SETS WITH DISTINCT SUMS OF PAIRS, LONG ARITHMETIC PROGRESSIONS, AND CONTINUOUS Mappings

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Dedicated to the memory of Jean-Pierre Kahane

Abstract. We show that if \( \varphi : \mathbb{R} \to \mathbb{R} \) is a continuous mapping and the set of nonlinearity of \( \varphi \) has nonzero Lebesgue measure, then \( \varphi \) maps bijectively a certain set that contains arbitrarily long arithmetic progressions onto a certain set with distinct sums of pairs.

1. Introduction

We say that a set \( E \) in the real line \( \mathbb{R} \) is a set with distinct sums of pairs if a relation \( \lambda_1 + \lambda_2 = \lambda_3 + \lambda_4 \) with \( \lambda_j \in E, \ 1 \leq j \leq 4, \) holds only in the trivial case when \( \lambda_1 = \lambda_3 \) and \( \lambda_2 = \lambda_4 \) or when \( \lambda_1 = \lambda_4 \) and \( \lambda_2 = \lambda_3 \).

By an arithmetic progression of length \( N \) we mean a set \( F \subseteq \mathbb{R} \) of the form \( F = \{x + ky, k = 1, 2, \ldots, N\} \), where \( x, y \in \mathbb{R} \) and \( y \neq 0 \).

It is natural to regard sets with distinct sums of pairs as thin sets. One of the properties of sets with distinct sums of pairs is as follows. Let \( B^p(\mathbb{R}) \), \( 1 \leq p \leq \infty \), denote the Besicovitch spaces of almost periodic functions (the definition of the spaces \( B^p(\mathbb{R}) \) and basic facts about them can be found in [1] and [10]). Recall that the norm \( \| \cdot \|_{B^p(\mathbb{R})} \) on \( B^p(\mathbb{R}) \) is defined by

\[
\| f \|_{B^p(\mathbb{R})} = \limsup_{T \to +\infty} \left( \frac{1}{2T} \int_{-T}^{T} |f(t)|^p \, dt \right)^{1/p}.
\]

It is well known that for \( 1 \leq p_1 \leq p_2 \leq \infty \) we have \( B^{p_1}(\mathbb{R}) \subseteq B^{p_2}(\mathbb{R}) \) with a natural relation for the norms: \( \| \cdot \|_{B^{p_1}(\mathbb{R})} \leq \| \cdot \|_{B^{p_2}(\mathbb{R})} \). In particular,

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\[ \|f\|_{B^2(\mathbb{R})} \leq \|f\|_{B^4(\mathbb{R})}. \]

Assume now that \( E \) is a set with distinct sums of pairs, and \( f \) is an \( E \)-polynomial, i.e., a function of the form

\[ f(t) = \sum_{\lambda \in E} c(\lambda) e^{i\lambda t}, \]

where only finitely many coefficients \( c(\lambda) \) are nonzero. Then we have

\[ \|f\|_{B^4(\mathbb{R})} \leq c \|f\|_{B^2(\mathbb{R})}, \]

where \( c > 0 \) does not depend on \( f \). This estimate is nearly obvious, it suffices to note that

\[ |f(t)|^2 = \sum_{\lambda_1, \lambda_2 \in E} c(\lambda_1) \overline{c(\lambda_2)} e^{i(\lambda_1 - \lambda_2)t} \]

and

\[ |f(t)|^4 = \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in E} c(\lambda_1) c(\lambda_2) c(\lambda_3) c(\lambda_4) e^{i(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)t}. \]

Since

\[ \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} e^{iat} \, dt = \begin{cases} 0 & \text{if } a \neq 0, \\ 1 & \text{if } a = 0, \end{cases} \]

it follows that

\[ \|f\|_{B^2(\mathbb{R})} = \left( \sum_{\lambda} |c(\lambda)|^2 \right)^{1/2} \quad \text{and} \quad \|f\|_{B^4(\mathbb{R})} \leq 2^{1/4} \left( \sum_{\lambda} |c(\lambda)|^2 \right)^{1/2}, \]

which yields (1) with \( c = 2^{1/4} \).

