Density fluctuations in Brans-Dicke inflation

Alexei A. STAROBINSKY\textsuperscript{1,2} and Jun’ichi YOKOYAMA\textsuperscript{1}

\textsuperscript{1} Yukawa Institute for Theoretical Physics, Kyoto University, Uji 611 (Japan)
\textsuperscript{2} Landau Institute for Theoretical Physics, Kosygina St. 2, Moscow 117334 (Russia)

Abstract

Spectrum of density perturbations in the Universe generated from quantum-gravitational fluctuations in slow-roll-over inflationary scenarios with the Brans-Dicke gravity is calculated. It is shown that after inflation the isocurvature mode of perturbations may be neglected as compared to the adiabatic mode, and that an amplitude of the latter mode is not significantly different from that in the Einstein gravity. However, the account of the isocurvature mode is necessary to obtain the quantitatively correct spectrum of adiabatic perturbations.

I. INTRODUCTION

Inflationary expansion in the early Universe not only is able to explain the observed degree of homogeneity, isotropy and flatness of the present-day Universe \cite{1} but can also account for the origin of small initial density perturbations which produce gravitationally bound objects (galaxies, quasars, etc.) and the large-scale structure of the Universe \cite{2}. Historically, among models of inflation making use of a scalar field (called an inflaton), the original model or the first-order phase transition model \cite{3} failed due to the graceful exit problem, which was taken over by the new \cite{4} and the chaotic \cite{5} inflation scenarios where the inflaton scalar field is slow rolling during the whole de Sitter (inflationary) stage. The latter property was shared by the alternative scenario with higher-derivative quantum gravity corrections \cite{6} (where the role of an inflaton is played by the Ricci scalar $R$) just from the beginning. Note that a simplified version of this scenario - the $R + R^2$ model - was even shown to be mathematically equivalent to some specific version of the chaotic scenario \cite{7} (see also a review in \cite{8}). In order to obtain a small enough amplitude of density perturbations in all these slow-roll-over models, the inflaton should be extremely weakly coupled to other fields. It is therefore not easy to find sound motivations to have such a scalar field in particle physics. (See, however, \cite{9} for recent improvements along this line.)

\*To be published in the proceedings for the fourth workshop on general relativity and gravitation edited by K. Maeda, T. Nakamura, and K. Nakao
Reflecting such a situation, the extended inflation \[10\] scenario was proposed several years ago to revive a GUT Higgs field as the inflaton by adopting non-Einstein gravity theories. Although the first version of the extended inflation model, which considers a first-order phase transition in the Brans-Dicke theory \[11\], resulted in failure again due to the graceful-exit problem \[12\], it triggered further study of more generic class of inflation models in non-Einstein theories, in particular extended chaotic inflation \[13\] where both the inflaton and the Brans-Dicke scalar fields are in the slow rolling regime during inflation. Note that the natural source of Brans-Dicke-like theories of gravity is the low-energy limit of the superstring theory \[14,15\] with the Brans-Dicke scalar being the dilaton.

Several analyses have been done on the density perturbations produced in extended new or chaotic inflation models \[16–20\], all of which made use of the constancy of the Bardeen’s gauge-invariant quantity \(\zeta\) \[21\] or its equivalent on super-horizon scales and match it directly to quantum field fluctuations at the moment of horizon crossing which would be the correct procedure in a single component inflationary model. However, such a treatment is not correct in the present case which contains two sources of quantum fluctuations, namely, the inflaton and the Brans-Dicke scalar field. In such a model not only adiabatic but also isocurvature perturbations are generated and \(\zeta\) does not remain constant for all modes.

In the present article we analyze density perturbations generated in both new and chaotic inflationary models in the Brans-Dicke gravity. Keeping the above point in mind and extending the method used in \[22,23\] to find spectra of all modes of adiabatic and isocurvature fluctuations in the multiple inflationary scenario in the Einstein gravity (with scalar fields interacting with each other through gravity only), we separate adiabatic and isocurvature modes at the inflationary stage and carefully evaluate their final amplitude. As a result, we arrive at a formula which is apparently very different from that used previously but find that constraints on the model parameters in both models remain approximately the same as in Einstein gravity if we express them in terms of the values of coupling constants at the end of inflation, contrary to the analysis by Berkin and Maeda \[16\].

We make use of the conformal transformation which transforms the original or the Jordan frame to the Einstein frame in which equations are somewhat simpler. We calculate the spectrum of fluctuations in the Einstein frame and then interpret the result to the Jordan frame.

The rest of the present article is organized as follows. In §2 we define the Lagrangian and perform conformal transformation to write down the equations of motion in the Einstein frame. Then in §3 solutions to these equations are given and they are interpreted in the Jordan frame. They are applied to both chaotic and new inflationary models in §4. Finally §5 is devoted to discussion and conclusion.

