]

Hence, in the roaming limit the two matrix elements \( F_{bb}^\Theta (\beta) \) and \( F_{ff}^\Theta (\beta) \) become equal, with their common value given by

\[
F_{bb}^\Theta (\beta) = F_{ff}^\Theta (\beta) = \frac{\sqrt{2\pi m^2}}{G(i\pi, 2\pi)} \sinh \frac{\beta}{2} \left[ \sinh \frac{\beta}{4} + i \cosh \frac{\beta}{4} \right] \hat{G}(\beta, 2\pi), \quad (7.15)
\]

which coincides with the one of the Sine-Gordon model, eq. (7.12). Therefore their contribution to the \( c \)-theorem sum rule is precisely the same as the Sine-Gordon model (in the case of the SSHG model, eq. (7.4), there is in fact a factor \( \frac{1}{2} \) with respect to the Sine-Gordon model, which is cancelled by the equality of the two Form Factors \( F_{bb}^\Theta \) and \( F_{ff}^\Theta \)).

### 7.2 \( C \)-theorem Sum Rule for Schoutens’s Model

In the analytic continuation \( \alpha \to -\alpha \), the \( S \)-matrix develops a pole in the physical strip located at \( \beta = 2\pi \alpha i \). As discussed in Section 5.1, the scattering theory for \( \alpha = \frac{1}{3} \) admits a consistent interpretation in terms of a multiplet made of a boson and a fermion, which are bound states of themselves. For this model, the operator \( \Theta(z, \bar{z}) \) has also a one-particle Form Factor \( F_b^\Theta \) in virtue of the graph of Figure 5.

In order to discuss the analytic continuation of the Form Factors \( F_{bb}^\Theta (\beta) \) and \( F_{ff}^\Theta (\beta) \) for \( \alpha \to -\alpha \) and to calculate the matrix element \( F_b^\Theta \), it is convenient to consider initially
a different representation of the functions \( f_\pm (\beta) \). This is given by

\[
f_+ (\beta) = \sinh \frac{\beta}{2} Z_+ (\beta) ;
\]
\[
f_- (\beta) = - \sinh \frac{\beta}{2} Z_- (\beta) ,
\]

with

\[
Z_+ (\beta) = \exp \left[ -2 \int_0^\infty \frac{dt}{t} \cosh (1 - 2\alpha) t \sin^2 \left( \frac{\beta - 2\pi i}{2\pi} \right) t \right] ; \quad (7.16)
\]
\[
Z_- (\beta) = \exp \left[ -2 \int_0^\infty \frac{dt}{t} \cosh (1 - 2\alpha) t \sin^2 \left( \frac{\beta}{2\pi} t \right) \right] .
\]

Their equivalence to the previous ones, eq. (6.20) is easily established by comparing their infinite product representations which for the function \( Z_- (\beta) \) is given by

\[
Z_- (\beta) = \prod_{k=0}^\infty \left| \frac{\Gamma \left( k + \frac{1}{2} + \frac{\alpha}{2} + i\frac{\beta}{4\pi} \right)}{\Gamma \left( k + \frac{3}{2} - \frac{\alpha}{2} + i\frac{\beta}{4\pi} \right) \Gamma \left( k + 1 + \frac{\alpha}{2} + i\frac{\beta}{4\pi} \right)} \right|^2 , \quad (7.17)
\]

while the one of \( Z_+ (\beta) \) is simply given by the identity \( Z_+ (\beta + 2\pi i) = Z_- (\beta) \). By using eq. (7.18) and the analogous for \( Z_+ (\beta) \), it is simply to see that under the analytic continuation \( \alpha \to -\alpha \), the functions \( f_\pm (\beta) \) change as follows

\[
f_+ (\beta) \to \hat{f}_+ (\beta) = - \frac{2\pi^2}{\cosh \frac{\beta}{2} - \cos \pi \alpha} Z_-^{-1} (\beta) ; \quad (7.19)
\]
\[
f_- (\beta) \to \hat{f}_- (\beta) = \frac{2\pi^2}{\cosh \frac{\beta}{2} + \cos \pi \alpha} Z_-^{-1} (\beta) .
\]

The analytic continuation of the two functions present therefore distinct physical properties: in fact, while the function \( \hat{f}_+ (\beta) \) develops a pole relative to the bound state, the other \( \hat{f}_- (\beta) \) does not have any singularity in the physical strip.

