The role of the field redefinition in noncommutative Maxwell theory

I. Frühwirth, J. M. Grimstrup, Z. Morsli, L. Popp, M. Schweda

Institut für Theoretische Physik, Technische Universität Wien
Wiedner Hauptstraße 8–10, A-1040 Wien, Austria

Abstract. We discuss $\theta$-deformed Maxwell theory at first order in $\theta$ with the help of the Seiberg-Witten (SW) map. With an appropriate field redefinition consistent with the SW-map we analyse the one-loop corrections of the vacuum polarization of photons. We show that the radiative corrections obtained in a previous work may be described by the Ward-identity of the BRST-shift symmetry corresponding to a field redefinition.
1 Introduction

Discussing noncommutative quantum field theories, especially noncommutative gauge field models, one has in principle many possibilities to formulate such models. The traditional approach widely used in many papers,\cite{1}, \cite{2}, \cite{3}, \cite{4}, is based on the fact that noncommutative gauge theory is realized as a theory on the set of ordinary functions by modifying the product of two functions in terms of the $\star$-product \cite{1}. The simplest candidate would be a noncommutative counterpart of QED. Due to the noncommutativity also the noncommutative $U(1)$ gauge field model (with and without matter) has a non-Abelian structure implying the usual BRST-quantization procedure involving the appearance of $\phi\pi$-ghost fields.

Unfortunately, the perturbative realization of such gauge field models develops unexpected new features: the so called UV/IR mixing emerging from nonplanar graphs. Explicitly, this can be seen by computing one-loop corrections of the vacuum polarization of the photon \cite{2}, \cite{3}. The evaluation of the finite part of the gauge field vacuum polarization shows the existence of singularities in the infrared limit. Those IR-singularities forbid Feynman graph computation of higher loop orders. Presently, one believes that such IR-singularities can perhaps be avoided by an appropriate field redefinition \cite{5}.

A second formulation of noncommutative gauge field models simultaneously incorporates the deformed product and the Seiberg-Witten (SW) map. The SW-map ensures the gauge equivalence between an ordinary gauge field model and its noncommutative counterpart \cite{1}, \cite{7}, \cite{8}, \cite{9}. In this sense it is possible to expand the noncommutative gauge field as series in the ordinary gauge field and the deformation parameter of the noncommutative space-time geometry $\theta^{\mu\nu}$.

Additionally, one has the possibility to formulate gauge field models on noncommutative spaces via covariant coordinates \cite{7}, \cite{8}, \cite{9}. Also in these approaches the use of the SW-map emerges quite naturally.

The present paper is devoted to study $U(1)$ noncommutative Yang-Mills ($U(1)$-NCYM) theory in the context of the SW-map for the simplest case allowing only a linear dependence on the deformation parameter $\theta^{\mu\nu}$. The corresponding $U(1)$-deformed gauge invariant action has been derived in \cite{9}, \cite{10} and recently been used very often to study physical consequences of such a deformed Maxwell theory \cite{12}, \cite{13}, \cite{14}, \cite{15}.

It is remarkable to note that in this simple $\theta$-deformed Maxwell theory (in its perturbative realization) at the one-loop level no IR-singularities in the above sense exist \cite{11}, \cite{18}. As is further explained in \cite{17} a consistent field redefinition (in agreement with the SW-map) allows to add to the gauge field...
of the $\theta$-deformed Maxwell theory further $\theta$-dependent and gauge invariant terms. Such additional terms are very useful for studying one-loop corrections of the vacuum polarization of the photon [17]. Since the interaction contains terms linear in $\theta$ and trilinear in the Abelian field strength one obtains terms proportional to the square of $\theta$ in the corresponding vacuum polarization at the one loop level. However, by redefining the relevant unphysical free parameters of the above mentioned field redefinition one is able to carry out the renormalization procedure in the usual sense at least for this simple case. More recently, it has been argued that the linear $\theta$-deformed QED (with the inclusion of fermions) leads to a nonrenormalizable theory [18] at the one-loop level.

The aim of this paper is to focus on the connection between the above mentioned field redefinition and the so-called BRST-shift symmetry described in [19], [11]. Especially, we try to explain that the perturbative corrections of the photon self-energy are compatible with the Ward-identity (WI) of the shift symmetry.

