Morse foliations of codimension one on the sphere $S^3$

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September 23, 2022

Abstract

Morse foliations of codimension one on the sphere $S^3$ are studied and the existence of special components for these foliations is derived. As a corollary the instability of Morse foliations of $S^3$ can be proven provided that all the leaves are not simply connected.

1 Introduction

In this paper we study Morse foliations $\mathcal{F}$ of codimension one and of class $C^\infty$ of the sphere $S^3$. Such a foliation $\mathcal{F}$ has finitely many singularities of conic or central type; outside these singularities $\mathcal{F}$ is a foliation in the usual sense. For each singularity $s$ there is an open neighborhood $U$ of $s$ homeomorphic to $\mathbb{R}^3$, such that the leaves of the induced foliation $\mathcal{F}|_U$ are defined as the level sets of a Morse function $f: U \to \mathbb{R}$. Such a neighborhood $U$ will be called a trivial neighborhood of $s$. The complete definition of Morse foliations will be given at the beginning of the next section.

Each Morse foliation can be transformed by a modification which will be referred to as a Morse modification. This modification is defined locally and moves conic singularities to nearby leaves. A Morse component $\mathcal{M}$ is a Morse foliation defined on the solid torus $D^2 \times S^1$ which contains one center, one conic singularity and all the non-singular leaves of $\mathcal{M}$ are either tori or 2-spheres.

The main result of this work is the following.

**Theorem 1** Let $\mathcal{F}$ be a Morse foliation of codimension one of $S^3$ which admits at least one non-simply connected leaf. Then, modulo Morse modifications, $\mathcal{F}$ contains either a Reeb component or a Morse component.

As a corollary of this theorem we may prove the following theorem which answers affirmatively a conjecture of Rosenberg and Roussarie [13].

**Theorem 2** The only stable Haefliger structures of $S^3$ are those defined by a Morse function having distinct critical values and whose level surfaces are all simply connected.
Generally, Morse foliations are an important generalization of foliations and their study has a long history, see for example [11], [5], [14], [3], [4], [12]. In particular, in [12] classical compact leaf theorems of Haefliger and Novikov are generalized (see for instance, Theorem 9.1). Nevertheless, the latter are proved under conditions on the set of singular points of a singular foliation which are strong and not related to our setup.

Despite the long time period, the theory of singular foliations on 3-manifolds did not grow as much as that of regular foliations. The reason is that each closed 3-manifold $M$ can be foliated by a regular foliation of codimension one. However, if $M$ is compact with $\partial M \neq \emptyset$ and if the genus of $\partial M$ is greater than one, then in order to foliate $M$ tangentially to the boundary we need to consider Morse foliations. On the other hand, Haefliger structures are an even greater generalization of foliations [7], [6] and they are quite useful to make a homotopy-theoretic approach to foliations. Below, we will see that Morse foliations are generic in the space of Haefliger structures of codimension one fact that makes the class of Morse foliations particular interesting.

The present work is organized as follows:

In section 2 definitions are given and basic notions are explained in order to fix the terminology. Typical examples of Morse foliations of $S^3$ are also constructed.

In section 3 we recall the definition of vanishing cycle and we generalize Novikov’s theorem for Morse foliations. We also introduce the notion of anti-vanishing cycle and examine its consequences.

In section 4 and 5 we prove Theorem 1. Actually, we prove that if $F$ does not contain a Reeb or a Morse component and if $F$ does not have trivial pair of singularities then $F$ contains either a truncated Reeb component or a truncated Morse component or $F$ has two centers and all the other leaves of it are 2-spheres. Notice that, a truncated Reeb or Morse component can be transformed to a Reeb or Morse component respectively by a Morse modification.

In section 6 we prove Theorem 2.

2 Definitions and Preliminaries

2.1 Morse foliations and Morse modification

First we will give the precise definition of Morse foliations of codimension one of a compact $n$-manifold.

**Definition 3** Let $M$ be a compact, orientable $n$-manifold of class $C^r$, $n \geq 2$ and $r \geq 2$. By a Morse foliation $F$ of codimension one of $M$ which is tangent to the boundary $\partial M$ if $\partial M \neq \emptyset$, we mean a decomposition of $M$ into a union of disjoint connected subsets $\{L_a\}_{a \in A}$, called the leaves of the foliation such that:

1. There are finitely many points $s_i$ in $M$ which have the following property: For each $s_i$ there exists an open neighborhood $U_i$ of $s_i$ in $M$ such that for each leaf $L_a$, the components of $U_i \cap L_a$ are described as level sets of a Morse function
Figure 1: The level surfaces in a trivial neighborhood of a conic singularity.

\[ f_i : U_i \to (-1,1) \] of class \( C^r \). Furthermore, the point \( s_i \) is the unique non-degenerate critical point of \( f_i \).

(2) For each point \( p \in M, p \neq s_i \) there exists an open neighborhood \( U \) of \( p \) such that for each leaf \( L_a \), the components of \( U \cap L_a \) are described as level sets of a submersion \( f_i : U \to (-1,1) \) of class \( C^r \).

The points \( s_i \) are called singularities of \( F \). If the set \( \{ s_i \} \) is empty then \( F \) is a foliation in the usual sense and will referred to as a regular foliation or simply as foliation of \( M \). The open neighborhood \( U_i \) of \( s_i \) will be called trivial, while the neighborhood \( U \) of \( p \neq s_i \) is called a local chart of \( F \).

In particular, for each \( s_i \) there is a system of coordinates \( x = (x_1, \cdots, x_n) \) on a neighborhood \( U_i \) of \( s_i \) such that

\[ f_i(x) = - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^n. \]

In each \( U_i \) with respect to the system of coordinates \( (x_1, \cdots, x_n) \), the singularity \( s_i \) is identified with the point \( (0, \cdots, 0) \), i.e. \( f_i^{-1}(0) = s_i \).

In both special cases \( f_i(x) = - \sum_{i=1}^{n} x_i^2 \) and \( f_i(x) = \sum_{i=1}^{n} x_i^n \), the singularity \( s_i \) is called a center. For the centers, the level surfaces are defined for each value in \((-1,0]\) or in \([0,1)\).

If a singularity is not a center it is called conic.

Existence of conic singularities on the boundary \( \partial M \) of \( M \) is analogously defined and are allowed.

In the following we consider Morse foliations \( F \) of codimension one and of
class $C^\infty$ of the sphere $S^3$. If the foliated 3-manifolds are not homeomorphic to $S^3$ we will always mention it.

Each leaf of $\mathcal{F}$ is one of the following three types:

- a regular leaf which is an orientable surface;
- a leaf with finitely many conic singularities $s_i$;
- a center.

A leaf containing a conic singularity will be called singular leaf.

We always assume that $\mathcal{F}$ is orientable which means that, if we remove from $S^3$ all the singularities of $\mathcal{F}$ then we take a 3-manifold $N$ such that $\mathcal{F}|_N$ is an orientable foliation of codimension one in the usual sense. Notice finally that the space of Morse foliations of $S^3$ is equipped with the $C^1$-topology.

Regular foliations of the sphere $S^3$ are extensively studied and they are dominated by Novikov’s theorem [9] which asserts that each orientable foliation of codimension one of $S^3$ has a Reeb component and thus non-compact leaves homeomorphic to $\mathbb{R}^2$. For Morse foliations the situation is (at least seemingly) completely different. We will construct below Morse foliations whose all leaves are compact, simply connected or not, as well as foliations without any compact leaf. However, we will see that, except the case where all the leaves are simply connected, two kinds of components appear. These components will be referred to as truncated Reeb or Morse components.

Let $s$ be a conic singularity and we will give below the definition of Morse modification around $s$. Notice that this modification is local in the sense that it takes place in a trivial neighborhood $U$ of $s$.

Consider a neighborhood $U$ of $s$ homeomorphic to $\mathbb{R}^3$ such that the leaves of the induced foliation $\mathcal{F}|_U$ are the level sets of a Morse function $f : U \to \mathbb{R}$ of index 1 or 2. Let $S_t = f^{-1}(t)$. Therefore, we have locally three types of level surfaces:

1. $S_t$ is a disjoint union of two open discs $D_t, D'_t$, for each $t \in (-\infty, 0)$;
2. $S_t$ is a cylindrical leaf homeomorphic to $S^1 \times (0,1)$, for each $t \in (0, \infty)$,
3. $S_0$ contains $s$ and will be referred to as a double cone containing $s$, see Figure 1.

**Definition 4** Let $U$ be a neighborhood of $s$, as above. By a Morse modification of $\mathcal{F}$ around $s$ we mean a local transformation which produces a new Morse foliation on $S^3$, such that one of the following two modifications (A) or (B) is performed.

(A) For some $t_0 \in (-\infty, 0)$ we consider points $p_{t_0} \subset D_{t_0}$ and $p'_{t_0} \subset D'_{t_0}$ and we glue $D_{t_0}$ and $D'_{t_0}$ by identifying $p_{t_0}$ with $p'_{t_0}$. Then, a double cone $S_{t_0}$ is constructed containing as conic singularity the point $p_{t_0} \equiv p'_{t_0}$. Each $S_t, t \in (-\infty, t_0)$ is a disjoint union of two open discs and each leaf $S_t, t \in (t_0, \infty)$ is cylindrical.

(B) For some $t_0 \in (0, \infty)$ we consider an essential simple curve $c_{t_0} \subset S_{t_0}$. By shrinking $c_{t_0}$ to a point $s_{t_0}$, the leaf $S_{t_0}$ is transformed to a double cone.
containing the point $s_{t_0}$ as a conic singularity. Each $S_t$, $t \in (-\infty, t_0)$ is a disjoint union of two open discs and each leaf $S_t$, $t \in (t_0, \infty)$ is cylindrical.

Obviously a Morse modification can be defined in any 3-manifold $M$ equipped with a Morse foliation $\mathcal{F}$ and the modification $(A)$ is the inverse of $(B)$ and vice versa.

**Remark 5** Obviously, modulo Morse modifications, we may always assume that a leaf of $\mathcal{F}$ does not contain more than one conic singularity. More precisely, the following property holds:

- For every Morse foliation $\mathcal{F}$ and every neighborhood $U$ of $\mathcal{F}$, with respect to the $C^1$- topology, there is a Morse foliation $\mathcal{F}'$ in $U$ such that every leaf of $\mathcal{F}'$ does not contain more than one conic singularity.

