ON THE THEORY OF 1-MOTIVES

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Abstract. This is an overview and a preview of the theory of mixed motives of level \( \leq 1 \) explaining some results, projects, ideas and indicating a bunch of problems.

Dedicated to Jacob Murre

Let \( k \) be an algebraically closed field of characteristic zero to start with and let \( S = \text{Spec}(k) \) denote our base scheme. Recall that Murre [46] associates to a smooth \( n \)-dimensional projective variety \( X \) over \( S \) a Chow cohomological Picard motive \( M^1(X) \) along with the Albanese motive \( M^{2n-1}(X) \). The projector \( \pi_1 \in CH^n(X \times X)_\mathbb{Q} \) defining \( M^1(X) \) is obtained via the isogeny \( \text{Pic}^0(X) \to \text{Alb}(X) \) between the Picard and Albanese variety, given by the restriction to a smooth curve \( C \) on \( X \) since \( \text{Alb}(C) = \text{Pic}^0(C) \) (such a curve is obtained by successive hyperplane sections). For a survey of classical Chow motives see [54] (cf. also [4]).

In the case of curves \( M^1(X) \) is the Chow motive of \( X \) refined from lower and higher trivial components, i.e., \( M^0(X) \) and \( M^2(X) \), such that, for smooth projective curves \( X \) and \( Y \)

\[
(1) \quad \text{Hom}(M^1(X), M^1(Y)) \cong \text{Hom}(\text{Pic}^0(X), \text{Pic}^0(Y))_\mathbb{Q}
\]

by Weil (see [58] Thm. 22 on p. 161] and also a remark of Grothendieck and Manin [11]). Furthermore, the semi-simple abelian category of abelian varieties up to isogeny is the pseudo-abelian envelope of the category of Jacobians and \( \mathbb{Q} \)-linear maps. Thus, such a theory of pure motives of smooth projective curves is known to be equivalent to the theory of abelian varieties up to isogeny, as pointed out by Grothendieck: one-dimensional (pure) motives are abelian varieties.

This formula [11] suggests that we may take objects represented by Pic-functors as models for larger categories of mixed motives of any kind of curves over arbitrary base schemes \( S \). However, non representability of Pic for open schemes, forces to refine our models. Let \( \overline{X} \) be a closure of \( X \)

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with divisor at infinity $X_\infty$, i.e., $X = \overline{X} - X_\infty$. For $X$ smooth we have that $\text{Pic}(X)$ is the cokernel of the canonical map $\text{Div}_\infty(\overline{X}) \to \text{Pic}(\overline{X})$ associating $D \mapsto O(D)$, for divisors $D$ on $\overline{X}$ supported at infinity. Thus we may set our models following Deligne [23] and Serre [55] as

$$(2) \quad [\text{Div}_0^0(\overline{X}) \to \text{Pic}^0(\overline{X})]$$

when $X$ is smooth over $S = \text{Spec}(k)$, by mapping algebraically equivalent to zero divisors at infinity to line bundles. Therefore, a vague definition of our categories of 1-motives $M$ can be envisioned as two terms complexes (up to quasi-isomorphisms) of the following kind

$$M := [L \to G]$$

where $L$ is discrete-infinitesimal and $G$ is continuous-connected. Moreover, we expect that a corresponding formula (1) would be available in the larger category of mixed motives.

1. On Picard functors

Let $\pi : X \to S$ and consider the Picard functor $T \mapsto \text{Pic}_{X/S}(T)$ on the category of schemes over $S$ obtained by sheafifying the functor $T \mapsto \text{Pic}(X \times_S T)$ with respect to the fppf-topology (= flat topology). This means that if $\pi : X \times_S T \to T$ then

$$\text{Pic}_{X/S}(T) := H^0_{\text{fppf}}(T, R^1\pi_*(\mathbb{G}_m |_{X \times_S T})).$$

If $\pi_*(O_X) = O_S$ or by reducing to this assumption, e.g., if $\pi$ is proper, the Leray spectral sequence along $\pi$ and descent yields an exact sequence

$$0 \to \text{Pic}(S) \to \text{Pic}(X) \to \text{Pic}_{X/S}(S) \to H^2_{\text{fppf}}(S, \mathbb{G}_m) \to H^2_{\text{fppf}}(X, \mathbb{G}_m).$$

Here the étale topology will suffice as $H^i_{\text{ét}}(-, \mathbb{G}_m) \cong H^i_{\text{fppf}}(-, \mathbb{G}_m)$ for all $i \geq 0$ by a theorem of Grothendieck (see [30 VI.5 p. 126 & VI.11 p. 171]). If there is a section of $\pi$ we then have that $\text{Pic}_{X/S}(S) \cong \text{Pic}(X)/\text{Pic}(S)$.

If we set $\pi : X \to S$ proper and flat over a base, the Picard fppf-sheaf $\text{Pic}_{X/S}$ would be possibly representable by an algebraic space only. For a general theory we should stick to algebraic spaces not schemes (see [19, 8.3]). However, as far as $S = \text{Spec}(k)$ is a field, we may just consider group schemes: by Grothendieck and Murre (see [15 and [19, 8.2]) we have that $\text{Pic}_{X/k}$ is representable by a scheme locally of finite type over $k$. As a group scheme $\text{Pic}^0$ usually stands for the connected component of the identity of Pic and $\text{Pic}^0_{X/k}$ is an abelian variety (known classically as the Picard variety, cf. [11 8.4]) as soon as $X$ is also smooth and $k$ has zero characteristic. Here $\text{NS}_X := \pi_0(\text{Pic}_{X/k})$ is finitely generated. In positive characteristic, for $X$ smooth and proper over $k$ perfect, the connected component of the identity endowed with its reduced structure $\text{Pic}^{0,\text{red}}_{X/k}$ is an abelian variety. More
informations, e.g., on the universal line bundle $\mathcal{P}$ on $X \times_S \text{Pic}_{X/S}$, can be obtained from [19, §8].

For example, if $X$ is a singular projective curve, in zero characteristic, and $\tilde{X}$ is the normalization of $X$ we then have an extension\(^1\)

$$0 \rightarrow V \oplus T \rightarrow \text{Pic}^0(X) \rightarrow \text{Pic}^0(\tilde{X}) \rightarrow 0$$

where $V = \mathbb{G}_a^r$ is a vector group and $T = \mathbb{G}_m^s$ is a torus. The additive part here is non homotopical invariant, that is, the semi-abelian quotient is homotopical invariant, e.g., consider the well known example of $X = \text{projective rational cusp}$: its first singular cohomology group is zero but $\text{Pic}^0_{X/k} = \mathbb{G}_a$. For proper schemes in zero characteristic, we can describe the semi-abelian quotient of Pic-functors as follows.

1.1. Simplicial Picard functors. Let $\pi : X_\bullet \rightarrow X$ be a smooth proper hypercovering of $X$ over $S = \text{Spec}(k)$. Recall that $X_\bullet$ is a simplicial scheme with smooth components obtained roughly as follows: $X_0$ is a resolution of singularities of $X$, $X_1$ is obtained by a resolution of singularities of $X_0 \times_X X_0$, etc. Such hypercoverings were introduced by Deligne [23] in characteristic zero (after Hironaka’s resolution of singularities) but are also available over a perfect field of positive characteristic (after de Jong’s theory [34]) by taking $X_0$ an alteration of $X$ (in this case $X_0 \rightarrow X$ is only generically étale). Actually, it is possible to refine such a construction, in characteristic zero, obtaining a (semi)simplicial scheme $X_\bullet$ such that $\dim(X_i) = \dim(X) - i$ so that the corresponding complex of algebraic varieties (in the sense of [9]) is bounded.

1.1.1. Denote $\mathbb{P}ic(X_\bullet) \cong \mathbb{H}^1_{fppf}(X_\bullet, \mathcal{O}^*_X) \cong \mathbb{H}^1_{ét}(X_\bullet, \mathcal{O}^*_X)$ the group of isomorphism classes of simplicial line bundles on $X_\bullet$, i.e., of invertible $\mathcal{O}_{X_\bullet}$-modules. Let $\mathbb{P}ic_{X_\bullet/S}$ be the associated fppf-sheaf on $S$. Over $S = \text{Spec}(k)$ such $\mathbb{P}ic_{X_\bullet/S}$ is also representable (see [11, A]). The canonical spectral sequence for the components of $X_\bullet$ yields the following long exact sequence of fppf-sheaves:

$$\frac{\text{Ker}((\pi_1)_* \mathcal{G}_{m,X_1} \rightarrow (\pi_2)_* \mathcal{G}_{m,X_2})}{\text{Im}((\pi_0)_* \mathcal{G}_{m,X_0} \rightarrow (\pi_1)_* \mathcal{G}_{m,X_1})} \hookrightarrow \mathbb{P}ic_{X_\bullet/S} \rightarrow \text{Ker}(\mathbb{P}ic_{X_0/S} \rightarrow \mathbb{P}ic_{X_1/S})$$

$$\rightarrow \frac{\text{Ker}((\pi_2)_* \mathcal{G}_{m,X_2} \rightarrow (\pi_3)_* \mathcal{G}_{m,X_3})}{\text{Im}((\pi_1)_* \mathcal{G}_{m,X_1} \rightarrow (\pi_2)_* \mathcal{G}_{m,X_2})}$$

where $\pi_i : X_i \rightarrow S$ are the structure morphisms. By pulling back along $\pi : X_\bullet \rightarrow X$ we have the following natural maps

$$\mathbb{P}ic_{X/S} \xrightarrow{\pi^*} \mathbb{P}ic_{X_\bullet/S} \rightarrow \text{Ker}(\mathbb{P}ic_{X_0/S} \rightarrow \mathbb{P}ic_{X_1/S}).$$

\(^1\)In the geometric case, i.e., when $k$ is algebraically closed, $\mathbb{P}ic_{X/k}(k) \cong \text{Pic}(X)$.
The most wonderful property of hypercoverings is cohomological descent that is an isomorphism

\[ H^*_{\text{ét}}(X, F) \cong H^*_{\text{ét}}(X, \pi^*(F)) \]

for any sheaf \( F \) on \( S_{\text{ét}} \) (as well as for other usual topologies). In particular, for the étale sheaf \( \mu_m \cong \mathbb{Z}/m \) of \( m \)-rooths of unity on \( S = \text{Spec}(k) \), \( k = \overline{k} \) and \((m, \text{char}(k)) = 1\), by (simplicial) Kummer theory (see [11, 5.1.2]) and cohomological descent we get the following commutative square of isomorphisms

\[
\begin{array}{ccc}
\mathbb{H}^1_{\text{ét}}(X, \mu_m) & \cong & \text{Pic}(X)_{m-\text{tor}} \\
\downarrow & & \downarrow \\
H^1_{\text{ét}}(X, \mu_m) & \cong & \text{Pic}(X)_{m-\text{tor}}
\end{array}
\]

The simplicial Néron-Severi group \( \text{NS}(X, \cdot) := \text{Pic}(X, \cdot)/\text{Pic}^0(X, \cdot) \) is finitely generated, therefore the Tate module of \( \text{Pic}(X, \cdot) \) is isomorphic to that of \( \text{Pic}^0(X, \cdot) \) and, by cohomological descent, to that of \( \text{Pic}^0(X) \). Moreover, \( \text{Pic}^0(X, \cdot) \) is the group of \( k \)-points of a semi-abelian variety, in which torsion points are Zariski dense.

1.1.2. **Scholium** ([11, 5.1.2]). If \( X \) is proper over \( S = \text{Spec}(k) \), \( k = \overline{k} \) of characteristic 0, and \( \pi : X, \to X \) is any smooth proper hypercovering, then

\[ \pi^* : \text{Pic}^0(X) \to \text{Pic}^0(X, \cdot) \]

is a surjection with torsion free kernel.

As a consequence, we see that the simplicial Picard variety \( \text{Pic}^0(X, \cdot) \) is the semi-abelian quotient of the connected commutative algebraic group \( \text{Pic}^0(X) \). Moreover, if \( X \) is semi-normal, then \( \pi_*(\mathcal{O}^*_X, \cdot) = \mathcal{O}^*_X \), and so \( \pi^* : \text{Pic}(X) \to \text{Pic}(X, \cdot) \) is injective, by the Leray spectral sequence for the sheaf \( \mathcal{O}^*_X, \cdot \) along \( \pi \); therefore, from [11.1.2] we get

\[ \text{Pic}^0(X) \cong \text{Pic}^0(X, \cdot) \cong \text{Ker}^0(\text{Pic}^0(X_0) \to \text{Pic}^0(X_1)) \]

whenever \( \text{Pic}(X) \to \text{Pic}(X_0) \) is also injective (here \( \text{Ker}^0 \) denotes the connected component of the identity of the kernel). Thus, if \( X \) is normal \( \text{Pic}^0(X) \) is an abelian variety which can be represented in terms of \( X_0 \) and \( X_1 \) only. If \( X \) is only semi-normal a similar argument applies and \( \text{Pic}^0(X) \cong \text{Pic}^0(X, \cdot) \) is semi-abelian.
1.1.3. Homotopical invariance of units and Pic, i.e.,\( H^i(X, \mathbb{G}_m) \cong H^i(A^1_S \times_S X, \mathbb{G}_m) \) for \( i = 0, 1 \) induced by the projection \( A^1_S \times_S X \rightarrow X \), is easily deduced for \( X \) smooth. Let \( A^1_S \times_S X \rightarrow X \) be the canonical projection; considering \( H^i(X, \mathbb{G}_m) \) we see that
\[
\mathbb{P}ic(X_\ast) \cong \mathbb{P}ic(A^1_S \times_S X_\ast)
\]
since \( X_\ast \) has smooth components. Therefore, the semi-abelian quotient of \( \mathbb{P}ic^0(X) \) is always homotopical invariant. By dealing with homotopical invariant theories we just need to avoid the additive factors, and \( \mathbb{P}ic^0 \) is the ‘motivic’ object corresponding to \( M^1 \) of proper (arbitrarily singular) \( S \)-schemes, i.e., \( \mathbb{P}ic^+ \) in the notation adopted in [11] (cf. [51] and also the commentaries below 3.1.1).

1.1.4. In positive characteristic \( p > 0 \) the picture is more involved and a corresponding Scholium 1.1.2 is valid up to \( p \)-power torsion only. However, the semi-abelian scheme \( \mathbb{P}ic^{0,\text{red}}(X_\ast) \) is independent of the choices of the hypercovering \( X_\ast \) (see [11, A.2]) furnishing a motivic definition of \( H^1_{\text{crys}} \) (described in [11], cf. 3.1.3 below).

1.2. Relative Picard functors. For a pair \((X,Y)\) consisting of a proper \( k \)-scheme \( X \) and a closed sub-scheme \( Y \) we have a natural long exact sequence

\[
H^0(X, \mathcal{O}_X^\ast) \rightarrow H^0(Y, \mathcal{O}_Y^\ast) \rightarrow \mathbb{P}ic(X,Y) \rightarrow \mathbb{P}ic(X) \rightarrow \mathbb{P}ic(Y)
\]
induced by the surjection of Zariski (or fppf) sheaves \( \mathbb{G}_{m,X} \rightarrow i_\ast \mathbb{G}_{m,Y} \) where \( i : Y \hookrightarrow X \) is the inclusion; here
\[
\mathbb{P}ic(X,Y) = \mathbb{H}^1(X, \mathbb{G}_{m,X} \rightarrow i_\ast \mathbb{G}_{m,Y})
\]
is the group of isomorphism classes of pairs \((\mathcal{L}, \varphi)\) such that \( \mathcal{L} \) is a line bundle on \( X \) and \( \varphi : \mathcal{L} \mid_Y \cong \mathcal{O}_Y \) is a trivialization on \( Y \) (see [11, §2]). For \((X,Y)\) as above the fppf-sheaf associated to the relative Picard functor
\[
T \rightsquigarrow \mathbb{P}ic(X \times_k T, Y \times_k T)
\]
is representable by a \( k \)-group scheme which is locally of finite type over \( k \) (cf. [11, A]). If \( \mathbb{P}ic^0(X) \) is abelian, e.g., \( X \) is normal, the sequence [11] yields a semi-abelian group scheme \( \mathbb{P}ic^0(X,Y) \) (cf. [11, 2.1.2]) which can be represented as an extension (say of \( k \)-points over \( k = \overline{k} \) of characteristic zero)

\[
\frac{H^0(Y, \mathcal{O}_Y^\ast)}{\text{Im} \ H^0(X, \mathcal{O}_X^\ast)} \hookrightarrow \mathbb{P}ic^0(X,Y) \rightarrow \text{Ker}^0(\mathbb{P}ic^0(X) \rightarrow \mathbb{P}ic^0(Y))
\]
where \( \mathbb{P}ic^0(X,Y) \) is the connected component of the identity of \( \mathbb{P}ic(X,Y) \), the \( k \)-torus is \( \text{Coker} ((\pi_X)_\ast \mathbb{G}_{m,X} \rightarrow (\pi_Y)_\ast \mathbb{G}_{m,Y}) \) where \( \pi_X : X \rightarrow \text{Spec} k \).
\( \pi_Y : Y \to \text{Spec } k \) are the structure morphisms and where \( \text{Ker}^0 \) denotes the connected component of the identity of the kernel (the abelian quotient is further described below).

1.2.1. For example, assume \( X \) proper (normal) and \( Y = \bigcup Y_i \), where \( Y_i \) are the (smooth) irreducible components of a reduced normal crossing divisor \( Y \).

