Robust weak Galerkin finite element methods for linear elasticity with continuous displacement trace approximation

Gang Chen†
School of Mathematics Sciences,
University of Electronic Science and Technology of China,
Chengdu 611731, China

Xiaoping Xie‡
School of Mathematics, Sichuan University, Chengdu 610064, China

Abstract
This paper proposes and analyzes a class of new weak Galerkin (WG) finite element methods for 2- and 3-dimensional linear elasticity problems. The methods use discontinuous piecewise-polynomial approximations of degrees \( k \geq 0 \) for the stress, \( k + 1 \) for the displacement, and a continuous piecewise-polynomial approximation of degree \( k + 1 \) for the displacement trace on the inter-element boundaries, respectively. After the local elimination of unknowns defined in the interior of elements, the WG methods result in SPD systems where the unknowns are only the degrees of freedom describing the continuous trace approximation. We show that the proposed methods are robust in the sense that the derived a priori error estimates are optimal and uniform with respect to the Lamé constant \( \lambda \). Numerical experiments confirm the theoretical results.

Keywords: linear elasticity; weak Galerkin method; robust a priori error estimate; strong symmetric stress

1 Introduction.
Let \( \Omega \subset \mathbb{R}^d \) (\( d = 2, 3 \)) be a polyhedral region with boundary \( \partial \Omega = \Gamma_D \cup \Gamma_N \), where \( \text{meas}(\Gamma_D) > 0 \) and \( \Gamma_D \cap \Gamma_N = \emptyset \). We consider the following linear isotropic elasticity model:

\[
\begin{cases}
  \mathcal{A}\sigma - \epsilon(u) = 0, \text{ in } \Omega, \\
  \nabla \cdot \sigma = f, \text{ in } \Omega, \\
  u = g_D, \text{ on } \Gamma_D, \\
  \sigma n = g_N, \text{ on } \Gamma_N.
\end{cases}
\]  

(1.1)

*This work was supported by National Natural Science Foundation of China (11771312) and Major Research Plan of National Natural Science Foundation of China (91430105).
†Email: 569615491@qq.com
‡Corresponding author. Email: xpxie@scu.edu.cn
Here $\sigma : \Omega \to \mathbb{R}^{d \times d}_{\text{sym}}$ denotes the symmetric $d \times d$ stress tensor field, $u : \Omega \to \mathbb{R}^d$ the displacement field, $\epsilon(u) = (\nabla u + (\nabla u)^T)/2$ the strain tensor, and $A\sigma \in \mathbb{R}^{d \times d}_{\text{sym}}$ the compliance tensor with

$$A\sigma = \frac{1}{2\mu} \left( \sigma - \frac{\lambda}{2\mu + d\lambda} \text{tr}(\sigma) I \right),$$

where $\lambda > 0, \mu > 0$ are the Lamé coefficients, $\text{tr}(\sigma)$ denotes the trace of $\sigma$, and $I$ is the $d \times d$ identity matrix. $f$ is the body force acting on $\Omega$, $n$ is the unit outward vector normal to $\Gamma_N$, and $g_D$ and $g_N$ are the surface displacement on $\Gamma_D$ and the surface traction on $\Gamma_N$, respectively.

It is well-known [56] that conforming finite element methods of displacement types suffer from a performance deterioration, called Poisson-locking, as the material becomes incompressible or, equivalently, the Lamé constant $\lambda$ tends to $\infty$.

When using a mixed finite element method based on Hellinger-Reissner variational principle to solve the model (1.1), the combination of stress and displacement approximation is required to satisfy two stability conditions, i.e. a coercivity condition and an inf-sup condition; see, e.g. [1, 2, 4, 3, 5, 6, 8, 13, 20, 26, 29, 30, 54, 55, 60]. These stability constraints usually lead to complicated construction of finite element combinations, in which the stress finite elements are of many degrees of freedom. In particular, when dealing with nearly incompressible materials, one needs some more-severe stability condition for the finite element combination so as to derive a uniformly stable mixed method which is free from Poisson-locking.

Due to the relaxation of function continuity, hybrid stress/strain finite element methods based on generalized variational principles [43, 45, 46, 44, 49, 51, 61, 64, 65] are a class of special mixed approaches that allow for piecewise-independent approximation to the stress solution, and thus lead to symmetric and positive definite (SPD) discrete systems of nodal displacements after the local elimination of the stress unknowns defined in the interior of the elements. We refer to [7, 9, 10, 62, 37] for the stability and error estimation of some robust 4-node hybrid stress/strain quadrilateral/rectangular elements that hold uniformly with respect to the the Lamé constant $\lambda$.

The hybridizable discontinuous Galerkin (HDG) framework, proposed in[21] for second order elliptic problems, provides a unifying strategy for hybridization of finite element methods. In the HDG model, the constraint of function continuity on the inter-element boundaries is relaxed by introducing Lagrange multipliers defined on the inter-element boundaries. Similar to the hybrid stress/strain finite element methods, by the local elimination of the unknowns defined in the interior of elements, the HDG method results in a SPD system where the unknowns are only the globally coupled degrees of freedom describing the introduced Lagrange multipliers. We refer to [22, 23, 34] for the analysis of several HDG methods for diffusion equations.

The first HDG method for linear elasticity was proposed in [52, 53] and analyzed in [27], where strongly symmetric stresses and piecewise-polynomial approximations of degree $k$ in all variables are used. The method converges optimally for the displacement and the antisymmetric part of the displacement gradient, but sub-optimally for the stress and the strain, i.e. the symmetric part of the displacement gradient, with 1/2- order loss of accuracy. In [48] an HDG
method was presented based on a strong symmetric stress formulation, where the piecewise-polynomial approximations of degrees $k(\geq 1)$, $k+1$ and $k$ are used for the stress, displacement and the numerical trace of the displacement, respectively. The method was shown to yield optimal convergence for both the displacement and stress approximations with the constants in the upper bounds of the errors depending on $\lambda$; In particular, a suboptimal convergence rate for the stress approximation was shown to hold uniformly with respect to $\lambda$. We refer to [25, 31, 42, 41] for several related HDG methods for linear elasticity, nonlinear elasticity and elasto-dynamics. We also refer to [24] for a local discontinuous Galerkin (LDG) method with strongly symmetric stresses for linear elasticity.

Closely related to the HDG framework is the weak Galerkin (WG) method developed in [57, 59] for second-order elliptic problems. The WG method is designed by using a weakly defined gradient operator over functions with discontinuity, and allows the use of totally discontinuous functions in the finite element procedure. Similar to the hybrid stress/strain and HDG methods, the WG scheme leads to a SPD system through the local elimination of unknowns defined in the interior of elements. We refer to [38, 40, 39, 15, 16, 17, 58] for applications of the WG method to some other partial differential equations, and refer to [19, 33, 36, 35] for fast solvers of WG methods.

In a very recent work [18], we developed a class of WG methods with strong symmetric stresses for the model (1.1) on polygon or polyhedral meshes, where discontinuous piecewise-polynomial approximations of degrees $k(\geq 1)$, $k+1$ are used for the stress, the displacement, respectively, and discontinuous piecewise-polynomial approximation of degree $k$ is used for the displacement trace on the inter-element boundaries. Optimal convergence rates for both the displacement and stress approximations are obtained which hold uniformly with respect to the Lamé constant $\lambda$. We note that the methods in [18], as well as that in [48], are not applicable to the lowest order case $k = 0$.

In this paper, we shall propose a class of new weak Galerkin method with strong symmetric stresses for the model (1.1) on polygon or polyhedral meshes. We use discontinuous piecewise-polynomial approximations of degrees $k(\geq 0)$ for the stress, $k+1$ for the displacement, and a continuous piecewise-polynomial approximation of degree $k+1$ for the displacement trace on the inter-element boundaries, respectively. Compared with the WG methods in [18], the new methods have the following features.

- They adopt continuous approximation for the displacement trace, while the methods in [18] use discontinuous trace approximation.
- They allow the lowest order case $k = 0$.
- They yield optimal convergence rates for both the displacement and stress approximations which are uniformly with respect to the Lamé constant $\lambda$.
- After the local elimination of unknowns defined in the interior of elements, the unknowns of the resultant SPD systems of the new WG method are only the degrees of freedom describing the continuous piecewise-polynomial approximation of the displacement trace on the inter-element boundaries. Especially, the SPD systems is are of smaller sizes than the corresponding systems of the methods in [18].
The rest of this paper is arranged as follows. In Section 2 we introduce notation and WG finite element schemes. Section 3 derives stability results of the methods. Sections 4 and 5 are devoted to the a priori error estimation for the displacement and stress approximations, respectively. Finally, Section 6 provides numerical results.

Throughout this paper, we use $a \lesssim b$ to denote $a \leq Cb$, where $C$ is a positive constant independent of the mesh parameters $h$, $h_T$, $h_E$ and the Lamé coefficients $\lambda$ and $\mu$.

2 WG finite element scheme

2.1 Notations and preliminary results

For any bounded domain $\Omega \subset \mathbb{R}^s$ ($s = d, d - 1$), let $H^m(\Omega)$ and $H^m_0(\Omega)$ denote the usual $m$th-order Sobolev spaces on $\Omega$, and $\| \cdot \|_{m, \Omega}$ denote the norm and semi-norm on these spaces. We use $(\cdot, \cdot)_{m, \Omega}$ to denote the inner product of $H^m(\Omega)$, with $(\cdot, \cdot)_{0, \Omega}$. When $\Omega = \Omega$, we denote $\| \cdot \|_{m, \Omega} := \| \cdot \|_{m, \Omega}$. In particular, when $\Omega \in \mathbb{R}^{d-1}$, we use $(\cdot, \cdot)_{\Omega}$ to replace $(\cdot, \cdot)_{\Omega}$. We note that bold face fonts will be used for vector (or tensor) analogues of the Sobolev spaces along with vector-valued (or tensor-valued) functions. For an integer $k \geq 0$, $P_k(\Omega)$ denotes the set of all polynomials defined on $\Omega$ with degree not greater than $k$. In addition, we introduce the following two spaces:

$$L^2_0(\Omega) := \{ v | v \in L^2(\Omega) := H^0(\Omega), (v, 1)_{\Omega} = 0 \},$$

$$H(\text{div}, \Omega) := \{ v | v \in [L^2(\Omega)]^d, \nabla \cdot v \in L^2(\Omega) \}.$$

Let $T_h = \bigcup \{ T \}$ be a shape regular partition (to be defined later) of the domain $\Omega$ consists of arbitrary polygons, such that each (open) boundary edge (or face) belongs either to $\Gamma_D$ or to $\Gamma_N$, and there should be at least two edges belong the interior of $\Gamma_S$ ($S = D, N$) if $\Gamma_S \neq \emptyset$. We note that $T_h$ can be a conforming partition or a nonconforming partition which allows hanging nodes.

For any $T \in T_h$, we let $h_T$ be the infimum of the diameters of circles (or spheres) containing $T$ and denote the mesh size $h := \max_{T \in T_h} h_T$. An edge (or face) $E$ on the boundary $\partial T$ of $T$ is called a proper edge (or face) if the endpoints (or vertexes) of the edge (or face) $E$ are the nodes of $T_h$ and no other nodes of $T_h$ are on $E$. See Figure 2.1 for example, $EF$, $FH$ and $HI$ are proper edges, while $EH$, $FI$ and $EI$ are not. Let $E_h = \bigcup \{ E \}$ be the union of all proper edges (faces) of $T \in T_h$. We denote by $h_E$ the length of edge $E$ if $d = 2$ and the infimum of the diameters of circles containing face $E$ if $d = 3$. For all $T \in T_h$ and $E \in E_h$, we denote by $n_T$ and $n_E$ the unit outward normal vectors along $\partial T$ and $E$, respectively.

The partition $T_h$ is called shape regular in the sense that assumptions M1-M2 hold true.

- **M1** (Star-shaped elements). There exists a positive constant $\theta_\ast$ such that the following holds: for each element $T \in T_h$, there exists a point $M_T \in T$ such that $T$ is star-shaped with respect to every point in the circle (or sphere) of center $M_T$ and radius $\theta_\ast h_T$. 

Figure 2.1: Demonstrating of proper edges in 2D

- **M2** (Edges or faces). There exist a positive constant \( l \) such that: every element \( T \in \mathcal{T}_h \), the distance between any two vertexes (include the hang nodes) is \( \geq l h_T \).

When \( d = 2 \), for every \( T \in \mathcal{T}_h \), we connect \( M_T \) and \( T \)'s vertexes (including hang nodes) to get a set of triangles, \( w(T) \); when \( d = 3 \), for every \( T, T' \in \mathcal{T}_h \) and every face \( E \subset \partial T \cap \partial T' \), we choose any vertex \( A \) on \( E \) and connect \( A \) to the rest of \( E \)'s vertexes (including hang nodes) to get a set of triangles, \( v(E) \), then we connect \( M_T \) and every \( v(E) \) \( (E \subset \partial T) \) to get a set of tetrahedrons, \( w(T) \). Set

\[ \mathcal{T}^*_h := \bigcup_{T \in \mathcal{T}_h} w(T). \tag{2.1} \]

We note that \( \mathcal{T}^*_h \) is shape regular due to **M1** and **M2**. Let \( \mathcal{E}^*_h \) be the union of all the edges (faces) of \( \mathcal{T}^*_h \). Notice that when \( d = 2 \), it holds \( \mathcal{E}^*_h = \mathcal{E}_h \).

For any nonnegative integer \( j \), define

\[ \mathbb{P}_j(T_h) := \{ q_h : q_h|_T \in \mathbb{P}_j(T), \forall T \in \mathcal{T}_h \}. \]

The space \( \mathbb{P}_j(T^*_h) \) is defined similarly.

By using the inverse inequality and trace inequality on the shape regular simplex mesh \( \mathcal{T}^*_h \), it is easy to get the following inverse inequality and trace inequality on shape regular polygon mesh \( \mathcal{T}_h \).

**Lemma 2.1.** For all \( T \in \mathcal{T}_h \) and any given nonnegative integer \( j \), the following inequalities hold true:

\[ |q_h|_{1,T} \lesssim h_T^{-1} \| q_h \|_{0,T}, \quad \forall q_h \in \mathbb{P}_j(T), \tag{2.2} \]

\[ \| q \|_{0,\partial T} \lesssim h_T^{-1/2} \| q \|_{0,T} + h_T^{1/2} | q |_{1,T}, \quad \forall q \in H^1(T). \tag{2.3} \]

For every \( T \in \mathcal{T}_h \), let \( \gamma_T := \frac{h_T}{\delta_T} \) be the chunkiness parameter of \( T \), where \( \delta_T \) denotes the supremum of the radius of a sphere with respect to which \( T \) is star-shaped. Then, in view of **M1**, we have \( 2 \leq \gamma_T \leq \frac{h_T}{\delta_T} = \theta^{-1} \), i.e. \( \gamma_T \) is independent of \( h_T \). Thus, by (Lemma 4.3.8, [14]) we obtain the following conclusion.
Lemma 2.2. For all $T \in \mathcal{T}_h$ and $v \in H^m(T)$ with $m \geq 1$, there exists $I_{m-1}v \in P_{m-1}(T)$ such that
\[ |v - I_{m-1}v|_{s,T} \lesssim h_{m-1}^s|v|_{m,T}, \quad \text{for } 0 \leq s \leq m. \quad (2.4) \]

In Appendix A we construct a modified Scott-Zhang interpolation operator, $I_{k+1}$, and derive some properties.

