Timelike Hopf Duality and Type IIA* String Solutions

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ABSTRACT

The usual T-duality that relates the type IIA and IIB theories compactified on circles of inversely-related radii does not operate if the dimensional reduction is performed on the time direction rather than a spatial one. This observation led to the recent proposal that there might exist two further ten-dimensional theories, namely type IIA* and type IIB*, related to type IIB and type IIA respectively by a timelike dimensional reduction. In this paper we explore such dimensional reductions in cases where time is the coordinate of a non-trivial $U(1)$ fibre bundle. We focus in particular on situations where there is an odd-dimensional anti-de Sitter spacetime $\text{AdS}_{2n+1}$, which can be described as a $U(1)$ bundle over $\tilde{\mathbb{C}}P^n$, a non-compact version of $\mathbb{C}P^n$ corresponding to the coset manifold $SU(n,1)/U(n)$. In particular, we study the $\text{AdS}_5 \times S^5$ and $\text{AdS}_7 \times S^4$ solutions of type IIB supergravity and eleven-dimensional supergravity. Applying a timelike Hopf T-duality transformation to the former provides a new solution of the type IIA* theory, of the form $\tilde{\mathbb{C}}P^2 \times S^1 \times S^5$. We show how the Hopf-reduced solutions provide further examples of “supersymmetry without supersymmetry.” We also present a detailed discussion of the geometrical structure of the Hopf-fibred metric on $\text{AdS}_{2n+1}$, and its relation to the horospherical metric that arises in the AdS/CFT correspondence.
1 Introduction

The duality symmetries of string theory and M-theory have been studied from a variety of viewpoints, in order to elucidate the interconnections between theories once thought to be distinct. The duality relations can be either at the perturbative level, as in the case of the T-duality that relates the type IIA and IIB string theories compactified on circles of inversely-related radius [1, 2], or non-perturbative, as in the strong-coupling limit of the type IIA string leading to the “opening out” of an eleventh dimension [3, 4].

In the case of a non-perturbative duality, the relation is necessarily non-perturbative, and strictly speaking can be regarded only as conjectural at the present time. Consequently, it is of interest to accumulate a body of evidence that supports the duality conjecture. One of the most useful tools in this regard has been the study of BPS-saturated solutions to the low-energy effective actions, which may be interpreted as solitonic states in the non-perturbative spectra of the theories (see, for example, [5]).

One of the commonly-employed techniques for uncovering the web of interrelations between the various theories is to study the spectrum of solutions corresponding to branes “wrapped around” non-trivial cycles in an internal compactifying manifold. The simplest such manifolds are tori. In this procedure, otherwise known as diagonal dimensional reduction, a $p$-brane in $D$ spacetime dimensions wrapped around an $n$-dimensional torus reduces to a $(p - n)$-brane in $D - n$ dimensions. Another possible reduction procedure involves first preparing a periodic array of $p$-branes in $D$ dimensions, so that from a large-distance standpoint the $p$-brane is effectively “smeared” uniformly over an $n$-dimensional submanifold of its $(D - p - 1)$-dimensional transverse space. These dimensions effectively correspond to Killing directions in the transverse space, and can then be wrapped around the $n$-torus. This procedure, in which the $p$-brane retains its spatial dimension $p$, while the spacetime dimension is again reduced from $D$ to $D - n$, is known as vertical dimensional reduction. One can also, of course, consider situations where the reduction on the $n$-torus is a mixture of diagonal and vertical dimensional reduction steps.

Another kind of reduction on a circle is also possible, in certain cases. The space transverse to a $p$-brane can be described in terms of hyperspherical polar coordinates, with a radial coordinate $r$ and angular coordinates on the foliating spheres. If the transverse space is even-dimensional, these spheres will be odd-dimensional, and then we can exploit the fact that the sphere $S^{2n+1}$ can be described as a $U(1)$ bundle over $CP^n$. The coordinate $\psi$ on the $U(1)$ fibre is a Killing direction, and the Killing vector $\partial/\partial\psi$ is of constant length in the sphere metric. It thus provides a suitable coordinate for an $S^1$ dimensional reduction.
This reduction mechanism, known as Hopf reduction \[6, 7, 8\], has been studied in a variety of contexts. It is akin to a vertical dimensional reduction, in that it is the dimension of the transverse space that is reduced. However, it is quite different from the usual vertical reduction in that $\partial/\partial \psi$ is a Killing direction even for a single isotropic $p$-brane.

In certain cases, such as the D3-brane of the type IIB theory, an important phenomenon is that as the horizon of the brane is approached, the spacetime geometry more and more nearly approaches the product of an anti-de Sitter (AdS) spacetime and a sphere. Specifically, the sphere corresponds to the foliating spheres of the transverse space, while the AdS spacetime is described in terms of foliating surfaces that are formed by the $p$-brane world-volume with the original radial coordinate $r$ now parameterising this sequence of spacetime foliations. In the case of the D3-brane, the near-horizon geometry is therefore AdS$_5 \times S^5$.

The description of AdS as a foliation of flat Minkowski spacetime surfaces is known as the horospherical construction of AdS. The structure of this near-horizon geometry plays a central rôle in the conjectured AdS/CFT correspondence, where the degrees of freedom in the bulk theory are mapped to those of a conformal field theory on the flat boundary of AdS $3, 10, 11$.

In \[7\], the Hopf reduction on the $U(1)$ fibres of the $S^5$ in this near-horizon geometry was investigated. One obtains a solution of the nine-dimensional theory with an AdS$_5 \times CP^2$ geometry. Redefining fields according to the map that relates the type IIB and type IIA variables in $D = 9$, one finds, upon lifting the solution back to $D = 10$, a solution of the type IIA theory with the geometry AdS$_5 \times S^1 \times CP^2$. In other words, the interchange of Kaluza-Klein and winding modes under T-duality has resulted in an “untwisting” of the $U(1)$ fibres in $S^5$. One of the intriguing features of this process is that one arrives at a solution of the type IIA theory that ostensibly has no supersymmetry, and indeed, no fermions at all. The reason for this is related to the fact that $CP^2$ does not admit an ordinary spin structure. Rather, it admits a generalised spin structure \[12\], meaning that all fermions must necessarily carry certain specific non-zero values of electric charge coupling to the $U(1)$ gauge field whose connection is responsible for the twisting of the $U(1)$ bundle. In other words, all the fermions in the nine-dimensional theory must be charged with respect to the Kaluza-Klein vector of the type IIB reduced theory. After the T-duality transformation, this becomes the winding vector, and so in consequence, from the ten-dimensional type IIA point of view the fermions would all have to carry non-zero winding-mode charge. They would all be associated with non-trivial winding states, and so would be seen only at the level of the full string theory, rather than in the low-energy
effective supergravity. Thus the AdS$ _5 \times S^1 \times CP^2$ solution of type IIA supergravity is actually a perfectly respectable BPS solution of the full string theory, which happens to look rather barren and unsatisfactory from the field-theory standpoint.

As is well known, the anti-de Sitter spacetimes AdS$ _n$ are closely analogous to the spheres $S^n$, with the only difference that they are described by constant-radius surfaces in a flat embedding space $\mathbb{R}^{n+1}$ with metric signature $(2, n-1)$ rather than $(0, n+1)$. By the same token, therefore, the anti-de Sitter spacetimes AdS$ _{2n+1}$ of odd dimension can be described in terms of $U(1)$ fibrations over certain spaces $\tilde{CP}^n$, which are non-compact versions of the usual complex projective spaces $CP^n$. The fibre coordinate is actually the time coordinate of AdS$ _{2n+1}$, so the non-compact $\tilde{CP}^n$ is still a space of Euclidean metric signature. It should be emphasised that this time coordinate on the fibre is periodic, and so the bundle is $U(1)$ even in this construction of the non-compact AdS spacetime.

Using this observation, we can now follow a similar strategy to the one used in [7], but now we can untwist the fibres of the AdS$ _5$ in the AdS$ _5 \times S^5$ solution of the type IIB theory. This will give a $\tilde{CP}^2 \times S^5$ solution of the Euclidean-signatured nine-dimensional theory obtained by time reduction of the type IIB theory. It is important, in this regard, that the time coordinate on the Hopf fibres is naturally periodic, as is appropriate for a Kaluza-Klein reduction accompanied by a truncation to the “massless” sector. (See, for example, [13, 14] for discussions of timelike reductions in supergravity.) As was shown in [14], the timelike reductions of the type IIA and type IIB theories in $D = 9$ are not related by (real) field redefinitions, and thus we cannot perform such a redefinition in $D = 9$ to obtain a solution of the usual type IIA theory in $D = 10$. However, as was shown in [15, 16], one can instead postulate the existence of a different theory in $D = 10$, called type IIA* in [15], which is related by timelike T-duality to the type IIB theory. The IIA* theory has a normal $(1,9)$ metric signature, but its Ramond-Ramond fields have kinetic terms of the “wrong” sign. Similarly, a type IIB* theory was also postulated in [15], related by timelike T-duality to the usual type IIA theory.

