Group Theoretical Formulation of Quantum Partial Search Algorithm

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Abstract

Searching and sorting used as a subroutine in many important algorithms. Quantum algorithm can find a target item in a database faster than any classical algorithm. One can trade accuracy for speed and find a part of the database (a block) containing the target item even faster, this is partial search. An example is the following: exact address of the target item is given by a sequence of many bits, but we need to know only some of them. More generally partial search considers the following problem: a database is separated into several blocks. We want to find a block with the target item, not the target item itself. In this paper we reformulate quantum partial search algorithm in terms of group theory.

1 Introduction

Database search is used as a subroutine in many important algorithms [15, 14, 15]. Grover discovered a quantum algorithm that searches faster than any classical algorithm [1]. It consists of repetition of the Grover iteration \( \hat{G}_1 \). We shall call it global iteration, see (5). The number of repetitions [queries to the oracle] is:

\[
j_{\text{full}} = \frac{\pi}{4} \sqrt{N} \quad \text{as} \quad N \to \infty
\]  

(1)

for a database with \( N \) entries. There is no faster quantum algorithm [7, 5, 4]. Nevertheless if we need less information then the search can be performed faster. For example if the exact address of the target item in a database is given by a sequence of many bits \( b_1 b_2 \ldots b_n \), but we want to know only three of them, we can do it faster then (1). This is an example of partial search. A partial search considers the following problem: a database is separated into \( K \) blocks of a size \( b = N/K \). We want to find a block with the target item, not the target item itself. The block with the target item is called the target block, all other blocks are non-target blocks. Grover and Radhakrishnan suggested a
quantum algorithm for a partial search in [8]. Partial search naturally arise in list matching [3]. Classical partial search takes \( \sim (N - b) \) queries, but quantum algorithm takes only \( \sim (\sqrt{N} - \text{coeff} \sqrt{b}) \) queries. Here coeff is a positive number, which has a limit, when number of items in each block is very large \( b \to \infty \). Grover-Radhakrishnan algorithm uses several global iterations \( \hat{G}_1 \) and then several local iteration \( \hat{G}_2 \), see (6). Grover-Radhakrishnan algorithm was simplified and clarified in [10] and optimized in [11] [the number of queries to the oracle was minimized, positive coeff was increased]. Other partial search algorithms were studied in [12]. The algorithm for blocks of finite size was formulated in [9]. Partial search algorithm for a database with multiple target items was formulated in [13].

2 The GRK Algorithm for Partial Search

To introduce the partial search algorithm it is useful to first remind the full Grover search. We shall consider a database with one target item. The Grover algorithm finds the target state \( |t\rangle \) among an unordered set of \( N \) states, which is called the database. In the classical case, the items in the database are labeled by a sequence of bits. In the quantum case these sequences label orthonormal basis in a linear space \( |x\rangle \). The search is performed by repeating global iteration which is defined in terms of two operators. The first changes the sign of the target state \( |t\rangle \) only:

\[
\hat{I}_t = \hat{I} - 2|t\rangle\langle t|, \quad \langle t|t\rangle = 1,
\]

where \( \hat{I} \) is the identity operator and \( |t\rangle\langle t| \) projects on the target item. The second operator,

\[
\hat{I}_{s_1} = \hat{I} - 2|s_1\rangle\langle s_1|,
\]

changes the sign of the uniform superposition of all basis states \( |s_1\rangle \)

\[
|s_1\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle, \quad \langle s_1|s_1\rangle = 1.
\]

The global iteration is defined as a unitary operator

\[
\hat{G}_1 = -\hat{I}_{s_1} \hat{I}_t.
\]

To describe partial search we need to introduce a local search. Local search is a search inside of each block done simultaneously in all blocks. In one block local search acts as

\[
\hat{G}_2 = -\hat{I}_{s_2} \hat{I}_t,
\]

where \( \hat{I}_t \) is the same operator appearing in the global search (2), but \( \hat{I}_{s_2} \) acts in each block as

\[
\hat{I}_{s_2}|_{\text{block}} = \hat{I}|_{\text{block}} - 2|s_2\rangle\langle s_2|, \quad |s_2\rangle = \frac{1}{\sqrt{b}} \sum_{\text{one block}} |x\rangle.
\]
Local search in the whole database is the direct sum of $\hat{I}_{s_2}$ with respect to all blocks.