Lemma 2.3. For any integer $k \geq 0$, there exists an interpolation operator $I_{k+1} : H^1(\Omega) \to H^1(\Omega) \cap \mathbb{P}_{k+1}(\mathcal{T}_h^*)$, such that the following properties hold:
\[ \langle I_{k+1}v, 1 \rangle_\Gamma = \langle v, 1 \rangle_\Gamma \quad \text{with } \Gamma = \Gamma_D, \Gamma_N, \quad \forall v \in H^1(\Omega), \quad (2.5) \]
\[ I_{k+1}v|_{\Gamma_D} = 0, \quad \forall v \in H^1_D(\Omega) := \{ v \in H^1(\Omega) : v|_{\Gamma_D} = 0 \}, \quad (2.6) \]
\[ |v - I_{k+1}v|_{s,T} \lesssim h^s_{m-1}|v|_{m,T}, \quad \forall v \in H^m(\Omega), \quad T \in \mathcal{T}_h^*, \quad (2.7) \]
\[ \|v - I_{k+1}v\|_{0,\partial T} \lesssim h^{m-1/2}_{m}|v|_{m,T}, \quad \forall v \in H^m(\Omega), \quad T \in \mathcal{T}_h^*, \quad (2.8) \]
where $s, m$ are integers satisfying $1 \leq m$ and $0 \leq s \leq m \leq k + 2$, and
\[ S(T) := \begin{cases} S'(T), & \text{if } k + 1 \geq d, \\ \bigcup_{T' \in \mathcal{T}_h^* \setminus T} T', & \text{if } k + 1 < d \end{cases} \quad (2.9) \]
with
\[ S'(T) := \bigcup_{T' \in \mathcal{T}_h^* \setminus T} T'. \quad (2.10) \]

Proof. The proofs of $(2.5)$-$\text{(2.7)}$ are given in Appendix A, and the estimate $(2.8)$ follows from $(2.7)$ and Lemma 2.1. \qed

2.2 Discrete weak gradient/divergence operators

We follow [40] to introduce the definitions of the discrete weak gradient/divergence operators. For $T \in \mathcal{T}_h^*$, denote by $\mathcal{V}(T)$ a space of weak functions on $T$ with
\[ \mathcal{V}(T) = \{ v = \{v_i, v_b\} : v_i \in L^2(T), v_b \in H^{1/2}(\partial T) \}. \quad (2.11) \]

Let $G(T) \subset H(\text{div}, T)$ be a local finite dimensional vector space. We define the discrete weak gradient operator $\nabla_{w,G,T} : \mathcal{V}(T) \to G(T)$ as follows.

Definition 2.4. For all $v \in \mathcal{V}(T)$, its discrete weak gradient $\nabla_{w,G,T}v \in G(T)$ satisfies the equation
\[ (\nabla_{w,G,T}v, \tau)_T = -(v_i, \nabla \cdot \tau)_T + (v_b, \tau \cdot n_{\partial T})_{\partial T}, \quad \forall \tau \in G(T). \quad (2.12) \]

Then we define the global discrete weak gradient operator $\nabla_{w,G}$ with
\[ \nabla_{w,G}|T = \nabla_{w,G,T}, \quad \forall T \in \mathcal{T}_h. \]
In particular, if $G(T) = [\mathbb{P}_r(T)]^d$, we write $\nabla_{w,r} := \nabla_{w,G}$. For a vector $v = (v_1, \cdots, v_d)^T \in [\mathcal{V}(T)]^d$, we define the weak gradient $\nabla_{w,r}v$ as
\[ \nabla_{w,r}v = (\nabla_{w,r}v_1, \cdots, \nabla_{w,r}v_d)^T. \]
Let $W(T)$ be a space of weak vector-valued functions on $T$ with

$$W(T) := \{ \mathbf{v} = (v_i, v_h) : v_i \in [L^2(T)]^d, v_h \cdot \mathbf{n}_T \in H^{-1/2}(\partial T) \},$$

(2.13)

and

$$G(T) \subset H^1(T)$$

be a local finite dimensional space. We define the discrete weak divergence operator $\nabla_{w,G,T} : W(T) \to G(T)$ as follows.

**Definition 2.5.** For any $\mathbf{v} \in W(T)$, its discrete weak divergence $\nabla_{w,G,T} \cdot \mathbf{v} \in G(T)$ satisfies the equation

$$\left( \nabla_{w,G,T} \cdot \mathbf{v}, \tau \right)_T = -\left( \mathbf{v}, \nabla \tau \right)_T + \left( \mathbf{v}_h \cdot \mathbf{n}_T, \tau \right)_G \quad \forall \tau \in G(T).$$

(2.14)

Then we define the global discrete weak divergence operator $\nabla_{w,G}$ with $\nabla_{w,G} : \nabla_{w,G,T} \cdot \mathbf{v} \in G(T)$.

In particular, if $G(T) = \mathbb{P}_r(T)$, we write $\nabla_{w,r} := \nabla_{w,G}$.

### 2.3 New WG finite element schemes

For any $T \in T_h$, $E \in \mathcal{E}_h$ and any nonnegative integer $j$, let $Q_j^d : L^2(\Omega) \to \mathbb{P}_j(\Omega)$ and $Q_j^b : L^2(E) \to \mathbb{P}_j(E)$ be the usual $L^2$ projection operators. Vector or tensor analogues of $Q_j^d$ and $Q_j^b$ are denoted by $Q_j^\tau$ and $Q_j^{\Sigma}$, respectively.

For any integer $k \geq 0$, we introduce the following finite dimensional spaces:

$$U_{hi} := \{ v_{hi} \in [L^2(\Omega)]^d : v_{hi}|_T \in [\mathbb{P}_{k+1}(T)]^d, \forall T \in T_h \},$$

(2.15)

$$U_{hb} := \{ v_{hb} \in [C^0(\mathcal{E}_h)]^d : v_{hb}|_E \in [\mathbb{P}_{k+1}(E)]^d, \forall E \in \mathcal{E}_h \},$$

(2.16)

$$\Sigma_{hi} := \{ \tau_{hi} \in [L^2(\Omega)]^d : T_{hi} = \tau_{hi} \text{ and } \tau_{hi}|_T \in [\mathbb{P}_h(T)]^d, \forall T \in T_h \}. $$

(2.18)

For simplicity we set

$$U_h := U_{hi} \times U_{hb}, \quad \bar{U}_h := U_{hi} \times U_{hb}$$

(2.19)

In the case of $k + 1 < d$, the following space is also required for stabilization:

$$\Sigma_{tr} := \{ \tau_{tr} \in C^0(\mathcal{E}_h/\partial \Omega) : \tau_{tr}|_E \in \mathbb{P}_1(E), \forall E \in \mathcal{E}_h/\partial \Omega \}. $$

(2.20)

Then the stabilized WG (new WG) finite element discretization of the elasticity model (1.1) is given in the following two cases:

(i) Case of $k + 1 < d$:

Find $\mathbf{u}_{hi} \in \Sigma_{hi}, \sigma_{tr} \in \Sigma_{tr}^r, u_{hb} \in U_{hb}^0$ such that

$$a_h(\sigma_{hi}, \tau_{hi}) - b_h(\tau_{hi}, u_{hi}) + z_h(\sigma_{hi}, \sigma_{tr}, \tau_{hi}, \tau_{hb}) = 0, \quad \forall \tau_{hi} \in \Sigma_{hi}, \tau_{tr} \in \Sigma_{tr},$$

(2.21)

$$-b_h(\sigma_{hi}, v_h) - s_h(u_{hi}, v_h) = (f, v_h) - (g_N, v_h|_{\Gamma_N}), \quad \forall v_h := \{ v_{hi}, v_{hb} \} \in U_h^0.$$  

(2.22)

(ii) Case of $k + 1 \geq d$:

Find $\mathbf{u}_{hi} \in \Sigma_{hi}, u_{hb} \in U_{hb}^0$ such that

$$a_h(\sigma_{hi}, \tau_{hi}) - b_h(\tau_{hi}, u_{hi}) = 0, \quad \forall \tau_{hi} \in \Sigma_{hi},$$

(2.23)

$$-b_h(\sigma_{hi}, v_h) - s_h(u_{hi}, v_h) = (f, v_h) - (g_N, v_h|_{\Gamma_N}), \forall v_h \in U_h^0.$$  

(2.24)
Here
\[ a_h(\sigma_{hi}, \tau_{hi}) := (A\sigma_{hi}, \tau_{hi}), \quad (2.25) \]
\[ b_h(\tau_{hi}, v_h) := (\tau_{hi}, \nabla_w v_h), \quad (2.26) \]
\[ s_h(u_h, v_h) := (\alpha(u_{hi} - u_{hh}), v_{hi} - v_{hh})_{\partial T_h}, \quad (2.27) \]
\[ z_h(\sigma_{hi}, \sigma_{hi}^{tr}, \tau_{hi}, \tau_{hi}^{tr}) := (\beta(\text{tr}(\sigma_{hi}) - \sigma_{hi}^{tr}), \text{tr}(\tau_{hi}) - \tau_{hi}^{tr})_{\partial T_h/\partial \Gamma}, \quad (2.28) \]
and the parameter \( \alpha \) and \( \beta \) are taken as
\[ \alpha|_E := 2\mu h_E^{-1} \quad \text{for any } E \in \mathcal{E}_h, \quad (2.29) \]
\[ \beta|_E := (2\mu)^{-1} h_E \quad \text{for any } E \in \mathcal{E}_h/\partial \Omega. \quad (2.30) \]

**Remark 2.1.** We note that the stabilization term \( z_h(\cdot, \cdot, \cdot) \) is the key to locking-free performance when \( k + 1 < d \).

**Remark 2.2.** The WG methods in [18] are based on the following formulations with \( k \geq 1 \). Find \( \sigma_{hi} \in \Sigma_{hi}, u_h = \{u_{hi}, u_{hh}\} \in \tilde{U}_h^{\text{rg}} := U_h \times \tilde{U}_h^{\text{rg}} \) such that
\[ a_h(\sigma_{hi}, \tau_{hi}) - b_h(\tau_{hi}, u_h) = 0, \quad \forall \tau_{hi} \in \Sigma_{hi}, \quad (2.31) \]
\[-b_h(\sigma_{hi}, v_h) - s_h(u_h, v_h) = (f, v_h) - (g_N, v_{hh})_{\Gamma_N}, \forall v_h \in \tilde{U}_h^0. \quad (2.32)\]
where
\[ \tilde{U}_h^0 := \{v_{hh} \in [L^2(\mathcal{E}_h)]^d : v_{hh}|_{\Gamma_h} = Q_h^0 g, \ v_{hh}|_E \in [P_k(E)]^d, \forall E \in \mathcal{E}_h \} \]
with \( g = g_D, 0 \), and
\[ s_h(u_h, v_h) := (\alpha(Q_h^0 u_{hi} - u_{hh}), Q_h^0 v_{hi} - v_{hh})_{\partial T_h}. \]
In fact, by following the routine of analysis in [18], the case of \( k = 0 \) for (2.31)-(2.32) also works if \( \tilde{U}_h^0 \) in this case is modified as
\[ \tilde{U}_h^0 := \{v_{hh} \in [L^2(\mathcal{E}_h)]^d : v_{hh}|_{\Gamma_h} = Q_h^0 g, \ v_{hh}|_E \in [P_1(E)]^d, \forall E \in \mathcal{E}_h \}. \]
As a result, for the WG scheme (2.31)-(2.32) with \( k \geq 0 \), the following error estimates hold true ([18]):
\[ \|\sigma - \sigma_{hi}\|_0 \lesssim h^{k+1}(|\sigma|_{k+1} + \mu|u|_{k+2}), \]
\[ \|\nabla u - \nabla_k u_{hi}\|_0 \lesssim h^{k+1}(\mu^{-1}|\sigma|_{k+1} + |u|_{k+2}). \]
In addition, under the regularity assumption (5.2), it holds
\[ \|u - u_{hi}\|_0 \lesssim h^{k+2}(\mu^{-1}|\sigma|_{k+1} + |u|_{k+2}). \]

To establish error estimates for the proposed WG schemes, we need the following approximation and stability results for the \( L^2 \)-projections \( Q_j^L \) and \( Q_j^B \) with nonnegative integer \( j \).

**Lemma 2.6.** [18] Let \( m \) be an integer with \( 1 \leq m \leq j + 1 \). For all \( T \in \mathcal{T}_h, \ E \in \mathcal{E}_h \), it holds
\[ \|Q_j^L v\|_{0,T} \leq \|v\|_{0,T}, \forall v \in L^2(T), \quad (2.33) \]
\[ \|Q_j^B v\|_{0,E} \leq \|v\|_{0,E}, \forall v \in L^2(E), \quad (2.34) \]
\[ \|v - Q_j^L v\|_{0,\partial T} \lesssim h^{-m/2}_T|v|_{m,T}, \forall v \in H^m(T), \quad (2.35) \]
\[ |v - Q_j^B v|_{s,T} \lesssim h^{-s}_T|v|_{m,T}, \forall v \in H^m(T), \quad 0 \leq s \leq m, \quad (2.36) \]
\[ \|\nabla^s(v - Q_j^L v)\|_{0,\partial T} \lesssim h^{-s-1/2}_T|v|_{m,T}, \forall v \in H^m(T), \quad 1 \leq s + 1 \leq m. \quad (2.37) \]
3 Stability

We first show the following inf-sup stability condition holds for the bilinear form $b_h(\cdot, \cdot)$.

**Theorem 3.1.** Denote $\epsilon_{w,k}(v_h) := (\nabla \cdot v_h + (\nabla \cdot v_h)^T)/2$. Then, for all $v_h \in U_h$, it holds

$$
\sup_{\tau_{hi} \in \Sigma_{hi}} \frac{b_h(\tau_{hi}, v_h)}{\|\tau_{hi}\|_0} \geq \|\epsilon_{w,k}(v_h)\|_0. \tag{3.1}
$$

**Proof.** The conclusion follows from the fact that $\epsilon_{w,k}(v_h) \in \Sigma_{hi}$ for any $v_h \in U_h$. \qed

To derive stability conditions for the bilinear form $a_h(\cdot, \cdot)$, we need the following lemma.

**Lemma 3.2.** For all $p_{hi} \in \mathbb{P}_k(T_h) \cap L^2(\Omega)$ if $\Gamma_N \neq \emptyset$, and all $p_{hi} \in \mathbb{P}_k(T_h) \cap L^2_0(\Omega)$ if $\Gamma_N = \emptyset$, it holds

$$
\|p_{hi}\|_0^2 \lesssim \begin{cases} 
\sum_{E \in E_h} h_E \|p_{hi}\|_{0,E}^2 + \left[ \sup_{v_{hi} \in X_{hi}} \frac{\|\nabla \cdot v_{hi} \cdot p_{hi}\|_{0,E}}{|v_{hi}|_{1,E}} \right]^2 & \text{if } k + 1 < d, \\
\left[ \sup_{v_{hi} \in X_{hi}} \frac{\|\nabla \cdot v_{hi} \cdot p_{hi}\|_{0,E}}{|v_{hi}|_{1,E}} \right]^2 & \text{if } k + 1 \geq d,
\end{cases} \tag{3.2}
$$

where

$$
X_{hi} := \{ v_{hi} \in [H^1_0(\Omega)]^d : v_{hi}|_E \in [\mathbb{P}_{k+1}(E)]^d, \forall E \in E_h \}. 
$$

**Proof.** From Theorem B.3 in Appendix B we know that the Stokes elements that are stable in $[H^1_0(\Omega)]^d \times L^2(\Omega)$ are also stable in $[H^1_0(\Omega)]^d \times L^2(\Omega)$.

