Research Article

Optimal Time-Consistent Investment and Reinsurance Strategy Under Time Delay and Risk Dependent Model

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1. Introduction

Since insurance companies have been allowed to enter the financial market for investing risk assets, the optimal investment strategy has become an important research topic in recent years. Many literature have studied the maximization of the utility of the terminal wealth or the minimization of the ruin probability of the insurer. Browne [1] uses the surplus process given by the diffusion risk model to study the investment problem of maximizing the utility of the terminal wealth and minimizing the ruin probability of an enterprise and obtains the explicit optimal solution. Hipp and Plum [2] apply the Cramer–Lundberg model to describe the insurance surplus process, based on the assumption that there is only one risky asset in the financial market and the time is discrete; the investment problem is studied. Wang et al. [3] use martingale approach to study the optimal portfolio selection of insurers under the criteria of mean-variance and constant absolute risk aversion utility maximization. For more similar literature, see Liu and Yang [4], Yang and Zhang [5], Wang [3], and Bai and Guo [6].

In addition to market risk, the insurer will also consider insurance risk. It is impossible to avoid insurance risk by investing in bonds and other assets in the market alone. However, reinsurance business provides a way for the insurer to avoid this risk. In recent years, this approach has been widely concerned. Reinsurance business mainly adopts two different forms of insurance: excess-of-loss reinsurance and proportional reinsurance. Promislow and Young [7] first investigate the proportional reinsurance and investment. Bauerle [8] considers proportional reinsurance and investment also, and the optimal explicit solution of the investment-reinsurance problem is obtained under the mean-variance criterion. Zeng and Li [9] also study proportional reinsurance and obtain the efficient frontier of the mean-variance under the multidimensional risky asset model. The stock price in the above model generally follows the geometric Brownian motion; the market price of stock-related risk is constant, but in the real market, stock price may have other characteristics, such as stochastic volatility. Liang et al. [10] used the Ornstein–Uhlenbeck process to characterize the instantaneous
return of stocks and obtained the optimal reinsurance and investment strategy. Gu et al. [11] investigate the excess-of-loss reinsurance-investment problem under the constant elasticity variance (CEV) model.

There are two deficiencies in the above literature that deserve further discussion. On the one hand, these literature implicitly assume that all insurance businesses of insurers are independent of each other, so they only study the investment and reinsurance of a single insurance business. However, in the real insurance market, there are often interdependencies between insurance businesses. For example, during the 2019-nCoV, medical claims and death claims often occur together. In order to depict this kind of dependency between different insurance businesses, the risk dependent model is proposed. The main works in this area are as follows. Yuen et al. [12], taking the expected utility maximization of the terminal wealth as the criterion, considered the optimal proportional reinsurance problem with multidimensional risk dependence by using the diffusion approach method. For the detailed process of diffusion approximate to the compound Poisson process, see Gandell [13]. Liang and Yuen [14], under the principle of variance premium, investigated the optimal proportional reinsurance of the Poisson model and diffusion approximation model. Ming et al. [15] derive the explicit expression of the optimal proportional reinsurance under the mean-variance criterion by using stochastic linear quadratic control. Considering the combination of investment and reinsurance, Bi et al. [16] obtains the optimal investment-reinsurance strategy for mean-variance under the diffusion approximation model. Bi and Chen [17], under the criterion of maximizing the expected utility of terminal wealth, arrived at the optimal investment and reinsurance strategies. On the other hand, most of the literature on optimal investment-reinsurance and other optimal control problems focus on time-delay free controlled systems. In fact, financial markets tend to rely on the past, Chang [18] considers the investment and consumption problems related to the return on risk assets and the historical performance. Federico [19] introduces the time-delay state process by considering the capital inflow/outflow related to performance. Peng et al. [20] and Yu et al. [21] study the optimal dividend policy based on observing the information of past time points to determine the behavior of the next moment. In fact, this is a discrete case of time delay. However, the stochastic control problems of systems with time-delay state may be infinite-dimensional in continuous cases; hence, it is difficult to find the analytical solutions. Only in some special cases, it is finite-dimensional and the problem has explicit solution. Elsanosi et al. [22], Øksendal and Sulem [23], and David [24] provide a theoretical basis for solving such problems. Shen and Zeng [25] first introduced the time delay in the investment and insurance problem. They introduced the inflow/outflow of capital in the wealth process of insurer and then depicted the wealth process of insurance companies through the stochastic delay differential equation (SDDE). After that Li [26] and Lai [27] studied the optimal investment-reinsurance problem with time delay under Heston and CEV models, respectively.

Inspired by the above research, this paper combines risk dependence with time delay to consider investment-reinsurance problem. The structure of the rest of this paper is as follows. In Section 2, the financial model framework of this paper is given, assuming that an insurer can invest in a risk-free asset and a risky asset, and in the case of two-dimensional dependent claim compound Poisson model and the introduction of the historical performance of the insurance company, the company’s wealth process with time delay is obtained. In Section 3, considering the mean-variance preference criterion, the time-inconsistent optimization problem is defined, and the extended HJB equation is obtained by using the stochastic control theory in the framework of game theory. In Section 4, under the principle of reinsurance expected premium, the explicit solutions of optimal investment and reinsurance strategies and their corresponding value functions are derived. In Section 5, the numerical calculation process of optimal investment and reinsurance strategies are introduced through numerical examples, and the influence of important model parameters on optimal strategy is analyzed. Section 6 concludes this paper.

2. The Model

Suppose that model is based on the probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)\) of information flow which satisfies the general assumptions of right continuity and completeness, where \(T\) is a finite constant, representing the operation cycle of an insurance company, and \(\{\mathcal{F}_t\}_{t \in [0,T]}\) is the sum of information available up to time \(t\). All stochastic processes involved in this paper are assumed to adapt to \(\mathcal{F}_t\).

Suppose an insurer has an insurance portfolio business, which is composed of two different insurance businesses, such as medical insurance and death insurance. Suppose that the random variable \(\{Y_i, i \geq 1\}\) represents the claim amount of the first type of insurance business; they are independent and have the same distribution function \(F(y)\). \(\{Z_i, i \geq 1\}\) represents the claim amount of the second type of insurance business; they are independent and have the same distribution function \(F_Z(z)\). We assume that if \(y \leq 0\), then \(F(y) = 0\). Otherwise, \(0 < F(y) \leq 1\). And also assume that if \(z \leq 0\), then \(F_Z(z) = 0\). Otherwise, \(0 < F_Z(z) \leq 1\). In addition, their moment generating functions \(M_Y(t)\) and \(M_Z(t)\) exist. The cumulative claim process of the two insurance businesses are as follows:

\[
C_1(t) = \sum_{i=1}^{N_1(t)} Y_i, \\
C_2(t) = \sum_{i=1}^{N_2(t)} Z_i, \tag{1}
\]

where \(N_1(t)\) and \(N_2(t)\) represent the number of claims for the first and second categories of insurance business up to time \(t\), respectively. And suppose \(\{Y_i, i \geq 1\}, \{Z_i, i \geq 1\}, \{N_1(t)\}_{t \geq 0}, \{N_2(t)\}_{t \geq 0}\) are independent of each other.

For different insurance businesses, it is assumed that they are interdependent as follows:

\[
C_1(t) = \sum_{i=1}^{N_1(t)} Y_i, \\
C_2(t) = \sum_{i=1}^{N_2(t)} Z_i, \tag{2}
\]
\[ N_1(t) = N_1(t) + N(t), \]
\[ N_2(t) = N_2(t) + N(t), \]

where \( N(t), N_1(t), \) and \( N_2(t) \) are three independent Poisson processes and the corresponding intensities are \( \lambda, \lambda_1, \) and \( \lambda_2, \) respectively. Therefore, the total claim amount of these two types of the insurance business is

\[ C(t) = C_1(t) + C_2(t) = \sum_{i=1}^{N_1(t) + N(t)} Y_i + \sum_{i=1}^{N_2(t) + N(t)} Z_i. \]

Suppose for arbitrary \( t \in (0, \zeta), \) \( E[Ye^{\psi}] \) and \( E[Ze^{\psi}] \) exist. And, for some \( \zeta \in (0, \infty), \) there are \( \lim_{t \to \infty} E[Ye^{\psi}] \to \infty \) and \( \lim_{t \to \infty} E[Ze^{\psi}] \to \infty. \)

For convenience of writing, we define

\[ a_1 = E[C_1(t)] = (\lambda + \lambda_1)\mu_Y, \]
\[ b_1 = \text{Var}[C_1(t)] = (\lambda + \lambda_1)\mu_{2Y}, \]
\[ a_2 = E[C_2(t)] = (\lambda + \lambda_2)\mu_Y, \]
\[ b_2 = \text{Var}[C_2(t)] = (\lambda + \lambda_2)\mu_{2Z}, \]

where \( \mu_Y = E[Y_1], \)
\[ \mu_{2Y} = E[Y_1^2], \]
\[ \mu_{1Z} = E[Z_1], \]
\[ \mu_{2Z} = E[Z_1^2]. \]

Considering the financial market, it is assumed that assets are traded continuously in time interval \([0, T],\) and tax and transaction costs are not considered. Suppose the insurer can invest wealth in the financial market composed of a risk-free asset and a risky asset. The risk-free asset price process \( (B(t)) \)

\[ dB(t) = rB(t)\,dt, \quad t \in [0, T], \]
\[ B(0) = 1. \]

The risky asset price process \( (S(t)) \) is as follows:

\[ dS(t) = S(t)[a_1\,dt + \sigma dW(t)], \quad t \in [0, T], \]
\[ S(0) = s_0, \] where \( r, \alpha, (\alpha > r), \) and \( \sigma > 0 \) are constants, representing risk-free interest rate, drift rate, and volatility, respectively. Define \( \alpha = \alpha_1 - r. \)

As usual, the surplus process from the insurer up to time \( t \) is defined as follows:

\[ R(t) = R_0 + ct - C(t), \]

where \( R_0 \) is the initial surplus and \( c \) is the premium rate. In addition, it is assumed that insurance companies can continuously reinsurance insurance business in a certain proportion to control business risk. We denote the retention ratio of categories 1 and 2 insurance business by \( q_1(t) \in [0, 1] \) and \( q_2(t) \in [0, 1]. \) When the claim occurs, the insurance company pays \( q_1(t)Y_i \) or \( q_2(t)Z_i, \) while the reinsurance company pays \( (1 - q_1(t))Y_i \) or \( (1 - q_2(t))Z_i, \) Let the reinsurance rate be \( \delta(q_1(t), q_2(t)) \) at time \( t. \)

