Analogue of the Poiseuille flow for incompressible polymeric fluid with volume charge. Asymptotics of the linearized problem spectrum

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Abstract. We deduce the formula of the asymptotics of the spectrum of the linear problem on the stability of stationary flows of an incompressible polymeric fluid with volume charge.

1. Introduction

In [1] stationary solutions of the electrohydrodynamic equations describing nonisothermic flows of an incompressible polymeric fluid with volume charge were studied. These stationary solutions are counterparts of the Poiseuille and Couette flows for the Navier-Stokes equations.

In this work we study equations derived by linearizing the system from [1] with respect to a basic solution describing stationary counterparts of Poiseuille flows. Our main achievement is the formula describing the asymptotics of the spectrum of this linear problem.

2. Preliminaries and main result

In [1] a mathematical model describing nonisothermic flow of an incompressible weakly conducting polymeric fluid with negative volume charge was proposed (for the description of polymeric media one used a new rheological model, see [2]). Formulation of the model in [1] was based on, for example, the results of [3], [4], [5] obtained for an inviscid incompressible fluid. In Cartesian coordinate system $x, y$ and in a dimensionless form (the process of obtaining this form is described in detail in [1]) this mathematical model can be written as follows:

$$u_x + v_y = 0,$$

$$\frac{du}{dt} + P_x = \frac{1}{Re} \{(Z a_{11})_x + (Z a_{12})_y\} - \sigma q \Phi_x,$$

$$\frac{dv}{dt} + P_y = \frac{1}{Re} \{(Z a_{12})_x + (Z a_{22})_y\} + G a \frac{Z}{\vartheta} - \sigma q \Phi_y,$$

$$\frac{da_{11}}{dt} - 2 A_1 u_x - 2 a_{12} u_y + \frac{L_{11}}{\tau_0(Z)} = 0,$$

$$\frac{da_{12}}{dt} - A_1 v_x - A_2 u_y + \tilde{K} \frac{a_{12}}{\tau_0(Z)} = 0,$$
\[
\frac{da_{22}}{dt} - 2a_2y - 2a_{12}v_x + \frac{L_{22}}{E_0(Z)} = 0, \quad (6)
\]

\[
\frac{dZ}{dt} = \frac{1}{Pr} \Delta_{x,y}Z, \quad (7)
\]

\[
\frac{dq}{dt} = b(q^2 - (\nabla \Phi, \nabla q)), \quad (8)
\]

\[
\Delta_{x,y} \Phi = -q. \quad (9)
\]

Here \( t \) is the time, \( u, v \) are the components of the velocity vector \( \vec{u} \), \( P \) is the pressure, \( a_{ij} \), \( i, j = 1, 2 \) are the components of the symmetric anisotropy tensor of second kind, \( Z = 1 + \theta \nu \), \( \theta = \frac{a_i}{l} \), \( \nu = \frac{T-T_0}{T_0} \) is the dimensionless deviation of the temperature \( T \) from the average temperature \( T_0 \), \( \theta \) is a constant having the same dimension as the temperature and described below, \( L_{22} = K_f a_{11} + \beta(a_{12}^2 + a_{12}^2), \) \( i = 1, 2 \); \( k, \beta \) \((0 < \beta < 1)\) are the phenomenological parameters of the rheological model (see [2]), \( K_l = W^{-1} + \frac{5}{2} I, \) \( \hat{k} = k - \beta, \) \( \hat{K}_l = K_l + \beta I, \) \( I = a_{11} + a_{22} \) is the first invariant of anisotropy tensor, \( \hat{A}_l = a_{11} + W^{-1}, \) \( i = 1, 2 \); \( \alpha_0(Z) = \frac{1}{E_A} \exp(\frac{E_A}{q}) = \frac{1}{\chi_0(Z)}, \) \( q \) is the density of the negative volume charge, \( \Phi \) is potential of electric field, \( \Delta_{x,y} \) is a characteristic length, \( \nu = \frac{Z}{a} \) is the Weissenberg number, \( A \) is the activation energy.

The independent and dependant variables \( t, x, y, u, v, P, a_{11}, a_{12}, a_{22}, q, \Phi \) in system (1)–(9) are scaled as \( \frac{t}{l}, \frac{x}{u_H}, \frac{y}{u_H}, \frac{W}{l}, \frac{E_A}{a_H}, \frac{\tau_H}{v_0}, \) where \( l \) is a characteristic length, \( u_H \) is a characteristic velocity, \( \rho(\text{const}) \) is the density, \( \varepsilon_0 \) is the electric constant, \( \varepsilon \) is the dielectric penetration, and the parameter \( V_0 \) is described below.

