Design of optimal disturbance cancellation controllers via modified loop transfer recovery

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The authors proposed a modified loop transfer recovery (LTR) method for designing the disturbance cancellation controllers where the disturbances were assumed to be step functions and the optimality of the controllers was not considered. This paper discusses the extension of their work to a more general class of disturbances with optimality consideration. It is assumed that the plant is minimum phase and the disturbances entering the plant input side are generated by function generators with unknown initial conditions. A quadratic performance index explicitly representing the disturbance cancellation requirement is introduced. The optimal disturbance cancellation controller minimizing the performance index is constructed by the separation principle. As a target for the LTR design, the optimal disturbance cancellation controller based on the measurement of the plant state is chosen. It is shown that the target feedback property can be recovered in the output feedback controller by a simple modification of the standard LTR procedure. A numerical example is presented to illustrate the effectiveness of the proposed optimal design.

Keywords: linear system; optimal disturbance cancellation; polynomial disturbances; sinusoidal disturbances; loop transfer recovery

1. Introduction

The authors discussed a loop transfer recovery (LTR) design of the disturbance cancellation controllers (Guo, Ishihara, & Takeda, 1996) where the disturbances were assumed to be step functions and the optimality of the controllers was not considered. It was pointed out that the standard LTR theory (e.g. Saberi, Chen, & Sannuti, 1993; Stein & Athans, 1987) could not directly be applied but that a procedure obtained by modifying the standard LTR procedure could be used for the disturbance cancellation controllers.

The extension of our earlier work to a more general class of disturbances has initially been discussed in our conference paper (Ishihara & Guo, 2012) with the optimality consideration. In this paper, the extension is discussed in more detail with a design example illustrating the effectiveness of the proposed optimal design. It is assumed that the plant is minimum phase and the disturbances, which enter the plant input side, are generated by function generators with unknown initial conditions. The class of disturbances includes sinusoids with known frequencies as well as polynomial functions of time such as steps and ramps. To guarantee the optimality, a quadratic performance index explicitly representing the disturbance cancellation requirement is introduced as in Ishihara and Guo (2008).

Assuming that the state of the plant and that of the disturbance model are perfectly measurable, we obtain the optimal control law by reducing the optimal disturbance cancellation problem to a standard linear-quadratic (LQ) problem (Anderson & Moore, 1990). The optimal output feedback controller is constructed by the separation principle with the use of the Kalman filter jointly estimating the state of the plant and that of the disturbance model. As a target for the LTR design, we choose the optimal disturbance cancellation controller including the optimal disturbance estimator based on the measurement of the plant state. The use of the optimal disturbance estimator ensures that the target has guaranteed large stability margins. It is shown that the target can be recovered by an extended version of the modified LTR procedure proposed in Guo et al. (1996).

It should be noted that, in the LTR design of LQG (Linear-Quadratic-Gaussian) controllers (e.g. Stein & Athans, 1987), the covariance parameters in the stochastic model and the weighting coefficients of the performance index are used as tuning parameters to achieve desired feedback property. Note also that, as in our earlier work (Guo et al., 1996), we use sensitivity matrices instead of loop transfer matrices to discuss the LTR.

This paper is organized as follows. The plant description is given in Section 2. The optimal output disturbance cancellation controller is constructed in Section 3. Section 4 discusses the modified LTR design. In Section 5, an illustrative numerical example is presented. Concluding remarks are given in Section 6.
2. Plant description
Consider a plant given by
\[ \dot{x}_p(t) = A_p x_p(t) + B_p u(t) + d(t), \quad y(t) = C_p x_p(t), \] (1)
where \( x_p(t) \) is an \( n_p \) dimensional state vector, \( u(t) \) is an \( m \) dimensional control input, \( y(t) \) is an \( m \) dimensional output and \( d(t) \) is an \( m \) dimensional disturbance vector described by
\[ d(t) = C_d x_d(t), \quad \dot{x}_d(t) = A_d x_d(t), \] (2)
where \( x_d(t) \) is an \( n_d \) dimensional state vector. The disturbance model (2) can be used to describe a fairly general class of persistent disturbances including steps, ramps and sinusoids with known frequency.

