MULTIPICITY OF SOLUTIONS FOR CRITICAL QUASILINEAR
SCHRÖDINGER EQUATIONS USING A LINKING STRUCTURE

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Abstract. It is established multiplicity of solutions for critical quasilinear
Schrödinger equations defined in the whole space using a linking structure.
The main difficulty comes from the lack of compactness of Sobolev embedding
into Lebesgue spaces. Moreover, the potential is bounded from below and
above by positive constants. In order to overcome these difficulties we employ
Lions Concentration Compactness Principle together with some fine estimates
for the energy functional restoring some kind of compactness.

1. Introduction. Quasilinear Schrödinger elliptic equations have been accepted as
a model in order to describe many physical phenomena. It is worthwhile to mention
that for the quasilinear elliptic problem

\[ i \partial_t z = -\Delta z + W'(x)z - \Delta l(|w|^2)l'(|z|^2)z - \tilde{g}(x, z)z, \quad x \in \mathbb{R}^N, \ t > 0, \]

where \( z \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{C}), \) \( W \in C(\mathbb{R}^N, \mathbb{R}) \) is a potential, \( l \) is a suitable real function,
\( \tilde{g} \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \) is a nonlinearity have been accepted as a model in many physical
phenomena depending of the function \( l \). For instance, assuming that \( l(t) = 1 \)
we obtain the classical semilinear Schrödinger equation, see [19]. For the case
\( l(t) = t \) the equation arises from fluid mechanics, plasma physics and dissipative
quantum mechanics, see [32, 24, 15, 17]. If \( l(t) = \sqrt{1 + t} \), the equation models the
propagation of a high-irradiance laser in a plasma as well as the self-channeling of a
high-power ultrashort laser in matter, see [18]. For further physical applications we
also refer the interesting reader to [6, 16]. In the simplest case \( l(t) = 1 \), we obtain
a semilinear equation and there exist a lot of papers working on existence, non-
existence, multiplicity and concentration behavior of solutions, see [37, 10, 8, 20, 22].

In the superfluid film case, choosing \( l(t) = t^{\alpha/2} \), for \( \alpha > 0 \), the problem also has
been extensively studied during the last years, see [28, 29, 26, 34, 39, 38] and
references therein. In the same way, we emphasize that the nonlinear term \( g \) can be
used to model several phenomena such as Bose-Einstein condensates, Bose-Fermi
mixture as well as in many others applications on nonlinear optics, plasma physics,
superfluid films and quantum mechanics, see for instance [6, 15, 16]. For more details

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on the physical background we also refer the reader to [5, 7, 31] and references therein. Notice that this condition includes the fluid mechanics, plasma physics and dissipative quantum mechanics case. On this subject we refer also the reader to some interesting works on quasilinear elliptic problems [1, 21, 35, 33, 41, 45].

Now, since we are interested in solitary wave solutions, namely we look for solutions with the special form $z(t, x) = \exp(-iEt)u(x)$, with $E \in \mathbb{R}$ and $z \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{C})$ where $u$ is real function, we are lead to consider the following equation

$$-\Delta u + V(x)u - \Delta |u^2|u' |u^2|u = g(x, u), \quad x \in \mathbb{R}^N$$

where $V = W - E$ is the new potential and $g(x, u) = \tilde{g}(x, |u|)u$, $x \in \mathbb{R}^N$, $u \in \mathbb{R}$ with $g \in C^0(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. More precisely, in the present work we shall consider the existence of weak solutions to the following quasilinear elliptic problem

$$-\Delta u + V(x)u - \Delta |u^2|u = \lambda q(x)u + K(x)|u|^{p-2}u + \theta \Gamma(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N \tag{2}$$

where $u \in H^1(\mathbb{R}^N), N \geq 3, 4 < p < 2 \cdot 2^*$, $V, K \in C(\mathbb{R}^N, \mathbb{R}), q \in L^\alpha(\mathbb{R}^N) \cap L^\beta(\mathbb{R}^N), \alpha > N/2$ and $2N/(N+2) < \beta \leq 2$, $K \in L^\gamma(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ where $\gamma = (2 \cdot 2^*/p)$ and $\Gamma \in L^\infty(\mathbb{R}^N)$. As usual $r'$ denotes the number $r' = r/(r-1)$ for each $r > 1$. Here we emphasize that the parameter $\lambda$ belongs to $(\lambda_j, \lambda_{j+1})$ where the sequence $(\lambda_j)_{j \in \mathbb{N}} \subset \mathbb{R}$ denotes the sequence of eigenvalues for the associated linear problem and $\theta > 0$ is a real parameter. Later on, we shall discuss some assumptions on the functions $V, q, K$ and $\Gamma$. For an easy reference we consider $g(x, t) = \lambda q(x)u + K(x)|u|^{p-2}u + \theta \Gamma(x)|u|^{2^*-2}u$ for each $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

In the present work we are interesting in to consider existence of weak solutions for the problem (1) choosing $l(t) = t$. More specifically, since we are leading to standing wave solutions we mention that $z(t, x) = \exp(-iEt)u(x)$, with $E \in \mathbb{R}$, is a solution for (1) if and only if $u$ solves the quasilinear elliptic problem (2). Notice also that from a standard mathematical point of view the problem (2) exhibits many difficulties. The first one comes from the fact that the standing wave solutions for (1) lead us to the following elliptic problem

$$-\Delta u + V(x)u - \Delta |u^2|u = g(x, u), \quad x \in \mathbb{R}^N. \tag{3}$$

which presents a quasilinear term. This term does not permit us to consider directly variational methods since the energy functional is not well defined. For further quasilinear elliptic problem we refer the reader to [1]. The second difficulty arises from the fact that the potential $V$ is not coercive which implies that the compact embedding are not available. Furthermore, the nonlinearity $g$ is a superlinear function with a critical behavior. In the same way, the nonlinearity term $g$ has an interaction with the linear problem. This iteration implies that our problem is asymptotically linear at the origin and superlinear at infinity. Under these conditions, using a changing of variable $f$ introduced in [9, 28], we turn the problem (3) into the following semilinear elliptic problem

$$-\Delta u + V(x)f(u)f'(u) = g(x, f(u))f'(u), \quad x \in \mathbb{R}^N.$$  

For this kind of problem we can look for an associated functional $J$ which is in $C^1$ class and its critical points allows us to find weak solutions to the elliptic problem (2). In order to overcome the difficulties listed just above we prove that the energy functional admits a linking structure. To the best our knowledge there is no results for quasilinear elliptic problems where the potential $V$ is bounded and the nonlinear term $g$ interacts with higher eigenvalues for the associated linear problem. Recall
that assuming that $g$ is superlinear at infinity and at the origin, that is, assuming the following conditions:
\[
\lim_{|t| \to 0} \frac{g(x,t)}{t} = 0 \quad \text{and} \quad \lim_{|t| \to \infty} \frac{g(x,t)}{t^3} = \infty, \quad \text{uniformly in } x \in \mathbb{R}^N
\]
the function $u = 0$ is a local minimum for the energy functional. In this case, the Mountain Pass Theorem can be applied proving several results concerning existence and multiplicity of solutions for quasilinear Schrödinger equations, see \[1, 9, 11, 13, 14, 27, 28, 29, 25\]. In the present work we shall consider the following case
\[
\lim_{|t| \to 0} \frac{g(x,t)}{t} = q(x) \quad \text{and} \quad \lim_{|t| \to \infty} \frac{g(x,t)}{t^3} = \infty, \quad \text{uniformly in } x \in \mathbb{R}^N
\]
holding for some function $q \in L^\alpha(\mathbb{R}^N)$ for some $\alpha > N/2$. In other words, we consider the case where the nonlinearity $g$ is asymptotically linear at the origin and superlinear at infinity. In fact, writing $g(x,t) = \lambda q(x)t + K(x)|t|^{p-2}t + \Theta \Gamma(x)|t|^{\alpha-2}t$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, we ensure the existence of weak solutions for the problem (2) under general assumptions on the functions $V, q, K$ and $\Gamma$.

The main contribution in this work is to consider quasilinear elliptic problems where the nonlinear term has an interaction with higher eigenvalues for the associated linear problem. Under these conditions the energy functional admits a linking structure which can be explored in order to ensure that problem (2) admits at least one nontrivial solution. Since we are looking for existence of solutions for quasilinear elliptic problems the quasilinear term $\Delta(u^2)u$ brings us serious difficulties in order to prove some geometric properties listed in the Linking Theorem. Furthermore, we consider the nonlinear term $g$ with a critical behavior which gives us many difficulties due to the lack of compactness. Hence the key point is to consider some estimates together the fact that the critical level for the associated functional belongs to a certain region where the compactness is recovered.

### 1.1. Assumptions and the main results.

In this work we shall consider existence of weak solutions for quasilinear elliptic equations where the nonlinear term $g$ interacts with higher eigenvalues for the associated linear problem. As was mentioned before we are interested in the critical case, that is, we consider $g$ with critical behavior at infinity. In order to establish our main result for the problem (2) we assume some extra hypotheses on $V, q, K$ and $\Gamma$. More specifically, we consider the following assumptions:

(V1) $V$ is continuous and there exist constants $V_0, V_\infty > 0$ such that $V_0 \leq V(x) \leq V_\infty$ for all $x \in \mathbb{R}^N$.

For the potential $q$ we shall assume the following assumption:

(q1) $q \in L^\alpha(\mathbb{R}^N) \cap L^\beta(\mathbb{R}^N)$ for some $\alpha > N/2$ and $2N/(N + 2) < \beta \leq 2$, $q(x) > 0$, a. e. $x \in \mathbb{R}^N$.

For the potential $K$ we shall consider the following hypothesis:

(K1) $K \in L^\gamma(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\gamma = (2 \cdot 2^*/p)'$, $K(x) > 0$ a. e. $x \in \mathbb{R}^N$.

For the function $\Gamma$ we assume the following condition:

(Γ1) $\Gamma \in L^\infty(\mathbb{R}^N)$, $\Gamma \geq 0$, a. e. $x \in \mathbb{R}^N$ with $\Gamma(x) \to 0$ as $|x| \to \infty$.

As was mentioned before the parameter $\lambda$ belongs to the interval $(\lambda_j, \lambda_{j+1})$ where $(\lambda_j)_{j \in \mathbb{N}} \subset \mathbb{R}$ denotes the sequence of eigenvalues for the weighted eigenvalue
problem:

\[-\Delta u + V(x)u = \lambda q(x)u, \quad x \in \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N). \quad (4)\]

It is not hard to see that the first eigenvalue for the problem (4) denoted by \(\lambda_1\) is positive for each \(q\) satisfying \((q_1)\), see for instance \([43, 44]\). The spectral theory for this kind of weighted eigenvalue problems follows using the fact that the Sobolev space \(H^1(\mathbb{R}^N)\) is compactly embedded into the weighted Lebesgue space \(L^2(\mathbb{R}^N, q)\), see Lemma 2.4 ahead. Furthermore, since our potential \(V\) is bounded from below and above by positive constants we observe that the spectrum for (4) can be characterized using some compact properties, see Lemma 2.4 ahead.

Recall that a function \(u \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\) is said to be a weak solution for the elliptic problem (2) provided that

\[
\int_{\mathbb{R}^N} \left[ (1 + 2u^2)\nabla u \nabla \phi dx + 2u|\nabla u|^2 \phi + V(x)u \phi \right] dx = \int_{\mathbb{R}^N} g(x, u) \phi dx
\]

holds for all \(\phi \in H^1(\mathbb{R}^N)\) with \(g(x, t) = \lambda q(x)t + K(x)|t|^{p-2}t + \theta \Gamma(x)|t|^{2^*-2}t, x \in \mathbb{R}^N, t \in \mathbb{R}.\)

It is important to emphasize that \(u \equiv 0\) is the trivial solution for the quasilinear elliptic problem (2). Hence, we shall consider existence of nontrivial solutions for the problem (2). This can be done using some fine estimates showing that the energy functional associated to the problem (2) has a linking structure. Furthermore, using Lion’s Concentrations Compactness Principle, we shall prove that the associated energy functional satisfies the well known Palais-Smale condition. Later on, we consider the definitions for Palais-Smale sequences and Palais-Smale condition. Now we can state our main result in the following form:

**Theorem 1.1.** Suppose \((V_1)\), \((q_1)\), \((K_1)\) and \((\Gamma_1)\) hold. Assume also that \(\lambda \in (\lambda_j, \lambda_{j+1})\) holds true for some \(j \geq 1\). Then for any \(k \in \mathbb{N}\) there exists \(\theta_k \in (0, \infty)\) such that the problem (2) has at least \(k\) pairs of nontrivial solutions for all \(\theta \in (0, \theta_k)\).

