Carbon nanotubes in an inhomogeneous transverse magnetic field: exactly solvable model

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Received 21 October 2013, revised 6 February 2014
Accepted for publication 7 February 2014
Published 3 March 2014

Abstract

A class of exactly solvable models describing carbon nanotubes in the presence of an external inhomogeneous magnetic field is considered. The framework of the continuum approximation is employed, where the motion of the charge carriers is governed by the Dirac–Weyl equation. The explicit solution of a particular example is provided. It is shown that these models possess nontrivial integrals of motion that establish \( N = 2 \) nonlinear supersymmetry in case of metallic and maximally semiconducting nanotubes. Remarkable stability of energy levels with respect to small fluctuations of longitudinal momentum is demonstrated.

Keywords: carbon nanotubes, Dirac equation, magnetic field, finite-gap systems
PACS numbers: 73.63.b, 75.70.Ak, 71.15.–m

(Some figures may appear in colour only in the online journal)

1. Introduction

Despite their structural simplicity, carbon nanotubes possess exciting physical properties [1, 2]. Besides their ultimate strength [3] and elasticity [4], they particularly excel in the variability of their electronic characteristics that make them an attractive material for the applications in electronic devices [5, 6]. Single-wall carbon nanotubes are small cylinders rolled up from a graphene strip with the shell being just one atom-thick. The gap between their valence and conduction band (positive and negative energies) depends on the radius of the nanotube and
on the orientation of the lattice in the shell. Most nanotubes are semiconducting, they have a nonzero gap in the spectrum with magnitude proportional to the inverse of the radius. The gap can be altered either by external fields or mechanical deformations.

The spectrum of carbon nanotubes in the presence of external magnetic and electric field was discussed in numerous works with the use of different techniques. For instance, the band structure of the nanotubes was considered within the framework of the tight-binding Hamiltonian [7–9]. In [10]4, it was shown that the metallic nanotubes can be turned into semiconducting ones (and vice versa) by an external homogeneous magnetic field, parallel with the axis of the nanotube. In [11], the effect of a homogeneous transverse field on the spectrum was analyzed in the low-energy approximation.

In this paper, we consider the single-wall carbon nanotube in the presence of an inhomogeneous external magnetic field. The inhomogeneous magnetic fields varying on the scale of nanometers were realized experimentally, e.g. by making structured patterns of a thin ferromagnet posed on the substrate with magnetization vector perpendicular to the surface [12]. The behavior of two-dimensional, both non-relativistic and relativistic, electron gas in presence of these fields was studied theoretically e.g. in [13, 14].

We will choose a magnetic field that is constant in the longitudinal direction (parallel with the axis of the nanotube), however, it has inhomogeneous transverse component (perpendicular to the axis). This kind of magnetic field with translational symmetry in one direction was also used in [13]. The setting is analyzed within the framework of the low-energy approximation. We show that in this case dynamics of the spin-up and spin-down components is described by the associated Lamé equation. The properties of the general finite-gap Hamiltonians are employed extensively in the analysis of this equation.

The paper is organized as follows. In the next section, we briefly review the general description of single-wall carbon nanotubes in the presence of an external magnetic field, explaining how the parallel and transverse fields affect the dynamics. In section 3, we will introduce a solvable model that allows for explicit solutions of the stationary equation. Then, in section 4, we show that the system possesses nontrivial symmetries that establish \( N = 2 \) nonlinear supersymmetry for metallic and maximally semiconducting nanotubes. Next, in section 5, we discuss the robustness of the energy levels with respect to a small fluctuation of the momentum in the longitudinal direction. The last section will be devoted to some comments and outlook.

2. Nanotubes in the external magnetic field: the low-energy approximation

We focus on the spectral properties of single-wall carbon nanotubes in the low-energy regime where the motion of quasi-particles is described by the Dirac–Weyl equation. Before considering the system in the external magnetic field, let us review briefly the case where the field is absent.

2.1. From graphene to nanotubes

The electronic properties of nanotubes can be easily deduced from the characteristics of planar graphene. The tight-binding Hamiltonian of graphene in the vicinity of one of the Dirac points, where the energy goes to zero, reduces to the two-dimensional massless Dirac–Weyl Hamiltonian [15]. The stationary equation in the \( x-z \) plane can be written as [11]

\[
H \Phi_\epsilon = (\pm i \sigma_2 \partial_\tau + i \sigma_1 \partial_x) \Phi_\epsilon = \epsilon \Phi_\epsilon, \quad \epsilon = \frac{E}{v_F \hbar}.
\] (2.1)

4 see also chapter E in [1] and references therein.
Here, $E$ is the energy and $v_F$ is the Fermi velocity that depends on the crystal characteristics (the hopping parameter) of graphene. The solution of (2.1) corresponding to a ‘scaled energy’ $\epsilon$ is given by means of plane waves

$$\Phi_n(x, z) = e^{i(k_x x + k_z z)} \left( \frac{i\sqrt{1/k_x - k_z}}{\sqrt{-1/k_x - k_z}} \right), \quad \epsilon = \pm \sqrt{k_x^2 + k_z^2},$$  \hspace{1cm} (2.2)

where $k_x$ and $k_z$ are constant momenta.

The nanotube can be created by gluing together the long sides of a graphene strip. We fix the coordinates such that $x$ goes in the circumference direction, $0 \leq x \leq 2\pi \rho_0$ with $\rho_0$ being the radius, while $z$ is parallel to the axis of the nanotube. Specific boundary conditions have to be prescribed at $x = 0$ and $x = 2\pi \rho_0$. Their explicit form depends on the orientation of the hexagonal lattice in the strip; they are quasi-periodic in general [1]. Thus, the solution of (2.1) has to satisfy the additional condition

$$\Phi_n(z, 2\pi \rho_0) = e^{i2\pi \omega} \Phi_n(z, 0).$$  \hspace{1cm} (2.3)

The spectral properties of nanotubes, and hence their electronic properties, will depend on the periodic condition (2.3). As we shall see in the following, the phase factor $\omega$ is vanishing for metallic nanotubes whereas it acquires nonzero value for semiconducting nanotubes.

