Electromagnetic Shocks in Strong Magnetic Fields

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We examine the propagation of electromagnetic radiation through a strong magnetic field using the method of characteristics. Owing to nonlinear effects associated with vacuum polarization, such waves can develop discontinuities analogous to hydrodynamical shocks. We derive shock jump conditions and discuss the physical nature of these non-linear waves.

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I. INTRODUCTION

The nonlinear properties of electromagnetic waves traveling through a magnetized vacuum is of particular interest in the study of neutron stars. Heisenberg and Euler [1] and Weisskopf [2] first derived nonlinear corrections to the Maxwell equations of the electromagnetic field. Lutzky and Toll [3] and Zheleznyakov and Fabrikant [4] applied the weak-field expansion to show that shock waves can develop in the electromagnetic field. Bialynicka-Birula [5] used the full expression for the nonlinear correction to the Lagrangian to study the generation of harmonics and other nonlinear phenomena in the propagation of EM radiation.

In this paper, we use the method of characteristics to study the evolution of waves governed by an arbitrary Lagrangian and then apply these techniques to the Heisenberg-Euler-Weisskopf-Schwinger Lagrangian [1,2,6]. We find a concise expression for the opacity to shocking for disturbances travelling through an arbitrarily strong magnetic field. After deriving a form for the energy-momentum tensor for an arbitrary electromagnetic Lagrangian, we use relativistic fluid mechanics to derive the shock jump conditions and the long-term evolution of electromagnetic disturbances propagating in magnetic fields.

II. DERIVING THE CHARACTERISTICS

We will use the method of characteristics to study the evolution of a disturbance of the electromagnetic field. In general, the relativistic Lagrangian \( \mathcal{L} \) of the electromagnetic field is a function of the two invariants of the field. We follow the notation of Lutzky and Toll [3] and Heisenberg and Euler [1] and define

\[
I = F_{\mu\nu} F^{\mu\nu} = 2 (|\mathbf{B}|^2 - |\mathbf{E}|^2)
\]

and

\[
K = -\left( \frac{1}{2} \epsilon_{\lambda\rho\mu\nu} F_{\lambda\rho} F_{\mu\nu} \right)^2 = -(4\mathbf{E} \cdot \mathbf{B})^2.
\]

As illustrated in Fig. 1, we choose coordinates so that the radiation is polarized in the \( z \)-direction and travels along the \( y \)-axis toward positive \( y \). The ambient magnetic field makes an angle \( \phi \) with the electric field, and the projection of the magnetic field into \( x - y \) plane makes an angle \( \theta \) with respect to the \( x \)-axis (magnetic field of the wave).

With these definitions, the invariants are

\[
I = 2 \left( (\tilde{B} \cos \theta \sin \phi + B)^2 + (\tilde{B} \sin \theta \sin \phi)^2 + (\tilde{B} \cos \phi)^2 - E^2 \right)
\]

\[
K = -(4E\tilde{B} \cos \phi)^2
\]

where \( \tilde{B} \) is the strength of the ambient magnetic field, \( E \) and \( B \) are the strengths of the electric and magnetic fields associated with the radiation.

We characterize the traveling wave by a vector potential with one non-zero component,

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\( A_z = \psi(y,t). \)  \( \quad (5) \)

We have assumed that the vector potential has only one independent component which allows us to treat the problem using characteristics. Unfortunately, we cannot follow the interaction between two polarizations which was treated by Bialynicka-Birula \[5\].

Using the coordinates, \( y \) and \( t \), Hamilton’s principle assumes the form

\[
\frac{\partial}{\partial y} \left( \frac{\partial L}{\partial B} \right) - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial E} \right) = 0.
\]  \( \quad (6) \)

where

\[
\psi_y = B \text{ and } \psi_t = -E,
\]  \( \quad (7) \)

and we have taken \( c = 1 \).

Since the Lagrangian is a function of \( I \) and \( K \) alone, we can rewrite this in terms of derivatives of \( L \) with respect to \( I \) and \( K \) with respect to \( \psi_y \) and \( \psi_t \) and finally \( \psi_{yy}, \psi_{yt}, \text{ and } \psi_{tt} \) by successive applications of the chain rule. Taking the partial derivatives yields an equation of the form,

\[
a \psi_{yy} + b \psi_{yt} + c \psi_{tt} = 0
\]  \( \quad (8) \)

where

\[
a = \frac{\partial^2 L}{\partial I^2} \left( \frac{\partial I}{\partial B} \right)^2 + \frac{\partial^2 L}{\partial K^2} \left( \frac{\partial K}{\partial B} \right)^2 + 2 \frac{\partial^2 L}{\partial I \partial K} \frac{\partial K}{\partial B} \frac{\partial I}{\partial B} + \frac{\partial L}{\partial I} \frac{\partial^2 I}{\partial B^2} + \frac{\partial L}{\partial K} \frac{\partial^2 K}{\partial B^2} \quad (9)
\]

\[
b = -2 \left[ \frac{\partial^2 L}{\partial I^2} \left( \frac{\partial I}{\partial E} \frac{\partial I}{\partial B} \right) + \frac{\partial^2 L}{\partial K^2} \left( \frac{\partial K}{\partial E} \frac{\partial K}{\partial B} \right) + \frac{\partial^2 L}{\partial I \partial K} \left( \frac{\partial I}{\partial E} \frac{\partial K}{\partial B} + \frac{\partial K}{\partial E} \frac{\partial I}{\partial B} \right) + \frac{\partial L}{\partial I} \frac{\partial^2 I}{\partial E \partial B} + \frac{\partial L}{\partial K} \frac{\partial^2 K}{\partial E \partial B} \right] \quad (10)
\]

