Dynamics of a Bright Soliton in Bose-Einstein condensates with Time-Dependent Atomic Scattering Length in an Expulsive Parabolic Potential

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We present a family of exact solutions of one-dimensional nonlinear Schrödinger equation, which describes the dynamics of a bright soliton in Bose-Einstein condensates with the time-dependent interatomic interaction in an expulsive parabolic potential. Our results show that, under the safe range of parameters, the bright soliton can be compressed into very high local matter densities by increasing the absolute value of atomic scattering length, which can provide an experimental tool for investigating the range of validity of the one-dimensional Gross-Pitaevskii equation. We also find that the number of atoms in the bright soliton keeps dynamic stability: a time-periodic atomic exchange is formed between the bright soliton and the background.

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With the experimental observation and theoretical studies of Bose-Einstein condensates (BECs)\(^3\), intensive interests have been paid to the nonlinear excitations of the atomic matter waves, such as dark\(^4\) and bright solitons\(^5\). It is believed that the atomic matter bright solitons are of primary importance for developing concrete applications of BEC in the future, so it is of interest to develop a new technique which allows us to construct a particular soliton with the assumed peak matter density. One possibility is to vary the interatomic interaction by means of external magnetic fields. Recent experiments have demonstrated that the variation of the effective scattering length, even including its sign, can be achieved by utilizing the so-called Feshbach resonance\(^7\). This offers a good opportunity for manipulation of atomic matter waves and nonlinear excitations in BEC\(^8\). In Ref.\(^9\), it has been demonstrated that the variation of nonlinearity of the Gross-Pitaevskii (GP) equation via Feshbach resonance provides a powerful tool for controlling the generation of bright and dark soliton trains starting from periodic waves. Besides, a sinusoidal variation of the scattering length has also been used to form patterns such as Faraday waves\(^11\), or as a means to maintain BECs without an external trap in two dimensions\(^12\).

In this letter, we present a thorough analysis on the dynamics of a bright soliton of BEC with time-varying atomic scattering length in an expulsive parabolic potential. Our study is greatly facilitated by the so-called Darboux transformation\(^13\), by which we can directly construct the exact solutions of one-dimensional (1D) nonlinear Schrödinger equation (NLSE). Under the safe range of parameters, the bright soliton in BEC can be compressed into very high local matter densities by increasing the absolute value of atomic scattering length with Feshbach resonance. During the compression of the bright soliton in BEC, the number of atoms in the bright soliton keeps dynamic stability and the exchange of the atoms between the bright soliton and the background of density exists.

At the mean-field level, the GP equation governs the evolution of the macroscopic wave function of a 3D BEC. In the physically important case of the cigar-shaped BEC, it is reasonable to reduce the GP equation into 1D NLSE\(^1\),\(^2\),\(^3\),\(^4\),\(^5\),\(^6\).

\begin{align}
    i \frac{\partial \psi (x,t)}{\partial t} + \frac{\partial^2 \psi (x,t)}{\partial x^2} + 2a(t)|\psi (x,t)|^2 \psi (x,t) + \frac{1}{4} \lambda x^2 \psi (x,t) &= 0. \quad (1)
\end{align}

In Eq. \(^1\), time \(t\) and coordinate \(x\) are measured in units \(2/\omega_\perp\) and \(a_\perp\), where \(a_\perp = (\hbar/m\omega_\perp)^{1/2}\) and \(a_0 = (\hbar/2m\omega_\parallel)^{1/2}\) are linear oscillator lengths in the transverse and cigar-axis directions, respectively. \(\omega_\perp\) and \(\omega_\parallel\) are corresponding harmonic oscillator frequencies, \(m\) is the atomic mass and \(\lambda = 2|\omega_\parallel/\omega_\perp| \ll 1\). The Feshbach-managed nonlinear coefficient reads \(a_\perp(t) = |a_\parallel(t)|/a_B = g_0 \exp (\lambda t)/(a_B\) is the Bohr radius\(^16\),\(^17\). The normalized macroscopic wave function \(\psi (x,t)\) is connected to the original order parameter \(\Psi (r,t)\) as follows,

\begin{align}
    \Psi (r,t) &= \frac{1}{\sqrt{2\pi a_B a_\perp}} \psi \left( \frac{x}{a_\perp}, \frac{\omega_\perp t}{2} \right) \times \exp \left( -i\omega_\perp t - \frac{y^2 + z^2}{2a_\perp} \right). \quad (2)
\end{align}

