COVARIANT REPRESENTATIONS OF $C^*$-ALGEBRAS AND THEIR COMPACT AUTOMORPHISM GROUPS

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ABSTRACT. Let $G$ be a compact group. Let $(\Gamma, \mu)$ be a standard Borel $G$-measure space. We show that the group action on $(\Gamma, \mu)$ is transitive if and only if it is ergodic. Using this result, we show that every irreducible covariant representation of a $C^*$-dynamical system $(A, G, \sigma)$ is induced from a stability group. In addition, we show that $(A, G, \sigma)$ satisfies strong-EHI.

1. INTRODUCTION

Let $G$ be a locally compact group, $A$ a $C^*$-algebra, and $\sigma$ a point-wise norm continuous homomorphism of $G$ into the automorphism group of $A$ then we call the triple $(A, G, \sigma)$ a $C^*$-dynamical system. Given a $C^*$-dynamical system we can construct the crossed product $C^*$-algebra $A \times_\sigma G$ that encodes the action of $G$ on $A$. It is well known that there exists a one to one correspondence between the set all covariant representations of the system $(A, G, \sigma)$ and the set of all $*$-representations of $A \times_\sigma G$. Therefore, the study of representations of $A \times_\sigma G$ is equivalent to that of covariant representations of $(A, G, \sigma)$.

Our goal is to study induced covariant representations of systems involving compact groups. The study of induced representations was initiated by Mackey in [9, 10] in the context of unitary representations of locally compact groups. Using Mackey’s approach Takesaki extended the theory to crossed products in [13]. Subsequently, Rieffel recast that theory in terms of Hilbert modules and Morita equivalence with [12]. It follows from Proposition 5.4 in [14] that the construction of induced representations for crossed products by Rieffel is equivalent to that of Takesaki.

The importance of induced representations arises from the fact that the fundamental structure of a crossed product $A \times_\sigma G$ is reflected in the structure of the orbit space for the $G$-action on Prim $A$ together with the sub-systems $(A, G_P, \sigma)$ where $G_P$ is the stability group at $P \in$ Prim $A$. In particular, one gets a complete description of the primitive ideal space and its topology for transformation group $C^*$-algebra $C_0(X) \times_H G$ when $G$ is

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abelian. In many important cases we also get a characterization of when $A \times_\sigma G$ is GCR or CCR. Williams presents all these results and more in his book [14].

Although induced representations have been studied extensively there remains a considerable gap in the theory. We outline below two questions for which answers are not known. Using structure theorems obtained in this paper we give a positive answer to both questions in the case of separable $C^*$-dynamical systems with compact groups.

One of the key ingredients in building the connection between Prim $A \times_\sigma G$ and the $G$-action on Prim $A$ is establishing that every primitive ideal of $A \times_\sigma G$ is induced from a stability group ([14]; p.235). The latter result was conjectured by Effros and Hahn, and systems for which the conjecture holds are called EH-regular. The proof that the Effros-Hahn conjecture holds is due Gootman, Rosenberg and Sauvageot and it is one of the major results in the theory (see Chapters 8 and 9 in [14] for the proof of the GRS theorem and its applications). There exists a stronger notion of EH-regularity namely the requirement that every irreducible representation of $A \times_\sigma G$ is induced from a stability group. The latter requirement is known to hold for many dynamical systems ([14]; Theorem 8.16) but the general case, to our knowledge, remains open.

Another natural question that arises when studying induced representations of a system $(A, G, \sigma)$ is when an irreducible representation of a sub-system $(A, G_P, \sigma)$ induces to an irreducible representation of $(A, G, \sigma)$. Following the nomenclature proposed by Echterhoff and Williams in [4], we say that $(A, G, \sigma)$ satisfies strong Effros-Hahn Induction Property (strong-EHI), if, for each primitive ideal $P$ of $A$ and a covariant irreducible representation $(\pi, U)$ of $(A, G_P, \sigma)$ such that $\ker(\pi) = P$ the corresponding induced representation of $(A, G, \sigma)$ is irreducible. A very nice summary of the results regarding the (strong)-EHI property can be found in [4].

In this paper we use Takesaki’s approach to the theory of induced representations for crossed products. As in [13] we will often assume basic countability conditions although most of the results in Section 3 do hold in greater generality. If $G$ is a second countable, locally compact group acting on a separable $C^*$-algebra $A$ then we call $(A, G, \sigma)$ a separable system.

In Section 2, we give the background about topological and Borel dynamical systems necessary for Section 3. In Section 3, we study Borel dynamical systems. In particular, we prove that if $G$ is a compact group and $(\Gamma, \mu)$ is an ergodic standard Borel $G$-measure space then $G$ acts transitively on $(\Gamma, \mu)$. Note that the last statement is not true in general. For instance, the action of $\mathbb{Z}$ on $\mathbb{T}$ by an irrational rotation is ergodic but it is not transitive.
In Section 4, we study covariant representation \((\pi, U)\) of a system \((A, G, \sigma)\) on a Hilbert space \(\mathcal{H}\). Given a covariant representation \((\pi, U)\) and a system of imprimitivity \(A\) for \((\pi, U)\) there exists an essentially unique standard Borel \(G\)-measure space \((\Gamma, \mu)\) such that \(L^\infty(\Gamma, \mu)\) is isomorphic to \(A\). If \(G\) acts ergodically on \(A\) then the corresponding action on \((\Gamma, \mu)\) is also ergodic (\cite{11}; Theorem 3). In particular, by the result mentioned in the previous paragraph \(G\) acts transitively on \((\Gamma, \mu)\) and we can identify the space \(\Gamma\) with the right coset space \(G_0/G\) for appropriate closed subgroup \(G_0\) of \(G\). We then build the induced covariant representation following Mackey’s construction (\cite{13}; Theorem 4.2). Our key result in this section is Theorem 10 regarding covariant factor representations of \(C^*\)-dynamical system with compact groups. This theorem extends a similar result in the context of finite groups obtained by Arias and Latremoliere (\cite{2}; Theorem 3.4). As a corollary of Theorem 10 we show that every irreducible representation of \((A, G, \sigma)\) is induced from a stability group.

