Koopman operator-based model reduction for switched-system control of PDEs

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Abstract

We present a new framework for optimal control of PDEs using Koopman operator-based reduced order models (K-ROMs). By introducing a finite number of constant controls, the dynamic control system is transformed into a set of autonomous systems and the corresponding optimal control problem into a switching time optimization problem. This way, a nonlinear infinite-dimensional control problem is transformed into a low-dimensional linear problem. Using a recent convergence result for Extended Dynamic Mode Decomposition (EDMD), we prove convergence of the K-ROM-based solution towards the solution of the full system. To illustrate the results, we consider the 1D Burgers equation and the 2D Navier–Stokes equations. The numerical experiments show remarkable performance concerning both solution times as well as accuracy.

1 Introduction

The increasing complexity of technical systems presents a great challenge for control. We often want to control system dynamics described by partial differential equations (PDEs). If the system is nonlinear – such as the Navier–Stokes equations for fluid flow –, this is particularly challenging, see, e.g., [BN15] for an overview. To this end, advanced control techniques such as Model Predictive Control (MPC) [GP17] or machine learning-based control [DBN17, Kut17] have gained more and more attention in recent years. In MPC, a system model is used to repeatedly compute an open-loop optimal control on a finite-time horizon which results in a closed-loop control behavior. This requires solving the open-loop problem in a very short time, which is in general infeasible for nonlinear PDEs when using a standard discretization approach such as finite element or finite volume methods.

To overcome this problem, reduced order modeling is frequently applied to replace the high fidelity model by a surrogate model that can be solved much faster, see [ASG01, BMS05] for overviews. Various methods exist for deriving such a surrogate model, the most common for nonlinear systems probably being Galerkin projection in combination with Proper Orthogonal Decomposition (POD)
Many researchers have dedicated their work to developing optimal control methods based on POD for which convergence towards the true optimum can be proved, either using the singular values associated with the POD modes [KV99, Row05, HV09, BBV16, BDPV17] or by trust-region approaches [Fah00, BC08, QGVW16, RTV17, Pei17]. An approach to closed-loop flow control using POD-based surrogate models has been developed in [PHA15].

The number of required POD modes grows rapidly with increasing complexity of the system dynamics so that Galerkin models can become infeasible. Often additional measures like calibration [CAF09] or the construction of special modes [NAM+03, NPM05] have to be taken. For fluid flows, large Reynolds numbers (i.e., turbulent flows) quickly completely prohibit the construction of efficient Galerkin models such that one has to rely on turbulence models [LCLR16].

An alternative approach to construct a reduced order model (ROM) is by means of the Koopman operator [Koo31], which is a linear but infinite-dimensional operator describing the dynamics of observables. This approach is particularly suited to be applied to sensor measurements, also in situations where the underlying system dynamics are unknown. A lot of work has been invested both to study the properties of the Koopman operator [MB04, Mez05, BMM12, Mez13] as well as to efficiently compute numerical approximations via Dynamic Mode Decomposition (DMD) or Extended Dynamic Mode Decomposition (EDMD) [Sch10, RMB+09, CTR12, TRL+14, WKR15, KGPS16, KKS16]. More recently, various attempts have been made to use ROMs based on the Koopman operator for control problems [PBK15, PBK16, BBPK16, KM16, KKB17]. In these approaches, the Koopman operator is approximated from an augmented state – consisting of the actual state and the control – in order to deal with the non-autonomous control system.

In this article, we propose an alternative approach for constructing both open and closed-loop control problems using Koopman operator-based ROMs (K-ROMs). The key idea is to interpret the dynamical control system as a switched dynamical system, where the state can be influenced by switching between different autonomous systems at each time step. By numerically approximating a Koopman operator for each of the different autonomous dynamical systems individually, the switched dynamics can be reproduced by switching between the respective Koopman operators. This results in a switching time problem for the open-loop case ([EWD03, EWA06, FMOB13], cf. also [VMS10, RV14] for a related concept based on the Perron–Frobenius operator), whereas the closed-loop approach is realized via MPC. Using recent convergence results for the Koopman operator [AM17, KM17], convergence to the true solution can be guaranteed.

The remainder of the article is structured as follows: In Section 2, we introduce the basic concepts for the Koopman operator and its numerical approximation as well as the relevant control techniques. Our new approach is then introduced in Section 3 and results are shown in Section 4 for an ODE problem, for the 1D Burgers equation, and for the 2D Navier–Stokes equations. We conclude with a short summary and possible future work in Section 5.

2 Preliminaries

In this section, the Koopman operator and its numerical approximation via EDMD are introduced. Convergence of EDMD towards the Koopman operator is discussed since we will rely on this result for our convergence statements. In the second part of the section, both Switching Time
Optimization (STO) and Model Predictive Control (MPC) are introduced. We will rely on STO for open-loop and on MPC for closed-loop control.

2.1 Koopman Operator and EDMD

2.1.1 Koopman operator

Let $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ be a discrete deterministic dynamical system defined on the state space $\mathcal{M}$ and let $f : \mathcal{M} \rightarrow \mathbb{R}$ be a real-valued observable of the system. Then the Koopman operator $K : \mathcal{F} \rightarrow \mathcal{F}$ with $\mathcal{F} = L^\infty(\mathcal{M})$, see [LM94, BMM12, Mez13, WKR15], which describes the evolution of the observable $f$, is defined by

$$(Kf)(y) = f(\Phi(y)).$$

The Koopman operator is linear but infinite-dimensional. Its adjoint, the Perron–Frobenius operator, describes the evolution of densities. The definition of the Koopman operator can be naturally extended to continuous-time dynamical systems as described in [LM94, BMM12]. Given an autonomous ODE of the form

$$\dot{y}(t) = G(y(t)),$$

the Koopman semigroup of operators $\{K^t\}$ is defined as

$$(K^t f)(y) = f(\Phi^t(y)),$$

where $\Phi^t$ is the flow map associated with $G$. In what follows, we will mainly consider discrete dynamical systems, given by the discretization of ODEs or PDEs. That is, $\Phi = \Phi^h$ for a fixed time step $h$.

The action of the Koopman operator can be decomposed into the Koopman eigenvalues, eigenfunctions, and modes. The eigenvalues and eigenfunctions are given by $K\phi_\ell = \lambda_\ell \phi_\ell$. Writing the so-called full state observable $g(y) = y$ in terms of the eigenfunctions, we obtain

$$g(y) = y = \sum_\ell \phi_\ell(y) \eta_\ell.$$

The vectors $\eta_\ell$ are called Koopman modes. Defining the Koopman operator to act componentwise for vector-valued functions, this leads to

$$(Kg)(y) = \Phi(y) = \sum_\ell \lambda_\ell \phi_\ell(y) \eta_\ell.$$

That is, once we know the eigenvalues, eigenfunctions, and modes, which can be estimated from simulation or measurement data, we can evaluate the dynamical system $\Phi$, without needing an analytic expression for the dynamics. One method to compute a numerical approximation of the Koopman operator from data is EDMD.

