CONSISTENCY OF QUANTUM BACKGROUND INDEPENDENCE

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ABSTRACT

We analyse higher order background independence conditions arising from multiple commutators of background deformations in quantum closed string field theory. The conditions are shown to amount to a vanishing theorem for $\Delta_S$ cohomology classes. This holds by virtue of the existence of moduli spaces of higher genus surfaces with two kinds of punctures. Our result is a generalisation of a previous genus zero analysis relevant to the classical theory.

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1. Introduction and Summary

It has been some time now since the background independence of string field was proven [1,2]. The infinitesimal background deformations were shown to be implemented as canonical transformations whose Hamiltonian functions were defined by moduli spaces of punctured Riemann surfaces with a single special puncture. It was realised in [3] that these $B^1$ spaces represented only the first order perturbations of the string background, and that higher order deformations implied the existence of moduli spaces $B^2$, $B^3$, ... of surfaces having more than one special puncture, antisymmetric under the exchange of special punctures. These were duly constructed in [3,4] for the classical case. Furthermore, these $B$-spaces have precisely the properties required to formulate string theory around non-conformal backgrounds, and such an action was constructed explicitly in [4]. The $B$-spaces do not depend upon any particular choice of string background and as such would be expected to play an important role in any manifestly background independent formulation of the theory. Arriving at such a formulation remains one of the major goals in string theory, and provides motivation for the present work.

The recent work of Zwiebach [3,4] dealt only with the classical closed string theory, whereas we would of course eventually wish to deal with the full quantum theory. Experience has shown that in string field theory, classical results usually generalise to the quantum case without too many complications, and this paper reaffirms this by finding the quantum generalisations of Zwiebach’s results.

This work is organised as follows. In §2 we review some properties of connections on the space of conformal field theories, and recall some basic facts about the string vertices, introducing some new notation for $B$-spaces which will be used throughout the paper. In §3, we follow [2] to derive the conditions for background independence when the Batalin-Vilkovisky density function $\rho$ is allowed some field-dependence. We then review the origin of general symplectic connections, and give the form of the Hamiltonian $B_\mu(\Gamma)$ implementing background deformations for general connections. We find that background independence to first order amounts to a $\Delta_S$-cohomology theorem for the action, being the natural generalisation of the antibracket-cohomology of the classical case. In §4 we consider the commutator of background deformations and show that background independence to second order implies a higher cohomology theorem. The consistency conditions are shown to be satisfied through the existence of spaces $B^2$ with two special punctures of all genera. In §5, we use the language of differential forms on the theory space manifold to efficiently derive $\Delta_S$-cohomology theorems to all
higher orders for the string action, and extend the complex of $\mathcal{B}$-spaces to include positive-dimensional moduli spaces of Riemann surfaces for all genera and all numbers of ordinary and special punctures compatible with the dimensionality requirement.

2. Review and Notation

In this section we will review a few definitions and results used in the present work.

2.1. Connections on the Space of Conformal Theories

We review here the formalism of [5,6]. A vector bundle is constructed over the space $M$ of CFTs by assigning a basis $|\Phi^x_i\rangle$ of states of the theory to each point $x = \{x^\mu\} = (x^1, \ldots, x^m)$ of $M$. The coordinates of the vector space at $x^\mu$ are denoted by $\psi = \{\psi^i\} = (\psi^1, \ldots, \psi^{2n})$. Given a connection $\Gamma^j_{\mu i}(x)$ on this bundle, the covariant derivative of sections, $\langle A(x)| = \sum_i a_i(x)\langle \Phi^i_x|$, and $|B(x)\rangle = \sum_i |\Phi^i_x\rangle b^i(x)$, on the bundle are defined by,

$$D_\mu(\Gamma)\langle A| \equiv \partial_\mu a_i \langle \Phi^i| - a_i \Gamma^i_{\mu j}\langle \Phi^j|$$

$$= \partial_\mu \langle A| - \langle A| \Gamma_\mu,$$  \hspace{1cm} (2.1)

where $\Gamma_\mu = \sum_i |\Phi^i_i\rangle \Gamma^i_{\mu j}\langle \Phi^j|$, and,

$$D_\mu(\Gamma) |B\rangle \equiv |\Phi^i_i\rangle \partial_\mu b^i + |\Phi^j_j\rangle \Gamma^j_{\mu i} b^i$$

$$= \partial_\mu |B\rangle + \Gamma_\mu |B\rangle.$$  \hspace{1cm} (2.2)

The covariant derivatives of functions on the bundle are defined by,

$$D_\mu(\Gamma)F \equiv \partial_\mu F - F \frac{\partial}{\partial |\Psi\rangle} \Gamma_\mu |\Psi\rangle,$$  \hspace{1cm} (2.3)

where we use Greek indices for theory space coordinates and Latin indices for symplectic coordinates and we introduce the notation,

$$\frac{\partial}{\partial |\Psi\rangle} \equiv \frac{\partial}{\partial |\Psi\rangle} \langle \Phi^i|.$$  \hspace{1cm} (2.4)

We will primarily be interested in symplectic connections for which the covariant derivatives
of the symplectic form and the sewing ket vanish,

\[ D_\mu(\Gamma) \langle \omega_{12} \mid S_{12} \rangle = 0. \]  

(2.5)

In particular this implies that the covariant derivative of a function of the type \( \langle A \mid \Psi \rangle \ldots \mid \Psi \rangle \)
defined by tensor sections \( \langle A(x) \rangle \) is given by,

\[ D_\mu \langle A \mid \Psi \rangle \ldots \mid \Psi \rangle = (D_\mu \langle A \mid \Psi \rangle \ldots \mid \Psi \rangle), \]

(2.6)

where covariant derivatives of sections are simply given by,

\[ D_\mu(\Gamma) \langle A \rangle = \partial_\mu \langle A \rangle - \sum_n \langle A \mid \Gamma^{(n)}_\mu \rangle, \]

(2.7)

with the label \( n \) referring to a state space in the tensor section.

### 2.2. Moduli Spaces of String Vertices and B Spaces

The string vertices \( V = \sum \hbar^g \mathcal{V}_{g,n} \) (where the sum is over \( n \geq 3 \) for genus zero, \( n \geq 1 \) for genus one, and \( n \geq 0 \) for all higher genera) satisfy the recursion relations (see for example Eqn.(2.22) of [2]),

\[ \partial V + \hbar \Delta V + \frac{1}{2} \{ V, V \} = 0. \]

(2.8)

The string action may then be written (Eqn.(3.35) of [2]),

\[ S = Q + \hbar S_{1,0} + f(V), \]

(2.9)

which satisfies the B-V master equation,

\[ \frac{1}{2} \{ S, S \} + \hbar \Delta S = 0. \]

(2.10)

The interpolating ‘B-spaces’ have been introduced (using various notations) in the papers [1,2,7,3,4], and are responsible for first and higher order infinitesimal background deformations of the theory via the canonical transformations implemented by their associated Hamiltonians. They also play an important rôle in the formulation of string field theory around non-conformal backgrounds.
We will define $\mathcal{B}_{g,n}^{\tilde{n}}$ as the moduli space of decorated Riemann surfaces of genus $g$ with $n$ ordinary punctures (each surrounded by a coordinate disk) and with $\tilde{n}$ special punctures. The $\mathcal{B}$-spaces are symmetrised with respect to labellings of the ordinary punctures and antisymmetrised with respect to labellings of the special punctures, and we recall that the space $\mathcal{B}_{g,n}^{\tilde{n}}$ has dimension $6g - 6 + 2n + 3\tilde{n}$. The spaces $\mathcal{B}_{g,n}^0$ with no special punctures will be identified with the usual string vertices $\mathcal{V}_{g,n}$. In the papers [1,2,7] the $\mathcal{B}$-spaces having a single special puncture with $n \geq 2$ at genus zero and $n \geq 1$ at higher genus were introduced, being responsible for implementing first order background deformations. In [3,4], higher order deformations of the classical theory were considered and it was found necessary to extend this complex to include the spaces $\mathcal{B}_{0,n}^{\tilde{n}}$ for all $n \geq 1$ and $\tilde{n} \geq 2$ which implemented them.

