Statistical Power-Law Spectra due to Reservoir Fluctuations

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LHC ALICE data are interpreted in terms of statistical power-law tailed \( p_T \) spectra. As explanation we derive such statistical distributions for particular particle number fluctuation patterns in a finite heat bath exactly, and for general thermodynamical systems in the subleading canonical expansion approximately. Our general result, \( q = 1 - 1/C + \Delta T^2/T^2 \), demonstrates how the heat capacity and the temperature fluctuation effects compete, and cancel only in the standard Gaussian approximation.

I. INTRODUCTION

Power-law tailed distributions occur in Nature numerous. The idea of a statistical – thermodynamical origin of these emerged already decades ago \([1,2]\). We have, however, long missed a "naturalness" argument connecting the basic principles of classical thermodynamics to the use of non-extensive entropy formulas by deriving canonical distributions of the one-particle energy. Although the observation has been made that the Tsallis and Rényi entropy formulas both lead to the cut power-law canonical distribution, and their use requires a constant heat capacity reservoir \([3]\), the \( q > 1 \) power-laws – featuring a negative power of a quantity larger than one – still seem unnatural.

In recent studies of ideal gases \([4,6]\) we investigated energy fluctuations in a subsystem – reservoir couple. They lead to Tsallis distribution with \( q = 1 - 1/C \) for ideal gas reservoirs, with \( C \) being the heat capacity of the total system.

Moreover, particle number fluctuations in the reservoir, either achieved naturally in a huge, inhomogenous heat bath or artificially by averaging the statistics over repeated events in high-energy experiments, lead to further effects \([7–10]\). We review in this paper how ideal fermionic and bosonic distributions, with binomial (BD) and negative binomial (NBD) distributions of the particle number, lead exactly to Tsallis power-law behavior with the parameters \( T = E/(n\bar{n}) \) and \( q = (n(n-1))/(n)^2 \), when the microcanonical ideal gas statistical factor, \( (1 - \omega/E)^n \) in one dimension for massless partons, \([13]\) is averaged over one of these distributions. The above \( q \), named as second factorial moment, \( F_2 \), was determined with respect to canonical suppression in Refs. \([11,12]\). For the binomial distribution one gets \( q = 1 - 1/k \), for the negative binomial \( q = 1 + 1/(k - 1) \).

We demonstrate by fits to recent ALICE data taken in LHC experiment \([13]\) that in the \( p_T \)-distribution of charged hadrons (dominated by pions) two Tsallis distributions emerge for the one-particle energy in a moving system, \( \omega = \gamma (m_T - v p_T) \) (with \( \gamma = 1/\sqrt{1-v^2} \) being the Lorentz factor and \( v \) a radial blast wave velocity, \( m_T = \sqrt{m^2 + p_T^2} \approx |p_T| \) the so called transverse mass). The softer parts, below \( p_T \approx 4 \text{ GeV/c} \), show a dependence on the participant number as expected from statistical considerations: bigger systems come closer to the Boltzmann–Gibbs prediction.

Our theoretical results on \( q \) and \( T \) expressed by the mean multiplicity and its variance in the reservoir for BD and NBD distributions also can be viewed as an approximation for arbitrary particle number distributions in the reservoir up to subleading (second) order in the canonical expansion \( \omega \ll E \). For non-ideal systems the general expansion up to second order delivers \( q = 1 - 1/C + \Delta T^2/T^2 \), a combined result with the heat capacity and the variance of the temperature of finite heat bath. These quantities seem to act against each other. Here the variance of the temperature is meant for the estimator \( 1/S'(E) \) of the thermodynamical temperature, the latter defined by \( 1/T = \langle S'(E) \rangle \). This way in the Gaussian approximation \( \Delta T^2/T^2 = 1/\sqrt{C} \) we regain \( q = 1 \) and verify the Boltzmann–Gibbs statistical factor. Part of this result has been derived and promoted by G. Wilk and Z. Wlodarczyk \((q = 1 + \Delta T^2/T^2)\) in recent years \([14–16]\). Instead of temperature fluctuations reservoir volume and particle number fluctuations were considered in recent publications \([5,10,17,18]\).

