W-algebras at the critical level

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Abstract. Let \( g \) be a complex simple Lie algebra, \( f \) a nilpotent element of \( g \). We show that (1) the center of the \( W \)-algebra \( W^{\text{cri}}(g, f) \) associated \((g, f)\) at the critical level coincides with the Feigin-Frenkel center of \(~\hat{g}~\), (2) the centerless quotient \( W_{\chi}(g, f) \) of \( W^{\text{cri}}(g, f) \) corresponding to an \( \mathfrak{g} \)-oper \( \chi \) on the disc is simple, and (3) the simple quotient \( W_{\chi}(g, f) \) is a quantization of the jet scheme of the intersection of the Slodowy slice at \( f \) with the nilpotent cone of \( g \).

1. Introduction

Let \( g \) be a complex simple Lie algebra, \( f \) a nilpotent element of \( g \), \( U(g, f) \) the finite \( W \)-algebra \([\text{PI}]\) associated with \((g, f)\). In \([\text{P2}]\) it was shown that the center of \( U(g, f) \) coincides with the center \( Z(g) \) of the universal enveloping algebra \( U(g) \) (Premet attributes the proof to Ginzburg).

Let \( W^{k}(g, f) \) be the (affine) \( W \)-algebra \([\text{FF3, KRW, KW}]\) associated with \((g, f)\) at level \( k \in \mathbb{C} \). One may \([\text{A3, DSK}]\) regard \( W^{k}(g, f) \) as a one-parameter chiralization of \( U(g, f) \). Hence it is natural to ask whether the analogous identity holds for the center \( Z(W^{k}(g, f)) \) of \( W^{k}(g, f) \), which is a commutative vertex subalgebra of \( W^{k}(g, f) \).

Let \( V^{k}(g) \) be the universal affine vertex algebra associated with \( g \) at level \( k \), \( Z(V^{k}(g)) \) the center of \( V^{k}(g) \). The embedding \( Z(V^{k}(g)) \hookrightarrow V^{k}(g) \) induces the vertex algebra homomorphism
\[
Z(V^{k}(g)) \rightarrow Z(W^{k}(g, f))
\]
for any \( k \in \mathbb{C} \). However, both \( Z(V^{k}(g)) \) and \( Z(W^{k}(g, f)) \) are trivial unless \( k \) is the critical level
\[
\text{cri} := -h^{\vee},
\]
where \( h^{\vee} \) is the dual Coxeter number of \( g \). Therefore the question one should ask is that whether the center \( Z(W^{\text{cri}}(g, f)) \) of the \( W \)-algebra at the critical level coincides with the Feigin-Frenkel center \([\text{FF4, PI}]\) \( Z(\hat{g}) := Z(V^{\text{cri}}(g)) \), which can be naturally considered as the space of functions on the space of \( L \)-opers \( \text{Op}_{L, \hat{g}}^{\text{SS}} \) of \( L \)-opers on the disc. Here \( \mathfrak{g} \) is the Langlands dual Lie algebra of \( g \).

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Theorem 1.1. The embedding \( \mathfrak{z} (\hat{\mathfrak{g}}) \hookrightarrow V^{\text{cri}} (\mathfrak{g}) \) induces the isomorphism

\[
\mathfrak{z} (\hat{\mathfrak{g}}) \cong Z (W^{\text{cri}} (\mathfrak{g}, f)).
\]

Moreover, \( W^{\text{cri}} (\mathfrak{g}, f) \) is free over \( \mathfrak{z} (\hat{\mathfrak{g}}) \), where \( \mathfrak{z} (\hat{\mathfrak{g}}) \) is regarded as a commutative ring with the \((-1)\)-product.

Theorem 1.1 generalizes a result of Feigin and Frenkel [FF4], who proved that \( \mathfrak{z} (\hat{\mathfrak{g}}) \cong W^{\text{cri}} (\mathfrak{g}, f_{\text{prin}}) \) for a principal nilpotent element \( f_{\text{prin}} \) of \( \mathfrak{g} \). It also generalizes a result of Frenkel and Gaitsgory [FG], who proved the freeness of \( V^{\text{cri}} (\mathfrak{g}) \) over \( \mathfrak{z} (\hat{\mathfrak{g}}) \).

Let \( G \) be the adjoint group of \( \mathfrak{g} \), \( S \) the Slodowy slice at \( f \) to \( \text{Ad} G.f \), \( N \) the nilpotent cone of \( \mathfrak{g} \). Set \( S = S \cap N \).

It is known [P1] that the scheme \( S \) is reduced, irreducible, and normal complete intersection of dimension \( \dim N - \dim \text{Ad} G.f \).

For \( \chi \in \text{Op}_{L^{\mathfrak{g}}}^{\text{reg}} \), let \( W^{\text{cri}} (\mathfrak{g}, f, \chi) \) be the quotient of \( W^{\text{cri}} (\mathfrak{g}, f) \) by the ideal generated by \( z - \chi (z) \) with \( z \in \mathfrak{z} (\hat{\mathfrak{g}}) \). Then any simple quotient of \( W^{\text{cri}} (\mathfrak{g}, f) \) is a quotient of \( W^{\text{cri}} (\mathfrak{g}, f, \chi) \) for some \( \chi \).

Theorem 1.2. For \( \chi \in \text{Op}_{L^{\mathfrak{g}}}^{\text{reg}} \), the vertex algebra \( W^{\text{cri}} (\mathfrak{g}, f, \chi) \) is simple. Its associated graded vertex Poisson algebra \( \text{gr} W^{\text{cri}} (\mathfrak{g}, f, \chi) \) is isomorphic to \( \mathbb{C} [S_{\infty}] \) as vertex Poisson algebras, where \( S_{\infty} \) is the infinite jet scheme of \( S \) and \( \mathbb{C} [S_{\infty}] \) is equipped with the level 0 vertex Poisson algebra structure.

Theorem 1.2 generalizes a result of Frenkel and Gaitsgory [FG], who proved the simplicity of the quotient of \( V^{\text{cri}} (\mathfrak{g}) \) by the ideal generated by \( z - \chi (z) \) for \( z \in \mathfrak{z} (\hat{\mathfrak{g}}) \).

