SYMPLECTIC ACTIONS AND CENTRAL EXTENSIONS

ANDREW BECKETT AND JOSÉ FIGUEROA-O’FARRILL

Abstract. We give a proof of the fact that a simply-connected symplectic homogeneous space \((M, \omega)\) of a connected Lie group \(G\) is the universal cover of a coadjoint orbit of a one-dimensional central extension of \(G\). We emphasise the rôle of symplectic group cocycles and the relationship between such cocycles, left-invariant presymplectic structures on \(G\) and central extensions of \(G\); in particular, we show that integrability of a central extension of \(g\) to a central extension of \(G\) is equivalent to integrability of a representative Chevalley-Eilenberg 2-cocycle of \(g\) to a symplectic cocycle of \(G\).



1. Introduction

Acknowledgements

2. Symplectic and hamiltonian actions

2.1. Basic definitions

2.2. The moment map and associated cocycles

2.3. Extending the Lie algebra

3. Symplectic cocycles

3.1. Differentiating group 1-cocycles

3.2. Symplectic cocycles and symplectic group cocycles

4. One-dimensional central extensions of Lie groups

4.1. Adjoint and coadjoint representations

4.2. Existence of extensions

5. Symplectic structures and central extensions

5.1. From the symplectic action to the central extension

5.2. Interpretation in terms of \(G\)-actions

5.3. Interpretation in terms of the affine \(G\)-action

5.4. Invariant presymplectic structures on groups, symplectic cocycles and central extensions

5.5. Exact homogeneous symplectic spaces

Appendix A. Lie group and Lie algebra cohomology

A.1. Group cohomology

A.2. Central extensions of Lie groups from group cohomology

A.3. Chevalley–Eilenberg cohomology

Appendix B. Structures on Lie groups

B.1. Invariant vector fields

B.2. Left-invariant de Rham complex

References
1. Introduction

In the study of dynamical systems invariant under the action of a Lie group $G$, one encounters the notion of an elementary system with symmetry $G$: namely, a symplectic manifold admitting a transitive action of $G$ via symplectomorphisms [1, §14] (or in the English translation [2]). Such spaces are also known as homogeneous symplectic $G$-spaces. These are the classical counterparts of free elementary particles with symmetry $G$, which are given by unitary irreducible representations of $G$. Such unitary irreducible representations often result from quantising elementary systems geometrically.

The paradigmatic examples of such elementary systems are the coadjoint orbits of $G$. Let $g$ be the Lie algebra of $G$ and $g^*$ its dual vector space. We shall let $\text{Ad} : G \to \text{GL}(g)$ denote the adjoint representation and $\text{ad} : g \to \text{End}(g)$ its linearisation at the identity. The coadjoint representation of $G$, denoted $\text{Ad}^* : G \to \text{GL}(g^*)$, is defined by $\text{Ad}^*_g \alpha = \alpha \circ \text{Ad}_g^{-1}$ for all $\alpha \in g^*$ and $g \in G$. Its linearisation at the identity is the coadjoint representation of the Lie algebra, denoted by $\text{ad}^* : g \to \text{End}(g^*)$ and defined by $\text{ad}^*_X \alpha = -\alpha \circ \text{ad}_X$ for all $\alpha \in g^*$ and $X \in g$.

Now fix $\alpha \in g^*$ and define $B_\alpha \in \wedge^2 g^*$ by $B_\alpha(X,Y) = \langle \alpha, [X,Y] \rangle$ for all $X,Y \in g$. The radical of the bilinear form $B_\alpha$ consists of all $X \in g$ such that $B_\alpha(X,Y) = 0$ for all $Y \in g$. This condition is equivalent to $\text{ad}^*_X \alpha = 0$, so that $X \in g_\alpha$, the Lie algebra of the stabiliser $G_\alpha := \{g \in G \mid \text{Ad}^*_g \alpha = \alpha \}$ of $\alpha$. It follows that $B_\alpha$ induces a symplectic inner product on $g/\mathfrak{g_\alpha}$. Introducing the coadjoint orbit $O_\alpha := \{\text{Ad}_g^* \alpha \mid g \in G\}$ of $\alpha$, we have an exact sequence of $G_\alpha$-modules

$$0 \longrightarrow g_\alpha \longrightarrow g \longrightarrow T_\alpha O_\alpha \longrightarrow 0,$$

where $G_\alpha$ acts on $T_\alpha O_\alpha$ via the linear isotropy representation. This shows that $g/\mathfrak{g_\alpha} \cong T_\alpha O_\alpha$ and we let $\omega_\alpha \in \wedge^2 T_\alpha^* O_\alpha$ be the symplectic inner product on $T_\alpha O_\alpha$ induced from $B_\alpha$. Since $\omega_\alpha$ is $G_\alpha$-invariant, the holonomy principle guarantees that $\omega_\alpha$ is the value at $\alpha$ of a $G$-invariant symplectic form $\omega \in \Omega^2(O_\alpha)$. This construction is due independently to Kirillov, Kostant and Souriau and we shall refer to the symplectic structure on $O_\alpha$ as $\omega_{KK}$.

It is not just that coadjoint orbits provide examples of homogeneous symplectic manifolds, but that in a way to be made precise below, any homogeneous symplectic manifold is locally isomorphic to a coadjoint orbit.

Indeed, let $(M,\omega)$ be a symplectic manifold admitting a transitive left action of a connected Lie group $G$ via symplectomorphisms. Let $X \in g$ and let $\xi_X \in \mathfrak{X}(M)$ denote the corresponding fundamental vector field. This defines a Lie algebra anti-homomorphism $\xi : g \to \mathfrak{X}(M)$. Since the symplectic form $\omega$ is $G$-invariant, the fundamental vector fields are symplectic: $\mathfrak{L}_{\xi_X} \omega = 0$ and by the Cartan formula, the one-forms $\iota_{\xi_X} \omega$ are closed. If $M$ is simply-connected (although this is typically too strong), these forms are exact, so that $\iota_{\xi_X} \omega = d\xi_X$ and we may always arrange for $\xi_X$ to be the image of $X \in g$ under a linear map $\varphi : g \to C^\infty(M)$. Dual to this map we have the moment map $\mu : M \to g^*$ defined by $(\mu(p),X) = \varphi_X(p)$. For the case of a coadjoint orbit $O_\alpha$ of $G$, the moment map relative to the Kirillov–Kostant–Souriau symplectic structure is simply the inclusion $1 : O_\alpha \to g^*$.

The moment map relates two spaces on which $G$ acts: the symplectic manifold $M$ and the dual of the Lie algebra, on which $G$ acts via the coadjoint representation. A natural question is whether this map is equivariant. Equivariance of the moment map is obstructed and, as shown by Atiyah and Bott [3, Section 6], the obstructions are captured by the $G$-equivariant cohomology of $M$: namely, the moment map is equivariant if and only if the symplectic form admits an equivariant closed extension.

For $G$ connected, equivariance of the moment map is equivalent to the cohomomorphism $\varphi : g \to C^\infty(M)$ being a Lie algebra homomorphism: $(\varphi_X,\varphi_Y) = \varphi_{[X,Y]}$ for all $X,Y \in g$, where $[-,-]$ is the Poisson bracket on $C^\infty(M)$. If this is the case, the image of $M$ under the moment map is a coadjoint orbit $O$ of $G$. The triple $(M,\omega,\mu)$ defines an object in the category of hamiltonian $G$-spaces (see [4, §5]), where a morphism between two objects $(M,\omega,\mu)$ and $(M',\omega',\mu')$ is a smooth map $\phi : M \to M'$ such that $\phi^*\omega' = \omega$ and $\phi^*\mu' = \mu$.

\footnote{Here and in the sequel we often use the dual pairing notation $\langle -,- \rangle : g^* \times g \to \mathbb{R}$ instead of viewing $g^*$ as $\text{Hom}(g,\mathbb{R})$.}
\(\phi^*\mu' = \mu\). Kostant [4, Proposition 5.1.1] proves that \(\phi\) is a \(G\)-equivariant covering. Since an equivariant moment map \(\mu : M \to \mathcal{O}\) obeys \(\mu^*\omega_{\text{KKS}} = \omega\), it defines a morphism between the hamiltonian \(G\)-spaces \((M, \omega, \mu)\) and \((\mathcal{O}, \omega_{\text{KKS}}, i)\), where \(i : \mathcal{O} \to g^*\) is the inclusion. It follows that \(\mu : M \to \mathcal{O}\) is a covering. If \(M\) is simply-connected, it is (symplectically as well as topologically) the universal cover of a coadjoint orbit of \(G\).

What about if \(\mu\) is not equivariant? Souriau [1, §11] showed that \(\mu\) is always equivariant relative to an affinisation of the coadjoint representation and hence if \(M\) is simply-connected, it is the universal cover of an affine orbit of \(G\) in \(g^*\).

To describe this, consider the failure of equivariance \(\theta(g, p) := \text{Ad}_g^* \mu(p) - \mu(g \cdot p)\), which defines a smooth function \(\theta : G \times M \to g^*\). It happens that for \(M\) connected, \(\theta\) does not depend on \(p\) (see Lemma 1) and hence, letting \(\text{pr}_1 : G \times M \to G\) denote the cartesian projection, \(\theta = \text{pr}_1^* \theta\) for some smooth group cocycle \(\theta : G \to g^*\). The cocycle condition

\[
\theta(g_1 g_2) = \text{Ad}_{g_2}^* \theta(g_2) + \theta(g_1) \quad \text{for all } g_1, g_2 \in G \tag{1.2}
\]

allows us to define a Lie group homomorphism \(p : G \to \text{Aff}(g^*)\) by \(\rho(g) \alpha = \text{Ad}_g^* \alpha - \theta(g)\) (the sign is conventional), relative to which \(\mu\) is equivariant by construction: \(\mu(g \cdot p) = \rho(p)\).

The group cocycle \(\theta\) linearises at the identity to \(d_0 \theta : g \to g^*\), which defines a Chevalley–Eilenberg 2-cocycle \(c \in \wedge^2 g^*\) via

\[
c(X, Y) = \langle (d_\theta^0)X, Y \rangle, \quad \text{for all } X, Y \in g. \tag{1.3}
\]

(See Lemma 2.) Following Souriau, one says that \(\theta\) is a symplectic cocycle. It follows that \(c(X, Y) = \langle \varphi_X, \varphi_Y \rangle - \varphi_{\langle X, Y \rangle}\), which is actually constant. If \(|c| = 0 \in H^2(g)\), so that \(c(X, Y) = -\langle \mu_0, [X, Y] \rangle\) for some \(\mu_0 \in g^*\), we can modify the moment map to \(\mu' = \mu - \mu_0\), which is now equivariant with respect to the coadjoint action: \(\mu'(g \cdot p) = \text{Ad}_g^* \mu'(p)\).

If \(|c| \neq 0\) in cohomology, it defines a nontrivial one-dimensional central extension \(\widehat{G}\) of \(g\). By the Lie correspondence we get a one-dimensional central extension \(\widehat{G}\) of the universal cover of \(G\), but it is not a priori clear that we get a central extension of \(G\) itself. In fact, there are known obstructions to a central extension of the Lie algebra of a Lie group to integrate to a central extension of that Lie group [5,6].

The purpose of this paper is to show that such obstructions are overcome for symplectic actions of connected Lie groups. More precisely, we give a proof of Theorem A below (see Theorem 15).

**Theorem A.** Let \((M, \omega)\) be a simply-connected symplectic manifold admitting a transitive action of a connected Lie group \(G\) by symplectomorphisms. Then \((M, \omega)\) is the universal cover of a coadjoint orbit of a one-dimensional central extension of \(G\).

This theorem has acquired an almost “folkloric” quality in that, despite being quoted often, to our knowledge there is no proof in the literature which does not make additional and, from our perspective, unwarranted assumptions. In fact, in a recent paper [7], in which it is shown that, in the diffeological category, every symplectic manifold is a coadjoint orbit of its group of hamiltonian diffeomorphisms, one can read that

the optimal result in the category of manifolds states that the symplectic manifold is, up to covering, an affine coadjoint orbit of the group.

The question of the construction of the central extension of \(G\) from a symplectic cocycle was addressed in [8]. There is no analogue of Theorem A in that paper, although they outline a procedure to integrate the symplectic cocycle to a central extension of the Lie group and illustrate it with two examples. To our knowledge, the first published statement of Theorem A is by Kirillov [9, §15]. (See also the more recent book [10].) Theorem 1 in Section 15.2 of [9] (or Proposition 4 in Section 1.4 of [10]) states a roughly equivalent result:

**Theorem B.** Every homogeneous symplectic manifold whose group of motions is a connected Lie group is locally isomorphic to an orbit in the coadjoint representation of the group \(G\) or a central extension of \(G\) with the aid of \(\mathbb{R}\).
The statement of the theorem notwithstanding, a closer look at the discussion preceding the theorem reveals that this is not the result which is proved. (The same applies to the discussion in [10, Section 1.4].) Kirillov starts with $G$ connected, but then passes to the universal covering group. He then considers a central extension $\hat{g}_1$ of the Lie algebra $g$ of $G$, giving a central extension $G_1$ of the universal cover of $G$. He then shows that $M$ is a homogeneous space of $G_1$. Hence the theorem which is actually proved in [9, §15.2] could be paraphrased as follows:

**Theorem C.** Every homogeneous symplectic manifold whose group of motions is a connected and simply-connected Lie group $G$ is locally isomorphic to an orbit in the coadjoint representation of $G$ or a one-dimensional central extension of $G$.

