Physical Equivalence on Non-Standard Spaces
and
Symmetries on Infinitesimal-Lattice Spaces

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Abstract

Equivalence in physics is discussed on the basis of experimental data accompanied by experimental errors. The introduction of the equivalence being consistent with the mathematical definition is possible only in theories constructed on non-standard number spaces by taking the experimental errors as infinitesimal numbers of the non-standard spaces. Following the idea for the equivalence (the physical equivalence), a new description of space-time in terms of infinitesimal-lattice points on non-standard real number space $^\ast \mathbb{R}$ is proposed. The infinitesimal-lattice space, $^\ast \mathcal{L}$, is represented by the set of points on $^\ast \mathbb{R}$ which are written by $l_n = n \ast \varepsilon$, where the infinitesimal lattice-spacing $\ast \varepsilon$ is determined by a non-standard natural number $^\ast \mathbb{N}$ such that $^\ast \varepsilon \equiv ^\ast \mathbb{N}^{-1}$. By using infinitesimal neighborhoods $(\text{Mon}(r \mid ^\ast \mathcal{L}))$ of real number $r$ on $^\ast \mathcal{L}$ we can make a space $^\ast \mathcal{M}$ which is isomorphic to $\mathcal{R}$ as additive group. Therefore, every point on $(^\ast \mathcal{M})^\mathbb{N}$ automatically has the internal confined-subspace $\text{Mon}(r \mid ^\ast \mathcal{L})$. A field theory on $^\ast \mathcal{L}$ is proposed. To determine a projection from $^\ast \mathcal{L}$ to $^\ast \mathcal{M}$, a fundamental principle based on the physical equivalence is introduced. The physical equivalence is expressed by the totally equal treatment for indistinguishable quantities in our observations. Following the principle, we show that $U(1)$ and $SU(N)$ symmetries on the space $(^\ast \mathcal{M})^\mathbb{N}$ are induced from the internal substructure $(\text{Mon}(r \mid ^\ast \mathcal{L}))^\mathbb{N}$. Quantized state describing configuration space is constructed on $(^\ast \mathcal{M})^\mathbb{N}$. By providing that the subspace $(\text{Mon}(r \mid ^\ast \mathcal{L}))^\mathbb{N}$ is local inertial system of general relativity, infinitesimal distance operators are consistently introduced. We see that Lorentz and general relativistic transformations are also represented by operators which involve the $U(1)$ and $SU(N)$ internal symmetries.

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1. Introduction

–Why are non-standard spaces needed in theories of physics?–

For our recognition derived from observations the judgment of equivalence between two or more phenomena plays a very important role. It is known that the equivalence is rigorously defined in mathematics in terms of the following three conditions:

(1) \( A \sim A \) (reflection)

(2) \( A \sim B \implies B \sim A \), (symmetry)

(3) \( A \sim B, \ B \sim C \implies A \sim C \). (transitivity)

In observations of physics, that is, in experiments, the equivalence (physical equivalence) can be described as follows:

Two phenomena \( A \) and \( B \) are equivalent,

if \( A \) and \( B \) coincide within the experimental errors.

It should be stressed that the physical equivalence is determined by the experimental errors. Furthermore we must recognize that there is no experiments accompanied by no error. We should consider that experimental errors are one of the fundamental observables in our experiments. It is quite hard to understand that there is no theory which involves any description of experimental errors, even though they are very fundamental observables. It is also hard to understand that the question whether such physical equivalence is compatible with the mathematical definition represented by the above three conditions had never been discussed. Let us discuss the question here. We easily see that the first two conditions, that is, reflection and symmetry are compatible with the physical equivalence based on experimental errors. We can, however, easily present examples which break the third condition (transitivity), that is to say, \( A \sim B \) and \( B \sim C \) are satisfied within their errors but \( A \) and \( C \) does not coincide within their errors. This arises from the fact that real numbers which exceed any real numbers can be made from repeated additions of a non-zero real number because of Archimedian property of real number space.
How can we introduce the physical equivalence in theories?

Consistent definition of the physical equivalence is allowed, only when experimental errors are taken as \textit{infinitesimal numbers} \cite{1} in non-standard spaces. This result comes from the fact that any non-zero real numbers cannot be made from any finite sum of infinitesimal numbers. Any repetitions of the transitivity, that is, repeated additions of any infinitesimal numbers does not lead any non-zero real numbers. We can describe the situation as follows;

\[ \forall \varepsilon \in \text{Mon}(0) \text{ and } \forall N \in \mathcal{N} \Rightarrow \varepsilon N \in \text{Mon}(0), \]

where Mon(0) and \(\mathcal{N}\), respectively, stand for the set of all infinitesimal numbers on non-standard spaces and the set of all natural numbers. From the above argument we can conclude that we must make theories, in which the physical equivalence based on experimental errors is described in terms of the mathematically consistent form, on a non-standard space. This is the reason why non-standard spaces are needed in the description of realistic theories based on the physical equivalence. It is once more stressed that such realistic theories must involve the fundamental observables, experimental errors, in the mathematically rigorous way.

An example for the introduction of the physical equivalence in quantum mechanics on non-standard space has been presented in the derivation of decoherence between quantum states for the description of quantum theory of measurements. \cite{2-4} In the theory not only the decoherence required for the wave function collapse but also that for microcanonical ensembles of statistical mechanics (principle of \textit{a priori} equal probabilities) have been simultaneously derived by the realization of the physical equivalence. Though we have many other interesting problems for the construction of theories on non-standard spaces, \cite{5-12} we shall investigate space-time structure and field theory, \cite{11,12} following the idea of the physical equivalence based on experimental errors, in this paper.

To help to see contents of this paper, we present a list of sections here:

1. Introduction
2. On observation of continuity of space-time
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Space-time structure has been studied as one of exciting theme in physics. Whether space-time is continuous (as represented by the set of real numbers \( \mathbb{R} \)) or discrete (as represented by the set of discrete lattice-points) is a fundamental question for the space-time structure. We may ask

"How can we experimentally verify the continuous property of space-time?"

As noted in the first section, we have no experiment accompanied by no error. Taking into account that experimental errors are fundamental observables in physical phenomena, we should understand that the continuity of space-time cannot be directly verified in any experiments. This means that a discrete space-time is sufficient to describe realistic space-time. We, however, know that translational and rotational invariances (including Lorenz invariance) with respect to space-time axes seems to be very fundamental concepts in nature and lattice spaces break them. This disadvantage seem to be very difficult to overcome on usual lattice spaces having a finite lattice-spacing between two neigh-
boring lattice-points. As was discussed in the introduction, experimental errors must be described in terms of infinitesimal numbers on non-standard spaces. On non-standard spaces[1] we can introduce infinitesimal lengths which are smaller than all real numbers except 0. It will be an interesting question whether we can overcome the disadvantage on lattice spaces defined by infinitesimal lattice-spacing. Actually such infinitesimal discreteness cannot be observed in our experiments, where all results must be described by real numbers. This fact indicates that such lattice space-time will possibly be observed as continuous structure. Hereafter we call lattice spaces discretized by infinitesimal numbers *latticespaces and they are denoted by *L.[1] That is to say, such a lattice space *L is constructed as the set of non-standard numbers which are separated by an infinitesimal lattice-spacing *ε on *R (the non-standard extension of R). Lattice-points on *L are defined by
\[ l_n = n *\varepsilon, \quad \text{for} \quad n \in *\mathbb{N} \]
where *\mathbb{N} stands for the non-standard extension of the set of natural numbers \mathbb{N}(=0, 1, 2, 3, \cdots) and consists of all natural numbers and non-standard natural numbers which are infinity. It is transparent that such *L do not contain many of real numbers. There is, however, a possibility that parts of infinitesimal neighborhoods of all real numbers are contained in *L, because it is known that the power of *L is same as that of R.[1] If it is true, there is a possibility that a space constructed from the set of all infinitesimal neighborhoods on *L will be isomorphic to R and translations and rotations on the space can be introduced as same as those on R.[11] In this paper we shall start from the investigation of properties of *L and examine the construction of a new theory on the space-time represented by *L. If we can succeed it, we shall construct a field theory on the new space.[12]

3. Short review of some fundamentals of non-standard space

Here we shall briefly review some fundamental languages of non-standard analysis,[1]
which are not familiar to physicists but needed in the argument of this paper. Readers
who are familiar to non-standard analysis may skip this section and go to the next section.

(i) Free ultra-filters
The set of real numbers \( \mathbb{R} \) can be extended to the set of numbers \( (\mathbb{R}') \) containing infinities in terms of free ultra-filters(\( \mathcal{F} \)) over \( \mathcal{N} \) (\( \mathcal{N} = (0,1,2,...) \) denotes the set of natural numbers). The free ultra-filters satisfy the following properties:

(a) \( \mathcal{N} \in \mathcal{F} \), \( \phi \) (empty set) \( \notin \mathcal{F} \),
(b) \( A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F} \),
(c) \( A \in \mathcal{F} \), \( A \subseteq B \implies B \in \mathcal{F} \),
(d) \( \mathcal{F} \) contains no finite set, (the filter having this property is called free),
(e) either \( E \in \mathcal{F} \) or \( \mathcal{N} - E \in \mathcal{F} \) for \( \forall E \subseteq \mathcal{N} \) (the filter having this property is called ultra-filter over \( \mathcal{N} \)).

Hereafter filters and ultra-filters always mean free ultra-filters.

(ii) Equivalence in terms of free ultra-filter and non-standard extension
We can construct the non-standard extension of \( \mathbb{R} \) by introducing an equivalence relation on sequences in \( \mathbb{R}^\mathcal{N} \) by means of an ultra-filter \( \mathcal{F} \). The equivalence relation, \( f \sim \mathcal{F} \), is defined as follows;

\[
f \sim_{\mathcal{F}} g
\]

if and only if \( \{ n \in \mathcal{N} | f(n) = g(n) \} \in \mathcal{F} \), where \( f \) and \( g \) are, respectively, represented by ultra-product

\[
f = \prod_{n \in \mathcal{N}} f(n), \quad g = \prod_{n \in \mathcal{N}} g(n).
\]

Note that the sequences associated with the equivalence relation may be expressed by using ultra-powers

\[
\prod_{n \in \mathcal{N}} f(n) / \sim_{\mathcal{F}}.
\]
We may write the non-standard extension of $\mathcal{R}$ in terms of the quotient space

$$^*\mathcal{R} = \mathcal{R}^\mathcal{N} / \sim_{\mathcal{F}}. \quad (4)$$

We also have non-standard extensions of $\mathcal{N}$, $\mathcal{Z}$ (the set of integers), $\mathcal{Q}$ (the set of rational numbers), $\mathcal{C}$ (the set of complex numbers) and so on, which are denoted as $^*\mathcal{N}$, $^*\mathcal{Z}$, $^*\mathcal{Q}$, $^*\mathcal{C}$ and so forth, respectively. It is shown that $\mathcal{R} \subset ^*\mathcal{R}$ and the magnitudes of the non-standard natural numbers, $\forall^*N \in ^*\mathcal{N} - \mathcal{N}$, are infinity.