To compute the two-particle FF of \( \Theta \) is not sufficient, however, to substitute the above functions \( \hat{f}_\pm (\beta) \) into eq. (6.33). In fact, we must also take into account the extra terms present in \( \hat{U} (\beta) \), eq. (5.9). This is a simple task, though, because we only have to introduce the function \( K (\beta) \equiv K_1 (\beta) K_2 (\beta) \), where \( K_1 (\beta) \) and \( K_2 (\beta) \) are solutions of the unitarity and crossing equations

\[
K_1 (\beta) = \left( \frac{\sinh \frac{\beta}{2} - i \sin \pi \alpha}{\sinh \frac{\beta}{2} + i \sin \pi \alpha} \right) K_1 (-\beta) ;
\]
\[
K_1 (\beta + 2\pi i) = K_1 (-\beta) ; \quad (7.20)
\]
\[
K_2 (\beta) = \left( \frac{\sin \pi \alpha + i \sin (2\pi \alpha)}{\sin \pi \alpha - i \sin (2\pi \alpha)} \right) K_2 (-\beta) ;
\]
\[
K_1 (\beta + 2\pi i) = K_2 (-\beta) .
\]
Their mixed representation expressions are given by

\[
K_1(\beta) = \prod_{k=0}^{N-1} \left[ \left( 1 + \left( \frac{i \pi - \beta}{4 \pi (k + \frac{3}{4} + \frac{\alpha}{2})} \right)^2 \right) \left( 1 + \left( \frac{i \pi - \beta}{4 \pi (k + \frac{3}{4} - \frac{\alpha}{2})} \right)^2 \right) \right]^{-1} \times \\
\times \exp \left[ -4 \int_0^\infty \frac{dt}{t} \cosh(1 - 2\alpha)t \sinh^2 \left( \frac{i \pi - \beta}{2\pi} \right) t \right] , \quad (7.21)
\]

and

\[
K_2(\beta) = \prod_{k=0}^{N-1} \left[ \left( 1 + \left( \frac{i \pi - \beta}{4 \pi (k + \frac{3}{4} + \frac{\alpha}{2})} \right)^2 \right) \left( 1 + \left( \frac{i \pi - \beta}{4 \pi (k + \frac{3}{4} - \frac{\alpha}{2})} \right)^2 \right) \right]^{k+1} \times \\
\times \exp \left[ 2 \int_0^\infty \frac{dt}{t} \cosh(1 - 2\alpha)t \sinh \left( \frac{1 - 2\alpha}{2} \right) t (N + 1 - Ne^{-2\alpha}) \right] e^{-2\alpha t} \sin^2 \left( \frac{i \pi - \beta}{2\pi} \right) t \right] , \quad (7.22)
\]

So, finally the two-particle FF of the operator \( \Theta(z, \bar{z}) \) of the Schoutens's model are given by

\[
F_{bb}^\Theta(\beta) = 2\pi m^2 \frac{\tilde{F}_{bb}(\beta)}{F_{bb}(i\pi)} ; \\
F_{ff}^\Theta(\beta) = 2\pi m^2 \frac{\tilde{F}_{ff}(\beta)}{F_{ff}(i\pi)} ,
\]

where

\[
\tilde{F}_{bb}(\beta) = \sinh \frac{\beta}{2} \left[ \cosh \frac{\beta}{4} \left( \frac{Z_+^{-1}(\beta)}{(\cosh \frac{\beta}{2} - \cos \pi \alpha)} + i \sinh \frac{\beta}{4} \left( \frac{Z_+^{-1}(\beta)}{(\cosh \frac{\beta}{2} + \cos \pi \alpha)} \right) \right) \right] G(\beta)K(\beta) ; \\
\tilde{F}_{ff}(\beta) = \sinh \frac{\beta}{2} \left[ \sinh \frac{\beta}{4} \left( \frac{Z_+^{-1}(\beta)}{(\cosh \frac{\beta}{2} - \cos \pi \alpha)} - i \cosh \frac{\beta}{4} \left( \frac{Z_+^{-1}(\beta)}{(\cosh \frac{\beta}{2} + \cos \pi \alpha)} \right) \right) \right] G(\beta)K(\beta) .
\]

The one-particle Form Factor \( F_b^\Theta \) can now be equivalently obtained either from the residue equation on \( F_{bb}^\Theta(\beta) \) or on \( F_{ff} \)

\[
-i \lim_{\beta \to \frac{2\pi i}{3}} \left( \beta - \frac{2\pi i}{3} \right) F_{bb}^\Theta(\beta) = \Gamma_{bb}^b F_b^\Theta ; \\
-i \lim_{\beta \to \frac{2\pi i}{3}} \left( \beta - \frac{2\pi i}{3} \right) F_{ff}^\Theta(\beta) = \Gamma_{ff}^b F_b^\Theta ,
\]

with the result

\[
F_b^\Theta = -i \frac{\pi}{\sqrt{2 \sqrt{3}}} \frac{Z_+^{(i\pi)}(\frac{2\pi i}{3})}{Z_+^{(2\pi i/3)}} G \left( \frac{2\pi i}{3} \right) K \left( \frac{2\pi i}{3} \right) = -1.6719(3) i \quad (7.25)
\]

