New solvability condition of 2-d nonlocal boundary value problem for Poisson’s operator on rectangle

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Abstract. Differential and difference interpretations of a nonlocal boundary value problem for Poisson’s equation in open rectangular domain are studied. New solvability conditions are obtained in respect of existence, uniqueness and a priori estimate of the classical solution. Second order of accuracy difference scheme is presented.

Keywords. Poisson’s operator, nonlocal boundary value problem, nonlocal boundary value condition, rectangular domain, difference scheme.

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1 Introduction

First of all, we note that a detailed overview on the nonlocal boundary value problem (NLBVP) that we consider in this paper is enclosed in [16, p. 38-39].

Let Π designate an open rectangle, i.e., Π = (0 < x < 1) × (0 < y < π). Our present paper deals with Poisson’s equation \( \Delta u(x, y) = f(x, y) \) in the rectangular domain Π where nonlocal boundary value condition (NLBVC) is represented by a linear combination of unknown solution values

\[ u(1, y) = \alpha_1 u(\xi_1, y) + \alpha_2 u(\xi_2, y) + \ldots + \alpha_m u(\xi_m, y) \]

for \( y \in [0, \pi] \), \( \xi_k \in (0, 1) \), \( k = 1, \ldots, m \) and \( u(x, y)|_{\partial \Pi \setminus \{x=1\}} = 0 \) is given on three sides of the rectangle boundary \( \partial \Pi \). Actually, herein the coefficients \( \alpha_k, k = 1, \ldots, m \) have an arbitrary sign. This kind of NLBVP was considered in [3] where the existence and uniqueness of classical solution were proved against the requirement

\[ \sum_{k=1}^{m} \frac{1}{2} (\alpha_k + |\alpha_k|) \leq 1, \]

but a priori estimate

\[ \|u\|_{W_2^2(\Pi)} \leq C \|f\|_{L_2(\Pi)} \]
was established for the same sign coefficients which satisfy the condition

\[-\infty < \sum_{k=1}^{m} \alpha_k \leq 1.\]

In addition, the second order of accuracy finite-difference scheme was offered on a uniform grid. In [5], the existence and uniqueness of classical solution were proved for a similar NLBVP in a rectangular domain when

\[\sum_{k=1}^{m} |\alpha_k| \leq |B_1|^{-1}\]

for \(0 < |B_1| < 1\), where the value \(|B_1|^{-1}\) could be an unboundedly large if \(\xi_m \to 0\), so that the unboundedness for \(\sum_{k=1}^{m} |\alpha_k|\) was revealed.

In [16], the differential and difference variants of NLBVP formulated in [3] were researched for the case when NLBVC encloses positive and negative coefficients together without failing. The condition of paper [3] on the coefficients in respect of NLBVC was improved, the well-posedness of the differential problem was established, a second order of accuracy approximation for the suggested difference scheme was proved.

In our present paper, we obtain a new condition that ensures the existence, uniqueness and a priori estimate of classical solution for the class of NLBVPs which was considered in [16]. Our new well-posedness condition for the differential problem reveals the unboundedness effect for the coefficients of NLBVC. In addition, herein, we improve the condition of [16] in respect of the difference problem and obtain a second order of accuracy for the difference scheme.

Before finishing this introduction, we note that for the NLBVP which we consider in our present paper, the most relevant references [1–15] from [16, p. 51-52] are included in the bibliography.

## 2 Differential problem

We consider NLBVP

\[
\begin{cases}
\Delta u(x, y) = f(x, y), \quad (x, y) \in \Pi, \\
u(x, 0) = u(x, \pi) = 0, \quad 0 \leq x \leq 1, \quad u(0, y) = 0, \quad 0 \leq y \leq \pi, \\
\ell[u](y) = 0, \quad 0 \leq y \leq \pi,
\end{cases}
\]

(1)

where

\[
\ell[u](y) \equiv u(1, y) - \sum_{r=1}^{n} \alpha_r u(\zeta_r, y) + \sum_{s=1}^{m} \beta_s u(\eta_s, y),
\]
0 < \zeta_1 < \ldots < \zeta_n < 1, \ 0 < \eta_1 < \ldots < \eta_m < 1, \ \zeta_r \neq \eta_s, \ \alpha_r > 0, \ \beta_s > 0, \ r = 1, \ldots, n, \ s = 1, \ldots, m. \ Further \ in \ this \ article, \ A \ denotes \ following \ conditions:

\begin{align*}
-\infty < \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s < \frac{\sinh 1}{\sinh \zeta_n} & \quad \text{when } \zeta_n < \eta_1; \\
\sum_{r=1}^{n} \alpha_r < \frac{\sinh 1}{\sinh \zeta_n} & \quad \text{when } \zeta_n > \eta_1.
\end{align*}

Naturally, the classical solution of NLBVP (1) is the function \( u(x, y) \) that belongs to \( C^2(\Pi) \cap C(\bar{\Pi}) \), satisfies the equation and all conditions of (1).