(iii) Definitions of $\leq$, $+$ and $\times$

We can introduce the order $\leq$ between two ultra-products $f$ and $g$ as follows;

if and only if $\{n \in \mathcal{N} | f(n) \leq g(n)\} \in \mathcal{F},$

$$f \leq g. \quad (5)$$

It is shown that $^*\mathcal{N}$, $^*\mathcal{Z}$, $^*\mathcal{Q}$ and $^*\mathcal{R}$ are totally ordered sets.

Sum and multiplication are defined by

$$f + g = \prod_{n \in \mathcal{N}} (f(n) + g(n)), \quad f \times g = \prod_{n \in \mathcal{N}} (f(n) \times g(n)). \quad (6)$$

An example of $^*N \in ^*\mathcal{N}$ is given by using the ultra-product

$$^*N = \prod_{n \in \mathcal{N}} (n + 1). \quad (7)$$

Following the order $\leq$ defined by free ultra-filters $\mathcal{F}$ and the properties (d) and (e) of $\mathcal{F}$, it is obvious that

$$^*N > N, \quad \text{for } \forall N \in \mathcal{N} \quad (8)$$

because the set $\{n \in \mathcal{N} | n + 1 \leq N\} \not\in \mathcal{F}$ is a finite set on $\mathcal{N}$, whereas that of $\{n \in \mathcal{N} | n + 1 > N\} \in \mathcal{F}$ is an infinite set on $\mathcal{N}$.

(iv) Standard part map (st-map)
We have a projection of every finite number(*r) of *R to a unique element(r) of R, which is called as the standard part map(st-map) and written by

\[ \text{st}(*r) = r. \] (9)

All infinitesimal numbers are mapped at zero.

(v) Monad of \( r \in R \) (Mon(\( r \)))

Each real number \( r \in R \) has its own infinitesimal neighborhood Mon(\( r \)) which is called monad of \( r \) and defined by the set of \(*r \in *R\) satisfying

\[ \text{st}(*r - r) = 0. \] (10)

In other words it may be represented by the set of \(*r \in *R\) satisfying \( \text{st}(*r) = r \). We see that Mon(0) contains all infinitesimals. Note that the map of all elements being finite into monads of real numbers is unique, that is, no element of \(*R\) cannot belong two or more monads simultaneously such that

\[ \text{Mon}(r) \cap \text{Mon}(r') = \phi, \quad \text{for } r \neq r' \text{ and } r, r' \in R. \]

(vi) Powers of \(*N\) and \(*R\)

Powers of \(*N\) and \(*R\) are same as that of \( R \), that is, \(*N\) and \(*R\) have the same continuous power as that of \( R \). In fact \(*R_{\mathbb{R}}/\text{Mon}(0) \cong R\) as field is shown,[1] where \(*R_{\mathbb{R}}\) is the set of elements of \(*R\), of which elements \(*r\) are satisfied by the relation \( \text{st}(*r) \in R \).

4. Infinitesimal-lattice spaces \(*L\)

Let us take a non-standard natural number

\[ *N \in *N - N, \] (11)
which is an infinity.[1] We take the closed set \([- *N/2, *N/2]\) on \(*\mathcal{R}\) and put \(( *N)^2 \) + 1 points with an equal spacing \(*\varepsilon = *N^{-1}\) on the set. For the convenience of the following discussions \(*N\) is chosen as \(*N/2 \in *\mathcal{N}\). The length between two neighboring points is \(*\varepsilon\) which is an infinitesimal, i.e. \(*\epsilon \in \text{Mon}(0)\). Let us consider the set of the infinitesimal lattice-points \(*\mathcal{L},[11]\) which consists of these \(( *N)^2 \) + 1 discrete points on the closed set. Lattice-points on \(*\mathcal{L}\) are written by

\[
    l_n = n *\varepsilon, \quad \text{(12)}
\]

where \(n \in *\mathcal{Z}\) and fulfil the relation

\[
    - ( *N)^2/2 \leq n \leq ( *N)^2/2. \quad \text{(13)}
\]

We put two end points as the same point, i.e.

\[
    l(-( *N)^2/2) = *N = l(-(- *N)^2/2) = -*N. \quad \text{(14)}
\]

This choice corresponds to the choice of periodic boundary which is required for the introduction of translations on \(*\mathcal{L}\). We may consider \(*\mathcal{L}\) as the set of \(( *N)^2\) points with the equal spacing \(*\varepsilon\) on the circle of the radius \(*N/2\pi\). From the process of the construction of \(*\mathcal{L}\) it is transparent that

\[
    *\mathcal{L} \nsubseteq \mathcal{R}. \quad \text{(15)}
\]

Actually it is obvious that all irrational numbers of \(\mathcal{R}\) are not contained in \(*\mathcal{L}\), because \(*N\) is taken as an element of \(*\mathcal{N}\) and \(n *\epsilon = n/ *N\) is an element of \(*\mathcal{Z}\).

Let us show a theorem:

\textit{Monads of all real numbers, Mon}(r) \(\forall r \in \mathcal{R}\), have their elements on \(*\mathcal{L}\).

\textit{Proof:} Take a real number \(r \in \mathcal{R}\). The number \(r\) is contained in the closed set \([- *N/2, *N/2]\) on \(*\mathcal{R}\), because \(*N\) is an infinity of \(*\mathcal{N}\) and then \([- *N/2, *N/2]\) \(\supset \mathcal{R}\). Since the lattice-points of \(*\mathcal{L}\) divide the closed set into \(( *N)^2\) regions of which length is \(*\varepsilon\), the real number \(r\) must be on a lattice-point or between two neighboring lattice-points.
whose distance is \( *\varepsilon \). We can, therefore, find out a non-standard integer \( N_r \) fulfilling the following relation;

\[
N_r *\varepsilon \leq r < (N_r + 1) *\varepsilon,
\]

where \(|N_r| \in *N \setminus N\). The difference \( r - N_r *\varepsilon \) is an infinitesimal number smaller than \( *\varepsilon \). Thus we can define the infinitesimal neighborhood of \( r \) on \( *\mathcal{L} \) such that

\[
\text{Mon}(r \mid *\mathcal{L}) \equiv \{l_n(r) = (N_r + n) *\varepsilon \mid n \in *\mathcal{Z}, n *\varepsilon \in \text{Mon}(0)\}.
\]

The relation

\[
\text{st}(l_n(r)) = r
\]

is obvious. The theorem has been proved. Hereafter we shall call \( \text{Mon}(r \mid *\mathcal{L}) \) and its elements \( l_n(r) \) monad lattice-space (\( *\mathcal{L}\)-monad) and monad lattice-points, respectively.

From the above argument we see that there is one-to-one correspondence between \( \mathcal{R} \) and

\[
*\mathcal{L}_{l(\mathcal{R})} \equiv \{l_0(r) \mid r \in \mathcal{R}\}
\]

(the set of \( l_0(r) \) for \( \forall r \in \mathcal{R} \)) with respect to the correspondence \( r \leftrightarrow l_0(r) \). Note also that from the definition of monad we have the relations

\[
\text{Mon}(r \mid *\mathcal{L}) \cap \text{Mon}(r' \mid *\mathcal{L}) = \phi, \quad \text{for } r \neq r', r, r' \in \mathcal{R}.
\]

Magnitudes of lattice-points contained in all of the monad lattice-space \( \text{Mon}(r \mid *\mathcal{L}) \) for \( \forall r \in \mathcal{R} \) are not infinity, because they are elements of monads of real numbers. We shall write the set of all these finite lattice-points by

\[
*\mathcal{L}_{\mathcal{R}} \equiv \{l_n(r) \mid r \in \mathcal{R}, n \in *\mathcal{Z}, n *\varepsilon \in \text{Mon}(0)\} = \cup_{r \in \mathcal{R}} \text{Mon}(r \mid *\mathcal{L}).
\]

The sets \( *\mathcal{L}_{\mathcal{R}} \) and \( \text{Mon}(0 \mid *\mathcal{L}) \) are additive groups. Note here that \( *\mathcal{L}_{l(\mathcal{R})} \) is not an additive group, because in general \( l_0(r) + l_0(r') \neq l_0(r + r') \) possibly happens, that is, \( N_{r+r'} \) is not always equal to \( N_r + N_{r'} \) but possibly equal to \( N_r + N_{r'} + 1 \). (See the definition of \( N_r \) given in (16).) It is apparent that

\[
*\mathcal{L}_{\mathcal{R}} = *\mathcal{L}_{l(\mathcal{R})} + \text{Mon}(0 \mid *\mathcal{L}) \quad \text{and} \quad *\mathcal{L}_{l(\mathcal{R})} \cap \text{Mon}(0 \mid *\mathcal{L}) = \{0\}.
\]

10
Let us introduce the quotient set of \( {^*\mathcal{L}_R} \) by Mon(0| \( {^*\mathcal{L}} \)) as

\[
{^*\mathcal{M}} \equiv {^*\mathcal{L}_R}/\text{Mon}(0| {^*\mathcal{L}}).
\]

From one-to-one correspondence between \( \mathcal{R} \) and \( {^*\mathcal{L}_l(\mathcal{R})} \) and the relations (20) we see that there is one-to-one correspondence between \( \mathcal{R} \) and \( {^*\mathcal{M}} \), and thus

\[
{^*\mathcal{M}} \cong \mathcal{R} \tag{21}
\]

as additive groups, where the addition on \( {^*\mathcal{M}} \) may be described by \( \text{st-map} \) of the addition on \( {^*\mathcal{L}_R} \) such that \( \text{st}(l_n(r) + l_m(r')) = r + r' \) for \( \forall l_n(r) \in \text{Mon}(r| {^*\mathcal{L}}) \) and \( \forall l_m(r') \in \text{Mon}(r'| {^*\mathcal{L}}) \) with \( r, r' \in \mathcal{R} \).