Consider the normalization \( \pi : \coprod Y_i \to Y \) and observe that \( \pi^* : \text{Pic}(Y) \to \oplus \text{Pic}(Y_i) \) is representable by an affine morphism (see \([11, 2.1.2]\)). Therefore

\[
\text{Ker}^0(\text{Pic}^0(X) \to \text{Pic}^0(Y)) = \text{Ker}^0(\text{Pic}^0(X) \to \oplus \text{Pic}^0(Y_i)).
\]

Moreover, for any such pair \((X, Y)\), we have that (cf. \([11, 2.2]\)) any relative Cartier divisor \( D \in \text{Div}(X, Y) \), i.e., as divisor on \( X \) such that the support \( \mid D \mid \cap Y = \emptyset \), provides \( [D] \in \text{Pic}(X, Y) \) where \( I \) denotes the tautological section of \( \mathcal{O}_X(D) \), trivializing it on \( X - \mid D \mid \).

Here a Cartier divisor \( D \in \text{Div}(X, Y) \) is algebraically equivalent to zero relative to \( Y \) if \([D] \in \text{Pic}^0(X, Y)\). Denote \( \text{Div}_{Z}^0(X, Y) \subset \text{Div}_{Z}(X, Y) \) the subgroup of relative divisors supported on a closed sub-scheme \( Z \subset X \) which are algebraically equivalent to zero relative to \( Y \). We also have a ‘motivic’ object

\[
[\text{Div}_{Z}^0(X, Y) \to \text{Pic}^0(X, Y)]
\]

which morally corresponds to \( M^1(X - Z, Y) \).

1.2.2. Starting from an open scheme \( X \) let \( \overline{X} \) be a closure of \( X \) with boundary \( X_\infty \), i.e., \( X = \overline{X} - X_\infty \). For \( Z = X_\infty \) and \( Y = \emptyset \) from the pair \((\overline{X}, \emptyset)\) we get \([2]\) and for \( Y = X_\infty \) we have \([\text{Div}^0_{Z}(\overline{X}, X_\infty) \to \text{Pic}^0(\overline{X}, X_\infty)]\) (cf. \([11, 2.2.1]\)).

1.3. **Higher Picard functors.** Let \( X \) be an equidimensional \( k \)-scheme. Let

\[
\text{CH}^p(X) := \text{Z}^p(X)/\equiv_{\text{rat}}
\]

be the Chow group of codimension \( p \)-cycles modulo rational equivalence. Recall that \( \text{CH}^{1}(X) = \text{Pic}(X) \) if \( X \) is smooth but the Chow functor \( T \leadsto \text{CH}^p(X \times_k T) \) for \( 1 < p \leq \dim(X) \) doesn’t provide a representable functor even in the case when \( X \) is smooth and proper over \( k = \overline{k} \).

1.3.1. To supply this defect several proposed generalizations have been investigated (see \([47, 40] \) and \([32]\)). Consider the sub-group \( \text{CH}^p(X)_{\text{alg}} \) of those cycles in \( \text{CH}^p(X) \) which are algebraically equivalent to zero and let \( \text{NS}^p(X) := \text{CH}^p(X)/\text{CH}^p(X)_{\text{alg}} \) denote the Néron-Severi group. Denote \( \text{CH}^p(X)_{\text{ab}} \) the sub-group of \( \text{CH}^p(X)_{\text{alg}} \) of those cycles which are abelian
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equivalent to zero, i.e., $CH^p(X)_{ab}$ is the intersection of all kernels of regular homomorphisms from $CH^p(X)_{alg}$ to abelian varieties (see [17] for definitions and references). The main question here is about the existence of an ‘algebraic representative’, i.e., a universal regular homomorphism from $CH^p(X)_{alg}$ to an abelian variety. In modern terms, one can rephrase it (equivalently or not) by asking if the homotopy invariant sheaf with transfers (see [57] for this notion) $CH^p_{X/k}$ associated to $X$ smooth is provided with a universal map to a 1-motivic sheaf (see [8] and [2], also 2.1.12 below). The abelian category $Shv_1(k)$ of 1-motivic étale sheaves is given by those homotopy invariant sheaves with transfers $F$ such that there is a map

$$G \rightarrow F$$

where $G$ is continuous-connected (e.g., semi-abelian); Ker $f$ and Coker $f$ are discrete-infinitesimal (e.g., finitely generated). The paradigmatic example is $F = \text{Pic}_{X/k}$ for $X$ a smooth $k$-variety (see [8]). Starting from $CH^p_{X/k}$ we may seek for

$$c^p : CH^p_{X/k} \rightarrow (CH^p_{X/k})^{(1)}$$

with $(CH^p_{X/k})^{(1)} \in Shv_1(k)$ universally. Remark that the key point is to provide a finite type object as such a universal Ind-object always exists (see [2]). Namely, $CH^p(X)_{alg}$ will be related to the ‘algebraic representative’.

1.3.2. Assume the existence of a universal regular homomorphism $\rho^p : CH^p(X)_{alg} \rightarrow A^p_{X/k}(k)$ to (the group of $k$-points) of an abelian variety $A^p_{X/k}$ defined over the base field $k$. This is given by Murre’s theorem for $p = 2$ (see [17]) and it is clear for $p = 1, \dim(X)$ by the theory of the Picard and Albanese varieties. We then quote the following functorial algebraic filtration $F^*_a$ on $CH^p(X)$ (cf. [5]):

- $F^0_aCH^p(X) = CH^p(X)$,
- $F^1_aCH^p(X) = CH^p(X)_{alg}$
- $F^2_aCH^p(X) = CH^p(X)_{ab}$, i.e., is the kernel of the universal regular homomorphism $\rho^p$ above,
- and the corresponding extension

$$(6) \quad 0 \rightarrow A^p_{X/k}(k) \rightarrow CH^p(X)/F^2_a \rightarrow NS^p(X) \rightarrow 0.$$ 

Remark that Bloch, Beilinson and Murre (see [35]) conjectured the existence of a finite filtration $F^*_m$ on $CH^p(X)_{Q}$ (with rational coefficients) such that $F^1_mCH^p(X)$ is given by $CH^p(X)_{hom}$, i.e., by the sub-group of those codimension $p$ cycles which are homologically equivalent to zero for some Weil cohomology theory. $F^*_mCH^p(X)$ should be functorial and compatible with the intersection pairing. The motivic filtration $F^*_m$ will be inducing the

\footnote{Note that $CH^p(X)_{alg}$ and $CH^p(X)_{ab}$ are divisible groups.}
algebraic (or 1-motivic) filtration $F^*_a$ somehow, e.g., $F^*_a = F^*_m \cap \text{CH}^p(X)_{\text{alg}}$ for $* > 0$.

Remark that we may even push further this picture by seeking for the 1-motivic algebraically defined extension of codimension $p$ cycles modulo numerical equivalence by $A^{p}_{X/k}(k)$ (which pulls back to (4), see also 3.2.4 below).

2. On 1-motives

A free 1-motive over $S$ (here $S$ is any base scheme) in Deligne’s definition (a 1-motif lisse cf. [23, §10]) is a complex $M := [L \rightarrow G]$ of $S$-group schemes where $G$ is semi-abelian, i.e., it is an extension of an abelian scheme $A$ by a torus $T$ over $S$, the group scheme $L$ is, locally for the étale topology on $S$, isomorphic to a finitely-generated free abelian constant group, and $u : L \rightarrow G$ is an $S$-homomorphism. A 1-motive $M$ can be represented in a diagram

$$
\begin{array}{cccc}
L & \downarrow & \\
0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0
\end{array}
$$

An effective morphism of 1-motives is a morphism of the corresponding complexes of group schemes (and actually of the corresponding diagrams). Any such a complex can be regarded as a complex of fppf-sheaves. Following the existing literature, $L$ is regarded in degree $-1$ and $G$ in degree 0 (however, for some purposes, e.g., in order to match the conventions in Voevodsky triangulated categories, it is convenient to shift $L$ in degree 0 and $G$ in degree 1, cf. [8]). We let $\mathcal{M}^f_1$ denote the category of Deligne 1-motives (cf. 2.9 below).

2.1. Generalities. It is easy to see that $\mathcal{M}^f_1$ has kernels and cokernels but images and coinages, in general, don’t coincide. For kernels, if $\text{Ker}^e(\phi) = \{\text{Ker}(f) \xrightarrow{u} \text{Ker}(g)\}$ is the kernel of $\phi = (f, g) : M \rightarrow M'$ as a map of complexes then $\text{Ker}^0(\phi) = \{\text{Ker}^0(f) \xrightarrow{u} \text{Ker}^0(g)\}$ is the pull-back of $\text{Ker}^0(g)$ along $u$, where $\text{Ker}^0(g)$ is the connected component of the identity of the kernel of $g : G \rightarrow G'$ and $\text{Ker}^0(f) \subseteq \text{Ker}(f)$.

Similarly, for cokernels, if $\text{Coker}^e(\phi) = \{\text{Coker}(f) \xrightarrow{\overline{u}} \text{Coker}(g)\}$ is the cokernel as complexes and $T$ is the torsion subgroup of $\text{Coker}(f)$, as group schemes, then

$$
\text{Coker}(\phi) = \{\text{Coker}(f)/T \rightarrow \text{Coker}(g)/\overline{\pi}(T)\}
$$

is a Deligne’s 1-motive which is clearly a cokernel of $\phi$. 

Associated to any 1-motive $M$ there is a canonical extension (as two terms complexes)

$$0 \to [0 \to G] \to M \to [L \to 0] \to 0 \tag{7}$$

2.1.1. Actually, a 1-motive $M$ is canonically equipped with an increasing weight filtration by sub-1-motives as follows:

$$W_i(M) = \begin{cases} 
M & i \geq 0 \\
[0 \to G] & i = -1 \\
[0 \to T] & i = -2 \\
0 & i \leq -3 
\end{cases}$$

In particular we have $\text{gr}^{-1}_W(M) = [0 \to A]$ and $\text{gr}^0_W(M) = [L \to 0]$.

2.1.2. For $S = \text{Spec}(k)$ a 1-motive $M = [L \xrightarrow{\pi} G]$ over $k$ (a perfect field) is equivalent to the given semi-abelian $k$-scheme $G$, a finitely generated free abelian group $L$ which underlies a $\text{Gal}(\overline{k}/k)$-module, a 1-motive $[\overline{L} \xrightarrow{\overline{\pi}} \overline{G}]$ over $\overline{k}$, such that $\overline{\pi}$ is $\text{Gal}(\overline{k}/k)$-equivariant, for the given module structure on $\overline{L}$, and the natural semi-linear action on $\overline{G} = G \times_k \overline{k}$. In fact, the morphism $\pi$ is determined uniquely by base change to $\overline{k}$, i.e., by the morphism $u_{\overline{k}} : L_{\overline{k}} \to G_{\overline{k}}$, which is $\text{Gal}(\overline{k}/k)$-equivariant.

2.1.3. It is easy to see that there are no non-trivial quasi-isomorphisms between Deligne 1-motives. Actually, there is a canonical functor $\iota : M_{1 \text{Fr}} \to D^b(S_{\text{fppf}})$ which is a full embedding into the derived category of bounded complexes of sheaves for the fppf-topology on $S$.

2.1.4. **Scholium (52 Prop.2.3.1).** Let $M$ and $M'$ be free 1-motives. Then

$$\text{Hom}_{M_{1 \text{Fr}}}(M, M') \cong \text{Hom}_{D^b(S_{\text{fppf}})}(\iota(M), \iota(M')).$$

**Proof.** The naive filtration of $M = [L \to G]$ and $M' = [L \to G']$ yields a spectral sequence

$$E_1^{p,q} = \bigoplus_{-i+j=p} \text{Ext}^q(iM, jM') \Rightarrow \text{Ext}^{p+q}(M, M')$$

yielding complexes $E_1^{*,-q}$

$$\text{Ext}^q(G, L') \to \text{Ext}^q(G, G') \oplus \text{Ext}^q(L, L') \to \text{Ext}^q(L, G')$$

where the left-most non-zero term is in degree -1. We see that $\text{Ext}^0(G, L') = \text{Hom}(G, L') = 0$ since $G$ is connected and $\text{Ext}(G, L') = 0$ since $L'$ is free.
Thus $E^{0,0}_2 = \text{Hom}_{\mathcal{M}_1^c}(M, M')$ is $\text{Ext}^0(M, M') = \text{Hom}_{D^b(S_{	ext{fppf}})}(\iota(M), \iota(M'))$.

\[ \square \]

2.2. **Hodge realization.** The Hodge realization $T_{\text{Hodge}}(M)$ of a 1-motive $M$ over $S = \text{Spec}(\mathbb{C})$ (see [23, 10.1.3]) is $(T_Z(M), W_*, F^*)$ where $T_Z(M)$ is the lattice given by the pull-back of $u : L \to G$ along $\exp : \text{Lie}(G) \to G$, $W_*$ is the integrally defined weight filtration

\[
W_i T(M) := \begin{cases} 
T_Z(M) & i \geq 0 \\
H_1(G) & i = -1 \\
H_1(T) & i = -2 \\
0 & i \leq -3 
\end{cases}
\]

and $F^*$ is the Hodge filtration defined by $F^0(T_Z(M) \otimes \mathbb{C}) := \text{Ker}(T_Z(M) \otimes \mathbb{C} \to \text{Lie}(G))$. Then we see that $T_{\text{Hodge}}(M)$ is a mixed Hodge structure and we have $\text{gr} W_{-1} T_{\text{Hodge}}(M) \cong H_1(A, \mathbb{Z})$ as pure polarizable Hodge structures of weight $-1$.

2.2.1. The functor

$$T_{\text{Hodge}} : \mathcal{M}_1^c(\mathbb{C}) \overset{\sim}{\to} \text{MHS}_1^c$$

is an equivalence between the category of 1-motives over $\mathbb{C}$ and the category of torsion free $\mathbb{Z}$-mixed Hodge structures of type

$$\{(0, 0), (0, -1), (-1, 0), (-1, -1)\}$$

such that $\text{gr} W_1$ is polarizable. Deligne (cf. [23, §10.1.3]) observed that such a $H \in \text{MHS}_1^c$ is equivalent to a 1-motive over the complex numbers. In fact, for $H \in \text{MHS}_1^c$ the canonical extension of mixed Hodge structures

\[
0 \to W_{-1}(H) \to H \to \text{gr}^W_0(H) \to 0
\]

yields an extension class map (cf. [21])

$$\epsilon_H : \text{Hom}_{\text{MHS}}(\mathbb{Z}, \text{gr}_0^W(H)) \to \text{Ext}_{\text{MHS}}(\mathbb{Z}, W_{-1}(H))$$

which provides a 1-motive with lattice $L := \text{gr}_0^W(H)\mathbb{Z}$ mapping to the semi-abelian variety with complex points $G(\mathbb{C}) := \text{Ext}_{\text{MHS}}(\mathbb{Z}, W_{-1}(H))$. Summarizing up, any 1-motive $M$ over $\mathbb{C}$ has a covariant Hodge realization

$$M \leadsto T_{\text{Hodge}}(M)$$

and the exact sequence [7] gives rise to the exact sequence [8] of Hodge realizations.

\[ ^3 \text{Note that } H_1(G) \text{ is the kernel of } \exp : \text{Lie}(G) \to G. \]
2.2.2. We have that \( T_{\text{Hodge}}([0 \to \mathbb{G}_m]) = \mathbb{Z}(1) \) is the Hodge structure (pure of weight \(-2\) and purely of type \((-1, -1)\)) provided by the complex exponential \( \exp : \mathbb{C} \to \mathbb{C}^* \), i.e., here \( T_{\mathbb{Z}}([0 \to \mathbb{G}_m]) \) is the free \( \mathbb{Z} \)-module on \( 2\pi \sqrt{-1} \). Recall that for \( H \in \text{MHS}_{\text{fr}}^1 \) we get \( H^\vee := \text{Hom}^\text{fr}(H, \mathbb{Z}(1)) \in \text{MHS}_{\text{fr}}^1 \) where \( \text{Hom} \) is the internal Hom in MHS (see [22]). We have that \( \mathbb{Z}^\vee = \mathbb{Z}(1) \).

Moreover \( (\ )^\vee := \text{Hom}(\ , \mathbb{Z}(1)) : (\text{MHS}_{\text{fr}}^1)_{\text{op}} \to \text{MHS}_{\text{fr}}^1 \) is an anti-equivalence providing \( \text{MHS}_{\text{fr}}^1 \) of a natural involution. We may set a contravariant Hodge realization given by

\[
M \rightsquigarrow T_{\text{Hodge}}(M) := T_{\text{Hodge}}(M)^\vee
\]

and an induced involution on \( \mathcal{M}_{\text{fr}}^1(\mathbb{C}) \) defined by the formula

\[
M^\vee := T_{\text{Hodge}}^{-1}(M).
\]

Actually, such an involution can be made algebraic (see 2.7 below) and is known as Cartier duality for 1-motives.

