A Generic Framework for Interesting Subspace Cluster Detection in Multi-attributed Networks

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ABSTRACT
Detection of interesting (e.g., coherent or anomalous) clusters has been studied extensively on plain or univariate networks, with various applications. Recently, algorithms have been extended to networks with multiple attributes for each node in the real-world. In a multi-attributed network, often, a cluster of nodes is only interesting for a subset (subspace) of attributes, and this type of clusters is called subspace clusters. However, in the current literature, few methods are capable of detecting subspace clusters, which involves concurrent feature selection and network cluster detection. These relevant methods are mostly heuristic-driven and customized for specific application scenarios.

In this work, we present a generic and theoretical framework for detection of interesting subspace clusters in large multi-attributed networks. Specifically, we propose a subspace graph-structured matching pursuit algorithm, namely, SG-Pursuit, to address a broad class of such problems for different score functions (e.g., coherence or anomalous functions) and topology constraints (e.g., connected subgraphs and dense subgraphs). We prove that our algorithm 1) runs in nearly-linear time on the network size and the total number of attributes and 2) enjoys rigorous guarantees (geometrical convergence rate and tight error bound) analogous to those of the state-of-the-art algorithms for sparse feature selection problems and subgraph detection problems. As a case study, we specialize SG-Pursuit to optimize a number of well-known score functions for two typical tasks, including detection of coherent dense and anomalous connected subspace clusters in real world networks. Empirical evidence demonstrates that our proposed generic algorithm SG-Pursuit performs superior over state-of-the-art methods that are designed specifically for these two tasks.

1 INTRODUCTION
With recent advances in hardware and software technologies, the huge volumes of data now being collected from multiple sources are naturally modeled as multi-attributed networks. For example, massive multi-attributed biological networks have been created by integrating gene expression data with secondary data such as pathway or protein-protein interaction data for improved outcome prediction of cancer patients [26]. Other examples include the multi-attributed networks that combine “Big data” (e.g., Twitter feeds) and traditional surveillance data for influenza studies [13] and the social networks that contain both the friendship relations and user attributes such as interests, frequencies of keywords mentioned in posts, and demographics [18].

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Figure 1: A social network with three attributes (age, PC games, and sport) in each user node and one potential coherent dense subspace cluster (highlighted in the shaded region and blue-colored texts) that has a coherent subset of attributes (age and PC game) and a dense subgraph of nodes (4, 5, 6, and 7). This cluster might be of interest to video game producers. (Adapt from [18])

Figure 2: A health surveillance network of emergency departments (EDs) with three attributes (counts of cases of three different ICD-9 disease symptoms [28], including cough, headache, chest pain) in each ED node and one potential anomalous connected subspace cluster (highlighted in the shaded region and blue-colored texts) that has a anomalous subset of attributes (cough and headache) and a connected subgraph of nodes (1, 2, 3, 5, and 6). The counts of these two attributes within the subgraph are abnormally higher than those outside the subgraph. In this scenario, the anomalous connected subspace cluster is used for disease outbreak detection.

As one of the major tasks in network mining, the detection of interesting clusters in attributed networks, such as coherent or anomalous clusters, has attracted a great deal of attention in many
applications, including medicine and public health [18, 28], law enforcement [43], cyber security [32], transportation [4], among others [36, 37, 39]. To deal with the multiple or even high-dimensional attributes, most existing methods either utilize all the given attributes [16, 31] or preform a unsupervised feature selection as a preprocessing step [41]. However, as demonstrated in a number of studies [17–19, 33, 34], clusters of interest in a multi-attribute network are often \textit{subspace clusters}, each of which is defined by a \textit{cluster of nodes} and a \textit{relevant subset of attributes}. For example, in social networks, it is very unlikely that people are similar within all of their characteristics [18]. In health surveillance networks, it is very rare that outbreaks of different disease types have identical symptoms [28]. In order to detect subspace clusters, it is required to conduct feature selection and cluster detection concurrently, as without knowing the true clusters of nodes, it is difficult to identify their relevant attributes, and vice versa.

In recent years, a limited number of methods have been proposed to detect \textit{subspace clusters}, which fall into two main categories, including detection of coherent dense subspace clusters and detection of anomalous connected subspace clusters. The methods for detecting \textit{coherent dense subspace clusters} search for subsets of nodes that show high similarity in subsets of their attributes and that are as well densely connected within the input network. Customized algorithms are developed for specific combinations of similarity functions of attributes (e.g., threshold based [18, 19] and pairwise distance based [34] functions) and density functions of nodes [17–19, 27, 34]. The methods for detecting \textit{anomalous connected subspace clusters} search for subsets of nodes that are significantly different from the other nodes on subsets of their attributes and that are as well connected (but not necessary dense) within the input network. The connectivity constraint ensures that the clusters of nodes reflect changes due to localized in-network processes. All the existing methods in this category consider a small set of neighborhoods (e.g., social circles and ego networks [33], subgraphs isomorphic to a query graph [20], and small-diameter subgraphs [28]), and identify anomalous subspace clusters \textit{among only these given neighborhoods}.

However, the aforementioned methods have two main limitations: 1) \textbf{Lack of generality}. All these methods are customized for specific score functions of attributes and topological constraints on clusters, and may be inapplicable if the functions or constraints are changed. As discussed in recent surveys [2], the definition of an interesting subgraph pattern, in which subspace clusters is a specific type, is meaningful only under a given context or application. There is a strong need of generic methods that can handle a broad class of score functions, such as parametric/nonparametric scan statistics functions [10], discriminative functions [37], and least square functions [12]; and topological constraints, such as the types of subgraphs aforementioned [18, 20, 28, 33, 34], compact subgraphs [39]; several methods for detecting coherent dense subspace clusters are tractable to large networks, but do not provide worst-case theoretical guarantees on the quality of the detected clusters.

This paper presents a novel generic and theoretical framework to address the above two main limitations of existing methods for a broad class of interesting subspace cluster detection problems. In particular, we consider the general form of subspace cluster detection as an optimization problem that has a general score function measuring the interestingness of a subset of features and a cluster of nodes, a sparsity constraint on the subset of features, and topological constraints on the cluster of nodes. We propose a novel subspace graph-structured matching pursuit algorithm, namely, \textit{SG-Pursuit}, to approximately solve this general problem in nearly-linear time. The key idea is to iteratively search for a close-to-optimal solution by solving easier subproblems in each iteration, including i) identification of topological-free clusters of nodes and a sparsity-free subset of attributes that maximizes the score function in a sub-solution-space determined by the gradient of the current solution; and ii) projection of the identified intermediate solution onto the solution-space defined by the sparsity and topological constraints. The contributions of this work are summarized as follows:

- \textbf{Design of a generic and efficient approximation algorithm for the subspace cluster detection problem}. We propose a novel generic algorithm, namely, \textit{SG-Pursuit}, to approximately solve a broad class of subspace cluster detection problems that are defined by different score functions and topological constraints in nearly-linear time. To the best of our knowledge, this is the first-known generic algorithm for such problems.

- \textbf{Theoretical guarantees and connections}. We present a theoretical analysis of the proposed \textit{SG-Pursuit} and show that
SG-Pursuit enjoys a geometric rate of convergence and a tight error bound on the quality of the detected subspace clusters. We further demonstrate that SG-Pursuit enjoys strong guarantees analogous to state-of-the-art methods for sparse feature selection in high-dimensional data and for subgraph detection in attributed networks.

- **Compressive experiments to validate the effectiveness and efficiency of the proposed techniques.** SG-Pursuit was specialized to conduct the specific tasks of coherent dense subspace cluster detection and anomalous connected subspace cluster detection on several real-world data sets. The results demonstrate that SG-Pursuit outperforms state-of-the-art methods that are designed specifically for these tasks, even that SG-Pursuit is designed to address general subspace cluster detection problems.

**Reproducibility:** The implementations of SG-Pursuit and baseline methods and the data sets are available via the link [15].

The rest of this paper is organized as follows. Section 2 introduces the proposed method SG-Pursuit and analyzes its theoretical properties. Section 3 discusses applications of our proposed algorithm for the tasks of coherent dense subspace cluster detection and anomalous connected subspace cluster detection. Experiments on several real-world benchmark datasets are presented in Section 4. Section 5 concludes the paper and describes future work.

## 2 METHOD SG-PURSUIT

In this section we first introduce the notation and define the problem of subspace cluster detection formally. Next, we present the algorithm SG-Pursuit and analyze its theoretical properties, including its convergence rate, error-bound, and time complexity.

### 2.1 Problem Formulation

We consider a multi-attributed network that is defined as $G = (V, E, w)$, where $V = \{1, \ldots, n\}$ is the ground set of nodes of size $n$, $E \subseteq V \times V$ is the set of edges, and the function $w: V \to \mathbb{R}^p$ defines a vector of attributes of size $p$ for each node $v \in V$; $w(v) \in \mathbb{R}^p$. For simplicity, we denote the attribute vector $w(v)$ by $w_0$.

We introduce two vectors of coefficients, including $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$, that will be optimized for detecting the most interesting subspace cluster in $G$, where $x$ identifies the cluster (subset) of nodes and $y$ identifies their relevant attributes. In particular, the vector $x$ refers to the vector of coefficients of the nodes in $V$. Each node $i \in V$ has a coefficient score $x_i$ indicating the importance of this node in the cluster of interest. If $x_i \neq 0$, it means that the node $i$ belongs to the cluster of interest. Similarly, the vector $y$ refers to the vector of coefficients of the $p$ attributes. Each attribute $f \in \{1, \ldots, p\}$ has a coefficient score $y_f$ indicating the relevance of this attribute to the clusters of interest. Let $\text{supp}(x)$ be the support set of indices of nonzero entries in $x$: $\text{supp}(x) = \{i \mid x_i \neq 0\}$. Then the support set $\text{supp}(x)$ represents the subset of nodes that belong to the cluster of interest. The support set $\text{supp}(y)$ represents the subset of relevant attributes. We define the feasible space of clusters of nodes as $\mathcal{M}(k) = \{S \mid S \subseteq V; |S| \leq k; G_S \text{ satisfies predefined topological constraints.}\}$.

Figure 3: An example function of $f(x, y)$ (negative squared error function (2)) for robust linear regression models that has been widely used in anomaly detection tasks [12, 38, 40, 42]. In this example, the vector $x$ is a vector of sparse coefficients of the nodes in the input network that must satisfy the topological constraints ($\text{supp}(k) = \{i\}$): the size of $\text{supp}(x)$ is at most 6. The residual vector $y$ is a sparse vector as defined by the constraint $\|y\|_0 \leq s$ and is used to identify anomalous attributes.

where $S$ refers to a subset of nodes in $V$, $G_S = (S, E \cap S \times S)$ refers to the subgraph induced by $S$, $|S|$ refers to the total number of nodes in $S$, and $k$ refers to an upper bound on the size of the cluster. The topological constraints can be any topological constraints on $G_S$, such as connected subgraphs [28, 33], dense subgraphs [18, 34], subgraphs that are isomorphic to a query graph [20], compact subgraphs [39], trees [25], and paths [6], among others.

Based on the above notations, we consider a general form of the subspace cluster detection problem as

$$\max_{x \in C_x, y \in C_y} f(x, y) \quad \text{s.t.} \quad \text{supp}(x) \in \mathcal{M}(k) \text{ and } \|y\|_0 \leq s,$$

where $f(x, y): \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ is a score function that measures the overall level of interestingness of the subspace clusters indicated by $x$ and $y$; $C_x \subseteq \mathbb{R}^n$ represents a convex set in the Euclidean space $\mathbb{R}^n$, $C_y \subseteq \mathbb{R}^p$ represents a convex set in the Euclidean space $\mathbb{R}^p$, $\mathcal{M}(k)$ refers to the feasible space of clusters of nodes as defined above, and $s$ refers to an upper bound on the number of attributes relevant to the subspace clusters of interest. The parameters $k$ and $s$ are predefined by the user. Let $\hat{x}$ and $\hat{y}$ be the solution to Problem (1). Denote by $\mathcal{S}$ the support set supp($\hat{x}$) that represents the most interesting cluster of nodes, and by $\mathcal{R}$ the support set supp($\hat{y}$) represents the subset of relevant attributes. The most interesting subspace cluster can then be identified as $(\mathcal{S}, \mathcal{R})$.

