On Defining Ideals or Subrings of Hall Algebras

with an appendix by Andrew Hubery

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Abstract

Let $A$ be a finitary algebra over a finite field $k$, and $A\text{-mod}$ the category of finite dimensional left $A$-modules. Let $\mathcal{H}(A)$ be the corresponding Hall algebra, and for a positive integer $r$ let $D_r(A)$ be the subspace of $\mathcal{H}(A)$ which has a basis consisting of isomorphism classes of modules in $A\text{-mod}$ with at least $r + 1$ indecomposable direct summands. If $A$ is the path algebra of the quiver of type $A_n$ with linear orientation, then $D_r(A)$ is known to be the kernel of the map from the twisted Hall algebra to the quantized Schur algebra indexed by $n + 1$ and $r$. For any $A$, we determine necessary and sufficient conditions for $D_r(A)$ to be an ideal and some conditions for $D_r(A)$ to be a subring of $\mathcal{H}(A)$. For $A$ the path algebra of a quiver, we also determine necessary and sufficient conditions for $D_r(A)$ to be a subring of $\mathcal{H}(A)$.

Key words: quiver, (twisted) Hall algebra.

1 Introduction

Let $k$ be a finite field with $q$ elements and $A$ a $k$-algebra. By an $A$-module we mean a finite dimensional left $A$-module. Denote by $A\text{-mod}$ the category of $A$-modules. Assume $A$ is finitary, i.e. $\text{Ext}^1(S_1, S_2)$ is a finite group for any two (not necessarily different) simple objects in $A\text{-mod}$ (cf. [7]). Let $v = q^{1/2}$. Define the Hall algebra $\mathcal{H}(A)$ to be the $\mathbb{Z}[v, v^{-1}]$-algebra with basis the set of isomorphism classes $[X]$ of modules in $A\text{-mod}$ and with multiplication given by

$$[M] \cdot [N] = \sum_{[X]} F_{M,N}^X [X]$$

where $F_{M,N}^X$ is the number of submodules $U$ of $X$ such that $U \cong N$ and $X/U \cong M$. For an $A$-module $M$, let $s(M)$ be the number of indecomposable direct summands of $M$. For an integer $r \geq 1$, let

$$D_r(A) = \mathbb{Z}[v, v^{-1}][\{[M] \in \mathcal{H}(A) | s(M) \geq r + 1\}].$$

For convenience we denote $D_1(A)$ by $D(A)$.

If in addition for any $A$-modules $M$, $N$, $\text{Ext}_A^i(M, N) = 0$ for $i \gg 0$, one can define the twisted Hall algebra $\mathcal{H}_*(A)$ to be the $\mathbb{Z}[v, v^{-1}]$-algebra with the same basis as $\mathcal{H}(A)$ and a twisted multiplication

$$[M] \ast [N] = v^{(M,N)} [M] \circ [N]$$

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where $\langle M, N \rangle = \sum_{i \geq 0} (-1)^i \dim_k \text{Ext}_A^i(M, N)$. One can check that $D_r(A)$ is an ideal (resp. subring) of $H_*(A)$ if and only if it is an ideal (resp. subring) of $H(A)$. We refer to [6, 7] for more on Hall algebras and twisted Hall algebras.

For positive integers $n$ and $r$, let $S_v(n+1, r)$ be the quantized Schur algebra of type $A_n$ and of degree $r$. R.M. Green [4] shows that there is a map from the twisted Hall algebra $H_*(kL_n)$ to $S_v(n+1, r)$ whose kernel is exactly $D_r(kL_n)$, where $L_n$ is the quiver of type $A_n$ with linear orientation. In particular, $D_r(kL_n)$ is an ideal of $H_*(kL_n)$ (and $H(kL_n)$) for all $r \geq 1$. This raises the question, for which algebras $A$, $D_r(A)$ is an ideal of $H(A)$, or weaker, a subring? In general, we have

**Theorem 1.1.** The following conditions are equivalent,

(i) $D(A)$ is an ideal of $H(A)$,

(ii) $A$ is serial, i.e. each indecomposable $A$-module is uniserial,

(iii) $D_r(A)$ is an ideal of $H(A)$ for all $r \geq 1$,

(iv) $D_r(A)$ is an ideal of $H(A)$ for some $r \geq 2$.

