A necessary condition for generic rigidity of bar-and-joint frameworks in $d$-space

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Abstract

A graph $G = (V, E)$ is $d$-sparse if each subset $X \subseteq V$ with $|X| \geq d$ induces at most $d|X| - \binom{d+1}{2}$ edges in $G$. Maxwell showed in 1864 that a necessary condition for a generic bar-and-joint framework with at least $d + 1$ vertices to be rigid in $\mathbb{R}^d$ is that $G$ should have a $d$-sparse subgraph with $d|X| - \binom{d+1}{2}$ edges. This necessary condition is also sufficient when $d = 1, 2$ but not when $d \geq 3$. Cheng and Sitharam strengthened Maxwell’s condition by showing that every maximal $d$-sparse subgraph of $G$ should have $d|X| - \binom{d+1}{2}$ edges when $d = 3$. We extend their result to all $d \leq 11$.

1 Introduction

A $d$-dimensional (bar-and-joint) framework is a pair $(G, p)$ where $G = (V, E)$ is a graph and $p : V \rightarrow \mathbb{R}^d$. It is a long standing open problem to determine when a given bar-and-joint framework is rigid i.e. every continuous motion of the points $p(v)$ which preserves the distances $\|p(u) - p(v)\|$ for all $uv \in E$ must also preserve the distances $\|p(u) - p(v)\|$ for all $u, v \in V$. It is not difficult to see that a 1-dimensional framework $(G, p)$ is rigid if and only if the graph $G$ is connected. Abbot [1] showed that the problem of determining rigidity is NP-hard for all $d \geq 2$ but the problem becomes more tractable if we assume that the framework is generic i.e. there are no algebraic dependencies between the coordinates of the points $p(v)$, $v \in V$.

Given a graph $G = (V, E)$, we can define a $|E| \times d|V|$ matrix, the $d$-dimensional rigidity matrix $R_d(G)$, whose entries are linear combinations of indeterminates representing the coordinates of the points $p(v)$, in such a way that a generic framework $(G, p)$ with at least $d + 1$ vertices is rigid if and only if the rank $r_d(G)$ of $R_d(G)$ is equal to $d|V| - \binom{d+1}{2}$. This naturally gives rise to a matroid on $E$, the $d$-dimensional rigidity matroid $R_d(G)$ in which a set of edges $F \subseteq E$ is independent if and only if the corresponding rows of $R_d(G)$ are linearly independent. We refer the reader to [10] for a precise definition of

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the rigidity matrix, the rigidity matroid, and other information on the topic of combinatorial rigidity.

Pollaczek-Geiringer \[9\] and subsequently Laman \[6\] characterized when a 2-dimensional generic framework is rigid (see also Lovász and Yemini \[7\]). Their characterization is based on the following concept. We say that a graph \( G = (V, E) \) is \( d \)-sparse if each \( X \subseteq V \) with \( |X| \geq d + 1 \) induces at most \( d|X| - \binom{d+1}{2} \) edges of \( G \). Maxwell \[8\] showed that being \( d \)-sparse is a necessary condition for the rows of \( R_d(G) \) to be linearly independent. Pollaczek-Geiringer and Laman showed that this condition is also sufficient when \( d = 2 \) and deduced that a 2-dimensional generic framework \( (G, p) \) is rigid if and only if it has a 2-sparse subgraph with \( 2|V| - 3 \) edges. Since every independent set of edges in \( R_2(G) \) can be extended to a base of \( R_2(G) \), Laman’s theorem implies that every maximal 2-sparse subgraph of \( G \) has the same number of edges.

It is known that the condition that \( H \) is a \( d \)-sparse subgraph of \( G \) is not sufficient for the edges of \( H \) to be independent in \( R_d(G) \) when \( d \geq 3 \). Indeed it is not even true that all maximal \( d \)-sparse subgraphs of \( G \) have the same number of edges when \( d \geq 3 \). On the other hand, Cheng and Sitharam \[3\] have shown that the number of edges in any maximal \( d \)-sparse subgraph of \( G \) does at least give an upper bound on \( r_d(G) \) when \( d = 3 \). The purpose of this paper is to prove a result, Theorem 3.3 below, which extends Cheng and Sitharam’s theorem to all values of \( d \leq 11 \).