\[
c = \frac{\partial^2 L}{\partial I^2} \left( \frac{\partial I}{\partial E} \right)^2 + \frac{\partial^2 L}{\partial K^2} \left( \frac{\partial K}{\partial E} \right)^2 + 2 \frac{\partial^2 L}{\partial I \partial K} \frac{\partial K}{\partial E} \frac{\partial I}{\partial E} + \frac{\partial L}{\partial I} \frac{\partial^2 I}{\partial E^2} + \frac{\partial L}{\partial K} \frac{\partial^2 K}{\partial E^2} \quad (11)
\]

\[
c = 4 \left[ 4E^2 \frac{\partial^2 L}{\partial I^2} - \frac{\partial L}{\partial I} + 8 \bar{B}^2 \cos^2 \phi \left( 32E^2 \bar{B}^2 \cos^2 \phi \frac{\partial^2 L}{\partial K^2} + 8E^2 \frac{\partial^2 L}{\partial I \partial K} - \frac{\partial L}{\partial K} \right) \right] \quad (12)
\]

This equation may be factored yielding

\[
\left( \frac{\partial}{\partial y} - \tau_+ \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial y} - \tau_- \frac{\partial}{\partial t} \right) \psi = -\psi_1 \left( \frac{\partial}{\partial y} - \tau_+ \frac{\partial}{\partial t} \right) \tau_- \quad (15)
\]

and

\[
\left( \frac{\partial}{\partial y} - \tau_- \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial y} - \tau_+ \frac{\partial}{\partial t} \right) \psi = -\psi_1 \left( \frac{\partial}{\partial y} - \tau_- \frac{\partial}{\partial t} \right) \tau_+ \quad (16)
\]

where
\[
\tau_{\pm} = -\frac{1}{\sigma_{\pm}} = -\frac{b \mp \sqrt{b^2 - 4ac}}{2a}
\]  \hspace{1cm} (17)

and \(\sigma_{\pm}\) is the speed of the travelling wave.

We define two new coordinates \((u, v)\) such that
\[
y = u + v \quad \text{and} \quad t = -\left(\tau_+ u + \tau_- v\right)
\]  \hspace{1cm} (18)

We now define the Riemann invariants,
\[
\psi_+ = \psi_y - \tau_+ \psi_t - \int_{v_0}^{v} \psi_t \left(\frac{\partial \tau_+}{\partial v}\right)_{v = v'} dv'
\]  \hspace{1cm} (19)

\[
\psi_- = \psi_y - \tau_- \psi_t - \int_{u_0}^{u} \psi_t \left(\frac{\partial \tau_-}{\partial u}\right)_{u = u'} du'
\]  \hspace{1cm} (20)

which are constant along curves with
\[
\frac{dt}{dy} = -\tau_+ \left(\psi_+, \psi_-; \bar{B}\right).
\]  \hspace{1cm} (21)

To obtain a cursory understanding of the characteristics of this equation, we expand each of the coefficients about \(E, B = 0\) to first order yielding,
\[
a = a_0 + a_B \psi_y - a_E \psi_t + \mathcal{O}(B^2)
\]  \hspace{1cm} (22)

\[
b = b_0 + b_B \psi_y - b_E \psi_t + \mathcal{O}(B^2)
\]  \hspace{1cm} (23)

\[
c = c_0 + c_B \psi_y - c_E \psi_t + \mathcal{O}(B^2).
\]  \hspace{1cm} (24)

The coefficients are
\[
a_0 = 4 \left[3 \left(\bar{B} \cos \theta \sin \phi\right)^2 \frac{\partial^2 \mathcal{L}}{\partial I^2} + \frac{\partial \mathcal{L}}{\partial I}\right]
\]  \hspace{1cm} (25)

\[
a_B = 16 \bar{B} \cos \theta \sin \phi \left[\frac{\partial^2 \mathcal{L}}{\partial I^2} + 4 \left(\bar{B} \cos \theta \sin \phi\right)^2 \frac{\partial^3 \mathcal{L}}{\partial I^3}\right]
\]  \hspace{1cm} (26)

\[
b_E = 32 \bar{B} \cos \theta \sin \phi \left(\frac{\partial^2 \mathcal{L}}{\partial I^2} + 8 \frac{\partial^2 \mathcal{L}}{\partial I \partial K} \bar{B}^2 \cos^2 \phi\right)
\]  \hspace{1cm} (27)

\[
c_0 = -4 \left(8 \frac{\partial \mathcal{L}}{\partial K} \bar{B}^2 \cos^2 \phi + \frac{\partial \mathcal{L}}{\partial I}\right)
\]  \hspace{1cm} (28)

\[
c_B = -\frac{b_E}{2}
\]  \hspace{1cm} (29)

and
\[
a_E = b_0 = b_B = c_E = 0.
\]  \hspace{1cm} (30)

Using this linearization we estimate the magnitude of the inhomogeneous term in Eq. (5) and Eq. (6) if we assume \(E_t \sim E\) and \(B_t \sim B\)
\[
-\psi_t \left(\frac{\partial}{\partial y} - \tau_\pm \frac{\partial}{\partial t}\right) \tau_\mp = \mathcal{O}(B^2)
\]  \hspace{1cm} (31)

so we can neglect it to first order.

