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Flocking Control of Groups of Mobile Autonomous Agents via Local Feedback *

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Abstract—This paper considers a group of mobile autonomous agents moving in Euclidean space with point mass dynamics. We introduce a set of coordination control laws that enable the group to generate the desired stable flocking motion. The control laws are a combination of attractive/repulsive and alignment forces. By using the control laws, all agent velocities asymptotically approach the desired velocity, collisions can be avoided between agents, and the final tight formation minimizes all agent potentials. Moreover, we prove that the velocity of the center of mass is always equal to the desired velocity or exponentially converges to the desired value. Furthermore, we study the motion of the group when the velocity damping is taken into account. In this case, we can properly modify the control laws to generate the same stable flocking motion. Finally, for the case that not all agents know the desired final velocity, we show that the desired flocking motion can still be obtained. Numerical simulations are worked out to illustrate our theoretical results.

Keywords—multi-agent systems, aggregation, cohesion, collision avoidance, coordination, flocking control, networked systems, information flow, local feedback, global collective behavior, swarm intelligence.

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1 Introduction

Flocking motion can be found everywhere in nature, e.g., flocking of birds, schooling of fish, and swarming of bees. Such collective behavior has certain advantages such as avoiding predators and increasing the chance of finding food. The study of collective emergent behavior of multiple mobile autonomous agents has attracted much attention in many fields such as biology, physics, robotics and control engineering. Understanding the mechanisms and operational principles in them can provide useful ideas for developing distributed cooperative control and coordination of multiple mobile autonomous agents/robots. Recently, distributed control/coordination of the motion of multiple dynamic agents/robots has emerged as a topic of major interest [1]–[4]. This is partly due to recent technological advances in communication and computation, and wide applications of multi-agent systems in many engineering areas including cooperative control of unmanned aerial vehicles (UAVs), scheduling of automated highway systems, coordination/formation of underwater vehicles, attitude alignment of satellite clusters and congestion control of communication networks [5]–[8]. There has been considerable effort in modelling and exploring the collective dynamics, and trying to understand how a group of autonomous creatures or man-made mobile autonomous agents/robots can cluster in formations without centralized coordination and control [9]–[25].

In order to generate computer animation of the motion of flocks, Reynolds [9] modelled a flying bird as an object moving in a three-dimensional environment based on the positions and velocities of its nearby flockmates, and introduced the following three rules (named steering forces) [9]:

1) Collision Avoidance: avoid collisions with nearby flockmates,
2) Velocity Matching: attempt to match velocity with nearby flockmates, and
3) Flock Centering: attempt to stay close to nearby flockmates.

Subsequently, Vicsek et al. [10] proposed a simple model of autonomous agents (i.e., points or particles). In the model, all agents move at a constant identical speed and each agent updates its heading as the average of the heading of itself with its nearest neighbors plus some additive noise. Moreover, the authors used numerical simulations to demonstrate that all agents eventually moved in the same direction, despite the absence of centralized coordination and control. In fact, Vicsek’s model can be viewed as a special case of Reynolds’s model, since it only considers the regulation of velocity matching. Jadbabie et al. [11] and Savkin [12] used two different methods to provide theoretical explanations for the observed behaviors in Vicsek’s model. Stimulated by the simulation results in [9], Tanner et al. [13]–[14] considered a group of mobile agents moving on the plane with double integrator dynamics. They introduced a set of control laws that enable the group to generate stable flocking motion, and provided theoretical justification. Nevertheless, it is perhaps more reasonable and realistic to take the agents’ masses into account and consider the point mass model in which each agent moves in high-dimensional space based on the Newton’s law. From [15], it is easy to see that, by using the control laws given in [13], the group’s final velocity relies solely on the initial velocities of all agents in the group. This means that these control laws cannot regulate the final speed and heading of the group. On the other hand, in reality, the motion of the group sometimes is inevitably influenced by some external factors. Hence, it is not enough to consider
only the interactions among agents. In some cases, the regulation of agents has certain purposes such as achieving desired common speed and heading or arriving at a desired destination. Therefore, the cooperation/coordination of multiple mobile agents with some virtual leaders is an interesting and important topic. There have been some papers dealing with this issue in the literature. For example, Leonard and Fiorelli [11] viewed reference points as virtual leaders used to manipulate the geometry of autonomous vehicle group and direct the motion of the group. [20] and [11] considered the cohesion/coordination of a group of mobile autonomous agents following an actual leader by the so-called nearest neighbor rules.

In this paper, we investigate the collective behavior of multi-agent systems in n-dimensional space with point mass dynamics. By viewing the external control signals (or “mission”) as virtual leaders, we show that all agents eventually move ahead at a desired common velocity and maintain constant distances between them. During the course of motion, each agent is influenced by the external control signal and the motion of other agents in the group. In order to generate the desired stable flocking, we introduce a set of control laws such that each agent regulates its position and orientation based on the desired velocity and the information of a fixed set of “neighbors”. The control laws are a combination of attractive/repulsive and alignment forces. By using the control laws, all agent velocities asymptotically approach the desired value, collisions can be avoided between agents, and the final tight formation minimizes all agent potentials. One salient feature of this paper is that the self-organized global behavior is achieved via local feedback, i.e. the desired emergent dynamics is produced through local interactions and information exchange between the dynamic agents.

This paper is organized as follows: In Section 2, we formulate the problem to be investigated. Some basic concepts and results in graph theory are provided in Section 3. By using some specific control laws, we analyze the system stability, the motion of the center of mass (CoM), and the convergence rate of the system in Section 4. We present some different control laws that can also generate the desired stable flocking motion in Section 5. For the case that not all agents know the desired velocity, we introduce a set of control laws and study the system stability in Section 6. Some numerical simulations are presented in Section 7. Finally, we briefly summarize our results in Section 8.

