On the Min-Max-Delay Problem: NP-completeness, Algorithm, and Int-Gap

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Abstract—We study a flow problem where a source streams information to a sink over a directed graph $G = (V, E)$ possibly using multiple paths to minimize the maximum end-to-end delay, denoted as Min-Max-Delay problem. Different from the maximum latency problem, our setting is unique in that communication over an edge incurs a constant delay within an associated capacity. We prove that Min-Max-Delay is weakly NP-complete, and demonstrate that it becomes strongly NP-complete if we require integer flow solution. We propose an optimal pseudo-polynomial time algorithm for Min-Max-Delay, with complexity of $O(\log(Nd_{\text{max}})/(N^{2}d_{\text{max}} \log(N^{2}d_{\text{max}})))$, where $N = \max(\{|V|, |E|\})$, $R$ is the flow rate requirement and $d_{\text{max}}$ is the maximum edge delay. Moreover, we propose a fully polynomial time approximation scheme with a time complexity of $O((L+\log N)(N^{2}\log R+NM\log(NM)))$, where $M = |E|(1+1/\epsilon)+1$ and $L$ is the bit complexity for representing $d_{\text{max}}$. Besides, in this paper we observe that the optimal Min-Max-Delay flow can only be achieved by fractional flow solution in some problem instances and in the worst case the Int-Gap, which is defined as the maximum delay gap between the optimal integer Min-Max-Delay flow and the optimal Min-Max-Delay flow, can be infinitely large. We also propose a near-tight bi-criteria bound of $(1-\epsilon, 1/\epsilon)$ for the Int-Gap. In the end, we show how the results introduced in this paper can be extended to the multi-commodity Min-Max-Delay problem case.

I. INTRODUCTION

Delay-sensitive network flows have strong applications in many domains, including communication networks, cyber-physical systems, transportation networks and evacuation planning [1]. In communication networks like video conferencing, the video delivery delay shall be no more than 250ms to ensure a good interactive conferencing experience [2]. In cyber-physical systems, shorter delay for control messages can improve control quality [3]. In transportation networks, timely delivery is critical to deliver perishable goods [4]. In evacuation planning, it is important to move all people from hazardous areas to safe areas as soon as possible [5].

There are mainly two different flow models for delay-sensitive flow problems: the flow-amount model and the flow-rate model. The flow-amount model is suitable to the applications where the flow is generated once, e.g., people to be moved in the evacuation planning, while the flow-rate model is suitable to the applications where the flow is generated continuously, e.g., the video streaming in video conferencing. For both models, two key delay-sensitive network flow problems either maximize the flow (amount or rate) subject to a maximum delay constraint or minimize the maximum delay subject to a flow (amount or rate) requirement. We summarize related studies in Tab. I.

For the flow-amount model, the first problem, called dynamic flow problem [6], is to maximize the flow amount to be delivered from a source to a sink within a given time horizon. [6] showed that it can be formulated as a min-cost flow problem and thus solved in polynomial time by various min-cost flow algorithms. The second problem, called quickest flow problem, is to minimize the time horizon to deliver a given amount of flow from a source to a sink and is also solvable in polynomial time [7].

For the flow-rate model, the first problem, called delay-constrained max-flow problem [8], is to maximize the flow rate to be sent from a source to a sink while the end-to-end delay is bounded above by a given delay constraint. The problem has been proved to be NP-complete [8]. The second problem, called maximum latency problem [9], is to minimize the maximum end-to-end delay that flow units experience from a source to a sink while satisfying a given flow rate requirement and has been proved to be NP-complete [9].

Compared with the maximum latency problem, this work targets a similar but different problem, denoted as Min-Max-Delay. Specifically, there are two key differences from the assumption of non-negative, non-decreasing and differentiable flow-dependent edge delay functions in the maximum latency problem [9], [13]: (i) Each edge has a capacity constraint such that assigned flow rate cannot exceed the given capacity; (ii) Edge transmission delay is a constant within the capacity and goes to infinity otherwise. Existing results on the maximum latency problem including the complexity analysis and approximation algorithms [9], [13] are not applicable to our work due to the two key differences. Our work has many applications. For example, in transportation network, the ground vehicle speed (or equivalently the time to pass the road) remains nearly constant before reaching a certain flow rate [14], [15].

In this paper, we make the following contributions:

- We prove that Min-Max-Delay is weakly NP-complete in Thm. IV-B based on the results in Sec. III-A and Sec. IV.
- In Thm. 2 of Sec. III-B we prove Min-Max-Delay becomes strongly NP-complete if each path can only have an integer flow.
- In Sec. IV we propose a simple binary-search algorithm, which can find the optimal solution to Min-Max-Delay in
where (2a) together with (2d) define our objective which is to minimize the maximum path delay for \( s \rightarrow t \) paths that carry positive flow rates (called flow-carrying paths). In fact, \( M(f) = M \) in the formulation (2). (2b) restricts that the source \( s \) sends \( R \) rate to the sink \( t \), and (2c) requires that the flow rate on edge \( e \) does not exceed its capacity \( c_e \). We use \( f_{\text{MM}}(R) \) to denote the optimal solution to the problem [2] namely the Min-Max-Delay flow given rate requirement \( R \). Formulation (2) is a convex optimization problem [17]. However, there exists a key difficulty to solve Min-Max-Delay from formulation (2): the number of paths (number of variables) could exponentially increase with respect to the input network size.

The integer Min-Max-Delay problem, denoted as Int-Min-Max-Delay, further requires that each path carries an integer flow rate, i.e., replacing (2c) by

\[
\text{vars. } x_p \in \mathbb{Z}^+, \forall p \in P.
\] (3)

Different from Min-Max-Delay flow which minimizes the maximum delay, another well-known delay-aware network flow in the literature is the system-optimal flow, which is defined as follows:

**Definition 1** (System-optimal flow). The system-optimal flow minimizes the total delay of the flow over all edges:

\[
\min \sum_{e \in E} \max_{x \neq 0} d_e \cdot x_e
\]

The flow is feasible in the sense that it meets the capacity constraint, flow conservation constraint, non-negative constraint as well as the rate requirement.

**Integer system-optimal flow** is the integer flow that minimizes total delay.

Given a rate requirement \( R \), we denote the system-optimal flow as \( f_{\text{SO}}(R) \) and the integer system-optimal flow as \( f_{\text{SO}}^+(R) \).

**III. HARDNESS RESULTS**

In Sec. [III-A] we prove that Min-Max-Delay is NP-complete based on the reduction from the NP-complete partition problem [16]. In Sec. [III-B] we further prove that Int-Min-Max-Delay is NP-complete in the strong sense based on the pseudo-polynomial transformation from the classic strongly NP-complete 3-partition problem [16].

**A. NP-completeness for Min-Max-Delay**

We first define partition and the partition problem below.

**Definition 2** (Partition). Given a non-empty set \( A \), its partition is a set of non-empty subsets such that each element in \( A \) is in exactly one of these subsets.

**Definition 3** (Partition Problem [16]). Given a set of \( n \) positive integers \( A = \{a_1, a_2, \ldots, a_n\} \) with sum \( \sum_{i \in A} a_i = 2b \). Is there a partition \( \{A_1, A_2\} \) of \( A \) such that \( \sum_{i \in A_1} a_i = \sum_{j \in A_2} a_j = b \)?
The partition problem is known to be NP-complete [16] (in the weak sense). We now leverage it to prove that Min-Max-Delay is NP-complete.

