FROM BIPOLAR EULER-POISSON SYSTEM TO UNIPOLAR EULER-POISSON SYSTEM IN THE PERSPECTIVE OF MASS

SHUAI XI\textsuperscript{1}, LIANG ZHAO\textsuperscript{2}

\textsuperscript{1}College of Mathematics and Systems Science, Shandong University of Science and Technonlogy, Qingdao 266590, P. R. China
\textsuperscript{2}School of Mathematical Sciences, Shanghai Jiao Tong University
Shanghai 200240, P. R. China

Abstract. The main purpose of this paper is to provide an effective procedure to study rigorously the relationship between unipolar and bipolar Euler-Poisson system in the perspective of mass. Based on the fact that the mass of an electron is far less than that of an ion, we amplify this property by letting $m_e/m_i \to 0$ and using two different singular limits to illustrate it, which are zero-electron mass limit and infinity-ion mass limit. We use the method of asymptotic expansion to handle the problem and find that the limiting process from bipolar to unipolar system is actually the process of decoupling, but not the vanishing of equations of the corresponding other particle.

Keywords: Euler-Poisson system; zero-electron mass limit; infinity-ion mass limit; unipolar; bipolar.

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1. Introduction

In the paper, we mainly discuss the fundamental relationship between the unipolar and bipolar system in the perspective of mass based on the famous Euler-Poisson system, which plays an important role in describing the movement of charged fluids (ions and electrons) in semi-conductors or plasmas. We consider an un-magnetized plasma consisting of electrons with charge $-1$ and ions with charge $+1$. More specifically, the scaled Euler-Poisson system in the $d$ dimension space $\mathbb{R}^d$ can be described as, with $e$ standing for the electrons and $i$ the
ions,
\[
\begin{align*}
\partial_t n_e + \text{div} (n_e u_e) &= 0, \\
m_e \partial_t (n_e u_e) + m_e \text{div} (n_e u_e \otimes u_e) + \nabla p_e (n_e) &= n_e \nabla \phi, \\
\partial_t n_i + \text{div} (n_i u_i) &= 0, \\
m_i \partial_t (n_i u_i) + m_i \text{div} (n_i u_i \otimes u_i) + \nabla p_i (n_i) &= -n_i \nabla \phi, \\
-\lambda^2 \Delta \phi &= n_i - n_e,
\end{align*}
\]
(1.1)

here for \( \nu = i, e \), \( n_\nu \) stand for the particle density and \( u_\nu \) the average velocity for ions and electrons respectively, \( \phi \) is the scaled electric potential. These are all functions of the position \( x \in \mathbb{R}^d \) and the time \( t > 0 \). The pressure functions \( p_\nu (n_\nu) \) are supposed to be smooth and strictly increasing for all \( n_\nu > 0 \). Usually, they are of the form
\[
p_\nu (n_\nu) = a_\nu^2 n_\nu^{\varepsilon_\nu}, \quad \nu = e, i,
\]
where \( \varepsilon_\nu \geq 1 \) and \( a_\nu > 0 \) are constants. The fluid is called isothermal if \( \varepsilon_e = 1 \) and adiabatic if \( \varepsilon_i > 1 \). The parameters \( m_\nu \) stand for the mass of electrons and ions respectively and \( \lambda \) is the scaled Debye length. For details of the scaling and physical background, we refer to [9] and the reference therein. In order to make \( \phi \) uniquely determined, we add a restriction condition
\[
\phi (x) \to 0, \quad \text{when} \quad |x| \to \infty.
\]

Physicians believe that the electrons can be regarded as background when studying the equations of ions because of the huge mass difference between them. That is to say, unipolar model was formerly derived from bipolar model by assuming that the mass of electrons can be neglected. However, this lacks rigorous proof. To study this, we amplify the relationship between the mass of ions and electrons by letting \( m_e / m_i \to 0 \) and use two different singular limits to illustrate it, which are zero-electron mass limit and infinity-ion mass limit. We will prove that the unipolar models are indeed the simplification of the bipolar models.

As is mentioned above, the study of the limit \( m_e / m_i \to 0 \) consists of two natural ways. One is to let \( m_i = 1 \) and \( m_e \to 0 \), which is the known-to-all **zero-electron mass limit**. The limit is based on the assumption that \( m_e \) can be ignored when \( m_i \) is fixed. Letting \( m_e \to 0 \) and \( m_i = 1 \) in (1.1), formally we get the system for ions
\[
\begin{align*}
\partial_t n_i + \text{div} (n_i u_i) &= 0, \\
\partial_t (n_i u_i) + \text{div} (n_i u_i \otimes u_i) + \nabla p_i (n_i) &= -n_i \nabla \phi, \\
-\lambda^2 \Delta \phi &= n_i - n_e,
\end{align*}
\]
(1.2)
and the system for electrons

\[
\begin{cases}
\partial_t n_e + \text{div} (n_e u_e) = 0, \\
\partial_t u_e + (u_e \cdot \nabla) u_e + \nabla P_e = 0,
\end{cases}
\tag{1.3}
\]

where \( P_e \) is a function of \( n_e \) and \( u_e \). At the same time, we can also obtain the Maxwell-Boltzmann relationship \[14\],

\[\nabla p_e(n_e) = n_e \nabla \phi,\]

which, together with (1.3), are used to replace \( n_e \) in (1.2), leading to the solvability of (1.2) (see the details in Section 2). We then take back \( n_e \) into (1.3) to solve for \( u_e \) and \( P_e \), which yields the formal limiting equations for the electrons and success in decoupling. Thus we get the unipolar model of ions (1.2) from the bipolar model (1.1).

Another way is to consider just the opposite, we set \( m_e = 1 \) and \( m_i \to \infty \). It is based on the fact that \( m_i \) turns to infinity when \( m_e \) is fixed. We call it **infinity-ion mass limit.** We let \( m_e = 1 \) and \( m_i \to +\infty \) in (1.1), which yields the formal limit system for electrons

\[
\begin{cases}
\partial_t n_e + \text{div} (n_e u_e) = 0, \\
\partial_t (n_e u_e) + \text{div} (n_e u_e \otimes u_e) + \nabla p_i(n_e) = n_e \nabla \phi, \\
- \lambda^2 \triangle \phi = n_i - n_e,
\end{cases}
\tag{1.4}
\]

and the system for ions

\[
\begin{cases}
\partial_t n_i + \text{div}(n_i u_i) = 0, \\
\partial_t u_i + (u_i \cdot \nabla) u_i = 0.
\end{cases}
\tag{1.5}
\]

It is easy to get the existence of (1.5) by the energy method, then we substitute \( n_i \) we have solved in (1.5) into (1.4). The solvability of (1.4) is guaranteed by Kato[11] and Majda[16]. Thus, the decoupling is success. That is to say we get the unipolar model of electrons (1.4) from the bipolar model (1.1). The details of the formal asymptotic analysis can be found in Section 2.

The main purpose of this paper is to provide an effective procedure to study rigorously the relationship between unipolar and bipolar systems in the perspective of mass. As to the zero-electron mass limit, many former works have been done (see [1], [2], [10] and [24]). They tended to believe that when letting \( m_e \to 0 \), the equations of the ions stay the same (see (1.2)), so it is rational to ignore the limiting process of the equations of the ions, and put emphasis on the equations of electrons. This is a misunderstanding. Although the system for ions (1.2) looks the same as the equations of ions in (1.1), the value of \( u_i \) and \( n_i \) are different, which actually are dependent on the parameter \( \varepsilon \equiv \sqrt{m_e/m_i} \). Thus, the system for ions (1.2) is only invariant in forms. It is improper to just ignore the effect of the ions, and only do the asymptotic analysis to the equations of electrons when considering the two-fluid
model. Thus, the limit process from bipolar to unipolar system is actually the process of decoupling, but not the vanishing of equations of the corresponding other particle.

The paper is organized as follows. In Section 2, we first introduce some basic lemmas and give the formal asymptotic analysis as well as the error estimates. The main results of this paper is Theorem 2.1 and Theorem 2.2 which are stated at the end of Section 2. Section 3 and Section 4 are devoted to detailed proof of Theorem 2.1 and Theorem 2.2 in the sense of zero-electron mass limit and the infinity-ion mass limit, respectively.

2. Preliminaries and main results

2.1. Notations and inequalities. In the following, we denote by $C$ a generic positive constant independent of $\varepsilon$. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ and $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{N}^d$, $\beta < \alpha$ stands for $\beta \neq \alpha$ and $\beta_j \leq \alpha_j$ for all $j = 1, \cdots, d$. We denote by $\| \cdot \|_s$, $\| \cdot \|$ and $\| \cdot \|_\infty$ the norm of the usual Sobolev spaces $H^s(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$, respectively. The inner product in $L^2(\mathbb{R}^d)$ is denoted by $\langle \cdot, \cdot \rangle$. Throughout the paper, we denote $\nu = e, i,$ and

$$\varepsilon = \sqrt{\frac{m_e}{m_i}}.$$

Lemma 2.1. (Moser-type calculus inequalities, see [12] and [16].) Let $s \geq 1$ be an integer. Suppose $u \in H^s(\mathbb{R}^d)$, $\nabla u \in L^\infty(\mathbb{R}^d)$ and $v \in H^{s-1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then for all $\alpha \in \mathbb{N}^d$ with $1 \leq |\alpha| \leq s$ and all smooth function $f$, we have $\partial^\alpha_x(\nabla u) - u \partial^\alpha_x v \in L^2(\mathbb{R}^d)$, $\partial^\alpha_x f(u) \in L^2(\mathbb{R}^d)$ and

$$\|\partial^\alpha_x(\nabla u) - u \partial^\alpha_x v\| \leq C_s \left( \|\nabla u\|_\infty \|\nabla^{|\alpha|-1} v\| + \|\nabla^{|\alpha|} u\| |v|_\infty \right),$$

$$\|\partial^\alpha_x f(u)\| \leq C_s (\|\nabla u\|_\infty + 1)^{|\alpha|-1} \|\nabla^{|\alpha|} u\|,$$

where the constant $C_s > 0$ depends on $\|u\|_\infty$ and $s$, and $C_s > 0$ depends only on $s$. Moreover, if $s > \frac{d}{2} + 1$, then the embedding $u \in H^s(\mathbb{R}^d) \hookrightarrow W^{1,\infty}(\mathbb{R}^d)$ is continuous and we have

$$\|\partial^\alpha_x(\nabla u) - u \partial^\alpha_x v\| \leq C_s \|\nabla u\|_{s-1} \|v\|_{s-1}.$$

Lemma 2.2. Let $s > \frac{d}{2} + 2$ be an integer and $d \leq 3$. For all $\alpha \in \mathbb{N}^d$ with $1 \leq |\alpha| \leq s$, if $u \in H^s(\mathbb{T}^d)$ and $v \in H^{|\alpha|}(\mathbb{T}^d)$, then

$$\|\partial^\alpha_x(\nabla u) - u \partial^\alpha_x v - \sum_{\alpha_1 \neq 0} m_{\alpha, \alpha_1} \partial^\alpha_{x_1} u \partial^\alpha_{x_2} v\| \leq \sum_{|\alpha - \gamma| \geq 2} m_{\alpha, \gamma} \|\nabla^{|\alpha| - \gamma} u \partial^\alpha_{x_1} v\|$$

$$\leq C_s (\|\nabla^2 u\|_\infty \|\nabla^{|\alpha|-2} v\| + \|\nabla^{|\alpha|} u\| |v|_\infty \|v\|_\infty)$$

$$\leq C_s \|\nabla u\|_{s-2} \|v\|_{s-2},$$

where $\alpha'$ is a multi-index and $\partial^\alpha_x = \partial^\alpha_{x_1} = \partial^\alpha_{x_2}$ and $\sum_{\alpha_1 \neq 0} m_{\alpha, \alpha_1} \partial^\alpha_{x_1} u \partial^\alpha_{x_2} v$ denotes the term related to the first order derivatives of $u$ by using the Leibniz Formulas.
Lemma 2.3. For any smooth function $u : \mathbb{R}^d \to \mathbb{R}^d$ and $\Phi : \mathbb{R}^d \to \mathbb{R}$, we have

$$|\langle u \Delta \Phi, \nabla \Phi \rangle| \leq C \|\nabla u\|_{\infty} \|\nabla \Phi\|^2,$$

where the constant $C > 0$ is independent of $u$ and $\Phi$.

The next result concerns the local existence of smooth solutions which can be easily obtained by employing the theory of Kato [11] for the symmetrizable hyperbolic system.

Proposition 2.1. Let $s > \frac{d}{2} + 1$ be an integer and $(n^{\nu_{0}, m_i}, u^{\nu_{0}, m_i}) \in H^s(\mathbb{R}^d)$ with $n^{\nu, m_i} \geq 2n$ for some given constant $n > 0$, independent of $m_e, m_i$. Then there exists $T_1^{m_e, m_i} > 0$ such that the Cauchy problem (2.1) has a unique smooth solution $(n^{\nu, m_i}, u^{\nu, m_i}, \phi^{m_e, m_i})$ defined in time interval $[0, T_1^{m_e, m_i}]$, satisfying $n^{m_e, m_i} \geq n$ and

$$(n^{\nu, m_i}, u^{\nu, m_i}) \in C \left([0, T_1^{m_e, m_i}]; H^s(\mathbb{R}^d)\right) \cap C^1 \left([0, T_1^{m_e, m_i}]; H^{s-1}(\mathbb{R}^d)\right),$$

$$\phi^{m_e, m_i} \in C \left([0, T_1^{m_e, m_i}]; H^{s+1}(\mathbb{R}^d)\right) \cap C^1 \left([0, T_1^{m_e, m_i}]; H^s(\mathbb{R}^d)\right).$$

In order to simply the later proof, we introduce the enthalpy function, defined as

$$h'_\nu(n) = \frac{p'_\nu(n)}{n} \quad \text{and} \quad h_\nu(1) = 0,$$

then for $n_\nu > 0$, system (1.11) can be rewritten into

$$\begin{cases}
\partial_t n_e + \text{div}(n_e u_e) = 0, \\
\partial_t u_e + (u_e \cdot \nabla) u_e + \frac{\nabla h_e(n_e)}{m_e} = \nabla \phi, \\
\partial_t n_i + \text{div}(n_i u_i) = 0, \\
\partial_t u_i + (u_i \cdot \nabla) u_i + \frac{\nabla h_i(n_i)}{m_i} = -\nabla \phi, \\
-\lambda^2 \Delta \phi = n_i - n_e, \\
t = 0: (n_\nu, u_\nu) = (n_\nu, u_\nu), \quad \nu = e, i.
\end{cases} \tag{2.1}$$

2.2. Asymptotic analysis for zero-electron mass limit ($\varepsilon \to 0, m_i = 1$).

2.2.1. Formal expansion. As to the zero-electron mass limit, setting $m_i = 1$, we look for an approximation of solution $(n^{\varepsilon, 1}_\nu, u^{\varepsilon, 1}_\nu, \phi^{\varepsilon, 1})$ to (2.1) in the form of power series. Assume that the initial data of $(n^{\varepsilon, 1}_\nu, u^{\varepsilon, 1}_\nu, \phi^{\varepsilon, 1})$ admit an asymptotic expansion with respect to $\varepsilon$, for $\nu = e, i$,

$$\left(n^{\varepsilon, 1}_\nu, u^{\varepsilon, 1}_\nu, \phi^{\varepsilon, 1}\right)(x) = \sum_{j \geq 0} \varepsilon^{2j} \left(\tilde{n}^{\varepsilon, j}_\nu, \tilde{u}^{\varepsilon, j}_\nu, \tilde{\phi}^{\varepsilon, j}\right)(x), \tag{2.2}$$

where $(\tilde{n}^{\varepsilon, j}_\nu, \tilde{u}^{\varepsilon, j}_\nu, \tilde{\phi}^{\varepsilon, j})_{j \geq 0}$ are sufficiently smooth, and the following ansatz:

$$\left(n^{\varepsilon, 1}_\nu, u^{\varepsilon, 1}_\nu, \phi^{\varepsilon, 1}\right)(t, x) = \sum_{j \geq 0} \varepsilon^{2j} \left(n^{\varepsilon, j}_\nu, u^{\varepsilon, j}_\nu, \phi^{\varepsilon, j}\right)(t, x). \tag{2.3}$$
In what follows, we use a formal expansion formula

\[
h_\nu \left( \sum_{i \geq 0} \varepsilon^{2i} n_\nu^i \right) = h_\nu \left( n_\nu^0 \right) + h_\nu \left( n_\nu^0 \right) \sum_{j \geq 1} \varepsilon^{2j} n_\nu^j + \sum_{j \geq 2} \varepsilon^{2j} h_\nu^{j-1} \left( (n_\nu^k)_{k \leq j-1} \right),
\]

where \( \{h_\nu^j\}_{j \geq 1} \) are smooth functions depending only on \( h_\nu \) and \( (n_\nu^k)_{k \leq j} \). For simplicity, from now on, we denote \((n_\nu^{e,1}, u_\nu^{e,1}, \phi^{e,1})\) and \((n_\nu^{\nu,j}, u_\nu^{\nu,j}, \phi^{\nu,j})_{j \geq 0}\) by \((n_\nu^e, u_\nu^e, \phi^e)\) and \((n_\nu^\nu, u_\nu^\nu, \phi^\nu)_{j \geq 0}\) in the section of zero-electron mass limit. Substituting the expansions (2.3) into system (2.1), we obtain

1. The leading profiles \((n_\nu^0, u_\nu^0, \phi^0)\) satisfy the following system

\[
\begin{align*}
\partial_t n_\nu^0 + \text{div}(n_\nu^0 u_\nu^0) &= 0, \\
\partial_t u_\nu^0 + (u_\nu^0 \cdot \nabla) u_\nu^0 + \nabla P_\nu^0 &= 0, \\
\partial_t n_\nu^i + \text{div}(n_\nu^i u_\nu^i) &= 0, \\
\partial_t u_\nu^i + (u_\nu^i \cdot \nabla) u_\nu^i + \nabla (h_\nu(n_\nu^i) + \phi^0) &= 0, \\
-\lambda^2 \Delta \phi^0 &= n_\nu^0 - n_\nu^e,
\end{align*}
\]

where

\[
P_\nu^0 = h_\nu'(n_\nu^e)n_\nu^1 - \phi^1,
\]

with the initial data

\[
(n_\nu^0, u_\nu^0) (0, x) = (\bar{n}_\nu^0, \bar{u}_\nu^0) (x), \quad x \in \mathbb{R}^d.
\]

Notice the \( \varepsilon^{-2} \) term

\[
\nabla h_\nu(n_\nu^0) - \nabla \phi^0 = 0,
\]

we deduce that \( n_\nu^0 = h_\nu^{-1}(\phi^0) \). Thus the equations for ions and the Poisson equation in (2.4) are actually the following unipolar Euler-Poisson system for ions

\[
\begin{align*}
\partial_t n_\nu^0 + \text{div}(n_\nu^0 u_\nu^0) &= 0, \\
\partial_t u_\nu^0 + (u_\nu^0 \cdot \nabla) u_\nu^0 + \nabla (h_\nu(n_\nu^0) + \phi^0) &= 0, \\
-\lambda^2 \Delta \phi^0 &= n_\nu^0 - h_\nu^{-1}(\phi^0).
\end{align*}
\]

The solvability of the Poisson equation can be found in [15], in which \( \phi^0 \) is expressed as a function of \( n_\nu^0 \). Thus the first two equations are hyperbolic system, of which the unique local smooth solution exists due to the famous work of Kato[11] and Majda[16]. At the same time \( n_\nu^0 \) is also known since it is a function of \( \phi^0 \), and \((u_\nu^e, P_\nu^0)\) satisfy the following incompressible Euler equations:

\[
\begin{align*}
\text{div} \left( n_\nu^0 u_\nu^0 \right) &= -\partial_t n_\nu^0, \\
\partial_t u_\nu^0 + (u_\nu^0 \cdot \nabla) u_\nu^0 + \nabla P_\nu^0 &= 0.
\end{align*}
\]
Then Cauchy problem \([2.4]\) with \([2.6]\) has then been solved, \(P^0_e\) is used to solve the \((n^1_e, \phi^1)\) by \([2.5]\).