The upshot of the above discussion is that if we lift the nine-dimensional $\tilde{CP}^2 \times S^5$ solution of the nine-dimensional Euclidean-signatured theory back to $D = 10$ after performing the redefinition to IIA variables, we will obtain an $S^1 \times \tilde{CP}^2 \times S^5$ solution not of the type IIA theory, but rather of the type IIA* theory postulated in [15]. The time coordinate

\footnote{It is clear that the fibre must be timelike in this construction, since the metric of any complex space such as $\tilde{CP}^n$ must necessarily have an \textit{even} number of timelike directions. Thus the single timelike direction of AdS$ _{2n+1}$ resides in the fibre, and not in the complex base-manifold.}
of this ten-dimensional solution lies along the $S^1$ direction. Like the previously-discussed $\text{AdS}_5 \times S^1 \times \mathbb{CP}^2$ solution of the usual type IIA theory, it is a solution that ostensibly lacks not only supersymmetry but also fermions, since $\mathbb{CP}^2$ also does not admit an ordinary spin structure. However, again this is an artefact of looking only at the field-theoretic tip of the string-theoretic iceberg, and in the full theory it should correspond to an honest BPS configuration.

In this paper, we explore some of the consequences of making Hopf reductions on the timelike fibre coordinate of odd-dimensional AdS spacetimes. We begin in section 2 by setting up our notation and using it to describe the ten-dimensional type IIA$^*$ theory of [15], and its relation to type IIB. In section 3, we describe the geometry of the non-compact $\mathbb{CP}^n$ spaces, including a very simple coordinatisation that is somewhat analogous to the horospherical coordinatisation of AdS itself. We also show how the odd-dimensional $\text{AdS}_{2n+1}$ metrics are described as $U(1)$ fibrations over $\mathbb{CP}^n$. We then discuss the geometrical structure of the Hopf-fibred description of $\text{AdS}_{2n+1}$, and its relation to the horospherical parameterisation. In section 4 we discuss the supersymmetry of the $\text{AdS}_5 \times S^5$ solution, and its T-duality related $S^1 \times \mathbb{CP}^2 \times S^5$ solution, making use of results for the Killing spinors in $\text{AdS}_5$ that are derived in an appendix. In section 5, we discuss the $\text{AdS}_7 \times S^4$ solution of eleven-dimensional supergravity, and relate it to a $\mathbb{CP}^3 \times S^4$ solution of the Euclidean-signatured ten-dimensional supergravity coming from the timelike reduction of $D = 11$. We discuss the supersymmetry of the ten-dimensional solution, using results for the Killing spinors of $\text{AdS}_7$ that are also obtained in the appendix.

2 Type IIB/IIA$^*$ T-duality

In this section we consider the case of timelike reductions and T-duality. The type IIB theory on a timelike circle of radius $R$ should be T-dual to some string theory on a timelike circle of radius $1/R$, so that the limit $R \to 0$ should give a T-dual string theory in $9 + 1$ non-compact dimensions. This T-duality cannot relate the IIB theory reduced on a timelike circle to the usual type IIA theory reduced on the dual timelike circle, since as was shown in [4], the usual IIB and IIA theories are not equivalent in $D = 9$, after timelike reductions. Instead, one is led to propose a new theory, called type IIA$^*$, in $D = 10$, whose timelike reduction is equivalent, after field redefinitions, to the timelike reduction of the usual type IIB theory [6]. We shall begin by setting up our notation and conventions.

Let us begin with the type IIB theory in $D = 10$. The bosonic equations of motion can
be derived from the Lagrangian

\[
\mathcal{L} = R \star 1 - \frac{1}{2} \ast d \phi \wedge d \phi - \frac{1}{2} e^{2 \phi} \ast d \chi \wedge d \chi - \frac{1}{2} e^{-\phi} \ast G^{NS} \wedge G^{NS} - \frac{1}{2} e^{\phi} \ast G^{RR} \wedge G^{RR} - \frac{1}{4} G^{(5)} \wedge G^{(5)} + \frac{1}{2} B^{(4)} \wedge dB^{NS}^{(2)} \wedge dB^{RR}^{(2)},
\]

where the various field strengths are defined by

\[
\begin{align*}
G^{NS}^{(3)} &= dB^{NS}^{(2)}, & G^{RR}^{(3)} &= dB^{RR}^{(2)} - \chi B^{NS}^{(2)}, \\
G^{(5)} &= dB^{(4)} + \frac{1}{2} B^{NS}^{(2)} dB^{RR}^{(2)} - \frac{1}{2} B^{RR}^{(2)} dB^{NS}^{(2)},
\end{align*}
\]

(2.1)

As described in [17], the self-duality of \(G^{(5)}\) is to be imposed here after varying the Lagrangian (2.1) to obtain the equations of motion. This can be done consistently, since the equation of motion for \(G^{(5)}\) turns out to be \(d \ast G^{(5)} = dB^{NS}^{(2)} dB^{RR}^{(2)}\), and the right-hand side is identical to the expression for the Bianchi identity for \(G^{(5)}\), following from (2.2).

Upon reduction on the time coordinate to \(D = 9\), this gives

\[
\mathcal{L}_9 = R \star 1 - \frac{1}{2} \ast d \phi \wedge d \phi - \frac{1}{2} \ast d \varphi \wedge d \varphi - \frac{1}{2} e^{2 \phi} \ast d \chi \wedge d \chi + \frac{1}{2} e^{-8 \alpha \varphi} \ast G^{(4)} \wedge G^{(4)}
\]

\[
- \frac{1}{2} e^{-\phi + 4 \alpha \varphi} \ast G^{NS}^{(3)} \wedge G^{NS}^{(3)} + \frac{1}{2} e^{-\phi - 12 \alpha \varphi} \ast G^{NS}^{(2)} \wedge G^{NS}^{(2)}
\]

\[
- \frac{1}{2} e^{\phi + 4 \alpha \varphi} \ast G^{RR}^{(3)} \wedge G^{RR}^{(3)} + \frac{1}{2} e^{\phi - 12 \alpha \varphi} \ast G^{RR}^{(2)} \wedge G^{RR}^{(2)} + \frac{1}{2} e^{16 \alpha \varphi} \ast \mathcal{F}^{(2)} \wedge \mathcal{F}^{(2)}
\]

(2.3)

\[
+ \frac{1}{2} B^{(3)} dB^{NS}^{(2)} dB^{RR}^{(2)} - \frac{1}{2} B^{(4)} dB^{NS}^{(1)} dB^{RR}^{(2)} + \frac{1}{2} B^{(4)} dB^{NS}^{(2)} dB^{RR}^{(1)}
\]

\[
- \frac{1}{2} \chi B^{(3)} dB^{NS}^{(2)} B^{NS}^{(2)},
\]

where

\[
\begin{align*}
G^{(1)} &= d \chi, & G^{RR}^{(1)} &= dB^{RR}^{(1)} - \chi dB^{NS}^{(1)}, \\
G^{NS}^{(2)} &= dB^{NS}^{(1)}, & G^{NS}^{(3)} &= dB^{NS}^{(2)} - B^{(1)} dB^{NS}^{(1)}, \\
G^{RR}^{(3)} &= dB^{RR}^{(2)} - \chi dB^{RR}^{(2)} - B^{(1)} dB^{RR}^{(1)} + \chi B^{(1)} dB^{NS}^{(1)}, \\
G^{(4)} &= dB^{(3)} - \frac{1}{2} B^{NS}^{(1)} dB^{RR}^{(2)} + \frac{1}{2} B^{NS}^{(2)} dB^{RR}^{(1)} + \frac{1}{2} B^{RR}^{(1)} dB^{NS}^{(2)} - \frac{1}{2} B^{RR}^{(2)} dB^{NS}^{(1)}, \\
G^{(5)} &= dB^{(4)} + \frac{1}{2} B^{NS}^{(2)} dB^{RR}^{(2)} - \frac{1}{2} B^{RR}^{(2)} dB^{NS}^{(2)} - B^{(3)} dB^{(3)} - \frac{1}{2} B^{NS}^{(1)} B^{(1)} dB^{NS}^{(2)}
\]

\[
- \frac{1}{2} B^{NS}^{(2)} B^{(1)} dB^{NS}^{(1)} + \frac{1}{2} B^{RR}^{(1)} B^{(1)} dB^{NS}^{(2)} + \frac{1}{2} B^{RR}^{(2)} B^{(1)} dB^{NS}^{(1)},
\]

\[
\mathcal{F}^{(2)} = dB^{(1)}.
\]

(2.4)

Note that the kinetic terms for \(G^{(4)}\), \(G^{NS}^{(2)}\), \(G^{RR}^{(2)}\) and \(\mathcal{F}^{(2)}\) all have signs that are reversed compared with the case of a standard spatial reduction.