Like the string vertices $\mathcal{V}$, the $\mathcal{B}$-spaces also satisfy recursion relations, and the ones which will be needed for the purposes of this paper are those satisfied by the spaces $\mathcal{B}^1$ (see for example Eqn.(3.20) of [7]),

$$\delta \mathcal{V} \mathcal{B}^1 = (\mathcal{K} - \mathcal{I}) \mathcal{V} + \mathcal{V}'_{0,3} + \hbar \Delta \mathcal{B}^1_{0,2} + \hbar \mathcal{I} \mathcal{V}_{1,1},$$

where $\mathcal{B}^1 = \sum_{g,n} \mathcal{B}^1_{g,n}$ and $\delta \mathcal{V} \equiv \partial + \hbar \Delta + \{\mathcal{V}, \cdot\}$. There is no space $\mathcal{B}_{1,0}^1$ so the terms involving objects of genus one and with one special puncture but no ordinary punctures have been extracted by hand to leave an equation which holds for all $(g,n)$.

Other useful formulae relating the various operators acting on moduli spaces and functions are collected in the Appendix for reference.

3. Background Independence Revisited

In this section, we review the derivation of the condition for quantum background independence [2]. This was originally carried out for the case where the B-V density $\rho$, being dependent only on $x$, was a section on the theory space bundle. Here we shall derive the background independence conditions for the more general case where $\rho = \rho(\psi, x)$ is allowed some field-dependence and hence is promoted to a function on the bundle. Given that the invariant B-V measure takes the form $d\mu_S \equiv \prod_i d\psi_i^\dagger \rho e^{2S/\hbar}$, there should be no problem in transferring some of the field-dependence of the action to the B-V density, and we verify this by analysing the transformation properties of our background independence condition under field-redefinitions. Nevertheless, since it is always possible to choose a frame in which $\rho$ is field-independent, we choose for simplicity to restrict our analysis to this case. After briefly
reviewing the origin of general symplectic connections \cite{2}, we write the explicit form of the Hamiltonian $B_\mu(\Gamma)$ implementing background deformations for general symplectic connections, and show that background independence to first order amounts to a ‘vanishing’ theorem for $\Delta_S$ cohomology classes.

3.1. Field-Independence of the Density $\rho$

Having chosen field/antifield coordinates $(\psi^i)$ on the symplectic manifold we can define the string field $|\Psi\rangle$ as usual by,

$$|\Psi\rangle = \sum_i |\Phi^x_i\rangle \psi^i_x.$$  

where the superscript $x$ denotes the theory space dependence. We will make $x$ implicit in the following. Let us investigate the effect of allowing the density $\rho = \rho(\psi, x)$ to have an explicit dependence on the field/antifield coordinates (though we are always able to choose coordinates such that the field-dependence vanishes). In the field-independent case, $\rho$ drops out of the expression for the delta operation, but this explicit $\rho$-dependence is reinstated on allowing the field-dependence, which may be determined as follows,

$$\Delta F = \frac{1}{2\rho} (-)^i \partial_i (\rho \omega^{ij} \partial_j F)$$
$$= \frac{1}{2} (-)^i \left( (\partial_i \ln \rho) \omega^{ij} \partial_j F + \partial_i (\omega^{ij} \partial_j F) \right)$$
$$= \left\{ \frac{1}{2} \ln \rho, F \right\} + \hat{\Delta} F,$$

where we have defined the $\rho$-independent hatted delta operation by,

$$\hat{\Delta} F \equiv \frac{1}{2} (-)^{F+1} \left( \frac{\overrightarrow{\partial}}{\partial |\Psi\rangle_1} \frac{\overrightarrow{\partial} F}{\partial |\Psi\rangle_2} \right) |S_{12}\rangle,$$  

(3.3)

and have used the notation,

$$\frac{\overrightarrow{\partial}}{\partial |\Psi\rangle} \equiv \langle \Phi' | \overrightarrow{\partial}.$$

(3.4)

We note that $\hat{\Delta}$, unlike $\Delta$, is in general neither a nilpotent nor a scalar operator.

We may now reconsider the consistency condition for local quantum background independence. This condition (Eqn.(4.16) of \cite{2}) was derived for the case of the canonical connection
\( \hat{\Gamma}_\mu \) of \([5]\), and reads,

\[
D_\mu(\hat{\Gamma})S = \hbar \Delta B_\mu + \{S, B_\mu\} + \frac{1}{2\hbar} \text{str} (\hat{\Gamma}_\mu - \partial_\mu \ln \rho) \tag{3.5}
\]

We would like to derive the general form where \( \rho = \rho(\psi, x) \) has a field/antifield coordinate dependence. Given the measures \( d\mu_x = \rho(\psi, x) \prod_i d\psi^i \) and \( d\mu_y = \rho(y, \psi) \prod_i d\psi^i \), we require the existence of a symplectic diffeomorphism \( F_{y,x}^* \) such that \( d\mu_x e^{2S_x/\hbar} = F_{y,x}^*(d\mu_y e^{2S_y/\hbar}) \).

Following Eqn.(4.9) of \([2]\) we know that,

\[
F_{y,x}^*(d\mu_y) = \frac{\rho(y, y)}{\rho(x, x)} \text{sdet} \left[ \frac{\partial \rho(y, x)}{\partial \rho(x, x)} \right] \cdot d\mu_x . \tag{3.6}
\]

so that the background independence condition becomes,

\[
\exp \left( \frac{2S(\psi, x)}{\hbar} \right) = \exp \left( \frac{2S(\psi, y)}{\hbar} \right) \cdot \frac{\rho(y, y)}{\rho(x, x)} \cdot \text{sdet} \left[ \frac{\partial \rho(y, x)}{\partial \rho(x, x)} \right] . \tag{3.7}
\]

If we consider infinitesimal diffeomorphisms, \( y = x + \delta x \), we may use Eqn.(4.11) of \([2]\),

\[
\psi^i_{x + \delta x} = F^i(\psi, x, x + \delta x) = \psi^i_x + \delta x^\mu f^i(\psi, x, x) + \mathcal{O}(\delta x^2) , \tag{3.8}
\]

to obtain,

\[
\frac{\rho(y, y)}{\rho(x, x)} \sim 1 + \frac{1}{\rho(x, x)} \frac{\partial \rho(x, x)}{\partial x^\mu} \delta x^\mu + \frac{1}{\rho(x, x)} \frac{\partial \rho(x, x)}{\partial \psi^i} f^i_\mu \delta x^\mu . \tag{3.9}
\]

So Eqn.(4.12) of \([2]\) is modified to,

\[
\frac{\partial S(\psi, x)}{\partial x^\mu} + \frac{\partial_r S(\psi, x)}{\partial \psi^i_x} f^i_\mu + \frac{1}{2\hbar} \left[ \frac{\partial \ln \rho}{\partial x^\mu} + \frac{\partial_r \ln \rho}{\partial \psi^i_x} f^i_\mu + \text{str} \left( \frac{\partial f^i_\mu}{\partial \psi^j} \right) \right] . \tag{3.10}
\]

