HYPERSURFACE SINGULARITIES IN POSITIVE CHARACTERISTIC.

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Abstract. The paper is motivated on the open problem of resolution of singularities in positive characteristic. The aim is to present a form of induction which is different from that used by Hironaka. In characteristic zero induction is formulated by restriction to smooth hypersurfaces (hypersurfaces of maximal contact). Our alternative approach, introduced here, replaces restrictions to smooth sub-schemes by generic projections on smooth schemes of smaller dimension. We introduce a generalization of the discriminant, and our result makes use of the elimination theory. In the case of fields of characteristic zero, elimination gives exactly the same information as the form of induction used by Hironaka.

The properties of this new form of elimination remain weaker in positive characteristic, than it does in characteristic zero, when it comes to resolution of singularities. But it opens the way to new invariants for this open problem.

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Part 1. Introduction.

Let $(S, M)$ denote a local ring, and fix a monic polynomial $f(Z)$, of degree $b$ in $S[Z]$. This defines a finite ring extension $S \subset S[Z]/\langle f(Z) \rangle$, and hence a finite morphism

\[(0.0.1) \quad Spec(S[Z]/\langle f(Z) \rangle) \xrightarrow{\sigma} Spec(S).\]

Date: Oct. 2006.

Key words and phrases. Singularities. Differential operators.

2000 Mathematics subject classification. 14E15.
It seems clear that a lot of information of this finite morphism is encoded in the coefficients of the monic polynomial \( f(Z) \). Higher differential operators in the variable \( Z \), applied to \( f(Z) \), also relate to ramification theoretical methods and to multiplicity theory.

The clarification of this point, initiated in [36], is the objective of Section 1; where we present the fundamental results on which our development will rely. The tools developed in this first section will open the way for explicit computation in the last section.

Briefly speaking, in the first section we produce a \textit{universal} morphism \( \pi \), on which the permutation group acts; and our invariants arise as an invariant subring by this group action. The discriminant is an element of the invariant subring.

The Weierstrass Preparation Theorem enables us to reduce the local study of hypersurfaces, embedded in a smooth \( d \)-dimensional scheme, to the study of finite (ramified) covers defined at a point of a smooth \((d-1)\)-dimensional scheme, as we indicate below.

We recall now how higher order differential operators relate to multiplicity theory, in order to motivate the development in the coming sections.

Let \( V \) be a smooth scheme over a perfect field \( k \), and let \( J \subset \mathcal{O}_V \) be a non-zero sheaf of ideals. For example take \( J \) to be the sheaf of ideals defining a hypersurface. Define a function, say

\[
ord_J : V \rightarrow \mathbb{Z},
\]

where \( \text{ord}_J(x) \) denotes the order of \( J_x \) at the local regular ring \( \mathcal{O}_{V,x} \). Let \( b \) denote the biggest value achieved by this function (the biggest order of \( J \)). The pair \( (J,b) \) is the \textit{object of interest} in resolution of singularities and in Log-principalization of ideals. There is a closed set attached to this pair in \( V \), namely the set of points where \( J \) has order \( b \). So if \( J \) is a locally principal ideal (defining a hypersurface), the closed set is the set of points of multiplicity \( b \) at the hypersurface.

For every non-negative integer \( s \), the sheaf of \( k \)-linear differential operators of order \( s \), say \( \text{Diff}^s_k \), is coherent and locally free over \( V \). There is a natural identification, say \( \text{Diff}^0_k = \mathcal{O}_V \), and for each \( s \geq 0 \) there is a natural inclusions \( \text{Diff}^s_k \subset \text{Diff}^{s+1}_k \).

If \( U \) is an affine open set in \( V \), each \( D \in \text{Diff}^s_k(U) \) is a differential operator: \( D : \mathcal{O}_V(U) \rightarrow \mathcal{O}_V(U) \). We define an extension of the sheaf of ideals \( J \subset \mathcal{O}_V \), say \( \text{Diff}^s_k(J) \), so that over the affine open set \( U \), \( \text{Diff}^s_k(J)(U) \) is the extension of \( J(U) \) obtained by adding all elements \( D(f) \), for all \( D \in \text{Diff}^s_k(U) \) and \( f \in J(U) \). One can check that \( \text{Diff}^0(J) = J \), and that \( \text{Diff}^s(J) \subset \text{Diff}^{s+1}(J) \) as sheaves of ideals in \( \mathcal{O}_V \). Let \( V(I) \subset V \) denote the closed set defined by an ideal \( I \subset \mathcal{O}_V \). So

\[
V(J) \supset V(\text{Diff}^1(J)) \supset \cdots \supset V(\text{Diff}^{s-1}(J)) \supset V(\text{Diff}^s(J)) \ldots
\]

It is simple to check that the order of \( J \) at the local regular ring \( \mathcal{O}_{V,x} \) is \( \geq s \) if and only if \( x \in V(\text{Diff}^{s-1}(J)) \).

The previous observations say that \( \text{ord}_J : V \rightarrow \mathbb{Z} \) is an upper-semi-continuous function, and that the highest order of \( J \) (at points \( x \in V \)) is \( b \) if \( V(\text{Diff}^b(J)) = \emptyset \) and \( V(\text{Diff}^{b-1}(J)) \neq \emptyset \).
There is a notion of transformation of pairs \((J, b)\), defined by monoidal transformations. Let

\[ V \xleftarrow{\pi} V_1 \supset H \]

denote the blow up of \(V\) at a smooth sub-scheme \(Y\), where \(\pi^{-1}(Y) = H\) is the exceptional hypersurface. If \(Y \subset V(\text{Diff} f^{b-1}(J))\) we say that \(\pi\) is \(b\)-permissible. In this case set

\[ J\mathcal{O}_{V_1} = I(H)^b J_1, \]

where \(I(H)\) is the sheaf of functions vanishing along the exceptional hypersurface \(H\).

If \(J\) has highest order \(b\), and if \(\pi\) is \(b\)-permissible, \(J_1\) also has at most order \(b\) at points of \(V_1\) (i.e., \(V(\text{Diff} f^b(J_1)) = \emptyset\)). If, in addition, \(J_1\) has no points of order \(b\), then we say that \(\pi\) defines a \(b\)-simplification of \(J\).

If \(V(\text{Diff} f^{b-1}(J_1)) \neq \emptyset\), let \(V_1 \xleftarrow{\pi_1} V_2\) denote a monoidal transformation with center \(Y_1 \subset V(\text{Diff} f^{b-1}(J_1))\). We say that \(\pi_1\) is \(b\)-permissible, and set

\[ J_1 \mathcal{O}_{V_2} = I(H_1)^b J_2. \]

So again \(J_2\) has at most points of order \(b\). If it does, define a \(b\)-permissible transformation at some smooth center \(Y_2 \subset V(\text{Diff} f^{b-1}(J_2))\).

For \(J\) and \(b\) as before, we define, by iteration, a \(b\)-permissible sequence:

\[ V \xleftarrow{\pi} V_1 \xleftarrow{\pi_1} V_2 \xleftarrow{\pi_2} \ldots V_{n-1} \xleftarrow{\pi_n} V_n, \]

and a factorization \(J_{n-1} \mathcal{O}_{V_n} = I(H_n)^b J_n\). Let \(H_i \subset V_n\) denote the strict transform of the exceptional hypersurface \(H_i \subset V_{i-1}, 1 \leq i \leq n\). Note that:

1) \(\{H, H_1, \ldots, H_{n-1}\}\) are components of the exceptional locus of \(V \leftarrow V_n\).

2) The total transform of \(J\) relates to \(J_n\) by an expression of the form:

\[ J \mathcal{O}_{V_n} = I(H)^{a_0} I(H_1)^{a_1} \cdots I(H_{n-1})^{a_0} J_n. \]

We say that this \(b\)-permissible sequence defines a \(b\)-simplification of \(J \subset \mathcal{O}_V\) if \(\cup H_i\) has normal crossings, and \(V(\text{Diff} f^{b-1}(J_n)) = \emptyset\) (i.e., \(J_n\) has order at most \(b - 1\) at \(V_n\)).

When \(k\) is a field of characteristic zero, and \(b\) is the highest order of a sheaf of ideals \(J \subset \mathcal{O}_V\), Hironaka proves that there is a \(b\)-simplification. A Log-principalization of \(J\) is achieved when \(J_n\) has at most order zero (i.e., when \(J_n = \mathcal{O}_{V_n}\)).

The key point for \(b\)-simplification, already used in Hironaka’s proof, is a form of induction. In fact, Hironaka proves \(b\)-simplification by induction on the dimension of the ambient space \(V\). To simplify matters, assume that \(J\) is locally principal, and let \(b\) denote the highest order of \(J\) along points in \(V\), which we take now to be a smooth scheme over a field of characteristic zero. Let

\[ \{\text{ord}_J \geq b\} \]

denote the closed set \(\{x \in V/\text{ord}_J(x) \geq b\}\) (or say = \(b\)).

Fix a closed point \(x \in \{\text{ord}_J \geq b\}\), and a regular system of parameters \(\{x_1, x_2, \ldots, x_n\}\) for \(\mathcal{O}_{V,x}\). For every \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\), set \(|\alpha| = \alpha_1 + \cdots + \alpha_n\), and

\[ \Delta^\alpha = \left( \frac{1}{\alpha_1!} \cdots \frac{1}{\alpha_n!} \right) \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}. \]
If $J_x$ is locally generated by $f \in \mathcal{O}_{V,x}$, then $f$ has order $b$ at $\mathcal{O}_{V,x}$, and

$$(\text{Diff}^{b-1}(J))_x = \langle f, \Delta^\alpha(f) / 0 \leq |\alpha| < b \rangle.$$  

The relevant property will be that the order of $(\text{Diff}^{b-1}(J))_x$ at $\mathcal{O}_{V,x}$ is one. This holds when $k$ is a field of characteristic zero.

Recall that, locally at $x$, $V(\text{Diff}^{b-1}(< f >)) = \{\text{ord} \geq b\}$ locally. One way to check that $(\text{Diff}^{b-1}(J))_x$ has order one at $\mathcal{O}_{V,x}$, is to check this at the completion $\mathcal{O}_{V,x}$, say $R = k'[[x_1, \ldots, x_n]]$. We may choose the system of parameters so that, for a suitable unit $u$:

$$uf = f_1 = Z^b + a_1 Z^{b-1} + \cdots + a_b \in S[Z],$$

where $S = k'[[x_1, \ldots, x_{n-1}]]$, and $Z = x_n$. As $k$ is a field of characteristic zero, $S[Z] = S[Z_1]$, where $Z_1 = Z + \frac{1}{b}a_1$, and we obtain a Tschirnhausen polynomial:

$$f_1 = Z_1^b + a'_1 Z_1^{b-2} + \cdots + a'_b$$

($a'_1 = 0$). Then:

A) $Z_1 \in \text{Diff}^{b-1}(f)$ (in fact $\frac{\partial^{b-1} f}{\partial x^n} \in \text{Diff}^{b-1}(f)$). In particular the ideal $\text{Diff}^{b-1}(f)$ has order one at $x$, and the closed set $\{\text{ord} \geq b\}$ is locally included in a smooth scheme of dimension $n - 1$.

B) (Elimination of one variable.) $\{\text{ord} f \geq b\}(\subset V(Z_1))$ can be described as

$$\{\text{ord} f \geq b\} = \cap_{2 \leq i \leq b} \{\text{ord} a'_i \geq i\}.$$  

C) (Stability: Elimination and monoidal transformations.) Both A), and the description in B), are preserved by $b$-permissible transformations.

We will not go into the details of A), B) and C). But let us point out that there is an elimination of one variable in (B). In fact the closed set $\{\text{ord} f \geq b\}$ defined in terms of $f$, is also described as $\cap_{2 \leq i \leq b} \{\text{ord} a'_i \geq i\}$, where now the $a'_i$ involve one variable less.

As it was previously indicated, A),B), and C), together, conform the essential reason and argument in resolution of singularities in characteristic zero. They rely entirely on the hypothesis of characteristic zero. For instance A) does not hold over fields of positive characteristic; so there is no way to formulate this form of induction over arbitrary fields (see also [18]).

On this paper. The objective of these notes is to report on a different approach to the elimination in B), that can be formulated over arbitrary fields, which we discuss below (see also [37]). Here the restriction to a smooth hypersurface in A) will be replaced by a projection, as in the Weierstrass Preparation Theorem.

In Section 1 we fix a finite morphism $\text{Spec}(S[Z]/(f(Z))) \rightarrow \text{Spec}(S)$, defined by $f(Z) = Z^b + a_1 Z^{b-1} + \cdots + a_b \in S[Z]$, and study a closed set, say $F$, in $\text{Spec}(S)$ over which the finite map is completely ramified, as we indicate below. The equations defining this closed set will be given by polynomial expressions on the coefficients. Among these equations is the discriminant of $f(Z)$.
This will lead us to the study of universal polynomials that can be evaluated on the coefficients of \( f(Z) \). These universal polynomials will be obtain via invariant theory, and they are to be thought of as generalized discriminants.

When evaluated on the coefficients of \( f(Z) \), these generalized discriminants define \( F \subset \text{Spec}(S) \). The main result in Section 1 is Theorem 1.16 which characterizes \( F \) in \( \text{Spec}(S) \) in terms of ramification theory. It is the biggest closed subset so that there is a natural bijections \( \sigma : \sigma^{-1}(F) \to F \). More precisely, it is the set of points over which the geometric fiber of \( \sigma \) is a unique point.

The connexion of this Theorem with multiplicity theory relies on the fact that the \( b \)-fold points of the hypersurface defined by \( f(Z) \) in \( \text{Spec}(S[Z]) \) are included in \( \sigma^{-1}(F) \). This property will ultimately ensure a form of compatibility of elimination with monoidal transformations, in the sense of C), addressed in the last Section.

Generalized discriminants are also endowed with a natural weight, a matter to be carefully discussed in Section 1. For example, if \( D(a_1,a_2,\ldots,a_b) \) denotes the discriminant of \( f(Z) \), then it is the evaluation on the \( a_i \)'s of a universal polynomial, say \( D(Y_1,Y_2,\ldots,Y_b) \), which is weighted homogeneous. Equations, as these, endowed with a weight, lead us to the notion of Rees algebras that we discuss below. Appendices 1) and 2) in Section 1 can be avoided in a first look at the paper.

In Section 2 and Section 3 we study the notion of Rees algebras. Here we fix \( V \), a smooth scheme over a perfect field \( k \). A Rees algebras will be of form \( G = \bigoplus_{k\geq 0} I_k W^k \), \( I_0 = \mathcal{O}_V \), and each \( I_k \) is an ideal in \( \mathcal{O}_V \). It is a graded subring of \( \mathcal{O}_V [W] \).

There are two reasons that justify our attention on these algebras:

i) There is a notion of resolution of Rees algebras. Moreover, the problem of resolution of singularities reduces to that of resolution of Rees algebras.

ii) Rees algebras can be naturally enriched by the action of higher order differentials.

As for ii) let us indicate simply that these algebras come with a grading. And it is with this grading that one can naturally extend them to new Rees algebras enriched with differential operators. To be precise, each \( G \) can be extended to say \( G \subset G' \) where now \( G' \) is what we call a differential Rees algebra, or simply a Diff-algebra. These are Rees algebras which are, in some sense, compatible with differential operators on \( V \). It will be shown that there is a smallest Diff-algebra containing \( G \), say \( G \subset Diff(G) \). This natural extension has fundamental properties. It is shown that the construction of a resolution of \( G \) is equivalent to a resolution of \( Diff(G) \). So, for the purpose of resolution of singularities, we may always assume that we start with a differential Rees algebra.

As a Rees algebra over \( V \) is a graded subring of \( \mathcal{O}_V [W] \), its integral closure is also a Rees algebra. If two Rees algebras over \( V \) have the same integral closure, then it is easy to check that the construction of a resolution of one is equivalent to a resolution of the other. For this reason it is natural to expect that invariants that aim to the definition of constrictive resolutions of Rees algebras should not distinguish one from the other. In \([35]\) it is proved that if \( G_1 \) and \( G_2 \) are Rees algebras over \( V \) with the same integral closure, then \( Diff(G_1) \) and
Diff(\mathcal{G})_2 also have the same integral closure. Due to the importance of this property, to be used in the further development, these results are discussed in Sections 2 and 3.

In Section 4 we develop our form of elimination of one variable, which is formulated at first in the context of differential Rees algebras. Let \mathcal{G} be a Diff-algebra over a smooth scheme \( V \) of dimension \( d \), and fix \( x \in \text{Sing}(\mathcal{G}) \). We consider here a suitable smooth morphisms \( \pi : V \to V^{(1)} \), defined at an étale neighborhood of \( x \in V \), where \( V^{(1)} \) is smooth of dimension \( d - 1 \). Then a new differential Rees algebra, say \( R_\mathcal{G} \), will be defined over the smooth \( d - 1 \)-dimensional scheme \( V^{(1)} \). So \( R_\mathcal{G} \subset \mathcal{O}_{V^{(1)}}[W] \) is defined in terms of \( \pi : V \to V^{(1)} \) and \( \mathcal{G} \). It has the property that \( \text{Sing}(\mathcal{G}) \subset V \) can be identified with \( \text{Sing}(R_\mathcal{G}) \) in \( V^{(1)} \) via \( \pi \). Here \( R_\mathcal{G} \) is called the elimination algebra.

If we take \( \mathcal{G} \) to be the Diff-algebra spanned by \( \mathcal{O}_V[fW^b] \), \( \text{Sing}(\mathcal{G}) \) is the set of points of multiplicity \( b \) in the hypersurface defined by \( f \), which can be identified with \( \text{Sing}(R_\mathcal{G}) \), defined now in a \((d - 1)\)-dimensional scheme. This is our approach to B).

The main properties of elimination algebras are collected in Theorem 4.11.

Section 5 is devoted to one of the main results in this paper, Theorem 5.5. It is shown there that the main invariant used for resolution of singularities in characteristic zero has a natural extension over perfect fields. Yet this invariant is new and has never been treated previously in the study of singularities over fields of positive characteristic.

The notion of stability of elimination (see (C) above) is addressed in Section 6. Results remain stronger over fields of characteristic zero, where we provide an alternative approach to induction for desingularization theorems. This form of elimination opens the way to new questions on resolution problems, also in characteristic zero; some suggestive properties in this sense are discussed. This section also includes some explicit calculation of elimination through examples.

A first look at Definition 4.10 and Theorem 4.11 is suggested right after reading Section 1, omitting Appendices 1 and 2 of this first section.

Differential Rees algebras appear in Wlodarczyk’s work \cite{38}, and play a central role in Kollár’s presentation in (\cite{26}), particularly with his notion of tuned ideals. Hironaka studies the relation of differential Rees algebras with integral closure of Rees algebras in \cite{21}, \cite{22}, \cite{23}, in connection with the theory of infinitely closed points. These connections, and various other aspects, are studied in detail within Kawanoue’s recent paper \cite{25}. See also \cite{35}, and \cite{37}.

I profited from discussions with Ana Bravo, Vincent Cossart, Marco Farinati, and Monique Lejeune.

1. Finite covers: multiplicity and ramification.

The aim in this paper is to introduce invariants of singularities over perfect fields. In particular we will refine the notion of multiplicity at points of a hypersurface embedded in a smooth scheme of dimension \( d \). Using the Weierstrass Preparation Theorem one can assume that the equation defining the hypersurface is given, locally, by a monic polynomial, in one variable, and coefficients in a smooth scheme of dimension \( d - 1 \). In this section we study relations of higher order differential with elimination theory.
Here we fix a positive integer $b$, and focus on a monic polynomial $f(Z) \in S[Z]$, where $S$ is a $k$-algebra. The objective is to study the ramification of the finite map in (0.0.1). Take $F(Z) = (Z - Y_1) \cdot (Z - Y_2) \cdots (Z - Y_b)$, in the polynomial ring in $b + 1$ variables $k[Y_1, \ldots, Y_b, Z]$. Consider the $k$-algebra $S = k[Y_1, \ldots, Y_b][b]$, ring of invariants where $S_b$ is the group of permutation of $b$ elements. Then $F(Z) \in S[Z]$ is monic in this $k$-algebra, and it is the universal monic polynomial of degree $b$ within the class of $k$-algebras.

The theory of elimination (of the variable $Z$) leads us to the study of the action of $S_b$ on a subring of $k[Y_1, \ldots, Y_b]$; namely on the $k$-algebra generated by all differences $Y_j - Y_k$. This provides a new ring of invariants which is the object of interest of Section 1, and a basic tool to be considered throughout the paper. Note that the discriminant of $F(Z)$ is an element of this invariant ring.

The section begins with the relation of this invariant ring with higher order differentials applied to $F(Z)$ (Remark 1.8); and the main result in this section is Theorem 1.16. Appendices 1) and 2) can be avoided in a first look at the paper.

1.1. Let $(S, M)$ denote a local ring, and fix a monic polynomial $f(Z)$, of degree $b$ in $S[Z]$. This defines a finite ring extension $S \subset S[Z]/\langle f(Z) \rangle$, and hence a finite morphism

$$\text{Spec}(S[Z]/\langle f(Z) \rangle) \xrightarrow{\pi} \text{Spec}(S)$$

This finite morphism is said to be purely ramified at $P \in \text{Spec}(S)$, when the geometric fiber at $P$ has a unique point. Equivalently, set $\overline{k(P)}$ an algebraically closed field extension of the residue field $k(P)$, then the morphism is purely ramified at $P$ if and only if the class of $f(Z)$ in $\overline{k(P)}[Z]$ has a unique root.

We begin by describing the set of prime ideals in $\text{Spec}(S)$ for which the finite extension is purely ramified. Our arguments, for this and further properties of this finite morphism, focus on two observations. Since the finite morphism is determined by the monic polynomial $f(Z) \in S[Z]$, many properties of the morphism should be encoded in the coefficients. On the other hand, changes of variables of the form $Z_1 = Z - s$, $s \in S$ do not affect the finite extension $S \subset S[Z]/\langle f(Z) \rangle$.

**Definition 1.2.** Consider the homomorphism of $S$-algebras, say $\text{Tay} : S[Z] \to S[Z, T]$, defined by setting $\text{Tay}(Z) = Z + T$. For each $F(Z) \in S[Z],$

$$\text{Tay}(F(Z)) = \sum_{k \geq 0} g_k(Z)T^k.$$

This defines, for each index $r \geq 0$, operators $\Delta^r : S[Z] \to S[Z]$, by setting $\Delta^r(F(Z)) = g_r(Z)$, or say

$$\text{Tay}(F(Z)) = \sum \Delta^r(F(Z))T^r.$$  

The $\Delta^r$ are differential operators of order $r$. These are $S$-linear operators; $\Delta^n(Z^n) = 1$ and $\Delta^r(Z^n) = 0$ for $r > n$.

Note that the morphism $\text{Tay}$, and the operators $\Delta^r$, are compatible with change of the base ring $S$. The following result is included simply to illustrate their link with ramification.
Lemma 1.3. Fix $f(Z) \in K[Z]$, monic of degree $b$, where $K$ is an algebraically closed field. The following are equivalent:

1) $f(Z) = (Z - \alpha)^b$ for some $\alpha \in K$.

2) The class of $\Delta^k(f(Z))$ in $K[Z]/\langle f(Z) \rangle$ is nilpotent for all integer $0 \leq k < b$.

Proof. That 1) implies 2) is clear.

Let $f(Z) = (Z - \alpha_1)^{\beta_1} \cdots (Z - \alpha_j)^{\beta_j}$ be the expression of the monic polynomial in terms of its $r$ different roots; so $\sum \beta_i = b$. We prove that 2) implies 1) by showing that the class of the ideal $\langle \Delta^k(f(Z)), 0 \leq k \leq b-1 \rangle \subset K[Z]$ is nilpotent in $K[Z]/\langle f(Z) \rangle$ only when $r = 1$. Assume that $r \geq 2$, and set $G(Z) = (Z - \alpha_2)^{\beta_2} \cdots (Z - \alpha_r)^{\beta_r}$, so $f(Z) = (Z - \alpha_1)^{\beta_1} \cdot G(Z)$, and $\beta_1 < b$. Now

$$\text{Tay}(f(Z)) = \text{Tay}((Z - \alpha_1)^{\beta_1}) \cdot \text{Tay}(G(Z)) \in K[Z, T],$$

and $\text{Tay}((Z - \alpha_1)^{\beta_1})$ is a monic polynomial of degree $\beta_1$ in $T$.

On the other hand, at $(K[Z]/\langle Z - \alpha_1 \rangle)[T] (= K[T])$, $\text{Tay}((Z - \alpha_1)^{\beta_1}) = T^{\beta_1}$ and

$$\text{Tay}(G(Z)) = G(\alpha_1) + \text{terms of degree } \geq 1 \text{ in } T,$$

where $G(\alpha_1)$ (the class of $G(Z)$ in $K[Z]/\langle Z - \alpha_1 \rangle$) is non-zero. This shows that $\Delta^{\beta_1}(f(Z)) \notin \langle Z - \alpha_1 \rangle$, and hence the class of $\Delta^{\beta_1}(F(Z))$ in the ring $K[Z]/\langle f(Z) \rangle$ is not nilpotent. \qed

1.4. The Lemma relates with a stronger question raised by E. Casas-Alvero: if either $f(Z) = (Z - \alpha)^b$ for some $\alpha \in K$, or $f(Z)$ and $\Delta^\alpha(f(Z))$ are coprime in $K[Z]$ for some $1 \leq \alpha \leq b-1$.