2 Problem Formulation

We consider a group of $N$ agents moving in an $n$-dimensional Euclidean space, each has point mass dynamics described by

$$
\dot{x}^i = v^i, \\
m_i \dot{v}^i = u^i, \quad i = 1, \ldots, N,
$$

(1)

where $x^i = (x_{i1}, \ldots, x_{in})^T \in \mathbb{R}^n$ is the position vector of agent $i$, $v^i = (v_{i1}, \ldots, v_{in})^T \in \mathbb{R}^n$ is its velocity vector, $m_i > 0$ is its mass, and $u^i = (u_{i1}, \ldots, u_{in})^T \in \mathbb{R}^n$ is the (force) control input acting on agent $i$. $x^{ij} = x^i - x^j$ denotes the relative position vector between agents $i$ and $j$.

Our objective is to make the entire group move at a desired velocity and maintain constant distances between the agents. Additionally, we choose the control laws such
that, during the course of motion, collisions can be avoided between agents, and the group final configuration minimizes all agent potentials. In what follows, we will investigate the motion of the agent group in two different cases, that is, we consider the group motion in ideal case (i.e., velocity damping is ignored) and nonideal case, respectively. For the two different cases, we propose two different control laws such that the entire group moves at a desired common velocity, and at the same time, collision-free can be ensured between agents, and the group final configuration minimizes all agent potentials.

We first consider the ideal case, that is, we ignore the velocity damping. In this case, in order to achieve our objective, we try to regulate each agent velocity to the desired velocity, reduce the velocity differences between agents, and at the same time, regulate their distances such that their potentials become minimum. Hence, we choose the control law $u^i$ for agent $i$ to be

$$u^i = \alpha^i + \beta^i + \gamma^i,$$

where $\alpha^i$ is used to regulate the potentials among agents, $\beta^i$ is used to regulate the velocity of agent $i$ to the weighted average of its flockmates, and $\gamma^i$ is used to regulate the momentum of agent $i$ to the desired final momentum (all to be designed later). $\alpha^i$ is derived from the social potential fields which is described by artificial social potential function $V^i$, a function of the relative distances between agent $i$ and its flockmates. Collision-free and cohesion in the group can be guaranteed by this term. $\beta^i$ reflects the alignment or velocity matching with neighbors among agents. $\gamma^i$ is designed to regulate the momentum among agents based on the external signal (the desired velocity). By using such a momentum regulation, we can obtain the explicit convergence rate of the CoM of the system.

**Remark 1:** The design of $\alpha^i$ and $\beta^i$ indicates that, during the course of motion, agent $i$ is influenced only by its “neighbors”, whereas $\gamma^i$ reflects the influence of the external signal on the agent motion.

Certainly, in some cases, the velocity damping can not be ignored. For example, objects moving in viscous environment and mobile objects with high speeds such as supersonic aerial vehicles, are subjected to the influence of velocity damping. Then, in this case, the model should be in the following form

$$\dot{x}^i = v^i,$$

$$m_i \dot{v}^i = u^i - k_i v^i,$$

where $k_i > 0$ is the “velocity damping gain”, $-k_i v^i$ is the velocity damping term, and $u^i$ is the control input for agent $i$. Here we assume that the damping force is in proportion to the magnitude of velocity. Moreover, since the “velocity damping gain” is determined by the shape and size of the object, the property of the medium and some other factors, we assume that the damping gains $k_i$, $i = 1, \cdots, N$ are not equal to each other. In order to achieve our objective, we need to compensate for the velocity damping. Hence, we modify the control law $u^i$ to be

$$u^i = \alpha^i + \beta^i + \gamma^i + k_i v^i.$$
3 Main Results

In this section, we investigate the stability properties of multiple mobile agents with point mass dynamics described in (1). We will present explicit control input in (2) for the terms $\alpha^i$, $\beta^i$, and $\gamma^i$. We will employ algebra and graph theory as basic tools for our discussion. Some concepts and results in graph theory are given in the Appendix.

Following [13], we make the following definitions and assumptions.

Definition 1 [13]: (Neighboring graph) The neighboring graph, $G = (\mathcal{V}, \mathcal{E})$, is an undirected graph consisting of
   
   i) a set of vertices, $\mathcal{V} = \{n_1, \cdots, n_N\}$, indexed by the agents in the group, and
   ii) a set of edges, $\mathcal{E} = \{(n_i, n_j) \in \mathcal{V} \times \mathcal{V} | n_j \sim n_i\}$, containing unordered pairs of vertices that represent the neighboring relations.

In this paper, we consider a group of mobile agents with fixed topology. We assume that the neighboring graph $G$ is connected, and hence does not change with time. Denote the set $\mathcal{N}_i = \{j | j \sim i\} \subseteq \{1, \cdots, N\}\{i\}$ which contains all neighbors of agent $i$. If agent $j$ is not a neighbor of agent $i$, we denote $j \not\sim i$.

Definition 2 [13]: (Potential function) Potential $V^{ij}$ is a differentiable, nonnegative, radially unbounded function of the distance $\|x^{ij}\|$ between agents $i$ and $j$, such that

   i) $V^{ij}(\|x^{ij}\|) \to \infty$ as $\|x^{ij}\| \to 0$,
   ii) $V^{ij}$ attains its unique minimum when agents $i$ and $j$ are located at a desired distance.

Functions $V^{ij}$, $i, j = 1, \cdots, N$ are the artificial social potential functions that govern the interindividual interactions. Cohesion and separation can be achieved by artificial potential fields [6]. In fact, cohesion can be ensured by the connectivity of the neighboring graph, but collision-free can only be guaranteed between interconnected agents. Collision can be avoided between all agents only when the neighboring graph is complete.