II. BASIC EQUATIONS

A. Conformal Transformation

It has been shown that with the help of a conformal transformation a wide class of non-Einstein gravity models can be recast in the action
\[ S = \int \mathcal{L} \sqrt{-g} d^4x = \int \left[ \frac{1}{2\kappa^2} \mathcal{R} + \frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - U(\chi) + \frac{1}{2} e^{-\gamma \kappa \chi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - e^{-\beta \kappa \chi} V(\phi) \right], \]  

(1)

where \( \kappa^2 = 8\pi G \), \( \beta \) and \( \gamma \) are constants, \( \chi \) and \( \phi \) are the dilaton and the inflaton fields, respectively [10]. For example, in the case of the Brans-Dicke theory [11], the original action in the Jordan frame

\[ S = \int \left[ \Phi_{BD} \frac{\mathcal{R}}{16\pi} + \frac{\omega^2}{16\pi \Phi_{BD}} \dot{g}^{\mu\nu} \partial_\mu \Phi_{BD} \partial_\nu \Phi_{BD} + \frac{1}{2} \dot{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \sqrt{-\dot{g}} d^4x, \]  

(2)

is transformed into (1) through the conformal transformation

\[ g_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu}, \quad \Omega^2 \equiv \frac{\kappa^2}{8\pi} \Phi_{BD} \equiv \exp \left( \frac{-\kappa \chi}{\sqrt{\omega + 3/2}} \right), \]  

(3)

with \( \beta = 2\gamma = \frac{2}{\sqrt{\omega + 3/2}} \) and \( U(\chi) = 0 \). Observations constrain \( \omega \) to be \( \omega > 500 \) [24], so that \( \beta < 0.09 \ll 1 \). Since the Brans-Dicke field \( \Phi_{BD} \) remains practically constant in the radiation or matter dominant stage (see below), \( \chi \) must be equal to zero at the end of inflation in order to reproduce the correct value of the gravitational constant today. Although we focus on the original Brans-Dicke theory in the present paper, we will not put the explicit values for \( \beta \) and \( \gamma \) below until the very end of the calculation in order to sustain applicability of our analysis to other non-Einstein gravity theories.

**B. Background equations in the Einstein frame**

Since the analysis is much simpler in the Einstein frame [1], we first investigate equations of motion derived from (1) and then transform the results into the original frame. Taking the background as the spatially flat Friedmann-Robertson-Walker spacetime,

\[ ds^2 = dt^2 - a(t)^2 dx^2, \]  

(4)

the scalar field equations for the homogeneous parts read

\[ \ddot{\chi} + 3H \dot{\chi} + \frac{\gamma \kappa}{2} e^{-\gamma \kappa \chi} \dot{\phi}^2 - \beta \kappa e^{-\beta \kappa \chi} V(\phi) = 0, \]  

(5)

\[ \ddot{\phi} + 3H \dot{\phi} - \gamma \kappa \dot{\chi} + e^{(\gamma - \beta) \kappa \chi} V'(\phi) = 0, \]  

(6)

with an overdot denoting time derivation. Because \( \beta, \gamma \ll 1 \) and \( V(\phi) \), being the inflaton’s potential, realizes slow roll over of \( \phi \), the above equations can be approximated as

\[ 3H \dot{\chi} = \beta \kappa e^{-\beta \kappa \chi} V(\phi), \]  

(7)

\[ 3H \dot{\phi} = -e^{(\gamma - \beta) \kappa \chi} V'(\phi), \]  

(8)

with

\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa^2}{3} e^{-\beta \kappa \chi} V(\phi), \]  

(9)
during inflation. We note that the above approximation is valid provided that the following inequalities are satisfied.

\[
\max \{e^{-\gamma \kappa \chi^2}, \chi^2 \} \ll e^{-\beta \kappa \chi} V(\phi), \\
|V'(\phi)| \ll 3ke^{-\gamma \kappa \chi/2} V(\phi), \\
V''(\phi)e^{(\gamma-\beta)\kappa \chi} \ll H^2.
\] (10)

Then the system (7)–(9) admits a solution as a function of the scale factor as [13,16]

\[
\chi = \frac{\beta}{\kappa} \ln a \equiv -\frac{\beta}{\kappa} z,
\] (11)

\[
\kappa^2 \int_{\phi_f}^{\phi} \frac{V(\varphi)}{V'(\varphi)} d\varphi = \frac{1-a^{\beta \gamma}}{\beta \gamma} = \frac{1-e^{-\beta \gamma z}}{\beta \gamma},
\] (12)

where a subscript \( f \) denotes the value of each quantity at the end of inflation and we have set \( a_f = 1 \).

In the case that \( \phi \) changes slowly compared with \( \chi \) and that \( V(\phi) \) can be regarded as a constant, evolution of the scale factor and \( \chi \) is given by

\[
a(t) = a_0 t^{2 \frac{\beta \gamma}{\kappa}}, \quad \chi(t) = \frac{1}{\kappa \beta} \ln \left( \frac{V(\phi) \kappa^2 \beta^4 t^2}{12} \right).
\] (13)

We thus have a power-law inflation with an extremely large power exponent.