**Theorem 1.** The decision version of Min-Max-Delay problem is NP-complete.

**Proof:** For any partition problem with set $A = \{a_1, a_2, ..., a_n\}$ and $b$ where any $a_i \in A$ is a positive integer and $\sum_{a_i \in A} = 2b$, we construct a graph $G''$ with $(2n + 1)$ nodes and $3n$ edges as in Fig. [1]. All edges in the graph have a capacity of 1. Each edge $(w_{i-1}, w_i)$ (denoted as dashed lines in the figure) has a delay of $a_i$ for all $i = 1, \ldots, n$, and all edges $(w_{i-1}, v_i)$ and $(v_i, w_i)$ (denoted as solid lines in the figure) have a delay of zero. Obviously it takes polynomial time to construct the graph $G''$ given any partition problem. We then consider the following decision problem of Min-Max-Delay: for graph $G'$ with source $s = w_0$, sink $t = w_n$, and flow rate requirement $R = 2$, is there any feasible flow $f$ such that the maximum delay $M(f) \leq b$?

Now we prove the partition problem answers “Yes” if and only if the decision version of Min-Max-Delay answers “Yes”.

**If Part.** If the decision problem of Min-Max-Delay answers “Yes”, then there exists a flow $f$ such that the maximum delay $M(f) \leq b$. Since $f$ is feasible, the total rate from $w_0$ to $w_n$ in $f$ is $R = 2$. Now due to the capacity constraint and flow conservation, all edges must exactly have a flow rate 1 to satisfy the requirement $R = 2$. The total delay in flow $f$ is

$$\sum_{p \in P} x^p d^p = \sum_{e \in E} x_e d_e = \sum_{e \in E} 1 \cdot d_e = \sum_{i=1}^n a_i = 2b.$$  

(4)

Since $M(f) \leq b$, we have

$$d^p \leq M(f) \leq b, \forall p \in P \text{ with } x^p > 0.$$  

(5)

Also, because the total flow rate is equal to 2, it is

$$2b = \sum_{p \in P} x^p d^p = \sum_{p \in P: x^p > 0} x^p d^p \leq b \cdot \sum_{p \in P: x^p > 0} x^p = 2b.$$  

(6)

As both ends in (6) are the same, it must be

$$d^p = b, \forall p \in P \text{ with } x^p > 0.$$  

(7)

Therefore, all flow-carrying paths have a path delay of $b$. We choose an arbitrary flow-carrying path $p$. Since all solid edges have a delay of 0, the path delay of $p$ is the delay of all dashed edges that belongs to $p$. We consider the set $A_1$ that contains $a_i$ if edge $(w_{i-1}, w_i) \in p$. Clearly, it holds that $\sum_{a_i \in A_1} a_i = b$. We then define $A_2 = A \setminus A_1$. It shall be $\sum_{a_i \in A_2} = \sum_{a_i \in A} - \sum_{a_i \in A_1} = 2b - b = b$. $A_1$ and $A_2$ are thus a partition of set $A$ and meet the requirement of the partition problem. Hence, the partition problem answers “Yes”.

**Only If Part.** If the partition problem answers “Yes”, then there exists a partition $A_1$ and $A_2$ such that $\sum_{a_i \in A_1} a_i = \sum_{a_i \in A_2} a_i = b$. We now construct two paths $p_1$ and $p_2$.

- $\forall i \in [1, n]$, if $a_i \in A_1$, we put edge $(w_{i-1}, w_i)$ into path $p_1$; otherwise, we put $(w_{i-1}, v_i)$ and $(v_i, w_i)$ into path $p_1$.
- Similarly, for any $i$, if $a_i \in A_2$, we put $(w_{i-1}, w_i)$ into $p_2$; otherwise, we put $(w_{i-1}, v_i)$ and $(v_i, w_i)$ into $p_2$.

Due to the definition of a partition (see Definition [2]), $A_1$ and $A_2$ are two disjoint sets, i.e., $A_1 \cap A_2 = \emptyset$. Thus, we can easily see that $p_1$ and $p_2$ are two disjoint $s-t$ paths, i.e., $p_1$ and $p_2$ do not share any common edge. We can see that $d^{p_1} = \sum_{a_i \in A_1} b$, and $d^{p_2} = \sum_{a_i \in A_2} b$. We then construct the flow $f$ with only two flow-carrying paths $p_1$ and $p_2$ and set $x^{p_1} = x^{p_2} = 1$. Since $p_1$ and $p_2$ are disjoint, the capacity constraint is satisfied. Also, since $x^{p_1} + x^{p_2} = R = 2$, the rate requirement is satisfied. Thus $f$ is a feasible flow with maximum delay $M(f) = b$. Therefore, the decision problem of Min-Max-Delay answers “Yes”.

Since the partition problem is NP-complete [16] and the reduction can be done in polynomial time, the decision problem of Min-Max-Delay is also NP-complete.

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**B. Strongly NP-completeness for Int-Min-Max-Delay**

If we further require that any path flow should be an integer, the corresponding flow problem becomes harder. In this subsection, we show that Int-Min-Max-Delay is NP-complete in the strong sense, based on the pseudo-polynomial transformation from the 3-Partition Problem [16].

**Definition 4** (3-Partition Problem [16]). Given a set of $n = 3(k > 0)$ positive integers $A = \{a_1, a_2, ..., a_n\}$ with sum $\sum_{a_i \in A} a_i = kb$ and $b/4 < a_i < b/2$ for each element $a_i$, is there a partition $\{A_1, A_2, ..., A_k\}$ of set $A$ such that for each subset $A_j$, $\sum_{a_i \in A_j} a_i = b$?

The 3-partition problem has been proved to be NP-complete in the strong sense in [16]. We now leverage it to show our Int-Min-Max-Delay problem is NP-complete in the strong sense.

**Theorem 2.** The decision version of Int-Min-Max-Delay problem is NP-complete in the strong sense.

**Proof:** The proof is similar to the proof for Thm. [1] Given any 3-partition problem, we construct a graph $G''$ as shown in Fig. [2]. In the figure, each dashed line $(w_{i-1}, w_i)$ has a positive delay of $a_i$ while all solid lines have a delay of 0. All edges have capacity 1. Obviously the way of constructing graph $G''$ satisfies requirements (b)-(d) in the definition of pseudo-polynomial transformation which is necessary to prove strongly NP-completeness introduced in [16] Definition 4]. Now we need to prove the requirement (a), namely the 3-partition problem answers “Yes” if and only if the decision problem of Int-Min-Max-Delay answers “Yes”.
Consider the following decision version for Int-Min-Max-Delay: for graph \( G^\prime \) with source \( s = w_0 \), sink \( t = w_n \), and flow rate requirement \( R = k \), is there any feasible integer flow \( f \) with maximum delay \( M(f) \leq b \)?