In the case that \( f = f_{\text{prin}} \) we have \( W^{\text{cri}} (\mathfrak{g}, f_{\text{prin}}) = \mathbb{C} [FF4] \), while \( S \) is a point, and so is \( S_{\infty} \). Theorem 1.2 implies that this is the only case that \( W^{\text{cri}} (\mathfrak{g}, f) \) admits finite-dimensional quotients.

In general little is known about the representations of \( W^{\text{cri}} (\mathfrak{g}, f) \). We have shown in [A4] that at least in type \( A \) the representation theory of \( W^{\mathfrak{k}} (\mathfrak{g}, f) \) is controlled by that of \( \mathfrak{h} \) at level \( k \) for any \( k \in \mathbb{C} \). Therefore the Feigin-Frenkel conjecture (see [AF]) implies that, at least in type \( A \), the character of irreducible highest weight representations of \( W^{\text{cri}} (\mathfrak{g}, f) \) should be expressed in terms of Lusztig’s periodic polynomial [Lus]. We plan to return to this in future work.

2. Associated graded vertex Poisson algebras

For a vertex algebra \( V \), let \( \{ F^p V \} \) be the Li filtration [Li],

\[
\text{gr} V = \bigoplus_p F^p V / F^{p+1} V
\]

the associated graded vertex Poisson algebra. The vertex Poisson algebra structure of \( \text{gr} V \) restricts to the Poisson algebra structure on Zhu’s Poisson algebra [Zhu]

\[
R_V := V / F^1 V \subset \text{gr} V.
\]

Moreover there is a surjective map

\[
(\text{R}_V)_{\infty} \to \text{gr} V
\]
of vertex Poisson algebras \cite{Li1, A5}. Here \( X_V = \text{Spec } R_V \), \((R_V)_\infty = \mathbb{C}[[(X_V)_\infty]]\), where \((X_V)_\infty\) denotes the infinite jet scheme of a scheme \( X \) of finite type, and \((R_V)_\infty\) is equipped with the level zero vertex Poisson algebra structure \cite{A5 2.3}.

Let \( D^{ch}(C') \) be the \( \beta \gamma \)-system of rank \( r \), that is, the vertex algebra generated by fields \( a_1(z), \ldots, a_r(z), a_1^*(z), \ldots, a_r^*(z) \), satisfying the following OPE’s:

\[
a_i(z)a_j(z)^* \sim \frac{\delta_{ij}}{z-w}, \quad a_i(z)a_j(z) \sim a_i^*(z)a_j^*(z) \sim 0.
\]

It is straightforward to see that \( R_{D^{ch}(C')} \cong \mathbb{C}[T^* C'] \) as Poisson algebras and that \( \mathfrak{h} \) gives the isomorphism

\[
(\mathfrak{h})_\infty \cong \mathfrak{g}^{\text{aff}}.
\]

Let \( \mathfrak{g}, \mathfrak{f} \) be as in Introduction, \( \text{rk } \mathfrak{g} \) the rank of \( \mathfrak{g} \), \( (\mid ) \) the normalized invariant bilinear form of \( \mathfrak{g} \). Let \( \mathfrak{s} = \{ e, h, f \} \) be an \( \mathfrak{sl}_2 \)-triple in \( \mathfrak{g} \), and let \( \mathfrak{g}_j = \{ x \in \mathfrak{g}; [h, x] = 2jx \} \) so that

\[
\mathfrak{g} = \bigoplus_{j \in \frac{1}{2} \mathbb{Z}} \mathfrak{g}_j.
\]

Fix a triangular decomposition \( \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n} \) such that \( h \in \mathfrak{h} \subset \mathfrak{g}_0 \) and \( \mathfrak{n} \subset \mathfrak{g}_{\geq 0} := \bigoplus_{j \geq 0} \mathfrak{g}_j \). We will identify \( \mathfrak{g} \) with \( \mathfrak{g}^* \) via \(( \mid )\).

The Slodowy slice to \( \text{Ad } G.f \) at \( f \) is by definition the affine subspace

\[
\mathcal{S} = f + \mathfrak{g}^c
\]

of \( \mathfrak{g} \), where \( \mathfrak{g}^c \) is the centralizer of \( e \) in \( \mathfrak{g} \). It is known \cite{CG} that the Kirillov-Kostant Poisson structure of \( \mathfrak{g}^* \) restricts to \( \mathcal{S} \).

Let \( \ell \) be an \( \text{ad } \mathfrak{h} \)-stable Lagrangian subspace of \( \mathfrak{g}_{1/2} \) with respect to the symplectic form \( \mathfrak{g}_{1/2} \times \mathfrak{g}_{1/2} \to \mathbb{C}, (x, y) \mapsto (f| [x, y]) \). Set

\[
\mathfrak{m} = \ell \oplus \bigoplus_{j \geq 1} \mathfrak{g}_j,
\]

and let \( M \) be the unipotent subgroup of \( G \) whose Lie algebra is \( \mathfrak{m} \), \( \mathfrak{m}^\perp = \{ x \in \mathfrak{g}; (x| y) = 0 \) for all \( y \in \mathfrak{m} \} \). Then \cite{CG} we have the isomorphism of affine varieties

\[
M \times \mathcal{S} \cong f + \mathfrak{m}^\perp, \quad (g, x) \mapsto \text{Ad}(g)(x).
\]

This induces the following isomorphism of jet schemes:

\[
M_\infty \times \mathcal{S}_\infty \cong (f + \mathfrak{m}^\perp)_\infty.
\]

Denote by \( I \) and \( I_\infty \) the defining ideals of \( f + \mathfrak{m}^\perp \) and \((f + \mathfrak{m}^\perp)_\infty \) in \( \mathfrak{g} \) and \( \mathfrak{g}_\infty \), respectively. By \cite{C} and \cite{M} we have

\[
\mathbb{C}[S] \cong (\mathbb{C}[[\mathfrak{g}]]/I)^M, \quad \mathbb{C}[S_\infty] \cong (\mathbb{C}[[\mathfrak{g}_\infty]]/I_\infty)^{M_\infty}.
\]

Let

\[
\mathfrak{g} = \mathfrak{g}[t, t^{-1}] \oplus \mathfrak{K} \oplus \mathfrak{D}
\]

be the affine Kac-Moody algebra associated with \( \mathfrak{g} \), where \( \mathfrak{K} \) is the central element and \( \mathfrak{D} \) is the degree operator. Set \( \mathfrak{g} = \mathfrak{g}[t, t^{-1}] \oplus \mathfrak{K} \), the derived algebra of \( \mathfrak{g} \).