C. Linear perturbations

We now turn to linear perturbation, which is most easily taken into account in a gauge-invariant manner using the longitudinal gauge,

\[
ds^2 = (1 + 2\Phi)dt^2 - a(t)^2(1 - 2\Psi)\delta_{ij}dx^i dx^j,
\] (14)

where \( \Phi \) and \( \Psi \) are gauge-invariant variables related with Bardeen’s \( \Phi_A \) and \( \Phi_H \) as \( \Phi = \Phi_A \) and \( \Psi = -\Phi_H \) [25]. Assuming an \( e^{ikx} \) spatial dependence, each Fourier mode satisfies the following equations of motion which are derived from the perturbed Einstein equations.

\[
\Phi = \Psi,
\] (15)

\[
\dot{\Phi} + H\Phi = \frac{\kappa^2}{2} (\dot{\chi} \delta \chi + e^{-\gamma \kappa \chi} \dot{\phi} \delta \chi),
\] (16)

\[
\ddot{\chi} + 3H \dot{\chi} + \left( \frac{k^2}{a^2} - \frac{(\gamma \kappa)^2}{2} e^{-\gamma \kappa \chi} \dot{\phi}^2 + (\beta \kappa)^2 e^{-\beta \kappa \chi} V(\phi) \right) \delta \chi + \gamma k e^{-\gamma \kappa \chi} \dot{\phi} \delta \phi \\
- \beta k e^{-\beta \kappa \chi} V'(\phi) \delta \phi = 2(\ddot{\chi} + 3H \dot{\chi}) \Phi + \dot{\Phi} \dot{\chi} + 3\dot{\Psi} \dot{\chi} + \gamma k e^{-\gamma \kappa \chi} \dot{\phi}^2 \Phi,
\] (17)
\[
\delta \phi + (3H - \gamma \kappa \chi) \delta \phi + \left( \frac{k^2}{a^2} + e^{(\gamma - \beta)\kappa \chi} V''(\phi) \right) \delta \phi - \gamma \kappa \phi \dot{\delta} \chi \\
+ (\gamma - \beta) \kappa V'(\phi) e^{(\gamma - \beta)\kappa \chi} \delta \chi = 2(\ddot{\phi} + 3H \dot{\phi})\Phi + \dot{\Phi} \dot{\phi} + 3\dot{\Psi} \dot{\chi} - 2\gamma \kappa \dot{\phi} \dot{\chi} \Phi,
\]

where \(\delta \chi\) and \(\delta \phi\) are gauge invariant fluctuation variables of the respective fields and we have suppressed the argument \(k\). Under the slow-roll approximations \(10\) the last two equations read

\[
\ddot{\delta \chi} + 3H \dot{\delta \chi} + \left( \frac{k^2}{a^2} + (\beta \kappa)^2 e^{-\beta \kappa \chi} V(\phi) \right) \delta \chi + \gamma \kappa e^{-\gamma \kappa \chi} \dot{\phi} \dot{\delta} \chi - \beta \kappa e^{-\beta \kappa \chi} V'(\phi) \delta \phi = 2\beta \kappa e^{-\gamma \kappa \chi} V(\phi) \Phi + \dot{\Phi} \dot{\chi} + 3\ddot{\Psi} \dot{\chi},
\]

\[
\ddot{\delta \phi} + 3H \dot{\delta \phi} + \left( \frac{k^2}{a^2} + e^{(\gamma - \beta)\kappa \chi} V''(\phi) \right) \delta \phi - \gamma \kappa \phi \dot{\delta} \chi + (\gamma - \beta) \kappa V'(\phi) e^{(\gamma - \beta)\kappa \chi} \delta \chi
\]
\]
\[
= -2e^{(\gamma - \beta)\kappa \chi} V'(\phi) \Phi + \dot{\Phi} \dot{\phi} + 3\ddot{\Psi} \dot{\phi}.
\]

III. GENERATION OF PERTURBATIONS

A. Solution of non-decreasing modes

We shall now solve eqs. \(21\)–\(23\). Inserting \(21\) to \(22\) we easily find \(3H \dot{\delta \chi} = 0\), or

\[
\delta \chi = \frac{\beta}{\kappa} Q_1 = \text{const.}
\]

Similarly \(23\) is transformed to

\[
3H \dot{\delta \phi} + e^{(\gamma - \beta)\kappa \chi} \left( V''(\phi) - \frac{V'(\phi)^2}{V(\phi)} \right) \delta \phi + \beta \gamma e^{(\gamma - \beta)\kappa \chi} V'(\phi) Q_1 = 0.
\]

The above equation can be solved taking \(\phi\) as an independent variable instead of \(t\). With the help of \(8\) we find
\[ 3H \frac{d}{dt} = 3H \phi \frac{d}{d\phi} = -e^{(\gamma - \beta)\kappa \chi} \frac{V'}{V} \frac{d}{d\phi}, \] (26)

so that (25) reads

\[ \frac{d}{d\phi} \delta \phi = \left( \frac{V''}{V'} - \frac{V'}{V} \right) \delta \phi + \beta \gamma Q_1. \] (27)

Its solution is given by

\[ \delta \phi = \frac{V'}{V(\phi)} \left( \beta \gamma Q_1 \int_\phi^\phi \frac{V(\varphi)}{V' (\varphi)} d\varphi + \frac{Q_2 - Q_1}{\kappa^2} \right) = -\frac{V''(\phi)}{\kappa^2 V(\phi)} (Q_1 e^{\gamma \kappa \chi} - Q_2), \] (28)

with \( Q_2 \) being another integration constant. We therefore find

\[ \Phi = \frac{\beta^2}{2} Q_1 + \frac{1}{2\kappa^2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2 (e^{\gamma \kappa \chi} Q_1 - Q_2). \] (29)

Thus we have obtained a generic solution of the system (21)–(23) containing two undetermined constants.

