Plane-Symmetric Vacuum Solutions with Null Singularities
for Inhomogeneous Models and Colliding Gravitational Waves

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Abstract

New exact vacuum solutions with various singularities in the plane-
symmetric spacetime are shown, and they are applied to the analysis of in-
homogeneous cosmological models and colliding gravitational waves. One of
the singularities can be true null singularities, whose existence was locally
clarified by Ori. These solutions may be interesting from the viewpoint of the
variety of cosmological singularities and the instability problem of Cauchy
horizons inside black holes.

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The important behavior of colliding gravitational waves was first shown in Khan and Penrose's pioneering work [1] in a plane-symmetric spacetime. The formulation for dynamics of colliding plane waves was given by Szekeres [2] in an extended form with non-parallel polarization, and the solution, which is impulsive similarly to Khan and Penrose's one, was derived by Nuktu and Halil [3].

The plane-symmetric inhomogeneous cosmological models were, on the other hand, studied by Gowdy [4] in a vacuum spacetime with compact invariant hypersurfaces, and models containing space-like and time-like singularities alternately were discussed by the present author [5].

Studies on impulsive colliding gravitational waves were developed by Chandrasekhar and his collaborators [6] who treated not only vacuum cases but also the cases with fluids and electromagnetic fields, and derived also the solutions with a time-like singularity. Yurtsever [7] derived solutions which tend to the Kasner solutions asymptotically, contain the Cauchy horizons, and are connected with the Schwarzschild and Weyl vacuum solutions.

The Cauchy horizons in the plane-symmetric spacetime were paid attention in similarity to the Cauchy horizons appearing in the neighborhood of the time-like singularity of the Ressner-Nordstrom black hole spacetime. Around the latter Cauchy horizon there is the extraordinary behavior of the spacetime such as the mass inflation [8] and infinite blue shift [9]. Recently Ori discussed the instability of the Cauchy horizon in plane-symmetric spacetimes in connection with that in the black hole solutions [10], and showed the possibility of a true null singularity in colliding gravitational waves [11].

In this paper we show new exact solutions representing plane-symmetric spacetimes with true null singularities, which are deeply related to Ori's local proof [11]. In §2, we show the basic equations to be solved, for the plane-symmetric spacetime s in the case of parallel polarization, and derive their new solutions with a kind of self-similarity. It is found by
analyzing one of the curvature invariants that in addition to the space-like or time-like singularity there appear true null singularities for a limited range of the model parameter. In §3, schemes of colliding waves are shown, corresponding to the above solutions, and in §4 concluding remarks are given. In Appendices A and B the tetrad components of curvature tensors and some basic formulas for hypergeometric functions are shown.

II. INHOMOGENEOUS MODEL

The line-element of plane-symmetric spacetimes with parallel polarization is given in the Rosen form

\[ ds^2 = -2e^{2a}dvdu + R\left[e^{2\gamma}dy^2 + e^{-2\gamma}dz^2\right], \quad (2.1) \]

where \(a, R\) and \(\gamma\) are functions of \(v\) and \(u\), and \(x^0 = v, x^1 = u, x^2 = y\) and \(x^3 = z\). The vacuum Einstein equations reduce to

\[ R_{,01} = 0, \quad (2.2) \]

\[ 2\gamma,_{01} + (\gamma,_{0}R_{,1} + \gamma,_{1}R_{,0})/R = 0, \quad (2.3) \]

\[ e^{2a} = |R_{,0}R_{,1}|R^{-1/2}e^{2I}, \quad (2.4) \]

where

\[ I \equiv \int \left[ (\gamma,_{0})^2/R_{,0} + (\gamma,_{1})^2/R_{,1} \right] R \quad (2.5) \]

with \(R,_{0} = \partial R/\partial x^0, R,_{1} = \partial R/\partial x^1\) and so on. The derivation of the above equations are shown in Appendix A, together with the components of the curvature tensor.

From Eq. (2.2) we have

\[ R = f(v) + g(u), \quad (2.6) \]
where \( f \) and \( g \) are arbitrary functions. If \( f'(v) > 0 \) and \( g'(u) > 0 \), we can put \( f(v) = v \) and \( g(u) = u \) without loss of generality. In the same way, if \( f'(v) > 0 \) and \( g'(u) < 0 \), we can put \( f(v) = v \) and \( g(u) = -u \).