On the other hand, we regard sets that contain arbitrarily long arithmetic progressions as very massive ones. Let

\[ \gamma_p(N) = \left\| \sum_{k=1}^{N} e^{ikt} \right\|_{L^p(\mathbb{T})}, \]

where \( \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) \) is the circle (\( \mathbb{Z} \) is the additive group of integers). Assume that \( \{a + kd, k = 1, 2, \ldots, N\} \) is a progression of length \( N \) contained in \( E \). Consider the polynomial \( f_{N,a,d}(t) = \sum_{k=1}^{N} e^{i(a+kd)t} \). Note that \( \|f_{N,a,d}\|_{B^p(\mathbb{R})} = \gamma_p(N) \). At the same time \( \gamma_4(N) \simeq N^{3/4}, \gamma_2(N) \simeq N^{1/2} \) (clearly, \( \gamma_p(N) \) behaves as the \( L^p(\mathbb{T}) \)-norm of the Dirichlet kernel \( D_N \) for large \( N \)). So, if \( E \) contains arbitrarily long arithmetic progressions then, in general, estimate (1) for \( E \)-polynomials \( f \) does not hold.

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We note that, since for a $2\pi$-periodic function its $B^p(\mathbb{R})$-norm coincides with the $L^p(\mathbb{T})$-norm, it follows that if a set $E$ with distinct sums of pairs is in $\mathbb{Z}$ then (1) has the form $\|f\|_{L^4(T)} \leq c\|f\|_{L^2(T)}$ (for any $E$-polynomial $f$), so, $E$ is a set of type $\Lambda(4)$ (see [11] for basic results on sets of type $\Lambda(p)$).

In this paper we consider continuous mappings $\varphi: \mathbb{R} \to \mathbb{R}$ and show that, with a possible exception for the case when $\varphi$ is of a very special form resembling that of a piecewise linear mapping, every continuous mapping $\varphi$ is singular in the sense that it maps bijectively a certain set that contains arbitrarily long arithmetic progressions (a massive set) onto a certain set with distinct sums of pairs (a thin set).

Our interest to the question of how thin the images of arithmetic progressions under continuous mappings can be is inspired by the proof of Beurling and Helson of their theorem [2] on the endomorphisms of the algebra of Fourier transforms of measures, and the work of Graham [3] on the mappings that preserve Sidon sets. We discuss these works in remarks at the end of this paper.

2. Statement of the result

We say that $t \in \mathbb{R}$ is a point of nonlinearity of a mapping $\varphi: \mathbb{R} \to \mathbb{R}$ if $t$ has no neighborhood in which $\varphi$ coincides with a linear function. The set of all such points is called the set of nonlinearity of $\varphi$ and is denoted by $E(\varphi)$. Clearly, the set $E(\varphi)$ is closed.

The result of this paper is the following theorem.

**Theorem.** Let $\varphi$ be a continuous self-mapping of $\mathbb{R}$. Suppose that $E(\varphi)$ has nonzero Lebesgue measure. Then there exist a set $A$ that contains arbitrarily long arithmetic progressions and a set $B$ with distinct sums of pairs such that $\varphi$ maps bijectively $A$ onto $B$.

Note that the case when $E(\varphi)$ is finite is trivial. Assume that $\varphi$ is piecewise linear and maps bijectively $A$ onto $B$. Then, if $A$ contains arbitrarily long arithmetic progressions, $B$ has the same property as well. The general case of mappings whose sets of nonlinearity have measure zero seems to be a difficult one. In particular the author does not know whether the classical Cantor staircase function is singular in the above arithmetical sense.

3. Statement of the main Lemma and deduction of the Theorem

Let $V$ be the family of the following seven vectors in $\mathbb{R}^4$:

$$v^1 = (1, 1, -1, -1), \quad v^2 = (1, 1, -2, 0), \quad v^3 = (1, -1, 0, 0).$$

1 For real functions of one variable, we use the terms “affine mapping” and “linear function” as synonyms.
\[ v^4 = (1, 1, 0, 0), \ v^5 = (2, 0, 0, 0), \ v^6 = (1, 1, -1, 0), \ v^7 = (1, 0, 0, 0). \]

By \( \mathbb{Z}_0^4 \) we denote the set of all vectors in \( \mathbb{R}^4 \) with pairwise different integer coordinates, and by \([1, N]^4\) the cube in \( \mathbb{R}^4 \) formed by all vectors \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \) satisfying \( 1 \leq x_j \leq N, 1 \leq j \leq 4 \).

The following lemma is the key assertion to the proof of the theorem.

**Lemma 1** (the main Lemma). Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a continuous mapping. Let \( S \subset \mathbb{R} \) be a finite set and \( N \geq 4 \) be an integer. Suppose that for every \( x, y \in \mathbb{R} \) there exist a vector \( v = (v_1, v_2, v_3, v_4) \in V \) and a vector \( k = (k_1, k_2, k_3, k_4) \in [1, N]^4 \cap \mathbb{Z}_0^4 \) such that

\[
\sum_{j=1}^{4} v_j \varphi(x + k_j y) \in S.
\]

Then \( E(\varphi) \) has Lebesgue measure zero.

