Box-constrained monotone $L_\infty$-approximations and Lipschitz-continuous regularized functions*

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March 21, 2019

Abstract

Let $f : [0, 1] \rightarrow [0, 1]$ be a nondecreasing function. The main goal of this work is to provide a regularized version, say $\tilde{f}_L$, of $f$. Our choice will be a best $L_\infty$-approximation to $f$ in the set of functions $h : [0, 1] \rightarrow [0, 1]$ which are Lipschitz-continuous, for a fixed Lipschitz norm bound $L$, and verify the boundary restrictions $h(0) = 0$ and $h(1) = 1$. Our findings allow to characterize a solution through a monotone best $L_\infty$-approximation to the Lipschitz regularization of $f$. This is seen to be equivalent to follow the alternative way of the average of the Pasch-Hausdorff envelopes. We include results showing stability of the procedure as well as directional differentiability of the $L_\infty$-distance to the regularized version. This problem is motivated within a statistical problem involving trimmed versions of distribution functions as to measure the level of contamination discrepancy from a fixed model.

Keywords: Contamination neighbourhoods, Kolmogorov distance, uniform norm, Lipschitz-continuous approximations, distribution functions, trimmed probabilities, Pasch-Hausdorff envelopes, Lipschitz regularization, robustness, directional differentiability.

A.M.S. classification: PRIMARY: 49J30. SECONDARY: 26A16, 62G35, 41A29.

1 Introduction.

Let us briefly motivate the problem. In his seminal paper [6], Huber introduced the contamination neighbourhood of a probability, becoming one of the very basis of Robust Statistics. An ($\alpha$-) contamination neighbourhood of a probability distribution $P_0$ is the set of probability distributions

$$V_\alpha(P_0) = \{(1 - \alpha)P_0 + \alpha Q : Q \in \mathcal{P}\},$$

where $\mathcal{P}$ is the set of all probability distributions in the space. Although it can be defined in a wholly general setting, throughout the paper $\mathcal{P}$ will be the set of probabilities on the (Borel) sets, $\beta$, of the real line $\mathbb{R})$. In this way, given an “ideal” model $P_0$, the vicinity includes those probabilities which are distorted versions of the model through gross or rounding errors: Given a particular value $\alpha_0 \in [0, 1)$, a probability $P$ in $V_{\alpha_0}(P_0)$ would generate samples with an approximate $(1 - \alpha_0) \times 100$ percentage of data coming from $P_0$. A dual point of view would consider that a such sample could be suitably “trimmed” as to obtain a right sample from the model. In fact, even $P_0$ would arose from an appropriate trimming of $P$.

The introduction of general trimmings of a probability goes back at least to [5]. A probability $\tilde{P} \in \mathbb{R}$ is said to be a trimming of level $\alpha \in [0, 1)$ of $P$ whenever there exists a down-weighting function $w$ such that $0 \leq w \leq 1$ and $\tilde{P}(B) = \frac{1}{1 - \alpha} \int_B w(x) P(dx)$ for all the sets $B \in \beta$. Equivalently,
it must be absolutely continuous w.r.t. \( P \), with Radon-Nykodim derivative bounded by \( \frac{1}{1-\alpha} \). The set of \( \alpha \)-trimmings of the probability distribution \( P \) will be denoted by \( R_\alpha(P) \):
\[
R_\alpha(P) = \{ \tilde{P} \in \mathcal{P} : \tilde{P} \ll P, \frac{d\tilde{P}}{dP} \leq \frac{1}{1-\alpha} \text{ P-a.s.} \},
\]
and the key link between (1) and (2), obtained in [2], is given by
\[
P \in \mathcal{V}_\alpha(P_0) \iff P_0 \in R_\alpha(P).
\]
The drawback of both approaches is that in a realistic statistical setting we know neither the value \( \alpha \) nor the “contaminated” distribution \( P \). We just dispose of an approximation \( \tilde{P} \) to \( P \); usually \( \tilde{P} \) will be the sample distribution associated to a data set, and our goal is to search for statistical evidence, on the basis of \( \tilde{P} \) for or against the hypothesis \( P \in \mathcal{V}_\alpha(P_0) \). For such task we can resort to a metric, \( d \), on \( \mathcal{P} \) and consider \( d(P_0, R_\alpha(\tilde{P})) \) as an estimator of \( d(P_0, R_\alpha(P)) \). With the introduction of this distance, adjusting the \( \alpha \) trim level accordingly, we measure to what extent our distribution, \( P \), can be considered as a contaminated version of the model, \( P_0 \). The success of this strategy will strongly depend on the suitability of the metric for this task. Our choice here is the Kolmogorov distance, \( d_K \), that for two probabilities \( P, Q \in \mathcal{P} \) is defined by the \( L_\infty \)-distance between their distribution functions \( F_P \) and \( F_Q \). In Section 2 we will present some alternative characterization of the set \( R_\alpha(P) \) as well as its main topological properties in this setting. In particular we will show (see Lemma 2.4) that, \( P \in \mathcal{V}_\alpha(P_0) \iff P_0 \in R_\alpha(P) \). The success of this strategy will strongly depend on the suitability of the metric for this task. Our choice here is the Kolmogorov distance, \( d_K \), that for two probabilities \( P, Q \in \mathcal{P} \) is defined by the \( L_\infty \)-distance between their distribution functions \( F_P \) and \( F_Q \). In Section 2 we will present some alternative characterization of the set \( R_\alpha(P) \) as well as its main topological properties in this setting. In particular we will show (see Lemma 2.4) that, by defining \( \Gamma = F_0(F^{-1}) \), \( F_0 \) and \( F \) being the distribution functions of \( P_0 \) and \( P \), with great generality, the following identity holds:
\[
d_K(P_0, R_\alpha(P)) = \min\{\|h - \Gamma\|, h \in \mathcal{C}_\alpha\}, \tag{4}
\]
where \( \mathcal{C}_\alpha := \{h : [0,1] \to [0,1] \text{ with } h(0) = 0, h(1) = 1, \text{ and } \|h\|_{Lip} \leq 1/1 - \alpha\} \). \tag{5}
Here, as will be used throughout, for any real valued mapping \( f : \mathbb{R} \to \mathbb{R} \) defined on a metric space \( (\mathbb{R}, d) \), with \( \|f\| \) and \( \|f\|_{Lip} \) we will denote the \( L_\infty \) and the Lipschitz norms:
\[
\|f\| = \sup_{x \in \mathbb{R}} |f(x)|, \quad \|f\|_{Lip} = \sup_{x, y \in \mathbb{R}} \frac{|f(x) - f(y)|}{d(x, y)}.
\]
Therefore, (1) translates the problem to finding a useful expression for a best \( L_\infty \)-approximation to a monotone function by Lipschitz-continuous functions verifying the boundary conditions \( h(0) = 0, h(1) = 1 \). This goal should include a computably feasible characterization of (4), as to be used for statistical purposes. Both objectives will be obtained in Theorem 2.5, although the proof will be given through Section 3. There, we will show how the Pasch-Hausdorff envelopes (see [7]) of a monotone function preserve monotonicity and provide the basis to build a best \( L_\infty \)-approximation verifying the boundary constraints. We will also relate this process with the alternative way of obtaining the Ubhaya’s monotone \( L_\infty \)-best approximation (see [9, 10]) to the Lipschitz regularization of the objective function. This approach is followed in Section 4. Finally, we must highlight our results on stability of the constrained regularization (see Proposition 2.2) as well as on directional differentiability of the \( L_\infty \)-distance to the regularized version (see Theorem 4.3), where the last approach is better suited. The relevance of this type of results on differentiability has been pointed out in [8], and recently highlighted in relation with statistical applications in [4]. In fact, these results provide a sound mathematical foundation allowing incoming statistical applications of the proposed methodology.

