Optimal Control and Numerical Methods for Hybrid Stochastic SIS Models

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Abstract

This work focuses on optimal controls of a class of stochastic SIS epidemic models under regime switching. By assuming that a decision maker can either influence the infectivity period or isolate infected individuals, our aim is to minimize the expected discounted cost due to illness, medical treatment, and the adverse effect on the society. In addition, a model with the incorporation of vaccination is proposed. Numerical schemes are developed by approximating the continuous-time dynamics using Markov chain approximation methods. It is demonstrated that the approximation schemes converge to the optimal strategy as the mesh size goes to zero. Numerical examples are provided to illustrate our results.

Key words. Controlled regime-switching diffusion; SIS epidemic model; Markov chain approximation; isolation; vaccination.

Brief Title. Control and Numerics for Hybrid Stochastic SIS Models

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1 Introduction

This work focuses on optimal controls and numerical methods for a class of stochastic epidemic models under regime switching. A SIS system stems from the compartmental models [18], which subdivides the population into susceptible (S) and infected (I) classes. The word SIS is an abbreviation of susceptible-infected-susceptible. That is, a susceptible individual becomes infected, and later on becomes susceptible again; see [1, 4, 18] for detailed discussions, and [13] and references therein for different applications. The models include in particular diseases being sexually transmitted and diseases being transmitted by bacteria, in which there is no permanent immunity. For such diseases, a promising model is the following classical deterministic SIS epidemic model

\[
\begin{aligned}
\frac{dS(t)}{dt} &= \left[ \mu N - \beta S(t)I(t) + \gamma I(t) - \mu S(t) \right] dt, \quad S(0) = s_0, \\
\frac{dI(t)}{dt} &= \left[ \beta S(t)I(t) - (\mu + \gamma) I(t) \right] dt, \quad I(0) = i_0,
\end{aligned}
\]

subject to \(S(t) + I(t) = N\), along with the initial values \(S(0) = s_0 > 0\) and \(I(0) = i_0 > 0\), where \(S(t)\) and \(I(t)\) are the numbers of susceptible and infected individuals at time \(t\) in a population of size \(N\), respectively; \(\mu\) and \(\gamma^{-1}\) are the average death rate and the average infectious period, respectively. The parameter \(\beta\) is the disease transmission coefficient with \(\beta = \lambda/N\) and \(\lambda\) being the disease contact rate of an infective individual. More specifically, \(\lambda\) is the per day average number of adequate contacts of an infective so that after an adequate contact with an infective, a susceptible individual becomes infected. There is a variety of applications of SIS models. To mention just a few, we refer to [12] for a white noise parameter perturbation of system (1.1), and [13] for a SIS model with Markovian switching; for SIS models with vaccination we refer to [15, 22, 34, 33]. All the papers above focus on the asymptotic properties of the diseases. The reader can also find related works on long-time behaviors of various epidemic models in [9, 26, 30].

Although SIS epidemic models and various epidemic models have been well studied, the work on optimal control of stochastic epidemic models is relatively scarce and the problem is largely open. The objective of such control problems is to identify effective strategies for minimizing the impacts of infectious diseases through a set of mixed control strategies including treatments, vaccination, isolation, and health promotion campaigns, etc. Most of the published papers focus on deterministic epidemics and the primary method applied to solve the associated control problems is Pontryagin’s maximum principle. To mention just a few, we refer to [25] for a work in the early days of this area, [2, 5, 6, 8, 14] for a deterministic SIR model with isolation and/or vaccination, [7, 28] for controlled epidemic network models. In [17], the authors studied two stochastic SIS models with external parameter noise and scaled additive noise to minimize the long term average costs. In [19], the authors investigated SIS meta-population models on networks by comparing different approaches numerically. The recent work [11] studies a SIS model under complete and incomplete observation of the state process by investigating the associated Hamilton-Jacobi-Bellman equation and the Kol-
mogorov forward equation respectively. Recently, increasing attention has been devoted to optimal control of the deterministic SIR model due to the appearance of the highly infectious diseases such as COVID-19; see [3, 10, 24]. To the best of our knowledge, not much attention has been given to stochastic hybrid SIS models to date. The main difficulties come from the complexity of the model. Even for the stochastic SIS without regime switching, the methods and results in [11] only works under a set of strict assumptions on the model and the cost function. In this work, we focus on a stochastic SIS model under Markovian switching. We develop numerical approximation schemes by approximating the continuous-time dynamics by Markov chains and then showing that the approximations converge to the correct optimal strategy as the mesh size goes to zero. Note that the addition of the Markovian switching is to account for environment changes that cannot be modeled by Brownian type of noise, but rather displaying jump behavior. We use the Markov chain approximation methodology developed by Kushner and Dupuis [21]; see also [29]. Motivated by recent developments in modeling of epidemics, we consider two different models. The first one is a direct extension of (1.1), in which the decision maker is able to control the recovery rate to some extent or isolate a part of the infected individuals. In the second model, we incorporate the vaccination into the formulation. That is, the decision maker is able to control the recovery rate and also to take a decision on the fraction of vaccinated individuals. Moreover, treating a general cost function, we take into account of any cost associated with the control and either the outbreak size or the infectious burden under the assumption that there are limited control resources.

The rest of the work is organized as follows. Section 2 begins with the problem formulation. Section 3 presents numerical algorithms based on the Markov chain approximation method. Section 4 focuses on a model with the incorporation of vaccination. In Section 5, we present several examples. Finally, the paper is concluded with some further remarks. To facilitate the reading, all proofs are placed in an appendix at the end of the paper in order not to interrupt the flow of presentation.

2 Formulation

Inspired by the recent trend in modeling using regime-switching models, we use a continuous-time Markov chain to model environment changes that are not covered in the usual diffusion models. Assume throughout the paper that both the Markov chain $\alpha(t)$ and the scalar standard Brownian motion $w(\cdot)$ are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}(t), \mathbb{P})$, where $\{\mathcal{F}(t)\}$ is a filtration satisfying the usual condition (i.e., right continuous, increasing, and $\mathcal{F}(0)$ containing all the null sets).

Started with the classical deterministic SIS epidemic model given in (1.1), we normalize
the population size to one by replacing $S$ and $I$ with $S/N$ and $I/N$, respectively. We obtain

$$\begin{aligned}
    dS(t) &= \left[\mu - \lambda S(t)I(t) + \gamma I(t) - \mu S(t)\right]dt, \quad S(0) = s_0, \\
    dI(t) &= \left[\lambda S(t)I(t) - (\mu + \gamma)I(t)\right]dt, \quad I(0) = i_0,
\end{aligned}$$

where $S(t)$ and $I(t)$ are the fraction of susceptible and infected individuals at time $t$, respectively. Throughout the paper, we use $i$ and $s$ to denote the state variables of $I(t)$ and $S(t)$, respectively, with $i_0$ and $s_0$ being the initial data. Taking into account the environmental noise, the system parameters $\mu$, $\lambda$, $\gamma$ may experience abrupt changes, which is modeled by a Markov chain $\alpha(\cdot)$ as in [13]

$$\begin{aligned}
    dS(t) &= \left[\mu_{\alpha(t)} - \lambda_{\alpha(t)} S(t)I(t) + \gamma_{\alpha(t)} I(t) - \mu_{\alpha(t)} S(t)\right]dt, \\
    dI(t) &= \left[\lambda_{\alpha(t)} S(t)I(t) - (\mu_{\alpha(t)} + \gamma_{\alpha(t)}) I(t)\right]dt, \\
    S(0) &= s_0, \quad I(0) = i_0, \quad \alpha(0) = \ell_0.
\end{aligned}$$

We suppose that $\alpha(\cdot)$ is a continuous time Markov chain taking values in a finite set $\mathcal{M} = \{1, 2, \ldots, m_0\}$ generated by $\Lambda = (\Lambda_{ij})_{m_0 \times m_0}$. The long time behavior of (2.2) has been studied in [13]. A key parameter of the system is the average number of contacts per day $\lambda_{\alpha(t)}$, which can be perturbed by white noise; that is, $\lambda_{\alpha(t)} \rightarrow \lambda_{\alpha(t)} + \sigma_{\alpha(t)} \tilde{w}(t)$ with $w(\cdot)$ being a scalar standard Brownian motion independent of $\alpha(\cdot)$. Thus,

$$\begin{aligned}
    dS(t) &= \left[\mu_{\alpha(t)} - \lambda_{\alpha(t)} S(t)I(t) + \gamma_{\alpha(t)} I(t) - \mu_{\alpha(t)} S(t)\right]dt - \sigma_{\alpha(t)} S(t)I(t)dw(t), \\
    dI(t) &= \left[\lambda_{\alpha(t)} S(t)I(t) - (\mu_{\alpha(t)} + \gamma_{\alpha(t)}) I(t)\right]dt + \sigma_{\alpha(t)} S(t)I(t)dw(t), \\
    S(0) &= s_0, \quad I(0) = i_0, \quad \alpha(0) = \ell_0.
\end{aligned}$$