We postpone the proof of this lemma to the next section; here we show how to derive the theorem from it.

Note that if \( E(\varphi) \) has nonzero measure and \( N \geq 4 \) is an integer, then, setting \( S = \{0\} \) and applying Lemma 1, we can find \( x, y \in \mathbb{R} \), such that

\[
\sum_{j=1}^{4} v_j \varphi(x + k_j y) \neq 0
\]

for all vectors \( v = (v_1, v_2, v_3, v_4) \in V \) and \( k = (k_1, k_2, k_3, k_4) \in [1, N]^4 \cap \mathbb{Z}_0^4 \).

Since the family \( V \) contains the vectors \( v^1, v^2 \) and \( v^3 \) (see (2)), it follows that the numbers \( \varphi(x + ky), \ k = 1, 2, \ldots, N \), are pairwise distinct and form a set with distinct sums of pairs. Thus, the arithmetic progression \( A = \{x + ky, \ k = 1, 2, \ldots, N\} \) of length \( N \) has the property that the image \( B = \varphi(A) \) of \( A \) under \( \varphi \) is a set with distinct sums of pairs, and \( \varphi \) maps bijectively \( A \) onto \( B \). The construction below allows us to accumulate this effect.

For an arbitrary set \( M \subseteq \mathbb{R} \), we define the set \( \gamma(M) \) by \(^2\)

\[
\gamma(M) = \{0\} \cup M \cup (M + M) \cup (M - M) \cup (M + M - M).
\]

By the assumption of the theorem, \( E(\varphi) \) has nonzero measure. Let \( N \geq 4 \). Let \( M \subseteq \mathbb{R} \) be a finite set. Using Lemma 1 with \( S = \gamma(M) \), we can find \( x, y \in \mathbb{R} \) such that

\[
\sum_{j=1}^{4} v_j \varphi(x + k_j y) \notin \gamma(M)
\]

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\(^2\)Throughout the paper we use the following notation: if \( E, F \subseteq \mathbb{R} \) and \( a \in \mathbb{R} \), then \( E + F = \{x + y : x \in E, y \in F\} \) and \( aE = \{ax : x \in E\} \).

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for all vectors $v = (v_1, v_2, v_3, v_4) \in V$ and $k = (k_1, k_2, k_3, k_4) \in [1, N]^4 \cap \mathbb{Z}_0^4$. This implies that the arithmetic progression $A = \{x + ky, k = 1, 2, \ldots, N\}$ of length $N$ has the following properties:

(a) $\varphi$ maps bijectively $A$ onto $B = \varphi(A)$;

(b) $B \cap M = \emptyset$;

(c) if $M$ is a set with distinct sums of pairs, then so is $B \cup M$.

To see this we essentially repeat the above argument. Indeed, since $0 \in \gamma(M)$ and $v^3 \in V$, we obtain (a). Since $M \subseteq \gamma(M)$ and $v^7 \in V$, we obtain (b). One can easily verify assertion (c) as well; for instance, the relation $b_1 + b_2 = m_2 + m_3$ cannot hold for $b_1, b_2 \in B$ and $m_1, m_2 \in M$, because $M + M \subseteq \gamma(M)$ and $V$ contains $v^4$ and $v^5$. The further routine verification is left to the reader.

Using this observation, we inductively construct a family of arithmetic progressions $A_n, n = 4, 5, \ldots$, where $A_n$ is of length $n$, as follows. Applying Lemma 1 with $S = \{0\}$, we find an arithmetic progression $A_4$ of length 4 such that $\varphi$ maps bijectively $A_4$ onto a set $B_4$ with distinct sums of pairs. Suppose that arithmetic progressions $A_n$, where $A_n$ is of length $n$, are already constructed for $n = 4, 5, \ldots, N$. Setting $M = B_4 \cup B_5 \cup \ldots \cup B_N$, we find an arithmetic progression $A_{N+1}$ of length $N + 1$ such that (see (a), (b), and (c))

(i) $\varphi$ maps bijectively $A_{N+1}$ onto $B_{N+1}$;

(ii) $B_{N+1}$ does not intersect $B_4 \cup B_5 \cup \ldots \cup B_N$;

(iii) $B_4 \cup B_5 \cup \ldots \cup B_N \cup B_{N+1}$ is a set with distinct sums of pairs.

