ENERGY-MOMENTUM TENSORS IN GAUGE THEORY

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Summary

In field theory on a fibre bundle \( Y \rightarrow X \), an energy-momentum current is associated to a lift onto \( Y \) of a vector field on \( X \). Such a lift by no means is unique and contains a vertical part. It follows that:

(i) there are a set of different energy-momentum currents;
(ii) the Noether part of an energy-momentum current is not taken away;
(iii) if a Lagrangian is not gauge-invariant, the energy-momentum fails to be conserved.

In gauge theory, classical fields are represented by sections of a fibre bundle \( Y \rightarrow X \), co-ordinated by \((x^\lambda, y^i, y_i^\lambda)\) (where \( y_i^\lambda \) are coordinates of derivatives of field functions). A Lagrangian on \( J^1Y \) is defined as a density

\[
L(x^\lambda, y^i, y_i^\lambda)d^nx, \quad n = \dim X.
\]

By gauge transformations are meant bundle automorphisms of \( Y \rightarrow X \). To study Lagrangian conservation laws, it suffices to consider 1-parameter groups \( G_u \) of gauge transformations. Their infinitesimal generators are projectable vector fields \( u = u^\lambda(x)\partial_\lambda + u^i(y)\partial_i \) on \( Y \rightarrow X \). A Lagrangian \( L \) is \( G_u \)-invariant iff its Lie derivative \( L_u L \) along \( u \) vanishes. The first variational formula states the canonical decomposition

\[
L_u L = (u^i - y^i_\mu u^\mu)\mathcal{E}_i d^n x - d_\lambda \mathcal{F}^\lambda u d^n x, \quad d_\lambda = \partial_\lambda + y_\lambda^i \partial_i + y_\lambda^i_\mu \partial^\mu_i,
\]

where \( \mathcal{E}_i = (\partial_i L - d_\lambda \partial^\lambda_i L) \) is the Euler–Lagrange operator and

\[
\mathcal{F}^\lambda_u = (u^\mu y^i_\mu - u^i)\partial^\lambda_i L - u^\lambda L
\]

is the current along \( u \). On the shell \( \mathcal{E}_i = 0 \), the first variational formula (1) leads to the weak identity

\[
L_u L \approx -d_\lambda[(u^\mu y^i_\mu - u^i)\partial^\lambda_i L - u^\lambda L].
\]

If the Lie derivative \( L_u L \) vanishes, we obtain the weak conservation law

\[
0 \approx -d_\lambda \mathcal{F}^\lambda_u
\]
Remark 1. It may happen that a current $\mathfrak{T}$ takes the form

$$\mathfrak{T}^\lambda = W^\lambda + d_\mu U^{\mu\lambda},$$

where the term $W$ vanishes on-shell ($W \approx 0$). Then one says that $\mathfrak{T}$ reduces to a superpotential $U$.

Remark 2. Background fields do not live in the dynamic shell $E_i = 0$ and, therefore, break Lagrangian conservation laws as follows. Let us consider the product $Y_{\text{tot}} = Y \times Y'$ of the above fibre bundle $Y$ of dynamic fields and a fibre bundle $Y'$, coordinated by $(x^\lambda, y^A)$, whose sections are background fields. A Lagrangian $L$ is defined on the total configuration space $J^1 Y_{\text{tot}}$. Let

$$u = u^\lambda(x) \partial_\lambda + u^A(x^\mu, y^B) \partial_A + u^i(x^\mu, y^B, y^j) \partial_i$$

be a projectable vector field on $Y_{\text{tot}}$ which also projects onto $Y'$ because gauge transformations of background fields do not depend on the dynamic ones. Substitution of (3) in (1) leads to the first variational formula in the presence of background fields

$$L_u L = (u^A - y^A u^\lambda) \partial_A L + \partial_\lambda L d_\lambda (u^A - y^A u^\mu) + (u^i - y^i u^\lambda) E_i - d_\lambda [(u^\mu y^i_\mu - u^i) \partial^\lambda L - u^\lambda L].$$

A total Lagrangian $L$ is usually invariant under gauge transformations of the product $Y \times Y'$. In this case, we obtain the weak identity

$$0 \approx (u^A - y^A u^\lambda) E_A - d_\lambda [(u^\mu y^i_\mu - u^i) \partial^\lambda L - u^\lambda L]$$

in the presence of background on the dynamic shell $E_i = 0$.