We estimate the Riemann invariants,
\[
\psi_+ = \psi_y - \tau_+ \psi_t + \mathcal{O}(B^2)
\]  \hspace{1cm} (32)

\[
= B + \tau_+ E + \mathcal{O}(B^2)
\]  \hspace{1cm} (33)

\[
\psi_- = \psi_y - \tau_- \psi_t + \mathcal{O}(B^2)
\]  \hspace{1cm} (34)

\[
= B + \tau_- E + \mathcal{O}(B^2)
\]  \hspace{1cm} (35)
Now taking the limit where \( E \) and \( B \) themselves may be neglected we have

\[
\sigma_\pm = \pm \sqrt{-a_0/c_0} = \pm \left[ \left( 4 (\dot{B} \cos \theta \sin \phi)^2 \frac{\partial^2 L}{\partial I^2} + \frac{\partial L}{\partial I} \right) / \left( 8 \dot{B}^2 \cos^2 \phi \frac{\partial L}{\partial K} + \frac{\partial L}{\partial I} \right) \right]^{1/2}
\]

where

\[
x = \left( \frac{\partial L}{\partial I} + 2 \dot{B}^2 \cos^2 \theta \sin^2 \phi \frac{\partial^2 L}{\partial I^2} + 4 \dot{B}^2 \cos^2 \phi \frac{\partial L}{\partial K} \right) / 2 \dot{B}^2 \left( \cos^2 \theta \sin^2 \phi \frac{\partial^2 L}{\partial I^2} - 2 \cos^2 \phi \frac{\partial L}{\partial K} \right)
\]

We can see from this result that \( \tau_+ = -\tau_- + \mathcal{O}(B^2) \), therefore we have for the two Riemann invariants

\[
\psi_\pm = B \pm \tau_+ E + \mathcal{O}(B^2)
\]

Fig. 3 depicts how two adjacent characteristic may intersect. Additionally, we see that \( \psi_- \) characteristics (i.e. lines along which \( \psi_- \) is constant) that originate from regions without wave fields cross the \( \psi_+ \) characteristics. Therefore, we can argue that \( \psi_- = B - \dot{\tau}_+ E + \mathcal{O}(B^2) = 0 \) throughout the region to the right of the antenna because the fields are zero in this region. We can use the same argument for the region to the left of the antenna and find that in general \( B = \tau_\pm E + \mathcal{O}(B^2) \) along the \( \psi_\pm \) characteristics. Furthermore, to first order, both \( \psi_\pm \) are constant along the characteristics; therefore, the slopes of the characteristics which depend only on \( \psi_+ \), \( \psi_- \) and the constant background field must be constants and the characteristics travel at a constant speed.

Using the figure as a guide, we estimate the distance over which two adjacent characteristics can travel before intersecting is given by

\[
\Delta y = c \left( \frac{\partial \tau_+}{\partial t} \right)^{-1} = \frac{-2a_0\tau_\pm c}{(a_B \tau_\pm^2 + c_B) B_t - 2c_B \tau_\pm E_t}.
\]

Here we have used that fact that \( \psi_+ \) characteristics travel with velocity \( -\tau_- = \tau_+ + \mathcal{O}(B) \).

To work with this equation further, we define an opacity due to shock formation and use \( B = \tau_\pm E \). We obtain

\[
\kappa = (\Delta y)^{-1} = \frac{a_B c_B a_0}{2a_0^2 \tau_\pm c} B_t \tag{40}
\]

\[
= \pm \frac{a_B c_B a_0}{2a_0^2} \sqrt{-\frac{a_0}{c}} B_t \tag{41}
\]

\[
= \pm 8 B \left( \frac{4 \partial^2 L}{\partial I^2} B^2 + \frac{\partial L}{\partial I} \right)^{-3/2} \left[ \frac{\partial^2 L}{\partial I^2} \frac{\partial L}{\partial I} + \left( \frac{2 \partial L}{\partial I} \frac{\partial^2 L}{\partial I \partial K} + 6 \frac{\partial^2 L}{\partial I^2} \frac{\partial L}{\partial K} \right) B^2_E \right. \\
+ \left. \left( \frac{\partial L}{\partial I} \frac{\partial^2 L}{\partial I^2} + \left( \frac{\partial^2 L}{\partial I^2} \right)^2 \right) \right]^{-1/2} B_E \] \\
\times \frac{8 \partial L}{\partial K} B_E + \frac{\partial L}{\partial I} \right)^{-1/2} B_t \tag{42}
\]

where we have defined

\[
\bar{B}_B = \dot{B} \cos \theta \sin \phi \text{ and } \bar{B}_E = \dot{B} \cos \phi.
\]

Where two of these characteristic curves intersect, a shock will form. To move further, we must choose the proper Lagrangian.

**III. THE NON-LINEAR LAGRANGIAN**

Heisenberg and Euler [1] and Weisskopf [2] independently derived the effective Lagrangian of the electromagnetic field using electron-hole theory. Schwinger [3] later redrew the same result using quantum electrodynamics. In rationalized electromagnetic units, the Lagrangian is given by
\( \mathcal{L} = -\frac{1}{4} I + \mathcal{L}_1 \)

\( \mathcal{L}_1 = \frac{e^2}{\hbar c} \int_0^\infty e^{-\zeta} \frac{d\zeta}{\zeta^3} \left\{ i \zeta^2 \sqrt{\frac{\mathcal{E}}{4}} \times \cos \left( \frac{\zeta_0}{\hbar c} \sqrt{\frac{\mathcal{F}}{2} + i \frac{\sqrt{\mathcal{K}}}{2}} \right) + \cos \left( \frac{\zeta_0}{\hbar c} \sqrt{\frac{\mathcal{F}}{2} - i \frac{\sqrt{\mathcal{K}}}{2}} \right) - \cos \left( \frac{\zeta_0}{\hbar c} \sqrt{\frac{\mathcal{F}}{2} - i \frac{\sqrt{\mathcal{K}}}{2}} \right) \right\} \) \]

where \( E_k = B_k = \frac{m^2 c^2}{\hbar} \). In the weak field limit Heisenberg and Euler give