2.2.3. Remark that \( \text{MHS}_{\text{fr}}^1 \subset \text{MHS}^1 \) where we just drop the assumption that the underlying \( \mathbb{Z} \)-module is torsion free and we have that the category \( \text{MHS}^1 \) is a thick abelian sub-category of (graded polarizable) mixed Hodge structures. In [9, §1] an algebraic description of \( \text{MHS}^1 \) is given (see 2.9 below). For \( H \in \text{MHS} \) let \( H_{(1)} \) denote the maximal sub-structure of the considered type (= largest 1-motivic sub-structure, for short) and let \( H^{(1)} \) be the largest 1-motivic quotient. For \( H' \in \text{MHS}_1 \) we clearly have

\[
\text{Hom}_{\text{MHS}}(H', H) = \text{Hom}_{\text{MHS}_1}(H', H_{(1)})
\]

and

\[
\text{Hom}_{\text{MHS}}(H, H') = \text{Hom}_{\text{MHS}_1}(H^{(1)}, H').
\]

In other words the embedding \( \text{MHS}_1 \subset \text{MHS} \) has right and left adjoints given by the functors \( H \mapsto H_{(1)} \) and \( H \mapsto H^{(1)} \) respectively. Moreover, it is quite well known to the experts that \( \text{Ext}_{\text{MHS}_1}^1 \) is right exact and the higher extension groups \( \text{Ext}_{\text{MHS}_1}^i \) \((i > 1)\) vanish since similar assertions hold in MHS (by Carlson [21]) and the objects of \( \text{MHS}_1 \) are stable by extensions in MHS. As a consequence, the derived category \( D^b(\text{MHS}_1) \) is a full subcategory of \( D^b(\text{MHS}) \).

2.3. Flat, \( \ell \)-adic and étale realizations. Let \( M = [L \to G] \) be a 1-motive over \( S \) which we consider as a complex of fppf-sheaves over \( S \) with \( L \) in degree \(-1\) and \( G \) in degree \( 0 \). Consider the cone \( M/m \) of the multiplication by \( m \) on \( M \). The exact sequence (7) of 1-motives yields a short exact sequence of cohomology sheaves

\[
0 \to H^{-1}(G/m) \to H^{-1}(M/m) \to H^{-1}(L[1]/m) \to 0
\]
as soon as $L$ is torsion-free, i.e., $H^{-2}(L[1]/m) = \text{Ker}(L \xrightarrow{m} L)$ vanishes, since multiplication by $m$ on $G$ connected is an epimorphism of fppf-sheaves, i.e., $H^0(G/m) = \text{Coker}(G \xrightarrow{m} G)$ vanishes. Here $H^{-1}(G/m) = m$-torsion of $G$ and $H^{-1}(L[1]/m) = L/m$ whence the sequence above is given by finite group schemes. The flat realization

$$T_{\mathbb{Z}/m}(M) := H^{-1}(M/m)$$

is a finite group scheme, flat over $S$, which is étale if $S$ is defined over $\mathbb{Z}[\frac{1}{m}]$. By taking the Cartier dual we also obtain a contravariant flat realization $T_{\mathbb{Z}/m}^\vee(M) := \text{Hom}_{\text{fppf}}(H^{-1}(M/m), G_m)$.

2.3.1. If $\ell$ is a prime number then the $\ell$-adic realization $T_{\ell}(M)$ is the inverse limit over $\nu$ of $T_{\mathbb{Z}/\ell^\nu}(M)$. We have $T_{\ell}([0 \to G_m]) = \mathbb{Z}_\ell(1)$ by the Kummer sequence. The $\ell$-adic realization of an abelian scheme $A$ is the $\ell$-adic Tate module of $A$. In characteristic zero then

$$\hat{T}(M) := \lim_{\longrightarrow} T_{\mathbb{Z}/m}(M) = \prod_{\ell} T_{\ell}(M)$$

is called the étale realization of $M$. For $S = \text{Spec}(k)$, $\hat{T}(M_{\mathbb{Z}})$, along with a natural action of $\text{Gal}(\overline{k}/k)$, is a (filtered) Galois module which is a free $\mathbb{Z}$-module of finite rank. Over $S = \text{Spec}(k)$ and $k = \overline{k}$ we just have

$$T_{\mathbb{Z}/m}(M)(k) = \frac{\{(x, g) \in L \times G(k) \mid u(x) = -mg\}}{\{(mx, -u(x)) \mid x \in L\}}.$$

2.3.2. If $k = \mathbb{C}$ we then have a comparison isomorphism $\hat{T}(M) \cong T_{\mathbb{Z}}(M) \otimes \hat{\mathbb{Z}}$ where $T_{\mathbb{Z}}(M)$ is the $\mathbb{Z}$-module underlying to $T_{\text{Hodge}}(M)$ (cf. [11 §1.3]).

2.4. Crystalline realization. Let $S_0$ be a scheme and $p$ a prime number such that $p$ is locally nilpotent on $S_0$. Now let $S_0 \hookrightarrow S_n$ be a thickening defined by an ideal with nilpotent divided powers. Actually, over $S_0 = \text{Spec}(k)$ a perfect field of characteristic $p > 0$ and $W(k)$ the Witt vectors of $k$ (with the standard divided power structure on its maximal ideal) a thickening $S_n = \text{Spec}(W_{n+1}(k))$ is given by the affine scheme defined by the truncated Witt vectors of length $n + 1$ (or equivalently by $W(k)/p^{n+1}$).

Suppose that $M_0 := [L_0 \to G_0]$ is a 1-motive defined over $S_0$. Consider

$$M_0[p^\infty] := \lim_{\longrightarrow} T_{\mathbb{Z}/p^\nu}(M_0)$$

the direct limit being taken, in terms of the explicit formula above, for $\mu \geq \nu$, by sending the class of a point $(x, g)$ in $L_0 \times G_0(k)$ to the class

\footnote{Note that for $p = 2$ the standard divided power structure of $W_n(k)$ is not nilpotent.}
of \((p^\mu \nu x, g)\). Such \(M_0[p^\infty]\) is a \(p\)-divisible (or Barsotti-Tate) group and the sequence \((\ref{eq:seq})\) yields the exact sequence

\[
0 \to G_0[p^\infty] \to M_0[p^\infty] \to L_0[p^\infty] \to 0
\]

where \(L_0[p^\infty] := L_0 \otimes \mathbb{Q}_p / \mathbb{Z}_p\). For \(M_0 := [0 \to A_0]\) an abelian scheme we get back the Barsotti-Tate group of \(A_0\).

2.4.1. Let \(\mathbb{D}\) be the contravariant Dieudonné functor from the category of \(p\)-divisible groups over \(S_0 = \text{Spec}(k)\) to the category of \(D_k\)-modules, for the Dieudonné ring \(D_k := \mathbb{W}(k)[F, V]/(FV = VF = p)\). This \(\mathbb{D}\) is defined as the module of homomorphisms from the \(p\)-divisible group to the group of Witt covectors over \(k\) and provides an anti-equivalence from the category of \(p\)-divisible groups over \(k\) to the category of \(D_k\)-modules which are finitely generated and free as \(\mathbb{W}(k)\)-modules (see \([26]\)).

For any such a thickening \(S_0 \hookrightarrow S_n\) the functor \(\mathbb{D}\) can be further extended to define a crystal on the nilpotent crystalline site on \(S_0\) that is (equivalently given by) the Lie algebra of the associated universal \(\mathbb{G}_a\)-extension of the dual \(p\)-divisible group, by lifting it to \(S_n\) (cf. \([22, 11]\)). Therefore, by taking \(\mathbb{D}(M_0[p^\infty])\) we further obtain a filtered F-crystal on the crystalline site of \(S_0\), associated to the Barsotti-Tate group \(M_0[p^\infty]\). Recall that (see \([11]\)) the category of filtered \(F\)-\(\mathbb{W}(k)\)-modules consists of finitely generated \(\mathbb{W}(k)\)-modules endowed with an increasing filtration and a \(\sigma\)-linear\(^5\) operator, the Frobenius \(F\), respecting the filtration. Filtered F-crystals are the objects whose underlying \(\mathbb{W}(k)\)-modules are free and there exists a \(\sigma^{-1}\)-linear operator, the Verschiebung \(V\), such that \(V \circ F = F \circ V = p\).

2.4.2. The crystalline realizations of \(M_0\) over \(S_0 = \text{Spec}(k)\) are the following filtered \(F\)-crystals (see \([11 \S 1.3]\) where are also called Barsotti-Tate crystals of the 1-motive \(M_0\) and cf. \([27]\) and \([36, 4.7]\)). The contravariant one is

\[
T^{\text{crys}}(M_0) := \lim_{\longrightarrow n} \mathbb{D}(M_0[p^\infty])(S_0 \hookrightarrow S_n)
\]

and the covariant is

\[
T^{\text{crys}}(M_0) := \lim_{\longleftarrow n} \mathbb{D}(M_0[p^\infty]^\vee)(S_0 \hookrightarrow S_n)
\]

where \(M_0[p^\infty]^\vee\) is the Cartier dual. It follows from \([10]\) that \(T^{\text{crys}}(M_0)\) admits Frobenius and Verschiebung operators and a filtration (respected by Frobenius and Verschiebung).

\(^5\)Here \(\sigma\) is the Frobenius on \(\mathbb{W}(k)\).
2.4.3. We get $T_{\text{crys}}([0 \to \mathbb{G}_m]) = \mathbb{W}(k)(1)$ which is the filtered $\mathbb{F}$-crystal $\mathbb{W}(k)$, with filtration $W_i = \mathbb{W}(k)$ if $i \geq -2$ and $W_i = 0$ for $i < -2$ and with the $\sigma$-linear operator $F$ given by $1 \mapsto 1$ and the $\sigma^{-1}$-linear operator $V$ defined by $1 \mapsto p$.

2.5. **De Rham realization.** The De Rham realization of a 1-motive $M = [L \to G]$ over a suitable base scheme $S$ is obtained via Grothendieck’s idea of universal $\mathbb{G}_a$-extensions (cf. [12, §4], [23, 10.1.7] and [1]). Consider $\mathbb{G}_a$ as a complex of $S$-group schemes concentrated in degree 0. If $G$ is any $S$-group scheme such that $\text{Hom}(G, \mathbb{G}_a) = 0$ and $\text{Ext}(G, \mathbb{G}_a)$ is a locally free $\mathcal{O}_S$-module of finite rank, the universal $\mathbb{G}_a$-extension is an extension of $G$ by the (additive dual) vector group $\text{Ext}(G, \mathbb{G}_a)^\vee$ (see [12]).

2.5.1. Now for any 1-motive $M = [L \to G]$ over $S$, we have $\text{Hom}(M, \mathbb{G}_a) = 0$, and by the extension (1) $\mathcal{E}xt(M, \mathbb{G}_a)$ is of finite rank. Thus we obtain a universal $\mathbb{G}_a$-extension $M^\natural$, in Deligne’s notation [23, 10.1.7], where $M^\natural = [L \to G^\natural]$ is a complex of $S$-group schemes which is an extension of $M$ by $\mathcal{E}xt(M, \mathbb{G}_a)^\vee$ considered as a complex in degree zero. Here we have an extension of $S$-group schemes

$$0 \to \mathcal{E}xt(M, \mathbb{G}_a)^\vee \to G^\natural \to G \to 0$$

such that $G^\natural$ is the push-out of the universal $\mathbb{G}_a$-extension of the semi-abelian scheme $G$ along the inclusion of $\mathcal{E}xt(G, \mathbb{G}_a)^\vee$ into $\mathcal{E}xt(M, \mathbb{G}_a)^\vee$. The canonical map $u^\natural : L \to G^\natural$ such that the composition

$$L \xrightarrow{u^\natural} G^\natural \to \mathcal{H}om(L, \mathbb{G}_a)^\vee$$

is the natural evaluation map. The De Rham realization of $M$ is then defined as

$$T_{\text{DR}}(M) := \text{Lie } G^\natural,$$

with the Hodge-De Rham filtration given by

$$F^0T_{\text{DR}}(M) := \text{Ker}(\text{Lie } G^\natural \to \text{Lie } G) \cong \mathcal{E}xt(M, \mathbb{G}_a)^\vee$$

2.5.2. Over a base scheme on which $p$ is locally nilpotent there is a canonical and functorial isomorphism (see [1, Prop. 1.2.8])

$$(M[p^\infty])^\natural \times_{M[p^\infty]} G[p^\infty] \xrightarrow{\sim} G^\natural \times_G G[p^\infty]$$

where $(M[p^\infty])^\natural$ also denotes the universal $\mathbb{G}_a$-extension of a Barsotti-Tate group. In particular, we have a natural isomorphism of Lie algebras

$$\text{Lie } (M[p^\infty])^\natural \xrightarrow{\sim} \text{Lie } G^\natural$$

Note that $G^\natural$ is not the universal $\mathbb{G}_a$-extension of $G$ unless $L = 0$. 
2.5.3. For $S_0$ a scheme such that $p$ is locally nilpotent and $M_0 = [L_0 \to G_0]$ a 1-motive over $S_0$, let $S_0 \hookrightarrow S$ be a locally nilpotent pd thickening of $S_0$. Let $M$ and $M'$ be two 1-motives over $S$ lifting $M_0$. We have proven (see [1, §3]) that there is a canonical isomorphism $M^2 \cong (M')^2$ showing that the universal $\mathbb{G}_a$-extension is crystalline. Define the crystal of (2-terms complexes of) group schemes $M_0^2$ on the nilpotent crystalline site of $S_0$ as follows

$$M_0^2(S_0 \hookrightarrow S) := M^2$$

which we called the universal extension crystal of a 1-motive (see [1, §3]).

Applying it to $M_0$ defined over $S_0 = \text{Spec}(k)$ a perfect field and $S_n = \text{Spec}(\mathbb{W}_{n+1}(k))$ we see that the De Rham realization is a crystal indeed. Actually (see [1, §4] for details) the formula (11) yields:

2.5.4. Scholium ([1, Thm. A']). There is a comparison isomorphism of $F$-crystals

$$T_{\text{crys}}(M_0) = T_{\text{DR}}(M)$$

for any (formal) lifting $M$ over $\mathbb{W}(k)$ of $M_0$ over $k$.

2.5.5. If $k = \mathbb{C}$ then the De Rham realization is also compatible with the Hodge realization; we have

$$T_{\text{DR}}(M) = T_{\text{Hodge}}(M) \otimes \mathbb{C}$$

as bifiltered $\mathbb{C}$-vector spaces, i.e., we have that $H_1(G^2, \mathbb{Z}) = H_1(G, \mathbb{Z})$ thus $T_C(M) := T_Z(M) \otimes \mathbb{C} \cong \text{Lie}G^2$ and $M^2 = [L \to T_C(M)/H_1(G, \mathbb{Z})]$, see [23, §10.1.8].

2.6. Paradigma. Let $X$ be a (smooth) projective variety over $k = \overline{k}$. Let $\text{Pic}_{X/k}$ be the Picard scheme and $\text{Pic}_{X/k}^{0,\text{red}}$ the connected component of the identity endowed with its reduced structure. Recall that $\text{NS}_X := \pi_0(\text{Pic}_{X/k})$ is finitely generated and $\text{Pic}_{X/k}^{0,\text{red}}$ is divisible. Recall that we always have

$$H^1_{\text{fppf}}(X, \mu_n) = \text{Pic}(X)_{n-\text{tor}}$$

Therefore

$$T_{\ell}([0 \to \text{Pic}_{X/k}^{0,\text{red}}]) = H^1_{\text{fppf}}(X, \mathbb{Z}_{\ell}(1))$$

If $\ell \neq \text{char}(k)$ then the étale topology will be enough.
2.6.1. Let Pic\(^{\natural}\)(X) be the group of isomorphism classes of pairs \((L, \nabla)\) where \(L\) is a line bundle on \(X\) and \(\nabla\) is an integrable connection on \(L\). In characteristic zero then there is the following extension

\[
0 \rightarrow H^0(X, \Omega^1_X) \rightarrow \text{Pic}^{\natural,0}(X) \rightarrow \text{Pic}^0(X) \rightarrow 0
\]

where Pic\(^{\natural,0}\) is the subgroup of those pairs \((L, \nabla)\) such that \(L \in \text{Pic}^0\). The above extension is the group of \(k\)-points of the universal \(G_a\)-extension of the abelian variety Pic\(^0\)(X/k), Lie Pic\(^0\)(X) = \(H^1(X, O_X)\) and

\[
\text{Lie Pic}^{\natural,0}(X) = H^1_{DR}(X/k)
\]
as \(k\)-vector spaces (as soon as the De Rham spectral sequence degenerates). Moreover, for \(k = \mathbb{C}\), the exponential sequence grants

\[
T_{\text{Hodge}}([0 \rightarrow \text{Pic}^0_{X/\mathbb{C}}]) = H^1(X, \mathbb{Z}(1)).
\]

2.6.2. In general, for an abelian \(S\)-scheme \(A\) (in any characteristics \(cf. \ [42, \S 4]\)) we have \((A^\vee)^{\natural} = \text{Pic}^{\natural,0}_{A/S}\), so that the dual of \(A\) has De Rham realization

\[
T_{\text{DR}}([0 \rightarrow A^\vee]) = H^1_{DR}(A/S)(1)
\]
where the twist (1) indicates that the indexing of the Hodge-De Rham filtration is shifted by 1 (\(cf. \ [11, \S 2.6.3]\)).

However, for \(X\) (smooth and proper) over a perfect field \(k\) of characteristic \(p > 0\), the \(k\)-vector space \(H^1_{DR}(X/k)\) cannot be recovered from the Picard scheme (as remarked by Oda \([48]\)). The subspace obtained via the Picard scheme is closely related to crystalline cohomology (see \([48, \S 5]\)).

