A lecture on Invariant Random Subgroups

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Invariant random subgroups (IRS) are conjugacy invariant probability measures on the space of subgroups in a given group $G$. They can be regarded both as a generalization of normal subgroups as well as a generalization of lattices. As such, it is intriguing to extend results from the theories of normal subgroups and of lattices to the context of IRS. Another approach is to analyse and then use the space $\text{IRS}(G)$ as a compact $G$-space in order to establish new results about lattices. This later approach has been taken in the work [7s12], that came to be known as the seven samurai paper. In this lecture I will explain this approach.

1 The Chabauty space of closed subgroups

Let $G$ be a locally compact group. We denote by $\text{Sub}_G$ the space of closed subgroups of $G$ equipped with the Chabauty topology. This topology is generated by sets of the following two types:

1. $O_1(U) := \{H \in \text{Sub}_G : H \cap U \neq \emptyset\}$ with $U \subset G$ an open subset, and
2. $O_2(K) := \{H \in \text{Sub}_G : H \cap K = \emptyset\}$ with $K \subset G$ a compact subset.

Exercise 1. Show that a sequence $H_n \in \text{Sub}_G$ converges to a limit $H$ iff

- for any $h \in H$ there is a sequence $h_n \in H_n$ such that $h = \lim h_n$, and
- for any sequence $h_{n_k} \in H_{n_k}$, with $n_{k+1} > n_k$, which converges to a limit, we have $\lim h_{n_k} \in H$.

Example 1.1. 1. $\text{Sub}_\mathbb{R} \sim [0, \infty]$. Indeed, every proper non-trivial closed subgroup of $\mathbb{R}$ is of the form $\alpha \mathbb{Z}$ for some $\alpha > 0$. When $\alpha \to 0$ the corresponding group tends to $\mathbb{R}$ and when $\alpha \to \infty$ it tends to $\{1\}$.

2. $\text{Sub}_{\mathbb{R}^2}$ is homeomorphic to the sphere $S^4$ (see [HP79]).
3. **Question:** What can you say about Sub$_G^n$ (cf. [Kio9])?

One direction which one should study is the case of semisimple Lie groups. Indeed, much is known about discrete and general closed subgroups of (semisimple) Lie groups, and it is possible to deduce information about the structure of Sub$_G$.

**Problem 1.2.** What can you say about Sub$_G$ for $G = \text{SL}_2(\mathbb{R})$?

While the structure of Sub$_G$ in general is highly complicated, we at least know that it is always compact:

**Proposition 1.3.** *(Exercise)* For every locally compact group $G$, the space Sub$_G$ is compact.

We can use Sub$_G$ in order to compactify certain sets of closed subgroups. For instance one can study the Chabauty compactification of the space of lattices in $G$. In particular, it is interesting to determine the points of that compactification:

**Problem 1.4.** Determine which subgroups of $\text{SL}_3(\mathbb{R})$ are limit of lattices.

This problem might be more accessible if we replace $\text{SL}_3(\mathbb{R})$ with a group for which the congruence subgroup property is known for all lattices.

Here is one simple, yet useful, result:

**Proposition 1.5.** Let $G$ be a connected simple Lie group. Then $G$ is an isolated point in Sub$_G$.

The idea is that if $H$ is sufficiently close to $G$ then it has points close to 1 whose logarithm generate the Lie algebra of $G$. This implies that the connected component of identity $H^\circ$ is normal in $G$, and as $G$ is simple, it is either trivial or everything. Thus, it is enough to show that $H^\circ$ is non-trivial, i.e. that $H$ is not discrete. This follows from:

**Theorem 1.6** *(Zassenhaus [Rag72])*Every Lie group $G$ admits an identity neighbourhood $U$ such that if $H \leq G$ is a discrete group generated by $H \cap U$ then $H$ is contained in a connected nilpotent subgroup of $G$.

For more details about Proposition 1.5 as well as other results in this spirit, see [7s12, Section 2].

**Exercise 2.** Consider $G = \text{SL}_n(\mathbb{Q}_p)$ and show that $G$ is an isolated point in Sub$_G$.

**Hint:** Use the following facts:

- $\text{SL}_n(\mathbb{Z}_p)$ is a maximal subgroup of $\text{SL}_n(\mathbb{Q}_p)$. 