We can construct the same quotient set where the zero point of the subset \( {^*\mathcal{L}_R} \) is taken at an arbitrary point of \( {^*\mathcal{L}} \). That is to say, by using the relative distance \( l_{N_m} \) between an arbitrary point \( l_m \) and the origin \( l_N \) as

\[
l_{N_m} \equiv l_m - l_N = (m - N) \, \varepsilon,
\]

we can proceed the same argument for constructing \( {^*\mathcal{M}} \). This means that \( {^*\mathcal{L}} \) contains infinite number of subsets which are congruent to \( {^*\mathcal{M}} \). When we consider \( {^*\mathcal{L}} \) on the circle with the radius \( N/2\pi \) on \( {^*\mathcal{R}^2} \), the angle of the sector including one \( {^*\mathcal{M}} \) is infinitesimal. This means that \( {^*\mathcal{M}} \) can be taken as a straight line on two dimensional real space \( \mathcal{R}^2 \), even if it is put on the circle of \( {^*\mathcal{R}^2} \).

Finally we summarize the notations newly introduced in this section for the convenience in the following discussions:

- \( {^*\mathcal{L}} \) = the set of all infinitesimal lattice-points, (infinitesimal lattice-space)
- \( \text{Mon}(r| {^*\mathcal{L}}) \) = the set of lattice points which are elements of Mon(\( r \)) for \( r \in \mathcal{R} \), (Monad lattice-space)
- \( {^*\mathcal{L}_R} \) = the set of lattice-points which are elements of Mon(\( r| {^*\mathcal{L}} \)) for \( \forall r \in \mathcal{R} \), (finite infinitesimal lattice-space)
- \( {^*\mathcal{M}} = {^*\mathcal{L}_R}/\text{Mon}(0| {^*\mathcal{L}}) \), (observed space).
5. Translations and rotations on $\ast\mathcal{M}$

Since $\ast\mathcal{M} \cong \mathcal{R}$ as additive groups has been proved in the last section, it is obvious that translations and rotations on $\ast\mathcal{M}$ can be taken as same as those on $\mathcal{R}$. We shall here study translations and rotations on the sub-lattice space $\ast\mathcal{L}_\mathcal{R}$ and construct them on $\ast\mathcal{M}$ explicitly. (See Ref. 11.)

5.1 Translations

In general a translation $\ast\hat{p}_m$ on $\ast\mathcal{L}$ is represented by the following map from $\ast\mathcal{L}$ to $\ast\mathcal{L}$:

$$ \ast\hat{p}_m l_n = l_{n+m}, \text{ for } n, m \in \ast\mathbb{Z}. $$

(23)

The displacement length by this translation is

$$ d_m \equiv l_{n+m} - l_n = l_m = m \ast\varepsilon. $$

(24)

Let us study only finite translations restricted by $\text{st}(d_m) \in \mathcal{R}$. (25)

Note that the subset of $\ast\mathcal{L}$, i.e., $\ast\mathcal{L}_\mathcal{R}$, is mapped on to $\ast\mathcal{L}_\mathcal{R}$ by these finite translations. We may, therefore, consider that these finite translations represent translations on $\ast\mathcal{L}_\mathcal{R}$. We also see that under these translations all the elements of $\text{Mon}(r | \ast\mathcal{L})$ are replaced on those of $\text{Mon}(r' | \ast\mathcal{L})$, where

$$ r' = \text{st}(r + d_m). $$

(26)

Let us show that the elements of $\text{Mon}(r | \ast\mathcal{L})$ and those of $\text{Mon}(r' | \ast\mathcal{L})$ have one-to-one correspondence. The displacement of $\text{Mon}(r | \ast\mathcal{L})$ to $\text{Mon}(r' | \ast\mathcal{L})$ for $r, r' \in \mathcal{R}$ is described by

$$ d_{rr'} = (N_{r'} - N_r) \ast\varepsilon, $$

(27)

where $N_r \ast\varepsilon \leq r < (N_r + 1) \ast\varepsilon$ and $N_{r'} \ast\varepsilon \leq r' < (N_{r'} + 1) \ast\varepsilon$ with $N_r, N_{r'} \in \ast\mathbb{Z}$. It is trivial that $\text{st}(d_{rr'}) = r' - r$ is finite. An element of $\text{Mon}(r | \ast\mathcal{L})$, $l_m(r) = (N_r + m) \ast\varepsilon$,
is replaced on an element of \( \text{Mon}(r'|L) \), \( l_m(r') = (N_{r'} + m) \varepsilon \), by the translation. Considering the inverse of the translation, which is described by the displacement \(-d_{rr'}\), one-to-one correspondence between \( \text{Mon}(r|L) \) and \( \text{Mon}(r'|L) \) is obvious.

We can change \( d_{rr'} \) by infinitesimal length \( \Delta d_k = k \varepsilon \) such that \( d_{rr',k} \equiv d_{rr'} + \Delta d_k \), where \( k \) must be taken as integers of \(*\mathbb{Z}\) satisfying the relation \( k \varepsilon \in \text{Mon}(0) \). Note that \( \Delta d_k \) does not depend on \( r \) and \( r' \). One-to-one correspondence is not affected by these infinitesimal changes. This fact means that all infinitesimal translations\( (*\hat{p}_0(*L_R)) \) described by \( \Delta d_k \) are mapped on the zero translation on \(*M\). Thus we see that all the translations from \( \text{Mon}(r|L) \) to \( \text{Mon}(r'|L) \) on \(*L_R\), which produce the displacement \( st(d_{rr'}) \) on \(*M\), are expressed by

\[
*\hat{p}_m(*L_R) \equiv *\hat{p}_m + *\hat{p}_0(*L_R),
\]

where \( m = N_{r'} - N_r \). The quotient of the set of finite transrates\( \{ *\hat{p}_m(*L_R) \} \) on \(*L\) by \(*\hat{p}_0(*L_R)\),

\[
\{ *\hat{p}_m(*L_R) \} / *\hat{p}_0(*L_R),
\]

represents translations on \(*M\), which corresponds to translations on \(R\).

Since translations in higher dimensional spaces are trivial, we do not discuss it here.

### 5.2 Rotations

Let us study rotations on two dimensional spaces \( (*L)^2 \), especially, rotations whose center is put at the origin of \( (*L)^2 \), i.e., \( \vec{l}_0(0) = (0, 0) \). A rotation in two dimensional real space \(R^2\), of which center is at the origin, is represented by one parameter, i.e., a rotation angle \( \theta \). Under the rotation a point on \(R^2\) written by \( \vec{r} = (r \cos \alpha, r \sin \alpha) \) is moved to \( \vec{r}' = (r \cos(\alpha + \theta), r \sin(\alpha + \theta)) \), where \( r \in \mathcal{R} \) and \( 0 \leq \alpha, \theta < 2\pi \). The difference between the two vectors is

\[
\vec{r}' - \vec{r} \equiv \vec{d}_\theta(\vec{r}) = (d^x_\theta(\vec{r}), d^y_\theta(\vec{r})),
\]

where

\[
d^x_\theta(\vec{r}) = r(\cos(\alpha + \theta) - \cos \alpha), \quad d^y_\theta(\vec{r}) = r(\sin(\alpha + \theta) - \sin \alpha).
\]
This means that the rotation of one point can be described by a displacement in the two dimensional space expressed by \( \tilde{d}_\theta(\vec{r}) \) such that

\[
\hat{R}_\theta \vec{r} \equiv \vec{r}' = \vec{r} + \tilde{d}_\theta(\vec{r}).
\]  

We see that the rotation \( \hat{R}_\theta \) for the rotation angle \( \theta \) in \( \mathbb{R}^2 \) can be described by a map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) producing the displacement \( \tilde{d}_\theta(\vec{r}) \) for every vector \( \vec{r} \).

On two dimensional infinitesimal-lattice subspace \((\ast \mathcal{L}_\mathcal{R})^2 \), let us consider a map \( \ast \hat{R}_\theta \) which transfers all vectors on \((\ast \mathcal{L}_\mathcal{R})^2 \) to those on \((\ast \mathcal{L}_\mathcal{R})^2 \) such that a vector \( \vec{l}_0(\vec{r}) = (l_0(r \cos \alpha), l_0(r \sin \alpha)) \) is transferred to \( \vec{l}_0(\vec{r}') = (l_0(r \cos (\alpha + \theta)), l_0(r \sin (\alpha + \theta))) \). \( l_0(x) \) is defined in (17.). The displacement vector produced by the map is given by

\[
\ast \hat{R}_\theta \vec{l}_0(\vec{r}) - \vec{l}_0(\vec{r}) \equiv \ast \tilde{d}_\theta(\vec{r}) = (\ast d_x^\theta(\vec{r}), \ast d_y^\theta(\vec{r})),
\]  

where

\[
\ast d_x^\theta(\vec{r}) = N_x^\theta(\vec{r}) \ast \varepsilon, \quad \ast d_y^\theta(\vec{r}) = N_y^\theta(\vec{r}) \ast \varepsilon.
\]  

In (33) the integers \( N_x^\theta(\vec{r}), N_y^\theta(\vec{r}) \in \ast \mathbb{Z} \) must fulfill the relations

\[
N_x^\theta(\vec{r}) \ast \varepsilon \leq r(\cos(\alpha + \theta) - \cos \alpha) < (N_x^\theta(\vec{r}) + 1) \ast \varepsilon,
\]

\[
N_y^\theta(\vec{r}) \ast \varepsilon \leq r(\sin(\alpha + \theta) - \sin \alpha) < (N_y^\theta(\vec{r}) + 1) \ast \varepsilon.
\]  

From one-to-one correspondence between \( \text{Mon}(r \cos \alpha | \ast \mathcal{L}) \) and \( \text{Mon}(r \cos(\alpha + \theta) | \ast \mathcal{L}) \) and that between \( \text{Mon}(r \sin \alpha | \ast \mathcal{L}) \) and \( \text{Mon}(r \sin(\alpha + \theta) | \ast \mathcal{L}) \) with respect to the above transfer (see the argument of §5.1), one-to-one correspondence between \( \text{Mon}(\vec{r} | \ast \mathcal{L}) \) and \( \text{Mon}(\vec{r} | \ast \mathcal{L}) \) is self-evident, where \( \text{Mon}(\vec{r} | \ast \mathcal{L}) \) is defined by the set of lattice-points \( \ast \vec{r} = (\ast x, \ast y) \) satisfying the relations \( \ast x \in \text{Mon}(r \cos \alpha | \ast \mathcal{L}) \) and \( \ast y \in \text{Mon}(r \sin \alpha | \ast \mathcal{L}) \). Since the magnitude of \( \vec{r} \) does not change under this map \( \ast \hat{R}_\theta \), i.e., \( |\vec{r}| = |\vec{r}'| \), all points on \((\ast \mathcal{L}_\mathcal{R})^2 \) are mapped on \((\ast \mathcal{L}_\mathcal{R})^2 \). We may consider that these maps \( \ast \hat{R}_\theta \) for \( \forall \theta \in \mathcal{R} \) \((0 \leq \theta < 2\pi) \) represent rotations for real angles on \((\ast \mathcal{L}_\mathcal{R})^2 \), which correspond to the rotations \( \hat{R}_\theta \) on \( \mathbb{R}^2 \).

