Two Theorems on Convergence of Schrödinger Means

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Abstract
Localization and convergence almost everywhere of Schrödinger means are studied.

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1 Introduction

For $f \in L^2(\mathbb{R}^n)$, $n \geq 1$ and $a > 1$ we set

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx, \quad \xi \in \mathbb{R}^n,$$

and

$$S_t f(x) = \int_{\mathbb{R}^n} e^{i\xi \cdot x} e^{it|\xi|^a} \hat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n, \quad t \geq 0.$$  

For $a = 2$ and $f$ belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ we set $u(x, t) = S_t f(x)/(2\pi)^n$. It then follows that $u(x, 0) = f(x)$ and $u$ satisfies the Schrödinger equation $i\partial u/\partial t = \Delta u$.

We introduce Sobolev spaces $H_s = H_s(\mathbb{R}^n)$ by setting

$$H_s = \{f \in \mathcal{S}'; \|f\|_{H_s} < \infty\}, \quad s \in \mathbb{R},$$

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where
\[ \| f \|_{H_s} = \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}. \]

In the case \( n = 1 \) it is well-known (see Sjölin [7] and Vega [9] and in the case \( a = 2 \) Carleson [3] and Dahlberg and Kenig [4]) that
\[ \lim_{t \to 0} (2\pi)^{-n} S_t f(x) = f(x) \quad (1) \]
after almost everywhere if \( f \in H_{1/4} \). Also it is known that \( H_{1/4} \) cannot be replaced by \( H_s \) if \( s < 1/4 \).

Assuming \( n \geq 2 \) and \( a = 2 \) Bourgain [1] has proved that (1) holds almost everywhere if \( f \in H_s \) and \( s > 1/2 - 1/4n \). On the other hand Bourgain [2] has proved that \( s \geq n/2(n + 1) \) is necessary for convergence for \( a = 2 \) and all \( n \geq 2 \). In the case \( n = 2 \) and \( a = 2 \), Du, Guth, and Li [5] proved that the condition \( s > 1/3 \) is sufficient. Recently Du and Zhang [6] proved that the condition \( s > n/2(n + 1) \) is sufficient for \( a = 2 \) and all \( n \geq 3 \).

In the case \( a > 1, n = 2 \), it is known that (1) holds almost everywhere if \( f \in H_{1/2} \) and in the case \( a > 1, n \geq 3 \), convergence has been proved for \( f \in H_s \) with \( s > 1/2 \) (see [7] and [9]).

If \( f \in L^2(\mathbb{R}^n) \) then \( (2\pi)^{-n} S_t f \to f \) in \( L^2 \) as \( t \to 0 \). It follows that there exists a sequence \((t_k)_{k=1}^{\infty}\) satisfying
\[ 1 > t_1 > t_2 > t_3 > \cdots > 0 \quad \text{and} \quad \lim_{k \to \infty} t_k = 0 \quad (2) \]
such that
\[ \lim_{k \to \infty} (2\pi)^{-n} S_{t_k} f(x) = f(x) \]
after almost everywhere.

We shall here study the problem of deciding for which sequences \((t_k)_{k=1}^{\infty}\) one has
\[ \lim_{k \to \infty} (2\pi)^{-n} S_{t_k} f(x) = f(x) \]
almost everywhere if \( f \in H_s \). We have the following result.

**Theorem 1** Assume \( n \geq 1 \) and \( a > 1 \) and \( s > 0 \). We assume that (2) holds and that
\[ \sum_{k=1}^{\infty} t_k^{2s/a} < \infty \quad \text{and} \quad f \in H_s(\mathbb{R}^n) \]. Then
\[ \lim_{k \to \infty} (2\pi)^{-n} S_{t_k} f(x) = f(x) \]
for almost every \( x \) in \( \mathbb{R}^n \).
Now assume $n = 1$, $a > 1$, and $0 \leq s < 1/4$. In Sjölin [8] we studied the problem if there is localization or localization almost everywhere for the above operators $S_t$ and the functions $f \in H_s$ with compact support, that is, do we have

$$\lim_{t \to 0} S_t f(x) = 0$$

everywhere or almost everywhere in $\mathbb{R}^n \setminus \text{supp}(f)$?

It is proved in [8] that there is no localization or localization almost everywhere of this type for $0 \leq s < 1/4$. In fact the following theorem was proved in Sjölin [8].

