Harmonic measures in embedded foliated manifolds

Pedro J. Catuogno\(^1\)  Diego S. Ledesma\(^2\)
Paulo R. Ruffino\(^3\)

Departamento de Matemática, Universidade Estadual de Campinas,
13.083-859- Campinas - SP, Brazil.

Abstract

We study harmonic and totally invariant measures in a foliated compact Riemannian manifold isometrically embedded in an Euclidean space. We introduce geometrical techniques for stochastic calculus in this space. In particular, using these techniques we can construct explicitly an Stratonovich equation for the foliated Brownian motion (cf. L. Garnett \cite{11} and others). We present a characterization of totally invariant measures in terms of the flow of diffeomorphisms of associated to this equation. We prove an ergodic formula for the sum of the Lyapunov exponents in terms of the geometry of the leaves.

Key words: foliated manifold, Brownian motion, stochastic flows of diffeomorphisms.

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1 Introduction

The main topic of this article is to study harmonic and totally invariant measures in a foliated compact Riemannian manifold \(M\) isometrically embedded in an Euclidean space. Our technique offers some tools for the geometrical analysis of stochastic processes in \(M\), in particular, it allows one to construct explicitly an Stratonovich equation for the foliated Brownian motion (as introduced in the literature by L. Garnett \cite{11}, see also \cite{4}, \cite{6} and references therein).

Next section introduces the geometrical background, in particular we present the tension \(\kappa\) in the tangent of the leaves which measures the difference between the divergent operator in \(M\) and the divergent operator in the leaves. Among others interesting properties, we prove that \(\kappa\) is related to the Gobillon-Vey class of the foliated space (Proposition \(\square\)). In Section 3 we put these geometrical tools to play with invariant measures: harmonic, totally invariant and holonomy invariant measures (when they exist). We present a characterization of totally invariant measures in terms of the flow of diffeomorphisms of our foliated Brownian system (Theorem \(\square\)), in the proof we use currents. Ergodic properties appears in the last section, where the main result is an ergodic formula for the sum of the Lyapunov exponents in terms of the geometry of the leaves: the tensor \(\kappa\) and the mean curvature \(H\) (Theorem \(\square\)).

\(^1\)E-mail: pedrojc@ime.unicamp.br. Research partially supported by CNPq 302.704/2008-6, 480.271/2009-7 and FAPESP 07/06896-5.
\(^2\)E-mail: Research supported by CNPQ, grant no. 142655/2005-8
\(^3\)Corresponding author, e-mail: ruffino@ime.unicamp.br. Research partially supported by CNPq 306.264/2009-9, 480.271/2009-7 and FAPESP 07/06896-5.
2 Foliated Geometry

We fix \((M, \mathcal{F}, g)\) a foliated Riemannian manifold without boundary. The foliation \(\mathcal{F}\) is given by the integrable subbundle \(E\) of tangent vectors to \(\mathcal{F}\).

We are interested in describing the foliated operators over \((M, \mathcal{F}, g)\) in terms of an isometric embedding of the Riemannian manifold \((M, g)\) in an Euclidean space \(\mathbb{R}^N\) as guaranteed by the classical Nash theorem. Let \(P : T\mathbb{R}^N|_M \to TM\) be defined by \(P(m, v) = P(m)v\), where \(P(m) : \mathbb{R}^N \to T_mM\) is the orthogonal projection. So, the Riemannian connection \(\nabla\) of \(M\) can be written as

\[
\nabla_V Y = PdY(V),
\]

for all sections \(V\) and \(Y\) in \(TM\).

The connection \(\nabla^E\) is defined on \(E\) in terms of the orthogonal projection \(\pi : TM \to E\) by

\[
\nabla^E_V Y = \pi \nabla V Y
\]

for all sections \(V \in TM\) and \(Y \in E\).

The elementary differential operators we are going to deal with in the calculus in a foliated manifolds are the following (cf. [3]):

**Definition 1.** Let \(f\) be a smooth function and \(X, Y\) sections of \(E\), we define the operators

a) foliated gradient \(\text{grad}^E f = \pi(\text{grad} f)\);

b) foliated divergence \(\text{div}^E Y = \text{Tr}_E g(\nabla^E Y, \cdot), \text{where Tr}_E\) is the trace on \(E\);

c) foliated Hessian \(\text{Hess}^E(f)(X, Y) = XY(f) - \nabla^E X Y f\);

d) foliated Laplacian \(\Delta^E f = \text{div}^E(\text{grad}^E f)\).