## 2 Sparse subgraphs

Let \( G = (V, E) \) be a graph and \( d \geq 1 \) be an integer. For \( X \subseteq V \) we use \( E_G(X) \) to denote the set, and \( i_G(X) \) the number, of edges of \( G \) joining pairs of vertices of \( X \). We simplify these to \( E(X) \) and \( i(X) \) when it is obvious to which graph we are referring. We may rewrite the condition for \( G \) to be \( d \)-sparse as \( i(X) \leq d|X| - \binom{d+1}{2} \) for all \( X \subseteq V \) with \( |X| \geq d \). (Note that if \( |X| \in \{d, d+1\} \) then we have \( i(X) \leq \binom{d+1}{2} = d|X| - \binom{d+1}{2} \) and the inequality holds trivially.) We will use the fact that the function \( i : 2^V \to \mathbb{Z} \) is supermodular i.e. \( i(X) + i(Y) \leq i(X \cup Y) + i(X \cap Y) \) for all \( X, Y \subseteq V \).

A subgraph \( H = (U, F) \) of a \( d \)-sparse graph \( G \) is \( d \)-critical if either \( |U| = 2 \) and \( |F| = 1 \), or \( |U| \geq d \) and \( |F| = d|X| - \binom{d+1}{2} \). The assumption that \( G \) is \( d \)-sparse implies that every \( d \)-critical subgraph of \( G \) is an induced subgraph. A \( d \)-critical component of \( G \) is a \( d \)-critical subgraph which is not properly contained in any other \( d \)-critical subgraph of \( G \).

**Lemma 2.1** Let \( G = (V, E) \) be a \( d \)-sparse graph and \( H_1 = (U_1, F_1), H_2 = (U_2, F_2) \) be distinct \( d \)-critical components of \( G \). Then \( |U_1 \cap U_2| \leq d - 1 \) and, if equality holds, then \( i_G(U_1 \cap U_2) = \binom{d-1}{2} \).

**Proof:** Suppose that \( |U_1 \cap U_2| \geq d - 1 \). When \( |U_1 \cap U_2| \geq d \) we have \( i(U_1 \cap U_2) \leq d|U_1 \cap U_2| - \binom{d+1}{2} \) since \( G \) is \( d \)-sparse. When \( |U_1 \cap U_2| = d - 1 \), we have \( i(U_1 \cap U_2) \leq \binom{d-1}{2} = d|U_1 \cap U_2| - \binom{d+1}{2} + 1 \) trivially. The maximality of \( H_1, H_2 \) and the definition of a \( d \)-critical component imply that \( |U_1|, |U_2| \geq d \), and \( d(|U_1| + |U_2|) - 2\binom{d+1}{2} = i_G(U_1) + i_G(U_2) \leq i_G(U_1 \cup U_2) + i_G(U_1 \cap U_2) \leq
Let \( k, t \) be non-negative integers, \( G = (V, E) \) be a graph and \( \mathcal{X} \) be a family of subsets of \( V \). We say that \( \mathcal{X} \) is \( t \)-thin if every pair of sets in \( \mathcal{X} \) intersect in at most \( t \) vertices. A \( k \)-hinge of \( \mathcal{X} \) is a set of \( k \) vertices which lie in the intersection of at least two sets in \( \mathcal{X} \). A \( k \)-hinge \( U \) of \( \mathcal{X} \) is closed in \( G \) if \( G[U] \) is a complete graph. We use \( \Theta_k(\mathcal{X}) \) to denote the set of all \( k \)-hinges of \( \mathcal{X} \). For \( U \in \Theta_k(\mathcal{X}) \), let \( d_X(U) \) denote the number of sets in \( \mathcal{X} \) which contain \( U \). Note that if \( G \) is \( t \)-thin then \( \Theta_k(\mathcal{X}) = \emptyset \) for all \( k \geq t + 1 \). Note also that \( \Theta_0(\mathcal{X}) = \{\emptyset\} \) and \( d_X(\emptyset) = |\mathcal{X}| \).

