On the Schwartz space isomorphism theorem for rank one symmetric space

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Abstract. In this paper we give a simpler proof of the $L^p$-Schwartz space isomorphism theorem for the class of functions of left $\delta$-type on a Riemannian symmetric space of rank one. Our treatment rests on Anker’s [2] proof of the corresponding result in the case of left $K$-invariant functions on $X$. Thus we give a proof which relies only on the Paley–Wiener theorem.

Keywords. $\delta$ Spherical transform; Helgason Fourier transform.

1. Introduction

Let $X$ be a rank one Riemannian symmetric space of noncompact type. We recall that such a space can be realized as $G/K$, where $G$ is a connected noncompact semisimple Lie group of real rank one with finite center and $K$ is a maximal compact subgroup of $G$. Anker [2], in his paper gave a remarkably short and elegant proof of the $L^p$-Schwartz space isomorphism theorem for $K$ bi-invariant functions on $G$ under the spherical Fourier transform for $(0 < p \leq 2)$. The result for $K$ bi-invariant functions was first proved by Harish-Chandra [6–8] (for $p = 2$) and Trombi and Varadarajan [12] (for $0 < p < 2$). Eguchi and Kowata [4] addressed the isomorphism problem for the $L^p$-Schwartz spaces on $X$. In [2], Anker has successfully avoided the involved asymptotic expansion of the elementary spherical functions, which has a crucial role in all the earlier works. In this paper, we have exploited Anker’s technique to obtain the isomorphism of the $L^p$-Schwartz space $(0 < p \leq 2)$ under Fourier transform for functions on $X$ of a fixed $K$-type.

Let $(\delta, V_\delta)$ be an unitary irreducible representation of $K$ of dimension $\delta$. Our basic $L^p$-Schwartz space $S^p_\delta(X)$ is a space of Hom$(V_\delta, V_\delta)$-valued $C^\infty$ functions, the Eisenstein integral $\Phi_{\lambda, \delta}(x)$ is a Hom$(V_\delta, V_\delta)$-valued entire function on $\mathbb{C}$ and $S_\delta(a_\epsilon^\ast)$ consists of analytic functions on the strip $a_\epsilon^\ast = \{\lambda \in \mathbb{C} | |\text{Im } \lambda | \leq \epsilon \}$. Anticipating these and other notations and definitions developed in §§2 and 3, we state the main result of the paper.

Theorem 1.1. For $0 < p \leq 2$ and $\epsilon = 2/p - 1$ the $\delta$-spherical transform $f \mapsto \tilde{f}$, where

$$\tilde{f}(\lambda) = d(\delta) \int_X \text{tr } f(x) \Phi_{\lambda, \delta}(x)^* dx,$$

(1.1)
is a topological vector space isomorphism between the spaces $S^p_3(X)$ and $S_3(a^*_+)$; with the inverse
\[ f(x) = \omega^{-1} \int_{\mathfrak{a}^*_+} \Phi_{\lambda, \delta}(x) \tilde{f}(\lambda)|c(\lambda)|^{-2} \, d\lambda. \] (1.2)

2. Preliminaries

The pair $(G, K)$ and $X$ are as described in the introduction. We let $G = KAN$ denote a fixed Iwasawa decomposition of $G$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ denote the Lie algebras of $G$, $K$, $A$ and $N$ respectively. We recall that dimension of $\mathfrak{a}^*_+$.

Let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$ and $\mathfrak{a}^*_+\subset \mathfrak{a}^*$ be its complexification. We identify $\mathfrak{a}$, $\mathfrak{a}^*$ with $\mathbb{R}$ and $\mathfrak{a}^*_\mathbb{C}$ with $\mathbb{C}$ using a normalization explained below. Let $H: g \mapsto H(g)$ and $A: g \mapsto A(g)$ be projections of $g \in \mathfrak{g}$ in $\mathfrak{a}$ in Iwasawa $KAN$ and $NAK$ decompositions respectively, that is any $g \in G$ can be written as $g = k \exp H(g)n = n' \exp A(g)k_1$. These two are related by $A(g) = -H^{-1}(g)$ for all $g \in G$. Let $M'$ and $M$ respectively be the normalizer and centralizer of $A$ in $K$. $M$ also normalizes $N$. Let $\mathcal{W} = M'/M$ be the Weyl group of $G$. Here $\mathcal{W} = \{\pm 1\}$. Let us choose and fix a system of positive restricted roots which we denote by $\Sigma^+$. The real number $\rho$ corresponds to $\frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ where $m_\alpha$ is the multiplicity of the root $\alpha$. With a suitable normalization of the basis of $\mathfrak{a}$ we can identify $\rho$ with 1. The positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ is identified with the positive real numbers. We denote $x^+$ to be the $\mathfrak{a}^+$ component of $x \in G$ for the Cartan decomposition $G = K^\mathfrak{a}^+ K = K(\exp \mathfrak{a}^+)K$ and let $|x| = x^+$. We have a basic estimate (Proposition 4.6.11 of [5]): there is a constant $c > 0$ such that
\[ |H(x)| \leq c|x| \quad \text{for} \quad x \in G. \] (2.1)

We note that, any function $f$ on $X$ can also be considered as a function on the group $G$ with the property $f(gk) = f(k)$, where $g \in G$ and $k \in K$. Let $x = ka_tk'$ where $a_t = \exp t \in A$, $t \in \mathfrak{a} \cong \mathbb{R}$. The Haar measure of $G$ for the Cartan decomposition is given by
\[ \int_H f(x) \, dx = \text{const} \int_K dk \int_{\mathfrak{a}^+} \Delta(t) \, dt \int_K dk' f(ka_tk'), \] (2.2)
where $\Delta(t) = \prod_{\alpha \in \Sigma^+} \sinh^{m_\alpha} \alpha(t)$. In the Iwasawa decomposition, $x = katn$, the Haar measure is
\[ \int_g f(x) \, dx = \text{const} \int_K dk \int_{\mathfrak{a}^+} e^{2t} \, dt \int_N df(ka_tn). \] (2.3)

In both (2.2) and (2.3) ‘const’ stands for positive normalizing constants for the respective cases.

