1. Introduction

The 2012 Ig Nobel Fluid Dynamics Prize was awarded to R. Krechetnikov and H. Mayer for their study of people walking while carrying a filled coffee mug [25]. They show that coffee spills so often because the sloshing mode with the lowest-frequency (most noticeable in practice) in a typical coffee mug tends to get excited during walking. Authors model oscillations of the coffee as appropriate mixed Steklov problem.

However, there is another reason for spilling from a mug: high spot on the boundary. The maximal elevation of the lowest-frequency liquid oscillation in a typical coffee mug is located on the boundary (see Figure 1a). This effect, proved rigorously by Kulczycki and Kwaśnicki [22], makes spilling even easier. On the other hand, in a bulbous snifter the lowest-frequency sloshing mode attains its maximal elevation (high spot) inside a snifter [22], reducing the risk of spilling (see Figure 1b).

The position of the high spot clearly depends on the container. Quite recently, this phenomenon has been studied by Faltinsen and Timokha [7]. The natural limiting case of bulbous containers is an infinite ocean, covered with ice, with a single round hole. The corresponding sloshing problem is known as the ice-fishing problem. Kozlov and Kuznetsov [19] showed that the maximal amplitude occurs approximately the third of the way from the boundary to the center of the hole, and the amplitude is over 50% larger than at the boundary.

Note that sloshing dynamics is very important from the engineering point of view. Tanks of liquid propelled rockets include carefully designed baffles to mitigate the free surface effect. Those baffles minimize sloshing, preventing jitter (course deviations). Similar prevention techniques are...
used in ships, where water ballast, or liquid cargo can lead to capsizing. Even improper fire fighting technique can sink a ship. Sloshing is also used to minimize bouncing of the roller hockey ball. Interestingly, certain amounts of water in a ball can lead to resonance effects.

In what follows we discuss only the high spot problem for sloshing. We begin with a formal definition of the model, present the explicit solutions for a cylindrical mug and plot some numerical approximations for harder cases. Then we describe a novel way of obtaining experimental evidence (see Section 5). We also present recently obtained rigorous results for solids of revolution. Finally we describe the relation of the high spot problem to the celebrated hot spots conjecture.

2. Mathematical model

Small oscillations of ideal (that is, inviscid, incompressible and heavy) liquid with no surface tension are accurately described by the linear water-wave theory (see Fox and Kuttler [9] and Troesch [30] for a historical overview, and [11, 12] for applications). In this framework, sloshing modes and frequencies correspond to solutions of the mixed Steklov eigenvalue problem ([18, 27])

$$\Delta \varphi = 0 \text{ in } W,$$

$$\frac{\partial \varphi}{\partial z} = \nu \varphi \text{ on } F,$$

$$\frac{\partial \varphi}{\partial n} = 0 \text{ on } B,$$

$$\int_F \varphi = 0,$$

where $W$ is the part of the container filled with liquid (in its mean position), $B$ is the wetted part of its boundary, and $F$ is the free surface of the liquid (see Figure 2). We choose a Cartesian coordinate system $(x, y, z)$ in which the $z$-axis is vertical and pointing upwards, and the free surface $F$ lies in the plane $z = 0$. The normal derivative on $B$ is denoted by $\frac{\partial}{\partial n}$.

Any solution of the problem (1–4) consists of an eigenfunction $\varphi$ and the corresponding eigenvalue $\nu$. In the language of hydrodynamics, there is a normal mode of the oscillations of the liquid such that the velocity of an element of liquid at point $(x, y, z)$ at time $t$ is equal to

$$\cos(\omega t + \alpha) \nabla \varphi(x, y, z),$$
where \( \frac{1}{2\pi} \omega = \frac{1}{2\pi} \sqrt{\nu g} \) is the frequency of liquid oscillation, \( g \) is the acceleration due to gravity and \( \alpha \) is an arbitrary constant.

For sufficiently regular domains it is known that the mixed Steklov problem (1–4) has a discrete sequence of eigenvalues

\[
0 < \nu_1 \leq \nu_2 \leq \nu_3 \leq \ldots \to \infty,
\]

and the corresponding modes \( \varphi_n \in H^1(W), \ n = 1, 2, 3, \ldots \), restricted to the free surface \( F \), form (together with a constant function) a complete orthogonal set in \( L^2(F) \). In hydrodynamics, the first eigenfunction corresponding to the least eigenvalue \( \nu_1 \) plays an important role: it has the smallest decay rate due to non-ideal effects for real-life liquids.