The paper is organized as follows: Sec. 2 is concerned with the presentation of the simplest $\theta$-deformed Maxwell theory (in terms of the usual Abelian gauge symmetry of the photon field). This is realized by starting with $U(1)$-NCYM and using the SW-map to lowest order in $\theta^{\mu\nu}$. In a next step one performs a field redefinition in consistency with the SW-map in order to derive the physical consequences leading to the WI for the BRST-shift symmetry.

In order to understand this BRST-shift symmetry we assume the existence of an Abelian gauge invariant action with an appropriate gauge fixing implemented by a multiplier field $B$:

$$\Gamma^{(0)}[A, B] = \Gamma_{\text{inv}}[A] + \Gamma_{\text{gf}}[B, A].$$

The description of the shift symmetry is based on the existence of the following redefinition

$$A_\mu = \tilde{A}_\mu + A_\mu^{(2)}(\tilde{A}),$$

where the upper index indicates quadratic dependence on $\theta^{\mu\nu}$ which will be needed in the future.

The derivation of the BRST-shift symmetry needs a new type of gauge fixing besides $\Gamma_{\text{gf}}[B, A]$. Here we follow [19], [11] adapted for our special
case \footnote{In consistency with the formulation of noncommutative gauge field models we use here \textit{\textdagger}} by defining:

\[
\Sigma = \Gamma^0[A, B] + \Gamma_{\text{shift}} + \Gamma_{\text{gf-shift}}
\]

\[
= \Gamma^0[A, B] + \int d^4x \int d^4y \bar{c}^\mu(x) \ast \frac{\delta \varphi_\mu(x)}{\delta A_\sigma(y)} \ast c_\sigma(y)
\]

\[
+ \int d^4x \Pi^\mu \ast (\varphi_\mu - A_\mu).
\]

where

\[
\varphi_\mu = \tilde{A}_\mu + \phi^{(2)}_\mu(\tilde{A}).
\]

The vectorial antighost \(\bar{c}^\mu\) and ghost field \(c^\mu\) carry a Faddeev-Popov ghost charge of \(-1\) and \(+1\), respectively.

Using (2) and (4) one gets

\[
\Sigma = \Gamma^0[A, B] + \int d^4x \bar{c}^\mu(x)c_\mu(x) + \int d^4x \int d^4y \bar{c}^\mu(x) \frac{\delta \phi^{(2)}_\mu(x)}{\delta A_\sigma(y)} c_\sigma(y)
\]

\[
+ \int d^4x \Pi^\mu \left(\phi^{(2)}_\mu - A^{(2)}_\mu(\tilde{A})\right).
\]

Since \(\phi^{(2)}_\mu(\tilde{A})\) and \(A^{(2)}_\mu(\tilde{A})\) are of order 2 in \(\theta\) no \(\ast\)-products are needed. The BRST-shift symmetry is defined as

\[
s\bar{c}^\mu = -\Pi_\mu, \quad s\tilde{A}_\mu = c_\mu = -sA^{(2)}_\mu, \quad sA_\mu = s\Pi_\mu = sc_\mu = 0. \quad (6)
\]

These transformations are manifestly nilpotent. However, eliminating the multiplier field \(\Pi_\mu\) via an algebraic equation of motion yields

\[
\frac{\delta \Sigma}{\delta \Pi_\mu} = \phi^{(2)}_\mu - A^{(2)}_\mu(\tilde{A}) = 0,
\]

and, additionally, from (3) follows

\[
\frac{\delta \Sigma}{\delta A_\mu} = \frac{\delta \Gamma^{(0)}}{\delta A_\mu} - \Pi_\mu = 0.
\]

This implies for the transformations (6)

\[
s\bar{c}^\mu = -\frac{\delta \Gamma^{(0)}}{\delta A_\mu}, \quad s\tilde{A}_\mu = c_\mu, \quad sc_\mu = 0. \quad (9)
\]
These equations show explicitly that off-shell nilpotency is lost and that the shift symmetry is no longer linear.

One has to introduce an unquantized external source to describe the non-linear transformation of $s\delta^\mu$. This will be used for the construction of the corresponding WI for the shift symmetry in sec. 2.