2.6.3. Let \(X\) be smooth and proper over a perfect field \(k\) of characteristic \(p > 0\). Let Pic\(^{crys,0}\)(X) be the sheaf on the fppf site on \(S_n = \text{Spec}(W_{n+1}(k))\) given by the functor associating to \(T\) the group of isomorphism classes of crystals of invertible \(\mathcal{O}^{crys}_{X\times S_n T/T}\)-modules (which are algebraically equivalent to 0 when restricted to the Zariski site). Such Pic\(^{crys}\) is the natural substitute of the previous functor Pic\(^{\natural}\) and we can think \(H^1_{crys}\) as Lie Pic\(^{crys}\) (see \(\Pi\) for details, \(cf. \ [16]\)). In fact, the \(S_n[\varepsilon]\)-points of Pic\(^{crys}\) reducing to the identity modulo \(\varepsilon\) are the infinitesimal deformations of \(\mathcal{O}^{crys}_{X/S_n}\). For \(A_0\), an abelian variety over \(S_0 = \text{Spec}(k)\), and an abelian scheme \(A_n\) over \(S_n\) lifting \(A_0\), the category of crystals of invertible \(\mathcal{O}^{crys}_{A_n \times S_n S_0[\varepsilon]/S_n[\varepsilon]}\)-modules over the nilpotent crystalline site of \(A_0 \times_{S_0} S_0[\varepsilon]\) relative to \(S_n[\varepsilon]\) is equivalent to the category of line bundles over \(A_n[\varepsilon]\) with integrable connection. Hence, we have an isomorphism of sheaves over the fppf site of \(S_n\)

\[
(A_n^\vee)^{\natural} \cong \text{Pic}^{\natural,0}_{A_n/S_n} \cong \text{Pic}^{crys,0}_{A_0/S_n}
\]
and passing to Lie, we get a natural isomorphism of \(\mathcal{O}_{S_n}\)-modules

\[
T^\text{crys}(A_0) \otimes \mathcal{O}_{S_n} \cong \text{Lie}(A_0^\vee)^{\text{t}} \cong \text{Lie} \, \text{Pic}^{\text{crys}, 0}_{A_0/S_n} \cong H^1_{\text{crys}}(A_0/S_n).
\]

2.6.4. By applying the previous arguments to the Albanese variety \(\text{Alb}(X) = (\text{Pic}^0_{X/k})^\vee = A_0\) we see that \(\text{Lie} \, \text{Pic}^{\text{crys}, 0}_{\text{Alb}(X)}\) can be identified to the Lie algebra of the universal extension of a (formal) lifting of \(\text{Pic}^0_{X/k}\) to the Witt vectors. The Albanese mapping is further inducing a canonical isomorphism\(^7\) (cf. [37, II.3.11.2], [10] and [1])

\[
\text{Lie} \, \text{Pic}^{\text{crys}, 0}_{\text{Alb}(X)} \cong \text{Lie} \, \text{Pic}^{\text{crys}, 0}(X).
\]

In conclusion, we have

\[
T^\text{crys}([0 \to \text{Pic}^0_{X/k}]) \cong H^1_{\text{crys}}(X/\mathbb{W}(k))
\]

for \(X\) a smooth proper \(k\)-scheme.

2.7. **Cartier duality.** For \(H = T_{H^{\text{ord}}}(M)\), \(H^\vee = \text{Hom}(H, \mathbb{Z}(1))\) is an implicit definition (see [2.2]) of the dual \(M^\vee\) of a 1-motive \(M\) over \(\mathbb{C}\). In general, Deligne [23, §10.2.11–13] provided an extension of Cartier duality to (free) 1-motives showing that it is compatible with such Hodge theoretic involution. The main deal here is the yoga of Grothendieck biextensions (see [44, 31 VII 2.1]) and [23 §10.2.1]).

2.7.1. A Grothendieck (commutative) biextension \(P\) of \(G_1\) and \(G_2\) by \(H\) is an \(H\)-torsor on \(G_1 \times G_2\) along with a structure of compatible isomorphisms of torsors \(P_{g_1, g_2} \cong P_{g_1'g_2'}\) and \(P_{g_1, g_2'} \cong P_{g_1g_2g_2'}\) (including associativity and commutativity) for all points \(g_1, g_1', g_2, g_2'\) of \(G_1\) and \(G_2\). Recall that an isomorphism class of a Grothendieck biextension (as commutative groups in a Grothendieck topos) can be essentially translated by the formula (see [31 VII 3.6.5])

\[
\text{Biext}(G_1, G_2; H) = \text{Ext}(G_1 \otimes^L G_2, H).
\]

Here we further have \(\text{Ext}(G_1 \otimes^L G_2, H) = \text{Ext}(G_1, \text{RHom}(G_2, H))\) and the canonical spectral sequence

\[
E^{p, q}_{2} = \text{Ext}^p(G_1, \text{Ext}^q(G_2, H)) \Rightarrow \text{Ext}^{p+q}(G_1, \text{RHom}(G_2, H))
\]

yields an exact sequence of low degree terms

\[
0 \to \text{Ext}(G_1, \mathcal{H}om(G_2, H)) \to \text{Biext}(G_1, G_2; H) \to \text{Hom}(G_1, \mathcal{E}xt(G_2, H))
\]

\[
\to \text{Ext}^2(G_1, \mathcal{H}om(G_2, H))
\]

If \(\mathcal{H}om(G_2, H) = 0\) then \(\partial : \text{Biext}(G_1, G_2; H) \cong \text{Hom}(G_1, \mathcal{E}xt(G_2, H))\).

In particular, for \(H = \mathbb{G}_m\) and \(G_2 = A\) an abelian scheme, since \(A^\vee = \)

\(^7\)Note that \(H^0(\text{Alb}(X), \Omega_X^1) \neq H^0(X, \Omega_X^1)\) in general, in positive characteristics.
$\mathcal{E}xt(A, \mathbb{G}_m)$ for abelian schemes, this isomorphism $\partial$ reduces to the more classical isomorphism (cf. [8, 4.1.3] and 2.8 below).

\[ \text{Hom}(\cdot, A^\vee) \xrightarrow{\cong} \text{Biext}(\cdot, A; \mathbb{G}_m) \]
given by $f \mapsto (f \times 1)^*P_A$ pulling back the (transposed) Poincaré $\mathbb{G}_m$-bixetension $P_A$ of $A$ and $A^\vee$, i.e., the functor $\text{Biext}(\cdot, A; \mathbb{G}_m)$ is representable by the dual abelian scheme.

If $G_1$ and $G_2$ are semi-abelian schemes we further have

\[ \text{Biext}(A_1, A_2; \mathbb{G}_m) \cong \text{Biext}(G_1, G_2; \mathbb{G}_m) \]

by pullback from the abelian quotients $A_1$ and $A_2$ (see [31, VIII 3.5-6]). Actually, we can regard bixensions of smooth connected group schemes (over a perfect base field) $G_1$ and $G_2$ by $\mathbb{G}_m$ as invertible sheaves on $G_1 \times G_2$ birigified with respect to the identity sections (see [31, VIII 4.3]).

2.7.2. Now let $M_i = [L_i \xrightarrow{u_i} G_i]$ for $i = 1, 2$ be two 2-terms complexes of sheaves. A biextension $(P, \tau, \sigma)$ of $M_1$ and $M_2$ by an abelian sheaf $H$ is given by (i) a Grothendieck biextension $P$ of $G_1$ and $G_2$ by $H$ and a pair of compatible trivializations, i.e., (ii) a biadditive section $\tau$ of the biextension $(1 \times u_2)^*(P)$ over $G_1 \times L_2$, and (iii) a biadditive section $\sigma$ of the biextension $(u_1 \times 1)^*(P)$ over $L_1 \times G_2$, such that (iv) the two induced sections $\tau|_{L_1 \times L_2} = \sigma|_{L_1 \times L_2}$ coincide.

Let $\text{Biext}(M_1, M_2; H)$ denote the group of isomorphism classes of bixensions. We still have the following fundamental formula (see [23, §10.2.1])

\[ \text{Biext}(M_1, M_2; H) = \text{Ext}(M_1 \otimes M_2, H) \]

here $\text{Ext}(M_1 \otimes M_2, H) = \text{Ext}(M_1, \text{RHom}(M_2, H))$ where $M_i$ is considered a complex of sheaves concentrated in degree $-1$ and $0$.

2.7.3. Let $M = [L \xrightarrow{u} G]$ be a 1-motive where $G$ is an extension of an abelian scheme $A$ by a torus $T$. The main goal is that the functor on 1-motives

\[ N \mapsto \text{Biext}(N, M; \mathbb{G}_m) \]
is representable, i.e., there is a Cartier dual $M^\vee = [T^\vee \xrightarrow{u^\vee} G^\vee]$ such that

\[ \text{Hom}(N, M^\vee) \xrightarrow{\cong} \text{Biext}(N, M; \mathbb{G}_m) \]
is given by pulling back the Poincaré bixextension generalizing (12). More precisely, it is given by $\varphi \mapsto (\varphi \times 1)^*P^\vee_M$, where the Poincaré $\mathbb{G}_m$-bixextension $P^\vee_M$ is simply obtained from that of $A$ and $A^\vee$ by further pullback to $G$ and $G^\vee$ according to the above (and below) description. See [23, 10.2.11] and [11, 1.5] for the construction of $M^\vee$ and [8, 4.1.1] for the representability (13). The Cartier dual can be described in the following way:
• For $M = [0 \to G]$ we have $M^\vee = [T^\vee \xrightarrow{u^\vee} A^\vee]$ where $T^\vee = \text{Hom}(T, \mathbb{G}_m)$ is the character group of $T$ and $u^\vee$ is the canonical homomorphism pushing out characters $T \to \mathbb{G}_m$ along the given extension $G$ of $A$ by $T$.

• For $M = [L \to A]$ we have $M^\vee = [0 \to G^u]$ where $G^u$ denote the group scheme which represents the functor associated to $\text{Ext}(M, \mathbb{G}_m)$. Here $\text{Ext}(M, \mathbb{G}_m)$ consists of extensions of $A$ by $\mathbb{G}_m$ together with a trivialization of the pull-back on $L$. In particular $[L \to 0]^\vee = \text{Hom}(L, \mathbb{G}_m)$.

• In general, the standard extension $M = [L \to G]$ of $M/W_{-2}M = [L \to A]$ by $W_{-2}M = [0 \to T]$ provides via $\text{Ext}(M/W_{-2}M, \mathbb{G}_m)$ the corresponding extension $G^u$ of $A^\vee$ by $\text{Hom}(L, \mathbb{G}_m)$ and a boundary map

$$u^\vee : \text{Hom}(W_{-2}M, \mathbb{G}_m) \to \text{Ext}(M/W_{-2}M, \mathbb{G}_m)$$

lifting $T^\vee \to A^\vee$ as above.

2.7.4. A biextension is also providing natural pairings in realizations (see [31, VIII 2] and [23, 10.2]). In fact, for Grothendieck biextensions we also have an exact sequence

$$0 \to \text{Ext}(G_1 \otimes G_2, H) \to \text{Biext}(G_1, G_2; H) \to \text{Hom}(\mathcal{T}or (G_1, G_2), H) \to \text{Ext}^2(G_1 \otimes G_2, H)$$

and a natural map $T_\ell(G_1) \otimes T_\ell(G_2) \to T_\ell(\mathcal{T}or (G_1, G_2))$ (see [31, VIII 2.1.13]) yielding a map

$$\text{Hom}(\mathcal{T}or (G_1, G_2), H) \to \text{Hom}(T_\ell(G_1) \otimes T_\ell(G_2), T_\ell(H))$$

which in turns, by composition, provides a map (see [31, VIII 2.2.3])

$$\text{Biext}(G_1, G_2; H) \to \text{Hom}(T_\ell(G_1) \otimes T_\ell(G_2), T_\ell(H)).$$

Similarly (non trivially! cf. [23, §10.2.3-9] and [13]) a biextension $P$ of 1-motives $M_1$ and $M_2$ by $H = \mathbb{G}_m$ provides the following pairings: $T_\ell(M_1) \otimes T_\ell(M_2) \to T_\ell(\mathbb{G}_m)$ and $T_{DR}(M_1) \otimes T_{DR}(M_2) \to T_{DR}(\mathbb{G}_m)$. This latter pairing on De Rham realizations is obtained by pulling back $P$ to a $\ell$-biextension $P^\ell$ of $M_1^\ell$ and $M_2^\ell$ by $\mathbb{G}_m$. The Poincaré biextension $\mathcal{P}_M$ of $M$ and $M^\vee$ by $\mathbb{G}_m$ is then providing compatibilities between the Cartier dual of a 1-motive and the Cartier dual of its realizations. Moreover, over a base such that $p$ is locally nilpotent, the Poincaré biextension is crystalline (see [11, 3.4]) providing the Poincaré crystal of biextensions $\mathcal{P}_0^\ell$ of $M_0^\ell$ and $(M_0^\vee)^\ell$ thus a pairing of $F$-crystals $T_{crys}(M_0) \otimes T_{crys}(M_0^\vee) \to T_{crys}(\mathbb{G}_m)$. We also have:
2.7.5. Scholium (23 10.2.3)]. If $M_1$ and $M_2$ are defined over $\mathbb{C}$ then there is a natural isomorphism

$$\text{Biext}(M_1, M_2; \mathbb{G}_m) \cong \text{Hom}_{\text{MHS}}(T_{\text{Hodge}}(M_1) \otimes T_{\text{Hodge}}(M_2), \mathbb{Z}(1))$$

Over $\mathbb{C}$, all these pairings on the realizations are deduced from Hodge theory.

2.8. Symmetric avatar. For a Deligne 1-motive $M = [L \xrightarrow{u} G]$ and its Cartier dual $M^\vee = [T^\vee \xrightarrow{u} G^\vee]$ the Poincaré biextension $\mathcal{P}_M = (\mathcal{P}_A, \tau, \sigma)$ of $M$ and $M^\vee$ by $\mathbb{G}_m$ is canonically trivialized on $L \times T^\vee$ by $\psi := \tau_{L \times T^\vee} = \sigma_{L \times T^\vee}$ given by the push-out map $\psi : G \to \chi_* G$ along the character $\chi : T \to \mathbb{G}_m$, i.e., we have

$$0 \to T \to G \to A \to 0$$

and $\psi(x, \chi) = \psi (\chi (u(x)) \in \chi_* G_{\mathfrak{m}(x)} = (\mathcal{P}_A)[\mathfrak{m}(x), \mathfrak{m}^\vee (\chi)]$ where $u : L \to G \to A$ and $\chi_* G = \mathfrak{m}^\vee (\chi)$. Actually, the data of $u : L \to L'$, $u' : T^\vee \to A^\vee$ and $\psi$ determine both $M$ and $M^\vee$ under the slogan

trivializations $\iff$ liftings

For example, for $\chi^1, \ldots, \chi^r$ a basis of $T^\vee$ we can regard $G$ as the the pull-back of $A$ diagonally embedded in $A^r$ as follows

$$0 \to T \to G \to A \to 0$$

and $(\psi(x, \chi^1), \ldots, \psi(x, \chi^r))$ provides a point of $G$ lifting $u(x)$.

2.8.1. The symmetric avatar can be abstractly defined as $(L \xrightarrow{u} A, L' \xrightarrow{u'} A', \psi)$ where $L$, $L'$ are lattices, $A'$ is dual to $A$ and $\psi : L \times L' \to (u \times u')^*(\mathcal{P}_A)$ is a trivialization of the Poincaré biextension when restricted to $L \times L'$ (cf. 23 10.2.12]). In order to make up a category we define morphisms between symmetric avatars by pairs of commutative squares such that the trivializations are compatible, i.e., a map

$$(L_1 \xrightarrow{u_1} A_1, L'_1 \xrightarrow{u'_1} A'_1, \psi_1) \to (L_2 \xrightarrow{u_2} A_2, L'_2 \xrightarrow{u'_2} A'_2, \psi_2)$$

is a map $f : A_1 \to A_2$ along with its dual $f' : A'_2 \to A'_1$ and a pair of liftings $g : L_1 \to L_2$ of $fu_1$ and $g' : L'_2 \to L'_1$ of $f'u'_2$ such that

$$\psi_1|_{L_1 \times L'_2} = \psi_2|_{L_1 \times L'_2}.$$
2.8.2. **Scholium ([23] 10.2.14]).** There is an equivalence of categories

\[ M \mapsto (L \xrightarrow{\overline{\pi}} A, T^\vee \xrightarrow{\overline{\psi}} A^\vee, \psi) : \mathcal{M}_1^{fr} \xrightarrow{\simeq} \mathcal{M}_1^{sym} \]

Under this equivalence Cartier duality is

\[(L \xrightarrow{\overline{\pi}} A, T^\vee \xrightarrow{\overline{\psi}} A^\vee, \psi) \mapsto (T^\vee \xrightarrow{\overline{\psi}} A^\vee, L \xrightarrow{\overline{\pi}} A, \psi^t).\]

2.8.3. For the sake of exposition we sketch how to construct a map of symmetric avatars out of any biextension (almost proving [13], see [8] 4.1 for more details). Let \((P, \tau, \sigma)\) be a \(\mathbb{G}_m\)-biextension of Deligne 1-motives \(M_1\) and \(M_2\). Translating via extensions, \(P\) corresponds to a map \(f : A_1 \to A_2^\vee\) and the trivialization \(\tau\) corresponds to a lifting \(g : L_1 \to T_2^\vee\) of \(f\). Here we have \(P \sim \overline{f\pi_1} \sim 0 \in \text{Hom}(L_1, \text{Ext}(G_2, \mathbb{G}_m)) = \text{Biext}(L_1, G_2; \mathbb{G}_m)\) where

\[0 \to \text{Hom}(G_2, \mathbb{G}_m) \to T_2^\vee \xrightarrow{\overline{\psi}} A_2^\vee \to \text{Ext}(G_2, \mathbb{G}_m) \to 0\]

granting the existence of \(g\) such that \(\overline{\pi_2}g = \overline{f\pi_1}\). Moreover \(\overline{f\pi_1} \sim [E] = 0 \in \text{Ext}(L_1 \otimes G_2, \mathbb{G}_m) = \text{Biext}(L_1, G_2; \mathbb{G}_m)\) and any section (= trivialization) of such trivial \(\mathbb{G}_m\)-extension \(E\) is exactly given by an element \(g \in \text{Hom}(L_1, T_2^\vee) = \text{Hom}(L_1 \otimes T_2, \mathbb{G}_m)\) as above. Since \(P^t\) corresponds to the dual \(f^\vee : A_2 \to A_1^\vee\) we have that \(\sigma\) also corresponds to a lifting \(g' : L_2 \to T_1^\vee\) of \(f^\vee \overline{\pi_2}\) yielding \(\overline{\pi'_2}g' = f^\vee \overline{\pi_2}\). Moreover, since \(P\) is a pull-back of \((f \times 1_{A_2})^*(\mathcal{P}_A_2)\) then the trivialization \(\tau\) is the pull-back along \(g \times 1 : L_1 \times G_2 \to T_2^\vee \times G_2\) of the canonical trivialization \(\psi_2^t\) on \(T_2^\vee \times G_2\) given by the identity.\(^8\) Since \(P\) is also a pull-back of \((1_{A_1} \times f^\vee)^*(\mathcal{P}_A_1)\) the trivialization \(\sigma\) on \(G_1 \times L_2\) is the pull-back of the canonical trivialization \(\psi_1\) on \(G_1 \times T_2^\vee\) along \(1 \times g' : G_1 \times L_2 \to G_1 \times T_2^\vee\). Thus, if we further pull-back to \(L_1 \times L_2\) we get

\[\psi_2^t \mid_{L_1 \times L_2} = \tau \mid_{L_1 \times L_2} = \sigma \mid_{L_1 \times L_2} = \psi_1 \mid_{L_1 \times L_2}\]

by assumption. We therefore get a map

\[(L_1 \xrightarrow{\overline{\pi}} A_1, T_1^\vee \xrightarrow{\overline{\psi}} A_1^\vee, \psi_1) \mapsto (T_2^\vee \xrightarrow{\overline{\psi}} A_2^\vee, L_2 \xrightarrow{\overline{\pi}} A_2, \psi_2^t)\]

which turns back a map \(M_1 \to M_2^\vee\).