Let $(\delta, V^\delta)$ be an unitary irreducible representation of $K$. Let $d(\delta)$ and $\chi_\delta$ stand for the dimension and character of the representation $\delta$. Let $V^M_\delta$ be the subspace of $V^\delta$ fixed under $\delta|_M$; i.e $V^M_\delta = \{v \in V^\delta | \delta(m)v = v, \forall m \in M\}$. Recall that as $G$ is of real rank one, the dimension of $V^M_\delta$ is 0 or 1 (see [11]). Let $\tilde{K}_M$ be the set of all equivalence classes of irreducible unitary representation $\delta$ of $K$ for which $V^M_\delta \neq [0]$. For our result we choose $\delta \in \tilde{K}_M$. We shall also fix an orthonormal basis $\{v_1, v_2, \ldots, v_{d(\delta)}\}$ of $V^\delta$ such that $v_1$ spans $V^M_\delta$. 
We shall denote $D(X)$ for the space of all $C^\infty$ functions on $X$ with compact support. For any function $f \in D(X)$, the Helgason Fourier transform (HFT) (III, §1 of [9]) $Ff$ is defined by

$$Ff(\lambda,kM) = \int_X f(x)e^{i(\lambda-1)H(x^{-1}k)} \, dx.$$ \hspace{1cm} (2.4)

Let us fix the notation $Ff(\lambda,kM) = Ff(\lambda,k)$. The inversion formula for HFT for $f \in D(X)$ is given by

$$f(x) = \frac{1}{\omega} \int_{a^*} \int_K Ff(\lambda,k)e^{-(i\lambda+1)H(x^{-1}k)}|c(\lambda)|^{-2}d\lambda dk.$$ \hspace{1cm} (2.5)

Here, $\omega = |W|$ is the cardinality of the Weyl group and $c(\lambda)$ is the Harish-Chandra $c$-function. For our purpose we shall need the following simple estimate on $c(\lambda)$: there exist constants $c, b > 0$ such that

$$|c(\lambda)|^{-2} \leq c(|\lambda| + 1)^b \hspace{1cm} \text{for} \hspace{1cm} \lambda \in a^*$$ \hspace{1cm} (2.6)

(see, [IV, Proposition 7.2 of [10]).

Let $D(X, \text{Hom}(V_\delta, V_\delta))$ be the space of all $C^\infty$ functions on $X$ taking values in $\text{Hom}(V_\delta, V_\delta)$ and with compact support.

Let $D^\delta(X) = \{ f \in D(X, \text{Hom}(V_\delta, V_\delta)) \mid f(k \cdot x) = \delta(k) f(x) \}$. We topologize $D^\delta(X)$ by the inductive limit topology of the spaces $D_R(X, \text{Hom}(V_\delta, V_\delta))$, where $R = 0, 1, 2, \ldots$. These are the spaces of functions on $X$ with support lying in the geodesic $R$-balls. Let $\delta$ be the contragradient representation of $\delta$. The class of functions $D^\delta(X) = \{ f \in D(X) \mid f = d(\delta) \chi_\delta * f \}$ is the space of all left $\delta$ type functions on $X$. Being a subspace of $D(X)$, $D^\delta(X)$ inherits the subspace topology of $D(X)$. We also notice that, for $f \in C^\infty(X)$ the function

$$f^\delta(x) = d(\delta) \int_K f(k \cdot x) \delta(k^{-1})dk$$ \hspace{1cm} (2.7)

is a $C^\infty$ map from $X$ to $\text{Hom}(V_\delta, V_\delta)$ satisfying

$$f^\delta(k \cdot x) = \delta(k) f^\delta(x).$$

The following lemma (III, Proposition 5.10 of [9]) shows that the two function spaces $D^\delta(X)$ and $D^\delta_\delta(X)$ are topologically isomorphic.

**Lemma 2.1** [9]. The map $Q: D^\delta(X) \rightarrow D^\delta_\delta(X)$ given by

$$Q: f \mapsto \text{tr} f$$

is a homeomorphism with the inverse given by $Q^{-1}(g) = g^\delta$ for $g \in D^\delta_\delta(X)$.

3. The $\delta$-spherical transform

Most of the material in this section can be retrieved from [9]. Here we will restructure the results in a form which is suitable for our purpose. In particular we will transfer the results from $D^\delta(X)$ to $D^\delta_\delta(X)$ using the homomorphism $Q$, defined in Lemma 2.1.
DEFINITION 3.1

For \( f \in D^\delta(X) \) the \( \delta \)-spherical transform \( \tilde{f} \) is given by

\[
\tilde{f}(\lambda) = d(\delta) \int_X \text{tr} f(x) \Phi_{\lambda, \delta}(x)^* dx, \quad \lambda \in \mathbb{C}
\]  

(3.1)

where, \( \Phi_{\lambda, \delta}(x) \) is the generalized spherical function (Eisenstein integral). Precisely,

\[
\Phi_{\lambda, \delta}(x) = \int_K e^{-i(\lambda + 1)H(x^{-1}k)} \delta(k) dk
\]  

(3.2)

and therefore, the adjoint of \( \Phi_{\lambda, \delta}(x) \) is

\[
\Phi_{\lambda, \delta}^*(x) = \int_K e^{i(\lambda - 1)H(x^{-1}k)} \delta(k^{-1}) dk.
\]  

(3.3)

The following is a list of some basic properties of the generalized spherical functions.

1. For \( k \in K \), \( \Phi_{\lambda, \delta}(kx) = \delta(k) \Phi_{\lambda, \delta}(x) \) and \( \Phi_{\lambda, \delta}(kx)^* = \Phi_{\lambda, \delta}(x)^* \delta(k^{-1}) \). For \( v \in V_\delta \) and \( m \in M \), \( \delta(m)(\Phi_{\lambda, \delta}(x)^* v) = \Phi_{\lambda, \delta}(x)^* v \). This shows that \( \Phi_{\lambda, \delta}^* \) is a Hom\((V_\delta, V_M^\delta)\)-valued function on \( X \).

2. Let \( L \) be the Laplace–Beltrami operator of \( X \). Then \( L \Phi_{\lambda, \delta} = -(\lambda^2 + 1)\Phi_{\lambda, \delta} \) (§1(6) of [9]).

3. Let \( U(g_{\mathbb{C}}) \) be the the universal enveloping algebra of \( G \). For any \( g_1, g_2 \in U(g_{\mathbb{C}}) \) there exist constants \( c_\delta = c_\delta(g_1, g_2, \delta), c_0 > 0, b = b(g_1, g_2) \) so that (see [3])

\[
\| \Phi_{\lambda, \delta}(g_1, x, g_2) \| \leq c_\delta(1 + |\lambda|^2) \varphi_0(x) e^{c_0|\text{Im} \lambda|(1 + |x|)}, \quad x \in X.
\]  

(3.4)

Here \( \| \cdot \| \) is the Hilbert–Schmidt norm.

4. If \( \delta \) is the trivial representation of \( K \) then \( \Phi_{\lambda, \delta}(x) \) reduces to the elementary spherical function

\[
\varphi_\delta(x) = \int_K e^{-i(\lambda + 1)H(x^{-1}k)} dk.
\]  

(3.5)

It satisfies the following estimates:

(i) For each \( H \in a^\tau \) and \( \lambda \in a^{\tau+} \),

\[
0 < \varphi_{-1, \delta}(\exp H) \leq e^{\lambda H} \varphi_0(\exp H),
\]  

(3.6)

where, \( \varphi_0(\cdot) \) is the elementary spherical function at \( \lambda = 0 \) (see Proposition 4.6.1 of [5]).

