ON GENERALIZED DERIVATIONS OF SEMIRINGS WITH INVOLUTION

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Abstract

In this paper we investigate some fundamental results on Jordan ideals, \(*\)-Jordan ideals, derivations and generalized derivations and hence establish some commutativity results for a certain class of semirings with involution.

Keywords: Inverse semirings; MA-semirings; Generalized derivations; \(*\)Jordan ideals

I. Introduction and Preliminaries

Involution is one of the important and fundamental concepts studied in functional analysis and algebra. For instance B*-algebra due to Rickart [III] and C*-algebra due to Segal [VIII] are now well known concepts which are defined with involution. Later on many algebraists used this idea in groups, rings and semirings (see [I], [II], [IV], [V], [VI],[IX], [X],[XI],[XIII], [XIV], [XV], [XVI],[XXII], [XXIII], [XXIV], [XXVI]).

Javed et al. [XXI] introduced the notion MA-semiring that is an additive inversesemiring satisfying \(A_2\) condition of Bandlet and Petrich [VII]. The concept of commutators along with derivations and certain additive mappings was further investigated and extended in [XXI], [XXV], [XXVII].

In this paragraph we compose some necessary definitions and preliminary concepts. By a semiring \(S\), we mean a semiring with absorbing zero \('0'\) in which addition is commutative. A semiring \(S\) is said to be additive inverse semiring if for each \(x \in S\) there is a unique \(x' \in S\) such that \(x + x' + x = x\) and \(x' + x + x' = x'\), where \(x'\) is
called the pseudo inverse of \( x \). An additive inverse semiring \( S \) is said to be a MA-semiring if it satisfies
\[
x + x' \in \mathbb{Z}(S), \quad \forall x \in S.
\]
In fact every ring is MA-semiring while converse may not be true. The following is one of the examples of MA-semirings which is not a ring. Such examples motivate us to generalize the results of ring theory in MA-semirings.

**Example 1.** [XXI] The set \( S = \{0, 1, 2, 3, 4, \ldots\} \) with addition \( \oplus \) and multiplication \( \circ \) respectively defined by
\[
a \oplus b = \text{sup}\{a, b\} \quad \text{and} \quad a \circ b = \text{inf}\{a, b\}
\]
is an MA-semiring. In fact \( S \) is a commutative prime MA-semiring.

Throughout the paper by a semiring \( S \) we mean an MA-semiring unless mentioned otherwise. We say \( S \) is prime if \( a \circ b = \{0\} \) implies that \( a = 0 \) or \( b = 0 \) and semiprime if \( a \circ a = \{0\} \) implies \( a = 0 \). R is 2-torsion free if for \( x \in S, \quad 2x = 0 \) implies \( x = 0 \). An additive mapping \( \ast : S \to S \) is involution if
\[
\ast \ast = \ast \quad \text{and} \quad \ast \ast = \ast.
\]
An element \( x \in S \) is hermitian if \( x^\ast = x \) and skew hermitian if \( x^\ast = x' \).

The notion of \( \ast \)-primeness and \( \ast \)-semiprimeness of MA-semirings is introduced in [XVII]. A semiring with involution is said to be \( \ast \)-prime if \( a \circ b = \{0\} \) implies \( a = 0 \) or \( b = 0 \) and \( \ast \)-semiprime if \( a \circ a = \{0\} \) implies \( a = 0 \).

**Example 2.** Let \( (S, +, \cdot) \) be an MA-semiring. Then the set \( S \) with addition \( ' + ' \) and multiplication \( \cdot \) defined as \( a \cdot b = b \cdot a \) forms an MA-semiring called the opposite MA-Semiring of \( S \). We usually denote it as \( S' \). Consider \( R = S \times S' \) with
\[
(a, b) \oplus (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b) \circ (c, d) = (ac, bd).
\]
Then \( (R, \oplus, \circ) \) forms an MA-semiring. Define \( \ast : R \to R \) by \( (x, y)^\ast = (y, x) \). Then \( \ast \) defines an involution on \( R \). The MA-semiring \( (R, \oplus, \circ) \) with involution \( \ast \) forms \( \ast \)-prime MA-semiring which is not prime.

This example shows that a prime MA-semiring with involution \( \ast \) is a \( \ast \)-prime MA-semiring but converse is not true in general.

