A NOTE FOR SOME PARABOLIC MULTILINEAR COMMUTATORS GENERATED BY A CLASS OF PARABOLIC MAXIMAL AND LINEAR OPERATORS WITH ROUGH KERNEL ON THE PARABOLIC GENERALIZED LOCAL MORREY SPACES

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Abstract. In this paper, we give the boundedness of some parabolic multilinear commutators generated by a class of parabolic maximal and linear operators with rough kernel and parabolic local Campanato functions on the parabolic generalized local Morrey spaces, respectively. Indeed, the results in this paper are extensions of some known results.

1. Introduction and main results

Let $S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \}$ denote the unit sphere on $\mathbb{R}^n \ (n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma (x')$, where $x'$ denotes the unit vector in the direction of $x$ and $\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq 1$ be fixed real numbers.

Note that for each fixed $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the function

$$F(x, \rho) = \sum_{i=1}^{n} \frac{x_i^2}{\rho^{2\alpha_i}}$$

is a strictly decreasing function of $\rho > 0$. Hence, there exists a unique $\rho = \rho (x)$ such that $F(x, \rho) = 1$. It is clear that for each fixed $x \in \mathbb{R}^n$, the function $F(x, \rho)$ is a decreasing function in $\rho > 0$. Fabes and Rivière showed that $(\mathbb{R}^n, \rho)$ is a metric space which is often called the mixed homogeneity space related to $\{\alpha_i\}_{i=1}^{n}$.

For $t > 0$, we let $A_t$ be the diagonal $n \times n$ matrix

$$A_t = \text{diag} [t^{\alpha_1}, \ldots, t^{\alpha_n}] = \begin{pmatrix} t^{\alpha_1} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t^{\alpha_n} \end{pmatrix}.$$
Let $\rho \in (0, \infty)$ and $0 \leq \varphi_{n-1} \leq 2\pi$, $0 \leq \varphi_i \leq \pi$, $i = 1, \ldots, n - 2$. For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, set

$$x_1 = \rho^{\alpha_1} \cos \varphi_1 \cdots \cos \varphi_{n-2} \cos \varphi_{n-1},$$
$$x_2 = \rho^{\alpha_2} \cos \varphi_1 \cdots \cos \varphi_{n-2} \sin \varphi_{n-1},$$
$$\vdots$$
$$x_{n-1} = \rho^{\alpha_{n-1}} \cos \varphi_1 \sin \varphi_2,$$
$$x_n = \rho^{\alpha_n} \sin \varphi_1.$$

Thus $dx = \rho^{n-1} J(x') d\rho d\sigma(x')$, where $\alpha = \sum_{i=1}^{n} \alpha_i$, $x' \in S^{n-1}$, $J(x') = \sum_{i=1}^{n} \alpha_i (x_i')^2$, $d\sigma$ is the element of area of $d\sigma$, such that $1 \leq J(x') \leq M$ such that $1 \leq J(x') \leq M$ and $x' \in S^{n-1}$.

Let $P$ be a real $n \times n$ matrix, whose all the eigenvalues have positive real part. Let $A_t = t \rho'$ ($t > 0$), and set $\gamma = tr P$. Then, there exists a quasi-distance $\rho$ associated with $P$ such that (see [3])

$$(1-1) \rho (A_t x) = t \rho (x), t > 0, \text{ for every } x \in \mathbb{R}^n,$$
$$(1-2) \rho (0) = 0, \rho (x - y) = \rho (y - x) \geq 0, \text{ and } \rho (x - y) \leq k (\rho (x - z) + \rho (y - z)),$$
$$(1-3) dx = \rho^{n-1} d\sigma (w) d\rho, \text{ where } \rho = \rho (x), w = A_{\rho^{-1}} x \text{ and } d\sigma (w) \text{ is a measure on the unit ellipsoid } \{ w : \rho (w) = 1 \}.$$

Then, $\mathbb{R}^n, \rho, dx$ becomes a space of homogeneous type in the sense of Coifman-Weiss (see [3]) and a homogeneous group in the sense of Folland-Stein (see [6]).

Denote by $E(x, r)$ the ellipsoid with center at $x$ and radius $r$, more precisely, $E(x, r) = \{ y \in \mathbb{R}^n : \rho (x - y) < r \}$. For $k > 0$, we denote $kE(x, r) = \{ y \in \mathbb{R}^n : \rho (x - y) < kr \}$. Moreover, by the property of $\rho$ and the polar coordinates transform above, we have

$$|E(x, r)| = \int_{\rho(x-y)<r} dy = v_\rho r^{\alpha_1 + \cdots + \alpha_n} = v_\rho r^\gamma,$$

where $|E(x, r)|$ stands for the Lebesgue measure of $E(x, r)$ and $v_\rho$ is the volume of the unit ellipsoid on $\mathbb{R}^n$. By $E^C(x, r) = \mathbb{R}^n \setminus E(x, r)$, we denote the complement of $E(x, r)$. If we take $\alpha_1 = \cdots = \alpha_n = 1$ and $P = I$, then obviously $\rho (x) = |x| = \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$, $\gamma = n$, $(\mathbb{R}^n, \rho) = (\mathbb{R}^n, |\cdot|)$, $E_t(x, r) = B(x, r)$, $A_t = t I$ and $J(x') \equiv 1$.

Moreover, in the standard parabolic case $P_0 = \text{diag} [1, \ldots, 1, 2]$ we have

$$\rho (x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \quad x = (x', x_n).$$

Note that we deal not exactly with the parabolic metric, but with a general anisotropic metric $\rho$ of generalized homogeneity, the parabolic metric being its particular case, but we keep the term parabolic in the title and text of the paper, the above existing tradition, see for instance [2].

Suppose that $\Omega(x)$ is a real-valued and measurable function defined on $\mathbb{R}^n$. Suppose that $S^{n-1}$ is the unit sphere on $\mathbb{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue surface measure $d\sigma$. Let $\Omega \in L_4(S^{n-1})$ with $1 < s \leq \infty$ be homogeneous
of degree zero with respect to $A_t$ ($\Omega(x)$ is $A_t$-homogeneous of degree zero), that is, $\Omega(A_t x) = \Omega(x)$, for any $t > 0$, $x \in \mathbb{R}^n$. We define $s' = \frac{1}{p'}$ for any $s > 1$.

One of the important problems on parabolic homogeneous spaces investigates the boundedness of parabolic linear operators satisfying the following size conditions \((1.1)\) and \((1.2)\). Therefore, in this paper, we consider parabolic linear operators $T^\rho_\Omega$ and $T^{\rho}_{\Omega, \alpha}$, $\alpha \in (0, \gamma)$ satisfying the size conditions for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \text{supp} f$, respectively

\begin{align}
\|T^\rho_\Omega f(x)\| &\leq c_0 \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{\rho(x-y)^{\gamma}} |f(y)| \, dy, \\
\|T^{\rho}_{\Omega, \alpha} f(x)\| &\leq c_0 \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{\rho(x-y)^{\gamma-\alpha}} |f(y)| \, dy,
\end{align}

where $c_0$ is independent of $f$ and $x$.

We point out that the conditions \((1.1)\) and \((1.2)\) in the case $\Omega \equiv 1$, $\alpha = 0$ and $P = I$ was first introduced by Soria and Weiss in [11]. Indeed, in 1944, Soria and Weiss developed Stein’s result [12] in the above shape. The conditions \((1.1)\) and \((1.2)\) are satisfied by many interesting operators in harmonic analysis, such as the parabolic Calderón–Zygmund operators, parabolic Carleson’s maximal operator, parabolic Hardy–Littlewood maximal operator, parabolic C. Fefferman’s singular integrals, parabolic R. Fefferman’s singular integrals, parabolic Ricci–Stein’s oscillatory singular integrals, parabolic the Bochner–Riesz means, the parabolic fractional integral operator(parabolic Riesz potential), parabolic fractional maximal operator, parabolic fractional Marcinkiewicz operator and so on (see [7, 8, 11] for details).

The parabolic fractional maximal function $M^{\rho}_{\Omega, \alpha} f$ and $T^{\rho}_{\Omega, \alpha} f$ by with rough kernels, $0 < \alpha < \gamma$, of a function $f \in L^{1,\text{loc}}(\mathbb{R}^n)$ are defined by

\begin{align}
M^{\rho}_{\Omega, \alpha} f(x) &= \sup_{t > 0} |E(x, t)|^{-1 + \frac{s'}{2}} \int_{E(x, t)} |\Omega(x-y)| |f(y)| \, dy, \\
T^{\rho}_{\Omega, \alpha} f(x) &= \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{\rho(x-y)^{\gamma-\alpha}} f(y) \, dy,
\end{align}

satisfy condition \((1.2)\). It is obvious that when $\Omega \equiv 1$, $M^{\rho}_{1, \alpha} = M^{\rho}_{\alpha}$ and $T^{\rho}_{1, \alpha} = T^{\rho}_{\alpha}$ are the parabolic fractional maximal operator and the parabolic fractional integral operator, respectively. If $P = I$, then $M^{\rho}_{\Omega, \alpha} = M_{\Omega, \alpha}$ and $T^{\rho}_{\Omega, \alpha} = T_{\Omega, \alpha}$ are the fractional maximal operator with rough kernel and fractional integral operator with rough kernel, respectively. It is well known that the parabolic fractional maximal and integral operators play an important role in harmonic analysis (see [2, 6, 8]).

We notice that when $\alpha = 0$, the above operators become the parabolic Calderón–Zygmund singular integral operator with rough kernel $T^{\rho}_{1, 0} = T^{\rho}_{1, 0}$ and the corresponding parabolic maximal operator with rough kernel $M^{\rho}_{1, 0} = M^{\rho}_{1}$:

\begin{align}
T^{\rho}_{1, 0} f(x) &= p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{\rho(x-y)^{\gamma}} f(y) \, dy,
\end{align}

of rough kernel, respectively. It is well known that the parabolic fractional maximal operator with rough kernel $M^{\rho}_{1, 0} = M^{\rho}_{1}$.
satisfy condition \((1.1)\). It is obvious that when \(\Omega \equiv 1\), \(T^P_\Omega \equiv T^P\) and \(M^P_\Omega \equiv M^P\) are the parabolic singular operator and the parabolic maximal operator, respectively. If \(P = I\), then \(M^P_\Omega \equiv M_\Omega\) is the Hardy-Littlewood maximal operator with rough kernel, and \(T^I_\Omega \equiv T_\Omega\) is the homogeneous singular integral operator. It is well known that the parabolic maximal and singular operators play an important role in harmonic analysis (see \([2, 6, 7, 14]\)).

On the other hand, let \(b\) be a locally integrable function on \(\mathbb{R}^n\), then for \(0 < \alpha < \gamma\), we define commutators generated by parabolic fractional maximal and integral operators with rough kernel and \(b\) as follows, respectively.

\[
M^P_{\Omega, b, \alpha}(f)(x) = \sup_{t > 0} |E(x, t)|^{-1} \int_{E(x, t)} |b(x) - b(y)| |\Omega(x - y)| |f(y)| dy,
\]

\[
[b, T^P_{\Omega, \alpha}]f(x) \equiv b(x)T^P_{\Omega, \alpha}f(x) - T^P_{\Omega, \alpha}(bf)(x) = \int_{\mathbb{R}^n} [b(x) - b(y)] \frac{\Omega(x - y)}{\rho(x - y)^{\gamma - \alpha}} f(y) dy.
\]

Similarly, for \(\alpha = 0\), we define commutators generated by parabolic maximal and singular integral operators by with rough kernels and \(b\) as follows, respectively.

\[
M^P_{\Omega, b}(f)(x) = \sup_{t > 0} |E(x, t)|^{-1} \int_{E(x, t)} |b(x) - b(y)| |\Omega(x - y)| |f(y)| dy,
\]

\[
[b, T^P_{\Omega}]f(x) \equiv b(x)T^P_{\Omega}f(x) - T^P_{\Omega}(bf)(x) = p.v. \int_{\mathbb{R}^n} [b(x) - b(y)] \frac{\Omega(x - y)}{\rho(x - y)^{\gamma}} f(y) dy.
\]

Because of the need for the study of partial differential equations (PDEs), Morrey \([10]\) introduced Morrey spaces \(M_{p, \lambda}\) which naturally are generalizations of Lebesgue spaces. We also refer to \([12]\) for the latest research on the theory of Morrey spaces associated with harmonic analysis.