As was shown in §5.1, \( \ast d_x^\theta(\vec{r}) \) and \( \ast d_y^\theta(\vec{r}) \) may be added by infinitesimal displacements as \( k_x \ast \varepsilon, k_y \ast \varepsilon \in \text{Mon}(0) \) for \( k_x, k_y \in \ast \mathbb{Z} \). Note that these infinitesimal displacements do
not depend on vectors $\vec{r}$. Maps producing these infinitesimal displacements represent no rotation on $(\mathcal{M})^2$ and we write the set of these maps by $\hat{R}_0(\mathcal{L}_\mathcal{R})$. An arbitrary rotation $\hat{R}_\theta(\mathcal{L}_\mathcal{R})$ on $(\mathcal{L}_\mathcal{R})^2$ is represented by a map producing the displacement $\hat{d}_\theta(\vec{r}) + \Delta \hat{d}_\theta(\vec{k})$ with $\Delta \hat{d}_\theta(\vec{k}) = (k^x, k^y)$ for each vector $\vec{l}_n(\vec{r})$. Actually $\hat{R}_\theta(\mathcal{L}_\mathcal{R})$ on $(\mathcal{L}_\mathcal{R})^2$, which produce rotations for the fixed angle $\theta \in \mathcal{R}$ on $(\mathcal{M})^2$, are expressed by the sum

$$\hat{R}_\theta(\mathcal{L}_\mathcal{R}) = \hat{R}_\theta + \hat{R}_0(\mathcal{L}_\mathcal{R}),$$  \hspace{1cm} (35)$$

where $\hat{R}_\theta$ is defined in (32). Thus the rotations on $(\mathcal{M})^2$ are represented by

$$\{\hat{R}_\theta(\mathcal{L}_\mathcal{R})\}/\hat{R}_0(\mathcal{L}_\mathcal{R}).$$  \hspace{1cm} (36)$$

where $\{\hat{R}_\theta(\mathcal{L}_\mathcal{R})\}$ stands for the set of rotations $\hat{R}_\theta(\mathcal{L}_\mathcal{R})$ on $(\mathcal{L}_\mathcal{R})^2$. It is obvious that these rotations make a group.

The extension of these rotations to those in higher dimensional spaces is trivial.

6. Confined fractal-like property of $\mathcal{L}$

We have shown that $\mathcal{M} \approx \mathcal{R}$. We, however, know that there is a large difference between them, that is, $\mathcal{M}$ is constructed from the monad lattice-spaces $\text{Mon}(r|\mathcal{L})$ which contain infinite number of different lattice-points on $\mathcal{L}_\mathcal{R}$. In fact the power of $\text{Mon}(r|\mathcal{L})$ can be not countable but continuous in general. Let us study the structure of $\text{Mon}(r|\mathcal{L})$ in more details. We can write the elements of $\text{Mon}(r|\mathcal{L})$ as

$$l_n(r) = (N_r + n) \varepsilon,$$  \hspace{1cm} (37)$$

where, even if $N_r$ is fixed, $n$ can be elements of $\mathcal{N} - \mathcal{N}$, which satisfy the relation $n \varepsilon \in \text{Mon}(0)$. There are a lot of different possibilities depending on the choice of the original non-standard natural number $\hat{N} \in \mathcal{N} - \mathcal{N}$. We shall here show two examples, that is, one has an infinite series of $\mathcal{M}$ and the other a finite series. (See Ref.11.)

6.1 Infinite series of $\mathcal{M}$
Define an infinite series of infinite non-standard natural numbers by the following ultra-products:

\[ *N_M \equiv \prod_{n \in \mathcal{N}} \alpha_n^{(M)}, \quad \text{for } M \in \mathcal{N} \quad (38) \]

where \( \alpha_n^{(M)} = 1 \) for \( 0 \leq n \leq M \) and \( \alpha_n^{(M)} = (n + 1)^{n-M} \) for \( n > M \). Following the definition of the order \( > \) for ultra-products, we see that all of \( *N_M \) are infinity and the order is given by

\[ *N_0 > *N_1 > *N_2 > \cdots. \quad (39) \]

Then we have an infinite series of infinitesimal numbers

\[ *\varepsilon_0 < *\varepsilon_1 < *\varepsilon_2 < \cdots, \quad (40) \]

where \( *\varepsilon_M \equiv (*N_M)^{-1} \). We can also prove that ratios

\[ *\lambda_M \equiv \frac{*N_{M-1}}{*N_M}, \quad \text{for } M \geq 1 \quad (41) \]

are infinities of \( *\mathcal{N} \). Since \( *N_0 \) is an element of natural numbers \( *\mathcal{N} - \mathcal{N} \), we can take

\[ *\varepsilon = *\varepsilon_0. \quad (42) \]

Here let us consider the following rescaling for the lattice points;

\[ l_n(r) - l_0(r) = n *\varepsilon_0 \equiv *\lambda_1^{-1}l_n^{(1)}, \quad (43) \]

where

\[ l_n^{(1)} = n *\varepsilon_1. \quad (44) \]

Note that \( l_n^{(1)} \) is independent of \( r \). Even if the relation \( n *\varepsilon_0 \in \text{Mon}(0) \) must be satisfied, the set of \( n \in *\mathcal{N} \) satisfying the relation contains non-standard integers such that

\[ n_m^{(1)} \equiv m \times *N_1 \in *\mathcal{Z}, \quad \text{for } \forall m \in \mathcal{Z}. \quad (45) \]

It is trivial that the relation is satisfied as

\[ n_m^{(1)} *\varepsilon_0 = m\lambda_1^{-1} \in \text{Mon}(0). \quad (46) \]
It is also obvious that
\[ n_m \varepsilon_1 = m \in \mathcal{Z}. \] (47)
Thus we can see that the set of \( \forall l_n^{(1)} \), \( \mathcal{L}_R^{(1)} \equiv \{ l_n^{(1)} = n \varepsilon_1 \mid n \in \mathcal{Z}, n \varepsilon_0 \in \text{Mon}(0) \} \), is an infinitesimal-lattice space with the lattice-length \( \varepsilon_1 \). In fact the set \( \mathcal{L}_R^{(1)} \) is constructed from the elements of Mon\((r| \mathcal{L})\) rescaled by the factor \( \lambda_1 \). From the facts that \( \mathcal{L}_R^{(1)} \) contains all integers, Archimedian property certifies the existence of natural numbers \( m \geq |r| \) for \( \forall r \in \mathcal{R} \) and the distance between two neighboring lattice-points is an infinitesimal number \( \varepsilon_1 \), we can find an element of \( \mathcal{N} - \mathcal{N}, N_r^{(1)} \), satisfying the relation
\[ N_r^{(1)} \varepsilon_1 \leq r^{(1)} < (N_r^{(1)} + 1) \varepsilon_1, \quad \text{for} \quad \forall r^{(1)} \in \mathcal{R}. \] (48)
Following the same argument for the construction of \( \mathcal{M} \) given in §4, we can introduce the monad of \( r^{(1)}, \text{Mon}(r^{(1)}| \mathcal{L}_R^{(1)}) \), by the set of the following lattice-points on \( \mathcal{L}_R^{(1)} \):
\[ l_n^{(1)}(r^{(1)}) = (N_r^{(1)} + n^{(1)}) \varepsilon_1, \] (49)
where \( n^{(1)} \in \mathcal{Z} \) and \( \text{st}(n^{(1)} \varepsilon_1) = 0 \) must be fulfilled. It is obvious that Mon\((r^{(1)}| \mathcal{L}_R^{(1)})\) contains an infinite number of elements. Now we can define \( \mathcal{M}^{(1)} \) by the set
\[ \mathcal{M}^{(1)} \equiv \mathcal{L}_R^{(1)}/\text{Mon}(0| \mathcal{L}_R^{(1)}). \] (50)
The relation
\[ \mathcal{M}^{(1)} \cong \mathcal{M} \cong \mathcal{R} \] (51)
as additive groups is obvious. Thus translations and rotations on \( N \)-dimensional space \( (\mathcal{M}^{(1)})^N \) are described as same as those of \( (\mathcal{M})^N \). We can conclude that every monad lattice-space Mon\((r| \mathcal{L})\) for \( \forall r \in \mathcal{R} \) contain the same space \( \mathcal{M}^{(1)} \) by means of the same scale transformation.