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1The state space $\mathcal{M}$ can either be a finite-dimensional (i.e., $\mathcal{M} \subseteq \mathbb{R}^d$) or an infinite-dimensional space [KM17].
2.1.2 EDMD

The following brief description of EDMD [WKR15, KKS16] is based on the review paper [KNK+17]. EDMD is a generalization of DMD [Sch10, TRL+14] and can be used to compute a finite-dimensional approximation of the Koopman operator, its eigenvalues, eigenfunctions, and modes. In contrast to DMD, EDMD allows arbitrary basis functions – which could be, for instance, monomials, Hermite polynomials, or trigonometric functions – for the approximation of the dynamics. We will sometimes not be able to observe the full (potentially infinite-dimensional) state of the system, but consider only a finite number of measurements, given by \( z = f(y) \in \mathbb{R}^q \). For a given set of basis functions \( \{\psi_1, \psi_2, \ldots, \psi_k\} \), we then define a vector-valued function \( \psi: \mathbb{R}^q \to \mathbb{R}^k \) by

\[
\psi(z) = \begin{bmatrix} \psi_1(z) & \psi_2(z) & \ldots & \psi_k(z) \end{bmatrix}^\top.
\]

If \( \psi(z) = z \), we obtain DMD as a special case of EDMD. We assume that we have either measurement or simulation data, written in matrix form as \( Z = \begin{bmatrix} z_1 & z_2 & \cdots & z_m \end{bmatrix} \) and \( \tilde{Z} = \begin{bmatrix} \tilde{z}_1 & \tilde{z}_2 & \cdots & \tilde{z}_m \end{bmatrix} \), where \( \tilde{z}_i = f(\Phi(y_i)) \). The data could either be obtained via many short simulations or experiments with different initial conditions or one long-term trajectory or measurement. If the data is extracted from one long trajectory, then \( \tilde{z}_i = z_{i+1} \). The data matrices are embedded into the typically higher-dimensional feature space by

\[
\Psi_Z = \begin{bmatrix} \psi(z_1) & \psi(z_2) & \cdots & \psi(z_m) \end{bmatrix} \quad \text{and} \quad \Psi_{\tilde{Z}} = \begin{bmatrix} \psi(\tilde{z}_1) & \psi(\tilde{z}_2) & \cdots & \psi(\tilde{z}_m) \end{bmatrix}.
\]

With these data matrices, we then compute the matrix \( K \in \mathbb{R}^{k \times k} \) defined by

\[
K^\top = \Psi_{\tilde{Z}}\Psi_Z^+ = (\Psi_{\tilde{Z}}\Psi_Z)^+(\Psi_Z\Psi_{\tilde{Z}})^+.
\]

The matrix \( K \) can be viewed as a finite-dimensional approximation of the Koopman operator. Let \( \xi_\ell \) be a left eigenvector of \( K^\top \), then an eigenfunctions of the Koopman operator is approximated by

\[
\varphi_\ell(z) = \xi_\ell^\top \psi(z).
\]

In order to approximate the Koopman modes, let \( \varphi(z) = [\varphi_1(z), \ldots, \varphi_k(z)]^\top \) be the vector-valued function containing the eigenfunction approximations and \( \Xi = [\xi_1, \xi_2, \ldots, \xi_k] \) the matrix that contains all left eigenvectors of \( K^\top \). We write the full state observable\(^2\) \( g \) in terms of the eigenfunctions as \( g(z) = B \psi(z) \), where \( B \in \mathbb{R}^{q \times k} \) is the corresponding coefficient matrix. From \( \varphi(z) = \Xi^\top \psi(z) \), we obtain

\[
g(z) = B \psi(z) = B (\Xi^\top)^{-1} \varphi(z) = \eta \varphi(z),
\]

where \( \eta = B (\Xi^\top)^{-1} \). The \( \ell \)th column vector of the matrix \( \eta \) thus represents the Koopman mode \( \eta_\ell \). With this, we can evaluate the dynamical system using the eigenvalues, eigenfunctions, and modes computed from data. The accuracy of this approximation depends strongly on the set of basis functions.

\(^2\)In our setting, the established term full state observable [WKR15] is a bit misleading since we only consider the observations \( z \) and thereby obtain a dynamical system for the observations only. In order to obtain the true full state observable, we can set \( z = y \).
Remark 2.1. The decomposition of the Koopman operator into eigenvalues, eigenfunctions, and modes is commonly used to analyze the system dynamics as well as predict the future state. In the situation we are presenting here, we can pursue an even simpler approach and obtain the update for the observable $z$ directly using $K$:

$$\psi(z_{i+1}) = K^\top \psi(z_i), \quad i = 0, 1, \ldots$$

From here, we can obtain $z_{i+1}$ using the projection matrix $B$.

2.1.3 Convergence of EDMD

First results showing convergence of the EDMD algorithm towards the Koopman operator have recently been proven in [AM17, KM17]. In short, the result states that – provided that the Koopman operator satisfies the Assumptions 2.2 and 2.3 below – as both the basis size $k$ as well as the number of measurements $m$ tend to infinity, the matrix $K$ obtained by EDMD converges to the Koopman operator. As we will utilize this result for our convergence analysis, it is stated below.

Denote by $K_{k,m}(= K^\top)$ the finite-dimensional approximation of the Koopman operator obtained by EDMD. Here, $k$ is the number of basis functions and $m$ the number of measurements. Before stating the two theorems required for the convergence, we introduce the following two assumptions.

Assumption 2.2. The basis functions $\psi_1, \ldots, \psi_k$ are such that

$$\mu\{z \in Z \mid c^\top \psi(z) = 0\} = 0$$

for all $c \in \mathbb{R}^k$, where $\mu$ is a given probability distribution according to which the data samples $f(y_1), \ldots, f(y_m)$ are drawn and $Z \subset \mathbb{R}^q$ is the space of all measurements.

This assumption ensures that the measure $\mu$ is not supported on a zero level set of a linear combination of the basis functions used, cf. [KM17] for details.