By the definition of $V^{ij}$, the total potential of agent $i$ can be expressed as

$$V^i = \sum_{j \in \mathcal{N}_i} V^{ij}(\|x^{ij}\|).$$

Agent dynamics in ideal case is different from that in nonideal case, i.e., agents have different motion equations in the two cases. Hence, in what follows, we will discuss the motion of the agent group in two different cases, respectively.

Note that, in this section, we always assume that all agents can receive the external signal, that is, they all know the desired final velocity. In the case that not all agents know the mission, we will discuss the flocking control problem in a separate section.

3.1 Ideal Case

In this case, we take the control law $u^i$ for agent $i$ to be

$$u^i = -\sum_{j \in \mathcal{N}_i} w_{ij}(v^i - v^j) - \sum_{j \in \mathcal{N}_i} \nabla_{x^i} V^{ij} - m_i(v^i - v^0),$$

where $v^0 \in \mathbb{R}^n$ is the desired common velocity and is a constant vector, $w_{ij} \geq 0$, $w_{ij} = w_{ji}$, and $w_{ii} = 0$, $i, j = 1, \cdots, N$ represent the interaction coefficients. And $w_{ij} > 0$ if agent $j$
is a neighbor of agent \( i \), and is 0 otherwise. We denote \( W = [w_{ij}] \). Thus, \( W \) is symmetric, and by the connectivity of the neighboring graph, \( W \) is irreducible.

### 3.1.1 Stability Analysis

Before presenting the main results of this paper, we first prove an important lemma.

**Lemma 1**: Let \( A \in \mathbb{R}^{n \times n} \) be any diagonal matrix with positive diagonal entries. Then

\[
(\text{Aspan}\{1\}^\perp) \cap \text{span}\{1\} = 0,
\]

where \( 1 = (1, \ldots, 1)^T \in \mathbb{R}^n \) is the space spanned by the vector \( 1 \), and \( \text{span}\{1\}^\perp \) is the orthogonal complement space of \( \text{span}\{1\} \).

**Proof**: Let \( p \in (\text{Aspan}\{1\}^\perp) \cap \text{span}\{1\} \). Then \( p \in \text{span}\{1\} \) and there is some \( q \in \text{span}\{1\}^\perp \) such that \( p = Aq \). It follows that \( q^T A q = q^T p = 0 \). Since \( A \) is positive definite by assumption, we have \( q = 0 \) and hence \( p = 0 \). \( \square \)

**Theorem 1**: By taking the control law in (6), all agent velocities in the group described in (1) asymptotically approach the desired common velocity, collision-free is ensured between neighboring agents, and the group final configuration minimizes all agent potentials.

This theorem becomes apparently true after Theorem 2 is proved, so we proceed to present Theorem 2.

We define the following error vectors:

\[
e_p^i = x^i - v^0 t,
\]

\[
e_v^i = v^i - v^0,
\]

where \( t \) is time variable and \( v^0 \) is the desired common velocity. \( e_p^i \) represents the relative position vector between the actual position of agent \( i \) and its desired position. \( e_v^i \) represents the velocity difference vector between the actual velocity and the desired velocity of agent \( i \). It is easy to see that \( \dot{e}_p^i = e_v^i \), and \( \dot{e}_v^i = \dot{v}^i \). Hence, the error dynamics is given by

\[
\dot{e}_p^i = e_v^i,
\]

\[
\dot{e}_v^i = \frac{1}{m_i} u^i, \quad i = 1, \ldots, N.
\]

By the definition of \( V^{ij} \), it follows that

\[
V^{ij}(\|x^{ij}\|) = V^{ij}(\|e_p^{ij}\|) := \bar{V}^{ij},
\]

where \( e_p^{ij} \triangleq e_p^i - e_p^j \), and hence \( \bar{V}^i = V^i \) and \( \nabla e_v^i \bar{V}^{ij} = \nabla x^i V^{ij} \). Thus, the control input for agent \( i \) in the error system has the following form

\[
u^i = - \sum_{j \in N_i} w_{ij} (e_v^i - e_v^j) - \sum_{j \in N_i} \nabla e_v^j \bar{V}^{ij} - m_i e_v^i.
\]

Consider the following positive semi-definite function

\[
J = \frac{1}{2} \sum_{i=1}^N \left( \bar{V}^i + m_i e_v^i e_v^i \right).
\]
It is easy to see that $J$ is the sum of the total artificial potential energy and the total kinetic energy of all agents in the error system. Define the level set of $J$ in the space of agent velocities and relative distances in the error system

$$
\Omega = \{(e^i_v, e^{ij}_p)|J \leq c\}.
$$

(12)

In what follows, we will prove that the set $\Omega$ is compact. In fact, the set $\{e^i_v, e^{ij}_p\}$ with $J \leq c$ ($c > 0$) is closed by continuity. Moreover, boundedness can be proved by connectivity. More specifically, from $J \leq c$, we have $\tilde{V}^{ij} \leq c$. Moreover, since the potential function $V^{ij}$ is radially unbounded, $\tilde{V}^{ij}$ is also radially unbounded, and there is a positive constant $d_{ij}$ such that $\|e^{ij}_p\| \leq d_{ij}$. Denote $d = \max_{j \in N_i} d_{ij}$. Since the neighboring graph is connected, there must be a path connecting any two agents $i$ and $j$, and its length does not exceed $N - 1$. Hence, we have $\|e^{ij}_p\| \leq (N - 1)d$. By similar analysis, we have $e^{iT}_v e^i_v \leq 2c/m_i$, thus $\|e^i_v\| \leq \sqrt{2c/m_i}$.