Proceeding, we obtain $A_n$ and $B_n$ for all $n = 4, 5, \ldots$. From the construction (see (i)−(iii)) it follows that $\varphi$ maps bijectively $\bigcup_{n=4}^{\infty} A_n$ onto $\bigcup_{n=4}^{\infty} B_n$, and the former set contains arbitrarily long arithmetic progressions whereas the latter is a set with distinct sums of pairs.

4. Proof of the main Lemma

To prove the main Lemma (Lemma 1) we need Lemmas 2 and 3 below.

Given a vector $k = (k_1, k_2, k_3, k_4) \in \mathbb{Z}^4$, a vector $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$, and a number $s \in \mathbb{R}$, we define a set $Q(k, v, s)$ by

\begin{equation}
Q(k, v, s) = \left\{ (x, y) \in \mathbb{R}^2 : \sum_{j=1}^{4} v_j \varphi(x + k_j y) = s \right\}.
\end{equation}

By an interval in $\mathbb{R}$ we always mean a nonempty interval.

**Lemma 2.** Suppose that the assumptions of Lemma 1 hold. Then, for any intervals $I, J \subseteq \mathbb{R}$, there exist intervals $I' \subseteq I$ and $J' \subseteq J$, vectors $k \in [1, N]^4 \cap \mathbb{Z}_0^4$ and $v \in V$, and a number $s \in S$ such that $I' \times J' \subseteq Q(k, v, s)$.
One easily proves this lemma as follows. Under the assumptions we have

$$\bigcup_{k \in [1, N] \cap \mathbb{Z}^4, v \in V, s \in S} Q(k, v, s) = \mathbb{R}^2.$$ 

So,

$$\bigcup_{k \in [1, N] \cap \mathbb{Z}^4, v \in V, s \in S} Q(k, v, s) \cap (I \times J) = I \times J.$$ 

Since \(\varphi\) is continuous, all sets \(Q(k, v, s)\) are closed. Without loss of generality we can assume that \(I\) and \(J\) are closed. Applying the Baire category theorem we complete the proof of Lemma 2.

Given a set \(F \subseteq \mathbb{R}\), we denote its closure by \(\overline{F}\). If \(F\) is measurable, then we use \(F^\circ\) to denote the set of points of density of \(F\), i.e., the set of all \(x \in F\) satisfying

$$\frac{|F \cap I(x, \delta)|}{|I(x, \delta)|} \to 1 \quad \text{as} \quad \delta \to +0.$$ 

Here \(|X|\) stands for the (Lebesgue) measure of a measurable set \(X \subseteq \mathbb{R}\) and \(I(x, \delta) = (x - \delta, x + \delta)\). As is known, almost all points of a measurable set are its points of density.

**Lemma 3.** Let \(E_1\) and \(E_2\) be measurable sets in \(\mathbb{R}\) such that \(\overline{E_1^\circ} \cap \overline{E_2^\circ}\) has nonzero Lebesgue measure. Then, for any positive integer \(N\) and any partition of the set \(\{1, 2, \ldots, N\}\) into two disjoint sets \(E_1\) and \(E_2\), there exists an arithmetic progression \(t_k = x + ky, k = 1, 2, \ldots, N\), of length \(N\) such that \(t_k \in E_1^\circ\) for \(k \in E_1\) and \(t_k \in E_2^\circ\) for \(k \in E_2\).

This combinatorial lemma was obtained by the author and A. Olevskii in [9, Lemma 1] (see also [8]). It plays one of the key roles in the proof of the \(M_p\)-version of the Beurling–Helson theorem (see Section 5, Remark 5). In a slightly weaker form it was obtained earlier by the same authors in the work [7] on idempotent Fourier multipliers.

We now proceed directly to the proof of Lemma 1. The proof is split into two steps.

**Step 1.** First, we obtain a weaker result; namely, we show that, under the assumptions of Lemma 1, the set \(E(\varphi)\) is nowhere dense.

Let \(\Delta = (a, b)\) be an arbitrary interval in \(\mathbb{R}\). Let us show that it contains a subinterval on which \(\varphi\) is linear; this will prove our claim. To this end consider the following two intervals \(I\) and \(J\):

$$I = \left(a, \frac{a+b}{2}\right), \quad J = \left(0, \frac{b-a}{2N}\right).$$

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Using Lemma 2, we find vectors $v = (v_1, v_2, v_3, v_4) \in V$ and $k = (k_1, k_2, k_3, k_4) 
 \in [1, N]^4 \cap \mathbb{Z}_4^4$, a number $s \in S$, and intervals $I' \subseteq I$ and $J' \subseteq J$ such that $I' \times J' \subseteq Q(k, v, s)$, i.e.,

$$\sum_{j=1}^{4} v_j \varphi(x + k_j y) = s \quad \text{for all} \ (x, y) \in I' \times J'.$$