A. The case when the hypersurfaces \( R = \text{const} \) are space-like

In this case we can assume \( f'(v) > 0 \) and \( g'(u) > 0 \), and so \( R = v + u \). Then Eqs. (2.3) and (2.3) lead to

\[
2\gamma_{,01} + (\gamma_{,0} + \gamma_{,1})/(v + u) = 0, \quad (2.7)
\]

and

\[
I = \int \left[ dv(\gamma_{,0})^2 + du(\gamma_{,1})^2 \right](v + u). \quad (2.8)
\]

Here let us assume the following form of \( \gamma : \)

\[
\gamma = |v|^\alpha |u|^{-1/2}A(v/u). \quad (2.9)
\]

This \( \gamma \) has a kind of self-similarity, because \( A \) is a function of only \( v/u \). The parameter \( \alpha \) is equal to \( 1/n \) in Ori’s paper [11]. If we substitute Eq. (2.9) into Eq. (2.7), we obtain an ordinary differential equation for \( A(p) \)

\[
p(p + 1)\frac{d^2A}{dp^2} + [(2 + \alpha)p + 1 + \alpha]\frac{dA}{dp} + \frac{1}{4}(1 + 2\alpha)A = 0. \quad (2.10)
\]

This is a hypergeometric differential equation. Its independent solutions are expressed using hypergeometric functions as follows and have singular points at \( p = 0, -1, \infty : \)

For the non-integer \( \alpha \), we have

\[
A = F(1/2, 1/2 + \alpha, 1 + \alpha; -p) \quad (2.11)
\]

and

\[
A = |p|^{-\alpha}F(1/2 - \alpha, 1/2, 1 - \alpha; -p). \quad (2.12)
\]

For the integer \( \alpha = 0, 1, 2, \cdots \),
\[ A = F(1/2, 1/2 + \alpha, 1 + \alpha; -p) \] (2.13)

and

\[ A = F(1/2, 1/2 + \alpha, 1 + \alpha; -p) \ln |p| + \tilde{F}(1/2, 1/2 + \alpha, 1 + \alpha; -p), \] (2.14)

where \( \tilde{F} = F_1 \) and \( F_2 \) for \( \alpha = 0 \) and \( \alpha \geq 1 \), respectively. The expressions of \( F, F_1 \) and \( F_2 \) are shown in Appendix B. From Eq. (2.8) we obtain for \( \alpha \neq 1/2 \)

\[ I = \frac{1}{2\alpha - 1} |u|^{2\alpha - 1} |p|^{2\alpha} (p + 1) \left[ \frac{1}{p} \left( \alpha A + pA' \right)^2 + \left( \frac{1}{2} A + pA' \right)^2 \right] \] (2.15)

and for \( \alpha = 1/2 \)

\[ I = \ln |u| \cdot (p + 1)^2 \left( \frac{1}{2} A + pA' \right)^2. \] (2.16)

From the two solutions (2.11) and (2.12) we have for \( |p| << 1 \)

\[ \gamma = |v|^\alpha |u|^{-1/2} \left[ 1 - \frac{1 + 2\alpha}{4(1 + \alpha)} \frac{v}{u} + \ldots \right] \] (2.17)

and

\[ \gamma = |u|^\alpha^{-1/2} \left[ 1 - \frac{1 - 2\alpha}{4(1 - \alpha)} \frac{v}{u} + \ldots \right], \] (2.18)

respectively.

If we use a formula

\[ F(a, b, c, -p) = \frac{\Gamma(b - a) \Gamma(c)}{\Gamma(b) \Gamma(c - a)} (-p)^{-a} F(a, a - c + 1, a - b + 1; -1/p) \]

\[ + \frac{\Gamma(a - b) \Gamma(c)}{\Gamma(a) \Gamma(c - b)} F(b, b - c + 1, b - a + 1; -1/p), \] (2.19)

it is found that the first solution (2.11) is a sum of the following two solutions:

\[ \gamma \propto |v|^\alpha^{-1/2} F(1/2, \alpha - 1/2, 1 - \alpha; -1/p) \] (2.20)

and

\[ \gamma \propto |u|^\alpha |v|^{-1/2} F(1/2 + \alpha, 1/2, 1 + \alpha; -1/p), \] (2.21)
and the second solution (2.12) is a sum of the following two solutions:

$$
\gamma \propto |v|^{\alpha - 1/2} F(1/2 - \alpha, 1/2, 1 - \alpha; -1/p)
$$

(2.22)

and

$$
\gamma \propto |u|^\alpha |v|^{-1/2} F(1/2, 1/2 + \alpha, 1 + \alpha; -1/p),
$$

(2.23)

This means that the solutions are symmetric for the conversion of \(v \rightarrow u\) and \(u \rightarrow v\).

Here let us examine the singular behavior in the limit of \(v \rightarrow 0\).