**Theorem 1.2.** Consider the following conditions,

(I) Each indecomposable $A$-module has simple socle,

(I') Each indecomposable $A$-module has simple top, where the top of a module is defined as the quotient of the module by its radical,

(II) Each indecomposable $A$-module has simple top or simple socle,

(III) $D_r(A)$ is a subring of $H(A)$ for all $r \geq 1$,

(IV) $D_r(A)$ is a subring of $H(A)$ for some $r \geq 2$,

(V) $D(A)$ is a subring of $H(A)$.

Then we have $[(I) \iff (I')] \iff (III) \iff (IV) \iff (III') \iff (V)$.

In the appendix by A.Hubery the equivalence (II) $\iff$ (V) is proved for $A$ being a finite dimensional algebra. We conjecture that this is true in general.

Let $Q$ be a (finite) quiver. Let $s, t$ be the maps sending a path to its starting vertex and terminating vertex respectively. Let $A = kQ$ be the path algebra of $Q$ over $k$, where the product $\alpha \beta$ of two paths $\alpha$ and $\beta$ of $Q$ is defined as the composition of $\beta$ and $\alpha$ if $t(\beta) = s(\alpha)$, and 0 otherwise. Then $A$ is finitary and $A\text{-mod}$ is equivalent to the category of finite dimensional representations of $Q$. We refer to [11, 12] for representation theory of quivers.

In the following, we will identify an $A$-module with the corresponding representation of $Q$. Let $H(Q) = H(A)$ be the Hall algebra, $D(Q) = D(A)$ and $D_r(Q) = D_r(A)$. Then we have the following theorem.

**Theorem 1.3.** (i) $D_r(Q)$ is an ideal of $H(Q)$ for all $r \geq 1$ if and only if $Q$ is a disjoint union of quivers of the form $L$ and $\Delta$, where

$$L_m = \begin{array}{c}
\bullet \\
1 & 2 & 3 & \cdots & m-1 & m
\end{array} \quad \text{and} \quad \Delta_n = \begin{array}{c}
\bullet \\
0 & 1 & 2
\end{array}$$

$m$ is a positive integer and $n$ is a nonnegative integer (the oriented cycle of $\Delta_0$ is a loop).

(ii) $D(Q)$ is a subring of $H(Q)$ if and only if $Q$ is a disjoint union of quivers of the form $L$, $\Delta$, $V$, and $\Lambda$, where

$$V_{m,x} = \begin{array}{c}
\bullet \\
1 & 2 & \cdots & x & \cdots & m-1 & m
\end{array} \quad \text{and} \quad \Lambda_{n,y} = \begin{array}{c}
\bullet \\
1 & 2 & \cdots & y & \cdots & n-1 & n
\end{array}$$

$m, n \geq 3$ are positive integers, $x \in \{2, \cdots, m-1\}$, $y \in \{2, \cdots, n-1\}$.
(iii) The following conditions are equivalent,
(a) $D_r(Q)$ is a subring of $\mathcal{H}(Q)$ for all $r \geq 1$,
(b) $D_r(Q)$ is a subring of $\mathcal{H}(Q)$ for some $r \geq 2$,
(c) $Q$ is a disjoint union of quivers of the form $L$, $\Delta$, and $V$, or a disjoint union of quivers of the form $L$, $\Delta$, and $\Lambda$.

When $\mathcal{H}(Q)$ is replaced by $\mathcal{H}^{nil}(Q)$, the subalgebra of $\mathcal{H}(Q)$ with basis the isomorphism classes of finite dimensional nilpotent representations of $Q$, and $D_r(Q)$ by $D_r^{nil}(Q) = D_r(Q) \cap \mathcal{H}^{nil}(Q)$, Theorem 1.1 still holds.