\(^8\)Note that for the Poincaré biextension we have that the resulting \(f, f^\vee, g\) and \(g'\) are all identities.
2.9. 1-motives with torsion. An effective 1-motive which admits torsion (see [9, §1] and [8]) is \( M = [L \xrightarrow{u} G] \) where \( L \) is a locally constant (for the étale topology) finitely generated abelian group and \( G \) is a semi-abelian scheme. Here \( L \) can be represented by an extension
\[
0 \to L_{\text{tor}} \to L \to L_{\text{fr}} \to 0
\]
where \( L_{\text{tor}} \) is finite and \( L_{\text{fr}} \) is free. An effective map from \( M = [L \xrightarrow{u} G] \) to \( M' = [L' \xrightarrow{u'} G'] \) is a commutative square and \( \text{Hom}_{\text{eff}}(M, M') \) denote the abelian group of effective morphisms. The corresponding category is denoted \( \mathcal{M}_1^{\text{eff}} \). We clearly have that \( \mathcal{M}_1^{\text{fr}} \subset \mathcal{M}_1^{\text{eff}} \). For \( M = [L \xrightarrow{u} G] \) we set
\[
\begin{align*}
M_{\text{fr}} &:= [L_{\text{fr}} \xrightarrow{u} G/u(L_{\text{tor}})] \\
M_{\text{tor}} &:= [\ker(u) \cap L_{\text{tor}} \to 0] \\
M_{\text{tf}} &:= [L/\ker(u) \cap L_{\text{tor}} \xrightarrow{u} G]
\end{align*}
\]
considered as effective 1-motives. We say that \( M \) is torsion if \( L \) is torsion and \( G = 0 \), \( M \) is torsion-free if \( \ker(u) \cap L_{\text{tor}} = 0 \) and free if \( L \) is free.

There are canonical effective maps \( M \to M_{\text{tf}}, M_{\text{tor}} \to M \) and \( M_{\text{tf}} \to M_{\text{fr}} \).

2.9.1. A quasi-isomorphism (q.i. for short) of 1-motives \( M \to M' \) is a q.i. of complexes of group schemes. Actually, an effective map of 1-motives \( M = [L \xrightarrow{u} G] \to M' = [L' \xrightarrow{u'} G'] \) is a q.i. of complexes if and only if we have that \( \ker(u) = \ker(u') \) and \( \text{coker}(u) = \text{coker}(u') \) and thus \( \ker \) and \( \text{coker} \) of \( L \to L' \) and \( G \to G' \) are equal. Then \( \text{coker}(G \to G') = 0 \), since it is connected and discrete, and \( \ker(G \to G') \) is a finite group. Therefore a q.i. of 1-motives is given by an isogeny \( G \to G' \) such that \( L \) is the pull-back of \( L' \), i.e.,
\[
\begin{array}{cccc}
0 & E & \to & G & \to & G' & \to & 0 \\
\parallel & u & \uparrow & \ker(u) & \cap & L_{\text{tor}} & \to & 0 \\
0 & E & \to & L & \to & L' & \to & 0
\end{array}
\]
where \( E \) is a finite group. We then define morphisms of 1-motives by localizing \( \mathcal{M}_1^{\text{eff}} \) at the class of q.i. and thus set
\[
\text{Hom}(M, M') := \lim_{\text{q.i.}} \text{Hom}_{\text{eff}}(\widehat{M}, M')
\]
where the limit is taken over q.i. \( \widehat{M} \to M \) as above. We then have a well-defined composition of morphisms of 1-motives (see [9, 1.2])
\[
\text{Hom}(M, M') \times \text{Hom}(M', M'') \to \text{Hom}(M, M'').
\]
In fact, for any effective morphism $\tilde{M} \to M'$ and any q.i. $\tilde{M}' \to M'$, there exists a q.i. $\tilde{M} \to \tilde{M}'$ together with an effective morphism $\tilde{M} \to M'$ making up a commutative diagram (such that $\tilde{M} \to \tilde{M}'$ is uniquely determined).

2.9.2. Denote the resulting category by $\mathcal{M}_1$, i.e., objects are effective 1-motives and morphisms from $M$ to $M'$ can be represented by a q.i. $\tilde{M} \to M$ and an effective morphism $\tilde{M} \to M'$. This category has been introduced in [9] and it is further investigated in [8]. The main basic facts are the following:

- $\mathcal{M}_1$ is an abelian category where exact sequences can be represented by effective exact sequences of two terms complexes;
- $\mathcal{M}^{fr}_1 \subset \mathcal{M}_1$ is a Quillen exact sub-category such that $M \mapsto M_{fr}$ is left-adjoint to the embedding, i.e., we have $\text{Hom}_{\text{eff}}(M_{fr}, M') = \text{Hom}(M, M')$ for $M \in \mathcal{M}_1$ and $M' \in \mathcal{M}^{fr}_1$.

Actually, we have

$$\text{Hom}_{\text{eff}}(M, M') = \text{Hom}(M, M')$$

for $M \in \mathcal{M}_1$ and $M' \in \mathcal{M}^{fr}_1$. Clearly, this is according with a corresponding Scholium [2.1.4] for the functor $\iota: \mathcal{M}_1 \to D^b(k_{fppf})$ which is still faithful but, in general, not full for effective morphisms. A key point in order to show that $\mathcal{M}_1$ is abelian is the following.

2.9.3. Scholium ([9 Prop. 1.3]). Any effective morphism $M \to M'$ can be factored as follows

$$M \longrightarrow M'$$

$$\downarrow \quad \uparrow$$

$$M \quad \tilde{M}$$

where $M \to \tilde{M}$ is an effective morphism such that the kernel of the morphism of semi-abelian varieties is connected, i.e., a strict morphism, and $\tilde{M} \to M'$ is a q.i.

For example, in the following canonical factorisation induced by [14]

$$M \longrightarrow M_{fr}$$

$$\downarrow \quad \uparrow$$

$$M_{tf}$$

the effective map $M \to M_{tf}$ is a strict epimorphism with kernel $M_{tor}$ and $M_{tf} \to M_{fr}$ is a q.i. We then always have a canonical exact sequence in $\mathcal{M}_1$

$$0 \to M_{tor} \to M \to M_{fr} \to 0$$
We further have that the Hodge realization (see [2.2] naturally extends to \( \mathcal{M}_1(\mathbb{C}) \) (see [9, Prop. 1.5]) and the functor

\[
T_{\text{Hodge}} : \mathcal{M}_1(\mathbb{C}) \to \text{MHS}_1
\]

is an equivalence between the category of 1-motives with torsion over \( \mathbb{C} \) and the category of \( \mathbb{Z} \)-mixed Hodge structures introduced in 2.2 above. Similarly, the other realizations extend to \( \mathcal{M}_1 \), e.g., (cf. [9, 8] and 2.3) let \( \mathcal{M}/\ell^\nu \) be the torsion 1-motive (= finite group) given by the cokernel of \( \ell^\nu : \mathcal{M} \to \mathcal{M} \) the effective multiplication by \( \ell^\nu \) which is fitting in an exact sequence (of finite groups)

\[
0 \to \ell^\nu M \to \ell^\nu L \to \ell^\nu G \to M/\ell^\nu \to L/\ell^\nu \to 0
\]

and set

\[
T_\ell(M) := \lim_{\nu} \mathcal{M}/\ell^\nu
\]

Remark that Cartier duality does not extends to \( \mathcal{M}_1 \): such category \( \mathcal{M}_1 \) is just an algebraic version of \( \text{MHS}_1 \).

2.9.4. In [8] (cf. 2.10 below) we also consider larger categories of non-connected 1-motives, e.g., \([L \to G]\) where \( G \) is a reduced group scheme locally of finite type over \( k \) such that \( G^0 \) is semi-abelian\(^9\) and \( \pi_0(G) \) is finitely generated. If \( M = [L \to G] \) is non-connected we get an effective 1-motive

\[
M^0 := [L^0 \to G^0]
\]

where \( L^0 \subseteq L \) is the subgroup of those elements mapping to \( G^0 \) and

\[
\pi_0(M) := [L/L_0 \hookrightarrow \pi_0(G)]
\]

is a discrete object.

2.10. 1-motives up to isogenies. For any additive category \( C \) denote \( C^\mathbb{Q} \) the \( \mathbb{Q} \)-linear category obtained from \( C \) by tensoring morphisms by \( \mathbb{Q} \).

Let \( C_1 := C^{[-1,0]}(\text{Shv}(k_{\acute{e}t})) \) be the category of complexes of étale sheaves of length 1 over \( \text{Spec} \, k \). Then \( C_1 \) and \( C_1^\mathbb{Q} \) are abelian categories. We may view \( \mathcal{M}_1^{\text{fr}} \) and \( \mathcal{M}_1^{\text{eff}} \) as full subcategories of \( C_1 \), hence \( \mathcal{M}_1^{\text{fr,}\mathbb{Q}} \) and \( \mathcal{M}_1^{\text{eff,}\mathbb{Q}} \) as a full subcategory of \( C_1^\mathbb{Q} \). The abelian category of 1-motives up to isogenies can be regarded via the following equivalences

\[
\mathcal{M}_1^{\text{fr,}\mathbb{Q}} \simeq \mathcal{M}_1^{\text{eff,}\mathbb{Q}} \simeq \mathcal{M}_1^\mathbb{Q}
\]

since torsion 1-motives vanish and q.i. of 1-motives are isomorphism in \( \mathcal{M}_1^{\text{eff,}\mathbb{Q}} \). Furthermore, let \( \mathcal{M}_1^{\text{nc}} \) be the full subcategory of \( C_1 \) consisting of non-connected 1-motives, i.e., complexes of the form \([L \to G]\) where \( L \)

\(^9\)Note that this condition can also be achieved by Murre’s axiomatic [11, Appendix A.1].
is finitely generated and $G$ is a commutative algebraic group whose connected component of the identity $G^0$ is semi-abelian (see [8]). We have that $\mathcal{M}^\text{nc}_1 \subset \mathcal{C}_1$ is an abelian (thick) subcategory of $\mathcal{C}_1$. For $M \in \mathcal{M}^\text{nc}_1$ we have that $M^0 \hookrightarrow M$ and $M^0 \rightarrow M^0_{fr}$ are isomorphisms in $\mathcal{M}^\text{nc,Q}_1$. Thus $\mathcal{M}^\text{fr,Q}_1 \simeq \mathcal{M}^\text{nc,Q}_1$ is an equivalence of abelian categories.

2.10.1. Scholium ([8, 1.1.3]). The category of Deligne 1-motives up to isogeny is equivalent to the abelian $\mathbb{Q}$-linear category given by complexes of étale sheaves $[L \rightarrow G]$ where $L$ is (locally constant) finitely generated and $G$ is a commutative algebraic group whose connected component of the identity $G^0$ is semi-abelian. Finally, this category $\mathcal{M}^\text{Q}_1$ is of cohomological dimension $\leq 1$, i.e., if $\text{Ext}^i(M, M') \neq 0$, for $M, M' \in \mathcal{M}^\text{Q}_1$, then $i = 0$ or $1$ ([49, Prop. 3.13]) and, clearly, the Scholium 2.1.4 holds for $\mathcal{M}^\text{Q}_1$ as well.

2.11. Universal realization and triangulated 1-motives. I briefly mention some results from [57], [49] and [8]. Considering the derived category of Deligne 1-motives up to isogeny we have a ‘universal realization’ in Voevodsky’s triangulated category of motives. Notably, this realization has a left adjoint: the ‘motivic Albanese complex’.

2.11.1. Recall that any abelian group scheme may be regarded as an étale sheaf with transfers (see [57] for this notion and cf. [43]). Moreover, a 1-motive $M = [L \rightarrow G]$ is a complex of étale sheaves where $L$ and the extension $G$ of $A$ by $T$ are clearly homotopy invariants. Thus a 1-motive $M$ gives rise to an effective complex of homotopy invariant étale sheaves with transfers, hence to an object of $\text{DM}^\text{eff}_{-\text{ét}}(k)$ (see [57, Sect. 3] for motivic complexes over a field $k$).

Regarding 1-motives up to isogeny Nisnevich sheaves will be enough as $\text{DM}^\text{eff}_{-\text{ét}}(k; \mathbb{Q}) \cong \text{DM}^\text{eff}_{-\text{ét}}(k; \mathbb{Q})$ is an equivalence of triangulated categories (see [57, Prop. 3.3.2] and [49, Th. 14.22]).

The triangulated category of effective geometrical motives $\text{DM}^\text{eff}_{\text{gm}}(k; \mathbb{Q})$ is the full triangulated sub-category of $\text{DM}^\text{eff}(k; \mathbb{Q})$ generated by motives of smooth varieties: here the motive of $X$ denoted $M(X) \in \text{DM}^\text{eff}(k)$ is defined in [57] by the Suslin complex $\mathcal{C}_X$ of the representable presheaf with transfers $L(X)$ associated to $X$ smooth over $k$. The motivic complexes provided by 1-motives up to isogeny actually belong to $\text{DM}^\text{eff}_{\text{gm}}(k; \mathbb{Q})$ (cf. [49] and [8]).

2.11.2. Scholium ([57, Sect. 3.4, on page 218] [49]). There is a fully faithful functor

$$\text{Tot} : D^b(\mathcal{M}^\text{Q}_1) \xrightarrow{\sim} d_{\leq} \text{DM}^\text{eff}_{\text{gm}}(k) \subseteq \text{DM}^\text{eff}_{\text{gm}}(k; \mathbb{Q})$$
whose essential image is the thick triangulated subcategory $d_{\leq 1} \text{DM}^\text{eff}_{\text{gm}}(k) \subseteq \text{DM}^\text{eff}(k)$ generated by motives of smooth varieties of dimension $\leq 1$.

Actually, in [8] we show that $D^b(\text{M}^{\text{fr}_1}) = D^b(\text{M}_1)$ and we also refine this embedding to an integrally defined embedding of $D^b(\text{M}_1)[1/p]$ (where $p$ is the exponential characteristic) into the étale version $\text{DM}^\text{eff}_{\text{gm, ét}}(k)$ of Voevodsky’s category. The homotopy t-structure on $\text{DM}^\text{eff}_{\text{gm, ét}}(k)$ induces a t-structure on $D^b(\text{M}_1) \cong d_{\leq 1} \text{DM}^\text{eff}_{\text{gm, ét}}(k)$ with heart the category $\text{Shv}_1(k)$ of 1-motivic sheaves. Here we also have that $\text{Tot}([0 \to \mathbb{G}_m]) = \mathbb{G}_m[-1] \cong \mathbb{Z}(1)$ (see [43 Th. 4.1] and [8]).

2.11.3. For $M \in \text{DM}^\text{eff}_{\text{gm}}$ there is an internal (effective) $\text{Hom}(M, -) \in \text{DM}^\text{eff}$ (see [57, 3.2.8]). Set

\begin{equation}
D_{\leq 1}(M) := \text{Hom}(M, \mathbb{Z}(1))
\end{equation}

for any object $M \in \text{DM}^\text{eff}_{\text{gm}}$. Actually, $D_{\leq 1}(M) \in d_{\leq 1} \text{DM}^\text{eff}_{\text{gm}}$ (see [8 3.1.1]) and restricted to $d_{\leq 1} \text{DM}^\text{eff}_{\text{gm}}$ is an involution (see [8 3.1.2]).