If we now separate from \( f^i_\mu \) the term proportional to the connection as follows,

\[
f^i_\mu \equiv -\hat{\Gamma}^i_{\mu j} \psi^j - B^i_\mu , \tag{3.11}
\]

and note that the condition that \( F^i \) be a symplectic map reduces to the condition that there exists an odd Hamiltonian \( B_\mu \) such that,

\[
B^i_\mu = \omega^{ij} \frac{\partial f^j_\mu}{\partial \psi^i} , \tag{3.12}
\]

then it is clear that the new \( \psi \)-dependence results in a shift of the \( \rho \)-dependent term of Eqn.(3.5)
as follows,
\[ \partial_\mu \ln \rho \rightarrow \partial_\mu \ln \rho - (\ln \rho) \frac{\partial}{\partial \mu} \bar{\Gamma}_i^{i\mu} \psi^j - (\ln \rho) \frac{\partial}{\partial \mu} \omega^{ij} \bar{\omega}_j^i B_\mu \]
\[ \rightarrow \partial_\mu \ln \rho - (\ln \rho) \frac{\partial}{\partial \mu} \bar{\Gamma}_i^{i\mu} \psi^j - \{ \ln \rho, B_\mu \} \]
\[ \rightarrow D_\mu (\bar{\Gamma}) (\ln \rho) - \{ \ln \rho, B_\mu \}. \]

where we have made use of Eqns. (4.11), (4.13) and (4.15) of [2]. The resulting background independence condition is,
\[ D_\mu (\bar{\Gamma}) (S + \frac{1}{2} \hbar \ln \rho) = \hbar \Delta B_\mu + \{ S, B_\mu \} + \hbar \Delta \bar{\Gamma}_\mu, \]

(3.14)

having used the fact that \( \frac{1}{2} \hbar \text{str} \bar{\Gamma}_\mu = \hbar \Delta \bar{\Gamma}_\mu \), (where \( \bar{\Gamma}_\mu \equiv -\frac{1}{2} (\omega_{12} | \Gamma_\mu^2 | \Psi_1 | \Psi_2) \)). While the form of the consistency conditions given in Eqn. (3.14) is useful for demonstrating the relationship between \( \rho \) and the action, it is, for our purposes, also convenient to write it in the form,
\[ D_\mu (\bar{\Gamma}) S = \hbar \Delta B_\mu + \{ S, B_\mu \} + \hbar \Delta \bar{\Gamma}_\mu - \frac{1}{2} \hbar D_\mu (\bar{\Gamma}) \ln \rho \]
\[ = \Delta_S B_\mu (\bar{\Gamma}) + \hbar \Delta \bar{\Gamma}_\mu - \frac{1}{2} \hbar D_\mu (\bar{\Gamma}) \ln \rho. \]

(3.15)

where we have introduced \( \Delta_S \cdot = \hbar \Delta_{d\mu_S} \cdot = \{ S, \cdot \} + \hbar \Delta \cdot = \{ S + \frac{1}{2} \ln \rho, \cdot \} + \hbar \Delta \cdot \). In B-V quantisation the action is taken to transform as a scalar under string field-redefinitions. It should therefore be possible to extract any scalar (and in general field-dependent) component from the action and absorb it into the B-V density as an additional factor. This procedure should leave the transformation properties of both the action and \( \rho \) unchanged, so that the requirement that B-V measure \( d\mu_S \) be an invariant remains satisfied. By analysing the transformation properties of Eqn. (3.15) we will now verify that it is indeed consistent for \( \rho \) to have such field-dependence.

Since the action is a scalar the LHS of Eqn. (3.15), as well as the first two terms on the RHS (in the first line of the equation) transform as scalars on the bundle. If the equation is to be consistent, we must require that the remaining terms also transform correctly. Let us see whether this is true.

Under a change of basis, \( | \Phi_i \rangle \rightarrow | \Phi_j \rangle N^j_i (x) \), where \( N^j_i \) is an invertible matrix, we must have \( \psi^i \rightarrow (N^{-1})^j_i \psi^j \). We know also that \( D_\mu | \Phi_i \rangle \psi^i \) and \( \partial_\mu \) are also invariants, and these facts
allow us to derive the transformation properties $\Gamma_{\mu i}^j \rightarrow \Gamma'_{\mu i}^j$ of the connection (see [6]),

$$D_\mu |\Phi_i\rangle \psi^i = |\Phi_i\rangle \partial_\mu \psi^i + |\Phi_j\rangle \Gamma_{\mu i}^{\prime j} \psi^i$$

$$\rightarrow |\Phi\rangle N \partial_\mu (N^{-1} \psi) + |\Phi\rangle N \Gamma_{\mu}^i N^{-1} \psi$$

$$= |\Phi\rangle \partial_\mu \psi + |\Phi\rangle N \partial_\mu N^{-1} \psi + |\Phi\rangle N \Gamma_{\mu}^i N^{-1} \psi,$$  \hspace{1cm} (3.16)

where we have adopted matrix notation for brevity. The invariance of $D_\mu |\Phi_i\rangle \psi^i$ then implies that with our conventions,

$$\Gamma'_{\mu i}^j = (N^{-1})^i_k \Gamma_{\mu l}^k N^l_j - (\partial_\mu N^{-1})^i_k N^l_j .$$  \hspace{1cm} (3.17)

Then the function $\Gamma_\mu = -\frac{1}{2} \langle \omega_1 | \Gamma_{\mu}^{(2)} | \Psi \rangle |\Psi\rangle_2$ associated to the connection transforms as,

$$\Gamma_\mu \rightarrow \Gamma_\mu - N(\partial_\mu N^{-1}) .$$  \hspace{1cm} (3.18)

The invariance of $d\mu S$ and the action $S$ implies that the density must transform according to,

$$\rho \rightarrow \rho \ sdet \left( \frac{\partial \psi^j}{\partial \psi^i} \right) = \rho \ sdet \ (N^{-1}) = \rho e^{-\text{str} \ ln (N^{-1})} .$$  \hspace{1cm} (3.19)

Also $\frac{\partial}{\partial i} \rightarrow \frac{\partial}{\partial j} N_i^j$ and therefore,

$$D_\mu \ln \rho - 2\Delta \Gamma_\mu = \partial_\mu \ln \rho - \ln \rho \hat{\partial}_j \Gamma_{\mu j}^{i} \psi^j - 2\Delta \Gamma_\mu$$

$$\rightarrow \partial_\mu \ln \rho - \partial_\mu \text{str} \ ln (N^{-1}) - \ln \rho \hat{\partial}_j N^{-1} \Gamma_{\mu j} \psi^j + \ln \rho \hat{\partial}_j N(\partial_\mu N^{-1}) \psi$$

$$- 2\Delta \Gamma_\mu + \text{str} \ (\partial_\mu N^{-1}) N - \ln \rho \hat{\partial}_j N(\partial_\mu N^{-1}) \psi$$

$$= \partial_\mu \ln \rho - \ln \rho \hat{\partial}_j \Gamma_{\mu j}^{i} \psi^j - 2\Delta \Gamma_\mu .$$  \hspace{1cm} (3.20)

This verifies that the last two terms on the RHS of Eqn.(3.15) transform as scalars, and we conclude that it is indeed consistent to allow $\rho$ some field-dependence. Nevertheless, given that frames always exist in which $\rho$ may be chosen to be field-independent, we will for simplicity restrict ourselves to this case in what follows. Choosing $\rho$ to be field-independent means that we may carry over directly the background independence condition (3.5), and also set $\Delta = \hat{\Delta}$. 
3.2. ORIGIN OF GENERAL SYMPLECTIC CONNECTIONS

Until now, we have employed the canonical connection $\hat{\Gamma}_\mu$. There is no real reason for restricting ourselves to this particular choice of connection and for the sake of generality we would like to express the background independence condition in terms of more general symplectic connections.

In order to understand how different connections are related to each other, we will review §6.2 of [2] which demonstrates how a particular choice of connection is related to a choice of three string vertex.