(2) At the \(\varepsilon^2\) term, we find

\[
\begin{aligned}
\partial_t n^1_e + \text{div} \left(n^0_e u^1_e + n^1_e u^0_e\right) &= 0, \\
\partial_t u^1_i + (u^0_e \cdot \nabla) u^1_i + (u^1_e \cdot \nabla) u^0_e + \nabla P^1_e &= 0, \\
\partial_t n^1_i + \text{div} \left(n^0_i u^1_i + n^1_i u^0_i\right) &= 0, \\
\partial_t u^1_i + (u^0_i \cdot \nabla) u^1_i + (u^1_i \cdot \nabla) u^0_i + \nabla (h'^1_i(n^0_i) n^1_i + \phi^1) &= 0, \\
- \lambda^2 \Delta \phi^1 &= n^1_i - n^1_e,
\end{aligned}
\]

with initial data

\[
(n^1_v, u^1_v)(0, x) = (\bar{n}^1_v, \bar{u}^1_v)(x), \quad x \in \mathbb{R}^d.
\]

Since \(P^0_e\) is known, we substitute \(\phi^1\) into the Poisson equation in \([2.10]\) with

\[
n^1_e = \frac{P^0_e - \phi^1}{h'^1_e(n^0_e)},
\]

which implies \(n^1_e\) is a function of \(n^1_i\). Now the equations for ions in \([2.10]\) turn out to be the following linear system

\[
\begin{aligned}
\partial_t n^1_i + \text{div} \left(n^0_i u^1_i + n^1_i u^0_i\right) &= 0, \\
\partial_t u^1_i + (u^0_i \cdot \nabla) u^1_i + (u^1_i \cdot \nabla) u^0_i + \nabla (h'^1_i(n^0_i) n^1_i + \phi^1) &= 0, \\
- \Delta \phi^1 &= n^1_i - \frac{P^0_e - \phi^1}{h'^1_e(n^0_e)}.
\end{aligned}
\]

for which we can get the unique solution \(n^1_i, u^1_i\) and \(\phi^1\), and thus \(n^1_e\). Also, \(u^1_e\) and \(P^1_e\) satisfy the following

\[
\begin{aligned}
\partial_t n^1_e + \text{div} \left(n^0_e u^1_e + n^1_e u^0_e\right) &= 0, \\
\partial_t u^1_e + (u^0_e \cdot \nabla) u^1_e + (u^1_e \cdot \nabla) u^0_e + \nabla P^1_e &= 0,
\end{aligned}
\]

in which \(P^1_e\) is used to solve \((u^2_e, \phi^2)\).

(3) For \(j \geq 2\), in general the profiles \((n^j_v, u^j_v, \phi^j)\) are obtained by induction. Assume that \((n^k_v, u^k_v, \phi^k)_{0 \leq k \leq j-1}\) are smooth and already determined in previous steps. Then \((n^j_v, u^j_v, \phi^j)\)
satisfy the linear system

\[
\begin{aligned}
\partial_t n^j_e + \text{div} (n^0_e u^j_e + n^j_0 u^0_e) &= - \sum_{k=1}^{j-1} \text{div} (n^k_e u^j_e), \\
\partial_t u^j_e + (u^0_e \cdot \nabla) u^j_e + (u^j_e \cdot \nabla) u^0_e + \nabla P^j_e &= - \sum_{k=1}^{j-1} (u^k_e \cdot \nabla) u^{j-k}, \\
\partial_t n^j_i + \text{div} (n^0_i u^j_i + n^j_0 u^0_i) &= - \sum_{k=1}^{j-1} \text{div} (n^k_i u^j_i), \\
\partial_t u^j_i + (u^0_i \cdot \nabla) u^j_i + (u^j_i \cdot \nabla) u^0_i + \nabla P^j_i &= - \sum_{k=1}^{j-1} (u^k_i \cdot \nabla) u^{j-k}, \\
- \lambda^2 \Delta \phi^j &= n^j_i - n^j_e, 
\end{aligned}
\]  

(2.12)

where

\[
\begin{aligned}
P^j_e &= h'(n^0_e) n^{j+1}_e + h_e^i ((n^k_e)_{k \leq j}) - \phi^{j+1}, \\
P^j_i &= h'(n^0_i) n^{j+1}_i + h_i^1 ((n^k_i)_{k \leq j-1}) + \phi^j, 
\end{aligned}
\]

with the initial data

\[
(n^0_\nu, u^0_\nu) (0, x) = (\bar{n}^j_\nu, \bar{u}^j_\nu) (x), \quad \nu = e, i.
\]  

(2.13)

Generally, we can get \((n^j_\nu, u^j_\nu, \phi^j)\) from \(P^{j-1}_e\) and the third to fifth equations in (2.12), and \((u^j_\nu, P^j_\nu)\) from the first two equations in (2.12).

**Proposition 2.2.** Assume that the initial data \((\bar{n}^j_\nu, \bar{u}^j_\nu, \bar{\phi}^j)_{j \geq 0}\) are sufficiently smooth with \(\bar{n}^0_\nu > 0\) in \(\mathbb{R}^d\). Then there exist the unique smooth profiles \((n^j_\nu, u^j_\nu, \phi^j)_{j \geq 0}\), solutions of the problems (2.4) with (2.6), (2.10) with (2.11) and (2.12) with (2.13) in the time interval \([0, T^n_\nu]\). In other words, there exists a unique asymptotic expansion up to any order of the form (2.3).

2.2.2. **Error estimates and main result.** Let \(m \in \mathbb{N}\) be a fixed integer and \((n^{m}_{\nu, \varepsilon}, u^{m}_{\nu, \varepsilon}, \phi^{m}_{\varepsilon})\) be the exact solution to problem (2.1) (with \(m_i = 1\)) defined in time interval \([0, T^n_{1, \varepsilon}]\). We denote by \((n^{m}_{\nu, \varepsilon}, u^{m}_{\nu, \varepsilon}, \phi^{m}_{\varepsilon})\) the approximate solution of order \(m\) defined in \([0, T^n_{1, \varepsilon}]\) by

\[
(n^{m}_{\nu, \varepsilon}, u^{m}_{\nu, \varepsilon}, \phi^{m}_{\varepsilon}) = \sum_{j=0}^{m} \varepsilon^{2j} \left( n^{j}_{\nu, \varepsilon}, u^{j}_{\nu, \varepsilon}, \phi^{j}_{\varepsilon} \right),
\]

where \((n^{j}_{\nu, \varepsilon}, u^{j}_{\nu, \varepsilon}, \phi^{j}_{\varepsilon})_{0 \leq j \leq m}\) are constructed in the previous subsection. The proof of the convergence of the asymptotic expansion (2.3) is to establish the limit

\[
(n^{\varepsilon}_{\nu, \varepsilon}, u^{\varepsilon}_{\nu, \varepsilon}, \phi^{\varepsilon}) - (n^{m}_{\nu, \varepsilon}, u^{m}_{\nu, \varepsilon}, \phi^{m}_{\varepsilon}) \longrightarrow 0,
\]

and its convergence rate as \(\varepsilon \to 0\) in a time interval independent of \(\varepsilon\), when the convergence holds at \(t = 0\).
For \( \nu = e, i \), define the remainders \( (R_{n_{\nu}}^{e,1,m}, R_{u_{\nu}}^{e,1,m}) \) by

\[
\begin{aligned}
\partial_t n_{\nu, \varepsilon}^m + \text{div} \left( n_{\nu, \varepsilon}^m u_{\nu, \varepsilon}^m \right) &= R_{n_{\nu}}^{e,1,m}, \\
\partial_t u_{\nu, \varepsilon}^m + \left( u_{\nu, \varepsilon}^m \cdot \nabla \right) u_{\nu, \varepsilon}^m + \frac{1}{\varepsilon^2} \nabla \left( h_e(n_{\nu, \varepsilon}^m) - \phi_{\varepsilon}^m \right) &= R_{u_{\nu}}^{e,1,m}, \\
\partial_t n_{i, \varepsilon}^m + \text{div} \left( n_{i, \varepsilon}^m u_{i, \varepsilon}^m \right) &= R_{n_{i}}^{e,1,m}, \\
\partial_t u_{i, \varepsilon}^m + \left( u_{i, \varepsilon}^m \cdot \nabla \right) u_{i, \varepsilon}^m + \nabla \left( h_i(n_{i, \varepsilon}^m) + \phi_{\varepsilon}^m \right) &= R_{u_{i}}^{e,1,m}, \\
- \lambda^2 \Delta \phi_{\varepsilon}^m &= n_{i, \varepsilon}^m - n_{\nu, \varepsilon}^m.
\end{aligned}
\]

(2.14)

It is clear that the convergence rate depends strongly on the order of the remainders with respect to \( \varepsilon \). Since the profiles \( (n_j^i, u_j^i, \phi_j)_{0 \leq j \leq m} \) are sufficiently smooth, we have

**Proposition 2.3.** If (2.4), (2.10) and (2.12) hold, then we can find \( \tilde{R}_{n_{\nu}}^{e,1,m} \), such that

\[
\text{div} \tilde{R}_{n_{\nu}}^{e,1,m} = R_{n_{\nu}}^{e,1,m},
\]

and for all integers \( m \geq 0 \) and \( s \geq 0 \), the remainders satisfy

\[
\sup_{0 \leq t \leq T_1} \| \left( \tilde{R}_{n_{\nu}}^{e,1,m}, R_{u_{\nu}}^{e,1,m}(t) \right) \|_s \leq C_m \varepsilon^{2m},
\]

(2.15)

where \( C_m > 0 \) is a constant independent of \( \varepsilon \).

**Proof.** By the definition of \( R_{n_{\nu}}^{e,1,m} \) in (2.14), we have

\[
R_{n_{\nu}}^{e,1,m} = \partial_t n_{\nu, \varepsilon}^m + \text{div} \left( n_{\nu, \varepsilon}^m u_{\nu, \varepsilon}^m \right)
\]

\[
= \sum_{j=0}^{m} \varepsilon^{2j} \partial_t n_j^\nu + \text{div} \left( \left( \sum_{j=0}^{m} \varepsilon^{2j} n_j^\nu \right) \left( \sum_{j=0}^{m} \varepsilon^{2j} u_j^\nu \right) \right)
\]

\[
= \partial_t n_0^\nu + \text{div} \left( n_0^\nu u_0^\nu \right) + \sum_{j=1}^{m} \varepsilon^{2j} \left( \partial_t n_j^\nu + \sum_{k=0}^{j} \text{div} \left( n_k^\nu u_j^{\nu,k} \right) \right)
\]

\[
+ \sum_{j=1}^{m} \varepsilon^{2j+2m} \left( \sum_{k=j}^{m} \text{div} \left( n_k^\nu u_{\nu}^{m+j-k} \right) \right)
\]

\[
= \text{div} \left( \sum_{j=1}^{m} \varepsilon^{2j+2m} \left( \sum_{k=j}^{m} n_k^\nu u_{\nu}^{m+j-k} \right) \right) = \text{div} \left( \tilde{R}_{n_{\nu}}^{e,1,m} \right),
\]

then

\[
\sup_{0 \leq t \leq T_1} \| \tilde{R}_{n_{\nu}}^{e,1,m}(t) \|_s \leq C_m \varepsilon^{2m+2}.
\]
By the definition of $R_{u_e}^{1,m}$, we have

$$R_{u_e}^{1,m} = \partial_t u_e^m + (u_e^m \cdot \nabla) u_e^m + \frac{1}{\epsilon^2} \nabla \left( h_e (n_e^m) - \phi_e^m \right)$$

$$= \sum_{j=0}^m \epsilon^{2j} \partial_t u_e^j + \left( \left( \sum_{j=0}^m \epsilon^{2j} u_e^j \right) \cdot \nabla \right) \sum_{j=0}^m \epsilon^{2j} u_e^j$$

$$+ \frac{1}{\epsilon^2} \nabla \left( h_e (n_e^0) + h_e' (n_e^0) \sum_{j=1}^{m-1} \epsilon^{2j} n_e^j + \sum_{j=2}^m \epsilon^{2j} h_e^{j-1} \left( (n_e^k)_{k \leq j-1} \right) \right) - \frac{1}{\epsilon^2} \sum_{j=0}^m \epsilon^{2j} \nabla \phi^j$$

$$= \frac{1}{\epsilon^2} \nabla (h_e (n_e^0) - \phi^0) + \left( \partial_t u_e^0 + (u_e^0 \cdot \nabla) u_e^0 + \nabla \left( h_e' (n_e^0) n_e^1 - \phi^0 \right) \right)$$

$$+ \sum_{j=1}^{m-1} \epsilon^{2j} \left( \partial_t u_e^j + \sum_{k=0}^j (u_e^k \cdot \nabla) u_e^{j-k} + \nabla \left( h_e' (n_e^0) n_e^j + h_e^{j-1} \left( (n_e^k)_{k \leq j-1} \right) \phi^j \right) \right)$$

$$+ O(\epsilon^{2m}),$$

and by the definition of $R_{u_i}^{1,m}$, we have

$$R_{u_i}^{1,m} = \partial_t u_i^m + (u_i^m \cdot \nabla) u_i^m + \nabla \left( h_i (n_i^m) + \phi_i^m \right)$$

$$= \sum_{j=0}^m \epsilon^{2j} \partial_t u_i^j + \left( \left( \sum_{j=0}^m \epsilon^{2j} u_i^j \right) \cdot \nabla \right) \sum_{j=0}^m \epsilon^{2j} u_i^j$$

$$+ \nabla \left( h_i (n_i^0) + h_i' (n_i^0) \sum_{j=1}^{m-1} \epsilon^{2j} n_i^j + \sum_{j=2}^m \epsilon^{2j} h_i^{j-1} \left( (n_i^k)_{k \leq j-1} \right) \right) + \sum_{j=0}^m \epsilon^{2j} \nabla \phi^j$$

$$= \partial_t u_i^0 + (u_i^0 \cdot \nabla) u_i^0 + h_i (n_i^0) + \nabla \phi^0$$

$$+ \epsilon^2 \left( \partial_t u_i^0 + (u_i^0 \cdot \nabla) u_i^1 + (u_i^1 \cdot \nabla) u_i^0 + \nabla \left( h_i' (n_i^0) n_i^1 + \phi^0 \right) \right)$$

$$+ \sum_{j=2}^m \epsilon^{2j} \left( \partial_t u_i^j + \sum_{k=0}^j (u_i^k \cdot \nabla) u_i^{j-k} + \nabla \left( h_i' (n_i^0) n_i^j + h_i^{j-1} \left( (n_i^k)_{k \leq j-1} \right) + \phi^j \right) \right)$$

$$+ O(\epsilon^{2m+2}).$$

Hence, (2.4), (2.11), (2.12), and the Maxwell-Boltzmann relationship (2.7), imply (2.15). \(\Box\)

The main result for the zero-electron mass limit is the following convergence result, of which the proof will be given in Section 3.

**Theorem 2.1** (The zero-electron mass limit). Under the conditions of Proposition 2.3, let \( s > \frac{d}{2} + 2 \) and \( m \in \mathbb{N} \) be integers. Assume

$$\left\| (n_e^{1,m} - n_e^{\epsilon,m}(0, \cdot), \epsilon (u_{\nu,0}^{\epsilon} - u_{\nu,0}(0, \cdot))) \right\|_{s} \leq C_1 \epsilon^{2m}, \quad (2.16)$$

where \( C_1 > 0 \) is a constant independent of \( \epsilon \). Then, for the isothermal fluid, there exists a constant \( C_2 > 0 \), which depends on \( T_1^e \) but is independent of \( \epsilon \), such that as \( \epsilon \to 0 \) we have

$$T_1^{e,1} \geq T_1^e$$

and for all integer \( 2m > s \), the solution \((n^{e,1}, u^{e,1}, \phi^{e,1})\), to the problem (2.11)
In what follows, we use a formal expansion defined by

$$\sup_{0 \leq t \leq T_1} \left\| \left( n_{\nu}^{\varepsilon,1} - n_{\nu,\varepsilon}^{e,m}, \varepsilon \left( u_{\nu}^{\varepsilon,1} - u_{\nu,\varepsilon}^{e,m} \right), \nabla \left( \phi^{\varepsilon,1} - \phi_{\nu,\varepsilon}^{e,m} \right) \right) \right\|_s \leq C_2 \varepsilon^{2m-s}.$$ \hspace{1cm} (2.17)

That is to say, the zero-electron-mass limit \( \varepsilon \to 0 \) of the bipolar Euler-Poisson system \((2.11)\) is the unipolar Euler-Poisson equations for ions \((2.8)\) and the incompressible Euler equations \((2.9)\).

