Decoherence for Markov chains

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Abstract
It is known that the subspace generated by the eigenvectors pertaining to the peripheral spectrum of any stochastic matrix is canonically equipped with a structure of a (finite-dimensional abelian) $C^*$-algebra under a canonical new product introduced by E.G. Effros and M.-D. Choi. We prove that the restriction of the action of such a stochastic matrix to this subspace is indeed a $*$-automorphism. The following new decoherence result is then established: any Markov chain encodes a conservative $C^*$-dynamical system, after isolation of the persistent part from the transient one. This result gives a partial answer to the general and currently unsolved decoherence problem for a relevant class of systems.

1. Introduction

The universally accepted concept of quantum decoherence was mathematically axiomatized by P. Blanchard and R. Olkiewicz. It is related to the properties of separation into the persistent and transient parts of a dissipative $C^*$-dynamical system, typically representing a ‘small’ system interacting with a huge reservoir, see [1].

Such kind of $C^*$-dynamical systems are described by a strongly continuous one-parameter ($C_0$ for short) semigroup. However, in order to capture most of the main properties, we can still consider discrete dynamics generated by Unital Completely Positive (UCP for short) linear maps. In all forthcoming analysis, we restrict the matter to the simpler picture described by such maps.

Indeed, let $(\mathcal{A}, \Phi)$ be a $C^*$-dynamical system consisting of a unital $C^*$-algebra $\mathcal{A}$ on which the UCP map $\Phi$ is acting. In this context, it is said that the decoherence takes place if the whole system can be split in some kind of direct sum as $\mathcal{A} = \mathcal{N}(\Phi) \bigoplus \mathcal{A}_o$, where

$$\mathcal{N}(\Phi) := \{ x \in \mathcal{A} \mid \Phi(x^*x) = \Phi(x^*)\Phi(x), \ \Phi(xx^*) = \Phi(x)\Phi(x^*) \},$$

often referred to as the multiplicative domain of $\Phi$, and

$$\mathcal{A}_o := \left\{ x \in \mathcal{A} \mid \lim_{n \to +\infty} \| \Phi^n x \| = 0 \right\}.$$
As it is explained in [8], a slightly different notion of decoherence seems to be strictly connected to the spectral properties of the involved UCP map $\Phi$. Apart from the case of $*$-automorphisms for which the decoherence is trivially satisfied, for the majority of the cases of interest, $\sigma(\Phi)$ is the whole disk, and thus it appears complicated to provide a splitting of $\mathcal{A}$ in a direct sum as above.\(^1\)

However, for the so-called gapped UCP maps, those for which the peripheral spectrum is topologically separated from the part lying inside the unit disk, a splitting of the involved algebra

$$
\mathcal{A} = P_\Phi \mathcal{A} + \left( I_{\mathcal{B}(\mathcal{A})} - P_\Phi \right) \mathcal{A}
$$

into the persistent and transient part is easily obtained using the holomorphic functional calculus, see [8], Section 3.

It is also straightforward to see that the persistent part $P_\Phi \mathcal{A}$ is a norm-closed subspace, trivially containing the identity of $\mathcal{A}$, closed under the adjoint but in general not under the product, i.e. it is merely an operator system, see e.g. [4] for more on this object.

Among some interesting situations of gapped UCP maps, there are certainly all those acting on finite-dimensional $C^*$-algebras. In this, apparently simple, situation for which a quite complete description of properties of UCP maps is still a formidable task, we point out that the persistent part can always be equipped with a new product, in general different from the original one, making such a persistent part a ‘genuine’ $C^*$-algebra. This crucial result is due to Choi and Effros, see [4], Theorem 3.1.

One could address the natural question whether the UCP map $\Phi$ under consideration, restricted to $P_\Phi \mathcal{A}$, the latter equipped with the Choi–Effros product, is indeed a $*$-automorphism, then providing a genuine conservative dynamical system. In this way, the decoherence would take place just by considering the persistent part with the new $C^*$-structure. In this regard, it should be noticed that some simple and natural examples do not enjoy the original notion of decoherence explained above, see [8], Section 6.

The solution to this intriguing problem is still open, even in the simplest finite-dimensional case. Yet, it would be of interest to provide an answer, at least for some specified class of UCP maps. Among them, there are the stochastic matrices associated to Markov chains, one of the most studied objects in modern mathematics.