**Lemma 2.2** Let \( H = (V, E) \) be a \( d \)-sparse graph, \( \mathcal{X} \) be a family of subsets of \( V \) such that \( H[V_i] \) is \( d \)-critical for all \( V_i \in \mathcal{X} \), and \( W \in \Theta_k(\mathcal{X}) \) for some \( 0 \leq k \leq d - 1 \). Suppose that \( |V_i| \geq d \) for all \( V_i \in \mathcal{X} \) with \( W \subseteq V_i \). Then

\[
(d-k) \sum_{U \in \Theta_{k+1}(\mathcal{X}) \atop W \subset U} (d_X(U)-1) - \sum_{U \in \Theta_{k+2}(\mathcal{X}) \atop W \subset U} (d_X(U)-1) \leq \binom{d+1-k}{2} (d_X(W)-1).
\]

**Proof:** Let \( d_X(W) = t \) and let \( V_1, V_2, \ldots, V_t \) be the sets in \( \mathcal{X} \) which contain \( W \). Let \( H_i = (V_i, E_i) = H[V_i] \) for \( 1 \leq i \leq t \). Let \( H' = \bigcup_{i=1}^{t} H_i \) and put \( H' = (V', E') \). Then

\[
|V'| = \sum_{i=1}^{t} |V_i| - k(t-1) - \sum_{U \in \Theta_{k+1}(\mathcal{X}) \atop W \subset U} (d_X(U)-1) \tag{1}
\]

since, for \( v \in V' \), if \( v \in W \) then \( v \) is counted \( t \) times in \( \sum_{i=1}^{t} |V_i| \), if \( v \in U \setminus W \) for some \( U \in \Theta_{k+1} \) with \( W \subset U \) then \( v \) is counted \( d_X(U) \) times in \( \sum_{i=1}^{t} |V_i| \), and all other vertices of \( V' \) are counted exactly once in \( \sum_{i=1}^{t} |V_i| \).

Similarly,

\[
|E'| \geq \frac{k}{2} (t-1) - \sum_{U \in \Theta_{k+1}(\mathcal{X}) \atop W \subset U} (d_X(U)-1) - \sum_{U \in \Theta_{k+2}(\mathcal{X}) \atop W \subset U} (d_X(U)-1) \tag{2}
\]

since, for \( e = xy \in E' \): if \( x, y \in W \) then \( e \) is counted \( t \) times in \( \sum_{i=1}^{t} |E_i| \) and there are at most \( \binom{k}{2} \) such edges; if \( x \in W \) and \( y \in U \setminus W \) for some \( U \in \Theta_{k+1} \) with \( W \subset U \) then \( e \) is counted \( d_X(U) \) times in \( \sum_{i=1}^{t} |E_i| \) and for each such \( y \) there are at most \( k \) choices for \( x \); if \( x, y \in U \setminus W \) for some \( U \in \Theta_{k+2} \) with \( W \subset U \) then \( e \) is counted \( d_X(U) \) times in \( \sum_{i=1}^{t} |E_i| \), and all other edges of \( E' \) are counted exactly once in \( \sum_{i=1}^{t} |E_i| \).

Since \( H' \subseteq H \), \( H' \) is \( d \)-sparse. Hence \( |E'| \leq d|V'| \left( \frac{d+1}{2} \right) \). We may substitute equations \([1]\) and \([2]\) into this inequality and use the fact that \( |E_i| = d|V_i| - \).
\[ (d+1) \] for all \( 1 \leq i \leq t \) to obtain
\[
(d-k) \sum_{W \subseteq U} (d_X(U) - 1) - \sum_{W \subseteq U} (d_X(U) - 1) \leq \left[ \binom{d+1}{2} + \binom{k}{2} - dk \right] (t-1) \]
\[ = \binom{d+1-k}{2} (t-1). \]

\[ \square \]