As in [11, 17, 19], we suppose that the natural development of the disease can be influenced by a decision maker. In particular, the decision maker is able to control the magnitude of the recovery rate to some extent. Such a control can be performed by increasing the treatment capacity or the efficiency of medication. To be more specific, $\gamma_{\alpha(t)}$ is the recovery rate without any action by the decision maker, while $\gamma_{\alpha(t)} + C(t)$ is the recovery rate at time $t$ if a control $C(t) \geq 0$ is applied at $t$. As in [19], $C(t)$ can be interpreted as the product $C(t) = \tilde{C}(t)\tilde{\zeta}$, where $\tilde{C}(t)$ is the proportion of the infected population treated at time $t$ and $\tilde{\zeta}$ is the treatment effectiveness. We can also regard $C(t)$ as per-capital rate of isolation, which has a direct effect only on infected individual. Thus, the model under consideration can be regarded as a standard isolation model; see [5, 14]. To indicate that we have limited resources to handle the epidemic, we suppose that $C(\cdot)$ takes values in a nonempty compact set $\mathcal{U}$ of $[0, \infty)$, where $0 \in \mathcal{U}$. The controlled version is given by

$$\begin{aligned}
    dS(t) &= \left[\mu_{\alpha(t)} - \lambda_{\alpha(t)} S(t)I(t) + \gamma_{\alpha(t)} I(t) - \mu_{\alpha(t)} S(t)\right]dt - \sigma_{\alpha(t)} S(t)I(t)dw(t), \\
    dI(t) &= \left[\lambda_{\alpha(t)} S(t)I(t) - (\mu_{\alpha(t)} + \gamma_{\alpha(t)} + C(t)) I(t)\right]dt + \sigma_{\alpha(t)} S(t)I(t)dw(t), \\
    S(0) &= s_0, \quad I(0) = i_0, \quad \alpha(0) = \ell_0.
\end{aligned}$$

Before proceeding further, we state the following result regarding the existence of a unique positive solution to (2.4).
Theorem 2.1. For any given initial value \((s_0, i_0, \ell_0) \in (0,1)^2 \times \mathcal{M}\) satisfying \(s_0 + i_0 = 1\) and \(C(t) \equiv c_0 \in \mathcal{U}\), equation (2.4) has a unique global solution \((S(t), I(t), \alpha(t)) \in (0,1)^2 \times \mathcal{M}\) and \(S(t) + I(t) = 1\) for any \(t \geq 0\) with probability one.

Given that \(I(t) + S(t) = 1\), \(I(t)\), the fraction of infected individuals, obeys the stochastic Lotka-Volterra model with Markovian switching given by

\[
dI(t) = I(t) \left[ \lambda_{\alpha(t)} - \mu_{\alpha(t)} - \gamma_{\alpha(t)} - \lambda_{\alpha(t)} I(t) - C(t) \right] dt + \sigma_{\alpha(t)} I(t) \left[ 1 - I(t) \right] dw(t). \tag{2.5}
\]

For convenience, we define

\[
b(i, \ell, c) = i(\lambda_\ell - \mu_\ell - \gamma_\ell - \lambda_\ell i - c), \quad a(i, \ell) = \sigma_i^2 i^2 (1 - i)^2.
\]

As in [5, 14], we choose the eradication time \(\tau\) as the terminal time of planning (that is, time horizon). Formally, let \(\xi\) be a positive constant such that \(\xi < 1\), then

\[
\tau = \inf \{ t \geq 0 : I(t) \leq \xi \}. \tag{2.6}
\]

To make the definition in (2.6) meaningful, we assume that the initial number of infected units \(I(0)\) is strictly greater than \(\xi\). Thus, \(\tau\) is the first time at which the state variable \(I\) drops to \(\xi\).

Let \(\mathcal{A}_{i,\ell}\) denote the collection of all admissible controls with initial value \((i, \ell) \in (0,1) \times \mathcal{M}\). A strategy \(C(\cdot)\) will be in \(\mathcal{A}_{i,\ell}\) if \(C(t)\) is \(\mathcal{F}(t)\)-adapted and \(C(t) \in \mathcal{U}\) for any \(t \geq 0\).

We suppose that increasing the recovery rate is costly and the treatment of the infected individuals also create additional costs. The cost is described by the bounded cost function \(F : [0,1] \times \mathcal{M} \times [0,\infty) \rightarrow [0,\infty)\). For a control strategy \(C(\cdot) \in \mathcal{A}_{i,\ell}\), we define the cost functional as

\[
J(i, \ell, C(\cdot)) := \mathbb{E}_{i,\ell} \int_0^\tau e^{-\delta t} F(I(t), \alpha(t), C(t)) dt, \tag{2.7}
\]

where \(\delta > 0\) is the discounting factor and \(\mathbb{E}_{i,\ell}\) denotes the expectation with respect to the probability law when the process \((I(t), \alpha(t))\) starts with initial condition \((i, \ell)\). The goal is to minimize the cost functional and find an optimal strategy \(C^*(\cdot)\) such that

\[
J(i, \ell, C^*(\cdot)) = V(i, \ell) := \inf_{C(\cdot) \in \mathcal{A}_{i,\ell}} J(i, \ell, C(\cdot)). \tag{2.8}
\]

Formally, the associated Hamilton-Jacobi-Bellman equation of the underlying problem is given by

\[
\min_{c \in \mathcal{U}} \left[ b(i, \ell, c) \frac{dV(i, \ell)}{di} + \frac{1}{2} a(i, \ell) \frac{d^2V(i, \ell)}{di^2} + \sum_{\ell=1}^{m_0} \Lambda_\ell V(i, \ell) + F(i, \ell, c) - \delta V(i, \ell) \right] = 0, \tag{2.9}
\]

for all \((i, \ell) \in [\xi,1] \times \mathcal{M}\), with the boundary condition \(V(\xi, \ell) = 0\) for any \(\ell \in \mathcal{M}\).
Remark 2.2. In the examples in Section 5, we will work on a specific cost function of the form

\[ F(i, \ell, c) = a_0 + a_1 i + a_2 i^2, \]

where \( a_0, a_1, \) and \( a_2 \) are the weighting factors, representing the cost per unit time of the components 1, \( i, i^2, \) respectively. In particular, \( \int_0^\tau a_0 dt \) is the cost due to the time period needed for outbreak eradication, while \( \int_0^\tau a_1 I(t) dt \) is the cost that infected individual creates for the society due to lost working hours and standard medical care, not including the treatment \( C(\cdot). \) \( \int_0^\tau a_2 I(t)C^2(t)dt \) is the cost of treating infected individuals. We assume that the cost is proportional to the number of infected individuals, and the cost per each patient depends quadratically on the treatment effort \( C(t). \) As in [5], if one is interested in minimizing the sum of the eradication time and the total epidemic size, one can use the cost function given by

\[ F(i, \ell, c) = a_0 + a_1 \lambda(1 - i) + a_2 i^2, \]

where \( a_0 \) and \( a_1 \) are the weighting factors, representing the cost per unit time of epidemic duration and the cost of a single new infection, respectively.

Our standing assumptions are as follows.

(A) 1. The system parameters \( \mu, \gamma, \lambda \) are all nonnegative for each \( \ell \in \mathcal{M}. \)
   2. The control set \( \mathcal{U} \) is a nonempty compact set in \([0, \infty)\) with \( 0 \in \mathcal{U}. \)
   3. For any \( \ell \in \mathcal{M}, \) the cost function \( F(\cdot, \ell, \cdot) \) is bounded, continuous, and nonnegative.

Remark 2.3. Under assumption (A), for any initial value \((i, \ell) \in (0, 1) \times \mathcal{M} \) and \( C(t) \equiv c \in \mathcal{U}, \) equation (2.5) has the unique global solution \((I(\cdot), \alpha(\cdot)) \in (0, 1) \times \mathcal{M} \) for all \( t \geq 0; \) see Theorem 2.1. It follows from the boundedness of \( F(\cdot) \) that \( V(i, \ell) < \infty \) for any \((i, \ell) \in (0, 1) \times \mathcal{M}. \)

Although our work is motivated by the presence of a Markovian switching random environment, our results and the numerical schemes we developed can also be used to the corresponding control problems of diffusion models without switching. In particular, one can simply take \( \mathcal{M} = \{1\}. \)

We are in a position to construct a numerical procedure for solving the optimal control problem.

3 Numerical Algorithm

Following the Markov chain approximation method in [21, 29], we construct a controlled Markov chain in discrete time to approximate the controlled switching diffusions.
3.1 Approximation Algorithm of the Combined Process

Let \( h > 0 \) be a discretization parameter for the continuous state variable. Define

\[
S_h := \{ i = kh : k \in \mathbb{Z}_+, k \leq 1/h \} \times \mathcal{M}.
\]

Let \( \{(I_n^h, \alpha_n^h) : n \in \mathbb{Z}_+\} \) be a discrete-time controlled Markov chain with state space \( S_h \) such that the controlled Markov chain well approximates the local behavior of the controlled diffusion \( (I(t), \alpha(t)) \). At any discrete-time step \( n \), the magnitude of the control component \( C_n^h \) must be specified. The space of controls is \( \mathcal{U} \). Let \( C^h = \{C_n^h\} \) be a sequence of controls. We denote by \( p^h((i, \ell), (i', \ell')|c) \) the transition probability from state \( (i, \ell) \) to another state \( (i', \ell') \) under the control \( c \). Denote \( \mathcal{F}_n^h = \sigma\{I_k^h, \alpha_k^h, C_k^h, k \leq n\} \).