Assumption 2.3. The following conditions hold:

1. The Koopman operator $K: \mathcal{F} \to \mathcal{F}$ is bounded.

2. The observables $\psi_1, \ldots, \psi_k$ defining $\mathcal{F}_k$ (i.e., the finite-dimensional representation of $\mathcal{F}$) are selected from a given orthonormal basis of $\mathcal{F}$, i.e., $(\psi_i)_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{F}$.

The convergence of $K_{k,m}$ to $K$ is now achieved in two steps. In the first step, convergence of $K_{k,m}$ to $K_k$ is shown as the number of samples $m$ tends to infinity. Here, $K_k$ is the projection of $K$ onto $\mathcal{F}_k$. The second step then yields convergence of $K_k$ to $K$ as the basis size $k$ increases.

Theorem 2.4 ([KM17]). If Assumption 2.2 holds, then we have with probability one for all $\phi \in \mathcal{F}_k$

$$\lim_{m \to \infty} \|K_{k,m} \phi - K_k \phi\| = 0,$$

where $\| \cdot \|$ is any norm on $\mathcal{F}_k$. In particular, we obtain

$$\lim_{m \to \infty} \|K_{k,m} - K_k\| = 0,$$
where \( \| \cdot \| \) is any operator norm and
\[
\lim_{m \to \infty} \text{dist}(\sigma(K_{k,m}), \sigma(K_k)) = 0,
\]
where \( \sigma(\cdot) \subset \mathbb{C} \) denotes the spectrum of an operator and \( \text{dist}(\cdot, \cdot) \) the Hausdorff metric on subsets of \( \mathbb{C} \).

**Theorem 2.5 ([KM17])**. Let Assumption 2.3 hold and define the \( L_2(\mu) \) projection of a function \( \phi \) onto \( F_k \) by
\[
P_k^\mu \phi = \arg \min_{f \in F_k} \| f - \phi \|_{L_2(\mu)}.
\]
Then the sequence of operators \( K_k P_k^\mu = P_k^\mu K P_k^\mu \) converges strongly to \( K \) as \( k \to \infty \).

Note that Assumptions 2.2 and 2.3 (in particular the boundedness of \( K \)) do not hold for all systems. However, we will assume throughout the remainder of this article that they are satisfied.

**2.2 Switching Time Optimization and Model Predictive Control**

We will now introduce the concepts of STO and MPC. The two approaches can be used to determine optimal switching between different autonomous dynamical systems. In Section 3, we will then design ROMs for different autonomous systems such that they fit into these frameworks. In both subsections, the goal is to solve an optimal control problem
\[
\min_{u \in U} J(y, u) = \min_{u \in U} \int_{t_0}^{t_e} L(y(t)) \, dt
\]
\[
\text{s.t. } \dot{y}(t) = G(y(t), u(t)),
\]
\[
y(0) = y^0,
\]
where \( y \in \mathcal{Y} \) is the system state, \( u \in \mathcal{U} \) is the control function, and \( G : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \) describes the system dynamics. For ease of notation, the objective function \( L : \mathbb{R}^d \to \mathbb{R} \) only depends on \( y \) explicitly. However, the framework presented here can be extended to objectives also depending on the control \( u \) in a straightforward manner. Problem (1) is constrained by an ordinary differential equation, but it could equally be formulated in terms of PDE constraints as this does not affect the framework we present below.

**2.2.1 Switching Time Optimization**

Switched systems are very common in engineering. They can be seen as a special case of hybrid systems which possess both continuous and discrete-time control inputs (cf. [ZA15] for a survey). The switched systems we want to consider here are characterized by \( n_c \) different (autonomous) right-hand sides. We introduce the switching sequence \( \tau \in \mathbb{R}^{p+2} \) with \( \tau_0 = t_0 \) and \( \tau_{p+1} = t_e \). The entries \( \tau_1, \ldots, \tau_p \) (with \( \tau_l \geq \tau_{l-1} \)) describe the time instants at which the right-hand side of the dynamical system is changed:
\[
\dot{y}(t) = G_j(y(t)) \quad \text{for } t \in [\tau_{l-1}, \tau_l),
\]
\[
y(0) = y^0,
\]
\[
j = l \mod n_c.
\]
The third line in (2) indicates that (following [EWD03]) the switching sequence of the system is predetermined, i.e., we switch from $G_0$ to $G_1$, from $G_1$ to $G_2$ and so on. Having reached the final system, we go back from $G_{n-1}$ to $G_0$.

Using this reformulation, the optimal control problem (1) can be written in terms of the switching instants:

$$
\min_{\tau \in \mathbb{R}^{p+2}} J(y) = \min_{\tau \in \mathbb{R}^{p+2}} \int_{t_0}^{t_e} L(y(t)) \, dt \quad \text{s.t.} \quad (2).
$$

Switched systems appear in many applications. Consider, for instance, a chemical reactor where a valve is either open or closed. Moreover, continuous control inputs may be approximated by a finite number of fixed control inputs. Motivated by this, switching time problems have been intensively studied in the literature, see, e.g., [EWD03, EWA06, FMOB13, SOBG16, SOBG17]. Consequently, there exist efficient methods to determine the optimal switching sequence using gradient-based and even second-order, Newton-type methods. Here, we use the gradient-based approach introduced in [EWD03], but we compute the gradient using a finite difference approximation.

**Example 2.6.** Let us consider the following example taken from [PBK16]:

$$
\dot{y}(t) = G(y(t), u(t)) = \begin{pmatrix} \mu y_1(t) \\ \lambda (y_2(t) - (y_1(t))^2) + u(t) \end{pmatrix},
$$

$$
y(0) = y^0.
$$

By restricting ourselves to $n_c$ constant controls, we can transform the control system into $n_c$ autonomous systems:

$$
\dot{y}(t) = G_j(y(t)) = \begin{pmatrix} \mu y_1(t) \\ \lambda (y_2(t) - (y_1(t))^2) \end{pmatrix} + \begin{pmatrix} 0 \\ u_j \end{pmatrix}, \quad j = 0, \ldots, n_c - 1,
$$

$$
y(0) = y^0.
$$

![Figure 1](image1.png)

**Figure 1:** (a) The trajectories of three different autonomous systems $(u_0 = 0, u_1 = 2, u_2 = -2)$ with starting point $y^0 = (1, 2)^\top$ and the trajectory of the switched system (dashed line) with switching according to (b).
The system dynamics of (4) (with $\mu = -0.05$ and $\lambda = -1$) are visualized in Figure 1 for $n_c = 3$ and a fixed switching sequence with 10 intervals.