In order to clarify physical meaning of the above solution, we should divide it to adiabatic and isocurvature modes. The latter mode is characterized by its vanishingly small contribution to the gravitational potential \( \Phi \), while the growing adiabatic mode can be described by the following universal expression.

\[ \Phi = C_1 \left( 1 - \frac{H}{a} \int_0^t a(t') dt' \right) \approx -C_1 \frac{\dot{H}}{H^2}, \] (30)

\[ \frac{\delta \chi}{\chi} = \frac{\delta \phi}{\phi} = \frac{C_1}{a} \int_0^t a(t') dt' \approx \frac{C_1}{H}, \] (31)

where \( C_1 \) is a constant and the latter approximate equality in each expression is satisfied during the inflationary stage.

From (7)–(9) we find,

\[ \dot{\chi} = \frac{\beta}{\kappa} H, \quad \dot{\phi} = -H e^{\gamma \kappa \chi} \frac{V'}{V}, \quad \frac{\dot{H}}{H^2} = \frac{\beta^2}{2} + \frac{e^{\gamma \kappa \chi}}{2\kappa^2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2, \] (32)

and it turns out that defining new constants \( C_1 \) and \( C_3 \) by \( Q_1 = C_1 - C_3 \) and \( Q_2 = -C_3 \) discriminates between adiabatic and isocurvature modes in the final result. That is,

\[ \frac{\delta \chi}{\chi} = \frac{C_1}{H} - \frac{C_3}{H}, \] (33)

\[ \frac{\delta \phi}{\phi} = \frac{C_1}{H} + \frac{C_3}{H} (e^{-\gamma \kappa \chi} - 1), \] (34)

\[ \Phi = -C_1 \frac{\dot{H}}{H^2} + C_3 \left[ \frac{1}{2\kappa^2} (1 - e^{\gamma \kappa \chi}) \left( \frac{V'(\phi)}{V(\phi)} \right)^2 - \frac{\beta^2}{2} \right]. \] (35)

In the above expressions, terms in proportion to \( C_1 \) and \( C_3 \) represent adiabatic and isocurvature modes, respectively. The isocurvature nature of the \( C_3 \)-terms is guaranteed by the fact that the second term in the right-hand-side of (33) is vanishingly small in the last stage of inflation when we have \( \chi \approx 0 \).
B. Quantum fluctuations

We shall next determine the constants $C_1$ and $C_3$ from amplitudes of quantum fluctuations of the scalar fields generated during the inflationary stage. Thanks to the inequalities (10), eqs. (19) and (20) can be approximated by equation of motion of a free massless scalar field in inflating background for $k \geq aH$ and even in the region $k < aH$ but with $H(t_k) \gg |\dot{H}(t_k)|(t - t_k)$ where $t_k$ is the time $k$-mode leaves the Hubble horizon during inflation. The standard quantization gives the well-known result, that is, the Fourier components of the fields can be represented in the form

$$
\delta \chi(k) = \frac{H(t_k)}{\sqrt{2k^3}} \epsilon_\chi(k), \quad \delta \phi(k) = \frac{H(t_k)}{\sqrt{2k^3}} e^{\gamma \kappa \chi(t_k)/2} \epsilon_\phi(k),
$$

where the exponential factor in the latter equality is present because $\phi$ has a non-canonical kinetic term in the action in the Einstein frame. Here $\epsilon_\chi(k)$ and $\epsilon_\phi(k)$ are classical random Gaussian quantities with the following averages.

$$
\langle \epsilon_\chi(k) \rangle = \langle \epsilon_\phi(k) \rangle = 0, \quad \langle \epsilon_i(k) \epsilon_j^*(k') \rangle = \delta_{ij} \delta(3)(k - k'), \quad i, j = \chi, \phi.
$$

We thus find

$$
C_1 = \left[ e^{\gamma \kappa \chi} \frac{H \delta \phi}{\phi} + (1 - e^{\gamma \kappa \chi}) H \frac{\delta \chi}{\chi} \right]_{t_k} = \frac{H^2(t_k)}{\sqrt{2k^3}} \left[ \frac{e^{\frac{3}{2} \gamma \kappa \chi}}{\phi} \epsilon_\phi(k) + \frac{1 - e^{\gamma \kappa \chi}}{\chi} \epsilon_\chi(k) \right]_{t_k},
$$

$$
C_3 = \left[ e^{\gamma \kappa \chi} \left( \frac{\delta \phi}{\phi} - \frac{\delta \chi}{\chi} \right) \right]_{t_k} = \frac{H^2(t_k)}{\sqrt{2k^3}} \left[ \frac{e^{\frac{3}{2} \gamma \kappa \chi}}{\phi} \epsilon_\phi(k) - \frac{\epsilon_\chi(k)}{\chi} \right]_{t_k}.
$$

Note the interesting fact that in the specific case of the "soft" inflation [26] ($\gamma \equiv 0$), the Brans-Dicke dilaton does not contribute to the adiabatic mode ($C_1$) at all.

