On some generalization of the normal distribution

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Abstract

In this paper we propose some new generalization of the normal distribution. We will call it the $\alpha$-normal (Gaussian) distribution. For $\alpha = 2$ it becomes the standard normal distribution. We show expression for moments and the differential entropy. We also calculate the (exponential) Orlicz norm of the standard $\alpha$-normal random variables.

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1 Notion of the $\alpha$-normal distribution

Several types of generalization of the normal distribution can be distinguished. For example, the first one is the exponential power distribution, a comprehensive description of which can be found in [8]. The second one is the log-normal distribution, but in the form presented by the authors in [5, Appendix 8]. Another example is the skew distributions; see e.g. [7]. The above probability distributions become normal distributions for certain parameter values.

One can also indicate other distributions related to the normal distribution, such as classical log-normal, folded normal, and inverse normal distributions but unlike the above mentioned generalizations these do not include the normal distribution as the special case.

In this paper, we would like to present a generalization of the normal distribution, which will be related, in some sense, to the Weibull distribution. For the same value of the shape parameter, it will have the same order of tail decay as the Weibull distribution.

The standard exponentially distributed random variable $X$ has exponential tail decay, that is, $\mathbb{P}(X \geq t) = \exp(-t)$ for $t \geq 0$. Consider a random variable $W_{\alpha,\lambda} := \lambda X^{1/\alpha}$ for some $\alpha, \lambda > 0$. Observe that for $t \geq 0$

$$\mathbb{P}(W_{\alpha,\lambda} \geq t) = \mathbb{P}(\lambda X^{1/\alpha} \geq t) = \mathbb{P}(X \geq (t/\lambda)^\alpha) = \exp\left(- (t/\lambda)^\alpha\right).$$

Let us note that $W_{\alpha,\lambda}$ has the two-parameter Weibull distribution with the shape parameter $\alpha$ and the scale parameter $\lambda$; compare [6, Ch.8]. We will call it as the $\text{Weibull}(\alpha, \lambda)$ random variable (i.e., $W_{\alpha,\lambda} \sim \text{Weibull}(\alpha, \lambda)$).
Similarly as in the case of the Weibull distribution we define a generalization of the normal distribution, which was announced in [9].

**Definition 1.1.** For the standard normally distributed random variable $G$ and positive number $\alpha$ define a symmetric random variable $G_{\alpha}$ such that $|G_{\alpha}|$ has the same distribution as $|G|^{2/\alpha}$. We will call $G_{\alpha}$ the standard $\alpha$-normal ($\alpha$-Gaussian) random variable.

The cumulative distribution function $F_{G_{\alpha}}(t)$ of the $\alpha$-Gaussian random variable $G_{\alpha}$, for $t > 0$, has the form

$$F_{G_{\alpha}}(t) = \mathbb{P}(G_{\alpha} \leq t) = 1 - \mathbb{P}(G_{\alpha} > t) = 1 - \frac{1}{2} \mathbb{P}(|G_{\alpha}| > t) \quad \text{(by symmetry of } G_{\alpha})$$

$$= 1 - \frac{1}{2} \mathbb{P}(|G|^{2/\alpha} > t) \quad \text{(by distribution of } G_{\alpha})$$

$$= 1 - \mathbb{P}(G > t^{2/\alpha}) = \mathbb{P}(G \leq t^{2/\alpha}) = \Phi(t^{2/\alpha}),$$

where $\Phi$ denotes the cumulative distribution function of the standard normal distribution.

Let us stress that for $\alpha = 2$ we get the density of the standard normal distribution.

**Density function description** (The following description and the graphic were made by the student Jacek Oszczepaliński).

The density function of the random variable $G_{\alpha}$ is even. One can calculate that, for $x > 0$, its derivative has the following form

$$f_{\alpha}(x) = \frac{\alpha}{2\sqrt{2\pi}} |x|^{\alpha/2} e^{-|x|^\alpha/2}.$$

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**Density function description** (The following description and the graphic were made by the student Jacek Oszczepaliński).

