A New Mathematical Modeling with Optimal Control Strategy for the Dynamics of Population of Diabetics and Its Complications with Effect of Behavioral Factors

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1. Introduction

Nowadays, diabetes is a chronic disease with a huge burden affecting individuals. According to the World Health Organization (WHO) [1], diabetes is a disorder characterized by the presence of problems in the insulin hormone, which naturally results from the pancreas to help the body use glucose and fat and store some of them. According to the American Diabetes Association (ADA) [2], diabetes mellitus is a group of metabolic diseases characterized by hyperglycemia resulting from defects in insulin secretion, insulin action, or both. It is known that the proper level of glucose in the blood after fasting eight hours should be less than 108 mg/dl, while the borderline is 126 mg/dl. If a person’s blood glucose level is 126 mg/dl and above, in two or more tests, then that person is diagnosed with diabetes. Diabetes is divided into several different types; some more prevalent than others. The most common type of diabetes in the general population is type 2 diabetes, and type 1 diabetes is more common in children, and gestational diabetes is a form of diabetes that can occur during pregnancy. According to the latest statistics from the International Diabetes Federation (IDF) and as reported in the 9th edition of the Atlas Diabetes 2017 [3], diabetes is a constantly growing disease. There are more than 370 million people with diabetes worldwide (8.5% of the adult population) and about 463 million people in prediabetes (6.5% of the adult population), and more than 625 million are expected to be affected by 2045.

Today, all countries of the world suffer from the high number of people with diabetes, which is increasing and expanding on the extreme level. When it is not treated well, all types of diabetes can lead to complications in many parts of the body, leading to an early death. When a diabetic knows how to control the level of glucose in the blood, this awareness plays a key role in reducing the serious complications of diabetes. According to IDF statistics, diabetes has serious and varied complications. For example, the risk of cardiovascular disease. Moreover, more than a third of diabetics have...
retinopathy which is the main cause of vision loss, in addition to the risk of kidney disease. In addition, the complications of diabetes are multiple and different depending on the degree of severity; there are complications that can be treated and others that have reached critical stages with which treatment is not beneficial. According to ADA [20], diabetes has economic and social burdens on the individual and society.

During the last decade, large mathematical models on diabetes have been developed to simulate, analyse, and understand the dynamics of a population of diabetics. In a related research work, Boutayeb and Chetouani [4] and Derouich et al. [5] introduced a mathematical model for the dynamics of the population of diabetics. And Kouidere et al. [6] proposed a discrete mathematical model highlighting the impact of the population of diabetes. And Pontryagin’s maximum principle. Numerical simulations through MATLAB are given in Section 4. Finally, we conclude the paper in Section 5.

2. A Mathematical Model

We consider a mathematical model $HPEDC_TC_5$ that describes the dynamics of a population of diabetics. We divide the population denoted by $N$ into six compartments.

2.1. Description of the Model. The graphical representation of the proposed model is shown in Figure 1.

The compartment $(H)$ are healthy people; the compartment $H$ is increased by $I$ (which is the recruitment rate of healthy people), and this compartment $H$ is decreased by $\theta_1$, $H(t)$ (the number of prediabetic people through genetic factor) and also decreased by $\theta_2$, $H(t)$ (the number of prediabetics through genetic factors) and decreased by the amount $\mu$ (natural mortality).

$$\frac{dH(t)}{dt} = I - (\mu + \theta_1 + \theta_2)H(t).$$

The compartment $(P)$ are people who are likely to have diabetes through genetic factors. The compartment $P$ is increased by $\theta_1H(t)$. This compartment $P$ is decreased by the amount $\mu$ (natural mortality) and also decreased by $\beta_1P(t)$ (the probability of developing diabetes) and by $\beta_3C_T(t)$ (the probability of developing diabetes at stage of complications).

$$\frac{dP(t)}{dt} = \theta_1H(t) - (\mu + \beta_1 + \beta_3)P(t).$$

**Figure 1:** The dynamics of a population of diabetics.
Compartment \((E)\) are people who are likely to have diabetes through the negative effect of lifestyle or psychological problem factors and others (these are people at risk of developing diabetes, such as those who are obese, overweight, gestational diabetes, or due to family and work problems, in addition to the person without diabetes). The compartment \(E\) increased by \(\theta_1 H(t)\) and decreased by \(\gamma E(t)\) (patients who become diabetics without complications because of the negative effect of lifestyle) and also decreased by \(\mu\) (natural mortality).

\[
\frac{dE(t)}{dt} = \theta_2 H(t) - (\mu + \gamma) E(t). \tag{3}
\]