**Lemma 2.1.** For \( x \in (0, 1) \) and \( t > 1 \) the following inequalities hold

\[
1 > \frac{\sinh x}{\sinh 1} > \frac{\sinh tx}{\sinh t}.
\]

**Proof.** Left side of inequality is obvious. Let we show that the other one holds. Let

\[
g(t) = \frac{\sinh tx}{\sinh t}
\]

for specified \( x \in (0, 1) \), then

\[
g'(t) = \left( \frac{\sinh tx}{\sinh t} \right)' = \frac{x \cosh tx \sinh t - \sinh tx \cosh t}{(\sinh t)^2}.
\]

Since

\[
\int \sinh at \sinh bt \ dt = \frac{1}{a^2 - b^2} \left( a \sinh bt \cosh at - b \sinh at \cosh bt \right)
\]

for \( a^2 \neq b^2 \),

\[
g'(t) = \frac{x \cosh tx \sinh t - \sinh tx \cosh t}{(\sinh t)^2} = \frac{x^2 - 1}{(\sinh t)^2} \int_0^t \sinh x \tau \sinh \tau \ d\tau.
\]

Since \( g'(t) < 0 \) for \( t > 0 \), \( g(t) \) strictly decreases, and therefore, for \( t > 1 \)

\[
\frac{\sinh x}{\sinh 1} > \frac{\sinh tx}{\sinh t}.
\]

Lemma 2.1 is proved. \( \Box \)
Theorem 2.2. Let \( f \in C(\Pi) \). If \( A \) holds, then classical solution of (1) exists, is unique and holds a priori estimate

\[
||u||_{W^2_2(\Pi)} \leq C||f||_{L_2(\Pi)}. \tag{2}
\]

Proof. First, we prove a priori estimate (2). We assume that classical solution exists. For \( k \in \mathbb{N} \) let us denote

\[
U_k(x) = \frac{\sqrt{2/\pi}}{\pi} \int_0^\pi u(x, y) \sin(ky) \, dy, \tag{3}
\]

\[
f_k(x) = \frac{\sqrt{2/\pi}}{\pi} \int_0^\pi f(x, y) \sin(ky) \, dy, \tag{4}
\]

so that using the equation \( \Delta u(x, y) = f(x, y) \) and conditions

\[
u(0, y) = 0, \quad u(1, y) = \sum_{r=1}^n \alpha_r u(\zeta_r, y) - \sum_{s=1}^m \beta_s u(\eta_s, y), \]

we see that \( U_k(x) \) satisfies the multipoint problem

\[
\begin{align*}
L[U_k](x) &= f_k(x), \quad 0 < x < 1, \\
U_k(0) &= 0, \quad \ell[U_k] = 0,
\end{align*} \tag{5}
\]

where

\[
L[U_k](x) \equiv U_k''(x) - k^2 U_k(x), \tag{6}
\]

\[
\ell[U_k] \equiv U_k(1) - \left( \sum_{r=1}^n \alpha_r U_k(\zeta_r) - \sum_{s=1}^m \beta_s U_k(\eta_s) \right). \tag{7}
\]

Letting \( U_k(x) = V_k(x) + W_k(x) \), where \( V_k(x) \) is the solution of

\[
\begin{align*}
L[V_k(x)] &= f_k(x), \quad 0 < x < 1, \\
V_k(0) &= 0, \quad V_k(1) = 0,
\end{align*} \tag{8}
\]

while \( W_k(x) \) is the solution of

\[
\begin{align*}
L[W_k(x)] &= 0, \quad 0 < x < 1, \\
W_k(0) &= 0, \quad \ell[W_k] = -\ell[V_k]. \tag{9}
\end{align*}
\]
In view of [3, p. 143], the solution of (8) holds the estimates

\[ \| V_k \|_{L^2[0,1]} \leq \frac{1}{k^2} \| f_k \|_{L^2[0,1]}, \]  
\[ \| V'_k \|_{L^2[0,1]} \leq \frac{1}{k} \| f_k \|_{L^2[0,1]}, \]  
\[ \| V''_k \|_{L^2[0,1]} \leq \| f_k \|_{L^2[0,1]}. \]

Since \( V_k(1) = 0 \), by virtue of Cauchy-Bunyakovskii inequality

\[ \left| \int_{\zeta_r}^1 (V_k(x)^2)' \, dx \right| = 2 \left| \int_{\zeta_r}^1 V_k(x) V'_k(x) \, dx \right| \leq 2 \| V_k \|_{L^2[0,1]} \| V'_k \|_{L^2[0,1]}, \]  
\[ \left| \int_{\eta_s}^1 (V_k(x)^2)' \, dx \right| = 2 \left| \int_{\eta_s}^1 V_k(x) V'_k(x) \, dx \right| \leq 2 \| V_k \|_{L^2[0,1]} \| V'_k \|_{L^2[0,1]}. \]

Since for \( \xi \in (0, 1) \)

\[ [V_k(\xi)]^2 = \int_{\xi}^1 ([V_k(x)]^2)' \, dx, \]

from (13)-(14), in view of (10)-(11), we get estimates

\[ |V_k(\zeta_r)| \leq \frac{\sqrt{2}}{k^{3/2}} \| f_k \|_{L^2[0,1]}, \quad |V_k(\eta_s)| \leq \frac{\sqrt{2}}{k^{3/2}} \| f_k \|_{L^2[0,1]} \]