In order to make the wave functions (2.2) compatible with the boundary condition (2.3), the momentum $k_x$ has to be restricted to the discrete values

$$k_x \equiv k_n = \frac{1}{\rho_0}(\omega + n), \quad n \in \mathbb{Z}.$$  \hspace{1cm} (2.4)

The energies (2.2) decay into a discrete set of values labeled by $n$, see figure 1 for illustration,

$$\epsilon_n(k_z) = \pm \frac{1}{\rho_0} (\omega + n)^2 + k_z^2.$$  \hspace{1cm} (2.5)

In particular, we have $\epsilon_n(0) = \pm \frac{1}{\rho_0} |\omega + n|$ for $k_z = 0$. The gap $\Delta(0)$ between positive and negative energies will depend on the explicit value of $\omega$,

$$\Delta(0) = 2|\epsilon_0(0)| = \frac{2}{\rho_0}|\omega|.$$  \hspace{1cm} (2.6)

For $\omega = 0$, the gap is vanishing and the nanotubes are metallic since an infinitesimal excitation is sufficient to move the electron from valence to conduction band. For $\omega \neq 0$ a gap is opened as the minimal distance $\Delta(0)$ is nonzero. The nanotube becomes semiconducting since an energy higher or equal to $\Delta(0)$ is needed to move the electron from negative to positive energy bands, which is the attribute of semiconductors. The formula (2.6) implies that the gap is proportional to the inverse of the radius of the nanotube $\rho_0$. 

Figure 1. The bands (2.5) for metallic and semiconducting nanotubes and some values of $n$: $\omega = 0$ (left), $\omega = 1/6$ (center) and $\omega = 1/2$ (right).
2.2. Nanotubes in external magnetic fields

Now, let us consider the nanotubes in the presence of an external magnetic field. The magnetic field can be decomposed into transverse and longitudinal components (perpendicular and parallel to the axis of the nanotube). It induces the Lorentz force which is perpendicular to the movement of the charged particles. As the particles are confined to the two-dimensional surface, only the part of the transverse component which is normal to the surface has an effect on the trajectories of the particles. The longitudinal component influences the dynamics as well, however, it manifests in a quite different manner.

For convenience, we introduce \( \phi = \frac{z}{\rho_0} \), where \( \phi \in [0, 2\pi] \). It constitutes, together with \( z \) (axis of the nanotube) and \( \rho_0 \) (the radius), the cylindrical coordinates which parametrize the nanotube. The effective magnetic field \( \mathbf{B} \) is induced by the vector potential tangent to the surface of the nanotube, \( \mathbf{A} = A_\phi \mathbf{n}_\phi + A_z \hat{\mathbf{z}} \), where \( \mathbf{n}_\phi \) and \( \hat{\mathbf{z}} \) are the unit tangent vectors in the circumference and the longitudinal direction, respectively. We will assume \( A_\phi \) to be a constant and \( A_z = A_z(\phi) \). The magnetic field is then

\[
\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{\rho_0} \partial_\phi A_z(\phi) \mathbf{n}_\rho + \frac{1}{\rho_0} A_\phi \mathbf{n}_\phi = B_\perp \mathbf{n}_\rho + B_\parallel \mathbf{n}_z. \tag{2.7}
\]

Taking into account the translational symmetry along the \( z \) axis, we look for eigenfunctions in the form \( \Phi(z, \phi) = e^{ik_zz} \Phi(\phi) \). Then, the massless Dirac–Weyl equation for the quasi-particle with minimal coupling to the magnetic field (2.7) takes the form

\[
\begin{bmatrix}
-\sigma_2 \left( -k_z + \frac{q}{c} \frac{A_z}{\hbar} \right) + \sigma_1 \left( \frac{1}{\rho_0} \partial_\phi + \frac{q}{c} \frac{A_\phi}{\hbar} \right)
\end{bmatrix} \Phi(\phi) = \epsilon \Phi(\phi) \tag{2.8}
\]

where its solutions are subject to the boundary condition (2.3).

We can see that the effect of \( A_\phi \) and \( A_z \) on the dynamics of the quasi-particles is qualitatively different. The component \( A_\phi \), corresponding to the longitudinal magnetic field, does not change the trajectory of a classical charged particle. However, as we will see now, it induces an additional phase shift of the wave functions, similarly to the Aharonov–Bohm effect. We can get rid of \( A_\phi \) in the equation (2.8) by extracting a suitable phase factor from the wave functions. Let us take

\[
\Phi(\phi) = e^{i\phi \frac{\pi \rho_0}{\hbar} A_\phi} \hat{\Phi}(\phi). \tag{2.9}
\]

Then we get the following equation for \( \hat{\Phi} \),

\[
\begin{bmatrix}
-\sigma_2 \left( -k_z + \frac{q}{c} \frac{A_z}{\hbar} \right) + \sigma_1 \frac{1}{\rho_0} \partial_\phi
\end{bmatrix} \hat{\Phi}(\phi) = \epsilon \hat{\Phi}(\phi). \tag{2.10}
\]

But now, the wave functions \( \hat{\Phi}(\phi) \) are subject to the modified boundary condition

\[
\hat{\Psi}(2\pi) = e^{i2\pi \delta} \hat{\Psi}(0), \quad \delta = \omega - \frac{q\rho_0}{c\hbar} A_\phi = \omega - \frac{2SB_0}{\Phi_0}, \tag{2.11}
\]

where \( S = \pi \rho_0^2 \) is the section area of the nanotube and \( \Phi_0 = \frac{c\hbar}{q} = \frac{2\pi c\hbar}{q} \). This shows that the phase \( \delta \) depends on the magnetic flux \( SB_0 \) that goes through the nanotube section.

In the current framework, we can easily understand the effect where the metallic nanotubes are converted into semiconducting and vice versa by the longitudinal magnetic field. Fixing \( A_z = 0 \), the energy gap of the nanotube can be obtained by substituting \( \delta \) instead of \( \omega \) into (2.6). Being interested in the lowest value of \( |\epsilon_n(0)| = \frac{1}{\rho_0} |\delta + n| \), we can consider \( \delta \in [0, 1/2] \) without loss of generality as the other values of \( \delta \) can be compensated by the integer \( n \). Now, the gap between the thresholds of the positive and negative energies becomes

\[
\Delta(0) = \frac{2}{\rho_0} \left| \omega - \frac{q\rho_0}{c\hbar} A_\phi \right|. \tag{2.12}
\]
By increasing \( A_\phi \) adiabatically, the metallic nanotube (\( \delta = 0 \)) turns to be semiconducting. After reaching the phase where it is maximally semiconducting (\( \delta = 1/2 \)), the gap decreases and the nanotube turns to be metallic again. This effect was called Aharonov–Bohm oscillation of carbon nanotubes [10].

3. Solvable model of the inhomogeneous magnetic field

Let the carbon nanotube be immersed into the external magnetic field (2.7) where \( B_1 \) is constant along the nanotube. The direction of the field can make an arbitrary angle with the axis of the nanotube. We will consider the configuration where

\[
A_z(\phi) = \rho_0 B_0 \left(1 + k^\prime\right) \frac{sn(\phi+\pi/2)K}{\pi} \frac{cn(\phi+\pi/2)K}{\pi} \frac{dn(\phi+\pi/2)K}{\pi} = \rho_0 B_0 a(\phi, k).
\]

Here, \( sn(x, k) \), \( cn(x, k) \) and \( dn(x, k) \) are Jacobi elliptic functions (the modulus \( k \) has been suppressed in (3.1) to simplify the notation) and \( K \) is for the complete elliptic integral \( K(k) = \int_0^\pi (1 - k^2 \sin^2 t)^{-1/2} dt \). The modular parameter is \( k \in [0, 1] \), the complementary modulus will be \( k' = \sqrt{1 - k^2} \) and the notation \( K = K(k) \) or \( K' = K(k') \) will be used.