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The series of the sum rule has alternating sign, with the first contribution given by the one-particle Form Factor

$$\Delta C^{(1)} = \frac{6}{\pi} (F_b^\Theta)^2 = -5.3387(4) \quad (7.26)$$

This quantity differs for a 1.6% from the theoretical value $\Delta C = -\frac{21}{4} = -5.25$. By also including the positive contribution of the two-particle FF, computed numerically

$$\Delta C^{(2)} = 0.09050(8) \quad , \quad (7.27)$$

the estimate of the central charge of the model further improves, $C = -5.2482(4)$, with a difference from the exact value of just 0.033%.

The Form Factors above determined for the trace operator $\Theta(x)$ can be used to estimate its two-point correlation function. The graph of this function is shown in Figure 6: note that this function diverges at the origin, in agreement with the positive conformal dimension $\Delta = 1/4$ of this operator, but it presents a non–monotonous behaviour for the alternating sign of its spectral series.

8 Fermionic Formulation of the $C$-theorem

In the OPE (4.9) of the fields $T(z)$ and $G(z)$ which are responsible for a superconformal symmetry, the central charge $C$ appears both in the short-distance singularity of $T(z)T(w)$ and $G(z)G(w)$. Therefore, going away from criticality by means of operators which preserve the supersymmetry of the critical point (but non necessarily its integrability), it should be possible to formulate a $C$-theorem for unitarity theories by looking at the fermionic sector. This is indeed the case, as proved in this section.

Let us consider the correlation functions of the operators $G(z, \bar{z})$ and $\chi(z, \bar{z})$ away from criticality. By taking into account the scaling dimensions of the operators, these correlators can be parameterised as

$$\langle G(z, \bar{z})G(0, 0) \rangle = \frac{H(z\bar{z})}{z^3} ;$$
$$\langle G(z, \bar{z})\chi(0, 0) \rangle = \frac{I(z\bar{z})}{z^2\bar{z}} ;$$
$$\langle \chi(z, \bar{z})\chi(0, 0) \rangle = \frac{L(z\bar{z})}{z\bar{z}^2} , \quad (8.1)$$

where the functions $H(z\bar{z})$, $I(z\bar{z})$ and $L(z\bar{z})$ depend on the scalar variable $x \equiv z\bar{z}$. In virtue of the conservation law

$$\partial_z G(z, \bar{z}) = \partial_{\bar{z}} \chi(z, \bar{z}) , \quad (8.2)$$
the above functions satisfy the following differential equations

\[
\dot{H} - \dot{I} = -2I; \quad \dot{I} - \dot{L} = I - L, \tag{8.3}
\]

where \(\dot{H} \equiv z\bar{z}\frac{dH}{dz}\bar{z}\) and similarly for the other functions. For the combination \(F = H + I - L\) we have therefore

\[
\dot{F} = -2L. \tag{8.4}
\]

At the critical points, the quantity \(F\) is proportional to the central charge \(C\), since

\[
H = \frac{2}{3} C; \quad I = L = 0. \tag{8.5}
\]

In order to conclude that the function \(F\) decreases along the Renormalization Group flow and to derive the corresponding sum-rule, it is necessary to analyse the positivity of the function \(L(z\bar{z})\), which is given by

\[
L(z\bar{z}) = (z\bar{z})\bar{z}\langle\chi(z, \bar{z})\chi(0, 0)\rangle. \tag{8.6}
\]

Although it seems a-priori difficult to prove the positivity of the above function for the explicit presence of the complex term \(\bar{z}\), it will become evident that this term is precisely required for the proof.

Let us start our analysis from the constraints provided by supersymmetry. By using eq. (4.19), we have

\[
\langle\Theta(z, \bar{z})\Theta(0, 0)\rangle = \langle\{\chi(z, \bar{z}), Q\}\{\chi(0, 0), Q\}\rangle = \langle\chi(z, \bar{z})Q^2\chi(0, 0)\rangle = \langle\chi(z, \bar{z})P_+\chi(0, 0)\rangle. \tag{8.7}
\]

The above equations implies that the spectral function of the correlator \(\langle\Theta(z, \bar{z})\Theta(0, 0)\rangle\) can be obtained from the one of \(\langle\chi(z, \bar{z})\chi(0, 0)\rangle\) by multiplying each term of the multi-particle expansion for the right momentum of the intermediate state. This is a positive quantity, since for a cluster of \(n\)-particles is given by

\[
P_+ = m \left(e^{\beta_1} + e^{\beta_2} + \cdots e^{\beta_n}\right). \tag{8.8}
\]

The positivity of the spectral series of \(\langle\Theta(z, \bar{z})\Theta(0, 0)\rangle\) for unitarity theories implies therefore the positivity of the spectral series of \(\langle\chi(z, \bar{z})\chi(0, 0)\rangle\) as well.