The universal affine vertex algebra \( V^k(\mathfrak{g}) \) associated with \( \mathfrak{g} \) at level \( k \in \mathbb{C} \) is the induced \( \mathfrak{g} \)-module \( U(\mathfrak{g}) \otimes V^k(\mathfrak{g}) \otimes \mathbb{C} \mathfrak{K} \), equipped with the natural vertex algebra structure (see e.g. \cite{Kac, FRZ}). Here \( \mathbb{C} \mathfrak{K} \) is the one-dimensional representation of \( \mathfrak{g}[t] \oplus \mathfrak{K} \) on which \( \mathfrak{g}[t] \) acts trivially and \( \mathfrak{K} \) acts as a multiplication by \( k \). The Li
filtration of $V^k(g)$ is essentially the same as the standard filtration of $U(g[t^{-1}]t^{-1})$ under the isomorphism $U(g[t^{-1}]t^{-1}) \cong V^k(g)$, see [A5]. We have

$$R_{V^k(g)} \cong \mathbb{C}[g^*]$$

(6) and (11) gives the isomorphism

$$\mathbb{C}[g^*] \xrightarrow{\text{vertex Poisson algebra structure of (9) and (1)}} \text{isomorphism}$$

and (11) gives the isomorphism

$$\mathbb{C}[g^*] \xrightarrow{\text{isomorphism}} \text{of vertex Poisson algebras. Let } \mathfrak{z}(\mathfrak{g}) \text{ be the Feigin-Frenkel center } Z(V_{\text{cris}}(\mathfrak{g})) \text{ as in Introduction. It is known } [FF] \text{ that the Li filtration of } V_{\text{cris}}(\mathfrak{g}) \text{ restricts to the Li filtration of } \mathfrak{z}(\mathfrak{g}). \text{ Moreover we have}$$

$$R_{\mathfrak{z}(\mathfrak{g})} \cong \mathbb{C}[g^*]^G,$$

(8) and (11) gives the isomorphism

$$R_{\mathfrak{z}(\mathfrak{g})} \cong \mathfrak{z}(\mathfrak{g}).$$

(Hence the vertex Poisson algebra structure of $\mathfrak{z}(\mathfrak{g})$ is trivial.) The isomorphisms (8) and (9) imply [BD1, EF] that

$$\text{gr } \mathfrak{z}(\mathfrak{g}) \cong \mathbb{C}((g^*)^G/_{G}) \cong \mathbb{C}[g^*]^G.$$
The \( W \)-algebra associated with \((g, f)\) at level \(k\) is by definition

\[
W_k^k(g, f) = H_{\tilde{f}}^{\tilde{g} + 0}(V^k(g)),
\]

which is naturally a vertex algebra because \(d\) is the zero mode of a odd field \(d(z)\) of \(C(V^k(g))\). We have \([\text{DSK}]\) that, for any \(k\),

\[
R_{W_k^k(g, f)} \cong C[S],
\]

and \([\text{A6}]\) gives the isomorphism

\[
C[S_{\infty}] \to \text{gr} W_k^k(g, f),
\]

see \([\text{A6}]\).

Let \(k = \text{cri}\). For \(z \in \mathfrak{z}(\mathfrak{g})\), we have \(dz = 0\), and the class of \(z\) belongs to the center \(Z(W_{\text{cri}}^k(g, f))\) of \(W_k^k(g, f)\). Hence the embedding \(\mathfrak{z}(\mathfrak{g}) \hookrightarrow V_{\text{cri}}^k(g)\) induces the vertex algebra homomorphism \(\mathfrak{z}(\mathfrak{g}) \to Z(W_{\text{cri}}^k(g, f)) \subset W_{\text{cri}}^k(g, f)\).

**Proposition 2.1.** The embedding \(\mathfrak{z}(\mathfrak{g}) \hookrightarrow V_{\text{cri}}^k(g)\) induces the embedding \(\mathfrak{z}(\mathfrak{g}) \hookrightarrow W_{\text{cri}}^k(g, f)\).

**Proof.** It is sufficient to show that it induces an injective homomorphism \(\text{gr} \mathfrak{z}(\mathfrak{g}) \to \text{gr} W_{\text{cri}}^k(g, f)\). Under the identification \(\mathfrak{g}\) and \([\text{III}]\), the induced map \(R_{V_{\text{cri}}}^k(g) \to R_{W_{\text{cri}}}^k(g, f)\) is identified the restriction map \(C[g]^*G \to C[S]\), and hence is injective \([\text{Kos}, \text{PT}]\). Therefore it induces the injective map \((R_{V_{\text{cri}}}^k(g))_{\infty} \hookrightarrow (R_{W_{\text{cri}}}^k(g, f))_{\infty}\), which is identical to the map \(\text{gr} \mathfrak{z}(\mathfrak{g}) \to \text{gr} W_{\text{cri}}^k(g, f)\).

By Proposition \([\text{A1}]\) we can define the quotient \(W_{\chi_{\text{cri}}}^k(g, f)\) of \(W_{\text{cri}}^k(g, f)\) for \(\chi \in \text{Op}_{\text{reg}}^k\) as in Introduction. Let

\[
W_{\text{res}}(g, f) = W_{\chi_{\text{cri}}}^k(g, f)
\]

and call it the **restricted \(W\)-algebra** associated with \((g, f)\). It is a graded quotient of \(W_{\text{cri}}^k(g, f)\).

**Remark 2.2.** Let \(\text{Zhu}(W_{\text{res}}^k(g, f))\) be the Ramond twisted Zhu algebra \([\text{DSK}]\) of \(W_{\text{res}}(g, f)\). Then from Proposition \([\text{A3}]\) below it follows that

\[
\text{Zhu}(W_{\text{res}}^k(g, f)) \cong U(g, f)/Z(g)^*U(g, f),
\]

where \(Z(g)^*\) is the argumentation ideal of \(Z(g)\).