### 2.2.2 Model Predictive Control

For real systems, it is often insufficient to determine a control input a priori. Due to the so-called plant-model mismatch – the difference between the dynamics of the real system and the model – the open-loop control input will not be able to control the system as desired or at least be non-optimal. Furthermore, disturbances cannot be taken into account by open-loop control strategies. A remedy to this issue is MPC [GP17], where open-loop problems are solved repeatedly on finite horizons (cf. Figure 2). Using a model of the system dynamics, an open-loop optimal control problem is solved in real-time over a so-called *prediction horizon* of length $p$:

$$\min_{u \in \mathbb{R}^p} \sum_{i=s}^{s+p-1} L(y_i)$$

s.t. $y_{i+1} = \Phi(y_i, u_{i-s+1})$ for $i = s, \ldots, s + p - 1$.

The first part of this solution is then applied to the real system while the optimization is repeated with the prediction horizon moving forward by one sample time. (The indexing $i - s + 1$ is required to account for the finite-horizon control and the infinite-horizon state.) For this reason, MPC is also referred to as *moving horizon control* or *receding horizon control*.

![Figure 2: Sketch of the MPC concept for a tracking problem. The step size is $h = t_{s+1} - t_s$.](image)

**Remark 2.7.** Note that – following [GP17] – the system dynamics are of discrete form. The motivation behind this is that the control is constant over each sample time interval. When dealing with continuous-time systems, this formulation can be regarded as flow map $\Phi^h$ (cf. Section 2.1) of the continuous dynamics. Since it is not important for the results in this article, we will from now on not distinguish between $\Phi$ and $\Phi^h$. 

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Similar to the previous section, we now replace the dynamical control system by \( n_c \) autonomous systems \( \Phi_0 \) to \( \Phi_{n_c-1} \) and thereby transform Problem (5) to a switching problem:

\[
\begin{align*}
\min_{\tau \in \{0, \ldots, n_c-1\}^p} \sum_{i=s}^{s+p-1} L(y_i) \\
\text{s.t.} \quad y_{i+1} = \Phi_{\tau_i}^{-1}(y_i) \quad \text{for} \quad i = s, \ldots, s + p - 1.
\end{align*}
\] (6)

In other words, each entry of \( \tau \) describes which system \( \Phi_{\tau_i} \) to apply in the \( i^{th} \) step. Due to the discrete-time dynamics, Problem (6) is now a combinatorial problem that can be solved using dynamic programming [BS15, XA00], for instance. Since the aim of the article is to introduce the K-ROM-based switched systems concept, we simply evaluate the objective for all possible \( \tau \) and select the optimal solution. For large values of \( n_c \) or \( p \), however, this is not feasible anymore.

### 3 Open and closed-loop control using K-ROMs

If the system dynamics \( G \) are known, then the open- and closed-loop techniques from the previous section can immediately be applied. However, if the underlying system dynamics are described by a PDE, then solving the problem numerically (e.g., with a finite element method) can quickly become very expensive such that real-time applicability is not guaranteed. Furthermore, there are many systems where the dynamics are not known explicitly. In both situations, we can use observations to approximate the Koopman operator and derive a linear system describing the dynamics of these observations. These could consist of (part of) the system state as well as arbitrary functions of the state such as the lift coefficient of an object within a flow field.

We want to use such a Koopman operator-based reduced order model (K-ROM) for both open- and closed-loop problems. Similar to reduced-basis approaches, this introduces a splitting into an offline phase and an online phase. During the offline phase, we have to collect data and compute reduced models. In the online phase, these models are then used to accelerate the solution of the optimization problems. Several approaches for Koopman operator-based control have recently been proposed, see e.g., [PBK15, PBK16, BBPK16, KKB17] for open-loop problems and [KM16] for closed-loop problems, where the authors also use MPC. All these approaches have in common that one single Koopman operator is computed for an augmented state \((y, u)\). This requires collecting data from a large number of state-control combinations and it can become difficult to represent these rich dynamics with one single Koopman operator.

We here propose an alternative approach where we compute \( n_c \) Koopman operators for the \( n_c \) different autonomous systems that have been introduced in Section 2.2:

\[(K_j f)(y) = f(\Phi_j(y)), \quad j = 0, \ldots, n_c - 1.\]

Using EDMD, we can compute an approximation of the individual Koopman operators and thereby derive discrete, linear dynamical systems for the observations \( z = f(y) \):

\[
\psi(z_{i+1}) = K_j^\top \psi(z_i), \quad j = 0, \ldots, n_c - 1.
\] (7)
These linear dynamics now replace the original differential equation (2) and using the convergence result for the Koopman operator (Theorems 2.4 and 2.5), we can show equality of the optimal solution, as we will see below. By this approach, we can significantly accelerate the computation which is due to the linearity of the model on the one hand and the restriction to observables instead of the full state $y$ on the other hand.

### 3.1 Switching Time Optimization

For the switching time optimization, the problem formulation has to be adapted since we are now restricted to the time step $h$ of the flow map $\Phi$ (i.e., the sample time between two consecutive snapshots of the sampled data). Consequently, the switched dynamics (2) are replaced by a discrete version:

$$
\psi(z_{i+1}) = K_j^\top \psi(z_i) \quad \text{for } i = \hat{\tau}_{l-1} h, \ldots, \hat{\tau}_l h,
$$

$$
j = l \mod n_c,
$$

and the switching problem (3) is replaced by an integer version:

$$
\min_{\hat{\tau} \in \mathbb{N}^{p+2}} J = \min_{\hat{\tau} \in \mathbb{N}^{p+2}} \sum_{i=0}^{k-1} L_K(\psi(z_i))
$$

s.t. (8),

where $L_K$ is the reduced objective function formulated with respect to the observables. We again have $p$ switching instants (with $\hat{\tau}_l \geq \hat{\tau}_{l-1}$, $l = 1, \ldots, p+1$) and $\hat{\tau}_0 = t_0/h$ and $\hat{\tau}_{p+2} = t_e/h$.

We now compare the two problem formulations (3) and (9), i.e., the switching time problem and the corresponding approximation using the K-ROM. First, we assume that the full objective function $L$ can be evaluated using only observations:

**Assumption 3.1.** $L(y(t)) = L_K(\psi(z_i))$ for all $t \in [t_0, t_e]$ and the corresponding $i = (t - t_0)/h$.

Note that in a practical setting, this is automatically satisfied since the objective function can only be evaluated using observations (e.g., sensor data) such that the objective $L$ has to be defined accordingly.

Assumption 3.1 is obviously not sufficient for the solutions to be identical since in Problem (9), we are restricted to the time grid defined by the sample time $h$. However, when restricting $\tau$ to multiples of $h$, the solutions do coincide.

