HEISENBERG GROUPS VIA ALGEBRA

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Abstract. We introduce a general class of Heisenberg groups motivated by applications of algebraic Fourier theory. Basic properties are examined from a homological perspective.

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1. Introduction

1.1. Terminological conventions and notation. We write \( \wp(S) \) for the power set of a set \( S \) and \( \wp(\omega)(S) \) for its finite version (the set of all finite subsets of \( S \)). Locally compact includes Hausdorff. Define bicharacter (left side additive, right side multiplicative!) and nondegenerate bicharacter. Column vectors \( K^n \), row vectors \( K_n \), the \( n \times n \) identity matrix \( I_n \). All vector space duals in the algebraic sense. Explain action as scalars versus operators. Category of groups is denoted by \( \text{Grp} \), abelian ones by \( \text{Ab} \). Remind of twisted modules \( M[\sigma] \) for arbitrary modules \( M \in \text{Mod}_R \) and homomorphism \( \sigma: R' \to R \), to be defined in detail. An involution is an automorphism (of groups or of \( K \)-algebras)
that is inverse to itself. Algebras generally assumed commutative but nonunital. We shall frequently employ the abbreviation \( x^- := x^{-1} \) for the inverse of an invertible element in a (multiplicatively written) monoid. Use \( \chi_S \) for the characteristic function of a set \( S \subseteq \mathbb{R}^n \), meaning \(+1\) within \( S \). Unit interval \( \mathbb{I} = [0,1] \). Explain notation \( \mathbb{P} = \{2,3,5,7,11,\ldots\} \) for the prime numbers and \( \mathbb{P}_* = \{1,2,3,5,7,11,\ldots\} \) for its extension by unity. Also introduce notation \( \mathbb{N}_{>0} = \{1,2,3,\ldots\} \) for the positive natural numbers. Write \( \mathbb{R}_{\geq 0} \) for the nonnegative reals, consequently \( \mathbb{R}_{>0} \) for the positive reals. The symmetric algebra over an \( R \)-module \( M \) is denoted by \( \text{Sym}(M) \cong T(M)/I(M) \), where \( I(M) \leq T(M) \) is the ideal generated by \( \{x \otimes y - y \otimes x \mid x,y \in M\} \). In case of ambiguity, the base ring \( R \) may be explicated in writing \( \text{Sym}_R(M) \cong T_R(M)/I_R(M) \). Note that for a set \( X \) one has \( R[X] = \text{Sym}_R(RX) \), where \( RX \) is the free \( R \)-module over the basis \( X \). The center of a group \( G \) is denoted by \( \mathcal{Z}G \). The dual of a poset \( P \) is denoted by \( \hat{P} \).

Given a \( R \)-module \( M \), an \( n \)-form is a bilinear map \( M^n \to R \); if \( n \) is suppressed, we take \( n = 2 \). An alternating form is a bilinear map \( \omega: M \oplus M \to R \) such that \( \omega(x,x) = 0 \) for all \( x \in M \); it is called a symplectic form if it is moreover nondegenerate in the sense that \( \omega(x,-): M \to M \) is injective for all \( x \in M \). We refer to the structure \((M,\omega)\) as an alternating or symplectic module, respectively. In this paper we will always deal with the case \( R = \mathbb{Z} \) so that \( M \) is an abelian group. In this case, a form is called a bicharacter, a nondegenerate one a duality, and an alternating duality induces the structure of a symplectic \( \mathbb{Z} \)-module. (This slightly awkward expression is necessary since the term symplectic group is already reserved for the linear transformations of a symplectic module that leave its symplectic form invariant.)

We will use the letter sequence ‘SES’ to stand for ‘short exact sequence’. Sometimes, when clear from context we write \( \bar{X} \) for the inverse image \( \pi^{-1}(X) \).

1.2. Remark. The material of this paper is to some extent coupled with that of its companion paper \([41]\), where the focus is on the structures actually arising in constructive analysis. Whenever we refer to specific places in the companion paper, we will use the shorthand \([\ldots]\) for \([41,\ldots]\).

2. Heisenberg Groups in Algebra

2.1. Nilquadratic Groups and Symplectic Forms. From a purely algebraic perspective, Heisenberg groups may be viewed as the simplest
nonabelian groups with bipartite internal symmetry. We shall make this vague characterization more precise in the next subsection.

Let us first recall some basic facts about nilpotent groups. A central series for a group $H$ may be defined as a normal series $(H_k)$ having central factors. More precisely, an increasing central series [39, §5.1] has $H_0 = 1$ and $H_{k+1}/H_k \leq \mathcal{Z}(H/H_k)$, whereas a decreasing central series is characterized [11, (2.4.5)] [44, Ex. 5.39] by $H_0 = H$ and $H_{k-1}/H_k \leq \mathcal{Z}(H/H_k)$. The center conditions are, respectively, equivalent to $[H, H_{k+1}] \leq H_k$ and $[H, H_{k-1}] \leq H_k$. It is known that: (i) If some increasing central series terminates with $H$, then all such series do so. (ii) If some decreasing central series terminates with 1, then all such series do so. (iii) These two conditions are equivalent and may be taken as the definition of a nilpotent group $H$, where increasing and decreasing central series may be turned into each other by reversal. The nilpotency class of $H$ is then defined as the minimal length of any central series.

Defining the higher centers [44, p. 113] by $Z_{k+1}/Z_k := \mathcal{Z}(H/Z_k)$ one obtains a canonical increasing central series with $Z_1 = \mathcal{Z}(G)$; it is called the upper central series since it majorizes all (increasing) central series [39, Prop. 5.1.9]. In a similar fashion, the lower commutator groups $A_{k+1} := [H, H_k]$ yield a decreasing central series with $A_1 = [H, H]$; it is called the lower central series since in minorizes all (decreasing) central series. For a nilpotent group $H$, the length of both $(A_k)$ and $(Z_k)$ is the nilpotency class of $H$.

From the viewpoint of group extensions, one may characterize nilpotent groups as those obtained from the trivial group via central extensions. Recall that a group extension $1 \rightarrow T \rightarrow H' \rightarrow H \rightarrow 1$ is central if the image of $T$ is contained in $\mathcal{Z}(H)$; we call it strictly central if the image is exactly $\mathcal{Z}(H)$. Now we declare $H = 1$ to be nilpotent of class 0, and we stipulate that $H'$ is nilpotent of class $r + 1$ if there exists a central extension $1 \rightarrow T \rightarrow H' \rightarrow H \rightarrow 1$ with $H$ nilpotent of class $r$. This characterization allows one to extract a central series from a given group $H$ of nilpotency class $r$ by unfolding the successive
central extensions:

\[
\begin{align*}
1 & \rightarrow H_1 \rightarrow H \rightarrow H/H_1 \rightarrow 1, \\
1 & \rightarrow H_2/H_1 \rightarrow H/H_1 \rightarrow H/H_2 \rightarrow 1, \\
1 & \rightarrow H_3/H_2 \rightarrow H/H_2 \rightarrow H/H_3 \rightarrow 1, \\
& \quad \vdots \\
1 & \rightarrow H/H_{r-1} \rightarrow H/H_{r-1} \rightarrow 1 \rightarrow 1,
\end{align*}
\]

so that \(1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_{r-1} \triangleleft H_r = H\) is an increasing central series for \(H\). Of course, one may as well proceed obversely, resulting in a decreasing central series.

Let us describe these relations in category-theoretic terms. Our starting point is \(\text{Nil}_r\), the collection of nilpotent groups of class at most \(r\), viewed as a full subcategory of \(\text{Grp}\). Central series of length \(r\) without repetition are taken as the objects of another category \(\text{Cnt}_r\). While it does not matter, we may fix increasing central series for definiteness. We define a morphism \((\varphi_0, \varphi_1, \ldots, \varphi_r)\) by the obvious commuting diagram

\[
\begin{array}{cccc}
1 = H_0 & \triangleleft & H_1 & \triangleleft \cdots \triangleleft H_{r-1} & \triangleleft H_r = H, \\
\downarrow \varphi_0 & & \downarrow \varphi_1 & & \downarrow \varphi_{r-1} & \downarrow \varphi_r \\
1 = H_0' & \triangleleft & H_1' & \triangleleft \cdots \triangleleft H_{r-1}' & \triangleleft H_r' = H,
\end{array}
\]

where \(\varphi_0: H_0 \rightarrow H_0', \ldots, \varphi_r: H_r \rightarrow H_r'\) are group homomorphisms. Clearly, \(\varphi_0 = 1\) and all \(\varphi_j (j < r)\) are determined by \(\varphi := \varphi_r\). There is an obvious functor \(\text{Cnt}_r \rightarrow \text{Nil}_r\) that maps any central series of a nilpotent group \(H\) to the bare group \(H\) and the morphism \((\varphi_0, \varphi_1, \ldots, \varphi_r)\) to \(\varphi = \varphi_r\). This functor is clearly dense (in fact, surjective on objects) and faithful (since \(\varphi\) is uniquely determined) and full (as one sees by taking for example upper central series), hence it constitutes an equivalence of categories. Note that \((\varphi_0, \varphi_1, \ldots, \varphi_r)\) also determines unique morphisms (in the category \(\text{SES}\) of short exact sequences) from each central extension in (1) to the corresponding central extensions for \(H'\).
Note that \( \text{Nil}_0 = \{1\} \) and \( \text{Nil}_1 = \text{Ab} \); we shall here be interested in the simplest noncommutative case—the *nilquadratic groups* \( \text{Nil}_2 \). In this case, there is only one row in (1) for a given \( H \in \text{Nil}_2 \), we shall write the corresponding central extension as \( 1 \to T \to H \to P \to 0 \). For reasons that will soon become clear, we refer to the abelian group \( T \) as a *torus* and to the abelian group \( P \) as a *phase space* of the nilquadratic group \( H \). Note that \( P \) is written additively, whereas \( T \) and of course \( H \) are written multiplicatively.

**Lemma 1.** Let \( G \in \text{Grp} \), and \( k_G : G \times G \to G, (x, y) \mapsto [x, y] \). Then \( k_G \) is left-linear \( \iff [G, G] \subseteq Z(G) \iff k_G \) is right-linear.

**Proof.** If \( [G, G] \subseteq Z(G) \) then

\[
[x, y][x, z] = xyx^{-1}y^{-1}x[x, z] = xyx^{-1}[x, z]y^{-1} = xyx^{-1}z^{-1}y^{-1} = [x, yz]
\]

\[
[x, z][y, z] = xzx^{-1}z^{-1}[y, z] = x[y, z]zx^{-1}z^{-1} = xyz^{-1}x^{-1}z^{-1} = [xy, z]
\]

i.e., \( k_G \) is bilinear.

Assume that \( k_G \) is left-linear. Take \( x, y, g \in G \), and set \( z := y^{-1}g^{-1}y \). Then

\[
z[x, y]g = zxy^{-1}y^{-1}(yzy^{-1})^{-1} = zxyx^{-1}y^{-1}y^{-1} = zxyx^{-1}z^{-1}y^{-1} = [z, x][z, y] = zy^{-1}x^{-1}y^{-1} = zg[x, y]
\]

\[\Rightarrow [x, y]g = g[x, y]. \text{ Thus } [G, G] \subseteq Z(G).\]

Now let \( [\cdot, \cdot] \) be right-linear, \( x, y, g \in G \). Then \( [x, g^{-1}y] = [x, g^{-1}][x, y], \) and therefore

\[
xg^{-1}yx^{-1}y^{-1}g = xg^{-1}x^{-1}g[x, y]
\]

\[
yx^{-1}y^{-1}g = x^{-1}g[x, y]
\]

\[
xyx^{-1}y^{-1}g = g[xy]
\]

\[
[x, y]g = g[x, y]
\]

Again, \( [G, G] \subseteq Z(G). \)

**Definition 2.** We say that a group \( G \) has *bilinear commutator* in case that the map \( k_G : G \times G \to G, (x, y) \mapsto [x, y] \) is bilinear.

**Fact 3.** Let \( H \) be a group. Then \( H \) is a central extension of an abelian group iff \( H/Z(H) \) is abelian iff \( [H, H] \leq Z(H) \) iff \( H \in \text{Nil}_2 \) iff \( H \) has bilinear commutator.

**Proof.** Assume \( 1 \to T \to H \to P \to 0 \) is a central extension of the abelian group \( P \), assuming \( T \subseteq H \) for simplicity. This yields abelian subgroups \( Z(H)/T \leq H/T \cong P \) whose quotient \( \frac{H/T}{Z(H)/T} \cong H/Z(H) \) is
abelian as well. Conversely, if we assume \( P := H/\mathcal{Z}(H) \) abelian, we have \( 1 \to \mathcal{Z}(H) \to H \to H/\mathcal{Z}(H) \to 0 \) is as a central extension of the abelian group \( P \). This takes care of the first equivalence; the second is immediate from the definition of \([H,H]\).

For showing the next equivalence, assume again that \( H/\mathcal{Z}(H) \) is abelian. Then the upper central series \( Z_0 = 1, Z_1 = \mathcal{Z}(H) \) ends with \( Z_2 = \pi^{-1} \mathcal{Z}(H/\mathcal{Z}(H)) = H \), where \( \pi \colon H \to H/\mathcal{Z}(H) \) is the canonical projection. Hence \( H \) is indeed nilpotent of class as most 2. Finally, assume now \( H \in \text{Nil}_2 \), and take an arbitrary central series \( 1 = H_0 \leq H_1 \leq H_2 = H \). Since the upper central series majorizes any other central series [44, Ex. 5.39], we get \( H_2 \leq Z_2 = \pi^{-1} \mathcal{Z}(H/\mathcal{Z}(H)) \) or \( \pi(H) \leq \mathcal{Z}(H/\mathcal{Z}(H)) \). But this implies that \( H/\mathcal{Z}(H) \) is abelian. The last point is an immediate consequence of Lemma 1. \( \square \)

**Lemma 4.**

1. \( G, H \in \text{Grp} \). Then \( G, H \in \text{Nil}_2 \iff G \times H \in \text{Nil}_2. \)
2. \( 1 \to N \to E \to G \to 1 \) SES in \( \text{Grp} \). Then \( E \in \text{Nil}_2 \iff N \in \text{Nil}_2 \) and \( G \in \text{Nil}_2. \)

**Proof.** If \( G, H \in \text{Nil}_2 \) then \( G' \subseteq Z(G) \wedge H' \subseteq Z(H) \). Therefore

\[
\left[ \begin{pmatrix} g_1 \\ h_1 \end{pmatrix}, \begin{pmatrix} g_2 \\ h_2 \end{pmatrix} \right] = \begin{pmatrix} g_1 g_2 g_1^{-1} g_2^{-1} \\ h_1 h_2 h_1^{-1} h_2^{-1} \end{pmatrix} = \begin{pmatrix} [g_1, g_2] \\ [h_1, h_2] \end{pmatrix} \in Z(G) \times Z(H) = Z(G \times H)
\]

that is, \( (G \times H)' \subseteq Z(G \times H) \).

Conversely, assume that \( G \times H \in \text{Nil}_2 \). Take \([g_1, g_2] \in G'\). Then

\[
\begin{pmatrix} [g_1, g_2] \\ [1, 1] \end{pmatrix} = \begin{pmatrix} [g_1] \\ [1, 1] \end{pmatrix} \in Z(G \times H) = Z(G) \times Z(H)
\]

and therefore \([g_1, g_2] \in Z(G)\). Similarly \( H \in \text{Nil}_2 \).

Consider the SES \( 1 \to N \to E \to G \to 1 \) in \( \text{Grp} \) with \( E \in \text{Nil}_2 \). Then \( E' \subseteq Z(E) \), and so

\[
G' = \pi(E)' = \pi(E') \subseteq \pi(Z(E)) \subseteq Z(\pi(E)) = Z(G)
\]

that means, \( G \in \text{Nil}_2 \). For \( r, s, r_1 \in N \)

\[
i([r, s] \cdot r_1) = [i(r), i(s)]i(r_1) = i(r_1)[i(r), i(s)] = i(r_1 \cdot [r, s])
\]

and so \([r, s]r_1 = r_1[r, s] \forall r, s, r_1 \in N \). Therefore \( N' \subseteq Z(N) \), that is, \( N \in \text{Nil}_2 \). \( \square \)

It is now easy to see that \( \text{Nil}_2 \) is also equivalent to the category \( \text{SES}_2 \) of central extensions of abelian groups; this is a full subcategory of \( \text{SES} \).
The extraction functor \( U : \text{SES}_2 \to \text{Nil}_2 \) maps a short exact sequence
\[ 1 \to T \to H \to P \to 0 \]
to the group homomorphism \( h : H \to H' \).

**Proposition 5.** We have an equivalence \( U : \text{SES}_2 \cong \rightarrow \text{Nil}_2 \).

**Proof.** The functor \( U \) is full and dense since any nilquadratic group \( H \) yields a canonical central extension \( 1 \to Z(H) \hookrightarrow H \to H/Z(H) \to 0 \), and any group homomorphisms \( H \to H' \) is obtained from a morphism \( (t,h,p) \) between such canonical extensions, where \( t : Z(H) \to Z(H') \) is the restriction of \( h \), and \( p : H/Z(H) \to H'/Z(H') \) the induced projection. To see that \( U \) is faithful, let \( T \hookrightarrow H \twoheadrightarrow P \) and \( T' \hookrightarrow H' \twoheadrightarrow P' \) be central extensions in \( \text{SES}_2 \) with two morphisms \( (t_1,h,p_1) \) and \( (t_2,h,p_2) \) between them. We have then \( t_1(c) = h_1(c) = t_2(c) \) for any \( c \in T \); since \( \iota \) is injective, this yields \( t_1 = t_2 \). Similarly, \( p_1 = p_2 \) follows from \( p_1\pi(u) = \pi'h(u) = p_2\pi(u) \) for \( u \in h \) because \( \pi \) is surjective. \( \square \)

Note that the equivalences \( \text{SES}_2 \cong \rightarrow \text{Cnt}_2 \cong \rightarrow \text{Nil}_2 \) act as surjections on objects—they provide progressively less information about a nilquadratic group \( H \). First one discards knowledge about how a specific torus is embedded in \( H \), and then also which torus is embedded. Since \( 1 \triangleleft T \triangleleft H \) is an increasing central series, the lower and upper central series impose bounds on the choice of the torus: One must have \( [H,H] \leq T \leq Z(H) \). Since the converse follows from Fact 3, we obtain the following characterization of central extensions in terms of their tori.

**Fact 6.** An exact sequence \( T \hookrightarrow H \twoheadrightarrow P \) describes a central extension of the abelian group \( P \) iff \( [H,H] \leq \iota(T) \leq Z(H) \).

According to Fact 3, there are two extreme cases of a central extension: The one corresponding to the lower central series \( H \triangleright [H,H] \triangleright 0 \) is given by \( [H,H] \hookrightarrow H \twoheadrightarrow H^{\text{ab}} \), and it has the abelianization for its phase space. On the other hand, the extension associated to the upper central series \( 1 \triangleleft Z(H) \triangleleft H \) is given by \( Z(H) \leftarrow H \twoheadrightarrow \text{Inn}(H) \), where the phase space may be taken as the group of inner automorphisms. We call an extension **strictly central** if it is of the latter type, meaning its torus is \( T = Z(H) \).
Example 7. This can be illustrated by a minor variation of the classical Heisenberg group (to be developed later in Example 70). Let us endow the set \( H := \mathbb{C}^\times \times \mathbb{R} \times \mathbb{R} \) with the group law

\[
\begin{align*}
    c(x, \xi) \cdot c'(x', \xi') &= cc'e^{ir\xi x'} (x + x', \xi + \xi'),
\end{align*}
\]

introducing the center convention of writing \((c, x, \xi) \in \mathbb{C}^\times \times \mathbb{R} \times \mathbb{R}\) in the form \(c(x, \xi)\) with center elements exposed. It is easy to see that \(Z(H) = \mathbb{C}^\times\) and \([H, H] = T\), via the embedding \(T \subset \mathbb{C}^\times \hookrightarrow H\) operating as \(c(x, \xi) \mapsto (c, 0, 0)\). Clearly, \(\mathbb{C}^\times\) and \(T\) are normal in \(H\), and one gets \(H/Z(H) \cong \mathbb{R} \oplus \mathbb{R}\) via \(c(x, \xi) \mathbb{C}^\times \leftrightarrow (x, \xi)\) and likewise also \(H/[H, H] \cong \mathbb{R}_{\geq 0} \oplus (\mathbb{R} \oplus \mathbb{R})\) with bijection \(c(x, \xi) T \leftrightarrow |c| (x, \xi)\). Therefore we obtain the \(\text{SES}_2\) morphism

\[
\begin{array}{ccccccccc}
    & 1 & \longrightarrow & T & \longrightarrow & H & \longrightarrow & \mathbb{R}_{>0} \oplus (\mathbb{R} \oplus \mathbb{R}) & \longrightarrow & 0, \\
    & \downarrow & & \downarrow & & & & \downarrow & & \\
    1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & H & \longrightarrow & \mathbb{R} \oplus \mathbb{R} & \longrightarrow & 0.
\end{array}
\]

Here the top sequence corresponds to the lower central series \(1 \triangleleft T \triangleleft H\), the bottom one to the upper central series \(1 \triangleleft \mathbb{C}^\times \triangleleft H\).

In a morphism (2) of central extension, the middle map \(h : H \to H'\) obviously determines the lateral maps \(t : T \to T'\) and \(p : P \to P'\) since \(\text{SES}_2 \to \text{Nil}_2\) is faithful as we have noted. The converse is not quite true: Having the maps \(t\) and \(p\) does not determine \(h\); an additional map is needed for fixing \(h\). To state the precise conditions, we use the language of group cohomology \([18]\). Let us write \(C^n(P, T)\) for the chain group of all functions \(\varphi : P^n \to T\) with the standard differential \(d^n : C^n(P, T) \to C^{n+1}(P, T)\) given by

\[
(d^n\varphi)(z_1, \ldots, z_{n+1}) := z_1 \cdot \varphi(z_2, \ldots, z_n)
\]

\[
+ \sum_{i=1}^{n} (-1)^i \varphi(z_1, \ldots, z_{i-1}, z_i z_{i+1}, \ldots, z_{n+1}) + (-1)^{n+1} \varphi(z_1, \ldots, z_n),
\]

for arbitrary extension. Since we consider here only central extensions, the action is trivial and the multiplication by \(z_1\) may be dropped.

Writing \(Z^n(P, T) := \ker(d^n)\) for the cocycles, \(B^n(P, T) := \text{im}(d^{n-1})\) for the coboundaries, the corresponding cohomology groups are given as usual by \(H^n(P, T) = Z^n(P, T)/B^n(P, T)\). Group operations on cycles are written additively. For functions \(t : T \to \tilde{T}\) and \(p : \tilde{P} \to P\) and a cochain \(\varphi \in C^n(P, T)\), we use the notation \(t_*(\varphi) := t \circ \varphi \in C^n(P, \tilde{T})\) and \(p^*(\varphi) := \varphi \circ (p \times \cdots \times p) \in C^n(\tilde{P}, T)\). We take the liberty of identifying the extension group \(H\) with \(T \times_{\gamma} P\), where \(\gamma \in Z^2(P, T)\) is
a suitable factor set for the extension. We assume normalized factor sets throughout.

**Proposition 8.** Let \( E: T \xrightarrow{i} H \xrightarrow{\pi} P \) and \( E': T' \xrightarrow{i'} H' \xrightarrow{\pi'} P' \) be central extensions with factor sets \( \gamma \) and \( \gamma' \), respectively. Then the Hom set \( \text{SES}_2(E, E') \) is in bijective correspondence with

\[
\{(t, p, \psi) \mid t \in \text{Hom}(T, T') \land p \in \text{Hom}(P, P') \land \psi \in C^1(P, T') \}
\]

\[
\land d^2(\psi) = t_\pi \gamma / p_\pi^* \gamma'
\]

such that \((t, p, \psi)\) corresponds to \((t, h, p)\): \( E \to E' \) with the middle map given by \( h(c, z) = (\psi(z) t(c), p(z)) \).

**Proof.** Assume first that \((t, h, p)\): \( E \to E' \) is a morphism in \( \text{SES}_2 \). Then we must have \( h(c, z) = (h_{T'}(c, z), h_{P'}(c, z)) \) for suitable maps \( h_{T'}: H \to T' \) and \( h_{P'}: H \to P' \). From \( \pi'h = \pi_\pi \) we obtain immediately that \( h_{P'}(c, z) = p(z) \) and from \( h\tau = \tau' t \) that \( h(c, 0) = (t(c), 0) \).

Then \((c, z) = (c, 0)(1, z)\) implies \( h(c, z) = (t(c), 0)(h_{T'}(1, z), p(z)) \), which is of the required form if we set \( \psi(z) := h_{T'}(1, z) \). Using the cocycles \( \gamma \) and \( \gamma' \), it is now easy to check that \( h(c, z) = (t(c) \psi(z), p(z)) \) is a homomorphism iff \( t \) and \( p \) are homomorphisms and

\[
\frac{t_\gamma(z, w)}{\gamma'(p(z), p(w))} = \frac{\psi(z) \psi(w)}{\psi(z + w)},
\]

which is the same as \( d^2(\psi) = \tau_\gamma / p^* \gamma' \). \( \square \)

**Example 9.** As a typical example of Proposition 8, consider the special case \( t = 1_T \) and \( p = 1_P \) on the group \( H = \mathbb{C}^\times \times \mathbb{R} \times \mathbb{R} \) of Example 7 with the 1-chain \( \psi: \mathbb{R} \times \mathbb{R} \to \mathbb{C}^\times \) given by \( \psi(x, \xi) := e^{i\pi x} e^{i\pi \xi} \). Since in this case \( \gamma = \gamma' \) and \( t = 1_T \), the 1-chain \( \psi \) must be a 1-cycle, which means a crossed homomorphism. Having trivial action, crossed homomorphisms are in fact plain homomorphisms. As a consequence, the condition \( d^1(\psi) = \tau_\gamma - p^* \gamma' \) is just that \( \psi \) be a homomorphism, which is indeed the case.

For a more crucial example of a \( \text{SES}_2 \) morphism, we refer to the twist map discussed below (Subsection 5.2).

One characteristic feature of nilquadratic groups is the symplectic structure they induce. We recall some basic facts, essentially following [10]. We call a subgroup \( H \) of a group \( G \) self-centralizing iff \( C_G(H) \subseteq H \).