• The Frattini subgroup of $\text{SL}_n(\mathbb{Z}_p)$ is open, i.e. of finite index.

**Example 1.7.** The circle group $S^1$ is not isolated in its Chabauty space $\text{Sub}_{S^1}$. Indeed, one can approximate $S^1$ by finite cyclic subgroups.

More generally,

**Exercise 3.** Show that if $G$ surjects on $S^1$, then $G$ is not isolated in $\text{Sub}_G$.

In fact, using Theorem 1.6 one can show:

**Proposition 1.8.** A connected Lie group is isolated in $\text{Sub}_G$ iff it does not surject on $S^1$.

Note that a connected Lie group $G$ does not surject on the circle iff its commutator $G'$ is dense in $G$. Such groups are called *topologically perfect*.

**Exercise 4.** Let $G$ be a Lie group. Deduce from Theorem 1.6 that if $H \in \text{Sub}_G$ is a limit of discrete groups, then the identity connected component $H^0$ of $H$ is nilpotent.

## 2 Invariant measures on $\text{Sub}_G$

The group $G$ acts on $\text{Sub}_G$ by conjugation and it is natural to consider the invariant measures on this compact $G$-space.

**Definition 2.1.** An Invariant Random Subgroup (hereafter IRS) is a Borel probability measure on $\text{Sub}_G$ which is invariant under conjugations.

### 2.1 First examples and remarks:

1. The Dirac measures correspond to normal subgroups. In view of that, one can regard IRS’s as a generalization of normal subgroup.

2. Let $\Gamma \leq G$ be a lattice (or more generally a closed subgroup of finite co-volume). The map $G \to \Gamma^G \subset \text{Sub}_G$, $g \mapsto g\Gamma g^{-1}$, factors through $G/\Gamma$. Hence we may push the invariant probability measure on $G/\Gamma$ to a conjugation invariant probability measure on $\text{Sub}_G$ supported on (the closure of) the conjugacy class of $\Gamma$. In view of that IRS’s also generalize ‘lattices’ or more precisely finite volume homogeneous spaces $G/\Gamma$ — as conjugated lattices give to the same IRS. We shall denote the IRS associated with (the conjugacy class of) $\Gamma$ by $\mu_\Gamma$.

   For instance let $\Sigma$ be a closed hyperbolic surface (of genus $\geq 2$) and normalize its Riemannian measure. Every unit tangent vector yields an embedding of $\pi_1(\Sigma)$
in $\text{PSL}_2(\mathbb{R})$. Thus the probability measure on the unit tangent bundle corresponds to an IRS of type (2) above.

(3) Let again $\Gamma \leq G$ be a lattice in $G$, and let $N < \Gamma$ be a normal subgroup of $\Gamma$. As in (2) the $G$-invariant probability measure on $G/\Gamma$ can be used to choose a random conjugate of $N$ in $G$. This is an IRS supported on the (closure of the) conjugacy class of $N$. More generally, every IRS on $\Gamma$ can be induced to an IRS on $G$. Intuitively, the random subgroup is obtained by conjugating $\Gamma$ by a random element from $G/\Gamma$ and then picking a random subgroup in the corresponding conjugate of $\Gamma$.

### 2.2 Connection with pmp actions

Let $G \act X$ be a probability measure preserving action. It can be shown that the stabilizer of almost every point in $X$ is closed in $G$. Moreover, the stabilizer map $X \to \text{Sub}_G$, $x \mapsto G_x$ is measurable, and hence one can push the measure $m$ to an IRS on $G$. In other words the random subgroup is the stabilizer of a random point in $X$.

This reflects the connection between invariant random subgroups and pmp actions. Moreover, the study of pmp $G$-spaces can be divided to

- the study of stabilizers (i.e. IRS),
- the study of orbit spaces

and the interplay between the two.

The connection between IRS and pmp actions goes also in the other direction:

**Theorem 2.2.** Let $G$ be a locally compact group and $\mu$ an IRS in $G$. Then there is a probability space $(X, m)$ and a measure preserving action $G \act X$ such that $\mu$ is the push-forward of the stabilizer map $X \to \text{Sub}_G$.

