On $\mathcal{H}^1$ and entropic convergence for contractive PDMP.

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Abstract

Explicit rate of convergence in variance (or more general entropies) is obtained for a class of Piecewise Deterministic Markov Processes such as the TCP process, relying on functional inequalities. A method to establish Poincaré (and more generally Beckner) inequalities with respect to a diffusion-type energy for the invariant law of such hybrid processes is developed.

1 Introduction

This work is devoted to the study of convergence to equilibrium for a class of Piecewise Deterministic Markov Process (PDMP). These hybrid processes, satisfying a deterministic differential equation between random jumps, have received much attention recently: we refer to [5] and the references therein for an overview of the topic. Ergodicity and, then, speed of convergence to the steady state are particularly studied. As far as this last point is concerned, coupling methods have recently proved efficient in order to get explicit rate of convergence in Wasserstein distances for PDMP (see [17, 7, 10, 26, 13] for instance, among many others). On the other hand, another classical approach to quantify ergodicity, based on functional inequalities, is hardly used, since the usual methods do not directly apply. Our aim is to adapt them (see also [33] in this direction).

Let $\Omega$ be an open set of $\mathbb{R}^d$. The dynamics is defined thanks to a vector field $b : \Omega \to \mathbb{R}^d$, a jump rate $\lambda : \Omega \to \mathbb{R}_+$, and a transition kernel $Q$ which will be seen either as a function from $\Omega$ to $\mathcal{P}(\Omega)$ the set of probability measures on $\Omega$, or as an operator on the space $\mathcal{C}(\Omega)$ of all continuous functions on $\Omega$. For $x \in \Omega$ and $t > 0$ let $\varphi_x(t)$ be the flow associated to $b$, namely the solution of
\[
\frac{\partial}{\partial t} \varphi_x(t) = b(\varphi_x(t)), \quad \varphi_x(0) = x.
\]
Starting at point $x$, the process $(X_t)_{t \geq 0}$ deterministically follows this flow up to its first jump time $T_x$ with law
\[
\mathbb{P}(T_x < s) = \int_0^s \lambda(\varphi_x(u)) e^{-\int_0^u \lambda(\varphi_x(w))dw}du = 1 - e^{-\int_0^s \lambda(\varphi_x(u))dw}.
\]
At time $T_x$, the process jumps according to the law $Q(\varphi_x(T_x))$, and starts anew from its new position. The infinitesimal generator of the process is
\[
L f(x) = b(x) \cdot \nabla f(x) + \lambda(x) (Qf(x) - f(x)),
\]
defined for smooth $f \in \mathcal{C}(\Omega)$. We note
\[
P_t f(x) = \mathbb{E}(f(X_t)|X_0 = x)
\]
the associated semi-group. The following assumptions hold throughout this work:
• the flow is well-defined and it stabilizes $\Omega$: if $x \in \Omega$ then $\varphi_x(t) \in \Omega$ for all $t > 0$.

• the process is non-explosive: there can’t be infinitely many jumps in a finite time interval, so that the process (and therefore the semi-group) is defined for all time.

• the functions $\lambda$ and $b$ are smooth; we write $J_b(x) = [\partial_i b_k(x)]_{1 \leq i, k \leq d}$ the Jacobian matrix of $b = (b_k)_{1 \leq k \leq d}$.

• The process admits a unique invariant law $\mu$, and $P_t$ is ergodic in the sense it converges weakly to $\mu$ as $t$ goes to infinity. Moreover The generator $L$ is well-defined on the set $A$ of all compactly supported smooth functions on $\Omega$, and $A$ is dense in $L^2(\mu)$.

The test functions will always belong to $A$ in order to keep the study at a formal level, all the forthcoming elementary definitions and calculations being licit in this framework. These strong assumptions allow us to focus only on the quantification of ergodicity. Note that the uniqueness of the invariant measure and the ergodicity of the process may often be proved by checking it is irreducible and admits a Lyapunov function (cf. [24]). Throughout this work the test functions will always belong to the set $A$, in order to keep the study at a formal level, all the forthcoming elementary definitions and calculations being licit in this framework.

We recall here some classical arguments (see [4] for a general introduction to functional inequalities and for the detailed proofs of the assertions in this paragraph). For $f \in A$, we write $\Gamma(f) = \frac{1}{2}L(f)^2 - fL f$ the carré du champ operator of $L$, $\Gamma(f,g)$ the corresponding symmetric bilinear operator obtained by polarization, and

$$\Gamma_2(f) = \frac{1}{2}L(\Gamma f) - \Gamma(f, Lf).$$

Writing $\psi(s) = P_s \Gamma(P_{t-s} f)$, from $\partial_t P_t f = LP_t f = P_t L f$ one gets

$$\psi'(s) = 2P_s \Gamma_2(P_{t-s} f).$$

Hence, if the Bakry-Emery (or $\Gamma_2$) criterion $\Gamma_2 > \rho \Gamma$ holds for some $\rho > 0$, the Gronwall Lemma yields $\psi(0) \leq e^{-2\rho t} \psi(t)$, namely

$$\Gamma(P_t f) \leq e^{-2\rho t} P_t \Gamma f.$$

For instance for the Ornstein-Uhlenbeck process with generator

$$L_{OU} f(x) = \Delta f(x) - \rho x. \nabla f(x),$$

this reads

$$|\nabla P_t f|^2 \leq e^{-2\rho t} P_t |\nabla f|^2,$$

where $|.|$ is the euclidian norm of $\mathbb{R}^d$. In fact, the sub-commutation (2) is equivalent to the Bakry-Emery criterion. Nevertheless the latter does not usually hold in our settings. That said, a simple adaptation of the $\Gamma_2$ argument will give, at least in the constant jump rate case, a gradient estimate similar to (3). In the following we denote by $A^*$ the usual transpose of a matrix $A$ and thus by $u^*v$ the scalar product of two vectors.

**Theorem 1.** Assume $\lambda$ is constant and $|\nabla Q f(x)|^2 \leq M(x)|\nabla f|^2(x)$ with $M$ such that

$$\forall (x, u) \in \Omega \times \mathbb{R}^d, \quad 2u^*J_b(x)u + \lambda(M(x) - 1)|u|^2 \leq -\eta|u|^2$$

for some $\eta \in \mathbb{R}$. Then for all $t > 0$, $f \in A$ and $x \in \Omega$,

$$|\nabla P_t f|^2(x) \leq e^{-\eta t} P_t |\nabla f|^2(x).$$

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Inequality (4) is a balance condition on the drift and the jumps, reminiscent of the condition on the curvature in [19]. More precisely, suppose |∇Qf(x)|² ≤ M(x)Q|∇f|²(x) for some function M on Ω. If M < 1, Q is a contraction of the Wasserstein distance (this will be detailed in Section 2); it means two particles that simultaneously jump can be coupled so that they get closer. More generally M measures how two such particles can be coupled in order for them not to get too far away from each other. On the other hand, Jₜ measures how two trajectories of the deterministic flow tends to get closer or to drift apart. Indeed, Theorem 1 implies energy, while W₂ implies the flow contracts the space in the neighborhood of all points of Ω.

We see that the condition u∗Jₜ(x)u < 0 for all (x, u) ∈ Ω × Rᵈ implies the flow contracts the space in the neighborhood of all points of Ω.

Note that by integrating Inequality (5) with respect to μ and writing

\[ W_t = \int |\nabla P_t f|^2 d\mu, \]

Theorem 1 implies \( W_t \leq e^{-\eta t} W_0 \) for all \( t > 0, f \in A \), which is equivalent to \( \partial_t W_t \leq -\eta W_t \) for all \( t > 0, f \in A \), or to \( (\partial_t W_t)_{t=0} \leq -\eta W_0 \) for all \( f \in A \).

In the non-constant jump rate case, under a condition similar to (4), we will prove there exists constants \( \beta > 0 \) and \( \eta \in R \) such that

\[ \partial_t W_t \leq -\eta W_t + 2\beta \mathcal{E}_t \]  

where \( \mathcal{E}_t \) is defined as

\[ \mathcal{E}_t = \int \Gamma (P_t f) d\mu. \]

Both \( W_t \) and \( \mathcal{E}_t \) are usually called energy; we may say \( W_t \) is the classical (or diffusion-like) energy, while \( \mathcal{E}_t \) is the markovian one. They coincide in the case of the Ornstein-Uhlenbeck process. The markovian energy usually appears in particular when one is concerned with the variance of \( P_t f \) with respect to \( \mu \),

\[ V_t = \int (P_t f)^2 d\mu - \left( \int P_t f d\mu \right)^2. \]

We say \( \mu \) satisfies a Poincaré (or spectral gap) inequality with respect to \( \Gamma \) if there exist a constant \( c > 0 \) such that \( V_0 \leq c \mathcal{E}_0 \) for all \( f \in A \). Since \( \partial_t V_t = -2\mathcal{E}_t \), such an inequality is equivalent to \( V_t \leq e^{-2t} V_0 \), namely to an exponential decay in \( L^2(\mu) \). The same goes for entropy and Gross log Sobolev inequality, or general \( \Phi \)-entropies (see [16] and Section 3 for some definitions), at least for diffusion processes.

For reversible processes (i.e. when \( L \) is symmetric in \( L^2(\mu) \)) there is a strong link between, on the one hand, Wasserstein distances and coupling and, on the other hand, variance (or entropy) and functional inequalities (see [6] [15] [29]); nevertheless PDMP are not reversible. Furthermore their invariant measures usually do not satisfy a Poincaré inequality for \( \Gamma \), which is non-local, not easy to handle, satisfying no chain rule (nevertheless, see [14] for a case in which such an inequality does indeed hold).

However, they may satisfy a diffusion-like Poincaré inequality of the form

\[ \forall f \in A \quad \int f^2 d\mu - \left( \int f d\mu \right)^2 \leq c \int |\nabla f|^2 d\mu, \]  

(7)
Moreover it is possible to get explicit values for $c \in A$. For addition to the previous difficulties (no Poincaré inequality for $\mu$-almost all $x \in \Omega$ and for all $u \in \mathbb{R}^d$,
$$u^* \left(2 J_b(x) + \frac{\nabla \lambda(x)(\nabla \lambda(x))^*}{\beta \lambda(x)} \right) u + \lambda(x) (M(x)-1) |u|^2 \leq -\eta |u|^2$$
for some constants $\eta, \beta > 0$. Then
$$W_t + \beta V_t \leq (W_0 + \beta V_0) e^{-\eta \frac{t}{1+\beta c}}.$$ 

Suppose such an inequality holds. Then, from inequality (4), if $\eta > 0$,
$$\partial_t (W_t + \beta V_t) \leq -\eta W_t \leq -\frac{\eta}{1+\beta c} (W_t + \beta V_t).$$
This yields:

**Theorem 2.** Assume the Poincaré inequality (7) holds, and $|\nabla Q f(x)|^2 \leq M(x)Q|\nabla f|^2(x)$ with $M$ such that for $\mu$-almost all $x \in \Omega$ and for all $u \in \mathbb{R}^d$,
$$u^* \left(2 J_b(x) + \frac{\nabla \lambda(x)(\nabla \lambda(x))^*}{\beta \lambda(x)} \right) u + \lambda(x) (M(x)-1) |u|^2 \leq -\eta |u|^2$$
for some constants $\eta, \beta > 0$. Then
$$W_t + \beta V_t \leq (W_0 + \beta V_0) e^{-\frac{\eta t}{1+\beta c}}.$$
2 Exponential decay

We keep the notations and assumptions of the introduction. In particular we study the semi-group \( (P_t)_{t \geq 0} \) with generator \( L \) defined by (1).

When \( A \) is a linear operator on \( \mathcal{A} \) and \( \phi \) is a bilinear symmetric one, for \( f, g \in \mathcal{A} \) we define
\[
\Gamma_{A,\phi}(f, g) = \frac{1}{2} (A\phi(f, g) - \phi(f, Ag) - \phi(Af, g)).
\]
With respect to \( f \), \( \Gamma_{A,\phi}(f, f) \) is quadratic, and linear with respect to \( A \) and \( \phi \). We will always note \( f \mapsto \phi(f) \) the quadratic form associated to a bilinear form \( f, g \mapsto \phi(f, g) \) and similarly we will always note \( f, g \mapsto q(f, g) \) the symmetric bilinear form associated by polarization to a quadratic form \( f \mapsto q(f) \) on \( \mathcal{A} \). Let
\[
\psi(s) = P_s \phi (P_{1-s} f), \quad s \in [0, t]
\]
which interpolates between \( \phi (P_t f) \) and \( P_t (\phi f) \). Then
\[
\psi'(s) = 2P_s \Gamma_{L,\phi} (P_{1-s} f).
\]
To prove Theorems 1 and 2 we should consider the applications to work with a weighted gradient \( \phi_a(f) = a|\nabla f|^2 \) with \( a > 0 \) a scalar field on \( \Omega \).