Hence,

\[ |\ell[V_k]| \leq \left( \sum_{r=1}^n \alpha_r + \sum_{s=1}^m \beta_s \right) \frac{\sqrt{2}}{k^{3/2}} \| f_k \|_{L^2[0,1]}. \]

Problem (9) has the solution

\[ W_k(x) = \mathcal{W}_k \frac{\sinh kx}{\sinh k}, \]

where

\[ \mathcal{W}_k = \frac{-\ell[V_k(x)]}{1 - (\sinh k)^{-1} \left( \sum_{r=1}^n \alpha_r \sinh k\zeta_r - \sum_{s=1}^m \beta_s \sinh k\eta_s \right)} \]

and since the denominator of the fraction in (18) is nonzero, moreover,

\[ 1 - (\sinh k)^{-1} \left( \sum_{r=1}^n \alpha_r \sinh k\zeta_r - \sum_{s=1}^m \beta_s \sinh k\eta_s \right) > 0. \]
Indeed,
\[
1 - \sum_{r=1}^{n} \frac{\alpha_r}{\sinh k} \sinh k_{r} + \sum_{s=1}^{m} \frac{\beta_s}{\sinh k} \sinh k_{s} \geq 1 - \sum_{r=1}^{n} \frac{\alpha_r}{\sinh k} \sum_{s=1}^{m} \frac{\beta_s}{\sinh k} \geq S_k
\]
for
\[
S_k = \begin{cases} 
1, & \text{if } -\infty < \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s \leq 0, \quad \zeta_n < \eta_1; \\
1 - (\sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s) \frac{\sinh k_{\zeta}}{\sinh k}, & \text{if } 0 < \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s, \quad \zeta_n < \eta_1; \\
1 - (\sum_{r=1}^{n} \alpha_r) \frac{\sinh k_{\zeta}}{\sinh k}, & \text{if } 0 < \sum_{r=1}^{n} \alpha_r, \quad \zeta_n > \eta_1.
\end{cases}
\]

By virtue of Lemma 1,
\[
\frac{\sinh \zeta_n}{\sinh 1} > \frac{\sinh k_{\zeta}}{\sinh k}
\]
then, in view of \( A \), we get that \( S_k \geq S_0 > 0 \) for
\[
S_0 = \begin{cases} 
1, & \text{when } -\infty < \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s \leq 0, \quad \zeta_n < \eta_1; \\
1 - (\sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s) \frac{\sinh \zeta_n}{\sinh 1}, & \text{when } 0 < \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s, \quad \zeta_n < \eta_1, \\
1 - (\sum_{r=1}^{n} \alpha_r) \frac{\sinh \zeta_n}{\sinh 1}, & \text{when } 0 < \sum_{r=1}^{n} \alpha_r, \quad \zeta_n > \eta_1.
\end{cases}
\]

Therefore,
\[
1 - (\sinh k)^{-1} \left( \sum_{r=1}^{n} \alpha_r \sinh k_{\zeta} - \sum_{s=1}^{m} \beta_s \sinh k_{\eta} \right) \geq S_0 > 0. \tag{20}
\]

Hence, in view of (16)-(20),
\[
|W_k(1)| \leq C_0 \frac{\sqrt{2}}{k^{3/2}} \| f_k(x) \|_{L^2[0,1]} \tag{21}
\]
for
\[
C_0 = \frac{1}{S_0} \left( \sum_{r=1}^{n} \alpha_r + \sum_{s=1}^{m} \beta_s \right).
\]
Since, in view of (17),

\[ W_k(x) = W_k(1) \frac{\sinh kx}{\sinh k} \]

is the explicit solution of (9), then

\[ \|W_k\|_{L^2[0,1]} \leq |W_k(1)| \left( \frac{\int_0^1 \sinh^2(kx)dx}{\sinh^2 k} \right)^{1/2}, \quad (22) \]

\[ \|W'_k\|_{L^2[0,1]} \leq k \cdot |W_k(1)| \left( \frac{\int_0^1 \cosh^2(kx)dx}{\sinh^2 k} \right)^{1/2}, \quad (23) \]

\[ \|W''_k\|_{L^2[0,1]} \leq k^2 \cdot |W_k(1)| \left( \frac{\int_0^1 \sinh^2(kx)dx}{\sinh^2 k} \right)^{1/2}. \quad (24) \]

Because

\[ \frac{\int_0^1 \sinh^2(kx)dx}{\sinh^2 k} \leq \frac{1}{k}, \quad \frac{\int_0^1 \cosh^2(kx)dx}{\sinh^2 k} \leq \frac{5}{2k}, \]

then, in view of (21), the inequalities (22), (23) and (24) result in

\[ \|W_k\|_{L^2[0,1]} \leq C_0 \sqrt{\frac{2}{k^2}} \|f_k\|_{L^2[0,1]}, \quad (25) \]

\[ \|W'_k\|_{L^2[0,1]} \leq C_0 \sqrt{\frac{5}{k}} \|f_k\|_{L^2[0,1]}, \quad (26) \]

\[ \|W''_k\|_{L^2[0,1]} \leq C_0 \sqrt{2} \|f_k\|_{L^2[0,1]}, \quad (27) \]