In fact, the assumption that $G$ is simply-connected is common in the literature, appearing also in Kostant [4, §5], Chu [11] and Sternberg [12], all of whom prove equivalent versions of Theorem C. One might be forgiven for thinking that nothing is lost by assuming ab initio that $G$ is simply-connected: after all, if a Lie group acts transitively on a manifold, so does its universal cover. Alas, we would like to argue that that the topology of $G$ matters. In applications, one is often trying to classify simply-connected homogeneous symplectic manifolds of a given Lie group $G$ and Theorem A says that we need to consider coadjoint orbits of a one-dimensional central extension of $G$. Such extensions are classified by the group cohomology $H^2_{\text{loc}}(G; \mathbb{R})$ (the topology of the kernel of the central extension ends up being of no consequence) whose cochains are smooth near the identity. Even if we restrict to the smooth group cohomology group $H^2(G; \mathbb{R})$, which classifies central extensions that are diffeomorphic to a product $G \times \mathbb{R}$, the van Est theorem [13] (see also the recent clear exposition in [14]) gives an isomorphism $H^2(G; \mathbb{R}) \cong H^2(g, \mathfrak{t})$, where $H^2(g, \mathfrak{t})$ is the Chevalley–Eilenberg cohomology of $g$ relative to the Lie algebra $\mathfrak{t}$ of the maximal compact subgroup $K \subset G$. Since the maximal compact subgroups of $G$ and of its universal cover need not coincide, the choice of $G$ does matter. Therefore Theorem C is weaker than what is needed in applications and we require the full strength of Theorem A.

The question of whether a central extension of the Lie algebra $g$ of a Lie group $G$ integrates to a central extension of $G$ has been studied by Tuynman and Wiegnerick [5] and later by Neeb [6], on whose results we rely. For the case of a one-dimensional central extension $\hat{g}$ of $g$ with Chevalley–Eilenberg cocycle $c \in \Lambda^2 g^*$, one can prove that $\hat{g}$ integrates to a one-dimensional central extension of $G$ if and only if the left-invariant presymplectic structure $\omega \in \Omega^2(G)$ associated to the cocycle $c$ admits a “moment map” for the left action of $G$ on itself; that is, if for every right-invariant vector field $p_X \in \mathfrak{X}(G)$, the closed one-form $\iota_{p_X} \omega$ is exact. It is this result which we shall exploit to show that the affine orbit of $G$ in $g^*$, arising as the image of the moment map of a symplectic $G$-action is a (linear) coadjoint orbit of a one-dimensional central extension $\hat{G}$ of $G$. We will do so by showing that the symplectic cocycle $\theta : G \rightarrow g^*$ gives us the desired moment map.

In the case where the symplectic manifold $(M, \omega)$ is pre-quantisable, so that (some multiple of) $\omega$ defines an integral class in de Rham cohomology, Neeb and Vizman [15] construct the central extension $\hat{G}$ of $G$ by pulling back the pre-quantum bundle to the group via the orbit map $G \rightarrow M$. This paper is organised as follows. In Section 2 we review the basic definitions of Lie group actions on symplectic manifolds, moment maps and hamiltonian $G$-spaces. We review the conditions for the equivariance of the moment map (Proposition 3) and state a theorem of Kostant’s (Theorem 4) concerning the nature of morphisms between hamiltonian $G$-spaces. We also review the notions of symplectic cocycles and the corresponding Chevalley–Eilenberg cocycles and discuss the Lie algebra extension resulting from a non-equivariant moment map. In Section 3 we go deeper into the relation between symplectic cocycles of a Lie group and their corresponding Chevalley–Eilenberg cocycles. In Section 4 we study one-dimensional central extensions of a Lie group $G$ and its coadjoint orbits, which can be interpreted as affine orbits of $G$ on $g^*$. We show how such a central extension gives rise to a symplectic cocycle of $G$ which agrees with the one defined by the affine orbits. We end the section recording the theorem of Neeb (Theorem 14) on the conditions for a central extension of $\mathfrak{g}$ to integrate to a central extension of $G$. Section 5 contains
our proof of Theorem A (Theorem 15), which we then reinterpret in a number of ways: in terms of actions of the original group $G$ instead of its central extension (Section 5.2); in terms of affine orbits in $\mathfrak{g}^*$ rather than coadjoint orbits of the central extension (Section 5.3); and lastly in terms of left-invariant presymplectic structures on Lie groups (Section 5.4). Finally, in Section 5.5, we show that the problem of existence of an appropriate central group extension simplifies significantly when one considers exact symplectic symmetric spaces; in particular, the central extension is geometrically trivial (diffeomorphic to a product) and can be explicitly constructed. We end the paper with two appendices on Lie groups, Lie algebras and their cohomology.

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2. Symplectic and Hamiltonian actions

2.1. Basic definitions. Let $(M,\omega)$ be a symplectic manifold equipped with a transitive, symplectic (left) action by a connected Lie group $G$. It follows that $M$ must also be connected. Each group element $g \in G$ defines a symplectomorphism $p \mapsto g \cdot p$ of $M$, the pushforward and pullback of which we denote by $g_\ast$ and $g^\ast$ respectively. The fundamental vector field associated to $X \in \mathfrak{g}$ is the vector field $\xi_X$ given by $(\xi_X)_p = \frac{d}{dt}|_{t=0} \exp(tX) \cdot p$. This assignment defines a linear map $\xi : \mathfrak{g} \to \mathcal{X}(M)$ given by $X \mapsto \xi_X$. The action being symplectic says that $g_\ast \xi_X = \xi_{g^{-1}X}$, which exists and is unique by the Lie algebra anti-homomorphism.

The action being symplectic says that $g_\ast \omega = \omega$ for all $g \in G$; infinitesimally, $\mathcal{L}_{\xi_X} \omega = 0$ for all $X \in \mathfrak{g}$. Using $d\omega = 0$ and Cartan’s magic formula $\mathcal{L}_{\xi_X} \omega = d\omega(\xi_X, \cdot) + d\omega(\cdot, \xi_X)$, we see that $\iota_{\xi_X} \omega$ is a closed one-form. Suppose that it is actually exact for all $X \in \mathfrak{g}$; that is, there exist some functions $\varphi_X \in C^\infty$ such that

$$\iota_{\xi_X} \omega = d\varphi_X.$$  \hfill (2.1)

Note that if $M$ is simply-connected, as we will later assume, then such functions do indeed exist. Each $\varphi_X$ is uniquely defined only up to addition of a constant, so we may assume without loss of generality that the map $\varphi : \mathfrak{g} \to C^\infty(M)$ sending $X \mapsto \varphi_X$ is linear. It is then uniquely defined only up to addition of a linear map $\mathfrak{g} \to \mathbb{R}$.

The symplectic form $\omega$ induces a Poisson bracket on $C^\infty(M)$: for $f, g \in C^\infty(M)$, we let $\chi_f$ be the hamiltonian vector field associated to $f$, i.e., the vector field satisfying $\iota_{\chi_f} \omega = df$, which exists and is unique by the nondegeneracy of $\omega$. The Poisson bracket of $f, g \in C^\infty(M)$ is given by

$$\{f, g\} = \omega(\chi_f, \chi_g) = -\mathcal{L}_{\chi_f} g.$$ \hfill (2.2)

In particular, note that $\chi_{\varphi_X} = \xi_X$ and

$$\{\varphi_X, \varphi_Y\} = \omega(\xi_X, \xi_Y) = -\mathcal{L}_{\xi_X} \varphi_Y.$$ \hfill (2.3)

2.2. The moment map and associated cocycles. We define a map $\mu : M \to \mathfrak{g}^*$ known as the moment map associated to the symplectic action by

$$\langle \mu(p), X \rangle = \varphi_X(p),$$ \hfill (2.4)

where $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the natural pairing. Dually, $\varphi$ is referred to as the comoment map. Changing the comoment map by the addition of a linear map $\mathfrak{g} \to \mathbb{R}$ (i.e., an element of $\mathfrak{g}^*$) corresponds to changing $\mu$ by the addition of the same element of $\mathfrak{g}^*$. 
Recalling that \( g^* \) comes equipped with a coadjoint action \( \text{Ad}^* : G \to \text{GL}(g^*) \), it is natural to ask whether \( \mu \) is (or can be defined to be) equivariant. We thus define a map \( \vartheta : G \times M \to g^* \) by
\[
\vartheta(g,p) = \text{Ad}^*_g \mu(p) - \mu(g \cdot p)
\] (2.5)
which measures the failure of \( \mu \) to be equivariant.

**Lemma 1.** The map \( \vartheta \) is independent of \( p \in M \) and so \( \vartheta = \text{pr}_1^* \Theta \) for some map \( \Theta : G \to g^* \), where \( \text{pr}_1 : G \times M \to G \) is the projection to the first factor. Moreover, \( \Theta \) satisfies
\[
\Theta(g_1 g_2) = \text{Ad}^*_g \Theta(g_2) + \Theta(g_1).
\] (2.6)

**Proof.** For the first claim, let \( \Theta_{g,X} \in C^\infty(M) \) be the function given by \( \Theta_{g,X}(p) = \langle \vartheta(g,p), X \rangle \) for \( g \in G \) and \( X \in g \). We will show that \( \Theta_{g,X} \in C^\infty(M) \) is locally constant (and therefore constant, since \( M \) is connected). We have
\[
\Theta_{g,X}(p) = \langle \text{Ad}^*_g \mu(p), X \rangle - \langle \mu(g \cdot p), X \rangle = \varphi_{\text{Ad}^{-1}_g} X(p) - \varphi_X(g \cdot p).
\] (2.7)
so we see that \( \Theta_{g,X} = \varphi_{\text{Ad}^{-1}_g} \theta - g^* \varphi_X \). Then
\[
\text{d} \Theta_{g,X} = \text{d} \varphi_{\text{Ad}^{-1}_g} \theta - g^* \text{d} \varphi_X = \iota_{\text{Ad}^{-1}_g} \text{d} \omega - g^* \iota_X \omega
\] (2.8)
so pairing with an arbitrary vector field \( \eta \in \mathfrak{g}(M) \), we find
\[
(\text{d} \Theta_{g,X})(\eta) = \omega(\xi_{\text{Ad}^{-1}_g} \theta, \eta) - \omega(\xi_X, g \cdot \eta) = \omega(\xi_{\text{Ad}^{-1}_g} \theta, \eta) - (g^* \omega)(g^{-1} \xi_X, \eta)
\] (2.9)
but since the action of \( G \) on \( (M, \omega) \) is symplectic \( (g^* \omega = \omega) \) and \( (g^{-1})^* \xi_X = \xi_{\text{Ad}^{-1}_g} X \), this vanishes, so indeed \( \text{d} \Theta_{g,X} = 0 \). We thus have a map \( \Theta : G \to g^* \) given by \( \Theta(g) = \text{Ad}^*_g \mu(p) - \mu(g \cdot p) \), or
\[
\Theta(g_1, X) = \varphi_{\text{Ad}^{-1}_g} X(p) - \varphi_X(g \cdot p)
\] (2.10)
for arbitrary \( p \in M \). For the second claim, we now compute directly
\[
\Theta(g_1 g_2) = \text{Ad}^*_{g_1 g_2} \mu(p) - \mu(g_1 g_2 p)
\]
\[
= \text{Ad}^*_{g_1} \text{Ad}^*_{g_2} \mu(p) - \mu(g_1 (g_2 p))
\]
\[
= \text{Ad}^*_{g_1} (\Theta(g_2) - \mu(g_2 p)) - (\text{Ad}^*_{g_1} \mu(g_2 p) - \Theta(g_1))
\]
\[
= \text{Ad}^*_{g_1} \Theta(g_2) + \Theta(g_1)
\] (2.11)
so \( \Theta \) satisfies (2.6) as claimed. \( \square \)

Equation (2.6) says that \( \Theta \) is a 1-cocycle for \( G \) with values in the coadjoint representation (see Appendix A.1). The following result shows in particular that \( \Theta \) is a symplectic group cocycle (see Section 3).

**Lemma 2.** The derivative of \( \Theta \) at the identity is a linear map \( \text{d}_e \Theta : g \to g^* \). The bilinear form \( c \) on \( g \) defined by \( c(X,Y) = \langle \text{d}_e \Theta(X), Y \rangle \) is alternating (i.e., it lies in \( \Lambda^2 g^* \)) and is a Chevalley–Eilenberg cocycle.

**Proof.** The first claim is immediate from the natural identifications \( g = T_e G \) and \( T_0 g^* \cong g^* \). Now let \( X, Y \in g \) and \( p \) be an arbitrary point in \( M \); using equation (2.10) gives
\[
c(X,Y) = \langle \text{d}_e \Theta(X,Y) = \left. \frac{d}{dt} \left( \varphi_{\text{Ad}^{-1}_g} \exp(-t X) \cdot Y(p) - \varphi_Y(\exp(tX)p) \right) \right|_{t=0}
\]
\[
= (-\varphi_{[X,Y]} + \xi_Y \varphi_Y)(p)
\]
\[
= (-\varphi_{[X,Y]} + \langle \varphi_X, \varphi_Y \rangle)(p).
\] (2.12)
Note that the LHS is independent of the point \( p \in M \), so \( \varphi_{[X,Y]} - \langle \varphi_X, \varphi_Y \rangle \) is constant on \( M \). Clearly this is skew-symmetric in \( X,Y \), hence \( c \in \Lambda^2 g^* \). Finally, we must show that \( \text{d}_e \text{Coc} = 0 \). This follows from the expression for \( c \) above and the Jacobi identities for \( g \) and for the Poisson bracket on \( C^\infty(M) \); alternatively, it is a consequence of \( \Theta \) being a symplectic cocycle (Lemma 8). \( \square \)
We now record a result linking the cohomology classes of the cocycles $\theta$ and $c$ to each other and to properties of the (co)moment map. This is a standard result, so we will not belabour the point.

