Matrix Roots of Eventually Positive Matrices

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Abstract

Eventually positive matrices are real matrices whose powers become and remain strictly positive. As such, eventually positive matrices are \textit{a fortiori} matrix roots of positive matrices, which motivates us to study the matrix roots of primitive matrices. Using classical matrix function theory and Perron-Frobenius theory, we characterize, classify, and describe in terms of the real Jordan canonical form the \textit{p}-th-roots of eventually positive matrices.

\textit{Keywords:} matrix function, eventually positive matrix, primitive matrix, matrix root, Perron-Frobenius theorem, Perron-Frobenius Property, stochastic matrix

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1. Introduction

A matrix $A \in M_n(\mathbb{R})$ is \textit{eventually positive (nonnegative)} if there exists a nonnegative integer \( p \) such that $A^k$ is entrywise positive (nonnegative) for all $k \geq p$. If $p$ is the smallest such integer, then $p$ is called the \textit{power index} of $A$ and is denoted by $p(A)$.

Eventually nonnegative matrices have been the subject of study in several papers \textsuperscript{[1, 2, 3, 4, 5, 6, 7, 8]} and it is well-known that the notions of eventual
positivity and nonnegativity are associated with properties of the eigenspace corresponding to the spectral radius.

A matrix \( A \in M_n(\mathbb{R}) \) has the *Perron-Frobenius property* if its spectral radius is a positive eigenvalue corresponding to an entrywise nonnegative eigenvector. The *strong Perron-Frobenius property* further requires that the spectral radius is simple; that it dominates in modulus every other eigenvalue of \( A \); and that it has an entrywise positive eigenvector.

Several challenges regarding the theory and applications of eventually nonnegative matrices remain unresolved. For example, eventual positivity of \( A \) is equivalent to \( A \) and \( A^T \) having the strong Perron-Frobenius property, however, the Perron-Frobenius property for \( A \) and \( A^T \) is a necessary but not sufficient condition for eventual nonnegativity of \( A \).

An eventually nonnegative (positive) matrix with power index \( p = p(A) \) is, *a fortiori*, a \( p \)-th-root of the nonnegative (positive) matrix \( A^p \). As a consequence, in order to gain more insight into the powers of an eventually nonnegative (positive) matrix, it is only natural to examine the roots of matrices that possess the (strong) Perron-Frobenius property. We begin this pursuit herein by characterizing the roots of matrices that possess the strong Perron-Frobenius property.

We proceed as follows: in Section 2, we recall results concerning matrix functions and, for the sake of completeness and clarity, we present facts needed to analyze a matrix function via the real Jordan canonical form; we also use the real Jordan canonical form to give alternate proofs for [9, Theorems 2.3 and 2.4]. In Section 3, we recall results from the Perron-Frobenius theory of nonnegative matrices and (eventually) positive matrices. We characterize the eventually positive roots of a general primitive matrix, and illustrate our main results via examples. We also present a related result concerning *eventually stochastic* matrices.

### 2. Matrix roots via the complex and real Jordan canonical form

We review some basic notions and results from the theory of matrix functions (for further results, see [10], [11, Chapter 9], or [12, Chapter 6]).

Let \( J_n(\lambda) \in M_n(\mathbb{C}) \) denote the \( n \times n \) Jordan block with eigenvalue \( \lambda \). For \( A \in M_n(\mathbb{C}) \), let \( J = Z^{-1}AZ = \bigoplus_{i=1}^s J_{n_i}(\lambda_i) = \bigoplus_{i=1}^s J_{n_i} \), where \( \sum n_i = n \), denote its Jordan canonical form. Denote by \( \lambda_1, \ldots, \lambda_s \) the *distinct* eigenvalues of \( A \), and, for \( i = 1, \ldots, s \), let \( m_i \) denote the *index* of \( \lambda_i \), i.e., the size of
the largest Jordan block associated with \( \lambda_i \). Denote by \( i \) the imaginary unit, i.e., \( i := \sqrt{-1} \).

**Definition 2.1.** Let \( f : \mathbb{C} \to \mathbb{C} \) be a function and let \( f^{(k)} \) denote the \( k \)th derivative of \( f \). The function \( f \) is said to be *defined on the spectrum of \( A \)* if the values

\[
f^{(k)}(\lambda_i), \quad k = 0, \ldots, m_i - 1, \quad i = 1, \ldots, s,
\]
called the *values of the function \( f \) on the spectrum of \( A \)*, exist.

**Definition 2.2 (Matrix function via Jordan canonical form).** If \( f \) is defined on the spectrum of \( A \in M_n(\mathbb{C}) \), then

\[
f(A) := Z f(J) Z^{-1} = Z \left( \bigoplus_{i=1}^{t} f(J_{n_i}) \right) Z^{-1},
\]

where

\[
f(J_{n_i}) := \begin{bmatrix} f(\lambda_i) & f'(\lambda_i) & \cdots & \frac{f^{(n_i-1)}(\lambda_i)}{(n_i-1)!} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \cdots & f'(\lambda_i) & f(\lambda_i) \end{bmatrix}.
\]

**Definition 2.3.** For \( z = r \exp(i \theta) \in \mathbb{C} \), where \( r > 0 \), and an integer \( p > 1 \), let

\[
z^{1/p} := r^{1/p} \exp(i \theta/p),
\]

and, for \( j \in \{0, 1, \ldots, p - 1\} \), define

\[
f_j(z) = z^{1/p} \exp(i 2\pi j/p) = r^{1/p} \exp(i [\theta + 2\pi j]/p),
\]
i.e., \( f_j \) is the \((j + 1)\)st-branch of the \( p \)th-root function.

Note that

\[
f_j^{(k)}(z) = \frac{1}{p^k} \prod_{i=0}^{k-1} (1 - ip) \left[ r^{(1-kp)/p} \exp(i[2\pi j + \theta(1 - kp)]/p) \right],
\]

where \( k \) is a nonnegative integer and the product \( \prod_{i=0}^{k-1} (1 - ip) \) is empty when \( k = 0 \).

Next we present several technical lemmas on the branches of the \( p \)th-root function.
Lemma 2.4. For \( z \in \mathbb{C} \), \( \Im(z) \neq 0 \), \( j, j' \in \{0, 1, \ldots, p-1\} \), and \( f_j^{(k)} \) as in (2.2), we have \( f_j^{(k)}(z) = \overline{f_j^{(k)}(\bar{z})} \) if and only if \( j + j' \equiv 0 \pmod{p} \).