In what follows and in order to avoid technical difficulties, we always assume that each leaf of a Morse foliation $\mathcal{F}$ contains at most one conic singularity. It is important to notice here that this assumption does not affect the dynamic behavior of $\mathcal{F}$.

### 2.2 The connected sum of foliations (see [13])
Let $D$ be an embedded 2-disc in $S^3$ which does not contain any singularity of $\mathcal{F}$. The definition of being $D$ in \textit{general position} with respect to $\mathcal{F}$ is standard for regular foliations of codimension one (see for example [2], Ch. 7, Def. 7.1.5) and it is readily generalized for Morse foliations.

Now, the connected sum of foliated 3-manifolds will be defined.

Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be two compact foliated 3-manifolds with non-empty boundary. Here we assume that each $\mathcal{F}_i$ is either a Morse foliation or a regular 2-dimensional foliation tangent to $\partial M_i$. Let $p_i \in \partial M_i$ and let $B_i$ be a neighborhood of $p_i$ in $M_i$ such that:

1. $B_i$ is homeomorphic to a closed 3-ball and there do not exist conic singularities or centers in $B_i$;
2. $\partial B_i$ is the union of two 2-discs $D_i$ and $E_i$ such that: $D_i \cap E_i = \partial D_i = \partial E_i$ and $E_i \subset \partial M_i$;
3. $D_i$ is in general position with respect to $\mathcal{F}_i$ and $\mathcal{F}_i|_{D_i}$ is a foliation by concentric circles of center $o_i$.

The induced foliation $\mathcal{F}_i|_{D_i}$ by concentric circles will be referred to as a \textit{trivial foliation} on the disc $D_i$.

From Reeb stability theorem [10] we deduce that $\mathcal{F}_i$ induces in $B_i - \{o_i\}$ a product foliation by 2-discs. The neighborhood $B_i$ will be referred to as a \textit{trivially foliated neighborhood} of $p_i$. If $N_i = (M_i - B_i) \cup D_i$, we may form the connected sum $M_1 \# M_2$ by identifying $N_1$ with $N_2$, via a diffeomorphism $f : D_1 \to D_2$ which sends $o_1$ to $o_2$ and the leaves of $\mathcal{F}_1|_{D_1}$ onto the leaves of $\mathcal{F}_2|_{D_2}$. In this way, a Morse foliation $\mathcal{F}_1 \# \mathcal{F}_2$ is defined on $M_1 \# M_2$: this foliation has one more conic singularity at the point $o_1 \equiv o_2$ and is tangent to $\partial(M_1 \# M_2)$. Henceforth, the connected sum of $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ will be denoted by $(M_1 \# M_2, \mathcal{F}_1 \# \mathcal{F}_2)$. Obviously $\partial(M_1 \# M_2) = \partial M_1 \# \partial M_2$, where $\partial M_i \# \partial M_2$ denotes the connected sum of surfaces $\partial M_1$ and $\partial M_2$. In Figure 2 and in a 2-dimensional slice, the steps of the operation of connected sum are depicted. This operation is defined in [13] by means of equations.
2.3 Examples of Morse foliations

Below we will give examples of Morse foliations of compact 3-manifolds.

(1) We may foliate the 3-dimensional ball $B^3$ by concentric spheres. Let denote this foliation by $S$. This is a Morse foliation with one center. By gluing two such 3-balls along their boundaries, we take a Morse foliation $\mathcal{F}$ on $S^3$ which has two centers and all others leaves are 2-spheres.

Similarly to the previous example, Morse foliations of $S^3$ can be constructed whose all leaves are compact and simply connected. For, we form the connected sum of finitely many $(B_i, S_i)$, where each $B_i$ is a 3-ball and each $S_i$ is a Morse foliation by concentric spheres. Let denote by $(\overline{B}, \mathcal{S})$ the resulting foliated manifold. Obviously $\overline{B}$ is a 3-ball and each leaf of $\mathcal{S}$ is compact and simply connected. Gluing two such copies $(\overline{B}_1, S_1)$, $(\overline{B}_2, S_2)$, by identifying the boundaries $\partial \overline{B}_1$ and $\partial \overline{B}_2$ via a diffeomorphism, we take a foliation $\mathcal{C}$ on $S^3$ whose all leaves are compact and simply connected. The foliation $\mathcal{C}$ will be referred to as a Morse foliation of $S^3$ with simply connected leaves.

(2) Let $(T, \mathcal{R})$ be a Reeb component, i.e. $T$ is homeomorphic to the solid torus $S^1 \times D^2$: all leaves in the interior of $T$ are planes and the boundary $\partial T$ is a toral leaf. Considering two Reeb components $(T_1, \mathcal{R}_1)$ and $(T_2, \mathcal{R}_2)$ we form the connected sum $(T_1 \# T_2, \mathcal{R}_1 \# \mathcal{R}_2)$. Obviously $P = T_1 \# T_2$ is a solid pretzel and $\mathcal{G} = \mathcal{R}_1 \# \mathcal{R}_2$ is Morse foliation whose all leaves in the interior of $P$ are non-compact. Finally, considering two such copies $(P, \mathcal{G})$ and $(P', \mathcal{G}')$ of foliated pretzels and gluing them properly along their boundaries we take a Morse foliation $\mathcal{F}$ on $S^3$ with a single compact leaf, say $L_0$, which is a surface of genus 2.

Now we may construct a Morse foliation on $S^3$ without compact leaves (see [13], Prop. 3.1). For this, we consider a 2-sphere $S$ embedded in $S^3$ such that:

a) $S = D \cup D'$, where $D, D'$ are 2-discs and $D \cap D' = c$ is a separating curve in $L_0$;

b) $D \subset P$, $D' \subset P'$ and the induced foliation $\mathcal{G}_{\mid D}$ (resp. $\mathcal{G}'_{\mid D'}$) is a trivial foliation by concentric circles.

Let $U$ be a neighborhood of $c$ in $S^3$, homeomorphic to $S^1 \times [-1, 1] \times [-1, 1]$ such that $\mathcal{F}_{\mid U}$ is a trivial product foliation of the form $S^1 \times [-1, 1] \times \{t\}$, $t \in [-1, 1]$ with $S^1 \times [-1, 1] \times \{0\} \subset L_0$ and $S^1 \times \{0\} \times [-1, 1] \subset S$. Let denote by $L_0^+$, $L_0^-$ the connected components of $L_0 - c$. By a small perturbation of $\mathcal{F}$ in $U$ we may destroy the compact leaf $L_0$ by leading $L_0^+$ inside $P'$ and $L_0^-$ inside $P$, see [13] for a formal description of this perturbation. In this way we take a foliation without compact leaves.

(3) Consider a Morse foliation $\mathcal{M}$ on the solid torus $T$ such that $\mathcal{M}$ has one center $c$ and one conic singularity $s$. The regular leaves of $\mathcal{M}$ are either spheres or tori parallel to $\partial T$. Moreover, $\mathcal{M}$ has a singular leaf $C$ containing the conic singularity $s$ and $C - s$ is homeomorphic to 2-sphere minus two points. The foliated manifold $(T, \mathcal{M})$ will be called a Morse component, see Figure 3. The singular leaf $C$ will be called a pseudo-torus. Let denote by $V$ the component of $T - C$ which is foliated by spherical leaves. By abusing the language, we will
say that the singular leaf \( C \) with all the spherical leaves of \( V \) define also a Morse component on the closure \( \overline{V} \) of \( V \); if we want to be more precise this component will be called a pseudo-Morse component.

Now, considering two copies of \( (T_i, M_i) \), \( i = 1, 2 \) of Morse components we form the connected sum \( (T_1 \# T_2, M_1 \# M_2) \). Obviously \( T_1 \# T_2 \) is a solid pretzel and \( M_1 \# M_2 \) has two centers and two conic singularities. Obviously, we may construct a Morse foliation on \( S^3 \) containing Morse components.

**Definition 6** Let \((T, R)\) (resp. \((T, M)\)) be a Reeb component (resp. Morse component). We remove the interior of a trivially foliated neighborhood \( B \) of some point \( p \in \partial T \) and let \( G = T - \text{int}(B) \). Then \((G, R_{|G})\) (resp. \((G, M_{|G})\)) will be called a truncated Reeb (resp. truncated Morse) component.

### 2.4 Elimination of trivial pairs of singularities

Now, we will describe a method that allow us to modify a Morse foliation \( \mathcal{F} \) by transforming the induced foliation \( \mathcal{F}_{|B} \) in certain 3-balls \( B \) which are trivially foliated by concentric spheres. In this manner the Morse foliation \( \mathcal{F} \) is transformed to a Morse foliation \( \mathcal{F}' \) which is simpler than \( \mathcal{F} \). On the other hand, proving Theorem 1 for \( \mathcal{F}' \) we may see that it remains valid for \( \mathcal{F} \).

First we will eliminate some pairs of singularities.

If \( c \) is a center of \( \mathcal{F} \) then around \( c \) there are spherical leaves. So, we assume that there exists a \( C^\infty \)-map \( g : [0, 1] \times S^2 \rightarrow S^3 \) such that:
1) \( g(\{0\} \times S^2) = c \),
2) \( g : (0, 1] \times S^2 \to S^3 \) is an embedding and \( g(\{t\} \times S^2) \), \( t \in (0, 1) \) are parallel spherical leaves,
3) \( g(\{1\} \times S^2) \) contains a conic singularity \( s \).

Let \( S = g(\{1\} \times S^2) \) and let \( V \) be the component of \( S^3 - S \) containing the center \( c \).

**Definition 7** We will say that the pair \((c, s)\) defines a trivial pair of singularities and \( \nabla = V \cup S \) will be called a trivial bubble.

Obviously, we may find discs \( D \) and \( E \) embedded in \( S^3 \) such that:

(i) \( F|_D \) is trivial;
(ii) \( E \) is contained in a leaf of \( F \), \( E \cap D \) is a leaf of \( F|_D \) and \( E \cup D \) bounds a 3-ball \( B \) containing \( \nabla \);
(iii) \( F|_B \) consists of a center \( c \), of spherical leaves, of discs parallel to \( E \) and of a singular leaf containing \( s \), see Figure 4(a).