As in [1], our main problem is that for finding solutions of the mathematical model (1)–(9) describing a flow of a weakly conducting polymeric fluid in a plane horizontal condenser. The condenser (channel) of width \( l \) is limited by horizontal walls or, in other words, by electrodes (see Figure 1) which are maintained under a constant temperature difference \( \theta \) and a constant potential difference \( V_0 \) applied to the electrodes (the cathode \( y = 0 \) and the anode \( y = l (l) \)).

In this case system (1)–(9) should be supplemented with the following boundary conditions (see [1]–[5]):

\[
\vec{u} = 0, \quad \nu = 1, \quad \Phi = 0, \quad q = -\lambda \Phi y, \quad A > 0 \text{ on the cathode } y = 0, \quad (10)
\]

\[
\vec{u} = 0, \quad \Phi = 1, \quad \nu = 0 \text{ on the anode } y = 1. \quad (11)
\]

The injected charge on the cathode \( (y = 0) \) is proportional to the magnitude of the electric field, and the injection is assumed to be weak (the positive coefficient \( A << 1 \)).

We will below study the linear boundary value problem obtained resulting from the linearization of the original problem (1)–(11) with respect to the stationary solution \( u \). This stationary solution is described in detail in [1] and for the case \( k = \beta (k = 0) \) has the following form:

\[
u(t, x, y) = \hat{v}(y) = \hat{D} \int_0^y F(s, \hat{C})(\hat{Z}(s) + \hat{C})ds, \]

\[
\hat{v}(y) = 0, \quad (12)
\]

\[
P(t, x, y) = \hat{P}(y) + P_0 - \hat{A}x, \quad \hat{P}(1) = 0, \quad (13)
\]

\[
\hat{P}(y) = \frac{1}{Re} (\hat{Z}(y)\hat{a}_{22}(y) - \hat{Z}(1/2)\hat{a}_{22}(1/2)) + Ga(y - \frac{y^2}{2} - \frac{3}{8}) + \sigma \lambda \alpha_2^2(y - \frac{1}{2}), \quad (14)
\]
\[ a_{11}(t, x, y) = \hat{a}_{11}(y) = \hat{a}_{22}(y) + \frac{2\hat{g}(y)}{A_2(y)}, \quad \hat{g} = \hat{a}_{12}, \quad \hat{A}_2 = W^{-1} + \hat{a}_{22}, \]
\[ a_{22}(t, x, y) = \hat{a}_{22}(y) = \frac{-1 + \sqrt{1 - \hat{G}(y)}}{2W}, \quad \hat{G} = 4\hat{W}^2\hat{q}, \quad \hat{W} = W\beta, \]
\[ a_{12}(t, x, y) = \hat{a}_{12}(y) = \frac{\hat{D}(C + \hat{Z}(y))}{\hat{\theta}\hat{Z}(y)}, \]
\[ Z(t, x, y) = \hat{Z}(y) = 1 + \hat{\theta}(1 - y), \]
\[ q(t, x, y) = \hat{q}(y) = -\frac{3A^2}{(1 + 2A\hat{y})^{\frac{3}{2}}((1 + 2A)^{\frac{3}{2}} - 1)}; \]
\[ \Phi(t, x, y) = \hat{\Phi}(y) = \frac{(1 + 2A\hat{y})^{\frac{3}{2}} - 1}{(1 + 2A)^{\frac{3}{2}} - 1}, \quad \text{where} \quad \frac{\delta(y)}{A_2(y)} \]
\[ F(y, \hat{C}) = \frac{\exp(E_A \hat{Z}(y) - 1)}{1 + W\hat{a}_{22}(y)}, \]

\[ \hat{D} = Re\hat{A}; \quad -\hat{A} \left( \hat{A} = \frac{\Delta P}{\rho u^2h} \right) \text{is the positive constant and the dimensional value} \Delta \hat{P} > 0 \text{is the dimensionless drop of pressure on the segment} h \text{ (see Figure 1)}, \quad \hat{P}_0 (= \text{const} > 0) \text{is the pressure on the channel’s axis} \ y = \frac{1}{2} \text{ for} \ x = 0, \quad \alpha = \hat{\Phi}'(0) = \frac{3A}{(1 + 2A)^{\frac{3}{2}} - 1}. \]

The constant \( \hat{C} \) can be found from the nonlinear equation
\[ \int_0^1 F(s, \hat{C})ds\hat{C} + \int_0^1 F(s, \hat{C})\hat{Z}(s)ds = 0. \]