The sequence \( C^h \) is said to be admissible if it satisfies the following conditions:

(a) \( C^h \) is \( \sigma\{I_0^h, \ldots, I_n^h, \alpha_0^h, \ldots, \alpha_n^h, C_0^h, \ldots, C_{n-1}^h\} \)-adapted,

(b) For any \( (i, \ell) \in S_h \), we have

\[
\mathbb{P}\{(I_{n+1}^h, \alpha_{n+1}^h) = (i, \ell) | \mathcal{F}_n^h\} = \mathbb{P}\{(I_{n+1}^h, \alpha_{n+1}^h) = (i, \ell) | I_n^h, \alpha_n^h, C_n^h\} = p^h((I_n^h, \alpha_n^h), (i, \ell)|C_n^h),
\]

(c) \( (I_n^h, \alpha_n^h) \in S_h \) for all \( n \in \mathbb{Z}_+ \).

The collection of all admissible control sequences for initial state \( (i, \ell) \) will be denoted by \( \mathcal{A}_{i,\ell}^h \). For each \( (i, \ell, c) \in S_h \times \mathcal{U} \), we define a family of the interpolation intervals \( \Delta t^h(i, \ell, c) \). The values of \( \Delta t^h(i, \ell, c) \) will be specified later. Then we define

\[
t_0^h = 0, \quad \Delta t_k^h = \Delta t^h(I_k^h, \alpha_k^h, C_k^h), \quad t_n^h = \sum_{k=0}^{n-1} \Delta t_k^h. \tag{3.1}
\]

For \( (i, \ell) \in S_h \) and \( C^h \in \mathcal{A}_{i,\ell}^h \), the cost functional for the controlled Markov chain is defined as

\[
J^h(i, \ell, C^h) = \mathbb{E} \sum_{k=1}^{\eta_h} e^{-\delta t_k^h} F(I_k^h, \alpha_k^h, C_k^h) \Delta t_k^h, \tag{3.2}
\]

with

\[
\eta_h = \inf\{ n \geq 0 : I_n^h \leq \xi \}.
\]

The value function of the controlled Markov chain is

\[
V^h(i, \ell) = \inf_{C^h \in \mathcal{A}_{i,\ell}^h} J^h(i, \ell, C^h). \tag{3.3}
\]

The corresponding dynamic programming equation for the discrete approximation is given by

\[
V^h(i, \ell) = \min_{c \in \mathcal{U}} \left[ e^{-\delta \Delta t^h(i, \ell, c)} \sum_{(i', \ell') \in S_h} V^h(i', \ell') p^h((i, \ell), (i', \ell')|c) + F(i, \ell, c) \Delta t^h(i, \ell, c) \right],
\]

for any \( (i, \ell) \in S_h \).
3.2 Transition Probabilities and Local Consistency

Let $\mathbb{E}^{\cdot,\cdot,\cdot}_{i,\ell,n}$, $\text{Cov}^{\cdot,\cdot,\cdot}_{i,\ell,n}$ denote the conditional expectation and covariance with given

$$\{I^h_k, \alpha^h_k, C^h_k, k \leq n, I^h_n = i, \alpha^h_n = \ell, C^h_n = c\},$$

respectively. Define

$$\Delta I^h_n = I^h_{n+1} - I^h_n.$$

Our objective is to define transition probabilities $p^h((i, \ell), (i', \ell')|c)$ so that the controlled Markov chain $\{(I^h_n, \alpha^h_n)\}$ is locally consistent with the controlled switching diffusion (2.5) in the sense that the following conditions hold:

$$\mathbb{E}^{h,c}_{i,\ell,n} \Delta I^h_n = a(i, \ell, c)\Delta t^h(i, \ell, c) + o(\Delta t^h(i, \ell, c)),
\text{Cov}^{h,c}_{i,\ell,n} \Delta I^h_n = a(i, \ell, c)\Delta t^h(i, \ell, c) + o(\Delta t^h(i, \ell, c)),
\mathbb{E}^{h,c}_{i,\ell,n} (\alpha^h_{n+1} = \ell') = \Lambda_{\ell\ell'} \Delta t^h(i, \ell, c) + o(\Delta t^h(i, \ell, c)) \quad \text{for} \quad \ell \neq \ell',
\mathbb{E}^{h,c}_{i,\ell,n} (\alpha^h_{n+1} = \ell) = 1 + \Lambda_{\ell\ell'} \Delta t^h(i, \ell, c) + o(\Delta t^h(i, \ell, c)),
\sup_{n, \omega} |\Delta I^h_n| \to 0 \quad \text{as} \quad h \to 0.
$$

Using the procedure in Inspired by [21], for $(i, \ell) \in S_h$, we define

$$Q_h(i, \ell, c) = a(i, \ell, c) + h|b(i, \ell)| - h^2\Lambda_{\ell\ell} + h,$n
$$p^h((i, \ell), (i + h, \ell)|c) = \frac{a(i, \ell)/2 + (b(i, \ell, c))^+ h}{Q_h(i, \ell, c)},
$$p^h((i, \ell), (i - h, \ell)|c) = \frac{a(i, \ell)/2 + (b(i, \ell, c))^- h}{Q_h(i, \ell, c)},
$$p^h((i, \ell), (i, \ell')|c) = \frac{h^2\Lambda_{\ell\ell'}}{Q_h(i, \ell, c)} \quad \text{for} \quad \ell \neq \ell',
$$p^h((i, \ell), (i, \ell)|c) = \frac{h}{Q_h(i, \ell, c)},
$$\Delta t^h(i, \ell, c) = \frac{h^2}{Q_h(i, \ell, c)}.$$n

where for a real number $r$, $r^+ = \max\{r, 0\}$, $r^- = -\min\{0, r\}$. Set $p^h((i, \ell), (i', \ell')|c) = 0$ for all unlisted values of $(i', \ell') \in S_h$. Assumption (A) guarantees that the transition probabilities in (4.7) are well-defined. Using the above transition probabilities, we can check that the local-consistency conditions of $\{(I^h_n, \alpha^h_n)\}$ in (3.4) are satisfied.

**Lemma 3.1.** The Markov chain $\{(I^h_n, \alpha^h_n)\}$ with transition probabilities $\{p^h(\cdot)\}$ defined in (4.7) is locally consistent with (2.5).

3.3 Continuous-Time Interpolation and Time Rescaling

To proceed, we construct a continuous-time interpolation of the approximating chain. For use in this construction, we define $n^h(t) = \max\{n : t_n^h \leq t\}, t \geq 0$. The discrete time
processes associated with the controlled Markov chain \( \{ (I_n^h, \alpha_n^h) \} \) are defined as follows. Let

\[
B_0^h = M_0^h = 0, \quad B_n^h = \sum_{k=0}^{n-1} \mathbb{E}_k^h \Delta I_k^h, \quad M_n^h = \sum_{k=0}^{n-1} (\Delta I_k^h - \mathbb{E}_k^h \Delta I_k^h), \quad n \geq 1.
\]  

(3.6)

The piecewise constant interpolation processes, denoted by

\[
I^h(\cdot), \alpha^h(\cdot), B^h(\cdot), M^h(\cdot), C^h(\cdot)
\]

are naturally defined as

\[
I^h(t) = I_{n^h(t)}^h, \quad \alpha^h(t) = \alpha_{n^h(t)}^h, \quad C^h(t) = C_{n^h(t)}^h, \quad B^h(t) = B_{n^h(t)}, \quad M^h(t) = M_{n^h(t)}, \quad t \geq 0.
\]

(3.7)

Define \( \mathcal{F}^h(t) = \sigma \{ I^h(s), \alpha^h(s), C^h(s) : s \leq t \} \). We have

\[
I^h(t) = i + B^h(t) + M^h(t).
\]

(3.8)

Recall that \( \Delta t_k^h = h^2/Q_h(I_k^h, \alpha_k^h, C_k^h) \). It follows that

\[
B^h(t) = \sum_{k=0}^{n^h(t)-1} b(I_k^h, \alpha_k^h, C_k^h) \Delta t_k^h
\]

\[
= \int_0^t b(I^h(u), \alpha^h(u), C^h(u)) du - \int_{n^h(t)}^t b(I^h(u), \alpha^h(u), C^h(u)) du
\]

\[
= \int_0^t b(I^h(u), \alpha^h(u), C^h(u)) du + \varepsilon_t^h(t),
\]

with \( \{ \varepsilon_t^h(\cdot) \} \) being an \( \mathcal{F}^h(t) \)-adapted process satisfying \( \lim_{h \to 0} \sup_{t \in [0, T_0]} \mathbb{E}|\varepsilon_t^h(t)| = 0 \) for \( T_0 \in (0, \infty) \).

Define \( \tau_h = t_{n^h}^h \). The cost functional from (3.2) can be rewritten as

\[
J^h(i, \ell, C^h) = \mathbb{E} \int_0^{\tau_h} e^{-\delta t} F(I^h(t), \alpha^h(t), C^h(t)) dt.
\]

(3.10)

### 3.4 Relaxed controls

For our analysis, it is more convenient to use the notion of relaxed controls. We first briefly recall the notion of “relaxed control”, which arises naturally in the weak convergence analysis for the approximation to the optimal control problems.

**Definition 3.2.** Let \( \mathcal{B}(\mathcal{U} \times [0, \infty)) \) be the \( \sigma \)-algebra of Borel subsets of \( \mathcal{U} \times [0, \infty) \). An admissible relaxed control or simply a relaxed control \( m(\cdot) \) is a measure on \( \mathcal{B}(\mathcal{U} \times [0, \infty)) \) such that

\[
m(\mathcal{U} \times [0, t]) = t \quad \text{for all} \quad t \geq 0.
\]
Given a relaxed control \( m(\cdot) \), there is a probability measure \( m_t(\cdot) \) defined on the \( \sigma \)-algebra \( B(U) \) such that \( m(dcdt) = m_t(dc)dt \).