In Section 5, we study covariant irreducible representation \((\pi, U)\) of \((A, G_P, \sigma)\) such that \(\ker(\pi) = P\), where \(P \in \text{Prim } A\) and show that in this case the representation \(\pi\) of \(A\) must be homogeneous. As a corollary, we get that \((A, G, \sigma)\) satisfies the strong-EHI property.

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2. Background

Suppose that \(G\) is a topological (resp. Borel) group; that is, \(G\) is a topological (resp. Borel) space and a group such that the map \((s, t) \in G \times G \mapsto s^{-1}t \in G\) is continuous (resp. Borelian). When \(G\) is a topological group, \(G\) is often considered as a Borel group equipped with the Borel structure determined by its topology. Let \(\Gamma\) be a topological (resp. Borel) space. Suppose that an anti homomorphism of \(G\) into the group of all homeomorphisms (resp. Borel-automorphisms) of \(\Gamma\) is given, denoting the homeomorphism (resp. Borel-automorphism) of \(\Gamma\) corresponding to \(s \in G\) by \(\gamma \in \Gamma \mapsto \gamma \cdot s \in \Gamma\). If the map: \((\gamma, s) \in \Gamma \times G \mapsto \gamma \cdot s \in \Gamma\) is continuous (resp Borelian), then \(\Gamma\) is said to be a topological (resp. Borel) \(G\)-space. By a measure \(\mu\) on a Borel space \(\Gamma\), we shall mean a complete measure determined by a \(\sigma\)-finite measure on the Borel sets of \(\Gamma\). For each \(s \in G\) define a measure \(s(\mu)\) on \(\Gamma\) by \(s(\mu)(E) = \mu(E \cdot s)\). We say that \(\mu\) is quasi-invariant if \(s(\mu)\) is equivalent to \(\mu\) for each \(s \in G\) and we call the measure space \((\Gamma, \mu)\) a \(G\)-measure space.
If a quasi-invariant measure $\mu$ on a Borel $G$-space $\Gamma$ satisfies the condition that $\mu(E) = 0$ or $\mu(\Gamma - E) = 0$ for every Borel set $E$ of $\Gamma$ with $\mu(E \triangle (E \cdot s)) = 0$ for every $s \in G$, then $\mu$ is said to be ergodic. Given a unital $C^*$-dynamical system $(A, G, \sigma)$ we say the group action is ergodic if the only fixed elements of $A$ under the group action are the scalars. Similarly, given a $W^*$-dynamical system $(A, G, \tau)$ (defined at the beginning of section 3) we say that action of $G$ on the von Neumann algebra $A$ is ergodic if the only fixed elements are the scalars. Note that if $(\Gamma, \mu)$ is a standard Borel $G$-measure space such that the corresponding action of $G$ on $L^\infty(\Gamma, \mu)$ given by $(\tau_s f)(\gamma) = f(\gamma \cdot s)$ is continuous in the strong operator topology (SOT) we can form a $W^*$-dynamical system $(L^\infty(\Gamma, \mu), G, \tau)$. In this case, $(\Gamma, \mu)$ is an ergodic $G$-measure space if and only if the action of $G$ on $L^\infty(\Gamma, \mu)$ is ergodic.

Remark 1. The continuity on the measure space can be expressed by the requirement that $\mu(E \triangle (E \cdot s)) \to 0$ as $s \to e$ for each measurable set $E$ with $\mu(E) < \infty$ ([3]; p.285).

Let $(\Gamma, \mu)$ be a Borel $G$-measure space. For each $\gamma \in \Gamma$ define $O_\gamma = \{\gamma \cdot s : s \in G\}$ to be the orbit of $\gamma$ under the group action. If there is $\gamma \in \Gamma$ such that $\mu(\Gamma - O_\gamma) = 0$ then $(\Gamma, \mu)$ is said to be transitive. Clearly transitivity implies ergodicity. As mentioned in the introduction the converse is not true in general.

3. Ergodic Actions of Compact Groups

Let $G$ be a locally compact group, $A$ a von Neumann algebra, and $\tau$ a point-wise SOT-continuous homomorphism of $G$ into the automorphism group of $A$ then we call the triple $(A, G, \tau)$ a $W^*$-dynamical system. It is well known that given a $W^*$-dynamical system $(A, G, \tau)$ the set $A^c$ of $x \in A$ such that the function $s \mapsto \tau_s x$ is norm continuous is a $G$-invariant $C^*$-algebra and it is $\sigma$-weakly dense in $A$ ([3]; Proposition III.3.2.4). Since $A^c$ is unital it follows from the Double Commutant Theorem that $A^c$ is SOT-dense in $A$. Using the proof of the Spectral Theorem we will show that $A^c$ is equivalent to a space of continuous functions on a second countable compact Hausdorff space.