The density function of the random variable $G_{\alpha}$ is even. One can calculate that, for $x > 0$, its derivative has the following form

$$f_{\alpha}'(x) = -\frac{\alpha^2}{4\sqrt{2\pi}} x^{\alpha/2} \left( x^\alpha - \frac{\alpha - 2}{\alpha} \right) e^{-1/2 x^\alpha}.$$

Let us observe that for $0 < \alpha < 2$ the density function has infinite negative slope at 0 ($\lim_{x \to 0^+} f_{\alpha}'(x) = -\infty$) and it is negative for any $x > 0$. If $\alpha = 2$ then $\lim_{x \to 0^+} f_{\alpha}'(x) = f_{\alpha}'(0) = 0$. If $2 < \alpha < 4$ then the slope at 0 is infinite positive, for $\alpha = 4 \lim_{x \to 0^+} f_{\alpha}'(x) = 2/\sqrt{\pi}$, and $f_{\alpha}'(0) = 0$ if $\alpha > 4$. In general, for any $2 < \alpha$ and $x > 0$ the slope of the density function is positive until the value $\sqrt[\alpha/2]{\frac{\alpha - 2}{\alpha}}$, $f_{\alpha}'(\sqrt[\alpha/2]{\frac{\alpha - 2}{\alpha}}) = 0$, and it is negative above $\sqrt[\alpha/2]{\frac{\alpha - 2}{\alpha}}$.

By the above we get that the form of the density function of the $\alpha$-normal distribution changes drastically with the value of $\alpha$. And so, for $0 < \alpha < 2$, the density
function has the vertical asymptote at zero. For $\alpha = 2$, we have the density of the standard normal distribution. For $\alpha > 2$, the density function have a local minimum at zero with a value zero and two maxima at $\pm \sqrt{\frac{\alpha - 2}{\alpha}}$.

The following graphic shows examples of $\alpha$-normal density functions depending on the shape parameter, such as $\alpha = 1$ (red), $\alpha = 2$ (blue), $\alpha = 3$ (purple) and $\alpha = 5$ (green).

![Figure 1: Density function $f_\alpha$ depending on the value of parameter $\alpha$.](image)

**Remark 1.2.** Because the cumulative distribution function of $G_\alpha$ has the form $F_{G_\alpha}(x) = \Phi(x^{1/\alpha})$ if $x \geq 0$ and $F_{G_\alpha}(x) = 1 - \Phi(|x|^{1/\alpha})$ if $x < 0$ then we see that $G_\alpha$ tends in distribution to Rademacher’s distribution as $\alpha \to \infty$.

**Moments and the moment generating function.** Since, for $G \sim \mathcal{N}(0, 1)$ and $p > 0$,

$$
\mathbb{E}(|G|^p) = \frac{2^{p/2}}{\sqrt{\pi}} \Gamma\left(\frac{p + 1}{2}\right),
$$

we immediately get

$$
\mathbb{E}(|G_\alpha|^p) = \mathbb{E}(|G|^{2p/\alpha}) = \frac{2^{p/\alpha}}{\sqrt{\pi}} \Gamma\left(\frac{p}{\alpha} + \frac{1}{2}\right).
$$

Thus the moment generating function of $G_\alpha$ equals

$$
\mathbb{E}\exp(sG_\alpha) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(4^{1/\alpha}s)^k}{k!} \Gamma\left(\frac{2k}{\alpha} + \frac{1}{2}\right).
$$

Now we compare the distribution of $G_\alpha$ with the distributions of the Weibull$(\alpha, \lambda)$ random variables for some $\lambda$’s. It is define that a random variable $X$ majorizes a random variable $Y$ in distribution, if there exists $t_0 \geq 0$ such that

$$
\mathbb{P}(|X| \geq t) \geq \mathbb{P}(|Y| \geq t),
$$

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for any $t > t_0$; see for instance [1, Def. 1.1.2].

**Proposition 1.3.** The standard $\alpha$-normal random variable majorizes the Weibull($\alpha, 1$) random variable and it is majorized by the Weibull($\alpha, 2^{1/\alpha}$) random variable.