Proof. Note that

\[
\begin{align*}
f_j^{(k)}(z) &= \overline{f_j^{(k)}(\bar{z})} \\
\iff \exp(i[2\pi j + \theta(1-kp)]/p) &= \exp(i[-2\pi j' + \theta(1-kp)]/p) \\
\iff [2\pi j + \theta(1-kp)]/p &= [-2\pi j' + \theta(1-kp)]/p + 2\pi \ell \\
\iff 2\pi j + \theta(1-kp) &= 2\pi(p\ell - j') + \theta(1-kp) \\
\iff j + j' &= \ell p \\
\iff j + j' &\equiv 0 \pmod{p},
\end{align*}
\]

where \( \ell \in \mathbb{Z} \). Finally, we remark that \( j + j' \equiv 0 \pmod{p} \iff j = j' = 0 \) or \( j = p - j' \).

Lemma 2.5. Let \( z = r \exp(i\pi), \ r > 0 \). For \( j, j' \in \{0, 1, \ldots, p-1\} \) and \( f_j^{(k)} \) as in (2.2), we have \( f_j^{(k)}(z) = \overline{f_j^{(k)}(\bar{z})} \) if and only if \( j + j' \equiv -1 \pmod{p} \).

Proof. Note that

\[
\begin{align*}
f_j^{(k)}(z) &= \overline{f_j^{(k)}(\bar{z})} \\
\iff \exp(i[2\pi j + \pi(1-kp)]/p) &= \exp(i[-2\pi j' - \pi(1-kp)]/p) \\
\iff [2\pi j + \pi(1-kp)]/p &= [-2\pi j' - \pi(1-kp)]/p + 2\pi \ell \\
\iff 2\pi j + \pi(1-kp) &= 2\pi(p\ell - j') - \pi(1-kp) \\
\iff 2\pi[(j + j') - (kp - 1)] &= 2\pi \ell p \\
\iff j + j' &\equiv (kp - 1) \pmod{p} \\
\iff j + j' &\equiv -1 \pmod{p},
\end{align*}
\]

where \( \ell \in \mathbb{Z} \). Finally, we remark that \( j + j' \equiv -1 \pmod{p} \iff \) or \( j + j' = p - 1 \).

The following theorem classifies all \( p \)-th-roots of a general nonsingular matrix [13, Theorems 2.1 and 2.2].

Theorem 2.6 (Classification of \( p \)-th-roots of nonsingular matrices). If \( A \in M_n(\mathbb{C}) \) is nonsingular, then \( A \) has precisely \( p^s \) \( p \)-th-roots that are expressible
as polynomials in $A$, given by

$$X_j = Z \left( \bigoplus_{i=1}^{t} f_{j_i}(J_{n_i}) \right) Z^{-1}, \quad (2.3)$$

where $j = (j_1, \ldots, j_t)$, $j_i \in \{0, 1, \ldots, p - 1\}$, and $j_i = j_k$ whenever $\lambda_i = \lambda_k$.

If $s < t$, then $A$ has additional $p$th-roots that form parameterized families

$$X_j(U) = ZU \left( \bigoplus_{i=1}^{t} f_{j_i}(J_{n_i}) \right) U^{-1} Z^{-1}, \quad (2.4)$$

where $U$ is an arbitrary nonsingular matrix that commutes with $J$ and, for each $j$, there exist $i$ and $k$, depending on $j$, such that $\lambda_i = \lambda_k$, while $j_i \neq j_k$.

In the theory of matrix functions, the roots given by (2.3) are called the primary roots of $A$, and the roots given by (2.4), which exist only if $A$ is derogatory (i.e., some eigenvalue appears in more than one Jordan block), are called the nonprimary roots [10, Chapter 1].

The next result provides a necessary and sufficient condition for the existence of a root for a general matrix, which is clearly satisfied by any nonsingular matrix (see [14]).

**Theorem 2.7** (Existence of $p$th-root). A matrix $A \in M_n(\mathbb{C})$ has a $p$th-root if and only if the “ascent sequence” of integers $d_1, d_2, \ldots$ defined by

$$d_i = \dim (\text{null} (A^i)) - \dim (\text{null} (A^{i-1}))$$

has the property that for every integer $\nu \geq 0$ no more than one element of the sequence lies strictly between $p\nu$ and $p(\nu + 1)$.

Before we state results concerning the matrix roots of a real matrix, we state some well-known results concerning the real Jordan canonical form for real matrices (see [15, Section 3.4], [12, Section 6.7]).

**Theorem 2.8** (Real Jordan canonical form). If $A \in M_n(\mathbb{R})$ has $r$ real eigenvalues (including multiplicities) and $c$ complex conjugate pairs of eigenvalues (including multiplicities), then there exists a real, invertible matrix $R \in M_n(\mathbb{R})$ such that

$$R^{-1}AR = J_R = \left[ \bigoplus_{k=1}^{r} J_{n_k}(\lambda_k) \bigoplus_{k=r+1}^{r+c} C_{n_k}(\lambda_k) \right],$$

where:
1. \( C_k(\lambda) := \begin{bmatrix} C(\lambda) & I_2 \\ C(\lambda) & \ddots \\ \vdots & \ddots & \ddots & I_2 \\ C(\lambda) & \end{bmatrix} \in M_{2k}(\mathbb{R}); \quad (2.5) \)

2. \( C(\lambda) := \begin{bmatrix} \Re(\lambda) & \Im(\lambda) \\ -\Im(\lambda) & \Re(\lambda) \end{bmatrix} \in M_2(\mathbb{R}); \quad (2.6) \)

3. \( \lambda_1, \ldots, \lambda_r \) are the real eigenvalues (including multiplicities) of \( A \); and

4. \( \lambda_{r+1}, \bar{\lambda}_{r+1}, \ldots, \lambda_{r+c}, \bar{\lambda}_{r+c} \) are the complex eigenvalues (including multiplicities) of \( A \).

Lemma 2.9. Let \( \lambda \in \mathbb{C} \) and suppose \( C_k(\lambda) \) and \( C(\lambda) \) are defined as in (2.5) and (2.6), respectively. If \( S_k := \left( \bigoplus_{i=1}^{k} S \right) \in M_{2k}(\mathbb{R}) \), where \( S := \begin{bmatrix} -i & -i \\ 1 & -1 \end{bmatrix} \), then

\[
S_k^{-1}C_k(\lambda)S_k = D_k(\lambda) := \begin{bmatrix} D(\lambda) & I_2 \\ D(\lambda) & \ddots \\ \vdots & \ddots & \ddots & I_2 \\ D(\lambda) & \end{bmatrix} \in M_{2k}(\mathbb{C}), \quad (2.7)
\]

where \( D(\lambda) := \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \).