Define
\[ G_p(s) \triangleq C_p (sI - A_p)^{-1} B_p. \] (3)

For the system given by Equations (1) and (2), we assume the following conditions:

A1: \((A_p, B_p, C_p)\) is a minimal realization and \( G_p(s) \) is nonsingular for almost all \( s \).

A2: \((A_p, B_p, C_p)\) is a minimum phase, that is, \( G_p(s) \) has no zero in the closed right half plane.

A3: \((C_d, A_d)\) is an observable pair.

A4: All the eigenvalues of \( A_d \) are in the closed left plane.

3. Optimal disturbance cancellation

3.1. Optimal controller

The optimal disturbance cancellation controller for the plant (1) is constructed based on the separation principle. First, we give the optimal disturbance cancellation control law under the assumption that the state and the disturbance are perfectly measurable.

**Proposition 1** Assume that the state vectors \( x_d(t) \) and \( x_p(t) \) in Equations (1) and (2) are measurable. Consider the quadratic performance index:

\[ J_d = \int_0^\infty \{ y'(t)Qy(t) + [u(t) + d(t)]'R[u(t) + d(t)] \} dt, \] (4)

where \( Q \) and \( R \) are positive definite matrices. Then the optimal controller minimizing (4) is given by

\[ u(t) = -Fx_p(t) - d(t), \] (5)

where \( F \) is the optimal feedback gain matrix of the optimal regulator problem for the plant \((A_p, B_p, C_p)\) with the standard quadratic performance index:

\[ J = \int_0^\infty \{ y'(t)Qy(t) + u'(t)Ru(t) \} dt, \] (6)

where \( Q \) and \( R \) are the same as the weighting matrices in Equations (4).

**Proof** Define a new control input as

\[ u_c(t) \triangleq u(t) + d(t). \] (7)

Using the new control input, we can rewrite the plant dynamics (1) as

\[ \dot{x}_p(t) = A_p x_p(t) + B_p u_c(t), \quad y(t) = C_p x_p(t). \] (8)

In addition, we can write the performance index (4) as

\[ J_d \triangleq \int_0^\infty [y'(t)Qy(t) + u_c'(t)Ru_c(t)] dt. \] (9)

Note that \( u_c(t) \) can take an arbitrary value by an appropriate choice of \( u(t) \) since the disturbance \( d(t) \) is perfectly measurable. Consequently, the optimal control problem for the plant (1) with the disturbance (2) can be reduced to the standard quadratic optimal problem under the assumption that \( x_d(t) \) and \( x_p(t) \) are perfectly measurable. The optimal control input \( u_c(t) \) is given as

\[ u_c(t) = -Fx_p(t), \] (10)

where \( F \) is the optimal feedback gain matrix of the plant \((A_p, B_p, C_p)\) minimizing the performance index (6). It follows from Equations (7) and (10) that the optimal control input \( u(t) \) for the disturbance cancellation problem is given by (5).

By the separation principle (e.g. Anderson & Moore, 1990), the output feedback disturbance cancellation controller for Equations (1) and (2) is obtained from Proposition 1 as

\[ u(t) = -F\hat{x}_p(t) - C_d \hat{x}_d(t), \] (11)

where \( \hat{x}_p(t) \) and \( \hat{x}_d(t) \) are optimal estimates of \( x_p(t) \) and \( x_d(t) \), respectively. The estimates are obtained by constructing a Kalman filter for the extended stochastic model:

\[ \dot{x}(t) = Ax(t) + Bu(t) + \hat{B}w(t), \quad y(t) = Cx(t) + v(t), \] (12)

where

\[ x(t) \triangleq \begin{bmatrix} x_d(t) \\ x_p(t) \end{bmatrix}, \quad A \triangleq \begin{bmatrix} A_d & 0 \\ B_p C_d & A_p \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 0 \\ B_p \end{bmatrix}, \] (13)

\( v(t) \) and \( w(t) \) are mutually independent white noise processes with covariance matrices \( V \) and \( W \), respectively, and \( \hat{B} \) is chosen such that \((A, \hat{B})\) is controllable. The choice of the matrix \( B \) will be discussed in Section 4.