A measurable function \(f \in L_p(\mathbb{R}^n), p \in (1, \infty)\), belongs to the parabolic Morrey spaces \(M_{p, \lambda, p}(\mathbb{R}^n)\) with \(\lambda \in (0, \gamma)\) if the following norm is finite:

\[
\|f\|_{M_{p, \lambda, p}} = \left( \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{r^\lambda} \int_{E(x, r)} |f(y)|^p dy \right)^{1/p},
\]

where \(E(x, r)\) stands for any ellipsoid with center at \(x\) and radius \(r\). When \(\lambda = 0\), \(M_{p, \lambda, p}(\mathbb{R}^n)\) coincides with the parabolic Lebesgue space \(L_{p, p}(\mathbb{R}^n)\).

If \(P = I\), then \(M_{p, \lambda, \Omega}(\mathbb{R}^n) \equiv M_{p, \lambda}(\mathbb{R}^n)\) and \(L_{p, \Omega}(\mathbb{R}^n) \equiv L_p(\mathbb{R}^n)\) are the classical Morrey and the Lebesgue spaces, respectively.

We now recall the definition of parabolic generalized local (central) Morrey space \(LM_{p, \lambda, p}(\Omega(x_0))\) in the following.

**Definition 1.** \([7, 8]\) (parabolic generalized local (central) Morrey space)

Let \(\varphi(x, r)\) be a positive measurable function on \(\mathbb{R}^n \times (0, \infty)\) and \(1 \leq p < \infty\). For
any fixed \( x_0 \in \mathbb{R}^n \) we denote by \( LM^{(x_0)}_{p, \varphi, P} \equiv LM^{(x_0)}_{p, \varphi, P}(\mathbb{R}^n) \) the parabolic generalized local Morrey space, the space of all functions \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) with finite quasinorm

\[
\|f\|_{LM^{(x_0)}_{p, \varphi, P}} = \sup_{r > 0} \varphi(x_0, r)^{-1} |E(x_0, r)|^{-\frac{1}{p}} \|f\|_{L^p(E(x_0, r))} < \infty.
\]

According to this definition, we recover the local parabolic Morrey space \( LM^{(x_0)}_{p, \varphi, P} \) and weak local parabolic Morrey space \( WLM^{(x_0)}_{p, \varphi, P} \) under the choice \( \varphi(x_0, r) = r^{-\frac{\lambda}{p}} \):

\[
LM^{(x_0)}_{p, \varphi, P} = LM^{(x_0)}_{p, \varphi, P} |_{\varphi(x_0, r) = r^{-\frac{\lambda}{p}}}, \quad WLM^{(x_0)}_{p, \varphi, P} = WLM^{(x_0)}_{p, \varphi, P} |_{\varphi(x_0, r) = r^{-\frac{\lambda}{p}}}. \]

Now, let us recall the definition of the space of \( LC^{(x_0)}_{p, \lambda, P} \) (parabolic local Campanato space).

**Definition 2.** [1] [3] Let \( 1 \leq p < \infty \) and \( 0 \leq \lambda < \frac{1}{p} \). A parabolic local Campanato function \( b \in L^p_{\text{loc}}(\mathbb{R}^n) \) is said to belong to the \( LC^{(x_0)}_{p, \lambda, P}(\mathbb{R}^n) \), if

\[
\|b\|_{LC^{(x_0)}_{p, \lambda, P}} = \sup_{r > 0} \left( \frac{1}{|E(x_0, r)|^{1+\lambda}} \int_{E(x_0, r)} |b(y) - b_{E(x_0, r)}|^p \, dy \right)^{\frac{1}{p}} < \infty,
\]

where \( b_{E(x_0, r)} = \frac{1}{|E(x_0, r)|} \int_{E(x_0, r)} b(y) \, dy \).

Define

\[
LC^{(x_0)}_{p, \lambda, P}(\mathbb{R}^n) = \left\{ b \in L^p_{\text{loc}}(\mathbb{R}^n) : \|b\|_{LC^{(x_0)}_{p, \lambda, P}} < \infty \right\}.
\]

Let \( b_i \) (\( i = 1, \ldots, m \)) be locally integrable functions on \( \mathbb{R}^n \), then the fractional parabolic multilinear commutators generated by parabolic fractional maximal and integral operators with rough kernel and \( \vec{b} = (b_1, \ldots, b_m) \) (parabolic local Campanato functions) are given as follows, respectively:

\[
\left[ \vec{b}, T^P_{\Omega, \alpha} \right] f(x) = \int \prod_{x_i = 1}^m |b_i(x) - b_i(y)| \frac{\Omega(x - y)}{\rho(x - y)^{\gamma - \alpha}} f(y) \, dy, \quad 0 < \alpha < \gamma,
\]

\[
M^P_{\Omega, b, \alpha} f(x) = \sup_{t > 0} |E(x, t)|^{-1 + \frac{\alpha}{p}} \int |b_i(x) - b_i(y)| |\Omega(x - y)| |f(y)| \, dy, \quad 0 < \alpha < \gamma.
\]

We notice that when \( \alpha = 0 \), the upper operators become the parabolic multilinear commutators generated by parabolic singular integral operators and the corresponding parabolic maximal operators with rough kernel and \( \vec{b} = (b_1, \ldots, b_m) \) as follows, respectively:

\[
\left[ \vec{b}, T^P_{\Omega} \right] f(x) = \int \prod_{x_i = 1}^m |b_i(x) - b_i(y)| \frac{\Omega(x - y)}{\rho(x - y)} f(y) \, dy,
\]

\[
M^P_{\Omega, b} f(x) = \sup_{t > 0} |E(x, t)|^{-1} \int |b_i(x) - b_i(y)| |\Omega(x - y)| |f(y)| \, dy.
\]
In [7, 8] the boundedness of a class of parabolic sublinear operators with rough kernel and their commutators on the parabolic generalized local Morrey spaces under generic size conditions which are satisfied by most of the operators in harmonic analysis has been investigated, respectively.

Inspired by [7, 8], our main purpose in this paper is to consider the boundedness of above operators \([\overrightarrow{b}, T_{\Omega}^P], M_{\Omega, \overrightarrow{b}}, [\overrightarrow{b}, T_{\Omega, \alpha}], M_{\Omega, \overrightarrow{b}, \alpha}\) on the parabolic generalized local Morrey spaces, respectively. But, the techniques and non-trivial estimates which have been used in the proofs of our main results are quite different from [7, 8]. For example, using inequality about the weighted Hardy operator \(H_w\) in [7, 8], in this paper we will only use the following relationship between essential supremum and essential infimum

\[
(1.3) \quad \left( \essinf_{x \in E} f(x) \right)^{-1} = \eess \frac{1}{f(x)},
\]

where \(f\) is any real-valued nonnegative function and measurable on \(E\) (see [13], page 143). Our main results can be formulated as follows.

**Theorem 1.** Suppose that \(x_0 \in \mathbb{R}^n, \Omega \in L_s(S^{n-1}), 1 < s \leq \infty,\) is \(A_t\)-homogeneous of degree zero. Let \(T_{\Omega}^P\) be a parabolic linear operator satisfying condition \((1.1)\). Let also, for \(1 < q, p, \alpha < \infty\) with \(\frac{1}{q} = \frac{\sum_{i=1}^{m} \frac{1}{p_i} + \frac{1}{p}}{m}\) and \(\overrightarrow{b} \in LC^{(x_0)}_{p_i, \lambda_i, p}(\mathbb{R}^n)\) for \(0 \leq \lambda_i < \frac{1}{\gamma}, i = 1, \ldots, m.\)

Let also, for \(s' \leq q\) the pair \((\varphi_1, \varphi_2)\) satisfies the condition

\[
(1.4) \quad \int_r^\infty \left(1 + \ln \frac{r}{t} \right)^m \ess \frac{\varphi_1(x_0, \tau)\tau^{\frac{m}{p}}}{t^{\frac{1}{p} - \frac{1}{p} \sum_{i=1}^{m} \lambda_i} + 1} \leq C \varphi_2(x_0, r),
\]

and for \(p < s\) the pair \((\varphi_1, \varphi_2)\) satisfies the condition

\[
\int_r^\infty \left(1 + \ln \frac{r}{t} \right)^m \ess \frac{\varphi_1(x_0, \tau)\tau^{\frac{m}{p}}}{t^{\frac{1}{p} - \frac{1}{p} \sum_{i=1}^{m} \lambda_i} + 1} \leq C \varphi_2(x_0, r)^{\frac{m}{p}},
\]

where \(C\) does not depend on \(r\).

Then, the operators \([\overrightarrow{b}, T_{\Omega}^P]\) and \(M^P_{\Omega, \overrightarrow{b}}\) are bounded from \(LM^{(x_0)}_{p, \varphi_2, p}\) to \(LM^{(x_0)}_{q, \varphi_1, P}\).

Moreover,

\[
(1.5) \quad \| [\overrightarrow{b}, T_{\Omega}^P] f \|_{LM^{(x_0)}_{q, \varphi_2, P}} \lesssim \prod_{i=1}^{m} \| \overrightarrow{b} \|_{LC^{(x_0)}_{p_i, \lambda_i, p}} \| f \|_{LM^{(x_0)}_{q, \varphi_2, P}},
\]

\[
(1.6) \quad \| [\overrightarrow{b}, M^P_{\Omega, \overrightarrow{b}}] f \|_{LM^{(x_0)}_{q, \varphi_2, P}} \lesssim \prod_{i=1}^{m} \| \overrightarrow{b} \|_{LC^{(x_0)}_{p_i, \lambda_i, p}} \| f \|_{LM^{(x_0)}_{q, \varphi_2, P}}.
\]

**Corollary 1.** Suppose that \(x_0 \in \mathbb{R}^n, \Omega \in L_s(S^{n-1}), 1 < s \leq \infty,\) is \(A_t\)-homogeneous of degree zero. Let \(T_{\Omega}^P\) be a parabolic linear operator satisfying condition \((1.1)\), bounded on \(L_p(\mathbb{R}^n)\) for \(1 < p < \infty.\) Let \(b \in LC^{(x_0)}_{p_2, \lambda, p}(\mathbb{R}^n), 0 \leq \lambda < \frac{1}{\gamma}\)
and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Let also, for $s' \leq p$ the pair $(\varphi_1, \varphi_2)$ satisfies the condition
\[
\int_r^\infty \left(1 + \ln \frac{t}{r}\right)^m \inf_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{n}{\gamma}} \frac{1}{t^{\frac{n}{\gamma} + 1 - \gamma \lambda}} \, dt \leq C \varphi_2(x_0, r),
\]
and for $p_1 < s$ the pair $(\varphi_1, \varphi_2)$ satisfies the condition
\[
\int_r^\infty \left(1 + \ln \frac{t}{r}\right)^m \inf_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{n}{\gamma}} \frac{1}{t^{\frac{n}{\gamma} + 1 - \gamma \lambda}} \, dt \leq C \varphi_2(x_0, r) r^{\frac{n}{\gamma}},
\]
where $C$ does not depend on $r$.