**If Part.** If the decision problem of Int-Min-Max-Delay answers “Yes”, then there exists a feasible integer flow \( f \) with maximal delay \( M(f) \leq b \). For any flow-carrying path \( p \) in \( f \), the flow rate \( x_p \) must be a positive integer and since all edges have unit capacity, the flow rate \( x_p \) must be 1. For a feasible integer flow, due to \( \sum_{p \in P} x_p = R = k \), there must exist exactly \( k \) flow-carrying paths. In addition, all \( k \) flow-carrying paths are disjoint in the sense that no edge will appear in two or more flow-carrying paths, because otherwise the edge capacity constraint will be violated.

Similar to the **If Part** in the proof of Thm.1, we can show that the flow rate of all edges is 1 and all flow-carrying paths have a path delay of \( b \).

Let us index all \( k \) flow-carrying paths by \( p_1, p_2, \ldots, p_k \). We can construct \( k \) sets \( A_1, A_2, \ldots, A_k \) as follows: for \( i \in [1, n] \), since the flow rate of edge \( (w_i, w_{i-1}) \) is 1, there must exists a path \( p_i \) containing edge \( (w_i, w_{i-1}) \), and then we put \( a_i \in A_i \). Since all paths are disjoint, we can easily see that \( \{A_1, A_2, \ldots, A_k\} \) is a partition of set \( A \). In addition, since any \( p_j \) has a path delay of \( b \), we also have \( \sum_{a_i \in A_j} a_i = b \) for any set \( A_j \). Therefore, the 3-partition problem answers “Yes”.

**Only If Part.** If the 3-partition problem answers “Yes”, then there exists a partition \( \{A_1, A_2, \ldots, A_k\} \) of set \( A \) such that \( \sum_{a_i \in A_j} a_i = b \) for any \( j \in [1, k] \). Similar to the **Only If Part** in the proof of Thm.1, we can show that the decision problem of Int-Min-Max-Delay also answers “Yes”.

As the 3-partition problem is strongly NPC-complete and our reduction is a pseudo-polynomial transformation, the decision version of Int-Min-Max-Delay is also strongly NPC-complete.

### IV. Optimal Pseudo-polynomial Time Algorithm

This section describes a pseudo-polynomial time algorithm which can find the optimal solution to the Min-Max-Delay problem. The algorithm runs in pseudo-polynomial time in the sense that corresponding time complexity in theory is polynomial w.r.t. the numerical value of the largest edge delay.

#### A. Related work

A closely-related problem to Min-Max-Delay is the Delay-Constrained maximum flow problem [12], denoted as DC-Max-Flow: for the same graph \( G \) and a given delay constraint \( T \), find the maximum flow (rate) such that the delay of any flow-carrying path does not exceed \( T \). Let us denote \( P^T \) as the set of all \( s \rightarrow t \) paths whose path delay does not exceed \( T \). Then DC-Max-Flow can be formulated as

\[
\begin{align*}
\text{max} & \quad \sum_{p \in P^T} x^p, \\
\text{s.t.} & \quad x_e = \sum_{p \in P^T, e \in p} x^p \leq c_e, \\
\text{vars.} & \quad x^p \geq 0, \forall p \in P^T.
\end{align*}
\]

Since the size of set \( P^T \) increases exponentially with respect to the network size, formulation [8] could have exponential number of variables. However, [12] shows that DC-Max-Flow can be solved with an edge-based flow formulation which has at most \( |E|T \) variables. They implicitly use the idea of expanding graph in the time range [13] by converting a delay-constrained max-flow problem in the original graph into a delay-unconstrained max-flow problem over the expanded graph. In [12], the authors only consider unit-delay edges, but it is straightforward to generalize their results to integer-delay edges. Due to the space limit, we omit the procedure of constructing the expanded graph and directly give the equivalent edge-based flow formulation for DC-Max-Flow as follows [12] Proposition 1.

\[
\begin{align*}
\text{max} & \quad \sum_{e \in \ln(t)} \sum_{d=0}^{T} x_{e}^{(d)} \\
\text{s.t.} & \quad \sum_{e \in \ln(v)} x_{e}^{(d)} = \sum_{e \in \text{Out}(v)} x_{e}^{(d+\text{d}_e)}, \\
& \quad \forall v \in V \setminus \{s, t\}, d \in [0, T] \\
& \quad \sum_{d=0}^{T} x_{e}^{(d)} \leq c_e, \quad \forall e \in E \\
& \quad \text{vars.} \quad x_{e}^{(d)} \geq 0, \forall e \in E, d \in [0, T]
\end{align*}
\]

where \( \ln(v) \) is the set of all incoming edges of node \( v \). Similarly, \( \text{Out}(v) \) is the set of all outgoing edges of node \( v \). \( x_{e}^{(d)} \) is the total flow rate that arrives at the sink \( t \) within the delay bound \( T \). Eqs. (9b) are the flow conservation constraints in the expanded graph. Note that by convention, for any edge \( e \in E \), we set \( x_{e}^{(d)} = 0 \) for \( d < 0 \) and \( d > T \), and for any edge \( e \in \text{Out}(s) \), we set \( x_{e}^{(d)} = 0 \) for all \( d \neq d_e \). Eqs. (9c) are the edge capacity constraints.

#### B. Proposed binary-search algorithm

We now show the relationship between our problem Min-Max-Delay and the problem DC-Max-Flow. For a graph \( G \), denote \( d^*(R) \) as the minimal maximum delay with rate requirement \( R \) (the optimal value of Min-Max-Delay). For the same graph \( G \) and a non-negative integer \( T \), denote \( r^*(T) \) as the maximum flow subject to a delay constraint \( T \) (the optimal value of DC-Max-Flow). We have the following lemma.

**Lemma 1.** \( d^*(R) \leq T \) if and only if \( r^*(T) \geq R \).

**Proof:** **If Part.** If \( r^*(T) \geq R \), then there exists a flow solution over \( P^T \), i.e., \( \tilde{f} = \{x^p : p \in P^T\} \) such that \( \sum_{p \in P^T} x^p \geq R \). We can thus decrease the flow solution \( f \) to construct another flow solution \( \hat{f} \) such that \( \sum_{p \in P^T} \hat{x}^p = R \). Since \( f \) satisfies the capacity constraints, \( \hat{f} \) must also satisfy the capacity constraints. Thus, \( \hat{f} \) is a feasible solution to Min-Max-Delay with rate requirement \( R \). In addition, since all flow-carrying paths in \( f \) belong to the set \( P^T \), we have \( d^*(R) \leq \hat{D}(\hat{f}) \leq T \).
**Only If Part.** If \(d^*(R) \leq T\), then there exists a flow solution \(f\) where the path delay of any flow-carrying path does not exceed \(T\). Thus all flow-carrying paths belong to \(p^T\) and \(f\) is also a feasible solution to DC-Max-Flow with a delay bound \(T\). Thus, \(r^*(T) \geq \sum_{p \in p^T} x_p^p = R\).

Lemma 1 suggests a simple binary-search algorithm to solve Min-Max-Delay optimally. Given a lower bound \(T_l\) (\(= 0\) initially) and an upper bound \(T_u\) (\(= |E|d_{\text{max}}\) initially) of the optimal maximum delay, in each iteration we solve the problem (9) with \(T = [(T_l + T_u)/2]\). We compare the optimal value of (9), i.e., \(r^*(T)\) with the rate requirement \(R\). If \(r^*(T) \geq R\), we update the upper bound as \(T_u = T\). Otherwise, if \(r^*(T) < R\), we update the lower bound as \(T_l = T + 1\). The algorithm terminates when \(T_l \geq T_u\). Lemma 1 clearly show that such a binary-search algorithm finds the optimal solution of Min-Max-Delay. We then present its time complexity.