Let us consider now the behaviour of the correlator \(\langle\chi(z, \bar{z})\chi(0, 0)\rangle\) as a function of the variables \(z\) and \(\bar{z}\). Since (8.7) can be written equivalently as

\[
\langle\Theta(z, \bar{z})\Theta(0, 0)\rangle = \langle\chi(z, \bar{z})\frac{M^2}{P_-}\chi(0, 0)\rangle, \tag{8.9}
\]
where $M^2$ is the invariant mass square of the cluster of the intermediate particles, this implies that $\langle \chi(z, \bar{z})\chi(0, 0) \rangle$ can be expressed as
\[
\langle \chi(z, \bar{z})\chi(0, 0) \rangle = \int \frac{d^2p}{2\pi} p_- \hat{\chi}(p^2)e^{-\frac{i}{2}(p_+ z + \bar{p}_- \bar{z})} = \frac{-2}{\pi} \frac{\partial}{\partial \bar{z}} \int \frac{d^2p}{2\pi} \hat{\chi}(p^2)e^{-\frac{i}{2}(p_+ z + \bar{p}_- \bar{z})} .
\]

The spectral density $\hat{\chi}(p^2)$ is given in terms of the matrix elements of the operator $\chi$ once we have factorised the left momentum of the intermediate states. According to the above discussion this is a positive quantity which depends only on the invariant mass square of the cluster of the intermediate particles. By using the identity
\[
\hat{\chi}(p^2) = \int_0^\infty da \left( p^2 - a^2 \right) \hat{\chi}(a^2) ,
\]
eq. (8.10) can be written then as
\[
\langle \chi(z, \bar{z})\chi(0, 0) \rangle = \frac{-1}{\pi} \frac{\partial}{\partial \bar{z}} \int_0^\infty da \hat{\chi}(a^2) K_0(a \sqrt{z \bar{z}}) .
\]
(8.11)

Since
\[
\frac{dK_0}{dx} = -K_1(x) ,
\]
by taking the derivative of the above expression we have
\[
\langle \chi(z, \bar{z})\chi(0, 0) \rangle = \frac{1}{\pi} \sqrt{\frac{z}{\bar{z}}} \int_0^\infty da a \hat{\chi}(a^2) K_1(a \sqrt{z \bar{z}}) .
\]
and therefore for the function $L$ we have
\[
L(z\bar{z}) = \frac{(z\bar{z})^{3/2}}{2\pi} \int_0^\infty da a \hat{\chi}(a^2) K_1(a \sqrt{z \bar{z}}) .
\]
(8.13)

Eq. (8.13) explicitly manifests the positivity of this function, Q.E.D.

The relative sum rule is easily obtained. In fact, since at the fixed points $H = \frac{2}{3}C$, we have
\[
C_{\text{in}} - C_{\text{fin}} = \frac{3}{2} \int d(z\bar{z}) \bar{z} \langle \chi(z, \bar{z})\chi(0, 0) \rangle = \frac{3}{2\pi} \int_0^\infty da a \hat{\chi}(a^2) \int_0^\infty dR R^2 K_1(a R) .
\]
(8.14)

Since
\[
\int dR R^2 K_1(a R) = \frac{2}{a^3} ,
\]
eq. (8.14) can be written as
\[
\Delta C = \frac{3}{\pi} \int_0^\infty \frac{da^2}{a^2} \hat{\chi}(a^2) .
\]
(8.15)
Let us consider now the two-particle contributions of the spectral function $\hat{\chi}(a^2)$. Since

$$F_{bf}^\chi(\beta_1, \beta_2) = \frac{1}{2 \cosh \frac{\beta}{2}} \left( \omega e^{-\beta_1/2} F_{bb}^\Theta(\beta) + \omega e^{-\beta_2/2} F_{ff}^\Theta(\beta) \right); \quad (8.16)$$

$$F_{fb}^\chi(\beta_1, \beta_2) = \frac{1}{2 \cosh \frac{\beta}{2}} \left( \omega e^{-\beta_2/2} F_{bb}^\Theta(\beta) - \omega e^{-\beta_1/2} F_{ff}^\Theta(\beta) \right),$$

we have

$$F_{bf}^\chi F_{bf}^\chi + F_{fb}^\chi F_{fb}^\chi = \left( e^{-\beta_1/2} + e^{-\beta_2/2} \right) \frac{|F_{bb}^\Theta|^2 + |F_{ff}^\Theta|}{4 \cosh^2 \frac{\beta}{2}}. \quad (8.17)$$