**Lemma 10.** Let \( H \leq G \) in \( \text{Grp} \). Then

1. \( C_G(H) \subseteq H \iff Z(H) = C_G(H) \);
2. \( C_G(H) \subseteq H \) and \( H \) abelian \iff \( H \) maximal abelian;
Proof. 1. $C_G(H) \subseteq H \implies Z(H) = C_G(H)$ is obvious. Trivially also $Z(H) = C_G(H) \implies C_G(H) = Z(H) \subseteq H$.

2. Assume $C_G(H) \subseteq H$ and $H \in \text{Ab}$. If $H \leq A \leq G$ with $A \in \text{Ab}$ then $A \subseteq C_G(H) \subseteq H$, hence $A = H$, i.e., $H$ is max. abelian. Conversely, assume $H$ being max. abelian. Take $x \in C_G(H)$. Then $H \cdot \langle x \rangle$ is an abelian group containing $H$, whence $x \in H$. Consequently $C_G(H) \subseteq H$ and $H \in \text{Ab}$.

3. is obvious from 1. and 2. □

Proposition 11. Let $E: T \xrightarrow{\iota} H \xrightarrow{\pi} P$ be an exact sequence of groups.

Then $E$ is central $\iff k_H$ factors $\xymatrix{ H \times H \ar[rr]_{\pi \times \pi} \ar[dr]_{\omega} & & H \ar[dl] \ar[d] \cr & P \times P & }$

Proof. Let $\varepsilon$ be central, $\pi(u) = \pi(u') \land \pi(v) = \pi(v')$. Then $u^{-1}u' = \iota(a)$, $v^{-1}v' = \iota(b)$ i.e., $u' = u\iota(a)$, $v' = v\iota(b)$

$[u', v'] = u\omega(v)b\iota(a)^{-1}u^{-1}\iota(b)^{-1}v^{-1} = uuv^{-1}v^{-1} = [u, v]$

Thus $\omega((\pi(u), \pi(v))) := [u, v]$ is well-defined. Conversely, assume that $[\bullet, \bullet]$ factors through $P \times P$. For $c \in T, e \in H$

$[\iota(c), e] = (\omega \circ \pi \times \pi)(\iota(c), e) = \omega(\pi\iota(c), \pi(e)) = \omega(1, \pi(e)) = \omega(\pi(1), \pi(e)) = [1, e] = 1$

Thus $\iota(T) \subseteq Z(H)$ i.e., $E$ is central (in particular, $T \in \text{Ab}$). □

Corollary 12. Consider central extensions of $P \in \text{Grp}$

$\varepsilon: 1 \rightarrow T \xrightarrow{\iota} E \xrightarrow{\pi} P \rightarrow 1, \varepsilon': 1 \rightarrow T \xrightarrow{j} F \xrightarrow{\rho} P \rightarrow 1$.

If $\varepsilon \sim \varepsilon'$ via $(1_T, \varphi, 1_P): \varepsilon \cong \varepsilon'$, then $\varphi' \circ \omega_\varepsilon = \omega_{\varepsilon'}$ (where $\varphi'$ is induced by $\varphi$).
\[ (u, v) \in F \times F, \quad u = \varphi(x), \quad v = \varphi(y). \]

\[ (\varphi' \circ \omega_\varepsilon \circ p \times p)(u, v) = \varphi(\omega_\varepsilon(p \varphi(x), p \varphi(y))) = \varphi(\omega_\varepsilon(\pi(x), \pi(y))) \]

\[ = \varphi[x, y] = [\varphi(x), \varphi(y)] = [u, v] \]

Thus, \( \varphi' \circ \omega_\varepsilon = \omega_\varphi'. \) \( \square \)

**Corollary 13.** Let \( P \in \text{Grp} \) and \( \varepsilon: T \xrightarrow{i} E \xrightarrow{\pi} P \) be central. Then

1. \( k_E: E \times E \rightarrow E' \) is bilinear \( \iff \omega_\varepsilon: P \times P \rightarrow E' \) is bilinear \( \iff E' \subseteq Z(E). \)
2. \( P \in \text{Ab} \implies \omega_\varepsilon: P \times P \rightarrow T \) is bilinear.

**Proof.**

\[ \omega_\varepsilon(\pi(u)\pi(v)) = [uv, w] \]

\[ \omega_\varepsilon(\pi(u)\pi(w))\omega_\varepsilon(\pi(v)\pi(w)) = [u, w][v, w]. \]

By Fact 3, point (1) and Fact 6

\( \omega_\varepsilon \) bilinear \( \iff k_E \) bilinear \( \iff E' \subseteq Z(E) \iff E \in \text{Nil}_2 \)

If \( P \in \text{Ab} \) then \( E' \subseteq i(T) \subseteq Z(E) \), hence \( \omega_\varepsilon \) is bilinear and may considered having values in \( T \)

\[ \omega_\varepsilon: P \times P \rightarrow T, \quad \omega_\varepsilon(\pi(u), \pi(v)) = i^{-1}[u, v]. \] \( \square \)

**Proposition 14.** Let \( G \leq P \in \text{Grp} \) and \( \varepsilon: T \xrightarrow{i} E \xrightarrow{\pi} P \) central. Then

1. \( C_E(\tilde{G}) = \pi^{-1}(G^\perp); \)
2. \( C_E(\tilde{G}) \subseteq \tilde{G} \iff G^\perp \subseteq G; \)
(3) $\tilde{G}$ maximal abelian $\iff G = G^\perp$.

Proof. 1. If $e \in C_E(\tilde{G})$ then $[e, \tilde{x}] = 1 \ \forall \tilde{x} \in \tilde{G}$, $\omega_\varepsilon(\pi(e), x) = 1 \ \forall x \in G$, $\pi(e) \in G^\perp$, $e \in \pi^{-1}(G^\perp)$. If, conversely, $e \in \pi^{-1}(G^\perp)$ then $\omega_\varepsilon(\pi(e), x) = 1 \ \forall x \in G$, $[e, \tilde{x}] = 1 \ \forall \tilde{x} \in \tilde{G}$, hence $e \in C_E(\tilde{G})$. 2. $C_E(\tilde{G}) \subseteq \tilde{G} \iff \pi^{-1}(G^\perp) \subseteq \pi^{-1}(G) \iff G^\perp \subseteq G$.

3. If $\tilde{G}$ max. abelian then, by Lemma 10, $\tilde{G} = C_E(\tilde{G})$ hence

$$G = \pi(\tilde{G}) = \pi C_E(\tilde{G}) = \pi \pi^{-1}(G^\perp) = G^\perp.$$ 

Conversely, if $G = G^\perp$ then

$$\tilde{G} = \pi^{-1}(G) = \pi^{-1}(G^\perp) = C_E(\tilde{G}).$$

hence, again by Lemma 10, $\tilde{G}$ is maximal abelian. \qed

The following theorem lists once more the relevant facts

**Theorem 15.** Let $\varepsilon : 1 \rightarrow T \xrightarrow{i} E \xrightarrow{\pi} P \rightarrow 1$ be central.

1. The commutator of $E$ factors through $P \times P$ 

$$E \times E \xrightarrow{[\cdot, \cdot]} E'. 

\pi \times \pi \downarrow \downarrow \omega

\downarrow \downarrow

P \times P$$

2. Let $G \leq P$. Then $\tilde{G}$ max. abelian $\iff G^\perp = G$.

3. If $G \leq P$ and $G^\perp = G$ then $G \in \text{Ab}$.

4. If $P = G \times \Gamma$ and $G^\perp = G$ and $\Gamma^\perp = \Gamma$ then $P \in \text{Ab}$ and $E' \leq i(T) \leq Z(E)$.

5. If $X \subseteq P$ then

$$X^\perp = \{u \mid \forall x \in X \omega(u, x) = 1\} = \{u \mid \forall x \in X \omega(x, u) = 1\}$$

is a subgroup of $P$.

**Definition 16.** Let $E : T \xrightarrow{i} H \xrightarrow{\pi} P$ be an extension in $\text{SES}_2$ and choose an arbitrary set-theoretic section $s$ of $\pi$. The **commutator form** $\omega_E : P \times P \rightarrow T$ is defined by $\omega_E(w, z) := i^{-1}[s(w), s(z)]$.

It is easy to see that $[s(w), s(z)] \in T$ and that $\omega_E(w, z)$ does not depend on the choice of $s$; hence $\omega_E$ is well-defined. Moreover, one checks that equivalent extensions have the same commutator form, so $\omega$ depends only on the cohomology class of any cocycle $\gamma \in Z^2(P, T)$ describing the equivalence class of $E$. Explicitly, one obtains the commutator form $\omega_E(w, z) = \gamma(w, z)/\gamma(z, w)$ in terms of the cocycle.

Without taking recourse to a section, one can in fact define a “commutator form” $[\cdot, \cdot] : H/\mathcal{Z}(H) \times H/\mathcal{Z}(H) \rightarrow [H, H]$ for an arbitrary group $H$. But it turns out that $[\cdot, \cdot]$ is bilinear precisely when $H$ is
nilquadratic; see Fact 2.4 in [10]. Since we are here only interested in the nilquadratic setting, the commutator form $\omega_E$ is in fact bilinear as we shall now check. Of course $\omega_E$ is always alternating in the sense that $\omega_E(z, z) = 1$ for all $z \in P$. As usual this implies that $\omega_E$ is antisymmetric, which here takes on the somewhat unusual form $\omega_E(w, z)\omega_E(z, w) = 1$. As an alternating bilinear form, we may thus think of the commutator form as $\omega_E \in \text{Hom}_\mathbb{Z}(\Lambda^2 P, T)$. But let us first provide the short proof of bilinearity.

**Proposition 17.** Let $E \colon T \xrightarrow{\iota} H \xrightarrow{\pi} P$ be an extension in $\text{SES}_2$. Then $\omega_E$ is bilinear.

**Proof.** We show $\omega(w_1 + w_2, z) = \omega(w_1, z)\omega(w_2, z)$ for $w_1, w_2, z \in P$. We may choose a section $s$ of $\pi$ with $s(w_1 + w_2) = s(w_1)s(w_2)$; as remarked above, the commutator form $\omega_E$ is independent of these choices. Let us then write $u_1 = s(w_1)$, $u_2 = s(w_2)$ and $v = s(z)$ for the corresponding elements over $\pi$. Since $[H, H] \leq Z(H)$ by Fact 3 we obtain

\[
[u_1, v][u_2, v] = u_1 vu_1^{-1}v^{-1}, u_2]v^{-1} = u_1 v [v^{-1}, u_2] u_1^{-1}v^{-1} = u_1 u_2 vu_2^{-1}u_1^{-1}v^{-1} = [u_1 u_2, v],
\]

which implies $\omega(w_1 + w_2, z) = \omega(w_1, z)\omega(w_2, z)$ as desired. By antisymmetry, we have also $\omega(w, z_1 + z_2) = \omega(w, z_1)\omega(w, z_2)$. \hfill $\square$

For the remainder of this section (and indeed the rest of the paper), we shall work with $\text{SES}_2$. Therefore we may work with the alternating form $\omega_E \in \text{Hom}_\mathbb{Z}(\Lambda^2 P, T)$ much in the same way as in classical symplectic geometry (where the abelian groups are finite-dimensional vector spaces, i.e. equipped with an additional scalar action).

In our context, the fundamental notion is the symplectic orthogonal of any subgroup $G \leq P$. We give the definition in a slightly more general setting; this will be useful for bringing out some parallels with the “natural orthogonal” of [38, § A.1].

**Definition 18.** Let $M, N, T$ be modules over a commutative unital ring $R$. For fixed bilinear form $\alpha \colon M \times N \to T$, the orthogonal of submodules $M_0 \leq M$ and $N_0 \leq N$ is given by $M_0^\perp := \{y \in N \mid M_0 \perp_\alpha y\}$ and $N_0^\perp := \{x \in M \mid x \perp_\alpha N_0\}$, respectively.

As usual, orthogonality relation is defined by $x \perp_\alpha y \iff \alpha(x, y) = 0$ for $(x, y) \in M \times N$. Moreover, we use the notation $M_0 \perp_\alpha y$ as shorthand for $\forall_{x \in M_0} x \perp_\alpha y$; similarly with $x \perp_\alpha N_0$.

Strictly speaking, we should have called $M_0^\perp$ and $N_0^\perp$ the left and right orthogonal, respectively. In applications, either membership in $M$ and $N$ serves to disambiguate the two orthogonals (Example 22 below)
or the orthogonality relation is in fact symmetric so that the two notions coincide (Examples 21 and 23 below). For principal rings like fields or \( \mathbb{Z} \), it is known—see for example [40, Thm. 11.4], which generalizes to this setting—that \( \perp \) is a symmetric relation iff \( \alpha \) is symmetric or alternating.

**Proposition 19.** If \( \alpha : M \times N \rightarrow T \) is a bilinear form, its orthogonals create an antitone Galois connection between the submodules of \( M \) and \( N \), with the biorthogonals \( \perp \perp \) as closure operators.

**Proof.** It is obvious that both orthogonals are antitone maps with respect to inclusion and that \( M_0 \leq M_0^{\perp \perp} \) as well as \( N_0 \leq N_0^{\perp \perp} \) holds. It follows that \( N_0 \leq M_0^{\perp \perp} \iff M_0 \leq N_0^{\perp \perp} \), so we have an antitone Galois connection. The statement about the biorthogonals is a general property of Galois connections [12, §7.27]. □

The fixed points of the closure operator are the closed submodules of the Galois connection. To be specific, let us write
\[
\text{Cl}(M) = \{ M_0 \leq M \mid M_0^{\perp \perp} = M_0 \}.
\]
\[
\text{Cl}(N) = \{ N_0 \leq N \mid N_0^{\perp \perp} = N_0 \}
\]
for the corresponding posets on the left and right side of the Galois connection. They are in fact not just posets but lattices.

**Proposition 20.** The posets \( \text{Cl}(M) \) and \( \text{Cl}(N) \) induced by the bilinear form \( \alpha : M \times N \rightarrow T \) are complete lattices with \( M_1 \land M_2 = M_1 \cap M_2 \) and \( M_1 \lor M_2 = (M_1 + M_2)^{\perp \perp} \) for \( \text{Cl}(M) \), similarly for \( \text{Cl}(N) \).

The orthogonal is a lattice anti-isomorphism \( \text{Cl}(M) \cong \text{Cl}(N) \).

**Proof.** Since \( M_0 \mapsto M_0^{\perp \perp} \) is a closure operator [12, §7.27i], its system of closed sets is a topped intersection structure [12, §7.4]. We conclude that \( \text{Cl}(M) \) is a complete lattice with operations as given [12, §2.32] and that \( M_0 \mapsto M_0^{\perp} \) is a lattice anti-isomorphism as claimed [12, §7.27i]. □

Whereas the lattice \( \text{Sub}(M) \) of all submodules of \( M \) is modular, the lattice \( \text{Cl}(M) \) is in general—see Example 21—not modular. Of course any sublattice of a modular lattice is again modular, but \( \text{Cl}(M) \) is not a sublattice of \( \text{Sub}(M) \) since its join operation differs in general.

It is clear that the lattice \( \text{Sub}(M) \) is bounded by 0 and \( M \). As to the lattice \( \text{Cl}(M) \), it is clear that \( M \) is still the maximal element since it is obviously closed. But 0 is closed and hence the minimal element of \( \text{Cl}(M) \) if and only if \( M^{\perp} = 0 \), which is equivalent to \( \alpha : M \times N \rightarrow T \) being non-degenerate (on the left). Since we always have \( 0^{\perp} = M \), the orthogonal will then act in between the global bounds 0 and \( M \) inclusively, swapping the two by its action.
It should also be noted that the orthogonal in general is *not a lattice endomorphism* on \( \text{Sub}(M) \) since there we have

\[
(M_1 + M_2)^\perp = M_1^\perp \cap M_2^\perp \quad \text{but only}
\]

\[
(M_1 \cap M_2)^\perp \geq M_1^\perp + M_2^\perp,
\]

and it is straightforward to give examples where the latter inclusion is strict (see Examples 21, 22 and 23 below). Since the left-hand side is closed, it may be replaced by \( M_1^\perp \lor M_2^\perp \) to strengthen the inequality.

As to be expected, the biorthogonal also fails to be a lattice endomorphism on \( \text{Sub}(M) \). But we do have the *biorthogonal inequalities*

\[
(M_1 \cap M_2)^{\perp \perp} \leq M_1^{\perp \perp} \cap M_2^{\perp \perp},
\]

\[
(M_1 + M_2)^{\perp \perp} \geq M_1^{\perp \perp} + M_2^{\perp \perp},
\]

where the first becomes equal iff the strengthened (5) does for \( M_1, M_2 \), and the second iff the original (5) does for \( M_1^\perp, M_2^\perp \). This asymmetry can be traced to the fact that intersections of closed submodules are again closed while sums of closed submodules generally are not (Proposition 20).

Let us now have a look how these relations play out in some important special settings. The orthogonals introduced in Examples 21, 22 and 23 are, respectively, called the *symmetric orthogonal*, the *natural orthogonal* and the *symplectic orthogonal*.

**Example 21.** The simplest examples is when \( M = N \) are inner product spaces over a field \( T = K \) with \( \alpha(x, y) = \langle x | y \rangle \) the *inner product*, where the orthogonal has its original geometric significance. By definition, this requires \( \alpha \) to be nondegenerate. This includes especially the setting of *Hilbert spaces*, where the closed subspaces in the sense of the Galois connection are precisely the closed subspaces in the sense of the topology \([21, \text{Cor.2.2.4}]\). The lattice \( \text{Cl}(H) \) is not modular if \( H \) is an infinite-dimensional Hilbert space \([37, \text{Prop. 4.4}]\); cf. also \([17, \text{Prob. 14}]\).

For a finite-dimensional vector space (which is naturally a Hilbert space when \( K = \mathbb{R} \)), all subspaces are closed. Otherwise, it is also easy to find examples where the inclusion (5) is strict: Take any disjoint dense subsets \( M_1 \) and \( M_2 \) of the separable real Hilbert space \( M = L^1(\mathbb{R}) \), for example the linear span \( M_1 \) of the Haar basis and \( M_2 = \mathbb{R}[x] \). Then \( M_1^{\perp \perp} = M = M_2^{\perp \perp} \) implies that both \( M_1^\perp \) and \( M_2^\perp \) collapse to \( M^\perp = 0 \), so that \( (M_1 \cap M_2)^\perp = 0^\perp = M \) is clearly larger than \( M_1^\perp \lor M_2^\perp = M_1^\perp + M_2^\perp \).

**Example 22.** If \( M \) and \( N \) are modules over \( T = R \) with a given bilinear form \( \alpha = b : M \times N \to R \), we recover the setting of \([38, \text{Prop.} 4.4]\).
§ A.1], which is particularly important when \( N = M^* = \text{Hom}_R(M, R) \) is the dual module and \( \alpha(x, \beta) = \beta(x) \) is the corresponding natural pairing. In the algebraic approach to boundary problems [43][42], the module \( M \) is in fact a vector space over a suitable field \( K \). If \( M \) is infinite-dimensional over \( K \), primal subspaces \( M_0 \leq M \) are always closed but there are many plenty dual subspaces \( N_0 \leq M^* \) that are not closed [27, §9.2/6]. Since in this case \( \text{Cl}(M) = \text{Sub}(M) \) is modular, its isomorphic twin \( \text{Cl}(M^*) \) is as well. Moreover, one has \( N_1 \vee N_2 = N_1 + N_2 \) in \( \text{Cl}(M^*) \), so the biquotient may be dropped for finite (but not for infinite) joins [27, §9.3/3].

The inclusion (5) is of course an identity in \( \text{Cl}(M) \) since primal subspaces are always closed and the finite join in \( \text{Cl}(M^*) \) is the sum of subspaces. But for dual subspaces \( N_1, N_2 \leq M^* \), one may construct a counterexample similar to the one in Example (21). For example, choose \( M = C^\infty(\mathbb{R}^2) \) and set

\[
N_1 = [\text{ev}_{\xi,\eta} \circ \partial_x | (\xi, \eta) \in \mathbb{R}^2] + [\text{ev}_{\eta,-1,\eta}], \\
N_2 = [\text{ev}_{\xi,\eta} \circ \partial_y | (\xi, \eta) \in \mathbb{R}^2] + [\text{ev}_{\xi,\xi-1}].
\]

Then one may check that again \( N_1 \perp = 0 = N_2 \perp \) so that \( N_1 \) and \( N_2 \) are “dense” disjoint subspaces of \( N^* \), and we obtain as before the strict inequality \( M = \overline{(N_1 \cap N_2)} \perp > N_1 \perp + N_2 \perp = 0 \).

Example 23. For our present purposes, an alternating form \( \alpha = \omega : P \times P \to T \) is given. Here \( P \) and \( T \) are abelian groups, i.e. modules over \( R = \mathbb{Z} \). As for the inner product spaces, the right and left orthogonal are identical since also here \( M = N \) and \( x \perp y \Leftrightarrow y \perp x \). As we shall see below (Example 24), there are in general many examples of non-closed subgroups.

We can see this, and various other properties, from the following example. Given a vector space \( V \) over a field \( K \) and using additive notation, the canonical symplectic structure \( \Omega_V : P \times P \to K \) on the phase space \( P := V \times V^* \) is given by \( \Omega_V(x, \xi | x', \xi') := \xi'(x) - \xi(x') \). It is easy to check that the symplectic orthogonal of a subspace \( A \times B \subseteq P \) is here given by \( (A \times B)\perp = B' \times A' \), where we write \( A' \) and \( B' \) for the natural orthogonal of Example 22, so as to avoid confusion with the symplectic orthogonal. Hence \( A \times B \in \text{Cl}(P) \) iff \( B \in \text{Cl}(V^*) \). Setting \( B_1 := 0 \times N_1 \) and \( B_2 := 0 \times N_2 \) for the space \( M = C^\infty(\mathbb{R}^2) \) used in Example 22, we obtain again an example where the inclusion (5) is in fact strict.

The symplectic orthogonal will be the crucial one for us. Thus fix a group \( P \) with alternating form \( \omega : P \times P \to T \). The following standard terminology for subgroups \( G \leq P \) is adopted from symplectic geometry.
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(where \( P \) is a vector space, having a scalar action in addition to the additive structure):

- **Symplectic**: \( G \cap G^\perp = 0 \iff \omega|_G \) nondegenerate
- **Isotropic**: \( G \leq G^\perp \iff \omega|_G = 1 \)
- **Coisotropic**: \( G^\perp \leq G \iff \tilde{\omega} \) nondegenerate on \( G/G^\perp \)
- **Lagrangian**: \( G = G^\perp \iff \omega \) isotropic and coisotropic

We denote the corresponding classes by \( \text{Sym}(P) \), \( \text{Iso}(P) \), \( \text{Co}(P) \), \( \text{Iso}(P) \cap \text{Co}(P) \), omitting reference to \( P \) where it is clear from the context. It is obvious that \( \text{Iso} \) and \( \text{Co} \) are, respectively, downward and upward closed. Beyond that, however, one observes some awkward asymmetries, ultimately due to the failure of closure for arbitrary subgroups \( G \leq P \). For example, one has \( G \in \text{Iso} \iff G^\perp \perp \in \text{Iso} \) whereas \( G \in \text{Co} \Rightarrow G^\perp \perp \in \text{Co} \), which in general cannot be upgraded to an equivalence. Moreover, we have \( G \in \text{Iso} \iff G^\perp \in \text{Iso} \) versus again just an implication \( G \in \text{Co} \Rightarrow G^\perp \in \text{Iso} \).

It should also be noted that \( \text{Iso} \) and \( \text{Co} \) are not sublattices of either \( \text{Sub} \) or \( \text{Cl} \); they are just meet and join sub-semilattices, respectively (since they are downward/upward closed). Hence the Lagrangian subgroups—which are of course symplectically closed—are neither: For \( G, H \in \text{Iso} \cap \text{Co} \) we have \( (G \cap H)^\perp = G^\perp + H^\perp = G + H \) and \( (G + H)^\perp = G^\perp \cap H^\perp = G \cap H \).

Let us illustrate these points for the canonical symplectic structure introduced in Example 23.

**Example 24.** As pointed above (Example 23), one has \( \mathcal{A}'' = \mathcal{A} \) for primal subspaces \( \mathcal{A} \leq V \), but in general only \( \mathcal{B}'' \geq \mathcal{B} \) for dual subspaces \( \mathcal{B} \leq V^* \). Hence \( \mathcal{A} \times \mathcal{B} \leq P \) is symplectically closed iff \( \mathcal{B} \) is orthogonally closed. In all cases, the subspaces \( \mathcal{A} \times 0 \) and \( 0 \times \mathcal{B} \) are isotropic, \( \mathcal{A} \times V^* \) and \( V \times \mathcal{B} \) coisotropic, \( \mathcal{A} \times \mathcal{A}' \) and \( \mathcal{B}' \times \mathcal{B}'' \) Lagrangian. For an example of a symplectic subspace, assume \( \mathcal{A} + \mathcal{B}^\perp = V \) and \( \mathcal{B} = \mathcal{B}'' \), for instance \( \mathcal{A} = \ker(T) \) for regular boundary problems \( (T, \mathcal{B}) \) over an integro-differential algebra \( (V, \partial, \int) \); then \( \mathcal{A} \times \mathcal{B} \) is a symplectic subspace. If \( \mathcal{B} \) is not orthogonally closed, then \( G := \mathcal{B}' \times \mathcal{B} \in \text{Iso} \setminus \text{Co} \) but we have \( G^\perp = G^{\perp \perp} = \mathcal{B}' \times \mathcal{B}'' \in \text{Iso} \cap \text{Co} \).

As to the lattice operations, take \( G = \mathcal{A} \times 0 \) and \( H = 0 \times \mathcal{B} \), again with \( \mathcal{A} + \mathcal{B}^\perp = V \). Then we know that \( G, H \in \text{Iso} \), but \( G + H = \mathcal{A} \times \mathcal{B} \) is symplectic and hence certainly not isotropic. Similarly, setting now \( G = \mathcal{A} \times V^* \) and \( H = V \times \mathcal{B} \) for the same \( \mathcal{A} \) and \( \mathcal{B} \) will ensure that we have \( G, H \in \text{Co} \) while \( G \cap H = \mathcal{A} \times \mathcal{B} \) is again symplectic and thus not coisotropic.