**Remark 2.1** Note that in (12) we can assign \( \hat{D} = 1 \) thanks to the choice of the characteristic parameter \( u_H = \frac{\Delta P}{k_0\hat{P}_0}. \)
The influence of the parameters $\bar{\theta}$ ($\bar{\theta} > 0$ is the heating from below and $\bar{\theta} < 0$ is the heating from above), and $E_A$ is shown in Figures 2 and 3. Moreover, in Figure 2 one has $k = 0$, $D = 1$, $W = 1$, $\beta = 0.5$, $E_A = 0.01$, $\bar{\theta} = -0.4, 0, 0.4, 8$ whereas in Figure 3 $k = 0$, $D = 1$, $W = 1$, $\beta = 0.5$, $\bar{\theta} = 2$, $E_A = 0.01, 1, 5$.

Figure 2. Curves: $1 \rightarrow \bar{\theta} = 8$, $2 \rightarrow \bar{\theta} = 0.4$, $3 \rightarrow \bar{\theta} = 0$, $4 \rightarrow \bar{\theta} = -0.4$.

Figure 3. Curves: $1 \rightarrow E_A = 0.01$, $2 \rightarrow E_A = 1$, $3 \rightarrow E_A = 5$.

Denoting small perturbations of the original values by capital letters and linearizing the original problem (1)–(11), we get

$$\vec{V}_t + \vec{B}\vec{V_x} + \vec{C}\vec{V_y} + \vec{R} + \vec{\Gamma} = 0,$$

$$\Delta_{x,y} \Omega = \frac{1}{Re} \{ \dot{Z}(a_{11} - a_{22})_{xx} + \frac{2\bar{\theta}\dot{\varphi}}{A_2} X_{xx} + 2(\dot{Z}a_{12} + \dot{a}_{12}\bar{\theta}X)_{xy} \} - 2\ddot{u}'v_x + GaX_y + \sigma M_y,$$
Here $\vec{V} = (u, v, a_{11}, a_{12}, a_{22})^T$, $\vec{B} = \hat{a}I_5 + \hat{B}_0$, $I_5$ is the unit matrix of fifth order, $\hat{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, 0)^T$, $X = \frac{Z_0}{\bar{y}}$, $Q = \frac{Q}{\bar{y}}$, $\Omega = P - \frac{(\bar{Z}_{0_{22}} + \bar{a}_{22}^2X)}{Re} + \sigma \hat{q} \Phi$, $M = \hat{q}'(\Phi - Q\hat{\Phi}')$,
\[
\dot{\vec{B}}_0 = \begin{pmatrix}
O_2 & -\alpha_0^2 & 0 & 0 \\
-2A_1 & 0 & 0 & 0 \\
0 & -\bar{A}_1 & 0 & 0 \\
0 & 0 & -2\bar{a}_{12} & 0 \\
\end{pmatrix}, \quad \hat{C} = \begin{pmatrix}
O_2 & 0 & -\alpha_0^2 & 0 \\
0 & 0 & 0 & 0 \\
-2\bar{a}_{12} & 0 & 0 & 0 \\
0 & -\bar{A}_1 & 0 & 0 \\
\end{pmatrix}, \quad \hat{R} = \begin{pmatrix}
0 & \hat{u} & 0 & 0 & 0 \\
0 & 0 & \bar{\theta} & 0 \end{pmatrix}, \quad \begin{pmatrix}
0 & \bar{a}_{11}^2 & \bar{a}_{12} & \bar{a}_{22} \\
0 & \bar{a}_{11} & 0 & 0 \\
0 & \bar{a}_{12} & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

$O_2$ and $O_3$ are zero matrices of second and third order respectively,
\[
\alpha_0^2 = \frac{Z(y)}{Re}, \quad R_{33} = \tilde{\chi}_0(\tilde{Z})(W^{-1} + 2\beta\hat{a}_{11}), \quad R_{44} = \tilde{\chi}_0(\tilde{Z})(W^{-1} + \beta\hat{I}), \\
R_{55} = \tilde{\chi}_0(\tilde{Z})(W^{-1} + 2\beta\hat{a}_{22}), \quad R_{34} = 2(-\hat{u}' + \beta\tilde{\chi}_0(\tilde{Z})\hat{a}_{12}), \\
\tilde{a}_{13} = \beta\tilde{\chi}_0(\tilde{Z})\tilde{a}_{12}, \quad \tilde{a}_{45} = -\hat{u}' + \beta\tilde{\chi}_0(\tilde{Z})\hat{a}_{12}, \quad R_{54} = 2\beta\tilde{\chi}_0(\tilde{Z})\hat{a}_{12}, \\
\tilde{a}_{11} + W^{-1}, \quad \tilde{I} = \tilde{a}_{11} + \tilde{a}_{22}, \\
\Gamma_1 = \Omega_x - \frac{\tilde{a}_{13}\hat{\theta}X_y}{Re} + \frac{\tilde{a}_{22}\hat{\theta}X_x}{Re X}, \\
\Gamma_2 = \Omega_y - \frac{\tilde{a}_{12}\hat{\theta}X_y}{Re} - GaX - \sigma M, \quad \tilde{\Gamma}_3 = r_1\hat{\theta}X, \quad \tilde{\Gamma}_4 = r_2\hat{\theta}X, \\
r_1 = 2\tilde{\chi}_0'(\tilde{Z})\frac{\tilde{a}_{12}(W^{-1} + \beta\hat{I})}{A_2}, \quad r_2 = \frac{\tilde{a}_{22}}{A_2}, \quad \tilde{\chi}_0'(\tilde{Z}) = \tilde{\chi}_0(\tilde{Z})\frac{E_4 + \tilde{Z}'}{2\tilde{Z}}.
\]