## 2 Deformed Maxwell theory—Consequences of the BRST-shift symmetry at the classical level

In order to derive the corresponding $\theta$-deformed Maxwell theory one starts with the $U(1)$-NCYM model \[7\], \[10\]:

$$\Gamma^{(0)}_{\text{inv}} = -\frac{1}{4} \int d^4x \hat{F}_{\mu\nu} \ast \hat{F}^{\mu\nu},$$

where the field strength in terms of the noncommutative $U(1)$ gauge field $\hat{A}_\mu$ is given by

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i [\hat{A}_\mu, \hat{A}_\nu]_M.$$

The Moyal bracket is defined by

$$[\hat{A}_\mu, \hat{A}_\nu]_M = \hat{A}_\mu \ast \hat{A}_\nu - \hat{A}_\nu \ast \hat{A}_\mu,$$

using the $\ast$-product

$$A(x) \ast B(x) = e^{i\theta^\mu_{\alpha\beta} \partial_\mu A(x+\alpha) B(x+\beta)} \bigg|_{\alpha=\beta=0},$$

where $\theta^\mu_{\alpha\beta}$ is the deformation parameter of the noncommutative geometry.

The action (10) is invariant under the infinitesimal noncommutative gauge transformation

$$\hat{\delta}_\lambda \hat{A}_\mu = \partial_\mu \hat{\lambda} - i \hat{A}_\mu \ast \hat{\lambda} + i \hat{\lambda} \ast \hat{A}_\mu \equiv \hat{D}_\mu \hat{\lambda}.$$

It was shown by Seiberg and Witten \[8\] that an expansion in $\theta^\mu_{\alpha\beta}$ leads to a map between the noncommutative gauge field $\hat{A}_\mu$ and the commutative gauge field $A_\mu$ as well as their respective gauge parameters $\hat{\lambda}$ and $\lambda$, known as the SW-map. To lowest order in $\theta$ one has in the Abelian case \[8\], \[9\], \[10\]

$$\hat{A}_\mu(A_\mu) = A_\mu - \frac{1}{2} A_\rho \left( \partial_\sigma A_\mu + F_{\sigma\mu} \right) + O(\theta^2),$$

$$\hat{\lambda}(A_\mu, \lambda) = \lambda - \frac{1}{2} \theta^\rho_{\sigma} A_\rho \partial_\sigma \lambda + O(\theta^2).$$
In (15) $F_{\sigma \mu}$ is the ordinary Abelian field strength given by

$$F_{\sigma \mu} = \partial_\sigma A_\mu - \partial_\mu A_\sigma.$$  \hfill (16)

Using (10), (11), (12), (13) and (15) one gets to lowest order in $\theta^\mu_\nu$, 

$$\Gamma_{\text{inv}} = \int d^4 x \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} \theta^{\rho \sigma} \left( F_{\mu \rho} F_{\nu \sigma} F^{\mu \nu} - \frac{1}{4} F_{\rho \sigma} F^{\mu \nu} F^{\mu \nu} \right) \right) + O(\theta^2),$$  \hfill (17)

which is invariant under the usual Abelian gauge transformation

$$\delta A_\mu = \partial_\mu \lambda.$$ \hfill (18)

The action (17) has in its full form, involving all orders of $\theta^\mu_\nu$, infinitely many interactions of infinitely high order in the field. Additionally, since $\theta^\mu_\nu$ has dimension $-2$, the model is power-counting nonrenormalizable in the traditional sense.

In order to quantize the model one introduces a Landau gauge fixing

$$\Gamma_{gf} = \int d^4 x B \partial^\mu A_\mu,$$  \hfill (19)

where $B$ is the multiplier field implementing the gauge $\partial^\mu A_\mu = 0$.

Then one establishes the shift symmetry according to (19), (11). As is explained in (17), the relevant field redefinition, compatible with the SW map, takes the following form:

$$A_\mu = \tilde{A}_\mu + A^{(2)}_\mu (\tilde{A}),$$  \hfill (20)

where the upper index again indicates that $A^{(2)}_\mu (\tilde{A})$ depends quadratically on $\theta^\mu_\nu$. Additionally, $A^{(2)}_\mu (\tilde{A})$ is gauge invariant with respect to (18)

$$\delta_\lambda A^{(2)}_\mu (\tilde{A}) = 0.$$  \hfill (21)

Terms linear in $\theta^\mu_\nu$ are excluded due to the topological nature of the corresponding action.