The structure of the output feedback controller is shown in Figure 1.
3.2. Sensitivity property

First, we give the transfer function matrix of the controller (11) in left factorization form as follows.

**Lemma 1** Consider the controller (11) with the Kalman filter for the stochastic model (12). Let $C_{uy}(s)$ denote the controller transfer function matrix of the output feedback controller from the output $y(t)$ to the control input $u(t)$. Define the partition of the Kalman filter gain matrix $K$ as

$$K = \begin{bmatrix} K_d \\ K_p \end{bmatrix}. \quad (14)$$

Then the controller transfer function matrix $C_{uy}(s)$ can be written as

$$C_{uy}(s) = -M^{-1}(s) N(s), \quad (15)$$

where

$$M(s) \triangleq [I + F(sI - A_p + K_p C_p)^{-1} B_p][I + C_d(sI - A_d)^{-1} K_d]$$

$$\times C_p(sI - A_p + K_p C_p)^{-1} B_p]^{-1}, \quad (16)$$

$$N(s) \triangleq F[I + (sI - A_p + K_p C_p)^{-1} B_p C_d(sI - A_d)^{-1} K_d C_p]^{-1}$$

$$\times [K_p + B_p C_d(sI - A_d)^{-1} K_d] - C_d(sI - A_d)^{-1} K_d$$

$$\times [I - C_p[I + (sI - A_p + K_p C_p)^{-1} B_p C_d(sI - A_d)^{-1}$$

$$\times K_d C_p]^{-1} (sI - A_p + K_p C_p)^{-1}$$

$$\times [K_p + B_p C_d(sI - A_d)^{-1} K_d]]. \quad (17)$$

**Proof** The expression can easily be obtained by straightforward matrix calculation.

Using the above lemma, we can obtain the following result on the sensitivity property.

**Proposition 2** Consider the output feedback controller (11) for the plant (1) and (2). Let $\Sigma(s)$ denote the sensitivity matrix at the plant input side. The sensitivity matrix can be factored as

$$\Sigma(s) = S(s) M(s), \quad (18)$$

where

$$S(s) \triangleq [I + F(sI - A_p)^{-1} B_p]^{-1} \quad (19)$$

is the sensitivity matrix of the standard LQ regulator and $M(s)$ is defined in Equation (16).

**Proof** The sensitivity matrix at the plant input side is defined as

$$\Sigma(s) = [M(s) + N(s) G_p(s)]^{-1} \quad (20)$$

where $G_p(s)$ is defined in Equation (3). It follows from Equations (15) and (20) that

$$\Sigma(s) = [M(s) + N(s) G_p(s)]^{-1} \quad (21)$$

Simple but somewhat tedious matrix calculation using Equations (16) and (17) yields

$$M(s) + N(s) G_p(s) = I + F(sI - A_p)^{-1} B_p. \quad (22)$$

The factorization (18) follows from Equations (21) and (22).

**Remark 1** Note that the eigenvalue of the matrix $A_d$ for the disturbance model (2) appears as a zero of $[I + C_d(sI - A_d)^{-1} K_d C_p(sI - A_p + K_p C_p)^{-1} B_p]^{-1}$ in the denominator matrix of Equation (16) provided $K_d \neq 0$. In the case that the matrix $A_d$ has an eigenvalue on the imaginary axis, the eigenvalue appears as a zero of the sensitivity matrix (18) under the assumptions A1–A4, which shows that the controller has the internal model for the disturbances. It is worth pointing out that the internal model is not assumed a priori but emerges as a result of minimization of the performance index (4).