II. \( p_T \) SPECTRA AT THE LHC

In high-energy physics the power-law tail in \( p_T \) spectra is traditionally fitted by cut power-laws, \((1 + a p_T)^{-b}\), conjectured to stem from the behavior of hadronization matrix elements. As a matter of fact, a statistical model also can be applied to the fragmentation functions which describe the yield of hadrons stemming from high-energy particle jets \([13,20]\). The real unknown is the soft part, with low \( p_T \) momenta; here thermal models are more fashionable.

It is therefore an intriguing question to decide whether there is a soft power-law, which can be naturally described and understood only by statistical phase-space considerations. The idea of a cut power-law as a thermal distribution, a characteristic consequence also from non-extensive thermodynamics, has been pursued by us since several years \([21,22]\). It is now for the first time that particle spectra over a wide \( p_T \) range are presented differentially for centrality classes \([13]\); such a presentation may inform about the multiplicity dependence of a heat reservoir in terms of thermal models.

In Fig. \([1]\) we display our fits to \( p_T \) spectra of charged hadrons in centrality classes. A break in the spectra is pronounced at high centralities (large participant numbers, \( N_{\text{part}} \), which must be positively correlated with the particle number in the fireball where the hadrons were born. Our fits have the lowest \( \chi^2 \) by making the soft–hard change around \( p_T \approx 4 \text{ GeV/c} \) for all centrality classes, therefore we think it...
is justified to talk about soft and hard power-laws separately.

The fit parameter $b$, connected to the parameter $q$ in Tsallis distribution as $b = 1/(q - 1)$, is plotted against $N_{\text{part}}$ in Fig. 2. The soft part shows a clear rising of the power $b$ with $N_{\text{part}}$, very characteristic to a statistical – thermal origin of a power-law. Contrary to this is the behavior of the hard spectra: the fitted power stays constant irrespective to the centrality, conjectured to vary with the size of the thermal bath. This is ‘naturally’ expected from QCD.

III. TEMPERATURE AND ENERGY FLUCTUATIONS

In this Section we turn to the theory of statistical power-law tailed distributions as canonical distributions in a thermal system connected to a heat reservoir with finite heat capacity. By fluctuation of temperature we mean the fluctuation of the estimator $1/S'(E)$ due to fluctuations of the energy $E$ in the reservoir. We are interested in the observable distribution of the one-particle energy, $\omega \ll E$, in the canonical limit.

Traditionally such thermodynamical fluctuations are treated in the Gaussian approximation. Based on the fundamental thermodynamic uncertainty relation, $\Delta E \cdot \Delta \beta = 1$ with $\beta = S'(E)$, it is easy to derive the characteristic scaled fluctuation of the temperature [23,24]. With any well peaked distribution of a random variable, $x$, the expectation value $a = \langle x \rangle$ is near to the value where the peak occurs. As a consequence the variance of any function, $f(x)$ in this approximation is related to the original variance by a Jacobi determinant: $\Delta f = |f'(a)| \Delta x$. Now we consider both $E$ and $\beta$ as functions of the temperature, $T$. We obtain $\Delta E = |C| \Delta T$ with $C = dE/dT$ being the definition of heat capacity, and $\Delta \beta = \Delta T/T^2$. Combining these two results one arrives at the classical formula $\Delta T/T = 1/\sqrt{|C|}$. The heat capacity $C$ is proportional to the heat bath size (volume, number of degrees of freedom) for large extensive systems.

There are, however, some deficiences in the Gaussian approximation. A Gauss distribution of $\beta$, given as $\omega(\beta) \propto \exp(-C(T\beta - 1)^2/2)$, allows for a finite probability for negative temperatures, and – even worse – its characteristic function, $\exp(-\beta \omega) = \exp(-\omega/T + \omega^2/2CT^2)$ is not integrable in $\omega$.

The next theoretical question is how to improve the canonical scheme beyond the Gauss approximation. We start our discussion with ideal gases. The one-particle energy, $\omega$, out of total energy, $E$, is distributed according to a statistical weight factor $(1 - \omega/E)^n$ [24]. The idea of superstatistics in general considers a distribution for the reservoir parameters $n$ and $E$ [25,26]. In high-energy experiments $E$ is typically controlled by the accelerator and does not vary much. However, $n$, the number of particles in the produced fireball scatters appreciably, which can be uncovered via the event-by-event detection of the spectra in $\omega$, as suggested in [29].