If $k + 1 \geq d$, by applying the stable Stokes-elements $(\mathbb{P}_k/\mathbb{P}_k)$ by [50, 47, 63, ?] on the barycenter refined mesh of $T_h^*$, the desired estimate (3.2) follows directly.

If $k + 1 < d$, we shall make use of the MINI element $\mathbb{P}_1^b/\mathbb{P}_1$ on the mesh $T_h$ with $\mathbb{P}_1^b = \mathbb{P}_1 \oplus \{ \text{bubbles} \}$. Note that in this case we have $k \leq 1$. From [32], there exists an interpolation operator $Q_1^b : \mathbb{P}_1(T_h^*) \to \mathbb{P}_1(T_h^*) \cap H^1(\Omega)$ such that, for all $q_{hi} \in \mathbb{P}_1(T_h^*)$,

$$
\sum_{T \in T_h^*} \|q_{hi} - Q_1^b q_{hi}\|_{0,T}^2 \lesssim \sum_{E \in E_h^*/\partial \Omega} h_E^{1/2} \|q_{hi}\|_{0,E}^2. \tag{3.3}
$$

Then, for all $p_{hi} \in \mathbb{P}_k(T_h) \cap L^2(\Omega)$ if $\Gamma_N \neq \emptyset$, and all $p_{hi} \in \mathbb{P}_k(T_h) \cap L^2_0(\Omega)$ if
\[ \Gamma_N = 0, \text{ we have} \]
\[ \| p_h \|_0^2 \lesssim \| p_h - Q_h p_h \|_0^2 + \| Q_h^* p_h \|_0^2 \]
\[ \lesssim \| p_h - Q_h p_h \|_0^2 + \left[ \sup_{v_h \in [P]_1(\Gamma \cap H^1_0(\Omega)) \setminus \emptyset} \frac{(\nabla \cdot v_h, Q_h^* p_h)}{|v_h|_1^2} \right] \]
\[ \lesssim \| p_h - Q_h p_h \|_0^2 + \left[ \sup_{v_h \in [P]_1(\Gamma \cap H^1_0(\Omega)) \setminus \emptyset} \frac{(\nabla \cdot v_h, p_h)}{|v_h|_1} \right]^2 \]
\[ \lesssim \| p_h - Q_h p_h \|_0^2 + \left[ \sup_{v_h \in [P]_1(\Gamma \cap H^1_0(\Omega)) \setminus \emptyset} \frac{(\nabla \cdot v_h, Q_h^* p_h - p_h)}{|v_h|_1} \right]^2 \]
\[ \lesssim \sum_{E \in S_h^F} h_E \| p_h \|_{0,E}^2 + \left[ \sup_{v_h \in X_h} \frac{(\nabla \cdot v_h, p_h)}{|v_h|_1} \right]^2 , \]
which completes the proof. \( \Box \)

For any \( \tau \in \mathbb{R}^{d \times d} \), let \( \tau_D := \tau - \frac{1}{d} tr(\tau) I \) denote its deviatoric tensor. Then we easily have the following identities: for all \( \sigma, \tau \in [L^2(\Omega)]^{d \times d} \) and \( v \in [H^1(\Omega)]^d \),
\[ (A\sigma, \tau) = \frac{1}{2\mu} (\sigma_D, \tau_D) + \frac{1}{d(d+2\mu)} (tr(\sigma), tr(\tau)), \] (3.4)
\[ \| \tau \|_0^2 = \| \tau_D \|_0^2 + \frac{1}{d} \| tr(\tau) \|_0^2 , \] (3.5)
\[ \nabla \cdot v I = d(\nabla v - (\nabla v)_D), \] (3.6)
\[ (\sigma, \tau) = (\sigma, \tau_D). \] (3.7)

In light of the relations (3.4)-(3.7) and Lemma 3.2, we can prove the following stability results.

**Theorem 3.3.** For all \( \sigma_h, \tau_h \in \Sigma_h \), it holds the continuity condition
\[ a_h(\sigma_h, \tau_h) \leq \frac{1}{2\mu} \| \sigma_h \|_0 \| \tau_h \|_0 . \] (3.8)

Moreover, it holds the coercivity condition
\[ \| \sigma_h \|_0^2 \lesssim \begin{cases} \mu a_h(\sigma_h, \sigma_h) + \mu \| \sigma_h \|_0^2 & \text{if } k + 1 < d , \\ \mu a_h(\sigma_h, \sigma_h) + \mu \| \sigma_h \|_0^2 & \text{if } k + 1 \geq d , \end{cases} \] (3.9)
for all \( \sigma_h \in \Sigma_h^m \) and \( (\sigma_h, \sigma_{hh}) \in \Sigma_h \times [L^2(\partial \Omega)_0]^{d \times d} \) satisfying
\[ \nabla w_{k+1} \cdot \{ \sigma_h, \sigma_{hh} \} = 0 , \] (3.10)
\[ \langle \sigma_{hh} n, v_{hh} \rangle_{\partial \Omega} = 0 , \forall v_{hh} \in U_{hh}^{D,0} , \] (3.11)
\[ tr(\sigma_h) \in L^2(\Omega) \text{ if } \Gamma_N = \emptyset . \] (3.12)
Proof. The inequality \((3.8)\) follows from the identities \((3.4)-(3.5)\) and the definition of \(\alpha_k(\cdot, \cdot)\).

For \(\sigma_{hi} \in \Sigma_{hi}\) satisfying \((3.12)\), by Lemma 3.2 we obtain

\[
\|tr(\sigma_{hi})\|_{0}^2 \lesssim \begin{cases} \sum_{T \in T_h/\partial \Omega} h_T\|tr(\sigma_{hi}) - \sigma_{hi}^{tr}\|_{0,T}^2 + \left[ \sup_{v_{hi} \in X_{hi}} \left( \frac{\sigma_{hi} v_{hi}(tr(\sigma_{hi}))}{\|v_{hi}\|_{1}} \right) \right]^2 & \text{if } k + 1 < d, \\
\sup_{v_{hi} \in X_{hi}} \left( \frac{\sigma_{hi} v_{hi}(tr(\sigma_{hi}))}{\|v_{hi}\|_{1}} \right)^2 & \text{if } k + 1 \geq d, 
\end{cases}
\]  

(3.13)

where in the case of \(k + 1 < d\) we have used the relation \(\|\sigma_{hi}^{tr}\|_E = 0\) for any \(E \in E_h/\partial \Omega\) and the shape-regularity of \(T_h\).

In view of \((3.6)\), it holds

\[
(\nabla \cdot v_{hi}, tr(\sigma_{hi})) = (\nabla \cdot v_{hi} I_{d \times d}, \sigma_{hi}) = d(\nabla v_{hi} - (\nabla v_{hi})_D, \sigma_{hi}).
\]

Since \(v_{hi} \in U_{bb}^{D,0}\) for all \(v_{hi} \in X_{hi}\), by the above relation, Green’s formula, the identity \((3.7)\), \((3.11)\), and the definition of weak divergence, we get

\[
(\nabla \cdot v_{hi}, tr(\sigma_{hi})) = -d(h_{hi}, \nabla \cdot \sigma_{hi}) + d(v_{hi}, \sigma_{hi} n)_{\partial T_h} - d(\nabla v_{hi}, (\sigma_{hi})_D)
\]

\[
= -d(Q_{k+1}^i v_{hi}, \nabla \cdot \sigma_{hi}) + d(v_{hi}, \sigma_{hi} n - \sigma_{hh} n)_{\partial T_h} - d(\nabla v_{hi}, (\sigma_{hi})_D)
\]

\[
= -d(Q_{k+1}^i v_{hi}, \nabla w_{k+1, \cdot} (\sigma_{hi}, \sigma_{hh})),
\]

\[
+ d(v_{hi} - Q_{k+1}^i v_{hi}, \sigma_{hi} n - \sigma_{hh} n)_{\partial T_h} - d(\nabla v_{hi}, (\sigma_{hi})_D).
\]

Then, from \((3.10)\) and Lemma 2.6 with \(m = 1\) it follows

\[
\frac{\|tr(\sigma_{hi})\|_{0}}{\|v_{hi}\|_{1}} \lesssim \mu^{1/2} \alpha^{-1/2} \|\sigma_{hi} n - \sigma_{hh} n\|_{0, \partial T_h} + \|\sigma_{hi} n\|_{0, \partial T_h},
\]

(3.14)

which, together with \((3.13)\) and \((3.4)\), yields the desired inequality \((3.9)\). \(\square\)

4 A priori error estimates

This section is devoted to the error estimation for the modified WG schemes \((2.21)-(2.22)\) and \((2.23)-(2.24)\).

Lemma 4.1. For all \(T \in T_h, \tau \in H(div, T), v \in [H^1(\Omega)]^d, \tau_{hi} \in [P_k(T)]^d,\) and \(v_h = \{v_{hi}, v_{hh}\} \in U_h,\) there holds

\[
\nabla w_{k+1} \cdot \{Q_k^i, Q_{k+1}^i \tau \} = Q_k^i \nabla \cdot \tau, \forall \tau \in H(div, T)
\]

(4.1)

\[
(\nabla w_{k} \{Q_{k+1}^i v, \tau_{hi} \}, \tau_{hi})_T = (\nabla v, \tau_{hi})_T + (\tau_{hi} v - v, \tau_{hi} n)_{\partial T_h}
\]

(4.2)

\[
\|e_h(v_{hi})\|_0 \lesssim \|e_{w,k}(v_{hi})\|_0 + \mu^{-1/2} \alpha^{1/2} \|Q_k^i v_{hi} - v_{hh}\|_{0, \partial T_h},
\]

(4.3)

where \(e_{w,k}(v_{hi}) := (\nabla w_{k} v_{hi} + (\nabla w_{k} v_{hi})^T) / 2,\) and \(e_h(v_{hi}) := (\nabla h v_{hi} + (\nabla h v_{hi})^T) / 2\) with \(\nabla h\) denoting the broken gradient operator with respect to \(T_h\).

Proof. For any \(T \in T_h, v_{hi} \in [P_{k+1}(T)]^d\) and \(\tau \in H(div, T),\) we use the definition of weak divergence, the orthogonality of \(Q_k^i, Q_{k+1}^i\) and \(Q_k^i\), and Green’s
holds the following error equations:

**Lemma 4.2.** Let 

\[ (\nabla_{w,k+1} \cdot \{ Q_{k+1}^i, Q_{k+1}^b \}, v_{hi})_T = - (Q_{k+1}^i, \nabla v_{hi})_T + (Q_{k+1}^b, v_{hi})_\partial T \]

\[ = - (\tau, \nabla v_{hi})_T + (\tau n, v_{hi})_\partial T \]

\[ = (\nabla \cdot \tau, v_{hi})_T \]

\[ = (Q_{k+1}^i \nabla \cdot \tau, v_{hi})_T, \]

which implies \( \nabla_{w,k+1} \cdot \{ Q_{k+1}^i, Q_{k+1}^b \} = Q_{k+1}^i \nabla \cdot \tau. \)

From the definition of weak gradient operator \( \nabla_{w,k} \), the projection property of \( Q_{k+1}^i \), and integration by parts, it follows, for all \( v \in [H^1(\Omega)]^d \), \( \tau_{hi} \in [P_k(T)]^{d \times d} \),

\[ (\nabla_{w,k} \{ Q_{k+1}^i, I_{k+1} v \}, \tau_{hi})_T = -(Q_{k+1}^i v, \nabla \cdot \tau_{hi})_T + (I_{k+1} v, \tau_{hi} n)_\partial T \]

\[ = -(v, \nabla \cdot \tau_{hi})_T + (v, \tau_{hi} n)_\partial T \]

\[ + (I_{k+1} v - v, \tau_{hi} n)_\partial T \]

\[ = (\nabla v, \tau_{hi})_T + (I_{k+1} v - v, \tau_{hi} n)_\partial T. \]

This proves (4.2).

For \( v_h = \{ v_{hi}, v_{hb} \} \in U_h \), we apply integration by parts, the definitions of \( \epsilon_h(v_{hi}), \epsilon_{w,k}(v_h) \), and \( \nabla_{w,k} v_h \) to get

\[ (\epsilon_{h}(v_{hi}), \epsilon_{h}(v_{hi})) = -(v_{hi}, \nabla_h \cdot \epsilon_{h}(v_{hi})) + (\epsilon_{h}(v_{hi}) n, v_{hi})_\partial T_n \]

\[ = (\epsilon_{w,k}(v_{hi}), \epsilon_{h}(v_{hi})) + (\epsilon_{h}(v_{hi}) n, v_{hi} - v_{hb})_\partial T_n. \]

As a result, the estimate (4.3) follows from Cauchy-Schwarz inequality and the inverse inequality.

Set

\[ \Sigma := \{ \tau \in [L^2(\Omega)]^{d \times d} : \tau^T = \tau \}, \quad U := \{ v \in [H^1(\Omega)]^d : v|_{\Gamma_D} = g_D \}. \]

**Lemma 4.2.** Let \( (\sigma, u) \in (\Sigma \cap H(\text{div}, \Omega)) \times U \) be the solution to the model (1.1). Then, for all \( \tau_{hi} \in \Sigma_{hi}, \tau_{hb} \in \Sigma_{hb} \), and \( v_h = \{ v_{hi}, v_{hb} \} \in U_{hi} \times U_{hb}^{D,0} \), it holds the following error equations:

\[ a_h(Q_{k+1}^i \sigma, \tau_{hi} - b_h(\tau_{hi}, \{ Q_{k+1}^i u, I_{k+1} u \}) = E_0(u, \tau_{hi}) \]

for \( k+1 \geq d \),

\[ a_h(Q_{k+1}^i \sigma, \tau_{hi}) + s_h(Q_{k+1}^i \sigma, I_{k+1} \text{tr} \sigma; \tau_{hi}, \tau_{hb}^{tr}) \]

\[ - b_h(\tau_{hi}, \{ Q_{k+1}^i u, I_{k+1} u \}) = E_1(\sigma, u; \tau_{hi}, \tau_{hb}^{tr}) \]

for \( k+1 < d \), and

\[ - b_h(Q_{k+1}^i \sigma, v_h) - s_h(\{ Q_{k+1}^i u, I_{k+1} u \}, v_h) = (f, v_h) - (g_N, v_{hb})_{\Gamma_N} + E_2(\sigma, u; v_h), \]

where

\[ E_0(u, \tau_{hi}) := - (I_{k+1} u - u, \tau_{hi} n)_{\partial T_n}, \]

\[ E_1(\sigma, u; \tau_{hi}, \tau_{hb}^{tr}) := - (I_{k+1} u - u, \tau_{hi} n)_{\partial T_n} \]

\[ + \langle \beta(\text{tr}(Q_{k+1}^i \sigma)) - I_{k+1} \text{tr}(\sigma), \text{tr}(\tau_{hi}) - \tau_{hb}^{tr} \rangle_{\partial T_n / \partial T_N}. \]

\[ E_2(\sigma, u; v_h) := - \langle \sigma (Q_{k+1}^i u - I_{k+1} u), v_h - v_{hb} \rangle_{\partial T_n} \]

\[ - \langle \sigma n - Q_{k+1}^i n, v_h - v_{hb} \rangle_{\partial T_n}. \]
Lemma 4.4. For any \( \Gamma \) where

\[
\begin{align*}
\mathbf{a}_h(Q^i_{k_{\sigma}}, \tau_{hi}) & = b_h(\tau_{hi}, \{Q^i_{k_{\sigma}+1}u, \mathcal{I}_{k+1}u\}) \\
& = (A\sigma - \nabla u, \tau_{hi}) - (\mathcal{I}_{k+1}u - u, \tau_{hi}n)\mathcal{T}_h \\
& = (A\sigma - \epsilon(u), \tau_{hi}) - (\mathcal{I}_{k+1}u - u, \tau_{hi}n)\mathcal{T}_h \\
& = - (\mathcal{I}_{k+1}u - u, \tau_{hi}n)\mathcal{T}_h,
\end{align*}
\]

i.e. (4.4) holds.