The vector potential (3.1) possesses some remarkable properties:

(a) For \( k = 0 \) and specific values of the intensity of the transverse field,

\[
B_0 = m \frac{k^2 K ch}{(1 + k')\pi \rho_0 q} = \frac{mk^2 K}{2\pi (1 + k') S} \Phi_0, \quad \Phi_0 = \frac{ch}{q} = \frac{2\pi ch}{q} \tag{3.2}
\]

the stationary equation (2.8) acquires the following form

\[
H\psi = \left(\sigma_1 \partial_1 - m k^2 \frac{sn(y, k) cn(y, k)}{dn(y, k)} \sigma_2\right) \psi = \epsilon \psi \tag{3.3}
\]

where we have introduced the following notation

\[
y = \frac{K(\phi + \pi/2)}{\pi}, \quad \epsilon = \frac{\pi}{K} \rho_0 \epsilon \equiv \frac{\pi}{K} \frac{\rho_0}{\nu F} E, \quad \hat{\psi} (\phi) = \psi (y).
\]

The Hamiltonian (3.3) is \( 2K \) periodic in the new variable \( y \). The solutions of the equation are subject to the boundary condition (2.11), where the phase factor \( \delta \) will also depend on \( B_1 \), see (2.7), (2.11). Thus, in the new variables, we have

\[
\psi (y + 2K) = e^{i2\pi \delta} \psi (y), \quad 0 \leq \delta \leq 1/2.
\]

The key point is that when \( m \) is an integer, the Hamiltonian in (3.3) can be classified as a finite-gap operator [18, 19] and its (formal) eigenfunctions can be found analytically in terms of the Jacobi (or Weirstrass) theta and zeta functions, see [20]. We will suppose that this is the case from now on.

(b) The potential function (3.1) approximates very well to an homogenous field which is given by \( A_\phi = \rho_0 B_0 \cos \phi \), see [11]. The smaller is \( k \), the smaller is the difference between \( \cos \phi \) and \( a(\phi, k) \),

\[
\lim_{k \to 0} a(\phi, k) = \cos \phi. \tag{3.6}
\]

Even for \( k \sim 0.7 \), the difference is of order \( 10^{-3} \), see figure 2 for illustration. Hence, the vector potential (3.1) gives rise to a magnetic field which fluctuates slightly around a constant value.
Equation (3.3) can be cast in the form

$$H \Psi = \begin{pmatrix} 0 & iA^\dagger \\ -iA & 0 \end{pmatrix} \Psi = \tilde{\epsilon} \Psi$$  

(3.7)

where

$$A^\dagger = \partial_y + mk^2 \frac{sn(y, k)cn(y, k)}{dn(y, k)}, \quad A = -\partial_y + mk^2 \frac{sn(y, k)cn(y, k)}{dn(y, k)}$$  

(3.8)

and $m$ is a fixed integer. Then, the stationary equation (3.3) can be diagonalized,

$$H^2 \Psi(y) \equiv \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \Psi(y) = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} \Psi(y)$$

$$= \begin{pmatrix} -\partial_y^2 + k^2C_m sn^2(y, k) + k^2C_{m+} \frac{cn(y, k)}{dn(y, k)} - k^2m^2 & 0 \\ 0 & -\partial_y^2 + k^2C_{m-} sn^2(y, k) + k^2C_{m-} \frac{cn(y, k)}{dn(y, k)} - k^2m^2 \end{pmatrix} \Psi(y)$$

$$= \tilde{\epsilon}^2 \Psi(y)$$  

(3.9)

where $C_m = m(m \pm 1)$. The corresponding second-order equations for the spin-up and down components of $\Psi(y)$ can be identified as the associated Lamé equations. It is worth noticing that the operators $H_1$ and $H_2$ are related by means of a shift of the coordinate in half of the period $K$: $H_1(y) = H_2(y + K)$. This implies in particular that the components of the spin or $\Psi = (\psi_1, \psi_2)^T$ satisfy $\psi_2(y + K) \sim \psi_1(y)$.

Let us notice that the quantum systems described by the stationary equations (3.3) and (3.9) with $y$ extending over the whole real line ($y \in \mathbb{R}$) and with integer valued $m$ were considered in the literature, see e.g. [18, 21] and references therein. The spectrum of these systems consists of a finite number of spectral bands where the inner-band energy levels are doubly degenerate while the band-edge states are singlets. On the contrary, the spectrum of our current model, which is confined on the finite interval, is purely discrete as the wave function are required to have a fixed value of $\delta$ in (3.5).

5 The operator $H^2$ can be considered as the supersymmetric Hamiltonian that, together with the supercharges $Q^\dagger = \frac{1}{2}(\sigma_1 - i\sigma_2)A^\dagger$ and $Q = \frac{1}{2}(\sigma_1 + i\sigma_2)A$, establishes the $sl(1|1)$ superalgebra [16]. Denoting $H_{SUSY} = H^2$, we have

$$[H_{SUSY}, Q^\dagger] = [H_{SUSY}, Q] = 0, \quad [Q, Q^\dagger] = 2H_{SUSY},$$

$$\{Q, Q^\dagger\} = \langle Q^\dagger, Q \rangle = 0.$$

(3.10)

The framework of the SUSY quantum mechanics was utilized recently in the analysis of a Dirac–Weyl system confined on the surface of the sphere [22].
The finite-gap system described by (3.3) possesses other interesting properties, e.g. it has a nontrivial integral of motion. It also exhibits a remarkable stability of the energy levels under the perturbation of small values of $k_z$. We will discuss these points later in sections 5 and 6. Now, let us step to the explicit solution of the stationary equation (3.3) for $m = 1$.