Let us mention eigenvectors and eigenvalues of global iteration, see [6]:

\[
\hat{G}_1|\psi^\pm_1\rangle = \lambda^\pm_1|\psi^\pm_1\rangle, \quad \lambda^\pm_1 = \exp[\pm 2i\theta_1],
\]

\[
|\psi^\pm_1\rangle = \frac{1}{\sqrt{2}}|t\rangle \pm \frac{i}{\sqrt{2}} \left( \sum_{x=0}^{N-1} \frac{|x\rangle}{\sqrt{(N-1)}} \right),
\]

where $\theta_1$ is

\[
\sin^2 \theta_1 = \frac{1}{N}
\]

In the case of local search $\hat{G}_2$ they are given by

\[
\hat{G}_2|\psi^\pm_2\rangle = \lambda^\pm_2|\psi^\pm_2\rangle, \quad \lambda^\pm_2 = \exp[\pm 2i\theta_2], \quad |\psi^\pm_2\rangle = \frac{1}{\sqrt{2}}|t\rangle \pm \frac{i}{\sqrt{2}}|\text{ntt}\rangle
\]

Amplitudes of all non-target items in the target blocks are the same. So we can only follow the amplitude of $|\text{ntt}\rangle$ which is the normalized sum of all non-target items in the target block:

\[
|\text{ntt}\rangle = \frac{1}{\sqrt{b-1}} \sum_{x \neq t \text{ target block}} |x\rangle, \quad \langle \text{ntt}|\text{ntt}\rangle = 1, \quad \langle \text{ntt}|t\rangle = 0.
\]

Here the angle $\theta_2$ is given by

\[
\sin^2 \theta_2 = \frac{K}{N} = \frac{1}{b}.
\]

2.1 Three Dimensional Space

Amplitudes of all items in non-target blocks are the same. So we can only follow the amplitude of $|u\rangle$. The vector $|u\rangle$ is given by

\[
|u\rangle = \frac{1}{\sqrt{b(K-1)}} \sum_{x \text{ all items in all non-target blocks}} |x\rangle, \quad \langle u|u\rangle = 1.
\]

Together with the target item $|t\rangle$ and $|\text{ntt}\rangle$ [see (11)] the unite vector $|u\rangle$ form a orthonormal basis in three dimensional linear space: $\langle t|u\rangle = \langle \text{ntt}|u\rangle = 0$.

For example the eigenvectors of global iterations $\hat{G}_1$ in Eq. (8) can be written as

\[
|\psi^\pm_1\rangle = \frac{1}{\sqrt{2}}|t\rangle \pm \frac{i}{\sqrt{2}} \left( \sum_{x=0}^{N-1} \frac{|x\rangle}{\sqrt{(N-1)}} \right)
\]
$$= \frac{1}{\sqrt{2}} |t\rangle \pm \frac{i}{\sqrt{2}} \left( \sqrt{\frac{b-1}{N-1}} |ntt\rangle + \sqrt{\frac{b(K-1)}{N-1}} |u\rangle \right).$$

Below we shall describe the partial search algorithm as a three dimensional matrix [29].