**Proposition 3.** Since $G$ is a connected Lie group, the following are equivalent:

1. The moment map $\mu$ can be chosen to be equivariant;
2. The cohomology class of $\theta$ is trivial;
3. The comoment map $\varphi$ can be chosen to be a Lie algebra homomorphism;
4. The cohomology class of $c$ is trivial.

If any (and therefore all) of these conditions hold, the action of $G$ on $M$ is said to be Hamiltonian.

**Proof.** (1 $\iff$ 2) If $\mu$ is equivariant then clearly $\theta = 0$. Conversely, if $\theta = \partial \mu_0$ for some $\mu_0 \in g$, for all $g \in G$ we have $\theta(g) = Ad^*_g \mu_0 - \mu_0$. But recall that $\theta(g) = \theta(g, p) = Ad^*_g \mu(p) - \mu(g \cdot p)$ for arbitrary $p \in M$, so letting $\mu'(p) = \mu(p) - \mu_0$, we have $Ad^*_g \mu'(p) = \mu'(g \cdot p)$.

(2 $\iff$ 4) By (2.12), we have $c(X, Y) = \{\varphi_X, \varphi_Y\} - \varphi_{[X, Y]}$ for all $X, Y \in g$. Clearly then, $c = 0$ if $\varphi$ is a Lie algebra homomorphism. Conversely, if $c = 0$ CE $\mu_0$, we have $c(X, Y) = -\mu_0([X, Y])$. Then setting $\varphi' = \varphi - \mu_0$, for all $X, Y \in g$, we have

\begin{align*}
\{\varphi'_X, \varphi'_Y\} - \varphi'_{[X, Y]} &= \{\varphi_X - \mu_0(X), \varphi_Y - \mu_0(Y)\} - \varphi_{[X, Y]} - \mu_0([X, Y]) \\
&= \{\varphi_X, \varphi_Y\} - \varphi_{[X, Y]} + \mu_0([X, Y]) \\
&= c(X, Y) + \mu_0([X, Y]) \\
&= 0.
\end{align*}

(1 $\implies$ 3) Let $p \in M$ and $X \in g$. We have $\mu(\exp(tX)p) = Ad^*_{\exp(tX)} \mu(p)$; pairing with $Y \in g$ yields

$$\langle \mu(\exp(tX)p), Y \rangle = \langle \mu(p), Ad_{\exp(-tX)} Y \rangle$$

or equivalently

$$\varphi_Y(\exp(tX)p) = \varphi_{Ad_{\exp(-tX)} Y}([p]) = \varphi_{\exp(-t \operatorname{Ad}_X Y)}(p).$$

Differentiating this expression at $t = 0$ yields $\partial_{\xi_X} \varphi_Y(p) = -\varphi_{\operatorname{Ad}_X Y}(p)$. But this is just

$$\{\varphi_X, \varphi_Y\} = \varphi_{[X, Y]}$$

after abstracting away $p$ and eliminating a sign.

(3 $\implies$ 1) Since $G$ is connected, it is sufficient to prove that $\mu$ is infinitesimally equivariant. For all $p \in M$ and $X \in g$,

$$\langle d_p \mu(\xi_X), Y \rangle = d_p \varphi_Y(\xi_X) = -\{\varphi_X, \varphi_Y\}(p) = -\varphi_{[X, Y]}(p) = -\langle \mu(p), [X, Y] \rangle = \langle \operatorname{Ad}_X^* \mu(p), Y \rangle,$n

so $d\mu(\xi_X) = \operatorname{Ad}_X^* \mu$, which is the infinitesimal form of $\mu(g \cdot p) = Ad^*_g \mu(p)$. \qed

**Remark.** The first three parts of the preceding proof do not require that $G$ be connected. Indeed, the only place where we have used this hypothesis in any of the above discussion is in the fourth part of the proof above, and implicitly in the proof of Lemma (1), although that only required the weaker assumption that $M$ is connected. It is already clear, then, that the symplectic cocycle $\theta$ captures important information about (co)moment maps in a more general setting, although there is more nuance if $G$ is not connected. We will however require that $G$ is connected in later sections, and we will also need to assume that $M$ is simply-connected. We also note that in the case where $G$ is connected, instead of proving that (1) $\iff$ (3) as above, we could instead invoke a general result about symplectic group cocycles (Proposition 10) to show that (2) $\iff$ (4).

If the $G$ action on a symplectic manifold $(M, \omega)$ is both Hamiltonian with moment map $\mu$ and transitive, we say that $(M, \omega, \mu)$ is a Hamiltonian $G$-space. Hamiltonian $G$-spaces form a category whose morphisms $(M, \omega, \mu) \to (M', \omega', \mu')$ consist of smooth maps $\phi : M \to M'$ such that $\phi^* \omega' = \omega$ and $\phi^* \mu' = \mu$. The paradigmatic examples of Hamiltonian $G$-spaces are the coadjoint orbits. Let $\mathcal{O} \subset g^*$ be a coadjoint
orbit with the Kirillov–Kostant–Souriau symplectic structure $\omega_{\text{KKS}}$. The coadjoint action of $G$ on $\mathcal{O}$ is hamiltonian with moment map $i: \mathcal{O} \to g^*$ the inclusion, making $(\mathcal{O}, \omega_{\text{KKS}}, i)$ into a hamiltonian $G$-space. We record for later use the following result of Kostant’s [4, Prop. 5.1.1].

**Theorem 4** (Kostant [4]). Let $\phi: M \to M'$ be a morphism of hamiltonian $G$-spaces $(M, \omega, \mu)$ and $(M', \omega', \mu')$, with $G$ a connected Lie group. Then $\phi$ is $G$-equivariant and a covering map.

Let $(M, \omega, \mu)$ be a hamiltonian $G$-space with $G$ connected, so that $M$ too is connected. Then the image of the moment map is a coadjoint orbit $O \subset g^*$ and we may restrict the codomain so that $\mu: M \to \mathcal{O}$.

**Proposition 5.** The moment map $\mu: M \to \mathcal{O}$ defines a morphism of hamiltonian $G$-spaces $(M, \omega, \mu)$ and $(\mathcal{O}, \omega_{\text{KKS}}, i)$.

**Proof.** We need only show that $\mu^* \omega_{\text{KKS}} = \omega$. Let $o \in M$ with $\mu(o) = \lambda \in g^*$ and let $\mathcal{O}$ be the coadjoint orbit of $\lambda$. Consider the following triangle

$$
\begin{array}{ccc}
G & & \mathcal{O} \\
\downarrow a_o & & \downarrow a_{\lambda} \\
M & \xrightarrow{\mu} & \mathcal{O}
\end{array}
$$

(2.18)

where $a_o: G \to M$, sending $g \mapsto g \cdot o$, and $a_{\lambda}: G \to \mathcal{O}$, sending $g \mapsto \text{Ad}_g^* \lambda$, are the orbit maps. All maps are surjections and, since $\mu$ is $G$-equivariant, the triangle commutes; that is, $\mu \circ a_o = a_{\lambda}$. Since both $\omega_{\text{KKS}}$ and $\omega$ are $G$-invariant, it is enough to show that $\mu^* \omega_{\text{KKS}}$ and $\omega$ agree at $o \in M$.

For every $X \in g$, let $\xi_X$ be the fundamental vector field on $\mathcal{O}$; we then have $\langle \xi_X \rangle_o = (a_o)_* X$ and $\langle \xi_X \rangle_{\lambda} = (a_{\lambda})_* X$. The images of $g$ under these maps span $T_o M$ and $T_{\lambda} \mathcal{O}$, respectively, since the orbit maps are surjections. Hence it is enough to show that $\mu^* \omega_{\text{KKS}}$ and $\omega$ agree on such tangent vectors. Also, by the chain rule and the commutativity of the diagram,

$$
\mu_* \langle \xi_X \rangle_o = (\mu \circ a_o)_* X = (a_{\lambda})_* X = \langle \xi_X \rangle_{\lambda}.
$$

(2.19)

Therefore on the one hand, we have

$$
\langle \mu^* \omega_{\text{KKS}}(\xi_X, \xi_Y) \rangle(o) = \omega_{\text{KKS}}(\mu_*, \xi_X, \mu_*, \xi_Y)(\lambda) = \omega_{\text{KKS}}(\xi_X, \xi_Y)(\lambda) = \langle \lambda, [X, Y] \rangle,
$$

(2.20)

and on the other hand,

$$
\langle \omega(\xi_X, \xi_Y) \rangle(o) = \langle \varphi_X, \varphi_Y \rangle \circ (a_o)_* X = \langle \mu_*(\varphi_X), [X, Y] \rangle = \langle \lambda, [X, Y] \rangle.
$$

(2.21)

As an immediate corollary of this result and Kostant’s Theorem 4, we have the following.

**Corollary 6.** Let $G$ be a connected Lie group and $(M, \omega, \mu)$ a simply-connected hamiltonian $G$-space. Then $M$ is (symplectically as well as topologically) the universal cover of a coadjoint orbit of $G$.

**Remark.** Another immediate corollary of Proposition 5 is that

$$
a_o^* \omega = a_o^* \mu^* \omega_{\text{KKS}} = (\mu \circ a_o)^* \omega_{\text{KKS}} = a_{\lambda}^* \omega_{\text{KKS}}.
$$

(2.22)

So that the two left-invariant 2-forms on $G$ resulting by pulling back the symplectic forms on $M$ and $\mathcal{O}$ via the orbit maps coincide.

**Remark.** In the general case, where the action is not necessarily hamiltonian, we can define an affine (not linear) action $\rho: G \to \text{Aff}(g^*)$ by

$$
\rho(g)\alpha = \text{Ad}_g^* \alpha - \theta(g).
$$

(2.23)

for $g \in G$, $\alpha \in g^*$. Indeed, the cocycle condition (2.6) is equivalent to $g_1 \cdot (g_2 \cdot \alpha) = (g_1 g_2) \cdot \alpha$. By the definition of $\theta$, the moment map $\mu$ is equivariant with respect to this action:

$$
\mu(g \cdot p) = \rho(g) \mu(p).
$$

(2.24)
In Section 2.3, we will show that this affine action can be viewed as the restriction to a hyperplane of a linear action of $G$ on the dual $\mathfrak{g}^*$ of a central extension $\mathfrak{g}$ of $\mathfrak{g}$. In Section 5.3, we will show that there is a $G$-invariant symplectic structure on the affine orbits and hence we will be able to rephrase our main result (Theorem 15) as a version of Corollary 6 for $(M,\omega)$ homogeneous symplectic $G$-space (but not necessarily hamiltonian) with affine orbits replacing the linear coadjoint orbits in the conclusion.

2.3. Extending the Lie algebra. The cocycle $c \in \Lambda^2\mathfrak{g}^*$ determines a one-dimensional central extension $\mathfrak{g}$ of $\mathfrak{g}$: as a vector space, we have $\mathfrak{g} = \mathfrak{g} \oplus \mathbb{R}$, and the Lie bracket is given by

$$\langle (X, u), (Y, v) \rangle = \langle [X, Y], c(X, Y) \rangle$$

(2.25)

where $X, Y \in \mathfrak{g}$, $u, v \in \mathbb{R}$. Note that this extension is trivial if and only if the action of $G$ on $M$ is hamiltonian. Even if it is not hamiltonian, and hence the comoment map $\varphi$ is not a Lie algebra homomorphism, it can be extended to a Lie algebra homomorphism $\tilde{\varphi} : \mathfrak{g} \to \mathcal{C}^\infty(M)$ defined as follows:

$$\tilde{\varphi}_{(X, u)}(p) = \varphi_X(p) + u$$

(2.26)

for $p \in M$, $(X, u) \in \mathfrak{g}$. Indeed, suppressing $p$ in the notation,

$$\tilde{\varphi}_{(X, u)}([Y, v]) = \tilde{\varphi}_{([X, Y], c(X, Y))} = \varphi_{[X, Y]} + c(X, Y) = \{\varphi_X, \varphi_Y\} = \{\tilde{\varphi}_{(X, u)}, \tilde{\varphi}_{(Y, v)}\}$$

(2.27)

where the last equality follows because $u, v$ are constants. Dually, we define an extended moment map $\tilde{\mu} : M \to \mathfrak{g}^*$ by

$$\langle \tilde{\mu}(p), (X, u) \rangle = \tilde{\varphi}_{(X, u)}(p) = \langle \mu(p), X \rangle + u$$

(2.28)

where we have used $\langle \cdot, \cdot \rangle$ to denote the dual pairing on $\mathfrak{g}$ as well as $\mathfrak{g}$. Note that $\tilde{\mu}(p) = (\mu(p), 1) \in \mathfrak{g} \oplus \mathbb{R}$.

Let us define a linear action of $G$ on $\mathfrak{g}$ by

$$g \cdot (X, u) = (\text{Ad}_g X, u - \langle \theta(g^{-1}), X \rangle)$$

(2.29)

where $\theta$ is the group cocycle appearing in Lemma 1. One can check that the inverse in the argument of $\theta$ is necessary to make this an action, and it is clearly linear. We have a dual action on $\mathfrak{g}^* \cong \mathfrak{g}^* \oplus \mathbb{R}$ defined by

$$\langle g \cdot (\alpha, \zeta), (X, u) \rangle = \langle (\alpha, \zeta), g^{-1} \cdot (X, u) \rangle$$

(2.30)

for $(X, u) \in \mathfrak{g}$, $(\alpha, \zeta) \in \mathfrak{g}^*$ and $g \in G$. A simple computation then shows that

$$g \cdot (\alpha, \zeta) = (\text{Ad}_g^* \alpha - \zeta \theta(g), \zeta).$$

(2.31)

The extended moment map is equivariant with respect to this action; indeed, we have

$$g \cdot \tilde{\mu}(p) = g \cdot (\mu(p), 1) = (\text{Ad}_g^* \mu(p) - \theta(g), 1) = (\mu(g \cdot p), 1) = \tilde{\mu}(g \cdot p).$$

(2.32)

Remark. The action (2.31) preserves the hyperplanes $\zeta = \text{constant}$; in particular, the action on the $\zeta = 1$ hyperplane is $g \cdot (\alpha, 1) = (\rho(g)\alpha, 1)$ where $\rho$ is the affine action defined by (2.23). We thus have an equivariant embedding of $\mathfrak{g}^*$ thought of as an affine $G$-space into $\mathfrak{g}^*$. The image of $\tilde{\mu}$ lies in this hyperplane and corresponds to the image of $\mu$ in $\mathfrak{g}^*$.