The restriction to a leaf of the operators \(\text{grad}^E, \text{div}^E\) and \(\Delta^E\) are the corresponding operators on the leaf with the induced metric. The following lemma is a natural consequence of the isometric embedding of \(M\) into \(\mathbb{R}^N\). In particular it gives a description of the foliated Laplacian \(\Delta^E\) as a sum of squares of vector fields over \(M\). This will be useful to get the foliated Brownian motion as a solution of a Stratonovich stochastic differential equation.

Let \(\{e_i : i = 1, \ldots, N\}\) be an orthonormal basis of \(\mathbb{R}^N\). We denote by \(\bar{X}_i\) the gradient vector field \(P e_i\) and by \(X_i\) the foliated gradient vector fields \(\pi \bar{X}_i\).

**Lemma 1.** Let \(f\) be a smooth function and \(X\) a section of \(E\). Then

a) \(\text{grad}^E f = \sum_{i=1}^N X_i f X_i\),

b) \(\text{div}^E(X) = \sum_{i=1}^N g(\nabla^E X_i X, X_i)\),

c) \(\Delta^E f = \sum_{i=1}^N X_i^2 f\).

**Proof.** We first observe that

\[
\sum_{i=1}^N X_i(m) \otimes X_i(m) = \sum_{i=1}^p u_i \otimes u_i, \tag{1}
\]

where the \(\{u_i\}\) is an orthonormal basis of \(E_m\).

Item (a) follows immediately from definition and the contraction of Equation (1) with \(df\).
For item (b) note that
\[ \text{div}_E(Y) = g(\nabla_Y E, \sum_{i=1}^{P} u_i \otimes u_i) \]
\[ = g(\nabla_Y E, \sum_{i=1}^{N} X_i(m) \otimes X_i(m)) \]
\[ = \sum_{i=1}^{N} g(\nabla_{X_i} E, X_i). \]

For item (c): By equation (1), we have that
\[ \Delta_E f = \sum_{i=1}^{N} g(\nabla_{X_i} \text{grad}_E f, X_i) \]
\[ = \sum_{i=1}^{N} X_i g(\text{grad}_E f, X_i) - g(\text{grad}_E f, \sum_{i=1}^{N} \nabla_{X_i} E, X_i) \]
Therefore we just need to prove that \( \sum_{i=1}^{N} \nabla_{X_i} E, X_i = 0 \). In order to do this we define the projectors
\[ \tilde{P} = \pi \circ P \]
\[ \tilde{Q} = I_{\mathbb{R}^N} - \tilde{P}. \]
Then \( \tilde{P} \circ \tilde{Q} = \tilde{Q} \circ \tilde{P} = 0 \). Using that
\[ \tilde{P} \circ d\tilde{P} = -\tilde{P} \circ d\tilde{Q} = d\tilde{P} \circ \tilde{Q}, \]
we obtain
\[ \sum_{i=1}^{N} \nabla_{X_i} E, X_i = \sum_{i=1}^{N} \pi P d\tilde{P}(X_i)(e_i) \]
\[ = \sum_{i=1}^{N} \tilde{P} d\tilde{P}(\tilde{P}(e_i))(e_i) \]
\[ = \sum_{i=1}^{N} d\tilde{P}(\tilde{P}(e_i)) \tilde{Q} e_i. \]
By the invariance of this expression with respect to the vectors \( e_i \), we have that \( d\tilde{P}(\tilde{P}(e_i)) \tilde{Q} e_i = 0 \), showing the result.

\[ \square \]

**Definition 2.** We define the tension \( \kappa \) as the unique section of \( E \) such that
\[ g(\kappa, X) = \text{div}_E(X) - \text{div}(X). \]
for all section \( X \) in \( E \).