2 The set of trimmings in the \( L_\infty \)-topological setting

Since probabilities on \( (\mathbb{R}, \beta) \) are determined by their distribution functions (d.f.’s in the sequel) and (1) and (2) can be equivalently stated in terms of the corresponding distribution functions, we will use the same notation \( R_\alpha(F) \) and \( \mathcal{V}_\alpha(F_0) \), with the same meanings as before, but defined in terms of distribution functions. On the other hand, the Kolmogorov distance between probabilities is defined
just through the $L_\infty$-distance between the corresponding d.f.'s, but we will often keep the notation $d_K$ for this distance.

The set $R_\alpha(F)$ can be also characterized, as shown in [1] (see also Proposition 2.2 in [2] for a more general result), in terms of the set of $\alpha$-trimmed versions of the uniform probability $U(0,1)$. Notice that this set is just $C_\alpha$, as defined in [3]. The parameterization, obtained through the composition of the functions $h$ and $F$: $F_h = h \circ F$ gives

$$R_\alpha(F) = \{F_h : h \in C_\alpha\}.$$  \hspace{1cm} (6)

We note that, as a consequence, the “trimmed Kolmogorov distance” from $F$ to $F_0$ is

$$d_K(F_0, R_\alpha(F)) := \inf_{\tilde{F} \in R_\alpha(F)} \|\tilde{F} - F_0\| = \inf_{h \in C_\alpha} \|h \circ F - F_0\|.$$  

The set $R_\alpha(F)$ is convex and also well behaved w.r.t. weak convergence of probabilities and widely employed probability metrics (see Section 2 in [2]). We show next that this also holds for $d_K$.

**Proposition 2.1** For $\alpha \in (0,1)$ and distribution functions $F$, $F_0, F_1, F_2, G_1$ and $G_2$, we have:

(a) $R_\alpha(F)$ is compact w.r.t. $d_K$.

(b) $d_K(F_0, R_\alpha(F)) = \min_{\tilde{F} \in R_\alpha(F)} \|\tilde{F} - F_0\| = \min_{h \in C_\alpha} \|h \circ F - F_0\|$.

(c) $|d_K(G_1, R_\alpha(F_1)) - d_K(G_2, R_\alpha(F_2))| \leq \|G_1 - G_2\| + \frac{1}{1-\alpha}\|F_1 - F_2\|$.

**Proof.**  
By the Ascoli-Arzelà Theorem, $C_\alpha$ is a compact subset of the space of continuous functions on $[0,1]$ endowed with the uniform norm. Hence, from any sequence of elements in $R_\alpha(F)$, say $\{h_n \circ F\}$ (recall [3]), we can extract a uniformly convergent subsequence $h_{n_j} \rightarrow h_0 \in C_\alpha$. But then, obviously, $h_{n_j} \circ F \rightarrow h_0 \circ F$ in $d_K$, which proves (a). Since, on the other hand,

$$\|h_1 \circ F - F_0\| - \|h_2 \circ F - F_0\| \leq \|h_1 \circ F - h_2 \circ F\| \leq \|h_1 - h_2\|,$$

we see that the map $h \mapsto \|h \circ F - F_0\|$ is continuous and, consequently, it attains its minimum in $R_\alpha(F)$, as claimed in (b). Finally, to check (c) we note that

$$|d_K(G_1, R_\alpha(F_1)) - d_K(G_1, R_\alpha(F_2))| \leq \sup_{h \in C_\alpha} \|G_1 - h \circ F_1\| - \|G_1 - h \circ F_2\|$$  \hspace{1cm} (7)

$$\leq \sup_{h \in C_\alpha} \|h \circ F_1 - h \circ F_2\| \leq \frac{1}{1-\alpha}\|F_1 - F_2\|$$

and

$$|d_K(G_1, R_\alpha(F_2)) - d_K(G_2, R_\alpha(F_2))| \leq \sup_{h \in C_\alpha} \|G_1 - h \circ F_2\| - \|G_2 - h \circ F_2\| \leq \|G_1 - G_2\|.$$  \hspace{1cm} (8)

Now, (7) and (8) yield (c). \hspace{1cm} $\blacksquare$

Proposition 2.1 guarantees the existence of optimal $L_\infty$-approximations to every distribution function $F_0$ by $\alpha$-trimmed versions of $F$:

There exists $\tilde{F} \in R_\alpha(F)$ such that $\|F_0 - \tilde{F}\| = d_K(F_0, R_\alpha(F))$. \hspace{1cm} (9)

It also shows, through [3], that for $\alpha \in [0,1)$

$$F \in V_\alpha(F_0) \quad \text{if and only if} \quad d_K(F_0, R_\alpha(F)) = 0.$$  \hspace{1cm} (10)

Moreover, by convexity of $R_\alpha(F)$, the set of optimally trimmed versions of $F$ associated to problem (9) is also convex. However, guarantying uniqueness of the minimizer (as it holds w.r.t. $L_2$-Wasserstein metric by Corollary 2.10 in [2]) is not possible here.

An additional consequence of Proposition 2.1 is the continuity of $d_K(F_0, R_\alpha(F))$ in $F_0$ and $F$. We quote this and some additional facts in our next result.
Proposition 2.2 For $\alpha \in [0,1]$, if $\{F_n\}$ and $F$ are d.f.’s such that $d_K(F_n, F) \to 0$, then:

a) for every $F \in R_\alpha(F)$, there exist $F_n \in R_\alpha(F_n), n \in \mathbb{N}$ such that $d_K(F_n, F) \to 0$.

b) if $F_n \in R_\alpha(F_n), n \geq 1$, then there exists some $d_K$-convergent subsequence $\{F_{n_k}\}$. If $F$ is the limit of such a subsequence, necessarily $F \in R_\alpha(F)$.

c) if, additionally, $\{G_n\}$ and $G$ are d.f.’s such that $d_K(G_n, G) \to 0$, then $d_K(G_n, R_\alpha(F_n)) \to d_K(G, R_\alpha(F))$ as $n, m \to \infty$.

Proof. To prove a), since $F = h \circ F$, with $h \in C_\alpha$, it suffices to consider $F_n := h \circ F_n \in R_\alpha(F_n)$ and recall that $h$ is Lipschitz. For b), we write $F_n = h_n \circ F_n$ and argue as in the proof of Proposition 2.1 to get a $d_K$-convergent subsequence $h_{n_k} \to h \in C_\alpha$ from which we easily get $d_K(h_{n_k} \circ F_{n_k}, h \circ F) \to 0$. Finally c) is a direct consequence of Proposition 2.1(c). □

By Polya’s uniform convergence theorem, if $F$ and $G$ are continuous and $\{F_n\}, \{G_n\}$ are sequences of d.f.’s which, respectively, weakly converge to $F, G$, then they also converge in the $d_K$-sense, therefore $d_K(G_n, R_\alpha(F_n)) \to d_K(G, R_\alpha(F))$ holds. Also, a direct application of the Glivenko-Cantelli theorem and item c) above guarantee the following strong consistency result.