What all this tells us is that the classes \( \text{Iso} \) and \( \text{Co} \) do not sit nicely in \( \text{Sub}(P) \). This is completely analogous to the case of the natural
orthogonal proposed in Example 22 above. In fact, one might view the symplectic orthogonal (on a vector space) as a kind of amalgamation of the natural orthogonal. The remedy taken for the latter—in particular for the algebraic approach to boundary problems [43][42]—is to restrict the dual subspaces $\mathcal{B} \leq V^*$ to the orthogonally closed ones [27, §9.3]. From the viewpoint of the Galois connection, a non-closed subspace $\mathcal{B}$ is just an arbitrary choice of “generators” for the actual object of interest: The invariant spaces taking part in the Galois connection, namely the pair $(\mathcal{B}', \mathcal{B}'')$ that stabilizes under successive orthogonals. In the same vein, it is apposite to discard the full subgroup lattice $\text{Sub}(P)$ in favor of the lattice $\text{Cl}(P)$ of symplectically closed groups expounded in Proposition 20.

Thus the orthogonal is a lattice anti-automorphism $\text{Cl} \rightarrow \text{Cl}$, inducing a semilattice anti-isomorphism between $\text{Iso}' := \text{Co}^\perp = \text{Iso} \cap \text{Cl}$ and $\text{Co}' := \text{Iso}^\perp = \text{Co} \cap \text{Cl}$ that stabilizes the Lagrangian subgroups $\text{Iso}' \cap \text{Co}' = \text{Iso} \cap \text{Co}$.

**Example 25.** As in the case of the inner orthogonal (Example 21), the lattice of closed subgroups is in general not modular. In fact, one may adopt the setting in a way similar to the canonical symplectic structure (Example 23). If $V$ is any vector space with inner product $\langle | \rangle$, we define the symplectic form $\Psi_V : P \times P \rightarrow K$ on $P := V \times V$ by the analogous relation $\Psi_V(x, \xi | x', \xi') = \langle \xi' | x \rangle - \langle \xi | x' \rangle$. Again we have $(A \times B)^\perp = B' \times A'$ for $A, B \leq V$, using the prime now to denote the inner orthogonal as opposed to the symplectic orthogonal $\perp$. It follows that $A \leq V$ is symmetrically closed iff $A \times 0$ is symplectically closed. If we choose for $V$ an infinite-dimensional Hilbert space, we have seen that $\text{Cl}(V)$ is not modular, hence the sublattice $V \times 0$ of $\text{Cl}(P)$ is also not modular, showing that $\text{Cl}(P)$ cannot be modular either. //

Recalling that we should restrict ourselves to symplectically closed subgroups, the isotropic and coisotropic semilattices may be characterized by the extremal properties, just as in classical symplectic geometry (where closure is not needed due to the hypothesis of finite dimension).

**Lemma 26.** Let $\omega$ be an alternating form on an abelian group $P$. Then a subgroup $G \leq P$ is Lagrangian iff $G$ is maximal in $\text{Iso}'(P)$ iff $G$ is minimal in $\text{Co}'(P)$.

**Proof.** We show first that $G$ is Lagrangian iff $G$ is maximal in $\text{Iso}(P)$. Assume that $G \leq P$ is Lagrangian. If $H \geq G$ is isotropic, we have $H \leq H^\perp \leq G^\perp = G$ and thus $G = H$. Hence $G$ is maximal isotropic. Similarly, if $H \leq G$ is coisotropic, we get $G = G^\perp \leq H^\perp \leq H$ so that $G = H$ and $G$ is seen to be minimal coisotropic. Next assume $G$
is a maximal isotropic subgroup of $P$; we must show that $G$ is then Lagrangian. If $G^\perp$ is larger than $G$, we take $x \in G^\perp \setminus G$ and let $H \leq P$ be the subgroup generated by $x$ and $G$. Each element of $H$ has the form $kx + g$ for $k \in \mathbb{Z}$ and $g \in G$, and we have

$$
\omega(kx + g, k'x + g') = \omega(x, x)^{kk'} \omega(x, g')^k \omega(g, x)^{k'} \omega(g, g') = 1
$$

since all four factors are unity: the first since $\omega(x, x) = 1$, the second and third because $x \in G^\perp$, and the fourth by the hypothesis that $G$ is isotropic. Since $\omega$ is trivial on $H$, this is a strictly larger isotropic group containing $G$, which contradicts the maximality of $G$. Hence we conclude that in fact $G^\perp = G$.

(Note that if $G$ is Lagrangian then $G$ is minimal in $\text{Co}(P)$; the proof is similar to the maximality statement. It is not clear to us if the converse is true. Since we do not need this for the characterization in $\text{Cl}$, we refrain from further investigations of this case.)

Now we show that maximality in $\text{Iso}(P)$ and in $\text{Iso}'(P)$ are equivalent for an arbitrary $G \in \text{Cl}(P)$. Given the former, assume now $H \in \text{Iso}'(P)$ satisfies $G \leq H$, then clearly $G = H$ by maximality in $\text{Iso}(P)$. Hence $G$ is also maximal in $\text{Iso}'(P)$. Conversely, assume this is the case. If $H \in \text{Iso}(P)$ is such that $G \leq H$, then also $G = G^\perp \leq H^\perp$. Since $H^\perp \in \text{Iso}'(P)$, we have $G = H^\perp$ by maximality in $\text{Iso}'(P)$ so that $H \leq H^\perp = G \leq H$ and hence $G = H$. Together with the above maximality result, we see thus that $G$ is Lagrangian iff $G$ is maximal in $\text{Iso}(P)$. The minimality characterization is now an immediate consequence since $G \mapsto G^\perp$ is a lattice anti-isomorphism $\text{Iso}' \cong \text{Co}'$.

As the preceding result confirms, Lagrangian subgroups (briefly called “Lagrangians”) of an abelian group $P$ reflect a significant and natural part of the symplectic structure. For our algebraic investigation of the Fourier transform, we shall in fact need an even richer structure—not just one but two Lagrangians, interlaced in a symmetric compound. Hence we call a pair of Lagrangians $(G, \Gamma)$ a Lagrangian bisection if they form a direct decomposition $G \bigoplus \Gamma = P$.

This is a straightforward generalization from the classical setting, where one just requires [6, p. 21] the so-called “transversality condition” $G \bigoplus \Gamma = P$. This may also be generalized: We call $(P, \omega)$ a symplectic $\mathbb{Z}$-module if $P$ is a symplectic subgroup of itself. In other words, $\omega$ is to be nondegenerate on all of $P$. In such cases (including the symplectic vector spaces of the classical setting), transversality is sufficient.

**Lemma 27.** Let $(P, \omega)$ be a symplectic $\mathbb{Z}$-module with two Lagrangians $G, \Gamma \leq P$ such that $G + \Gamma = P$. Then $(G, \Gamma)$ is a Lagrangian bisection.
Proof. We need only show \( G \cap \Gamma = 0 \), so assume \( z \in G \cap \Gamma \). Since we have \( G^\perp = G \) and \( \Gamma^\perp = \Gamma \), this implies that \( z \) is orthogonal to any element of \( G \) and also orthogonal to any element of \( \Gamma \). Now if \( x + \xi \) with \( x \in G \) and \( \xi \in \Gamma \) is an arbitrary element of \( P \), we obtain that \( \omega(z, x + \xi) = \omega(z, x) \omega(z, \xi) = 1 \). Hence \( z \) is orthogonal to all of \( P \), which implies \( z = 0 \) by the nondegeneracy of \( \omega \).

We shall now apply the “symplectic machinery” for investigating nilquadratic extensions \( E: T \overset{\iota}{\hookleftarrow} H \overset{\pi}{\twoheadrightarrow} P \). Indeed, we know from Proposition 17 that they always come with an alternating form \( \omega_E \). It is thus to be expected that certain properties of \( H \) will be reflected in the symplectic structure induced by \( \omega_E \). In particular, the subgroup lattice of \( H \) will become visible as a kind of mirror image in \((P, \omega_E)\). Certain subgroup types turn out to be prominent in this context: Recall that a subgroup \( H_0 \) of a group \( H \) is called self-centralizing if \( C_H(H_0) \leq H_0 \) or equivalently if \( C_H(H_0) = Z(H_0) \). A self-centralizing abelian subgroup is the same as a maximal abelian subgroup; such subgroups can also be characterized by \( C_H(H_0) = H_0 \). Most importantly, there is a counterpart of Lagrangian bisections: For any group \( H \) with a distinguished abelian subgroup \( \hat{T} \), we call \((\tilde{G}, \tilde{\Gamma})\) an abelian bisection over \( \hat{T} \) if \( \tilde{G} \) and \( \tilde{\Gamma} \) are maximal abelian subgroups of \( H \) such that \( \tilde{G} \cap \tilde{\Gamma} = \hat{T} \) and \( \tilde{G} \tilde{\Gamma} = H \).

**Theorem 28.** Let \( E: T \overset{\iota}{\hookleftarrow} H \overset{\pi}{\twoheadrightarrow} P \) be an extension in \( \text{SES}_2 \) with symplectic form \( \omega_E \in \text{Hom}_{\mathbb{Z}}(\Lambda^2 P, T) \). Then \( G \mapsto \overset{\pi}{\tilde{G}} := \pi^{-1}(G) \) is a monotone bijection between subgroups of \( P \) and those subgroups of \( H \) that contain \( \hat{T} := \iota T \), with the following properties:

1. The group \( H \) is abelian iff \( \omega_E \) is trivial.
2. The extension \( E \) is strictly central iff \( \omega_E \) is nondegenerate.
3. The subgroup \( \overset{\pi}{\tilde{G}} \) is abelian iff \( G \) is isotropic.
4. The subgroup \( \overset{\pi}{\tilde{G}} \) is self-centralizing iff \( G \) is coisotropic.
5. The subgroup \( \overset{\pi}{\tilde{G}} \) is maximal abelian iff \( G \) is Lagrangian.
6. We have \( \overset{\pi}{\tilde{G}}_1 \overset{\pi}{\tilde{G}}_2 = H \) iff \( G_1 + G_2 = P \).
7. We have \( \overset{\pi}{\tilde{G}}_1 \cap \overset{\pi}{\tilde{G}}_2 = \overset{\pi}{\hat{T}} \) iff \( G_1 \cap G_2 = 0 \).
8. We get an abelian bisection \((\overset{\pi}{\tilde{G}}, \overset{\pi}{\tilde{\Gamma}})\) over \( \overset{\pi}{\hat{T}} \) iff \((G, \Gamma)\) forms a Lagrangian bisection.

**Proof.** \( G \mapsto \pi^{-1}(G) \) is a monotone bijection between arbitrary subgroups of \( P \) and those subgroups of \( H \) that contain \( T \). We now go through all items in order.

1. If \( H \in \text{Ab} \), then \( \omega_E(w, z) = \iota^{-1} [s(w), s(z)] = 1 \) for \( w, z \in P \). Conversely, assume \( \omega_E \) is trivial. Then \( [s(w), s(z)] = 1 \) for all
$w, z \in P$. Since the section $s$ is arbitrary, this means $[u, v] = 1$ for all $u, v \in H$.

(2) The nondegeneracy of $\omega_E$ means $\forall_{w \in P} [s(w), s(z)] = 1$ implies $z = 0$. By the arbitrariness of the section $s$, the premise is equivalent to $s(z) \in \mathcal{Z}(H)$ and then—for the same reason—to $\pi^{-1}(z) \subseteq \mathcal{Z}(H)$. Assuming nondegeneracy, let us now prove strict centrality. Taking $c \in \mathcal{Z}(H)$ and setting $z = \pi(c)$, we have $\pi^{-1}(z) \subseteq \mathcal{Z}(H)$ since $c' \in \pi^{-1}(z)$ implies $\pi(c'/c) = 1$ and hence $c'/c \in \hat{T} \leq \mathcal{Z}(H)$ by the exactness of $E$ and Fact 6. But then $c' \in \mathcal{Z}(H)$, so we have indeed $\pi^{-1}(z) \subseteq \mathcal{Z}(H)$, so the hypothesis of strict centrality yields $z = 0$ or $c \in \ker(\pi) = \hat{T}$.

For the converse, we assume $E$ is strictly central. Taking any $z \in P$ with $\pi^{-1}(z) \subseteq \mathcal{Z}(H) = \hat{T}$, we must show $z = 0$. Since $\pi$ is surjective, $z = \pi(c)$ for some $c \in \pi^{-1}(z) \subseteq \hat{T}$, hence $c = \iota(t)$ for some $t \in T$. But then $z = \pi(\iota(t)) = 0$ by the exactness of $E$.

(3) Isotropy of $G \leq P$ is equivalent to $\omega_E|_G : G \times G \to T$ being trivial. Since $\hat{T} \leq \hat{G}$, we may restrict the exact sequence to

$\hat{E} : T \to \hat{G} \ orchestrate P,$

and we have $\omega_{\hat{E}} = \omega_E|_G$. Now the claim follows from Item (1).

(4) The condition $C_H(\hat{G}) \leq \hat{G}$ amounts to requiring for all $u \in H$ that $\forall_{v \in \hat{G}} [u, v] = 1$ implies $u \in \hat{G}$, which is in turn equivalent to requiring for all $z \in P$ and all sections $s$ the implication

$\forall_{v \in \hat{G}} [s(z), v] = 1 \implies z \in G.$

It is easy to see that $\hat{G} = \hat{T}s(G)$, therefore the antecedent of the above implication is equivalent to $\forall_{w \in G} \forall_{c \in T} [s(z), \iota(c) s(w)] = 1$.

Since $\hat{T}$ commutes with all of $H$, the factor $\iota(c)$ and the quantifier over $c$ may be dropped, so the antecedent of the implication is actually equivalent to $\forall_{w \in G} [s(z), s(w)] = 1$, which is the same as $z \in G^\perp$.

(5) This follows from (3) and (4). Alternatively, it may also be inferred from (3) in conjunction with Lemma 26. (In the proof of the latter lemma, it has been observed that Lagrangian subgroups may also be characterized as the maximal isotropic ones in the full subgroup lattice.)

(6) Assume that $G_1 + G_2 = P$. For arbitrary $u \in H$, we must show that $u \in \hat{G}_1 \hat{G}_2$. By the hypothesis $G_1 + G_2 = P$, we have $\pi(u) = z_1 + z_2$ for suitable $(z_1, z_2) \in G_1 \times G_2$. Since $\pi$ is surjective, there are $u_1, u_2 \in H$ with $\pi(u_1) = z_1$ and $\pi(u_2) = z_2$. Using the exactness of $E$, we conclude from $\pi(u_1 u_2/u) = 0$.
that \( u_1 u_2 / u = \iota(c) \) for some \( c \in T \), and we obtain the required representation \( u = (\iota(c^{-1}) u_1) u_2 \in \hat{G}_1 \hat{G}_2 \). Conversely, assume that \( \hat{G}_1 \hat{G}_2 = H \). Taking \( z \in P \) arbitrary, we must show that \( z \in G_1 + G_2 \). We may pick any \( u \in H \) with \( \pi(u) = z \) and then choose \( u_1 \in \hat{G}_1 \) and \( u_2 \in \hat{G}_2 \) such that \( u = u_1 u_2 \). Then \( z = \pi(u_1) + \pi(u_2) \in \hat{G}_1 + \hat{G}_2 \).

(7) Assuming \( G_1 \cap G_2 = 0 \), take any \( u \in \hat{G}_1 \cap \hat{G}_2 \). Then \( \pi(u) = 0 \), and we obtain \( u \in \hat{T} \) since \( E \) is exact. Conversely, if \( \hat{G}_1 \cap \hat{G}_2 = \hat{T} \) and \( z \in G_1 \cap G_2 \) we have to show that \( z = 0 \). Taking any \( u \in H \) with \( \pi(u) = z \), it is clear that \( u \in \hat{G}_1 \) and \( u \in \hat{G}_2 \), so we obtain \( u = \iota(c) \) for some \( c \in T \). But then \( z = \pi(\iota(c)) = 0 \) as claimed.

(8) This follows from Items (6), (7) and (5).

\[ \square \]

The crucial property of Heisenberg groups is the existence of an abelian bisection or, equivalently by Theorem 28(8), the existence of a Lagrangian bisection in its phase space. While this is a strong requirement, we should not expect such a bisection to be unique as one can see even from the standard example in classical symplectic geometry.

**Example 29.** The classical Heisenberg group \( H_1(\mathbb{R}) \) may be defined as the group upper triangular of \( 3 \times 3 \) matrices

\[
\left\{ \begin{pmatrix} 1 & \xi & c \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mid (\xi, x) \in \mathbb{R}^2, c \in \mathbb{T} \right\}
\]

with the multiplication law given by

\[
\begin{pmatrix} 1 & \xi & c \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \xi' & c' \\ 0 & 1 & x' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \xi + \xi' & cc' \langle \xi | x' \rangle \beta \\ 0 & 1 & x + x' \\ 0 & 0 & 1 \end{pmatrix},
\]

under the bilinear form \( \beta: \mathbb{R} \times \mathbb{R} \to \mathbb{T} \) with \( \langle \xi | x \rangle_\beta := e^{ix\xi} \). It is known that \( H_1(\mathbb{R}) \) is nilquadratic, with phase space \( P = H_1(\mathbb{R})/\mathbb{T} \cong \mathbb{R}^2 \) via \( \left( \begin{smallmatrix} 1 \\ \xi \\ c \end{smallmatrix} \right) \mathbb{T} \leftrightarrow (\xi, x) \) and symplectic form \( \omega(\xi, x; \xi', x') = \langle \xi | x' \rangle_\beta / \langle \xi' | x' \rangle_\beta \), easily seen to be nondegenerate. The standard Lagrangian bisection used in Physics is then \( (\hat{G}, \hat{\Gamma}) \) with \( \hat{G} = \mathbb{R} \times 0 \leq P \) and \( \hat{\Gamma} = 0 \times \mathbb{R} \leq P \).

But there are may other ones, for example \( \hat{G} = \{ (\xi, -\xi) \in P \mid \xi \in \mathbb{R} \} \) with \( \hat{\Gamma} \) as before.

We shall now continue working with the category \( \text{Cnt}_2 \) of central extensions \( 1 \triangleleft T \triangleleft H \), where the torus \( T \) is somehow “fixed in the background”. It plays the role of the “dualizing object”, like \( K \) for \( K \)-vector
spaces or $\mathbb{T}$ for LCA groups. We shall thus refer to $H$ as a nilquadratic group over $T$. When $H'$ is another nilquadratic group over $T$, we speak of a group homomorphisms $\varphi: H \rightarrow H'$ over $T$ if $\varphi|_T = 1_T$. We call such morphisms toroidal when the torus $T$ is clear from the context. One may think of this as a generalized sort of $T$-linearity.

**Remark 30.** Note that the integral group ring $\mathbb{Z}[H]$ may be endowed with the natural map $\mathbb{Z}[T] \rightarrow \text{End}_\mathbb{Z}(\mathbb{Z}[H])$ given by the action $c \cdot \sum_{i \in \mathbb{Z}} n_i h_i = \sum_{i \in \mathbb{Z}} n_i (ch_i)$; thus $\mathbb{Z}[H]$ is a (in general) noncommutative algebra over the ring $\mathbb{Z}[T]$. Identifying the groups with their integral group rings, the group homomorphisms $H \rightarrow H'$ appear as ring homomorphisms, the toroidal ones as algebra homomorphisms.

We come now to the central definition of this section—*Heisenberg groups in their most general form* (in the scope of this paper).

**Definition 31.** A Heisenberg group over $T$ is a nilquadratic group $H$ together with an abelian bisection over $\hat{T}$.

We have $\hat{T} \leq Z(H)$ for any Heisenberg group $H$ over $\hat{T}$. Indeed, for a given abelian bisection $(\hat{G}, \hat{\Gamma})$ of $H$, any element $u \in H$ may be written as $u = \bar{x}\xi$ with $(\bar{x}, \xi) \in \hat{G} \times \hat{\Gamma}$ so that $cu = uc$ for $c \in \hat{T} = \hat{G} \cap \hat{\Gamma}$ since $c$ commutes with both $\bar{x}$ and $\xi$. The inclusion $\hat{T} \leq Z(H)$ is one half of the characterization of central extensions in Fact 6. The other half being supplied by the following Lemma, we see that

$$(6) \quad 1 \rightarrow T \hookrightarrow H \twoheadrightarrow H/T \rightarrow 0$$

is in fact a central extension.

**Lemma 32.** If $H$ is a Heisenberg group over $T$, we have $[H, H] \leq T$.

**Proof.** Let $(\hat{G}, \hat{\Gamma})$ be an abelian bisection for $H$. We show $[u, u'] \in T$ for arbitrary $u, u' \in H$. Since $\hat{G}$ and $\hat{\Gamma}$ generate $H$, we have $u = \bar{x}\xi$ and $u' = \bar{x}'\xi'$ for suitable $\bar{x}, \bar{x}' \in \hat{G}$ and $\xi, \xi' \in \hat{\Gamma}$. Hence we obtain

$$\begin{align*}
[u, u'] &= \bar{x}\xi \bar{x}'\xi' \xi^{-1}\bar{x}^{-1}\xi'^{-1}\bar{x}'^{-1} \\
&= (\bar{x}\xi \bar{x}'\xi' \xi^{-1}\bar{x}^{-1})(\bar{x}'\bar{x}\xi^{-1}\bar{x}^{-1})(\bar{x}'\bar{x}\xi^{-1}\bar{x}^{-1}),
\end{align*}$$

using the commutativity of $\hat{G}$ in the third parenthesis. Note that each of the four factors in this expression is a conjugate of an element in the subgroup $\hat{\Gamma}$. Since the latter is a maximal abelian subgroup of the nilpotent group $H$, we know [44, Thm. 5.40] that it is a normal subgroup. Hence each conjugate and therefore $[u, u']$ itself is contained in $\Gamma$. By symmetry, one obtains that $[u, u']$ is likewise contained in $\hat{G}$. Using $\hat{G} \cap \hat{\Gamma}$, this establishes that $[u, u'] \in T$. \hfill $\square$
A key property of Heisenberg groups is that they come with a nice factorization of their elements, which is unique relative to a choice of section.

**Lemma 33.** In Heisenberg a group \( (H, \tilde{G}, \tilde{\Gamma}) \) over \( T \), every element has a decomposition \( u = c\tilde{x}\tilde{\xi} \) with \( c \in T \) and \( (\tilde{x}, \tilde{\xi}) \in \tilde{G} \times \tilde{\Gamma} \), which is unique if \( \tilde{x}, \tilde{\xi} \in s(P) \) for a fixed section \( s \) of the quotient map \( H \rightarrow H/T \).

If \( s \) is normalized, we have \( u \in \tilde{G} \) iff \( \tilde{\xi} = 1 \) and \( u \in \tilde{\Gamma} \) iff \( \tilde{x} = 1 \).

**Proof.** Let \( u \in H \) be given. Since \( \tilde{G} \) and \( \tilde{\Gamma} \) generate \( H \), we can write the element as \( u = \tilde{x}_0\tilde{\xi}_0 \) with \( (\tilde{x}_0, \tilde{\xi}_0) \in \tilde{G} \times \tilde{\Gamma} \). Using the exact sequence \( E : T \hookrightarrow H \twoheadrightarrow H/T \), we set \( \tilde{x} = s\pi(\tilde{x}_0) \) and \( \tilde{\xi} = s\pi(\tilde{\xi}_0) \). Then exactness of \( E \) yields \( \tilde{x}_0/\tilde{\xi}_0/\tilde{\xi} \in \ker(\pi) = T \) so that \( \tilde{x} = c\tilde{x}, \tilde{\xi} = c\tilde{\xi} \) for suitable \( c, \tilde{c}, \tilde{\xi} \in T \). Defining \( c = \tilde{c}\xi, \tilde{\xi} \), which establishes the existence of a decomposition of the required form.

Now let us show uniqueness. Assuming \( u = c'\tilde{x}'\tilde{\xi}' \) is another such decomposition, we get \( (c/c')(\tilde{x}/\tilde{x}') = \tilde{\xi}/\tilde{\xi} \in \tilde{G} \cap \tilde{\Gamma} = T \). Hence we may write \( \tilde{x}' = c_x\tilde{x} \) and \( \tilde{\xi}' = c_\xi\tilde{\xi} \) for suitable constants \( c_x, c_\xi \in T \). Since each of \( \tilde{x}, \tilde{x}_0, \tilde{\xi}, \tilde{\xi}_0 \) must be in \( s(P) \), let us write \( \tilde{x} = s(z), \tilde{x}' = s(z') \) and \( \tilde{\xi} = s(z), \tilde{\xi}' = s(z') \) for some \( z, z', \zeta, \zeta' \in P \). Applying \( \pi \) to the equalities \( \tilde{x}' = c_x\tilde{x}, \tilde{\xi}' = c_\xi\tilde{\xi} \) yields \( z' = z, \zeta' = \zeta \) by \( c_x, c_\xi \in T = \ker(\pi) \). But then we have also \( \tilde{x}' = \tilde{x}, \tilde{\xi}' = \tilde{\xi} \), which forces \( c' = c \) and thus uniqueness.

Assume \( s \) is normalized. If \( u = c\tilde{x}\tilde{\xi} \in \tilde{G} \), we get \( \tilde{\xi} = u(c\tilde{x})^{-1} \in \tilde{G} \). But then \( \xi \in \tilde{G} \cap \tilde{\Gamma} = T \), and we get a second decomposition \( u = (c\tilde{\xi})\tilde{x}1 \) since \( 1 \in s(P) \). By the uniqueness of decomposition, this yields \( \xi = 1 \). Conversely, it is clear that \( u = c\tilde{x} \in \tilde{G} \). The statement on membership in \( \tilde{\Gamma} \) follows by symmetry. \( \square \)

For turning Heisenberg groups into a category, it turns out to be more convenient not to fix the way the torus \( T \) is embedded. In view of the equivalences \( \text{SES}_2 \cong \text{Cunt}_2 \cong \text{Nil}_2 \) mentioned before Fact 6, our definition of the Heisenberg category \( \text{Hei}(\bullet) \) shall be based on \( \text{SES}_2 \) rather than \( \text{Nil}_2 \).