Recall that under our hypotheses the energy functional does not admits \(u = 0\) as a local minimum. On this subject we refer the reader to the recent works \([43, 30]\). In \([43]\) the authors considered quasilinear elliptic problems where the nonlinearity \(g\) is superlinear or asymptotically linear at the origin where was proved existence and multiplicity of solutions provided that potential \(V\) is coercive. In \([30]\) was considered existence of positive solutions for quasilinear elliptic problems where the nonlinear term \(g\) is superlinear at infinity and at the origin and the potential \(V\) satisfies \(\{x \in \mathbb{R}^N : V(x) \leq M\}\) has finite Lebesgue measure for any \(M > 0\). Using the last hypothesis we recover the compactness for the embedding of \(\{x \in \mathbb{R}^N : V(x) \leq M\}\) into the Lebesgue space \(L^p(\mathbb{R}^N)\) for each \(p \in [2, 2^*)\), see \([3]\). For related results we refer also the reader to \([41, 42]\) where it was treated quasilinear elliptic problems with subcritical and critical assumptions, respectively. In the present work, using the fact that \(V\) is bounded, the previous embedding for the Sobolev spaces into Lebesgue spaces are only continuous. More precisely, there is no compact embedding in our setting.

Notice that for our main problem the nonlinear term \(g\) is asymptotically at the origin and superlinear at infinity. More precisely, the nonlinearity \(g\) is presents also the critical behavior at infinity. To the best our knowledge there is no results on quasilinear elliptic problems such as (2) where the potential \(V\) is bounded from below and above and the associated energy functional does not admit \(u \equiv 0\) as a
local minimum. In fact, assuming that \( \lambda \in (\lambda_j, \lambda_{j+1}) \) the trivial solution \( u \equiv 0 \) can be a saddle point for the energy functional. One more time, the lack of compactness exhibits an obstacle in order to apply variational procedures. In this case we prove that the critical level for the associate functional is less than a suitable positive constant for each \( \theta \in (0, \theta_k) \) where \( \theta_k > 0 \). Hence our main result complements the aforementioned results by considering existence of solutions for quasilinear elliptic problem defined in the whole space with bounded potentials in such way that the nonlinear term \( q \) is critical.

In order to clarify our assumptions we consider some examples as prototypes for the functions \( V, q, K \) and \( \Gamma \).

**Example 1.1.** It is important to mention that the constant potential \( V \), i.e, assuming that \( V(x) = V_0 \) for all \( x \in \mathbb{R}^N \) and \( V_0 > 0 \) the hypothesis \((V_1)\) is satisfied. Here we also consider the non-constant potential \( V(x) = V_0 + \frac{a_1}{1 + a_2 |x|^2} \) where \( V_0, a_1, a_2 > 0 \). Clearly, the function \( V \) satisfies our assumption \((V_1)\) with \( \inf_{x \in \mathbb{R}^N} V(x) = V_0 \) and \( V_\infty = \sup_{x \in \mathbb{R}^N} V(x) = V_0 + a_1 \). Similarly, we take

\[
K(x) = \frac{b_1}{(1 + b_2 |x|^2)^{q_2}}
\]

where \( b_1, b_2 > 0 \) and \( q \geq 1 \) where \( \gamma = (2 \cdot 2^*/p)' \). Using the Coarea formula the function \( K \) belongs to \( L^\gamma (\mathbb{R}^N) \) proving that hypothesis \((K_1)\) holds for \( \gamma = (2 \cdot 2^*/p)' \). Here was used the fact that \( p \in (4, 2 \cdot 2^*) \). Furthermore, we choose

\[
q(x) = \frac{c_1}{(1 + c_2 |x|^2)^{q_2}}
\]

where \( c_1 > 0, c_2 > 0 \) and \( q_2 > N/4 \). Hence, by using the Coarea formula, \( q \) is in \( L^\alpha (\mathbb{R}^N) \cap L^\beta (\mathbb{R}^N) \) for \( \alpha > \max(N/2, N/(2q_2)), \beta \in (2N/(N + 2), 2] \) showing that \((q_1)\) is verified.

**Example 1.2.** Here we put \( V(x) = V_\infty - \frac{a_1}{1 + a_2 |x|^2} \) where \( V_\infty > a_1 > 0, a_2 > 0 \). It is easy to see that \( V_0 = \inf_{x \in \mathbb{R}^N} V(x) = V_\infty - a_1 > 0 \) and \( V_\infty = \sup_{x \in \mathbb{R}^N} V(x) \). Analogously, we define the function

\[
K(x) = b_1 \exp (-b_2 |x|^2), \quad x \in \mathbb{R}^N
\]

where \( b_1 > 0, b_2 > 0 \). In the same way, we consider

\[
q(x) = c_1 \exp (-c_2 |x|^2), \quad x \in \mathbb{R}^N
\]

where \( c_1 > 0 \) and \( c_2 > 0 \). One more time using by using the Coarea formula we see that \( q \in L^\alpha (\mathbb{R}^N) \cap L^\beta (\mathbb{R}^N) \) for any \( \alpha > N/2, \beta \in (2N/(N + 2), 2] \). Analogously, we also obtain \( K \) is in \( L^\gamma (\mathbb{R}^N) \) for \( \gamma = ((2 \cdot 2^*/p)') \). As a consequence, the functions \( V, q, K \) verify our hypotheses \((V_1), (q_1)\) and \((K_1)\), respectively.

1.2. **Outline.** The remainder of this paper is organized as follows: In the forthcoming Section 2 we consider the variational framework for the quasilinear elliptic problem (2). Section 3 is devoted to the proof of linking geometry for the associated energy functional. In Section 4 we consider the behavior Palais-Smale sequences for the energy functional together with some compactness results proving our main theorem.
1.3. Notation. Let us introduce the following basic notations used in the current work:

- \( C, \tilde{C}, C_1, C_2, \ldots \) denote positive constants (possibly different).
- \( a_n(1) \) denotes a sequence which converges to 0 as \( n \to \infty \).
- The norm in \( L^q(\mathbb{R}^N) \) and \( L^\infty(\mathbb{R}^N) \), will be denoted respectively by \( \| \cdot \|_q \) and \( \| \cdot \|_\infty \).
- The variational framework.

1.3. Notation. Let us introduce the following basic notations used in the current work:

- \( \mathcal{E}_2 \) denotes a sequence which converges to 0 as \( n \to \infty \).
- The norm in \( \mathcal{E} = H^1(\mathbb{R}^N) \) is denoted by \( \| \cdot \|_2 \).
- The variational framework.

2. The variational framework. Throughout this paper the working space is given by the Sobolev space \( \mathcal{E} = H^1(\mathbb{R}^N) \) endowed with norm

\[
\| u \| := \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx \right)^{1/2}, \forall \, u \in \mathcal{E}.
\]

This norm is equivalent to the usual norm in \( \mathcal{E} \) defined by

\[
\| u \|_2 := \left( \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx \right)^{1/2}, \forall \, u \in \mathcal{E}.
\]

Notice that the norm \( \| \cdot \| \) is induced by the inner product

\[
\langle u, v \rangle_\mathcal{E} := \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv) \, dx, \forall \, u, v \in \mathcal{E}.
\]

For our purpose we need to consider a version of the Symmetric Mountain Pass Theorem which is useful in order to get our main result in the present work. As a first step we recall some definitions and notations. Let \( \mathcal{E} \) a Banach space endowed with the norm \( \| \cdot \| \). Consider a functional \( J : \mathcal{E} \to \mathbb{R} \) of \( C^1 \) class. Recall that a sequence \( (u_n)_{n \in \mathbb{N}} \in \mathcal{E} \) is said to be a Palais-Smale sequence at the level \( c \in \mathbb{R} \), in short (PS)\(_c\) sequence, whenever \( J(u_n) \to c \) and \( J'(u_n) \to 0 \) as \( n \to \infty \). The functional \( J \) satisfies the Palais-Smale condition at the level \( c \in \mathbb{R} \), in short (PS)\(_c\) condition, whenever any (PS)\(_c\) sequence possesses a convergent subsequence. When \( J \) satisfies the (PS)\(_c\) condition at any level \( c \in \mathbb{R} \) we say purely that \( J \) satisfies the (PS) condition.

Now, we shall consider the Symmetric Mountain Pass Theorem introduced in [2] which can be stated as follows:

**Theorem 2.1.** Let \( \mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \), where \( \mathcal{E} \) is a Banach space and \( \mathcal{E}_1 \) is finite dimensional. Suppose that \( J \in C^1(\mathcal{E}, \mathbb{R}) \) is an even functional satisfying \( J(0) = 0 \). Assume also that

\[
\begin{align*}
(J_1) \text{ there exists a constant } \rho > 0 \text{ such that } J(v) \geq 0 \text{ for all } v \in \partial \mathcal{B}_\rho \cap \mathcal{E}_2; \\
(J_2) \text{ there exists a subspace } \mathcal{W} \text{ of } \mathcal{E} \text{ satisfying } \dim \mathcal{E}_1 < \dim \mathcal{W} < \infty. \text{ Assume also that there exists } M > 0 \text{ such that } \max_{v \in \mathcal{W}} J(v) < M; \\
(J_3) \text{ the functional } J \text{ satisfies (PS)}_c \text{ for } 0 \leq c \leq M \text{ where } M > 0 \text{ was given by } (J_2).
\end{align*}
\]

Then \( J \) possesses at least \( \dim \mathcal{W} - \dim \mathcal{E}_1 \) pairs of nontrivial critical points.
For related results concerning on Linking Theorems we infer the reader to [41, 36, 40, 46]. At this stage, we write \( E = E_1 \oplus E_2 \) where
\[
E_1 := \text{span}\{\phi_1, \ldots, \phi_j\} \quad \text{and} \quad E_2 := E_1^\perp.
\]
The functions \( \phi_i \) are eigenfunctions of the linear eigenvalue problem (4) associated to eigenvalue \( \lambda_i \) for each \( l \geq 1 \), respectively. Notice that the subspaces \( E_1 \) and \( E_2 \) are orthogonal and \( \text{dim}(E_1) < \infty \). The first eigenvalue \( \lambda_1 \) is characterized by
\[
\lambda_1 := \inf_{u \in E \setminus \{0\}} \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx \mid \int_{\mathbb{R}^N} q(x)u^2 = 1 \right\}. \tag{5}
\]
More generally, for each \( l \geq 2 \), the eigenvalue \( \lambda_l \) is characterized by
\[
\lambda_l := \inf_{u \in F_l \setminus \{0\}} \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx \mid \int_{\mathbb{R}^N} q(x)u^2 = 1 \right\}, \tag{6}
\]
where \( F_l = \{ u \in E : \langle u, \phi_i \rangle_E = 0, \forall i \in \{1, 2, \ldots, l-1\} \} \). In the same way, we shall consider the weighted Lebesgue space \( L^2(\mathbb{R}^N, q) \) given by
\[
L^2(\mathbb{R}^N, q) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \int_{\mathbb{R}^N} q(x)u^2 \, dx < \infty \right\}
\]
where \( q \in L^\alpha(\mathbb{R}^N) \) for some \( \alpha > N/2 \), \( q(x) > 0 \), a.e. \( x \in \mathbb{R}^N \). Notice also that \( L^2(\mathbb{R}^N, q) \) is a Banach space endowed with the natural norm
\[
\|u\|_{L^2(\mathbb{R}^N, q)} = \left( \int_{\mathbb{R}^N} q(x)u^2 \, dx \right)^{1/2}, \quad u \in L^2(\mathbb{R}^N, q).
\]
Now we shall give some compact embedding for the Sobolev space \( E \). In order to do that we need to discuss some properties for the space \( L^2(\mathbb{R}^N, q) \). Firstly, we consider an auxiliary result given in [44] which can be rewritten in the following form:

**Lemma 2.2** (Compact embedding [44]). *Let \( q \in L^\alpha(\mathbb{R}^N) \) be a fixed function with \( \alpha > N/2 \). Then the embedding of \( E \) into \( L^2(\mathbb{R}^N, q) \) is compact.*

In what follows we shall consider also the following weighted Lebesgue space \( L^\frac{p}{4}(\mathbb{R}^N, K) \) given by
\[
L^\frac{p}{4}(\mathbb{R}^N, K) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \int_{\mathbb{R}^N} K(x)|u|^\frac{p}{4} \, dx < \infty \right\}
\]
where \( 4 < p < 2 \cdot 2^* \). This space is endowed with the norm
\[
\|u\|_{L^\frac{p}{4}(\mathbb{R}^N, K)} = \left( \int_{\mathbb{R}^N} K(x)|u|^\frac{p}{4} \, dx \right)^{2/p}.
\]
Under these conditions, we consider the following compact embedding for Sobolev spaces.

**Lemma 2.3.** *Suppose that \( K \) satisfies \((K_1)\). Then the embedding of \( E \) into \( L^\frac{p}{4}(\mathbb{R}^N, K) \) is compact.*

**Proof.** Notice that Hölder’s inequality implies
\[
\int_{\mathbb{R}^N} K(x)|u|^\frac{p}{4} \, dx \leq \left( \int_{\mathbb{R}^N} |K(x)|^\gamma \, dx \right)^{1/\gamma} \left( \int_{\mathbb{R}^N} |u|^{2^*} \, dx \right)^{p/2^*}
\]
where \( \gamma = (2 \cdot 2^*/p)^* \). Hence, using that \( E \) is continuously embedded into \( L^{2^*}(\mathbb{R}^N) \), we also mention that \( E \) is continuously embedded into \( L^{\frac{p}{4}}(\mathbb{R}^N, K) \). In order to
established that the previous embedding is compact we shall prove that for any sequence \((u_n)_{n\in\mathbb{N}} \subset E\) in such way that \(u_n \to 0\) satisfies \(u_n \to 0\) in \(L^2_{\text{loc}}(\mathbb{R}^N, K)\).