The set of hermitian elements is denoted by \( H \) and skew hermitian elements by \( K \) and \( S = H \cup K \). An additive mapping \( d : S \to S \) is a derivation if
\[
d(x \cdot y) = d(x) \cdot y + x \cdot d(y).
\]
An additive mapping \( F : S \to S \) is a generalized derivation associated with a derivation \( d \) if \( F(x \cdot y) = F(x) \cdot y + x \cdot d(y) \) (see [9]). The commutator is defined as \( [x, y] = xy + yx \). By Jordan product we mean \( x \circ y = xy + yx \) for all \( x, y \in S \). An additive subsemigroup \( G \) of \( S \) is called Jordan ideal if \( x \circ r \in G, \quad \forall x \in G, r \in S \). The concept of \( \ast \)-Jordan ideals for semirings is defined in [XVIII]. A Jordan ideal \( G \) is \( \ast \)-Jordan if \( G = G^\ast \). A mapping \( f : S \to S \) is
Commuting if \( \left[ f(x), x \right] = 0, \forall x \in S \). A mapping \( f: S \rightarrow S \) is Centralizing if \( \left[ \left[ f(x), x \right], r \right] = 0, \forall x, r \in S \). Following identities are very useful for sequel: For all \( x, y, z \in S \):

1. \( [x, xy] = x[x, y] \)
2. \( [xy, z] = x[y, z] + [x, z]y \)
3. \( [x, yz] = [x, y]z + y[x, z] \)
4. \( [x, y] + [y, x] = y(x + x') = x(y + y') \)
5. \( (xy)' = x'y = xy' \)
6. \( [x, y'] = [x', y] = [x', y'] \)
7. \( x \circ (y + z) = x \circ y + x \circ z \)

For more details one can see [XXI], [XXV].

Following results can be established very easily which are very useful for proving the main results:

**Lemma 1.** [XVIII] Let \( S \) be a semiring and \( G \) be a nonzero Jordan ideal of \( S \). Then \( 2[S, S]G \subseteq G \) and \( 2G[S, S] \subseteq G \).

**Lemma 2.** [XVIII] Let \( G \) be a *-Jordan ideal of a 2-torsion free *-prime semiring \( S \). If \( xGy = \{0\} = x'Gy \) then \( x = 0 \) or \( y = 0 \).

**Lemma 3.** [XVIII] Let \( G \) be a Jordan ideal of a 2-torsion free prime semiring \( S \). If \( \{G, G\} = \{0\} \) then \( \{G, S\} = \{0\} \) and hence \( G \subseteq Z(S) \).

**Remark 1.** If \( aG = \{0\} \) or \( Ga = \{0\} \), then \( a = 0 \). For if \( aG = \{0\} \) then \( aGa = \{0\} = aGa' \). By Lemma 2 \( a = 0 \). Every *-prime semiring is semiprime. If \( S \) is *-prime and for \( x \in Z(S) \), \( x^2 = 0 \) then \( x = 0 \). For if \( x^2 = 0 \) with \( x \in Z(S) \), then \( xSx = \{0\} \). As \( S \) is semiprime and therefore \( x = 0 \).

**Lemma 4.** [XIX] Let \( S \) be a 2-torsion free *-prime semiring and \( G \) be a nonzero *-Jordan ideal of \( S \). If \( \{G, S\} = \{0\} \), then \( S \) is commutative.

**Lemma 5.** [XIX] Let \( S \) be a 2-torsion free semiring with involution *, \( G \) be a nonzero *-Jordan ideal of \( S \) and \( d \) a nonzero derivation on \( G \) such that \( \left[ \left[ d(x), x \right], r \right] = 0, \forall x \in G, r \in S \). Also assume that \( d \) commutes with *. If \( S \) is a *-prime, then \( S \) is commutative.
Lemma 6.[XVIII] Let $S$ be a 2-torsion free $\ast$ - prime semiring and $G$ be a nonzero $\ast$ - Jordan ideal of $S$. If $d$ is derivation of $S$ such that $d(G) = \{0\}$, then either $d = 0$ or $[G,S] = \{0\}$ and hence $G \subseteq Z(S)$.

The aim of this paper is to extend the results for generalized derivation of rings established in [XIII]. In the upcoming section, we establish some fundamental results on Jordan ideals, $\ast$-Jordan ideals, derivations and generalized derivations. Inconsequent results on commutativity of MA-Semirings are obtained.