Proof. Proceed by induction on \( k \), the number of \( 2 \times 2 \)-blocks; when \( k = 1 \) one readily obtains

\[
S^{-1}C(\lambda)S = \frac{1}{2} \begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix} \begin{bmatrix} \Re(\lambda) & \Im(\lambda) \\ -\Im(\lambda) & \Re(\lambda) \end{bmatrix} \begin{bmatrix} -i & -i \\ 1 & -1 \end{bmatrix} = D(\lambda).
\]

Now assume the assertion holds for all matrices of the form (2.5) of dimension \( 2(k - 1) \). Note that the matrices in the product \( S_k^{-1}C_k(\lambda)S_k \) can be partitioned as

\[
\begin{bmatrix} S_{k-1}^{-1} & Z \\ Z^T & S_{k-1} \end{bmatrix} \begin{bmatrix} C_{k-1}(\lambda) & Y \\ Z^T & C(\lambda) \end{bmatrix} \begin{bmatrix} S_{k-1} & Z \\ Z^T & S \end{bmatrix}
\]

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where $Z \in M_{2(k-1),2}(\mathbb{R})$ is a rectangular zero matrix, $Y = \begin{bmatrix} Z_2 \\ Z_2 \\ \vdots \\ Z_2 \\ I_2 \end{bmatrix} \in M_{2(k-1),2}(\mathbb{R})$, and $Z_2$ is the $2 \times 2$ zero matrix. With the above partition in mind, and following the induction hypothesis, we obtain

$$S^{-1}C_k(\lambda)S = \begin{bmatrix} D_{k-1}(\lambda) & S^{-1}_{k-1}YS \\ Z^T & S^{-1}C(\lambda)S \end{bmatrix}$$

and note that $S^{-1}_{k-1}YS = Y$ and $S^{-1}C(\lambda)S = D(\lambda)$. \hfill \Box

**Lemma 2.10.** Let $\lambda \in \mathbb{C}$ and suppose $D_k(\lambda)$ and $D(\lambda)$ are defined as in Lemma 2.9. If $P_k$ is the permutation matrix given by

$$P_k = \begin{bmatrix} e_1 & e_3 & \ldots & e_{2k-1} & e_2 & e_4 & \ldots & e_{2k} \end{bmatrix} \in M_{2k}(\mathbb{R}),$$

where $e_i$ denotes the canonical basis vector in $\mathbb{R}^n$ of appropriate dimension, then

$$P_k^T D_k(\lambda) P_k = J_k(\lambda) \oplus J_k(\bar{\lambda}).$$

**Proof.** Proceed by induction on $k$: the base-case when $k = 1$ is trivial, so we assume the assertion holds for matrices of the form (2.7) of dimension $2(k-1)$. If $z_n$ denotes the $n \times 1$ zero vector, then

$$D(\lambda) = \begin{bmatrix} \lambda & e_2^T \\ z_{2(k-1)} & D_{2(k-1)}(\bar{\lambda}) & e_{2(k-1)} \\ 0 & z_{2(k-1)}^T & \bar{\lambda} \end{bmatrix},$$

and if $\hat{P}$ is the permutation matrix defined by

$$\hat{P} := \begin{bmatrix} 1 & z_{2(k-1)}^T \\ z_{2(k-1)} & P_{2(k-1)} & z_{2(k-1)} \\ 0 & z_{2(k-1)}^T & 1 \end{bmatrix} \in M_{2k}(\mathbb{R}),$$

then, following the induction-hypothesis,

$$\hat{P}^T D(\lambda) \hat{P} = \begin{bmatrix} \lambda & z_{k-1}^T & e_1^T & 0 \\ z_{k-1} & J_{k-1}(\bar{\lambda}) & Z_{k-1} & z_{k-1} \\ z_{k-1}^T & J_{k-1}(\lambda) & e_{k-1} \\ 0 & z_{k-1}^T & \bar{\lambda} \end{bmatrix}. \hfill (2.9)$$
A permutation-similarity by the matrix \( \bar{P} \) defined by
\[
\bar{P} := \begin{bmatrix}
1 & & \\
& I_{k-1} & \\
& 1 & \\
\end{bmatrix} \in M_{2k}(\mathbb{R})
\]
brings the matrix in the right-hand-side of (2.9) to the desired form. The proof is completed by noting that
\[
\hat{P} \bar{P} = \begin{bmatrix}
e_1 & e_2 & \ldots & e_{2(k-1)} & e_3 & \ldots & e_{2k-1} & e_{2k}
\end{bmatrix} \begin{bmatrix}
1 & & \\
& I_{k-1} & \\
& 1 & \\
\end{bmatrix} = P_k,
\]
since right-hand multiplication by \( \bar{P} \) permutes columns 2 through \( k \) with columns \( k + 1 \) through \( 2k - 1 \).

**Corollary 2.11.** Let \( \lambda \in \mathbb{C}, \lambda \neq 0 \), and let \( f \) be a function defined on the spectrum of \( J_k(\lambda) \oplus J_k(\bar{\lambda}) \). For \( j \) a nonnegative integer, let \( f^{(j)}(\lambda) \) denote \( f^{(j)}(\lambda) \). If \( C_k(\lambda) \) and \( C(\lambda) \) are defined as in (2.5) and (2.6), respectively, then
\[
f(C_k(\lambda)) = \begin{bmatrix}
C(f_\lambda) & C(f'_\lambda) & \ldots & C\left(\frac{f_{(k-1)}^{(k-1)}}{(k-1)!}\right) \\
& C(f_\lambda) & \ddots & \\
& \ddots & \ddots & C(f'_\lambda) \\
& & & C(f_\lambda)
\end{bmatrix} \in M_{2k}(\mathbb{R})
\]
if and only if \( f^{(j)}(\lambda) = f^{(j)}(\bar{\lambda}) \).

**Proof.** Following Lemmas 2.9 and 2.10,
\[
P_k^T S_k^{-1} C_k(\lambda) S_k P_k = J_k(\lambda) \oplus J_k(\bar{\lambda}).
\]
Since \( f(A) = f(X^{-1}AX) \) (10, Theorem 1.13(c)), \( f(A \oplus B) = f(A) \oplus f(B) \) (10, Theorem 1.13(g)), and \( f^{(j)}(\lambda) = f^{(j)}(\bar{\lambda}) \) for all \( j \), applying \( f \) to (2.10) yields
\[
P_k^T S_k^{-1} f(C_k(\lambda)) S_k P_k = f(J_k(\lambda)) \oplus f(J_k(\bar{\lambda})) = f(J_k(\lambda)) \oplus f(J_k(\lambda)).
\]
Hence,

\[ S_k^{-1} f(C_k(\lambda)) S_k = P_k \left[ f(J_k(\lambda)) \oplus f(J_k(\lambda)) \right] P_k^T \]

\[ = \begin{bmatrix}
D(f_{\lambda}) & D(f'_{\lambda}) & \cdots & D\left(\frac{f^{(k-1)}}{(k-1)!}\right) \\
D(f_{\lambda}) & \cdots & & \\
& \cdots & & \\
& & & \cdots
\end{bmatrix}
\]

and

\[ f(C_k(\lambda)) = S_k \begin{bmatrix}
D(f_{\lambda}) & D(f'_{\lambda}) & \cdots & D\left(\frac{f^{(k-1)}}{(k-1)!}\right) \\
D(f_{\lambda}) & \cdots & & \\
& \cdots & & \\
& & & \cdots
\end{bmatrix} S_k^{-1} \]

\[ = \begin{bmatrix}
C(f_{\lambda}) & C(f'_{\lambda}) & \cdots & C\left(\frac{f^{(k-1)}}{(k-1)!}\right) \\
C(f_{\lambda}) & \cdots & & \\
& \cdots & & \\
& & & \cdots
\end{bmatrix}
\]