Keeping the discs \( D \) and \( E \) fixed we may replace the foliation \( F|_B \) by a foliation by parallel discs such that \( F|_D \) stays invariant, see Figure 4(b). In this way we may remove the singularities \( c \) and \( s \).

Notice here that the elimination of a trivial pair of singularities is also defined in Proposition 6.2 of [12].

Finally, it is not difficult to see that any Morse foliation of \( S^3 \) with compact simply connected leaves, is led to a foliation with two centers and spherical leaves by removing successively all trivial pairs of singularities. In fact, eliminating all these trivial pairs we obtain a foliation with all its leaves simply connected and without conic singularities. Thus our assertion follows.

### 2.5 Elimination of bubbles

In this paragraph we assume that \( F \) does not have trivial pair of singularities.

Contrary to the regular foliations, if we allow singularities we may construct in the interior of a 3-ball \( B \) non-trivial Morse foliations which are tangent to the boundary \( \partial B \) i.e. \( F|_B \) does not consist of concentric 2-spheres parallel to \( \partial B \).

Let \( U \) be a local chart of \( F \) i.e. the closure \( \overline{U} \) of \( U \) is homeomorphic to \( D^2 \times [0, 1] \) and the leaves of \( F|_{\overline{U}} \) are of the form \( D^2 \times \{t\} \), \( t \in [0, 1] \). To the foliation \( F \) we may perform the following modification. In the interior of some \( D_{t_0} = D^2 \times \{t_0\} \) we pick a point, say \( s \), and we join to \( D_{t_0} \) a 2-sphere \( S \subset U \) so that \( S \cap D_{t_0} = \{s\} \). Obviously, \( S \) defines a 3-ball \( B \) in \( U \). In the interior of \( B \) we consider a non-trivial Morse foliation. We also assume that all the other discs \( D_t \) with \( t \neq t_0 \) remain invariants under an isotopy taking place in \( U \). In this way, a new Morse foliation \( F' \) is constructed which has \( s \) as a conic singularity; in Figure 5(a) and 5(b) a 2-dimensional section of the whole process is drawn. The 3-ball \( B \) will be called a (non-trivial) bubble of \( F' \), provided that the foliation \( F'|_B \) is non-trivial and we will say that \( F' \) is constructed from \( F \) by adding a bubble to the leaf \( L_{t_0} \).

We will define now the notion of bubble of \( F \) in a more general way.
Definition 8 Let $S$ be a subset of a singular leaf $L_0$ of $\mathcal{F}$ such that $S$ is homeomorphic to $S^2$ and $S$ contains a conic singularity $s$. Let $W$ be the closure of the component of $S^3 - S$ which does not contain $L_0 - S$ and we assume that $\mathcal{F}|_W$ is non-trivial. The 3-ball $W$ will be called a bubble of $\mathcal{F}$ and the interior $\text{Int}(W)$ of $W$ will be called the interior of the bubble. The 2-sphere $S$ will be called the boundary of $W$ and it will be said that $S$ defines the bubble $W$.

Now $W$ can be eliminated in two steps. First we foliate $\text{Int}(W)$ by concentric 2-spheres by adding a center $c$. We take in this way a foliation $\mathcal{F}'$ such that the pair of singularities $(s, c)$ forms a trivial pair. Hence, in a second step, we can eliminate $(s, c)$ as it is described above. If $\mathcal{F}''$ is the resulting foliation, then we will say that $\mathcal{F}''$ results from $\mathcal{F}$ by eliminating (or removing) the bubble $W$. Below by the term bubble we mean always a non-trivial one.

It is worthy to notice that the procedure of eliminating the bubbles of $\mathcal{F}$ creates a new Morse foliation $\mathcal{F}_0$, which is not well defined in the sense that $\mathcal{F}_0$ depends on the order in which the bubbles are eliminated. For instance, let $W_1, W_2$ be two bubbles such that $S_2 = \partial W_2$ is contained in the interior of $W_1$ and $S_1 = \partial W_1$ is contained in the interior of $W_2$. If we eliminate $W_1$ (resp. $W_2$) obtaining a foliation $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) then the bubble $W_2$ (resp. $W_1$) disappears also. Obviously $\mathcal{F}_1 \neq \mathcal{F}_2$. However, we will show that the order in which the bubbles are eliminated does not affect our main results as they are stated in Theorems 1 and 2.

The following proposition can be easily proven.

Proposition 9 Let $\mathcal{F}$ be a Morse foliation containing and assume that $\mathcal{F}$ does not have bubbles and admits at least one non-spherical leaf. If $\mathcal{F}$ has at least one spherical leaf then $\mathcal{F}$ contains a pseudo-toral leaf provided that each leaf of $\mathcal{F}$ contains at most one conic singularity.

Proof. By Reeb stability theorem three cases can occur. Either $\mathcal{F}$ has two
centers and all its leaves are homeomorphic to $S^2$ or $\mathcal{F}$ has a bubble or $\mathcal{F}$ has a pseudo-toral leaf $T$. By hypothesis the first and second cases are excluded. Therefore our result follows immediately.

### 2.6 Truncated bubbles

In the following we will introduce the notion of **truncated bubble**. The elimination of such bubbles is also necessary in order to simplify further and understand better a Morse foliation.

We start with the following definitions.

**Definition 10** Let $s$ be a conic singularity of $\mathcal{F}$ and let $D$ be a 2-disc such that (see Figure 6):

1. $\mathcal{F}|_D$ is a trivial foliation. The leaves of $\mathcal{F}|_D$, which are simple closed curves (s.c.c.), are denoted by $c_t$, $t \in (0, 1]$.

2. If we denote by $L_t$ the leaf of $\mathcal{F}$ containing $c_t$, $t \in (0, 1]$, then $c_t$ is the boundary of a 2-disc $D_t \subset L_t$ with $D_t \cap D = c_t$ for each $t \in (0, 1/2]$. Furthermore, $s \in \text{Int}(D_{1/2})$ while $D_t$ does not contain any conic singularity for $t \in (0, 1/2)$.

3. The curves $c_t$ are not homotopic to a constant in $L_t$ for each $t \in (1/2, 1]$.

4. The curve $c_1$ has trivial holonomy in its both sides.

If $D$ satisfies the above conditions (1)-(4) then $D$ will be called a **perfect disc** associated to $s$ or simply a perfect disc. Also, $s$ will be called the singularity corresponding to the perfect disc $D$.

Hence, to each perfect disc corresponds a conic singularity and vice-versa.
Now we are able to give the following definition.

**Definition 11** Let $W$ be a 3-ball in $S^3$ and let $D_i$, $i = 1, \ldots, n$ be disjoint 2-discs in $\partial W$ such that:

1. For each $i$, $D_i$ is a perfect disc with corresponding conic singularity $s_i$, such that $s_i \neq s_j$ for $i \neq j$.
2. If $F = \partial W - (\cup_i D_i)$ then $F$ does not contain any conic singularity.
3. Assuming that all $s_i$ are contained in $W$, there does not exist a bubble $S$ in $W$.

Then $W$ will be called a truncated bubble and the set $\partial_t W = \partial W - (\cup_i D_i)$ will be called the tangent boundary of $W$.

Let $W$ be a truncated bubble and let $\vec{n}$ be the unit normal vector to $F$. The condition (3) in the Definition above ensures that there exists at least one perfect disc $D_k \subset \partial W$ and one perfect disc $D_l \subset \partial W$ such that $\vec{n}$ is directed inward $D_k$ and outward $D_l$. Indeed, if our statement was not true, then we may deduce from Reeb’s stability theorem [10] that a bubble would exist in $W$. Therefore the discs $D_i$ are separated in the discs $D_1, \ldots, D_m$ and the discs $D_{m+1}, \ldots, D_n$, such that $\vec{n}$ is directed inward on $D_i$ for $i = 1, \ldots, m$ and outward on $D_j$ for $j = m + 1, \ldots, D_n$, see Figure 7.

Apparently, we may isotope each $D_i$ to a perfect disc, keeping $\partial D_i$ on the same leaf, so that all $s_i$ are outside $W$, see Figure 7. Thus, in Definition 11 we have assumed that all $s_i$ are contained in $W$ so that the hypothesis (3) of the definition makes sense.

In order to eliminate the truncated bubble $W$ we may perform appropriate Morse modifications around each conic singularity $s_i$ so that the truncated bubble $W$ disappears and in its place two bubbles $S_1$ and $S_2$ are created. Notice that $S_1$ should be contained in the interior of $S_2$ and $S_2$ should be contained in the interior of $S_1$. Now, we may eliminate the created bubbles as previously. That is, eliminating the bubble $S_1$ (resp. $S_2$) the bubble $S_2$ (resp. $S_1$) is also eliminated since $S_2$ (resp. $S_1$) is contained in the interior of $S_1$ (resp. $S_2$).

In a next paragraph we will need the following property of a truncated bubble. There is one disc $E_1$ which contains a conic singularity $s_{i_1}$, $1 \leq i_1 \leq m$ and one disc $E_2$ which contains a conic singularity $s_{i_2}$, $m + 1 \leq i_2 \leq n$ such that, if $\partial E_1 = c_1$ and $\partial E_2 = c_2$ then the curves $c_1$ and $c_2$ bound an annulus $C$ contained in a neighborhood of $\partial_t W$ and $F_{|C}$ is a family of parallel closed leaves, see Figure 7.

**Remark 12** From the discussion above, if $F$ is a Morse foliation then we may construct a new Morse foliation $F'$ which does not contain neither trivial pair of singularities nor bubbles nor truncated bubbles. In fact, first we remove all trivial pair of singularities, second we eliminate all bubbles and third we eliminate all truncated bubbles. In the second and third step of our procedure we remark that new bubbles/truncated bubbles are not created. So $F'$ results after finitely many steps.
Figure 7: A truncated bubble.
2.7 Haefliger structures

Finally, in the next few lines of this paragraph we will relate Haefliger structures with Morse foliations in order to indicate that the latter are naturally considered.