One can straightforwardly check that the derivative of the action (2.29) is the adjoint representation of $\mathfrak{g}$ restricted to $\mathfrak{g}$. One might then wonder whether the central extension $\mathfrak{g}$ of $\mathfrak{g}$ can be “integrated” to a central extension $\mathfrak{G}$ of $\mathfrak{g}$ in such a way that the actions of $G$ on $\mathfrak{g}$ and its dual described above arise naturally as restrictions of the adjoint and coadjoint actions of $\mathfrak{G}$. We will show that this is the case.

3. Symplectic cocycles

Earlier we saw a moment map for a symplectic action giving rise to a group cocycle $\theta$ with values in the coadjoint representation which could be differentiated and curried to produce a Chevalley–Eilenberg cocycle $c$. We will set aside for now the symplectic action and discuss some properties of such cocycles abstractly.
3.1. Differentiating group 1-cocycles. We begin by deriving a useful identity. Let \( \theta : G \to g^* \) be a group 1-cocycle with values in the coadjoint representation; that is, a smooth map which satisfies
\[
\theta(g_1 g_2) = \text{Ad}^*_g \theta(g_2) + \theta(g_1).
\]
(3.1)

For \( X \in g, \ g \in G \) and \( t \in \mathbb{R} \), the cocycle condition gives us two different expressions for \( \theta(\exp(tX)g) \):
\[
\theta(\exp(tX)g) = \text{Ad}^*_{\exp(tX)} \theta(g) + \theta(\exp(tX)),
\]
(3.2)
\[
\theta(\exp(tX)g) = \theta(g \exp(t\text{Ad}_{g^{-1}}X)) = \text{Ad}^*_g \theta(\exp(t \text{Ad}_{g^{-1}}X)) + \theta(g);
\]
(3.3)
so for \( Y \in g \), we have
\[
\langle \theta(\exp(t\text{Ad}_{g^{-1}}X)), \text{Ad}_{g^{-1}}Y \rangle + \langle \theta(g), Y \rangle - \langle \theta(g), \text{Ad}_{\exp(-tX)} Y \rangle - \langle \theta(\exp(tX)), Y \rangle = 0,
\]
(3.4)
and differentiating this with respect to \( t \) at \( t = 0 \) gives
\[
\langle \theta(g), [X,Y] \rangle = \langle d_e \theta(X), Y \rangle - \langle d_e \theta(\text{Ad}_{g^{-1}}X), \text{Ad}_{g^{-1}}Y \rangle.
\]
(3.5)

Lemma 7. If \( \theta \) is a cocycle in \( C^1(G; g^*) \) then \( d_e \theta \) is a cocycle in \( C^1(g; g^*) \).

Proof. We set \( g = \exp(tZ) \) in equation (3.5) and differentiate with respect to \( t \) at \( t = 0 \) to find
\[
\langle d_e \theta(Z), [X,Y] \rangle = -\left. \frac{d}{dt} \langle d_e \theta(\text{Ad}_{\exp(-tZ)}X), \text{Ad}_{\exp(-tZ)}Y \rangle \right|_{t=0}
\]
\[
= -\left. \frac{d}{dt} \langle d_e \theta(\text{Ad}_{\exp(-tZ)}X), Y \rangle \right|_{t=0} - \frac{d}{dt} \langle d_e \theta(X), \text{Ad}_{\exp(-tZ)}Y \rangle \bigg|_{t=0}
\]
\[
= \langle d_e \theta([X,Z]), Y \rangle + \langle d_e \theta(X), [Z,Y] \rangle.
\]
(3.6)

We then abstract \( Y \) and rearrange to find
\[
ad^*_Z(d_e \theta(X)) - ad^*_X(d_e \theta(Z)) - d_e \theta([Z,X]) = 0,
\]
(3.7)

hence the claim. \( \square \)

3.2. Symplectic cocycles and symplectic group cocycles. A cochain \( \phi \in C^1(g; g^*) \) is called symplectic if \( \langle \phi(X), Y \rangle = -\langle \phi(Y), X \rangle \). There is a one-to-one correspondence between the space of symplectic cochains \( C^1_{\text{sympl}}(g; g^*) \) and \( C^2(g) \) given by \( \phi \mapsto \tilde{\phi} \) where \( \tilde{\phi}(X,Y) := \langle \phi(X), Y \rangle \). Furthermore, \( \phi \) is a cocycle if and only if \( \tilde{\phi} \) is, and all coboundaries in \( C^1(g; g^*) \) are symplectic, so we can form the symplectic cohomology \( H^1_{\text{sympl}}(g; g^*) \), which is isomorphic to \( H^2(g) \). In light of Lemma 7, we say that a cocycle \( \theta \in C^1(G; g^*) \) is a symplectic (group) cocycle if \( d_e \theta \in C^1(g; g^*) \) is a symplectic cocycle; we will denote the space of such cocycles by \( Z^1_{\text{sympl}}(G; g^*) \). We already saw in Lemma 2 that moment maps for symplectic actions give rise to such cocycles.

Lemma 8. Suppose that \( \theta \) is a symplectic group cocycle. Then the bilinear form \( c \in C^2(g) = \Lambda^2 g^* \) defined by \( c(X,Y) := \langle d_e \theta(X), Y \rangle \) is the cocycle corresponding to \( d_e \theta \) under the isomorphism \( C^1_{\text{sympl}}(g; g^*) \cong C^2(g) \), and furthermore we have
\[
\langle \theta(g), [X,Y] \rangle = c(X,Y) - (\text{Ad}^*_g c)(X,Y)
\]
(3.8)
for all \( g \in G \) and \( X,Y \in g \), where \( (\text{Ad}^*_g c)(X,Y) := c(\text{Ad}^*_{g^{-1}} X, \text{Ad}^*_{g^{-1}} Y) \).

Proof. The first claim follows from the discussion of symplectic cochains and cocycles above since \( c = \tilde{d}_e \theta \), but for completeness we will show that \( c \) is indeed a cocycle. From the calculation (3.6) in the preceding proof, we have
\[
c(Z, [X,Y]) = c([Z,X], Y) + c(X, [Z,Y]),
\]
(3.9)
so, using the skew-symmetry of \( c \),
\[
c([X,Y], Z) + c([Z,X], Y) + c([Y,Z], X) = 0.
\]
(3.10)
Thus \( c \) is indeed a cocycle in \( C^2(g) \). The second claim follows from (3.5). \( \square \)
The proofs of the final results of this section use some notational conventions which may appear odd but are chosen so as to agree with the notation of Sections 5.1 and 5.4. It also uses right-invariant vector fields, which are introduced in Appendix B.1.

**Lemma 9.** Let θ be a symplectic group cocycle and c the corresponding Chevalley-Eilenberg cocycle. For each $X \in \mathfrak{g}$, define a function $\Phi_X \in C^\infty(G)$ by $\Phi_X(g) = -\langle \theta(g), X \rangle$. Then for all $X, Y \in \mathfrak{g}$ and $g \in G$,
\[
(L_{\rho^g}, \Phi_X)(g) = (Ad^*_g c)(X, Y)
\] (3.11)
where $L$ is the Lie derivative.

**Proof.** Using cocycle condition (3.1) and Lemma 8 (or more directly, recalling the calculations at the start of Section 3.1),
\[
(L_{\rho^g}, \Phi_X)(g) = \left. \frac{d}{dt} \langle \theta(\exp(tY)g), X \rangle \right|_{t=0} = (Ad^*_g c)(X, Y).
\] (3.12)

**Proposition 10.** Let θ be a symplectic group cocycle for a connected Lie group G and c the induced Chevalley–Eilenberg cocycle. Then $\theta = 0$ if and only if $[c] = 0$.

**Proof.** First suppose that $\theta = \partial \alpha$ for some $\alpha \in \mathfrak{g}^*$. Then
\[
c(X, Y) = \langle d\theta(X), Y \rangle = \left. \frac{d}{dt} \left[ \langle \alpha, Ad_{\exp(-tX)}Y \rangle - \langle \alpha, Y \rangle \right] \right|_{t=0} = -\langle \alpha, [X, Y] \rangle.
\] (3.13)
so $c = \partial_{\text{CE}} \alpha$. Now assume $c = \partial_{\text{CE}} \alpha$ and let $\theta' = \theta - \partial \alpha$. Clearly $\theta'$ is also a symplectic group cocycle in the same cohomology class as $\theta$. We will show that $\theta' = 0$. As in Lemma 9, to each $X \in \mathfrak{g}$ we assign functions $\Phi_X, \Phi'_X \in C^\infty(G)$ defined by $\Phi_X(g) = -\langle \theta(g), X \rangle$ and $\Phi'_X(g) = -\langle \theta'(g), X \rangle$; then
\[
\Phi'_X(g) = \Phi_X(g) + \langle \partial \alpha(g), X \rangle.
\] (3.14)
For all $X, Y \in \mathfrak{g}$, we thus have
\[
L_{\rho^g} \Phi'_X = L_{\rho^g} \Phi_X + L_{\rho^g} \langle \partial \alpha, X \rangle.
\] (3.15)
where we understand $\langle \partial \alpha, X \rangle$ as the smooth function $g \mapsto \langle \partial \alpha(g), X \rangle$. On the one hand, by Lemma 9 we have $(L_{\rho^g}, \Phi_X)(g) = (Ad^*_g c)(X, Y) = -\langle Ad^* \alpha, [X, Y] \rangle$, while on the other,
\[
(L_{\rho^g}, \langle \partial \alpha, X \rangle)(g) = \left. \frac{d}{dt} \langle Ad_{\exp(tY)g}^* \alpha, X \rangle \right|_{t=0} = \frac{d}{dt} \left[ \langle Ad_{\exp(tY)^g}^* \alpha, X \rangle \right]_{t=0} = \langle Ad^*_g \alpha, [X, Y] \rangle.
\] (3.16)
The assumption on c therefore gives us $L_{\rho^g} \Phi'_X = 0$, so since the values of vector fields $\rho^g$ span the tangent space at every point in G, we find that $d\Phi'_X = 0$. Since G is connected, this shows that $\Phi'_X$ is a constant for all $X \in \mathfrak{g}$, so $\theta'$ is constant as a map $G \to \mathfrak{g}^*$. But $\theta'$ is a cocycle, so it vanishes at the identity and therefore everywhere.

**Proposition 11.** Let $\theta, \theta'$ be two symplectic group cocycles for a connected group G which induce the same Chevalley–Eilenberg cocycle $c$. Then $\theta = \theta'$.

**Proof.** By Lemma 9 we have
\[
L_{\rho^g} \Phi'_X = L_{\rho^g} \Phi_X = (Ad^*_g c)(X, Y),
\] (3.17)
and so $L_{\rho^g} (\Phi'_X - \Phi_X) = 0$. By a similar argument to the previous proof, this shows that the cocycle $\theta' - \theta$ is constant and therefore zero, so $\theta = \theta'$.

We finish this section by noting that the preceding pair of results show that the map $Z^1_{\text{sympl}}(G; \mathfrak{g}^*) \to Z^2(\mathfrak{g})$ taking a symplectic group cocycle to the corresponding Chevalley–Eilenberg cocycle is injective and restricts to an isomorphism of the subspaces of coboundaries (note that all coboundaries in $Z^1(\mathfrak{g}; \mathfrak{g})$ are symplectic), so there is an induced injective map $H^1_{\text{sympl}}(G; \mathfrak{g}^*) \hookrightarrow H^2(\mathfrak{g})$. 

4. One-dimensional central extensions of Lie groups

In Section 2.3, we used data obtained from a symplectic action of a Lie group $G$ to construct a central extension $\hat{g}$ of the Lie algebra of $G$ as well as actions of $G$ on $\hat{g}$ and on its dual $\hat{g}^*$ and asked whether such actions of $G$ might result from the adjoint and coadjoint representations of a central extension of $G$ itself. We now consider the existence of such group extensions and their adjoint and coadjoint representations. We start with a definition.

**Definition 12.** Let $G$ be a connected Lie group. A one-dimensional central extension of $G$ is a short exact sequence of Lie groups

$$1 \longrightarrow K \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1.$$  \hspace{1cm} (4.1)

where $K$ is a one-dimensional abelian group whose image lies in the centre of $\hat{G}$. If we do not wish to specify $K$, we may say that $\hat{G}$ is a one-dimensional central extension of $G$.