In a standard spatial reduction, the above nine-dimensional fields would be equivalent, after appropriate redefinitions, to those coming from the spatial reduction of the type IIA
theory. In Table 1 below, we indicate how the various potentials should be related. However, to indicate the changes resulting from performing the reduction on the time, direction, we underline those fields that arise with the “non-standard” sign for their kinetic terms. Our notation for the reduction of the gauge potentials $A_{(3)}$, $A_{(2)}$ and $A_{(1)}$ of the type IIA* theory to $D = 9$ is that each $A_{(n)}$ reduces to give $A_{(n)}$ and $A_{(n−1)}$.  

| R-R fields | $A_{(3)}$ | $A_{(3)}$ | $A_{(2)}$ | $B_{(3)}$ | $B_{(4)}$ | $B_{(2)}$ |
|------------|----------|----------|----------|---------|---------|---------|
| IIA*       | $A_{(3)}$ | $A_{(3)}$ | $A_{(2)}$ | $A_{(2)}$ | $B_{(3)}$ | $B_{(4)}$ |
| IIB        | $B_{(3)}$ | $B_{(4)}$ | $B_{(2)}$ | $B_{(2)}$ |
| NS-NS fields | $g_{MN}$ | $A_{(1)}$ | $A_{(1)}$ | $B_{NS}^{(1)}$ | $B_{NS}^{(2)}$ |
| D = 10     | $A_{(1)}$ | $A_{(1)}$ | $A_{(1)}$ | $B_{NS}^{(1)}$ | $B_{NS}^{(2)}$ |
| D = 9      | $A_{(1)}$ | $A_{(1)}$ | $A_{(1)}$ | $B_{NS}^{(1)}$ | $B_{NS}^{(2)}$ |
| T-duality  | $A_{(1)}$ | $A_{(1)}$ | $A_{(1)}$ | $B_{NS}^{(1)}$ | $B_{NS}^{(2)}$ |
| D = 10     | $A_{(1)}$ | $A_{(1)}$ | $A_{(1)}$ | $B_{NS}^{(1)}$ | $B_{NS}^{(2)}$ |
| D = 9      | $A_{(1)}$ | $A_{(1)}$ | $A_{(1)}$ | $B_{NS}^{(1)}$ | $B_{NS}^{(2)}$ |
| $g_{MN}$  | $g_{MN}$ | $g_{MN}$ | $g_{MN}$ | $g_{MN}$ | $g_{MN}$ |

Table 1: Gauge potentials of type IIA* and IIB theories in $D = 10$ and $D = 9$

The remarkable observation that allowed Hull to hypothesise the existence of a new type IIA* string theory is that after making the identifications of fields in $D = 9$, as indicated in Table 1, it turns out that the fields of the type IIA formulation in $D = 9$ can be reassembled into fields in $D = 10$, where the fields $A_{(3)}$ and $A_{(1)}$ of the RR sector in $D = 10$ have the non-standard sign for their kinetic terms. This theory, which is called the type IIA* theory, therefore has the low-energy effective Lagrangian

$$\mathcal{L} = R \ast 1 - \frac{1}{2} \ast d\phi \wedge d\phi + \frac{1}{2} e^{\frac{\phi}{2}} \ast F_{(2)} \wedge F_{(2)} - \frac{1}{2} e^{-\phi} \ast F_{(3)} \wedge F_{(3)} + \frac{1}{2} e^{\frac{3\phi}{2}} \ast F_{(4)} \wedge F_{(4)}$$

$$- \frac{1}{2} dA_{(3)} \wedge dA_{(3)} \wedge A_{(2)}, \quad (2.5)$$

It should be emphasised that the type IIA* theory in $D = 10$ has a normal $(1,9)$ spacetime signature.

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It was not a priori obvious that this should have been possible, since it requires that the signs of the kinetic terms of the nine-dimensional fields after transforming to the type IIA language must be exactly correct in order to allow them to reassemble covariantly as ten-dimensional fields.
For completeness, we note that by the same token, one can also propose a type IIB* theory, which again differs from the usual type IIB theory only in having the non-standard signs for the kinetic terms of its RR fields [15, 16]. It follows that the type IIA* and type IIB* theories are themselves related by an ordinary spacelike T-duality.

Our interest will be in starting from a solution of the usual type IIB theory, with the geometry $\text{AdS}_5 \times S^5$, and performing a timelike reduction to $D = 9$. After redefining fields as in Table 1, we can then lift the solution back to $D = 10$ to obtain a solution of the type IIA* theory. In particular, our timelike reduction will be of a slightly unusual kind, in which time is the fibre coordinate on $\text{AdS}_5$ viewed as a Hopf bundle over $\widetilde{CP}^2$, where $\widetilde{CP}^2$ is a certain non-compact form of the complex projective 2-space. Such a reduction is a timelike analogue of the spacelike Hopf reductions that were considered in [7]. Having arrived at a $\widetilde{CP}^2 \times S^5$ solution in $D = 9$, we may lift it back to $D = 10$ to give a $\widetilde{CP}^2 \times S^1 \times S^5$ solution of the type IIA* theory.

Let us begin with the $\text{AdS}_5 \times S^5$ solution of the type IIB theory. This solution involves just the metric tensor and the self-dual 5-form field strength $G_{(5)}$ of the type IIB theory, whose relevant equations of motion can be written simply as

$$R_{MN} = \frac{1}{96} G_{MPQRS} G_{NPQRS},$$

$$G_{(5)} = * G_{(5)}. \quad (2.6)$$

In the absence of the other fields of the theory, we have simply $G_{(5)} = dB_{(4)}$. We may find a solution on $\text{AdS}_5 \times S^5$ of the form

$$ds_{10}^2 = ds_{\text{AdS}_5}^2 + ds_{S^5}^2,$$

$$G_{(5)} = 4m \epsilon_{\text{AdS}_5} + 4m \epsilon_{S^5}, \quad (2.7)$$

where $\epsilon_{\text{AdS}_5}$ and $\epsilon_{S^5}$ are the volume forms for the metrics $ds_{\text{AdS}_5}^2$ and $ds_{S^5}^2$ on $\text{AdS}_5$ and $S^5$ respectively, $m$ is a constant, and the metrics on $\text{AdS}_5$ and $S^5$ satisfy

$$R_{\mu\nu} = -4m^2 g_{\mu\nu}, \quad R_{mn} = 4m^2 g_{mn} \quad (2.8)$$

respectively, where the indices run from 1 to 5. Since the unit $\text{AdS}_5$ has metric $d\Omega^2_{\text{AdS}_5}$ with Ricci tensor $R_{mn} = -4 \bar{g}_{mn}$, it follows that we can write

$$ds_{\text{AdS}_5}^2 = \frac{1}{m^2} d\Omega^2_{\text{AdS}_5}. \quad (2.9)$$

By Hopf fibered $\text{AdS}_5$ over $\widetilde{CP}^2$, we can write the line element as

$$ds_{\text{AdS}_5}^2 = \frac{1}{m^2} d\Sigma_4^2 - \frac{1}{4m^2} (d\tau + \tilde{A})^2, \quad (2.10)$$

where $\Sigma_4$ is the 4-surface of $\text{AdS}_5$, and $\tilde{A}$ is a 1-form on $\widetilde{CP}^2$. This setup allows us to consider the timelike reduction of the $\text{AdS}_5 \times S^5$ solution to $D = 9$, leading to a solution of the type IIA* theory with a particular timelike geometry on $\widetilde{CP}^2 \times S^5$. The details of this reduction and the resulting solution would involve a careful analysis of the equations of motion and boundary conditions, as well as the interplay between the different fields and geometries involved in the reduction process.
where $d\tilde{\Sigma}^2_4$ is the metric on the “unit” $\widetilde{CP}^2$, and $d\tilde{A} = 4J$, where $J$ is the Kähler form on $\widetilde{CP}^2$.

We may now perform a timelike dimensional reduction of this solution to $D = 9$, along the $U(1)$ fiber parameterized by $\tau$. Comparing with the general Kaluza-Klein prescription (with the dilaton set to zero), for which

$$ds^2_{10} = ds^2_9 - (dt + B_{(1)})^2,$$

$$G_{(5)}(x,t) = G_{(5)}(x) + G_{(4)}(x) \wedge (dt + B_{(1)}(x)),$$

(2.11)

where $t = \frac{1}{2m}\tau$ and $B_{(1)} = \frac{1}{2m}\tilde{A}$, we see, from the fact that the volume forms of the unit AdS$_5$ and $\widetilde{CP}^2$ are related by $\Omega_{AdS_5} = \frac{1}{2}(d\tau + \tilde{A}) \wedge \tilde{\Sigma}_4$, that the solution will take the 9-dimensional form

$$ds^2_9 = ds^2_{S^5} + \frac{1}{m^2} d\tilde{\Sigma}^2_4,$$

$$F_{(4)} = \frac{4}{m^3} \tilde{\Sigma}_4, \quad F_{(2)} = \frac{2}{m} J.$$

(2.12)

We note that in the dimensional reduction of the 5-form of the type IIB theory, its self-duality translates into the statement that the fields $G_{(5)}$ and $H_{(4)}$ in $D = 9$ must satisfy $G_{(4)} = *G_{(5)} = F_{(4)}$.