On the other hand, Cartier duality for 1-motives $M \mapsto M^\vee$ is an exact functor and extends to $D^b(\text{M}_1)$. A key ingredient of [8] is that, under Tot, Cartier duality is transformed into the involution $M \mapsto \text{Hom}(M, \mathbb{Z}(1))$ on $d_{\leq 1} \text{DM}^\text{eff}_{\text{gm}}(k; \mathbb{Q})$ given by the internal (effective) Hom above.

2.11.4. **Scholium** ([8 4.2]). We have a natural equivalence of functors

\[ \eta : ( )^\vee \xrightarrow{\sim} \text{Tot}^{-1} D_{\leq 1} \text{Tot} \]

i.e., under the equivalence Tot we have

\[ D^b(\text{M}_1^Q) \xrightarrow{\cong} d_{\leq 1} \text{DM}^\text{eff}_{\text{gm}}(k; \mathbb{Q}) \]

Regarding Tot as the universal realization functor we expect that any other realization of $D^b(\text{M}_1^Q)$ (hence of $\text{M}_1^Q$) will be obtained from a realization of $\text{DM}^\text{eff}_{\text{gm}}(k; \mathbb{Q})$ by composition with Tot and Cartier duality will be interchanging homological into cohomological theories.

2.11.5. We show in [8] that Tot has a left adjoint $\text{LAlb}$. Dually, composing with Cartier duality, we obtain $\text{RPic}$. In order to construct $\text{LAlb}$, let

\[ d_{\leq 1} := D^2_{\leq 1} : \text{DM}^\text{eff}_{\text{gm}}(k; \mathbb{Q}) \rightarrow d_{\leq 1} \text{DM}^\text{eff}_{\text{gm}}(k; \mathbb{Q}) \]

denote the functor

\begin{equation}
d_{\leq 1}(M) = \text{Hom}(\text{Hom}(M, \mathbb{Z}(1)), \mathbb{Z}(1)) \in d_{\leq 1} \text{DM}^\text{eff}_{\text{gm}}(k; \mathbb{Q}).
\end{equation}
The evaluation map yields a canonical map $a_M : M \to d_{\leq 1}(M)$ that induces an isomorphism

$$\text{Hom}(d_{\leq 1}M, M') \simeq \text{Hom}(M, M')$$

for $M \in \text{DM}^{\text{eff}}_{\text{gm}}(k; \mathbb{Q})$ and $M' \in d_{\leq 1}\text{DM}^{\text{eff}}_{\text{gm}}(k; \mathbb{Q})$. In fact, $M' = D_{\leq 1}(N)$ for some $N \in d_{\leq 1}\text{DM}^{\text{eff}}_{\text{gm}}(k; \mathbb{Q})$ and if $C$ is the cone of $a_M$ then

$$\text{Hom}(C, M') = \text{Hom}(C, D_{\leq 1}(N)) = \text{Hom}(C, \text{Hom}(N, \mathbb{Z}(1)))$$

$$= \text{Hom}(C \otimes N, \mathbb{Z}(1)) = \text{Hom}(N \otimes C, \mathbb{Z}(1)) = \text{Hom}(N, D_{\leq 1}(C)) = 0$$

since $D_{\leq 1}^3 = D_{\leq 1}$.

2.11.6. **Scholium** ([8, Sect. 2.2]). Define

$$\text{LAlb} : \text{DM}^{\text{eff}}_{\text{gm}}(k; \mathbb{Q}) \to \text{D}^b(M^Q_1)$$

as the composition of $d_{\leq 1} := D_{\leq 1}^2$ in (16) and $\text{Tot}^{-1}$. It is left adjoint to the embedding

$$\text{Tot} : \text{D}^b(M^Q_1) \hookrightarrow \text{DM}^{\text{eff}}_{\text{gm}}(k; \mathbb{Q})$$

and $M \mapsto a_M$ is the unit of this adjunction.

The Cartier dual of $\text{LAlb}$ is $R\text{Pic} = \text{Tot}^{-1}D_{\leq 1}$.

2.11.7. These functors provide natural complexes of 1-motives (up to isogeny) of any algebraic variety $X$ over a field $k$ if $\text{char}(k) = 0$ (for $X$ smooth and $k$ perfect even if $\text{char}(k) > 0$). Their basic properties are investigated in [8]. We have:

- $\text{LAlb}(X) := \text{LAlb}(M(X))$ the **homological Albanese complex** which is covariant on $X$ and, e.g., it is homotopy invariant and satisfies Mayer-Vietoris;
- $\text{LAlb}^c(X) := \text{LAlb}(M^c(X))$ the **Borel-Moore Albanese complex** which is covariant for proper morphisms and

$$\text{LAlb}(X) = \text{LAlb}^c(X)$$

if $X$ is proper;
- $\text{LAlb}^*(X) := \text{LAlb}(M(X)^*(n)[2n])$ the **cohomological Albanese complex** of $X$ purely $n$-dimensional, which is contravariant for maps between varieties of the same dimension and

$$\text{LAlb}^c(X) = \text{LAlb}^*(X)$$

if $X$ is smooth (by motivic Poincaré duality $M^c(X) = M(X)^*(n)[2n]$, see [57 Th. 4.3.2]);

and the Cartier duals:
• \( \text{RPic}(X) := \text{RPic}(M(X)) \) the cohomological Picard complex which is contravariant in \( X \);
• \( \text{RPic}^c(X) := \text{RPic}(M^c(X)) \) the compactly supported Picard complex such that
  \[
  \text{RPic}(X) = \text{RPic}^c(X)
  \]
if \( X \) is proper;
• \( \text{RPic}^*(X) := \text{RPic}(M(X)^*(n)[2n]) \) the homological Picard complex of \( X \) purely \( n \)-dimensional, which is covariant for maps between varieties of the same dimension and
  \[
  \text{RPic}^c(X) = \text{RPic}^*(X)
  \]
if \( X \) is smooth.

Remark that the unit
\[
(17) \quad a_X : M(X) \rightarrow \text{TotLAlb}(X)
\]
provide a universal map in \( \text{DM}_{\text{eff}}(k; \mathbb{Q}) \), the motivic Albanese map, which is an isomorphism if \( \dim(X) \leq 1 \) and it refines the classical Albanese map and the less classical map in [56].

2.12. **1-motives with additive factors.** In order to keep care of non homotopical invariant theories we do have to include additive factors. This is also suitable in order to include, in the 1-motivic world, the universal \( \mathbb{G}_a \)-extension \( M^\natural \) of a Deligne 1-motive \( M \). In order to make Cartier duality working we cannot simply take \([L \rightarrow G]\) where \( L \) is (free) finitely generated and \( G \) is a (connected) algebraic group: the Cartier dual of \( G \) is the formal group \( \widehat{G}_a \), i.e., the connected formal additive \( k \)-group (see [26] and [28] for formal groups). Laumon [38] introduced a generalization of Deligne’s 1-motives in the following sense.

2.12.1. **Laumon’s 1-motives over a field \( k \) of characteristic zero** are given by

\[
M := [F \xrightarrow{u} G]
\]
where \( F \) is a torsion free formal group and \( G \) is a connected algebraic group, i.e., \( F \) has a presentation by a splitting extension

\[
0 \rightarrow F^0 \rightarrow F \rightarrow F_{\text{ét}} \rightarrow 0
\]
where \( F_{\text{ét}} \) étale over \( k \) is further assumed torsion free (which means \( F_{\text{ét}}(\overline{k}) = \mathbb{Z}^r \)) and \( F^0 \) is infinitesimal (that is given by a finite number of copies of \( \widehat{\mathbb{G}_a} \)) and \( G \) has a presentation

\[
0 \rightarrow T + V \rightarrow G \rightarrow A \rightarrow 0
\]
where $T$ is a $k$-torus, $V$ is a $k$-vector group and $A$ is an abelian variety. The map $u : F \to G$ is any map of abelian fppf-sheaves so that an effective map $M \to M'$ is given by a map of complexes concentrated in degrees $-1$ and $0$. Let $\mathcal{M}^{a,fr}_1$ denote this category.

2.12.2. Recall [28, 2.2.2] that we have an antiequivalence between (affine) algebraic groups and (commutative) formal groups, and, moreover, the following formula (see [38, 5.2.1]) holds: if such a formal group $F$ has Cartier dual $F^\vee$ and $A$ has dual $\text{Pic}^0(A) = A^\vee$ then

$$\text{Hom}(F, A) = \text{Ext}(A^\vee, F^\vee).$$

Note that if $F = F^0$ is infinitesimal then $F^\vee := \text{Lie}(F)^\vee$ (= dual $k$-vector space of the Lie algebra) and the extension associated to $F \to A$ is here obtained from the universal $\mathbb{G}_a$-extension $\text{Pic}^0$ of $A^\vee$ by push-out along $H^0(A, \Omega^1_A) = \text{Lie}(A)^\vee \to \text{Lie}(F)^\vee$. The Cartier dual (cf. [27]) of $M = [F \to G]$ is given by an extension $G^u$ of $A^\vee$ by $F^\vee$ associated to the composite $F \to G \to A$ and a lifting of $u^\vee : (T + V)^\vee \to A^\vee$ to $G^u$ yielding

$$M^\vee := [(T + V)^\vee \to G^u]$$

Moreover the Poincaré biextension of $M$ and $M^\vee$ by $\mathbb{G}_m$ is obtained by pull-back from that of $A$ and $A^\vee$ as usual (see [38, 5.2] for details).

2.12.3. We have the following paradigmatic examples (cf. [2.6] and [38, 5.2.5]). If $X$ is a proper $k$-scheme then $[0 \to \text{Pic}^0_{X/k}]$ is a 1-motive defined by the Picard functor whose Cartier dual (= the homological Albanese 1-motive) is $[F \to \text{Alb}(X)]$ where $\text{Alb}(X) = \text{Coker}(\text{Alb}(X_1) \to \text{Alb}(X_0))$ is dual to the abelian quotient of $\text{Pic}^0_{X/k}$, $F_{et} = \mathbb{Z}^r$ is the character group of the torus, see [31], and $F^0 = \hat{\mathbb{G}}^d_a$ corresponds to $d$-copies of $\mathbb{G}_a$ in $\text{Pic}^0_{X/k}$. Let $A$ be an abelian variety and let $[0 \to \text{Pic}^+_{A/k}]$ the 1-motive determined by the universal $\mathbb{G}_a$-extension of the dual $A^\vee$. The Cartier dual is $[\hat{A} \to A]$ where $\hat{A}$ is the the completion at the origin of $A$. For $A = \text{Alb}(X)$ and $X$ smooth proper over $k$ (of zero characteristic) we have so described the Cartier dual of $[0 \to \text{Pic}^+_{X/k}]$.

2.12.4. It seems possible to modify such a category, as we did (see [2.9] for Deligne 1-motives, in order to include torsion, obtaining an abelian category. Just consider effective 1-motives $M = [F \to G]$ where $F$ is any formal group, so that $F_{et}$ may have torsion. However Cartier duality doesn’t extend (here $F^\vee$ would be any, also non connected, algebraic group). Let $\mathcal{M}^1_{1}$ denote this category. Similarly (cf. Scholium 2.10.1) the category of Laumon 1-motives up to isogeny is equivalent to the abelian $\mathbb{Q}$-linear category given
by complexes of sheaves \([F \to G]\) where \(F\) is a formal group and \(G\) is a commutative algebraic group.

2.12.5. A related matter is the Hodge theoretic counterpart of Laumon’s 1-motives over \(\mathbb{C}\) providing a generalized Hodge structure catching such additive factors (see \([3]\)).

Provisionally define a formal Hodge structure (of level \(\leq 1\)) as follows. A formal group \(H\) and a two steps filtration on a \(\mathbb{C}\)-vector space \(V\), i.e., \(H = H^0 \times H_{\text{ét}}\), \(H_{\text{ét}} = \mathbb{Z}^r + \text{torsion}\), \(H^0 = \hat{\mathbb{C}}^d\) and \(V^0 \subseteq V^1 \subseteq V = \mathbb{C}^n\), along with a mixed Hodge structure on the étale part, i.e., say \(H_{\text{ét}} \in \text{MHS}_1\) for short, and a map \(v : H \to V\). Regarding the induced map \(v_{\text{ét}} : H_{\text{ét}} \to V\) we require the following conditions: for \(H_\mathbb{C} := H_{\text{ét}} \otimes \mathbb{C}\) with Hodge filtration \(F^0_{\text{Hodge}}\) and \(c : H_{\text{ét}} \to H_{\mathbb{C}}/F^0_{\text{Hodge}}\) the canonical map, the following

\[
\begin{array}{ccc}
H_{\text{ét}} & \xrightarrow{v_{\text{ét}}} & V \\
\downarrow c & & \downarrow \text{pr} \\
H_{\mathbb{C}}/F^0_{\text{Hodge}} & \cong & V/V^0
\end{array}
\]

commutes in such a way that \(v_{\text{ét}}\) yields an isomorphism \(H_{\mathbb{C}}/F^0_{\text{Hodge}} \cong V/V^0\) restricting to an isomorphism \(W_{-2}H_{\mathbb{C}} \cong V^1/V^0\).

Denote \((H, V)\) for short such a structure and let \(\text{FHS}_1\) denote the category whose objects are \((H, V)\) and the (obvious) morphisms given by commutative squares compatibly with the data and preserving the conditions \([18]\), e.g., inducing a map of mixed Hodge structures on the étale parts. Here we then get a forgetful functor \((H, V) \mapsto H_{\text{ét}}\) from \(\text{FHS}_1\) to \(\text{MHS}_1\), left inverse of the embedding \(H \mapsto (H, H_{\mathbb{C}}/F^0_{\text{Hodge}})\). Actually we can define \((H, V)_{\text{ét}} := (H_{\text{ét}}, V/V^0)\) and say that a formal Hodge structure is étale if \((H, V) = (H, V)_{\text{ét}}, \) i.e., if \(H^0 = V^0 = 0\). The full subcategory \(\text{FHS}^\text{ét}_1\) of étale structures is then equivalent to \(\text{MHS}_1\) via the forgetful functor and the functor \((H, V) \mapsto (H, V)_{\text{ét}}\) is a left inverse of the inclusion \(\text{FHS}^\text{ét}_1 \subset \text{FHS}_1\).

Remark that \((H, V)\) with \(H_{\text{ét}}\) pure of weight zero exists if and only if \(V = V^1 = V^0\). Thus if \(v\) restricts to a map \(v^0 : H^0 \to V^0\) then \((H^0, V^0)\) is a formal substructure of \((H, V)\) and we have a ‘non canonical’ extension

\[
0 \to (H^0, V^0) \to (H, V) \to (H, V)_{\text{ét}} \to 0
\]

Say that \((H, V)\) is connected if \((H, V)_{\text{ét}} = 0\) and that it is special if \((H^0, V^0) := (H, V)^0\) is a substructure of \((H, V)\) or, equivalently, \((H, V)_{\text{ét}}\) is a quotient of \((H, V)\): the above extension \((19)\) is then characterizing special structures.

2.12.6. Extending Deligne’s Hodge realization (cf. \([2, 2]\)) for a given 1-motive \(M = [F \to G]\) consider the pull-back \(T_f(F)\) of \(F \to G\) along \(\text{Lie}(G) \to G\).
Here $T_f(F)$ is a formal group and the canonical map $T_f(F) \to \text{Lie}(G)$ provides the ‘formal Hodge realization’ of $M$

$$T_f(M) := (T_f(F), \text{Lie}(G))$$

as follows.

For $M = [F \to G]$ over $k$ let $V(G) := \mathbb{G}_a^n \subseteq G$ be the additive factor and display $G$ as follows

$$(20) \quad 0 \to V(G) \to G \to G_\times \to 0$$

where $G_\times$ is the semi-abelian quotient. We have that $\text{Lie}(G)$ is the pull-back of $\text{Lie}(G_\times)$ and $H_1(G) = H_1(G_\times)$. Moreover $F = F^0 \times_k F_{\text{ét}}$ (canonically) and we can set $M_{\text{ét}} := [F_{\text{ét}} \to G_\times]$. The functor $M \mapsto M_{\text{ét}}$ is a left inverse of the inclusion of Deligne’s 1-motives. We have that $T_f(F)_{\text{ét}}$ is an extension of $F_{\text{ét}}$ by $H_1(G_\times)$ so that, by construction, the formal group $T_f(F)$ has canonical extension

$$0 \to F^0 \to T_f(F) \to T_{\mathbb{Z}}(M_{\text{ét}}) \to 0$$

where $T_{\mathbb{Z}}(M_{\text{ét}})$ is the $\mathbb{Z}$-module of the usual Hodge realization (see 2.2) providing the formula $T_f(F)_{\text{ét}} = T_{\mathbb{Z}}(M_{\text{ét}}) = \text{the pullback of } F_{\text{ét}} \hookrightarrow F$ along $T_f(F) \to F$.