A \( C^r \)-Haefliger structure \( \mathcal{H} \) on a manifold \( M \) can be considered as a generalized foliation on \( M \) with a certain type of singularities, see \([7], [6]\) for more details. Suppose that \( \mathcal{H} \) is a codimension \( q \) \( C^r \)-Haefliger structure. Then, we may associate to \( \mathcal{H} \), a \( q \)-dimensional \( C^r \)-vector bundle \( E \) over \( M \), a section \( i : M \rightarrow E \) and a \( C^r \)-foliation \( \mathcal{G} \) defined in a neighborhood of \( i(M) \) and transverse to the fibers. The triple \((E, M, i)\) is called the graph of \( \mathcal{H} \) and determines it.

In particular, if we consider a Haefliger structure \( \mathcal{H} \) of codimension one on a closed 3-manifold \( M \), then generically, the section \( i : M \rightarrow E \) is in general position with respect to \( \mathcal{G} \), which implies that the contact points are finitely many singularities of a Morse function. In this way, a codimension 1, Morse foliation \( F \) on \( M \) is associated to \( \mathcal{H} \). Furthermore, it follows that in this particular case, Morse foliations defined above are generic in the space of Haefliger structures. This fact that makes the class of Morse foliations quite interesting.

3 Vanishing and anti-vanishing cycle

In this section we recall the definition of vanishing cycle and we will generalize Novikov’s theorem for Morse foliations. In contrast to the vanishing cycle the notion of “anti-vanishing cycle” is introduced. We always assume that each leaf contains at most one conic singularity and that \( F \) does not contain trivial pair of singularities.

The following definition is borrowed from \([13]\) and is less general than the initial one in \([9]\) because we assume that the curves \( a_t \) below are simple.

**Definition 13** Let \( f : S^1 \times [0, 1] \rightarrow S^3 \) be a smooth map such that:

1. \( f_t(S^1) = a_t \) is a simple curve contained in a leaf \( L_t \) for each \( t \in [0, 1] \), where \( f_t(x) = f(x, t) \);
2. for each \( x \in S^1 \), the arc \( f_t(x) \) is transverse to \( F \);
3. the curve \( a_0 \) is not homotopic to a constant in \( L_0 \);
4. \( a_t \) is nullhomotopic in \( L_t \), \( 0 < t \leq 1 \).

The curve \( f_0(S^1) = a_0 \) will be called a vanishing cycle on the leaf \( L_0 \).

In order to state the next theorem we need the following definition.

**Definition 14** A generalized Reeb component is a Morse foliation obtained by adding finitely many bubbles to the leaves of a Reeb component.

**Definition 15** Let \( \mathcal{R} \) be a Reeb component. If we add finitely many bubbles to the leaves of \( \mathcal{R} \) then we take a Morse foliation which will be called a generalized Reeb component.
Figure 8: Appearance of a truncated bubble.
Novikov in [9] proved that if \((M, F)\) is a closed, orientable 3-manifold foliated by a regular, orientable foliation \(F\) of codimension one and if some leaf of \(F\) contains a vanishing cycle then \(F\) contains a Reeb component. Below we will show that a similar result is also valid for Morse foliations of \(S^3\). Thus, we have:

**Theorem 16** Let \(f : S^1 \times [0, 1] \to S^3\) be a smooth map defining a vanishing cycle for the Morse foliation \(F\) of \(S^3\). Then the foliation \(F\) contains a generalized Reeb component.

**Proof.** A first complete proof of this theorem for regular foliations is given in [13]. We follow it to prove Novikov’s theorem for Morse foliations. Below, we will sketch the steps of the proof.

Let \(a_0 = f(S^1 \times \{0\})\) be the vanishing cycle defined by \(f\) and let \(a_t\) be contained in a leaf \(L_t\) for each \(t\). First of all, if \(U_i\) is any bubble of \(F\) which does not contain \(a_0\) in its interior then we eliminate \(U_i\). In this way, the map \(f\) of the theorem which defines \(a_0\) is not affected and furthermore, the lack of bubbles, which do not contain \(a_0\) in their interiors, allow us to apply the steps of proof of Lemma 1 of [13]. Thus, as in [13], the following claim can be proved (see Lemma 1 in [13]).

**Claim 1.** There is an immersion \(F : D^2 \times (0, 1] \to M\) such that,
1. for each \(t \in (0, 1]\), \(F_t(D^2)\) is contained in a leaf \(L_t\) and \(F_t\) is an embedding,
2. \(F_t|_{\partial D^2} = f_t\),
3. for each \(x \in D^2\), the curve \(t \to F_t(x)\) is normal to the foliation.

As in [13], if this claim was not valid then a spherical leaf would exist. Indeed, the set of \(t \in (0, 1]\) for which \(F_t\) can be defined to satisfy conditions (1)-(3) is open by Reeb’s stability theorem. So, it suffices to show that this set is also closed. Suppose \(t_0 \in (0, 1]\) such that \(F_t\) is defined for \(t_0 < t \leq 1\). The map \(f_{t_0} : S^1 \to L_{t_0}\) is an embedding and is nullhomotopic hence \(f_{t_0}\) extends to an embedding \(F_{t_0} : D^2 \to L_{t_0}\). Now displace \(F_{t_0}(D^2)\) by the orthogonal trajectory field to the leaves \(L_t\) for some \(t > t_0\). Let denote by \(D_t\) this displaced disc in \(L_t\). Then there are two possibilities, either \(F_t(D^2) = D_t\) or \(F_t(D^2) \cap D_t = f_t(S^1)\).

In the first case, \(F_t\) is extended continuously on \([t_0, 1]\). In the second case, by Proposition 9, \(L_t\) should be either a spherical leaf or a pseudo-torus. But we have eliminated all bubbles which do not contain \(a_0\) in their interior. Hence \(L_t\) cannot be a spherical leaf. On the other hand, if \(L_t\) was a pseudo-torus then for \(\varepsilon > 0\) arbitrarily small the curve \(f_{t+\varepsilon}(S^1)\) should not be null homotopic in \(L_{t+\varepsilon}\). This gives a contradiction which proves Claim 1.

The other steps of the proof, as they are expressed in Lemmata 2-6 of [13], can be repeated verbatim and so Novikov’s Theorem follows for Morse foliations. Obviously, if we add again all eliminated bubbles we obtain a generalized Reeb component. ■

**Remark 17** In the proof above, we have assumed that each \(f_t\) is an embedding. For regular foliations of \(S^3\) the theorem is valid without this extra-hypothesis [9], [2]. Novikov’s theorem could also be proven in its full generality, but for the purpose of the present paper we don’t need the stronger version of the theorem.
Definition 18 Let \( f : S^1 \times [0, 1] \to S^3 \) be an embedding such that:

1. \( f_t(S^1) = a_t \) is a simple curve contained in a leaf \( L_t, \ 0 \leq t \leq 1 \), where \( f_t(x) = f(x, t) \);
2. for each \( x \in S^1 \), the arc \( f_t(x) \) is transverse to \( \mathcal{F} \);
3. the curves \( a_t \) are nullhomotopic in \( L_t, \ 0 \leq t \leq 1/2 \);
4. the curves \( a_t \) are not homotopic to a constant in \( L_t, \ 1/2 < t \leq 1 \).

The curve \( f_{1/2}(S^1) = a_{1/2} \) will be called an anti-vanishing cycle on \( L_{1/2} \).

The following terminology will be used below.

Let \( E \) be a 2-disc and \((p, q)\) be a pair of points in the interior of \( E \). Let \( E' \) be the space obtained by identifying \( p \) with \( q \). Then \( E' \) will be called a pseudo-disc and the point \( p \equiv q \) will be called a double point of \( E' \).

We have the following lemma.

Lemma 19 Let \( f : S^1 \times [0, 1] \to S^3 \) be a smooth family of embedded circles \( f_t(S^1) \) which define an anti-vanishing cycle \( f_{1/2}(S^1) \subset L_{1/2} \). Then either there exists a perfect disc \( D \) associated to some conic singularity \( s \) such that all \( f_t(S^1) \), \( t \in [0, 1] \) are leaves of \( \mathcal{F}_D \) or, the appearance of \( f_{1/2}(S^1) \) implies the existence of a truncated Morse component.

Proof. Let \( f_{1/2}(S^1) = a \). Since \( a \) is a simple closed curve, nullhomotopic in \( L_{1/2} \), it bounds a simply connected subset \( \Delta \subset L_{1/2} \) which is either a 2-disc or a pseudo-disc with a double point. Since \( f_t(S^1) \) is not nullhomotopic in \( L_t \), \( 1/2 < t \leq 1 \) we deduce the existence of a conic singularity \( s \in \Delta \) which in the latter case is the double point of the pseudo-disc \( \Delta \). In the first case, and since we have assumed that \( L_0 \) contains a single singularity (see Remark[3]), it follows that we may find a trivially foliated neighborhood \( U \) of \( s \) in \( S^3 \) such that \( \Delta \subset U \). Therefore, there is in \( U \) a perfect disc \( D \) associated to \( s \) such that \( f_{1/2}(S^1) \) is a leaf of \( \mathcal{F}_D \). In the second case, the existence of a pseudo-disc is equivalent with the existence of truncated Morse/pseudo-Morse component. \( \blacksquare \)
From the previous proof we have that each anti-vanishing cycle $a$ on a leaf $L$ determines a conic singularity $s$ which will be referred to as the singularity determined by the anti-vanishing cycle $a$. The singularity $s$ is unique provided that each leaf contains at most one conic singularity.

Two anti-vanishing cycles $a, a'$ which determine the same conic singularity will be referred to as equivalent. Obviously $a$ and $a'$ must belong to the same leaf $L$ of $F$.

Finally, the following terminology will be used in the next paragraph:

**Terminology**: Let $D$ be 2-disc embedded in $S^3$ which does not contain any conic singularity of $F$. We assume that $D$ is in general position with respect to $F$, $\partial D$ belongs to a leaf $L$ of $F$ and $\partial D$ is a non-nullhomotopic curve in $L$. We assume that $a$ is an anti-vanishing cycle on a leaf $L_a$ of $F$ such that: $a$ is the boundary of a sub-disc $A \subset D$ and all the leaves of $F_A$ are closed curves, nullhomotopic in the leaves of $F$ where they belong. In this case, we will say that the anti-vanishing cycle $a$, as well, any anti-vanishing cycle $a'$ equivalent to $a$, appears in $D$ or that $a$, equivalently $a'$, is contained in $D$. Obviously, there exists at least one center $c$ of $F|_D$ in the interior of $A$ surrounded by $a$.