**Theorem A** There exist two disjoint compact intervals $I$ and $J$ in $\mathbb{R}$ and a function $f$ which belongs to $H_s$ for all $s < 1/4$, with the properties that $(\text{supp } f) \subset I$ and for every $x \in J$ one does not have

$$\lim_{t \to 0} S_t f(x) = 0.$$

Let $\omega$ be a continuous and decreasing function on $[0, \infty)$. Assume that $\omega(t) \to 0$ as $t \to \infty$. Set

$$H_\omega = \{ f \in \mathcal{S}'; \| f \|_{H_\omega} < \infty \}$$

where

$$\| f \|_{H_\omega} = \left( \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^{s/2} \omega(|\xi|) d\xi \right)^{1/2}$$

We have the following result.

**Theorem 2** The function $f$ in theorem A can be chosen so that $f \in H_\omega$.

Theorem 2 shows that the sufficient condition $f \in H_{1/4}$ for convergence almost everywhere and localization almost everywhere of Schrödinger means is very sharp. In the case $a = 2$ Theorem 2 was obtained in 2009 (unpublished). After proving Theorem 2 we shall use Theorem 1 to make a remark on the Schrödinger means $S_t f(x)$ for the function $f$ which was constructed in [8] to prove Theorem A.

## 2 Proofs

In the proof of Theorem 1 we shall need the following lemma.

**Lemma 1** Assume $n \geq 1$, $a > 1$, $0 < s < 1$, and $0 < \delta < 1$. Set

$$m(\xi) = \frac{e^{i\delta|\xi|^a} - 1}{(1 + |\xi|^2)^{s/2}}, \quad \xi \in \mathbb{R}^n.$$
Then one has

\[ \|m\|_\infty \leq C \delta^{s/a} \]

where the constant C does not depend on \( \delta \), and \( \|m\|_\infty \) denotes the norm of \( m \) in \( L^\infty(\mathbb{R}^n) \).

**Proof of Lemma 1.** We shall write \( A \lesssim B \) if there is a constant \( C \) such that \( A \leq CB \).

In the case \( |\xi| \geq \delta^{-1/a} \) one has

\[ |\xi|^s \geq \delta^{-s/a} \quad \text{and} \quad |m(\xi)| \lesssim \frac{1}{|\xi|^s} \leq \delta^{s/a}. \]

Then assume \( 0 \leq |\xi| \leq 1 \). We obtain

\[ |m(\xi)| \lesssim \delta |\xi|^a \leq \delta \leq \delta^{s/a}. \]

In the remaining case \( 1 < |\xi| < \delta^{-1/a} \) one obtains

\[ |m(\xi)| \lesssim \delta |\xi|^a |\xi|-s/a = \delta \delta^{-s/a} = \delta^{-1+s/a} = \delta^{s/a} \]

and the proof of Lemma 1 is complete. \( \square \)

We shall then give the proof of Theorem 1.

**Proof of Theorem 1.** We may assume \( 0 < s < 1 \). We set

\[ h_k(x) = (2\pi)^{-n} S_k f(x) - f(x), \quad x \in \mathbb{R}^n, \text{ for } k = 1, 2, 3, \ldots \]

We have \( f \in H_s \) and we define \( g \) by taking

\[ \widehat{g}(\xi) = \widehat{f}(\xi)(1 + |\xi|^2)^{s/2}. \]

It then follows that \( g \in L^2(\mathbb{R}^n) \).

We have

\[ S_k f(x) = \int e^{ix \cdot \xi} e^{it_k |\xi|^a} (1 + |\xi|^2)^{-s/2} \widehat{g}(\xi) d\xi \]

and

\[ f(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} (1 + |\xi|^2)^{-s/2} \widehat{g}(\xi) d\xi. \]
Hence
\[
h_k(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} (e^{it_k|\xi|^a} - 1)(1 + |\xi|^2)^{-s/2} \hat{g}(\xi) d\xi
\]
\[
= (2\pi)^{-n} \int e^{ix \cdot \xi} m(\xi) \hat{g}(\xi) d\xi,
\]
where
\[
m(\xi) = (e^{it_k|\xi|^a} - 1)(1 + |\xi|^2)^{-s/2}.
\]
According to Lemma 1 we have \(\|m\|_\infty \lesssim t_k^{s/a}\) and applying the Plancherel theorem we obtain
\[
\|h_k\|_2^2 = c \int |m(\xi) \hat{g}(\xi)|^2 d\xi \lesssim \|m\|_\infty^2 \int |\hat{g}(\xi)|^2 d\xi \lesssim t_k^{2s/a} \|f\|_{H_s}^2.
\]
It follows that
\[
\sum_1^\infty \int |h_k|^2 dx \lesssim \left( \sum_1^\infty t_k^{2s/a} \right) \|f\|_{H_s}^2 < \infty
\]
and applying the theorem on monotone convergence one also obtains
\[
\int \left( \sum_1^\infty |h_k|^2 \right) dx < \infty.
\]
We conclude that \(\sum_1^\infty |h_k|^2\) is convergent almost everywhere and hence \(\lim_{k \to \infty} h_k(x) = 0\) and
\[
\lim_{k \to \infty} (2\pi)^{-n} S_{tk} f(x) = f(x)
\]
almost everywhere. \(\square\)

