Ground State and Tkachenko Modes of a Rapidly Rotating Bose–Einstein Condensate in the Lowest Landau Level State

E.B. Sonin
Racah Institute of Physics, Hebrew University of Jerusalem, Jerusalem 91904, Israel

(Dated: March 23, 2022)

The Letter considers the ground state and the Tkachenko modes for a rapidly rotating Bose–Einstein condensate (BEC), when its macroscopic wave function is a coherent superposition of states analogous to the lowest Landau levels of a charge in a magnetic field. As well as in type II superconductors close to the critical magnetic field \( H_{c2} \), this corresponds to a periodic vortex lattice. The exact value of the shear elastic modulus of the vortex lattice, which was known from the old works on type II superconductors, essentially exceeds the values calculated recently for BEC. This is important for comparison with observation of the Tkachenko mode in the rapidly rotating BEC.

PACS numbers: 03.75.Kk, 67.40.Vs

A rapidly rotating Bose-Einstein condensate (BEC) of cold atoms is now a subject of intensive experimental and theoretical investigations. It is well known that rotation gives rise to a regular triangular vortex lattice. At moderate rotation speed this is a lattice of vortex lines (points in the 2D case) with the core size of the order of coherence length \( \xi \), which essentially smaller than the intervortex distance \( b = \sqrt{\frac{\kappa}{\sqrt{3}}\Omega} \). Here \( \Omega \) is the angular velocity of rotation and \( \kappa = \hbar/m \) is the circulation quantum. One can call it the Vortex Line Lattice (VLL) regime. With increasing \( \Omega \), the vortex lattice becomes more and more dense and eventually enters the regime, in which the vortex cores start to overlap, i.e., \( \xi \) becomes larger than the \( b \). This regime is analogous to the mixed state of a type-II superconductor close to the second critical magnetic field \( H_{c2} \sim \Phi_0/\xi^2 \) (\( \Phi_0 \) is the magnetic flux quantum), at which the transition to the normal state takes place. However, in a rotating BEC there is no phase transition at corresponding “critical” angular velocity \( \Omega_{c2} \sim \kappa/\xi^2 \). Instead the crossover to the new regime takes place: At \( \Omega > \Omega_{c2} \) all atoms condensate in a state, which is a coherent superposition of single-particle states similar to the Lowest Landau Levels (LLL) of a charged particle in a magnetic field (the LLL regime). An important method of investigation of the vortex structure is studying its collective modes. Coddington et al. were able to detect the Tkachenko modes (transverse sound in the vortex lattice) in the VLL regime experimentally. Recently they increased the rotation speed in the attempt to reach the LLL regime. They revealed softer Tkachenko modes as was predicted by the theory for the LLL regime.

Theyoretical study of the Tkachenko mode requires good knowledge of the equilibrium state. A number of papers addressed this issue using the analogy with the quantum Hall effect. They started from the LLL wave functions for noninteracting particles in a trapping potential and switched interaction on after it. For a regular vortex lattice this yielded the Gaussian density profile \( \Psi(\mathbf{r}) \), but it was unstable with respect to small distortions of the lattice, which transformed the Gaussian profile to the inverted parabola (Thomas-Fermi distribution). This Letter suggests another strategy. One can start from an infinite periodical vortex lattice in an infinite uniform liquid, neglecting first the trapping potential but taking into account interaction. The exact wave function for this state was found in the classical work by Abrikosov for type II superconductors close to \( H_{c2} \), and later it was generalized for an arbitrary unit cell of the vortex lattice. If a trapping potential is added, the vortex lattice (as well as the liquid density) ceases to be uniform. But as far as modulation by the trapping potential is smooth (the cloud size is much larger than the intervortex distance) it does not distort the lattice essentially (except for the extreme periphery of the cloud) and one can use the thermodynamic potential derived for a uniform vortex lattice. The suggested approach is especially useful for investigation of the Tkachenko mode since for the infinite uniform lattice the shear elastic modulus can be calculated exactly and for the vortex lattice in type II superconductors it was done many years ago. The previous calculations of the shear modulus in BEC yielded the values smaller than the exact one. This difference is important for comparison with recent experiments on Tkachenko modes.