On the other hand the formula (20) allows the introduction of terms with a quadratic dependence already in the classical action. Such terms are needed for the one-loop renormalization procedure of the vacuum polarization of photons.

\textsuperscript{2}In (17) one finds the most general $A^{(2)}_\mu (\tilde{A})$. 

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\[ \text{In} \]
In the spirit of \([19], [11]\) one defines now for the deformed Maxwell theory, eq. \([17]\), the corresponding shift-action in the following way, see formula \(\|\)
\[
\Gamma_{\text{shift}} = \int d^4 x \int d^4 y \bar{c}^\mu(x) \ast \frac{\delta A_\mu(x)}{\delta \tilde{A}_\sigma(y)} \ast c_\sigma(y)
\]
\[
= \int d^4 x \bar{c}^\mu(x) c_\mu(x) + \int d^4 x \int d^4 y \bar{c}^\mu(x) \frac{\delta A_\mu^{(2)}(x)}{\delta \tilde{A}_\sigma(y)} c_\sigma(y),
\]
where \(\bar{c}^\mu\) and \(c_\mu\) are the vectorial shift ghost and antighost fields, respectively. Due to the fact that \(A_\mu^{(2)}\) is already of second order in \(\theta\) one can neglect the stars in the second term of \(\|\)\.

Thus, the total action of the model under consideration ready for quantization is given by
\[
\Gamma^{(0)} = \Gamma_{\text{inv}}^{(0)} + \Gamma_{\text{gf}} + \Gamma_{\text{shift}}
\]
\[
= \int d^4 x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \theta^{\rho\sigma} \left( F_{\mu\rho} F_{\nu\sigma} F^{\mu\nu} - \frac{1}{4} F_{\mu\sigma} F_{\rho\nu} F^{\mu\nu} \right) \right)
\]
\[
+ \int d^4 x B \partial^\mu A_\mu + \int d^4 x \bar{c}^\mu(x) c_\mu(x) + \int d^4 x \int d^4 y \bar{c}^\mu(x) \frac{\delta A_\mu^{(2)}(x)}{\delta \tilde{A}_\sigma(y)} c_\sigma(y). 
\]

More explicitly, one has
\[
\Gamma^{(0)} = \int d^4 x \left( -\frac{1}{4} \left( \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - 4 \partial^\mu \tilde{F}^{\mu\nu} A_\nu^{(2)} \right) - \frac{1}{2} \theta^{\rho\sigma} \left( \tilde{F}_{\mu\rho} \tilde{F}_{\nu\sigma} \tilde{F}^{\mu\nu} - \frac{1}{4} \tilde{F}_{\mu\sigma} \tilde{F}_{\rho\nu} \tilde{F}^{\mu\nu} \right) \right)
\]
\[
+ \int d^4 x B \partial^\mu \tilde{A}_\mu + B \partial^\mu A_\nu^{(2)}
\]
\[
+ \int d^4 x \bar{c}^\mu(x) c_\mu(x) + \int d^4 x \int d^4 y \bar{c}^\mu(x) \frac{\delta A_\mu^{(2)}(x)}{\delta \tilde{A}_\sigma(y)} c_\sigma(y),
\]
where \(\tilde{F}_{\alpha\beta}\) is the usual Abelian field strength in terms of \(\tilde{A}_\beta\).

\(^{3}\text{In \cite{[22]} we have used } \int d^4 x \bar{c}^\mu(x) \ast c_\mu(x) = \int d^4 x \bar{c}^\mu(x) c_\mu(x).\)
From eq. (9) the BRST-shift symmetry of the action (24) is given by

\[ s\bar{c}^\tau = \frac{\delta \Gamma(0)}{\delta A_\tau} \bigg|_{A_\mu = \bar{A}_\mu + A_\mu^{(2)}(\bar{A})} \]

\[ = \partial_\mu F^{\mu \tau} + \theta^{\tau \sigma} \partial_\mu (F_{\nu \sigma} F^{\mu \nu}) - \theta^{\rho \sigma} \partial_\rho (F_{\nu \sigma} F^{\tau \nu}) + \theta^{\rho \sigma} \partial_\rho (F^{\mu \rho} F^{\tau \sigma}) \]