The relation (4.5) follows from (4.4) and the definition of \( z_h(\cdot, \cdot, \cdot) \).

The thing left is to show (4.6). From integration by parts and the definitions of the weak gradient \( \nabla w, k \) and the projection \( Q^i_k \), it follows

\[
\begin{align*}
- b_h(Q^i_k\sigma, v_h) & = (\nabla_h \cdot Q^i_k\sigma, v_h) - (Q^i_k\sigma n, v_{hh})\mathcal{T}_h \\
& = -(Q^i_k\sigma, \nabla_h v_h) - (Q^i_k\sigma n, v_{hh} - v_{hi})\mathcal{T}_h \\
& = (\sigma \cdot \sigma, v_h) - (\sigma n, v_{hh})\mathcal{T}_h - (Q^i_k\sigma n, v_{hh} - v_{hi})\mathcal{T}_h,
\end{align*}
\]

which, together with (1.1), the relation \( (\sigma n, v_{hh})\mathcal{T}_h = (\sigma n, v_{hh})_{\mathcal{T}_h} \) and the definition of \( s_h(\cdot, \cdot, \cdot) \), yields the relation (4.6).

\[
\square
\]

Introduce the space of (infinitesimal) rigid motions on \( \Omega \):

\[
RM(\Omega) = \{ a + b\eta : a \in \mathbb{R}^d, \eta \in so(d) \},
\]

where \( so(d) \) is the Lie algebra of anti-symmetric \( d \times d \) matrices. The space \( RM(\Omega) \) is precisely the kernel of the strain tensor. We recall the Piecewise Korn’s inequality as follows.

**Lemma 4.3.** \([12]\) (Piecewise Korn’s inequality) Let \( \mathcal{T}_h \) be a shape-regular decomposition of \( \Omega \), then for any \( v_p \in [H^1(\mathcal{T}_h)]^d \) it holds

\[
\|\nabla_h v_p\|_0^2 \lesssim \|\mathbf{e}_h(v_p)\|_0^2 + \sup_{m \in RM(\Omega), \|m\|_{so(d)} = 1} \int_{E \in \mathcal{E}_h, \partial \Omega} \int_{E \in \mathcal{E}_h, \partial \Omega} h_E^{-1} \|Q^i_p[v_p]\|_{0,\mathcal{E}_h}^2 + \sum_{E \in \mathcal{E}_h, \partial \Omega} h_E^{-1} \|Q^i_p[v_p]\|_{0,\mathcal{E}_h}^2,
\]

where \( \Gamma_e \) is a measurable subset of \( \partial \Omega \) with a positive \( d - 1 \)-dimensional, and \( [v_p]_E \) denote the jump of \( v_p \) over \( E \in \mathcal{E}_h \).

By Lemma 4.3, we can prove the following lemma.

**Lemma 4.4.** For any \( v_h = \{v_{hi}, v_{hh}\} \in U^0_h \), it holds

\[
\|\nabla_h v_{hi}\|_0^2 \lesssim \|\mathbf{e}_h(v_{hi})\|_0^2 + \sum_{T \in \mathcal{T}_h} h_T^{-1} \|v_{hi} - v_{hh}\|_{0,\partial T}^2.
\]

**Proof.** For \( v_h = \{v_{hi}, v_{hh}\} \in U^0_h \), we apply Lemma 4.3, Cauchy-Schwarz in-
Lemma 4.5. For \( (\sigma, u) \in (\Sigma \cap [H^{k+1}(\Omega)]^{d \times d}) \times (U \cap [H^{k+2}(\Omega)]^d) \) and \( v_h = (v_{hi}, v_{hb}) \in U_h^0 \), it holds

\[
|E_0(u, \tau_{hi})| \lesssim h^{k+1}|u|_{k+2}\|\tau_{hi}\|_0, \quad (4.15)
\]

\[
|E_2(\sigma, u; v_h)| \lesssim h^{k+1}(\mu^{-1/2}|\sigma|_{k+1} + \mu^{1/2}|u|_{k+2})\|v_h\|_{0,0} \quad (4.16)
\]

and, for \( k + 1 < d \),

\[
|E_1(\sigma, u; \tau_{hi}, \tau_{hb}^T)| \lesssim h^{k+1} \left( \mu^{-1/2}|\sigma|_{k+1} + \mu^{1/2}|u|_{k+2} \right)
\]

\[
\times \left( \mu^{1/2}\|\tau_{hi}\|_0 + \|\beta^{1/2}(tr(\tau_{hi}) - \tau_{hb}^T)|\|_{\partial T_h}\right), \quad (4.17)
\]

Proof. The estimates \(4.15\)-(4.16) follow from Cauchy-Schwarz inequality, the inverse inequality, Lemmas 2.3 and 2.6.

Similarly, we have

\[
|E_1(\sigma, u; \tau_{hi}, \tau_{hb}^T)| \lesssim h^{k+1}|u|_{k+2}\|\tau_{hi}\|_0
\]

\[
+ \mu^{-1/2}h^2|\sigma|_{k+1}\|\beta^{1/2}(tr(\tau_{hi}) - \tau_{hb}^T)|\|_{\partial T_h}\Omega}
\]

\[
\lesssim h^{k+1}|u|_{k+2}\|\tau_{hi}\|_0
\]

\[
+ \mu^{-1/2}h^{k+1}|\sigma|_{k+1}\|\beta^{1/2}(tr(\tau_{hi}) - \tau_{hb}^T)|\|_{\partial T_h}\Omega}. \quad (4.18)
\]

where we have used the fact \( h^2 \leq h^{k+1} \) for \( h < 1 \) and \( k + 1 < d \) with \( d = 2, 3 \).

This completes the proof.

\[ \square \]

Lemma 4.6. Let \((\sigma, u) \in (\Sigma \cap [H^{k+1}(\Omega)]^{d \times d}) \times (U \cap [H^{k+2}(\Omega)]^d)\) be the solution to the model (1.1), and let \((\sigma_{hi}, \sigma_{hb}^T, u_h := \{u_{hi}, u_{hb}\}) \in \Sigma_{hi} \times \Sigma_{hb} \times U_h^0\) and \( (\sigma_{hi}, u_h) \in \Sigma_{hi} \times U_h^0 \) be the solutions to the WG schemes (2.21)-(2.22) and (2.23)-(2.24), respectively. Then it holds

\[
a_h(\xi_{hi}, \xi_{hb}^T) + s_h(tr(\xi_{hi}^T), \xi_{hb}^T; tr(\xi_{hi}^T), \xi_{hb}^T) + s_h(\xi_{hi}^T, \xi_{hb}^T) \lesssim h^{2k+2}(\mu^{-1}|\sigma_{k+1}^2 + \mu|u_{k+2}^2) \quad (4.19)
\]

for \( k + 1 < d \), and

\[
a_h(\xi_{hi}^T, \xi_{hb}^T) + s_h(\xi_{hi}^T, \xi_{hb}^T) \lesssim h^{2k+2}(\mu^{-1}|\sigma_{k+1}^2 + \mu|u_{k+2}^2) \quad (4.20)
\]
for $k + 1 \geq d$, where

$$\xi^i_{hi} := Q_k^i \sigma - \sigma_{hi}, \quad \xi^r_{hi} := \mathcal{I}_1 tr(\sigma) - \sigma^r_{hi}, \quad \xi^u := \{\xi^i_{hi}, \xi^u_{hi}\}$$  \hspace{1cm} (4.21)

with

$$\xi^u_{hi} := Q_{k+1}^i u - u_{hi}, \quad \xi^u_{hb} := I_{k+1} u - u_{hb}.$$

Proof. From the relations (2.21)-(2.24) and (4.4)-(4.6), we have, for all $\tau_{hi} \in \Sigma_{hi}, \tau^r_{hb} \in \Sigma^0_{hb},$ and $v_h = \{v_{hi}, v_{hb}\} \in U^0_h$,

$$a_h(\xi^r_{hi}, \tau_{hi}) + z_h(\xi^r_{hi}, \xi^r_{hi}, \tau^r_{hi}) - b_h(\tau_{hi}, \xi^u) = E_1(\sigma, u; \tau_{hi}, \tau^r_{hi}) \quad \text{if } k + 1 < d,$$  \hspace{1cm} (4.22)

$$a_h(\xi^r_{hi}, \tau_{hi}) - b_h(\tau_{hi}, \xi^u) = E_0(u_h; \tau_{hi}) \quad \text{if } k + 1 \geq d,$$  \hspace{1cm} (4.23)

$$-b_h(\xi^r_{hi}, v_h) - s_h(\xi^u, v_h) = E_2(\sigma, u; v_h).$$  \hspace{1cm} (4.24)

Then the desired estimates (4.19)-(4.20) follow from Lemmas 4.4-4.5 and Young’s inequality.

Lemma 4.7. Under the conditions of Lemma 4.6, let $\sigma_{hb} \in [L^2(\partial T_h)]^{d \times d}$ satisfy

$$\langle \sigma_{hi} n \rangle_{\partial T} := (\sigma_{hi} n)_{\partial T} - \alpha(u_{hi} - u_{hb}), \quad \forall T \in \mathcal{T}_h.$$  \hspace{1cm} (4.25)

Then it holds

$$\nabla_{w, k+1} \cdot \{Q_k^i \sigma - \sigma_{hi}, Q_{k+1}^i \sigma - \sigma_{hb}\} = 0,$$  \hspace{1cm} (4.26)

$$\langle Q_{k+1}^i (\sigma n) - \sigma_{hb} n, v_{hb} \rangle_{\partial T_h} = 0, \forall v_{hb} \in U^{D, 0}_{hb},$$  \hspace{1cm} (4.27)

$$tr(Q_k^i \sigma - \sigma_{hi}) \in L^2_0(\Omega) \text{ if } \Gamma_N = \emptyset.$$  \hspace{1cm} (4.28)

Proof. We first show (4.26). Taking $v_{hi} = 0$ and $v_{hb} \in U^{D, 0}_{hb}$ in (2.22) or (2.24), we get

$$\langle \sigma_{hi} n, v_{hb} \rangle_{\partial T_h} + \langle \alpha(u_{hi} - u_{hb}), v_{hb} \rangle_{\partial T_h} = -\langle g_N, v_{hb} \rangle_{\Gamma_N},$$  \hspace{1cm} (4.29)

which, together with (4.25), yields

$$\langle \sigma_{hb} n, v_{hb} \rangle_{\partial T_h} = \langle g_N, v_{hb} \rangle_{\Gamma_N}.$$  \hspace{1cm} (4.30)

On one hand, for all $v_{hi} \in U_{hi}, v_{hb} \in U^{D, 0}_{hb}$, by the definitions of the discrete weak divergence and weak gradient, we have

$$\langle \nabla_{w,k+1} \cdot (\sigma_{hi}, \sigma_{hb}), v_{hi} \rangle = -\langle \sigma_{hi}, \nabla_{w,k} v_{hi} \rangle + \langle \sigma_{hb} n, v_{hi} \rangle_{\partial T_h}$$

$$= \langle \nabla_h \cdot \sigma_{hi}, v_{hi} \rangle + \langle \sigma_{hb} n - \sigma_{hi} n, v_{hi} \rangle_{\partial T_h}$$

$$= \langle \sigma_{hi}, \nabla_{w,k} v_{hi} \rangle + \langle \sigma_{hb} n - \sigma_{hi} n, v_{hi} \rangle_{\partial T_h}$$

$$+ \langle \sigma_{hi} n, v_{hb} \rangle_{\partial T_h},$$

which, together with (4.30), (4.25), and (2.24), implies

$$\langle \nabla_{w,k+1} \cdot (\sigma_{hi}, \sigma_{hb}), v_{hi} \rangle = -\langle \sigma_{hi}, \nabla_{w,k} v_{hi} \rangle + \langle \sigma_{hb} n - \sigma_{hi} n, v_{hi} - v_{hb} \rangle_{\partial T_h} + \langle g_N, v_{hb} \rangle_{\Gamma_N}$$

$$= -\langle \sigma_{hi}, \nabla_{w,k} v_{hi} \rangle - \langle \alpha(u_{hi} - u_{hb}), v_{hi} - v_{hb} \rangle_{\partial T_h} + \langle g_N, v_{hb} \rangle_{\Gamma_N}$$

$$= (f, v_{hi}).$$  \hspace{1cm} (4.31)

15
On the other hand, from the commutativity property (4.1) and the fact $\nabla \cdot \sigma = f$ in (1.1), it follows

$$(\nabla w, k+1 \cdot \{Q_i^k \sigma, Q_{k+1}^b \sigma\}, v_{hi}) = (Q_{k+1}^b \nabla \cdot \sigma, v_{hi}) = (\nabla \cdot \sigma, v_{hi}) = (f, v_{hi}),$$

which, together with (4.1), yields (4.26).

By (4.30) and the boundary condition $\sigma n = g_N$ on $\Gamma_N$ we easily get

$$\langle Q_{k+1}^b (\sigma n) - \sigma_{hb} n, v_{hb} \rangle_{\partial T_n} = \langle \sigma n, v_{hb} \rangle_{\partial T_n} - \langle g_N, v_{hb} \rangle_{\Gamma_N} = 0,$$

i.e. (4.27) holds.

The work left is to prove (4.28). When $\Gamma_N = \emptyset$, we use the first relation in (1.1) to get

$$\frac{1}{2\mu + d} (tr(Q_i^k \sigma), 1) = \frac{1}{2\mu + d} (tr(\sigma), 1) = (A \sigma, I) = (\epsilon(u), I) = \langle g_D, n \rangle_{\partial \Omega},$$

(4.32)

Since

$$\langle I, \nabla w, k u_h \rangle = \langle I_{k+1} g_D, n \rangle_{\partial \Omega},$$

taking $\tau_{hi} = I$ in (2.23) yields

$$\frac{1}{2\mu + d} (tr(\sigma_{hi}), 1) = (A \sigma_{hi}, I) = \langle I_{k+1} g_D, n \rangle_{\partial \Omega},$$

(4.33)

which, together with (2.5), leads to the desired relation (4.28) for the case of $k + 1 \geq d$.