We consider a 2D rotating BEC in a parabolic trapping potential characterized by the frequency \( \omega_1 \). In the Gross-Pitaevskii theory the Gibbs thermodynamic potential is

\[
G = -\mu|\psi|^2 + \frac{\hbar^2}{2m} \left( -i \nabla - \frac{2\pi}{\kappa} \vec{v}_0 \right)^2 |\psi|^2 + \frac{g}{2} |\psi|^4 + \frac{m(\omega_1^2 - \Omega^2)\nu^2}{2} |\psi|^2. \tag{1}
\]

Here \( \psi \) is the BEC wave function, \( \mu \) is the chemical potential, \( g \) is the interaction constant, and \( \vec{v}_0 = [\Omega \times \vec{r}] \) is the velocity of solid body rotation. The Gibbs potential is invariant with respect to the gauge transformation.
\[ \psi \rightarrow \psi e^{i\theta}, \quad \vec{v}_0 \rightarrow \vec{v}_0 + (\kappa/2\pi) \vec{\nabla} \phi. \]

At rapid rotation the potential \( m\Omega^2 r^2/2 \) of centrifugal forces nearly compensates the trapping potential \( m\omega_z^2 r^2/2 \). Though stability requires that \( \omega_\perp > \Omega \), at the first stage of the analysis one can assume that \( \omega_\perp = \Omega \) and BEC is infinite in size. Then the Gibbs potential Eq. [11] is invariant with respect to translation accompanied by the gauge transformation, which corresponds to the shift of the rotation axis.

As well as for type-II superconductors close to \( H_c2 \), in zero-order approximation one can neglect interaction in the LLL regime \( \xi \gg b \). Then the linearized Schrödinger equation is similar to that for a charged particle in a uniform magnetic field:

\[ \mu \psi = -\frac{\hbar^2}{2m} \left[ \left( \frac{\partial}{\partial x} - i \frac{2\pi v_{0x}}{\kappa} \right)^2 + \left( \frac{\partial}{\partial y} - i \frac{2\pi v_{0y}}{\kappa} \right)^2 \right] \psi. \]

At \( \mu = \hbar \Omega \) it has a solution, which corresponds to the lowest Landau level:

\[ \psi_k \propto \exp \left[ i k x - \frac{(y - y_k)^2}{2 l^2} \right], \]

where \( l^2 = \kappa/4\pi \Omega \) and \( y_k = -l^2 k \). The solution is given for the gauge with \( \vec{v}_0(-2\Omega y, 0) \). The frequency \( 2\Omega \) is the analog of the cyclotron frequency \( \omega_c = eH/mc \) for an electron in a magnetic field. If we consider a square \( L \times L \) with periodic boundary conditions, then \( k = -2\pi n/L \) with the integer \( n \). Using the condition \( 0 < y_k < L \), one can see that the integer \( n \) should vary from zero to the integer closest to \( L^2/2\pi l^2 \). This is the total number of LLL states, and the density of the LLL states is \( 1/2\pi l^2 \). All these states are orthogonal each other and have the same energy. But degeneracy is lifted by taking into account the interaction energy. The solution, which corresponds to the periodic vortex lattice with one quantum per lattice unit cell, is [12]

\[ \psi = \sum_n C_n \exp \left[ i n k x - \frac{(y + \frac{1}{2} n k)^2}{2 l^2} \right], \]

where \( C_{n+1} = C_n \exp(2\pi ib \cos \alpha/a) \), \( a, b \), and the angle \( \alpha \) are the parameters of the unit lattice cell (see Fig. 1).