1.5. On the general strategy. Fix a ring $k$ and consider $S$ in the class of $k$-algebras. We recall that there is a universal monic polynomial of degree $b$ within this class. In our further applications $k$ will be a field, but to some extent the form of elimination we discuss here works over arbitrary rings, in fact over the integers.

Let $k[Y_1, \ldots, Y_b]$ be a polynomial ring over $k$. We will denote by $R_b$ the ring of symmetric polynomials in $b$ variables with coefficients in $k$. Here $s_{b,1}, \ldots, s_{b,b}$ will denote the symmetric polynomials, where each $s_{b,i}$ is homogeneous of degree $i$ in $k[Y_1, \ldots, Y_b]$. Therefore $R_b = k[s_{b,1}, \ldots, s_{b,b}]$ is a graded subring in $k[Y_1, \ldots, Y_b]$. Set

$$F_b(Z) = (Z - Y_1) \cdot (Z - Y_2) \cdots (Z - Y_b) \in k[Y_1, \ldots, Y_b][Z],$$

the generic polynomial of degree $b$. Recall that

$$F_b(Z) = Z^b - s_{b,1}Z^{b-1} + \cdots + (-1)^b s_{b,b} \in R_b[Z],$$

and note that every monic polynomial of degree $b$, say $f(Z) = Z^b + a_1Z^{b-1} + \cdots + a_b \in S[Z]$, over a $k$-algebra $S$, is obtained from $F_b(Z) \in R_b[Z]$ by a unique morphism $R_b \to S$.

Let $S_b$ denote the symmetric group acting on $k[Y_1, \ldots, Y_b]$ in the usual manner, so that

(1.5.1) $$R_b = k[s_{b,1}, \ldots, s_{b,b}] = k[Y_1, \ldots, Y_b]^{S_b}.$$ We shall show, via Galois correspondence, that there are natural $R_b$-isomorphisms, or say identifications:

$$R_b[Z]/\langle F_b(Z) \rangle = R_b[Y_j],$$
for every index $j$, where $R_b[Y_j]$ is a subring of $k[Y_1, \ldots, Y_b]$. Many properties of a finite extension $S \subset S[Z]/(f(Z))$, defined by a monic polynomial $f(Z) \in S[Z]$, can be expressed in terms of the coefficients, and are independent of changes of the form $Z_1 = Z - s$, $s \in S$. Such is the case of the ramification locus of the induced finite morphism. In fact, the ramification is described by the discriminant, which is invariant by those changes of $Z$.

Changes of variables in $S[Z]$ are more general, namely of the form $Z_1 = uZ - s$, where $s \in S$ and $u$ is a unit of $S$. We indicate below why it suffices, for our purpose, to restrict to changes of the form $Z_1 = Z - s$, at the universal level, in order to obtain, on each specific $S[Z]$, invariants for arbitrary changes $Z_1 = uZ - s$.

Note that the action of $S_b$ on $k[Y_1, \ldots, Y_b]$ restricts to an action on the subring

$$k[Y_2 - Y_1, \ldots, Y_b - Y_1] = k[Z_{i,j}]$$

where $Z_{i,j} = Y_i - Y_j$, $1 \leq i, j \leq b$. In what follows we denote by

$$(1.5.2) \quad \overline{R}_b = k[Y_2 - Y_1, \ldots, Y_b - Y_1]^{S_b},$$

the subring of invariants by this action. So

$$\overline{R}_b = R_b \cap k[Y_2 - Y_1, \ldots, Y_b - Y_1](\subset k[Y_1, \ldots, Y_b]).$$

Elements of $\overline{R}_b$, are elements in $R_b$, that provide, for every monic polynomial $f(Z) \in S[Z]$ of degree $b$, equations on the coefficients which are independent of changes of the form $Z_1 = Z - s$, $s \in S$. Our aim is to study generators of $\overline{R}_b$, and also its weighted structure as subring of the graded ring $k[Y_1, Y_2, \ldots, Y_b]$. It is also a graded subring of $R_b$. Here $R_b$ is mapped to $S$, so that $F_b(Z)$ defines $f(Z) = Z^b + a_1Z^{b-1} + \cdots + a_b \in S[Z]$ as above. In particular, an homogenous element $H$ of degree, say $r$, in $\overline{R}_b$, maps to an element, say $h$, in $S$, which is a polynomial expression on the coefficients $a_i$ of $f(Z)$. However, in our development we will want to recall the degree of $H$. To this end, in [1.39] we will add a dummy variable $W$, and we assign to $H$ the element $h \cdot W^r$ in the ring $S[W]$. The algebra $\overline{R}_b$ is generated by finitely many homogeneous elements in $R_b$, say $\overline{R}_b = k[H_1, \ldots, H_s]$, where each $H_i$ is homogeneous of degree, say $r_i$. We will consider, by a change of the base ring $\overline{R}_b \rightarrow S$, the $S$ subalgebra of $S[W]$ generated by $\{h_1 \cdot W^{r_1}, \ldots, h_s \cdot W^{r_s}\}$, namely

$$S[h_1 \cdot W^{r_1}, \ldots, h_s \cdot W^{r_s}].$$

This is a Rees algebra, a direct sum of ideals in $S$, say $\bigoplus_{r \geq 0} I_r W^r$. Now for each index $r$, the ideal $I_r$ is generated by polynomials on the coefficients of $f(Z)$. They are clearly invariants by a change of variable $Z_1 = Z - s$, $s \in S$, as each $h_i$ is invariant by such change. But the point is that each ideal $I_r$ will be independent of any change $Z_1 = uZ - s$ in $S[Z]$. This last property will rely on the fact that the previous (universal) $H_i$ are weighted homogeneous on the symmetric functions $s_{h,i}$.

Therefore the grading of these invariant rings play a central role in our further discussion. $\overline{R}_b$ is just one example of a graded ring that will arise. A first look into [1.15] and the
formulation of Theorem 1.16 is suggested to understand the connection of these invariant rings with the notion of ramification.

**Remark 1.6.** 1) Notice that
\[
\dim (k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b]) = \dim \left( (k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b])^{S_b} \right) = b - 1.
\]
2) Identify \( S_{b-1} \) with the subgroup of permutation in \( S_b \) fixing \( Y_1 \), so
\[
(k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b])^{S_b} = ((k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b])^{S_{b-1}})^{S_b}.
\]
3) \((k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b])^{S_{b-1}} \subset k[Y_1, \ldots, Y_b]^{S_{b-1}} = k[s_{b,1}, \ldots, s_{b,b}][Y_1].
\]
We check that \( k[Y_1, \ldots, Y_b]^{S_{b-1}} = k[s_{b,1}, \ldots, s_{b,b}][Y_1] \). The inclusion
\[
k[Y_1, \ldots, Y_b]^{S_{b-1}} \supset k[s_{b,1}, \ldots, s_{b,b}][Y_1]
\]
is clear. Let \( t_{b-1,1}, \ldots, t_{b-1,b-1} \) denote the symmetric polynomials in \( b - 1 \) variables \( Y_2, \ldots, Y_b \), and note that \( k[Y_1, \ldots, Y_b]^{S_{b-1}} = k[Y_1, t_{b-1,1}, \ldots, t_{b-1,b-1}] \). For the other inclusion check that \( s_{b,1} = Y_1 + t_{b-1,1}; s_{b,2} = Y_1 t_{b-1,1} + t_{b-1,2}; \ldots s_{b,b-1} = Y_1 t_{b-1,b-2} + t_{b-1,b-1}; \) and \( s_{b,b} = Y_1 t_{b-1,b-1} \).

4) Notice that
\[
(k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b])^{S_{b-1}} = k[t_{b-1,1}(Y_1 - Y_j), \ldots, t_{b-1,b-1}(Y_1 - Y_j)],
\]
where, as before, \( t_{b-1,1}, \ldots, t_{b-1,b-1} \) denote the symmetric polynomials in \( b - 1 \) variables, evaluated here at the elements \( \{ Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b \} \).

**1.7.** Define the morphism \( Tay : R_b[Z] = k[s_{b,1}, \ldots, s_{b,b}][Z] \to R_b[Z, T] \), and \( \Delta^{(\alpha)} : R_b[Z] \to R_b[Z] \), as usual, by setting \( Tay(F_b(Z)) = F_b(Z + T) = \sum \Delta^{(\alpha)}(F_b(Z))T^\alpha \).

In what follows recall the natural identification
\[
k[s_{b,1}, \ldots, s_{b,b}][Y_1] = k[s_{b,1}, \ldots, s_{b,b}][Z]/(F_b(Z)),
\]
and that the \( \Delta^{(\alpha)} \) operators are, in a natural sense, compatible with change of base rings \( R_b \to S \) within the class of algebras over \( k \).

**Remark 1.8.** Since \( F_b(Z) = (Y_1 - Z) \cdot (Y_2 - Z) \cdots (Y_b - Z) \in k[Y_1, \ldots, Y_b][Z] \),
\[
F_b(T + Z) = (T + (Y_1 - Z)) \cdot (T + (Y_2 - Z)) \cdots (T + (Y_b - Z)) \in k[Y_1, \ldots, Y_b][Z, T].
\]

Let \( F^{(\alpha)}(Z) \) denote the element \( \Delta^{(\alpha)}(F_b(Z)) \). The coefficients of this polynomial in the variable \( T \), are the symmetric polynomials evaluated on the elements \( Z - Y_j, 1 \leq j \leq b \). So
\[
F^{(\alpha)}_b(Z) = (-1)^{b-e} s_{b,b-e}(Z - Y_1, Z - Y_2, \ldots, Z - Y_b).
\]
Here we extend the action of \( S_b \), acting on the variables \( Y_j \), acting on the variables \( Y_j \), setting \( \sigma(Z) = Z \). Note that
\[
k[Z - Y_1, \ldots, Z - Y_b]^{S_b} = k[[F^{(e)}_b(Z), e = 0, 1, \ldots, b - 1]],
\]
and that each \( F^{(e)}_b(Z) \) is homogeneous. Observe that \( (Z - Y_1) - (Z - Y_j) = Y_j - Y_1 \), so \( \overline{R}_b \) is also a subring of \( k[[F^{(e)}_b(Z), e = 0, 1, \ldots, b - 1]] \) (1.5.2).
Lemma 1.9. Let the setting be as in [1.6 4). Then
\[ k[t_{b-1,1}(Y_1 - Y_j), \ldots, t_{b-1,b-1}(Y_1 - Y_j)] = k[F_b^{(1)}(Y_1), \ldots, F_b^{(b-1)}(Y_1)](\subset k[s_{b,1}, \ldots, s_{b,b}][Y_1]). \]

In fact: \( F_b^{(e)}(Y_1) = (-1)^b e t_{b-1,b-e}(Y_1 - Y_j), \ 1 \leq e \leq b - 1. \)

Proof. This is a consequence of [1.8.1] Note also that the equality
\[ s_{b,b-e}(Y_1 - Y_1, Y_1 - Y_2, \ldots, Y_1 - Y_b) = t_{b-1,b-e}(Y_1 - Y_2, \ldots, Y_1 - Y_b) \]
follows from the definition of symmetric polynomials. \( \square \)

Corollary 1.10. 1)
\[ (k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b])^{\leq b} = k[F_b^{(1)}(Y_1), \ldots, F_b^{(b-1)}(Y_1)](\subset k[s_{b,1}, \ldots, s_{b,b}][Y_1]). \]

2) \( \overline{R}_b = k[F_b^{(1)}(Y_1), \ldots, F_b^{(b-1)}(Y_1)] \cap k[s_{b,1}, \ldots, s_{b,b}], \) as subrings of \( k[s_{b,1}, \ldots, s_{b,b}][Y_1]. \)

1.11. Let \( k[Y_2 - Y_1, \ldots, Y_b - Y_1] \) be graded as subring of \( k[Y_1, \ldots, Y_b] \) with the usual grade. Since the action of \( S_b \) preserves degrees, it follows that \( \overline{R}_b \) and \( k[F_b^{(1)}(Y_1), \ldots, F_b^{(b-1)}(Y_1)] \) are graded subrings of \( k[Y_2 - Y_1, \ldots, Y_b - Y_1] \). So we may assume that \( \overline{R}_b = k[G_{b,1}, \ldots, G_{b,r_b}], \)
where the generators \( G_{b,i} \) are homogeneous polynomials. Furthermore,
\[ \overline{R}_b = k[F_b^{(1)}(Y_1), \ldots, F_b^{(b-1)}(Y_1)] \cap k[s_{b,1}, \ldots, s_{b,b}] = k[F_b^{(1)}(Y_1), \ldots, F_b^{(b-1)}(Y_1)]^{S_b}. \]

So \( \overline{R}_b \subset k[F_b^{(1)}(Y_1), \ldots, F_b^{(b-1)}(Y_1)] \) is a finite extension of graded rings. Therefore
\[ \langle G_{b,1}, \ldots, G_{b,r_b} \rangle \subset \langle F_b^{(1)}(Y_1), \ldots, F_b^{(b-1)}(Y_1) \rangle \]
are homogeneous ideals in \( k[F_b^{(1)}(Y_1), \ldots, F_b^{(b-1)}(Y_1)] \), and
\[ \sqrt{\langle G_{b,1}, \ldots, G_{b,r_b} \rangle} = \langle F_b^{(1)}(Y_1), \ldots, F_b^{(b-1)}(Y_1) \rangle \]
in the ring \( k[F_b^{(1)}(Y_1), \ldots, F_b^{(b-1)}(Y_1)] \) (see [1.13]). This, together with the inclusion
\[ k[F_b^{(1)}(Y_1), \ldots, F_b^{(b-1)}(Y_1)] \subset k[s_{b,1}, \ldots, s_{b,b}][Y_1] \]
in \( (1.10), 1 \), show that
\[ \sqrt{\langle G_{b,1}, \ldots, G_{b,r_b} \rangle} = \sqrt{\langle F_b^{(1)}(Y_1), \ldots, F_b^{(b-1)}(Y_1) \rangle} \]
as ideals in \( k[s_{b,1}, \ldots, s_{b,b}][Y_1] \) (see [1.12]).

Remark 1.12. Fix a ring \( A \) and two ideals \( I_1, I_2. \) If \( I_1 \subset I_2, \) then for every ring homomorphism \( A \rightarrow B, \) \( I_1B \subset I_2B. \) A similar property holds if \( \sqrt{T_1} = \sqrt{T_2} \) in \( A. \) We set \( R_b = k[s_{b,1}, \ldots, s_{b,b}] = k[Y_1, \ldots, Y_b]^{S_b}, \) and the universal polynomial of degree \( b: \)
\[ F_b(Z) = Z^b - s_{b,1}Z^{b-1} + \cdots + (-1)^b s_{b,b} \in R_b[Z]. \]
Any monic polynomial of degree \( b \), say \( f(Z) = Z^b - a_1Z^{b-1} + \cdots + (-1)^b a_b \in R[Z] \), over a \( k \)-algebra \( R \), is obtained from \( F_b(Z) \in R_b[Z] \), by the \( k \)-algebra homomorphism \( R_b \to R \) defined by mapping \( s_{b,i} \) to \( a_i \). In particular, there is a natural homomorphism

\[
R_b[Z]/\langle F_b(Z) \rangle \to R[Z]/\langle f(Z) \rangle
\]

defined by the base change \( R_b \to R \), and the result in 1.11 ensures that

\[
\sqrt{\langle G_{b,1}(a_1,\ldots,a_b),\ldots,G_{b,r_b}(a_1,\ldots,a_b) \rangle} = \sqrt{\langle f^{(1)}(Z),\ldots,f^{(b-1)}(Z) \rangle}
\]
as ideals in \( R[Z]/\langle f(Z) \rangle \).

Remark 1.13. Let \( F_1 \subset F_2 \) be an inclusion of finitely generated \( \mathbb{N} \)-graded algebras over a field \( k \). Let \( \text{max}_1 \) and \( \text{max}_2 \) denote the irrelevant maximal ideals of \( F_1 \) and \( F_2 \) respectively. Note that if \( F_1 \subset F_2 \) is a finite extension, then \( \sqrt{\text{max}_1 F_2} = \text{max}_2 \).

Remark 1.14. Fix a finitely \( \mathbb{N} \)-graded algebra, say \( k[H_1,\ldots,H_r] \), where each \( H_i \) is homogeneous of degree \( n_i \), and a local regular \( k \)-algebra \((S,M)\). We say that an homomorphism of \( k \)-algebras \( \phi : k[H_1,\ldots,H_r] \to S \) preserves degrees, if, for every homogeneous element \( H \in k[H_1,\ldots,H_r] \) of degree \( n \):

\[
\nu_S(\phi(H)) \geq n,
\]

where \( \nu_S \) denotes the order at the local regular ring.

This property holds if and only if \( \nu_S(\phi(H_i)) \geq n_i, 1 \leq i \leq r \). This follows from the fact that every homogeneous element \( H, \) of degree \( n \), can be expressed as \( H = G(H_1,\ldots,H_r) \), where \( G(Z_1,\ldots,Z_r) \in k[Z_1,\ldots,Z_r] \) is weighted homogeneous of degree \( n \), where each \( Z_i \) is considered with degree \( n_i \).

1.15. Set \( \overline{R}_b(= k[Y_1 - Y_2, Y_1 - Y_3,\ldots,Y_1 - Y_b])^{s_b} = k[G_{b,1},\ldots,G_{b,r_b}] \); and assume that each generator \( G_{b,i} \) is homogeneous of degree \( n_i \), as polynomial in \( k[Y_1,\ldots,Y_b] \). So an homogeneous element \( H \in \overline{R}_b \) of degree \( m \) in \( k[Y_1,\ldots,Y_b] \), is also a weighted homogeneous polynomial of degree \( m \) in the \( G_{b,i} \)'s, provided each \( G_{b,i} \) is given weight \( n_i \).

The same argument applies for the inclusion \( \overline{R}_b \subset k[s_1,\ldots,s_b] \), so \( H \) is also a weighted homogeneous polynomial of degree \( m \) in the \( s_i \)'s, provided each \( s_i \) is given weight \( i \).

Let \((S,M)\) denote, as above, a local regular ring; and recall that \( \phi : \overline{R}_b \to S \) preserves degrees if and only if \( \nu_S(\phi(G_{b,i})) \geq n_i \), for \( 1 \leq i \leq r_b \) (1.11). Consider a regular \( k \)-algebra \( R \). A monic polynomial \( f(Z) = Z^b + a_1Z^{b-1} + \cdots + a_{b-1}Z + a_b \in R[Z] \) defines a hypersurface in the regular scheme \( \text{Spec}(R[Z]) \), and a finite morphism

\[
\text{Spec}(R[Z]/\langle f(Z) \rangle) \xrightarrow{\pi} \text{Spec}(R).
\]

Let \( Q \) be a point of this hypersurface. Set \( P = \pi(Q) \), and \( S = RP \) (local regular ring).

We claim that if the hypersurface has multiplicity \( b \) at \( Q \), then \( \nu_S(G_{b,i}(a_1,a_2,\ldots,a_b)) \geq n_i \), for \( 1 \leq i \leq r_b \). In order to prove this claim we use Zariski’s multiplicity formula (see [39], Corollary 1, page 299). It follows that if \( Q \) is a \( b \)-fold point of this hypersurface, there is a suitable change of coordinate \( Z_1 = Z - s, s \in S \), so that: \( f(Z) = Z_1^b + c_1Z_1^{b-1} + \cdots + c_{b-1}Z_1 + c_b \in S[Z] \), and \( \nu_S(c_i) \geq i \). In fact, the multiplicity formula, applied to the finite extension
$S \subset B = S[Z]/\langle f(Z) \rangle$, ensures that $B$ is local (i.e., $B = B_Q$) and the residue fields of the local rings $B$ and $S$ are the same. In particular, the class of $Z$ in the residue field of $B$ is the class of some element $s \in S$.

Recall that $k[s_{b,1},\ldots,s_{b,b}]$ is a graded subalgebra of $k[Y_1,\ldots,Y_b]$, and each $s_{b,i}$ is homogeneous of degree $i$. The morphism $k[s_{b,1},\ldots,s_{b,b}] \to S$, defined by mapping $(-1)^i s_{b,i}$ to $c_i$, maps $G_{b,i}(s_{b,1},\ldots,s_{b,b})$ to $G_{b,i}(c_1,\ldots,c_b)$. As this morphism $k[s_{b,1},\ldots,s_{b,b}] \to S$ preserves degrees, it follows that $\nu_S(G_{b,i}(c_1,\ldots,c_b)) \geq n_i$.

On the other hand $G_{b,i}(c_1,\ldots,c_b) = G_{b,i}(a_1,\ldots,a_b)$ since these functions are invariant by these changes of the coordinate $Z$, hence $\nu_S(G_{b,i}(a_1,\ldots,a_b)) \geq n_i$. (1.4)

**Theorem 1.16.** Let $R$ be a $k$ algebra, $f(Z) = Z^b + a_1 Z^{b-1} + \ldots + a_{b-1} Z + a_b \in R[Z]$, and set $\text{Spec}(R[Z]/\langle f(Z) \rangle) \to \text{Spec}(R)$. Then:

i) $V(\langle G_{b,1}(a_1,\ldots,a_b),\ldots,G_{b,r_b}(a_1,\ldots,a_b) \rangle)$ is the set of points in $\text{Spec}(R)$ where the finite morphism is purely ramified (1.7).

ii) If $R$ is regular, and $Q \in V(\langle f(Z) \rangle)$ is a point of multiplicity $b$ of this hypersurface in $\text{Spec}(R[Z])$, then

$$\nu_S(G_{b,i}(a_1,\ldots,a_b)) \geq n_i$$

for $1 \leq i \leq r_b$, where $S = R_P$, $P = \pi(Q)$.

**Proof.** i) Note that $\text{Spec}(S[Z]/\langle f \rangle) \to \text{Spec}(S)$ arises from $\text{Spec}(R_b[Z]/\langle f_b(Z) \rangle) \to \text{Spec}(R_b)$ by the change of base rings $\phi : R_b \to S$, where $\phi$ is a $k$ algebra morphism, and $\phi((-1)^i s_{b,i}) = a_i$. So, as it was indicated in (1.12) it suffices to prove the claim for the universal case.

In (1.11) we show that $\langle G_{b,1},\ldots,G_{b,r_b} \rangle$ and $\langle F_b^{(1)}(Y_1),\ldots,F_b^{(b-1)}(Y_1) \rangle$ have the same radical ideal, as ideals in $R_b[Z]/\langle f_b(Z) \rangle = k[s_{b,1},\ldots,s_{b,b}][Y_1])$. Fix a prime $P \subset k[s_{b,1},\ldots,s_{b,b}]$ and set $\{Q_1,\ldots,Q_s\}$ the primes in $R_b[Z]/\langle f_b(Z) \rangle$, over $P$. Let $K$ be an algebraic closure of the residue field of $(R_b)_P$, and argue as in Lemma (1.3) If $P$ contains the ideal $\langle G_{b,1},\ldots,G_{b,r_b} \rangle$ in $R_b$, then any $Q_i$ contains $\langle F_b^{(1)}(Y_1),\ldots,F_b^{(b-1)}(Y_1) \rangle$ as ideals in $R_b[Z]/\langle f_b(Z) \rangle$, so Lemma (1.3) asserts that the fiber over $P$ is purely ramified. Conversely, if $P$ does not contain the ideal $\langle G_{b,1},\ldots,G_{b,r_b} \rangle$ in $R_b$, then $\langle F_b^{(1)}(Y_1),\ldots,F_b^{(b-1)}(Y_1) \rangle$ (in $R_b[Z]/\langle f_b(Z) \rangle$) is not contained in any $Q_i$. In this case Lemma (1.3) asserts that the morphism is not purely ramified at $P$.

ii) Proved in (1.13) \qedsymbol

**Appendix 1: On normality and graded structure of $\overline{R_b}$.**

In the coming sections we will study invariants that arise from the graded structure of the rings $\overline{R_b} = k[Y_2 - Y_1,\ldots,Y_b - Y_1][s_b]$ and $k[Z - Y_1,\ldots,Z - Y_b][s_b]$.

We now introduce a graded subring of $\overline{R_b}$, whose integral closure is $\overline{T_b}$. This subring will be particularly useful for explicit computation of invariants (see examples of the last Section 6). This appendix can be omitted in a first look at the paper.

**Remark 1.17.** We have studied graded subrings of $k[Y_1,\ldots,Y_b, Z]$, which we consider with the usual grade. In particular an element of a graded subring is homogeneous if and only if it is homogeneous in $k[Y_1,\ldots,Y_b, Z]$. A subring of this ring is graded when it is generated by
homogeneous elements. Since $\overline{R}_b$ is the subring of $S_b$ invariants in $k[Y_2 - Y_1, \ldots, Y_b - Y_1]$, 
\[ \overline{R}_b = k[Y_2 - Y_1, \ldots, Y_b - Y_1] \cap R_b, \]
as subrings of $k[Y_1, \ldots, Y_b]$ (see \[1.5.1\]). In particular $\overline{R}_b$ is an intersection of normal rings, so it is normal.