Meanwhile let us remind the GRK algorithm for partial search. The partial search of [11] creates a vector

$$|d\rangle = \hat{G}_1 \hat{G}_2^2 \hat{G}_1^2 |s_1\rangle.$$  (15)

Note that this algorithm uses a sequence of global-local operators. The final operation $\hat{G}_1$ is necessary since $\hat{G}_2$ acts trivially on $|u\rangle$, i.e. $\langle u | \hat{G}_2 = \langle u |$. The final state $|d\rangle$ should have zero amplitudes of each item in non-target blocks, in other words it should satisfy

$$\langle u | d \rangle = 0,$$

which means that a measurement will reveal the position of the target block.

We consider large blocks $b = N/K \to \infty$. The number of blocks $K$ is a finite number ($K = 2, 3, \ldots$), it is an important parameter in the algorithm. It is useful to introduce an angle $\gamma$ defined by

$$\sin(\gamma) = \frac{1}{\sqrt{K}}, \quad 0 < \gamma \leq \frac{\pi}{4}.$$

In the limit $b \to \infty$ it was shown in [8] that the total number of iterations grows as

$$j_1 = \frac{\pi}{4} \sqrt{N} - \eta_K \sqrt{\frac{N}{K}}, \quad j_2 = \alpha_K \sqrt{\frac{N}{K}},$$

where the coefficients $\eta_K$ and $\alpha_K$ have a well defined limit. The total number of queries to the oracle is $j_1 + j_2 + 1 \to \pi/4 \sqrt{N} - (\eta_K - \alpha_K) \sqrt{N/K}$.

One of the authors found the optimal values of $(\eta_K, \alpha_K)$ (see [11]) such that total number of queries to the oracle is minimal:

$$\tan \left( \frac{2\eta_K}{\sqrt{K}} \right) = \frac{\sqrt{3K - 4}}{K - 2}, \quad \cos (2\alpha_K) = \frac{K - 2}{2(K - 1)}.$$

### 3 $O(3)$ Group Formulation of GRK

The GRK is the fastest among partial search algorithms, which use the sequence $\hat{G}_1 \hat{G}_2^j \hat{G}_1^j$. Partial search algorithms using some other sequences were considered in [12], but no acceleration was found. In order to prove that the GRK algorithm is the fastest among all partial search algorithms, which use other sequences of $\hat{G}_1$ and $\hat{G}_2$ one has to prove the following:

**Conjecture:** Start from an arbitrary vector $|\phi\rangle$ in the three dimensional space with the basis $(|t\rangle, |ntt\rangle, |u\rangle)$. If the sequence of local-global-local-global operations can find the target block

$$\langle u | \hat{G}_i^{j_1} \hat{G}_2^{j_2} \hat{G}_1^{j_2} \hat{G}_2^{j_1} | \phi \rangle = 0,$$
then there exists a global-local-global sequence such that

\[ \langle u|\hat{G}_1^{j_3} \hat{G}_2^{j_2} \hat{G}_1^{j_1} |\phi\rangle = 0, \]

it finds the target block faster, i.e. \( \hat{j}_3 + \hat{j}_2 + \hat{j}_1 \leq \hat{j}_1 + \hat{j}_2 + \hat{j}_3 + \hat{j}_4 \). ✠

If the conjecture is true then GRK is the fastest among partial search algorithms using arbitrary sequence of local and global searches.

**Proof:** Assume a general algorithm constructed with global \( \hat{G}_1 \) and local \( \hat{G}_2 \) iterations with \( n \) letters given by

\[ \langle u|\hat{G}_1^{j_1} \hat{G}_2^{j_2-1} \hat{G}_1^{j_1-2} \hat{G}_2^{j_2-3} \hat{G}_1^{j_1-3} \hat{G}_2^{j_2-5} \cdots \hat{G}_1^{j_1} \hat{G}_2^{j_2} \hat{G}_1^{j_1} |s_1 \rangle = 0, \] (16)

where \( j_i \geq 0 \) for \( i \leq n-1 \) and \( j_n > 0 \). In particular we can have \( j_1 = 0 \), so the sequence can also start with \( \hat{G}_2 \). The final operation has to be \( \hat{G}_1^{j_1} \) since \( \hat{G}_2 \) acts trivially on \( |u\rangle \), i.e. \( \langle u|\hat{G}_2 = \langle u \rangle \). Equation (16) can be written as