Let \( X(t) \) denote the wealth process of insurance companies at time \( t, \) \( p_1(t) \) denote the amount of capital invested in the risky asset, and then \( X(t) - p_1(t) \) denote the amount of wealth invested in the risk-free asset. The investment-reinsurance strategy \( \pi(t) = (p_1(t), q_1(t), q_2(t)) \) will be applied by the insurer. Given an investment-reinsurance strategy \( \pi(t), \) the wealth process \( \{X^\pi(t)\} \) of an insurer satisfies the following stochastic differential equation:

\[ dX^\pi(t) = [rX^\pi(t) + ap_1(t) + (c - \delta(q_1(t), q_2(t)))]\,dt \]
\[ + \sigma p_1(t)dW(t) - q_1(t)C_1(t) - q_2(t)dC_2(t). \]

Next, we consider the influence of historical performance on the wealth process. Suppose that \( f(t, X^\pi(t) - L^\pi(t), X^\pi(t) - M^\pi(t)) \) represents the inflow/outflow of capital, then the wealth process of insurers with time delay is given by the following stochastic delay differential equation (SDDE):

\[ dX^\pi(t) = [rX^\pi(t) + ap_1(t) + (c - \delta(q_1(t), q_2(t)))]\,dt \]
\[ + \sigma p_1(t)dW(t) - f(t, X^\pi(t) - L^\pi(t), X^\pi(t) - M^\pi(t))\,dt \]
\[ - q_1(t)dC_1(t) - q_2(t)dC_2(t). \] To make the problem easier to deal with, consider a linear capital inflow/outflow function, that is,

\[ f(t, X^\pi(t) - L^\pi(t), X^\pi(t) - M^\pi(t)) \]
\[ = y_1 \left( X^\pi(t) - L^\pi(t) \right) + y_2 \left( X^\pi(t) - M^\pi(t) \right) \]
\[ = y_1 \left( X^\pi(t) - \frac{L^\pi(t)}{\int_{-h}^0 e^{\mu s} \,ds} \right) + y_2 \left( X^\pi(t) - M^\pi(t) \right) \]
\[ = (y_1 + y_2)X^\pi(t) - \overline{y}_1 L^\pi(t) - \overline{y}_2 M^\pi(t), \]

where \( y_1 > 0 \) and \( y_2 > 0 \) are constants, \( \overline{y}_1 = y_1/\int_{-h}^0 e^{\mu s} \,ds. \) \( L^\pi(t) = \int_{-h}^0 e^{\mu s}X^\pi(t + u)\,du, \)
\[ \overline{L}^\pi(t) = L^\pi(t)/\int_{-h}^0 e^{\mu s} \,du, \]
\[ M^\pi(t) = X^\pi(t - h) \] represent the integrated, average, and point by point delay information of wealth process in time interval \([t - h, t]. \) \( A(0) \) and \( h(\geq 0) \) are given average parameters and delay parameters, respectively. Note that \( \overline{L}^\pi(t) \) is defined as the weighted average value of wealth process \( X^\pi(\cdot) \) in time interval \([t - h, t], \) and the exponential decay factor \( e^{\mu u}(u \in [-h, 0]) \) represents the weight. When \( h = 1, \) \( X^\pi(t) - \overline{L}^\pi(t) \) and \( X^\pi(t) - M^\pi(t) \) represent the average gain or loss and absolute gain or loss of wealth of insurers in the last operating cycle. Because the inflow/outflow of capital is closely related to the past performance of the wealth process. If the past performance is good, the company will give part of its earnings to shareholders or give bonuses to the management, which shows the inflow of capital, i.e., \( f > 0. \) At this time, \( X^\pi(t) > \overline{L}^\pi(t) \) and \( X^\pi(t) > M^\pi(t). \) On the contrary, if the past performance of the insurance company is not good, the company needs additional funding to achieve the predetermined goal. This shows capital inflow, i.e., \( f < 0 \) when \( X^\pi(t) < \overline{L}^\pi(t) \) and \( X^\pi(t) < M^\pi(t). \) Therefore, the function \( f(\cdot, \cdot, \cdot) \) considers the average and absolute performance of the wealth process in \([t - h, t]. \)
Substituting (10) into (9), the following stochastic delay differential equation (SDDE) is obtained:

\[
\begin{align*}
\frac{d^\pi X(t)}{dt} &= [(r - y_1(t) - y_2)X(t) + \bar{Y}_1(t)L^\pi(t) + y_2M^\pi(t) + \alpha p_1(t) + (c - \delta(q_1(t), q_2(t)))]\,dt \\
&+ \sigma p_1(t)\,dW(t) - q_1(t)\,dC_1(t) - q_2(t)\,dC_2(t), \\
\frac{dL^\pi(t)}{dt} &= \left[X^\pi(t) - AL^\pi(t) - e^{-tA}M^\pi(t)\right]\,dt.
\end{align*}
\]

Furthermore, suppose \(X^\pi(t) = x_0\), \(\forall t \in [-h, 0]\), which can be interpreted as that the insurance company has the initial wealth of \(x_0\) at \(-h\). There is no business operation during \([-h, 0]\), and the wealth has no change. The integrated delay wealth initial value can be calculated to get \(L^\pi(0) = x_0/A\left(1 - e^{-Ah}\right)\).

**Definition 1** (admissible strategy). For any fixed \(t \in [0, T]\), an investment-reinsurance strategy \(\pi(t) = (p_1(t), q_1(t), q_2(t))\) is said to be admissible if (i) \((p_1(t), q_1(t), q_2(t))\) is \(\mathcal{F}_t\), progressively measurable, (ii) for \(t \in [0, T]\), \(q_1(t) \in [0, 1]\), \(q_2(t) \in [0, 1]\), and \(E\left[\int_0^t p_1(t)\,dt\right] < 0\), and (iii) the SDDE (11) has a unique strong solution \(X()\) such that \(E[\sup_{0 \leq t \leq T} |X(t)|^2] < \infty\). Let \(\Pi\) be the set of all admissible investment-reinsurance strategy.

### 3. Optimization Problem

To take historical operating performance into account, the insurer will focus on both terminal wealth \(X^\pi(T)\) and historical average operating performance \(L^\pi(T)\); thus, the following objective function is defined:

\[
J(t, x, l; \tilde{\pi}) = \sup_{\pi \in \Pi} E_{t,x,l,m}[X^\pi(T) + \beta L^\pi(T)] - \frac{\omega}{2} \text{Var}_{t,x,l,m}[X^\pi(T) + \beta L^\pi(T)],
\]

where risk aversion coefficient \(\omega(> 0)\) and delay parameter \(\tilde{\beta}(\in [0, 1])\) are constants. \(E_{t,x,l,m}[\cdot]\) and \(\text{Var}_{t,x,l,m}[\cdot]\) represent conditional expectation and conditional variance based on \(X^\pi(t) = x, L^\pi(t) = l\), and \(M^\pi(t) = m\), respectively. \(\tilde{\beta}(\in [0, 1])\) is the weight of \(L^\pi(t)\), indicating the degree of terminal wealth affected by historical average performance. If we write \(\beta = \tilde{\beta}/\int_0^t e^{\beta u}\,du\), then \(X^\pi(t) + \beta L^\pi(t) = \bar{X}^\pi(t) + \beta L^\pi(t)\). In addition, according to Chang [18], delay optimal control problem is generally an infinite-dimensional problem. In order to obtain the optimal solution, some additional conditions will be attached. We assume that the value function \(V()\) is only related to \(x\) and \(l\), but \(L^\pi(t)\) is related to \(M^\pi(t)\); in order to make \(V()\) only depend on \((t, x, l)\), the problem can obtain the optimal solution, and we assume the following conditions hold:

\[
\begin{align*}
\gamma_2 &= \tilde{\beta}e^{-Ah}, \\
\bar{Y}_1 &= A\tilde{\beta} = (r - y_1 - y_2 + \beta)\beta.
\end{align*}
\]

Therefore, this paper aims at the following optimization problems:

\[
J(t, x, l; \tilde{\pi}) = \sup_{\pi \in \Pi} E_{t,x,l,m}[X^\pi(T) + \beta L^\pi(T)] - \frac{\omega}{2} \text{Var}_{t,x,l,m}[X^\pi(T) + \beta L^\pi(T)]
\]

\[
= E_{t,x,l,m}[F(X^\pi(T) + \beta L^\pi(T))] + G(E_{t,x,l,m}[X^\pi(T) + \beta L^\pi(T)]),
\]

where \(F(x) = x - \omega/2x^2\) and \(G(x) = \omega/2x^2\).

**Remark 1**

(i) According to Shen and Zeng [25], condition (13) can be regarded as exogenous technical conditions that need to be determined in advance by the insurance company. Firstly, the average delay wealth \(L^\pi(t)\) and point by point delay wealth \(M^\pi(t)\) are determined by selecting the average parameter \(A\) and delay time \(h\). Secondly, it selects the weight \(\beta\). Finally, it calculates the weight ratios \(y_2 = \tilde{\beta}e^{-Ah}\) and \(y_1 = (\beta \int_0^t e^{\beta u}\,du)/(1 + \tilde{\beta} \int_0^t e^{\beta u}\,du)\) \((r - y_1 - y_2 + A)\) of historical performance \(X^\pi(t) - L^\pi(t)\) and \(X^\pi(t) - M^\pi(t)\) according to the two assumptions in (13) and adjusts the inflow/outflow of capital accordingly.

(ii) Because there is a nonlinear function of the expectation of the terminal value wealth in the variance term, problem (14) is time inconsistent, which leads to the failure of Bellman’s optimal principle. Many works of literature deal with the mean-variance problem by setting a precommitment, so the optimal strategy obtained are time-inconsistent. However, for a rational decision maker, time consistency is often not negligible. Rational decision makers hope that the equilibrium strategy they find is not only optimal at this time but also optimal in the future with the evolution of time, that is to say, the equilibrium strategy is time consistent. Therefore, for problem (14), this paper aims to find the equilibrium strategy.
Definition 2. Consider a control law \( \tilde{\pi}(t), t \in [0, T] \). Choose arbitrarily \( \tilde{\pi} \in \Pi \), \( t > 0 \), and \( \epsilon > 0 \) and define the control law \( \tilde{\pi}_\epsilon \):

\[
\tilde{\pi}_\epsilon(u) = \begin{cases} 
\tilde{\pi}, & t \leq u < t + \epsilon, \\
\tilde{\pi}(u), & t + \epsilon \leq u \leq T.
\end{cases}
\]  

(15)

We call that \( \tilde{\pi} \) is an equilibrium strategy if \( \lim_{\epsilon \to 0} \inf (J(t, x, l; \tilde{\pi}) - J(t, x, l; \pi_\epsilon)/\epsilon) \geq 0 \) for any \( t \) and \( \tilde{\pi} \). If the equilibrium strategy \( \tilde{\pi} \) exists, the equilibrium value function is defined as \( V(t, x, l) = J(t, x, l; \tilde{\pi}) \).