4. Modified LTR design

The design of the optimal output feedback disturbance cancellation controller requires the determination of $F$, $K_d$ and $K_p$ such that design specifications are satisfied. Since the stabilizability of the extended model (12) consisting of the plant and the disturbance model is not guaranteed, the classical LTR method cannot directly be applied. To overcome the difficulty, the modified LTR method has been proposed in Guo et al. (1996) for the step disturbances. However, they have not considered the use of the optimal target. In this section, we discuss the modified LTR design using the optimal target for a more general class of disturbances. The target and the recovery procedure of the modified LTR design are discussed in the following subsections.
4.1. Target controller design

As a target of the design, we choose the optimal disturbance cancellation controller for the case where the plant state $x_p(t)$ is measurable but the disturbance state $x_d(t)$ is not. By the separation principle, the target controller is obtained from Equation (5) as

$$u(t) = -Fx_p(t) - C_d\hat{x}_d(t), \quad (23)$$

where $F$ is the optimal feedback gain matrix minimizing the performance index (6) and $\hat{x}_d(t)$ is the optimal estimate of $x_d(t)$ based on the observation of $x_p(t)$.

The optimal estimate $\hat{x}_d(t)$ is obtained from the relation:

$$\dot{x}_d(t) = A_d\hat{x}_d(t), \quad (24)$$

$$\dot{x}_p(t) - A_p x_p(t) - B_p u(t) = B_p C_d \hat{x}_d(t), \quad (25)$$

where the left side of Equation (25) is available from the measurement of $x_p(t)$. Note that the observation relation (25) includes redundant rows. Multiplying the both sides of Equation (25) by $(B'_p B_p)^{-1}B'_p$ from the left, we obtain

$$(B'_p B_p)^{-1}B'_p \dot{x}_p(t) - (B'_p B_p)^{-1}B'_p A_p x_p(t) - u(t) = C_d x_d(t). \quad (26)$$

From Equations (24) and (26), we can construct the stochastic model as

$$\dot{x}_d(t) = A_d x_d(t) + B_d w_d(t),$$

$$(B'_p B_p)^{-1}B'_p \dot{x}_p(t) - (B'_p B_p)^{-1}B'_p A_p x_p(t) - u(t) = C_d x_d(t) + v_d(t), \quad (27)$$

where $w_d(t)$ and $v_d(t)$ are mutually independent zero-mean white noise processes with the covariance matrices $W_d$ and $V_d$, respectively, and $B_d$ is chosen such that $(A_d, B_d)$ is controllable. Using the Kalman filter theory, we can obtain the optimal estimate $\hat{x}_d(t)$ by the optimal disturbance estimator based on the observation of $x_p(t)$ as

$$\dot{\hat{x}}_d(t) = A_d \hat{x}_d(t) + \tilde{K}_d [(B'_p B_p)^{-1}B'_p \dot{x}_p(t) - (B'_p B_p)^{-1}B'_p A_p x_p(t) - u(t) - C_d \hat{x}_d(t)], \quad (28)$$

where $\tilde{K}_d$ is the optimal estimator gain matrix given by

$$\tilde{K}_d = \tilde{P}_d C_d V_d^{-1}, \quad (29)$$

with $\tilde{P}_d$ satisfying the Riccati equation:

$$A_d \tilde{P}_d + \tilde{P}_d A_d^T - \tilde{P}_d C_d V_d^{-1} C_d^T \tilde{P}_d + B_d W_d B_d^T = 0. \quad (30)$$

The structure of the target control system is shown in Figure 2.

**Remark 2** Note that the target controller is used only in the design process but is not used as a real-time controller. Although the optimal disturbance estimator (28) includes time derivative of $x_p(t)$, the differentiation is acceptable in the design process. In software such as Simulink, the time derivative of $x_p(t)$ is easily obtained from the input of the integrator generating $x_p(t)$ in the block diagram representation of the target control system.

For the target controller (23), we have the following result.

**Lemma 2** Consider the controller (23) with the optimal disturbance estimator (28). The controller transfer function matrix $\tilde{C}_{uy}(s)$ from $x_p(t)$ to $u(t)$ can be written as

$$\tilde{C}_{uy}(s) = -\tilde{M}^{-1}(s)\tilde{N}(s), \quad (31)$$

where

$$\tilde{M}(s) \triangleq I - C_d(sI - A_d + \tilde{K}_d C_d)^{-1}\tilde{K}_d, \quad (32)$$

$$\tilde{N}(s) \triangleq F + C_d(sI - A_d + \tilde{K}_d C_d)^{-1}\tilde{K}_d (B'_p B_p)^{-1}B_p (sI - A_p). \quad (33)$$

**Proof** The expression can easily be obtained by straightforward matrix calculation. ■

Using the above lemma, we can obtain the following result for the target sensitivity property.