C. Adiabatic modes

From (30) curvature perturbation due to primordially adiabatic fluctuation is given by

$$
\Phi_{ad} = \left[ 1 + \frac{2}{3(1 + w)} \right]^{-1} C_1, \quad w \equiv \frac{p}{\rho},
$$

with $p$ and $\rho$ being total pressure and energy density, respectively. since $\Phi$ is related to the gauge-invariant density fluctuation $\delta \rho^{(c)}/\rho$, which reduces to $\delta \rho/\rho$ in the comoving gauge, as

$$
\frac{\delta \rho^{(c)}}{\rho} = \frac{2}{3} \left( \frac{k}{H a} \right)^2 \Phi,
$$

we find

$$
\frac{\delta \rho^{(c)}}{\rho} = f \left[ e^{\gamma \kappa \chi} H \frac{\delta \phi}{\phi} + (1 - e^{\gamma \kappa \chi}) H \frac{\delta \chi}{\chi} \right]_{t_k},
$$

where $f$ is the constant of proportionality.
at the horizon crossing \((k = aH)\), where \(f\) is a constant equal to 4/9 during radiation domination and to 2/5 during matter domination (we omit the minus sign here because it may be absorbed into the stochastic variables \(\delta \phi\) and \(\delta \chi\)).

The above formula is quite different from that used by the previous authors [16–20], who did not account for the isocurvature mode and its mixing with the adiabatic mode during inflation, and concluded that a density fluctuation at the second horizon crossing is approximately equal to \(\delta \rho / (\rho + p)\) evaluated at the first horizon crossing during inflation. That is, up to a factor of order of unity,

\[
\frac{\delta \rho}{\rho} \simeq \frac{H(|\dot{\chi}|\delta \chi + e^{-\gamma \kappa \chi} |\dot{\phi}| \delta \phi)}{\dot{\chi}^2 + e^{-\gamma \kappa \phi^2}} \bigg|_{t_h} \tag{incorrect}, \label{43}
\]

which gives

\[
\frac{\delta \rho}{\rho} \simeq \begin{cases} 
\frac{H|\dot{\chi}|}{|\dot{\phi}|} & \text{for } |\dot{\chi}| > |\dot{\phi}| e^{-\gamma \kappa \chi/2} \\
\frac{H|\dot{\phi}|}{|\dot{\chi}|} & \text{for } |\dot{\chi}| < |\dot{\phi}| e^{-\gamma \kappa \chi/2}
\end{cases} \tag{incorrect}. \label{44}
\]

Note also that we would have reached the above formula \((43)\) or its analogue in the Jordan frame [20], had we put \(C_3 = 0\) neglecting the isocurvature mode in \((35)\). Clearly this is not justifiable because it results in imposing a redundant constraint between \(\delta \chi\) and \(\delta \phi\) in eqs. \((33)\) and \((34)\). The difference between the expressions \((42)\) and \((43)\) is especially large near the points where \(V''(\phi) = 0\). Then Eq. \((42)\) predicts much larger adiabatic perturbations than Eq. \((43)\).

We are interested in the amplitude of density fluctuations about 40 \(\sim 60\) e-folds before the end of inflation, or at \(z \simeq 40 \sim 60\), corresponding to large-scale structures within the horizon scale today. From \((11)\) the corresponding value of \(\chi\) ranges \(\kappa \chi = -40\beta \sim -60\beta\), for which we have \(e^{\gamma \kappa \chi} \simeq 1.2\) with \(\omega \simeq 500\). Thus the above incorrect formula and ours give practically opposite results with respect to the two cases \(|\dot{\chi}| \gtrsim |\dot{\phi}|\) and \(|\dot{\chi}| \lesssim |\dot{\phi}|\). In some models of inflation, unless \(\beta\) is too small, we find \(|\dot{\phi}| < |\dot{\chi}|\), so that the adiabatic fluctuation is given by

\[
\frac{\delta \rho}{\rho} \simeq e^{\gamma \kappa \chi} \frac{H|\dot{\chi}|}{|\dot{\phi}|}, \tag{45}
\]

which gives practically the same result as in the Einstein gravity. On the other hand, in the limit \(\omega \rightarrow \infty\) or \(\beta, \gamma \rightarrow 0\), we find \(e^{\gamma \kappa \chi} = 1\) so that the second term in \((42)\) does not contribute to adiabatic fluctuation and we again find

\[
\frac{\delta \rho}{\rho} \simeq \frac{H|\dot{\phi}|}{|\dot{\chi}|}, \tag{46}
\]

in agreement with the result in the Einstein gravity.

### D. Isocurvature modes

Isocurvature fluctuations can be significant if some ingredient of the model is essentially decoupled from the usual matter and constitutes a part of dark matter. In our model, the Brans-Dicke dilaton is a candidate of such an ingredient.
During inflation, its fractional comoving energy perturbation satisfies the inequality,
\[
\frac{\delta \rho^{(c)}}{\rho_\chi} < \frac{2\delta \rho^{(c)}}{\rho_\chi + p_\chi} = 2 \frac{\partial}{\partial \chi} \left( \frac{\delta \chi}{\chi} \right) - 2\Phi = \frac{C_3}{\kappa^2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2 < C_3.
\] (47)

Since $C_3$ is of the same order of $C_1$, which gives the amplitude of adiabatic perturbation at the second horizon crossing, fractional density fluctuation in $\chi$ is smaller than the adiabatic one. Furthermore, its contribution to the total energy density fluctuation, $\delta \rho^{(c)}/\rho$ turns out to be much smaller than the adiabatic counterpart, because we find $\rho_\chi = \mathcal{O}(\omega^{-1}) \rho \ll \rho$ in the post inflationary universe as will be seen below. We, therefore, conclude that isocurvature fluctuations are negligible in this model.