The converse follows from noting that, for \( \mu, \nu \in \mathbb{C} \), the product

\[ S \begin{bmatrix} \mu & 0 \\ 0 & \nu \end{bmatrix} S^{-1} = \begin{bmatrix} -i & -i \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \mu & 0 \\ 0 & \nu \end{bmatrix} \frac{1}{2} \begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix} \]

\[ = \frac{1}{2} \begin{bmatrix} -i\mu & -i\nu \\ \mu & -\nu \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -1 \end{bmatrix} \]

\[ = \frac{1}{2} \begin{bmatrix} \mu + \nu & i(\nu - \mu) \\ i(\mu - \nu) & \mu + \nu \end{bmatrix} \]

is real if and only if \( \bar{\nu} = \mu \).

Moreover, from our analysis, it also follows that, in general,

\[ f(C_k(\lambda)) = \begin{bmatrix}
f(C_{\lambda}) & f'(C_{\lambda}) & \cdots & \frac{f^{(k-1)}(C_{\lambda})}{(k-1)!} \\
f(C_{\lambda}) & \cdots & & \\
& \cdots & & \\
& & & \cdots
\end{bmatrix} \in M_{2k}(\mathbb{C}), \]
which bears a striking resemblance to (2.1).

Corollary 2.12. If $\lambda \in \mathbb{C}$, $\Im(\lambda) \neq 0$ and

$$F_j(C_k(\lambda)) := S_k P_k \begin{bmatrix} f_{j_1}(J_k(\lambda)) & 0 \\ 0 & f_{j_2}(J_k(\bar{\lambda})) \end{bmatrix} P_k^T S_k^{-1} \in M_{2k}(\mathbb{C}), \quad (2.11)$$

where $j = (j_1, j_2)$ and $j_1, j_2 \in \{0, 1, \ldots, p - 1\}$, then

$$F_j(C_k(\lambda)) = \begin{bmatrix} C(f_{j_1}(\lambda)) & C(f'_{j_1}(\lambda)) & \ldots & C\left(\frac{f_{j_1}^{(k-1)}(\lambda)}{(k-1)!}\right) \\ \vdots & \ddots & \ddots & \vdots \\ C(f_{j_1}(\lambda)) & \ldots & C(f'_{j_1}(\lambda)) & C(f_{j_1}(\lambda)) \end{bmatrix} \in M_{2k}(\mathbb{R})$$

if and only if $j_1 = j_2 = 0$ or $j_1 = j_2 - p$.

**Proof.** Follows from Lemma 2.4 and Corollary 2.11.

Corollary 2.13. If $\lambda = r \exp(i\pi) \in \mathbb{C}$, where $r > 0$, and $F_j$ is defined as in (2.11), then

$$F_j(C_k(\lambda)) = \begin{bmatrix} C(f_{j_1}(\lambda)) & C(f'_{j_1}(\lambda)) & \ldots & C\left(\frac{f_{j_1}^{(k-1)}(\lambda)}{(k-1)!}\right) \\ \vdots & \ddots & \ddots & \vdots \\ C(f_{j_1}(\lambda)) & \ldots & C(f'_{j_1}(\lambda)) & C(f_{j_1}(\lambda)) \end{bmatrix} \in M_{2k}(\mathbb{R})$$

if and only if $j_1 + j_2 = p - 1$.

**Proof.** Follows from Lemma 2.5 and Corollary 2.11.

The next theorem provides a necessary and sufficient condition for the existence of a real $p$th-root of a real $A$ (see [9, Theorem 2.3]) and our proof utilizes the real Jordan canonical form.

Theorem 2.14 (Existence of real $p$th-root). A matrix $A \in M_n(\mathbb{R})$ has a real $p$th-root if and only if it satisfies the ascent sequence condition specified in Theorem 2.7 and, if $p$ is even, $A$ has an even number of Jordan blocks of each size for every negative eigenvalue.
Proof. Case 1: $p$ is even. Following Theorem 2.8, there exists a real, invertible matrix $R$ such that

$$A = R \begin{bmatrix} J_0 & J_+ & J_- \end{bmatrix} R^{-1}$$

where $J_0$ collects the singular Jordan blocks; $J_+$ collects the Jordan blocks with positive real eigenvalues; $J_-$ collects Jordan blocks with negative real eigenvalues; and $C$ collects blocks of the form (2.5) corresponding to the complex conjugate pairs of eigenvalues of $A$.

By hypothesis, if $J_k(\lambda)$ is a submatrix of $J_-$, it must appear an even number of times; for every such pair of blocks, it follows that

$$P_k [J_k(\lambda) \oplus J_k(\lambda)] P_k^T = C_k(\lambda),$$

where $P_k$ is defined as in (2.8). Thus, there exists a permutation matrix $P$ such that

$$A = RP^T P \begin{bmatrix} J_0 & J_+ & J_- \end{bmatrix} P^T P R^{-1} = \tilde{R} \begin{bmatrix} J_0 & J_p & \tilde{C} \end{bmatrix} \tilde{R}^{-1},$$

where $\tilde{R} = RP^T$, and $C$ collects all the blocks of the form (2.5).

Since the ascent sequence condition holds for $A$ it also holds for $J_0$, so $J_0$ has a $p$th-root $W_0$, and $W_0$ can be taken real in view of the construction given in [14, Section 3]; clearly, there exists a real matrix $W_+$ such that $W_+^p = J_+$ and, following Corollaries 2.12 and 2.13, there exists a real matrix $W^c$ such that $W^c_p = \tilde{C}$. Hence, the matrix $X = \tilde{R}[W_0 \oplus W_+ \oplus W_c] \tilde{R}^{-1}$ is a real $p$th-root of $A$.

Conversely, if $A$ satisfies the ascent sequence condition and has an odd number of Jordan blocks corresponding to a negative eigenvalue, then the process just described can not produce a real matrix $p$th-root, as one of the Jordan blocks can not be paired, so that the root of such a block is necessarily complex.

Case 2: $p$ is odd. Follows similarly to the first case since real roots can be taken for $J_0$, $J_+$, $J_-$, and $C$. \qed
We now present an analog of Theorem 2.6 for real matrices.