**Lemma 2.3** Let \( H = (V,E) \) be a \( d \)-sparse graph, \( \mathcal{X} \) be a family of subsets of \( V \) such that \( H[V_i] \) is \( d \)-critical and \( |V_i| \geq d \) for all \( V_i \in \mathcal{X} \). Put \( a_k = \sum_{U \in \Theta_k(\mathcal{X})} (d_X(U) - 1) \) for \( 0 \leq k \leq d \). Then for all \( 0 \leq k \leq d - 2 \) we have:

(a) \( (d-k)(k+1)a_{k+1} - \binom{k+2}{2} a_{k+2} \leq \binom{d+1-k}{2} a_k; \)

(b) \( (d-k)a_{k+1} - (k+1)a_{k+2} \leq \binom{d+1}{k+2} (|\mathcal{X}| - 1); \)

(c) if \( \mathcal{X} \) is \( (d-1) \)-thin, \( d(d-k)a_{k+1} \leq (k + 2)(d - k - 1) \frac{d+1}{k+2} (|\mathcal{X}| - 1). \)

**Proof:** Part (a) follows by summing the inequality in Lemma 2.2 over all \( W \in \Theta_k \), and using the facts that \( a_0 = |\mathcal{X}| - 1 \). Hence suppose that \( k \geq 1 \). Then (a) gives
\[
2(d-k)a_{k+1} - 2(k+1)a_{k+2} \leq \frac{(d-k+1)(d-k)}{k+1} a_k - k a_{k+2}. \quad (3)
\]
We may also use (a) to obtain
\[
ka_{k+2} \geq \frac{k(d-k)}{k+2} \left( 2a_{k+1} - \frac{d-k+1}{k+1} a_k \right). \quad (4)
\]
Substituting (4) into (3) and using induction we obtain
\[
(d-k)a_{k+1} - (k+1)a_{k+2} \leq \frac{d-k}{k+2} \left[ (d-k+1)a_k - ka_{k+1} \right] \leq \frac{d-k}{k+2} \binom{d+1}{k+1} (|\mathcal{X}| - 1) = \binom{d+1}{k+2} (|\mathcal{X}| - 1). \]
We prove (c) by induction on \( d - k \). When \( d - k = 2 \), (c) follows by putting \( k = d - 2 \) in (b) and using the fact that \( a_d = 0 \) since \( \mathcal{X} \) is \((d - 1)\)-thin. Hence suppose that \( d - k \geq 3 \). Then (b) gives

\[
d(d - k)a_{k+1} \leq d\binom{d+1}{k+2}(|\mathcal{X}| - 1) + d(k + 1)a_{k+2}.
\]

We may now apply induction to \( a_{k+2} \) to obtain

\[
d(d - k)a_{k+1} \leq (k + 2)(d - k - 1)\binom{d+1}{k+2}(|\mathcal{X}| - 1).
\]

\[\blacksquare\]

**Theorem 2.4.** Let \( H = (V, E) \) be a \( d \)-sparse graph, \( \mathcal{X} \) be a \((d - 1)\)-thin family of subsets of \( V \) such that \( H[\mathcal{X}] \) is \( d \)-critical and \(|\mathcal{X}| \geq d \) for all \( X \in \mathcal{X} \). For each \( X \in \mathcal{X} \) let \( \theta_k(X) \) be the number of \( k \)-hinges of \( X \) contained in \( X \). Then:

(a) \( \theta_1(X) \leq 2d - 1 \) for some \( X \in \mathcal{X} \);

(b) \( \theta_2(X) \leq (d - 2)(d + 1) - 1 \) for some \( X \in \mathcal{X} \);

(c) \( \theta_{d-1}(X) \leq d \) for some \( X \in \mathcal{X} \).

**Proof:**

We first prove (a). Putting \( k = 0 \) in Lemma 2.3(c) we obtain

\[
d \sum_{U \in \Theta_1(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) \leq (d - 1)(d + 1)(|\mathcal{X}| - 1).
\]

Since \( d_{\mathcal{X}}(U) \geq 2 \) for all \( U \in \Theta_1(\mathcal{X}) \) we have \( d_{\mathcal{X}}(U) - 1 \geq d_{\mathcal{X}}(U)/2 \) and hence (5) gives

\[
\sum_{U \in \Theta_1(\mathcal{X})} d_{\mathcal{X}}(U) < 2d|\mathcal{X}|.
\]

This tells us that the average number of 1-hinges in a set in \( \mathcal{X} \) is strictly less than \( 2d \).