\[ \langle u|\hat{G}_1^{j_1} \hat{G}_2^{j_2-1} \hat{G}_1^{j_1-2} \hat{G}_2^{j_2-3} \hat{G}_1^{j_1-3} \hat{G}_2^{j_2-5} \cdots \hat{G}_1^{j_1} \hat{G}_2^{j_2} \hat{G}_1^{j_1} |\phi\rangle = 0, \]

where \( |\phi\rangle = \hat{G}_1^{j_1-4} \cdots \hat{G}_2^{j_2} \hat{G}_1^{j_1} |s_1 \rangle \) Using the conjecture, this can be reduced to

\[ \langle u|\hat{G}_1^{j_1-1} \hat{G}_2^{j_2-2} \hat{G}_1^{j_1-3} |\phi\rangle = 0, \]

where \( \hat{j}_1-1 + \hat{j}_2-2 + \hat{j}_1-3 \leq \hat{j}_1 + \hat{j}_1-1 + \hat{j}_2-2 + \hat{j}_1-4 \). We now have another sequence

\[ \langle u|\hat{G}_1^{j_1-1} \hat{G}_2^{j_2-2} \hat{G}_1^{j_1-3} \hat{G}_2^{j_2-5} |\phi\prime\rangle = 0, \]

where \( |\phi\prime\rangle = \hat{G}_1^{j_1-4} \hat{G}_2^{j_2-5} |\phi\prime\rangle \). We can put two powers of \( \hat{G}_1 \) together

\[ \langle u|\hat{G}_1^{j_1-1} \hat{G}_2^{j_2-2} \hat{G}_1^{j_1-3} \hat{G}_2^{j_2-5} |\phi\prime\rangle = 0, \]

and use the conjecture again, starting with \( |\phi\prime\rangle \). After several iterations, (16) will be reduced to

\[ \langle u|\hat{G}_1^{j_1} \hat{G}_2^{j_2} \hat{G}_1^{j_1} |s_1 \rangle = 0. \] (17)

It was proved in [12] that the GRK algorithm is the fastest among all possible algorithms in the form (17). This means that in order to prove that GRK algorithm is the fastest among all partial search algorithms consisting of arbitrary sequence of of \( \hat{G}_1 \) and \( \hat{G}_2 \) it is enough to prove the conjecture. ▼

- Let us reduce partial search to \( O(3) \) group.

In subsection 2.1 we explained that the partial search algorithm acts naturally in three dimensional space with the orthonormal basis: target item \( |t\rangle \), non-target items in the target block \( |ntt\rangle \) and all items in non-target blocks \( |u\rangle \). Search operations are rotations in three dimensional space spanned by these three vectors. All the vectors involved in present quantum search problem can be written in this basis as

\[ |V \rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \] (18)
In the above equation \((a, b, c)\) are the real coefficients in the base defined by \((|t\rangle, |ntt\rangle, |u\rangle)\), meaning
\[
|V\rangle = a|t\rangle + b|ntt\rangle + c|u\rangle
\] (19)
For example, the initial state \[4\] can be written as:
\[
|s_1\rangle = \begin{pmatrix} \sin \gamma \sin \theta_2 \\ \sin \gamma \cos \theta_2 \\ \cos \gamma \end{pmatrix}
\] (20)
and the local uniform state \[7\] is
\[
|s_2\rangle = \begin{pmatrix} \sin \theta_2 \\ \cos \theta_2 \\ 0 \end{pmatrix}
\] (21)
From this basic relations and the definitions of \(\hat{G}_2\) we can calculate its \(j_2\) power of local search:
\[
\hat{G}_2^{j_2} = \begin{pmatrix} \cos(2j_2\theta_2) & \sin(2j_2\theta_2) & 0 \\ -\sin(2j_2\theta_2) & \cos(2j_2\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] (22)
The ordering of eigenvectors is \(|t\rangle, |ntt\rangle\) and \(|u\rangle\). The matrix has three eigenvectors:
\[
\hat{G}_2^{j_2} |v^\pm_{2}\rangle = \exp(\pm\theta_2 j_2) |v^\pm_{2}\rangle, \quad \hat{G}_2^{j_2} |v^0_{2}\rangle = |v^0_{2}\rangle
\] (23)