Thus let us call \( E : T \hookrightarrow H \twoheadrightarrow P \in \text{SES}_2 \) a Heisenberg extension if \( H \) is a Heisenberg group over \( \tilde{T} := \iota(T) \). The objects of \( \text{Hei}(\bullet) \) are then triples \( \mathcal{H} = (E, \tilde{G}, \tilde{\Gamma}) \) such that \( E \) is a Heisenberg extension with a choice of abelian bisection \( (\tilde{G}, \tilde{\Gamma}) \) for \( H \). If \( \mathcal{H}' = (E', \tilde{G}', \tilde{\Gamma}') \) is another object of \( \text{Hei}(\bullet) \) based on the Heisenberg extension \( E' : T' \hookrightarrow H' \twoheadrightarrow P' \), a morphism of \( \text{Hei}(\bullet) \) from \( \mathcal{H} \) to \( \mathcal{H}' \) is a morphism \( (t, h, p) : E \rightarrow E' \) such that \( h(\tilde{G}) \leq \tilde{G}' \) and \( h(\tilde{\Gamma}) \leq \tilde{\Gamma}' \).
The functor \( \Pi_{\text{Hei}} : \text{Hei}(\bullet) \to \text{Ab} \) mapping \((T \xrightarrow{\iota} H \xrightarrow{\pi} P, \tilde{G}, \tilde{\Gamma})\) to \(T\) makes \(\text{Hei}(\bullet)\) into a category over \(\text{Ab}\). As usual, we denote the fibers by \(\text{Hei}(T) := \Pi_{\text{Hei}}^{-1}(T)\). Given \(\mathcal{H}, \mathcal{H}' \in \text{Hei}(T)\), we will also apply the common notation \(\text{Hom}_T(\mathcal{H}, \mathcal{H}') := \Pi_{\text{Hei}}^{-1}(1_T)\) and its generalization \(\text{Hom}_t(\mathcal{H}, \mathcal{H}') := \Pi_{\text{Hei}}^{-1}(t)\) for a group homomorphism \(t : T \to T'\). Clearly, we have \(\text{Hom}_T(\mathcal{H}, \mathcal{H}') = \text{Hom}_{1_T}(\mathcal{H}, \mathcal{H}')\) for any \(T \in \text{Ab}\).

For any Heisenberg extension \(E : T \xrightarrow{\iota} H \xrightarrow{\pi} P\) we have \(\hat{T} \leq \mathcal{Z}(H)\) as we have seen before Lemma 32. If \(E\) is a strictly central extension so that \(\hat{T} = \mathcal{Z}(H)\), we will call \(E\) a nondegenerate Heisenberg extension (and \(H\) a nondegenerate Heisenberg group). The full subcategory of such \((E, \tilde{G}, \tilde{\Gamma})\) is denoted by \(\text{Hei}^\times(\bullet)\), and its fiber over \(T\) by \(\text{Hei}^\times(T)\).

**Proposition 34.** The category \(\text{Hei}(\bullet)\) is fibered over \(\text{Ab}\) and contains \(\text{Hei}^\times(\bullet)\) as a fibered subcategory.

**Proof.** Given any group homomorphism \(t : T \to T'\) in \(\text{Ab}\) and an object \(\mathcal{H} \in \text{Hei}(T)\), we must find a distinguished object \(t_\ast[\mathcal{H}] \in \text{Hei}(T')\) and a distinguished morphism \(t_\ast : \mathcal{H} \to t_\ast[\mathcal{H}]\) over \(t\) such that the following universal property is satisfied [5, §12.1.1]: For any morphism \(k \in \text{Hom}_s(\mathcal{H}, \mathcal{K})\) over a group homomorphism \(s : T \to S\) and for any other group homomorphism \(s' : T' \to S\) such that \(s' \circ t = s\) there is a unique \(k' \in \text{Hom}_s(t_\ast[\mathcal{H}], \mathcal{K})\) with \(k' \circ t_\ast = k\). One may call \(t_\ast[\mathcal{H}]\) the direct image of \(\mathcal{H}\) over \(t\), and the morphism \(t_\ast : \mathcal{H} \to t_\ast[\mathcal{H}]\) the corresponding extension of scalars; this terminology is inspired by the well-known opfibration in the module category. (In category jargon, \(t_\ast\) is usually called an “opcartesian morphism” while \(\Pi_{\text{Hei}} : \text{Hei}(\bullet) \to \text{Ab}\) is known as an “opfibration” over \(\text{Ab}\). Sometimes the alternative terms “co-cartesian morphism” and “co-fibration” are in use.)

Given \(\mathcal{H} = (E, \tilde{G}, \tilde{\Gamma})\) with the Heisenberg extension \(E : T \xrightarrow{\iota} H \xrightarrow{\pi} P\) and \(t : T \to T'\) in \(\text{Ab}\), we define the direct image as \(t_\ast[H] := (T' \times H)/Z\) with \(Z := \{(tc, uc^{-1}) \mid c \in T\}\). We wrap this group into the central extension \(t_\ast[E] : T' \xrightarrow{\iota'} t_\ast[H] \xrightarrow{\pi'} P\), with homomorphisms \(t'(c') := (c', 1) Z\) and \(\pi'(c', u) Z := \pi(u)\). Note that \(t_\ast[E]\) coincides with the forward induced extension of [8, I.1], for the special case of central extensions considered here.

Let us now show that \(t_\ast[H]\) is a Heisenberg group over \(\hat{T}'\) with abelian bisection \((\tilde{G}', \tilde{\Gamma}')\) given by \(\tilde{G}' := (T' \times \tilde{G})/Z\) and \(\tilde{\Gamma}' := (T' \times \tilde{\Gamma})/Z\). We split the work into the following tasks:

(1) It is easy to see that \(\tilde{G}'\) and \(\tilde{\Gamma}'\) are maximal abelian in \(t_\ast[H]\).

One way to show this is via the bijective correspondence between the subgroups of \(t_\ast[H]\) and those of \(T' \times H\) containing \(Z\); another method is to show \(C_{t_\ast[H]}(\tilde{G}') = \tilde{G}'\) and \(C_{t_\ast[H]}(\tilde{\Gamma}') = \tilde{\Gamma}'\).
(2) We verify that \( \tilde{G}' \tilde{\Gamma}' = t_*[H] \). Indeed, for any \((c', u) Z \in t_*[H] \) we can find \( c \in T \) and \((\tilde{x}, \tilde{\xi}) \in \tilde{G} \times \tilde{\Gamma} \) with \( u = \iota(c) \tilde{x} \tilde{\xi} \), using Lemma 33 with any section of \( \pi : H \to P \). But then we obtain \((c', u) Z = (c', \tilde{x}) Z \cdot (tc, \tilde{\xi}) Z \), so the subgroups \( \tilde{G}' \) and \( \tilde{\Gamma}' \) do generate \( t_*[H] \).

(3) Finally, we must show \( \tilde{G}' \cap \tilde{\Gamma}' = \tilde{T}' \), where the embedded torus is \( \tilde{T}' := \iota'(T') = (T' \times \tilde{T})/Z \) as one sees immediately. Inclusion from right to left being clear, assume \((c', u) Z \in \tilde{G}' \cap \tilde{\Gamma}' \); we must show \((c', u) Z \in (T' \times T)/Z \). The hypothesis means that \((c', u) Z = (d', \tilde{x}) Z \) and \((c', u) Z = (e', \tilde{\xi}) Z \) for \( d', e' \in T' \) and \((\tilde{x}, \tilde{\xi}) \in \tilde{G} \times \tilde{\Gamma} \). Then \((d'/e', \tilde{x}/\tilde{\xi}) \in Z \) implies \( d' = t(c) e' \) and \( \tilde{\xi} = \iota(c) \tilde{x} \in \tilde{G} \cap \tilde{\Gamma} = \tilde{T} \) for some \( c \in T \). Thus we obtain as claimed \((c', u) Z = (e', \tilde{\xi}) Z \in (T' \times \tilde{T})/Z \).

Thus we have \( t_*[\mathcal{H}] := (t_*[E], \tilde{G}', \tilde{\Gamma}') \in \text{Hei}(T') \), and the next step is to exhibit a morphisms \( t_* : \mathcal{H} \to t_*[\mathcal{H}] \) of \( \text{Hei}(\bullet) \) with the requisite universal property. Following the convention of naming \( \text{SES}_2 \) morphisms by their middle maps (confer Proposition 5), we define \( t_* : H \to t_*[H] \) by \( u \mapsto (1, u) Z \). It is immediate from the definition of the group \( t_*[H] \) that \( t_* \iota = \iota't_* \) and \( \pi = \pi' t_* \) is also obvious. Hence we have

\[
\begin{array}{ccc}
T & \xrightarrow{\iota} & H \\
\downarrow t_* & & \downarrow \pi P \\
T' & \xrightarrow{\iota'} & t_*[H] \\
\downarrow s_* & & \downarrow \pi' P \\
S & \xrightarrow{\kappa} & K \\
\downarrow s & & \downarrow \rho R \\
T' & \xrightarrow{s'} & S \\
\end{array}
\]

which means \( t_* = (t, t_*1_P) \) is a morphism of \( \text{SES}_2 \). Since we also have \( t_*(\tilde{G}) \leq \tilde{G}' \) and \( t_*(\tilde{\Gamma}) \leq \tilde{\Gamma}' \), we see that \( t_* : \mathcal{H} \to t_*[\mathcal{H}] \) is indeed a morphism of \( \text{Hei}(\bullet) \). It remains to show the universal property.

Thus let \( k \in \text{Hom}_*(\mathcal{H}, \mathcal{K}) \) be a morphism from \( \mathcal{H} \) to another object \( \mathcal{K} = (S \xrightarrow{\kappa} K \xrightarrow{\rho} R, \tilde{L}, \tilde{\Lambda}) \) over \( s : T \to S \), and let \( s' : T' \to S \) be some homomorphism in \( \text{Ab} \) such that \( s' \circ t = s \). These hypotheses amount to the commutative diagram

\[
\begin{array}{ccc}
T' & \xrightarrow{s} & S \\
\downarrow s' & & \downarrow \kappa \\
T & \xrightarrow{t} & H \\
\downarrow \iota & & \downarrow \pi P \\
\end{array}
\]

still following the convention of identifying \( \text{SES}_2 \) morphisms with their middle maps. Using the commutativity of the triangle and left square of (7), we see that the map \( T' \times H \to K \) given by \( (c', u) \mapsto \kappa(s'c')k(u) \) annihilates \( Z \), hence descends to a homomorphism \( k' : t_*[\mathcal{H}] \to K \).
One sees immediately that $k'\nu' = \kappa s'$, while using the left commutative square of (7) establishes $pk' = r\pi'$. Hence $(s', k', r)$ is a SES$_2$ morphism, which we write also as $k'$ by our middle-map convention. For seeing that $k'$ is in fact a morphism of $\text{Hei}(\bullet)$, it remains to ensure that $k'((\tilde{G}') \leq \tilde{L}$ and $k'((\tilde{\Gamma}') \leq \tilde{\Lambda}$). For the former, take any $(c', \tilde{x}) Z \in \tilde{G}'$. Then we have $k'((c', \tilde{x}) Z) = \kappa(s'c') k(\tilde{x}) \in \tilde{L}$ because $\kappa(s'c') \in \tilde{S} \leq \tilde{L}$ and $k(\tilde{x}) \in \tilde{L}$ from the fact that $(\tilde{L}, \tilde{\Lambda})$ is an abelian bisection of $K$.

The argument for $k'((\tilde{\Gamma}') \leq \tilde{\Lambda}$ is analogous. We have thus established $k' \in \text{Hom}_\sigma(t_\ast[H], K)$ as a morphism of \text{Hei}(\bullet), and it is easy to see that $(s', k', r) \circ (t, t_\ast, 1_p) = (s, k, r)$.

Having established existence of the morphism $k'$, our last task is to prove its uniqueness. So assume $k'' \in \text{Hom}_\sigma(t_\ast[H], K)$ is any morphism of \text{Hei}(\bullet) with $k''t_\ast = k$, and take $\tilde{u} = (c', u) Z \in t_\ast[H]$. We compute now $k''(\tilde{u})$ as

$$k''((c', 1) Z) \cdot k''((1, u) Z) = k''((c', u) Z) = \kappa(s'c') k(u) = k'(\tilde{u}),$$

using the main hypothesis $k''t_\ast = k$ on $k''$ and the fact that $k''\nu' = \kappa s'$, which follows from $k''$ being a morphism over $s'$.

The statement about $\text{Hei}^\times(\bullet)$ is easy to verify: It is clear that $\Pi_{\text{Hei}}$ restricts to a functor $\text{Hei}^\times(\bullet) \to \text{Ab}$. Since $\text{Hei}^\times(\bullet)$ is defined as a full subcategory of $\text{Hei}(\bullet)$, it suffices to show that $t_\ast[H] \in \text{Hei}^\times(T')$ whenever $H \in \text{Hei}^\times(T)$ and $t : T \to T'$ in $\text{Ab}$. An easy computation shows that $\mathcal{Z}((T' \times H)/Z) = (T' \times \mathcal{Z}(H))/Z$. Hence $\mathcal{Z}(H) = T'$ implies $\mathcal{Z}((T' \times H)/Z) = (T' \times T)/Z = T'$, as required. 

Note that the construction $(t, H) \mapsto (t_\ast[H] \to t_\ast[H])$ of Proposition 34 provides a cleavage but not a splitting [5, 12.1.3] of the fibered category $\Pi_{\text{Hei}}: \text{Hei}(\bullet) \to \text{Ab}$ since $(1_T)_\ast : H \Longrightarrow (T \times H)/Z$ is only an isomorphism, not the identity $1_H$. Similarly, the composition of scalar extensions produces a codomain that is only isomorphic but not equal to the codomain produced by the scalar extension of the composite. Nevertheless, any fibered category yields a family of pushforward functors [5, 12.1.8] by the universal property mentioned in the proof of Proposition 34: In our case, each morphism $t : T \to T'$ gives rise to the functor $\text{Hei}(t) : \text{Hei}(T) \to \text{Hei}(T')$ defined on objects by $H \mapsto t_\ast[H]$ and on morphisms $h \in \text{Hom}_T(H_1, H_2)$ as the unique $\text{Hei}(T')$ morphism $h' \in \text{Hom}_{T'}(t_\ast[H_1], t_\ast[H_2])$ that makes the diagram

$$
\begin{array}{ccc}
H_1 & \xrightarrow{t_\ast} & t_\ast[H_1] \\
\downarrow h & & \downarrow h' \\
H_2 & \xrightarrow{t_\ast} & t_\ast[H_2]
\end{array}
$$
commute. However, since the underlying cleavage is not a splitting in our case, the assignment \( \text{Hei}: \text{Ab} \to \text{Cat} \) is not a functor, just a morphism of graphs [31, §II.7]. Of course, similar remarks hold for the fibered subcategory \( \text{Hei}^\times: \text{Ab} \to \text{Cat} \) of nondegenerate Heisenberg objects. Note also that \( T_0 := (T = T \to 0, T, T) \) is an initial object of \( \text{Hei}^\times(T) \subset \text{Hei}(T) \) for each \( T \in \text{Ab} \), with the assignment \( T \mapsto T_0 \) defining a cartesian section.

The next step in our algebraic investigation of Heisenberg groups is to provide a way of constructing them in terms of bilinear forms \( \beta: G \times \Gamma \to T \) with \( G, \Gamma, T \in \text{Ab} \). Such a bilinear form naturally induces the left right curried homomorphism \( G \to \hat{\Gamma} := \text{Hom}(\Gamma, T) \) and \( \Gamma \to \hat{G} := \text{Hom}(G, T) \), respectively. Their kernels \( G_0 \) and \( \Gamma_0 \) are accordingly called the left kernel and the right kernel of \( \beta \). The bilinear form \( \beta \) is called nondegenerate if the unilateral kernels are trivial and proper if they are complemented (there are subgroups \( G_1 \leq G \) and \( \Gamma_1 \leq \Gamma \) such that \( G = G_0 \dot{\cup} G_1 \) and \( \Gamma = \Gamma_0 \dot{\cup} \Gamma_1 \)).

Bilinear forms may be collected into the bilinear category \( \text{Bil}(\bullet) \), with the following morphisms: If \( \beta: G \times \Gamma \to T \) and \( \beta': G' \times \Gamma' \to T' \) are bilinear forms, and if \( g: G \to G' \), \( \gamma: \Gamma \to \Gamma' \) and \( t: T \to T' \) are group homomorphisms, we have \((g \times \gamma, t): \beta \to \beta'\) whenever \( \beta' \circ (g \times \gamma) = \beta \circ t \) and \( g(G_0) \leq G_0' \), \( \gamma(\Gamma_0) \leq \Gamma_0' \). In other words, \( \text{Bil}(\bullet) \) is the subcategory of the comma category \( \times \downarrow 1 \) that respects the kernels (where we view the direct product of abelian groups as a functor \( \times: \text{Ab}^2 \to \text{Ab} \) and the identity as a functor \( 1: \text{Ab} \to \text{Ab} \)). The full subcategories of \( \text{Bil}(\bullet) \) consisting of nondegenerate forms and proper forms are, respectively, denoted by \( \text{Bil}^\ast(\bullet) \) and \( \text{Bil}^\parallel(\bullet) \). We have the obvious inclusions \( \text{Bil}^\ast(\bullet) \subset \text{Bil}^\parallel(\bullet) \subset \text{Bil}(\bullet) \).

Like \( \text{Hei}(\bullet) \), the bilinear category is also fibered over \( \text{Ab} \), but this time we have in fact a split fibration. Hence let \( \Pi_{\text{Bil}}: \text{Bil}(\bullet) \to \text{Ab} \) be the obvious projection functor mapping an object \( \beta: G \times \Gamma \to T \) of \( \text{Bil}(\bullet) \) to \( T \in \text{Ab} \) and a morphism \((g \times \gamma, t)\) of \( \text{Bil}(\bullet) \) to its base morphism \( t \in \text{Ab}(T, T') \). The pushforward \( t_*[\beta] \) is the bilinear form \( t\beta: G \times \Gamma \to T' \), with the extension of scalars \( t_*: \beta \to t_*[\beta] \) given by \((1_G \times 1_\Gamma, t)\). It is easy to check that the opfibration axioms [5, §12.1.1] are satisfied, together with the splitting axioms [5, §12.1.3] for the cleavage \((\beta, t) \mapsto t_*\). Thus \( T \mapsto \text{Bil}(T) \) is a genuine functor \( \text{Ab} \to \text{Cat} \) by [5, Prop. 12.1.8], and we write \( \text{Bil}(T) := \Pi_{\text{Bil}}^{-1}(T) \) for its fibers.

It will be noted, however, that \( \text{Bil}^\ast(\bullet) \) is not fibered since pushing forward a bilinear form under \( t = 0 \) clearly renders it degenerate (but restricting to monomorphisms in the base category \( \text{Ab} \) of \( \text{Bil}(\bullet) \) does lead to a fibered subcategory of nondegenerate bilinear
It is also straightforward to check that \( \pi \) obviously have no complement. We can now give the promised construction of the Heisenberg group of a bilinear form. So let \( \beta: G \times \Gamma \to T \) be a bilinear form. We will show that the group \( H := TG \times \Gamma \) is a Heisenberg group over the expanded torus \( T_2 := TG_0 \times \Gamma_0 \), with an abelian bisection given by \((TG \times \Gamma_0, TG_0 \times \Gamma)\). Writing \( P := G/G_0 \times \Gamma_0/G_0 \) for the prospective phase space, we therefore set

\[
(8) \quad \mathfrak{f}(\beta) := (T_2 \ni t \mapsto H \xrightarrow{\pi} P, TG \times \Gamma_0, TG_0 \times \Gamma),
\]

where \( \iota \) is the inclusion and \( \pi \) the map \( c(x, \xi) \mapsto (xG_0, \xi\Gamma_0) \). Furthermore, if \((g \times \gamma, t)\) is a \(\text{Bil}(\bullet)\) morphism from \( \beta: G \times \Gamma \to T \) to another bilinear form \( \beta': G' \times \Gamma' \to T' \), we define

\[
(9) \quad \mathfrak{f}(g \times \gamma, t) := (t_2, h, p)
\]

whose components are the group homomorphisms

\[
\begin{align*}
H & \xrightarrow{h} H', \\
\beta \quad & \\
\frac{c(x, \xi)}{c(xG_0, \xi\Gamma_0)} & \quad \frac{tc(gx, \gamma\xi)}{(xG_0, \xi\Gamma_0)} \mapsto ((gx)G_0', (\gamma\xi)\Gamma_0).
\end{align*}
\]

and the torus map \( t_2: T_2 \to T_2' \) induced by restriction of \( h: H \to H' \). Let us now verify that \( \mathfrak{f} \) is a functor.

**Proposition 35.** We have a functor \( \mathfrak{f}: \text{Bil}(\bullet) \to \text{Hei}(\bullet) \) given by Equations (8) and (9); its image is contained in \( \text{Hei}^* (\bullet) \).

**Proof.** Let us first check that \( \mathfrak{f}(\beta) \) is an object of \( \text{Hei}(\bullet) \). It is evident that \( E: T_2 \ni t \mapsto H \xrightarrow{\pi} P \) is exact, hence \( H \) is nilquadratic (Proposition 5). By a short computation one obtains the associated commutator form

\[
\omega_E(xG_0, \xi\Gamma_0; yG_0, \eta\Gamma_0) = \beta(\xi, y)/\beta(\eta, x).
\]

It is easy to check that \( G/G_0 \times 0 \) and \( 0 \times \Gamma/\Gamma_0 \) are Lagrangian subgroups of \( P \) with respect to \( \omega_E \). Since they obviously form a direct decomposition of \( P \), we obtain a Lagrangian bisection \( (G/G_0 \times 0, 0 \times \Gamma/\Gamma_0) \) of \( P \).

It is also straightforward to check that \( \pi^{-1}(G/G_0 \times 0) = TG \times \Gamma_0 \) and \( \pi^{-1}(0 \times \Gamma/\Gamma_0) = TG_0 \times \Gamma \), hence we conclude by Item 8 of Theorem 28 that \((TG \times \Gamma_0, TG_0 \times \Gamma)\) is indeed an abelian bisection of \( H \) over the expanded torus \( T_2 \). It is also easy to see that \( \mathcal{Z}(H) = T_2 \), so \( H \) is nondegenerate as claimed.

We turn now to the morphisms. Assuming \((g \times \gamma, t)\) is a morphism of \( \text{Bil}(\bullet) \), we must verify that (9) yields a morphism \((t_2, h, p)\) of \( \text{Hei}(\bullet) \). First of all, it is clear that \( h: H \to H' \) does restrict to \( t_2: T_2 \to T_2' \)
since we have \( g(G_0) \leq G_0 \) and \( \gamma(\Gamma_0) \leq \Gamma_0 \) from the fact that \( \text{Bil}(\bullet) \) morphisms respect the left and right kernels (this is also the reason why \( p: P \to P' \) is well-defined). This takes care of the left commutative square needed for showing \((t\sharp, h, p)\) to be a \( \text{SES}_2 \) morphism; the commutativity of the right square follows immediately from the definitions of the maps involved. It remains to show that \( h \) respects the abelian bi-sections, meaning \( h(TG \times \Gamma_0) \leq T'G' \times \Gamma_0 \) and \( h(TG_0 \times \Gamma) \leq T'G'_0 \times \Gamma' \); this follows again directly from the fact that the morphisms of \( \text{Bil}(\bullet) \) respect the left and right kernels.

Once it is clear that \( H \) is well-defined on morphisms, it is trivial to check that it is indeed a functor \( \text{Bil}(\bullet) \to \text{Hei}(\bullet) \) since identities and compositions are obviously preserved by \( H \).

The way back from a Heisenberg group to “its” bilinear form is easier in the sense that we need not modify the torus. Hence we can describe this reverse construction as a functor within the same fibers.

**Proposition 36.** Fix \( T \in \text{Ab} \). The function \( \mathfrak{B}: \text{Hei}(T) \to \text{Bil}(T) \) mapping \((H, \tilde{G}, \tilde{\Gamma})\) to the commutator form \( \tilde{G}/T \times \tilde{\Gamma}/T \to T \) is a functor with image contained in \( \text{Bil}(T) \).

**Proof.** We have seen that any Heisenberg group \((H, \tilde{G}, \tilde{\Gamma}) \in \text{Hei}(T)\) induces a central extension \( E \) as in \((6)\). By Proposition 17, its commutator form \( \omega_E: P \times P \to T \) on the phase space \( P := H/T \) is then bilinear. Its restriction \( \beta \) to the subgroups \( \tilde{G}/T \) and \( \tilde{\Gamma}/T \) of \( P \) is bilinear as well, hence \((\beta, \tilde{G}/T, \tilde{\Gamma}/T) \in \text{Bil}(T)\).

Letting \((H', \tilde{G}', \tilde{\Gamma'})\) be another Heisenberg group over \( T \), consider a Heisenberg morphisms \( h: H \to H' \). We define \( g: \tilde{G}/T \to \tilde{G}'/T \) and \( \gamma: \tilde{\Gamma}/T \to \tilde{\Gamma}'/T \) as the homomorphisms induced by \( h \) on the corresponding subgroups; these are well-defined since \( h \) respects the abelian bisections and fixes the torus \( T \). For checking that \( \beta' \circ (g \times \gamma) = \beta \), we compute

\[
\beta' \circ (g \times \gamma)(\tilde{x}T, \tilde{\xi}T) = \beta'(h(\tilde{x})T, h(\tilde{\xi})T) = [h(\tilde{x}), h(\tilde{\xi})] = h[\tilde{x}, \tilde{\xi}] = [\bar{x}, \bar{\xi}] = \beta(\tilde{x}T, \tilde{\xi}T),
\]

using the fact that commutators are fixed by \( h \) since \([H, H] \leq T\) by centrality of \( E \).

Let us now check that \( \beta \) is nondegenerate. Using the fact that \( \tilde{\Gamma} \) is maximal abelian, it is easy to compute the left kernel of \( \beta \) as

\[
(\tilde{G}/T)_0 = \frac{\tilde{G} \cap C_H(\tilde{\Gamma})}{T} = \frac{\tilde{G} \cap \tilde{\Gamma}}{T} = T/T = 0 \in \tilde{G}/T.
\]
By the symmetry of the argument, the right kernel $(\tilde{\Gamma}/T)_0$ is likewise trivial, so $\beta$ is indeed nondegenerate. Hence $\mathfrak{B}(h) := (g, \gamma)$ automatically respects the one-sided kernels of $\beta$ and $\beta'$ and thus is a morphism in $\text{Bil}(T)$. Since it is obvious that unitality/composition of morphisms in $\text{Hei}(T)$ corresponds to unitality/composition in $\text{Bil}(T)$, this completes the proof that $\mathfrak{B} : \text{Hei}(T) \to \text{Bil}(T)$ is a functor.

The functor $\mathfrak{B} : \text{Hei}(T) : \text{Bil}(T)$ of Proposition 36 will also be written as $\mathfrak{B}_T$ if the dependence on the torus needs to be made explicit. Likewise, we shall write $\mathfrak{S}_T$ for the construction of the Heisenberg group from a given bilinear form in $\text{Bil}(T)$ in Proposition 35. For making this "construction" into a proper functor, we have to view its codomain as a category comprised of all Heisenberg groups over arbitrary tori. The key to realize this is once again a fibration whose model is that of the module category.