(ii) For all \( g \in G \), \( 0 < \varphi_0(g) \leq 1 \) (Proposition 4.6.3 of [5]) and for \( t \in a^\tau \),

\[
e^{-t} \leq \varphi_0(\exp t) \leq q(1 + t)e^{-t}
\]  

(3.7)

for some \( q > 0 \) (see [1] for a sharper estimate).
5. We have already noticed that $V^M_\delta$ is 1-dimensional. For $\lambda \in \mathfrak{a}_C^*$, $\delta \in \hat{K}_M$ and $x \in X$, the linear functional $\Phi_{\lambda, \delta}(x)|_{V^M_\delta}$ is a scalar multiplication. The elementary spherical function $\phi_\delta$ is related to $\Phi_{\lambda, \delta}$ in the following way (see III, Corollary 5.17 of [9]):

$$\Phi_{\lambda, \delta}(x)|_{V^M_\delta} = Q_{\delta}(\lambda)^{-1} (D_{\lambda, \delta}(\varphi_{\lambda}))(x), \quad (3.8)$$

where $D_{\lambda, \delta}$ is a certain constant coefficient differential operator and $Q_{\delta}(\lambda)$ is a constant real coefficient polynomial in $i\lambda$. An explicit expression for the polynomial $Q_{\delta}$ is available in III, §2 of [9].

6. For each $a \in A$, the functions $\lambda \mapsto Q_{\delta}(\lambda)\Phi_{\lambda, \delta}(a)$ and $\lambda \mapsto Q_{\delta}(\lambda)^{-1}\Phi_{\lambda, \delta}(a)^*$ are even holomorphic functions on $\mathfrak{a}_C^*$ (see III, Theorem 5.15 of [9]).

It follows from 1 and 6 above that for $f \in D^\delta(X)$, $\lambda \mapsto Q_{\delta}(\lambda)f(\lambda)$ is a Hom($V_{\delta}$, $V^M_\delta$)-valued even function on $\mathbb{C}$.

The HFT and the $\delta$-spherical transform of a function $f \in D^\delta(X)$ are related in the following manner.

**DEFINITION 3.2**

Let $\delta \in \hat{K}_M$, $f \in D(X)$ and $Ff$ be its HFT. Then let us define the $\delta$-projection $(Ff)^\delta$ of $Ff$ by

$$ (Ff)^\delta(\lambda, k) = d(\delta) \int_K Ff(\lambda, k) \delta(k^{-1}) dk. \quad (3.9) $$

As noted earlier for $f \in D(X)$, its $\delta$-projection $f^\delta \in D^\delta(X)$. Each of its matrix entry is a member of $D(X)$. We define the HFT of $f^\delta$ by

$$ F(f^\delta)(\lambda, k) = \int_X f^\delta(x) e^{i(\lambda - 1)(x - k)} dx. \quad (3.10) $$

This is nothing but the usual HFT at each matrix entry of $f^\delta$.

**PROPOSITION 3.3**

For $f \in D(X)$ and $\delta \in \hat{K}_M$ the following are true:

1. $(Ff)^\delta(\lambda, k) = \delta(k)(Ff)^\delta(\lambda, e)$.
2. $ F(f^\delta)(\lambda, k) = (Ff)^\delta(\lambda, k).$  

**Proof.** It is clear from the definition that $(Ff)^\delta(\lambda, k) = \delta(k)(Ff)^\delta(\lambda, e)$.

The following straightforward calculation using Fubini’s theorem proves the second assertion.

$$ F(f^\delta)(\lambda, kM) = \int_X f^\delta(x) e^{i(\lambda - 1)(x - k)} dx,$$

$$ = d(\delta) \int_X \left\{ \int_K f(k_1 x) \delta(k_1^{-1}) dk_1 \right\} e^{i(\lambda - 1)(x - k)} dx,$$
The next lemma relates the $\delta$-spherical transform defined in (3.1) with the HFT.

**Lemma 3.4.** If $f \in D^\delta(X)$ and $\delta \in \hat{K}_M$, then $F f(\lambda, e) = \tilde{f}(\lambda)$.

**Proof.** For any $f \in D^\delta(X)$, by Lemma 2.1, $f(x) = d(\delta) \int_K \text{tr} f(kx) \delta(k^{-1}) dk$. From the definition of HFT (2.4) we get

$$F f(\lambda, e) = \int_X f(x) e^{(i\lambda - 1)H(x^{-1})} dx,$$

$$= \int_X d(\delta) \int_K \text{tr} f(kx) \delta(k^{-1}) dk e^{(i\lambda - 1)H(x^{-1})} dx.$$

Substituting $kx = y$ we have

$$F f(\lambda, e) = d(\delta) \int_X \text{tr} f(y) \int_K e^{(i\lambda - 1)H(y^{-1}k)} \delta(k^{-1}) dk,

= d(\delta) \int_X \text{tr} f(y) \Phi_{\lambda, \delta}(y)^* dy,

= \tilde{f}(\lambda).$$

**Lemma 3.5.** The inversion formula for the $\delta$-spherical transform $f \mapsto \tilde{f}$ is given by the following: For each $f \in D^\delta(X)$,

$$f(x) = \frac{1}{\omega} \int_{q^*} \Phi_{\lambda, \delta}(x) \tilde{f}(\lambda) |c(\lambda)|^{-2} d\lambda. \quad (3.11)$$

Moreover,

$$\int_X \|f(x)\|^2 dx = \frac{1}{w} \int_{q^*} \|\tilde{f}(\lambda)\|^2 |c(\lambda)|^{-2} d\lambda. \quad (3.12)$$

Here, the norm $\|\cdot\|$ is the Hilbert–Schmidt norm.
Proof. We use the inversion formula for the HFT (2.5), Proposition 3.3 and Lemma 3.4 to obtain
\[
f(x) = \frac{1}{\omega} \int_{\mathfrak{a}^*} \int_{K} \mathcal{F} f(\lambda, k) e^{-(i\lambda+1)H(x^{-1}k)}|c(\lambda)|^{-2} d\lambda dk
\]
\[
= \frac{1}{\omega} \int_{\mathfrak{a}^*} \int_{K} \delta(k) \mathcal{F} f(\lambda, e) e^{-(i\lambda+1)H(x^{-1}k)}|c(\lambda)|^{-2} d\lambda dk
\]
\[
= \frac{1}{\omega} \int_{\mathfrak{a}^*} \left( \int_{K} e^{-(i\lambda+1)H(x^{-1}k)} \delta(k) dk \right) \mathcal{F} f(\lambda, e) |c(\lambda)|^{-2} d\lambda
\]
\[
= \frac{1}{\omega} \int_{\mathfrak{a}^*} \Phi_{\lambda, \delta}(x) \tilde{f}(\lambda) |c(\lambda)|^{-2} d\lambda.
\]
As the HFT (2.4) of a function \( f \in \mathcal{D}(X) \) is defined entry-wise, it is clear that the Plancherel formula for Helgason Fourier transform is as follows:
\[
\int_X \|f(x)\|^2 dx = \frac{1}{\omega} \int_{\mathfrak{a}^*} \int_{K} \|\tilde{f}(\lambda, k)\|^2 |c(\lambda)|^{-2} d\lambda dk.
\]
(3.13)
Using the relation \( \tilde{f}(\lambda, k) = \delta(k) \tilde{f}(\lambda) \) together with the Schur’s orthogonality relation, the formula (3.12) can be deduced from (3.13).