Where the eigenvectors can be written as
\[
|v^\pm_{2}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}, \quad |v^0_{2}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\] (24)

In the same way, \(j_1\) repetitions of the global iterations \[5\] is \({}^1\)
\[
\hat{G}_1^{j_1} =
\]
\[
\begin{pmatrix} c(2j_1\theta_1), & s(2j_1\theta_1)s(\gamma), & s(2j_1\theta_1)c(\gamma) \\ -s(2j_1\theta_1)s(\gamma), & (\gamma c^2(\gamma) + c(2j_1\theta_1)s^2(\gamma)), & s(\gamma)c(\gamma) (1 + c(2j_1\theta_1)) \\ -s(2j_1\theta_1)c(\gamma), & s(\gamma)c(\gamma) (1 + c(2j_1\theta_1)), & (-1)^{j_1}c^2(\gamma) + c(2j_1\theta_1)c^2(\gamma) \end{pmatrix}
\]
This is a simplified asymptotic expression valid in the limit of large blocks \(b \rightarrow \infty\). The matrix has three eigenvectors
\[
\hat{G}_1^{j_1} |v^\pm_{1}\rangle = \exp(\pm\theta_1 j_1) |v^\pm_{1}\rangle, \quad \hat{G}_1^{j_1} |v^0_{1}\rangle = (-1)^{j_1} |v^0_{1}\rangle
\] (26)
\({}^1\)here we use \(c(\cdot) = \cos(\cdot)\) and \(s(\cdot) = \sin(\cdot)\)
where the eigenvectors are

\[ |v^\pm_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm i \sin \gamma \\ \pm i \cos \gamma \end{pmatrix}, \quad |v^0_1\rangle = \begin{pmatrix} 0 \\ \cos \gamma \\ -\sin \gamma \end{pmatrix}. \tag{27} \]

\[ |v_{01}\rangle = \begin{pmatrix} 0 \\ 0 \cos \gamma \\ -\sin \gamma \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 + \frac{3K - 4}{K(K - 1)} \end{pmatrix}, \tag{28} \]

It is also possible to represent the whole GRK algorithm as a matrix

\[ \hat{G}_{GRK} = \hat{G}_1 \hat{G}_2 \hat{G}_1^* = \begin{pmatrix} 0 & 1/2 + \sqrt{\frac{3K - 4}{3K - 4}} \\ 0 & 0 \end{pmatrix}, \tag{29} \]

It has the form,

\[ \begin{pmatrix} 0 & a & b \\ 0 & b & -a \\ -1 & 0 & 0 \end{pmatrix}, \tag{30} \]

where \(a\) and \(b\) satisfy \(a^2 + b^2 = 1\) which shows that the GRK matrix is an element of the group \(O(3)\) \(\heartsuit\).