By the symmetry of $\tilde{V}^{ij}$ with respect to $e^{ij}_p$ and by $e^{ij}_p = -e^{ji}_p$, it follows that

$$
\frac{\partial \tilde{V}^{ij}}{\partial e^j_p} = \frac{\partial \tilde{V}^{ij}}{\partial e^i_p} = -\frac{\partial \tilde{V}^{ij}}{\partial e^j_p},
$$

(13)

and therefore

$$
\frac{d}{dt} \sum_{i=1}^N \frac{1}{2} \tilde{V}^i = \sum_{i=1}^N \nabla e^i_p \tilde{V}^i \cdot e^i_v.
$$

Theorem 2: By taking the control law in (10), all agent velocities in the system described in (9) asymptotically approach zero, collision-free is ensured between neighboring agents, and the group final configuration minimizes all agent potentials.

Proof: Choosing the positive semi-definite function $J$ defined as in (11) and calculating the time derivative of $J$ along the solution of the error system (9), we have

$$
\dot{J} = - \sum_{i=1}^N \sum_{j \in N_i} w_{ij} e^{iT}_v (e^i_v - e^j_v) - \sum_{i=1}^N m_i e^{iT}_v e^i_v
$$

(14)

$$
= -e^{iT}_v (L \otimes I_n) e^i_v - e^{iT}_v (M \otimes I_n) e^i_v,
$$

where $e_v = (e^1_v, \cdots, e^N_v)^T$ is the stack vector of all agent velocity vectors in the error system; $L = [l_{ij}]$ with

$$
l_{ij} = \begin{cases} 
-w_{ij}, & i \neq j, \\
\sum_{k=1, k \neq i}^N w_{ik}, & i = j;
\end{cases}
$$

(15)

$M = \text{diag}(m_1, \cdots, m_N)$; $I_n$ is the identity matrix of order $n$ and $\otimes$ stands for the Kronecker product.

By the definition of matrix $L$, it is easy to see that $L$ is symmetric, each row sum is equal to 0, the diagonal entries are positive, and all the other entries are nonpositive. By matrix theory [27], all eigenvalues of $L$ are nonnegative. Hence, matrix $L$ is positive semi-definite. By the connectivity of the neighboring graph and the symmetry of matrix $L$, it follows that $L$ is irreducible and the eigenvector associated with the single zero eigenvalue
is $1_N$. On the other hand, it is known that the identity matrix $I_n$ has an eigenvalue $\mu = 1$ of $n$ multiplicity and $n$ linearly independent eigenvectors

$$ p^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad p^2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad p^n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. $$

By matrix theory \cite{27}, the eigenvalues of $L \otimes I_n$ are nonnegative, $\lambda = 0$ is an eigenvalue of multiplicity $n$ and the associated eigenvectors are

$$ q^1 = [p^1T, \ldots, p^nT]^T, \quad \ldots, \quad q^n = [p^nT, \ldots, p^1T]^T. $$

Furthermore, it is easy to see that matrix $M$ is positive definite and hence $-e_v^T(M \otimes I_n)e_v \leq 0$. Thus $\dot{J} \leq 0$, and $\dot{J} = 0$ implies that $e_v^1 = e_v^2 = \cdots = e_v^n$ and they all must equal zero. This occurs only when $v^1 = v^2 = \cdots = v^n = 0$, that is, the vector $e_{vk} = (e_{vk}^1, \ldots, e_{vk}^n)$ $(k = 1, \ldots, n)$, which is composed of all the corresponding $k$th components $e_{vk}^1, \ldots, e_{vk}^n$ of $e_v^1, \ldots, e_v^n$, is contained in span$\{1\}$, where $1 = (1, \ldots, 1)^T \in \mathbb{R}^n$ and each entry $e_{vk}^i$ of $e_{vk}$ equals zero. It follows that $\dot{e}_{ij}^k = 0, \forall (i,j) \in N \times N$.

We use LaSalle’s invariance principle \cite{28} to establish convergence of the system trajectories to the largest positively invariant subset of the set defined by $E = \{e_v | \dot{J} = 0\}$. In $E$, the agent velocity dynamics in the error system is

$$ \dot{e}_v^i = \frac{1}{m_i} u^i = -\frac{1}{m_i} \sum_{j \in N_i} \nabla e_p^{ij} \tilde{V}^{ij} = -\frac{1}{m_i} \nabla e_p^{ij} \tilde{V}^{ij} $$

and therefore it follows that

$$ \dot{e}_v = -(M^{-1}B \otimes I_n) \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix}, $$

where $M^{-1} = \text{diag}(\frac{1}{m_1}, \ldots, \frac{1}{m_N})$ is the inverse of matrix $M$, and matrix $B$ is the incidence matrix of the neighboring graph. Hence

$$ \dot{e}_{vk} = -(M^{-1}B)[\nabla e_{pk}^{ij} \tilde{V}^{ij}]_k, \quad k = 1, \ldots, n. $$

Thus, $\dot{e}_{vk} \in \text{range}(M^{-1}B), \ k = 1, \ldots, n$. By matrix theory and by the connectivity of the neighboring graph $G$, we have

$$ \text{range}(M^{-1}B) = M^{-1}\text{range}B = M^{-1}\text{range}(BB^T) = M^{-1}\text{span}\{1\} $$

and therefore

$$ \dot{e}_{vk} \in M^{-1}\text{span}\{1\}, \quad k = 1, \ldots, n. \ \ \ \ (17) $$

In any invariant set of $E$, by $e_{vk} \in \text{span}\{1\}$, we have

$$ \dot{e}_{vk} \in \text{span}\{1\}. \ \ \ \ (18) $$

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Furthermore, by Lemma 1, we get from (17) and (18) that
\[ \dot{e}_{vk} \in (M^{-1}\text{span}\{1\})^\perp \cap \text{span}\{1\} \equiv 0, \quad k = 1, \ldots, n. \]
Thus, in steady state, all agent velocities in the error system no longer change and equal zero, and moreover, from (16), the potential \( \tilde{V}_i \) of each agent \( i \) is minimized. Collision-free can be ensured between neighboring agents since otherwise it will result in \( \tilde{V}_i \to \infty \). □

From the proof of Theorem 2, it follows that, in steady state, all agent actual velocities no longer change and are equal to the desired velocity.