As illustrated in Figure 3, an example score function $f(x, y)$ is a negative squared error function for robust linear regression that has been widely used in anomaly detection tasks [12, 38, 40, 42]:

$$f(x, y) = -\|c - WT x - y\|^2_2,$$

where $x \in C_x := \mathbb{R}^n$, $y \in C_y := \mathbb{R}^p$, $c \in \mathbb{R}^p$ refers to a vector of observed response values, and $W = [w_1, w_2, \ldots, w_n]^T \in \mathbb{R}^{n \times p}$. The residual vector $y$ is used to identify anomalous attributes, and its sparsity $s$ is usually much smaller than $p$ (the total number of
attributes). There are also applications where both \(x\) and \(y\) need to be vectors of positive coefficients [42]: \(C_x := \mathbb{R}^{+n}\) and \(C_y := \mathbb{R}^{+p}\).

Remark 1. There are scenarios where \(x\) is considered as a vector of binary values, instead of numerical coefficients, and the resulting problem becomes a discrete optimization problem that is NP-hard in general and does not have known solutions. In this case, by relaxing the input domain of \(x\) from \([0, 1]^n\) to the convex set \(C_x := [0, 1]^n\) and replacing the score function \(f(x, y)\) with its tight concave surrogate function, the resulting relaxed problem becomes a special case of Problem (1). In particular, when the cost function is a supermodular function of \(x\), a tight concave surrogate function can be obtained based on Lovasz extensions, such that the solutions to the relaxed problem are identical to the solutions to the original discrete optimization problem. In addition, the same equivalence also holds for a number of popular non-convex functions that are non-supermodular, such as Hinge and Squared Hinge functions, and their tight concave surrogate functions have been studied in recent work [9, 40].

Remark 2. Problem (1) considers the detection of the most interesting subspace cluster in a multi-attribute network. There are applications, where top \(k\) most interesting subspace clusters are of interest, where \(k\) is predefined by the user. In this case, the \(k\) clusters can be identified one-by-one, repeatedly, by solving Problem (1) for each subspace cluster and deflating the attribute data to remove information captured by previously extracted subspace clusters.

2.2 Head and Tail Projections on \(\mathbb{M}(k)\)

Before we present our proposed algorithm SG-Pursuit, we first introduce two major components related to the support of the topological constraints "\(\text{supp}(x) \in \mathbb{M}(k)\)" including head and tail projections. The key idea is that, suppose we are able to find a good intermediate solution \(\hat{x}\) that does not satisfy this constraint, these two types of projections will be used to find good approximations of \(\hat{x}\) in the feasible space defined by \(\mathbb{M}(k)\).

- **Tail Projection** \((T(x))\) [22]: Find a \(S \subseteq \mathbb{V}\) such that
  \[
  \|x^* - x_S\|_2 \leq c_T \cdot \min_{S' \in \mathbb{M}(k)} \|x^* - x_{S'}\|_2,
  \]
  where \(c_T \geq 1\), and \(x_S\) is the restriction of \(x\) to indices in \(S\): we have \((x_S)_i = x_i\) for \(i \in S\) and \((x_S)_i = 0\) otherwise. When \(c_T = 1\), \(T(x)\) returns an optimal solution to the problem: \(\min_{S' \in \mathbb{M}(k)} \|x^* - x_{S'}\|_2\). When \(c_T < 1\), \(T(x)\) returns an approximate solution to this problem with the approximation factor \(c_T\).

- **Head Projection** \((H(x))\) [22]: Find a \(S \subseteq \mathbb{V}\) such that
  \[
  \|x_S\|_2 \geq c_H \cdot \max_{S' \in \mathbb{M}(k)} \|x_{S'}\|_2,
  \]
  where \(c_H \in [0, 1]\). When \(c_H = 1\), \(H(x)\) returns an optimal solution to the problem: \(\max_{S' \in \mathbb{M}(k)} \|x_{S'}\|_2\). When \(c_H < 1\), \(H(x)\) returns an approximate solution to this problem with the approximation factor \(c_H\).

It can be readily proved that, when \(c_T = 1\) and \(c_H = 1\), both \(T(x)\) and \(H(x)\) return the same subset \(S\), and the corresponding vector \(x_S\) is an optimal solution to the standard projection oracle in the traditional projected gradient descent algorithm [7]:

\[
\arg\min_{x' \in \mathbb{R}^n} \|x^* - x'\|_2 \text{ s.t. } \text{supp}(x') \in \mathbb{M}(k),
\]

which is NP-hard in general for popular topological constraints, such as connected subgraphs and dense subgraphs [35]. However, when \(c_T > 1\) and \(c_H < 1\), \(T(x)\) and \(H(x)\) return different approximate solutions to the standard projection problem (5). Although the head and tail projections are NP-hard problems when \(c_T = 1\) and \(c_H = 1\), these two projections can often be implemented in nearly-linear time when we allow relaxations on \(c_T\) and \(c_H\): \(c_T > 1\) and \(c_H < 1\). For example, when the topological constraints considered in \(\mathbb{M}(k)\) is that: "\(\mathcal{G}_S\) is a connected subgraph", where \(S\) is a specific cluster of nodes, the resulting head and tail projections can be implemented in nearly-linear time with the parameters: \(c_T = \sqrt{7}\) and \(c_H = \sqrt{1/4}[22]\). The impact of these two parameters on the performance of SG-Pursuit will be discussed in Section 2.4.

As discussed above, the head and tail projections can be considered as two different approximations to the standard projection problem (5). It has been demonstrated that the joint utilization of both head and tail projections is critical in design of approximate algorithms for network-related optimization problems [11, 21, 22, 44].

2.3 Algorithm Details

We propose a novel Subspace Graph-structured matching Pursuit algorithm, namely, SG-Pursuit, to approximately solve Problem (1) in nearly-linear time. The key idea is to iteratively search for a close-to-optimal solution by solving easier subproblems in each iteration \(i\), including i) identification of the intermediate solution \((b^i_1, b^i_2)\) that maximizes the score function \(f(x, y)\) in a solution-space determined by the partial derivatives of the function on the current solution, including \(\nabla_x f(x^i, y^i)\) and \(\nabla_y f(x^i, y^i)\), and ii) projection of the intermediate solution \((b^i_1, b^i_2)\) to the feasible space defined by the topological constraints: "\(\text{supp}(x) \in \mathbb{M}(k)\)" and the sparsity constraint: "\(\|y\|_0 \leq s\)". The projected solution \((x^{i+1}, y^{i+1})\) is then the updated intermediate solution returned by this iteration.

The main steps of SG-Pursuit are shown in Algorithm 1. The procedure generates a sequence of intermediate solutions \((x^0, y^0), (x^1, y^1), \ldots\) from an initial solution \((x^0, y^0)\). At the \(i\)-th iteration, the first step (Line 6) calculates the partial derivative \(\nabla_x f(x^i, y^i)\) and then identifies a subset of nodes via head projection that returns a support set with the head value at least a constraint factor \(c_T\) of the optimal head value: "\(T(x) = H(\nabla_x f(x^i, y^i))\)". The support set \(\Gamma_y\) can be interpreted as the directions where the nonconvex set "\(\text{supp}(x) \in \mathbb{M}(k)\)" is located, within which pursuing the maximization over \(y\) will be most effective. The second step (Line 7) identifies the 2s nodes of the partial derivative vector \(\nabla_y f(x^i, y^i)\) that have the largest magnitude that are chosen as the directions in which pursuing the maximization on \(y\) will be most effective:

\[
\Gamma_y = \arg\max_{R \subseteq \{1, \ldots, p\}} \{||\nabla_y f(x^i, y^i)_R||_2 : ||R||_0 \leq 2s\},
\]

where \(\nabla_y f(x^i, y^i)_R\) refers to the projected vector in the subspace defined by the subset \(R\). Denote by \(w\) the projected vector \(\{\nabla_y f(x^i, y^i)_R\}_R\). We then have \(w_l = [\nabla_y f(x^i, y^i)_l]_l\), the \(i\)-th entry in the gradient vector \(\nabla_y f(x^i, y^i)\), if \(i \in R\); otherwise, \(w_l = 0\). The subsets \(\Gamma_x\) and \(\Gamma_y\) are then merged in Line 8 and Line 9 with the supports of the current estimates "\(\text{supp}(x^i)\)" and "\(\text{supp}(y^i)\)", respectively, to obtain "\(\Omega_x = \Gamma_x \cup \text{supp}(x^i)\)" and "\(\Omega_y = \Gamma_y \cup \text{supp}(y^i)\)". The combined support sets define a subspace of \(x\) and \(y\) over which
the function $f(x, y)$ is maximized to produce an intermediate solution in Line 10:

$$(b_{x}^{i}, b_{y}^{i}) = \arg\max_{x \in C_{x}, y \in C_{y}} f(x, y) \text{ s.t. supp}(x) \subseteq \Omega_{x}, \text{supp}(y) \subseteq \Omega_{y}. $$

Then a subset of nodes are identified via tail projection of $b_{x}^{i}$ in Line 11: $\Psi_{x}^{i+1} = T(b_{x}^{i})$, that returns a support set with the tail value at most a constant $c_{T}$ times larger than the optimal tail value. A subset of attributes of size $s$ that have the largest magnitude are chosen in Line 12 as the subset of relevant attributes:

$$\Psi_{y}^{i+1} = \arg\max_{R \subseteq \{1, \ldots, p\}} \{||b_{y}^{i}\|_{2} : ||R||_{0} \leq s\}. $$

As the final steps of this iteration (Line 13 and Line 14), the estimates $x^{i+1}$ and $y^{i+1}$ are updated as the restrictions of $b_{x}^{i}$ and $b_{y}^{i}$ on the support sets $\Psi_{x}^{i+1}$ and $\Psi_{y}^{i+1}$, respectively: "$x^{i+1} = [b_{x}^{i}]_{\Psi_{x}^{i+1}}$" and "$y^{i+1} = [b_{y}^{i}]_{\Psi_{y}^{i+1}}$". These steps are conducted to ensure that the estimates $x^{i+1}$ and $y^{i+1}$ returned by each iteration always satisfy the sparsity and topological constraints, respectively. After the termination of the iterations, Line 17 identifies the subspace cluster: $C = (\Psi_{x}^{T}, \Psi_{y}^{T})$, where $\Psi_{x}^{T}$ represents the subset (cluster) of nodes and $\Psi_{y}^{T}$ represents the subset of relevant attributes.