II. Main Results

Lemma 7. Let $G$ be a $\ast$-Jordan ideal of a 2-torsion free $\ast$-prime semiring $S$. If $[G,S] = \{0\}$ then $S$ is commutative.

Proof. By the hypothesis we have $\forall r \in S, j \in G$

\[ [j,r] = 0 \]  \hspace{1cm} (2.1)

Since $G$ is a Jordan ideal, replacing $j$ by $j \circ s, s \in S$ in (2.1), we get $[js + sj, r] = 0, \forall r \in S$ and using (2.1) again we get $2j[s,r] = 0$ and hence by the 2-torsion freeness of $S$, we have $j[s,r] = 0$ and hence

\[ jS[s,r] = \{0\} \]  \hspace{1cm} (2.2)

Since $G = G^*$, therefore from (2.2) we have

\[ j^*S[s,r] = \{0\} \]  \hspace{1cm} (2.3)

Since $G$ is nonzero, by the $\ast$-primeness of $S$, we have $[s,r] = 0, \forall r, s \in S$ and hence $S$ is commutative.

Theorem 1. Let $S$ be a 2-torsion free $\ast$-prime inverse semiring and $G$ a nonzero $\ast$-Jordan ideal and subsemiring of $S$. If $d$ is derivation of $S$ such that $d(j^2) = 0, \forall j \in G$, then $d = 0$.

Proof. By the hypothesis for all $j \in G$, we have

\[ d(j)j + jd(j) = d(j^2) = 0 \]  \hspace{1cm} (2.4)

In (2.4) replacing $j$ by $j + k, j, k \in G$ and using it again we get

\[ d(j \circ k) = d(jk + kj) = 0 \]  \hspace{1cm} (2.5)

and hence $d(j)k + jd(k) + d(k)j + kd(j) = 0$ which further implies that
In (2.6) replacing \( k \) by \( j \circ k \), \( j, k \in G \) and using (2.5), we get \( d(j) \circ (j \circ k) = 0 \) and hence \( d(j)(jk) + d(j)(kj) = 0 \), which further implies

\[
d(j)(jk) + (jk)d(j) + d(j)(kj) + (kj)d(j) = 0
\]  
(2.7)

From (2.4), we have

\[
d(j)j = \frac{d(j)}{j} - j
\]

and hence

\[
d(j)j + d(j)j + kj + (kj)d(j) = 0
\]

which further implies

\[
d(j)(j) + j(j) + k(j) + (k)(j)d(j) = 0
\]

and hence

\[
d(j)(j) + j(j) + k(j) + (k)(j)d(j) = 0
\]

In view of Lemma 1, replacing \( k \) by \( 2[k, r, uv], r \in S, k, u, v \in G \), we get

\[
[d(j), 2k[r, uv], j] = 0,
\]

which further gives

\[
2k[d(j)[r, uv], j] + 2d(j), r, uv, j] = 0
\]  
(2.9)

In (2.9), taking \( k = j \), we get

\[
2j[d(j)[r, uv], j] + 2[d(j), j, r, uv, j] = 0
\]

and hence

\[
j[d(j), 2[r, uv], j] + 2d(j), j, r, uv, j] + 2[d(j), r, uv, j] = 0.
\]

In view of Lemma 1, as \( 2[r, uv] = 2u[r, v] + 2[r, u]v \in G \), therefore using (2.8), we get

\[
2[d(j), j][r, uv, j] = 0,
\]

and hence by the 2-torsion freeness of \( S \), we obtain

\[
d(j), j][r, uv, j] = 0
\]  
(2.10)

In (2.10) replacing \( r \) by \( jr \), \( j \in G, r \in S \), we get \([d(j), j][jr, uv], j] = 0\), and hence

\[
[d(j), j][jr, uv, j] + [j, uv][r, j] = 0
\]

which further gives

\[
[d(j), j][jr, uv, j] + [d(j), j][j, uv][r, j] = 0
\]  
(2.11)