Remark 2: Only when the neighboring graph is complete, collision avoidance between all agents can be guaranteed with the control laws above.

Remark 3: If we take the control law for agent \( i \) to be
\[ u^i = -\sum_{j \in N_i} (v^i - v^j) - \sum_{j \in N_i} \nabla x^i V^{ij} - m_i (v^i - v^0), \tag{19} \]
we can also get the same conclusion as in Theorem 1. Here, we still consider the error system (9). In fact, if we take the same Laypunov function \( J \) defined as in Theorem 2 and take the control law in (19), we obtain that
\[ \dot{J} = -e_v^T (\overline{L} \otimes I_n) e_v - e_v^T (M \otimes I_n) e_v, \]
where \( \overline{L} = \Delta - \mathcal{A} \) is the Laplacian matrix of the neighboring graph, \( \Delta = \text{diag}(N_1, \ldots, N_N) \), \( N_i \) denotes the valance of vertex \( i \) in the graph, and \( \mathcal{A} \) is the adjacency matrix of the graph. Using a similar analysis method as in Theorem 2, we can obtain the same conclusion of stable flocking. Note that, in comparison with Theorem 2, we have the following difference on the decaying rates of the energy function \( J \)
\[ -e_v^T (L \otimes I_n) e_v + e_v^T (\overline{L} \otimes I_n) e_v = -e_v^T (\overline{L} \otimes I_n) e_v, \]
where \( \overline{L} = \overline{[l_{ij}]} \) with
\[ \overline{l}_{ij} = \begin{cases} -w_{ij} + 1, & i \neq j \text{ and } j \sim i, \\ 0, & i \neq j \text{ and } j \sim i, \\ \sum_{k=1, k \neq i}^N w_{ik} - N_i, & i = j. \end{cases} \]
It is easy to see that, by using the different control laws in (3) and (19), the decaying rates of the total energy \( J \) may be different. Hence, the interaction coefficients \( w_{ij} \) can influence the convergence rate of the system.

3.1.2 The Motion of the Center of Mass

In what follows, we will analyze the motion of the center of mass (CoM) of system (11).

The position vector of the CoM in system (11) is defined as
\[ x^* = \frac{\sum_{i=1}^N m_i x^i}{\sum_{i=1}^N m_i}. \]
Thus, the velocity vector of the CoM is
\[ v^* = \frac{\sum_{i=1}^{N} m_i v_i}{\sum_{i=1}^{N} m_i}. \]

By using control law (6), we obtain
\[ \dot{v}^* = \frac{-1}{\sum_{i=1}^{N} m_i} \sum_{i=1}^{N} \left[ \sum_{j \in N_i} w_{ij} (v_i - v_j) + \sum_{j \in N_i} \nabla x_i V_{ij} + m_i (v_i - v^0) \right]. \]

By the symmetry of matrix \( W \) and the symmetry of function \( V_{ij} \) with respect to \( x_{ij} \), we get
\[ \dot{v}^* = -v^* + v^0. \] (20)

Suppose the initial time \( t_0 = 0 \), and \( v^*(0) = v^*_0 \). By solving (20), we get
\[ v^* = v^0 + (v^*_0 - v^0) e^{-t}. \]

Thus, it follows that, if \( v^*_0 = v^0 \), then the velocity of the CoM is invariant and equals \( v^0 \) for all the time; if \( v^*_0 \neq v^0 \), then the velocity of the CoM exponentially converges to the desired velocity \( v^0 \) with convergence exponent 1.

Therefore, from the analysis above, we have the following theorem.

**Theorem 3:** By taking the control law in (6), if the initial velocity of the CoM is equal to the desired velocity, then it is invariant for all the time; otherwise it will exponentially converge to the desired velocity.

**Remark 4:** Note that, by the calculation above, we can see that the velocity variation of the CoM does not rely on the neighboring relations or the magnitudes of the interaction coefficients. Even if the neighboring graph is not connected, the velocity of the CoM still equals the desired velocity or exponentially converges to it, and the final velocities of all connected agent groups equal the desired velocity as well. However, in this case, the distance between disconnected subgroups might be very far.

### 3.1.3 Convergence Rate Analysis

From the discussion above, we know that the coupling coefficients can influence the convergence rate of system (1). In what follows, we will present some qualitative analysis of the influence of the weights on the convergence rate of the system.

Let us again consider the dynamics of the error system. From the analysis in Theorem 2, we know that \( \dot{J} \leq 0 \), and \( \dot{J} = 0 \) occurs only when \( e^1_v = e^2_v = \ldots = e^N_v = 0 \), that is, only when all agents have reached the desired velocity. In other words, if there exists one agent whose velocity is different from the desired velocity, then the energy function \( J \) is strictly monotone decreasing with time. Of course, before the group forms the final tight configuration, there might be the case that all agent velocities have reached the desired value, but due to the regulation of the potentials among neighboring agents, it instantly changes into the case that not all agents have the desired velocity. Hence, the decaying rate of energy is equivalent to the convergence rate of the system. It is easy to see that, when not all agents have reached the desired velocity, for any solution of the error
system, $e_v$ must be in the subspace spanned by the eigenvectors of $L \otimes I_n$ corresponding to the nonzero eigenvalues. Thus, from (13), we have $\dot{J} \leq -\lambda_2 e_v^T e_v - e_v^T (M \otimes I_n) e_v$, where $\lambda_2$ denotes the second smallest real eigenvalue of matrix $L$. Therefore, we have the following conclusion: The convergence rate of the system relies on the second smallest real eigenvalue of matrix $L$ defined as in (15) as well as agent masses, and it is always not faster than the convergence rate of the CoM. Furthermore, if the initial velocity of the CoM is not equal to the desired velocity, then the fastest convergence rate of the system does not exceed the exponential convergence rate with convergence exponent 1.