**Proposition 3** Consider the target control system consisting of the plant (1) and the optimal controller (23) with the optimal disturbance state estimator (28). Let $\tilde{\Sigma}(s)$ denote the sensitivity matrix at the plant input side. Then the sensitivity matrix is factored as

$$\tilde{\Sigma}(s) = S(s)\tilde{M}(s), \quad (34)$$

where $S(s)$ is the sensitivity matrix defined in Equation (19) and $\tilde{M}(s)$, which is given in Equation (32), can be regarded as the sensitivity matrix for the optimal disturbance estimator (28).
The target sensitivity matrix at the plant input side is defined as
\[ \bar{\Sigma}(s) \triangleq [I - \bar{C}_w(s)(sI - A_p)^{-1}B_p]^{-1}, \quad (35) \]
where \( \bar{C}_w(s) \) is defined in Equation (31). It follows from Equations (31) and (35) that
\[ \bar{\Sigma}(s) = [\bar{M}(s) + \bar{N}(s)(sI - A_p)^{-1}B_p]^{-1}\bar{M}(s). \quad (36) \]

It readily follows from Equations (32) and (33) that
\[ \bar{M}(s) + \bar{N}(s)(sI - A_p)^{-1}B_p = I + F(sI - A_p)^{-1}B_p. \quad (37) \]

The factorization (34) follows from Equations (36) and (37). By the matrix inversion lemma, we can write \( \bar{M}(s) \) as
\[ \bar{M}(s) = [I + C_d(sI - A_d)^{-1}\bar{K}_d]^{-1}, \quad (38) \]
which shows that Equation (38) is the sensitivity matrix for the estimation error dynamics of the optimal disturbance estimator (28).

Using the above result, we can show that the target controller provides guaranteed robustness property as in the target for the standard LTR design.

**Proposition 4** For the single-input-single-output (SISO) case \( m = 1 \), the target sensitivity function (34) satisfies:
\[ |\bar{\Sigma}(j\omega)| \leq 1 \quad \text{for all } \omega, \quad (39) \]
which guarantees that the target control system has the large gain margins (infinite gain margin and phase margin more than 60 degrees) for any choice of the optimal feedback gain matrix \( F \) and the optimal estimator gain matrix \( \bar{K}_d \).

**Proof** Note that the optimal disturbance estimator (28) is constructed using the Kalman filter theory. By the well-known Kalman inequality for optimal LQ regulators and Kalman filters, it is guaranteed that
\[ |S(j\omega)| \leq 1 \quad \text{and} \quad |\bar{M}(j\omega)| \leq 1 \quad \text{for all } \omega. \quad (40) \]

The inequality (39) follows from Equations (34) and (40).

### 4.2. Recovery procedure

The target feedback property is recovered in the output feedback controller by the following recovery procedure.

**Proposition 5** Assume that the target controller is determined, that is, the optimal feedback gain matrix \( F \) and the optimal disturbance estimator matrix \( \bar{K}_d \) are fixed. Consider the optimal disturbance cancellation controller using the Kalman filter gain matrix \( K \) determined for the stochastic model (12) and (13) with the covariance matrices \( V = I \) and \( W = \sigma^2I \), where \( \sigma \) is a positive scalar, and
\[ \bar{B} \triangleq \begin{bmatrix} \bar{K}_d \\ B_p \end{bmatrix}. \quad (41) \]

Then, as \( \sigma \) tends to infinity, the sensitivity matrix \( \Sigma(s) \) defined in Equation (18) for the output feedback controller approaches the target sensitivity matrix \( \bar{\Sigma}(s) \) defined in Equation (34).