The present note is indeed devoted to prove that any Markov chain encodes a genuine conservative dynamical system, after separating the persistent part from the transient part, the latter exponentially vanishing in the limit taken on the iterations of the involved stochastic matrix. In the concrete situation, up to measurement errors, this happens after a finite number of iterations, depending on the size of the so-called mass gap, i.e. the distance between the peripheral spectrum and the part lying inside the unit disk.\(^2\)

2. Preliminaries

Basic notation. If it is not otherwise specified, in the present note we only deal with everywhere defined linear maps between vector spaces. All $C^*$-algebras $\mathcal{A}$ are unital with $I \equiv I_{\mathcal{A}}$.

For the Banach algebra $\mathcal{B}(\mathcal{A})$ consisting of all bounded operators acting on the $C^*$-algebra $\mathcal{A}$, we also put $I := I_{\mathcal{B}(\mathcal{A})}$.

For involutive algebras $\mathcal{C}_i$, $i = 1, 2$, a map $\Psi : \mathcal{C}_1 \to \mathcal{C}_2$ is said to be selfadjoint or real if $\Psi(x^*) = \Psi(x)^*$ for every $x \in \mathcal{C}_1$. 

For the $C^*$-algebra $\mathfrak{A}$, the map $\Phi : \mathfrak{A} \to \mathfrak{A}$ is said to be completely positive if $\Phi \otimes \text{id}_{\mathcal{M}_n(C)} := \Phi_n : \mathcal{M}_n(\mathfrak{A}) \to \mathcal{M}_n(\mathfrak{A})$ is positive for each $n = 1, 2, \ldots$. In particular, it is positive if $\Phi_1$ is. It is unital if $\Phi(I) = I$. If $\mathfrak{A}$ is abelian, then complete positivity coincides with positivity, see e.g. [12], Theorem 1.2.4.

For a self-map $T \in \mathcal{B}(\mathcal{X})$ on the Banach space $\mathcal{X}$, the peripheral spectrum is $\sigma_{\text{ph}}(T) := \{\lambda \in \sigma(T) \mid |\lambda| = \text{spr}(T)\}$, $\text{spr}(T)$ being the spectral radius. Such a map $T$ is said to be gapped if it presents the so-called mass-gap, that is $\text{dist}(\sigma_{\text{ph}}(T), \sigma(T) \setminus \sigma_{\text{ph}}(T)) > 0$. In the case of UCP maps $\Phi$, $\text{spr}(\Phi) = 1 = \|\Phi\|$, and thus $\sigma_{\text{ph}}(T)$ is contained in the unit circle $\mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Moreover, if the UCP map $\Phi$ is gapped, then $\mathfrak{A} = P_\Phi \mathfrak{A} + Q_\Phi \mathfrak{A}$ with $Q_\Phi := \frac{1}{2\pi i} \int_{\gamma} (\lambda I - \Phi)^{-1} d\lambda$, $P_\Phi := I - Q_\Phi$. Here, $\gamma$ is a counterclockwise Jordan curve inside the open unit disk surrounding $\sigma(\Phi) \setminus \sigma_{\text{ph}}(\Phi)$.

Therefore, in the case of gapped UCP maps, $\lim_n \|\Phi^n(x)\| = 0$, $x \in Q_\Phi \mathfrak{A}$, see [8], Proposition 3.1.

**Jordan morphisms.** To reduce the matter to our setting, we deal only with Jordan algebras made of the selfadjoint part $\mathfrak{A}_{\text{sa}}$ of a $C^*$-algebra $\mathfrak{A}$ equipped with the Jordan product given by

$$\mathfrak{A}_{\text{sa}} \ni a, b \mapsto a \circ b := \frac{1}{2} (ab + ba) \in \mathfrak{A}_{\text{sa}}.$$ 

A selfadjoint map $\Phi : \mathfrak{A} \to \mathfrak{B}$ between $C^*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ is said an order-isomorphism if it is invertible and $\Phi$ and $\Phi^{-1}$ are both positive. If $\mathfrak{A}$ coincides with $\mathfrak{B}$, we speak of order-automorphism.

For the convenience of the reader, we report the following result, crucial for our analysis.

**Theorem 2.1 ([12], Thm. 2.1.3):** A unital selfadjoint map $\Phi : \mathfrak{A} \to \mathfrak{B}$ is an order-isomorphism if and only if $\Phi |_{\mathfrak{A}_{\text{sa}}} : \mathfrak{A}_{\text{sa}} \to \mathfrak{B}_{\text{sa}}$ is a Jordan isomorphism.