Hence, in view of (10)-(12),

\[ \|U_k\|_{L^2[0,1]} \leq C_1 \frac{1}{k^2} \|f_k\|_{L^2[0,1]}, \quad (28) \]

\[ \|U'_k\|_{L^2[0,1]} \leq C_2 \frac{1}{k} \|f_k\|_{L^2[0,1]}, \quad (29) \]

\[ \|U''_k\|_{L^2[0,1]} \leq C_3 \|f_k\|_{L^2[0,1]}, \quad (30) \]

where \( C_1 = C_3 = 1 + C_0 \sqrt{2}, \; C_2 = 1 + C_0 \sqrt{5}. \) Therefore, in view of [3, p. 142-143], we have

\[ \sum_{k=1}^{\infty} \int_0^1 U_k^2(x)dx \leq C_1^2 \|f\|_{L^2(\Pi)}^2, \]

\[ \sum_{k=1}^{\infty} \int_0^1 (U'_k(x))^2 dx \leq \frac{1}{k^2} C_2^2 \|f\|_{L^2(\Pi)}^2, \]

\[ \sum_{k=1}^{\infty} \int_0^1 (U''_k(x))^2 dx \leq C_3^2 \|f\|_{L^2(\Pi)}^2. \]
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\[
\sum_{k=1}^{\infty} \int_{0}^{1} (U_k''(x))^2 \, dx \leq C_3^2 \|f\|_{L_2(\Pi)}^2,
\]

so that (28)-(30) result [3, p. 142-143] in

\[
\|u\|_{W_2^2(\Pi)} \leq C_1 \|f\|_{L_2(\Pi)},
\]

(31)

\[
\|u_{xx}\|_{W_2^2(\Pi)} \leq C_2 \|f\|_{L_2(\Pi)},
\]

(32)

\[
\|u_{xy}\|_{W_2^2(\Pi)} \leq C_3 \|f\|_{L_2(\Pi)}.
\]

(33)

In view of (32), from the equation \( \Delta u(x, y) = f(x, y) \) we get

\[
\|u_{yy}\|_{W_2^2(\Pi)} \leq C_4 \|f\|_{L_2(\Pi)}.
\]

(34)

Finally, a priori estimate (2) results from (31)-(34). Since, the uniqueness of classical solution follows from (2), then the existence results from Fredholm’s property [2] which is inherent to the problem (1). Theorem 2.2 is proved.

\[\square\]

**Corollary 2.3.** Let \( f \in C(\Pi) \), \( n = m \) and \( \zeta_r < \eta_r \), \( r = 1, \ldots, n \). If

\[
\sum_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} = 0,
\]

or if

\[
0 < \sum_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} < \frac{\sinh 1}{\sinh \zeta_p}
\]

(35)

for \( p \leq n \), so that \( \frac{(\alpha_p - \beta_p) + |\alpha_p - \beta_p|}{2} > 0 \), but \( \frac{(\alpha_{p+i} - \beta_{p+i}) + |\alpha_{p+i} - \beta_{p+i}|}{2} = 0 \) for \( 1 < i \leq n - p \) (if such \( i \) does not exists we put \( p = n \)), then classical solution of (1) exists, is a unique and holds a priori estimate (2).

**Proof.** In view of (3)-(7), we find that \( U_k(x) \) satisfies the multipoint problem (5)

\[
\begin{cases}
L[U_k(x)] = f_k(x), & 0 < x < 1, \\
U_k(0) = 0, \ell[U_k] = 0,
\end{cases}
\]

where

\[
\ell[U_k] \equiv U_k(1) - \sum_{r=1}^{n} (\alpha_r U_k(\zeta_r) - \beta_r U_k(\eta_r)).
\]

(36)
Put \( U_k(x) = V_k(x) + W_k(x) \), where \( V_k(x) \) is the solution of (8), \( W_k(x) \) is the solution of (9). Similar to the proof of Theorem 2.2, estimates (10)-(12) hold, then estimates (13)-(15) hold for \( r = s \). Hence, in view of (15),

\[
|\ell[V_k]| \leq \left( \sum_{r=1}^{n} (\alpha_r + \beta_r) \right) \frac{\sqrt{2}}{k^{3/2}} \|f_k(x)\|_{L^2[0,1]}.
\]

(37)

In view of (17)-(18),

\[
W_k = \frac{-\ell[V_k]}{1 - (\sinh k)^{-1} \sum_{r=1}^{n} (\alpha_r \sinh k\zeta_r - \beta_r \sinh k\eta_r)}.
\]

(38)

Noting that the denominator of the fraction \( W_k \) is nonzero, we have

\[
1 - \frac{\sum_{r=1}^{n} (\alpha_r \sinh k\zeta_r - \beta_r \sinh k\eta_r)}{\sinh k} \geq 1 - \frac{\sum_{r=1}^{n} (\alpha_r - \beta_r) \sinh k\zeta_r}{\sinh k} \geq S_k
\]

for

\[
S_k = \left\{ \begin{array}{ll}
1, & \text{if} \quad \sum_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} = 0, \\
1 - \left( \sum_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} \right) \frac{\sinh k\zeta_p}{\sinh k}, & \text{if} \quad \sum_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} > 0.
\end{array} \right.
\]