**Theorem 3.** The binary search has a pseudo-polynomial time complexity of \(O((\log(Nd_{\text{max}})(N^3d_{\text{max}})^2)(\log R + N^2d_{\text{max}}\log(N^2d_{\text{max}})))\).

**Proof:** Our binary search scheme terminates in \(O(\log T)\) iterations where \(T = |E|d_{\text{max}}\) is the initial upper bound for the minimal maximum delay. In each iteration, we need to solve the linear program (9). Obviously the number of variables in (9) is \(O(|E|T)\). The conservation constraint (9b) is formulated for \(v \in V \setminus \{s,t\}\) and \(d \in [0,T]\). Thus the number of conservation constraint is \(O(|V|T)\). Clearly the number of capacity constraint (9c) is \(|E|\). Overall, the number of constraints in formulation (9) is \(O(NT)\).

[19] has proposed an algorithm to solve a linear program within time \(O((g + h)^{1.5}hL)\) where \(g\) is the number of constraints, \(h\) is the number of variables, and \(L\) denotes the standard “bit complexity” of the linear program. Therefore, it takes \(O((NT)^2.5(\log R + N T \log(NT)))\) time to solve formulation (9) based on the calculation of \(L\) and the Hadamard’s inequality. Since \(T \leq |E|d_{\text{max}}\), the overall time complexity is \(O((\log(Nd_{\text{max}})(N^3d_{\text{max}})^2)(\log R + N^2d_{\text{max}}\log(N^2d_{\text{max}})))\). A direct result of Thm. 3 as follows.

**Theorem 4.** Min-Max-Delay is weakly NP-complete.

**Proof:** It follows from Thm. 1 and Thm. 5.

Note that the linear program (9) returns an edge-based flow with the optimal maximum delay \(T^*\). Thus, we should do flow decomposition on the expanded graph with \(T^*\) to be the end-to-end delay bound in order to get a path-based flow. The obtained path-based flow must have the same maximum delay of \(T^*\) since the delay for any \(s-t\) path is bounded above by \(T^*\) in the expanded graph. The number of nodes and edges in the expanded graph are \(O(NT^*)\). And since the flow decomposition in the graph \(\hat{G}(V,E)\) has a time complexity of \(O(|E|(|V| + |E|))\), the time complexity to get the path-based flow by decomposition in this work will be \(O(N^4d_{\text{max}}^2)\), which does not affect the overall time complexity to solve Min-Max-Delay according to Thm. 3.

**V. Fully Polynomial Time Approximation Scheme**

This section proposes a FPTAS to solve Min-Max-Delay approximately with a \((1 + \epsilon)\) approximation ratio guaranteed and a fully polynomial time complexity.

**A. Related work**

[13] has proposed the system-optimal flow which minimizes the total delay to be a \(\gamma(L)\)-approximate solution for the general Min-Max-Delay problem where edge delay is a non-negative, non-decreasing convex function w.r.t the assigned flow rate and no capacity constraints are involved in the problem. For the constant edge delay function, calculated \(\gamma(L)\) is 1, leading to an algorithm (the algorithm that calculates the system-optimal flow) with the optimal solution guaranteed. However, it is a totally different story when edge capacity constraints are involved which is the problem setting in this paper.

In fact, with capacity constraints involved, the maximum delay gap between the system-optimal flow and the Min-Max-Delay flow can be infinitely large. Use Fig. 4 as an example. Suppose \(R = (n - 1)/(n - 2) > 1\). In this example, the maximum delay of the Min-Max-Delay flow is 1 according to Lema. 7. For the system-optimal flow, it will place 1 flow rate on each lower edge and 1/(n - 2) flow rate on each upper edge. Then the system-optimal flow that only uses two paths, one containing all upper dashed edges and one containing all lower solid edges, will have a maximum delay of \(n - 1\). Thus, the maximum delay gap between the system-optimal flow and the Min-Max-Delay flow in this example is \(n - 1\) and can be infinitely large as \(n\) increases.

Overall, existed approximate algorithm is not applicable for the Min-Max-Delay problem introduced in this paper.

**B. Proposed FPTAS**

First, given a delay \(T\), we give an algorithm Test(\(\cdot\)) which must return a feasible flow if \(T \geq M(f_{MM}(R))\) within a fully polynomial time.

**Algorithm 1 Test(G, s, t, R, T, \epsilon)**

1: **input:** \(G = (V, E)\), \(R\), \(s\), \(t\), \(T\), \(\epsilon\)
2: **output:** \(f_{A}(R)\)
3: **procedure**
4: \(\Delta_T = T\epsilon / |E|\)
5: \(\hat{d}_e = [d_e / \Delta_T], \forall e \in E\)
6: \(\bar{T} = \lceil T / \Delta_T \rceil + |E|\)
7: \(\hat{G}(V,E)\) \(= (\tilde{G}(V,E)\) with each \(d_e\) replaced by \(\hat{d}_e\)
8: \(f_{A}(R)\) = Binary\_Search(G, s, t, R, \bar{T})
9: **return** \(f_{A}(R)\)
10: **end procedure**

Proposed Test(\(\cdot\)) is shown in Algorithm 1. First each edge delay is updated (line 5) to be the multiple of appropriately defined \(\Delta_T\) (line 4) to guarantee a fully polynomial time complexity of the algorithm. Then we can define a new network \(\hat{G}\) as the network \(G\) with each edge delay \(d_e\) replaced by the updated edge delay \(\hat{d}_e\). Besides, \(T\) is also updated to \(\bar{T}\) based on \(\Delta_T\) to make sure that \(\bar{T}\) is the upper bound for the maximum delay of \(f_{MM}(R)\) in the new network \(\hat{G}\) if \(T\) is the upper bound for the maximum delay of \(f_{MM}(R)\) in \(G\) (line 6). Finally, the expected solution flow \(f_{A}(R)\) is
calculated by solving the Min-Max-Delay problem using the same binary search algorithm introduced in Sec. [IV-B] on the newly defined network G.

With Lem.2 Algorithm 1 can be proved to return a feasible flow if $M(f_{MM}(R)) \leq T$ within fully polynomial time.

**Lemma 2.** For a feasible flow $f$, we use $M_G(f)$ to denote the maximum delay of $f$ in the network $G$. Without loss of generality, $M(f)$ is same to $M_G(f)$ since $G$ is the input network to our Min-Max-Delay problem. It is true that $M_G(f) \leq \Delta_T \cdot M_G(f)$ where $G$ and $\Delta_T$ is defined in Algorithm 7.