**E. Density fluctuations in the physical (Jordan) frame**

We can relate the amplitude of density fluctuations in the Einstein frame with that in the Jordan frame using the conformal transformation [27]. First we note that gauge-invariant variables in both frames are related with each other as
\[
\hat{\Phi} = \Phi - \frac{\delta \Omega}{\Omega}, \quad \hat{\Psi} = \Psi + \frac{\delta \Omega}{\Omega},
\] (48)

where $\delta \Omega = \frac{\kappa}{2} \delta \chi \Omega$ is gauge-invariant perturbation of $\Omega$. Using the equalities
\[
\dot{a} = \Omega^{-1} a, \quad \dot{d} = \omega^{-1} dt, \quad \dot{H} = \frac{\dot{a}_i}{\dot{a}} = \Omega \left( H - \frac{\dot{\Omega}}{\Omega} \right),
\] (49)

one can write (30) and (31) in terms of physical variables as
\[
\Phi = C_1 \left[ 1 - \left( \dot{H} + \frac{\Omega_i}{\Omega} \right) \frac{1}{\dot{a} \Omega^2} \int \dot{a} \Omega^2 d\dot{t}' \right],
\] (50)

\[
\frac{\delta \chi}{\chi} = \frac{C_1}{\dot{a} \Omega} \int \dot{a} \Omega^2 d\dot{t}'.
\] (51)

We, therefore, find
\[
\hat{\Phi} = C_1 \left[ 1 - \left( \dot{H} + 2 \frac{\Omega_i}{\Omega} \right) \frac{1}{\dot{a} \Omega^2} \int \dot{a} \Omega^2 d\dot{t}' \right],
\] (52)

\[
\hat{\Psi} = C_1 \left[ 1 - \frac{\dot{H}}{\dot{a} \Omega^2} \int \dot{a} \Omega^2 d\dot{t}' \right].
\] (53)

The above two equations are the desired formula to relate density fluctuations in the Einstein frame and those in the Jordan frame through $C_1$. In practice, however, since the Brans-Dicke field varies extremely slowly in the post-inflationary universe, one can regard $\Omega$ as a constant in these equations. For example, in the matter dominated stage, we find [11]
\[
\Phi_{BD}(\hat{t}) = \Phi_{BDO} \left( \frac{\hat{t}}{t_0} \right)^{\frac{2}{3(\omega - 1)}},
\] (54)

\[
\dot{a}(\hat{t}) = \dot{a}_0 \left( \frac{\hat{t}}{t_0} \right)^{\frac{2 + 2\omega - 1}{3 + 4\omega - 2}},
\] (55)
Hence $\Omega_t/\Omega$ is smaller than $\dot{H}$ by a factor less than 1/500. Thus we may simply conclude that the adiabatic fluctuations are described by the same formula in both frames.

Using (54), we can estimate energy density of $\chi$ in the Einstein frame in this stage as

$$\rho_\chi = \frac{1}{2} \dot{\chi}^2 = \frac{2\omega + 3}{(3\omega + 5)^2} \frac{1}{\kappa^2 t^2} \simeq \frac{1}{6\omega} \rho.$$  

(56)

Thus we may conclude the isocurvature perturbation due to $\chi$ is not important as stated in the previous subsection.

**IV. APPLICATION TO SPECIFIC INFLATIONARY MODELS**

Having developed general considerations, we now apply the above results to two specific inflation models, namely chaotic inflation [5] and new inflation [4], and examine constraints on their model parameters as well as the spectral shape.

**A. Chaotic inflation**

Here we consider two different potentials $V(\phi) = \frac{1}{2} m^2 \phi^2$ and $V(\phi) = \frac{1}{2} \lambda \phi^4$ which are denoted collectively as $V(\phi) = \frac{n}{n} \phi^n$. In this model, inflation occurs at large $\phi$ and it is terminated when $|\dot{\phi}/\phi|$ becomes as large as $H$ at $\phi = \phi_f = \sqrt{n}/\kappa$. Since we are interested in the last 60 e-foldings in the inflationary stage, we may approximate (12) as

$$\kappa^2 \int_{\phi_f}^{\phi} V(\varphi) \frac{d\varphi}{V'(\varphi)} = \frac{\kappa^2}{2n} \left( \phi^2 - \frac{n}{\kappa^2} \right) \simeq z.$$  

(57)

The error in the last expression is only about 12% with $z = 60$, $\beta = 2\gamma = 0.09$. The e-folding number, $z_k$, when the comoving wave-number $k$ leaves the Hubble radius during inflation satisfies

$$\frac{k}{k_f} = e^{\left(1 - \frac{\beta^2}{2}\right) z_k} \left(2z_k\right)^\frac{\beta}{2}, \quad \text{for} \quad 1 \ll z_k \lesssim 60.$$  