We are interested in the coupling constant-independent $O(\bar{\hbar})$ terms in the background independence consistency condition. For the case of field-independent $\rho$, the $O(\bar{\hbar})$ condition is just Eqn.(6.2) of [2],

$$\partial_\mu S_{1,0} = -\frac{1}{2} \partial_\mu \ln \rho + \Delta \hat{\Gamma}_\mu + f_\mu(\Delta \hat{B}_{1,0}^1) + f_\mu(V_{1,1}).$$  \hspace{1cm} (3.21)\]

Now $\hat{B}_{1,0}^1$ interpolates from $T\mathcal{V}_{0,3}$, which is the three string vertex with one special puncture, to the auxiliary string vertex $\mathcal{V}_{0,3}'$, so the above equation seems to involve singular tori $\Delta \mathcal{V}_{0,3}'$. The way this is avoided is to introduce a new vertex $\tilde{\mathcal{V}}_{0,3}$ with one special puncture and two ordinary punctures such that $\Delta \tilde{\mathcal{V}}_{0,3}$ is not singular. We can now introduce a new space $\hat{B}_{0,2}^1$ which interpolates between $\mathcal{V}_{0,3}'$ and the vertex $\tilde{\mathcal{V}}_{0,3}$ and use this to define a new symplectic connection $\Gamma_{\mu}(\tilde{\mathcal{V}}_{0,3})$ by,

$$\langle \omega_{12} | \Gamma^{(1)}_{\mu} \rangle = \langle \omega_{12} | \hat{\Gamma}^{(1)}_{\mu} \rangle + \int_{\hat{B}_{0,2}^1} \langle \Omega^{(0)}_{1,1} | \hat{\Phi}_{\mu} \rangle,$$  \hspace{1cm} (3.22)

Absorbing these changes into the canonical connection and $\hat{B}_{0,2}^1$, we make the replacement in Eqn.(3.21),

$$\Delta \hat{\Gamma}_\mu + f_\mu(\Delta \hat{B}_{1,0}^1) = \Delta \Gamma_\mu + \int_{\Delta(\hat{B}_{0,2}^1)} \langle \Omega^{(0)}_{1,1} | \hat{\Phi}_{\mu} \rangle = \Delta \Gamma_\mu + f_\mu(\Delta \hat{B}_{0,2}^1 + \Delta \hat{B}_{0,2}^1),$$  \hspace{1cm} (3.23)

The integral is actually path independent, which allows us to define a moduli space $\hat{B}_{0,2}^1(\Gamma)$ satisfying $f_\mu(\Delta \hat{B}_{0,2}^1(\Gamma)) = f_\mu(\Delta \hat{B}_{0,2}^1 + \Delta \hat{B}_{0,2}^1)$ interpolating from $T\mathcal{V}_{0,3}$ to $\tilde{\mathcal{V}}_{0,3}$ in such a way that it completely avoids the vertex $\mathcal{V}_{0,3}'$, so that the resulting expression avoids any singularities.
We see then that the supertrace of a choice of symplectic connection \( \Gamma_\mu \) is determined by a choice of three-string vertex \( \tilde{V}_{0,3} \). Alternatively, a choice of symplectic connection determines \( \tilde{V}_{0,3} \) which in turn allows us to choose \( B_{0,2}^{1}(\Gamma) \).

### 3.3. Background Independence and \( \Delta S \)-Cohomology

Having recalled in some detail the origin of generalised connections, we may now return to our covariant analysis of background independence in terms of arbitrary connections. As noted in [3], the connection \( \hat{\Gamma}_\mu \) is a reference symplectic connection and can be shifted as long as we preserve the symplectic nature of the connection. Writing the canonical connection in terms of another symplectic connection, \( \hat{\Gamma}_\mu = \Gamma_\mu - \delta \Gamma_\mu \) we find,

\[
D_\mu(\hat{\Gamma})S = D_\mu(\hat{\Gamma} + \delta \Gamma)S + \{S, \delta \Gamma_\mu\},
\]

\[
\Delta \hat{\Gamma}_\mu = \Delta(\Gamma_\mu - \delta \Gamma_\mu),
\]

with \( \partial_\mu \ln \rho \) invariant. Rearranging terms, we may write the condition for background independence Eqn.(3.5) in terms of a general symplectic connection as,

\[
D_\mu(\Gamma)S = \Delta S B_\mu(\Gamma) + \hbar \Delta \Gamma_\mu - \frac{1}{2} \hbar \partial_\mu \ln \rho,
\]

where the Hamiltonian \( B_\mu(\Gamma) \) for deformations via general connections is,

\[
B_\mu(\Gamma) = B_\mu(\hat{\Gamma}) - \delta \Gamma_\mu = B_\mu(\hat{\Gamma}) - (\Gamma_\mu - \hat{\Gamma}_\mu).
\]

Clearly \( B_\mu(\Gamma) + \Gamma_\mu \) is invariant under shifts of the connection, so this final expression actually holds with respect to any reference connection \( \tilde{\Gamma}_\mu \), which may replace the canonical connection \( \hat{\Gamma}_\mu \). It follows from (3.26) that,

\[
\Delta S(D_\mu(\Gamma)S) = 0,
\]

since both \( \Delta \Gamma_\mu \) and \( \partial_\mu \ln \rho \) are field-independent. If we now require uniqueness of the master action, we can use the condition (3.28) to derive a cohomology theorem as follows.

Suppose we have a master action \( S \) satisfying the master equation. If we now perturb this action slightly by \( \lambda^\mu D_\mu S \), the new action will also satisfy the master equation,

\[
\frac{1}{2}\{S + \lambda^\mu D_\mu S, S + \lambda^\mu D_\mu S\} + \hbar \Delta(S + \lambda^\mu D_\mu S) = \lambda^\mu \Delta_S(D_\mu S) = 0.
\]

We already know from Eqn.(3.28) that these these marginal deformations are \( \Delta S \)-closed. The discussion in §6 of [2] still applies here, so that with an appropriate choice of basis we have
\[ \text{str } \Gamma_{\mu} = \partial_{\mu} \ln \rho, \text{ leaving the simplified condition,} \]
\[ D_{\mu}(\Gamma)S = \Delta S B_{\mu}, \quad (3.30) \]
which tells us that the marginal deformations are also exact. We are left to conclude that the requirement of uniqueness of the master action reduces to a cohomology theorem for the master action in that \( D_{\mu}S \), which being \( \Delta S \)-closed, must also be \( \Delta S \)-exact. We will come back to this point in more detail in §5. Let us now examine the commutator of deformations.

4. The Commutator of Background Deformations

In this section we will take a second covariant derivative of (3.26) and demonstrate the existence of a \( \Delta S \)-closed ‘field strength’ \( H_{\mu\nu} \), which in its turn will imply the existence of a Hamiltonian \( B_{\mu\nu} \) from uniqueness of the master action.