Set \(S = S \cap \mathcal{N}\) as in Introduction. By restricting \([\text{II}]\) and \([\text{IV}]\), we obtain the isomorphisms

\[
\begin{align*}
M \times S &\to (f + \mathfrak{m}^\perp) \cap \mathcal{N}, \\
M_{\infty} \times S_{\infty} &\to ((f + \mathfrak{m}^\perp) \cap \mathcal{N})_{\infty} = (f + \mathfrak{m}^\perp)_{\infty} \cap \mathcal{N}_{\infty}.
\end{align*}
\]

**Proposition 2.3.** We have the following.

(i) \(H_{\tilde{f}}^{\tilde{g} + 1}(V_{\text{res}}^k(g)) = 0\) for \(i \neq 0\) and \(H_{\tilde{f}}^{\tilde{g} + 0}(V_{\text{res}}^k(g)) \cong W_{\text{res}}(g, f)\) as vertex algebras.

(ii) \(W_{\chi_{\text{cri}}}^k(g, f)\) is free over \(\mathfrak{z}(\mathfrak{g})\).

(iii) \(R_{W_{\text{res}}^k(g, f)} \cong C[S]\) as Poisson algebras and \(\text{gr} W_{\text{res}}(g, f) \cong C[S_{\infty}]\) as vertex Poisson algebras.
Let Proposition 2.5. as $H$ 4.4.3, (10) and (14) we obtain proved in $[A6]$. Let $\lambda $ highest weight of $W$ Remark 2.4. By the vanishing result of $H$ 4.4.3, (15) (i) follows from (15). (iii) By (i), the Li filtration of $V$ as $\hat{\mathfrak{g}}$-modules, and the freeness of $V^\text{cri} (\mathfrak{g})$ over $\hat{\mathfrak{g}}$ implies that $$\text{gr}^i V^\text{cri} (\mathfrak{g}) \cong V_{\text{res}}(\mathfrak{g}) \otimes_{\mathbb{C}} \hat{\mathfrak{g}}(\mathfrak{g})$$ as $V^\text{cri} (\mathfrak{g}) \otimes_{\mathbb{C}} \hat{\mathfrak{g}}(\mathfrak{g})$-modules. The vanishing of $H_f^\text{cri} (\mathfrak{g})$ implies that the spectral sequence associated with the filtration $\{\Gamma^p V^\text{cri} (\mathfrak{g})\}$ collapses at $E_1 = E_\infty$ and that $$(15) \quad \text{gr}^i H_f^\text{cri}(\mathfrak{g}) \cong H_f^\text{cri}(\mathfrak{g})\otimes_{\mathbb{C}} \mathfrak{g}(\mathfrak{g})$$ as $V^\text{cri} (\mathfrak{g}) \otimes_{\mathbb{C}} \mathfrak{g}(\mathfrak{g})$-modules. This proves the second assertion. (ii) follows from (15). Hence Proposition 2.5 implies that the filtration $\{\Gamma^p V^\text{cri} (\mathfrak{g})\}$ coincides with the Li filtration of $W_{\text{res}}(\mathfrak{g}, f)$. But this can be seen as in the same manner as $[A6]$ Theorem 4.4.4. 

**Remark 2.4.** By the vanishing result of $[Gin]$, the same argument proves the freeness of $U(\mathfrak{g}, f)$ over $\mathcal{Z}(\mathfrak{g})$.

**Proposition 2.5.** Let $\lambda \in \text{Op}^\text{reg}_{\mathfrak{g}}$. Then $H_f^\text{cri}(\mathfrak{g})^\text{cri} = 0$ for $i \neq 0$ and $H_f^\text{cri}(\mathfrak{g}) = W^\text{cri} (\mathfrak{g}, f)$. We have $\text{gr}^i W^\text{cri} (\mathfrak{g}, f) \cong \mathbb{C}[S_\infty]$ as vertex Poisson algebras.

**Proof.** We have proved the assertion in the case that $\chi = \chi_0$ in Proposition 2.3. The general case follows from this by the following argument: Consider the conformal weight filtration of a vertex algebra $V$ defined in [ACM 3.3.2], which we denote by $\{E_p V\}$. Then $\text{gr}^E V^\text{cri} (\mathfrak{g})$ is a $\mathfrak{g}$-module isomorphic to $V_{\text{res}}(\mathfrak{g})$ for any $\lambda \in \text{Op}^\text{reg}_{\mathfrak{g}}$. Hence Proposition 2.3 implies that $$(16) \quad \text{gr}^E H_f^\text{cri}(\mathfrak{g})^\text{cri} = \begin{cases} W_{\text{res}}(\mathfrak{g}, f) & \text{for } i = 0 \\ 0 & \text{for } i \neq 0. \end{cases}$$ This completes the proof. 

**3. BRST cohomology of restricted Wakimoto modules**

Let $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$, the Cartan subalgebra of $\hat{\mathfrak{g}}$, $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C} \delta \oplus \mathbb{C} \Lambda_0$ the dual of $\mathfrak{h}$, where $\delta$ and $\Lambda_0$ are the elements dual to $D$ and $K$, respectively. Put $\widehat{\mathfrak{h}}^*_k = \{ \lambda \in \mathfrak{h}^* ; \lambda(K) = k \}$. For $\lambda \in \widehat{\mathfrak{h}}^*_k$, let $F^\text{res}_\lambda$ be the restricted Wakimoto module $[FF1]$ with highest weight $\lambda$. Set $M^\text{res}_\lambda := F^\text{res}_\lambda$. 

\[ *\text{In }[FT] \text{ } F^\text{res}_\lambda \text{ is denoted by } W_{\lambda/\ell}. \]
The module $M_{g}$ has a natural vertex algebra structure, and
$M_{g} \cong \mathcal{D}^{ch}(\mathbb{C}^{\text{dim} n})$ as vertex algebras. There is a vertex algebra homomorphism $V^{\text{crit}}(g) \rightarrow M_{g}$, which induces the vertex algebra homomorphism

$$\omega : V_{\text{res}}(g) \rightarrow M_{g},$$

see [FT]. The map $\omega$ is injective because $V_{\text{res}}(g)$ is simple.