**Solutions for the configuration $m = 1$ and $k_z = 0$.**

Fixing $m = 1$ in the field intensity (3.2), the stationary equation (3.3) acquires the following form

$$H \Psi(y) = \left( i \sigma_1 \partial_y - k^2 \frac{sn(y, k) cn(y, k)}{dn(y, k)} \sigma_2 \right) \Psi(y) = \tilde{\epsilon} \Psi(y). \quad (3.11)$$

In this case, the diagonal operators of the decoupled equation in (3.9) take the simple form of Lamé equations

$$\begin{pmatrix} -\partial_y^2 + 2k^2 sn^2(y + K, k) - k^2 & 0 \\ 0 & -\partial_y^2 + 2k^2 sn^2(y, k) - k^2 \end{pmatrix} \Psi(y) = \tilde{\epsilon}^2 \Psi(y). \quad (3.12)$$

To find $\Psi$, it is sufficient to solve just one of the Lamé equations in (3.12). Thus, for instance, take that of spin-down component

$$H_2 \psi = \left( -\partial_y^2 + 2k^2 sn^2(y, k) - k^2 \right) \psi = \tilde{\epsilon}_2^2 \psi. \quad (3.13)$$

The corresponding potential $V_2(y)$ is plotted in figure 3. Two independent solutions $\psi_{\pm}(y)$ of (3.13) can be found in a closed form [20]

$$\psi_{\pm}(y) = \frac{\mathcal{H}(y \pm \alpha)}{\Theta(y)} e^{\mp i p(\alpha)} e^{\mp i \pi y / K}, \quad \alpha \in \mathbb{C}, \quad (3.14)$$

where $\mathcal{H}(y)$ and $\Theta(y)$ are theta functions and $\zeta(\alpha)$ is the Jacobi zeta function [23]. These theta functions satisfy $\Theta(y + 2K) = \Theta(y)$ and $\mathcal{H}(y + 2K) = -\mathcal{H}(y)$. The solutions (3.14) correspond to the eigenvalue $\tilde{\epsilon}_2^2$ which depends on the parameter $\alpha$ in the following manner,

$$\tilde{\epsilon}_2^2 = dn^2(\alpha, k). \quad (3.15)$$

For the sake of convenience, we divide the wave functions into a periodic part multiplied by a phase factor. Having in mind (anti-)periodicity of the theta functions, we can write

$$\psi_{\pm}(y) = u_{\pm}(y) \exp(i p(\alpha)), \quad u_{\pm}(y + 2K) = u_{\pm}(y) \quad (3.16)$$

where

$$u_{\pm}(y) = \left( \frac{\mathcal{H}(y \pm \alpha)}{\Theta(y)} e^{\mp i \pi y / 2K} \right), \quad p(\alpha) = -i \zeta(\alpha) + \frac{\pi}{2K}. \quad (3.17)$$

![Figure 3. Lamé potentials $V_2$ for different values of $k$. As $k \to 0$ the intensity of the potential decreases due to the $k^2$ term in (3.13), $V_2(y) \to 0$. The unit for $y$ is taken $K/\pi$.](image)
The two independent (unnormalized) solutions $\Psi_{\pm}(y)$ for $H$ in (3.11) can be written with the help of $A^\dagger$ (3.8) in the following manner

$$\Psi_{\pm}(y) = \begin{pmatrix} A^\dagger \psi_{\pm}(y) \\ \psi_{\pm}(y) \end{pmatrix} = \begin{pmatrix} i(A^\dagger \mp p(\alpha))u_{\pm}(y) \\ u_{\pm}(y) \end{pmatrix} e^{\mp iy p(\alpha)}. \tag{3.18}$$

Since differentiation of a function does not alter its periodicity, the spin-up component in (3.18) has the same phase factor as $\psi_{\pm}(y)$ and it has to be proportional to $\psi_{\pm}(y + K)$, $A^\dagger \psi_{\pm}(y) \sim \psi_{\pm}(y + K)$. For the special case $\tilde{v} = 0$, the general solution is given by

$$\Psi_0(y) = \begin{pmatrix} \beta_2 \dn(y + K,k) \\ \beta_1 \dn(y,k) \end{pmatrix}, \quad H \Psi_0(y) = 0, \tag{3.19}$$

where $\beta_1$ and $\beta_2$ are arbitrary constants.

The admissible solutions $\Psi_{\pm}(y)$ and $\Psi_0(y)$ have to comply with the boundary condition (3.5). Substituting (3.16) into (3.5), we find that the values of the phase function $p(\alpha)$ in (3.18) has to satisfy the following equation

$$\pm p(\alpha) = \frac{\pi}{K}(\delta + l), \quad l \in \mathbb{Z}. \tag{3.20}$$

We can see immediately that only the real values of $p(\alpha)$ are acceptable. Keeping in mind the definition (3.17) of the phase factor $p(\alpha)$, the latter condition implies that only the values of $\alpha$ such that $\text{Re}(\tilde{\zeta}(\alpha)) = 0$ are admissible. After some computations (see appendix A) we find that the parameter $\alpha$ can be restricted to the following two intervals:

$$\alpha = K + i \eta \quad \text{or} \quad \alpha = i \eta, \quad \eta \in [0, 2K^*]. \tag{3.21}$$

For these values of $\alpha$, the equation (3.20) takes the following form

$$p(\alpha) = -\zeta(\eta, k') + \frac{\dn(\eta, k') \sn(\eta, k')}{\cn(\eta, k')} + s(\alpha) \frac{k'^2 \sn(\eta, k')}{\cn(\eta, k') \dn(\eta, k')}$$

$$- \frac{\pi \eta}{2K K'} + \frac{\pi}{2K} = \pm \frac{\pi}{K}(\delta + l) \tag{3.22}$$

where $s(\alpha) = 0$ for $\alpha = i \eta$ and $s(\alpha) = 1$ for $\alpha = K + i \eta$.

This transcendental equation must be solved numerically in order to find the admissible values of $\eta$, once $\delta$ and $l \in \mathbb{Z}$ are fixed. Some insight can be provided by a graphical solution of the equation, see figure 4 for illustration. The solutions of $\eta$ are given by the intersection of the horizontal lines which represent the rhs. of (3.22) with the dashed curve of the function in the lhs. of (3.22). For these solutions of $\eta$, the corresponding energy can be found on the solid curve.

The central gap between positive and negative energies $\epsilon$ in the absence of transverse field ($A_2(\phi) = 0$) is equal to $2\delta$, which corresponds to the distance of the two straight lines in figure 4 (on the left side). The central gap of the system gets shrunk when the transverse magnetic field is switched on; the corresponding positive (negative) energy curve lies below (above) the intersection of the phase curve with the horizontal line, see figure 4. The positive and negative energies for different values of $k$ are represented in figure 5.