4.1. The Commutator Conditions

We shall begin with Eqn. (3.26) which expresses the background independence condition in terms of a general symplectic connection,
\[ D_{\mu}(\Gamma)S = \Delta S B_{\mu}. \quad (4.1) \]
Before proceeding, we derive the useful identity \([D_{\mu}, \Delta]F = 0\) (for arbitrary functions \( F \)) which holds if the connection is symplectic. From their definitions we have,
\[ D_{\mu}F \equiv \partial_{\mu}F - F \Gamma_{\mu}^{ij} \psi_{ij} = \partial_{\mu}F - \Gamma_{\mu}^{ij} \psi_{ij} \partial_{i} F, \quad (4.2) \]
\[ \Delta F \equiv \frac{1}{2}(-)^{i} \partial_{i}(\omega^{ij} \partial_{j} F) = \frac{1}{2}(-)^{i+j} \omega^{ij} \partial_{i} \partial_{j} F. \quad (4.3) \]
The commutator is calculated thus,
\[ [D_{\mu}, \Delta]F = \partial_{\mu}(\frac{1}{2}(-)^{i+j} \omega^{ij} \partial_{i} \partial_{j} F) - \Gamma_{\mu}^{k} \psi^{l} \partial_{k} (\frac{1}{2}(-)^{i+j} \omega^{ij} \partial_{i} \partial_{j} F) \]
\[ - \frac{1}{2}(-)^{i+j} \omega^{ij} \partial_{i} \partial_{j} (\partial_{\mu}F - \Gamma_{\mu}^{k} \psi^{l} \partial_{k} F) \]
\[ = \frac{1}{2}(-)^{i+j} \partial_{\mu} \omega^{ij} \partial_{i} \partial_{j} F + (-)^{i+j} \Gamma_{\mu}^{k} \psi^{l} \omega^{ij} \partial_{i} \partial_{j} \partial_{k} F \]
\[ + (-)^{i+j} \Gamma_{\mu}^{k} \omega^{ij} \partial_{i} \partial_{j} \partial_{k} F \]
\[ = \frac{1}{2}(-)^{i+j} \partial_{\mu} \omega^{ij} \partial_{i} \partial_{j} F + (-)^{i+j} \Gamma_{\mu}^{k} \omega^{ij} \partial_{i} \partial_{j} \partial_{k} F \]
\[ = \frac{1}{2}(-)^{i+j} (\partial_{\mu} \omega^{ij} + \Gamma_{\mu \nu}^{ij} \partial_{\nu} F - (-)^{(i+1)(j+1)} \Gamma_{\mu \nu}^{ij} \partial_{\nu} \partial_{k} F) \partial_{i} \partial_{j} F \equiv 0, \quad (4.4) \]
as the vanishing of the expression in brackets is precisely the condition for the connection to
be symplectic. (Note that the last two terms in the brackets are actually equal, though we have chosen to separate the terms as above in order to make the symplectic identity explicit).

Another result we will need is the following,

\[ \Delta R_{\mu\nu} = \frac{1}{2} (-)^i (\partial_{\mu} \Gamma^i_{\nu} - \partial_{\nu} \Gamma^i_{\mu} + \Gamma^i_{\mu j} \Gamma^j_{\nu} - \Gamma^i_{\nu j} \Gamma^j_{\mu}) \]

\[ = \frac{1}{2} (-)^i (\partial_{\mu} \Gamma^i_{\nu} - \partial_{\nu} \Gamma^i_{\mu}) \]

\[ = \partial_{\mu} \Delta \Gamma_{\nu} - \partial_{\nu} \Delta \Gamma_{\mu} \]

\[ = D_{\mu} \Delta \Gamma_{\nu} - D_{\nu} \Delta \Gamma_{\mu} , \]

(4.5)

where we have have used the fact that \( \Delta \Gamma_{\mu} \) is field-independent.

Additionally, the field-independence of \( \rho \) allows us to ignore its mixed covariant derivatives,

\[ [D_{\mu}, D_{\nu}] \ln \rho = [\partial_{\mu}, \partial_{\nu}] \ln \rho = 0 . \]

(4.6)

We are now ready to take a second covariant derivative of Eqn.(4.1). Making the connection \( \Gamma_{\mu} \) implicit we have,

\[ [D_{\mu}, D_{\nu}] S = D_{\mu} D_{\nu} S - D_{\nu} D_{\mu} S \]

\[ = D_{\mu} \Delta S B_{\nu} - D_{\nu} \Delta S B_{\mu} + h(D_{\mu} \Delta \Gamma_{\nu} - D_{\nu} \Delta \Gamma_{\mu}) \]

\[ = D_{\mu} \{ S, B_{\nu} \} - D_{\nu} \{ S, B_{\mu} \} + h(D_{\mu} \Delta B_{\nu} - D_{\nu} \Delta B_{\mu}) + h \Delta R_{\mu\nu} \]

\[ = \{ D_{\mu} S, B_{\nu} \} - \{ D_{\nu} S, B_{\mu} \} + \{ S, D_{\mu} B_{\nu} - D_{\nu} B_{\mu} \} \]

\[ + h \Delta (D_{\mu} B_{\nu} - D_{\nu} B_{\mu}) + h([D_{\mu}, \Delta] B_{\nu} - [D_{\nu}, \Delta] B_{\mu}) + h \Delta R_{\mu\nu} . \]

(4.7)

where we have made use of Eqns.(4.5) and (4.6). The result proven above allows us to discard the commutators \([D_{\mu}, \Delta]\) so that,

\[ [D_{\mu}, D_{\nu}] S = \{ \{ S, B_{\mu} \} + h \Delta B_{\mu}, B_{\nu} \} - \{ \{ S, B_{\nu} \} + h \Delta B_{\nu}, B_{\mu} \} \]

\[ + \Delta_S (D_{\mu} B_{\nu} - D_{\nu} B_{\mu}) + h \Delta R_{\mu\nu} \]

\[ = \{ \{ S, B_{\mu} \}, B_{\nu} \} - \{ \{ S, B_{\nu} \}, B_{\mu} \} + h \{ \Delta B_{\mu}, B_{\nu} \} - h \{ \Delta B_{\nu}, B_{\mu} \} \]

\[ + \Delta_S (D_{\mu} B_{\nu} - D_{\nu} B_{\mu}) + h \Delta R_{\mu\nu} \]

\[ = \{ S, \{ B_{\mu}, B_{\nu} \} \} + h \Delta \{ B_{\mu}, B_{\nu} \} + \Delta_S (D_{\mu} B_{\nu} - D_{\nu} B_{\mu}) + h \Delta R_{\mu\nu} \]

\[ = \Delta_S (\{ B_{\mu}, B_{\nu} \} + D_{\mu} B_{\nu} - D_{\nu} B_{\mu}) + h \Delta R_{\mu\nu} . \]

(4.8)

But we know that the action of the commutator is related to the antibracket with the curvature,

\[ [D_{\mu}, D_{\nu}] S = -\{ S, R_{\mu\nu} \} \]

\[ = -\Delta_S R_{\mu\nu} + h \Delta R_{\mu\nu} . \]

(4.9)
Combining Eqns.(4.8) and (4.9) we find the consistency condition,

$$\Delta S H_{\mu\nu} = 0 ,$$

(4.10)

where we have introduced the field strength,

$$H_{\mu\nu} \equiv \{ B_\mu, B_\nu \} + D_\mu B_\nu - D_\nu B_\mu + R_{\mu\nu} .$$

(4.11)

Of course this equation must be satisfied since the condition results directly from Eqn.(3.26), which has already been solved explicitly. Eqn.(4.10) implies that a perturbed master action

$$S' = S + \lambda^{\mu\nu} H_{\mu\nu}$$

also satisfies the master equation,

$$\frac{1}{2} \{ S', S' \} + \hbar \Delta S' = \frac{1}{2} \{ S + \lambda^{\mu\nu} H_{\mu\nu}, S + \lambda^{\mu\nu} H_{\mu\nu} \} + \hbar \Delta (S + \lambda^{\mu\nu} H_{\mu\nu})$$

$$= \frac{1}{2} \{ S, S \} + \hbar \Delta S + \{ S, \lambda^{\mu\nu} H_{\mu\nu} \} + \hbar \lambda^{\mu\nu} H_{\mu\nu}$$

$$= \lambda^{\mu\nu} \{ S, H_{\mu\nu} \} + \hbar \lambda^{\mu\nu} \Delta H_{\mu\nu}$$

$$= \lambda^{\mu\nu} \Delta S H_{\mu\nu}$$

$$= 0 .$$

(4.12)

The hope that the master action be unique up to gauge transformations would require that the perturbed action be merely a field redefined version of the original. This is so if there exists a Hamiltonian $B_{\mu\nu}$ such that,

$$H_{\mu\nu} = \Delta S B_{\mu\nu} .$$

(4.13)

So we see that the existence of $B_{\mu\nu}$ or alternatively, uniqueness of the string action, implies a (higher) cohomology theorem for the string action which in turn implies quantum background independence of the string action with respect to commutators of deformations.