The fact that $V_{\text{res}}(g)$ is a vertex subalgebra of $M_{g}$ implies that $d(z)$ can be considered as a field on $M_{g}$. Hence $H^{\bullet+0}_{\mathfrak{f}}(M_{g})$ is also naturally a vertex algebra. Thus applying the functor $H^{\bullet+0}_{\mathfrak{f}}(\mathcal{D})$ to (17), we obtain the vertex algebra homomorphism

$$\omega_{\mathcal{D}} : \mathcal{D}_{\text{res}}(g, f) = H^{\bullet+0}_{\mathfrak{f}}(V_{\text{res}}(g)) \rightarrow H^{\bullet+0}_{\mathfrak{f}}(M_{g}).$$

**Proposition 3.1.** The vertex algebra homomorphism $\omega_{\mathcal{D}}$ is injective. In fact it induces an injective homomorphism $\text{gr} \mathcal{D}_{\text{res}}(g, f) \hookrightarrow \text{gr} H^{\bullet+0}_{\mathfrak{f}}(M_{g})$ of vertex Poisson algebras.

In order to prove Proposition 3.1 we first describe the homomorphism

$$\bar{\omega} : \text{gr} V_{\text{res}}(g) \rightarrow \text{gr} M_{g}$$

induced by $\omega$. Recall that $\text{gr} V_{\text{res}}(g) \cong \mathbb{C}[\mathcal{N}_{\infty}]$ and $R_{V_{\text{res}}(g)} \cong \mathbb{C}[\mathcal{N}]$, see [10]. Let $\mathcal{B}$ be the set of Borel subalgebras in $\mathfrak{g}$, or the flag variety of $\mathfrak{g}$. Denote by $U$ be the big cell, i.e., the unique open $N$-orbit in $\mathcal{B}$, where $N$ is the unipotent subgroup of $G$ corresponding to $\mathfrak{n}$. Let $T^{*}\mathcal{B}$ be the cotangent bundle of $\mathcal{B}$, $\pi : T^{*}\mathcal{B} \rightarrow \mathcal{B}$ the projection,

$$\tilde{U} = \pi^{-1}(U).$$

By construction [FT] we have

$$R_{M_{g}} \cong \mathbb{C}[\tilde{U}], \quad \text{gr} M_{g} \cong (R_{M_{g}})_{\infty} = \mathbb{C}[\tilde{U}_{\infty}],$$

and the homomorphism

$$\bar{\omega} |_{R_{V_{\text{res}}(g)}} : R_{V_{\text{res}}(g)} \rightarrow R_{M_{g}}$$

may be identified with the restriction $\tilde{U} \rightarrow \mathcal{N}$ of the Springer resolution

$$\mu : T^{*}\mathcal{B} \rightarrow \mathcal{N}.$$

This in particular shows that $\bar{\omega}$ is also injective. Indeed, $\bar{\omega} |_{R_{V_{\text{res}}(g)}}$ is injective because it is the composition of the isomorphism $\mu^{*} : \mathbb{C}[\mathcal{N}] \cong \Gamma(T^{*}\mathcal{B}, \mathcal{O}_{T^{*}\mathcal{B}})$, with the restriction map $\Gamma(T^{*}\mathcal{B}, \mathcal{O}_{T^{*}\mathcal{B}}) \rightarrow \Gamma(\tilde{U}, \mathcal{O}_{T^{*}\mathcal{B}}) = \mathbb{C}[	ilde{U}]$. Hence it induces an injection $(R_{V_{\text{res}}(g)})_{\infty} \rightarrow (R_{M_{g}})_{\infty}$, and this is identical to $\bar{\omega}$.

**Remark 3.2.** Let $\mathcal{D}^{ch}_{\mathcal{B}}$ be the sheaf of chiral differential operators [GMS2, MSV, BD2] on $\mathcal{B}$, which exists uniquely [GMS1, AG]. It is a sheaf of vertex algebras on $\mathcal{B}$, and we have

$$R_{\mathcal{D}^{ch}_{\mathcal{B}}} \cong \pi_{*}\mathcal{O}_{T^{*}\mathcal{B}}, \quad \text{gr} \mathcal{D}^{ch}_{\mathcal{B}} \cong (\pi_{\infty})_{*}\mathcal{O}_{(T^{*}\mathcal{B})_{\infty}},$$

where $R_{\mathcal{D}^{ch}_{\mathcal{B}}}$ and gr $\mathcal{D}^{ch}_{\mathcal{B}}$ are the corresponding sheaves of Zhu’s Poisson algebras respectively, and $\pi_{\infty} : (T^{*}\mathcal{B})_{\infty} \rightarrow \mathcal{B}$ is the projection. We have

$$\mathcal{D}^{ch}_{\mathcal{B}}(U) \cong M_{g}.$$
as vertex algebras. The homomorphism (17) lifts to a vertex algebra homomorphism

\[ \omega_{\text{res}}: V_{\text{res}}(g) \to \Gamma(B, D^b), \]

which is in fact an isomorphism [ACM].

Next we describe the vertex Poisson algebra structure of \( \text{gr} H_f^{\infty i_0}(M_g) \). Let

\[ \tilde{S} = \mu^{-1}(S), \]

the Slodowy variety. It is known [Gin] that \( \tilde{S} \) is a smooth, connected symplectic submanifold of \( T^*B \) and the morphism \( \mu: \tilde{S} \to S \) is a symplectic resolution of singularities. As explained in [Gin], \( \tilde{S} \) can be also obtained by means of the Hamiltonian reduction: Let

\[ \tilde{\mu}: T^*B \to m^* \]

be the composition of \( T^*B \xrightarrow{\beta_N} N \hookrightarrow g^* \) with the restriction map \( g^* \to m^* \). Then \( \tilde{\mu} \) is the moment map for the \( M \)-action and the one point \( M \)-orbit \( f \in m^* \) is a regular value of \( \tilde{\mu} \). Let

\[ \Sigma = \tilde{\mu}^{-1}(f). \]

Then \( \Sigma \) is a reduced smooth connected submanifold of \( T^*B \), and the action map gives an isomorphism

\[ M \times \tilde{S} \to \Sigma, \]

and we get that

\[ \tilde{S} \cong \Sigma/M. \]

By (22), we obtain the jet scheme analogue

\[ M_\infty \times S_\infty \to \Sigma_\infty. \]

Let

\[ V = \tilde{S} \cap \tilde{U}. \]

Because \( \tilde{U} \) is \( M \)-stable, by restricting (22) and (24) we obtain the isomorphisms

\[ M \times V \to \Sigma \cap \tilde{U}, \]

\[ M_\infty \times V_\infty \to \Sigma_\infty \cap \tilde{U}_\infty = (\Sigma \cap \tilde{U})_\infty. \]

Note that \( \Sigma \cap \tilde{U} \) is an open dense subset of \( \Sigma \), and \( V \) is an open dense subset of \( \tilde{S} \).