With the given probability space, we say that \( m(\cdot) \) is an admissible relaxed stochastic control for \( (w(\cdot), \alpha(\cdot)) \) or \( (m(\cdot), w(\cdot), \alpha(\cdot)) \) is admissible, if (i) for each fixed \( t \geq 0 \), \( m_t(\cdot) \) is a random variable taking values in \( R(U \times [0, \infty)) \), and for each fixed \( \omega \), \( m(\cdot, \omega) \) is a deterministic relaxed control; (ii) the function defined by \( m(A \times [0, t]) \) is \( F(t) \)-adapted for any \( A \in B(U) \). As a result, with probability one, there is a measure \( m_\ell(\cdot, \omega) \) on the Borel \( \sigma \)-algebra \( B(U) \) such that \( m(dcdt) = m_\ell(dc)dt \).

**Remark 3.3.** For a sequence of controls \( C^h = \{C^h_n : n \in \mathbb{Z}_+\} \), we define a sequence of equivalent relaxed controls as follows. First, we set \( m^h_{\ell}(dc) = \delta_{C^h_n}(dc) \), where \( \delta_{C^h_n}(\cdot) \) is the probability measure concentrated at \( C^h_n \). Then \( m^h(\cdot) \) is defined by \( m^h(dcdt) = m^h_t(dc)dt \). That is,

\[
m^h(B \times [0, t]) = \int_0^t \left( \int_B \delta_{C^h_n(u)}(dc) \right) du, \quad B \in B(U) \quad \text{and} \quad t \geq 0.
\]

Let \( R(U \times [0, \infty)) \) denote the space of all relaxed controls on \( U \times [0, \infty) \). Then \( R(U \times [0, \infty)) \) can be metrized using the Prohorov metric in the usual way as in [21, pp. 263-264]. With the Prohorov metric, \( R(U \times [0, \infty)) \) is a compact space. It follows that any sequence of relaxed controls has a convergent subsequence. Moreover, a sequence \( \eta_n \in R(U \times [0, \infty)) \) converges to \( \eta \in R(U \times [0, \infty)) \) if and only if for all continuous functions with compact support \( \Psi(\cdot) \) on \( U \times [0, \infty) \),

\[
\int_{U \times [0, \infty)} \Psi(c, t)\eta_n(dc, dt) \to \int_{U \times [0, \infty)} \Psi(c, t)\eta(dc, dt).
\]

Note that for a sequence of ordinary controls \( C^h = \{C^h_n : n \in \mathbb{Z}_+\} \), the associated relaxed control \( m^h(dcdt) \) belongs to \( R(U \times [0, \infty)) \). Note also that the limits of the “relaxed control representations” of the ordinary controls might not be ordinary controls, but only relaxed controls.

With the notion of relaxed control given above, we can write (3.10) as

\[
J^h(i, \ell, C^h) = J^h(i, \ell, m^h(\cdot)) = \mathbb{E} \int_0^\tau e^{-\delta t} F(I^h(t), \alpha^h(t), c) m^h_t(dc)dt. \tag{3.11}
\]

Note also that the value function defined in (2.8) can be rewritten as

\[
V(i, \ell) = \inf \{J(i, \ell, m(\cdot)) : m(\cdot) \text{ is an admissible relaxed control}\},
\]

where

\[
J(i, \ell, m(\cdot)) := \mathbb{E}_{i, \ell} \int_0^\tau e^{-\delta t} F(I(t), \alpha(t), c) m_t(dc)dt.
\]
3.5 Convergence Results

The proof of the next lemma can be obtained similar to that of [32, Theorem 3.1].

**Lemma 3.4.** The process \( \{\alpha^h(\cdot)\} \) converges weakly to \( \alpha(\cdot) \), which is a Markov chain with generator \( \Lambda = (\Lambda_{i\ell}) \).

The following theorems establish the convergence of the approximating Markov chain to the original controlled switching diffusion as well as the convergence of the value functions. To facilitate the reading, all proofs are placed in an appendix at the end of the paper.

**Theorem 3.5.** Let the chain \( \{(I^n_h, \alpha^n_h)\} \) be constructed with transition probabilities defined in (4.7), \( I^h(\cdot), \alpha^h(\cdot), B^h(\cdot), M^h(\cdot), \tau^h(\cdot) \) be the continuous-time interpolation defined in (3.6)-(3.7), \( \{C^n_h\} \) be an admissible control, and \( m^h(\cdot) \) be the relaxed control representation of \( \{C^n_h\} \). Then the following assertions hold.

(a) The sequence
\[
H^h(\cdot) = (I^h(\cdot), \alpha^h(\cdot), B^h(\cdot), M^h(\cdot), m^h(\cdot), \tau^h)
\]
is tight. As a result, \( (H^h)_{h>0} \) has a weakly convergent subsequence with limit
\[
H(\cdot) = (I(\cdot), \alpha(\cdot), B(\cdot), M(\cdot), m(\cdot), \tau).
\]
Moreover, \( I(\cdot), B(\cdot), M(\cdot) \) have continuous paths with probability one.

(b) For any \( t \geq 0 \),
\[
B(t) = \int_0^t b(I(u), \alpha(u), c)m_u(dc)du. \tag{3.12}
\]

(c) \( M(\cdot) \) is a continuous \( \mathcal{F}(t) \)-martingale with quadratic variation \( \int_0^t a(I(u), \alpha(u))du \). Thus, there is an \( \mathcal{F}(t) \)-standard Brownian motion \( w(t) \), in which we might have to augment the probability space such that
\[
M(t) = \int_0^t \sigma(I(u), \alpha(u))dw(u). \tag{3.13}
\]

(d) The limit processes satisfy
\[
I(t) = i + B(t) + M(t), \quad t \geq 0. \tag{3.14}
\]

Our main convergence result is given below.

**Theorem 3.6.** Let \( V^h(i, \ell) \) and \( V(i, \ell) \) be value functions defined in (3.3) and (2.8), respectively. Then \( V^h(i, \ell) \to V(i, \ell) \) as \( h \to 0 \).
4 A Stochastic SIS Model with Vaccination

Building on our approach of the SIS models, we consider a SIS epidemic model with vaccination proposed in [15, 22]; see also [33, 34] for closely related models. We assume that $S(t)$, $I(t)$, and $\mathcal{V}(t)$ are the fraction of infectious, susceptible, and vaccinated individuals at time $t$, respectively. The evolution of $(S(\cdot), I(\cdot), \mathcal{V}(\cdot))$ is given by

\[
\begin{align*}
    ds(t) &= \left[\mu(1 - q(t)) - \lambda S(t)I(t) - (\mu + p(t))S(t) + \gamma I(t) + \varepsilon \mathcal{V}(t)\right]dt, \\
    dI(t) &= \left[\lambda S(t)I(t) - (\mu + \gamma)I(t)\right]dt, \\
    d\mathcal{V}(t) &= \left[\mu q(t) + p(t)S(t) - (\mu + \varepsilon)\mathcal{V}(t)\right]dt,
\end{align*}
\]

(4.1)

where $q(\cdot)$ is the fraction of vaccinated for newborns, $p(\cdot)$ is the proportional coefficient of vaccinated for the susceptible, $\varepsilon$ is the rate of losing their immunity for vaccinated individuals. The other parameters are understood as in the preceding sections; that is, $\mu$ and $\gamma^{-1}$ are the average death rate and the average infectious period respectively, $\lambda$ is the disease contact rate of an infective individual. Note that if $\mathcal{V}(0) = p(t) = q(t) = 0$ for $t \geq 0$, then $\mathcal{V}(t) = 0$ for any $t \geq 0$ and the system (4.1) reduces to (2.1).

Using the same methods as in the preceding sections, we obtain the following controlled SIS system under regime switching in random environment

\[
\begin{align*}
    ds(t) &= \left[\mu_{\alpha(t)}(1 - q(t)) - \lambda_{\alpha(t)}S(t)I(t) - (\mu_{\alpha(t)} + p(t))S(t) + \gamma_{\alpha(t)}I(t) + \varepsilon_{\alpha(t)}\mathcal{V}(t)\right]dt \\
    &\quad - \sigma_{\alpha(t)}S(t)I(t)dw(t), \\
    dI(t) &= \left[\lambda_{\alpha(t)}S(t)I(t) - (\mu_{\alpha(t)} + \gamma_{\alpha(t)} + C(t))I(t)\right]dt + \sigma_{\alpha(t)}S(t)I(t)dw(t), \\
    d\mathcal{V}(t) &= \left[\mu_{\alpha(t)}q(t) + p(t)S(t) - (\mu_{\alpha(t)} + \varepsilon_{\alpha(t)})\mathcal{V}(t)\right]dt,
\end{align*}
\]

(4.2)

where $\alpha(\cdot)$ is a Markov chain taking values in $\mathcal{M} = \{1, 2, \ldots, m_0\}$ and $w(\cdot)$ is a scalar standard Brownian motion independent of $\alpha(\cdot)$. We suppose that $C(\cdot)$ takes values in a nonempty compact set $U$ of $[0, \infty)$. Suppose $\mathcal{V}_p$ and $\mathcal{V}_q$ are nonempty compact subsets of $[0, 1]$. Moreover, $p(\cdot)$ and $q(\cdot)$ takes the values in $\mathcal{V}_p$ and $\mathcal{V}_q$, respectively. Compared to the related formulations in [5, 14], we also consider the possibility that vaccinated individuals lose their immunity. We suppose that the initial value is $(s_0, i_0, v_0) \in (0, 1)^2 \times [0, 1)$ satisfying $s_0 + i_0 + v_0 = 1$. Thus, we exclude the trivial case $i_0 = 0$ or unrealistic cases such as $i_0 = 1$ and $s_0 = 0$. A fundamental result on the existence of a unique global solution of (4.2) is given below.