**Lemma 4.** 1. For all \( f \in \mathcal{A} \)
\[
\Gamma_{b^*\nabla,\phi_a}(f) = \frac{b^*\nabla a}{2a} \phi_a(f) - a(\nabla f)^* J_b \nabla f.
\]

2. Suppose there exists a function \( M \) on \( \Omega \) such that, for all \( f \in \mathcal{A} \), \( \phi_a(Q_f) \leq MQ(\phi_a(f)) \), and let \( I \) be the identity operator on \( \mathcal{A} \). Then for all \( f \in \mathcal{A} \)
\[
\Gamma_{\lambda (Q-I),\phi_a}(f) \geq -a(\nabla f)^* (\nabla \lambda) (Q_f - f) + \frac{\lambda}{2} (1 - M) \phi_a(f).
\]

**Proof.** First we note that
\[
\nabla (b^* \nabla f) = J_b \nabla f + H_f b
\]
with \( H_f(x) = [\partial_i \partial_k f(x)]_{1 \leq i, k \leq d} \) the Hessian of \( f \), and
\[
b^* \nabla (a|\nabla f|^2) = (b^* \nabla a) |\nabla f|^2 + 2ab^* H_f \nabla f
\]
Thus
\[
\Gamma_{b^*\nabla,\phi_a}(f) = \frac{1}{2} b^* \nabla (a|\nabla f|^2) - a(\nabla f)^* \nabla (b^* \nabla f)
\]
\[
= \frac{1}{2} (b^* \nabla a) |\nabla f|^2 - a(\nabla f)^* J_b \nabla f.
\]
As far as the second point is concerned,
\[
\Gamma_{\lambda (Q-I),\phi_a}(f) = \frac{1}{2} \lambda (Q(\phi_a(f)) - \phi_a(f)) - a(\nabla f)^* (\nabla \lambda) (Q_f - f) - \lambda a (\nabla f)^* (\nabla Q_f - \nabla f)
\]
\[
\geq \frac{\lambda}{2} \left( Q(\phi_a(f)) + \phi_a(f) - 2\sqrt{\phi_a(f)}\phi_a(Q_f) \right) - a(\nabla f)^* (\nabla \lambda) (Q_f - f).
\]
We conclude by
\[
2\sqrt{\phi_a(f)}\phi_a(Q_f) \leq 2\sqrt{M\phi_a(f)\phi_a(f)} \leq M\phi_a(f) + Q(\phi_a(f)).
\]
\[\square\]
We can now state the following:

**Theorem 5.** Assume $\lambda$ is constant and there exist a function $M$ on $\Omega$ and a constant $\eta \in \mathbb{R}$ such that, for all $f \in \mathcal{A}$, $\phi_a (Qf) \leq M Q (\phi_a (f))$ and

$$\forall (x, u) \in \Omega \times \mathbb{R}^d, \quad 2u^* J_b (x) u + \left( \lambda (M(x) - 1) - \frac{b^* \nabla a(x)}{a(x)} + \eta \right) \|u\|^2 \leq 0.$$  

Then

$$\phi_a (P_t f) \leq e^{-\eta t} P_t (\phi_a (f)).$$

In particular with $a = 1$ we retrieve Theorem 1.

**Proof.** From Lemma 4, since in the constant rate case $\nabla \lambda = 0$,

$$\Gamma_{L, \phi_a} (f) \geq -a (\nabla f)^* J_b \nabla f + a \left( \frac{b^* \nabla a}{2a} + \frac{\lambda}{2} (1 - M) \right) |\nabla f|^2 \geq \frac{\eta}{2} \phi_a (f).$$

Hence if $\psi (s) = P_s \phi_a (P_{t-s} f)$,

$$\psi' (s) = 2 P_s \Gamma_{L, \phi_a} (P_{t-s} f) \geq \eta \psi (s)$$

and $\psi (t) \geq e^{\eta t} \psi (0)$, which concludes. 

Remark that we did note use the ergodicity of the process here, and that $\eta$ can be negative.

This commutation between the semigroup and the gradient leads to a contraction in Wasserstein distance. More precisely, define on $\Omega$ the distance associated to the weighted gradient $D = \sqrt{a \nabla}$ by

$$d(x, y) = \inf \left\{ \int_0^1 \frac{|\gamma' (s)|}{\sqrt{a (\gamma (s))}} ds, \gamma : [0, 1] \to \Omega, \text{ smooth, } \gamma (0) = x, \gamma (t) = y \right\}$$

and the associated Wasserstein distance between two probability laws $\nu_1, \nu_2$ having a finite $p^{th}$ moment (i.e. for which there exists a $x_0 \in \Omega$ with $\nu_i [d^p (., x_0)] < \infty$) by

$$W_{d, p} (\nu_1, \nu_2) = \inf_{X \sim \nu_1, Y \sim \nu_2} \left( \mathbb{E} [d^p (X, Y)] \right)^{\frac{1}{p}}.$$  

A function $f$ will be called $\kappa$-Lipschitz with respect to $D$ if $\forall x, y \in \Omega$,

$$f (x) - f (y) \leq \kappa d(x, y).$$

This is equivalent for a smooth function to $\|Df\|_\infty \leq \kappa$. At fixed $x$, the function $y \mapsto d(x, y)$ is 1-Lipschitz with respect to $D$. Since the metric space $(\Omega, d)$ is locally diffeomorphic to $(\mathbb{R}^d, |\cdot|)$, thanks to Rademacher’s theorem, this means a $\kappa$-Lipschitz function $g$ is differentiable almost everywhere, with $\|Dg\|_\infty \leq \kappa$. We have the Kantorovich-Rubinstein dual representation (see [38])

$$W_{d, 1} (\nu_1, \nu_2) = \sup \{ \nu_1 f - \nu_2 f, \|Df\|_\infty \leq 1 \},$$

where we use the operator notation $\nu f = \int f d\nu$.

Recall that by duality a Markov semi-group acts on the right on probability laws by

$$(\nu_1 P_t) f := \nu_1 (P_t f).$$
If \( P_t \) were absolutely continuous with respect to the Lebesgue measure for \( t > 0 \) - which is not the case for a PDMP since for all time \( t \) there is a non-zero probability that the process hasn’t jumped yet - the gradient estimate of Theorem 5 would yield, from [30 Theorem 2.2], a contraction of the \( W_{d,2} \) distance:

\[
W_{d,2}(\nu_1 P_t, \nu_2 P_t) \leq e^{-\frac{t}{2}} W_{d,2}(\nu_1, \nu_2).
\]

Instead of trying to adapt Kuwada’s result, since our work is more concerned about variance than Wasserstein distance, we will only state the weaker result:

**Corollary 6.** In the settings of Theorem 5 for all laws \( \nu_1, \nu_2 \) with finite first moment, if \( P_t \nu_1 \) and \( P_t \nu_2 \) still have finite first moment,

\[
W_{d,1}(\nu_1 P_t, \nu_2 P_t) \leq e^{-\frac{t}{2}} W_{d,1}(\nu_1, \nu_2).
\]

**Proof.** Theorem 5 yields the weaker gradient estimate

\[
\|DP_t f\|_\infty \leq e^{-\frac{t}{2}} P_t \|Df\|_\infty = e^{-\frac{t}{2}} \|Df\|_\infty.
\]

This implies the \( W_{d,1} \) decay, thanks to the Kantorovich-Rubinstein dual representation.\( \square \)

Note that the invariant measure does not intervene neither in Theorem 5 nor in Corollary 6 so that its existence and uniqueness are not necessary. Besides, on a complete space, a contraction of the Wasserstein distance would imply ergodicity, from [18 Theorem 5.23].

We won’t push the analysis further concerning the Wasserstein distance, but refer to the study in [7] of the TCP process where an exponential decay is first obtained for a distance equivalent to \( d(x, y) = \sqrt{|x - y|} \) and then is transposed to \( d(x, y) = |x - y|^p \) via moments estimates and Hölder inequality. For further considerations on gradient-semigroup commutation, one shall consult [12 3 30].

We now turn to the non-constant jump rate case. Let \( a \) be a non-negative scalar field on \( \Omega \). Throughout all the text we will say a probability measure \( \nu \) satisfies a weighted Poincaré inequality with constant \( c \) and weight \( a \), if for all \( f \in \mathcal{A} \)

\[
\nu f^2 - (\nu f)^2 \leq c \nu (\nabla f)^2.
\]  

Let \( V_t = \mu (P_t f)^2 - (\mu f)^2 \) and \( W_t = \mu \phi_a(P_t f) \).

**Theorem 7.** Assume that \( \mu \) satisfies the weighted Poincaré inequality (9) with constant \( c \) and weight \( a \), that \( \mu \)-almost everywhere \( \lambda > 0 \) and that there exist a function \( M \) and constants \( \eta, \beta > 0 \) such that for \( \mu \)-almost all \( x \in \Omega \), for all \( f \in \mathcal{A} \) and for all \( u \in \mathbb{R}^d \), \( \phi_a(Qf) \leq MQ(\phi_a(f)) \) and

\[
\begin{align*}
\|a \nabla (\lambda(x))^* \| &
\leq \lambda(x) (M(x) - 1) - \frac{b^* \nabla a(x)}{a(x)} + \eta, \\
\end{align*}
\]

Then

\[
W_t + \beta V_t \leq e^{-\frac{nt}{\mu + \beta}} (W_0 + \beta V_0),
\]

and

\[
W_t \leq (1 + \beta c) e^{-\frac{nt}{\mu + \beta}} W_0.
\]

**Proof.** Since \( \mu \) is the invariant measure of the process, \( \mu Lg = 0 \) for all \( g \in \mathcal{A} \). In particular if \( \phi \) is a quadratic form on \( \mathcal{A} \), \( \mu (L \phi(f)) = 0 \) and

\[
\begin{align*}
\partial_t (\mu (\phi(P_t f))) &= 2 \mu (\phi(P_t f), LP_t f) \\
&= -2 \mu \Gamma_{L, \phi} (P_t f).
\end{align*}
\]
In particular
\[ \partial_t W_t = -2\mu \Gamma_{L,\phi_a}(P_t f). \]

From Lemma 4
\[ \Gamma_{\lambda(Q-I),\phi_a}(f) \geq \eta \frac{1}{2} \phi_a(f) - \beta (Qf - f)^2. \]
Again from Lemma 4 and from Inequality (10),
\[ \Gamma_{L,\phi_2}(f) \geq \eta \frac{1}{2} \phi_a(f) - \beta \frac{1}{2} (Qf - f)^2. \]

On the other hand, if \( \phi_2(f) = f^2 \) then \( \Gamma_{L,\phi_2} \) is the usual carré du champ operator. From the Leibniz rule, \( \Gamma_{b \cdot V,\phi_2} f = 0 \), so that
\[ \partial_t V_t = -2\mu \Gamma_{(Q-I),\phi_2}(P_t f) \]
\[ = -\mu \lambda \left( (P_t f)^2 + (P_t f)^2 - 2(P_t f)(QP_t f) \right) \]
\[ \leq -\mu \lambda (P_t f - P_t f)^2 \]
the last inequality being a consequence of the Cauchy-Schwarz inequality for \( Q \). At the end of the day, we get
\[ \partial_t (W_t + \beta V_t) \leq -\eta W_t \]
and, thanks to the weighted Poincaré inequality (9),
\[ \partial_t (W_t + \beta V_t) \leq -\frac{\eta}{1 + \beta c} (W_t + \beta V_t), \]
which yields the first assertion. Then
\[ W_t \leq W_t + \beta V_t \leq e^{-\frac{\eta t}{1 + \beta c}} (W_0 + \beta V_0) \leq (1 + \beta c) e^{-\frac{\eta t}{1 + \beta c}} W_0. \]

Note that \( \eta \) could depend on \( x \), so that the weight that intervenes in the Poincaré inequality may be different from \( a \). For instance for the TCP with linear rate on \( \mathbb{R}_+ \) (Example 4.4), one could consider \( a(x) = x \) and \( \eta(x) = -\kappa - \alpha x \) for some \( \kappa, \alpha > 0 \). Then it would be sufficient to prove an inequality with weight \( \tilde{a}(x) = 1 + x \), which is weaker than both the classical inequality with constant weight and the inequality with weight \( a \).