(39)

By virtue of Lemma 2.1,

\[
1 > \frac{\sinh \zeta_p}{\sinh 1} > \frac{\sinh k\zeta_p}{\sinh k},
\]

and then \( S_k \geq S_0 > 0 \) for

\[
S_0 = \left\{ \begin{array}{ll}
1, & \text{if} \quad \sum_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} = 0, \\
1 - \left( \sum_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} \right) \frac{\sinh \zeta_p}{\sinh 1}, & \text{if} \quad \sum_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} > 0.
\end{array} \right.
\]

(39)

In view of corollary conditions, \( S_k \geq S_0 > 0 \). Therefore,

\[
1 - (\sinh k)^{-1} \sum_{r=1}^{n} (\alpha_r \sinh k\zeta_r - \beta_r \sinh k\eta_r) \geq S_0 > 0.
\]

Hence, in view of (17) and (36)-(39),

\[
|W_k(1)| \leq \frac{\sum_{r=1}^{n} (\alpha_r + \beta_r)}{S_0} \frac{\sqrt{2}}{k^{3/2}} \|f_k(x)\|_{L^2[0,1]},
\]
i.e., (21) holds for \( C_0 = S_0^{-1} \sum_{r=1}^{n} (\alpha_r + \beta_r) \). Then (22)-(34) hold similarly as in Theorem 2.2. Finally, a priori estimate (2) results from (31)-(34). Since the uniqueness of classical solution follows from (2), then the existence results from Fredholm’s property [2] which is inherent to the problem (1). Corollary 2.3 is proved.

**Note 2.1.** To prove Theorem 2.2 and Corollary 2.3, the fulfillment of condition \( \mathcal{A} \) and (35) is required correspondingly. Obviously, these conditions cover the condition \( S \leq 1 \), where

\[
S = \begin{cases} 
\sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s & \text{if } \zeta_n < \eta_1, \\
\sum_{r=1}^{n} \alpha_r & \text{if } \zeta_n > \eta_1, \\
\sum_{r=1}^{n} (\alpha_r - \beta_r) + |\alpha_r - \beta_r| & 2 
\end{cases}
\]

The condition \( S \leq 1 \) was required (see [16, p. 39-44]) to prove the well-posedness of NLBVP (1). Obviously, irrespective of \( \zeta_n \) and \( \zeta_p \) location, this result also follows from Theorem 2.2 and Corollary 2.3 correspondingly. In addition, for any value \( S > 1 \), by virtue of Theorem 2.2, we can define an open interval for the location of \( \zeta_n \), i.e.,

\[
0 < \zeta_n < \arcsinh(S^{-1} \sinh 1),
\]

so that the NLBVP (1) remains well-posed. Similarly, by virtue of Corollary 2.3, for any \( S > 1 \) we can define an interval for \( \zeta_p \), i.e.,

\[
0 < \zeta_p < \arcsinh(S^{-1} \sinh 1),
\]

so that the NLBVP (1) remains well-posed.

**Note 2.2.** Actually, the requirement \( \mathcal{A} \), as well the condition (35), reveals the unboundedness effect, i.e., the corresponding value \( S \) could be an arbitrarily large positive real number that depends on \( \zeta_n \to 0 \), or on \( \zeta_p \to 0 \), correspondingly, but nevertheless the NLBVP (1) remains well-posed.

**Note 2.3.** By virtue of Theorem 2.2, we can improve the condition of well-posed solvability for formulated in [3, p. 140] NLBVP (1) and write it as following:

\[
\sum_{k=1}^{m} \alpha_k^+ < \frac{\sinh 1}{\sinh \xi_p},
\]

where \( \alpha_k^+ = 2^{-1}(\alpha_k + |\alpha_k|) \) and \( p \) is the largest subindex of \( \xi_k, \ k = 1, \ldots, m \), so that \( \alpha_p > 0 \) (we assume that there is at least one \( \alpha_k, \ k = 1, \ldots, m \) which has positive value), but \( \alpha_{p+i} \leq 0, \ 1 < i \leq n-p \ (p = n \ if \ such \ i \ does \ not \ exists) \).
3 Difference problem

We consider difference interpretation of NLBVP (1)

\[
\begin{aligned}
\Lambda Y &= Y_{xx} + Y_{yy} = f(x, y), \quad (x_i, y_j) \in \Pi, \\
Y|_{y=0} &= Y|_{y=\pi} = 0, \quad x_i \in [0, 1), \quad Y|_{x=0} = 0, \quad y_j \in [0, \pi], \\
\mathcal{L}Y &= \sum_{r=1}^{n} \alpha_r \left( Y_{i_r, j_r} - \frac{(i_{\zeta_r} + 1)h_1 - \zeta_r}{h_1} + Y_{i_r, j_r} + \frac{\zeta_r - i_{\zeta_r}h_1}{h_1} \right) - \sum_{s=1}^{m} \beta_s \left( Y_{i_s, j} - \frac{(i_{\eta_s} + 1)h_1 - \eta_s}{h_1} + Y_{i_s, j} - \frac{\eta_s - i_{\eta_s}h_1}{h_1} \right) - YN_{i,j} = 0,
\end{aligned}
\]