The second rescaling by using \( \mathcal{N}_2 \) is carried out by following the same procedure presented above. We can perform the second rescaling by
\[ l_n^{(2)} \equiv \lambda_2 (t_n^{(1)}(r^{(1)}) - t_0^{(1)}(r^{(1})), \] (52)

17
The derivations of \( \ast \mathcal{L}_R^{(2)} \) and \( \ast \mathcal{M}^{(2)} \equiv \ast \mathcal{L}_R^{(2)}/\text{Mon}(0| \ast \mathcal{L}_R^{(2)}) \) are same as those given in the previous argument, and then we have

\[
\ast \mathcal{M}^{(2)} \cong \ast \mathcal{M}^{(1)} \cong \ast \mathcal{M} \cong \mathcal{R}. \quad (53)
\]

By using the infinite series of \( \ast N_M \) we can proceed the same argument for the construction of \( \ast \mathcal{M}^{(M)} \) and thus we obtain the infinite series of sets isomorphic to \( \mathcal{R} \) as additive group such that

\[
\mathcal{R} \cong \ast \mathcal{M} \cong \ast \mathcal{M}^{(1)} \cong \ldots \cong \ast \mathcal{M}^{(M)} \cong \ldots. \quad (54)
\]

### 6.2 Finite series of \( \ast \mathcal{M} \)

We definite a finite series of infinite numbers

\[
\ast N_l^L \equiv \prod_{n \in \mathcal{N}} (n + 1)^{L-l}, \quad \text{for } l = 0, 1, 2, \ldots, L - 1 \quad (55)
\]

where \( \ast N_l^L \in \ast \mathcal{N} - \mathcal{N} \). We also see that

\[
\ast \lambda_l \equiv \frac{\ast N_{l-1}^L}{\ast N_l^L} = \prod_{n \in \mathcal{N}} (n + 1) \in \ast \mathcal{N} - \mathcal{N}. \quad (56)
\]

Following the same argument as that of the infinite series, we can construct a finite series of sets isomorphic to \( \mathcal{R} \) as additive group

\[
\mathcal{R} \cong \ast \mathcal{M} \cong \ast \mathcal{M}^{(1)} \cong \ldots \cong \ast \mathcal{M}^{(L-1)}. \quad (57)
\]

We have many different examples for deriving such series. Note that \( \text{Mon}(r| \ast \mathcal{L}) \) does not have the structure discussed above, if \( \ast N \) defined by (7) is taken, that is, the case for \( L = 1 \) in the above argument.

From the above arguments we understand that the set of finite lattice-points on \( \ast \mathcal{L} \), i.e., \( \ast \mathcal{L}_R \) contains series of spaces \( \ast \mathcal{M}^{(M)} \cong \mathcal{R} \), which are constructed by means of relevant series of rescalings. Thus we may say that \( \ast \mathcal{L}_R \) has a property similar to so-called fractal property. Note that the scaling parameter \( \ast \lambda_n \) in the rescaling

\[
2_{(n-1)}^{(n)}(r) - 2_{(n-1)}^{(n)}(r) = \ast \lambda_n (2_{(n-1)}^{(n-1)}(r) - 2_{(n-2)}^{(n-1)}(r)), \quad (58)
\]
is infinity. So the similarity between fractal property and the structure of \( ^*\mathcal{L}_R \) cannot directly define on \( \mathcal{R} \). We may consider that the infinitesimal fractal-like property of \( ^*\mathcal{L}_R \) cannot directly be observed on \( ^*\mathcal{M} \). This means that the infinitesimal fractal-like property is confined on \( ^*\mathcal{M} \). Note here that \( ^*\mathcal{L} \) itself contains infinite number of the same sets as \( ^*\mathcal{L}_R \). Then we may say that \( ^*\mathcal{L} \) itself has the infinitesimal fractal-like property.

7. Construction of fields on \( ^*\mathcal{M} \)

Let us construct fields on \( ^*\mathcal{M} \). In the construction of field theory on \( ^*\mathcal{M} \) we follow the next two fundamental principles:[12]

(I) All definitions and evaluations should be carried out on the original space \( ^*\mathcal{L} \).

(II) In definitions of any kinds of physical quantities on \( ^*\mathcal{M} \), all the fields contained in the same monad lattice-space Mon(\( r\mid^*\mathcal{L} \)) should be treated equivalently. (Principle of physical equivalence)

It should be noted that the principle (I) means that theories which we will make on \( ^*\mathcal{L} \) is generally not the same as any extensions of standard theories which have been constructed on \( \mathcal{R} \). The principle (I) also tells us that all physical expectation values on \( \mathcal{R} \) are obtained by taking standard part maps (maps from \( ^*\mathcal{R} \) to \( \mathcal{R} \))[1] of results calculated on \( ^*\mathcal{L} \). The principle (II) is considered as the realization of the equivalence for indistinguishable quantities in quantum mechanics on non-standard space.[3] This principle, principle of physical equivalence, determines projections of physical systems defined on \( ^*\mathcal{L} \) to those defined on \( ^*\mathcal{M} \). Taking account of the fact that all points contained in the same monad lattice-space Mon(\( r\mid^*\mathcal{L} \)) cannot be experimentally distinguished, the equivalent treatment with respect to all quantities defined on these indistinguishable points is a natural requirement in the construction of theories on \( ^*\mathcal{M} \).
7.1 Fields on $^*\mathcal{L}$

Let us define two fields $A(m)$ and $\bar{A}(m)$ on every lattice point $r(m)$ on $^*\mathcal{L}$, which follow the commutation relations

$$[A(m), \bar{A}(m')] = \delta_{mm'} \quad \text{and} \quad \text{others} = 0. \quad (59)$$

The vacuum $|^0\rangle = \prod_m |0\rangle_m$ and the dual vacuum $<^0| = \prod_m m < \bar{0}\rangle$ are defined by

$$A(m)|0\rangle_m = 0 \quad \text{and} \quad m < \bar{0}|\bar{A}(m) = 0 \quad (60)$$

with $m < \bar{0}|0\rangle_m = 1$. The fields $A(m)$ and $\bar{A}(m)$ operate only on the vacuum $|0\rangle_m$ and the dual vacuum $m < \bar{0}|$. Following the principle (I), all expectation values are imposed to be calculated on $^*\mathcal{L}$ such that

$$<^0|\hat{O}(\{A\}, \{\bar{A}\})|^0\rangle \in ^*\mathcal{R},$$

where $\hat{O}$ is operator constructed from the sets of the fields $A(m)$ and $\bar{A}(m)$. Physical values are obtained by the standard part map as

$$\text{st}(<^0|\hat{O}|^0\rangle) \in \mathcal{R}.$$

7.2 Fields on $^*\mathcal{M}$

Following principle of physical equivalence (principle (II)), we define fields at every point on $^*\mathcal{M}$ as the following equivalent sum over all fields contained in $\text{Mon}(r|^*\mathcal{L})$;

$$\varphi([r]) \equiv ^* \sum_l A(N_r + l)/\sqrt{^* \sum_l 1},$$

$$\bar{\varphi}([r]) \equiv ^* \sum_l \bar{A}(N_r + l)/\sqrt{^* \sum_l 1}, \quad (61)$$

where $^* \sum_l \equiv \sum_{l,^* \epsilon \text{Mon}(0)}$ and hereafter $[r]$ in $\varphi([r])$ always means the fact that the equivalent sum over $\text{Mon}(r|^*\mathcal{L})$ expressed by $^* \sum_l$ is carried out in the definition of $\varphi([r])$. Here the equivalent sum is just the expression of principle of physical equivalence. We
can easily evaluate the commutation relation

\[ [\varphi([r]), \bar{\varphi}([r'])] = *\delta_{rr'} = \begin{cases} 1 & \text{(for } r' = r), \\ 0 & \text{(for } r' \neq r). \end{cases} \quad (62) \]

Note that \( r, r' \in \mathcal{R} \) but \(*\delta_{rr'}\) is not equal to the usual Dirac delta function \( \delta(r - r')\).

Complex fields on \(*\mathcal{M}\), which are represented by linear combinations certifying the same weight for all fields contained in \(\text{Mon}(r|*\mathcal{L})\), are generally written by

\[
\varphi([r]; k) = *\sum_l e^{i\theta_l^k(r)} \frac{A(N_r + l)}{\sqrt{*\sum_l 1}}, \\
\bar{\varphi}([r]; k) = *\sum_l e^{-i\theta_l^k(r)} \frac{\bar{A}(N_r + l)}{\sqrt{*\sum_l 1}}, \quad (63)
\]

where

\[ \theta_l^k(r) = \theta_k(r) + 2\pi lk/ *\sum_l 1 \]

with the constraint \(*\varepsilon k \in \text{Mon}(0)\) for non-standard integers \(k\). They satisfy the commutation relations

\[ [\varphi([r]; k), \bar{\varphi}([r']; k')] = *\delta_{rr'}\delta_{kk'} \]

and others = 0. These fields are the Fourier components for the fields on \(\text{Mon}(r|*\mathcal{L})\) and their component number is same as that of \(A(N_r + l)\) and \(\bar{A}(N_r + l)\) included in \(\text{Mon}(r|*\mathcal{L})\), because the constraint for \(k\), that is, \(*\varepsilon k \in \text{Mon}(0)\), is same as that for \(l\). Note that \(\varphi([r])\) and \(\bar{\varphi}([r])\) given in (61) correspond to the above fields with \(k = 0\) and \(\theta_0(r) = 0\). Experimentally the differences of the wave numbers \(k\) are not observable, because their wave lengths are infinitesimal. It is stressed that fields on one point of \(*\mathcal{M}\) have infinite degrees of freedom. General fields on \(*\mathcal{M}\) are described by functions of these fields such that

\[ \phi([r]) = f(\{\varphi([r]; k)\}, \{\bar{\varphi}([r]; k)\}). \quad (64) \]

7.3 Extension to \(N\)-dimensional space
Extension of the above consideration to $N$-dimensional space $(*\mathcal{M})^N$ is trivial. Note that one should not confuse the $N$ for the $N$-dimensions of the space with the $\mathcal{N}_r$ for the lattice number corresponding to $r$ of $\mathcal{R}$ (see (16)) in the following discussions. Every point of $(*\mathcal{L})^N$ is represented by a $N$-dimensional vector

$$\vec{r}^N(\vec{m}) \equiv (r_1(m_1), \ldots, r_N(m_N)),$$

where $r_i(m_i) = *\varepsilon(N + l_i)$ with $st(*\varepsilon N_r) = r_i \in \mathcal{R}$ and $*\varepsilon l_i \in \text{Mon}(0)$ for $i = 1, \ldots, N$. Fields with $N$-components at $\vec{r}^N(\vec{m})$, $A_j(\vec{m})$ and $\bar{A}_k(\vec{m})$ ($j, k = 1, \ldots, N$), are defined by the following commutation relations;

$$[A_j(\vec{m}), \bar{A}_k(\vec{m}')] = \delta_{jk} \prod_{i=1}^{N} \delta_{m_i, m_i'}$$

(65)

for $j, k = 1, \ldots, N$ and others = 0. We may consider that these $N$ number of fields describe the $N$ oscillators of a lattice point corresponding to $N$ different directions of the space. The fields on $(*\mathcal{M})^N$ are described as follows;

$$\varphi_j([\vec{r}^N]; \vec{k}^N) = *\sum_{l_1} \cdots *\sum_{l_N} e^{i \sum_{s=1}^{N} \varphi_{l_1}^{(s)}(\vec{r}^N)} A_j(N_r + l_1, \ldots, N_r + l_N)/(\sum_{l}^N)^{1/2}$$

(66)

and similar to $\varphi_j([\vec{r}^N]; \vec{k})$. We again have the commutation relations

$$[\varphi_j([\vec{r}^N]; \vec{k}^N), \varphi_l([\vec{r}^N]; \vec{k}'^N)] = \delta_{jl} \prod_{i=1}^{N} (\delta_{r_i, r_i'} \delta_{k_i, k_i'})$$

(67)

and others = 0.