Choose an infinitely differentiable nonnegative function $\rho$ on $\mathbb{R}$ so that $\text{supp} \rho \subseteq [-1, 1]$ and $\int_{\mathbb{R}} \rho(x) dx = 1$. For each $\varepsilon > 0$, we set $\rho_\varepsilon(t) = \frac{1}{\varepsilon} \rho\left(\frac{t}{\varepsilon}\right)$. We have $\text{supp} \rho_\varepsilon \subseteq [-\varepsilon, \varepsilon]$. Consider the convolution $\varphi_\varepsilon = \varphi \ast \rho_\varepsilon$:

$$\varphi_\varepsilon(u) = \int_{\mathbb{R}} \varphi(u - t) \rho_\varepsilon(t) dt.$$  

Obviously $\varphi_\varepsilon$ is infinitely differentiable. Note also that since $\varphi$ is continuous, it follows that $\varphi_\varepsilon$ converges pointwise to $\varphi$ as $\varepsilon \rightarrow +0$.

Let $I''$ be the interval concentric with $I'$ and of length three times smaller than that of $I'$. We set $\varepsilon_0$ equal to the length of $I''$. For any $t$ with $|t| < \varepsilon_0$ and any point $(x, y) \in I'' \times J'$, we have $(x - t, y) \in I' \times J'$; therefore (see (5)),

$$\sum_{j=1}^{4} v_j \varphi(x - t + k_j y) = s.$$

Thus, we see that

$$\sum_{j=1}^{4} v_j \varphi_\varepsilon(x + k_j y) = s \quad \text{for all} \ (x, y) \in I'' \times J' \text{ and } 0 < \varepsilon < \varepsilon_0.$$

Differentiating this relation three times, namely, taking the derivatives $\frac{\partial^3}{\partial x^q \partial y^{3-q}}$, $0 \leq q \leq 3$, we obtain

$$\sum_{j=1}^{4} k_j^q v_j \varphi_\varepsilon'''(x + k_j y) = 0, \quad 0 \leq q \leq 3, \ x \in I'', \ y \in J'.$$

Since $k_j$, $1 \leq j \leq 4$, are pairwise different (recall that $k \in \mathbb{Z}_4^4$), it follows that the matrix $\{k_j^q\}_{1 \leq j \leq 4, 0 \leq q \leq 3}$ has nonzero determinant. Since not all $v_j$ vanish, we see that there exists a $j_0$ (we can take, e.g., $j_0 = 1$, see (2)) such that $\varphi_\varepsilon'''(x + k_{j_0} y) = 0$ for all $x \in I''$ and $y \in J'$. Thus, if $0 < \varepsilon < \varepsilon_0$, then $\varphi_\varepsilon'''(t) = 0$ for all $t \in I'' + k_{j_0} J'$. Hence, $\varphi_\varepsilon$ is a polynomial of degree at most 2 on the interval $\tilde{\Delta} = I'' + k_{j_0} J'$. Letting $\varepsilon \rightarrow +0$, we see that $\varphi$ coincides with a polynomial $P$ of degree at most 2 on $\tilde{\Delta}$. Since $k \in [1, N]^4$, we have $1 \leq k_{j_0} \leq N$, whence $\tilde{\Delta} = I'' + k_{j_0} J' \subseteq I + k_{j_0} J \subseteq \Delta$ (see (4)).
Let us show that the degree of the polynomial $P$ is strictly less than 2. We repeat part of the argument used above, this time for the interval $\tilde{\Delta} = (\tilde{a}, \tilde{b})$ instead of $\Delta = (a, b)$. Namely, we consider the following intervals $\tilde{I}$ and $\tilde{J}$:

$$\tilde{I} = (\tilde{a}, \frac{\tilde{a} + \tilde{b}}{2}), \quad \tilde{J} = (0, \frac{\tilde{b} - \tilde{a}}{2N}).$$

Using Lemma 2, we find vectors $v = (v_1, v_2, v_3, v_4) \in V$, $k = (k_1, k_2, k_3, k_4) \in [1, N]^4 \cap \mathbb{Z}_4$, a number $s \in S$, and intervals $\tilde{I}' \subseteq \tilde{I}$ and $\tilde{J}' \subseteq \tilde{J}$ such that $\tilde{I}' \times \tilde{J}' \subseteq Q(k, v, s)$, i.e.,

$$\sum_{j=0}^{4} v_j \varphi(x + k_j y) = s \quad \text{for all } (x, y) \in \tilde{I}' \times \tilde{J}'. \tag{6}$$