**4.1. Adjoint and coadjoint representations.** Let $G$ be a one-dimensional central extension of $G$. We denote by $\text{Ad} : G \to \text{GL}(g)$ and $\hat{\text{Ad}} : \hat{G} \to \text{GL}(\hat{g})$ the adjoint representations of $G$ and $\hat{G}$ respectively; the coadjoint representations are denoted $\text{Ad}^* : G \to \text{GL}(g^*)$ and $\hat{\text{Ad}}^* : \hat{G} \to \text{GL}(\hat{g}^*)$. Let us identify $K$ with its image in $\hat{G}$ and regard $G$ as $\hat{G}/K$. Since $K$ is central, its coadjoint action on $\hat{g}$ is trivial so $\hat{\text{Ad}}$ factors through $G$ to give a representation of $G$ on $\hat{g}$, also denoted $\hat{\text{Ad}}$, such that $\hat{\text{Ad}}_g = \hat{\text{Ad}}_{\hat{g}}$ where $\hat{g}$ is any lift of $g \in G$; that is, there exists a dotted arrow making the following diagram commute:

$$\begin{array}{ccc}
\hat{G} & \xrightarrow{\hat{\text{Ad}}} & \text{GL}(\hat{g}) \\
\text{G = } \hat{G}/K & \xrightarrow{\hat{\text{Ad}}} & \\
\end{array}$$  \hspace{1cm} (4.2)

Of course, $K$ also acts trivially via the coadjoint action which thus factors through $G$ similarly to give a representation $\hat{\text{Ad}}^* : G \to \text{GL}(\hat{g}^*)$ dual to $\hat{\text{Ad}} : \hat{G} \to \text{GL}(\hat{g})$. On the other hand, the adjoint and coadjoint representations of $G$ give rise to representations of $\hat{G}$ by pulling back along the quotient map:

$$\begin{array}{ccc}
\hat{G} & \xrightarrow{\hat{\text{Ad}}} & \text{GL}(\hat{g}) \\
\text{G = } \hat{G}/K & \xrightarrow{\hat{\text{Ad}}} & \\
\end{array}$$  \hspace{1cm} (4.3)

One can check that the canonical map $\hat{g} \to g$ and the dual embedding $g^* \hookrightarrow \hat{g}^*$ are $G$-equivariant (equivalently $\hat{G}$-equivariant). We shall refer to $\hat{\text{Ad}}$ and $\hat{\text{Ad}}^*$ as the factored adjoint and coadjoint actions of $G$ on $\hat{g}$ and $\hat{g}^*$, respectively.

We now pick a complement to $\text{Lie}(K) = \mathfrak{k} \cong \mathbb{R}$ in $\hat{g}$ and identify it with $g$ so that we have a (not necessarily invariant) splitting of vector spaces $\hat{g} = g \oplus \mathbb{R}$. Since $K$ is central, $\hat{\text{Ad}}$ acts trivially on the $\mathbb{R}$ factor. Using this and equivariance of the quotient map, we conclude that the hatted adjoint action is of the form

$$\hat{\text{Ad}}_g(X, u) := (\text{Ad}_g X, u - \langle \theta(g^{-1}), X \rangle)$$  \hspace{1cm} (4.4)

for some map $\theta : G \to g^*$, where the reasons for the sign and the inversion in the argument will become clear soon.

The splitting $\hat{g} = g \oplus \mathbb{R}$ induces a dual splitting $\hat{g}^* = g^* \oplus \mathbb{R}$ such that the natural embedding $g^* \hookrightarrow \hat{g}^*$ is given by $\alpha \mapsto (\alpha, 0)$. The natural paring between $\hat{g}$ and $\hat{g}^*$ is then

$$\langle (\alpha, \zeta), (X, u) \rangle = \langle \alpha, X \rangle + \zeta u$$  \hspace{1cm} (4.5)
where \( \alpha \in \mathfrak{g}^* \), \( X \in \mathfrak{g} \) and \( \zeta, u \in \mathbb{R} \). The factored coadjoint action can be explicitly computed as follows:

\[
\left\langle \hat{\text{Ad}}_g^*(\alpha, \zeta), (X, u) \right\rangle := \left\langle (\alpha, \zeta), (\text{Ad}_{g^{-1}} X, u - (\theta(g), X)) \right\rangle \\
= \left\langle (\alpha, \zeta), (\text{Ad}_{g^{-1}} X) + \zeta u - \zeta \theta(g), X \right\rangle \\
= \left\langle \text{Ad}_g^* \alpha - \zeta \theta(g), X \right\rangle + \zeta u.
\]

(4.6)

Abstracing away \((X, u)\), we thus have

\[
\hat{\text{Ad}}_g^*(\alpha, \zeta) = (\text{Ad}_g^* \alpha - \zeta \theta(g), \zeta).
\]

(4.7)

**Lemma 13.** The map \( \theta : G \to \mathfrak{g}^* \) in equation (4.7) is a symplectic group cocycle, and the associated Chevalley–Eilenberg cocycle \( c \in \Lambda^2 \mathfrak{g}^* \) (see Lemma 8) is the one determining the presentation of \( \hat{\mathfrak{g}} \) as an extension of \( \mathfrak{g} \). Furthermore, their cohomology classes do not depend on the splitting of \( \hat{\mathfrak{g}} \).

**Proof.** Let \( g_1, g_2 \in G \). Since \( \hat{\text{Ad}}^* \) is a representation, \( \hat{\text{Ad}}^*_{g_1 g_2} = \hat{\text{Ad}}^*_{g_1} \hat{\text{Ad}}^*_{g_2} \) so we see that

\[
(\text{Ad}_{g_1}^* \alpha - \zeta \theta(g_1), \zeta) = (\hat{\text{Ad}}^*_{g_1} \left( (\text{Ad}_{g_2}^* \alpha - \zeta \theta(g_2), \zeta) \right) \\
= (\text{Ad}_{g_1}^* \text{Ad}_{g_2}^* \alpha - \zeta \text{Ad}_{g_2}^* \theta(g_2) - \zeta \theta(g_1), \zeta)
\]

(4.8)

for all \( \alpha \in \mathfrak{g}^* \) and \( \zeta \in \mathbb{R} \). Then we use that \( \text{Ad}^* \) is a representation and abstract \( \zeta \) to find

\[
\theta(g_1 g_2) = \text{Ad}_{g_1}^* \theta(g_2) + \theta(g_1),
\]

(4.9)

thus \( \theta \) is a cocycle. We now differentiate (4.4) to find

\[
[(X, u), (Y, v)] = ([X, Y], c(X, Y))
\]

(4.10)

where \( c(X, Y) := (\text{d}_c \theta(X), Y) \); in particular, \( \theta \) is symplectic as required. If a different splitting of \( \hat{\mathfrak{g}} \) is chosen then \([c] = [c']\), since the cohomology class of \( c \) is determined by the isomorphism class of the extension. Now note that \( \theta' - \theta \) is a symplectic group cocycle and its cohomology class is trivial by Proposition 10 since \([c' - c] = 0\), so \([\theta'] = [\theta] \).

**Remark.** Note that the action (4.7) preserves the affine hyperplanes \( \zeta = \text{constant} \), so in particular if we identify \( \mathfrak{g}^* \) with the affine hyperplane \( \zeta = 1 \) in \( \hat{\mathfrak{g}}^* \), we obtain an affine action \( \rho \) of \( G \) on \( \mathfrak{g}^* \) given by

\[
\rho(g) \alpha = \text{Ad}_g^* \alpha - \theta(g).
\]

(4.11)

This is the same formula as (2.23), except there the cocycle \( \theta \) arose from a symplectic action rather than from the coadjoint action of a central group extension. The actions of \( G \) on \( \hat{\mathfrak{g}} \) and \( \hat{\mathfrak{g}}^* \) defined in Section 2.3 are also of the same form as the hatted adjoint and coadjoint representations here, which again prompts the question of whether those actions do indeed arise from a central group extension. The next section contains the key result we need to answer this question.

**4.2. Existence of extensions.** We now present a necessary and sufficient criterion for the existence of central extensions of Lie groups due to Neeb [6]. We use the notation for left- and right-translations and left- and right-invariant vector fields on \( G \) introduced in Appendix B.1.

Let \( c \in \Lambda^2 \mathfrak{g}^* \) be a Chevalley–Eilenberg 2-cocycle for \( \mathfrak{g} \). We showed in Section 2.3 that this defines a one-dimensional central extension of \( \mathfrak{g} \), and recalling the isomorphism between the Chevalley–Eilenberg complex for \( \mathfrak{g} \) and the left-invariant de Rham complex on \( G \) discussed in Appendix B.2, there exists a closed, left-invariant 2-form on \( G \) defined by \( \Omega_g = \lambda_g^* c \), or equivalently,

\[
\Omega(\lambda_X, \lambda_Y)(g) = c(X, Y)
\]

(4.12)

for all \( g \in G \). We may now state Neeb’s result.
Theorem 14 (Neeb [6]). Let $G$ be a connected Lie group, $c \in \Lambda^2 \mathfrak{g}^*$ a Chevalley–Eilenberg 2-cocycle of the Lie algebra $\mathfrak{g}$ defining a one-dimensional central extension $0 \to \mathbb{R} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0$ and $\Omega \in \Omega^2(G)$ the corresponding left-invariant closed 2-form. Then there exists a one-dimensional central extension $\tilde{G}$ of $G$ integrating the above central extension if and only if for any $X \in \mathfrak{g}$ there exists a function $\Phi_X \in C^\infty(G)$ satisfying

$$t_{\rho_X} \Omega = d\Phi_X. \quad (4.13)$$

One might observe that equation (4.13) is similar to (2.1); indeed, if we interpret $\Omega$ as a presymplectic structure on $G$, this equation says that $X \mapsto \Phi_X$ is a comoment map for the left action of $G$ on itself. Indeed, although Neeb does not interpret the equation in this way in [6], the earlier work of Tuynman and Wiegerinck [5] upon which Neeb builds does. We will return to this point of view in Section 5.4.

Remark. Neeb's result is actually slightly stronger than the above, although we will not need the stronger version; fixing a connected one-dimensional Lie group $K$, functions $\Phi_X$ satisfying (4.13) exist if and only if there is a central extension $\tilde{G}$ of $G$ by $K$ integrating the extension of Lie algebras. Note, however, that the theorem above does not say that there is a unique such extension, only that one exists – even for a fixed choice of kernel group $K$.

5. SYMPLECTIC STRUCTURES AND CENTRAL EXTENSIONS

Finally we bring the preceding discussion together and present our main result. We will again use the results and notation of Appendix B freely.

5.1. From the symplectic action to the central extension. Let us recapitulate some of the main points of the preceding discussion. In Section 2, we considered a symplectic homogeneous $G$-space $M$, with $G$ a connected Lie group, with a moment map $\mu : M \to \mathfrak{g}^*$ and comoment map $\varphi : \mathfrak{g} \to C^\infty(M)$. We now assume that $M$ is simply-connected, so such maps exist, but we do not assume that the action is hamiltonian. By Lemma 1, the failure of equivariance of the moment map is measured by a group cocycle $\theta : G \to \mathfrak{g}^*$ which is symplectic by Lemma 2 and gives rise to a Chevalley–Eilenberg 2-cocycle $c \in \Lambda^2 \mathfrak{g}^*$ and thus to a closed, left-invariant 2-form $\Omega$ on $G$ given by equation (4.12). In Section 2.3, we used $c$ to construct a central extension $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$, upon which $G$ acts via an action (2.29) determined by $\theta$. There is a dual action (2.31) of $G$ on $\widetilde{\mathfrak{g}}^*$ with respect to which the extended moment map $\tilde{\mu} : M \to \mathfrak{g}^*$ defined by equation (2.28) is equivariant. Dual to this, we have an extended comoment map $\tilde{\varphi} : \widetilde{\mathfrak{g}} \to C^\infty(M)$ defined by equation (2.26) which is a Lie algebra homomorphism. One is naturally lead to ask whether there exists a central group extension $\tilde{G}$ of $G$ integrating the algebra extension, and furthermore whether the actions of $G$ on $\widetilde{\mathfrak{g}}$ and $\mathfrak{g}^*$ arise from the adjoint and coadjoint representations of $G$, and finally whether $\tilde{\mu}$ is a moment map for an action of $\tilde{G}$ on $M$. Theorem 14 allows us to finally answer all of these questions in the affirmative.

Theorem 15. Let $G$ be a connected Lie group and $(M, \omega)$ be a simply-connected symplectic homogeneous manifold of $G$. Then there exists a one-dimensional central extension $\tilde{G}$ of $G$ such that $M$ is the universal cover of a coadjoint orbit of $\tilde{G}$.

Proof. Recall the functions $\Phi_X \in C^\infty(G)$ given by $\Phi_X(g) = -\langle \theta(g), X \rangle$ which we first saw in Lemma 9. We will show that these satisfy the condition (4.13) in Theorem 14. Recall that the right-invariant vector fields span $\mathfrak{X}(G)$ as a $C^\infty(G)$-module, so it is sufficient to show that

$$\Omega(\rho_X, \rho_Y) = \mathcal{L}_{\rho_X} \Phi_X. \quad (5.1)$$

By Lemma 9, we have $\mathcal{L}_{\rho_X} \Phi_X(g) = (Ad_g^* c)(X, Y)$; on the other hand, since $\langle \rho_X \rangle_g = (\lambda_{\text{Ad}_g^{-1}} x) g$ at $g \in G$,

$$\Omega(\rho_X, \rho_Y)(g) = \Omega(\lambda_{\text{Ad}_g^{-1}} x, \lambda_{\text{Ad}_g^{-1}} y)(g) = c(Ad_g^{-1} X, Ad_g^{-1} Y) = (Ad_g^* c)(X, Y). \quad (5.2)$$
So we indeed have (4.13), and by Theorem 14, there exists a one-dimensional central extension\( \hat{G} \) of\( G \) integrating the central extension\( \hat{c} \) of\( c \). The action of\( G \) on\( M \) pulls back to\( \hat{G} \) along the quotient map\( \pi: \hat{G} \to G \); for\( g \in \hat{G} \) and\( p \in M \), we have\( g \cdot p = \pi(g) \cdot p \), and clearly this action is symplectic and transitive.