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2 Proof of the theorems

Assume $0 \to N \to X \to M \to 0$ is an exact sequence in $A$-mod, i.e. $[X]$ is a summand (up to scalar) of $[M] \circ [N]$, then the socle $soc(N)$ of $N$ and the top $top(M)$ of $M$ are direct summands of $soc(X)$ and $top(X)$ respectively.

Lemma 2.1. The following conditions are equivalent,
(I) Each indecomposable $A$-module has simple socle,
(VI) $D_r(A)$ is a left ideal of $\mathcal{H}(A)$ for all $r \geq 1$,
(VII) $D_r(A)$ is a left ideal of $\mathcal{H}(A)$ for some $r \geq 1$.

Proof. (I)$\Rightarrow$(VI) Assume $r \geq 1$, and $[M] \in D_r(A)$. Then by (I) $soc(M)$ has at least $r + 1$ direct summands. Therefore if $X$ is an extension of some $A$-module $N$ by $M$, then $soc(X)$ has at least $r + 1$ direct summands, and hence $X$ has at least $r + 1$ direct summands by (I). Therefore $D_r(A)$ is a left ideal of $\mathcal{H}(A)$.

(VII)$\Rightarrow$(I) Suppose on the contrary that there exists an indecomposable module $M$ such that $soc(M)$ is decomposable. Let $r \geq 1$ be any integer and $N$ an indecomposable $A$-module. Then $0 \to soc(M) \oplus N^{\oplus r-1} \to M \oplus N^{\oplus r-1} \to M/soc(M) \to 0$ is an exact sequence, and $soc(M) \oplus N^{\oplus r-1}$ has at least $r + 1$ direct summands but $M \oplus N^{\oplus r-1}$ has exactly $r$ indecomposable direct summands. Thus $D_r(A)$ is not a left ideal of $\mathcal{H}(A)$ for all $r \geq 1$, contradicting (VII). □

Dually, we have

Lemma 2.2. The following conditions are equivalent,
(I') Each indecomposable $A$-module has simple top,
(VI') $D_r(A)$ is a right ideal of $\mathcal{H}(A)$ for all $r \geq 1$,
(VII') $D_r(A)$ is a right ideal of $\mathcal{H}(A)$ for some $r \geq 1$.

Proof of Theorem 1.1:
It follows from Lemma 2.1 and Lemma 2.2 since a module which has simple top and simple socle is uniserial. □

Proof of Theorem 1.2:
[[(I) or (I')] $\Rightarrow$ (III)] It follows from Lemma 2.1 and Lemma 2.2

(IV) $\Rightarrow$ [(I) or (I')] Suppose $M$, $N$ are two indecomposable $A$-modules such that $soc(M)$ is not simple and $top(N)$ is not simple. Let $r \geq 2$ be any integer. Then $0 \to soc(M) \oplus \cdots \oplus soc(M) \oplus N^{\oplus r-1}$ is an exact sequence, and $soc(M) \oplus N^{\oplus r-1}$ has at least $r + 1$ direct summands but $M \oplus N^{\oplus r-1}$ has exactly $r$ indecomposable direct summands. Thus $D_r(A)$ is not a left ideal of $\mathcal{H}(A)$ for all $r \geq 1$, contradicting (IV). □

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Corollary 2.3. Let $M$ be a summand, but $M$ is not a subring of $H$. Let $D$ be a left ideal or a right ideal of $H$. If $D$ is decomposable, then $D$ is not a subring of $H(A)$, contradicting (IV).

(II)⇒(V) Let $M$, $N$ be two decomposable $A$-modules. In particular, $top(M)$ and $soc(N)$ are decomposable modules. Let $X$ be an extension of $M$ by $N$. Then both $top(X)$ and $soc(X)$ are decomposable. Since by (II) each indecomposable $A$-module has simple top or simple socle, $X$ is decomposable. □

Corollary 2.3. Let $r \geq 2$ be an integer, then $D_r(A)$ is a subring of $H(A)$ if and only if $D_r(A)$ is a left ideal or a right ideal of $H(A)$.