**Algorithm 1 SG-Pursuit**

1. **Input**: Network instance $\mathcal{G}$ and the parameters, including $k$ (maximum number of nodes in the subspace cluster) and $s$ (maximum size of selected features).
2. **Output**: The vectors of coefficients of nodes and attributes, including $x^{i}$ and $y^{i}$, and the identified subspace cluster $C$.
3. $\epsilon = 0.0001 \%$ The termination criterium of the iterations
4. $i = 0$ $x^{i}, y^{i} = \text{initial vectors}$
5. **repeat**
6. $\Gamma_{x} = H(\nabla_{x} f(x^{i}, y^{i}))$
7. $\Gamma_{y} = \arg\max_{R \subseteq \{1, \ldots, p\}} \{||\nabla_{y} f(x^{i}, y^{i})||_{2} : ||R||_{0} \leq 2s\}$
8. $\Omega_{x} = \Gamma_{x} \cup \text{supp}(x^{i})$
9. $\Omega_{y} = \Gamma_{y} \cup \text{supp}(y^{i})$
10. $(b_{x}^{i}, b_{y}^{i}) = \arg\max_{x \in C_{x}, y \in C_{y}} f(x, y) \text{ s.t. supp}(x) \subseteq \Omega_{x}, \text{supp}(y) \subseteq \Omega_{y}$
11. $\Psi_{x}^{i+1} = T(b_{x}^{i})$
12. $\Psi_{y}^{i+1} = \arg\max_{R \subseteq \{1, \ldots, p\}} \{||b_{y}^{i}\|_{2} : ||R||_{0} \leq s\}$
13. $x^{i+1} = [b_{x}^{i}]_{\Psi_{x}^{i+1}}$
14. $y^{i+1} = [b_{y}^{i}]_{\Psi_{y}^{i+1}}$
15. $i = i + 1$
16. **until** $||x^{i} - x^{i-1}|| \leq \epsilon$ and $||y^{i} - y^{i-1}|| \leq \epsilon$
17. $C = (\Psi_{x}^{T}, \Psi_{y}^{T})$.
18. **return** $x^{i}, y^{i}, C$.

### 2.4 Theoretical Analysis

In order to demonstrate the accuracy and efficiency of SG-Pursuit, we require that the score function $f(x, y)$ satisfies the Restricted Strong Concavity/Smoothness (RSC/RSS) condition as follows:

**Definition 2.1 (Restricted Strong Concavity/Smoothness (RSC/RSS)).** A score function $f$ satisfies the $(\mathcal{M}(k), s, \gamma^{-}, \gamma^{+})$-RSC/RSS if, for every $x, x' \in \mathbb{R}^{n}$ and $y, y' \in \mathbb{R}^{p}$ with $\text{supp}(x) \subseteq \mathcal{M}(2k)$, $\text{supp}(x') \subseteq \mathcal{M}(2k)$, $|\text{supp}(y)| \leq 2s$, and $|\text{supp}(y')| \leq 2s$, the following inequalities hold:

$$\frac{\gamma^{-}}{2} \left(\|x - x'\|_{2}^{2} + \|y - y'\|_{2}^{2}\right) \leq f(x, y) - f(x', y') - \nabla_{x} f(x, y)^{T}(x - x') - \nabla_{y} f(x, y)^{T}(y - y') \leq \frac{\gamma^{+}}{2} \left(\|x - x'\|_{2}^{2} + \|y - y'\|_{2}^{2}\right). \quad (6)$$

The RSC/RSS condition basically characterizes cost functions that have quadratic bounds on the derivative of the objective function when restricted to the graph-structured vector $x$ and the sparsity-constrained vector $y$. When the score function $f$ is a quadratic function of $x$ and $y$, RSC/RSC condition degenerates to the restricted isometry property (RIP) that is well-known in the field of compressive sensing. For example, we consider the negative squared error function (2) as discussed in Section 2.1: $f(x, y) = -\|W x - y\|_{2}^{2}$. Let $W = [W_{1}, \ldots, W_{l}]$, where $l$ is a $p$ by $p$ identity matrix. Let $z = [x^{T}, y^{T}]^{T}$. The RSC/RSC condition can then be reformulated as the RIP condition:

$$(1 - \delta)\|z\|_{2}^{2} \leq \|W z\|_{2}^{2} \leq (1 + \delta)\|z\|_{2}^{2},$$

where $\gamma^{+} = 2(1 + \delta)$, $\gamma^{-} = 2(1 - \delta)$, and $\delta \in [0, 1]$ is the standard parameter as defined in RIP. However, the RIP condition in this example is different from the traditional RIP condition in that the components of $z$, including $x$ and $y$, must satisfy the constraints related to $\mathcal{M}(k)$ and the sparsity $s$ as described in Definition 2.1.

**Theorem 2.2.** If the score function $f$ satisfies the property $(\mathcal{M}(k), s, \gamma^{-}, \gamma^{+})$-RSC/RSS, then for any true $(x^{*}, y^{*}) \in \mathbb{R}^{n} \times \mathbb{R}^{p}$, the iterations of the proposed algorithm SG-Pursuit satisfy the inequality

$$||r_{x}^{i+1}||_{2}^{2} + ||r_{y}^{i+1}||_{2}^{2} \leq \alpha \left(||r_{x}^{i}||_{2}^{2} + ||r_{y}^{i}||_{2}^{2}\right) + \beta \epsilon_{x} + \epsilon_{y}$$

where $r_{x}^{i+1} = x^{i+1} - x^{i}$, $r_{y}^{i+1} = y^{i+1} - y^{i}$, $\alpha_{0} = c_{T}(1 - \rho)$, $\rho = \sqrt{\frac{1 - \left(\frac{1}{\beta_{0}}\right)}{1 - \frac{\gamma_{2}}{\gamma_{1}}}}$, $\beta = \frac{c_{T}(1 - \rho)}{1 - \frac{\gamma_{2}}{\gamma_{1}}} = \frac{\sqrt{\gamma_{2} - 2\gamma_{1}}}{\gamma_{2}}$, $\epsilon_{x} = \max_{k \in \mathcal{M}(2k)} ||\nabla f_{x}(x^{*}, y^{*})||_{2}$, and $\epsilon_{y} = \max_{k \in \mathcal{M}(2k)} ||\nabla f_{y}(x^{*}, y^{*})||_{2}$.

**Proof.** See the Appendix A for details.

**Theorem 2.3.** Let $(x^{*}, y^{*})$ the optimal solution to Problem (1) and $f$ be a score function that satisfies the $(\mathcal{M}(k), s, \gamma^{-}, \gamma^{+})$-RSC/RSS property. Let $T$ and $H$ be the tail and head projections with $c_{T}$ and $c_{H}$ such that $0 < a < 1$. Then after $t = \left\lfloor \frac{\log \left(\frac{||x^{*}||_{2}^{2} + ||y^{*}||_{2}^{2}}{\epsilon_{x} + \epsilon_{y}}\right)}{\log\frac{1}{a}} \right\rfloor$ iterations, SG-Pursuit returns a single estimate $(\hat{x}, \hat{y})$ satisfying

$$||\hat{x} - x^{*}||_{2} + ||\hat{y} - y^{*}||_{2} \leq c(\epsilon_{x} + \epsilon_{y}),$$

where $c = (1 + \frac{\beta}{1 - \gamma_{2}^{2}})$ is a fixed constant. Moreover, SG-Pursuit runs in time

$$O \left( (T_{1} + T_{2} + p \log p) \log \left(\frac{||x^{*}||_{2}^{2} + ||y^{*}||_{2}^{2}}{(\epsilon_{x} + \epsilon_{y})} \right) \right),$$

where $T_{1}$ is the time complexity of one execution of the subproblem in Line 10 in SG-Pursuit and $T_{2}$ is the time complexity of one execution of the head and tail projections.

In particular, when the connectivity constraint or a density constraint is considered as the topological constraint on the feasible clusters of nodes in $\mathcal{M}(k)$, there exist efficient algorithms for the head
and tail projections that have the time complexity $O(\|\mathbf{x}\|^3 n)$ [22].

When $s$ and $k$ are fixed small constants with respect to $n$, the subproblem in Line 10 in $\text{SG-Pursuit}$ can be solved in nearly linear time in practice using convex optimization algorithms, such as the project gradient descent algorithm. Therefore, under these conditions, for coherent dense subgraph detection and connected anomalous subspace cluster detection problems, $\text{SG-Pursuit}$ has a nearly-linear time complexity on the network size $n$ and the cardinality of attributes $p$:

$$O\left( (\|\mathbf{x}\|^3 n + p \log p) \log \left( (\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)/\epsilon\right) \right). \tag{9}$$

Proof. From Theorem 2.2, the following inequality can be obtained via an inductive argument:

$$\|\mathbf{x}^i - \mathbf{x}^j\|^2 + \|\mathbf{y}^j - \mathbf{y}^i\|^2 \leq \alpha^i \left( \|\mathbf{x}^i\|^2 + \|\mathbf{y}^j\|^2 \right) + \beta (\epsilon_x + \epsilon_y) \sum_{j=0}^{i} \alpha^j.$$

For $i = \log \left( \frac{\|\mathbf{x}^i\|^2 + \|\mathbf{y}^i\|^2}{\epsilon_x + \epsilon_y} \right) / \log \frac{1}{\alpha}$, we have $\alpha^i \left( \|\mathbf{x}^i\|^2 + \|\mathbf{y}^i\|^2 \right) \leq (\epsilon_x + \epsilon_y)$.

The geometric series $\sum_{j=0}^{i} \alpha^j$ can be bounded by $\frac{1}{1-\alpha}$. The error bound (7) can be obtained by coming the preceding inequalities.

The time complexity of the subproblem in Line 10 is denoted by $O(T_1)$, and the time complexities of both head and tail projections are denoted by $O(T_2)$. The total time complexity to solve the subproblem in Line 7 is $O(p \log p)$, as the exact solution can be obtained by sorting the entries in $\nabla_y f(x^j, y^j)$ in a descending order based their absolute values, and then returning the indices of the top 2 $s$ entries. Similarly, the time complexity to solve the subproblem in Line 12 is $O(p \log p)$. As the total number of iterations is $O(p \log p)$, the time complexity specified in Equation (8) can be calculated. Accordingly, when $T_1$ is bounded by $O(n \log n)$ and $T_2 = |\mathbf{E}| \log^3 n$, the nearly-linear time complexity specified in Equation (9) can be obtained.

Theorem 2.3 shows that $\text{SG-Pursuit}$ enjoys a geometric rate of convergence and the estimation error is determined by the multiplier of $(\epsilon_x + \epsilon_y)$, where $\epsilon_x = \max_{x \in \mathcal{E} \cup \{0\}} \|\nabla f(x, y^*; x)\|_2$ and $\epsilon_y = \max_{y \in \mathcal{Y}} \|\nabla f(x^*, y)\|_2$. The shrinkage rate $\alpha$ controls the convergence rate of $\text{SG-Pursuit}$. In particular, if the true $x^*$ and $y^*$ are sufficiently close to an unconstrained maximum of $f$, then the estimation error is negligible because both $\epsilon_x$ and $\epsilon_y$ have small magnitudes. Especially, in the idea case where $\epsilon_x = \epsilon_y = 0$, it is guaranteed that we can obtain the true $x$ and $y$ to arbitrary precision. Note that we can make $\gamma^+$ and $\gamma^-$ as close as we desire, such that $\gamma^+ / \gamma^- = 1$, since this assumption only affects the measurement bound by a constant factor. In this case, in order to ensure that $\gamma = (c_T + 1)\sqrt{2 - 2c_T H} < 1$, the factors $c_T$ and $c_H$ should satisfy the inequality: $c_T^2 + 1 - 1/(2c_T)$. As proved in [21], the fixed factor $c_H$ of any given algorithm for the head projection can be boosted to any arbitrary constant $c_H < 1$, such that the above condition can be satisfied. This indicates the flexibility of designing approximate algorithms for head and tail projections in order to ensure the geometric convergence rate of $\text{SG-Pursuit}$.

