Improved Determination of the $b$ Quark Mass from Spectroscopy

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Abstract. Using recently evaluated contributions (including a novel one calculated here), we present updated values for the pole mass and $\overline{\text{MS}}$ mass of the $b$ quark: $m_b = 5022 \pm 58$ MeV, for the pole mass, and $\bar{m}_b(m_b) = 4286 \pm 36$ MeV for the $\overline{\text{MS}}$ one. These values are accurate including, respectively, $O(\alpha_s^3 \log \alpha_s)$ and $O(\alpha_s^2)$ corrections and, in both cases, leading orders in the ratio $m_c^2/m_b^2$.

One of the sources of information for the quark masses is quarkonium spectroscopy. By evaluating the $\bar{b}b$ potential including relativistic and radiative corrections, as well as leading nonperturbative effects,\cite{1,2,3} and using this in a perturbative expansion, it has been possible to find values of the pole quark masses with increasing accuracy\cite{2,3,4}. In this note we will go up to fourth and leading fifth order, in the approximation of neglecting “light” ($u$, $d$, $s$) quark masses and to leading order (actually, $O(\alpha_s m_c^2/m_b^2)$) in the $c$ quark mass. The connection with the $\overline{\text{MS}}$ mass has been known for some time to one and two loops\cite{5}; very recently, a three loop evaluation has been completed. Coupling this with the pole mass evaluations, we now have an order $\alpha_s^3$ result for the $\overline{\text{MS}}$ mass. We review here briefly this.

1. $m_b - \bar{m}_b(\bar{m}_b)$ connection

Write, for a heavy quark,

$$m(\bar{m}) \equiv m/\{1 + \delta_1 + \delta_2 + \delta_3 + \cdots\};$$

(1a)

$m$ here denotes the pole mass, and $\bar{m}$ is the $\overline{\text{MS}}$ one. One has

$$\delta_1 = C_F \frac{\alpha_s(\bar{m})}{\pi}, \quad \delta_2 = c_2 \left( \frac{\alpha_s(\bar{m})}{\pi} \right)^2, \quad \delta_3 = c_3 \left( \frac{\alpha_s(\bar{m})}{\pi} \right)^3.$$  

(1b)

Here $\alpha_s$ is to be calculated to three loops:

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 L} \left\{ 1 - \frac{\beta_1 \log L}{\beta_0^2 L} + \frac{\beta_1^2 \log^2 L}{\beta_0^4 L^2} - \frac{\beta_1^3}{\beta_0^6} \right\}$$

with

$$L = \log \frac{\mu^2}{\Lambda^2}; \quad \beta_0 = 11 - \frac{2}{3} n_f, \quad \beta_1 = 102 - \frac{38}{3} n_f, \quad \beta_2 = \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{34} n_f^2.$$  

The coefficient $c_2$ has been evaluated by Gray et al.\cite{5}, and reads

$$c_2 = -K + 2C_F,$$  

(1c)

$$K = K_0 + \sum_{i=1}^{n_f} \Delta \left( \frac{m_i}{m} \right), \quad K_0 = \frac{1}{9} \pi^2 \log 2 + \frac{7}{18} \pi^2 - \frac{1}{2} \zeta(3) + \frac{3673}{288} - \left( \frac{1}{18} \pi^2 + \frac{771}{171} \right) (n_f + 1)$$

$$\simeq 16.11 - 1.04 n_f; \quad \Delta(\rho) = \frac{4}{3} \left[ \frac{1}{2} \pi^2 \rho - \frac{3}{2} \rho^2 + \cdots \right].$$  

(1d)