Then, the operators $[b, T^P_{\Omega}]$ and $M^P_{\Omega, b}$ are bounded from $LM^{(x_0)}_{p_1, \varphi_1, p}$ to $LM^{(x_0)}_{p_2, \varphi_2, p}$. Moreover,
\[
\| [b, T^P_{\Omega}]f \|_{LM^{(x_0)}_{p, \varphi_2, p}} \lesssim \| b \|_{LC^{(x_0)}_{p_2, \lambda, p}} \| f \|_{LM^{(x_0)}_{p_1, \varphi_1, p}},
\]
\[
\| M^P_{\Omega, b}f \|_{LM^{(x_0)}_{p, \varphi_2, p}} \lesssim \| b \|_{LC^{(x_0)}_{p_2, \lambda, p}} \| f \|_{LM^{(x_0)}_{p_1, \varphi_1, p}}.
\]

**Theorem 2.** Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is $A_t$-homogeneous of degree zero. Let $T^P_{\Omega, \alpha}$ be a parabolic linear operator satisfying condition (1.9).

Let also $0 < \alpha < \gamma$ and $1 < q_1 < p_1$, $p_1 < \frac{1}{\alpha}$ with $\frac{1}{q} = \sum_{i=1}^m \frac{1}{p_i} + \frac{1}{p_1}$, $\frac{1}{q_1} = \frac{1}{q} - \frac{\alpha}{\gamma}$ and $\vec{b} \in LC^{(x_0)}_{p_1, \lambda, p} (\mathbb{R}^n)$ for $0 \leq \lambda_i < \frac{1}{p_i}$, $i = 1, \ldots, m$.

Let also, for $s' \leq q$ the pair $(\varphi_1, \varphi_2)$ satisfies the condition
\[
(1.7) \quad \int_r^\infty \left(1 + \ln \frac{t}{r}\right)^m \inf_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{n}{\gamma}} \frac{1}{t^{\frac{n}{\gamma} + 1 - \gamma \lambda}} \, dt \leq C \varphi_2(x_0, r),
\]
and for $q_1 < s$ the pair $(\varphi_1, \varphi_2)$ satisfies the condition
\[
(1.8) \quad \int_r^\infty \left(1 + \ln \frac{t}{r}\right)^m \inf_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{n}{\gamma}} \frac{1}{t^{\frac{n}{\gamma} + 1 - \gamma \lambda}} \, dt \leq C \varphi_2(x_0, r) r^{\frac{n}{\gamma}},
\]
where $C$ does not depend on $r$.

Then, the operators $[\vec{b}, T^P_{\Omega, \alpha}]$ and $M^P_{\Omega, \vec{b}, \alpha}$ are bounded from $LM^{(x_0)}_{p_1, \varphi_1, p}$ to $LM^{(x_0)}_{p_2, \varphi_2, p}$.

Moreover,
\[
\| [\vec{b}, T^P_{\Omega, \alpha}]f \|_{LM^{(x_0)}_{q_1, \varphi_2, p}} \lesssim \prod_{i=1}^m \| \vec{b} \|_{LC^{(x_0)}_{p_i, \lambda_i, p}} \| f \|_{LM^{(x_0)}_{p_1, \varphi_1, p}},
\]
\[
\| M^P_{\Omega, \vec{b}, \alpha}f \|_{LM^{(x_0)}_{q_1, \varphi_2, p}} \lesssim \prod_{i=1}^m \| \vec{b} \|_{LC^{(x_0)}_{p_i, \lambda_i, p}} \| f \|_{LM^{(x_0)}_{p_1, \varphi_1, p}}.
\]

**Corollary 2.** Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is $A_t$-homogeneous of degree zero. Let $T^P_{\Omega, \alpha}$ be a parabolic linear operator satisfying condition (1.9) and bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$. Let $0 < \alpha < \gamma$, $1 < p < \frac{\gamma}{\alpha}$,
Lemma 1. Let \( b \in LC^{(x_0)}_{p,\lambda,p} (\mathbb{R}^n) \), \( 0 \leq \lambda < \frac{1}{p} \), \( \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} \), \( \frac{1}{q} = \frac{1}{p} - \frac{\lambda}{p_1} + \frac{\lambda}{p_2} \). Let also, for \( s' \leq p \) the pair \((\varphi_1, \varphi_2)\) satisfies the condition

\[
\int_r^\infty \left( 1 + \ln \frac{t}{r} \right) \frac{\text{essinf}_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^\gamma}{t^{\frac{1}{p_1}} \gamma + 1 - \gamma \lambda} \, dt \leq C \varphi_2(x_0, r),
\]

and for \( q_1 < s \) the pair \((\varphi_1, \varphi_2)\) satisfies the condition

\[
\int_r^\infty \left( 1 + \ln \frac{t}{r} \right) \frac{\text{essinf}_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^\gamma}{t^{\frac{1}{p_1}} \gamma + 1 - \gamma \lambda} \, dt \leq C \varphi_2(x_0, r)^{\frac{\gamma}{s}},
\]

where \( C \) does not depend on \( r \).

Then, the operators \([b, T^p_{\Omega,\alpha}]\) and \( M^p_{\Omega, b, \alpha} \) are bounded from \( L^{(x_0)}_{p_1, \varphi_1, p} \) to \( L^{(x_0)}_{q, \varphi_2, p} \).

Moreover,

\[
\| [b, T^p_{\Omega,\alpha}] f \|_{L^{(x_0)}_{q, \varphi_2, p}} \lesssim \| b \|_{LC^{(x_0)}_{p_1, \varphi_1, p}} \| f \|_{L^{(x_0)}_{p_1, \varphi_1, p}},
\]

\[
\| M^p_{\Omega, b, \alpha} f \|_{L^{(x_0)}_{q, \varphi_2, p}} \lesssim \| b \|_{LC^{(x_0)}_{p_1, \varphi_1, p}} \| f \|_{L^{(x_0)}_{p_1, \varphi_1, p}}.
\]

At last, throughout the paper we use the letter \( C \) for a positive constant, independent of appropriate parameters and not necessarily the same at each occurrence. By \( A \lesssim B \) we mean that \( A \leq CB \) with some positive constant \( C \) independent of appropriate quantities. If \( A \lesssim B \) and \( B \lesssim A \), we write \( A \approx B \) and say that \( A \) and \( B \) are equivalent.

2. Some Lemmas

To prove the main results (Theorems 1 and 2), we need the following lemmas.

Lemma 1. [7, 8] Let \( b \) be a parabolic local Campanato function in \( LC^{(x_0)}_{p,\lambda,p} (\mathbb{R}^n) \), \( 1 \leq p < \infty \), \( 0 \leq \lambda < \frac{1}{q} \), and \( r_1, r_2 > 0 \). Then

\[
(2.1) \quad \left( \frac{1}{|E(x_0, r_1)|^{1 + \lambda \gamma}} \int_{E(x_0, r_1)} \left| b(y) - b_{E(x_0, r_2)} \right|^p \, dy \right)^{\frac{1}{p}} \leq C \left( 1 + \ln \frac{r_1}{r_2} \right) \| b \|_{LC^{(x_0)}_{p,\lambda,p}},
\]

where \( C > 0 \) is independent of \( b \), \( r_1 \) and \( r_2 \).

From this inequality (2.1), we have

\[
(2.2) \quad \| b_{E(x_0, r_1)} - b_{E(x_0, r_2)} \|_{L^p(E)} \leq C \left( 1 + \ln \frac{r_1}{r_2} \right) \| E(x_0, r_1) \|^{\lambda \gamma} \| b \|_{LC^{(x_0)}_{p,\lambda,p}},
\]

and it is easy to see that

\[
(2.3) \quad \| b - b_E \|_{L^p(E)} \leq C \left( 1 + \ln \frac{r_1}{r_2} \right) r^{\frac{\gamma}{s}} \| b \|_{LC^{(x_0)}_{p,\lambda,p}}.
\]

Lemma 2. Suppose that \( x_0 \in \mathbb{R}^n \), \( \Omega \in L_s(S^{n-1}) \), \( 1 < s \leq \infty \), is \( A_\ast \)-homogeneous of degree zero. Let \( T^p_{\Omega} \) be a parabolic linear operator satisfying condition (1.1). Let also \( 1 < q, p_i, p < \infty \) with \( \frac{1}{q} = \sum_{i=1}^m \frac{1}{p_i} + \frac{1}{p} \) and \( b \in LC_{p_i,\lambda_i,p}^{(x_0)}(\mathbb{R}^n) \) for \( 0 \leq \lambda_i < \frac{1}{q} \),
\(i = 1, \ldots, m\). Then, for \(s' \leq q\) the inequality

\[(2.4)
\]

\[
\|\langle \mathbf{b}, T^P_{\Omega} \rangle f \|_{L^q(E(x_0, r))} \lesssim \prod_{i=1}^m \| \mathbf{b} \|_{L^{\infty}(x_0, r)} \quad r^{\frac{q}{2}} \int_0^\infty \left(1 + \ln \frac{t}{r} \right)^m \frac{\|f\|_{L^p(E(x_0,t))}}{t^{\frac{q}{2} - \sum_{i=1}^m \lambda_i} + 1} dt
\]

holds for any ellipsoid \(E(x_0, r)\) and for all \(f \in L^p_{\text{loc}}(\mathbb{R}^n)\). Also, for \(p < s\) the inequality

\[
\|\langle \mathbf{b}, T^P_{\Omega} \rangle f \|_{L^q(E(x_0, r))} \lesssim \prod_{i=1}^m \| \mathbf{b} \|_{L^{\infty}(x_0, r)} \quad r^{\frac{q}{2} - \frac{s}{p}} \int_0^\infty \left(1 + \ln \frac{t}{r} \right)^m \frac{\|f\|_{L^p(E(x_0,t))}}{t^{\frac{q}{2} - \frac{s}{p} - \sum_{i=1}^m \lambda_i} + 1} dt
\]

holds for any ellipsoid \(E(x_0, r)\) and for all \(f \in L^p_{\text{loc}}(\mathbb{R}^n)\).

**Proof.** Without loss of generality, it is sufficient to show that the conclusion holds for \(\langle \mathbf{b}, T^P_{\Omega} \rangle f = [(b_1, b_2), T^P_{\Omega}] f\). Let \(1 < q, p_i < \infty\) with \(\frac{1}{q} = \sum_{i=1}^m \frac{1}{p_i} + \frac{1}{p}\) and \(\mathbf{b} \in L^{\infty}(x_0, \lambda_i, p)_{\text{loc}}(\mathbb{R}^n)\) for \(0 \leq \lambda_i < \frac{1}{q}, i = 1, \ldots, m\). Set \(E = E(x_0, r)\) for the parabolic ball (ellipsoid) centered at \(x_0\) and of radius \(r\) and \(2kE = E(x_0, 2kr)\). We represent \(f\) as

\[(2.5)
\]

\[
f = f_1 + f_2, \quad f_1(y) = f(y) \chi_{2kE}(y), \quad f_2(y) = f(y) \chi_{(2kE)^c}(y), \quad r > 0
\]

and thus have

\[
\|[(b_1, b_2), T^P_{\Omega}] f_1\|_{L^q(E)} \leq \|[(b_1, b_2), T^P_{\Omega}] f_1\|_{L^q(E)} + \|[(b_1, b_2), T^P_{\Omega}] f_2\|_{L^q(E)} =: F + G.
\]

Let us estimate \(F + G\), respectively.