We now supplement system (13)–(17) with the boundary conditions
\[
y = 0 \text{ (cathode): } u = X = \Phi = 0, \quad Q = -\frac{1}{A\alpha} \Phi_y, \quad \Omega_y = \frac{1 + \tilde{\theta}}{Re} (a_{12})_x - \sigma \alpha^2 A^2 Q, \quad (18)
\]
\[
y = 1 \text{ (anode): } u = X = \Phi = 0, \quad \Omega_y = (a_{12})_x Re - \frac{\sigma \alpha^2 A^2}{1 + 2A} Q, \quad (19)
\]
and some initial data. Initial data must fulfill equations (14), (17), the boundary conditions (18), (19) as well as the incompressibility condition
\[
\text{div} \vec{u} = 0, \quad \vec{u} = (u, v).
\]

We will below search for solutions of the linear problem formulated above in the special form
\[
\begin{cases}
\tilde{V} = \exp(\lambda t + i\omega x)\tilde{V}(y), \\
(\Omega, X, Q, \Phi) = \exp(\lambda t + i\omega x)(\Omega^*(y), X^*(y), Q^*(y), \Phi^*(y)), \\
\lambda = \eta + i\xi; \quad \xi, \omega \in R^1
\end{cases}
\]
(20)

(the hats are omitted below).

Let us assume that
\[
\frac{\gamma_0}{\alpha_0} \neq \frac{1}{b\Phi}, \quad y \in [0, 1].
\]

We are now in a position to formulate for the obtained spectral problem our main result on the asymptotic behavior of the spectrum points as $|\lambda| \rightarrow \infty$ for a fixed parameter $\omega$. 

\[5\]
Theorem 2.1 If condition (21) holds, then the eigenvalues of the boundary value problem arising under the search of harmonic (with respect to \( x \)) solutions in form (20) for problem (13)–(19) (with some initial data) have the following asymptotic representation:

\[
\lambda_k = -\frac{1}{2} \int_0^1 \frac{d\xi}{\alpha_0} \left[ \int_0^1 (d_{11} - d_{22})d\xi + 2k\pi \right] + O\left(\frac{1}{|k|}\right), \quad \gamma_2^2 = \frac{1}{A_2},
\]

where \( k \) is an integer, \(|k| \to +\infty\), and the quantities \( d_{11}, d_{22} \) are defined as follows

\[
d_{11} = \frac{\gamma_0}{\alpha_0} (\hat{u} + \hat{a}_{12}\alpha_0\gamma_0)i\omega - \frac{1}{2Re}\frac{\bar{\theta}}{\alpha_0^2} + \frac{\gamma_0}{\alpha_0} R_{43}\hat{a}_{12} + \frac{\gamma_0}{\alpha_0} \frac{R_{44}}{2} + \frac{\gamma_0}{\alpha_0} \hat{A}_2 + \frac{1}{\alpha_0} \frac{A_2}{2} (\alpha_0\gamma_0)^{\prime}.
\]

\[
d_{22} = -\frac{\gamma_0}{\alpha_0} (\hat{u} - \hat{a}_{12}\alpha_0\gamma_0)i\omega \quad - \quad \frac{\theta}{2Re}\frac{\bar{\theta}}{\alpha_0^2} - \frac{\gamma_0}{\alpha_0} R_{43}\hat{a}_{12} - \frac{R_{44}\gamma_0}{2\alpha_0} + \frac{\hat{A}_2\gamma_0}{\alpha_0} (\alpha_0\gamma_0)^{\prime}.
\]

The formulated result together with full proof will be published in [11].

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