For the case of $k + 1 < d$, we take $\tau_{hi} = I$ in (2.21) to get

$$\frac{1}{2\mu + d} (tr(\sigma_{hi}), 1) = (A \sigma_{hi}, I) = \langle I_{k+1} g_D, n \rangle_{\partial \Omega},$$

(4.34)

$$\langle \beta(tr(\sigma_{hi}) - \sigma_{hb}^r), \tau_{hb}^r \rangle_{\partial T_n / \partial \Omega} = 0, \forall \tau_{hb}^r \in \Sigma_{hb}^r,$$

which means

$$\langle \beta(tr(\sigma_{hi}) - \sigma_{hb}^r), \tau_{hb}^r \rangle_{\partial T_n / \partial \Omega} = 0,$$

(4.35)

since $d \in \Sigma_{hb}^r$. As a result, (4.28) follows from (2.5) and (4.32)-(4.34).

Finally, we shall derive the following error estimates for the stress and displacement approximations.

**Theorem 4.8.** Let $\sigma, u \in (\Sigma \bigcap [H^{k+1}(\Omega)]^{d \times d}) \times (U \bigcap [H^{k+2}(\Omega)]^d)$ be the solution to the model (1.1), and let $(\sigma_{hi}, \sigma_{hb}^r, u_{hi}) \in \Sigma_{hi} \times \Sigma_{hb}^r \times U_h^0$ and $(\sigma_{hi}, u_{hi}) \in \Sigma_{hi} \times U_h^0$ be the solutions to the WG schemes (2.21)-(2.22) and (2.23)-(2.24), respectively. Then it holds the following error estimates:

$$\|\sigma - \sigma_{hi}\|_0 \lesssim h^{k+1}(|\sigma|_{k+1} + \mu |u|_{k+2}),$$

(4.36)

$$\|\nabla u - \nabla_h u_{hi}\|_0 \lesssim h^{k+1}(\mu^{-1} |\sigma|_{k+1} + |u|_{k+2}).$$

(4.37)
Proof. Let $\sigma_{hb}$ be the same as in (4.25), and set

$$
\xi_{hb} := Q_{k+1}^i(\sigma n) \otimes n - \sigma_{hb},
$$

where

$$
\mathbf{s} \otimes \mathbf{r} := \begin{pmatrix} s_1r_1 & \cdots & s_1r_d \\ \vdots & & \vdots \\ s_dr_1 & \cdots & s_dr_d \end{pmatrix}
$$

for $\mathbf{s} = (s_1, \ldots, s_d)^T, \mathbf{r} = (r_1, \ldots, r_d)^T$.

Recall from (4.21) that

$$
\xi_{i_h}^\tau = Q_i^\tau \sigma - \sigma_{hi}, \quad \xi_{i_h}^{\tau r} = I_i tr(\sigma) - \sigma_{hb}, \quad \xi_n^i = \{Q_{k+1}^i u - u_{hi}, I_{k+1} u - u_{hb}\}.
$$

Then, in view of (2.27), Lemma 2.3, and Lemma 2.6, we have

$$
\|\alpha^{-1/2}(\xi_{hi}^\tau n - \xi_{hb} n)\|_0,\partial T_h^2
= \|\alpha^{-1/2}(Q_i^\tau \sigma n - Q_{k+1}^i(\sigma n) - \alpha^{-1/2}(\sigma_{hi} n - \sigma_{hb} n))\|_0,\partial T_h^2
= \|\alpha^{-1/2}(Q_i^\tau \sigma n - Q_{k+1}^i(\sigma n)) - \alpha^{-1/2}(u_{hi} - u_{hb})\|_0,\partial T_h^2
= \|\alpha^{-1/2}(Q_i^\tau \sigma n - Q_{k+1}^i(\sigma n)) - \alpha^{-1/2}(u_{hi} - u_{hb})\|_0,\partial T_h
\leq \|\alpha^{-1/2}((Q_{k+1}^i u - u_{hi}) - (I_{k+1} u - u_{hb}))\|_0,\partial T_h
+ \|\alpha^{-1/2}(Q_i^\tau \sigma n - Q_{k+1}^i(\sigma n)n)\|_0,\partial T_h
+ \|\alpha^{-1/2}(Q_{k+1}^i u - I_{k+1} u)\|_0,\partial T_h
\lesssim s_h(\xi_{hi}^\tau, \xi_{hi}^{\tau r}) + h^{k+2}(\mu^{-1}\sigma_{k+1}^2 + \mu|u|_{k+2}). \tag{4.37}
$$

Thus, from Lemma 4.7 and Theorem 3.3 it follows

$$
\|\xi_{hi}^\tau\|_0 \lesssim \begin{cases} 
\mu_a(\xi_{hi}^\tau, \xi_{hi}^{\tau r}) + \mu\alpha^{-1/2}(\xi_{hi}^\tau n - \xi_{hb} n)\|_0,T_h^2, + \mu\beta^{1/2}(tr(\xi_{hi}^\tau) - \xi_{hb}^{\tau r})_0,\partial T_h^2, & \text{if } k + 1 < d,
\mu_a(\xi_{hi}^\tau, \xi_{hi}^{\tau r}) + \mu\alpha^{-1/2}(\xi_{hi}^\tau n - \xi_{hb} n)\|_0,\partial T_h^2, & \text{if } k + 1 \geq d,
\end{cases}
$$

which, together with (4.37), Lemma 4.6 and the approximation property of $Q_i^k$, yields the desired estimate (4.35).

The thing left is to show (4.36). From the relation (4.22) we have, for all $\tau_{hi} \in \Sigma_{hi},$

$$
a_h(\xi_{hi}^\tau, \tau_{hi}) + z_h(\xi_{hi}^\tau, \xi_{hi}^{\tau r}; \tau_{hi}, 0) - b_h(\tau_{hi}, \xi_{hi}^\tau) = E_1(\sigma, u; \tau_{hi}, 0) \text{ if } k + 1 < d,
$$

which, together with (4.23), (4.3), Theorem 3.1, Theorem 3.3, Lemmas 4.5-4.6, and the inverse inequality, indicates

$$
\|\varepsilon_h(\xi_{hi}^\tau)\|_0 \lesssim \|\varepsilon_{w,k}(\xi_{hi}^\tau)\|_0 + \mu^{-1/2}\|\alpha^{1/2}(\xi_{hi}^\tau n - \xi_{hb} n)\|_0,\partial T_h
\lesssim \sup_{\tau_{hi} \in \Sigma_{hi}} \frac{b_h(\tau_{hi}, \xi_{hi}^\tau)}{\|\tau_{hi}\|_0} + \mu^{-1/2}s_h(\xi_{hi}^\tau, \xi_{hi}^{\tau r})^{1/2}
\lesssim h^{k+1}(\mu^{-1}|\sigma_{k+1} + |u|_{k+2}).
$$

As a result, the desired estimate (4.36) follows from Lemma 4.4, Lemma 4.6 and the triangle inequality. \qed
5 \(L^2\) error estimation for displacement approximation

In order to derive the \(L^2\) error estimation for the displacement approximation \(u_{hi}\), we shall perform Aubin-Nitsche duality argument based on the following auxiliary problem:

\[
\begin{align*}
A\Psi - \epsilon(\Phi) &= 0 & \text{in } \Omega, \\
\nabla \cdot \Psi &= \xi_{hi}^0 & \text{in } \Omega, \\
\Phi &= 0 & \text{on } \Gamma_D, \\
\Psi n &= 0 & \text{on } \Gamma_N. \\
\end{align*}
\]

(5.1)

Here \(\xi_{hi}^0\) is the same as in (4.21), i.e. \(\xi_{hi}^0 = Q_{k+1}^i u - u_{hi}\). In addition, we assume the following regularity estimate holds:

\[
|\Psi|_1 + \mu |\Phi|_2 \lesssim \|\xi_{hi}^0\|_a. 
\]

(5.2)

**Lemma 5.1.** For \((\sigma, u) \in (\Sigma \cap [H^{k+1}(\Omega)]^{d \times d}) \times (U \bigcap [H^{k+2}(\Omega)]^d)\), it holds

\[
\begin{align*}
|E_0(u; Q_k^i \Psi)| &\lesssim h^{k+2} |u|_{k+2} |\Psi|_1, \\
|E_0(\Phi; \xi_{hi}^0)| &\lesssim h^{k+2} (|\sigma|_{k+1} + \mu |u|_{k+2}) |\Phi|_2, \\
|E_1(\sigma, u; Q_k^i \Psi, I_k tr(\Psi))| &\lesssim h^{k+2} (|\sigma|_{k+1} + \mu |u|_{k+2}) |\Psi|_1, \\
|E_1(\Phi, \Psi; \xi_{hi}^0, \xi_{hb}^0)| &\lesssim h^{k+2} (|\sigma|_{k+1} + \mu |u|_{k+2}) (\mu^{-1}|\Psi|_1 + |\Phi|_2), \\
|E_2(\sigma, u; (Q_{k+1}^i \Phi, I_{k+1} \Phi))| &\lesssim h^{k+2} (|\sigma|_{k+1} + \mu |u|_{k+2}) (\mu^{-1}|\Psi|_1 + |\Phi|_2), \\
|E_2(\Psi, \Phi; \xi_{hi}^0)| &\lesssim h^{k+2} (|\sigma|_{k+1} + \mu |u|_{k+2}) (\mu^{-1}|\Psi|_1 + |\Phi|_2),
\end{align*}
\]

where \(E_0()\), \(E_1()\) and \(E_2()\) are defined in (4.7)-(4.9), and \(\xi_{hi}^0, \xi_{hb}^0\) and \(\xi_{hi}^0 = \xi_{hi}^i = \xi_{hi}^0 \cup \xi_{hb}^0\) are the same as in (4.21).

**Proof.** Since the proofs of (5.3) and (5.4) follow from those of (5.5) and (5.6), respectively, we only prove (5.5)-(5.8).

From \([\mathcal{I}_{k+1} u - u, Q_k^i \Psi n]|_{\Gamma_D} = 0, Q_k^i \Psi n|_{\Gamma_N} = \Psi n|_{\Gamma_N} = 0\), and Lemma 2.6, it follows

\[
|E_1(\sigma, u; Q_k^i \Psi, I_k tr(\Psi))| \leq |(\mathcal{I}_{k+1} u - u, Q_k^i \Psi n - Q_k^i \Psi n)|_{\partial \Omega} \\
+ |(\beta (tr(Q_k^i \sigma) - I_k tr(\sigma)) + (\alpha (Q_k^i \Psi n - Q_k^i \Psi n)|_{\partial \Omega})| \\
\lesssim h^{k+2} (|\sigma|_{k+1} + \mu |u|_{k+2}) (\mu^{-1}|\Psi|_1, \\
\]

i.e. (5.5) holds. Similarly, we have

\[
|E_2(\sigma, u; (Q_{k+1}^i \Phi, I_{k+1} \Phi))| \leq |(\alpha Q_{k+1}^i u - I_{k+1} u, Q_{k+1}^i \Phi - I_{k+1} \Phi)|_{\partial \Omega} \\
+ |(\sigma n - Q_{k+1}^i \sigma n, Q_{k+1}^i \Phi - I_{k+1} \Phi)|_{\partial \Omega} \\
\lesssim h^{k+2} (|\sigma|_{k+1} + \mu |u|_{k+2}) |\Phi|_2,
\]

i.e. (5.7) holds. In light of Lemma 4.6 and the approximation properties of \(Q_k^i\), \(I_k\) and \(I_{k+1}\), we get

\[
|E_1(\Psi, \Phi; \xi_{hi}^0, \xi_{hb}^0)| \leq |(\mathcal{I}_{k+1} \Phi - \Phi, \xi_{hi}^0 n)|_{\partial \Omega} \\
+ |(\beta (tr(Q_k^i \Psi) - I_k tr(\Psi)) + (\alpha (\xi_{hi}^0 - \xi_{hb}^0)|_{\partial \Omega})| \\
\lesssim h^{k+2} (|\sigma|_{k+1} + \mu |u|_{k+2}) (\mu^{-1}|\Psi|_1 + |\Phi|_2)
\]

(5.9)
and

\[ |E_2(\Psi, \Phi; \xi_h^\kappa)| \leq |\alpha(Q_{k+1}^i \Phi - \mathcal{I}_{k+1} \Phi, \xi_h^\kappa - \xi_h^\kappa)|\sigma_h | + |\langle \Psi_n - \Psi_k \sigma_n, \xi_h^\kappa - \xi_h^\kappa \rangle |\sigma_h | \leq h^{k+2}(|\sigma_{k+1} + \mu| \Phi_{k+2}|2|) \tag{5.10} \]

i.e. (5.6) and (5.8) hold.

We are now ready to show the \(L^2\)-error estimation for the displacement approximation \(u_{hi}\).

**Theorem 5.2.** Let \((\sigma, u) \in (\Sigma \cap H^{k+1}(\Omega)^{d \times d}) \times (U \cap H^{k+2}(\Omega)^d)\) be the solution to the model (1.1), and let \((\sigma_{hi}, \sigma_{hi}^tr, u_h := \{u_{hi}, u_{hi}^h\}) \in \Sigma_{hi} \times \Sigma_{hi}^tr \times U_h^D\) and \((\sigma_{hi}, u_h) \in \Sigma_{hi} \times U_h^D\) be the solutions to the WG schemes (2.21)-(2.22) and (2.23)-(2.24), respectively. Then, under the regularity assumption (5.2), it holds

\[ \|u - u_{hi}\|_0 \lesssim h^{k+2}(\mu^{-1}|\sigma|_{k+1} + |u|_{k+2}). \tag{5.11} \]

**Proof.** We only prove (5.11) for \(k + 1 < d\), since the case of \(k + 1 \geq d\) follows similarly. Similar to the proof of Lemma 4.2, from (5.1) we can easily obtain, for all \(\tau_h \in \Sigma_{hi}, \tau_{tr}^h \in \Sigma_{hi}^tr, v_h = \{v_{hi}, v_{hi}^h\} \in U_h^D\),

\[ a_h(Q_j^1 \Psi, \tau_h) + s_h(Q_j^1 \Phi, \mathcal{I}_j \tau(\Psi); \tau_{hi}; \tau_{hi}^tr) - b_h(\tau_h, \{Q_j^1 \Phi, \mathcal{I}_j \tau(\Psi)\}) = E_1(\Psi, \Phi; \tau_h, \tau_{hi}, \tau_{hi}^tr), \tag{5.12} \]

Taking \(v_h = \xi_h^\kappa\) in (5.13), by (4.22), (5.12) and (4.24) we have

\[ \|\xi_h^\kappa\|_0^2 = -b_h(Q_j^1 \Psi, \xi_h^\kappa) - s_h(\{Q_j^1 \Phi, \mathcal{I}_j \tau(\Psi)\}, \xi_h^\kappa) - E_2(\Psi, \Phi; \xi_h^\kappa) \]

which, together with Lemma 5.1 and the regularity (5.2), yields

\[ \|\xi_h^\kappa\|_0^2 \lesssim h^{k+2}(\mu^{-1}|\sigma|_{k+1} + |u|_{k+2})(|\Psi|_1 + \mu|\Phi|_2) \leq h^{k+2}(\mu^{-1}|\sigma|_{k+1} + |u|_{k+2})\|\xi_h^\kappa\|_0. \]

As a result, the desired estimate follows from the triangle inequality and the approximation property of \(Q_{k+1}^i\). \(\square\)
6 Numerical examples

In this section, we provide several numerical examples to verify our theoretical results. All tests are programmed in C++ using the Eigen [28] library.