**DEFINITION 3.6**

A \( C^\infty \) function \( \psi \) on \( \mathfrak{a}^*_c \) with values in \( \text{Hom}(V_\delta, V_M^\delta) \), is said to be of **exponential type** \( R \) if there exists a constant \( R \geq 0 \) such that for each \( N \in \mathbb{Z}^+ \),
\[
\sup_{\lambda \in \mathfrak{a}^*_c} e^{-R|\text{Im}\lambda|(1 + |\lambda|)^N} \|\psi(\lambda)\| < +\infty.
\]

We denote the space of \( C^\infty \) function from \( \mathfrak{a}^*_c \rightarrow \text{Hom}(V_\delta, V_M^\delta) \) of exponential type \( R \) by \( \mathcal{H}^R(\mathfrak{a}^*_c) \). Let \( \mathcal{H}(\mathfrak{a}^*_c) = \bigcup_{R > 0} \mathcal{H}^R(\mathfrak{a}^*_c) \). We state the following topological Paley–Wiener theorem for the \( K \)-types. The proof of this theorem follows from III, Theorem 5.11 of [9] and Lemma 2.1.

**Theorem 3.7.** The \( \delta \)-spherical transform defined in Definition 3.1 is a homeomorphism between the spaces \( \mathcal{D}(X) \) and \( \mathcal{P}_\delta(\mathfrak{a}^*_c) \), where
\[
\mathcal{P}_\delta(\mathfrak{a}^*_c) = \{ F \in \mathcal{H}(\mathfrak{a}^*_c) | (Q^\delta)^{-1} \cdot F \text{ is an even entire function} \}.
\]
Here \( Q^\delta(\lambda) \) is the polynomial in \( i\lambda \) with real coefficients introduced in (3.8).

Let \( \mathcal{P}_0(\mathfrak{a}^*_c) \) denote the set of all even functions in \( \mathcal{H}(\mathfrak{a}^*_c) \), with the relative topology. Let \( h \in \mathcal{P}_0(\mathfrak{a}^*_c) \). By definition, \( h \) is a \( \text{Hom}(V_\delta, V_M^\delta) \)-valued function. As \( V_M^\delta \) is of dimension 1 so we can write \( h = (h_1, \ldots, h_{d(\delta)}) \), where each of \( h_i \) satisfies the following conditions:

(i) it is of exponential type,
(ii) it is entire,
(iii) it is an even function.
Let \( \mathcal{D}(K \setminus X) \) and \( \mathcal{D}(K \setminus X, \text{Hom}(V_\delta, V_\delta^M)) \) denote the left \( K \)-invariant, compactly supported, \( C^\infty \) functions on \( X \) taking values respectively in \( \mathbb{C} \) and \( \text{Hom}(V_\delta, V_\delta^M) \). The spherical transform of \( \phi \in \mathcal{D}(K \setminus X) \) is defined by \( \phi \mapsto \int_X \phi(x) \varphi_\lambda(x^{-1}) \, dx \). For the class \( \mathcal{D}(K \setminus X, \text{Hom}(V_\delta, V_\delta^M)) \) we define it entry-wise. From the Paley–Wiener theorem for the spherical transform [5], there exists one \( f_i \in \mathcal{D}(K \setminus X) \) so that \( h_i(\lambda) = \int_G f_i(x) \varphi_\lambda(x^{-1}) \, dx \). Therefore \( \mathcal{P}_0(\alpha_\cdot^\cdot) \) is the image of \( \mathcal{D}(K \setminus X, \text{Hom}(V_\delta, V_\delta^M)) \) under the spherical transform. The following lemma shows that the Paley–Wiener (PW) spaces \( \mathcal{P}_\delta(\alpha_\cdot^\cdot) \) and \( \mathcal{P}_0(\alpha_\cdot^\cdot) \) are homeomorphic.

**Lemma 3.8 (III, Lemma 5.12 of [9]).** The mapping

\[
\psi(\lambda) \mapsto Q_\lambda(\lambda) \psi(\lambda)
\]

is a homeomorphism of \( \mathcal{P}_0(\alpha_\cdot^\cdot) \) onto \( \mathcal{P}_\delta(\alpha_\cdot^\cdot) \).

**Lemma 3.9.** Any \( f \in \mathcal{D}_\delta(X) \) can be written as \( f(x) = D^\delta \phi(x) \), where \( \phi \in \mathcal{D}(K \setminus X, \text{Hom}(V_\delta, V_\delta^M)) \) and \( D^\delta \) is a certain constant coefficient differential operator.

**Proof.** Let \( f \in \mathcal{D}_\delta(X) \). Then \( \tilde{f} \in \mathcal{P}_0(\alpha_\cdot^\cdot) \). Therefore by Lemma 3.8, the map \( \lambda \mapsto \Phi(\lambda) = Q_\lambda(\lambda)^{-1} \tilde{f}(\lambda) \) is in \( \mathcal{P}_0(\alpha_\cdot^\cdot) \). By the PW theorem for the spherical function we get one \( \phi \in \mathcal{D}(K \setminus X, \text{Hom}(V_\delta, V_\delta^M)) \) such that

\[
\phi(x) = \frac{1}{w} \int_{a^\cdot} \varphi_\lambda(x) \Phi(\lambda) |\epsilon(\lambda)|^{-2} \, d\lambda,
\]

where \( \varphi_\lambda(\cdot) \) is an elementary spherical function. Now applying the differential operator \( D^\delta \) (see 3.8)) on both sides of (3.15) we get

\[
(D^\delta \phi)(x) = \frac{1}{w} \int_{a^\cdot} \Phi_{\lambda, \cdot}(x) Q_\lambda(\lambda) \Phi(\lambda) |\epsilon(\lambda)|^{-2} \, d\lambda,
\]

\[
= \frac{1}{w} \int_{a^\cdot} \Phi_{\lambda, \cdot}(x) \tilde{f}(\lambda) |\epsilon(\lambda)|^{-2} \, d\lambda,
\]

\[
= f(x).
\]

We shall denote the Hilbert–Schmidt norm of an operator by \( \|H\| \).