### 3 Functional inequalities for PDMP

This section is devoted to the obtention of the Poincaré inequality (9) and of slightly more general functional inequalities for \( \mu \) the invariant measure of the process \( (X_t)_{t \geq 0} \) with generator (1).
3.1 Confining operators

The variance is a way among others to quantify the distance to equilibrium. In this section we suppose that for all \( f \in \mathcal{A} \) the so-called \( p \)-entropies

\[
\text{Ent}_p f = \frac{\mu f^2 - \left( \mu f^\frac{2}{p} \right)^p}{p-1} \quad \text{for } p \in (1,2],
\]

\[
\text{Ent}_1 f = \mu \left( f^2 \log f^2 \right) - \left( \mu f^2 \right) \log \left( \mu f^2 \right)
\]

are well-defined. We say that \( \mu \) satisfies a Beckner’s inequality \( B(p,c) \) if

\[
\forall f \in \mathcal{A}, \quad \text{Ent}_p f \leq c \mu |\nabla f|^2.
\] (11)

For \( p = 2 \) this is the Poincaré inequality, for \( p = 1 \) this is the Gross log Sobolev one. Since \( \text{Ent}_p f \) is non increasing with \( p \in (1,2] \) (see [32]; note that we took the definitions of [11]), \( B(p,c) \) implies \( B(q,c) \) whenever \( q \geq p \). On the other hand by Jensen inequality \( \text{Ent}_p f \) is non decreasing with \( p \in [1,2] \). In particular all Beckner’s for \( p \in (1,2] \) are equivalent up to some factor. For the global study of these inequalities and of more general \( \Phi \)-entropies, we refer to [16] and [11].

For \( \alpha \in [0,1] \) we say \( \mu \) satisfies a generalized Poincaré inequality \( I(\alpha,c) \) if

\[
\forall f \in \mathcal{A}, \forall p \in (1,2], \quad (1-p)^{1-\alpha} \text{Ent}_p f \leq c \mu |\nabla f|^2.
\] (12)

For \( \alpha = 0 \) this is still the Poincaré inequality, for \( \alpha = 1 \) this is the log Sobolev one, and for \( \alpha \in (0,1) \) this is an interpolation between these two cases which implies the following concentration property: there exists a constant \( L > 0 \) such that for any borellian set \( A \) with \( \mu(A) \geq \frac{1}{2} \), if \( A_t \) is the set of points at distance at most \( t \) from \( A \), then \( \mu(A_t) \geq 1 - e^{Lt^\frac{2}{p-2}} \) (see [32]). To prove \( I(\alpha,c) \) is equivalent to prove \( B(p,c(1-p)^{\alpha-1}) \) for all \( p \in (1,2) \).

In this section, for the sake of simplicity, we won’t consider weighted inequalities such as the weighted Poincaré inequality [9]. Everything would work the same, and, at least in dimension one, a weighted inequality can be seen as a non-weighted one through a change of variable (see an application in Section 4.4).

Remark that if \( \mu \) satisfies \( B(p,c) \) for \( p \in [1,2] \), then it satisfies a Poincaré inequality. In this case, providing the inequality (10) of Theorem 7 holds, \( W_t \) decays exponentially fast, and

\[
\text{Ent}_p P_t f \leq c W_t \leq c(1+\beta c)e^{-\frac{m}{1+\lambda c}} W_0.
\]

Let \( \psi : \Omega \to \Omega \) be a smooth function with Jacobian matrix \( J_\psi \), and let \( |J_\psi| \) be the euclidian operator norm of \( J_\psi \), namely

\[
|J_\psi| = \sup \left\{ |J_\psi u|, \ u \in \mathbb{R}^d, \ |u| = 1 \right\}.
\]

We say \( \psi \) is \( \gamma \)-Lipschitz (where \( \gamma \in \mathbb{R}_+ \)) if for all \( x \in \Omega \), \( |J_\psi(x)| \leq \gamma \). It is clear that in this case when the law of a random variable \( Z \) satisfies \( B(p,c) \) then the law of \( \psi(Z) \) satisfies \( B(p,\gamma^2 c) \). In order to get Beckner’s inequalities for the invariant law of a PDMP we will prove a generalization of this fact, based on an initial idea of Malrieu and Talay [36].

Let \( H \) be a Markov kernel on \( \Omega \).

**Definition 8.** Let \( c, \gamma > 0, p \in [1,2] \). We say that \( H \) is \((c,\gamma,p)\)-confining if both the following conditions are satisfied:

- sub-commutation: \( \forall f \in \mathcal{A}, \forall x \in \Omega, \)

\[
|\nabla \left( H f^\frac{2}{p} \right)^\frac{p}{2} (x) | \leq \gamma H |\nabla f|^2(x).
\] (13)
• Local Beckner’s inequality: \( \forall f \in A, \forall x \in \Omega, \)
\[
\frac{Hf^2(x) - \left( Hf^\gamma \right)^p (x)}{p-1} \leq cH|\nabla f|^2(x). \tag{14}
\]
If \( \gamma < 1 \) we say \( H \) is \((c, \gamma, p)\)-contractive. When there is no ambiguity for \( p, \) \( H \) will simply be called confining (or contractive) if there exist \( c, \gamma > 0 \) satisfying both conditions.

Examples:

• Let \( \psi \) be a \( \gamma \)-Lipschitz function and \( Hf(x) = f(\psi(x)) \). The sub-commutation (13) is clear, and the local inequality (14) holds with \( c = 0 \), since \( H(x) \) is a Dirac mass.

• The sub-commutation is always satisfied with \( \gamma = 0 \) if \( H(x) = \nu \) is a constant kernel, namely is a probability on \( \Omega \), so that \( \nu \) is confining if it satisfies a Beckner’s inequality.

• if \( N \) is a standard Gaussian vector on \( \mathbb{R}^d \) and \((B_t)_{t \geq 0} \) a Brownian motion on \( \mathbb{R}^d \) then
\[
K_t f(x) = \mathbb{E} (f(x + B_t)) = \mathbb{E} \left( f(x + \sqrt{t} N) \right)
\]
is \((t, 1)\)-confined for the usual gradient and \( p = 1 \) (see [4]). If the Brownian motion is replaced by an elliptic diffusion, a sub-commutation is given by its Bakry-Emery curvature.

• Remark this definition could be extended to a Markov kernel \( H : \Omega_1 \to \mathcal{P}(\Omega_2) \) with \( \Omega_1 \subset \mathbb{R}^d \) and \( \Omega_2 \subset \mathbb{R}^n \). For instance if \( \varphi \) is the flow associated to a vector fields \( b \) on \( \Omega_1 \) then \( Hf(t) = f(\varphi_x(t)) \) is a Markov kernel from \( \mathbb{R}_+ \) to \( \mathcal{P}(\Omega_1) \), and \( \partial_t Hf = H(b^* \nabla f) \).

Here is maybe our most important, although very simple result:

**Lemma 9.** For \( i = 1, 2 \), let \( H_i \) be a \((c_i, \gamma_i, p)\)-confining Markov kernel on \( \Omega \).

1. Then \( H_1 H_2 \) is a \((c_2 + \gamma_2 c_1, \gamma_1 \gamma_2, p)\)-confining Markov kernel.
2. If \( \nu \in \mathcal{P}(\Omega) \) satisfies \( \mathcal{B}(p, c) \) then \( \nu H_2 \) satisfies \( \mathcal{B}(p, c_2 + \gamma_2 c) \).
3. If \( H \) is \((c, \gamma, p)\)-contractive and if the Markov chain generated by \( H \) is ergodic, meaning that \( H^n \) converges weakly to some \( \nu \in \mathcal{P}(\Omega) \) when \( n \) goes to infinity, then the invariant law \( \nu \) satisfies \( \mathcal{B}(p, c(1 - \gamma)^{-1}) \).

**Proof.** Let \( p \in (1, 2] \) (the case \( p = 1 \) is similar and already treated in [17]). First,
\[
\left| \nabla \left( H_1 H_2 f^\gamma \right) \right|^2 \leq \gamma_1 H_1 \left( \left| \nabla \left( H_2 f^\gamma \right) \right|^2 \right) \leq \gamma_1 \gamma_2 H_1 H_2 |\nabla f|^2
\]
and
\[
\frac{H_1 H_2 f^2 - \left( H_1 H_2 f^\gamma \right)^p}{p-1} = \frac{1}{p-1} \left( H_1 \left[ H_2 f^2 - \left( H_2 f^\gamma \right)^p \right] + H_1 \left( H_2 f^\gamma \right)^p - \left( H_1 H_2 f^\gamma \right)^p \right)
\]
\[
\leq c_2 H_1 H_2 |\nabla f|^2 + \frac{1}{p-1} \left( H_1 g^2 - \left( H_1 g^\gamma \right)^p \right) \quad \text{with } g = \left( H_2 f^\gamma \right)^{\frac{1}{2}}
\]
\[
\leq c_2 H_1 H_2 |\nabla f|^2 + c_1 H_1 |\nabla g|^2
\]
\[
\leq (c_2 + \gamma_2 c_1) H_1 H_2 |\nabla f|^2.
\]
The second point is obtained from the first one by considering \( H_1 = \nu \). Concerning the third assertion, by induction from the first one we get for all \( n \in \mathbb{N} \)
\[
\frac{H^n f^2 - \left( H^n f^\gamma \right)^p}{p-1} \leq c \left( \sum_{k=0}^{n} \gamma^k \right) H^n |\nabla f|^2.
\]

The weak convergence of \( H^n \) to \( \nu \) concludes. \( \square \)

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Example: Let \((E_k)_{k \geq 0}\) be an i.i.d. sequence of standard exponential variables, and \((X_k)_{k \geq 0}\) be the Markov chain on \(\mathbb{R}_+\) defined by \(X_{k+1} = X_k + E_k\). Its transition operator is

\[
P f(x) = E \left( f \left( \frac{x + E_0}{2} \right) \right).
\]

Clearly \((P f)'(x) = \frac{1}{2} P (f'')(x)\), so that \(|(P f)'|^2 \leq \frac{1}{2} P |f'|^2\). On the other hand \(P(x)\), the law of \(\frac{x + E_k}{2}\), is the image by a \(\frac{1}{2}\)-Lipschitz transformation of the exponential law \(\mathcal{E}(1)\), which satisfies a Poincaré inequality \(\mathcal{B}(2, 4)\) (cf. Theorem [4] Theorem 6.2.2] for instance). Thus \(P\) is \((2, \frac{1}{2}, 2)\)-contractive. On the other hand it is clear the chain is irreducible, it admits \(C = [0, 3]\) as a small set and \(V(x) = x + 1\) as a Lyapunov function (since \(PV(x) = \frac{1}{4} V(x) + 1_{x < 3}\) so that it is ergodic (see [24] for definitions and proof). According to Lemma [9] the invariant measure satisfies a Poincaré inequality \(\mathcal{B}(2, \frac{4}{3})\).

### 3.2 The embedded chain

Recall \(X = (X_t)_{t \geq 0}\) is a process on \(\Omega\) with generator given by [1]. Let \((S_k)_{k \geq 0}\) be the jump times of \(X\) and let \(Z_k = X_{S_k}\). The Markov chain \((Z_k)_{k \geq 0}\) is called the embedded chain associated to \(X\).

For \(s \in [S_k, S_{k+1})\), \(X_s = \varphi_{Z_k}(s - S_k)\) where we recall \(\varphi_x(t)\) is the flow associated to the vector field \(b\). Since

\[
d \frac{d}{dt} (f(\varphi_x(t))) = (b^\ast \nabla f)(\varphi_x(t)),
\]

we shall say that a function \(f\) is non-decreasing (resp. constant, concave, etc.) along the flow if \(t \mapsto f(\varphi_x(t))\) is non-decreasing (resp. constant, etc.) for all \(x \in \Omega\); in other word if \(b^\ast \nabla f \geq 0\) (resp. = 0, etc.).

Conditionally to the event \(Z_k = x\), the inter-jump time \(T_k = S_{k+1} - S_k\) has a density

\[
p_x(t) = \lambda(\varphi_x(t)) e^{-\int_0^t \lambda(\varphi_x(s)) ds}
\]

on \(\mathbb{R}^+\). We assume the inter-jump times are a.s. finite (which is clear if \(\liminf_{t \to \infty} \lambda(\varphi_x(t)) > 0\) for all \(x\)), and define

\[
K f(x) = \int_0^{\infty} f(\varphi_x(t)) p_x(t) dt = E(f(\varphi_x(T_k))|Z_k = x).
\]

Then \(P = K Q\) is the transition operator for the chain \(Z\).

Transferring properties from \(X\) to \(Z\), or the converse, is far from obvious. In fact it is quite easy to find counter-examples for which one is ergodic and not the other (see examples 34.28 and 34.33 of [21]). In [20] this problem is solved with the definition of another embedded chain by adding observation points at constant rate. That being said, in the following we won’t delve into this issue, and simply assume \(Z\) has a unique invariant law \(\mu_e\) (which can often be proved under conditions of irreducibility, aperiodicity and existence of a Lyapunov function). In this case we can express \(\mu\) from \(\mu_e\).