2.3. Asymptotic analysis for infinity-ion mass limit \( (\varepsilon \to 0, m_e = 1) \).

2.3.1. Formal expansion. As to the infinity-ion mass limit, setting \( m_e = 1 \), we look for an approximation of solution \( \left( n_{\nu}^{1,\frac{1}{2}}, u_{\nu}^{1,\frac{1}{2}}, \phi^{1,\frac{1}{2}} \right) \) to \((2.1)\) in the form of power series. Assume that the initial data of \( \left( n_{\nu}^{1,\frac{1}{2}}, u_{\nu}^{1,\frac{1}{2}}, \phi^{1,\frac{1}{2}} \right) \) admit an asymptotic expansion with respect to \( \varepsilon \), for \( \nu = e, i \),

$$\left( n_{\nu,0}^{1,\frac{1}{2}}, u_{\nu,0}^{1,\frac{1}{2}}, \phi_{0,\varepsilon}^{1,\frac{1}{2}} \right) (x) = \sum_{j \geq 0} \varepsilon^{2j} \left( \tilde{n}_{\nu}^{i,j}, \tilde{w}_{\nu}^{i,j}, \tilde{\phi}_{0,\varepsilon}^{i,j} \right) (x),$$ \hspace{1cm} (2.18)

where \( \left( \tilde{n}_{\nu}^{i,j}, \tilde{w}_{\nu}^{i,j}, \tilde{\phi}_{0,\varepsilon}^{i,j} \right) \) are sufficiently smooth, and the following ansatz:

$$\left( n_{\nu}^{1,\frac{1}{2}}, u_{\nu}^{1,\frac{1}{2}}, \phi^{1,\frac{1}{2}} \right) (t, x) = \sum_{j \geq 0} \varepsilon^{2j} \left( n_{\nu}^{i,j}, u_{\nu}^{i,j}, \phi^{i,j} \right) (t, x).$$ \hspace{1cm} (2.19)

In what follows, we use a formal expansion defined by

$$h_{\nu} \left( \sum_{i \geq 0} \varepsilon^{2i} n_{\nu}^{i} \right) = h_{\nu} (n_{\nu}^{0}) + h_{\nu}' (n_{\nu}^{0}) \sum_{j \geq 1} \varepsilon^{2j} n_{\nu}^{j} + \sum_{j \geq 2} \varepsilon^{2j} h_{\nu}^{j-1} \left( (n_{\nu}^{k})_{k \leq j-1} \right),$$

where \( \{ h_{\nu}^{j} \}_{j \geq 1} \) are smooth functions depending only on \( (n_{\nu}^{k})_{k \leq j} \). For simplicity, from now on, we denote \( \left( n_{\nu}^{1,\frac{1}{2}}, u_{\nu}^{1,\frac{1}{2}}, \phi^{1,\frac{1}{2}} \right) \) and \( (n_{\nu}^{i,j}, u_{\nu}^{i,j}, \phi^{i,j})_{j \geq 0} \) by \( (n_{\nu}^{e}, u_{\nu}^{e}, \phi^{e}) \) and \( (n_{\nu}^{i,j}, u_{\nu}^{i,j}, \phi^{i,j})_{j \geq 0} \) in the infinity-ion mass limit. Substituting the expansions \((2.19)\) into system \((2.1)\), we obtain

(1) The leading profiles \( (n_{\nu}^{e}, u_{\nu}^{e}, \phi^{0}) \) satisfy the following system

$$\begin{align*}
\partial_{t} n_{i}^{0} + \text{div}(n_{i}^{0} u_{i}^{0}) &= 0, \\
\partial_{t} u_{i}^{0} + (u_{i}^{0} \cdot \nabla) u_{i}^{0} &= 0, \\
\partial_{t} n_{e}^{0} + \text{div}(n_{e}^{0} u_{e}^{0}) &= 0, \\
\partial_{t} u_{e}^{0} + (u_{e}^{0} \cdot \nabla) u_{e}^{0} + \nabla h(n_{e}^{0}) &= \nabla \phi^{0}, \\
-\lambda^2 \Delta \phi^{0} &= n_{i}^{0} - n_{e}^{0},
\end{align*}$$ \hspace{1cm} (2.20)

with the initial data

$$\left( n_{\nu}^{0}, u_{\nu}^{0} \right) (0, x) = \left( \tilde{n}_{\nu}^{0}, \tilde{u}_{\nu}^{0} \right) (x), \quad x \in \mathbb{R}^d.$$ \hspace{1cm} (2.21)
Through energy method, it is easy to get the unique solution \((n^0_1, u^0_1)\) of the following system.

\[
\begin{align*}
\partial_t n^0_1 + \text{div}(n^0_1 u^0_1) &= 0, \\
\partial_t u^0_1 + (u^0_1 \cdot \nabla) u^0_1 &= 0.
\end{align*}
\] (2.22)

Since \(n^0_1\) is known, we can see that the third to fifth equation in (2.20) is actually the decoupled unipolar Euler-Poisson system for electrons,

\[
\begin{align*}
\partial_t n^0_e + \text{div}(n^0_e u^0_e) &= 0, \\
\partial_t u^0_e + (u^0_e \cdot \nabla) u^0_e + \nabla h(n^0_e) &= \nabla \phi^0, \\
-\lambda^2 \Delta \phi^0 &= n^0(t, x) - n^0_e,
\end{align*}
\] (2.23)

and thus \((n^0_e, u^0_e, \phi^0)\) is known due to Kato\[11\] and Majda\[16\].

(2) In general, for \(j \geq 1\), the profiles \((n^k_j, u^k_j, \phi^j)\) are obtained by induction. Assume that \((n^k_r, u^k_r, \phi^k)_{0 \leq k \leq j-1}\) are smooth and already determined in previous steps. Then \((n^k_j, u^k_j, \phi^j)\) satisfy the linear system

\[
\begin{align*}
\partial_t n^k_j + \text{div} \left( n^k_j u^j_i + n^j_i u^k_i \right) &= - \sum_{k=1}^{j-1} \text{div} \left( n^k_i u^{j-k}_i \right), \\
\partial_t u^j_i + (u^j_i \cdot \nabla) u^j_i + (u^j_i \cdot \nabla) u^0_i + \nabla P^j_i &= - \sum_{k=1}^{j-1} (u^k_i \cdot \nabla) u^{j-k}_i, \\
\partial_t n^j_e + \text{div} \left( n^j_e u^j_e + n^j_e u^0_e \right) &= - \sum_{k=1}^{j-1} \text{div} \left( n^j_e u^{j-k}_e \right), \\
\partial_t u^j_e + (u^j_e \cdot \nabla) u^j_e + (u^j_e \cdot \nabla) u^0_e + \nabla P^j_e &= - \sum_{k=1}^{j-1} (u^k_e \cdot \nabla) u^{j-k}_e, \\
-\lambda^2 \Delta \phi^j &= n^j_i - n^j_e,
\end{align*}
\] (2.24)

where

\[
\begin{align*}
P^j_e &= h'(n^0_e)n^j_e + h^{j-1} \left( (n^k_e)_{k \leq j-1} \right) - \phi^j, \\
P^j_i &= h'(n^0_i)n^{j-1}_i + h^{j-2} \left( (n^k_i)_{k \leq j-2} \right) + \phi^{j-1},
\end{align*}
\]

with the initial data

\[
(n^j_e, u^j_e)(0, x) = (\bar{n}^j_e, \bar{u}^j_e)(x), \quad \nu = e, i.
\] (2.25)

Generally, we can get \((n^j_i, u^j_i)\) from the first two equations in (2.24), and then insert \(u^j_i\) into the third to fifth equations in (2.24) to get \((n^j_e, u^j_e, \phi^j)\).

**Proposition 2.4.** Assume that the initial data \((\bar{n}^j_i, \bar{u}^j_i, \bar{\phi}^j)_{j \geq 0}\) are sufficiently smooth with \(\bar{n}^0_e > 0\) in \(\mathbb{R}^d\), then there exist the unique smooth profiles \((n^j_i, u^j_i, \phi^j)_{j \geq 0}\), solutions of the problems (2.20) with (2.21) and (2.24) with (2.25) in the time interval \([0, T^*_1]\). That is to say there exists a unique asymptotic expansion up to any order of the form (2.19).
2.3.2. **Error estimates and main result.** Let $m \in \mathbb{N}$ be a fixed integer and $(n^\nu_\varepsilon, u^\nu_\varepsilon, \phi^\nu_\varepsilon)$ be the exact solution to problem (2.1) defined in time interval $[0, T^1_\varepsilon]$. We denote by $(n^m_{\nu,\varepsilon}, u^m_{\nu,\varepsilon}, \phi^m_\varepsilon)$ the approximate solution of order $m$ defined in $[0, T^1_\varepsilon]$ by

$$
(n^m_{\nu,\varepsilon}, u^m_{\nu,\varepsilon}, \phi^m_\varepsilon) = \sum_{j=0}^{m} \varepsilon^{2j} (n^j_\nu, u^j_\nu, \phi^j),
$$

where $(n^j_\nu, u^j_\nu, \phi^j)_{0 \leq j \leq m}$ are constructed in the previous subsection. The convergence of the asymptotic expansion (2.3) is to establish the limit

$$
(n^\nu_\varepsilon, u^\nu_\varepsilon, \phi^\nu_\varepsilon) - (n^m_{\nu,\varepsilon}, u^m_{\nu,\varepsilon}, \phi^m_\varepsilon) \to 0,
$$

and its convergence rate as $\varepsilon \to 0$ in a time interval independent of $\varepsilon$, when the convergence holds at $t = 0$.

For $\nu = \varepsilon, i$, define the remainders $(R^1_{n_\nu}, R^1_{u_\nu})$ by

$$
\partial_t n^m_{\nu,\varepsilon} + \text{div} (n^m_{\nu,\varepsilon} u^m_{\nu,\varepsilon}) = R^1_{n_\nu},
$$

$$
\partial_t u^m_{\nu,\varepsilon} + (u^m_{\nu,\varepsilon} \cdot \nabla) u^m_{\nu,\varepsilon} + \nabla (h(n^m_{\nu,\varepsilon}) - \phi^m_\varepsilon) = R^1_{u_\nu},
$$

$$
\partial_t n^m_{i,\varepsilon} + \text{div} (n^m_{i,\varepsilon} u^m_{i,\varepsilon}) = R^1_{n_i},
$$

$$
\partial_t u^m_{i,\varepsilon} + (u^m_{i,\varepsilon} \cdot \nabla) u^m_{i,\varepsilon} + \varepsilon^2 \nabla (h(n^m_{i,\varepsilon}) + \phi^m_\varepsilon) = R^1_{u_i},
$$

$$
- \lambda^2(\xi) = n^m_{i,\varepsilon} - n^m_{\varepsilon,\varepsilon},
$$

where $\lambda(\xi)$ is a constant independent of $\varepsilon$. Since the profiles $(n^j_\nu, u^j_\nu, \phi^j)_{0 \leq j \leq m}$ are sufficiently smooth, a straightforward computation gives the following result.

**Proposition 2.5.** If (2.4), (2.10) and (2.12) hold, then for all integers $m \geq 0$ and $s \geq 0$, the remainders satisfy

$$
\sup_{0 \leq t \leq T^1_\varepsilon} \| (R^1_{n_\nu}, R^1_{u_\nu}) (t) \|_s \leq C_m \varepsilon^{2m+2},
$$

(2.26)

where $C_m > 0$ is a constant independent of $\varepsilon$.

The proof is similar to Proposition 2.3, we omit it here. The main result for the infinity-ion mass limit is the following convergence result, of which the proof will be given in Section 4.

**Theorem 2.2 (The infinity-ion mass limit).** Under the conditions of Proposition 2.5, Let $s > \frac{d}{2} + 1$ and $m \in \mathbb{N}$ be integers. Assume

$$
\| (n^{1/2}_{\nu,0} - n^{1/2}_{\nu,\varepsilon} (0, \cdot), \frac{1}{\varepsilon} (u^{1/2}_{\nu,0} - u^{1/2}_{\nu,\varepsilon} (0, \cdot)) \|_s \leq C_1 \varepsilon^{2m+2},
$$

(2.27)
where $C_1 > 0$ is a constant independent of $\varepsilon$. Then there exists a constant $C_2 > 0$, which depends on $T_1^*$ but is independent of $\varepsilon$, such that as $\varepsilon \to 0$ we have $T_1^{1+\varepsilon} \geq T_1^*$ and for all integer $m \geq 0$, the solution \( (n_{\nu, \varepsilon}^{1+\varepsilon}, u_{\nu, \varepsilon}^{1+\varepsilon}, \phi_{\nu, \varepsilon}^{1+\varepsilon}) \), to the problem (2.1) satisfies

\[
\sup_{0 \leq t \leq T_1} \left\| \left( n_{\nu, \varepsilon}^{1+\varepsilon} - n_{\nu, \varepsilon}^{i,m}, \frac{1}{\varepsilon} (u_{\nu, \varepsilon}^{1+\varepsilon} - u_{\nu, \varepsilon}^{i,m}), \nabla (\phi_{\nu, \varepsilon}^{1+\varepsilon} - \phi_{\nu, \varepsilon}^{i,m}) \right) \right\|_{L^q} \leq C_2 \varepsilon^{2m+2}. \tag{2.28}
\]

That is to say, the infinity-ion mass limit $\varepsilon \to 0$ of the bipolar Euler-Poisson system (2.1) is the unipolar Euler-Poisson equations for electrons (2.23) and the equations (2.22).

**Remark 2.1.** We mention the difference of condition needed for zero-electron mass limit and the infinity-ion mass limit. In Theorem 2.1, we require the fluid to be isothermal and the integer $s > \frac{d}{2} + 2$, which is like the situation in [4]. And in Theorem 2.2, the isothermal condition is not needed, and $s > \frac{d}{2} + 1$.

### 3. Proof of Theorem 2.1

#### 3.1. Energy estimates.

In this section, we continue to use $(n_{\nu}^e, u_{\nu}^e, \phi^e)$ and $(n_{\nu}^i, u_{\nu}^i, \phi^i)_{j \geq 0}$ to substitute $(n_{\nu}^{e,1}, u_{\nu}^{e,1}, \phi^{e,1})$ and $(n_{\nu}^{i,j}, u_{\nu}^{i,j}, \phi^{i,j})_{j \geq 0}$. The exact solution $(n_{\nu}^e, u_{\nu}^e, \phi^e)$ is defined in time interval $[0, T_{1}^{e,1}]$ and the approximate solution $(n_{\nu, \varepsilon}^m, u_{\nu, \varepsilon}^m, \phi_{\nu, \varepsilon}^m)$ in time interval $[0, T_{1}^{e,1}]$, with $T_{1}^{e,1} > 0$ and $T_{1}^{i,1} > 0$. Let

\[ T_{2}^{e,1} = \min (T_{1}^{e,1}, T_{1}^{i,1}) > 0, \]

then the exact solution and the approximate solution are both defined in time interval $[0, T_{2}^{e,1}]$. In this time interval, we denote

\[
(N_{\nu}^{e,1}, U_{\nu}^{e,1}, \Phi^{e,1}) \triangleq (n_{\nu}^e - n_{\nu, \varepsilon}^m, u_{\nu}^e - u_{\nu, \varepsilon}^m, \phi^e - \phi_{\nu, \varepsilon}^m), \quad \nu = e, i. \tag{3.1}
\]

For simplicity, we denote $(N_{\nu}^{e,1}, U_{\nu}^{e,1}, \Phi^{e,1}, R_{e,1}^{e,m})$ by $(N_{\nu}^e, U_{\nu}^e, \Phi^e, R_{e,m})$ in this section. It is easy to check that the variable $(N_{\nu}^e, U_{\nu}^e)$ satisfy

\[
\begin{align*}
\partial_t N_{\nu}^e + (U_{\nu}^e + u_{\nu, \varepsilon}^m) \cdot \nabla N_{\nu}^e + \left( N_{\nu}^e + n_{\nu, \varepsilon}^m \right) \text{div} U_{\nu}^e &= - \left( N_{\nu}^e \text{div} u_{\nu, \varepsilon}^m + U_{\nu}^e \nabla n_{\nu, \varepsilon}^m \right) - R_{n_{\nu}, \varepsilon}^e, \\
\varepsilon \partial_t U_{\nu}^e + \varepsilon \left( (U_{\nu}^e + u_{\nu, \varepsilon}^m) \cdot \nabla \right) U_{\nu}^e + \frac{1}{\varepsilon} h'_{\nu} (N_{\nu}^e + n_{\nu, \varepsilon}^m) \nabla N_{\nu}^e &= -\varepsilon (U_{\nu}^e \cdot \nabla) u_{\nu, \varepsilon}^m - \frac{1}{\varepsilon} (h'_\nu (N_{\nu}^e + n_{\nu, \varepsilon}^m) - h'_\nu (n_{\nu, \varepsilon}^m)) \nabla n_{\nu, \varepsilon}^m + \frac{1}{\varepsilon} \nabla \Phi^e - \frac{1}{\varepsilon} R_{u_{\nu}, \varepsilon}^e, \\
\partial_t N_{\nu}^i + (U_{\nu}^i + u_{\nu, \varepsilon}^m) \cdot \nabla N_{\nu}^i + \left( N_{\nu}^i + n_{\nu, \varepsilon}^m \right) \text{div} U_{\nu}^i &= - \left( N_{\nu}^i \text{div} u_{\nu, \varepsilon}^m + U_{\nu}^i \nabla n_{\nu, \varepsilon}^m \right) - R_{n_{\nu}, \varepsilon}^i, \\
\partial_t U_{\nu}^i + ((U_{\nu}^i + u_{\nu, \varepsilon}^m) \cdot \nabla) U_{\nu}^i + h'_\nu (N_{\nu}^i + n_{\nu, \varepsilon}^m) \nabla N_{\nu}^i &= - (U_{\nu}^i \cdot \nabla) u_{\nu, \varepsilon}^m - (h'_\nu (N_{\nu}^i + n_{\nu, \varepsilon}^m) - h'_\nu (n_{\nu, \varepsilon}^m)) \nabla n_{\nu, \varepsilon}^i - \nabla \Phi^i - R_{u_{\nu}, \varepsilon}^i, \\
\left( N_{\nu}^e, U_{\nu}^e \right)_{t=0} &= \left( n_{\nu,0}^e - n_{\nu, \varepsilon}^m, 0, u_{\nu,0}^e - u_{\nu, \varepsilon}^m (0, \cdot) \right).
\end{align*}
\]
coupled with the Poisson equation for $\Phi^\epsilon$

$$-\lambda^2 \Delta \Phi^\epsilon = N_i^\epsilon - N_e^\epsilon, \quad \lim_{|x| \to +\infty} \Phi^\epsilon(x) = 0. \quad (3.3)$$

For simplicity, we let $\lambda = 1$. Set

$$W_e^\epsilon = \begin{pmatrix} N_e^\epsilon \\ \varepsilon U_e^\epsilon \end{pmatrix}, \quad W_i^\epsilon = \begin{pmatrix} N_i^\epsilon \\ U_i^\epsilon \end{pmatrix},$$

$$H_{e,\epsilon}^1 = \begin{pmatrix} N_e^\epsilon \text{div} u_{e,\epsilon}^m + U_e^\epsilon \nabla n_{e,\epsilon}^m \\ \varepsilon (U_e^\epsilon \cdot \nabla) u_{e,\epsilon}^m + \frac{1}{\varepsilon} \left( h'_e(N_e^\epsilon + n_{e,\epsilon}^m) - h'_e(n_{e,\epsilon}^m) \right) \nabla n_{e,\epsilon}^m \end{pmatrix},$$

$$H_{i,\epsilon}^1 = \begin{pmatrix} N_i^\epsilon \text{div} u_{i,\epsilon}^m + U_i^\epsilon \nabla n_{i,\epsilon}^m \\ (U_i^\epsilon \cdot \nabla) u_{i,\epsilon}^m + \left( h'_i(N_i^\epsilon + n_{i,\epsilon}^m) - h'_i(n_{i,\epsilon}^m) \right) \nabla n_{i,\epsilon}^m \end{pmatrix},$$

$$H_{e,\epsilon}^2 = \begin{pmatrix} 0 \\ \frac{1}{\varepsilon} \nabla \Phi^\epsilon \end{pmatrix}, \quad H_{i,\epsilon}^2 = \begin{pmatrix} 0 \\ -\nabla \Phi^\epsilon \end{pmatrix},$$

$$R_e^\epsilon = \begin{pmatrix} R_e^{\epsilon} \\ \frac{1}{\varepsilon} R_{ue}^\epsilon \end{pmatrix}, \quad R_i^\epsilon = \begin{pmatrix} R_i^{\epsilon} \\ R_{ui}^\epsilon \end{pmatrix},$$

and for $j = 1, 2, 3$,

$$A_j^e(n_e^\epsilon, u_e^\epsilon) = \begin{pmatrix} u_{e,j}^\epsilon \\ \frac{1}{\varepsilon} n_e^\epsilon e_j^\top \\ \frac{1}{\varepsilon} h'_e(n_e^\epsilon) e_j u_{e,j}^\epsilon \end{pmatrix},$$

$$A_j^i(n_i^\epsilon, u_i^\epsilon) = \begin{pmatrix} u_{i,j}^\epsilon \\ n_i^\epsilon e_j^\top \\ h'_i(n_i^\epsilon) e_j u_{i,j}^\epsilon \end{pmatrix},$$

where $(e_1, e_2, e_3)$ is the canonical basis of $\mathbb{R}^3$ and $I_3$ is the $3 \times 3$ unit matrix, thus $(3.2)$ can be written as

$$\partial_t W^\epsilon + \sum_{j=1}^3 A_j^\nu(n_{\nu}^\epsilon, u_{\nu}^\epsilon) \partial_{x_j} W^\epsilon = -H_{\nu,\epsilon}^1 + H_{\nu,\epsilon}^2 - R^\epsilon_{\nu}, \quad \nu = e, i. \quad (3.4)$$

with the initial data

$$t = 0 : \quad W^\epsilon_{\nu,0} = W^\epsilon_{\nu,0}, \quad \nu = e, i, \quad (3.5)$$

in which

$$W_{e,0}^\epsilon = \begin{pmatrix} N_e^\epsilon(0, \cdot) \\ \varepsilon U_e^\epsilon(0, \cdot) \end{pmatrix} = \begin{pmatrix} n_{e,0}^\epsilon - n_{e,\epsilon}^m(0, \cdot) \\ \varepsilon (u_{e,0}^\epsilon - u_{e,\epsilon}^m(0, \cdot)) \end{pmatrix},$$

$$W_{i,0}^\epsilon = \begin{pmatrix} N_i^\epsilon(0, \cdot) \\ U_i^\epsilon(0, \cdot) \end{pmatrix} = \begin{pmatrix} n_{i,0}^\epsilon - n_{i,\epsilon}^m(0, \cdot) \\ u_{i,0}^\epsilon - u_{i,\epsilon}^m(0, \cdot) \end{pmatrix}. $$
System (3.4)-(3.5) for \( W^\varepsilon \) is symmetrizable hyperbolic when \( n^\varepsilon_\nu > 0 \). Indeed, since the density \( n^0 \) of the leading profile satisfies

\[
n^0 \geq C > 0, \quad n^m_{\nu, \varepsilon} - n^0 = O(\varepsilon),
\]

and \( N^\varepsilon_\nu \) is small for small \( \varepsilon \), which we will prove later, so we have

\[
n^\varepsilon_\nu > 0, \quad \text{for } \nu = e, i.
\]