**Proposition 3.3.** We have \( H_f^{\infty i_0}(M_g) = 0 \) for \( i \neq 0 \), \( R_{H_f^{\infty i_0}(M_g)} \cong \mathbb{C}[V] \) and \( \text{gr} H_f^{\infty 0}(M_g) \cong \mathbb{C}[V_\infty] \).

**Proof.** By [A2] 3.7, the differential \( d \) decomposes as \( d = dt + dx \) with \( (dt^2) = (dx^2) = \{dt^2, dx\} = 0 \). It follows that there is a spectral sequence \( E_r \Rightarrow H_f^{\infty 0}(M_g) \) such that \( d_0 = dt \) and \( d_1 = dx \). By [A2] Remark 3.7.1 we have

\[ H^*(C(M_g), dt^2) \cong H^{\infty 0}(L_{g \geq 0}, M_g \otimes S(g_{1/2}[t^{-1}]t^{-1})), \]

where \( L_{g \geq 0} = g[t, t^{-1}] \) and \( S(g_{1/2}[t^{-1}]t^{-1}) \) is considered as a \( L_{g \geq 0} \)-module through the identification \( S(g_{1/2}[t^{-1}]t^{-1}) \cong U(L_{g \geq 0})/U(L_{g < 0})(L_{g \geq 1} + g_{1/2}[t]) \). Here, \( L_{g \geq 1} = \bigoplus_{j \geq 1} g_{j}[t, t^{-1}] \).
Because $M_\lambda$ is free over $n[t^{-1}]t^{-1}$ and cofree over $n[t]$ by construction, it follows by [Vor] Theorem 2.1 that $H_{f}^{\infty+i}(L_{g^{-1}0}, M_\lambda \otimes S(g_{1/2}[t^{-1}]t^{-1})) = 0$ for $i \neq 0$. Hence the spectral sequence collapses at $E_1 = E_\infty$ and we get that

$$H_{f}^{\infty+i}(M_\lambda) \cong \begin{cases} H^0(C(M_\lambda), d^0) & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

(28)

This proves the first assertion.

Let $g C(M_\lambda)$ be the associated graded complex of $C(M_\lambda)$ with respect to the Li filtration of $C(M_\lambda)$. Then as in [A6] Theorem 4.3.3 we find that

$$H^i(g C(M_\lambda)) \cong \begin{cases} (g M_\lambda/I_\infty g M_\lambda)^{M_\infty} \cong \mathbb{C}[V_\infty] & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

(29)

By (29), (28) and [A6] Proposition 4.4.3, the spectral sequence associated with the Li filtration of $C(M_\lambda)$ converges to $H_{f}^{\infty+i}(M_\lambda)$. Hence we have $gr^K H_{f}^{\infty+i}(M_\lambda) \cong \mathbb{C}[V_\infty], \text{ where } gr^K H_{f}^{\infty+i}(M_\lambda)$ is the associated graded vertex algebra with respect to the filtration $\{K^p H_{f}^{\infty+i}(M_\lambda)\}$ induced by the Li filtration of $C(M_\lambda)$. As in the proof of Proposition 2.3 we see that $\{K^p H_{f}^{\infty+i}(M_\lambda)\}$ coincides with the Li filtration of $H_{f}^{\infty+i}(M_\lambda)$. This proves the last assertion, which restricts to the second assertion.

□

**Proof of Proposition 3.1** Let $\mu_{|V} : V \to S$ be the restriction of the resolution $\mu_{|S} : S \to S$. By Propositions 2.3 and 3.3

$$\mu_{|V}^* : \mathbb{C}[S] \to \mathbb{C}[V]$$

(30)

can be identified with the homomorphism $R_{\mathcal{W}_{res}(g,f)} \to R_{H_{f}^{\infty+i}(M_\lambda)}$ induced by the vertex algebra homomorphism $\omega_{\mathcal{W}} : \mathcal{W}_{res}(g,f) \to H_{f}^{\infty+i}(M_\lambda)$. Thus, by Propositions 2.3 and 3.3 it is sufficient to show that (30) is injective. But (30) is the composition of $\mu^* : \mathbb{C}[S] \to \Gamma(S, \mathcal{O}_S)$ with the restriction map $\Gamma(S, \mathcal{O}_S) \to \mathbb{C}[V]$. Hence it is injective as required.

□

**4. Proof of Theorems 1.1 and 1.2**

For $\lambda \in \widehat{h}_k^*$, let $M_\lambda$ be the Verma module with highest weight $\lambda$, $M_\lambda^*$ its contragredient dual.

**Proposition 4.1.** Suppose that $H_{f}^{\infty+i}(M_\lambda^*) = 0$ for $i \neq 0$. Then $H_{f}^{\infty+i}(M_\lambda^*)$ is a cocyclic $\mathcal{W}^k(g,f)$-module with the cocyclic vector $v^*_\lambda$, where $v^*_\lambda$ is the image of the cocyclic vector of $M_\lambda^*$.

**Proof (outline).** By the argument of [A1 §6], [A2 §7], [A3 §7], we can construct a subcomplex $C'$ of $C(M_\lambda^*)$ with the following properties:

(i) $H^i(C') = 0$ for $i \neq 0$;

(ii) $C'$ is a $\mathcal{W}^k(g,f)$-submodule of $C(M_\lambda^*)$ containing $v^*_\lambda$, and moreover,

$H^0(C')$ is a cocyclic $\mathcal{W}^k(g,f)$-module with the cocyclic vector $v^*_\lambda$;

(iii) The character of $H^0(C')$ coincides with the character of $H_{f}^{\infty+i}(M_\lambda^*)$.
Because $H^0(C')$ is cocyclic the above property (iii) forces that the map $H^0(C') \to H^1_f(\mathcal{M}^\lambda_\alpha)$ is an injection. But $H^0(C')$ and $H^1_f(\mathcal{M}^\lambda_\alpha)$ have the same character. Therefore it must be an isomorphism. \hfill \Box