4 Existence of truncated Reeb components
In this paragraph we make the following assumptions:

1. \( \mathcal{F} \) does not have all its leaves compact and simply connected;
2. \( \mathcal{F} \) does not have trivial pairs of singularities;
3. \( \mathcal{F} \) does not have bubbles or truncated bubbles;
4. \( \mathcal{F} \) does not have a vanishing cycle;
5. \( \mathcal{F} \) does not have a Morse component or a truncated Morse component.

From Remark 12 such a foliation \( \mathcal{F} \) can be obtained after eliminating successively all trivial pair of singularities, all bubbles and all truncated bubbles. Under these assumptions we will prove that \( \mathcal{F} \) has a truncated Reeb component.

Notice that this component appears in the Example 2 of Section 2. The basic idea for this proof is to capture the conic singularities inside trivially foliated 3-balls and study their complement.

The lack of bubbles and truncated bubbles give us a better insight into the study of \( \mathcal{F} \). From Proposition 16 the lack of vanishing cycles implies the lack of Reeb/singular Reeb components. Also, Novikov’s theorem [9] implies that if \( \mathcal{F} \) does not have a vanishing cycle then \( \mathcal{F} \) must have central or conic singularities.

Finally, from Lemma 19, the lack of Morse/truncated Morse components implies that there do not exist pseudo-discs with double points contained in some singular leaves of \( \mathcal{F} \). Thus, if \( a \) is an anti-vanishing cycle on a leaf \( L \), it follows that there exists a 2-disc \( D \subset L \) containing a conic singularity in its interior with \( \partial D = a \).

We need the following.

**Lemma 20** Let \( D \) be an embedded disc such that:

1. \( \mathcal{F}|_{D} \) is a trivial foliation of concentric circles \( d_{t} \subset L_{t} \), \( t \in [0,1] \), where \( L_{t} \) is a leaf of \( \mathcal{F} \) and \( d_{0} \) degenerates to a point of \( D \);
2. \( d_{1} \approx 0 \) in \( L_{1} \) i.e. \( d_{1} = \partial D \) is a non-nullhomotopic curve in \( L_{1} \).

Then there exists \( t_{0} \in (0,1) \) such that: \( d_{t} \approx 0 \) in \( L_{t} \) for each \( t \in [0,t_{0}] \) i.e. all the curves \( d_{t} \) are nullhomotopic in \( L_{t} \) for each \( t \in (0,t_{0}] \), while \( d_{t} \approx 0 \) in \( L_{t} \) for each \( t \in (t_{0},1] \).

**Proof.** Let \( t_{0} = \sup \{ t \in (0,1) : d_{t} \text{ is nullhomotopic in } L_{t} \text{ for } t \leq t_{0} \} \).

Obviously a such \( t_{0} \) exists because \( \mathcal{F} \) does not have a vanishing cycle on any of its leaves. Therefore there exist discs, say \( E_{t_{0}} \), such that: \( \partial E_{t_{0}} = d_{t_{0}} \) and \( E_{t} \subset L_{t} \) for each \( t \in (0,t_{0}] \). Let \( D_{t} \) be the subdisc of \( D \) bounded by \( d_{t} \) for each \( t \in (0,1) \). Obviously \( E_{t_{0}} \cap D \) consists of finitely many leaves of \( \mathcal{F}|_{\text{Int}(D_{t_{0}})} \). If \( E_{t_{0}} \cap D \neq d_{t_{0}} \), we denote by \( d_{t_{0}}^{c} \) the leaf of \( \mathcal{F}|_{\text{Int}(D_{t_{0}})} \) which is closest to \( d_{t_{0}} \) in \( D_{t_{0}} \). The disc \( E_{t_{0}} \) must contain a conical singularity, say \( s_{0} \), see Figure 8. Notice that, if \( E_{t_{0}} \cap D \neq d_{t_{0}} \) and \( E_{t_{0},t_{0}^{c}} \subset E_{t_{0}} \) is the annulus bounded by the curves \( d_{t_{0}}^{c} \) and \( d_{t_{0}} \), then \( s_{0} \in E_{t_{0},t_{0}^{c}} \) otherwise \( \mathcal{F} \) should have a bubble.

Now, let \( t_{1} = \inf \{ r \in (t_{0},1) : d_{r} \sim 0 \text{ in } L_{r} \text{ and } d_{r} \sim 0 \text{ in } L_{r} \text{ for each } t \in (t_{0},r) \} \). Apparently, a such \( t_{1} \) always exists if the lemma is not valid. On the other hand, there exists a disc \( E_{t_{1}} \subset L_{t_{1}} \) with \( \partial E_{t_{1}} = d_{t_{1}} \) and \( E_{t_{1}} \) must contain a conic singularity \( s_{1} \). We may see that \( s_{1} \neq s_{0} \) since \( d_{t} \sim 0 \) in \( L_{t} \) for each \( t \in (t_{0},t_{1}) \). Also, if we denote by \( A = d_{t_{1}} - \text{Int}(D_{t_{0}}) \) then the interior of \( E_{t_{1}} \) cannot intersect \( A \) but it can intersect \( \text{Int}(D_{t_{0}}) \), see Figure 8.
Claim: If there exists a $t_1 \in (t_0, 1)$ as above, then $\mathcal{F}$ has a truncated bubble. Proof of Claim. We may replace $D_{t_0}$ with a disc $D'_{t_0}$ with $\partial D'_{t_0} = d_{t_0}$ such that (see Figure 8):

1. $\mathcal{F}|_{D'_{t_0}}$ is trivial;
2. $D_{t_0} \cap D'_{t_0} = d_{t_0}$ and all the curves of $\mathcal{F}|_{D'_{t_0}}$ are contractible in the leaves of $\mathcal{F}$ where they belong;
3. $E_{t_1} \cap D_{t_0} = \emptyset$ and $E_{t_0} \cap D'_{t_0} = d_{t_0}$.

Let $S_{t_1} = E_{t_1} \cup A \cup D'_{t_0}$. Then $S_{t_1}$ is a 2-sphere and let $B_{t_1}$ be the 3-ball bounded by $S_{t_1}$ and containing some neighborhood $V(s_1) \subset L_{t_1}$ of $s_1$ which is homeomorphic to a double cone.

In a trivial neighborhood $U$ of $s_1$ we may find first a simple closed curve $e$ contained in some leaf $L$ of $\mathcal{F}|_{B_{t_1}}$, and second, an annulus $C \subset L \cap \text{Int}(B_{t_1})$ such that:

1. $e$ is not nullhomotopic in $L$;
2. $\partial C = e \cup d_\varepsilon$, where $t' \in (t_1 - \varepsilon, t_1)$ for $\varepsilon$ sufficiently small and $d_\varepsilon$ is a leaf of $\mathcal{F}|_{A}$.

Now we may consider a perfect disc $D_e \subset U$ with $\partial D_e = e$ and the perfect disc $D_\varepsilon \subset D_{t_1}$ with $\partial D_\varepsilon = d_\varepsilon$. Obviously the sphere $D_e \cup C \cup D_\varepsilon$ is the boundary of a truncated bubble, say $W$, contained in $B_{t_1}$. In this way we take a contradiction which proves our claim.

Finally, the claim above gives a contradiction to the assumption (3) and thus our lemma is proven. ■

We need the following definitions.

**Definition 21** Let $L_0$, $L_1$ be two surfaces in $S^3$ such that:

1. $L_0$, $L_1$ are homeomorphic and each one is a subset of a leaf of $\mathcal{F}$;
2. there exists a trivial foliated product $V = L \times [0, 1]$ in $S^3$ such that, each $L \times \{t\}$ is a subset of a leaf of $\mathcal{F}$ with $L \times \{0\} = L_0$, $L \times \{1\} = L_1$.

Then the surfaces $L_0$, $L_1$ will be called parallels in $\mathcal{F}$ and $\mathcal{F}|_V$ will be referred to as band of leaves of $\mathcal{F}$.

**Definition 22** Let $B$ be a 3-ball in $S^3$ and let $D_i$, $i = 1, ..., n$ be disjoint 2-discs in $\partial B$ such that:

- For each $i$, $\mathcal{F}|_{D_i}$ is a trivial foliation.
- For each $i$, if $d_i = \partial D_i$, then $d_i$ is non-contractible in the leaf of $\mathcal{F}$ where it belongs and it has trivial holonomy.
- If $S = \partial B - (\cup_i D_i)$ then $S$ does not contain any conic singularity.
- After performing a finite number of Morse modifications, if necessary, each leaf of $\mathcal{F}|_B$ is transformed to a leaf parallel to $S$.

The pair $(B, \mathcal{F}|_B)$ will be referred to as a trivially foliated ball with spots $D_i$. If $B$ has three spots then $B$ will be referred to as a solid pair of pants. The set $\partial_i B = \partial W - (\cup_i D_i)$ will be called the tangent boundary of $B$. 20
Figure 11: A solid pair of pants with $s = s_1 = s_2$. 
Figure 12: In a solid pair of pants the hypothesis $s'_1 = s'_2$ implies $s'_1 \neq s$.

From Lemma 20, each spot of a trivially foliated ball is a perfect disc. Therefore in the definition above the three first conditions are the same with the conditions (1)-(3) of Definition 11. However, a trivially foliated ball with spots is not a special case of a truncated bubble. The reason is that, if $\overrightarrow{n}$ denotes the unit normal vector to $\mathcal{F}$, then $\overrightarrow{n}$ is directed either inward or outward $D_i$ for each $i$. Also, the conic singularities which are associated to each spot $D_i$ are not necessarily distinct.

Definition 23 Let $B$, $B'$ be trivially foliated balls with spots. We will say that $B$ and $B'$ are equivalent if,
(a) $B' \subset B$;
(b) $\partial B$ is parallel to $\partial B'$ in $\mathcal{F}$.