Since all points $x + k_j y$ with $x \in \tilde{I}'$, $y \in \tilde{J}'$ and $j = 1, 2, 3, 4$, are in $\tilde{\Delta}$ and since $\varphi$ coincides with $P$ on $\tilde{\Delta}$, from (6) it follows that

$$\sum_{j=0}^{4} v_j P(x + k_j y) = s \quad \text{for all } (x, y) \in \tilde{I}' \times \tilde{J}'. \tag{7}$$

Assuming that the degree of $P$ equals 2, we have $P'' \equiv \text{const} \neq 0$. Twice differentiating relation (7), that is, taking the derivatives $\frac{\partial^2}{\partial x^q \partial y}$, $0 \leq q \leq 2$, we see that

$$\sum_{j=0}^{4} k_j^q v_j = 0, \quad 0 \leq q \leq 2.$$ 

Thus, the vector $v$ belongs to the kernel of the matrix

$$M(k) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ k_1 & k_2 & k_3 & k_4 \\ k_2^2 & k_2^2 & k_3^2 & k_4^2 \end{pmatrix}.$$ 

It remains to observe that this is impossible: none of the vectors of the family $V$ (see (2)) belongs to $\ker M(k)$ whenever $k_1, k_2, k_3, k_4$ are pairwise different positive integers. The verification is left to the reader. Thus we see that $P$ is a linear function. This completes the proof of our claim that $E(\varphi)$ is nowhere dense.

**Step 2.** Now, we show that, under the assumptions of Lemma 1, the set $E(\varphi)$ has measure zero. The core of this step is the combinatorial Lemma 3.

Let $\Omega$ denote the family of all open intervals complementary to $E(\varphi)$, i.e., the family of connected components of the complement $\mathbb{R} \setminus E(\varphi)$ (recall
that the set $E(\varphi)$ is closed). For every interval $I \in \Omega$, we have $\varphi(t) = P_I(t)$, $t \in I$, where $P_I$ is a linear function.

Now, suppose that, contrary to the assertion of Lemma 1, the set $E(\varphi)$ has nonzero measure. Let $E$ be the set of accumulation points of $E(\varphi)$. Under assumption that $E(\varphi)$ has nonzero measure, the same holds for $E$.

Note that if $x_0 \in E$, then (since $E(\varphi)$ is closed and nowhere dense) any neighborhood of $x_0$ contains infinitely many intervals $I \in \Omega$ with the property that the corresponding functions $P_I$ are pairwise different. Indeed, otherwise, the point $(x_0, \varphi(x_0)) \in \mathbb{R}^2$ has a neighborhood $J$ such that the piece $J \cap G$ of the graph $G$ of $\varphi$ is contained in a finite union of straight lines, which is possible only if $x_0$ is an isolated point of $E(\varphi)$ or does not belong to $E(\varphi)$ at all.

For each $n = 1, 2, \ldots$, consider the open $1/n$-neighborhood of the set $E$. Let $\Delta^n_k$, $k = 1, 2, \ldots$, be the family of all connected components of this neighborhood. We renumber the intervals $\Delta^n_k$, $n = 1, 2, \ldots$, $k = 1, 2, \ldots$, as $\Delta_1$, $\Delta_2$, $\ldots$. Each of the intervals $\Delta_j$ contains an accumulation point of the set $E(\varphi)$. Hence, it contains infinitely many intervals $I$ complementary to $E(\varphi)$ with the property that the corresponding functions $P_I$ are pairwise different.

We choose an interval $I_1 \in \Omega$ contained in $\Delta_1$. Having chosen intervals $I_1, I_2, \ldots, I_j \in \Omega$ contained in $\Delta_1, \Delta_2, \ldots, \Delta_j$, respectively, we choose an interval $I_{j+1} \in \Omega$ so that $I_{j+1} \subseteq \Delta_{j+1}$ and none of nontrivial linear combinations of $P_{I_{j+1}}, P_{I_1}, P_{I_2}, \ldots, P_{I_j}$ with coefficients $0, \pm 1, \pm 2$ can be identically equal to $s$ whenever $s \in S$ (by a nontrivial linear combination we mean a combination not all of whose coefficients are zero). Clearly, such an interval always exists, because $S$ is finite and there are only finitely many linear combinations of the functions $P_{I_1}, P_{I_2}, \ldots, P_{I_j}$ with coefficients $0, \pm 1, \pm 2, \pm 1/2$. Proceeding by induction, we obtain intervals $I_m \in \Omega$, $m = 1, 2, \ldots$, with the following two properties: firstly, the intervals $I_m$ accumulate to $E$, i.e., any neighborhood of any point of $E$ contains an interval that belongs to the family $\{I_m\}$, and, secondly, no nontrivial linear combination of the corresponding linear functions $P_{I_m}$, $m = 1, 2, \ldots$, with coefficients $0, \pm 1, \pm 2$ can be identically equal to $s$ whenever $s \in S$.