Lemma 2. Let $N$ be a masa on a Hilbert space $K$ and $\zeta \in K$ be a cyclic separating vector for $N$. Suppose $M$ is a unital $C^*$-subalgebra of $N$ such that $M \zeta = K$. Then there exists a compact Hausdorff space $Y$ and a finite Borel measure $\nu$ and a unitary $V : K \to L^2(Y, \nu)$ such that $V M V^* = L^\infty(Y, \nu)$ and $V M V^* = C(Y)$.

Proof. Let $\rho : M \to C(Y)$ be the Gelfand isomorphism. Define a positive linear functional $\phi$ on $M$ by $\phi(x) = \langle x \zeta, \zeta \rangle$. Then there is a finite positive
Borel measure \( \nu \) on \( Y \) such that

\[
\phi(x) = \int_Y \rho(x)d\nu
\]

for all \( x \in M \).

Let \( \pi_\phi : M \to \mathcal{B}(L^2(Y, \nu)) \) be the corresponding GNS representation with \( 1_Y \) as the cyclic vector. Since \( \zeta \) is a separating vector then the map \( V : M\zeta \to \pi_\phi(M)1_Y \) given by \( V(x\zeta) = \pi_\phi(x)1_Y \) is well defined. Clearly, \( V \) is an isometry. Hence we can extend \( V \) to a unitary from \( \mathcal{K} \) onto \( L^2(Y, \nu) \).

Moreover, \( \pi_\phi(x) = VxV^* \) for all \( x \in M \) so that \( VMV^* = \pi_\phi(M) = C(Y) \). To see that \( VNV^* = L^\infty(Y, \nu) \) let \( x_1 \in M \) and \( x_2 \in N \) then

\[
(Vx_1V^*)(Vx_2V^*) = (Vx_2V^*)(Vx_1V^*).
\]

So \( (Vx_2V^*) \subseteq (VMV^*)' = (C(Y))' = L^\infty(Y, \nu) \). Conversely, if \( T \in L^\infty(Y, \nu) \subseteq (VMV^*)' \) then \( T(VxV^*) = (VxV^*)T \), for all \( x \in N \). So \( x(V^*TV) = (V^*TV)x \), for all \( x \in N \). Thus \( V^*TV \in N' = N \) and \( T = V(V^*TV)V^* \in VN^* \).

\[\square\]

Let \((\Gamma, \mu)\) be a standard Borel \( G \)-measure space where the group action is continuous in the appropriate sense. Consider the corresponding \( W^* \)-dynamical system \((L^\infty(\Gamma, \mu), G, \tau)\). Then \( L^\infty(\Gamma, \mu)^c \) is SOT-dense in \( L^\infty(\Gamma, \mu) \). Let \( \zeta \in L^2(\Gamma, \mu) \) be a cyclic, separating vector for \( L^\infty(\Gamma, \mu) \). Then we can apply Lemma 2 to \( N = L^\infty(\Gamma, \mu), M = L^\infty(\Gamma, \mu)^c \), and \( \zeta \).

**Corollary 3.** Let \( G \) be a locally compact group and let \((\Gamma, \mu)\) be a standard Borel \( G \)-measure space. Then there is a compact Hausdorff space \( Y \) together with a finite positive Borel measure \( \nu \) and a unitary \( V : L^2(\Gamma, \mu) \to L^2(Y, \nu) \) such that \( VL^\infty(\Gamma, \mu)V^* = L^\infty(Y, \nu) \) and \( VL^\infty(\Gamma, \mu)^cV^* = C(Y) \).

Consider the \( W^* \)-dynamical system \((L^\infty(Y, \nu), G, \tau')\) where \( \tau'_s(VfV^*) = V(\tau_s f)V^* \) for all \( s \in G \) and \( f \in L^\infty(\Gamma, \mu) \). Then by construction we get that \( L^\infty(Y, \nu)^c = C(Y) \). In particular, \((C(Y), G, \tau')\) is a \( C^* \)-dynamical system. Hence there is an action of \( G \) on \( Y \) so that \((Y, G)\) is a topological \( G \)-space and

\[
(\tau'_s f)(y) = f(y \cdot s)
\]

for all \( y \in Y, s \in G \) and \( f \in C(Y) \) ([14]; Proposition 2.7). We would like to show that the above equality holds for all functions in \( L^\infty(Y, \nu) \).

**Lemma 4.** In the above situation, let \( s \in G \) and \( g \in L^\infty(Y, \nu) \). Then

\[
(\tau'_s g)(y) = g(y \cdot s)
\]

for almost all \( y \in Y \).

**Proof.** Let \( g \) be in the unit ball of \( L^\infty(Y, \nu) \) then by the Kaplansky Density Theorem there is a sequence \((f_i)\) in the unit ball of \( C(Y) \) such that \( f_i \to g \) in \( \sigma \)-SOT. Since \( \nu \) is finite, a simple computation shows that there is a
subsequence $f_{i_j}$ converging to $g$ almost everywhere. So without loss of
generality we can assume that $f_i \to g$ almost everywhere. In particular,
\[ f_i(y \cdot s) \to g(y \cdot s) \]
for almost all $y \in Y$. Since automorphisms of von Neumann algebras are
$s$-strong continuous then $f_i \to g$ in $s$-SOT implies $\tau'_s f_i \to \tau'_s g$ in $s$-SOT.
By the same argument as above,
\[ (\tau'_s f_i)(y) \to (\tau'_s g)(y) \]
for almost all $y \in Y$. Since $f_i \subseteq C(Y)$ then $(\tau'_s f_i)(y) = f_i(y \cdot s)$ for all
$y \in Y$ and $i$. It follows $(\tau'_s g)(y) = g(y \cdot s)$ for almost all $y \in Y$. \(\square\)

**Corollary 5.** Let $(Y, \nu)$ be as in Lemma 4. Then $\nu$ is a quasi-invariant
measure.

**Proof.** Let $X$ be a Borel subset of $Y$. Then $\nu(X) = 0 \iff \chi_X = 0 \iff
\tau'_s(\chi_X) = 0 \iff \chi_{X \cdot s} = 0 \iff \nu(X \cdot s) = 0$. \(\square\)

Suppose the action of $G$ on $L^\infty(\Gamma, \mu)$ is ergodic then the action of $G$ on
$C(Y)$ must also be ergodic. In general, as mentioned in the introduction,
ergodic actions are far from being transitive. However, if $G$ is a compact
group the two notions coincide. To this end, we need the following fact
which was initially proved by Albeverio ([1]; Lemma 2.1) but we offer
a different proof.