**Remark 5**: Note that when the group has achieved the final steady state, the control input above equals zero.

### 3.2 Nonideal Case

Sometimes, the velocity damping should not be ignored. Then, in this case, in order to make the group generate the desired stable flocking motion, the velocity damping need to be cancelled by some terms in the control laws. Hence, we modify the control law as in (4), where $\alpha^i$, $\beta^i$, and $\gamma^i$ are defined as in (6), that is, the control law acting on agent $i$ is

$$u^i = -\sum_{j \in N_i} w_{ij} (v^i - v^j) - \sum_{j \in N_i} \nabla x_i V^i_{ij} - m_i (v^i - v^0) + k_i v^i.$$  \hspace{2cm} (21)

Then, the total force acting on agent $i$ is

$$u^i = -\sum_{j \in N_i} w_{ij} (v^i - v^j) - \sum_{j \in N_i} \nabla x_i V^i_{ij} - m_i (v^i - v^0).$$

All the results in ideal case can be analogously extended to the nonideal case. Namely, following Theorems 1 and 2, we can easily obtain the desired stable flocking motion, that is, when the velocity damping is taken into account, by using control law (21), all agent velocities in the group described in (3) asymptotically approach the desired value, collision-free can be ensured between neighboring agents, and the group final configuration minimizes all agent potentials. Furthermore, following Theorem 3 and the convergence rate analysis above, we conclude that the convergence rate of the system relies on the interaction coefficients and agent masses, and when the initial velocity of the CoM is not equal to the desired velocity, the fastest convergence rate of the system does not exceed the exponential convergence rate with convergence exponent 1.

Note also that, because the velocity damping is cancelled by some terms in the control law, the velocity damping cannot influence the convergence rate of system (3).

**Remark 5**: In steady state, the group keeps on moving at a desired velocity. During this period, the control laws’ role is only to cancel the velocity damping.

### 4 Discussions on Various Control Laws

In the sections above, we introduced a set of control laws that enable the group to generate the desired stable flocking motion. However, it should be clear that control law (6) is not the unique control law to produce the desired motion for the group. In this section, we
provide some more useful control laws. For simplicity, we only present the control laws for the group moving in the ideal case, since in the nonideal case, we only need to add the terms $k_i v^i (i = 1, \ldots, N)$ to cancel the velocity damping.

In the sequel, we will propose three different control laws that can achieve our control objective. The analysis and proofs are quite similar for these control laws, so we only present the control laws and their corresponding Lyapunov functions.

1) In the control laws above, $\gamma^i$ is used to regulate the momentum of agent $i$. However, we can also use $\gamma^i$ to directly regulate the velocity of agent $i$ to the desired value. Hence, we take the control law acting on agent $i$ to be

$$u^i = -\sum_{j \in N_i} w_{ij} (v^j - v^i) - \sum_{j \in N_i} \nabla_{x^i} V_{ij} - (v^i - v^0). \tag{22}$$

We still consider the error system (9) and choose Lyapunov function (11). By similar calculation, we get

$$\dot{J} = -e_v^T (L \otimes I_n) e_v - e_v^T e_v.$$ Using the same analysis method as in Theorem 2, we obtain that $\dot{J} \leq 0$, and $\dot{J} = 0$ implies that $e_v^1 = e_v^2 = \cdots = e_v^N = 0$. The rest analysis is similar to Theorem 2, thus is omitted.

Remark 6: Note that, the control law in (22) can make the group generate the desired stable flocking motion. But we cannot explicitly estimate the convergence rate of the CoM by using this control law.

2) Suppose that $\alpha^i$ and $\beta^i$ rely on agent $i$’s mass. The control law acting on agent $i$ has the following form

$$u^i = -\sum_{j \in N_i} m_i w_{ij} (v^j - v^i) - \sum_{j \in N_i} m_i \nabla_{x^i} V_{ij} - m_i (v^i - v^0). \tag{23}$$

In this case, for the error system (9), we choose the following Lyapunov function

$$J = \frac{1}{2} \sum_{i=1}^N \left( \bar{V}^i + e_v^i e_v^i \right). \tag{24}$$

By similar calculation, we have

$$\dot{J} = -e_v^T (L \otimes I_n) e_v - e_v^T e_v.$$ Following the analysis method in Theorem 2, we can show that the desired stable flocking motion will be achieved.

Definition 3: Define the center of the system of agents as $\bar{x} = (\sum_{i=1}^N x^i)/N$.

Definition 4: The average velocity of all agents is defined as $\bar{v} = (\sum_{i=1}^N v^i)/N$.

It is obvious that the velocity of the system center is just the average velocity of all agents.

Using the control law in (23), we have $\dot{v} = -\bar{v} + v^0$. Suppose the initial time $t_0 = 0$ and $\bar{v}(0) = \bar{v}_0$. We get

$$\bar{v} = v^0 + (\bar{v}_0 - v^0) e^{-t}.$$
It is obvious that, if $\tau_0 = v^0$, then the velocity of the system center is equal to the desired velocity $v^0$ for all the time, and if $\tau_0 \neq v^0$, then the velocity of the system center exponentially converges to the desired velocity with convergence exponent 1.

3) Suppose that $\alpha^i$ and $\beta^i$ rely on agent $i$’s mass, and $\gamma^i$ is used to regulate the velocity of agent $i$ to the desired velocity. The control law $u^i$ is then taken to be

$$u^i = -\sum_{j \in N_i} m_i w_{ij} (v^i - v^j) - \sum_{j \in N_i} m_i \nabla x_i V_{ij} - (v^i - v^0). \quad (25)$$

We consider the error system (9) and choose the corresponding Lyapunov function (24). Then,

$$\dot{J} = -e_v^T (L \otimes I_n)e_v - e_v^T (M^{-1} \otimes I_n)e_v,$$

where $M^{-1}$ is the inverse of matrix $M$. The rest analysis is similar, thus is omitted.