**Proof** Let \( K(\sigma) \) denote the Kalman filter gain matrix for the stated stochastic model. Using the Popov-Belevitch-Hautus test (e.g. Anderson & Moore, 1990), we can show that the pair \( (A, \bar{B}) \) is stabilizable, \( (C, A) \) is observable and that the invariant zero of the realization \( (A, B, C) \) are in the open left plane under the condition A1–A4. These results guarantee that the Kalman filter gain matrix \( K(\sigma) \) satisfies the asymptotic property
\[ \lim_{\sigma \to \infty} \sigma^{-1}K(\sigma) = \bar{B}, \quad (42) \]
which justifies that the Kalman filter gain matrix \( K(\sigma) \) for sufficiently large \( \sigma \) can be written as
\[ K(\sigma) = \begin{bmatrix} K_d(\sigma) \\ K_p(\sigma) \end{bmatrix} = \sigma \begin{bmatrix} \bar{K}_d \\ B_p \end{bmatrix}. \quad (43) \]

Using the above expression, we can obtain the asymptotic expressions of the two transfer function matrices in \( M(s) \) defined in Equation (16) as
\[ [sI - A_p + K_p(\sigma)C_p]^{-1}B_p = (sI - A_p)^{-1}B_p[I + \sigma C(sI - A)^{-1}B_p]^{-1} \rightarrow 0 \quad (\sigma \to \infty), \quad (44) \]
\[ K_d(s)[sI - A_p + K_p(\sigma)C_p]^{-1}B_p = \sigma \bar{K}_d(sI - A_p)^{-1}B[I + \sigma C(sI - A)^{-1}B]^{-1} \rightarrow \bar{K}_d(\sigma \to \infty). \quad (45) \]

From Equations (16), (18), (32) and (34), we have \( M(s) \rightarrow \bar{M}(s) \) as \( \sigma \to \infty \), which implies that \( \Sigma(s) \rightarrow \bar{\Sigma}(s) \) as \( \sigma \to \infty. \)

**Remark 3** Note that the modified recovery procedure differs from the original (e.g. Saberi et al., 1993) in the point that it requires the optimal estimator gain matrix \( \bar{K}_d \) used in the target. Despite the difference, the proposed procedure retains the philosophy of the original LTR design.

### 5. Illustrative example

In this section, a numerical example is presented to illustrate the design procedure proposed in the preceding
section. Consider a SISO plant described by Equation (1) with

\[
A_P = \begin{bmatrix} -2.0 & -1.0 & -0.5 \\ 2.0 & 0 & 0 \\ 0 & 1.0 & 0 \end{bmatrix},
\]

\[
B_P = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}, \quad C_P = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},
\]

which is a minimal realization of the transfer function given by

\[
G(s) = \frac{1}{(s+1)(s^2+s+1)}. \quad (47)
\]

It is assumed that the sinusoidal disturbance with the angular frequency \( \omega = \pi \) (rad/sec) enters the plant input side. The disturbance model is given by Equation (2) with

\[
A_d = \begin{bmatrix} 0 & -\pi^2 \\ 1 & 0 \end{bmatrix}, \quad C_d = \begin{bmatrix} 0 & 1 \end{bmatrix}. \quad (48)
\]

The set-up of the control system for this example is shown in Figure 3 where the reference input is inserted with the pre-compensator:

\[
T = \left[C_p(-A_p + B_pF)^{-1}B_p\right]^{-1}. \quad (49)
\]

In the subsequent discussion on the time response, the unit step signal is applied as a reference at \( t = 0 \) and the test disturbance signal

\[
d(t) = \begin{cases} 0 & (0 \leq t < 10), \\ 10 \sin \pi t & (10 \leq t \leq 24), \\ 0 & (t > 24) \end{cases}, \quad (50)
\]

is injected to check the disturbance rejection capability.

5.1. Target controller design

First, we determine the target controller by appropriate choice of the optimal feedback gain matrix \( F \) and the optimal gain matrix \( \bar{K}_d \) for the disturbance estimator. The reference input with pre-compensator (49) is included as in the output feedback case shown in Figure 3.