**Lemma 6.** Let $G$ be a compact group. Let $X$ be a compact, Hausdorff topo-
logical $G$-space. Suppose the action of $G$ on $C(X)$ given by $(\sigma_s f)(x) = f(x \cdot s)$ is ergodic, i.e. the only $G$
invariant functions are the constant func-
tions. Then the action of $G$ on $X$ is transitive. Moreover, there exists a
closed subgroup $G_0$ of $G$ such that the right coset space $G_0/G$ with the
quotient topology is homeomorphic to $X$.

**Proof.** For each $x \in X$ define the orbit of $x$ to be $O_x = \{x \cdot s : s \in G\}$. Since
the map $s \mapsto x \cdot s$ is continuous from $G \to X$ and $G$ is compact then
$O_x$ is compact for each $x \in X$. In particular, $O_x$ is closed for each $x \in X$.

Fix $x_0 \in X$. Suppose there is $x_1 \in X - O_{x_0}$ then $O_{x_0}$ and $O_{x_1}$ are disjoint
closed subsets of $X$. By Urysohn’s Lemma there exists a continuous function $f : X \to [0, 1]$ such that $f(x_0 \cdot s) = 0$ for all $s \in G$ and $f(x_1 \cdot s) = 1$
for all $s \in G$. Define a function $g : X \to [0, 1]$ by $g(x) = \int_G f(x \cdot s)dm(s)$. We want to show that $g$ is continuous. To this end, let $\epsilon > 0$ be given;
extend $f$ to $\overline{f} : X \times G \to [0, 1]$ by defining $\overline{f}(x, s) = f(x \cdot s)$. Then $\overline{f}$ is
continuous function with compact support so we can find a finite open cover
$\{F_i \times G_i\}_{i=1}^n$ of $X \times G$ such that $|f(x \cdot s) - f(y \cdot t)| < \epsilon$ whenever $(x, s)$
and $(y, t)$ are both in $F_i \times G_i$ for some $i = 1, \ldots, n$. Given any $x \in X$ define
$F_x = \bigcap \{F_i : x \in F_i\}$. It is not hard to check that $|f(x \cdot s) - f(y \cdot s)| < \epsilon$ for
all $y \in F_x$ and $s \in G$. Then $|g(x) - g(y)| \leq \int_G |f(x \cdot s) - f(y \cdot s)|dm(s) \leq \epsilon$ for all $y \in F_x$. It follows that $g$ is continuous.

Moreover, $g$ is $G$-invariant and hence must be constant on $X$. But $g(x_0) = 0$ and $g(x_1) = 1$, contradiction. It follows that $O_{x_0} = X$.

To prove the second part of the statement let $G_{x_0} = \{ s \in G : x_0 \cdot s = x_0 \}$. Then $G_{x_0}$ is a closed subgroup of $G$ and the right coset space $G_{x_0}/G$ is compact in the quotient topology. Moreover, it is easy to see that the map $G_{x_0} \cdot s \mapsto x_0 \cdot s$ is a continuous bijection from $G_{x_0}/G$ onto $X$. Since $G_{x_0}/G$ is compact and $X$ is Hausdorff it follows that $G_{x_0}/G$ is in fact homeomorphic to $X$. □

**Corollary 7.** Let $G$ be a second countable compact group. Let $X$ be a compact, Hausdorff topological $G$-space. Suppose the action of $G$ on $C(X)$ given by $(\sigma_s f)(x) = f(x \cdot s)$ is ergodic. Then $X$ is a second countable topological space.

Applying Lemma 6 to $(C(Y), G, \tau')$ we see that $G$ acts transitively on $Y$. We are now ready to prove our main result.

**Theorem 8.** Let $G$ be a second countable, compact group. Let $(\Gamma, \mu)$ be a standard Borel $G$-measure space. Suppose the action of $G$ on $(\Gamma, \mu)$ is ergodic and the corresponding action of $G$ on $L^\infty(\Gamma, \mu)$ is SOT-continuous. Then $G$ acts transitively on $(\Gamma, \mu)$.

**Proof.** We know by Lemma 2 that there is a compact, Hausdorff space $Y$ together with a probability measure $\nu$ and a unitary $V : L^2(\Gamma, \mu) \to L^2(Y, \nu)$ such that $VL^\infty(\Gamma, \mu)V^* = L^\infty(Y, \nu)$ and $VL^\infty(\Gamma, \mu)V^* = C(Y)$. We define the action of $G$ on $L^\infty(Y, \nu)$ as in Lemma 4 then $(Y, \nu)$ becomes a Borel $G$-measure space by Corollary 5. Since $G$ is a second countable, compact group then $Y$ is a second countable topological space by Corollary 7. In particular, $(Y, \nu)$ is a standard Borel $G$-measure space.