Compartment \((D)\) is the number of diabetics without complications. The compartment \(D(t)\) increased by the amount \(\beta_1 P(t)\) and by the amount \(\gamma E(t)\). This compartment \(D\) is decreasing by \(\mu\) (natural mortality) and \(\alpha_1\) (the rate of negative impact of ordinary people on diabetics without complications through improper nutrition, or practical or family social problems) and also decreased by \(\beta_2 D(t)\) (the probability of a diabetic person developing a complication) and also decreased by \(\alpha_2 D(t)\) (the number of diabetics whose serious complications because of a sudden shock).

\[
\frac{dD(t)}{dt} = \beta_1 P(t) + \gamma E(t) - \alpha_1 \frac{D(t) E(t)}{N} - (\mu + \beta_2 + \alpha_2) D(t). \tag{4}
\]

Compartment \((C_T)\) is the number of diabetics with treatable complications. The compartment \(C_T\) increased by \(\beta_3 P(t)\) and by \(\beta_4 D(t)\) and also increased by \(\alpha_1\) and and decreased by \(\alpha_2\) (the rate negative impact of ordinary people on diabetics with treatable complications through improper nutrition, or practical or family social problems) and decreased by \(\eta_1 C_T(t)\) (the number of people developing of diabetics with complications can be treated to serious complications) and also by \(\mu\) (natural mortality).

\[
\frac{dC_T(t)}{dt} = \beta_3 P(t) + \beta_4 D(t) + \alpha_1 \frac{D(t) E(t)}{N} - \alpha_2 \frac{C_T(t) E(t)}{N} - (\mu + \eta_1) C_T(t). \tag{5}
\]

Compartment \((C_S)\) is the number of diabetics with serious complications, they are the diabetics who have serious complications such as retinopathy or renal failure. The compartment \(C_S\) is increased by \(\eta_2 D(t)\) and also by \(\eta_1 C_T(t)\) and by \(\alpha_2\). This compartment \(C_S\) decreased by \(\delta\) (mortality rate due to complications) and also by \(\mu\) (natural mortality).

\[
\frac{dC_S(t)}{dt} = \eta_2 D(t) + \eta_1 C_T(t) + \alpha_2 \frac{C_T(t) E(t)}{N} - (\mu + \delta) C_S(t). \tag{6}
\]

Hence, we present the diabetic model by the following system of differential equations:

\[
\begin{align*}
\frac{dH(t)}{dt} & = I - (\mu + \theta_1 + \theta_2) H(t), \\
\frac{dP(t)}{dt} & = \theta_1 H(t) - (\mu + \beta_1 + \beta_2) P(t), \\
\frac{dE(t)}{dt} & = \theta_2 H(t) - (\mu + \gamma) E(t), \\
\frac{dD(t)}{dt} & = \beta_1 P(t) + \gamma E(t) - \alpha_1 \frac{D(t) E(t)}{N} - (\mu + \beta_2 + \alpha_2) D(t), \\
\frac{dC_T(t)}{dt} & = \beta_3 P(t) + \beta_4 D(t) + \alpha_1 \frac{D(t) E(t)}{N} - \alpha_2 \frac{C_T(t) E(t)}{N} - (\mu + \eta_1) C_T(t), \\
\frac{dC_S(t)}{dt} & = \eta_2 D(t) + \eta_1 C_T(t) + \alpha_2 \frac{C_T(t) E(t)}{N} - (\mu + \delta) C_S(t),
\end{align*}
\]

with \(H(0) \geq 0, P(0) \geq 0, E(0) \geq 0, D(0) \geq 0, C_T(0) \geq 0, C_S(0) \geq 0\).

2.2. Positivity of Solutions

**Theorem 1.** If \(H(0) \geq 0, P(0) \geq 0, E(0) \geq 0, D(0) \geq 0, C_T(0) \geq 0, \) and \(C_S(0) \geq 0\), the solutions \(H(t), P(t), E(t), D(t), C_T(t), C_S(t)\) of system (7) are positive for all \(t \geq 0\).

**Proof.** It follows from the first equation of system (7) that

\[
\frac{dH(t)}{dt} = I - (\mu + \theta_1 + \theta_2) H(t) \geq -(\mu + \theta_1 + \theta_2) H(t) \tag{8}
\]

\[
\frac{dH(t)}{dt} + (\mu + \theta_1 + \theta_2) H(t) \geq 0
\]
Both sides in the last inequality are multiplied with \( \exp ((\mu + \theta_1 + \theta_2) t) \).