This was proved in [AGV14] for discrete groups and in [7s12, Theorem 2.4] for general $G$. The first thing that comes to mind is to take the given $G$ action on $(\text{Sub}_G, \mu)$, but then the stabilizer of a point $H \in \text{Sub}_G$ is $N_G(H)$ rather than $H$. To correct this one can consider the larger space $\text{Cos}_G$ of all cosets of all closed subgroups, as a measurable $G$-bundle over $\text{Sub}_G$. Defining an appropriate invariant measure on $\text{Cos}_G \times \mathbb{R}$ and replacing each fiber by a Poisson process on it, gives the desired probability space.

### 2.3 Topology

We shall denote by $\text{IRS}(G)$ the space of IRS on $G$ equipped with the $w^*$-topology. By Alaoglu’s theorem $\text{IRS}(G)$ is compact.
2.4 Existance

An interesting yet open question is whether this space is always non-trivial.

**Question 2.3.** Does every non-discrete locally compact group admit a non-trivial IRS?

A counterexample, if exists, should in particular be a simple group without lattices. Currently the only known such example is the Neretin group and some close relatives (see [BCGM12]). The question whether the Neretin group admits non-trivial IRS has two natural sub-questions:

**Question 2.4.**

1. Does the Neretin group admits a (non-discrete) closed subgroup of finite co-volume?

2. Does the Neretin group admits a non-trivial discrete IRS, i.e. an IRS with respect to which a random subgroup is a.s. discrete?

**Remark 2.5.** There are many discrete groups without nontrivial IRS, for instance $\text{PSL}_n(\mathbb{Q})$.

3 IRS and lattices

Viewing IRS as a generalization of lattices there are two directions toward which one is urged to go:

1. Extend classical theorems about lattices to general IRS.

2. Use the compact space $\text{IRS}(G)$ in order to study its special ‘lattice’ points.

Remarkably, the approach (2) turns out to be quite fruitful in the theory of asymptotic properties of lattices. We shall see later on an example of how rigidity properties of $G$-actions yield interesting data of the geometric structure of locally symmetric spaces $\Gamma \backslash G/K$ when the volume tends to infinity.

Here is another direction in the spirit of (2), this time with a fixed volume:

3.1 The IRS compactification of moduli spaces

Let $\Sigma$ be a closed surface of genus $\geq 2$. As we have seen in 2.1(2), every hyperbolic structure on $\Sigma$ corresponds to an IRS in $\text{PSL}_2(\mathbb{R})$. Taking the closure in IRS$(G)$ of the set of hyperbolic structures on $\Gamma$, one obtains an interesting compactification of the moduli space of $\Sigma$.

**Problem 3.1.** Analyse the IRS compactification of $\text{Mod}(\Sigma)$.

Let us now describe two important results in the spirit of (1):
3.2 Borel density theorem

**Theorem 3.2** (Borel density theorem for IRS, [7s12]). Let $G$ be a connected non-compact simple Lie group. Let $\mu$ be an IRS on $G$ without atoms. Then a random subgroup is $\mu$-a.s. discrete and Zariski dense.

Note that since $G$ is simple, the only possible atoms are at the trivial group $\{1\}$ and at $G$. Since $G$ is an isolated point in $\text{Sub}_G$, it follows that

$$\text{DIRS}(G) := \{\mu \in \text{IRS}(G) : \text{a $\mu$-random subgroup is a.s. discrete}\}$$

is a compact space. We shall refer to the points of $\text{DIRS}(G)$ as discrete IRS.

In order to prove Theorem 3.2 one first observes that there are only countably many conjugacy classes of non-trivial finite subgroups in $G$, hence the measure of their union is zero with respect to any non-atomic IRS. Then one can apply the same idea as in Furstenberg’s proof of the classical Borel density theorem [Fu76]. Indeed, taking the Lie algebra of $H \in \text{sub}_G$ as well as of its Zariski closure induce measurable maps

$$H \mapsto \text{Lie}(H), \quad H \mapsto \text{Lie}(\overline{H})$$

and since a non-trivial Grassman variety associated to subspaces of $\text{Lie}(G)$ does not carry an $\text{Ad}(G)$-invariant measure (because $G$ is noncompact) one deduces that $\text{Lie}(H) = 0$, i.e. that $H$ is discrete, and $\text{Lie}(\overline{H}) = \text{Lie}(G)$ almost surely (see [7s12] for more details).