We next prove (b). Putting \( k = 1 \) in Lemma 2.3(c) we obtain

\[
\sum_{U \in \Theta_2(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) \leq (d - 2)(d + 1)(|\mathcal{X}| - 1)/2.
\]

We can now proceed as in (a).

Finally we prove (c). Putting \( k = d - 2 \) in Lemma 2.3(c) gives

\[
2 \sum_{U \in \Theta_{d-1}(\mathcal{X})} (d_{\mathcal{X}}(U) - 1) \leq (d + 1)(|\mathcal{X}| - 1).
\]

We can now proceed as in (a).

\[\blacksquare\]

The bounds given in Theorem 2.4 (a), (b) are close to being best possible. To see this consider the graph \( H = H_1 \cup H_2 \cup \ldots \cup H_m \) where \( H_i = (V_i, E_i) \) is
Proof: The definition of a $d$-critical, $H_i \cap H_j = K_{d-1}$ for $i - j \equiv \pm 1 \mod m$ and otherwise $H_i \cap H_j = \emptyset$. Then $H$ is $d$-sparse when $m$ is sufficiently large, $\mathcal{X} = \{V_1, V_2, \ldots, V_m\}$ is $(d-1)$-thin and we have $\theta_1(V_i) = 2d - 2$ and $\theta_2(V_i) = (d - 1)(d - 2)$ for all $V_i \in \mathcal{X}$. We do not know whether (e) is close to best possible for large $d$. It is conceivable that there always exists a set $X \in \mathcal{X}$ with $\theta_{d-1}(X) \leq 2$.

3 Main result

In order to prove our main theorem we will need the following result from [4].

Lemma 3.1 Let $G = (V, E)$ be a graph such that $E$ is a non-rigid circuit in $R_d(G)$. Then $|E| \geq d(d + 9)/2$. \hfill \blacksquare

Let $G = (V, E)$ be a graph and $\mathcal{X}$ be a family of subsets of $V$. We say that $\mathcal{X}$ is a cover of $G$ if every set in $\mathcal{X}$ contains at least two vertices, and every edge of $G$ is induced by at least one set in $\mathcal{X}$.

Lemma 3.2 Let $G = (V, E)$ be a graph, $H = (V, F)$ be a maximal $d$-sparse subgraph of $G$, and $H_1, H_2, \ldots, H_m$ be the $d$-critical components of $H$. Let $X_i$ be the vertex set of $H_i$ for $1 \leq i \leq m$. Then $\mathcal{X} = \{X_1, X_2, \ldots, X_m\}$ is a $(d - 1)$-thin cover of $G$ and each $(d - 1)$-hinge of $\mathcal{X}$ is closed in $H$.

Proof: The definition of a $d$-critical subgraph implies that each $H_i$ has at least two vertices and that every edge of $H$ belongs to at least one $H_i$. Thus $\mathcal{X}$ is a cover of $H$. To see that $\mathcal{X}$ also covers $G$ we choose $e = uv \in E \setminus F$. The maximality of $H$ implies that $H + e$ is not $d$-sparse. Hence $\{u, v\}$ is contained in some $d$-critical subgraph of $H$. Thus $\mathcal{X}$ also covers $G$. The facts that $\mathcal{X}$ is $(d - 1)$-thin and that each $(d - 1)$-hinge of $\mathcal{X}$ is closed follow from Lemma 2.1. \hfill \blacksquare

We refer to the $(d - 1)$-thin cover of $G$ described in Lemma 3.2 as the $H$-critical cover of $G$. Note that the definition of a $d$-critical set implies that each set in the $H$-critical cover has size two or has size at least $d$.

Theorem 3.3 Let $G = (V, E)$ be a graph, $d \leq 11$ be an integer and $H = (V, F)$ be a maximal $d$-sparse subgraph of $G$. Then $r_d(G) \leq |F|$.