We now perform the T-duality transformation to the fields of the $D = 9$ reduction of the type IIA$^*$ theory. The relation between the IIB and the IIA$^*$ fields is shown in Table 1. Thus in the IIA$^*$ notation, we have the nine-dimensional configuration

$$ds^2_{9} = ds^2_{S^5} + \frac{1}{m^2} d\tilde{\Sigma}^2_4,$$

$$F_{(4)} = \frac{4}{m^3} \tilde{\Sigma}_4, \quad F_{(2)} = \frac{2}{m} J.$$

(2.13)

The crucial point is that the 2-form field strength $F_{(2)}$ of the IIA$^*$ variables is no longer a Kaluza-Klein field coming from the metric; rather, it comes from the dimensional reduction of the 3-form field strength in $D = 10$. Indeed, if we lift the solution (2.13) back to $D = 10$, we have the type IIA$^*$ configuration

$$ds^2_{10} = ds^2_{S^5} + \frac{1}{m^2} d\tilde{\Sigma}^2_4 - dt^2,$$

$$F_{(4)} = \frac{4}{m^3} \tilde{\Sigma}_4, \quad F_{(3)} = \frac{2}{m} J \wedge dt.$$

(2.14)

The solution has the topology $\widetilde{CP}^2 \times S^1 \times S^5$. This should be contrasted with the topology $AdS_5 \times S^5$ for the original $D = 10$ solution in the type IIB framework. Thus the T-duality transformation in $D = 9$ has “unravelled” the twisting of the $U(1)$ fiber bundle over $\widetilde{CP}^2$, leaving us with a direct product $\widetilde{CP}^2 \times S^1$ in the type IIA$^*$ description.
In order to study the geometry of this solution in more detail, we shall now present an explicit construction of the $\widetilde{CP}^2$, and more generally, $\widetilde{CP}^n$, metrics.

3 $\widetilde{CP}^n$ and the Hopf fibration of $\text{AdS}_{2n+1}$

3.1 Fubini-Study metric on $\widetilde{CP}^n$, and the coset $SU(n,1)/U(n)$

We shall begin by constructing the metric on $\text{AdS}_{2n+1}$, written as a $U(1)$ bundle over $\widetilde{CP}^n$. This will closely parallel the standard construction of the Fubini-Study metric on the usual compact $CP^n$.

To begin, we introduce $(n+1)$ complex coordinates $Z^a$ on $C^{n+1}$, where $0 \leq a \leq n$. We also introduce the metric $\eta_{ab}$, defined by

$$\eta_{ab} = \text{diag} (-1, 1, 1, \ldots, 1).$$ (3.1)

The line element on $C^{n+1}$ may be taken to be:

$$ds^2 = \eta_{ab} dZ^a d\bar{Z}^b,$$ (3.2)

and is obviously invariant under $SU(n,1)$. We then impose the $SU(n,1)$-invariant constraint

$$\eta_{ab} Z^a \bar{Z}^b = -1,$$ (3.3)

which restricts the metric to the $\text{AdS}_{2n+1}$ manifold.

Splitting the indices $a = (\ell, 0)$, with $1 \leq \ell \leq n$, we now introduce inhomogeneous complex coordinates $\zeta^\ell$, defined by

$$\zeta^\ell = Z^\ell / Z^0, \quad 1 \leq \ell \leq n.$$ (3.4)

The line element (3.2), subject to the constraint (3.3), can then be expressed in terms of the coordinates $\zeta^\ell$ and $Z^0$ as follows:

$$ds^2 = -|dZ^0|^2 + |Z^0|^2 \zeta^\ell d\bar{\zeta}^\ell + |\zeta|^2 |dZ^0|^2 + Z^0 \zeta^\ell d\bar{\zeta}^\ell d\bar{Z}^0 + \bar{Z}^0 \zeta^\ell d\bar{\zeta}^\ell dZ^0,$$ (3.5)

where $|\zeta|^2$ denotes $\zeta^\ell \bar{\zeta}^\ell$. One can rewrite (3.5) by completing the square, as

$$ds^2 = -\left|\frac{dZ^0}{Z^0} - |Z^0|^2 \zeta^\ell d\bar{\zeta}^\ell\right|^2 + |Z^0|^2 d\zeta^\ell d\bar{\zeta}^\ell + |Z^0|^4 \zeta^\ell \zeta^m d\zeta^\ell d\bar{\zeta}^m.$$ (3.6)

This is just like a Kaluza-Klein metric, with the first term corresponding to the extra dimension. We may think of the $U(1)$ coordinate as being $\tau$, where $Z^0 = |Z^0| e^{i \frac{\tau}{2}}$. Thus we
can read off the metric on the base space, orthogonal to \(\partial/\partial \tau\), by dropping the first term in (3.10), giving us the line element

\[
ds^2_{\widetilde{CP}^n} = |Z^0|^2 \, d\zeta^i \, d\bar{\zeta}^i + |Z^0|^4 \, \zeta^j \, d\zeta^i \, d\bar{\zeta}^j.
\] (3.7)

Noting from (3.3) and (3.4) that we have \(|Z^0|^2 = (1 - |\zeta|^2)^{-1}\), we see that the metric on the base space is

\[
ds^2_{\widetilde{CP}^n} = \frac{d\zeta^\ell \, d\bar{\zeta}^\ell}{1 - |\zeta|^2} + \frac{\zeta^m \, d\zeta^\ell \, d\bar{\zeta}^m}{(1 - |\zeta|^2)^2}.
\] (3.8)

This is the metric on \(\widetilde{CP}^n\), and is analogous to the standard Fubini-Study metric on the standard compact complex projective space \(CP^n\).

A convenient parameterisation for \(\widetilde{CP}^n\) can be obtained by introducing the \(2n + 1\) real coordinates \((\tau, \phi, \chi, x_i, y_i)\), and writing the homogeneous coordinates of \(\widetilde{CP}^n\) as

\[
Z^0 = e^{\frac{i}{2} \tau} \left( \cosh \frac{1}{2} \phi + \frac{1}{8} e^{\frac{i}{2} \phi} (4i \chi - 2i x_i y_i + x_i^2 + y_i^2) \right),
\]

\[
Z^n = e^{\frac{i}{2} \tau} \left( \sinh \frac{1}{2} \phi - \frac{1}{8} e^{\frac{i}{2} \phi} (4i \chi - 2i x_i y_i + x_i^2 + y_i^2) \right),
\]

\[
Z^i = \frac{1}{4} e^{\frac{i}{2} (\phi + i \tau)} (x_i + i y_i), \quad 1 \leq i \leq n - 1.
\] (3.9)

It is easily verified that these \(Z^a\) satisfy the constraint (3.3). Substituting into (3.2), we immediately obtain the metric on \(AdS_{2n+1}\), in the form

\[
d\Omega^2_{AdS_{2n+1}} = -\frac{1}{4} (d\tau + e^\phi (d\chi - x_i \, dy_i))^2 + \frac{1}{4} d\phi^2 + \frac{1}{4} e^\phi (dx_i^2 + dy_i^2) + \frac{1}{4} e^{2\phi} (d\chi - x_i \, dy_i)^2.
\] (3.10)

Note that this is an \(AdS_{2n+1}\) metric of “unit radius,” since it is defined by \(Z^a \, \tilde{Z}^b \eta_{ab} = -1\).

As before, by projecting orthogonally to the orbits of \(\partial/\partial \tau\), we obtain the metric on \(\widetilde{CP}^n\), which now takes the simple real form

\[
d\Sigma^2_{2n} = \frac{1}{4} d\phi^2 + \frac{1}{4} e^\phi (dx_i^2 + dy_i^2) + \frac{1}{4} e^{2\phi} (d\chi - x_i \, dy_i)^2.
\] (3.11)

In fact, the parameterisation of the \(\widetilde{CP}^n\) metric in terms of the real coordinates \((\phi, \chi, x_i, y_i)\) coincides precisely with a coset parameterisation in an “upper-triangular” gauge. To see this, let us define the \((n + 1) \times (n + 1)\) matrix \(E_a^b\), which has zeroes everywhere except for a 1 at row \(a\) and column \(b\). Let \(H = E_0^0 - E_n^n\). Then \(H\), together with \(E_0^n\), \(E_0^i\) and \(E_i^n\), where \(1 \leq i \leq n - 1\), form the generators of the solvable Lie algebra of \(SU(n, 1)\). (See, for example, [18, 19, 20, 21, 22] for discussions of solvable Lie algebras.) In other words, the generators \(E_0^i\) and \(E_i^n\) constitute the entire subset of positive-root generators that have non-vanishing weights under the Cartan generator \(H\). (The generator \(H\) is the non-compact Cartan generator; all the others are compact in \(SU(n, 1)\).) By exponentiating
the solvable Lie algebra generators, we obtain a gauge-fixed parameterisation of points in
the coset $SU(n,1)/U(n)$. Let us define the coset representative

$$
V = e^{\frac{1}{2} \phi H} e^{-i \chi E_0^n} \left( \prod_i e^{\frac{1}{\sqrt{2}} \xi_i (E_0^i + E_i^n)} \right) \left( \prod_i e^{-\frac{1}{\sqrt{2}} \eta_i (E_0^i - E_i^n)} \right),
$$

(3.12)

where the terms in the product are ordered such that factors with larger index values $i$ sit
to the right of those with smaller $i$.