3.3 Stuck–Zimmer rigidity theorem

Perhaps the first result about IRS and certainly one of the most remarkable, is the Stuck–Zimmer rigidity theorem, which can be regarded as a (far reaching) generalisation of Margulis normal subgroup theorem.

**Theorem 3.3.** [SZ94] Let $G$ be a connected simple Lie group of real rank $\geq 2$. Then every ergodic pmp action of $G$ is either (essentially) free or transitive.

In view of Theorem 2.2 one can read Theorem 3.3 as: *every non-atomic ergodic IRS in $G$ is of the form $\mu_{\Gamma}$ for some lattice $\Gamma \leq G$.*

**Remark 3.4.**

1. Stuck and Zimmer proved the theorem for the wider class of higher rank semisimple groups with property $(T)$. The situation for certain groups, such as $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ is still unknown.

2. Recently A. Levit [Le14] proved the analog result for analytic groups over non-archimedean local fields.
In order to see that Margulis normal subgroup theorem is a consequence of Theorem 3.3, suppose that $\Gamma \leq G$ is a lattice and $N \triangleleft \Gamma$ is a normal subgroup in $\Gamma$. As described in Example (3) above there is an IRS $\mu$ supported on (the closure of) the conjugacy class of $N$. Since $G \curvearrowright G/\Gamma$ is ergodic and $(\text{Sub}_G, \mu)$ is a factor, it is also ergodic. In view of Theorem 2.2 there is an ergodic pmp $G$-space $(X, m)$ such that the stabilizer of an $m$-random point is a $\mu$-random conjugate of $N$. By Theorem 3.3 $X$ is transitive, i.e. $X = G/N$. It follows that $G/N$ carries a $G$-invariant probability measure. Thus $N$ is a lattice, hence of finite index in $\Gamma$.

Obviously, this reasoning also goes in the other direction — If $G$ has a lattice with an infinite index normal subgroup $N \triangleleft \Gamma$ one obtains an ergodic pmp space for which almost any stabilizer is a conjugate of $N$.

We shall now give a more interesting example, in the lack of rigidity:

### 3.4 Samet’s example — an exotic IRS in $\text{PSL}_2(\mathbb{R})$

**Example 3.5.** Let $A, B$ be two copies of a torus with 2 open discs removed. We chose hyperbolic metrics on $A$ and $B$ so that all 4 boundary components are totally geodesic circles of the same length, and that $A$ admits a closed geodesic of length much smaller than the injectivity radius at any point of $B$. We may agree that one boundary component of $A$ (resp. of $B$) is ‘on the left side’ and the other is ‘on the right side’, and fix a special point on each boundary component, in order to specify a gluing pattern of a ‘left’ copy and a ‘right’ copy, each of either $A$ or $B$.

![Figure 1: The hyperbolic building blocks](image)

Now consider the space $\{A, B\}^\mathbb{Z}$ with the Bernoulli measure $(\frac{1}{2}, \frac{1}{2})^\mathbb{Z}$. Any element $\alpha$ in this space is a two sided infinite sequence of $A$’s and $B$’s and we can glue copies of $A, B$ ‘along a bi-infinite line’ following this sequence. This produces a random surface $M^\alpha$. Choosing a probability measure on the unit tangent bundle of $A$ (resp. of $B$) we define an IRS in $\text{PSL}_2(\mathbb{R})$ as follows. First choose $M^\alpha$.
randomly, next choose a point and a unit tangent vector in the copy of $A$ or $B$
which lies at the place $M_0^{\alpha}$ (above 0 in the sequence $\alpha$), then take the fundamental
group of $M^\alpha$ according to the chosen point and direction.