**$C^*$-dynamical systems.** A discrete $C^*$-dynamical system is simply a triple $(\mathfrak{A}, \Phi, M)$, where $\mathfrak{A}$ is a $C^*$-algebra, $\Phi$ a UCP map acting on $\mathfrak{A}$ via its powers, and $M$ is the monoid $\mathbb{N}$ or $\mathbb{Z}$. By definition, the case relative to the group $\mathbb{Z}$ corresponds to $\Phi$ being a $*$-automorphism. In this context, we talk about microscopically reversible $C^*$-dynamical systems.\(^3\) Alternative terminologies are ‘conservative’, ‘Hamiltonian’ and ‘unitary’ systems. We will treat instead the dissipative cases, when $M = \mathbb{N}$ and $\Phi$ is in general not invertible. The simplified notation $(\mathfrak{A}, \Phi)$ will stand for $(\mathfrak{A}, \Phi, \mathbb{N})$.

Given a Banach space $X$, let $T \in \mathcal{B}(X)$ such that $\text{spr}(T) = 1$. It is customary to set the space of the almost periodic elements of $T$ as

$$\text{AP}(T) := \text{closed linear span of all eigenvectors pertaining to the peripheral eigenvalues of } T,$$

see e.g. [6] for a standard situation.

Then, we point out the following

**Remark 2.1:** Let $\Phi : \mathfrak{A} \to \mathfrak{A}$ be an irreducible UCP map. By [9], Proposition 3.2, $\text{AP}(\Phi) \subset P_\Phi(\mathfrak{A})$ is a $C^*$-subalgebra of $\mathfrak{A}$, and the restriction $\Phi |_{\text{AP}(\mathfrak{A})}$ of $\Phi$ to $\text{AP}(\mathfrak{A})$ is automatically a $*$-automorphism.
Here, we are using the notion of irreducibility in [9], Definition 2.2. We see in Proposition 2.2 below that such a definition of irreducibility is equivalent to the analogous one used for positive matrices.

**Stochastic matrices.** A stochastic matrix \( S \in \mathbb{M}_n(\mathbb{R}) \) is nothing but a UCP map acting on \( \mathbb{C}^n \), and thus for all entries, \( S_{ij} \geq 0 \). The \( C^* \)-dynamical systems \((\mathbb{C}^n, S)\) describe the dynamics of the so-called **Markov chains** (e.g. [11]). The structure of a stochastic matrix is briefly outlined in Section 4. Here, we recall their basic properties.

Since the probability at each step of the transition (i.e. after the repeated application of \( S \) on vectors of \( \mathbb{C}^n \)) must be conserved, we have that \( 1 := \left[ \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right] \) is a right eigenvector of \( S \) with eigenvalue 1: \( S 1 = 1 \).

On the other hand, any non negative row-vector \( [\pi_1 \pi_2 \cdots \pi_n] \) with \( \sum_{i=1}^{n} \pi_i = 1 \), which is also a left eigenvector corresponding to the eigenvalue 1, is a **stationary distribution** for the associated Markov chain. The multiplicities of the left and right eigenvalue 1 coincide.

A positive, and in particular a stochastic, matrix \( A \) of order \( n \) is said to be **irreducible** if there exists no permutation-matrix \( P \) of order \( n \) such that \( PAP^{-1} = \begin{bmatrix} A_1 & B \\ O & A_2 \end{bmatrix} \), where \( A_i \) is a square-matrix of order \( n_i \) with \( 1 \leq n_i < n \), see e.g. [10,11].

For the convenience of the reader, we show that the definition of irreducibility provided in Definition 2.2 of [9] coincides with the previous one for stochastic matrices.

**Proposition 2.2:** Let \( A \) be a positive matrix. Then, it is irreducible if and only if the only \( A \)-invariant faces of the positive cone \( \mathbb{C}^n_+ = \bigoplus_{j=1}^{n} \mathbb{R}^+ e_j \) are \( \{0\} \) and the whole \( \mathbb{C}^n_+ \).