$m_i$ are the (pole) masses of the quarks strictly lighter than $m$, and $n_f$ is the number of these. For the $b$ quark case, $n_f = 4$ and only the $c$ quark mass has to be considered; we will take $m_c = 1.8$ GeV (see Table 1 below) for the calculations.
The coefficient $c_3$ was recently calculated by Melnikov and van Ritbergen,[6] where the exact expression may be found. Neglecting now the $m_i$, 

$$c_3 \simeq 190.389 - 26.6551 n_f + 0.652694 n_f^2. \quad (1e)$$

For the $b$, $c$ quarks, with $\alpha_s$ as given below,

$$\begin{align*}
\delta_1 (b) &= 0.090, \quad \delta_1 (c) = 0.137, \\
\delta_2 (b) &= 0.045, \quad \delta_2 (c) = 0.108, \\
\delta_3 (b) &= 0.029; \quad \delta_3 (c) = 0.125. \quad (2)
\end{align*}$$

From these values we conclude that, for the $c$ quark, the series has started to diverge at second order, and it certainly diverges at order $\alpha_s^3$. For the $b$ quark the series is at the edge of convergence for the $\alpha_s^3$ contribution.

Take now as input parameters

$$\Lambda(n_f = 4, \text{three loops}) = 0.283 \pm 0.035 \text{ GeV} \quad [\alpha_s(M_Z^2) \simeq 0.117 \pm 0.024]$$

(ref. 7) and for the gluon condensate, very poorly known, the value $\langle \alpha_s G^2 \rangle = 0.06 \pm 0.02 \text{ GeV}^4$. From the mass of the $\Upsilon$ particle we have a very precise determination for the pole mass of the $b$ quark. This determination is correct to order $\alpha^3_s$ and including leading $O(m_b^3/m_c^2)$ and leading nonperturbative corrections as well as the $\alpha_s^3$ corrections proportional to $\log \alpha_s$; the details of it will be given below. With the renormalization point $\mu = m_b C_F \alpha_s$, we have,

$$m_b = 5022 \pm 43 \text{ (A)} \mp 5 \langle \langle \alpha_s G^2 \rangle \rangle \pm 35 \text{ (other th. uncert.)} \pm 38 \text{ (other th. uncert.)} = 5022 \pm 58 \text{ MeV}. \quad (3a)$$

Here we append (A) to the error induced by that of $\Lambda$, and likewise $\langle \langle \alpha_s G^2 \rangle \rangle$ tags the error due to that of the condensate. The error labeled (other th. uncert.) includes also the error evaluated in ref. 8; the rest is as in ref. 3.

Using the three loop relation (1) of the pole mass to the $\overline{\text{MS}}$ mass we then find

$$\bar{m}_b(\bar{m}_b) = 4284 \pm 7 \text{ (A)} \mp 5 \langle \langle \alpha_s G^2 \rangle \rangle \pm 35 \text{ (other th. uncert.)} = 4284 \pm 36 \text{ MeV}. \quad (3b)$$

The slight dependence of $\bar{m}$ on $\Lambda$ when evaluated in this way was already noted in ref. 2.

There is another way of obtaining $\bar{m}$, which is to express directly the mass of the $\Upsilon$ in terms of it, using Eq. (1) and the order $\alpha_s^3$ formula for the $\Upsilon$ mass in terms of the pole mass (see e.g. ref. 2). One finds, for $n_f = 4$, and neglecting $m_c^2/m_b^2$,

$$M(\Upsilon) = 2\bar{m}(\bar{m}) \left\{ 1 + C_F \frac{\alpha_s(\bar{m})}{\pi} + 7.559 \left( \frac{\alpha_s(\bar{m})}{\pi} \right)^2 + [66.769 + 18.277 (\log C_F + \log \alpha_s(\bar{m}))] \left( \frac{\alpha_s(\bar{m})}{\pi} \right)^3 \right\}. \quad (4a)$$

(One could add the leading nonperturbative contributions to (4a) à la Leutwyler–Voloshin in the standard way; see e.g. refs. 2, 3, 9). This method has been at times advertised as improving the convergence, allegedly because the $\overline{\text{MS}}$ mass does not suffer from nearby renormalon singularities. But a close look to (4a) does not seem to bear this out. To an acceptable $O(\alpha_s^4)$ error we can replace $\log(\alpha_s(\bar{m}))$ by $\log(\alpha_s(M(\Upsilon)/2))$ above. With $\Lambda$ as before (4a) then becomes

$$M(\Upsilon) = 2\bar{m}_b(\bar{m}_b) \left\{ 1 + C_F \frac{\alpha_s(\bar{m})}{\pi} + 7.559 \left( \frac{\alpha_s(\bar{m})}{\pi} \right)^2 + 43.502 \left( \frac{\alpha_s(\bar{m})}{\pi} \right)^3 \right\}. \quad (4b)$$