Now assume \(n = 1\) and \(a > 1\). We set
\[
m(\xi) = e^{i|\xi|^a}, \quad \xi \in \mathbb{R},
\]
and let \(K\) denote the Fourier transform of \(m\) so that \(K \in \mathcal{S}'(\mathbb{R})\). According to Sjölin [8] p.142, \(K \in C^\infty(\mathbb{R})\) and there exists a number \(\alpha > 0\) such that
\[
|K(x)| \lesssim 1 + |x|^\alpha \text{ for } x \in \mathbb{R}
\]
For \(t > 0\) it is then clear that
\[
e^{it|\xi|^a} = m(t^{1/a} \xi)
\]
has the Fourier transform
\[ K_t(y) = t^{-1/a} K(t^{-1/a} y). \]

It follows that \( S_t f(x) = K_t \ast f(x) \) for \( f \in L^2(\mathbb{R}^m) \) with compact support. We let \( \tilde{g} \) denote the inverse Fourier transform of \( g \) and choose \( g \in \mathcal{S}(\mathbb{R}) \) such that \( \text{supp} \tilde{g} \subset (-1, 1) \) and \( \tilde{g}(0) \neq 0 \). We set
\[ f_v(x) = e^{-ix/v^2} \tilde{g}(x/v), \quad 0 < v < 1, \quad x \in \mathbb{R}. \]

According to [7], p.143, one has \( \hat{f}_v(\xi) = v g(v \xi + 1/v) \) and
\[ \| f_v \|_{H^s} \lesssim v^{1/2 - 2s} \text{ for } 0 < s < 1/4. \]

We shall state three lemmas from [8].

**Lemma 2** There exist positive numbers \( c_0, \delta \) and \( v_0 \) such that
\[ |S_{xv^{2\alpha - 2}/a} f_v(x)| \geq c_0 \]
for \( 0 < v < v_0 \) and \( 0 < x < \delta \).

In the remaining part of this paper \( \delta \) and \( v_0 \) are given by Lemma 2. We may also assume that \( \delta < 1 \).

**Lemma 3** For \( 0 < v < \min(v_0, \delta/4) \), \( 0 < t < 1 \), and \( \delta/2 < x < \delta \) one has
\[ |S_t f_v(x)| \lesssim \frac{v}{t^\gamma} \]
where \( \gamma = (1 + \alpha)/a > 0 \).

**Lemma 4** For \( 0 < v < \min(v_0, \delta/4) \), \( 0 < t < 1 \), and \( \delta/2 < x < \delta \) one has
\[ |S_t f_v(x)| \lesssim \frac{t}{v^\beta} \]
where \( \beta = 2a \).

We shall use these lemmas to prove Theorem 2.

**Proof of Theorem 2.** Now let \( v_1 \) satisfy \( 0 < v_1 < \min(v_0, \delta/4) \) and set \( \epsilon_k = 2^{-k}, \) \( k = 1, 2, 3, \ldots \)

We also set \( \mu = \max((2a - 2)\gamma, \beta/(2a - 2)) \) and choose \( v_k, k = 2, 3, 4, \ldots \), such that \( 0 < v_k \leq \epsilon_k v_{k-1}^\mu \) and
\[ \sum_{k=1}^{\infty} \sqrt{\omega(1/v_k^{1/2})} < \infty. \]
We then set $f = \sum_{k=1}^{\infty} f_{v_k}$ and shall prove that $f \in H_{\omega_0}$.