It follows from Mackey’s Theorem 5 in [11] that there are invariant Borel subsets $Y' \subseteq Y$ and $\Gamma' \subseteq \Gamma$ and a Borel isomorphism $\theta : Y' \to \Gamma'$ such that

1. $\mu(\Gamma - \Gamma') = \nu(Y - Y') = 0$,
2. $\theta(y \cdot s) = \theta(y) \cdot s$ for all $y \in Y'$, $s \in G$.

To show that $\Gamma'$ is an orbit of $G$ let $\gamma_1, \gamma_2 \in \Gamma'$. Let $y_1, y_2 \in Y'$ such that $\theta(y_i) = \gamma_i$. By Lemma 6 we know that $G$ acts transitively on $Y'$ so there is $s \in G$ such that $y_1 \cdot s = y_2$. It follows $\gamma_2 = \theta(y_2) = \theta(y_1 \cdot s) = \theta(y_1) \cdot s = \gamma_1 \cdot s$ which completes the proof. □

4. **COVARIANT REPRESENTATIONS OF SEPARABLE $C^*$-ALGEBRAS AND THEIR COMPACT AUTOMORPHISM GROUPS**

In this section we will assume $(A, G, \sigma)$ is a separable system and all Hilbert spaces are separable. A covariant representation of $(A, G, \sigma)$ on a
Hilbert space \( \mathcal{H} \) is a pair \((\pi, U)\) where \( \pi \) is a non-degenerate representation of \( A \) on \( \mathcal{H} \) and \( U \) is a SOT-continuous homomorphism of \( G \) into the unitary group of \( B(\mathcal{H}) \) such that 
\[
U(s)\pi(a)U(s)^* = \pi(\sigma_s a)
\]
for all \( a \in A \) and \( s \in G \).

Let \( G_0 \) be a closed subgroup of \( G \) and denote \( G_0/G \) to be the corresponding right coset space endowed with the quotient topology. Let \( (\pi_0, U_0) \) be a covariant representation of \((A, G_0, \sigma)\) on a separable Hilbert space \( \mathcal{H}_0 \). Then following Mackey’s construction of induced representations we can construct a new covariant representation \((\pi, U)\) of \((A, G, \sigma)\), which is called the induced covariant representation.

In general, if \( G \) is a locally compact group then \( G_0/G \) does not always admit a \( G \)-invariant measure so the construction of induced representations for groups involves the use of a quasi-invariant measure \( \mu \) on \( G_0/G \). However, if \( G \) is a compact group there exists a unique, up to scalar multiple, \( G \)-invariant Radon measure on \( G_0/G \) ([6]; Corollary 2.51). Since the induced representation is independent, up to unitary equivalence, of the choice of the quasi-invariant measure ([9]; Theorem 2.1) the construction of the induced representation is considerably simplified.

We now describe induced covariant representations following the construction given in [13]. Let \( G_0 \) be a closed subgroup of a compact group \( G \) and let \( (\pi_0, U_0) \) be a covariant representation of \((A, G_0, \sigma)\) on a separable Hilbert space \( \mathcal{H}_0 \). Let \( \mu \) be a fixed \( G \)-invariant measure on \( G_0/G \). Let \( \mathcal{H} \) denote the induced representation space then \( \mathcal{H} \) is the space of all \( \mathcal{H}_0 \) valued functions \( \xi \) on \( G \) satisfying the following conditions:

1. \( \langle \xi(s), h_0 \rangle \) is Borel function of \( s \) for all \( h_0 \in \mathcal{H}_0 \).
2. \( \xi(ts) = U_0(t)\xi(s) \) for all \( t \in G_0 \) and all \( s \in G \).
3. \( \int_{G_0/G} \langle \xi(s), \xi(s) \rangle d\mu(\overline{s}) < \infty \).

Define \( U \) to be the homomorphism of \( G \) into the unitary group of \( B(\mathcal{H}) \) given by:
\[
(U(t)\xi)(s) = \xi(st)
\]
for all \( \xi \in \mathcal{H} \) and \( s, t \in G \). And for each \( a \in A \) define an operator \( \pi(a) \) on \( \mathcal{H} \) by:
\[
(\pi(a)\xi)(s) = \pi_0(\sigma_s a)\xi(s)
\]
for all \( \xi \in \mathcal{H} \) and \( s \in G \). Then \( (\pi, U) \) is easily checked to be a covariant representation of \((A, G, \sigma)\):
\[
U(t)\pi(a)U(t^{-1})\xi(s) = (\pi(a)U(t^{-1})\xi)(st)
= \pi_0(\sigma_{st} a)(U(t^{-1})\xi)(st)
= \pi_0(\sigma_{st} a)\xi(s) = \pi(\sigma_t a)\xi(s)
\]
for all \(s, t \in G\) and \(a \in A\). Since the \(G\)-invariant measure \(\mu\) is unique up to a scalar multiple (\cite{5}; Theorem 2.49) the induced representation is independent of the choice of the measure.

Let \((\pi, U)\) be a covariant representation of \((A, G, \sigma)\) on \(\mathcal{H}\). We say that \((\pi, U)\) is irreducible if the only operators that commute with \(\pi(a)\) and \(U(s)\) for all \(a \in A, s \in G\) are the scalars.

Following \cite{13} we define a system of imprimitivity for \((\pi, U)\) to be a commutative von Neumann algebra \(\mathbb{A}\) acting on \(\mathcal{H}\) such that:

1. \(\mathbb{A} \subseteq \pi(A)'\).
2. \(U(s)\mathbb{A}U(s)^* = \mathbb{A}\) for all \(s \in G\).