**Proof:** Suppose that \( A \) is irreducible, and there exists a face \( F := \bigoplus_{j \in J} \mathbb{R}^+ e_j \) for some \( J \neq \emptyset, \{1, \ldots, n\} \) which is invariant under \( A \), that is \( A(F) \subset F \). Consider any permutation \( P \) which reorders the canonical basis as \( \{e_i\}_{i \in J} \cup \{e_i\}_{i \in J^c} \). Then, \( PAP^{-1} = \begin{bmatrix} A_1 & B \\ O & A_2 \end{bmatrix} \) with \( A_i \) a square matrix of size \( 1 \leq m_i = |J| \leq n - 1 \) and \( A_2 \) a square matrix of size \( 1 \leq m_2 \leq n - 1 \), which is a contradiction.

Suppose now that there exists a permutation matrix \( P \) such that \( PAP^{-1} = \begin{bmatrix} A_1 & B \\ O & A_2 \end{bmatrix} \) as above. Consider the set \( \{e_i\}_{i \in J} := \{Pe_i\}_{i=1}^{m_1} \). Then, \( \bigoplus_{j \in J} \mathbb{R}^+ e_j \) is a non-trivial face of \( \mathbb{C}^n_+ \) which is invariant under \( A \), which is again a contradiction. \[\square\]

### 3. On UCP maps on finite-dimensional \( C^* \)-algebras

The present section is devoted to basic results, perhaps well known to the experts, on which is based the forthcoming analysis for stochastic matrices. We report those for the convenience of the reader.

**Proposition 3.1:** Let \( \Phi : \mathfrak{A} \to \mathfrak{A} \) be a UCP map on the finite-dimensional \( C^* \)-algebra \( \mathfrak{A} \). Then there exists a subsequence of natural numbers \( (n_j) \subset \mathbb{N} \) such that \( \lim_j \Phi^{n_j} = P_\Phi \).
**Proof:** By using the Jordan decomposition of $\Phi$, we have

$$\Phi = \sum_{\lambda \in \sigma_{\text{ph}}(\Phi)} \lambda E_\lambda + Q_\Phi \Phi,$$

with $E_\lambda E_\mu = \delta_{\lambda, \mu} E_\lambda$, $\sum_{\lambda \in \sigma_{\text{ph}}(\Phi)} E_\lambda = P_\Phi$.

Since $\sigma_{\text{ph}}(\Phi)$ is a finite subset of the circle (not necessarily a finite subgroup), there exists a subsequence $(n_j)_j$ of natural numbers such that $\lim_j \lambda^{n_j} = 1$ for each $\lambda \in \sigma_{\text{ph}}(\Phi)$. Then, by Fidaleo et al. [8], Proposition 3.1, and the choice of the subsequence $(n_j)_j$, we get

$$\lim_j \Phi^{n_j} = \lim_j \left( \sum_{\lambda \in \sigma_{\text{ph}}(\Phi)} \lambda E_\lambda + Q_\Phi \Phi \right)^{n_j} = \lim_j \left( \sum_{\lambda \in \sigma_{\text{ph}}(\Phi)} \lambda E_\lambda \right)^{n_j} + \lim_j \left( Q_\Phi \Phi^{n_j} \right) = \lim_j \sum_{\lambda \in \sigma_{\text{ph}}(\Phi)} \lambda^{n_j} E_\lambda = \sum_{\lambda \in \sigma_{\text{ph}}(\Phi)} \lim_j \lambda^{n_j} E_\lambda = P_\Phi. \quad \blacksquare$$

We also report the well-known fact of the mean ergodicity of such maps.

**Proposition 3.2:** Let $\Phi : \mathfrak{A} \to \mathfrak{A}$ be a UCP map on the finite-dimensional C*-algebra $\mathfrak{A}$. Then, $\lim_n \left( \frac{1}{n} \sum_{k=0}^{n-1} \Phi^k \right) = E_1$, the projection onto the fixed-point subspace $\mathfrak{A}^\Phi$. Moreover, $E_1$ is a UCP map.

**Proof:** By performing the same calculations in the above proof, we get

$$\lim_n \left( \frac{1}{n} \sum_{k=0}^{n-1} \Phi^k \right) = E_1 + \sum_{\lambda \in \sigma_{\text{ph}}(\Phi) \setminus \{1\}} \frac{1}{1 - \lambda^n} \lim_n \left( 1 - \lambda^n \right) E_\lambda = E_1,$$

since $|1 - \lambda^n| \leq 2$. For the proof of the second part of the statement see [7], Theorem 2.1. \hfill \blacksquare

Here, we report the main theorem concerning the injective operator systems in the finite-dimensional setting.