![Figure 2: A random surface](image)

As the $\mathbb{Z}$-action on the Bernoulli space of sequences is ergodic one sees that
the corresponding IRS is also ergodic. It can be shown that almost surely the
corresponding group is not contained in a lattice in $\text{PSL}_2(\mathbb{R})$. Analog constructions
can be made in $\text{SO}(n,1)$ for all $n$'s (see [7s12, Section 13]).

### 3.5 Soficity of IRS

The following question can be asked for any locally compact group $G$, however I
find the 3 special cases of $G = \text{SL}_2(\mathbb{R}), \text{SL}_2(\mathbb{Q}_p)$ and $\text{Aut}(T)$ particularly intriguing:

**Question 3.6.** Is every discrete IRS a weak-$*$ limit of ones supported on lattices?

In view of the connection between IRS’s and the Benjamini–Schramm topology
which we are about to describe, the case of $G = \text{Aut}(T)$ is strongly related to the
famous Aldous–Lyons conjecture that every unimodular network is sofic [AL07]
(i.e. a limit of ones corresponding to finite graphs).

**Exercise 5.** Show that the same question for $G = F_n$, the discrete rank $n$ free
group, is equivalent to the question whether every $n$-generated group is sofic.

### 4 The Gromov–Hausdorff topology

Given a compact metric space $X$, the Hausdorff distance $\text{Hd}_X(A,B)$ between two
closed subsets is defined as

$$\text{Hd}_X(A,B) := \inf\{\epsilon : N_\epsilon(A) \supset B \text{ and } N_\epsilon(B) \supset A\},$$

where $N_\epsilon(A)$ is the $\epsilon$-neighborhood of $A$. The space $2^X$ of closed subsets of $X$
equipped with the Hausdorff metric, is compact.

Given two compact metric spaces $X, Y$, the Gromov distance $\text{Gd}(X,Y)$ is defined as

$$\text{Gd}(X,Y) := \inf_Z\{\text{Hd}_Z(i(X), j(Y))\},$$
over all compact metric spaces \( Z \) admitting isometric copies \( i(X), j(Y) \) of \( X,Y \) respectively. If \((X,p), (Y,q)\) are pointed compact metric spaces, i.e. ones with a chosen point, we define the Gromov distance

\[
Gd((X,p), (Y,q)) := \inf_Z \{Hd_Z(i(X), j(Y)) + d_Z(i(p), j(q))\}.
\]

The Gromov–Hausdorff distance between two pointed proper (not necessarily bounded) metric spaces \((X,p), (Y,q)\) can be defined as

\[
GHd((X,p), (Y,q)) := \sum_{n \in \mathbb{N}} \frac{1}{2^n} Gd((B_X(n), p), (B_Y(n), q)),
\]

where \(B_X(n)\) is the ball of radius \(n\) around \(p\).

## 5 The Benjamini–Schramm topology

Let \( \mathcal{M} \) be the space of all (isometry classes of) pointed proper metric spaces equipped with the Gromov–Hausdorff topology. This is a huge space and for many applications it is enough to consider compact subspaces of it by bounding the geometry. That is, let \( f(\epsilon, r) \) be an integer valued function defined on \((0, 1) \times \mathbb{R}_{>0}\), and let \( \mathcal{M}_f \) consist of those spaces for which, for every \( r > 0 \), no \( f(\epsilon, r) + 1 \) points in the \( r \)-ball around the special point form an \( \epsilon \)-discrete set. Then \( \mathcal{M}_f \) is a compact subspace of \( \mathcal{M} \).

In many situations one prefers to consider some variants of \( \mathcal{M} \) which carries more information about the spaces. For instance when considering graphs, it may be useful to add colors and directions to the edges, and the distance between rooted coloured graphs remembers the coloring. Another example is smooth Riemannian manifolds, in which case it is better to consider framed manifolds, i.e. manifold with a chosen point and a chosen frame at the tangent space at that point. In that case, one replace the Gromov–Hausdorff topology by the ones determined by \((\epsilon, r)\) relations (see [7s12, Section 3] for details), which remembers also the directions from the special point.