From the viewpoint of stability, 3D and 1D equations are very different. For a true 1D system, one does not expect the collapse of the system with increasing number of atoms\(^5\). However, it happens that a realistic 1D limit is not a true 1D system, with the density of particles still increasing due to the strong restoring forces in the perpendicular directions. To avoid the collapse of the bright soliton\(^15\), we must restrict our study of BEC to the safe range of parameters, in which the system becomes...
effectively 1D, i.e. the energy of two body interactions is much less than the kinetic energy in the transverse direction $\Delta \psi^2 = \frac{a_1}{\xi^2} \sim N|a_s|/a_0 \ll 1$ ($\xi$ is the healing length). The bright soliton in BECs has been created with the parameters of $N \approx 10^3$, $\omega_\perp = 2\pi \times 700\, \text{Hz}$ and $\omega_\parallel = 2\pi \times 7\, \text{Hz}$, $a_{f\text{inal}} = -4a_B$ for $\tilde{t} \approx 14$, which provides the safe range of parameters. With the same experimental conditions in Ref. [4] and $a_s(t = 0) = -0.25a_B$, we can calculate $\epsilon^2 = \frac{a_1}{\xi^2} \sim N|a_s|/a_0 = 9.5 \times 10^{-3} \ll 1$. Then, the scattering length is increased in the form of $a(t) = g_0 \exp(\lambda t)$. After at least up to 50 dimensionless units of time, the absolute value of the atomic scattering length turns to $|a_\pi(t)| = 0.8a_B$, corresponding to $\epsilon^2 = \frac{a_1}{\xi^2} \sim N|a_s|/a_0 = 3 \times 10^{-2} \ll 1$. Under the above conditions, the system is effectively 1D. So the safe range of parameters can be described as follows: (1) with the same experimental conditions in Ref. [4]; (2) ramp up the scattering length in the form of $a(t) = g_0 \exp(\lambda t)$ within 50 dimensionless units of time. We also have to specify the terminology long-time dynamics. A unity of time, $\Delta t = 1$, in the dimensionless variables corresponds to $2/\omega_\perp$ real seconds. This means, for example, that for a BEC in a trap with transversal size of the order of $a_\perp \approx 1.5\, \mu\text{m}$, a unity of the dimensionless time corresponds to $5.0 \times 10^{-3}\, \text{s}$. The lifetime of a BEC in today’s experiments is of the order of 1s, which is about 200 in our dimensionless units.

The so-called ‘seed’ solution of Eq. (1) can be chosen as follows,

$$
\psi_0(x, t) = A_c \exp\left[\frac{\lambda t}{2} + i\varphi_c\right],
$$

where $\varphi_c = k_0 x \exp(\lambda t) - \frac{\Lambda^2}{2} + \frac{(2a_B A_c^2 - k_0^2) \exp(2\lambda t) - 1}{4\lambda}$ and $A_c$ and $\varphi_c$ are the arbitrary real constants. We perform the Darboux transformation

$$
\psi_1 = \psi_0 + \frac{2\sqrt{2}\delta}{\varphi_c} \exp(-\lambda t/2 - \Lambda x^2/4),
$$

to obtain the new solution of Eq. (1) by taking Eq. (4) as the seed. Then we obtain the exact solution of Eq. (1) as follows:

$$
\psi = \left[A_c + A_s \left(\gamma \cosh \theta + \cos \varphi\right) i a_s \sinh \theta + \beta \sin \varphi\right] \exp\left[\frac{\lambda t}{2} + i\varphi_c\right],
$$

where

$$
\theta = -\frac{[k_0 + k_s] \Delta R - \sqrt{g_0} A_s \Delta I}{2\lambda} \exp(2\lambda t) - 1]
+\Delta \psi^2 \exp(\lambda t),
$$

$$
\varphi = -\frac{[k_0 + k_s] \Delta I + \sqrt{g_0} A_s \Delta R}{2\lambda} \exp(2\lambda t) - 1]
+\Delta \psi^2 \exp(\lambda t),
$$