From (2.8) taking \( k = j \), we have \([d(j), j][j] = j[d(j), j]\). Hence (2.11) becomes

\[
j[d(j), j][r, uv, j] + [d(j), j][j, uv][r, j] = 0
\]

and using (2.10) again, we get

\[
[d(j), j][j, uv][r, j] = 0
\]

and hence

\[
[d(j), j][j, uv][r, j] = 0
\]

and using (2.10) again
In (2.12) replacing $r$ by $rd(j)$ and using (2.12) again, we get 
\[ [d(j), j][j, uv][r, j] = 0 \] 
and hence multiplying by $[j, uv]$ from the right, we get 
\[ [d(j), j][j, uv]S[d(j), j][j, uv] = \{0\}. \] 
By remark 1, $S$ is semiprime. Hence we obtain 
\[ [d(j), j][j, uv] = 0, \] 
which further gives 
\[ \sum_{j, d(j)} d(j, j)u[j, v] = 0 \] 
(2.13)

In view of Lemma 1, replacing $v$ by $2v[r, s], r, s \in S, v \in G$ in (2.13), we get 
\[ 2[d(j), j][j, uv][r, s] = 0 \] 
and hence 
\[ 2[d(j), j][j, uv][r, s] + 2[d(j), j][j, u]v[r, s] = 0. \] 
Using (2.13) again, we get 
\[ 2[d(j), j][j, uv][r, s] = 0 \] 
and by the 2-torsion freeness of $S$, we have 
\[ [d(j), j]uG[j, r, s] = \{0\} \] 
(2.14)

In particular for 
\[ j \in G \cap S_{d(j)}(S), \] 
we have 
\[ [d(j), j]uG[j, r, s] = 0 \] 
(2.15)

In view of Lemma 2, using (2.14) and (2.15), we obtain either 
\[ [d(j), j]u = 0 \] 
or 
\[ [j, [r, s]] = 0. \] 
Firstly suppose that 
\[ [j, [r, s]] = 0 \] 
(2.16)

In (2.16) replacing $r$ by $rj$ and using it again, we get 
\[ [j, r][j, s] = 0 \] 
(2.17)

In (2.17) replacing $s$ by $sr$ and using it again, we have 
\[ [j, r]S[j, r] = \{0\} \] 
and in view of remark 1 since $S$ is semiprime, therefore 
\[ [j, r] = 0 \] 
and replacing $r$ by $d(j), j \in G \cap S_{d(j)}(S)$, we get 
\[ [j, d(j)] = 0 \] 
(2.18)
It is easy to see that for any $j \in G \cap S_a(S)$, $j + j^*$, $j^* + j^* \in G \cap S_a(S)$. In (2.18) replacing $j$ by $j + j^*$, we get

$$\left[ d(j), j^* \right] + [d(j), j^*] + [d(j^*), j] + [d(j^*), j^*] = 0, \forall j \in G \text{ and hence }$$

$$\left[ d(j), j^* \right] + [d(j^*), j] = [d(j^*), j]^* + [d(j^*), j] (2.19)$$

Replacing $j$ by $j + j^*$ in (2.18), we get

$$\left[ d(j), j^* \right] + [d(j), j^*] + [d(j^*), j] + [d(j^*), j^*] = 0 \quad (2.20)$$

Using (2.19) into (2.20) and hence 2-torsion freeness of $S$, we get

$$\left[ d(j), j^* \right] + [d(j^*), j] = 0 \quad (2.21)$$

In (2.14), replacing $j$ by $j^*$, we have $[d(j^*), j^*] u G[j^*, [r, s]] = \{0\}$ and using (2.21), we obtain

$$\left[ d(j), j^* \right] u G[j^*, [r, s]] = \{0\} \quad (2.22)$$

In view of Lemma 2, using (2.14) and (2.22), we get either $\left[ d(j), j^* \right] u = 0$ or $[j, [r, s]] = 0$, that is either $\left[ d(j), j^* \right] G = \{0\}$ or $[j, [r, s]] = 0$. In view of remark 1, we have either $\left[ d(j), j^* \right] = 0$ or $[j, [r, s]] = 0$. If $[j, [r, s]] = 0$, then as above this assumption implies that $\left[ d(j), j^* \right] = 0, \forall j \in G$ and by Lemma 5, we have either $d = 0$ or $G \subseteq Z(S)$. If $G \subseteq Z(S)$, then $j \circ r = 2 jr \in G, \forall j \in G, r \in S$. In (2.5) replacing $k$ by $2 jr, j \in G, r \in S$, we get $d(2 j r + 2 j r) = 0$. Simplifying and using 2-torsion freeness of $S$, we get $d(\left( j \circ j \right) r) = 0$ and hence $d(\left( j \circ j \right) r + (j \circ j) d(r)) = 0$. Using (2.5), we get $(j \circ j) d(r) = 0$ and therefore $2 j j d(r) = 0$. As $S$ is 2-torsion free, we obtain $j^2 d(r) = 0$ and therefore