We define the Benjamini–Schramm space \( \mathcal{BS} = \text{Prob}(\mathcal{M}) \) to be the space of all Borel probability measures on \( \mathcal{M} \) equipped with the weak-* topology. Given \( f \) as above, we set \( \mathcal{BS}_f := \text{Prob}(\mathcal{M}_f) \). Note that \( \mathcal{BS}_f \) is compact.

The name of the space is chosen to hint that this is the same topology induced by ‘local convergence’, introduced by Benjamini and Schramm in [BS01], when restricting to measures on rooted graphs. Recall that a sequence of random rooted bounded degree graphs converges to a limiting distribution if for every \( n \) the statistics of the \( n \) ball around the root (i.e. the probability vector corresponding to the finitely many possibilities for \( n \)-balls) converges to a limit.
The case of general proper metric spaces can be described similarly. A sequence \( \mu_n \in \mathcal{BS} \) converges to a limit \( \mu \) iff for any compact pointed ‘test-space’ \( M \in \mathcal{M} \), any \( r \) and arbitrarily small \( \epsilon > 0 \), the \( \mu_n \) probability that the \( r \) ball around the special point is ‘\( \epsilon \)-close’ to \( M \) tends to the \( \mu \)-probability of the same event.

An example of a point in \( \mathcal{BS} \) is a measured metric space, i.e. a metric space with a Borel probability measure. A particular case is a finite volume Riemannian manifold — in which case we normalize the Riemannian measure to be one, and then randomly choose a point and a frame.

Thus a finite volume locally symmetric space \( M = \Gamma \backslash G / K \) produces both a point in the Benjamini–Schramm space and an IRS in \( G \). This is a special case of a general analogy that I’ll now describe. Given a symmetric space \( X \), let us denote by \( \mathcal{M}(X) \) the space of all pointed (or framed) complete Riemannian manifolds whose universal cover is \( X \), and by \( \mathcal{BS}(X) = \text{Prob}(\mathcal{M}(X)) \) the corresponding subspace of the Benjamini–Schramm space.

Let \( G \) be a non-compact simple Lie group with maximal compact subgroup \( K \leq G \) and an associated Riemannian symmetric space \( X = G / K \). There is a natural map

\[
\{ \text{discrete subgroups of } G \} \to \mathcal{M}(X), \quad \Gamma \mapsto \Gamma \backslash X.
\]

It can be shown that this map is continuous, hence inducing a continuous map

\[
\text{DIRS}(G) \to \mathcal{BS}(X).
\]

It can be shown that the later map is one to one, and since \( \text{DIRS}(G) \) is compact, it is an homeomorphism to its image (see \[7s12, Corollary 3.4\]).

**Exercise 6** (Invariance under the geodesic flow). Given a tangent vector \( \mathbf{v} \) at the origin (the point corresponding to \( K \)) of \( X = G / K \), define a map \( \mathcal{F}_\mathbf{v} \) from \( \mathcal{M}(X) \) to itself by moving the special point using the exponent of \( \mathbf{v} \) and applying parallel transport to the frame. This induces a homeomorphism of \( \mathcal{BS}(X) \). Show that the image of \( \text{DIRS}(G) \) under the map above is exactly the set of \( \mu \in \mathcal{BS}(X) \) which are invariant under \( \mathcal{F}_\mathbf{v} \) for all \( \mathbf{v} \in T_K(G / K) \).

Thus we can view geodesic-flow invariant probability measures on framed locally \( X \)-manifolds as IRS on \( G \) and vice versa, and the Benjamini–Schramm topology on the first coincides with the IRS-topology on the second.

**Exercise 7.** Show that the analogy above can be generalised, to some extent, to the context of general locally compact groups. Given a locally compact group \( G \), fixing a right invariant metric on \( G \), we obtain a map \( \text{Sub}_G \to \mathcal{M}, \quad H \mapsto G / H \), where the metric on \( G / H \) is the induced one. Show that this map is continuous and deduce that it defines a continuous map \( \text{IRS}(G) \to \mathcal{BS} \).

\[1\]This doesn’t mean that it happens for all \( \epsilon \).
For the sake of simplicity we shall now forget ‘the frame’ and consider pointed $X$-manifolds, and $\mathcal{BS}(X)$ as probability measures on such. We note that while for general Riemannian manifolds there is a benefit for working with framed manifolds, in the world of locally symmetric spaces of non-compact type, pointed manifolds, and measures on such, behave nicely enough.