4.2. Analysis of Gauge Freedom of $B_{\mu\nu}$

Having postulated the existence of the object $B_{\mu\nu}$, let us now explore the extent to which it is uniquely defined.

It was shown for the classical case in [3] that shifting the connection whilst retaining the symplectic property does not alter $H_{\mu\nu}$. An identical argument which we need not repeat here shows that this statement also holds in the quantum case, so that $B_{\mu\nu}$ does not depend on the particular choice of symplectic connection.
From the nilpotency of $\Delta_S$, any shift of $B_\mu$ by a $\Delta_S$-trivial object will clearly also satisfy the background independence condition Eqn.(3.26),

$$B_\mu \rightarrow B_\mu + \Delta_S\lambda_\mu.$$  \hspace{1cm} (4.14)

This results in a corresponding shift in $H_{\mu\nu}$ given by,

$$H_{\mu\nu} \rightarrow H_{\mu\nu} + \{\Delta_S\lambda_\mu, B_{\nu}\} + \{B_\mu, \Delta_S\lambda_\nu\} + D_\mu(\Delta_S\lambda_\nu) - D_\nu(\Delta_S\lambda_\mu)$$

$$\rightarrow H_{\mu\nu} + \Delta_S\{\lambda_\mu, B_{\nu}\} + \{\lambda_\mu, \Delta_S B_{\nu}\} + \Delta_S\{B_\mu, \lambda_\nu\} - \{\Delta_S B_\mu, \lambda_\nu\}$$

$$+ D_\mu(\{S, \lambda_\nu\} + \hbar \Delta \lambda_\nu) - D_\nu(\{S, \lambda_\mu\} + \hbar \Delta \lambda_\mu)$$

$$\rightarrow H_{\mu\nu} + \Delta_S\{B_\mu, \lambda_\nu\} - \{B_\nu, \lambda_\mu\} - \{\Delta_S B_\mu, \lambda_\nu\} + \{\Delta_S B_\nu, \lambda_\mu\}$$

$$+ \hbar (D_\mu \Delta \lambda_\nu - D_\nu \Delta \lambda_\mu) + \{D_\mu S, \lambda_\nu\} + \{S, D_\mu \lambda_\nu\} - \{D_\nu S, \lambda_\mu\} - \{S, D_\nu \lambda_\mu\}.$$ \hspace{1cm} (4.15)

We can now use Eqn.(4.1) and the fact that $D_\mu$ and $\Delta$ commute,

$$H_{\mu\nu} \rightarrow H_{\mu\nu} + \Delta_S\{B_\mu, \lambda_\nu\} - \{B_\nu, \lambda_\mu\} + \{S, D_\mu \lambda_\nu - D_\nu \lambda_\mu\} + \hbar \Delta (D_\mu \lambda_\nu - D_\nu \lambda_\mu)$$

$$+ \{\Delta_S B_\mu, \lambda_\nu\} - \{\Delta_S B_\nu, \lambda_\mu\} - \{\Delta_S B_\nu, \lambda_\mu\} + \{\Delta_S B_\nu, \lambda_\mu\}$$

$$\rightarrow H_{\mu\nu} + \Delta_S\{B_\mu, \lambda_\nu\} - \{B_\nu, \lambda_\mu\} + \{D_\mu \lambda_\nu - D_\nu \lambda_\mu\} + \{S, D_\mu \lambda_\nu\} - \{D_\nu S, \lambda_\mu\} - \{S, D_\nu \lambda_\mu\}.$$ \hspace{1cm} (4.16)

where $D_\mu \equiv \{B_\mu, \cdot\} + D_\mu$ is the ‘gauge covariant derivative’ introduced in [3]. Given that we seek $B_{\mu\nu}$ such that $H_{\mu\nu} = \Delta_S B_{\mu\nu}$, the shift $B_\mu \rightarrow B_\mu + \Delta_S\lambda_\mu$ must correspond to a non-trivial gauge freedom,

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + D_\mu \lambda_\nu - D_\nu \lambda_\mu.$$ \hspace{1cm} (4.17)

There is also of course a trivial gauge freedom under $B_{\mu\nu} \rightarrow B_{\mu\nu} + \Delta_S\lambda_{\mu\nu}$. We will give a detailed interpretation of these in the sequel.

4.3. Consistency Conditions and Recursion Relations for Moduli Spaces

We will now examine explicitly the consistency conditions derived in §4.1,

$$D_\mu B_{\nu} - D_\nu B_\mu + \{B_\mu, B_{\nu}\} + R_{\mu\nu} = \Delta_S B_{\mu\nu}.$$ \hspace{1cm} (4.18)

The aim of the present section is to show that the Hamiltonian $B_{\mu\nu}$ is the function associated to some moduli spaces of Riemann surfaces of any genus and with two special punctures. We will follow the analysis of [3] to derive the recursion relations which must be satisfied by the higher genus moduli spaces that define the Hamiltonian $B_{\mu\nu}$.
We therefore take $B_{\mu\nu}$ to be a Hamiltonian of the form,

$$B_{\mu\nu} = - f_{\mu\nu}(B^2) = - \int_{B^2} \langle \Omega_{12} | \hat{O}_\mu \rangle_1 | \hat{O}_\nu \rangle_2,$$

(4.19)

where $B^2$ is a sum of moduli spaces of surfaces with two special punctures, now extended to include higher genus terms.

In terms of moduli spaces the right hand side of Eqn.(4.18) may be written,

$$\Delta_S B_{\mu\nu} = -\{S, f_{\mu\nu}(B^2)\} - \hbar \Delta f_{\mu\nu}(B^2)$$

$$= -\{Q + f(V) + \hbar S_{1,0}, f_{\mu\nu}(B^2)\} + \hbar f_{\mu\nu}(\Delta B^2)$$

$$= f_{\mu\nu}(\partial B^2 + \{V, B\} + \hbar \Delta B^2)$$

$$= f_{\mu\nu}(\delta V B^2).$$

(4.20)

Recalling (from §3.2 of [3]) that the left hand side of (4.18) is independent of the connection, we may simply apply the results derived in §6 of [3] for the canonical connection (noting of course that $B^1$ now includes the spaces of higher genus) which tells us that,

$$D_\mu B_\nu - D_\nu B_\mu + \{B_\mu, B_\nu\} + R_{\mu\nu} = f_{\mu\nu}(T_{0,1}^2 + (K - I)B^1 - \frac{1}{2}\{B^1, B^1\}).$$

(4.21)

Putting these together, Eqn.(4.18) becomes,

$$f_{\mu\nu}(T_{0,1}^2 + (K - I)B^1 - \frac{1}{2}\{B^1, B^1\}) = f_{\mu\nu}(\delta V B^2).$$

(4.22)

This equation will be satisfied if,

$$\delta V B^2 = T_{0,1}^2 + (K - I)B^1 - \frac{1}{2}\{B^1, B^1\}.$$ 

(4.23)

This has the same form as the classical formula, except that $\delta V$ now contains in its definition the additional operator $\hbar \Delta$.