For \([(b_1, b_2), T^P_{\Omega}] f_1\) (\(x\)), we have the following decomposition,

\[
[(b_1, b_2), T^P_{\Omega}] f_1(x) = (b_1(x) - (b_1)_E)(b_2(x) - (b_2)_E) T^P_{\Omega} f_1(x)
\]

\[
- (b_1(\cdot) - (b_1)_E) T^P_{\Omega} ((b_2(\cdot) - (b_2)_E) f_1)(x)
\]

\[
+ (b_2(x) - (b_2)_E) T^P_{\Omega} ((b_1(x) - (b_1)_E) f_1)(x)
\]

\[
- T^P_{\Omega} ((b_1(\cdot) - (b_1)_E)(b_2(\cdot) - (b_2)_E) f_1)(x).
\]

Hence, we get

\[F = \|[(b_1, b_2), T^P_{\Omega}] f_1\|_{L^q(E)} \lesssim
\]

\[
\|((b_1 - (b_1)_E)(b_2(x) - (b_2)_E) T^P_{\Omega} f_1)\|_{L^q(E)}
\]

\[
+ \|((b_1 - (b_1)_E) T^P_{\Omega} ((b_2 - (b_2)_E) f_1))\|_{L^q(E)}
\]

\[
+ \|((b_2 - (b_2)_E) T^P_{\Omega} ((b_1 - (b_1)_E) f_1))\|_{L^q(E)}
\]

\[
+ \|T^P_{\Omega} ((b_1 - (b_1)_E)(b_2 - (b_2)_E) f_1))\|_{L^q(E)}
\]

\[= F_1 + F_2 + F_3 + F_4.
\]

One observes that the estimate of \(F_2\) is analogous to that of \(F_3\). Thus, we will only estimate \(F_1, F_2\) and \(F_4\).
To estimate $F_1$, let $1 < q, \tau < \infty$, such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{p_1}$. Then, using Hölder’s inequality and by the boundedness of $T_{D1}^p$ on $L_p$ (see Theorem 1.2. in [1]) it follows that:

$$F_1 = \| (b_1 - (b_1)_E) (b_2 (x) - (b_2)_E) T_{D1}^p f_1 \|_{L_q(E)}$$

$$\lesssim \| (b_1 - (b_1)_E) (b_2 (x) - (b_2)_E) \|_{L_{p_1}(E)} \| T_{D1}^p f_1 \|_{L_p(E)}$$

$$\lesssim \| b_1 - (b_1)_E \|_{L_{p_1}(E)} \| b_2 - (b_2)_E \|_{L_{p_2}(E)} \| f \|_{L_p(2kE)}$$

$$\lesssim \| b_1 - (b_1)_E \|_{L_{p_1}(E)} \| b_2 - (b_2)_E \|_{L_{p_2}(E)} \int_0^\infty \| f \|_{L_p(E(x_0,t))} \frac{dt}{t^{\frac{2}{p} + 1}}.$$  

From Lemma [1] it is easy to see that

$$\| b_1 - (b_1)_E \|_{L_{p_1}(E)} \leq C \| b_1 \|_{LC_{\gamma_1}} \| b_2 \|_{LC_{\gamma_2}} \| f \|_{L_{\gamma}}$$

and

$$\| b_1 - (b_1)_E \|_{L_{p_1}(2kE)} \leq \| b_1 - (b_1)_E \|_{L_{p_1}(E)} + \| (b_1)_E \|_{L_{p_1}(2kE)}$$

for $i = 1, 2$. Hence, by (2.7) we get

$$F_1 \lesssim \| b_1 \|_{LC_{\gamma_1}} \| b_2 \|_{LC_{\gamma_2}} \int_0^\infty \left( 1 + \ln \frac{t}{r} \right)^2 \frac{1}{t^{\frac{2}{p} + 1}} \| f \|_{L_{p}(E(x_0,t))} \frac{dt}{t^{\frac{2}{p} + 1}}.$$  

To estimate $F_2$, let $1 < \tau < \infty$, such that $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, similar to the estimates for $F_1$, we have

$$F_2 \lesssim \| b_1 - (b_1)_E \|_{L_{p_1}(E)} \| T_{D1}^p ((b_2 (\cdot) - (b_2)_E) f_1) \|_{L_\tau(E)}$$

$$\lesssim \| b_1 - (b_1)_E \|_{L_{p_1}(E)} \| (b_2 (\cdot) - (b_2)_E) f_1 \|_{L_{p_2}(E)}$$

$$\lesssim \| b_1 - (b_1)_E \|_{L_{p_1}(E)} \| b_2 - (b_2)_E \|_{L_{p_2}(2kE)} \| f \|_{L_p(2kE)}$$

where $1 < k < \infty$, such that $\frac{1}{k} = \frac{1}{p_2} + \frac{1}{p} = \frac{1}{p}. \frac{1}{\tau}$. By (2.7) and (2.8), we get

$$F_2 \lesssim \| b_1 \|_{LC_{\gamma_1}} \| b_2 \|_{LC_{\gamma_2}} \int_0^\infty \left( 1 + \ln \frac{t}{r} \right)^2 \frac{1}{t^{\frac{2}{p} + 1}} \| f \|_{L_\tau(E(x_0,t))} \frac{dt}{t^{\frac{2}{p} + 1}}.$$  

In a similar way, $F_3$ has the same estimate as above, so we omit the details. Then we have that

$$F_3 \lesssim \| b_1 \|_{LC_{\gamma_1}} \| b_2 \|_{LC_{\gamma_2}} \int_0^\infty \left( 1 + \ln \frac{t}{r} \right)^2 \frac{1}{t^{\frac{2}{p} + 1}} \| f \|_{L_\tau(E(x_0,t))} \frac{dt}{t^{\frac{2}{p} + 1}}.$$  

Now let us consider the term $F_4$. Let $1 < q, \tau < \infty$, such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{p_1}$, $\frac{1}{\tau} = \frac{1}{p_1} + \frac{1}{p_2}$. Then by the boundedness of $T_{D1}^p$ on $L_p$ (see Theorem 1.2. in [1]),
Hölder’s inequality and (2.8), we obtain
\[ F_4 = \left\| T_{\Omega}^p \left( (b_1 - (b_1)_E) (b_2 - (b_2)_E) f_1 \right) \right\|_{L^q(E)} \]
\[ \lesssim \left\| (b_1 - (b_1)_E) (b_2 - (b_2)_E) f_1 \right\|_{L^q(E)} \]
\[ \lesssim \left\| (b_1 - (b_1)_E) (b_2 - (b_2)_E) f_1 \right\|_{L^r(E)} \]
\[ \lesssim \left\| b_1 - (b_1)_E \right\|_{L^p_1(2kE)} \| b_2 - (b_2)_E \right\|_{L^p_2(2kE)} \| f \|_{L^p_2(2kE)} \]
\[ \lesssim \left\| b_1 \right\|_{L^{C(x_0)}} \| b_2 \|_{L^{C(x_0)}} \frac{2}{7} \int_{2kr}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\| f \|_{L^p(E(x_0,t))}}{t^{\gamma(\lambda_1 + \lambda_2) + 1}} dt. \]

Combining all the estimates of \( F_1, F_2, F_3, F_4 \); we get
\[ F = \left\| (b_1, b_2), T_{\Omega}^p f_1, f_2 \right\|_{L^q(E)} \lesssim \left\| b_1 \right\|_{L^{C(x_0)}} \| b_2 \|_{L^{C(x_0)}} \frac{2}{7} \int_{2kr}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\| f \|_{L^p(E(x_0,t))}}{t^{\gamma(\lambda_1 + \lambda_2) + 1}} dt. \]

Now, let us estimate \( G = \left\| (b_1, b_2), T_{\Omega}^p f_2 \right\|_{L^q(E)}. \) For \( G \), it’s similar to (2.6) we also write
\[ G = \left\| (b_1, b_2), T_{\Omega}^p f_2 \right\|_{L^q(E)} \lesssim \left\| (b_1 - (b_1)_E) (b_2 - (b_2)_E) T_{\Omega}^p f_2 \right\|_{L^q(E)} \]
\[ + \left\| (b_1 - (b_1)_E) (b_2 - (b_2)_E) f_2 \right\|_{L^q(E)} \]
\[ + \left\| (b_2 - (b_2)_E) T_{\Omega}^p ((b_1 - (b_1)_E) f_2) \right\|_{L^q(E)} \]
\[ + \left\| T_{\Omega}^p ((b_1 - (b_1)_E) (b_2 - (b_2)_E) f_2) \right\|_{L^q(E)} \]
\[ \equiv G_1 + G_2 + G_3 + G_4. \]

To estimate \( G_1 \), let \( 1 < q, r < \infty \), such that \( \frac{1}{q} = \frac{1}{r} + \frac{1}{p}, \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} \). Then, using Hölder’s inequality and by (2.4), we have
\[ G_1 = \left\| (b_1 - (b_1)_E) (b_2 - (b_2)_E) T_{\Omega}^p f_2 \right\|_{L^q(E)} \]
\[ \lesssim \left\| (b_1 - (b_1)_E) (b_2 - (b_2)_E) \right\|_{L^r(E)} \| T_{\Omega}^p f_2 \|_{L^p(E)} \]
\[ \lesssim \left\| b_1 - (b_1)_E \right\|_{L^p_1(E)} \| b_2 - (b_2)_E \|_{L^p_2(E)} \frac{2}{7} \int_{2kr}^{\infty} \| f \|_{L^p(E(x_0,t))} t^{-\frac{\gamma}{p} - 1} dt \]
\[ \lesssim \left\| b_1 \right\|_{L^{C(x_0)}} \| b_2 \|_{L^{C(x_0)}} \frac{2}{7} \int_{2kr}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\| f \|_{L^p(E(x_0,t))}}{t^{\gamma(\lambda_1 + \lambda_2) + 1}} dt, \]

where in the second inequality we have used the following fact:

It is clear that \( x \in E, y \in (2kE)^C \) implies
\[ \frac{1}{2k} \rho (x_0 - y) \leq \rho (x - y) \leq \frac{3k}{2} \rho (x_0 - y). \]
Hence, we get
\[ |T_{11}^P f_2 (x)| \leq 2^\gamma c_1 \int_{(2kE)^C} \frac{|f (y)| |\Omega (x-y)|}{\rho (x_0 - y)^\gamma} dy. \]

By Fubini’s theorem, we have
\[
\int_{(2kE)^C} \frac{|f (y)| |\Omega (x-y)|}{\rho (x_0 - y)^\gamma} dy \approx \int_{(2kE)^C} \frac{|f (y)| |\Omega (x-y)|}{\rho (x_0 - y)^\gamma} dt \frac{dy}{\gamma+1},
\]
\[
\approx \int_{2kr}^{\infty} \int_{2kr \leq \rho (x_0 - y) \leq t} |f (y)| |\Omega (x-y)| dy \frac{dt}{\gamma+1}.
\]

Applying Hölder’s inequality, we get
\[
\int_{(2kE)^C} \frac{|f (y)| |\Omega (x-y)|}{\rho (x_0 - y)^\gamma} dy \lesssim \int_{2kr}^{\infty} \|f \|_{L_p (E(x_0, t))} \|\Omega (x-\cdot)\|_{L_s (E(x_0, t))} |E (x_0, t)|^{1 - \frac{\gamma}{p} - \frac{\gamma}{s}} \frac{dt}{\gamma+1}.
\]

For \( x \in E (x_0, t) \), notice that \( \Omega \) is \( A_t \)-homogenous of degree zero and \( \Omega \in L_s (S^{n-1}), s > 1 \). Then, we obtain
\[
\left( \int_{E (x_0, t)} |\Omega (x-y)|^s dy \right)^{\frac{1}{s}} \lesssim \left( \int_{E (x_0, t)} |\Omega (z)|^s dz \right)^{\frac{1}{s}} \lesssim \left( \int_{E (0, t)} |\Omega (z)|^s dz \right)^{\frac{1}{s}} \lesssim \left( \int_{E (0, 2t)} |\Omega (z)|^s dz \right)^{\frac{1}{s}} = \left( \int_{S^{n-1}}^{2t} |\Omega (z')|^s d\sigma (z') r^{n-1} dr \right)^{\frac{1}{s}} = C \|\Omega\|_{L_s (S^{n-1})} |E (x_0, 2t)|^{\frac{s}{p}}.
\]

Thus, by (2.11), it follows that:
\[
|T_{11}^P f_2 (x)| \lesssim \int_{2kr}^{\infty} \|f \|_{L_p (E(x_0, t))} \frac{dt}{\gamma+1}.
\]
Moreover, for all $p \in [1, \infty)$ the inequality

$$\| T^p_{b_2} f_2 \|_{L^p(E)} \lesssim \frac{1}{2k r} \int_{kr}^{\infty} \| f \|_{L^p(E(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}$$

is valid.