Proposition 2.3 Let $\alpha \in [0,1)$ and $\{F_n\}$ be the sequence of empirical d.f.’s based on a sequence $\{X_n\}$ of independent random variables with distribution function $F$. If $\{G_n\}$ is any sequence of distribution functions $d_K$-approximating the d.f. $G$ (i.e. $d_K(G_n, G) \to 0$), then:

$$d_K(G_n, R_\alpha(F_n)) \to d_K(G, R_\alpha(F)), \text{ as } n, m \to \infty, \text{ with probability one.}$$

Given a d.f. $F$, we write $F^{-1}$ for the associated quantile function (or left continuous inverse function), namely, $F^{-1}(t) := \inf\{x \mid t \leq F(x)\}$. We recall that if $U$ is a uniformly distributed $U(0,1)$ random variable, $F^{-1}(U)$ has d.f. $F$. Similarly, if $X$ has a continuous d.f. $F$, the composed function $F_0 \circ F^{-1}$ is the quantile function associated to the r.v. $Y = F_0(X)$ . As we show next, under some regularity assumptions $d_K(F_0, R_\alpha(F))$ can be expressed in terms of the function $F_0 \circ F^{-1}$. We will see later the usefulness of this fact both for the asymptotic analysis and the practical computation of $d_K(F_0, R_\alpha(F))$ when $F_n$ is an empirical d.f. based on a data sample $x_1, \ldots, x_n$. Recall that then $F_n(x) := \frac{1}{n} \sum_{i=1}^{n} I_{(\sim \infty, x]}(x_i)$.

Lemma 2.4 Let $\alpha \in [0,1)$. If $F, F_0$ are continuous d.f.’s and $F$ is additionally strictly increasing then

$$d_K(F_0, R_\alpha(F)) = \min_{h \in C_\alpha} \|h - F_0 \circ F^{-1}\| \text{ and } d_K(F_0, R_\alpha(F_n)) = \min_{h \in C_\alpha} \|h - F_0 \circ F_n^{-1}\|.$$ 

Proof. For the first identity observe that

$$\|h \circ F - F_0\| = \sup_{x \in \mathbb{R}} |h(F(x)) - F_0(x)| = \sup_{F(x) \in [0,1]} |h(F(x)) - F_0(F^{-1}(F(x)))|
= \sup_{t \in [0,1]} |h(t) - F_0(F^{-1}(t))| = \|h - F_0(F^{-1})\|.$$

On the other hand, if $x(i), i = 1, \ldots, n$, denote the ordered sample associated to $x_1, \ldots, x_n$ (the same set of values but ordered in nondecreasing sense) and

$$t_0 = 0, \quad t_i = \frac{i}{n}, \quad h_i = h(F_n(x(i))) = h(t_i), \quad \text{and} \quad F_{0,i} = F_0(x(i)), \quad 1 \leq i \leq n.$$

Taking into account that $h(F_n)$ and $F_0(F_n^{-1})$ are piecewise constant while $F_0$ and $h$ are non decreasing and continuous, we obtain

$$\|h(F_n) - F_0\| = \max_{1 \leq i \leq n} \max \left( F_{0,i} - h_{i-1}, h_i - F_{0,i} \right) = \|h - F_0(F_n^{-1})\|,$$
and the other identity follows from Proposition 2.1 part (b).

Our final result in this section provides a simple representation of \( \min_{h \in C_\alpha} \| h - F_0 \circ F^{-1} \| \) (hence, of \( d_K(F_0, R_\alpha(F)) \)). In this statement we assume that \( \Gamma \) is a nondecreasing function taking values in \([0,1]\) (which is always the case if \( \Gamma = F_0 \circ F^{-1} \)). Note that taking right and left limits at 0 and 1, respectively, we can assume that \( F_0 \circ F^{-1} \) is a nondecreasing (and left continuous) function from \([0,1]\) to \([0,1]\).

**Theorem 2.5** Let \( \alpha \in [0,1] \). Assume \( \Gamma : [0,1] \to [0,1] \) is a nondecreasing function. Define \( G(t) = \Gamma(t) - \frac{t}{1-\alpha} \), \( U(t) = \sup_{t \leq s \leq 1} G(s) \), \( L(t) = \inf_{0 \leq s \leq t} G(s) \) and

\[
\tilde{h}_\alpha(t) = \max \left( \min \left( \frac{U(t) + L(t)}{2}, 0 \right), \frac{-\alpha}{1-\alpha} \right).
\]

Then,

\[
\min_{h \in C_\alpha} \| h - \Gamma \| = \| \tilde{h}_\alpha - \Gamma \|.
\]

The proof of this result will be developed in Section 3. In fact Theorem 2.3 is just a rephrasing of this result. A look at that Theorem shows that \( h_\alpha = \tilde{h}_\alpha + \frac{\alpha}{1-\alpha} \) is an element of \( C_\alpha \) such that \( \| h_\alpha - \Gamma \| = \min_{h \in C_\alpha} \| h - \Gamma \| \), that is, \( h_\alpha \) is an optimal trimming function in the sense described above. We recall that we do not claim uniqueness of this minimizer, but this particular choice allows to compute \( d_K(F_0, R_\alpha(F)) \) for sample d.f.'s. Moreover, Theorem 2.5 even provides a simple way for the computation of \( d_K(F_0, R_\alpha(F)) \) for theoretical distributions. Let us see an illustration of this use.

**Example 2.1 (Trimmed Kolmogorov distances in the Gaussian model.)** Consider the case \( F_0 = \Phi, F = \Phi(\cdot - \mu)/\sigma \), where \( \Phi \) denotes the standard normal d.f., \( \mu \in \mathbb{R} \) and \( \sigma > 0 \). Here we have \( H^{-1}(t) := F_0 \circ F^{-1}(t) = \Phi(\mu + \sigma F^{-1}(t)) \). We note that \( w(t) := (H^{-1})'(t) \leq 1/(1 - \alpha) \) if and only if \( p(\Phi^{-1}(t)) \geq 0 \), where

\[
p(x) = (\sigma^2 - 1)x^2 + 2\mu x + \mu^2 - 2 \log((1 - \alpha)\sigma).
\] (11)

To avoid cumbersome computations we focus on the cases \( \sigma = 1, \mu \neq 0 \) and \( \mu = 0, \sigma \neq 1 \).

If \( \sigma = 1 \) and \( \mu > 0 \) then \( p \) is linear with positive slope and we see that \( w(t) \leq 1/(1 - \alpha) \) if and only if \( t \geq t_0 := \Phi(\frac{\mu}{\sigma} + \frac{1}{\mu} \log(1 - \alpha)) \). This means that \( G(s) = H^{-1}(s) - s/(1 - \alpha) \) is increasing in \([0,t_0]\) and decreasing in \([t_0,1]\). Since, \( H^{-1}(0) = 0 \), we have that, \( \tilde{h}_\alpha(t) = 0 \) for \( t \in [0,t_1] \), where \( t_1 \in (t_0,1) \) is the (unique) solution to \( G(t_1) = 0 \), and \( \tilde{h}_\alpha(t) = G(t) \) for \( t \in [t_1,1] \). We conclude that \( d_K(R_\alpha(N(\mu,1)), N(0,1)) = G(t_0) \). The case \( \mu < 0 \) can be handled similarly to obtain

\[
d_K(R_\alpha(N(\mu,1)), N(0,1)) = \Phi \left( \frac{|\mu|}{\sigma} + \frac{1}{|\mu|} \log(1 - \alpha) \right) - \frac{1}{1-\alpha} \Phi \left( -\frac{|\mu|}{\sigma} + \frac{1}{|\mu|} \log(1 - \alpha) \right), \quad \mu \neq 0.
\] (12)

We focus now on the case \( \mu = 0 \). If \( \sigma^2 < 1 \), \( p \) is a parabola with negative leading coefficient and discriminant \( \Delta^2 = 8(\sigma^2 - 1)\log(\sigma(1 - \alpha)) > 0 \). Hence, \( p(x) \) is positive for \( x \in (x_a,x_b) \) with

\[
x_a = -\frac{\Delta}{2(\sigma^2 - 1)}, \quad x_b = \frac{\Delta}{2(\sigma^2 - 1)}.
\]