**Theorem 2.15** *(Classification of $p$th-roots of nonsingular real matrices).* Let $F_k$ be defined as in (2.11). If $A \in M_n(\mathbb{R})$ is nonsingular, then $A$ has precisely $p^s$ primary $p$th-roots, given by

$$X_j = R \left[ \bigoplus_{k=1}^r f_{j_k}(J_{n_k}(\lambda_k)) \quad 0 \right] \bigoplus_{k=r+1}^{r+c} F_{j_k}(C_{n_k}(\lambda_k)) R^{-1}, \quad (2.12)$$

where $j = (j_1, \ldots, j_r, j_{r+1}, \ldots, j_{r+c})$, $j_k = (j_{k_1}, j_{k_2})$ for $k = r + 1, \ldots, r + c$, and $j_i = j_k$ whenever $\lambda_i = \lambda_k$.

If $s < t$, then $A$ has additional nonprimary $p$th-roots that form parameterized families of the form

$$X_j(U) = RU \left[ \bigoplus_{k=1}^r f_{j_k}(J_{n_k}(\lambda_k)) \quad 0 \right] \bigoplus_{k=r+1}^{r+c} F_{j_k}(C_{n_k}(\lambda_k)) U^{-1}R^{-1}, \quad (2.13)$$

where $U$ is an arbitrary nonsingular matrix that commutes with $J_R$, and for each $j$ there exist $i$ and $k$, depending on $j$, such that $\lambda_i = \lambda_k$ while $j_i \neq j_k$.

**Proof.** Following Theorem 2.8, there exists a real, invertible matrix $R$ such that

$$R^{-1}AR = J_R = \left[ \bigoplus_{k=1}^r J_{n_k}(\lambda_k) \bigoplus_{k=r+1}^{r+c} C_{n_k}(\lambda_k) \right];$$

if $T = \left[ I \bigoplus_{k=r+1}^{r+c} S_{n_k} P_{n_k} \right]$, then, following Lemmas 2.9 and 2.10, it follows that

$$T^{-1}J_RT = J = \left[ \bigoplus_{k=1}^r J_{n_k}(\lambda_k) \bigoplus_{k=r+1}^{r+c} \left[ J_{n_k}(\lambda_k) \oplus J_{n_k}(\bar{\lambda}_k) \right] \right].$$

Following Theorem 2.6, $J_R$ has $p^s$ primary roots given by

$$T \left[ \bigoplus_{k=1}^r f_{j_k}(J_{n_k}(\lambda_k)) \bigoplus_{k=r+1}^{r+c} \left[ f_{j_{k_1}}(J_{n_k}(\mu_k)) \oplus f_{j_{k_2}}(J_{n_k}(\bar{\mu}_k)) \right] \right] T^{-1}$$

$$= \left[ \bigoplus_{k=1}^r f_{j_k}(J_{n_k}(\lambda_k)) \bigoplus_{k=r+1}^{r+c} \left[ f_{j_{k_1}}(J_{n_k}(\mu_k)) \oplus f_{j_{k_2}}(J_{n_k}(\bar{\mu}_k)) \right] \right],$$

where $j_k = (j_{k_1}, j_{k_2})$ for $k = r + 1, \ldots, r + c$, which establishes (2.12).
If $A$ is derogatory, then $J_R$ has additional roots of the form

$$TW\left[\bigoplus_{k=1}^{r} f_{jk}(J_{nk}(\lambda_k)) \bigoplus_{k=r+1}^{r+c} f_{jk_1}(J_{nk}(\mu_k)) \oplus f_{jk_2}(J_{nk}(\bar{\mu}_k))\right]W^{-1}T^{-1},$$

where $W$ is any matrix that commutes with $J$. Note that

$$TW\left[\bigoplus_{k=1}^{r} f_{jk}(J_{nk}(\lambda_k)) \bigoplus_{k=r+1}^{r+c} f_{jk_1}(J_{nk}(\mu_k)) \oplus f_{jk_2}(J_{nk}(\bar{\mu}_k))\right]W^{-1}T^{-1} = U\left[\bigoplus_{k=1}^{r} f_{jk}(J_{nk}(\lambda_k)) \bigoplus_{k=r+1}^{r+c} F_{jk}(C_{nk}(\lambda_k))\right]U^{-1},$$

where $U = TWT^{-1}$. Following [12, Theorem 1, §12.4], $U$ is an arbitrary, nonsingular matrix that commutes with $J_R$, which establishes (2.13). \(\square\)

The next theorem identifies the number of real primary $p$th-roots of a real matrix (c.f. [9, Theorem 2.4]) and our proof utilizes the real Jordan canonical form.

**Corollary 2.16.** Let the nonsingular real matrix $A$ have $r_1$ distinct positive real eigenvalues, $r_2$ distinct negative real eigenvalues, and $c$ distinct complex-conjugate pairs of eigenvalues. If $p$ is even, there are (a) $2^r p^c$ real primary $p$th-roots when $r_2 = 0$; and (b) no real primary $p$th-roots when $r_2 > 0$. If $p$ is odd, there are $p^c$ real primary $p$th-roots.

**Proof.** Following Theorem 2.8, there exists a real, invertible matrix $R$ such that

$$R^{-1}AR = J_R = \left[\bigoplus_{k=1}^{r} J_{nk}(\lambda_k) \bigoplus_{k=r+1}^{r+c} C_{nk}(\lambda_k)\right].$$

Case 1: $p$ is even. If $r_2 > 0$, then $A$ does not possess a real primary root, since $A$ must have an even number of Jordan blocks of each size for every negative eigenvalue, and the same branch of the $p$th-root function must be selected for every Jordan block containing the same negative eigenvalue. If $r_2 = 0$, then, following Corollary 2.12, for every complex-conjugate pair of eigenvalues, there are $p$ choices such that $F_{jk}(C_{nk}(\lambda_k))$ is real. For every real eigenvalue, there are two choices such that $f_{jk}(J_{nk}(\lambda_k))$ is real, yielding $2^r p^c$ real primary roots.
Case 2: $p$ is odd. The matrix $X_j$ is real provided that the principal-branch of the $p$th-root function is chosen for every real eigenvalue. Similar to the first case, there are $p$ choices such that $F_{j_k}(C_{n_k}(\lambda_k))$ is real, yielding $p^c$ real primary roots.

The following theorem extends Theorem 2.6 to include singular matrices (see [9, Theorem 2.6]).

**Theorem 2.17 (Classification of $p$th-roots).** Let $A \in M_n(\mathbb{C})$ have the Jordan canonical form $Z^{-1}AZ = J = J_0 \oplus J_1$, where $J_0$ collects together all the Jordan blocks corresponding to the eigenvalue zero and $J_1$ contains the remaining Jordan blocks. If $A$ possesses a $p$th-root, then all $p$th-roots of $A$ are given by $A = Z(X_0 \oplus X_1)Z^{-1}$, where $X_1$ is any $p$th-root of $J_1$, characterized by Theorem 2.6 and $X_0$ is any $p$th-root of $J_0$.