Remark 7: Note that, using the control law in (25), the convergence rates of the CoM and the system center both cannot be explicitly estimated.

From the analysis above, we conclude that the control law in (6) is the best one among the various control laws. On the one hand, control law (6) can be given certain physical explanations, on the other hand, the corresponding Lyapunov function has certain physical meaning. More importantly, by using the control law in (6), the convergence rate of the CoM of the system can be accurately estimated.

5 Extensions and Discussions

In this section, we investigate the case that not all agents know the desired velocity. We assume that the neighboring graph is connected and the neighboring relations are fixed.

We first divide the group into two subgroups. Subgroup One consists of all agents that can detect the reference signal, i.e., all agents who know the desired velocity belong to Subgroup One. Subgroup Two contains all agents that can not detect the reference signal. Hence, each agent in Subgroup One regulates its state based on the reference signal and the information of its “neighbors”, whereas each agent in Subgroup Two regulates its state only based on its “neighbors”.

We assume that there exists at least one agent who knows the desired velocity. In the case that there is no external signal acting on the group, the collective dynamic behaviors of the agent group have been analyzed in [15].

Without loss of generality, suppose that agent $i$ ($i = 1, \cdots, N_1$) ($1 \leq N_1 < N$) are contained in Subgroup One, and agent $j$ ($j = N_1 + 1, \cdots, N$) are contained in Subgroup Two. The control law acting on agent $i$ in Subgroup One is taken to be

$$u^i = -\sum_{k \in N_i} w_{ik} (v^i - v^k) - \sum_{k \in N_i} \nabla x_i V^{ik} - m_i (v^i - v^0),$$

and the control law acting on agent $j$ in Subgroup Two is taken to be

$$u^j = -\sum_{k \in N_j} w_{jk} (v^j - v^k) - \sum_{k \in N_j} \nabla x_j V^{jk}.$$
We can lump the two equations above into one

\[ u^i = - \sum_{j \in \mathcal{N}_i} w_{ij} (v^i - v^j) - \sum_{j \in \mathcal{N}_i} \nabla x^i V^{ij} - h_i m_i (v^i - v^0) \]  

for \( i = 1, \cdots, N \), where \( h_i \) is defined as

\[ h_i = \begin{cases} 1, & \text{if agent } i \text{ is contained in Subgroup One}, \\ 0, & \text{if agent } i \text{ is contained in Subgroup Two}. \end{cases} \]

We still consider the error system (11). Using control law (26) and taking Lyapunov function (11), we have

\[ \dot{J} = -e^T_v (L \otimes I_n) e_v - e^T_v (\hat{M} \otimes I_n) e_v, \]

where \( L, e^i_v, \) and \( I_n \) are defined as before, \( \hat{M} = \text{diag}(h_1 m_1, \cdots, h_N m_N) \). By the definition of \( h_i \) and \( m_i \), it follows that matrix \( \hat{M} \) is positive semi-definite. Following similar analysis as in the previous sections, we can conclude that the desired stable flocking motion can be achieved.

**Remark 8:** If there exists only one agent in the group who can detect the external reference signal, the group can still generate the desired stable flocking motion. This is of practical interest in control of multi-agent systems.

**Remark 9:** Even if only one agent in the group cannot detect the external reference signal, it is difficult to explicitly estimate the convergence rate of the CoM.

It should be noted that there is no actual leader among agents, all agents play the same role. However, we can view the external reference signal as a virtual leader.

The results in this section suggest that, if we want to control a group of mobile agents to move at a given velocity, we only need to send our mission signal to any one of them. Then the signal can be propagated through the neighboring interactions. This is of practical interest in control of multiple mobile robots or a large population of animals (think how do you pass through a crowds of people? and how the shepherding dog steer a large group of sheep back home?).

### 6 Simulations

In this section, we will present some numerical simulations for the system described by (1) in order to illustrate the theoretic results obtained in the previous sections.

These simulations are performed with ten agents moving on the plane whose initial positions, velocities and the neighboring relations are set randomly, but they satisfy: 1) all initial positions are set within a ball of radius \( R = 15[m] \) centered at the origin, 2) all initial velocities are set with arbitrary directions and magnitudes within the range of \((0, 10)[m/s]\), and 3) the neighboring graph is connected. All agents have different masses and they are set randomly in the range of \((0, 1)[kg]\).

The following simulations are all performed with the same group, and the group has the same initial state, including all agent initial positions, velocities, and the fixed neighboring...
relations between agents. However, different control laws are taken in the form of (8) or (26) with the explicit potential function

\[ V^{ij} = 5 \ln \|x^{ij}\|^2 + \frac{5}{\|x^{ij}\|^2}, \quad i, j = 1, \cdots, 10. \]

The interaction coefficient matrix \( W \) is generated randomly such that \( w_{ii} = 0 \), \( w_{ij} = w_{ji} \), and the nonzero \( w_{ij} \)s satisfy \( 0 < w_{ij} < 1 \) for all \( i, j = 1, \cdots, 10 \). We run all simulations for 250 seconds.

Fig. 1 presents the group initial state including the initial positions and velocities of all agents, and the neighboring relations between agents. Figs. 2–4, 5–7 and 8–10 show the motion trajectories of all agents, the final configurations of the group, and all agent velocities in three different simulations, respectively. Figs. 11 and 12 depict the motion trajectories of the CoM in the three simulations, whereas Figs. 13 and 14 are the velocity curves of the CoM. Note that, in the velocity curve figures, the solid arrow indicates the tendency of velocity variation, and the meanings of the other arrows, dashed lines, and solid lines are all presented in the figures.