**Lemma 10 (Theorem 34.31 of [21], p.123).** Assume \(C_e = \mu_e (\frac{1}{\lambda}) = \mu_e \left[ \int_0^{\infty} e^{-\int_0^t \lambda(\varphi_x(s)) ds} dt \right] < \infty\). Then

\[
\mu f = C_e^{-1} \mu_e K \left( \frac{f}{\lambda} \right).
\]

In other words, \(\mu = \nu_e \tilde{K}\) where

\[
\tilde{K} f(x) = \frac{1}{K(\frac{1}{\lambda})(x)} K \left( \frac{f}{\lambda} \right)(x),
\]

\[
\nu_e f = \frac{1}{C_e} \mu_e \left[ f K \left( \frac{1}{\lambda} \right) \right].
\]
In the following we will always assume the condition $C < \infty$ holds, so that $\nu_e$ and $\tilde{K}$ are well defined.

Here is our plan: from Lemma 9 we may establish a Beckner’s inequality for $\mu_e$ by proving the operator $P$ is contractive. By perturbative results on functional inequalities (see [16] or Appendix) this may give an inequality for $\nu_e$. Finally, again from Lemma 9 we may transfer the inequality from $\nu_e$ to $\mu$ by proving the operator $\tilde{K}$ is confining.

The rest of this section will thus enlight some general facts which will later help us (mostly in dimension 1) prove $K$ and $\tilde{K}$ are confining. It is strongly inspired by the work of Chafaï, Malrieu and Paroux [17], in which a log-Sobolev inequality is proved for the invariant measure of the embedded chain of a particular PDMP, the TCP with linear rate (see example 1.4).

Until the end of this section we suppose $\lambda > 0$ almost everywhere. Then

$$t \mapsto \Lambda_x(t) := \int_0^t \lambda(\varphi_x(u)) \, du$$

is invertible for all $x \in \Omega$. Since we assumed the jump times are $a.s.$ finite, necessarily, for all $x \in \Omega$, $\Lambda_x(t) \to \infty$ as $t \to \infty$. Remark that

$$\Lambda_{\varphi_x(s)}(t) = \Lambda_x(t + s) - \Lambda_x(s)$$

which yields both

$$b(x)^* \nabla_x (\Lambda_x(t)) = \frac{d}{ds} \bigg|_{s=0} (\Lambda_{\varphi_x(s)}(t)) = \lambda(\varphi_x(t)) - \lambda(x)$$

and, taking $u = \Lambda_{\varphi_x(s)}(t)$ in $t + s = \Lambda_x^{-1}(\Lambda_{\varphi_x(s)}(t) + \Lambda_x(s))$,

$$\Lambda_{\varphi_x(s)}^{-1}(u) = \Lambda_x^{-1}(u + \Lambda_x(s)) - s.$$  \hfill (17)

If $X_0 = x$ and if $T_x$ is the next time of jump then

$$E = \int_0^{T_x} \lambda(\phi_x(u)) \, du$$

is independent from $X_0$, and has a standard exponential law. In other words $T_x \overset{\text{dist}}{=} \Lambda_x^{-1}(E)$, and $T_{\varphi_x(t)} \overset{\text{dist}}{=} \Lambda_{\varphi_x(t)}^{-1}(\Lambda_x(T_x))$.

**Lemma 11.** If $\lambda$ is non-decreasing along the flow, then for all $x \in \Omega$ and $t > 0$, the law of $T_{\varphi_x(t)}$ is the image of the law of $T_x$ by a $1$-Lipschitz function.

**Proof.** Let $x \in \Omega$ and $t > 0$. For $s > 0$ we note $G(s) = \Lambda_{\varphi_x(t)}^{-1}(\Lambda_x(s))$, so that $T_{\varphi_x(t)} \overset{\text{dist}}{=} G(T_x)$. From $\frac{d}{dt} (\Lambda_x(u)) = \lambda(\varphi_x(u))$, we get

$$G'(s) = \frac{\lambda(\varphi_x(s))}{\lambda(\varphi_x(t)\left(\Lambda_{\varphi_x(t)}^{-1}(\Lambda_x(s))\right))} = \frac{\lambda(\varphi_x(s))}{\lambda(\varphi_x(t + G(s)))}$$

From the relation (17) and the fact that $\Lambda_x$ (hence $\Lambda_x^{-1}$) is non-decreasing,

$$t + G(s) = \Lambda_x^{-1}(\Lambda_x(s) + \Lambda_x(t)) \geq \Lambda_x^{-1}(\Lambda_x(s)) = s.$$  

Thus $\lambda(\varphi_x(s)) \leq \lambda(\varphi_x(t + G(s)))$ and $|G'(s)| \leq 1$. \hfill $\square$
The assumption that the jump rate is non-decreasing along the flow is natural in several applications where the role of the jump mechanism is to counteract a deterministic trend (growth/fragmentation models for cells [13], TCP dynamics [17], etc.). In this context, the more the system is driven away by the flow, the more it is likely to jump. From a mathematical point of view, thanks to Lemma [11] a Beckner’s inequality for the law $K(x)$ may be transferred to $K(\varphi_x(t))$ for all $t > 0.$

In fact this is also true for $\tilde{K}.$ Let $\tilde{T}_x$ be a random variable on $\mathbb{R}_+$ with density $\frac{e^{-\lambda_x(t)}}{\int_0 e^{-\lambda_x(w)}dw},$ so that

$$\tilde{K}f(x) = \mathbb{E}\left[f(\varphi_x(\tilde{T}_x)) \right].$$

**Lemma 12.** If $\lambda$ is non-decreasing along the flow, then for all $x \in \Omega$ and $t > 0,$ the law of $\tilde{T}_{\varphi_x(t)}$ is the law of $\tilde{T}_{\varphi_x(t)}$ conditionnaly to the event $\tilde{T}_x > t$ exactly as the law of $T_{\varphi_x(t)}$ is the law of $T_x - t$ conditionnaly to the event $T_x > t.$ We need to find a jump rate which define $\tilde{T}_x$ as the jump time of a Markov process.

Let $e^{-V(s)} ds$ be a positive probability density on $\mathbb{R}_+$, assume $V$ is convex and let

$$r(t) = \frac{e^{-V(t)}}{\int_0 e^{-V(s)}ds}.$$ 

Note that $r(t) = \frac{d}{dt} \left(-\ln \int_t e^{-V(s)}ds\right),$ so that

$$e^{-\int_0 r(s)ds} = \int_t e^{-V(s)}ds.$$ 

Differentiating this equality yields

$$r(t)e^{-\int_0 r(s)ds} = e^{-V(t)}.$$ 

We want to prove $r$ is non-decreasing. From the convexity of $V,$

$$r(t) = \frac{e^{-V(t)}}{\int_t e^{-V(s)}ds} = \frac{\int_t e^{-V(s)}ds}{\int_t e^{-V(s)}ds} \geq V'(t).$$

As a consequence,

$$r'(t) = r(t) (r(t) - V'(t)) \geq 0.$$ 

In the case of $\tilde{T}_x,$ if $\lambda$ is non-decreasing along the flow then $V(t) = \Lambda_x(t) - \ln \int_0 e^{-\Lambda_x(w)}dw$ is convex, so that the corresponding $r$ is non-decreasing and Lemma [11] applies.

**Lemma 13.** For all $f \in \mathcal{A}, x \in \Omega,$

$$b(x)^\star \nabla (Kf)(x) = \lambda(x)K \left(\frac{b^\star \nabla f}{\lambda}\right)(x).$$ 

In particular if $\lambda$ is non-decreasing along the flow, $|b^\star \nabla (Kf)| \leq K|b^\star \nabla f|.$

**Proof.** From the representation

$$Kf(x) = \mathbb{E}\left[f(\varphi_x(T_x))\right] = \mathbb{E}\left[f(\varphi_x(\Lambda_x^{-1}(E)))\right],$$
we compute (recall $f \in A$ is smooth and compactly supported)

$$b(x)^* \nabla (Kf)(x) = \left. \frac{d}{ds} \right|_{s=0} (Kf(\varphi_x(s)))$$

$$= \mathbb{E} \left( \frac{d}{ds} \bigg|_{s=0} f \left[ \varphi_{\varphi_x}(s) \left( \Lambda_x^{-1}(E) \phi \right) \right] \right)$$

$$= \mathbb{E} \left( \frac{d}{ds} \bigg|_{s=0} f \left[ \varphi_x \left( s + \Lambda_x^{-1}(E) \phi \right) \right] \right)$$

$$= \mathbb{E} \left( \frac{d}{ds} \bigg|_{s=0} f \left[ \varphi_x \left( \Lambda_x^{-1}(E + \Lambda_x(s)) \right) \right] \right)$$

$$= \mathbb{E} \left( \Lambda_x(0) \left( \Lambda_x^{-1}(E) \right)' \left( b^* \nabla f \right) \left[ \varphi_x \left( \Lambda_x^{-1}(E) \right) \right] \right)$$

If $\lambda$ is non-decreasing along the flow, $\lambda(\varphi_x(t)) \geq \lambda(x)$ for all $t \geq 0$. □

**Lemma 14.** Let $h(x) = \int_0^\infty e^{-\Lambda_x(u)} du$. Then for all $f \in A$, $x \in \Omega$,

$$b(x)^* \nabla \left( \tilde{K} f \right)(x) = \frac{\tilde{K} (b^* \nabla f)(x) - \tilde{K} (b^* \nabla f)(x)}{h(x)}.$$ 

In particular if $\lambda$ is non-decreasing along the flow $|b^* \nabla \tilde{K} f|(x) \leq \tilde{K} |b^* \nabla f|(x)$.

**Proof.**

$$\tilde{K} f(x) = \mathbb{E} \left[ f \left( \varphi_x \left( \tilde{T}_x \right) \right) \right] .$$

Note that $F_x(t) = \int_0^t \int_0^{e^{-\Lambda_x(u)}} \frac{d s}{d u} du$ the cumulative function of $\tilde{T}_x$ is invertible. Let $U$ be a uniform random variable on $[0, 1]$. Then

$$\tilde{K} f(x) = \mathbb{E} \left[ f \left( \varphi_x \left( F_x^{-1}(U) \right) \right) \right]$$

$$\Rightarrow b(x)^* \nabla \tilde{K} f(x) = \mathbb{E} \left[ \frac{d}{d s} \bigg|_{s=0} f \left( \varphi_x \left( s + F_x^{-1}(U) \right) \right) \right]$$

$$= \mathbb{E} \left[ \left( 1 + b(x)^* \nabla \varphi_x \left( F_x^{-1}(U) \right) \right) \left( b^* \nabla f \right) \left( \varphi_x \left( F_x^{-1}(U) \right) \right) \right] ,$$

If $u \in [0, 1]$, from $\nabla_x \left( F_x \left( F_x^{-1}(u) \right) \right) = \nabla_x (u) = 0$ we get

$$b(x)^* \nabla_x \left( F_x^{-1}(u) \right) = \frac{-b(x)^* \nabla_x \left( F_x^{-1}(u) \right)}{F_x'(F_x^{-1}(u))} . \quad (18)$$

On the one hand $F_x'(t) = \frac{e^{-\Lambda_x(t)}}{\int_0^t e^{-\Lambda_x(u)} du}$, On the other hand from Equality (16) we compute

$$b(x)^* \nabla_x (F_x(t))(t) = \int_0^t (\Lambda(x) - \Lambda(\varphi_x(s))) e^{-\Lambda_x(s)} ds + F_x(t) \int_0^\infty (\Lambda(\varphi_x(w)) - \Lambda(x)) e^{-\Lambda_x(w)} dw$$

$$= \frac{\lambda x F_x(t) + \int_0^t [e^{-\Lambda_x(s)}]_0^t - \Lambda(x) F_x(t) - F_x(t) \int_0^\infty e^{-\Lambda_x(w)} dw}{\int_0^\infty e^{-\Lambda_x(w)} dw}$$

Relation (18) yields

$$1 + b(x)^* \nabla_x \left( F_x^{-1}(u) \right) = 1 - \frac{1 + e^{-\Lambda_x(F_x^{-1}(u))}}{e^{-\Lambda_x(F_x^{-1}(u))}}$$

$$= \frac{e^{\Lambda_x(t)}(1 - F_x(t))}. \quad (14)$$
Lemma 16.

\[ e^{\lambda_s(t)}(1 - F_{x}(t)) = e^{\lambda_s(t)} \int_{t}^{\infty} e^{-\lambda_s(s)} ds \]

\[ = \int_{0}^{\infty} e^{-\lambda_s(w+t) + \lambda_t(t)} dw \]

\[ = \int_{0}^{\infty} e^{-\lambda_s(v)} dv \]

Bringing the pieces together, we have proved

\[ b(x)^* \nabla (Kf)(x) = E \left[ \frac{h(\phi_x(F_{x}^{-1}(U)))}{h(x)} (b^* \nabla f)(\phi_x(F_{x}^{-1}(U))) \right] \]

\[ = \frac{\tilde{K}(bb^* f)(x)}{h(x)} \]

When \( \lambda \) is non-decreasing along the flow, from [16], \( x \mapsto \lambda_x(t) \) is non-decreasing along the flow for all \( t \geq 0 \), and \( h(\phi_x(t)) \leq h(x) \).

\[
\]

4 Examples

4.1 The TCP with constant rate

A simple yet instructive example on \( \mathbb{R}_+ \) is the TCP with constant rate of jump with generator

\[ Lf(x) = f'(x) + \lambda (E(f(Rx)) - f(x)) \]

where \( R \) is a random variable on \([0,1)\) and \( \lambda > 0 \) is constant. It is a simple growth/fragmentation model, or obtained by renormalizing a pure fragmentation model (cf. [27] for instance). In [34, 31], ergodicity is proved and it is shown the moments of the invariant measure \( \mu \) are all finite; so instead of the set of compactly supported smooth functions, \( A \) may be chosen as the set of smooth functions for which all derivatives grow at most polynomially at infinity.