Under this condition, we remember the following assertion

Let \(B_R = \{x \in \mathbb{R}^N : |x| \leq R\}\) be the closed ball with radius \(R\) centered at the origin. Using the fact that \(K \in L^\gamma(\mathbb{R}^N)\) for \(\gamma = (2 \cdot 2^*/p)'\), we obtain for each \(\tau > 0\) that there exists \(R_\tau > 0\) such that

\[
\left(\int_{\mathbb{R}^N \setminus B_R} |K(x)|^\gamma dx\right)^{1/\gamma} < \tau, \text{ for all } R > R_\tau. \tag{7}
\]

Therefore, choosing \(R > R_\tau\), we deduce that

\[
\int_{\mathbb{R}^N} K(x)|u_n|^2 dx = \int_{B_R} K(x)|u_n|^2 dx + \int_{\mathbb{R}^N \setminus B_R} K(x)|u_n|^2 dx = o_n(1) + \int_{\mathbb{R}^N \setminus B_R} K(x)|u_n|^2 dx.
\]

Applying the Hölder’s inequality and (7) we infer that

\[
\int_{\mathbb{R}^N \setminus B_R} K(x)|u_n|^2 dx \leq \left(\int_{\mathbb{R}^N \setminus B_R} |K(x)|^\gamma dx\right)^{1/\gamma} \left(\int_{\mathbb{R}^N \setminus B_R} |u_n|^{2^*} dx\right)^{1/2^*} \lesssim \epsilon.
\]

Here we emphasize that \(\epsilon > 0\) is arbitrary which it does not depend on \(n \in \mathbb{N}\). It follows from the last estimate that

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} K(x)|u_n|^2 dx \leq \epsilon.
\]

The last assertion says that \(u_n \to 0\) in \(L^2(\mathbb{R}^N, K)\). This finishes the proof. \(\Box\)

From now on we shall consider some variational inequalities for the eigenvalue problem (4). These inequalities are powerful tools in order to provide the behavior of our energy functional at the origin and at infinity. The key point is to ensure that \(E\) is compact embedded into \(L^2(\mathbb{R}^N, q)\), see Lemma 2.2. For further results on spectral theory involving compact self-adjoint operators we infer the interested reader to [4, 12]. In order to apply this spectral theory define the continuous operator \(S : L^2(\mathbb{R}^N, q) \to H^1(\mathbb{R}^N)\) given by \(S(h) = u\) where \(u\) is the unique weak solution for the problem

\[-\Delta u + V(x)u = q(x)h, \quad x \in \mathbb{R}^N, h \in L^2(\mathbb{R}^N, q).\]

Define also \(T = S \circ i : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)\) where \(i : H^1(\mathbb{R}^N) \to L^2(\mathbb{R}^N, q)\) is the inclusion operator. It is not hard to see that \(u \in H^1(\mathbb{R}^N) \setminus \{0\}\) is a weak solution for (4) if and only if \(T(u) = \frac{1}{\lambda} u\) where \(\lambda > 0\). Furthermore, by using Lemma 2.2, we know that \(T\) is a compact self-adjoint operator. Hence we can use the spectral analysis for compact self-adjoint operators. Now, we consider the following result:

**Lemma 2.4.** Suppose that \((V_1)\) and \((q_1)\) holds. Then we obtain the following variational inequalities:

(i) There holds

\[
\lambda_1 \int_{\mathbb{R}^N} q(x)u^2 dx \leq \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx, \quad u \in E. \tag{8}
\]
(ii) There holds
\[ \lambda_{j+1} \int_{\mathbb{R}^N} q(x)u^2 \, dx \leq \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx, \quad u \in E_2. \] (9)

(iii) There holds
\[ \lambda_j \int_{\mathbb{R}^N} q(x)u^2 \, dx \geq \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx, \quad u \in E_1. \] (10)

Proof. Initially we shall consider the proof for item (i). It is sufficient to prove that \( \lambda_1 \) given in (5) is attained. In order to do that we consider a minimizer sequence \((u_n)_{n \in \mathbb{N}} \subset E\) such that
\[ \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) \, dx \to \lambda_1 \quad \text{and} \quad \int_{\mathbb{R}^N} q(x)u_n^2 = 1. \] (11)
Clearly, \((u_n)_{n \in \mathbb{N}}\) is a bounded sequence in \( E \). Hence, up to a subsequence, there exists \( u \in E \) such that \( u_n \rightharpoonup u \) in \( E \). Now, by using that \( E \) is compact embedded into \( L^2(\mathbb{R}^N, q) \), we infer that \( u_n \to u \) in \( L^2(\mathbb{R}^N, q) \). As a consequence, we know that \( u_n(x) \to u(x) \) a.e in \( \mathbb{R}^N \) and there exists a nonnegative function \( h \in L^1(\mathbb{R}^N) \) such that \( q(x)u_n^2 \leq h \) holds for any \( n \in \mathbb{N} \). Hence, by applying Lebesgue Dominated Convergence Theorem together with (11), we get \( \int_{\mathbb{R}^N} q(x)u^2 = 1 \). In particular, we obtain that \( u \neq 0 \). Now, taking into account that the norm \( \| \cdot \| \) is weakly lower semicontinuous, we deduce that
\[ \lambda_1 \leq \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) \, dx = \lambda_1. \]
Therefore, we obtain that \( \lambda_1 \) is attained and \( \lambda_1 > 0 \). This assertion completes the proof of item (i). The proof of item (ii) follows using the same ideas discussed just above using (6) instead of (5). In this way, by using the fact that \( \dim H^1(\mathbb{R}^N) = \infty \), we obtain a sequence of eigenvalues \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \ldots \) which satisfies \( \lambda_j \to \infty \) as \( j \to \infty \), see [4, 12]. Without any loss of generality we assume that the associated eigenfunctions \( \phi_i \) are normalized in \( L^2(\mathbb{R}^N, q) \) for each \( i \in \mathbb{N} \). Furthermore, by using the expression (6), the eigenfunctions \( \phi_i \) are orthogonal both \( L^2(\mathbb{R}^N, q) \) and \( H^1(\mathbb{R}^N) \). Notice also that \( (\phi_j/\|\phi_j\|)_{j \in \mathbb{N}} \) is an orthonormal sequence in \( H^1(\mathbb{R}^N) \). It is easy to check that the sequence \( (\phi_j/\|\phi_j\|)_{j \in \mathbb{N}} \) is a Hilbert basis for \( H^1(\mathbb{R}^N) \), that is, the sequence \( (\phi_j/\|\phi_j\|)_{j \in \mathbb{N}} \) is an orthonormal total set in \( H^1(\mathbb{R}^N) \).
In fact, writing \( H^1(\mathbb{R}^N) = W \bigoplus W^\perp \) with \( W = \bigoplus_{j=1}^{\infty} H(\lambda_j) \), where we consider the finite dimensional sets
\[ H(\lambda_j) = \{ u \in H^1(\mathbb{R}^N), -\Delta u + V(x)u = \lambda_j q(x)u \ \text{in} \ \mathbb{R}^N \}. \]
It is sufficient to show that \( W^\perp = \{ 0 \} \). In order to do that we argue by contradiction assuming that \( W^\perp \neq \{ 0 \} \). Hence, using the same arguments employed just above, we can find another eigenvalue \( \lambda^* > 0 \) given by
\[ \lambda^* := \inf_{w \in W^\perp \setminus \{ 0 \}} \left\{ \int_{\mathbb{R}^N} (|\nabla w|^2 + V(x)w^2) \, dx \mid \int_{\mathbb{R}^N} q(x)w^2 = 1 \right\}. \]
Furthermore, using again the compact embedding \( E \) into \( L^2(\mathbb{R}^N, q) \) we know that \( \lambda^* > 0 \) is attained. In particular, there exists \( w \in W^\perp \setminus \{ 0 \} \) such that
\[ -\Delta w + V(x)w = \lambda^* q(x)w \ \text{in} \ \mathbb{R}^N \] holds true showing that \( \lambda^* \) is an eigenvalue which was not listed before. This is a contradiction proving that \( W^\perp = \{ 0 \} \). Hence \( H^1(\mathbb{R}^N) = \bigoplus_{j=1}^{\infty} H(\lambda_j) \) proving that \( (\phi_j/\|\phi_j\|)_{j \in \mathbb{N}} \) is a Hilbert basis for \( H^1(\mathbb{R}^N) \).
For the proof of item (iii) we write \( u = \sum_{i=1}^{j} \alpha_i \phi_i, u \in E_1 \), where \( \alpha_i \in \mathbb{R}, i = 1, \ldots, j \) and \( \phi_i \) denotes the eigenfunctions for the weighted eigenvalue problem given in (4). Using the fact that \( \phi_i \) is a weak solution for (4) and taking \( \phi_i \) as test function we obtain that

\[
\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx = \sum_{i=1}^{j} \alpha_i^2 \int_{\mathbb{R}^N} (|\nabla \phi_i|^2 + V(x)\phi_i^2) \, dx
\]

\[
= \sum_{i=1}^{j} \alpha_i^2 \lambda_i \int_{\mathbb{R}^N} q(x)\phi_i^2 \, dx
\]

\[
\leq \lambda_j \sum_{i=1}^{j} \alpha_i^2 \int_{\mathbb{R}^N} q(x)\phi_i^2 \, dx = \lambda_j \int_{\mathbb{R}^N} q(x)u^2 \, dx
\]

Here was used also the fact that the eigenfunctions \( \phi_i \) are orthogonal both \( L^2(\mathbb{R}^N, q) \) and \( H^1(\mathbb{R}^N) \). This ends the proof. \( \square \)

From a standard variational point of view the problem (2) is formally the Euler-Lagrange equation associated to the functional

\[
I_\theta(u) = \frac{1}{2} \int_{\mathbb{R}^N} (1 + 2u^2)|\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 \, dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} q(x)u^2 \, dx
\]

\[
- \frac{1}{p} \int_{\mathbb{R}^N} K(x)|u|^p \, dx - \frac{\theta}{2} \int_{\mathbb{R}^N} \Gamma(x)|u|^{2^*} \, dx.
\]

(12)

It is easy to check that any critical points \( u \in E \cap L^\infty(\mathbb{R}^N) \) for the functional \( I_\theta \) are weak solutions to the problem (2), see \([9, 33]\). However, the nonlinear term \( \int_{\mathbb{R}^N} u^2|\nabla u|^2 \, dx \) is not well defined in \( E \). More specifically, there exists \( u \in E \) in such way that the last integral is infinity. Hence, following some ideas introduced in \([9, 28]\), we reformulate the problem (2) by using the change of variable \( f : \mathbb{R} \rightarrow \mathbb{R} \) given by

\[
\begin{align*}
    f'(t) &= \frac{1}{\sqrt{1 + 2f^2(t)}}, \\
    f(0) &= 0.
\end{align*}
\]

At this moment we present the main properties of the function \( f \) given just above. These properties are very useful in the present work allowing to control the behavior the energy functional at infinity and at the origin. In order to do that we consider the following properties for \( f \):

**Lemma 2.5.** The function \( f : \mathbb{R} \rightarrow \mathbb{R} \) satisfies the following properties:

- \((f_1)\) \( f \) is uniquely determined, \( f \) is an odd function, \( f \in C^\infty \) and invertible.
- \((f_2)\) \( 0 < f'(t) \leq 1 \), for all \( t \in \mathbb{R} \).
- \((f_3)\) \( |f(t)| \leq |t| \), for all \( t \in \mathbb{R} \).
- \((f_4)\) \( \lim_{t \to 0} \frac{f(t)}{t} = 1 \).
- \((f_5)\) \( \lim_{t \to \infty} \frac{f(t)}{\sqrt{t}} = 2^{1/4} \).
- \((f_6)\) \( \frac{f(t)}{2} \leq tf'(t) \leq f(t), \text{ for all } t > 0 \).
- \((f_7)\) \( |f(t)| \leq 2^{1/4} \sqrt{|t|}, \text{ for all } t \in \mathbb{R} \).
There exists \( k > 0 \) such that

\[
|f(t)| \geq \begin{cases} 
  k|t|, & |t| \leq 1 \\
  k|t|^{1/2}, & |t| \geq 1 
\end{cases}
\]

\( f(t) \) \( t \in \mathbb{R} \) \( \leq 2^{-1/2} \) for all \( t \in \mathbb{R} \).

\( f(t) |f(t)f'(t)| \leq 2^{-1/2} \) for all \( t \in \mathbb{R} \).

\( f(t) \lim_{t \to \infty} |f(t)|^p \) \( t^2 \rightarrow \infty \) for all \( p > 4 \).