We obtain

\[
\exp ((\mu + \theta_1 + \theta_2) t) \cdot \frac{dH(t)}{dt} + (\mu + \theta_1 + \theta_2) \exp ((\mu + \theta_1 + \theta_2) t) \cdot H(t) \geq 0,
\]

then

\[
\frac{d}{dt} (\exp ((\mu + \theta_1 + \theta_2) t) \cdot H(t)) \geq 0. \tag{10}
\]

Integrating this inequality from 0 to \( t \) gives

\[
\int_0^t \frac{d}{ds} (\exp ((\mu + \theta_1 + \theta_2) s) \cdot H(s)) ds \geq 0, \tag{11}
\]

then

\[
H(t) \geq H(0) \exp ((\mu + \theta_1 + \theta_2) t) \Rightarrow H(t) > 0. \tag{12}
\]

Similarly, we prove that \( P(t) \geq 0, E(t) \geq 0, D(t) \geq 0, C_T(t) \geq 0, \) and \( C_S(t) \geq 0. \)

2.2.1. Boundedness of the Solutions

**Theorem 2.** The set \( \Omega = \{(H, P, E, D, C_T, C_S) \in \mathbb{R}^6 : 0 \leq H + P + E + D + C_T + C_S \leq I/\mu \} \) is positively invariant under system (7) with initial conditions \( H(0) \geq 0, P(0) \geq 0, E(0) \geq 0, D(0) \geq 0, C_T(0) \geq 0, \) and \( C_S(0) > 0. \)

**Proof.** By adding the equations of system (7), we obtain

\[
\frac{dN}{dt} = I - \mu N - \delta C_S \leq I - \mu N \Rightarrow N(t) \leq \frac{I}{\mu} + N(0) e^{-\mu t}, \tag{13}
\]

where \( N(0) \) represents the initial values of the total population.

Thus, \( \limsup_{t \to \infty} N(t) = (I/\mu). \) It implies that the region \( \Omega \) is a positively invariant set for system (7). So, we only need to consider the dynamics of the system on the set \( \Omega. \)

2.2.2. Existence of Solutions

**Theorem 3.** The system (7) that satisfies a given initial condition \( (H(0), P(0), E(0), D(0), C_T(0), C_S(0)) \) has a unique solution.

**Proof.** Let

\[
\varphi(X) = \left( \begin{array}{c}
\frac{dH(t)/dt}{dt} \\
\frac{dP(t)/dt}{dt} \\
\frac{dE(t)/dt}{dt} \\
\frac{dD(t)/dt}{dt} \\
\frac{dC_T(t)/dt}{dt} \\
\frac{dC_S(t)/dt}{dt}
\end{array} \right),
\]

so the system (7) can be rewritten in the following form:

\[
\varphi(X) = X = AX + B(X),
\]

where

\[
A = \begin{pmatrix}
- (\mu + \theta_1 + \theta_2) & 0 & 0 & 0 & 0 & 0 \\
\theta_1 & - (\mu + \beta_1 + \beta_3) & 0 & 0 & 0 & 0 \\
\theta_2 & 0 & - \mu + \gamma & 0 & 0 & 0 \\
0 & \beta_1 & \gamma & - (\mu + \beta_2 + \beta_3) & 0 & 0 \\
0 & \beta_3 & 0 & \beta_2 & - (\mu + \delta) & 0 \\
0 & 0 & 0 & \eta_2 & \eta_1 & - (\mu + \delta)
\end{pmatrix},
\]

\[
B(X) = \begin{pmatrix}
I \\
0 \\
0 \\
- \alpha_1 \frac{D(t)E(t)}{N} \\
\alpha_1 \frac{D(t)E(t)}{N} - \alpha_2 \frac{C_T(t)E(t)}{N} \\
\alpha_2 \frac{C_T(t)E(t)}{N}
\end{pmatrix}.
\]
The second term on the right-hand side of (15) satisfies

\[ |B(X_1) - B(X_2)| \leq M(|D_1(t) - D_2(t)| + |C_{T1}(t) - C_{T2}(t)|), \]
\[ |B(X_1) - B(X_2)| \leq M \cdot \|X_1 - X_2\|, \]

(17)

where \( M = 2(1/\mu)(|a_1/N| + |a_2/N|) \).