From the commutation relations $[E_a^b, E_c^d] = \delta_c^b E_a^d - \delta_a^c E_c^b$, it follows that

$$
dV^V^{-1} = \frac{1}{2} d\phi H + \frac{1}{\sqrt{2}} e^{\frac{1}{2} \phi} (dx_i - i dy_i) E_0^i + \frac{1}{\sqrt{2}} e^{\frac{1}{2} \phi} (dx_i + i dy_i) E_i^n - i e^\phi (d\chi - x_i dy_i) E_0^n.
$$

(3.13)

The coset metric can then be written as

$$
 ds^2 = \frac{1}{8} \text{tr} ((dV^V^{-1})_\perp)^2,
$$

(3.14)

where $(dV^V^{-1})_\perp = dV V^{-1} + (dV V^{-1})^\dagger$. An alternative way of writing the coset metric is

$$
 ds^2 = \frac{1}{8} \text{tr} (\mathcal{M}^{-1} d\mathcal{M})^2,
$$

(3.15)

where $\mathcal{M} = V^\dagger V$. It is straightforward to verify that the metric (3.14) or (3.13) coincides precisely with (3.11). Note that the $SU(n,1)$ symmetry of the metric can be seen from
the Iwasawa decomposition, which implies that there exists a $U(n)$ compensating transformation $O$ such that $V' = O V \Lambda$ is again in the exponentiation of the solvable Lie algebra, where $\Lambda$ is an arbitrary constant $SU(n,1)$ matrix.

### 3.2 The boundary of AdS$_{2n+1}$

In the AdS/CFT correspondence, the conformal field theory is defined on the boundary
of AdS described in terms of horospherical coordinates $[9, 10, 11]$. To be precise, the
horospherical metric on AdS$_{d+1}$ can be written as

$$
 d\Omega^2_{\text{AdS}_{d+1}} = d\rho^2 + \eta_{\mu\nu} e^{2\rho} d\xi^\mu d\xi^\nu,
$$

(3.16)

where $\eta_{\mu\nu}$ is the Minkowski metric in $d$ dimensions. Thus the boundary at fixed $\rho$ is flat
$d$-dimensional Minkowski spacetime. In the AdS/CFT correspondence, one considers the
boundary where $\rho$ tends to infinity.

In order to relate our description of AdS$_{2n+1}$ to this horospherical parameterisation, it is
useful first to recall how the metric (3.16) on AdS$_{d+1}$ is embedded in flat $\mathbb{R}^{2,d}$. Introducing
real coordinates $X$, $Y$ and $W^\mu$ in $\mathbb{R}^{2,d}$, we have

$$
 X + Y = e^\rho, \quad X - Y = e^{-\rho} + \xi^\mu \xi_\mu e^\rho, \quad W^\mu = \xi^\mu e^\rho,
$$

(3.17)
where \(-X^2 + Y^2 + W^\mu W_\mu = -1\). Comparing with the complex coordinates \((Z^0, Z^i, Z^n)\)
given in (3.14), which satisfy the \(\mathbb{R}^{2,2n}\) constraint (3.3) (with \(d = 2n\)), we see that we can
make the identifications

\[
Z^0 = X + i W^0, \quad Z^n = Y + i W^1, \quad Z^i = W^{2i} + i W^{2i+1}, \quad 1 \leq i \leq n - 1.
\] (3.18)

For general values of the AdS\(_{2n+1}\) coordinates, this identification implies that there will
be rather complicated relations between the horospherical description using \((\rho, \xi^\mu)\), and the
Hopf-fibred description using \((\tau, \phi, \chi, x^i, y^i)\). These can be expressed as follows:

\[
\begin{align*}
\tan \frac{1}{2} \tau &= \xi^0 + \xi^1, \quad e^{\frac{1}{2} \phi} = \sec \frac{1}{2} \tau \ e^\rho, \\
x^i &= 2 \cos \frac{1}{2} \tau \left( \cos \frac{1}{2} \tau \xi^{2i} + \sin \frac{1}{2} \tau \xi^{2i+1} \right), \\
y^i &= 2 \cos \frac{1}{2} \tau \left( \cos \frac{1}{2} \tau \xi^{2i+1} - \sin \frac{1}{2} \tau \xi^{2i} \right), \\
\chi &= \cos^2 \frac{1}{2} \tau \left( \xi^0 - \xi^1 \right) + \frac{1}{2} x_i y_i - \frac{1}{2} \sin \tau \left( \xi^\mu \xi_\mu + e^{-2\rho} \right).
\end{align*}
\] (3.19)

In particular, we see that large positive values of \(\rho\) correspond to large positive values of
\(\phi\), although the \(\rho = \text{constant}\) surfaces do not coincide with \(\phi = \text{constant}\) surfaces unless \(\tau\) is
also fixed. Note that the time coordinate \(\tau\) of the Hopf fibred description is related to the
light-cone coordinate \(\xi^+ \equiv \xi^0 + \xi^1\) of the horospherical description.

Let us now consider the boundary metric at fixed \(\phi\) in detail. From (3.10), we see
that when \(\phi\) is large and is held fixed, the AdS\(_{2n+1}\) metric approaches \(\frac{1}{4} e^\phi \, ds^2\), where the
boundary metric \(ds^2\) is given by

\[
ds^2 = -2d\tau \ (d\chi - x_i \, dy_i) + dx_i^2 + dy_i^2.
\] (3.20)

Unlike the \(\rho = \text{constant}\) boundary in the horospherical description, this is not a flat metric.
However, it is not hard to show that its Weyl tensor vanishes, and hence it is conformally
flat. Indeed, if we define the conformally-rescaled boundary metric \(d\tilde{s}^2 = \Omega^2 \, ds^2\), then we
can show that \(d\tilde{s}^2\) will be flat if \(\Omega\) is taken to be

\[
\Omega = \frac{1}{\cos(\frac{1}{2} \tau + \alpha)} ,
\] (3.21)

where \(\alpha\) is any constant. This can in fact be understood from the relation between \(\phi\) and
\(\rho\) given in (3.13).

It is appropriate to make some further remarks about the relation between various
different descriptions of anti-de Sitter spacetime. In particular, one might wonder how it
can be that the time coordinate $\tau$ in the Hopf-fibred description (3.10) of the AdS$_{2n+1}$ metric is periodic, whilst the time coordinate $\xi^0$ in the horospherical metric (3.16) is not. Of course this question is not a new one that occurs only because of our use of a Hopf fibration to describe AdS$_{2n+1}$; the same issue arises if one uses “standard” coordinates of a traditional kind in AdS, as described, for example, in [23]. Generalising to $D$ dimensions, we can write the standard metric on AdS$_D$ as

$$d\Omega^2_{\text{AdS}} = -\cosh^2 r \, dt^2 + dr^2 + \sinh^2 r \, d\Omega^2_{S_{D-2}},$$

(3.22)

where $d\Omega^2_{S_{D-2}}$ denote the metric on a unit $(D-2)$-sphere. This can be embedded in a flat $(D+1)$ dimensional spacetime $\mathbb{R}^{2,D-1}$ with coordinates $(T, S, U^a)$ as follows:

$$T = \cosh r \, \cos t, \quad S = \cosh r \, \sin t, \quad U^a = \sinh r \, u^a,$$

(3.23)

where the $(D-1)$ quantities $u^a$ satisfy $u^a u^a = 1$, so that the metric on the unit $(D-2)$-sphere is given by $d\Omega^2_{S_{D-2}} = du^a du^a$ subject to this constraint. The coordinates of $\mathbb{R}^{2,D-1}$ thus satisfy $-T^2 - S^2 + U^a U^a = -1$. It is manifest from (3.23) that the AdS$_D$ metric (3.22) is given by $d\Omega^2_{\text{AdS}} = -dT^2 - dS^2 + dU^a dU^a$. Again we see that the time coordinate $t$ in the AdS metric is periodic. One can easily express the coordinates $(t, r, u^a)$ of the AdS metric (3.22) in terms of those of the horospherical metric (3.16) by making appropriate identifications of the corresponding embedding coordinates, for example by taking $T = X$ and $S = W^0$, with all the remaining coordinates $U^a$, which are spacelike, equated with the $Y$ and remaining $W^\mu$ ($\mu > 0$) coordinates of the horospherical embedding.

The global structure of the horospherical metric on AdS was discussed in detail in [24]. In particular, it was shown that the horospherical coordinates cover only one half of AdS. This is easily seen by noting from (3.17) that the positivity of $e^\rho$ implies that only the region $X + Y > 0$ of the AdS hyperboloid in $\mathbb{R}^{2,D-1}$ is covered by the horospherical parameterisation. From this observation, the dichotomy between the non-periodic time in the horospherical parameterisation, and the periodic time of the standard or the Hopf-fibred parameterisations, becomes understandable, since the latter two descriptions cover the whole of AdS. In fact the $\rho =$constant boundary of the horospherical description really corresponds to the intersection of a 45-degree null hyperplane with the hyperboloid in $\mathbb{R}^{2,D-1}$, and the time coordinate $\xi^0$ parameterises an infinite arc within the intersection.