The following observation plays a key role in our study: the amplitude of the oscillations of the liquid described by (5) on the free surface is proportional to \( \frac{\partial \varphi}{\partial z} \), which, by (2), is equal to \( \nu \varphi \). Hence, the location of the high spot coincides with the location of the maximum of \( |\varphi| \) on \( F \), where \( \varphi \) corresponds to \( \nu_1 \).

### 3. CYLINDRICAL MUG — A SIMPLE DOMAIN WITH EXPLICIT SOLUTION

Krechetnikov and Mayer [25] modelled a mug by a cylinder

\[
W = \{(x, y, z) : \ x^2 + y^2 < 1, \ z \in (-h, 0)\},
\]

with the unit disk as free surface

\[
F = \{(x, y, 0) : \ x^2 + y^2 < 1\}.
\]

In this particularly nice case, the solutions of the mixed Steklov problem (1–4) are known explicitly. There are two linearly independent eigenfunctions corresponding to the least eigenvalue \( \nu_1 = \nu_2 \). In cylindrical coordinates \( x = r \cos \theta, \ y = r \sin \theta \), these sloshing modes are given by

\[
\varphi_1 = J_1(j_{1,1} r) \cosh(j_{1,1}(z + h)) \sin \theta, \tag{6}
\]

\[
\varphi_2 = J_1(j_{1,1} r) \cosh(j_{1,1}(z + h)) \cos \theta, \tag{7}
\]

where \( J_1 \) is the Bessel function of the first kind, and \( j_{1,1} \approx 1.8412 \) is the first zero of \( J_1' \). Figure 3a shows a few level sets of some linear combination of \( \varphi_1 \) and \( \varphi_2 \). Note that values on the top surface translate into liquid elevations.

Clearly \( \varphi_1(x, y, 0) \) is an odd and increasing function of \( y \), and attains extreme values (high spots) at boundary points \((0, 1)\) and \((0, -1)\). Similarly, \( \varphi_2 \) has its high spots at \((1, 0)\) and \((-1, 0)\). In fact, any linear combination of \( \varphi_1 \) and \( \varphi_2 \) will have high spots on the boundary, as on Figure 3a.

### 4. NUMERICAL APPROXIMATIONS

Finite Elements Methods can provide approximate positions for high spots for more complicated domains. Using FEniCS (fenicsproject.org, [26]) we implemented a cylindrical mug and we obtained level sets clearly indicating that the extremal points are on the boundary (see Figure 3a). Of course the eigenfunctions are explicit in this case, hence we could just plot the exact solutions from Section 3.

As a second example, we tried a grain silo shaped container, as on Figure 3b. The high spot is clearly inside. This fact can actually be rigorously proved. See the discussion in Section 6.

Finally, we implemented a trough with a hexagonal cross-section and a small length (cf. [21]). Figure 3c shows this trough with the level sets of the lowest-frequency mode. Clearly, the maximum is not on the boundary.
Figure 3. Numerical solutions: level sets of fundamental eigenfunctions.

Figure 4. Sloshing in a trough

However, this trough and its lowest-frequency mode are essentially two dimensional. We exploited this reduction in dimension to obtain more accurate numerical profiles of the free surface shown on Figure 4b. The blue curve corresponds to a trapezoidal trough that is 50\% wider at the top than at the bottom, and the maximum is on the boundary (as in coffee mug). Other profiles correspond to troughs from Figure 4a (cf. Figure 3c) with different slopes. Generally, the maximum moves toward the center as the sloped part approaches horizontal line. The high spot also becomes more pronounced. Recently, similar numerical calculations have been made in [7] for spherical tanks (see Figure 2, ibid.).

5. Physical Experiment

We have also performed experiments with bulbous and cylindrical containers. We tried to photograph the oscillations, and this proved difficult in any conventional way. Nonlinear effects caused by relatively large amplitude of oscillations and non-ideal nature of water are definitely noticeable. These problems are compounded by the existence of two modes for the lowest frequency. This phenomenon causes whirling of the fluid if both modes are present and shifted in phase.