In the sequel we fix our attention on quivers. We observe the following facts.

Assume $Q$, $Q'$ are two quivers and there is an exact functor $F : kQ$-$mod \to kQ$-$mod$ preserving indecomposability of representations. Then for any positive integer $r$ the space $D_r(Q')$ is not a subring (resp. an ideal) of $H(Q')$ implies that $D_r(Q)$ is not a subring (resp. an ideal) of $H(Q)$.

(i) Let $Q'$ be a subquiver of $Q$, i.e. the sets of vertices and arrows of $Q'$ are subsets of those of $Q$. Then a representation of $Q'$ can be regarded as a representation of $Q$ by putting zero vector spaces and zero maps on vertices and arrows of $Q$ which are different from those of $Q'$ respectively. This defines a (covariant) exact functor $F : kQ'$-$mod \to kQ$-$mod$ which preserves the indecomposability of representations and sends nilpotent representations to nilpotent representations.

(ii) Let $Q$ be a quiver and $Q^{op}$ the quiver with the same underlying diagram as $Q$ but opposite orientation. To a representation of $Q$ we associate a representation of $Q^{op}$ by taking the dual of all vector spaces and all maps. This defines a (contravariant) exact functor from $kQ$-$mod$ to $kQ^{op}$-$mod$ which preserves indecomposability and sends nilpotent representations to nilpotent representations. In particular, $D_r(Q)$ is a subring (resp. an ideal) of $H(Q)$ if and only if $D_r(Q^{op})$ is a subring (resp. an ideal) of $H(Q^{op})$. If we replace $H$ by $H^{nil}$ and $D_r$ by $D_r^{nil}$, the statement is true.

Theorem 2.4. Let $Q$ be a connected quiver, and $L$, $\Delta$, $V$ and $\Lambda$ as in Theorem 1.3.

(i) The following conditions are equivalent,
(a) $D(Q)$ is an ideal of $H(Q)$,
(b) $D_r(Q)$ is an ideal of $H(Q)$ for all $r \geq 1$,
(c) $Q$ is of the form $L$ or $\Delta$.

(ii) The following conditions are equivalent,
(a) $D(Q)$ is a subring of $H(Q)$,
(b) $D_r(Q)$ is a subring of $H(Q)$ for all $r \geq 1$,
(c) $Q$ is of the form $L$, $\Delta$, $V$, or $\Lambda$.

To prove Theorem 2.4 we need the following Lemmas 2.5, 2.6.

Lemma 2.5. Let $Q_1 = 1 \xrightarrow{2} \cdots \xrightarrow{n} \cdots \xrightarrow{n-1} n$. Then $D(Q_1)$ is not a subring of $H(Q_1)$.
Lemma 2.6. Let $Q_2$ be a quiver of type $D_4$, i.e. the underlying graph of $Q_2$ is

Then $D(Q_2)$ is not a subring of $\mathcal{H}(Q_2)$.

Proof. Let $M$ be the indecomposable module with dimension vector $(1, 2, 1, 1)$. Then either both $\text{rad}(M)$ and $M/\text{rad}(M)$ are decomposable, or both $\text{soc}(M)$ and $M/\text{soc}(M)$ are decomposable. □

Lemma 2.7. Let $Q_3 = \begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
1 & & 2
\end{array}$ be the Kronecker quiver. Then $D(Q_3)$ is not a subring of $\mathcal{H}(Q_3)$.

Proof. Consider the indecomposable module $M$ with dimension vector $(3, 2)$. Then both $\text{soc}(M)$ and $M/\text{soc}(M)$ are decomposable. □

Lemma 2.8. Let $Q_4 = \begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
1 & & 2
\end{array}$, $Q_5 = \begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
1 & & 2
\end{array}$, $Q_6 = \begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
1 & & 2
\end{array}$, $Q_7 = \begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
1 & & 2
\end{array}$. Then $D(Q_i)$ is not a subring of $\mathcal{H}(Q_i)$, $i = 4, 5, 6, 7$.