**DEFINITION 3.10 (The \( L^p \)-Schwartz space on \( X \))**

For every \( 0 < p \leq 2 \), \( \mathbf{D}, \mathbf{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C}) \) and \( q \in \mathbb{N} \cup \{0\} \) we define a semi-norm on \( f \in C^\infty(X, \text{Hom}(V_\delta, V_\delta^M)) \) by

\[
\nu_{\mathbf{D}, \mathbf{E}, q}(f) = \sup_{x \in G} \|f(D, x, E)\| \varphi_0(x)^{-2/p}(1 + |x|)^q.
\]

Let \( S^p(X) \) be the space of all functions in \( C^\infty(X, \text{Hom}(V_\delta, V_\delta^M)) \) such that \( \nu_{\mathbf{D}, \mathbf{E}, q}(f) < \infty \) for all \( \mathbf{D}, \mathbf{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C}) \) and \( q \in \mathbb{N} \cup \{0\} \). We topologize \( S^p(X) \) by means of the seminorms \( \nu_{\mathbf{D}, \mathbf{E}, q} \), \( \mathbf{D}, \mathbf{E} \in \mathcal{U}(\mathfrak{g}_\mathbb{C}) \), \( q \in \mathbb{N} \cup \{0\} \).
Then \( S^p(X) \) is a Frechet space and \( D(X, \text{Hom}(V_\delta, V_\delta)) \) is a dense subspace of \( S^p(X) \). Let \( S_0^\delta(X) \) be the subspace of \( S^p(X) \) consisting of the left \( \delta \) type \( \text{Hom}(V_\delta, V_\delta) \)-valued functions in \( S^p(X) \). Then clearly \( D^\delta(X) \) is a dense subspace in \( S_0^\delta(X) \).

**Remark 3.11.** Let \( D^\delta(X) \) be the Schwartz space of scalar-valued \( \delta \) type functions. Recall that \( D^\delta(X) \) is dense in \( S_0^\delta(X) \). Therefore the homeomorphism \( Q \) defined in Lemma 2.1 between \( D^\delta(X) \) and \( S_0^\delta(X) \) extends to a homeomorphism between the corresponding Schwartz spaces \( S_0^\delta(X) \) and \( D^\delta(X) \).

We shall now define the Schwartz space \( S_0^\delta(a^*_\epsilon) \) containing the Paley–Wiener space \( P_\delta(a^*_C) \) as follows.

**DEFINITION 3.12**

Let \( S_0^\delta(a^*_\epsilon) \) be the class of functions on \( a^*_\epsilon \) taking values in \( \text{Hom}(V_\delta, V_M^\delta) \) and satisfying the following conditions:

1. \( h \) is analytic in the interior of the strip \( a^*_\epsilon \).
2. \( h \) extends continuously to the boundary of the strip \( a^*_\epsilon \).
3. \((Q_\delta)^{-1}h\) is even and analytic in the interior of the strip \( a^*_\epsilon \).
4. For each positive integer \( r \) and for each symmetric polynomial \( P \) on \( a^*_\epsilon \),
   \[
   \tau_r(P, h) = \sup_{\lambda \in \text{Int} a^*_\epsilon} \left\| \frac{d}{d\lambda} h(\lambda) \right\| (1 + |\lambda|)^r < +\infty.
   \] (3.18)

\( P(\partial\lambda) \) is the differential operator obtained by replacing the variable \( \lambda \) by \( d/d\lambda \).

The topology given by the countable family of seminorms \( \tau_r, P \) makes \( S_0^\delta(a^*_\epsilon) \) a Frechet space.

The condition (3.18) can also be written in the form

\[
\tau_{r, t}(h) = \sup_{\lambda \in \text{Int} a^*_\epsilon} \left\| \left( \frac{d}{d\lambda} \right)^t \left( \frac{d}{d\lambda} \right)^{t-1} h(\lambda) \right\| (1 + |\lambda|)^m < +\infty.
\]

Let \( S_0(a^*_\epsilon) \) be the class of all even functions on \( a^*_\epsilon \) taking values in \( \text{Hom}(V_\delta, V_M^\delta) \) satisfying conditions (1), (2) and (4) of Definition 3.12. Then \( S_0(a^*_\epsilon) \) becomes a Frechet space with the seminorms \( \tau_r, P \). Clearly, \( P_\delta(a^*_C) \subset S_0(a^*_\epsilon) \).

**Lemma 3.13.** The map

\[
h(\lambda) \mapsto Q_\delta(\lambda)h(\lambda)
\]

is a homeomorphism from \( S_0(a^*_\epsilon) \) onto \( S_0^\delta(a^*_\epsilon) \)

**Proof.** Let \( h \in S_0(a^*_\epsilon) \). Then

\[
\sup_{\lambda \in \text{Int} a^*_\epsilon} \left\| \left( \frac{d}{d\lambda} \right)^t Q_\delta(\lambda)h(\lambda) \right\| (1 + |\lambda|)^m \\
\leq \sum_{t_i} c_{t_i} \sup_{\lambda \in \text{Int} a^*_\epsilon} \left\| \left( \frac{d}{d\lambda} \right)^{t_i} Q_\delta(\lambda) \right\| \left( \frac{d}{d\lambda} \right)^{t_i - t_0} h(\lambda) \left( 1 + |\lambda| \right)^m,
\]

\[
\leq \sum c_{t_i} \sup_{\lambda \in \text{Int} a^*_\epsilon} \left\| \left( \frac{d}{d\lambda} \right)^{t_i - t_0} h(\lambda) \right\| (1 + |\lambda|)^m.
\]
The constants $c_{i}$ and the positive integers $m_{i}$ are dependent on $\delta$. On the other hand, if $g \in S_{\delta}(a_{\epsilon}^{*})$ then $\psi(\lambda) = g(\lambda)/Q^{i}(\lambda)$ satisfies the conditions (1) and (2) of Definition 3.12. As $g \in S_{\delta}(a_{\epsilon}^{*})$, by (3), $\psi$ is an even function. We need to establish (4) of Definition 3.12 to conclude $\psi \in S_{0}(a_{\epsilon}^{*})$.