The energies are given by (3.15) in terms of $\alpha$. According to (3.21) there are two types of admissible values of $\alpha$. For the type $\alpha = K + i \eta$, we get the first energy band,

$$\tilde{\epsilon}(K + i \eta, k) = \dn(K + i \eta, k) = k' \frac{\cn(\eta, k')}{\dn(\eta, k')}, \quad \eta \in [0, 2K^*]. \tag{3.23}$$

The second energy band is obtained with $\alpha = i \eta$,

$$\tilde{\epsilon}(i \eta, k) = \dn(i \eta, k) \frac{\dn(\eta, k')}{\cn(\eta, k')}, \quad \eta \in [0, 2K^*]. \tag{3.24}$$
Figure 4. Left: The function $K_p(\alpha)$ for $\alpha = K + i\eta$ (thin dashed line), $\epsilon(\alpha)$ (thick line) and $\pm \delta$ (horizontal thin dashed line). Right: The functions $K_p(\alpha)$ (dashing) and $\epsilon(\alpha)$ (continuous) as a function of the parameter $\eta$ for the case $\alpha = K + i\eta$ (central band) and $\alpha = i\eta$ (upper and lower bands). The discrete values $\pm (\delta + l)$ are represented by thin black lines for $l = 0, \pm 1, \pm 2, \pm 3$. We have taken $k = \sqrt{0.5}$ and $\delta = \frac{1}{3}$. The unit of the horizontal axis is $K'$. 

Figure 5. Energies as a function of the phase $p(\alpha)$ for different values of $k$. The continuous straight line is for the null field corresponding to the value $k = 0$. 

From the formulas (3.23) and (3.24), we can get a qualitative insight into the behavior of the energy in dependence on the parameter $\alpha$. First, $\tilde{\epsilon}(\alpha)$ is positive for $\eta \in (0, K')$ while it acquires negative values for $\eta \in (K', 2K')$. In the endpoints of the intervals, we have $\tilde{\epsilon}(K) = k'$, $\tilde{\epsilon}(K + iK') = 0$ and $\tilde{\epsilon}(K + 2iK') = -k'$ while $\tilde{\epsilon}(0) = -\tilde{\epsilon}(2iK') = 1$ and $\tilde{\epsilon}(iK'\mp) = \pm \infty$. See figure 6 for the illustration of these properties.

4. Nonlinear supersymmetry of Dirac Hamiltonian

Due to the specific form of the potential term, the Hamiltonian (3.3) with integer $m$ belongs to the family of finite-gap operators that is associated with the stationary Ablowitz–Kaup–Newell–Segur hierarchy of nonlinear differential equations. The finite-gap systems possess an integral of motion $\mathcal{Y}$ given in terms of a higher-order differential operator. Its existence is deeply related with the theory of the integrable systems. It forms, together with the Hamiltonian, the
celebrated Lax pair [19]. Its role in the description of both non-relativistic and relativistic periodic systems was discussed extensively in the literature, see e.g. [18, 21, 24].

In the case of the Hamiltonian (3.3), the symmetry operator acquires the following explicit form [21]

$$[H, Y] = 0, \quad Y = \begin{pmatrix} 0 & Y(y) \\ Y(y)^\dagger & 0 \end{pmatrix},$$  \hspace{1cm} (4.1)

where

$$Y(y) = \frac{dn^{m+1}(y, k)}{cn^{2m+1}(y, k)} \left( c_n^2(y, k) \frac{dn(y, k)}{dn(y, k)} \right)^{2m} \frac{dn^{m-1}(y, k)}{cn^{2m-1}(y, k)}.$$  \hspace{1cm} (4.2)

The operator $\hat{Y}$ is uniquely determined by its kernel. It annihilates $2m$ formal eigenstates of $H$ corresponding to the eigenvalues $\tilde{\epsilon}_i (i \in 1, \ldots, 2m)$. These states are antiperiodic and given in terms of Jacobi elliptic functions. The operator satisfies the following remarkable relation

$$\hat{Y}^2 = \prod_{i=1}^{2m} (H^2 - \tilde{\epsilon}_i^2),$$  \hspace{1cm} (4.3)

where the right-hand side is identified as the spectral polynomial of the finite-gap system, see [21] for details.

Besides (4.1), the Hamiltonian (3.3) formally commutes with the operator $\Gamma = \sigma_3 \hat{R}$, where $\hat{R} \hat{R} = -y$ (there holds $cn(-y) = cn(y)$, $sn(-y) = -sn(y)$ and $dn(-y) = dn(y)$). It suggests that the symmetries of the system can be conveniently treated in the framework of $N = 2$ superalgebra, graded by $\Gamma = \sigma_3 \hat{R}$ and generated by the bosonic Hamiltonian (3.3) and two fermionic supercharges,

$$Q_1 = \hat{Y}, \quad Q_2 = i\sigma_3 \hat{R} \hat{Y}, \quad [H, \Gamma] = \{ Q_a, \Gamma \} = 0.$$  \hspace{1cm} (4.4)

Indeed, there hold the following relations

$$[H, Q_a] = 0, \quad [Q_a, Q_b] = 2\delta_{ab} \prod_{i=1}^{2m} (H^2 - \tilde{\epsilon}_i^2), \quad a, b = 1, 2.$$  \hspace{1cm} (4.5)

Notice that the superalgebra differs from the standard supersymmetry as discussed in quantum mechanics [16] where all the relations are linear in the generators. Here, the $N = 2$ superalgebra (4.5) is nonlinear as the anticommutator of the supercharges is a polynomial of second order.
in the Hamiltonian [17]. Let us notice that this structure was discussed for periodic systems in [18]. The supersymmetry of non-relativistic systems based on nonlocal supercharges and graded by the parity operator was also discussed in [21], [25] or in [26].

However, when the system is confined on the finite interval, the superalgebra (4.4)–(4.5) does not exist in general. The problem is that the supercharges does not comply with the (3.5) which fixes the required value of the quasi-momentum.

Let us illustrate the situation for \( m = 1 \). First, let us show that \( Q_1 \) preserves (3.5). It has the following explicit form

\[
y = \sigma_1 \left( \partial_y^2 + \frac{k^2}{2 \, dn(y, k)^2} + \frac{1}{2} \, dn(y, k)^2 \right) - i k^2 \sigma_2 \sqrt{\frac{sn(y, k)cn(y, k)}{dn(y, k)}} \partial_y \sqrt{\frac{sn(y, k)cn(y, k)}{dn(y, k)}}.
\]

(4.6)

Its action on the wave functions (3.18) can be concluded directly from (4.3) and (3.17). As the operator \( y \) is \( 2K \) periodic and because derivation of a function maintains its periodicity, we can write

\[
y \Psi_{\pm}(y) = \tilde{u}_{\pm}(e^{\mp ip(\alpha)}y), \quad \tilde{u}_{\pm}(y + 2K) = \tilde{u}_{\pm}(y).
\]

(4.7)

Since \( y \Psi_{\pm} \) corresponds to the same energy as \( \Psi_{\mp} \) (here, \( H \Psi_{\mp} = \tilde{e} \Psi_{\mp} \)) and satisfies the same boundary condition (it has the same phase factor), it must be proportional to \( \Psi_{\mp} \). Having in mind (4.3), we can conclude

\[
y \Psi_{\mp} = \pm \sqrt{\prod_{i=1}^{2}(\tilde{e}^2 - \tilde{e}_i^2)} \Psi_{\pm} = \pm \sqrt{\prod_{i=1}^{2}(\tilde{e}^2 - 1)(\tilde{e}^2 - k^2)} \Psi_{\pm}.
\]

(4.8)

Hence, \( Q_1 = y \) preserves quasi-momentum of the wave functions.