1.18. The ring $k[s_{b,1}, \ldots, s_{b,b}][Y_1]$ is a free module of rank $b$ over $k[s_{b,1}, \ldots, s_{b,b}]$. Multiplication by an element $\Theta \in k[s_{b,1}, \ldots, s_{b,b}][Y_1]$ defines an endomorphism, say $\psi_{\Theta}$, with characteristic polynomial, say 
\[ \psi_{\Theta}(V) = V^b + h_1 V^{b-1} + \cdots + h_b \in k[s_{b,1}, \ldots, s_{b,b}][V]. \]

Lemma 1.19. We claim that, if $\Theta = a_0 + a_1 Y_1 + \ldots a_{b-1} Y_1^{b-1}$, 
\[ \psi_{\Theta}(V) = \prod_{1 \leq j \leq b} (V - (a_0 + a_1 Y_j + \ldots a_{b-1} Y_j^{b-1})). \]

(i.e., the coefficients $h_i$ are, up to sign, the symmetric polynomials evaluated at the elements $a_0 + a_1 Y_j + \ldots a_{b-1} Y_j^{b-1}$).

Proof. The proof follows from the observations:
1) $\psi_{\Theta}(\Theta) = 0$.
2) There is an isomorphism of $k[s_{b,1}, \ldots, s_{b,b}]$-modules, say 
\[ \beta_j : k[s_{b,1}, \ldots, s_{b,b}][Y_1] \to k[s_{b,1}, \ldots, s_{b,b}][Y_j] \quad \beta_j(Y_1) = Y_j. \]

So the characteristic polynomial of $\Theta$ in $k[s_{b,1}, \ldots, s_{b,b}][Y_1]$, is the same as that of $\beta_j(\Theta)$ in $k[s_{b,1}, \ldots, s_{b,b}][Y_j]$.

Note that $S_b$ is the Galois group of the extension $k[s_{b,1}, \ldots, s_{b,b}] \subset k[Y_1, \ldots, Y_b]$. An element $\sigma \in S_b$, such that $\sigma(Y_1) = Y_j$, induces $\beta_j$ when restricted to $k[s_{b,1}, \ldots, s_{b,b}][Y_1]$.

Corollary 1.20. If $\Theta = a_0 + a_1 Y_1 + \ldots a_{b-1} Y_1^{b-1}$ is weighted homogeneous in $k[s_{b,1}, \ldots, s_{b,b}][Y_1]$ (i.e., if $\Theta$ is homogeneous as element of $k[Y_1, \ldots, Y_b]$), then the coefficients of the characteristic polynomial $\psi_{\Theta}(V)$ are weighted homogeneous in $k[s_{b,1}, \ldots, s_{b,b}]$ (i.e., are homogeneous in $k[Y_1, \ldots, Y_b]$).

In fact, the action of $S_b$ preserves degrees in $k[Y_1, \ldots, Y_b]$, and each $\beta_j$ is a restriction of an element in $S_b$.

Let $F_b^{(e)}(Y_1)$ denote the class of $\Delta^e(F_b)(Z)$ in $k[s_{b,1}, \ldots, s_{b,b}][Y_1]$. Namely, set $F_b^{(e)}(Y_1) = \Delta^e(F_b)(Y_1)$ \( \square \).

Definition 1.21. Let $H_{F_b}$ be the $k$-subalgebra of $k[s_{b,1}, \ldots, s_{b,b}]$ generated by the coefficients of the $b-1$ characteristic polynomials, say $\psi_{F_b^{(e)}(Y_1)}(V)$, for $1 \leq e \leq b-1$.

Lemma 1.22. $H_{F_b}$ is included in $k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b]$, and it is a graded subalgebra of $k[s_{b,1}, \ldots, s_{b,b}]$. 


The product of these polynomials is a polynomial of degree $\sum_{i=1}^{r}c_i$, and note that $t^{b-1, b-r}(Y_1 - Y_j)$ is homogeneous in $K[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b]$. The coefficients of the characteristic polynomial of $F^{(e)}_b(Y_1)$ are symmetric polynomials on $\{F^{(e)}_b(Y_j)/1 \leq j \leq b\}$, so they are also homogeneous elements in $k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b]$. The second part of the claim follows from the corollary.

**Proposition 1.23.** $\overline{R}_b$ is the integral closure of the graded ring $H_{F_b}$ in $k[s_{b,1}, \ldots, s_{b,b}]$.

**Proof.** Observe that $(k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b])^S_b = ((k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b])^{S_{b-1}})^S_b$, and recall that:

- a) $(k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b])^{S_{b-1}} = k[F^{(1)}_b(Y_1), \ldots, F^{(b-1)}_b(Y_1)] \cap k[s_{b,1}, \ldots, s_{b,b}][Y_1]$;
- b) $(k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b])^{S_b} = k[F^{(1)}_b(Y_1), \ldots, F^{(b-1)}_b(Y_1)] \cap k[s_{b,1}, \ldots, s_{b,b}]$, as subrings of $k[s_{b,1}, \ldots, s_{b,b}][Y_1]$.

Lemma 1.22 shows that the coefficients of $\psi^{(e)}_{F_b}(V)(Y_1)$ are in the ring $\overline{R}_b$. In fact, in $k[Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b] \cap k[s_{b,1}, \ldots, s_{b,b}] = \overline{R}_b$.

Finally use b) to show that $H_{F_b} \subset k[F^{(1)}_b(Y_1), \ldots, F^{(b-1)}_b(Y_1)]$ and notice that this is a finite ring extension. In fact, the elements $F^{(e)}_b(Y_1)$ are integral over $H_{F_b}$ since they satisfy the characteristic polynomial (i.e., they are roots of their own characteristic polynomials).

The claim follows now from:

1. $H_{F_b} \subset k[F^{(1)}_b(Y_1), \ldots, F^{(b-1)}_b(Y_1)] \cap k[s_{b,1}, \ldots, s_{b,b}]$.
2. $k[F^{(1)}_b(Y_1), \ldots, F^{(b-1)}_b(Y_1)]$ is a finite extension of $H_{F_b}$, and
3. $k[F^{(1)}_b(Y_1), \ldots, F^{(b-1)}_b(Y_1)] \cap k[s_{b,1}, \ldots, s_{b,b}] = \overline{R}_b$ is integrally closed in $k[s_{b,1}, \ldots, s_{b,b}]$.

As for iii): it was indicated in (1.17) that $\overline{R}_b$ is normal. Note that the quotient field $\overline{R}_b$ is a subfield of $k(Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b)$, whereas the quotient field of $k[s_{b,1}, \ldots, s_{b,b}]$ is a subfield of $k(Y_1, Y_2, Y_3, \ldots, Y_b) = k(Y_1 - Y_2, Y_1 - Y_3, \ldots, Y_1 - Y_b)(Y_1)$, which is purely transcendental over the latter. □

**On multi-graded structures.**

We have studied invariants of a monic polynomial, motivated by elimination theory, which were related to differential operators. However, elimination applies also to more then one polynomial. An example of this is the resultant of two polynomial. Here we generalize the previous discussion to the case of several polynomials. This will lead us to different graded rings, and also to extensions of graded rings. Of special interest for further applications is the case in which these ring extensions are integral (i.e., finite extensions).

**1.24.** Fix positive integers $c_1, \ldots, c_r$, and set $b = \sum c_i$. For each index $i$ define $F_{c_i} = (Z - Y_1^{(i)})(Z - Y_2^{(i)}) \cdots (Z - Y_{c_i}^{(i)}) \in k[Y_1^{(i)}, \ldots, Y_{c_i}^{(i)}][Z]$.

The product of these polynomials is a polynomial of degree $b$ in $Z$, say:

$$F_{c_1}(Z) \cdot F_{c_2}(Z) \cdots F_{c_r}(Z) = F_b(Z),$$
as polynomial in $k[Y_1^{(1)}, \ldots, Y_{c_1}^{(1)}, Y_1^{(2)}, \ldots, Y_{c_2}^{(2)}, \ldots, Y_1^{(r)}, \ldots, Y_{c_r}^{(r)}][Z]$, which we naturally identify with $k[Y_1, \ldots, Y_b][Z]$, as $c_1 + \cdots + c_r = b$.

In other words, identify $k[Y_1^{(1)}, \ldots, Y_{c_1}^{(1)}, Y_1^{(2)}, \ldots, Y_{c_2}^{(2)}, \ldots, Y_1^{(r)}, \ldots, Y_{c_r}^{(r)}]$ with $k[Y_1, \ldots, Y_b]$. The permutation group $S_{c_i}$ acts on $k[Y_1^{(i)}, \ldots, Y_{c_i}^{(i)}]$, and there is an inclusion $S_{c_1} \times S_{c_2} \times \cdots \times S_{c_r}$ in $S_b$. The main arguments rely on two simple observations. First, that the permutation groups $S_{c_i}$ and $S_b$ act linearly on $k[Y_1^{(i)}, \ldots, Y_{c_i}^{(i)}]$ and $k[Y_1, \ldots, Y_b]$ respectively, which asserts that the invariant rings are graded. Second, that the inclusion of finite groups $S_{c_1} \times S_{c_2} \times \cdots \times S_{c_r}$ in $S_b$ provides a finite extension of invariant rings.

1.25. The permutation group $S_{c_i}$ acts on $k[Y_1^{(i)}, \ldots, Y_{c_i}^{(i)}]$, and the inclusion $S_{c_1} \times S_{c_2} \times \cdots \times S_{c_r}$ in $S_b$ is such that:

$$(k[Y_1^{(1)}, \ldots, Y_{c_1}^{(1)}, \ldots, Y_1^{(r)}, \ldots, Y_{c_r}^{(r)}])^{S_{c_1} \times S_{c_2} \times \cdots \times S_{c_r}} = \otimes_i k[Y_1^{(i)}, \ldots, Y_{c_i}^{(i)}]^{S_{c_i}}.$$ 

The finite group $S_{c_1} \times S_{c_2} \times \cdots \times S_{c_r}$ also acts on the graded subalgebra

$$(1.25.1) \quad A_{c_1, \ldots, c_r} = k[Y_2^{(1)} - Y_1^{(1)}, \ldots, Y_{c_1}^{(1)} - Y_1^{(1)}, \ldots, Y_1^{(r)} - Y_1^{(1)}, \ldots, Y_{c_r}^{(r)} - Y_1^{(1)}]$$

in a way that preserves the usual degrees. Therefore, the ring of invariants, say

$$(1.25.2) \quad \overline{R}_{c_1, \ldots, c_r} = A_{c_1, \ldots, c_r}^{S_{c_1} \times S_{c_2} \times \cdots \times S_{c_r}}$$

is a finitely generated $k$-algebra, and a graded subring of $k[Y_1^{(1)}, \ldots, Y_{c_1}^{(1)}, \ldots, Y_1^{(r)}, \ldots, Y_{c_r}^{(r)}]$. Set

$$k[Y_1^{(i)}, \ldots, Y_{c_i}^{(i)}]^{S_{c_i}} = k[s_1^{(i)}, \ldots, s_{c_i}^{(i)}].$$

The graded rings $k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}]$ and $A_{c_1, \ldots, c_r}$ are both polynomial rings, and graded subrings of $k[Y_1^{(1)}, \ldots, Y_{c_1}^{(1)}, \ldots, Y_1^{(r)}, \ldots, Y_{c_r}^{(r)}]$. As both are normal, so is

$$\overline{R}_{c_1, \ldots, c_r} = A_{c_1, \ldots, c_r} \cap k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}].$$

Since $\overline{R}_{c_1, \ldots, c_r}$ is graded, it is generated by weighted homogeneous polynomials, say

$$(1.25.3) \quad G_{c_1, \ldots, c_r}^l = G_{c_1, \ldots, c_r}^l(s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}), \quad 1 \leq l \leq n_{c_1, \ldots, c_r},$$

for some positive integer $n_{c_1, \ldots, c_r}$.

**Theorem 1.26.** Let $R$ be a $k$-algebra, and let $\text{Spec}(R[Z]/(f(Z))) \xrightarrow{\nu} \text{Spec}(R)$ be the natural finite morphism, where

$$f(Z) = f_1(Z) \cdot f_2(Z) \cdots f_r(Z), \quad f_i(Z) = Z^{c_i} + a_1^{(i)} Z^{c_i-1} + \cdots + a_{c_i-1}^{(i)} Z + a_c^{(i)} \in R[Z],$$

and $c_1 + c_2 + \cdots + c_r = b$. Then:

i) $V((G_{c_1, \ldots, c_r}^l(a_1^{(1)}, \ldots, a_{c_1}^{(1)}, \ldots, a_1^{(r)}, \ldots, a_{c_r}^{(r)})/1 \leq l \leq n_{c_1, \ldots, c_r}))$ is the set of points in $\text{Spec}(R)$ where the finite morphism is purely ramified.

ii) If $R$ is regular, and $Q \in V((f(Z)))$ is a point of multiplicity $b$ of this hypersurface, then

$$\nu_S(G_{c_1, \ldots, c_r}^l(a_1^{(1)}, \ldots, a_{c_1}^{(1)}, \ldots, a_1^{(r)}, \ldots, a_{c_r}^{(r)}) \geq \deg G_{c_1, \ldots, c_r}^l.$$
for $1 \leq l \leq n_{c_1,\ldots,c_r}$, where $S = R_P$, $P = \pi(Q)$.

Proof. i) Set $f(Z) = Z^b + h_1 Z^{b-1} + \ldots + h_{b-1} Z + h_b \in R[Z]$. According to Theorem 1.16 the purely ramified locus is the closed set in $\text{Spec}(R)$ defined by the ideal spanned by all $G_i(h_1,\ldots,h_b)$, where the $G_i$ are homogeneous generators of the subring of $S_b$ invariants in $A_{c_1,\ldots,c_r}$ in (1.25.1).

Here the elements $G^l_{c_1,\ldots,c_r}$ span the subring of $S_{c_1} \times S_{c_2} \times \cdots \times S_{c_r}$-invariants (1.25.3). The inclusion $S_{c_1} \times S_{c_2} \times \cdots \times S_{c_r} \subset S_b$ defines a finite extension of invariant rings, and both are generated by homogeneous elements, say $k\{G_i\}_{1 \leq i \leq n_b} \subset k\{G^l_{c_1,\ldots,c_r}\}_{1 \leq l \leq n_{c_1,\ldots,c_r}}$.

So the ideal in $k\{G^l_{c_1,\ldots,c_r}\}_{1 \leq l \leq n_{c_1,\ldots,c_r}}$ spanned by all elements $G_i$, is included, and has the same radical, as that spanned by the $G^l_{c_1,\ldots,c_r}$'s (1.13). As it was indicated in 1.12 this property is preserved by arbitrary homomorphisms of $k$-algebras. This proves (i).

ii) Follows by the same argument used in Theorem 1.16 (ii).

\[ \begin{array}{l}
\medskip
1.27. \text{We have defined } F_b(Z) = (Z - Y_1) \cdot (Z - Y_2) \cdots (Z - Y_b) \in k[Y_1, \ldots, Y_b][Z]. \text{ In 1.8 it was shown that the coefficients of } \medskip
\medskip
\medskip
Tag(F(Z)) = F_b(T + Z) = (T + (Z - Y_1)) \cdot (T + (Z - Y_2)) \cdots (T + (Z - Y_b)) \in k[Y_1, \ldots, Y_b][Z, T], \text{ in the variable } T, \text{ are the symmetric polynomials evaluated on the elements } Z - Y_j, 1 \leq j \leq b. \text{ Namely, } F_b^{(e)}(Z) = (-1)^{b-e} \mathfrak{s}_{b-b-e}(Z - Y_1, Z - Y_2, \ldots, Z - Y_b), \text{ where } F_b^{(e)}(Z) \text{ denotes } \Delta^{(e)}(F(Z))(\text{see 1.8.1}), \text{ and } \medskip
\medskip
k[Z - Y_1, \ldots, Z - Y_b]_{\mathfrak{n}_b} = k\{F_b^{(e)}(Z), e = 0, 1, \ldots, b - 1\}. \medskip
\medskip
\text{Fix, as in 1.24 positive integers } c_1, \ldots, c_r \text{ so that } c_1 + \cdots + c_r = b, \text{ and set, for each index } \medskip
i, \text{ } F_{c_i}(Z) = \prod_{1 \leq j \leq c_i} (Z - Y_j^{(i)}) \in k[Y_1^{(i)}, \ldots, Y_{c_i}^{(i)}][Z]. \text{ The product of these polynomials is of } \medskip
\text{degree } b \text{ in } Z, \text{ say: } \medskip
\text{ } F_{c_1}(Z) \cdot F_{c_2}(Z) \cdots F_{c_r}(Z) = F_b(Z), \text{ in } k[Y_1^{(1)}, \ldots, Y_{c_1}^{(1)}, Y_1^{(2)}, \ldots, Y_{c_2}^{(2)}, \ldots, Y_1^{(r)}, \ldots, Y_{c_r}^{(r)}][Z], \text{ which we identified with } k[Y_1, \ldots, Y_b][Z], \text{ as } c_1 + \cdots + c_r = b. \text{ In fact, we identify } k[Y_1, \ldots, Y_{c_1}, Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_{c_2}^{(2)}, \ldots, Y_1^{(r)}, \ldots, Y_{c_r}^{(r)}] \text{ with } \medskip
k[Y_1, \ldots, Y_b], \text{ and hence: } \medskip
k[Z - Y_1, \ldots, Z - Y_b] = k\{Z - Y_j^{(i)}\}_{1 \leq i \leq r, 1 \leq j \leq c_i}. \medskip
\text{Set, as before, } F_b^{(\alpha)}(Z) = \Delta^{(\alpha)}(F_b(Z)) \text{ and } F_{c_i}^{(\alpha)}(Z) = \Delta^{(\alpha)}(F_{c_i}(Z))(1.17). \\end{array} \]

Proposition 1.28. 1) For $0 \leq \alpha \leq b - 1$, and $0 \leq \beta \leq c_i - 1$, each $F_b^{(\alpha)}(Z)$ is homogeneous of degree $b - \alpha$, and each $F_{c_i}^{(\beta)}(Z)$ is homogeneous of degree $c_i - \beta$ in the graded ring $k[Y_1^{(1)}, \ldots, Y_{c_1}^{(1)}, Y_1^{(2)}, \ldots, Y_{c_2}^{(2)}, \ldots, Y_1^{(r)}, \ldots, Y_{c_r}^{(r)}][Z] = k[Y_1, \ldots, Y_b][Z].$

2) There is a finite inclusion of graded subrings of $k[Y_1, \ldots, Y_b][Z]$, defined by

\[ k\{F_b^{(e)}(Z), e = 0, 1, \ldots, b - 1\} \subset k\{F_{c_i}^{(\beta)}(Z), 1 \leq i \leq r, 0 \leq \beta \leq c_i\}. \]

3) The two algebras in 2) are also graded subalgebras in $k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}][Z]$. 
Proof. 1) follows from [2.6.1] and 2) from the fact that such rings are the invariants of the finite groups \( S_{c_1} \times S_{c_2} \times \cdots \times S_{c_r} \subset S_b \); acting on \( k[Z - Y_1, \ldots , Z - Y_b] = k[[Y_1^{(i)}, \ldots , Y_c^{(i)}]_{1 \leq i \leq r, 1 \leq j \leq c_i}]. \) \( \square \)

**Proposition 1.29.** The graded algebra \( \overline{R}_{c_1, \ldots , c_r} \) in [1.25.3] is a graded subring of \( k[[F_{c_i}^{(\beta)}(Z), 1 \leq i \leq r, 0 \leq \beta \leq c_i]]. \)

**Proof.** Note that \( A_{c_1, \ldots , c_r} \) in [1.25.1] can be expressed as a subring of \( k[[Z - Y_j^{(i)}; 1 \leq i \leq r, 1 \leq j \leq c_i]] \) by setting \( Y_j^{(i)} - Y_1^{(i)} = (Z - Y_1^{(1)}) - (Z - Y_j^{(i)}). \)

**Corollary 1.30.** Let \( H_N \in \overline{R}_{c_1, c_2, \ldots , c_r} \) be homogeneous of degree \( N \). Then there is a polynomial in variables \( \{W_{c_1}, W_{c_1}^{(1)}, W_{c_2}^{(2)}, \ldots , W_{c_1}^{(c_1-1)}, \ldots , W_{c_2}, W_{c_2}^{(1)}, \ldots , W_{c_r}^{(c_r-1)}\} \), and coefficients in \( k \), say

\[
G \in k[W_{c_1}, W_{c_1}^{(1)}, W_{c_2}^{(2)}, \ldots , W_{c_1}^{(c_1-1)}, \ldots , W_{c_2}, W_{c_2}^{(1)}, \ldots , W_{c_r}^{(c_r-1)}],
\]

such that

\[
G(F_{c_1}(Z), F_{c_2}^{(1)}(Z), F_{c_2}^{(2)}(Z), \ldots , F_{c_1}^{(c_1-1)}(Z), \ldots , F_{c_r}(Z), F_{c_2}^{(1)}(Z), F_{c_2}^{(2)}(Z), \ldots , F_{c_r}^{(c_r-1)}(Z)) = H_N.
\]

Furthermore, we may assume that \( G \) is weighted homogeneous of degree \( N \), if we assign degree \( c_i - j \) to the variable \( W_{c_i}^{(j)}. \)

**Multi-graded structures and elimination.**

1.31. We now extend Corollary [1.10] and Proposition [1.23] to the context of several monic polynomials. Notice that

\[
F_{c_i}(Z) = \prod_{1 \leq j \leq c_i} (Z - Y_j^{(i)}) \in k[Y_1^{(i)}, \ldots , Y_c^{(i)}][Z]
\]

has coefficients in \( k[Y_1^{(i)}, \ldots , Y_c^{(i)}]_{S_{c_i}} = k[s_1^{(i)}, \ldots , s_c^{(i)}]; \) in fact

\[
F_{c_i}(Z) = Z^{c_i} + (-1)^{s_1^{(i)}} Z^{c_i-1} + \ldots + (-1)^{s_{c_i-1}^{(i)}} Z + (-1)^{s_c^{(i)}}
\]

Set \( i = 1 \) and let \( S_{c_i-1} \) denote the subgroup of \( S_{c_i} \), consisting of those elements fixing \( Y_1^{(1)}. \) Observe that

\[
k[s_1^{(1)}, \ldots , s_c^{(1)}][Y_1^{(1)}] = k[Y_1^{(1)}, \ldots , Y_{c_i}^{(1)}]_{S_{c_i-1}}
\]

can be identified with \( k[s_1^{(1)}, \ldots , s_c^{(1)}][Z]/\langle F_{c_1}(Z) \rangle. \)

On the other hand

\[
k[s_1^{(1)}, \ldots , s_c^{(1)}, s_1^{(r)}, \ldots , s_c^{(r)}][Y_1^{(1)}] = (k[Y_1^{(1)}, \ldots , Y_{c_i}^{(1)}, Y_1^{(r)}, \ldots , Y_{c_r}^{(r)}])_{S_{c_i-1} \times S_{c_2 - \cdots \times S_r}}
\]

can be identified with

\[
k[s_1^{(1)}, \ldots , s_c^{(1)}, s_1^{(r)}, \ldots , s_c^{(r)}][Z]/\langle F_{c_1}(Z) \rangle.
\]

The natural inclusion \( A_{c_1, \ldots , c_r} \subset k[Y_1^{(1)}, \ldots , Y_{c_i}^{(1)}, \ldots , Y_1^{(r)}, \ldots , Y_{c_r}^{(r)}] \) (see [1.25.1]) shows that

\[
A_{c_1, \ldots , c_r}^{S_{c_1-1} \times S_{c_2 - \cdots \times S_r} \subset k[s_1^{(1)}, \ldots , s_c^{(1)}, s_1^{(r)}, \ldots , s_c^{(r)}][Z]/\langle F_{c_1}(Z) \rangle.
\]
Let \( F^{(α)}(Z) \) denote \( Δ^{(α)}(F(Z)) \) (as in \([\text{1.7}])

**Lemma 1.32.** The ring

\[
A_{c_1,...,c_r}^{S_{c_1-1} × S_{c_2} ×...× S_{cr}} = k[y_2^{(1)} - y_1^{(1)}, ..., y_{c_1}^{(1)} - y_1^{(1)}, ..., y_r^{(r)} - y_1^{(1)}, ..., y_{c_r}^{(r)} - y_1^{(1)}]_{S_{c_1-1} × S_{c_2} ×...× S_{cr}}
\]

is a graded \( k \)-algebra and a subring of

\[
k[s_1^{(1)}, ..., s_{c_1}^{(1)}, ..., s_r^{(r)}][Y_1^{(1)}] = k[s_1^{(1)}, ..., s_{c_1}^{(1)}, ..., s_r^{(r)}][Z]/⟨F_{c_1}(Z)⟩,
\]
generated by the class of the elements

\[
\{F_{c_1}(Z), F_{c_1}(Z)^{(1)}, F_{c_1}(Z)^{(2)}, ..., F_{c_1}(Z)^{(c_1-1)}, ..., F_{c_r}(Z), F_{c_r}(Z)^{(1)}, F_{c_r}(Z)^{(2)}, ..., F_{c_r}(Z)^{(c_r-1)}\}
\]
in this quotient ring. In other words,

\[
A_{c_1,...,c_r}^{S_{c_1-1} × S_{c_2} ×...× S_{cr}} = k[F_{c_1}(Y_1^{(1)}), F_{c_1}(Y_1^{(1)}), F_{c_2}(Y_1^{(1)}), ..., F_{c_1-1}(Y_1^{(1)}), ..., F_{c_r}(Y_1^{(1)}), F_{c_r}(Y_1^{(1)}), F_{c_r}(Y_1^{(1)}), ..., F_{c_r-1}(Y_1^{(1)})].
\]

**Corollary 1.33.** 1) \( \overline{R}_{c_1,...,c_r} \subset R_{c_1,...,c_r} \)

is a finite extension of graded rings.