Remark 3. ($\text{Connections to existing methods}$) $\text{SG-Pursuit}$ is a generalization of the GraSP (Gradient Support Pursuit) method [8], a state-of-the-art method for general sparsity-constrained optimization problems, and the Graph-MP method [11], a state-of-the-art method for general graph-structured sparse optimization problems. In particular, when we fix $x$ and only update $y$ in the steps of $\text{SG-Pursuit}$, $\text{SG-Pursuit}$ then degenerates to GraSP. When we fix $x$ and only update $y$ in the steps of $\text{SG-Pursuit}$, $\text{SG-Pursuit}$ then degenerates to Graph-MP. Surprisingly, even that $\text{SG-Pursuit}$ concurrently optimizes $x$ and $y$, its convergence rate is of the same order as those of Graph-MP and GraSP under the RSS/RSC property.

3 EXAMPLE APPLICATIONS

In this section, we specialize $\text{SG-Pursuit}$ to address two typical subspace cluster detection problems in multi-attributed networks, including coherent dense subspace cluster detection and anomalous connected subspace cluster detection. The former searches for subsets of nodes that show high similarity in subsets of their attributes and that are as well densely connected within the input network. The coherence score function, as shown in Table 2, is defined as the log likelihood ratio function, $\log \frac{\text{Prob}([\mathbf{H}])}{\text{Prob}([\mathbf{H}])}$, that corresponds to the hypothesis testing framework:

- Under the null $(\mathbf{H}_0)$, $w_{i,j} \sim N(0, 1), \forall i \in \mathcal{V}, j \in \{1, \cdots, p\}$, where $w_{i,j}$ refers to the observed value of the $j$-th attribute of node $i$;
- Under the alternative $H_1(x, y)$, $w_{i,j} \sim N(\mu_j, 1)$, if $x_i = 1$ and $y_j = 1$; otherwise, $w_{i,j} \sim N(0, \sigma)$, where $x \in \{0, 1\}^p$, $y \in \{0, 1\}^p$, and $x_i = 1$ indicates that node $i$ belongs to the cluster, $y_j = 1$ indicates that the attribute $j$ belongs to the subset of coherent attributes. Each coherent attribute $j$ has a different mean parameter $\mu_j$ and the variance $\sigma$ should be less than 1, the variance of an incoherent attribute, in order to ensure the coherence of its observations. $\sigma$ is set 0.01 by default.

The latter (anomalous connected subspace cluster detection) searches for subsets of nodes that are significantly different from the other nodes on subsets of their attributes and that are as well connected within the input network. The elevated mean scan statistic, as shown in Table 2, is defined as the log likelihood ratio function that corresponds to a hypothesis testing framework: the same as the above, except that 1) "coherent" is replaced by "anomalous", 2) the mean of each anomalous attribute is greater than (or more anomalous than) 0, the mean of a normal attribute, and the standard deviation of each anomalous attribute is set to 1 (the same variance of a normal attribute). The Fisher test statistic function is considered when each $w_{i,x_{i,j}}$ represents the level of anomalous (e.g., negative log p-value) of the $j$-th attribute of node $i$, and $x_i W y$ represents the overall level of anomalous of $x$ and $y$. A large class of scan statistic functions for anomaly detection can be transformed to the Fisher test statistic function using a 2-step procedure as proposed in [35]. The negative squared error is considered as the score function of anomalous subspace cluster detection in a regression setting and is introduced in Section 2.1.
This section thoroughly evaluates the performance of our proposed methods based on different combinations of the following parameters: 1) the number of incoherent clusters, 2) the number of coherent attributes, 3) the total number of attributes, and 4) cluster size. We set these parameters to 9, 10, 100, 30, respectively, by default. Note that we set the size of all coherent and incoherent clusters to 30, as GARMeR is not scalable to detection of clusters of size larger than 30. We generated one coherent dense cluster and multiple incoherent dense clusters in each synthetic graph.

2. Real-world data. We used five public benchmark real-world attributed network datasets, including DBLP, Arxiv, Genes, IMDB, and DFB (German soccer premier league data), which are available from and described in details in [1]. The basic statistics of these five datasets are provided in Table 3, with the numbers of nodes ranging from 100 to 11,989; the numbers of edges ranging from 1,106 to 119,258; and the number of attributes ranging from 5 to 300.

3. Implementation and parameter tuning: The implementations of FocusCO and GAMeR are publicly released by authors 3. FocusCO requires an exemplar set of nodes and has a trade-off parameter γ that is used in learning of feature weights. We have tried the representative values of γ: {0.0, 0.0001, 0.001, 0.01, 0.1, 0.5, 1.0, 2.0, 3.0, 4.0, 5.0, 10.0, 15.0, 20.0, 25.0, 30.0, 50.0, 100.0} on the graphs, and identified the best value γ = 1 (with the largest overall F-measure), which is also the default value used in FocusCO. In order to make FocusCO the best competitive to our method, we used a random set of 90% nodes in each coherent dense subspace cluster as the input exemplar set of nodes. FocusCO estimates a weight for each attribute that characterizes the importance of this attribute, and returns the top s attributes with the largest weights as the set of coherent attributes, and set s to the true number of coherent attributes. GAMeR has four main parameters, including sGMP (the minimum number of coherent attributes), γsGMP (the minimum threshold on density), and min (the minimum cluster size), and w (the maximum width that control the level of coherence). We followed the recommended strategies by the authors and identified the best parameter values for FocusCO and GAMeR. In particular, these parameters for synthetic data sets were set as follows:

- nout (minimum cluster size): {0.5s, s}, where s is the size of the true coherent cluster.
- smin (the minimum number of coherent attributes): {0.5ks, k}, where k is the number of the true coherent attributes.
- γ (the minimum threshold on density): 0.35 (the density of the true coherent cluster).
- w (the maximum width that controls the level of coherence): 0.1, which is around 3 times σ, where σ = √0.001 is the standard deviation of coherent attributes.

For the five real-world data sets, as the ground truth labels are unavailable, we followed the recommended strategies by the authors and identified the best parameter values for GAMeR [17, 18]. We tried different combinations of the four major parameters and returned the best results: nout = {2, 3, 4, 5, 10, 15, 20}, s = {0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9}, and w = {0.01, 0.1, 0.5, 10, 0.05, 0.1, 0.05, 0.5, 10, 0.01}.

We did not consider other related papers that focus on different objectives rather than only cluster density (e.g., the normalized subspace graph cut objective as considered in [19]) and also their implementations are not publicly available.

### 4. EXPERIMENTS

This section thoroughly evaluates the performance of our proposed method on the quality of the detected subspace clusters and runtime on synthetic and real-world networks. The experimental code and data sets are available from the Link [15] for reproducibility.

#### 4.1 Coherent dense subgraph detection

4.1.1 Experimental design. We compared SG-Pursuit with two representative methods, including GARMeR [18] and FocusCO [34].

1. Generation of synthetic graphs: We used the same generator of synthetic coherent and dense subgraphs as used in the state-of-the-art FocusCO method [34], except that the standard deviation (std) of coherent attributes was set to 0.001, instead of 0.001, which makes the detection problem more challenging. The settings of the other parameters used in FocusCO include: pIN = 0.35 (density of edges in each cluster) and pOUT = 0.1 (density of edges between clusters). We will compare the performance of different methods based on different combinations of the following parameters: 1) the number of incoherent clusters, 2) the number of coherent attributes, 3) the total number of attributes, and 4) cluster size. We set these parameters to 9, 10, 100, 30, respectively, by default. Note that we set the size of all coherent and incoherent clusters to 30, as GARMeR is not scalable to detection of clusters of size larger than 30. We generated one coherent dense cluster and multiple incoherent dense clusters in each synthetic graph.

2. Real-world data. We used five public benchmark real-world attributed network datasets, including DBLP, Arxiv, Genes, IMDB, and DFB (German soccer premier league data), which are available from and described in details in [1]. The basic statistics of these five datasets are provided in Table 3, with the numbers of nodes ranging from 100 to 11,989; the numbers of edges ranging from 1,106 to 119,258; and the number of attributes ranging from 5 to 300.

3. Implementation and parameter tuning: The implementations of FocusCO and GARMeR are publicly released by authors 3. FocusCO requires an exemplar set of nodes and has a trade-off parameter γ that is used in learning of feature weights. We have tried the representative values of γ: {0.0, 0.0001, 0.001, 0.01, 0.1, 0.5, 1.0, 2.0, 3.0, 4.0, 5.0, 10.0, 15.0, 20.0, 25.0, 30.0, 50.0, 100.0} on the graphs, and identified the best value γ = 1 (with the largest overall F-measure), which is also the default value used in FocusCO. In order to make FocusCO the best competitive to our method, we used a random set of 90% nodes in each coherent dense subspace cluster as the input exemplar set of nodes. FocusCO estimates a weight for each attribute that characterizes the importance of this attribute, and returns the top s attributes with the largest weights as the set of coherent attributes, and set s to the true number of coherent attributes. GARMeR has four main parameters, including sGMP (the minimum number of coherent attributes), γsGMP (the minimum threshold on density), and min (the minimum cluster size), and w (the maximum width that controls the level of coherence). We followed the recommended strategies by the authors and identified the best parameter values for FocusCO and GARMeR. In particular, these parameters for synthetic data sets were set as follows:

- nout (minimum cluster size): {0.5s, s}, where s is the size of the true coherent cluster.
- s (the minimum number of coherent attributes): {0.5ks, k}, where k is the number of the true coherent attributes.
- γ (the minimum threshold on density): 0.35 (the density of the true coherent cluster).
- w (the maximum width that controls the level of coherence): 0.1, which is around 3 times σ, where σ = √0.001 is the standard deviation of coherent attributes.

For the five real-world data sets, as the ground truth labels are unavailable, we followed the recommended strategies by the authors and identified the best parameter values for GARMeR [17, 18]. We tried different combinations of the four major parameters and returned the best results: nout = {2, 3, 4, 5, 10, 15, 20}, s = {0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9}, and w = {0.01, 0.1, 0.5, 10, 0.05, 0.1, 0.05, 0.5, 10, 0.01}.

We did not consider other related papers that focus on different objectives rather than only cluster density (e.g., the normalized subspace graph cut objective as considered in [19]) and also their implementations are not publicly available.

### 4. Settings of our proposed method SG-Pursuit

We used the following score function to detect the most coherent dense subspace cluster in each synthetic graph: f(x, y) = \( x^T \left[ W \odot (W - \frac{1}{n}I) \right] y - \frac{16}{n^2} (x^T W x - 1 \sum_i \frac{1}{n} y_i^2) - \frac{1}{2} \|y\|^2_2 + \lambda \frac{x^T \Lambda x}{1 + x^T x} \), where \( \Lambda \) is the adjacency matrix of the input graph, and \( \lambda \) is a tradeoff parameter to balance to coherence score (See Table 2) and the density score \( \frac{x^T Ax}{1 + x^T x} \). The parameter \( \lambda \) was set to 5. We applied projected gradient descent to solve the subproblem in Line 10 of SG-Pursuit. The parameters \( k \) (upper bound of the cluster size) and \( s \) (upper bound on coherent attributes) were set to the true
5. Evaluation metrics. Each synthetic graph has a single true coherent dense subspace cluster (a combination of a subset of nodes and a subset of attributes) and the task was to detect this cluster. We reported the F-measures of the subsets of nodes and attributes for each competitive method. We note that FocusCO and GAMer may return multiple candidate clusters in an input graph, and in this case we return the cluster with the highest F-measure in order to make fair comparisons. We generated 50 synthetic graphs for each setting and reported the average F-measure and running time. For the five real-world attributed network datasets, where no ground truth is given, we considered three major measures, including average cluster density, average cluster size, and average coherence distance. The average cluster density is defined as the average degree of nodes within the K subspace clusters identified, where K is predefined. The coherence distance of a specific subspace cluster is defined as the average Euclidean distance between the nodes in this cluster based on the subset of attributes selected. The average coherence distance is the average of the coherence distances of the K subspace clusters. A combination of a high average cluster density, a high average cluster size, and a low average coherence distance indicates a high overall quality of the clusters detected.