Proof. Sending $f_1 \begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
1 & & 2
\end{array} V_1 \xleftarrow{f_2} V_2$ to $V_1 \xleftarrow{f_1} \begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
1 & & 2
\end{array} V_2$ yields an exact functor from $kQ_6\text{-mod}$ to $kQ_7\text{-mod}$ which preserves the indecomposability of representations. Moreover, $Q_4$ and $Q_5$ are opposite to each other and so are $Q_6$ and $Q_7$. Therefore it is enough to prove the statement for $Q_6$. We have

$\begin{bmatrix}
0 \\
k \circ 0 \\
k \circ -k
\end{bmatrix} \circ \begin{bmatrix}
k^3 \circ 0 \\
\circ 0 \\
k^3 \circ 0
\end{bmatrix} = \begin{bmatrix}
k^4 \circ g_2 \\
\circ 0 \\
k^4 \circ g_2
\end{bmatrix} + \text{others}
$

where $f = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$, $g_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$, $g_2 = \begin{pmatrix}
0 \\
1 \\
0 \\
1
\end{pmatrix}$.

Both modules on the left hand side are decomposable. Let $M$ denote the module on the right hand side. Then

$\text{End}(M) = \{ (A, B) | A = \begin{pmatrix}
a & 0 & 0 & 0 \\
b & a & 0 & 0 \\
c & b & a & -b \\
d & 0 & 0 & a
\end{pmatrix}, B = a, \text{ where } a, b, c, d \in k \}.$

It has a unique maximal ideal $\{ (A, B) | a = 0 \}$, so it is local. Therefore, $M$ is indecomposable. □

Lemma 2.9. Let $Q_8 = \begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
1 & & 2
\end{array}$. Then $D(Q_8)$ is not a subring of $\mathcal{H}(Q_8)$.

Proof. We have

$\begin{bmatrix}
0 \\
k^2 \circ 0 \\
k^2 \circ 0
\end{bmatrix} \circ \begin{bmatrix}
k^2 \circ 0 \\
\circ 0 \\
k^2 \circ 0
\end{bmatrix} = \begin{bmatrix}
k^4 \circ g_1 \\
\circ 0 \\
k^4 \circ g_1
\end{bmatrix} + \text{others}$

5
where \( g_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \) and \( g_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \).

Both modules on the left hand side are decomposable. Let \( M \) denote the module on the right hand side. Then

\[
\text{End}(M) = \left\{ A \mid A = \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ d & c & a & 0 \\ e & f & b & a \end{pmatrix} \right\}, \text{ where } a, b, c, d, e, f \in k \}
\]

It has a unique maximal ideal \( \{A | a = 0 \} \), so it is local. Hence \( M \) is indecomposable. \( \square \)

**Proof of Theorem 2.4:**

Let \( Q \) be a connected quiver. It is known that \( kQ \) is serial if and only if \( Q \) is of the form \( L \) or \( \Delta \). Thus (i) follows from Theorem 1.1. Now let us prove (ii).

If \( Q \) is of the form \( L, \Delta, V \) or \( \Lambda \), then Theorem 1.2 [(I) or (I')] holds, and hence \( D_r(Q) \) is a subring of \( \mathcal{H}(Q) \) for all \( r \geq 1 \). By Theorem 1.2 it remains to show that if \( Q \) is not of the form \( L, \Delta, V \) or \( \Lambda \) then \( D(Q) \) is not a subring of \( \mathcal{H}(Q) \). We prove case by case.

**Case 1.** \( Q \) is of type \( A_n \) but not of the form \( L, V, \) or \( \Lambda \). Then \( Q \) has a subquiver of the form \( Q_1 \). Therefore by Lemma 2.5 we have that \( D(Q) \) is not a subring of \( \mathcal{H}(Q) \).

**Case 2.** \( Q \) is of type \( A_n \) but not of the form \( \Delta \). Then there exists an exact functor from \( kQ \)-mod to \( kQ \)-mod which preserves indecomposability of representations (cf. [3] [5]). Therefore by Lemma 2.7 \( D(Q) \) is not a subring of \( \mathcal{H}(Q) \).

