Anomalous Hall effect for semiclassical chiral fermions

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A B S T R A C T

Semiclassical chiral fermions manifest the anomalous spin-Hall effect: when put into a pure electric field they suffer a side jump, analogous to what happens to their massive counterparts in non-commutative mechanics. The transverse shift is consistent with the conservation of the angular momentum. In a pure magnetic field, instead, spiraling motion is found. Motion in Hall-type perpendicular electric and magnetic fields is also studied.

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Semiclassical massless chiral fermions have attracted considerable recent interest [1–14]. The model in [4], for example, describes a spin-\(1/2\) system with positive helicity and energy by the phase-space action

\[
S = \int \left( (p + eA) \cdot \frac{dx}{dt} - (|p| + e\phi) - a \cdot \frac{dp}{dt} \right) dt,
\]

where also involves an additional “momentum-dependent vector potential” \(a(p)\) for the “Berry monopole” in \(p\)-space [15],

\[
\nabla_p \times a = \Theta = \frac{\hat{p}}{2|p|},
\]

where \(\hat{p}\) is the unit vector \(\hat{p} = p/|p|\). Here \(A(x)\) and \(\phi(x)\) are ordinary vector and scalar potentials and \(e\) is the electric charge. Variation of the chiral action (1) yields the equations of motion for position \(x\) and momentum \(p\) \(\neq 0\) in three-space,

\[
\begin{aligned}
\frac{dx}{dt} &= \hat{p} + eE \times \Theta + (\Theta \cdot \hat{p})eB, \\
\frac{dp}{dt} &= eE + e\hat{p} \times B + e^2(E \cdot B)\Theta.
\end{aligned}
\]

where \(E\) and \(B\) are the electric and magnetic field, respectively, and \(m = 1 + e\Theta \cdot B\) is an effective mass.

These equations are strongly reminiscent of those in non-commutative mechanics [15,16] which allowed us to propose a simple mechanical model for the anomalous Hall effect, i.e., the transverse shift observed in some ferromagnetic materials in the absence of any magnetic field [17,18].

The aim of this Note is to prove a similar result also for massless fermions.

(i) Let us first study what happens within the chiral model in a constant pure electric field, \(B = 0\), \(E = E\hat{y}\), \(E = \) const. The equations of motion (2) then become,

\[
\begin{aligned}
\frac{dx}{dt} &= \hat{p} + eE \times \Theta, \\
\frac{dp}{dt} &= eE.
\end{aligned}
\]

Thus \(p(t) = p_0 + eEt\). Choosing, for example, the initial momentum \(p_0 = e\lambda\hat{x}\) and \(x = 0\) for initial position, we get,

\[
\begin{aligned}
x(t) &= \text{argsh} t, \\
y(t) &= \sqrt{1 + t^2} - 1, \\
z(t) &= -\frac{1}{2eE} \frac{t}{\sqrt{1 + t^2}}
\end{aligned}
\]

cf. Fig. 1. The remarkable result is that, due to the anomalous velocity term in Eq. (3), the particle follows a 3D trajectory even for planar initial conditions: for large values of \(|t|\), the motion is approximately a chain curve \(y(x) = \cosh x - 1\) (as it would be for \(\Theta = 0\) lying, for \(t = \pm \infty\), in the planes \(z(\pm \infty) = \mp (2eE)^{-1}\), respectively. With \(|t|\) approaching 0, however, the trajectory abruptly leaves the initial plane of the motion and suffers a transverse shift perpendicular to \(E\) and the initial momentum,

\[
\Delta z = z(\infty) - z(-\infty) = -\frac{1}{2eE}.
\]

Thus, the chiral model provides a (semi)classical description of the anomalous Hall effect.

A similar behavior was observed before for the non-relativistic dispersion relation \(E = p^2/2\) [18]. It is also reminiscent of the “side jump” for the scattering of free chiral fermions [9], as well as of the optical Hall effect [19].

The shift formula (5) is consistent with the conservation of the angular momentum. The constant electric field breaks the full
rotational symmetry to $O(2)$ of rotations around the direction (chosen to be $\hat{y}$) of $E$; $|E|$ times the associated angular momentum is

$$\ell \equiv E \cdot (x \times p) + \frac{E \cdot p}{2|p|}.$$  

(6)

Then $\ell = eE^2z(t) + \frac{eE}{2\sqrt{1+t^2}} = 0$ for $t = 0$ and thus for any $t$; therefore $eE\Delta z = -1$.

(iii) Let us assume instead that we have a constant pure magnetic field, $E = 0$ and $B = B\hat{z}$, $B = \text{const}$. Then the chiral equations of motion (2) reduce to

$$\begin{align*}
\frac{dx}{dt} &= \hat{p} + \frac{1}{2}\frac{eB}{|p|^2} \hat{z} \quad \text{where} \quad m = 1 + \frac{e}{|\hat{p}|}B, \\
\frac{dp}{dt} &= e\hat{p} \times B \\
|p|, \hat{p} \cdot B &= B \cos \theta \quad \text{and} \quad p_z = \hat{p} \cdot \hat{z} \quad \text{and thus also} \quad m \quad \text{are constants of the motion.}
\end{align*}$$  

(7)

If the effective mass does not vanish, $m \neq 0$, then $p$ precesses around the $z$-axis with angular velocity $\omega = -eB/|p|m$, $p(t) = (p_0 e^{-ieB/|p|m}t, p_2)$ where $p_0 = p_{x0} + ip_{y0}$. The anomalous term in the upper relation merely adds to the drift along the $B$ direction,

$$\begin{align*}
x(t) &= \left(\frac{ip_0}{eB} e^{-ieB/|p|m}t, z(t)\right), \\
z(t) &= \left(\cos \theta \frac{eB}{m} + \frac{eB}{2|m^2}t\right) + z_0.
\end{align*}$$  

(8)

This cork-screw-like spiraling motion is reminiscent of the one found for heavy ions [2].