Thus, it follows that the function \( \phi \) is uniformly Lipschitz continuous, and the restriction on \( H(t) \geq 0, P(t) \geq 0, E(t) \geq 0, D(t) \geq 0, C_T(t) \geq 0, \) and \( C_s(t) \geq 0, \) we see that a solution of the system (7) exists [22].

\[
\begin{align*}
\frac{dH(t)}{dt} &= I - (\mu + \theta_1 + \theta_2)H(t), \\
\frac{dP(t)}{dt} &= \theta_1 H(t) - (\mu + \beta_1 + \beta_3)P(t), \\
\frac{dE(t)}{dt} &= \theta_2 H(t) - \mu E(t) - \gamma(1 - u_4(t))E(t), \\
\frac{dD(t)}{dt} &= \beta_1 P(t) + \gamma(1 - u_4(t))E(t) - \alpha_1(1 - u_2(t)) - \frac{D(t)E(t)}{N} - (\mu + \beta_2 + \eta_2)D(t) + u_1(t)C_T(t), \\
\frac{dC_T(t)}{dt} &= \beta_2 P(t) + \beta_2 D(t) + \alpha_1(1 - u_2(t)) - \frac{D(t)E(t)}{N} - \alpha_2(1 - u_3(t)) - \frac{C_T(t)E(t)}{N} - (1 + \theta_1 + \theta_2)C_T(t), \\
\frac{dC_s(t)}{dt} &= \eta_2 D(t) + \eta_1 C_T(t) + \alpha_3(1 - u_3(t)) - \frac{C_s(t)E(t)}{N} - (1 + \theta_1 + \theta_2)C_s(t).
\end{align*}
\]

(19)

The problem is to minimize the objective functional

\[
J(u_1, u_2, u_3, u_4) = C_T(T_f) - D(T_f) - E(T_f) \\
+ \int_0^{T_f} C_T(t) - D(t) - E(t) + \frac{A}{2} u_1^2(t) \\
+ \frac{B}{2} u_2^2(t) + \frac{F}{2} u_3^2(t) + \frac{G}{2} u_4^2(t) \, dt,
\]

where \( A > 0, B > 0, F > 0, \) and \( G > 0 \) are the cost coefficients; they are selected to weigh the relative importance of \( u_1(t), u_2(t), \) \( u_3(t), \) and \( u_4(t) \) at time \( t, \) and \( T_f \) is the final time.

In other words, we seek the optimal controls \( u_1^*(t), u_2^*(t), u_3^*(t), \) and \( u_4^*(t) \) such that

\[
J(u_1^*, u_2^*, u_3^*, u_4^*) = \min_{u_1, u_2, u_3, u_4 \in U} J(u_1, u_2, u_3, u_4),
\]

(20)

where \( U \) is the set of admissible controls defined by

\[
U = \{ (u_1, u_2, u_3, u_4) / 0 \leq u_{1 \text{ min}} \leq u_1(t) \leq u_{1 \text{ max}} \}
\]

(22)

4. The Optimal Control: Existence and Characterization

We first show the existence of solutions of the system (18), thereafter, we will prove the existence of optimal control.

4.1. Existence of an Optimal Control

**Theorem 4.** Consider the control problem with system (18). There exists an optimal control \( (u_1^*, u_2^*, u_3^*, u_4^*) \in U^4 \) such that

\[
J(u_1^*, u_2^*, u_3^*, u_4^*) = \min_{u_1, u_2, u_3, u_4 \in U} J(u_1, u_2, u_3, u_4).
\]

(23)
Proof. The existence of the optimal control can be obtained using a result by Fleming and Rishel [23], checking the following steps:

(i) It follows that the set of controls and corresponding state variables is not empty. We will use a simplified version of an existence result ([24], Theorem 7.1.1).

(ii) $J(u_1, u_2, u_3, u_4)$ is convex in $U$.

(iii) The control space $U = \{ (u_1, u_2, u_3, u_4, u_1, u_2, u_3, u_4) \}$ is measurable, $0 \leq u_1_{\text{min}} \leq u_1(t) \leq u_1_{\text{max}} \leq 1$, $0 \leq u_2_{\text{min}} \leq u_2(t) \leq u_2_{\text{max}} \leq 1$, and $0 \leq u_3_{\text{min}} \leq u_3(t) \leq u_3_{\text{max}} \leq 1$, $0 \leq u_4_{\text{min}} \leq u_4(t) \leq u_4_{\text{max}} \leq 1/t \in [0, T_f]$ is convex and closed by definition.

(iv) All the right-hand sides of equations of system are continuous, bounded above by a sum of bounded control and state, and can be written as a linear function of $u$, $v$, and $w$ with coefficients depending on the time and state.