6.1 A 2D example

Let $\Omega = (0,1) \times (0,1)$. We consider the homogeneous Dirichlet boundary condition, and the exact solution $(u, \sigma)$ is of the following form:

\[
\begin{align*}
    u &= \begin{pmatrix} 
        \sin 2\pi y (-1 + \cos 2\pi x) + \frac{1}{1+\lambda} \sin \pi x \sin \pi y \\
        \sin 2\pi x (1 - \cos 2\pi y) + \frac{1}{1+\lambda} \sin \pi x \sin \pi y 
    \end{pmatrix}, \\
    \sigma &= \begin{pmatrix} 
        2\mu \frac{du_1}{dx} + \lambda \left( \frac{du_1}{dx} + \frac{du_2}{dy} \right), & \mu \left( \frac{du_1}{dy} + \frac{du_2}{dx} \right), \\
        \mu \left( \frac{du_1}{dy} + \frac{du_2}{dx} \right), & 2\mu \frac{du_2}{dy} + \lambda \left( \frac{du_1}{dx} + \frac{du_2}{dy} \right)
    \end{pmatrix},
\end{align*}
\]

where $\mu = 1$ and $\lambda = 1, 10^3, 10^6$. Two types of meshes are used (cf. Figures 6.2-6.3).

Figure 6.2: Triangle meshes
Numerical results of the displacement and stress approximations are listed in Tables 6.1-6.2 for the proposed new WG methods (2.21)-(2.22) and (2.23)-(2.24) with $k = 0, 1, 2$. We can see that the methods yield optimal convergence rates that are uniformly with respect to the Lamé constant $\lambda$, as is conformable to the theoretical results.

For comparison we also list in Table 6.1 some results computed by the WG scheme (2.31)-(2.32), denoted by WG*, with $k = 0$ (cf. Remark 2.2). We note that for a fixed $k$, the new method is of fewer degrees of freedom (DOF) than the corresponding WG* (after the local elimination). We refer to Table 6.1(c) for the numbers of DOF for the two methods with $k = 0$. 

Figure 6.3: Ladder-shaped meshes
Table 6.1: Numerical results for $k = 0$ on triangle meshes for a 2D example

(a) Displacement error $\|u - u_{hi}\|_0 / \|u\|_0$

| Method | $\lambda = 10^0$ | $\lambda = 10^1$ | $\lambda = 10^6$ |
|--------|------------------|------------------|------------------|
|    | $\lambda = 10^0$ | $\lambda = 10^1$ | $\lambda = 10^6$ |
| new WG $2^2 \times 2^2$ | 2.5118E-01 | 2.9130E-01 | 2.9137E-01 |
| $2^3 \times 2^3$ | 1.1458E-01 | 1.5561E-01 | 1.5572E-01 |
| $2^4 \times 2^4$ | 4.1279E-02 | 6.1521E-02 | 6.1579E-02 |
| $2^5 \times 2^5$ | 1.2346E-02 | 1.9154E-02 | 1.9172E-02 |
| $2^6 \times 2^6$ | 3.3351E-03 | 5.2839E-03 | 5.2892E-03 |
| $2^7 \times 2^7$ | 8.6194E-04 | 1.3782E-03 | 1.3796E-03 |
| $2^8 \times 2^8$ | 2.1870E-04 | 3.5089E-04 | 3.5126E-04 |

(b) Stress error $\|\sigma - \sigma_{hi}\|_0 / \|\sigma\|_0$

| Method | $\lambda = 10^0$ | $\lambda = 10^1$ | $\lambda = 10^6$ |
|--------|------------------|------------------|------------------|
|    | $\lambda = 10^0$ | $\lambda = 10^1$ | $\lambda = 10^6$ |
| new WG $2^2 \times 2^2$ | 4.6433E-01 | 4.4568E-01 | 4.4551E-01 |
| $2^3 \times 2^3$ | 1.3023E-01 | 1.3145E-01 | 1.3145E-01 |
| $2^4 \times 2^4$ | 3.4483E-02 | 3.6053E-02 | 3.6059E-02 |
| $2^5 \times 2^5$ | 8.7824E-03 | 9.3198E-03 | 9.3220E-03 |
| $2^6 \times 2^6$ | 2.2064E-03 | 2.3522E-03 | 2.3528E-03 |
| $2^7 \times 2^7$ | 5.5226E-04 | 5.8949E-04 | 5.8961E-04 |
| $2^8 \times 2^8$ | 1.3811E-04 | 1.4746E-04 | 1.4736E-04 |

(c) Numbers of DOF of different methods (after local elimination)

| Method | $2^2 \times 2^2$ | $2^3 \times 2^3$ | $2^4 \times 2^4$ | $2^5 \times 2^5$ | $2^6 \times 2^6$ | $2^7 \times 2^7$ | $2^8 \times 2^8$ |
|--------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| new WG | 75 | 243 | 867 | 3267 | 12675 | 49923 | 198147 |
| WG*    | 224 | 832 | 3200 | 12544 | 49664 | 197632 | 788480 |
Table 6.2: Numerical results for new WG methods with $k = 1$ on triangular meshes and $k = 2$ on Ladder-shaped meshes: a 2D example

(a) Displacement error $\|\mathbf{u} - \mathbf{u}_h\|_0/\|\mathbf{u}\|_0$

| Method | $\lambda = 10^0$ | $\lambda = 10^3$ | $\lambda = 10^6$ |
|--------|-----------------|-----------------|-----------------|
| Mesh   | Error | Rate | Error | Rate | Error | Rate |
| $k = 1$: triangular meshes | | | | | | |
| $2^2 \times 2^2$ | 5.2634E-02 | 2.86 | 6.2573E-02 | 2.84 | 6.2624E-02 | 2.84 |
| $2^3 \times 2^3$ | 7.2373E-03 | 3.05 | 1.1972E-03 | 2.86 | 1.1987E-03 | 2.86 |
| $2^4 \times 2^3$ | 1.1056E-04 | 3.05 | 1.6295E-04 | 2.86 | 1.6328E-04 | 2.86 |
| $2^5 \times 2^3$ | 1.3369E-05 | 3.05 | 2.1207E-05 | 2.94 | 2.1263E-05 | 2.94 |
| $2^6 \times 2^3$ | 1.6481E-06 | 3.05 | 2.6839E-06 | 2.98 | 2.6930E-06 | 2.98 |
| $k = 2$: Ladder-shaped meshes | | | | | | |
| $2^2 \times 2^2$ | 2.4966E-02 | 2.5643E-02 | 2.5643E-02 | | | |
| $2^3 \times 2^3$ | 1.7134E-03 | 3.96 | 1.7813E-03 | 3.95 | 1.7813E-03 | 3.95 |
| $2^4 \times 2^3$ | 6.9419E-06 | 3.99 | 7.2747E-06 | 3.98 | 7.2752E-06 | 3.98 |
| $2^5 \times 2^3$ | 4.3538E-07 | 3.99 | 4.5666E-07 | 3.99 | 4.6030E-07 | 3.99 |

(b) Stress error $\|\mathbf{\sigma} - \mathbf{\sigma}_h\|_0/\|\mathbf{\sigma}\|_0$

| Method | $\lambda = 10^0$ | $\lambda = 10^3$ | $\lambda = 10^6$ |
|--------|-----------------|-----------------|-----------------|
| Mesh   | Error | Rate | Error | Rate | Error | Rate |
| $k = 1$: triangular meshes | | | | | | |
| $2^2 \times 2^2$ | 1.5162E-01 | 1.86 | 6.4509E-02 | 1.76 | 6.4643E-02 | 1.76 |
| $2^3 \times 2^3$ | 4.1901E-02 | 1.74 | 2.2094E-02 | 1.77 | 2.2151E-02 | 1.77 |
| $2^4 \times 2^3$ | 1.2563E-02 | 1.75 | 7.6549E-03 | 1.53 | 7.6789E-03 | 1.53 |
| $2^5 \times 2^3$ | 1.0091E-03 | 1.88 | 2.3014E-03 | 1.73 | 2.3098E-03 | 1.73 |
| $2^6 \times 2^3$ | 2.5899E-04 | 1.96 | 6.1511E-04 | 1.99 | 6.1749E-04 | 1.99 |
| $k = 2$: Ladder-shaped meshes | | | | | | |
| $2^2 \times 2^2$ | 4.5306E-02 | 1.86 | 7.317E-02 | 1.76 | 7.3222E-02 | 1.76 |
| $2^3 \times 2^3$ | 5.8426E-03 | 1.96 | 6.1579E-03 | 1.97 | 6.1587E-03 | 1.97 |
| $2^4 \times 2^3$ | 7.0054E-03 | 3.00 | 7.7307E-04 | 2.99 | 7.7318E-04 | 2.99 |
| $2^5 \times 2^3$ | 9.0101E-05 | 3.00 | 6.8514E-05 | 3.00 | 6.8755E-05 | 3.00 |
| $2^6 \times 2^3$ | 1.1348E-05 | 3.00 | 1.2046E-05 | 3.00 | 1.2066E-05 | 3.00 |

6.2 A 3D example

Let $\Omega = (0, 1) \times (0, 1) \times (0, 1)$ be subdivided into simplicial meshes (cf. Figure 6.4). We consider the homogeneous Dirichlet boundary condition, and the exact
solution \((u, \sigma)\) is of the following form:

\[
\begin{align*}
    u_1 &= 200(x - x^2)^2(2y^3 - 3y^2 + y)(2z^3 + 3z^2 + z), \\
    u_2 &= -100(y - y^2)^2(2x^3 - 3x^2 + x)(2z^3 - 3z^2 + z), \\
    u_3 &= -100(z - z^2)^2(2y^3 - 3y^2 + y)(2x^3 - 3x^2 + x), \\
    \sigma_{11} &= 400\mu(2x^3 - 3x^2 + x)^2(2y^3 - 3y^2 + y)(2z^3 + 3z^2 + z), \\
    \sigma_{22} &= -200\mu(2x^3 - 3x^2 + x)^2(2y^3 - 3y^2 + y)(2z^3 + 3z^2 + z), \\
    \sigma_{33} &= -200\mu(2x^3 - 3x^2 + x)^2(2y^3 - 3y^2 + y)(2z^3 + 3z^2 + z), \\
    \sigma_{12} = \sigma_{21} &= 2\mu\left(\frac{du_1}{dy} + \frac{du_2}{dx}\right), \\
    \sigma_{13} = \sigma_{31} &= 2\mu\left(\frac{du_1}{dz} + \frac{du_3}{dx}\right), \\
    \sigma_{23} = \sigma_{32} &= 2\mu\left(\frac{du_2}{dz} + \frac{du_3}{dy}\right),
\end{align*}
\]

where \(\mu = 0.5\) and \(\lambda = 10^6, 10^3, 10^6\).

Numerical results of the new WG methods with \(k = 0, 1\) are listed in Table 6.4. We can see that the methods yield uniformly optimal convergence rates, as is conformable to the theoretical results.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6_4.png}
\caption{3D simplicial mesh: \(1 \times 1 \times 1\) (six elements)}
\end{figure}
Table 6.3: Numerical results for $k = 0$ on simplicial meshes for a 3D example

| Method   | Mesh     | $\lambda = 10^0$ | Error | Rate | $\lambda = 10^3$ | Error | Rate | $\lambda = 10^6$ | Error | Rate |
|----------|----------|------------------|-------|------|------------------|-------|------|------------------|-------|------|
|          | $\parallel u - u_h \parallel_0$ | 4 x 4 x 4 | 5.2896E-01 |  | 5.2877E-01 |  | 5.2877E-01 | | |
|          |          | 8 x 8 x 8 | 2.3132E-01 | 1.19 | 2.3085E-01 | 1.20 | 2.3085E-01 | 1.20 | |
|          |          | 16 x 16 x 16 | 7.0628E-02 | 1.71 | 7.0193E-02 | 1.72 | 7.0192E-02 | 1.72 | |
|          | $\parallel u_h \parallel_0$ | 32 x 32 x 32 | 1.8673E-02 | 1.92 | 1.8463E-02 | 1.93 | 1.8463E-02 | 1.93 | |
|          |          | 36 x 36 x 36 | 1.4813E-02 | 1.97 | 1.4640E-02 | 1.97 | 1.4640E-02 | 1.97 | |
|          |          | 40 x 40 x 40 | 1.2033E-02 | 1.97 | 1.1891E-02 | 1.97 | 1.1890E-02 | 1.97 | |
|          |          | 44 x 44 x 44 | 9.9662E-03 | 1.98 | 9.8478E-03 | 1.98 | 9.8476E-03 | 1.98 | |
|          |          | 48 x 48 x 48 | 8.3882E-03 | 1.98 | 8.2891E-03 | 1.98 | 8.2889E-03 | 1.98 | |

Table 6.4: Numerical results for $k = 1$ on simplicial meshes for a 3D example

| Method   | Mesh     | $\lambda = 10^0$ | Error | Rate | $\lambda = 10^3$ | Error | Rate | $\lambda = 10^6$ | Error | Rate |
|----------|----------|------------------|-------|------|------------------|-------|------|------------------|-------|------|
|          | $\parallel u - u_h \parallel_0$ | 4 x 4 x 4 | 8.6049E-02 |  | 8.7240E-02 |  | 8.7245E-02 | | |
|          |          | 8 x 8 x 8 | 1.1316E-02 | 0.67 | 1.1347E-01 | 0.67 | 1.1347E-01 | 0.67 | |
|          |          | 16 x 16 x 16 | 2.7874E-01 | 0.88 | 2.7906E-01 | 0.88 | 2.7906E-01 | 0.88 | |
|          |          | 32 x 32 x 32 | 1.4725E-01 | 0.97 | 1.4296E-01 | 0.97 | 1.4296E-01 | 0.97 | |
|          | $\parallel u - u_h \parallel_0$ | 36 x 36 x 36 | 1.2710E-01 | 0.99 | 1.2728E-01 | 0.99 | 1.2728E-01 | 0.99 | |
|          |          | 40 x 40 x 40 | 1.1453E-01 | 0.99 | 1.1468E-01 | 0.99 | 1.1469E-01 | 0.99 | |
|          |          | 44 x 44 x 44 | 1.0421E-01 | 0.99 | 1.0434E-01 | 0.99 | 1.0434E-01 | 0.99 | |
|          |          | 48 x 48 x 48 | 9.5592E-02 | 0.99 | 9.5705E-02 | 0.99 | 9.5706E-02 | 0.99 | |

References

[1] D. N. Arnold and G. Awanou, Rectangular mixed finite elements for elasticity, Math. Models Methods Appl. Sci., 15 (2005), pp. 1417–1429.

[2] D. N. Arnold, F. Brezzi, and J. Douglas, Jr., PEERS: a new mixed finite element for plane elasticity, Japan J. Appl. Math., 1 (1984), pp. 347–367.

[3] D. N. Arnold, J. Douglas, Jr., and C. P. Gupta, A family of higher order mixed finite element methods for plane elasticity, Numer. Math., 45 (1984), pp. 1–22.