Let $\Delta_+$ be the set of positive roots of $\mathfrak{g}, W$ the Weyl group of $\mathfrak{g}, \rho = \sum_{\alpha \in \Delta_+} \alpha/2, \rho^\vee = \sum_{\alpha \in \Delta_+} \alpha^\vee/2$. Denote by $\hat{\Delta}_+^{\mathbb{R}}$ the set of positive real roots of $\hat{\mathfrak{g}}$. The set $\Delta_+$ is naturally considered as a subset of $\hat{\Delta}_+^{\mathbb{R}}$. Let $\hat{W} = W \rtimes P^\vee$ the extended affine Weyl group of $\hat{\mathfrak{g}}$, where $P^\vee \subset \mathfrak{h}$ is the set of coweights of $\mathfrak{g}$. We denote by $t_\mu$ the element of $\hat{W}$ corresponding $\mu \in P^\vee$. Set
\[ \hat{P}^+_k = \{ \lambda \in \hat{\mathfrak{g}}^*_k; \lambda(\alpha^\vee) \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in \Delta_+ \} . \]

**Proposition 4.2.** Suppose that $k + h^\vee \not\in \mathbb{Q}_{> 0}$. Then, for $\lambda \in \hat{P}^+_k$, $\mathcal{M}^\lambda_\alpha$ is free over $n[t^{-1}]t^{-1}$.

**Proof.** By the assumption
\[ (\lambda + \hat{\rho}, \alpha^\vee) \not\in \mathbb{N} \text{ for } \alpha \in \hat{\Delta}_+^{\mathbb{R}} \text{ such that } \hat{\alpha} \in -\Delta_+ , \]
where $\hat{\rho} = \rho + h^\vee \Lambda_0$. Hence $(\lambda + \hat{\rho}, \alpha^\vee) \not\in \mathbb{N}$ for all $\alpha \in \hat{\Delta}_+^{\mathbb{R}} \cap t_{-n\alpha^\vee}(-\hat{\Delta}_+^{\mathbb{R}})$, $n \in \mathbb{N}$. By [A1] Theorem 3.1, this implies that $\mathcal{M}^\lambda_\alpha$ is cofree over the subalgebra $\bigoplus_{\alpha \in \hat{\Delta}_+^{\mathbb{R}} \cap t_{-n\alpha^\vee}(-\hat{\Delta}_+^{\mathbb{R}})} \hat{\mathfrak{g}}_\alpha \subset n_- [t]t$, or equivalently, $\mathcal{M}^\lambda_\alpha$ is free over $\bigoplus_{\alpha \in \hat{\Delta}_+^{\mathbb{R}} \cap t_{-n\alpha^\vee}(-\hat{\Delta}_+^{\mathbb{R}})} n[t^{-1}]t^{-1}$. Here $\hat{\mathfrak{g}}_\alpha$ is the root space of $\hat{\mathfrak{g}}$ of root $\alpha$. Now we have $n[t^{-1}]t^{-1} = \lim_{n \to \infty} \bigoplus_{\alpha \in -\hat{\Delta}_+^{\mathbb{R}} \cap t_{-n\alpha^\vee}(-\hat{\Delta}_+^{\mathbb{R}})} \hat{\mathfrak{g}}_\alpha$. Therefore $\mathcal{M}^\lambda_\alpha$ is free over $n[t^{-1}]t^{-1}$ as required. \hfill \Box

**Remark 4.3.** Proposition 4.2 implies that $\mathcal{M}^\lambda_\alpha$ with $\lambda \in \hat{P}^+_k, k + h^\vee \not\in \mathbb{Q}_{> 0}$, is isomorphic to the Wakimoto module $F^\lambda_\alpha$ with highest weight $\lambda$, see the proof of Proposition 3.3.

**Proposition 4.4.** Suppose that $k + h^\vee \not\in \mathbb{Q}_{> 0}$ and let $\lambda \in \hat{P}^+_k$. Then we have $H^1_f(\mathcal{M}^\lambda_\alpha) = 0$ for $i \neq 0$ and $H^1_f(\mathcal{M}^\lambda_\alpha)$ is cocyclic $\mathcal{W}^k(\mathfrak{g}, f)$-module with the cocyclic vector $v^\lambda_\alpha$.

**Proof.** $\mathcal{M}^\lambda_\alpha$ is cofree over $n[t]$ by definition, and is free over $n[t^{-1}]t^{-1}$ by Proposition 4.2. Hence one can apply the proof of Proposition 3.3 to obtain the vanishing assertion. The rest follows from Proposition 4.1. \hfill \Box

Now let $k = cri$. For $\lambda \in \hat{\mathfrak{h}}^*_\text{cri}$, let $\mathcal{M}^\text{res}^\lambda_\alpha$ be the restricted Verma module [AF] with highest weight $\lambda$, $(\mathcal{M}^\text{res}^\lambda_\alpha)^*$ its contragredient dual. The module $\mathcal{M}^\text{res}^\lambda_\alpha$ is defined as follows: Let $p^{(1)}, \ldots, p^{(l)}$ be a set of homogeneous generators of $\mathfrak{g}(\bar{\lambda})$ as a differential algebra (which is the same as a commutative vertex algebra), so that $R_{\text{cr}}(\hat{\mathfrak{g}}) = \mathbb{C}[\bar{p}^{(1)}, \ldots, \bar{p}^{(l)}]$, where $\bar{p}^{(i)}$ is the image of $p^{(i)}$ in $R_{\text{cr}}(\hat{\mathfrak{g}}) = \mathbb{C}[\hat{\mathfrak{g}}]^G$. Let $Y(p_i, z) = \sum_{n \in \mathbb{Z}} p^{(i)}_n z^{-n-\Delta_i}$ be the field corresponding to $p^{(i)}$, where $\Delta_i$ is the degree of the polynomial $\bar{p}^{(i)}$. Then
\[ Z_\pm = \mathbb{C}[p^{(i)}_n; i = 1, \ldots, l, \pm n > 0] \]
can be regarded as a polynomial ring, which acts on any $V^{\text{cri}}(\hat{\mathfrak{g}})$-module. According to Feigin and Frenkel [FF2] [F2], $Z_-$ acts on $\mathcal{M}^\lambda_\alpha$ freely. By definition,
\[ \mathcal{M}^\text{res}^\lambda_\alpha = \mathcal{M}^\lambda_\alpha/Z_- \mathcal{M}^\lambda_\alpha, \]
where \( Z^*_+ \) is the argumentation ideal of \( Z_- \). Dually,
\[
(M^{\text{res}}_\lambda)^* = \{ m \in M^*_\lambda; Z^*_+ m = 0 \},
\]
where \( Z^*_+ \) is the argumentation ideal of \( Z_+ \).