\( \mathcal{L} \approx -\frac{1}{4} I + E_k^2 \frac{e^2}{\hbar c} \left[ \frac{1}{E_k^4} \left( \frac{1}{180} I^2 - \frac{7}{720} K \right) + \frac{1}{E_k^6} \left( \frac{13}{5040} K I - \frac{1}{630} I^3 \right) \right] \) \]

We define a dimensionless parameter \( \xi \) to characterize the field strength \( (I) \)

\( \xi = \frac{1}{E_k} \sqrt{\frac{I}{2}} \)

and use the analytic expansion of this Lagrangian for small \( K \) derived by Heyl and Hernquist \[\[]:

\( \mathcal{L}_1 = \mathcal{L}_1(I, 0) + K \frac{\partial \mathcal{L}_1}{\partial K} \bigg|_{K=0} + \frac{K^2}{2} \frac{\partial^2 \mathcal{L}_1}{\partial K^2} \bigg|_{K=0} + \cdots \)

The first two terms of this expansion are given by

\( \mathcal{L}_1(I, 0) = \frac{e^2}{\hbar c^2} X_0 \left( \frac{1}{\xi} \right) \) 

\( \frac{\partial \mathcal{L}_1}{\partial K} \bigg|_{K=0} = \frac{e^2}{\hbar c^2} X_1 \left( \frac{1}{\xi} \right) \)

where

\( X_0(x) = 4 \int_0^{x/2 - 1} \ln(\Gamma(v + 1))dv + \frac{1}{3} \ln \left( \frac{1}{x} \right) + 2 \ln 4\pi - (4 \ln A + \frac{5}{3} \ln 2) \)

\( - \left[ \ln 4\pi + 1 + \ln \left( \frac{1}{x} \right) \right] x + \left[ \frac{3}{4} + \frac{1}{2} \ln \left( \frac{2}{x} \right) \right] x^2 \)

\( X_1(x) = -2 X_0(x) + x X_0^{(1)}(x) + \frac{2}{3} X_0^{(2)}(x) - \frac{2}{9} \frac{1}{x^2} \)

and

\( X_0^{(n)}(x) = \frac{d^n X_0(x)}{dx^n} \)

\( \ln A = \frac{1}{12} - \zeta(1)(-1) \approx 0.2488. \)

where \( \zeta(1)(x) \) is the first derivative of the Riemann Zeta function.

Using these definitions we can derive the various partial derivatives important for shock formation

\( \frac{\partial \mathcal{L}}{\partial I} = -\frac{1}{4} + \frac{e^2}{\hbar c} \left[ \frac{1}{2} X_0 \left( \frac{1}{\xi} \right) - \frac{1}{4} X_0^{(1)} \left( \frac{1}{\xi} \right) \xi^{-1} \right] \)

\( \frac{\partial \mathcal{L}}{\partial K} = \frac{1}{288 \hbar c} E_k^{-2} \left[ -2 - 18 X_0 \left( \frac{1}{\xi} \right) - 6 X_0^{(2)} \left( \frac{1}{\xi} \right) \right] \xi^{-2} + 9 X_0^{(1)} \left( \frac{1}{\xi} \right) \xi^{-3} \)

\( \frac{\partial^2 \mathcal{L}}{\partial I^2} = \frac{1}{16 \hbar c^2} E_k^{-2} \left[ - X_0^{(1)} \left( \frac{1}{\xi} \right) \xi^{-3} + X_0^{(2)} \left( \frac{1}{\xi} \right) \xi^{-4} \right] \)

\( \frac{\partial^2 \mathcal{L}}{\partial I \partial K} = \frac{1}{384 \hbar c} E_k^{-4} \left[ \left( 12 X_0 \left( \frac{1}{\xi} \right) - 4 X_0^{(2)} \left( \frac{1}{\xi} \right) \right) \xi^{-4} - \left( 3 X_0^{(1)} \left( \frac{1}{\xi} \right) + 2 X_0^{(3)} \left( \frac{1}{\xi} \right) \right) \xi^{-5} - 3 X_0^{(2)} \left( \frac{1}{\xi} \right) \xi^{-6} \right] \)

\( \frac{\partial^3 \mathcal{L}}{\partial I^3} = \frac{1}{64 \hbar c} E_k^{-4} \left[ 3 X_0^{(1)} \left( \frac{1}{\xi} \right) \xi^{-5} - 3 X_0^{(2)} \left( \frac{1}{\xi} \right) \xi^{-6} - X_0^{(3)} \left( \frac{1}{\xi} \right) \xi^{-7} \right] \)
IV. THE OPACITY TO SHOCKING

Using the results of the previous section we can expand the opacity ($\kappa$) to order $e^2/\hbar c$, which results in a substantial simplification

$$\kappa = -32\frac{B_B \bar{B}_B}{c} \left( \frac{\partial^2 \mathcal{L}}{\partial \Omega^2} + \frac{\partial^3 \mathcal{L}}{\partial \Omega^3} \frac{B_B^2}{c} + 2 \frac{\partial^2 \mathcal{L}}{\partial \Omega \partial K} \frac{B_E^2}{c} \right) + \mathcal{O} \left[ \left( \frac{e^2}{\hbar c} \right)^2 \right]$$