(58)

We can express the amplitude of curvature perturbation on scale $l = \frac{2\pi}{k}$ as

$$\dot{\Phi}(l) = \left[ 1 + \frac{2}{3(1 + w)} \right]^{-1} \frac{\sqrt{2\kappa^3 |C_1|^2}}{2\pi} \left[ \left( \frac{2n\beta}{3n} \frac{e^{\beta(\gamma-\beta)z_k} - 1}{\beta} \right) + \frac{e^{-\beta^2 z_k}}{\beta} \right], \quad \text{for} \quad 1 \ll z_k \lesssim 60.$$  

(59)

Since the large-angular-scale anisotropy of background radiation due to the Sachs-Wolfe effect is given by $\delta T/T = \dot{\Phi}/3$, we can normalize the value of $\lambda_n$ by the COBE observation [28]. For $\beta = 2\gamma = 0.09$, we find
\[
\frac{\delta T}{T} = \frac{1}{3} \hat{\Phi}(z_k \simeq 60) \simeq \begin{cases} 
\frac{9 m}{32 \sqrt{\lambda_4}} & \text{for } n = 2 \\
\frac{32 \sqrt{\lambda_4}}{\sqrt{\lambda_4}} & \text{for } n = 4
\end{cases}
\]
\[
= 1.1 \times 10^{-5}
\]  

Note that our normalization of the background solution \( a_f = 1, \chi_f = 0 \) (see Eqs. (11,12)) means that \( m \) and \( \lambda_4 \) are values of physical parameters ("coupling constants") at the end of inflation. However, since \( \chi \) generally grows only as logarithm of \( t \) after inflation and is even constant during the whole radiation-dominated stage in the Brans-Dicke case \( \beta = 2\gamma \), their present values are not significantly different from those at the end of inflation. So, it is natural to use just the latter values when comparing non-Einstein gravity models having variable coupling constants with models based on the Einstein gravity. We find

\[
m = 1 \times 10^{13}\text{GeV}, \quad n = 2, \quad (62)
\]
\[
\lambda_4 = 1 \times 10^{-13}, \quad n = 4 \quad (63)
\]

which is no different from the values obtained assuming the Einstein gravity \[29\]. Since the behavior of the system approaches to that in the Einstein gravity as we increase \( \omega \), we can conclude that in Brans-Dicke theory, model parameters of the inflaton’s potential should take the same value as in the Einstein gravity.

**B. New inflation**

Next we consider new inflation with a potential

\[
V(\phi) = V_0 - \frac{\lambda}{4} \phi^4,
\]

for which we find, from (12),

\[
\kappa^2 \int_{\phi_0}^{\phi_f} \frac{V(\phi)}{V'(\phi)} d\phi \simeq \frac{\kappa^2 V_0}{2\lambda} \left( \frac{1}{\phi^2} - \frac{1}{\phi^2_f} \right) \simeq \frac{\kappa^2}{2\lambda \phi^2} \simeq z.
\]  

We can again express the amplitude of curvature fluctuation as a function of \( z_k \), which is now related with \( k \) as,

\[
\frac{k}{k_f} = e^{\left(1 - \frac{\delta^2}{2}\right)z_k} H_f.
\]  

\[
\hat{\phi}(l) = \left[ 1 + \frac{2}{3(1 + w)} \right]^{-1} \left[ e^{-\frac{1}{2}\beta(\beta-\gamma)z_k} \sqrt{\frac{\lambda}{3}(2z_k)} \frac{\lambda H_f e^{-\delta^2 z_k}}{\beta} (e^{\beta \gamma z_k} - 1) \right].
\]  

with \( H_f \equiv \sqrt{\frac{\kappa^2}{3} V_0} \). Taking \( \beta = 2\gamma = 0.09 \) again, it predicts the amplitude of \( \delta T/T \) to be compared with COBE data as

\[
\frac{\delta T}{T}(z_k \simeq 60) \simeq 21\sqrt{\lambda} + 0.4 \frac{H_f}{M_{Pl}}.
\]  

\[11\]
Since $H_f$ should also satisfy

$$\frac{H_f}{M_{Pl}} \lesssim 10^{-5}$$  \hfill (69)

to suppress long-wave gravitational radiation of quantum origin \cite{30}, we find

$$\lambda \lesssim 2 \times 10^{-13}$$  \hfill (70)

from (68). Again its amplitude is practically no different from the case of the Einstein gravity.

V. CONCLUSION

In the present paper we have investigated density perturbations generated during inflation in the Brans-Dicke theory, paying attention to the fact that quantum fluctuations of both the inflaton and the Brans-Dicke dilaton fields contribute to them. This implies that not only adiabatic (curvature) but also isocurvature perturbations exist and that the simple conservation formula of a perturbation outside the Hubble radius cannot be used.