**Proof.** Suppose $P_f$ is the flow-carrying path set of $f$. Since the only difference between $G$ and $f$ is the edge delay, $P_f$ is same for the flow $f$ in $G$ and in $G$.

$$M_G(f) = \max_{p \in P_f} \sum_{e \in p} d_e \leq \max_{p \in P_f} \sum_{e \in p} [\Delta_T[d_e/\Delta_T]]$$

$$= \Delta_T \cdot \max_{p \in P_f} \sum_{e \in p} d_e = \Delta_T \cdot M_G(f)$$

**Lemma 3.** If in Algorithm 7 we have $T \geq M(f_{MM}(R))$, then Algorithm 7 must return a feasible flow.

**Proof.** Now we prove if $M(f_{MM}(R)) \leq T$, then $M_G(f_{MM}(R)) \leq T$. Proof is shown below:

$$M_G[f_{MM}(R)] = \max_{p \in P_{f_{MM}(R)}} \sum_{e \in p} [d_e/\Delta_T]$$

$$\leq \max_{p \in P_{f_{MM}(R)}} \sum_{e \in p} (d_e/\Delta_T + 1)$$

$$\leq \sum_{p \in P_{f_{MM}(R)}} \sum_{e \in p} (d_e/\Delta_T) + |E|$$

$$= \frac{1}{\Delta_T} \cdot \max_{p \in P_{f_{MM}(R)}} \sum_{e \in p} d_e + |E|$$

$$= \frac{1}{\Delta_T} \cdot M_G[f_{MM}(R)] + |E| \leq \Delta_T \cdot M_G[f_{MM}(R)] + |E| \leq T$$

**Algorithm 2 FPTAS to solve Min-Max-Delay**

1. input: $G = (V, E)$, $R, s, t, \epsilon$
2. output: $f_\epsilon(R)$
3. procedure
   4. $T_U = |E| \cdot d_{max}$, $T_L = 0$, $T^* = T_U$
   5. while $T_U \geq T_L$
      6. $T = [(T_U + T_L)/2]$, $f = \text{NULL}$
      7. $f = \text{Test}(G, s, t, R, T)$
      8. if $f = \text{NULL}$ then
         9. $T_L = T + 1$
      10. else
          11. $f_\epsilon(R) = f$, $T^* = T$, $T_U = T - 1$
      12. end if
   13. end while
   14. return $f_\epsilon(R)$
5. end procedure

Algorithm 2 is our proposed FPTAS to solve Min-Max-Delay, which is a simple binary search procedure on the Test() described in Algorithm 1. Note that we use $T^*$ (line 11) to denote the input $T$ to get $f_\epsilon(R)$ using Test().

**Lemma 5.** In the end of Algorithm 2 it holds that $T^* \leq M(f_{MM}(R))$.

**Proof.** Obviously initially $T_U = |E| \cdot d_{max}$ is an upper bound for $M(f_{MM}(R))$ since the number of edges on any flow-carrying paths can be assumed to be bounded above by $|E|$ in $f_{MM}(R)$ without loss of generality.

It is clear that if we can prove Test$(G, s, t, R, T, \epsilon)$ must return a feasible flow for any $T \geq M(f_{MM}(R))$, Lem.5 will hold since Algorithm 2 is a binary search procedure on Test().

According to Lem.3, obviously it is true that a feasible flow must be returned for any $T \geq M(f_{MM}(R))$. Therefore, Lem.5 is correct.

**Theorem 5.** Algorithm 2 is a $(1 + \epsilon)$-approximate algorithm for the Min-Max-Delay problem, namely

$$M[f_\epsilon(R)] \leq (1 + \epsilon)M[f_{MM}(R)]$$

**Proof.**

$$M_G[f_\epsilon(R)] \leq \Delta_T \cdot M_G(f_\epsilon(R)) \leq \Delta_T \cdot M_G[f_{MM}(R)]$$

$$= \Delta_T \cdot \max_{p \in P_{f_{MM}(R)}} \sum_{e \in p} [d_e/\Delta_T]$$

$$\leq \max_{p \in P_{f_{MM}(R)}} \sum_{e \in p} (d_e/\Delta_T + 1)$$

$$\leq \max_{p \in P_{f_{MM}(R)}} \sum_{e \in p} (d_e/\Delta_T) + \Delta_T \cdot |E|$$

$$= M_G[f_{MM}(R)] + \Delta_T \cdot |E|$$

$$= M_G[f_{MM}(R)] + \epsilon T^* \leq (1 + \epsilon)M_G[f_{MM}(R)]$$
Inequality $(a)$ holds due to Lem. 2. Inequality $(b)$ is true because $f_{\epsilon}(R)$ is the Min-Max-Delay flow which minimizes the maximum delay in $G$. Inequality $(c)$ is true due to Lem. 5.

**Theorem 6.** Algorithm 2 has a fully polynomial time complexity of $O(L + \log N)\log(M)(NM^{2.5}(\log R + NM \log(NM)))$ where $M = |E|(1 + 1/\epsilon) + 1$ and $L$ is the bit complexity for representing $d_{\text{max}}$.

**Proof.** Assume we need $L$ bits to represent $d_{\text{max}}$ in the computer, namely $d_{\text{max}} = O(2^L)$. According to Lem. 4 in each iteration of Algorithm 2, corresponding time complexity is $O(\log(M)(NM^{2.5}(\log R + NM \log(NM))))$. Due to the binary search procedure, we need at most $O(\log(|E|d_{\text{max}}))$ iterations. Therefore, overall we can get the proposed time complexity.

VI. FRACTIONAL FLOW VS. INTEGER FLOW

We have shown that Min-Max-Delay, which allows fractional path flows, is NP-complete in the weak sense, but Int-Min-Max-Delay, which only allows integer path flows, is NP-complete in the strong sense. This section further presents another key difference: the optimum of Min-Max-Delay, namely the maximum delay of the Min-Max-Delay flow, may be better than the optimum of Int-Min-Max-Delay, namely the maximum delay of the Int-Min-Max-Delay flow.

A famous integral flow theorem holds for the classic max-flow problem: there must exist an optimal integral flow if each edge has integral capacity [10]. However, when edge delay is involved, Min-Max-Delay does not have similar integral flow theorems in spite of the integral capacity, delay, and flow rate requirement, namely the Min-Max-Delay flow must be fractional in some problem instances. Here in this paper we define the Int-Gap as the maximum delay ratio between the Int-Min-Max-Delay flow and Min-Max-Delay flow which maybe fractional.

A. A simple example with Int-Gap $= 9/8$

In Fig. 3 we use the nice example from [12, Fig. 2] as a building block. We place two such blocks ($v$-related block and $w$-related block) side by side disjointly and then connect them to a source $s$ and a sink $t$. The capacity of edges $(s, v_1), (s, w_1), (v_6, t)$ as well as $(w_6, t)$ is 2 and the capacity of all other edges is 1. We set the delay to be 1 for all edges and the rate requirement to be 3.

For the subnetwork from $v_1$ to $v_6$ (and its symmetric subnetwork from $w_1$ to $w_6$), there are totally 5 different paths (shown in Tab. II). For the DC-Max-Flow problem shown in [12], given a maximum delay constraint of 6, the subnetwork has a fractional maximum flow of 1.5: $x^D = 0.5$, $x^W = 0.5$. While the integer maximum flow is 1 due to capacity constraint. Since in Fig. 3 the symmetric $v$-related subnetwork and $w$-related subnetwork are placed side by side, given $R = 3$ in our Min-Max-Delay problem, the optimal flow will route 1.5 flow rate to each of the subnetwork, resulting in a maximum delay of 8. However, for Int-Min-Max-Delay, one of the subnetworks shall have a flow rate of 2, and the corresponding maximum delay is 9. Therefore, the optimal fractional flow solution could be better (i.e., with smaller maximum delay) than the optimal integral solution.