For any \( a \in [0,1] \) there holds

\[
a^2 f^2(t) \leq f^2(at) \leq a f^2(t), \quad t \in \mathbb{R}.
\]

For any \( a \in [1, \infty) \) there holds

\[
a^2 f^2(t) \leq f^2(at) \leq a f^2(t), \quad t \in \mathbb{R}.
\]

The proof of items \((f_1) - (f_{12})\) can be viewed in \([9, 13, 14, 44]\). In what follows taking into account the change of variables \( u = f(v) \) in the functional given in \((12)\) we obtain the following auxiliary functional

\[
J_\theta(v) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)f^2(v))\,dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} q(x)f^2(v)\,dx \\
- \frac{1}{p} \int_{\mathbb{R}^N} K(x)|f(v)|^p\,dx - \frac{\theta}{2\cdot 2^*} \int_{\mathbb{R}^N} \Gamma(x)|f(v)|^{2^*} \,dx
\]

which is well defined in the usual space Sobolev \( E \). Furthermore, applying hypotheses \((V_1), (q_1), (K_1), (\Gamma_1)\) and Lemma 2.5 we deduce that \( J_\theta \in C^1(E, \mathbb{R}) \).

Hence, any critical points of the functional \( J_\theta \) correspond precisely to the weak solutions for the following semilinear elliptic problem

\[
- \Delta v + V(x)f(v)f'(v) = g(x, f(v))f'(v), x \in \mathbb{R}^N, \quad v \in H^1(\mathbb{R}^N).
\]

where \( g(x,t) = \lambda q(x)t + K(x)|t|^{p-2}t + \theta \Gamma(x)|t|^{2^*-2}t \), \( x \in \mathbb{R}^N, t \in \mathbb{R} \). Under these conditions we need to ensure existence of critical points \( v \) for the functional \( J_\theta \).

It is not hard to see that \( u = f(v) \in E \cap L^\infty(\mathbb{R}^N) \) is a weak solution for \((2)\) if and only if \( v \) is a critical point for the auxiliary functional \( J_\theta \), see \([9, 33]\). Therefore, we need to find critical points \( v \in E \cap L^\infty(\Omega) \) for the functional \( J_\theta \).

3. The linking structure. In this section we shall prove the linking structure for the energy functional \( J_\theta \) given in \((13)\). The basic idea is to guarantee that \( J_\theta \) has the linking geometry described in Theorem 2.1. Furthermore, we recover some compactness properties required in variational methods. Firstly, we consider the following result:

**Proposition 1.** Suppose \((V_1), (q_1), (K_1)\) and \((\Gamma_1)\). Assume also that \( \lambda \in (\lambda_j, \lambda_{j+1}) \) for some \( j \geq 1 \). Then there exist \( \rho > 0 \) and \( \sigma > 0 \) in such way that \( J_\theta(v) \geq \sigma \) for all \( v \in \partial B_\rho \cap E_2 \).

**Proof.** Firstly, we claim that there exist \( \rho > 0 \) and \( \delta > 0 \) in such way that

\[
J_\theta(v) \geq \delta \|v\|^2, \quad \forall \ v \in \partial B_\rho \cap E_2.
\]

Indeed, arguing for contradiction, choosing \( \rho = 1/n \) we find a sequence \((v_n)_{n \in \mathbb{N}} \subset E_2 \) satisfying \( \|v_n\| = 1/n \) such that

\[
J_\theta(v_n) \leq \frac{1}{n} \|v_n\|^2, \quad \forall \ n \in \mathbb{N}.
\]
Here was used the fact that \((\nabla w_n)^2 + V(x)f^2(v_n))dx \leq \frac{\lambda}{2} \int_{\mathbb{R}^N} q(x)f^2(v_n)dx - \frac{1}{p} \int_{\mathbb{R}^N} K(x)|f(v_n)|^pdx - \frac{\theta}{2 \cdot 2^*} \int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^{2^*}dx.

(17)

Define the normalized sequence \(w_n = v_n/\|v_n\|\). Then, up to a subsequence, we see that

\[ w_n \rightharpoonup w \quad \text{weakly in } E_2, \]

\[ w_n \to w \quad \text{strongly in } L^s_{{\text{loc}}}(\mathbb{R}^N), \]

\[ w_n(x) \to w(x), \quad \text{a. e. in } \mathbb{R}^N, \]

hold true for any \(1 \leq s < 2^*.\)

Now, dividing (17) by \(\|v_n\|^2\) and using (K1) and (Γ1), we see also that

\[ \frac{1}{n} \left[ \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(x)f^2(v_n)) \frac{f^2(v_n)}{v_n^2} w_n^2 \right] dx \leq \frac{1}{n} \left[ \frac{\lambda}{2} \int_{\mathbb{R}^N} q(x)f^2(v_n)w_n^2 dx \right] - \frac{1}{p} \|K\|_\infty \int_{\mathbb{R}^N} |f(v_n)|^p w_n^2 dx - \frac{\theta}{2 \cdot 2^*} \|\Gamma\|_\infty \int_{\mathbb{R}^N} |f(v_n)|^{2^*} w_n^2 dx.

Hence, the last estimate says that

\[ \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(x)f^2(v_n)) \frac{f^2(v_n)}{v_n^2} w_n^2 dx \leq \frac{1}{n} + \frac{\theta}{2 \cdot 2^*} \|\Gamma\|_\infty \int_{\mathbb{R}^N} |f(v_n)|^{2^*} w_n^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}^N} q(x)f^2(v_n)w_n^2 dx + \frac{\|K\|_\infty}{p} \int_{\mathbb{R}^N} |f(v_n)|^p w_n^2 dx.

Furthermore, we mention that

\[ \int_{\mathbb{R}^N} |f(v_n)|^l w_n^2 dx \to 0 \quad \text{for all } 4 < l \leq 2 \cdot 2^*. \]

(18)

In fact, using the fact that \(4 < l \leq 2 \cdot 2^*\), we observe that Lemma 2.5 (f7) implies

\[ \int_{\mathbb{R}^N} |f(v_n)|^l w_n^2 dx \leq 2^{l/2} \int_{\mathbb{R}^N} |v_n|^{(l-4)/2}w_n^2 dx. \]

Notice also that \(|v_n|^{(l-4)/2} \in L^s(\mathbb{R}^N)\) for \(s = l/(l-4) > 1\). It follows from H"older's inequality that

\[ \int_{\mathbb{R}^N} |v_n|^{(l-4)/2}w_n^2 dx \leq \|v_n\|_{l/2}^{(l-4)/2} \|w_n\|_l^2 \to 0. \]

Here was used the fact that \((w_n)_{n \in \mathbb{N}}\) is bounded and \(v_n \to 0\) in \(L^s(\mathbb{R}^N)\) for any \(s \in [2, 2^*]\). This implies that (18) is now satisfied. As a consequence, using Lemma 2.5 (f4), we obtain

\[ \int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(x)f^2(v_n)) \frac{f^2(v_n)}{v_n^2} w_n^2 dx \leq \lambda \int_{\mathbb{R}^N} q(x)w_n^2 dx + o_n(1). \]
According to Lemma 2.4, by using the fact that the \((w_n)_{n \in \mathbb{N}}\) is a normalized sequence, we infer that
\[
1 + \int_{\mathbb{R}^N} V(x) \left( \frac{f^2(v_n)}{v_n^2} - 1 \right) w_n^2 dx = \int_{\mathbb{R}^N} \left( |\nabla w_n|^2 + V(x) \frac{f^2(v_n)}{v_n^2} \right) w_n^2 dx \\
\leq \lambda \frac{1}{\lambda_{j+1}} + o_n(1). \tag{19}
\]
Now, we also claim that
\[
\int_{\mathbb{R}^N} V(x) \left( \frac{f^2(v_n)}{v_n^2} - 1 \right) w_n^2 dx = o_n(1) \tag{20}
\]
Assuming the claim just above and using (19) we obtain that \(\lambda \geq \lambda_{j+1}\). This is a contradiction proving that (15) is now verified. Hence we obtain that
\[
J_\theta(v) \geq \sigma > 0, \text{ for all } v \in \partial B_\rho \cap E_2, \sigma = \delta \rho^2. \tag{21}
\]

It remains to prove the claim (20). In order to do that we choose \(\delta_1 > 0\) writing the following identity
\[
\int_{\mathbb{R}^N} V(x) \left( \frac{f^2(v_n)}{v_n^2} - 1 \right) w_n^2 dx = \int_{A_n} V(x) \left( \frac{f^2(v_n)}{v_n^2} - 1 \right) w_n^2 dx + \int_{B_n} V(x) \left( \frac{f^2(v_n)}{v_n^2} - 1 \right) w_n^2 dx \tag{22}
\]
where \(A_n := \{ x \in \mathbb{R}^N | |v_n(x)| \leq \delta_1 \}\) and \(B_n := \{ x \in \mathbb{R}^N | |v_n(x)| > \delta_1 \}\). Therefore, we show that each integrals on the right hand in the identity just above goes to zero as \(n \to \infty\). Firstly, we observe that
\[
|B_n| = o_n(1). \tag{23}
\]
Indeed, using the fact that \(v_n \to 0\) in \(L^2(\mathbb{R}^N)\), we have that
\[
|B_n| = \frac{1}{\delta_1^2} \int_{B_n} \delta_1^2 dx \leq \frac{1}{\delta_1^2} \int_{B_n} |v_n|^2 dx \to 0. \tag{24}
\]
Thus, using (V1) and Lemma 2.5 \((f_3)\), the integral over \(B_n\) is estimated as follows
\[
\left| \int_{B_n} V(x) \left( \frac{f^2(v_n)}{v_n^2} - 1 \right) w_n^2 dx \right| \leq \int_{B_n} \left| V(x) \left( \frac{f^2(v_n)}{v_n^2} - 1 \right) w_n^2 \right| dx \\
\leq 2V_\infty \int_{B_n} w_n^2 dx.
\]
Now, choosing \(r = 2^*/2\), the Hölder’s inequality and (23) imply that
\[
\int_{B_n} w_n^2 dx \leq |B_n|^{1/r'} \left( \int_{B_n} \frac{1}{w_n^{2^*}} dx \right)^{2/2^*} \to 0
\]
It follows from the last estimates that
\[
\int_{B_n} V(x) \left( \frac{f^2(v_n)}{v_n^2} - 1 \right) w_n^2 dx \to 0
\]
Now, we consider the integral over the set \(A_n\). In order to do that, using Lemma 2.5 \((f_3)\), we observe that for each \(\epsilon > 0\) there exists \(\delta > 0\) satisfying the following estimate
\[
\left| \frac{f^2(t)}{t^2} - 1 \right| \leq \epsilon, \text{ for any } |t| \leq \delta.
\]
As a consequence, taking $\delta_1 = \delta$, we infer that
\[
\left| \int_{A_n} V(x) \left( \frac{f^2(v_n)}{v_n^2} - 1 \right) w_n^2 dx \right| \leq V_\infty \int_{A_n} \left( \frac{f^2(v_n)}{v_n^2} - 1 \right) w_n^2 dx \\
\leq \epsilon V_\infty \int_{A_n} w_n^2 dx \leq \epsilon V_\infty \int_{\mathbb{R}^N} w_n^2 dx.
\]
Now, using the fact that $E$ is continuously embedded into $L^2(\mathbb{R}^N)$, we have that there exists a constant $C > 0$ such that
\[
\left| \int_{A_n} V(x) \left( \frac{f^2(v_n)}{v_n^2} - 1 \right) w_n^2 dx \right| \leq C \epsilon.
\]
Putting all these estimates together we observe that (22) implies
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} V(x) \left( \frac{f^2(v_n)}{v_n^2} - 1 \right) w_n^2 dx \leq C \epsilon.
\]
Hence, by using the fact that $\epsilon > 0$ is arbitrary, the proof of claim (20) follows. This ends the proof. \hfill \Box