Thus $(T_f(F), \text{Lie}(G)) \in \text{FHS}_1$ where $T_f(F)_{\text{ét}}$ is the underlying group of the Hodge structure $T_{\text{Hodge}}(M_{\text{ét}})$, the filtration $V(G) \subseteq V(G) + \text{Lie}(T) \subseteq \text{Lie}(G)$ is the two steps filtration and the condition (18) is provided by construction (see 2.2) since $T_{\mathbb{C}}(M_{\text{ét}}) := T_{\mathbb{Z}}(M_{\text{ét}}) \otimes \mathbb{C} \cong \text{Lie}(G^2)$ (see 2.5) here $M_{\text{ét}}^2 = [F_{\text{ét}} \to G^2]$ is the universal $\mathbb{G}_a$-extension of $M_{\text{ét}}$, i.e.,

$$T_{\mathbb{Z}}(M_{\text{ét}}) \quad \overset{v_{\text{ét}}}{\longrightarrow} \quad \text{Lie}(G) \quad \overset{pr}{\downarrow} \quad \text{Lie}(G_\times)$$

commutes and $W_{-2}T_{\mathbb{C}}(M_{\text{ét}}) \cong \text{Lie}(T)$.

Moreover, if $u$ restricted to $F^0$ is mapped to $V(G)$ we can further set $V(M) := [F^0 \to V(G)]$ fitting in an extension

$$(21) \quad 0 \to V(M) \to M \to M_{\text{ét}} \to 0$$

providing a ‘non canonical’ extension of the 1-motive (cf. 19), here (21) becomes (19) by applying $T_f$. Note that $M_{\text{ét}}$ is pure of weight zero if and only if $T_f(M) = M$. For example, if $W \overset{u}{\to} V$ is a linear map between $\mathbb{C}$-vector spaces, and $M = \widehat{W} \overset{u}{\to} V$ is the induced 1-motive (here $\widehat{W}$ is the formal completion at the origin, cf. [38, 5.2.5]) then $T_f(M) = M$. 
There is an equivalence of categories $M \sim T_f(M) : \mathcal{M}^{a,fr}(\mathbb{C}) \overset{\sim}{\longrightarrow} \text{FHS}_1^{fr}$ between Laumon’s 1-motives and torsion free formal Hodge structures (of level $\leq 1$) providing a diagram

$$
\begin{array}{ccc}
\mathcal{M}_1^{fr}(\mathbb{C}) & \overset{\sim}{\rightarrow} & \text{MHS}_1^{fr} \\
\updownarrow & & \updownarrow \\
\mathcal{M}_1^{a,fr}(\mathbb{C}) & \overset{\sim}{\rightarrow} & \text{FHS}_1^{fr}
\end{array}
$$

Regarding duality for $(H,V) = T_f(M)$ such that $H_{\text{et}}$ is free, we can argue cheaply defining it as follows

$$T_f(M)^\vee := T_f(M^\vee)$$

After Cartier duality (cf. [38, 5.2]) it is easy to check that this is a self duality extending the one on MHS$^{fr}_1$. For example $H_{\text{et}}^\vee := \text{Hom}(H_{\text{et}}, \mathbb{Z}(1)) = (H,V)^\vee_0$ as usual and the Cartier dual of $(H,V)$ with $H_{\text{et}}$ of weight zero such that (21) splits $M = V(M) \oplus M_{\text{et}}$, is obtained as follows: $V(M)^\vee$ is given by $\hat{V}^\vee \rightarrow \text{Lie}(H^0)^\vee$ obtained from the induced map $\text{Lie}(H^0) \rightarrow V$, taking the dual vector space map $V^\vee \rightarrow \text{Lie}(H^0)^\vee$ and its completion $\hat{V}^\vee \rightarrow V^\vee$ at the origin$^{10}$ thus

$$(H^0 \times \mathbb{Z}(0)^{\oplus r}, V)^\vee = (\hat{V}^\vee \times \mathbb{Z}(1)^{\oplus r}, \text{Lie}(H^0)^\vee \times G_a^{\oplus r})$$

where the map $\mathbb{Z}(1)^{\oplus r} \rightarrow G_a^{\oplus r}$ is canonically induced from the exponential map, e.g., in particular $(\mathbb{Z}(0),0)^\vee = (\mathbb{Z}(1), G_a)$.

2.12.8. Formal Hodge structures$^{11}$ would pitch in the following diagram

$$
\begin{array}{ccc}
\text{Deligne’s 1-motives} & \overset{T_{\text{Hodge}}}{\rightarrow} & \text{MHS} \\
\updownarrow & & \updownarrow \\
\text{Laumon’s 1-motives} & \overset{T_f}{\rightarrow} & \text{FHS}
\end{array}
$$

where

- FHS would be a rigid tensor abelian category which is an enlargement of MHS and $H \mapsto H_{\text{et}}$ would yield a functor from FHS to MHS, left inverse of the embedding;

---

$^{10}$Note that in characteristic zero there is a canonical equivalence of categories between Lie algebras and infinitesimal formal groups.

$^{11}$We here mean to deal with arbitrary Hodge numbers. However, for the sake of brevity, no more details on FHS are provided: generalizing our definition above it’s not that difficult but it’s more appropriate to treat such a matter separately.
• $T_f$ would be fully faithful so that under the realizations Cartier duality corresponds to a canonical $\text{Hom}(\cdot, \mathbb{Z}_a(1))$ involution.\footnote{Here we clearly have the candidate $T_f([0 \to \mathbb{G}_m]) := \mathbb{Z}_a(1)$ and a formal version of Scholium 2.7.5 should be conceivable.}

2.12.9. Similarly define other realizations, e.g., see [6] where we obtain the sharp De Rham realization $T_\sharp$. For example, if $F^0 = 0$ we can describe $T_\sharp$ out of the universal $\mathbb{G}_a$-extension $M^\sharp_{\text{ét}}$ (see 2.5), defining the algebraic group $G^\sharp$ by pull-back via (20) as an extension

$$0 \to \text{Ext}(M_{\text{ét}}, \mathbb{G}_a)^\vee \to G^\sharp \to G \to 0$$

taking $\text{Lie}(G^\sharp)$. In this case we thus obtain a canonical extension

$$0 \to V(G) \to M^\sharp \to M^\sharp_{\text{ét}} \to 0$$

and we can relate to $(H, V) = T_f(M)$, where $H^0 = F^0 = 0$, passing to Lie algebras, by the following pull-back diagram

\[
\begin{array}{cccccc}
0 & \to & F^0_{\text{Hodge}} & \to & H_C & \to & H_C/F^0_{\text{Hodge}} & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & \text{Ext}(M_{\text{ét}}, \mathbb{G}_a)^\vee & \to & \text{Lie}(G^\sharp) & \to & \text{Lie}(G) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
V(G) & \to & 0
\end{array}
\]

where $H_{\text{ét}} = T_\sharp(Z_{\text{ét}})$, $H_C = \text{Lie}(G^\sharp)$, $H_C/F^0_{\text{Hodge}} = \text{Lie}(G_x)$, $V^0 = V(G) \subseteq V = \text{Lie}(G)$. Set $H^\sharp_C := \text{Lie}(G^\sharp)$ and by the universal property we get an induced map $c^\sharp : H_Z \to H^\sharp_C$ providing a splitting of the projection $H^\sharp_C \to H_C$ and the diagram above can be translated in the following diagram

\[
\begin{array}{ccc}
H_C & \to & H_C/F^0_{\text{Hodge}} \\
\uparrow & & \uparrow \\
H^\sharp_C & \to & V \\
\uparrow & & \uparrow \\
H_C
\end{array}
\]

(22)
2.12.10. Remark that, from a different point of view Bloch and Srinivas [18] proposed a category of enriched Hodge structures EHS whose objects are pairs \( E := (H, V) \) where \( H \) is a mixed Hodge structure and \( V \) is a diagram (not a complex) \( \cdots = V_{a+1} = V_a \to V_{a-1} \to \cdots \to V_0 \to 0 \to 0 \cdots \) of \( \mathbb{C} \)-vector spaces such that \( V_i \to H_{\mathbb{C}}/F^i \) (compatibly with the diagram) and there is a map \( H_{\mathbb{C}} \to V_a \) such that \( H_{\mathbb{C}} \to V_a \to H_{\mathbb{C}}/F^a \) is the identity, thus \( F^a = 0 \). There is a canonical functor \( E \mapsto H \) to MHS (with a right adjoint). It is not difficult to see that EHS is equivalent to FHS via \((22)\). Sharp De Rham realization also clearly provides an enriched Hodge structure, e.g., via \((22)\). In fact, we can refine the construction \((22)\) obtaining a functor

\[ T_s^s : \text{FHS}_1^s \to \text{EHS}_1 \]

by sending

\[ (H, V) \mapsto T_s^s(H, V) := (H_{\text{ét}}, H_{\mathbb{C}}^{\text{ét}} \to V) \]

where \( H_{\text{ét}}^{\text{ét}} \) (along with the splitting) is just obtained by pull-back when \( H^0 = 0 \) (see \([6]\) for the precise statements and further properties).

3. On 1-motivic (co)homology

In the previous section 2 we have provided realizations as covariant and contravariant functors from categories of 1-motives to categories of various kind of structures. Here we draft a picture (which goes back to the algebraic geometry constructions of section 1) providing 1-motives, i.e., 1-motivic cohomology, whose realizations are the ‘1-motivic part’ of various existing (or forthcoming) homology and cohomology theories.

3.1. Albanese and Picard 1-motives. Let \( X \) be a complex algebraic variety and let \( H^*(X, \mathbb{Z}) \) be the mixed Hodge structure on the singular cohomology of the associated analytic space. Denote \( H_{(1)}^*(X, \mathbb{Z}(\cdot)) \subseteq H^*(X, \mathbb{Z}(\cdot)) \) the largest substructure and \( H^*(X, \mathbb{Z}(\cdot))^{(1)} \) the largest quotient in MHS_1 (cf. \(2.2\)).

3.1.1. Deligne’s Conjecture \((23, 10.4)\). Let \( X \) be a complex algebraic variety of dimension \( \leq n \). There exist algebraically defined 1-motives whose Hodge realizations over \( \mathbb{C} \) are \( H_{(1)}^i(X, \mathbb{Z}(1))_{\text{Fr}}, H^i(X, \mathbb{Z}(i))^{(1)}_{\text{Fr}} \) for \( i \leq n \) and \( H^i(X, \mathbb{Z}(n))^{(1)}_{\text{Fr}} \) for \( i \geq n \) and similarly for \( \ell \)-adic and De Rham realizations.

The results contained in \([22], [11], [50], [51], [9]\) and \([1]\) show some cases of this conjecture. Over a field \( k, \text{char}(k) = 0 \), with the notation of \([11]\):
• Pic\(^+\)(X) which reduces to \([2]\) if X is smooth (or to the simplicial Pic\(^0\)
if X is proper) provides an algebraic definition of \(H^1_{(1)}(X, \mathbb{Z}(1))^\text{fr} = H^1(X, \mathbb{Z}(1))\);
• Alb\(^+\)(X) is an algebraic definition of \(H^{2n-1}(X, \mathbb{Z}(n))^\text{fr}\) for \(n = \dim(X)\);
• Pic\(^-\)(X) = Alb\(^+\)(X)\(^\vee\) is an algebraic definition of \(H_{2n-1}(X, \mathbb{Z}(1-n))\);
• Alb\(^-\)(X) = Pic\(^+\)(X)\(^\vee\) which reduces to Serre Albanese if X is smooth, is an algebraic definition of \(H^1(X, \mathbb{Z})^\text{fr}\).

Moreover, in \([9]\) we have constructed effective 1-motives with torsion:
• Pic\(^+\)(X, i) for \(i \geq 0\) providing an algebraic definition of \(H^{i+1}_{(1)}(X, \mathbb{Z}(1))^\text{fr}\) up to isogeny.

Actually, for Y a closed subvariety of X we have Pic\(^+\)(X, Y; i) (= \(M_{i+1}(X, Y)\) in the notation of \([9]\)) such that Pic\(^+\)(X, 0; 0) = Pic\(^+\)(X). These Pic\(^+\)(X, Y; i) are obtained using appropriate ‘bounded resolutions’ which also provide a canonical integral weight filtration W on the relative cohomology \(H^*(X, Y; \mathbb{Z})\) (see \([9\text{, }2.3]\)).

3.1.2. Scholium \(([9\text{, }0.1])\). There exists a canonical isomorphism of mixed Hodge structures
\[
\phi^\text{fr}_*: T_{H^*}(\text{Pic}^+(X, Y; i))^\text{fr} \longrightarrow W_0 H^{i+1}_{(1)}(X, Y; \mathbb{Z}(1))^\text{fr}
\]
and similarly for the l-adic and de Rham realizations.

This implies Deligne’s conjecture on \(H^*_{(1)}(X, \mathbb{Z}(1))^\text{fr}\) up to isogeny and in cohomological degrees \(\leq 2\) even without isogenies by dealing with such 1-motives with torsion. The conjecture without isogeny is reduced to
\[
H^*_{(1)}(X, Y; \mathbb{Z})^\text{fr} = W_2 H^*_{(1)}(X, Y; \mathbb{Z})^\text{fr}.
\]

Here the semiabelian part of Pic\(^+\)(X, Y; i) yields \(W_1 H^1_{(1)}(X, Y; \mathbb{Z}(1))^\text{fr}\) and the torus corresponds to \(W_2 H^1_{(1)}(X, Y; \mathbb{Z}(1))^\text{fr}\).

3.1.3. In \([1]\) we have also formulated a corresponding statement \([3.1.1]\) for the crystalline realization. Recall that de Jong \([34\text{, }p.\ 51-52]\) proposed a definition of crystalline cohomology\(^{13}\) forcing cohomological descent. Let X be an algebraic variety, over a perfect field \(k\), de Jong’s theory \([34]\) provide a pair \((X, Y, )\) where X is a smooth proper simplicial scheme, \(Y, \) is a normal crossing divisor in \(X, \) and \(X, -Y, \) is a smooth proper hypercovering of \(X.\) Set \(H^*_{\text{crys}}(X/\mathbb{W}(k)) := H^*_{\text{logcrys}}(X, \log Y, )\) where \((X, \log Y, )\) here denotes the simplicial logarithmic structure on \(X, \) determined by \(Y, \) (see

\(^{13}\)Note that we can also deal with rigid cohomology and everything here can be rephrased switching crystalline to rigid.)
The question here (cf. [34]) is that $H_{\text{crys}}^*(X/W(k))$ is not a priori well-defined. Similarly to [9, 2.3] we may also expect a weight filtration $W^*$ on the crystalline cohomology $H_{\text{crys}}^*((X,Y)/W(k))$ of a pair $(X,Y)$. Over a perfect field, using de Jong’s resolutions, it is easy to obtain a suitable construction of $\text{Pic}^+(X,Y;i)$ such that $\text{Pic}^+(X,\emptyset;0) = \text{Pic}^+(X)$ as above. In [1, Appendix A] we have shown that $\text{Pic}^+(X)$ is really well-defined and independent of the choices of resolutions or compactifications. However, it is not clear, for $i > 0$, if $\text{Pic}^+(X,Y;i)$ is integrally well-defined: the 1-motive is well-defined up to $p$-power isogenies in characteristic $p$ by [1, A.1.1] and a variant of [9, Thm. 3.4] for $\ell$-adic realizations with $\ell \neq p$.

3.1.4. **Crystalline Conjecture** ([1, Conj. C]). Let $H_{\text{crys},(1)}^*((X,Y)/W(k))$ denote the submodule of $W_2H_{\text{crys}}^*((X,Y)/W(k))$ whose image in $\text{gr}_W^2$ is generated by the image of the discrete part of $\text{Pic}^+(X,Y;i)$ under a suitable cycle map. Then there is a canonical isomorphism (eventually up to $p$-power isogenies)

$$T_{\text{crys}}(\text{Pic}^+(X,Y;i)) \cong H_{\text{crys},(1)}^{i+1}((X,Y)/W(k))(1)$$

of filtered $F^*W(k)$-modules (i.e., we expect a crystalline analogue of [7.7.7]).

We can show this statement for $i = 0$ and $Y = \emptyset$ (see [1, Thm. B']). The corresponding general statement for De Rham cohomology over a field of characteristic zero is [9, Thm. 3.5].

3.1.5. According with the program in [8] these $\text{Pic}^+(X,Y;i)$ would get linked to Voevodsky’s theory of triangulated motives as follows. The covariant functor $M: Sm/k \to \text{DM}^{\text{eff}}_{\text{gm}}(k)$ from the category of smooth schemes of finite type over $k$ (a field admitting resolution of singularities) extends to all schemes of finite type (see [57, §4.1]). Thus the motivic Albanese complex $\text{LAlb}(X)$ and the motivic Picard complex $\text{RPic}(X)$ are well-defined for any such scheme $X$ (and similarly for the other complexes, see 2.11.7). Consider the (co)homology 1-motives (up to isogenies) $H_i(\text{LAlb}(X)) := L_i\text{Alb}(X)$ and $H^i(\text{RPic}(X)) := R^i\text{Pic}(X)$ for $i \in \mathbb{Z}$. We have that

$$R^i\text{Pic}(X) = L_i\text{Alb}(X)^\vee$$

by motivic Cartier duality (see Scholium 2.11.4).