**Theorem 4.1.** For any given initial value $(s_0, i_0, v_0, l_0) \in (0, 1)^2 \times [0, 1) \times \mathcal{M}$ satisfying $s_0 + i_0 + v_0 = 1$, $C(t) \equiv c_0 \in U$, $p(t) \equiv p_0 \in \mathcal{V}_p$ and $q(t) \equiv q_0 \in \mathcal{V}_q$, the equation (4.2) has a unique global solution $(S(t), I(t), \mathcal{V}(t), \alpha(t)) \in (0, 1)^2 \times [0, 1) \times \mathcal{M}$ and $S(t) + I(t) + \mathcal{V}(t) = 1$ for any $t \geq 0$ with probability one.

Since $S(t) + I(t) + \mathcal{V}(t) = 1$ for any $t \geq 0$, we need only consider the last two equations;
that is,
\[
\begin{aligned}
&dI(t) = \left[ \lambda_{\alpha(t)} (1 - I(t) - \mathcal{W}(t)) I(t) - \left( \mu_{\alpha(t)} + \gamma_{\alpha(t)} + C(t) \right) I(t) \right] dt \\
&+ \sigma_{\alpha(t)} (1 - I(t) - \mathcal{W}(t)) I(t) dw(t), \\
&d\mathcal{W}(t) = \left[ \mu_{\alpha(t)} q(t) + p(t) (1 - I(t) - V(t)) - (\mu_{\alpha(t)} + \varepsilon_{\alpha(t)}) \mathcal{W}(t) \right] dt.
\end{aligned}
\] (4.3)

Let $\mathcal{A}_{i,v,\ell}$ denote the collection of all admissible controls with initial value $(i,v,\ell) \in (0,1) \times [0,1) \times \mathcal{M}$ satisfying $0 < i + v \leq 1$; that is,
\[
I(0) = i, \quad \mathcal{W}(0) = v.
\]

Then Theorem 4.1 indicates that $\mathcal{A}_{i,v,\ell} \neq \emptyset$. The control component is $(C(\cdot),p(\cdot),q(\cdot))$. A strategy $(C(\cdot),p(\cdot),q(\cdot))$ will be in $\mathcal{A}_{i,v,\ell}$ if $(C(t),p(t),q(t))$ is $\mathcal{F}(t)$-adapted and
\[
(C(t),p(t),q(t)) \in U \times \mathcal{V}_p \times \mathcal{V}_q \quad \text{for any } t \geq 0.
\]
The cost is described by the bounded cost function $F : [0,1] \times \mathcal{M} \times U \times \mathcal{V}_p \times \mathcal{V}_q \to [0,\infty)$. For a control strategy $(C(\cdot),p(\cdot),q(\cdot)) \in \mathcal{A}_{i,v,\alpha}$, we define the cost functional as
\[
J(i,v,\ell,C(\cdot),p(\cdot),q(\cdot)) := \mathbb{E}_{i,v,\ell} \int_0^T e^{-\delta t} F(I(t),\alpha(t),C(t),p(t),q(t)) dt,
\] (4.4)

where $\delta > 0$ is the discounting factor and $\mathbb{E}_{i,v,\ell}$ denotes the expectation with respect to the probability law when the process $(I(t),\mathcal{W}(t),\alpha(t))$ starts with initial condition $(i,v,\ell)$. The goal is to minimize the cost functional and find an optimal strategy $(C^*(\cdot),p^*(\cdot),q^*(\cdot))$ such that
\[
J(i,v,\ell,C^*(\cdot),p^*(\cdot),q^*(\cdot)) = V(i,v,\ell) := \inf_{(C(\cdot),p(\cdot),q(\cdot)) \in \mathcal{A}_{i,v,\alpha}} J(i,v,\ell,C(\cdot),p(\cdot),q(\cdot)).
\] (4.5)

We use the same method as in the preceding section to construct a discrete-time controlled Markov chain $\{(I^h_n,\mathcal{W}^h_n,\alpha^h_n) : n \in \mathbb{Z}_+\}$ with the state space
\[
T_h := \{(k_1h,k_2h) : k \in \mathbb{Z}_+, k_1 + k_2 \leq 1/h \} \cap [0,1]^2 \times \mathcal{M}.
\]
At any discrete-time step $n$, the magnitude of the control component $(C^h_n,p^h_n,q^h_n)$ must be specified. The space of controls is $U \times \mathcal{V}_p \times \mathcal{V}_q$. Let $(C^h_n,p^h_n,q^h_n) = \{(C^h_n,p^h_n,q^h_n)\}$ be a sequence of controls. Denote $\mathcal{F}^h_n = \sigma\{I^h_k,\mathcal{W}^h_k,\alpha^h_k,C^h_k,p^h_k,q^h_k : k \leq n\}$.

The sequence $(C^h,n,p^h,q^h)$ is said to be admissible if it satisfies the following conditions:

(a) $(C^h_n,p^h_n,q^h_n)$ is $\sigma\{I^h_0,\mathcal{W}^h_0,\alpha^h_0,\ldots,I^h_n,\mathcal{W}^h_n,\alpha^h_n,C^h_0,p^h_0,q^h_0,\ldots,C^h_{n-1},p^h_{n-1},q^h_{n-1}\}$-adapted,

(b) For any $(i,v,\ell) \in T_h$, we have
\[
\mathbb{P}\{(I^h_{n+1},\mathcal{W}^h_{n+1},\alpha^h_{n+1}) = (i,v,\ell) | \mathcal{F}^h_n\} = \mathbb{P}\{(I^h_{n+1},\mathcal{W}^h_{n+1},\alpha^h_{n+1}) = (i,v,\ell) | I^h_n,\mathcal{W}^h_n,\alpha^h_n,C^h_n,p^h_n,q^h_n\}
\]
\[
= p^h((I^h_n,\mathcal{W}^h_n,\alpha^h_n),(i,v,\ell)|C^h_n,p^h_n,q^h_n),
\]
The class of all admissible control sequences \((C^h, p^h, q^h)\) for initial state \((i, v, \ell)\) will be denoted by \(\mathcal{A}^h_{i,v,\ell}\). We need to define the transition probabilities \(p^h((i, v, \ell), (i', v', \ell')|c, p, q)\) so that the controlled Markov chain \(\{(I^h_n, \mathcal{Y}^h_n, \alpha^h_n) : n \in \mathbb{Z}_+\}\) is locally consistent with respect to the controlled diffusion \((4.2)\).

To proceed, we denote

\[
\begin{align*}
b_1(i, v, \ell, c) &= \lambda_i(1 - i - v) - (\mu_\ell + \gamma_\ell + c)i, \\
b_2(i, v, \ell, c, p, q) &= \mu_\ell q + p(1 - i - v) - (\mu_\ell + \varepsilon_\ell)v.
\end{align*}
\]

In particular, for \((i, v, \ell) \in T_h\) and \((c, p, q) \in U \times V_p \times V_q\), we define

\[
\begin{align*}
Q_h(i, v, \ell, c, p, q) &= a_1(i, v, \ell) + h|b_1(i, v, \ell, c)| + h|b_2(i, v, \ell, c, p, q)| - h^2\Lambda_\ell + h, \\
p^h((i, v, \ell), (i + h, v, \ell)|c, p, q) &= \frac{a_1(i, v, \ell)/2 + (b_1(i, v, \ell, c))^+ h}{Q_h(i, v, \ell, c, p, q)}, \\
p^h((i, v, \ell), (i - h, v, \ell)|c, p, q) &= \frac{a_1(i, v, \ell)/2 + (b_1(i, v, \ell, c))^+ h}{Q_h(i, v, \ell, c, p, q)} - h, \\
p^h((i, v, \ell), (i, v + h, \ell)|c, p, q) &= \frac{(b_2(i, v, \ell, c, p, q))^+ h}{Q_h(i, v, \ell, c, p, q)}, \\
p^h((i, v, \ell), (i, v - h, \ell)|c, p, q) &= \frac{(b_2(i, v, \ell, c, p, q))^+ h}{Q_h(i, v, \ell, c, p, q)} - h, \\
p^h((i, v, \ell), (i, v', \ell)|c, p, q) &= \frac{h\Lambda_\ell}{Q_h(i, v, \ell, c, p, q)} - h, \\
p^h((i, v, \ell), (i, v, \ell')|c, p, q) &= \frac{h^2\Lambda_\ell}{Q_h(i, v, \ell, c, p, q)} \text{ for } \ell \neq \ell', \\
p^h((i, v, \ell), (i, v, \ell)|c, p, q) &= \frac{h^2}{Q_h(i, v, \ell, c, p, q)}, \quad \Delta^h(i, v, \ell, c, p, q) = \frac{h^2}{Q_h(i, v, \ell, c, p, q)}.
\end{align*}
\]

For \((i, v, \ell) \in T_h\) and \((C^h, p^h, q^h) \in \mathcal{A}^h_{i,v,\ell}\), the cost functional for the controlled Markov chain is given by

\[
J^h(i, v, \ell, C^h, p^h, q^h) = E \sum_{k=1}^{\eta_h} e^{-\delta k} F(i^h_k, \alpha^h_k, C^h_k, p^h_k, q^h_k) \Delta^h_k.
\]

The value function is

\[
V^h(i, v, \ell) = \inf_{(C^h, p^h, q^h) \in \mathcal{A}^h_{i,v,\ell}} J^h(i, v, \ell, C^h, p^h, q^h).
\]

The main convergence result in this case is given below.

