APPROXIMATE COHOMOLOGY

DAVID KAZHDAN AND TAMAR ZIEGLER

Abstract. Let $k$ be a field, $G$ be an abelian group and $r \in \mathbb{N}$. Let $L$ be an infinite dimensional $k$-vector space. For any $m \in \text{End}_k(L)$ we denote by $r(m) \in [0, \infty]$ the rank of $m$. We define by $R(G, r, k) \in [0, \infty]$ the minimal $R$ such that for any map $A : G \to \text{End}_k(L)$ with $r(A(g' + g'') - A(g') - A(g'')) \leq r$, $g', g'' \in G$ there exists a homomorphism $\chi : G \to \text{End}_k(L)$ such that $r(A(g) - \chi(g)) \leq R(G, r, k)$ for all $g \in G$.

We show the finiteness of $R(G, r, k)$ for the case when $k$ is a finite field, $G = V$ is a $k$-vector space $V$ of countable dimension. We actually prove a generalization of this result.

In addition we introduce a notion of Approximate Cohomology groups $H^d_{\mathbb{Z}}(V, M)$ (which is a purely algebraic analogue of the notion of $\epsilon$-representation (9)) and interpret our result as a computation of the group $H^1_{\mathbb{Z}}(V, M)$ for some $V$-modules $M$.

1. Introduction

Let $k$ be a field, $G$ be an abelian group and $r \in \mathbb{N}$. Let $L$ be an infinite dimensional $k$-vector space, $\text{End}(L) = \text{End}_k(L)$ and $M \subset \text{End}(L)$ be the subspace of operators of finite rank. For any $m \in \text{End}(L)$ we denote by $r(m) \in [0, \infty]$ the rank of $m$. We define by $R(G, r, k) \in [0, \infty]$(correspondingly $R^d(G, r, k)$) the minimal number $R$ such that for any map $A : G \to \text{End}(L)$(correspondingly a map $A : G \to M$), $g \to m_g$ with $r(A(g' + g'') - A(g') - A(g'')) \leq r$, $g', g'' \in G$ there exists a homomorphism $\chi : G \to \text{End}(L)$ such that $r(A(g) - \chi(g)) \leq R(G, r, k)$ for all $g \in G$.

It is easy to see that in the case when $G = \mathbb{Z}$ and $\text{char}(k) \neq 2$ we have $R(\mathbb{Z}, 1, \mathbb{F}) \leq 2$. We sketch the proof: one studies the rank $\leq 1$ operators $r_{m,n} = A(m + n) - A(m) - A(n)$. Since $r_{0,0}$ of rank $\leq 1$, we replace $r_{m,n}$ by $r_{n,m} - r_{0,0}$ and can then assume that $r_{0,0} = 0$. Under this condition one must show that $r_{m,n}$ is a coboundary of rank one operators. The operators $r_{n,m}$ satisfy the equation $r_{a,-a} = r_{a+c,-a} + r_{a,c}$. From this one can deduce that either there is a subspace of codimension 1 in the kernel of all three operators, or a subspace of dimension 1 containing the image of all three. One shows inductively that this property holds for all operators $r_{m,n}$. Unfortunately we don’t know whether $R(\mathbb{Z}, 2, \mathbb{C}) < \infty$.

In this paper we first show that $R^d(V, r, k) < \infty$ in the case when $k = \mathbb{F}_p$ and $G$ is a $k$-vector space $V$ of countable dimension and then show that $R(V, r, k) = R^d(V, r, k)$.

Actually we prove the analogous bound in a more general case when $M$ is replaced by the space of tensors $L^1 \otimes L^2 \otimes \ldots \otimes L^n$. To simplify the exposition we assume that $L_i = L^\vee$ and that $\text{Im}(A)$ is contained in the subset $\text{Sym}^d(L^\vee) \subset L^\vee \otimes^n$ of symmetric tensors. In other words we consider map $A : V \to M^d$ where $M^d$ is isomorphic to the space of homogeneous polynomials of degree $d$ on $L$. We denote by $N^d \subset M^d$ the subspace of multilinear polynomials.

Now some formal definitions.

The second author is supported by ERC grant ErgComNum 682150.
Definition 1.1 (Filtration). Let $M$ be an abelian group. A filtration $\mathcal{F}$ on $M$ is an increasing sequence of subsets $M_i, 1 \leq i \leq \infty$ of $M$ such that for each $i, j$ there exists $c(i, j)$ such that $M_i + M_j \subset M_{c(i, j)}$ and $M = \bigcup M_i$. Two filtrations $M_i, M_i'$ on $M$ are equivalent if there exist functions $a, b: \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $M_i \subset M_{a(i)}$ and $M_i' \subset M_{b(i)}$.