Suppose that there exists a 1-form \( \omega \) in \( M, ||\omega|| = 1 \) determining a transversaly oriented codimension 1 foliation by \( E = \text{Ker}(\omega) \), i.e. form \( \omega \) satisfies \( \omega \wedge d\omega = 0 \). The integrability of \( E \) guarantees the existence of a 1-form \( \alpha \) such that \( d\omega = \alpha \wedge \omega \). The 1-form \( \alpha \) determines the Godbillon-Vey class of the foliation by \( \text{gv}(\omega) = [\alpha \wedge d\alpha] \in H^3_{dR}(M) \), see Godbillon and Vey [12], Moerdijk and Mrčun [15] or Walczak [19]. We are going to prove that the tension \( \kappa \) is related to the Godbillon-Vey class:
Proposition 1. With the notation above,

\[ g(v(\omega)) = [\alpha^\flat \wedge d\alpha^\flat]. \]

Proof. Denote by \( \eta = \omega^\sharp \) the nowhere vanishing vector field associated to \( \omega \). The following traces vanish

\[ \sum_{i=1}^{N} g(\tilde{X}_i, \eta) g(\pi(\nabla_{\eta} X_i), \tilde{X}_i) = 0 \]

and

\[ \sum_{i=1}^{N} g(\tilde{X}_i, \eta) g(\nabla_{\pi \tilde{X}_i} X, \eta) = 0. \]

Writing each gradient vector field \( \tilde{X}_i \) as

\[ \tilde{X}_i = X_i + \eta_i, \]

where \( \eta_i = g(\tilde{X}_i, \eta) \), then, for any \( X \in \Gamma(E) \)

\[ \text{div}(X) = \sum_{i=1}^{N} (g(\nabla_{X_i} X, X_i) + g(\nabla_X X, N_i) + g(\nabla_{N_i} X, X_i) + g(\nabla_N X, N_i)) \]

Thus

\[ \text{div}(X) - \text{div}_E(X) = g(X, -\nabla_N N) \]

so, \( \kappa = -\nabla_N N \). On the other hand, since \( d\omega(X, Y) = 0 \) for all \( X, Y \in E \) and \( d\omega(N, N) = 0 \) then

\[ d\omega(X, N) = -\omega([X, N]) = g(N, \nabla_N X) = g(\kappa, X). \]

Since \( d\omega(X, N) = \alpha(X) \) we get that \( \alpha = \kappa^\flat \).

We denote by \( \nu(E) \) the normal bundle of \( E \) with respect to \( \mathbb{R}^N \), that is

\[ \nu(E) = \{(x, v), \ x \in M, \ v \in \mathbb{R}^N, \ \text{such that} \ v \perp E_x}\}. \]

Definition 3. The second fundamental form \( \alpha \in \Gamma(E^* \otimes E^* \otimes \nu(E)) \) is the unique \( \nu(E) \)-valued bilinear form satisfying

\[ <\alpha(X, Y), N> = g(\tilde{P}dN(X), Y) \]

for all \( X, Y \in \Gamma(E) \) and \( N \in \Gamma(\nu(E)) \).

The mean curvature is defined by \( H = Tr_E(\alpha) \).

Lemma 2. For all \( v \in \mathbb{R}^N \) we have that

\[ <H, v> = -\text{div}_E(\tilde{P}(v)). \]
Proof. We observe that
\[ \nabla^E_v X_i = - \tilde{P} d(\tilde{Q} e_i)(v). \]
In fact, consider the decomposition
\[ e_i = X_i + \tilde{Q} e_i. \]
Taking the directional derivative with respect to \( v \) and the projection \( \tilde{P} \) to \( E \) we have
\[ \tilde{P} d(\tilde{P} e_i)(v) + \tilde{P} d(\tilde{Q} e_i)(v) = 0. \]
Using this formula, we find that
\[
\text{div}_E(X_i) = \sum_{j=1}^{N} g(\nabla^E_{X_j} X_i, X_j) \\
= - \sum_{j=1}^{N} g(\tilde{P} d(\tilde{Q} e_i)(X_j), X_j) \\
= - \sum_{j=1}^{N} < \tilde{Q} e_i, \alpha(X_j, X_j) > \\
= - < H, e_i > + < H, \tilde{P} e_i > .
\]
Using that \( < H, \tilde{P}(e_i) > = 0 \), the result follows by linearity.