Finally in this section, it is worth commenting on the nature of the periodic times in the standard parameterisation and the Hopf-fibred parameterisation of AdS. In (3.22), it is evident that $\partial/\partial t$ is a Killing vector, and hence in principle one could perform a timelike
reduction on this circle. However, the result would be rather inelegant, since the radius of the circle depends on the radial coordinate \( r \) of the AdS metric. By contrast, in the timelike Hopf reduction that we consider in this paper, the timelike Killing vector \( \partial/\partial \tau \) of the AdS metric (3.10) is of constant length, and so the reduction circle is of uniform radius over the entire AdS spacetime. This gives a much better-behaved interpretation to the Kaluza-Klein reduction. It is analogous to the distinction between reduction on a circle of latitude, versus a Hopf circle, in a spacelike reduction on a Killing direction in a sphere \([6, 7, 8]\).

4 Supersymmetry of the \( \text{AdS}_5 \times S^5 \) solution

In section 2 we showed how the \( \text{AdS}_5 \times S^5 \) solution of the type IIB theory could be dimensionally reduced on the timelike Hopf fibres of \( \text{AdS}_5 \), viewed as \( U(1) \) bundle over \( \widetilde{CP}^2 \). In section 3, we showed how the Fubini-Study metric on \( \widetilde{CP}^2 \) could be constructed in a convenient coordinate system, given by (3.11) with just a single pair of coordinates \( x \) and \( y \):

\[
d\Sigma_4^2 = \frac{1}{4} d\phi^2 + \frac{1}{4} e^\phi (dx^2 + dy^2) + \frac{1}{4} e^{2\phi} (d\chi - x \, dy)^2 . \tag{4.1}
\]

The metric on the unit \( \text{AdS}_5 \) is then, from (3.10), given by

\[
d\Omega^2_{\text{AdS}_5} = -\frac{1}{4}(d\tau + e^\phi (d\chi - x \, dy))^2 + \frac{1}{4} d\phi^2 + \frac{1}{4} e^\phi (dx^2 + dy^2) + \frac{1}{4} e^{2\phi} (d\chi - x \, dy)^2 . \tag{4.2}
\]

The relevant equation governing the supersymmetry of the \( \text{AdS}_5 \times S^5 \) solution is the Killing spinor equation, which in terms of the unit \( \text{AdS}_5 \times S^5 \) spaces reduces to the conditions

\[
D_\mu \epsilon = \frac{1}{2} \Gamma_\mu \epsilon , \quad D_m \epsilon = \frac{1}{2} \Gamma_m \epsilon , \tag{4.3}
\]

in \( \text{AdS}_5 \) and \( S^5 \) respectively. Note that in writing these equations, we have made a specific choice of orientation for the solution. With other choices, the right-hand sides of these equations could be multiplied by factors of \((-1)\).

When we make the Kaluza-Klein reduction to \( D = 9 \), a truncation to the massless sector involves taking all the fields to be independent of the reduction coordinate, which in our case will be the time coordinate. Consequently, this truncation will project out any fields, including the Killing spinors, that are time-dependent. The relevant Killing spinors of \( \text{AdS}_5 \) are given in equations (A.17) in the appendix. We see that there are four of them, and that they all have specific non-trivial dependence on the time coordinate \( \tau \). Thus they will all be projected out in the Kaluza-Klein reduction and truncation process. Note that the same thing will happen if the opposite orientation choice for the solution is taken; the Killing
spinors in AdS$_5$ that satisfy the equation $D_\mu \epsilon = -\frac{1}{2} \Gamma_\mu \epsilon$ are given in (A.18), and they too all have non-trivial time dependence.

In fact not merely the Killing spinors, but all spinors are projected out in the Kaluza-Klein reduction and truncation procedure, in this Hopf reduction on AdS$_5$. The reason for this is that the non-compact $\tilde{C}P^2$ manifold, like the usual compact $C$P$^2$, does not admit an ordinary spin structure. It does allow generalised spinors, but these must all carry electric charge and be minimally coupled to the Kaluza-Klein vector potential $A_{(1)}$. Such states, with the necessary electric charges, are in fact precisely non-zero modes in the Kaluza-Klein spectrum. Thus the full Kaluza-Klein spectrum of fermionic states in $D = 9$ will be associated with non-zero modes of the massive sectors of the theory.

The T-duality between the type IIB and type IIA* theories can be seen among the massless modes, at the level of the field theories. Namely, the massless fields from the $S^1$ reduction of type IIB supergravity map over into massless fields of type IIA* supergravity, as we discussed in section 2. The T-duality will also be visible in the full string spectra, where in $D = 9$ we include massive modes arising both from the Kaluza-Klein towers of states, and also from the massive string excitations. The massive Kaluza-Klein states and string states interchange under T-duality. Since, as we have described, the fermions in the $D = 9$ type IIB field theory picture are all contained in the massive Kaluza-Klein sector, it follows that after T-dualising to the type IIA* picture, they will be associated with massive string states. These will not be seen at the level of the field theory in the type IIA* picture, and so from a field-theoretic standpoint the fermions, and in particular the supersymmetry, will appear to be "lost" in the Hopf dualisation procedure. It will, eventually, be regained in the full string picture.

The upshot of this is that after dualising to the type IIA* theory in $D = 9$, we have a solution of the form $\tilde{C}P^2 \times S^5$ that appears to lack not only supersymmetry, but also any fermions at all. Likewise, after lifting back to $D = 10$ we will have a type IIA* solution of the form $\tilde{C}P^2 \times S^1 \times S^5$ without supersymmetry or fermions. Nonetheless this solution, from the bosonic point of view, satisfies all the conditions for being a BPS state, since it is merely a T-duality transformation of a known BPS solution, namely the AdS$_5 \times S^5$ solution of type IIB.
5 Hopf reduction of $\text{AdS}_7 \times S^4$

In [25] it was shown that the $\text{AdS}_4 \times S^7$ solution of $D = 11$ supergravity could be re-interpreted as an $\text{AdS}_4 \times CP^3$ solution of the type IIA theory in $D = 10$, by performing a dimensional reduction on the Hopf fibres of $S^7$. In [3], this solution was discussed from the viewpoint of M-theory and the type IIA string.

Here, we may carry out a somewhat analogous Hopf reduction of the $\text{AdS}_7 \times S^4$ solution of the $D = 11$ supergravity, where now we perform the dimensional reduction on the timelike Hopf fibres of $\text{AdS}_7$ viewed as a $U(1)$ bundle over $\widetilde{CP}^3$.

The $\text{AdS}_7 \times S^4$ solution is obtained by taking

$$
\begin{align*}
\text{ds}_{11}^2 &= \text{ds}^2(\text{AdS}_7) + \text{ds}^2(S^4), \\
F_{(4)} &= 6m \epsilon_{(4)},
\end{align*}
$$

(5.1)

where $m$ is a constant and $\epsilon_{(4)}$ is the volume form on the $S^4$ metric $\text{ds}^2(S^4)$. Substituting into the eleven-dimensional equations of motion, we find that the Ricci tensors for the $\text{AdS}_7$ and $S^4$ metrics are given by

$$
\begin{align*}
R_{\mu\nu} &= -6m^2 g_{\mu\nu}, \\
R_{mn} &= 12m^2 g_{mn}.
\end{align*}
$$

(5.2)

Thus in terms of standard unit metrics $d\Omega_{\text{AdS}_7}^2$ and $d\Omega_{S^4}^2$ on $\text{AdS}_7$ and $S^4$, we therefore have

$$
\text{ds}_{11}^2 = \frac{1}{m^2} d\Omega_{\text{AdS}_7}^2 + \frac{1}{4m^2} d\Omega_{S^4}^2.
$$

(5.3)

From the discussion in section 3, we can therefore write this as

$$
\text{ds}_{11}^2 = -\frac{1}{4m^2} (d\tau + e^\phi (d\chi - x_i dy_i))^2 + \frac{1}{m^2} d\Sigma_6^2 + \frac{1}{4m^2} d\Omega_{S^4}^2,
$$

(5.4)

where $d\Sigma_6^2$ is the metric on $\widetilde{CP}^3$, given in (3.11).