[4] D. N. Arnold and R. S. Falk, A new mixed formulation for elasticity, Numer. Math., 53 (1988), pp. 13–30.
[5] D. N. Arnold and R. Winther, *Mixed finite elements for elasticity*, Numer. Math., 92 (2002), pp. 401–419. 2

[6] ———, *Nonconforming mixed elements for elasticity*, Math. Models Methods Appl. Sci., 13 (2003), pp. 295–307. Dedicated to Jim Douglas, Jr. on the occasion of his 75th birthday. 2

[7] Y. Bai, Y. Wu, X. Xie, *Superconvergence and recovery type a posteriori error estimation for hybrid stress finite element method*, Sci. China Math., 59 (9) (2016), pp. 1835–1850 2

[8] D. Boffi, F. Brezzi, and M. Fortin, *Mixed finite element methods and applications*, vol. 44 of Springer Series in Computational Mathematics, Springer, Heidelberg, 2013. 2

[9] D. Braess, *Enhanced assumed strain elements and locking in membrane problems*, Comput. Methods Appl. Mech. Engrg., 165 (1998), pp. 155–174. 2

[10] D. Braess, C. Carstensen, and B. D. Reddy, *Uniform convergence and a posteriori error estimators for the enhanced strain finite element method*, Numer. Math., 96 (2004), pp. 461–479. 2

[11] J. H. Bramble, R. D. Lazarov, and J. E. Pasciak, *Least-squares methods for linear elasticity based on a discrete minus one inner product*, Comput. Methods Appl. Mech. Engrg., 191 (2001), pp. 727–744. 34

[12] S. C. Brenner, *Korn’s inequalities for piecewise $H^1$ vector fields*, Math. Comp., 73 (2004), pp. 1067–1087. 13

[13] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, vol. 15 of Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991. 2

[14] S. C. Brenner and L. R. Scott, *The mathematical theory of finite element methods*, vol. 15 of Texts in Applied Mathematics, Springer-Verlag, New York, 1994. 5

[15] G. Chen, M. Feng, and X. Xie, *A class of robust WG/HDG finite element method for convection-diffusion-reaction equations*, J. Comput. Appl. Math., 315 (2017), pp. 107–125. 3

[16] ———, *Robust globally divergence-free weak galerkin methods for Stokes equations*, J. Comput. Math., 5 (2016), pp. 549–572. 3

[17] G. Chen and M. Feng, *A $C^0$-weak galerkin finite element method for fourth order elliptic problems*, Numer. Methods Part. Differ. Equat., 32 (3)(2016), pp. 1090–1104. 3

[18] G. Chen and X. Xie, *A robust weak galerkin finite element method for linear elasticity with strong symmetric stresses*, Comput. Meth. Appl. Mat., 16(3) (2016), pp. 389–408. 3, 8

[19] L. Chen, J. Wang, Y. Wang, and X. Ye, *An auxiliary space multigrid preconditioner for the weak galerkin method*, 70 (4) (2015), pp. 330-344. 3
[20] S.-C. Chen and Y.-N. Wang, *Conforming rectangular mixed finite elements for elasticity*, J. Sci. Comput., 47 (2011), pp. 93–108.  

[21] B. Cockburn, J. Gopalakrishnan, R. Lazarov, *Unified hybridization of discontinuous Galerkin, mixed, and conforming Galerkin methods for second order elliptic problems*, SIAM J. Numer. Anal., 47 (2009), 1319–1365.  

[22] B. Cockburn, J. Gopalakrishnan, F. J. Sayas, *A projection-based error analysis of HDG methods*, Math. Comp., 79 (2010), 1351-1367.  

[23] B. Cockburn, W. Qiu, K. Shi, *Conditions for superconvergence of HDG methods for second-order elliptic problems*, Math. Comp., 81 (2012), 1327-1353.  

[24] B. Cockburn, D. Schötzau, and J. Wang, *Discontinuous Galerkin methods for incompressible elastic materials*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 3184–3204.  

[25] B. Cockburn and K. Shi, *Superconvergent HDG methods for linear elasticity with weakly symmetric stresses*, IMA J. Numer. Anal., 33 (2013), pp. 747–770.  

[26] F. de Veubeke B M, *Displacement and equilibrium models in the finite element method*, in: Zienkiewicz o c, holister g, eds., Stress Analysis, (1965), pp. 145–197.  

[27] G. Fu, B. Cockburn, and H. Stolarski, *Analysis of an HDG method for linear elasticity*, International Journal for Numerical Methods in Engineering, 102 (2015), pp. 551–575.  

[28] B. J. Gaël Guennebaud et al., *Eigen 3.2.8*, (2016), http://eigen.tuxfamily.org.  

[29] J. Hu and Z.-C. Shi, *Lower order rectangular nonconforming mixed finite elements for plane elasticity*, SIAM J. Numer. Anal., 46 (2007/08), pp. 88–102.  

[30] C. Johnson and B. Mercier, *Some equilibrium finite element methods for two-dimensional elasticity problems*, Numer. Math., 30 (1978), pp. 103–116.  

[31] H. Kabaria, A. J. Lew, and B. Cockburn, *A hybridizable discontinuous galerkin formulation for non-linear elasticity*, Computer Methods in Applied Mechanics and Engineering, 283 (2015), pp. 303 – 329.  

[32] O. A. Karakashian and F. Pascal, *Convergence of adaptive discontinuous galerkin approximations of second-order elliptic problems*, SIAM J. Numerical Analysis, 45 (2007), pp. 641–665.  

[33] B. Li and X. Xie, *A two-level algorithm for the weak Galerkin discretization of diffusion problems*, Journal of Computational and Applied Mathematics, 287 (2015), pp. 179 – 195.
[34] B. Li, X. Xie, Analysis of a family of HDG methods for second order elliptic problems, J. Comput. Appl. Math., (2016), pp. 37–51. 2

[35] B. Li, X. Xie, BPX preconditioner for nonstandard finite element methods for diffusion problems, SIAM J. Numer. Anal., 54(2)(2016), pp. 1147–1168. 3

[36] B. Li, X. Xie, S. Zhang, Analysis of a two-level algorithm for HDG methods for diffusion problems, Commun. Comput. Phys., 19 (2016), pp. 1435–1460. 3

[37] B. Li, X. Xie, S. Zhang, New convergence analysis for assumed stress hybrid quadrilateral finite element method, Discrete and Continuous Dynamical Systems series B, 22(7), 2017 pp. 2831-2856. 2

[38] Q. H. Li and J. Wang, Weak Galerkin finite element methods for parabolic equations, Numer. Methods Partial Differential Equations, 29 (2013), pp. 2004–2024. 3

[39] L. Mu, J. Wang, and X. Ye, Weak Galerkin finite element methods for the biharmonic equation on polytopal meshes, Numer. Methods Partial Differential Equations, 30 (2014), pp. 1003–1029. 3

[40] L. Mu, J. Wang, X. Ye, S. Zhang, A weak galerkin finite element method for the Maxwell equations, J. Sci. Comput., 65 (2015), pp 363–386. 3, 6

[41] N. C. Nguyen and J. Peraire, Hybridizable discontinuous Galerkin methods for partial differential equations in continuum mechanics, J. Comput. Phys., 231 (2012), pp. 5955–5988. 3

[42] N. C. Nguyen, J. Peraire, and B. Cockburn, High-order implicit hybridizable discontinuous Galerkin methods for acoustics and elastodynamics, J. Comput. Phys., 230 (2011), pp. 3695–3718. 3

[43] T. H. H. Pian, Derivation of element stiffness matrices by assumed stress distributions, AIAA Journal, 2 (1964), pp. 1333–1336. 2

[44] ——, State-of-the-art development of hybrid/mixed finite element method, Finite Elements in Analysis and Design, 21 (1995), pp. 5 – 20. Mixed and Hybrid Finite Element Methods Part I. 2

[45] T. H. H. Pian and K. Sumihara, Rational approach for assumed stress finite elements, International Journal for Numerical Methods in Engineering, 20 (1984), pp. 1685–1695. 2

[46] E. Punch and S. Atluri, Development and testing of stable, invariant, isoparametric curvilinear 2- and 3-d hybrid-stress elements, Computer Methods in Applied Mechanics and Engineering, 47 (1984), pp. 331 – 356. 2

[47] J. Qin, On the Convergence of Some Low Order Mixed Finite Elements for Incompressible Fluids, Ph.D. Thesis, Penn State University, Department of Mathematics, 1994. 9

28
[48] W. Qiu and K. Shi, An HDG method for linear elasticity with strong symmetric stresses, arXiv preprint, arXiv:1312.1407, (2013). 2, 3

[49] B. Reddy, J. Simo, Stability and convergence of a class of enhanced strain methods, SIAM J. Numer. Anal. 32 (6) (1995),1705-728. 2

[50] L.R. Scott and M. Vogelius, Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials, Math. Mod. Num. Anal., 19 (1985), 111-143. 9

[51] J.C. Simo, M.S. Rifai, A class of mixed assumed strain methods and the method of incompatible modes, Int. J. Numer. Methods Engrg. 29 (8) (1990) 1595-1638. 2

[52] S.-C. Soon, Hybridizable discontinuous galerkin methods for solid mechanics, Ph.D. Thesis, University of Minnesota, (2008). 2

[53] S.-C. Soon, B. Cockburn, and H. K. Stolarski, A hybridizable discontinuous Galerkin method for linear elasticity, Internat. J. Numer. Methods Engrg., 80 (2009), pp. 1058–1092. 2

[54] R. Stenberg, On the construction of optimal mixed finite element methods for the linear elasticity problem, Numer. Math., 48 (1986), pp. 447–462. 2

[55] ______, A family of mixed finite elements for the elasticity problem, Numerische Mathematik, 53 (1988), pp. 513–538. 2

[56] M. Vogelius, An analysis of the p-version of the finite element method for nearly incompressible materials. Uniformly valid, optimal error estimates, Numer. Math., 41 (1983), pp. 39–53. 2

[57] J. Wang and X. Ye, A weak Galerkin finite element method for second-order elliptic problems, J. Comput. Appl. Math., 241 (2013), pp. 103–115. 3

[58] R. Wang, X. Wang, Q. Zhai, and R. Zhang, A Weak Galerkin Finite Element Scheme for solving the stationary Stokes Equations, J. Comput. Appl. Math., 302 (2016), pp. 171–185. 3

[59] ______, A weak Galerkin mixed finite element method for second order elliptic problems, Math. Comp., 83 (2014), pp. 2101–2126. 3

[60] X. Xie and J. Xu, New mixed finite elements for plane elasticity and Stokes equations, Sci. China Math., 54 (2011), pp. 1499–1519. 2

[61] X. Xie and T. Zhou, Optimization of stress modes by energy compatibility for 4-node hybrid quadrilaterals, Internat. J. Numer. Methods Engrg., 59 (2004), pp. 293–313. 2

[62] G. Yu, X. Xie, and C. Carstensen, Uniform convergence and a posteriori error estimation for assumed stress hybrid finite element methods, Comput. Methods Appl. Mech. Engrg., 200 (2011), pp. 2421–2433. 2

[63] S. Zhang, A new family of stable mixed finite elements for the 3d Stokes equations, Math. Comput., 74 (2005), 543-554. 9
Appendices

A Modified Scott-Zhang interpolation for vectors

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a polyhedral region with boundary $\partial \Omega = \Gamma_D \cup \Gamma_N$, where $\text{meas}(\Gamma_D) > 0$ and $\Gamma_D \cap \Gamma_N = \emptyset$.

Let $\mathcal{T}_h$ be a shape regular simplicial subdivision of $\Omega$ with maximum mesh size $h$, such that each (open) boundary edge (face) belongs either to $\Gamma_D$, or to $\Gamma_N$. Furthermore, we assume that there should be at least 2 edges (or faces) on $\Gamma_S$ if $\Gamma_S \neq \emptyset$, where $S = D, N$.

We shall construct an interpolation operator $I_{k+1} : W^{l,p}(\Omega) \to W^{l,p}(\Omega) \cap \mathbb{P}_{k+1}(\mathcal{T}_h)$, where

$$l \geq 1 \text{ if } p = 1 \text{ and } l > 1/p \text{ otherwise.} \quad (A.1)$$

To this end, let $N_h = \{A_i\}_{i=1}^N$ be the set of all interpolation nodes of $\mathcal{T}_h$ and $\{\phi_i\}_{i=1}^N$ be the corresponding nodal basis of $\mathbb{P}_{k+1}(\mathcal{T}_h) \cap H^1(\Omega)$. And let $\{A_{j,i}\}_{i=0}^M \subset N_h$ be the set of vertexes of edge (or face) $E \subset \Gamma_N$, and $\{A_{M,i}\}_{i=1}^S \subset N_h$ be the set of interior nodes of edge (or face) $E \subset \Gamma_N$. If $d = 2$ the points $A_{j,i}$ and $A_{j+1,i}$ should be the adjoint point of $\Gamma_D$ and $\Gamma_N$, and $A_{j,i}$, $(i = 1, 2, \cdots, M)$ is between $A_{j-1,i}$ and $A_{j+1,i}$.

The interpolation operator $I_{k+1}$ is defined as follows: For any $v \in W^{l,p}(\Omega)$, given by

$$I_{k+1}v := \sum_{A_i \in N_h} I_{k+1}v(A_i) \phi_i. \quad (A.2)$$

Here for any node $A_i$, the value $I_{k+1}v(A_i)$ is determined by the following way, i.e. (1)-(4).

(1) If $A_i$ is an interior point of some $d$-simplex $T \in \mathcal{T}_h$, then let $\{A_{i,j}\}_{j=1}^{n_0}$ be the set of nodal points in $T$ and $\{\phi_{i,j}\}_{j=1}^{n_0}$ be the corresponding nodal basis, and let $\{\psi_{i,j}\}_{j=1}^{n_0}$ be the $L^2(T)$-dual basis of $\{\phi_{i,j}\}_{j=1}^{n_0}$ satisfying

$$\langle \phi_{i,j}, \psi_{i,k} \rangle_T = \delta_{jk}, \quad (A.3)$$

where $\delta_{jk}$ is the Kronecker delta. In this case, we define

$$I_{k+1}v(A_i) := \langle v, \psi_{i,1} \rangle_T. \quad (A.4)$$
Then we define

$$\langle \phi_{i,j}, \psi_{i,k} \rangle_E = \delta_{jk}. \quad (A.5)$$

Thus, we define

$$I_{k+1} v(A_i) := \langle v, \psi_{i,1} \rangle_E. \quad (A.6)$$

(3) For the rest of $A_i \in {\mathcal N}_h$, we select a $(d-1)$-simplex $E$ such that $A_i \in \overline{E}$, subject only to the restriction

$$E \subset \Gamma_D \text{ if } A_i \in \Gamma_D. \quad (A.7)$$

Let $\{A_{i,j}\}_{j=1}^{n_1} (A_{i,1} = A_i, n_1 = C_{k+1}^{d+1})$ be the set of nodal points in $E$ and $\{\phi_{i,j}\}_{j=1}^{n_1}$ be the corresponding nodal basis, and let $\{\psi_{i,j}\}_{j=1}^{n_1}$ be the $L^2(E)$-dual basis of $\{\phi_{i,j}\}_{j=1}^{n_1}$ satisfying

$$\langle \phi_{i,j}, \psi_{i,k} \rangle_E = \delta_{jk}. \quad (A.8)$$

Then we define

$$I_{k+1} v(A_i) := \langle v, \psi_{i,1} \rangle_E. \quad (A.9)$$

(4) We need to modify the interpolation conditions of $I_{k+1} v$ at some nodes $A_i$ in $\partial \Omega$ for the three cases (4a)-(4c) below. We note that the (4b)-(4c) are corresponding to the case of $k+1 < d$.