(1) Non-integer \(\alpha\)

For the first solution (2.11) or (2.17) in the case of \(\alpha < 1/2\), we find \((-I) \propto |v|^{2\alpha - 1} \rightarrow \infty\) from Eq.(2.15) and \(\Phi \propto |v|^{6\alpha - 3}|u|^{-4} \rightarrow \infty\), which comes from the product \((\gamma_0)^3 \cdot (\gamma_1)^3\) in Eq.(A19), so that we have the invariant curvature \(R \rightarrow \infty\). If \(\alpha = 1/2\), we find that \(I\) and \(\Phi\) are finite, so that \(R\) also is finite, and if \(\alpha > 1/2\), \(I \propto |v|^{2\alpha - 1} \rightarrow 0\) and \(\Phi\) is finite, so that \(R\) is finite.

The second solution (2.12) or (2.18) is regular at \(v \simeq 0\) and so \(I\) and \(R\) are finite in the limit of \(v \rightarrow 0\).

It is concluded therefore that only the first solution given by (2.11) or (2.17) for \(\alpha < 1/2\) has the true null singularity. The singular behavior in the limit of \(u \rightarrow 0\) is the same as that in the limit of \(v \rightarrow 0\).

As was shown in Eqs. (2.20) \sim (2.23), the first and second solutions at \(v \simeq 0\) correspond to the first and second ones at \(u \simeq 0\), but this not the one-to-one correspondence. The second solution at \(v \simeq 0\) corresponds to both solutions at \(u \simeq 0\), and the second solution at \(u \simeq 0\) corresponds to both solutions at \(v \simeq 0\).

As for the individual tetrad components, there are divergent components even for \(\alpha > 1/2\). For example, \(R_{(0202)}\) and \(R_{(0303)}\) diverge in the limit of \(v \rightarrow 0\) for \(\alpha < 2\), because they include \(\gamma_{00} \propto |v|^\alpha - 2\). This divergence is a kind of weak singularities.

(2) Integer \(\alpha = 0, 1, 2, \cdots\)

If \(\alpha = 0\), we find that \(\gamma_0\) is finite, so that \((-I), \Phi,\) and \(R\) from the first solution (2.13) are finite. In the second solution (2.14), the term \(\ln |p|\) gives the infinity to \((-I), \Phi,\) and \(R\) in the
limit of \( v \to 0 \) and \( u \to 0 \). If \( \alpha \geq 1 \), on the other hand, \( \gamma_0 \) is finite, in both solutions (2.13) and (2.14), so that \( \mathcal{R} \) remains to be finite. The true null singularities appear, therefore, only for \( \alpha = 0 \), if \( \alpha \) is integer.

Next let us consider the behavior of the solutions at \( R \simeq 0 \). If we put \( q \equiv p + 1 \), then Eq. (2.10) reduces to

\[
q(q - 1) \frac{d^2A}{dq^2} + [(2 + \alpha)q - 1] \frac{dA}{dq} + \frac{1}{4} (1 + 2\alpha) A = 0,
\]

and the following two solutions are obtained:

\[
A = F(1/2, 1/2 + \alpha, 1; q) \tag{2.25}
\]

and

\[
A = F(1/2, 1/2 + \alpha, 1; q) \ln |q| + F_1(1/2, 1/2 + \alpha, 1; q). \tag{2.26}
\]

At \( q \simeq 0 \), we have \( F = 1 + 0(q) \) and \( F_1 = 0(q) \), so that the first solution (2.25) is finite and the second (2.26) diverges as \( \ln |q| \to -\infty \) for \( q \to 0 \). At \( R \propto q \simeq 0 \) with finite \( v \) and \( u \), the first solution gives finite \( \gamma \) and \( I \), and so \( e^{\gamma} \propto R^{-1/2} \) from Eqs. (2.4), (2.15) and (2.16). If we define \( Q \) and \( T \) as \( Q = v - u \) and \( dT = e^{\gamma} dR \) (or \( R \propto T^{4/3} \)), we obtain

\[
-2e^{2\alpha} dvdu = e^{2\alpha} (-dR^2 + dQ^2) \propto -dT^2 + T^{-2/3} dQ^2 \tag{2.27}
\]

in the neighborhood of \( R = 0 \). Accordingly the solution reduces to the Kasner solution with the Kasner parameter \((-1/3, 2/3, 2/3)\) in the limit of \( R \to 0 \). The invariant curvature diverges as \( \mathcal{R} \simeq 1/R \to \infty \) (cf. Eq.(A19)).

