IMPROVED CONSTRAINED SCHEME FOR THE EINSTEIN EQUATIONS: AN APPROACH TO THE UNIQUENESS ISSUE

Jérôme Novak (Jerome.Novak@obspm.fr)

Laboratoire Univers et Théories (LUTH)  
CNRS / Observatoire de Paris / Université Paris-Diderot

based on collaboration with  
I. Cordero-Carrión, P. Cerdá-Durán, H. Dimmelmeier,  
J.L. Jaramillo and É. Gourgoulhon.

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Plan

1. Introduction
2. CFC and FCF
3. Non-uniqueness problem
4. A cure in CFC
5. New constrained formulation
1. **Introduction**

2. **CFC and FCF**

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**3+1 FORMALISM**

Decomposition of spacetime and of Einstein equations

**Evolution Equations:**

\[
\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_\beta K_{ij} = -D_i D_j N + NR_{ij} - 2NK_{ik}K^k_j + N[KK_{ij} + 4\pi((S - E)\gamma_{ij} - 2S_{ij})]
\]

\[
K^{ij} = \frac{1}{2N} \left( \frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i \right).
\]

**Constraint Equations:**

\[
R + K^2 - K_{ij} K^{ij} = 16\pi E,
\]

\[
D_j K^{ij} - D^i K = 8\pi J^i.
\]

\[
g_{\mu\nu} \, dx^\mu \, dx^\nu = -N^2 \, dt^2 + \gamma_{ij} \left( dx^i + \beta^i \, dt \right) \left( dx^j + \beta^j \, dt \right)
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**Free vs. Constrained Formulations**

As in electromagnetism, if the constraints are satisfied initially, they remain so for a solution of the evolution equations.

**Free Evolution**
- Start with initial data verifying the constraints,
- Solve only the 6 evolution equations,
- Recover a solution of all Einstein equations.

⇒ Apparition of constraint violating modes from round-off errors. Considered cures:
- Using of constraint damping terms and adapted gauges (many groups).
- Solving the constraints at every time-step (efficient elliptic solver?).
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Conformal flatness condition (CFC)

and

Fully constrained formulation (FCF)
Conformal flatness condition

Within 3+1 formalism, one imposes that:

$$\gamma_{ij} = \psi^4 f_{ij}$$

with $f_{ij}$ the flat metric and $\psi(t, x^1, x^2, x^3)$ the conformal factor.

First devised by Isenberg in 1978 as a waveless approximation to GR, it has been widely used for generating initial data, ...

Set of 5 non-linear elliptic PDEs ($K = 0$)

\[
\Delta \psi = -2\pi \psi^{-1} \left( E^* + \frac{\psi^6 K_{ij} K^{ij}}{16\pi} \right),
\]

\[
\Delta (N\psi) = 2\pi N\psi^{-1} \left( E^* + 2S^* + \frac{7\psi^6 K_{ij} K^{ij}}{16\pi} \right),
\]

\[
\Delta \beta^i + \frac{1}{3} \nabla^i \nabla_j \beta^j = 16\pi N\psi^{-2} (S^*)^i + 2\psi^{10} K^{ij} \nabla_j \frac{N}{\psi^6}.
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Fully Constrained Formulation

Bonazzola et al. (2004)

With no approximation: \( \tilde{\gamma}^{ij} = \psi^4 \gamma^{ij} \) and the choice of generalized Dirac gauge (and maximal slicing)

\[
\nabla_j \tilde{\gamma}^{ij} = \nabla_j h^{ij} = 0. \quad (\tilde{\gamma}^{ij} = f^{ij} + h^{ij})
\]

\(\Rightarrow\) very similar equations to the CFC system + evolution equations for \( \tilde{\gamma}^{ij} \):

\[
\frac{\partial K^{ij}}{\partial t} - \mathcal{L}_\beta K^{ij} = N D_k D^k h^{ij} - D^i D^j N + S^{ij},
\]

\[
\frac{\partial h^{ij}}{\partial t} - \mathcal{L}_\beta h^{ij} = 2N K^{ij}.
\]