With this, let

\[
A^0_e(n^\varepsilon_e) = \begin{pmatrix} h'_e(n^\varepsilon_e) & 0 \\ 0 & n^\varepsilon_eI_3 \end{pmatrix}, \quad A^0_i(n^\varepsilon_i) = \begin{pmatrix} h'_i(n^\varepsilon_i) & 0 \\ 0 & n^\varepsilon_iI_3 \end{pmatrix},
\]

and for \( j = 1, 2, 3 \),

\[
\tilde{A}^j_e(n^\varepsilon_e, u^\varepsilon_e) = A^0_e(n^\varepsilon_e) A^j_e(n^\varepsilon_e, u^\varepsilon_e) = \begin{pmatrix} h'_e(n^\varepsilon_e) u^{\varepsilon, j}_{e, j} & \frac{1}{\varepsilon} p'_e(n^\varepsilon_e) e_j \\ \frac{1}{\varepsilon} p'_e(n^\varepsilon_e) e_j & n^\varepsilon_e u^{\varepsilon, j}_{e, j}I_3 \end{pmatrix},
\]

\[
\tilde{A}^j_i(n^\varepsilon_i, u^\varepsilon_i) = A^0_i(n^\varepsilon_i) A^j_i(n^\varepsilon_i, u^\varepsilon_i) = \begin{pmatrix} h'_i(n^\varepsilon_i) u^{\varepsilon, j}_{i, j} & p'_i(n^\varepsilon_i) e_j \\ p'_i(n^\varepsilon_i) e_j & n^\varepsilon_i u^{\varepsilon, j}_{i, j}I_3 \end{pmatrix},
\]

then \( A^0_e \) is positively definite and \( \tilde{A}^j_e \) is symmetric for all \( 1 \leq j \leq 3 \). Thus, the theorem of Kato for the local existence of smooth solutions can also be applied to (3.4)-(3.5).

By standard arguments, to prove Theorem 2.1, it suffices to establish uniform estimates of \( W^\varepsilon \) with respect to \( \varepsilon \). Since \( W^\varepsilon \in C([0, T^\varepsilon_1]; H^s(\mathbb{R}^3)) \), the function \( t \to \|W^\varepsilon\|_s \) is continuous on \([0, T^\varepsilon_1]\). From (2.16) and \( m \geq 1 \), there exists \( T^\varepsilon_1 \in (0, T^\varepsilon_1) \) such that

\[
\|W^\varepsilon\|_s \leq C, \quad \forall t \in [0, T^\varepsilon_1],
\]

provided that \( \varepsilon > 0 \) is bounded by a constant. If \( s \geq 3 \), the imbedding from \( H^s(\mathbb{R}^3) \) to \( W^{1, \infty}(\mathbb{R}^3) \) is continuous. Then we have

\[
\|W^\varepsilon\|_{W^{1, \infty}(\mathbb{R}^3)} \leq C, \quad \forall t \in [0, T^\varepsilon_1].
\]

In order to prove \( T^\varepsilon_1 \geq T^\varepsilon_1 \), we need to show that there exists a constant \( \mu > 0 \) such that

\[
\sup_{0 \leq t \leq T^\varepsilon} \|W^\varepsilon\|_s \leq C\varepsilon^\mu.
\]

3.1.1. \( L^2 \)-estimates. In what follows, we always assume that the conditions of Theorem 2.1 hold.

**Lemma 3.1.** For all \( t \in [0, T^\varepsilon_1] \) and sufficiently small \( \varepsilon > 0 \), we have

\[
\frac{d}{dt} \left( \sum_{\nu=e, i} \langle A^0_\nu(n^\varepsilon_\nu) W^\varepsilon_\nu, W^\varepsilon_\nu \rangle + \|\nabla \Phi^\varepsilon\|^2 \right) \\
\leq C \sum_{\nu=e, i} \|W^\varepsilon_\nu\|^2 + \|\nabla \Phi^\varepsilon\|^2 + C\varepsilon^{4m}. \tag{3.7}
\]
Proof. Step 1: Taking the inner product of the electron equations in (3.4) with \(2A_e^0(n_e^\epsilon)W_e^\epsilon\) in \(L^2(\mathbb{R}^3)\), we obtain the following energy equality for \(W_e^\epsilon\)

\[
\frac{d}{dt} \langle A_e^0(n_e^\epsilon)W_e^\epsilon, W_e^\epsilon \rangle = \langle \text{div}A_e(n_e^\epsilon, u_e^\epsilon)W_e^\epsilon, W_e^\epsilon \rangle - 2 \langle A_e^0(n_e^\epsilon)W_e^\epsilon, H_{e,e}^1 \rangle \\
+ 2 \langle A_e^0(n_e^\epsilon)W_e^\epsilon, H_{e,e}^2 \rangle - 2 \langle A_e^0(n_e^\epsilon)W_e^\epsilon, R_e^\epsilon \rangle, \tag{3.8}
\]

where

\[
\text{div}A_e(n_e^\epsilon, u_e^\epsilon) = \partial_t A_e^0(n_e^\epsilon) + \sum_{j=1}^3 \partial_{x_j} \tilde{A}_e^j(n_e^\epsilon, u_e^\epsilon). 
\]

Now we deal with the right-hand side of (3.8) term by term. From the mass conservation equation \(\partial_t n_e^\epsilon = -\text{div}(n_e^\epsilon u_e^\epsilon)\), obviously we have

\[
\langle \partial_t A_e^0(n_e^\epsilon)W_e^\epsilon, W_e^\epsilon \rangle = \langle A_e^0(n_e^\epsilon)'\text{div}(n_e^\epsilon u_e^\epsilon)W_e^\epsilon, W_e^\epsilon \rangle \\
= -\langle h'_e(n_e^\epsilon)\text{div}(n_e^\epsilon u_e^\epsilon)N_e^\epsilon, N_e^\epsilon \rangle - \epsilon^2 \langle \text{div}(n_e^\epsilon u_e^\epsilon)U_e^\epsilon, U_e^\epsilon \rangle \\
\leq C\|W_e^\epsilon\|^2, \tag{3.9}
\]

then in view of the expression of \(\tilde{A}_e^j(W_e^\epsilon)\), we obtain

\[
\langle \partial_{x_j} \tilde{A}_e^j(n_e^\epsilon, u_e^\epsilon)W_e^\epsilon, W_e^\epsilon \rangle = \langle \partial_{x_j}(h'_e(n_e^\epsilon)u_e^\epsilon)N_e^\epsilon, N_e^\epsilon \rangle + 2\langle N_e^\epsilon \partial_{x_j}(p'_e(n_e^\epsilon)e_j), U_e^\epsilon \rangle \\
+ \epsilon^2 \langle \partial_{x_j}(n_e^\epsilon u_e^\epsilon)U_e^\epsilon, U_e^\epsilon \rangle,
\]

in which we have

\[
\langle \partial_{x_j}(h'_e(n_e^\epsilon)u_e^\epsilon)N_e^\epsilon, N_e^\epsilon \rangle + \epsilon^2 \langle \partial_{x_j}(n_e^\epsilon u_e^\epsilon)U_e^\epsilon, U_e^\epsilon \rangle \leq C\|W_e^\epsilon\|^2,
\]

and

\[
2 \sum_{j=1}^d \langle N_e^\epsilon \nabla p'_e(n_e^\epsilon)e_j, U_e^\epsilon \rangle = 2\langle N_e^\epsilon \nabla p'_e(n_e^\epsilon), U_e^\epsilon \rangle,
\]

therefore,

\[
\sum_{j=1}^d \langle \partial_{x_j} \tilde{A}_e^j(n_e^\epsilon, u_e^\epsilon)W_e^\epsilon, W_e^\epsilon \rangle \leq C\|W_e^\epsilon\|^2 + \langle N_e^\epsilon \nabla p'_e(n_e^\epsilon), U_e^\epsilon \rangle. \tag{3.10}
\]

It follows from (3.9) and (3.10) that

\[
\langle \text{div}A_e(n_e^\epsilon, u_e^\epsilon)W_e^\epsilon, W_e^\epsilon \rangle \leq C\|W_e^\epsilon\|^2 + 2\langle N_e^\epsilon \nabla p'_e(n_e^\epsilon), U_e^\epsilon \rangle. \tag{3.11}
\]
For the remaining terms without $H^2_{e,e}$ in the right hand side of (3.8), it can be treated as

$$-2\langle A^0_e(n^e) W^e, H^1_{e,e} \rangle - 2\langle A^0_e(n^e) W^e, R^e \rangle = -2\langle h'_e(n^e) N^e_e \nabla u^m_{e,e} + U^e_e \cdot \nabla n^m_{e,e} + R^e_{u e,m}, N^e_e \rangle - 2\langle n^e \left[ 2(U^e_e \cdot \nabla) u^m_{e,e} + (h'_e(N^e_e + n^m_{e,e}) - h'_e(n^e_e)) \nabla n^m_{e,e} + R^e_{u e,m} \right], U^e_e \rangle$$

$$= -2\langle h'_e(n^e) N^e_e \nabla u^m_{e,e} + N^e_e \rangle - 2\langle n^e \left[ h'_e(N^e_e + n^m_{e,e}) - h'_e(n^e_e) \nabla n^m_{e,e} + U^e_e \right] \rangle - 2\langle h'_e(n^e) R^e_{u e,m}, N^e_e \rangle$$

$$\leq -2\langle h'_e(n^e) N^e_e \nabla n^m_{e,e}, U^e_e \rangle - 2\langle n^e \left( h'_e(n^e_e) - h'_e(n^m_{e,e}) \right) \nabla n^m_{e,e}, U^e_e \rangle + C\|W^e_e\|^2 + C\left( \|R^e_{u e,m}\|^2 + \|R^e_{u e,m}\|^2 \right).$$

For the term containing $H^2_{e,e}$ in the right hand side of (3.8), a direct calculation gives

$$2\langle A^0_e(n^e) W^e, H^2_{e,e} \rangle = 2\langle (n^e U^e_e, \nabla \Phi^e) \rangle.$$

Back to (3.8), combining the above three estimates yield

$$\frac{d}{dt}\langle A^0_e(n^e) W^e, W^e \rangle \leq C\|W^e_e\|^2 + 2\langle n^e U^e_e, \nabla \Phi^e \rangle + C\left( \|R^e_{u e,m}\|^2 + \|R^e_{u e,m}\|^2 \right) + 2r^e, \quad (3.12)$$

where the remaining term

$$r^e = \langle N^e_e \nabla p'_e(n^e) - h'_e(n^e) N^e_e \nabla n^m_{e,e} + n^e \left( h'_e(n^m_{e,e}) - h'_e(n^e) \right) \nabla n^m_{e,e}, U^e_e \rangle.$$

Note that

$$n^e = N^e_e + n^m_{e,e}, \quad p''_e(n^e) = h'_e(n^e) + n^e h''(n^e). \quad (3.13)$$

When $N^e_e$ is small, we have

$$h'_e(n^m_{e,e}) - h'_e(n^e) = -h''_e(n^e) N^e_e + \frac{1}{2} h''_e(n^e - \theta N^e_e)(N^e_e)^2, \quad \text{with } \theta \in [0, 1], \quad (3.14)$$

then,

$$N^e_e \nabla p'_e(n^e) - h'_e(n^e) N^e_e \nabla n^m_{e,e} + n^e (h'_e(n^m_{e,e}) - h'_e(n^e)) \nabla n^m_{e,e}$$

$$= N^e_e p''_e(n^e) \nabla N^e_e + N^e_e p''_e(n^e) \nabla n^m_{e,e} - h'_e(n^e) N^e_e \nabla n^m_{e,e} - n^e h''_e(n^e) N^e_e \nabla n^m_{e,e}$$

$$+ \frac{1}{2} n^e h''_e(n^e - \theta N^e_e)(N^e_e)^2 \nabla n^m_{e,e}$$

$$= \frac{1}{2} p''_e(n^e) \nabla (N^e_e)^2 + \frac{1}{2} n^e h''_e(n^e - \theta N^e_e)(N^e_e)^2 \nabla n^m_{e,e}. $$
Therefore,

\[
    r^\varepsilon = \frac{1}{2} \langle p_e^{m\varepsilon}(n_e^\varepsilon) \nabla (N_c^\varepsilon)^2, U_e^\varepsilon \rangle + \frac{1}{2} \langle n_e^\varepsilon b_e^m(n_e^\varepsilon - \theta N_c^\varepsilon)(N_c^\varepsilon)^2 \nabla n_e^m, U_e^\varepsilon \rangle
    = -\frac{1}{2} \langle (N_c^\varepsilon)^2, \text{div} (p_e^{m\varepsilon}(n_e^\varepsilon) U_e^\varepsilon) \rangle + \frac{1}{2} \langle n_e^\varepsilon b_e^m(n_e^\varepsilon - \theta N_c^\varepsilon)(N_c^\varepsilon)^2 \nabla n_e^m, U_e^\varepsilon \rangle
    \leq C \|N_c^\varepsilon\|^2.
\]

This together with (3.12) yields

\[
    \frac{d}{dt} \langle A_i^0(n_e^\varepsilon) W_i^\varepsilon, W_i^\varepsilon \rangle \leq C \|W_i^\varepsilon\|^2 + 2 \langle n_e^\varepsilon U_e^\varepsilon, \nabla \Phi^\varepsilon \rangle + C \left( \|R_e^{\varepsilon,m}\|^2 \right) \left( \|R_e^{\varepsilon,m}\|^2 \right). \tag{3.15}
\]

Step 2: Taking the inner product of the ion equations in (3.4) with \(2 A_i^0(n_e^\varepsilon) W_i^\varepsilon\) in \(L^2(\mathbb{R}^3)\), we obtain the following energy equality for \(W_i^\varepsilon\)

\[
    \frac{d}{dt} \langle A_i^0(n_e^\varepsilon) W_i^\varepsilon, W_i^\varepsilon \rangle = -2 \langle A_i^0(n_e^\varepsilon) W_i^\varepsilon, H_i^{1\varepsilon} \rangle + 2 \langle A_i^0(n_e^\varepsilon) W_i^\varepsilon, H_i^{2\varepsilon} \rangle
    - 2 \langle A_i^0(n_e^\varepsilon) W_i^\varepsilon, R_i^\varepsilon \rangle + \langle \text{div} A_i(n_e^\varepsilon, u_i^\varepsilon) W_i^\varepsilon, W_i^\varepsilon \rangle, \tag{3.16}
\]

where

\[
    \text{div} A_i(n_e^\varepsilon, u_i^\varepsilon) = \partial_t A_i^0(n_e^\varepsilon) + \sum_{j=1}^{3} \partial_{x_j} A_i^j(n_e^\varepsilon, u_i^\varepsilon),
\]

which are treated term by term as follows. Notice the expressions of \(A_i^0\), \(\text{div} A_i\) and \(H_i^{1,2\varepsilon}\), we have

\[
    \left| \langle A_i^0(n_e^\varepsilon) W_i^\varepsilon, H_i^{1\varepsilon} \rangle \right| \leq C \|W_i^\varepsilon\|^2, \\
    \left| \langle A_i^0(n_e^\varepsilon) W_i^\varepsilon, R_i^\varepsilon \rangle \right| \leq C \|W_i^\varepsilon\|^2 + C \|R_i^\varepsilon\|^2 \leq C \|W_i^\varepsilon\|^2 + C\varepsilon^{4m}, \\
    \left| \langle \text{div} A_i(n_e^\varepsilon, u_i^\varepsilon) W_i^\varepsilon, W_i^\varepsilon \rangle \right| \leq C \|\text{div} A_i(n_e^\varepsilon, u_i^\varepsilon) \|_\infty \|W_i^\varepsilon\|^2 \leq C \|W_i^\varepsilon\|^2.
\]

For the term containing \(H_i^{2\varepsilon}\) in the right hand side of (3.16), a direct calculation gives

\[
    2 \langle A_i^0(n_e^\varepsilon) W_i^\varepsilon, H_i^{2\varepsilon} \rangle = -2 \langle n_i^\varepsilon U_i^\varepsilon, \nabla \Phi^\varepsilon \rangle.
\]

Inserting the above four estimates into (3.16), we get

\[
    \frac{d}{dt} \langle A_i^0(n_e^\varepsilon) W_i^\varepsilon, W_i^\varepsilon \rangle \leq -2 \langle n_i^\varepsilon U_i^\varepsilon, \nabla \Phi^\varepsilon \rangle + C \|W_i^\varepsilon\|^2 + C\varepsilon^{4m}. \tag{3.17}
\]
Step 3: Summing (3.15) and (3.17) for all $|\alpha| \leq s$, we obtain
\[
\frac{d}{dt} \sum_{\nu = e, i} \langle A^0_{\nu}(n^\varepsilon_{\nu}) W^\varepsilon_{\nu}, W^\varepsilon_{\nu} \rangle \leq 2 \langle n^\varepsilon_i U^e_i, \nabla \Phi^\varepsilon \rangle - 2 \langle n^\varepsilon_i U^i_i, \nabla \Phi^\varepsilon \rangle + C \sum_{\nu = e, i} \|W^\varepsilon_{\nu}\|^2 + C \varepsilon^{4m} = 2 \langle (n^\varepsilon_i u^e_i - n^m_i u^m_i, u^m_i, u^m_i), \nabla \Phi^\varepsilon \rangle + 2 \langle (n^\varepsilon_i u^e_i - n^m_i u^m_i), \nabla \Phi^\varepsilon \rangle + C \sum_{\nu = e, i} \|W^\varepsilon_{\nu}\|^2 + C \varepsilon^{4m},
\]
in which we have
\[
2 \langle (n^\varepsilon_i u^e_i - n^m_i u^m_i), \nabla \Phi^\varepsilon \rangle = -2 \langle \text{div} \left(n^\varepsilon_i u^e_i - n^m_i u^m_i\right), \Phi^\varepsilon \rangle = 2 \langle \partial_i \left(N^\varepsilon_i - N^m_i\right) + \left(R^e_{\nu} - R^m_{\nu}\right), \Phi^\varepsilon \rangle = -\frac{d}{dt} \|\nabla \Phi^\varepsilon\|^2 - 2 \langle \tilde{R}^e_{\nu} - \tilde{R}^m_{\nu}, \nabla \Phi^\varepsilon \rangle \leq -\frac{d}{dt} \|\nabla \Phi^\varepsilon\|^2 + \|\nabla \Phi^\varepsilon\|^2 + C \varepsilon^{4m},
\]
and
\[
2 \langle N^\varepsilon_i u^m_i - N^m_i u^m_i, \nabla \Phi^\varepsilon \rangle = 2 \langle N^\varepsilon_i u^m_i, \nabla \Phi^\varepsilon \rangle + 2 \langle N^\varepsilon_i - N^m_i, u^m_i, \nabla \Phi^\varepsilon \rangle = 2 \langle \nabla \Phi^\varepsilon, \nabla \Phi^\varepsilon \rangle - 2 \langle \Delta \Phi^\varepsilon u^m_i, \nabla \Phi^\varepsilon \rangle \leq \|N^\varepsilon_i\|^2 + \|\nabla \Phi^\varepsilon\|^2.
\]
Combining these estimates yields (3.7).