Denote the union of the intervals $I_m$, $m = 1, 2, \ldots$, by $U$. Observe that the sets $E^\circ$ and $U^\circ$ (the sets of points of density of $E$ and $U$, respectively) satisfy $E^\circ \cap U^\circ \supseteq E^\circ$. Hence, $E^\circ \cap U^\circ$ has nonzero measure. Using Lemma 3, we find an arithmetic progression $t_k = a + kd, k = 1, 2, \ldots, 2N$, of length $2N$ such that its terms with odd indices belong to $E$ and those with even indices belong to $U$. Consider only the terms with even indices. Clearly they form a progression $x_0 + k y_0$, $k = 1, 2, \ldots, N$, of length $N$ with the property that no two different terms of this progression belong to the same interval of the family $\{I_m\}$. We consider now only those intervals of the family $\{I_m\}$ which contain a point of this progression. For $k = 1, 2, \ldots, N$ denote the interval of the family $\{I_m\}$ that contains the point $x_0 + k y_0$ by $H_k$. Let $P_k$, $k = 1, 2, \ldots, N$, be the corresponding linear functions, i.e., $P_k = P_{H_k}$.
Clearly, if $I$ and $J$ are sufficiently small neighborhoods of the points $x_0$ and $y_0$, respectively, then for all $x \in I$ and $y \in J$ we have $x + ky \in H_k$, $k = 1, 2, \ldots, N$. We fix these $I$ and $J$.

Again applying Lemma 2, we find vectors $v = (v_1, v_2, v_3, v_4) \in V$ and $k = (k_1, k_2, k_3, k_4) \in [1, N]^4 \cap \mathbb{Z}_0^4$, a number $s \in S$, and intervals $I' \subseteq I$ and $J' \subseteq J$ such that

$$\sum_{j=1}^{4} v_j \varphi(x + k_j y) = s \quad \text{for all } (x, y) \in I' \times J'.$$

This implies

$$\sum_{j=1}^{4} v_j P_{k_j}(x + k_j y) = s \quad \text{for all } (x, y) \in I' \times J'.$$

Clearly, if an affine function of two variables identically equals $s$ on a rectangle in $\mathbb{R}^2$, then it identically equals $s$ in the entire plane $\mathbb{R}^2$. Thus,

$$\sum_{j=1}^{4} v_j P_{k_j}(x + k_j y) = s, \quad (x, y) \in \mathbb{R}^2.$$

Setting $y = 0$, we see that

$$\sum_{j=1}^{4} v_j P_{k_j}(x) = s, \quad x \in \mathbb{R},$$

which is impossible because the coordinates of each vector $v \in V$ are $0, \pm1$ or $\pm2$ and not all of them are zero (see (2)). This proves Lemma 1 and, thereby, the theorem.

5. Remarks

1. The theorem proved in this paper admits a generalisation for mappings $\varphi : I \to \mathbb{R}$, where $I$ is an interval in $\mathbb{R}$. Indeed, without loss of generality, we can assume that $I$ is a closed interval, $I = [a, b]$. It suffices to consider a continuous extension of $\varphi$ constant on the rays $(-\infty, a]$ and $(b, +\infty)$, and apply the original version of the theorem to the extension.

2. The following assertion on affine copies of $\mathbb{Z}$ holds: If $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous nowhere linear (that is, nonlinear on every interval) mapping, then there exists an affine copy $a\mathbb{Z} + b$ of $\mathbb{Z}$ such that $\varphi$ maps bijectively $a\mathbb{Z} + b$ onto a certain set with distinct sums of pairs. This can be proved by a modification of the first step in the proof of the main Lemma (without
use of the second step based on Lemma 3). Indeed, let $V'$ be a family of the first three vectors $v^1, v^2, v^3$ defined in (2). Assuming that the assertion on copies is not true, we have (see (3))

$$\bigcup_{k \in \mathbb{Z}_0^4; v \in V'} Q(k, v, 0) = \mathbb{R}^2.$$ 

Using categorical considerations (similar to those in the proof of Lemma 2) we obtain two intervals $I, J \subseteq \mathbb{R}$, a vector $k = (k_1, k_2, k_3, k_4) \in \mathbb{Z}_0^4$, and a vector $v = (v_1, v_2, v_3, v_4) \in V'$ such that

$$\sum_{j=1}^{4} v_j \varphi(x + k_j y) = 0 \quad \text{for all } (x, y) \in I \times J.$$ 

Choosing an interval $I'$ concentric with $I$ and of strictly smaller length than that of $I$, and repeating the argument of the first step in the proof of the main Lemma, we obtain that $\varphi$ is linear on $I' + k_{j_0} J$, which contradicts the assumption on $\varphi$.