Note that (2) implies that \(G\) acts by automorphisms on \(\mathbb{A}\). Moreover, since \(U\) is assumed to be strongly continuous, then for each \(x \in \mathbb{A}\) the map \(s \mapsto U(s)xU(s)^*\) is continuous in the strong operator topology. If the only \(G\) invariant elements of \(\mathbb{A}\) are scalars then \(\mathbb{A}\) is called an ergodic system of imprimitivity. In particular, if \((\pi, U)\) is an irreducible covariant representation then \(\mathbb{A}\) is always an ergodic system of imprimitivity. Given a system of imprimitivity \(\mathbb{A}\) for \((\pi, U)\), not necessarily ergodic, there exists a standard Borel \(G\)-measure space \((\Gamma, \mu)\) and an isomorphism \(i\) of the algebra \(L^\infty(\Gamma, \mu)\) onto \(\mathbb{A}\) such that

\[
U(s)i(f)U(s)^* = i(\tau_s f)
\]

for each \(f \in L^\infty(\Gamma, \mu)\) and \(s \in G\) where \((\tau_s f)(\gamma) = f(\gamma \cdot s^{-1})\) (\cite{11}; Theorem 4). In the above situation we say that the system of imprimitivity \(\mathbb{A}\) for \((\pi, U)\) is based on the \(G\)-measure space \((\Gamma, \mu)\) with respect to \(i\). Note that the ergodicity of the system of imprimitivity \(\mathbb{A}\) is equivalent to that of the action of \(G\) on \((\Gamma, \mu)\) (\cite{11}; Theorem 3). As in \cite{13} we say that a system of imprimitivity \(\mathbb{A}\) is transitive if the corresponding Borel \(G\)-measure space is transitive. It follows from Theorem 5 in \cite{11} that the definition of transitivity is independent of the choice of \(G\)-space \((\Gamma, \mu)\). Moreover, if a system of imprimitivity for a covariant representation is transitive then by (\cite{10}; Theorem 6.1), the associated \(G\)-measure space \((\Gamma, \mu)\) can be identified with the right coset space \(G_0/G\) of a closed subgroup of \(G\) together with a \(G\)-invariant measure on \(G_0/G\).

If \(\mathbb{A}\) is an ergodic system of imprimitivity for \((\pi, U)\) on a Hilbert space \(\mathcal{H}\) then we can assume \(\mathcal{H} = L^2(\Gamma, \mu) \otimes \mathcal{H}_0\) and \(\mathbb{A} = L^\infty(\Gamma, \mu) \otimes I_{\mathcal{H}_0}\) (\cite{10}; Theorem 5.2). Moreover, the action of \(\mathbb{A}\) on \(\mathcal{H}\) is given by

\[
(i(f)\xi)(\gamma) = f(\gamma)\xi(\gamma)
\]

for all \(f \in L^\infty(\Gamma, \mu)\) and \(\xi \in L^2(\mathcal{H}_0, \Gamma, \mu)\). In addition, there exists a \(\text{Rep}(A : \mathcal{H}_0)\)-valued measurable function \(\gamma \in \Gamma \mapsto \pi_\gamma \in \text{Rep}(A : \mathcal{H}_0)\) such that

\[
(\pi(a)\xi)(\gamma) = \pi_\gamma(a)\xi(\gamma)
\]
for each $a \in A$, $\xi \in \mathcal{H}$ and almost all $\gamma \in \Gamma$. Since the action $G$ on $\mathbb{A}$ is continuous in the strong operator topology of $\mathcal{B}(\mathcal{H})$ then the corresponding action of $G$ on $L^\infty(\Gamma, \mu)$ is also continuous in the strong operator topology of $\mathcal{B}(L^2(\Gamma, \mu))$. Using Theorem 8 we obtain the following result.

Corollary 9. Let $(\pi, U)$ be an irreducible covariant representation of a separable system $(A, G, \sigma)$ where $G$ is compact. Suppose $\mathbb{A}$ is a system of imprimitivity for $(\pi, U)$ then $\mathbb{A}$ is transitive.

A natural choice for a system of imprimitivity for $(\pi, U)$ is the center of the commutant of $\pi(A)$, which we denote by $Z(\pi(A)')$. In particular, if $(\pi, U)$ is a factor representation then $Z(\pi(A)')$ is an ergodic system of imprimitivity for $(\pi, U)$. In this case, $(\pi, U)$ is particularly easy to describe. Combining Theorem 8 in Section 3 and Theorem 5.2 in [13] we obtain the following result.

Theorem 10. Let $(\pi, U)$ be a factor (resp. irreducible) representation of a separable system $(A, G, \sigma)$ where $G$ is compact. Then there exists a unique closed subgroup $G_0$ of $G$ and a unique covariant representation $(\pi_0, U_0)$ of the subsystem $(A, G_0, \sigma)$ such that $(\pi, U)$ is induced by $(\pi_0, U_0)$, where the uniqueness is up to equivalence. Moreover,

1. $(\pi_0, U_0)$ is a factor (resp. irreducible) representation.
2. $\pi_0$ is a factor representation.
3. There is an isomorphism $i : L^\infty(G_0/G, \mu) \rightarrow Z(\pi(A)')$ given by $(i(f)\xi)(s) = f(\gamma)s\xi(s)$.