In ideal reservoirs $n$ particles are distributed among $k$ phase-space cells: fermions $\binom{k}{n}$, bosons $\binom{n+k}{n}$ ways. The binomial and negative binomial distributions can be derived by considering a subspace $(n,k)$ out of $(N,K)$ in the limit $K \to \infty$ and $N \to \infty$ while $f = N/K$ is fixed.

$$F_{n,k}(f) := \lim_{K \to \infty} \frac{\binom{k}{n} \binom{K-k}{N-n}}{\binom{K}{N}} = \binom{k}{n} f^n (1 - f)^{k-n}. \quad (1)$$
\[ B_{n,k}(f) := \lim_{K \to \infty} \left( \frac{n}{N-n} \right)^k \frac{n^{N-n}}{N^{N-n}} \frac{(n+k)^k}{(N-K-k)^{N-K}} = \left( \frac{n+k}{n} \right) f^n (1 + f)^{-n-k-1}. \] (2)

These distributions are normalized based on the binomial expansion of \((a+b)^k\) and \((b-a)^{-k-1}\), respectively.

Assuming a typical fireball in high-energy experiments, \(E\) is fixed and \(n\) fluctuates according to NBD. The ideal gas statistical weight factor, describing the complement phase-space for reservoir configurations, becomes \(\omega_{\text{BD}}\):

\[ \sum_{n=0}^{\infty} \left( 1 - \frac{\omega}{E} \right)^n B_{n,k}(f) = \left( 1 + f \frac{\omega}{E} \right)^{-k}. \] (3)

Note that \(\langle n \rangle = \frac{(k+1)f}{1}\) for NBD. Then with \(T = E/\langle n \rangle\) and \(q = 1 + 1/(k+1)\) we get

\[ \left( 1 + (q-1) \frac{\omega}{T} \right)^{-\frac{1}{q-1}}. \] (4)

This is exactly the statistical weight factor which provides the \(q > 1\) Tsallis–Pareto distribution \(\omega_{\text{TBD}}\). Similarly in a fermionic reservoir \(E\) is fixed, \(n\) is distributed according to BD. We obtain

\[ \sum_{n=0}^{\infty} \left( 1 - \frac{\omega}{E} \right)^n B_{k}(f) = \left( 1 + f \frac{\omega}{E} \right)^k. \] (5)

Note that \(\langle n \rangle = kf\) for BD. Then with \(T = E/\langle n \rangle\) and \(q = 1 - 1/k\) we again get a Tsallis–Pareto distribution, but now with \(q < 1\). In the \(k \gg n\) limit (low occupancy in phase-space) the particle distribution in the reservoir becomes Poissonian in both cases. The result is exactly the Boltzmann–Gibbs weight factor with \(T = E/\langle n \rangle\):

\[ \sum_{n=0}^{\infty} \left( 1 - \frac{\omega}{E} \right)^n \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle} = e^{-\langle n \rangle} \frac{\omega}{E}. \] (6)

We note that NBD distributions are observed experimentally, a nice analysis of heavy ion data are given by the PHENIX group \([30]\). In all the three above cases

\[ T = \frac{E}{\langle n \rangle} \quad \text{and} \quad q = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2}. \] (7)

Now we turn to the ideal statistical weight factor with general finite reservoir fluctuations. In the canonical approach we expand for small \(\omega \ll E\) and view the Tsallis–Pareto distribution as an approximation:

\[ \left( 1 + (q-1) \frac{\omega}{T} \right)^{-\frac{1}{q-1}} = 1 - \frac{\omega}{T} + q \frac{\omega^2}{2T^2} - \ldots \] (8)

on the one hand and

\[ \left( 1 - \frac{\omega}{E} \right)^n = 1 - \langle n \rangle \frac{\omega}{E} + \langle n(n-1) \rangle \frac{\omega^2}{2E^2} - \ldots \] (9)

on the other hand. To match up to subleading canonical order, it follows in general:

\[ T = \frac{E}{\langle n \rangle} \quad \text{and} \quad q = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2}. \] (10)