This solution yields the thermodynamic potential of the infinite BEC in the LLL regime:

\[ G = (-\mu + \hbar \Omega)n + \frac{\beta}{2} n^2, \]

where \( n = \langle |\psi|^2 \rangle \) is the average particle density and the parameter [12]

\[ \beta = \frac{\langle |\psi|^4 \rangle}{\langle |\psi|^2 \rangle^2} = \sqrt{\sigma} \left\{ |\theta_2(0, e^{2\pi i\zeta})|^2 + |\theta_2(0, e^{2\pi i\zeta})|^2 \right\} \]

characterizes dependence on lattice parameters \( a, b \), and \( \alpha \) via the complex parameter

\[ \zeta = b/a = \rho + i \sigma. \]

Here

\[ \theta_2(z, q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \cos(2n + 1)z \]

are theta functions [14]. The minimum of the interaction energy correspond to the triangular vortex lattice with \( \beta = 1.1596, a = b = 2l\sqrt{\pi/3}, \alpha = \pi/3 \). According to Eq. [11] the Gibbs potential has a minimum at the particle density \( n = (\mu - \hbar \Omega)/\beta g \). The vortex density \( n_v = 1/2\pi l^2 \) is equal to the density of LLL states.

Let us now take into account the parabolic trapping potential. In equilibrium the usual Thomas-Fermi condition takes place:

\[ \mu(r) + m \left( \frac{\omega_z^2 - \Omega^2}{2} \right)^2 = \mu(0), \]

where \( \mu(0) \) is the chemical potential in the center of the trap, \( r = 0 \). This gives the inverted parabola for the density distribution in the trap:

\[ n(r) = n(0) - \frac{m(\omega_z^2 - \Omega^2)^2}{2 \beta g}. \]

The condition \( n(R) = 0 \) yields the cloud radius \( R = \zeta_c(0)/\sqrt{2/(\omega_z^2 - \Omega^2)} \), where \( \zeta_c(0) = \sqrt{\beta g n(0)} \) is the sound velocity in the center \( r = 0 \) of the rotating BEC.

The authors of Refs. [3, 5] also received the inverted-parabola density profile. In their approach it looks as the effect of small distortions of the vortex lattice, which essentially transforms the Gaussian density profile predicted by Ho [2] for an ideally regular vortex lattice in a trap. But sensitivity of the Gaussian profile to small distortions is nothing more than evidence that the Gaussian profile is unstable. Small distortions are inevitable.
in a finite vortex cluster (see Sec. V.B in Ref. [16]), but it does not mean that they are crucial for bulk properties of a macroscopic vortex crystal. Our approach yields the inverted-parabola density profile without paying attention to small distortions, because the Gaussian profile does not appear at any stage of the analysis.

Let us consider restrictions on existence of the LLL regime. First, the energy of the lowest Landau level, $\hbar \Omega$, should exceed the interaction energy per particle, $\beta g n$ (but $\beta$ is close to unity and is not essential for an order-of-magnitude estimation). This yields the inequality $n \ll \hbar \Omega/g$, which is equivalent to the inequalities $\xi \gg b$ or $c_T \gg c_s$ where $c_s = \sqrt{\hbar g}$ and $c_T = \sqrt{\hbar \Omega/8\pi}$ are the sound and the Tkachenko mode velocity in the VLL regime. Second, the BEC with a regular vortex lattice exists as far as the filling factor $n/n_v$ (the number of particles per vortex) exceeds unity (see below), i.e., the inequality $n \gg \Omega/\kappa$ is required. The two inequalities are compatible for a weakly interacting bose gas with $g \ll \hbar^2/m$. Since $g \sim \hbar^2 a_s/m l_s$, it is necessary that the oscillator length $l_z$ for the trapping potential localizing the BEC cloud along the rotation axis exceeds the scattering length $a_s$. One can rewrite these inequalities in terms of the total number of particles $N = \pi n v R^2$. Since $R^2 = 2\beta g n/m (\omega_1^2 - \Omega^2)$, the order-of-magnitude estimation for density is $n \sim \sqrt{m n (\omega_1^2 - \Omega^2)/g}$. Then the LLL regime takes place if

$$\frac{\Omega^2}{\omega_1^2 - \Omega^2} \frac{g}{\hbar^2} \ll N \ll \frac{\Omega^2}{\omega_1^2 - \Omega^2} \frac{\hbar^2}{g m}.$$  \hspace{1cm} (11)

One more condition is the presence of many vortices in the cloud: $\pi n v R^2 \gg 1$. This yields the inequality

$$N \gg \frac{\omega_1^2 - \Omega^2}{\Omega^2} \frac{\hbar^2}{g m}.$$  \hspace{1cm} (12)

This is compatible with the previous inequalities only for rapid rotation when $\omega_1^2 - \Omega^2 \ll \Omega^2$.

