Correct Use of Percent Coefficient of Variation (%CV) Formula for Log-Transformed Data

Abstract
The coefficient of variation (CV) is a unitless measure typically used to evaluate the variability of a population relative to its standard deviation and is normally presented as a percentage [1]. When considering the percent coefficient of variation (%CV) for log-transformed data, we have discovered the incorrect application of the standard %CV form in obtaining the %CV for log-transformed data. Upon review of various journals, we have noted the formula for the %CV for log-transformed data was not being applied correctly. This communication provides a framework from which the correct mathematical formula for the %CV can be applied to log-transformed data.

Keywords: Coefficient of variation; Log-transformation; Variances; Statistical technique

Abbreviations: CV: Coefficient of Variation; %CV: CV x 100%

Introduction
The percent coefficient of variation, %CV, is a unitless measure of variation and can be considered as a “relative standard deviation” since it is defined as the standard deviation divided by the mean multiplied by 100 percent:

\[ \%CV = 100\% \frac{\sigma}{\mu} \]  \hspace{1cm} (1)

This formula (1) holds true for non-transformed data. The %CV calculation will be different mathematically depending on the mean and variance of the transformation.

Table 1: Recreation of portions of Table 5 from Hatzakis et al. (2016) and the correct calculation of lognormal %CV.

| Level     | N  | Mean | Total SD | Published Incorrect %CV | Correct %CV |
|-----------|----|------|----------|-------------------------|-------------|
| 5.00E+01  | 41 | 1.66 | 0.144    | 8.67                    | 34.1        |
| 1.00E+02  | 74 | 1.82 | 0.180    | 9.91                    | 43.3        |
| 1.00E+03  | 81 | 2.75 | 0.112    | 4.08                    | 26.2        |
| 1.00E+04  | 81 | 3.81 | 0.067    | 1.77                    | 15.5        |
| 1.00E+05  | 81 | 4.96 | 0.067    | 1.35                    | 15.5        |
| 1.00E+06  | 78 | 6.00 | 0.055    | 0.92                    | 12.7        |
| 1.00E+07  | 81 | 6.89 | 0.062    | 0.90                    | 14.3        |

If the untransformed %CV is used on log-normal data, the resulting %CV will be too small and give an overly optimistic, but incorrect, view of the performance of the measured device.

For example, Hatzakis et al. [1], Table 1, showed an assessment of inter-instrument, inter-operator, inter-day, inter-run, intra-run and total variability of the Aptima HIV-1 Quant Dx in various HIV-1 RNA concentrations. In Table 1, below, we recreate their total SD and %CV columns (the latter for which they use Formula (1), and calculate the correct log-normal %CV from Formula (7) below. From the Table 1, it can be seen that using the incorrect %CV formula for lognormally distributed data will give abnormally smaller %CVs.
To estimate variances of transformations of raw values, we use a statistical technique called the method of moments. Table 2 shows the variances standard deviations and %CVs for the untransformed and log-transformation one may consider.

Table 2: Variances, SDS and %CV of log-transformation.

| Transformation          | $SD[f(x)]$ | $SD[f(x)]$ | %CV | %CV |
|-------------------------|------------|------------|-----|-----|
| None: $X$               | $Var(x)$   | $\sqrt{Var(x)}$ | $%CV=100\% \frac{\sigma}{\mu}$ | $%CV(Y)=100\% \sqrt{\frac{ln(10)^2}{1}}$ |
| Log: $log_{10}X$ or $ln(X)$ | $\frac{Var(x)}{ln(10).E(x)^2}$ | $\frac{\sqrt{Var(x)}}{ln(10).E(x)}$ |  | |

$\sigma$ = standard deviation; $\mu$ = mean; $ln(*)$ = natural logarithm; $\sigma_{log}$ = standard deviation of the log-transformed data; $E(x)$ is the expected value of x.

%CV for the log-Normally distributed random variable (RV)

We show the derivation of the percent coefficient of variation (%CV) for a log-normally distributed random variable. The coefficient of variation for log-normally distributed random variable $Y=ln(X)$ is estimated using the following formula:

$$%CV(Y)=100\% \sqrt{\frac{[ln(10)]^2}{1}}$$

Or its equivalent $log_{10}(x) = \frac{log_5(x)}{log_5(b)}$

Where $ln$ is the natural log and $\sigma^2$ is the variance. The derivation of the formulae follows.

Since the random variable $X$ is log-normally distributed, then $Y=ln(X)$ is distributed as a Normal probability distribution with mean $\mu$ and variance $\lambda^2$, that is, $Y \sim N(\mu, \lambda^2)$.

Now, the moment generating function for a Normal probability distribution is [3]:

$$M(t)=E(e^{tY})=e^{\mu t+\frac{\lambda^2 t^2}{2}}$$

Therefore, it follows by substitution:

$$C(\lambda)=SD(Y) = SD(Y) \left[ \frac{E(e^{tY})}{E(e^{tY})} \right]^2 = \sqrt{M(2)-M(0)} \frac{\sqrt{e^{2\lambda^2}e^{2\mu^2}-2e^{\mu^2+\lambda^2}}}{\sqrt{e^{\lambda^2}2}} \left( \frac{1}{2} \right)$$

using the general statistical property that defines the variance as

$$Var(Y) = E\left[ (Y-E[Y])^2 \right] = E\left( Y^2 \right) - \left[ E(Y) \right]^2$$

such that the standard deviation becomes

$$SD(Y) = \sqrt{E\left( Y^2 \right) - \left[ E(Y) \right]^2}$$

(5)

To simplify expression (5), above, we use the logarithm base change rule result [4] that shows $log_a(x)=\frac{log_b(x)}{log_b(a)}$ for any logarithm base $b$ and $c$. If $b=10$ and $c$ = the "natural log base e" = $e$, then

$$log_{10}(X)=log_5(X) \frac{ln(Y)}{ln(10)}$$

(6)

since $Y=ln(X)$ and, given that $Y$ is distributed as a Normal probability distribution with mean $\mu$ and variance $\lambda^2$, that is, $Y \sim N(\mu, \lambda^2)$, this implies that $\lambda^2=[ln(10)]^2 \sigma^2$ [using the statistical property that $VAR(aX) = a^2 \cdot VARX$ where $a$ is a constant and $X$ is a random variable].

Next, substituting this result into the formula for the %CV involving $\lambda$ and multiplying by 100% we obtain the final %CV expression:

$$%CV(Y)=100\% \sqrt{\frac{ln(10)^2}{1}}$$

(7)

Conclusion

The authors have shown that it is easy for the researcher to be confused with respect to which is the correct formula to use for log-transformed data when calculating the percent coefficient of variation (%CV). When using the incorrect formula, the researcher may be faced with abnormally low %CV values. With that in mind,
the authors have shown the correct formula to use for calculating %CV for log-transformed data.

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Conflict of Interest

No conflict of interest.

Reference

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