To obtain the relative contribution for $\hat{\chi}(a^2)$ we have just to disregard the factor

$$\left( e^{-\beta_1/2} + e^{-\beta_2/2} \right)$$

relative to the left component of the momentum, so that

$$\hat{\chi}(a^2) = 2\pi \int_{\beta_1 > \beta_2} \frac{d\beta_1}{2\pi} \frac{d\beta_2}{2\pi} \delta(a - m\cosh \beta_1 - m\cosh \beta_2) \delta(m\sinh \beta_1 + m\sinh \beta_2) \times \frac{|F_{bb}^\Theta(\beta_1 - \beta_2)|^2 + |F_{ff}^\Theta(\beta_1 - \beta_2)|^2}{4 \cosh^2 \frac{\beta_1 - \beta_2}{2}} = \quad (8.18)$$

$$= \frac{1}{16\pi} \int_{-\infty}^{\infty} \frac{d\beta}{\cosh^4 \beta} \delta(a - 2m\cosh \beta) \left[ |F_{bb}^\Theta(2\beta)|^2 + |F_{ff}^\Theta(2\beta)|^2 \right]$$

Inserting this expression into eq. (8.13), we finally have

$$\Delta C^{(2)} = \frac{3}{8\pi^2 m^4} \int_{0}^{\infty} \frac{d\beta}{\cosh^4 \beta} \left[ |F_{bb}^\Theta(2\beta)|^2 + |F_{ff}^\Theta(2\beta)|^2 \right]. \quad (8.19)$$

i.e. the sum rule relative to the fermionic case employs the same integrand as the bosonic case and therefore provides the same results as before.

9 Summary and Conclusions

In this paper we have analysed several aspects of supersymmetric models from the point of view of the $S$-matrix approach. The simplest supersymmetric scattering theory discussed in Section 2 has a natural interpretation and identification in terms of the first model described by the Roaming Series. We have determined the lowest Form Factors of the trace operators of the super stress-energy tensor and we have discussed the novelty present in this calculation. Several checks of their validity have been presented in Section 7, in particular a remarkable saturation of the $c$-theorem sum rule obtained for the simplest Schoutens’s model. We have finally obtained a fermionic version of the $c$-theorem by employing the supersymmetric properties of the correlators. It would be extremely interesting to explore further the properties of these supersymmetric theories and obtain
closed expressions for Form Factors of other operators and with an arbitrary number of
external particles.

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Appendix A

In this appendix we collect some useful formulas of the function $G(\beta)$ used in Section 6.

The function $G(\beta)$ is the minimal solution of the equations
\[
G(\beta) = U(\beta)G(-\beta) ;
G(i\pi - \beta) = G(i\pi + \beta),
\] (A.1)
where $U(\beta)$ is the function entering the $S$-matrix of the SShG model, eq. (5.3). Its explicit expression is given by
\[
G(\beta) = \exp \left[ -\int_0^\infty \frac{dt}{t} \sinh \alpha t \sinh(1 - \alpha) t \sin \frac{\beta t}{2\pi} \right]. \tag{A.2}
\]

For large value of $\beta$, this function tends to a constant given by
\[
\lim_{\beta \to \infty} G(\beta) = \exp \left[ -\frac{1}{2} \int_0^\infty \frac{dt}{t} \sinh \alpha t \sinh(1 - \alpha) t \sin \frac{\beta t}{2\pi} \right]. \tag{A.3}
\]

For the purposes of numerical calculation, it can be written in a more convenient way as follows. First of all, notice that the function $U(\beta)$ can be factorised as
\[
U(\beta) = U_1(\beta)U_2(\beta), \tag{A.4}
\]
where
\[
U_1(\beta) = \exp \left[ -i \int_0^\infty \frac{dt}{t} \frac{(1 - \cosh t)(1 + \cosh(1 - 2\alpha) t)}{\sinh^2 t} \sin \frac{\beta t}{\pi} \right]; \tag{A.5}
\]
\[
U_2(\beta) = \exp \left[ -i \int_0^\infty \frac{dt}{t} \frac{\cosh(1 - 2\alpha) t}{\cosh t} \sin \frac{\beta t}{\pi} \right].
\]