Corollary 1. The following formulae hold:
\[
\sum_{i=1}^{N} \text{div}_E(X_i) X_i = 0 \quad (3)
\]
and
\[
||H||^2 = - \sum_{i=1}^{N} X_i \text{div}_E(X_i). \quad (4)
\]
Proof. Formula (3) follows by substituting Equation (2) in
\[
\sum_{i=1}^{N} \text{div}_E(X_i) X_i = - \sum_{i=1}^{N} < H, \tilde{Q} e_i > \tilde{P} e_i = 0.
\]
For the proof of formula (4) we calculate \( ||H||^2 \) and use Lemma 2.
\[ ||H||^2 = \sum_{i=1}^{N} <H, e_i>^2 \]
\[ = \sum_{i=1}^{N} \text{div}_E(X_i)^2 \]
\[ = \sum_{i,j,k=1}^{N} g(\nabla^E_{X_j} X_i, X_j) g(\nabla^E_{X_k} X_i, X_k) \]
\[ = \sum_{k=1}^{N} g \left( \nabla_{X_k} \left( \sum_{i=1}^{N} \text{div}_E(X_i) X_i \right), X_k \right) - \sum_{i=1}^{N} X_i(\text{div}_E(X_i)) \]
\[ = - \sum_{i=1}^{N} X_i \text{div}_E(X_i). \]

3 Invariant and totally invariant measures

A construction of a foliated Brownian motion (FoBM) with drift can be obtained via a Stratonovich SDE using the gradient vector fields \( X_1, \ldots, X_N \) defined before:

\[
\begin{aligned}
    dX &= V(X) \, dt + \sum_{i=1}^{N} X_i(X) \, \delta B^i \\
    X_0 &= x_0 \in M,
\end{aligned}
\]

where \( V \) is a section of \( E \) and \((B^1, \ldots, B^N)\) is the standard Brownian motion on \( \mathbb{R}^N \) based on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}, P)\). Lemma \( \square \) guarantees that the infinitesimal generator associated to the process \( X_t \) is given by \( \mathcal{L} = V + \frac{1}{2} \Delta_E \).

Alternative construction of a foliated Brownian motion is given via projection on \( M \) of a diffusion generated by standard vector fields in the orthonormal frame bundle, see [3].

Let \( T_t \), with \( t \geq 0 \), be the Markov semigroup of operators associated to FoBM with drift acting on \( B^L_b \), the space of bounded measurable functions which are leafwise smooth. A measure \( \mu \) is invariant if \( \int_M T_t f \, d\mu = \int_M f \, d\mu \) for all \( f \in B^L_b \). It is equivalent to \( \int (\mathcal{L} f) \, d\mu = 0 \). The assumption of compactness of \( M \) guarantees the existence of invariant measures for foliated diffusions.

A point \( x \) in \( M \) is called recurrent for the process \( X \) if for all open neighborhoods \( U \) of \( x \) we have \( \mathbb{P}\{\omega \in \Omega, X_{t_k}(\omega) \in U \text{ for a sequence } t_k \to \infty\} = 1 \). A subset \( U \) of \( M \) is said to be saturated if it is the union of all the leaves passing through points of \( U \), i.e.

\[ \bigcup_{x \in U} L_x \subseteq U. \]

Next proposition says that the support of any invariant measure is a saturated set.

**Proposition 2.** Let \( X \) be a foliated Brownian motion with drift. The support of an invariant measure \( \mu \) is a saturated Borel set contained in the set of recurrent points.
Proof. Denote by $P_t(x,dy)$ the family of transition probabilities of the process. For $x \in \text{supp}(\mu)$, by the action of $T^*_t$ in $\delta_x$ we have that
\[
\text{supp}(P_t(x,dy)) \subseteq \text{supp}(T^*_t \mu) = \text{supp}(\mu),
\]
and $L_x \subseteq \text{supp}(P_t(x,dy))$ for any $t > 0$ since the diffusion is nondegenerate in the leaves. Hence
\[
\bigcup_{x \in \text{supp}(\mu)} L_x \subseteq \text{supp}(\mu).
\]
The result follows by the fact that $\text{supp}(\mu)$ is contained in the closure of the subset of recurrent points of $M$ (Kliemann [13, Lemma 4.1]).