(4a) If $k+1 \geq d$, then, for any edge (or face, $d-1$-simplex) $E \subset \Gamma_S (S = D$ or $N)$, there always exists at least one interior node in $E$. We choose one such node, $A_i$, and replace the corresponding interpolation condition of $I_{k+1} v(A_i)$ in (2) or (3) by

$$\langle I_{k+1} v, 1 \rangle_E = \langle v, 1 \rangle_E. \quad (A.10)$$

(4b) If $k+1 = d-1$, then, for any open $(d-2)$-simplex $P \subset \Gamma_S (S = D$ or $N)$, there exists only one node $A_i$ inside $P$. Let $\{A_{i,j}\}_{j=1}^{M}$ denote the set of all nodes inside all the $(d-2)$-simplexes in $\Gamma_S$. For any $A_i \in \{A_{i,j}\}_{j=1}^{M}$, there exist exactly two $(d-1)$-simplexes $E_{i,1} \subset \Gamma_S$ and $E_{i,2} \subset \Gamma_S$ that are adjoined at $A_i$. Set $w(A_i) := E_{i,1} \cup E_{i,2}$. We choose a set of nodes $\{A_{j,k}\}_{i=1}^{L} \subset \{A_{i,j}\}_{j=1}^{M}$ such that

$$\mu_{d-1}(w(A_{j,k}) \cap w(A_{j,k}')) = 0, j \neq k, \quad (A.11)$$

$$\bigcup_{A_{j,k} \in \{A_{j,k}\}_{i=1}^{L}} w(A_{j,k}) = \Gamma_S. \quad (A.12)$$

Then the interpolation condition of $I_{k+1} v$ at each $A_i \in \{A_{i,j}\}_{i=1}^{L}$ is replaced by

$$\langle I_{k+1} v, 1 \rangle_{E_{i,1}} + \langle I_{k+1} v, 1 \rangle_{E_{i,2}} = \langle v, 1 \rangle_{E_{i,1}} + \langle v, 1 \rangle_{E_{i,2}}. \quad (A.13)$$
(4c) If \( k + 1 = 1 \) and \( d = 3 \), then let \( \{ A_{ij} \}_{j=1}^M \) denote the set of all nodes inside \( \Gamma_S \) \( (S = D \text{ or } N) \). For any \( A_i \in \{ A_{ij} \}_{j=1}^M \), there exist \( l_i \) \((d-1)\)-simplexes, \( \{ E_{ij} \}_{j=1}^{l_i} \), in \( \Gamma_S \) that adjoined at \( A_i \). Set \( w(A_i) := \bigcup_{E_{ij} \in \{ E_{ij} \}_{j=1}^{l_i}} \overline{E_{ij}} \). We choose a set of nodes \( \{ A_{ij} \}_{j=1}^M \subset \{ A_{ij} \}_{j=1}^M \) such that

\[
\mu_{d-1}(w(A_{ij}) \bigcap w(A_{jk})) = 0, j \neq k, \quad (A.14)
\]

\[
\bigcup_{A_{ij} \in \{ A_{ij} \}_{j=1}^M} w(A_{ij}) = \Gamma_S. \quad (A.15)
\]

Then the interpolation condition of \( I_{k+1} v \) at each \( A_i \in \{ A_{ij} \}_{j=1}^M \) is replaced by

\[
\sum_{E_{ij} \in \{ E_{ij} \}_{j=1}^{l_i}} \langle I_{k+1} v, 1 \rangle_{E_{ij}} = \sum_{E_{ij} \in \{ E_{ij} \}_{j=1}^{l_i}} \langle v, 1 \rangle_{E_{ij}}, \quad (A.16)
\]

Based on interpolation conditions in (1)-(4), we know that the interpolation polynomial \( I_{k+1} v \in W^{l,p}(\Omega) \cap \mathbb{P}_{k+1}(T_h) \) is uniquely determined.

In view of the definition of the operator \( I_{k+1} \), i.e. (A.2), we easily have the following results.

**Lemma A.1.** For any \( v \in W^{l,p}(\Omega) \), there holds

\[
\langle I_{k+1} v, 1 \rangle_{\Gamma_S} = \langle v, 1 \rangle_{\Gamma_S}, \quad S = D, N, \quad (A.17)
\]

\[
I_{k+1} v|_{\Gamma_D} = 0 \text{ if } v|_{\Gamma_D} = 0. \quad (A.18)
\]

In particular,

\[
I_{k+1} v = v, \quad \forall v \in \mathbb{P}_{k+1}(T_h) \cap W^{l,p}(\Omega). \quad (A.19)
\]

**Lemma A.2.** For any \( v \in W^{l,p}(\Omega) \) and \( T \in T_h \), there holds

\[
\| I_{k+1} v \|_{m,q,T} \lesssim \sum_{r=0}^{l} h_T^{r-m+d/q-d/p} |v|_{r,p,S(T)}, \quad (A.20)
\]

where

\[
S(T) = S'(T) := \text{interior}(\bigcup \{ T' | T' \cap T \neq \emptyset, T' \in T_h \}) \quad (A.21)
\]

if \( k + 1 \geq d \), and

\[
S(T) = \text{interior}(\bigcup \{ T' | T' \cap S(T) \neq \emptyset, T' \in T_h \}) \quad (A.22)
\]

if \( k + 1 < d \).

**Proof.** For \( v \in W^{l,p}(\Omega) \) and \( A_i \in N_h \), by (1) we have

\[
|I_{k+1} v(A_i)| \leq \|v\|_{0,1,T} \|\psi_{i,1}\|_{0,\infty,T} \leq (h_T^{-1})^0 (h_T^{-d/1})^0 |v|_{0,1,T} h_T^{-d} \leq \|\hat{v}\|_{l,p,T} \leq \sum_{r=0}^{l} h_T^{-d/p} |v|_{r,p,T}. \quad (A.23)
\]
From (2)-(3) we get
\[
|I_{k+1} v(A_i)| \leq \|v\|_{0,1,E} \|\psi_i\|_{0,\infty,E}
\leq (h_T^{-1/4} h_T^{d/2}) \|v\|_{0,1,E} h_T^{-(d-1)}
\lesssim \|v\|_{l,p,T}
\lesssim \sum_{r=0}^l h_T^{r-d/p} |v|_{r,p,T}.
\] (A.24)

For the case (4a), \(I_{k+1} v\) on \(E\) has the form
\[
I_{k+1} v|_E = I_{k+1} v(A_i) \phi_i + \sum_{j=1}^{d+1} I_{k+1} v(A_{i,j}) \phi_{i,j}.
\] (A.25)

Therefore
\[
\langle I_{k+1} v, 1 \rangle_E = I_{k+1} v(A_i) \langle \phi_i, 1 \rangle_E + \sum_{j=1}^{d+1} I_{k+1} v(A_{i,j}) \langle \phi_{i,j}, 1 \rangle_E,
\] (A.26)

which leads to
\[
I_{k+1} v(A_i) = \frac{1}{\langle \phi_i, 1 \rangle_E} \langle I_{k+1} v, 1 \rangle_E - \sum_{j=1}^{d+1} I_{k+1} v(A_{i,j}) \frac{\langle \phi_{i,j}, 1 \rangle_E}{\langle \phi_i, 1 \rangle_E}
= \frac{1}{\langle \phi_i, 1 \rangle_E} \langle v, 1 \rangle_E - \sum_{j=1}^{d+1} I_{k+1} v(A_{i,j}) \frac{\langle \phi_{i,j}, 1 \rangle_E}{\langle \phi_i, 1 \rangle_E}.
\] (A.27)

So
\[
|I_{k+1} v(A_i)| \lesssim \frac{1}{|E|} \langle v, 1 \rangle_E - \sum_{j=1}^{d+1} |I_{k+1} v(A_{i,j})|
\lesssim h_T^{-(d-1)} |v|_{0,1,E} + \sum_{r=0}^l h_T^{r-d/p} |v|_{r,p,S(T)}
\lesssim \sum_{r=0}^l h_T^{r-d/p} |v|_{r,p,S(T)}.
\] (A.28)

Similarly, for the cases (4b)-(4c), it holds
\[
I_{k+1} v(A_i) \lesssim \sum_{r=0}^l h_T^{r-d/p} |v|_{r,p,S(T)}.
\] (A.29)

As a result, from (A.2) it follows
\[
\|I_{k+1} v\|_{m,q,T} \leq \sum_{i=1}^{n_1} |I_{k+1} v(A_i)| \|\psi_i\|_{m,q,T}
\lesssim \sum_{r=0}^l h_T^{r-m+d-q-d/p} |v|_{r,p,S(T)}.
\] (A.30)

This completes the proof. \(\square\)
In light of (A.19), Lemma A.2, and the triangle inequality, we easily obtain the following approximation result.

**Lemma A.3.** For any \( v \in W^{1,p}(\Omega) \) and \( 0 \leq m \leq k + 2 \), we have

\[
\| v - \mathcal{I}_{k+1} v \|_{m,p,T} \lesssim \sum_{r=0}^{m} h_{T}^{r-m} \inf_{w \in [P_{k+1}(T_{h})] \cap H^{r}(\Omega)^{d}} \| v - w \|_{r,p,T}. \tag{A.31}
\]

**B Inf-sup conditions on \([H^{1}_{D}(\Omega)]^{d} \times L^{2}(\Omega)\)**

We first cite a result from [11]:

**Lemma B.1.** Assume that \( \text{meas}(\Gamma_{D}) > 0 \). Then there exists a positive \( C \) such that

\[
\sup_{v \in [H^{1}_{0}(\Omega)]^{d}} \frac{(\nabla \cdot v, q)}{\| v \|_{1}} \geq C \| q \|_{0}, \quad \forall q \in L^{2}(\Omega). \tag{B.1}
\]

**Lemma B.2.** Assume that \( \text{meas}(\Gamma_{D}) > 0 \). Then it holds the inf-sup condition

\[
\sup_{v_{h} \in [P_{1}(T_{h})] \cap [H^{1}_{D}(\Omega)]^{d}} \frac{(\nabla \cdot v_{h}, p)}{|v_{h}|_{1}} \geq \| p \|_{0}, \quad \text{for any constant } \mathfrak{p}. \tag{B.2}
\]

**Proof.** From (B.1) and Lemma 2.3 it follows

\[
\| \mathfrak{p} \|_{0} \lesssim \sup_{v \in [H^{1}_{0}(\Omega)]^{d}} \frac{(\nabla \cdot v, \mathfrak{p})}{|v|_{1}}
\]

\[
= \sup_{v \in [H^{1}_{0}(\Omega)]^{d}} \frac{(\nabla \cdot \mathcal{I}_{1} v, \mathfrak{p})}{|v|_{1}} + \sup_{v \in [H^{1}_{0}(\Omega)]^{d}} \frac{(\nabla \cdot (v - \mathcal{I}_{1} v), \mathfrak{p})}{|v|_{1}}
\]

\[
= \sup_{v \in [H^{1}_{0}(\Omega)]^{d}} \frac{(\nabla \cdot \mathcal{I}_{1} v, \mathfrak{p})}{|v|_{1}}
\]

\[
\lesssim \sup_{v \in [H^{1}_{0}(\Omega)]^{d}} \frac{(\nabla \cdot \mathcal{I}_{1} v, \mathfrak{p})}{|\mathcal{I}_{1} v|_{1}},
\]

which yields the desired result. \( \square \)

**Theorem B.3.** Let \( V_{h} \subset [H^{1}_{D}(\Omega)]^{d} \) and \( Q_{h} \subset L^{2}(\Omega) \) be two finite dimensional spaces such that \([P_{1}(T_{h})] \cap [H^{1}_{D}(\Omega)]^{d} \subset V_{h}\) and

\[
\sup_{v_{h} \in V_{h} \cap [H^{1}_{D}(\Omega)]^{d}} \frac{(\nabla \cdot v_{h}, p_{h})}{|v_{h}|_{1}} \gtrsim p_{h} \| p_{h} \|_{0}, \forall p_{h} \in Q_{h} \cap L^{2}_{0}(\Omega). \tag{B.3}
\]

Then the following inf-sup condition holds:

\[
\sup_{v_{h} \in V_{h}} \frac{(\nabla \cdot v_{h}, p_{h})}{|v_{h}|_{1}} \gtrsim p_{h} \| p_{h} \|_{0}, \forall p_{h} \in Q_{h}. \tag{B.4}
\]

**Proof.** From (B.2) and (B.3) we know that, for all \( p_{h} \in Q_{h} \), there exists \( v_{1} \in [P_{1}(T_{h})] \cap [H^{1}_{D}(\Omega)]^{d}, v_{2} \in V_{h} \cap [H^{1}_{D}(\Omega)]^{d}, \) and a positive constant \( C_{0} \), independent of \( p_{h}, v_{1}, v_{2} \), and the mesh size \( h \), such that

\[
(\nabla \cdot v_{1}, \mathfrak{p}_{h}) = \| \mathfrak{p}_{h} \|_{0}^{2}, \quad |v_{1}|_{1} \leq C_{0} \| \mathfrak{p}_{h} \|_{0}, \tag{B.5}
\]

\[
(\nabla \cdot v_{2}, p_{h} - \mathfrak{p}_{h}) = \| p_{h} - \mathfrak{p}_{h} \|_{0}^{2}, \quad |v_{2}|_{1} \leq C_{0} \| p_{h} - \mathfrak{p}_{h} \|_{0}, \tag{B.6}
\]
where $p_h = (p_h, 1)$. Then we can take $v_h = (v_1 + \tilde{\alpha} v_2) \in V_h$ with
\begin{align*}
(\nabla \cdot v_h, p_h) &= (\nabla \cdot v_1, p_h) + \tilde{\alpha} (\nabla \cdot v_2, p_h) \\
&= (\nabla \cdot v_1, p_h) + (\nabla \cdot v_1, p_h - p_h) + \tilde{\alpha} (\nabla \cdot v_2, p_h - p_h) \\
&\geq \|p_h\|_0^2 + \tilde{\alpha} \|p_h - p_h\|_0^2 - C_0 \|p_h\|_0 \|p_h - p_h\|_0 \\
&\geq \frac{1}{2} \|p_h\|_0^2 + (\tilde{\alpha} - \frac{C_0^2}{2}) \|p_h - p_h\|_0^2 \\
&\geq \frac{1}{4} \|p_h\|_0^2,
\end{align*}
where $\tilde{\alpha} = \frac{C_0^2}{2} + \frac{1}{2}$. On the other hand,
\begin{align*}
|v_h|_1 &\leq |v_1|_1 + \tilde{\alpha} |v_2|_1 \\
&\leq C_0 (\|p_h\|_0 + \tilde{\alpha} \|p_h - p_h\|_0) \lesssim \|p_h\|_0,
\end{align*}
which, together with (B.7), implies the desired conclusion. \hfill \Box