In view of Remark 30, we expect a way of restricting and extending scalars, just as for modules, where a ring homomorphism $f : R \to S$ induces an extension functor $f_* : \text{Mod}_R \to \text{Mod}_S$ with $M \mapsto M \otimes_R S$ and a restriction functor $f^* : \text{Mod}_S \to \text{Mod}_R$ with the action on $f^*(N)$ defined by $\lambda \cdot x = f(\lambda) x$ for $x \in N$ and $\lambda \in R$, such that extension is left adjoint to restriction $[32, \text{Exc. XV.5}]$. For our present setting we shall only need the extension functor, which may be derived as a special case of the so-called Calculus of Induced Extensions $[8, \text{I.1}]$.

**Theorem 37.** Defining the pair of functors $\Phi : \text{Bil}(\bullet) \to \text{Hei}(\bullet)$ and $\Psi : \text{Hei}(\bullet) \to \text{Bil}(\bullet)$ by

$$\Phi(g, G, \Gamma, T) = (TG \times \Gamma, TG_0 \times \Gamma, T(G_0 \times \Gamma_0),$$

$$\Psi(H, \tilde{G}, \tilde{\Gamma}, T) = (\omega|_{\tilde{G}/T \times \tilde{\Gamma}/T}, \tilde{G}/T, \tilde{\Gamma}/T, T),$$

we obtain an adjunction $\Phi \dashv \Psi$.

**Proof.** Next we check that $\Phi$ is a functor. Hence assume $g : G \to G'$ and $\gamma : \Gamma \to \Gamma'$ are homomorphisms with $\beta' \circ (g \times \gamma) = \beta$ and $g(G_0) \leq G'_0$, $\gamma(G_0) \leq G'_0$. Then we define $\varphi := \Phi(g, \gamma)$ to be the map $c(x, \xi) \mapsto c(gx, \gamma \xi)$. We must first check that $\varphi$ fixes the torus $T$.

We need for all $\mathcal{H} = (H, G, \Gamma) \in \text{Hei}(T)$ an arrow $\alpha_H : \mathcal{H} \to \Phi \Psi(\mathcal{H})$ universal from $\mathcal{H}$ to $\Phi$. First of all, we need a group homomorphism $H \to T(\tilde{G}/T) \times (\tilde{\Gamma}/T)$. Fixing an arbitrary set-theoretic section $s$ of the projection $\pi : H \to P := H/T$, we employ Lemma 33 in defining the required map by $\alpha(c \tilde{x} \tilde{\xi}) = c(\tilde{x}T, \tilde{\xi}T)$ for $c \in T$ and $(\tilde{x}, \tilde{\xi}) \in \tilde{G} \times \tilde{\Gamma}$ such that $\tilde{x}, \tilde{\xi} \in s(P)$. For checking that $\alpha$ is a homomorphism, note that $c \tilde{x} \tilde{\xi} \cdot c' \tilde{x}' \tilde{\xi}' = (c'c)(\tilde{x} \tilde{\xi})(\tilde{x}' \tilde{\xi}')$ where $c := [\tilde{\xi}, \tilde{x}] \in [H, H] \leq T$ by Fact 6. Applying $\alpha$ thus yields $(c'c)(\tilde{x} \tilde{\xi})(\tilde{x}' \tilde{\xi}') \in T(\tilde{G}/T) \times (\tilde{\Gamma}/T)$, which
equals the product of \( \alpha(c \tilde{x} \tilde{\xi}) = c(\tilde{x}T, \tilde{\xi}T) \) and \( \alpha(c' \tilde{x}' \tilde{\xi}') = c'(\tilde{x}'T, \tilde{\xi}'T) \) by the definition of multiplication in \( T(\tilde{G}/T) \times (\tilde{\Gamma}/T) \) via the bilinear form \( \langle \cdot \rangle : (\tilde{G}/T) \times (\tilde{\Gamma}/T) \to T \) given by \( \langle \tilde{\xi}T|\tilde{x}'T \rangle = [\tilde{\xi}, \tilde{x}'] = \tilde{c} \).

Next we must check that \( \alpha_H \) is consistent with the abelian bisections. In other words, we need to ensure

\[
\alpha(\tilde{G}) \leq T(\tilde{G}/T) \times (\tilde{\Gamma}/T)_0 \quad \text{and} \quad \alpha(\tilde{\Gamma}) \leq T(\tilde{G}/T)_0 \times (\tilde{\Gamma}/T).
\]

Since the left and right kernels vanish, this means the elements of \( \alpha(\tilde{G}) \) and \( \alpha(\tilde{\Gamma}) \), respectively, must have the form \( c(\tilde{x}T, 0) \) and \( c(0, \tilde{\xi}T) \), with \( c \in T \) and \( \tilde{x} \in \tilde{G}, \tilde{\xi} \in \tilde{\Gamma} \), which follows immediately from the fact that

\[
\tilde{G} = \{ c\tilde{x} | c \in \tilde{T}, \tilde{x} \in \tilde{G} \cap s(P) \}, \\
\tilde{\Gamma} = \{ c\tilde{\xi} | c \in \tilde{T}, \tilde{\xi} \in \tilde{\Gamma} \cap s(P) \}.
\]

So here the inclusions (10) are, in fact, equalities. \( \square \)

For our treatment of Heisenberg groups and the Fourier transform, we start with an abelian group \( P \), along with its commutator form \( \omega_E \in \text{Hom}_G(\Lambda^2 P, T) \). Before we come to this we provide the theory within a more general setting.

### 3. Heisenberg Groups with nonabelian Phase Groups

We will now provide the general concepts for abelian torus group \( T \) but with possibly nonabelian groups \( G, \Gamma \). Therefore we will prefer multiplicative notation in all three groups during this exposition. We start with a short repetition of the basic concepts in the general environment.

If \( X \) is an arbitrary set, we write \( S_X \) for the symmetric group on \( X \).

#### 3.1. Bilinear forms with values in an abelian group

We fix a set-map \( \beta : \Gamma \times G \to T \) between groups \((\Gamma, \cdot), (G, \cdot)\) and the abelian group \((T, \cdot)\)—the torus. We write \( \langle \xi|x \rangle := \beta(\xi, x) \).

**Definition 38.**

1. \( 0\beta := \Gamma_0 := \{ \xi \in \Gamma | \forall x \in G \langle \xi|x \rangle = 1 \} \)
2. \( \beta_0 := G_0 := \{ x \in G | \forall \xi \in \Gamma \langle \xi|x \rangle = 1 \} \)
3. \( X \subseteq \Gamma \implies X^\perp := \{ g \in G | \langle X|g \rangle = 1 \} \)
4. \( Y \subseteq G \implies Y^\perp := \{ \xi \in \Gamma | \langle \xi|Y \rangle = 1 \} \)

\( \perp \) denotes the \( \beta \)-orthogonal. Derived from \( \beta : \Gamma \times G \to T \) there are the phase maps:
Definition 39.
\[ \langle \beta \rangle : \Gamma \rightarrow (T \times G)^{T \times G}, \quad \langle \beta \rangle (c, x) = (c \langle \xi \rangle x, x) \]
(11) \[ \triangleright \beta : G \rightarrow (T \times \Gamma)^{T \times \Gamma}, \quad \triangleright \beta (x, c, \xi) = (c \langle \xi \rangle x, \xi). \]

Their invariants are
\[ \text{Inv}(\langle \beta \rangle) := \{(c, x) \in T \times G \mid \forall \xi \in \Gamma \langle \beta \rangle (c, x) = (c, x)\} \]
\[ \text{Inv}(\triangleright \beta) := \{(c, \xi) \in T \times \Gamma \mid \forall x \in G \triangleright \beta (x, c, \xi) = (c, \xi)\}. \]

We consider \((T \times G)^{T \times G}\) and \((T \times \Gamma)^{T \times \Gamma}\) as monoids with composition of maps.

Definition 40. Given a map \(\beta : \Gamma \times G \rightarrow T\). We call \(\beta\)
- left linear \(\iff\) \(\forall \xi, \eta \forall x \langle \xi \eta \rangle x = \langle \xi \rangle \langle \eta \rangle x\)
- right linear \(\iff\) \(\forall \xi \forall x, y \langle \xi xy \rangle = \langle \xi x \rangle \langle \xi y \rangle\)
- bilinear \(\iff\) \(\beta\) is left-and right-linear
- duality \(\iff\) \(\beta\) is bilinear and \(G_0 = \Gamma_0 = 0\).

Proposition 41. Given \(\beta : \Gamma \times G \rightarrow T\). Then
\[ \text{Inv}(\langle \beta \rangle) = T \times \beta_0 \text{ and } \text{Inv}(\triangleright \beta) = T \times 0\beta \]
\[ \langle \beta \rangle : \Gamma \rightarrow S_{T \times G} \text{ and } \triangleright \beta : G \rightarrow S_{T \times \Gamma} \]
- left linear \(\iff\) \(\langle \beta \rangle \in \text{hom}(\Gamma, S_{T \times G}) \iff G \triangleright \beta \in \text{Aut}(T \times \Gamma)\)
- right linear \(\iff\) \(\Gamma \langle \beta \rangle \in \text{Aut}(T \times G) \iff \triangleright \beta \in \text{hom}(G, S_{T \times \Gamma})\)
- bilinear \(\iff\) \(\langle \beta \rangle \in \text{hom}(\Gamma, \text{Aut}(T \times G)) \iff \triangleright \beta \in \text{hom}(\Gamma, \text{Aut}(T \times \Gamma))\)
- left linear \(\iff\) \(\Gamma_0 = \text{ker}(\langle \beta \rangle) \text{ and } \beta(\bullet, x) \in \Gamma^* \forall x \in G\)
- right linear \(\iff\) \(G_0 = \text{ker}(\triangleright \beta) \text{ and } \beta(\xi, \bullet) \in G^* \forall \xi \in \Gamma\)
- bilinear \(\iff\) \(\beta_1 : \Gamma \rightarrow G^*, \xi \mapsto \beta(\xi, \bullet) \text{ and } \beta_2 : G \rightarrow \Gamma^*, x \mapsto \beta(\bullet, x)\)

are group homomorphisms.

Thus, for bilinear \(\beta\) the phase maps (11) define group actions
\[ \Gamma \times (T \times G) \overset{\bullet}{\rightarrow} T \times G \text{ and } G \times (T \times \Gamma) \overset{\bullet}{\rightarrow} T \times \Gamma \]
(12) \[ \xi \bullet (c, x) = (c \langle \xi \rangle x, x) \quad x \bullet (c, \xi) = (c \langle \xi \rangle x, \xi) \]

A map \(\beta : \Gamma \times G \rightarrow T\) comes together with its ‘dual map’ \(\beta^0 : G \times \Gamma \rightarrow T\), connected by the diagram
\[
\begin{array}{ccc}
\langle \xi, x \rangle & \overset{\beta}{\longrightarrow} & \Gamma \times G \\
\downarrow & & \downarrow \\
\langle x, \xi \rangle & \overset{\beta^0}{\longrightarrow} & G \times \Gamma
\end{array}
\]
i.e., \(\langle x | \xi \rangle^0 := \beta^0(x, \xi) = \beta(\xi, x) = \langle \xi | x \rangle\). Obviously, \(\beta^0\) is left-linear iff \(\beta\) is right-linear, thus \(\beta^0\) is bilinear iff \(\beta\) is so. Supplying the superscript
'0' to all items related to $\beta^0$ we obtain:

\[
\begin{align*}
\Gamma & \xrightarrow{\beta^0} (T \times G)^{T \times G} \\
G & \xrightarrow{\beta^0} (T \times \Gamma)^{T \times \Gamma} \\
\Gamma & \xrightarrow{\beta^0} (T \times G)^{T \times G}
\end{align*}
\]

\[
\begin{align*}
\beta^0_\beta (\xi)(c, x) &= (c \langle x | \xi \rangle^0, x) = (c \langle \xi | x \rangle, x) = \beta_\beta (c, x) \\
\beta^0_\gamma (x)(c, \xi) &= (c \langle x | \xi \rangle^0, \xi) = (c \langle \xi | x \rangle, \xi) = \beta_\gamma (x, c)
\end{align*}
\]

i.e., $\beta^0_\beta = \beta$ and $\beta^0_\gamma = \beta$. Therefore it is enough to pay attention to the action $\beta$.

Let $\Gamma, G$ be groups, $T$ an abelian group, and $\beta: \Gamma \times G \to T$ bilinear. Evidently then, $X^\perp$ and $Y^\perp$ are groups, that means, the $\beta$-orthogonal takes values in the lattice of subgroups and the composition $\perp \circ \perp$, being idempotent, monotone and increasing is a hull operator.

### 3.2. The Heisenberg Group of a Bilinear Form.

**Definition 42.** Let $(\Gamma, \cdot), (G, \cdot) \in \text{Grp}$, $(T, \cdot) \in \text{Ab}$ and $\beta: \Gamma \times G \to T$ bilinear. The **Heisenberg group induced by** $\beta$ is the semi-direct product wrt. the (left) group action $\Gamma \times (T \times G) \to T \times G$ resulting from $\beta_\beta: \Gamma \to \text{Aut}(T \times G)$ (cf. 12). We write $H(\beta) = (T \times G) \rtimes \Gamma$.

We call a group $X$ a **Heisenberg group** iff

\[
\exists \Gamma, G \in \text{Grp} \exists T \in \text{Ab} \exists \beta: \Gamma \times G \to T \text{ bilinear } X \cong H(\beta).
\]

$X$ is a proper **Heisenberg group** if

\[
\exists \Gamma, G, T \in \text{Ab} \exists \beta: \Gamma \times G \to T \text{ duality } X \cong H(\beta).
\]

Let $c, d \in T$, $x, y \in G$, $\xi, \eta \in \Gamma$. The basic computation rules are:

\[
(c, x, \xi) \cdot (d, y, \eta) = ((c, x)(\xi \cdot (d, y)), \xi \eta) = ((c, x)(d \langle \xi | y \rangle), \xi \eta) = (cd(\xi \rangle, xy, \xi \eta)
\]

Then $1 = (1, 1, 1)$ and $(c, x, \xi)^{-1} = (\overrightarrow{\xi | x}_c, x^{-1}, \xi^{-1})$.

There are embeddings $t: T \to H(\beta)$, $g: G \to H(\beta)$, $\gamma: \Gamma \to H(\beta)$. We abbreviate their values by setting $c = (c, 1, 1)$, $x = (1, x, 1)$, $\xi = (1, 1, \xi)$.
Then the following identities hold in $H(\beta)$:

\[
\begin{align*}
    cx &= (c, x, 1) = xc \\
    c\xi &= (c, 1, \xi) = \xi c \\
    x\xi &= (1, x, \xi) \quad \text{and} \quad \xi x = (\langle \xi \vert x \rangle, x, \xi) = (\xi \vert x)x\xi \\
    cx\xi &= (c, x, \xi)
\end{align*}
\]

Thus, $T$ is central in $H(\beta)$ and each element $h \in H(\beta)$ has a unique factorization

\[
h = cx\xi \quad (c \in T, x \in G, \xi \in \Gamma).
\]

**Proposition 43.** Commutator and center in $H(\beta)$ are

\[
[(c, x, \xi), (d, y, \eta)] = \left( \frac{\langle \xi \vert y \rangle}{\langle \eta \vert x \rangle}, [x, y], [\xi, \eta] \right)
\]

(13) \quad $Z(H(\beta)) = T \times (G_0 \cap Z(G)) \times (\Gamma_0 \cap Z(\Gamma))$

Consequently $\langle \xi \vert x \rangle = [\xi, x] \quad (\xi \in \Gamma, x \in G)$.

Note that the center of $H(\beta)$ is indeed a direct product of abelian groups $Z(H(\beta)) = T \oplus (G_0 \cap Z(G)) \oplus (\Gamma_0 \cap Z(\Gamma))$. We are concerned with the two exact sequences

\[
\begin{align*}
    1 \rightarrow T \times G \rightarrow H(\beta) \rightarrow \Gamma \rightarrow 0 \\
    1 \rightarrow T \rightarrow H(\beta) \pi \rightarrow G \times \Gamma \rightarrow 0
\end{align*}
\]

the first, coming from the semidirect product, is splitting. The second one is a central extensions with the abelian kernel $T$ hence inducing the trivial action on $T$. We set

\[
\varepsilon(\beta) = 1 \rightarrow T \rightarrow H(\beta) \pi \rightarrow P \rightarrow 1 \quad \text{with} \quad P = G \times \Gamma.
\]

Since $\varepsilon(\beta)$ is central, it corresponds to a cohomology class $\text{cls}(\gamma) \in H^2(P, T)$, where $T$ is considered as trivial $P$-module. For computing a cocycle $\gamma$ we can use the ‘standard section’ of $\pi$

\[
s_0: P \rightarrow H(\beta), \quad s_0(x, \xi) = (1, x, \xi).
\]

**Lemma 44.** Each cocycle $\gamma \in Z^2(P, T)$ has the form

\[
\gamma((x, \xi), (y, \eta)) = \langle \xi \vert y \rangle h(x, \xi)h(y, \eta) \quad \text{where} \quad h \text{is an arbitrary function} \quad P \rightarrow T.
\]

In particular, the cocycle $\gamma_0$ induced by the standard section is

\[
\gamma_0 = \beta \circ (\pi_T \times \pi_G)
\]

where $\pi_\bullet$ denote the respective projection $P \rightarrow \Gamma, P \rightarrow G$. 


Proof.
\[ \gamma_0((x, \xi), (y, \eta)) = i^{-1}(s_0(x, \xi)s_0(y, \eta)s_0((x, \xi)(y, \eta))^{-1}) \]
\[ = i^{-1}((1, x, \xi)(1, y, \eta)(1, xy, \xi\eta)^{-1}) \]
\[ = i^{-1}((\langle \xi|y\rangle, xy, \xi\eta)((\xi\eta|xy\rangle, (xy)^{-1}, (\xi\eta)^{-1}))\]
\[ = \langle \xi|y\rangle \langle \xi\eta|xy\rangle \langle \xi\eta|(xy)^{-1} \rangle \]
\[ = \langle \xi|y\rangle \]

This shows that \[ \gamma_0 = \beta \circ (\pi_G \times \pi_G) \]

If \( \gamma \in Z^2(P, T) \) is an arbitrary cocycle then there is a function \( h \in T^P \) s.t. \( \gamma = \gamma_0 \cdot \partial^2 h \). Therefore

\[ \gamma((x, \xi), (y, \eta)) = \gamma_0((x, \xi), (y, \eta)) \cdot (\partial^2 h)((x, \xi), (y, \eta)) = \langle \xi|y\rangle \frac{h(x, \xi)h(y, \eta)}{h(xy, \xi\eta)} \]

Of course this can be seen also by using an arbitrary section, which must be of the form \( s(x, \xi) = (h(x, \xi), x, \xi) \)

The following trivial observations will be useful.

Lemma 45.

(1) \( H \leq G \) and \( \Delta \leq \Gamma \), \( \varphi: T \times H \times \Delta \to H(\beta), (c, x, \xi) \mapsto cx\xi \).

Then \( \varphi \) is a homomorphism iff \( \langle \Delta|H \rangle = 1 \).

(2) If \( \Gamma \) and \( G \) both have bilinear commutator, then so has \( H(\beta) \).

Definition 46. An SES \( \varepsilon: 1 \to T \to E \to P \to 1 \) (with \( T \in \text{Ab} \)) is called Heisenberg extension provided that \( \exists \) a factorization \( P = G \times \Gamma, \exists \) a bilinear \( \beta: \Gamma \times G \to T \) s.t. \( \varepsilon \) is equivalent with \( \varepsilon(\beta) \). The sequence \( \varepsilon \) is then a proper Heisenberg extension in case \( T, P \in \text{Ab} \) and \( \beta \) is a duality.

Corollary 47. Let \( \varepsilon: 1 \to T \to E \to P \to 1 \) be a Heisenberg extension. Then \( E \in \text{Nil}_2 \iff P \in \text{Nil}_2 \).

Proof. There is a factorization \( P = G \times \Gamma \), a bilinear \( \beta: \Gamma \times G \to T \) and an isomorphism \( \varphi \) s.t.

\[
\begin{array}{cccccc}
1 & \longrightarrow & T & \longrightarrow & E & \longrightarrow & P & \longrightarrow & 1 \\
\| & & \| & & \| & & \| & & \| \\
1 & \xrightarrow{\cong} & T & \xrightarrow{\varphi} & H(\beta) & \longrightarrow & G \times \Gamma & \longrightarrow & 1 \\
\end{array}
\]

commutes. By Corollary 4.2, if \( E \in \text{Nil}_2 \) then \( P \in \text{Nil}_2 \). Conversely, assume \( P \in \text{Nil}_2 \). Then \( G, \Gamma \in \text{Nil}_2 \) (by Lemma 4.1. By Fact 3, \( \Gamma \) and \( G \) have a bilinear commutator. By Lemma 45 (2) \( H(\beta) \) has a bilinear commutator, hence \( H(\beta) \in \text{Nil}_2 \). It follows that \( E \in \text{Nil}_2 \). \( \square \)
Proposition 48.

(1) \( \omega: P \times P \rightarrow H(\beta)' \), \( \omega((x, \xi), (y, \eta)) = ((\xi y^{-1}, [x, y], [\xi, \eta]) \mapsto \langle x, \xi \rangle \times [y, \eta] \). 

(2) \( G^\perp = Z(G) \times \Gamma_0 \leq P \) and \( \Gamma^\perp = G_0 \times Z(\Gamma) \leq P \)

(3) \( \tilde{G} = T \times G \leq H(\beta) \) and \( \tilde{\Gamma} = T \times \Gamma \leq H(\beta) \)

(4) \( \tilde{G} \cap \tilde{\Gamma} = T \) and \( \tilde{G} \cdot \tilde{\Gamma} = H(\beta) \)

(5) \( C_{H(\beta)}(\tilde{G}) = T \times Z(G) \times \Gamma_0 \) and \( C_{H(\beta)}(\tilde{\Gamma}) = T \times G_0 \times Z(\Gamma) \).

Proof.

1. \( \omega((x, \xi), (y, \eta)) = \omega(\pi(c, x, \xi), \pi(d, y, \eta)) = [(c, x, \xi), (d, y, \eta)] = (\langle x, \xi \rangle, [x, y], [\xi, \eta]) \).

2. \( (x, \xi) \in G^\perp \iff \forall_{y \in G} \omega((x, \xi), (y, 1)) = 1 \)

\( \iff \forall_{y \in G} \left( \langle x, \xi \rangle \in [x, y], [\xi, 1] \right) = 1 \)

\( \iff \forall_{y \in G} (\langle x, \xi \rangle = 1 \land [x, y] = 1) \iff \xi \in \Gamma_0 \land x \in Z(G) \)

If \( (x, \xi) \in G^\perp \) and \( (y, \eta) \in G \times \Gamma \) then

\( (y, \eta)(x, \xi)(y, \eta)^{-1} = (yx y^{-1}, \eta \xi \eta^{-1}) = (x, \eta \xi \eta^{-1}) \).

For arbitrary \( z \in G \), \( \langle \eta \xi \eta^{-1} | z \rangle = \langle \xi | z \rangle = 1 \), hence \( \eta \xi \eta^{-1} \in \Gamma_0 \). Therefore \( (y, \eta)(x, \xi)(y, \eta)^{-1} = (x, \eta \xi \eta^{-1}) \in Z(G) \times \Gamma_0 = G^\perp \), which shows that \( G^\perp \leq P \).

\( (x, \xi) \in \Gamma^\perp \iff \forall_{\eta \in \Gamma} \omega((x, \xi), (1, \eta)) = 1 \iff \forall_{\eta \in \Gamma} \left( \langle x, \xi \rangle \in [x, 1], [\xi, \eta] \right) = 1 \)

\( \iff \forall_{\eta \in \Gamma} (\langle x, \xi \rangle = 1 \land [x, \eta] = 1) \iff x \in \Gamma_0 \land \xi \in Z(\Gamma) \)

If \( (x, \xi) \in \Gamma^\perp \) and \( (y, \eta) \in G \times \Gamma \) then

\( (y, \eta)(x, \xi)(y, \eta)^{-1} = (yx y^{-1}, \eta \xi \eta^{-1}) = (yx y^{-1}, \xi) \), and again

for arbitrary \( \rho \in \Gamma \), \( \langle \rho | yx y^{-1} \rangle = \langle \rho | x \rangle = 1 \), hence \( (yx y^{-1}, \xi) \in G_0 \times Z(\Gamma) \), hence also \( \Gamma^\perp \leq P \).

3. As \( \tilde{G} = \pi^{-1}(G) \) it is clear that it is normal in \( H(\beta) \); the same for \( \tilde{\Gamma} \). \( \tilde{G} = \{(c, x, \xi) \mid (x, \xi) \in G \} = \{(c, x, \xi) \mid \xi = 1 \} \). Multiplying two elements from \( \tilde{G} \) shows that \( \tilde{G} = T \times G \). Similarly

\( \tilde{\Gamma} = \{(c, x, \xi) \mid x = 1 \} = T \times \Gamma. \)
4. \((c, x, \xi) \in \widetilde{G} \cap \widetilde{\Gamma} \iff \xi = 1 \wedge x = 1\), hence \(\widetilde{G} \cap \widetilde{\Gamma} = T\).
\((c, x, \xi) = (c, x, 1)(1, 1, \xi) \in \widetilde{G} \cdot \widetilde{\Gamma} \) demonstrates the last point.

5. By Proposition 14 \(C_{H(\beta)}(\widetilde{G}) = \pi^{-1}(G^\perp)\) and \(C_{H(\beta)}(\widetilde{\Gamma}) = \pi^{-1}(\Gamma^\perp)\).

If \((c, x, \xi) \in C_{H(\beta)}(\widetilde{G})\) then \((x, \xi) \in G^\perp = Z(G) \times \Gamma_0\), then \((c, x, \xi) \in T \cdot (Z(G) \times \Gamma_0)\). By Lemma 45(1), \(T \times Z(G) \times \Gamma_0 \longrightarrow T \cdot (Z(G) \times \Gamma_0)\), \((c, x, \xi) \mapsto cx\xi\) is an embedding, whence \(T \cdot (Z(G) \times \Gamma_0) = T \times Z(G) \times \Gamma_0\), and so \((c, x, \xi) \in T \times Z(G) \times \Gamma_0\).

If, conversely, \((c, x, \xi) \in T \times Z(G) \times \Gamma_0\) then \(\pi(c, x, \xi) = (x, \xi) \in Z(G) \times \Gamma_0 = G^\perp\), hence \((c, x, \xi) \in \pi^{-1}(G^\perp) = C_{H(\beta)}(\widetilde{G})\), which shows the first equality. The same arguments provide also \(C_{H(\beta)}(\widetilde{\Gamma}) = T \times G_0 \times Z(\Gamma)\). \(\square\)

In order to bring together the 2 different approaches to Heisenberg groups we introduce the following concept.