$$j^2 G d(r) = \{0\} \quad (2.23)$$

As $G = G^*$ and $(j^*)^2 = (j^2)^*$, therefore we have

$$(j^2)^* G d(r) = \{0\} \quad (2.24)$$
Since $G$ is nonzero, in view of Lemma 2 using (2.23) and (2.24), we obtain $d = 0$. Secondly suppose that $[d(j), j]u = 0$. Then following the same process as above we will get the same conclusion.

**Theorem 2.** Let $G$ be a $*$-Jordan ideal of a 2-torsion free $*$-prime semiring $S$. If $F$ is an additive mapping satisfying $\left[ [F(G), G], S \right] = \{0\}$ then $F(G \cap Z(S)) \subseteq Z(S)$.

**Proof.** By hypothesis for all $j, k \in G, r \in S$ we have
\[
\left[ [F(j), k], r \right] = 0 \tag{2.25}
\]
In particular for all $j \in G \cap Z(S)$
\[
\left[ [F(j), k], r \right] = 0 \tag{2.26}
\]
In view of Lemma 1, replacing $k$ by $2k[v, u], v, u \in S, k \in G$ in (2.26), we get
\[
\left[ [F(j), 2k[v, u]], r \right] = 0 \quad \text{and by the 2-torsion freeness of } S, \quad \text{we have}
\]
\[
\left[ [F(j), k[v, u]], r \right] + [k[F(j), [v, u]], r] = 0 \tag{2.27}
\]
Replacing $u$ by $uv, v, u \in S$ in (2.27) and using it again, we get
\[
\left[ [F(j), k[v, uv]], r \right] + [k[F(j), [v, uv]], r] = 0 \quad \text{and so}
\]
\[
\left[ [F(j), k[v, u]], r \right] + [k[F(j), [v, k]], r] = 0 \quad \text{which further gives}
\]
\[
\left[ [F(j), k[v, u]], r \right] + [k[F(j), [v, u]], r] + [k[v, u][F(j), v]], r] = 0 \tag{2.28}
\]
Replacing $v$ by $F(j), j \in G \cap Z(S)$ and $u$ by $k \in G$ in (2.28), we get
\[
\left[ [F(j), k[F(j), F(j)], r] + [k[F(j), [F(j), k]]F(j), r]
\]
\[
+ [k[F(j), k[F(j), F(j)], r] = 0.
\]
Since $k[F(j), F(j)] \in Z(S)$, therefore
\[
\left[ [F(j), k[F(j), k]]F(j), r] + [k[F(j), [F(j), k]]F(j), r]
\]
\[
+ [k[F(j), F(j)][F(j), k], r] = 0.
\]
Using (2.26), we have
\[
\left[ [F(j), k[F(j), k]]F(j), r] + [k[F(j), [F(j), k]]F(j), r] = 0 \quad \text{and hence}
\]
\[
[F(j), k][F(j), r] = 0 \quad \text{(since by (2.26) } [F(j), k] \in Z(S)) \text{ and replacing}
\]
$r$ by $k \in G$ we get $[F(j), k]^3 = 0$. Since by (2.26) $[f(j), k] \in Z(S)$, therefore

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\[ \left[ F(j), k \right]^3 u \left[ F(j), k \right] \left[ F(j), k \right] = \{0\} \quad (2.29) \]

Since \( \left( \left[ F(j), k \right] \left[ F(j), k \right] \right)^* \) = \( \left[ F(j), k \right] \left[ F(j), k \right] \), therefore

\[ \left[ F(j), k \right]^3 u \left( \left[ F(j), k \right] \left[ F(j), k \right] \right)^* = \{0\} \quad (2.30) \]