When combined, reduce to a wave-like (strongly hyperbolic) operator on \( h^{ij} \), with no incoming characteristics from a black hole excision boundary (Cordero-Carrión et al. (2008)).
**Fully constrained formulation**

**Motivations for the FCF:**

- Easy to use CFC initial data for an evolution using the constrained formulation,
- Evolution of two scalar fields: the rest of the tensor $h^{ij}$ can be reconstructed using the gauge conditions.
  $\iff$ dynamical degrees of freedom of the gravitational field.
- Elliptic systems have good stability properties (what about uniqueness?).
- Constraints are verified!

+ the generalized Dirac gauge gives the property that $h^{ij}$ is asymptotically transverse-traceless
$\Rightarrow$ straightforward extraction of gravitational waves . . .
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Non-uniqueness problem
Spherical collapse of matter

We consider the case of the collapse of an unstable relativistic star, governed by the equations for the hydrodynamics

\[
\frac{1}{\sqrt{-g}} \left[ \frac{\partial \sqrt{\gamma} U}{\partial t} + \frac{\partial \sqrt{-g} F^i}{\partial x^i} \right] = Q,
\]

with \( U = (\rho W, \rho h W^2 v_i, \rho h W^2 - P - D) \).

At every time-step, we solve the equations of the CFC system (elliptic)
\Rightarrow \text{exact in spherical symmetry! (isotropic gauge)}

- During the collapse, when the star becomes very compact, the elliptic system would no longer converge, or give a wrong solution (wrong ADM mass).
- Even for equilibrium configurations, if the iteration is done only on the metric system, it may converge to a wrong solution.
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**Collapse of gravitational waves**

Using FCF (full 3D Einstein equations), the same phenomenon is observed for the collapse of a gravitational wave packet.

- Initial data: vacuum spacetime with Gaussian gravitational wave packet,
- if the initial amplitude is sufficiently large, the waves collapse to a black hole.
- As in the fluid-CFC case, the elliptic system of the FCF suddenly starts to converge to a wrong solution.

\[ \Rightarrow \text{effect on the ADM mass computed from } \psi \text{ at } r = \infty. \]
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⇒ effect on the ADM mass computed from $\psi$ at $r = \infty$.
In the *extended conformal thin sandwich* approach for initial data, the system of PDEs is the same as in CFC.

Pfeiffer & York (2005) have numerically observed a parabolic branching in the solutions of this system for perturbation of Minkowski spacetime.

Some analytical studies have been performed by Baumgarte *et al.* (2007), which have shown the genericity of the non-uniqueness behavior.

*from* Pfeiffer & York (2005)
A cure in the CFC case
**Origin of the Problem**

In the simplified non-linear scalar-field case, of unknown function $u$

$$\Delta u = \alpha u^p + s.$$ 

Local uniqueness of solutions can be proven using a maximum principle:

- If $\alpha$ and $p$ have the same sign, the solution is locally unique.

In the CFC system (or elliptic part of FCF), the case appears for the Hamiltonian constraint:

$$\Delta \psi = -2\pi \psi^5 E - \frac{1}{8} \psi^5 K_{ij} K^{ij};$$

Both terms (matter and gravitational field) on the r.h.s. have wrong signs.
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Approximate CFC
also in Saijo (2004)

Let $L, V^i \mapsto (LV)^{ij} = \nabla^i V^j + \nabla^j V^i - \frac{2}{3} f^{ij} \nabla_k V^k$.

In CFC, $K^{ij} = \psi^{-4} \tilde{A}^{ij}$, with $\tilde{A}^{ij} = \frac{1}{2N} (L \beta)^{ij}$,

here $K^{ij} = \psi^{-10} \hat{A}^{ij}$, with $\hat{A}^{ij} = (LX)^{ij} + \hat{A}^{ij}_{TT}$.