For a UCP self-map $\Phi : \mathfrak{A} \to \mathfrak{A}$ of a finite-dimensional C*-algebra $\mathfrak{A}$, define $\mathfrak{A}_\Phi := (P_\Phi \mathfrak{A}, 1_{\mathfrak{A}}, *, \circ, \| \|_{\mathfrak{A}})$ as the operator system $P_\Phi \mathfrak{A}$ equipped with the binary operation $a \circ b := P_\Phi(ab)$.

**Theorem 3.1 (Choi–Effros):** The operator system $\mathfrak{A}_\Phi$ is in fact a C*-algebra.

**Proof:** See Theorem 3.1 of [4]. \hfill \blacksquare
4. Stochastic matrices and associated persistent dynamical systems

The present section starts with some useful facts on stochastic matrices. The second half is devoted to the main result of the present note.

Indeed, let $S$ be a stochastic matrix as defined in Section 2. Up to row-column permutations, any such matrix $S$ has the following canonical form (e.g. [11], Proposition 8.8 and the discussion after Proposition 9.2)

$$S = \begin{bmatrix}
B_{00} & B_{01} & \cdots & B_{0n} \\
0 & B_{11} & 0 & \cdots & 0 \\
\vdots & 0 & B_{22} & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & B_{nn}
\end{bmatrix}. \quad (1)$$

Here, $B_{00}$ is a square strictly sub-stochastic matrix associated to the transient indices (which is the empty matrix if and only if the subset of such transient indices is empty), while the square-block matrices $\{B_{11}, \ldots, B_{nn}\}$ are irreducible. Each of them is the transition matrix of an ergodic component of the Markov chain.

The above is said to be the reduced form of $S$ and, if there are no transient indices, $S$ is said to be completely reducible.

The following theorem collects some properties which are useful in the sequel.

**Theorem 4.1:** Let us consider the canonical form (1) of the stochastic matrix $S$. Then,

(i) for $j = 1, \ldots, n$, $\sigma_{ph}(B_{jj})$ is a subgroup of the circle group $\mathbb{T}$, and the multiplicity of all peripheral eigenvalues is always 1;

(ii) $\sigma(S) = \bigcup_{j=0}^n \sigma(B_{jj})$;

(iii) $\sigma_{ph}(S) = \bigcup_{j=1}^n \sigma_{ph}(B_{jj})$.

**Proof:** (i) Follows by Theorem I.6.5 in [11], where $\sigma_{ph}(B_{jj})$ coincides with the $d_j$-th roots of the unity, $d_j$ being the index of imprimitivity of $B_{jj}$, see [11], Section I.9.

(ii) is well known, see e.g. [13], Section 2.3.

(iii) follows from (ii) because $\sigma_{ph}(B_{00}) = \emptyset$. Indeed, if $B_{00}$ had an eigenvalue $\lambda$ with $|\lambda| = 1$, Proposition I.9.3 in [11] would not hold. ■

Given a stochastic matrix $S \in \mathbb{M}_n(\mathbb{R})$, the associated Markov chain is nothing else than a commutative finite-dimensional $C^*$-dynamical system $(\mathbb{C}^n, S)$, where the matrix $S$ generates, via its non negative powers, the action of the monoid $\mathbb{N}$.

By Theorem 3.1, with $\mathfrak{A} := \mathbb{C}^n$, the linear space $\mathfrak{A}_S = P_S \mathbb{C}^n$ is in fact an abelian $C^*$-algebra with the new Choi-Effros product $\circ$. We now show the main result of the present section: $(\mathfrak{A}_S, S|_{\mathfrak{A}_S})$ provides a genuine conservative $C^*$-dynamical system, and thus the action of $S|_{\mathfrak{A}_S}$ can now be extended to negative powers.