Deformation of the triangular lattice should increase its energy. On the basis of the elasticity theory for a 2D crystal with hexagonal symmetry \[13\], the elastic energy should be

$$E_{el} = C_1 (u_{xx} + u_{yy})^2 + C_2 ((u_{xx} - u_{yy})^2 + 4u_{xy}^2)$$  \hspace{1cm} (13)

where $C_1$ is the compressional modulus, $C_2 = mn c_s^2/2$ is the shear modulus of the vortex lattice, $c_T$ is the Tkachenko wave velocity in the LLL regime, and $u\delta = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i)$ are components of the deformation tensor. Since the parameter $\beta$ does not depend on the vortex density, the compressional modulus vanishes: $C_1 = 0$ \[9\].

In order to find the shear modulus we deform the triangular lattice as shown in Fig. \[1\]. The variation of the complex parameter $\zeta$ is proportional to the shear deformation: $\delta \zeta = \delta \rho = 2 u_{xy} \sin \alpha$. Expanding the expression Eq. \[9\] for $\beta$ and comparing the term $\propto \delta \rho^2$ to the thermodynamic potential, Eq. \[5\], with the elastic energy Eq. \[13\], one obtains the value of the shear modulus:

$$C_2 = \frac{g n^2 \partial^2 \beta}{4 \partial \rho^2} \sin \alpha = 0.1191 g n^2 = 0.1027 m n c_s^2.$$  \hspace{1cm} (14)

This exactly agrees with the numerous calculations of the shear modulus $c_{gs} = 2C_2$ for the flux lattice in type II superconductors close to the critical field $H_{c2}$ \[13\]. But in this work it was calculated anew since Eq. \[14\] yields the result 2 times larger than the value by Sinova et al \[3\] and 10 times larger than the value $(81/80\pi^4) m n c_s^2$ received by Baym \[4\]. This numerical difference is important for interpretation of the experiment (see below).

One can proceed with the analysis of the Tkachenko-modes spectrum on the LLL regime using the same hydrodynamic equations as in the VLL regime \[16\], \[17\], \[13\], but with $c_s$ and $c_T$ replaced by $c_\gamma$ and $c_\gamma$. However, since the LLL regime is possible only for very rapid rotation when $\omega_\perp - \Omega \ll \omega_\perp$ the effect of high BEC compressibility is always strong (see the comment \[16\] in Ref. \[3\]) transforming the Tkachenko mode spectrum to quadratic: $\omega = c_\gamma c_T k^2/2\Omega^2$. In this limit one can use the analytic expression for the Tkachenko eigenmodes of a finite BEC cloud in a parabolic trap derived for the VLL regime in Ref. \[16\]: $\omega_\perp = \gamma_\perp$. Here $\gamma_\perp$ is the value of the reduced frequency $\tilde{\omega} = \sqrt{3/8\pi \omega_R^2/c_T}$ and $s = 2\sqrt{2\Omega/\omega_\perp^2 - \Omega^2}$. The numbers $\gamma_1$ depend on the number $i$ of the eigenmode. The two lowest eigenmodes correspond to $\gamma_1 = 7.17$ and $\gamma_2 = 16.9$. Using the expression $R = \sqrt{2c_s(0)/\sqrt{\omega_\perp^2 - \Omega^2}}$ for the cloud radius one obtains $\omega_i = \sqrt{\pi/3\gamma_1(c_T/c_s)(\omega_\perp^2 - \Omega^2)/\gamma_1}$.