On the other hand, for the estimates used in $G_2, G_3$, we have to prove the below inequality:

(2.12)

$$| T^p_{b_2} ((b_2 - (b_2)_E) f_2) (x) | \lesssim \| b_2 \|_{L^p_{\rho_2, \lambda_2, p}} \int_{kr}^{\infty} \left( 1 + \ln \frac{t}{r} \right) t^{-\frac{n}{p} + \gamma \lambda_2 - 1} \| f \|_{L^p(E(x_0,t))} dt.$$

Indeed, when $s' \leq q$, for $x \in E$, by Fubini’s theorem and applying Hölder’s inequality and from (2.22), (2.23), (2.41) we have

$$\int | T^p_{b_2} ((b_2 - (b_2)_E) f_2) (x) | \lesssim \int \| b_2 \|_{L^p_{\rho_2, \lambda_2, p}} | \Omega (x-y) | \| f (y) \|_{p(x-y)^{\gamma}} dy$$

$$\lesssim \int \| b_2 \|_{L^p_{\rho_2, \lambda_2, p}} | \Omega (x-y) | \| f (y) \|_{p(x-y)^{\gamma}} dy$$

$$\lesssim \int \int \left| b_2 (y) - (b_2)_E \right| | \Omega (x-y) | | f (y) | dy dt$$

$$\lesssim \int \int \left| b_2 (y) - (b_2)_E \right| | \Omega (x-y) | | f (y) | dy dt$$

$$\lesssim \int \int \left| b_2 (y) - (b_2)_E \right| | \Omega (x-y) | | f (y) | dy dt$$

$$\lesssim \int \int \left| b_2 (y) - (b_2)_E \right| | \Omega (x-y) | | f (y) | dy dt$$

This completes the proof of inequality (2.12).
Let \( 1 < \tau < \infty \), such that \( \frac{1}{q} = \frac{1}{p_1} + \frac{1}{\tau} \). Then, using Hölder’s inequality and from (2.12) and (2.3), we get
\[
G_2 = \| (b_1 - (b_1)_E) T_\Omega^p \|_{L^q(E)} \| (b_2 - (b_2)_E) f_2 \|_{L^q(E)} \\
\leq \| b_1 - (b_1)_E \|_{L^{p_1}(E)} \| T_\Omega^p \|_{L^{\infty}(E)} \| (b_2 - (b_2)_E) f_2 \|_{L^q(E)} \\
\leq \| b_1 \|_{L^{C_{\gamma_0}}(\rho_1, \lambda_1, p)} \| b_2 \|_{L^{C_{\gamma_0}}(\rho_2, \lambda_2, p)} \frac{r^2}{2kr} \int \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\| f \|_{L^p(E(x_0,t))}}{t^{\gamma(\lambda_1 + \lambda_2) + 1}} \, dt.
\]
Similarly, \( G_3 \) has the same estimate above, so here we omit the details. Then the inequality
\[
G_3 = \| (b_2 - (b_2)_E) T_\Omega^p \|_{L^q(E)} \| (b_1 - (b_1)_E) f_2 \|_{L^q(E)} \\
\leq \| b_1 \|_{L^{C_{\gamma_0}}(\rho_1, \lambda_1, p)} \| b_2 \|_{L^{C_{\gamma_0}}(\rho_2, \lambda_2, p)} \frac{r^2}{2kr} \int \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\| f \|_{L^p(E(x_0,t))}}{t^{\gamma(\lambda_1 + \lambda_2) + 1}} \, dt
\]
is valid.

Now, let us estimate \( G_4 = \| T_\Omega^p \|_{L^q(E)} \). It’s similar to the estimate of (2.12), for any \( x \in E \), we also write
\[
\| T_\Omega^p \|_{L^q(E)} = \| (b_1 - (b_1)_E) \|_{L^{p_1}(E)} \| (b_2 - (b_2)_E) f_2 \|_{L^q(E)} \\
\leq \int \int \int_{2krE(x_0,t)} \left| b_1 (y) - (b_1)_E(x_0,t) \right| \left| b_2 (y) - (b_2)_E(x_0,t) \right| |\Omega(x-y)| |f(y)| dy \frac{dt}{t^{\gamma(\lambda_1 + \lambda_2) + 1}} \\
+ \int \int \int_{2krE(x_0,t)} \left| (b_1)_E(x_0,t) \right| \left| (b_2)_E(x_0,t) - (b_2)_E(x_0,r) \right| |\Omega(x-y)| |f(y)| dy \frac{dt}{t^{\gamma(\lambda_1 + \lambda_2) + 1}} \\
+ \int \int \int_{2krE(x_0,t)} \left| (b_1)_E(x_0,t) - (b_2)_E(x_0,t) \right| \left| b_2 (y) - (b_2)_E(x_0,t) \right| |\Omega(x-y)| |f(y)| dy \frac{dt}{t^{\gamma(\lambda_1 + \lambda_2) + 1}} \\
+ \int \int \int_{2krE(x_0,t)} \left| (b_1)_E(x_0,t) - (b_2)_E(x_0,r) \right| \left| (b_2)_E(x_0,t) - (b_2)_E(x_0,r) \right| |\Omega(x-y)| |f(y)| dy \frac{dt}{t^{\gamma(\lambda_1 + \lambda_2) + 1}} \\
= G_{41} + G_{42} + G_{43} + G_{44}.
\]
Let us estimate \( G_{41}, G_{42}, G_{43}, G_{44} \), respectively.

Firstly, to estimate \( G_{41} \), similar to the estimate of (2.12), we get
\[
G_{41} \leq \| b_1 \|_{L^{C_{\gamma_0}}(\rho_1, \lambda_1, p)} \| b_2 \|_{L^{C_{\gamma_0}}(\rho_2, \lambda_2, p)} \frac{r^2}{2kr} \int \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\| f \|_{L^p(E(x_0,t))}}{t^{\gamma(\lambda_1 + \lambda_2) + 1}} \, dt.
\]
Secondly, to estimate \( G_{42} \) and \( G_{43} \), from (2.12), (2.22) and (2.3), it follows that
\[
G_{42} \leq \| b_1 \|_{L^{C_{\gamma_0}}(\rho_1, \lambda_1, p)} \| b_2 \|_{L^{C_{\gamma_0}}(\rho_2, \lambda_2, p)} \frac{r^2}{2kr} \int \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\| f \|_{L^p(E(x_0,t))}}{t^{\gamma(\lambda_1 + \lambda_2) + 1}} \, dt,
\]
and
\[
G_{43} \leq \| b_1 \|_{L^{C_{\gamma_0}}(\rho_1, \lambda_1, p)} \| b_2 \|_{L^{C_{\gamma_0}}(\rho_2, \lambda_2, p)} \frac{r^2}{2kr} \int \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\| f \|_{L^p(E(x_0,t))}}{t^{\gamma(\lambda_1 + \lambda_2) + 1}} \, dt.
\]
Finally, to estimate $G_{44}$, similar to the estimate of (2.12) and from (2.2) and (2.3), we have

$$G_{44} \lesssim \|b_1\|_{L^C(x_0)} \|b_2\|_{L^C(x_0)} \int_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \frac{\|f\|_{L^p(E(x_0,t))}}{t^\gamma(\lambda_1 + \lambda_2) + 1} \, dt.$$

By the estimates of $G_{4j}$ above, where $j = 1, 2, 3$, we know that

$$|T^{F}_{\Omega}(f_1)(b_1 - (b_1)_E)(b_2 - (b_2)_E)(x)| \lesssim \|b_1\|_{L^C(x_0)} \|b_2\|_{L^C(x_0)} \int_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \frac{\|f\|_{L^p(E(x_0,t))}}{t^\gamma(\lambda_1 + \lambda_2) + 1} \, dt.$$

Then, we have

$$G_4 = \|T^{F}_{\Omega}(f_1)(b_1 - (b_1)_E)(b_2 - (b_2)_E)(x)| \lesssim \|b_1\|_{L^C(x_0)} \|b_2\|_{L^C(x_0)} \int_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \frac{\|f\|_{L^p(E(x_0,t))}}{t^\gamma(\lambda_1 + \lambda_2) + 1} \, dt.$$

So, combining all the estimates for $G_1, G_2, G_3, G_4$, we get

$$G = \|(b_1, b_2), T^{F}_{\Omega}(f_1)\|_{L^q(E)} \lesssim \|b_1\|_{L^C(x_0)} \|b_2\|_{L^C(x_0)} \int_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \frac{\|f\|_{L^p(E(x_0,t))}}{t^\gamma(\lambda_1 + \lambda_2) + 1} \, dt.$$

Thus, putting estimates $F$ and $G$ together, we get the desired conclusion

$$\|b_1, b_2, T^{F}_{\Omega}(f_1)\|_{L^q(E)} \lesssim \|b_1\|_{L^C(x_0)} \|b_2\|_{L^C(x_0)} \int_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \frac{\|f\|_{L^p(E(x_0,t))}}{t^\gamma(\lambda_1 + \lambda_2) + 1} \, dt.$$

For the case of $p < s$, we can also use the same method, so we omit the details. This completes the proof of Lemma 2.

**Lemma 3.** Suppose that $x_0 \in \mathbb{R}^n$, $\Omega \in L_\alpha(S^{n-1})$, $1 < s < \infty$, is $A_\alpha$-homogeneous of degree zero. Let $T^{F}_{\Omega,\alpha}$ be a parabolic linear operator satisfying condition (1.3).

Let also $0 < \alpha < \gamma$ and $1 < q_1, q_1, p, q_1 > 0$ with $\frac{1}{r} = m \sum_{i=1}^{m} \frac{1}{p_i} + \frac{1}{q_1} = \frac{1}{q} - \frac{\alpha}{\gamma}$ and $\vec{b} \in L^C_{p_1, \lambda_1, p_1}(\mathbb{R}^n)$ for $0 \leq \lambda_i < \frac{\alpha}{\gamma}$, $i = 1, \ldots, m$.

Then, for $s' \leq q$ the inequality

$$(2.13)$$

$$\|\vec{b}, T^{F}_{\Omega,\alpha}(f)\|_{L^q(E(x_0,r))} \lesssim \prod_{i=1}^{m} \|\vec{b}\|_{L^C(x_0)} \int_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right)^m \frac{\|f\|_{L^p(E(x_0,t))}}{t^\gamma(\sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \frac{1}{p_i}) + 1} \, dt$$

holds for any ellipsoid $E(x_0, r)$ and for all $f \in L^p_{loc}(\mathbb{R}^n)$. Also, for $q_1 < s$ the inequality

$$(2.13)$$

$$\|\vec{b}, T^{F}_{\Omega,\alpha}(f)\|_{L^q(E(x_0,r))} \lesssim \prod_{i=1}^{m} \|\vec{b}\|_{L^C(x_0)} \int_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right)^m \frac{\|f\|_{L^p(E(x_0,t))}}{t^\gamma(\sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \frac{1}{p_i}) + 1} \, dt$$

holds for any ellipsoid $E(x_0, r)$ and for all $f \in L^p_{loc}(\mathbb{R}^n)$. 