Comparing with the standard timelike Kaluza-Klein reduction ansatz from $D = 11$ to $D = 10$, for which

$$
\begin{align*}
\text{ds}_{10}^2 &= -e^{-\frac{1}{2}\phi} (dt + A_{(1)})^2 + e^{\frac{3}{2}\phi} d\text{ds}_{10}^2, \\
\tilde{F}_{(4)} &= F_{(4)} + F_{(3)} \wedge (dt + A_{(1)}),
\end{align*}
$$

(5.5)

we see that the $\text{AdS}_7 \times S^4$ solution can be interpreted as a solution of the Euclidean-signatured $D = 10$ supergravity with fields given by

$$
\begin{align*}
\text{ds}_{10}^2 &= \frac{1}{m^2} d\Sigma_6^2 + \frac{1}{4m^2} d\Omega_{S^4}^2, \\
F_{(4)} &= \frac{3}{8m^3} \Omega_{(4)}, \\
F_{(3)} &= 0, \\
F_{(2)} &= \frac{2}{m} J, \\
\varphi &= 0.
\end{align*}
$$

(5.6)
Here, $\Omega_4$ is the volume form of the unit 4-sphere. The time coordinate $t$ of the eleven-dimensional metric is related to the coordinate $\tau$ on the Hopf fibres by $t = \tau/(2m)$.

The AdS$_7 \times S^4$ solution of $D = 11$ supergravity has maximal (i.e., $N = 4$) supersymmetry from the viewpoint of seven-dimensional gauged supergravity. However, the Kaluza-Klein Hopf reduction to $D = 10$ involves discarding all the modes that have dependence on the time coordinate $t$. From the results for the Killing spinors on AdS$_7$ given in the appendix, we see that of the eight Killing spinors on AdS$_7$ they either all depend upon $\tau$, or else just 2 out the 8 depend upon $\tau$, depending on the sign choice in the Killing spinor equation. This means that in the Kaluza-Klein reduction, the ten-dimensional $\widetilde{\mathbb{CP}^3} \times S^4$ solution will have either no supersymmetry, or $N = 3$ supersymmetry, from the $D = 7$ viewpoint, depending upon the orientation-choice associated with the Hopf fibration. This is analogous to what was seen in the Hopf reductions of AdS$_4 \times S^7$, which gave either $N = 0$ or $N = 6$ four-dimensional supersymmetry, depending on the orientation [25, 3].

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Appendices

A Killing Spinors in AdS

In order to study the supersymmetry of the solutions obtained by performing Hopf reductions on the time coordinate in AdS, it is useful to construct the Killing spinors in the relevant AdS$_{2n+1}$ backgrounds, viewed as $U(1)$ bundles over $\widetilde{\mathbb{CP}^n}$. Thus, in particular, we are interested in the cases of AdS$_5$ and AdS$_7$ in this paper.

To begin, we obtain the spin connection for the $\widetilde{\mathbb{CP}^n}$ metric given in (3.11). To this end, we define a vielbein basis as follows:

\begin{align}
  e^0 &= \frac{1}{2} d\phi , \\
  e^0' &= \frac{1}{2} e^{\phi} (d\chi - x_i dy^i) , \\
  e^i &= \frac{1}{2} e^{\frac{1}{2}\phi} dy^i , \\
  e^i' &= \frac{1}{2} e^{\frac{1}{2}\phi} dx^i .
\end{align}

(A.1)

(Our notation for the labelling of tangent-space indices is implicitly defined by these expressions.) The spin connection $\omega_{ab}$, defined by $de^a = -\omega^a_b \wedge e^b$ and $\omega_{ab} = -\omega_{ba}$, is then
given by

\[
\begin{align*}
\omega_{00'} &= -2e^{0'}, & \omega_{0i} &= -e^i, & \omega_{0'i'} &= -e^{i'}, \\
\omega_{0'i} &= e^{i'}, & \omega_{0'i'} &= -e^i, & \omega_{i'i'} &= -\delta_{ij} e^{0'}. \\
\end{align*}
\]

(A.2)

Note that the 2-form

\[
J = e^0 \wedge e^{0'} + e^i \wedge e^{i'}
\]

(A.3)

is closed, as can be seen from the fact that locally \( J = \frac{1}{2} de^{0'} \). In fact \( J \) is the Kähler form on \( \widetilde{CP}^n \).

As we saw in section 3, the metric on AdS\(_{2n+1}\) can be written as a \( U(1) \) bundle over \( \widetilde{CP}^n \), as in (3.10). If we introduce the natural vielbein basis for this AdS metric, namely

\[
\hat{e}^\tau = \frac{1}{2} (d\tau + e^\phi (d\chi - x_i dy_i)), \quad \hat{e}^a = e^a,
\]

(A.4)

where \( a = (0, 0', i, i') \) runs over the indices of the \( \widetilde{CP}^n \) vielbein \( e^a \) defined above, the the spin connection for AdS\(_{2n+1}\) is given by

\[
\begin{align*}
\hat{\omega}_{0i} &= -e^i, & \hat{\omega}_{0'i'} &= -e^{i'}, & \hat{\omega}_{00'} &= e^{0'}, & \hat{\omega}_{i'i'} &= e^{i'}, \\
\hat{\omega}_{0'i} &= e^{i'}, & \hat{\omega}_{0'i'} &= -e^i, & \hat{\omega}_{00'} &= -e^0, & \hat{\omega}_{i'i'} &= -e^i, \\
\hat{\omega}_{00'} &= \hat{\epsilon}' - 2e^{0'}, & \hat{\omega}_{i'i'} &= \delta_{ij} (\hat{\epsilon}' - e^{0'}). \\
\end{align*}
\]

(A.5)

Note that the potential term \( e^\phi (d\chi - x_i dy_i) \) in the vielbein \( \hat{e}^\tau \), which is responsible for the topological twist of the \( U(1) \) fibres, is nothing but \( 2e^{0'} \), and \( \frac{1}{2} e^{0'} \) is the potential for the Kähler form \( J \) on \( \widetilde{CP}^n \).

The Killing spinor equation for this AdS metric, which has “unit radius,” is

\[
D_M \epsilon^\pm = \pm \frac{1}{2} \Gamma_M \epsilon^\pm,
\]

(A.6)

where \( \epsilon^\pm \) is the Killing spinor, and \( D_M \) is the covariant derivative on AdS, \( D_M \equiv \partial_M + \frac{1}{4} \omega_M^{AB} \Gamma_{AB} \). Killing spinors exist for each choice of sign in (A.6), although in a specific supergravity solution, having established orientation conventions, only one particular choice will correspond to unbroken supersymmetries.

From the expressions in (A.1), (A.4) and (A.3) for the vielbeins and spin connection of AdS\(_{2n+1}\), we therefore find that the Killing spinor equation can be written as

\[
\begin{align*}
\partial_\tau \epsilon^\pm &= \frac{1}{4} (\pm \Gamma_\tau - \Gamma_{00'} - \Gamma_{ii'}) \epsilon^\pm \\
\partial_0 \epsilon^\pm &= \frac{1}{4} (\pm \Gamma_0 - \Gamma_{0'i'}) \epsilon^\pm \\
\partial_{0'} \epsilon^\pm &= \frac{1}{4} e^\phi (\Gamma_0 \pm 1) (\Gamma_\tau + \Gamma_{0'}) \epsilon^\pm
\end{align*}
\]

(A.7) (A.8) (A.9)
\[ \partial_i \epsilon^\pm = \frac{1}{4} e^{\frac{i}{2} \phi} [(\Gamma_0 \pm 1) (\Gamma_i - e^{\frac{i}{2} \phi} A_4 (\Gamma_{0'} + \Gamma_{\tau})) - \Gamma_{i'} (\Gamma_{0'} + \Gamma_{\tau})] \epsilon^\pm \] (A.10)
\[ \partial_{i'} \epsilon^\pm = \frac{1}{4} e^{\frac{i}{2} \phi} [(\Gamma_0 \pm 1) \Gamma_{i'} + \Gamma_i (\Gamma_{0'} + \Gamma_{\tau})] \epsilon^\pm . \] (A.11)

In order to make explicit computations of the Killing spinors, we need to choose a representation for the \( \Gamma \)-matrices. In terms of the standard Pauli matrices \( \sigma_i \), we may define the \( \Gamma \) matrices in Euclidean dimension \( D = 2n \) as
\[
\begin{align*}
\Gamma_1 &= \sigma_2 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_3 , \\
\Gamma_2 &= \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_3 , \\
\Gamma_3 &= 1 \otimes \sigma_2 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_3 , \\
\Gamma_4 &= 1 \otimes \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_3 , \\
\Gamma_5 &= 1 \otimes 1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_3 \otimes \sigma_3 , \\
& \quad \ldots , \\
\Gamma_{2n-1} &= 1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_2 , \\
\Gamma_{2n} &= 1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_1 .
\end{align*}
\] (A.12)

In an odd dimension \( D = 2n + 1 \), with Lorentzian signature, we use the above construction for the Dirac \( \Gamma \)-matrices of \( 2n \) dimensions, and take \( \Gamma_{2n+1} \) (in the timelike direction) to be
\[ \Gamma_{2n+1} = i \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_3 . \] (A.13)

The one-to-one correspondence between these indices and the ones we used for our parameterisation of AdS\(_{2n+1} \) will be taken to be
\[
\begin{align*}
\Gamma_{2p-1} &\rightarrow \Gamma_{p'} , & \Gamma_{2p} &\rightarrow \Gamma_p , & 1 \leq p \leq n-1 , \\
\Gamma_{2n-1} &\rightarrow \Gamma_{0'} , & \Gamma_{2n} &\rightarrow \Gamma_0 , & \Gamma_{2n+1} &\rightarrow \Gamma_{\tau} .
\end{align*}
\] (A.14)