Applying Theorem 5 with \( J_0 = 0 \), \( M = E(R^2) \) and \( a = 1 \), we get

Proposition 15. for all \( f \in A \),

\[ \|(P_t f)'\|^2 \leq e^{-\lambda(1 - E(R^2))t} P_t |f'|^2. \]

Corrolary 6 then yields a contraction at rate \( \lambda (1 - E(R^2)) \) of the Wasserstein distance \( \mathcal{W}_1(\nu_1, \nu_2) \). In fact by coupling two processes starting at different points to have the same jump times and the same factor \( R \) at each jump, one get that for any \( p \geq 1 \), the \( \mathcal{W}_p \) distance decays at rate \( \lambda p^{-1}(1 - E(R^p)) \) (see [17]), and those rates are optimal (see [35]). In particular \( \lambda (1 - E(R^2)) \) is the rate of decay of \( \mathcal{W}_2 \).

Let

\[ Kf = \int_{0}^{\infty} f(x+s) e^{-\lambda s} ds. \]

Obviously \((Kf)' = K(f')\). Moreover the exponential law \( \mathcal{E}(1) \) satisfies a Poincaré inequality \( \mathcal{B}(2,4) \), so that by the change of variable \( z \mapsto z/\lambda \), \( \mathcal{E}(\lambda) \) satisfies \( \mathcal{B}(2,4\lambda^{-2}) \). Finally, \( K(x) \) is the image of \( \mathcal{E}(\lambda) \) by the translation \( u \mapsto u + x \), which is a 1-Lipschitz transformation. As a conclusion,

Lemma 16. The operator \( K \) is \((4\lambda^{-2}, 1, 2)\)-confining.
As far as the jump operator $Qf(x) = \mathbb{E}(f(Rx))$ is concerned, we have already used the sub-commutation

$$((Qf)' )^2 \leq \mathbb{E}(R^2) Q(\mu f')^2.$$  

However a local Poincaré inequality \cite{14} for $Q(x)$ would mean $\forall f \in \mathcal{A}, \ x > 0,$

$$\mathbb{E}(f^2(Rx)) - (\mathbb{E}(f(Rx)))^2 \leq c\mathbb{E}((f'/(R)) \mathbb{E}((g_x'(R))^2)$$

with $g_x(r) = f(rx).$ This implies the law of $R$ satisfies $B(2, c\mathbb{E}^{-2})$ for all $x > 0,$ hence $B(2, 0),$ which means $R$ is deterministic. Indeed, when $R$ is deterministic, the local inequality always holds:

**Lemma 17.** If $R = \delta$ a.s. with a constant $\delta \in [0, 1)$ then $Q$ is $(0, \delta^2, p)$-contractive.

When $R$ is random, what prevent to straightforwardly use our argument is the possibility of arbitrarily little concentrated jump, for instance with uniform law on $(0, x)$ for any $x.$ It’s a shame because if, say, $R$ is uniform on $(0, \frac{1}{2},)$ it means when the process jumps it is at least divided by 2 but can be even much more contracted. In particular its invariant measure should be more concentrated near zero than the process with $R = \frac{1}{2}$ a.s. for which, as we will see, the invariant measure satisfies a Poincaré inequality. This illustrates a limit of our procedure.

**Proposition 18.** If $R = \delta$ is deterministic then $\mu$ satisfies the Poincaré inequality

$$\forall f \in \mathcal{A}, \quad \mu(f - \mu f)^2 \leq \frac{4}{\lambda^2(1 - \delta^2)} \mu(f')^2.$$  

As a consequence,

$$\forall f \in \mathcal{A}, \quad \mu(P_t f - \mu f)^2 \leq \frac{4e^{-\lambda(1 - \delta^2)t}}{\lambda(1 - \delta^2)} \mu(f')^2.$$  

*Proof.* Since $K$ and $Q$ are confining, from Lemma \cite{9} $P = KQ$ is $(4\lambda^{-2}\delta^2, \delta^2, 2)$-confining and $\mu_e$ the invariant measure of the embedded chain satisfies $B \left(2, \frac{4\delta^2}{\sqrt{1 - \delta^2}} \right).$ From Lemma \cite{10} $\mu = \mu_e \kappa$ and so by Lemma \cite{9} again $\mu$ satisfies $B \left(2, \frac{4\delta^2}{\sqrt{1 - \delta^2}} \right).$ The second inequality is a consequence of this Poincaré inequality and of Proposition \cite{15} \hfill $\square$

In fact in this example the spectrum of the generator in $L^2(\mu)$ is explicit: there are polynomial eigenfunctions, and since the tail of $\mu$ is exponential, polynomials are dense in $L^2(\mu)$ and these eigenfunctions are the only one in $L^2(\mu).$ The eigenvalues are $l_k = \lambda(\mathbb{E}[R^k] - 1)$ with $k \in \mathbb{Z}^+.$ The convergence rate of the $L^2$-norm obtained in Proposition \cite{18} for a deterministic $R$ appears to be $\frac{1}{2}|l_2|$ and not the spectral gap $|l_1|,$ and of course

$$\frac{1}{2}|l_2| = \lambda\mathbb{E}\left[(1 - R)\frac{1 + R}{2}\right] \leq \lambda\mathbb{E}(1 - R) = |l_1|.$$  

Nevertheless $\frac{1}{2}|l_1| \leq |l_2|/2$ so we get the right rate up to a factor 1/2.

Note that, contrary to the Ornstein-Uhlenbeck case (see p.81 of \cite{4}), a pointwise Poincaré inequality of the form

$$\forall x > 0, \ f \in \mathcal{A}, \quad P_t f^2(x) - (P_t f)^2(x) \leq c(t) P_t (f')^2(x).$$  

(19)

cannot hold. Indeed $P_t(x)$ is the mix of a Dirac mass at $x + t$ and of a smooth density with support included in $[0, \delta x + t].$ If one consider $f \in \mathcal{A}$ which is constant equal to 0 on $[0, \delta x + t]$
and constant equal to 1 in the neighborhood of \( x + t \) then the left part of (19) is non-zero while the right part vanishes. More precisely if there have been \( n \) jumps during the time \( t \), then \( X_t \in I_n := [\delta^n(x + t), \delta^n x + t] \), so that the support of \( P_t(x) \) is included in \( \bigcup_{n \geq 0} I_n \). To control the variance of \( P_t f \), one need to control the variations of \( f \) inside each interval \( I_n \), which is done with \( P_t(f')^2 \), but also the variations between two different intervals, which may be done with \( P_t(Qf - f)^2 \).

**Proposition 19.** If \( R = \delta \) is deterministic then \( P_t \) satisfies the local inequality: \( \forall x, t > 0, f \in \mathcal{A}, \)

\[
P_t f^2(x) - (P_t f)^2(x) \leq \lambda(t + t^2) P_t(Qf - f)^2(x) + \lambda t^2 (1 + t)^2 P_t(f')^2(\delta x).
\]

**Proof.** Let \( \phi(f)(x) = (f(\delta x) - f(x))^2 \). We compute

\[
\Gamma_{\partial_x, \phi} f(x) = (f(\delta x) - f(x))(\delta f'(\delta x) - f'(x)) - (f(\delta x) - f(x))(f'(\delta x) - f'(x)) = -(1 - \delta)(f(\delta x) - f(x)) f'(\delta x)
\]

and

\[
\Gamma_{\lambda(Q-I), \phi} f(x) = \frac{\lambda}{2} \left( (f(\delta x) - f(x))^2 + (f(\delta^2 x) - f(\delta x))^2 - 2(f(\delta x) - f(x))(f(\delta^2 x) - f(\delta x)) \right)
\]

\[
= \frac{\lambda}{2} \left( f(\delta^2 x) + f(x) - 2f(\delta x) \right)^2.
\]

Hence for all \( \beta > 0 \),

\[
\Gamma_L, \phi f(x) \geq -(1 - \delta)(f(\delta x) - f(x)) f'(\delta x)
\]

\[
\geq -\frac{1 - \delta}{2} \left( \frac{1}{\beta} \phi(f)(x) + \beta (f'(\delta x))^2 \right).
\]

With \( \psi(s) = P_s \phi(P_{t-s} f)(x) \) and \( \beta = 1 + s \) we roughly get

\[
\psi'(s) \geq - \left( \frac{1}{1 + s} \psi(s) + (1 + s) P_s ((P_{t-s} f)'(\delta x))^2 \right)
\]

\[
\Rightarrow \frac{d}{ds} \left( (1 + s) \psi(s) \right) \geq -(1 + s)^2 P_s ((P_{t-s} f)'(\delta x))^2.
\]

From Proposition 15 this yields

\[
\psi(0) \leq (1 + t) \psi(t) + \left( \int_0^t (1 + s)^2 e^{-\lambda(1 - \delta^2)(t-s)} ds \right) P_t(f')^2(\delta x)
\]

\[
\Rightarrow \phi(P_t f)(x) \leq (1 + t) P_t \phi(f)(x) + t(1 + t)^2 P_t(f')^2(\delta x).
\]

With \( \phi_2(f) = f^2 \) we now consider the usual carré-du-champs operator

\[
\Gamma_{L, \phi_2} f(x) = \frac{\lambda}{2} \left( f^2(\delta x) - f^2(x) \right) - \lambda f(x)(f(\delta x) - f(x))
\]

\[
= \frac{\lambda}{2} \phi(f)(x),
\]

so that

\[
P_t f^2(x) - (P_t f)^2(x) = \lambda \int_0^t \psi(s) ds \\
\leq \lambda(t + t^2) P_t \phi(f)(x) + \lambda t^2 (1 + t)^2 P_t(f')^2(\delta x).
\]

\(\square\)
It would be natural to expect an inequality of the form
\[ P_t f^2(x) - (P_t f)^2(x) \leq c(t) P_t ((Qf - f)^2 + (f')^2) \]
with a bounded \( t \mapsto c(t) \). We could prove such an inequality in the same way as Lemma 9 if we had a function \( \gamma \) such that
\[ (QP_t f - P_t f)^2 \leq \gamma(t) P_t (Qf - f)^2 \] (20)
and a time \( t \) for which \( \gamma(t) < 1 \). Note that in the present case the Bakry-Emery criterion is not satisfied, which implies that an inequality of the form (20) cannot hold with \( \gamma'(0) < 0 \) or, equivalently, with \( \gamma(t) = e^{-rt} \) for some \( r > 0 \).

4.2 The storage model
Let \( U \) be a positive random variable, and consider the generator on \( \mathbb{R}_+ \)
\[ Lf(x) = -xf'(x) + \lambda(\mathbb{E}[f(x + U)] - f(x)) \).
This is, in a sense, the converse of the TCP: the jumps send the process away from 0 and the flow brings it back. Applying Theorem 5 with \( M = 1 \), \( a = 1 \) and \( J_b = -1 \), we get
\[ |\nabla P_t f|^2 \leq e^{-2t} P_t |\nabla f|^2. \] (21)
Besides in this case it is easy to obtain a Wasserstein decay, as the distance \( s \) between two processes starting at different point and coupled to have the same jump times and the same \( U \) at each jump satisfies \( s' = -s \), and such a decay implies (21) (see [30]; the converse is not clear, since \( P_t \) is a mix of a Dirac mass and a smooth density).
To prove a Beckner’s inequality, the same problem arises as in the previous example with a random \( R \): here the law \( K(x) \), namely the law of \( e^{-T x} \) with \( T \) an exponential random variable, can be as little concentrated as possible when \( x \) goes to infinity, so that \( K \) does not satisfy a local Beckner’s inequality (14).