Proposition 2. Suppose (V1), (q1), (K1) and (\Gamma1). Consider the following set $W_k := \langle \phi_1, \ldots, \phi_j, \phi_{j+1}, \ldots, \phi_{j+k} \rangle$ which is a subspace of $E$ with $k \geq 1$. Then for each $k$ there exists a constant $M_k > 0$ where $M_k$ does not depending of $\theta$ such that
\[
\max_{v \in W_k} J_\theta(v) \leq M_k.
\]
Proof. Firstly, by using the condition (\Gamma1), we infer that
\[
J_\theta(v) \leq J_\theta(v) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla v|^2 + V(x) f^2(v) \right) dv - \frac{\lambda}{2} \int_{\mathbb{R}^N} q(x) f^2(v) dv - \frac{1}{p} \int_{\mathbb{R}^N} K(x) |f(v)|^p dv.
\]
Let $k \in \mathbb{N}$ be an integer number satisfying $k \geq 1$. Hence is sufficient to show that there exists a positive constant $M_k$ such that
\[
\max_{v \in W_k} J_\theta(v) \leq M_k.
\]
Under these conditions, we claim that there exists $R_k > 0$ such that for all $v \in W_k$ with $\|v\| > R_k$ satisfies that $J_\theta(v) \leq 0$. Indeed, arguing by contradiction, there exists a sequence $(v_n)_{n \in \mathbb{N}} \subset W_k$ such that, up to a subsequence, $\|v_n\| \to +\infty$ as $n \to \infty$ such that $J_\theta(v_n) > 0$ for all $n \in \mathbb{N}$. Define $w_n = v_n/\|v_n\|$ for all $n \in \mathbb{N}$. Then, up to a subsequence, we obtain
\[
w_n \to w \neq 0 \quad \text{strongly in } W_k,
\]
\[
w_n \to w \quad \text{strongly in } L^s(\mathbb{R}^N),
\]
\[
w_n(x) \to w(x), \quad \text{a. e. in } \mathbb{R}^N,
\]
hold for any $2 \leq s \leq 2^*$. As $J_\theta(v_n) > 0$ for all $n \in \mathbb{N}$, we mention also that
\[
0 < \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla v_n|^2 + V(x) f^2(v_n) \right) dv - \frac{\lambda}{2} \int_{\mathbb{R}^N} q(x) f^2(v_n) dv - \frac{1}{p} \int_{\mathbb{R}^N} K(x) |f(v_n)|^p dv.
\]
Using (q1) and applying the Lemma 2.5 (f3), we obtain that
\[
0 < \frac{1}{2} \|v_n\|^2 - \frac{1}{p} \int_{\mathbb{R}^N} K(x) |f(v_n)|^p dv.
\]
Dividing (28) by $\|v_n\|^2$ we also mention that

$$0 < \frac{1}{2} - \frac{1}{p} \int_{\mathbb{R}^N} K(x) \frac{|f(v_n)|^p}{v_n^2} w_n^2 dx.$$  

Using the estimates given in (27), we can rewrite the last inequality as follows

$$\int_{|w\neq 0|} K(x) \frac{|f(v_n)|^p}{v_n^2} w_n^2 dx < \frac{p}{2}$$

As $|v_n(x)| \to +\infty$ a.e in $\mathbb{R}^N$ as $n \to \infty$, using the Fatou’s Lemma and Lemma 2.5 ($f_{10}$) in the last inequality, we reach in a contradiction. Thus, there exists $R_k > 0$ such that $J_0(v) \leq 0$ for all $v \in W_k$ with $\|v\| > R_k$. At this moment, using the continuous functional $J_0$ and the fact that $W_k$ be finite dimensional, we deduce that there exists a constant $M_k > 0$ which does not depending on $\theta$ in such way that $\max_{v \in W_k} J_0(v) \leq M_k$. Therefore, $\max_{v \in W_k} J_0(v) \leq M_k$. This the finishes the proof. □

4. The behavior of Palais-Smale sequences. In this section we shall consider the behavior of Palais-Smale sequences for our energy functional $J_\theta$ where $\theta > 0$. The main idea is to control the critical energy level for $J_\theta$ using the parameter $\theta > 0$. Firstly, we shall consider the following result:

**Proposition 3.** Suppose that ($V_1$), ($q_1$), ($K_1$) and ($\Gamma_1$) hold. Suppose also that $\lambda \in (\lambda_j, \lambda_{j+1})$ for some $j \in \mathbb{N}$ with $j \geq 1$. Then any $(PS)_c$ sequence $(v_n)_{n \in \mathbb{N}}$ for the energy functional $J_\theta$ is bounded in $E$.

**Proof.** Let $(v_n)_{n \in \mathbb{N}}$ be a $(PS)_c$ sequence for $J_\theta$. Given any $\phi \in E$ we mention that

$$\langle J_\theta'(v_n), \phi \rangle = \int_{\mathbb{R}^N} (\nabla v_n \nabla \phi + V(x)f(v_n)f'(v_n)\phi) dx - \lambda \int_{\mathbb{R}^N} q(x)f(v_n)f'(v_n)\phi dx$$

$$- \int_{\mathbb{R}^N} K(x)|f(v_n)|^{p-2} f(v_n)f'(v_n)\phi dx$$

$$- \theta \int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^{2^* - 2} f(v_n)f'(v_n)\phi dx.$$  

Applying the Lemma 2.5 we can check that $f(v_n)/f'(v_n) \in E$. This can be done using Lemma 2.5 together the fact that $f(t) > 0$ for each $t > 0$. More precisely, we observe that

$$\left\| \frac{f(v)}{f'(v)} \right\| \leq C \|v\|, \quad v \in E,$$

holds true for some constant $C > 0$. Hence, choosing $\phi = f(v_n)/f'(v_n)$ as a testing function, we get

$$\langle J_\theta'(v_n), \frac{f(v_n)}{f'(v_n)} \rangle \leq 2 \int_{\mathbb{R}^N} |\nabla v_n|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)f^2(v_n)dx - \lambda \int_{\mathbb{R}^N} q(x)f^2(v_n)dx$$

$$- \int_{\mathbb{R}^N} K(x)|f(v_n)|^p dx - \theta \int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^{2^* - 2} dx.$$  

(29)

As a consequence, taking into account that $(v_n)_{n \in \mathbb{N}}$ is a $(PS)_c$ sequence, we infer that

$$C_1 + o_n(1) \|v_n\| \geq J_\theta(v_n) - \frac{1}{\gamma} \left\langle J_\theta'(v_n), \frac{f(v_n)}{f'(v_n)} \right\rangle$$
\[ \geq \left( \frac{1}{2} - \frac{2}{\gamma} \right) \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)f^2(v_n)) \, dx \]
\[
- \lambda \left( \frac{1}{2} - \frac{1}{\gamma} \right) \int_{\mathbb{R}^N} q(x)f^2(v_n) \, dx
- \theta \left( \frac{1}{2} - 2\gamma \right) \int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^{1+2\gamma} \, dx,\]

holds true for any \( 4 < \gamma < p \) where \( C > 0 \) is a constant. Thus, we rewrite the last estimate as follows
\[
C_1 + o_n(1)\|v_n\| \geq \left( \frac{1}{2} - \frac{2}{\gamma} \right) \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)f^2(v_n)) \, dx
- \lambda \left( \frac{1}{2} - \frac{1}{\gamma} \right) \int_{\mathbb{R}^N} q(x)f^2(v_n) \, dx.
\]

Here was used one more time that \( 4 < \gamma < p \) and \( \Gamma(x) > 0 \) for each \( x \in \mathbb{R}^N \).

From now on, we argue by contradiction. Suppose that, up to a subsequence, we have \( \|v_n\| \to \infty \) as \( n \to \infty \). Define the normalized sequence \( w_n = v_n/\|v_n\| \) for all \( n \in \mathbb{N} \). Therefore, \( \|w_n\| = 1 \) showing the following statements
\[
w_n \rightharpoonup w \ \text{weakly in } E, \\
w_n \to w \ \text{strongly in } L^s_{\text{loc}}(\mathbb{R}^N), \\
w_n(x) \to w(x), \ \text{a. e. in } \mathbb{R}^N, 
\]
for any \( 1 \leq s < 2^*. \) Using (30) and dividing by \( \|v_n\|^2 \), we obtain also that
\[
\int_{\mathbb{R}^N} \left( |\nabla w_n|^2 + \frac{V(x)f^2(v_n)}{\|v_n\|^2} \right) \, dx \leq C \int_{\mathbb{R}^N} q(x)\frac{f^2(v_n)}{\|v_n\|^2} \, w_n^2 \, dx + o_n(1)
\]
where \( C > 0 \). Now we claim that
\[
\int_{\mathbb{R}^N} q(x)\frac{f^2(v_n)}{\|v_n\|^2} \, w_n^2 \, dx = o_n(1).
\]

At this stage, assuming the claim given just above, it follows from (31) that
\[
\int_{\mathbb{R}^N} \left( |\nabla w_n|^2 + \frac{V(x)f^2(v_n)}{\|v_n\|^2} \right) \, dx = o_n(1).
\]

As a consequence, we obtain the following limits
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{V(x)f^2(v_n)}{\|v_n\|^2} \, dx = 0.
\]

In particular, using the norm in \( D^{1,2}(\mathbb{R}^N) \) is weakly lower semicontinuous, we have that \( w = 0 \). However, we recall that \( (w_n)_{n \in \mathbb{N}} \) is a normalized sequence. Hence applying (33) we get
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V(x)w_n^2 \, dx = 1.
\]

Now, we consider the following sets
\[
A_n := \{ x \in \mathbb{R}^N \mid |w_n| \leq 1 \} \quad \text{and} \quad B_n := \{ x \in \mathbb{R}^N \mid |w_n| > 1 \}.
\]

Then, we write
\[
\int_{\mathbb{R}^N} V(x)w_n^2 \, dx = \int_{A_n} V(x)w_n^2 \, dx + \int_{B_n} V(x)w_n^2 \, dx.
\]
The main idea is to prove that each integral on the right side in the identity just above goes to zero as \( n \to \infty \). Firstly, using that \((v_n)_{n \in \mathbb{N}}\) is a \((PS)_c\) sequence for functional \( J_\theta \), taking into account hypotheses \((K_1), (\Gamma_1)\) and \((29)\) we obtain that

\[
C_1 + o_n(1)\|v_n\| \geq J_\theta(v_n) - \frac{1}{4} \left\langle J_\theta'(v_n), \frac{f(v_n)}{f'(v_n)} \right\rangle \geq \frac{1}{4} \int_{\mathbb{R}^N} V(x)f^2(v_n)dx - \frac{\lambda}{4} \int_{\mathbb{R}^N} q(x)f^2(v_n)dx.
\]

The last inequality can be rewritten in the following form

\[
\int_{\mathbb{R}^N} V(x)f^2(v_n)dx \leq \lambda \int_{\mathbb{R}^N} q(x)f^2(v_n)dx + C_2 + o_n(1)\|v_n\|\tag{36}
\]

for some positive constant \( C_2 > 0 \). Dividing \((36)\) by \( \|v_n\| \) and using the Lemma 2.5 \((f_7)\), we mention that

\[
\int_{\mathbb{R}^N} \frac{V(x)f^2(v_n)}{\|v_n\|}dx \leq 2^{1/2}\lambda \int_{\mathbb{R}^N} q(x)|w_n|dx + o_n(1)\tag{37}
\]

Using the condition \( (q_1) \) and the continuous embedding of \( E \) into \( L^{\beta'}(\mathbb{R}^N) \), where \( 2N/(N+2) < \beta \leq 2 \), we deduce that \( w_n \to 0 \) in \( L^{\beta'}(\mathbb{R}^N) \) as \( n \to \infty \). Thus, we obtain the following convergence

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} q(x)|w_n|dx = 0.
\]

Hence, by using the previous convergence and \((37)\), we also obtain that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{V(x)f^2(v_n)}{\|v_n\|}dx = 0. \tag{38}
\]

From now on, looking for the estimates around the sets \( A_n \) and \( B_n \), we assume without any loss of generality that \( \|v_n\| > 1 \) for all \( n \in \mathbb{N} \). Thus, applying the Lemma 2.5, \((f_8)\) and \((f_{11})\), and \((38)\), we get

\[
0 \leq \int_{A_n} V(x)w_n^2dx \leq \frac{1}{k} \int_{A_n} V(x)f^2(w_n)dx \leq \frac{1}{k} \int_{\mathbb{R}^N} V(x)f^2(v_n)dx \to 0 \tag{39}
\]

as \( n \to \infty \). Now, using the condition \( (V_1) \), continuous embedding of \( D^{1,2}(\mathbb{R}^N) \) into \( L^{2^*}(\mathbb{R}^N) \) and \((33)\), we have that

\[
0 \leq \int_{B_n} V(x)w_n^2dx \leq V_\infty \int_{B_n} |w_n|^{2^*}dx \leq V_\infty \int_{\mathbb{R}^N} |w_n|^{2^*} \leq C \left( \int_{\mathbb{R}^N} |\nabla w_n|^2dx \right)^{2^*/2} \to 0 \tag{40}
\]

as \( n \to \infty \). Finally, putting together \((39)\), \((40)\) and \((35)\), we infer that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V(x)w_n^2dx = 0.
\]

This is a contradiction with \((34)\) proving that the sequence \((v_n)_{n \in \mathbb{N}}\) is now bounded.