Note that the target controller has the two-degree-of-freedom structure as in the output feedback case. It is reasonable to determine \( F \) first setting \( \bar{K}_d = 0 \) and then determine \( \bar{K}_d \). Let us assume that the optimal feedback gain matrix \( F \) corresponding to the performance index (6) with \( R = 1 \) and \( Q = 100 \) provides satisfactory response to the step reference signal. Then \( \bar{K}_d \) is determined considering the disturbance cancellation capability. To determine \( \bar{K}_d \) by Equations (29) and (30), we set:

\[
B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad W_d = q, \quad V_d = 1,
\]

where \( q \) is a scalar tuning parameter.

For \( q = 10, 10^3 \) and \( 10^4 \), the magnitude characteristics of the target sensitivity function (34) are shown in Figure 4. Note that a sharp dip exists at the angular frequency of the disturbance, which reflects that the target controller has the internal model of the disturbance. The sensitivity is reduced at the low-frequency region as \( q \) increases.

The time response of the target to the unit step reference and the test disturbance (50) is shown in Figure 5. It is seen that the disturbance cancellation capability is improved as \( q \) is increased.

The stability margins of the target are summarized in Table 1, which confirms that the target has the infinite gain margin and the phase margin more than 60 degrees irrespective of \( q \). For this illustrative example, we choose \( \bar{K}_d \) corresponding to \( q = 10^3 \) as the disturbance estimator gain matrix for the target.

5.1. Target controller design

Figure 3. Control system set-up for the example.
5.2. Target recovery

The second step determines the Kalman filter gain matrix $K$ by the formal procedure using the $\sigma$ introduced in Proposition 5. Consider the output feedback disturbance cancellation controller (11). The feedback gain matrix $F$ is the same as in the target and the estimate of the extended state $x(t)$ is obtained as the Kalman filter gain matrix for the stochastic model (12) where the matrix $\bar{K}_d$ determined in the first step is included in $\bar{B}$ defined in Equation (41). For $\sigma = 10^4$, $10^8$ and $10^{12}$, the magnitude characteristics of the sensitivity matrix (18) and the time response for the unit step reference and the test disturbance (50) are shown in Figures 6 and 7, respectively. It is confirmed numerically that the sensitivity characteristics and the time response approach those of the target as $\sigma$ is increased. The stability margins of the output feedback controller are summarized in Table 2. It is seen that the stability margins are improved as $\sigma$ is increased.

### Table 1. Target stability margins.

| $Q$  | Gain margin (dB) | Phase margin (degree) |
|------|------------------|-----------------------|
| 10   | $\infty$        | 63.9                  |
| $10^3$ | $\infty$     | 63.8                  |
| $10^4$ | $\infty$     | 84.0                  |

### Table 2. Stability margins of the output feedback case.

| $\rho$  | Gain margin (dB) | Phase margin (degree) |
|---------|------------------|-----------------------|
| $10^3$ | 6.83             | 37.7                  |
| $10^6$ | 10.1             | 51.8                  |
| $10^{12}$ | 15.9          | 79.3                  |

As shown above, the proposed design procedure provides flexible design with a small number of tuning parameters taking account of stability margins.

6. Conclusions

Our earlier work (Guo et al., 1996) has been extended to a more general class of disturbances with optimality consideration. The optimal disturbance cancellation controller based on the measurement of the plant state is chosen as the target with the guaranteed stability margins. It has been shown that the target feedback property can be recovered in the output feedback controller by the extended version of the modified LTR procedure proposed in our earlier work. A numerical example has been presented to show...
that the proposed optimal design provides efficient tuning of the disturbance cancellation capability with attention to the stability margins. The result of this paper can be extended to non-minimum phase plants using the partial LTR technique used in Ishihara, Guo, and Takeda (2005).

For the disturbance cancellation at the plant output side, a completely different approach is required. Some fundamental issues have been discussed in Ishihara and Guo (2011, 2013a) for step disturbances. Recently, it has been extended to sinusoidal and polynomial disturbances in the conference papers (Ishihara & Guo, 2013b, 2013c).

It is an interesting future problem to develop a new type of predictive controllers (Maciejowski, 2002; Mosca, 1995) based on the disturbance cancellation technique with additional practical constraints.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

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