**Definition 49.** \(E \in \text{Grp}\). A pair of subgroups \(K, N \leq E\) is called a **normal splitting** of \(E\) provided that

1. \(K\) and \(N\) are normal in \(E\);
2. \(K N = E\) and \(K \cap N \subseteq Z(E)\);
3. \(\exists x \leq E\) \(K = (K \cap N) \times X\);
4. \(\exists y \leq E\) \(N = (K \cap N) \times Y\).

A normal splitting is called **abelian splitting** when in addition \(K\) and \(N\) are maximal abelian.

**Theorem 50.**

1. Let \(T \in \text{Ab}\) and \(\beta : \Gamma \times G \longrightarrow T\) bilinear. Then \(\widetilde{G}, \widetilde{\Gamma}\) is a normal splitting of \(H(\beta)\).
2. If \(T, \Gamma, G \in \text{Ab}\) and \(\beta\) is a duality then \(\widetilde{G}, \widetilde{\Gamma}\) is an abelian splitting of \(H(\beta)\).

**Proof.** 1. is obvious from Proposition 48. For point 2., take \(x, y \in G\):

\[\omega(x, y) = \omega((x, 1), (y, 1)) = [(1, x, 1), (1, y, 1)] = \left\langle \frac{[y]}{[1]}, [x, y], [1, 1] \right\rangle = 1.\]

This shows that \(G \subseteq G^\perp\). Take \(p = (x, \xi) \in G^\perp \leq P\). Then \(\forall y \in G\) \(\omega((x, \xi), (y, 1)) = 1\). Therefore

\[1 = [(1, x, \xi), (1, y, 1)] = \left\langle \frac{[y]}{[1]}, [x, y], [\xi, 1] \right\rangle = \left\langle [\xi]y, 1, 1 \right\rangle\]

and so \(\forall y \in G\) \(\langle ([\xi])y = 1, \) hence, \(\xi = 1\). Therefore \(p = (x, 1) \in G\). Consequently \(G^\perp = G\). By Proposition 14, \(\widetilde{G}\) is maximal abelian. The same arguments apply to \(\Gamma\). \(\square\)
Theorem 51. Assume that $E \in \text{Grp}$, $K, N \trianglelefteq E \land E = KN \land K \cap N \subseteq Z(E)$. Set $T = K \cap N$ and $P = E/T$ so that $\varepsilon : 1 \to T \xrightarrow{i} E \xrightarrow{\pi} P \to 1$ is central and $\omega : P \times P \to E'$ exists. Moreover let $G := \pi(K)$ and $\Gamma = \pi(N)$. Then $P = G \times \Gamma$ and $\omega|_{G \times \Gamma}$ provides a bilinear map $\beta : \Gamma \times G \to T$. If, in addition, $(K, N)$ constitutes a normal splitting, then $E \cong H(\beta)$ and the resulting central sequence $\varepsilon(\beta)$ is equivalent with $\varepsilon$. If $(K, N)$ is even an abelian splitting then $G, \Gamma \in \text{Ab}$ and $\beta : \Gamma \times G \to T$ is a duality.

Proof. Obviously $G, \Gamma \trianglelefteq P$, $\bar{G} = \pi^{-1}(K) = K$, $\bar{\Gamma} = \pi^{-1}(N) = N$.

For $a \in P$ we have $a = \pi(e) = \pi(kn) = \pi(k)\pi(n) \in G\Gamma$, i.e. $P = G\Gamma$.

$a \in G \cap \Gamma \implies a = \pi(e), e \in \pi^{-1}(G) = \pi^{-1}(K) = K$ and $e \in \pi^{-1}(\Gamma) = \pi^{-1}(N) = N$. $\implies e \in K \cap N = T$. Therefore $a = \pi(e) = 1$, i.e. $G \cap \Gamma = \{1\}$. This shows that $P = G \times \Gamma$.

Let $\beta := \omega|_{\Gamma \times G}$. Take $\xi \in \Gamma, x \in G$. Then $\xi = \pi(n), x = \pi(k)$ ($n \in N, k \in K$).

$$\beta(\xi, x) = \omega(\pi(n), \pi(k)) = [n, k] = nkn^{-1}k^{-1} \in K \cap N = T$$

thus $\beta$ takes values in $T$. Using normality of $N, K$ and centrality of $K \cap N$:

$$\beta(\xi, x) \beta(\eta, x) = \omega(\pi(m), \pi(k)) \omega(\pi(n), \pi(k)) = [m, k][n, k] = mkn^{-1}k^{-1}nkn^{-1}k^{-1}$$

$$= mkm^{-1}[k^{-1}, n]k^{-1} = m[k^{-1}, n]m^{-1}k^{-1} = mkk^{-1}nkn^{-1}m^{-1}k^{-1}$$

$$= mkn^{-1}m^{-1}k^{-1} = [mn, k] = \omega(\pi(mn), \pi(k))$$

$$= \omega(\pi(m)\pi(n), \pi(k)) = \beta(\eta, x)$$

$$\beta(\xi, x) \beta(\xi, y) = \omega(\pi(n), \pi(k)) \omega(\pi(n), \pi(l)) = [n, k][n, l] = nkn^{-1}k^{-1}nln^{-1}l^{-1}$$

$$= nkn^{-1}nln^{-1}l^{-1}k^{-1} = nkn^{-1}l^{-1}k^{-1} = [n, kl] = \omega(\pi(n), \pi(kl))$$

$$= \omega(\pi(n), \pi(k)\pi(l)) = \beta(\xi, xy)$$

Therefore $\beta : \Gamma \times G \to T$ is bilinear and we can build $\varepsilon(\beta)$. Now assume there are groups $X, Y$ complementary to $T$ in $K$ and $T$ respectively

$$K = T \times X, N = T \times Y.$$ 

Obviously $\pi|_X : X \cong G$, so let $s = (\pi|_X)^{-1}$. Similarly, $\pi|_Y : Y \cong \Gamma$ and we set $t = (\pi|_Y)^{-1}$. Now define $\varphi(c, x, \xi) = c \cdot s(x) \cdot t(\xi)$ ($c \in T, x \in G, \xi \in \Gamma$).

For the last point assume that $K$ and $N$ are maximal abelian. Then $G = \pi(K)$ and $\Gamma = \pi(N)$ are abelian. Because $K = \bar{G}$ and $N = \bar{\Gamma}$, Proposition 14 yields $G = G^\perp$ and $\Gamma = \Gamma^\perp$. If $\beta(\xi, x) = 1 \ \forall x \in G$
then $\omega(\xi, x) = 1$ $\forall x \in G$ whence $\xi \in \Gamma \cap G \perp = \Gamma \cap G = 1$. Similarly $\beta(\xi, x) = 1$ $\forall \xi \in \Gamma$ implies $x = 1$. Consequently $\beta$ is a duality. \hfill \Box

4. Characterization of Heisenberg Extensions

For dealing with the cohomological conditions it is necessary to return to abelian phase groups. This is owed to the requisiteness that the omega-form induced by a Heisenberg extension $\varepsilon(\beta)$ should take values in the torus (as opposed to $[H(\beta), H(\beta)]$).

Therefore, for the rest of the paper, we assume that all groups $T, G, \Gamma$ be abelian, and we write $G$ and $\Gamma$ in additive notation (the torus $T$ shall stay multiplicative.) The symbol $\star$ will denote the contravariant functor $\text{hom}(\bullet, T)$.

Consider a fixed extension problem $1 \rightarrow i \rightarrow T \rightarrow E \rightarrow \pi \rightarrow G \oplus \Gamma \rightarrow 0$. Given a factorization $P = G \oplus \Gamma$ and a bilinear map $\beta \in (\Gamma \otimes G)^\star$, there is the associated Heisenberg group $H(\beta) = (T \times G) \rtimes \Gamma$, its operations now written as

\[
(c, x, \xi) \cdot (d, y, \eta) = (cd\langle \xi | y \rangle, x + y, \xi + \eta)\\
1 = (1, 0, 0)\\
(c, x, \xi)^{-1} = \left(\frac{\langle \xi | x \rangle}{c}, -x, -\xi\right)
\]

and the corresponding Heisenberg extension $\varepsilon(\beta)$

\[
\varepsilon(\beta) : 1 \rightarrow T \rightarrow H(\beta) \rightarrow G \oplus \Gamma \rightarrow 0
\]

with $i(c) = (c, 0, 0)$ and $\pi(c, x, \xi) = (x, \xi)$. Since $\varepsilon(\beta)$ is a central extension of an abelian group, the commutator form takes values in $T$. As the commutator is $[(c, x, \xi), (d, y, \eta)] = \left(\frac{\langle \xi | y \rangle}{\langle \eta | x \rangle}, 0, 0\right)$, the corresponding form is

\[
\omega: P \times P \rightarrow \omega((x, \xi), (y, \eta)) = \frac{\langle \xi | y \rangle}{\langle \eta | x \rangle}.
\]

With the standard section $s_0: P \rightarrow H(\beta)$, $(x, \xi) \mapsto (1, x, \xi)$ we obtain the 2-cocycle

\[
\gamma_0: P \times P \rightarrow T, \quad \gamma_0((x, \xi), (y, \eta)) = \langle \xi | y \rangle.
\]

Thus, each $\gamma \in Z^2(P, T)$ is $\gamma = \gamma_0 \cdot (\partial^2 h)$ for some $h \in T^P$, i.e.,

\[
\gamma((x, \xi), (y, \eta)) = \langle \xi | y \rangle h(x, \xi) h(y, \eta) h(x + y, \xi + \eta).
\]
The standard cocycle $\gamma_0$ is bilinear and it factors through $\Gamma \otimes G$: With projections $\pi^1: P \to G$, $\pi^2: P \to \Gamma$ we get

\[
\begin{array}{ccc}
\Gamma \otimes G & \xrightarrow{\pi^2 \otimes \pi^1} & P \otimes P \\
\beta & \downarrow & \downarrow \\
& T & \\
\end{array}
\]

By left exactness of $\text{hom}(\bullet, T)$ the map $(\pi^2 \otimes \pi^1)^*$ is a monomorphism. Because bilinear forms are specific 2-cocycles we obtain the embedding

\[
\varphi = (\Gamma \otimes G)^* \xrightarrow{(\pi^2 \otimes \pi^1)^*} (P \otimes P)^* \xrightarrow{} Z^2(P, T)
\]

$\varphi(\beta)((x, \xi), (y, \eta)) = \beta(\xi \otimes y) = \langle \xi | y \rangle$.

**Definition 52.** Let $P = G \oplus \Gamma$ in $\text{Ab}$. Then $H^2_{\Gamma \times G}(P, T)$ denotes the set of those cohomology classes in $H^2(P, T)$ that contain a bilinear cocycle which factors through $\Gamma \otimes G$:

\[
H^2_{\Gamma \times G}(P, T) = \text{im}(\varphi) + B^2(P, T)
\]

**Lemma 53.** Let $\beta: \Gamma \times G \to T$ be bilinear, and $h: G \times \Gamma \to T$ an arbitrary map of sets. If

\[
(15) \quad \forall \xi, \eta \in \Gamma \forall x, y \in G \quad \beta(\xi, y) = \frac{h(x, \xi)h(y, \eta)}{h(x + y, \xi + \eta)}
\]

then $\beta$ is the constant map $\beta = 1$.

**Proof.** Spezializing $x = \eta = 0$ in (15) gives

\[
(16) \quad \beta(\xi, y) = \frac{h(0, \xi)h(y, 0)}{h(y, \xi)}
\]

$\xi = y = 0$ in (15) yields

\[
1 = \frac{h(x, 0)h(0, \eta)}{h(x, \eta)}
\]

and therefore $h(x, \eta) = h(x, 0)h(0, \eta) \forall \eta \in \Gamma$, $\forall x \in G$. Rephrasing this identity:

\[
h(y, \xi) = h(y, 0)h(0, \xi) \forall \xi \in \Gamma \forall y \in G
\]

and plugging into (16) gives

\[
\beta(\xi, y) = \frac{h(0, \xi)h(y, 0)}{h(y, 0)h(0, \xi)} = 1.
\]

\[\square\]

**Proposition 54.** $\text{im}(\varphi) \cap B^2(P, T) = 0$. 
Proof. Take γ ∈ im(ϕ) ∩ B²(P, T). Then γ = ϕ(β), hence
γ((x, ξ), (y, η)) = ϕ(β)((x, ξ), (y, η)) = β(ξ ⊗ y)
Since γ is a coboundary, ∃h: P → T with γ = ∂²h. Therefore
β(ξ ⊗ y) = γ((x, ξ), (y, η)) = \frac{h(x, ξ)h(y, η)}{h(x + y, ξ + η)}
Lemma 53 yields β = 1, hence γ = 1. □

Corollary 55.

(17) \( H^2_{Γ×G}(P, T) ≅ (Γ ⊗ G)^* \).

Proof.

\[
H^2_{Γ×G}(P, T) = \frac{\text{im}(ϕ) + B²(P, T)}{B²(P, T)} ≅ \frac{\text{im}(ϕ)}{\text{im}(ϕ) \cap B²(P, T)} = \text{im}(ϕ)
\]

Consequently (Γ ⊔ G)^* ≅ H^2_{Γ×G}(P, T). □

As a consequence we obtain the following result.

Proposition 56. \( P = Γ ⊗ G \). Then \( β ↦ \text{cls}(ε(β)) \) defines an isomorphism

\( (Γ ⊗ G)^* ≅ H^2_{Γ×G}(P, T) \).

Proof. Obvious. □

When γ is a cocycle in \( Z²(P, T) \), the function \( \text{cls}(γ) ↦ ω \) with
\( ω(x ∧ y) = \frac{γ(x, y)}{γ(y, x)} \) is a homomorphism \( H²(P, T) → (P ∧ P)^* = Ω²(P, T) \),
as can be verified directly computing the cocycle identity or by representing γ by a set-theoretic cross-section w.r.t. a central extension. From the appropriate instance of the universal coefficient theorem

\[
1 → \text{Ext}¹(P, T) → H²(P, T) \overset{q}{→} Ω²(P, T) → 0
\]
it is known that q is an epimorphism—in fact the sequence splits. There is a special case where a splitting can be obtained easily.

Lemma 57. Assume the abelian group T is uniquely 2-divisible, i.e., each element has a unique square root. Then

\( σ: ω ↦ \text{cls}(\sqrt{ω}) \)

is a cross section of \( H²(P, T) \overset{q}{→} Ω²(P, T) \).

Proof. Since ω is bilinear and \( \sqrt{ω} = P ∧ P \overset{ω}{→} T \overset{\sqrt{ω}}{=} T \) it is plain that \( \sqrt{ω} \), considered as map \( P × P → T \) is in \( Z²(P, T) \). Obviously \( \sqrt{ω} · μ = \sqrt{ω} · \sqrt{μ} \), and therefore also

\( \text{cls}(\sqrt{ω} · μ) = \text{cls}(\sqrt{ω}) · \text{cls}(\sqrt{μ}) \)
which demonstrates that $\sigma$ is a homomorphism.

\[
(q \circ \sigma)(\omega)(x \land y) = q(\text{cls}(\sqrt{\omega}))(x \land y) = \frac{\sqrt{\omega}(x \land y)}{\sqrt{\omega}(y \land x)}
\]

\[
= \sqrt{\omega}(x \land y) \cdot \sqrt{\omega}(x \land y)^{-1} = \sqrt{\omega}(x \land y)^2
\]

therefore $q \circ \sigma = 1_{\Omega^2(P,T)}$. \hfill \square

A uniquely 2-divisible abelian group is a module over the localization $\mathbb{Z}_2 = \mathbb{Z}[\frac{1}{2}]$. For this reason it will be written additively in the following Proposition.

**Proposition 58.** Let $\chi: A \to T$ be a homomorphism, where $(A, +) \in \text{Ab}$ is uniquely 2-divisible. If $\omega \in \chi^*(\Omega^2(P,T))$ then we can compute a cocycle $\gamma \in Z^2(P,T)$ with $q(\text{cls}(\gamma)) = \omega$.

**Proof.** Consider the following diagram

\[
\xymatrix{
H^2(P, A) \ar[r]^{q_A} & \Omega^2(P, A) \\
H^2(P, T) \ar[r]^{q_T} & \Omega^2(P, T) \\
\ar[u]_{\chi^*} \ar[u]_{\chi^*}
}
\]

\[
= (\chi^* \circ q_A)(\text{cls}(\mu))(x \land y) = (\chi \circ q_A(\text{cls}(\mu)))(x \land y) = \chi(\mu(x, y) - \mu(y, x))
\]

\[
= (\chi \circ \mu)(x, y) = \chi^*(\mu)(x, y)
\]

\[
= (q_T \circ \chi^*)(\text{cls}(\mu))(x \land y)
\]

shows commutativity. Choose $\mu \in \Omega^2(P, A)$ with $\chi^*(\mu) = \omega$. By Lemma 57, $\frac{1}{2}\mu$ is a cocyle in $Z^2(P, A)$ that maps to $\mu$, i.e., $q_A(\text{cls}(\frac{1}{2}\mu)) = \mu$. Therefore

\[
q_T \left( \chi^* \left( \text{cls} \left( \frac{1}{2}\mu \right) \right) \right) = \chi^* \left( q_A \left( \text{cls} \left( \frac{1}{2}\mu \right) \right) \right) = \chi^*(\mu) = \omega.
\]

\hfill \square

We can give a slight generalization of Lemma 57.

Consider the torus $(T, \cdot) \in \text{Ab}$ and write $T^2$ for the image of the map $T \to T$, $x \mapsto x^2$. Assume that the set of roots of 1 in $T$ has a complement, i.e.

\[
\exists S \leq T \left( \ker(x^2) \cdot S = T \land \ker(x^2) \cap S = \{1\} \right).
\]
This means, the SES \( \varepsilon^2 : 1 \rightarrow \ker(x^2) \rightarrow T \xrightarrow{x^2} T^2 \rightarrow 1 \) splits and there is a cross section, i.e., a homomorphism \( r : T^2 \rightarrow T \) s.t. \( r(c)^2 = c \ \forall c \in T^2 \). So \( r \) is a partial square-root function for \( T \).

**Proposition 59.** Assume the torus has the property mentioned above. Given an abelian group \((P,+)\) and let \( \omega \in (P\wedge P)^* \) be such that \( \text{im}(\omega) \in T^2 \). So \( \gamma := r \circ \omega^\wedge \in Z^2(P,T) \) and \( q(\text{cls}(\gamma)) = \omega \).

**Proof.** Since \( \omega^\wedge \) is bilinear and \( r \) is a homomorphism, \( \gamma \) is bilinear, hence a cocycle.

\[
q(\text{cls}(\gamma))(x \wedge y) = \gamma(x,y) = r(\omega(x \wedge y)) = r\left(\frac{\omega(x \wedge y)}{r(\omega(y \wedge x))}\right) = r(\omega(x \wedge y)) = \omega(x \wedge y)
\]

The association of a bilinear form to its Heisenberg extension is functorial:

**Definition 60.**

1. We write \( \text{Bi} \) for the comma category \( \otimes \downarrow \text{Ab} \). That is, \( \text{Bi} \) has bilinear forms \( \beta : \Gamma \times G \rightarrow T \) as objects—all three groups \( \Gamma, G, T \) being abelian. A morphism \( \beta \rightarrow \beta' \) is a triple of homomorphisms \((\gamma, g, t)\) that commutes the diagram

\[
\begin{array}{ccc}
\Gamma \times G & \xrightarrow{\beta} & T \\
\gamma \times g \downarrow & & \downarrow t \\
\Gamma' \times G' & \xrightarrow{\beta'} & T'
\end{array}
\]

The full subcategory of \( \text{Bi} \) consisting of dualities on abelian groups as objects is denoted by \( \text{Du} \).

2. The objects of the category \( \text{Hei} \) are central extensions \( \varepsilon \) of abelian groups \( G \oplus \Gamma \); a morphism \( \varepsilon \rightarrow \varepsilon' \) is a quadrupel
\((t, e, g, \gamma)\) of homomorphisms commuting the diagram

\[
\begin{array}{c}
\varepsilon : 1 \longrightarrow T \xrightarrow{i} E \xrightarrow{\pi} G \oplus \Gamma \longrightarrow 0 \\
\varepsilon' : 1 \longrightarrow T' \xrightarrow{i'} E' \xrightarrow{\pi'} G' \oplus \Gamma' \longrightarrow 0
\end{array}
\]

\[
\begin{array}{c}
\varepsilon : 1 \longrightarrow T \xrightarrow{i} E \xrightarrow{\pi} G \oplus \Gamma \longrightarrow 0 \\
\varepsilon' : 1 \longrightarrow T' \xrightarrow{i'} E' \xrightarrow{\pi'} G' \oplus \Gamma' \longrightarrow 0
\end{array}
\]

We write \(\text{Hei}\) for the full subcategory of \(\overline{\text{Hei}}\) whose objects are central extensions of abelian groups \(G \oplus \Gamma\) in such a way that \(\widetilde{G}, \widetilde{\Gamma}\) provide an abelian splitting.

The functor \(\varepsilon : \text{Bi} \longrightarrow \overline{\text{Hei}}\) acts as

\[
\beta \xrightarrow{(\gamma, g, t)} \beta' \xrightarrow{\varepsilon} \varepsilon(\beta) \xrightarrow{(t, t \times g \times \gamma, \gamma)} \varepsilon(\beta')
\]

The functor \(\beta : \overline{\text{Hei}} \longrightarrow \text{Bi}\) acting on objects as \(\beta(\varepsilon) = \omega(\varepsilon)|_{\Gamma \times G}\), is given by

\[
\varepsilon \xrightarrow{(t, t \times g \times \gamma)} \varepsilon' \xrightarrow{\beta} \omega(\varepsilon)|_{\Gamma \times G} \xrightarrow{(\gamma, g, t)} \omega(\varepsilon')|_{\Gamma' \times G'}
\]

so the value of the morphism \((t, e, g, \gamma)\) is the commutative diagram (18).

Choosing liftings \((\widetilde{\xi}, \widetilde{x}) \in E \times E\) we obtain \((t \circ \beta(\varepsilon))(\xi, x) = ti^{-1}[\widetilde{\xi}, \widetilde{x}]\)

hence \(i'(t \circ \beta(\varepsilon))(\xi, x) = i'ti^{-1}[\widetilde{\xi}, \widetilde{x}] = eii^{-1}[\widetilde{\xi}, \widetilde{x}] = [\varepsilon(\xi), e(\widetilde{x})]\).

Since \(\pi'(e(\widetilde{\xi})) = (\pi' \circ e)(\widetilde{\xi}) = (g \oplus \gamma \circ \pi)(\widetilde{\xi}) = (g \oplus \gamma)(\xi) = \gamma(\xi)\) and similar \(\pi'(e(\widetilde{x})) = g(x)\), we get

\(i'(\beta(\varepsilon) \circ \gamma \times g)(\xi, x) = i'(\omega(\varepsilon')|_{\Gamma' \times G'}(\gamma(\xi), g(x))) = [e(\widetilde{\xi}), e(\widetilde{x})]\)

Thus \(t \circ \beta(\varepsilon) = \beta(\varepsilon') \circ \gamma \times g\), i.e., \((\gamma, g, t)\) is indeed a morphism in \(\text{Bi}\).

**Proposition 61.** \(\beta \circ \varepsilon = 1_{\text{Bi}}\).

**Proof.** The commutator form \(\omega(\varepsilon(\beta))\) for a bilinear \(\beta : \Gamma \times G \longrightarrow T\) is

\[
\omega(\varepsilon(\beta)) : G \oplus \Gamma \times G \oplus \Gamma \longrightarrow T, \quad ((x, \xi), (y, \eta)) \mapsto \frac{\langle \xi | y \rangle}{\langle \eta | x \rangle}
\]

and therefore \(\beta(\varepsilon(\beta)) = \omega(\varepsilon(\beta))|_{\Gamma \times G} = \beta\). On morphisms we get

\[
\beta \xrightarrow{(\gamma, g, t)} \beta' \xrightarrow{\varepsilon} \varepsilon(\beta) \xrightarrow{(t, t \times g \times \gamma, \gamma)} \varepsilon(\beta') \xrightarrow{\beta} \beta \xrightarrow{(\gamma, g, t)} \beta'
\]

which proves the statement. \(\square\)

**Proposition 62.** Consider \(\varepsilon : 1 \longrightarrow T \xrightarrow{i} E \xrightarrow{\pi} G \oplus \Gamma \longrightarrow 0 \in \overline{\text{Hei}},\)

and let \(\widetilde{G}, \widetilde{\Gamma}\) denote the inverse images in \(E\) of \(G, \Gamma\) respectively. If \(\widetilde{G}, \widetilde{\Gamma}\) provide a normal splitting, then \(\varepsilon \sim \varepsilon(\beta(\varepsilon))\).
Proof. Due to the splitting concept (cf. Definition 49) there are normal subgroups \(X, Y \leq E\) with
\[
\tilde{G} = i_\varepsilon(T) \cdot X, \tilde{\Gamma} = i_\varepsilon(T) \cdot Y.
\]
It follows that restriction of \(\pi_\varepsilon\) to \(X\) and \(Y\) are isomorphisms
\[
\pi_\varepsilon|_{X}: X \cong G, \pi_\varepsilon|_{Y}: Y \cong \Gamma.
\]
Let \(g := (\pi_\varepsilon|_{X})^{-1}\), \(\gamma := (\pi_\varepsilon|_{Y})^{-1}\), and define the map
\[
\varphi: H(\beta(\varepsilon)) \rightarrow E, \quad \varphi(c, x, \xi) = i_\varepsilon(c)g(x)\gamma(\xi) \quad (c \in T, x \in G, \xi \in \Gamma).
\]
Then \(\varphi\) is a homomorphism:
\[
\varphi((c, x, \xi)(d, y, \eta)) = \varphi(c d \cdot \omega(\varepsilon)(\xi, y), x + y, \xi + \eta)
\]
\[
= i_\varepsilon(c d \cdot \omega(\varepsilon)(\xi, y)) \cdot g(x + y) \cdot \gamma(\xi + \eta)
\]
\[
= i_\varepsilon(c d) \cdot i_\varepsilon(\varepsilon)^{-1}[\gamma(\xi), g(y)] \cdot g(x)g(y)\gamma(\xi)\gamma(\eta)
\]
\[
= i_\varepsilon(c) \cdot i_\varepsilon(d) \cdot \gamma(\xi), g(y) \cdot g(x)g(y)\gamma(\xi)\gamma(\eta)
\]
which comes to the same. Obviously \(\varphi \circ i = i_\varepsilon\) and \(\pi_\varepsilon \circ \varphi = \pi\). This demonstrates the equivalence of the central extensions \(\varepsilon\) and \(\varepsilon(\beta(\varepsilon))\).

\[\square\]

**Corollary 63.** Restriction of the functors \(\varepsilon\) and \(\beta\) to the full subcategories \(\text{Du}\) and \(\text{Hei}\) respectively results in an equivalence \(\text{Du} \simeq \text{Hei}\).