Here below, we give a crucial preliminary

**Lemma 4.1:** With the above notation, $S|_{\mathfrak{A}_S}$ is an order-automorphism.
Proof: As in the proof of Proposition 3.1, we get $S[\mathfrak{A}_S]=\sum_{\lambda\in\sigma_{ph}(S)}\lambda E_{\lambda}$. Since by Theorem 4.1, $\sigma_{ph}(S)=\bigcup_{j=1}^{n} \sigma_{ph}(B_{jj})$ and $\sigma_{ph}(B_{jj})$ is the cyclic subgroup of the circle consisting of the $d_j$-th roots of the unity, $(S[\mathfrak{A}_S])^{\text{l.c.m.}\{d_1,\ldots,d_n\}} = \text{id}_{\mathfrak{A}_S}$. Therefore, $(S[\mathfrak{A}_S])^{-1} = (S[\mathfrak{A}_S])^{\text{l.c.m.}\{d_1,\ldots,d_n\}^{-1}}$ which is manifestly positive.\footnote{\textit{7}}

Theorem 4.2: With the above notation, $S[\mathfrak{A}_S]$ is a $\ast$-automorphism of $\mathfrak{A}_S$.

Proof: We only need to check that, for $a, b \in \mathfrak{A}_S$, $S(a \circ b) = S(a) \circ S(b)$.

By Lemma 4.1, $S[\mathfrak{A}_S]$ is an order-automorphism and thus, by Theorem 2.1, it provides a Jordan isomorphism when restricted to its selfadjoint part. Since $\mathfrak{A}_S$ is abelian, $a \cdot b = \frac{1}{2}(a \circ b + b \circ a) = a \circ b,$

for every selfadjoint elements $a, b \in \mathfrak{A}_S$, and thus $S(a \circ b) = S(a \bullet b) = S(a) \bullet S(b) = S(a) \circ S(b).$

We have then recognized that $S[\mathfrak{A}_S]$ preserves the product when restricted to its selfadjoint part. But $x, y \in \mathfrak{A}_S$ can be written as combinations of two selfadjoint elements and, since the algebra under consideration is abelian, $xy$ is written as a combination of four selfadjoint elements. By linearity, $S[\mathfrak{A}_S]$ preserves the product on the whole $\mathfrak{A}_S$ and the assertion follows.\footnote{\textit{8}}

Remark 4.2: Since $S[\mathfrak{A}_S]$ is invertible, we get a reversible $\mathbb{C}\ast$-dynamical system $(\mathfrak{A}_S, S[\mathfrak{A}_S], \mathbb{Z}).$

We end the section with some considerations. By taking into account Proposition 2.2, point 2 in Proposition 3.2 of [9], and lastly (iii) in Theorem 4.1, we conclude that in the irreducible cases, hence in all completely reducible ones, the Choi–Effros product coincides with the original one. On the other hand, we know that there are examples, necessarily admitting transient indices, for which the original product must be changed, see e.g. [8], Section 6. Therefore, one might conclude that the cases for which the original product should be changed are connected with the presence of transient indices.

Unfortunately, also this conjecture does not hold in general. For instance, when all the imprimitivity indices $d_j, j = 1, 2, \ldots, n$, of the square-block matrices in (1) are $1$, $\sigma_{ph}(S) = \{1\}$ with multiplicity $n$ and the original product need not to be changed. All in all, the cases for which the original product might be replaced with the Choi–Effros one are to be found among those with a nonempty set of transient indices and at least one ergodic imprimitive component $B_{j_{ojo}}$.

5. Some simple noncommutative examples

The present section is devoted to some noncommutative explanatory examples for which the restriction of the original UCP map to the persistent part, the last equipped with the Choi–Effros product, provides indeed a $\ast$-automorphism.
The first example we want to briefly discuss is the UCP map on the full matrix algebra $\mathbb{M}_n(\mathbb{C})$ obtained by pulling over the action of a stochastic matrix $S \in \mathbb{M}_n(\mathbb{R})$ through the conditional expectation $E_n : \mathbb{M}_n(\mathbb{C}) \to D_n \subset \mathbb{M}_n(\mathbb{C})$, $D_n$ being the Maximal Abelian Sub-Algebra consisting of the diagonal elements. Explicitly, $\Phi = S E_n$, and $P_\Phi = P S E_n$. Therefore, $\sigma (\Phi) = \sigma (S) \cup \{0\}$, where 0 appears with multiplicity $n(n-1) + m$, $m$ being its multiplicity in $S$.

It is easy to recognize that $\mathbb{M}_n(\mathbb{C})/\Phi \sim \mathbb{C}^n_S$, and the conservative dynamical system $(\mathbb{M}_n(\mathbb{C})/\Phi, \Phi \lceil \mathbb{M}_n(\mathbb{C})/\Phi)$ associated to $\Phi$ is $*$-isomorphic to that of $(\mathbb{C}^n_S, S \lceil \mathbb{C}^n_S)$ both equipped, possibly, with the modified Choi–Effros product.