According to Definition 2, the equilibrium strategy is time consistent. For simplicity, we denote that \( \mathcal{C}^{1,1}([0, T] \times \mathbb{R} \times \mathbb{R}) \) is once continuously differentiable on \( [0, T] \), \( \phi(t, \cdot, \cdot) \) is twice continuously differentiable on \( \mathbb{R} \), and \( \phi(\cdot, \cdot, \cdot) \) is once continuously differentiable on \( \mathbb{R} \).

To provide verification theorem and derive conveniently extended HJB equation, for \( \forall (t, x, l) \in [0, T] \times \mathbb{R} \times \mathbb{R} \), \( \forall \phi \in \mathcal{C}^{1,1}([0, T] \times \mathbb{R} \times \mathbb{R}) \), and given control law \( \pi \), we define variational operator as follows:

\[
\mathcal{L}^\pi \phi(t, x, l) = \phi_t(t, x, l) + [(r - \gamma_1 - \gamma_2)x + \gamma_1 l + \gamma_2 m + \alpha p_1(t) + (c - \delta(q_1(t), q_2(t)))] \phi_x(t, x, l) \\
+ \left( x - \beta l - e^{-\alpha t} m \right) \phi_t(t, x, l) + \frac{1}{2} \sigma^2 p_1(t) \phi_{xx}(t, x, l) \\
+ \lambda_1 \mathbb{E} \left[ \phi(t, x - q_1(t)Y_i, l) - \phi(t, x, l) \right] + \lambda_2 \mathbb{E} \left[ \phi(t, x - q_2(t)Z_i, l) - \phi(t, x, l) \right] \\
+ \lambda \mathbb{E} \left[ \phi(t, x - q_1(t)Y_i - q_2(t)Z_i, l) - \phi(t, x, l) \right].
\]  

(16)

The following theorem provides verification for the extended HJB equation in problem (14).

Theorem 1 (verification theorem). For problem (14), we assume that there exist two real-valued functions \( V(t, x, l), g(t, x, l) \in \mathcal{C}^{1,1}([0, T] \times \mathbb{R} \times \mathbb{R}) \) satisfying the following extended HJB equation:

\[
\sup_{\pi \in \Pi} \left\{ \mathcal{L}^\pi V(t, x, l) - \frac{\omega}{2} \mathcal{L}^\pi g^2(t, x, l) + \omega g(t, x, l) \mathcal{L}^\pi g(t, x, l) \right\} = 0,
\]

\( \mathcal{L}^\pi g(t, x, l) = 0, \)

\( \tilde{\pi} = \arg \sup_{\pi \in \Pi} \left\{ \mathcal{L}^\pi V(t, x, l) - \frac{\omega}{2} \mathcal{L}^\pi g^2(t, x, l) + \omega g(t, x, l) \mathcal{L}^\pi g(t, x, l) \right\}, \)

\( V(T, x, l) = x + \beta l, \quad g(T, x, l) = x + \beta l. \)

Then, \( J(t, x, l; \tilde{\pi}) = V(t, x, l), \quad E_{t+1}[X_\pi(T) + \beta X(T)] = g(t, x, l), \) and \( \tilde{\pi} \) is an equilibrium investment-reinsurance strategy.

The proof process of Theorem 1 is similar to that of Björk et al. [28], so it is omitted here.
operator (16), the extended HJB (17) can be expanded as follows:

\[
\begin{align*}
\sup_{\pi(t \in \mathbb{I})} & \left\{ V_\pi(t, x, l) + \left[ (r - \gamma_2) x + p_\pi (t) x + \gamma_2 m + \alpha \pi_1 (t) + (c - \delta (q_1(t), q_2(t))) \right] V_\pi(t, x, l) \right. \\
+ & \frac{1}{2} \sigma_1^2 \pi_1^2(t) \left[ V_{xx}(t, x, l) - \omega g_2^2(t, x, l) \right] + \left( x - AL - e^{-\alpha l} m \right) V_\pi(t, x, l) + \lambda_1 \left\{ E \left[ V(t, x - q_1(t) Y, l) - V(t, x, l) \right] \right. \\
- & \frac{\omega}{2} E \left[ g^2(t, x - q_1(t) Y, l) - g^2(t, x, l) \right] + \omega g(t, x, l) E \left[ g(t, x - q_1(t) Y, l) - g(t, x, l) \right] \bigg) \\
+ & \lambda_1 \left\{ E \left[ V(t, x - q_1(t) Y, l) - V(t, x, l) \right] - \frac{\omega}{2} E \left[ g^2(t, x - q_1(t) Y, l) - g^2(t, x, l) \right] \bigg) \\
+ & \omega g(t, x, l) E \left[ g(t, x - q_2(t) Z, l) - g(t, x, l) \right] \bigg) \\
= & 0, \quad t \in [0, T],
\end{align*}
\]

\[
\begin{align*}
g_\pi(t, x, l) + \left[ (r - \gamma_2(t) x + p_\pi(t) x + \gamma_2 m + \alpha \pi_1(t) + (c - \delta (\hat{q}_1(t), \hat{q}_2(t))) \right] \\
g_\pi (t, x, l) + \frac{1}{2} \sigma_1^2 \pi_1^2(t) g_{xx}(t, x, l) + \left( x - AL - e^{-\alpha l} m \right) g_\pi(t, x, l) + \lambda_1 E \left[ g(t, x - \hat{q}_1(t) Y, l) - g(t, x, l) \right] \\
+ \lambda_1 E \left[ g(t, x - \hat{q}_1(t) Y, l) - g(t, x, l) \right] + \lambda E \left[ g(t, x - \hat{q}_1(t) Y, l - \hat{q}_2(t) Z, l) - g(t, x, l) \right] = 0, \quad t \in [0, T],
\end{align*}
\]

\[
\begin{align*}
V(T, x, l) = x + \beta L, \\
g(T, x, l) = x + \beta L.
\end{align*}
\]

Suppose that the solution of the above extended HJB equation has the following structure:

\[
\begin{align*}
\begin{cases}
V(t, x, l) &= H_1(t)(x + \beta L) + F_1(t), \\
g(t, x, l) &= H_2(t)(x + \beta L) + F_2(t),
\end{cases}
\end{align*}
\]

(19)

with the boundary condition \( H_1(T) = H_2(T) = 1 \) and \( F_1(T) = F_2(T) = 0 \).

Differentiating \( V \) and \( g \) with respect to \( t, x, \) and \( l \), we obtain

\[
\begin{align*}
E[V(t, x - q_1(t) Y, l) - V(t, x, l)] &= -\mu_1 q_1(t) H_1(t), \\
E[V(t, x - q_2(t) Z, l) - V(t, x, l)] &= -\mu_2 q_2(t) H_1(t), \\
E[V(t, x - q_1(t) Y, l - q_2(t) Z, l) - V(t, x, l)] &= -\mu_1 q_1(t) H_1(t) - \mu_2 q_2(t) H_1(t), \\
E[g^2(t, x - q_1(t) Y, l) - g^2(t, x, l)] &= \left[ \mu_2 q_1^2(t) - 2 \mu_1 q_1(t) (x + \beta L) \right] H_2^2(t) - 2 \mu_1 q_1(t) H_2(t) F_2(t), \\
E[g^2(t, x - q_2(t) Z, l) - g^2(t, x, l)] &= \left[ \mu_2 q_2^2(t) - 2 \mu_1 q_2(t) (x + \beta L) \right] H_2^2(t) - 2 \mu_1 q_2(t) H_2(t) F_2(t), \\
E[g^2(t, x - q_1(t) Y, l - q_2(t) Z, l) - g^2(t, x, l)] &= \left[ \mu_2 q_1^2(t) + \mu_2 q_2^2(t) + 2 \mu_2 q_1(t) q_2(t) - 2 \mu_1 q_1(t) \mu_1 q_2(t) (x + \beta L) \right], \\
H_2^2(t) - 2 \mu_1 q_1(t) H_2(t) F_2(t), \\
E[g(t, x - q_1(t) Y, l) - g(t, x, l)] &= -\mu_1 q_1(t) H_2(t), \\
E[g(t, x - q_2(t) Z, l) - g(t, x, l)] &= -\mu_2 q_2(t) H_2(t), \\
E[g(t, x - q_1(t) Y, l - q_2(t) Z, l) - g(t, x, l)] &= -\mu_1 q_1(t) H_2(t) - \mu_2 q_2(t) H_2(t).
\end{align*}
\]

(20)

Through simple calculation, we can also obtain
Putting the above results back into (18), we can arrive at

\[
\begin{align*}
\sup_{\omega} \left[ H'_1(t) (x + \beta l) + F'_1(t) + \psi(p_1, q_1, q_2) H_1(t) - \frac{\omega}{2} \sigma^2 p_1^2(t) H_2^2(t) - \frac{\omega}{2} \left( b_1 q_1^2(t) + b_2 q_2^2(t) \right) H_2^2(t) - \omega \lambda \mu_{1y} \mu_{1z} q_1(t) q_2(t) H_2^2(t) \right] &= 0, \\
H'_2(t) (x + \beta l) + F'_2(t) + \psi(p_1, q_1, q_2) H_2(t) &= 0, \\
H_1(T) &= H_2(T), \\
F_1(T) &= F_2(T) = 0,
\end{align*}
\]

where

\[
\psi(p_1, q_1, q_2) = (r - \gamma_1 - \gamma_2 + \beta)x + (\gamma_1 - A \beta) l + (c - \delta(q_1(t), q_2(t))) - a_1 q_1(t) - a_2 q_2(t). 
\]

(23)

According to \( \gamma_2 = \beta e^{-Ah} \), we have

\[
\psi(p_1, q_1, q_2) = (r - \gamma_1 (1 - q_1(t))) a_1 + (1 + \eta_1) + (1 + \eta_2) (1 - q_2(t)) a_2, 
\]

(24)

For the convenience of writing, let

\[
\begin{align*}
\frac{\partial h}{\partial p_1} &= \frac{\partial h}{\partial q_1} = \frac{\partial h}{\partial q_2} = 0, \\
\frac{\partial^2 h}{\partial p_1^2} &= \frac{\partial^2 h}{\partial p_1 \partial q_1} = \frac{\partial^2 h}{\partial p_1 \partial q_2} = \frac{\partial^2 h}{\partial q_1^2} = \frac{\partial^2 h}{\partial q_2^2} = 0, \\
\frac{\partial h}{\partial \eta_1} &= \frac{\partial h}{\partial \eta_2} = \frac{\partial h}{\partial \lambda} = 0, \\
\frac{\partial h}{\partial \mu_{1y}} &= \frac{\partial h}{\partial \mu_{1z}} = \frac{\partial h}{\partial \mu_{2y}} = \frac{\partial h}{\partial \mu_{2z}} = \frac{\partial h}{\partial \lambda} = 0.
\end{align*}
\]

(25)

(26)

\[
\begin{align*}
4. \text{Optimal Time-Consistent Strategy}
\end{align*}
\]

This section assumes that the reinsurance premium rate is calculated by the expected premium principle, i.e.,

\[
\delta(q_1(t), q_2(t)) = (1 + \eta_1)(1 - q_1(t)) a_1 + (1 + \eta_2)(1 - q_2(t)) a_2,
\]

(27)

where \( \eta_1 \) and \( \eta_2 \) are the reinsurer’s safety loading of the insurance business.