It remains to prove the claim \((32)\). In order to do that, we split the proof for this claim into two cases. In the first one we consider the case \( w \equiv 0 \). Under this condition we mention also that

\[
0 \leq \int_{\mathbb{R}^N} q(x)\frac{f^2(v_n)}{|v_n|^2}w_n^2dx \leq \int_{\mathbb{R}^N} q(x)w_n^2dx = o_n(1).
\]
Here was used the fact that the embedding of $E$ into $L^2(\mathbb{R}^N, q)$ is compact, see Lemma 2.2. For the second case we assume that $||w \neq 0|| > 0$. As a consequence, we observe that $|v_n(x)| = |w_n(x)||v_n| \to \infty$ in $|w \neq 0|$. Hence, applying Lemma 2.5 ($f_2$), we infer also that

$$\lim_{n \to \infty} \frac{f^2(v_n)}{v_n^2} = 0 \text{ a. e. in } [w \neq 0].$$

Using one more time Lemma 2.2, we have that $w_n \to w$ in $L^2(\mathbb{R}^N, q)$. Thus, we observe that $w_n(x) \to w(x)$ a. e. in $\mathbb{R}^N$. Furthermore, there exists $\phi : \mathbb{R}^N \to \mathbb{R}$ with $\phi \in L^2(\mathbb{R}^N, q)$ such that $q(x)w_n^2 \leq q(x)\phi^2$ in $\mathbb{R}^N$. According to Lemma 2.5 ($f_3$) we also mention that

$$q(x)\frac{f^2(v_n)}{|v_n|^2}w_n^2 \leq q(x)\phi^2 \text{ and } q\phi^2 \in L^1(\mathbb{R}^N).$$

Thus, applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\int_{\mathbb{R}^N} q(x)\frac{f^2(v_n)}{|v_n|^2}w_n^2 dx = o_n(1)$$

Putting all these estimates together the proof for the desired claim follows. This ends the proof. \hfill \Box

**Lemma 4.1.** For all constant $M > 0$ there exists $\Lambda > 0$ such that

$$F_M := \{v \in E, \ J_\theta(v) \leq M \text{ and } J'_\theta(v) = 0\}$$

is bounded by constant $\Lambda$ which does not depend of $\theta$, that is, $\|v\| \leq \Lambda$ for each $u \in F_M$.

**Proof.** The proof follows arguing by contradiction. Assume that there exists a positive constant $M$ such that the set $F_M$ is unbounded. Thus, we choose a sequence $(v_n)_{n \in \mathbb{N}} \subset F_M$ such that, up to a subsequence, $\|v_n\| \to +\infty$ as $n \to \infty$. For each $\gamma \in [4, p)$ we deduce that

$$M \geq J_\theta(v_n) - \frac{1}{\gamma} \left\langle J'_\theta(v_n), \frac{f(v_n)}{f'(v_n)} \right\rangle$$

$$\geq \left(\frac{1}{2} - \frac{2}{\gamma}\right) \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)f^2(v_n)) \, dx - \lambda \left(\frac{1}{2} - \frac{1}{\gamma}\right) \int_{\mathbb{R}^N} q(x)f^2(v_n) \, dx$$

$$- \left(\frac{1}{p} - \frac{1}{\gamma}\right) \int_{\mathbb{R}^N} K(x)|f(v_n)|^p \, dx - \theta \left(\frac{1}{2} - \frac{2}{p} - \frac{1}{\gamma}\right) \int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^2 \, dx.$$

Using hypotheses $(K_1)$ and $(\Gamma_1)$, we rewrite the last inequality as follows

$$M \geq \left(\frac{1}{2} - \frac{2}{\gamma}\right) \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)f^2(v_n)) \, dx - \lambda \left(\frac{1}{2} - \frac{1}{\gamma}\right) \int_{\mathbb{R}^N} q(x)f^2(v_n) \, dx.$$

Dividing the last estimate by $\|v_n\|^2$ and arguing as was done in the proof of Proposition 3 we observe that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx = 0 \text{ and } \lim_{n \to \infty} \int_{\mathbb{R}^N} V(x)f^2(v_n) \, dx = 0,$$

where $w_n = v_n/\|v_n\|$ for all $n \in \mathbb{N}$. In particular, using the fact that the norm in $D^{1,2}(\mathbb{R}^N)$ is weakly lower semicontinuous, we have that $w = 0$. One more time,
using the same ideas discussed in the proof of Proposition 3, we obtain that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V(x) w_n^2 \, dx = 0. \tag{42}
\]
This is a contradiction due the fact the sequence \((w_n)_{n \in \mathbb{N}}\) is normalized. Hence the set \(F_M\) is bounded for all \(M > 0\).

**Remark 1.** It is easy to verify that Lemma 4.1 ensures that there exists a constant \(\hat{\Lambda} > 0\) such that
\[
\int_{\mathbb{R}^N} q(x) f^2(v) \, dx \leq \hat{\Lambda}
\]
holds for all \(v \in F_M\).

**Lemma 4.2.** Suppose that \((V_1), (q_1), (K_1)\) and \((\Gamma_1)\) are satisfied. Let \((v_n)_{n \in \mathbb{N}}\) be a \((PS)_c\) sequence for \(J_\theta\) where \(c < M_k\) where \(M_k\) was obtained by Proposition 2. Assume also that \(v_n \rightharpoonup v\) in \(E\) as \(n \to \infty\). Then there exists \(\theta_k > 0\) such that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^{2^*} \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} \Gamma(x)|f(v)|^{2^*} \, dx \tag{43}
\]
holds true for all \(\theta \in (0, \theta_k)\).

**Proof.** Fix \(R > 0\) which will be chosen later. It is easy to see that
\[
\int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^{2^*} \, dx = \int_{B_R} \Gamma(x)|f(v_n)|^{2^*} \, dx + \int_{\mathbb{R}^N \setminus B_R} \Gamma(x)|f(v_n)|^{2^*} \, dx. \tag{44}
\]
Now we prove that \(\Gamma(x)|f(v_n)|^{2^*} \to \Gamma(x)|f(v)|^{2^*}\) strongly in \(L^1(B_R)\) as \(n \to \infty\). Note that, using the fact of \(v_n \rightharpoonup v\), we obtain
\[
\begin{align*}
v_n & \to v \quad \text{strongly in } L^s(B_R), \\
v_n(x) & \to v(x) \quad \text{a.e. in } B_R, \\
f(v_n(x)) & \to f(v(x)) \quad \text{a.e. in } B_R,
\end{align*} \tag{45}
\]
hold for any \(1 \leq s < 2^*\). Applying Lions Concentration Compactness Principle [23] to \((f^2(v_n))_{n \in \mathbb{N}}\) on \(B_R\), that there exist two nonnegative measures denoted by \(\mu\) and \(\nu\) and an enumerable index set \(\mathcal{K}\), positive constants \(\{\mu_k\}_{k \in \mathcal{K}}\) and \(\{\nu_k\}_{k \in \mathcal{K}}\) and a collection points \(\{x_k\}_{k \in \mathcal{K}} \subset B_R\) such that

1. \(\Gamma(x)\nu = \Gamma(x)|f(v)|^{2^*} + \sum_{k \in \mathcal{K}} \Gamma(x_k)\nu_k \delta_{x_k}\);
2. \(\mu \geq |\nabla (f^2(v))|^2 + \sum_{k \in \mathcal{K}} \mu_k \delta_{x_k}\);
3. \(\mu_k \geq S(\nu_k)^{2^*/2}\),

where \(\delta_{x_k}\) is the Dirac measure at \(x_k\). The constant \(S > 0\) is the best constant of Sobolev for embedding of \(D^{1,2}(\mathbb{R}^N)\) into \(L^{2^*}(\mathbb{R}^N)\).

At this stage, we claim that \(\nu_k = 0\) for all \(k \in \mathcal{K}\). Indeed, let \(x_k\) be a singular point of the measures \(\mu\) and \(\nu\). Define the cutoff function \(\phi \in C^\infty_0(\mathbb{R}^N)\) given by
\[
\phi(x) = \begin{cases} 
1 \text{ in } B_p(x_k), \\
0 \text{ in } \mathbb{R}^N \setminus B_{2p}(x_k), \\
\phi \geq 0, \quad |\nabla \phi| \leq 1/p \text{ in } B_{2p}(x_k) \setminus B_p(x_k),
\end{cases} \tag{46}
\]
where $B_{\rho}(x_k)$ is the ball in $\mathbb{R}^N$ centered in $x_k$ with radius $\rho > 0$. Under these conditions, we consider the test functions $\psi = \phi(v_n)/f'(v_n)$ proving that

$$
\langle J'_\theta(v_n), \psi \rangle = \int_{\mathbb{R}^N} \nabla v_n \nabla \psi + V(x)f^2(v_n)\phi dx - \lambda \int_{\mathbb{R}^N} q(x)f^2(v_n)\phi dx - \int_{\mathbb{R}^N} K(x)|f(v_n)|^p\phi dx - \theta \int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^{2-2*}\phi dx
$$

$$
\int_{\mathbb{R}^N} \frac{f(v_n)}{f'(v_n)} \nabla v_n \nabla \phi dx + \int_{\mathbb{R}^N} (1 + 2f^2(v_n)f'(v_n)^2)|\nabla v_n|^2\phi dx + \int_{\mathbb{R}^N} V(x)f^2(v_n)\phi dx - \lambda \int_{\mathbb{R}^N} q(x)f^2(v_n)\phi dx - \int_{\mathbb{R}^N} K(x)|f(v_n)|^p\phi dx - \theta \int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^{2-2*}\phi dx.
$$

(47)

By definition of the function $\psi$ and using the Lemma 2.5 there exists a constant $C > 0$ in such way that $\|\psi\| \leq C\|v_n\|$. Using this and the fact that $(v_n)_{n \in \mathbb{N}}$ be a $(PS)_c$ sequence we mention also that $\langle J'_\theta(v_n), \psi \rangle = o_n(1)\|v_n\|$. Thus, we rewrite (47) as follows

$$
o_n(1)\|v_n\| = \int_{\mathbb{R}^N} \frac{f(v_n)}{f'(v_n)} \nabla v_n \nabla \phi dx + \int_{\mathbb{R}^N} (1 + 2f^2(v_n)f'(v_n)^2)|\nabla v_n|^2\phi dx + \int_{\mathbb{R}^N} V(x)f^2(v_n)\phi dx - \lambda \int_{\mathbb{R}^N} q(x)f^2(v_n)\phi dx - \int_{\mathbb{R}^N} K(x)|f(v_n)|^p\phi dx - \theta \int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^{2-2*}\phi dx.
$$

As a consequence, by using Lions Concentration Compactness Principle, we infer that

$$
\int_{B_R} |\nabla f^2(v_n)|^2\phi dx \to \int_{B_R} \phi d\mu
$$

and

$$
\int_{B_R} \Gamma(x)|f(v_n)|^{2-2*}\phi dx \to \int_{B_R} \Gamma(x)\phi d\nu \text{ as } n \to \infty. (49)
$$

At this moment, as $x_k$ is a singular point of measure $\nu$ and taking into account the continuity of $f$, we also infer that $\|f(v_n)|_{B_{2\rho}(x_k) \setminus \{x_k\}} \to \infty$ as $\rho \to 0$. Thus, we write $1 + 2f^2(v_n)f'(v_n)^2 = 2 + o(\rho)$ on $B_{2\rho}(x_k)$ with $\rho$ small enough. Applying Lemma 2.5 $(f_9)$ we observe that $|\nabla f^2(v)|^2 \leq 2|\nabla v|^2$ for all $v \in E$. Using the previous estimates together with (47) and (48), (49) follows that

$$
\int_{B_R} \phi d\mu - \theta \int_{B_R} \Gamma(x)\phi d\nu
$$

$$
= \lim_{n \to \infty} \left\{ \int_{B_R} |\nabla f^2(v_n)|^2\phi dx - \theta \int_{B_R} \Gamma(x)|f(v_n)|^{2-2*}\phi dx \right\}
$$

$$
\leq \lim_{n \to \infty} \left\{ \int_{B_R} 2|\nabla v_n|^2\phi dx - \theta \int_{B_R} \Gamma(x)|f(v_n)|^{2-2*}\phi dx \right\}
$$

$$
\leq \lim_{n \to \infty} \int_{B_R} (1 + 2f^2(v_n)f'(v_n)^2 - o(\rho))|\nabla v_n|^2\phi dx
$$

$$
- \theta \lim_{n \to \infty} \int_{B_R} \Gamma(x)|f(v_n)|^{2-2*}\phi dx
$$
\[
= \lim_{n \to \infty} \left\{ - \int_{B_R} \frac{f'(v_n)}{f''(v_n)} \nabla v_n \nabla \phi \, dx - \int_{B_R} V(x) f^2(v_n) \phi \, dx \right\} \\
+ \lim_{n \to \infty} \left\{ \lambda \int_{B_R} q(x) f^2(v_n) \phi \, dx + \int_{B_R} K(x) |f(v_n)|^p \phi \, dx \right\} \\
+ \lim_{n \to \infty} \left\{ -o(\rho) \int_{B_R} 2 |\nabla v_n|^2 \phi \, dx + o_n(1) \|v_n\phi\| \right\}
\]

(50)

From now on, we shall prove that each the term given in the right side in (50) goes to zero as \( \rho \to 0 \). Firstly, we observe that \((f(v_n)/f''(v_n)) \nabla \phi \in L^2(\mathbb{R}^N) \). Indeed, using Lemma 2.5, \((f_3)\) and \((f_7)\), we have

\[
|\frac{f(v_n)}{f''(v_n)} \nabla \phi|^2 = \left( f^2(v_n) + 2f^4(v_n) \right) |\nabla \phi|^2 \leq C \rho^{-2} |v_n|^2
\]

for some constant \( C > 0 \). Therefore, \((f(v_n)/f''(v_n)) \nabla \phi \in L^2(\mathbb{R}^N)\). Hence

\[
\left( \int_{B_R} \left| \frac{f'(v_n)}{f''(v_n)} \nabla \phi \right|^2 \, dx \right)^{1/2} \leq C_1 \rho^{-1} \left( \int_{B_R} |v_n|^2 \, dx \right)^{1/2}.
\]