In Fig. 1, the solid lines represent the neighboring relations and the dotted arrows represent the initial velocity vectors. Figs. 2–4 describe the group state in the case that the motion of the group is not influenced by any external signal and only relies on the interactions between agents. It can be seen from them that, during the course of motion, all agents regulate their positions to minimize their potentials, regulate their velocities to reduce the differences, and move ahead with a steady state configuration. Moreover, the final common velocity is equal to the initial velocity of the CoM of the system.

When we send a signal to the group and try to make all agents move at a desired velocity, Figs. 5–7 show the results in our simulation with the control laws taken in the form of (8), whereas Figs. 8–10 show the simulation results with the control laws taken in the form of (26) and with the assumption that there is only one agent who knows the desired velocity. It can be seen from them that all agents regulate their positions to minimize their potentials and eventually move ahead with a steady state configuration. Figs. 7 and 10 are the velocity curves, and they distinctly demonstrate that all agent velocities asymptotically approach the desired velocity.

Note that the final configurations of the group are different in the three simulations. This is because, during the course of motion, each agent regulates its position only based on the information of its “neighbors” in the group, hence collisions cannot be avoided between the agents having no neighboring relations.

Fig. 11 shows the motion trajectories of the CoM in the simulations where the star represents the initial position of the CoM, and Fig. 12 is the magnification of the trajectories of the CoM at the initial time. Fig. 13 shows the velocity curves of the CoM where the star represents the initial velocity of the CoM, and Fig. 14 is the magnification of the velocity curves of the CoM at the initial time. In these four figures, (a), (b), and (c) represent the corresponding states of the CoM in the three simulations, respectively. It can be seen from them that, when there is no external signal acting on the group, the velocity of the CoM is always invariant and is equal to the final common velocity, otherwise, the velocity of the CoM converges to the desired velocity. Apparently, the convergence rate of the CoM is faster than the convergence rate of the system.
Hence, numerical simulations also indicate that, by using the control law in (6), the desired stable flocking motion can be achieved.

7 Conclusions

We have investigated the collective behavior of multiple dynamic agents moving in high-dimensional space with point mass dynamics, and presented some control laws which ensure the group to generate the desired stable flocking motion. The group dynamic properties are characterized in two different cases. When the velocity damping is negligible, using a set of coordination control laws, we can make the group generate the desired stable flocking motion. The control laws are a combination of attractive/repulsive and alignment forces, and they ensure that all agent velocities asymptotically approach the desired velocity, collisions are avoided between neighboring agents, and the final tight formation minimizes all agent potentials. Moreover, we showed that, when the initial velocity of the center of mass is not equal to the desired velocity, it will exponentially converge to the desired velocity. When the velocity damping is taken into account, we can properly modify the control laws in order to generate the desired stable flocking. Subsequently, we investigated the motion of the group in the case that not all agents know the desired final velocity, and showed that the desired stable flocking motion can still be achieved by our control laws. Finally, numerical simulations were worked out to further illustrate our theoretical results. Our method is general, integrating both algebraic theory and graph theory, and is applicable to dealing with more complex agent dynamics, information topology and interaction mechanisms.

8 Appendix: Graph Theory Preliminaries

In this section, we briefly summarize some basic concepts and results in graph theory that have been used in this paper. More comprehensive discussions can be found in [29].

A graph \( G \) consists of a vertex set \( V = \{n_1, n_2, \ldots, n_m\} \) and an edge set \( E = \{(n_i, n_j) : n_i, n_j \in V\} \), where an edge is an unordered pair of distinct vertices of \( V \). If \( n_i, n_j \in V \), and \((n_i, n_j) \in E\), then we say that \( n_i \) and \( n_j \) are adjacent or that \( n_j \) is a neighbor of \( n_i \), and denote this by writing \( n_j \sim n_i \). A graph is called complete if every pair of vertices are adjacent. The valence of vertex \( n_i \) of \( G \) is defined as the number of edges of \( G \) which are incident with \( n_i \), where an edge is incident with vertex \( n_i \) if one of the two vertices of the edge is \( n_i \). The adjacency matrix of \( G \) is an \( m \times m \) matrix whose \( ij \)-th entry is 1 if \((n_i, n_j) \) is one of \( G \)'s edges and is 0 if it is not. A path of length \( r \) from \( n_i \) to \( n_j \) in a graph is a sequence of \( r + 1 \) distinct vertices starting with \( n_i \) and ending with \( n_j \) such that consecutive vertices are adjacent. If there exists a path between any two vertices of \( G \), then \( G \) is connected.

An oriented graph is a graph together with a particular orientation, where the orientation of a graph \( G \) is the assignment of a direction to each edge, so edge \((n_i, n_j)\) is an directed edge from \( n_i \) to \( n_j \). The incidence matrix \( B \) of an oriented graph \( G \) is the \( \{0, \pm 1\} \)-matrix with rows and columns indexed by the vertices and edges of \( G \), respectively, such that the \( ij \)-entry is equal to 1 if edge \( j \) is ending on vertex \( n_i \), -1 if edge \( j \) is beginning...
with vertex $n_i$, and 0 otherwise. Define the Laplacian matrix of $\mathcal{G}$ as $L(\mathcal{G}) = BB^T$. It follows that $L(\mathcal{G}) = \Delta - A$, where $A$ is the adjacency matrix of undirected graph $\mathcal{G}$ and $\Delta$ is a diagonal matrix whose $i$th diagonal element is the valence of vertex $n_i$ in the graph. $L$ is always positive semi-definite. Moreover, for a connected graph, $L$ has a single zero eigenvalue, and the associated right eigenvector is $1_m$.

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