(60)

$$= -\frac{B_B \bar{B}_B}{c B_k^2} \frac{e^2}{\hbar c} \left\{ 2 \left[ -X_0^{(1)} \left( \frac{1}{\xi} \right) \xi^{-3} + X_0^{(2)} \left( \frac{1}{\xi} \right) \xi^{-4} \right] + \frac{1}{2} \left( \frac{\bar{B}_E}{B_k} \right) \frac{B_B}{B_k} \frac{e^2}{\hbar c} \left[ 3X_0^{(1)} \left( \frac{1}{\xi} \right) \xi^{-5} - 3X_0^{(2)} \left( \frac{1}{\xi} \right) \xi^{-6} - X_0^{(3)} \left( \frac{1}{\xi} \right) \xi^{-7} \right] \right\}$$

$$+ \frac{1}{6} \left( \frac{\bar{B}_E}{B_k} \right)^3 \left[ 12X_0^{(1)} \left( \frac{1}{\xi} \right) - 4X_0^{(2)} \left( \frac{1}{\xi} \right) \right] \xi^{-4} \left[ 3X_0^{(1)} \left( \frac{1}{\xi} \right) + 2X_0^{(3)} \left( \frac{1}{\xi} \right) \right] \xi^{-5} - 3X_0^{(2)} \left( \frac{1}{\xi} \right) \xi^{-6} \right\}$$

$$+ \mathcal{O} \left[ \left( \frac{e^2}{\hbar c} \right)^2 \right]$$

(61)

where we have focussed on the propagation of the $\psi_\perp$ characteristics. The results for the $\psi_\parallel$ characteristics are identical in magnitude and follow from symmetry.

From the form of Eq. 42, we see that the opacity is zero for waves traveling with their magnetic field vectors perpendicular to the external field ($\perp$ mode). This result agrees with Bialynicka-Birula’s analysis [5] who found that although a wave in the $\parallel$ mode readily generates waves in the $\perp$ mode, a wave in the $\perp$ mode does not change to first order. These selection rules result from the $CP$ invariance of QED and may be gleaned from the selection rules for photon splitting [6].

Because our analysis tracks the evolution of only a single mode, we will calculate the opacity in the limit where the magnetic field of the wave is parallel to the external field. Waves in the $\parallel$ mode generate higher harmonics in the $\parallel$ mode but none in the $\perp$ mode. In this limit,

$$\bar{B}_B = \bar{B} \text{ and } \bar{B}_E = 0$$

(62)

and Eq. 42 simplifies further,

$$\kappa = \frac{1}{2} \frac{e^2}{\hbar c} \left[ X_0^{(1)} \left( \frac{1}{\xi} \right) \xi^{-2} - X_0^{(2)} \left( \frac{1}{\xi} \right) \xi^{-3} + X_0^{(3)} \left( \frac{1}{\xi} \right) \xi^{-4} \right] \frac{1}{c B_k} + \mathcal{O} \left[ \left( \frac{e^2}{\hbar c} \right)^2 \right]$$

(63)

We define a dimensionless auxiliary function $F(\xi)$ to characterize the opacity due to shocking

$$\kappa = -\frac{1}{c B_k} \frac{e^2}{\hbar c} \frac{B_B}{B_k} = -F(\xi) l_B^{-1}$$

(64)

We define $l_B$ to be the characteristic length over which the magnetic field of the wave would change by $B_k$. $l_B$ is positive for sections of the wave where the magnetic field strength increases as its passes a stationary observer, or equivalently in the frame of the wave itself, $l_B$ is negative in sections where the field strength decreases in the direction of propagation.

The function $F(\xi)$ may be expanded in the weak-field limit ($\xi < 0.5$) yielding

$$F(\xi) = -\frac{e^2}{\hbar c} \frac{1}{\xi} \sum_{j=1}^{\infty} 2^{2j} B_2(j+1) \frac{j+1}{2j+1} \xi^{2j} + \mathcal{O} \left[ \left( \frac{e^2}{\hbar c} \right)^2 \right]$$

(65)

$$= \frac{e^2}{\hbar c} \left[ \frac{16}{45} \xi - \frac{32}{35} \xi^2 + \cdots \right] + \mathcal{O} \left[ \left( \frac{e^2}{\hbar c} \right)^2 \right]$$

(66)

where $B_2$ denotes the 0th Bernoulli number. In the strong-field limit ($\xi > 0.5$), we obtain

$$F(\xi) = \frac{e^2}{\hbar c} \left[ \frac{2}{3} \frac{1}{\xi^2} + \frac{1}{2} (\ln \pi - 2 - \ln \xi) \frac{1}{\xi^2} + \frac{2}{\xi} \sum_{j=3}^{\infty} (-1)^{j-1} \frac{4(j-1) - j^2}{2} \frac{1}{j-1} \xi^{-j} \right] + \mathcal{O} \left[ \left( \frac{e^2}{\hbar c} \right)^2 \right]$$

(67)

where $\zeta(x)$ is the Riemann Zeta function and we have used the expansions of Heyl and Hernquist [7].

Note from Fig. 3 that $F(\xi)$ is positive for all field strengths. Thus, from examination of Eq. 64 we see that the opacity is positive (shocks will develop) in regions where the magnetic field is increasing toward the direction of propagation in the frame of the wave. $F(\xi)$ also reaches a maximum near the critical field strength.
V. THE PHYSICAL SHOCK: JUMP CONDITIONS

We expect a shock to develop when and where the value of the Riemann invariant is discontinuous. From Eq. 35 and using the result $B = \tau \pm E$, we see that the invariants are simply the electric and magnetic field strengths associated with the wave. Fig. 4 schematically depicts the evolution of the wave. The shock begins to form at an optical depth of one where the field of the wave becomes discontinuous.