**Case 3.** \( Q \) has a proper subquiver which is of the form \( \Delta \). Then \( Q \) has a double-loop or a subquiver of type \( D_4 \), or a subquiver of the form \( Q_4, Q_5, Q_6, \) or \( Q_7 \). Therefore it follows from Lemma 2.8 and Lemma 2.9 that \( D(Q) \) is not a subring of \( \mathcal{H}(Q) \).

**Case 4.** Otherwise. Then \( Q \) has a subquiver of type \( D_4 \) or a Kronecker subquiver. Thus by Lemma 2.6 and Lemma 2.7 \( D(Q) \) is not a subring of \( \mathcal{H}(Q) \). \( \square \)

**Proof of Theorem 1.3:**

(i) It follows, by Theorem 1.1, from the fact that \( kQ \) is serial if and only if \( Q \) is a disjoint union of quivers of the form \( L \) and \( \Delta \).

(ii) Assume \( D(Q) \) is a subring of \( \mathcal{H}(Q) \). It follows from Theorem 2.8 that \( Q \) is a disjoint union of quivers of the form \( L, \Delta, V \) and \( \Lambda \). If \( Q \) is such a quiver, then each \( kQ \)-module has simple top or simple socle. By Theorem 1.2 \( D(Q) \) is a subring of \( \mathcal{H}(Q) \).

(iii) Assume \( Q \) is a disjoint union of quivers of the form \( L, \Delta, V \) and \( \Lambda \). Then each \( kQ \)-module has simple socle if and only if \( Q \) is a disjoint union of quivers of the form \( L, \Delta, \) and \( \Lambda \). The desired result follows from Theorem 1.2 \( \square \)

We can follow the same procedure to prove

**Theorem 2.10.** (i) \( D^\text{nil}(Q) \) is an ideal of \( \mathcal{H}^\text{nil}(Q) \) for any integer \( r \geq 1 \) if and only if \( Q \) is a disjoint union of quivers of the form \( L \) and \( \Delta \).

(ii) \( D^\text{nil}(Q) \) is a subring of \( \mathcal{H}^\text{nil}(Q) \) if and only if \( Q \) is a disjoint union of quivers of the form \( L, \Delta, V, \) and \( \Lambda \).

(iii) The following conditions are equivalent,

(a) \( D^\text{nil}(Q) \) is a subring of \( \mathcal{H}^\text{nil}(Q) \) for any integer \( r \geq 1 \),

(b) \( D^\text{nil}(Q) \) is a subring of \( \mathcal{H}^\text{nil}(Q) \) for some integer \( r \geq 2 \).
Q is a disjoint union of quivers of the form L, Δ, and V, or a disjoint union of quivers of the form L, Δ, and Λ.

3 Appendix by Andrew Hubery

Let $k$ be a finite field and $A$ a finite dimensional $k$-algebra. Let $\mathcal{H}(A)$ denote the Ringel-Hall algebra of $A$ and define $D(A)$ to the be subspace of all decomposable modules.

**Theorem 3.1.** $D(A)$ is a subring of $\mathcal{H}(A)$ if and only if every indecomposable $A$-module has either simple top or simple socle.

We shall use the following characterisation, due to Tachikawa [8].

**Theorem 3.2.** Every indecomposable $A$-module has simple top or simple socle if and only if
1. every indecomposable projective module has radical a sum of at most two uniserial modules (and dually for indecomposable injective modules), and
2. if an indecomposable projective has decomposable socle, then the injective envelopes of these simples is uniserial (and dually for injectives).

**Proof.** It is clear that if every indecomposable has simple top or simple socle, then $D(A)$ is a subring, since every extension of decomposable modules must remain decomposable. Let us therefore assume that $D(A)$ is a subring, and write $Q$ for the valued quiver of $A$.