8. **Internal symmetries on $(*\mathcal{M})^N$ induced from the confined substructure $(\text{Mon}(r| *\mathcal{L}))^N$**

Symmetries on $(*\mathcal{M})^N$ which is induced from the internal substructure $(\text{Mon}(r| *\mathcal{L}))^N$ are expressed by transformations $U_T$ which keep all expectation values unchanged such that

$$< *\bar{0}| \hat{O}(\{A\}, \{\bar{A}\})|*0 > = < *\bar{0}| U_T^{-1} U_T \hat{O}(\{A\}, \{\bar{A}\}) U_T^{-1} U_T |*0 >.$$

22
In general the transformation $U_T$ will be represented by maps of fields $A_j(\vec{m})$ ($\tilde{A}_j(\vec{m})$) to a linear combination of the fields $A_k(\vec{m})$ ($\tilde{A}_k(\vec{m})$) $(k = 1, \cdots, N)$ on $^*\mathcal{L}$. If the operators $U_T$ do not change the structure of $(^*\mathcal{M})^N$, they can represent symmetries on $(\text{Mon}(r \mid ^*\mathcal{L}))^N$.

8.1 Transformation operators on internal subspaces $(\text{Mon}(r \mid ^*\mathcal{L}))^N$

Let us start from the construction of transformation operators on an internal subspace contained in a point on $(^*\mathcal{M})^N$ corresponding to a point $\vec{r}^N = (r_1, \ldots, r_N)$ on $\mathcal{R}^N$. The transformations map fields $A_j(\vec{r}^N(\vec{m}))$ ($\tilde{A}_j(\vec{r}^N(\vec{m}))$) on every lattice-point $\vec{r}^N(\vec{m}) = (N_{r_1} + l_1, \ldots, N_{r_N} + l_N)$ to linear combinations of fields $A_k(N_{r_1} + l_1', \ldots, N_{r_N} + l_N')$ ($\tilde{A}_k(N_{r_1} + l_1', \ldots, N_{r_N} + l_N')$) $(k = 1, \ldots, N)$ on the lattice-points of the same subspace. Following principle of physical equivalence (principle (II)), we construct the following $N^2$-number of operators $\hat{T}_{jk}(\vec{r}^N)$ on $(^*\mathcal{M})^N$, which are again defined by the equivalent sum over all fields contained in the $N$-dimensional subspace $(\text{Mon}(r \mid ^*\mathcal{L}))^N$ as

$$\hat{T}_{jk}(\vec{r}^N) = \sum_{l_1} \cdots \sum_{l_N} A_j(N_{r_1} + l_1, \ldots, N_{r_N} + l_N)A_k(N_{r_1} + l_1', \ldots, N_{r_N} + l_N').$$

(68)

We easily obtain commutation relations

$$[\hat{T}_{jk}(\vec{r}^N), A_l(\vec{r}'^N(\vec{m}))] = -\left(\prod_{i=1}^{N} *\delta_{r_ir_i'}\right)\delta_{jl}A_k(\vec{r}'^N(\vec{m})), $$

$$[\hat{T}_{jk}(\vec{r}^N), \tilde{A}_l(\vec{r}'^N(\vec{m}))] = \left(\prod_{i=1}^{N} *\delta_{r_ir_i'}\right)\delta_{kl}\tilde{A}_j(\vec{r}'^N(\vec{m})), $$

$$[\hat{T}_{jk}(\vec{r}^N), \hat{T}_{lm}(\vec{r}'^N)] = \left(\prod_{i=1}^{N} *\delta_{r_ir_i'}\right)(\delta_{kl}\hat{T}_{jm}(\vec{r}^N) - \delta_{jm}\hat{T}_{lk}(\vec{r}'^N)). $$

(69)

These operators $\hat{T}_{jk}$ can be recomposed into the following generators;

(1) $U(1)$-generator:

$$\hat{J}_0 = \sum_{j=1}^{N} \hat{T}_{jj}. $$

(70)

(2) $SU(N)$-generators:

$$\hat{J}_L = \sum_{j=1}^{L+1} g_j \hat{T}_{jj}, \quad \text{for } L = 1, \ldots, N - 1 $$

(71)
with the traceless condition \( \sum_{j=1}^{L+1} g_j = 0 \) and

\[
\hat{J}^{(1)}_{jk} = \hat{T}_{jk} + \hat{T}_{kj}, \quad \hat{J}^{(2)}_{jk} = \frac{1}{i}(\hat{T}_{jk} - \hat{T}_{kj}),
\]

(72)

for \( j \neq k \).

For instance, we can represent them by well-known matrices including Pauli spin matrices \( \vec{\sigma} \) for \( N = 2 \) case as

\[
\hat{J}_0 \Rightarrow 1, \quad \hat{J}_1 \Rightarrow \sigma_3, \quad \hat{J}^{(1)}_{12} \Rightarrow \sigma_1 \quad \text{and} \quad \hat{J}^{(2)}_{12} \Rightarrow \sigma_2.
\]

Now it is trivial that operators given by

\[
\langle \cdots \rangle = \exp\left[i \sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_{jk}(\vec{r}^N)\hat{T}_{jk}(\vec{r}^N)\right] (73)
\]

with \( \forall \alpha_{jk}(\vec{r}^N) \in \mathcal{C} \) (the set of complex numbers) produce maps of all fields on the subspace \( (\text{Mon}(r|\ast \mathcal{L}))^N \) to linear combinations of the fields on the same subspace.

From the construction procedure of \( \hat{T}_{jk} \) it is obvious that the operators do not break the structure of \((\ast \mathcal{M})^N\). Note also that \( U \) does not change the vacuum and the dual vacuum, because

\[
\forall \hat{T}_{jk}|^*0 > = \langle ^*0 | \forall \hat{T}_{jk} = 0.
\]

### 8.2 Symmetries on \((\ast \mathcal{M})^N\)

Operators on \((\ast \mathcal{M})^N\) can be defined by products of \( U(\alpha(\vec{r}^N)) \) as

\[
U_T(\{\alpha\}) = \prod_{i=1}^{N} \ast \prod_{N_{r_i}} U(\{\alpha(\vec{r}^N)\}),
\]

(74)

where \( \ast \prod_{N_{r_i}} \) stand for the product with respect to \( \forall N_{r_i} \) with the constraint \( \text{st}(\ast \varepsilon N_{r_i}) = r_i \in \mathcal{R} \). (See the definition of \( \ast \varepsilon N_{r_i} \) given in (16).) It is interesting that the transformations produced by \( U_T(\{\alpha\}) \) are generally local transformations on our observed space \((\ast \mathcal{M})^N\) because the parameters \( \{\alpha\} \) can depend on the position \( \vec{r}^N \), whereas they are global ones on the internal subspace \((\text{Mon}(r|\ast \mathcal{L}))^N\). Note that \( U_T \) does not change the vacuum and the dual vacuum.
Let us show a few realistic transformations included in $U_T$.

(a) $U(1)$ transformation:

$$U_0(\vec{r}^N) = \exp[i\alpha_0(\vec{r}^N)\hat{J}_0([\vec{r}^N])]$$  \hspace{1cm} (75)

for $\text{st}(\alpha_0) \in \mathcal{R}$. It is an interesting problem to investigate whether this $U(1)$ symmetry can be the $U(1)$ symmetry of electro-weak gauge theory or the solution of so-called $U(1)$ problem in hadron dynamics.

(b) $SU(N)$ transformation:

$$U_N(\vec{r}^N) = \exp[i\left\{\sum_{L=1}^{N-1} \alpha_L(\vec{r}^N)\hat{J}_L([\vec{r}^N]) + \sum_{j=1}^{k-1} \sum_{i=1}^{N} \sum_{j=2}^{2} \sum_{i=1}^{2} \alpha_{ij}(\vec{r}^N)\hat{J}_{ij}(\vec{r}^N)\right\}]$$  \hspace{1cm} (76)

for $\text{st}(\forall \alpha_L)$, $\text{st}(\forall \alpha_{ij}) \in \mathcal{R}$. It is an interesting proposal that three color components of QCD may be identified by those of $U_3(\vec{r}^3)$ for three spatial dimensions.

9. Quantized configuration space and infinitesimal distances

Here we study configuration space describing $^*\mathcal{M}$, which will be useful in the investigations of general relativity and gravitations.