(v) The integrand in the objective functional, $C_T(t) - D(t) - E(t) + (A/2)u_1(t) + (B/2)u_2(t) + (F/2)u_3(t) + (G/2)u_4(t)$, is clearly convex on $U$.

(vi) It rests to show that there exist constants $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \theta, \epsilon > 0$ and $\xi$ such that $C_T(t) - D(t) - E(t) + (A/2)u_1(t) + (B/2)u_2(t) + (F/2)u_3(t) + (G/2)u_4(t)$ satisfies

$$C_T(t) - D(t) - E(t) + A/2 u_1(t) + B/2 u_2(t) + F/2 u_3(t) + G/2 u_4(t) \geq -\zeta_1 |u_1(t)|^\epsilon + \zeta_2 |u_2(t)|^\epsilon + \zeta_3 |u_3(t)|^\epsilon + \zeta_4 |u_4(t)|^\epsilon. \quad (24)$$

The state variables are being bounded; let $\zeta_1 = \sup_{t \in [0, T_f]} (C_T(t) - D(t) - E(t)), \zeta_2 = A, \zeta_3 = B, \zeta_4 = F, \zeta_5 = G$ and $\zeta = 2$; then, it follows that

$$C_T(t) - D(t) - E(t) + A/2 u_1(t) + B/2 u_2(t) + F/2 u_3(t) + G/2 u_4(t) \geq -\zeta_1 |u_1(t)|^\epsilon + \zeta_2 |u_2(t)|^\epsilon + \zeta_3 |u_3(t)|^\epsilon + \zeta_4 |u_4(t)|^\epsilon. \quad (25)$$

Then, from Fleming and Rishel [23], we conclude that there exists an optimal control.

4.2. Characterization of the Optimal Control. In order to derive the necessary conditions for the optimal control, we apply Pontryagin’s maximum principle to the Hamiltonian $H$ at time $t$ defined by

$$H(t) = C_T(t) - D(t) - E(t) + A/2 u_1(t) + B/2 u_2(t) + F/2 u_3(t) + G/2 u_4(t) + \sum_{i=1}^{6} \lambda_i(t) f_i(H, P, E, D, C_T, C_5). \quad (26)$$

where $f_i$ is the right side of the difference equation of the $i$th state variable.

Theorem 5. Given the optimal controls $(u_1^*, u_2^*, u_3^*, u_4^*)$ and the solutions $H^*, P^*, E^*, D^*, C_T^*$, and $C_5^*$ of the corresponding state system (18), there exist adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ satisfying

$$
\begin{align*}
\lambda_1' &= \lambda_1 (\mu + \theta_1 + \theta_2) - \lambda_2 \theta_1 - \lambda_3 \theta_2, \\
\lambda_2' &= \lambda_2 (\mu + \beta_1 + \beta_2) - \lambda_3 \beta_1 - \lambda_4 \beta_2, \\
\lambda_3' &= \lambda_3 (\mu + \gamma_1 + \gamma_2) - \lambda_4 \gamma_1 - \lambda_5 \gamma_2, \\
\lambda_4' &= \lambda_4 (\mu + \delta) - \lambda_5 \eta_1 - \lambda_6 \eta_2, \\
\lambda_5' &= \lambda_5 (\mu + \eta_1) - \lambda_6 \eta_2, \\
\lambda_6' &= \lambda_6 (\mu + \eta_1) - \lambda_7 \eta_2.
\end{align*}
$$

With the transversality conditions at time $T_f$, $\lambda_1(T_f) = 0, \lambda_2(T_f) = 0, \lambda_3(T_f) = 1, \lambda_4(T_f) = 1, \lambda_5(T_f) = -1$, and $\lambda_6(T_f) = 0$.

Furthermore, for $t \in [0, T_f]$, the optimal controls $u_1^*, u_2^*, u_3^*$, and $u_4^*$ are given by

$$
\begin{align*}
\begin{cases}
\begin{align*}
u_1^* &= \min \left( 1, \max \left( 0, \frac{\lambda_3 - \lambda_4}{A} C_T(t) \right) \right), \\
u_2^* &= \min \left( 1, \max \left( 0, \frac{\lambda_1 x_1}{B} \times \frac{D(t) E(t)}{N} \right) \right), \\
u_3^* &= \min \left( 1, \max \left( 0, \frac{\lambda_2 x_2}{F} \times \frac{C_T(t) E(t)}{N} \right) \right), \\
u_4^* &= \min \left( 1, \max \left( 0, \frac{\lambda_3 - \lambda_4}{G} E(t) \right) \right),
\end{align*}
\end{cases}
\end{align*}
$$