The addition of a drift in the foliated Laplacian preserves Liouville type theorem for harmonic functions in foliated spaces (Garnett [11]):

**Corollary 2.** Let $X$ be a FoBM with drift and $\mu$ an invariant measure. Any function $f \in B^L$ satisfying $\mathcal{L}f = 0$ is constant on every leaf $\mu$- a.s..

**Proof.** Given such a function $f$,
\[
\int_M ||\text{grad}_E f||^2 d\mu = \int_M (\mathcal{L}(f^2) - 2f \mathcal{L}f) \ d\mu = 0.
\]
Thus $||\text{grad}_E f|| = 0$ on a saturated set and therefore $f$ is leafwise constant on the $\text{supp}(\mu)$.

Assume that the bundle $E$ defining the foliation is oriented such that there exists a volume form on the leaves $\chi_E \in \Omega^p(M)$ and $\nu \in \Gamma(\Lambda^p E)$ with $\chi_E(\nu) = 1$. A probability measure $\mu$ on $M$ defines a $p$-current $\psi_\mu : \Gamma(\Lambda^p E) \to \mathbb{R}$ which for $\alpha \in \Gamma(\Lambda^p E)$ is given by:
\[
\psi_\mu(\alpha) := \int_M \alpha(\nu) \ d\mu.
\]
A measure $\mu$ is called **totally invariant** if the associated $p$-current $\psi_\mu$ is a foliated cycle, that is $L_X \psi_\mu = 0$ for any $X \in \Gamma(E)$ (Candel [5], Garnett [11], Sullivan [17]). In terms of a foliated atlas, an alternatively description of a totally invariant measure $\mu$ is via the product of the volume measure on the leaves $\chi_E$ and a holonomy invariant measure $\nu$ in the following sense:
\[
\int f \ d\mu = \sum_{\alpha \in \mathcal{U}} \int_{S_\alpha} \left( \int_P \lambda_\alpha f \chi_E \right) \ d\nu(P)
\]
where $\lambda_\alpha$ is a partition of unity subordinated to a foliated atlas $\mathcal{U} = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$, $P$ are plaques in $U_\alpha$ and $S_\alpha$ are transversal in $U_\alpha$ (see Plante [16, p.330], Candel [4, p.235]).

The following theorem characterizes totally invariant measures in terms of stochastic flows.

**Theorem 1.** A measure $\mu$ is totally invariant if and only if its associated $p$-current $\psi_\mu$ is invariant by the flow of the gradient foliated Brownian motion for each $\omega$ a.s..
where $\hat{p}$ is a $\psi$ action of Corollary 3. The group action of the flow $\phi_t$ in the $p$-current $\phi_t\mu$ associated to a measure $\mu$ is a $p$-current $\phi_t\mu$ associated to a measure $\mu_t$. In fact, direct calculation shows that $\mu_t = \phi_t*(\det_E(\phi_t\mu))$, where $\det_E(\phi_t\mu) = \chi_E(\phi_t\mu)\psi_t(\psi_t\mu)$ is the determinant in the leaf. Hence, the action of the flow $\phi_t$ in the $p$-currents induces an action of the flow in the space of measures given by $\phi_t\star\mu = \mu_t$. Denoting by $\theta_t$ the canonical shift in the probability space $\Omega$, the cocycle property of the flow implies that

\begin{equation}
\phi_t\alpha = \alpha + \int_0^t \phi_s^* L_V \alpha \, ds + \sum_{i=1}^N \int_0^t \phi_s^* L_{X_i} \alpha \, \delta B^i_t
\end{equation}

where $\delta$ denotes the backward Stratonovich integral.

Let $\mu$ be a totally invariant measure. By definition, the integrands of the last part of Equation (6) vanishes for any $p$-form $\alpha$. Hence $\psi_\mu(\phi_t^*\alpha) = \psi_\mu(\alpha)$ a.s..