**Theorem 3.2.** Let Assumptions 2.2, 2.3, and 3.1 be satisfied. Consider the switching time optimization problem (3) and the corresponding approximation (9) using the K-ROM. Furthermore, consider for (3) the additional constraint that

$$
\frac{\tau_l}{h} \in \mathbb{N} \quad \text{for } l = 1, \ldots, p.
$$

Then the optimal solutions of (3) and (9) are identical, i.e.,

$$
h \hat{\tau}^* = \tau^*.
$$
Proof. The first two assumptions yield convergence of EDMD to the Koopman operator, cf. Theorems 2.4 and 2.5. Consequently, we have
\[ K_j^T \psi(z_i) = \psi(z_{i+1}) = \psi(f(y_{i+1})) = \psi(f(\Phi_j(y_i))) \quad \text{for} \ j = 0, \ldots, n_c - 1, \] (11)
i.e., the dynamics of observations of the full model and the K-ROM are identical.

Due to the additional constraint (10), we can change the optimization variable to \( \tau = \tau/h \) and reformulate Problem (3):
\[
\begin{align*}
\min_{\tau \in \mathbb{N}^{p+2}} & \int_{t_0}^{t_e} L(y(t)) \, dt \\
\text{s.t.} & \quad (2).
\end{align*}
\] (12)
By Assumption 3.1, the objective functions of the two problems are identical and due to the equality of the dynamics (Equation (11)), we obtain
\[ \hat{\tau}^* = \arg \min (9) = \arg \min (12) = \frac{1}{h} \arg \min (3) = \frac{\tau^*}{h}, \]
which completes the proof.

Remark 3.3. The requirement (10) obviously has an impact on the solution. We will see in an example in Section 4 that if it is omitted, the solutions are not identical, but remain close. In the MPC case, this additional constraint is not required.

3.2 Model Predictive Control

In the closed-loop setting, the result is very similar to the open-loop version. In fact, due to the discrete formulation of the MPC problem (5) – and the corresponding switching formulation (6) – we obtain an even stronger result since we do not have to restrict the solution to a subset of the feasible set of the original problem.

The reduced version of Problem (6) is obtained by replacing the objective function as well as the system dynamics by the K-ROM formulations:
\[
\begin{align*}
\min_{\hat{\tau} \in \{0, \ldots, n_c - 1\}^p} & \sum_{i=s}^{s+p-1} L_K(\psi(z_i)) \\
\text{s.t.} & \quad \psi(z_{i+1}) = K_{\hat{\tau}_{i+1}}^T \psi(z_i) \quad \text{for} \ i = s, \ldots, s + p - 1.
\end{align*}
\] (13)
Using again the assumption that the objective functions \( L \) and \( L_K \) are identical (Assumption 3.1), we obtain the following result.

Theorem 3.4. Consider Problem (6) and the corresponding approximation (13) using the K-ROM and let Assumptions 2.2, 2.3, and 3.1 be satisfied. Then the optimal solutions of (6) and (13) are identical, i.e.,
\[ \hat{\tau}^* = \tau^*. \]

Proof. The proof is analog to Theorem 3.2. \qed
Algorithm 1 (K-ROM-based MPC)

Require: EDMD approximations of $n_c$ Koopman operators; prediction horizon length $p \in \mathbb{N}$.

1: for $i = 0, 1, 2, \ldots$ do.
2: Obtain measurements of the current system state $z_i = f(y_i)$.
3: Predict the initial condition $z_{i+1}$ for the next MPC optimization problem using the currently active K-ROM based on (7).
4: Solve Problem (13) with initial condition $z_{i+1}$ on the prediction horizon of length $p$.
5: At $t = (i + 1)h$, apply the first entry of the solution, i.e., $\hat{\tau}_*$, to the system.
6: end for

The MPC procedure now follows classical approaches as discussed in Section 2.2.2. It is summarized in Algorithm 1. In order to achieve real-time applicability, Problem (13) (Step 4) has to be solved within the sample time $h$. Note that the Koopman operator can also be used to predict the initial condition of the next optimization problem (Step 3), i.e., it serves as a state estimator [Sim06]. For state prediction, accuracy plays an important role and due to the convergence result of the Koopman operator, excellent prediction accuracy can be guaranteed.

4 Results

We now illustrate the results using several examples of varying complexity. We will first revisit the ODE problem (4) for the switching time optimization. In the MPC framework, we will then consider control of the 1D Burgers equation and of the incompressible 2D Navier–Stokes equations. For the latter problems, the K-ROM predictions become too inaccurate after several steps such that open loop control cannot be realized.

Table 1: Sampling of data for the three problems.

| Problem          | ODE (4)       | 1D Burgers   | 2D NSE       |
|------------------|---------------|--------------|--------------|
| $h$              | $0.04$        | $0.5$        | $0.25$       |
| Sampling         | Separate trajectories for the individual $K_j$ (cf. Figure 1 (a)) | Two simulations (60s) with different initial conditions $y^0$, fixed switching sequence (similar to Figure 1 (b)) | One simulation (900s) with random switching (concerning both switching time and order) |
| Basis $\psi$     | Monomials (order 3) | Monomials (order 3) | Monomials (order 4) |
| K-ROM speed-up   | $\approx 8$  | $\approx 25$ | $\approx 4000$ |

The approaches for sampling the data and for the EDMD approximations of the respective Koopman operators $K_j$ for $j = 0, \ldots, n_c - 1$ are summarized in Table 1. In the first case, data is sampled individually for the respective autonomous dynamics, whereas in the other two cases, long-term simulations are split according to the active system for the individual snapshots. For
the sake of simplicity, we use a set of basis functions comprising monomials of order up to 3 or 4, respectively. However, choosing problem-dependent basis functions might significantly improve the accuracy of the K-ROMs.

Table 1 also shows the speed-up achieved by the K-ROM. We observe an acceleration by several orders, especially for systems that are expensive to evaluate. The reason is that the K-ROM is linear and its size only depends on the dimension $q$ of the observable $z$ and the size $k$ of the basis $\psi$. It is completely independent of the numerical discretization of the domain. Furthermore, the time steps of the K-ROM can be much larger than the step size for the numerical solution of the PDE.

### 4.1 Switching Time Optimization

For the switching time optimization, we revisit Problem (4) from Example 2.6 and compare the performance of the full control problem (3) with the K-ROM approximation (9) for a tracking type objective, i.e., we want the system state to follow a prescribed trajectory $y^{\text{opt}}(t)$:

$$J = \sum_{i=0}^{k-1} (y_{i,2} - y_{i,2}^{\text{opt}})^2.$$ 

Note that we have dropped the requirement $\tau_l/h \in \mathbb{N}$. Hence, we do not observe identity of the two optimal solutions. However, when considering this constraint, our numerical results confirm Theorem 3.2.