Let us write\( \xi_{(X,u)} \) for the fundamental vector field on\( M \) of the\( \hat{G} \)-action generated by\( (X,u) \in \hat{g} = \mathfrak{g} \oplus \mathbb{R} \). But then, since\( (0,u) \in \ker \pi \), we have\( \xi_{(X,u)} = \xi_X \), where the latter is the fundamental vector field for the\( G \)-action as before. Now recall the extended comoment map\( \hat{\phi}: \hat{g} \to M \) given by\( \hat{\phi}_{(X,u)} = \phi_X + u \). We have

\[
d\hat{\phi}_{(X,u)} = d\phi_X = \iota_{\xi_X} \omega = \iota_{\hat{\xi}_{(X,u)}} \omega, \tag{5.3}
\]

so\( \hat{\phi} \) is indeed a comoment map. We already showed (see (2.27)) that it is a Lie algebra homomorphism, so (since\( \hat{G} \) is connected) we conclude that\( (M,\omega,\hat{\mu}) \) is a hamiltonian\( \hat{G} \)-space. Corollary 6, with\( \hat{G} \) now playing the rôle of\( G \), then shows that\( \hat{\mu} : M \to \hat{\Omega} \) is a covering of a coadjoint orbit\( \hat{\Omega} \) and, since\( M \) is simply-connected, it is the universal cover.

\[\Box\]

5.2. Interpretation in terms of\( G \)-actions. In the preceding proof, after proving the existence of the central extension\( \hat{G} \) of\( G \), we passed from the action of the original group\( G \) to the action of\( \hat{G} \), which we showed was hamiltonian, and thus we were able to use results about hamiltonian\( G \)-spaces for a concise proof. It is possible, however, to instead work entirely with\( G \)-spaces by factoring the adjoint and coadjoint actions of\( G \) through\( \hat{G} \) as in Section 4.1, and it is instructive to see how the situation looks from this point of view, especially since it makes the rôle of the symplectic cocycle\( \theta \) more explicit.

Recall that, by construction,\( \hat{g} = \mathfrak{g} \oplus \mathbb{R} \) as vector spaces with\( c \) as the Chevalley–Eilenberg cocycle associated to the splitting, and by equation (4.7) the factored coadjoint action is given by

\[
(\alpha, \zeta) \mapsto (\hat{\text{Ad}}^*_g \alpha - \hat{\theta}(g), \zeta). \tag{5.4}
\]

The symplectic cocycle\( \theta \) appearing here is not a priori the same as the one arising from the moment map\( \mu \), but since both integrate\( c \), they are in fact the same by Proposition 11. This action is thus identical to the one defined by equation (2.31), so the extended moment map\( \hat{\mu} \) is\( G \)-equivariant with respect to\( \hat{\text{Ad}}^* \) by the calculation (2.32). Similarly, the action defined by (2.29) is just the factored adjoint action. This answers the questions posed at the beginning of Section 5.1 explicitly.

Let us now recall the proof of Proposition 5 and borrow some of its notation, albeit with some differences since\( (M,\omega,\mu) \) is not a hamiltonian\( G \)-space here. We fix an arbitrary point\( o \in M \) and let\( \lambda = \hat{\mu}(o) = (\mu(o),1) \). Since the action of\( G \) is transitive on\( M \), any point in\( M \) can be expressed as\( g \cdot o \) for some\( g \in G \), and we have

\[
\hat{\mu}(g \cdot o) = \hat{\text{Ad}}^*_g \hat{\mu}(o) = \hat{\text{Ad}}^*_g \lambda, \tag{5.5}
\]

so in particular the image of\( \hat{\mu} \), denoted by\( \hat{\Omega} \) above, is the orbit of\( \lambda \) under the factored coadjoint action. Let\( a_o : G \to M \) and\( a_{\lambda} : G \to \hat{\Omega} \) be the orbit maps\( g \mapsto g \cdot o \) and\( g \mapsto \hat{\text{Ad}}^*_g \lambda \) respectively. The\( G \)-equivariance of\( \hat{\mu} \) means that the following triangle of\( G \)-equivariant maps commutes:

\[
G \xrightarrow{a_\lambda} \hat{\Omega} \xleftarrow{\hat{\mu}} M \tag{5.6}
\]

Let us denote the fundamental vector fields of the\( G \)-action on\( \hat{\Omega} \) by\( \xi_X \). We have\( (\xi_X)_o = (a_o)_X \) and\( (\xi_X)_\lambda = (a_{\lambda})_X \) for each\( X \in \mathfrak{g} \), and these values span the tangent spaces of\( M \) and\( \hat{\Omega} \) respectively. Similarly to the computations in the proof of Proposition 5, we can compute the values at\( o \) of the
pullback of the Kirillov–Kostant–Souriau structure $\tilde{\omega}_{KKS}$ on $\tilde{\mathcal{O}}$ and of the symplectic form $\omega$ on $M$:

$$\omega(\xi_X, \xi_Y)(\alpha) = \omega_{KKS}(\xi_X, \xi_Y)(\lambda) = \langle \lambda, ([X, 0], [Y, 0]) \rangle = \langle \mu(\alpha), [X, Y] \rangle + c(X, Y),$$

(5.7)

$$\omega(\xi_X, \xi_Y)(\alpha) = \langle \varphi_X, \varphi_Y \rangle(\alpha) = \varphi_{|X|Y}(\alpha) + c(X, Y) = \langle \mu(\alpha), [X, Y] \rangle + c(X, Y),$$

(5.8)

which shows that $\tilde{\mu} : M \to \tilde{\mathcal{O}}$ is a symplectomorphism (as usual, it is sufficient to compute at one point by $G$-invariance and transitivity), and thus a homomorphism of homogeneous symplectic $G$-spaces. Since $\tilde{\mu}^*\tilde{\omega}_{KKS} = \omega$ on $M$, commutativity of the diagram (5.6) implies that we also have $\alpha^*\omega = \alpha^*\tilde{\omega}_{KKS}$ on $G$. One might ask whether the orbit maps are also symplectic in the sense that this invariant 2-form is also equal to $\Omega$. By $G$-invariance and since $G$ acts transitively on itself, it suffices to compute at the identity. We have

$$\Omega - \alpha^*\omega)(X, Y)(e) = c(X, Y) - \omega(\xi_X, \xi_Y)(\alpha)$$

$$= c(X, Y) - \langle \mu(\alpha), [X, Y] \rangle - c(X, Y)$$

$$= -\langle \mu(\alpha), [X, Y] \rangle.$$  

(5.9)

Thus $(\Omega - \alpha^*\omega)e = \partial G\mu(\alpha)$, so the orbit maps are not symplectic on the nose but we have $[\Omega] = [\alpha^*\omega]$ in de Rham cohomology. However, since we have freedom to redefine $\mu$ by addition of an element in $g^*$, we can without loss of generality assume that $\mu(\alpha) = 0$, and then the orbit maps are indeed symplectic. Note that with this assumption, we also have $\lambda = (0, 1) \in \tilde{g}^*$; this is the element denoted $\lambda$ in [6].

### 5.3. Interpretation in terms of the affine $G$-action.

Recall the affine action $\rho : G \to \text{Aff}(g^*)$ given by $\rho(g)\alpha = Ad_{g}^{\lambda} \alpha - \theta(g)$ (equations (2.23) and (4.11)). We noted in previous remarks that there is a $G$-equivariant embedding of $g^*$ in $\tilde{g}^*$ as the hyperplane $\{g, 1\}$, with the action $\rho$ on $g$ and the factored coadjoint action on $\tilde{g}^*$; $\tilde{Ad}_{g}^{\lambda}(\alpha, 1) = (\rho(g)\alpha, 1)$. We now interpret Theorem 15 in terms of this KK structure.

Let $\alpha \in g$ and let $O_{\alpha}^{\text{aff}}$ be the orbit of $\alpha \in g$ under the affine action $\rho$. Completely analogous to the KK structure on a coadjoint orbit, there is a natural invariant symplectic structure on this orbit. Let $G_{\alpha}^{\text{aff}}$ be the stabiliser of $\alpha$ under $\rho$ and let $g_{\alpha}^{\text{aff}}$ be its Lie algebra, so

$$G_{\alpha}^{\text{aff}} = \{ g \in G \mid Ad_{g}^{\lambda} \alpha - \theta(g) = \alpha \}$$

(5.10)

and

$$g_{\alpha}^{\text{aff}} = \{ X \in g \mid \langle \alpha, [X, Y] \rangle + c(X, Y) = 0 \quad \forall Y \in g \}.$$  

(5.11)

Since $G_{\alpha}^{\text{aff}}$ fixes $\alpha$, the pushforward by the action of an element $g \in G_{\alpha}^{\text{aff}}$ preserves $T_{\alpha}O_{\alpha}^{\text{aff}}$, hence we obtain an action of $G_{\alpha}^{\text{aff}}$ on $T_{\alpha}O_{\alpha}^{\text{aff}}$. Denoting by $\xi_{\alpha}$ the fundamental vector field on $O_{\alpha}^{\text{aff}}$ associated to $X \in g$, there is a linear map $g \to T_{\alpha}O_{\alpha}^{\text{aff}}$ given by $X \mapsto \langle \xi_{\alpha}, X \rangle$, the kernel of which is $g_{\alpha}^{\text{aff}}$. Since $\rho(g)\xi_{\alpha} = \xi_{Ad_{g}^{\lambda} X}$ for all $g \in G$, in particular this map is $G_{\alpha}^{\text{aff}}$-equivariant, there is a short exact sequence of $G_{\alpha}^{\text{aff}}$-modules

$$0 \longrightarrow g_{\alpha}^{\text{aff}} \longrightarrow g \longrightarrow T_{\alpha}O_{\alpha}^{\text{aff}} \longrightarrow 0.$$  

(5.12)

There exists a $G_{\alpha}^{\text{aff}}$-invariant 2-form on $g$ given by $(X, Y) \mapsto \langle [X, Y], [X, Y] \rangle + c(X, Y)$ which by (5.11) descends to a nondegenerate, $G_{\alpha}^{\text{aff}}$-invariant 2-form on $T_{\alpha}O_{\alpha}^{\text{aff}}$ $\equiv g_{\alpha}^{\text{aff}}/g_{\alpha}^{\text{aff}}$, and by the holonomy principle this in turn extends to a nondegenerate, $G$-invariant 2-form $\omega_{\text{aff}}$ on $O_{\alpha}^{\text{aff}}$. One can then verify that $d\omega_{\text{aff}} = 0$, so $\omega_{\text{aff}}$ is a symplectic form. One can also check that the inclusion $i : O_{\alpha}^{\text{aff}} \to g^*$ is a moment map, although unlike in the case of coadjoint orbits, it is not equivariant (with the coadjoint action on $g^*$), so $(O_{\alpha}^{\text{aff}}, \omega_{\text{aff}}, i)$ is not a hamiltonian $G$-space.

The embedding $g^* \hookrightarrow \tilde{g}^* = g^* \oplus R$ restricts to an isomorphism of homogeneous $G$-spaces $O_{\alpha}^{\text{aff}} \to \tilde{\mathcal{O}}(\alpha, 1)$, where the latter space is the coadjoint orbit of $(\alpha, 1)$. We can compute

$$\tilde{\omega}_{KKS}(\xi_X, \xi_Y)(\alpha, 1) = \langle \langle \alpha, 1 \rangle, ([X, 0], [Y, 0]) \rangle = \langle \langle \alpha, 1 \rangle, ([X, Y], c(X, Y)) \rangle = \langle \alpha, [X, Y] \rangle + c(X, Y)$$

(5.13)

so the isomorphism is also a symplectomorphism.
Setting $\alpha = \mu(\alpha)$, we have $[\alpha, 1] = \lambda = \tilde{u}(\alpha)$, so we see that the affine orbit $\mathcal{O}^{\operatorname{aff}}_{\lambda}$ is isomorphic to the coadjoint orbit of $\lambda$ as a homogeneous symplectic $G$-space. We can thus interpret Theorem 15 as saying that $M$ covers an affine orbit of $G$ as a homogeneous symplectic $G$-space.

5.4. Invariant presymplectic structures on groups, symplectic cocycles and central extensions. The lynchpin in the proof of Theorem 15 is the symplectic group cocycle $\theta$ arising from the moment map for the action of $G$ on $M$. We also saw in Section 4.1 that factoring the coadjoint action of a central extension of $G$ through $G$ itself gives rise to a symplectic group cocycle (after splitting the central extension of algebras), and that this was the same cocycle when the central extension was defined by $c = d_\epsilon \theta$. Forgetting the action of $G$ on $M$, what we have implicitly shown is the following.

**Theorem 16.** Let $G$ be a connected Lie group and

$$0 \longrightarrow \mathbb{R} \longrightarrow \hat{g} \longrightarrow g \longrightarrow 0 \tag{5.14}$$

a central extension of the Lie algebra $\mathfrak{g}$ of $G$. This integrates to a one-dimensional central extension of the group $G$

$$1 \longrightarrow K \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1 \tag{5.15}$$

if and only if there exists a symplectic group cocycle $\theta \in C^1(G; \mathfrak{g}^*)$ for which the cohomology class of the Chevalley–Eilenberg 2-cocycle $c$ of $g$ given by $c(X,Y) = \langle d_\epsilon \theta(X), Y \rangle$ corresponds to the extension (5.14).

We saw at the end of Section 3 that the map assigning a Chevalley–Eilenberg 2-cocycle $c$ to a symplectic group cocycle $\theta$ induces an injection $H^1_{\operatorname{symp}}(G, \mathfrak{g}^*) \hookrightarrow H^2(\mathfrak{g})$. Recalling that $H^2(\mathfrak{g})$ classifies the one-dimensional central extensions of $G$, we now see that $H^1_{\operatorname{symp}}(G, \mathfrak{g}^*)$ (or more precisely its image in $H^2(\mathfrak{g})$) classifies the integrable extensions.