\[ - \frac{1}{4} \theta^{\rho \tau} \partial_\rho \left( F_{\mu \nu} F^{\mu \nu} \right) - \frac{1}{2} \theta^{\rho \sigma} \partial_\mu \left( F_{\rho \sigma} F^{\mu \tau} \right) - \partial^\tau B \]

\[ = \partial_\mu (\tilde{F}^{\mu \tau} + \partial^\mu A^{(2)}(2) - \partial^\tau \Lambda^{(2)}(2) + \theta^{\tau \sigma} \partial_\mu (\tilde{F}_{\nu \sigma} \tilde{F}^{\mu \nu}) - \theta^{\rho \sigma} \partial_\rho (\tilde{F}_{\nu \sigma} \tilde{F}^{\tau \nu}) \]

\[ + \theta^{\rho \sigma} \partial_\mu (\tilde{F}^\mu \tilde{F}^{\tau}) - \frac{1}{4} \theta^{\rho \tau} \partial_\rho (\tilde{F}_{\mu \nu} \tilde{F}^{\mu \nu}) - \frac{1}{2} \theta^{\rho \sigma} \partial_\mu (\tilde{F}_{\rho \sigma} \tilde{F}^{\mu \tau}) - \partial^\tau B =: \partial_\mu (\tilde{F}^{\mu \tau} + \partial^\mu A^{(2)}(2) - \partial^\tau \Lambda^{(2)}(2) - \partial^\tau B + \mathcal{F}^{(1)\tau}(\bar{A}), \]

\[ s\bar{A}_\mu = c_\mu, \]

\[ sc_\mu = sB = 0, \]

where

\[ \mathcal{F}^{(1)\tau}(\bar{A}) := \theta^{\tau \sigma} \partial_\mu (\tilde{F}_{\nu \sigma} \tilde{F}^{\mu \nu}) - \theta^{\rho \sigma} \partial_\rho (\tilde{F}_{\nu \sigma} \tilde{F}^{\tau \nu}) + \theta^{\rho \sigma} \partial_\rho (\tilde{F}^\mu \tilde{F}^{\tau}) \]

\[ - \frac{1}{4} \theta^{\rho \tau} \partial_\rho \left( \tilde{F}_{\mu \nu} \tilde{F}^{\mu \nu} \right) - \frac{1}{2} \theta^{\rho \sigma} \partial_\mu \left( \tilde{F}_{\rho \sigma} \tilde{F}^{\mu \tau} \right). \]

One has to comment at this point that the BRST-shift symmetry for the vectorial antighost field \( \bar{c}^\mu \) is highly nonlinear. Additionally, as is explained in the introduction, the off-shell nilpotency is also lost:

\[ s^2 \bar{c}^\mu \neq 0. \]

Since the transformation of the antighost field \( \bar{c}^\mu \) contains nonlinear expressions one must introduce an external unquantized source \( \rho_\mu \) for the term \( \mathcal{F}^{(1)\mu}(\bar{A}) \). This implies a further piece in the action (21)

\[ \Gamma_{ext} = \int d^4x \rho_\mu \mathcal{F}^{(1)\mu}(\bar{A}), \]

where \( \rho_\mu \) is gauge invariant.

The new total action becomes therefore

\[ \Gamma^{(0)} = \int d^4x \left( - \frac{1}{4} \left( \tilde{F}_{\mu \nu} \tilde{F}^{\mu \nu} - 4 \partial^\mu \tilde{F}^{\mu \nu} A^{(2)\nu} \right) \right) \]

\[ - \frac{1}{2} \theta^{\rho \sigma} \left( \tilde{F}_{\mu \nu} \tilde{F}^{\rho \sigma} \tilde{F}^{\mu \nu} - \frac{1}{4} \tilde{F}_{\rho \sigma} \tilde{F}_{\mu \nu} \tilde{F}^{\mu \nu} \right) \right) + \int d^4x \left( B \partial^\mu \bar{A}_\mu + \tilde{A}_\mu^{(2)}(2) \right) \]