Since \( S \) is \(+\)-prime, using (2.29) and (2.30), we get either

\[ \left[ F(j), k \right]^3 = 0 \] or \[ \left[ F(j), k \right] \left[ F(j), k \right] = 0 \]. Firstly suppose that

\[ \left[ F(j), k \right] \left[ F(j), k \right] = 0 \]. Then

\[ \left[ F(j), k \right]^3 u \left[ F(j), k \right] = \{0\} \quad (2.31) \]

Again using the relation \( \left[ F(j), k \right]^3 = 0 \), we have

\[ \left[ F(j), k \right]^3 u \left[ F(j), k \right] = \{0\} \quad (2.32) \]

Since \( S \) is \(+\)-prime, using (2.31) and (2.32), we get either

\[ \left[ F(j), k \right]^3 = 0 \] or \( \left[ F(j), k \right] = 0 \), \( \forall j \in G \cap Z(S), k \in G \). If \( \left[ F(j), k \right]^3 = 0 \), then clearly

\[ \left[ F(j), k \right]^3 u \left[ F(j), k \right] = \{0\} \quad (2.33) \]

By the remark 1 \( S \) is semiprime, therefore equation (2.33) yields

\[ \left[ F(j), k \right] = 0 \quad (2.34) \]

In view of Lemma 1 replacing \( k \) by \( 2k \left[ r, u \right], k \in G, r, u \in S \) in (2.34), we get

\[ 2 \left( \left[ F(j), k \right] \left[ F(j), k \right] + k \left[ F(j), k \right] \left[ r, u \right] \right) = 0 \]. By the 2-torsion freeness of \( S \), using

(2.34), we get \( k \left[ F(j), \left[ r, u \right] \right] = 0 \) and so \( G \left[ F(j), \left[ r, u \right] \right] = \{0\} \). In view of remark 1, we get

\[ \left[ F(j), \left[ r, u \right] \right] = 0 \quad (2.35) \]

In (2.35) replacing \( r \) by \( rz, z \in G \), and using 2-torsion freeness of \( S \) and (2.35) again, we get

\[ \left[ z, u \right] \left[ F(j), v \right] = 0, \forall j \in G \cap Z(S), z \in G, v, u \in S \quad (2.36) \]

Replacing \( v \) by \( wv, w, v \in S \) in (2.35) and using (2.35) again, we obtain
Since $G = G^*$, therefore we can write
\[
[z,u]S[F(j),v] = \{0\}
\] (2.38)

By the $*$-primeness of $S$, using (2.37) and (2.38), we get either
\[
[z,u] = 0 \quad \text{or} \quad j \in G \cap Z(S)
\]
and hence $F(G \cap Z(S)) \subseteq Z(S)$. Secondly, if
\[
[z,u] = 0 \quad \text{i.e.} \quad [G,S] = \{0\}. \quad \text{Replacing} \quad k \quad \text{by} \quad 2i, \quad i \in G, r \in S \quad \text{in} \quad (2.34), \quad \text{we get}
\]
\[
2[F(j),r] = 0 \quad \text{and by the 2-torsion freeness of} \quad S, \quad \text{we have} \quad G[F(j),r] = \{0\}. \quad \text{By the remark 1, we have} \quad F(j,r) = 0. \quad \text{Hence} \quad F(G \cap Z(S)) \subseteq Z(S). \quad \text{This completes the theorem.}
\]

**Theorem 3.** Let $G$ be a $*$-Jordan ideal of a 2-torsion free $*$-prime semiring $S$. If $d$ is a nonzero derivation and $F$ a generalized derivation associated with $d$ centralizing on $G$ then $S$ is commutative.

**Proof.** Suppose that $G \cap Z(S) = \{0\}$. by Lemma 1, one can easily check that for all $j \in G$,
\[
4[F(j),j]^2 = (4jF(j))j + 4j(F(j))^2 + F(j)j + 4j^2(F(j))^2
\]
therefore $4[F(j),j]^2 \in G$. Since $F$ is centralizing on $G$, therefore
\[
\forall j \in G, p \in S, [F(j),j], p = 0 \quad \text{which means that} \quad [F(j),j] \in Z(S) \quad \text{and hence} \quad 4[F(j),j]^2 \in Z(S). \quad \text{By our supposition} \quad G \cap Z(S) = \{0\}, \quad \text{therefore by the 2-torsion freeness of} \quad S, \quad \text{we have} \quad [F(j),j]^2 = 0, \quad \text{which further gives}
\]
\[
[F(j),j] = 0
\] (2.39)