Equivalently, \( w(t) \leq 1/(1 - \alpha) \) if and only if \( t_a := \Phi(x_a) \leq t \leq t_b := \Phi(x_b) \). This means that \( G \) is increasing in \([0,t_a]\), decreasing in \([t_a,t_b]\), increasing in \([t_b,1]\), \( G(0) = 0 \) and \( G(1) = -\alpha/(1 - \alpha) \). Arguing as before, we have \( \tilde{h}_\alpha(t) = \min(G(t),0) \) for \( 0 \leq t \leq \frac{1}{2}, \tilde{h}_\alpha(t) = \max(G(t),-\frac{\alpha}{1-\alpha}) \) for \( \frac{1}{2} \leq t \leq 1 \), \( \tilde{h}_\alpha(t_a) = 0 \) and \( \tilde{h}_\alpha(t_b) = -\frac{\alpha}{1-\alpha} \). We conclude that

\[
d_K(R_\alpha(N(\mu,\sigma^2)), N(0,1)) = G(t_a) - \tilde{h}_\alpha(t_a) = \tilde{h}_\alpha(t_b) - G(t_b).
\]

Hence,

\[
d_K(R_\alpha(N(0,\sigma^2)), N(0,1)) = \Phi \left( \frac{\sigma \Delta}{\sigma^2 - 1} \right) - \frac{1}{1-\alpha} \Phi \left( \frac{\Delta}{\sigma^2 - 1} \right), \quad \text{if } \sigma < 1.
\]

If \( 1 \leq \sigma \leq 1/(1 - \alpha) \) then we have that \( w(t) \leq 1/(1 - \alpha) \) for all \( t \) and \( h_0 = H^{-1} \in C_\alpha \). In particular, \( d_K(R_\alpha(N(0,\sigma^2)), N(0,1)) = 0 \).

Finally, we consider the case \( \sigma > 1/(1 - \alpha) \). In this case \( p \) is positive for \( x \not\in [x_a,x_b] \) with

\[
x_a = -\frac{\Delta}{2(\sigma^2 - 1)}, \quad x_b = \frac{\Delta}{2(\sigma^2 - 1)}.
\]

This means that \( (H^{-1})'(t) > \frac{1}{1-\alpha} \) for \( t \in (t_a,t_b) \) with \( t_a = \Phi(x_a), t_b = \Phi(x_b) \).
\[ \Phi(x_b). \] Therefore, \( G \) is decreasing in \([0, t_a]\), increasing in \([t_a, t_b]\), decreasing in \((t_b, 1]\), \( G(0) = 0 \) and \( G(1) = -\alpha/(1 - \alpha) \). Hence, \( \tilde{h}_\alpha(t) = \max(G(t), \frac{G(t) + G(t_a)}{2}, 0 \leq t \leq t_a, \tilde{h}_\alpha(t) = \frac{G(t_a) + G(t_b)}{2}, t_a \leq t \leq t_b, \tilde{h}_\alpha(t) = \min(G(t), \frac{G(t) + G(t_b)}{2}). \) \( t_b \leq t \leq 1. \) In particular, \( d_K(R_\alpha(N(0, \sigma^2)), N(0, 1)) = h_\alpha(t_a) - G(t_a) = G(t_b) - \tilde{h}_\alpha(t_b) = \frac{1}{2}(G(t_b) - G(t_a)), \) that is,

\[
d_K(R_\alpha(N(0, \sigma^2)), N(0, 1)) = \Phi\left(\frac{\sigma \Delta}{\sigma \Delta + 1}\right) - \Phi\left(\frac{\sigma \Delta}{\sigma \Delta - 1}\right), \quad \text{if } \sigma > \frac{1}{1 - \alpha}.
\]

\[ \Box \]

3 Best \( L_\infty \) -approximations by Lipschitz-continuous functions with box constraints

In this section we refresh the notation. The role of \( 1/(1 - \alpha) \) will be played now by a generic Lipschitz constant \( L \); our \( \Gamma \) will be substituted by a bounded function \( f : \mathbb{R} \to \mathbb{R} \), where \((\mathbb{R}, d)\) is (at least at the beginning) a general metric space, while we maintain \([0, 1]\) as the range of values. We will also use the notation \( x \vee y \) (resp. \( x \wedge y \)) for the maximum (resp. minimum) of both numbers (or functions). Regarding the Lipschitz norm, recall the trivial inequalities

\[ \| f \wedge g \|_{\text{Lip}}, \| f \vee g \|_{\text{Lip}} \leq \| f \|_{\text{Lip}} \vee \| g \|_{\text{Lip}}. \quad (13) \]

The first lemma collects some basic properties on the role of the Pasch-Hausdorff envelopes of a function to obtain a Lipschitz-continuous best \( L_\infty \)-approximation with constrained Lipschitz constant. For the sake of completeness, we will also include a simple proof.

Lemma 3.1 For a function \( f : \mathbb{R} \to [0, 1] \), given a constant \( L \geq 0 \), let us consider

\[ f_{L,1}(x) := \inf_{y \in \mathbb{R}}(f(y) + Ld(x, y)), \quad f_{L,2}(x) := \sup_{y \in \mathbb{R}}(f(y) - Ld(x, y)). \]

(i) This defines functions \( f_{L,1}, f_{L,2} : \mathbb{R} \to \mathbb{R} \) such that \( 0 \leq f_{L,1} \leq f_{L,2} \leq 1 \).

(ii) \( f_{L,1} \) is the pointwise largest function \( g : \mathbb{R} \to \mathbb{R} \) satisfying \( g \leq f \) and \( \| g \|_{\text{Lip}} \leq L \). Likewise \( f_{L,2} \) is the pointwise smallest function \( g : \mathbb{R} \to \mathbb{R} \) satisfying \( g \geq f \) and \( \| g \|_{\text{Lip}} \leq L \).

(iii) The average \( f_L := (f_{L,1} + f_{L,2})/2 \) satisfies \( \| f_L \|_{\text{Lip}} \leq L \) and

\[ \| g - f_L \| \geq \| f_L - f \| = \| f_{L,2} - f_{L,1} \| \]

for any function \( g : \mathbb{R} \to \mathbb{R} \) such that \( \| g \|_{\text{Lip}} \leq L \).

Proof. Part (i) follows directly from the definitions of \( f_{L,1} \) and \( f_{L,2} \), because, for every \( x \in \mathbb{R} \):

\[ \inf_{y \in \mathbb{R}} f(y) \leq f_{L,1}(x) \leq f(x) + Ld(x, x) = f(x) = f(x) - Ld(x, x) \leq f_{L,2}(x) \leq \sup_{y \in \mathbb{R}} f(y). \]

To address part (ii) observe that, for arbitrary \( x_1, x_2, y \in \mathbb{N} \), the triangle inequality for the distance implies \( |Ld(x_1, y) - Ld(x_2, y)| \leq Ld(x_1, x_2) \), leading to the inequalities

\[ |f_{L,j}(x_2) - f_{L,j}(x_1)| \leq Ld(x_1, x_2) \quad \text{for } j = 1, 2, \]

thus to \( \| f_{L,j} \|_{\text{Lip}} \leq L, j = 1, 2. \) Now, if \( g : \mathbb{R} \to \mathbb{R} \) satisfies \( g \leq f \) and \( \| g \|_{\text{Lip}} \leq L \), then for \( x, y \in \mathbb{R} \):

\[ g(x) \leq g(y) + Ld(x, y) \]

with equality if \( x = y \). Hence

\[ g(x) = \inf_{y \in \mathbb{R}} (g(y) + Ld(x, y)) \leq \inf_{y \in \mathbb{R}} (f(y) + Ld(x, y)) = f_{L,1}(x). \]
Analogously, it follows from \( g \geq f \) and \( \|g\|_{\text{Lip}} \leq L \) that \( g \geq f_{L,2} \), proving (ii).