### 3.1 Reformulation in terms of \(SO(3)\) Group

We see in (28) that the matrix corresponding to the operator \(\hat{G}_1\) has a reflection if \(j_1\) is odd. This fact makes the analysis of the algorithm difficult, since algorithms with even and odd powers of \(\hat{G}_1\) have a different behavior \([12]\). To overcome this problem, we can reformulate the algorithm in such a way that it will use only even powers of \(\hat{G}_1\). To do that we have to introduce auxiliary search defined by

\[ \hat{G}_a^j = \hat{G}_1 \hat{G}_2 \hat{G}_1^*. \tag{31} \]

Now it is necessary to show how the introduction of this new operator is done inside the algorithm. We consider a general sequence \([10]\) of \(\hat{G}_1\) and \(\hat{G}_2\)

\[ \langle u|\hat{G}_1^{j_n} \hat{G}_2^{j_{n-1}} \hat{G}_1^{j_{n-2}} \hat{G}_2^{j_{n-3}} \cdots \hat{G}_1^{j_3} \hat{G}_2^{j_2} \hat{G}_1^*|s_1\rangle = 0, \tag{32} \]

We can always make the total number of \(\hat{G}_1\) factors \((j_n + j_{n-2} + \cdots + j_3 + j_1)\) to be an even number. We can add one extra factor \(\hat{G}_1\) in the beginning using the fact that \(\hat{G}_1|s_1\rangle = |s_1\rangle + O(1/\sqrt{b})\):

\[ \langle u|\hat{G}_1^{j_n} \hat{G}_2^{j_{n-1}} \hat{G}_1^{j_{n-2}} \hat{G}_2^{j_{n-3}} \cdots \hat{G}_1^{j_3} \hat{G}_2^{j_2} \hat{G}_1^* \hat{G}_1|s_1\rangle = 0, \tag{33} \]

We can now consider only sequences with even total number of \(\hat{G}_1\) factors. Individual powers of \(\hat{G}_1\) can still be odd. Each odd power of \(\hat{G}_1\) we represent as even multiplied by one \(\hat{G}_1\) factor. Since the total number of \(\hat{G}_1\) factors is even,
single $\hat{G}_1$ can only occur in pairs. This means that in the string of operators we can choose single $\hat{G}_1$'s such that we have
\[
\langle u | \cdots \hat{G}_1 \hat{G}_2 \cdots \hat{G}_2 \hat{G}_1 \cdots | s_1 \rangle = 0,
\]
where between the two factors of $\hat{G}_2$ and $\hat{G}_2$ we have only even powers of $\hat{G}_1$'s. Now one uses the definition of $\hat{G}_a$ given in Eq. 31 and rewrites Eq. 34 as
\[
\langle u | \cdots \hat{G}_a \hat{G}_2 \cdots \hat{G}_2 \hat{G}_a \cdots | s_1 \rangle = 0.
\]
Here we used $\hat{G}_2^2 = 1 + O(1/\sqrt{b})$.

A general algorithm will now be a sequence of the three operators global search $\hat{G}_1$ raised in even powers, local search $\hat{G}_2$ and auxiliary search $\hat{G}_a$:
\[
\langle u | \hat{G}_1^{j_1} \hat{G}_2^{j_2-1} \hat{G}_a^{j_3-2} \cdots \hat{G}_2^2 \hat{G}_a^2 \hat{G}_1 \cdots | s_1 \rangle = 0.
\]

Using the matrix description given above we can calculate explicitly a power of auxiliary search $\hat{G}_a$:
\[
\hat{G}_a^j =
\begin{pmatrix}
c(2j_a \theta_2) & -c(2\gamma)s(2j_a \theta_2) & s(2\gamma)s(2j_a \theta_2) \\
c(2\gamma)s(2j_a \theta_2) & s^2(2\gamma) + c^2(2\gamma)c(2j_a \theta_2) & s(2\gamma)c(2\gamma)(1 - c(2j_a \theta_2)) \\
-s(2\gamma)s(2j_a \theta_2) & s(2\gamma)c(2\gamma)[1 - c(2j_a \theta_2)] & c^2(2\gamma) + s^2(2\gamma)c(2j_a \theta_2)
\end{pmatrix}
\]
(36)

Its spectrum is
\[
\hat{G}_a^j |u_2^\pm\rangle = \exp(\pm 2i\theta_2 j_2) |u_2^\pm\rangle, \quad \hat{G}_2^j |u_2^0\rangle = |u_2^0\rangle
\]
(37)