Let us verify the consistency of Eqn.(4.23) by checking that $\delta V$ acting on the RHS vanishes. Consider each term separately we find,

$$\delta V T_{0,1}^2 = \partial T_{0,1}^2 + \{V, T_{0,1}^2\} + \hbar \Delta T_{0,1}^2 = T_{0,1}^2 + \{V, T_{0,1}^2\},$$

(4.24)
\[ \delta_Y(\mathcal{K} - \mathcal{I})B^1 = [\delta_Y, \mathcal{K} - \mathcal{I}]B^1 + (\mathcal{K} - \mathcal{I})\delta_Y B^1 \]
\[ = \{V'_{0,3} + (\mathcal{K} - \mathcal{I})V, B^1\} + (\mathcal{K} - \mathcal{I})(V'_{0,3} + (\mathcal{K} - \mathcal{I})V + h\Delta B^1_{0,2} + h\mathcal{I}V_{1,1}) \]
\[ = \{V'_{0,3} + (\mathcal{K} - \mathcal{I})V, B^1\} - \mathcal{I}V'_{0,3} - \{V, \mathcal{I}V_{1,1}\} + h\mathcal{K}\Delta B^1_{0,2} + h\mathcal{K}\mathcal{I}V_{1,1}, \]
\[ \delta_Y(-\frac{1}{2}\{B^1, B^1\}) = -\{\delta_Y B^1, B^1\} = -\{V'_{0,3} + (\mathcal{K} - \mathcal{I})V, B^1\} \] (4.26)

We remind ourselves from the discussion of §3.3 that \( B^1_{0,2} \) interpolates between \( \mathcal{I}V_{0,3} \) and some vertex \( \tilde{V}_{0,3} \) determined by the choice of connection, so that there is no longer the unwanted singularity associated with \( \Delta V'_{0,3} \). So Eqn.(4.23) is consistent if,
\[ \mathcal{K}(\Delta B^1_{0,2} + \mathcal{I}V_{1,1}) = 0. \] (4.27)

Both terms in this expression consists of the operator \( \mathcal{K} \) acting on a torus or tori with a single special puncture. It is fairly simple to see why each term must vanish.

Consider a torus \( T^1_{1,0} \) with a single special puncture. The operator \( \mathcal{K} \) adds another special puncture over the remainder of the surface of the torus, antisymmetrising with respect to the two punctures. The translational symmetry of the torus means that for any relative position of the two punctures of any torus in \( \mathcal{K}T^1_{1,0} \) there will be another torus with the two punctures with positions reversed. The antisymmetrising property of \( \mathcal{K} \) ensures that this pair of twice-punctured tori will occur with the opposite sign and cancel. This pairwise cancellation means that each term in Eqn.(4.27) vanishes, thus verifying the consistency of Eqn.(4.23)and thereby Eqn.(4.19).

5. \( \Delta_S \)-Cohomology Classes and Theory Space Geometry

Having introduced \( B \)-spaces with two special punctures in the previous section, we will outline in this section how uniqueness of the master action implies the existence of \( B \)-spaces with more than two punctures. We will show that the quantum generalisation of the analysis [4] has an efficient description in terms of differential forms on the theory space manifold which is parametrised by the marginal operators.

Let us consider a basis of marginal states, \( \{ |\tilde{O}_1 \rangle, \ldots, |\tilde{O}_N \rangle \} \), where \( N \) may be infinite. For a given point \( S \) in the theory space manifold, these form a basis of the tangent space \( T_*M_S \). This \( N \)-dimensional theory space manifold \( M_S \) may therefore be parametrised locally
by coordinates \( \{x^1, \ldots, x^N\} \). We will use the following notation for differential forms on \( T^*M_S \),
\[
A_{(n)} \equiv \frac{1}{n!} A_{\mu_1 \ldots \mu_n} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}.
\]  
(5.1)

In this language the \( B_\mu \) are components of a one-form ‘gauge field’ \( B_{(1)} = B_\mu dx^\mu \), and the \( H_{\mu\nu} \) are components of a two-form field strength \( H_{(2)} = \frac{1}{2} H_{\mu\nu} dx^\mu \wedge dx^\nu \) such that \( H_{(2)} = \frac{1}{2} \Delta S B_{(2)} = \Delta S B_{\mu\nu} dx^\mu \wedge dx^\nu \).

It is convenient also to introduce a kind of ‘gauge-covariant exterior derivative’ \( D = D + \{B, \cdot\} \) (not to be confused with the actual exterior derivative \( d \) on \( T^*M_S \)), which is defined by,
\[
DA_{(n)} \equiv DA_{(n)} + \{B_{(1)}, A_{(n)}\}
\]
\[
= \frac{1}{n!} D_{\mu_0} A_{\mu_1 \ldots \mu_n} \, dx^{\mu_0} \wedge \cdots \wedge dx^{\mu_n}
\]
\[
= \frac{1}{n!} (D_{\mu_0} A_{\mu_1 \ldots \mu_n} + \{B_{\mu_0}, A_{\mu_1 \ldots \mu_n}\}) \, dx^{\mu_0} \wedge \cdots \wedge dx^{\mu_n}.
\]  
(5.2)

If we define the antibracket of two forms by,
\[
\{A_{(n)}, C_{(m)}\} = \frac{1}{n!m!} \{A_{\mu_1 \ldots \mu_n}, C_{\mu_{n+1} \ldots \mu_{n+m}}\} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \wedge dx^{\mu_{n+1}} \wedge \cdots \wedge dx^{\mu_{n+m}},
\]  
(5.3)

then \( D \) has the property,
\[
D\{A_{(n)}, C_{(m)}\} = \{DA_{(n)}, C_{(m)}\} + (-)^n \{A_{(n)}, DC_{(m)}\}.
\]  
(5.4)

As a useful identity, we show that \( D \) commutes with \( \Delta_S \),
\[
[D, \Delta] A_{(n)} = \Delta(DA_{(n)} + \{B_{(1)}, A_{(n)}\}) + \{S, DA_{(n)} + \{B_{(1)}, A_{(n)}\}\} - D\Delta S A_{(n)}
\]
\[
= D\Delta A_{(n)} + \{\Delta B_{(1)}, A_{(n)}\} + \{B_{(1)}, \Delta A_{(n)}\} + D\{S, A_{(n)}\}
\]
\[
- \{DS, A_{(n)}\} + \{B_{(1)}, \{S, A_{(n)}\}\} + \{\{S, B_{(1)}\}, A_{(n)}\} - D\Delta S A_{(n)}
\]
\[
= D\Delta S A_{(n)} - \{DS, A_{(n)}\} + \{\Delta S B_{(1)}, A_{(n)}\} - D\Delta S A_{(n)}
\]
\[
= 0.
\]  
(5.5)

Another useful property is the following,
\[
DDA_{(n)} = \frac{1}{n!} D_{\mu_0} D_{\mu_1} A_{\mu_1 \ldots \mu_n} \, dx^{\mu_0} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}
\]
\[
= \frac{1}{2n!} [D_{\mu_0}, D_{\mu_1}] A_{\mu_1 \ldots \mu_n} \, dx^{\mu_0} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}
\]
\[
= \frac{1}{2n!} (H_{\mu_0\mu_1}, A_{\mu_1 \ldots \mu_n}) \, dx^{\mu_0} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}
\]
\[
= \{H_{(2)}, A_{(n)}\}.
\]  
(5.6)

where we have made use of the identity \([D_\mu, D_\nu] = \{H_{\mu\nu}, \cdot\}\) (note the sign correction to
Refs. [3] and [4]).