**Proof.** If \(\varepsilon \in \text{Hei}\) then \(\varepsilon\) is central with \((\tilde{G}, \tilde{\Gamma})\) as an abelian splitting. By Theorem 51, \(\beta(\varepsilon)\) is a duality, hence \(\beta(\varepsilon) \in \text{Du}\).

If \(\beta \in \text{Du}\) the it is a duality \(\Gamma \times G \xrightarrow{\beta} T\) on abelian groups \(\Gamma, G, T\). \(\varepsilon(\beta)\) is then the central extension
\[
1 \rightarrow T \rightarrow H(\beta) \rightarrow G \oplus \Gamma \rightarrow 0
\]
which has \((\tilde{G}, \tilde{\Gamma})\) as abelian splitting (due to Theorem 50). Therefore, by its very definition, \(\varepsilon(\beta) \in \text{Hei}\). Consequently, \(\varepsilon\) and \(\beta\) restrict to
\[
\begin{array}{rcl}
\text{Du} & \xrightarrow{\varepsilon'} & \text{Hei} \\
\beta' & \xrightarrow{} & \text{Hei}
\end{array}
\]
From Proposition 61 we get \(\beta' \circ \varepsilon' = 1_{\text{Du}}\). Since an abelian splitting is in particular a normal splitting, we get from Proposition 62 that \(\varepsilon \sim \varepsilon'(\beta'(\varepsilon))\) \(\forall \varepsilon \in \text{Hei}\). In particular, \(\varepsilon \cong (\varepsilon' \circ \beta')(\varepsilon)\) in the category.
Heisenberg groups via algebra

Therefore the functors $\varepsilon', \beta'$ provide an equivalence. Plainly then they yield also a mutually adjoint situation.

We conclude this section with an observation concerning dualities.

**Lemma 64.** Let $F$ be a field of zero characteristic. For a bilinear form $(\langle \cdot, \cdot \rangle): V \times V \to F$ on a vector space, let $(\langle \cdot \rangle): V \times V \to T$ be its associated bicharacter with respect to any fixed standard character $\chi: (F, +) \to (T, \cdot)$. Then $(\langle \cdot \rangle)$ is nondegenerate iff $(\langle \cdot, \cdot \rangle)$ is nondegenerate.

**Proof.** Write $\alpha: V \times V \to F$ and $\beta = \chi \circ \alpha: V \times V \to T$ for the bilinear forms. For the left and right kernels we have $\ker(\alpha) \subseteq \ker(\beta)$ and $\ker(\beta) \subseteq \ker(\alpha)$ (cf. Definition 38), therefore if $\beta$ is a duality then so is $\alpha$.

Now let $\alpha$ be a duality, i.e. $\ker(\alpha) = 0$ and assume $\forall_y \beta(v, y) = 1$. $\chi$ being standard means $\ker(\chi)$ is the prime ring of $F$, which is $\mathbb{Z}$, hence $\forall_y \alpha(v, y) \in \mathbb{Z}$.

Assume $\exists_y \alpha(v, y) \neq 0$. Let $k := \min \{ n \in \mathbb{Z}_{>0} \mid \exists_y \alpha(v, y) = n \}$ and set $\alpha(v, y_0) = k$. Then

$$\alpha\left(v, \frac{y_0}{2}\right) + \alpha\left(v, \frac{y_0}{2}\right) = \alpha(v, y_0) = k$$

whence $\alpha\left(v, \frac{y_0}{2}\right) = \frac{k}{2} < k$ a contradiction. Therefore $\forall_y \alpha(v, y) = 0$ and since $\alpha$ is a duality, $v = 0$. Consequently $\ker(\alpha) = 0$. The same happens with the right kernels. Therefore $\beta$ is a duality. \qed

5. **Heisenberg Groups with a View towards Fourier Theory**

The crucial object to obtain a Heisenberg group is a Lagrangian bisection, which we generalize from the classical setting [6, p. 21] as a pair $(G, \Gamma)$ of transverse Lagrangian subgroups, meaning $G, \Gamma \leq \text{Iso}(P) \cap \text{Co}(P)$ such that $G + \Gamma = P$. It is a simple observation that one has then a direct sum decomposition, provided the commutator form is nondegenerate.

**Lemma 65.** Assume $P \in \text{Ab}$. If $(G, \Gamma)$ is a Lagrangian bisection of $P \in \text{Ab}$, then $G + \Gamma = P$.

**Proof.** We need only show $G \cap \Gamma = 0$, so assume $x \in G \cap \Gamma$. Since $G^\perp = G$ and $\Gamma^\perp = \Gamma$, this implies that $x$ is orthogonal to any element of $G$ and also orthogonal to any element of $\Gamma$. Hence we have $\omega()$ \qed

If $\gamma$ is a factor set of $T \to H \to P$, strict centrality may be characterized by the condition that $\forall_{w \in P} \gamma(z, w) = \gamma(w, z)$ implies $z = 0$. As this condition is invariant under cohomological equivalence, the strictly
central extensions form a well-defined subset \( H_1^2(P, T) \subset H^2(P, T) \) that is, however, not a subgroup. Moreover, abelian extensions correspond to symmetric factor sets; identifying the corresponding cohomology groups in the sense that \( \text{Ext}^1(P, T) \leq H^2(P, T) \), we obtain the invariance property \( H_1^2(P, T) + \text{Ext}^1(P, T) = H_1^2(P, T) \). It is interesting to consider an example of this.

**Example 66.** The central extension \( \mathbb{Z}_N \rightarrow H \rightarrow \mathbb{Z}_N \oplus \mathbb{Z}_N \) is defined by the semidirect product \( H := (\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes \mathbb{Z}_N \), where \( \mathbb{Z}_N \) acts on \( \mathbb{Z}_N \times \mathbb{Z}_N \) via addition in the second factor. In other words, the group operation in \( H \) is defined by

\[
c(x, \xi) \cdot c'(x', \xi') = (c + c' + \xi + x') (x + x', \xi + \xi')
\]

for \( c(x, \xi), c'(x', \xi') \in (\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes \mathbb{Z}_N \), continuing with the center convention introduced after Equation (3). As we shall see later (Example 72), the group \( H \) is an important specimen of a discrete Heisenberg group. The insertion \( \iota \) is here defined by \( c \mapsto c(0,0) \), and the projection \( \pi \) by \( c(x, \xi) \mapsto (x, \xi) \). Note that \( H \) has indeed center \( \mathbb{Z}_N \), so the given extension is strictly central.

Let us now “perturb” the Heisenberg group \( H \) by adding the abelian extension \( \mathbb{Z}_N \rightarrow A \rightarrow \mathbb{Z}_N \oplus \mathbb{Z}_N \), where the torus \( \mathbb{Z}_N \) is embedded into \( A := \mathbb{Z}_N^2 \oplus \mathbb{Z}_N \) via \( \iota'(k + N\mathbb{Z}) := (kn + N^2\mathbb{Z}, nN\mathbb{Z}) \) with the corresponding projection \( \pi'(m + N^2\mathbb{Z}, n + N\mathbb{Z}) := (m + N\mathbb{Z}, n + N\mathbb{Z}) \).

Note that this extension is not trivial in \( \text{Ext}^1(\mathbb{Z}_N \oplus \mathbb{Z}_N, \mathbb{Z}_N) \) since \( A \) is not isomorphic to a direct sum of \( \mathbb{Z}_N \) and \( \mathbb{Z}_N \oplus \mathbb{Z}_N \). It is easy to see that the cocycle of the central extension \( H \) is given by \( \gamma(x; \xi; x', \xi') = \xi + x' \), that of the abelian extension \( A \) by

\[
\alpha(m + N\mathbb{Z}, n + N\mathbb{Z}; m' + N\mathbb{Z}, n' + N\mathbb{Z}) = \left[ \frac{m \bmod N + m' \bmod N}{N} \right] + N\mathbb{Z}
\]

This extension corresponds to the “natural section” \( s' : \mathbb{Z}_N \oplus \mathbb{Z}_N \rightarrow A \) of \( \pi' \) given by \( s'(m + N\mathbb{Z}, n + N\mathbb{Z}) := (m \bmod N + N^2\mathbb{Z}, n + N\mathbb{Z}) \), which is of course not a homomorphism since the abelian extension \( A \) is not trivial, as noted before. (Without the modding operation, the map would not be well-defined; using \( mN + N^2\mathbb{Z} \) in place of \( m \bmod N + N^2\mathbb{Z} \) yields a homomorphism that is not a section of \( \pi' \).)

For adding the two cocycles, it is practical to use positive representatives \( 0, 1, \ldots, N - 1 \) of \( \mathbb{Z}_N \). Moreover, we shall suppress the arguments \( n + N\mathbb{Z} \) and \( n' + N\mathbb{Z} \) of \( \alpha \) since they are irrelevant. Under these conventions, the two cocycles are given by

\[
\gamma(x, \xi; x', \xi') = (\xi + x') \bmod N \quad \text{and} \quad \alpha(x; x') = \frac{x + x' - (x + x') \bmod N}{N}.
\]
Using Iverson’s bracket convention [15, (2.5)], one can write the abelian cocycle as \( \alpha(x; x') = [x + x' \geq N] \), revealing it as the carry bit \( \varepsilon_{x,x'} \) of addition modulo \( N \). Reverting to coset notation, the operation

\[
c(x, \xi) \cdot c'(x', \xi') = (c + c' + \xi + x' + \varepsilon_{x,x'}) (x + x', \xi + \xi')
\]

in the sum group \( \tilde{H} := T \times_{\gamma + \alpha} P \) is indeed a perturbation of the one given above for \( H \). Since the commutator form of the corresponding central extension for \( \tilde{H} \) is still \( \omega(x, \xi) = \gamma(x, \xi) - \gamma(\xi, x) \) as for \( H \), we see that the assignment of commutator forms to (equivalence classes of) strictly central extensions is not injective. In other words, the mapping \( H^2(P, T) \to \text{Hom}(\Lambda^2 P, T) \) is not an injection.

Having a symplectic structure \( \omega \) on the phase space \( P \) of a central extension \( T \hookrightarrow H \to P \), it is natural to consider its isotropic subgroups \( G \leq P \) with respect to \( \omega \). As usual, these are defined by the condition \( G \leq G^\perp \), where one defines the symplectic orthogonal by \( G^\perp := \{ z \in P \mid \forall w \in G \omega(z, w) = 1 \} \). Equivalently, one may also require that \( \omega \) restricts to \( 1 \) on \( G \). Similarly, a coisotropic subgroup \( \Gamma \leq P \) is defined by \( \Gamma^{\perp} \leq \Gamma \); equivalently, \( \omega \) descends to a nondegenerate symplectic form on the quotient \( P/\Gamma \). A subgroup that is both isotropic and coisotropic is called Lagrangian. As in the case of vector spaces, this is equivalent to being maximal isotropic or minimal coisotropic (with respect to inclusion).

Motivate “our” definition of Heisenberg groups as a natural algebraic analog of Mumford’s definition in [34]. But also clarify the contrast to “Mumford groups” in the sense of [10, Def. 5.1]: The stipulation of a bijective instead of injective map into the dual (yielding a strong rather than a weak symplectic duality!) is too strong for the algebraic setting. It can only be realized in the topological category where one uses the continuous dual instead of any purely algebraic dual.

The motivation of Mumford is also very clear: Heisenberg groups in his sense characterize the cases where essentially unique irreducible representations exist, as established in [34, Thm. 1.2ii] and more explicitly in [48, Thm. 1.4]. Of course here one always thinks of unitary representations, as in the classical Stone–von Neumann representation of the standard Heisenberg group on \( L^2(\mathbb{R}^n) \). In the special case \( G = \mathbb{R}^n \), Mumford’s notion of Heisenberg group leads to the important topic of harmonic analysis in phase space [14], via the so-called symplectic Fourier transform. Note that here a \emph{representation} is defined as an equivalence class under unitary isomorphism; its representatives are then called the \emph{realizations} of this representation. In the
case of locally compact abelian groups, there is a bijective correspondence between irreducible representations and characters [13, §4.1]. In the nonabelian case, irreducible (unitary) representations constitute the so-called unitary dual, which is isomorphic to the spectrum of the group algebra $C^*(G)$.

Using the various characterizations, we can now exhibit a series of infinite groups that are nilquadratic but fail to be Heisenberg groups.

**Example 67.** The free nilquadratic group on $s$ generators [4] may be defined as the group $N_s := N_{2,s}$ generated by $\{x_i, x_{ij} \mid 1 \leq i < j \leq s\}$ subject to the relations $[x_i, x_j] = x_{ij}$, and $[x_{ij}, x_k] = [x_{ij}, x_{kl}] = 1$. It is easy to see that every element $x \in N_s$ has the unique representation

$$x = \left( \prod_{1 \leq i \leq s} x_i^{\alpha_i} \right) \left( \prod_{1 \leq i < j \leq s} x_{ij}^{\beta_{ij}} \right)$$

with arbitrary exponents $\alpha_i, \beta_{ij} \in \mathbb{Z}$. Writing $s' := \binom{s}{2}$, it is clear that $[N_s, N_s] = \mathbb{Z}(N_s) = \langle x_{ij} \mid 1 \leq i < j \leq s \rangle \cong \mathbb{Z}^{s'}$ with quotient the abelianization $N_s^{ab} = N_s/[N_s, N_s] = \langle [x_i] \mid 1 \leq i \leq s \rangle \cong \mathbb{Z}^s$. Hence one obtains the central extension

$$1 \rightarrow [N_s, N_s] \rightarrow N_s \rightarrow N_s^{ab} \rightarrow 1$$

whose commutator form $\omega_s : N_s^{ab} \times N_s^{ab} \rightarrow [N_s, N_s]$ may be computed via commutator identities and $[N_s, N_s] = \mathbb{Z}(N_s)$ to yield

$$\omega_s(x_1^{\alpha_1} \cdots x_s^{\alpha_s}, x_1^{\bar{\alpha}_1} \cdots x_s^{\bar{\alpha}_s}) = \prod_{i,j=1}^{s} [x_i, x_j]^{\alpha_i \bar{\alpha}_j} = \prod_{1 \leq i < j \leq s} x_{ij}^{2\alpha_i \bar{\alpha}_j},$$

where the indexed brackets denote the antisymmetrizer (as customary in physics).

Assume $A, B \leq N_s$ are abelian subgroups such that $\langle A \cup B \rangle = N_s$ and $A \cap B = \mathbb{Z}(N_s) = [N_s, N_s]$. For any $x_i (1 \leq i \leq s)$ we have either $x_i \in A$ or $x_i \in B$, but not both since $x_i \not\in [N_s, N_s] = \langle x_{ij} \rangle$. Hence we may, without loss of generality, reorder the generators so that $x_1, \ldots, x_t \in A \setminus B$ and $x_{t+1}, \ldots, x_s \in B \setminus A$. Hence we must have

$$A = \langle x_i, x_{ij} \mid 1 \leq i \leq t \wedge 1 \leq i < j \leq s \rangle,$$

$$B = \langle x_i, x_{ij} \mid t + 1 \leq i \leq s \wedge 1 \leq i < j \leq s \rangle.$$

In the degenerate case $s = 1$ with $N_1 \cong \mathbb{Z}$, this just means $A \cong \mathbb{Z}$ and $B \cong \mathbb{Z}$. For $s = 2$ we have the subgroups $A = \langle x_1, x_{12} \rangle \leq N_2$ and $B = \langle x_2, x_{12} \rangle \leq N_2$. Therefore we conclude that $N_2$ is in fact a Heisenberg group. However, for $s > 2$ the free nilpotent group $N_s$
cannot be a Heisenberg group because at least one of $A$, $B$ must contain two generators $x_i$ and thus cannot be abelian. This settles also the following questions: Given a nilquadratic group $H$ with torus $T$, is there always an abelian bisection over $T$? Equivalently (confer Theorem 28), given an abelian group $P$ with a symplectic form $\omega: P \times P \to T$, is there always a Lagrangian bisection? Obviously, the answer to both questions is no.

For introducing Fourier transforms, it is advantageous to envisage a somewhat wider concept of Heisenberg modules/algebras. The basic ideas go back to David Mumford’s *magnum opus* on theta functions [34].

As before we consider a central extension of the abelian group $P$ with kernel the torus $T$

\[(20) \quad 1 \longrightarrow T \overset{\iota}{\longrightarrow} H \overset{\pi}{\longrightarrow} P \longrightarrow 0,\]

It is of crucial importance to observe that the *Heisenberg twist* $J$ is a morphism of $\text{Heis}$. Indeed, given any duality $\beta: H(\beta) \to H(\beta^\top)$, we have the $\text{Nil}_2$-morphism

\[
\begin{array}{cccccc}
1 & & T & & H(\beta) & & G \oplus \Gamma & & 0 \\
& & \downarrow J & & \downarrow j & & \\
1 & & T & & H(\beta^\top) & & \Gamma \oplus G & & 0,
\end{array}
\]

where $j: G \oplus \Gamma \to \Gamma \oplus G$ is the map introduced in Subsection 5.2. It should be noted that $J$ does not descend to $\text{Du}$, meaning there is no morphism $\tilde{J}: \beta \to \beta^\top$ in $\text{Du}$ with $J = H(\tilde{J})$.

Just as for $H(\beta)$, there is also a generalized concept of *Heisenberg module* for $H \in \text{Heis}$. Recall that an action $\eta$ of $H$ on a $K$-vector space $S$ may be written as a group homomorphism $\eta: H \to \text{Aut}_K(S)$. Since the torus extends via $\varepsilon_T: T \to K^\times$ into the multiplicative group of the field and the latter acts naturally on $S$ via its scalar action $\Delta: K^\times \to \text{Aut}_K(S)$, there is an induced natural action $\tau = \Delta \circ \varepsilon_T: T \to \text{Aut}_K(S)$. Since $\Delta$ is a faithful action, we may identify $\Delta(K^\times)$ with a subgroup of $\text{Aut}_K(S)$. We obtain a morphism of short exact sequences

\[(21) \quad 1 \longrightarrow T \overset{\iota}{\longrightarrow} H \overset{\pi}{\longrightarrow} P \longrightarrow 0 \]

\[
\begin{array}{cccccc}
& & \downarrow \varepsilon_T & & \downarrow \eta & & \downarrow \xi \\
1 & & K^\times \overset{\Delta}{\longrightarrow} \text{Aut}_K(S) & & \text{Aut}_K(S)/K^\times & & 0,
\end{array}
\]
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where the quotient action \( \zeta \) is induced from \( \eta \) via \( \zeta_z(s) := \eta_{\sigma(z)}(s) K^\times \)
for \( z \in P \) and \( s \in S \). Again, the choice of section \( \sigma : P \to H \) does not influence the definition of \( \zeta \). For a given homomorphism \( \eta : H \to \text{Aut}_K(S) \), the diagram (21) is thus commutative as soon as one has \( \eta_c(s) = \varepsilon_T(c) s \) for all \( c \in T \) and \( s \in S \). In other words, the torus \( T \) must act naturally through \( \varepsilon_T \), which is exactly what we have required of a Heisenberg module over \( H(\beta) \) in Definition [17].

**Definition 68.** Let \( T \) be a torus for \( K \) with torus map \( \varepsilon_T : T \to K^\times \), and fix a Heisenberg group \( H \in \text{Heis} \) with central extension (20). Then a **Heisenberg module** is a \( K \)-vector space \( S \) with a Heisenberg action \( \eta : H \to \text{Aut}_K(S) \), meaning the diagram (21) commutes.

Our new definition of Heisenberg module thus includes Heisenberg modules over \( H(\beta) \) in the previous sense. Heisenberg morphism over \( H \in \text{Heis} \) are defined as before to be equivariant maps \( \Phi : S \to S' \). In other words, we require \( \Phi : \eta_h = \eta_h' \circ \Phi \) for all \( h \in H \), where \( \eta : H \to \text{Aut}_K(S) \) and \( \eta' : H \to \text{Aut}_K(S') \) are the corresponding Heisenberg actions.

For generalizing the notion of Heisenberg algebra to arbitrary \( H \in \text{Heis} \), we need a bit more structure on the Heisenberg group \( H \). Recall that in the case \( H = H(\beta) \), we had to distinguish between the scalars \( TG \) and the operators \( \Gamma \). Since we do not require the phase space to split into \( G \) and \( \Gamma \), we should not insist on any particular decomposition into scalars and operators. We must therefore enrich the concept of Heisenberg group.

We have \( H(\beta) = TG \rtimes \Gamma \) and \( H(\beta^\top) = \Gamma \rtimes TG \), with the twist map given by \( (cx, \xi) \mapsto (\xi, \frac{c}{\langle x|\xi \rangle} x) \) in this setting. Pulling phase factors out, this is equivalent to \( J(x, \xi) = (\xi, x)/\langle x|\xi \rangle \). The product \( (x, c, \xi)(\tilde{x}, \tilde{c}, \tilde{\xi}) \) of both \( H(\beta) \) and \( H(\beta^\top) \) may be described on the set \( G \times T \times \Gamma \) as componentwise in the outer factors with extra factor \( \langle x|\xi \rangle \) and \( \langle x|\xi \rangle \), respectively. We have the following diagram:

\[
\begin{array}{ccc}
TG \rtimes \Gamma & \xrightarrow{J} & \Gamma \times TG \\
0 & \xrightarrow{} & 0 \\
\end{array}
\]

Furthermore, we have for the opposite algebra \( KH(\beta)^o = KH(\beta^\top) \). Hence we may identify right \( H(\beta^\top) \)-modules with left \( H(\beta) \)-modules.

The procedure is as follows: We fix a duality \( \beta : G \times \Gamma \) and call \( TG \) the **Heisenberg scalars** and \( \Gamma \) the **Heisenberg operators**. This refers both
to $H(\beta)$ and to $H(\beta^\top)$. Now we define a left/right Heisenberg algebra as a left/right $TG$-algebra with $\Gamma$ as its group of operators.

One checks that the twist $J_\beta: H(\beta) \to H(\beta^\top)$ induces a ring isomorphism $J_\beta: H_K(\beta) \to H_K(\beta^\top)$ given by $\lambda(x, \xi) \mapsto \lambda T\langle x | \xi \rangle^{-1}(-\xi, x)$ with inverse $\lambda(x, \xi) \mapsto \lambda T\langle x | \xi \rangle^{-1}(x, -\xi)$. Furthermore, we have an isomorphism $\tau: H_K(\beta^\top) \to H_K(\beta)^o$ given by $\lambda(x, \xi) \mapsto \lambda(x, \xi)$. We may thus identify $H_K(\beta^\top)$ with $H_K(\beta)^o$, preferring to keep the position-before-momentum order in $H(\beta) = TG \rtimes \Gamma$ everywhere and composing the twist map with $\tau$ to obtain the new twist $J_\beta: H_K(\beta) \to H_K(\beta)^o$, which is now an anti-homomorphism and in fact an involution. In other words, $H_K(\beta)$ is an involution algebra (also known as $*$-algebra) over the coefficient field $K$, endowed with the trivial involution. It is easy to see that its self-adjoint elements are given by $K G \leq H_K(\beta)$.

We think of the Fourier transform $F: S \to \Sigma$ as a Heisenberg morphism between a left $H(\beta)$-algebra $S$ and a right $H(\beta)$-algebra $\Sigma$. The composition $F_\Sigma \circ F_S = P$ should be understood as in the following diagram:

\[
\begin{array}{ccc}
\beta S & \xrightarrow{F_S} & \Sigma_\beta \\
\downarrow J^* & & \downarrow J^* \\
\beta \Sigma & \xrightarrow{F_\Sigma} & S_\beta \\
\downarrow J^* & & \downarrow J^* \\
\beta S & & \beta S
\end{array}
\]

Using moon-phase notation, the parity behavior is then deduced via $F_\Sigma F_S(h \cdot s) = F_\Sigma(F_S h \cdot s) = F_\Sigma F_S F_S h = h \cdot F_\Sigma F_S$. The point is that $H(\beta)$ is allowed to vary in the category of all Heisenberg algebras (left or right), but it is fixed within a Fourier singlet/doublet.

References to be included are [10], [30], [36], [35], [48].

Two subspaces are called transverse if they span the given vector space, and, following [6, p. 21], we define a Lagrangian bisection of a symplectic vector space $V$ as a pair of transverse Lagrangian subspaces $L, L'$; then automatically $L + L' = V$. Each Lagrangian bisection corresponds uniquely to a symplectic isomorphism $V \cong L \oplus L^*$. This was called “double polarization” earlier since this is more than a polarization but less than a choice of basis. The notion of Lagrangian bisection is akin to an orthogonal decomposition in a euclidean space. One should, however, note the following two crucial differences: (1) While the orthogonal complement of a subspace is always unique, there are in general many choices for symplectic complement (see the reference for “double polarization” given at another place). Hence one needs two
choices. (2) The two pieces are isomorphic to each other—at least in the vector space case: Choosing a basis in each, one may perform

From Wikipedia article “Symplectic Vector Space”: Formally, the symmetric algebra of \( V \) is the group algebra of the dual, \( \text{Sym}(V) := K[V^*] \), and the Weyl algebra is the group algebra of the (dual) Heisenberg group \( W(V) = K[H(V^*)] \). Since passing to group algebras is a contravariant functor, the central extension map \( H(V) \to V \) becomes an inclusion \( \text{Sym}(V) \to W(V) \).

5.1. **Generalized Heisenberg Groups.** Given any locally compact abelian group \( G \), its characters in the classical sense are the continuous homomorphisms from \( G \) into the complex torus \( \mathbb{T} := \mathbb{S}^1 \subset \mathbb{C} \). The collection \( \widehat{\Gamma} \) of all characters is then again an abelian group known as the dual group of \( G \), and the famous Pontryagin duality theorem asserts that the natural pairing \( \varpi : G \times \widehat{\Gamma} \to \mathbb{T} \) is a nondegenerate bicharacter; confer for example Theorem 1.7.2 of [45].