As to part (iii), let \( \epsilon := \|g - f\| \). Then \( \|g + \epsilon\|_{\text{Lip}} = \|g\|_{\text{Lip}} \) and \( g - \epsilon \leq f \leq g + \epsilon \). Consequently, by part (ii),

\[
g - \epsilon \leq f_{L,1} \leq f \leq f_{L,2} \leq g + \epsilon
\]

This implies that

\[
\|f_L - f\| = (f - f_L) \vee (f_L - f) \leq (f_{L,2} - f_L) \vee (f_L - f_{L,1}) = \frac{f_{L,2} - f_{L,1}}{2} \leq \epsilon,
\]

whence

\[
\|f_L - f\| \leq \frac{\|f_{L,2} - f_{L,1}\|}{2} \leq \|g - f\|.
\]

Since \( \|f_L\|_{\text{Lip}} \leq \|f_{L,1}\|_{\text{Lip}} / 2 + \|f_{L,2}\|_{\text{Lip}} / 2 \leq L \), taking \( g = f_L \) gives the announced equality \( \|f_L - f\| = \|f_{L,2} - f_{L,1}\| / 2 \). \( \square \)

When \( \mathbb{R} \) is a real interval and \( f \) is non-decreasing, the functions \( f_{L,1} \) and \( f_{L,2} \) in Lemma 6.1 share also that property and can be alternatively expressed in terms of the Ubhaya’s monotone envelopes of the function \( f(x) - Lx \). This is the content of the following lemma.

**Lemma 3.2** Let \( \mathbb{R} \) be a real interval, equipped with the usual distance \( d(x, y) = |x - y| \). If \( f : \mathbb{R} \to [0, 1] \) is non-decreasing, then the functions \( f_{L,1}, f_{L,2} \) in Lemma 3.1 are non-decreasing too, and for arbitrary \( x \in \mathbb{R} \) and \( j = 1, 2 \),

\[
f_{L,j}(x) = \gamma_{L,j}(x) + Lx,
\]

where \( \gamma_{L,j}, j = 1, 2 \) are the non-increasing functions

\[
\gamma_{L,1}(x) := \inf_{y \in \mathbb{R} : y \leq x} (f(y) - Ly) \quad \text{and} \quad \gamma_{L,2}(x) := \sup_{y \in \mathbb{R} : y \geq x} (f(y) - Ly).
\]

In particular,

\[
\|f_{L,2} - f_{L,1}\| = \|\gamma_{L,2} - \gamma_{L,1}\| = \sup_{y, x \in \mathbb{R} : y \leq x} (f(x) - f(y) - L(x - y)). \tag{14}
\]

**Proof.** The representations of \( f_{L,1} \) and \( f_{L,2} \) in terms of \( \gamma_{L,1} \) and \( \gamma_{L,2} \) follow from the fact that for arbitrary \( x, y \in \mathbb{R} \),

\[
\begin{align*}
f(y) + Ld(x, y) & = \begin{cases} f(y) + L(x - y) = f(y) - Ly + Lx & \text{if } y \leq x \\ \geq f(x) = f(x) - Lx + Lx & \text{if } y \geq x,
\end{cases} \\
f(y) - Ld(x, y) & = \begin{cases} f(y) - L(y - x) = f(y) - Ly + Lx & \text{if } y \geq x \\ \leq f(x) = f(x) - Lx + Lx & \text{if } y \leq x,
\end{cases}
\end{align*}
\]

where the inequalities follow from \( f \) being non-decreasing. Note that both functions \( \gamma_{L,1} \) and \( \gamma_{L,2} \) are non-increasing, but adding the term \( Lx \) to them leads to non-decreasing functions: For \( x_1, x_2 \in \mathbb{R} \) with \( x_1 < x_2 \), isotonicity of \( f \) implies that

\[
f_{L,2}(x_1) = \sup_{y \geq x_2} (f(y) - Ly + Lx_1) \vee \sup_{x_1 \leq y \leq x_2} (f(y) - Ly + Lx_1)
\]

\[
\leq (f_{L,2}(x_2) - Lx_2 + Lx_1) \vee f(x_2)
\]

\[
\leq f_{L,2}(x_2),
\]

and

\[
f_{L,1}(x_2) = \inf_{y \leq x_1} (f(y) - Ly + Lx_2) \wedge \sup_{x_1 \leq y \leq x_2} (f(y) - Ly + Lx_2)
\]

\[
\geq (f_{L,1}(x_2) + Lx_2 - Lx_1) \wedge f(x_1)
\]

\[
\geq f_{L,1}(x_1),
\]

\[7\]
Similarly, if \( f \) is non-decreasing, we have

\[
\{ f(x) + L(1-x) - 1 \leq f(x) + L(1-x) - 1 \}
\]

which is non-decreasing and verifies \( \tilde{f}_L(0) = 0 \) and \( \tilde{f}_L(1) = 1 \) and \( \| f_L \|_{\text{Lip}} \leq L \), and for arbitrary functions \( g : [0,1] \to \mathbb{R} \) with \( g(0) = 0 \) and \( g(1) = 1 \) and \( \| g \|_{\text{Lip}} \leq L \).

**Proof.** Let us begin noting that both expressions for \( \tilde{f}_L \) are trivially equivalent from the relations between \( \gamma_{L,j} \) and \( f_{L,j} \).

That \( \tilde{f}_L \) verifies the required properties easily follows from the preceding lemmas (recall also inequalities (13)). Let then \( g : [0,1] \to \mathbb{R} \) with \( \| g \|_{\text{Lip}} \leq L \). Also by the precedent lemmas,

\[
\| g - f \| \geq \| f_L - f \| = \sup_{0 \leq y, x \leq 1} (f(x) - f(y) - L(x-y))/2.
\]

Under the additional constraint that \( g(0) = 0 \), for arbitrary \( x \in [0,1] \),

\[
f(x) - g(x) = f(x) - (g(x) - g(0)) \geq f(x) - Lx,
\]

whence

\[
\| g - f \| \geq \sup_{0 \leq x \leq 1} (f(x) - Lx) = f_{L,2}(0).
\]

Analogously, the additional constraint \( g(1) = 1 \) implies that

\[
f(x) - g(x) = f(x) + (g(1) - g(x)) - 1 \leq f(x) + L(1-x) - 1,
\]

whence

\[
-\| g - f \| \leq \inf_{0 \leq x \leq 1} (f(x) + L(1-x)) - 1 = f_{L,1}(1) - 1.
\]

These considerations show that for any function \( g : [0,1] \to \mathbb{R} \) verifying the conditions \( g(0) = 0 \), \( g(1) = 1 \) and \( \| g \|_{\text{Lip}} \leq L \),

\[
\| g - f \| \geq \| f_L - f \| \lor f_{L,2}(0) \lor (1 - f_{L,1}(1)).
\]

The function \( \tilde{f}_L \) satisfies the previous constraints on \( g \), too, so

\[
\| \tilde{f}_L - f \| \geq \| f_L - f \| \lor f_{L,2}(0) \lor (1 - f_{L,1}(1)).
\]