The results without enforcing $\tau_l/h \in \mathbb{N}$ are shown in Figure 3. We observe that as we increase the number of switches, the distance in $y^2$ between the full problem and the K-ROM approximation decreases. More importantly, we observe a significant speed-up (by a factor of approximately 50), cf. Figure 3 (f). This is due to the linearity of the K-ROM as well as an increased step length by a factor of 40.

### 4.2 Model Predictive Control

**1D Burgers equation.** As a second example, we consider the 1D Burgers equation:

$$\dot{y}(x,t) - \nu \Delta y(x,t) + y(x,t) \nabla y(x,t) = u_j(x), \quad \text{for} \quad j \in \{0, 1, 2\},$$

$$y(x,0) = y^0(x),$$

with periodic boundary conditions and $\nu = 0.01$. We have directly transformed the system with a distributed control $u(x,t)$ into three autonomous systems with time-independent shape functions $u_j$ (see Figure 4 (a)).

In contrast to the ODE case, we do not observe the entire state here, but only certain points in space (the black dots in Figure 4 (a)), i.e.,

$$z = f(y(x,t)) = (y(0,t), y(0.5,t), y(1,t), y(1.5,t)).$$
and we construct the K-ROM for these observations from data. Following Assumption 3.1, we formulate the tracking type objective function in terms of the observables only, which yields the

- (a) Optimal trajectory for $y_2$ of the ODE (blue solid line) and the K-ROM (orange dashed line) for $n_{\text{switch}} = 5$ with $J = 2.57 - T = 2.1\text{ sec}$ and $J = 2.21 - T = 46.4\text{ sec}$.
- (b) Optimal trajectory for $y_2$ of the ODE (blue solid line) and the K-ROM (orange dashed line) for $n_{\text{switch}} = 10$ with $J = 1.11 - T = 3.2\text{ sec}$ and $J = 0.92 - T = 150.5\text{ sec}$.
- (c) Optimal trajectory for $y_2$ of the ODE (blue solid line) and the K-ROM (orange dashed line) for $n_{\text{switch}} = 20$ with $J = 0.53 - T = 7.4\text{ sec}$ and $J = 0.37 - T = 263.8\text{ sec}$.
- (d) Optimal trajectory for $y_2$ of the ODE (blue solid line) and the K-ROM (orange dashed line) for $n_{\text{switch}} = 30$ with $J = 0.40 - T = 18.7\text{ sec}$ and $J = 0.21 - T = 510.7\text{ sec}$.
- (e) Comparison of the objective function values depending on the number of switching instants $p$.
- (f) The computing times corresponding to (e).

**Figure 3:** (a) to (d) Optimal trajectories of $y_2$ of the ODE (blue solid line) and the K-ROM (orange dashed line) for different numbers of switching instants ($p = 5, 10, 20, 30$). The variable $y_1$ is shown in black. (e) Comparison of the objective function values depending on the number of switching instants $p$. (f) The computing times corresponding to (e).
The results of the K-ROM-based MPC are shown in Figure 4. We see that by using a low-dimensional linear model for the observations, we are able to control the PDE very accurately. This is even more remarkable since we have severely restricted the control space to only three inputs.

**2D Navier–Stokes equations.** As a final example, we consider the flow of a fluid around a cylinder described by the 2D incompressible Navier–Stokes equations at a Reynolds number of $Re = 100$.
The system is controlled via rotation of the cylinder, i.e., $u(t)$ is the angular velocity. The uncontrolled system possesses a periodic solution, the well-known von Kármán vortex street.

We now follow the same procedure as in the previous example. Instead of observing the full

![Figure 5:](image-url)

(a) Sketch of the problem setting. The domain is denoted by $\Omega$ and the boundary of the cylinder by $\Gamma_{cyl}$. (b) Sequence of the active autonomous dynamics obtained by the K-ROM MPC approach. (c) The corresponding trajectories of the observations $f((y(\cdot,t), p(\cdot,t)))$. The reference trajectory which the lift coefficient (black line) has to track is marked by red stars. The drag is shown in blue and the six horizontal velocities are colored according to the sensor positions shown in (d). (d) Snapshot of the full system simulation for the sequence shown in (b). The sensor positions for the observations are marked by the colored dots.
state, we observe the lift $C_l$ and the drag $C_d$ of the cylinder:

$$C_l(t) = \int_{\Gamma_{cyl}} p_2(x,t) \, dx, \quad C_d(t) = \int_{\Gamma_{cyl}} p_1(x,t) \, dx,$$

where $p_i$ is the projection of the pressure onto the $i^{th}$ spatial direction. Additionally, we observe the vertical velocity at six different positions ($x_1, \ldots, x_6$) in the cylinder wake (see Figure 5 (d)):

$$z = f((y(\cdot, t), p(\cdot, t)) = (C_l(t), C_d(t), y_2(x_1, t), \ldots, y_2(x_6, t))^\top.$$

The goal is to control the lift by rotating the cylinder. We transform the non-autonomous system into three autonomous ones with constant cylinder rotations $u_0 = 0$, $u_1 = 2$, $u_2 = -2$ and approximate the three corresponding Koopman operators. Since the lift coefficient is one of the observables, we simply have to track the corresponding entry of $z$ in the MPC problem:

$$\min_{\tau \in \{0, 1, 2\}^p} \sum_{i=s}^{s+p-1} (z_i - z_{i}^{\text{opt}})^2.$$

The switching sequence obtained by the K-ROM-based MPC algorithm is shown in Figure 5 (b), the corresponding dynamics of the K-ROM in Figure 5 (c). We see that the algorithm can successfully track the desired lift trajectory (shown as red stars) using only three different control inputs and a linear reduced model. We also observe some divergence, for example between seconds 9 and 10. This may be due to the fact that the system simply cannot follow the prescribed desired state using only three different inputs. An alternative (and more likely) explanation is that the data (collected from one long-term simulation with random switching) was not rich enough to achieve convergence for all three Koopman operator approximations. As a consequence, predictions are not sufficiently accurate which results in the selection of the incorrect control input. Furthermore, we have not verified Assumptions 2.2 and 2.3. This will be the subject of further research. In the first case, one could simply add a few more controls. However, this will lead to an exponential increase in the number of possible solutions for the MPC problem such that special care has to be taken to maintain real-time applicability. In the second case, one could adopt ideas from [HWR14]: during system operation, additional data is collected and then used to regularly update the Koopman operator approximations.