Let us choose a compact subset $C$ of $a_{\epsilon}^{*}$ containing all the zeros of $Q^{i}(\lambda)$ in the strip $a_{\epsilon}^{*}$ such that $|Q^{i}(\lambda)| \geq \alpha$ for all $\lambda \in a_{\epsilon}^{*} \setminus C$, where $\alpha$ is a positive constant.

\[
\sup_{\lambda \in \text{int } a_{\epsilon}^{*}} \left| \left( \frac{d}{dx} \right)^{j} \psi(\lambda) \right| (1 + |\lambda|)^{m} \leq \sup_{\lambda \in C} \left| \left( \frac{d}{dx} \right)^{j} \frac{g(\lambda)}{Q^{i}(\lambda)} \right| (1 + |\lambda|)^{m} + \sup_{\lambda \in \text{int } a_{\epsilon}^{*} \setminus C} \frac{\|\beta(\lambda)\left( \frac{d}{dx} \right)^{j} g(\lambda)\|}{|Q^{i}(\lambda)|^{2}} \leq k_{1} + \frac{k_{2}}{\alpha} \sup_{\lambda \in \text{int } a_{\epsilon}^{*}} \left| \left( \frac{d}{dx} \right)^{j} g(\lambda) \right| (1 + |\lambda|)^{m+1} < +\infty,
\]

where $\beta(\lambda)$ is a polynomial in $\lambda$. This concludes the proof. □

It follows from above that any $h \in S_{\delta}(a_{\epsilon}^{*})$ can be written as $Q^{i}(\lambda)g(\lambda)$ where $g \in S_{0}(a_{\epsilon}^{*})$ and vice-versa.

Let $g = (g_{1}, \ldots, g_{d(\delta)}) \in S_{0}(a_{\epsilon}^{*})$. Then each scalar-valued function $g_{i}$ belongs to the Schwartz space $S(a_{\epsilon}^{*})$ containing the Paley–Wiener space $P(a_{\epsilon}^{*})$ under the spherical Fourier transform.

**PROPOSITION 3.14**

The Paley–Wiener space $P_{\delta}(a_{\epsilon}^{*})$ is a dense subspace of $S_{\delta}(a_{\epsilon}^{*})$.

**Proof.** We have seen in Lemma 3.13 that any $h = (h_{1}, \ldots, h_{d(\delta)}) \in P_{\delta}(a_{\epsilon}^{*})$ can be written as $Q^{i} \cdot (g_{1}, \ldots, g_{d(\delta)})$, where each $g_{i}$ belongs to the Paley–Wiener space $P(a_{\epsilon}^{*})$ under the spherical Fourier transform. We recall that $P(a_{\epsilon}^{*})$ is dense in $S(a_{\epsilon}^{*})$. Let $H = (H_{1}, \ldots, H_{d(\delta)}) \in S_{\delta}(a_{\epsilon}^{*})$. Then $H = (Q^{i}G_{1}, \ldots, Q^{i}G_{d(\delta)})$ where $G = (G_{1}, \ldots, G_{d(\delta)}) \in S_{0}(a_{\epsilon}^{*})$, i.e., each $G_{i} \in S(a_{\epsilon}^{*})$. Then there exists a sequence $[G_{i}^{*}]$ in $P_{0}(a_{\epsilon}^{*})$ converging to $G_{i}$. Hence $\{Q^{i} \cdot g_{i}^{*}\}$ converges to $Q^{i} \cdot G_{i}$ in the topology of $S(a_{\epsilon}^{*})$. Therefore, the sequence $\{Q^{i} \cdot (g_{i}^{*}, \ldots, g_{d(\delta)}^{*})\} \subset P_{\delta}(a_{\epsilon}^{*})$ converges to $(Q^{i}G_{1}, \ldots, Q^{i}G_{d(\delta)})$ in $S_{\delta}(a_{\epsilon}^{*})$. This completes the proof. □

**4. Proof of Theorem 1.1**

**Lemma 4.1.** Let $f \in S_{0}^{\delta}(X)$. Then its $\delta$-spherical transform $\tilde{f}$ is an analytic function in the interior of the strip $a_{\epsilon}^{*}$.

**Proof.** For any function $f : X \mapsto \text{Hom}(V_{\delta}, V_{\delta}^{M})$, it is easy to show that $|\text{tr } f(x)| \leq \|f(x)\|$ for all $x \in X$. As $f \in S_{0}^{\delta}(X)$, from (3.17), we conclude that for each $D, E \in \mathcal{U}(g_{\mathcal{C}})$ and $n \in \mathbb{Z}^{+} \cup \{0\}$,

\[
\sup_{x \in X} \|\text{tr } f(D, x, E)\|(1 + |x|)^{n} \psi_{0}^{-2/p}(x) < +\infty. \tag{4.1}
\]
Using (4.1) and the estimate (3.4) one can show that the integral in Definition 3.1 of the \( \delta \)-spherical transform converges absolutely for \( \lambda \in a^+_\ast \).

A standard application of Morera’s theorem together with Fubini’s theorem shows that \( \lambda \mapsto \tilde{f}(\lambda) \) is analytic in the interior of the strip \( a^+_\ast \).

**Lemma 4.2.** For \( f \in S^0_\phi(X) \) and for each \( t, n \in \mathbb{Z}^+ \cup \{0\} \), there exists a positive integer \( m \) and \( n \) such that

\[
\sup_{\lambda \in \text{Int} a^+_\ast} \left\| \left( \frac{d}{d\lambda} \right)^t (1 + \lambda^2)^n \tilde{f}(\lambda) \right\| \leq c \sup_{x \in X} \|L^n f(x)\|(1 + |x|)^m \psi_0^{-2/p}(x),
\]

where \( c \) is a positive constant.

**Proof.** From (3.1) we have

\[
\left( \frac{d}{d\lambda} \right)^t (1 + \lambda^2)^n \tilde{f}(\lambda) = \left( \frac{d}{d\lambda} \right)^t \left( d(\delta) \int_X \text{tr} f(x)(1 + \lambda^2)^n \Phi_{\lambda,\delta}(x) \right) dx
\]

\[
= \left( \frac{d}{d\lambda} \right)^t \left( d(\delta) \int_X \text{tr} f(x)(-L)^n \Phi_{\lambda,\delta}(x) \right) dx.
\]

(4.2)

where the last equality follows from (2) of the discussion following Definition 3.1. Using integration by parts we get from above that

\[
\left( \frac{d}{d\lambda} \right)^t (1 + \lambda^2)^n \tilde{f}(\lambda) = \left( i \right)^t d(\delta) \int_X \int_K (H(x^{-1}k))^{(i(\lambda^{-1})H(x^{-1}k))} e(i\lambda^{-1}H(x^{-1}k)) \delta(k^{-1}) dk dx
\]

\[
= \left( i \right)^t d(\delta) \int_X \int_K (H(x^{-1}k))^{(i(\lambda^{-1})H(x^{-1}k))} L^n \text{tr} f(x) e(i\lambda^{-1}H(x^{-1}k)) \delta(k^{-1}) dk dx
\]

\[
= \left( i \right)^t d(\delta) \int_K \int_X (H(x^{-1}k))^{(i(\lambda^{-1})H(x^{-1}k))} L^n \text{tr} f(x) e(i\lambda^{-1}H(x^{-1}k)) \delta(k^{-1}) dx dk.
\]