**Theorem 4.2.** Let \(V(i, v, \ell)\) and \(V^h(i, v, \ell)\) be the value functions defined in \((4.5)\) and \((4.9)\), respectively. Then \(V^h(i, v, \ell) \to V(i, v, \ell)\) as \(h \to 0\).

## 5 Examples

Throughout this section, we suppose the discounting factor is \(\delta = 0.05\). Also, we work with \(\mathcal{M} = \{1, 2\}\) and \(\xi = 0.02\).
Example 5.1. We start with the model given by (2.5). That is,

\[ dI(t) = I(t)[\lambda_\alpha(t) - \mu_\alpha(t) - \gamma_\alpha(t) - \lambda_\alpha(t)I(t) - C(t)]dt + \sigma_\alpha(t)I(t)[1 - I(t)]dw(t). \]  

(5.1)

For \((i, \ell) \in (\xi, 1] \times \mathcal{M}\), we take the initial control \(C_0(i, \ell) \equiv \max \mathcal{U}\) and set the initial value \(V_0^h(i, \ell) \equiv \int_0^\infty e^{-dt}F(1, \ell, C_0)dt\). We outline how to find the sequence \(V_n^h(\cdot)\) as follows. For each \((i, \ell) \in S_h\) and control \(c \in \mathcal{U}\), we compute

\[ V_{n+1}^h(i, \ell | c) = e^{-\delta t} \sum_{(i', \ell') \in S_h} V_n^h(i', \ell')p^h((i, \ell), (i', \ell') | c) + F(i, \ell, c)\Delta t^h(i, \ell, c). \]

Note also that if \(i \leq \xi\), then \(V_n^h(i, \ell) = 0\) for any \(\ell\) and \(n\). We choose the control \(C_{n+1}^h(i, \ell)\) and record an improved value \(V_{n+1}^h(i, \ell)\) by

\[ C_{n+1}^h(i, \ell) = \text{arcmin}_{c \in \mathcal{U}} V_{n+1}^h(i, \ell | c), \quad V_{n+1}^h(i, \ell) = V_{n+1}^h(i, \ell | C_{n+1}^h(i, \ell)). \]

The iterations stop as soon as the increment \(V_n^h(\cdot) - V_{n+1}^h(\cdot)\) reaches a predetermined tolerance level. We set the error tolerance to be \(10^{-8}\). Our specific example is motivated by [13, Example 6.2.1]. The system parameters are given by

\[ \mu_1 = 0.45, \quad \mu_2 = 0.05, \quad \gamma_1 = 0.35, \quad \gamma_2 = 0.15, \quad \lambda_1 = 2, \quad \lambda_2 = 2.4, \quad \sigma_1 = 0, \quad \sigma_2 = 1. \]

Suppose that the generator \(\Gamma\) of the Markov chain \(\alpha(\cdot)\) is given by

\[ \Lambda_{11} = -1, \quad \Lambda_{12} = 1, \quad \Lambda_{21} = 1, \quad \Lambda_{22} = -1. \]

The set of controls is given by \(\mathcal{U} = \{k/5 : 0 \leq k \leq 15\}\). Thus, there are 16 control levels.

In the first example, we consider the cost function

\[ F(i, \ell, c) = 1 + \ell i + \ell i c^2; \]

(5.2)

see Remark 2.1 for its interpretation. The value function and optimal control is shown in Figure 1. In each regime \(\ell\), there is a level \(L(\ell)\) such that the optimal control is the maximum control effort for \(i < L(\ell)\). It appears that the value function in regime 2 is higher than that in regime 1, while the optimal control in regime 2 is strictly smaller than that in regime 1 possibly because of the higher cost function in regime 2; that is \(F(i, 2, c) > F(i, 1, c)\) for any \((i, c) \in (0, 1) \times \mathcal{U}\).

In the second experiment, we consider the cost function

\[ F(i, \ell, c) = 1 + 2\lambda_\ell(1 - i) + i c^2, \]

(5.3)

aiming at minimizing the sum of the eradication time and the total epidemic size. The value function and optimal control is shown in Figure 2. The optimal control in each regime has a similar shape as that in the preceding experiment. Although \(F(i, 2, c) > F(i, 1, c)\) for any
Figure 1: Value function (left) and optimal control (right) when $F(i, \ell, c) = 1 + \ell i + \ell i c^{2}$.

Figure 2: Value function (left) and optimal control (right) when $F(i, \ell, c) = 1 + 2\lambda(1-i)i + ic^{2}$.

$(i, c) \in (0, 1) \times \mathcal{U}$, the optimal control for $i \in (0.4, 0.6)$ in regime 2 is higher than that in regime 1.
Figure 3: Value function (left) and optimal control (right) when $F(i, \ell, c) = 1 + \ell i + \ell ic$

In the third experiment, we consider the cost function

$$F(i, \ell, c) = 1 + \ell i + \ell ic. \quad (5.4)$$

The value function and optimal control is shown in Figure 3. The results in Figure 3 tells us that the control cost is small enough so that we should apply the maximum possible control in any regime.

**Example 5.2.** We consider the model with vaccination given by (4.2); that is,

$$\begin{align*}
\dot{dI}(t) &= \left[\lambda_{\alpha(t)}S(t)I(t) - (\mu_{\alpha(t)} + \gamma_{\alpha(t)} + C(t))I(t)\right]dt + \sigma_{\alpha(t)}S(t)I(t)dw(t), \\
\dot{d\mathcal{W}}(t) &= \left[\mu_{\alpha(t)}q(t) + p(t)S(t) - (\mu_{\alpha(t)} + \varepsilon_{\alpha(t)})\mathcal{W}(t)\right]dt. \quad (5.5)
\end{align*}$$

We use the same parameters and the control set as in the preceding exam. In addition,

$$F(i, \ell, c, p, q) = 1 + \ell i + 2\ell ic^2 + (0.1)p + (0.1)q,$$

$$\mathcal{V}_p = \mathcal{V}_q = \{0.2k : 0 \leq k \leq 4\}.$$

We take the initial control

$$C_0(i, v, \ell) \equiv \max U, \quad p_0(i, v, \ell) \equiv \max \mathcal{V}_p, \quad q_0(i, v, \ell) \equiv \max \mathcal{V}_q,$$
and set the initial value $V_0(i,v,\ell) \equiv \int_0^{\infty} e^{-\delta t} F(1, 2, C_0, p_0, q_0) dt$ for $(i,v,\ell) \in T_h$. We outline how to find the sequence of costs of $V_n(\cdot)$ as follows. For each $(i,v,\ell) \in T_h$, and control $(c,p,q) \in \mathcal{U} \times \mathcal{V}_p \times \mathcal{V}_q$, we compute

\[
V_{n+1}(i,v,\ell | c,p,q) = e^{-\delta \Delta^h(i,v,\ell,c,p,q)} \sum_{(i',v',\ell') \in T_h} V_n(i',v',\ell') p^h((i,v,\ell),(i',v',\ell)|c,p,q) + F(i,\ell,c,p,q) \Delta^h(i,v,\ell,c,p,q).
\]
Then we choose the control and record an improved value $V_{n+1}^h(i, v, \ell)$ by

$$
(C_{n+1}^h(i, v, \ell), p_{n+1}^h(i, v, \ell), q_{n+1}^h(i, v, \ell)) = \text{arccmin}_{(c, p, q) \in \mathcal{U} \times \mathcal{V}_p \times \mathcal{V}_q} V_{n+1}^h(i, v, \ell | c, p, q)
$$

and

$$
V_{n+1}^h(i, v, \ell) = V_{n+1}^h(i, v, \ell | C_{n+1}^h(i, v, \ell), p_{n+1}^h(i, v, \ell), q_{n+1}^h(i, v, \ell)).
$$

The iterations stop as soon as the increment $V_{n+1}^h(\cdot) - V_n^h(\cdot)$ reaches the tolerance level. Intuitively, the shape of the optimal control will depend on how large the coefficients of the cost function are. The value function and optimal control are displayed in Figures 4, 5, 6, and 7.

It can be seen that with the presence of an effective vaccination, the value function shown in Figure 4 is much lower than that of Figure 1 in Example 1. Moreover, Figure 5 reveals that one should use a lower control $C(\cdot)$ compared to the case with no vaccination. Figures 6 and 7 provide the optimal vaccination plan. It appears that the optimal vaccination plan depends on the regime of the environment. The numerical studies emphasize the importance of effective vaccines with low costs in controlling epidemics.

![Figure 6](image_url): The optimal control $p(\cdot)$ in regime 1 (left) and in regime 2 (right)

### 6 Further Remarks

This paper focused on numerical methods for optimal control of SIS epidemic models. We considered two models incorporating treatments, isolation, and vaccination. Using the Markov chain approximation method, we are able to treat a hybrid model affected by two types of environmental fluctuations with a general cost consideration. The convergence of
the algorithms was proved. Several numerical examples were used to demonstrate the performance of our algorithm. Some interesting questions deserve further investigation. One can study a set of mixed controls consisting of vaccination, isolation, and social distancing, etc. One can also apply the approach in this work to investigate a variety of control problems of epidemic models, which becomes an urgent issue due to COVID-19. Thus, the techniques and the simulation study might be of interests to researchers in various disciplines.