Proof. Similar to the proof of Lemma 2, it is sufficient to show that the conclusion holds for \( m = 2 \). Let \( 0 < \alpha < \gamma \) and \( 1 < q, q_1, p, p < \frac{\gamma}{\alpha} \) with \( \frac{1}{q} = \sum_{i=1}^{m} \frac{1}{p_i} + \frac{1}{p} \), \( \frac{1}{q_1} = \frac{1}{q} - \frac{\alpha}{\gamma} \), and \( \overrightarrow{b} \in LC_{P,\lambda_i,\gamma}(\mathbb{R}^n) \) for \( 0 \leq \lambda_i < \frac{1}{\gamma} \), \( i = 1, \ldots, m \). As in the proof of Lemma 2 we split \( f = f_1 + f_2 \) in form (2.5) and have

\[
\|[(b_1, b_2), T_{\Omega, \alpha}^P] f_1\|_{L_{q_1}(E)} \leq \|[(b_1, b_2), T_{\Omega, \alpha}^P] f_1\|_{L_{q_1}(E)} + \|[(b_1, b_2), T_{\Omega, \alpha}^P] f_2\|_{L_{q_1}(E)} =: A + B.
\]

Let us estimate \( A + B \), respectively.

For \( [(b_1, b_2), T_{\Omega, \alpha}^P] f_1 \), we have the following decomposition,

\[
[(b_1, b_2), T_{\Omega, \alpha}^P] f_1(x) = (b_1(x) - (b_1)_E)(b_2(x) - (b_2)_E) T_{\Omega, \alpha}^P f_1(x) - (b_1(\cdot) - (b_1)_E) T_{\Omega, \alpha}^P ((b_2(\cdot) - (b_2)_E) f_1)(x) + (b_2(x) - (b_2)_E) T_{\Omega, \alpha}^P ((b_1(x) - (b_1)_E) f_1)(x) - T_{\Omega, \alpha}^P ((b_1(\cdot) - (b_1)_E)(b_2(\cdot) - (b_2)_E) f_1)(x).
\]

Hence, we get

\[
A = \|[(b_1, b_2), T_{\Omega, \alpha}^P] f_1\|_{L_{q_1}(E)} \lesssim \|[(b_1 - (b_1)_E)(b_2(x) - (b_2)_E) T_{\Omega, \alpha}^P f_1]\|_{L_{q_1}(E)} + \|[(b_1 - (b_1)_E) T_{\Omega, \alpha}^P ((b_2 - (b_2)_E) f_1)]\|_{L_{q_1}(E)} + \|[(b_2 - (b_2)_E) T_{\Omega, \alpha}^P ((b_1 - (b_1)_E) f_1)]\|_{L_{q_1}(E)} + \|T_{\Omega, \alpha}^P ((b_1 - (b_1)_E)(b_2 - (b_2)_E) f_1)\|_{L_{q_1}(E)}
\]

(2.14) \equiv A_1 + A_2 + A_3 + A_4.

One observes that the estimate of \( A_2 \) is analogous to that of \( A_3 \). Thus, we will only estimate \( A_1, A_2 \) and \( A_4 \).

To estimate \( A_1 \), let \( 1 < \frac{\gamma}{2}, \frac{\gamma}{p} < \infty \), such that \( \frac{1}{q} = \frac{1}{2} - \frac{\gamma}{2p}, \frac{1}{q_1} = \frac{1}{2} + \frac{1}{p_1} + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p} \). Then, using Hölder’s inequality and by the boundedness of \( T_{\Omega, \alpha}^P \) from \( L_p \) into \( L_{q_1} \) (see Theorem 0.1 in [8]) and by (2.7) it follows that:

\[
A_1 = \|[(b_1 - (b_1)_E)(b_2(x) - (b_2)_E) T_{\Omega, \alpha}^P f_1]\|_{L_{q_1}(E)} \lesssim \|b_1 - (b_1)_E\|_{L_{p_1}(E)} \|b_2 - (b_2)_E\|_{L_{p_2}(E)} \|f\|_{L_{\gamma}(E)} \int_{2kr}^{\infty} \frac{dt}{t^{\frac{\gamma}{p} + 1 - \alpha}}
\]

\[
\lesssim \|b_1\|_{LC_{p_1,\lambda_1,\gamma}} \|b_2\|_{LC_{p_2,\lambda_2,\gamma}} \left(r^\gamma (\frac{1}{2} + \frac{1}{p_1} + \frac{1}{p} + \frac{1}{p_2}) \int_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right)^{2} t^{\frac{\gamma}{p} \gamma(\lambda_1 + \lambda_2) - 1 + \alpha} \|f\|_{L_p(E(x_0,t))} dt \right)
\]

\[
\lesssim \|b_1\|_{LC_{p_1,\lambda_1,\gamma}} \|b_2\|_{LC_{p_2,\lambda_2,\gamma}} \int_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right)^{2} \|f\|_{L_p(E(x_0,t))} dt.
\]
To estimate $A_2$, let $1 < \tau < \infty$, such that $\frac{1}{\tau} = \frac{1}{p_1} + \frac{1}{\tau}$. Then, similar to the estimates for $A_1$, we have

$$A_2 = \left\| \left( (b_1 - (b_1)_E) \right)T_{\Omega,\alpha}^P ((b_2 - (b_2)_E) f_1) \right\|_{L_{q_1}(E)}$$

$$\lesssim \left\| (b_1 - (b_1)_E) \right\|_{L_{p_1}(E)} \left\| T_{\Omega,\alpha}^P ((b_2 - (b_2)_E) f_1) \right\|_{L_\tau(E)}$$

$$\lesssim \left\| (b_1 - (b_1)_E) \right\|_{L_{p_1}(E)} \left\| (b_2 - (b_2)_E) f_1 \right\|_{L_k(E)}$$

$$\lesssim \left\| (b_1 - (b_1)_E) \right\|_{L_{p_1}(E)} \left\| (b_2 - (b_2)_E) \right\|_{L_{p_2}(2kE)} \left\| f \right\|_{L_p(2kE)},$$

where $1 < k < \frac{2\alpha}{\alpha}$, such that $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p_2}$. By (2.7) and (2.8), we get

$$A_2 \lesssim \left\| b_1 \right\|_{L_{p_1}(E)} \left\| b_2 \right\|_{L_{p_2}(E)} \frac{1}{2kr} \int_0^\infty \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\left\| f \right\|_{L_p(E(x_0,t))}}{t^{\frac{1}{p_1} - \gamma(1+\lambda_2) - \gamma(\frac{1}{p_1} + \frac{1}{p_2})^{-1}} + dt.$$

In a similar way, $A_3$ has the same estimate as above, so we omit the details. Then we have that

$$A_3 \lesssim \left\| b_1 \right\|_{L_{p_1}(E)} \left\| b_2 \right\|_{L_{p_2}(E)} \frac{1}{2kr} \int_0^\infty \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\left\| f \right\|_{L_p(E(x_0,t))}}{t^{\frac{1}{p_1} - \gamma(1+\lambda_2) - \gamma(\frac{1}{p_1} + \frac{1}{p_2})^{-1}} + dt.$$

Now let us consider the term $A_4$. Let $1 < q, \tau < \frac{2\alpha}{\alpha}$, such that $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, \( \frac{1}{\tau} = \frac{1}{p_1} + \frac{1}{p_2} \) and $\frac{1}{\tau} = \frac{1}{q} - \frac{1}{2}$. Then by the boundedness of $T_{\Omega,\alpha}^P$ from $L_q$ into $L_{q_1}$ (see Theorem 0.1 in [8]), Hölder’s inequality and (2.8), we obtain

$$A_4 = \left\| T_{\Omega,\alpha}^P \left( (b_1 - (b_1)_E) (b_2 - (b_2)_E) f_1 \right) \right\|_{L_{q_1}(E)}$$

$$\lesssim \left\| (b_1 - (b_1)_E) \right\|_{L_{p_1}(E)} \left\| (b_2 - (b_2)_E) \right\|_{L_{p_2}(E)} \left\| f_1 \right\|_{L_q(E)}$$

$$\lesssim \left\| (b_1 - (b_1)_E) \right\|_{L_{p_1}(E)} \left\| (b_2 - (b_2)_E) \right\|_{L_{p_2}(2kE)} \left\| f \right\|_{L_p(2kE)}$$

$$\lesssim \left\| b_1 \right\|_{L_{p_1}(E)} \left\| b_2 \right\|_{L_{p_2}(E)} \frac{1}{2kr} \int_0^\infty \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\left\| f \right\|_{L_p(E(x_0,t))}}{t^{\frac{1}{p_1} - \gamma(1+\lambda_2) - \gamma(\frac{1}{p_1} + \frac{1}{p_2})^{-1}} + dt.$$

Combining all the estimates of $A_1, A_2, A_3, A_4$, we get

$$A = \left\| (b_1, b_2), T_{\Omega,\alpha}^P f_1 \right\|_{L_{q_1}(E)} \lesssim \left\| b_1 \right\|_{L_{p_1}(E)} \left\| b_2 \right\|_{L_{p_2}(E)} \frac{1}{2kr} \int_0^\infty \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\left\| f \right\|_{L_p(E(x_0,t))}}{t^{\frac{1}{p_1} - \gamma(1+\lambda_2) - \gamma(\frac{1}{p_1} + \frac{1}{p_2})^{-1}} + dt.$$

Now, let us estimate $B = \left\| (b_1, b_2), T_{\Omega,\alpha}^P f_2 \right\|_{L_{q_1}(E)}$. For $B$, it’s similar to (2.7.4) we also write

$$B = \left\| (b_1, b_2), T_{\Omega,\alpha}^P f_2 \right\|_{L_{q_1}(E)} \lesssim$$

$$\left\| (b_1 - (b_1)_E) \right\|_{L_{p_1}(E)} \left\| (b_2 - (b_2)_E) \right\|_{L_{p_2}(E)} \left\| f_2 \right\|_{L_{q_1}(E)}$$

$$+ \left\| (b_1 - (b_1)_E) \right\|_{L_{p_1}(E)} \left\| (b_2 - (b_2)_E) \right\|_{L_{p_2}(E)} \left\| f_2 \right\|_{L_{q_1}(E)}$$

$$\equiv B_1 + B_2 + B_3 + B_4.$$
To estimate $B_1$, let $1 < p_1, p_2 < \frac{2}{\alpha}$, such that $\frac{1}{q_1} = \frac{1}{p} + \frac{1}{q_2}$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{\gamma}$. Then, using Hölder’s inequality and by (2.7), we have

$$B_1 = \left\| (b_1 - (b_1)_E) (b_2(x) - (b_2)_E) T_{\Omega}^{p_1, \alpha} f_2 \right\|_{L_q(E)}$$

$$\lesssim \left\| (b_1 - (b_1)_E) (b_2 - (b_2)_E) \right\|_{L_{p_1}(E)} \left\| T_{\Omega}^{p_1, \alpha} f_2 \right\|_{L_{q_1}(E)}$$

$$\lesssim \left\| b_1 - (b_1)_E \right\|_{L_{p_1}(E)} \left\| b_2 - (b_2)_E \right\|_{L_{p_2}(E)} \int_{2kr}^\infty \int \left( 1 + \ln \frac{t}{r} \right)^2 t^{-\gamma + \lambda_1 + \lambda_2 - 1 + \alpha} \frac{df}{L_p(E(y, t))} dt$$

where in the second inequality we have used the following fact:

When $s' \leq \gamma$, by (2.9), Fubini’s theorem, Hölder’s inequality and (2.11), we have

$$|T_{\Omega}^{p_1, \alpha} f_2 (x)| \leq c_0 \int_{(2kE)^c} |\Omega(x - y)| \frac{|f(y)|}{\rho(x_0 - y)} \gamma^{-\alpha} dy$$

$$\approx \int_{2kr}^\infty \int \left| \Omega(x - y) \right| \frac{|f(y)|}{\rho(x_0 - y)} \gamma^{-\alpha} dt$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$\left\| T_{\Omega}^{p_1, \alpha} f_2 \right\|_{L_p(E)} \lesssim \frac{dt}{t^{\frac{\gamma}{p} + 1}}$$

is valid.