The second solution (2.26) may have a dominant contribution because of the large term \( \ln |q| \). It gives \( \gamma \simeq \mu \ln R \) and so \( e^{\gamma} \propto R^\mu \), where \( \mu \equiv |v|^{\alpha-1/2} \). From Eq. (2.15) we obtain \( I \simeq \mu^2 \ln |q| \) and so \( e^{\gamma} \propto R^{\alpha^2/4}e^I \propto R^{\alpha^2-1/4} \). In this case, \( T \) defined by \( dT = e^{\gamma} dR \) gives the relation \( T \propto R^{\alpha^2+3/4} \), so that we have

\[
e^\gamma \propto T^{\zeta_1}, \ R^{1/2} e^{\gamma} \propto T^{\zeta_2}, \ R^{1/2} e^{-\gamma} \propto T^{\zeta_3}, \tag{2.28}
\]

where
\[ \zeta_1 = (\mu^2 - 1/4)/(\mu^2 + 3/4), \quad \zeta_2 = (1/2 + \mu)/(\mu^2 + 3/4), \quad \zeta_3 = (1/2 - \mu)/(\mu^2 + 3/4). \]  

These exponents satisfy the Kasner relation

\[ \sum_{i=1}^{3} \zeta_i = \sum_{i=1}^{3} (\zeta_i)^2 = 1. \]  

Accordingly the solution is found to be a generalized Kasner solution at \( R \simeq 0 \) with the Kasner parameters depending on a spatial variable as \( \mu = |v|^{\alpha-1/2} \simeq |Q/2|^{\alpha-1/2}, \) where \( Q \simeq 2v \) at \( R \simeq 0 \). The singularities at \( R = 0 \) are space-like ones.

In the limit of \( R \to \infty \) and \( |v| \to \infty \) (with \( v/u = \) const), the two solutions (2.11) and (2.14) satisfy the relation \( \gamma \propto R^{\alpha-1/2} \), and so we find for \( \alpha <, =, \) and \( > 1/2 \) that \( \gamma \) vanishes, is finite and diverges, respectively. For \( \alpha < 1/2 \), \( I \) in Eq. (2.13) vanishes and we have the Kasner behavior of \((-1/3, 2/3, 2/3)\), as in the case \( R \to 0 \). For \( \alpha = 1/2 \), \( \gamma \) and \( I \) remain to be constant and we have the Kasner behavior of \((-1/3, 2/3, 2/3)\). For \( \alpha > 1/2 \), \( \gamma \) diverges but \( I \) diverges more strongly as \( I \propto R^{2\alpha-1} \), so that the spacetime reduces to the Minkowskian spacetime with the Kasner parameter \((1, 0, 0)\).

The above treatment is applicable to the half-plane \( R = v + u > 0 \). To the half-plane \( v + u < 0 \), we transform \( v \) and \( u \) as \( v \to -v \) and \( u \to -u \). Then we have \( R = -(v + u) > 0 \), and the same conclusion is derived in the same way.

**B. The case when the hypersurfaces \( R = \text{const} \) have time-like sections**

Now let us consider the case \( R = v - u \). Then Eqs. (2.3) and (2.4) lead to

\[ 2\gamma_{,01} + (-\gamma_{,0} + \gamma_{,1})/(v - u) = 0, \]  

and

\[ I = \int [dv(\gamma_{,0})^2 - du(\gamma_{,1})^2](v - u). \]  

If we assume \( \gamma \) in the form of

\[ \gamma = |v|^\alpha |u|^{-1/2} B(v/u), \]  

\[ 8 \]
we obtain an ordinary differential equation $B(p)$:

$$p(p - 1) \frac{d^2 B}{dp^2} + [(2 + \alpha)p - 1 - \alpha] \frac{dB}{dp} + \frac{1}{4}(1 + 2\alpha)B = 0. \quad (2.34)$$

This equation reduces to Eq. (2.10) by putting $B(p) = A(-p)$, so that the solutions are given by Eqs. (2.11) $\sim$ (2.14) in which $p$ is transformed to $-p$. It is shown similarly to the case A that the null singularities appear at $v = 0$ and $u = 0$. They are true only for non-integer $\alpha < 1/2$ and integer $\alpha = 0$.

At $R \simeq 0$, the equation for $B$ with respect to $q \equiv p - 1$ leads to

$$q(q + 1) \frac{d^2 B}{dq^2} + [(2 + \alpha)q + 1] \frac{dB}{dq} + \frac{1}{4}(1 + 2\alpha)B = 0. \quad (2.35)$$

This equation reduces to Eq. (2.24) by putting $B(q) = A(-q)$, so that the solutions are given by Eqs. (2.25) and (2.26) in which $q$ is transformed to $-q$. In the limit of $R \to 0$, we find from the first solution that $\gamma$ is finite and $e^a \propto R^{-1/2}$, and so we obtain

$$-2e^{2a}dvdu = e^{2a}(-dQ^2 + dR^2) \propto -X^{-2/3}dQ^2 + dX^2 \quad (2.36)$$

in the neighborhood of $R = 0$, where $Q = v + u$ and $dX = e^a dR$ (or $R \propto X^{4/3}$). Accordingly the solution tends to the Kasner-type one with the Kasner parameter $(-1/3, 2/3, 2/3)$, in which the time-like singularity appears in the limit of $R \to 0$ or $X \to 0$, and the invariant curvature diverges as $R \simeq 1/R \to \infty$ (cf. Eq.(A19)).