(40)

where same as in the differential problem we require $0 < \zeta_1 < \ldots < \zeta_n < 1$, $0 < \eta_1 < \ldots < \eta_m < 1$, $\zeta_r \neq \eta_s$, $\alpha_r > 0$, $\beta_s > 0$, $r = 1, \ldots, n$, $s = 1, \ldots, m$, and additionally, we define the numbers $i_{\zeta_r}$ and $i_{\eta_s}$ by corresponding inequalities $i_{\zeta_r}h_1 \leq \zeta_r < (i_{\zeta_r} + 1)h_1$ for $r = 1, \ldots, n$ and $i_{\eta_s}h_1 \leq \eta_s < (i_{\eta_s} + 1)h_1$ for $s = 1, \ldots, m$, at least we put $\zeta_0 = \eta_0 = 0$, $\zeta_{n+1} = \eta_{m+1} = 1$, $h_1 = 1/N_1$, $h_2 = \pi/N_2$ and require $h_1 \leq c_0h_2$, $c_0 = \text{const}$ add $h_1 < \theta$, $\theta = \frac{1}{2} \min \{\zeta_{r+1} - \zeta_r, r = 0, 1, \ldots, n; \eta_{s+1} - \eta_s, s = 0, 1, \ldots, m; |\zeta_r - \eta_s|, r = 1, \ldots, n, s = 1, \ldots, m\}$.

Let $\overline{A}$ denotes the condition:

\[
-\infty < \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s < \left(1 + \frac{4}{\pi}\right)^{1-\zeta_n - \theta} \quad \text{when} \quad \zeta_n < \eta_1,
\]

\[
\sum_{r=1}^{n} \alpha_r < \left(1 + \frac{4}{\pi}\right)^{1-\zeta_n - \theta} \quad \text{when} \quad \zeta_n > \eta_1.
\]

**Theorem 3.1.** Let $f(x, y)$ so that $u(x, y) \in C^{(4)}(\Pi)$ is a solution of NLBVP (1) when the condition $\overline{A}$ holds. If, additionally, the condition $\overline{A}$ holds too, then difference solution of (40) approximates $u(x, y)$ by the second order of accuracy in terms of $h = \sqrt{h_1^2 + h_2^2}$, $h_2 \to 0$ in each of the difference metrics $C$, $W_2^2$.

**Proof.** We denote $z = Y - u$, then $z$ satisfies the difference problem

\[
\begin{aligned}
\Lambda z &= f - \Lambda u = F, \quad (ih_1, jh_2) \in \Pi, \\
z|_{x=0} &= z|_{y=0} = z|_{y=\pi} = 0, \quad \mathcal{L}z = -\mathcal{L}u.
\end{aligned}
\]

(41)

For this problem $F = O(h^2)$ and $\mathcal{L}u = O(h^2)$ [10, p. 81, 229]. Put $z = \bar{z} + \hat{z}$, where $\bar{z}$ is the solution of

\[
\begin{aligned}
\Lambda \bar{z} &= 0, \quad (ih_1, jh_2) \in \Pi, \\
\bar{z}|_{x=0} &= \bar{z}|_{y=\pi} = 0, \quad \mathcal{L}\bar{z} = -\mathcal{L}u,
\end{aligned}
\]

(42)
and \( \tilde{z} \) is the solution of
\[
\begin{cases}
\Lambda \tilde{z} = F, \ (ih_1, jh_2) \in \Pi, \\
\tilde{z}|_{x=0} = \tilde{z}|_{y=0} = \tilde{z}|_{y=\pi} = 0, \ L\tilde{z} = 0.
\end{cases}
\] (43)

Further, to estimate \( \tilde{z} \) we use [10, p. 113] the orthogonal system of mesh functions \( \{\sin(ky)\}_{k=1}^{k=N_2-1} \), so that from the representation
\[
\tilde{z} = \sum_{k=1}^{N_2-1} \tilde{z}_k \sin(ky), \ y = jh_2, \ j = 0, 1, \ldots, N_2
\]
it follows, that \( \tilde{z}_k, \ k = 1, \ldots, N_2 - 1 \) is the difference solution of the problem
\[
\begin{cases}
\Lambda_1 \tilde{z}_k - \lambda_k \tilde{z}_k = 0, \\
\tilde{z}_k|_{x=0} = 0, \ L\tilde{z}_k = -Q_k,
\end{cases}
\] (44)
where \( \Lambda_1 \tilde{z}_k = \tilde{z}_{xx}, \ \lambda_k = 4h_2^{-2} \sin^2(kh_2), \ Q_k = (Lu)_k \) and, in view of [3, p. 142-143],
\[
\tilde{z}_k|_{x_i=ih_1} = A_k \sinh(i \ln q_k),
\]
\[
A_k = -Q_k/\mathcal{L}[\sinh(i \ln q_k)], \ i = 0, \ldots, N_1,
\]
\[
q_k = 1 + \lambda_k h_1^2/2 + \sqrt{\lambda_k h_1^4 + \lambda_k^2 h_1^2}/4.
\]