Substituting the above formula into (24), we have

\[
\psi(p_1, q_1, q_2) = (r - \gamma_1 - \gamma_2 + \beta)x + (\gamma_1 - A \beta) l + (c - \delta(q_1(t), q_2(t))) - a_1 q_1(t) - a_2 q_2(t). 
\]

(28)

To facilitate derivation, we rewrite (25) as

\[
\begin{align*}
\frac{\partial^2 h}{\partial q_1^2} &= \frac{\partial^2 h}{\partial q_2^2} = \frac{\partial^2 h}{\partial q_1 \partial q_2} = \frac{\partial^2 h}{\partial p_1^2} = \frac{\partial^2 h}{\partial \eta_1^2} = \frac{\partial^2 h}{\partial \eta_2^2} = \frac{\partial^2 h}{\partial \lambda^2} = \frac{\partial^2 h}{\partial \mu_{1y}^2} = \frac{\partial^2 h}{\partial \mu_{1z}^2} = \frac{\partial^2 h}{\partial \mu_{2y}^2} = \frac{\partial^2 h}{\partial \mu_{2z}^2} = 0.
\end{align*}
\]

(29)

From (29), we obtain the following Hessian matrix:
By solving the above equations, we can obtain

\[
B = \begin{bmatrix}
\sigma^2 & 0 & 0 \\
0 & b_1 & \lambda_{\mu_{1Y}1Z} \\
0 & \lambda_{\mu_{1Y}1Z} & b_2
\end{bmatrix},
\]

where

\[
\mathbf{C} \cdot \mathbf{B} \cdot \mathbf{C}' = c_1^2\sigma^2 + c_1^2b_1 + c_2^2b_2 + 2c_2c_1\lambda_{\mu_{1Y}1Z}
\]

\[
= c_1^2\sigma^2 + c_1^2(\lambda_1 + \lambda)\mu_{1Z} + c_2^2(\lambda_2 + \lambda)\mu_{2Z} + 2c_2c_1\lambda_{\mu_{1Y}1Z}
\]

\[
= c_1^2\sigma^2 + c_1^2\lambda E[(Y_j)^2] + c_2^2\lambda E[(Z_i)^2] + \lambda \left[ c_2^2 E[(Y_j)^2] + c_1^2 E[(Z_i)^2] + 2c_2c_1 E[Y_j] E[Z_i] \right]
\]

\[
\geq c_1^2\sigma^2 + c_1^2\lambda E[(Y_j)^2] + c_2^2\lambda E[(Z_i)^2] + \lambda \left[ c_2^2 E[Y_j] + c_1^2 E[Z_i] \right] > 0.
\]

So, matrix \( \mathbf{B} \) is positive definite.

From (29), we have

\[
\begin{cases}
aH_1(t) - \omega \sigma^2 p_1(t) H_2^2(t) = 0, \\
a_1\eta_1 H_1(t) - \omega b_1 q_1(t) H_2^2(t) - \omega \lambda_{\mu_{1Y}1Z} q_2(t) H_2^2(t) = 0, \\
a_2\eta_2 H_1(t) - \omega b_2 q_2(t) H_2^2(t) - \omega \lambda_{\mu_{1Y}1Z} q_1(t) H_2^2(t) = 0.
\end{cases}
\]

By solving the above equations, we can obtain

\[
H_1'(t) (x + \beta l) + F_1'(t) + [(r - \gamma_1 - \gamma_2 + \beta)x + (\bar{\gamma}_1 - A\beta)l]H_1(t)
\]

\[
+ \left[ c - a_1 (1 + \eta_1) - a_2 (1 + \eta_2) \right] H_1(t) + \left[ a\bar{p}_1(t) + a_1\eta_1\bar{q}_1(t) + a_2\eta_2\bar{q}_2(t) \right] H_1(t)
\]

\[
- \frac{\omega}{2} \left[ \sigma^2 \bar{p}_1^2(t) + (b_1\bar{q}_1^2(t) + b_2\bar{q}_2^2(t)) + 2\lambda_{\mu_{1Y}1Z} \bar{q}_1(t)\bar{q}_2(t) \right] H_2^2(t) = 0,
\]

\[
H_2'(t) (x + \beta l) + F_2'(t) + [(r - \gamma_1 - \gamma_2 + \beta)x + (\bar{\gamma}_1 - A\beta)l]H_2(t)
\]

\[
+ \left[ \alpha\bar{p}_1(t) + c - a_1 (1 + \eta_1) - a_2 (1 + \eta_2) \right] H_1(t) + a_1\eta_1\bar{q}_1(t) + a_2\eta_2\bar{q}_2(t) \right] H_2(t) = 0.
\]
\[
\begin{align*}
H_1'(t) + (r - y_1 - y_2 + \beta)H_1(t) &= 0, \\
H_1(T) &= 1,
\end{align*}
\] (39)

\[
\begin{align*}
H_2'(t) + (r - y_1 - y_2 + \beta)H_2(t) &= 0, \\
H_2(T) &= 1,
\end{align*}
\] (40)

\[
\begin{align*}
F_1'(t) + [c - a_1(1 + \eta_1) - a_2(1 + \eta_2)]H_1(t) + [a\tilde{p}_1(t) + a_1\eta_1\tilde{q}_1(t) + a_2\eta_2\tilde{q}_2(t)H_1(t) \\
- \frac{\omega}{2} \left[ a^2\tilde{p}_1^2(t) + b_1\tilde{q}_1^2(t) + b_2\tilde{q}_2^2(t) + 2\lambda\mu_{1Y}\mu_{1Z}\tilde{q}_1(t)\tilde{q}_2(t) \right]H_2^2(t) &= 0, \\
F_1(T) &= 0,
\end{align*}
\] (41)

\[
\begin{align*}
F_2'(t) + [a\tilde{p}_1(t) + c - a_1(1 + \eta_1) - a_2(1 + \eta_2) + a_1\eta_1\tilde{q}_1(t) + a_2\eta_2\tilde{q}_2(t)]H_2(t) &= 0, \\
F_1(T) &= 0.
\end{align*}
\] (42)

By solving the above equations, we have

\[
\begin{align*}
H_1(t) &= H_2(t) = e^{(r - y_1 - y_2 + \beta)(T - t)}, \\
F_1(t) &= \frac{c - a_1(1 + \eta_1) - a_2(1 + \eta_2)}{r - y_1 - y_2 + \beta} \left[ e^{(r - y_1 - y_2 + \beta)(T - t) - 1} + \frac{1}{\omega} \left[ a_1\eta_1D_1 + a_2\eta_2D_2 - \frac{1}{2}b_1D_1^2 - \frac{1}{2}b_2D_2^2 - \lambda\mu_{1Y}\mu_{1Z}D_1D_2 + \frac{\alpha^2}{2\sigma^2} \right] \right] (T - t), \\
F_2(t) &= \frac{c - a_1(1 + \eta_1) - a_2(1 + \eta_2)}{r - y_1 - y_2 + \beta} \left[ e^{(r - y_1 - y_2 + \beta)(T - t) - 1} + \frac{1}{\omega} \left[ a_1\eta_1D_1 + a_2\eta_2D_2 + \frac{\alpha^2}{\sigma^2} \right] \right] (T - t).
\end{align*}
\] (43)

According to the above discussion, the following proposition can be obtained.

**Proposition 1.** For problem (14), the time-consistent investment-reinsurance strategy in set \( \bar{\Pi} \) is as follows:

\[
\begin{align*}
\tilde{p}_1(t) &= \frac{\alpha}{\sigma^2\omega} e^{-(r - y_1 - y_2 + \beta)(T - t)}, \\
\tilde{q}_1(t) &= \frac{a_1\eta_1b_2 - a_2\eta_2\lambda\mu_{1Y}\mu_{1Z}}{(b_1b_2 - \lambda^2\mu^2_{1Y}\mu^2_{1Z})\omega} e^{-(r - y_1 - y_2 + \beta)(T - t)}, \\
\tilde{q}_2(t) &= \frac{a_2\eta_2b_1 - a_1\eta_1\lambda\mu_{1Y}\mu_{1Z}}{(b_1b_2 - \lambda^2\mu^2_{1Y}\mu^2_{1Z})\omega} e^{-(r - y_1 - y_2 + \beta)(T - t)}.
\end{align*}
\] (44)

The corresponding equilibrium function is

\[
V(t, x, l) = H_1(t) (x + \beta l) + F_1(t),
\] (45)

where \( H \) and \( F \) are given by (43). Let \( t_1 = T - (1/r - y_1 - y_2 + \beta)\ln(D_1/\omega) \) for \( \omega \leq D_1 \leq \omega e^{(r - y_1 - y_2 + \beta)T} \). Let \( t_1 = T - (1/r - y_1 - y_2 + \beta)\ln(D_2/\omega) \) for \( \omega \leq D_2 \leq \omega e^{(r - y_1 - y_2 + \beta)T} \). For \( D_1 < \omega \) (\( D_2 < \omega \)), we set \( t_1 = T \) (\( t_2 = T \)). And for \( D_1 > \omega e^{(r - y_1 - y_2 + \beta)T} \) (\( D_2 > \omega e^{(r - y_1 - y_2 + \beta)T} \)), we set \( t_1 = 0 \) (\( t_2 = 0 \)). To make sure that the optimal reinsurance strategies satisfy \( q_1(t) \in [0, 1] \) and \( q_2(t) \in [0, 1] \), we introduce the following lemma.