**Proposition 4.5.** Let \( \lambda \in \hat{P}^+_\text{cri} \).

(i) The embedding \((M^{\text{res}}_\lambda)^* \hookrightarrow M^*_\lambda\) induces an embedding \( H^{\hat{\lambda}}_f H^{\hat{\lambda}}_f +0 ((M^{\text{res}}_\lambda)^*) \hookrightarrow H^{\hat{\lambda}}_f H^{\hat{\lambda}}_f +0 (M^*_\lambda)\).

(ii) The \( W^{\text{cri}}(g, f)\)-module \( H^{\hat{\lambda}}_f H^{\hat{\lambda}}_f +0 ((M^{\text{res}}_\lambda)^*) \) is cocyclic with the cocyclic vector \( e^\lambda_\lambda \).

**Proof.** (1) Let \( \Gamma^0 M^*_\lambda = 0, \Gamma^p M^*_\lambda = \{ m \in M^*_\lambda; (Z^*_+)^p m = 0 \} \) for \( p \geq 1 \). Then \( \{ \Gamma^p M^*_\lambda \} \) defines an increasing filtration of \( M^*_\lambda \) as a \( \hat{g} \)-module, and the freeness of \( M^*_\lambda \) over \( Z_- \) implies that
\[
gr^\Gamma M^*_\lambda \cong (M^{\text{res}}_\lambda)^* \otimes D(Z_+),
\]
as \( V^k(g) \otimes Z_+ \)-modules, where \( D(Z_+) \) is the restricted dual of \( Z_+ \).

Now \( (M^{\text{res}}_\lambda)^* \) is free over \( n[t]^{-1}t^{-1} \) and cofree over \( n[t] \) by Proposition 4.6. Thus we see as in the proof of Proposition 4.3 that \( H^{\hat{\lambda}}_f H^{\hat{\lambda}}_f +i ((M^{\text{res}}_\lambda)^*) = 0 \) for \( i \neq 0 \). This shows that the spectral sequence corresponding to the filtration \( \Gamma^p M^*_\lambda \) collapses at \( E_1 = E_\infty \) and we get that
\[
gr^\Gamma H^{\hat{\lambda}}_f H^{\hat{\lambda}}_f +0 (M^*_\lambda) \cong H^{\hat{\lambda}}_f H^{\hat{\lambda}}_f +0 ((M^{\text{res}}_\lambda)^*) \otimes D(Z_+).
\]
In particular,
\[
H^{\hat{\lambda}}_f H^{\hat{\lambda}}_f +0 ((M^{\text{res}}_\lambda)^*) \cong \Gamma^1 H^{\hat{\lambda}}_f H^{\hat{\lambda}}_f +0 (M^*_\lambda) = \{ c \in H^{\hat{\lambda}}_f H^{\hat{\lambda}}_f +0 (M^*_\lambda); Z^*_+ c = 0 \}.
\]
This proves (1). (2) follows from (1) and Proposition 4.3.

**Proposition 4.6.** For \( \lambda \in \hat{P}^+_\text{cri} \), the restricted Wakimoto module \( F^{\text{res}}_\lambda \) is isomorphic to \((M^{\text{res}}_\lambda)^*\).

**Proof.** By [ACM] 6.2.2, \( F^{\text{res}}_\lambda \) is cocyclic with the cocyclic vector \(|\lambda|\), where \(|\lambda|\) is the highest weight vector. Hence its contragredient dual \((F^{\text{res}}_\lambda)^*\) is cyclic, and the natural \( \hat{g} \)-module homomorphism \( M^*_\lambda \rightarrow (F^{\text{res}}_\lambda)^* \) is surjective. Because \( Z_- \) acts trivially on \((F^{\text{res}}_\lambda)^*\), this factors through the surjective homomorphism \( M^{\text{res}}_\lambda \rightarrow (F^{\text{res}}_\lambda)^* \). Since \((F^{\text{res}}_\lambda)^*\) and \( M^{\text{res}}_\lambda \) are the same character, it must be an isomorphism. By duality, this proves the assertion.

**Proof of Theorem 1.2.** We have already shown the assertion on the associated graded vertex Poisson algebras in Proposition 2.5. It remains to prove the simplicity.

First, let \( \chi = \chi_0 \). By Propositions 4.4 and 4.6, \( W^{\text{cri}}(g, f) \) is a submodule of \( H^{\hat{\lambda}}_f H^{\hat{\lambda}}_f +0 ((M^{\text{res}}_{\chi_0 \text{cri}})^*) \). On the other hand, \( H^{\hat{\lambda}}_f H^{\hat{\lambda}}_f +0 ((M^{\text{res}}_{\chi_0 \text{cri}})^*) \) is cocyclic by Proposition 4.3, and the image of the vacuum vector \( 1 \) of \( W^{\text{cri}}(g, f) \) equals to the cocyclic vector of \( H^{\hat{\lambda}}_f H^{\hat{\lambda}}_f +0 ((M^{\text{res}}_{\chi_0 \text{cri}})^*) \) up to nonzero constant multiplication. Hence \( W^{\text{cri}}(g, f) \) is also cocyclic, with the cocyclic vector \( 1 \). Therefore \( W^{\text{cri}}(g, f) \) must be simple.

Next, let \( \chi \) be arbitrary. Let \( \{ E_p W^{\text{cri}}_\chi(g, f) \} \) be the conformal filtration of \( W^{\text{cri}}_\chi(g, f) \) as in the proof of Proposition 2.5. Then (10) shows that \( gr^E W^{\text{cri}}_\chi(g, f) \cong \)
\( \mathcal{W}_{\text{res}}(g, f) \) as \( \mathcal{W}^{\text{cri}}(g, f) \)-modules, which is simple. Therefore \( \mathcal{W}^{\text{cri}}(g, f) \) is also simple. This completes the proof. \( \Box \)

**Proof of Theorem 1.1** The first assertion follows immediately from Proposition 2.1 and Theorem 1.2. The freeness assertion has been proved in Proposition 2.3.

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