It remains to prove the reverse inequality. For \( x \in [0,1] \), we have to distinguish three cases: If \( 1 - L + Lx \leq f_L(x) \leq Lx \), then \( \tilde{f}_L(x) = f_L(x) \), so \( |\tilde{f}_L(x) - f(x)| \leq \| f_L - f \| \). If \( f_L(x) > Lx \), then \( \tilde{f}_L(x) = Lx \), and

\[
f(x) - \tilde{f}_L(x) = \begin{cases} f(x) - Lx \leq f_{L,2}(0), \\ > f(x) - f_L(x) \geq -\| f_L - f \|. 
\end{cases}
\]

Similarly, if \( f_L(x) < 1 - L + Lx \), then \( \tilde{f}_L(x) = Lx \), and

\[
f(x) - \tilde{f}_L(x) = \begin{cases} f(x) + L(1-x) - 1 \geq f_{L,1}(1) - 1, \\ < f(x) - f_L(x) \leq \| f_L - f \|. 
\end{cases}
\]
Let $\gamma_{L,j} = f_{L,j} - Lx$ hold, all the functions $f_{L,j}, \gamma_{L,j}$ are absolutely continuous so $\{\gamma_L \leq 1 - L\}$, $\{\gamma_L \geq 0\}$, $\{\gamma_L \in [1 - L, 0]\}$ are compact sets and continuous functions attain their maximum values on these sets. This allows to get alternative expressions for (15) as given in the following theorem. We note that here and throughout we use the convention that the max over an empty set equals $-\infty$.

**Theorem 3.4** Let $f : [0, 1] \rightarrow [0, 1]$ be non-decreasing and continuous and assume the notation in Theorem 3.3. Then the following alternative expressions for (15) hold:

$$\|f - \tilde{f}_L\| = \max \left( \max_{x \in T_1} (f(x) - Lx), \max_{x \in T_2} (1 - L + Lx - f(x)), \frac{1}{2} \max_{1 - L \leq \gamma_L(x) \leq 0} (\gamma_{L,2}(x) - \gamma_{L,1}(x)) \right)$$  \hspace{1cm} (16)

$$\|f - \tilde{f}_L\| = \max \left( \max_{x \in T_1} (f(x) - Lx), \max_{x \in T_2} (1 - L + Lx - f(x)), \frac{1}{2} \max_{(y,x) \in T_3} (f(x) - f(y) - L(x - y)) \right)$$  \hspace{1cm} (17)

Here, we used the notation $T_1 = \{x \in [0, 1] : \gamma_L(x) \geq 0\}$, $T_2 = \{x \in [0, 1] : \gamma_L(x) \leq 1 - L\}$, $T_3 = \{(y,x) : 0 \leq y \leq x \leq 1, 1 - L \leq \frac{1}{2}(f(y) + f(x)) - L(y + x) \leq 0\}$.

Once we know Theorem 3.3, a proof of this result would take advantage of the fact that the right-hand side in (16) is upper bounded by the same expression with the unrestricted maxima, which, by (14) is just the right-hand side in (15) when $f$ is continuous. However, with some additional effort we can obtain a more general result that does not require the monotonicity assumption on the objective function and opens a way to address the directional differentiability of the functional $f \rightarrow \|f - \tilde{f}_L\|$. Both goals will be carried through the following section.

### 4 Best $L_\infty$-approximations by monotone functions with box constraints

The following theorem gives appropriate characterizations of the best approximation of a bounded function (in uniform norm) by monotone functions with a box constraint. Without this constraint, the best approximation by monotone functions in the $L_\infty$-norm has been considered in [9, 10], with results that cover the case $A = -\infty, B = \infty$ in Theorem 4.1 below. Notice that this theorem, based on Uhhaya’s envelopes, would also provide an (arguably more involved) alternative proof for Theorem [3,3]. Notice that the function $G$ plays the role of the transformed function, $f(x) - Lx$ (the difference of two nondecreasing functions) in the previous section, while the scope here is general.

**Theorem 4.1** Assume $G : [0, 1] \rightarrow \mathbb{R}$ is a bounded function and $-\infty \leq A \leq B \leq \infty$. Define $U(x) = \sup_{x \leq y \leq 1} G(y)$, $L(x) = \inf_{0 \leq y \leq x} G(y)$, $\tilde{G}(x) = (L(x) + U(x))/2$ and $G_{A,B} = \max(\min(G(x), B), A)$.

Then $U, L, \tilde{G}$ and $G_{A,B}$ are nonincreasing, $L(x) \leq G(x) \leq U(x)$ and for every nonincreasing $h : [0, 1] \rightarrow [A, B]$ we have

$$\|G - G_{A,B}\| \leq \|G - h\|.$$  \hspace{1cm} (18)

Furthermore, if $G$ is continuous then $U, L, \tilde{G}$ and $G_{A,B}$ are also continuous and

$$\|G - G_{A,B}\| = \max \left( \max_{G(x) \geq B} (G(x) - B), \max_{G(x) \leq A} (A - G(x)), \frac{1}{2} \max_{A \leq G(x) \leq B} (U(x) - L(x)) \right) \hspace{1cm} (19)$$

where $T_1 = \{x \in [0, 1] : \tilde{G}(x) \geq B\}$, $T_2 = \{x \in [0, 1] : \tilde{G}(x) \leq A\}$ and $T_3 = \{(y,x) : 0 \leq y \leq x \leq 1, A \leq \frac{1}{2}(G(y) + G(x)) \leq B\}.$
Proof. The bounds $L(x) \leq G(x) \leq U(x)$ are obvious, and also the fact that $U$ and $L$ are nonincreasing (hence, also $G$ and $\bar{G}_{A,B}$).

• Next, consider some nonincreasing $h : [0,1] \to [A,B]$ and $x \in [0,1]$. Since $L(x) \leq G(x) \leq U(x)$, we have that $G(x) = \bar{G}(x)$ whenever $U(x) = L(x)$. Hence, if $U(x) = L(x) \in [A,B]$ we have $\bar{G}_{A,B}(x) = G(x)$ and, consequently,

$$0 = |\bar{G}_{A,B}(x) - G(x)| \leq \|h - G\|.$$ 

• Obviously, $\bar{G}_{A,B}(x) = B$ if $U(x) = L(x) > B$ and we still have that 

$$|\bar{G}_{A,B}(x) - G(x)| \leq |h(x) - G(x)| \leq \|h - G\|$$ 

and similarly for the case $U(x) = L(x) < A$.

• It remains to deal with the case $U(x) > L(x)$. For every $\varepsilon > 0$ there exist $x_a \in [0,x]$, $x_b \in [x,1]$ such that $G(x_a) < L(x) + \varepsilon$ and $G(x_b) > U(x) - \varepsilon$. If $\bar{G}(x) > B$ then $\bar{G}_{A,B}(x) = B$. Using again that $L(x) \leq G(x) \leq U(x)$ we see that $|\bar{G}_{A,B}(x) - G(x)| \leq U(x) - B < G(x_b) - B + \varepsilon \leq |G(x_b) - h(x_b)| + \varepsilon$ for small enough $\varepsilon$, showing that $|\bar{G}_{A,B}(x) - G(x)| \leq \|h - G\|$.

Similarly, if $\bar{G}(x) < A$ we conclude that $|\bar{G}_{A,B}(x) - G(x)| \leq \|h - G\|$.

Finally, assume that $U(x) > L(x)$ and $\bar{G}(x) \in [A,B]$. Since $h$ is nonincreasing we have that $h(x_a) \geq h(x_b)$ and, consequently,

$$\|h - G\| \geq \max(|h(x_a) - G(x_a)|, |h(x_b) - G(x_b)|) \geq \frac{G(x_b) - G(x_a)}{2} \geq |\bar{G}_{A,B}(x) - G(x)| - 2\varepsilon$$

for $\varepsilon$ small enough. This completes the proof of (15).