**Lemma 2.** For \( \lambda, \mu_{1Y}, \mu_{1Z}, a_1, a_2, b_1, \) and \( b_2 \) given in (4), the following inequality holds:

\[
\frac{\lambda\mu_{1Y}\mu_{1Z}a_2}{a_1b_2} \leq \frac{\lambda\mu_{1Y}\mu_{1Z}a_3 + b_1a_2}{a_1b_2 + \lambda\mu_{1Y}\mu_{1Z}a_1} \leq \frac{b_1a_2}{a_1b_2 + \lambda\mu_{1Y}\mu_{1Z}a_1}.
\] (46)

**Proof.** Using Cauchy – Schwarz inequality, we can easily get \( b_1 > \lambda\mu_{1Y}\mu_{1Z} \) and \( b_2 > \lambda\mu_{1Y}\mu_{1Z} \) and then we can obtain

\[
\frac{\lambda\mu_{1Y}\mu_{1Z}a_2}{b_2a_1} \leq \frac{b_1a_2}{b_2a_1} \leq \frac{b_1a_2}{a_1b_2 + \lambda\mu_{1Y}\mu_{1Z}a_1}.
\] (47)

In addition, for any positive number \( d_1, d_2, d_3, \) and \( d_4 \), if \( (d_1/d_2) \leq (d_3/d_4) \), then \( (d_1/d_2) \leq (d_1 + d_2 + d_3 + d_4)/(d_3 + d_4) \). In combination with inequality (47), inequality (46) is easily proved.

From Lemma 2, we will investigate the optimal results in the following four cases:

Case 1: \( \eta_2 < (\lambda\mu_{1Y}\mu_{1Z}a_2/b_2a_1)\eta_1 \)

Case 2: \( (\lambda\mu_{1Y}\mu_{1Z}a_2/b_2a_1)\eta_1 \leq \eta_2 < (\lambda\mu_{1Y}\mu_{1Z}a_2 + b_1 + \lambda\mu_{1Y}\mu_{1Z}a_1)\eta_1 \)

Case 3: \( (\lambda\mu_{1Y}\mu_{1Z}a_2 + b_1a_2/a_1b_2 + \lambda\mu_{1Y}\mu_{1Z}a_1)\eta_1 \leq \eta_2 \leq (b_1a_2/\lambda\mu_{1Y}\mu_{1Z}a_1)\eta_2 \)
Case 4: $\eta > (b_1 a_2 / \lambda \mu_{1Y} \mu_{1Z} a_1)$

Next, the optimal time-consistent strategy $\pi^*(t) = (p^*_1, q^*_1)(t)$ is considered in admissible strategy set $\Pi$. The corresponding value function $V(t, x, l)$ is discussed. In order to have a clear classification discussion, it is assumed that $r = -\gamma_1 - \gamma_2 + \beta \geq 0$.

Case 1: In this case, we have $\tilde{q}_1(t) < 0$ and $\tilde{q}_1(t) \geq 0$; thus, $q^*_1(t) = 0$. Let $h_1(p_1, q_2) = h(p_1, 0, q_2)$. By substituting $q^*_2(t) = 0$ into (28) and maximizing function $h_1(p_1, q_2)$, we can get the maximum point:

$$\tilde{p}_1(t) = \frac{\alpha}{\omega \sigma^2} e^{-(r-\gamma_1-\gamma_2+\beta)(T-t)},$$

$$\tilde{q}_1(t) = \frac{a_2 \eta_2}{\omega b_2} e^{-(r-\gamma_1-\gamma_2+\beta)(T-t)}. \quad (48)$$

Let $t_3 = T - (1/r - \gamma_1 - \gamma_2 + \beta) \ln(\eta_1 a_2 / \omega b_2)$. For $0 \leq t \leq t_3$, it is easy to see $\tilde{q}_2(t) \leq 1$, and then we have $\pi^*(t) = (\tilde{p}_1(t), 0, \tilde{q}_2(t))$. Putting $(\tilde{p}_1(t), 0, \tilde{q}_2(t))$ into (41) and (45), we obtain

$$V(t, x, l) = Q_1(t, x, l) + Q_2(t) + R_1, \quad (49)$$

where

$$Q_1(t, x, l) = e^{(r-\gamma_1-\gamma_2+\beta)(T-t)} (x + \beta t - a_1(1 + \eta_1) - a_2(1 + \eta_2)) \frac{r - \gamma_1 - \gamma_2 + \beta}{\omega}$$

$$+ \frac{2}{\omega b_2} \left( e^{(r-\gamma_1-\gamma_2+\beta)(T-t)} - 1 \right) + \frac{\alpha^2}{2 \omega^2} (T-t), \quad (50)$$

$$Q_2(t) = \frac{a_2^2 \eta_2}{2 \omega b_2} (T-t), \quad (51)$$

where $R_1$ is a constant whose value will be determined in a later calculation.

For $t_3 < t \leq T$, we have $\pi^*(t) = (\tilde{p}_1(t), 0, 1)$. Substituting it into (41) and (45), we can obtain

$$V(t, x, l) = Q_1(t, x, l) + Q_3(t) + Q_4(t), \quad (52)$$

where

$$Q_3(t) = \frac{a_2 \eta_2}{\omega b_2} \left( e^{(r-\gamma_1-\gamma_2+\beta)(T-t)} - 1 \right), \quad (53)$$

$$Q_4(t) = \frac{a_2 \eta_2}{4(r - \gamma_1 - \gamma_2 + \beta)} \left( e^{2(r-\gamma_1-\gamma_2+\beta)(T-t)} - 1 \right). \quad (54)$$

To make the value function $V(t, x, l)$ continuous, let $Q_2(t_3) + R_1 = Q_3(t_3) + Q_4(t_3); then,$

$$R_1 = Q_3(t_3) + Q_4(t_3) - Q_2(t_3). \quad (55)$$

Case 2: In this case, we have $\tilde{q}_1(t) \geq 0, \tilde{q}_2(t) \geq 0$ and $D_1 \leq D_2$, and it is easy to see $t_2 \leq t_1$. For $0 \leq t \leq t_2$, we have $\tilde{q}_1(t) \leq 1, \tilde{q}_2(t) \leq 1$, and thus $\pi^*(t) = (\tilde{p}_1(t), \tilde{q}_1(t), \tilde{q}_2(t))$. Substituting it into (41) and (45), we can derive

$$V(t, x, l) = Q_1(t, x, l) + Q_3(t) + Q_5(t) + R_2, \quad (56)$$

where

$$Q_5(t) = \frac{1}{\omega} \left( a_1 \eta_1 D_1 + a_2 \eta_2 D_2 - \frac{1}{2} b_1 D_1^2 - \frac{1}{2} b_2 D_2^2 - \lambda \mu_{1Y} \mu_{1Z} D_1 D_2 \right) (T-t). \quad (57)$$

For $t \geq t_5$, we have $\tilde{q}_2(t) > 1, and thus q^*_2(t) = 1$. Let $h_2(p_1, q_2) = h(p_1, q_2, 1)$. Putting $q^*_2(t) = 1$ into (28) and maximizing function $h_2(p_1, q_2)$, we can get the maximum point:

$$\tilde{p}_1(t) = \frac{\alpha}{\omega \sigma^2} e^{-(r-\gamma_1-\gamma_2+\beta)(T-t)}, \quad (58)$$

$$\tilde{q}_1(t) = \frac{a_1 \eta_1 e^{-(r-\gamma_1-\gamma_2+\beta)(T-t)} - \omega \lambda \mu_{1Y} \mu_{1Z} \omega b_1}{\omega b_1}. \quad (59)$$

Let $t_4 = T - (1/r - \gamma_1 - \gamma_2 + \beta) \ln(a_1 \eta_1 / \omega b_1 + \lambda \mu_{1Y} \mu_{1Z})$ and $t_5 = T - (1/r - \gamma_1 - \gamma_2 + \beta) \ln(a_1 \eta_1 / \omega (b_1 + \lambda \mu_{1Y} \mu_{1Z}))$. It is easy to see that $t_4 \leq t_3 \leq t_5$. For $t_3 < t \leq t_5$, we have $\pi^*(t) = (\tilde{p}_1(t), \tilde{q}_1(t), 1)$. Inserting it into (41) and (45), we can obtain

$$V(t, x, l) = Q_1(t, x, l) + Q_3(t) + Q_6(t) + Q_7(t) + R_3, \quad (60)$$

where

$$Q_6(t) = \frac{a_2^2 \eta_2}{2 \omega b_1} \left( T-t \right) - \frac{\lambda \mu_{1Y} \mu_{1Z} \omega b_1}{b_1 (r - \gamma_1 - \gamma_2 + \beta)} \left( e^{-(r-\gamma_1-\gamma_2+\beta)(T-t)} - 1 \right), \quad (61)$$

$$Q_7(t) = \left( \frac{\omega \lambda^2 \mu_{1Y} \mu_{1Z}^2}{4 b_1 (r - \gamma_1 - \gamma_2 + \beta)} - \frac{\omega b_2}{4 (r - \gamma_1 - \gamma_2 + \beta)} \right) \left( \left[ e^{2(r-\gamma_1-\gamma_2+\beta)(T-t)} - 1 \right] \right). \quad (62)$$

For $t_5 < t \leq T$, we have $\tilde{q}_1(t) > 1, and thus q^*_1(t) = (\tilde{p}_1(t), 1, 1)$. Putting it into (41) and (45), we can arrive at
\[ V(t, x, l) = Q_1(t, x, l) + Q_3(t) + Q_6(t) + Q_9(t), \quad (62) \]

where

\[ Q_6(t) = \frac{a_1 \eta_1}{r - y_1 - y_2 + \beta} \left( e^{(r - y_1 - y_2 + \beta)(T - t)} - 1 \right), \quad (63) \]

\[ Q_9(t) = -\frac{\omega (b_1 + b_2 + 2 \lambda \mu_{1Y} \mu_{1Z})}{4(r - y_1 - y_2 + \beta)} \left( e^{2(r - y_1 - y_2 + \beta)(T - t)} - 1 \right), \quad (64) \]

Let

\[ Q_5(t_2) + R_2 = Q_3(t_2) + Q_6(t_2) + Q_7(t_2) + R_3, \]
\[ Q_3(t_2) + Q_6(t_3) + Q_7(t_2) + R_3 = Q_3(t_3) + Q_6(t_3), \]
\[ + Q_8(t_3) + Q_9(t_3). \quad (65) \]

then

\[ R_3 = Q_3(t_3) + Q_6(t_3) - Q_3(t_5) - Q_6(t_5) - Q_7(t_5), \quad (66) \]
\[ R_2 = Q_3(t_2) + Q_6(t_2) + Q_7(t_2) + Q_3(t_3) \]
\[ + Q_6(t_5) + Q_8(t_5) - Q_3(t_5) - Q_6(t_5) - Q_7(t_5) - Q_5(t_3) \quad (67). \]

Case 3: in this case, we have \( \tilde{q}_1(t) \geq 0, \tilde{q}_2(t) \geq 0 \). And \( D_1 \geq D_2, so t_1 \leq t_2 \).