**Proof.** It is known that the tails of the Gaussian random variable can be estimated from above in the following way

$$P(|G| \geq t) \leq \exp\left(-\frac{t^2}{2}\right)$$

for any $t \geq 0$; see for instance [2, Prop.2.2.1]). Hence for the $\alpha$-normal random variable we get

$$P(|G_{\alpha}| \geq t) = P(|G| \geq t^{\alpha/2}) \leq \exp\left(-\frac{t}{\sqrt{2\pi}}\right)^{\alpha/2}.$$

Let us observe that the right hand side is the tails of the Weibull($\alpha, 2^{1/\alpha}$) random variable. It means that the Weibull($\alpha, 2^{1/\alpha}$) random variable majorizes the $\alpha$-normal random variable.

In the same source [2, Prop.2.2.1)] one can find the following lower estimate of the tails of the Gaussian random variable

$$P(|G| \geq t) \geq \frac{1}{t \sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right)$$

for $t \geq 1$. Because $\sqrt{2\pi t} \exp(-t^2/2)$ tends to 0 as $t \to \infty$ then there exists $t_0$ such that $\sqrt{2\pi t} \exp(-t^2/2) \leq 1$ for $t \geq t_0$. It gives $\exp(-t^2/2) \leq 1/\sqrt{2\pi t}$ and, in consequence,

$$P(|G| \geq t) \geq \frac{1}{t \sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \geq \exp(-t^2)$$

for $t \geq t_0$. By the above

$$P(|G_{\alpha}| \geq t) = P(|G|^{2/\alpha} \geq t) = P(|G| \geq t^{2/\alpha}) \geq \exp(-t^{\alpha}) = P(W_{\alpha,1} \geq t),$$

for $t \geq t_0^{2/\alpha}$, which means that the $\alpha$-normal random variable majorizes the Weibull($\alpha, 1$) random variable.

Although the the standard $\alpha$-normal distribution is comparable to the Weibull distribution in the above sense, it is significantly different. We will show it on the example of the entropy function. Recall that the differential entropy of the two-parameter Weibull distribution is given by the formula

$$H(W_{\alpha,\lambda}) = \gamma \left(1 - \frac{1}{\alpha}\right) + \ln \left(\frac{\lambda}{\alpha}\right) + 1,$$

where $\gamma$ is the Euler-Mascheroni constant. In the following proposition we derive a formula on the differential entropy for $G_{\alpha}$.

\[\square\]
Proposition 1.4. The differential entropy of $G_\alpha$ has the following form

$$H(G_\alpha) = \left(\frac{1}{\alpha} - \frac{1}{2}\right)(\gamma + \ln 2) + \ln \frac{2\sqrt{2\pi}}{\alpha} + \frac{1}{2},$$

where $\gamma$ denotes the Euler-Mascheroni constant.

Proof. By the definition of the differential entropy and the form of density $f_\alpha$ of $G_\alpha$ we get

$$H(G_\alpha) = -\int_{-\infty}^{\infty} f_\alpha(x) \ln f_\alpha(x) dx$$

$$= -\int_{-\infty}^{\infty} f_\alpha(x) \left[ \ln \frac{\alpha}{2\sqrt{2\pi}} + \left(\frac{\alpha}{2} - 1\right) \ln |x| - \frac{1}{2} |x|^\alpha \right] dx$$

$$= \ln \frac{2\sqrt{2\pi}}{\alpha} + \left(1 - \frac{\alpha}{2}\right) \int_{-\infty}^{\infty} \ln |x| f_\alpha(x) dx + \frac{1}{2} \int_{-\infty}^{\infty} |x|^\alpha f_\alpha(x) dx. \quad (2)$$

Let us observe that

$$\int_{-\infty}^{\infty} |x|^\alpha f_\alpha(x) dx = E|G_\alpha|^\alpha = EG^2 = 1. \quad (3)$$

The integral

$$\int_{-\infty}^{\infty} \ln |x| f_\alpha(x) dx = \frac{\alpha}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \ln |x| |x|^\alpha/2 - 1 e^{-|x|^{\alpha/2}} dx. \quad (4)$$

Substituting $u = x^{\alpha/2} (x > 0)$ we obtain

$$\int_{-\infty}^{\infty} \ln |x| |x|^\alpha/2 - 1 e^{-|x|^{\alpha/2}} dx = \frac{8}{\alpha^2} \int_{0}^{\infty} e^{-\frac{1}{2} u^2} \ln u du. \quad (5)$$