3.1.2. Higher order estimates. Let $\alpha \in \mathbb{N}^3$ with $1 \leq |\alpha| \leq s$. Applying $\partial^\alpha_x$ to (3.4), we get
\[
\partial_t \partial^\alpha_x W^\varepsilon_{\nu} + \sum_{j=1}^{3} A^j_{\nu}(n^\varepsilon_{\nu}, u^\varepsilon_{\nu}) \partial_{x_j} \partial^\alpha_x W^\varepsilon_{\nu} = -\partial^\alpha_x (H^1_{\nu} - H^2_{\nu} + R^\varepsilon_{\nu}) + J^\alpha_{\nu, \varepsilon}, \quad \nu = e, i, \quad (3.18)
\]
where
\[
J^\alpha_{\nu, \varepsilon} = \sum_{j=1}^{3} \left( A^j_{\nu}(n^\varepsilon_{\nu}, u^\varepsilon_{\nu}) \partial_{x_j} \partial^\alpha_x W^\varepsilon_{\nu} - \partial^\alpha_x \left( A^j_{\nu}(n^\varepsilon_{\nu}, u^\varepsilon_{\nu}) \partial_{x_j} W^\varepsilon_{\nu} \right) \right).
\]

Lemma 3.2. For all $t \in [0, T^\varepsilon_{\nu}]$ and sufficiently small $\varepsilon > 0$, we have
\[
\frac{d}{dt} \left( \sum_{\nu = e, i} \langle A^0_{\nu}(n^\varepsilon_{\nu}) \partial^\alpha_x W^\varepsilon_{\nu}, \partial^\alpha_x W^\varepsilon_{\nu} \rangle + \|\nabla \partial^\alpha_x \Phi^\varepsilon\|^2 \right) \leq C \sum_{\nu = e, i} \|W^\varepsilon_{\nu}\|_{|\alpha|}^2 + \|\nabla \Phi^\varepsilon\|_{|\alpha|}^2 + \frac{C}{\varepsilon^2} \|W^\varepsilon_{\nu}\|_{|\alpha|-1}^2 + C \varepsilon^{4m}. \quad (3.19)
\]
Proof. Step1: Taking the inner product of the electrons equations in (3.18) with $2A^0_e (n^e) \partial^2_x W^e$ in $L^2(\mathbb{R}^3)$ yields the following energy equality for $\partial^2_x W^e$

$$\frac{d}{dt} \langle A^0_e(n^e)\partial^0_x W^e, \partial^0_x W^e \rangle = \langle \text{div} A_e(n^e, u^e)\partial^0_x W^e, \partial^0_x W^e \rangle$$

$$+ 2\langle A^0_e(n^e)\partial^0_x W^e, \partial^0_x H^1_e + \partial^2_x R^e \rangle$$

$$- 2\langle A^0_e(n^e)\partial^0_x W^e, \partial^0_x H^1_e, \partial^2_x R^e \rangle$$

$$+ 2\langle A^0_e(n^e)\partial^0_x W^e, \partial^0_x H^2_e \rangle + 2\langle A^0_e(n^e)\partial^0_x W^e, J^a_{e,\epsilon,\epsilon} \rangle,$$

(3.20)

which are treated term by term as follows. First, similarly to (3.11), it is easy to get

$$|\langle \text{div} A_e(n^e, u^e)\partial^0_x W^e, \partial^0_x W^e \rangle| \leq C \|W^e\|^2_{|a|} + \langle \partial^0_x N^e \nabla p^e(n^e), \partial^0_x U^e \rangle.$$  

(3.21)

For the terms without $H^2_{e,\epsilon}$ and $J^a_{e,\epsilon,\epsilon}$ in the right hand side of (3.20), a straightforward calculation yields

$$-2\langle A^0_e(n^e)\partial^0_x W^e, \partial^0_x H^1_e \rangle$$

$$-2\langle A^0_e(n^e)\partial^0_x W^e, \partial^0_x H^2_e \rangle + 2\langle A^0_e(n^e)\partial^0_x W^e, J^a_{e,\epsilon,\epsilon} \rangle,$$

in which we have

$$-2\langle h^e(n^e)\partial^0_x (N^e \text{div} u^m_{e,\epsilon}), \partial^0_x N^e \rangle \leq C \left( \|W^e\|^2_{|a|} + \|R^e_{m,\epsilon}\|^2_{|a|} \right),$$

$$-2\langle n^e\partial^0_x (u^e \cdot \nabla) u^m_{e,\epsilon} + R^e_{u,\epsilon}, \partial^0_x U^e \rangle \leq C \left( \|W^e\|^2_{|a|} + \|R^e_{u,\epsilon}\|^2_{|a|} \right).$$

Besides, applying the Moser-type inequalities, we have

$$-2\langle h^e(n^e)\partial^0_x (U^e \cdot \nabla) u^m_{e,\epsilon}, \partial^0_x U^e \rangle \leq C \left( \|N^e\|^2_{|a|} + \|U^e\|^2_{|a|} \right),$$

and

$$2\langle n^e\partial^0_x (h^e(n^e) - h^e(n^m_{e,\epsilon}))\nabla u^m_{e,\epsilon}, \partial^0_x U^e \rangle \leq C \left( \frac{1}{\varepsilon^2} \|N^e\|^2_{|a|} + \|U^e\|^2_{|a|} \right).$$

The above four estimates imply

$$-2\langle A^0_e(n^e)\partial^2_x W^e, \partial^2_x H^1_e \rangle$$

$$\leq \frac{C}{\varepsilon^2} \left( \|N^e\|^2_{|a|} + \|U^e\|^2_{|a|} \right) + C \left( \|R^e_{m,\epsilon}\|^2_{|a|} + \|R^e_{u,\epsilon}\|^2_{|a|} \right)$$

$$+ C \left( \|N^e\|^2_{|a|} + \|U^e\|^2_{|a|} \right) - 2\langle h^e(n^e)\partial^0_x N^e \nabla u^m_{e,\epsilon}, \partial^0_x U^e \rangle$$

$$-2\langle n^e\partial^0_x (h^e(n^e) - h^e(n^m_{e,\epsilon}))\nabla u^m_{e,\epsilon}, \partial^0_x U^e \rangle.$$  

(3.22)
For the term containing $J_{e,v}^\alpha$ in the right hand side of (3.20), for $j = 1, 2, 3$, we have

$$
\begin{align*}
\langle A^0_e(n^e_v)\partial_x^\alpha W^e_v, \partial_x^\alpha (A^1_e(n^e_v, u^e_v)\partial_x W^e_v) - A^1_e(n^e_v, u^e_v)\partial_x^\alpha (\partial_x W^e_v) \rangle \\
= & \langle h'_e(n^e_v) (\partial_x^\alpha (u^e_{e,j} \partial_x, N^e_v) - u^e_{e,j} \partial_x^\alpha \partial_x, N^e_v) , \partial_x^\alpha N^e_v \rangle \\
& + \varepsilon^2 \langle n^e_v (\partial_x^\alpha (u^e_{e,j} \partial_x U^e_{e,j}) - u^e_{e,j} \partial_x^\alpha U^e_{e,j}) , \partial_x^\alpha U^e_{e,j} \rangle \\
& + \langle n^e_v (\partial_x^\alpha (h'_e(n^e_v) \partial_x, N^e_v) - h'_e(n^e_v) \partial_x^\alpha \partial_x, N^e_v) , \partial_x^\alpha U^e_{e,j} \rangle \\
& + \langle h'_e(n^e_v) (\partial_x^\alpha (n^e_v \partial_x U^e_{e,j}) - n^e_v \partial_x^\alpha \partial_x U^e_{e,j}) , \partial_x^\alpha N^e_v \rangle \\
= & \langle h'_e(n^e_v) (\partial_x^\alpha (u^e_{e,j} \partial_x, N^e_v) - u^e_{e,j} \partial_x^\alpha \partial_x, N^e_v) , \partial_x^\alpha N^e_v \rangle \\
& + \varepsilon^2 \langle n^e_v (\partial_x^\alpha (u^e_{e,j} \partial_x U^e_{e,j}) - u^e_{e,j} \partial_x^\alpha U^e_{e,j}) , \partial_x^\alpha U^e_{e,j} \rangle \\
& + \langle n^e_v (\partial_x^\alpha (h'_e(n^e_v) \partial_x, N^e_v) - h'_e(n^e_v) \partial_x^\alpha \partial_x, N^e_v) - \sum_{\alpha_i \neq 0} C_{\alpha_i} \partial_x n^e_v h'_e(n^e_v) \partial_x^\alpha \partial_x, N^e_v) , \partial_x^\alpha U^e_{e,j} \rangle \\
& + \sum_{\alpha_i \neq 0} C_{\alpha_i} \left( \left( n^e_v \partial_x h'_e(n^e_v) \partial_x^\alpha \partial_x, N^e_v, \partial_x^\alpha U^e_{e,j} \right) + \left( h'_e(n^e_v) \partial_x, n^e_v \partial_x^\alpha \partial_x, U^e_{e,j}, \partial_x^\alpha N^e_v \right) \right) \\
\leq & \left( \frac{C_{\alpha} \|N^e_v\|_{2}}{\|\alpha\|_{2}} + \frac{C_{\varepsilon}}{\varepsilon} (\|U^e_v\|_{2}^{2} + \|U^e_v\|_{0}^{2}) + \varepsilon^2 \right) \left( \|U^e_v\|_{2}^{2} + \|U^e_v\|_{0}^{2} \right) \\
& + \sum_{\alpha_i \neq 0} C_{\alpha_i} \left( \left( n^e_v \partial_x h'_e(n^e_v) \partial_x^\alpha \partial_x, N^e_v, \partial_x^\alpha U^e_{e,j} \right) + \left( h'_e(n^e_v) \partial_x, n^e_v \partial_x^\alpha \partial_x, U^e_{e,j}, \partial_x^\alpha N^e_v \right) \right) ,
\end{align*}
\tag{3.23}
$$

where $\alpha^i$ is also a multi-index and $\partial_x, \partial_x^\alpha = \partial_x^\alpha$. The last term of (3.23) can be estimated as

$$
\begin{align*}
\sum_{\alpha_i \neq 0} C_{\alpha_i} \left( \left( n^e_v \partial_x h'_e(n^e_v) \partial_x^\alpha \partial_x, N^e_v, \partial_x^\alpha U^e_{e,j} \right) + \left( h'_e(n^e_v) \partial_x, n^e_v \partial_x^\alpha \partial_x, U^e_{e,j}, \partial_x^\alpha N^e_v \right) \right) \\
= \sum_{\alpha_i \neq 0} C_{\alpha_i} \left( \left( n^e_v \partial_x h'_e(n^e_v) \partial_x^\alpha \partial_x, N^e_v, \partial_x^\alpha U^e_{e,j} \right) + \left( h'_e(n^e_v) \partial_x, n^e_v \partial_x^\alpha \partial_x, U^e_{e,j}, \partial_x^\alpha N^e_v \right) \right) \\
& - \sum_{\alpha_i \neq 0} C_{\alpha_i} \left( \partial_x h'_e(n^e_v) \partial_x^\alpha \partial_x, N^e_v, \partial_x^\alpha U^e_{e,j} \right) \\
\leq & \sum_{\alpha_i \neq 0} C_{\alpha_i} \left( \partial_x p'_e(n^e_v) \partial_x^\alpha \partial_x, U^e_{e,j}, \partial_x^\alpha N^e_v \right) + C \left( \|N^e_v\|_{2}^{2} + \|U^e_v\|_{2}^{2} \right) .
\end{align*}
$$

Note that $p'_e(n^e_v) = a^2_e$, we get
\[ \langle A_e^0 (n_e^\varepsilon) \partial_x^3 W_e^\varepsilon, J_{e,\varepsilon}^m \rangle \leq C \| W_e^\varepsilon \|_{[\alpha]}^2 + \frac{C}{\varepsilon^2} \| W_e^\varepsilon \|_{[\alpha]-1}^2. \] (3.24)

As to the term containing \( H_e^2 \) in the right hand side of (3.20), a direct calculation gives

\[ 2 \langle A_e^0 (n_e^\varepsilon) \partial_x^3 W_e^\varepsilon, \partial_x^3 H_e^2 \rangle = 2 \langle (n_e^\varepsilon) \partial_x^3 U_e^\varepsilon, \nabla \partial_x^3 \Phi^\varepsilon \rangle. \]

Therefore, (3.21)-(3.24) yield

\[
\frac{d}{dt} \langle A_e^0 (n_e^\varepsilon) \partial_x^3 W_e^\varepsilon, \partial_x^3 W_e^\varepsilon \rangle \leq \frac{C}{\varepsilon^2} \left( \| N_e^\varepsilon \|_{[\alpha]-1}^2 + \| U_e^\varepsilon \|_{[\alpha]-1}^2 \right) + C \left( \| R_m^e \|_{[\alpha]}^2 + \| R_m^\varepsilon \|_{[\alpha]}^2 \right) + 2 \langle (n_e^\varepsilon) \partial_x^3 U_e^\varepsilon, \nabla \partial_x^3 \Phi^\varepsilon \rangle + C \left( \| N_e^\varepsilon \|_{[\alpha]}^2 + \| U_e^\varepsilon \|_{[\alpha]}^2 \right) + 2r_\alpha^\varepsilon, \tag{3.25}
\]

where

\[ r_\alpha^\varepsilon = \langle \partial_x^3 N_e^\varepsilon \nabla p_e^\varepsilon (n_e^\varepsilon) - h_e^\varepsilon (n_e^\varepsilon) \partial_x^3 N_e^\varepsilon \nabla n_{m,e}^\varepsilon + n_e^\varepsilon \partial_x^3 (h_e^\varepsilon (n_{m,e}^\varepsilon) - h_e^\varepsilon (n_e^\varepsilon)) \nabla n_{m,e}^\varepsilon, \partial_x^3 U_e^\varepsilon \rangle, \]

which can be estimated in a same way as in the proof of Lemma 3.1. Indeed, by (3.13)-(3.14), we have

\[
\partial_x^3 N_e^\varepsilon \nabla p_e^\varepsilon (n_e^\varepsilon) - h_e^\varepsilon (n_e^\varepsilon) \partial_x^3 N_e^\varepsilon \nabla n_{m,e}^\varepsilon + n_e^\varepsilon \partial_x^3 (h_e^\varepsilon (n_{m,e}^\varepsilon) - h_e^\varepsilon (n_e^\varepsilon)) \nabla n_{m,e}^\varepsilon
\]

\[ = \partial_x^3 N_e^\varepsilon p_e^\varepsilon (n_e^\varepsilon) \nabla N_e^\varepsilon + n_e^\varepsilon \partial_x^3 (h_e^\varepsilon (n_{m,e}^\varepsilon) - h_e^\varepsilon (n_e^\varepsilon)) \nabla n_{m,e}^\varepsilon
\]

\[ = \partial_x^3 N_e^\varepsilon p_e^\varepsilon (n_e^\varepsilon) \nabla N_e^\varepsilon + n_e^\varepsilon \partial_x^3 (h_e^\varepsilon (n_{m,e}^\varepsilon) - h_e^\varepsilon (n_e^\varepsilon)) \nabla n_{m,e}^\varepsilon
\]

\[ = \partial_x^3 N_e^\varepsilon p_e^\varepsilon (n_e^\varepsilon) \nabla N_e^\varepsilon + n_e^\varepsilon \partial_x^3 (h_e^\varepsilon (n_{m,e}^\varepsilon) - h_e^\varepsilon (n_e^\varepsilon)) \nabla n_{m,e}^\varepsilon
\]

\[ = \partial_x^3 N_e^\varepsilon p_e^\varepsilon (n_e^\varepsilon) \nabla N_e^\varepsilon + n_e^\varepsilon \partial_x^3 (h_e^\varepsilon (n_{m,e}^\varepsilon) - h_e^\varepsilon (n_e^\varepsilon)) \nabla n_{m,e}^\varepsilon
\]

by the Moser-type inequalities, we have

\[
\| \partial_x^3 N_e^\varepsilon \nabla p_e^\varepsilon (n_e^\varepsilon) - h_e^\varepsilon (n_e^\varepsilon) \partial_x^3 N_e^\varepsilon \nabla n_{m,e}^\varepsilon + n_e^\varepsilon \partial_x^3 (h_e^\varepsilon (n_{m,e}^\varepsilon) - h_e^\varepsilon (n_e^\varepsilon)) \nabla n_{m,e}^\varepsilon \| \leq C \| N_e^\varepsilon \|_{[\alpha]}^2 + C \| N_e^\varepsilon \|_{[\alpha]-1} + C \| N_e^\varepsilon \|_{[\alpha]}^2
\]

\[ \leq C \| N_e^\varepsilon \|_{[\alpha]}^2 + C \| N_e^\varepsilon \|_{[\alpha]-1}. \]

It follows that

\[ 2r_\alpha^\varepsilon \leq C \left( \| N_e^\varepsilon \|_{[\alpha]}^2 + \| N_e^\varepsilon \|_{[\alpha]-1} \right) \| U_e^\varepsilon \|_{[\alpha]} \]

\[ \leq C \left( \| N_e^\varepsilon \|_{[\alpha]}^2 + \varepsilon^2 \| U_e^\varepsilon \|_{[\alpha]}^2 \right) + \frac{C}{\varepsilon^2} \| N_e^\varepsilon \|_{[\alpha]-1}^2. \tag{3.26}
\]

Combining (3.25) and (3.26), we obtain

\[
\frac{d}{dt} \langle A_e^0 (n_e^\varepsilon) \partial_x^3 W_e^\varepsilon, \partial_x^3 W_e^\varepsilon \rangle \leq 2 \langle (n_e^\varepsilon) \partial_x^3 U_e^\varepsilon, \nabla \partial_x^3 \Phi^\varepsilon \rangle + \| W_e^\varepsilon \|_{[\alpha]}^2 \| + \frac{C}{\varepsilon^2} \| W_e^\varepsilon \|_{[\alpha]-1}^2 + C \left( \| R_m^e \|_{[\alpha]}^2 + \| R_m^\varepsilon \|_{[\alpha]}^2 \right). \tag{3.27}
\]

From bipolar Euler-Poisson to unipolar one
Step 2: Taking the inner product of the ion equations in (3.18) with $2A_i^0 (n_i^\varepsilon) \partial_x^\alpha W_i^\varepsilon$ in $L^2(\mathbb{R}^3)$ yields the following energy equality for $\partial_x^\alpha W_i^\varepsilon$

$$\frac{d}{dt} \langle A_i^0 (n_i^\varepsilon) \partial_x^\alpha W_i^\varepsilon, \partial_x^\alpha W_i^\varepsilon \rangle = -2\langle A_i^0 (n_i^\varepsilon) \partial_x^\alpha W_i^\varepsilon, \partial_x^\alpha H_{i,\varepsilon}^1 \rangle + 2\langle A_i^0 (n_i^\varepsilon) \partial_x^\alpha W_i^\varepsilon, \partial_x^\alpha H_{i,\varepsilon}^1 \rangle - 2\langle A_i^0 (n_i^\varepsilon) \partial_x^\alpha W_i^\varepsilon, \partial_x^\alpha R_i^\varepsilon \rangle + 2\langle A_i^0 (n_i^\varepsilon) \partial_x^\alpha W_i^\varepsilon, J_{i,\varepsilon}^\alpha \rangle + \langle \text{div} A_i (n_i^\varepsilon, u_i^\varepsilon) \partial_x^\alpha W_i^\varepsilon, \partial_x^\alpha W_i^\varepsilon \rangle. \quad (3.28)$$