For \( 0 \leq t \leq t_1 \), we have \( \pi^* (t) = (\tilde{p}_1(t), \tilde{q}_1(t), \tilde{q}_2(t)) \). Substituting it into (41) and (45), we can obtain

\[ V(t, x, l) = Q_1(t, x, l) + Q_3(t) + Q_6(t) + Q_9(t), \quad (68) \]

For \( t \geq t_1 \), we have \( \tilde{q}_1(t) \geq 1 \), and thus \( q_1^* (t) = 1 \). Denote by \( h_1(p_1, q_1) \) the function \( h(p_1, q_1, \bar{q}_2) \) in (28). By maximizing \( h_1(p_1, q_1) \), we derive

\[ \tilde{p}_1(t) = \frac{\alpha}{\omega \sigma^2} e^{-\left( r - y_1 - y_2 + \beta \right)(T - t)}, \]
\[ \tilde{q}_2(t) = \frac{a_2 \eta_2 e^{-\left( r - y_1 - y_2 + \beta \right)(T - t)} - \omega \lambda \mu_{1Y} \mu_{1Z}}{\omega b_2}, \quad (69) \]

Let \( t_6 = T - (1/r - y_1 - y_2 + \beta) \ln(a_2/\omega b_2/\lambda \mu_{1Y} \mu_{1Z}) \) and \( t_7 = T - (1/r - y_1 - y_2 + \beta) \ln(a_2/\omega b_2 + \lambda \mu_{1Y} \mu_{1Z}) \).

It is easy to see that \( t_6 \leq t_1 \leq t_7 \).

For \( t_1 < t \leq t_7 \), we have \( \pi^* (t) = (\tilde{p}_1(t), \tilde{q}_1(t), \tilde{q}_2(t)) \). By substituting it into (41) and (45), we can obtain

\[ V(t, x, l) = Q_1(t, x, l) + Q_6(t) + Q_{10}(t) + Q_{11}(t) + R_5, \quad (70) \]

where

\[ Q_{10}(t) = \frac{a_2^2 \eta_2^2}{\omega b_2} (T - t) - \frac{\lambda \mu_{1Z}}{b_2 (r - y_1 - y_2 + \beta)} \left( e^{(r - y_1 - y_2 + \beta)(T - t)} - 1 \right), \quad (71) \]

\[ Q_{11}(t) = \left( \frac{\omega \lambda^2 \mu_{1Y} \mu_{1Z}^2}{4b_2 (r - y_1 - y_2 + \beta)} - \omega b_1 \right) \left( e^{(r - y_1 - y_2 + \beta)(T - t)} - 1 \right). \quad (72) \]

For \( t_7 < t \leq T \), we have \( \pi^* (t) = (\tilde{p}_1(t), 1, 1) \). Putting it into (41) and (45), we can obtain

\[ V(t, x, l) = Q_1(t, x, l) + Q_3(t) + Q_6(t) + Q_9(t), \quad (73) \]

Let

\[ Q_5(t_1) + R_1 = Q_8(t_1) + Q_{10}(t_1) + Q_{11}(t_1) + R_5, \]
\[ Q_8(t_1) + Q_{10}(t_1) + Q_{11}(t_1) + R_5 \]
\[ = Q_5(t_1) + Q_8(t_1) + Q_9(t_1). \quad (74) \]

We derive

\[ R_5 = Q_3(t_3) + Q_6(t_3) + Q_9(t_3) - Q_{10}(t_3) - Q_{11}(t_3). \quad (75) \]

\[ R_4 = Q_4(t_1) + Q_{10}(t_1) + Q_{11}(t_1) + Q_7(t_1) + Q_8(t_1) \]
\[ + Q_9(t_1) - Q_8(t_1) - Q_{10}(t_1) - Q_{11}(t_1) - Q_5(t_1). \quad (76) \]

Case 4: in this case, we have \( \tilde{q}_1(t) \geq 0 \) and \( \tilde{q}_3(t) < 0 \), and thus \( q_1^* (t) = 0 \). Let \( h_1(p_1, q_1) = h(p_1, q_1, q_2) \). By maximizing \( h_1(p_1, q_2) \), we arrive at

\[ \tilde{p}_1(t) = \frac{\alpha}{\omega \sigma^2} e^{-\left( r - y_1 - y_2 + \beta \right)(T - t)}, \]
\[ \tilde{q}_1(t) = \frac{a_1 \eta_1}{\omega b_2} e^{-\left( r - y_1 - y_2 + \beta \right)(T - t)}. \quad (77) \]

Let \( t_8 = T - (1/r - y_1 - y_2 + \beta) \ln(a_1/\omega b_1) \).
For $0 \leq t \leq t_8$, we have $\pi^* (t) = (\bar{p}_1 (t), \bar{q}_1 (t), 0)$. Inserting it into (41) and (45), we can derive
\[
V (t, x, l) = Q_1 (t, x, l) + Q_{12} (t) + R_6,
\]
where
\[
Q_{12} (t) = \frac{d_1^2 \eta_2^2}{2 \omega b_1} (T - t). \tag{79}
\]
For $t_8 < t \leq T$, we have $\pi^* (t) = (\bar{p}_1 (t), 1, 0)$. Putting it into (41) and (45), we can obtain
\[
V (t, x, l) = Q_1 (t, x, l) + Q_8 (t) + Q_{13} (t), \tag{80}
\]
where
\[
Q_{13} (t) = -\frac{\omega b_1}{4 (r - \gamma_1 - \gamma_2 + \beta)} \left[ e^{2 (r - \gamma_1 - \gamma_2 + \beta) (T - t) - 1} \right]. \tag{81}
\]
Let
\[
Q_{12} (t) + R_6 = Q_8 (t) + Q_{13} (t). \tag{82}
\]
We have
\[
R_6 = Q_8 (t) + Q_{13} (t) - Q_{12} (t). \tag{83}
\]
From the above discussion, we can get the following theorem.

**Theorem 2.** Assuming $r - \gamma_1 - \gamma_2 + \beta \geq 0$, the optimal time-consistent investment and reinsurance strategies for problem (14) are as follows:

(i) If Case 1 holds, the optimal investment-reinsurance strategies for model (14) are
\[
(p_{11}^* (t), q_{11}^* (t), q_{12}^* (t)) = \begin{cases} 
\left( \frac{\alpha}{\sigma^2 \omega} e^{(r - \gamma_1 - \gamma_2 + \beta) (T - t)}, 0, \frac{a_2 \eta_2^2 - (r - \gamma_1 - \gamma_2 + \beta) e^{(r - \gamma_1 - \gamma_2 + \beta) (T - t)}}{\omega b_2} \right), & 0 \leq t \leq t_3, \\
\left( \frac{\alpha}{\omega b_2} e^{(r - \gamma_1 - \gamma_2 + \beta) (T - t)}, 0, 1 \right), & t_3 < t \leq T, 
\end{cases}
\]  
and the value function is given by
\[
V (t, x, l) = \begin{cases} 
Q_1 (t, x, l) + Q_2 (t) + R_1, & 0 \leq t \leq t_3, \\
Q_1 (t, x, l) + Q_3 (t) + Q_4 (t), & t_3 < t \leq T, 
\end{cases}
\]  

(ii) If Case 2 holds, the optimal investment-reinsurance strategies for model (14) are
\[
(p_{11}^* (t), q_{11}^* (t), q_{12}^* (t)) = \begin{cases} 
\left( \frac{\alpha}{\sigma^2 \omega} e^{(r - \gamma_1 - \gamma_2 + \beta) (T - t)}, a_1 \eta_1 b_1 - a_2 \eta_2 \lambda \mu_1 \mu_1 \mu_1 e^{-(r - \gamma_1 - \gamma_2 + \beta) (T - t)}, \right. & a_2 \eta_2 b_1 - a_1 \eta_1 \lambda \mu_1 \mu_1 \mu_1 \mu_1 \mu_1 e^{-(r - \gamma_1 - \gamma_2 + \beta) (T - t)}, \right) & 0 \leq t \leq t_2, \\
\left( \frac{\alpha}{\omega b_1} e^{(r - \gamma_1 - \gamma_2 + \beta) (T - t)} - \frac{\omega \lambda \mu_1 \mu_1 \mu_1}{\omega b_1}, 1 \right), & t_2 < t \leq t_5, \\
\left( \frac{\alpha}{\sigma^2 \omega} e^{(r - \gamma_1 - \gamma_2 + \beta) (T - t)}, 1, 1 \right), & t_5 < t \leq T, 
\end{cases}
\]  
and the value function is given by
\[
V (t, x, l) = \begin{cases} 
Q_1 (t, x, l) + Q_2 (t) + R_1, & 0 \leq t \leq t_3, \\
Q_1 (t, x, l) + Q_3 (t) + Q_4 (t), & t_3 < t \leq T, 
\end{cases}
\]
Remark 2. respectively.

(iii) If Case 3 holds, the optimal investment-reinsurance strategies for model (14) are

\[
V(t,x,l) = \begin{cases}
Q_3(t,x,l) + Q_5(t) + R_2, & 0 \leq t \leq t_2, \\
Q_3(t,x,l) + Q_4(t) + Q_7(t) + R_5, & t_2 < t \leq t_5, \\
Q_3(t,x,l) + Q_5(t) + Q_8(t) + Q_9(t), & t_5 < t \leq T,
\end{cases}
\]

\[
(p_1^*(t), q_1^*(t), q_2^*(t)) = \left\{ \begin{array}{l}
\frac{\alpha}{\sigma^2} e^{-(r-y_{12}+\beta)(T-t)} - \frac{a_1 \eta_1 \beta_1 - a_2 \beta_1 \lambda \mu_{12} Z e^{-(r-y_{12}+\beta)(T-t)}}{(b_1 b_2 - \lambda^2 \mu_{12}^2)\omega} \\
\frac{a_2 \eta_2 b_1 - a_1 \beta_1 \lambda \mu_{12} Z e^{-(r-y_{12}+\beta)(T-t)}}{(b_1 b_2 - \lambda^2 \mu_{12}^2)\omega} , & 0 \leq t \leq t_1, \\
\frac{\alpha}{\sigma^2} e^{-(r-y_{12}+\beta)(T-t)} - \frac{a_2 \beta_2 e^{-(r-y_{12}+\beta)(T-t)}}{\omega b_2} , & t_1 < t \leq t_7, \\
\frac{\alpha}{\sigma^2} e^{-(r-y_{12}+\beta)(T-t)} , & t_7 < t \leq T,
\end{array} \right.
\]

and the value function is given by

\[
V(t,x,l) = \begin{cases}
Q_1(t,x,l) + Q_5(t) + R_2, & 0 \leq t \leq t_1, \\
Q_1(t,x,l) + Q_8(t) + Q_{10}(t) + Q_{11}(t) + R_5, & t_1 < t \leq t_7, \\
t_7 < t \leq T,
\end{cases}
\]

\[
(p_1^*(t), q_1^*(t), q_2^*(t)) = \left\{ \begin{array}{l}
\frac{\alpha}{\sigma^2} e^{-(r-y_{12}+\beta)(T-t)} - \frac{a_1 \eta_1 \beta_1 - a_2 \beta_1 \lambda \mu_{12} Z e^{-(r-y_{12}+\beta)(T-t)}}{(b_1 b_2 - \lambda^2 \mu_{12}^2)\omega} , & 0 \leq t \leq t_8, \\
\frac{\alpha}{\sigma^2} e^{-(r-y_{12}+\beta)(T-t)} , & t_8 < t \leq T,
\end{array} \right.
\]

and the value function is given by

\[
\begin{align*}
Q_1(t,x,l) + Q_{12}(t) + R_6, & \quad 0 \leq t \leq t_8, \\
Q_1(t,x,l) + Q_8(t) + Q_{13}(t), & \quad t_8 < t \leq T,
\end{align*}
\]

where \( Q_{12}(t), Q_{13}(t), \) and \( R_6 \) are given by (79)–(83), respectively.