We will now shift to a slightly different perspective which will serve to unify a few different themes already present in this work and prove the above theorem more concisely. Recall that, given a Lie algebra extension (5.14), choosing a splitting (as vector spaces) yields a Chevalley–Eilenberg cocycle $c \in \Lambda^2\mathfrak{g}$ and a closed, left-invariant two-form $\Omega$ on $G$ corresponding to $c$. A different choice of splitting gives rise a cocycle and 2-form in the same cohomology classes $[c]$ and $[\Omega]$, respectively.

We now return to the observation following Theorem 14 and emphasise that $\Omega$ is a left-invariant closed 2-form on $G$; that is, a left-invariant presymplectic structure, making $G$ into a presymplectic homogeneous space for itself. The fundamental vector fields of this action are the right-invariant vector fields $\rho_X$. With this point of view, let us compare the equations (4.13) and (2.1) and note that we can perform the same analysis on $(G, \Omega)$ as we did on $(M, \omega)$, mutatis mutandis, the only major difference being that there is no Poisson bracket on $G$ in general since $\Omega$ is possibly degenerate. That is, where they exist, we can choose the $\Phi_X$ such that the map $\Phi : \mathfrak{g} \rightarrow C^\infty(G)$ is linear and consider it as a comoment map with corresponding moment map $\mu : G \rightarrow \mathfrak{g}^*$, both unique up to addition of an element in $\mathfrak{g}^*$. Continuing with the analogy, we discover that (as in Lemmas 1 and 2) there exists a symplectic group cocycle $\theta : G \rightarrow \mathfrak{g}^*$ satisfying

$$\theta(g_1, g_2) = \mu(g_2) - \mu(g_1)g_2 \tag{5.16}$$

for all $g_1, g_2 \in G$ (in particular the RHS does not depend on $g_2$). Note, however, that since $\mu$ is defined only up to an element of $\mathfrak{g}^*$, we may assume without loss of generality that $\mu(\epsilon) = 0$, in which case we find

$$\theta(g) = \operatorname{Ad}_g^* \mu(\epsilon) - \mu(g) = -\mu(g); \quad \tag{5.17}$$

in particular, the moment map $\mu$ is itself a cocycle and we have $\Phi_X(g) = -\langle \theta(g), X \rangle$. It is simple to check now that the Chevalley–Eilenberg cocycle obtained by taking the derivative of $\theta$ is the cocycle $c$:

$$d_\epsilon \theta(X,Y) := \langle d_\epsilon \theta(X), Y \rangle = \frac{d}{dt} \Phi_Y(\exp(tX))|_{t=0} = -\langle L_{\rho_X} \Phi_Y(\epsilon) \rangle = \Omega(\rho_X, \rho_Y)(\epsilon) = c(X,Y). \tag{5.18}$$

Notice that if we do not assume that $\mu(\epsilon) = 0$, the claim is true up to a coboundary, i.e., $[d_\epsilon \theta] = \epsilon$. 


Recalling Proposition 3, one might ask what it means for the action of $G$ on itself to be “hamiltonian” with respect to $\Omega$. Note that, since we do not have a Poisson bracket on $G$, we cannot ask if $\Phi$ is a Lie algebra homomorphism as we did with the comoment map of Section 2; we may however analogously ask if $\Phi_{[X,Y]} = \Omega(\rho_X, \rho_Y)$ for all $X, Y \in \mathfrak{g}$. The following result, along with Proposition 10, tells us that the action of $G$ on $(G, \Omega)$ is hamiltonian if and only if the Lie algebra extension is trivial.

**Proposition 17.** Where it exists, the comoment map $\Phi$ can be chosen such that $\Phi_{[X,Y]} = \Omega(\rho_X, \rho_Y)$ for all $X, Y \in \mathfrak{g}$ if and only if $[c] = 0$.

**Proof.** By the analogous calculation to (2.12) (or by Lemma 8), we have

$$c(X,Y) = \Omega(\rho_X, \rho_Y)(g) - \Phi_{[X,Y]} = (\text{Ad}^* c)(X,Y) - \Phi_{[X,Y]}(g).$$

Clearly, if $\Phi_{[X,Y]} = \Omega(\rho_X, \rho_Y)$ then $c(X,Y) = 0$. Conversely, if $c = \partial_{CE} \alpha$ for some $\alpha \in \mathfrak{g}^*$ then we have

$$\Omega(\rho_X, \rho_Y) = \Phi_{[X,Y]} = c(X,Y) = -\alpha([X,Y]).$$

We see that $\Phi'$ defined by $\Phi'_X(g) = \Phi_X(g) - \alpha(X)$ satisfies both $\tau_{\rho_X} \Omega = d\Phi_X'$ and $\Phi'_{[X,Y]} = \Omega(\rho_X, \rho_Y)$. □

**Proof of Theorem 16.** Let us choose a splitting of the algebra extension and fix notation as above. The calculations in the first part of the proof of Theorem 15 show that if $\theta : G \to \mathfrak{g}^*$ is a symplectic group cocycle satisfying $c(X,Y) = (\text{Ad}, \theta(X), Y)$, then the functions $\Phi_X \in C^\infty$ given by $\Phi_X(g) = -\langle \theta(g), X \rangle$ satisfy equation (4.13) of Theorem 14, and so by that theorem there exists a one-dimensional central extension $\tilde{G}$ of $G$ integrating the extension of Lie algebras.

On the other hand, suppose that the group extension (5.15) exists. Then by Theorem 14, for each element $X \in \mathfrak{g}$ there exists a map $\Phi_{X} \in C^\infty$ satisfying equation (4.13). We argued above that without loss of generality, these maps can be chosen such that $\Phi_{X}(g) = -\langle \theta(g), X \rangle$ for some symplectic group cocycle $\theta$ integrating $c$.

Note that by Proposition 11, the symplectic cocycle $\theta$ appearing in the second part of the proof above is the same as the one arising from the factored coadjoint action $\text{Ad}^*$ in equation (4.7).

### 5.5. Exact homogeneous symplectic spaces

Let $(M, \omega)$ be a homogeneous symplectic $G$-space for which $\omega = d\eta$ for some $\eta \in \Omega^1(M)$. In such a case, we call $\eta$ the symplectic potential, and the pair $(M, \eta)$ is an **exact homogeneous symplectic G-space**. We will now show that a central extension of $G$ satisfying the conclusion of Theorem 15 can be constructed directly without appeal to Neeb’s Theorem 14, and that it takes a particularly simple form.

For an exact homogeneous symplectic $G$-space, the $G$-invariance condition $g^* \omega = \omega$ can be rewritten as

$$d(g^* \eta - \eta) = 0. \tag{5.21}$$

If the one-form $g^* \eta - \eta$ is exact for all $g \in G$ (in particular if $M$ is simply-connected), for each $g \in G$ there exists a function $\Psi_g \in C^\infty(M)$ such that

$$g^* \eta - \eta = d\Psi_g. \tag{5.22}$$

We can assume without loss of generality that $\Psi_g$ depends smoothly on $g$, in the sense that the function $\Psi : G \times M \to \mathbb{R}$ given by $\Psi(g, p) = \Psi_g(p)$ is smooth. Indeed, for any $X \in \mathfrak{g}$, evaluating (5.22) on the fundamental vector field $\xi_X$ at a point $p \in M$, we find

$$(\mathcal{L}_{\xi_X} \Psi_g)(p) = \eta(g, \xi_X(p)) gp - \eta(\xi_X(p)) = \eta(\xi_{Ad_g} X)(gp) - \eta(p) = (F(g,p), X) \tag{5.23}$$

where the last equality defines the smooth map $F \in C^\infty(G \times M; g^*)$. It follows that $\Psi$ is smooth up to the addition of a function $f : G \to H^0(M) = \mathbb{R}$, but this is exactly the freedom we have in the choice of $\Psi_g$ in equation (5.22).

Taking $\Psi$ to be smooth, the map $\Psi : g \to C^\infty(M), X \to \Psi_X$ defined by $\Psi_X(p) = \frac{d}{dt} \Psi_{\exp(tX)}(p)|_{t=0}$ satisfies

$$\mathcal{L}_{\xi_X} \eta = d\Psi_X, \tag{5.24}$$
and so we can define a comoment map $\varphi : g \to C^\infty(M)$, $X \mapsto \varphi_X$ by

$$\varphi_X = \Psi_X - \eta(\xi_X);$$  \hspace{1cm} (5.25)

indeed, we have

$$d\varphi_X = d\Psi_X - d(\eta(\xi_X)) = \mathcal{L}_{\xi_X}\eta - d\iota_{\xi_X}\eta = \iota_{\xi_X}d\eta = \iota_{\xi_X}\omega.$$  \hspace{1cm} (5.26)

**Lemma 18.** The map $\gamma : G \times G \to C^\infty(M)$ given by

$$\gamma(g_1, g_2) := \Psi_{g_2} + g_2^*\Psi_{g_1} - \Psi_{g_1} g_2.$$  \hspace{1cm} (5.27)

takes values in the constant functions, and can thus be considered as a (smooth) map $\gamma : G \times G \to \mathbb{R}$. Furthermore, $\gamma$ satisfies the equation

$$\gamma(g_2, g_3) + \gamma(g_1, g_2 g_3) = \gamma(g_1, g_2) + \gamma(g_1 g_2, g_3)$$  \hspace{1cm} (5.28)

for all $g_1, g_2, g_3 \in G$.

**Proof.** For all $g_1, g_2, (g_1 g_2)^*\eta = g_2^* g_1^*\eta$, thus

$$d\gamma(g_1, g_2) = d\Psi_{g_2} + g_2^* d\Psi_{g_1} - d\Psi_{g_1 g_2} = g_2^*\eta - \eta + g_2^*(g_1^*\eta - \eta) - ((g_1 g_2)^*\eta - \eta) = 0,$$  \hspace{1cm} (5.29)

so since $M$ is connected, this shows that $\gamma(g_1, g_2)$ is constant. The second claim can be easily verified using the definition of $\gamma$. \hfill \square

The last part of the above lemma says that $\gamma$ is a cocycle in $C^2(G) = C^2(G; \mathbb{R})$. Recalling the defining equation (5.22) for $\Psi$, we note that it is unique only up to addition of a map in $C^\infty(G)$; $d\Psi'_g = d\Psi_g$ if and only if there exists some $f : G \to \mathbb{R}$ such that $\Psi'_g(p) = \Psi_g(p) + f(g)$ for all $p \in M, g \in G$. The corresponding change in $\gamma$ is

$$\gamma'(g_1, g_2) = \gamma(g_1, g_2) + f(g_1) + f(g_2) - f(g_1 g_2) = (g + \delta f)(g_1, g_2)$$  \hspace{1cm} (5.30)

where $\delta$ denotes the differential of the group cochain complex, so $[\gamma'] = [\gamma] \in H^2(G)$. The action of $G$ on $(M, d\eta)$ thus determines an element of $H^2(G)$, which itself corresponds to a one-dimensional central extension

$$1 \longrightarrow \mathbb{R} \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1.$$  \hspace{1cm} (5.31)

See Appendix A.2 for more details. We can describe this group extension explicitly, but let us first make a convenient choice for $\gamma$. We note that $\Psi_e$ is constant by equation (5.22) since $M$ is connected, and $\gamma(g, e) = \gamma(e, g) = \Psi_e$ for all $g \in G$ by (5.27). Using the freedom to redefine $\Psi$, we can assume without loss of generality that $\Psi_e = 0$, whence $\gamma(g, e) = \gamma(e, g) = 0$. We then construct the group $\hat{G}$ as follows: as a manifold, $\hat{G} = G \times \mathbb{R}$, and the group multiplication is given by

$$(g_1, u_1) \cdot (g_2, u_2) = (g_1 g_2, u_1 + u_2 + \gamma(u_1, u_2))$$  \hspace{1cm} (5.32)

with identity $\hat{e} = (e, 0)$ and inverses $(g, u)^{-1} = (g^{-1}, -u - \gamma(g, g^{-1}))$. The morphisms in the short exact sequence above are given by the natural inclusion and projection.

**Lemma 19.** For all $g \in G$ and $X, Y \in g$,

$$\langle \theta(g), X \rangle = \frac{d}{dt} \langle \gamma(g, \exp(tAd_g^{-1}X)), \exp(tX), g \rangle \bigg|_{t=0},$$  \hspace{1cm} (5.33)

$$c(X, Y) = \frac{\partial}{\partial s} \frac{\partial}{\partial t} \langle \gamma(\exp(sX), \exp(tY)), \exp(sY), \exp(tX) \rangle \bigg|_{s=0, t=0}.$$

where $\theta$ and $c$ are the cocycles associated to the comoment map $\varphi$ defined by equation (5.25).
Proof. For the first formula, we can use the definition of $\gamma$, (5.27), to write
\[
\frac{d}{dt}\gamma(\exp(tX), g)\bigg|_{t=0} = g^*\omega_X - \frac{d}{dt}\psi_{\exp(tX)g}\bigg|_{t=0},
\]
(5.35)
and using the definition of $\theta$ gives (5.33). The second formula follows from the first by differentiating and using the fact that $\gamma(e, g) = 0$. \qed

**Proposition 20.** The central extension of $G$ defined by $\gamma$ integrates the central extension of $\mathfrak{g}$ defined by $c$, and thus suffices for the central extension $\hat{G}$ appearing in Theorem 15, where we take $K = \mathbb{R}$.