We first consider those indecomposable modules of Loewy length two. There are no valued arrows with valuation $(a, b)$ for $ab \geq 3$, so in particular, there is no Kronecker subquiver, and hence no vertex with a double loop. Also, there are at most two arrows starting at each vertex, and if there are two such arrows, then they are both unvalued. Dually for arrows ending at a given vertex. Finally, we can have no subquiver of the form 

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· ← · → · ←
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We now consider indecomposable modules of Loewy length three. Suppose we have a vertex $i$ with an arrow $\alpha$ ending at $i$ and two arrows $\beta$ and $\gamma$ starting at $i$. We know from the above that both $\beta$ and $\gamma$ are unvalued, but $\alpha$ may be valued. We show that there can be no sincere indecomposable module for this subquiver.

It is sufficient to consider the valued graph

```
1 (a,b) i 2
     \downarrow \quad \downarrow
     \quad \downarrow \quad \downarrow
     \quad \downarrow \quad \downarrow
     \quad \downarrow \quad \downarrow
8 3
```

The corresponding $k$-species $\Lambda$ is given by

$$
\Lambda = \begin{pmatrix}
F & 0 & 0 & 0 \\
H & G & 0 & 0 \\
H & H & G & 0 \\
H & H & 0 & G
\end{pmatrix},$

where $F/k$, $G/k$ and $H/k$ are field extensions of degrees $a$, $b$ and $ab$ respectively, and $G_HF$ has the natural bimodule structure. Note that as $ab \leq 2$, then either $a = 1$, so $F = k$ and $G = H$, or $b = 1$, so $F = H$ and $G = k$. Now, $\text{rad}^2 \Lambda \cong H \oplus H$ and hence the only possible relations are the zero relations $\beta\alpha = 0$ or $\gamma\alpha = 0$. Thus, if there exists a sincere
indecomposable module, then there are no relations and mod $\Lambda$ embeds into mod $A$. This gives a contradiction, since $D(A)$ is not a subring of $H(\Lambda)$.

The dual argument works whenever there are two arrows ending at $i$ and an arrow starting at $i$.

It follows that for each indecomposable projective $P$, rad $P$ is the sum of at most two modules. These modules must be uniserial, since if not, then we are in the situation above for some vertex $i$: that is, there exists an arrow ending at $i$, two arrows starting at $i$ and no zero relations, a contradiction.

Now let $P$ be an indecomposable projective module such that soc $P$ is decomposable. Write rad $P = U_1 + U_2$ as a sum of two uniserial modules and let $j$ be the vertex corresponding to $S = \text{soc} U_1$. Suppose that $I(S)$ is not uniserial. Then the module $M = P/\text{rad} U_2$ is indecomposable and we claim that there exists an (unvalued) arrow $\alpha : i \to j$ such that \( \alpha \cdot M = 0 \).

If the socles of $U_1$ and $U_2$ are non-isomorphic, this is clear, so suppose that soc $P \cong S^2$. Then $P$ has Loewy length at least three and we may assume that $U_2$ has Loewy length at least two. This proves the claim.

Now, there is a natural non-split extension of $S_i$ by $M$ yielding an indecomposable with decomposable radical and decomposable top, a contradiction.

We clearly have the dual statements involving indecomposable injective modules, and hence the conditions of Tachikawa’s Theorem are fulfilled.

As a corollary, we extend Theorem 3.2 to all hereditary algebras.

**Corollary 3.3.** Let $A$ be a connected hereditary $k$-algebra. Then $D(A)$ is a subring if and only if the quiver of $A$ is either an oriented cycle, of type $\tilde{A}$ and having either a unique sink or a unique source, or of type $B$ or $C$ with a linear orientation.

**Proof.** Assume that $A$ is not an oriented cycle. We know from [3] that if $A$ is of tame representation type, then there exists an embedding into mod $A$ of the module category for some tame bimodule. This has valuation $(a, b)$ with $ab = 4$, a contradiction. Thus $A$ must be representation finite. Now, by the previous arguments and Tachikawa’s criteria, $A$ must be of type $\tilde{A}$, $B$ or $C$ with the required restrictions on the orientation. $\square$

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