2) Let \( H_N \in \overline{R}_{c_1,...,c_r} \) be homogeneous of degree \( N \). There is a polynomial in variables

\[
\{W_{c_1}, W_{c_1}^{(1)}, W_{c_1}^{(2)}, ..., W_{c_1}^{(c_1-1)}, ..., W_{c_r}, W_{c_r}^{(1)}, ..., W_{c_r}^{(c_r-1)}\},
\]
say \( G \in k[W_{c_1}, W_{c_1}^{(1)}, W_{c_1}^{(2)}, ..., W_{c_1}^{(c_1-1)}, ..., W_{c_r}, W_{c_r}^{(1)}, ..., W_{c_r}^{(c_r-1)}] \), such that

\[
G(F_{c_1}(Y_1^{(1)}), F_{c_1}(Y_1^{(1)}), F_{c_2}(Y_1^{(1)}), ..., F_{c_1-1}(Y_1^{(1)}), ..., F_{c_r}(Y_1^{(1)}), F_{c_r}(Y_1^{(1)}), F_{c_r}(Y_1^{(1)}), ..., F_{c_r-1}(Y_1^{(1)})) = H_N.
\]

Furthermore, we may assume that \( G \) is weighted homogeneous of degree \( N \), if we assign weight (degree) \( c_i - j \) to the variable \( W_{c_i}^{(j)} \).

**Proof.** 1) Both rings are defined as subrings of invariants of the finite groups

\[
\mathbb{S}_{c_1-1} × \mathbb{S}_{c_2} ×...× \mathbb{S}_{c_r} \subset \mathbb{S}_{c_1} × \mathbb{S}_{c_2} ×...× \mathbb{S}_{c_r}
\]
acting linearly on \( A_{c_1,...,c_r} \) (see \([1.25.2] \) and \([1.32] \).

2) Same argument as in \([1.30] \). It follows from 1) and the fact that each \( F_{c_i}^{(j)}(Y_1) \) is homogeneous of degree \( c_i - j \).

**Remark 1.34.** If, instead of \( \mathbb{S}_{c_1-1} × \mathbb{S}_{c_2} ×...× \mathbb{S}_{c_r} \) we consider \( \mathbb{S}_{c_1} × \mathbb{S}_{c_2-1} × \mathbb{S}_{c_3} ×...× \mathbb{S}_{c_r} \), the same argument shows that

\[
A_{c_1,...,c_r}^{\mathbb{S}_{c_1} × \mathbb{S}_{c_2-1} × \mathbb{S}_{c_3} ×...× \mathbb{S}_{c_r}}
\]
is a graded \( k \)-algebra and a subring of

\[
k[s_1^{(1)}, ..., s_{c_1}^{(1)}, ..., s_r^{(r)}][Y_1^{(2)}] = k[s_1^{(1)}, ..., s_{c_1}^{(1)}, ..., s_r^{(r)}][Z]/⟨F_{c_2}(Z)⟩,
\]
generated by the class of the elements

\[
\{F_{c_1}(Z), F_{c_1}(Z)^{(1)}, F_{c_1}(Z)^{(2)}, ..., F_{c_1}(Z)^{(c_1-1)}, ..., F_{c_r}(Z), F_{c_r}(Z)^{(1)}, F_{c_r}(Z)^{(2)}, ..., F_{c_r}(Z)^{(c_r-1)}\}
\]
in this quotient ring.
Appendix 2: On normality of multi-graded invariant ring.
In this part we extend Lemma [1.22] and Proposition [1.23] to the case of several polynomials. Proofs are straightforward generalizations.

1.35. The ring \( k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}][Y_1^{(1)}] \) is a free module of rank \( b \) over \( k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}] \). Multiplication by an element \( \Theta \in k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}][Y_1^{(1)}] \) defines an endomorphism, say \( \phi_\Theta \), with characteristic polynomial, say
\[
\psi_\Theta(V) = V^b + h_1 V^{b-1} + \cdots + h_b \in k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}][V].
\]

Definition 1.36. Let \( H_{F_1, F_2, \ldots, F_r} \) be the \( k \)-subalgebra of \( k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}] \) generated by the coefficients of the characteristic polynomials
\[
\psi_{F_i^{(e)}(Y_1)}(V), \quad 1 \leq e \leq c_i - 1, \quad 1 \leq i \leq r.
\]

Lemma 1.37. \( H_{F_1, F_2, \ldots, F_r} \) is graded subalgebra both of \( A_{c_1, c_2, \ldots, c_r} \) [1.25.1], and of \( k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}] \).

Proposition 1.38. \( \overline{R}_{c_1, c_2, \ldots, c_r} \) is the integral closure of the graded ring \( H_{F_1, F_2, \ldots, F_r} \) in \( k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}] \).

Keeping track of the grade (the variable \( W \)), and elimination algebras.

1.39. Let \( G \) be an \( \mathbb{N} \)-graded ring. Add a variable \( W \), say \( G[W] \), and grade this new algebra with \( W \) of degree one. We define now what we call a graded inclusion, say \( G \to G[W] \), as follows: If \( G = \sum_{k \geq 0} [G]_k \), define \( \sum_{k \geq 0} [G]_k W^k \) as subalgebra in \( G[W] \). Note that if \( G = \sum_{k \geq 0} [G]_k \subset G' = \sum_{k \geq 0} [G']_k \) is a finite extension of \( \mathbb{N} \)-graded algebras, then \( \sum_{k \geq 0} [G]_k W^k \subset G' = \sum_{k \geq 0} [G']_k W^k \) is also finite. If \( G \) is a \( k \)-algebra generated by elements, say \( \{H_1, \ldots, H_s\} \), where each \( H_i \) is homogeneous of degree \( d_i \); then the graded inclusion of \( G \) is the \( k \)-subalgebra in \( G[W] \) generated, over \( k \), by \( \{H_1 W^{d_1}, \ldots, H_s W^{d_s}\} \).

1.40. Set \( T = k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}] \). We attach to the subalgebras \( \overline{R}_{c_1, c_2, \ldots, c_r} \) and \( H_{F_1, F_2, \ldots, F_r} \) two graded \( k \)-subalgebras included in a polynomial ring \( T[W] \), as in [1.39].

Now set \( U^{(1)}_{c_1, c_2, \ldots, c_r} \) (set \( U^{(2)}_{c_1, c_2, \ldots, c_r} \) as the \( T \)-subalgebra of \( T[W] \) generated by such \( k \)-subalgebra. In other words, the algebras generated over \( T \) by all elements of the form
\[
H(s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}), W^{d_H},
\]
where \( H \) is an homogeneous element of degree \( d_H \) in \( \overline{R}_{c_1, c_2, \ldots, c_r} \) (in \( H_{F_1, F_2, \ldots, F_r} \)).

We can also express \( U^{(1)}_{c_1, c_2, \ldots, c_r} = \sum I_k^{(1)} W^k(\subset T[W]) \), and \( U^{(2)}_{c_1, c_2, \ldots, c_r} = \sum I_k^{(2)} W^k(\subset T[W]) \), where \( I_k^{(1)} \) and \( I_k^{(2)} \) are ideals in \( T \).

Lemma [1.37] ensures that \( I_k^{(2)} \subset I_k^{(1)} \) for each positive index \( k \). Namely, that
\[
U^{(2)}_{c_1, c_2, \ldots, c_r} \subset U^{(1)}_{c_1, c_2, \ldots, c_r}.
\]
Furthermore, Prop 1.38 asserts that this extension is finite.

Note that both algebras $U^{(1)}_{c_1, c_2, \ldots, c_r}$ and $U^{(2)}_{c_1, c_2, \ldots, c_r}$ are finitely generated over $T$. This follows from the fact that the graded algebras $\overline{R}_{c_1, c_2, \ldots, c_r}$ and $H_{F_1, F_2, \ldots, F_r}$ are finitely generated.

For example $U^{(1)}_{c_1, c_2, \ldots, c_r}$ is the subring of $T[W]$ generated over $T$ by:

$$\mathcal{F} = \{G_{c_1, \ldots, c_r}(s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}) \cdot W^{\deg G_{c_1, \ldots, c_r}}, \ 1 \leq l \leq n_{c_1, \ldots, c_r} \} \quad (\text{see 1.25.3}).$$

Remark 1.41. $\overline{R}_{c_1, \ldots, c_r}$ and $H_{F_1, F_2, \ldots, F_r}$ are graded subalgebra both of $A_{c_1, c_2, \ldots, c_r}$ (1.25.1), and of $T$, which in turn is a graded ring in $k[Y_1^{(1)}, \ldots, Y_{c_1}^{(1)}, Y_1^{(r)}, \ldots, Y_{c_r}^{(r)}]$. The algebras $U^{(2)}_{c_1, c_2, \ldots, c_r} \subset U^{(1)}_{c_1, c_2, \ldots, c_r}$ are defined in $T[W]$, where the variable $W$ keeps track of the degree.

Specialization: From universal to concrete.

Definition 1.42. Fix a $k$-algebra $S$ and monic polynomials

$$f_i(Z) = Z^{c_i} - a_1^{(i)} + \cdots + (-1)^{c_i} a_{c_i}^{(i)} \in S[Z], \quad 1 \leq i \leq r.$$

Set $F_{c_1}, F_{c_2}, \ldots, F_{c_r}$ and $\overline{R}_{F_{c_1}, F_{c_2}, \ldots, F_{c_r}}$ as before. So $F_{c_i}(Z) \in k[s_1^{(i)}, \ldots, s_{c_i}^{(i)}][Z]$, and each $f_i(Z)$ is obtained from $F_{c_i}(Z)$ by the change of base rings $\pi : k[s_1^{(i)}, \ldots, s_{c_i}^{(i)}] \to S$ defined by setting

$$\pi(s_j^{(i)}) = a_j^{(i)}.$$

In this way the polynomials $f_i(Z) \in S[Z], \ 1 \leq i \leq r$ define a morphism of $k$-algebras:

$$(1.42.1) \quad T = k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}] \to S,$$

which extends to a morphism $T[W] \to S[W]$. We define the elimination algebra, say

$$(1.42.2) \quad \overline{R}_{f_{c_1}, f_{c_2}, \ldots, f_{c_r}} (\subset S[W])$$

as the subalgebra of $S[W]$ generated by the image of $U^{(1)}_{c_1, c_2, \ldots, c_r} = \sum I_k^{(1)} W^k (\subset T[W])$.

In the same way we define

$$(1.42.3) \quad \mathcal{H}_{f_{c_1}, f_{c_2}, \ldots, f_{c_r}} (\subset S[W])$$

as the subalgebra of $S[W]$ generated by the image of $U^{(2)}_{c_1, c_2, \ldots, c_r} = \sum I_k^{(2)} W^k \subset T[W])$.

The previous observations show that

$$(1.42.4) \quad \mathcal{H}_{f_{c_1}, f_{c_2}, \ldots, f_{c_r}} \subset \overline{R}_{f_{c_1}, f_{c_2}, \ldots, f_{c_r}}$$

and that this ring extension is finite. Note also that both are finitely generated, for instance, the elimination algebra $\overline{R}_{f_{c_1}, f_{c_2}, \ldots, f_{c_r}}$ is spanned over $S$ by the finite set

$$\mathcal{F} = \{G_{c_1, \ldots, c_r}(s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}) \cdot W^{\deg G_{c_1, \ldots, c_r}}, \ 1 \leq l \leq n_{c_1, \ldots, c_r} \} \quad (1.25.3).$$

1.43. Set $T = k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}]$ as above. Prop 1.28 asserts that there is a finite extension, say: $k[[F_{c_i}^{(\beta)}(Z), e = 0, 1, \ldots, b - 1]] \subset k[[F_{c_i}^{(\beta)}(Z), 1 \leq i \leq r, 0 \leq \beta \leq c_i]]$ of graded $k$-subalgebras of (the graded algebra) $T[Z] = k[s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_1^{(r)}, \ldots, s_{c_r}^{(r)}][Z]$. This
defines a finite extension of $k$-subalgebras in $T[Z,W]$ \text{(1.39)}. Finally, as each such $k$-algebra extends to an algebra over $T[Z]$, there is a finite extension of $T[Z]$-algebras:

\[(1.43.1) \quad T[Z][\{f_b^{(e)}(Z)W^{b-e}, 0 \leq e \leq b-1\}] \subset T[Z][\{f_{c_i}^{(e)}(Z)W^{c_i-e}, 1 \leq i \leq r, 0 \leq e \leq c_i\}]\]

Consider, as in \text{(1.42)}, a ring $S$ and monic polynomials $f_{c_i}(Z) = Z^{c_i} - a_1^{(i)} + \cdots + (-1)^{c_i}a_{c_i}^{(i)} \in S[Z], \quad 1 \leq i \leq r$. Set $b = c_1 + \cdots + c_r$, and $f_b(Z) = f_{c_1}(Z) \cdot f_{c_2}(Z) \cdots f_{c_r}(Z)$, which is monic of degree $b$. The morphism in \text{(1.42.1)}, and \text{(1.43.1)}, define a finite inclusion

\[(1.43.2) \quad S[Z][\{f_b^{(e)}(Z)W^{b-e}, 0 \leq e \leq b-1\}] \subset S[Z][\{f_{c_i}^{(e)}(Z)W^{c_i-e}, 1 \leq i \leq r, 0 \leq e \leq c_i\}]\]

Similarly, the inclusion in Prop\text{[1.29]} yields a (non-finite) inclusion of the elimination algebra

\[(1.43.3) \quad \overline{K}_{f_{c_1}, f_{c_2}, \ldots, f_{c_r}} \subset S[Z][\{f_{c_i}^{(e)}(Z)W^{c_i-e}, 1 \leq i \leq r, 0 \leq e \leq c_i\}]\]

Where the left hand side is a subalgebra of $S[W]$, and the variable $Z$ has been eliminated.

2. Rees algebras and relative differential Rees algebras.

In this section a class of $\mathbb{N}$-graded algebras is introduced, which extend the notion of Rees rings of ideals. The geometric application of these algebras will lead us to consider them only up to integral closure (not to distinguish algebras with the same integral closure). Special attention will be devoted to finite extensions among these algebras. In \text{[2.4]} we begin the study algebras with a natural action of differential operators.

The results in this Section appear in Sections 1 of \text{[35]}. They are included here because of their relevance in all the further discussion.

2.1. Fix a noetherian ring $B$, and a sequence of ideals $\{I_k\}, k \geq 0$, which fulfill the conditions:

i) $I_0 = B$, and
ii) $I_k \cdot I_s \subset I_{k+s}$.

This defines a graded subring $\bigoplus_{k \geq 1} I_k \cdot W^k$ of the polynomial ring $B[W]$. We say that $\bigoplus I_k \cdot W^k$ is a Rees algebra if this subring is a (noetherian) finitely generated $B$-algebra.

Remark 2.2. 1) Examples of Rees algebras are the Rees rings of an ideal $I \subset B$, where $I_k = I^k$ for each $k \geq 1$. These are the Rees algebras generated, as $B$-algebras, in degree one.

2) Whenever $\bigoplus I_k \cdot W^k \subset \subset B[W]$ is a Rees algebra, define $\bigoplus I_k' \cdot W^k$ by setting

$I_k' = \sum_{r \geq k} I_r$.

If $\bigoplus I_k \cdot W^k$ is generated by $\mathcal{F} = \{g_{n_i}W^{n_i}, 1 \leq i \leq m, n_i > 0\}$. Namely, if:

$\bigoplus I_k \cdot W^k = B[\{g_{n_i}W^{n_i}\}_{g_{n_i}W^{n_i} \in \mathcal{F}}]$,

then $\bigoplus I_k' \cdot W^k$ is generated by the finite set $\{g_{n_i}W^{n_i'}, 1 \leq i \leq m, 1 \leq n_i' \leq n_i\}$.

Note that $I_k' \supset I_{k+1}'$, and that $\bigoplus I_k \cdot W^k \subset \bigoplus I_k' \cdot W^k$ is a finite extension. It suffices to check that given an element $g \in I_k$, then $g \cdot W^{k-1}$ is integral over $\bigoplus I_k \cdot W^k$. Notice that $g \in I_k \Rightarrow g^{k-1} \in I_{k(k-1)} \Rightarrow g^k \in I_{k(k-1)}$. 
so \( g \cdot W^{k-1} \) fulfills the equation \( Z^k = (g^k \cdot W^{k(k-1)}) = 0 \).

Up to integral closure we may assume that a Rees algebra has the additional condition:

iii) \( I_k \supset I_{k+1} \).

**2.3.** In what follows we define a Rees algebra, say \( \bigoplus_{n \geq 0} I_n W^n \) in \( B[W] \), by fixing a set of generators, say \( \mathcal{F} = \{ g_n W^{n_i}/n_i > 0, 1 \leq i \leq m \} \). So if \( f \in I_n \), then \( f = F_n(g_{n_1}, \ldots, g_{n_m}) \), where \( F_n(Y_1, \ldots, Y_m) \) is weighted homogeneous in \( m \) variables, and each \( Y_j \) has degree \( n_j \).

**2.4.** Let \( B = S[Z] \) be a polynomial ring, and let \( \text{Tay} : B \rightarrow B[U] \) be the \( S \)-homomorphism defined by setting \( \text{Tay}(Z) = Z + U \). For \( f(Z) \in B \) set \( \text{Tay}(f(Z)) = \sum_{\alpha \geq 0} \Delta^\alpha(f(Z)) U^\alpha \).

The operators \( \Delta^\alpha \) are \( S \)-differential operators (\( S \) linear). Furthermore, for every positive integer \( N \), \( \{ \Delta^\alpha, 0 \leq \alpha \leq N \} \) is a basis of the \( B \)-module of \( S \)-differential operators on \( B \), of order \( \leq N \).

**Definition 2.5.** Set \( B = S[Z] \) as before, a polynomial ring over a noetherian ring \( S \). A Rees algebra \( \bigoplus I_k \cdot W^k \subset B[W] \) is a differential Rees algebra, relative to \( S \), when:

i) \( I_k \supset I_{k+1} \) for \( k \geq 0 \).

ii) For all \( n > 0 \) and \( f \in I_n \), and every index \( 0 \leq j \leq n \) and every \( S \)-differential operator of order \( \leq j \), say \( D_j : D_j(f) \in I_{n-j} \).

**Remark 2.6.** Let \( \text{Diff}_S^N(B) \) denote the module of \( S \)-differential operators of order at most \( N \). Then \( \text{Diff}_S^N(B) \subset \text{Diff}_S^{N+1}(B) \subset \ldots \). For this reason it is natural to require condition (i) in our previous definition. Note also that 2.4 asserts that (ii) can be reformulated as:

ii') For all \( n > 0 \) and \( f \in I_n \), and for every index \( 0 \leq \alpha \leq n \): \( \Delta^\alpha(f) \in I_{n-\alpha} \).

In fact, (i)+(ii) is equivalent to (i)+(ii'):

**Theorem 2.7.** Fix \( B = S[Z] \) as before, and a finite set \( \mathcal{F} = \{ g_n W^{n_i}, n_i > 0, 1 \leq i \leq m \} \), with the following properties:

a) For \( 1 \leq i \leq m \), and every \( n'_i, 0 < n'_i \leq n_i \); \( g_n W^{n'_i} \in \mathcal{F} \).

b) For \( 1 \leq i \leq m \), and for every index \( 0 \leq \alpha < n_i \); \( \Delta^\alpha(g_n)W^{n_i-\alpha} \in \mathcal{F} \).

Then the \( B \)-subalgebra of \( B[W] \), generated by \( \mathcal{F} \) over the ring \( B \), a differential Rees algebra relative to \( S \).

**Proof.** Condition (i) in Def 2.5 is by 2.2.2. Let \( I_N \subset B \) be the homogeneous component of degree \( N \) of the \( B \) subalgebra generated by \( \mathcal{F} \). We prove that for all \( h \in I_N \), and \( 0 \leq \alpha \leq N \), \( \Delta^\alpha(h) \in I_{N-\alpha} \). The ideal \( I_N \subset B \) is generated by all elements of the form

\[
(2.7.1) \quad H_N = g_{n_{i_1}} \cdot g_{n_{i_2}} \cdots g_{n_{i_p}} \quad n_{i_1} + n_{i_2} + \cdots n_{i_p} = N,
\]

with the \( g_{n_{i}} W^{n_{i}} \in \mathcal{F} \) not necessarily different.

Since the operators \( \Delta^\alpha \) are linear, it suffices to prove that \( \Delta^\alpha(a \cdot H_N) \in I_{N-\alpha} \), for \( a \in B \), \( H_N \) as in 2.7.1 and \( 0 \leq \alpha \leq N \). We proceed in two steps, by proving:

1) \( \Delta^\alpha(H_N) \in I_{N-\alpha} \).
2) \( \Delta^\alpha(a \cdot H_N) \in I_{N-\alpha} \).
We first prove 1). Set $\text{Tay} : B = S[Z] \to B[U]$, as in [2.4]. Consider, for every $g_{n_{i}}W^{n_{i}} \in \mathcal{F}$,
\[
\text{Tay}(g_{n_{i}}) = \sum_{\beta \geq 0} \Delta^{\beta}(g_{n_{i}})U^{\beta} \in B[U].
\]

Hypothesis (b) states that for $0 \leq \beta < n_{i}$, $\Delta^{\beta}(g_{n_{i}})W^{n_{i} - \beta} \in \mathcal{F}$. On the one hand
\[
\text{Tay}(H_{N}) = \sum_{\alpha \geq 0} \Delta^{\alpha}(H_{N})U^{\alpha},
\]
and, on the other hand
\[
\text{Tay}(H_{N}) = \text{Tay}(g_{n_{i_{1}}}) \cdot \text{Tay}(g_{n_{i_{2}}}) \cdots \text{Tay}(g_{n_{i_{p}}})
\]
in $B[U]$. So for a fixed $\alpha$ ($0 \leq \alpha \leq N$), $\Delta^{\alpha}(H_{N})$ is a sum of elements of the form:
\[
\Delta^{\beta_{1}}(g_{n_{i_{1}}}) \cdot \Delta^{\beta_{2}}(g_{n_{i_{2}}}) \cdots \Delta^{\beta_{p}}(g_{n_{i_{p}}}), \quad \sum_{1 \leq s \leq p} \beta_{s} = \alpha.
\]

Therefore it suffices to show that each of these summands is in $I_{N - \alpha}$. Note here that
\[
\sum_{1 \leq s \leq p} (n_{i_{s}} - \beta_{s}) = N - \alpha,
\]
and that some of the integers $n_{i_{s}} - \beta_{s}$ might be zero or negative. Set
\[
G = \{ r : 1 \leq r \leq p, \text{ and } n_{i_{r}} - \beta_{r} > 0 \}.
\]
So $N - \alpha = \sum_{1 \leq s \leq p}(n_{i_{s}} - \beta_{s}) \leq \sum_{r \in G}(n_{i_{r}} - \beta_{r}) = M$.

Hypothesis (b) ensures that $\Delta^{\beta_{r}}(g_{n_{i_{r}}}) \in I_{n_{i_{r}} - \beta_{r}}$ for every index $r \in G$, in particular:
\[
\Delta^{\beta_{1}}(g_{n_{i_{1}}}) \cdot \Delta^{\beta_{2}}(g_{n_{i_{2}}}) \cdots \Delta^{\beta_{p}}(g_{n_{i_{p}}}) \in I_{M}.
\]
Finally, since $M \geq N - \alpha$, $I_{M} \subset I_{N - \alpha}$, and this proves Case 1).

For Case 2), fix $0 \leq \alpha \leq N$. We claim that $\Delta^{\alpha}(a \cdot H_{N}) \in I_{N - \alpha}$, for $a \in B$ and $H_{N}$ as in [2.7.1] At the ring $B[U]$, $\text{Tay}(a \cdot H_{N}) = \sum_{\alpha \geq 0} \Delta^{\alpha}(a \cdot H_{N})U^{\alpha}$, and, on the other hand
\[
\text{Tay}(a \cdot H_{N}) = \text{Tay}(a) \cdot \text{Tay}(H_{N}).
\]
This shows that $\Delta^{\alpha}(a \cdot H_{N})$ is a sum of terms of the form $\Delta^{\alpha_{1}}(a) \cdot \Delta^{\alpha_{2}}(H_{N}), \alpha_{i} \geq 0$, and $\alpha_{1} + \alpha_{2} = \alpha$. In particular $\alpha_{2} \leq \alpha$; and by Case 1), $\Delta^{\alpha_{2}}(H_{N}) \in I_{N - \alpha_{2}}$. On the other hand $N - \alpha_{2} \geq N - \alpha$, so $\Delta^{\alpha_{2}}(H_{N}) \in I_{N - \alpha}$, and hence $\Delta^{\alpha}(a \cdot H_{N}) \in I_{N - \alpha}$.

