Universality of the Einstein theory of gravitation

Jerzy Kijowski
Center for Theoretical Physics, Polish Academy of Sciences, Al. Lotników 32/46; 02-668 Warszawa, Poland

We show that generalizations of general relativity theory, which consist in replacing the (linear in scalar curvature) Hilbert Lagrangian $L_{\text{Hilbert}} = \frac{1}{16\pi} \sqrt{|g|} R$, by a generic scalar density $L = L(g_{\mu\nu}, R^\lambda_{\mu\nu\kappa})$ depending upon the metric $g_{\mu\nu}$ and the whole curvature tensor $R^\lambda_{\mu\nu\kappa}$, are equivalent to the conventional Einstein theory for a (possibly) different metric tensor $\tilde{g}_{\mu\nu}$ and (possibly) a different set of matter fields.

More precisely, consider a „generalized” theory of gravity, based on an invariant Lagrangian density:

$$L = L(g_{\mu\nu}, R^\lambda_{\mu\nu\kappa}, \Gamma^\lambda_{\mu\nu}, \varphi, \partial \varphi) , \quad (1)$$

where $\Gamma^\lambda_{\mu\nu}$ is a Levi-Civita connection of the metric $g_{\mu\nu}$ and $R^\lambda_{\mu\nu\kappa}$ denotes its Riemann tensor, whereas $\varphi$ denotes some matter fields. The following mathematical statement can be proved:

**Theorem 1:** There exists a one-to-one change of variables:

$$(g, \varphi) \iff (\tilde{g}, \varphi, \phi) , \quad (2)$$

and a new matter Lagrangian:

$$\tilde{L}_{\text{Matter}} = \tilde{L}_{\text{Matter}}(\tilde{g}, \partial \tilde{g}, \varphi, \partial \varphi, \partial \phi) , \quad (3)$$

such that $(g, \varphi)$ satisfy field equations derived from the Lagrangian (1) if and only if the corresponding fields $(\tilde{g}, \varphi, \phi)$ satisfy the conventional „Einstein + matter” equations, derived from the conventional Hilbert variational principle:

$$\tilde{\mathcal{L}} := L_{\text{Hilbert}}(\tilde{g}) + \tilde{L}_{\text{Matter}} . \quad (4)$$

In particular, equations for the new metric $\tilde{g}$ are of the second differential order: $G^\mu\nu(\tilde{g}) = 8\pi \tilde{T}^\mu\nu$, with

$$\tilde{T}^\mu\nu := \frac{2}{\sqrt{\tilde{g}}} \frac{\delta \tilde{L}_{\text{Matter}}}{\delta \tilde{g}^\mu\nu} . \quad (5)$$

whereas equations for the old metric $g$, derived from the original Lagrangian density (1), were of the fourth order. Also matter field equations are of the second differential order because $\tilde{L}_{\text{Matter}}$ depends upon first derivatives only.

Recently, we have been able to extend our result to higher order Lagrangians (i.e. admitting covariant derivatives of the Riemann tensor up to order $k$).
The particular case of a Lagrangian $L$ which depends non-linearly upon the Ricci tensor, but does not depend upon the Weyl tensor, were first considered by Stephenson and Higgs (see [1]) and later analyzed thoroughly by many authors (cf. [2]). Equivalence of such theories with the standard General Relativity Theory was proved already long ago (see e.g. [3]). In particular, the Sacharov’s non-linear Lagrangian containing the $R^2$ term (see [4]) is equivalent to the standard GR interacting with a scalar field (see [5] and also [3]).

But there is a renewed interest in such (or even more radical) generalizations of the Einsteinian theory of gravity (see e.g. [6]). In this context our result can be summarized as follows: generalizations of the theory of gravity, based on non-conventional Lagrangians are equivalent to the conventional (i.e. Einsteinian) theory of gravity, interacting with non-conventional matter fields. Einstein theory is, therefore, universal!

References

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