By (2.15) and (3.3), it is clear that

$$\left| \langle A_i^0 (n_i^\varepsilon) \partial_x^\alpha W_i^\varepsilon, \partial_x^\alpha H_{i,\varepsilon}^1 \rangle \right| \leq C \| W_i^\varepsilon \|_{[\alpha]}^2, \quad (3.29)$$

$$\left| \langle A_i^0 (n_i^\varepsilon) \partial_x^\alpha W_i^\varepsilon, \partial_x^\alpha R_i^\varepsilon \rangle \right| \leq C \| W_i^\varepsilon \|_{[\alpha]}^2 + C \| R_i^\varepsilon \|_{[\alpha]}^2 \leq C \| W_i^\varepsilon \|_{[\alpha]}^2 + C \varepsilon^{2(m+1)}, \quad (3.30)$$

and

$$\left| \langle \text{div} A_i (n_i^\varepsilon, u_i^\varepsilon) \partial_x^\alpha W_i^\varepsilon, \partial_x^\alpha W_i^\varepsilon \rangle \right| \leq C \| \text{div} A_i (n_i^\varepsilon, u_i^\varepsilon) \|_{\infty} \| W_i^\varepsilon \|_{[\alpha]}^2 \leq C \| W_i^\varepsilon \|_{[\alpha]}^2. \quad (3.31)$$

For the term containing $J_{i,\varepsilon}^\alpha$ in the right hand side of (3.28), we write $J_{i,\varepsilon}^\alpha$ as

$$J_{i,\varepsilon}^\alpha = \sum_{j=1}^3 \left( A_i^j (n_i^\varepsilon, u_i^\varepsilon) - A_i^j (n_{i,\varepsilon}^m, u_{i,\varepsilon}^m) \right) \partial_{x_j} \partial_x^\alpha W_i^\varepsilon$$

$$- \sum_{j=1}^3 \partial_x^\alpha \left( (A_i^j (n_i^\varepsilon, u_i^\varepsilon) - A_i^j (n_{i,\varepsilon}^m, u_{i,\varepsilon}^m)) \partial_{x_j} W_i^\varepsilon \right)$$

$$+ \sum_{j=1}^3 \left( A_i^j (n_{i,\varepsilon}^m, u_{i,\varepsilon}^m) \partial_{x_j} \partial_x^\alpha W_i^\varepsilon - \partial_x^\alpha \left( A_i^j (n_{i,\varepsilon}^m, u_{i,\varepsilon}^m) \partial_{x_j} W_i^\varepsilon \right) \right),$$

then applying Moser-type inequalities to $J_{i,\varepsilon}^\alpha$ together with (3.3) yields

$$\left| \langle A_i^0 (n_i^\varepsilon) \partial_x^\alpha W_i^\varepsilon, J_{i,\varepsilon}^\alpha \rangle \right| \leq C \| W_i^\varepsilon \|_{[\alpha]}^2. \quad (3.32)$$

Since concerning the $L^2(\mathbb{R}^3)$ estimate, we may write the term containing $\partial_x^\alpha H_{i,\varepsilon}^2$ as

$$\langle A_i^0 (n_i^\varepsilon) \partial_x^\alpha W_i^\varepsilon, \partial_x^\alpha H_{i,\varepsilon}^2 \rangle = -2\langle n_i^\varepsilon \partial_x^\alpha U_i^\varepsilon, \nabla \partial_x^\alpha \Phi^\varepsilon \rangle. \quad (3.33)$$

Inserting (3.29)–(3.33) into (3.28), we get

$$\frac{d}{dt} \langle A_i^0 (n_i^\varepsilon) \partial_x^\alpha W_i^\varepsilon, \partial_x^\alpha W_i^\varepsilon \rangle \leq -2\langle n_i^\varepsilon \partial_x^\alpha U_i^\varepsilon, \nabla \partial_x^\alpha \Phi^\varepsilon \rangle + C \| W_i^\varepsilon \|_{[\alpha]}^2 + C \varepsilon^{2m}. \quad (3.34)$$
Step 3: Summing (3.27) and (3.34) for all $|\alpha| \leq s$, we obtain
\[
\frac{d}{dt} \sum_{\nu=\epsilon, i} \langle A^\nu_\alpha (n^\nu_\epsilon) \partial_x^\alpha W^\nu_\epsilon, \partial_x^\alpha W^\nu_\epsilon \rangle \\
\leq 2\langle n^\epsilon_\nu \partial_x^\alpha U^\epsilon_\nu, \nabla \partial_x^\alpha \Phi^\epsilon \rangle - 2\langle n^i_\nu \partial_x^\alpha U^i_\nu, \nabla \partial_x^\alpha \Phi^\epsilon \rangle \\
+ C \sum_{\nu=\epsilon, i} \|W^\nu_\epsilon\|^2_{|\alpha|} + \frac{C}{\epsilon^2} \|W^\nu_\epsilon\|^2_{|\alpha|-1} + C\epsilon^{4m} \\
= 2\langle \partial_x^\alpha (n^\epsilon_\nu U^\epsilon_\nu), \nabla \partial_x^\alpha \Phi^\epsilon \rangle - 2\langle \partial_x^\alpha (n^i_\nu U^i_\nu), \nabla \partial_x^\alpha \Phi^\epsilon \rangle \\
+ C \sum_{\nu=\epsilon, i} \|W^\nu_\epsilon\|^2_{|\alpha|} + \frac{C}{\epsilon^2} \|W^\nu_\epsilon\|^2_{|\alpha|-1} + C\epsilon^{4m} \\
+ 2\langle (n^\epsilon_\nu U^\epsilon_\nu) - \partial_x^\alpha (n^\epsilon_\nu U^\epsilon_\nu), \nabla \partial_x^\alpha \Phi^\epsilon \rangle - 2\langle (n^i_\nu U^i_\nu) - \partial_x^\alpha (n^i_\nu U^i_\nu), \nabla \partial_x^\alpha \Phi^\epsilon \rangle \\
= 2\langle \partial_x^\alpha (n^\epsilon_\nu u^\epsilon_{\nu, e} - n^m_{\epsilon, \nu} u^m_{\epsilon, e}), - \partial_x^\alpha (n^i_\nu u^i_{\nu, e} - n^m_{i, \nu} u^m_{i, e}), \nabla \partial_x^\alpha \Phi^\epsilon \rangle \\
+ 2\langle \partial_x^\alpha (n^\epsilon_\nu u^\epsilon_{\nu, e} - n^m_{\epsilon, \nu} u^m_{\epsilon, e}), \nabla \partial_x^\alpha \Phi^\epsilon \rangle \\
+ C \sum_{\nu=\epsilon, i} \|W^\nu_\epsilon\|^2_{|\alpha|} + C\epsilon^{4m} + 2\langle \partial_x^\alpha n^\epsilon_\nu U^\epsilon_\nu - \partial_x^\alpha (n^\epsilon_\nu U^\epsilon_\nu), \nabla \partial_x^\alpha \Phi^\epsilon \rangle \\
- 2\langle (n^i_\nu U^i_\nu) - \partial_x^\alpha (n^i_\nu U^i_\nu), \nabla \partial_x^\alpha \Phi^\epsilon \rangle,
\]
in which we have
\[
2\langle \partial_x^\alpha (n^\epsilon_\nu u^\epsilon_{\nu, e} - n^m_{\epsilon, \nu} u^m_{\epsilon, e}), \partial_x^\alpha (n^i_\nu u^i_{\nu, e} - n^m_{i, \nu} u^m_{i, e}), \nabla \partial_x^\alpha \Phi^\epsilon \rangle \\
= -2\langle \text{div} (\partial_x^\alpha (n^\epsilon_\nu u^\epsilon_{\nu, e} - n^m_{\epsilon, \nu} u^m_{\epsilon, e})), \text{div} (\partial_x^\alpha (n^i_\nu u^i_{\nu, e} - n^m_{i, \nu} u^m_{i, e})), \partial_x^\alpha \Phi^\epsilon \rangle \\
= 2\langle \partial_t (\partial_x^\alpha n^\epsilon_\nu - \partial_x^\alpha N^\epsilon_\nu) + ( \partial_x^\alpha \tilde{R}^\epsilon_{\nu} - \partial_x^\alpha \tilde{R}^\epsilon_{\nu}), \partial_x^\alpha \Phi^\epsilon \rangle \\
= \frac{d}{dt} \| \nabla \partial_x^\alpha \Phi^\epsilon \|^2 - 2\langle \partial_x^\alpha \tilde{R}^\epsilon_{\nu} - \partial_x^\alpha \tilde{R}^\epsilon_{\nu}, \nabla \partial_x^\alpha \Phi^\epsilon \rangle \\
\leq \frac{d}{dt} \| \nabla \partial_x^\alpha \Phi^\epsilon \|^2 + \| \nabla \partial_x^\alpha \Phi^\epsilon \|^2 + C\epsilon^{4m},
\]
and
\[
2\langle \partial_x^\alpha (N^\epsilon_\nu u^m_{\epsilon, e} - N^\epsilon_\nu u^m_{\epsilon, e}), \nabla \partial_x^\alpha \Phi^\epsilon \rangle \\
= 2\langle \partial_x^\alpha (N^\epsilon_\nu (u^m_{\epsilon, e} - u^m_{\epsilon, e})), \nabla \partial_x^\alpha \Phi^\epsilon \rangle + 2\langle \partial_x^\alpha ((N^\epsilon_\nu - N^\epsilon_\nu) u^m_{\epsilon, e}), \nabla \partial_x^\alpha \Phi^\epsilon \rangle \\
= 2\langle \partial_x^\alpha (N^\epsilon_\nu (u^m_{\epsilon, e} - u^m_{\epsilon, e})), \nabla \partial_x^\alpha \Phi^\epsilon \rangle - 2\langle \triangle \partial_x^\alpha \Phi^\epsilon u^m_{\epsilon, e}, \nabla \partial_x^\alpha \Phi^\epsilon \rangle \\
+ 2\langle \triangle \partial_x^\alpha \Phi^\epsilon u^m_{\epsilon, e} - \partial_x^\alpha (\triangle \Phi^\epsilon u^m_{\epsilon, e}), \nabla \partial_x^\alpha \Phi^\epsilon \rangle \\
\leq \|N^\epsilon_\nu\|^2_{|\alpha|} + \| \nabla \Phi^\epsilon \|^2_{|\alpha|}.
\]
Finally, the Moser-type inequalities and the fact $n_\nu = n^m_{\nu, e} + N^\epsilon_\nu \leq C$, for $\nu = i, e$ imply
\[
2\langle (n^\epsilon_\nu U^\epsilon_\nu) - \partial_x^\alpha (n^\epsilon_\nu U^\epsilon_\nu), \nabla \partial_x^\alpha \Phi^\epsilon \rangle - 2\langle \partial_x^\alpha (n^i_\nu U^i_\nu), \nabla \partial_x^\alpha \Phi^\epsilon \rangle \\
\leq \frac{C}{\epsilon^2} \|W^\epsilon_\nu\|^2_{|\alpha|-1} + C \|W^\epsilon_\nu\|^2_{|\alpha|-1} + \| \nabla \Phi^\epsilon \|^2_{|\alpha|}.
Combining the above four inequalities yields (3.19).

3.2. Proof of Theorem 2.1. It suffices to prove $T^\varepsilon_{1,1} \geq T^\varepsilon_1$, i.e. $T^\varepsilon_{2,1} = T^\varepsilon_1$. By the definitions of $T^\varepsilon_1$, $T^\varepsilon_{1,1}$, $T^\varepsilon_{2,1}$ and $T^\varepsilon_1$, we have $T^\varepsilon_{1,1} \leq T^\varepsilon_{2,1} \leq T^\varepsilon_1$. According to (3.6), we may replace $T^\varepsilon_{1,1}$ by $T^\varepsilon_{1,1} \in (0, T^\varepsilon_1]$ such that $[0, T^\varepsilon_{1,1}]$ is the maximum time interval on which $W^\varepsilon$ exists and satisfies (3.6), i.e.

$$\|W^\varepsilon\|_s^2 \leq C, \quad \forall t \in [0, T^\varepsilon_{1,1}],$$

for some constant $C > 0$. We want to prove $T^\varepsilon_{1,1} = T^\varepsilon_1$.

We deal with (3.19) by induction for $1 \leq |\alpha| \leq s$. In view of the $L^2$ estimate, we assume

$$\|W^\varepsilon_{\nu}\|_{|\alpha|-1}^2 \leq C\varepsilon^{2(2m+1-|\alpha|)}.$$  (3.35)

Then, (3.19) becomes

$$\frac{d}{dt} \left( \sum_{\nu=\varepsilon,d} \langle A^0(n^\varepsilon_{\nu})\partial_x n^\varepsilon_{\nu} \partial_x W^\varepsilon_{\nu}, \partial_x W^\varepsilon_{\nu} \rangle + \|\nabla \partial_x \Phi^\varepsilon\|^2 \right) \leq C \sum_{\nu=\varepsilon,d} \|W^\varepsilon_{\nu}\|_{|\alpha|}^2 + \|\nabla \Phi^\varepsilon\|_{|\alpha|}^2 + C\varepsilon^{2(2m-|\alpha|)} + C\varepsilon^{4m}.\quad (3.36)$$

Together with Lemma 3.1 we have

$$\sup_{0 \leq t \leq T^\varepsilon_{1,1}} \|W^\varepsilon(t)\|_s^2 \leq C\varepsilon^{2(2m-s)}.$$  

In particular,

$$\|W^\varepsilon(T^\varepsilon_{1,1})\|_s^2 \leq C\varepsilon^{2(2m-s)}.$$  

If $T^\varepsilon_{1,1} < T^\varepsilon_1$, we apply the theorem of Kato for the local existence of smooth solutions with initial data $W^\varepsilon(T^\varepsilon_{1,1})$. Consequently, there exist $T^\varepsilon_{1,1} > T^\varepsilon_{1,1}$ and a smooth solution $W^\varepsilon \in C([0, T^\varepsilon_{1,1}]; H^s(\mathbb{R}^3))$ of (3.4)–(3.5). When $2m > s$ and $\varepsilon$ is sufficiently small, we always have $\varepsilon^{2(2m-s)} < C$ for all fixed constant $C > 0$. Since the function $t \to \|W^\varepsilon(t)\|_s$ is continuous on $[T^\varepsilon_{1,1}, T^\varepsilon_{1,1}]$, there exists $T^\varepsilon_{1,1} \in (T^\varepsilon_{1,1}, T^\varepsilon_{1,1}]$ such that

$$\|W^\varepsilon(t)\|_s^2 \leq C, \quad t \in [0, T^\varepsilon_{1,1}].$$

This is contradictory to the maximality of $T^\varepsilon_{1,1}$. Thus, we have proved $T^\varepsilon_{1,1} = T^\varepsilon_1$, which implies that $T^\varepsilon_{2,1} \geq T^\varepsilon_1$.  \qed
4. Proof of Theorem 2.2

4.1. Energy estimates. In this section, we continue to use \((n^{e}_{\nu}, u^{e}_{\nu}, \phi^{e})\) and \((n^{i}_{\nu}, u^{i}_{\nu}, \phi^{i})\) to replace \((n_{\nu}^{1/\varepsilon}, u_{\nu}^{1/\varepsilon}, \phi_{\nu}^{1/\varepsilon})\) and \((n_{\nu}^{i,j}, u_{\nu}^{i,j}, \phi_{\nu}^{i,j})\). The exact solution \((n^{e}_{\nu}, u^{e}_{\nu}, \phi^{e})\) is defined in time interval \([0, T^{1/\varepsilon}_{1}]\) and the approximate solution \((n^{m}_{\nu,e}, u^{m}_{\nu,e}, \phi^{m}_{\varepsilon})\) in time interval \([0, T^{1/\varepsilon}_{1}]\), with \(T^{1/\varepsilon}_{1} > 0\) and \(T^{1/\varepsilon}_{1} > 0\). Let

\[
T^{1/\varepsilon}_{2} = \min \left( T^{1/\varepsilon}_{1}, T^{1/\varepsilon}_{1} \right) > 0,
\]

then the exact solution and the approximate solution are both defined in time interval \([0, T^{1/\varepsilon}_{2}]\). In this time interval, we denote

\[
\left( N^{1/\varepsilon}_{\nu}, U^{1/\varepsilon}_{\nu}, \Phi^{1/\varepsilon} \right) \triangleq \left( n^{e}_{\nu} - n^{m}_{\nu,e}, u^{e}_{\nu} - u^{m}_{\nu,e}, \phi^{e} - \phi^{m}_{\varepsilon} \right), \quad \nu = e, i, \tag{4.1}
\]

For simplicity, we denote \(\left( N^{1/\varepsilon}_{\nu}, U^{1/\varepsilon}_{\nu}, \Phi^{1/\varepsilon}, R^{1/\varepsilon,m} \right)\) by \((N^{e}_{\nu}, U^{e}_{\nu}, \Phi^{e}, R^{e,m})\) in this section. It is easy to check that the variable \((N^{e}_{\nu}, U^{e}_{\nu})\) satisfy

\[
\begin{aligned}
\partial_{t} N^{e}_{\nu} + & (U^{e}_{\nu} + u^{m}_{\nu,e}) \cdot \nabla N^{e}_{\nu} + (N^{e}_{\nu} + n^{e}_{\nu,e}) \text{div} U^{e}_{\nu} = - (N^{e}_{\nu} \text{div} u^{m}_{\nu,e} + U^{e}_{\nu} \nabla n^{m}_{\nu,e}) - R^{e,m}_{\nu}, \\
\partial_{t} U^{e}_{\nu} + & \left( (U^{e}_{\nu} + u^{m}_{\nu,e}) \cdot \nabla \right) U^{e}_{\nu} + h'(N^{e}_{\nu} + n^{m}_{\nu,e}) \nabla N^{e}_{\nu} \\
& = - (U^{e}_{\nu} \cdot \nabla) u^{m}_{\nu,e} - (h'(N^{e}_{\nu} + n^{m}_{\nu,e}) - h'(n^{m}_{\nu,e})) \nabla n^{m}_{\nu,e} + \nabla \Phi^{e} - R^{e,m}_{u_{\nu}}, \\
\partial_{t} N^{i}_{\nu} + & (U^{i}_{\nu} + u^{m}_{i,e}) \cdot \nabla N^{i}_{\nu} + (N^{i}_{\nu} + n^{m}_{i,e}) \text{div} U^{i}_{\nu} = - (N^{i}_{\nu} \text{div} u^{m}_{i,e} + U^{i}_{\nu} \nabla n^{m}_{i,e}) - R^{e,m}_{n_{\nu}}, \\
\frac{1}{\varepsilon} \partial_{t} U^{i}_{\nu} + & \frac{1}{\varepsilon} \left( (U^{i}_{\nu} + u^{m}_{i,e}) \cdot \nabla \right) U^{i}_{\nu} + \varepsilon h'(N^{i}_{\nu} + n^{m}_{i,e}) \nabla N^{i}_{\nu} \\
& = - \frac{\varepsilon}{\varepsilon} (U^{i}_{\nu} \cdot \nabla) u^{m}_{\nu,e} - \varepsilon (h'(N^{i}_{\nu} + n^{m}_{i,e}) - h'(n^{m}_{i,e})) \nabla n^{m}_{i,e} - \varepsilon \nabla \Phi^{e} - \varepsilon R^{e,m}_{u_{\nu}}, \\
(N^{e}_{\nu}, U^{e}_{\nu}) \big|_{t=0} &= \left( n^{e}_{\nu,0} - n^{m}_{\nu,e}(0, \cdot), u^{e}_{\nu,0} - u^{m}_{\nu,e}(0, \cdot) \right),
\end{aligned}
\]

with the Poisson equation for \(\Phi^{e}\)

\[
- \Delta \Phi^{e} = N^{e}_{i} - N^{e}_{c}, \quad \lim_{|x| \to +\infty} \Phi^{e}(x) = 0. \tag{4.3}
\]