For the sake of comparison, we record $|B$-times$|$ the conserved angular momentum along the magnetic field,

$$\ell = B \cdot x \times p + \frac{1}{2}B \cdot \hat{p} + \frac{e}{2}(B \times x)^2.$$  

(9)

The presence of the last term here is required by the equations of motion (7).

When $|p|^2 = -(eB/2)\cos\theta$, then the effective mass vanishes, $m = 0$, and the system becomes singular; it requires reduction, as does its planar counterpart [20]. Eq. (7) implies that the momentum is vertical, $\theta = 0$ [and therefore $p_z^2 = -(eB/2)$, which in turn requires $eB < 0$]. Then the upper equation is identically satisfied, leaving $x(t)$ undetermined.

(iii) Let us consider what happens when the fields are combined. For simplicity, we only consider perpendicularly electric and magnetic fields, $E = E\hat{y}$, $B = B\hat{z}$, when a massive particle would perform Hall motion. The equations of motion read

$$\begin{align*}
m\ddot{x} &= eB \frac{p_x}{|p|} + eE \frac{p_x}{2|p|^3} \\
m\ddot{y} &= \frac{p_y}{|p|} \quad \text{m} \quad \ddot{z} = -eB \frac{p_x}{|p|} + eE \\
m\ddot{z} &= \frac{p_z}{|p|} - \frac{eB}{2} \frac{p_x}{|p|^3} + eB \frac{e}{2|p|^2} \quad \text{m} \quad \ddot{z} = 0 \quad \text{m} = 1 + \frac{eBp_z}{2|p|^2}.
\end{align*}$$  

(10)

Therefore the $p$-diagram (analogous to the hodograph) is a curve in the horizontal plane $p_z = \text{const}$. Combining the other equations,

$$\frac{dp_x}{dt} = eB \frac{dy}{dt} \quad \Rightarrow \quad p_x = p_{x0} + eB(y - y_0).$$  

(11)

Then, imitating the elementary derivation of energy conservation, we multiply the $p$-equations with $p$ to infer

$$\frac{1}{2m} \frac{d}{dt}(|p|^2) = eEp_y.$$  

(12)

Reinserting into (10) yields

$$\frac{d|p|}{dt} = \frac{E}{B} \frac{dp_x}{dt} \quad \Rightarrow \quad |p| = \frac{E}{B} (p_x - p_{x0}) + |p|_{x0}.$$  

(13)

Dividing $\dot{p}_x$ by $\dot{p}_y$ allows us to deduce the “$p$-hodograph”;

$$\begin{align*}
(B^2 - E^2)p_x - 2E(B|p|_0 - Ep_{x0})p_x + B^2p_y^2 &= (Bp_0 - Ep_{x0})^2 - B^2p_{z0}^2,
\end{align*}$$  

(14)

where $|p|_0, p_{x0}$ are constants of integration. Eq. (14) describes a conic section, namely an ellipse-parabola/hyperbola, depending on $|E|$ being smaller/equal/larger as $|B|$. Bounded $p$-hodographs arise therefore in strong magnetic fields, $|E| < |B|$.

Coming to motion in real space, from (11) we infer that

$$y(t) = \frac{1}{eB} (p_y(t) - p_{y0}) + y_0.$$  

(15)

so that $y(t)$ oscillates when $p_y(t)$ does so, i.e., for $|B| > |E|$.
For the sake of further simplification, we assume henceforth that $p_z = 0$; then
\[ x(t) = \frac{E}{B} t - \frac{1}{eB} (p_y(t) - p_{y0}) + x_0, \]
which is a Hall-type motion perpendicularly to both the electric and magnetic fields, combined with that of $p_y(t)$. This motion is thus unbounded, except in the purely magnetic case $E = 0$.

For the $z$-equation,
\[ \dot{z} = -eE \frac{p_x}{2|p(t)|^3} + \frac{eB}{2|p(t)|^2}, \]
we only found numerical solutions, shown in Figs. 3 and 4. The manifest asymmetry under $B \to -B$ comes from that the system is invariant under the simultaneous changes $B \to -B$ and $p \to -p$ only and here we kept $p(0)$ fixed.

Finite transverse shift, $\Delta z < \infty$, requires the electric field to dominate, $|E| \geq |B|$, since then $|p|$ is unbounded by (14), so that $\dot{z} \to 0$ by (17). But the value of the initial momentum plays a role also, and we have not been able to derive an exact formula like (5), let alone a precise threshold between bounded or unbounded transverse shifts.

Both numerical and analytical study confirm that for $B \to 0$ and for $E = 0$, respectively, the purely electric and magnetic cases are recovered as expected.

**Conclusion**

In this Note, we studied the motion of a semiclassical massless chiral fermion in a constant electromagnetic field. In a dominating electric field the system suffers a finite transverse shift $\Delta z < \infty$ as in the anomalous Hall effect [17], and similar to what was found before in non-commutative mechanics [18]. If the magnetic field dominates, the transverse shift becomes unbounded, $\Delta z = \infty$. The motion parallel to $E$ is oscillatory when $B^2 - E^2 > 0$ and unbounded when $B^2 - E^2 \leq 0$. Perpendicular to both $E$ and $B$ the motion is unbounded except in a pure magnetic field, when it spirals as shown in Fig. 2.

We mention that the same problem can also be studied within the framework proposed in [12]. The motions are substantially different, underlining the importance of the coupling rule one adopts [21].
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