As in fluid shocks, dissipative processes prevent the field strengths from becoming double valued. We use the Maxwell equal area prescription (12) to calculate the shock profile after the characteristic analysis indicates that the field strengths become double valued. We start with a sinusoidal wavefront,

$$B(y,t) = -B_0 \sin(y - \sigma_t) = -B \sin \nu_0$$

and obtain the following equation for the characteristics

$$\nu(\tau) = \nu_0 + \tau \sin \nu_0$$

in the frame of the wave. Furthermore, for convenience we use the optical depth to shock formation as the time unit.

From Fig. 4 and Eq. 69 we see that the wave evolves symmetrically about $\nu = \pi$. The position of the shock is given by the location which divides the double-valued regions into equal areas. By symmetry this occurs at $\nu = \pi$. The wavefront at $\tau = 2$ is constructed in this manner.

To determine the dissipation of energy by the shock, we calculate the mean power of the wave

$$P = \frac{1}{\pi} \int_0^\pi cB^2 d\nu = \frac{1}{\pi} \sigma_+ B_0^2 \int_0^\pi \sin^2 \nu_0 d\nu$$

$$= \frac{B_0^2}{\pi} \int_{\nu=0}^{\nu=\pi} \sin^2 \nu_0 (1 + \tau \cos \nu_0) d\nu_0$$

$$= \frac{B_0^2}{\pi} \left( \frac{\nu_{0,s}}{2} - \frac{1}{4} \sin 2\nu_{0,s} + \frac{1}{3} \tau \sin^3 \nu_{0,s} \right)$$

where $\nu_{0,s}$ is the smallest solution of

$$\pi = \nu_{0,s} + \tau \sin \nu_{0,s}$$

That is, the shock is located at $\nu_s = \pi$. For $\tau \leq 1$ the only real solution to Eq. 73 is $\nu_{0,s} = \pi$. Therefore, before the shock forms the mean power in the wave is simply $1/2\sigma_+ B_0^2$.

Unfortunately, in general this equation can only be solved numerically; however, two limits exist which can be treated analytically. As the shock just begins to develop $\nu_0 \approx \nu$ near the shock, so $\nu_{0,s} \approx \pi$. If we expand Eq. 73 about $\nu_{0,s} = \pi$, we obtain

$$\nu_{0,s} \approx \sqrt{6\tau - 1}$$

$$P = \sigma_+ B_0^2 \left[ \frac{1}{2} - \frac{8}{5\pi} \sqrt{6(\tau - 1)^{5/2}} + \frac{45}{77} \sqrt{6(\tau - 1)^{7/2}} + O(\tau - 1)^{9/2} \right]$$

We find that as soon as the shock forms at $\tau = 1$, the shock begins to dissipate energy from the wave. Additionally, the dissipation does not begin abruptly. This first two terms in this expansion are accurate to $\sim 1\%$ for $\tau - 1 \lesssim 0.2$.

At late times, we can find a solution to Eq. 73 such that $\nu_s \approx 0$. Here we obtain

$$\nu_{0,s} \approx \frac{\pi}{\tau + 1}$$

$$P \approx \frac{\pi^2}{3} \frac{1}{(\tau + 1)^2}$$

This expression is accurate to one percent for $\tau > 12$. The upper panel of Fig. 5 depicts the energy dissipation soon after the shock forms. It is apparent that the dissipation begins smoothly. The lower panel shows the late evolution.

Here, in the preceding analysis we have assumed that the field gradients are small both in our linearization and in our selection of the Heisenberg-Euler Lagrangian. Our linearization, specifically the assumption that the gradient of the fields is small relative to the fields breaks down when
\[ \lambda_e \frac{\partial B}{\partial z} \sim B \]  

(78)

where \( \lambda_e \) is the Compton wavelength of the electron. The Heisenberg-Euler Lagrangian also breaks down when the field changes by \( B_k \) over scales similar to \( \lambda_e \). In this limit, one must use more powerful techniques, such as Schwinger’s proper-time method \( [3] \) to determine the effective Lagrangian.

When the field changes dramatically over scales similar to \( \lambda_e \), our fluid approximation breaks down. We estimate the thickness of the shock to be approximately \( \lambda_e \). To further understand the properties of the shock, we derive the jump conditions across the shock discontinuity.

We move to the rest-frame of the shock and insist on the continuity of the dual to the field tensor and the energy-momentum tensor:

\[ \partial_\nu F^{\mu\nu} = 0 \]  

(79)

where \( F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\delta\gamma} F_{\delta\gamma} \) (e.g. \( [9] \)) and

\[ \partial_\nu \equiv \frac{\partial}{\partial x^\nu}. \]  

(80)

The first condition follows from the gauge invariance of the fields. The second condition represents the conservation of energy and momentum. These jump conditions are equivalent to those used by Boillat \( [10] \) who insisted that the dual of field tensor be continuous and that the Euler-Lagrange condition be satisfied.