and

$$
\alpha = \frac{\sqrt{g_0} A_c (k_0 - k_s + \Delta I)}{\Lambda},
$$

$$
\beta = 1 - \frac{2g_0 A_c^2}{\Lambda},
$$

$$
\gamma = \frac{\sqrt{g_0} A_c (\Delta R - \sqrt{g_0} A_s)}{\Lambda},
$$

with

$$
\Delta = \sqrt{\left[-\sqrt{g_0} A_s + i (k_s - k_0)^2 - 4g_0 A_c^2\right] \Delta R + i\Delta_I},
$$

$$
\Lambda = g_0 A_c^2 + \left(\frac{\Delta R - \sqrt{g_0} A_s^2}{4}\right) + \left(\frac{k_s - k_0 + \Delta I}{4}\right),
$$

where $k_s$ is the arbitrary real constant. On the one hand, when $A_c = k_0 = 0$, Eq. (1) reduces to the well-known soliton solution: $\psi_s = A_s \text{sech} \theta \exp(\lambda t/2 + i\varphi_s)$, where $\theta_s = -\sqrt{g_0} \exp(\lambda t) A_s x + \sqrt{g_0} k_s A_s \exp(2\lambda t - 1) / \lambda$ and $\varphi_s = \varphi_c - g_0 A_c^2 \exp(2\lambda t - 1) / 2\lambda$. On the other hand, when the amplitude of this soliton vanishes ($A_s = 0$), Eq. (4) reduces to Eq. (3). Thus, Eq. (4) represents a bright soliton embedded in the background. Considering the dynamics of the bright soliton on the background, the length $2L$ of the spatial background must be very large compared to the scale of the soliton. In the real experiment [3], the length of background of BEC can be reached at least $2L = 370\, \mu\text{m}$. At the same time, in Fig. 1, the width of the bright soliton is about $2L = 2 \times 1.4\, \mu\text{m} = 2.8\, \mu\text{m}$ (a unity of coordinate, $\Delta x = 1$ in the dimensionless variables, corresponds to $a_\perp = (\hbar/m\omega) \times 1/2 = 1.4\, \mu\text{m}$). So, we indeed have $l \ll L$, a necessary condition for realizing our soliton in experiment.

By utilizing the properties of Eq. (1), we demonstrate that the manipulation of the scattering length can be used to compre ss a bright soliton of BEC into an assumed peak matter density. It has been reported that an abrupt change of the scattering length can lead to the splitting of the soliton with generating the new solitons. The fragmentation of the soliton obviously decreases the numbers of atoms of the original soliton, which is undesirable for application [2]. However, in Eq. (4), the change of the scattering length preserves the soliton from splitting new ones. For simplicity, we assume $k_0 = k_s$ in Eq. (7) and only consider the case of $A_c^2 > 4A_s^2$, for in the case of $A_c^2 < 4A_s^2$, a small perturbation for Eq. (1) may lead to the modulation instability [21]. On the conditions above, Eq. (4) can be deduced into the following form:

$$
\psi = \left[-A_c + \delta_2 \delta \cos \varphi - iA_c \sin \varphi\right] \exp(\lambda t/2 + i\varphi_c),
$$

where

$$
\theta = \sqrt{g_0} \delta_2 \exp(\lambda t) - \sqrt{g_0} k_0 \delta_2 \exp(2\lambda t - 1) / \lambda,
$$

$$
\varphi = -\frac{g_0 A_c \delta_2 \exp(2\lambda t - 1)}{2\lambda},
$$

$$
\delta_2 = \frac{A_c^2 - 4A_s^2}{2\lambda}.
$$
For a better understanding, we plot Eq. (8) in Fig. 1, which shows the dynamics of the Feshbach resonance managed bright soliton in the expulsive parabolic potential given by Eq. (8). The parameters are given as follows: \( \lambda = 2 \times 10^{-2}, g_0 = \sqrt{2}, A_c = 1, A_s = 2.4, k_0 = 0.03 \), which is less than \( |a_{final}| = 4a_B \). This means that during the process of compressing the bright soliton, the stability of soliton and the validity of 1D approximation can be kept as displayed in Fig. 1. Therefore, the phenomena discussed in this letter should be observable within the current experimental capability.