Remark 2. (i) Since

where \( Q_5(t), Q_6(t), Q_7(t), Q_8(t), Q_9(t), R_3, \) and \( R_2 \) are given by (57)–(67), respectively.

(iv) If Case 4 holds, the optimal investment-reinsurance strategies for model (14) are

\[
V(t,x,l) \quad \text{is a continuous function for any} \quad (t,x,l) \in [0,T] \times \mathbb{R} \times \mathbb{R}. \quad \text{Furthermore},
\]

\[
\begin{align*}
Q_2(t_3) + R_1 & = Q_3(t_3) + Q_4(t_3), \\
Q_3(t_2) + R_2 & = Q_3(t_2) + Q_6(t_2) + Q_7(t_3) + R_5, \\
Q_3(t_2) + Q_8(t_3) + Q_2(t_5) + R_3 & = Q_4(t_5) + Q_6(t_5) + Q_8(t_3), \\
Q_2(t_7) + R_4 & = Q_4(t_7) + Q_{10}(t_7) + Q_{11}(t_7) + R_5, \\
Q_8(t_7) + Q_{10}(t_7) + Q_{11}(t_7) + R_5 & = Q_5(t_7) + Q_6(t_7) + Q_9(t_7), \\
Q_{12}(t_8) + R_6 & = Q_6(t_8) + Q_{13}(t_8), \\
\end{align*}
\]
\[
\begin{align*}
Q'_k(t_2) &= Q'_k(t_2) + Q'_k(t_2), \\
Q'_l(t_2) &= Q'_l(t_2) + Q'_l(t_2), \\
Q'_s(t_2) &= Q'_s(t_2) + Q'_s(t_2), \\
Q'_1(t_2) &= Q'_1(t_2) + Q'_1(t_2) + Q'_1(t_2), \\
Q'_2(t_2) &= Q'_2(t_2) + Q'_2(t_2), \\
Q'_3(t_2) &= Q'_3(t_2) + Q'_3(t_2), \\
Q'_4(t_2) &= Q'_4(t_2) + Q'_4(t_2) + Q'_4(t_2), \\
Q'_5(t_2) &= Q'_5(t_2) + Q'_5(t_2) + Q'_5(t_2).
\end{align*}
\]

which includes that \( V(t, x, l) \) is a classical solution to the extended HJB (18).

(ii) According to Theorem 2, the investment and reinsurance strategy of the insurer is not directly affected by the average parameter \( A \) and the delay time \( h \), but according to (13), the average parameter \( A \) and the delay time \( h \) have an indirect influence on the investment and reinsurance strategy of insurance companies.

(iii) Note that, in the classification discussion of Theorem 2, in order to make the classification clear, we assume that \( r - \gamma_1 - \gamma_2 + \beta \geq 0 \). For \( r - \gamma_1 - \gamma_2 + \beta < 0 \), we can also make a similar discussion.

When \( A = h = \beta = \gamma_1 = \gamma_2 = 0 \), problem (14) degenerates to the case without time delay.

**Corollary 1.** Without time delay, the optimal time-consistent investment and reinsurance policies of problem (14) are as follows:

(i) If Case 1 holds, the optimal investment-reinsurance strategies for problem (14) are

\[
(p^*_1(t), q^*_1(t), q^*_2(t)) = \begin{cases} \\
\left( \frac{a}{w\sigma^2}e^{-r(T-t)}, 0, \frac{a_2\eta_2}{wb_2}e^{-r(T-t)} \right), & 0 \leq t \leq t_3, \\
\left( \frac{a}{w\sigma^2}e^{-r(T-t)}, 0, 1 \right), & t_3 < t \leq T,
\end{cases}
\]

and the value function is given by

\[
V(t, x, l) = \begin{cases} \\
\bar{Q}_1(t, x, l) + \bar{Q}_2(t) + R_1, & 0 \leq t \leq t_3, \\
\bar{Q}_1(t, x, l) + \bar{Q}_3(t) + \bar{Q}_4(t), & t_3 < t \leq T,
\end{cases}
\]

where

\[
\begin{align*}
\bar{Q}_1(t, x, l) &= e^{r(T-t)}(x + \beta l) + \frac{c - a_1(1 + \eta_1) - a_2(1 + \eta_2)}{r} \left( e^{r(T-t)} - 1 \right) + \frac{a^2}{2w\sigma^2} (T - t), \\
\bar{Q}_2(t) &= \frac{a_2\eta_2}{2wb_2} (T - t), \\
\bar{Q}_3(t) &= \frac{a_2\eta_2}{r} \left( e^{r(T-t)} - 1 \right), \\
\bar{Q}_4(t) &= \frac{wb_2}{4r} \left( e^{r(T-t)} - 1 \right), \\
\bar{R}_1 &= \bar{Q}_3(t_3) + \bar{Q}_4(t_3) - \bar{Q}_2(t_3).
\end{align*}
\]
(ii) If Case 2 holds, the optimal investment-reinsurance strategies for problem (14) are

\[
(p_1^*(t), q_1^*(t), q_2^*(t)) = \begin{cases} 
\left( \frac{a}{\sigma^2 \omega} e^{-r(T-t)} \left( \frac{1}{b_1 b_2 - \lambda^2 \mu_{1Y} \mu_{1Z} \omega} \right), a_i \eta_1 b_2 - a_i \eta_1 \lambda \mu_{1Y} \mu_{1Z} \omega^{-r(T-t)} \right), & 0 \leq t \leq t_2, \\
\left( \frac{\alpha}{\sigma^2 \omega} e^{-r(T-t)} \left( - \frac{a_i \eta_1}{b_1} \right), t_2 < t \leq t_5, \\
\left( \frac{\alpha}{\sigma^2 \omega} e^{-r(T-t)}, 1, 1 \right), & t_5 < t \leq T,
\end{cases}
\]

and the value function is given by

\[
V(t, x, l) = \begin{cases} 
\tilde{Q}_1(t, x, l) + \tilde{Q}_5(t) + R_2, & 0 \leq t \leq t_2, \\
\tilde{Q}_1(t, x, l) + \tilde{Q}_3(t) + \tilde{Q}_5(t) + R_3, & t_2 < t \leq t_5, \\
\tilde{Q}_1(t, x, l) + \tilde{Q}_5(t) + \tilde{Q}_6(t) + \tilde{Q}_5(t), & t_5 < t \leq T,
\end{cases}
\]

where

\[
\begin{align*}
\tilde{Q}_5(t) & = \frac{1}{\omega} \left( a_i \eta_1 D_1 + a_i \eta_2 D_2 - \frac{1}{2} b_1 D_1^2 - \frac{1}{2} b_2 D_2^2 - \lambda \mu_{1Y} \mu_{1Z} D_1 D_2 \right) (T-t), \\
\tilde{Q}_6(t) & = \frac{a_i \eta_1^2}{2 \omega b_1} (T-t) - \frac{\lambda a_i \eta_1 \mu_{1Y} \mu_{1Z}}{b_1 r} \left( e^{r(T-t)} - 1 \right), \\
\tilde{Q}_7(t) & = \left( \frac{\omega \lambda^2 \mu_{1Y} \mu_{1Z}^2}{4 b_1 r} - \frac{\omega b_2}{4 r} \right) (e^{r(T-t)}), \\
\tilde{Q}_8(t) & = \frac{a_i \eta_1}{r} (e^{r(T-t)} - 1), \\
\tilde{Q}_9(t) & = -\frac{\omega (b_1 + b_2 + 2 \lambda \mu_{1Y} \mu_{1Z})}{4 r} \left( e^{r(T-t)} - 1 \right), \\
\tilde{R}_3 & = \tilde{Q}_3(t_5) + \tilde{Q}_6(t_5) - \tilde{Q}_5(t_5), \\
\tilde{R}_2 & = \tilde{Q}_3(t_2) + \tilde{Q}_6(t_2) + \tilde{Q}_7(t_2) + \tilde{Q}_8(t_2) + \tilde{Q}_9(t_2) - 2 \tilde{Q}_5(t_5) - \tilde{Q}_6(t_5) - \tilde{Q}_7(t_5) - \tilde{Q}_8(t_5) - \tilde{Q}_9(t_5).
\end{align*}
\]
(iii) If Case 3 holds, the optimal investment-reinsurance strategies for problem (14) are

\[
(p_1^*(t), q_1^*(t), q_2^*(t)) = \begin{cases} 
\left( \frac{\alpha}{\sigma_1^2 \omega} e^{-r(T-t)} \omega b_1, \frac{\alpha_2 \eta_2 \mu_1 \mu_1 Z e^{-r(T-t)}}{b_1 b_2 - \lambda \mu_1 \mu_1 Z} \omega b_2 \right), & 0 \leq t \leq t_1, \\
\left( \frac{\alpha}{\sigma_1^2 \omega} e^{-r(T-t)}, 1, \frac{\alpha_2 \eta_2 \mu_1 \mu_1 Z e^{-r(T-t)}}{\omega b_2} \right), & t_1 < t \leq t_7, \\
\left( \frac{\alpha}{\sigma_1^2 \omega} e^{-r(T-t)}, 1, 0 \right), & t_7 < t \leq T,
\end{cases}
\]

and the value function is given by

\[
V(t, x, l) = \begin{cases} 
\bar{Q}_1(t, x, l) + \bar{Q}_5(t) + R_4, & 0 \leq t \leq t_1, \\
\bar{Q}_1(t, x, l) + \bar{Q}_6(t) + \bar{Q}_{10} + \bar{Q}_{11}(t) + R_9, & t_1 < t \leq t_7, \\
\bar{Q}_6(t), & t_7 < t \leq T,
\end{cases}
\]

where

\[
\bar{Q}_{10}(t) = \frac{a_2^2 \eta_2^2}{2 \omega b_2} (T-t) - \frac{\lambda a_2 \eta_2 \mu_1 \mu_1 Z}{b_1 r} (e^{r(T-t)} - 1),
\]

\[
\bar{Q}_{11}(t) = \left( \frac{\omega \lambda \mu_1 \mu_1 Z}{4 b_2 r} - \frac{\omega b_1}{4r} \right) (e^{r(T-t)}).
\]