(28)
Proof. The Hamiltonian is defined as follows:
\[
\begin{align*}
\tilde{H}(t) &= C_T(t) - D(t) - E(t) + \frac{A}{2} u_1^2(t) + \frac{B}{2} u_2^2(t) + \frac{F}{2} u_3^2(t) \\
+ G &\sum_{i=1}^{\delta} \lambda_i(t) f_i(H, P, E, D, C_T, C_3),
\end{align*}
\]

where
\[
\begin{align*}
f_1(H, P, E, D, C_T, C_3) &= I - (\mu + \theta_1 + \theta_2) H(t), \\
f_2(H, P, E, D, C_T, C_3) &= \theta_1 H(t) - (\mu + \beta_1 + \beta_3) P(t), \\
f_3(H, P, E, D, C_T, C_3) &= \theta_2 H(t) - \mu E(t) + \gamma(1 - u_4(t)) E(t),
\end{align*}
\]

For \( t \in [0, T_i] \), the adjoint equations and transversality conditions can be obtained by using Pontryagin’s maximum principle \([13, 25]\) such that
\[
\begin{align*}
\lambda_1' &= -\frac{\partial \tilde{H}}{\partial H} = \lambda_1(\mu + \theta_1 + \theta_2) - \lambda_2 \theta_1 - \lambda_3 \theta_2, \\
\lambda_2' &= -\frac{\partial \tilde{H}}{\partial P} = \lambda_2(\mu + \beta_1 + \beta_3) - \lambda_4 \beta_1 - \lambda_5 \beta_3, \\
\lambda_3' &= -\frac{\partial \tilde{H}}{\partial E} = 1 + \lambda_3[\mu + \gamma(1 - u_4(t))] \\
- \lambda_4 \left[ \gamma(1 - u_4(t)) - \alpha_1(1 - u_2(t)) \frac{D(t)}{N} \right] \\
- \lambda_5 \left[ \alpha_2(1 - u_3(t)) \frac{D(t)}{N} - \alpha_3(1 - u_3(t)) \frac{C_T(t)}{N} \right] \\
- \lambda_6 \left[ \alpha_4(1 - u_3(t)) \frac{C_T(t)}{N} \right],
\end{align*}
\]
Table 1: Parameter values used in numerical simulation.

| Parameter | Value in month⁻¹ | Description |
|-----------|-------------------|-------------|
| µ         | 0.02              | Natural mortality |
| δ         | 0.001             | Mortality rate due to complications |
| β₁        | 0.2               | The probability of developing diabetes |
| β₂        | 0.08              | The probability of a diabetic person developing a complication |
| β₃        | 0.01              | The probability of developing diabetes at stage of complications |
| α₁        | 0.4               | The rate of negative impact on diabetics without complications |
| γ         | 0.06              | Rate of patients become diabetic without complications through lifestyle |
| N         | 2000000           | Denote the incidence of healthy people |
| a₂        | 0.6               | The rate of negative impact on diabetics with treatable complications |
| θ₁        | 0.1               | Rate of prediabetic people through genetic factor |
| θ₂        | 0.2               | Rate of prediabetic people through lifestyle factor |
| η₁        | 0.6               | The probability of a diabetic person developing a serious complication |
| η₂        | 0.3               | Rate of diabetics whose serious complications are because of a sudden shock |

Table 2: Evolution of number of diabetics without control after 120 days.

| Population without control after 120 days | Without control |
|------------------------------------------|-----------------|
| Diabetics without complications          | 3.05 × 10⁶      |
| Diabetics with treatable complications    | 1.49 × 10⁷      |
| Diabetics with serious complications      | 1.0 × 10⁸       |

\[
\lambda'₄ = -\frac{\partial H}{\partial D} = -1 - \lambda_4 \left[ \alpha_1 (1 - u_2(t)) \frac{E(t)}{N} + (\mu + \beta_2 + \eta_2) \right] \\
- \lambda_5 \left[ \beta_2 + \alpha_1 (1 - u_2(t)) \frac{E(t)}{N} - \lambda_6 \eta_2, \right]
\]

\[
\lambda'₅ = -\frac{\partial H}{\partial C_T} = -1 - \lambda_4 u_1(t) + \lambda_5 \left[ -\alpha_2 (1 - u_3(t)) \frac{E(t)}{N} - (\mu + \eta_1 + u_1(t)) \right] - \lambda_6 \left[ \eta_1 + \alpha_3 (1 - u_3(t)) \frac{E(t)}{N} \right]
\]

\[
\lambda'₆ = -\frac{\partial H}{\partial C_S} = \lambda_6 (\mu + \delta).
\]