For clarity, in contrast to the analysis of the preceding sections, we examine the energy-momentum and the field tensors of the combined wave and constant background field. Using the techniques outlined in \( [11] \), we find the energy-momentum tensor for non-linear electrodynamics. The canonical tensor (\( \tilde{\Theta}^{\mu\nu} \)) is constructed from the Lagrangian by means of a Legendre transformation,

\[ \tilde{\Theta}^{\mu\nu} = \frac{\partial L}{\partial \left[ \partial_\mu A_\rho \right]} \delta^{\nu} A_\rho - g^{\mu\nu} L \]  

(81)

where \( A_\rho \) is the potential four-vector of the electromagnetic field. We construct the more familiar symmetrized energy-momentum tensor (\( \Theta^{\mu\nu} \)) by subtracting a total divergence,

\[ \Theta^{\mu\nu} = \frac{\partial L}{\partial \left[ \partial_\mu A_\rho \right]} \delta^{\nu} A_\rho - g^{\mu\nu} L - \partial_\rho \left( \frac{\partial L}{\partial \left[ \partial_\mu A_\rho \right]} A^\nu \right) \]  

(82)

\[ = \frac{\partial L}{\partial \left[ \partial_\mu A_\rho \right]} \delta^{\nu} A_\rho - g^{\mu\nu} L - \frac{\partial L}{\partial \left[ \partial_\mu A_\rho \right]} \partial_\rho \delta^{\nu} A_\rho - \delta^{\nu} A_\rho \frac{\partial L}{\partial \left[ \partial_\mu A_\rho \right]} \]  

(83)

\[ = \frac{\partial L}{\partial \left[ \partial_\mu A_\rho \right]} F^{\nu}_\rho - g^{\mu\nu} L \]  

(84)

where the final term of Eq. 83 is zero by the Euler-Lagrange condition. Using the definitions of \( I \) and \( K \) we obtain,

\[ \Theta^{\mu\nu} = 4 \frac{\partial L}{\partial I} F^{\mu}_\rho F^{\nu}_\rho - 8 J \frac{\partial L}{\partial K} F^{\mu}_\rho \]  

(85)

where \( J = F^{\mu\nu} F^{\mu\nu}, \) the linear case, yields,

\[ \Theta^{\mu\nu} = -4 \frac{\partial L}{\partial I} F^{\mu\rho} F^{\nu}_\rho - g^{\mu\nu} \left( L - 2 K \frac{\partial L}{\partial K} \right). \]  

(86)

which for \( L = -\frac{1}{4} F^{\mu\sigma} F_{\mu\sigma} \), the linear case, yields

\[ \Theta^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + F^{\mu\rho} F^{\nu}_\rho. \]  

(87)

in agreement with Itzykson and Zuber’s result \( [11] \).

When determining shock jump conditions for a fluid, one calculates the velocity of the discontinuity relative to the rest frame of the fluid. In analogy to a fluid, we can associate the rest frame of the electromagnetic field with the
frame in which the energy-momentum tensor is diagonal. This frame exists if either of the two invariants \((I\) and \(K)\) is non-zero. In this frame, the electric and magnetic fields are parallel and their magnitudes are given by

\[
B^2 = \frac{1}{4} \left( I \pm \sqrt{I^2 - K} \right),
\]

\[
E^2 = \frac{1}{4} \left( -I \pm \sqrt{I^2 - K} \right)
\]

where the + sign is chosen for \(I > 0\) and – for \(I < 0\). If we take the fields to point along the \(x\)-axis, we obtain

\[
\Theta_{00} = -\Theta_{11} = -4 \frac{\partial \mathcal{L}}{\partial I} E^2 - \mathcal{L} + 2K \frac{\partial \mathcal{L}}{\partial K},
\]

\[
\Theta_{22} = \Theta_{33} = -4 \frac{\partial \mathcal{L}}{\partial I} B^2 + \mathcal{L} - 2K \frac{\partial \mathcal{L}}{\partial K}.
\]

To apply this to the geometry of the previous sections (Fig. 1), we will assume that \(K = 0\) and that the shock front is parallel to the \(x - z\) plane and traveling toward increasing \(y\).

The geometry leads to the jump conditions:

\[
[\Theta^{02}] = 0, [\Theta^{22}] = 0 \text{ and } [\mathcal{F}^{12}] = 0.
\]

across the shock. We calculate the components of the energy-momentum tensor in the shock frame through a boost. If \(K \neq 0\), the Lorentz transformation from the rest frames to the shock frame would include a rotation as well. By boosting the rest frames into the shock frame we get the following jump conditions

\[
[\Theta^{02}] = 2\sqrt{\gamma^2 v} \frac{\partial \mathcal{L}}{\partial I} = 0
\]

\[
[\Theta^{22}] = \left[ \mathcal{L} - 2\sqrt{\gamma^2 v} \frac{\partial \mathcal{L}}{\partial I} \right] = 0
\]

\[
[\mathcal{F}^{12}] = [\sqrt{I/2\gamma v}] = 0
\]

where \(v\) is the speed that the diagonalizing frame is moving relative to the shock and \(\gamma = (1 - v^2)^{-1/2}\). The three conditions are physically conservation of energy and momentum flux, and the continuity of the electric field parallel to the surface of shock. Analogous jump conditions are given by [12] for relativistic fluid shocks. If we define,

\[
e = w - p = -\mathcal{L}, \ p = \mathcal{L} - 2 \frac{\partial \mathcal{L}}{\partial I} I, \ w = -2 \frac{\partial \mathcal{L}}{\partial I} I \text{ and } u = -\sqrt{I}
\]

we obtain the following jump conditions

\[
[\Theta^{02}] = [\gamma^2 v w] = 0
\]

\[
[\Theta^{22}] = [p + \gamma^2 v^2 w] = 0
\]

\[
[\mathcal{F}^{12}] = [\gamma v n] = 0
\]