Arguing as in [8, pp. 145–147], it follows from Lemmas 2, 3, and 4 that with $t_k(x) = x v_k^{2a-2}/a$ one has

$$|S_{t_k}(x) f(x)| \geq c_0 > 0$$

for $\delta/2 < x < \delta$ and $k \geq k_0$. Hence we do not have $\lim_{t \to 0} S_t f(x) = 0$ in the interval $(\delta/2, \delta)$. Taking $I = [-v_1, v_1]$ and $J \subset (\delta/2, \delta)$ we have $\text{supp } f \subset I$ and for every $x \in J$ one does not have $\lim_{t \to 0} S_t f(x) = 0$. Thus Theorem 2 follows. It remains to prove that $f \in H_{\omega_0}$.

We have

$$\|f_v\|_{H_\omega}^2 = \int |\hat{f}_v(\xi)|^2 (1 + \xi^2)^{1/4} \omega(|\xi|) d\xi \lesssim I_1 + I_2,$$

where

$$I_1 = \int_{-1}^{1} |\hat{f}_v(\xi)|^2 d\xi \leq C v^2$$

and

$$I_2 = \int |\hat{f}_v(\xi)|^2 |\xi|^{1/2} \omega(|\xi|) d\xi.$$

It follows that

$$I_2 = \int v^2 |g(\xi + 1/v)|^2 |\xi|^{1/2} \omega(|\xi|) d\xi$$

$$= \int v^{1/2} |g(\eta + 1/v)|^2 |\eta|^{1/2} \omega(v^{-1/2} \eta) \eta =$$

$$= v^{1/2} \int |g(\xi)|^2 |\xi - 1/v|^{1/2} \omega(v^{-1/2} \xi) \leq C v^{1/2}$$

$$\times \int |\xi - 1/v| \leq v^{1/2}$$

$$+ C v^{1/2} \int |g(\xi)|^2 (|\xi|^{1/2} + v^{-1/2}) \omega(v^{-1/2}) d\xi$$

$$\leq C v^{3/4} + C \omega(v^{-1/2}).$$

Hence

$$\|f_v\|_{H_\omega}^2 \lesssim v^{3/4} + \omega(v^{-1/2}), \quad 0 < v < 1,$$

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and 
\[ \|f_v\|_{H_\omega} \lesssim v^{3/8} + \sqrt{\omega(v^{-1/2})}. \]

We have \( f = \sum_1^\infty f_{v_k} \) and it follows that 
\[ \|f\|_{H_\omega} \leq \sum_1^\infty \|f_{v_k}\|_{H_\omega} \lesssim \sum_1^\infty v_k^{3/8} + \sum_1^\infty \omega(v_k^{-1/2}) < \infty \]

since \( v_k \leq \epsilon_k \).

We conclude that \( f \in H_\omega \) and the proof of Theorem 2 is complete. \( \square \)

Remark 1 In Sjölin [8] the function \( f \) in Theorem A is given by the formula 
\[ f = \sum_1^\infty f_{v_k}, \]

where \( v_k \) is defined by taking \( 0 < v_1 < \min(v_0, \delta/4) \) and \( v_k = \epsilon_k v_k^{\mu} \) for \( k = 2, 3, 4, \ldots \). Here \( \epsilon_k = 2^{-k} \) and \( \mu > 0 \) is given in the proof of Theorem 2. Also let the intervals \( I \) and \( J \) be defined as in the proof of Theorem 2. We then set \( t_k(x) = x v_k^{2a-2}/a \) for \( x \in J \) and \( k = 1, 2, 3, \ldots \)

It is proved in [8] that for every \( x_0 \in J \)
\[ \lim_{k \to \infty} S_{t_k(x_0)} f(x_0) = 0. \] (3)

We now fix \( x_0 \in J \) and shall use Theorem 1 to prove that although (3) holds one also has
\[ \lim_{k \to \infty} S_{t_k(x_0)} f(x) = 0 \text{ for almost every } x \in J. \] (4)

We have \( v_k < \epsilon_k \) and it follows that 
\[ 0 < t_k(x_0) \leq \epsilon_k^{2a-2} \]

and 
\[ \sum_1^\infty (t_k(x_0))^{2s/a} \leq \sum_1^\infty 2^{-k(2a-2)2s/a} < \infty \]

for \( 0 < s < 1/4 \). Also \( f \in H_s \) for \( 0 < s < 1/4 \) and (4) follows from an application of Theorem 1.
Remark 2 In the case $a = 2$ one has $\mu = 2$ and $v_k = \epsilon_k v_{k-1}^2$, and we also have $0 < v_1 < 1/4$. It can be proved that it follows that

$$v_k = 4 \cdot 2^{k-d}$$

where $d$ is a constant and $d > 2$.

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