We now present a simple example concerning a class of UCP maps on $\mathbb{M}_2(\mathbb{C})$. For the structure of completely positive maps between matrix algebras, see [3].

Indeed, let $\Phi : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})$ be a UCP map of the form

\[
\begin{align*}
\Phi (\cdot) &= V_1^* (\cdot) V_1 + V_2^* (\cdot) V_2, \\
V_1^* V_1 + V_2^* V_2 &= I,
\end{align*}
\]

where the partial isometries $V_1, V_2$ have the form

\[
\begin{align*}
V_1 (\cdot) &= \langle \cdot | e_1 \rangle x, \\
V_2 (\cdot) &= \langle \cdot | e_2 \rangle y.
\end{align*}
\]

Here, $\{e_1, e_2\}$ is the canonical basis of $\mathbb{C}^2$ and $x, y \in \mathbb{C}^2$ arbitrary vectors with norm 1 which can be written as

\[
x = e^{i\theta} \cos \alpha \ e_1 + e^{i\theta} \sin \alpha \ e_2, \quad y = e^{i\gamma} \cos \beta \ e_1 + e^{i\gamma} \sin \beta \ e_2.
\]

It follows that

\[
\Phi \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \cos^2 \alpha + \frac{a_{12} + a_{21}}{2} \sin 2\alpha + a_{22} \sin^2 \alpha & 0 \\ 0 & a_{11} \cos^2 \beta + \frac{a_{12} + a_{21}}{2} \sin 2\beta + a_{22} \sin^2 \beta \end{bmatrix},
\]

and thus the matrix $M_\Phi$ of $\Phi$ w.r.t. the basis $\{e_{11}, e_{22}, e_{12}, e_{21}\}$ of $\mathbb{M}_2(\mathbb{C})$ is

\[
M_\Phi = \begin{bmatrix}
cos^2 \alpha & \sin^2 \alpha & \frac{1}{2} \sin 2\alpha & \frac{1}{2} \sin 2\alpha \\
\cos^2 \beta & \sin^2 \beta & \frac{1}{2} \sin 2\beta & \frac{1}{2} \sin 2\beta \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix} S & B \\ O & O \end{bmatrix}.
\]

It follows that $\sigma (M_\Phi) = \sigma (S) \cup \{0\}$ and $\sigma_{ph}(M_\Phi) = \sigma_{ph}(S)$. 

Suppose now that \( \lambda \in \sigma(M_{\Phi}) \setminus \{0\} \). With \( \xi = \begin{bmatrix} a_{11} \\ a_{22} \end{bmatrix} \) and \( \eta = \begin{bmatrix} a_{12} \\ a_{21} \end{bmatrix} \), \( M_{\Phi} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \lambda \begin{bmatrix} \xi \\ \eta \end{bmatrix} \) reads

\[
S\xi + B\eta = \lambda \xi & \lambda \eta = 0,
\]

which leads to

\[
\eta = 0 \& S\xi = \lambda \xi.
\]

Therefore, even if the situation of the latter example is different from the former one, at the same time we have \( M_2(\mathbb{C}) \sim \mathbb{C}_S^2 \), with \( (M_2(\mathbb{C}), \Phi|_{M_2(\mathbb{C})}) \) being \(*\)-isomorphic to \((\mathbb{C}_S^2, S|_{\mathbb{C}_S^2})\).

### 6. Concluding remarks

In classical physics, time evolution is described by a suitable differential equation, and studying the possible superposition of a persistent and a transient part turns out to be very natural.

In all these systems, such an analysis simply describes a partition of relevant physical quantities into the part which is vanishing (transient part) and the one surviving (persistent part) when \( t \to +\infty \), provided such a splitting can be performed.\(^8\) This means that, for the investigation of the long-time behaviour, only the persistent part is substantial and, obviously, only the properties of the surviving portion of the original dynamical system are encoded by the (surviving, i.e. restricted) time evolution.

As a useful example when building electrical measurement tools, we mention the forced RLC circuit.\(^9\) To summarize, in the classical situation, such a simplified notion of decoherence consisting in the splitting into a transient and a persistent part is well understood.