3. Main Results

We now focus on the $p$th-roots of primitive matrices and matrices possessing the strong Perron-Frobenius property.

Recall that a matrix $A \in M_n(\mathbb{R})$ is reducible if $n \geq 2$ and there is a permutation matrix $P$ such that

$$A = P^T \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} P$$

where $A_{11}$ and $A_{22}$ are square, nonempty submatrices. A matrix $A$ is irreducible if it is not reducible. A matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ is said to be (entrywise) nonnegative (respectively, positive), denoted $A \geq 0$ (respectively, $A > 0$), if $a_{ij} \geq 0$ (respectively, $a_{ij} > 0$) for all $1 \leq i, j \leq n$. Recall from the Introduction that a matrix $A \in M_n(\mathbb{R})$ is eventually positive (nonnegative) if there exists a nonnegative integer $p$ such that $A^k$ is entrywise positive (nonnegative) for all $k \geq p$. If $p$ is the smallest such integer, then $p$ is called the power index of $A$ and is denoted by $p(A)$.

We recall the Perron-Frobenius theorem for positive matrices (see [15, Theorem 8.2.11]).

**Theorem 3.1.** If $A \in M_n(\mathbb{R})$ is positive, then

(a) $\rho := \rho(A) > 0$;

(b) $\rho \in \sigma(A)$;
(c) there exists a positive vector $x$ such that $Ax = \rho x$;
(d) $\rho$ is a simple eigenvalue of $A$.
(e) $|\lambda| < \rho$ for every $\lambda \in \sigma(A)$ such that $\lambda \neq \rho$.

There are nonnegative matrices containing entries that are zero that satisfy Theorem 3.1. Recall that a nonnegative matrix $A \in M_n(\mathbb{R})$ is said to be primitive if it is irreducible and has only one eigenvalue of maximum modulus. The conclusions to Theorem 3.1 apply to primitive matrices (see [15, Theorem 8.5.1]), and the following theorem is a useful characterization of primitivity (see [15, Theorem 8.5.2]).

**Theorem 3.2.** If $A \in M_n(\mathbb{R})$ is nonnegative, then $A$ is primitive if and only if $A^k > 0$ for some $k \geq 1$.

One can verify that the matrix $\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$ possesses properties (a) through (e) of Theorem 3.1 is irreducible, but obviously contains a negative entry. This motivates the following concept.

**Definition 3.3.** A matrix $A \in M_n(\mathbb{R})$ is said to possess the strong Perron-Frobenius property if $A$ possesses properties (a) through (e) of Theorem 3.1.

The following theorem characterizes the strong Perron-Frobenius property (see [16, Lemma 2.1], [4, Theorem 1], or [5, Theorem 2.2]).

**Theorem 3.4.** A real matrix $A$ is eventually positive if and only if $A$ and $A^T$ possess the strong Perron-Frobenius property.

We now present our main results.

**Theorem 3.5.** Let the nonsingular primitive matrix $A$ have $r_1$ distinct positive real eigenvalues, $r_2$ distinct negative real eigenvalues, and $c$ distinct complex-conjugate pairs of eigenvalues. If $p$ is even, there are (a) $2^{r_1-1}p^c$ eventually positive primary $p$th-roots when $r_2 = 0$; and (b) no eventually positive primary $p$th-roots if $r_2 > 0$. If $p$ is odd, there are $p^c$ eventually positive primary $p$th-roots.

**Proof.** If $r_2 > 0$, then, following Theorem 2.16, the matrix $A$ does not have a real primary $p$th-root, hence, a fortiori, it can not have an eventually positive primary $p$th-root.
If \( r_2 = 0 \), then, following Theorems 2.8 and 3.1, there exists a real, invertible matrix \( R \) such that

\[
A = \begin{bmatrix} x & R' \end{bmatrix} \begin{bmatrix} \rho \bigoplus_{k=2}^{r} J_{n_k}(\lambda_k) \\ \bigoplus_{k=r+1}^{r+c} C_{n_k}(\lambda_k) \end{bmatrix} \begin{bmatrix} y^T \\ (R^{-1})' \end{bmatrix}, \tag{3.1}
\]

where \( R = \begin{bmatrix} x & R' \end{bmatrix} \in M_n(\mathbb{R}) \), \( R^{-1} = \begin{bmatrix} y \\ (R^{-1})' \end{bmatrix} \in M_n(\mathbb{R}) \), \( x > 0 \) is the right Perron-vector, and \( y > 0 \) is the left Perron-vector.

Because \( |f_j(z)| < |f_{j'}(z')| \) for all \( j, j' \in \{0, 1, \ldots, p - 1\} \), any primary \( p \)th-root \( X \) of the form

\[
X_j = R \begin{bmatrix} \sqrt[p]{\rho} \\ \bigoplus_{k=2}^{r} f_{j_k}(J_{n_k}(\lambda_k)) \\ \bigoplus_{k=r+1}^{r+c} F_{j_k}(C_{n_k}(\lambda_k)) \end{bmatrix} R^{-1}
\]

inherits the strong Perron-Frobenius property from \( A \). Since \( f(A)^T = f(A^T) \) ([10, Theorem 1.13(b)]), a similar argument demonstrates that \( X^T \) inherits the strong Perron-Frobenius property from \( A^T \).

Case 1: \( p \) is even. For all \( k = 2, \ldots, r \), there are two possible choices such that \( f_{j_k}(J_{n_k}(\lambda_k)) \) is real; following Corollary 2.12 for all \( k = r + 1, \ldots, r + c \), there are \( p \) choices such that \( F_{j_k}(C_{n_k}(\lambda_k)) \) is real. Thus, there are \( 2^{r_1 - 1} p^c \) possible ways to select \( X \) and \( X^T \) to be real.

Case 2: \( p \) is odd. For all \( k = 2, \ldots, r \), the principal-branch of the \( p \)th-root function must be selected so that \( f_{j_k}(J_{n_k}(\lambda_k)) \) is real and, similar to the previous case, there are \( p \) choices such that \( F_{j_k}(C_{n_k}(\lambda_k)) \) is real. Hence, there are \( p^c \) ways to select \( X \) and \( X^T \) real.

In either case, following Theorem 3.4, the matrices \( X \) and \( X^T \) are eventually positive.