9.1 Quantization of configuration space

Let us start from 1-dimensional space. We can construct position operator

$$\hat{r} \cdot \mathcal{M} = \sum_{N_r} r \hat{T}_r,$$  \hspace{1cm} (77)

where $\sum_{N_r}$ stands for the sum over $\forall N_r$ with the constraint $\text{st}(^*\varepsilon N_r) = r \in \mathcal{R}$ and

$$\hat{T}_r = \sum_{l} \tilde{A}(N_r + l)A(N_r + l).$$  \hspace{1cm} (78)

Following principle of physical equivalence, $\hat{T}_r$ is expressed by the equivalent sum with respect to all fields in the same monad lattice-space $\text{Mon}(r \cdot ^*\mathcal{L})$. Note that $r$ in (77) can be replaced by $r + a_r \cdot ^*\varepsilon$ with the constant $\text{st}(a_r \cdot ^*\varepsilon) = 0$ for $\forall r \in \mathcal{R}$. The eigenstate of
\(\hat{r} \cdot \mathcal{M}\) for the eigenvalue \(r\) is written by

\[
|r > \cdot \mathcal{M} \equiv \varphi([r])|*0 > .
\] (79)

Hereafter we call them monad states. The relation

\[
\hat{r} \cdot \mathcal{M} | r > \cdot \mathcal{M} = r | r > \cdot \mathcal{M}
\]

is trivial. If one does not want to have 0 eigenvalue for \(r = 0\), \(r + a_r \cdot \varepsilon\) can be used instead of \(r\) in the definition of \(\hat{r} \cdot \mathcal{M}\). The monad states \(| r > \cdot \mathcal{M}\) are quite similar to the ket states of usual quantum mechanics except the normalization condition

\[
\cdot \mathcal{M} < r | r' > \cdot \mathcal{M} = \delta_{rr'},
\]

where \(\cdot \mathcal{M} < r | =< *0| \prod_{r} \varphi([r])\). It is noted that every monad state \(| r > \cdot \mathcal{M}\) has its own internal substructure \(\text{Mon}(r| \cdot \mathcal{L})\).

Now we can define the quantized states for our configuration space as follows;

\[
| *\mathcal{M} > \equiv \prod_{r} | r > \cdot \mathcal{M}, \quad < *\mathcal{M}| \equiv \prod_{r} *\mathcal{M} < r |.
\] (80)

On these states the position operator \(\hat{r} \cdot \mathcal{M}\) is represented by a diagonal operator and then we can consider that the base state \(| *\mathcal{M} >\) describes our configuration space, which is normalized as \(< *\mathcal{M}| *\mathcal{M} > = 1\).

Extension to \(N\)-dimension is trivial. A component of the position-vector operator can be defined as same as that of the 1-dimensinal case, e.g., for the \(i\)th component

\[
\hat{r}_i \cdot \mathcal{M} = \sum_{N_{r1}} \cdots \sum_{N_{rN}} r_i \hat{T}_i([\vec{r}^N]),
\] (81)

where

\[
\hat{T}_i([\vec{r}^N]) = \sum_{l_1} \cdots \sum_{l_N} \tilde{A}_i(N_{r1} + l_1, ..., N_{rN} + l_N)A_i(N_{r1} + l_1, ..., N_{rN} + l_N)
\] (82)

for \(i = 1, 2, ..., N\). The \(N\)-dimensional configuration state is expressed by

\[
| *\mathcal{M}^N > = \prod_{j=1}^{N} (\prod_{N_{r1}} \cdots \prod_{N_{rN}} \varphi_j([\vec{r}^N]))|*0 > .
\] (83)
9.2 Infinitesimal distance

Infinitesimal relative distance operators are definable only on the internal subspace $\text{Mon}(r|L)$ such that

$$d\hat{r}(\Delta l) \equiv \hat{r}(N_r + l) - \hat{r}(N_r + l'),$$

where $\Delta l \equiv l - l'$ and

$$\hat{r}(N_r + k) \equiv *\varepsilon(N_r + l)\bar{A}([r])A(N_r + k)$$

with the definition

$$\bar{A}([r]) \equiv *\sum_l \bar{A}(N_r + l),$$

which follows principle of physical equivalence. The monad states $|r > *M$ are the eigenstates of $\hat{r}(N_r + l)$ and $d\hat{r}(\Delta l)$. We actually obtain

$$d\hat{r}(\Delta l)|r > *M = *\varepsilon\Delta l|r > *M.$$  

We can write squared distance operators in the $N$-dimensional space as

$$(d\hat{s})^2(\vec{r}^N) = d\hat{r}_\mu(\Delta \vec{l}^N)g^{\mu\nu}d\hat{r}_\nu(\Delta \vec{l}^N),$$

where the sums over $\mu$ and $\nu$ from 1 to $N$ are neglected,

$$d\hat{r}_\mu(\Delta \vec{l}^N) = \hat{r}_\mu(N_{r_1} + l_1, ..., N_{r_N} + l_N) - \hat{r}_\mu(N_{r_1} + l'_1, ..., N_{r_N} + l'_N)$$

with $\hat{r}_\mu(N_{r_1} + l_1, ..., N_{r_N} + l_N) = *\varepsilon(N_{r_\mu} + l_\mu)\bar{A}_\mu([\vec{r}^N])A_\mu(N_{r_1} + l_1, ..., N_{r_N} + l_N)$ and $\Delta \vec{l}^N = (l_1 - l'_1, ..., l_N - l'_N)$. If the metric operator $g^{\mu\nu}$ is taken as Minkowski metric, the internal subspace $(\text{Mon}(r|L))^N$ just represents so-called local inertial system in general relativity. We have the equations

$$d\hat{r}_\mu(\Delta \vec{l}^N)|r > *M = *\varepsilon\Delta l_\mu|\vec{r}^N > *M,$$

$$ (d\hat{s})^2(\vec{r}^N)|\vec{r}^N > = *\varepsilon^2\Delta l_\mu g^{\mu\nu}\Delta l_\nu|\vec{r}^N > *M.$$  

27
The expectation value of \((d\hat{s})^2\) is calculated as follows;

\[
(ds)^2 = \cdot _{\mathcal{M}}< \vec{r}^{N}|(d\hat{s})^2(\vec{r}^{N})|\vec{r}^{N} > \cdot _{\mathcal{M}}.
\]

The same expectation value of the squared distance operator can be obtained in terms of the expectation value with respect to the configuration state \(\cdot _{\mathcal{M}^N}\). It is transparent that transformations keeping \((ds)^2\) unchanged are represented by \(U(\{\alpha(\vec{r}^{N})\})\) given in (73).

10. Translations, Rotations and Lorentz and general relativistic transformations

Let us study symmetries on the configuration space, which keep all expectation values unchanged such that

\[
< \cdot _{\mathcal{M}^N}|U^{-1}U\hat{\mathcal{O}}(\{\vec{A}\},\{A\})U^{-1}U| \cdot _{\mathcal{M}^N} >.
\]

Note that the configuration state \(\cdot _{\mathcal{M}^N}\), the dual state \(< \cdot _{\mathcal{M}^N}|\) and operators are transformed as follows;

\[
\cdot _{\mathcal{M}^N} \rightarrow U| \cdot _{\mathcal{M}^N} >, \quad < \cdot _{\mathcal{M}^N}| \rightarrow < \cdot _{\mathcal{M}^N}|U^{-1} , U\hat{\mathcal{O}}(\ldots)U^{-1}.
\]

10.1 Translational invariance on \((\cdot _{\mathcal{M}})^N\)

The operator which replaces \(|r\rangle\) with \(|r+\Delta\rangle\) for \(\Delta \in \mathcal{R}\) is obtained as

\[
\hat{p}_r(\Delta) = \cdot _{\mathcal{M}}\sum_l \vec{A}(N_r + l)A(N_r + l).
\]

(88)

We have \(\hat{p}_r(\Delta)|0\rangle = 0\). Then we can define the translation operator by

\[
\hat{P}(\Delta) = \cdot \prod_{N_r} \hat{p}_r(\Delta) :,
\]

(89)

where : \ldots : means the normal product used in usual field theory, in which all creation operators \((\vec{A}_j(m))\) must put on the left-hand side of all annihilation operators \((A_j(m))\).
We see that $\hat{P}(\Delta)$ transforms the configuration state $|*\mathcal{M}>$ to the isomorphic space, that is,

$$\hat{P}(\Delta)|*\mathcal{M}>=\equiv |*\mathcal{M}>$$

for $\forall \Delta \in \mathcal{R}$.

Let us study the invariance of expectation values

$$<*\mathcal{M}|\hat{\Omega}(\{\bar{A}\},\{A\})|*\mathcal{M}>.$$ 

Taking account of the definitions of $|*\mathcal{M}>: \equiv \prod \varphi([r])|*0>$ and $<*\mathcal{M}|=\equiv <*0|\prod \varphi([r])$ and the fact that all the fields commute each other except $A$ and $\bar{A}$ on the same lattice-point, the number of $A$ and that of $\bar{A}$ on the same lattice-point must be same in operators having non-vanishing expectation values on $|*\mathcal{M}>$. This means that every term of such operators must be written by the product of powers such as $(\bar{A}A)^n$ with $n \in \mathcal{N}$ for all pairs of $A$ and $\bar{A}$ on the same lattice-point. On the other hand we easily see that the products of $\bar{A}A$ on the same lattice-point commute with $\hat{P}(\Delta)$ such that

$$[\bar{A}A, \hat{P}(\Delta)] = 0$$

for $\forall \Delta \in \mathcal{R}$. Now we can conclude that operators having non-vanishing expectation values commute with the translation operators, that is,

$$[\hat{\Omega}(\{\bar{A}\},\{A\}), \hat{P}(\Delta)] = 0.$$  

Translational invariance is certified for physically meaningful operators as

$$<*\mathcal{M}|\hat{P}(-\Delta)\hat{\Omega}(\ldots)\hat{P}(\Delta)|*\mathcal{M}> = <*\mathcal{M}|\hat{P}(-\Delta)\hat{\Omega}(\ldots)\hat{P}(\Delta)|*\mathcal{M}> = <*\mathcal{M}|\hat{\Omega}(\ldots)|*\mathcal{M}>$$

because of the relation $<*\mathcal{M}|\hat{P}(-\Delta)\hat{P}(\Delta) = <*\mathcal{M}|$.

The extension of the above argument to the $N$-dimensional spaces is trivial. Note also that the translations cannot be generated by the operators $U_T$ given in (74).

10.2 Rotations
Rotational invariance can be introduced only for subspaces whose metric $g_{\mu\nu}$ have the same sign like $SO(3)$ subspace of $SO(3,1)$. Generators for the rotations in $(j,k)$-plane are given by

$$\hat{J}_{jk} = \hat{T}_{jk} - \hat{T}_{kj}. \quad (94)$$

In general rotation operators are described by

$$U_R(\{\theta\}) = e^{i \sum_{(j,k)} \theta_{jk} \hat{J}_{jk}}. \quad (95)$$

It is transparent that $U_R$ for $st(\forall \theta_{jk}) \in \mathcal{R}$ are unitary operators and generate rotations on the subspace.