Applying this expression to the LLL regime one should replace $c_s$ and $c_T$ with $\bar{c}_s = \sqrt{3}c_s$ and $\bar{c}_T$ given by Eq. \[13\]. This yields the Tkachenko eigenfrequencies $\omega_i \approx 0.8633\gamma_i(\omega_\perp - \Omega)$. The experiments on rapid rotation \[2\] roughly agree with expected linear dependence of the first Tkachenko mode on $\omega_\perp - \Omega$, but the slope of the experimental line is about 4 times less than our prediction for the LLL regime. It could be an evidence that the experiment has not yet reached the LLL limit. Since experimental values of $(\omega_\perp - \Omega)/\omega_\perp$ look small enough, apparently in order to approach to the LLL limit further, the experiment should be done with a smaller number of atoms.

What should happen with the LLL regime when the filling factor $n/n_v$ approaches unity? Possible answers to this question are now intensively studied by theorists numerically and analytically \[3\], \[4\], \[13\]. The following simple discussion certainly cannot be a substitute of these investigations, but hopefully would be useful for better qualitative understanding of possible scenarios. At low $n/n_v$ boson condensation is destroyed and a single macroscopic wave function cannot describe all atoms anymore \[3\]. For an illustration let us construct a state of the Bose system, which successfully competes with the BEC state.
at low filling factors. The single-particle state

\[ \psi_i = \frac{1}{\sqrt{2\pi l}} \exp \left[ -\frac{(x - X_i)^2 + (y - Y_i)^2}{4l^2} \right] \]  

(15)
corresponds to the classical Larmor orbit of a charged particle centered in the point with coordinates \((X_i, Y_i)\). It is a solution of Eq. (2) for the gauge \(\vec{v}_0(-\Omega y, \Omega x)\) and belongs to the space of the LLL states. Let us assume that any LLL state given by Eq. (15) cannot have more than one atom (the number \(N\) of them is much less than the number of LLL states). Then the wave function of \(N\) bosons can be written as

\[ \Psi = \frac{1}{N!} \sum_P \prod_{i=1}^{N} \psi_i(x_i, y_i) . \]  

(16)

Here \(i\) is the number of the particle (subscript of \(x\) and \(y\)) and the number of the occupied site \((X_i, Y_i)\) (subscript of \(\psi\)). In contrast to BEC, all atoms are in different states, and proper symmetrization (summation over all permutations \(P\)) should be done. Strictly speaking the states \(\psi_i\) given by Eq. (15) are not orthogonal. Therefore the wave function \(\Psi\) requires some normalization factor. But for small filling factors overlapping of the states is weak and an additional normalization factor is exponentially close to unity. The next step is to calculate the interaction energy for this state, Only interaction between closest neighbors is essential. The cross interaction term between them is

\[ \frac{g}{2} \int \, dx \, dy \, |\psi_i|^2 |\psi_{i+1}|^2 = \frac{g}{8\pi l^2} e^{-4r_0^2/l^2} . \]  

(17)

Here \(r_0\) is the distance between two sites. Assuming the triangular-lattice ordering of occupied sites, \(r_0 = \sqrt{2/\sqrt{3}n}\). Collecting all terms of this value we receive for the total interaction energy integrated over the whole space:

\[ E_{\text{cr}} = \frac{6N!}{N!} \frac{g}{8\pi l^2} e^{-4r_0^2/l^2} = \frac{3}{2} \frac{gNn_v}{2} n e^{-\pi n_v/\sqrt{3}n} . \]  

(18)

Though we used the properly symmetrized wave function of \(N\) bosons, in fact statistics is not important and the same energy can be derived for an unsymmetrized function. Comparing the energy Eq. (18) with the total energy of the BEC state, \(E_B = \frac{g}{2} \beta Nn\), one sees that at small filling factors \(n/n_v \ll 1\) the BEC state has a larger energy. The transition between two states is expected at \(n/n_v \sim 1\), but the present calculation becomes inaccurate there. This estimation certainly cannot pretend to be an evidence that some crystal structure appears at small \(n/n_v\), but it demonstrates that in this limit the BEC is not the equilibrium state.