Proof. The adjoint action of $\hat{G}$ on $\hat{\mathfrak{g}}$ is given by
\[
\hat{\text{Ad}}_{(g,u)}(Y,v) = \frac{d}{dt}(g,u)\exp(t(Y,v))(g,u)^{-1}\bigg|_{t=0}
\]
\[
=\frac{d}{dt}(g,u)(\exp(tY), tv)(g^{-1},-u-\gamma(g,g^{-1}))\bigg|_{t=0}
\]
\[
=\frac{d}{dt}(g\exp(tY)g^{-1}, tv+\gamma(\exp(tY), g^{-1}) - \gamma(g,g^{-1}) + \gamma(g, \exp(tY)g^{-1}))\bigg|_{t=0},
\]
(5.37)
and using the identity $\exp(tY)g^{-1} = g^{-1}\exp(t\text{Ad}_g Y)$ and the cocycle condition for $\gamma$, we find that
\[
-\gamma(g, g^{-1}) + \gamma(g, \exp(tY)g^{-1}) = \gamma(gg^{-1}, \exp(t\text{Ad}_g Y)) - (g^{-1}, \exp(t\text{Ad}_g Y))
\]
(5.38)
and so, using equation (5.33),
\[
\hat{\text{Ad}}_{(g,u)}(Y,v) = (\text{Ad}_g Y, v - \theta(g^{-1})).
\]
(5.39)
Differentiating once again with respect to the group element, we find
\[
[(X,u), (Y,v)] = \text{ad}_{[X,u]}(Y,v) = ([X,Y], c(X,Y)),
\]
(5.40)
so $\hat{\mathfrak{g}}$ is the Lie algebra of $\hat{G}$. We thus see that the short exact sequence expressing $\hat{G}$ as a central extension of $G$ by $\mathbb{R}$ integrates the one expressing $\hat{\mathfrak{g}}$ as a central extension of $\mathfrak{g}$. \qed

We note that equation (5.39) closely resembles (4.4); in fact, $\hat{\text{Ad}}_g = \hat{\text{Ad}}_{(g,u)}$ where the former is the “factored” coadjoint representation, we they are in fact the same equation. We thus see that the symplectic cocycle $\theta$ arising from the moment map is manifestly the one arising from the coadjoint action of $\hat{G}$ (equation 4.4); we saw in Section 5.2 that the same held in the general case (where $\omega$ was not exact), but there we needed to use a uniqueness result (Proposition 11).

Finally, we can explicitly demonstrate the observation in the remark following Theorem 14 that the central extension $\hat{G}$ integrating the algebra extension is not unique, and that we have a choice of the kernel group. For any integer $n$, let $\pi_n : \mathbb{R} \to \mathbb{R}/n\mathbb{Z} \cong S^1$ be the canonical Lie group morphism and let $(\pi_n)_* : C^*(G) \to C^*(G; S^1)$ be the map $\phi \to \pi_n \circ \phi$. One can easily check that this commutes with the differential, inducing maps $(\pi_n)_* : H^*(G) \to H^*(G; S^1)$. In particular, $\gamma_n := \pi_n \circ \gamma$ is an element in $Z^2(G; S^1)$ and thus defines a (geometrically trivial) central extension $\hat{G}_n$ of $G$ by $S^1$. Now, if $\hat{G}$ is the extension by $\mathbb{R}$ defined by $\gamma$, letting $\Pi_n : \hat{G} \to \hat{G}_n$ be the map $\Pi_n(g,u) = (g, [u])$, we get the commutative diagram with exact rows and covering maps for vertical arrows
\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{R} & \longrightarrow & \hat{G} & \longrightarrow & G & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & S^1 & \longrightarrow & \hat{G}_n & \longrightarrow & G & \longrightarrow & 1.
\end{array}
\]
(5.41)
In particular, the lower row is also a central extension of \( G \) integrating the Lie algebra extension.

**Appendix A. Lie group and Lie algebra cohomology**

Here we give a brief overview of the cohomology theories used in this paper.

A.1. **Group cohomology.** Let \( G \) be a Lie group, \( g = \text{Lie} \, G \) its Lie algebra and \( \rho: G \to \text{GL}(V) \) a representation of \( G \). The *cochain complex for \( G \) with values in \( (V, \rho) \) (or simply with values in \( V \) where there is no ambiguity) is the complex \( (C^*(G; V), \partial) \) with cochain spaces

\[
C^p(G; V) = C^\infty(G^p, V),
\]

(A.1)

where \( G^p = G \times G \times \cdots \times G \) (under product), and differential \( \partial: C^p(G; V) \to C^{p+1}(G; V) \) given by

\[
\partial \psi(g_1, g_2, \ldots, g_{p+1}) = \rho(g_1)\psi(g_2, g_3, \ldots, g_{p+1}) + \sum_{i=1}^p (-1)^i \psi(g_1, g_2, \ldots, g_1g_{i+1}, \ldots, g_{p+1})
\]

(A.2)

The cohomology \( H^*(G; V) \) of this complex is known as the *cohomology of \( G \) with values in \( V \). In this paper, the group cohomology theories we are concerned with take values either in the coadjoint representation \( \mathfrak{g}^* \) or trivial representation \( (\mathbb{R}, 1) \); for the latter, we write \( C^*(\mathfrak{g}) := C^*(\mathfrak{g}; \mathbb{R}) \) and \( H^*(\mathfrak{g}) := H^*(\mathfrak{g}; \mathbb{R}) \). Now let \( \mathfrak{a} \) be a connected abelian Lie group whose group operation is denoted \(+\). We define the *cochain complex for \( G \) with values in \( \mathfrak{a} \) to be the complex \( (C^*(G; \mathfrak{a}), \partial) \), with cochain spaces

\[
C^p(G; \mathfrak{a}) = C^\infty(G^p, \mathfrak{a})
\]

(A.3)

and differential \( \partial: C^p(G; \mathfrak{a}) \to C^{p+1}(G; \mathfrak{a}) \) given by the same formula (A.2), except that the first term is simply \( \psi(g_2, g_3, \ldots, g_{p+1}) \) (without the \( \rho(g_1) \) action). Note that we do not require an action of \( G \) on \( \mathfrak{a} \). The cohomology \( H^*(G; \mathfrak{a}) \) of this complex is the *cohomology of \( G \) with values in \( \mathfrak{a} \).

A.2. **Central extensions of Lie groups from group cohomology.** Let \( G \) be a connected Lie group and \( \mathfrak{a} \) a connected abelian Lie group. Given a cocycle \( \gamma \in C^2(G; \mathfrak{a}) \), one can define a Lie group structure on the product manifold \( G \times \mathfrak{a} \) by

\[
(g_1, a_1) \cdot (g_2, a_2) = (g_1g_2, a_1 + a_2 + \gamma(g_1, g_2)).
\]

(A.4)

The cocycle condition for \( \gamma \) is equivalent to associativity of this product, the identity element is

\[
\widehat{e} = (e, a_0)
\]

(A.5)

where \( a_0 = -\gamma(e, e) = -\gamma(e, e) = -\gamma(e, e) \) – the latter equalities follow from the cocycle condition for \( \gamma \), and inverses are given by

\[
(g, a)^{-1} = (g^{-1}, a_0 - a - \gamma(g, g^{-1})).
\]

(A.6)

We denote \( G \times \mathfrak{a} \) equipped with this group structure by \( \widehat{G} \). The map \( \iota: A \to G \times \mathfrak{a} \) given by \( \iota(a) = (e, a_0 + a) \) and the natural projection \( \pi: G \times \mathfrak{a} \to G \) are Lie group homomorphisms, and they give us a short exact sequence

\[
1 \longrightarrow A \overset{\iota}{\longrightarrow} \widehat{G} \overset{\pi}{\longrightarrow} G \longrightarrow 1.
\]

(A.7)

where the image of \( \iota \) is central; that is, \( \widehat{G} \) is a central extension of \( G \) by \( A \). If a different cocycle \( \gamma' \in C^2(G; \mathfrak{a}) \) is chosen, the central extensions they define are equivalent if and only if \( \gamma' - \gamma \) is a coboundary. Thus we have an injective map from \( H^2(G; \mathfrak{a}) \) to the set of equivalence classes of central extensions of \( G \) by \( A \). Let us describe the image of this map.

Any central extension of \( G \) by \( A \) is a fibre bundle (in particular a principal \( A \)-bundle) over \( G \) with fibre \( A \), but the central extension associated to an element in \( H^2(G; \mathfrak{a}) \) is manifestly trivial \( (\widehat{G} \cong G \times \mathfrak{a}) \) as a fibre bundle. We will call such an extension *geometrically trivial*. Conversely, given any geometrically trivial
central extension, one can show that, in any trivialisation, the group structure takes the form (A.4) for some cocycle $\gamma \in C^2(G; A)$. If a different trivialisation is chosen, one obtains a different cocycle $\gamma'$ but $[\gamma'] = [\gamma]$ in $H^2(G; A)$, allowing us to assign an element of $H^2(G; A)$ to any geometrically trivial central extension. This shows that the map from $H^2(G; A)$ to the set of geometrically trivial central extensions is a bijection.

A.3. Chevalley–Eilenberg cohomology. Let $\rho : g \rightarrow gl(V)$ be a representation of the Lie algebra $g$ on the real vector space $V$. The (Chevalley–Eilenberg) cochain complex for $g$ with values in $(V, \rho)$ is the complex $(C^\bullet(g; V), \delta_{CE})$ with cochain spaces

$$C^p(g; V) = \text{Hom}(A^p g, V)$$

and differential $\delta_{CE} : C^p(g; V) \rightarrow C^{p+1}(g; V)$ given by

$$(\delta_{CE}\psi)(X_1, X_2, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} \rho(X_i)\psi(X_1, X_2, \ldots, \widehat{X_i}, \ldots, X_{p+1})$$

$$+ \sum_{i<j} (-1)^{i+j} \psi([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{p+1}),$$

where the $\widehat{\ }$ adorning a symbol denotes its omission. The cohomology $H^\bullet(g; V)$ of this complex is the (Chevalley–Eilenberg) cohomology of $g$ with values in $(V, \rho)$ (or simply with values in $V$).

We will be particularly interested in 2-cocycles of $g$ with values in the trivial representation. We write $C^\bullet(g) := C^\bullet(g; \Bbb{R})$ and $H^\bullet(g) := H^\bullet(g; \Bbb{R})$ for cochains and cohomology in the trivial representation.

Appendix B. Structures on Lie groups

B.1. Invariant vector fields. Let $G$ be a Lie group and $g$ its Lie algebra, thought of as the tangent space $T_e G$ at the identity. For each $X \in g$, we let $\lambda_X, \rho_X \in \mathcal{X}(G)$ be the left- and right-invariant vector fields associated to $X$ respectively:

$$\lambda_X = [L_g] \cdot X = \frac{d}{dt} g \exp(tX) \bigg|_{t=0},$$

$$\rho_X = [R_g] \cdot X = \frac{d}{dt} \exp(tX) g \bigg|_{t=0}$$

where $L_g, R_g$ are left- and right-translation respectively. The maps $\lambda, \rho : g \rightarrow \mathcal{X}(G)$ given by $X \mapsto \lambda_X$ and $X \mapsto \rho_X$ are injective, and

$$\lambda_{[X,Y]} = [\lambda_X, \lambda_Y]$$

$$\rho_{[X,Y]} = [-\rho_X, \rho_Y].$$

for all $X, Y \in g$; that is, $\lambda$ is a Lie algebra homomorphism and $\rho$ is an anti-homomorphism. Furthermore, composing these maps with evaluation at a point $g \in G$, we find that $X \mapsto [\lambda_X]_g$ and $X \mapsto [\rho_X]_g$ are isomorphisms $g \rightarrow T_g G$; in particular, the values of left-invariant vector fields span every tangent space, as do those of the right-invariant vector fields, and the sets of left- and right-invariant vector fields both span $\mathcal{X}(G)$ as a $C^\infty(G)$-module.

Lemma 21. For each $X \in g$ and $g \in G$, we have

$$[\lambda_X]_g = (\rho_{Ad_g} X) g,$$

$$[\rho_X]_g = [\lambda_{Ad_{g^{-1}} X}] g.$$  \hspace{1cm} (B.3)

Proof. We have

$$[\lambda_X]_g = (L_g)_* X = (R_g)_* (R_{g^{-1}})_* (L_g)_* X = (R_g)_* Ad_g X = \rho_{Ad_g} X,$$

and similarly for the other claim. \hfill $\Box$
B.2. Left-invariant de Rham complex. For any differential form \( \omega \in \Omega^p(G) \), we have \( L^*_g[\omega] = d(\Omega_1^*(G)) \); thus the De Rham complex \( (\Omega^*(G), d) \) of \( G \) has a subcomplex \( (\Omega^*_LI(G), d) \) consisting of left-invariant differential forms. Left-invariant forms are determined by their value at the identity – in particular \( \omega_g = L^*_g[\omega] \) and these values lie in the Chevalley–Eilenberg cochain space \( C^*(g) = \Lambda^g \); furthermore one can show that \( d \omega = L^*_g[\omega] \). It follows from these observations that \( (\Omega^*_LI(G), d) \cong (\Omega^*(G), d) \). This observation will be used in particular in Section 5.1. In the case that \( G \) is compact, the natural inclusion of the left-invariant de Rham complex into the full de Rham complex induces an isomorphism in cohomology \( H^*_LI(G) \cong H^*_GR(G) \), so we have \( H^*_GR(G) \cong H^*(g) \).

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MAXWELL INSTITUTE AND SCHOOL OF MATHEMATICS, THE UNIVERSITY OF EDINBURGH, JAMES CLERK MAXWELL BUILDING, PETER GUTHRIE TAIT ROAD, EDINBURGH EH9 3FD, SCOTLAND, UNITED KINGDOM
Email address, AB: abeckett[at]ed.ac.uk, ORCID: 0000-0002-7287-3156
Email address, JMF: j.m.figueras[at]ed.ac.uk, ORCID: 0000-0002-9308-9360