Neglecting $\hat{A}^{ij}_{TT}$, we can solve in a hierarchical way:

1. Momentum constraints $\Rightarrow$ linear equation for $X^i$ from the actually computed hydrodynamic quantity $S^*_j = \psi^6 S_j$,
2. Hamiltonian constraint $\Rightarrow \Delta \psi = -2\pi \psi^{-1} E^* - \psi^{-7} \hat{A}^{ij} \hat{A}^{ij}/8$,
3. linear equation for $N\psi$,
4. linear equation for $\beta$, from the definitions of $\hat{A}^{ij}$.

It can be shown that the error made neglecting $\hat{A}^{ij}_{TT}$ falls within the error of CFC approximation.
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APPLICATION

AXISYMMETRIC COLLAPSE TO A BLACK HOLE

Using the code CoCoNuT combining Godunov-type methods for the solution of hydrodynamic equations and spectral methods for the gravitational fields.

- Unstable rotating neutron star initial data, with polytropic equation of state,
- approximate CFC equations are solved every time-step.
- Collapse proceeds beyond the formation of an apparent horizon;
- Results compare well with those of Baoititi et al. (2005) in GR, although in approximate CFC.

Other test: migration of unstable neutron star toward the stable branch.
New constrained formulation
**New Constrained Formulation**

**Evolution Equations**

In the general case, one cannot neglect the TT-part of $\hat{A}^{ij}$ and one must therefore evolve it numerically.

| Sym. Tensor | Longitudinal Part | Transverse Part |
|-------------|-------------------|-----------------|
| $\hat{A}^{ij}$ | $(LX)^{ij}$ | $+ \hat{A}^{ij}_{TT}$ |
| $h^{ij}$ | 0 (gauge) | $+ h^{ij}$ |

The evolution equations are written only for the transverse parts:

\[
\frac{\partial \hat{A}^{ij}_{TT}}{\partial t} = \left[ \mathcal{L}_\beta \hat{A}^{ij} + N \psi^2 \Delta h^{ij} + S^{ij} \right]_{TT},
\]

\[
\frac{\partial h^{ij}}{\partial t} = \left[ \mathcal{L}_\beta h^{ij} + 2N \psi^{-6} \hat{A}^{ij} - (L\beta)^{ij} \right]_{TT}.
\]
New constrained formulation

If all metric and matter quantities are supposed known at a given time-step.

1. Advance hydrodynamic quantities to new time-step,
2. advance the TT-parts of $\hat{A}^{ij}$ and $h^{ij}$,
3. obtain the logitudinal part of $\hat{A}^{ij}$ from the momentum constraint, solving a vector Poisson-like equation for $X^i$ (the $\Delta^i_{jk}$'s are obtained from $h^{ij}$):

$$\Delta X^i + \frac{1}{3} \nabla^i \nabla_j X^j = 8\pi (S^*)^i - \Delta^i_{jk} \hat{A}^{jk},$$

4. recover $\hat{A}^{ij}$ and solve the Hamiltonian constraint to obtain $\psi$ at new time-step,
5. solve for $N\psi$ and recover $\beta^i$. 
**Summary - Perspectives**

- We have presented, implemented and tested an approach to cure the uniqueness problem in the elliptic part of Einstein equations;
- This problem was appearing in the CFC approximation to GR and in the constrained formulation;
- Based on previous works (e.g. by Saijo (2004)) in the CFC case, it has been generalized to the fully constrained case (full GR).
  \[\Rightarrow\] the accuracy has been checked: the additional approximation does not introduce any new errors.

The numerical codes are present in the LORENE library: [http://lorene.obspm.fr](http://lorene.obspm.fr), publicly available under GPL.

Future directions:
- Implementation of the new FCF and tests in the case of gravitational wave collapse;
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