In order to examine convergence in $\mathcal{BS}(X)$ its enough to use as ‘test-space’ balls in locally $X$-manifolds. Moreover, since $X$ is non-positively curved, a ball in an $X$-manifold is isometric to a ball in $X$ iff it is contractible.

For an $X$-manifold $M$ and $r > 0$, we denote by $M_{\geq r}$ the $r$-thick part in $M$:

$$M_{\geq r} := \{ x \in M : \text{InjRad}_M(x) \geq r \},$$

where $\text{InjRad}_M(x) = \sup \{ \epsilon : B_M(x, \epsilon) \text{ is contractible} \}$.

Note that since $X$ is a homogeneous space, all probability measure on $X$ correspond to the same point in $\mathcal{BS}(X)$, and we shall denote this point by $X$. We have the following simple characterisation of convergence to $X$:

**Proposition 5.1.** A sequence $M_n$ of finite volume $X$-manifolds $BS$-converges to $X$ iff

$$\frac{\text{vol}(M_{n,\geq r})}{\text{vol}(M_n)} \to 1,$$

for every $r > 0$.

6 Higher rank and rigidity

Suppose now that $G$ is a non-compact simple Lie group of real rank at least 2. The following result from [7s12] can be interpreted as ‘large manifolds are almost everywhere fat’:

**Theorem 6.1.** Let $M_n = \Gamma_n \backslash X$ be a sequence of finite volume $X$-manifolds with $\text{vol}(M_n) \to \infty$. Then $M_n \to X$ in the Benjamini–Schramm topology.

This means that for any $r$ and $\epsilon$ there is $V(r, \epsilon)$ such that if $M$ is an $X$-manifold of volume $v \geq V(r, \epsilon)$ then $\frac{\text{vol}(M_{\geq r})}{v} \geq 1 - \epsilon$ (see Figure 4).

Using the dictionary from the previous section we may reformulate Theorem 6.1 in the language of IRS:

**Theorem 6.2.** Let $\Gamma_n \leq G$ be a sequence of lattices with $\text{vol}(G/\Gamma_n) \to \infty$ and denote by $\mu_n$ the corresponding IRS. Then $\mu_n \to \delta_1$.

The proof makes use of the equivalence between the two formulations. The main ingredient in the proof are Stuck-Zimmer rigidity theorem [3.3] and Kazhdan’s property (T), which will be used at two places.
Recall that by Kazhdan’s theorem, $G$ has property (T). This implies that a limit of ergodic measures is ergodic:

**Theorem 6.3 ([GW97]).** Let $G$ be a group with property (T) acting by homeomorphisms on a compact Hausdorff space $X$. Then the set of ergodic $G$-invariant probability Borel measures on $X$ is $w^*$-closed.

The idea is that if $\mu_n$ are probability measures converging to a limit $\mu$ and $\mu$ is not ergodic, then there is a continuous function $f$ on $X$ which, as a function in $L_2(\mu)$ is $G$-invariant, almost orthogonal to the constants and with norm 1. Thus for large $n$ we have that $f$ is almost invariant in $L_2(\mu_n)$, almost orthogonal to the constants and with norm almost 1. Since $G$ has property (T) it follows that there is an invariant $L_2(\mu_n)$ function close to $f$, so $\mu_n$ cannot be ergodic.

Let now $\mu_n$ be a sequence as in the formulation 6.2 and let $\mu$ be a weak-$*$ limit of $\mu_n$. Our aim is to show that $\mu = \delta_1$. Up to replacing $\mu_n$ by a subnet, we may suppose that $\mu_n \to \mu$. Let $M_n = \Gamma_n \backslash X$ be the corresponding manifolds, as in 6.1. By Theorem 6.3 we know that $\mu$ is ergodic. The following result is a consequence of Theorem 6.3.