**Corollary 2.8.** A Rees algebra in $B[W]$, generated over $B$ by
\[
\mathcal{F} = \{ g_{n_{i}}W^{n_{i}}, n_{i} > 0, 1 \leq i \leq m \},
\]
extends to a smallest differential Rees algebra relative to $S$, which is generated by
\[
\mathcal{F}' = \{ \Delta^{\alpha}(g_{n})W^{n_{i} - \alpha} ; \text{ for all } g_{n_{i}}W^{n_{i}} \in \mathcal{F} \text{ and } 0 \leq \alpha < n_{i}' \leq n_{i} \}.
\]
2.9. Of particular interest in our development will be the case $B = S[Z]$ where $S$ is a local regular ring. In particular both $S$ and $S[Z]$ will be unique factorization domains. We will consider graded subalgebras in $B[W]$, and always up to integral closure within this ring.

Assume that $\bigoplus I_k \cdot W^k \subset B[W]$ is a differential Rees algebra relative to $S$. If, for some positive integer $k$ there is a polynomial, say $f(Z) \in I_k$, which is monic of degree, say $a < k$, then $\Delta_{a-k}(f(Z)) = 1$ In this case so $W^{a-k} \subset \bigoplus I_k \cdot W^k$, and the integral closure of this algebra is all $B[W]$.

Assume now that for some positive integer $b$, there is a monic polynomial of degree $b$, say $f_b(Z) \in I_b$. Let $f_b^{(e)}$ denote $\Delta^e(f_b)$, note that $S[Z][\{f_b^{(e)}(Z)W^{b-e}, 0 \leq e \leq b-1\}] \subset \bigoplus I_k \cdot W^k$.

If, in addition, there is a factorization of $f_b(Z)$, of the form: $f_b(Z) = f_{c_1}(Z) \cdot f_{c_2}(Z) \cdots f_{c_i}(Z)$, where each factor $f_{c_i}(Z)$ is a monic polynomial of degree $c_i$, then [1.43.2] asserts that

$$S[Z][\{f_b^{(e)}(Z)W^{b-e}, 0 \leq e \leq b-1\}] \subset S[Z][\{f_{c_i}^{(e)}(Z)W^{c_i-e}, 1 \leq i \leq r, 0 \leq e \leq c_i\}]$$

is finite. In particular, each element $f_{c_i}(Z)W^{c_i}$ is integral over $\bigoplus I_k \cdot W^k$.

3. Differential Rees algebras on smooth schemes

Since the outstanding Theorem of Hironaka was published in [20], an important effort was done to simplify the proof of this Theorem. One of the major steps in this direction was achieved by Jean Giraud in [15] and [16]. This simplification grows from the new technics, which were introduced there, involving differential operators.

In Section 3 of [35] it is proved that every Rees algebra can be naturally extended, in a unique manner, to a new Rees algebra enriched by the action of differential operators. These are called Differential Rees algebras, which are to be discussed in this section. For the ease of the exposition we also recall some properties of these extensions, introduced in [35], as they are to be used in the coming sections.

3.1. Let $V$ be a smooth scheme over a field. A sequence of coherent ideals on $V$, say $\{I_n\}_{n \in \mathbb{N}}$, such that $I_0 = \mathcal{O}_V$, and $I_k \cdot I_s \subset I_{k+s}$, defines a graded sheaf of algebras $\bigoplus_{n \geq 0} I_n \cdot W^n \subset \mathcal{O}_V[W]$.

We say that this algebra is a Rees algebra over $V$, if there is an open covering of $V$ by affine open sets $\{U_i\}$, so that $\bigoplus_n I_n(U_i)W^n \subset \mathcal{O}_V(U_i)[W]$ is a finitely generated $\mathcal{O}_V(U_i)$-algebra. In what follows $V$ will denote a smooth scheme of a perfect field $k$, and $\text{Diff}^r_k(V)$, or simply $\text{Diff}^r_k$, denotes the locally free sheaf of $k$-linear differential operators of order at most $r$.

**Definition 3.2.** 1) A Rees algebra $\mathcal{G}$ defined by $\{I_n\}_{n \in \mathbb{N}}$ is a differential Rees algebra, relative to the field $k$, or simply a $\text{Diff}$-algebra, if:

i) $I_n \supset I_{n+1}$.

ii) There is open covering of $V$ by affine open sets $\{U_i\}$, and for every $D \in \text{Diff}^r(U_i)$, and $h \in I_n(U_i)$, $D(h) \in I_{n-r}(U_i)$, provided that $n \geq r$.

2) We sometimes fix a smooth morphism $\beta : V \to V'$, of smooth schemes over $k$. In this case $\mathcal{G}$ is said to be a relative differential algebra, or a $\beta$-differential algebra, when ii) holds only for those $D \in \text{Diff}_\beta^r(U_i)$, namely for $\beta$-linear differential operators.
The abbreviated notation of "Diff-algebra" omits reference to the underlying field $k$, but this will be clear from the context. Due to the local nature of the definition, we reformulate this notion in terms of $k$-algebras.

**Definition 3.3.** In what follows $R$ will denote a smooth algebra over a field, or a localization of such algebra at a closed point (a regular local ring). A Rees algebra is defined by a sequences of ideals $\{I_k\}_{k \in \mathbb{N}}$ such that:
1) $I_0 = R$, and $I_k \cdot I_s \subseteq I_{k+s}$.
2) $\bigoplus I_k W^k$ is a finitely generated $R$-algebra.

We say that the Rees algebra is a differential Rees algebra relative to $k$, or simply a Diff-algebra, if, in addition to the previous conditions:
3) $I_n \supset I_{n+1}$.
4) given $D \in \text{Diff}_k^r(R)$, $D(I_n) \subseteq I_{n-r}$.

**Theorem 3.4.** Fix a smooth scheme $V$ over a perfect field $k$. Assume that $G = \bigoplus I_k \cdot W^k$ is a Rees algebra over $V$. Then there is a smallest extension of it to a differential Rees algebra relative to the field $k$.

The Theorem says that a Rees algebra $G$, over a smooth scheme $V$ extends to a smallest Diff-algebra (i.e., included in every other Diff-algebra containing it). The latter is called the Diff-algebra spanned by $G$. We refer here to Th 3.4 in [35] for the proof. Let us indicate that it follows easily from the argument for the one-variable case (Th 2.7). Fix a closed point $x \in V$, and a regular system of parameters $\{x_1, \ldots, x_n\}$ for $\mathcal{O}_{V,x}$, then smoothness of $V$ locally at $x$ asserts that there is a ring homomorphism, say:

$$
\text{Tay}: \mathcal{O}_{V,x} \rightarrow \mathcal{O}_{V,x}[\langle T_1, \ldots, T_n \rangle], \quad \text{Tay}(f) = \sum_{\alpha \in (\mathbb{Z})^n} \Delta^\alpha(f) T^\alpha
$$

where $\text{Tay}(x_i) = x_i + T_i$. Furthermore, as the underling field $k$ is perfect, $\{\Delta^\alpha, \alpha \in (\mathbb{N})^n, 0 \leq |\alpha| \leq c\}$ generate the $\mathcal{O}_{V,x}$-module $\text{Diff}_k^c(\mathcal{O}_{V,x})$.

The proof of the previous Theorem shows that, at a suitable affine neighborhood of $x$ in $V$, where say $\bigoplus I_k \cdot W^k$ is generated by $\mathcal{F} = \{g_{n_i} W^{n_i}, n_i > 0, 1 \leq i \leq m\}$, then

$$
(3.4.1) \quad \mathcal{F}' = \{\Delta^\alpha (g_{n_i}) W^{n_i'-\alpha}/g_{n_i} W^{n_i}, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{N})^n, 0 \leq |\alpha| < n_i' \leq n_i\}
$$
generates the smallest extension of $\bigoplus I_k \cdot W^k$ to a Diff-algebra (relative to $k$). In particular, a Rees ring of an ideal (all $n_i = 1$ in $\mathcal{F}$) is a Diff-algebra.

**Corollary 3.5.** Given inclusions of Rees algebras, say

$$
\mathcal{G} = \bigoplus I_n W^n \subset \mathcal{G}' = \bigoplus I'_n W^n \subset \mathcal{G}'' = \bigoplus I''_n W^n,
$$

where $\mathcal{G}''$ is the Diff-algebra spanned by $\mathcal{G}$, then $\mathcal{G}''$ is also the Diff-algebra spanned by $\mathcal{G}'$.

Differential Rees algebras and singular locus.
3.6. The notion Diff-algebras, over a smooth $k$-scheme $V$, is closely related to the notion of order at the local regular rings of $V$ when $k$ is a perfect field. Recall that the order of a non-zero ideal $I$ at a local regular ring $(R, M)$ is the highest integer $b$ for which $I \subset M^b$.

If $I \subset O_V$ is a sheaf of ideals, $V(Diff^{b-1}_k(I))$ is the closed set of points of $V$ where the ideal has order at least $b$. We analyze this fact locally at a closed point $x$.

Let $\{x_1, \ldots, x_n\}$ be a regular system of parameters for $O_{V,x}$, and consider the differential operators $\Delta^\alpha$, defined on $O_{V,x}$ in terms of these parameters, as in the Theorem 3.4. So at $x$,

$$(Diff^{b-1}_k(I))_x = \langle \Delta^\alpha(f)/f \in I, 0 \leq |\alpha| \leq b-1 \rangle.$$

One can now check at $O_{V,x}$, or at the ring of formal power series $\hat{O}_{V,x}$, that $Diff^{b-1}_k(I)$ is a proper ideal if and only if $I$ has order at least $b$ at the local ring.

The operators $\Delta^\alpha$ are defined globally at a suitable neighborhood $U$ of $x$. So if $\bigoplus I_k \cdot W^n \subset O_V[W]$ is a Diff-algebra, and $x \in V$ is a closed point, the Diff-algebra $\bigoplus (I_n)_x \cdot W^n \subset O_{V,x}[W]$ will be properly included in $O_{V,x}[W]$, if and only, for each index $k \in \mathbb{N}$, the ideal $(I_k)_x$ has order at least $k$ at the local regular ring $O_{V,x}$.

Definition 3.7. The singular locus of a Rees algebra $G = \bigoplus I_n \cdot W^n \subset O_V[W]$, will be

$$Sing(G) = \cap_{r \geq 0} V(Diff^{r-1}_k(I_r))(\subset V).$$

It is the set of points $x \in V$ for which all $(I_r)_x$ have order at least $r$ (at $O_{V,x}$).

Remark 3.8. Assume that $f \in (I_r)_x$ has order $r$ at $O_{V,x}$. Then, locally at $x$, $Sing(G)$ is included in the set of points of multiplicity $r$ (or say, $r$-fold points) of the hypersurface $V(\langle f \rangle)$.

In fact $Diff^{r-1}_k(f) \subset Diff^{r-1}_k(I_r)$, and the closed set defined by the first ideal is that of points of multiplicity $r$.

Proposition 3.9.  

1. If $G$ is a Rees algebra generated over $O_V$ by $F = \{g_i W^{n_i}, n_i > 0, 1 \leq i \leq m\}$, then $Sing(G) = \cap V(Diff^{n_i}(\langle g_i \rangle))$.

2. If $G = \bigoplus I_n \cdot W^n$ and $G' = \bigoplus I'_n \cdot W^n$ are Rees algebras with the same integral closure (e.g. if $G \subset G'$ is a finite extension), then $Sing(G) = Sing(G')$.

3. Let $G'' = \bigoplus I''_n \cdot W^n$ be the extension of $G$ to a Diff-algebra, as defined in Theorem 3.4, then $Sing(G) = Sing(G'')$.

4. For every $Diff$-algebra $G'' = \bigoplus I''_n \cdot W^n$, $Sing(G'') = V(I''_n)$.

5. Let $G'' = \bigoplus I''_n \cdot W^n$ be a Diff-algebra. For every index $r$, $Sing(G'') = V(I''_r)$.

Proof. 1) We have formulated 1) with a global condition on $V$, however this is always the case locally. In fact, there is a covering of $V$ by affine open sets, so that the restriction of $G$ is generated by finitely many elements. Let $U$ be such open set, so $G(U) = \bigoplus I_k(U) \cdot W^k$ is generated by $F = \{g_i W^{n_i}, n_i > 0, 1 \leq i \leq m\}$, $g_i \in O(U)$. The claim is that $y \in Sing(G) \cap U$ if and only if the order of $g_i$, at $O_{V,y}$ is at least $n_i$, for $1 \leq i \leq m$.

The condition is clearly necessary. Conversely, if $G = \bigoplus I_n = O_{V,\{g_i W^{n_i}\}_{g_i \in F}}$, and each $g_i$ has order at least $n_i$ at $O_{V,y}$, then $I_n$ (generated by weighted homogeneous expressions on the $g_i$’s) has order at least $n$ at $O_{V,y}$. 


2) Rees algebras are, locally, finitely generated over $\mathcal{O}_V$. This ensures that there is a (infinite) semigroup $\mathbb{M}$ in $\mathbb{N}$, so that for every $n \in \mathbb{M}$ both $I_n$ and $I'_n$ have the same integral closure. We may also define both singular loci as the points where these ideals have order at least $n$. The equality in 2) holds because the order of an ideal in a local regular ring, is the same as the order of the integral closure ([39] Appendix 3).

3) We argue as in 1), here we may also assume that there is $x \in U$, a regular system of parameters $\{x_1, \ldots, x_n\}$ at $x$, and differential operators $\Delta^\alpha$ as in the Theorem [3.4] defined globally at $U$. The Diff-algebra $\mathcal{G}''$ in Theorem [3.4] is defined by

$$\mathcal{F}' = \{\Delta^\alpha(g_n)W^{n_i-\alpha}/g_nW^{n_i} \in \mathcal{F}, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{N})^n, \text{ and } 0 \leq |\alpha| < n_i\}.$$ 

Note finally that if the order of $g_n$ at a local ring is $\geq n_i$, then the order of $\Delta^\alpha(g_n)$ is $\geq n_i - |\alpha|$.

4) The inclusion $\text{Sing}(\mathcal{G}'') \subset V(I''_1)$ holds, by definition, for every Rees algebra. On the other hand, the hypothesis ensures that $\text{Diff}r^{-1}(I''_r) \subset I''_1$, so $\text{Sing}(\mathcal{G}'') \supset V(I''_1)$.

5) Follows from 4).

4. Simple differential algebras and projections

This work, as a whole, is motivated by the problem of resolution of singularities over arbitrary fields. Generally speaking, the open problem of resolution of singularities over a perfect field reduces to that of constructing resolutions of Rees algebras. A second fundamental reduction of the open problem of resolution is that one can take, as starting point, a Rees algebra which is, in addition, a differential Rees algebra.

We do not define here resolution of Rees algebras, neither do we address these two fundamental reductions in this work, as the subject has been thoroughly studied. But let us indicate that this already justifies our particular attention on differential Rees algebras.

So, summarizing, the problem of embedded resolution over perfect fields reduces to that of resolution of differential Rees algebras. This is what we know, but what we do not know, at least at the moment, is to define resolutions of this kind of Rees algebras.

The hope is to prove the latter by using some form of induction, a strategy which works in characteristic zero. The form of induction we are searching for leads us firstly to the discussion of the $\tau$-invariant in [4.2] In fact, the $\tau$-invariant is a positive integer, and the larger this integer is, the simpler it is to construct a resolution.

This Section aims to the presentation of an entirely new form of induction. It is sustained, essentially, on elimination theory, and makes use of the algebras that were introduced in Definition [4.10] The properties of these are gathered in Theorem [4.11] which is the main result in this Section.

Hironaka’s Theorem of resolution of singularities in characteristic zero uses a form of induction which is based on a reformulation of the problem but now in a smooth hypersurface. So his induction is on the dimension on the ambient space. The form of induction we present
here is different from that of Hironaka, but leads to the same result over fields of characteristic zero. Here the restriction to a smooth hypersurface is replaced by the elimination of one variable. Both approaches make use of Hironaka’s notion of \( \tau \)-invariants. This is a powerful invariant attached to a point, as it indicates the total number of variables that can be eliminated, at least in a neighborhood of the point (see also \([5,11]\).

Fix a Rees algebra over a smooth scheme \( V \), say \( \mathcal{G} = \bigoplus I_k \cdot W^k (\subset \mathcal{O}_V[W]) \), and a closed point \( x \in \text{Sing}(\mathcal{G}) \). Let \( \mathcal{G}_x = \bigoplus I_k \cdot W^k (\subset R[W]) \) be the localization at \( R = \mathcal{O}_{V,x} \). Observe that the Diff-algebra spanned by \( \mathcal{G} \) induces, at \( x \in V \), the Diff-algebra spanned by \( \mathcal{G}_x \).

**Definition 4.1.** A Rees algebra \( \mathcal{G} = \bigoplus I_k \cdot W^k (\subset \mathcal{O}_V[W]) \) is said to be simple at a point \( x \in \text{Sing}(\mathcal{G}) \), if, for some \( n \), the order of \( I_n \) is \( n \) at the local ring \( \mathcal{O}_{V,x} \).

**4.2.** Here \( R = \mathcal{O}_{V,x} \) is a local regular ring. The graded algebra of the maximal ideal, say \( \text{gr}_M(R) \) is a polynomial ring. Recall that \( \text{Spec}(\text{gr}_M(R)) \) is the tangent space of \( V \) at \( x \). Attach to \( \mathcal{G} \) an homogeneous ideal in \( \text{gr}_M(R) \), called the initial (or tangent) ideal of \( \mathcal{G}_x \), spanned by \( I_{nk}(I_k) \), for all index \( k \). It defines a closed set in \( \text{Spec}(\text{gr}_M(R)) \) called the tangent cone of \( \mathcal{G}_x \).

The tangent ideal of a Rees algebra at a closed point \( x \in \text{Sing}(\mathcal{G}) \) is zero unless \( x \) is a simple point. In this case \( \text{gr}_M(R) = k'[Z_1, \ldots, Z_n] \) (polynomial ring in \( n \) variables), where \( k' \) denotes the residue field at \( x \). In the case in which \( \mathcal{G} \) is a Diff-algebra it is easy to check that the tangent ideal is closed when applying homogeneous differential operators of the form \( \Delta^\alpha \), for every multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \), where these operators are defined by taking Taylor expansions in terms of the variables \( Z_i \). In other words, if \( H \) is an homogeneous element of degree \( N \) in the tangent ideal, then \( \Delta^\alpha(H) \) is homogeneous of degree \( N - |\alpha| \), and belongs to the tangent ideal.

Homogeneous ideals with this property were studied in \([24]\). If \( k' \) is a field of characteristic zero, then the ideals with this property are exactly those generated by linear forms, which we may take to be \( Z_1, \ldots, Z_\tau \). If \( k' \) is a field of characteristic \( p \), the ideals with this property are generated by elements of the form

\[
(l_1, \ldots, l_{s_0}, l_{s_0+1}, \ldots, l_{s_1}, \ldots, l_{s_r-1}, l_{s_r},)
\]

where each \( l_j \) is a linear combination of powers \( Z_j^{p^i} \), for \( s_t \leq j \leq s_{t+1} \); and no \( l_j \) is in the ideal spanned by the previous elements. When \( k' \) is a perfect field, one can take \( l_j = Z_j^{p^0} \), for \( s_t \leq j \leq s_{t+1} \).

It is said that the initial ideal (defining the tangent cone at the point) is spanned by a flag of Frobenius-linear ideals in powers of the characteristic. The tangent cone is a subspace of the tangent space in characteristic zero, and an additive subgroup in arbitrary characteristic.

There are two important invariants defined by the tangent cone at the point:

1) the integer \( s_r \), usually called the invariant \( \tau \) of the singularity (at the point \( x \)), and
2) the smallest integer \( e_0 \) so that \( p^{e_0} \) is the smallest power which arises in the description of the elements \( l_i \) in the previous flag. When \( \mathcal{G} \) is a Diff-algebra the order (at \( R \)) of \( I_n \) is \( n \) if and only if \( n \) is a multiple of \( p^{e_0} \).

We refer to the work of T. Oda (\([29, 30, \text{and 28}]\)) where these notions are studied. Related to this notion, but different and original, is that introduced by Kawanoue in \([25], \text{and 26}\).
consisting on graded sub-algebras of tangent algebra \( gr_M(R) \) (as opposed the previous discussion, based on homogeneous ideals in \( gr_M(R) \)).

4.3. Algebras generated by monic polynomials.

Take a Rees algebra and a closed point \( x \) as above, and let \( q \) be an integer such that, for all natural number \( m \), \( I_{mq} \) has order \( mq \) at the local regular ring \( R = \mathcal{O}_{V,x} \). Take now the completion, so assume that \( R \) is complete.

Once we fix a regular system of parameters for \( R \), say \( \{ z, x_1, \ldots, x_{d-1} \} \), then one can set \( gr_M(R) = k'[Z, X_1, \ldots, X_{d-1}] \) where \( Z \) is the initial form of \( z \), and each \( X_i \) is the initial form of \( x_i \). Moreover, after enlarging the base field one can take the regular system of parameters so that the line \( V(<X_1, \ldots, X_{d-1}>) \) is transversal to the tangent cone in \( \text{Spec}(gr_M(R)) \). The Weierstrass Preparation Theorem ensures that, for some index \( n = mq \), there is an element \( f \in I_n \) of order \( n \), that multiplied by a suitable unit of \( R \), can be expressed as a monic polynomial of degree \( n \) in \( S[z] \). Here \( S \) is a regular local ring with coordinates \( \{ x_1, \ldots, x_{d-1} \} \), and say

\[
F(z) = z^n + a_1z^{n-1} + \cdots + a_n.
\]

Observe that in this case

\[
I_{(n)} := I_n \cap S[z]
\]

is an ideal spanned by monic polynomials of degree \( n \). To check this, set

\[
A = S[z]/\langle F \rangle = R/\langle f \rangle,
\]

and note that each \( g \in I_n \) has a class, say: \( b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n \). On the other hand

\[
G(z) = (z^n + a_1z^{n-1} + \cdots + a_n) + (b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n)
\]

is a monic polynomial in \( I_n \cap S[z] \), and all monic polynomials arising in this manner span \( I_{(n)} \). In this case \( I_n = I_{(n)} R \), thus \( I_n \) is generated by monic polynomials of degree \( n \) in \( S[z] \).

Note here that after an enlargement of the base field, \( \bigoplus I_k \cdot W^k(\subset R[W]) \) can be generated by elements in \( S[z][W] \). In fact, let \( \mathcal{F} = \{ g_{n,i} W^{n_i}, n_i > 0, 1 \leq i \leq m \} \subset R[W] \) be a set of generators of this graded \( R \) subalgebra of \( R[W] \). We can always take the system of coordinates so that, up to multiplication by a unit in \( R \), each \( g_{n,i} \) is a monic polynomial in \( z \), of some degree, say \( m_i \).

A remarkable fact about simple Rees algebras at a local regular ring \( R = \mathcal{O}_{V,x} \) of a point \( x \in \text{Sing}(G) \), is that, up to integral closure, they can be generated by monic polynomials in \( S[z] \), say

\[
\mathcal{F} = \{ G_{n,i} W^{n_i}, n_i > 0, 1 \leq i \leq m \} \subset S[Z][W],
\]

where now each \( G_{n,i} \) is monic of degree \( n_i \) in \( S[z] \). In fact, if we choose \( n \) to be divisible by all \( n_i \), it is clear that \( \bigoplus I_k \cdot W^k \) is a finite extension of \( G' = R[I_n W^n](\subset R[W^n]) \); and, as it was indicated above, \( I_n \) can be generated by monic polynomials of degree \( n \) in \( S[Z] \).

Since \( \bigoplus I_k \cdot W^k \) is a noetherian subalgebra of \( R[W] \), we may assume that so is

\[
\bigoplus I_{(k)} \cdot W^k(\subset S[Z][W]).
\]
This observation, together with the discussion in [1.32], where elimination algebras where defined, will lead us to a form of elimination for Diff-algebras, locally at simple points.

**On Rees algebras and integral extensions.**

We discuss here the concept of finite extensions of Rees algebras, as it will be used later in the definition of elimination (or projection of Diff-algebras). Fix a noetherian ring $B$ and ideals defining a Rees algebra $\bigoplus_{k \geq 0} I_k \cdot W^k \subset (C[B,W])$ as in [2.1]. Set

$$\bigoplus_{k \geq 0} I_k \cdot W^k = B[[I_n W^n, n \geq 0]].$$

An inclusion of Rees algebras $B[[I_n W^n, n \geq 0]] \subset C[[J_n W^n, n \geq 0]]$, is defined by a ring extension $B \subset C$, and an inclusion of ideals $I_n \subset J_n$ for each $n$. They arise in various ways:

- Given a Rees algebra $B[[I_n W^n, n \geq 0]]$ and a positive integer $m$ define

  $$V^{(m)}(B[[I_n W^n, n \geq 0]]) = \bigoplus_{n \geq 0} I_m W^{mn} \subset B[[I_n W^n, n \geq 0]].$$

  An inclusion, which is an integral extension, is that of:

  $$V^{(m)}(B[[I_n W^n, n \geq 0]]) \subset B[[I_n W^n, n \geq 0]].$$

- Let $A \subset B$ be a ring extension, and $B[[I_n W^n, n \geq 0]]$ a Rees algebra. An inclusion of Rees algebras arises by setting

  $$B[[I_n W^n, n \geq 0]] \cap A[W] \subset B[[I_n W^n, n \geq 0]],$$

  where the left hand side is a graded subring of $A[W]$. Finally, given rings $B \subset B'$, then:

  $$B[[I_n W^n, n \geq 0]] \subset B'[[[I'_n W^n, n \geq 0]],$$

  where $I'_n = I_n B'$, also defines a graded extension.