Doing the change variable \( y = \rho R^{-1} x + x_k \) with \( x \in B_R \), we obtain that

\[
\left( \int_{B_R} |v_n|^2 \, dx \right)^{1/2} = \rho^{N/2} R^{-N/2} \left( \int_{B_{\rho}(x_k)} |v_n|^2 dy \right)^{1/2}.
\]

(51)

Using (51) and Hölder’s inequality it follows that

\[
\lim_{\rho \to 0} \left( \lim_{n \to \infty} \int_{B_R} \left| \frac{f'(v_n)}{f''(v_n)} \nabla v_n \nabla \phi \right| \, dx \right) = 0.
\]

(52)

Now, by using using \((V_1)\), Lemma 2.5 \((f_7)\) together with Hölder’s Inequality, we obtain also that

\[
\int_{B_R} V(x) f^2(v_n) \phi \, dx \leq 2^{1/2} V_{\infty} \int_{B_R} |v_n| \phi \, dx \\
\leq 2^{1/2} V_{\infty} \left( \int_{B_R} |v_n|^2 \right)^{1/2} \left( \int_{B_R} |\phi|^2 \, dx \right)^{1/2}.
\]

(53)

Doing the variable change \( y = 2\rho R^{-1} x + x_k \), \( x \in B_R \), we infer that

\[
\left( \int_{B_R} |\phi|^2 \, dx \right)^{1/2} = 2^{N/2} \rho^{N/2} R^{-N/2} \left( \int_{B_{2\rho}(x_k)} |\phi|^2 \, dy \right)^{1/2} \to 0
\]

(54)
as \( \rho \to 0 \). Using that \((v_n)_{n \in \mathbb{N}}\) is bounded and applying (54) in (53) we deduce that

\[
\lim_{\rho \to 0} \left( \lim_{n \to \infty} \int_{B_R} V(x) f^2(v_n) \phi \, dx \right) = 0.
\]

(55)
One more time, by using again Lemma 2.5 \((f_7)\), Hölder’s Inequality and the variable change \(y = 2\rho R^{-1} x + x_k\), we infer also that

\[
\int_{B_R} q(x)f^2(v_n)\phi dx \leq 2^{1/2} \int_{B_R} q(x)|v_n|^2 \phi dx
\]

\[
\leq 2^{1/2} ||q||_\alpha \left( \int_{B_R} |v_n|^{2\alpha^*} dx \right)^{1/2\alpha^*} \left( \int_{B_R} |\phi|^{2\alpha^*} dx \right)^{1/2\alpha^*}
\]

\[
\leq 2^{1/2} ||q||_\alpha \left( \int_{\mathbb{R}^N} |v_n|^{2\alpha^*} dx \right)^{1/2\alpha^*} \left( \int_{B_R} |\phi|^{2\alpha^*} dx \right)^{1/2\alpha^*}
\]

\[
\leq C_\rho^{N/2\alpha^*} ||q||_\alpha ||v_n||^{2\alpha^*} \left( \int_{B_{2\rho}(x_k)} |\phi|^{2\alpha^*} dx \right)^{1/2\alpha^*}
\]

where \(C > 0\) is constant which does not depend of \(n\) and \(\rho\) where \(1/\alpha + 1/\alpha^* = 1\). Using that \((v_n)_{n \in \mathbb{N}}\) is bounded and applying the estimates above we observe that

\[
\lim_{n \to \infty} \left( \lim_{\rho \to 0} \int_{B_R} q(x)f^2(v_n)\phi dx \right) = 0.
\]

(56)

Furthermore, by using Lemma 2.5 \((f_7)\), \((K_1)\), Hölder’s inequality and variable change \(y = 2\rho R^{-1} x + x_k\), we infer that

\[
\int_{B_R} K(x)|f(v_n)|^p \phi dx \leq 2^{p/4} \|K\|_\infty \int_{B_R} |v_n|^{p/2} \phi dx
\]

\[
\leq 2^{p/4} \|K\|_\infty \left( \int_{B_R} |v_n|^{2^*} dx \right)^{p/2 \cdot 2^*} \left( \int_{B_R} |\phi|^{r^*} dx \right)^{1/r^*}
\]

\[
\leq C_1 \rho^{N/r^*} \left( \int_{\mathbb{R}^N} |v_n|^{2^*} dx \right)^{p/2 \cdot 2^*} \left( \int_{B_{2\rho}(x_k)} |\phi|^{r^*} dx \right)^{1/r^*}
\]

where \(C_1 > 0\) is a constant that does not depend of \(n\) and \(\rho\). Using again that \((v_n)_{n \in \mathbb{N}}\) is bounded the previous estimates ensure that

\[
\lim_{n \to \infty} \left( \lim_{\rho \to 0} \int_{B_R} K(x)|f(v_n)|^p \phi dx \right) = 0.
\]

(57)

Since \(0 \leq \phi \leq 1, x \in \mathbb{R}^N\) and \((v_n)_{n \in \mathbb{N}}\) is bounded we also observe that

\[
\lim_{\rho \to 0} \left( o(\rho) \int_{B_R} |\nabla v_n|^2 \phi dx \right) = 0.
\]

(58)

Finally, putting together the estimates (52), (55),(56), (57) and (58) in (50), we deduce that

\[
\lim_{\rho \to 0} \left( \int_{B_R} \phi d\mu - \Theta \int_{B_R} \Gamma(x) \phi d\nu \right) \leq 0.
\]

As a consequence, we obtain that \(\mu_k - \Theta \Gamma(x_k) \nu_k \leq 0\). Using (III), we infer also that \(S\nu_k^{2^*/2^*} \leq \mu_k \leq \Theta \Gamma(x_k)|\nu_k|\). Therefore, we see that \(\nu_k \left( \Theta \Gamma(x_k) - S\nu_k^{2^*/2^*} \right) \geq 0\). It follows from the last estimate that \(\nu_k = 0\) or \(\Theta \Gamma(x_k) \geq S\nu_k^{2^*/N}\). Assume that \(\Theta \Gamma(x_k) > S\nu_k^{2^*/N}\) holds. Then
\[ \nu_k \geq S^{N/2} / (\theta \Gamma(x_k))^{N/2} \]. Hence, applying Fatou’s Lemma, we observe that
\[
M_k \geq \lim_{n \to \infty} \left( J_{\theta}(v_n) - \frac{1}{4} \left\langle J_{\theta}(v_n), \frac{f(v_n)}{f'(v_n)} \right\rangle \right) \\
\geq -\frac{\lambda}{4} \int_{\mathbb{R}^N} q(x) f^2(v) dx + \theta \frac{1}{2N} \Gamma(x_k) \nu_k \\
\geq -\frac{\lambda}{4} \int_{\mathbb{R}^N} q(x) f^2(v) dx + \frac{1}{2N} \frac{S^{N/2}}{(\theta \Gamma(x_k))^{(N-2)/2}} \\
\geq -\frac{\lambda}{4} \int_{\mathbb{R}^N} q(x) f^2(v) dx + \frac{1}{2N} \frac{S^{N/2}}{(\theta \| \Gamma \|_{\infty})^{(N-2)/2}}.
\]

Now, taking into account Remark 1, using the estimates above we infer that
\[
M_k \geq -\frac{\lambda \Lambda}{4} + \frac{1}{2N} \frac{S^{N/2}}{\theta \| \Gamma \|_{\infty}^{(N-2)/2}}.
\] (59)

Define the following positive number
\[
\theta_k := \left[ \frac{2N \| \Gamma \|_{(N-2)/2}^{(N-2)/2}}{S^{N/2}} \left( M_k + \frac{\lambda \Lambda}{4} \right) \right]^{2/(2-N)}.
\]

As a consequence, by taking \( \theta \in (0, \theta_k) \) we reach in a contradiction with (59). Therefore, the last statement says that \( \nu_k = 0 \), holds true for all \( k \in \mathcal{K} \) provided that \( \theta \in (0, \theta_k) \). Under these conditions, we see that
\[
\lim_{n \to \infty} \int_{B_R} \Gamma(x)|f(v_n)|^{2-2^*} dx = \lim_{n \to \infty} \int_{B_R} \Gamma(x)|f(v)|^{2-2^*} dx.
\] (60)

Now, by using hypothesis (\( \Gamma_1 \)), for each \( \epsilon > 0 \) we choose \( R > 0 \) in such a way that
\[
\int_{\mathbb{R}^N \setminus B_R} \Gamma(x)|f(v_n)|^{2-2^*} dx \leq \epsilon
\]
As a consequence, by using Fatou’s Lemma, we infer also that
\[
\int_{\mathbb{R}^N \setminus B_R} \Gamma(x)|f(v)|^{2-2^*} dx \leq \epsilon.
\]

Therefore, by using the previous estimate and (60), we see that
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^{2-2^*} dx \leq \int_{\mathbb{R}^N} \Gamma(x)|f(v)|^{2-2^*} dx + \epsilon
\]
holds true for all \( \epsilon > 0 \). This ends the proof.

**Lemma 4.3.** Suppose that \((V_1), (q_1), (K_1)\) and \((\Gamma_1)\) are satisfied. Let \( (v_n)_{n \in \mathbb{N}} \) be a Palais-Smale Sequence at the level \( c < M_k \) for functional \( J_{\theta} \). Assume also that \( v_n \rightharpoonup v \) weakly in \( E \) and \( \theta \in (0, \theta_k) \) where \( \theta_k \) is given by Lemma 4.2. Then we obtain the following statements:

(i) Given \( \epsilon > 0 \), there exist \( R_\epsilon > 0 \) such that for all \( R > R_\epsilon \)
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R} (|\nabla v_n|^2 + V(x)f^2(v_n)) dx \leq \epsilon.
\] (61)
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R} (|\nabla v_n|^2 + V(x)f'(v_n)f(v_n)) dx \leq \epsilon.
\] (62)
(ii) There holds
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V(x)f^2(v_n)dx = \int_{\mathbb{R}^N} V(x)f^2(v)dx. \quad (63)
\]

(iii) The sequence \((v_n)_{n \in \mathbb{N}}\) converges strongly to \(v\) in \(E\). In other words, the functional \(J_0\) satisfies the Palais-Smale condition.

Proof. Define the cutoff function \(\phi \in C^\infty(\mathbb{R}^N)\) given by
\[
\phi(x) = \begin{cases} 
1, & \text{in } \mathbb{R}^N \setminus B_{2R_1}, \\
0, & \text{in } B_{R_1}, \\
\phi \geq 0, & \text{in } B_{2R_1} \setminus B_{R_1},
\end{cases} \quad (64)
\]
where \(R_1 > 0\) will be chosen later. Now, we choose as test function \(\psi = \phi f(v_n)/f'(v_n)\). Thus
\[
\langle J_0'(v_n), \psi \rangle = \int_{\mathbb{R}^N} \frac{f(v_n)}{f'(v_n)} \nabla v_n \nabla \phi dx + \int_{\mathbb{R}^N} \left(1 + 2f^2(v_n)f'(v_n)^2\right) |\nabla v_n|^2 \phi dx
\]
\[
+ \int_{\mathbb{R}^N} V(x)f^2(v_n)\phi dx - \lambda \int_{\mathbb{R}^N} q(x)f^2(v_n)\phi dx
\]
\[
- \int_{\mathbb{R}^N} K(x)|f(v_n)|^p\phi dx - \theta \int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^{2-2^*} \phi dx.
\]
(65)

As \((v_n)_{n \in \mathbb{N}}\) is a bounded (PS)_c sequence for functional \(J_0\) we observe that \(\langle J_0'(v_n), \psi \rangle = o_n(1)\). Using (q1), (K1) and (\(\Gamma_1\)), for any \(\epsilon > 0\) there exists \(R_\epsilon > 0\) such that for all \(R > R_\epsilon\) there hold
\[
\int_{\mathbb{R}^N \setminus B_R} q(x)f^2(v_n)dx \leq \frac{\epsilon}{4}, \quad \int_{\mathbb{R}^N \setminus B_R} K(x)|f(v_n)|^p dx \leq \frac{\epsilon}{4}
\]
and
\[
\int_{\mathbb{R}^N \setminus B_R} \Gamma(x)|f(v_n)|^{2-2^*} dx \leq \frac{\epsilon}{4}.
\]
(66)

At the same time, by using Lemma 2.5 \((f_3), (f_7)\) and \((64)\), we deduce the following estimates
\[
\left| \frac{f(v_n)}{f'(v_n)} \nabla \phi \right|^2 \leq \left| \frac{f(v_n)}{f'(v_n)} \right|^2 |\nabla \phi|^2 \leq \frac{C_2}{R_1^2} |v_n|^2 \quad (68)
\]
where \(C_2 > 0\) does not depend on \(n\) and \(R_1\). It follows from the last estimate that
\[
\left( \int_{\mathbb{R}^N} \left| \frac{f(v_n)}{f'(v_n)} \nabla \phi \right|^2 dx \right)^{1/2} \leq \frac{C_2^{1/2}}{R_1} \left( \int_{\mathbb{R}^N} |v_n|^2 dx \right)^{1/2}.
\]
(69)

Using (69) and Hölder’s inequality we have that
\[
\int_{\mathbb{R}^N} \left| \frac{f(v_n)}{f'(v_n)} \nabla v_n \nabla \phi \right| dx \leq \frac{C_2^{1/2}}{R_1} \left( \int_{\mathbb{R}^N} |v_n|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \right)^{1/2}.
\]

Under these conditions, by using the fact that \((v_n)_{n \in \mathbb{N}}\) is bounded in \(E\), we choose \(R_1\) in such a way that \(R_1 > R_\epsilon\) where \(R_\epsilon\) given in (66) satisfying
\[
\int_{\mathbb{R}^N} \left| \frac{f(v_n)}{f'(v_n)} \nabla v_n \nabla \phi \right| dx \leq \frac{\epsilon}{4}.
\]
(70)
As a consequence, putting together (64), (65), (66), (67) and (70), we deduce also that
\[
\int_{\mathbb{R}^N \setminus B_{2R_1}} (|\nabla v_n|^2 + V(x) f^2(v_n)) \, dx \leq \epsilon + o_n(1).
\]
Therefore, for each $R > 2R_1$ the estimate given in (61) is now satisfied.