**Proposition 6.4.** The only ergodic IRS on $G$ are $\delta_G, \delta_1$ and $\mu_\Gamma$ for $\Gamma \leq G$ a lattice.
Proof. Let $\mu$ be an ergodic IRS on $G$. By Theorem 2.2 $\mu$ is the stabilizer of some pmp action $G \rhd (X, m)$. By Theorem 3.3 the latter action is either essentially free, in which case $\mu = \delta_1$, or transitive, in which case the (random) stabilizer is a subgroup of co-finite volume. The Borel density theorem implies that in the latter case, the stabilizer is either $G$ or a lattice $\Gamma \leq G$.

Thus, in order to prove Theorem 6.2, we have to exclude the cases $\mu = \delta_G$ and $\mu = \mu_{\Gamma}$. The case $\mu = \delta_G$ is impossible since $G$ is an isolated point in $\text{Sub}_G$ (see 1.5). Let us now suppose that $\mu = \mu_{\Gamma}$ for some lattice $\Gamma \leq G$ and aim towards a contradiction. For this, we will adopt the formulation of 6.1. Thus we suppose that $M_n \to M = \Gamma \setminus X$.

Recall that Property (T) of $G$ implies that there is a lower bound $C > 0$ for the Cheeger constant of all finite volume $X$-manifolds. For our purposes, the Cheeger constant of a manifold $M$ can be defined as the infimum of

$$\frac{\text{vol}(N_1(S))}{\min\{\text{vol}(M_i)\}},$$

where $S$ is a subset which disconnects the manifold, $N_1(S)$ is its 1-neighbourhood, and $M_i$ is chosen from the connected pieces of $M \setminus S$.

Since $M = \Gamma \setminus X$ has finite volume we may pick a point $p \in M$ and $r$ large enough so that the volume of the $r-1$ ball around $p$ in $M$ is greater than $\text{vol}(M)(1-C)$ (note that if $M$ is compact we may even take a ball that covers $M$). In particular, when taking $S = \{x \in M : d(x, p) = r\}$ we have that $\frac{\text{vol}(N_1(S))}{\text{vol}(M)} < C$. This on itself does not contradict property (T) since the complement of $B_M(p, r+1)$ is very small.

Now since $M_n$ converges to $M$ in the BS-topology, it follows that for large $n$, a random point $q$ in $M_n$ with positive probability satisfies that

$$\frac{\text{vol}(B_{M_n}(q, r+1) \setminus B_{M_n}(q, r-1))}{\text{vol}(B_{M_n}(q, r-1))} < C.$$ 

Bearing in mind that $\text{vol}(M_n) \to \infty$, we get that for large $n$, the complement $M_n \setminus B_{M_n}(q, r+1)$ has arbitrarily large volume, and in particular greater than $\text{vol}(B_{M_n}(q, r-1))$. Now, this contradicts the assumption that $C$ is the Cheeger constant of $X$.

Note that Theorems 6.1 and 6.2 can also be formulated as:

- The set of extreme points in IRS(G) (the ergodic IRS) is closed and equals $\{\delta_G, \delta_1, \mu_{\Gamma}, \Gamma \leq G \text{ a lattice}\}$ and its unique accumulation point is $\delta_1$, or as:

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• The space of geodesic flow invariant probability measures on $\mathcal{M}(X)$ is compact and convex, its extreme points are the finite volume $X$-manifolds and the space $X$, and the later being the only accumulation point.

Finally let us note that Theorem 6.1 has many applications in the theory of asymptotic invariants, and in particular $L_2$-invariant, of arithmetic groups and locally symmetric manifolds. Most of the work [7s12] is dedicated for such asymptotic results and our main new ingredient is Theorem 6.1. For instance, one quite immediate application is that if $M_n$ is a sequence of uniformly discrete (i.e. with a uniform lower bound on the injectivity radius) $X$-manifolds with volume $\text{vol}(M_n) \to \infty$ then the normalised betti numbers converge to a limit

$$\frac{b_i(M_n)}{\text{vol}(M_n)} \to b^{(2)}_i(X),$$

and the limit $b^{(2)}_i(X)$ is computable, and vanishes for $i \neq \text{dim}(X)/2$ (cf. [7s11]).

Figure 4: The Cheeger constant of $M_n$ is too small.
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