### 10.3 Lorentz transformations

Position operator for one point on $(\mathcal{M})^N$ corresponding to $\vec{r}^N$ on $\mathcal{R}^N$ is given by

$$\hat{r}_j([\vec{r}^N]) = r_j \varphi_j([\vec{r}^N]) \varphi_j([\vec{r}^N]), \quad \text{for } j = 1, ..., N. \quad (96)$$

The expectation value of squared distance from the origin are evaluated as

$$(\vec{r}^N)^2 = <\mathcal{M}^N|\hat{r}_\mu([\vec{r}^N])g^{\mu\nu}\hat{r}_\nu([\vec{r}^N])|\mathcal{M}^N>, \quad (97)$$

where the metric tensors $g^{\mu\nu}$ are taken as Minkowski metric tensors.

Let us study the simplest case for $N = 2$. The metric tensors are chosen such that

$$g^{11} = -g^{22} = 1 \quad \text{and} \quad g^{12} = g^{21} = 0.$$

Transformations

$$U_L(a) = \prod_{j=1}^{N} * \prod_{N_{r_j}} e^{-a \hat{J}_{12}^{(1)}([\vec{r}^N])} \quad (98)$$

with the constraint $st(a) \in \mathcal{R}$ (see (73) and (74)) generate 2-dimensional Lorentz transformations which are expressed in 2-dimensional matrices as

$$U_L(a) = \begin{pmatrix} \cosh a & -\sinh a \\ -\sinh a & \cosh a \end{pmatrix}.$$
Generalization for the $N$-dimensions can be performed by using combinations of $U_L(a)$ with the rotations.

### 10.4 General relativistic transformations

We have many different types of transformations which keep the squared distance $(\mathbf{r}^N)^2$ invariant but generally do not the metric tensors invariant, while Lorentz transformations keep both of them invariant. They are described by the transformations $U_T(\{\alpha\})$ given in (74), where the parameters $\{\alpha\}$ should be chosen such that all the axes are real after the translations. Of course, all the parameters must be finite. In such transformations we have different types of vectors corresponding to covariant and contravariant tensors in general coordinate transformations. The difference between them is expressed as follows;

$$U_G\hat{r}_\mu|*\mathcal{M}^N>,$$ for covariant vectors

$$<*_\mathcal{M}^N|\hat{r}_\mu g^{\mu\nu}U_G^{-1},$$ for contravariant vectors.  \hfill(99)

A simple example representing dilatation transformations are described by

$$D_d = e^{-\sum_{j=1}^N a_j(\mathbf{R}^N)\hat{r}_{jj}(\mathbf{R}^N)},$$ \hfill(100)

which transforms as

$$U_d\hat{r}_\mu|*\mathcal{M}^N >= e^{-a_\mu(\mathbf{R}^N)}\hat{r}_\mu|*\mathcal{M}^N >,$$

$$<*_\mathcal{M}^N|\hat{r}_\nu g^{\nu\mu}U_d^{-1} = <*_\mathcal{M}^N|\hat{r}_\nu g^{\nu\mu}e^{a_\mu(\mathbf{R}^N)}.$$

Note again that $U_G(\{\alpha(\mathbf{R}^N)\})$ is global on the subspace $(\text{Mon}(r|*\mathcal{L}))^N$, even though it is generally local on observed space $(*_\mathcal{M})^N$. We understand that all the transformations described by $U_T$ of (74) can include general relativistic transformations. This fact implies that general relativistic transformations are generally represented by local non-abelian transformations.

### 11. Remarks on fermionic oscillators
In this section we shall comment that instead of bosonic fields $A(m)$ and $\bar{A}(m)$ we can construct similar field theory by using fermionic fields $C(m)$ and $\bar{C}(m)$ which satisfy anticommutation relations $[C(m), \bar{C}(m)]=1$ and commutation relations $[C(m), C(m')]_{-} = [\bar{C}(m), \bar{C}(m')]_{-} = 0$ for $m \neq m'$.

As far as operators $\hat{T}_{j,k}(\hat{r}^N)$ presented in (68) are concerned, we can define them by the replacement of $A$ and $\bar{A}$ with $C$ and $\bar{C}$, respectively. And we get the same commutation relations given in (69). This means that all the arguments of the internal symmetries performed in the bosonic oscillator case are completely accomplished in the fermionic oscillator case. That is to say, as far as the internal symmetries are concerned, there is no difference between the bosonic and the fermionic cases. Furthermore we can easily understand that not only $U_T$ but also all other operators written by the products of $\bar{A}$ and $A$ like $\hat{T}_r$, $\hat{r}$ and $\hat{p}_r$ can be defined in the replacement of $\bar{A} A$ with $\bar{C} C$ and they have the same properties as discussed in the bosonic case.

Difference between them appears in the construction of realistic fields from $\varphi([r]; k)$. Namely products of more than the non-standard natural number $\sum_l 1$ with respect to the fields $\varphi([r]; k)$ vanish for the fermionic case, whereas there is no such restriction in the bosonic case. We may say that the concept of antiparticles will be introduced more easily in the fermionic case by using occupation and unoccupation numbers of lattice-points of the monad lattice-space $\text{Mon}(r| \mathcal{L})$.

Anyhow the selection of the bosonic or the fermionic or both like supersymmetric is still open question at present.

12. Concluding remarks

We have constructed a field theory on the quantized space-time by using infinitesimal-lattice space $(\mathcal{L})^N$. In this scheme the internal subspace $(\text{Mon}(r| \mathcal{L}))^N$ and the symmetry transformation $U_T$ induced from the subspace are uniquely determined, when we construct the field theory on $(\mathcal{M})^N \cong \mathcal{R}^N$. Since all definitions and evaluations are imposed to be
done on \((\ast \mathcal{L})^N\), we can perform them in terms of \(\ast\)-finite sum in non-standard analysis. In fact we need not introduce any Dirac \(\delta\)-functions. In this scheme we can carry out all evaluations on configuration space, not on Fock space in usual field theory. This fact is an interesting advantage in the investigation of quantum gravity, as was seen in the introduction of the infinitesimal relative distance and the local inertial system. In order to investigate this model in more detail an inevitable problem is introducing equation of motions on \((\ast \mathcal{M})^N\), which will be represented by difference equation on \(\text{Mon}(r\mid \ast \mathcal{L})\). It is also interesting to study relations between the general field \(\phi([r])\) and observed fields like leptons, quarks, gauge fields and etc. To carry out these works we have to investigate the symmetries described by \(U_T\) more precisely.

Finally I would like to present the global view of theory on non-standard space once more. The fundamental concept is introducing the equivalence based on experimental errors (physical equivalence) into theories in a mathematically consistent logic, which is allowed only on non-standard spaces. On the spaces the physical equivalence determine projections from non-standard spaces to observed spaces \(\mathcal{R}^N\), which are described by filters in non-standard theory. In fact the filters determine topologies, because they determine the structure of the monad space and then that of the observed space.[1] This means that we observe physical phenomena which depend on the errors, that is to say, we have to answer the following questions to determine the worlds which we observe in experiments:

Which quantities are taken as observables accompanied by errors in experiments?

How large are the errors?

We have to understand that in an experiment we are allowed to peep only through a filter which is determined by the physical equivalence based on the errors of the experiment. Theories on observed spaces, which explain experimental results, of course have to depend on the filters which determine the projections of the theory on the non-standard space to theories on the observed spaces, even if the theory is unique on the non-standard space. Actually we have presented some different filters, for instance,

filters with \(h \in \text{Mon}(0)\) derives classical limits,[6,7]
filters with $h \in \text{Mon}(0)$ but $hN \not\in \text{Mon}(0)$ does macroscopic limit,[8,9]
filters with $g$(coupling of objects with heat baths) $\in \text{Mon}(0)$ does microcanonical ensembles of statistical mechanics.[2,3,4,10]

We see that those filters derive different monad spaces and then different observed spaces (different theories). In any quantum mechanical systems experimental errors are mainly determined by the characters of measurement apparatus, even though the errors are produced from the interactions between objects and apparatus. We have to determine the filters in analyzing the schemes of the apparatus which produce the errors in the experiments. Here I would like again to repeat that we cannot perform any expriments which are not accompanied by any errors. Therefore we have always to take account of phenomena hidden behind experimental errors, when we make theories in our observed spaces.

References
[1] A. Robinson, Non Standard Analysis (North-Holland, Amsterdam, 1970).
M. Saito, Ultra-Products and Nonstandard Analysis (Tokyo-Tosyo, Tokyo, 1976) (in Japanese).
S. Albeverio et al., Nonstandard Method in Stochastic Analysis and Mathematical Physics (Academic Press, New York, 1985).
[2] T. Kobayashi, Nuovo Cim. 113B (1998) 1407.
[3] T. Kobayashi, Proceedings of 5th Wigner Symposium, edited by P. Kasperkovitz and D. Grau (World Scientific, Singapoe, 1998) 518.
[4] T. Kobayahi, Symmetries in Science X, edited by B.Gruber and M. Ramek (Plenum Press, New York and London, 1998) 153.
[5] M. O. Farrukh, J. Math. Phys. 16(1975) 177.
[6] T. Kobayashi, Symmetries in Science VII, edited by B.Gruber and T. Otsuka, (Plenum Press, New York, 1994) 287.
[7] T. Kobayashi, Nuovo Cim. 110B (1995) 61.
[8] T. Kobayashi, Nuovo Cim. 111B (1996) 227.

[9] T. Kobayashi, Proceedings of the Fourth International Conference on Squeezed States and Uncertainty Relations, edited by A. Han, (NASA Conference Publication 3322, 1996) 301.

[10] T. Kobayashi, Phys. Lett. A, 207(1995) 320; 210(1996) 241; 222(1996) 26.

[11] T. Kobayashi, Translations, Rotations and Confined Fractal Property on Infinitesimal-Lattice Spaces, preprint of University of Tsukuba (1997).

[12] T. Kobayashi, Talk in the XI International Conference on Problems of Quantum Field Theory, July, 1998, Dubna, Russia (to appear in the Proceedings).