Let us now proceed to show how to recursively construct in a simple manner the \( n \)-form field strengths \( H_{(n)} \) and gauge fields \( B_{(n)} \) for all \( n \geq 2 \). Treating the modified action \( \tilde{S} \equiv S + \frac{1}{2} \hbar \ln \rho \) as a zero-form and \( B_\mu \) and \( \Gamma_\mu \) as components of one-forms, we may write the background independence conditions Eqn.(3.26) as,

\[
\mathcal{D} \tilde{S} = \Delta (B_{(1)} + \Gamma_{(1)}).
\]  

(5.7)

Acting once again with the gauge-covariant exterior derivative,

\[
\mathcal{D} \mathcal{D} \tilde{S} = \mathcal{D}_\mu \Delta (B_\nu + \Gamma_\nu) dx^\mu \wedge dx^\nu
\]

\[
= \Delta H_{(2)}.
\]  

(5.8)

But we know from Eqn.(5.6) that \( \mathcal{D} \mathcal{D} \tilde{S} = \{H_{(2)}, \tilde{S}\} \), from which immediately follows the result we derived earlier (now written in terms of forms),

\[
\Delta_S H_{(2)} = 0.
\]  

(5.9)

Now, by the same argument which was used in Eqn.(3.29), we know that we can add to the action any \( \Delta_S \)-closed function to get a new action also satisfying the master equation. The hope that the master action be unique implies no non-trivial \( \Delta_S \)-cohomology, which leads us naturally to the requirement that \( H_{(2)} \) be \( \Delta_S \)-exact,

\[
H_{(2)} = \Delta_S B_{(2)}.
\]  

(5.10)

We have already shown that the Hamiltonian \( B_{\mu\nu} \) may be obtained from moduli spaces of surfaces with two special punctures.

The procedure to construct higher forms goes as follows. Let us define an auxiliary three-form,

\[
H'_{(3)} = \mathcal{D} B_{(2)}.
\]  

(5.11)

Acting on this with \( \Delta_S \), we find,

\[
\Delta_S H'_{(3)} = \Delta_S \mathcal{D} B_{(2)} = \mathcal{D} \Delta_S B_{(2)} = \mathcal{D} H_{(2)} = 0,
\]  

(5.12)

where we have used Eqns.(5.5), (5.10) and finally the Bianchi identity for \( H_{\mu\nu} \), Eqn.(2.26) of
This means we can simply choose our $\Delta_S$-closed three-form field strength to be,

$$H(3) \equiv H'(3) = DB(2),$$

which is a condensed way of expressing the analogous classical result Eqn.(2.27) of [4]. Once again, uniqueness of the master action requires the existence of a corresponding three-form gauge field such that,

$$H(3) = \Delta_S B(3).$$

Finding the four-form field strength is still simple, albeit not quite as trivial. We first define an auxiliary four-form,

$$H'(4) = DB(3).$$

Acting with $\Delta_S$ gives a long chain of identities,

$$\Delta_S H'(4) = \Delta_S DB(3) = D\Delta_S B(3) = DB(3) = DDB(2)$$

$$= \{H(2), B(2)\} = \{\Delta_S B(2), B(2)\} = \frac{1}{2}\Delta_S \{B(2), B(2)\}. \quad (5.16)$$

From this we can extract the $\Delta_S$-closed four-form field strength,

$$H(4) \equiv DB(3) - \frac{1}{2}\{B(2), B(2)\}. \quad (5.17)$$

Note that this simplified expression agrees with Eqn.(2.33) of [4]. By repeating the same procedure that is, defining an auxiliary $p$-form by $H'(p) = DB_{(p-1)}$, and then acting upon it with $\Delta_S$ to eventually extract a $\Delta_S$-closed $p$-form $H(p)$, we may construct all higher $n$-forms $H(n) = \Delta_S B(n)$ ad nauseam. We will refer to this last equality as the $n$-th vanishing theorem for $\Delta_S$ cohomology classes. Indeed one can shown by induction that the general formula is,

$$H(n) = DB_{(n-1)} + \frac{1}{2}\sum_{m=2}^{n-2} (-1)^{m+1}\{B(m), B(n-m)\} \quad (n > 2). \quad (5.18)$$

Generalising the results which have already been demonstrated for $B_\mu$ and $B_{\mu\nu}$, we can assume that the antisymmetric coefficients of the $n$-form Hamiltonians $B(n)$ are given by functions (with the appropriate marginal state insertions), of moduli spaces with $n$ special
punctures,

\[ B_{\mu_1 \cdots \mu_n} = -f_{\mu_1 \cdots \mu_n}(B^n) = -\frac{1}{n!\bar{n}!} \int_{B^n} \langle \Omega_{\bar{1} \cdots \bar{n}} | \hat{\mathcal{O}}_{\mu_1} \rangle_{\bar{1}} \cdots | \hat{\mathcal{O}}_{\mu_n} \rangle_{\bar{n}}. \]  

(5.19)

We note that \( B^n = \sum_{g,k \geq 0} B^n_{g,k} \) extends over a complete set of positive-dimensional moduli spaces of punctured Riemann surfaces for all genera, and all numbers of ordinary punctures compatible with the dimensionality requirement.

Just as before, these \( B^n \)-spaces may be explicitly constructed using their recursion relations, but we should defer this task until the sequel [8], when we have before us the complete \( \mathcal{B} \)-complex and corresponding recursion relations.

In summary, what he have shown is that the requirement of unique physics implies the need for background independence. The statement of uniqueness and background independence at the \( p \)-th order of deformations implies the \( p \)-th vanishing theorem for the \( \Delta_S \) cohomology class of the master action, so that uniqueness and background independence to all orders implies a set of \( \Delta_S \) cohomology vanishing theorems for the master action \( S \). In addition, the space of equivalent theories related by marginal deformations is simply the equivalence class (contained in the manifold \( M_S \)) of the \( \Delta_S \)-cohomology of which the action \( S \) is a representative.

6. Conclusion

In this paper we have generalised the work [3] of Zwiebach from the classical to the quantum case, showing that uniqueness and background independence of the master action to all orders implies a set of cohomology vanishing theorems for the closed string action, and have postulated the existence of all punctured higher genus interpolating moduli spaces \( \mathcal{B}^n_{g,n} \) of positive dimension.

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We collect here for reference some useful formulae used in this paper. We assume that the states $|O_\mu\rangle$ are BRST-closed.

\[
\{A, B\} = -(-)^{(A+\bar{n}_A+1)(B+\bar{n}_B+1)}\{B, A\}, \quad (A.1)
\]

\[
(-)^{(A+\bar{n}_A+1)(C+\bar{n}_C+1)}\{A, B, C\} + \text{cycl.} = 0, \quad (A.2)
\]

\[
(-)^{(A+\bar{n}_A+1)(C+\bar{n}_C+1)}\{A, \{B, C\}\} + \text{cycl.} = 0, \quad (A.3)
\]

\[
\Delta \{A, B\} = \{\Delta A, B\} + (-)^{A+\bar{n}_A+1}\{A, \Delta B\}, \quad (A.4)
\]

\[
\Delta S = 0, \quad (A.5)
\]

\[
\Delta S \{A, B\} = \{\Delta S A, B\} + (-)^{A+\bar{n}_A+1}\{A, \Delta S B\}, \quad (A.6)
\]

\[
D_\mu \{A, B\} = \{D_\mu A, B\} + \{A, D_\mu B\}, \quad (A.7)
\]

\[
f_{\mu\nu}(\{V, A\}) = -\{f(V), f_{\mu\nu}(A)\}, \quad (A.8)
\]

\[
\delta_V \delta_V A = 0, \quad (A.9)
\]

\[
[\delta_V, \mathcal{K}] A = (-)^{A+\bar{n}_A}\{V_0', A\}, \quad (A.10)
\]

\[
[\delta_V, T] A = (-)^{A+\bar{n}_A}\{TV, A\}. \quad (A.11)
\]

\[
KK A = 0, \quad (A.12)
\]

\[
(KT + TK) A = \{A, T^2_{0,1}\}. \quad (A.13)
\]

\[
\Pi A = 0. \quad (A.14)
\]

\[
\partial T^2_{0,1} = TV_0', \quad (A.15)
\]
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