Landau and Lifshitz [12] give the velocities of the diagonalizing frames relative to the discontinuity

\[
v_1 = \sqrt{\frac{p_2 - p_1}{e_2 - e_1} \frac{e_2 - e_1}{e_2 - e_1 + p_2}}
\]

\[
v_2 = \sqrt{\frac{p_2 - p_1}{e_2 - e_1} \frac{e_2 - e_1}{e_2 - e_1 + p_1}}
\]

and the equation of the shock adiabatic

\[
\left( \frac{w_1}{n_1} \right)^2 - \left( \frac{w_2}{n_2} \right)^2 + (p_2 - p_1) \left( \frac{w_1}{n_1^2} + \frac{w_2}{n_2^2} \right) = 0
\]

where the subscripts 1 and 2 denote conditions on either side of the shock. The equation for the shock adiabatic is automatically satisfied to first order.
Taking Eq. 100 and Eq. 101 and assuming that the shock strength is a linear perturbation on the background field we get

\[ v_{1,2} = \sqrt{\left( \frac{2I}{\partial I^2} + \frac{\partial L}{\partial I} \right) \left/ \frac{\partial L}{\partial I} \right\} - \frac{I}{2} \Delta I + \frac{1}{2} I \Delta I \left/ \frac{\partial L}{\partial I} \right\} + \mathcal{O} \left( \frac{B^2}{B^2} \right) \}, \]  

(103)

\[ = \sigma_+ \left[ 1 + (1 \pm 1) \right] \frac{1 - \sigma^2}{4} \Delta I \left/ \frac{\partial L}{\partial I} \right\} + \mathcal{O} \left( \frac{B^2}{B^2} \right), \]  

(104)

\[ = \sigma_+ \left[ 1 + (1 \pm 1) \right] \frac{1 - \sigma^2}{2} \Delta B \left/ \frac{B}{B} \right\} - 4 \hat{B}^3 \Delta B \left/ \frac{\partial L}{\partial I} \right\} + \mathcal{O} \left( \frac{B^2}{B^2} \right), \]  

(105)

where \( \Delta I = I_1 - I_2 > 0, \Delta B = B_1 - B_2 > 0 \) and we have used Eq. 36 to simplify the expression.

It is more useful to find the speed at which the shocks travel relative to the external field. In general, the rest frame travels at a velocity that satisfies

\[ \frac{\mathbf{V}}{1 + |\mathbf{V}|^2} = \frac{\mathbf{E} \times (\mathbf{B} + \mathbf{B})}{|\mathbf{E}|^2 + |\mathbf{B} + \mathbf{B}|^2} \]  

(106)

For \( K = 0 \), the rest frame of the electromagnetic field travels at

\[ v_{1,2} = \mp \frac{\mathbf{E}}{\mathbf{B}} + \mathcal{O} \left( \frac{\mathbf{B}^2}{\mathbf{B}^2} \right) = \mp \sigma \frac{\mathbf{B}}{\mathbf{B}} + \mathcal{O} \left( \frac{\mathbf{B}^2}{\mathbf{B}^2} \right) \]  

(107)

in the \( y \)-direction relative to external field behind and ahead of the shock, respectively.

By summing the velocities of the rest frames relative to the external field and the shock relative to the rest frames, we find that the shock travels at the speed of light in the medium relative to the external field

\[ v_{1,2} = \sigma + \mathcal{O} \left( \frac{\mathbf{B}^2}{\mathbf{B}^2} \right) \]  

(108)

so we find in agreement with Boillat [10] that the Heisenberg-Euler Lagrangian does not admit shocks to first order in the strength of the discontinuity, if shocks are strictly defined as discontinuities that travel at a speed other that the speed of “sound” in the medium.

VI. CONCLUSIONS

We have developed a relativistic fluid dynamic description of the electromagnetic field to derive the characteristic equations for electromagnetic waves in the presence of a strong external magnetic field. By using analytic expressions for the effective Lagrangian of QED, we have obtained simple expressions to estimate the opacity of waves to shocking for arbitrary magnetic field strengths. Furthermore, by deriving a concise expression for the energy-momentum tensor for an arbitrary electromagnetic Lagrangian, we have calculated the shock jump conditions and find that the discontinuities travel at the speed of light in the medium relative to the external magnetic field. By extending this fluid dynamic description, we follow the eventual decay of an electromagnetic disturbance travelling through an intense magnetic field.

For shocks to form from electromagnetic waves, not only is a strong external field required but also a source of coherent radiation. A prime location for electromagnetic shocks is the vicinity of a neutron star with field strengths approaching and exceeding the critical value and coherent electromagnetic Alfven waves. The study of the nonlinear corrections to the propagation of radiation through a plasma is beyond the scope of this work; however, we expect shock formation to be a hallmark of the nonlinear corrections of quantum electrodynamics and possibly an important process in the energy transmission near neutron stars.

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FIG. 1. Illustration of the field geometry
FIG. 2. Characteristics and Shock Formation
FIG. 3. The figure depicts the auxiliary function $F(\xi)$ as a function of $\xi$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3.png}
\end{figure}
FIG. 4. Evolution of a wave toward shock formation in the frame of the wave. The upper panel traces the wave in the comoving frame. The lower panel traces the power spectrum of the wave. The successive lines denote original wave, the wave at an optical depth of one-half, at an optical depth of one and at an optical depth of two. The bold line shows a power spectrum of $\nu^{-2}$.
FIG. 5. The evolution of the power carried by the wave before and after shock formation.