Proof: We proceed by contradiction. Suppose the theorem is false and choose a counterexample $(G, H)$ such that $|E|$ is as small as possible. Let $H_1, H_2, \ldots, H_m$ be the $d$-critical components of $H$ where $H_i = (V_i, F_i)$ for $1 \leq i \leq m$. Then $X_0 = \{V_1, V_2, \ldots, V_m\}$ is the $H$-critical cover of $G$.

Choose a cover $\mathcal{X}$ of $G$ such that $\mathcal{X} \subseteq X_0$ and $|\mathcal{X}|$ is as small as possible. Note that $X_0$, and hence also $\mathcal{X}$, are $(d - 1)$-thin. For each $V_i \in \mathcal{X}$, let $F_i^*$ be the set of all edges $uv \in F_i$ such that $\{u, v\}$ is a 2-hinge of $\mathcal{X}$, and let $E_i$ be the set of edges of $G$ induced by $V_i$.

Claim 3.4 If $e = uv \in E$ satisfies $r_d(G) = r_d(G - e)$, then $\{u, v\}$ is a 2-hinge of $\mathcal{X}$.
Proof: First suppose that \( e \in E \setminus F \). Since \( H \) is a maximal \( d \)-sparse subgraph of \( G - e \), the minimality of \( |E| \) gives \( r_d(G - e) \leq |F| \). Since \( r_d(G) = r_d(G - e) \) this gives a contradiction.

Thus we can assume that \( e \in F \). Let \( d_X(e) \) be the number of \( V_i \in \mathcal{X} \) such that \( e \in F_i \). Since \( H - e \) is a \( d \)-sparse subgraph of \( G - e \), we may choose a maximal \( d \)-sparse subgraph \( H' = (V, F') \) of \( G - e \) which contains \( H - e \). Let \( V_i \in \mathcal{X} \). If \( e \notin F_i \), then no edge of \( E_i \setminus F_i \) can be in \( F' \), since \( H_i \) is \( d \)-critical. On the other hand, if \( e \in F_i \), then at most one edge of \( E_i \setminus F_i \) can be in \( F' \), since \( |F_i - e| = d|V_i| - \binom{d+1}{2} - 1 \). These observations imply that \( |F'| \leq |F| - 1 + d_X(e) \). By the minimality of \( |E| \) we have \( r_d(G - e) \leq |F'| \), and hence \( r_d(G) = r_d(G - e) \leq |F| - 1 + d_X(e) \). Combining this with \( r_d(G) > |F| \) gives \( d_X(e) \geq 2 \).

We next show that \( F_i^* \) is dependent in \( \mathcal{R}_d(G) \) for all \( V_i \in \mathcal{X} \). Suppose this is not the case. Then \( E_i \) is independent in \( \mathcal{R}_d(G) \) by Claim 3.3. Thus \( E_i \) can have at most \( d|V_i| - \binom{d+1}{2} \) edges. Since \( H_i \) is \( d \)-critical, this gives \( E_i = F_i \). The minimality of \( \mathcal{X} \) implies that \( F_i \neq F_i^* \) and hence we may choose an edge \( e \in F_i \setminus F_i^* \). Since \( F_i = E_i \), all edges of \( G - e \) which are induced by \( V_i \) are in \( H - e \). Since each \( V_j \in \mathcal{X} - V_i \) induce a \( d \)-critical subgraph of \( H - e \), we conclude that \( H - e \) is a maximal \( d \)-sparse subgraph of \( G - e \). The minimality of \( |E| \) now gives \( r_d(G - e) \leq |F - e| = |F| - 1 \). Since \( e \notin F_i^* \), Claim 3.4 gives \( r_d(G) = r_d(G - e) + 1 \leq |F| - 1 + d_X(e) \). This contradicts the choice of \( G \) and implies that \( F_i^* \) is dependent in \( \mathcal{R}_d(G) \) for all \( V_i \in \mathcal{X} \).