**4.4.** Let $B[[I_n W^n, n \geq 0]]$ be a Rees algebra, and assume that $A \subset B$ is a finite extension of rings. In this case one could expect that $B[[I_n W^n, n \geq 0]]$ to be a finite extension of the intersection algebra [1.33]. Example 4.6 shows that this is not so in general. However this will be the case for our notion of elimination for Diff-algebras (see Theorem 4.11).

**Remark 4.5.** 1) The extension (4.3.1) is integral, and so is (4.3.3) when $B \subset B'$ is integral.

2) The extension $B[[I_n W^n, n \geq 0]] \subset C[[J_n W^n, n \geq 0]]$ is integral if and only if

$$V^{(m)}(B[[I_n W^n, n \geq 0]]) \subset V^{(m)}(C[[J_n W^n, n \geq 0]])$$

is integral for some $m$.

**Proof.** 1) is clear; 2) follows from the finiteness in (4.3.2).

**Example 4.6.** Set $A = k[x_1, \ldots, x_n]_M \subset B = A[Z]/\langle f(Z) \rangle$, where $M = \langle x_1, \ldots, x_n \rangle$, and $f(Z)$ is a monic polynomial of degree $e$, and $f(Z) \in \langle M, Z \rangle$. Let $M$ and $N$ denote the maximal ideals of $A$ and $B$, and assume that $In(f(Z)) \in gr_{M,Z}(A[Z]) = k[X_1, \ldots, X_n, Z]$ is such that $\{In(f), X_1, \ldots, X_n\}$ is a regular sequence.
In this case $gr_M(S) \to gr_N(B)(= gr_{(M,Z)}/(In(f)))$ is flat. Note that:

i) $\bigoplus N^k \cdot W^k \cap A[W] = \bigoplus M^k \cdot W^k$.

ii) The ring extension (4.3.3) is not finite in this case, unless $f(Z) \in (M, Z)^e$.

To prove i), use the fact that flatness ensures an inclusion $gr_M(S) \subset gr_N(B)$.

**On differential Rees algebras and projections.**

4.7. Fix a Diff-algebra $G$ over a smooth scheme $V$, and a closed point $x \in Sing(G)$. Define $G_x = \bigoplus I_k \cdot W^k(\subset \mathcal{O}_{V,x}[W])$ by localization, which, in a natural sense, is also a Diff-algebra, since differential operators also act on $\mathcal{O}_{V,x}$. Assume that $x \in Sing(G)$ is a simple point, and hence that $G_x$ is a simple Diff-algebra.

We make use of the following handy observations:

**Observation 1):** Fixed a point $x$ in a smooth scheme $V$ of dimension $d$. It is simple to define smooth schemes $V'$, of dimension $d - 1$, together with a smooth morphisms, say $\pi : V \to V'$.

The construction of such morphisms require restrictions to étale neighborhoods of $x$.

**Observation 2):** If $G$ is a Diff-algebra over $V$, and if $\pi : V \to V'$ is a smooth morphism over $k$, then $G$ is also a differential algebra relative to this morphism (see Def 3.2).

As for Observation 1) recall that $(V, x)$ is an étale neighborhood of, say $(\mathbb{A}^d, x')$, where $d$ is the dimension of $V$ at $x$. Étale maps are of course smooth, so the claim reduces to the affine case. The underlying field $k$ is assumed to be perfect. So after suitable separable extension one can assume that both $x \in V$ and $x' \in \mathbb{A}^d$ are rational points over $k$. In this case the assertion is simple, as one can we can even take linear projections on smaller dimensional affine schemes.

Set $S \subset \mathcal{O}_{V, \pi(x)}$, so $S \subset \mathcal{O}_{V,x}$ is the inclusion of regular rings defined as in 4.3 by a transversal projection $\pi$. Set $R = S[Z]_{<M_S,Z>}(\subset \mathcal{O}_{V,x})$. Choose elements $f_{c_i}(Z) \in I_{(c_i)}$, monic of degree $c_i$ in $S[Z]$, $1 \leq i \leq r$, and set $\mathcal{F} = \{f_{c_i}(Z)W^{c_i}; 1 \leq i \leq r\} \subset S[Z][W]$. So $S[Z] \subset R$, and

\[(4.7.1) S[Z][\{f_{c_i}(Z)W^{c_i}; 1 \leq i \leq r\}] \subset G_x.\]

Moreover, the discussion in 4.3 ensures that such inclusion can be taken to be a finite extension.

Observation 2) ensures that, as $G_x$ is a differential Rees algebra relative to the structure field $k$, it is also closed by differentials relative to $S$; and hence

\[S[Z][\{\Delta^\alpha(f_{c_i})W^{\alpha - \alpha}/f_{c_i}W^{\alpha} \in \mathcal{F}, \text{ and } 0 \leq \alpha < c_i\}] \subset G_x (2.8).\]

Recall that the elimination algebra $\overline{R}_{f_{c_1}, f_{c_2}, \ldots, f_{c_r}}$ (1.42.2) is defined as the graded $S$ subalgebra of $S[W]$, generated by $\mathcal{F} = \{G^l_{c_1, \ldots, c_r}(a_1^{(1)}, \ldots, a_{c_1}^{(1)}, \ldots, a_1^{(r)}, \ldots, a_{c_r}^{(r)})W^{deg G^l_{c_1, \ldots, c_r}}\}$ (notation as in Th 1.26), where $G^l_{c_1, \ldots, c_r}$ runs among the generators of the algebra $\overline{R}_{c_1, \ldots, c_r}$ (1.25.2), and $\text{deg } G^l_{c_1, \ldots, c_r}$ is the degree of the weighted homogeneous polynomial $G^l_{c_1, \ldots, c_r}$.\]
Here $S$ is a regular local ring, and $\text{Sing}(R_{f_1, \ldots, f_r})$ is the closed set in $\text{Spec}(S)$ defined by the points where each element $G^d_{e_1, \ldots, e_r}(a^{(1)}_1, \ldots, a^{(1)}_1, \ldots, a^{(r)}_1, \ldots, a^{(r)}_r)$ has order at least $\deg G^d_{e_1, \ldots, e_r}$ (Prop 3.9 1).

**Lemma 4.8.** Let $G_x = \bigoplus I_k \cdot W^k(\subset O_{V_x}[W])$ (localization at $x \in \text{Sing}(G)$), be a simple Rees algebra at the point, and set $R = S[Z,_{M_S,Z}]$ for a suitable transversal projection. Assume that $G_x$ is a differential algebra relative to $S$ (e.g., that $G_x$ is a Diff-algebra)(see 2.5).

Choose elements $f_{c_i} \in I_{c_i}$ which are monic polynomials of degree $c_i$ in $S[Z]$ and order $c_i$ at $R = S[Z,_{M_S,Z}]$, for $1 \leq i \leq r$. Set $f(Z) = f_{c_1} \cdot f_{c_2} \cdots f_{c_r} \in S[Z]$, $b = c_1 + \cdots + c_r$, and $\pi: \text{Spec}(S[Z]/\langle f(Z) \rangle) \to \text{Spec}(S)$. Then:

1) Locally at $x$ the closed set $\text{Sing}(G)$ is included in the set of points of multiplicity $b$ of the hypersurface $V(\langle f(Z) \rangle)$.

2) $\pi(\text{Sing}(G_x)) \subset \text{Sing}(R_{f_1, f_2, \ldots, f_r})$

**Proof.** Note that $f(Z) \in I_b$, is an element of order $b$ in $R = S[Z,_{M_S,Z}]$ and hence in $O_{V,x}$, so 1) holds by Remark 3.8. In order to prove 2) it suffices to show that the set of points of multiplicity $b$ of $V(\langle f(Z) \rangle)$ map into $\text{Sing}(R_{f_1, f_2, \ldots, f_r})$. This follows from part ii) in Theorem 1.26 and Prop 3.9 1. \qed

**Lemma 4.9.** Assume here that $G = \bigoplus I_k \cdot W^k$ is a Diff-algebra, and set, as before a simple point $x \in \text{Sing}(G)$. Then elements $f_{c_1}, \ldots, f_{c_r}$ can be chosen, as in the previous Lemma, so that

$$\pi(\text{Sing}(G_x)) = \text{Sing}(R_{f_1, f_2, \ldots, f_r}).$$

**Proof.** It suffices to prove that, for suitable $f_{c_1}, f_{c_2}, \ldots, f_{c_r}$ as above, $\text{Sing}(R_{f_1, f_2, \ldots, f_r}) \subset \pi(\text{Sing}(G))$. For technical reasons, to be used later, we first choose and fix an element $f_{c_1}$ of order $c_1$ in $I_{c_1}$. Since $G_x$ is finitely generated, there is an integer $n_0$ so that $I_{n_0}$ has order $n_0$ at $O_{V,x}$, and such that $G_x$ is an integral extension of $O_{V,x}[I_{n_0}W^{n_0}](\subset O_{V,x}[W])$. In this case $V(I_{n_0}) = \text{Sing}(G_x)$ (Prop 3.9 5), and there are elements $f_{c_2}, \ldots, f_{c_r} \in I_{n_0}$, all of order $n_0$, that generate the ideal $I_{n_0}$ in a neighborhood of the point.

Recall that $\text{Sing}(R_{f_1, f_2, \ldots, f_r})$ consists of all primes $P$ in $S$, such that, at $S_P$:

$$\nu_{s_P}(G^d_{e_1, \ldots, e_r}(a^{(1)}_1, \ldots, a^{(1)}_1, \ldots, a^{(r)}_1, \ldots, a^{(r)}_r)) \geq \deg G^d_{e_1, \ldots, e_r}$$

(see 4.7).

So if $P \in \text{Sing}(R_{f_1, f_2, \ldots, f_r})$, Theorem 1.26 i) asserts that

$$\pi(\text{Spec}(S[Z]/\langle f(Z) \rangle) \to \text{Spec}(S)$$

is purely ramified over $P$. In particular $\pi^{-1}(P) = Q$ (is a unique point in $\text{Spec}(S[Z]/\langle f(Z) \rangle)$). Here $f(Z) = f_{c_1} \cdot f_{c_2} \cdots f_{c_r}$, and $\text{Spec}(S[Z]/\langle f_{c_1}(Z) \rangle)$ is closed in $\text{Spec}(S[Z]/\langle f(Z) \rangle)$, and maps surjectively into $\text{Spec}(S)$. Identify $Q$ with a prime in $S[Z]$, say $Q$ again, so

$$Q \in V(\langle f_{c_1} \rangle) \cap V(\langle f_{c_2} \rangle) \cap \cdots \cap V(\langle f_{c_r} \rangle).$$

In particular,

$$\langle f_{c_2}, \ldots, f_{c_r} \rangle = I_{n_0} \subset Q,$$
so \( Q \in V(I_{no}) = Sing(\mathcal{G}) \), and hence \( P \in \pi(Sing(\mathcal{G})) \) as was to be shown. \( \square \)

**Definition 4.10.** We now define the elimination algebra of \( \mathcal{G} \) relative to a transversal projection \( \pi \), under the assumption that \( \mathcal{G} \) is a \( \pi \)-relative differential algebra \([3,2]\).

The elimination algebra, say: \( R_\mathcal{G} \), is defined as the smallest subalgebra of \( S[W] \), containing all (elimination) algebras \( \overline{R}_{f_{c_1},f_{c_2},...,f_{c_r}} \), for all choices of \( r \), and of elements \( f_{c_i}(Z) \in I_{(c_i)} \) monic of degree \( c_i \) in \( S[Z] \). Note that, as graded subalgebra, it can be expressed in terms of ideals \( J_k \) in \( S \), namely:

\[
R_\mathcal{G} = \bigoplus J_k \cdot W^k (\subset S[W])
\]

for suitable ideals \( J_k \) in \( S \).

There is a natural inclusion of graded algebras \( \overline{R}_{f_{c_1},f_{c_2},...,f_{c_r}} \supset \overline{R}_{f_{c_2},...,f_{c_r}} \). So if we fix \( f_{c_1}(Z) \in I_{(c_1)} \), monic of degree \( c_1 \) in \( S[Z] \), we may also define \( R_\mathcal{G} \) as the smallest subalgebra containing all those of the form \( \overline{R}_{f_{c_2},...,f_{c_r}} \) (including the fixed element \( f_{c_1} \)).

On the other hand we can define \( B = S[Z]/(f_{c_1}(Z)) \), and consider the algebra induced by restriction of \( \mathcal{G} \), say:

\[
\mathcal{G} = \bigoplus I_k \cdot W^k (\subset B[W]),
\]

where \( I_k = I_k B \).

**Theorem 4.11.** Fix an algebra \( \mathcal{G} \) over a smooth scheme \( V \), a closed point \( x \in Sing(\mathcal{G}) \), and \( \pi : V \rightarrow V' \) transversal at \( x \). Set \( R = \mathcal{O}_{V,x} \) and \( S = \mathcal{O}_{V',\pi(x)} \) as before, namely \( R = S[Z]_{< M_S Z>} \). Assume that \( \mathcal{G}_x \) is a differential algebra relative to \( S \) (e.g., that \( \mathcal{G}_x \) is a Diff-algebra) \([2,3]\).

Fix \( f_{c_1}(Z) \in I_{c_1} \), monic of degree \( c_1 \) in \( S[Z] \). Set \( B = S[Z]/(f_{c_1}(Z)) \), \( \mathcal{G} \subset B[W] \) as above, and \( \pi : Spec(S[Z]/(f_{c_1}(Z))) \rightarrow Spec(S) \) (restriction of \( \pi \)). Then, locally at \( x \):

(i) \( Sing(\mathcal{G}) \subset V(Diff^{c_1-1}(f_{c_1}(Z))) \), and \( \pi(Sing(\mathcal{G})) \subset Sing(R_\mathcal{G}) \). Moreover,

\[
\pi(Sing(\mathcal{G})) = Sing(R_\mathcal{G})
\]

if \( \mathcal{G} \) is a Diff-algebra.

(ii) The elimination algebra \( R_\mathcal{G} \) is included in \( \mathcal{G} \cap S[W] \) (as subalgebras of \( S[W] \)).

(iii) \( \mathcal{G} \) is integral over \( R_\mathcal{G} \) (in particular \( \mathcal{G} \cap S[W] \) is integral over \( R_\mathcal{G} \)).

(iv) The algebra \( \mathcal{G} \cap S[W] \) is, up to integral closure, independent of the choice of \( f_{c_1}(Z) \in I_{c_1} \).

(v) If \( \mathcal{G} \subset \mathcal{G}' \) is a finite extension, then \( R_\mathcal{G} \subset R_{\mathcal{G}'} \) is also finite.

**Proof.** i) The first inclusion is \([3.8]\) The equality follows from Lemmas \([1.8]\) and \([1.9]\).

ii) It suffices to show that each algebra \( \overline{R}_{f_{c_1},f_{c_2},...,f_{c_r}} \) (\( f_{c_1} \) as above) is included in \( \mathcal{G} \cap S[W] \) as graded algebra. This will follow, on the one hand from \([1.33]\) and, on the other hand, by the fact that \( \mathcal{G} \) is closed by the action of differential operators in the variable \( Z \). In fact, recall that \( \overline{R}_{f_{c_1},f_{c_2},...,f_{c_r}} \) was defined in terms of the universal polynomials \( F_{c_1}, F_{c_2}, ..., F_{c_r} \), and the Rees algebra \( \overline{R}_{c_1,c_2,...,c_r} \) (see \([1.42,2]\) and \([1.40]\)). Fix an homogeneous element of degree \( m \),
say $G_m$, in $R_{e_1,e_2,\ldots,e_r}$. $G_m$ can be expressed as a polynomial in $\{F_{e_i}^{(j)}, 0 \leq j \leq c_i, 1 \leq i \leq r\}$, $(F_{e_i}^{(j)}$ defined in terms of differential operators) with coefficients in the field $k$. Furthermore, such expression of $G_m = G_m(F_{e_i}^{(j)})$ is weighted homogeneous, provided $F_{e_i}^{(j)}$ is given weight $c_i - j$ (see Corollary 1.33.2).

The elements $f_{c_1}, f_{c_2}, \ldots, f_{c_r}$ are defined from $F_{c_1}, F_{c_2}, \ldots, F_{c_r}$ by specialization, and each $f_{c_i}$ is homogeneous of degree $c_i$ in $G$ (1.42.2). Therefore the elements $f_{c_i}^{(j)}$ (defined in terms of differential operators) are homogeneous of degree $c_i - j$ in the Diff-algebra $G$; and $G_m(f_{c_i}^{(j)})$ (image of $G_m(F_{e_i}^{(j)})$) is homogeneous of degree $m$ in $G$. This proves that $R_{f_{c_1}, f_{c_2}, \ldots, f_{c_r}} \subset G \cap S[W]$ as graded algebras, since $R_{f_{c_1}, f_{c_2}, \ldots, f_{c_r}}$ is the graded algebra generated by all $G_m(f_{c_i}^{(j)})$.

iii) Choose a positive integer $n_0$ with two conditions. First, that $V^{(n_0)}(G)$ be a usual Rees ring defined by the ideal $I_{n_0}$ (i.e., $V^{(n_0)}(G) = \bigoplus I_{n_0}^k W^k n_0$). And second, that the order of $I_{n_0}$ at the local ring $R$ is $n_0$. Such choice of $n_0$ and $I_{n_0}$ is possible since $G$ is finitely generated and simple.

The ideal $I_{n_0}$ can be generated by elements of order $n_0$ in the local regular ring $R$; and, replacing $R$ by its completion, we may assume that it is generated by monic polynomials, say $f_2(Z), \ldots, f_r(Z)$ in the variable $Z$ ($I_{n_0} = < f_2(Z), \ldots, f_r(Z) >$).

Recall that $B = S[Z]/\langle f_{c_1} \rangle$, and set $\mathcal{T}_n = I_n B$, and $\mathcal{T}_i \in I_n B$ as the class of $f_i$. In order to prove that $\mathcal{G}$ is finite over the subalgebra $R_G$, it suffices to prove that the elements $\mathcal{T}_i$ are integral over $R_G$ (see Remark 4.5). Observe that:

a) the elements $F_2(Y_1), 1 \leq i \leq r$ are integral over $R_{e_1,e_2,\ldots,e_r}$ (see Corollary 1.33.1)).

b) $F_2(Y_1), 1 \leq i \leq r$ are elements in $k\{s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_{r}^{(r)}, \ldots, s_{c_r}^{(r)}\}[Z]/ < F_{c_i}(Z) >$ (see Lemma 1.32).

c) $B$ and $R_{f_{c_1}, f_{c_2}, \ldots, f_{c_r}}$ are defined from $k\{s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_{r}^{(r)}, \ldots, s_{c_r}^{(r)}\}[Z]/ < F_{c_i}(Z) >$ and from $R_{e_1,e_2,\ldots,e_r}$ by base change: $k\{s_1^{(1)}, \ldots, s_{c_1}^{(1)}, \ldots, s_{r}^{(r)}, \ldots, s_{c_r}^{(r)}\} \rightarrow S(1.42).

This shows that the elements $\mathcal{T}_i$ are integral over $R_{f_{c_1}, f_{c_2}, \ldots, f_{c_r}}$ and hence over $R_G$.

(iv) Follows from (iii), since $R_G$ is defined independently of the choice of $f_{c_1}(Z) \in I_{c_1}$.

(v) The image at $B = S[Z]/\langle f_{c_1}(Z) \rangle$ defines $\mathcal{G} \subset \mathcal{G}'$, which is also a finite extension, so the claim follows from (iii). \hfill $\Box$

The elimination algebra $R_G$ has been defined as a direct limit of algebras $R_{f_{c_1}, f_{c_2}, \ldots, f_{c_r}}$ (1.42.2). We can also define $H_G$ as a direct limit of $H_{f_{c_1}, f_{c_2}, \ldots, f_{c_r}} \subset S[W]$ (1.42.3).

**Corollary 4.12.** The singular locus of the Diff-algebra $G_x = \bigoplus I_k \cdot W^k$ maps bijectively to $\text{Sing}(R_G)$, which coincides with $\text{Sing}(H_G)$. In fact $R_G$ is a finite extension of $H_G$.

**Proof.** The claim follows from the finiteness of the extension in (1.42.4). \hfill $\Box$

**Theorem 4.13.** Fix a Rees algebra $G$, which is simple at a closed point $x \in \text{Sing}(G)$, and $\pi : V \rightarrow V'$ transversal at $x$. Set $R = O_{V,x}$ and $S = O_{V',\pi(x)}$ as before, namely $R = S[Z]/_{M_S, Z}$. 


and assume that $G_x = \bigoplus I_k \cdot W^k (\subset R[W])$ is a differential Rees algebra relative to $S$. The elimination algebra $R_G (\subset S[W])$ is a graded subalgebra of $G_x (\subset R[W])$, via the inclusion $S[W] \subset R[W]$.

Proof. Recall, as in the proof of the previous corollary, the definition of $R_G$ in terms of algebras $R_{f_{e_1}, f_{e_2}, \ldots, f_{e_r}}$ (4.10). Recall also that $R_{f_{e_1}, f_{e_2}, \ldots, f_{e_r}}$ is the pull-back of $R_{f_1, f_2, \ldots, f_r}$ (see 4.40). The claim follows from Prop 1.29 and Corollary 1.30 which show that $R_{f_{e_1}, f_{e_2}, \ldots, f_{e_r}}$ is generated by elements which are weighted homogeneous on elements on $\{\Delta^e (f_{e_i}) W^{e_i - e}, 0 \leq e \leq c_i - 1\}$; and hence by homogeneous elements in $G$.

Corollary 4.14. 1) If $\{f_{e_i}(Z) W^{c_i}; 1 \leq i \leq r\} \subset S[Z][W]$ are chosen so that $S[Z][\{f_{e_i}(Z) W^{c_i}; 1 \leq i \leq r\}] \subset G_x$ is a finite extension, then $R_G$ and $R_{f_{e_1}, f_{e_2}, \ldots, f_{e_r}}$ have the same integral closure (see 4.74)

2) If $G$ is, in addition, a Diff-algebra, then the Rees algebra $R_G$ is also a Diff-algebra.

Proof. Claim 1) follows from the two Theorems. As for 2) note that the inclusion $S \subset S[Z] \subset \mathcal{O}_{V,x}$ provides an inclusion of higher order differentials on $S$, as differentials on $\mathcal{O}_{V,x}$. □

5. On differential invariants and projections.

The local ring $\mathcal{O}_{V,x}$ of the smooth scheme $V$ at $x$, is regular. There is a well defined notion of order for ideals in a local regular ring. So a sheaf of ideals, say $I \subset \mathcal{O}_V$, defines a function on $V$ with values in the integers, by considering, at each point $x \in V$ the order of the ideal at $\mathcal{O}_{V,x}$. We first extend this notion of order to the case of Rees algebras $G$ over $V$. In Theorem 5.3, which is the main result in this Section, we study the behavior of the order with our notion of elimination of one variable. This elimination is defined in terms of a projection. There can be different projections and the Theorem studies the order function of the elimination algebras for different projections.

In Proposition 5.12 we show that Hironakas $\tau$-invariant has the expected behavior in positive characteristic when considering elimination of one variable (as the known behavior of the $\tau$-invariant in characteristic zero).

5.1. The notion of Rees algebras $G = \bigoplus_{k \geq 1} I_k \cdot W^k$ parallels that of idealistic exponents in [21], and the notion of singular locus $Sing(G)$, is the natural analog of that defined for idealistic exponents. In what follows assume that $I_k$ is non-zero for some index $k > 0$.

We recall the definition of a function, which is the natural reformulation of that defined by Hironaka for idealistic exponents. We follow here the presentation of section 6 in [37]. Fix $x \in Sing(G)$. Given $f_n W^n \in I_n W^n$, $f_n \neq 0$, set $ord_x (f_n) = \frac{\nu_x (f_n)}{n} \in \mathbb{Q};$ called the order of $f_n$ (weighted by $n$), where $\nu_x$ denotes the order at the local regular ring $\mathcal{O}_{V,x}$. As $x \in Sing(G)$ it follows that $ord_x (f_n) \geq 1$. Define

$$ord_x (G) = \inf \{ord_x (f_n); f_n W^n \in I_n W^n\};$$

or equivalently, $ord_x (G) = \inf_{n \geq 0} \{\frac{\nu_x (f_n)}{n}\}$. In general $ord_x (G) \geq 1$ for all $x \in Sing(G)$; and a Rees algebra $G$ is simple at $x$ if and only if $ord_x (G) = 1$ (4.4).
Proposition 5.2. (1) If \( G \) is generated over \( \mathcal{O}_V \) by \( F = \{ g_i W^{n_i} : n_i > 0, 1 \leq i \leq m \} \), then \( \text{ord}_x(G) = \inf \{ \text{ord}_x(g_i) : 1 \leq i \leq m \} \). And if \( N \) is a common multiple of all \( n_i, 1 \leq i \leq m \), then \( \text{ord}_x(G) = \frac{\nu_x(I_n)}{N} \).