Correspondingly, $G$ can be factorised as
\[
G(\beta) = G_1(\beta)G_2(\beta), \tag{A.6}
\]
where
\[
G_1(\beta) = \prod_{k=0}^{N-1} \left[ \frac{p_k^2(\beta, 0)p_k^2(\beta, 1 - 2\alpha)}{p_k^2(\beta, 1)p(\beta, 2\alpha)p_k(\beta, 2 - 2\alpha)} \right]^{(k+1)/(k+2)} \times \tag{A.7}
\]
\[
\exp \left[ \int_0^\infty \frac{dt}{t} \frac{(1 - \cosh t)(1 + \cosh(1 - 2\alpha) t)}{\sinh^3 t} d_N(t) \sin \frac{\beta t}{2\pi} \right] .
\]
and
\[
G_2(\beta) = \prod_{k=0}^{N-1} \left[ \left( 1 + \frac{(i\pi - \beta)}{(4k + 1 + 2\alpha)\pi} \right)^2 \left( 1 + \frac{(i\pi - \beta)}{(4k + 3 - 2\alpha)\pi} \right)^2 \right]^{\frac{1}{2}} \times \tag{A.8}
\]
\[
\times \exp \left[ \int_0^\infty \frac{dt}{t} \frac{\cosh(1 - 2\alpha) t e^{-4Nt}}{\cosh t \sinh t} \sin \frac{\beta t}{2\pi} \right],
\]
with
\[ p_k(\beta, x) = \left[ \left( 1 + \left( \frac{(i\pi - \beta)}{2k + 3 + x} \right)^2 \right) \left( 1 + \left( \frac{(i\pi - \beta)}{2k + 3 - x} \right)^2 \right) \right] , \]
and
\[ d_N(t) = \frac{1}{2} \left[ (N + 1)(N + 2) - 2N(N + 2)e^{-2t} + N(N + 1)e^{-4t} \right] e^{-2Nt} \]

Appendix B

In this appendix we briefly describe the reason of the inconsistency of the Form Factors found by Ahn [24] for \( F^{_{bb}} \) and \( F^{_{ff}} \) of the SShG model, referring the reader to his original paper for all notations used by this author.

The Form Factors found by Ahn are those of formula (4.24) of [24], i.e.
\[
F^{_{bb}}(\beta) = 2\pi m^2 \left[ \frac{(F^{_{min}} + F^{_{min}})}{2} + \frac{(F^{_{min}} - F^{_{min}})}{2} \cosh \frac{\beta}{2} \right] ; \quad (B.1)
\]
\[
F^{_{ff}}(\beta) = 2\pi m^2 \left( \frac{F^{_{min}} + F^{_{min}}}{2} \right) \sinh \frac{\beta}{2} , \quad (B.2)
\]
where
\[
F_{+}(\beta) = \exp \left[ \int_0^\infty \frac{dt}{t} \frac{f_{+}(t)}{\sinh t} \sin^2 \left( \frac{(i\pi - \beta)t}{2\pi} \right) \right] ; \quad (B.3)
\]
\[
F_{-}(\beta) = \exp \left[ \int_0^\infty \frac{dt}{t} \frac{f_{-}(t)}{\sinh t} \sin^2 \left( \frac{(i\pi - \beta)t}{2\pi} \right) \right] ; \quad (B.4)
\]
and
\[
f_{\pm}(t) = \frac{(1 - \cosh t)(1 + \cosh((1 - 2 | \alpha |)t))}{\sinh^2 t} \pm \frac{\cosh(1 - 2 | \alpha |)t}{\cosh t} \] (B.4)

By construction, the functions \( F_{\pm}(\beta) \) considered by Ahn satisfied the unitarity equations relative to the eigenvalues of the S-matrix in the \( F = -1 \) sector and the crossing equations in the usual way, namely
\[
F_{+}(\beta + 2\pi i) = F_{+}(-\beta) ; \quad (B.5)
\]
\[
F_{-}(\beta + 2\pi i) = F_{-}(-\beta) . \quad (B.5)
\]

As discussed in Section 6, the crossing properties of the Form Factors in the \( F = -1 \) on the other hand are quite different. This has an important consequence on the validity of

\[ ^8 \text{They slightly differ from ours. The reader should also be aware that there are several misprints in the paper [24].} \]
Consider in fact the transformation $\beta \rightarrow \beta + 2\pi i$ in the first of (B.1). It is easy to see that the Form Factor of the trace operator $\Theta$ does not satisfy in this case the equation

$$F^{\Theta}_{bb}(\beta + 2\pi i) = F^{\Theta}_{bb}(-\beta),$$

expressing the locality of this operator.