Denote \( \mathcal{D} = \mathcal{L}[\sinh(i \ln q_k)] \). By acting \( \mathcal{L} \) on \( \sinh(i \ln q_k) \) in the denominator of the fraction for \( A_k \), we get
\[
-D \geq \sinh(N_1 \ln q_k) - \sum_{r=1}^{n} \alpha_r \sinh((i\zeta_n + 1) \ln q_k) + \sum_{s=1}^{m} \beta_s \sinh(i\eta_1 \ln q_k).
\] (45)

Hence,
\[
-D \geq \sinh(N_1 \ln q_k) - S \sinh((i\zeta_n + 1) \ln q_k)
\] (46)
for
\[
S = \begin{cases}
0, & \text{if } -\infty < \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s \leq 0, \ \zeta_n < \eta_1, \\
\sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s, & \text{if } 0 < \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s, \ \zeta_n < \eta_1, \\
\sum_{r=1}^{n} \alpha_r, & \text{if } \zeta_n > \eta_1.
\end{cases}
\]

Then
\[
-D \geq \sinh(N_1 \ln q_k) \left[ 1 - S \frac{\sinh(i\zeta_n + 1) \ln q_k)}{\sinh(N_1 \ln q_k)} \right],
\] (47)
therefore,

\[ -D \geq \sinh(N_1 \ln q_k)
\left[ 1 - S \frac{q_k^{i\zeta_n + 1} - q_k^{-(i\zeta_n + 1)}}{q_k^{N_1} - q_k^{-N_1}} \right]. \]

Since \( q_k \geq 1 \), we get

\[ \frac{q_k^{i\zeta_n + 1} - q_k^{-(i\zeta_n + 1)}}{q_k^{N_1} - q_k^{-N_1}} \leq \frac{q_k^{i\zeta_n + 1} [1 - q_k^{-2(i\zeta_n + 1)}]}{q_k^{N_1} [1 - q_k^{-2N_1}]} \leq \frac{q_k^{i\zeta_n + 1}}{q_k^{N_1}}. \]

Since \( h_1 < \theta \) for \( \theta = \frac{1}{2} \min\{\zeta_r + 1 - \zeta_r, \ r = 0, n, \eta_{s+1} - \eta_s, \ s = 0, m\} \), for specified \( \delta = 1 - \zeta_n - \theta \) the inequality \( \zeta_n + h_1 \leq 1 - \delta \) holds. Hence, \( i\zeta_n + 1 \leq h_1^{-1}(1 - \delta) \). Then

\[ \frac{q_k^{i\zeta_n + 1} - q_k^{-(i\zeta_n + 1)}}{q_k^{N_1} - q_k^{-N_1}} \leq \frac{q_k^{N_1(1-\delta)}}{q_k^{N_1}} \leq \frac{1}{q_k^{N_1\delta}}. \]  \( (48) \)

Therefore,

\[ -D \geq \left( 1 - S \frac{1}{q_k^{N_1\delta}} \right) \sinh(N_1 \ln q_k). \]  \( (49) \)

Since

\[ q_k^{N_1} \geq (1 + \sqrt{\lambda_1} h_1)^{N_1} \geq (1 + \sqrt{\lambda_1} h_1)^{N_1} \geq (1 + \sqrt{\lambda_1}) \geq 1 + \frac{4}{\pi}, \]  \( (50) \)

we have

\[ -D \geq \left[ 1 - S \frac{1}{(1 + 4/\pi)^\delta} \right] \sinh(N_1 \ln q_k), \]  \( (51) \)

so that

\[ -D \geq C \sinh(N_1 \ln q_k) \]  \( (52) \)

for

\[ C = \begin{cases} 
1, & \text{if } -\infty < \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s \leq 0, \zeta_n < \eta_1, \\
1 - (1 + 4/\pi)^{-\delta} \left( \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s \right), & \text{if } 0 < \sum_{r=1}^{n} \alpha_r - \sum_{s=1}^{m} \beta_s, \zeta_n < \eta_1, \\
1 - (1 + 4/\pi)^{-\delta} \sum_{r=1}^{n} \alpha_r, & \text{if } \zeta_n > \eta_1.
\end{cases} \]

In summary, since the condition \( \bar{A} \) holds,

\[ -\mathcal{L} [\sinh(i \ln q_k)] \geq C \sinh(N_1 \ln q_k) > 0. \]  \( (53) \)
Finally, in view of (53), by virtue of [3, 150-151], we obtain the estimates

\[
\max_{i,j} |\hat{z}_{ij}| = O(h^2), \quad ||\hat{z}||_{W^2} = O(h^2), \quad \max_{i,j} |\tilde{z}_{ij}| = O(h^2), \quad ||\tilde{z}||_{W^2} = O(h^2).
\]