3.1.6. **LAlb - RPic Hypothesis** (cf. [8]). We assume the following picture (up to isogeny):

- $T_{\text{Hodge}}(L_i\text{Alb}(X)) = H_i(X,\mathbb{Z})^{(1)}_{\text{tr}} = 1$-motivic singular homology mixed Hodge structure;
• \( T_{\text{Hodge}}(L_i \text{Alb}^r(X)) = H_i^{BM}(X, \mathbb{Z})^{(1)}_{\text{fr}} \) = 1-motivic Borel-Moore homology mixed Hodge structure;
• \( T_{\text{Hodge}}(L_i \text{Alb}^r(X)) = H^{2n-i}(X, \mathbb{Z}(n))^{(1)}_{\text{fr}} \) = 1-motivic Tate twisted singular cohomology mixed Hodge structure of \( X \) \( n \)-dimensional;

and dually:
• \( T_{\text{Hodge}}(R^1 \text{Pic}(X)) = H^i(X, \mathbb{Z}(1))_{\text{fr}} \) = 1-motivic singular cohomology mixed Hodge structure;
• \( T_{\text{Hodge}}(R^1 \text{Pic}^\ast(X)) = H^i_c(X, \mathbb{Z}(1))_{\text{fr}} \) = 1-motivic compactly supported cohomology mixed Hodge structure;
• \( T_{\text{Hodge}}(R^1 \text{Pic}^\ast(X)) = H_{2n-i}(X, \mathbb{Z}(1-n))_{\text{fr}} \) = 1-motivic Tate twisted singular homology of \( X \) \( n \)-dimensional.

Similar statements for \( \ell \)-adic, De Rham and crystalline realizations are also workable (providing a positive answer to 3.1.1 and 3.1.4).

It is not difficult (see [8]) to compute these 1-motivic (co)homologies for \( X \) smooth or a singular curve. We recover in this way Deligne-Lichtenbaum motivic (co)homology of curves (cf. [23] and [39]). The picture above also recover the previously mentioned Picard and Albanese 1-motives as follows

\[
\text{L}_1 \text{Alb}(X) = \text{Alb}^{-}(X) \quad \text{L}_1 \text{Alb}^r(X) = \text{Alb}^{+}(X)
\]

and

\[
R^1 \text{Pic}(X) = \text{Pic}^{+}(X) \quad R^1 \text{Pic}^\ast(X) = \text{Pic}^{-}(X).
\]

Finally, for \( i \geq 1 \), we should get a formula like that

\[
R^i \text{Pic}(X, Y) = \text{Pic}^{+}(X, Y; i-1) \quad L_i \text{Alb}(X, Y) = \text{Alb}^{-}(X, Y; i-1)
\]

with the obvious meaningful notation adopted above.

3.2. **Hodge 1-motives.** We shortly explain the point of view developed in [5] extending Deligne’s philosophy 3.1.1 to algebraic cycles in higher codimension (cf. 1.3.1). The starting point is by looking at the side of 3.1.2 which provides a Lefschetz theorem on \((1, 1)\)-classes, \( i.e. \), in degrees \( > 1 \). See also [10].

3.2.1. Define \( \text{NS}^+(X, Y; i) \) for \( i \geq 0 \) as the quotient of \( \text{Pic}^+(X, Y; i+1) \) by its toric part and consider the extension

\[
0 \to W_{-2} \to \text{Pic}^+(X, Y; i+1) \to \text{NS}^+(X, Y; i) \to 0
\]

It follows from 3.1.2 up to isogeny

\[
T_{\text{Hodge}}(\text{NS}^+(X, Y; i)) = W_0 H^{2+i}_1(X, Y; \mathbb{Z}(1))/W_{-2}
\]
given by the extension\[0 \to \gr_1^W \to W_2H^{2+i}(X,Y;\mathbb{Z})/W_0 \to \gr_2^W \to 0\]
pulling back (1,1)-classes in $\gr_2^W$ (and twisting by $\mathbb{Z}(1)$). We may call
NS$(X,Y;i)$ the Hodge-Lefschetz 1-motive since, e.g., if $X$ is smooth proper
and $Y = \emptyset$ we obtain $\NS^+(X;0) = \NS(X)$ and $\NS^+(X;i) = 0$ for $i \neq 0$.

3.2.2. Set $H := H^{2p+i}(X,Y;\mathbb{Z})$ for a fixed $p \geq 1$ and $i \geq -1$ and consider
\[0 \to \gr_{2p-1}^W H \to W_{2p}H/W_{2p-2}H \to \gr_{2p}^W H \to 0\]
given by the integral weight filtration (see [9]). Consider the integral $(p,p)$-
classes $H_{Z}^{p,p} := \Hom_{\text{MHS}}(\mathbb{Z}(-p), \gr_{2p}^W H)$ and the associated intermediate jacobians $J^{p}(H) := \Ext(\mathbb{Z}(-p), \gr_{2p-1}^W H)$ which is just a complex torus if $p > 1$
(see [21]). Consider the largest abelian subvariety $A^{p}(H)$ of the torus $J^{p}(H)$
which corresponds to the maximal polarizable substructure of $\gr_{2p-1}^W H$ purely
of types $\{(p-1,p), (p,p-1)\}$. Define the group of Hodge cycles $H^{p}(H)$ as the
preimage in $H_{Z}^{p,p}$ of $A^{p}(H)$ under the extension class map $e^{p} : H_{Z}^{p,p} \to J^{p}(H)$.
Define the Hodge 1-motive by
\[e^{p} : H^{p}(H) \to A^{p}(H)\]
and the corresponding mixed Hodge structure $H^{h} \in \text{MHS}_{1}$.

3.2.3. **Anodyne Hodge Conjecture** (cf. [5 2.3.4]). Let $X$ be an algebraic variety
and $Y$ a closed subvariety defined over a perfect field $k$. There exist algebraically defined 1-motives with torsion $\Xi^{i,p}(X,Y) \in \mathcal{M}_{1}(k)$
whose Hodge realization over $k = \mathbb{C}$ are $H^{2p+i}(X,Y;\mathbb{Z})^{h} \in \text{MHS}_{1}$, i.e.,
here $H^{2p+i}(X,Y;\mathbb{Z})$ is the associated mixed Hodge structure (for $p \geq 1$ and
$i \geq -1$) so that
\[T_{\text{Hodge}}(\Xi^{i,p}(X,Y)) \cong H^{2p+i}(X,Y;\mathbb{Z})^{h}\]
and similarly for $\ell$-adic, De Rham and crystalline realizations.

For $p = 1$ this is ‘almost’ true (≈ Deligne’s conjecture 3.1.1 and 3.1.2
but 3.1.4) and it follows from 3.2.1 and $\Xi^{1,1}(X,Y) = \NS^+(X,Y;i)$. One
can also easily formulate a homological version of 3.2.3. Recall that for $\overline{X}$
smooth proper purely $n$-dimensional and $Y + Z$ normal crossing divisors on
$\overline{X}$ (in particular when $X = \overline{X} - Z$ and $Y \cap Z = \emptyset$) we have
\[H^{2p+i}(\overline{X} - Z;\mathbb{Z}(p)) \cong H_{2r-1}(\overline{X} - Y, Z;\mathbb{Z}(-r)) \quad (p = n - r)\]
as mixed Hodge structures (see [11 2.4.2]). For $X$ smooth and proper (here
we assume that $Y = Z = \emptyset$ and $X = \overline{X}$) we get $H^{2p+i}(X,\mathbb{Z})^{h} \neq 0$ if and
only if $i = -1,0$ and 3.2.3 reduces to the quest of an algebraic definition
of $A^{p} \subseteq J^{p}$ or $H_{Z}^{p,p}$ respectively. Classical Grothendieck-Hodge conjecture
then provides candidates up to isogeny.
3.2.4. For $X$ a smooth proper $\mathbb{C}$-scheme we can consider $J_p^a(X) \subseteq J^p(X)$ the image of $CH^p(X)_{\text{alg}}$ (cf. 3.2) under the Abel-Jacobi map: the usual Grothendieck-Hodge conjecture claims that $J_p^a(X)$ is the largest abelian variety in $J^p(X)$, i.e., that $A^p = J_p^a$ (up to isogeny) and $H^{2p-1}(X, \mathbb{Z})^h$ is algebraically defined via the coniveau filtration. Similarly, the image of $\text{NS}^p(X)$ generates $H^{p,p}_Z$ (with \(Q\)-coefficients). In the most wonderful world (mathematics!) the 1-motivic sheaf $(CH^p_X)^{(1)}$ in 3.3 could make the job providing an algebraically defined extension of $H^{p,p}_Z$ by $J_p^a$ compatibly with (6) (here $J_p^a$ would also coincide with the universal regular quotient of $CH^p(X)_{\text{alg}}$ when $X$ is smooth and proper).

If $X$ is only proper then let $\pi : X_\ast \to X$ be a resolution and consider the Chow groups of each component $X_i$ of $X_\ast$ (which are proper and smooth). Let $(\text{NS}^p)^\bullet$ and $(J_p^a)^\bullet$ denote the complexes induced by the simplicial structure and similarly to (6) we obtain an extension of $(\text{NS}^p)^\bullet$ by $(J_p^a)^\bullet$. By taking homology groups we then get boundary maps

$$\lambda^i_a : H^i((\text{NS}^p)^\bullet) \to H^{i+1}((J_p^a)^\bullet).$$

3.2.5. **Hodge Conjecture (3.2.4).** The boundary map $\lambda^i_a$ behave well with respect to the extension class map $e^p$ yielding a motivic cycle class map, i.e., the following diagram

$$
\begin{array}{ccc}
H^i((\text{NS}^p)^\bullet) & \xrightarrow{\lambda^i_a} & H^{i+1}((J_p^a)^\bullet) \\
\downarrow & & \downarrow \\
H^{2p+i}(X)^{p,p} & \xrightarrow{e^p} & J^p(H^{2p+i}(X))
\end{array}
$$

commutes.\(^{14}\) The image 1-motive (up to isogeny) is the Hodge 1-motive $\Xi^i_{p}(X)$ corresponding to $H^{2p+i}(X, \mathbb{Z})^h$.

Moreover, one might then guess that the complex of 1-motivic sheaves $(CH^p_{X_\ast})^{(1)}$ would provide such Hodge 1-motive directly.

3.3. **Non-homotopical invariant theories.** A typical problem occurring with homotopical invariant theories attached to singular varieties is that they do not catch some informations coming from the singularities. In general, the cohomological Picard 1-motive $\text{Pic}^+(X)$ of a proper scheme $X$ is given by the semi-abelian quotient of $\text{Pic}^0(X)$ (see Scholium 1.1.2). Loosing its additive components we loose informations, e.g., we don’t see cusps. In order to reach the full picture here we have to enlarge our target to Laumon’s 1-motives at least. A natural guess is that our 1-motives are only

\(^{14}\)Note that all maps in the square are canonically defined.
the étale part of Laumon’s 1-motives, i.e., there exists Pic$_a^+(X, Y; i) \in M_1^a$

such that

$$\text{Pic}_a^+(X, Y; i) \otimes = \text{Pic}^+(X, Y; i)$$

and similarly RPic$_a(X) \in D^b(M_1^a)$ (cf. 3.1.6), $\Xi_{i, \text{p}}^a(X) \in M_1^a$ such that

$$\Xi_{i, \text{p}}^a(X) \otimes = \Xi_{\text{p}}(X) \otimes$$

Their geometrical sources are additive Chow groups and their universal regular quotients, cf. [17] and, by the way, see [24] for a construction of an additive version of the cohomological Albanese $\text{Alb}_a^+(X)$ of a projective variety $X$, i.e., here $\text{Alb}_a^+(X) \otimes = \text{L}_1 \text{Alb}^a(X)$, etc. as above. Similarly, for $X$ quasi-projective, we expect a formal part defining $\text{Alb}_a^+(X)$ as a Laumon 1-motive.

3.3.1. The forthcoming theories are sharp cohomology theories, e.g., $\sharp$-singular cohomology $X \sim H^*_x(X) \in \text{FHS}$ for $X$ over $\mathbb{C}$, $\sharp$-De Rham cohomology $H^*_x,\text{-DR}(X)$ of $X$ $k$-algebraic over a field $k$ of zero characteristic and $\sharp$-crystalline cohomology in positive characteristics, which are non-homotopical invariant theories. Sample

$$H^1_x(X) := T_x(\text{Pic}^+_a(X))$$

Here $\text{Pic}^+_a(X) = [0 \rightarrow \text{Pic}_a^0(X)]$ if $X$ is proper: in this case, define the group scheme $\text{Pic}^x$ by the following pull-back square (cf. [22], 2.6.1 1.1.2 and [11, 4.5])

$$\begin{array}{c}
\text{Pic}^x(X, \ast) \rightarrow \text{Pic}(X, \ast) \\
\uparrow \\
\text{Pic}^x(X) \rightarrow \text{Pic}(X)
\end{array}$$

(23)

such that

- $\text{Ker}(\text{Pic}^x, \ast(0) \rightarrow \text{Pic}(0)) = H^0(X, \Omega^1_{X, \ast})$ and
- $\text{Ker}(\text{Pic}^x, \ast(0) \rightarrow \text{Pic}(0))$ is the additive subgroup $\subseteq \text{Pic}(0)$;

then

$$H^1_x,\text{-DR}(X) := \text{Lie} \text{Pic}^x,\ast(0)$$

so that $H^1_x,\text{-DR}(X)$ is an extension of $H^1_{\text{DR}}(X)$ by the additive part of $\text{Pic}^0(X)$.

$^{15}$Note that such Laumon 1-motives should rather be visible from a triangulated viewpoint! There should be a “sharp” cohomological motive $M_4(X)$ in a triangulated category $\text{DM}_4$, related to Voevodsky category of motivic complexes, with a realisation in $D^b(\text{FHS})$. The conjectural formalism for motivic complexes should be translated for $\sharp$-motivic complexes.
3.3.2. Similarly, remark that (see [18]) for \( E = (H, V) \in \text{EHS} \) there is a surjection
\[
\text{Ext}_{\text{EHS}}(\mathbb{Z}(0), E) \to \text{Ext}_{\text{MHS}}(\mathbb{Z}(0), H)
\]
and the kernel of this map is a vector space if \( H = H^{2r-1}(X, \mathbb{Z}(r)) \) and \( V_i = H^{2r-1}(X, \mathcal{O}_X \to \cdots \to \Omega^1_{X}(r)) \), where \( X \) is a proper \( \mathbb{C} \)-scheme; in particular, if \( X \) is the cuspidal curve then \( \text{Ext}_{\text{EHS}}(\mathbb{Z}(0), E) \) is the additive group \( \mathbb{G}_a = \text{Pic}^0(X) \).

3.4. **Final remarks.** Hoping to have puzzled the reader enough to proceed on these matters I would finally remark that this exposition is far from being exhaustive.

3.4.1. For example, for \( S = \text{Spec}(R) \) where \( R \) is a complete discrete valuation ring and \( K \) its function field, a 1-motive over \( K \) with good reduction (resp. potentially good reduction) is defined (in [52]) by the property of yielding a 1-motive over \( R \) (resp. after a finite extension of \( K \)). To any 1-motive \( M = [L \to G] \) over \( K \) is canonically associated (see [52] for details) a strict 1-motive over \( K \), i.e., \( M' = [L' \to G'] \), such that \( G' \) has potentially good reduction, a quasi-isomorphism \( M'_\text{rig} \to M_\text{rig} \) in the derived category of bounded complexes of fppf-sheaves on the rigid site of \( \text{Spec}(K) \), producing a canonical isomorphism \( T_\ell(M') \cong T_\ell(M) \) between the \( \ell \)-adic realizations, for any prime \( \ell \). For a strict 1-motive \( M = [L \to G] \) over \( K \), the geometric monodromy \( \mu : L \times T^\vee \to \mathbb{Q} \) (where \( T^\vee \) is the character group of the torus \( T \subseteq G \)) is defined by valuating the trivialization of the Poincaré biextension. The geometric monodromy is zero if and only if \( M \) has potentially good reduction. This theme is further investigated in [14].

3.4.2. The employ of 1-motives in arithmetical geometry is well testified, e.g., see [20], [53], [25], [33] and [36]. Note that in [7] we also investigate \( L \)-functions with respect to Mordell-Weil and Tate-Shafarevich groups of 1-motives. Also the theme of 1-motivic Galois groups is afforded. For \( M \) a 1-motive over a field \( k \) of zero characteristic let \( M^\otimes \) be the Tannakian subcategory generated by \( M \) in suitable mixed realisations (hopefully mixed motives). The motivic Galois group of \( M \), denoted \( \text{Gal}_{\text{mot}}(M) \), is the fundamental group of \( M^\otimes \). The group \( \text{Gal}_{\text{mot}}(M) \) has an induced weight filtration \( W_s \) and the unipotent radical \( W_{-1} \text{Gal}_{\text{mot}}(M) \) has a nice characterisation (see [15] for details). Furthermore, Fontaine’s theory relating \( p \)-adic mixed Hodge structures over a finite extension \( K \) of \( \mathbb{Q}_p \) to mixed motives would provide categories of 1-motives over the \( p \)-adic field \( K \) (see [27]).

3.4.3. Passing from 1-motives to 2-motives is conceivable but (even conjecturally) harmless. A general guess is that there should be abelian categories
\[
\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \cdots \subseteq \mathcal{M}
\]
where $\mathcal{M}_0 = \text{Artin motives}$, $\mathcal{M}_1 = 1$-motives and further on we have categories of $n$-motives $\mathcal{M}_n$ which can be realized as Serre subcategories of cohomological dimension $\leq n$ of the abelian category $\mathcal{M}$ of mixed motives. Assuming the existence of $\mathcal{M}$ a source of inspiration is [20], [57] and [12]: such $\mathcal{M}_n$ would be somehow ‘generated’ by motives of varieties of dimension $\leq n$ and $M(X)$, the motive of $X$ smooth and projective, decomposes as $\oplus M^i(X)[-i]$ where $M^i(X) \in \mathcal{M}_i$ such that $M^i(X) = M^{2d-i}(X)$ for $d = \dim(X)$.

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\(^{16}\)Note that work in progress and preliminary versions of my papers are firstly published on the web and currently updated, e.g., browsing from my home page.
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