Linearizing (2.39) and using (2.39) again, we get for all $j, k \in G$
\[
[F(j),k] + [F(k),j] = 0
\] (2.40)

In view of Lemma 1, replacing $k$ by $4jz^2, z, j \in G$ in (2.40), we get
\[
4[F(j),zj^2] + [F(j),z]j^2 = 0 \quad \text{and by the 2-torsion freeness of} \quad S, \quad \text{we get on simplification} \quad ([F(j),z] + [F(z),j]j^2 + z[F(j),j]^2 + [zd(j^2),j] = 0. \quad \text{using}
\]

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(2.40), we get $z[F(j), j^2] + [zd(j^2), j] = 0$ and hence

$$z[F(j), j] + j[F(j), j] + [zd(j^2), j] = 0.$$  

Using (2.39) again, we get

$$z[d(j^2), j] + [z, j]d(j^2) = 0$$  \hspace{1cm} (2.41)

In view of Lemma 1, replacing $z$ by $2[p, q]z, q, p \in S, z \in G$ in (2.41), we get

$$2[p, q]z[d(j^2), j] + 2[p, q]z, j]d(j^2) = 0.$$  

Since $S$ is 2-torsion free, therefore on simplification, we get

$$[p, q]z[d(j^2), j] + [p, q][z, j]d(j^2) + [p, q], j]zd(j^2) = 0.$$  

And hence

$$[p, q]z[d(j^2), j] + [z, j]d(j^2) + [p, q], j]zd(j^2) = 0.$$  

Using (2.41) again, we get

$$[p, q], j]zd(j^2) = 0$$  \hspace{1cm} (2.42)

For any $j \in G \cap S, (S)$, we have

$$[[p, q], j]Gd(j^2) = \{0\}$$  \hspace{1cm} (2.43)

In view of Lemma 2, using (2.42) and (2.43), we get either $[[p, q], j]Gd(j^2) = \{0\}$ or $d(j^2) = 0$, $p, q \in S, j \in G \cap S, (S)$. Suppose that

$$[[p, q], j] = 0$$  \hspace{1cm} (2.44)

In (2.44) replacing $q$ by $jt$, $t \in q$ and using (2.44) again, we get

$$[p, j][t, j] = 0$$  \hspace{1cm} (2.45)

In (2.45) replacing $t$ by $tq, q \in S$ and using (2.45) again, we get

$$[p, j]S[t, j] = \{0\}$$  \hspace{1cm} (2.46)

Replacing $p$ by $t$ in (2.46) we get $[t, j]S[t, j] = \{0\}$ and using the remark 1, we get

$$[t, j] = 0, \forall t, \in S, j \in G \cap S, (S).$$

This shows that $j \in Z(S)$ and hence $j \in G \cap Z(S)$. But since $G \cap Z(S) = \{0\}$, therefore $j = 0$ and so $d(j^2) = 0$.