On the other hand, for the estimates used in $B_2, B_3$, we have to prove the below inequality:

(2.15)

$$\left| T_{\Omega}^{p_1, \alpha} ((b_2 - (b_2)_E) f_2) (x) \right| \lesssim \| b_2 \|_{L_{p_2}(\alpha)} \int_{2kr}^\infty \left( 1 + \ln \frac{t}{r} \right)^{t^{-\gamma + \lambda_2 - 1 + \alpha} \frac{df}{L_p(E(y, t))} dt.$$
Indeed, when \( s' \leq q \), for \( x \in E \), by Fubini's theorem and applying Hölder's inequality and from (2.2), (2.3), (2.11) we have

\[
|T_{2}^{p}((b_2 (\cdot) - (b_2)_{E}) f_2)(x)| \lesssim \int_{(2kE)^{c}} |b_2 (y) - (b_2)_{E}||\Omega (x - y)| \frac{|f(y)|}{r(x-y)} dy
\]

\[
\lesssim \int_{(2kE)^{c}} |b_2 (y) - (b_2)_{E}||\Omega (x - y)| |f(y)| dy
\]

\[
\approx \int_{2k^r < r < \rho} \int_{(x_0,y) < t} \int_{E(x_0,t)} |b_2 (y) - (b_2)_{E}||\Omega (x - y)| |f(y)| dy \frac{dt}{t^{\frac{1}{\gamma} - \frac{1}{p} - \frac{1}{q}}} 
\]

\[
\lesssim \int_{2k^r} \int_{(x_0,y) < t} \int_{E(x_0,t)} |b_2 (y) - (b_2)_{E}||\Omega (x - y)| |f(y)| dy \frac{dt}{t^{\frac{1}{\gamma} - \frac{1}{p} - \frac{1}{q}}} 
\]

\[
\lesssim \int_{2k^r} \left\|b_2 (\cdot) - (b_2)_{E}(x_0,t)\right\|_{L_{p_2}(E(x_0,t))} \left\|\Omega (\cdot) - (\Omega)_{E}(x_0,t)\right\|_{L_{\gamma}(E(x_0,t))} \left\|f\right\|_{L_{\gamma}(E(x_0,t))} |E (x_0, t)|^{1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{\gamma}} \frac{dt}{t^{\frac{1}{\gamma} - \frac{1}{p} - \frac{1}{q}}} 
\]

\[
\lesssim \int_{2k^r} \left\|b_2 (\cdot) - (b_2)_{E}(x_0,t)\right\|_{L_{p_2}(E(x_0,t))} \left\|f\right\|_{L_{\gamma}(E(x_0,t))} t^{-1 - \frac{1}{p} + \frac{1}{\gamma}} dt 
\]

\[
\lesssim \int_{2k^r} \left\|b_2 (\cdot) - (b_2)_{E}(x_0,t)\right\|_{L_{p_2}(E(x_0,t))} \left\|f\right\|_{L_{\gamma}(E(x_0,t))} t^{-1 - \frac{1}{p} + \frac{1}{\gamma}} dt 
\]

This completes the proof of inequality (2.15).

Let \( 1 < r < \infty \), such that \( \frac{1}{q_1} = \frac{1}{p_1} + \frac{1}{r} \) and \( \frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p} - \frac{1}{q} \). Then, using Hölder’s inequality and from (2.15) and (2.3), we get

\[
B_2 = \left\|b_1 - (b_1)_{E}\right\|_{L_{q_1}(E)} T_{p_1}^{p_2}((b_2 - (b_2)_{E}) f_2)_{L_{q_1}(E)} \lesssim \left\|b_1 - (b_1)_{E}\right\|_{L_{p_1}(E)} \left\|T_{p_1}^{p_2}((b_2 (\cdot) - (b_2)_{E}) f_2)\right\|_{L_{r}(E)} 
\]

\[
\lesssim \left\|b_1 - (b_1)_{E}\right\|_{L_{p_1}(E)} \left\|b_2\right\|_{L_{C}^{(x_0)}_{p_2,\lambda_2,p}} \left\|f\right\|_{L_{r}(E(x_0,t))} t^{-\frac{1}{q_1} + \frac{1}{\gamma}} + \frac{1}{\gamma} \alpha dt 
\]

\[
\lesssim \left\|b_1\right\|_{L_{C}^{(x_0)}_{p_1,\lambda_1,p}} \left\|b_2\right\|_{L_{C}^{(x_0)}_{p_2,\lambda_2,p}} \left\|f\right\|_{L_{r}(E(x_0,t))} t^{-\frac{1}{r}} \frac{\gamma}{\gamma - \lambda_2 + \frac{1}{p} - \frac{1}{q}} dt. 
\]
Similarly, $B_3$ has the same estimate above, so here we omit the details. Then the inequality

$$B_3 = \left\| (b_2 - (b_2)_E) T_{T_{1}, \alpha}^p ((b_1 - (b_1)_E) f_2) \right\|_{L_{q_1}(E)}$$

$$\lesssim \| b_1 \|_{L_C(x_0)} \| b_2 \|_{L_C(x_0)} \int_{2kr}^\infty \left( 1 + \ln \frac{r}{t} \right)^2 \frac{\| f \|_{L_p(E(x_0, t))}}{t^{\frac{2}{n} - \gamma(\lambda_1 + \lambda_2) - \gamma}\left(\frac{1}{2p_1} + \frac{1}{2p_2}\right) + 1} \, dt$$

is valid.

Now, let us estimate $B_4 = \| T_{T_{1}, \alpha}^p ((b_1 - (b_1)_E) (b_2 - (b_2)_E) f_2) \|_{L_{q_1}(E)}$. It's similar to the estimate of (2.15), for any $x \in E$, we also write

$$\| T_{T_{1}, \alpha}^p ((b_1 - (b_1)_E) (b_2 - (b_2)_E) f_2) (x) \|_{L_{q_1}(E)}$$

$$\lesssim \int_{2krE(x_0, t)}^\infty \int b_1(y) - (b_1)_E(x_0, t) \left| b_2(y) - (b_2)_E(x_0, t) \right| \| \Omega(x - y) \| \| f(y) \| \, dy \, \frac{dt}{r^{\gamma}}$$

$$+ \int_{2krE(x_0, t)}^\infty \int b_1(y) - (b_1)_E(x_0, t) \left| (b_2)_E(x_0, t) - (b_2)_E(x_0, r) \right| \| \Omega(x - y) \| \| f(y) \| \, dy \, \frac{dt}{r^{\gamma}}$$

$$+ \int_{2krE(x_0, t)}^\infty \int (b_1)_E(x_0, t) - (b_2)_E(x_0, r) \left| b_2(y) - (b_2)_E(x_0, t) \right| \| \Omega(x - y) \| \| f(y) \| \, dy \, \frac{dt}{r^{\gamma}}$$

$$+ \int_{2krE(x_0, t)}^\infty \int (b_1)_E(x_0, t) - (b_2)_E(x_0, r) \left| (b_2)_E(x_0, t) - (b_2)_E(x_0, r) \right| \| \Omega(x - y) \| \| f(y) \| \, dy \, \frac{dt}{r^{\gamma}}$$

$$\equiv B_{41} + B_{42} + B_{43} + B_{44}.$$

Let us estimate $B_{41}$, $B_{42}$, $B_{43}$, and $B_{44}$, respectively.

Firstly, to estimate $B_{41}$, similar to the estimate of (2.15), we get

$$B_{41} \lesssim \| b_1 \|_{L_C(x_0)} \| b_2 \|_{L_C(x_0)} \int_{2kr}^\infty \left( 1 + \ln \frac{r}{t} \right)^2 \frac{\| f \|_{L_p(E(x_0, t))}}{t^{\frac{2}{n} - \gamma(\lambda_1 + \lambda_2) - \gamma}\left(\frac{1}{2p_1} + \frac{1}{2p_2}\right) + 1} \, dt.$$

Secondly, to estimate $B_{42}$ and $B_{43}$, from (2.15), (2.2), and (2.3), it follows that

$$B_{42} \lesssim \| b_1 \|_{L_C(x_0)} \| b_2 \|_{L_C(x_0)} \int_{2kr}^\infty \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\| f \|_{L_p(E(x_0, t))}}{t^{\frac{2}{n} - \gamma(\lambda_1 + \lambda_2) - \gamma}\left(\frac{1}{2p_1} + \frac{1}{2p_2}\right) + 1} \, dt,$$

and

$$B_{43} \lesssim \| b_1 \|_{L_C(x_0)} \| b_2 \|_{L_C(x_0)} \int_{2kr}^\infty \left( 1 + \ln \frac{r}{t} \right)^2 \frac{\| f \|_{L_p(E(x_0, t))}}{t^{\frac{2}{n} - \gamma(\lambda_1 + \lambda_2) - \gamma}\left(\frac{1}{2p_1} + \frac{1}{2p_2}\right) + 1} \, dt.$$

Finally, to estimate $B_{44}$, similar to the estimate of (2.15) and from (2.2) and (2.3), we have

$$B_{44} \lesssim \| b_1 \|_{L_C(x_0)} \| b_2 \|_{L_C(x_0)} \int_{2kr}^\infty \left( 1 + \ln \frac{t}{r} \right)^2 \frac{\| f \|_{L_p(E(x_0, t))}}{t^{\frac{2}{n} - \gamma(\lambda_1 + \lambda_2) - \gamma}\left(\frac{1}{2p_1} + \frac{1}{2p_2}\right) + 1} \, dt.$$
By the estimates of $B_j$ above, where $j = 1, 2, 3$, we know that

$$|T_{\Omega, \alpha}^p ((b_1 - (b_1)_E)(b_2 - (b_2)_E) f_2)(x)| \lesssim \|b_1\|_{L_{p_1, \lambda_1, p}^{(q_0)}} \|b_2\|_{L_{p_2, \lambda_2, p}^{(q_0)}} \int_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \times \frac{\|f\|_{L_p(E(x_0, t))}}{t^{-\gamma(\lambda_1 + \lambda_2) - \gamma \left(\frac{1}{p_1} + \frac{1}{p_2}\right) + 1}} dt. $$

Then, we have

$$B_1 = \|T_{\Omega, \alpha}^p ((b_1 - (b_1)_E)(b_2 - (b_2)_E) f_2)\|_{L_{q_1}(E)} \lesssim \|b_1\|_{L_{p_1, \lambda_1, p}^{(q_0)}} \|b_2\|_{L_{p_2, \lambda_2, p}^{(q_0)}} \int_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right)^2 \times \frac{\|f\|_{L_p(E(x_0, t))}}{t^{-\gamma(\lambda_1 + \lambda_2) - \gamma \left(\frac{1}{p_1} + \frac{1}{p_2}\right) + 1}} dt. $$