From the second solution we obtain similarly the generalized Kasner-type solution expressed by $e^\gamma \propto R^\mu$ and $e^a \propto R^{-1/4}e^t \propto R^{2a-1/4}$, where $\mu \equiv |v|^{a-1/2}$. In this case, a spatial variable $X$ defined by $dX = e^a dR$ gives the relation $X \propto R^{a+3/4}$, so that we have

$$e^a \propto X^{\zeta_1}, \quad R^{1/2}e^\gamma \propto X^{\zeta_2}, \quad R^{1/2}e^{-\gamma} \propto X^{\zeta_3}, \quad (2.37)$$

where $\zeta_1, \zeta_2,$ and $\zeta_3$ are defined by Eq. (2.29). Accordingly the hypersurface $R = 0$ gives the Kasner-type time-like singularity and the Kasner parameter depends on $Q \simeq 2v$, which is a time variable in the case B.

In the limit of $R \to \infty$ and $|v| \to \infty$ (with $v/u = \text{const}$), the two solutions (2.11) and (2.14), in which $p$ was replaced by $-p$, satisfy the relation $\gamma \propto R^{a-1/2}$, and so we find for
\( \alpha <, =, \text{and} > 1/2 \) that \( \gamma \) vanishes, is finite and diverges, respectively. For \( \alpha < 1/2 \), \( I \) in Eq.(2.15) vanishes and we have the Kasner behavior of \((-1/3, 2/3, 2/3)\), as in the case \( R \to 0 \).

In the above we considered the half-plane with \( R = v - u > 0 \). To the half-plane \( v - u < 0 \), we have to transform \( v \) and \( u \) as \( v \to -v \) and \( u \to -u \), and use \( R = -v + u > 0 \). The conclusion about singularities does not depend on this transformation. The positions of singularities are shown in Figs. 1 and 2 for \( R = \pm(v + u) \) and \( R = \pm(v - u) \), respectively.

**III. COLLIDING GRAVITATIONAL WAVES**

Now, corresponding to two cases A and B in the previous sections, let us consider a spacetime containing colliding gravitational waves, which consists of the no-wave region (I), the outgoing-wave region (II), the incoming-wave region (III), and the colliding-waves region (IV). For the connection of these four regions at \( v = v_0 \) and \( u = u_0 \), we use the Penrose prescription in which \( v \) and \( u \) in the metric are replaced by

\[
(v - v_0) \theta(v - v_0) + v_0
\]

and

\[
(u - u_0) \theta(u - u_0) + u_0,
\]
where \( \theta(x) \) is 1 and 0 for \( x > 0 \) and \( < 0 \), respectively.

In the case A, the wave scheme is shown in Fig. 3, in which colliding waves are in the region IV within a triangle acd. The true space-like singularity is in the line ac, and for \( \alpha < 1/2 \) the true null singularities are in two lines be and bf, in which one of these two lines can be weak singularity due to the second solution.

In the case B, the scheme is shown in Fig. 4. Similarly colliding waves are in the region IV within a triangle acd. The time-like singularity ac and null singularities bf and be are only in the region IV. In the regions II and III weak singularities appear in the extended part of the above lines, because \( R \) vanishes there.

In the region III, \( R, \gamma, \) and \( a \) are functions of only \( v \), so that \( R, 1 = \gamma, 1 = a, 1 = 0 \). For \( R = v \pm u_0 \) or \( R = -(v \pm u_0) \), Eqs. (2.11), (2.12), (2.13), and (2.14) are satisfied by an arbitrary function \( \gamma(v) \), and Eq. (A11) gives

\[
e^a = R^{-1/4} \exp \left[ \int R(\gamma, 0)^2 dv/R, 0 \right]. \tag{3.3}
\]

The tetrad components of curvature tensor vanish, except for

\[
R_{0202} = -R_{0303} = e^{-2a} \left[ -\frac{3}{2R} \gamma, 0 + 2R(\gamma, 0)^3 - \gamma, 00 \right]. \tag{3.4}
\]