Set

\[
W^{e}_{\nu} = \begin{pmatrix} N^{e}_{\nu} \\ U^{e}_{\nu} \end{pmatrix}, \quad W^{e}_{i} = \begin{pmatrix} N^{e}_{i} \\ U^{e}_{i} \end{pmatrix},
\]

\[
H^{1}_{e,e} = \begin{pmatrix} N^{e}_{\nu} \text{div} u^{m}_{\nu,e} + U^{e}_{\nu} \nabla n^{m}_{\nu,e} \\ (U^{e}_{\nu} \cdot \nabla) u^{m}_{\nu,e} + (h'(N^{e}_{\nu} + n^{m}_{\nu,e}) - h'(n^{m}_{\nu,e})) \nabla n^{m}_{\nu,e} \end{pmatrix},
\]

\[
H^{1}_{i,e} = \begin{pmatrix} N^{i}_{\nu} \text{div} u^{m}_{i,e} + U^{i}_{\nu} \nabla n^{m}_{i,e} \\ \frac{1}{\varepsilon} (U^{i}_{\nu} \cdot \nabla) u^{m}_{i,e} + \varepsilon (h'(N^{i}_{\nu} + n^{m}_{i,e}) - h'(n^{m}_{i,e})) \nabla n^{m}_{i,e} \end{pmatrix},
\]

\[
H^{2}_{e,e} = \begin{pmatrix} 0 \\ \nabla \Phi^{e} \end{pmatrix}, \quad H^{2}_{i,e} = \begin{pmatrix} 0 \\ -\varepsilon \nabla \Phi^{e} \end{pmatrix},
\]
\[
R^\varepsilon_e = \begin{pmatrix} R_{n\varepsilon_e}^\varepsilon & \varepsilon R_{u\varepsilon_e}^\varepsilon \\ R_{u\varepsilon_e}^\varepsilon \end{pmatrix}, \quad R^\varepsilon_i = \begin{pmatrix} R_{n\varepsilon_i}^\varepsilon \\ \varepsilon R_{u\varepsilon_i}^\varepsilon \end{pmatrix},
\]

and for \(j = 1, 2, 3\),

\[
A^\varepsilon_e (n^\varepsilon_{e}, u^\varepsilon_{e}) = \begin{pmatrix} u^\varepsilon_{e,j} & n^\varepsilon_{e} e_j^\top \\ h^\varepsilon_{e} (n^\varepsilon_{e}) e_j & u^\varepsilon_{e,j} I_3 \end{pmatrix},
\]

\[
A^\varepsilon_i (n^\varepsilon_{i}, u^\varepsilon_{i}) = \begin{pmatrix} u^\varepsilon_{i,j} & \varepsilon n^\varepsilon_{i} e_j^\top \\ \varepsilon h^\varepsilon_{i} (n^\varepsilon_{i}) e_j & u^\varepsilon_{i,j} I_3 \end{pmatrix},
\]

where \((e_1, e_2, e_3)\) is the canonical basis of \(\mathbb{R}^3\) and \(I_3\) is the \(3 \times 3\) unit matrix. Thus the equations of (4.2) can be written as

\[
\partial_t W^\varepsilon_\nu + \sum_{j=1}^{3} A^\varepsilon_{\nu} (n^\varepsilon_{\nu}, u^\varepsilon_{\nu}) \partial_{x_j} W^\varepsilon_\nu = -H^1_{\nu, e} + H^2_{\nu, e} - R^\varepsilon_\nu, \quad \nu = e, i, \quad (4.4)
\]

with the initial data is

\[
t = 0 : \quad W^\varepsilon_\nu = W^\varepsilon_{\nu,0}, \quad \nu = e, i, \quad (4.5)
\]

where

\[
W^\varepsilon_{e,0} = \begin{pmatrix} N^\varepsilon_{e} (0, \cdot) \\ \varepsilon U^\varepsilon_{e} (0, \cdot) \end{pmatrix} = \begin{pmatrix} n^\varepsilon_{e,0} - n^m_{e,\varepsilon} (0, \cdot) \\ u^\varepsilon_{e,0} - u^m_{e,\varepsilon} (0, \cdot) \end{pmatrix},
\]

\[
W^\varepsilon_{i,0} = \begin{pmatrix} N^\varepsilon_{i} (0, \cdot) \\ U^\varepsilon_{i} (0, \cdot) \end{pmatrix} = \begin{pmatrix} n^\varepsilon_{i,0} - n^m_{i,\varepsilon} (0, \cdot) \\ \frac{1}{\varepsilon} (u^\varepsilon_{i,0} - u^m_{i,\varepsilon} (0, \cdot)) \end{pmatrix}.
\]

System (4.4) - (4.5) for \(W^\varepsilon_\nu\) is symmetrizable hyperbolic when \(n^\varepsilon_{\nu} > 0\). Indeed, since the density \(n^0\) of the leading profile satisfies

\[n^0 \geq C > 0, \quad n^m_{\nu, \varepsilon} - n^0 = O(\varepsilon),\]

and \(N^\varepsilon_{\nu}\) is small for small \(\varepsilon\), which we will prove later, so we have

\[n^\varepsilon_{\nu} > 0, \quad \text{for} \ \nu = e, i.\]

With this, let

\[
A^0_e (n^\varepsilon_{e}) = \begin{pmatrix} h^\varepsilon_{e} (n^\varepsilon_{e}) & 0 \\ 0 & n^\varepsilon_{i} I_3 \end{pmatrix}, \quad A^0_i (n^\varepsilon_{i}) = \begin{pmatrix} h^\varepsilon_{i} (n^\varepsilon_{i}) & 0 \\ 0 & n^\varepsilon_{i} I_3 \end{pmatrix}
\]

and for \(j = 1, 2, 3\),

\[
\tilde{A}^\varepsilon_e (n^\varepsilon_{e}, u^\varepsilon_{e}) = A^0_e (n^\varepsilon_{e}) A^\varepsilon_e (n^\varepsilon_{e}, u^\varepsilon_{e}) = \begin{pmatrix} h^\varepsilon_{e} (n^\varepsilon_{e}) u^\varepsilon_{e,j} & p^\varepsilon_{e} (n^\varepsilon_{e}) e_j^\top \\ p^\varepsilon_{e} (n^\varepsilon_{e}) e_j & n^\varepsilon_{e,j} I_3 \end{pmatrix},
\]

\[
\tilde{A}^\varepsilon_i (n^\varepsilon_{i}, u^\varepsilon_{i}) = A^0_i (n^\varepsilon_{i}) A^\varepsilon_i (n^\varepsilon_{i}, u^\varepsilon_{i}) = \begin{pmatrix} h^\varepsilon_{i} (n^\varepsilon_{i}) u^\varepsilon_{i,j} & \varepsilon p^\varepsilon_{i} (n^\varepsilon_{i}) e_j^\top \\ \varepsilon p^\varepsilon_{i} (n^\varepsilon_{i}) e_j & n^\varepsilon_{i,j} I_3 \end{pmatrix},
\]
then for \( n^\varepsilon_i > 0 \), \( A^0_\nu \) is positively definite and \( \tilde{A}^j_i \) is symmetric for all \( 1 \leq j \leq 3 \). Thus, the theorem of Kato for the local existence of smooth solutions can also be applied to (4.4)-(4.5).

4.1.1. \( L^2 \)-estimates. In what follows, we always assume that the conditions of Theorem 2.2 hold.

**Lemma 4.1.** For all \( t \in \left[0, T^{1,1}\right] \) and sufficiently small \( \varepsilon > 0 \), we have

\[
\frac{d}{dt} \left( \sum_{\nu=\varepsilon, i} \langle A^0_\nu (n^\varepsilon_i) W^\varepsilon_i, W^\varepsilon_i \rangle + \| \nabla \Phi \|^2 \right) \leq C \sum_{\nu=\varepsilon, i} \| W^\varepsilon_i \|^2 + \| \nabla \Phi \|^2 + C \varepsilon^{4(m+1)}. \tag{4.6}
\]

**Proof.** Step 1: Taking the inner product of the ion equations in (4.4) with \( 2A^0_\nu (n^\varepsilon_i) W^\varepsilon_i \) in \( L^2(\mathbb{R}^3) \), we obtain the following energy equality for \( W^\varepsilon_i \)

\[
\frac{d}{dt} \langle A^0_\nu (n^\varepsilon_i) W^\varepsilon_i, W^\varepsilon_i \rangle = \langle \text{div} A_i (n^\varepsilon_i, u^\varepsilon_i) W^\varepsilon_i, W^\varepsilon_i \rangle - 2 \langle A^0_\nu (n^\varepsilon_i) W^\varepsilon_i, H^1_{i\varepsilon} \rangle \\
+ 2 \langle A^0_\nu (n^\varepsilon_i) W^\varepsilon_i, H^2_{i\varepsilon} \rangle - 2 \langle A^0_\nu (n^\varepsilon_i) W^\varepsilon_i, R_i \rangle, \tag{4.7}
\]

where

\[
\text{div} A_i (n^\varepsilon_i, u^\varepsilon_i) = \partial_i A^0_i (n^\varepsilon_i) + \sum_{j=1}^{3} \partial_{x_j} \tilde{A}^j_i (n^\varepsilon_i, u^\varepsilon_i),
\]

Now we deal with each term on the right-hand side of (4.7). First, from the mass conservation law \( \partial_t n^\varepsilon_i = -\text{div}(n^\varepsilon_i u^\varepsilon_i) \), we have

\[
\langle \partial_t A^0_i (n^\varepsilon_i) W^\varepsilon_i, W^\varepsilon_i \rangle = -\langle A^0_i (n^\varepsilon_i) \text{div}(n^\varepsilon_i u^\varepsilon_i) W^\varepsilon_i, W^\varepsilon_i \rangle \\
= -\langle h^\varepsilon_i (n^\varepsilon_i) \text{div}(n^\varepsilon_i u^\varepsilon_i) N^\varepsilon_i, N^\varepsilon_i \rangle - \frac{1}{\varepsilon^2} \langle \text{div}(n^\varepsilon_i u^\varepsilon_i) N^\varepsilon_i, U^\varepsilon_i \rangle \\
\leq C \| W^\varepsilon_i \|^2,
\]

and in view of the expression of \( \tilde{A}^j_i (W^\varepsilon_i) \), we obtain

\[
\langle \partial_{x_j} \tilde{A}^j_i (n^\varepsilon_i, u^\varepsilon_i) W^\varepsilon_i, W^\varepsilon_i \rangle = \langle \partial_{x_j} (h^\varepsilon_i (n^\varepsilon_i) u^\varepsilon_i) N^\varepsilon_i, N^\varepsilon_i \rangle + 2 \langle N^\varepsilon_i \partial_{x_j} (p^\varepsilon_i (n^\varepsilon_i) i_j), U^\varepsilon_i \rangle \\
+ \frac{1}{\varepsilon^2} \langle \partial_{x_j} (n^\varepsilon_i u^\varepsilon_i) U^\varepsilon_i, U^\varepsilon_i \rangle,
\]

in which

\[
\langle \partial_{x_j} (h^\varepsilon_i (n^\varepsilon_i) u^\varepsilon_i) N^\varepsilon_i, N^\varepsilon_i \rangle + \frac{1}{\varepsilon^2} \langle \partial_{x_j} (n^\varepsilon_i u^\varepsilon_i) U^\varepsilon_i, U^\varepsilon_i \rangle \leq C \| W^\varepsilon_i \|^2,
\]

and

\[
2 \sum_{j=1}^{d} \langle N^\varepsilon_i \partial_{x_j} (p^\varepsilon_i (n^\varepsilon_i) i_j), U^\varepsilon_i \rangle = \langle N^\varepsilon_i \nabla p^\varepsilon_i (n^\varepsilon_i), U^\varepsilon_i \rangle \leq C \| U^\varepsilon_i \|^2 + C \| N^\varepsilon_i \|^2 \leq C \| W^\varepsilon_i \|^2,
\]
Therefore,
\[
\sum_{j=1}^{d} \langle \partial_{x_j} \tilde{A}_{ij}^\varepsilon(n_i^\varepsilon, u_i^\varepsilon), W_i^\varepsilon, W_i^\varepsilon \rangle \leq C \|W_i^\varepsilon\|^2. \tag{4.9}
\]
It follows from (4.8) and (4.9) that
\[
\langle \text{div} A_i(n_i^\varepsilon, u_i^\varepsilon), W_i^\varepsilon, W_i^\varepsilon \rangle \leq C \|W_i^\varepsilon\|^2. \tag{4.10}
\]
For the remaining terms without $H_{i,\varepsilon}^2$ in the right hand side of (4.7), we have
\[
-2\langle A_i^0(n_i^\varepsilon) W_i^\varepsilon, H_{i,\varepsilon}^1 \rangle - 2\langle A_i^0(n_i^\varepsilon) W_i^\varepsilon, R_i^\varepsilon \rangle
\]
\[
= -2\langle h_i'(n_i^\varepsilon)(N_i^\varepsilon \text{div} u_{i,\varepsilon}^m + U_i^\varepsilon \cdot \nabla n_{i,\varepsilon}^m + R_i^{e,m}), N_i^\varepsilon \rangle
\]
\[
- 2\langle n_i^\varepsilon \left( \varepsilon^2 (U_i^\varepsilon \cdot \nabla) u_{i,\varepsilon}^m + (h_i'(N_i^\varepsilon + n_{i,\varepsilon}^m) - h_i'(n_{i,\varepsilon}^m)) \nabla n_{i,\varepsilon}^m + R_{ui}^{e,m} \right), U_i^\varepsilon \rangle
\]
\[
= -2\langle h_i'(n_i^\varepsilon)(N_i^\varepsilon \text{div} u_{i,\varepsilon}^m, N_i^\varepsilon \rangle - \frac{2}{\varepsilon^2} \langle n_i^\varepsilon \left( U_i^\varepsilon \cdot \nabla) u_{i,\varepsilon}^m, U_i^\varepsilon \rangle - 2\langle h_i'(n_i^\varepsilon)R_{ni,\varepsilon}^{e,m}, N_i^\varepsilon \rangle
\]
\[
- 2\langle n_i^\varepsilon R_{ni,\varepsilon}^{e,m}, U_i^\varepsilon \rangle - 2\langle h_i'(n_i^\varepsilon)N_i^\varepsilon \nabla n_{i,\varepsilon}^m, U_i^\varepsilon \rangle - 2\langle n_i^\varepsilon \left( h_i'(n_i^\varepsilon) - h_i'(n_{i,\varepsilon}^m)) \nabla n_{i,\varepsilon}^m, U_i^\varepsilon \rangle,
\]
in which
\[
- 2\langle h_i'(n_i^\varepsilon)N_i^\varepsilon \text{div} u_{i,\varepsilon}^m, N_i^\varepsilon \rangle - \frac{2}{\varepsilon^2} \langle n_i^\varepsilon \left( U_i^\varepsilon \cdot \nabla) u_{i,\varepsilon}^m, U_i^\varepsilon \rangle \leq C \|W_i^\varepsilon\|^2,
\]
\[
\langle - h_i'(n_i^\varepsilon)N_i^\varepsilon \nabla n_{i,\varepsilon}^m, U_i^\varepsilon \rangle - 2\langle n_i^\varepsilon \left( h_i'(n_i^\varepsilon) - h_i'(n_{i,\varepsilon}^m) \nabla n_{i,\varepsilon}^m, U_i^\varepsilon \rangle \leq C \|W_i^\varepsilon\|^2,
\]
and
\[
-2\langle h_i'(n_i^\varepsilon)R_{ni,\varepsilon}^{e,m}, N_i^\varepsilon \rangle - 2\langle n_i^\varepsilon R_{ni,\varepsilon}^{e,m}, U_i^\varepsilon \rangle \leq C \|W_i^\varepsilon\|^2 + C \left( \|R_{ni,\varepsilon}^{e,m}\|^2 + \|R_{ui,\varepsilon}^{e,m}\|^2 \right).
\]
As for the term containing $H_{i,\varepsilon}^2$ in (4.7), a direct calculation gives
\[
2\langle A_i^0(n_i^\varepsilon) W_i^\varepsilon, H_{i,\varepsilon}^2 \rangle = -2\langle n_i^\varepsilon U_i^\varepsilon, \nabla \Phi^\varepsilon \rangle.
\]
Finally, using (4.7), (4.10) and the four estimates above yield
\[
\frac{d}{dt} \langle A_i^0(n_i^\varepsilon) W_i^\varepsilon, W_i^\varepsilon \rangle \leq C \|W_i^\varepsilon\|^2 - 2\langle n_i^\varepsilon U_i^\varepsilon, \nabla \Phi^\varepsilon \rangle + C \left( \|R_{ni,\varepsilon}^{e,m}\|^2 + \|R_{ui,\varepsilon}^{e,m}\|^2 \right). \tag{4.11}
\]

**Step 2:** Similar to what we have done in the previous section, taking the inner product of the electron equations in (4.4) with $2A_i^0(n_i^\varepsilon) W_i^\varepsilon$ in $L^2(\mathbb{R}^3)$, we have
\[
\frac{d}{dt} \langle A_i^0(n_i^\varepsilon) W_i^\varepsilon, W_i^\varepsilon \rangle = -2\langle A_i^0(n_i^\varepsilon) W_i^\varepsilon, H_{i,\varepsilon}^1 \rangle + 2\langle A_i^0(n_i^\varepsilon) W_i^\varepsilon, H_{i,\varepsilon}^2 \rangle
\]
\[
- 2\langle A_i^0(n_i^\varepsilon) W_i^\varepsilon, R_i^\varepsilon \rangle + \langle \text{div} A_e(n_i^\varepsilon, u_i^\varepsilon) W_i^\varepsilon, W_i^\varepsilon \rangle,
\]
where
\[
\text{div} A_e(n_i^\varepsilon, u_i^\varepsilon) = \partial_t A_e^0(n_i^\varepsilon) + \sum_{j=1}^{3} \partial_{x_j} A_e^0 (n_i^\varepsilon, u_i^\varepsilon).
\]
The estimates are all the same as we did in the zero-electron mass limit, since both of them do not have the parameters $\varepsilon$ and is only different in notations, we omit the proof. Indeed, we have

$$\frac{d}{dt}\langle A_0^\nu(n_\varepsilon^\nu W_\varepsilon^\nu, W_\varepsilon^\nu) \rangle \leq -2\langle n_\varepsilon^\nu U_\varepsilon^\nu, \nabla \Phi^\varepsilon \rangle + C \|W_\varepsilon^\nu\|^2 + C\varepsilon^{4(m+1)}. \quad (4.12)$$

Step 3: Summing (4.11) and (4.12) for all $|\alpha| \leq s$, following the same procedure as the $L^2$-estimate in the previous section, we obtain (4.16). \qed