5 Conclusion

We have presented a framework for open- and closed-loop control using Koopman operator-based reduced order models. By transforming the non-autonomous control system into a (small) number of autonomous systems with fixed control inputs, the control problem is turned into a switching time problem. The approach enables us to control infinite-dimensional nonlinear systems using finite-dimensional, linear surrogate models. Using a recent convergence result for EDMD, we can prove optimality of the obtained solution. The numerical results show excellent performance, both considering the accuracy as well as the computing time. It will therefore be of great interest to further explore the limits and possibilities of this approach.
Further directions of research are the analysis of stability properties of the K-ROM-based MPC method. Due to the identity of the optimal solutions, we expect that stability results can be carried over from previous results without reduced order modeling. As already mentioned, it will be interesting to abandon the separation into offline and online phase and investigate the influence of regular updates using streaming data [HWR14]. In situations where more multiple control inputs are required, it might become challenging to maintain real-time applicability. In this situation, special techniques from linear switching time optimization [SOBG16] have to be exploited. It would also be of interest to further study the influence of the assumptions on the Koopman operator and whether convergence can be improved by choosing basis functions tailored to the system’s dynamics (e.g., via dictionary learning [LDBK17]). Finally, it would be interesting to implement the proposed approach in systems where multiple objectives are of interest, see, e.g., [PSOB+17]. In this situation, one could observe the individual objectives such that tracking single objectives as well as compromises can easily be achieved.

References

[AM17] H. Arbabi and I. Mezić. Ergodic theory, dynamic mode decomposition and computation of spectral properties of the Koopman operator. arXiv:1611.06664v6, pages 1–30, 2017.

[ASG01] A. C. Antoulas, D. C. Sorensen, and S. Gugercin. A survey of model reduction methods for large-scale systems. Contemporary Mathematics, 280:193–220, 2001.

[BBPK16] S. L. Brunton, B. W. Brunton, J. L. Proctor, and J. N. Kutz. Koopman invariant subspaces and finite linear representations of nonlinear dynamical systems for control. PLoS ONE, 11(2):1–19, 2016.

[BBV16] S. Banholzer, D. Beermann, and S. Volkwein. POD-Based Bicritical Optimal Control by the Reference Point Method. In 2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations, pages 210–215, 2016.

[BC08] M. Bergmann and L. Cordier. Optimal control of the cylinder wake in the laminar regime by trust-region methods and POD reduced-order models. Journal of Computational Physics, 227(16):7813–7840, 2008.

[BDPV17] D. Beermann, M. Dellnitz, S. Peitz, and S. Volkwein. Set-Oriented Multiobjective Optimal Control of PDEs using Proper Orthogonal Decomposition. Submitted, 2017.

[BMM12] M. Budisic, R. Mohr, and I. Mezić. Applied Koopmanism. Chaos, 22, 2012.

[BMS05] P. Benner, V. Mehrmann, and D. C. Sorensen, editors. Dimension Reduction of Large-Scale Systems. Springer Berlin Heidelberg New York, 2005.

[BN15] S. L. Brunton and B. R. Noack. Closed-Loop Turbulence Control: Progress and Challenges. Applied Mechanics Reviews, 67(5):1–48, 2015.

[BS15] R. E. Bellmann and E. D. Stuart. Applied dynamic programming. Princeton University Press, 2015.

[CAF09] L. Cordier, B. Abou El Majd, and J. Favier. Calibration of POD reduced-order models using Tikhonov regularization. International Journal for Numerical Methods in Fluids, 63(2):269–296, 2009.
K. K. Chen, J. H. Tu, and C. W. Rowley. Variants of Dynamic Mode Decomposition: Boundary Condition, Koopman, and Fourier Analyses. *Journal of Nonlinear Science*, 22(6):887–915, apr 2012.

T. Duriez, S. L. Brunton, and B. R. Noack. *Machine Learning Control Taming Nonlinear Dynamics and Turbulence*. Springer, 2017.

M. Egerstedt, Y. Wardi, and H. Axelsson. Transition-Time Optimization for Switched-Mode Dynamical Systems. *IEEE Transactions on Automatic Control*, 51(1):110–115, 2006.

M. Egerstedt, Y. Wardi, and F. Delmotte. Optimal Control of Switching Times in Switched Dynamical Systems. In *42nd IEEE International Conference on Decision and Control (CDC)*, pages 2138–2143, 2003.

M. Fahl. *Trust-region Methods for Flow Control based on Reduced Order Modelling*. Phd thesis, University of Trier, 2000.

K. Flaßkamp, T. Murphey, and S. Ober-Blöbaum. Discretized Switching Time Optimization Problems. In *12th European Control Conference*, pages 3179–3184, 2013.

L. Grüne and J. Pannek. *Nonlinear Model Predictive Control*. Springer International Publishing, 2 edition, 2017.

M. Hinze and S. Volkwein. Proper Orthogonal Decomposition Surrogate Models for Nonlinear Dynamical Systems: Error Estimates and Suboptimal Control. In P. Benner, D. C. Sorensen, and V. Mehrmann, editors, *Reduction of Large-Scale Systems*, volume 45, pages 261–306. Springer Berlin Heidelberg, 2005.

M. S. Hemati, M. O. Williams, and C. W. Rowley. Dynamic Mode Decomposition for Large and Streaming Datasets. *Physics of Fluids*, 26(111701):1–6, 2014.

S. Klus, P. Gelß, S. Peitz, and C. Schütte. Tensor-based dynamic mode decomposition. *Submitted (preprint: arXiv:1606.06625)*, 2016.

E. Kaiser, J. N. Kutz, and S. L. Brunton. Data-driven discovery of Koopman eigenfunctions for control. *arXiv:1707.01114*, 2017.

S. Klus, P. Koltaï, and C. Schütte. On the numerical approximation of the Perron-Frobenius and Koopman operator. *Journal of Computational Dynamics*, 3(1):51–79, 2016.

M. Korda and I. Mezić. Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control. *arXiv:1611.03537*, pages 1–16, 2016.

M. Korda and I. Mezić. On Convergence of Extended Dynamic Mode Decomposition to the Koopman Operator. *arXiv:1703.04680*, pages 1–18, 2017.

S. Klus, F. Nüske, P. Koltaï, H. Wu, I. Kevrekidis, C. Schütte, and F. Noé. Data-driven model reduction and transfer operator approximation. *ArXiv e-prints*, 2017.

B. O. Koopman. Hamiltonian Systems and Transformations in Hilbert Space. *Proceedings of the National Academy of Sciences*, 17(5):315–318, 1931.