Therefore, \( \max_{i,j} |z_{ij}| = O(h^2), ||z||_{W^2} = O(h^2). \) Theorem 3.1 is proved. \( \square \)

**Corollary 3.2.** Let \( n = m, \ z_r < \eta_r, \ r = 1, ..., n. \) Let \( f(x, y) \) and so that \( u(x, y) \in C^4(\overline{\Omega}) \) is a solution of NLBVP (1) when condition (35) holds for

\[
2^{-1} \sum_{r=1}^n (\alpha_r - \beta_r + |\alpha_r - \beta_r|) > 0.
\]

If

\[
0 < \sum_{r=1}^n \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} < \left(1 + \frac{4}{\pi}\right)^{1-\zeta_p-\theta}
\]

for \( 1 \leq p \leq n, \) so that \( \frac{(\alpha_p - \beta_p) + |\alpha_p - \beta_p|}{2} > 0, \) but \( \frac{(\alpha_{p+1} - \beta_{p+1}) + |\alpha_{p+1} - \beta_{p+1}|}{2} = 0 \)

for all \( 1 < i \leq n - p \) (if such \( i \) does not exist, we put \( p = n), \) then the solution of (40) approximates \( u(x, y) \) by the second order of accuracy in terms of \( h = \sqrt{h_1^2 + h_2^2}, \) \( h_2 \to 0 \) in each of the difference metrics \( C, \ W^2_2. \)

**Proof.** In view of (41)-(45), for \( \mathcal{D} = \mathcal{L}[\sinh(i \ln q_k)] \) we obtain the inequality

\[
-\mathcal{D} \geq \sinh(N_1 \ln q_k) - \sum_{r=1}^n \alpha_r \sinh((i\zeta_r + 1) \ln q_k) + \sum_{r=1}^n \beta_r \sinh(i\eta_r \ln q_k).
\]

Since \( i\zeta_r + 1 < i\eta_r, \ r = 1, \ldots, n, \) we get

\[
-\mathcal{D} \geq \sinh(N_1 \ln q_k) - \sum_{r=1}^n (\alpha_r - \beta_r) \sinh((i\zeta_r + 1) \ln q_k).
\]

Hence,

\[
-\mathcal{D} \geq \left[ 1 - \sum_{r=1}^n (\alpha_r - \beta_r) \left( \frac{q_k^{-N_1} - q_k^{-(i\zeta_r + 1)}}{q_k^{-N_1} - q_k^{-N_1}} \right) \right] \sinh(N_1 \ln q_k).
\]

Also,

\[
-\mathcal{D} \geq \left[ 1 - S \frac{q_k^{i\zeta_p + 1} - q_k^{-(i\zeta_p + 1)}}{q_k^{-N_1} - q_k^{-N_1}} \right] \sinh(N_1 \ln q_k)
\]

for

\[
S = \sum_{r=1}^n \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2}.
\]
By analogy with (48), for $q_k \geq 1$ and $\delta = 1 - \zeta_p - \theta$, we get

$$\frac{q_k^{-i\zeta_p+1} - q_k^{-(i\zeta_p+1)}}{q_k^{N_1} - q_k^{-N_1}} \leq \frac{1}{q_k^{N_1} \delta}$$

(56)

since the inequalities $\zeta_p + h_1 \leq 1 - \delta$ and $i\zeta_p + 1 \leq h_1^{-1}(1 - \delta)$ hold. In view of (50) and (55)-(56), the analogies of (51)-(53) hold for

$$C = 1 - (1 + 4/\pi)^{-\delta} \left( \sum_{r=1}^{n} \frac{(\alpha_r - \beta_r) + |\alpha_r - \beta_r|}{2} \right).$$

In view of (53), similar to Theorem 3.1, we obtain

$$\max_{i,j} |\tilde{z}_{ij}| = O(h^2), \quad ||\tilde{z}||_{W^2} = O(h^2), \quad \max_{i,j} |\hat{z}_{ij}| = O(h^2), \quad ||\hat{z}||_{W^2} = O(h^2),$$

and therefore, $\max_{i,j} |z_{ij}| = O(h^2), \quad ||z||_{W^2} = O(h^2)$. Corollary 3.2 is proved.

4 Conclusion

In this paper we used an approach which is based on modified methods of papers [3] and [16].

The basic result of our paper demonstrates new conditions on the well-posedness of NLBVP (1) (see Theorem 2.2 and Corollary 2.3). The newness of the condition $A$ and (35) is shown in Note 2.1. As it is shown in Note 2.2, condition $A$, as well as the requirement (35), reveals the unboundedness effect for the value $S$, which is specified by corresponding values of the coefficients in NLBVC of the differential problem (1).

The difference interpretation of NLBVP (1) is proposed by the finite-difference scheme (40). In Theorem 3.1, under the condition $A$, and in Corollary 3.2 under the requirement (54), correspondingly, we proved the second order of accuracy approximation for smooth classical solution of NLBVP (1) on a uniform grid with sufficiently small step. The required new condition $A$ and the inequality (54) covers the condition $S \leq 1$ which was used by the author earlier in the paper [16, p. 45-48] to obtain the second order of accuracy approximation.

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