Hence we conclude that

$$d(j^2) = 0, \forall j \in G \cap S, (S)$$  \hspace{1cm} (2.47)
Since \( \forall j \in G \) we have \( j' + j^* , j + j^* \in G \cap Sd_\ast (S) \), therefore replacing
\( j \) by \( j' + j^* \) in (2.47), we get for all \( j \in G \)
\[
\begin{align*}
d\left( j^2 + j'^2 \right) &= d\left( jj^* + j^* j \right), \tag{2.48}
\end{align*}
\]
Similarly replacing \( j \) by \( j + j^* \) in (2.47), we get
\[
\begin{align*}
d\left( j^2 + j'^2 \right) + d\left( jj^* + j^* j \right) &= 0 \tag{2.49}
\end{align*}
\]
Using (2.48) into (2.49) and the 2-torsion freeness of \( S \), we get \( d(j^2 + j'^2) = 0 \) and hence
\[
\left( d\left( j^2 \right) \right)' = d(j'^2) \tag{2.50}
\]
In (2.42), replacing \( j \) by \( j^* \), we get \([ [p, q], j^* ] Gd\left( j'^2 \right) = \{0\} , \forall j \in G , p , q \in S \). Using (2.50), we get
\[
\begin{align*}
[[p, q], j^*] Gd\left( j^2 \right) &= \{0\} \tag{2.51}
\end{align*}
\]
In view of Lemma 2, using (2.42) and (2.51), we get either \([ [p, q], j ] = 0 \) or \( d\left( j^2 \right) = 0 \). If \( d\left( j^2 \right) = 0 \) then by theorem 1 \( d = 0 \) , a contradiction. On the other hand if \([ [p, q], j ] = 0 \) then as above \( j = 0 \) implying that \( G = \{0\} \), a contradiction. Hence we must have \( G \cap Z(S) \neq \{0\} \). Let \( 0 \neq w \in G \cap Z(S) \). Since \( F \) is centralizing, for all \( j \in G , q \in S \) therefore
\[
[[F(j), j], q] = 0 \tag{2.52}
\]
Linearizing (2.52) and using (2.52) again, we get for all \( j , k \in G , q \in S \)
\[
[[F(j), k] + [F(k), j], q] = 0 \tag{2.53}
\]
Replacing \( k \) by \( w \circ wp = 2w^2 p \) in (2.53), we get
\[
\begin{align*}
[[F(j), 2w^2 p] + [F(2w^2), p] + 2w^2 d(p), j], q] &= 0 \text{ In view of Theorem 2 and the fact that } 2w^2 \in G \cap Z(S), \text{ we have}
2w^2 \left[ [F(j), p], q \right] + F(2w^2) ([p, j], q] + 2w^2 \left[ d(p), j \right], q] &= 0. \text{ Replacing } p \text{ by } j \text{ and using (2.52) again, we get}
\end{align*}
\]
\[
\begin{align*}
F(2w^2) \left[ [j, j], q \right] + 2w^2 \left[ d(j), j \right], q] &= 0 \tag{2.54}
\end{align*}
\]
Using the fact $[j, j] = [j, j]$, from (2.54), we get
\[ F\left(2w^2\right)[[j, j]] + 2w^2[d(j), [j, j], q] = 0 \]
and hence
\[ F\left(2w^2\right)[[j, j], q] = 2w^2[d(j), [j, j], q] \quad (2.55) \]
Using (2.55) into (2.54) and the 2-torsion freeness of $S$, we get
\[ w^2[[d(j), [j, j], q] = 0 \]
and hence for all $j \in G, q \in S$
\[ w^2 S[[d(j), [j, j], q] = \{0\} \quad (2.56) \]
since $G = G^*$, therefore
\[ \left(w^2\right)^* S[[d(j), [j, j], q] = \{0\} \quad (2.57) \]
Since $w^2 \neq 0$ and $S$ is $*$-prime, therefore using (2.56) and (2.57), we get
\[ [[d(j), [j, j], q] = 0 \]
from lemma 5, we have $[G, S] = \{0\}$ and by lemma 7 $S$ is commutative.

**Remark 2.** In theorem 3, if we replace the generalized derivation by derivation then result follows.

**Corollary 1.** Let $d$ be a nonzero derivation of a 2-torsion free $*$-prime semiring $S$ and $F$ a generalized derivation associated $d$ centralizing on $S$. Then $S$ is commutative.

**Proof.** For $G = S$, it follows from Theorem 3.

**III. Concluding Remarks**

This article presents some criteria to induce commutativity in additive inverse semirings with involution via generalized derivations. In particular we used centralizing generalizations of derivations on $*$-Jordan Ideals to enforce commutativity in additive inverse semirings. The commutativity of algebraic structures paves the way towards the convenience in calculations. Therefore this research is useful which enforces commutativity by derivations and opens the door of further research in this area by using different types of additive mappings. We propose some open problems as follows:

1. Let $S$ be a 2-torsion free $*$-prime semiring and $F$ a generalized derivation of $S$. If $F$ is centralizing on a $*$-Lie ideal. Is $S$ commutative?
2. Let $S$ be a 2-torsion free $*$-semiprimes semiring and $F$ a generalized derivation of $S$. If $F$ is centralizing on a $*$-Jordan ideal. Is $S$ commutative?
3. Let $S$ be a 2-torsion free *-semiprimesemiring and $F$ a multiplicative generalized derivation $F(xy) = xF(y) + \alpha(x)y$, where $\alpha$ is an additive mapping. If $F$ iscentralizing on a *-Jordan ideal. Is $S$ commutative?

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