Instead, we photographed a reflection of a dotted piece of paper on a very slightly disturbed surface of the liquid (see Figure 5d). This approach is consistent with the infinitesimal nature of the mathematical sloshing model. We would disturb the water, then wait until the surface became still to the naked eye. Then, the long exposure photograph would allow us to obtain mostly blurred images, with just a few clearly visible dots. Except at local extrema of the sloshing amplitude, planes tangent to the liquid surface oscillate, creating a path for each dot. On the other hand, at the
Figure 5. Photographic experiment

extremal point the tangent plane is always horizontal, and the corresponding dot is sharp. Note that even this experiment is susceptible to mixing of the modes, hence whirling effect (see Figure 5b).

Figure 5a shows the image obtained for a bulbous container (a fish bowl), while Figure 5c the image for a conical tank (a cocktail glass). There are clearly two extremal points away from the boundary in the bulbous container. On the other hand we obtained almost the same blurred paths for all dots in the conical container.

6. Rigorous Results

Finally, we discuss rigorous results for solids of revolution. Let \( W \) be the 3-dimensional domain obtained by rotating a profile \( D \) (Figure 6b) around the \( z \)-axis (Figure 6a). Usually the free surface \( F \) is a disk in the plane \( z = 0 \). Such containers were considered in [22, 11, 16].
We assume that the modes corresponding to $\nu_1$ are antisymmetric. Many domains have this property. Nevertheless, strange examples of rotationally symmetric fundamental eigenfunctions exist, for example for a profile from Figure 6c.

Under the antisymmetry assumption, there are two linearly independent antisymmetric eigenfunctions corresponding to the least eigenvalue $\nu_1 = \nu_2$. In cylindrical coordinates they have the form

$$\varphi_1 = \psi(r, z) \cos \theta, \quad \varphi_2 = \psi(r, z) \sin \theta,$$

where $\psi(r, z)$ is a function defined on $D$.

By [22, Theorems 1.1 and 1.2], if $W$ is a convex solid of revolution contained in the infinite cylinder $\{(x, y, z) : (x, y, 0) \in F\}$ (the last property is often called John’s condition), then indeed $\nu_1 = \nu_2$ correspond to antisymmetric eigenfunctions $\varphi_1$ and $\varphi_2$, and the high spots of the modes described by $\varphi_1$ and $\varphi_2$ are located on the boundary, e.g. for a profile from Figure 6b. On the other hand, [22, Proposition 1.3] (see also [7]) asserts that if $B$ and $F$ form an obtuse angle then $\varphi_1(x, y, 0), \varphi_2(x, y, 0)$ attain their maxima inside $F$, as shown on Figure 1b and Figure 3b.

The proof of [22, Theorem 1.2] is based on the technique of domain deformation, and uses ideas of D. Jerison and N. Nadirashvili [13]. They studied related hot spots conjecture (posed by J. Rauch in 1974). Similarity between the high and hot spots is not an accident. It showcases the deep connection between sloshing problems and classical Laplace spectral problems with Neumann boundary conditions. We describe some of these in the next section.

7. RELATION TO THE CLASSICAL SPECTRAL PROBLEMS

Traditionally, Laplace eigenvalues are intuitively understood via heat flow. The Dirichlet eigenvalues of a domain $D$ govern the heat on a plate $D$ with zero temperature on the boundary, while the Neumann eigenvalues give the insulated plate. This explains the name hot spots conjecture. The famous paper “Can one hear the shape of drum?” by Kac[17] emphasizes another practical interpretation for Dirichlet spectrum. The corresponding eigenfunctions give the shapes of the drum membrane vibrating in one of the characteristic frequencies. The Neumann spectrum is harder to explain this way, as one would need a membrane that is free to move up and down, but is horizontal near the boundary. This behavior can however be attributed to the free surface of an ideal liquid in a container. Therefore, sloshing not only generalizes the Neumann spectral problem, but also provides physical intuition. In particular the hot spots problem might as well be called the high spot problem for a container with vertical walls.
Roughly speaking, the hot spots conjecture states that in a thermally insulated domain, for “typical” initial conditions, the hottest point will move towards the boundary of the domain as time passes. The mathematical formulation of the hot spots conjecture is the following: the extrema of every fundamental eigenfunction of the Neumann eigenvalue problem for a domain $D$ are located on the boundary of $D$. The conjecture was proved for sufficiently regular planar domains by Bañuelos and Burdzy [1], Jerison and Nadirashvili [13], but disproved for some domains with holes by Burdzy and Werner [5] and just one hole by Burdzy [4]. See Nature article [28] for more information, and Terence Tao’s Polymath project [29] for current developments.