In our algebraic setup, we start from a duality \( \beta : G \times \Gamma \to T \), which we define as a nondegenerate bicharacter from the abelian groups \( G \) and \( \Gamma \) to an arbitrary torus \( T \) over a fixed ground field \( K \). By definition, a torus over \( K \) is a (multiplicatively written) abelian group \( T \) equipped with an action \( \ast : T \to \text{Aut}_K(K) \). In other words, we have the action laws \( 1 \ast \lambda = \lambda \) and \((cd) \ast \lambda = c \ast (d \ast \lambda)\) as well as the linearity laws \( c \ast (\lambda + \mu) = c \ast \lambda + c \ast \mu \) and \( c \ast (\lambda \mu) = (c \ast \lambda)\mu = \lambda(c \ast \mu) \), for all \( c,d \in T \) and \( \lambda, \mu \in K \). Equivalently, the action \( \ast \) may be described by the torus map \( \varepsilon_T : (T, \cdot) \to (K^\times, \cdot) \) with \( \varepsilon_T(c) := c \ast 1_K \) since we have \( c \ast \lambda = \varepsilon_T(c) \lambda \). We shall often suppress the action symbol \( \ast \), provided this does not give rise to confusion. Moreover, \( \beta(x, \xi) \) will usually be written as \( \langle x|\xi \rangle_\beta \), with the index \( \beta \) omitted when the context makes it clear.

In the classical case cited above, \( G \) and \( \Gamma := \widehat{G} \) as well as \( K = \mathbb{C} \) and \( T = \mathbb{T} \) all carry a topology, which we discard to focus on the algebraic and algorithmic aspects. Given any duality \( \beta : G \times \Gamma \to T \), we shall refer to the elements of \( G \) as positions and to those of \( \Gamma \) as momenta; this is motivated by the fundamental example of symplectic duality (Example 73 below). Due to the nondegeneracy of \( \beta \), we can identify positions \( x \in G \) with their position characters \( \langle x|\cdot \rangle : \Gamma \to T \) given by \( \xi \mapsto \langle x|\xi \rangle \), momenta \( \xi \in \Gamma \) with their momentum characters \( \langle \cdot|\xi \rangle : G \to T \), \( x \mapsto \langle x|\xi \rangle \). Since \( G \) and \( \Gamma \) do not have any specific topology, one may view them as discrete groups. Consequently, their dual groups \( \widehat{G} = \text{Hom}(G, T) \) and \( \widehat{\Gamma} = \text{Hom}(\Gamma, T) \) consist of all homomorphisms, and nondegeneracy means one has embeddings \( G \hookrightarrow \widehat{G} \) and \( \Gamma \hookrightarrow \widehat{\Gamma} \) for encoding positions and momenta by their characters.
Example 69. One of the most famous examples in the classical setting of $K = \mathbb{C}$ and $T = \mathbb{T}$ is given by the duality $T \times \mathbb{Z} \to \mathbb{T}$ defined by $\langle x | \xi \rangle := x^\xi$. It is clear that this may be extended to $\mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{T}$ with $\langle x | \xi \rangle := x_1^{\xi_1} \cdots x_n^{\xi_n}$. We refer to this example as the classical torus duality $\langle \mathbb{T}^n | \mathbb{Z}^n \rangle$. The same term will also be used for its conjugate duality $\langle \mathbb{Z}^n | \mathbb{T}^n \rangle$.

Example 70. The most important special case of Example 69 is given by the complex field $K = \mathbb{C}$ with the classical torus $T = \mathbb{T}$ and vector spaces $V = V' = \mathbb{R}^n$ having their canonical Euclidean structure. Since the latter is also equivalent to the natural paring $\mathbb{R}^n \times \mathbb{R}_n \to \mathbb{R}$, we shall use the dot notation for both.

Consider $K = \mathbb{C}$, torus $T = \mathbb{T}$ and vector spaces $V = V' = \mathbb{R}^n$. Via the standard character $\chi(c) = e^{irc}$, one obtains the bicharacter $\langle x | \xi \rangle = e^{ix \cdot \xi}$. We shall refer to this example as the standard vector duality $\langle \mathbb{R}^n | \mathbb{R}^n \rangle$, which may also be presented as $\langle \mathbb{R}^n | \mathbb{R}_n \rangle$ under the natural pairing. Note that here the underlying scalar field $F = \mathbb{R}$ of the vector spaces is distinct from the ground field $K = \mathbb{C}$ of the duality.

Example 71. If $V = F^n$ is a vector space over a Galois field $F = GF(q)$ with $q = p^m$ ($m \in \mathbb{N}$) elements, we may again use the usual dot product as a bilinear form. Using the multiplicative cyclic group $\langle \zeta_p \rangle \subset \mathbb{Q}(\zeta_p)$ as in the proof of Lemma 64, a standard character $\chi_a : GF(q) \to \langle \zeta_p \rangle$ is given by $c \mapsto c^{aq}$, with $a \in \mathbb{Z}_p^\times$ chosen arbitrarily and tr: $GF(q) \to \mathbb{Z}_p$ the trace map tr $c := c + c^p + \cdots + c^{p^m-1}$; for details regarding characters on Galois fields see [7]. With the induced duality $\langle x | \xi \rangle = \chi_a(x \cdot \xi)$, we call this example the modular vector duality $\langle GF(q)^n | GF(q)^n \rangle$.

As is well known, the torus duality gives rise to Fourier series and the vector duality to Fourier integrals. There is another well-known breed of Fourier transforms, known as the discrete Fourier transform; it is based on the following important duality.

Example 72. Given any $N \in \mathbb{N}$, let us introduce the following two concrete realizations of the cyclic group of order $N$. The first is the common representation $\mathbb{Z}_N := \{0, 1, \ldots, N-1\}$ with addition modulo $N$, the second is $\mathbb{T}_N := N^{-1}\mathbb{Z}_N = \{0, \frac{1}{N}, \ldots, \frac{N-1}{N}\}$ with addition modulo 1. The duality $\mathbb{T}_N^n \times \mathbb{Z}_N^n \to \mathbb{C}$ is now defined by $\langle x | \xi \rangle := e^{i\xi \cdot \xi}$. This can be obtained from the classical torus duality in two steps: First one restricts to the bicharacter $\langle | \rangle : \mathbb{T}_N^n \times \mathbb{Z}_N^n \to \mathbb{C}$ via the embedding $\mathbb{T}_N^n \hookrightarrow \mathbb{T}^n$, $\xi \mapsto e^{i\xi}$. Then one applies the usual universal construction [28, §I.9] to make the bicharacter into a duality by taking the quotient on the right modulo the right kernel $(N \mathbb{Z})^n$; nothing is
needed on the left since the left kernel is trivial. The resulting duality will be called the cyclic duality \( \langle \mathbb{T}^n_N \mid \mathbb{Z}^n_N \rangle \).

Of course, one may equally well form the conjugate duality \( \langle \mathbb{Z}^n_N \mid \mathbb{T}^n_N \rangle \). But unlike their infinite relatives, the dualities \( \langle \mathbb{T}^n_N \mid \mathbb{Z}^n_N \rangle \) and \( \langle \mathbb{Z}^n_N \mid \mathbb{T}^n_N \rangle \) are not only similar but actually the same (i.e. isomorphic): Since the cyclic groups \( \mathbb{Z}^n_N \) and \( \mathbb{T}^n_N \) are the same abstract group \( \mathbb{Z}/N \), both are one and the same duality \( \langle (\mathbb{Z}/N)^n \mid (\mathbb{Z}/N)^n \rangle \), given \[13\], Thm. 4.5d] by

\[
\langle k + N\mathbb{Z}^n \mid l + N\mathbb{Z}^n \rangle = e^{i\tau(k\cdot l)/N}
\]

for \( k, l \in \mathbb{Z}^n \). Nevertheless, it can be worthwhile to distinguish the two realizations of this duality.

The different nature of \( \mathbb{T}^n_N \) and \( \mathbb{Z}^n_N \) can also be seen in the context of normalizing Haar measure \( \mu \). While such a choice is per se immaterial, it must be consistent between the primal and dual group for the inversion theorem to hold [45, §1.5.1]. For compact groups \( G \), the canonical choice is to set \( \mu(G) = 1 \), while for discrete groups \( G \) one sets \( \mu(\{x\}) = 1 \) for all points \( x \in G \). But since \( (\mathbb{Z}/N)^n \) happens to be both discrete and compact, one must decide whether to impose the discrete or the compact normalization on the primal group \( (\mathbb{Z}/N)^n \), so that the other normalization is then conferred onto its dual, which is again \( (\mathbb{Z}/N)^n \). From the above construction, it is clear that the natural choice is to endow \( \mathbb{T}^n_N \) with the compact and \( \mathbb{Z}^n_N \) with the discrete normalization. In other words, we have \( \mu(\mathbb{T}^n_N) = 1 \) and \( \mu(\mathbb{Z}^n_N) = N^n \), thus also \( \mu(\{x\}) = 1/N^n \) for \( x \in \mathbb{T}^n_N \) but \( \mu(\{x\}) = 1 \) for \( x \in \mathbb{Z}^n_N \).

As is well-known, the classical definition [16, Ex. 14.6], [19], [51, §13], [46, §1.1] of the Heisenberg group is as a matrix Lie group

\[
H_n(K) = \left\{ \begin{pmatrix} 1 & \xi & c \\ 0 & I_n & x \\ 0 & 0 & 1 \end{pmatrix} \mid \xi \in K_n, x \in K^n, c \in K \right\},
\]

where the underlying field \( K \) is usually either \( \mathbb{R} \) or \( \mathbb{C} \). It is also customary to cast (23) in terms of symplectic vector spaces (see Example 73). What is not so well-known is that this setup can be generalized to arbitrary LCA groups under Pontryagin duality \( \varpi \). This has been exhibited forcefully by André Weil in his lucid article [50].

One may interpret (4) as a schematic matrix group like (23), but now with \( x \in G, \xi \in \Gamma \) and \( c \in T \). Here one should keep in mind that the addition of the upper right matrix elements corresponds to the (multiplicatively written!) group operation in \( T \). Furthermore, one
agrees that a left matrix element $\xi$ multiplies with a right matrix element $x$ to yield $\langle x|\xi \rangle$; all other products are trivial since they involve 0 or 1.

For the standard vector duality $\langle \mathbb{R}^n|\mathbb{R}^n \rangle$ of Example 70, the link to the matrix group (23) can be made more precise by reframing the Heisenberg group in terms of a symplectic vector space $V$. This is the formulation commonly used in more advanced treatments of the Heisenberg group [9, §5.1], [14], [29].

Example 73. Let us take $G = V$ and $\Gamma = V^*$ for a vector space and its dual over a field $F$, writing the natural pairing as $\langle \rangle : V \times V^* \rightarrow F$. Then $Z := V \oplus V^*$ is a symplectic vector space under the canonical symplectic form $\omega: Z \times Z \rightarrow F$ with $\omega(x,\xi; \tilde{x},\tilde{\xi}) = (\tilde{x}|\xi) - (x|\tilde{\xi})$. Using Lemma 64, we define the duality $\langle x|\xi \rangle := \chi(x|\xi)$. We shall refer to this as the symplectic duality $\langle V|V^* \rangle$. While such dualities are associated to the Hamiltonian phase space $T^*V = V \oplus V^* = Z$, the abstract vector dualities $\langle V|V \rangle$ of Example 67 are linked to the Lagrangian phase space $TV = V \oplus V$. In both cases, there is an evident physical interpretation [33, §1.1], where elements of $V$ denote positions while its tangent/cotangent vectors are the corresponding velocities/momenta (this generalizes to the nonlinear case where the configuration space is a manifold $M$ rather than a vector space $V$, with Lagrangian phase space $TM$ and Hamiltonian phase space $T^*M$ enjoying the same interpretation). The Hamiltonian case is the important since it naturally leads to quantization, replacing the commutative algebra of classical observables $C^\infty(T^*V)$ by the noncommutative algebra $\mathcal{H}(L^2(V))$ of self-adjoint (alias “Hermitian”) operators on the Hilbert space $L^2(V)$. The observables position and momentum are then quantized [16, §3.5] into the position operator $f(x) \mapsto x f(x)$ and its canonically conjugate momentum operator $f(x) \mapsto \partial f/\partial x)$. This is intimately linked to the unitary irrep of the Heisenberg group (see below at the end of the example).

There are two flavors of Heisenberg group on the symplectic vector space $Z$, depending on whether or not one regards the (double) polarization $Z = V \oplus V^*$ as part of the given data. In general, a polarization of a symplectic vector space $(Z,\omega)$ is a choice of Lagrangian subspace $V \leq Z$. Such subspaces $V$ exist in abundance—they comprise the so-called Lagrange-Grassmann manifold—and even for fixed $V$, there are plenty [29, Prop. A6.1.6] of Lagrangian complements $V'$, each of which may be identified with the dual space $V^*$. For any Lagrangian decomposition $Z = V \oplus V'$, the bases of $V$ are in bijective correspondence with the symplectic bases of $Z$. 
(a) So if one does have the double polarization $Z = V \oplus V^*$, one may define the \textit{polarized symplectic Heisenberg group} $H(V, V^*) := K \times Z$ with multiplication given by

$$(u, x, \xi) \cdot (u', x', \xi') := (u + u' + (x'|\xi), x + x', \xi + \xi').$$

In the standard case $V = \mathbb{R}^n$, this is the definition of [14, p. 19], where $H(V, V^*)$ is written $H^{\text{pol}}_n$; this yields the matrix Lie group (23).

In general, $\pi := \chi \times 1_Z$ is an epimorphism $H(V, V^*) \to H(\beta)$, $(u, x, \xi) \mapsto \chi(u)(x, \xi)$ with $\ker(\pi) = \ker(\chi) \times Z$. Hence the Heisenberg group of Definition [2] is essentially $H(V, V^*)$, modulo the prime ring. Some sources [1, Exc. 5.1-4] define the Heisenberg group as $H(\beta)$, but restricted to the symplectic duality $\beta = \chi \circ (\cdot | \cdot)$.

(b) Using only the symplectic structure $(Z, \omega)$, one is led to the \textit{apolar symplectic Heisenberg group} $H(Z, \omega)$; confer for example [47, (8)].

As a set, we have again $H(Z, \omega) := K \times Z$ but with the group law

$$(t, z) \cdot (t', z') := (t + t' + \frac{1}{2} \omega(z, z'), z + z').$$

In the standard case $V = \mathbb{R}^n$, this coincides with [14, p. 19], where the notation $H_n$ is employed for $H(Z, \omega)$. Here the conventional factor of $1/2$ is motivated by exponentiating the canonical commutators of Hamiltonian mechanics [14, (1.15)]. Given an arbitrary Lagrangian decomposition $Z = V \oplus V^*$, one obtains the isomorphism $\Psi: H(Z, \omega) \cong H(V, V^*)$ defined by $\Psi(t, x, \xi) := (t + \frac{1}{2} (x|\xi), x, \xi)$.

Thus the polarized and the symplectic Heisenberg group are the same—provided we have picked a polarization. For our definition of the abstract Heisenberg group $H(\beta)$, this is the case and will be of importance for our further development (confer Definition [17]). In the special case of the symplectic duality $(V|V^*)$, one may thus refer to $H(\beta)$ as the “symplectic Heisenberg group”.

The symplectic Heisenberg group has two important representations: There is (up to isomorphism) just one

\begin{remark}
Since the polarized symplectic Heisenberg group $H(V, V^*)$ may be viewed as the additive counterpart of $H(\beta)$, one might ask for an apolar variant $H_\Omega(\beta)$ of the latter having as its additive counterpart the apolar symplectic Heisenberg group $H(Z, \omega)$. Since we do not need this $H_\Omega(\beta)$ for our treatment of Heisenberg modules, we will only mention it briefly here.

We define $H_\Omega(\beta) := T \times Z$ as a set and endow it with the group law $(c, z) \cdot (\tilde{c}, \tilde{z}) := (c \tilde{c} \Omega(z, \tilde{x})^{1/2}, z + \tilde{z})$. Here $\Omega^{1/2} : Z \times Z \to T$ is the
\end{remark}
multiplicative symplectic form with \( \Omega(x, \xi; \tilde{x}, \tilde{\xi})^{1/2} := \langle \tilde{x} | \xi \rangle^{1/2} / \langle x | \tilde{\xi} \rangle^{1/2} \)
where \( \langle | \rangle^{1/2} := \chi(\underline{\underline{\chi}}) \): \( Z \to T \) is the “square root duality”. As a consequence, we have \( \chi(\omega/2) = \Omega^{1/2} \) as the square root of the given symplectic form. Note that in the classical setting \( K = \mathbb{C}, T = S^1 \subset \mathbb{C} \) with \( \chi(x) = e^{i\tau \xi} \), the determination of the square root corresponds to choosing the principal branch.

We have again an epimorphism \( \pi_\Omega = \chi \times 1_Z : H(Z, \omega) \to H_\Omega(\beta) \), and the isomorphism \( \Psi : H(Z, \omega) \to H(V, V^*) \) introduced in Example 73 lies above the “multiplicative” isomorphism \( \Psi_\Omega : H_\Omega(\beta) \to H(\beta) \) given by \( \Psi_\Omega(c, x, \xi) := (\langle x | \xi \rangle^{1/2} c, x, \xi) \), in the sense that \( \pi_\Psi = \Psi_\Omega \pi_\Omega \). Despite these analogies, there is a crucial difference between the Heisenberg group \( H(\beta) \) of Definition \[2\] and its unpolarized variant \( H_\Omega(\beta) \) in that the former is a semidirect product while the latter is not.

The Heisenberg group over the standard vector duality \( \langle \mathbb{R}^n | \mathbb{R}_n \rangle \) could be called the \textit{kinematic Heisenberg group} since it underpins physical kinematics, both classical (Hamilton’s equations) and quantum (Heisenberg equation). Indeed, the Schrödinger representation \( \rho_h : H_n \to \mathcal{H}(L^2(\mathbb{R}^n)) \) yields the latter; it is parametrized by the Planck constant \( h \) whose so-called semi-classical limit \( h \to 0 \) leads to Hamiltonian mechanics. See also [19], [20] for more on the representation theory of \( H_n \).

If one dislikes the idea of constants tending to zero (though one might interpret this as taking place in a hypothetical sequence of universes with progressively less significant quantum effects), the framework of Plain Mechanics (p-mechanics) offers an alternative viewpoint [22]: Before specializing to any quantum or classical (or hyperbolic quantum) version, physical systems are described in the “plain” setting of the Heisenberg group \( H_n \), using \( (L^1(H_n), *) \) as the algebra of observables (including in particular the Hamiltonian). A so-called universal equation rules the evolution of the system, which transforms to the Heisenberg equation under the representation \( \rho_h \) and to Hamilton’s equations under \( \rho_0 \). In this framework, one may develop corresponding brackets [23], notions of state [24] and a detailed mechanical theory [25]. In a newer presentation [26, IV.1], p-mechanics is treated in a more geometric context where the elliptic / parabolic / hyperbolic cases correspond to different number rings (complex / dual / double numbers), which in turn lead to different physical frameworks (quantum / classic / hyperbolic quantum).

It would be nice to reformulate the last results in term of suitable tori. For example, the classical case should correspond to the torus \( T_\varepsilon := \{1 + b_\varepsilon \mid b \in \mathbb{R} \} \) of the dual numbers \( \mathbb{R}_\varepsilon := \mathbb{R}[\varepsilon \mid \varepsilon^2 = 0] \).
with the duality $\langle \mathbb{R}^n | \mathbb{R}^m \rangle_\varepsilon$ given by $\langle x | \xi \rangle_\varepsilon := 1 + \varepsilon (x \cdot \xi)$. Its $L^2$ representation (encoded in the $L^2$ Fourier doublet) should then yield the classical Hamilton’s equations just as the standard duality yields the Heisenberg equation. In an even more ambitious enterprise, one might try to develop a general theory of “universal equations” for a given duality that yields classical and quantum kinematics as special cases. For this purpose, some tools developed in [2], [3] may be of help.

**Example 75.** For another instance of the construction $R_* R_+ \times R_+$, take the ring $R = \mathbb{Z}_2$. Hence we consider the duality $\beta: \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ defined by $\beta(m, n) = mn$. Conforming to our conventions, we write the torus multiplicatively via $\mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, \([c] \mapsto (−1)^c\), meaning $[0] \leftrightarrow +1$, $[1] \leftrightarrow −1$. As usual, we often express the values $±1$ just by the sign. The duality is then given by $\beta(m, n) = (−1)^{mn}$. We will show that $H(\beta)$ is the dihedral group $D_4$, the symmetry group of the square (which we assume centered in the origin with axis-parallel sides). If $t$ denotes the counter-clockwise $90^\circ$ turn and $r$ the reflection in the vertical axis, we obtain the presentation

$$D_4 = \langle t, r \mid t^4 = r^2 = 1, rt = t^3 r \rangle = \{1, t, t^2, r, tr, t^2 r, t^3 r\},$$

Where $tr, t^2 r, t^3 r$ may be respectively interpreted as reflections in the anti-diagonal $x + y = 0$, horizontal $y = 0$, and diagonal $x − y = 0$. We choose $Z(D_4) = \{1, t^2\}$ as our torus $T = \mathbb{Z}_2$, which enforces the identification $1 \leftrightarrow +1$, $t^2 \leftrightarrow −1$. For the position group $G = \mathbb{Z}_2$ we take $\{1, r\}$, leading to the identification $1 \leftrightarrow [0]$, $r \leftrightarrow [1]$; for the momentum group $\Gamma = \mathbb{Z}_2$ we use $\{1, tr\}$ with identification $1 \leftrightarrow [0]$, $tr \leftrightarrow [1]$. In these terms, the duality $\beta: \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$, $(m, n) \mapsto mn$ sends $(r, tr)$ to $−1$ and all other pairs to $+1$.

| $D_4$ | $H(\beta)$ | $D_4$ | $H(\beta)$ |
|-------|-------------|-------|-------------|
| 1     | +00         | $r$   | +10         |
| $t$   | −11         | $tr$  | +01         |
| $t^2$ | −00         | $t^2 r$ | −10         |
| $t^3$ | +11         | $t^3 r$ | −01         |

For defining a group isomorphism $D_4 \cong H(\beta)$, we construct first the unique homomorphism on the free group with $1 \mapsto +00$, $t \mapsto −11$, $r \mapsto +10$. As one sees immediately, this homomorphism annihilates the relators $t^4, r^2, trtr$ and thus yields a homomorphism $\iota: D_4 \to H(\beta)$. Computing all other elements in terms of the generators $t$ and $r$, one will verify that $\iota$ is given as in the table above. Since this is obviously a bijection, it provides us with the required isomorphism $D_4 \cong H(\beta)$. //
5.2. The Heisenberg Twist. In Example 73 we have seen how the symplectic dualities \( \langle V | V^* \rangle \) arise naturally on any given vector space (relative to a standard character). As usual, the symplectic structure \( \omega: Z \times Z \rightarrow F \) induces a map \( j: Z \rightarrow Z^* \), given by \( j(x, \xi) = (-\xi, x) \) in the case \( Z = V \oplus V^* \). Obviously, \( j^2: Z \rightarrow Z \) is the involution \( (x, \xi) \mapsto -(x, \xi) \), so we have in particular \( j^4 = 1_Z \).

**Remark 76.** In case \( F = \mathbb{R} \), the map \( j \) is akin to a complex structure, except that the latter maps a vector space to itself rather than to its dual. Indeed, the standard symplectic form on the Hamiltonian phase space \( Z = T^* V = V \oplus V^* \) is *formally* identical to the standard complex structure on the Lagrangian phase space \( L = TV = V \oplus V \), namely the analogous map \( j: L \rightarrow L \) with \( j(x, y) = (-y, x) \).

We will now forge the map \( j \) into an isomorphism between Heisenberg groups. So let \( \beta: G \times \Gamma \rightarrow T \) be a duality. Generalizing the case of symplectic vector spaces, we define first

\[
(24) \quad j: \quad G \times \Gamma \rightarrow G \times G, \quad (x, \xi) \mapsto (-\xi, x).
\]

Writing \( \beta^\top: G \times \Gamma \rightarrow T \) for the *transposed duality* \( \beta^\top(x, \xi) := \beta(x, \xi) \), we want to obtain a group isomorphism \( J: H(\beta) \rightarrow H(\beta^\top) \). Since phase factors are only added for capturing the modulation twists, we expect \( J \) to act trivially on \( T \); thus we are only concerned with the action of \( J \) on \( G \times \Gamma \cong 1 \times (G \times \Gamma) \). Assuming further that \( J \) coincides with \( j \) on \( G \cong G \times 0 \) and \( \Gamma \cong 0 \times \Gamma \), we obtain

\[
J(x, \xi) = J((x, 0)(0, \xi)) = J(x, 0) J(0, \xi) = (0, x)(-\xi, 0) = \langle x | -\xi \rangle (-\xi, x) = j(x, \xi)/\langle x | \xi \rangle
\]

as our definition of the Heisenberg twist \( J: H(\beta) \rightarrow H(\beta^\top) \). More explicitly, this yields now the map \( J: T \times G \times \Gamma \rightarrow T \times G \times G \) defined by \( c(x, \xi) \mapsto c/\langle x | \xi \rangle (-\xi, x) \). This shows that the passage from the abelian groups \( G \times \Gamma \) and \( \Gamma \times G \) to the nonabelian groups \( H(\beta) \) and \( H(\beta^\top) \) just imposes the additional phase factor \( \langle x | \xi \rangle^{-1} \). It is easy to check that \( J: H(\beta) \rightarrow H(\beta^\top) \) is indeed an isomorphism of groups and that \( J^2: H(\beta) \rightarrow H(\beta) \) is again an involution, namely \( J^2 = 1 \times j^2 \).

**Remark 77.** The inverted twist map \( J^{-1}: H(\beta^\top) \rightarrow H(\beta) \) is given by \( c(\xi, x) \mapsto c/\langle x | \xi \rangle (x, -\xi) \); it should not be confused with the twist map \( H(\beta^\top) \rightarrow H(\beta) \) intrinsic to the transposed Heisenberg group, acting via \( c(\xi, x) \mapsto c/\langle x | \xi \rangle (-x, \xi) \). Indexing the twist map to its duality, this shows that \( J^{-1}_\beta \neq J^\top_\beta \) while we have of course \( J^2_\beta = J^{2\top}_\beta =: J^2 \) and in fact \( J^{-1}_\beta = J^\top_\beta \circ J^2 \). In the sequel, it will always be clear from the context which duality is being referred to; hence we shall suppress the
index on $J$. If there is any danger of confusion, the duality in question will be specified *expressis verbis*.

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