(2) If \( G \) and \( G' \) are Rees algebras with the same integral closure (e.g., if \( G \subset G' \) is a finite extension), then, for all \( x \in \text{Sing}(G)(= \text{Sing}(G')) \): \( \text{ord}_x(G) = \text{ord}_x(G') \).

(3) Let \( G'' = \bigoplus I''_n \cdot W^n \) be the extension of \( G \) to a differential Rees algebra relative to \( k \), as defined in Theorem 3.4, then for all \( x \in \text{Sing}(G)(= \text{Sing}(G'')) \): \( \text{ord}_x(G) = \text{ord}_x(G'') \).

All assertions are easy to check. The assumption in (1) holds at an affine open set of \( V \).

5.3. Fix a Diff-algebra \( G = \bigoplus_{k \geq 1} I_k \cdot W^k \) over a smooth scheme \( V \) and a closed point \( x \in \text{Sing}(G) \). Assume that \( x \) is a simple point, or equivalently, that \( \text{ord}_x(G) = 1 \). In such a case there must be an index \( n \) so that \( \nu_x(I_n) = n \) (order of \( I_n \) at \( \mathcal{O}_{V,x} \)). In other words, there is an homogeneous element of degree \( n \), say \( f_n \cdot W^n \in G = \bigoplus_{k \geq 1} I_k \cdot W^k \), so that \( f_n \) has order \( n \) at \( \mathcal{O}_{V,x} \). We claim now that if \( G \) is integrally closed (i.e., equal to its integral closure in \( \mathcal{O}_V[W] \)), then \( f_n \) can be chosen to be analytically irreducible at the local regular ring \( \mathcal{O}_{V,x} \). This would show, in particular, that \( f_n(Z) \in S[Z] \) can be chosen to be irreducible in Theorem 4.11. This fact, already interesting in itself, will be used in our forthcoming Theorem 5.5.

Note here that if \( G' \) denotes the integral closure of \( G \) in \( \mathcal{O}_V[Z] \), then \( G \subset G' \) is a finite extension. In Theorem 6.12 of [35], it is proved that if \( G \subset G' \) is a finite extension of Rees algebras, there is an inclusion of the Diff-algebras spanned by each of them, and this extension is also finite. In particular, if \( G \) is already a Diff-algebra, its integral closure is also a Diff-algebra.

Recall that for any smooth morphism \( \pi_1 : V \to V^{(1)} \), \( G \) is also a differential Rees algebra relative to \( \pi_1 \) (see Observation (2) in 4.7). This relative differential structure is as much as we use in order to define the elimination algebra \( R_G \) (4.10).

Let us first discuss the claim, that \( f_n(Z) \in S[Z] \) can be chosen to be irreducible, under the assumption that \( G \) is an \( S \)-relative differential algebra. Set \( S = \mathcal{O}_{V^{(1)},x} \), and let \( m_S \) be the maximal ideal. Suppose that a polynomial \( S[Z] \) is defined so that \( S[Z] \subset \mathcal{O}_{V,x} \), and \( S[Z]_{(m_S, Z)} \subset \mathcal{O}_{V,x} \) is an étale extension of local rings.

Assume that there is an element \( f_n W^n \in G \), with \( f_n \) of order \( n \) at \( \mathcal{O}_{V,x} \), and that up to multiplication by a unit \( f_n = F_n(Z) \in S[Z] \) is a monic polynomial of degree \( n \).

Set \( F_n(Z) = G_{r_1} \cdot G_{r_2} \cdots G_{r_s} \), a product of irreducible polynomials in \( S[Z] \), where each \( G_{r_i}(Z) \) is monic of degree \( r_i < n \). In this case, it follows from 1.43 that each \( G_{r_i}(Z) \cdot W^{r_i} \) is in the integral closure of \( G \). As we assume that \( G \) is integrally closed, \( G_{r_i}(Z) \cdot W^{r_i} \in G \) and \( G_{r_i}(Z) \) has order \( r_i < n \) at \( \mathcal{O}_{V,x} \). If \( r_i = 1 \) then \( G_{r_i}(Z) \) is analytically irreducible in this local ring.

We prove our claim, in full generality, by using the previous argument. At this point we shall make use of the Weierstrass Preparation Theorem, on the one hand, and also use the fact that, locally at \( x \in V \), there are plenty of smooth morphisms as \( \pi_1 : V \to V^{(1)} \). Moreover, we show here that constructing such morphisms is very simple, at least over perfect fields.
Assume that \( f_n \) has order \( n \) at \( \mathcal{O}_{V,x} \), and let \( f_n = g_{r_1} \cdot g_{r_2} \cdots g_{r_s} \) be a factorization as a product of irreducible elements at the henselization of \( \mathcal{O}_{V,x} \), say \( R \). Then, and after replacing \( V \) by a suitable étale neighborhood of \( x \), we may assume that such factorization holds at \( \mathcal{O}_{V,x} \), and that each \( g_{r_i} \) is analytically irreducible. Notice that if \( r_i \) denotes the multiplicity of each \( g_{r_i} \) at the local regular ring, then \( r_1 + r_2 + \cdots + r_s = n \).

The Weierstrass Preparation Theorem holds at henselian local rings. We claim now that after taking again a suitable étale neighborhood, one can construct a subring \( S \subset \mathcal{O}_{V,x} \), and a polynomial ring \( S[Z] \subset \mathcal{O}_{V,x} \), so that up to multiplication by units, \( f_n = F_n(Z) \) and each \( g_{r_i} = G_{r_i}(Z) \), where \( F_n(Z) = G_{r_1} \cdot G_{r_2} \cdots G_{r_s} \) is a product of irreducible polynomials in \( S[Z] \), and each \( G_{r_i}(Z) \) is monic of degree \( r_i \). To check this claim note first that one can choose a regular system of parameters, say \( \{x_1, \ldots, x_d\} \) for \( \mathcal{O}_{V,x} \), so that \( f_n \) also has order \( n \) when restricted to \( \mathcal{O}_{V,x}/(x_1, \ldots, x_{d-1}) \). Since \( V \) is smooth over \( k \), \( \mathcal{O}_{V,x} \) contains the polynomial ring \( k[x_1, \ldots, x_d] \).

As \( k \) is a perfect field, and \( x \) is a closed point, after suitable finite extension of the base field, \( (V, x) \) is an étale neighborhood of the affine space \( \mathbb{A}^d \) at the origin (see [2]). So there is an inclusion of local rings, say

\[
k[x_1, \ldots, x_{d-1}]/(x_1, \ldots, x_{d-1}) \subset \mathcal{O}_{V,x},
\]

and a smooth morphism, say \( V \to \mathbb{A}^{d-1} \), defined at a suitable neighborhood of \( x \). Let \( X \subset V \) be the hypersurface \( V((f_n)) \), and set \( X \to \mathbb{A}^{d-1} \) by restriction. Notice that \( x \) is an isolated point in the fiber over the origin of \( \mathbb{A}^{d-1} \). Under these conditions an expression \( F_n(Z) = G_{r_1} \cdot G_{r_2} \cdots G_{r_s} \), as above, can be defined in \( S[Z] \), where \( S \) denotes the henselization of \( k[x_1, \ldots, x_{d-1}]/(x_1, \ldots, x_{d-1}) \) (see [31]).

As each \( G(i)(Z) \) involves finitely many coefficients in \( S \), we may also assume that all statements hold at a suitable étale neighborhood of \( \mathbb{A}^{d-1} \) at the origin. Take, finally, an étale neighborhood of the closed point \( x \in V \) containing the previous local ring.

Therefore, at a suitable étale neighborhood of \( x \) we may assume that \( G_{r_i}(Z) \cdot W^{r_i} \) is in the integral closure of \( \mathcal{G} \) (see (1.43.2)). This proves the claim. Namely that if we pass from \( \mathcal{G} \) to its integral closure, we may assume that the element \( f_{c_1}(Z) \), in Theorem 4.11 is irreducible in \( S[Z] \) and, moreover, analytically irreducible in \( \mathcal{O}_{V,x} \).

5.4. Given a Diff-algebra \( \mathcal{G} \), a simple point \( x \in \text{Sing}(\mathcal{G}) \), and a general projection, then an elimination algebra \( \mathcal{R}_\mathcal{G} \) has been defined over a regular scheme \( \text{Spec}(S) \). The previous discussion shows how to construct transversal smooth morphisms over a field. In fact, let \( d \) denote the dimension of the smooth scheme \( V \), then, after restriction to a suitable étale neighborhood of \( x \), there is a smooth scheme \( V^{(1)} \) of dimension \( d-1 \), and a smooth morphism \( \pi_1 : V \to V^{(1)} \), so that \( \mathcal{R}_\mathcal{G}^{(1)} \) (4.10) can be defined at \( V^{(1)} \).

Furthermore, we may take \( f_{c_1}(Z) \) to be a global section at a neighborhood of \( x \), and define \( \mathcal{R}_\mathcal{G}^{(1)} \subset \mathcal{O}_{V^{(1)}}[W] \). Moreover, if \( \mathcal{G} \) is a Diff-algebra, then \( \pi_1(\text{Sing}(\mathcal{G})) = \text{Sing}(\mathcal{R}_\mathcal{G}^{(1)}) \). The setting of Theorem 4.11 holds at every closed point \( y \in \text{Sing}(\mathcal{G}) \) by taking \( S = \mathcal{O}_{V^{(1)}, \pi_1(y)} \).

As \( \text{Sing}(\mathcal{G}) \) is included in the set of points of multiplicity \( c_1 \) of the hypersurface defined by \( f_{c_1}(Z) \), it follows that the components of \( \text{Sing}(\mathcal{G}) \) of codimension one in \( V \) are open and
closed in $\text{Sing}(\mathcal{G})$, and furthermore, they are also smooth. In fact this is the case for the highest multiplicity locus of a hypersurface in a smooth scheme (see [12], Ex. 13.11, where a characteristic free proof is suggested).

We will assume here that $\text{Sing}(\mathcal{G})$ has codimension at least 2 locally at the simple point. This ensures that $\mathcal{R}_\mathcal{G}^{(1)}$ is a direct sum of non-zero ideals in $S = \mathcal{O}_{V^{(1)}, \pi_1(y)}$, and in this case $\text{ord}_{\pi_1(x)}(\mathcal{R}_\mathcal{G}^{(1)})$ is defined (5.1).

We took, as stating point, that $\mathcal{G}$ was a Diff-algebra. This of course ensures that for any smooth morphism $\pi_1 : V \to V^{(1)}$, $\mathcal{G}$ as also a differential Rees algebra relative to $\pi_1$ (see Observations 2) in [4,7]). However this latter condition is all we need to define $\mathcal{R}_\mathcal{G}^{(1)}$.

**Theorem 5.5.** Fix a Diff-algebra $\mathcal{G}$, a simple closed point $x \in \text{Sing}(\mathcal{G})$, and assume that the local codimension of this closed set (in $V$) is at least 2. Let $\pi_1 : V \to V^{(1)}$ and $\pi_2 : V \to V^{(2)}$ be, as above, two morphisms of smooth schemes ($\text{dim}(V^{(1)}) = \text{dim}(V^{(2)}) = d - 1$); defining elimination algebras, say $\mathcal{R}_\mathcal{G}^{(1)} \subset \mathcal{O}_{V^{(1)}}[W]$ and $\mathcal{R}_\mathcal{G}^{(2)} \subset \mathcal{O}_{V^{(2)}}[W]$. Then,

$$\text{ord}_{\pi_1(x)}(\mathcal{R}_\mathcal{G}^{(1)}) = \text{ord}_{\pi_2(x)}(\mathcal{R}_\mathcal{G}^{(2)}).$$

This shows that $\text{ord}_{\pi(x)}(\mathcal{R}_\mathcal{G})$ is an invariant of $x \in \text{Sing}(\mathcal{G})$ (i.e., independent of the projection $\pi$). We will introduce some notation, and discuss some preliminary results before we address the proof in 5.10.

5.6. Let $k[[x_1, \ldots, x_d]]$ be the ring of formal power series over a field $k$; and let $f_c$ an irreducible element of multiplicity $c$. Let $B$ denote the quotient $k[[x_1, \ldots, x_d]]/\langle f_c \rangle$, which is a domain with quotient field, say $L$. The Weierstrass Preparation Theorem asserts that for a sufficiently general choice of coordinates we may assume that, up to multiplication by a unit, $f_c$ is a monic polynomial of degree $c$ in the variable $x_d$, and that the class of $x_1, \ldots, x_{d-1}$ in $B$ span a reduction of the maximal ideal.

Here $B$ is a finite extension, and a free module of rank $c$, over the subring of formal power series, say $S = k[[x_1, \ldots, x_{d-1}]]$. Let $K$ denote the total quotient field of $S$, and note that $L$ is a finite extension of degree $c$ over $K$.

For each discrete valuation ring, say $V$, in $L$, we consider the restriction, say

$$V_K = V \cap K.$$

Identify the group of values of $V$ with the integers $\mathbb{Z}$, and define the ramification index of $V$ over $V_K$ to be the index in $\mathbb{Z}$ of the subgroup of values of $V_K$.

Let $\text{Spec}(B) \leftarrow F$ denote the normalized blowup of $B$ at the maximal ideal. Let $H_1, H_2, \ldots, H_l$ denote the irreducible exceptional hypersurfaces of $F$, and let $V_1, V_2, \ldots, V_l$ denote the discrete valuation rings in $L$, where each $V_i$ is the local ring of the normal scheme $F$ at the generic point of the hypersurface $H_i$.

Let $\text{Spec}(S) \leftarrow Y$ denote the blow up of $S$ at the maximal ideal (i.e., the quadratic transformation). So here $Y$ is regular, and has only one exceptional hypersurface, say $h$, let $V_S$ be the local ring of $Y$ at the generic point of $h$. $V_S$ is the valuation at $K$ which extends the order at the local regular ring $S$, for elements in $S$.  

Lemma 5.7. For each discrete valuation ring \( V_i \), \( i = 1, \ldots, l \) as above, the ramification index of \( V_i \) over \( (V_i)_K \) is the order at \( V_i \) of the maximal ideal of \( B \).

Proof. Let \( \text{Spec}(B) \leftarrow \mathcal{Y} \) denote the fiber product of \( \text{Spec}(S) \leftarrow Y \) with the finite morphism \( \text{Spec}(B) \rightarrow \text{Spec}(S) \). Note that \( \text{Spec}(B) \leftarrow \mathcal{Y} \) is also the blow up of \( B \) at the ideal spanned by the elements \( x_1, \ldots, x_{d-1} \), say \( \langle x_1, \ldots, x_{d-1} \rangle B \). So \( \mathcal{Y} \) is a finite extension of \( Y \), and the total quotient field of \( \mathcal{Y} \) is \( L \). Since \( \langle x_1, \ldots, x_{d-1} \rangle B \) is a reduction of the maximal ideal of \( B \), it follows that \( F \) is the normalization of \( \mathcal{Y} \), and that \( (V_i)_K = V_S \) for all \( i = 1, \ldots, l \). Furthermore, as the ideal spanned by \( x_1, \ldots, x_{d-1} \) has order one at \( V_S \), it also follows that the ramification index of \( V_i \) over \( V_S \) is the order of the ideal spanned by \( x_1, \ldots, x_{d-1} \) at \( V_i \). But this is the order of the maximal ideal of \( B \) at \( V_i \). In fact, the maximal ideal and \( \langle x_1, \ldots, x_{d-1} \rangle B \) have the same integral closure.

Remark 5.8. Fix notation as above, and let \( e_i \) denote the ramification index of \( V_i \) over \( (V_i)_K = V_S \), for \( i = 1, \ldots, l \). Let \( J \) be an ideal in the local regular ring \( S \). The order of \( J \) at \( S \) is the valuation of \( J \) at \( V_S \), say \( b \in \mathbb{Z} \). It follows from Lemma 5.7 that the order of the extended ideal \( J B \) at the valuation \( V_i \) is the integer \( b \cdot e_i \).

Corollary 5.9. The ramification index of each \( V_i \) over \( (V_i)_K \) is independent of the choice of \( x_1, \ldots, x_{d-1} \) (i.e., of \( S = k[[x_1, \ldots, x_{d-1}]] \subset B \)), as far as \( \langle x_1, \ldots, x_{d-1} \rangle B \) is a reduction of the maximal ideal of \( B \).

5.10. Proof of Theorem 5.3. Here \( \mathcal{G} = \bigoplus I_k \cdot W^k(\subset \mathcal{O}_V[W]) \) is a Diff-algebra and \( x \in \text{Sing}(\mathcal{G}) \) is a simple point. Set \( \pi_1 : V \rightarrow V^{(1)} \) and \( \mathcal{R}^{(1)}_\mathcal{G} = \bigoplus J_k^{(1)} \cdot W^k(\subset \mathcal{O}_{V^{(1)}}[W]) \) for suitable ideals \( J_k^{(1)} \) in \( \mathcal{O}_{V^{(1)}} \). Let \( \mathcal{G}_x = \bigoplus (I_k)_x \cdot W^k(\subset \mathcal{O}_{V,x}[W]) \) be the localization at \( x \in V \). As the point \( x \) is simple there must be an index \( c \) and an element \( f_c \in (I_c)_x \) of order \( c \) at \( \mathcal{O}_{V,x} \). At a suitable étale neighborhood of \( x \) and \( \pi_1(x) \), \( \pi_1 \) induces a finite morphism from the subscheme defined by \( \langle f_c \rangle \) to \( V^{(1)} \). In particular, a finite morphism

\[
\pi_1 : \text{Spec}(\mathcal{O}_{V,x}/\langle f_c \rangle) \rightarrow \text{Spec}(\mathcal{O}_{V^{(1)},x}).
\]

The Weierstrass Preparation Theorem asserts that (at a suitable étale neighborhood), setting \( S = \mathcal{O}_{V^{(1)},x} \), there is an inclusion of regular local rings, say \( R = S[Z]_{<M_S,Z>} \subset \mathcal{O}_{V,x} \), and a monic polynomial of degree \( c \), say \( f_c \in S[Z] \), so that

\[
B = \mathcal{O}_{V,x}/\langle f_c \rangle = S[Z]/\langle f_c(Z) \rangle.
\]

Let \( \overline{\mathcal{G}} = \bigoplus \overline{T}_k \cdot W^k(\subset B[W]) \) be the algebra induced by restriction of \( \mathcal{G} \), where \( \overline{T}_k = I_k B \).

Theorem 4.11 (iii), states that up to integral closure, the localization of \( \mathcal{R}^{(1)}_\mathcal{G} \) at \( \pi_1(x) \) is \( \overline{\mathcal{G}} \cap S[W] \). In particular \( \text{ord}_{\pi_1(x)}(\mathcal{R}^{(1)}_\mathcal{G}) = \text{ord}_{\pi_1(x)}(\overline{\mathcal{G}} \cap S[W]) \) (5.2, (2)).

If \( \mathcal{G} \subset \mathcal{G}' \) is a finite extension of Diff-algebras in \( \mathcal{O}_V[W] \), then the restrictions to \( B[W] \), say \( \mathcal{G} \subset \mathcal{G}' \), is also a finite extension. Therefore \( \mathcal{G} \cap S[W] \subset \mathcal{G}' \cap S[W] \) is a finite extension, and hence \( \text{ord}_{\pi_1(x)}(\mathcal{G} \cap S[W]) = \text{ord}_{\pi_1(x)}(\mathcal{G}' \cap S[W]) \) (5.2, (2)).

As the integral closure of a Diff-algebra is a Diff-algebra (5.3), we may assume, for the proof of this Theorem, that \( \mathcal{G} \) is integrally closed. In particular, as it was indicated in (5.3),
we may assume here that \( f_c \in I_c \) is chosen to be analytically irreducible. In other words, that the completion of local ring \( B \) is irreducible.

Here \( \pi_1 : V \to V^{(1)} \) and \( \pi_2 : V \to V^{(2)} \) are defined in a neighborhood of \( x \in V \). Set \( R = \mathcal{O}_{V,x} \). At the tangent space of \( x \in V \), say \( \text{Spec}(gr_M(R)) \), the tangent cone of the differential algebra \( \mathcal{G} \) is a linear subspace, say \( T \), in \( \text{Spec}(gr_M(R)) \) (see 4.2). As \( \pi_1 \) and \( \pi_2 \) are smooth, their differential maps define one dimensional subspaces, say \( \ker(d(\pi_1)) \) and \( \ker(d(\pi_2)) \), in \( \text{Spec}(gr_M(R)) \). As both smooth morphisms are transversal to \( \mathcal{G} \) at \( x \): \( T, \ker(d(\pi_1)) \) and \( \ker(d(\pi_2)) \), are three subspaces in general position. Using this fact we conclude that the analytically irreducible element \( f_c \) can be taken to be transversal to both smooth morphisms.

Set \( S_1 = \mathcal{O}_{V^{(1)},\pi_1} \) and \( S_2 = \mathcal{O}_{V^{(2)},\pi_2} \). Both play the role of \( S \) in the previous discussion. Set now, by localization at \( \pi_1(x) \):

\[
(\mathcal{R}_G^{(i)})_{\pi_1(x)} = \bigoplus J_k^{(i)} \cdot W^k (\subset S_i[W]) \quad (i = 1, 2).
\]

Each \( \mathcal{R}_G^{(i)} \) is a Rees algebras over the smooth scheme \( V^{(i)} \). The claim is that

\[
\text{ord}_{\pi_1(x)}(\mathcal{R}_G^{(1)}) = \text{ord}_{\pi_2(x)}(\mathcal{R}_G^{(2)}).
\]

We can choose an integer \( N \), so that both \( \text{ord}_{\pi_1(x)}(\mathcal{R}_G^{(i)}) = \nu_{\pi_1(x)}(J_N^{(i)})/N \), for \( i = 1, 2 \) (see Prop 5.2(1)). In particular it suffices to show that \( \nu_{\pi_1(x)}(J_N^{(1)}) = \nu_{\pi_2(x)}(J_N^{(2)}) \), where \( \nu_{\pi_1(x)} \) denotes the order at the local regular ring \( S_i \) (i=1,2).

Within the local ring \( B \) there are two regular local rings \( S_1 \) and \( S_2 \). Theorem 4.11(iii) asserts that the inclusions

\[
(\mathcal{R}_G^{(i)})_{\pi_1(x)} = \bigoplus J_k^{(i)} \cdot W^k \subset \mathcal{G} = \bigoplus T_k \cdot W^k (\subset B[W])
\]

are both finite, for \( i = 1, 2 \). We can choose the integer \( N \) so that, in addition to the previous conditions, \( V^{(1)}(\mathcal{G}), \nu_{\pi_1(x)}(\mathcal{R}_G^{(1)}) \), and \( V^{(2)}(\mathcal{R}_G^{(2)}) \) are all Rees rings of ideals. As \( V^{(1)}(\mathcal{R}_G^{(1)})_{\pi_1(x)} \subset V^{(2)}(\mathcal{R}_G^{(2)})_{\pi_2(x)} \) is a finite extension of Rees rings (see Remark 4.4), it follows that \( J_N^{(1)} \cdot B \), and \( T_N \) have the same integral closure. A similar argument proves that the three ideals: \( J_N^{(1)} \cdot B, J_N^{(2)} \cdot B, \) and \( T_N \) have the same integral closure in \( B \).

Fix notation as in 5.6 where \( \text{Spec}(B) \leftarrow F \) denotes the normalized blow up of \( B \) at the maximal ideal, and \( V_1, \ldots, V_l \) are valuation rings corresponding to the irreducible exceptional hypersurfaces in \( F \). Set \( S = S_1 \) and let \( \text{Spec}(S) \leftarrow Y \) and \( V_S \) also as in 5.6.

Fix a valuation ring among \( V_1, \ldots, V_l \), say \( V_1 \). As \( V_1 \) dominates the local domain \( B \), and \( J_N^{(1)} \cdot B \), and \( T_N \) have the same integral closure in \( B \), it follows that both ideals have the same valuation at \( V_1 \).

Let \( e_1 \) denote the ramification index of \( V_1 \) over \( V_S \). Remark 5.8 says that the valuation of the ideal \( J_N^{(1)} \cdot B \) at the valuation ring \( V_1 \) is \( \nu_{V_1}(J_N^{(1)}) \cdot e_1 \). Finally Corollary 5.9 ensures that

\[
\nu_{V_1}(J_N^{(1)}) \cdot e_1 = \nu_{V_1}(J_N^{(2)}) \cdot e_1,
\]

as both coincide with the order of \( T_N \) at \( V_1 \). In particular \( \nu_{V_1}(J_N^{(1)}) = \nu_{V_1}(J_N^{(2)}) \). \( \square \)
On Hironaka’s $\tau$-invariant and projections.