4.3 The TCP with increasing rate
Consider the generator on \( \mathbb{R}_+ \)
\[ Lf(x) = f'(x) + \lambda(x)(f(\delta x) - f(x)) . \] (22)
We have already studied the constant rate case. Before tackling the case of \( \lambda(x) = x \), we consider in this section an intermediate difficulty, with the following assumptions: \( \lambda \) is non-decreasing, \( \lambda(0) = \lambda_* > 0 \), and \( \ln \lambda \) is a \( \kappa \)-Lipschitz function. Let \( \beta = \frac{1-\delta^2}{2e^c} \), so that
\[ \frac{(\lambda')^2}{\beta \lambda} - \lambda (1 - \delta^2) = \frac{\lambda (1 - \delta^2)}{2} \left( \frac{(\lambda')^2}{\lambda \lambda_*^2 (1 - \delta^2)} - 2 \right) \leq -\frac{\lambda_* (1 - \delta^2)}{2} . \]
In other word, Inequality (10) holds with \( \eta = \frac{\lambda_* (1 - \delta^2)}{2} \) and \( a = 1 \). To apply Theorem 2 we also need to prove a Poincaré inequality.

Lemma 20. The operators
\[ K f(x) = \int_0^\infty f(x+t) \lambda(x+t)e^{-\int_0^t \lambda(x+s)ds} dt \]
Lemma 31, which requires an upper bound on the median $m_K$. Thus we only need to prove the inequality holds for $x$ and $\tilde{x}$.

**Proof.** The sub-commutation (13) is a direct consequence of Lemma 13 and 14, since the rate of jump is non-decreasing and $b = 1$. On the other hand $K(x)$ (resp $\tilde{K}(x)$) is the law of $x + T_x$ (resp. $x + \tilde{T}_x$) which is from Lemma 11 the image by a 1-Lipschitz function of $T_0$ (resp. $\tilde{T}_0$). Thus we only need to prove the inequality holds for $K(0)$ and $\tilde{K}(0)$.

For the case of $K(0)$, denote by $F(t) = 1 - e^{-\Lambda_0(t)}$ the cumulative function of $T_0$. Then, if $E$ is a standard exponential random variable,

$$T_0 \overset{\text{dist}}{=} F^{-1}(1 - e^{-E}) = \Lambda_0^{-1}(E).$$

Since $\Lambda_0^{-1}$ is a non-decreasing concave function with $(\Lambda_0^{-1})'(0) = \frac{1}{\lambda^*}$, $T_0$ is a $\frac{1}{\lambda^*}$-Lipschitz transformation of $E$, whose law satisfies the Poincaré inequality $B(2, 4)$.

In Lemma 12, we saw the cumulative function of $\tilde{T}_0$ is $t \mapsto 1 - e^{-\int_0^t r(s) ds}$ with an increasing function $r$ defined by

$$r(t) = \frac{e^{-\Lambda_0(t)}}{\int_t^\infty e^{-\Lambda_0(s)} ds}.$$

The previous argument shows $\tilde{T}_0$ is a $\frac{1}{r(0)}$-Lipschitz transformation of $E$, and

$$r(0) = \frac{1}{\int_0^\infty e^{-\Lambda_0(s)} ds} \geq \frac{1}{\int_0^\infty e^{-\lambda_* s} ds} = \lambda_*.$$

**Remark:** in fact if moreover $\lambda(x) \geq k(1 + x)^q$ for some $k > 0$ and $q \in [0, 1]$, the laws of $T_0$ and $\tilde{T}_0$ satisfy some generalized Poincaré inequality $\mathcal{I}(\alpha, c)$ with $\alpha = \frac{2q}{1+q}$ (see [8, Theorem 3] and [16]), or in other words the Beckner’s inequalities $B(p, c(1 - p)^{\alpha - 1})$ for all $p \in (1, 2]$. By the previous arguments, $K$ and $\tilde{K}$ are $(c(1 - p)^{\alpha - 1}, 1, p)$-confining for all $p \in (1, 2]$.

**Corollary 21.** The invariant measure $\mu$ of the process satisfies a Poincaré inequality $B(2, c)$ with

$$c = \frac{4\delta^2}{\lambda_*^2} \left( 1 + \frac{\lambda}{\lambda_* (1 - \delta)} \left( \frac{2\delta^2}{\lambda_* (1 - \delta)} \right) \right).$$

**Proof.** It is clear the jump operator $Q$ is $(0, \delta^2, 2)$-contractive, so that from Lemma 2 $P = KQ$ is $(\frac{4\delta^2}{\lambda_*^2}, 1, 2)$-contractive, and $\mu_\epsilon$ the invariant measure of the embedded chain associated with the process satisfies a Poincaré inequality $B \left( 2, \frac{4\delta^2}{\lambda_*^2 (1 - \delta)} \right)$. Let

$$h(x) = K \left( \frac{1}{\lambda} \right)(x) = \int_0^\infty e^{-\int_0^t \lambda(x + u) du} ds.$$

It is a non-increasing function with $h(0) \leq \int_0^\infty e^{-\lambda_* s} ds = \frac{1}{\lambda_*}$. In order to prove the perturbation $\nu_\epsilon$ of $\mu_\epsilon$, defined by $\nu_\epsilon(f) = \frac{1}{\mu_\epsilon(\Omega)} \mu_\epsilon(f h)$, satisfies a Poincaré inequality, we will use Lemma 31 which requires an upper bound on the median $m_\epsilon$ of $\mu_\epsilon$. Note that it is possible to couple a process $X$ with rate $\lambda$ and a process $Z$ with constant rate $\lambda_*$ so that, if they start at the same point, the first one will always stay below the second one: suppose such a
coupling \((X, Z)\) has been defined up to a jump time \(T_k\) of \(X\). Then both process increases linearly up to the next jump time \(T_{k+1}\) of \(X\). At time \(T_{k+1}\), \(X\) jumps, but \(Z\) jumps only with probability \(\lambda_k^*\), else it does not move. In other words the jump part of the generator of \(Z\) is thought as

\[
\lambda_k^*(f(\delta x) - f(x)) = \lambda(x) \left( \left( \frac{\lambda_*}{\lambda(x)} f(\delta x) \right) + \left( 1 - \frac{\lambda_*}{\lambda(x)} \right) f(x) \right) - f(x)
\]

Such a coupling proves \(m_c\) is less than the median of the invariant law of the process with constant rate \(\lambda_*\). Let \(Z_\infty\) be a random variable with this invariant law, so that, if \(E\) is a standard exponential random variable,

\[
Z_\infty \overset{\text{dist.}}{=} \delta \left( Z_\infty + \frac{1}{\lambda_*} E \right)
\]

Hence from Markov’s inequality, \(m_c \leq \frac{2\delta}{\lambda_* (1 - \delta)}\), and

\[
h(m_c) \geq \int_0^\infty e^{-\lambda(m_c)s} ds \geq \frac{1}{\lambda \left( \frac{2\delta}{\lambda_* (1 - \delta)} \right)}.
\]

Finally, from Lemma 31 \(\nu_e\) satisfies a Poincaré inequality with constant

\[
c' = 32 \frac{\delta^2 \lambda \left( \frac{2\delta}{\lambda_* (1 - \delta)} \right)}{\lambda_*^2 (1 - \delta^2)},
\]

and since \(\tilde{K}\) is confining, from Lemma \(\eta\) \(\mu = \nu_e \tilde{K}\) satisfies such an inequality with constant

\[
c = \frac{4\delta^2}{\lambda_*^2} + c'.
\]

**Remark:** if, again, \(\lambda(x) \geq k(1 + x)^q\) for some \(k > 0\) and \(q \in [0, 1]\), these arguments prove the invariant measure satisfies a generalized Poincaré inequality \(I(\alpha, c)\) for some \(c > 0\) and \(\alpha = \frac{\eta}{q+1}\). Thus the invariant measure inherits the concentration properties of the law of the jump time \(T_0\): the logarithm of its density tail is (at most) of order \(-x^{q+1}\).

Let \((P_t)_{t \geq 0}\) be the semi-group associated to the generator (23) and for \(f \in \mathcal{A}\) let \(W_t = \mu ((P_t f)^2)\) and \(V_t = \mu (P_t f - \mu f)^2\). We have proved Theorem 7 holds:

**Corollary 22.** If \(\lambda\) is increasing with \(\lambda(0) = \lambda_* > 0\) and \(\ln \lambda\) is \(\kappa\)-Lipschitz then

\[
W_t + \beta V_t \leq (W_0 + \beta V_0)e^{-\frac{\eta}{1 + \beta c t}}.
\]

with

\[
\eta = \frac{\lambda_* (1 - \delta^2)}{2}, \\
\beta = \frac{1 - \delta^2}{2\kappa^2}, \\
c = \frac{4\delta^2}{\lambda_*^2} \left( 1 + 8 \frac{\lambda \left( \frac{2\delta}{\lambda_* (1 - \delta)} \right)}{(1 - \delta^2) \lambda_*} \right).
\]
4.4 The TCP with linear rate

In this section,

\[ L f(x) = f'(x) + x (f(\delta x) - f(x)), \]

where \( \delta \in [0, 1) \), and we will prove Proposition 3. We keep the general notations for \((P_t)_{t\geq 0}\), \(Q\), \(\lambda\) and \(\mu\) (for the proof of ergodicity, see [25]), and write \( \text{Ent} = \mu (f^2 \ln f^2) - \mu(f^2) \ln \mu(f^2) \).

To cope with the rate of jump that vanishes at the origin, we will apply Theorem 7 with a weight \(a\) that behaves linearly near 0. More precisely, let

\[ a(x) = 1 - e^{-(1-\delta)x}, \]
\[ \phi_a(f) = a|f'|^2, \]
\[ W_t = \mu (\phi_a (P_t f)). \]

Lemma 23. Suppose \( \mu \) satisfies the weighted Poincaré inequality

\[ \forall f \in \mathcal{A}, \quad \mu (f - \mu f)^2 \leq c \mu (\phi_a (f)) \]

for some \( c > 0 \), and let

\[ \eta = \left( \frac{3 + \sqrt{5}}{2} - 1 \right)^{-1} + \ln \left( \frac{3 + \sqrt{5}}{2} \right) \simeq 1.58. \]

Then for all \( \beta > \eta^{-1} \), \( t > 0 \) and \( f \in \mathcal{A} \),

\[ W_t \leq e^{-(1-\delta) \frac{\eta - 1}{1 + \beta \mu} (1 + \beta c) W_0}. \]

Proof. Note that \( a \) is a concave function, so that

\[ a(\delta x) = a(\delta x + (1-\delta)0) \geq \delta a(x) + (1-\delta) a(0) = \delta a(x). \]

Thus

\[ \phi_a(Qf)(x) = a(x)\delta^2 |f'(\delta x)|^2 \leq \delta a(\delta x)|f'(\delta x)|^2 = \delta Q(\phi_a(f))(x). \]

Hence, from Lemma 3 for any \( \beta > 0 \),

\[ 2 \Gamma_{L, \phi_a} (f) \geq \frac{a'}{a} \phi_a(f) - 2a f'(Qf - f) + \lambda (1-\delta) \phi_a(f) \]
\[ \geq \left( \frac{a'}{a} + \lambda(1-\delta) - \frac{a}{\lambda \beta} \right) \phi_a(f) - \beta \lambda (Qf - f)^2. \]

Note that \( a(x) \leq (1-\delta) x = (1-\delta) \lambda (x) \), so that

\[ \frac{a'(x)}{a(x)} + \lambda(x)(1-\delta) - \frac{a(x)}{\lambda(x) \beta} \geq (1-\delta) \left( \frac{1}{e^{(1-\delta) x} - 1} + x - \frac{1}{\beta} \right) \]
\[ \geq (1-\delta) \left( \frac{1}{e^x - 1} + x - \frac{1}{\beta} \right). \]

The function \( g(x) = \frac{1}{e^x - 1} + x \) goes to \(+\infty\) at 0 and \(+\infty\) and admits a unique positive critical point for which

\[ e^x = (e^x - 1)^2 \]
\[ \Rightarrow x = \ln \left( \frac{3 + \sqrt{5}}{2} \right). \]
Hence for all $x > 0$, $g(x) \geq g \left( \ln \left( \frac{3 + \sqrt{5}}{2} \right) \right) = \eta$

$$2\Gamma_{L, \phi_a}(f) \geq (1 - \delta) \left( \eta - \frac{1}{\beta} \right) \phi_a(f) - \beta \lambda Q(f - f)^2.$$ 