Another option at \(n/n_v \sim 1\) is vortex melting without destruction of BE condensation \([3, 4, 19]\). The plane Tkachenko mode \(u(k)e^{ikx-i\omega t}\), where \(u(k)\) is the amplitude of vortex displacements, has the energy \(\sim L^2u(k)^2mn\sigma^2k^2\) in the square \(L \times L\). Considering the quantum melting at zero temperature this energy is equal to \(\hbar \omega(k)/2\). The average vortex displacement squared is obtained by integration of \(u(k)^2\) over the Brillouin zone of the vortex lattice:

\[ \langle u^2 \rangle \sim L^2 \int_{0}^{1/b} dk \, u(k)^2 \sim \int_{0}^{1/b} dk \, \frac{\hbar \omega(k)}{mn\sigma^2k^2} \sim \frac{1}{n} \]  

(19)

According to the Lindemann criterion, quantum melting is expected, when \(\sqrt{(u^2)}\) approaches to the intervortex distance \(b \sim 1/\sqrt{n_v}\). This corresponds to the filling factor \(n/n_v\) of order unity. So vortex-lattice melting can precede destruction of the BEC.

In summary, the ground state of a rapidly rotating Bose-Einstein condensate is analyzed in the LLL regime using the exact wave function known for type II superconductors close to the upper critical magnetic field \(H_c2\). The analysis yields the inverted-parabola density distribution in a parabolic trap and the exact value of the shear elastic modulus of the vortex lattice in the LLL regime, which exceeds the values received in the previous calculations for BEC. This has an impact on interpretation of recent experiments on very rapid rotating BEC.

I thank Alexander Fetter for interesting stimulating discussions of the LLL regime. Ernest-Helmut Brandt helped a lot to my search of literature on vortex elasticity in type II superconductors.

[1] I. Coddington, P. Engels, V. Schweikhard, and E.A. Cornell, Phys. Rev. Lett. 91, 100402 (2003).
[2] V. Schweikhard, I. Coddington, P. Engels, V. P. Mogen-dorff, and E.A. Cornell, Rev. Lett. 92, 040404 (2004).
[3] J. Sinova, C.B. Hanna, and A.H. MacDonald, Phys. Rev. Lett. 89, 030403 (2002).
[4] G. Baym, Phys. Rev. A 69, 043618 (2004).
[5] T.-L. Ho, Phys. Rev. Lett. 87, 060403 (2001).
[6] G. Watanabe, G. Baym, and C.J. Pethick, Phys. Rev. Lett. 93, 190401 (2004).
[7] N.R. Cooper, S. Komineas, and N. Read, Phys. Rev. A 70, 033604 (2004).
[8] A. Aftalion, X. Blanc, and J. Dalibard, cond-mat/0410665.
[9] S.A. Gifford and G. Baym, Phys. Rev. A 70, 033602 (2004).
[10] A.A. Abrikosov, Zh. Eksp. Teor. Fiz. 32, 1442 (1957) [Sov. Phys.-JETP 5, 1174 (1957)].
[11] W.H. Kleiner, L.M., Roth, and S.H. Austin, Phys. Rev. 133, A1226 (1964).
[12] D. Saint-James and G. Schweikhard, Type II superconductiv- ity, (Pergamon Press, Oxford, 1969).
[13] R. Labusch, Phys. Stat. Sol. 32, 439 (1969); E.H. Brandt, Phys. Stat. Sol. 36, 381 (1969). See E.H. Brandt, Phys. Rev. B 71, 04521 (2005), for more references. In order to receive Eq. (18) from these papers one should use the relation \(gn^2 = (H_{c2} - H)^2/8\pi\kappa^2\), which follows from the Ginzburg-Landau theory in the limit \(\kappa = \lambda/\xi \to \infty\).
[14] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).

[15] L.D. Landau and E.M. Lifshitz, *Theory of elasticity* (Pergamon Press, Oxford, 1986).

[16] E.B. Sonin, Rev. Mod. Phys. 59, 87 (1987).

[17] G. Baym, Phys. Rev. Lett. 91, 110402 (2003).

[18] E.B. Sonin, Phys. Rev. A 71, 011603(R) (2005).

[19] N.R. Cooper, N.K. Wilkin, and J.M.F. Gunn, Phys. Rev. Lett. 87, 120405 (2001).