A Proofs of results

Proof of Theorem 2.1. The proof is a special case of that of Theorem 4.1 given below when we take \( \mathcal{V}(0) = p(t) = q(t) = 0 \) for any \( t \geq 0 \) with probability one. □

Proof of Theorem 4.1. Since the coefficients of (4.2) are locally Lipschitz continuous, there is a unique local solution \((I(t), \mathcal{V}(t), \alpha(t))\) on \( t \in [0, \zeta) \), where \( \zeta \) is the explosion time. Now, by (4.2), \( S + I + \mathcal{V} \) satisfies

\[
\begin{cases}
    d(S + I + \mathcal{V})(t) = \mu_{\alpha(t)}(1 - S(t) - I(t) - \mathcal{V}(t))dt,
    \\
    S(0) + I(0) + \mathcal{V}(0) = 1,
\end{cases}
\]

which implies

\[
S(t) + I(t) + \mathcal{V}(t) = 1 \quad \text{for} \quad t \in [0, \zeta). \tag{A.1}
\]

Let \( k_0 \) be a sufficiently large positive integer such that \( s_0, i_0 \in (1/k_0, 1) \). For each \( k \geq k_0 \), we define

\[
\tau_k := \inf \left\{ t \geq 0 : I(t) \leq \frac{1}{k} \quad \text{or} \quad S(t) \leq \frac{1}{k} \right\}. \tag{A.2}
\]
Clearly the sequence \( \{ \tau_k \} \) is monotonically increasing. Let \( \tau_\infty := \lim_{t \to \infty} \tau_k \). Then \( \tau_\infty \leq \zeta \). It suffices to show that \( \tau_\infty = \infty \) with probability one. If this were false, there would exist a \( T > 0 \) and \( \varepsilon > 0 \) such that \( P\{ \tau_\infty \leq T \} > \varepsilon \). Therefore we can find some \( k_1 \geq k_0 \) such that

\[
P\{ \tau_k \leq T \} > \varepsilon \quad \text{for} \quad k \geq k_1. \tag{A.3}
\]

We fix \( \omega \in \Omega \). For \( t \in [0, \tau_\infty) \), by (A.1) and (A.2), \( I(t) \in (0,1] \). We have from the third equation in (4.2) that

\[
\mathcal{V}(t) \leq v_0 + \int_0^t \left( \mu_\alpha(u)q_0 + p_0 - [\mu_\alpha(u) + \varepsilon_\alpha(u) + p_0] \mathcal{V}(u) \right) du \leq v_0 + K_1 \int_0^t (1 - \mathcal{V}(u)) du,
\]

\[
\mathcal{V}(t) \geq v_0 + \int_0^t \left( \mu_\alpha(u)q_0 - [\mu_\alpha(u) + \varepsilon_\alpha(u) + p_0] \mathcal{V}(u) \right) du \geq v_0 - K_2 \int_0^t \mathcal{V}(u) du,
\]

where \( K_1 \) and \( K_2 \) are nonnegative numbers. Since \( v_0 \in [0,1) \) it follows from (A.4) that \( \mathcal{V}(t) \in [0,1) \) for any \( t \in [0, \tau_\infty) \). Combining this fact and (A.1), we have

\[
(S(t), I(t), \mathcal{V}(t)) \in (0,1)^2 \times [0,1) \quad \text{for} \quad t \in [0, \tau_\infty) \quad \text{with probability one.}
\]

To proceed, we consider the function

\[
\Phi(s, i) = \frac{1}{t} + \frac{1}{s}, \quad (s, i) \in (0,1)^2,
\]

and define

\[
\mathcal{L}\Phi(s, i, v, \ell) = \left( \mu_\ell(1 - q_0) - \lambda_\ell si - (\mu_\ell + p_0)s + \gamma_\ell i + \varepsilon_\ell v \right) \frac{\partial \Phi}{\partial s}(s, i)
\]

\[
+ \left( \lambda_\ell si - (\mu_\ell + \gamma_\ell + c_0)i \right) \frac{\partial \Phi}{\partial i}(s, i) + \frac{1}{2} \frac{\partial^2 \Phi}{\partial s^2}(s, i) \sigma_1^2 s^2 \ell^2 + \frac{1}{2} \frac{\partial^2 \Phi}{\partial i^2}(s, i) \sigma_2^2 s^2 \ell^2.
\]

Note that for our study, it suffices to consider \( \Phi \) to be independent of the switching states. We have \( \Phi(s, i) > 0 \) and

\[
\frac{\partial \Phi}{\partial s}(s, i) = -\frac{1}{s^2}, \quad \frac{\partial \Phi}{\partial i}(s, i) = -\frac{1}{i^2},
\]

\[
\frac{\partial^2 \Phi}{\partial s^2}(s, i) = \frac{2}{s^3}, \quad \frac{\partial^2 \Phi}{\partial i^2}(s, i) = \frac{2}{i^3}.
\]

It follows that

\[
\mathcal{L}\Phi(s, i, v, \ell) = -\left( \mu_\ell(1 - q_0) - \lambda_\ell si - (\mu_\ell + p_0)s + \gamma_\ell i + \varepsilon_\ell v \right) \frac{1}{s^2}
\]

\[
- \left( \lambda_\ell si - (\mu_\ell + \gamma_\ell + c_0)i \right) \frac{1}{i^2} + \frac{1}{2} \frac{2}{s^3}(s, i) \sigma_1^2 s^2 \ell^2 + \frac{1}{2} \frac{2}{i^3}(s, i) \sigma_2^2 s^2 \ell^2. \tag{A.5}
\]

\[
\leq K\Phi(s, i),
\]
for some positive constant $K$ independent of $(s, i, v, l)$. By Dynkin’s formula, we obtain for $k \geq k_1$ and $t \in [0, T]$ that

$$
\mathbb{E} \Phi(S(\tau_k \wedge t), I(\tau_k \wedge t)) = \Phi(s_0, i_0) + \mathbb{E} \int_0^{\tau_k \wedge t} \mathcal{L} \Phi(S(u), I(u), \mathcal{W}(u), \alpha(u)) du
$$

\[
\leq \Phi(s_0, i_0) + K \mathbb{E} \int_0^{t} \Phi(S(u \wedge \tau_k), I(u \wedge \tau_k)) du.
\]

The Gronwall inequality yields that

$$
\mathbb{E} \Phi(S(s \wedge \tau_k), I(\tau_k \wedge T)) \leq \Phi(s_0, i_0)e^{KT}.
$$

(A.6)

Note that for each $\omega \in \{\tau_k \leq T\}$, $I(\tau_k(\omega)) \leq \frac{1}{k}$ or $S(\tau_k(\omega)) \leq \frac{1}{k}$. It follows from the definition of $\Phi(\cdot, \cdot)$ that $\Phi(S(\tau_k(\omega)), I(\tau_k(\omega))) \geq k$. In view of (A.3) and (A.6), we obtain

$$
\Phi(s_0, i_0)e^{KT} \geq k\varepsilon,
$$

leading to a contradiction as $k \to \infty$. Thus, $\zeta = \tau_\infty = \infty$ with probability one. The conclusion follows. □

The proofs below are motivated by [21, 29]. Let $D[0, \infty)$ denote the space of functions that are right continuous and have left-hand limits endowed with the Skorohod topology. All the weak convergence analysis will be on this space or its $k$-fold products $D^k[0, \infty)$ for appropriate $k$. We will provide a sketch of the proofs and refer to the well-known references for the details.

**Proof of Theorem 3.5.** (a) The tightness of $\{\alpha^h(\cdot)\}$ is obvious by Lemma 3.4. For other components, we use the tightness criteria in [20, p. 47]. Specifically, a sufficient condition for tightness of a sequence of processes $\zeta^h(\cdot)$ with paths in $D[0, \infty)$ is that for any $T_0, \rho \in (0, \infty)$,

$$
\mathbb{E}_h |\zeta^h(t + s) - \zeta^h(t)|^2 \leq \mathbb{E}_h^\gamma(h, \rho) \quad \text{for all } s \in [0, \rho], \quad t \leq T_0.
$$

$$
\lim_{\rho \to 0} \limsup_{h \to 0} \mathbb{E}_h^\gamma(h, \rho) = 0.
$$

The detailed proof of the tightness is standard; see [29]. Thus, $H^h(\cdot)$ is tight. As a result, a subsequence of $H^h(\cdot)$ converges weakly to the limit $H(\cdot) = (I(\cdot), \alpha(\cdot), B(\cdot), M(\cdot), m(\cdot), \tau)$. Since the sizes of jumps of $I^h(\cdot)$, $B^h(\cdot)$, and $M^h(\cdot)$ go to zero as $h \to 0$, then $I(\cdot)$, $B(\cdot)$, and $M(\cdot)$ have continuous paths with probability one.

(b) follows from (a) and (3.9).

(c) Let $\mathbb{E}_h^t$ denote the expectation conditioned on $\mathcal{F}^h(t)$. By the definition of $M^h(\cdot)$, it is an $\{\mathcal{F}^h(t)\}$-martingale with quadratic variation process

$$
\int_0^t a(I^h(u), \alpha^h(u)) du + \varepsilon_3^h(t),
$$
where \( \{\varepsilon^h_3(\cdot)\} \) is an \( \{\mathcal{F}^h(t)\} \)-adapted process satisfying \( \lim_{h \to 0} \sup_{t \in [0,T_0]} \mathbb{E} |\varepsilon^h_3(t)| = 0 \) for \( T_0 \in (0, \infty) \).