Now, let us focus on the grading operator \( \Gamma \),

\[
\Gamma \Psi_{\pm} = \sigma_3 \tilde{R} \left( \frac{1}{2} (A^+ \mp p(\alpha))u_{\pm}(y) \right) e^{\mp iyp(\alpha)}
\]

\[
= \sigma_3 \left( \frac{1}{2} (A^+ \pm p(\alpha))u_{\pm}(y) \right) e^{\mp iyp(\alpha)} = \Psi_{\mp}
\]

(4.9)

where we used \( \tilde{R} u_{\pm}(y) = -u_{\mp}(y) \) and \( \tilde{R} A^+ = -A^+ \tilde{R} \). The operator \( \Gamma \) changes the sign of the phase factor, which collides with the boundary condition (3.5) in general. However, there are exceptional values of quasi-momentum that are preserved by \( \Gamma \). The exception occurs when both \( p(\alpha) = \frac{\pi}{K}(\delta + l_1) \) and \( -p(\alpha) = \frac{\pi}{K}(\delta + l_2) \) for some integers \( l_1 \) and \( l_2 \). Then the change of sign in phase factor does not violate the boundary condition. This holds true for

\[
\delta = -\frac{l_1 + l_2}{2}.
\]

(4.10)

Hence, \( \Gamma \) is a good symmetry of the system provided that \( \delta \) acquires (semi-)integer values. For \( \delta = 0 \) or \( \delta = 1/2 \), the operator \( \Gamma \) commutes with \( H \) but anticommutes with \( y \).

Below, we discuss the two specific configurations where the \( N = 2 \) nonlinear supersymmetry exists, distinguished by the values of \( \delta \):

4.1. Metallic nanotubes: \( \delta = 0 \)

The wave functions have to be periodic. It allows for existence of the ground states with zero energy. The explicit form of the two zero modes is

\[
\Psi_{0 \pm} = \begin{pmatrix} dn(y, k) \\ (\pm dn(y + K, k) \end{pmatrix}.
\]

(4.11)
They satisfy the following relations,
\[ Q_a \Psi_{0\pm} = \pm k' \Psi_{0\pm}, \quad H \Psi_{0\pm} = 0, \quad a = 1, 2. \]

There is no central gap in the spectrum of the nanotubes. As discussed in [11] the zero modes are protected by the standard supersymmetry presented in section 2, where the Hamiltonian (3.11) plays the role of supercharge.

### 4.2. Maximally semiconducting nanotubes: \( \delta = 1/2 \)

The wave functions are required to be antiperiodic. In this regime, the nanotubes are maximally semiconducting. The states \( \Psi_{1\pm} \) corresponding to the threshold of the positive and negative energy spectrum, \( \tilde{\epsilon} = \pm k' \), are annihilated by the supercharge \( Q_1 \) (and \( Q_2 \)). Together with other two physical states \( \Psi_{2\pm} \), corresponding to \( \tilde{\epsilon} = \pm 1 \), they form the kernel of \( Q_1 \). The explicit form of these states is
\[
\Psi_{1\pm} = \begin{pmatrix} \pm i n(y + K, k) \\ c n(y, k) \end{pmatrix}, \quad \Psi_{2\pm} = \begin{pmatrix} \pm i s n(y + K, k) \\ s n(y, k) \end{pmatrix},
\]
where
\[
H \Psi_{1\pm} = \pm k' \Psi_{1\pm}, \quad \Psi_{1\pm}(y + 2K) = -\Psi_{1\pm}(y),
\]
\[
H \Psi_{2\pm} = \pm \Psi_{2\pm}, \quad \Psi_{2\pm}(y + 2K) = -\Psi_{2\pm}(y),
\]
and
\[
Q_a \Psi_{b\pm} = 0, \quad a, b \in \{1, 2\}.
\]

### 5. Spectral stability in the \( k_z \neq 0 \) regime

Up to now, we considered the system where the longitudinal momentum \( k_z \) was vanishing. In the current section, we will extend the analysis to \( k_z \neq 0 \) with the use of perturbation theory.

Let us consider the spectrum of the following operator
\[
H_{k_z} = H + k_z \sigma_z,
\]
where \( H \) is the finite-gap Hamiltonian (3.3). Let us suppose that \( k_z \) is small enough to justify considering the term \( k_z \sigma_z \) as a small perturbation. The first-order correction \( \Delta \tilde{\epsilon} \) to the energy \( \tilde{\epsilon} \), where \( \Psi_{\tilde{\epsilon}} = (\psi_{\tilde{\epsilon}}(y), \xi_{\tilde{\epsilon}}(y))^T e^{i\tilde{\epsilon}y} \), satisfies \( (H - \tilde{\epsilon}) \Psi_{\tilde{\epsilon}} = 0 \), is given by
\[
\Delta \tilde{\epsilon} = k_z \int_{-K}^{K} \Psi_{\tilde{\epsilon}}^\dagger \sigma_2 \Psi_{\tilde{\epsilon}} \, dy = 2k_z \int_{-K}^{K} \text{Im}(\overline{\psi_{\tilde{\epsilon}}(y)} \xi_{\tilde{\epsilon}}(y)) \, dy
\]
\[
= -2k_z \int_{-K}^{K} \text{Im}(\beta \overline{\xi_{\tilde{\epsilon}}(y)} \xi_{\tilde{\epsilon}}(y + K)) \, dy.
\]

In the second line, we used the fact that the upper component \( \psi_{\tilde{\epsilon}} \) is proportional to \( \xi_{\tilde{\epsilon}} \) up to the shift of the coordinate, \( \psi_{\tilde{\epsilon}}(y) = \beta \xi_{\tilde{\epsilon}}(y + K) \) (see the corresponding comment below (3.9)).

The formula (5.2) suggests that \( \Delta \tilde{\epsilon} = 0 \) when both \( \psi_{\tilde{\epsilon}} \) and \( \xi_{\tilde{\epsilon}} \) are real. This happens for the zero mode (i.e. \( \delta = 0 \)); the stationary equation is decoupled and the upper and the lower component of the zero mode can be fixed as real functions.