4.1.2 Quality Analysis. 1) Synthetic data with ground truth labels. The comparison on F-measures among the three competitive methods is shown in Figure 5, by varying total number of irrelevant attributes, number of incoherent clusters, and cluster size variance. The results indicate that SG-Pursuit significantly outperformed FocusCO and GAMer with more than 15 percent marginal improvements in overall on F-measures of the detected nodes and the detect coherent attributes. As shown in Figure 4(c), when...
the cluster size increases, the F-measure of FocusCO consistently increases. In particular, we observed that when the cluster size is above 150, FocusCO achieved F-measure close to 1.0. In addition, when the standard deviation of coherent attributes decreases (in the shown Figures, we fixed this to $\sqrt{0.001}$), FocusCO performed better for large cluster sizes. To summarize, SG-Pursuit was more robust to FocusCO on low levels of coherence and small cluster sizes.

2) **Real-world data.** As the real-world datasets do not have ground truth labels, we can not apply FocusCO since it requires a predefined subset of ground truth nodes. Hence, we focus on the comparison between SG-Pursuit and GAMer with different predefined numbers of clusters ($K = 5, 10, 15, 20$). As shown in Table 3, SG-Pursuit was able to identify subspace clusters with the three major measures coherently better than those of the clusters returned by GAMer in most of the settings. GAMer was able to identify clusters with densities larger than those detected by SG-Pursuit, but with much smaller cluster sizes and much large coherence distances.

### 4.1.3 Scalability analysis
The comparison on running times of competitive methods is shown in Figure 5 with respect to varying numbers of attributes and nodes. The results indicate that SG-Pursuit was faster than both FocusCO and GAMer over several orders of magnitude. The running time of FocusCO was independent on the number of attributes, but increases quadratically on the number of nodes (graph size). The running time GAMer increases quadratically on both numbers of attributes and nodes.

### 4.2 Anomalous connected cluster detection

#### 4.2.1 Experimental design
We considered two representative methods, including AMEN [33] and SODA [20].

1. **Data sets:** 1) **Chicago Crime Data.** A data set of crime data records in Chicago was collected form the official website “https://data.cityofchicago.org/” from 2010 to 2014 that has $1,515,241$ crime records in total, each of which has the location information (latitude and longitude), crime category (e.g., BATTERY, BURGLARY, THEFT), and description (e.g., “aggravated domestic battery: knife / cutting inst”). There are 35 different crime categories in total. We collected the census-tract-level graph in Chicago from the same website that has 46,357 nodes (census tracts) and 168,020 edges in total, and considered the frequency of each keyword in the descriptions of crime records as an attribute. There are 121 keywords in total that are non-stop-words and have frequencies above 10,000, which are considered as attributes. In order to generate a ground-truth anomalous connected cluster of nodes, we picked a particular crime type (BATTERY or BURGLARY), identified a connected subgraph of size 100 via random walk, and then removed the crime records of this particular category in all nodes outside this subgraph, which generated a rare category as an anomalous category. This subgraph was considered as an anomalous cluster for crime records in categories that are different from this specific category, and the keywords that are specifically relevant to this category were considered as ground-truth anomalous attributes. We tried this process 50 times to generate 50 anomalous connected clusters, and manually identified 22 keywords relevant to BATTERY and 5 keywords relevant to BURGLARY as anomalous attributes.

2) **Yelp Data.** A Yelp reviews data set was publicly released by Yelp for academic research purposes. All restaurants and reviews in the U.S. from 2014 to 2015 were considered, which includes 25,881 restaurants and 686,703 reviews. The frequencies of $1,151$ keywords in the reviews that are non-stop-words and have frequencies above 5,000 are considered as attributes. We generated a geographic network of restaurants (nodes), in which each restaurant is connected to its 10 nearest restaurants, and there are 244,012 edges in total. We used the sample strategy as in the Chicago Crime Data to generate 50 ground-truth anomalous connected clusters of size 100 for the specific category “Mexican”.

2. **Implementation and parameter tuning:** The implementation of AMEN and SODA are publicly released by the authors. Their parameters were tuned by the recommended strategies by the authors. In particular, both methods require the definition of candidate neighborhoods for scanning. A neighborhood is defined a subset that includes a focus node and the nodes who are $k$-step nearest neighbors to the focus node. If $k = 1$, a neighborhood is also called an ego network. We considered the possible values $k \in \{1, 2, 3, 4, 5\}$. Therefore, there were 50 candidate neighborhoods in total for each of the two methods with the sizes ranging around 10 to 300 nodes. For our proposed method SG-Pursuit, we considered the elevated mean statistic function as defined in Table 2. The upper bound of cluster size $k$ was set to 100. The upper bound of number of attributes $s$ was set to 22 for BATTERY related anomalous clusters and 5 for BURGLARY related anomalous clusters.

#### 4.2.2 Quality and Scalability Analysis
The detection results of the competitive methods on the Chicago Crime Data are shown in Table 4. The results indicate that SG-Pursuit outperformed SODA and AMEN on F-Measure of nodes with more than 20% marginal improvements, and on F-measure of attributes with around 15% marginal improvements. The running time of SG-Pursuit was less than those of SODA and AMEN on several orders of magnitude. The results of our method on Yelp Data contain three parts: 1) The quality of returned clusters: The F-measure of the returned clusters is 0.31 with the precision 0.314 and the recall 0.309; 2) The top 10 most frequent keyword pairs, i.e. (frequency, keyword), returned are (21, “tacos”), (21, “asada”),(20, “taco”),(19, “salsa”), (19, “level”), (15, “vegas”) , (14, “mexican”), (14, “item”) , (14, “beans”), and (13, “worth”), where the frequency of a keyword refers to the number of times that this keyword occurs in the anomalous subspace clusters detected by SG-Pursuit. 6 out of 10 keywords are related with “Mexican”, which demonstrates that our method can identify the related keywords on the specified category; 3) The running time of our algorithm was 6.98 minutes. We were not able to obtain results from AMEN and SODA after running several hours. These baseline methods cannot handle graphs that have more than 10,000 nodes and 1,000 attributes.

### 5 CONCLUSIONS
This paper presents SG-Pursuit, a novel generic algorithm to subspace cluster detection in multi-attributed networks that runs in nearly-linear time and provides rigorous guarantees, including a geometrical convergence rate and a tight error bound. Extensive

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2 Available at [http://www.yelp.com/dataset_challenge](http://www.yelp.com/dataset_challenge)

3 Available at [https://github.com/phanen/amen/tree/master/amen](https://github.com/phanen/amen/tree/master/amen) and [https://github.com/manavsi19/subgraph-outlier-detection](https://github.com/manavsi19/subgraph-outlier-detection)
Table 3: Analysis of five real-world datasets for coherent dense subspace cluster (subgraph) detection.

| Dataset | Node | Edge | Attribute | Top-K | Avg. Cluster density | Avg. Cluster Size | Avg. Coherence Distance |
|---------|------|------|-----------|-------|----------------------|------------------|------------------------|
| DFB     | 100  | 1106 | 5         | 5     | 9.69                 | 10.8             | 0.13                   |
|         |      |      |           | 10    | 9.94                 | 11.0             | 0.13                   |
|         |      |      |           | 15    | 11.03                | 12.07            | 0.12                   |
|         |      |      |           | 20    | 10.55                | 11.6             | 0.12                   |
| DBLP    | 774  | 1757 | 20        | 5     | 4.12                 | 3.2              | 5.2                    |
|         |      |      |           | 10    | 3.84                 | 3.5              | 4.7                    |
|         |      |      |           | 15    | 3.25                 | 3.7              | 7.3                    |
|         |      |      |           | 20    | 3.34                 | 3.17             | 7.6                    |
| IMDB    | 862  | 4388 | 21        | 5     | 3.24                 | 4.96             | 4.6                    |
|         |      |      |           | 10    | 3.17                 | 5.6              | 10.0                   |
|         |      |      |           | 15    | 3.52                 | 5.79             | 4.27                   |
|         |      |      |           | 20    | 3.36                 | 2.78             | 5.37                   |
| Genes   | 2900 | 8264 | 115       | 5     | 3.76                 | 6.92             | 24.8                   |
|         |      |      |           | 10    | 3.74                 | 5.91             | 24.7                   |
|         |      |      |           | 15    | 3.82                 | 5.58             | 23.09                  |
|         |      |      |           | 20    | 3.85                 | 5.36             | 21.83                  |
| Arxiv   | 11989| 119258| 300       | 5     | 11.53                | 4.16             | 15.8                   |
|         |      |      |           | 10    | 9.65                 | 4.24             | 10.0                   |
|         |      |      |           | 15    | 9.03                 | 4.23             | 11.27                  |
|         |      |      |           | 20    | 8.72                 | 4.21             | 10.7                   |

Table 4: Chicago Crime data (Fm refers to F-Measure).

| Methods   | type         | Node Fm | Attribute Fm | Running Time (s) |
|-----------|--------------|---------|--------------|------------------|
| SODA      | BATTERY      | 0.476   | 0.146        | 7,997,893        |
| AMEN      | BATTERY      | 0.363   | 0.818        | 3,835,589        |
| SG-Pursuit| BATTERY      | 0.683   | 0.955        | 73,998           |
|           | BATTERY      | 0.583   | 1.000        | 37,538           |

Experiments demonstrate the effectiveness and efficiency of our algorithms. For the future work, we plan to generate our algorithm to subspace cluster detection in heterogeneous networks.

A PROOF OF THEOREM 2.2

Proof. Let $\phi^{+1} = x^{i+1} - x^*$ and $\phi^{-1} = y^{i+1} - y^*$. Then the component $\|\phi^+_y\|_2 + \|\phi^-_y\|_2$ is upper bounded as

$$\|\phi^+_y\|_2 + \|\phi^-_y\|_2 = \|\phi^{i+1} - x^*\|_2 + \|y^{i+1} - y^*\|_2 = \|\phi^+_y\|_2 + \|\phi^-_y\|_2 \leq \|\phi^+_y\|_2 + \|\phi^-_y\|_2 \leq c_T \|\phi^+_y\|_2 + \|\phi^-_y\|_2 = \sqrt{2} \|\phi^+_y\|_2 + \|\phi^-_y\|_2,$$

where the first inequality follows from the use of the triangle inequality; the second equality follows from the fact that $x^{i+1} = [b_y]_{x^{i+1}}$ and $y^{i+1} = [b_y]_{y^{i+1}}$; the second inequality follows from the definition of the tail projection oracle $T(\cdot)$ and the fact that $\Psi^{+1} = T(b_y^*)$, $\Psi^{-1} = \arg \max_{\rho \leq (1, \ldots, \rho)} \|\phi^+_y\|_2 + \|\phi^-_y\|_2$; $\supp(x^*) \in \mathbb{R}(k)$ and $\|y^*\|_2 \leq s$; and the last inequality follows from the fact that $c_T \geq 1$. Recall that $[b_y]_{y^{i+1}}$ refers to the projected vector in the subspace defined by the subset $\Psi^{i+1}$. Denote by $w_i$ the projected vector $[b_y]_{y^{i+1}}$, $w_i$ is defined as: $w_i = [b_y]_{i}$, the $i$-th entry in the vector $b_i$, if $i \in \Psi^{i+1}$; otherwise, $w_i = 0$.