From now on we will consider classes $[B]$ of trivially foliated balls $B$ and we will not distinguish two elements which belong in the same class. Thus, we denote by $B$ the whole class $[B]$ if no misunderstandings are caused.

Definition 24 Let $B$ be a 3-ball in $(S^3, \mathcal{F})$ such that $(B, \mathcal{F}|_B)$ is a trivially foliated ball with $n$ spots $D_i$. Let $D_1$, $D_2$ be two such spots in $\partial B$ and let $c_1$, $c_2$ be two closed leaves of $\mathcal{F}|_{D_1}$, $\mathcal{F}|_{D_2}$ respectively. If there is an annulus $C$ with $\partial C = c_1 \cup c_2$ and if $C$ is contained in leaf of $\mathcal{F}$, then we will say that the spot $D_1$ (resp. $D_2$) is captured or that the pair of spots $D_1$, $D_2$ are captured between them.

If a spot $D_i$ is not captured by another spot of $B$ then $D_i$ will be called free.

Now we are able to prove the following proposition.
Proposition 25 Let $B$ be a trivially foliated ball with spots $D_i$, $i = 1, \ldots, k$. If two of the spots of $B$ are captured between them then $\mathcal{F}$ has a truncated Reeb component.

Proof. We assume that the spots $D_1, D_2$ are captured between them. Then we may find a disc $D$ embedded in $B$, in general position with respect to $\mathcal{F}$ such that:

1) $\mathcal{F}|_D$ has the form of Figure 9;
2) $\partial D, \partial D_1, \partial D_2$ are the boundaries of a pair of pants $P \subset \partial_i B \subset L_0$, where $L_0$ is a leaf of $\mathcal{F}$.

Thus, in a standard way, we may isotope $D$ in $B$ such that $\mathcal{F}|_D$ is trivial.

Let denote by $B_0 \subset B$ the 3-ball with $\partial B_0 = P \cup D_1 \cup D_2 \cup D$. On each disc $D, D_1, D_2$ there is respectively an anti-vanishing cycle, say $a, a_1, a_2$, because by hypothesis, there do not exist vanishing cycles. From the proof of Lemma 19 the anti-vanishing cycle $a$ (resp. $a_1, a_2$) determines a conic singularity $s$ (resp. $s_1, s_2$).

Now we claim that $s = s_1 = s_2$. In fact, assuming that $s \neq s_1$ we will get a contradiction. To prove it, we may assume that $s \in D, s_1 \in D_1$ and thus we have the configuration of Figure 10. But then $\mathcal{F}$ must have a bubble which is impossible from the assumption (3) at the beginning of the section.

Thus, we have that $s = s_1 = s_2$ and without loss of generality, we may assume that we have the configuration of Figure 11. Since the spots $D_1, D_2$ are captured between them, there exists an annulus $C$ with $\partial C = c_1 \cup c_2$ such that: $C \cap B_0 = c_1 \cup c_2$ where $c_i$ is a leaf of $\mathcal{F}|_{D_i}$, $i = 1, 2$ and each $c_i$ is a non-contractible curve in the leaf $L$ of $\mathcal{F}$, where it belongs. We denote by $D'_i$ the subdisc of $D_i$ bounded by $c_i, i = 1, 2$. Then $S = C \cup D'_1 \cup D'_2$ is a 2-sphere and we denote by $A$ the 3-ball bounded by $S$ such that $A \cap B_0 = D'_1 \cup D'_2$. Since $\mathcal{F}|_{D'_1}$ is trivial we have that all the leaves in the interior of $A$ are either annuli or discs and no leaf of $\mathcal{F}|_A$ contains a conic singularity.

On the other hand, by applying a Morse modification around $s$ the pair of pants $P$ can be transformed to a cylinder $P_1$ such that:

In $P_1$ the curve $\partial D$ is contractible in the leaf of $\mathcal{F}$ where it belongs while the curves $\partial D_1, \partial D_2$, as well as, the curves $c_1$ and $c_2$ are not nullhomotopic in the leaves of $\mathcal{F}$ where they belong respectively.

Let $B_1$ be the 3-ball with $\partial B_1 = P_1 \cup D_1 \cup D_2$ and $B_1 \cap A = D'_1 \cup D'_2$. If we denote by $\mathcal{F}_1$ the Morse foliation obtained after performing the previous Morse modification, we deduce by our construction, that each leaf of $\mathcal{F}|_{B_1}$ is also non-singular.

Examining now the foliation $\mathcal{F}|_{D'_1}$ (or $\mathcal{F}|_{D'_2}$) we derive the existence of a leaf $a$ of $\mathcal{F}|_{D'_1}$ which defines a vanishing cycle on some leaf, say $L_0$ of $\mathcal{F}$. This follows from the following two facts: (1) all the leaves of $\mathcal{F}|_{A \cup B_1}$ are non-singular and (2) any leaf $L'$ of $\mathcal{F}|_A$ homeomorphic to a disc is a part of a leaf $L''$ of $\mathcal{F}_1$ which stays always in $A \cup B_1$. The latter follows easily from Lemma 20. Finally, from Proposition 10 the vanishing cycle $a$ implies the existence of a Reeb component for $\mathcal{F}_1$ and thus $\mathcal{F}$ has a truncated Reeb component. \qed
We will describe now an algorithm which allow us to construct truncated Reeb components.

Let $D_0$ be a perfect disc and let $c_0 = \partial D_0$. We distinguish the following two cases:

(I) There exists a solid pair of pants $B$ with three spots $D_0$, $D_1$, $D_2$.

(II) There does not exist such a solid pair of pants $B$.

We proceed as follows:

In the case (II) we do nothing and our procedure stops.

In the case (I) two of the spots $D_0$, $D_1$, $D_2$ could be captured between them but at least one will be free. Without loss of generality, we assume that $D_1$ is a free spot and we repeat the same procedure with $D_1$ in the place of $D_0$. That is, for the free spot $D_1$ we examine if there is a solid pair of pants $B'$ with three spots $D_1$, $D_1'$, $D_2$ such that: $B' \cap B = D_1$.

If such a $B'$ exists, then we set $B_1 = B \cup B'$ and thus a trivially foliated 3-ball $B_1$ with 4 spots in $S^3$ is obtained. In this case we will say that $B$ can be developed by adding a solid pair of pants. The spots of $B_1$ are the discs $D_0$, $D_2$, $D_1'$, $D_2'$. This procedure will be referred to as the addition of a trivially foliated solid pair of pants to $B$ along $D_1$ and the free spot $D_1$ will be called an inessential free spot. Otherwise, $D_1$ will be called essential free spot.

In the following, we distinguish two cases: either our procedure can be continued, which means that we may add one more trivially foliated solid pair of pants to $B_1$ along some essential free spot of $B_1$ or, $B_1$ cannot be developed further.

Let $B$ be a trivially foliated ball with spots. Then from Lemma 20 each spot of $B$ is a perfect disc. Therefore, to each spot of $B$ corresponds a conic singularity. Let $s_i$, $i = 1, 2, \ldots, k$ be the conic singularities which correspond to the spots of $B$. Without loss of generality we may assume that each $s_i$ belongs in the interior of $B$. These singularities $s_i$ will be called conic singularities of $B$.

We need the following lemma.

**Lemma 26** Let $B$ be a trivially foliated 3-ball in $S^3$ with $n$ spots. We assume that $n_1$ of them are free and $n_2$ are captured. If we add to $B$ a trivially foliated solid pair of pants then we take a 3-ball $B'$ with, either $n_1 - 1$ free spots or with $n_1 - 3$ free spots or $B'$ has one more conic singularity with respect to $B$.

**Proof.** By adding a trivially foliated solid pair of pants $B_0$ to $B$ along a free spot of $B$, say $D_0$, we obtain a new 3-ball $B'$ with two more spots, say $D_1'$, $D_2'$. If only one of $D_1'$, $D_2'$ is captured then $B'$ has necessarily $n_1 - 1$ free spots. Indeed, assuming for example that $D_1'$ is a captured spot and that $D_2'$ is free, we deduce the existence of a free spot of $B$, say $D$, such that $D_1'$ is captured by $D$. Therefore, we have destroyed two free spots of $B$: first the spot $D$ and second the spot $D_0$ by adding $B_0$. On the other hand, $B'$ has a new free spot, with respect to $B$, the spot $D_2'$. Therefore $B'$ has $n_1 - 1$ free spots. If both $D_1'$, $D_2'$ are captured between them, then $B'$ has again $n_1 - 1$ free spots because the free spot $D_0$ of $B$ is destroyed. If both $D_1'$, $D_2'$ are captured but each one is captured with a spot of $B$ then $B'$ has $n_1 - 3$ captured spots, since we have destroyed three free spots of $B$. 

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We assume now that both the spots $D'_1, D'_2$ are free. Then on each $D'_i$ there is a leaf $c_i$ of $F_{|D'_i}$ which defines an anti-vanishing cycle. Therefore, from Lemma [10], there is a disc $\Delta_i$ such that: $\partial \Delta_i = c_i$, $\Delta_i$ is contained in a leaf $L_i$ of $F$ and $\Delta_i$ contains a conic singularity $s'_i$. We have the followings:

(i) For each spot, say $D'_i$ of $B'$ it is $\Delta_i \cap D' = \emptyset$.

Indeed, if not, the spot $D'_i$ should not be free.

(ii) If $s'_1 = s'_2$ then $s'_1 \neq s$ for each conic singularity $s$ of $B$.

Indeed, this case leads to the configuration of Figure 12. Obviously, $s'_1 \neq s$ for each conic singularity $s$ of $B$.

(iii) If $s'_1 \neq s'_2$ then some $s'_i$ should coincide with some conic singularity $s$ of $B$.

Indeed, if neither $s'_1$ nor $s'_2$ coincides with a conic singularity $s$ of $B$ then $F$ should contain a spherical leaf. Now, from Proposition [9] since there are no bubbles, the existence of a spherical leaf implies the existence of a pseudo-toral leaf $T$. Furthermore, since the 3-ball $B'$ is trivially foliated, the existence of $T$ implies the existence of a Morse/truncated Morse component. But this is not allowed by our hypothesis. Therefore our lemma is proven.