The robustness of energy levels can occur for nonzero levels as well, however, the specific properties of the system play more important role here. Suppose we deal with anti-periodic boundary condition, i.e. \( \delta = 1/2 \), which corresponds to the case of maximally semiconducting nanotubes. Then the lower component \( \xi_{\tilde{\epsilon}}(y) \) satisfies \( \xi_{\tilde{\epsilon}}(y + 2K) = -\xi_{\tilde{\epsilon}}(y) \). Next, we suppose
that \( \xi_i(y) \) is either even or odd with respect to the parity, i.e. \( \xi_i(-y) = \tau \xi_i(y) \), where \( \tau = \pm 1 \). Then we can write

\[
\int_{-K}^{K} \Im(\beta \xi_i(y) \xi_i(y + K)) \, dy = -\int_{-K}^{K} \Im(\beta \xi_i(y) \xi_i(y - K)) \, dy
\]

\[
= - \int_{-K}^{K} \Im(\beta \xi_i(y) \xi_i(y + K)) \, dy,
\]

where we made the substitution \( y \to -y \) in the first step and then employed the parity and antiperiodicity of the wave functions. Since the left and the right hand side of the equality (5.3) differ just in sign, the integral has to be vanishing. Hence, the energy corresponding to the spinor with the required properties is robust with respect to small fluctuations of \( k_z \). The states possessing required properties can be found by purely algebraic means in the kernel of the operator \( \gamma \) that is formed by \( 2m \) anti-periodic states \( \Psi_i, i = 1, \ldots, 2m \), see [21].

Considering the specific case where \( m = 1 \), one can prove much stronger statement: the first-order correction (5.2) is vanishing for all energy levels. Let us sketch the proof briefly, referring to appendix B for more details. First, it is convenient to find explicitly the coefficient \( \beta = \frac{\psi(\gamma)}{\xi(\gamma + K)} \). It is \( \beta = \imath \psi(\alpha i \kappa) \). Next, there holds the following relation

\[
\hat{K}(\gamma(y + \alpha)\gamma(y + \alpha + K)) = -\hat{\gamma}(y + \alpha)\gamma(y + \alpha + K)
\]

(5.4)

for \( \alpha = \imath n \) and \( \alpha = K + \imath n \). Substituting the explicit form of the wave function (3.17) into the integral (5.2), we get

\[
\Delta \hat{\varepsilon} = -2k_c \int_{-K}^{K} \Im(\beta \xi_i(y) \xi_i(y + K)) \, dy = -2k_c \int_{-K}^{K} \Re \left( \frac{\hat{\gamma}(y + \alpha)\gamma(y + \alpha + K)}{\Theta(y)\Theta(y + K)} \right) \, dy = 0.
\]

(5.5)

Here we used the fact that \( \xi(\alpha) \) is purely imaginary and substituted the explicit form of \( \beta \). The last equality is obtained since the integrand is an odd function of \( y \), which follows from (5.4). The relation (5.5) manifests that the energy levels of \( m = 1 \) case are robust with respect to small fluctuations of the longitudinal momentum \( k_z \) for any value of \( \delta \).

6. Conclusions

In the current paper, we have considered the continuum model of the single-wall carbon nanotubes in the presence of a specific transverse magnetic field (3.1)–(3.2). The model is quite flexible as it also allows to control the inhomogeneity of the field by the modular parameter \( k \). For \( k \to 0 \), the field tends to the homogeneous one while for \( k \to 1 \) it diverges from the constant value, see figures 2 and 3. The current model might be also useful in the situation where the carbon nanotube is in the presence of homogeneous magnetic field, but the profile of the nanotube is no longer circular but slightly flattened so that the normal projection of the homogeneous magnetic field could be described by (3.1)–(3.2).

The case represented by the two-gap Hamiltonian (3.11) has been elaborated in detail. The wave functions and energies were found for the vanishing longitudinal momentum, \( k_z = 0 \). The energies of the system were given as a solution of the transcendental equation (3.22). We checked that the model manifests remarkable stability with respect to the fluctuations \( k_z \neq 0 \) of the longitudinal momentum.

The considered systems belong to the family of finite-gap operators characteristic by existence of a nontrivial integral of motion (4.1). We showed that they give rise \( N = 2 \) nonlinear supersymmetry (4.5) for metallic and maximally semiconducting nanotubes.
In this article, we restricted ourselves to the dynamics in the vicinity of a single Dirac point. When both Dirac points are taken into account, the effective Hamiltonian is $4 \times 4$ block-diagonal operator. The $2 \times 2$ operators on its diagonal describe the system at each of the two Dirac points. In fact, the operator in (3.11) would correspond to one of them.

Recently, the extended framework was used in the analysis of carbon nanotubes with finite-gap configurations of the pseudo-magnetic field generated by the axial twist [24]. It was observed that when both Dirac points are taken into account, there emerges a $so(3) \oplus u(1)$ algebra of integrals of motion for the system with time-reversal symmetry. The question arises whether a similar algebraic structure could be detected in the current system when both Dirac points would be considered. Despite that our current setting lacks time-reversal symmetry (it is violated by the external magnetic field), we suppose that the algebra could exist for any strength of the parallel magnetic field $B_{||}$ as long as $\omega = 0$.

The treatment here presented opens a number of interesting questions. As the wave functions of the Lamé Hamiltonian are known explicitly for any integer $m$, it suggests that it could be feasible to extend our results to the systems with a generic, integer valued, coupling constant $m$. In particular, we have in mind the existence of the hidden $N = 2$ nonlinear supersymmetry or the spectral stability of the system. Our present results based on finite-gap Hamiltonian provide a good basis for construction of other solvable configurations of external magnetic field. We have in mind to apply Darboux transformations [27] in the construction of new solvable Hamiltonians from the systems presented in this paper. Let us notice in this context that this technique proved to be very useful in the analysis of twisted carbon nanotubes [28]. The results supplied along this paper system can also serve as a test field for the analysis of a wider class of configurations with the use of rigorous methods like those applied recently in the study of low-dimensional non-relativistic systems with external magnetic field [29]. However, these considerations go beyond the scope of the present work.

Acknowledgments

This work is partially supported by the Spanish MEC (FIS2009-09002) and by the GAČR grant P203/11/P038 of the Czech Republic.