We substitute \( x^{-1}k = y^{-1} \) and use \( L \text{tr} f(y) = \text{tr} (L f)(y) \) to obtain

\[
\left( \frac{d}{d\lambda} \right)^t (1 + \lambda^2)^n \tilde{f}(\lambda) = \left( i \right)^t d(\delta) \int_K \int_X (H(y^{-1}))^{(i(\lambda^{-1})H(y^{-1}))} \text{tr} (L^n f)(ky) e(i\lambda^{-1}H(y^{-1}) \delta(k^{-1}) dk dy.
\]
Note that $L^n f$ is again a function of left $\delta$ type. Therefore from above we get

$$
\left( \frac{d}{d\lambda} \right)^t \{ (1 + \lambda^2)^n \tilde{f}(\lambda) \} \\
= (i)^t \int_X H(y^{-1}) e^{i(\lambda - 1)H(y^{-1})} \left\{ d(\delta) \int_K \text{tr} (L^n f)(ky) \delta(k^{-1}) dk \right\} dy,
$$

$$
= (i)^t \int_X (H(y)^{-1})L^n f(y) e^{i(\lambda - 1)H(y^{-1})} dy \quad \text{(by (2.7))}
$$

$$
= (i)^t \int_X (H(y))L^n f(y^{-1}) e^{i(\lambda - 1)H(y)} dy.
$$

(4.3)

We use the Iwasawa decomposition $G = KAN$ and write $y = kan$, where $r \in a$ and $\exp r = a$, to obtain

$$
\left( \frac{d}{d\lambda} \right)^t \{ (1 + \lambda^2)^n \tilde{f}(\lambda) \} \\
= c(i)^t \int_K \int_a \int_N L^n f(n^{-1}a_{\lambda}^{-1}k^{-1}) (H(ka_n)) e^{i(\lambda - 1)H(ka_n)} dk e^{2\pi i r} dr dn
$$

$$
= (i)^t \int_a \int_N L^n f((a_{\lambda}n)^{-1}) r e^{i(\lambda + 1)r} dr dn.
$$

(4.4)

From (4.4) we get the following norm inequality.

$$
\left\| \left( \frac{d}{d\lambda} \right)^t \{ (1 + \lambda^2)^n \tilde{f}(\lambda) \} \right\| \leq c \int_a \int_N \| L^n f((a_{\lambda}n)^{-1}) \| |r| e^{i|\text{Im}\lambda + 1| r} dr dn.
$$

(4.5)

As $f \in S^p_\delta(X)$, for each $m \in \mathbb{Z}^+$ we have $\| L^n f((a_{\lambda}n)^{-1}) \| \leq v_{m} L_n^p (f) (1 + |(a_{\lambda}n)^{-1}|)^{-m} \phi_{0}^{2/p}((a_{\lambda}n)^{-1})$ where $v$ is as defined in (3.17). Using (2.1) we get from above

$$
\left\| \left( \frac{d}{d\lambda} \right)^t \{ (1 + \lambda^2)^n \tilde{f}(\lambda) \} \right\| \\
\leq c_1 v_{m} L_n^p (f) \int_a \int_N (1 + |(a_{\lambda}n)|)^{-m} \phi_{0}^{2/p}((a_{\lambda}n)^{-1}) (1 + |r|)^t e^{i|\text{Im}\lambda + 1| r} dr dn
$$

$$
\leq c_1 v_{m} L_n^p (f) \int_a \int_N (1 + |(a_{\lambda}n)|)^{-m} (\phi_{0}^{2/p}((a_{\lambda}n)^{-1}) e^{i|\text{Im}\lambda + 1| H(a_{\lambda}n)} dr dn
$$

$$
= c_1 v_{m} L_n^p (f) \int_G (1 + |x|)^t \phi_{0}^{2/p}((x)^{-1}) e^{i|\text{Im}\lambda| - 1)H(x)} dx.
$$

(4.6)

For convenience we denote $c_1 v_{m} L_n^p (f)$ by $c_v$. We use the Cartan decomposition $G = K A^* K$ and write $x = k_1 \exp |x| k_2$ and decompose the integral (4.6) as follows:
\[ c_0 \int_{K} \int_{a^+} (1 + |k_1 \exp |x|k_2|)^{-m+t} \phi_0^{2/p}(\exp |x^{-1}|) \]
\[ \times e^{(\operatorname{Im} \lambda - 1)H(\exp |x|k_2)\Delta(|x|)dx}dk_2 \]
\[ = c_0 \int_{a^+} \int_{K} (1 + |x|)^{-m+t} \phi_0^{2/p}(\exp |x|) \]
\[ \times e^{(\operatorname{Im} \lambda - 1)H(\exp |x|k_2)\Delta(|x|)dx}dk_2, \]

as \(|x^{-1}| = |x|\) and \(|k_1 \exp |x|k_2| = |x|\). Using (3.6) the expression above is
\[ \leq c_0 \int_{a^+} (1 + |x|)^{-m+t} \phi_0^{2/p}(\exp |x|) \]
\[ \times \left\{ \int_{K} e^{i(\operatorname{Im} \lambda - 1)H(\exp |x|k_2)\Delta(|x|)dx} \right\} |\lambda - 1|H(\exp |x|k_2)\Delta(|x|)dx]. \quad (4.7) \]

We take \(\lambda \in \operatorname{Int}a^*_\epsilon\), i.e., \(|\operatorname{Im} \lambda| < \epsilon = \left( \frac{2}{p} - 1 \right)\). Using the estimate (3.7) we get
\[ \leq c_0 \int_{a^+} (1 + |x|)^{-m+t} \phi_0^{2/p}(\exp |x|) \exp |x| \Delta(|x|)dx, \quad \text{(see (2.2))}. \quad (4.8) \]

Choosing a suitably large \(m\), we see that the integral in (4.8) converges (Lemma 11 of [8]). Hence, we conclude that
\[ \sup_{\lambda \in \operatorname{Int}a^*_\epsilon} \left\| \left( \frac{d}{d\lambda} \right)^t (1 + \lambda^2)^{\frac{n}{2}} \tilde{f}(\lambda) \right\| \leq \text{const} \nu_{L,m}(f). \quad (4.9) \]

This completes the proof of the lemma.

**Lemma 4.3.** The \(\delta\)-spherical transform \(f \mapsto \tilde{f}\) is a continuous injection of \(S^\delta_{\phi}(X)\) into \(S_\delta(a^*_\epsilon)\).

**Proof.** From Lemma 4.1, Lemma 4.2 and (6) of the discussion below Definition 3.1, we conclude that if \(f \in S^\delta_{\phi}(X)\) then \(\tilde{f} \in S_\delta(a^*_\epsilon)\). Also the transform \(f \mapsto \tilde{f}\) is continuous. The fact that \(f \mapsto \tilde{f}\) is injective is a consequence of the Plancherel formula for the HFT (III, Theorem 1.5 of [9]).