A trivial foliated ball $B$ with spots will be called completely expanded if all the free spots of $B$ are essential.

Let now $B, B'$ be two disjoint trivially foliated balls with spots and let $D, D'$ be two spots on $B, B'$ respectively. We will say that $B, B'$ can be added through the spots $D, D'$ if there are leaves $c, c'$ of $F_{|D}, F_{|D'}$ respectively such that: $c, c'$ belong on the same leaf $L$ of $F$ and there exists an annulus $A \subset L$ with $\partial A = c \cup c'$. Otherwise, we will say that $B$ and $B'$ cannot be added.

In the case that $B, B'$ can be added through the spots $D, D'$ then, after performing a finite number of Morse modifications, a new trivially foliated ball $B_0$ is created and we will say that $B_0$ contains both $B$ and $B'$. Therefore, modulo Morse modifications, we deduce from Lemma [20] that there exists a maximum number of trivially foliated balls with spots, say $B_1, \ldots, B_k$, which are disjoint among them, i.e. $B_i \cap B_j = \emptyset$ for $i \neq j$, and none of them can be added to another, in other words, for each $i$, all the free spots of $B_i$ are essential.

For each $i$, we assume that $B_i$ has $n_i$ essential free spots. Henceforth we will denote by $B = \{B_i\}$ this family.

Now we are able to prove the following theorem.

**Theorem 27** The Morse foliation $F$ contains a truncated Reeb component.

**Proof.** In order to fix the notation, if $N$ is a compact manifold the set $C^\infty(N, S^3)$ of $C^r$ maps from $N$ to $S^3$ for $r \geq 2$, will be equipped with the strong topology, (see [8], Ch. 2). Roughly speaking, a neighborhood of some $f \in C^\infty(N, S^3)$ consists of all maps $g \in C^\infty(N, S^3)$ which are close to $f$ together with their derivatives of order $k, k = 0, \ldots, r$, at each point of $N$. A neighborhood of $f$ consisting of all such $g$ will be denoted by $U_\varepsilon(f(N))$ or by $U_\varepsilon(f)$. The sphere $S^3$ is also equipped with its standard metric.

By performing Morse modifications on $F$ we obtain a foliation $F_0$ for which there exists the family $B = \{B_i\}$ defined above, that is, each $B_i$ is a trivially
foliated, completely expanded ball with spots and $B_i \cap B_j = \emptyset$ for $i \neq j$. We may also assume that each conic singularity of $\mathcal{F}$ is contained in the interior of some $B_i$. Obviously each $B_i$ has at least two spots. Our hypothesis implies that there do not exist disjoint elements $B_i, B_j \in \mathcal{B}$ which can be added. We claim that the existence of two captured spots, say $D, D'$, on the same $B_i$ implies the existence of a truncated Reeb component for $\mathcal{F}$. To prove our claim, we deduce from Proposition 25 the existence of a truncated Reeb component, say $C_0$, for the foliation $\mathcal{F}_0$ and we will show that $\mathcal{F}$ should also have a truncated Reeb component. Indeed, in the proof of Proposition 25 which ensures the existence of $C_0$, a solid pair of pants $B$ is depicted in some $B_i$ with the following features: $B$ has two spots captured between them and there exists a common conic singularity $s$ associated to each spot of $B$. Obviously, the inverse Morse modifications, which lead $\mathcal{F}_0$ back to $\mathcal{F}$ affect $C_0$, if and only if these Morse modifications are applied around $s$. But even in this case, it is clear that $C_0$ is transformed to a truncated Reeb component $C$ of $\mathcal{F}$.

Therefore, in order to prove the theorem we will show that the existence of the family $\mathcal{B} = \{B_i\}$ and the hypothesis that all the spots of $B_i$ are essential free spots leads to a contradiction.

Let denote by $D_1, D_2, ..., D_n$ the spots of $B_i$, $i = 1, 2, ..., k$, and by $a_j$ the anti-vanishing cycle which appears in the spot $D_i$. We pick an arbitrary element of the family $\mathcal{B}$, say $B_1$. Let $W_1 = S^3 - Int(B_1)$, $S_1 = \partial W_1 = \partial B_1$ and $c_i = \partial D_i$.

We consider a $C^\infty$-embedding $H : D^2 \times [0, 1] \to W_1$ such that:
(1) $H(\partial D^2 \times [0, 1]) \subset S_1$ and $H(Int(D^2) \times [0, 1]) \subset Int(W_1)$.
(2) If $h_t = H(\cdot, t)$, the disc $D_t = h_t(D^2)$ is transverse to $S_1$ along $d_t = \partial D_t$ for each $t \in [0, 1]$.

Obviously, $D_t$ separates $W_1$ and the orientation of $[0, 1]$ permit us to talk about the left hand side and the right hand side of $D_t$ in $W_1$.

Furthermore, we may assume that the embedding $H$ satisfies:
(3) Each $h_t$ contains on the left hand component of $S_1 - d_t$ only the curve $c_1$;
(4) The disk $D_0$ is in general position with respect to $\mathcal{F}$ and there is a unique anti-vanishing cycle $a_0 \subset D_0$ i.e. $a_0$ is contained in $D_0$ and it is equivalent to the anti-vanishing cycle $a_1$. Also, the disk $D_1$ is in general position with respect to $\mathcal{F}$ and every anti-vanishing cycles $a_j$ contained in $D_1$ is equivalent to the anti-vanishing cycles $a_1$, $j = 2, ..., n_1$ respectively.

The property (4) above can be obtained by choosing $D_0$ (resp. $D_1$) in a sufficient thin tubular neighborhood $U$ of $\partial W_1$ in $W_1$ with respect to the standard metric of $S^3$.

Obviously the following statement is valid. With respect to the strong topology defined above, there is $\varepsilon_0 > 0$ such that: each disc $E_0$ (resp. $E_1$), which is $\varepsilon_0$-close to $D_0$ (resp. to $D_1$) is in general position with respect to $\mathcal{F}$ and each anti-vanishing cycle appeared in $E_0$ (resp. $E_1$) is equivalent to the anti-vanishing cycle $a_0$ (resp. to some anti-vanishing cycle $a_1$).

Now, let

- $I = \{t \in [0, 1] \mid \text{for which there exists an } \varepsilon_t > 0 \text{ such that: if } E_t \text{ is a 2-disc} \}$
in general position with respect to \( \mathcal{F} \) and \( \varepsilon_t \)-close to \( D_t \) with \( \partial E_t \subset S_1 \), then in \( E_t \) appears a single anti-vanishing cycle which is equivalent to \( a_0 \).

Since we have assumed that all the spots of any \( B_i \), and hence of \( B_1 \), are free and essential we deduce that \( \mathcal{I} \neq \emptyset \) and if \( t_0 = \max \mathcal{I} \) then \( t_0 < 1 \).

**Claim 1:** \( t_0 \notin \mathcal{I} \).

**Proof of Claim 1.** We assume that \( t_0 \in \mathcal{I} \). Then we consider a \( t' \) on the right hand side of \( t_0 \) such that \( D_{t'} \) is \( \varepsilon_{t_0}/2 \) close to \( D_{t_0} \). It follows that, all the discs \( E_t \) which are in general position with respect to \( \mathcal{F} \) and \( \varepsilon_{t_0}/4 \)-close to \( D_{t'} \) have a single anti-vanishing cycle equivalent to \( a_0 \). This contradicts the definition of \( t_0 \) as maximum of \( \mathcal{I} \) and proves Claim 1.

**Claim 2:** There is a neighborhood \( U_\varepsilon(h_{t_0}) \) of \( h_{t_0} \) (denoted also by \( U_\varepsilon(D_{t_0}) \)) such that: if \( g : D^2 \to S^3 \) is an embedding in general position with respect to \( \mathcal{F} \) with \( g(D^2) \in U_\varepsilon(D_{t_0}) \) and \( g(\partial D^2) \subset \partial B_1 \) then all the anti-vanishing cycles which appear in \( g(D^2) \) are equivalent.

**Proof of Claim 2.** In this proof we need the following terminology. If \( V \) is a trivial neighborhood of some conic singularity of \( \mathcal{F} \) we will denote by \( \partial_1 V \) the two perfect discs in the boundary of the closure \( \overline{V} \) of \( V \). We consider now a finite number of open subsets \( U_i \) of \( S^3 \) which cover \( D_{t_0} \) such that: each \( U_i \) is either a local chart of \( \mathcal{F} \) or a trivial neighborhood of some conic singularity of \( \mathcal{F} \) and \( D_{t_0} \) intersects each \( U_i \). Furthermore, since the set of singularities of \( \mathcal{F} \) is finite we may assume that, if \( U_k \) is any trivial neighborhood of a conic singularity then \( \partial_1 U_k \cap U_j = \emptyset \) for each \( j \neq k \). Let \( U = \cup U_i \). Obviously, we may choose an \( \varepsilon > 0 \) such that if \( g, g' \in U_\varepsilon(D_{t_0}) \) then the discs \( g(D^2) \) and \( g'(D^2) \) are contained in \( U \) and both intersect each \( U_i \). From the properties of the cover \( \{ U_i \}_i \) and in particular, since \( \partial_1 U_k \cap U_j = \emptyset \) for each trivial neighborhood \( U_k \) and each \( U_j \) with \( j \neq k \), we deduce that the anti-vanishing cycles which appear in \( g(D^2) \) and \( g'(D^2) \) are equivalent. In fact, for each trivial neighborhood \( U_k \), the foliation \( \mathcal{F}_{U_k} \) induces the same anti-vanishing cycle on \( g(D^2) \) and on \( g'(D^2) \). This proves Claim 2.

Now from Claim 2, we may find positive numbers \( \varepsilon_t \) and \( \varepsilon_{t'} \) and discs \( E_t \in U_{\varepsilon_t}(D_t) \), \( E_{t'} \in U_{\varepsilon_{t'}}(D_{t'}) \) on the left and the right side of \( D_{t_0} \) respectively, which are in general position with respect to \( \mathcal{F} \) and which have equivalent anti-vanishing cycles. Therefore, we get a contradiction to the definition of the set \( \mathcal{I} \). On the other hand, we may always assume that all the free spot of \( B_1 \) (as well as, of any \( B_i \)) are essential. Therefore all the spots of \( B_1 \) cannot be free. In other words, there are spots of \( B_1 \) which are captured between them. Therefore from Proposition 28 we get our theorem. ■

### 5 Proof of the main Theorem

Let \( \mathcal{F} \) be a Morse foliation on \( S^3 \) allowed to have bubbles, truncated bubbles, as well as, vanishing cycles.