On the other hand, assume that for any $p$-form $\alpha$ in $M$ we have that $\psi_\mu(\phi_t^*\alpha) = \psi_\mu(\alpha)$ a.s.. Equation (6) and Doob-Meyer decomposition implies that $\psi_\mu(L_V\alpha) = 0$ and $\psi_\mu(L_X\alpha) = 0$ for $i = 1, \ldots, N$. We have to prove that $\psi_\mu(L_X\alpha) = 0$ for all $X \in \Gamma(E)$ and all $p$-form $\alpha$ in $M$. Any $p$-form $\alpha$ can be written as $\alpha = f\chi_E + \beta$ with $\beta$ a $p$-form such that $\beta(v) = 0$.

We have that $\psi_\mu(L_X\beta) = 0$ since

$L_X\beta(v) = X(\beta(v)) - \sum_{j=1}^p g([X, v_j], v_j) \beta(v) = 0$

for a local expression of $v = v_1 \wedge \cdots \wedge v_p$ in terms of orthonormal sections in $\Gamma(E)$.

Let $X = \sum_{i=1}^N a_iX_i$ for some smooth functions $a_i$. We have that

$\psi_\mu(L_X(f\chi_E)) = \psi_\mu((X(f) + f\text{div}_E(X)) \chi_E) = \sum_{i=1}^N \psi_\mu(a_iX_i(f)\chi_E + X_i(a_i)f\chi_E + a_ifL_{X_i}\chi_E) = \sum_{i=1}^N \psi_\mu(L_{X_i}(f\chi_E)) = 0$.

The group action of the flow $\phi_t$ on the measures $\mu_t = \phi_t*(\det_E(\phi_t\mu))$ satisfies the cocycle property

$\phi_s(\theta_t\omega) = \mu_s(\omega) = \mu_{t+s}(\omega)$.

A deterministic measure $\mu$ is totally invariant if and only if it is a fixed point of the action of $\phi_t$ a.s..
Proof. The formula follows immediately from the group action of \( \phi_t \) on \( p \)-currents. Totally invariance comes from Theorem 1.

4 Ergodic Measures

In this section we study the support of ergodic invariant probability measures in \( M \) for the foliated Brownian motion with drift. A minimal set \( K \) is a closed nonempty saturated set with the property that if \( K' \subseteq K \) is again a nonempty closed saturated set, then \( K = K' \). A transitive set is a minimal set such that there exists at least one dense leaf, i.e. the transitive sets are closures of the leaves. Lemma 3 below implies that the support of ergodic measures always contains a minimal set.

Lemma 3. Let \( \mu \) be an ergodic invariant measure for the foliated Brownian motion with drift. The support \( \text{supp}(\mu) \) is a transitive set. Moreover for any minimal set \( K \) we have that \( \mu(K) = 0 \) or \( \mu(K) = 1 \).

Proof. For \( \mu \)-almost every point \( x \in \text{supp}(\mu) \) we have the weak limit:

\[
\mu(dy) = \lim_{t \to \infty} \frac{1}{t} \int_0^t P_s(x, dy) \, ds,
\]

with

\[
\text{supp}(\mu) = \bigcup_{t>0} \text{supp}(P_t(x, dy)).
\]

But for all \( t > 0 \), \( \text{supp}(P_t(x, dy)) = \overline{L_x} \), the leaf through \( x \).

The second statement is straightforward by invariance of \( K \) and ergodicity of \( \mu \).

Corollary 4. Let \( K \subset M \) be a minimal set. There always exists an ergodic measure supported on \( K \).

Proof. By compactness there always exists an ergodic measure \( \mu \) with support contained in \( K \). Lemma 3 implies that \( \text{sup}(\mu) = K \).

4.1 Application to stable foliations:

Consider the foliation of \( M \) given by a strongly stable diffeomorphism \( \phi : M \to M \), i.e. the leaf through a point \( x \) of \( M \) is given by

\[
L_x = \{ y \in M, \ d(\phi^n(x), \phi^n(y)) \to 0 \ as \ n \to \infty \}. 
\]

A diffeomorphism \( \phi : M \to M \) is conservative if, for all nonempty measurable subset \( A \subset M \), we have that

\[
\phi^{-j}(A) \cap \phi^{-k}(A) \neq \emptyset 
\]

for all \( j, k \in \mathbb{N} \cup \{0\} \). A function \( f \) on \( M \) which is invariant by \( \phi \), i.e. \( f = f \circ \phi \) is constant in the leaves \( \nu \)-a.s. for any measure \( \nu \) which is \( \phi \)-invariant, see Y. Coudene [9]. We have the following criteria to \( \nu \)-ergodicity of \( \phi \):
Proposition 3. Let $\phi : M \to M$ be a strongly stable conservative transformation which preserves a probability measure $\nu$. If $\nu$ is equivalent to an ergodic harmonic probability measure $\mu$ (w.r.t. FoBM) then $\nu$ is $\phi$-ergodic.