**Example 3.6.** We demonstrate Theorem 3.5 via an example. Consider the matrix

\[
A = \frac{1}{5} \begin{bmatrix} 16 & 16 & 6 & 11 & 1 \\ 7 & 12 & 12 & 12 & 7 \\ 9 & 4 & 14 & 9 & 14 \\ 8 & 8 & 13 & 13 \\ 10 & 10 & 10 & 5 & 15 \end{bmatrix}.
\]
Note that $A = RJR^{-1}$, where

$$\begin{align*}
J_R &= \begin{bmatrix}
10 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & -1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & -1 & 1
\end{bmatrix},
R &= \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & -1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & -1
\end{bmatrix},
\end{align*}$$

and

$$R^{-1} = \frac{1}{5} \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & -4 & 1 & 1 & 1 \\
1 & 1 & -4 & 1 & 1 \\
1 & 1 & 1 & -4 & 1 \\
1 & 1 & 1 & 1 & -4
\end{bmatrix}.$$

Because $\sigma (A) = \{10, 1 + i, 1 + i, 1 - i, 1 - i \}$, following Theorem 3.5, $A$ has eight primary matrix square-roots, of which the matrices

$$X_j = R \begin{bmatrix}
\sqrt{10} & 0 & 0 & 0 & 0 \\
0 & 1.0987 & 0.4551 & 0.3884 & -0.1609 \\
0 & -0.4551 & 1.0987 & 0.1609 & 0.3884 \\
0 & 0 & 0 & 1.0987 & 0.4551 \\
0 & 0 & 0 & -0.4551 & 1.0987
\end{bmatrix} R^{-1}$$

$$= \begin{bmatrix}
1.6668 & 1.0232 & 0.1130 & 0.4738 & -0.1145 \\
0.2762 & 1.3749 & 0.7313 & 0.6646 & 0.1153 \\
0.3939 & -0.0612 & 1.4926 & 0.5548 & 0.7823 \\
0.3217 & 0.3217 & 0.3217 & 1.4204 & 0.7768 \\
0.5037 & 0.5037 & 0.5037 & 0.0486 & 1.6024
\end{bmatrix},$$

where $j = (0, 0, 0)$, and
\[
X_j' = R \begin{pmatrix}
\sqrt{10} & 0 & 0 & 0 & 0 \\
0 & -1.0987 & -0.4551 & -0.3884 & 0.1609 \\
0 & 0.4551 & -1.0987 & -0.1609 & -0.3884 \\
0 & 0 & 0 & -1.0987 & -0.4551 \\
0 & 0 & 0 & 0.4551 & -1.0987 \\
\end{pmatrix} R^{-1}
\]

\[
= \begin{pmatrix}
-0.4019 & 0.2417 & 1.1519 & 0.7911 & 1.3794 \\
0.9887 & -0.1100 & 0.5336 & 0.6003 & 1.1496 \\
0.8710 & 1.3261 & -0.2276 & 0.7101 & 0.4826 \\
0.9432 & 0.9432 & 0.9432 & -0.1555 & 0.4881 \\
0.7612 & 0.7612 & 0.7612 & 1.2163 & -0.3375 \\
\end{pmatrix},
\]

where \( j' = (0, (1, 1)) \), are eventually positive square-roots of \( A \).

The following question arises from Example 3.6: are \( X_j \) and \( X_j' \) the only eventually positive square-roots of \( A \)? This is answered in the following result, which yields an explicit description of the eventually positive primary roots of a nonsingular primitive matrix \( A \).

**Theorem 3.7.** Let \( A \) be a nonsingular, primitive matrix and \( X_j \) be any primary \( p \)th-root of \( A \) of the form

\[
X_j = R \left[ f_{j_1}(\rho) \bigoplus_{k=2}^{r} f_{j_k}(J_{n_k}(\lambda_k)) + \bigoplus_{k=r+1}^{r+c} F_{j_k}(C_{n_k}(\lambda_k)) \right] R^{-1},
\]

where \( j = (j_1, \ldots, j_r, j_{r+1}, \ldots, j_{r+c}) \) and \( j_k = (j_{k_1}, j_{k_2}) \), for \( k = r+1, \ldots, r+c \).

If \( p \) is odd, then \( X_j \) is eventually positive if and only if

1. \( j_1 = 0 \);
2. \( j_k = 0 \) for all \( k = 2, \ldots, r \); and
3. \( j_k = (0, 0) \) or \( j_k = (j_{k_1}, p - j_{k_1}) \) for all \( k = r + 1, \ldots, r + c \).

If \( p \) is even, then \( X_j \) is eventually positive if and only if

1. \( j_1 = 0 \);
2. \( j_k = 0 \) or \( j_k = p/2 \) for all \( k = 2, \ldots, r \); and
3. \( j_k = (0, 0) \) or \( j_k = (j_{k_1}, p - j_{k_1}) \) for all \( k = r + 1, \ldots, r + c \).
Proof. We demonstrate necessity as sufficiency is shown in the proof of Theorem 3.5.

To this end, we demonstrate the contrapositive. Case 1: $p$ is odd. If the principal branch of the $p$th-root function is not selected for the Perron eigenvalue, then $X_j$ cannot possess the strong Perron-Frobenius property; if $j_k \neq 0$ for some $k \in \{2, \ldots, r\}$, then $f_{j_k}(J_{n_k}(\lambda_k))$ is not real so that $X_j$ cannot be real; similarly, $X_j$ is not real if $j_k \neq (0,0)$ or $j_k \neq (j_{k_1}, p - j_{k_1})$ for some $k \in \{r + 1, \ldots, r + c\}$.

Case 2: $p$ is even. Result is similar to the first case, but we note that, without loss of generality, we may assume that $A$ does not have any negative eigenvalues (else it cannot possess a primary root).

Theorem 3.8. Let $A$ be a primitive, nonsingular, derogatory matrix that possesses a real root, and let $X_j(U)$ be any nonprimary root of $A$ of the form

\[
X_j(U) = RU \left[ f_{j_1}(\rho) \bigoplus_{k=2}^r f_{j_k}(J_{n_k}(\lambda_k)) \bigoplus_{k=r+1}^{r+c} F_{j_k}(C_{n_k}(\lambda_k)) \right] U^{-1} R^{-1},
\]

where $j = (j_1, \ldots, j_r, j_{r+1}, \ldots, j_{r+c})$ and $j_k = (j_{k_1}, j_{k_2})$, for $k = r + 1, \ldots, r + c$. If $p$ is even, then $X_j(U)$ is eventually positive if and only if

1. $j_1 = 0$;
2. $j_k = 0$, or $j_k = p/2$, for all $k = 2, \ldots, r$;
3. $j_k = (0,0)$ or $j_k = (j_{k_1}, p - j_{k_1})$ for all $k = r + 1, \ldots, r + c$;
4. $U$ is selected to be real and nonsingular; and
5. Jordan blocks containing negative eigenvalues are transformed, via a permutation matrix, to blocks of the form \((2.5)\) (see proof of Theorem 2.14) and branches for these blocks are selected in accordance with Corollary 2.13 subject to the constraint that for each $j$, there exist $i$ and $k$, depending on $j$, such that $\lambda_i = \lambda_k$, while $j_i \neq j_k$.