The function $\varphi$ is a sloshing mode in the cylinder

$$W = \{ (x, y, z) : (x, y) \in D, \ z \in (-h, 0) \}$$

if and only if $\psi(x, y) = \varphi(x, y, 0)$ is a Neumann eigenfunction on $D$ (see Figure 7). Furthermore

$$\varphi(x, y, z) = \psi(x, y) \cosh(\sqrt{\mu}(z + h))$$

and $\nu = \sqrt{\mu} \tanh(\sqrt{\mu} h)$, where $\mu$ is the Neumann eigenvalue corresponding to $\psi$ and $\nu$ is the sloshing eigenvalue corresponding to $\varphi$ by [20, Proposition 3.1]. This directly embeds the Neumann problem in the sloshing problem. Due to the very explicit connection many results and conjectures for Neumann spectrum can be quickly translated to the sloshing language. In particular, the high spots problem (1–4) generalizes the hot spots conjecture. Note that we already discussed a special case of a mug in Section 3. In (6-7) one may recognize Neumann eigenfunctions of the disk, given by the product of a Bessel and a trigonometric function.

By the above identification and the results of [1, 13], for any convex domain $D$ with two orthogonal axes of symmetry (e.g. ellipses), the high spots of the fundamental sloshing modes for cylindrical tanks with cross-section $D$ are located on the boundary. Note that the counterexample for the hot spots conjecture given by Burdzy and Werner [5] is rather similar to the example from Figure 6c. Indeed, their domain consists of a disk and a larger annulus, connected using spokes. The profile from Figure 6c shows two deep containers with a disk and an annulus as free surfaces, connected using extremely shallow portion. In both cases one would like to have two disjoint subdomains, even though that is not allowed. In the high spot problem there are more ways to make the domain connected due to one extra dimension. For that reason it is much easier to construct intuitive counterexamples for high spots problem.

The connection between Neumann problem and sloshing was also used to conjecture bounds for the eigenvalues. In particular the Dirichlet to Neumann eigenvalue comparisons [15, 10, 8] was already extended to mixed Steklov eigenvalues [2]. Furthermore, bounds for spectral functionals for Neumann eigenvalues naturally extend to the cylindrical sloshing case (see [14, Section 13]).
opening a whole new area of research for general containers. Finally, sloshing problem can be used to tackle, seemingly unrelated, spectral problems for nonlocal fractional Laplacian [3, 23, 24].

**References**

[1] **R. Bañuelos and K. Burdzy**, *On the “hot spots” conjecture of J. Rauch*, J. Funct. Anal. 164 (1999), no. 1, 1–33. MR1694534

[2] **R. Bañuelos, T. Kulczycki, I. Polterovich, and B. Siudeja**, *Eigenvalue inequalities for mixed Steklov problems*, Operator theory and its applications, Amer. Math. Soc. Transl. Ser. 2, vol. 231, Amer. Math. Soc., Providence, RI, 2010, pp. 19–34. MR2758960

[3] **R. Bañuelos and T. Kulczycki**, *The Cauchy process and the Steklov problem*, J. Funct. Anal. 211 (2004), no. 2, 355–423. MR2056835

[4] **K. Burdzy**, *The hot spots problem in planar domains with one hole*, Duke Math. J. 129 (2005), no. 3, 481–502. MR2169871

[5] **K. Burdzy and W. Werner**, *A counterexample to the “hot spots” conjecture*, Ann. of Math. (2) 149 (1999), no. 1, 309–317. MR1680567

[6] **K. Burdzy and W. Werner**, *No triple point of planar Brownian motion is accessible*, Ann. Probab. 24 (1996), no. 1, 125–147. MR1387629

[7] **O. M. Faltinsen and A. N. Timokha**, *Analytically approximate natural sloshing modes for a spherical tank shape*, J. Fluid Mech. 703 (2012), 391–301. MR2949919

[8] **N. Filonov**, *On an inequality for the eigenvalues of the Dirichlet and Neumann problems for the Laplace operator*, Algebra i Analiz 16 (2004), no. 2, 172–176. MR2068346

[9] **D. W. Fox and J. R. Kuttler**, *Sloshing frequencies*, Z. Angew. Math. Phys. 34 (1983), no. 5, 668–696. MR723140

[10] **L. Friedlander**, *Remarks on Dirichlet and Neumann eigenvalues*, Amer. J. Math. 117 (1995), no. 1, 257–262. MR1314466

[11] **I. Gavrilyuk, M. Hermann, I. Lukovsky, O. Solodun and A. Timokha**, *Natural sloshing frequencies in rigid truncated conical tanks*, Eng. Comput. 25 (2008), 518-540.