Proof. Let $A \subset M$ be such that $\phi^{-1}(A) = A$, hence for $f = 1_A$ we have that

$$f^* = \lim_{t \to \infty} \frac{1}{t} \int_0^t T_s f(x) \, ds = \int_M f \, d\mu \quad \mu - \text{a.s.}$$

On the other hand by the Coudene’s result mentioned above [9] we have that $f$ is constant in the leaves, hence $T_t f = f$, $\nu$-a.s. therefore $f^* = f$ $\nu$-a.s..

\[\Box\]

4.2 Lyapunov Exponents

Let $\phi_t$ be the stochastic flow associated to the foliated Brownian motion with drift of Equation (5) and consider $\mu$ an ergodic invariant probability measure in $M$. The sum of the Lyapunov exponents $\lambda_\Sigma(x)$ at a point $x \in M$ including multiplicity is given by the limit

$$\lambda_\Sigma(x) = \lim_{t \to \infty} \frac{1}{t} \ln |\det(\phi_{ts}(x))|$$

which exists and is constant $\mathbb{P} \times \mu$-almost surely for $(\omega, x) \in \Omega \times M$ according to (multiplicative) ergodic theorems for stochastic flows. Itô formulae for the logarithm of this determinant have been obtained by various authors:

$$\ln(|\det(\phi_{ts}(x))|) = \sum_{i=1}^N \int_0^t \text{div}(X_i)(\phi_s(x)) \, dB^i_s$$

$$+ \int_0^t \left(\text{div}(V) + \frac{1}{2} \sum_{i=1}^N X_i \text{div}(X_i)\right)(\phi_s(x)) \, ds.$$

Birkhoff’s theorem in the skew-product flow (Furtenberg-Khasminskii type argument) leads to the Baxendale’s ergodic formula:

$$\lambda_\Sigma = \int_M \left(\text{div}(V) + \frac{1}{2} \sum_{i=1}^N X_i \text{div}(X_i)\right) d\mu \quad \mathbb{P} \times \mu - \text{a.s.} \tag{9}$$

See e.g. Chapell [8], Arnold [1] and many references therein.

We have the following expression which involves the geometry of the leaves for these ergodic theorems.

**Theorem 2.** Let $\mu$ be an ergodic probability measure for a gradient foliated Brownian motion with drift $V$. Then the sum of the Lyapunov exponents is given by

$$\lambda_\Sigma = -\frac{1}{2} \int_M \left(||H||^2 - \text{div}_E(2V - \kappa) + 2g(\kappa, V)\right) d\mu.$$

Proof. Using the formula of Corollary [1] and the definition of the tensor $\kappa$ we have that
$$\lambda_\Sigma = \frac{1}{2} \int_M \left( 2 \text{div}(V) + \sum_{i=1}^N X_i \text{div}_E(X_i) - \text{div}_E(\kappa) \right) d\mu$$

$$= \frac{1}{2} \int_M \left( \sum_{i=1}^N X_i \text{div}_E(X_i) + \text{div}_E(2V - \kappa) - 2g(\kappa, V) \right) d\mu$$

$$= -\frac{1}{2} \int_M (||H||^2 - \text{div}_E(2V - \kappa) + 2g(\kappa, V)) \ d\mu.$$

\[\square\]

**Corollary 5.** If the ergodic measure \(\mu\) is harmonic totally invariant then the sum of the Lyapunov exponents depends only on the second fundamental form of the leaves:

$$\lambda_\Sigma = -\frac{1}{2} \int_M ||H||^2 \ d\mu.$$

**Proof.** With \(V = 0\), use that \(\int_M \text{div}_E X \ d\mu = 0\), for all \(X \in \Gamma(E)\) see [3, Thm 4.3]. \(\square\)

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