(iv) If Case 4 holds, the optimal investment-reinsurance strategies for problem (14) are

\[
(p_1^*(t), q_1^*(t), q_2^*(t)) = \begin{cases} 
\left( \frac{\alpha}{\sigma_1^2 \omega} e^{-r(T-t)} \omega b_1, 0 \right), & 0 \leq t \leq t_8, \\
\left( \frac{\alpha}{\sigma_1^2 \omega} e^{-r(T-t)}, 1, 0 \right), & t_8 < t \leq T,
\end{cases}
\]

and the value function is given by

\[
\begin{cases} 
\bar{Q}_1(t, x, l) + \bar{Q}_{12}(t) + R_6, & 0 \leq t \leq t_8, \\
\bar{Q}_1(t, x, l) + \bar{Q}_8(t) + \bar{Q}_{13}(t), & t_8 < t \leq T,
\end{cases}
\]

5. Numerical Simulations

In this section, Example 1 will be used to illustrate the specific numerical calculation process of finding the optimal
that the optimal time-consistent investment strategy. From Figure 1, we can see

\[ \text{Table 1: Optimal time-consistent strategy in } \tilde{\Pi}. \]

| t  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|----|----|----|----|----|----|----|----|----|----|
| \( \hat{p}_1 \) | 3.8978 | 4.6503 | 5.5481 | 6.6192 | 7.8971 | 9.4218 | 11.2407 | 13.4109 | 16.0000 |
| \( \hat{q}_1 \) | 0.4213 | 0.5026 | 0.5997 | 0.7155 | 0.8536 | 1.0184 | 1.2150 | 1.4496 | 1.7294 |
| \( \hat{q}_2 \) | 0.6019 | 0.7181 | 0.8567 | 1.0221 | 1.2194 | 1.4548 | 1.7357 | 2.0708 | 2.4706 |

time-consistent strategy, and Example 2 will be used to analyze the influence of important parameters on the optimal time-consistent strategy. Assuming that the claim amount \( Y_i \) and \( Z_i \) are exponentially distributed with parameters \( \xi_1 \) and \( \xi_2 \), respectively, then \( \mu_{1Y} = 1/\xi_1, \mu_{1Z} = 1/\xi_2 \), \( b_1 = 2(\lambda + \lambda_1)/\xi_1^2 \), and \( b_2 = 2(\lambda + \lambda_2)/\xi_2^2 \).

Example 1. Let \( \eta_1 = \eta_2 = 0.7, \xi_1 = 2, \xi_2 = 3, \lambda = 2, \lambda_1 = 3, \lambda_2 = 5, \alpha_1 = 0.5, \sigma = 0.2, T = 8, r = 0.18, \beta = 0.1, \ \lambda = 0.15, \) and \( h = 0.2 \) and according to Remark 1, we can calculate \( \gamma_1 = 0.0064 \) and \( \gamma_2 = 0.0970 \), and thus \( r - \gamma_1 - \gamma_2 + \beta = 0.1765 > 0 \). According to the above model parameters, Table 1 can be calculated.

From Table 1, for \( t \geq 5 \), we have \( \hat{q}_2(t) > 1 \). According to the analysis of Theorem 2, it is easy to see that \( t_2 = T - (1/r - \gamma_1 - \gamma_2 + \beta)\ln(D_1/\omega) = 2.8762 \) and \( t_5 = T - (1/r - \gamma_1 - \gamma_2 + \beta)\ln(a_1/\omega(b_1 + \lambda_1\mu_{1Y})) = 6.8029 \), \( t_2 < t \leq t_5 \), and hence \( \hat{q}_1(t) = a_1 \eta_1 e^{-(r \gamma_1 - \gamma_2 - \beta)(T - t)} = \omega \lambda_1 \mu_{1Z}/\omega b_1 \).

For \( t_2 < t \leq T \), we have \( \hat{q}_1(t) = 1 \). So, recalculate Table 1 to obtain Table 2.

Example 2. If there is no special description in this example, the basic parameter values are as follows: \( \eta_1 = \eta_2 = 0.7, \xi_1 = 2, \xi_2 = 3, \lambda = 2, \lambda_1 = 3, \lambda_2 = 4, \alpha_1 = 0.5, \sigma = 0.2, \) \( r = 0.18, \lambda = 0.1, \beta = 0.1, h = 0.2, \) and \( \omega = 0.5 \).

Figures 1 and 2 depict the influence of risk aversion parameter \( \omega \) and delay parameter \( \beta \) on the optimal time-consistent investment strategy. From Figure 1, we can see that the optimal time-consistent investment strategy \( p_1(t) \) decreases with the increase of risk aversion parameter \( \omega \), that is, the higher the risk aversion degree of the insurer is, the less the amount of risk investment will be. Because parameter \( \beta \) includes the information of average parameter \( \lambda \) and delay \( h \), it is a comprehensive time-delay parameter, so we only analyze \( \beta \). Figure 2 shows that the larger the delay parameter \( \beta \) is, the larger the number of investment in risky assets will be. Note that if \( \beta = 0 \), then the insurer decision-making is only based on the current information, so it may take short-term risk-taking behavior for the immediate possible high return. For \( \beta > 0 \), when the insurer is making decision, the comprehensive performance in the past period will be taken into account. Insurer focuses on information in a period when making decisions. According to (12), the greater the value of \( \beta \), the greater the proportion of average

| t  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|----|----|----|----|----|----|----|----|----|----|
| \( p_1^* \) | 1.9205 | 2.5032 | 3.2628 | 4.2528 | 5.5433 | 7.2253 | 9.4177 | 12.2753 | 16.0000 |
| \( q_1^* \) | 0.4213 | 0.5026 | 0.5997 | 0.4458 | 0.3577 | 0.6911 | 0.8502 | 1.0000 | 1.0000 |
| \( q_2^* \) | 0.6019 | 0.7181 | 0.8567 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

\[ \text{Figure 1: The effect of risk aversion parameter } \omega \text{ on } p_1. \]

\[ \text{Figure 2: The effect of delay parameter } \beta \text{ on } p_1. \]
wealth in performance measurement. That is, the insurer can change the inflow/outflow of the insurer’s capital by adjusting the size of the parameter beta, thus changing the risk faced by the insurer. The bigger the beta, the smaller the risk, so the insurer will consider increasing the number of risky assets.

Figures 3–6 depict the influence of risk aversion coefficient \( \omega \) and delay parameter \( \beta \) on two types of insurance reinsurance. According to Figures 3 and 4, \( q_1(t) \) and \( q_2(t) \) decrease with respect to \( \omega \). The higher the risk aversion degree of the insurer, the more reinsurance he will buy to reduce his risk, so the retention ratio of \( q_1(t) \) and \( q_2(t) \) will be reduced. Figures 5 and 6 show that the retention ratio \( q_1(t) \) (\( q_2(t) \)) increase with respect to the parameter \( \beta \). As the impact of \( \beta \) on investment strategy \( p_1 \). The larger the \( \beta \), the stronger the insurer’s ability to adjust capital inflow/outflow, that is, the stronger the insurer’s risk control ability. To a certain extent, the profitability of the insurer will be stronger, so the insurer will reduce the purchase of reinsurance, and the proportion of reinsurance retention \( q_1(t) \) (\( q_1(t) \)) will increase. This is consistent with economic reality, which the more information investors observe, the more profit they will make.

Figures 7–9 depict the effect of the claim intensity \( \lambda_1, \lambda_2 \), and \( \lambda \) on reinsurance. In Figure 7, the larger the \( \lambda_1 \) is, the larger the \( q_1(t) \) is and the smaller the \( q_2(t) \) is. Because the
larger the $\lambda_1$ is, the greater the expected claim amount of the first type of insurance business will be, so the insurer will purchase more reinsurance for the first type of insurance business and reduce the proportion of retained insurance $q_1(t)$. At this time, $\lambda_2$ will remain unchanged, that is, the expected claim amount of the second type of insurance business will remain unchanged. Based on the consideration of constant total risk and more profits, the insurer will increase the retention ratio $q_2(t)$ of reinsurance. A similar analysis can explain why; with the increase of $\lambda_2$, $q_1(t)$ decreases and $q_2(t)$ increases in Figures 8 and 9 which shows that the retention ratios $q_1(t)$ and $q_2(t)$ of the two types of insurance businesses decrease with the increase of lambda. Because the larger the lambda is, the greater the expected claim amount of the two types of insurance businesses will be. Therefore, in order to control the risk within a certain range, the insurer will buy more reinsurance for the two types of insurance businesses and reduce the retention ratio $q_1(t)$ and $q_2(t)$.

6. Conclusion

In this paper, we study the optimal investment-reinsurance problem with delay and risk dependence under the mean-variance preference criterion. Considering the time-delay effect and risk dependence, we obtain the extended HJB equation with delay based on the time delay stochastic control framework and the equilibrium stochastic control method. The results show that the optimal time-consistent investment and reinsurance strategy will be affected by the time delay effect. The larger the capital flow related to the historical business performance, the greater the risk faced by the insurance company. In a prudent attitude, the insurer will reduce the amount invested in a risk asset and reduce the reinsurance retention ratio of all insurance businesses. In addition, risk dependence is linked by common risk shock sources. The greater the risk common shock intensity is, the smaller the reinsurance retention ratio will be. From the numerical analysis results, we can see not only the numerical calculation process of the optimal strategy but also the intuitive verification of the above conclusions.

In this paper, we study the risk assets under geometric Brownian motion. To better simulate the real financial market, the following research will consider the introduction of CEV, Heston, and other stochastic volatility models, Vasicek, CIR, and other stochastic interest rate models.

Data Availability

The data in this paper can be used publicly.

Conflicts of Interest

The authors declare that they have no conflicts of interest.
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