By Theorem 2.4(b) we may choose \( V_i \in \mathcal{X} \) such that \( |F_i^*| \leq (d - 2)(d+1) - 1 \). Since \( F_i^* \) is dependent in \( \mathcal{R}_d(G) \), it contains a circuit of \( \mathcal{R}_d(G) \). This circuit cannot be rigid, since \( H \) is \( d \)-sparse. Lemma 3.1 now gives \( \frac{d^2}{2} - 9d \leq |F_i^*| \leq (d - 2)(d+1) - 1 \) which implies that \( d \geq 12 \).

We have the following immediate corollary.

**Corollary 3.5** Let \( d \leq 11 \) be an integer and \( G = (V, E) \) be a graph with \( |V| \geq d+1 \). If \( G \) is generically rigid in \( \mathbb{R}^d \) then every maximal \( d \)-sparse subgraph of \( G \) has \( d|V| - \binom{d+1}{2} \) edges.

### 4 Closing remarks

1. Given a graph \( G \), let \( s_d(G) \) be the minimum number of edges in a maximal \( d \)-sparse subgraph of \( G \). Theorem 3.3 tells us that \( r_d(G) \leq s_d(G) \) when \( d \leq 11 \). We can use the following operation to construct graphs for which strict inequality holds. Given two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) with \( V_1 \cap V_2 = \{u, v\} \) and \( E_1 \cap E_2 = \{uv\} \), we refer to the graph \( G = G_1 \cup G_2 \) as the parallel connection of \( G_1 \) and \( G_2 \) along the edge uv.

The graph \( G \) obtained by taking the parallel connection of two copies of \( K_5 \) along an edge uv and then deleting uv, is 3-sparse and is not rigid in \( \mathbb{R}^3 \). Hence \( s_3(G) = |E(G)| = 18 > 17 = r_3(G) \). On the other hand we may improve the upper bound on \( r_3(G) \) in this example by considering the graph \( H = G + uv \).
A maximal $3$-sparse subgraph of $H$ which contains $uv$ has $17$ edges. Thus we have $17 = r_3(G) \leq r_3(H) \leq s_3(H) = 17$.

More generally, for any graph $G$, let $s^*_d(G) = \min\{s_d(H) : G \subseteq H\}$. Then $r_d(G) \leq s^*_d(G)$ for all $d \leq 11$. The following example shows that strict inequality can also hold in this inequality. Let $G$ be obtained from $K_5$ by taking parallel connections with $10$ different $K_5$’s along each of the edges of the original $K_5$. We have $r_3(G) = 89$. On the other hand, $s_3(G) = 90$ (obtained by taking a maximal $3$-sparse subgraph which contains nine of the edges of the original $K_5$). Furthermore we have $s_3(H) \geq r_3(H) > r_3(G)$ for all graphs $H$ which properly contain $G$. Thus $s^*_3(G) = 90 > r_3(G)$.

2. For fixed $d$, we can use network flow algorithms to test whether a graph is $d$-sparse in polynomial time, see for example [2]. This means we can greedily construct a maximal $d$-sparse subgraph $H$ of a graph $G$ in polynomial time and hence obtain an upper bound on $r_d(G)$ when $d \leq 11$ via Theorem 3.3. We do not know whether $s_d(G)$ or $s^*_d(G)$ can be determined in polynomial time.

3. We believe that the conclusion of Theorem 3.3 should be valid for all $d$. However the graph $G$ given in the example at the end of Section 2 shows that our proof technique will not give this: $G$ is $d$-sparse and we have $\theta_3(V_i) = (d-1)(d-2)$ for all $V_i$ in the $G$-critical cover of $G$. On the other hand, the lower bound on the number of edges in a non-rigid circuit in $R_d(G)$ given by Lemma 3.1 is $\frac{d(d-1)}{2}$, so we cannot use it to deduce that the set of $2$-hinges in some $G|V_i$ is $R_d$-independent when $d \geq 15$. One way to get round this problem would be to show that the $d$-critical components in a $d$-sparse graph form a cover which is ‘iteratively independent’ i.e. we can order the vertex sets of these components as $V_1, V_2, \ldots V_m$ such that the set of $2$-hinges of $\{V_1, V_2, \ldots, V_i\}$ which belong to $V_i$ is $R_d$-independent for all $2 \leq i \leq m$. We refer the reader to [5] for more information on iteratively independent covers.

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