Point out the following properties of currents.

(i) $\mathfrak{T}_{u+u'} = \mathfrak{T}_u + \mathfrak{T}_{u'}$.

(ii) Any projectable vector field $u$ on $Y$ projected onto the vector field $\tau = u^\lambda \partial_\lambda$ on $X$ is written as the sum $u = \bar{\tau} + u_V$ of some lift $\bar{\tau}$ of $\tau$ onto $Y$ and the vertical vector field $u_V = u - \bar{\tau}$ on $Y$.

(iii) The current along a vertical vector field $u = u^i \partial_i$ on $Y$ is the Noether current $\mathfrak{T}_u^\lambda = -u^i \partial^\lambda L$.

(iv) The current $\mathfrak{T}_{\bar{\tau}}$ along a lift $\bar{\tau}$ onto $Y$ of a vector field $\tau = \tau^\lambda \partial_\lambda$ on $X$ is said to be the energy-momentum current.

It follows from the items (i) – (iv) that any current can be represented by a sum of an energy-momentum current and a Noether one.

Different lifts $\bar{\tau}$ and $\bar{\tau}'$ onto $Y$ of a vector field $\tau$ on $X$ lead to distinct energy-momentum currents $\mathfrak{T}_{\bar{\tau}}$ and $\mathfrak{T}_{\bar{\tau}'}$, whose difference $\mathfrak{T}_{\bar{\tau}} - \mathfrak{T}_{\bar{\tau}'}$ is the Noether current along the vertical vector field $\bar{\tau} - \bar{\tau}'$ on $Y$. The problem is that, in general, there is no canonical lift onto $Y$ of vector fields on $X$, and one can not take the Noether part away from an energy-momentum current.
There exists the category of so called natural bundles $T \to X$, exemplified by tensor bundles, which admit the canonical lift $\bar{\tau}$ onto $T$ of any vector field $\tau$ on $X$. This is the case of space-time symmetries and gravitation theory. Such a lift is the infinitesimal generator of a 1-parameter group of general covariant transformations of $T$. The corresponding energy-momentum current $\mathcal{T}_{\bar{\tau}}$ is reduced to the generalized Komar superpotential. Other energy-momentum currents differ from $\mathcal{T}_{\bar{\tau}}$ in the Noether ones, but they fail to be conserved because almost all gravitation Lagrangians are not invariant under vertical (non-holonomic) gauge transformations.

Let us focus on field models on non-natural bundles $Y \to X$, i.e., they possess internal symmetries. Then a vector field on $X$ gives rise to $Y$ by means of a connection on $Y \to X$. A connection on a fibre bundle $Y \to X$ is defined as a section $\Gamma$ of the affine jet bundle $J^1Y \to X$, and is represented by the tangent-valued form

$$
\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_i^\lambda \partial_i)
$$

It follows that connections on $Y \to X$ make up an affine space modelled on the space of soldering forms $\sigma = \sigma_i^\lambda dx^\lambda \otimes \partial_i$. In particular, the difference of two connections $\Gamma - \Gamma'$ is a soldering form. Given a connection $\Gamma$ (5), a vector field $\tau = \tau_\lambda \partial_\lambda$ on $X$ gives rise to the projectable vector field

$$
\Gamma \tau = \tau_\lambda (\partial_\lambda + \Gamma_i^\lambda \partial_i)
$$
on a fibre bundle $Y$. The corresponding current

$$
\mathcal{T}^\lambda_{\Gamma \tau} = \tau^\mu T^\lambda_\mu = \tau^\mu [(y^\mu_i - \Gamma^\mu_i_\lambda)\partial_\lambda \mathcal{L} - \delta^\lambda_\mu \mathcal{L}]
$$
is called the energy-momentum current, while $T^\lambda_\mu$ is said to be the energy-momentum tensor with respect to a connection $\Gamma$.