Here we take the functional form

\[
\gamma = |v|^\alpha |u_0|^{-1/2} A(v/u_0), \tag{3.5}
\]

where \( A(p) \) and \( A(-p) \) are given by Eqs. (2.11), (2.12), corresponding to the cases A and B, respectively. Then in the limit of \( R \rightarrow 0 \) (with finite values of \( v \)), we have \( e^a \propto R^{-1/4} \) and \( R_{0202} \propto R^{-1/2} \rightarrow \infty \), and in the limit of \( v \rightarrow 0 \), we have for \( \alpha \neq 1/2 \)

\[
e^a \propto \exp \left[ -\frac{\alpha^2}{2\alpha - 1} |v|^{2\alpha - 1} \right], \tag{3.6}
\]

\[
R_{0202} \propto \exp \left[ -\frac{2\alpha^2}{2\alpha - 1} |v|^{2\alpha - 1} \right]|v|^{-2} \tag{3.7}
\]

and we have for \( \alpha = 1/2 \)
\[ e^a \propto |v|^{1/4}, \quad (3.8) \]

\[ R_{(0202)} \propto v^{-2}, \quad (3.9) \]

where we assumed \( A(0) = 1 \). Accordingly, \( R_{(0202)} (= R_{(0303)}) \) diverges for \( \alpha < 2 \), but since \( \text{cal} R \) vanishes, these divergences give only weak singularities.

In the connection at the boundaries of \( v = v_0 \) and \( u = u_0 \), \( \gamma \) is continuous, but \( \gamma_0 \) and \( \gamma_1 \) are not continuous, because for example \( \gamma_0 = 0 \) and \( \neq 0 \) at \( v > v_0 \) and \( v < v_0 \), respectively. Accordingly, \( R_{(0202)} (= R_{(0303)}) \) has the delta function \( \delta(v-v_0) \) and \( \delta(u-u_0) \) along the lines \( v = v_0 \) and \( u = u_0 \), respectively. Moreover, since \( \gamma \propto |v|^{\alpha} |u|^{-1/2} \) and \( |u|^{\alpha} |v|^{-1/2} \) and \( \gamma_0 \) and \( \gamma_1 \) are not continuous at \( v = 0 \) and \( u = 0 \), \( R_{(0202)} (= R_{(0303)}) \) has the delta function \( \delta(v) \) and \( \delta(u) \) along the lines \( v = 0 \) and \( u = 0 \).

**Fig.3**

**Fig.4**

**IV. CONCLUDING REMARKS**

In the previous sections we derived exact solutions representing inhomogeneous vacuum spacetimes with space-like and null singularities or time-like and null singularities, and applied them to the problem of colliding gravitational waves. While these null singularities may be closely related to the instability of Cauchy horizons, we must consider the stability of this null singularities themselves. For instance, there are questions whether they can coexist.
together with the electromagnetic fields and fluidal matter, and whether they can appear in colliding waves with non-parallel polarization. These problems will be discussed in separate papers.

The physical meaning of the important parameter $\alpha$ may be a change rate of the shear, because it is included as a power index in the expression of the Kasner parameter.

The above spacetimes are cosmologically interesting in that near the space-like or time-like singularity they include strong gravitational waves and at the same time show exactly the velocity-dominated [13] and anti-Newtonian behavior [14], in which the Kasner parameters depend on a spatial or time variable, respectively.

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**APPENDIX A: TETRAD COMPONENTS OF THE CURVATURE TENSOR AND THE RICCI TENSOR**

The following tetrads are used:

$$
e^{\mu}_{(0)} = \delta^{\mu}_{0} e^{-a}, \quad e^{\mu}_{(1)} = \delta^{\mu}_{1} e^{-a}, \quad e^{\mu}_{(2)} = \delta^{\mu}_{2} R^{-1/2} e^{-\gamma}, \quad e^{\mu}_{(3)} = \delta^{\mu}_{3} R^{-1/2} e^{\gamma}
$$

which satisfy the condition

$$g_{\mu\nu} e^{\mu}_{(\alpha)} e^{\nu}_{(\beta)} = \zeta_{\alpha\beta}, \quad (A1)$$

where the non-zero components of $(\zeta_{\alpha\beta})$ are $\zeta_{01} = \zeta_{10} = -1$ and $\zeta_{22} = \zeta_{33} = 1$.

The tetrad components of curvature tensor are given by $R_{\alpha\beta\gamma\delta} \equiv e^{\mu}_{(\alpha)} e^{\nu}_{(\beta)} e^{\lambda}_{(\gamma)} e^{\sigma}_{(\delta)} R_{\mu\nu\lambda\sigma}$ and their components are expressed as

$$R_{(0101)} = -2 a_{01} e^{-2a}, \quad (A3)$$
\[ R_{(0202)} = e^{-2\alpha} \left[ -\left( \frac{1}{2} R_{00}/R + \gamma_{00} \right) + a_{0}(R_{0}/R + 2\gamma_{0}) + \frac{1}{4}(R_{0}/R)^2 - \gamma_{0}R_{0}/R - (\gamma_{0})^2 \right], \]  