B. How large can the Int-Gap be?

Previous section introduces a simple example with Int-Gap of 9/8. An interesting open question is what is the largest Int-Gap in one Min-Max-Delay problem instance? This section proposes a way to construct a Min-Max-Delay problem instance with infinite large Int-Gap.

There are two useful lemmas for the network shown in Fig. 4 where source is $a_1$ and sink is $a_n$.

**Lemma 6.** The Int-Min-Max-Delay flow in Fig. 4 given $R = 2$ has a maximum delay of $\left\lfloor \frac{n-1}{2} \right\rfloor$.

**Proof.** Because of the unit capacity constraint for each edge, any flow-carrying path will be assigned a unit flow rate in the integer flow from $a_1$ to $a_n$.

Because of the rate requirement $R = 2$, there are exactly two flow-carrying paths in the Int-Min-Max-Delay flow and each of them carries a unit flow rate.

Moreover, because of the capacity constraint, the two flow-carrying paths are disjoint, namely they share no edges.

Therefore, since the Int-Min-Max-Delay flow minimizes the maximum flow-carrying path delay among all integer flows, the two flow-carrying paths will have path delay of $\left\lfloor \frac{n-1}{2} \right\rfloor$ and $\left\lfloor \frac{n-1}{2} \right\rfloor$ respectively in the Int-Min-Max-Delay flow, leading to a maximum delay of $\left\lfloor \frac{n-1}{2} \right\rfloor$. 

---

**TABLE II**

| Path label | Path | Path Delay |
|------------|------|------------|
| A          | (v_1, v_7, v_6, v_4, v_9, v_10, v_6) | 7       |
| B          | (v_1, v_7, v_6, v_4, v_9, v_10, v_6) | 5       |
| C          | (v_1, v_2, v_3, v_4, v_5, v_6)      | 5       |
| D          | (v_1, v_2, v_3, v_4, v_5, v_6)      | 6       |
| E          | (v_1, v_2, v_3, v_4, v_5, v_6)      | 5       |
Lemma 7. The Min-Max-Delay flow in Fig. 4 given $R = \frac{n-1}{n-2}$ has a maximum delay of 1.

Proof. First it is straightforward that the Min-Max-Delay flow in Fig. 4 given $R = \frac{n-1}{n-2}$ has a maximum delay no smaller than 1 since $R > 1$ and there always exist some fractional flow rate using upper dashed edges.

Next, we will explicitly construct a feasible flow in the network with unit maximum delay to prove Lem. 7. Note that there are $(n-1)$ different $a_1 - a_n$ paths containing exactly one dashed edge. We then place $1/(n-2)$ flow rate on each of these $(n-1)$ paths with the rate requirement $R = (n-1)/(n-2)$ satisfied. The assigned flow rate on each upper dashed edge is $1/(n-2)$ which is no larger than the unit capacity and the assigned flow rate on each lower solid edge is $(n-2) \times 1/(n-2) = 1$ which is no larger than the unit capacity. Overall the constructed flow is feasible. Since each path only contains one dashed edge, the maximum delay of the constructed flow is 1.

Now we use Fig. 4 to construct a more complex network using infinite large Int-Gap $R = n-1$. There are totally $(n-2)$ network $G^*$'s. Each $s - G^*$ and $G^* - t$ edge has a capacity of 2 and a delay of 0. $G^*$ is the same network shown in Fig. 4 $n$ is assumed to be odd.

Lemma 8. There exists a Min-Max-Delay problem instance where Int-Gap is infinite large.

Proof. Lem. 8 is proved using the network in Fig. 5.

First, we prove that the Min-Max-Delay flow has a maximum delay of 1.

It is straightforward that the Min-Max-Delay flow has a maximum delay no smaller than 1 since the feasible flow with a maximum delay of 0 will have a maximum flow rate of $(n-2)$ which do not meet the rate requirement $R = n-1$.

Now we can route $(n-1)/(n-2)$ flow rate to each of the $(n-2)$ network $G^*$'s with the rate requirement satisfied. According to Lem. 7 the minimal maximum delay for $(n-1)/(n-2)$ flow rate to pass $G^*$ is 1. Thus in this way we have found a feasible flow with the maximum delay of 1 in Fig. 5 and it is the Min-Max-Delay flow.

Next we prove that the Int-Max-Max-Delay flow has a maximum delay of $\lceil (n-1)/2 \rceil$.

Since the rate requirement $R = n-1$ is strictly larger than $n-2$ which is the number of $G^*$'s, for any feasible integer flow, there will be at least one $G^*$ who is assigned a flow rate of 2. According to Lem. 6 corresponding minimal maximum delay to pass the $G^*$ is $\lceil (n-1)/2 \rceil$. Thus, the Int-Max-Max-Delay flow will have a maximum delay of $\lceil (n-1)/2 \rceil$.

Overall, the Int-Gap in Fig. 5 is $\lceil (n-1)/2 \rceil$. Since $n$ can be any positive large integer, the gap is infinite large.

C. Bi-criteria upper bounding the Int-Gap

With the help of a new defined flow $f_{SO}^\epsilon$, which deletes a certain amount of flow rate from the slowest flow-carrying paths in the integer system-optimal flow which is a special case of the classic min-cost flow [1]. this section presents a bi-criteria bound $(1 - \epsilon, 1/\epsilon)$ for the Int-Gap and gives corresponding tightness analysis.

We note the integrality flow theorem holds for min-cost flow as shown in [1]. As a special case, the system-optimal flow defined in this work also has such an integrality property:

Theorem 7 (Thm. 9.10 [1]). If all edge capacities, edge delays and rate requirement are integer, the system-optimal flow problem always has an integer system-optimal flow.

Thm. 7 also means if all problem inputs are integers, the total delay of $f_{SO}(R)$ and $f_{SO}^\epsilon(R)$ are equal.

Now we define $f_{SO}^\epsilon$ using Algorithm 3.

Algorithm 3 Finding $f_{SO}^\epsilon((1-\epsilon)R)$

1: input: $G = (V, E), R, s, t, \epsilon \in (0, 1)$
2: output: $f_{SO}^\epsilon((1-\epsilon)R)$
3: procedure
4: $f_{SO}^\epsilon(R) =$ Integer-System-Optimal-Flow($G, R, s, t$)
5: $x_{\text{delete}} \epsilon R \quad //the rate to be deleted$
6: while $x_{\text{delete}} > 0$ do
7: Find the slowest flow-carrying path $p_l$
8: if $x_{p_l} > x_{\text{delete}}$ then
9: $x_{p_l} = x_{p_l} - x_{\text{delete}}$
10: $x_{\text{delete}} = 0$
11: else
12: $x_{\text{delete}} = x_{\text{delete}} - x_{p_l}$
13: $x_{p_l} = 0$
14: end if
15: end while
16: $f_{SO}^\epsilon((1-\epsilon)R)$ is defined by the updated flow
17: return $f_{SO}^\epsilon((1-\epsilon)R)$
18: end procedure

First, we obtain a path-based integer system-optimal flow $f_{SO}^\epsilon(R)$ given rate requirement $R$ (Line 4). Then, the algorithm deletes $\epsilon R$ rate iteratively from $f_{SO}^\epsilon(R)$ (Lines 6–15).