**Remark 3.** The difference $\mathcal{T}_{\Gamma \tau} - \mathcal{T}_{\Gamma' \tau}$ of energy momentum currents with respect to different connections $\Gamma$ and $\Gamma'$ is the Noether current along the vertical vector field $\Gamma \tau - \Gamma' \tau = \tau^\lambda (\Gamma_i^\lambda - \Gamma'_i^\lambda) \partial_i$.

**Remark 4.** Let $Y \to X$ be a trivial bundle and $\Gamma$ a flat connection on it. There is a coordinate system on $Y$ such that $\Gamma_i^\lambda = 0$. Then the energy-momentum tensor with respect to $\Gamma$ reduces to the familiar canonical energy-momentum tensor. The latter, however, is not preserved under gauge transformations, and it is not defined on a non-trivial bundle.

Let us study energy-momentum conservation laws in gauge theory of principal connections on a principal bundle $P \to X$ with a structure Lie group $G$. These connections are sections of the fibre bundle $C = J^1P/G \to X$, coordinated by $(x^\lambda, a^\lambda_\mu)$, and are identified with gauge potentials. Their configuration space is the jet manifold $J^1C$ coordinated by $(x^\lambda, a^\lambda_\mu, a^\lambda_{\mu\lambda})$. It admits the canonical splitting

$$
a^\tau_{\lambda\mu} = \frac{1}{2} (\mathcal{F}^\tau_{\lambda\mu} + \mathcal{S}^\tau_{\lambda\mu}) = \frac{1}{2} (a^\tau_{\lambda\mu} + a^\tau_{\mu\lambda} - c^\tau_{pq} a^p_{\lambda\mu} a^q_\lambda) + \frac{1}{2} (a^\tau_{\lambda\mu} - a^\tau_{\mu\lambda} + c^\tau_{pq} a^p_\lambda a^q_\mu).
$$
Gauge transformation in gauge theory on a principal bundle $P \to X$ are automorphism of $P \to X$ which are equivariant under the canonical action of the structure group $G$ on $P$ on the right. They induce the automorphisms of the bundle of connections $C$ whose generators read

$$
\xi = \xi^\lambda \partial_\lambda + (\partial_\mu \xi^\tau + c_{pq}^r \partial_\mu \xi^q - a^r_\lambda \partial_\mu \xi^\lambda) \partial_\mu^r,
$$

(6)

where $\xi^\tau$ are functions on $X$ which play the role of gauge parameters.

Let $L$ be a Lagrangian on $J^1C$. One usually requires of $L$ to be invariant under vertical gauge transformations with the generators

$$
\xi^\tau = (\partial_\mu \xi^\tau + c_{pq}^r \partial_\mu \xi^q) \partial_\mu^r.
$$

Hence, $L$ is a function of the strength $F$. Then the Noether current

$$
\Sigma_\xi^\lambda = -(\partial_\mu \xi^\tau + c_{pq}^r \partial_\mu \xi^q) \partial_\mu^\lambda L
$$

is conserved. It reduces to the superpotential form

$$
\Sigma_\xi^\lambda = \xi^\tau \varepsilon_\tau^\lambda + d_\mu U^\mu \lambda, \quad U^\mu \lambda = \xi^\rho \partial_\rho^\lambda \mu L.
$$

Given a principal connection $B$ on $P \to X$, there exists the lift

$$
\tilde{\tau}_B = \tau^\lambda \partial_\lambda + [\partial_\mu (\tau^\lambda B^\alpha_\lambda) + c_{pq}^r \partial_\mu (\tau^\lambda B^p_\alpha) - a^r_\lambda \partial_\mu \tau^\lambda] \partial_\mu^r.
$$

(7)

of a vector field $\tau$ on $X$ onto the bundle of connections $C \to X$. It is a generator (6) of gauge transformations of $C$ with the gauge parameters $\xi^\tau = \tau^\lambda B^\alpha_\lambda$.