The problem becomes evident as one tries to apply the $c$-theorem sum rule. If this is applied by using the above formulas and equation (7.4), one gets the results reported in Table 2 which are definitely in disagreement with the theoretical result $C = 3/2$. The results reported in the original Table 1 of the paper by Ahn were actually obtained by means of an unjustified step at this stage of his calculations, namely by plugging negative values for the quantity $|\alpha|$ which appears in all formulas of Ahn’s paper (see, for instance eq. (B.4) above). In this case, although the first values reported by Ahn seem in reasonable agreement with the $c$-theorem sum rule, if that calculation had been pursued for higher values of the coupling constant, the results would have been quite unsatisfactory, as shown by the unreasonable small values obtained for the central charge for certain cases and a weird increasing behaviour around $|\alpha| = -0.5$, see Table 3.
References

[1] B. Berg, M. Karowski, P. Weisz, *Phys. Rev. D* **19** (1979), 2477; M. Karowski, P. Weisz, *Nucl. Phys. B* **139** (1978), 445; M. Karowski, *Phys. Rep.* **49** (1979), 229;

[2] F.A. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory* (World Scientific) 1992, and references therein.

[3] F.A. Smirnov, *Nucl. Phys. B* **453** (1995), 807.

[4] H. Babujian, A. Fring, M. Karowski, A. Zapletal, *Exact Form Factors in Integrable Quantum Field Theories: the Sine-Gordon Model*, hep-th/9805183.

[5] G. Delfino, P. Simonetti and J.L. Cardy, *Phys. Lett. B* **387** (1996), 327.

[6] J.L. Cardy and G. Mussardo, *Nucl. Phys. B* **340** (1990), 387; O. Babelon and D. Bernard, *Phys. Lett. B* **288** (1992), 113.

[7] A. Koubek and G. Mussardo, *Phys. Lett. B* **311** (1993), 193.

[8] C. Ahn, G. Delfino and G. Mussardo, *Phys. Lett. B* **317** (1993), 573.

[9] A. Fring, G. Mussardo and P. Simonetti, *Nucl. Phys. B* **393** (1993), 413; G. Mussardo and P. Simonetti, *Int. J. Mod. Phys. A* **9** (1994), 3307.

[10] A. Koubek, *Phys. Lett. B* **346** (1995), 275; *Nucl. Phys. B* **428** (1994), 655; *Nucl. Phys. B* **435** (1995), 703.

[11] G. Delfino and G. Mussardo, *Nucl. Phys. B* **455** [FS] (1995), 724; G. Delfino and P. Simonetti *Phys. Lett. B* **383** (1996), 450.

[12] T. Oota, *Two-point correlation functions in perturbed minimal models* hep-th/9804050.

[13] V.E. Korepin, N.A. Slavnov, *The determinant representation for quantum correlation functions of the sinh-Gordon model*, hep-th/9801046.

[14] V.P. Yurov and A.B. Zamolodchikov, *Int. J. Mod. Phys. A* **6** (1991), 3419.

[15] A.B. Zamolodchikov, *Nucl. Phys. B* **348** (1991), 619.

[16] J.L. Cardy and G. Mussardo, *Nucl. Phys. B* **410** [FS] (1993), 451.

[17] G. Delfino and G. Mussardo, *Phys. Lett. B* **324** (1994), 40.
[18] G. Delfino, *Phys. Lett.* B 419 (1998), 291; G. Delfino and J. L. Cardy, *Nucl. Phys.* B 519 (1998), 551.

[19] F. Lesage, H. Saleur and S. Skorik, *Nucl. Phys.* B 474 (1996) 602; A. Leclair, F. Lesage, S. Sachdev and H. Saleur, *Nucl. Phys.* B 482 (1996) 579.

[20] K. Schoutens, *Nucl. Phys.* B 344 (1990), 665.

[21] T. Hollowood and E. Mavrikis, *Nucl. Phys.* B 484 (1997), 631.

[22] J.L. Cardy and G. Mussardo, *Phys. Lett.* B 225 (1989), 275.

[23] S. Ferrara, L. Girardello and S. Sciuto, *Phys. Lett.* B 76 (1978), 303; T. Marinucci and S. Sciuto, *Nucl. Phys.* B 156 (1979), 144.

[24] C. Ahn, *Nucl. Phys.* B 422 (1994), 449.

[25] M. Bershadsky, V. Knizhnik and M. Teilman, *Phys. Lett.* B 151 (1985), 31; D. Friedan, Z. Qiu and S.H. Shenker, *Phys. Lett.* 151 (1985), 37.