So, combining all the estimates for $B_1, B_2, B_3, B_4$, we get

$$B = \|[b_1, b_2], T_{\Omega, \alpha}^p f\|_{L_{q_1}(E)} \lesssim \|b_1\|_{L_{p_1, \lambda_1, p}^{(q_0)}} \|b_2\|_{L_{p_2, \lambda_2, p}^{(q_0)}} \frac{\|f\|_{L_p(E(x_0, t))}}{t^{-\gamma(\lambda_1 + \lambda_2) - \gamma \left(\frac{1}{p_1} + \frac{1}{p_2}\right) + 1}} dt. $$

Thus, putting estimates $A$ and $B$ together, we get the desired conclusion

$$\|[b_1, b_2], T_{\Omega, \alpha}^p f\|_{L_{q_1}(E)} \lesssim \|b_1\|_{L_{p_1, \lambda_1, p}^{(q_0)}} \|b_2\|_{L_{p_2, \lambda_2, p}^{(q_0)}} \frac{\|f\|_{L_p(E(x_0, t))}}{t^{-\gamma(\lambda_1 + \lambda_2) - \gamma \left(\frac{1}{p_1} + \frac{1}{p_2}\right) + 1}} dt. $$

For the case of $q_1 < s$, we can also use the same method, so we omit the details. This completes the proof of Lemma 3. □

3. Proofs of the main results

3.1. Proof of Theorem 1. We consider (1.5) firstly. Since $f \in L_{p', \varphi_1}^{(q_0)}$, by (1.3) and it is also non-decreasing, with respect to $t$, of the norm $\|f\|_{L_p(E(x_0, t))}$, we get

$$\frac{\|f\|_{L_p(E(x_0, t))}}{\varphi_1(x_0, t)^{\frac{1}{p}}} \leq \text{esssup}_{0 < t < \tau < \infty} \frac{\|f\|_{L_p(E(x_0, t))}}{\varphi_1(x_0, \tau)^{\frac{1}{p}}} \leq \|f\|_{L_{p, \varphi_1}^{(q_0)}},$$

(3.1)
For $s' \leq q < \infty$, since $(\varphi_1, \varphi_2)$ satisfies (1.4), we have
\[
\int_{r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^{m} \left\|f\right\|_{L_p(E(x_0,t))} \frac{\gamma^{\frac{1}{p}}}{t^{\left(\frac{1}{p} - \sum_{i=1}^{m} \lambda_i\right)+1}} dt \\
\leq \int_{r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^{m} \left\|f\right\|_{L_p(E(x_0,t))} \frac{\essinf_{t<r<\infty} \varphi_1(x_0,\tau)\tau^{\frac{1}{p}}}{t^{\left(\frac{1}{p} - \sum_{i=1}^{m} \lambda_i\right)+1}} dt \\
\leq C\left\|f\right\|_{LM_{p,\varphi,\rho}} \int_{r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^{m} \left\|f\right\|_{L_p(E(x_0,t))} \frac{\essinf_{t<r<\infty} \varphi_1(x_0,\tau)\tau^{\frac{1}{p}}}{t^{\left(\frac{1}{p} - \sum_{i=1}^{m} \lambda_i\right)+1}} dt \\
\leq C\left\|f\right\|_{LM_{p,\varphi,\rho}} \varphi_2(x_0, r).
\] (3.2)

Then by (2.4) and (3.2), we get
\[
\left\| [\vec{b}, T_{\Omega}^P] f \right\|_{LM_{p,\varphi,\rho}} = \sup_{r>0} \varphi_2(x_0, r)^{-1} \left| E(x_0, r) \right|^{-\frac{1}{p}} \left\| [\vec{b}, T_{\Omega}^P] f \right\|_{L_q(E(x_0, r))} \\
\leq C \prod_{i=1}^{m} \left\| \vec{b} \right\|_{LC_{p,\lambda_i,\rho}} \sup_{r>0} \varphi_2(x_0, r)^{-1} \\
\times \int_{r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^{m} \left\|f\right\|_{L_p(E(x_0,t))} \frac{\essinf_{t<r<\infty} \varphi_1(x_0,\tau)\tau^{\frac{1}{p}}}{t^{\left(\frac{1}{p} - \sum_{i=1}^{m} \lambda_i\right)+1}} dt \\
\leq C \prod_{i=1}^{m} \left\| \vec{b} \right\|_{LC_{p,\lambda_i,\rho}} \left\|f\right\|_{LM_{p,\varphi,\rho}}.
\]

For the case of $p < s$, we can also use the same method, so we omit the details. Thus, we finish the proof of (1.5).

We are now in a place of proving (1.6) in Theorem 1.

**Remark 1.** The conclusion of (1.6) is a direct consequence of the following Lemma 4 and (1.5). In order to do this, we need to define an operator by
\[
[\vec{b}, T^P_{\Omega}] (|f|)(x) = \int_{\mathbb{R}^n} \prod_{i=1}^{m} \left| b_i(x) - b_i(y) \right| \frac{|\Omega(x-y)|}{|\rho(x-y)|} |f(y)| dy,
\]
where $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is $A_t$-homogeneous of degree zero in $\mathbb{R}^n$.

Using the idea of proving Lemma 2 in [4] (see also [9]), we can obtain the following pointwise relation:

**Lemma 4.** Let $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be $A_t$-homogeneous of degree zero. Then we have
\[
M^P_{\Omega, \vec{b}} f(x) \leq [\vec{b}, T^P_{\Omega}] (|f|)(x) \quad \text{for } x \in \mathbb{R}^n.
\]
In fact, for any $t > 0$, we have
\[
[\mathbf{b}, \hat{T}^P_{\Omega}] (|f|) (x) \geq \int \prod_{i=1}^{m} \left[ |b_i (x) - b_i (y)| \right] \frac{|\Omega(x - y)|}{\rho(x - y)} |f(y)| dy
\]
\[
\geq \frac{1}{t^{\gamma}} \int \prod_{E(x, t)}^{m} \left[ |b_i (x) - b_i (y)| \right] |\Omega(x - y)| |f(y)| dy.
\]
Taking the supremum for $t > 0$ on the inequality above, we get
\[
[\mathbf{b}, \hat{T}^P_{\Omega}] (|f|) (x) \geq M^P_{\Omega, b} f(x) \quad \text{for } x \in \mathbb{R}^n.
\]
From the process proving (1.5), it is easy to see that the conclusions of (1.5) also hold for $[\mathbf{b}, \hat{T}^P_{\Omega}]$. Combining this with Lemma 4, we can immediately obtain (1.6), which completes the proof.

3.2. **Proof of Theorem 2.** Similar to the proof of Theorem 1, we consider (1.8) firstly.

For $s' \leq q < \infty$, since $(\varphi_1, \varphi_2)$ satisfies (1.7) and by (3.1), we have
\[
\int \left( 1 + \ln \frac{t}{r} \right)^m \frac{\|f\|_{L^p(E(x, t))}}{\gamma \left( \frac{1}{\eta_1} - \sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \frac{1}{\rho_i} \right) + 1} \, dt
\]
\[
\leq \int \left( 1 + \ln \frac{t}{r} \right)^m \frac{\|f\|_{L^p(E(x, t))}}{\gamma \left( \frac{1}{\eta_1} - \sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \frac{1}{\rho_i} \right) + 1} \, dt
\]
\[
\leq C \|f\|_{LM^{(\gamma, \eta)}_{p, p, p}} \int \left( 1 + \ln \frac{t}{r} \right)^m \frac{\text{essinf}_{t < \tau < \infty} \varphi_1 (x_0, \tau)^{\gamma}}{\gamma \left( \frac{1}{\eta_1} - \sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \frac{1}{\rho_i} \right) + 1} \, dt
\]
\[
\leq C \|f\|_{LM^{(\gamma, \eta)}_{p, p, p}} \varphi_2 (x_0, r).
\]
Then by (3.3) and (3.3), we get
\[
\left\| [\mathbf{b}, \hat{T}^P_{\Omega, \alpha}] f \right\|_{LM^{(\gamma, \eta)}_{\alpha, 2, 2}} = \sup_{r > 0} \varphi_2 (x_0, r)^{-1} |E(x_0, r)|^{-\frac{1}{\gamma}} \left\| [\mathbf{b}, \hat{T}^P_{\Omega, \alpha}] f \right\|_{L^{q_1}(E(x_0, r))}
\]
\[
\leq C \prod_{i=1}^{m} \left\| b \right\|_{L^c(\gamma, \eta, \lambda, r)} \sup_{r > 0} \varphi_2 (x_0, r)^{-1}
\]
\[
\times \int \left( 1 + \ln \frac{t}{r} \right)^m \frac{\|f\|_{L^p(E(x, t))}}{\gamma \left( \frac{1}{\eta_1} - \sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \frac{1}{\rho_i} \right) + 1} \, dt
\]
\[
\leq C \prod_{i=1}^{m} \left\| b \right\|_{L^c(\gamma, \eta, \lambda, r)} \|f\|_{LM^{(\gamma, \eta)}_{p, p, p}}.
\]
For the case of $q_1 < s$, we can also use the same method, so we omit the details. Thus, we finish the proof of (1.8).
We are now in a place of proving (1.9) in Theorem 2.

**Remark 2.** The conclusion of (1.9) is a direct consequence of the following Lemma 5 and (1.8). In order to do this, we need to define an operator by

\[
\hat{b} \cdot \hat{T}_{[\Omega],\alpha}^P (|f|) (x) = \int_{\mathbb{R}^n} \prod_{i=1}^m |b_i(x) - b_i(y)| \frac{|\Omega(x-y)|}{\rho(x-y)^{s-\alpha}} |f(y)| dy \quad 0 < \alpha < \gamma,
\]

where \( \Omega \in L_s(S^{n-1}), 1 < s \leq \infty, \) is \( A_t \)-homogeneous of degree zero in \( \mathbb{R}^n \).

Using the idea of proving Lemma 2 in [4] (see also [9]), we can obtain the following pointwise relation:

**Lemma 5.** Let \( 0 < \alpha < \gamma \) and \( \Omega \in L_s(S^{n-1}), 1 < s \leq \infty, \) be \( A_t \)-homogeneous of degree zero. Then we have

\[
M_{\Omega, b, \alpha}^P f(x) \leq \hat{b} \cdot \hat{T}_{[\Omega],\alpha}^P (|f|) (x) \quad \text{for } x \in \mathbb{R}^n.
\]

In fact, for any \( t > 0 \), we have

\[
\hat{b} \cdot \hat{T}_{[\Omega],\alpha}^P (|f|) (x) \geq \int_{\rho(x-y)<t} \prod_{i=1}^m |b_i(x) - b_i(y)| \frac{|\Omega(x-y)|}{\rho(x-y)^{s-\alpha}} |f(y)| dy \\
\geq \frac{1}{t^{\gamma-\alpha}} \int_{E(x,t)} \prod_{i=1}^m |b_i(x) - b_i(y)| \frac{|\Omega(x-y)|}{\rho(x-y)^{s-\alpha}} |f(y)| dy.
\]

Taking the supremum for \( t > 0 \) on the inequality above, we get

\[
\hat{b} \cdot \hat{T}_{[\Omega],\alpha}^P (|f|) (x) \geq M_{\Omega, b, \alpha}^P f(x) \quad \text{for } x \in \mathbb{R}^n.
\]

From the process proving (1.8), it is easy to see that the conclusions of (1.8) also hold for \( \hat{b} \cdot \hat{T}_{[\Omega],\alpha}^P \). Combining this with Lemma 5, we can immediately obtain (1.9), which completes the proof.

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