Where the eigenvectors can be written as
\[
|u_2^\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp i \cos(2\gamma) \\ \pm i \sin(2\gamma) \end{pmatrix}, \quad |u_2^0\rangle = \begin{pmatrix} 0 \\ \sin(2\gamma) \\ \cos(2\gamma) \end{pmatrix}.
\]
(38)

To summaries in this section: we eliminated reflection by introducing a new element $\hat{G}_a$. In new formulation of the GRK algorithm (see (35) only even powers of $\hat{G}_1$ appears. These means that in formulae $\hat{G}_3$ and $\hat{G}_4$ $j_1$ can be replaced by even number, so $(-1)^{j_1} = 1$.

## 4 Lie Algebra Relations

The introduction of a third operator $\hat{G}_a$ simplifies the analysis of general algorithms, since now we do not have to take into account reflections. Now the algorithm consists of a sequence of even powers of $\hat{G}_1$ and integer powers of $\hat{G}_2$ and $\hat{G}_a$. Each of these operators [searches ] is an element of $SO(3)$. This is a simplification, since now we are dealing with connected component to the identity element. But $\hat{G}_a$ is dependent on the other two operators $\hat{G}_1$. 


Any element of the SU(2) group can be written in terms of rotations around two linearly independent unite vectors (see the Appendix). This is also true for SO(3) group. To find general relations among our three operators it is useful to first see what is the Lie Algebra relation.

Using the matrix form of \( \hat{G}_1 \) given by (25) we can compute its expression for small powers:

\[
\hat{G}_1^2 = I + 4\theta_2 T_G_1 = I + 4\theta_2 \left( \begin{array}{ccc}
0 & \sin^2(\gamma) & \sin(\gamma)\cos(\gamma) \\
-\sin^2(\gamma) & 0 & 0 \\
-\sin\gamma\cos\gamma & 0 & 0 \\
\end{array} \right) \tag{39}
\]

The same calculation can be done with \( \hat{G}_2 \)

\[
\hat{G}_2^j = I + 2\theta_2 T_G_2 = I + 2\theta_2 \left( \begin{array}{ccc}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right), \tag{40}
\]

and \( \hat{G}_a \)

\[
\hat{G}_a^j = I + 2\theta_2 T_G_a = I + 2\theta_2 \left( \begin{array}{ccc}
0 & -\cos(2\gamma) & \sin(2\gamma) \\
\cos(2\gamma) & 0 & 0 \\
-\sin(2\gamma) & 0 & 0 \\
\end{array} \right), \tag{41}
\]

note that we used the relation \( \theta_1 = \sin(\gamma) \theta_2 \) to simplify the above equations. The relation follows from the definition of the angles in the limit of \( b \to \infty \).

\( T_{G_1}, T_{G_2}, \) and \( T_{G_a} \) are as elements of Lie Algebra generators corresponding to our searches. Using their matrix expressions we see that they satisfy the linear relation

\[
T_{G_a} + T_{G_2} - 2T_{G_1} = 0 \tag{42}
\]

which explicitly shows that the Lie algebra elements describing global, local and auxiliary searches are linearly dependent. In the next section we shall rise this relation into the group, see (45).

5 \( SU(2) \) Formulation

Let us formulate partial search in terms of \( su(2) \) algebra and later \( SU(2) \) group. The transition to \( SU(2) \) group makes the manipulation of the group elements algebraically easier.