Now, we shall prove the convergence given in (62). One more time, we consider the cutoff function $\phi$ given in (64). Define the auxiliary function $\psi = \phi v_n$. It is not hard to see that

\[
\langle J_\phi'(v_n), \psi \rangle = \int_{\mathbb{R}^N} \left( v_n \nabla v_n \nabla \phi + |\nabla v_n|^2 \phi \right) \, dx + \int_{\mathbb{R}^N} V(x) f(v_n) f'(v_n) v_n \phi \, dx \\
- \lambda \int_{\mathbb{R}^N} q(x) f(v_n) f'(v_n) v_n \phi \, dx \\
- \int_{\mathbb{R}^N} K(x) |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n \phi \, dx \\
- \theta \int_{\mathbb{R}^N} \Gamma(x) |f(v_n)|^{2+2^*-2} f(v_n) f'(v_n) v_n \phi \, dx
\]

(71)

Under these conditions, by using Lemma 2.5 ($f_n$) and hypotheses $(q_1)$, $(K_1)$ and $(\Gamma_1)$ there exists $R$ large enough in such way that

\[
\int_{\mathbb{R}^N \setminus B_R} q(x) f(v_n) f'(v_n) v_n \, dx \leq \frac{\epsilon}{4}, \\
\int_{\mathbb{R}^N \setminus B_R} K(x) |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n \, dx \leq \frac{\epsilon}{4}, \\
\int_{\mathbb{R}^N \setminus B_R} \Gamma(x) |f(v_n)|^{2+2^*-2} f(v_n) f'(v_n) v_n \phi \, dx \leq \frac{\epsilon}{4}
\]

(72)

holds true for all $\epsilon > 0$. Furthermore, by using Hölder inequality we mention that
\[
\int_{\mathbb{R}^N} |\nabla v_n||v_n| \, dx \leq \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^N} |v_n|^2 \, dx \right)^{1/2} \leq C_3,
\]
for some constant $C_3 > 0$ which does not depend on $n$. Here was used that the sequence $(v_n)_{n \in \mathbb{N}}$ is bounded. According to the previous estimates together with (64) we deduce that
\[
\int_{\mathbb{R}^N} |\nabla v_n \phi v_n| \, dx \leq \frac{C}{R_1} \int_{\mathbb{R}^N} |\nabla v_n||v_n| \, dx \leq \frac{\epsilon}{4},
\]
holds for all $R_1 > 0$ large enough. At this stage, taking $R_1$ large enough and using (64), (71), (73) and (70), we infer also that
\[
\int_{\mathbb{R}^N \setminus B_{2R_1}} (|\nabla v_n|^2 + V(x) f(v_n) f'(v_n) v_n) \, dx \leq \epsilon + o_n(1).
\]
As a consequence, we obtain that (62) is now verified for all $R > 2R_1$. This finishes the proof of item $(i)$.

Now, we shall proceed by proving the item $(ii)$. Notice that, by using item $(i)$, we get
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus B_R} V(x) f^2(v_n) \, dx \leq \epsilon.
\]
Moreover, applying Fatou’s Lemma we see that
\[
\int_{\mathbb{R}^N \setminus B_R} V(x) f^2(v) dx \leq \varepsilon. \tag{74}
\]
Notice also that \(v_n \to v\) in \(L^2(B_R)\) as \(n \to \infty\). Hence, by using (V1), Lemma 2.5 (f3) and Lebesgue’s dominated convergence theorem we infer that
\[
\lim_{n \to \infty} \int_{B_R} V(x) f^2(v_n) dx = \int_{B_R} V(x) f^2(v) dx.
\tag{75}
\]
Under these conditions we write
\[
\int_{\mathbb{R}^N} V(x) (f^2(v_n) - f^2(v)) \, dx = \int_{\mathbb{R}^N \setminus B_R} V(x) (f^2(v_n) - f^2(v)) \, dx
\]
\[+ \int_{B_R} V(x) (f^2(v_n) - f^2(v)) \, dx,
\tag{76}
\]
which together with (74) and (75) imply that
\[
\limsup_{n \to \infty} \left| \int_{\mathbb{R}^N} V(x) (f^2(v_n) - f^2(v)) \, dx \right| \leq 2 \varepsilon.
\tag{77}
\]
As \(\varepsilon > 0\) can be chosen arbitrarily small the proof of (63) is now finished. These facts conclude the proof of (ii).

In what follows we shall prove the item (iii). Taking into account that \((v_n)\) is a \((PS)_c\) sequence for functional \(J_\lambda\), we write the following equality
\[
\int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx = - \int_{\mathbb{R}^N} V(x) f(v_n) f'(v_n) v_n \, dx + \lambda \int_{\mathbb{R}^N} q(x) f(v_n) f'(v_n) v_n \, dx
\]
\[+ \int_{\mathbb{R}^N} K(x)|f(v_n)|^{p-2} f'(v_n) v_n \, dx
\]
\[+ \theta \int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^{2^{*} - 2} f(v_n) f'(v_n) v_n \, dx + o_n(1).
\tag{78}
\]
Now, by using (62) and the same arguments employed in the proof item (ii), we obtain that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V(x) f(v_n) f'(v_n) v_n \, dx = \int_{\mathbb{R}^N} V(x) f(v) f'(v) v \, dx.
\tag{79}
\]
According to Lemma 2.2, Lemma 2.3 and Lebesgue Dominated Convergence Theorem we deduce the following convergences
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} q(x) f(v_n) f'(v_n) v_n \, dx = \int_{\mathbb{R}^N} q(x) f(v) f'(v) v \, dx; \tag{80}
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x)|f(v_n)|^{p-2} f'(v_n) v_n \, dx = \int_{\mathbb{R}^N} K(x)|f(v)|^{p-2} f'(v) v \, dx.
\tag{81}
\]
Now, we claim that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \Gamma(x)|f(v_n)|^{2^{*} - 2} f'(v_n) v_n \, dx = \int_{\mathbb{R}^N} \Gamma(x)|f(v)|^{2^{*} - 2} f'(v) v \, dx.
\tag{82}
\]
Indeed, applying the Lemma 4.2 and using the uniform convexity of the Lebesgue space \(L^{2^{*}}(\mathbb{R}^N)\), we infer also that \(\Gamma(x)^{1/2^{*}}|f(v_n)| \to \Gamma(x)^{1/2^{*}}|f(v)|\) in
$L^{2,2^*}(\mathbb{R}^N)$ as $n \to \infty$. Thus, there exists a function $h \in L^{2,2^*}(\mathbb{R}^N)$ such that
\[ \Gamma(x)^{1/2-2^*}|f(v_n)| \leq h(x) \quad \text{a. e. } x \in \mathbb{R}^N \] and for all $n \in \mathbb{N}$. Using these estimates it follows from Lemma 2.5 \((f_7)\) that
\[ \Gamma(x)|f(v_n)|^{2-2^*}f(v_n)f'(v_n)v_n \leq \left[ \Gamma(x)^{1/(2-2^*)}|f(v_n)| \right]^{2-2^*} \leq |h(x)|^{2-2^*} \in L^1(\mathbb{R}^N). \]
As a consequence, by using one more time Lebesgue Dominated Convergence Theorem the proof of claim given in \((82)\) is finished.

At this stage we shall proceed to prove that $J_\theta$ satisfies the $(PS)_c$ for all $c \in (0,M_k)$. Recall that $(v_n)_{n\in \mathbb{N}}$ is a $(PS)_c$ sequence for functional $J_\theta$. Here we mention that $v$ is a critical point for $J_\theta$. The main idea here is to use a density argument, that is, taking into account that $C_0^\infty(\mathbb{R}^N)$ is dense in $E$ it is sufficient to show that $\langle J'_\theta(v), \psi \rangle = 0$ holds true for any $\psi \in C_0^\infty(\mathbb{R}^N)$. This can be done using the fact that $v_n \rightharpoonup v$ in $E$ and $E$ is embedded compactly into $L^{s}_{loc}(\mathbb{R}^N)$ for each $s \in [1,2^*)$. In particular, we know that $\langle J'_\theta(v), v \rangle = 0$ for any $\theta > 0$. The last identity allows us to write
\[
\int_{\mathbb{R}^N} |\nabla v|^2 \, dx = -\int_{\mathbb{R}^N} V(x)f(v)f'(v)v \, dx + \lambda \int_{\mathbb{R}^N} q(x)f(v)f'(v)v \, dx
+ \int_{\mathbb{R}^N} K(x)|f(v)|^{p-2}f(v)f'(v)v \, dx
+ \theta \int_{\mathbb{R}^N} \Gamma(x)|f(v)|^{2-2^*-2}f(v)f'(v)v \, dx.
\]
On the other hand, by using \((78)\), \((83)\) and taking into account \((79)-(82)\), we infer that
\[
\lim_{n \to \infty} \left( \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx - \int_{\mathbb{R}^N} |\nabla v|^2 \, dx \right) = 0.
\]
As a consequence, by using Brezis Lieb’s Lemma together with \((84)\), we see that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla(v_n - v)|^2 \, dx = 0.
\]
Now, we need to prove that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} V(x)(v_n - v)^2 \, dx = 0.
\]
as $n \to \infty$ where $C > 0$ is a constant. Here we was used also the convergence given in (86). Under these conditions, by using Lemma 2.5 ($f_0$), we write

$$\int_{B_n} V(x)(v_n-v)^2dx \leq C_1 \int_{B_n} V(x)f^2(v_n-v)dx \leq C_1 \int_{\mathbb{R}^N} V(x)f^2(v_n-v)dx.$$ \hspace{1cm} (89)

Since $f^2$ is convex function we mention that $f^2(t-s) \leq 4(f^2(t) + f^2(s))$ holds true for all $s,t \in \mathbb{R}$, for instance see [14, 13]. Using the item ii) we know that $0 \leq V(x)f^2(v_n) \leq h$ holds for some $h \in L^1(\mathbb{R}^N)$. Hence $V(x)f^2(v_n-v) \leq 4V(x)f^2(v_n) + 4V(x)f^2(v) \leq 4h + 4V(x)f^2(v) \in L^1(\mathbb{R}^N)$. Using the last estimate and Lebesgue Convergence Theorem together with (89) we deduce that

$$\lim_{n \to \infty} \int_{B_n} V(x)(v_n-v)^2dx = 0.$$ \hspace{1cm} (90)

Therefore, by using (88) and (90) the limit given in (86) is now satisfied. Under these conditions, we deduce that $||v_n-v|| \to 0$ as $n \to \infty$. This finishes the proof. \hfill \Box

The Proof of Theorem 1.1 completed. Initially, by using Proposition 1, there exist $\rho > 0$ and $\sigma > 0$ in such way that $J_\sigma(v) \geq \sigma$ for all $v \in \partial B_\rho \cap E_2$. As a consequence, we obtain that the condition ($J_1$) in Theorem 2.1 is satisfied. Now, considering the subspace $W_k := \langle \phi_1, \ldots, \phi_j, \phi_{j+1}, \ldots, \phi_{j+k} \rangle \subset E$, $k \in \mathbb{N}$ where $k \geq 1$, it follows from Proposition 2 that there exists a constant $M_k > 0$ where $M_k$ does not depend on $\theta$ in such way that $\max_{v \in W_k} J_\theta(v) \leq M_k$. Hence, the condition ($J_2$) given in Theorem 2.1 is verified. Furthermore, for each $c \in (0, M_k)$ and $\theta \in (0, \theta_k)$ the Lemma 4.3 says that the energy functional $J_\theta$ satisfies the $(PS)_C$ condition. Thus, the condition ($J_3$) given in Theorem 2.1 is also satisfied. Hence, we apply Theorem 2.1 which ensures that the problem (14) has at least $k$ pairs of nontrivial solutions. In particular, we obtain that the problem (2) has at least $k$ pairs of nontrivial solutions. This ends the proof of Theorem 1.1. \hfill \Box

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