In terms of these conventions, we find the following results for the explicit forms of the Killing spinors in AdS\(_3 \), AdS\(_5 \) and AdS\(_7 \):

### A.1 Killing Spinors for AdS\(_3 \)

The set of Killing spinor corresponding to the positive sign in (A.6) is given by
\[
\epsilon_1^+ = \left( -e^{-\frac{\phi}{2}} - i \chi e^{\frac{\phi}{2}} \right) , \quad \epsilon_2^+ = \left( e^{\frac{\phi}{2}} \right) ,
\] (A.15)

The other set, corresponding to the negative sign in (A.6), is instead given by
\[
\epsilon_1^- = e^{\frac{\phi}{2}} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) , \quad \epsilon_2^- = e^{\frac{\phi}{2}} \tau \left( \begin{array}{c} 0 \\ 1 \end{array} \right) .
\] (A.16)
A.2 Killing Spinors for AdS\(_5\)

The set of Killing spinor corresponding to the positive sign in (A.6) is given by

\[ \epsilon_1^+ = e^{-\frac{i \phi}{4}} \begin{pmatrix} \frac{1}{2} e^{\frac{1}{2} \phi} (y - ix) \\ \frac{1}{2} e^{\frac{1}{2} \phi} (y - ix) \\ 1 \\ 0 \end{pmatrix}, \]

\[ \epsilon_3^+ = e^{-\frac{i \phi}{4}} \begin{pmatrix} \frac{1}{2} e^{\frac{1}{2} \phi} e^{\frac{i \chi}{2}} (-y + ix) + e^{\frac{i \chi}{2}} \left[ i \chi + \frac{1}{2} i y x - \frac{1}{4}(y^2 + x^2) \right] \\ \frac{1}{2} e^{\frac{1}{2} \phi} e^{\frac{i \chi}{2}} \left[ i \chi + \frac{1}{2} i y x - \frac{1}{4}(y^2 + x^2) \right] \\ -(y + ix) \\ 0 \end{pmatrix}, \]

\[ \epsilon_2^+ = e^{-\frac{i \phi}{4}} \begin{pmatrix} 0 \\ \frac{1}{2} e^{\frac{1}{2} \phi} \\ 0 \\ 1 \end{pmatrix}, \]

\[ \epsilon_4^+ = e^{-\frac{3 i \phi}{4}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \]

The other set of Killing spinors corresponding to the negative sign is instead given by

\[ \epsilon_1^- = e^{\frac{i \phi}{4}} \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} e^{\frac{1}{2} \phi} (y + ix) \\ \frac{1}{2} e^{\frac{1}{2} \phi} (y + ix) \end{pmatrix}, \]

\[ \epsilon_3^- = e^{\frac{i \phi}{4}} \begin{pmatrix} 0 \\ (y - ix) \\ e^{-\frac{1}{2} \phi} + e^{\frac{1}{2} \phi} \left[ i \chi + \frac{1}{2} i y x - \frac{1}{4}(y^2 + x^2) \right] \\ e^{-\frac{1}{2} \phi} + e^{\frac{1}{2} \phi} \left[ i \chi + \frac{1}{2} i y x + \frac{1}{4}(y^2 + x^2) \right] \end{pmatrix}, \]

\[ \epsilon_2^- = e^{\frac{i \phi}{4}} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} e^{\frac{1}{2} \phi} \\ \frac{1}{2} e^{\frac{1}{2} \phi} \end{pmatrix}, \]

\[ \epsilon_4^- = e^{\frac{-3 i \phi}{4}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \]

A.3 Killing Spinors for AdS\(_7\)

We find that the set of Killing spinor corresponding to the positive sign in (A.6) is given by
\[e_1^+ = e^{-\frac{i}{2}r} \begin{pmatrix} \frac{1}{2} e^{\frac{1}{2}\phi} (y_1 - ix_1) \\ \frac{1}{2} e^{\frac{1}{2}\phi} (y_1 - ix_1) \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2^+ = e^{-\frac{i}{2}r} \begin{pmatrix} \frac{1}{2} e^{\frac{1}{2}\phi} (y_2 - ix_2) \\ \frac{1}{2} e^{\frac{1}{2}\phi} (y_2 - ix_2) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_3^+ = e^{-\frac{i}{2}r} \begin{pmatrix} -e^{-\frac{1}{2}\phi} + e^{\frac{1}{2}\phi} \left[ -i\chi + \frac{1}{2}i(y_1 x_1 + y_2 x_2) - \frac{1}{4}(y_1^2 + y_2^2 + x_1^2 + x_2^2) \right] \\ e^{-\frac{1}{2}\phi} + e^{\frac{1}{2}\phi} \left[ -i\chi + \frac{1}{2}i(y_1 x_1 + y_2 x_2) - \frac{1}{4}(y_1^2 + y_2^2 + x_1^2 + x_2^2) \right] \\ -(y_2 + ix_2) \\ 0 \\ -(y_1 + ix_1) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \]

\[e_4^+ = e^{-\frac{i}{2}r} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_6^+ = e^{\frac{i}{2}r} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ e^{\frac{1}{2}\phi} \\ e^{\frac{1}{2}\phi} \\ 0 \end{pmatrix}, \quad (A.19) \]

\[e_7^+ = e^{\frac{i}{2}r} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -\frac{1}{2} e^{\phi} (y_2 + ix_2) \\ -\frac{1}{2} e^{\phi} (y_2 + ix_2) \end{pmatrix}, \quad e_8^+ = e^{\frac{i}{2}r} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \frac{1}{2} e^{\phi} (y_1 + ix_1) \\ \frac{1}{2} e^{\phi} (y_1 + ix_1) \end{pmatrix} \]
The other set, corresponding to the negative sign in (A.6), is instead given by

\[
\epsilon^+_5 = e^{\frac{i}{2} \tau} \begin{pmatrix}
0 \\
0 \\
0 \\
(y_1 - ix_1) \\
0 \\
-(y_2 - ix_2) \\
-e^{-\frac{1}{2} \phi} + e^{\frac{1}{2} \phi}[-i \chi + \frac{1}{2}i(y_1x_1 + y_2x_2) + \frac{1}{4}(y_1^2 + y_2^2 + x_1^2 + x_2^2)] \\
e^{-\frac{1}{2} \phi} + e^{\frac{1}{2} \phi}[-i \chi + \frac{1}{2}i(y_1x_1 + y_2x_2) + \frac{1}{4}(y_1^2 + y_2^2 + x_1^2 + x_2^2)]
\end{pmatrix}.
\]

\[
\epsilon^-_1 = e^{i \tau}, \quad \epsilon^-_8 = e^{-i \tau}, \quad \epsilon^-_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -e^{\frac{1}{2} \phi} \\ e^{-\frac{1}{2} \phi} \\ e^{\frac{1}{2} \phi} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^-_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]

\[
\epsilon^-_4 = \begin{pmatrix}
0 \\
(y_1 - ix_1) \\
-\frac{1}{2}e^{\frac{1}{2} \phi}(y_1 - ix_1)(y_2 + ix_2) \\
\frac{1}{2}e^{\frac{1}{2} \phi}(y_1 - ix_1)(y_2 + ix_2) \\
e^{-\frac{1}{2} \phi} - e^{\frac{1}{2} \phi}[-i \chi + \frac{1}{2}i(y_1x_1 + y_2x_2) + \frac{1}{4}(y_1^2 - y_2^2 + x_1^2 - x_2^2)] \\
e^{-\frac{1}{2} \phi} + e^{\frac{1}{2} \phi}[-i \chi + \frac{1}{2}i(y_1x_1 + y_2x_2) + \frac{1}{4}(y_1^2 - y_2^2 + x_1^2 - x_2^2)] \\
-(y_2 + ix_2) \\
0
\end{pmatrix}.
\]
\[
\epsilon^2_6 = \begin{pmatrix}
0 \\
1 \\
-\frac{1}{2}e^{\frac{1}{2}\phi}(y_2 + ix_2) \\
-\frac{1}{2}e^{\frac{1}{2}\phi}(y_1 + ix_1) \\
\frac{1}{2}e^{\frac{1}{2}\phi}(y_1 + ix_1) \\
0 \\
0 
\end{pmatrix}, \quad \epsilon^6_7 = \begin{pmatrix}
0 \\
0 \\
\frac{1}{2}e^{\frac{1}{2}\phi}(y_1 - ix_1) \\
-\frac{1}{2}e^{\frac{1}{2}\phi}(y_1 - ix_1) \\
\frac{1}{2}e^{\frac{1}{2}\phi}(y_2 - ix_2) \\
\frac{1}{2}e^{\frac{1}{2}\phi}(y_2 - ix_2) \\
0 
\end{pmatrix}, \quad (A.20)
\]

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