Let $G$ be a finite group and $(\pi, U)$ be an irreducible representation of $(A, G, \sigma)$. Then we know by the above theorem that $(\pi, U)$ is induced from an irreducible representation $(\pi_0, U_0)$ of $(A, G_0, \sigma)$ where $\pi_0$ is a factor representation. Define an action of $G_0$ on the commutant of $\pi_0(A)$ by $\pi_0(T) = U_0(s)TU_0(s)^*$ for all $s \in G_0$ and $T \in \pi_0(A)'$. Since $G$ is finite and acts ergodically on $\pi_0(A)'$ then $\pi_0(A)'$ must be finite dimensional. It follows that $\pi_0$ is a direct sum of finitely many equivalent irreducible representations. Consequently, Theorem 10 can be viewed as a generalization of a similar result for finite groups obtained by Arias and Latremoliere (2; Theorem 3.4).

Let $P$ be a primitive ideal of $A$ and define $G_P := \{s \in G : \sigma_sP = P\}$. Note that $G_P$ is a closed subgroup of $G$. Applying Theorem 10 we get the following corollary.

Corollary 11. Let $(\pi, U)$ be an irreducible representation of $(A, G, \sigma)$. Then there exists a primitive ideal $P$ of $A$ and a covariant representation $(\pi_P, U_P)$ of the subsystem $(A, G_P, \sigma)$ such that $(\pi, U)$ is induced by $(\pi_P, U_P)$. Moreover, $\ker \pi_P = P$. 
Proof. By Theorem \[10\] there exists a closed subgroup \( G_0 \) of \( G \) and a unique covariant representation \((\pi_0, U_0)\) of the subsystem \((A, G_0, \sigma)\) such that \((\pi, U)\) is induced by \((\pi_0, U_0)\). Since \( A \) is separable and \( \pi_0 \) is a factor representation \( \ker(\pi_0) \in \text{Prim} \ A \). Let \( P := \ker \pi_0 \) then \( G_0 \subseteq G_P \). We take \((\pi_P, U_P)\) to be the representation of \((A, G_P, \sigma)\) induced by the representation \((\pi_0, U_0)\) of the subsystem \((A, G_0, \sigma)\).

Moreover, it follows from Lemma \[12\] in the next section that \( \ker \pi_P \cap \sigma_r(\ker \pi_0) = \bigcap_{r \in G_P} \sigma_r P = P \).

We note that the above corollary generalizes the GRS Theorem \[7\] in the case of compact groups.

5. Strong EHI

In this section we continue working with a separable system \((A, G, \sigma)\) where \( G \) is a compact group. Let \( \pi \) be a representation of \( A \) on a separable Hilbert space \( \mathcal{H} \). If \( E \) is a projection in the commutant \( \pi(A)' \) of \( \pi \) then we denote \( \pi^E \) to be the subrepresentation of \( \pi \) acting on \( E\mathcal{H} \).

Let \( G_0 \) be a closed subgroup of \( G \) and \((\pi_0, U_0)\) be a covariant representation of \((A, G_0, \sigma)\) on \( \mathcal{H}_0 \). Let \((\pi, U)\) be the covariant representation of \((A, G, \sigma)\) on \( \mathcal{H} \) induced by \((\pi_0, U_0)\) then there is a natural family of projections in \( \pi(A)' \) associated with Borel subsets of \( G_0/G \). Consider the map \( i : L^\infty(G_0/G, \mu) \to \pi(A)' \) given by \( (i(f))\xi(s) = f(\pi)\xi(s) \). For each nonzero Borel subset \( E \) of \( G_0/G \) we denote \( \pi^E \) to be the subrepresentation of \( \pi \) acting on \( i(\chi_E)\mathcal{H} \).

Lemma 12. In the above situation, let \( Q := \ker \pi_0 \). If \( F \) is an open subset of \( G_0/G \) then \( \ker \pi^F = \bigcap_{s \in g^{-1}(F)} \sigma_{s^{-1}}Q \).

Proof. Note that the quotient map \( q : G \to G_0/G \) is continuous and open. Let \( F \) be an open subset of \( G_0/G \) and suppose there is an \( a \in A \) such that \( a \notin \bigcap_{s \in g^{-1}(F)} \sigma_{s^{-1}}Q \) then we will show that \( \pi^F(a) \neq 0 \). To this end, let \( s \in g^{-1}(F) \) such that \( \pi_0(\sigma_s a) \neq 0 \). Choose a unit vector \( h \in \mathcal{H}_0 \) and \( \epsilon > 0 \) so that

\[
\|\pi_0(\sigma_s a)h\| \geq 2\epsilon
\]

Then as in the proof of Lemma 6.19 in \[14\] we can construct a function \( \xi \in C_0(G, \mathcal{H}_0) \subseteq \mathcal{H} \) such that

\[
\|\xi(s) - h\| \leq \epsilon/\|a\|
\]

It follows that \( \|\pi_0(\sigma_s a)\xi(s) - \pi_0(\sigma_s a)h\| \leq \|\pi_0(\sigma_s a)\| \cdot \|\xi(s) - h\| \leq \|a\| \cdot (\epsilon/\|a\|) = \epsilon \). By the reverse triangle inequality we get

\[
\|\pi_0(\sigma_s a)\xi(s)\| \geq \epsilon
\]
Since \( \pi_0(\sigma_s a) \rightarrow \pi_0(\sigma_s a) \) whenever \( s_j \rightarrow s \) and \( \xi \in C_0(G, \mathcal{H}_0) \) there exists an open neighborhood \( F_s \subseteq G_0/G \) of \( G_0s \) such that
\[
\|\pi_0(ta)\xi(t)\| > \epsilon/2
\]
for all \( t \in q^{-1}(F_s) \). Then \( \pi^E(a)(\chi_{q^{-1}(F_s \cap E)}\xi) \neq 0 \).