Following the proof of Theorem 7 with $V_t = \mu (P_t f - \mu f)^2$, this yields

$$W_t' + \beta V_t' \leq -(1 - \delta) \left( \eta - \frac{1}{\beta} \right) W_t,$$

and, thanks to the weighted Poincaré inequality, if $\beta \eta > 1$,

$$W_t' + \beta V_t' \leq -(1 - \delta) \left( \eta - \frac{1}{\beta} \right) (W_t + \beta V_t)$$

$$\Rightarrow W_t + \beta V_t \leq e^{-(1 - \delta) \left( \eta - \frac{1}{\beta} \right) t} (W_0 + \beta V_0).$$

Finally,

$$W_t \leq W_t + \beta V_t \leq e^{-(1 - \delta) \left( \eta - \frac{1}{\beta} \right) t} (W_0 + \beta V_0) \leq e^{-(1 - \delta) \left( \eta - \frac{1}{\beta} \right) t} (1 + \beta \epsilon) W_0.$$

Remark that $h(\beta) = \frac{\beta \eta - 1}{\beta + \beta \epsilon}$ goes to 0 when $\beta$ either goes to $\eta^{-1}$ or to $+ \infty$, and admits a unique positive critical point for which

$$\beta^2 c \eta - 2 \beta c - 1 = 0$$

$$\Rightarrow \beta = c \left( 1 + \sqrt{1 + \frac{1}{c}} \right).$$

**Corollary 24.** Suppose $\mu$ satisfies the weighted inequalities, for all $f \in A$,

$$\mu (f - \mu f)^2 \leq c_1 \mu (\phi_a(f)),$$

$$\text{Ent} f \leq c_2 \mu (\phi_a(f))$$

for some $c_1, c_2 > 0$, and let $\eta$ be such as defined in Lemma 23. Then for all $\beta > \eta^{-1}, t > 0$ and $f \in A$,

$$\text{Ent} P_t f \leq c_2 e^{-(1 - \delta \left( \frac{1}{\beta} \right)) \frac{t}{1 + \beta c_1}} (1 + \beta c_1) \mu(f')^2.$$  

**Proof.** From Lemma 23 and the fact $\alpha \leq 1$,

$$\text{Ent} P_t f \leq c_2 W_t \leq c_2 e^{-(1 - \delta \left( \frac{1}{\beta} \right)) \frac{t}{1 + \beta c_1}} (1 + \beta c_1) W_0 \leq c_2 e^{-(1 - \delta \left( \frac{1}{\beta} \right)) \frac{t}{1 + \beta c_1}} (1 + \beta c_1) \mu(f')^2.$$  

Thus, to prove Proposition 3 it only remains to prove a weighted log-Sobolev inequality holds. In order to simplify some upcoming computations, we consider an intermediate weight $\alpha(x) = 1 - e^{-\sqrt{2}x}$. Then, by concavity, $\alpha(x) = \alpha \left( \frac{1 - \delta}{\sqrt{2}} x \right) \geq \frac{1 - \delta}{\sqrt{2}} \alpha(x)$. If we prove a log-Sobolev inequality (24) holds with weight $\alpha$, it implies such an inequality with weight $\alpha$. Let

$$\psi(x) = \int_0^x \frac{1}{\sqrt{\alpha(y)}} dy.$$
It is a concave, non-decreasing, one-to-one function. If \( Z \) is a random variable with law \( \mu \) and \( Y = \psi(Z) \), then
\[
\mathbb{E} \left( f^2(Z) \ln f^2(Z) \right) - \mathbb{E} \left( f^2(Z) \right) \ln \mathbb{E} \left( f^2(Z) \right) \leq c \mathbb{E} \left( (\alpha(Z)f')^2(Z) \right)
\]
\[
\iff \mathbb{E} \left( g^2(Y) \ln g^2(Y) \right) - \mathbb{E} \left( g^2(Y) \right) \ln \mathbb{E} \left( g^2(Y) \right) \leq c \mathbb{E} \left( (g')^2(Y) \right)
\]
with \( g(y) = f(\psi^{-1}(y)) \). As a consequence we will study the Markov process \( \psi(X) = (\psi(X_i))_{i \geq 0} \), where \( X = (X_i)_{i \geq 0} \) has generator \( \mathcal{L}^{\mu} \), and prove a classical non-weighted log-Sobolev for the invariant measure of this twisted process, which will imply the weighted log-Sobolev assumed in Corollary \( \ref{corollary:weighted_log_sobolev} \).

We still denote by \( Q \) the jump kernel of \( X \), so that
\[
Q_\alpha g(z) = g \left( \psi \left( \delta \psi^{-1}(z) \right) \right)
\]
is the jump kernel of \( \psi(X) \). Let \( K_\alpha \) and \( \tilde{K}_\alpha \) be such as defined in Section \( \ref{section:weighted_log_sobolev} \) but corresponding to the process \( \psi(X) \).

**Lemma 25.** For all \( g \in \mathcal{A} \),
\[
| (Q_\alpha g)' | \leq \sqrt{\delta} Q_\alpha |g'|
\]
\[
| (K_\alpha g)' | \leq K_\alpha |g'|
\]
\[
| (\tilde{K}_\alpha g)' | \leq \tilde{K}_\alpha |g'|.
\]

**Proof.** By concavity, \( \alpha(\delta x) \geq \delta \alpha(x) \) for all \( x \geq 0 \). Thus
\[
(Q_\alpha g)'(z) = \delta \left( \psi^{-1} \right)'(z) \psi'(\delta \psi^{-1}(z)) Q_\alpha g'(z)
\]
\[
= \frac{\delta \left( \psi^{-1} \right)'(z)}{\sqrt{\alpha(\delta \psi^{-1}(z))}} Q_\alpha g'(z)
\]
\[
\leq \frac{\sqrt{\delta} \left( \psi^{-1} \right)'(z)}{\sqrt{\alpha(\psi^{-1}(z))}} Q_\alpha |g'|(|z|)
\]
\[
= \sqrt{\delta} Q_\alpha |g'|(|z|).
\]

On the other hand the vector field associated to \( \psi(X) \) is \( b_\alpha(z) = \frac{1}{\sqrt{\alpha(\psi^{-1}(z))}} \), and the rate of jump is non-decreasing along the flow. Hence, according to Lemma \( \ref{lemma:finite_jumps} \)
\[
b_\alpha |(K_\alpha g)' | \leq K_\alpha \left( b_\alpha |g'| \right)
\]
(and according to Lemma \( \ref{lemma:finite_jumps_times} \) the same goes for \( \tilde{K}_\alpha \)). Note that the support of both probability measure \( K_\alpha(z) \) and \( \tilde{K}_\alpha(z) \) is \([z, \infty]\), and that \( b_\alpha \) is non-increasing along the flow, so that
\[
|(K_\alpha g)'(z)| \leq K_\alpha \left( \frac{b_\alpha |g'|}{b_\alpha(z)} \right) \leq K_\alpha \left( |g'| \right)(z)
\]
(and the same goes for \( \tilde{K}_\alpha \)). □

**Lemma 26.** For any \( z > 0 \), the law \( K_\alpha(z) \) (resp. \( \tilde{K}_\alpha(z) \)) can be obtained from \( K_\alpha(0) \) (resp. \( \tilde{K}_\alpha(0) \)) through a 1-Lipschitz transformation.

**Proof.** Let \( T_x \) be the first time of jump of \( X \) starting from \( x \). According to Lemma \( \ref{lemma:finite_jumps_times} \) there exists a 1-Lipschitz functions \( G \) such that \( T_x \overset{\text{dist}}{=} G(T_0) \). Note that \( K_\alpha(\psi(x)) \) is the law of \( \psi(x + T_x) \). Let \( H(z) = \psi \left( x + G \left( \psi^{-1}(z) \right) \right) \), so that \( \psi(x + T_x) \overset{\text{dist}}{=} H \left( \psi(T_0) \right) \). We compute
\[
| H'(z) | = | g' \left( \psi^{-1}(z) \right) \left( \psi^{-1} \right)'(z) \psi'(x + G \left( \psi^{-1}(z) \right)) | \leq \frac{\psi' \left( x + G \left( \psi^{-1}(z) \right) \right)}{\psi(\psi^{-1}(z))}.
\]
Now $\psi$ is concave, and in the proof of Lemma 11 we have seen that $x + G(s) \geq s$ for all $s \geq 0$; hence $|H'(z)| \leq 1$ for all $z \geq 0$.

Similarly, let $\tilde{T}_x$ be a random variable on $\mathbb{R}_+$ with density $\frac{e^{-\tilde{f}_0'(x-s)} ds}{\int_0^\infty e^{-\tilde{f}_0'(x-s)} ds}$, so that $\tilde{K}_\alpha(\psi(x))$ is the law of $\psi(x + \tilde{T}_x)$. From Lemma 12 there exists a 1-Lipschitz functions $\tilde{G}$ such that $\tilde{T}_x \dist \tilde{G}(\tilde{T}_0)$, and the previous argument concludes. \hfill \Box

**Lemma 27.** Both $K_\alpha(0)$ and $\tilde{K}_\alpha(0)$ satisfies $B(1,1)$.

**Proof.** If $T_0$ is the first time of jump starting from 0 then $K_\alpha(0)$ is the law of $\psi(T_0)$. For any $f \in A$,

$$K_\alpha f(0) = \int_0^\infty f(u) u e^{-\frac{u^2}{2}} du$$

$$= \int_0^\infty f(z) e^{-\frac{(\psi(z))^2}{2} + \ln(\psi(z)) + \frac{1}{2} \ln(\alpha(z))} dz.$$

On the other hand, if $N$ is a standard gaussian variable then $K_\alpha(0)$ is the law of $\psi(|N|)$, and for all $f \in A$

$$\tilde{K}_\alpha f(0) = \int_0^\infty f(u) \left( \frac{\pi}{2} \right)^{-\frac{1}{2}} e^{-\frac{u^2}{2}} du$$

$$= \int_0^\infty f(z) e^{-\frac{(\psi(z))^2}{2} + \ln(\psi(z)) + \frac{1}{2} \ln(\alpha(z))} dz.$$

For $\varepsilon \in \{0, 1\}$, let $V_\varepsilon(z) = \frac{1}{2} (\psi^{-1}(z))^2 - \varepsilon \ln(\psi^{-1}(z)) - \frac{1}{2} \ln(\alpha^{-1}(z))$; we want to prove $V_\varepsilon$ is strictly convex. Writing $x = \psi^{-1}(z)$, we compute $\partial_x(z) = \sqrt{\alpha(x)}$ and

$$V'_\varepsilon(z) = \sqrt{\alpha(x)} \left( x - \frac{\varepsilon}{x} - \frac{\alpha'(x)}{2\alpha(x)} \right)$$

$$V''_\varepsilon(z) = \frac{\alpha'(x)}{2} \left( x - \frac{\varepsilon}{x} - \frac{\alpha'(x)}{2\alpha(x)} \right) + \alpha(x) \left( 1 + \frac{\varepsilon}{x^2} - \frac{\alpha''(x)}{2\alpha(x)} + \frac{1}{2} \left( \frac{\alpha'(x)}{\alpha(x)} \right)^2 \right)$$

$$= \varepsilon \left( \frac{\alpha(x)}{x^2} - \frac{\alpha'(x)}{2x} \right) + \frac{\alpha'(x)x}{2} + \frac{(\alpha'(x))^2}{4\alpha(x)} + \alpha(x) - \frac{1}{2} \alpha''(x).$$

As a first step, note that $V''_1(z) \geq V''_0(z)$: indeed, $V''_1(z) - V''_0(z) = \frac{j(x)}{x^2}$ with

$$j(y) = \alpha(y) - \frac{y}{2} \alpha'(y)$$

$$\Rightarrow \quad j'(y) = \frac{1}{2} \alpha'(y) - \frac{y}{2} \alpha''(y) > 0$$

(since $\alpha$ is non-decreasing and concave). Since $j(0) = 0$, it implies $j(y) \geq 0$ for all $y \geq 0$, in other words $V''_1(z) \geq V''_0(z)$. On the other hand,

$$V''_0(z) \geq \alpha(x) - \frac{1}{2} \frac{\alpha''(x)}{\alpha(x)}$$

$$\geq 1,$$

the last line being the very reason we decided to work with $\alpha$ rather than with $\alpha$. As a consequence, both $K_\alpha(0)$ and $\tilde{K}_\alpha(0)$ satisfies $B(1,1)$ (see for instance [4, Theorem 5.4.7], applied to the diffusion with generator $\partial_x^2 - V_\varepsilon \partial_x$). \hfill \Box

To sum up the consequences of the previous results,
Corollary 28.

1. The operators $K_\alpha$ and $\tilde{K}_\alpha$ are $(1,1,1)$-confining and the operator $Q_\alpha$ is $(0,\sqrt{\alpha},1)$-contractive.

2. The invariant measure $\nu_\alpha$ of the embedded chain associated to $\psi(X)$ satisfies $B\left(1,\frac{\sqrt{\alpha}}{1-\sqrt{\alpha}}\right)$.