[12] **R. Ibrahim**, *Liquid sloshing dynamics*, Theory and applications (Cambridge University Press), Cambridge, UK, 2005.

[13] **D. Jerison and N. Nadirashvili**, *The “hot spots” conjecture for domains with two axes of symmetry*, J. Amer. Math. Soc. 13 (2000), no. 4, 741–772. MR1775736

[14] **R. Laugesen and B. Siudeja**, *Sharp spectral bounds on starlike domains*, to appear in J. Spect. Theory, ArXiv:1209.1058

[15] **H. A. Levine and H. F. Weinberger**, *Inequalities between Dirichlet and Neumann eigenvalues*, Arch. Rational Mech. Anal. 94 (1986), no. 3, 193–208. MR846060

[16] **I. A. Lukovskiǐ, M. Y. Barnyak, and A. N. Komarenko**, *Approximate methods for solving problems of the dynamics of a bounded volume of fluid*, “Naukova Dunika”, Kiev, 1984. MR780750

[17] **M. Kac**, *Can one hear the shape of a drum?*, Amer. Math. Monthly 73 (1966), no. 4, part II, 1–23. MR0201237

[18] **N. D. Kopachevsky and S. G. Krein**, *Operator approach to linear problems of hydrodynamics. Vol. 1*, Operator Theory: Advances and Applications, vol. 128, Birkhäuser Verlag, Basel, 2001, Self-adjoint problems for an ideal fluid. MR1694534

[19] **V. Kozlov and N. Kuznetsov**, *The ice-fishing problem: the fundamental sloshing frequency versus geometry of holes*, Math. Methods Appl. Sci. 27 (2004), no. 3, 289–312. MR2023011

[20] **T. Kulczycki and N. Kuznetsov**, *‘High spots’ theorems for sloshing problems*, Bull. Lond. Math. Soc. 41 (2009), no. 3, 495–505. MR2506833

[21] **T. Kulczycki and N. Kuznetsov**, *On the ‘high spots’ of fundamental sloshing modes in a trough*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 467 (2011), no. 2129, 1491–1502. MR2782167

[22] **T. Kulczycki and M. Kwaśnicki**, *On high spots of the fundamental sloshing eigenfunctions in axially symmetric domains*, Proc. Lond. Math. Soc. (3) 105 (2012), no. 5, 921–952. MR2997042

[23] **T. Kulczycki, M. Kwaśnicki, J. Malecki, and A. Stos**, *Spectral properties of the Cauchy process on half-line and interval*, Proc. Lond. Math. Soc. (3) 101 (2010), no. 2, 589–622. MR2679702

[24] **M. Kwaśnicki**, *Eigenvalues of the fractional Laplace operator in the interval*, J. Funct. Anal. 262 (2012), no. 5, 2379–2402. MR2876409
[25] H. Mayer, R. Krechetnikov, Walking with coffee: Why does it spill?, Phys. Rev. E 85 (2012).
[26] A. Logg, K.-A. Mardal, and G. Wells, Automated solution of differential equations by the finite element method. The FEniCS book., Lecture Notes in Computational Science and Engineering 84. Berlin: Springer. xiii, 723 p. (2012).
[27] N. N. Moiseev, Introduction to the theory of oscillations of liquid-containing bodies, Advances in Applied Mechanics, Vol. 8, Academic Press, New York, 1964, pp. 233–289. MR0167074
[28] I. Stewart, Mathematics: Holes and hot spots, Nature 401 (1999), 863–865.
[29] T. Tao et al., Polymath7: The hot spots conjecture, work in progress, michael-nielsen.org/polymath1/index.php?title=The_hot_spots_conjecture.
[30] B. A. Troesch, Free oscillations of a fluid in a container, Boundary problems in differential equations, Univ. of Wisconsin Press, Madison, Wis., 1960, pp. 279–299. MR0114437

T. Kulczycki and M. Kwaśnicki, Institute of Mathematics and Computer Science, Wrocław University of Technology, Wybrzeże Wyspiąńskiego 27, 50-370 Wrocław, Poland.

B. Siudeja, Department of Mathematics, University of Oregon, Eugene, OR 97403.
E-mail address: Tadeusz.Kulczycki@pwr.wroc.pl
E-mail address: Mateusz.Kwasnicki@pwr.wroc.pl
E-mail address: siudeja@uoregon.edu