Discovering the energy-momentum current along the lift (7), we assume that a Lagrangian $L$ of gauge theory depends on a background metric on $X$. This metric is a section of the tensor bundle $\hat{\sqrt{\mu}}T X$ coordinated by $(x^\lambda, \sigma^{\mu\nu})$. Following Remark 2, we define $L$ on the total configuration space $J^1Y = J^1(C \times \hat{\sqrt{\mu}}T X)$. Given a vector field $\tau$ on $X$, there exists its canonical lift

$$
\tilde{\tau}_g = \tau^\lambda \partial_\lambda + (\partial_\nu \tau^\alpha \sigma^{\nu\beta} + \partial_\nu \tau^\beta \sigma^{\nu\alpha}) \partial_\alpha^\beta
$$

(8)

onto the tensor bundle $\hat{\sqrt{\mu}}T X$. Combining (7) and (8) gives the lift

$$
\tilde{\tau}_Y = [\tau_g - a^r_\lambda \partial_\mu \tau^\lambda \partial_\mu^r] + [\partial_\mu (\tau^\lambda B^\alpha_\lambda) + c_{pq}^r \partial_\mu (\tau^\lambda B^p_\alpha) - a^r_\lambda \partial_\mu \tau^\lambda] \partial_\mu^r
$$

of a vector field $\tau$ on $X$ onto the product $Y$. The first term in this expression is a local generator of general covariant transformations, while the second one is that of vertical gauge transformations.

Let a total Lagrangian $L$ be invariant under general covariant transformations and vertical gauge transformations. Then using the formula (4), we obtain the weak identity

$$
0 \approx \partial_\lambda \tau^\mu \partial_\mu^\lambda \sqrt{|g|} - \tau^\mu \{_\mu^\beta \partial_\beta^\lambda \sqrt{|g|} - d_\lambda \Sigma_\xi^\lambda,
$$

(9)
where $t_\mu^\lambda$ is the metric energy-momentum tensor, $\{\mu^\beta_\lambda\}$ are the Christoffel symbols of a background metric $g$, and

$$\check{T}_B^\lambda = \left[ \partial_\nu^\lambda \mathcal{L}(\tau^\mu a_\mu^\nu + \partial_\nu \tau^\mu a_\mu^\nu) - \tau^\lambda \mathcal{L} \right] + \left[ -\partial_\nu^\lambda \mathcal{L}(\partial_\nu (\tau^\mu B_\mu^\nu) + c^r_{qp}a^q_\nu(\tau^\mu B_\mu^p) \right]$$

(10)

is the energy-momentum current along the vector field (7). If $L$ is the Yang–Mills Lagrangian, a simple computation brings (7) into the familiar covariant conservation law

$$\nabla_\lambda (t_\mu^\lambda \sqrt{|g|}) \approx 0,$$

(11)

independent of a connection $B$. All other energy-momentum conservation laws differ from (11) in a superpotential term $d_\mu d_\lambda U^{\mu \lambda}$.

If a Lagrangian $L$ is not gauge-invariant, no energy-momentum is conserved. For instance, let

$$L = \frac{1}{2k} a^m_{\alpha \beta}(\mathcal{F}_{\lambda \mu} - \frac{1}{3} c_{pq} a^p_\lambda a^q_\mu) d^3x$$

be the Chern–Simons Lagrangian. It is not gauge-invariant, but its Euler–Lagrange operator is so. Then we obtain the conservation law

$$0 \approx -d_\lambda [\check{T}_B^\lambda + \frac{1}{k} c^m_{\alpha \lambda}(\tau^\alpha B_\mu^\nu a_\mu^\nu)],$$

where $\check{T}_B$ is the energy-momentum current (10) along the vector field $\tau_B$. Thus, the energy-momentum current of the Chern–Simons model is not conserved, but there exists another conserved quantity.

Another interesting example is a Lagrangian in the generating functional of quantum gauge theory. It is not gauge-invariant, but is BRST-invariant. The corresponding energy-momentum current is conserved, but it contains ghost fields.

References

G.Sardanashvily, Energy-momentum conservation laws in gauge theory with broken invariance, E-print arXiv: [hep-th/0203275](http://arxiv.org/abs/hep-th/0203275).

L.Mangiarotti and G.Sardanashvily, *Connections in Classical and Quantum Field Theory* (World Scientific, Singapore, 2000).

G.Sardanashvily, Stress-energy-momentum tensors in constraint field theories, *J. Math. Phys.* 38 (1997) 847.

G.Sardanashvily, Stress-energy-momentum conservation law in gauge gravitation theory, *Class. Quant. Grav.* 14 (1997) 1371.