The next lemma is an extension of the inversion formula given in Lemma 3.5 for the Schwartz class functions.
Lemma 4.4. Let $h \in S_\delta(a^*_\epsilon)$. Then the inversion $Ih$ given by

$$Ih(x) = \frac{1}{\omega} \int_{a^*} \Phi_{\lambda,\delta}(x)h(\lambda)|c(\lambda)|^{-2}d\lambda.$$ 

is a left $\delta$-type $C^\infty$ function on $X$ taking values in $\text{Hom}(V_\delta, V_\delta)$.

**Proof.** Let us take any derivative $D$ of $X$. For any $D \in \mathcal{U}(g_C)$,

$$Ih(D; x) = \frac{1}{\omega} \int_{a^*} \Phi_{\lambda,\delta}(D; x)h(\lambda)|c(\lambda)|^{-2}d\lambda. \quad (4.10)$$

Therefore,

$$\|Ih(D; x)\| \leq c \int_{a^*} \|\Phi_{\lambda,\delta}(D; x)\| |h(\lambda)|((1 + |\lambda|)^b d\lambda \leq c_\delta \int_{a^*} (1 + |\lambda|)^{b_n + b - n} \psi_0(x) d\lambda,$$

(using estimate (3) of the discussion below Definition 3.1 and (3.18))

$$\leq c_\delta \int_{a^*} (1 + |\lambda|)^{b_n + b - n} d\lambda.$$

We choose $n$ sufficiently large so that the last integral on the right-hand side exists. Hence, $Ih(D; x)$ exists for every $D$. Therefore $Ih$ is a $C^\infty$ function on $X$. As $\Phi_{\lambda,\delta}(x)$ is of left $\delta$ type, so is $Ih$. \hfill \Box

Lemma 4.5. If $h \in S_\delta(a^*_\epsilon)$, then $Ih \in \mathcal{S}_p^\delta(X)$.

**Proof.** We consider the spaces $\mathcal{P}_\delta(a^*_C)$ and $\mathcal{D}^\delta(X)$ equipped with the topologies of the Schwartz spaces $S_\delta(a^*_\epsilon)$ and $S_\delta^p(X)$ respectively. It is clear from the Paley–Wiener theorem that $I$ maps $\mathcal{P}_\delta(a^*_C)$ onto $\mathcal{D}^\delta(X)$. We shall show that $I$ is a continuous map from $\mathcal{P}_\delta(a^*_C)$ onto $\mathcal{D}^\delta(X)$ in these topologies. Let $h \in \mathcal{P}_\delta(a^*_C)$ and $Ih = f \in \mathcal{D}^\delta(X)$. We have to show that for any seminorm $\nu$ on $\mathcal{D}^\delta(X)$ there exists a seminorm $\tau$ on $P(a^*_\epsilon)$ so that

$$\nu(f) \leq c_\delta \tau(h),$$

where $c_\delta$ is a constant depending only on $\delta$.

Let $D \in \mathcal{U}(g_C)$ and $n \in \mathbb{Z}^+$. We consider $f$ as a right $K$-invariant function on the group $G$. Let

$$\nu_{D,M}(f) = \sup_{x \in G} \|Df(x)\|(1 + |x|^n)^{-2/p} \psi_0(x). \quad (4.11)$$

From Lemmas 3.8 and 3.9 we know that $f(x) = D^\delta \phi(x)$, where $\phi$ is a $K$ bi-invariant function on $G$ and $h(\lambda) = Q^\delta(\lambda)\Phi(\lambda)$. Here $\Phi$ is the spherical Fourier transform of $\phi$. Hence from (4.11) we have

$$\nu_{D,M}(f) = \sup_{x \in G} \|DD^\delta \phi(x)\|(1 + |x|^n)^{-2/p} \psi_0(x) = \nu_{D,D^\delta,n}(\phi). \quad (4.12)$$
By the isomorphism of the $K$ bi-invariant functions in the Schwartz space (see [2]), for each $D \in U(g)$, $D' \in U(g)$ and for each $n \in \mathbb{Z}^+$ there exists $m_\delta, t \in \mathbb{Z}^+$ and a positive constant $c_\delta$ so that,

$$
\sup_{x \in G} \|DD'\phi(x)\|(1 + |x|)^{n}\varphi_0^{-2/p}(x) \leq c_\delta \sup_{\lambda \in \text{Int} a_\ast^\delta} \left\| \left(\frac{d}{d\lambda}\right)^t \Phi(\lambda) \right\| (1 + |\lambda|)^{m_\delta}.
$$

(4.13)

Now by Lemma 3.13, for $t, m_\delta \in \mathbb{Z}^+$ there exists $t_1, m_1 \in \mathbb{Z}^+$ such that

$$
\sup_{\lambda \in a_\ast^\delta} \left\| \left(\frac{d}{d\lambda}\right)^{t_1} \Phi(\lambda) \right\| (1 + |\lambda|)^{m_1} \leq c_\delta' \sup_{\lambda \in \text{Int} a_\ast^\delta} \left\| \left(\frac{d}{d\lambda}\right)^{m_1} h(\lambda) \right\| (1 + |\lambda|)^{m_1} = c_\delta' \tau_{t_1,m_1}(h) < +\infty.
$$

Hence, $\nu_{\delta,n}(f) \leq c_\delta' c_\delta \tau_{t_1,m_1}(h)$. The positive constants $c_\delta$ and $c_\delta'$ are dependent on $|\delta|$. The positive integer $m_1$ can be made independent of the $\delta \in \hat{K}_M$ chosen. This shows that the inversion $I$ is a continuous linear transformation on a dense subset $P_\delta(a_\ast^\epsilon)$ of $S_\delta(a_\ast^\epsilon)$ onto $D^\delta(X)$ (The surjectivity follows from Theorem 3.7.)

Let us now take $h \in S_\delta(a_\ast^\epsilon)$. As $P_\delta(a_\ast^\epsilon)$ is dense in $S_\delta(a_\ast^\epsilon)$, there exists a Cauchy sequence $\{h_n\} \subset P_\delta(a_\ast^\epsilon)$ converging to $h$. Then by what we have proved above, we can get a Cauchy sequence $\{f_n\} \subset D^\delta(X)$ such that $\tilde{f}_n = h_n$. As $S^\delta_0(X)$ is a Frechet space, the sequence converge to some $f \in S^\delta_0(X)$. Clearly, $f = Ih$. This completes the proof. 

Finally, Lemmas 3.7, 4.3 and 4.5 together show that the $\delta$-spherical transform is a surjection onto $S_\delta(a_\ast^\epsilon)$ and that $I: S_\delta(a_\ast^\epsilon) \rightarrow S^\delta_0(X)$ is continuous. That is, the $\delta$-spherical transform is a topological isomorphism between the spaces $S^\delta_0(X)$ and $S_\delta(a_\ast^\epsilon)$. This proves Theorem 1.1.

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