For, \( t \in [0, T_f] \) the optimal controls \( u'_1, u'_2, u'_3, \) and \( u'_4 \) can be solved from the optimality condition,

\[
\begin{align*}
\frac{\partial H}{\partial u_1} &= 0, \\
\frac{\partial H}{\partial u_2} &= 0, \\
\frac{\partial H}{\partial u_3} &= 0, \\
\frac{\partial H}{\partial u_4} &= 0.
\end{align*}
\]

That are

\[
\begin{align*}
-\frac{\partial H}{\partial u_1} &= -Au_1(t) + (\lambda_2 - \lambda_4) C_T(t) = 0, \\
-\frac{\partial H}{\partial u_2} &= -Bu_2(t) + \alpha_1 (\lambda_5 - \lambda_4) \frac{D(t)E(t)}{N} = 0, \\
-\frac{\partial H}{\partial u_3} &= -Fu_3(t) + \alpha_2 (\lambda_6 - \lambda_5) C_T(t) \frac{E(t)}{N} = 0, \\
-\frac{\partial H}{\partial u_4} &= -Gu_4(t) + (\lambda_4 - \lambda_3) \gamma E(t) = 0.
\end{align*}
\]

We have

\[
\begin{align*}
u_1(t) &= \frac{(\lambda_2 - \lambda_4)}{A} C_T(t), \\
u_2(t) &= \alpha_1 \times \left( (\lambda_5 - \lambda_4) / B \right) \times \left( D(t)E(t) / N \right), \\
u_3(t) &= \alpha_2 \times \left( (\lambda_6 - \lambda_5) / F \right) \times \left( C_T(t)E(t) / N \right), \\
u_4(t) &= \frac{(\lambda_4 - \lambda_3)}{G} \times \gamma E(t).
\end{align*}
\]

By the bounds in \( U \) of the controls, it is easy to obtain \( u'_1, u'_2, u'_3, \) and \( u'_4 \) and are given in (13–16) the form of system and in the form of system (18).

5. Numerical Simulation

In this section, we present the results obtained by solving numerically the optimality system (18). In our control problem, we have initial conditions for the state variables and terminal conditions for the adjoints. That is, the optimality
The reason of this increase was justified by the fact that the number of diabetics with treatable complications will become diabetics without complications. For improving the system is a two-point boundary value problem with separated boundary conditions at times step 0 ≤ i ≤ Ti.

We solve the optimality system by an iterative method with forward solving of the state system followed by backward solving of the adjoint system. We start with an initial guess for the controls at the first iteration, and then before the next iteration, we update the controls by using the characterization. We continue until convergence of successive iterates is achieved. A code is written and compiled in MATLAB using the following data.

Different simulations can be carried out using various values of parameters. In the present numerical approach, we use the following parameters values taken from [6].

Since control and state functions are on different scales, the weight constant value is chosen as follows: \( A = 10000, B = 10000, F = 10000, \) and \( G = 10000 \) and with the initial value of \( H(0) = 14000000, P(0) = 66600000, E(0) = 13000000, D(0) = 6200000, C_F(0) = 4500000, \) and \( C_E(0) = 2000000 \) (Figure 2).

After the parameter values (Tables 1 and 2), we noted that diabetics without complications after 120 months decreased from \( 6.2 \times 10^6 \) to \( 3.05 \times 10^6 \) (Figure 2). This transformation is due to three main things: first, it is by the genetic factors. Second, due to the negative impact of behavioral factors on the patient (nutrition pattern and psychological and moral problems) and the third by sudden shock (family problem, work problem), we noted that diabetics with treatable complications are increasing. Indeed, we noted that the number of the transition becomes from \( 4.5 \times 10^6 \) to \( 1.49 \times 10^7 \) (Figure 2) and, as mentioned above, has disease progression for diabetics without complications, and also a sudden shift in the potential for people diagnosed with diabetics by means of genetics and with negative impact of behavioral factors.

We noted that diabetics with serious complications are increasing and that the number of the transition becomes from \( 2 \times 10^6 \) to \( 1 \times 10^9 \) (Figure 1) and, as mentioned above, has disease progression for diabetics without complications by sudden shock and with negative impact of behavioral factors, and by developing the disease of diabetics with treatable complications.

In this formulation, there are initial conditions for the state variables and terminal conditions for the adjoints.