\[ + \int d^4x \bar{c}^\mu(x)c_\sigma(x) + \int d^4x \int d^4y \bar{c}^\mu(x)\frac{\delta A^{(2)}(x)}{\delta A_\sigma(y)} c_\sigma(y) \]

\[ + \int d^4x \rho_\mu \mathcal{F}^{(1)\mu}(\bar{A}). \]
Now we are able to characterize the symmetry content of the BRST-shift symmetry by the following nonlinear WI:

\[ S(\hat{\Gamma}^{(0)}) = \int d^4 x \left( \left( \partial_\rho (\tilde{F}^{\rho \mu} + \partial^\rho A^{(2)\mu} - \partial^\mu A^{(2)\rho}) - \partial^\mu B + \frac{\delta \hat{\Gamma}^{(0)}}{\delta \tilde{\varepsilon}^\rho} \right) \frac{\delta \hat{\Gamma}^{(0)}}{\delta \tilde{\varepsilon}^\mu} + c_\mu \frac{\delta \hat{\Gamma}^{(0)}}{\delta A^\mu} \right) = 0. \]  

(30)

Eq. (30) will be the key for the understanding of the radiative corrections of the 2-point vertex-functional at the one-loop-level [17]. Additionally, our model is also characterized by the gauge symmetry (18) with

\[ \delta \lambda \tilde{c}^\mu = \delta \lambda c^\mu = \delta \lambda B = 0. \]  

(31)

This ordinary gauge invariance is described by the following WI operator

\[ W_\lambda = \int d^4 x \partial_\mu \lambda \frac{\delta}{\delta A_\mu(x)} \]  

(32)

and, as usual, the gauge symmetry is broken by the gauge fixing. This leads to

\[ W_\lambda \hat{\Gamma}^{(0)} = \int d^4 x B \Box \lambda \neq 0. \]  

(33)

By functional differentiation with respect to \( \lambda(y) \) one obtains the local version of (33)

\[ W(x)\hat{\Gamma}^{(0)} = -\partial_\mu \frac{\delta \hat{\Gamma}^{(0)}}{\delta A_\mu(x)} = \Box B(x) \neq 0. \]  

(34)

Due to the fact that this breaking is linear in the quantum field B, there do not arise any problems for the discussion of the gauge symmetry at higher orders of perturbation theory [20], [21].

We would like to point out that our model is characterized by two symmetries at the classical level: the gauge symmetry and the BRST-shift symmetry. This implies the existence of two WI’s, (30) and (33). These WI’s have severe consequences for the computation of the 2-point vertex functional.

From eq. (34) follows immediately the well-known transversality condition

\[ \partial_\mu \frac{\delta^2 \hat{\Gamma}^{(0)}}{\delta A_\mu(x) \delta A^\rho(y)} \bigg|_0 = 0, \]  

(35)
where the subscript indicates vanishing classical fields. However, the WI (30) furnishes a further possibility to calculate
\[
\Pi_{\mu\rho} = \frac{\delta^2 \hat{\Gamma}^{(0)}}{\delta A^\mu(x)\delta A^\rho(y)} \bigg|_0.
\]

The result obtained by direct calculation (functional variation) must be consistent with the result emerging from the WI (30).

However, one has to stress that all considerations done in this section are purely classical—i.e. in the tree approximation.

Therefore, one has to study the consequences of (30). Functional differentiation with respect to \(\tilde{A}^\rho(z)\) and \(c_\mu(y)\) of \(S(\hat{\Gamma}^{(0)})\) gives:
\[
\frac{\delta^2 S(\hat{\Gamma}^{(0)})}{\delta e^\mu(y)\delta \tilde{A}^\rho(z)} \bigg|_0 = \left(\Box g_{\rho\sigma} - \partial_\rho \partial_\sigma\right)(z) \frac{\delta^2 \hat{\Gamma}^{(0)}}{\delta e^\mu(y)\delta \tilde{c}_\sigma(z)} \bigg|_0 + \frac{\delta^2 \hat{\Gamma}^{(0)}}{\delta \tilde{A}^\rho(z)\delta A^\mu(y)} \bigg|_0 + \int d^4x \frac{\delta A^{(2)\lambda}(x)}{\delta A^\rho(z)} \left(\Box g_{\lambda\sigma} - \partial_\lambda \partial_\sigma\right)(x) \frac{\delta^2 \hat{\Gamma}^{(0)}}{\delta e^\mu(x)\delta \tilde{c}_\sigma(z)} \bigg|_0 = 0. \tag{37}
\]