**Theorem 28** Assuming that \( \mathcal{F} \) does not have trivial pair of singularities then \( \mathcal{F} \) has either a Morse/truncated Morse component or a Reeb/truncated Reeb
component. Therefore, modulo a Morse modification, \( F \) has either a Morse or a Reeb component.

**Proof.** From Theorem \[16\] we may assume that \( F \) does not have Morse or truncated Morse components and vanishing cycles. If \( F \) does not also have bubbles and truncated bubbles the proof follows from Theorem \[27\]. If \( F \) has bubbles but not truncated bubbles then we pick an innermost bubble \( B \); that is, in the interior of \( B \) there does not exist another bubble of \( F \). Now, we may apply our method of work in \( B \) and prove, as in the previous paragraph, that in \( B \) there is a Reeb/truncated Reeb component.

From now on we assume that \( F \) has also truncated bubbles. Thus, if \( W \) is a truncated bubble of \( F \) then the configuration of Figure 7 appears and the notation of paragraph of Section 2 will be used. Therefore, for \( W \) we may modify locally \( F \) in a neighborhood of annulus \( C \) so that the disc \( E_1 \) fits with the disc \( E_2 \) so that \( c_1 = c_2 \). Applying for any truncated bubble \( W \) a such modification on \( F \), which will be referred to as a corrective movement, we get a new foliation, say \( F' \). It is obvious now that \( F' \) does not have truncated bubbles. The union \( E_1 \cup E_2 \) is homeomorphic to \( S^2 \), it contains two conic singularities, say \( s_1, s_2 \), and will be referred to as a *special bubble with two conic singularities*. Therefore after applying finitely many, arbitrary small Morse modifications around \( s_1 \) or \( s_2 \) we may assume that each leaf contains at most one singularity. Let denote by \( F'' \) the obtained foliation after applying these last Morse modifications on \( F' \).

Now let \( B_1 \) be an innermost bubble of \( F'' \). In \( B_1 \) we may prove that there exists Reeb/truncated Reeb component as before. If \( B_1 \) existed as an innermost bubble of \( F \) then we deduce that \( F \) has a Reeb/truncated Reeb component. Hence we assume that \( B_1 \) is an innermost bubble created after applying the corrective movements and the Morse modifications described above.

We assume first that in the interior of \( B_1 \) there is a Reeb component \( R \) and let \( T = \partial R \) be the toral leaf of \( R \). Then applying the inverse Morse modifications which lead \( F'' \) back to \( F' \) it is clear that at most one of these modifications affect the Reeb component \( R \). Therefore \( R \) either remains a Reeb component of \( F' \) or it is transformed to a truncated Reeb component of \( F' \). On the other hand, we may see that the inverse of corrective movements which lead \( F' \) back to \( F \) do not affect the Reeb component \( R \). This results from the fact that the annuli \( C \) in a neighborhood of which the corrective movements take place, do not touch the torus \( T \). Therefore, \( F \) has a Reeb/truncated Reeb component.

Finally, we assume that in the interior of \( B_1 \) there is a truncated Reeb component \( K \). Then in \( B_1 \) must appear the configuration of Figure 11. That is, a solid pair of pants \( P \) with two captured spots and a conic singularity \( s \) in \( P \). Therefore, at most one of the inverse Morse modifications which lead \( F'' \) back to \( F' \), may take place in a neighborhood of \( s \). This proves as before that \( K \) remains a truncated Reeb component of \( F \). ■

If to the leaves of a Morse/truncated Morse component or to the leaves of a Reeb/truncated Reeb component we add finitely many trivial bubbles the components will be called *singular*.
As a corollary of the Theorem above we have.

**Corollary 29** Assuming that \( F \) does not have all its leaves simply connected then \( F \) has either a singular Morse/truncated Morse component or a singular Reeb/truncated Reeb component. Therefore, modulo Morse modifications, \( F \) has either a singular Morse component or a singular Reeb component.

**Proof.** If we remove all trivial pair of singularities of \( F \) and we apply Theorem 28 we obtain the result. \( \blacksquare \)

## 6 Stability of Morse foliations

The goal of this section is to prove the following theorem:

**Theorem 30** Let \( F \) be a Morse foliation on \( S^3 \). Then \( F \) is \( C^1 \)-instable unless all the leaves of \( F \) are compact and simply connected and each leaf contains at most one conic singularity.

**Proof.** First, we remark that if a foliation \( F \) has a leaf \( L \) containing more than one conic singularity then by a Morse modification we may obtain a foliation \( F' \) arbitrary close to \( F \) in the \( C^1 \)-topology, such that each leaf of \( F' \) has at most one conic singularity. This implies that a Morse foliation \( F \) in order to be stable should contain at most one conic singularity in the same leaf.

Now we will show that if \( F \) satisfies the assumptions of the theorem then it is structural stable. Indeed, from Reeb stability theorem \([10]\), it follows that if \( L \) is a non-singular leaf of \( F \) diffeomorphic to \( S^2 \) then \( L \) has a tubular neighborhood \( V \) homeomorphic to \( S^2 \times \mathbb{R} \) such that the interior of \( V \) is foliated by the product foliation \( S^2 \times \{t\}, \ t \in [0,1] \) and the leaves \( L_{\pm1} \) passing from the points \( S^2 \times \{\pm1\} \) respectively are singular. Since \( L_1 \) (resp. \( L_{-1} \)) is simply connected we have that \( L_1 \) is homeomorphic to two copies of \( S^2 \) attached to a point. Therefore, Reeb stability theorem can be applied beyond the leaf \( L_1 \) (resp. \( L_{-1} \)). Since \( F \) has finitely many singularities our statement follows.

In the following we assume that \( F \) contains leaves which are non-compact and simply connected and we will prove that \( F \) is instable. From Corollary 29 we have that if \( F \) does not have a singular Morse/truncated Morse component then \( F \) has a singular Reeb/truncated Reeb component. Without loss of generality, we may assume that all the previous components are not singular i.e. their leaves do not contain trivial bubbles. Indeed, one may verify that our proof below is applied for singular components.

So, we assume first that \( F \) has a Morse component \( V \). Let \( T \) be the boundary leaf of \( V \) which is either a torus or a pseudo-torus. Without loss of generality we may assume that \( T \) is a torus and that there exists a neighborhood \( B \) of \( T \) such that \( F \) induces on \( B \) a band of leaves i.e. \( F|_B \) is the product foliation \( T \times \{t\}, \ t \in [0,1] \). Indeed, if such a band does not exist then by a Morse modification we may construct a new foliation \( F' \) which \( C^1 \)-close to \( F \) and which contains a band of leaves. Thus \( F \) is instable. On the other hand, if \( F \) contains a band
of leaves around $T$ then we will show that $\mathcal{F}$ is again instable. Indeed, it is a standard fact that a stable foliation cannot contain any band since the product foliation on $B$ can be $C^1$-approximated, rel $\partial(T^2 \times [0, 1])$ by a foliation with a finite number of compact leaves.

In the case that $\mathcal{F}$ has a generalized Morse component then the same proof can be applied and it always results that $\mathcal{F}$ is instable. For instance, by a Morse modification the generalized Morse component is transformed to Morse component, say $\mathcal{V}$, and $\mathcal{F}$ is transformed to a new foliation, say $\mathcal{F}'$. As we have shown, the existence of $\mathcal{V}$ implies that $\mathcal{F}'$ is instable and hence the initial foliation $\mathcal{F}$ is instable too. Therefore, we may assume in the following that $\mathcal{F}$ does not have neither Morse components nor truncated Morse components

Now, without loss of generality, we may assume that $\mathcal{F}$ has only Reeb components. Indeed, by applying finitely many Morse modifications, $\mathcal{F}$ is transformed to new Morse foliation, say $\mathcal{F}'$, which has only Reeb components. Obviously, if we show that $\mathcal{F}'$ is instable then the same will be true for $\mathcal{F}$.

Claim 1: One of the toral leaves, say $T$, of some Reeb component of $\mathcal{F}'$ should be flat, i.e. all the elements of the holonomy group of $T$ have $\infty$ contact with the identity $[13]$.

The proof of this claim is similar to the proof of the corresponding statement (see $[13]$, Proof of Theorem 1.1). We have to notice only that the method of erasing Reeb components, as it is explained in $[13]$, if it is applied for Morse foliations does not create neither Morse components nor foliations with all the leaves compact and simply connected. This proves that in our case, all Reeb components of $\mathcal{F}'$ cannot be erased so some toral leaf $T$ is flat.

Claim 2: The leaf $T$ can be thickened, i.e. $\mathcal{F}'$ can be $C^1$-approximated by a foliation $\mathcal{F}''$ such that $\mathcal{F}'' = \mathcal{F}'$ outside a tubular neighborhood $V$ of $T$, and $\mathcal{F}''$ has a band of toral leaves.

The proof of this claim is identical to the proof of Lemma a in $[13]$.

As we noted above, the band $T \times [0, 1]$ of toral leaves can be $C^1$-approximated, rel $\partial(T \times [0, 1])$ by a foliation with a finite number of compact leaves. Therefore $\mathcal{F}'$ can be $C^1$-approximated by a Morse foliation $\mathcal{G}'$ for which there does not exist a homeomorphism $h$ sending $\mathcal{G}'$ to $\mathcal{F}'$. Now, going back to $\mathcal{F}$ by applying to $\mathcal{F}'$ the inverse Morse modifications, it is easy to see that by means of $\mathcal{G}'$, we may construct a foliation $\mathcal{G}$ which is $C^1$-close to $\mathcal{F}$ and which is not topological equivalent to $\mathcal{F}$.

Remark 31 Theorem 2 can be easily derived from the previous theorem since Morse foliations are generic in the space of Haefliger structures.

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