With the arrival of quantum mechanics, the precise axiomatization of the measurement process assumed a fundamental role, and thus things appear much more complicated. One of the axioms contemplates that the set of observable quantities is modelled by some suitable Jordan algebra. Since the structure of a Jordan algebra is very far from being completely understood, to provide many significant models and avoid many technical troubles, it is usually assumed that such a Jordan algebra is taken as the selfadjoint part of a \( C^* \)-algebra without referring to any experimental evidence, see e.g. [5], Section 2.

By coming back to the universally accepted (quantum) decoherence, on one hand it takes place when the whole system can be described by the superposition of the multiplicative domain, automatically a \( C^* \)-algebra under its own multiplicative operation, and the remaining part which disappears in the long-time behaviour. On the other hand, as it happens for gapped UCP maps, there are very simple examples which do not satisfy this standard definition of decoherence. However, the part pertaining to the peripheral spectrum can still be separated by the remaining one: the former provides a dynamical (not necessarily \( C^* \)) system which survives, and the latter disappears in the long-time behaviour.

The aim of the present paper was to show that Markov chains, a relevant class of commutative examples, encode a conservative \( C^* \)-dynamical system after isolating the persistent part from the transient one, and equipping the former with the Choi–Effros product. Such a conservative dynamical system is in general larger than that consisting merely of the
multiplicative domain. This result appears as a relevant step to provide a partial answer to the general and currently unsolved decoherence problem. Therefore, it would be of interest either to prove or to disprove in the full generality in which sense a dissipative system admits a decomposition as that described above.

The second alternative, that is to provide an example of dissipative system for which the persistent dynamical system is not described by a \( C^* \)-algebra at all, has also a relevance for the foundation of physics. In fact, it would imply the existence of perfectly meaningful dynamical systems whose observables are not described by the selfadjoint part of any \( C^* \)-algebra.

As suggested in the present paper, such examples should be searched among dissipative noncommutative finite-dimensional dynamical systems. Even in this quite simple situation, such a project seems to be a formidable task, and the noncommutative examples described in Section 5 are an attempt towards this direction.

**Notes**

1. If one deals with \( C_0 \) semigroup \( e^{-tA} \), the case of the whole closed unit disc for the spectrum of \( \Phi \) would correspond for the generator \( A \) to \( \sigma(A) \) being equal to the whole right half of the complex plane.
2. The terminology ‘mass-gap’ comes from physical motivations since such a distance is dimensionally equivalent to a mass.
3. Physical systems that are macroscopically irreversible, but microscopically reversible, typically describe temperature states (or, equivalently, states satisfying the Kubo–Martin–Schwinger boundary condition) since their dynamics is generated by unitary operators, see e.g. [2].
4. In the language of Markov chains, the transient indices are associated to the so-called transient (or inessential) ‘states’.
5. If the index of imprimitivity \( d_j \) of a block \( B_{jj} \) in the reduced form (1) is 1, then \( B_{jj} \) is said to be primitive.
6. As witnessed by the matrix

\[
\begin{bmatrix}
1/2 & 1/4 & 1/4 \\
0 & 2/3 & 1/3 \\
0 & 0 & 1
\end{bmatrix},
\]

\( B_{00} = \begin{bmatrix}
1/2 & 1/4 \\
0 & 2/3
\end{bmatrix} = \begin{bmatrix}
\sigma(B_{00}) & B_{10}^* \\
0 & \sigma(B_{00})^*
\end{bmatrix}
\]

can be further reduced. Yet, \( \sigma(B_{00}) = \{1/3, 2/3\} = \sigma(B_{00}^0) \cup \sigma(B_{00}^1) \).
7. If all blocks \( B_{jj}, j = 1, \ldots, n, \) are primitive, that is \( d_j = 1, \) then on one hand \( S[\mathbb{A}_S] = \text{id}_{\mathbb{A}_S}, \) while on the other hand \( \text{l.c.m.}\{d_1, \ldots, d_n\} - 1 = 0 \) which means \( (S[\mathbb{A}_S])^{-1} = \text{id}_{\mathbb{A}_S} = (S[\mathbb{A}_S])^{\text{l.c.m.}\{d_1, \ldots, d_n\} - 1} \).
8. Here, the parameter \( t \) describes the time evolution.
9. Here, \( R\ L\) and \( C \) stand for ‘resistance’ ‘inductance’ and ‘capacity’, respectively.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

**Funding**

The authors acknowledge the MIUR ‘Dipartimenti di Eccellenza’ awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006. The authors are members of the Italian research group GNAMPA (Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica).
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