3. Consider the algebra $B(\mathbb{R})$ of Fourier transforms of measures on $\mathbb{R}$. According to the well-known Beurling–Helson theorem [2], if $\varphi$ is a real continuous function on $\mathbb{R}$ such that

$$(8) \quad \|e^{int}\varphi\|_{B(\mathbb{R})} = O(1), \quad n \in \mathbb{Z},$$

then $\varphi$ is linear. This theorem has a version for the Wiener algebra $A(\mathbb{T})$ of absolutely convergent Fourier series on the circle $\mathbb{T}$, which is due to Kahane [5, Ch. VI]. In particular, the Beurling–Helson theorem implies that only trivial (i.e., linear) changes of variable are allowable in $B(\mathbb{R})$ (the same holds for $A(\mathbb{T})$). An essential point in the proof of the Beurling–Helson theorem is the observation that condition (8) implies that $\varphi$ cannot map bijectively long arithmetic progressions onto sets which are mod $2\pi$-independent over integers. (Subsequently, Kahane [6] gave a proof based on completely different argument.) So, the question of how thin the images of arithmetic progressions under continuous mappings can be traces back to Beurling and Helson.

4. A closed set $E \subseteq \mathbb{R}$ is called a Helson set if any continuous function on $E$ vanishing at infinity is the restriction to $E$ of the Fourier transform of a function in $L^1(\mathbb{R})$. Equivalently (see [4]), $E$ is a Helson set if, given a measure $\mu$ on $\mathbb{R}$, we have $\|\mu\|_{M(\mathbb{R})} \leq c \|\hat{\mu}\|_{L^\infty(\mathbb{R})}$, where $\|\mu\|_{M(\mathbb{R})}$ is the variation of $\mu$ and $c > 0$ does not depend on $\mu$. In particular, taking for $\mu$ a linear combination of point masses, we see that

$$\sum_{k=1}^{N} |c_k| \leq c \left\| \sum_{k=1}^{N} c_k e^{i\lambda_k t} \right\|_{L^\infty(\mathbb{R})}$$

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for any $N$, any pairwise different $\lambda_1, \lambda_2, \ldots, \lambda_N \in E$, and any (complex) numbers $c_k, k = 1, 2, \ldots, N$. Countable Helson sets are called Sidon sets. In [3] Graham considered self-mappings of $\mathbb{R}$ which take Sidon sets to Sidon sets. He called a mapping $\varphi$ countably piecewise affine if the set of $x$ such that $\varphi$ is affine in a neighborhood of $x$ is dense in $\mathbb{R}$ (which in our terms is the same as to say that $E(\varphi)$ is nowhere dense). Graham showed that if a self-homeomorphism $h$ of $\mathbb{R}$ has the property that the image $h(E)$ of any Sidon set $E$ is a Sidon set, then $h$ is countably piecewise affine with a finite number of slopes. The use of categorical argument and subsequent use of convolution in the proof of the Theorem of the present paper were suggested by this work of Graham. We also note that Graham conjectured that his result on Sidon sets can be supplemented by the assertion that the set of nonlinearity of $h$ has measure zero. It is very plausible that this is indeed the case and perhaps it can be proved by an argument similar to that used at the second step of the proof of Lemma 1 of the present paper.

5. Mappings $\varphi$ whose sets of nonlinearity have measure zero also appeared in relation with analogues of the Beurling–Helson theorem for the algebras $M_p(\mathbb{R})$ of Fourier multipliers (it is well-known that $M_1(\mathbb{R}) = M_\infty(\mathbb{R}) = B(\mathbb{R})$ and the corresponding norms coincide). As it turned out [9] (see also [8]), if $\varphi: \mathbb{R} \to [0, 2\pi]$ is a measurable function such that $\|e^{in\varphi}\|_{M_p(\mathbb{R})} = O(1), n \in \mathbb{Z}$, for some $p, 1 < p < \infty$, $p \neq 2$, then $E(\varphi)$ has measure zero and the set of distinct slopes of $\varphi$ is finite.

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