By [4, 4.333] we have that

$$\int_{0}^{\infty} e^{-\frac{1}{2} u^2} \ln u du = \frac{1}{4}(\gamma + \ln 2)\sqrt{2\pi} \quad (6)$$

Summing up (6), (5), (1), (3) and substituting into (2) we obtain

$$H(G_\alpha) = \left(\frac{1}{\alpha} - \frac{1}{2}\right)(\gamma + \ln 2) + \ln \frac{2\sqrt{2\pi}}{\alpha} + \frac{1}{2}. \quad \square$$

Let us note that for $\alpha = 2$ we obtain the differential entropy of the standard Gaussian density.
2 Orlicz norm of the $\alpha$-normal distribution

Let us emphasize that the Weibull random variables form the model examples of random variables with $\alpha$-sub-exponential tail decay. We say that a random variable $X$ has $\alpha$-sub-exponential tail decay if there exist two constant $c, C$ such that for $t \geq 0$ it holds

$$P(|X| \geq t) \leq c \exp \left(-\frac{t}{C}\right)\alpha.$$

Since

$$P(W_{\alpha,\lambda} \geq t) = \exp \left(-\frac{t}{\lambda}\right)\alpha,$$

the Weibull random $W_{\alpha,\lambda}$ has $\alpha$-sub-exponential tail decay with $c = 1$ and $C = \lambda$. Whereas the estimate (1) means that $G_\alpha$ has such tail decay with $c = 1$ and $C = 2^{1/\alpha}$.

The property of $\alpha$-sub-exponential tail decay can be equivalently expressed in terms of so-called (exponential) Orlicz norms. Recall that for any random variable $X$ define the $\psi_\alpha$-norm

$$\|X\|_{\psi_\alpha} := \inf \{K > 0 : \mathbb{E}\exp(|X|/K) \leq 2\};$$

according to the standard convention $\inf \emptyset = \infty$. We will call the above functional $\psi_\alpha$-norm but let us emphasize that only for $\alpha \geq 1$ it is a proper norm. For $0 < \alpha < 1$ it is so-called quasi-norm. It does not satisfy the triangle inequality (see Appendix A in [3] for more details).

One can easily observe that $\|X\|_{\psi_\alpha} = \|X\|_{\psi_\alpha}$ and, moreover, one can easily check that, for $\alpha, \beta > 0$, $\|X^\beta\|_{\psi_\alpha} = \|X\|_{\psi_{\alpha\beta}}^{\beta}$; see Lemma 2.3 in [9].

Since the closed form of the moment generating function of random variable $G^2$ is known, we can calculate the $\psi_\alpha$-norm of $\alpha$-normal random variable $G_\alpha$. Because $G^2$ has $\chi^2_1$-distribution with one degree of freedom whose moment generating function is $\mathbb{E}\exp(sG) = (1 - 2s)^{-1/2}$ for $s < 1/2$ then

$$\mathbb{E}\exp(G^2/K^2) = (1 - 2/K^2)^{-1/2},$$

which is less or equal 2 if $K \geq \sqrt{8/3})$. It gives that $\|G\|_{\psi_2} = \sqrt{8/3}$. The $\psi_2$-norm of $|G|$ is equal to $\psi_2$-norm of $G$. By Lemma 2.3 in [9] and the definition of $\alpha$-normal distribution we get

$$\|G_\alpha\|_{\psi_\alpha} = \|G_\alpha^2/\alpha\|_{\psi_\alpha} = \|G_\alpha^2/\alpha = (8/3)^{1/\alpha}.$$

Remark 2.1. Using the closed form of the moment generating function of the standard exponential random variable and the above mentioned definition of the two-parameter Weibull distribution, similarly as for the standard $\alpha$-Gaussian random variable, one can obtain its $\psi_\alpha$-norm $\|W_{\alpha,\lambda}\|_{\psi_\alpha} = \lambda 2^{1/\alpha}$.

Remark 2.2. Although the Weibull($\alpha, \lambda$) random variables provide model examples of random variables with $\alpha$-sub-exponential tail decay (they are model elements of spaces generated by the $\psi_\alpha$-norms), it can nevertheless be argued that the standard $\alpha$-Gaussian variables play a central role among these variables (in these spaces).
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