5.11. We now discuss a property of elimination of one variable which parallels well-known properties that hold for inductive arguments used in desingularization over fields of characteristic zero. To clarify this point recall that the elimination of a variable, when passing from a Diff-algebra $\mathcal{G}$ to the Diff-algebra $\mathcal{R}_G$, is defined locally at a point $x \in \text{Sing}(\mathcal{G})$ when this point is simple. For simple points we have defined an homogeneous tangent ideal and a positive integer $\tau = s_r$ \textbf{[4.2.1]}. In the case of characteristic zero $s_0 = s_r$, and the algebra $\mathcal{R}_G$ will also be simple, unless $\tau = s_0 = 1$. In fact if $\tau > 1$ at $x \in \text{Sing}(\mathcal{G})$, the invariant $\tau'$ in the elimination algebra is $\tau - 1$. The following result shows that this also holds in the context of positive characteristic.

**Proposition 5.12.** If in the previous setting $\tau(\mathcal{G}_x) > 1$, then the elimination algebra $\mathcal{R}_G$ is a simple Diff-algebra.

**Proof.** In what follows we assume that $R$ is the completion of the local ring at a closed point. Set $\mathcal{G}_x = \oplus I_n W^n \subset R[W]$. After change of base field, which does not affect our arguments, we may assume that the closed point is rational over a perfect field $k$. Let $\{z, x_1, \ldots, x_{d-1}\}$ be a regular system of parameters for $M$ (the maximal ideal of $R$), and set $gr_M(R) = k[Z, X_1, \ldots, X_{d-1}]$ the graded ring where the variables are the initial forms of the parameters. Over perfect fields we may assume that the ideal of the tangent cone is generated by $p$-th powers of linear forms. Assume, for simplicity, that there are two elements, say $Z^{p^r}, X^{p^r}_1 \in k[Z, X_1, \ldots, X_{d-1}]$, in the tangent ideal. For a suitable $p$-th power, say $n$, there is an element $F_n \in I_n (n = p^r)$, and an element $g_n \in I_n$, such that:

i) $In_{p^r}(F_{p^r}) = Z^{p^r}$, and

ii) $In_{p^r}(g_{p^r}) = X^{p^r}_1$.

The Weierstrass Preparation Theorem allows us to assume that $F_{p^r} = F_{p^r}(z) \in S[z]$, is a monic polynomial in the variable $z$, where $(S, N)$ is the formal power series ring, and the maximal ideal $N$ is generated by the regular system of parameters $\{x_1, \ldots, x_{d-1}\}$. In fact, multiplication by a unit modifies the initial form by multiplication by a non-zero constant in the field. Set

$$F_{p^r}(z) = z^{p^r} + a_1 z^{p^r-1} + \cdots + a_{p^r} \in S[z],$$

and note that $\nu_S(a_i) > i$, for $1 \leq i \leq p^r$. The surjection $S[z] \to B = S[z]/ < F_{p^r}(z) >$ defines a graded ring $\overline{\mathcal{G}}_x = \oplus I_n W^n \subset B[W]$, where $I_n = I_n B$. In particular, the class of $g_{p^r}$, say $\overline{g}_{p^r}$, is an element of degree $p^r$. Set

$$\overline{g}_{p^r} = b_1 z^{p^r-1} + b_2 z^{p^r-2} \cdots + b_{p^r} \in S[z]/ < F_{p^r}(z) > .$$

We claim that $\nu_S(b_i) > i$ for $1 \leq i \leq p^r - 1$, and $In_N(b_{p^r}) = X^{p^r}_1$. This can be checked at the formal power series ring $R$. In fact the class is obtained by replacing the powers $z^N$, for all $N > p^r$, in the formal expression of $g_{p^r}$, by smaller powers. This is done by means of the relation:

$$z^{p^r} = -a_1 z^{p^r-1} - \cdots - a_{p^r}.$$

But $-a_1 z^{p^r-1} - \cdots - a_{p^r} \in M^{p^r+1}$, so this operation does not affect the initial form of $g_{p^r}$.
The ring $S[z]/<F^{p^e}(z)>$ is a free $S$ module, and multiplication by $p^e$ defines an $S$-linear endomorphism. So a characteristic polynomial is assigned to it. The norm, say $|p^e| = \prod_{1 \leq i \leq p^e} (b_1z_i^{p^e-1} + b_2z_i^{p^e-2} \cdots + b_{p^e})$, is defined formally as this product, where each $z_i$ is (formally) the image of $z$ by the different embeddings. We view each factor $(b_1z_i^{p^e-1} + b_2z_i^{p^e-2} \cdots + b_{p^e})$ with the same weight as $p^e$. The product is an $S$-linear combination on the symmetric functions on $z_i$, which in turn are weighted functions on the coefficients $a_i$. Since $\nu_S(a_i) > i$, the order of these elements is higher than the expected order, so

$$I_{n_S} (|b_1z_i^{p^e-1} + b_2z_i^{p^e-2} \cdots + b_{p^e}|) = X_1^{(p^e)^2},$$

and the weight of the norm $|p^e| \in S$ is precisely $(p^e)^2$ (i.e., $|p^e|W^{(p^e)^2} \in \mathcal{H}_G$).

This shows that $\mathcal{H}_G$ is simple, and hence that $\mathcal{R}_G$ is simple (4.12). □

**Remark 5.13.** In most resolution problems ideals in smooth schemes are considered up to integral closure. The natural analog, in the context of Rees algebras, is to consider them up to integral closure (see Section 5 in [37]). For this reason it is important to introduce invariants of Rees algebras, or of Diff-algebras, which coincide for algebras with the same integral closure.

Proposition 5.12, together with Theorem 4.11 and Prop 5.2 (2), already show that the $\tau$-invariant of a Diff-algebra at a singular point, is the same as that of its integral closure. In particular the $\tau$-invariant is well defined up to integral closure.

The following example shows that this is not the case for the invariant $\epsilon_0$, attached to a singular point of a Diff-algebra in 4.12.

**Example 5.14.** Set $G_1 = X^2 + Y^5$, $G_2 = X, G_3 = X$, and $F = X^4 + X^2Y^5$ in $R = k[X,Y]$, where $k$ is a field of characteristic two. All polynomials are monic in $X$, and $F = G_1 \cdot G_2 \cdot G_3$. Let $\mathcal{G} = \bigoplus I_k \cdot W^k(\subset R[W])$ denote Diff-algebra spanned by the Rees algebra, generated over $R$, by the element $F \cdot W^4$, as in Theorem 3.4. It follows easily from (3.4.11) that the $\epsilon_0$ invariant at the origin is 2, namely that $I_4$ has order $4(= p^2)$ at the origin; but the order of $I_i < 4$, for $i = 1,2,3$, at such point. Let $\mathcal{G}' = \bigoplus I_k \cdot W^k(\subset R[W])$ denote the integral closure. The discussion in 5.3 shows that $X = G_1 \in I_1$, so the $\epsilon_0$ invariant of $\mathcal{G}'$ at the origin is zero.

6. Monoidal transformations and differential Rees algebras. Examples.

**Definition 6.1.** Let $\mathcal{G} = \bigoplus I_k \cdot W^k(\subset \mathcal{O}_V)$ be a Rees algebra over a smooth scheme $V$. A monoidal transformation with smooth center $Y \subset V$, say $V \leftarrow V_1$, is said to be permissible for $\mathcal{G}$ if $Y \subset \text{Sing}(\mathcal{G})$. The exceptional locus is a smooth hypersurface, say $H \subset V_1$. Since $I_n$ has order at least $n$ along $Y$, $I_n \mathcal{O}_V \subset I(H)^n$, for $n \geq 0$. In particular, there is a factorization, say $I_n \mathcal{O}_{V_1} = I_n \cdot I(H)^n$, for a unique sheaf of ideals $I_n(1)$, which we call the weighted transform of $I_n$. The weighted transform of $\mathcal{G}$ will be the Rees algebra:

$$(\mathcal{G})_1 = \bigoplus I_n(1) \cdot W^k(\subset \mathcal{O}_{V_1}[W]).$$
Assume that, after restriction to an affine open set of $V$, $\mathcal{G}$ is generated by $\mathcal{F} = \{g_n, W^{n_i}, n_i > 0, 1 \leq i \leq m\}$. The total transform of $g_{n_i}$ is an element of $I(H)^{n_i}$ (i.e., vanishes along the hypersurface $H$ with order at least $n_i$). The total transform of $\mathcal{G}$ is the algebra generated by the same $\mathcal{F}$, but now as sub-algebra in $\mathcal{O}_V[W]$.

There is an open covering of $V_1$, so that locally $g_{n_i} \mathcal{O}_{V'} = < g'_{n_i} > \cdot I(H)^{n_i}$, for a principal ideal spanned by some $g'_{n_i}$. This defines $g'_{n_i}$ locally, and up to a unit. Every such $g'_{n_i}$ will be called a weighted transform of $g_{n_i}$.

**Proposition 6.2.** Let $\mathcal{G} = \bigoplus I_k \cdot W^k (\subset \mathcal{O}_V[W])$ be a Rees algebra over $V$, generated by elements $\{g_{N_1}W^{N_1}, \ldots, g_{N_s}W^{N_s}\}$; and let $V \leftarrow V_1$ be a permissible monoidal transformation. Then the weighted transform of $\mathcal{G}$ is generated by $\{g'_{N_1}W^{N_1}, \ldots, g'_{N_s}W^{N_s}\}$, where each $g'_{N_i}$ is a weighted transform of $g_{N_i}$.

(see [37] Prop. 1.3).

**Remark 6.3.** If $\mathcal{G} = \bigoplus I_k \cdot W^k \subset \mathcal{G}' = \bigoplus I_k \cdot W^k$ is a finite extension of graded algebras in $\mathcal{O}_V[W]$, then $\text{Sing}(\mathcal{G}) = \text{Sing}(\mathcal{G}')$ ([37] Prop. 3.9.2)), and there is an inclusion of the weighted transforms, say $(\mathcal{G})_1 \subset (\mathcal{G}')_1$ which is a finite extension of algebras over $V_1$.

**Remark 6.4.** It is convenient to pass from an arbitrary Rees algebra to a Diff-algebra, namely to the differential extension. In fact for the sake of resolution replacing a Rees algebra by Diff-algebra it spans can be taken for free. On the other hand, our previous results show that Diff-algebras have very powerful properties. The differential extension does not affect the singular locus, so a permissible transformation for one is permissible for both. However the weighted transform of a Diff-algebra is not necessarily a Diff-algebra. But still elimination algebras are well defined. This fundamental property, which is based on the stability of transversality, will be discussed in [6.7]

Suppose that a hypersurface $X$, embedded in a smooth scheme $V$, has points of multiplicity $b$ and no point of multiplicity $b + 1$. In order to study the b-fold points we consider the Rees algebra, say $\mathcal{G} = \mathcal{O}_V[I(X)W^b]$. Note that $\text{Sing}(\mathcal{G})$ is the closed set of points where the hypersurface has multiplicity $b$. Let $V \leftarrow V_1$ be a monoidal transformation at $Y \subset \text{Sing}(\mathcal{G})$. Let $\mathcal{G}_1$ be the weighted transform of $\mathcal{G}$, and set $X_1$ as the strict transform of $X$. Then $\mathcal{G}_1$ is the Rees algebra $\mathcal{O}_{V_1}[I(X_1)W^b]$, so $\text{Sing}(\mathcal{G}_1)$ is the set of $b$-fold points of $X_1$.

Consider again the situation at $V$. We pass from an ordinary Rees algebra $\mathcal{G}$ to the Diff-algebra that it spans, say $\mathcal{G}'$. The monoidal transform $V \leftarrow V_1$ defines a transform of $\mathcal{G}'$, say $\mathcal{G}'_1$, and clearly $\mathcal{G}_1 \subset \mathcal{G}'_1$.

Our interest is on the $b$-fold points of $X_1$, namely on $\text{Sing}(\mathcal{G}_1)$. We can consider the Diff-algebra generated by $\mathcal{G}_1$. It is therefore clear, that in a step by step argument, we would like to relate the Diff-algebra spanned by $\mathcal{G}_1$, with that spanned by $\mathcal{G}'_1$. In Theorem 6.6 we address this form of compatibility of transformations with extensions by differential operators.

**6.5.** Fix $\mathcal{G} = \bigoplus I_n W^n \subset \mathcal{G}' = \bigoplus I'_n W^n \subset \mathcal{G}'' = \bigoplus I''_n W^n$, over $V$ as in Corollary 3.5, so that $\mathcal{G}''$ is the Diff-algebra spanned by $\mathcal{G}$. Then $\text{Sing}(\mathcal{G}) = \text{Sing}(\mathcal{G}') = \text{Sing}(\mathcal{G}'')$ (Prop 3.9). In particular, a monoidal transformation $V \leftarrow V_1$, at a smooth center $Y \subset \text{Sing}(\mathcal{G})$, defines weighted transforms, say $\mathcal{G}^{(1)}$, $\mathcal{G}'^{(1)}$, and $\mathcal{G}''^{(1)}$. 
The following result is the so-called Giraud’s Lemma, formulated here in the context of Rees algebras.

**Theorem 6.6.** Let \( \mathcal{G} \subset \mathcal{G}' \subset \mathcal{G}''(\subset \mathcal{O}_V[W]) \) be an inclusion of Rees algebras as above, so that \( \mathcal{G}'' \) is the Diff-algebra spanned by \( \mathcal{G} \). Fix a monoidal transformation \( V \leftarrow V_1 \) with center \( Y \subset \text{Sing}(\mathcal{G}) \). Then

1) \( \mathcal{G}^{(1)} \subset \mathcal{G}'^{(1)} \subset \mathcal{G}''^{(1)}(\subset \mathcal{O}_{V_1}[W]) \), and
2) all three Rees algebras in 1) span the same Diff-algebras over \( V_1 \).

In particular, the condition on the inclusion \( \mathcal{G} \subset \mathcal{G}' \) in Corollary 3.5 is preserved by weighted transformations of Rees algebras.

**Elimination and monoidal transformations.**

6.7. Consider the localization, say \( \mathcal{G}_x = \bigoplus I_k \cdot W^k(\subset \mathcal{O}_{V,x}[W]) \), of a Diff-algebra \( \mathcal{G} \) at a simple closed point \( x \in \text{Sing}(\mathcal{G}) \). Since the Diff-algebra is simple, there is an index \( c_1 \) and an element \( f_{c_1} \in I_{c_1} \) of order \( c_1 \) at \( \mathcal{O}_{V,x} \). At a suitable étale neighborhood, a smooth transversal morphism \( \beta : V \rightarrow V' \) can be defined on a smooth scheme \( V' \), \( \dim V' = \dim V - 1 \).

At a suitable étale neighborhood we can also assume that \( f_{c_1} \) is a monic polynomial of degree \( c_1 \) in \( S[Z] \), and of order \( c_1 \) in \( R = S[Z]_{<M_x,Z>} \), where \( S = \mathcal{O}_{V',x} \), \( R \subset \mathcal{O}_{V,x} \), and the inclusion is étale. This defines by restriction a finite morphism, say \( \beta \) again:

\[ \beta : \text{Spec}(S[Z]/(f_{c_1}(Z))) \rightarrow \text{Spec}(S), \]

together with an elimination algebra \( \mathcal{R}_\mathcal{G} \subset S[W] \).

Here we view \( X = \text{Spec}(S[Z]/(f_{c_1}(Z))) \) as a hypersurface in \( V \). Theorem 4.11 (i), asserts that, locally at \( x \), \( \text{Sing}(\mathcal{G}) \) is included in the \( c_1 \)-fold points of this hypersurface (i.e., in \( \text{V}((\text{Diff}^{k-1}(f_{c_1}(Z)))) \)). Moreover, as \( \mathcal{G} \) is a Diff-algebra, \( \beta(\text{V}((\text{Diff}^{k-1}(f_{c_1}(Z)))) \equiv \text{Sing}(\mathcal{R}_\mathcal{G}) \).

Under these conditions the multiplicity formula asserts \( \beta \) is one to one on the closed set \( \text{V}((\text{Diff}^{k-1}(f_{c_1}(Z)))) \). Furthermore, if \( Y \) is a closed and smooth in \( \text{V}((\text{Diff}^{k-1}(f_{c_1}(Z)))) \), then \( Y \) is isomorphic to \( \beta(Y)(\subset \text{Spec}(S)) \) (see 39, Corollary 1, page 299). So both \( Y \) in \( V \), and \( \beta(Y) \) in \( \text{Spec}(S) \), are regular centers. Let now \( V \leftarrow V_1 \), and \( \text{Spec}(S) \leftarrow U \), denote the monoidal transformations at \( Y \) and \( \pi(Y) \) respectively. As \( \pi(Y) \subset \text{Sing}(\mathcal{R}_\mathcal{G}) \), \( \text{Spec}(S) \leftarrow U \) defines a weak transform, say:

\[ (\mathcal{R}_\mathcal{G})_1 \subset \mathcal{O}_U[W] \]

Let \( X' \) denote the strict transform of \( X \). The hypersurface \( X' \) has at most points of multiplicity \( c_1 \). Let \( F(\subset X') \) denote the closed set of points of multiplicity \( c_1 \). It is well known that locally at a neighborhood of \( F \), there is a smooth morphism \( \beta_1 : V_1 \rightarrow U \) that defines a commutative square. Moreover, it defines, by restriction, finite morphism say \( \beta : X' \rightarrow U \), compatible with \( \beta \). As the center \( Y \) was chosen in \( \text{Sing}(\mathcal{G}) \), a weighted transform, say

\[ \mathcal{G}_1 = \bigoplus I_{n_{i_1}} \cdot W^k(\subset \mathcal{O}_V[W]) \]

is defined. As \( I(X) \subset I_{c_1} \), it follows that \( I(X') \subset I_{c_1}^{(1)} \) (where \( I(X) \) and \( I(X') \) are the sheaves of ideals defining \( X \) and \( X' \) respectively). In particular \( \text{Sing}(\mathcal{G}_1) \subset F \) (points of multiplicity
Example 6.9. The following example illustrates the general fact that \( G \) spanned by \( R \). Here \( S \) can check now that the projection in or say \( (6.8.2) \) is non-zero. One can check that \( \beta_1 \) is transversal to \( X' \) and to \( G_1 \) at any point \( y \in \text{Sing} (G_1) \subset F \) mapping to \( x \). In can be proved that \( \beta_1 \) defines an elimination algebra, say \( R_{G_1} \), and moreover:

\[
R_{G_1} = (R_G)_1.
\]

To be precise, we mean here that both algebras have the same integral closure. This provides a natural commutativity of elimination with monoidal transformations, sustained on the so-called stability of transversality by monoidal transformations. This property is not addressed in this work. The point is that \( G_1 \) is no longer a Diff-algebra, but it is a \( \beta_1 \)-differential algebra, which is all we need to define \( R_{G_1} \).

In particular we can only guarantee that \( \beta_1 (\text{Sing} (G_1)) \subset \text{Sing} (R_{G_1}) \).

6.8. Here \( (R_G)_1 \) is the transform of \( R_G \) by one monoidal transformation. If we could ensure that \( \text{Sing} (R_G)_1 = \beta_1 (\text{Sing} (G_1)) \), we could identify the singular locus of \( G_1 \) (i.e., of \( G_1' \)) with the singular locus of the transform of \( R_G \). If furthermore, this link between \( G \) and \( R_G \) is preserved by every sequence of monoidal transformations, then we have achieved a way of representing the singular locus of \( G \) which is stable by monoidal transformations (see property (C), of stability of elimination, in the Introduction). So this would lead to resolution of singularities.

This will be the case when the invariant \( e_0 \) attached to the simple point \( x \) in \( \{1,2\} \) is zero. In this case \( f_{c_1} \) can be taken to be a monic polynomial of degree 1, defining therefore a smooth hypersurface of maximal contact, and then \( R_G \) is naturally identified with the restriction of \( G \) at such smooth hypersurface. In such case

\[
(6.8.1) \quad \text{Sing} ((R_G)_1) (= \text{Sing} (R_{G_1})) = \beta_1 (\text{Sing} (G_1)).
\]

And furthermore, a similar result holds for every sequence of, say \( r \) transformations, namely \( (R_G)_r = R_{G_r} \). In particular:

\[
(6.8.2) \quad \text{Sing} ((R_G)_r) = \text{Sing} (R_{G_r}) = \beta_r (\text{Sing} (G_r)).
\]

This is not the case, at least in general, when the invariant \( e_0 \) is non-zero.

Example 6.9. The following example illustrates the general fact that \( (R_G)_1 \) and \( R_{G_1} \) coincide.

Set \( R = \mathbb{Q}[Y, Z] \). Consider the Rees algebra in \( R[W] \) generated (over \( R \)) by the element \( (Z^2 + Y^5)W^2 \). Let \( G = \bigoplus I_k \cdot W^k (\subset R[W]) \) be the Diff-algebra spanned by this Rees algebra. According to Theorem 3.3 and formula (3.3.1), \( G \) is generated by \( 2ZW, 5Y^4W, (Z^2 + Y^5)W^2 \), or say \( \{ZW, Y^4W\} \). As \( Z \in I_1 \), we can choose \( c_1 = 1 \) and \( f_{c_1} (Z) = Z \) in Theorem 4.11. One can check now that the projection in \( S = k[Y] \), namely \( R_G \) is generated by \( \{Y^4W\} \).

Consider the quadratic transformation at the relevant chart \( \mathbb{Q}[Y, Z_1] \), where \( Y \cdot Z_1 = Z \). Here \( G_1 \) is generated by \( \{Z_1W, Y^3W\} \), which is also a Diff-algebra. The elimination algebra, say \( R_{G_1} \), is the S subalgebra in \( S[W] \) generated by \( \{Y^3W\} \). Finally check that \( R_{G_1} = (R_G)_1 \).

There is yet another natural observation. If \( (6.8.2) \) modified by taking \( G_r' \) as the Diff-algebra spanned by \( G_r \), then Theorem 4.11 (i), says that \( \text{Sing} (R_{G_r'}) = \beta_r (\text{Sing} (G_r')) \). Resolution of
singularity would be achieved if we could show that

$$\text{Sing}(\mathcal{R}_{G'}) = \text{Sing}(\mathcal{R}_{G}).$$

Unfortunately this does not hold in positive characteristic.

**Example 6.10.** The following is an example of a pathology that can only occur in positive characteristic. Namely that $(\mathcal{R}_{G})_1$ and $\mathcal{R}_{G'}$ do not span Diff-algebras with the same integral closure, where $\mathcal{G}'_i$ denotes the Diff-algebra spanned by $\mathcal{G}_1$.

Fix a field $k$ of characteristic two, and set $R = k[Y, Z]$. Let $\mathcal{G} = \bigoplus I_k \cdot W^k (\subset R[W])$ be, as before, the Diff-algebra generated by $(Z^2 + Y^5)W^2$. Here $\mathcal{G}$ is generated by $\{Y^4W, (Z^2 + Y^5)W^2\}$. The elimination in $S = k[Y]$, namely $\mathcal{R}_{\mathcal{G}}$, is generated by $\{Y^8W^2\}$, so up to integral closure it is generated $\{Y^4W\}$. To check this fact consider

$$\beta : \text{Spec}(S[Z]/(f_{c_1}(Z))) \to \text{Spec}(S),$$

where $f_{c_1} = Z^2 + Y^5$. This is a purely inseparable extension, and $\mathcal{R}_{\mathcal{G}}$ is (up to integral closure) generated by the coefficients of the characteristic polynomial of multiplication by $Y^4$ in the free $S$-module $B = S[Z]/(f_{c_1}(Z))$ (see Corollary 6.12). In this case $\mathcal{H}_{\mathcal{G}}$ is generated by $Y^8W^2$, so up to integral closure $\mathcal{R}_{\mathcal{G}}$ is generated by $Y^4W$.

Consider the quadratic transformation at the relevant chart $k[Y, Z_1]$, where $Y \cdot Z_1 = Z$. Here $\mathcal{G}_1$ is generated by $\{Y^3W, (Z_1^2 + Y^3)W^2\}$.

Let $\mathcal{G}'_1$ be the Diff-algebra spanned by $\mathcal{G}_1$. Then $\mathcal{R}_{\mathcal{G}_1}$ is, up to integral closure, generated by $\{Y^2W\}$, which is already a Diff-algebra. So $\mathcal{R}_{\mathcal{G}_1}$ is a Diff-algebra, and it is different from the Diff-algebra spanned by $(\mathcal{R}_G)_1$. In fact $(\mathcal{R}_G)_1$ is generated by $\{Y^3W\}$.

6.11. The situation in 6.10, in which $(\mathcal{R}_G)_1$ and $\mathcal{R}_{\mathcal{G}_1}$ do not span the same Diff-algebra, can only occur when the invariant $e_0$ is not zero. In this case $p = 2$, and $e_0 = 1$ at the origin, which is a singular point of Diff-algebra $\mathcal{G} = \bigoplus I_k \cdot W^k (\subset R[W])$.

An interesting case, also computable from our invariants, is that of $Z^3 + X^{13}Z + X^{16}$ in characteristic 3. The invariant $e_0$ at the origin is 1. This curve is analytically irreducible, the singularity is resolved with five quadratic transformations.

In this case $\mathcal{R}_{\mathcal{G}_1}$ spans the same Diff-algebra as $(\mathcal{R}_G)_1$ for the first two transformations (for $i = 0, 1$), but not for the next 3 quadratic transformations.

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