To check continuity of $U$ note that for $0 \leq y < x \leq 1$ $U(y) = \max(U(x), \max_{y \leq z \leq x} G(z))$. Now, given $\varepsilon > 0$ we can fix $\delta > 0$ such that $|G(x) - G(y)| \leq \varepsilon$ whenever $|y - x| \leq \delta$. But then $|U(y) - U(x)| \leq \varepsilon$ if $|y - x| \leq \delta$, proving continuity of $U$. $L$ can be handled similarly. As a consequence we see that $\bar{G}$ and $\bar{G}_{A,B}$ are also continuous.

Now, to prove the first equality in the statement we take $x \in [0,1]$ and consider first the case $x \in T_1$. Note that, necessarily, $U(x) \geq B$, $U(x) - B \geq B - L(x)$ and $\bar{G}_{A,B}(x) = B$.

• If $\bar{G}(x) \geq B$ then $|\bar{G}(x) - \bar{G}_{A,B}(x)| = G(x) - B$.

• Assume, on the contrary, that $\bar{G}(x) < B$. Set $x_+ = \inf\{y \leq x : G(y) = U(x)\}$. By continuity, $\bar{G}(x_+) = U(x_+)$.

Now, if $\bar{G}(x_+) \geq B$ then $\bar{G}(x_+) - B = U(x_+) - B \geq B - L(x) \geq B - G(x) = |G(x) - \bar{G}_{A,B}(x)|$. If, on the contrary, $\bar{G}(x_+) < B$, then there exists $x' \in [x,x_+]$ such that $\bar{G}(x') \in (A,B)$. But we must have $U(x') = U(x) = U(x_+)$ and $L(x') < L(x)$ and, consequently, we have that

$$|G(x) - \bar{G}_{A,B}(x)| = B - G(x) \leq B - L(x) \leq \frac{U(x) - L(x)}{2} < \frac{U(x') - L(x')}{2}.$$ 

Summarizing, we see that

$$\max_{G(x) \geq B} |G(x) - \bar{G}_{A,B}(x)| \leq \max_{G(x) \geq B} (G(x) - B), \frac{1}{2} \max_{A \leq G(x) \leq B} (U(x) - L(x)).$$ 

(20)

Similarly,

$$\max_{G(x) \leq A} |G(x) - \bar{G}_{A,B}(x)| \leq \max_{G(x) \leq A} (A - G(x)), \frac{1}{2} \max_{A \leq G(x) \leq B} (U(x) - L(x)).$$ 

(21)

and, obviously, if $G(x) \in [A,B]$ then $\bar{G}_{A,B}(x) = G(x)$ and $|G(x) - \bar{G}_{A,B}(x)| \leq \frac{1}{2}(U(x) - L(x))$, which implies that

$$\max_{A \leq G(x) \leq B} |G(x) - \bar{G}_{A,B}(x)| \leq \frac{1}{2} \max_{A \leq G(x) \leq B} (U(x) - L(x)).$$ 

(22)
Now combining (20), (21) and (22) we see that

\[ \|G - \tilde{G}_{A,B}\| \leq \max \left( \max_{G(x) \geq B} (G(x) - B), \max_{G(x) \leq A} (A - G(x)), \frac{1}{2} \max_{A \leq G(x) \leq B} (U(x) - L(x)) \right). \]

Assume now that \( x_0 \) is such that \( \tilde{G}(x_0) \geq B \). Then \( \tilde{G}_{A,B}(x_0) = B \) and \( G(x_0) - B \leq |G(x_0) - \tilde{G}_{A,B}(x_0)| \). This implies \( \max_{G(x) \geq B} (G(x) - B) \leq \|G - \tilde{G}_{A,B}\| \).

Similarly, \( \max_{G(x) \leq A} (A - G(x)) \leq \|G - \tilde{G}_{A,B}\| \).

Finally, suppose \( x_0 \) is such that \( \tilde{G}(x_0) \in [A, B] \) and

\[ U(x_0) - L(x_0) = \max_{G(x) \in [1, B]} (U(x) - L(x)) \geq \max \left( \max_{G(x) \geq B} (G(x) - B), \max_{G(x) \leq A} (A - G(x)) \right). \]

- If \( U(x_0) = L(x_0) \) then

\[ \|G - \tilde{G}_{A,B}\| = \max \left( \max_{G(x) \geq B} (G(x) - B), \max_{G(x) \leq A} (A - G(x)), \frac{1}{2} \max_{A \leq G(x) \leq B} (U(x) - L(x)) \right) = 0. \]

- If \( U(x_0) > L(x_0) \) then we set \( x_+ = \inf \{ y \in [x_0, 1] : G(y) = U(x_0) \} \). Then \( U(y) = U(x_0) \) for \( y \in [x_0, x_+] \) and

\[ G(x_+) = U(x_+) = U(x_0). \]

Set \( x_- = \sup \{ y \in [0, x_0] : G(y) = L(x_0) \} \). We have \( L(y) = L(x_0) = G(x_-) \) for \( y \in [x_-, x_0] \). We claim that

\[ L(y) = L(x_0) \quad \text{for} \quad y \in [x_0, x_+]. \] (23)

To check (23) note that, if \( \tilde{G}(x_0) > A \) and (23) fails then we could find \( y \in [x_0, x_+] \) with \( L(y) < L(x_0) \), \( \tilde{G}(y) \in (A, B) \) and \( U(y) - L(y) > U(x_0) - L(x_0) \), while if \( \tilde{G}(x_0) = A \) and (23) fails then \( G(y) < L(x_0) \) for some \( y \in (x_0, x_+) \), \( \tilde{G}(y) < A \) and \( A - L(y) > A - L(x_0) = \frac{1}{2}(U(x_0) - L(x_0)) \), against the assumption on \( x_0 \).

Hence, from (23) we conclude that \( \tilde{G}(x_+) = \tilde{G}(x_0) \in [A, B] \) and \( |\tilde{G}(x_+) - \tilde{G}_{A,B}(x_+)| = \frac{1}{2}(U(x_0) - L(x_0)) \), showing that \( \frac{1}{2}(U(x_0) - L(x_0)) \leq \|G - \tilde{G}_{A,B}\| \). Combining the last estimates we see that the first equality in (19) holds.

For the second identity we note that arguing as above we see that \( U(x_0) - L(x_0) = G(x) - G(y) \) for some \( (y, x) \in \mathcal{T}_3 \) if \( \tilde{G}(x_0) \in [A, B] \). Assume, on the other hand, that \( (y_0, x_0) \in \mathcal{T}_3 \) satisfies

\[ \frac{1}{2}(G(x_0) - G(y_0)) \geq \max \left( \max_{G(x) \geq B} (G(x) - B), \max_{G(x) \leq A} (A - G(x)) \right). \]

- We consider first the case \( \frac{1}{2}(G(y_0) + G(x_0)) \in (A, B) \).

We claim that \( U(x_0) = G(x_0) \) since, otherwise, there exists \( x' > x_0 \) such that \( \frac{1}{2}(G(y_0) + G(x')) \in (A, B) \) and \( G(x') > G(x_0) \) and this would imply \( G(x') - G(y_0) > G(x_0) - G(y_0) \), contradicting maximality of \( (y_0, x_0) \). Similarly we see that \( G(y_0) = L(y_0) \) and also that \( L(x) = L(y_0) \) for \( x \in [y_0, x_0] \). Hence, \( G(x_0) - G(y_0) = U(x_0) - L(x_0) \) and \( \tilde{G}(x_0) \in (A, B) \).