This does not look particularly convergent, and is certainly not an improvement over the expression using the pole mass, where one has for the choice $\mu = C_F m_b \alpha_s$, and still neglecting the masses of quarks lighter than the $b$,

$$M(\Upsilon) = 2m_b \left\{ 1 - 2.193 \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 - 24.725 \left( \frac{\alpha_s(\mu)}{\pi} \right)^3 - 458.28 \left( \frac{\alpha_s(\mu)}{\pi} \right)^4 - 897.93 \log \alpha_s \left( \frac{\alpha_s}{\pi} \right)^5 \right\}. \quad (5)$$
To order three, (5) is actually better\(^1\) than (4b). What is more, logarithmic terms appear in (4) at order \(\alpha_s^3\), while for the pole mass expression they first show up at \(\alpha_s^5\). Finally, the direct formula for \(M(T)\) in terms of the \(\overline{\text{MS}}\) mass presents the extra difficulty that the nonperturbative contribution becomes larger than what one has for the expression in terms of the pole mass \((\sim 80 \text{ against } \sim 9 \text{ MeV})\), because of the definition of the renormalization point. With the purely perturbative expression (4) plus leading nonperturbative (gluon condensate) correction one finds the value \(\bar{m}_b(\bar{m}_b) = 4167 \text{ MeV}\), rather low.

2. Improved determination of \(m_b\)

Eq. (5) was deduced neglecting the masses of all quarks lighter than the \(b\). The influence of the nonzero mass of the \(c\) quark, the only worth considering, will be evaluated now. To leading order it only contributes to the \(b\bar{b}\) potential through a \(c\)-quark loop in the gluon exchange diagram (diagram \(f_2\) in ref. 2). The momentum space potential generated by a nonzero mass quark through this mechanism is then, in the nonrelativistic limit,

\[ \bar{V}_{\text{mass}} = -\frac{8C_FT_F\alpha_s^2}{k^2} \int_0^1 dx x(1-x) \log \frac{m_c^2 + x(1-x)k^2}{\mu^2}. \]  

(6)

We expand in powers of \(m_c^2/k^2\). The zeroth term is already included in (5). The first order correction is

\[ \delta_{c,\text{mass}}\bar{V} = -\frac{8C_FT_F\alpha_s^2m_c^2}{k^2}. \]  

(7)

In x-space,

\[ \delta_{c,\text{mass}}V = \frac{C_FT_F\alpha_s^2m_c^2}{\pi} r. \]  

(8)

This induces the shift in the mass of the \(\Upsilon\) of

\[ \delta_{c,\text{mass}}M(\Upsilon) = \frac{3T_F\alpha_s}{\pi} \frac{m_c^2}{m_b^2} m_b, \]

so Eq. (5) is modified to

\[ M(\Upsilon) = 2m_b \left\{ 1 - 2.193 \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 - 24.725 \left( \frac{\alpha_s(\mu)}{\pi} \right)^3 - 458.28 \left( \frac{\alpha_s(\mu)}{\pi} \right)^4 \right. \]

\[ \left. + 897.93 [\log \alpha_s] \left( \frac{\alpha_s}{\pi} \right)^5 + \frac{3T_F\alpha_s}{2\pi} \frac{m_c^2}{m_b^2} \right\}. \]  

(8)