J. N. Kutz. Deep learning in fluid dynamics. *Journal of Fluid Mechanics*, 814:1–4, 2017.

K. Kunisch and S. Volkwein. Control of the Burgers Equation by a Reduced-Order Approach Using Proper Orthogonal Decomposition. *Journal of Optimization Theory and Applications*, 102(2):345–371, 1999.
[LCLR16] S. Lorenzi, A. Cammi, L. Luzzi, and G. Rozza. POD-Galerkin method for finite volume approximation of Navier-Stokes and RANS equations. *Computer Methods in Applied Mechanics and Engineering*, 311:151–179, 2016.

[LDBK17] Q. Li, F. Dietrich, E. M. Bollt, and I. G. Kevrekidis. Extended dynamic mode decomposition with dictionary learning: a data-driven adaptive spectral decomposition of the Koopman operator. *arXiv:1707.00225v1*, 2017.

[LM94] A. Lasota and M. C. Mackey. *Chaos, fractals, and noise: Stochastic aspects of dynamics*, volume 97 of *Applied Mathematical Sciences*. Springer, 2nd edition, 1994.

[MB04] I. Mezić and A. Banaszuk. Comparison of systems with complex behavior. *Physica D: Nonlinear Phenomena*, 197:101–133, 2004.

[Mez05] I. Mezić. Spectral Properties of Dynamical Systems, Model Reduction and Decompositions. *Nonlinear Dynamics*, 41:309–325, 2005.

[Mez13] I. Mezić. Analysis of Fluid Flows via Spectral Properties of the Koopman Operator. *Annual Review of Fluid Mechanics*, 45:357–378, 2013.

[NAM+03] B. R. Noack, K. Afanasiev, M. Morzyński, G. Tadmor, and F. Thiele. A hierarchy of low-dimensional models for the transient and post-transient cylinder wake. *Journal of Fluid Mechanics*, 497:335–363, 2003.

[NPM05] B. R. Noack, P. Papas, and P. A. Monkewitz. The need for a pressure-term representation in empirical Galerkin models of incompressible shear flows. *Journal of Fluid Mechanics*, 523:339–365, 2005.

[PBK15] J. L. Proctor, S. L. Brunton, and J. N. Kutz. Dynamic mode decomposition with control. *SIAM Journal on Applied Dynamical Systems*, 15(1):142–161, 2015.

[PBK16] J. L. Proctor, S. L. Brunton, and J. N. Kutz. Generalizing Koopman Theory to allow for inputs and control. *arXiv:1602.07647v1*, 2016.

[Pei17] S. Peitz. *Exploiting Structure in Multiobjective Optimization and Optimal Control*. PhD thesis, Paderborn University, 2017.

[PHA15] L. Pyta, M. Herty, and D. Abel. Optimal Feedback Control of the Incompressible Navier-Stokes-Equations using Reduced Order Models. In *IEEE 54th Annual Conference on Decision and Control (CDC)*, pages 2519–2524, 2015.

[PSOB+17] S. Peitz, K. Schäfer, S. Ober-Blöbaum, J. Eckstein, U. Köhler, and M. Dellnitz. A Multiobjective MPC Approach for Autonomously Driven Electric Vehicles. In *Proceedings of the 20th IFAC World Congress* (preprint: *arXiv:1610.08777*), 2017.

[QGVW16] E. Qian, M. Grepl, K. Veroy, and K. Willcox. A Certified Trust Region Reduced Basis Approach to PDE-Constrained Optimization. *ACDL Technical Report TR16-3*, 2016.

[RMB+09] C. W. Rowley, I. Mezić, S. Bagheri, P. Schlatter, and D. S. Henningson. Spectral analysis of nonlinear flows. *Journal of Fluid Mechanics*, 641:115–127, 2009.

[Row05] C. W. Rowley. Model Reduction for Fluids, Using Balanced Proper Orthogonal Decomposition. *International Journal of Bifurcation and Chaos*, 15(3):997–1013, 2005.

[RTV17] S. Rogg, S. Trenz, and S. Volkwein. Trust-Region POD using A-Posteriori Error Estimation for Semilinear Parabolic Optimal Control Problems. *http://kops.uni-konstanz.de/handle/123456789/38240*, 2017.

[RV14] A. Raghunathan and U. Vaidya. Optimal Stabilization Using Lyapunov Measures. *IEEE Transactions on Automatic Control*, 59(5):1316–1321, 2014.
[Sch10] P. J. Schmid. Dynamic mode decomposition of numerical and experimental data. *Journal of Fluid Mechanics*, 656:5–28, 2010.

[Sim06] D. Simon. *Optimal state estimation: Kalman, H∞, and nonlinear approaches*. John Wiley & Sons, 2006.

[Sir87] L. Sirovich. Turbulence and the dynamics of coherent structures part I: coherent structures. *Quarterly of Applied Mathematics*, XLV(3):561–571, 1987.

[SOBG16] B. Stellato, S. Ober-Blöbaum, and P. J. Goulart. Optimal Control of Switching Times in Switched Linear Systems. In *IEEE 55th Conference on Decision and Control*, pages 7228–7233, 2016.

[SOBG17] B. Stellato, S. Ober-Blöbaum, and P. J. Goulart. Second-Order Switching Time Optimization for Switched Dynamical Systems. *IEEE Transactions on Automatic Control*, PP(99):1–8, 2017.

[TRL+] J. H. Tu, C. W. Rowley, D. M. Luchtenburg, S. L. Brunton, and J. N. Kutz. On Dynamic Mode Decomposition: Theory and Applications. *Journal of Computational Dynamics*, 1(2):391–421, 2014.

[TV09] F. Tröltzsch and S. Volkwein. POD a-posteriori error estimates for linear-quadratic optimal control problems. *Computational Optimization and Applications*, 44(1):83–115, 2009.

[VMS10] U. Vaidya, P. G. Mehta, and U. V. Shanbhag. Nonlinear Stabilization via Control Lyapunov Measure. *IEEE Transactions on Automatic Control*, 55(6):1314–1328, 2010.

[WKR15] M. O. Williams, I. G. Kevrekidis, and C. W. Rowley. A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition. *Journal of Nonlinear Science*, 25(6):1307–1346, 2015.

[XA00] X. Xu and P. Antsaklis. A dynamic programming approach for optimal control of switched systems. In *Proceedings of the 39th IEEE Conference on Decision and Control*, pages 1822–1827, 2000.

[ZA15] F. Zhu and P. J. Antsaklis. Optimal control of hybrid switched systems: A brief survey. *Discrete Event Dynamic Systems*, 25(3):345–364, 2015.