From (29) one gets additionally
\[
\frac{\delta^2 \hat{\Gamma}^{(0)}}{\delta c^\mu(y)\delta \tilde{c}_\sigma(z)} \bigg|_0 = \delta^\mu_\sigma \delta(y - z) + \frac{\delta A^{(2)\lambda}(z)}{\delta A^\mu(y)} \delta^\sigma_\mu. \tag{38}
\]

At order \(\theta^2\) one has therefore
\[
\frac{\delta^2 \hat{\Gamma}^{(0)}}{\delta A^\rho(z)\delta A^\mu(y)} \bigg|_0 = \left(\Box g_{\rho\mu} - \partial_\rho \partial_\mu\right)(z) \delta(y - z) - \left(\Box g_{\rho\sigma} - \partial_\rho \partial_\sigma\right)(z) \frac{\delta A^{(2)\sigma}(z)}{\delta A^\mu(y)}
\]
\[
- \left(\Box g_{\lambda\mu} - \partial_\lambda \partial_\mu\right)(y) \frac{\delta A^{(2)\lambda}(y)}{\delta A^\rho(z)}. \tag{39}
\]

The result (39) is fully transversal—a consequence of the gauge symmetry. Since \(A^{(2)\mu}\) is of order \(\theta^2\), (39) yields the well known result for the 2-point free vertex functional for \(\theta^{\rho\sigma} = 0\)
\[
\frac{\delta^2 \hat{\Gamma}^{(0)}}{\delta A^\rho(z)\delta A^\mu(y)} \bigg|_0 = -\left(\Box g_{\rho\mu} - \partial_\rho \partial_\mu\right)(z) \delta(y - z). \tag{40}
\]

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In the tree approximation there only exists a linear dependence on $\theta^{\rho\sigma}$—therefore $A^{(2)\mu}$ is not needed in the action,

$$\Gamma_{\text{inv}} = \int d^4 x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \theta^{\rho\sigma} \left( F_{\mu\rho} F_{\nu\sigma} F^{\mu\nu} - \frac{1}{4} F_{\rho\sigma} F_{\mu\nu} F^{\mu\nu} \right) \right)$$

$$+ \int d^4 x B \partial^\mu A_{\mu},$$

which is needed to calculate eq. (40).

The additional terms proportional to $\delta \Delta^{(2)(x)}$ in (39) become useful if one considers one-loop corrections.

The result (39) is easily reproduced by direct twofold functional derivation of the action (29) with respect to the gauge field $\tilde{A}_\mu(x)$.

### 3 $\theta$-deformed Maxwell theory: one-loop corrections

If one considers only the photon sector of the noncommutative Maxwell theory the relevant action\cite{17} is given by:

$$\Gamma^{(0)} = \Gamma^{(1)} + \int d^4 x \partial_\mu \tilde{F}^{\mu\nu} A^{(2)}_{\nu}$$

$$= \int d^4 x \left( -\frac{1}{4} \left( \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - 4 \partial_\mu \tilde{F}_{\mu\nu} \tilde{A}^{(2)\nu} \right) \right)$$

$$- \frac{1}{2} \theta^{\rho\sigma} \left( \tilde{F}_{\mu\rho} \tilde{F}_{\nu\sigma} \tilde{F}^{\mu\nu} - \frac{1}{4} \tilde{F}_{\rho\sigma} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \right)$$

where $\Gamma^{(1)}$ denotes terms of order 0 and 1 in $\theta$. Eq. (12) follows from (29).