Recall that $\Omega_\gamma = \Gamma_\gamma \cup \supp(x^*)$ and $\Omega_{\mu} = \Gamma_{\mu} \cup \supp(y^*)$. The component $\|((x^* - b_x)_{\Omega_\gamma})_2 + (y^* - b_y)_{\Omega_{\mu}}\|_2$ is upper bounded as

$$\begin{align*}
\|((x^* - b_x)_{\Omega_\gamma})_2 + (y^* - b_y)_{\Omega_{\mu}}\|_2 &= \frac{x - x^*}{(y - y^*)^\top} \left[\nabla f(b_x, b_y)_{\Omega_\gamma} + \frac{y^*(1 - \nabla f(x^*, y^*)_{\Omega_{\mu}})}{(y - y^*)^\top} \nabla f(x^*, y^*)_{\Omega_{\mu}}\right] \\
&= \frac{(b_x - x^*)_2}{(y^* - b_y)_{\Omega_{\mu}}} \\
&\leq \sqrt{2} \rho \left(\|b_x - x^*\|_2 + (y^* - y^*)\right) \\
&\leq \sqrt{2} \rho \left(\|b_x - x^*\|_2 + (y^* - y^*)\right)
\end{align*}$$
where the second equality follows from the fact that \((b_x, b_y)\) is the optimal solution to the sub-problem in Line 10 of Algorithm 1 and hence \(\| \nabla f(b_x, b_y) \|_{\Omega_Z} = 0\) and \(\| \nabla f(b_x, b_y) \|_{\Omega_Z} = 0\); the first inequality follows from (1) the fact that \((w, v) \leq \| w \|_2 \| v \|_2\) for any vectors \(w\) and \(v\); and (2) the inequality (16) in Lemma A.2 by letting \(\xi = \frac{\| y \|_2}{\sqrt{2} \rho}\), given that \(\text{supp}(b_x), \text{supp}(x^* + \epsilon y) \in \mathcal{M}(2k), \Omega_x \subseteq \mathcal{M}(4k), \) and \(\|\Omega_y\| \leq 4\); and the last inequality follows from the definitions of \(\epsilon_x\) and \(\epsilon_y\) in Theorem 2.2 and the fact that \(\Omega_x \in \mathcal{M}(2k)\) and \(\|\Omega_y\|_0 \leq 3\):

\[
\epsilon_x = \max_{S \in \mathcal{M}(2k)} \| \nabla f_{\epsilon}(x^*(\cdot), y^*) \|_S^2,
\]
and

\[
\epsilon_y = \max_{R \subseteq \{1, \ldots, p\}, |R| \leq 3s} \| \nabla f_{\epsilon}(x^*(\cdot), y^*) \|_R^2.
\]

It follows that

\[
\| x^* - b_x \|_2 + \| y^* - b_y \|_2 \leq \| x^* - b_x \|_\Omega_x^2 + \| y^* - b_y \|_\Omega_y^2 + \| x^* - b_x \|_\Omega_x^2 + \| y^* - b_y \|_\Omega_y^2 \leq \sqrt{2} \rho \left( \| b_x - x^* \|_2 + \| b_y - y^* \|_2 \right) + \frac{y}{(y^*)^2} (\epsilon_x + \epsilon_y) + \| x^* - b_x \|_\Omega_x^2 + \| y^* - b_y \|_\Omega_y^2 \|_2,
\]

where the first inequality follows from the use of triangle inequality, and the second inequality follows from the inequality obtained above. After rearrangement, we obtain

\[
\| x^* - b_x \|_2 + \| y^* - b_y \|_2 \leq \frac{1}{1 - \sqrt{2} \rho} \left( \| x^* - b_x \|_\Omega_x^2 + \| y^* - b_y \|_\Omega_y^2 \right) + \frac{y}{(y^*)^2} (\epsilon_x + \epsilon_y) = \frac{1}{1 - \sqrt{2} \rho} \left( \| x^* - b_x \|_\Omega_x^2 + \| y^* - b_y \|_\Omega_y^2 \right) + \frac{y}{(y^*)^2} (\epsilon_x + \epsilon_y) = \frac{1}{1 - \sqrt{2} \rho} \left( \| x^* - b_x \|_\Omega_x^2 + \| y^* - b_y \|_\Omega_y^2 \right) + \frac{y}{(y^*)^2} (\epsilon_x + \epsilon_y)
\]

where the first equality follows from the fact that \(\text{supp}(b_x) \in \Omega_x\) and \(\text{supp}(b_y) \in \Omega_y\) and hence \(\| b_x \|_{\Omega_x^2} = 0\) and \(\| b_y \|_{\Omega_y^2} = 0\); the second equality follows from the fact that \(\text{supp}(x^*) \subseteq \Omega_x\) and \(\text{supp}(y^*) \subseteq \Omega_y\), and hence \(\| x^* \|_{\Omega_x^2} = 0\) and \(\| y^* \|_{\Omega_y^2} = 0\); and the last inequality follows from the fact that \(\Omega_x \leq \Omega_x\) and \(\Omega_y \leq \Omega_y\), and hence \(\Omega_x^2 \subseteq \Omega_x^2\) and \(\Omega_y^2 \subseteq \Omega_y^2\).

Combining the above inequalities, we obtain

\[
\| r^i_{x+} \|_2 + \| r^i_{y+} \|_2 \leq \frac{c_H+1}{1 - \sqrt{2} \rho} \left( \| r^i_{x+} \|_\Omega_x^2 + \| r^i_{y+} \|_\Omega_y^2 \right) + \frac{y}{(y^*)^2} (\epsilon_x + \epsilon_y).
\]

From Lemma A.1, we have

\[
\|\| r^i_{x+} \|_\Omega_x \|_2 + \| r^i_{y+} \|_\Omega_y \|_2 \|_2 \leq \sqrt{1 - \alpha_0^2} \| x^* \|_\Omega_x \|_2 + \frac{\alpha_0 \beta_0}{1 - \alpha_0^2} \| y^* \|_\Omega_y \|_2 \|_2 \leq \sqrt{1 - \alpha_0^2} \| x^* \|_\Omega_x \|_2 + \frac{\alpha_0 \beta_0}{1 - \alpha_0^2} \| y^* \|_\Omega_y \|_2 \|_2 \leq \sqrt{1 - \alpha_0^2} \| x^* \|_\Omega_x \|_2 + \frac{\alpha_0 \beta_0}{1 - \alpha_0^2} \| y^* \|_\Omega_y \|_2 \|_2 \leq \sqrt{1 - \alpha_0^2} \| x^* \|_\Omega_x \|_2 + \frac{\alpha_0 \beta_0}{1 - \alpha_0^2} \| y^* \|_\Omega_y \|_2 \|_2 \]
where the first inequality follows from the inequality (10); the second and third inequalities follow from the use of the triangle inequality; the second equality follows from the fact that \(\supp(r^2) \subseteq \Gamma_x \cup \Theta_x\) and \(\supp(r^2) \subseteq \Gamma_x \cup \Theta_x\); the fourth inequality follows from the bound (15) in Lemma A.2, given that \(\supp(r^2) \subseteq \Gamma_x \cup \Theta_x \subseteq \mathbb{m}(3k)\) and \(\supp(r^2) \subseteq \left| \Gamma_y \cup \Theta_y \right| \leq 4s\); and the last inequality follows from the definitions of \(\varepsilon_x\) and \(\varepsilon_y\), given that \(\Gamma_x \in \mathbb{m}(k)\) and \(\|\Phi_y\| \leq 2s\).

Combining the above two upper bounds and grouping terms, we have
\[
\|r^x_i, r^y_i\|_2 \geq a_0 \|r^x_i, r^y_i\|_2 - \beta_0(\varepsilon_x + \varepsilon_y),
\]
where \(a_0 = c_H(1-\rho) - \rho = c_H + 1\), \(\rho = \sqrt{1 + (\xi y^2)^2 - 2\xi y^2}\), and \(\beta_0 = (c_H + 1)^\xi\). Let \(\xi = y^2 / (y^2)^2\), then \(\rho = \sqrt{1 - (y^2 / y^2)^2}\).

We assume that \(\delta\) is small enough such that \(c_H \geq \frac{\rho}{\gamma - \rho}\) and \(a_0 > 0\). We consider two cases:

**Case 1:** The value of \(\|r^x_i, r^y_i\|_2\) satisfies the condition:
\[
a_0 \|r^x_i, r^y_i\|_2 \leq \beta_0(\varepsilon_x + \varepsilon_y).
\]
Then we have
\[
\|r^x_i, r^y_i\|_2 \geq \frac{\beta_0(\varepsilon_x + \varepsilon_y)}{a_0}
\]

**Case 2:** The value of \(\|r^x_i, r^y_i\|_2\) satisfies the condition:
\[
a_0 \|r^x_i, r^y_i\|_2 \geq \beta_0(\varepsilon_x + \varepsilon_y).
\]
Rewriting the inequality, we get
\[
\|r^x_i, r^y_i\|_2 \geq \left(\alpha - \frac{\beta_0(\varepsilon_x + \varepsilon_y)}{\|r^x_i, r^y_i\|_2}\right)
\]
and
\[
\|r^x_i, r^y_i\|_2 \geq \left(\alpha - \frac{\beta_0(\varepsilon_x + \varepsilon_y)}{\|r^x_i, r^y_i\|_2}\right)^2
\]
Moreover, we also have
\[
\left\| [r_x^i, r_y^i]_2 \right\|^2_2 = \left\| [l_x^i, l_y^i]_2 \right\|^2_2 + \left\| [r_x^i, l_y^i]_2 \right\|^2_2.
\]
and
\[
\left\| [l_x^i, l_y^i]_2 \right\|^2_2 = \left\| [r_x^i, r_y^i]_2 \right\|^2_2 - \left\| [l_x^i, r_y^i]_2 \right\|^2_2.
\]
Therefore, we obtain
\[
\left\| [r_x^i, r_y^i]_2 \right\|^2_2 \leq \left\| [r_x^i, r_y^i]_2 \right\|^2_2\left(1 - \frac{\beta_0 (\varepsilon_x + \varepsilon_y)}{\left\| [r_x^i, r_y^i]_2 \right\|^2_2}\right)^2.
\]
We can simplify the right hand side using the following geometric argument, adapted from [24]. Denote \( \omega_0 = \alpha_0 - \beta_0 (\varepsilon_x + \varepsilon_y) \). Then, \( 0 < \omega_0 < 1 \) because \( \beta_0 \left\| [r_x^i, r_y^i]_2 \right\|^2_2 \geq \beta_0 (\varepsilon_x + \varepsilon_y) \) and \( \alpha_0 < 1 \).
For a free parameter \( 0 \leq \omega \leq 1 \), a straightforward calculation yields
\[
\sqrt{1 - \omega_0^2} \leq 1 - \frac{\omega}{\sqrt{1 - \omega^2}} = 1 - \frac{\omega_0}{\sqrt{1 - \omega^2}}\omega_0.
\]
Therefore, substituting into the bound for \( \left\| [r_x^i, r_y^i]_2 \right\|^2_2 \), we get
\[
\left\| [r_x^i, r_y^i]_2 \right\|^2_2 \leq \left(1 - \frac{\omega_0}{\sqrt{1 - \omega^2}}\right) \left(1 - \frac{\omega_0}{\sqrt{1 - \omega^2}}\right) \frac{\alpha_0}{\sqrt{1 - \omega_0^2}} (\varepsilon_x + \varepsilon_y)
\]
\[+ \frac{\alpha_0 \beta_0 (\varepsilon_x + \varepsilon_y)}{\sqrt{1 - \omega_0^2}}.
\]

The proof of the Lemma is complete. \( \square \)

**Lemma 2.2 (Properties of RSS/RSC).** If \( f \) satisfies the \((d,h),(s, \gamma, \gamma')\)-RSS/RSC, then for every \( x, x' \in \mathbb{R}^n \) and \( S_x \in \mathbb{H}(d,k) \) with \( \text{supp}(x), \text{supp}(x') \in \mathbb{H}(2k) \), and every \( y, y' \subseteq \mathbb{R}^m \) with \( \text{supp}(y), \text{supp}(y') \leq 2s \) and \( |S_y| \leq 4s \), the following inequalities hold:

- **Part 1:**
  \[
  y^+(\|x - x'\|_2^2 + \|y - y'\|_2^2) \leq -\left(\langle \nabla_x f(x, y), l_s_x \rangle_{S_x} - \langle \nabla_x f(x', y'), l_s_x \rangle_{S_x}, x - x' \right) - \left(\langle \nabla_y f(x, y), l_s_y \rangle_{S_y} - \langle \nabla_y f(x', y'), l_s_y \rangle_{S_y}, y - y' \right) \leq y^+(\|x - x'\|_2^2 + \|y - y'\|_2^2).
  \]

- **Part 2:**
  \[
  (\|\nabla_x f(x, y) - \nabla_x f(x', y')\|_2^2 + \|\nabla_y f(x, y) - \nabla_y f(x', y')\|_2^2) \leq (y^+)^2(\|x - x'\|_2^2 + \|y - y'\|_2^2).
  \]

- **Part 3:**
  For any \( \xi \leq 2 \frac{\rho}{\gamma_2} \), we have
  \[
  \left\| \xi \langle \nabla_x f(x, y), l_s_x \rangle_{S_x} - \xi \langle \nabla_x f(x', y'), l_s_x \rangle_{S_x}, (x - x') \right\|^2_2 + \left\| \xi \langle \nabla_y f(x, y), l_s_y \rangle_{S_y} - \xi \langle \nabla_y f(x', y'), l_s_y \rangle_{S_y}, (y - y') \right\|^2_2 \leq \rho^2(\|x - x'\|_2^2 + \|y - y'\|_2^2).
  \]

where \( \rho = \sqrt{1 + (y^+)^2 - 2\gamma^2} \). The condition \( \xi \leq 2 \frac{\rho}{\gamma_2} \) ensures that \( \rho \leq 1 \). In particular, if \( \xi = \frac{-\rho}{\gamma_2} \), then \( \rho = \sqrt{1 - (y^+/\gamma)^2} \).

**Proof.** The proofs of the inequalities in the four parts are stated as follows:

- **Part 1:** Recall that \( \text{supp}(x - x') \subseteq S_x \) and \( \text{supp}(y - y') \subseteq S_y \). By adding two copies of the inequalities (6) with \( (x, y) \) and \( (x', y') \) as described in Definition 2.1, we have

  \[
  y^-\left(\|x - x'\|_2^2 + \|y - y'\|_2^2\right) \leq -\left(\langle \nabla_x f(x, y), l_s_x \rangle_{S_x} - \langle \nabla_x f(x', y'), l_s_x \rangle_{S_x}, x - x' \right) - \left(\langle \nabla_y f(x, y), l_s_y \rangle_{S_y} - \langle \nabla_y f(x', y'), l_s_y \rangle_{S_y}, y - y' \right) = -\left(\langle \nabla_x f(x, y) - \nabla_x f(x', y'), l_s_x \rangle_{S_x}, x - x' \right) - \left(\langle \nabla_y f(x, y) - \nabla_y f(x', y'), l_s_y \rangle_{S_y}, y - y' \right) \leq y^+(\|x - x'\|_2^2 + \|y - y'\|_2^2).
  \]

where
\[
\langle \nabla_x f(x, y) - \nabla_x f(x', y'), l_s_x \rangle_{S_x} = \langle \nabla_x f(x, y) - \nabla_x f(x', y'), [x - x']_{S_x} \rangle_{S_x}, x - x' \rangle = \langle \nabla_y f(x, y) - \nabla_y f(x', y'), [y - y']_{S_y} \rangle_{S_y}, y - y' \rangle.
\]
• Part 2: By Theorem 2.1.5 in [29], we have
\[-(\nabla_x f(x, y) - \nabla_x f(x', y'), x - x') - (\nabla_y f(x, y) - \nabla_y f(x', y'), y - y') \geq \frac{1}{\gamma^2} \left( \|\nabla_x f(x, y) - \nabla_x f(x', y')\|^2 + \left\| \nabla_y f(x, y) - \nabla_y f(x', y') \right\|^2 \right)\] 

We then have
\[\left( \|\nabla_x f(x, y) - \nabla_x f(x', y')\|^2 + \|\nabla_y f(x, y) - \nabla_y f(x', y')\|^2 \right)^{1/2} \geq \left( \|x - x'\|^2 + \|y - y'\|^2 \right)^{1/2}\] 
\[\geq \left( \|\nabla_x f(x, y) - \nabla_x f(x', y')\|^2 + \|\nabla_y f(x, y) - \nabla_y f(x', y')\|^2 \right)^{1/2}\] 
\[\geq -(\nabla_x f(x, y) - \nabla_x f(x', y'), x - x') - (\nabla_y f(x, y) - \nabla_y f(x', y'), y - y') \geq \frac{1}{\gamma^2} \left( \|\nabla_x f(x, y) - \nabla_x f(x', y')\|^2 + \|\nabla_y f(x, y) - \nabla_y f(x', y')\|^2 \right)\] 
The above inequalities indicate that
\[\left( \|x - x'\|^2 + \|y - y'\|^2 \right)^{1/2} \geq \frac{1}{\gamma^2} \left( \|\nabla_x f(x, y) - \nabla_x f(x', y')\|^2 + \|\nabla_y f(x, y) - \nabla_y f(x', y')\|^2 \right)^{1/2}\] 

We then obtain
\[\left( \|\nabla_x f(x, y) - \nabla_x f(x', y')\|^2 + \|\nabla_y f(x, y) - \nabla_y f(x', y')\|^2 \right) \leq (\gamma^4)^2 \left( \|x - x'\|^2 + \|y - y'\|^2 \right)\] 

• Part 3: Combining the two bounds (13) and (14) and grouping terms, we get
\[\left\| \xi [\nabla_x f(x, y)]_{y_k} - \xi [\nabla_x f(x', y')]_{y_k} - (x - x') \right\|^2 + \left\| \xi [\nabla_y f(x, y)]_{y_k} - \xi [\nabla_y f(x', y')]_{y_k} - (y - y') \right\|^2 = \] 
\[\left\| \xi [\nabla_x f(x, y)]_{y_k} - \xi [\nabla_x f(x', y')]_{y_k} \right\|^2 + \|x - x'\|^2 \right) - 2\xi [\nabla_x f(x, y)]_{y_k} - \xi [\nabla_x f(x', y')]_{y_k} - (x - x') + \|\xi [\nabla_y f(x, y)]_{y_k} - \xi [\nabla_y f(x', y')]_{y_k} + \|y - y'\|^2 \right) - 2\xi [\nabla_y f(x, y)]_{y_k} - \xi [\nabla_y f(x', y')]_{y_k} - (y - y') \leq \] 
\[\left( 1 + \xi^2 \gamma^4 \right) \left( \|x - x'\|^2 + \|y - y'\|^2 \right) - 2\xi [\nabla_x f(x, y)]_{y_k} - \xi [\nabla_x f(x', y')]_{y_k} - (x - x') - 2\xi [\nabla_y f(x, y)]_{y_k} - \xi [\nabla_y f(x', y')]_{y_k} - (y - y') \leq \] 
\[\left( 1 + \xi^2 \gamma^4 \right) \left( \|x - x'\|^2 + \|y - y'\|^2 \right) - 2\xi \gamma^{-2} \left( \|x - x'\|^2 + \|y - y'\|^2 \right) = \] 
\[\left( 1 + \xi^2 \gamma^4 \right) \left( \|x - x'\|^2 + \|y - y'\|^2 \right) \tag{18} \] 
where the first inequality follows from the bound (14), and the last inequality follows from the bound (13). By combining the inequality (18) and the following inequality
\[\left\| \xi [\nabla_x f(x, y)]_{y_k} - \xi [\nabla_x f(x', y')]_{y_k} - (x - x') \right\|^2 + \left\| \xi [\nabla_y f(x, y)]_{y_k} - \xi [\nabla_y f(x', y')]_{y_k} - (y - y') \right\|^2 \leq \] 
\[2\left( \xi [\nabla_x f(x, y)]_{y_k} - \xi [\nabla_x f(x', y')]_{y_k} - (x - x') \right)^2 + 2\left( \xi [\nabla_y f(x, y)]_{y_k} - \xi [\nabla_y f(x', y')]_{y_k} - (y - y') \right)^2, \tag{19} \] 
we have
\[\left\| \xi [\nabla_x f(x, y)]_{y_k} - \xi [\nabla_x f(x', y')]_{y_k} - (x - x') \right\|^2 + \left\| \xi [\nabla_y f(x, y)]_{y_k} - \xi [\nabla_y f(x', y')]_{y_k} - (y - y') \right\|^2 \leq \] 
\[\sqrt{2 \left( 1 - 2\xi \gamma^{-2} \right) \left( \|x - x'\|^2 + \|y - y'\|^2 \right)} \tag{20} \]

• Part 4: Let \( \xi = \frac{1}{(y')^2} \) and \( \rho = 1 + (\xi \gamma^2)^2 = 2 - 2\xi \gamma^2\). We have
\[\xi^2 \left\| \nabla_x f(x, y) \right\|_{y_k} - \left\| \nabla_x f(x', y') \right\|_{y_k} \right) \right) \] 

By combining the inequality (18) and the following inequality:
\[\xi^2 \left\| \nabla_x f(x, y) \right\|_{y_k} - \left\| \nabla_x f(x', y') \right\|_{y_k} \right) \] 

we conclude that
\[\frac{1 - \rho}{\xi} \left( \|x - x'\|^2 + \|y - y'\|^2 \right)^{1/2} \leq \] 
\[\left( \left\| \nabla_x f(x, y) \right\|_{y_k} - \left\| \nabla_x f(x', y') \right\|_{y_k} \right) \] 
\[\left\| \nabla_y f(x, y) \right\|_{y_k} - \left\| \nabla_y f(x', y') \right\|_{y_k} \right) \] 

\[\left( \|x - x'\|^2 + \|y - y'\|^2 \right)^{1/2}. \tag{21} \]

\[\square\]

B PROOF OF THEOREM 3.1

B.1 Negative squared error function
Recall that the negative squared error function has the form:
\[f(x, y) = -\frac{1}{x^T x} \left( -\frac{1}{2} |x|^2 - \frac{1}{2} |y|^2 \right), \] 

where \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^p \). The following Lemma discusses the RSS/RSC property of the negative squared error function.
**Lemma B.1 (Negative squared error function).** Let $I_n$ and $I_p$ be the identity matrices of sizes $n \times n$ and $p \times p$, respectively. If the attribute matrix $W \in \mathbb{R}^{n \times p}$ satisfies the condition: $WW^T \preceq bI_n \preceq I_n$ and $WTW \preceq b^2I_p \preceq I_p$, for every $x \in [0,1]^n$ and $y \in [0,1]^p$, such that $sup(x) \in M(k)$ and $|y|_0 \leq s$, then the negative squared error function satisfies the $(b(k), s, y^-, y^+)$-RSS/RSC, where $y^- = 1$ and $y^+ = \max \left(2b + 2\sqrt{b} + 1, 2\sqrt{b}\right)$.