- In the case \( \frac{1}{2}(G(y_0) + G(x_0)) = B \) we have that necessarily \( G(x_0) \geq B \) and, arguing as above, we see that \( G(y_0) = L(y) \) for all \( y \in [y_0, x_0] \). This implies that \( G(x_0) \geq B \) and \( \frac{1}{2}(G(x_0) - G(y_0)) = G(x_0) - B \).

- Arguing similarly for the case \( \frac{1}{2}(G(y_0) + G(x_0)) = A \) we conclude that the second equality in (19) holds.\[ \square \]
Remark 4.2 The sets of optimizers within $T_1, T_2$ and $T_3$ in Lemma 4.1 play an important role in the next results. For convenience, we denote $T_1 = \{x_0 \in T_1 : G(x_0) = G_{A,B}(x_0)\}$, $T_2 = \{x_0 \in T_2 : A - G(x_0) = \|G_{A,B}(x_0)\|\}$ and $T_3 = \{y_0, x_0 \in T_3 : \frac{1}{2}(G(x_0) - G(y_0)) = \|G_{A,B}(x_0)\|\}$. A look at the proof of Lemma 4.1 shows that if $x_0 \in T_1$ then $G$ has a local maximum at $x_0$ and a local minimum if $x_0 \in T_2$. Also, if $(y_0, x_0) \in T_3$ then $G$ has a local maximum at $x_0$ and a local minimum at $y_0$.

Our next result addresses the directional differentiability of the functional $G \rightarrow \|G - G_{A,B}\|$ that appeared in the last theorem. This kind of result typically allows to obtain efficiency and asymptotic distributional behaviour of functionals in the statistical setting (see e.g. [1]).

Theorem 4.3 Assume $G, J : [0,1] \rightarrow \mathbb{R}$ are continuous functions and $r_n > 0$ is a sequence of real numbers such that $r_n \rightarrow \infty$. Define $G_n = G + \frac{1}{r_n}$ and consider $G_n, G_{A,B}$ as in Theorem 4.1 and $G_{A,B,n}$ built in the same way as $G_{A,B}$ but from $G_n$. Assume further that $T_1, T_2$ and $T_3$ are as in Remark 4.2 and that there is no $x \in T_1$ with $G(x) = B$, no $x \in T_2$ with $G(x) = A$ and no $(y,x) \in T_3$ with $\frac{1}{2}(G(x) + G(y)) \in \{A,B\}$. Then

$$r_n(\|G_n - G_{A,B,n}\| - \|G - G_{A,B}\|) \rightarrow \max \left( \max_{x \in T_1} J(x), \max_{x \in T_2} (-J(x)) \right) \frac{1}{2} \max_{(y,x) \in T_3} (J(x) - J(y)).$$

Proof. We use the notation $U, L$ from Theorem 4.1 and write $U_n, L_n, G_n, T_{n,i}$ for the corresponding objects coming from $G_n$. Observe that $\|U_n - U\| \leq \|J\|/r_n \rightarrow 0$ and similarly, $\|G_n - G\| \rightarrow 0$. Assume that $x \in T_1$. By assumption and the last convergence we have that $G_n(x) > B$ for large enough $n$ and, therefore, $\|G_n - G_{A,B,n}\| \geq (G_n(t) - B)$. But this implies

$$r_n(\|G_n - G_{A,B,n}\| - \|G - G_{A,B}\|) \geq r_n((G_n(x) - B) - (G(x) - B)) = J(x).$$

Arguing similarly for $T_2$ and $T_3$ we conclude that

$$\liminf r_n(\|G_n - G_{A,B,n}\| - \|G - G_{A,B}\|) \geq \max \left( \max_{x \in T_1} J(x), \max_{x \in T_2} (-J(x)) \right) \frac{1}{2} \max_{(y,x) \in T_3} (J(x) - J(y)).$$

For the upper bound assume $x_n \in T_{n,1}$ (that is, $x_n \in T_{n,1}$ such that $G_n(x_n) - B = \|G_n - G_{A,B,n}\|$). By compactness, taking subsequences if necessary, we can assume that $x_n \rightarrow x_0$ for some $x_0 \in [0,1]$ with $G(x_0) \geq B$ and $G(x_0) - B = \|G - G_{A,B}\|$. But this means that $x_0 \in T_1$. Hence, by assumption $G(x_0) > B$ and, consequently, $G(x_n) > B$ for large enough $n$. In this case $\|G - G_{A,B}\| \geq (G(x_n) - B)$, which implies that

$$r_n(\|G_n - G_{A,B,n}\| - \|G - G_{A,B}\|) \leq r_n((G_n(x_n) - B) - (G(x_n) - B)) = J(x_n) \rightarrow J(x_0).$$

With the same argument applied to $T_2$ and $T_3$ we conclude that

$$\limsup r_n(\|G_n - G_{A,B,n}\| - \|G - G_{A,B}\|) \leq \max \left( \max_{x \in T_1} J(x), \max_{x \in T_2} (-J(x)) \right) \frac{1}{2} \max_{(y,x) \in T_3} (J(x) - J(y))$$

and complete the proof. □

Specializing the last results for $G(x) = f(x) - Lx$, where $f$ is nondecreasing, $L \geq 1$ a constant, and $A = 1 - L$, $B = 0$, we can obtain a first result on the directional differentiability of the functional $f \rightarrow \|f - \tilde{f}_L\|$ considered in Section 3. Note that now, recovering the notation in that section, the relevant sets are $T_1, T_2$ and $T_3$ as defined in Theorem 4.4 and $T_1 = \{x_0 \in T_1 : f(x_0) - Lx_0 = \|f - \tilde{f}_L\|\}$, $T_2 = \{x_0 \in T_2 : 1 - L + Lx_0 - f(x_0) = \|f - \tilde{f}_L\|\}$ and $T_3 = \{y_0, x_0 \in T_3 : \frac{1}{2}(f(x_0) - f(y_0) - L(x_0 - y_0)) = \|f - \tilde{f}_L\|\}$. Theorem 4.4 translates then to the following immediate corollary.
Corollary 4.4 (Directional differentiability.) Let \( f, f_n : [0, 1] \to \mathbb{R} \) be nondecreasing functions, \( r_n > 0 \) a sequence of real numbers such that \( r_n \to \infty \) and \( r_n (f_n - f) \to J \) pointwise, where \( J : [0, 1] \to \mathbb{R} \) is a continuous function. Assume further that \( f \) is continuous, that \( T_1, T_2 \) and \( T_3 \) are as above and that there is no \( x \in T_1 \) with \( \gamma_L(x) = 0 \), no \( x \in T_2 \) with \( \gamma_L(x) = 1 - L \) and no \( (y, x) \in T_3 \) with \( \frac{1}{2} (f(x) + f(y) - L(x + y)) \in \{1 - L, 0\} \). Let \( \tilde{f}_{n,L}, \tilde{f}_L \) respectively denote the best \( L_\infty \)-approximations to \( f_n \) and \( f \) by Lipschitz-continuous functions \( h : [0, 1] \to \mathbb{R} \) with \( \|h\|_{\text{Lip}} \leq L \) and verifying \( h(0) = 0, h(1) = 1 \), as in Theorem 3.3. Then

\[
\begin{align*}
    r_n (\|f_n - \tilde{f}_{n,L}\| - \|f - \tilde{f}_L\|) & \to \max \left( \max_{x \in T_1} J(x), \max_{x \in T_2} (-J(x)), \frac{1}{2} \max_{(y, x) \in T_3} (J(x) - J(y)) \right).
\end{align*}
\]

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