Finally we consider a general system with general reservoir fluctuations. Expanding for small \(\omega \ll E\)

\[ \left\langle e^{S(E-\omega)-S(E)} \right\rangle_{\omega \ll E} = \left\langle e^{-\omega S'(E) + \omega^2 S''(E)/2 - \ldots} \right\rangle_{\omega \ll E} = 1 - \omega \left\langle S'(E) \right\rangle + \frac{\omega^2}{2} \left\langle S'(E)^2 + S''(E) \right\rangle - \ldots \] (11)

Compare this with the expansion eq.\([8]\) of the Tsallis distribution: In the view of the above we interpret the parameters as

\[ \frac{1}{T} = \left\langle S'(E) \right\rangle, \quad q = \frac{\left\langle S'(E)^2 + S''(E) \right\rangle}{\left\langle S'(E) \right\rangle^2}. \] (12)

Here \(\left\langle S''(E) \right\rangle = -1/CT^2\) follows from the definition of the heat capacity of the reservoir, \(1/C = dT/\langle E \rangle\). Summarizing these results we understand that the parameter \(q\) has opposite sign contributions from \(\left\langle S'(E)^2 \right\rangle - \left\langle S'(E) \right\rangle^2\) and from \(\left\langle S''(E) \right\rangle\). In general

\[ q = 1 + \frac{\Delta T^2}{T^2} - \frac{1}{C}. \] (13)

to subleading canonical order. With this formula \(q > 1\) and \(q < 1\) are both possible and for temperature fluctuations with Gaussian variance, \(\Delta T^2/T^2 = 1/\sqrt{C}\), one has \(q = 1\).

In summary we studied the mechanism behind the occurrence of statistical power-law distributions in high-energy particle creation, exploiting the role of reservoir fluctuations. First we demonstrated that ALICE charged hadron \(p_T\)-spectra feature soft, statistical Tsallis-distributions besides the traditionally known hard QCD based power-law. Observing that the power, \(b = 1/(q - 1)\), in our fits increases with increasing participant number, \(N_{\text{part}}\), we concluded that bigger fireballs come closer to the conventional thermal model exponential distribution \((q \to 1)\).

Seeking for a theoretical explanation we analyzed scenarios with ideal gas reservoirs formed at fix total energy, \(E\), but fluctuating particle number \(n\). We have clearly concluded that \(q > 1\) power-laws occur with the NBD distribution \((q = 1 + 1/(k+1))\) and for the Poissonian distribution of \(n\) exactly the Boltzmann–Gibbs exponential formula \((q = 1)\) follows.

For a general fluctuation pattern of \(n\) the Tsallis–Pareto form is only an approximation, but it goes beyond the traditional exponential. In general \(T = E/\langle n \rangle\) and \(q = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2}\). This interprets the parameter \(q\) as the scaled second factorial moment for ideal gas reservoirs with arbitrary particle number fluctuations.

Finally for non-ideal reservoirs, described by an equation of state, \(S(E)\), we also derived the meaning of the Tsallis parameters \(T\) and \(q\) (cf. eq.\([12]\)). This is one of the main novel
results of this paper.

Our formula demonstrates that in general heat capacity and temperature variance effects compete with each other; at exact balance the traditional Boltzmann–Gibbs thermodynamics is restored. In this case the scaled fluctuations follow the traditional inverse square root law.

As an outlook we shall consider that for cases when the entropy is non-extensive the concept of reservoir has to be investigated in more depth. First steps towards such an analysis included the construction of a deformed entropy formula without, however, discussing fluctuations in the reservoir \[31\]. The generalization of that procedure with reservoir fluctuations, as discussed in this paper, will be presented in a forthcoming publication. Since such an entropy concept also has to satisfy basic thermodynamical requirements for a general equilibrium state, a deformed entropy formula is not arbitrary \[32\].

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[33] This is \(\exp(S(E - \omega) - S(E))\), the complement phase-space factor for ideal gases. Since each exponential grows like \(x^n\), their ratio delivers the formula.
[34] This statistical weight factor is to be multiplied with a one-particle phase-space factor, \(\rho(\omega)\), which however does not depend on \(n\) and \(E\).
[35] One has to note that in fact zero particles in the reservoir are unrealistic. Partial sums starting not at \(n = 0\) can be obtained by derivations with respect to \(\omega\).
[36] Unfortunately different conventions are in use for the parameter \(q\), some papers in fact apply \(2 - q\) at the same place.