**Proof.** Let $x' = x + \Delta_x$ and $y' = y + \Delta_y$, such that $sup(x)$, $sup(x') \in M(k)$ and $|y|_0, |y'|_0 \leq s$. Denote $g(x', y', x, y) = f(x,y) - f(x',y') - \nabla_x f(x,y)^T(x-x') - \nabla_y f(x,y)^T(y-y')$. The component $g(x', y', x, y)$ can be upper bounded as

$$g(x', y', x, y) = \|W\Delta_x + \Delta_y\|_2^2 + \frac{1}{2}\Delta_y^T\Delta_y \geq \frac{y^+}{2} \left(\|\Delta_x\|_2^2 + \|\Delta_y\|_2^2\right),$$

where $y^+ = \max \left(2b + 2\sqrt{b} + 1, 2\sqrt{b}\right)$. The component $g(x', y', x, y)$ can also be lower bounded as

$$g(x', y', x, y) = \|W\Delta_x + \Delta_y\|_2^2 + \frac{1}{2}\Delta_y^T\Delta_y \geq 0.5\left(\|\Delta_x\|_2^2 + \|\Delta_y\|_2^2\right) = \frac{y^-}{2} \left(\|\Delta_x\|_2^2 + \|\Delta_y\|_2^2\right),$$

where $y^- = 1$. $\square$

**B.2 Fisher's test statistic function**

Recall that $w_i$ refers to the vector of observations of the $p$ attributes at node $i$, the attribute matrix $W$ is defined as $W = [w_1, \cdots, w_n]^T$, and the Fisher's test statistic is defined as

$$f(x, y) = x^TWy - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|y\|^2,$$

where we consider the soft values of $x$ and $y$: $x \in [0,1]^n$ and $y \in [0,1]^p$. We consider the relaxed input domains $[0,1]^n$ and $[0,1]^p$ for $x$ and $y$, instead of their original domains $[0,1]^n$ and $[0,1]^p$, respectively, such that our proposed algorithm $\text{SG-Purit}$ can be applied to optimize this score function. The following Lemma discusses the RSS/RSC property of the Fisher's test statistic function:

**Lemma B.2 (Fisher's test statistic).** Let $I_n$ and $I_p$ be the identity matrices of sizes $n \times n$ and $p \times p$, respectively. If the attribute matrix $W \in \mathbb{R}^{n \times p}$ satisfies the condition: $WW^T \preceq bI_n \preceq I_n$ and $WTW \preceq b^2I_p \preceq I_p$, for every $x \in [0,1]^n$ and $y \in [0,1]^p$, such that $sup(x) \in M(k)$ and $|y|_0 \leq s$, then the Fisher's test statistic function satisfies the $(b(k), s, y^-, y^+)$-RSS/RSC, where $y^- = \min \{1 - b^0, 1 - b^1\}$ and $y^+ = 2$.

**Proof.** Let $x' = x + \Delta_x$ and $y' = y + \Delta_y$, such that $sup(x)$, $sup(x') \in M(k)$ and $|y|_0, |y'|_0 \leq s$. Denote $g(x', y', x, y) = f(x,y) - f(x',y') - \nabla_x f(x,y)^T(x-x') - \nabla_y f(x,y)^T(y-y')$. The component $g(x', y', x, y)$ can be upper bounded as

$$g(x', y', x, y) = \|W\Delta_x + \Delta_y\|_2^2 + \frac{1}{2}W^T\Delta_x + \frac{1}{2}W^T\Delta_x = \frac{y^+}{2} \left(\|\Delta_x\|^2 + \|\Delta_y\|^2\right),$$

where $y^+ = \max \left(2b + 2\sqrt{b} + 1, 2\sqrt{b}\right)$. The component $g(x', y', x, y)$ can also be lower bounded as

$$g(x', y', x, y) = -\Delta_x^TW\Delta_y + \frac{1}{2}\Delta_y^T\Delta_x + \frac{1}{2}\Delta_y^T\Delta_y \geq \frac{y^-}{2} \left(\|\Delta_x\|^2 + \|\Delta_y\|^2\right),$$

where $y^- = \min \{1 - b^0, 1 - b^1\}$. By combining the inequalities (23) and (24), we obtain

$$f(x', y') - f(x, y) - \nabla_x f(x,y)^T(x' - x) - \nabla_y f(x,y)^T(y - y') \geq \frac{1}{2} \min \left(1 - b^1, 1 - b^0\right) \left(\|\Delta_x\|^2 + \|\Delta_y\|^2\right).$$

By combining the inequalities (22) and (24), we get

$$\frac{y^+}{2} \left(\|\Delta_x\|^2 + \|\Delta_y\|^2\right) \leq g(x', y', x, y) \leq \frac{y^-}{2} \left(\|\Delta_x\|^2 + \|\Delta_y\|^2\right).$$
where $y^- = \min \{1-b^0, 1-b^1\}$ and $y^+ = 2$. \hfill \Box

In the above lemma, it is required that $b^0$ and $b^1$ are less than 1. Given that $x \in [0, 1]^n$ and $y \in [0, 1]^p$, the attribute matrix $W$ can be normalized such that $b^0, b^1 \leq 1$.

B.3 Logistic function

Recall that the logistic function is defined as

$$f(x, y) = \sum_{i=1}^{p} \left( y_i \log(g(x^T w_i) + (1 - y_i) \log(1 - g(x^T w_i)) \right) - \frac{1}{2} ||x||^2 - \frac{1}{2} ||y||^2,$$

where $w_i = [w_i(1), \cdots, w_i(n)]^T$ is the vector of observations of the $i$-th attribute at the $n$ nodes in $\mathbb{N}$, $w_i(j)$ is the observation of the $i$-th attribute at node $j$, $x \in \mathbb{R}^n$ is the vector of the weights (coefficients) of the $n$ nodes in $\mathbb{N}$, and $y \in [0, 1]^p$ is the vector of soft binary variables that indicate the anomalousness of the $p$ attributes, and the $i$-th attribute is anomalous if $y_i > 0$.

**Lemma B.3 (Logistic function).** Let $I_n$ and $I_p$ be the identity matrices of sizes $n \times n$ and $p \times p$, respectively. If the attribute matrix $W \in \mathbb{R}^{n \times p}$ satisfies the condition: $WW^T \leq b^0 I_n \leq I_n$ and $W^TW \leq b^1 I_p \leq I_p$, for every $x \in [0, 1]^n$ and $y \in [0, 1]^p$, such that $\operatorname{supp}(x) \in \mathbb{M}(k)$ and $\|y\|_0 \leq s$, then the logistic function satisfies the $(h(k), s, y^-, y^+)-\text{RSS/RSC}$, where $y^- = \min \{1-b^0, 1-b^1\}$ and $y^+ = \max \{2b^0, 1+2\}$.

**Proof.** It suffices to prove the RSS/RSC property of the logistic function if the following inequalities hold:

$$y^- I_{n+p} \leq -\nabla_{x, y}^2 f(x, y) \leq y^+ I_{n+p}, \tag{26}$$

where $\nabla_{x, y}^2 f(x, y)$ is the Hessian matrix of $f(x, y)$, and $I_{n+p}$ is an identity matrix of size $n+p$ by $n+p$.

The first-order derivatives of the score function $f(x, y)$ has the following forms:

$$\nabla_y f(x, y) = (\log(g(x^T w_1)), \cdots, \log(g(x^T w_p))^T - (1 - g(x^T w_1)), \cdots, (1 - g(x^T w_p)) \right) - y$$

and

$$\nabla_x f(x, y) = [(1 - g(x^T w_1)) w_1, \cdots, (1 - g(x^T w_p)) w_p] y + [(g(x^T w_1), \cdots, g(x^T w_p)) w_p] (1 - y) - x.$$

The second-order derivatives of the score function has the following forms:

$$\nabla_x^2 f(x, y) = \begin{cases} (g(x^T w_1) (1 - g(x^T w_1)) w_1, \cdots, g(x^T w_p) (1 - g(x^T w_p)) w_p \right) - y \\ (1 - g(x^T w_1)) (1 - g(x^T w_1)) w_1, \cdots, g(x^T w_p) (1 - g(x^T w_p)) w_p) (1 - y) - I_n, \end{cases}$$

$$\nabla_{x, y}^2 f(x, y) = \begin{cases} (1 - g(x^T w_1)) w_1, \cdots, (1 - g(x^T w_p)) w_p) + (g(x^T w_1), \cdots, g(x^T w_p) w_p = [w_1, \cdots, w_p], \end{cases}$$

where $I_n$ and $I_p$ refer to the identity matrices of sizes $n$ by $n$ and $p$ by $p$, respectively. For every $\Delta_x$ and $\Delta_y$, such that $\operatorname{supp}(\Delta_x) \in \mathbb{M}(k)$ and $\|\Delta_y\|_0 \leq s$, we obtain

$$\Delta_x \nabla_{x, y}^2 f(x, y) \Delta_y = \sum_{i=1}^{p} g(x^T w_i) (1 - g(x^T w_i)) \Delta_x^T w_i w_i^T \Delta_x + \Delta_x^T \Delta_y,$$

$$\Delta_x \nabla_x f(x, y) \Delta_x = -\Delta_x^T [w_1, \cdots, w_p] \Delta_y,$$

and

$$\Delta_y \nabla_x^2 f(x, y) \Delta_y = \Delta_y^T \Delta_y. \tag{27}$$

It follows that

$$-\Delta_x \Delta_y^T \nabla_{x, y}^2 f(x, y) \Delta_x \Delta_y = -\Delta_x \nabla_{x, y}^2 f(x, y) \Delta_x \Delta_y = -\Delta_x \Delta_x^T \Delta_y \Delta_y = -2\Delta_x \Delta_x \Delta_y \Delta_y \leq$$

$$\Delta_x^T WW^T \Delta_x - 2\Delta_x^T W \Delta_y + \Delta_x^T \Delta_x + \Delta_y^T \Delta_y \leq$$

$$\Delta_x^T WW^T \Delta_x + \Delta_y^T \Delta_y + \Delta_x^T W \Delta_y + \Delta_y^T \Delta_x + \Delta_x^T \Delta_y \leq$$

$$(2b^0 + 1) ||\Delta_x||^2 + 2 ||\Delta_y||^2 \leq$$

$$(2b^0 + 1, 2) \left( ||\Delta_x||^2 + ||\Delta_y||^2 \right)$$

where the first inequality follows from the fact that $0 \leq g(x^T w_i) \leq 1$, the second and third inequalities follow from the use of the triangle inequality, and the third inequality follows from the assumed property of the attribute matrix $W: WW^T \leq b^0 I_n$. The component $-\Delta_x \Delta_y^T \nabla_{x, y}^2 f(x, y) \Delta_x \Delta_y$ can lower bounded as

$$-\Delta_x \Delta_y^T \nabla_{x, y}^2 f(x, y) \Delta_x \Delta_y = -\Delta_x \Delta_x^T \Delta_y \Delta_y = -2\Delta_x \Delta_y \leq$$

$$\Delta_x \Delta_x + \Delta_y \Delta_y = \Delta_x \Delta_x \Delta_y - \Delta_x \Delta_y \leq$$

$$(1 - b^0) \Delta_x \Delta_y,$$
A refined lower bound can be obtained as

$$\mathbf{A} = \mathbf{A} \mathbf{f}_{\text{in}} \text{obtained as}$$

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A re/fined lower bound can be obtained as

$$\nabla^2 f(x,y) \mathbf{A} \mathbf{x} = \nabla^2 f(x,y) \mathbf{A} \mathbf{x} = 2 \mathbf{A} \nabla_x, y f(x,y) \mathbf{A} \mathbf{y} \geq 2 \nabla^2 W \mathbf{A} \mathbf{y} + \nabla^2 \mathbf{A} \mathbf{y} \geq \mathbf{A} \nabla \mathbf{A} \mathbf{y} - \mathbf{A} \nabla \mathbf{A} \mathbf{y} - \mathbf{A} \nabla \mathbf{A} \mathbf{y} \geq \nabla^2 \mathbf{A} \mathbf{y} \geq (1 - \beta) \mathbf{A} \mathbf{y} \mathbf{y}.$$
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