[26] D.A. Kastor, E.J. Martinec and S.H. Shenker, *Nucl. Phys.* B 316 (1989), 590.

[27] Al.B. Zamolodchikov, *Resonance Factorized Scattering and Roaming Trajectories*, ENS-LPS-335 (1991).

[28] M.J. Martins, *Phys. Lett.* B 304 (1993), 111.

[29] A.B. Zamolodchikov, Al.B. Zamolodchikov, *Ann.Phys.* 120 (1979), 253.

[30] A.B. Zamolodchikov, *JETP Lett.* 43 (1986), 730.

[31] J.L. Cardy, *Phys. Rev. Lett.* 60 (1988), 2709.
**Table Caption**

**Table 1.** The two-particle contribution to the sum rule of the $c$-theorem for the SShG model.

**Table 2.** The two-particle contribution to the $c$-theorem sum rule relative to the Ahn’s Form Factors.

**Table 3.** The two-particle contribution to the $c$-theorem sum rule relative to the Ahn’s Form Factors for negative values of $|\alpha|$. 
| $\alpha$ | $\frac{\Delta^2}{4\pi}$ | $\Delta c^{(2)}$ | precision\% |
|---------|-------------------|-----------------|-------------|
| $\frac{1}{100}$ | $\frac{1}{99}$ | 1.49968 | 0.0213 |
| $\frac{3}{100}$ | $\frac{3}{97}$ | 1.49741 | 0.1726 |
| $\frac{1}{20}$ | $\frac{1}{19}$ | 1.49349 | 0.4340 |
| $\frac{1}{10}$ | $\frac{1}{9}$ | 1.47955 | 1.3633 |
| $\frac{3}{20}$ | $\frac{3}{17}$ | 1.46333 | 2.4446 |
| $\frac{1}{5}$ | $\frac{1}{4}$ | 1.44742 | 3.5053 |
| $\frac{3}{10}$ | $\frac{3}{7}$ | 1.42109 | 5.2606 |
| $\frac{1}{2}$ | $\frac{2}{3}$ | 1.40480 | 6.3466 |
| $\frac{1}{2}$ | 1 | 1.39935 | 6.7100 |

Table 1
| $\alpha$ | $\frac{\lambda^2}{4\pi}$ | $\Delta c^{(2)}$ |
|--------|-----------------|----------------|
| $\frac{1}{100}$ | $\frac{1}{99}$ | 1.51481 |
| $\frac{3}{100}$ | $\frac{3}{97}$ | 1.54328 |
| $\frac{1}{20}$ | $\frac{1}{19}$ | 1.57019 |
| $\frac{1}{10}$ | $\frac{1}{9}$ | 1.63080 |
| $\frac{3}{20}$ | $\frac{3}{17}$ | 1.68227 |
| $\frac{1}{5}$ | $\frac{1}{4}$ | 1.72524 |
| $\frac{3}{10}$ | $\frac{3}{7}$ | 1.78833 |
| $\frac{2}{5}$ | $\frac{2}{3}$ | 1.82442 |
| $\frac{1}{2}$ | 1 | 1.83616 |

Table 2
| $|\alpha|$ | $\Delta c^{(2)}$ |
|---|---|
| $-0.001$ | $1.4985$ |
| $-0.005$ | $1.49246$ |
| $-0.01$ | $1.48485$ |
| $-0.02$ | $1.46951$ |
| $-0.03$ | $1.45413$ |
| $-0.05$ | $1.42369$ |
| $-0.1$ | $1.35190$ |
| $-0.15$ | $1.28762$ |
| $-0.20$ | $1.23038$ |
| $-0.33$ | $1.14079$ |
| $-0.40$ | $1.14552$ |
| $-0.50$ | $1.58825$ |

Table 3
Figure Captions

Figure 1. Renormalization Group Flows described by the Roaming Models.

Figure 2. Bound states and residues in some amplitudes of the Schoutens’s S-matrix.

Figure 3. Complex zeros of the Roaming Models.

Figure 4. Logarithm of the correlation function $m^{-4} \langle \Theta(x) \Theta(0) \rangle$ versus $mr$ for $\alpha = 0$ (full line) and for $\alpha = 0.5$ (dashed line).

Figure 5. Bound state recursive equation for the form factor $F_2$.

Figure 6. Correlation function $m^{-4} \langle \Theta(x) \Theta(0) \rangle$ versus $mr$ of the first Schoutens’s model, obtained with the first two terms of the spectral series.
Figure 1
Figure 2
Figure 3
Figure 4
Figure 5
Figure 6