If $p$ is odd, then $X_j(U)$ is eventually positive if and only if

1. $j_1 = 0$;
2. $j_k = 0$ for all $k = 2, \ldots, r$;
3. $j_k = (0,0)$ or $j_k = (j_{k_1}, p - j_{k_1})$ for all $k = r + 1, \ldots, r + c$; and
4. $U$ is selected to be real and nonsingular; subject to the constraint that for each $j$, there exist $i$ and $k$, depending on $j$, such that $\lambda_i = \lambda_k$, while $j_i \neq j_k$. 

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Example 3.9. We demonstrate Theorem 3.8 via an example. Consider the matrix

\[
A = \begin{bmatrix}
32 & 32 & 14 & 23 & 5 & 32 & 14 & 23 & 5 \\
17 & 26 & 26 & 26 & 17 & 17 & 17 & 17 & 17 \\
19 & 10 & 28 & 19 & 28 & 19 & 19 & 19 & 19 \\
18 & 18 & 18 & 27 & 27 & 18 & 18 & 18 & 18 \\
20 & 20 & 20 & 11 & 29 & 20 & 20 & 20 & 20 \\
17 & 17 & 17 & 17 & 17 & 26 & 26 & 26 & 17 \\
19 & 19 & 19 & 19 & 19 & 10 & 28 & 19 & 28 \\
18 & 18 & 18 & 18 & 18 & 18 & 27 & 27 & 27 \\
20 & 20 & 20 & 20 & 20 & 20 & 20 & 11 & 29
\end{bmatrix}.
\]

Note that \( A = R J_R R^{-1} \), where

\[
J_R = \begin{bmatrix}
20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}
\]

\[
R = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}.
\]
and

\[
R^{-1} = \frac{1}{9} \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -8 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -8 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -8 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -8 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -8 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -8 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -8 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -8 \\
\end{bmatrix}.
\]

If \( j = (0, (0, 0), (1, 1)) \), then, following Theorem 3.8, any matrix of the form

\[
X_j(U) = RU \begin{bmatrix}
\sqrt{20} & F_{(0,0)}(C_2(1+i)) \\
& F_{(1,1)}(C_2(1+i))
\end{bmatrix} R^{-1} U^{-1},
\]

where

\[
U = \begin{bmatrix}
u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & u_2 & u_3 & 1 & 0 & u_4 & u_5 & 1 & 0 \\
0 & -u_3 & u_2 & 0 & 1 & -u_5 & u_4 & 0 & 1 \\
0 & 0 & 0 & u_2 & u_3 & 0 & 0 & u_4 & u_5 \\
0 & 0 & 0 & -u_3 & u_2 & 0 & 0 & -u_5 & u_4 \\
0 & u_6 & u_7 & 1 & 0 & u_8 & u_9 & 1 & 0 \\
0 & -u_7 & u_6 & 0 & 1 & -u_9 & u_8 & 0 & 1 \\
0 & 0 & 0 & u_6 & u_7 & 0 & 0 & u_8 & u_9 \\
0 & 0 & 0 & -u_7 & u_6 & 0 & 0 & -u_9 & u_8 \\
\end{bmatrix} \in M_9(\mathbb{R}),
\]

det \((U) \neq 0\), is an eventually positive square-root of \( A \).

Next, we present an analog of Theorem 2.17 for real matrices.

**Theorem 3.10.** Let the primitive matrix \( A \in M_n(\mathbb{R}) \) have the real Jordan canonical form \( R^{-1}AR = J_\mathbb{R} = J_0 \oplus J_1 \), where \( J_0 \) collects all the singular Jordan blocks and \( J_1 \) collects the remaining Jordan blocks. If \( A \) possesses a real root, then all eventually positive \( p \)th-roots of \( A \) are given by \( A = R(X_0 \oplus X_1)R^{-1} \), where \( X_1 \) is any \( p \)th-root of \( J_1 \), characterized by Theorem 3.7 or Theorem 3.8 and \( X_0 \) is a real \( p \)th-root of \( J_0 \).
Remark 3.11. It should be clear that Theorems 3.5, 3.7, 3.8, and 3.10 remain true if the assumption of primitivity is replaced with eventually positivity.

Recall that for $A \in M_n(\mathbb{C})$ with no eigenvalues on $\mathbb{R}^-$, the principal $p$th-root, denoted by $A^{1/p}$, is the unique $p$th-root of $A$ all of whose eigenvalues lie in the segment $\{z : -\pi/p < \arg(z) < \pi/p\}$ \cite[Theorem 7.2]{10}. In addition, recall that a nonnegative matrix $A$ is said to be stochastic if $\sum_{j=1}^n a_{ij} = 1$, for all $i = 1, \ldots, n$.

In \cite{9}, being motivated by discrete-time Markov-chain applications, two classes of stochastic matrices were identified that possess stochastic principal $p$th-roots for all $p$. As a consequence, and of particular interest, for these classes of matrices the twelfth-root of an annual transition matrix is itself a transition matrix. A (monthly) transition matrix that contains a negative entry or is complex is meaningless in the context of a model, however the following remark demonstrates that, under suitable conditions, a matrix-root of a primitive stochastic matrix will be eventually stochastic (i.e., eventually positive with row sums equal to one).

Eventual stochasticity may be useful in the following manner: consider, for example, the application of discrete-time Markov chains in credit risk: let $P = [p_{ij}] \in M_n(\mathbb{R})$ be a primitive transition matrix where $p_{ij} \in [0, 1]$ is the probability that a firm with rating $i$ transitions to rating $j$. Such matrices are derived via annual data, and the twelfth-root of such a matrix would correspond to a monthly transition matrix if the entries are nonnegative.

In application, the twelfth-root would be used for forecasting purposes (e.g., to estimate the likelihood that a firm with credit-rating $i$, transitions to rating $j$, $m$ months into the future). Thus, if $R$ is the twelfth-root of $P$ and $R^m \geq 0$, then $r_{ij}^{(m)}$ is a candidate for the aforementioned probability. Moreover, it is well-known that $n^2 - 2n + 2$ is a sharp upper-bound for the primitivity index of a primitive matrix (see, e.g., \cite[Corollary 8.5.9]{15}) so that eventual stochasticity is feasible in application.

Remark 3.12. Theorems 3.5, 3.7, 3.8, and 3.10 remain true if stochasticity is added as an assumption on the matrix $A$ and the conclusion of eventually positivity is replaced with eventually stochasticity.

We conclude with the following remark.

Remark 3.13. If $A$ is an eventually positive matrix matrix, then $B := A^q$
is eventually positive for all $q \in \mathbb{N}$: thus, all the previous results hold for rational powers of $A$.

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