Appendix A. Computation of the phase of the eigenfunctions

Using the appropriate formulas for the Jacobi zeta functions [23], we can find

$$
\zeta(\alpha, k) = \zeta(Re(\alpha), k) + k^2 \frac{s_t s c d}{c_t^2} + \frac{\pi}{2} \frac{Im(\alpha)}{K(k)K(k')} \frac{ds_t}{c_t} \tag{A.1}
$$

$$
+ i \left( -\zeta(Im(\alpha), k') - \frac{\pi}{2} \frac{Im(\alpha)}{K(k)K(k')} + \frac{ds_t}{c_t} \right) \tag{A.2}
$$

$$
- i k^2 \frac{s_t^2 s d_t}{c_t (c_t^2 + k^2 s_t^2)} \tag{A.3}
$$

where we abbreviated

$$
s = sn(Re(\alpha), k), \quad c = cn(Re(\alpha), k), \quad d = dn(Re(\alpha), k), \tag{A.4}
$$

$$
s_t = sn(Im(\alpha), k'), \quad c_t = cn(Im(\alpha), k'), \quad d_t = dn(Im(\alpha), k'). \tag{A.5}
$$
In order to have \( p(\alpha) \) real valued (see (3.17)), the real part of (A.1) has to be eliminated. It can be done by fixing \( \Re(\alpha) \) as an integer multiple of \( K \), i.e.

\[
\alpha = nK + in, \quad \eta \in \mathbb{R}, \quad n \in \mathbb{Z}.
\]  

(A.6)

Then there holds \( \zeta(iK, k) = 0 \) and \( sc = 0 \) and the real part of (A.1) vanishing. We can write

\[
p(\alpha) = -\zeta(\eta, k') + \frac{dn(\eta, k')sn(\eta, k')}{cn(\eta, k')} + s(n) \frac{k^2 sn(\eta, k')}{cn(\eta, k')dn(\eta, k')} = \frac{\pi \eta}{2KK'} + \frac{\pi}{2K'}
\]

(A.7)

where \( s(n) = \frac{(1 - (-1)^n)}{2} \). The relation (3.17) together with (3.20) determine the relevant values of \( \alpha \). The physically relevant values of \( \alpha \) (or \( \eta \)) are given by the following equation

\[
p(\alpha) = -\zeta(\eta, k') + \frac{dn(\eta, k')sn(\eta, k')}{cn(\eta, k')} + \epsilon(n) \frac{k^2 sn(\eta, k')}{cn(\eta, k')dn(\eta, k')} = \pm \frac{\pi}{K} (\delta + l),
\]

(A.8)

where \([.\] denotes the integer part of a number and \( \eta = \eta_{\text{mod}(2K')} + 2K'[\frac{\pi}{2K}]. \) Without loss of generality, we can fix \( \alpha = K + in \) or \( \alpha = in \) with \( \eta \in (0, 2K') \).

**Appendix B. Stability of energy levels**

The function \( \mathfrak{H}(y) \) is \( 2K \) antiperiodic and odd with respect to the parity operator \( \hat{R} \), \( \mathfrak{H}(y + 2K) = -\mathfrak{H}(y), \hat{R}\mathfrak{H}(y) = \mathfrak{H}(-y) = -\mathfrak{H}(y) \). The eta function is \( 2K \) periodic and even with respect to the parity, \( \Theta(y + 2K) = \Theta(y), \hat{R}\Theta(y) = \Theta(-y) = \Theta(y) \). Next, there holds \( \mathfrak{H}(y + \alpha) = \mathfrak{H}(y + \alpha') \) provided that \( y \) is real. These properties can be obtained directly from the definition, see [23].

There holds the following relation between the up and down components of the eigenfunction \( \Psi_\xi = (\psi, \xi)^T \),

\[
\psi(y) = \beta \xi(y + K), \quad \xi(y + K) = -A(y).
\]

(B.1)

where \( \beta \) is a complex number that should be specified. There also holds

\[
\frac{i}{\epsilon} A\xi = \psi, \quad \frac{1}{\epsilon^2} AA\xi = \xi, \quad A\psi(y + K) = -A(y).
\]

(B.2)

Then we can write

\[
\xi(y) = \frac{1}{\epsilon^2} A\xi (y) = -\frac{i}{\epsilon} A\psi(y) = -\frac{i}{\epsilon} A\beta \xi(y + K)
\]

\[
\xi(y + K) = \beta^2 \xi(y + 2K)
\]

so that \( \beta^2 = \frac{\xi(y)}{\xi(y + 2K)} \). Keeping in mind that \( \xi = \frac{\hat{\zeta}(y + \alpha)}{\sqrt{\Theta(y + 2K)}}, \) we can write

\[
\beta^2 = \frac{\xi(y)}{\xi(y + 2K)} = \frac{\hat{\zeta}(y + \alpha)\Theta(y + 2K) e^{\xi(y + 2K)}}{\Theta(y + \alpha + 2K) e^{\xi(y + \alpha)}} = -e^{2\xi(y + K)}
\]

(B.4)

Hence, we get \( \beta = i\epsilon^{\xi(y + K)} \).

Now, we can simplify the integrand \( I \) of the first-order energy correction,

\[
I = \langle \psi, \xi \rangle = \epsilon^2 \int \left( \frac{\hat{\zeta}(y)\xi(y + K)}{\xi(y + 2K)} - i\beta \xi(y + K)\xi(y) \right) = -\Im(\beta \xi(y + K)\xi(y)).
\]

(B.5)
by inserting explicit form of $\xi$ and substituting the explicit value for $\beta$. We get

\[
I = \text{Im} \left( \frac{i e^{i(\alpha)y} H(y + \alpha) \Theta(y) \Theta(y + K)}{\Theta(y) \Theta(y + K)} e^{-i(\alpha)(y+K)} \right)
\]

\[
= \text{Im} \left( \frac{H(y + \alpha) \Theta(y) \Theta(y + K)}{\Theta(y) \Theta(y + K)} \right) = \text{Re} \left( \frac{H(y + \alpha) \Theta(y) \Theta(y + K)}{\Theta(y) \Theta(y + K)} \right)
\]

Now, we will show that the integrand $I$ is odd with respect to the parity. Since $\Theta(y)$ is $2K$ periodic and even with respect to the parity, it is sufficient to show that

\[
\hat{R}(H(y + \alpha) \Theta(y + \alpha + K)) = -\hat{R}(H(y + \alpha) \Theta(y + K)).
\]

Taking $\alpha = K + i\eta$, we can write

\[
\hat{R}(H(y + \alpha) \Theta(y + \alpha + K)) = H(-y + \bar{\eta}) \Theta(-y + \eta + 2K)
\]

\[
= -H(-y - K - i\eta) \Theta(-y + i\eta + 2K)
\]

\[
= H(y + K + i\eta) \Theta(-y + i\eta - 2K)
\]

\[
= -H(y + \alpha) \Theta(y - i\eta + 2K) = -\hat{R}(H(y + \alpha) \Theta(y + \bar{\eta} + K))
\]

\[
= -\hat{R}(H(y + \alpha) \Theta(y + \alpha + K)) \quad (B.6)
\]

Similar result can be obtained for $\alpha = i\eta$. We showed that the integrand $I$ is an odd function. Hence, the first energy correction is vanishing identically,

\[
\int_{-K}^{K} \langle \bar{\psi} \xi \rangle \sigma_2 \left( \begin{array}{c} \psi \\ \xi \end{array} \right) dy = 0. \quad (B.7)
\]

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