In each iteration, we find the slowest flow-carrying path $p_i$, i.e., the path that has the maximum path delay among all paths with a positive flow rate, and then delete the right amount of rate
from it. Specifically, in the case that \( p_t \) carries enough flow rate for deletion (Line 7), then its rate is deleted for as much as needed, and the iteration terminates. Otherwise, the rate of \( p_t \) is all deleted and the iteration continues. The iteration terminates when \( \epsilon R \) rate is deleted in total, and the resulting flow is assigned to \( f_{SO^+}(1-\epsilon)R \).

**Lemma 9.** Following Algorithm 3 for \( f_{SO^+}(1-\epsilon)R \), we have:

\[
T[f_{SO^+}(1-\epsilon)R] + \epsilon R \cdot M[f_{SO^+}(1-\epsilon)R] \leq T(f_{SO^+}(R)).
\]

**Proof.** According to Algorithm 3, \( f_{SO^+}(1-\epsilon)R \) is obtained by iteratively deleting \( \epsilon R \) flow rate from \( f_{SO^+}(R) \). Suppose that there are in total \( K \) iterations to get \( f_{SO^+}(1-\epsilon)R \). We use \( f_k \) to represent the resulting flow at the beginning of the \( k \)-th iteration (or equivalently, at the end of the \( (k-1) \)-th iteration) for \( 1 \leq k \leq K + 1 \). Obviously, \( f_1 = f_{SO^+}(R) \), \( f_{K+1} = f_{SO^+}(1-\epsilon)R \).

We denote \( P_k \) as the set of flow-carrying paths in flow \( f_k \), and \( p_k \in P_k \) as the slowest flow-carrying path in \( f_k \), i.e., the flow with the maximum delay among all flow-carrying paths. According to Algorithm 3 in the \( k \)-th iteration in the while loop, we delete some flow rate, say \( x_k > 0 \), from \( p_k \). The flow after deletion is \( f_{k+1} \).

For the maximum delay, since all edge delays are non-negative integers, the path delay of all flow-carrying paths cannot increase with reduced flow rate. Thus, the maximum delay cannot increase, i.e.,

\[
M(f_{k+1}) \leq M(f_k),
\]

implying that

\[
M(f_{SO^+}(1-\epsilon)R) \leq M(f_{SO^+}(R)).
\]

Considering the total delay, for any \( k \), we have

\[
T(f_k) = \sum_{e \in P_k} [x_e d_e] + \sum_{e \in p_k} [x_e d_e]
= \sum_{e \in P_k \setminus p_k} [x_e d_e] + \sum_{e \in p_k} [x_e + x_k d_e] + x_k \sum_{e \in p_k} d_e
\]

\[
= \sum_{e \in P_k \setminus p_k} [x_e d_e] + \sum_{e \in P_k \setminus p_k} [(x_e - x_k) d_e] + x_k M(f_k) \]

\[
\geq T(f_{k+1}) + x_k M(f_{SO^+}(1-\epsilon)R)).
\]

In (13), equality (a) holds because \( \sum_{e \in p_k \setminus p_k} d_e \) is the path delay of the slowest-carrying path \( p_k \), i.e., the maximum delay \( M(f_k) \) of \( f_k \). Equality (b) holds because flow \( f_{k+1} \) is the resulting flow when flow \( f_k \) deletes \( x_k \) rate in path \( p_k \). Inequality (c) comes from (11) and \( f_{K+1} = f_{SO^+}(1-\epsilon)R \). We then do summation for (13) over \( k \in [1, K] \), and get

\[
T[f_{SO^+}(R)] = T(f_1)
\geq T(f_{K+1}) + \sum_{k=1}^{K} x_k \cdot M[f_{SO^+}(1-\epsilon)R]
= T[f_{SO^+}(1-\epsilon)R] + \epsilon R \cdot M[f_{SO^+}(1-\epsilon)R],
\]

which proves (10).

**Lemma 10.** Compared with the Min-Max-Delay flow given rate requirement \( R \), denoted by \( f_{MM}(R) \), the maximum delay of \( f_{SO^+}(1-\epsilon)R \) is upper bounded by a ratio of \( 1/\epsilon \):

\[
M(f_{SO^+}(1-\epsilon)R) \leq M(f_{MM}(R))/\epsilon
\]

**Proof.** Considering the definition of \( f_{SO^+}(R) \) and Thm. 7 we have

\[
T(f_{SO^+}(R)) = T(f_{SO^+}(R)) \leq T(f_{MM}(R)) \leq R \cdot M(f_{MM}(R)).
\]

Leveraging inequality (10) in Lem. 9 it is

\[
M(f_{SO^+}(1-\epsilon)R) \leq \frac{T(f_{SO^+}(R)) - T(f_{SO^+}(1-\epsilon)R)}{\epsilon R} \leq \frac{R \cdot M(f_{MM}(R))}{\epsilon} = \frac{M(f_{MM}(R))}{\epsilon}.
\]

Here inequality (a) follows (15).

Now we give the bi-criteria bound \((1-\epsilon, 1/\epsilon)\) for the Int-Gap of our Min-Max-Delay problem:

**Theorem 8.** Considering the Min-Max-Delay problem where all edge capacities, edge delays and rate requirement are non-negative integer, the Int-Gap is upper bounded by a bi-criteria ratio of \((1-\epsilon, 1/\epsilon)\), namely:

\[
|f_{MM+}^*[1-\epsilon, 1/\epsilon]| = (1-\epsilon) \cdot |f_{MM}(R)|,
\]

\[
M(f_{MM+}^*[1-\epsilon, 1/\epsilon]) \leq M(f_{MM}(R))/\epsilon
\]

where \( f_{MM+}^*[1-\epsilon, 1/\epsilon] \) is the Int-Max-Max-Delay flow given rate requirement \((1-\epsilon)R\), \( f_{MM}(R) \) is the Min-Max-Delay flow given rate requirement \( R \) and \( \epsilon \in (0, 1), R \in Z^+ \).

Moreover, the bound \((1-\epsilon, 1/\epsilon)\) is near tight in the sense that there exists a problem instance with a bi-criteria Int-Gap of at least \((1-\epsilon, (1/\epsilon) - 1)/2)\).

**Proof.** Inequality (17) is obvious:

\[
|f_{MM+}^*[1-\epsilon, 1/\epsilon]| = (1-\epsilon)R \geq (1-\epsilon) \cdot |f_{MM}(R)|
\]

Now we prove the inequality (18).

Due to that \( f_{MM+}^*[1-\epsilon, 1/\epsilon] \) is the optimal solution for the problem Int-Max-Max-Delay given rate requirement \((1-\epsilon)R\), \( f_{MM+}^*[1-\epsilon, 1/\epsilon] \) should have the minimal maximum delay among all feasible integer flows with rate requirement \((1-\epsilon)R\). Thus we have

\[
M(f_{MM+}^*[1-\epsilon, 1/\epsilon]) \leq M(f_{SO^+}(1-\epsilon)R)
\]

Now according to Lem. 10 we can prove the inequality (18):

\[
M(f_{MM+}^*[1-\epsilon, 1/\epsilon]) \leq M(f_{SO^+}(1-\epsilon)R) \leq M(f_{MM}(R))/\epsilon
\]

Next we prove the near-tightness.