\( (A4) \)

\[ R_{(0303)} = e^{-2\alpha} \left[ -\left( \frac{1}{2} R_{00}/R - \gamma_{00} \right) + a_{0}(R_{0}/R - 2\gamma_{0}) + \frac{1}{4}(R_{0}/R)^2 + \gamma_{0}R_{0}/R - (\gamma_{0})^2 \right], \]

\( (A5) \)

\[ R_{(0212)} = e^{-2\alpha} \left[ -\left( \frac{1}{2} R_{01}/R + \gamma_{01} \right) + \frac{1}{4} R_{0}R_{1}/R^2 - \frac{1}{2}(\gamma_{0}R,_{1} + \gamma,_{1}R_{0})/R - (\gamma_{0}\gamma,_{1}) \right], \]  

\( (A6) \)

\[ R_{(0313)} = e^{-2\alpha} \left[ -\left( \frac{1}{2} R_{01}/R - \gamma_{01} \right) + \frac{1}{4} R_{0}R_{1}/R^2 + \frac{1}{2}(\gamma_{0}R,_{1} + \gamma,_{1}R_{0})/R - (\gamma_{0}\gamma,_{1}) \right], \]  

\( (A7) \)

\[ R_{(2323)} = e^{-2\alpha} \left[ R_{0}R_{1}/R^2 - 4\gamma_{0}\gamma,_{1} \right], \]  

\( (A8) \)

where \( R_{0} = \partial R/\partial x^{0} \), \( R_{01} = \partial^2 R/\partial x^{0}\partial x^{1} \), etc. The components \( R_{(1212)}, R_{(1313)} \) and \( R_{(1213)} \) are derived from \( R_{(0202)}, R_{(0303)} \) and \( R_{(0203)} \), respectively, by replacing \( \partial/\partial x^{0} \) and \( \partial/\partial x^{1} \) each other, as \( R_{0} \rightarrow R_{1} \) and \( R_{1} \rightarrow R_{0} \). Other components such as \( R_{(0102)}, R_{(0112)} \) etc vanish.

The tetrad components of the Ricci tensor are given by \( R_{(\gamma\delta)} \equiv \zeta^{\alpha\beta}R_{(\alpha\gamma\beta\delta)} \) and their non-zero components are

\[ \frac{1}{2}(R_{(22)} + R_{(33)}) = e^{-2\alpha}R_{01}/R = 0, \]  

\( (A9) \)

\[ \frac{1}{2}(R_{(22)} - R_{(33)}) = e^{-2\alpha}\left[ 2\gamma_{01} + (\gamma_{0}R,_{1} + \gamma,_{1}R_{0})/R \right] = 0, \]  

\( (A10) \)

\[ R_{(00)} = e^{-2\alpha}\left[ R_{00}/R + 2a_{0}R_{0}/R + \frac{1}{2}(R_{0}/R)^2 - 2(\gamma_{0})^2 \right] = 0, \]  

\( (A11) \)

\[ R_{(11)} = e^{-2\alpha}\left[ R_{11}/R + 2a_{1}R_{1}/R + \frac{1}{2}(R_{1}/R)^2 - 2(\gamma_{1})^2 \right] = 0, \]  

\( (A12) \)

\[ R_{(01)} = e^{-2\alpha}\left[ R_{01}/R - 2a_{01} + \frac{1}{2}(R_{1}R_{0})/R^2 - 2\gamma_{0}\gamma,_{1} \right] = 0. \]  

\( (A13) \)

From Eqs.\((A11)\) \( A(12)\), the equations for \( a_{0} \) and \( a_{1} \) derived and it is shown that their differentiations with respect to \( x^{1} \) and \( x^{0} \), respectively give the same expression for \( a_{01} \)
consistent with Eq. (A13), when we use Eqs. (A9) and (A10). Accordingly Eqs. (A11) and (A12) are integrable for $a$, as in Eq. (2.5) in the text.

If we use the Einstein equations (A9) $\sim$ (A13) and the conditions $R_0 = \text{const}$ and $R_1 = \text{const}$, we obtain

$$R_{(0101)} = -R_{(2323)} = e^{-2a} \left[-\frac{1}{2} R_0 R_1/R^2 + 2\gamma_0\gamma_1\right] \tag{A14}$$

$$R_{(0202)} = -R_{(0303)} = e^{-2a} (R_0)^2 \left[-\frac{3}{2R} \gamma_0/R_0 + 2R(\gamma_0/R_0)^3 - \gamma_{00}/(R_0)^2\right] \tag{A15}$$

$$R_{(1212)} = -R_{(1313)} = e^{-2a} (R_1)^2 \left[-\frac{3}{2R} \gamma_1/R_1 + 2R(\gamma_1/R_1)^3 - \gamma_{11}/(R_1)^2\right] \tag{A16}$$

$$R_{(0212)} = -R_{(0313)} = -\frac{1}{2} e^{-2a} \left[\frac{1}{2} R_0 R_1/R^2 + 2\gamma_0\gamma_1\right] \tag{A17}$$