Working in the Einstein frame, which is connected with the original frame through a conformal transformation, we have obtained expressions for growing adiabatic and isocurvature modes and have shown that only the former is important after inflation. Then we have corrected the previously used formula based on the use of incorrect matching of superhorizon perturbations with quantum fluctuations at horizon (Hubble radius) crossing. Since both the inflaton and the dilaton turn out to contribute to the perturbation with the same order of magnitude, the amplitude itself is not significantly altered compared with the case of inflation in the Einstein gravity.

However, our constraints on the model parameters turned out to be more severe than those obtained by Berkin and Maeda \cite{16}. This is because they have taken the initial value of $\chi$ rather arbitrary, which might be appropriate to the case of original soft inflation model in which inflaton’s potential is multiplied by an exponential potential by hand \cite{26}. On the other hand, we, concentrating on the Brans-Dicke model, had to normalize its value to be zero after inflation to reproduce the correct gravitational constant today.

In conclusion, values of inflaton’s coupling parameters in the Brans-Dicke gravity (if taken at the end of inflation) which are required to produce the correct amplitude of present adiabatic perturbations are practically no different from those in the Einstein gravity in the case of both chaotic and new inflationary models.

ACKNOWLEDGMENTS

A. S. is grateful to Profs. Y. Nagaoka and J. Yokoyama for their hospitality at the Yukawa Institute for Theoretical Physics, Kyoto University where this project was started. A. S. was supported in part by the Russian Foundation for Basic Research, Project Code 93-02-3631, and by Russian Research Project “Cosmomicrophysics”. J. Y. acknowledges support by the Japanese Grant-in-Aid for Scientific Research Fund of Ministry of Education, Science, and Culture, No. 06740216.
REFERENCES

[1] For a review of inflation see, e.g. A.D. Linde, Particle Physics and Inflationary Cosmology (Harwood, 1990).
[2] S.W. Hawking, Phys. Lett. 115B, 295 (1982); A.A. Starobinsky, Phys. Lett. 117B, 175 (1982); A.H. Guth and S-Y. Pi, Phys. Rev. Lett. 49, 1110 (1982).
[3] A.H. Guth, Phys. Rev. D23, 347 (1981); K. Sato, Mon. Not. R. astr. Soc. 195, 467 (1981).
[4] A.D. Linde, Phys. Lett. 108B, 389 (1982); A. Albrecht and P.J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982).
[5] A.D. Linde, Phys. Lett. 129B, 177 (1983).
[6] A.A. Starobinsky, Phys. Lett. 91B, 99 (1980).
[7] B. Whitt, Phys. Lett. 145B, 176 (1984).
[8] S. Gottlöber, V. Müller, H.-J. Schmidt and A.A. Starobinsky, Int. J. Mod. Phys. D1, 257 (1992).
[9] H. Murayama, H. Suzuki, T. Yanagida, and J. Yokoyama, Phys. Rev. Lett. 70, 1912 (1993); Phys. Rev. D50, R2356 (1994).
[10] D. La and P.J. Steinhardt, Phys. Rev. Lett. 62, 376 (1989).
[11] C. Brans and R.H. Dicke, Phys. Rev. 24, 925 (1961).
[12] D. La, P.J. Steinhardt, and E.W. Bertschinger, Phys. Lett. B 231, 231 (1989); E. Weinberg, Phys. Rev. D40, 3950 (1989).
[13] A.D. Linde, Phys. Lett. 238B, 160 (1990).
[14] E.S. Fradkin and A.A. Tseytlin, Phys. Lett. 158B, 316 (1985); Nucl. Phys. B261, 1 (1985).
[15] C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, Nucl. Phys. B262, 593 (1985); C.G. Callan, I.R. Klebanov and M.J. Perry, Nucl. Phys. B278, 78 (1986).
[16] A.L. Berkin and K. Maeda, Phys. Rev. D44, 1691 (1991).
[17] J. McDonald, Phys. Rev. D44, 2314 (1991).
[18] S. Mollerach and S. Matarrese, Phys. Rev. D45, 1961 (1992).
[19] N. Deruelle, C. Gundlach and D. Langlois, Phys. Rev. D46, 5337 (1992).
[20] J. García-Bellido, A.D. Linde and D.A. Linde, Phys. Rev. D50, 730 (1994); J. García-Bellido, Nucl. Phys. B423, 221 (1994).
[21] J.M. Bardeen, P.J. Steinhardt, and M.S. Turner, Phys. Rev. D28, 679 (1983).
[22] A.A. Starobinsky, JETP Lett. 42, 152 (1985).
[23] D. Polarski and A.A. Starobinsky, Nucl. Phys. B385, 623 (1992); Phys. Rev. D51, 6123 (1994).
[24] R.D. Reasenberg et al. Astrophys. J. Lett. 234, L219 (1979).
[25] J.M. Bardeen, Phys. Rev. D22, 1882 (1980).
[26] A.L. Berkin, K. Maeda and J. Yokoyama, Phys. Rev. Lett. 65, 141 (1990).
[27] J. Hwang, Class. Quant. Grav. 7, 1613 (1990).
[28] G.F. Smoot, et al. Astrophys. J. Lett. 396, L1 (1992).
[29] D.S. Salopek, Phys. Rev. Lett. 69, 3602 (1992).
[30] V.A. Rubakov, M.V. Sazhin, and A.V. Veryaskin, Phys. Lett. 115B, 189 (1982).