Furthermore, based on Eq. (8), we find that when \( \sin \theta = 0 \), the peak matter density of bright soliton arrives at the maximum

\[
|\psi|^2 = \exp(\lambda t) \left( A_s^2 + \frac{\delta_2 A_s}{A_s - 2A_c \cos \varphi} \right)
\]

and when \( \cosh \theta = \frac{A_s}{A_c \cos \varphi} - \frac{A_c \cos \varphi}{A_s} \), the peak matter density of bright soliton arrives at the minimum

\[
|\psi|^2 = \exp(\lambda t) \left( A_s^2 - \frac{A_s^2 \delta_2^2 \cos^2 \varphi}{A_s^2 - 4A_c^2 \cos^2 \varphi} \right)
\]

This means that the bright soliton can only be squeezed into the assumed peak matter density between the minimum and maximum values. In order to investigate the stability of the bright soliton against the variation of the scattering length in the expulsive parabolic potential, we obtain

\[
\lim_{L \to \infty} \int_{-L}^{+L} |\psi(x, t)|^2 - |\psi(\pm L, t)|^2 \, dx = \frac{2\delta_2}{\sqrt{g_0}}
\]

which is the exact number of the atoms in the bright soliton against the background described by Eq. (8). This indicates that during the process of the compression of the bright soliton, the number of atoms in the bright soliton keeps invariant. In contrast, the quantity

\[
\kappa = \lim_{L \to \infty} \int_{-L}^{+L} |\psi(x, t) - \psi(\pm L, t)|^2 \, dx
\]

\[
= \frac{2\delta_2}{\sqrt{g_0}} (1 + A_c M \cos \varphi), \quad (13)
\]

counts number of atoms in both the bright soliton and background under the condition of \( \psi(\pm L, t) \neq 0 \). Eq. (13) displays that a time-periodic atomic exchange is formed between the bright soliton and the background. As shown in Fig. 2(a), in the case of zero-background, i. e. \( A_c = 0 \), there will not be the exchange of atoms. However, in the case of nonzero-background as shown in the Figs. 2 (b) and (c), the exchange of atoms between the bright soliton and the background becomes more quickly with increasing the absolute value of the scattering length. The conclusion can be made that the number of atoms in the bright soliton in BEC keeps dynamic stability against the variation of the scattering length.
length in the expulsive parabolic potential. The consideration above implies that we should construct the bright soliton in BEC on the background of density described by Eq. (3). A question arises about the possibility of creation of such a state experimentally. We notice that Eq. (15) will take the particular form at the time is: (a) and (b) \( t = [0, 5] \), (c) \( t = [20, 25] \). The parameters are given as follows: \( \lambda = 0.02 \), \( g_0 = 1 \), \( A_s = 4.8 \), (a) \( A_c = 0 \), (b) and (c) \( A_c = 2.3 \).

In conclusion, we present a family of exact solutions of the nonlinear Schrödinger equation with the time-varying parameter. It is possible to squeeze a bright soliton of BEC into the assumed peak matter density, which can provide an experimental tool for investigating the range of validity of the 1D GP equation. We also find that the number of atoms in the bright soliton keeps dynamic stability: the exchange of atoms between the bright soliton and the background becomes more quickly with increasing the absolute value of the scattering length. Recent developments of controlling the scattering length in the experiments allow for the experimental investigation of our prediction in the future.

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[19] Considering the following UV-pair \( \phi_x = U \phi \), \( \phi_t = V \phi \), \( \phi = (\phi_1 \phi_2)^T \) where \( U = \zeta J + P \), \( V = 2 i \zeta \xi J + \ldots \) and \( J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \), \( P = \begin{bmatrix} 0 & i \sqrt{g_0 Q} \\ -i \sqrt{g_0 Q} & 0 \end{bmatrix} \), \( W = \begin{bmatrix} i g_0 |Q|^2 & -\sqrt{g_0} \lambda x Q \xi - i g_0 Q \theta \xi \\ \sqrt{g_0} \lambda x Q \xi + i g_0 Q \theta \xi & -i g_0 |Q|^2 \end{bmatrix} \). Here the overbar denotes the complex conjugate and \( \psi = Q \exp \left( -i \lambda x^2 / 4 - \lambda J / 2 \right) \). From the compatibility condition \( U_t - V_x + [U, V] = 0 \) one can derive Eq. (4) in the
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