Proof. The sub-commutation property has been showed in Lemma 25, and the local inequality is clear for $Q_\alpha$ which is deterministic, and is a consequence of Lemma 26 and 27 for $K_\alpha$ and $\tilde{K}_\alpha$.

From Lemma 9 the transition operator of the embedded chain associated to $\psi(X)$, $P_\alpha = K_\alpha Q_\alpha$, is $(\sqrt{\alpha},\sqrt{\alpha},1)$-confining, conclusion follows again from Lemma 9.

The last step of our procedure is the study of a perturbation of $\nu_\alpha$. Since the rate of jump of $Z = \psi(X)$ at point $z$ is $\lambda_\alpha(z) = \psi^{-1}(z)$ and the operator $K_\alpha$ is such that $K_\alpha f(\psi(x)) = \mathbb{E}(f(\psi(x + T_x)))$, according to Lemma 10 we have to study the function $g$ defined by

$$g(\psi(x)) = K_\alpha \left(\frac{1}{\lambda_\alpha}\right)(\psi(x)) = \mathbb{E}\left(\frac{1}{x + T_x}\right).$$

Lemma 29. The function $g$ is decreasing, and $\ln g$ is $\frac{2}{\sqrt{\pi}}$-Lipschitz.

Proof. Let

$$h(x) = \mathbb{E}\left(\frac{1}{x + T_x}\right) = \int_0^\infty e^{-\int_0^t (x + u)du}dt = \int_0^\infty e^{-t^2 - 2xt}dt,$$

so that $g(z) = h(\psi^{-1}(z))$. Since $h$ is decreasing and $\psi^{-1}$ is increasing, $g$ is decreasing. Moreover, as $|\ln g^n(z)| = \sqrt{\alpha (\psi^{-1}(s))} |(\ln h)'(\psi^{-1}(z))|$ and $\alpha \leq 1$, it is sufficient to prove $\ln h$ is $\frac{2}{\sqrt{\pi}}$-Lipschitz. From $h(x) = e^{x^2} \int_x^\infty e^{-s^2}ds$, one gets

$$h'(x) = -1 + 2xh(x)$$

$$\Rightarrow (\ln h)'(x) = -\frac{1}{h(x)} + 2x$$

$$\Rightarrow (\ln h)''(x) = \frac{(\ln h)'(x)}{h(x)} + 2$$

$$\Rightarrow \left(\frac{1}{h}(\ln h)\right)'(x) = \frac{2}{h(x)} \geq 0$$

$$\Rightarrow (\ln h)'(x) \geq \frac{h(x)}{h(0)}(\ln h)'(0)$$

$$\geq \frac{h'(0)}{h(0)} = -\frac{1}{h(0)}.$$

On the other hand $(\ln h)' \leq 0$; thus $\ln g$ is $\frac{1}{h(0)}$ Lipschitz, and

$$h(0) = \int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}.$$

We have in mind to apply to $\nu_\alpha$ and $g$ the perturbation Lemma 31 of the Appendix. In that aim we need to bound $g(m_\alpha)$, where $m_\alpha$ is the median of $\nu_\alpha$ and $\nu_\alpha(g^{-1})$. In fact, note that $\nu_\alpha$, which is the invariant measure of the embedded chain associated to the process $\psi(X)$, is also the image through the function $\psi$ of the invariant measure of $\mu_e$, the invariant measure of the embedded chain associated to the initial process $X$. In particular if $m_e$ is the median of $\mu_e$ then $m_\alpha = \psi(m_e)$. Keeping the notation $h(x) = g(\psi(x))$, we have $g(m_\alpha) = h(m_e)$ and $\nu_\alpha(g^{-1}) = \mu_e(h^{-1})$.

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Lemma 30. We have

\[
\frac{h(0)}{h(m_e)} \leq 3 \left( 1 + \frac{\delta}{\sqrt{1 - \delta^2}} \right)
\]

and

\[
\mu_e(h^{-1}) \leq \frac{6}{5} \left( 2 + \frac{\delta \sqrt{2}}{\sqrt{1 - \delta^2}} \right).
\]

Proof. Recall that, keeping the notations of Section 3.2 if \( T_x \) is the first time of jump of the process starting from \( x \) and \( E \) is a standard exponential variable, then \( T_x \overset{\text{dist}}{=} \Lambda_x^{-1}(E) \). In the present case \( \Lambda_x(t) = \int_0^t (x + u)du \), so that \( T_x \overset{\text{dist}}{=} \sqrt{x^2 + 2E} - x \). In particular if \( Y \) is a random variable with measure \( \mu_e \), \( Y \overset{\text{dist}}{=} \delta \sqrt{Y^2 + 2E} \), or in other words \( Y^2 \overset{\text{dist}}{=} \delta^2 (Y^2 + 2E) \), so that

\[
(1 - \delta^2)E(Y^2) = 2\delta^2 E(E) = 2\delta^2.
\]

From this,

\[
P(Y \geq t) \leq \frac{\delta^2}{(1 - \delta^2)t^2},
\]

which implies

\[
m_e \leq \frac{2\delta}{\sqrt{1 - \delta^2}}.
\]

Moreover

\[
h(x) = \mathbb{E} \left( \frac{1}{x + T_x} \right)
\]
\[
\geq \frac{1}{x + 2} P(T_x \leq 2)
\]
\[
\geq \frac{1}{x + 2} P(T_0 \leq 2)
\]
\[
= \frac{1}{x + 2} P(\sqrt{2E} \leq 2)
\]
\[
= \frac{1}{x + 2} (1 - e^{-2}) \geq \frac{5}{6(x + 2)}.
\]

Hence

\[
\frac{h(0)}{h(m_e)} \leq \frac{\sqrt{\pi}}{2} \times \frac{6}{5} (m_e + 2)
\]
\[
\leq 3 \left( 1 + \frac{\delta}{\sqrt{1 - \delta^2}} \right).
\]

Finally, if \( Y \) is a random variable with measure \( \mu_e \),

\[
\mu_e(h^{-1}) = \mathbb{E} \left( \frac{1}{h(Y)} \right)
\]
\[
\leq \frac{6}{5} \mathbb{E}(Y + 2)
\]
\[
\leq \frac{6}{5} \left( 2 + \sqrt{\mathbb{E}(Y^2)} \right).
\]

We can now bring the pieces together.
proof of Proposition We have proved in Corollary that satisfies a log-Sobolev inequality. From Lemma and the perturbation defined by also satisfies such an inequality. From Lemma the invariant measure of is , and it also satisfies a log-Sobolev inequality since (Corollary). It means , the invariant measure of , satisfies a weighted log-Sobolev inequality

As we have already noticed, so that the conditions of Corollary are fulfilled, and Proposition is proved.

Appendix

Monotonous perturbation on the half-line

Let be a probability measure on with a positive smooth density (still denoted by ), and be a positive smooth function on such that . We define , the perturbation of by , by for all bounded . Let be the median of , defined by for .

The aim of this section is to prove the following:

Lemma 31. Suppose is non-increasing and .

1. If satisfies the Poincaré inequality , then satisfies with

2. If is Lipschitz and satisfies the log-Sobolev inequality and with for all

Remark: actually as far as point 2 is concerned the monotonicity of is only needed to get the explicit estimate of as soon as satisfies a log-Sobolev inequality and is Lipschitz, satisfies a log-Sobolev inequality (see ).

Moreover when satisfies a log-Sobolev inequality and is Lipschitz, satisfies for all (see [1]), so that is finite.

proof of point 1. According to Muckenhoupt work (see Theorem p. 99), a probability with density is finite when is the median of and

Furthermore, in that case, the optimal (namely the smallest such that holds) is such that

In the present case, for all ,

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Moreover when satisfies a log-Sobolev inequality and is Lipschitz, satisfies for all (see [1]), so that is finite.
and for all $x \leq m$
\[
\int_0^x g(t)\nu(t)dt \int_x^m \frac{1}{g(t)\nu(t)}dt \leq \int_0^x g(0)\nu(t)dt \int_x^m \frac{1}{g(m)\nu(t)}dt \leq \frac{2g(0)}{g(m)}c_1.
\]

Hence $\nu_g$ satisfies $B(2, c_2)$ with
\[
c_2 \leq 4\inf_{\alpha > 0} B_\alpha(\nu g) \leq 4B_m(\nu g) \leq \frac{8g(0)}{g(m)}c_1.
\]

\[\Box\]

**proof of point 2.** Following a computation of Aida and Shigekawa ([2]), we apply the inequality $B(1, c_1)$, namely
\[
\forall f \in \mathcal{A}, \quad \nu (f^2 \ln f^2) \leq 2c_1\nu(f')^2 + (\nu f^2) \ln (\nu f^2),
\]
to the function $f\sqrt{g}$, which reads
\[
\forall f \in \mathcal{A}, \quad \nu_g (f^2 \ln f^2) + \nu_g (f^2 \ln g) \leq 2c_1\nu_g (f' + \frac{g'}{2g})^2 + (\nu_g f^2) \ln (\nu_g f^2). \quad (25)
\]

From the inequality $(a + b)^2 \leq 2a^2 + 2b^2$ and the assumption on $\ln g$,
\[
\nu_g (f' + \frac{g'}{2g})^2 \leq 2\nu_g (f')^2 + \frac{1}{2}\kappa^2 \nu_g (f^2).
\]

On the other hand, from the Young inequality $st \leq s\ln s - s + e^t$ applied with $s = \varepsilon f^2$ and $t = -\varepsilon^{-1} \ln \left(\frac{g(0)}{g(0)}\right)$ for any $\varepsilon > 0$,
\[
-\nu_g (f^2 \ln g) = -\nu_g \left(f^2 \ln \left(\frac{g}{g(0)}\right)\right) - \ln g(0)\nu_g (f^2)
\]
\[
\leq \varepsilon \nu_g (f^2 \ln f^2) - (\varepsilon (1 - \ln \varepsilon) + \ln g(0)) \nu_g (f^2) + \nu_g \left(\frac{g(0)}{g}\right)^{\frac{1}{2}}.
\]

Thus Inequality (25) yields
\[
(1 - \varepsilon)\nu_g (f^2 \ln f^2) \leq 4c_1\nu_g (f')^2 + (c_1\kappa^2 - \varepsilon (1 - \ln \varepsilon) - \ln g(0)) \nu_g (f^2) + \nu_g \left(\frac{g(0)}{g}\right)^{\frac{1}{2}}
\]
\[+ \nu_g (f^2) \ln \nu_g (f^2).
\]

Thanks to Gross’ Lemma (2.2 of [28]), this implies (for $\varepsilon < 1$)
\[
\nu_g (f^2 \ln f^2) - \nu_g (f^2) \ln \nu_g (f^2) \leq \frac{4c_1}{1 - \varepsilon} \nu_g (f')^2 + \gamma \nu_g (f^2) \quad \text{(26)}
\]

with
\[
\gamma = \frac{c_1\kappa^2 - \varepsilon (1 - \ln \varepsilon) - \ln g(0)}{1 - \varepsilon} + \frac{\varepsilon}{1 - \varepsilon} \left(\ln \left(\frac{\nu_g \left(\frac{g(0)}{g}\right)}{\varepsilon}\right)\right)
\]
\[
= \frac{c_1\kappa^2 + \varepsilon \ln \nu_g \left(g^{-\frac{1}{2}}\right)}{1 - \varepsilon}.
\]
It is classical to retrieve a log-Sobolev inequality from Inequality (26) and a Poincaré inequality, thanks to the following inequality (see [22], p.146): if $h = f - \nu g f$,

$$
\nu_g \left( f^2 \ln f^2 \right) - \nu_g(f^2) \ln \nu_g(f^2) \leq \nu_g \left( h^2 \ln h^2 \right) - \nu_g(h^2) \ln \nu_g(h^2) + 2 \nu_g(h^2).
$$

Together with Inequality (26) applied to $h$, and since $h' = f'$,

$$
\nu_g \left( f^2 \ln f^2 \right) - \nu_g(f^2) \ln \nu_g(f^2) \leq \frac{4 c_1}{1 - \varepsilon} \nu_g(f'f^2) + (\gamma + 2) \nu_g((f - \nu g f)^2).
$$

Since $\nu$ satisfies $B(1, c_1)$ it also satisfies $B(2, 2c_1)$. Thus, according to point 1 of Lemma 31, $\nu_g$ satisfies $B \left( 2, 16 \frac{g(0)}{g(m)} c_1 \right)$, which means

$$
\nu_g \left( f^2 \ln f^2 \right) - \nu_g(f^2) \ln \nu_g(f^2) \leq 2 \left( \frac{2}{1 - \varepsilon} + \frac{g(0)}{g(m)}(2 + \gamma) \right) c_1 \nu_g(f'f^2).
$$

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