Clearly, \( \bigcap_{s \in q^{-1}(E)} \sigma_s^{-1}Q \subseteq \pi^E \). \( \square \)

We call \( \pi \) a homogeneous representation if \( \ker \pi^E = \ker \pi \) for every nonzero projection \( E \in \pi(A)' \). It follows from Lemma G.3 in [14] that \( \pi \) is a homogeneous representation if \( \ker \pi^E = \ker \pi \) for every nonzero projection \( E \in \pi(A)' \cap \pi(A)^P \). A structure theory developed by Effros in [5] allows us to decompose arbitrary representations into a direct integral of homogeneous representations that has very useful properties. In particular, the following result is due to Echterhoff and Williams [4]:

**Theorem 13.** Let \((A, G, \sigma)\) be a separable system. Suppose that \( \rho \) is a homogeneous representation of \( A \) with \( \ker \rho = P \), and that \( \rho \times_\sigma V \) is an irreducible covariant representation of \( A \times_\sigma G_P \). Then the representation of \( A \times_\sigma G \) induced by \( \rho \times_\sigma V \) is irreducible.

We would like to use Theorem 13 to prove the strong-EHI property for separable systems involving compact groups. To this end we prove the following theorem.

**Theorem 14.** Let \((A, G, \sigma)\) be a separable system where \( G \) is a compact group. Suppose \( P \) is a primitive ideal of \( A \) and \((\pi, U)\) is an irreducible covariant representation of \((A, G_P, \sigma)\) such that \( \ker \pi = P \). Then \( \pi \) is a homogeneous representation of \( A \).

**Proof.** Note that \( G_P \) is a closed subgroup of \( G \), so \( G_P \) is compact. We know by Theorem 10 that there exists a closed subgroup \( G_0 \) of \( G_P \) and an irreducible covariant representation \((\pi_0, U_0)\) of the subsystem \((A, G_0)\) such that \((\pi, U)\) is induced by \((\pi_0, U_0)\). Moreover, there is an isomorphism \( i : L^\infty(G_0/G_P, \mu) \rightarrow Z(\pi(A)') \) given by \( (i(f))\xi(s) = f(\pi(x))\xi(s) \). Let \( E \) be a Borel subset of \( G_0/G_P \) of nonzero measure. Let \( \mathcal{H} \) be the representation space of \((\pi, U)\) and denote \( \pi^E \) to be the subrepresentation of \( \pi \) acting on \( i(\chi_E)\mathcal{H} \). Then it is enough to show that \( \ker \pi^E = \ker \pi \).

Denote \( Q := \ker \pi_0 \). If \( F \) is an open subset of \( G_0/G_P \) denote \( F' := \{s^{-1} : s \in q^{-1}(F)\} \). Then by Lemma 12 \( \ker \pi^F = \bigcap_{s \in F'} \sigma_s Q \). Since \( G_0/G_P \) is compact there is \( \{t_j\}_{1 \leq j \leq n} \subseteq G_P \) such that \( G_P = \bigcup t_j F' \). Then \( P = \ker \pi = \bigcap_{t \in G_P} \sigma_t Q = \bigcap \sigma_j (\bigcap_{s \in F'} \sigma_s Q) = \bigcap \sigma_j (\ker \pi^F) \). Since \( P \) is a prime ideal and \( P \) is \( G_P \)-invariant it follows that \( P = \ker \pi^F \). In particular, \( \|\pi^F(a)\| = \|\pi(a)\| \) for all \( a \in A \).

Now let \( K \) be a compact subset of \( G_0/G_P \) of nonzero measure. By a simple compactness argument we can find \( G_0s \in K \) such that every open
neighborhood of $G_0$ intersects with $K$ in a set of positive measure. Then by the arguments similar to Lemma 12 it follows that $\ker \pi^K \subseteq \ker \pi_0 \circ s$. We want to show that $\ker \pi_0 \circ s \subseteq \ker \pi$. To this end, suppose $\pi_0(\sigma_s a) = 0$ and let $\epsilon > 0$ be given. Since $\pi_0(\sigma_s a) \to 0$ whenever $s_j \to s$ we can find an open neighborhood $F'$ of $s$ in $G_P$ such that $\|\pi_0(\sigma_t a)\| < \epsilon$ for all $t \in F'$. Then $\|\pi(a)\| = \|\pi_0(F')(a)\| < \epsilon$. Thus $\pi(a) = 0$ as claimed. It follows $\ker \pi^K = P$.

Finally, if $E$ a nonzero Borel subset of $G_0/G_P$ then we can choose a compact subset $K \subseteq E$ such that $\mu(K) > 0$. Suppose $\pi^E(a) = 0$ then $\pi^K(a) = 0$. It follows $\|\pi(a)\| = \|\pi^K(a)\| = 0$. So $\ker \pi^E = P$. $\square$

Combining Theorem 13 and Theorem 14 we obtain the following important corollary.

**Corollary 15.** Let $(A, G, \sigma)$ be a separable $C^*$-dynamical system where $G$ is compact. Then $(A, G, \sigma)$ satisfies the strong-EHI property.

As mentioned in the introduction it remains unknown whether the strong-EHI property holds for an arbitrary $C^*$-dynamical system. We can inquire about a weaker property of $C^*$-dynamical systems, called simply the EHI property, where we ask every primitive ideal of $A \times_\sigma G$ to be induced from a stability group (see [4]). However, even with an additional assumption that $G$ is amenable it is not known whether all separable $C^*$-dynamical systems satisfy the EHI property.

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