Assume that \( |1/\epsilon| - 1 = n - 1 \). Consider Fig. 3 as an example where there are totally \( k(n-2) \) G’s. Suppose \( R = k(n-1) \). \( k \) is a non-negative integer which satisfies the requirement: \((1-\epsilon)R = k(1-\epsilon)(n-1) \in Z^+\).

For \( f_{MM}(R) \), we can route \((n-1)/(n-2)\) rate to each of the \( G^* \), leading to a maximum delay of 1 which has been proved in Lem. 7.
For \( f_{\text{MM}}^*[(1 - \epsilon)R] \), since \( 1/\epsilon > [1/\epsilon] - 1 = n - 1 \) which implies that \( \epsilon < 1/(n - 1) \), we have

\[
(1 - \epsilon)R > k \left( 1 - \frac{1}{n - 1} \right)(n - 1) = k(n - 2)
\] (19)

Because we have assumed the number of \( G^* \)s in the example is \( k(n - 2) \), considering the inequality (19), we have to route a rate of 2 using at least one \( G^* \), leading to a maximum delay of \( [n - 1]/2 \) which has been proved in Lem. 6.

Overall in the example we have

\[
\text{Int-Gap} = \left\lceil \frac{n - 1}{2} \right\rceil \geq \frac{n - 1}{2} = \frac{1}{2} \left[ 1/\epsilon \right] - 1
\]

which proves the near-tightness.

VII. REMARKS ON MULTI-COMMODITY FLOW PROBLEM

This section discusses how our previous results can be applied to multi-commodity case.

A. Multi-commodity flow problem definitions

Instead of a single source-sink pair, in the multi-commodity flow problem there are \( k \geq 1 \) source-sink pairs: \( s_i - t_i, 1 \leq i \leq k \). We denote \( P_i \) as the path set from \( s_i \) to \( t_i \) and \( P = \bigcup_{i=1}^{k} P_i \). \( R_i \) is the rate requirement for \( s_i - t_i \).

Similar to the maximum delay definition for a multi-commodity flow in [13], in this work, the maximum delay \( \mathcal{M}(\cdot) \) for the multi-commodity flow \( f \) is defined:

\[
\mathcal{M}(f) \triangleq \max_{p \in P, \lambda^p \geq 0} d_p^f
\] (20)

The Multi-Min-Max-Delay problem has the following formulation:

\[
\begin{align*}
\min_{x^p} & \quad M \\
\text{s.t.} & \quad \sum_{p \in P_i} x^p = R_i, \quad 1 \leq i \leq k, \\
& \quad x^p \leq c_e, \quad \forall e \in E, \\
& \quad x^p (d_p^f - M) \leq 0, \quad \forall p \in P, \\
\text{vars.} & \quad x^p \geq 0, \quad \forall p \in P.
\end{align*}
\] (21)

Multi-Min-Max-Delay formulation (21) is similar to the Min-Max-Delay formulation (2) except that rate requirement (21b) is imposed on each source-sink pair and assigned flow rate on each edge for all source-sink pairs should jointly respect the edge capacity constraint (21c).

Similar to the multi-commodity system-optimal flow problem in [13], in this work the multi-commodity system-optimal flow problem which tries to find a feasible flow with minimal total delay has the following formulation:

\[
\begin{align*}
\min_{x^p} & \quad \sum_{e \in E} d_e x_e \\
\text{s.t.} & \quad \sum_{p \in P_i} x^p = R_i, \quad 1 \leq i \leq k, \\
& \quad x^p \leq c_e, \quad \forall e \in E, \\
\text{vars.} & \quad x^p \geq 0, \quad \forall p \in P.
\end{align*}
\] (22)

Note that the multi-commodity system-optimal flow problem is a special case of the classic min-cost multi-commodity flow problem.

B. Optimal pseudo-polynomial time algorithm and hardness

Given a maximum delay bound \( T \), we can find whether there exists a feasible multi-commodity flow by solving a similar linear program as formulation (9):

\[
\begin{align*}
\text{max} \quad & \quad 1 \\
\text{s.t.} \quad & \quad \sum_{e \in \text{In}(s_i)} x^{(d)}_{e,i} = R_i, 1 \leq i \leq k \\
& \quad \sum_{e \in \text{Out}(v)} x^{(d)}_{e,i} = \sum_{e \in \text{Out}(v)} x^{(d+d_e)}_{e,i}, \\
& \quad \forall v \in V \setminus \{ s_i, t_i \}, d \in [0, T], 1 \leq i \leq k \\
& \quad \sum_{d=0}^{k} \sum_{i=1}^{k} x^{(d)}_{e,i} \leq c_e, \quad \forall e \in E \\
\text{vars.} \quad & \quad x^{(d)}_{e,i} \geq 0, \quad \forall e \in E, d \in [0, T], 1 \leq i \leq k
\end{align*}
\] (23)

where \( x^{(d)}_{e,i} \) is the total flow rate that experiences a delay of \( d \) after passing edge \( e \) from the source \( s_i \) and should go to the sink \( t_i \). Inequality (23b) is the rate requirement for each source-sink pair. The flow conservation constraints (23c) for a node is imposed on each source-sink pair. Besides, the flow rate variables for all source-sink pairs on a particular edge should jointly respect the edge capacity constraint (23d). Note that by convention, for any edge \( e \in E \) and any \( i \in [1, k] \), we set \( x^{(d)}_{e,i} = 0 \) for \( d < 0 \) and \( d > T \), and for any edge \( e \in \text{Out}(s_i) \), we set \( x^{(d)}_{e,i} = 0 \) for all \( d \neq d_e \).

Now we can do a binary search scheme on the linear program (23) to find the minimal maximum delay bound \( T^* \) supporting a feasible multi-commodity flow which meets flow conservation, capacity, non-negative constraints as well as the rate requirement, namely we solve the Multi-Min-Max-Delay optimally. Following the same procedure as used in Thm. 3 we can prove the time complexity should be \( O(\log(Nd_{\text{max}})(N^2(kd_{\text{max}})^2/M)(\log R + N^2kd_{\text{max}}\log(N^2kd_{\text{max}}))) \).

Multi-Min-Max-Delay is NP-complete since Min-Max-Delay, a special case of the Multi-Min-Max-Delay problem, has been proved to be NP-complete in Thm. 1. Moreover, Multi-Min-Max-Delay is weakly NP-complete since we have proposed a binary search scheme to solve the problem optimally with a pseudo-polynomial time algorithm.

Int-Multi-Min-Max-Delay is NP-complete in the strong sense since Int-Min-Max-Delay, a special case of the Int-Multi-Min-Max-Delay problem, has been proved to be strongly NP-complete in Thm. 2.
Under a rate requirement constraint, we prove it is weakly NP-complete, and propose a pseudo-polynomial time algorithm to find the optimal solution. We also show that if integer solution is required, the problem becomes strongly NP-complete. An example is given to show that the optimal fraction flow could have smaller maximum delay than the optimal integral flow in Min-Max-Delay.

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