4.1.2. Higher order estimates. Let $\alpha \in \mathbb{N}^3$ with $1 \leq |\alpha| \leq s$. Applying $\partial_\nu^\alpha$ to (4.14), we get

$$\partial_\nu^\alpha W_\nu^\varepsilon + \sum_{j=1}^3 A_j^\nu(n_\nu^\nu, u_\nu^\nu) \partial_{x_j} \partial_\nu^\alpha W_\nu^\varepsilon = -\partial_\nu^\alpha \left(H_1^{\nu,\varepsilon} - H_2^{\nu,\varepsilon} + R_\nu^{\varepsilon}\right) + J_{\nu,\varepsilon}^\alpha, \quad \nu = e, i, \quad (4.13)$$

where

$$J_{\nu,\varepsilon}^\alpha = \sum_{j=1}^3 \left( A_j^\nu(n_\nu^\nu, u_\nu^\nu) \partial_{x_j} \partial_\nu^\alpha W_\nu^\varepsilon - \partial_\nu^\alpha \left(A_j^\nu(n_\nu^\nu, u_\nu^\nu) \partial_{x_j} W_\nu^\varepsilon\right) \right).$$

Lemma 4.2. For all $t \in [0, T^{1/\gamma}]$ and sufficiently small $\varepsilon > 0$, we have

$$\frac{d}{dt}\left(\sum_{\nu=\varepsilon, i} \langle A_0^\nu(n_\nu^\varepsilon \partial_\nu^\alpha W_\nu^\varepsilon, \partial_\nu^\alpha W_\nu^\varepsilon) \rangle + \|\nabla \partial_\nu^\alpha \Phi^\varepsilon\|^2\right)$$

$$\leq C \sum_{\nu=\varepsilon, i} \|W_\nu^\varepsilon\|_{|\alpha|}^2 + \|\nabla \Phi^\varepsilon\|_{|\alpha|}^2 + C\varepsilon^{4(m+1)}. \quad (4.14)$$

Proof. Step 1: Taking the inner product of the ion equations in (4.13) with $2A_0^\nu(n_\nu^\varepsilon \partial_\nu^\alpha W_\nu^\varepsilon$ in $L^2(\mathbb{R}^3)$ yields the following energy equality for $\partial_\nu^\alpha W_\nu^\varepsilon$

$$\frac{d}{dt}\langle A_0^\nu(n_\nu^\varepsilon \partial_\nu^\alpha W_\nu^\varepsilon, \partial_\nu^\alpha W_\nu^\varepsilon) \rangle = \langle \text{div} A_0^\nu(n_\nu^\varepsilon, u_\nu^\varepsilon) \partial_\nu^\alpha W_\nu^\varepsilon, \partial_\nu^\alpha W_\nu^\varepsilon \rangle$$

$$-2\langle A_0^\nu(n_\nu^\varepsilon) \partial_\nu^\alpha W_\nu^\varepsilon, \partial_\nu^\alpha H_1^{\nu,\varepsilon} + \partial_\nu^\alpha R_\nu^{\varepsilon}\rangle$$

$$+2\langle A_0^\nu(n_\nu^\varepsilon) \partial_\nu^\alpha W_\nu^\varepsilon, \partial_\nu^\alpha H_2^{\nu,\varepsilon}\rangle + 2\langle A_0^\nu(n_\nu^\varepsilon) \partial_\nu^\alpha W_\nu^\varepsilon, J_{\nu,\varepsilon}^\alpha \rangle, \quad (4.15)$$

which are treated term by term as follows. First, similarly to (4.10), it is easy to get

$$\|\langle \text{div} A_0(n_\nu^\varepsilon, u_\nu^\varepsilon) \partial_\nu^\alpha W_\nu^\varepsilon, \partial_\nu^\alpha W_\nu^\varepsilon \rangle\| \leq C \|W_\nu^\varepsilon\|_{|\alpha|}^2 + \langle \partial_\nu^\alpha N_\nu^\varepsilon \nabla p_\nu^\varepsilon(n_\nu^\varepsilon), \partial_\nu^\alpha U_\nu^\varepsilon \rangle. \quad (4.16)$$
For the terms without $H_{i,\epsilon}^2$ and $J_{i,\epsilon}^n$ in the right hand side of (4.15), a straightforward calculation yields
\[
-2\langle A_i^0(n_i^\epsilon)\partial_x^\alpha W_i^\epsilon, \partial_x^\alpha H_{i,\epsilon}^1 + \partial_x^\alpha R_i^\epsilon \rangle \\
= -2\langle h_i^\prime(n_i^\epsilon)\partial_x^\alpha (N_i^\epsilon \text{div} u_{i,\epsilon}^m) + \partial_x^\alpha R_{i,n}^m, \partial_x^\alpha N_i^\epsilon \rangle - 2\langle h_i^\prime(n_i^\epsilon)(\partial_x^\alpha N_i^\epsilon \nabla n_{i,\epsilon}^m), \partial_x^\alpha U_i^\epsilon \rangle \\
-2\langle h_i^\prime(n_i^\epsilon)(\partial_x^\alpha (U_i^\epsilon \cdot \nabla n_{i,\epsilon}^m) - \partial_x^\alpha U_i^\epsilon \cdot \nabla n_{i,\epsilon}^m), \partial_x^\alpha N_i^\epsilon \rangle \\
-2\langle n_i^\epsilon \partial_x^\alpha \left[ \frac{1}{\epsilon^2} (U_i^\epsilon \cdot \nabla) u_{i,\epsilon}^m + R_{u,i}^m \right], \partial_x^\alpha U_i^\epsilon \rangle - 2\langle n_i^\epsilon \partial_x^\alpha (h_i^\prime(n_i^\epsilon) - h_i^\prime(n_{i,\epsilon}^m))\nabla n_{i,\epsilon}^m, \partial_x^\alpha U_i^\epsilon \rangle \\
-2\langle n_i^\epsilon \partial_x^\alpha ((h_i^\prime(n_i^\epsilon) - h_i^\prime(n_{i,\epsilon}^m))\nabla n_{i,\epsilon}^m) - \partial_x^\alpha (h_i^\prime(n_i^\epsilon) - h_i^\prime(n_{i,\epsilon}^m))\nabla n_{i,\epsilon}^m, \partial_x^\alpha U_i^\epsilon \rangle,
\]
to which applying the Moser-type inequalities yields
\[
-2\langle h_i^\prime(n_i^\epsilon)(\partial_x^\alpha (N_i^\epsilon \text{div} u_{i,\epsilon}^m) + \partial_x^\alpha R_{i,n}^m), \partial_x^\alpha N_i^\epsilon \rangle \leq C \left( \|W_i^\epsilon\|_{|\alpha|}^2 + \|R_{i,n}^m\|_{|\alpha|}^2 \right), \\
-2\langle h_i^\prime(n_i^\epsilon)(\partial_x^\alpha (U_i^\epsilon \cdot \nabla n_{i,\epsilon}^m) - \partial_x^\alpha U_i^\epsilon \cdot \nabla n_{i,\epsilon}^m), \partial_x^\alpha N_i^\epsilon \rangle \leq C \|W_i^\epsilon\|_{|\alpha|}^2, \\
-2\langle n_i^\epsilon \partial_x^\alpha \left[ \frac{1}{\epsilon^2} (U_i^\epsilon \cdot \nabla) u_{i,\epsilon}^m + R_{u,i}^m \right], \partial_x^\alpha U_i^\epsilon \rangle \leq C \left( \|W_i^\epsilon\|_{|\alpha|}^2 + \|R_{u,i}^m\|_{|\alpha|}^2 \right), \\
-2\langle n_i^\epsilon \partial_x^\alpha ((h_i^\prime(n_i^\epsilon) - h_i^\prime(n_{i,\epsilon}^m))\nabla n_{i,\epsilon}^m) - \partial_x^\alpha (h_i^\prime(n_i^\epsilon) - h_i^\prime(n_{i,\epsilon}^m))\nabla n_{i,\epsilon}^m, \partial_x^\alpha U_i^\epsilon \rangle \leq C \|W_i^\epsilon\|_{|\alpha|}^2,
\]
and
\[
-2\langle n_i^\epsilon \partial_x^\alpha ((h_i^\prime(n_i^\epsilon) - h_i^\prime(n_{i,\epsilon}^m))\nabla n_{i,\epsilon}^m) - \partial_x^\alpha (h_i^\prime(n_i^\epsilon) - h_i^\prime(n_{i,\epsilon}^m))\nabla n_{i,\epsilon}^m, \partial_x^\alpha U_i^\epsilon \rangle \\
\leq C \left( \|N_i^\epsilon\|_{|\alpha|-1}^2 + \|U_i^\epsilon\|_{|\alpha|}^2 \right) \leq C \|W_i^\epsilon\|_{|\alpha|}^2.
\]
These estimates imply
\[
-2\langle A_i^0(n_i^\epsilon)\partial_x^\alpha W_i^\epsilon, \partial_x^\alpha H_{i,\epsilon}^1 + \partial_x^\alpha R_i^\epsilon \rangle \leq C \left( \|W_i^\epsilon\|_{|\alpha|}^2 + \|R_{i,n}^m\|_{|\alpha|}^2 + \|R_{u,i}^m\|_{|\alpha|}^2 \right). \tag{4.17}
\]
For the term containing $J_{i,\epsilon}^n$ in the right hand side of (4.15), we have
\[
\langle A_i^0(n_i^\epsilon)\partial_x^\alpha W_i^\epsilon, \partial_x^\alpha (A_i^1(n_i^\epsilon, u_i^\epsilon)\partial_x W_i^\epsilon) - A_i^1(n_i^\epsilon, u_i^\epsilon)\partial_x^\alpha (\partial_x W_i^\epsilon) \rangle \\
= \langle h_i^\prime(n_i^\epsilon)\partial_x^\alpha (u_{i,j}^\epsilon \partial_x^\alpha N_i^\epsilon), \partial_x^\alpha N_i^\epsilon \rangle \\
+ \frac{1}{\epsilon^2} \langle n_i^\epsilon (\partial_x^\alpha (u_{i,j}^\epsilon \partial_x U_{i,j}^\epsilon) - u_{i,j}^\epsilon \partial_x^\alpha U_{i,j}^\epsilon), \partial_x^\alpha U_{i,j}^\epsilon \rangle \\
+ \langle n_i^\epsilon (\partial_x^\alpha (h_i^\prime(n_i^\epsilon)\partial_x N_i^\epsilon) - h_i^\prime(n_i^\epsilon)\partial_x^\alpha N_i^\epsilon), \partial_x^\alpha U_{i,j}^\epsilon \rangle \\
+ \langle h_i^\prime(n_i^\epsilon) (\partial_x^\alpha (n_i^\epsilon \partial_x U_{i,j}^\epsilon) - n_i^\epsilon \partial_x^\alpha U_{i,j}^\epsilon), \partial_x^\alpha N_i^\epsilon \rangle \\
\leq C \|W_i^\epsilon\|_{|\alpha|}^2 + C \left( \|N_i^\epsilon\|_{|\alpha|-1}^2 + \|U_i^\epsilon\|_{|\alpha|}^2 \right) + C \left( \|N_i^\epsilon\|_{|\alpha|}^2 + \|U_i^\epsilon\|_{|\alpha|-1}^2 \right),
\]
which implies
\[
\left| \langle A_i^0(n_i^\epsilon)\partial_x^\alpha W_i^\epsilon, J_{i,\epsilon}^n \rangle \right| \leq C \|W_i^\epsilon\|_{|\alpha|}^2. \tag{4.18}
\]
For the term containing $H_{i,e}$ in the right hand side of (4.7), a direct calculation gives
\[
2 \langle A_i^e (n_e^\varepsilon) \partial_x^\varepsilon W_{i,e}^\varepsilon, \partial_x^\varepsilon H_{i,e}^\varepsilon \rangle = 2 \langle n_e^\varepsilon \partial_x^\varepsilon U_i^e, \nabla \partial_x^\varepsilon \Phi^\varepsilon \rangle.
\]
(4.19)

Therefore, using (4.15), (4.16)-(4.19) yield
\[
\frac{d}{dt} \langle A_i^0 (n_e^\varepsilon) \partial_x^\varepsilon W_{i,e}^\varepsilon, \partial_x^\varepsilon W_{i,e}^\varepsilon \rangle \\
\leq 2 \langle n_e^\varepsilon \partial_x^\varepsilon U_i^e, \nabla \partial_x^\varepsilon \Phi^\varepsilon \rangle + C ||W_{i,e}^\varepsilon||^2_{|\alpha|} + C \left( ||R_{n_{m},n_{e}}^\varepsilon||_{|\alpha|} + ||R_{u_{m},u_{e}}^\varepsilon||_{|\alpha|} \right).
\]
(4.20)

Step 2: Taking the inner product of the electrons equations in (4.3) with $2 A_i^0 (n_e^\varepsilon) \partial_x^\varepsilon W_{i,e}^\varepsilon$ in $L^2(\mathbb{R}^3)$ yields the following energy equality for $\partial_x^\varepsilon W_{i,e}^\varepsilon$
\[
\frac{d}{dt} \langle A_i^0 (n_e^\varepsilon) \partial_x^\varepsilon W_{i,e}^\varepsilon, \partial_x^\varepsilon W_{i,e}^\varepsilon \rangle = -2 \langle A_i^0 (n_e^\varepsilon) \partial_x^\varepsilon W_{i,e}^\varepsilon, \partial_x^\varepsilon H_{i,e}^1 \rangle + 2 \langle A_i^0 (n_e^\varepsilon) \partial_x^\varepsilon W_{i,e}^\varepsilon, \partial_x^\varepsilon H_{i,e}^2 \rangle \\
-2 \langle A_i^0 (n_e^\varepsilon) \partial_x^\varepsilon W_{i,e}^\varepsilon, \partial_x^\varepsilon R_{i,e}^\varepsilon \rangle + 2 \langle A_i^0 (n_e^\varepsilon) \partial_x^\varepsilon W_{i,e}^\varepsilon, \partial_x^\varepsilon \Phi^\varepsilon \rangle \\
+ \langle \text{div} A_i (n_e^\varepsilon, u_e^\varepsilon) \partial_x^\varepsilon W_{i,e}^\varepsilon, \partial_x^\varepsilon W_{i,e}^\varepsilon \rangle.
\]

The estimates are all the same as we did in the zero-electron mass limit, since both of them do not have the parameters $\varepsilon$ and is only different in notations, we omit the proof. Indeed, we have
\[
\frac{d}{dt} \langle A_i^0 (n_e^\varepsilon) \partial_x^\varepsilon W_{i,e}^\varepsilon, \partial_x^\varepsilon W_{i,e}^\varepsilon \rangle \leq -2 \langle n_e^\varepsilon \partial_x^\varepsilon U_i^e, \nabla \partial_x^\varepsilon \Phi^\varepsilon \rangle + C ||W_{i,e}^\varepsilon||^2_{|\alpha|} + C \varepsilon^{4(m+1)}.
\]
(4.21)

Step 3: Summing (4.20) and (4.21) for all $|\alpha| \leq s$, following the same procedure as the higher order estimates in the previous section, we obtain (4.14).

\[\square\]

4.2. Proof of Theorem 2.2. The rest of the proof is also based on the continuous method, which is similar as what we did in zero-electron mass limit, we omit it here.

REFERENCES

[1] G. Ali and L. Chen, The zero-electron-mass limit in the Euler-Poisson system for both well-and ill-prepared initial data, Nonlinearity, 24 (2011) 2745.
[2] G. Ali, L. Chen, A. Jüngel and Y. J. Peng, The zero-electron-mass limit in the hydrodynamic model for plasmas, Nonlinear Analysis: Theory, Methods & Applications, 72 (2010) 4415–4427.
[3] C. Besse, P. Degond, F. Deluzet, J. Claudel, G. Gallice and C. Tessieras, A model hierarchy for ionospheric plasma modeling, Math. Models Methods Appl. Sci. 14 (2004) 393–415.
[4] Y. Brenier, N. Mauser and M. Puel, Incompressible Euler and e-MHD as scaling limits of the Vlasov-Maxwell system, Commun. Math. Sci. 1 (2003) 437–447.
[5] F. Chen, Introduction to Plasma Physics and Controlled Fusion, Vol. 1, (Plenum Press, 1984).
[6] G. Q. Chen, J. W. Jerome and D. Wang, Compressible Euler-Maxwell equations, (In Proceedings of the Fifth International Workshop on Mathematical Aspects of Fluid and Plasma Dynamics (Maui, HI, 1998) 29 (2000) 311-331).
[7] P. Germain and N. Masmoudi, Global existence for the Euler-Maxwell system, *Ann. Sci. e. Norm.* **47** (2014) 469-503.

[8] Y. Guo, A. Ionescu and B. Pausader, The Euler-Maxwell two-fluid system in 3d. *arXiv preprint arXiv:1303.1060* (2013).

[9] A. Jüngel and Y. J. Peng, A hierarchy of hydrodynamic models for plasmas. zero-electron-mass limits in the drift-diffusion equations, (In *Annales de l’Institut Henri Poincare - Non Linear Analysis*, **17** (2000) 83–118).

[10] A. Jüngel and Y. J. Peng, *Zero-relaxation-time Limits in the Hydrodynamic Equations for Plasmas Revisited*, (Citeseer, 1998).

[11] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, *Arch. Rational Mech. Anal.* **58** (1975) 181–205.

[12] S. Klainerman and A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, *Comm. Pure Appl. Math.* **34** (1981) 481–524.

[13] Y. Li, Y. J. Peng and Y. G. Wang, From two-fluid Euler-Poisson equations to one-fluid Euler equations. *Asymptotic Analysis*, **85** (2013) 125–148.

[14] Y. Li, Y. J. Peng and S. Xi, Rigorous derivation of a Boltzmann relation from isothermal Euler-Poisson systems, *J. Math. Phys.* **59** (2018) 123501.

[15] C. Liu, Z. Guo and Y. J. Peng, Global stability of large steady-states for an isentropic Euler-Maxwell system in $\mathbb{R}^3$, preprint.

[16] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables, Vol. 53*. (Springer Science & Business Media, 1984).

[17] Y. J. Peng, Stability of non-constant equilibrium solutions for Euler-Maxwell equations, *J. Math. Pures Appl.* **103** (2015) 39–67.

[18] Y. J. Peng and S. Wang, Convergence of compressible Euler-Maxwell equations to compressible Euler-Poisson equations, *Chin. Ann. Math. Ser. B*, **28** (2007) 583–602.

[19] Y. J. Peng and S. Wang, Convergence of compressible Euler-Maxwell equations to incompressible Euler equations, *Comm. Partial Differential Equations*, **33** (2008), 349–376.

[20] Y. J. Peng and S. Wang, Rigorous derivation of incompressible e-MHD equations from compressible Euler-Maxwell equations. *SIAM J. Math. Anal.*, **40** (2008) 540–565.

[21] Y. J. Peng and S. Wang, Asymptotic expansions in two-fluid compressible Euler-Maxwell equations with small parameters, *Discrete Contin. Dyn. Syst.* **23** (2009) 415–433.

[22] Y. J. Peng and S. Wang, Convergence of compressible Euler-Poisson equations to incompressible type Euler equations, *Asymptotic Analysis*, **41** (2005) 141-160.

[23] H. Rishbeth and O. Garriott, *Introduction to ionospheric physics*, (IEEE Transactions on Image Processing, 1969).

[24] J. Xu and T. Zhang, Zero-electron-mass limit of Euler-Poisson equations, *Discrete Contin. Dyn. Syst.* **33** (2013) 4743–4768.

[25] J. Yang and S. Wang, The non-relativistic limit of Euler-Maxwell equations for two-fluid plasma, *Nonlinear Anal.* **72** (2010) 1829–1840.