This produces the value quoted in (3a). Note that the (new) correction of order \(m_c^2/m_b^2\) is responsible for a shift in \(m_b\) of

\[ \delta_{c,\text{mass}}m_b = -35 \text{ MeV}, \]

substantially larger than the \(\alpha_s^5\log \alpha_s\) correction evaluated by Brambilla et al.\(^4\) which, for the renormalization point \(\mu = m_b C_F \alpha_s\), gives

\[ \delta_{[\alpha_s^5 \log \alpha_s]}m_b = \frac{1}{2}m_b[C_F + \frac{3}{2}C_A]C_F \alpha_s^5 (\log \alpha_s)/\pi \simeq -8 \text{ MeV}. \]

We collect in the table the determinations of the \(b\) quark mass based on spectroscopy, to increasing accuracy. The stability of the numerical values of the pole mass is remarkable: the pole masses all lie within each other error bars. The \(\overline{\text{MS}}\) ones show more spread.

\(^1\) The convergence of Eq. (5) is still improved if one solves exactly the purely coulombic part of the static potential, as was done in refs. 2, 3, where we send for details. For example, the \(O(\alpha_s^4)\) term becomes \(-232.12(\alpha_s/\pi)^4\). This is the method we used to get the values of \(m_b\) here.
TY: Titard and Ynduráin\cite{2}. $O(\alpha_s^2)$ plus $O(\alpha_s^0)$ for $m$; $O(\alpha_s^2)$ for $\bar{m}$. Rescaled for $\Lambda(n_f = 4) = 283$ MeV.
PY: Pineda and Ynduráin\cite{3}. Full $O(\alpha_s^4)$ for $m$; $O(\alpha_s^2)$ for $\bar{m}$. Rescaled for $\Lambda(n_f = 4) = 283$ MeV.
Here: This calculation. $O(\alpha_s^4)$, $O(\alpha_s m_c^2/m_b^2)$ and $O(\alpha_s^2 \log \alpha_s)$ for $m$; $O(\alpha_s^2)$ and $O(\alpha_s^2 m_c^2/m_b^2)$ for $\bar{m}$. Values not given for the $c$ quark, as the higher order terms are as large as the leading ones.

We finally remark that the values of $m_b$ quoted e.g. in the Table 1 were not obtained solving Eq. (8), but solving exactly the coulombic part of the interaction, and perturbing the result (see refs. 2, 3 for details). We also note that, in the determinations of $m_b$, the new pieces, $O(\alpha_s m_c^2/m_b^2)$ and $O(\alpha_s^2 \log \alpha_s)$, have been evaluated to first order; in particular, we have included the corresponding shifts in the central values, not in the errors. If we included these, the errors would decrease by some 7%.

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References

1. W. Fischler, Nucl. Phys. B 129, 157 (1977); A. Billoire, Phys. Lett. B 92, 343 (1980); S. N. Gupta and S. Radford, Phys. Rev. D 24, 2309 (1981) and (E) D 25, 3430 (1982); S. N. Gupta, S. F. Radford and W. W. Repko, ibid D 26, 3305 (1982); M. Peter, Phys. Rev. Lett. 78, 602 (1997); Y. Schröder, Phys. Lett. B447, 321 (1999).
2. S. Titard and F. J. Ynduráin, Phys. Rev. D 49, 6007 (1994) and D 51, 6348 (1995).
3. A. Pineda and F. J. Ynduráin, Phys. Rev. D 58, 3003 (1998), and hep-ph/9812371, 1998, in press in Phys. Rev.
4. N. Brambilla et al., Phys. Lett. B 470, 215 (1999).
5. R. Coquereaux, Phys. Rev. D 23, 1365 (1981); R. Tarrach, Nucl. Phys. B 183, 384 (1981); N. Gray et al., Z. Phys. C 48, 673 (1990).
6. K. Melnikov and T. van Ritbergen, hep-ph/9912391.
7. See J. Santiago and F. J. Ynduráin, Nucl. Phys. B 563, 45 (1999), and work quoted there.
8. W. Lucha and F. F. Schöberl, UWTThPh-1999-77 (hep-ph/0001191).
9. M. B. Voloshin, Nucl. Phys. B 154, 365 (1979) and Sov. J. Nucl. Phys. 36, 143 (1982); H. Leutwyler, Phys. Lett. B 98, 447 (1981).