By the Burkholder-Gundy inequality, there is a positive constant \( K \) such that
\[
\mathbb{E} |M^h(t)|^2 \leq K \mathbb{E} \left| \int_0^t a(I^h(u), \alpha^h(u)) du + \varepsilon^h_3(t) \right| \leq K(t + 1),
\]
for any \( h > 0 \). Thus, the family \( \{M^h(t) : h > 0\} \) is uniformly integrable. For any \( \rho > 0 \),
\[
\mathbb{E}^h_t (M^h(t + \rho) - M^h(t)) = 0,
\]
\[
\mathbb{E}^h_t [M^h(t + \rho) - M^h(t)] [M^h(t + \rho) - M^h(t)]' = \mathbb{E}^h_t \int_t^{t+\rho} a(I^h(u), \alpha^h(u)) du + \varepsilon^h_4(\rho),
\]
(A.7)
where \( \mathbb{E} |\varepsilon^h_4(\rho)| \to 0 \) as \( h \to 0 \). To characterize \( M^h(\cdot) \), let \( r \) be an arbitrary integer, \( t > 0 \), \( \rho > 0 \) and \( \{t_k : k \leq r\} \) be such that \( t_k \leq t < t + \rho \) for each \( k \). Let \( \Psi(\cdot) \) be a real-valued and continuous function of its arguments with compact support. Then in view of (A.7), we have
\[
\mathbb{E} \Psi(H^h(t_k), k \leq r) [M^h(t + \rho) - M^h(t)] = 0,
\]
(A.8)
and
\[
\mathbb{E} \Psi(H^h(t_k), k \leq r) \left( [M^h(t + \rho) - M^h(t)] [M^h(t + \rho) - M^h(t)]' - \int_t^{t+\rho} a(I^h(u), \alpha^h(u)) du \right) = \varepsilon^h_5(\rho),
\]
(A.9)
where \( \mathbb{E} |\varepsilon^h_5(\rho)| \to 0 \) as \( h \to 0 \). By using the Skorohod representation, letting \( h \to 0 \) in (A.8), we obtain
\[
\mathbb{E} \Psi(H(t_k), k \leq r) [M(t + \rho) - M(t)] = 0.
\]
(A.10)
Since \( M(\cdot) \) has continuous paths with probability one, (A.10) implies that \( M(\cdot) \) is a continuous \( \mathcal{F}(\cdot) \)-martingale. Moreover, (A.9) gives us that
\[
\mathbb{E} \Psi(H^h(t_k), k \leq r) \left( [M(t + \rho) - M(t)] [M(t + \rho) - M(t)]' - \int_t^{t+\rho} a(I(u), \alpha(u)) du \right) = 0,
\]
(A.11)
which gives us the quadratic variation of \( M(\cdot) \). The conclusion follows from the martingale representation theorem (see [16, Theorem 3.4.2]).

(d) follows immediately from the results in (a), (b), and (c). \( \square \)

**Proof of Theorem 3.6 and Theorem 4.2.** The proofs are modifications of that for [29, Theorem 7]. Hence, we omit the details for brevity. \( \square \)
References

[1] R.M. Anderson, R.M. May, Infectious diseases of humans: dynamics and control, (1992) Oxford university press.

[2] H. Behncke, Optimal control of deterministic epidemics. Optimal Control Appl. Methods 21 (2000), no. 6, 269-285.

[3] P.A. Bliman, M. Duprez, Y. Privat, N. Vauchelet, Optimal immunity control by social distancing for the SIR epidemic model, (2020) arXiv preprint arXiv:2006.05733.

[4] F. Brauer, Mathematical epidemiology: Past, present, and future. Infectious Disease Modelling, 2(2) (2017), 113-27.

[5] L. Bolzoni, Luca, E. Bonacini, R. Della Marca, Rossella, M. Groppi, Optimal control of epidemic size and duration with limited resources, Math. Biosci. 315 (2019), 108-232.

[6] L. Bolzoni, E. Bonacini, C. Soresina, M. Groppi, Time-optimal control strategies in SIR epidemic models, Math. Biosci. 292 (2017), 86-96.

[7] L. Chen, J. Sun, Optimal vaccination and treatment of an epidemic network model, Phys. Lett. A 378 (2014), no. 41, 3028-3036.

[8] D. Clancy, A.B. Piunovskiy, An explicit optimal isolation policy for a determinstic epidemic model, Appl. Math. Comput. 163 (2005), no. 3, 1109-1121.

[9] N.T. Dieu, D.H. Nguyen, N.H. Du, G. Yin, Classification of asymptotic behavior in a stochastic SIR model, SIAM J. Appl. Dyn. Syst. 15 (2016), no. 2, 1062-1084.

[10] R. Elie, E. Hubert, G. Turinici, Contact rate epidemic control of COVID-19: an equilibrium view, (2020) arXiv preprint arXiv:2004.08221.

[11] P. Grandits, R. M. Kovacevic, V. M. Veliov, Optimal control and the value of information for a stochastic epidemiological SIS-model, J. Math. Anal. Appl. 476 (2019), no. 2, 665-695.

[12] A. Gray, D. Greenhalgh, L. Hu, X. Mao, J. Pan, A stochastic differential equation SIS epidemic model, SIAM J. Appl. Math. 71 (2011), no. 3, 876-902.

[13] A. Gray, D. Greenhalgh, X. Mao, J. Pan, The SIS epidemic model with Markovian switching, J. Math. Anal. Appl. 394 (2012), no. 2, 496-516.

[14] E. Hansen, T. Day, Optimal control of epidemics with limited resources, J. Math. Biol. 62 (2011), no. 3, 423-451.

[15] J. Li, Z. Ma, Global analysis of SIS epidemic models with variable total population size, Math. Comput. Modelling 39 (2004), no. 11-12, 1231-1242.

[16] I. Karatzas, S. Shreve, Brownian Motion and Stochastic Calculus, Springer-Verlag, New York, 1988.

[17] R.M. Kovacevic, Stochastic contagion models without immunity: their long term behaviour and the optimal level of treatment, CEJOR Cent. Eur. J. Oper. Res. 26 (2018), no. 2, 395-421.

[18] W.O. Kermack, A.G. McKendrick, A contribution to the mathematical theory of epidemics. Proceedings of the royal society of london, Series A, Containing papers of a mathematical and physical character, 115(772) (1927), 700-721.

[19] A. Krause, L. Kurowski, K. Yawar, R.A. Van Gorder, Stochastic epidemic metapopulation models on networks: SIS dynamics and control strategies, J. Theoret. Biol. 449 (2018), 35-52.

[20] H.J. Kushner, Approximation and weak convergence methods for random processes, with applications to stochastic systems theory (Vol. 6), (1984) MIT press.
[21] H.J. Kushner, P.G. Dupuis, *Numerical methods for stochastic control problems in continuous time* (Vol. 24), (2013) Springer.

[22] J. Li, Z. Ma, Qualitative analyses of SIS epidemic model with vaccination and varying total population size, *Math. Comput. Modelling* 35 (2002), no. 11-12, 1235–1243.

[23] J. Li, Z. Ma, Stability analysis for SIS epidemic models with vaccination and constant population size, *Discrete Contin. Dyn. Syst. Ser. B* 4 (2004), no. 3, 635-642.

[24] A. Lesniewski, Epidemic control via stochastic optimal control, (2020) arXiv preprint arXiv:2004.06680.

[25] R. Morton, K.H. Wickwire, On the optimal control of a deterministic epidemic. *Advances in Appl. Probability* 6 (1974), 622-635.

[26] D. Nguyen, G. Yin, and C. Zhu, Long-term analysis of a stochastic SIRS model with general incidence rates, *SIAM J. Appl. Math.*, 80 (2020), 814-838.

[27] N.N. Nguyen, G. Yin, Stochastic partial differential equation SIS epidemic models: modeling and analysis, *Commun. Stoch. Anal.* 13 (2019), no. 3, Art. 8, 22 pp.

[28] V.M. Preciado, M. Zargham, C. Enyioha, A. Jadbabaie, G. Pappas, Optimal vaccine allocation to control epidemic outbreaks in arbitrary networks, In 52nd IEEE conference on decision and control (2013), 7486-7491.

[29] Q.S. Song, G. Yin, Z. Zhang, Numerical methods for controlled regime-switching diffusions and regime-switching jump diffusions, *Automatica J. IFAC* 42 (2006), no. 7, 1147-1157.

[30] T.D. Tuong, D.H. Nguyen, N.T. Dieu, K. Tran, Extinction and permanence in a stochastic SIRS model in regime-switching with general incidence rate, *Nonlinear Anal. Hybrid Syst.* 34 (2019), 121-130.

[31] G. Yin, C. Zhu, *Hybrid Switching Diffusions: Properties and Applications*, Springer, New York, 2010.

[32] G. Yin, Q. Zhang, G. Badowski, Discrete-time singularly perturbed Markov chains: aggregation, occupation measures, and switching diffusion limit, *Adv. in Appl. Probab.* 35 (2003), no. 2, 449-476.

[33] X. Zhang, D. Jiang, A. Alsaedi, T. Hayat, Stationary distribution of stochastic SIS epidemic model with vaccination under regime switching, *Appl. Math. Lett.* 59 (2016), 87-93.

[34] Y. Zhao, D. Jiang, The threshold of a stochastic SIS epidemic model with vaccination, *Appl. Math. Comput.* 243 (2014), 718-727.