That is, the optimality system is a two-point boundary value problem with separated boundary conditions at time steps \( i = 0 \) and \( i = T_f \). We solve the optimality system by an iterative method with forward solving of the state system followed by backward solving of the adjoint system. We start with an initial guess for the controls at the first iteration, and then before the next iteration, we update the controls by using the characterization.

We continue until convergence of successive iterates is achieved.

The proposed control strategy in this work helps to achieve several objectives.

5.1. Strategy A. In this strategy, we applied two controls \( u_1(t) \) and \( u_2(t) \) in order to reduced the number of diabetics with treatable complications to diabetics without complications. Through Figure 2, we noted that after applied different strategies, the number of diabetics with treatable complications decreased from \( 1.49 \times 10^7 \) to \( 1.26 \times 10^7 \) by the end of the strategy (Figure 3).

The reason of this increase was justified by the fact that the number of diabetics with treatable complications will become diabetics without complications. For

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**Table 3: Evolution of the number of diabetics with two controls \( u_1(t) \) and \( u_2(t) \) after 120 days.**

| Population without control after 120 days | Without control \( u_i(t) \) | With two controls \( u_i(t) \) |
|------------------------------------------|-----------------------------|-------------------------------|
| Diabetics without complications \( 1.26 \times 10^6 \) | \( 1.78 \times 10^7 \) | \( 2.14 \times 10^7 \) |
| Diabetics with treatable complications \( 10^7 \) | \( 2.14 \times 10^7 \) | \( 2.14 \times 10^7 \) |

**Figure 3:** The evolution of the number of diabetics with and without complications without controls.
effectiveness of this strategy, we added the elements of follow-up and psychological support and education about the negative impact of behavioral factors which are represented in the proposed strategy by the optimal controls variables \( u_1(t), u_2(t) \), and \( u_3(t) \) (Figure 3 and Table 3), combining follow-up and psychological support with treatment and education results in an obvious decreased in the number of diabetics with treatable complications.

5.2. Strategy C: Control with Awareness Program, Treatment, and Psychological Support with Follow-Up. We combined three optimal controls \( u_1(t), u_2(t), \) and \( u_3(t) \).

In this strategy, the three optimal controls \( u_1(t), u_2(t), \) and \( u_3(t) \) are activated at the same time, in order to reduced the number of diabetics with treatable complications to diabetics without complications (Figure 4).

In this strategy (Figure 4 and Table 4), we used three controls optimal \( u_1(t), u_2(t), \) and \( u_3(t) \). That is, we combined the previous two strategies to achieved better results that represented treatment, and psychological support with follow-up, and also awareness program through education and media for lower the negative impact of behavioral factors.

In Figure 4 and Table 4, we observe that the number of diabetics with treatable complications is decreasing from \( 1.5 \times 10^7 \) to \( 1.28 \times 10^7 \), and also, the number of diabetics with serious complications is decreasing from \( 10^8 \) to \( 7.32 \times 10^7 \).

5.3. Strategy D: Prevention and Protection from Diabetes. We use only the optimal control \( u_4(t) \).

In this strategy, we focus the effort of the awareness campaign to reduce the negative impact of behavioral factors (Figure 5).

In this strategy, we used control \( u_4(t) \) (Figure 5 and Table 5); the objective of this control \( u_4(t) \) is to raise awareness campaigns for this target group on the risks of diabetes and its complications as cardiovascular disease, blindness, kidney failure, and lower limb amputation, with tracking healthy and balanced diet program. After Figure 5 and Table 5, we observed that the number of diabetics without complications is decreasing from \( 1.29 \times 10^6 \) to \( 10^6 \).

Remark 6. We could also merge multiple assemblies as \( (u_1(t), u_2(t), u_3(t)), \) and \( (u_1(t), u_2(t), u_3(t), u_4(t)) \) and thus get a variety of results.

6. Conclusion

In this paper, we formulated a mathematical model of populations of diabetics, having six compartments: prediabetics through the genetics effects and others by behavioral factors, diabetics without complications, and diabetics with treatable and serious complications, in order to minimize the number of diabetics with treatable complications, and reduce the effect of behavioral factors. We also introduced four controls which, respectively, represent awareness program through education and media, treatment, and psychological support with follow-up. We applied the results of the control theory, and we managed to obtain the characterizations of the optimal controls. The numerical simulation of the obtained results showed the effectiveness of the proposed control strategies.
Data Availability

The disciplinary data used to support the findings of this study have been deposited in the Network Repository (http://www.networkrepository.com).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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