One of curvature invariants $R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$ is proportional to

$$\mathcal{R} = R_{(0202)} R_{(1212)} + (R_{(0212)})^2 + (R_{(0101)})^2 \tag{A18}$$

Using Eqs. (A14) $\sim$ (A16) and Eq. (2.4) we obtain

$$\mathcal{R} = R e^{-4I} \Phi \tag{A19}$$

with

$$\Phi = \left[\gamma_{00}/(R_0)^2 - 2R(\gamma_0/R_0)^3 + \frac{3}{2R} \gamma_0/R_0\right] \left[\gamma_{11}/(R_1)^2 - 2R(\gamma_1/R_1)^3\right]
+ \frac{3}{2R} \gamma_1/R_1\right] + 2\left[-\frac{1}{2R^2} + 2\left(\frac{\gamma_0\gamma_1}{R_0 R_1}\right)^2\right] + \left[\frac{1}{2R^2} + 2\left(\frac{\gamma_0\gamma_1}{R_0 R_1}\right)^2\right]. \tag{A20}$$

**APPENDIX B: HYPERGEOMETRIC FUNCTIONS**

Formula (2.19) for the hypergeometric functions and their following expressions in series are found in the texts on special functions [12].

$$F(a, b, c; z) = 1 + \sum_{k=1}^{\infty} \frac{a(a+1)\cdots(a+k-1)b(b+1)\cdots(b+k-1)}{k!\ c(c+1)\cdots(c+k-1)} z^k, \tag{B1}$$
\[ F_1(a, b, 1; z) = \sum_{k=1}^{\infty} \frac{a(a + 1) \cdots (a + k - 1)b(b + 1) \cdots (b + k - 1)}{(k!)^2} \times \left[ \sum_{l=0}^{k-1} \left( \frac{1}{a + l} + \frac{1}{b + l} - \frac{1}{l + 1} \right) \right] z^k, \tag{B2} \]

\[ F_2(a, b, c; z) = (-1)^c(c - 1)! z^{k-1} \times \sum_{k=0}^{c-2} \frac{(-1)^k(c - k - 2)! z^k}{(a - 1)(a - 2) \cdots (a - c + k + 1)(b - 1)(b - 2) \cdots (b - c + k + 1)} \]
\[ + \sum_{k=1}^{\infty} \frac{a(a + 1) \cdots (a + k - 1)b(b + 1) \cdots (b + k - 1)}{k! c(c + 1) \cdots (c + k - 1)} \times \left[ \sum_{l=0}^{k-1} \left( \frac{1}{a + l} + \frac{1}{b + l} - \frac{1}{c + l} - \frac{1}{l + 1} \right) \right] z^k, \tag{B3} \]
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FIGURES

FIG. 1. Inhomogeneous vacuum models with a space-like singularity at \( R = 0 \) and null singularities at \( v = u = 0 \). The null singularities are true only for \( \alpha < 1/2 \). Two figures are shown in case when (a) \( R = v + u \) and (b) \( R = -(v + u) \).

FIG. 2. Inhomogeneous vacuum models with a time-like singularity at \( R = 0 \) and null singularities at \( v = u = 0 \). The null singularities are true only for \( \alpha < 1/2 \). Two figures are shown in case when (a) \( R = v - u \) and (b) \( R = -v + u \).

FIG. 3. Colliding gravitational waves with a space-like singularity and null singularities. In regions I(\( v < v_0 \) and \( u < u_0 \)), II(\( v < v_0 \) and \( u > u_0 \)), III(\( v > v_0 \) and \( u < u_0 \)), and IV(\( v > v_0 \) and \( u > u_0 \)), there are the Minkowskian spacetime, the outgoing wave, the incoming wave, and colliding waves, respectively. These singularities are true only in the region IV.

FIG. 4. Colliding gravitational waves with a time-like singularity and null singularities. These singularities are true only in the region IV in which \( v > v_0 \) and \( u > u_0 \). (a) Null singularities are in the lines be and bf. (b) The limiting case when one of null singularities (bf) and a null boundary (ad) are overlapped.
\[ R = 0 \]

Out going

\[ u = 0 \]

\[ R > 0 \]

\[ v = 0 \]

Incoming

\[ u = 0 \]

\[ R = v - u \] (a)

\[ R = -v + u \] (b)

\[ v = 0 \]

\[ R > 0 \]

\[ u = 0 \]