From Eqs. (39), (40) and (41) we see that the Lie algebra generators are linear combinations of standard generators \( T_z \) and \( T_y \) of the \( so(3) \) Lie algebra, see [14]. Any three dimensional vector \( \mathbf{v} \) can be mapped to two dimensional matrices

\[
V \to v = a\sigma_x + b\sigma_y + c\sigma_z,
\]

using Pauli matrices

\[
\sigma_x = \left( \begin{array}{cc}
0 & 1 \\
1 & 0 \\
\end{array} \right), \quad \sigma_y = \left( \begin{array}{cc}
0 & -i \\
i & 0 \\
\end{array} \right), \quad \sigma_z = \left( \begin{array}{cc}
1 & 0 \\
0 & -1 \\
\end{array} \right). \tag{43}
\]
We can map so(3) algebra to su(2) algebra by replacing the generators $T_x$, $T_y$ and $T_z$

\[
T_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad T_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

by $\frac{i}{2}\sigma_x$, $\frac{i}{2}\sigma_y$ and $\frac{i}{2}\sigma_z$ correspondingly, so

\[
R = e^{a^T} \to u = e^{\frac{a}{2}\sigma},
\]

and rotation act on vectors,

\[
RV \to uVu^{-1}
\]

The actual SU(2) to O(3) correspondence is given by

\[
u(\vec{r} \cdot \vec{\sigma})u^{-1} = (R\vec{r}) \cdot \vec{\sigma}
\]

Here $\vec{r}$ is a vector with real components. Elements of partial search can be mapped to SU(2) group in the following way:

\[
\hat{G}_1^{j_1} \to u_1^{j_1} = \begin{pmatrix} \cos(j_1\theta_1) + i \sin(\gamma) \sin(j_1\theta_1) & -\cos(\gamma) \sin(j_1\theta_1) \\ \cos(\gamma) \sin(j_1\theta_1) & \cos(j_1\theta_1) - i \sin(\gamma) \sin(j_1\theta_1) \end{pmatrix}
\]

\[
\hat{G}_2^{j_2} \to u_2 = \begin{pmatrix} e^{ij_2\theta_2} & 0 \\ 0 & e^{-ij_2\theta_2} \end{pmatrix}
\]

\[
\hat{G}_a^{j_a} \to u_a^{j_a} = \begin{pmatrix} \cos(j_a\theta_2) - i \cos(2\gamma) \sin(j_a\theta_2) & -\sin(2\gamma) \sin(j_a\theta_2) \\ \sin(2\gamma) \sin(j_a\theta_2) & \cos(j_a\theta_2) + i \cos(2\gamma) \sin(j_a\theta_2) \end{pmatrix}
\]

So we mapped partial search in SU(2) group. The elements $u_1$, $u_2$ and $u_a$ of SU(2) group are dependent. Using the Appendix we found algebraic relation between group elements describing global, local and auxiliary searches:

\[
u_1^{j_1} u_a^{-j_2} u_1^{j_1} = u_2^{j_2}, \quad \text{(45)}
\]

here $\sin(\gamma) \tan(j_2\theta_2) = \tan(j_1\theta_1)$. Corresponding Lie algebraic relation is (42).

6 Conclusion

In this paper we formulated the partial search algorithms in terms of group theory. We think that it will be useful for proof of optimality of GRK algorithm in wide class of partial search algorithms.

7 Acknowledgment

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8 Appendix

Arbitrary element of $SU(2)$ group can be written using only rotation around two different axis, see page 176 of the book [13]. We shall represent rotation axis by a unit vector. If we define $R_{\bar{n}}(\lambda)$ as a rotation around the unit vector $\bar{n}$ by an angle $\lambda$:

$$R_{\bar{n}}(\lambda) = \exp\{-i\frac{\lambda}{2}(\bar{n} \cdot \vec{\sigma})\}$$

A rotation around any axis by any angle (arbitrary element of $SU(2)$ group $R$) can be represented as sequential rotations around two fixed axis $\bar{n}$ and $\bar{m}$:

$$R = R_{\bar{n}}(\lambda)R_{\bar{m}}(\theta)R_{\bar{n}}(\gamma),$$

Here $\bar{n}$ and $\bar{m}$ are two linearly independent unit vectors and $(\lambda,\theta,\gamma)$ are three real numbers (angles).

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