In order to compensate the one-loop selfenergy corrections one needs the explicit form of $A^{(2)}_{\mu}$\cite{17}:

$$A^{(2)}_{\mu} = \kappa_1^{(2)} g^{\alpha\gamma} g^{\beta\delta} g^{\lambda\rho} g^{\sigma\tau} \theta_{\alpha\beta} \theta_{\gamma\delta} \partial_\lambda \partial_\rho \partial_\sigma \tilde{F}_{\tau\mu}$$

$$+ \kappa_2^{(2)} g^{\sigma\gamma} g^{\beta\lambda} g^{\delta\rho} g^{\sigma\tau} \theta_{\alpha\beta} \theta_{\gamma\delta} \partial_\lambda \partial_\rho \partial_\sigma \tilde{F}_{\tau\mu}$$

$$+ \kappa_3^{(2)} g^{\beta\sigma} g^{\gamma\tau} g^{\alpha\lambda} g^{\delta\rho} \theta_{\mu\beta} \theta_{\gamma\delta} \partial_\alpha \partial_\rho \partial_\sigma \tilde{F}_{\sigma\tau}$$

$$+ \kappa_4^{(2)} g^{\gamma\tau} g^{\beta\delta} g^{\alpha\lambda} g^{\rho\sigma} \theta_{\mu\beta} \theta_{\gamma\delta} \partial_\alpha \partial_\rho \partial_\sigma \tilde{F}_{\sigma\tau},$$

which is gauge invariant.

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4 Here, the vectorial ghost, the antighost, the $B$ field and the external sources are assumed to be zero.
Inserting (43) into (42) one gets for the terms quadratic in $\theta_{\mu\nu}$

$$
\Gamma^{(0)} = \Gamma^{(1)} + \int d^4x \tilde{A}_\mu \left( g^{\mu\nu} \Box - \partial^\mu \partial^\nu \right) \left( \kappa^{(2)} \left( \theta^2 \Box^2 + \kappa^{(2)} \tilde{\Box} \right) + \kappa^{(2)} \partial^\mu \partial^\nu \tilde{\partial} \right)
$$

$$
+ \kappa^{(2)} \tilde{\partial}^\mu \tilde{\partial}^\nu \Box^2 + \kappa^{(2)} \left( \theta^{\mu\alpha} \theta_{\alpha}^\nu \Box^2 + \left( \tilde{\partial}^\mu \partial^\nu + \tilde{\partial}^\nu \partial^\mu \right) \Box^2 + \theta^2 \theta_{\alpha}^\mu \tilde{\partial} \right) \tilde{A}_\nu, \quad (44)
$$

where $\Box = \partial^\alpha \partial_\alpha$, $\tilde{\partial}^\alpha = \theta^{\alpha\beta} \partial_\beta$, $\tilde{\partial}^\alpha = \theta^{\alpha\beta} \tilde{\partial}_\beta$, $\tilde{\Box} = \tilde{\partial}^\alpha \tilde{\partial}_\alpha$ and $\theta^2 = \theta^{\alpha\beta} \theta_{\alpha\beta}$.

With the help of (43) one verifies by direct calculation that (44) and (39) are consistent.

At the one-loop level this means that the shift symmetry controls the radiative corrections of the perturbative calculation.

In order to cancel the one-loop divergences one performs the following renormalization of $\kappa^{(2)}_1$, $\kappa^{(2)}_2$, $\kappa^{(2)}_3$ and $\kappa^{(2)}_4$ [17]:

$$
\kappa^{(2)}_1 \rightarrow \kappa^{(2)}_1 - \frac{1}{16} \frac{\hbar}{(4\pi)^2 \epsilon}, \quad \kappa^{(2)}_2 \rightarrow \kappa^{(2)}_2 + \frac{1}{20} \frac{\hbar}{(4\pi)^2 \epsilon},
$$

$$
\kappa^{(2)}_3 \rightarrow \kappa^{(2)}_3 + \frac{1}{60} \frac{\hbar}{(4\pi)^2 \epsilon}, \quad \kappa^{(2)}_4 \rightarrow \kappa^{(2)}_4 + \frac{1}{8} \frac{\hbar}{(4\pi)^2 \epsilon}.
$$

However, one has to stress that (45) represent unphysical renormalizations because the $\kappa^{(2)}_i$ parametrize the field redefinition (3) and (23).

### 4 Conclusion

In this paper we have demonstrated the usefulness of the BRST-shift symmetry in connection with the renormalization program of the vacuum polarization of the $\theta$-deformed Maxwell theory at the one-loop level. Gauge symmetry and BRST-shift symmetry can be implemented consistently. Unfortunately the non-Abelian extension is plagued by several difficulties.

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