Definition 1.2 (Algebraic rank filtration). Let $k$ be a field. Fix $d \geq 2$ and consider the $k$-vector space $M = M^d$ of homogeneous polynomials $P$ of degree $d$ in variables $x_j, j \geq 1$. For a non-zero homogeneous polynomial $P$ on a $k$-vector space $W, P \in k[W^\vee]$ of degree $d \geq 2$ we define the rank $r(P) = r$, where $r$ is the minimal number $r$ such that it is possible to write $P$ in the form

$$P = \sum_{i=1}^r l_i R_i,$$

where $l_i, R_i \in \bar{k}[W^\vee]$ are homogeneous polynomials of positive degrees (in $\bar{k}$ this is called the $h$-invariant). We denote by $A_d$ the filtration on $M$ such that $M_n$ is the subset of polynomials $P$ with $r(P) < n$. For the $k$-space $P^d$ of non-homogeneous polynomials of degree $\leq d$ define the rank similarly, and denote by $B_d$ the corresponding filtration.

Definition 1.3 (Finite rank homomorphisms). Let $V$ be a countable vector space over $k$. We say that a linear map $P: V \to M^d$ is of finite rank if we can write $P$ as a sum $\sum_{k=1}^{d-1} P_k$ where each $P_k$ is a finite sum $P_k = \sum Q_j R_j$ where $Q_j \in M^k$ and $Q_j$ is a linear map from $V$ to $M^{d-k}$. Denote $\text{Hom}(V, M^d)$ the subspace of $\text{Hom}(V, M^d)$ of finite rank maps.

Now we can formulate our main result.

Theorem 1.4. For any finite field $k = \mathbb{F}_p, d < p - 1, r \geq 0$ there exists $R = R(r, k, d)$ such that for any map $A: V \to M = M^d, \ r(A(v' + v'') - A(v') - A(v'')) \leq r, \ v', v'' \in V$

there exists a homomorphism $\chi_A: V \to M$ such that $r(A(v) - \chi_A(v)) \leq R$. Moreover the homomorphism $\chi_A$ is unique up to an addition of a homomorphism of a finite rank.

Question 1.5. a) Is there a bound on $R$ independent of $k$? Moreover, does there exist $c(d)$ such that $R(V, r, k, d) \leq c(d)r$?

b) Could we drop the condition $d < p - 1$ if $\text{Im}(A) \subset N^d$?

Remark 1.6. Theorem 1.4 does not hold for $p = d = 2$, see [8] for a function from $\mathbb{F}_2^n$ to the space of quadratic forms over $\mathbb{F}_2$ such that $f(u + v) - f(u) - f(v)$ is of rank $\leq 3$ but for $n$ sufficiently large $f$ does not differ from a linear function by a function taking values in bounded rank quadratics. In the low characteristic case the same proof shows that obstructions come from non-classical polynomials see Remark 2.4. In the case when $G$ is a finite cyclic group, $k$ any field, $d = 2$, one can show that $C(1, k) \leq 2$.

We can reformulate Theorem 1.4 as an example of a computation of approximate cohomology groups.

Definition 1.7 (Approximate cohomology). (1) Let $M$ be an abelian group, and let $\mathcal{F} = \{M_i\}$ be a filtration on $M$. 
(2) Let $G$ be a discrete group acting on $M$ preserving the subsets $M_i$. A cochain $r : G^n \rightarrow M$ is an approximate $n$-cocycle if $\text{Im}(\partial r) \subset M_i$ for some $i \in \mathbb{Z}_+$. It is clear that the set $Z^n_F$ of approximate $n$-cocycles is a subgroup of the group $C^n$ of $n$-chains which depends only on the equivalence class of a filtration $\mathcal{F}$.

(3) A cochain $r : G^n \rightarrow M$ is an approximate $n$-coboundary if there exists an $n - 1$-cochain $t \in C^{n-1}$ such that $\text{Im}(r - \partial t) \subset M_i$ for some $i \in \mathbb{Z}_+$. It is clear that the set $B^n_F$ of approximate $n$-coboundaries is a subgroup of $Z^n_F$.

(4) We define $H^n_F = Z^n_F/B^n_F$.

(5) Since any cocycle is an approximate cocycle and any coboundary is an approximate coboundary we have a morphism $a^n_F : H^n(G, M) \rightarrow H^n_F(G, M)$.

In this paper we consider the case when the group $V$ acts trivially on $M$. So the group $Z^1(V, M)$ of 1-cocycles coincides with the group $\text{Hom}(V, M)$ of linear maps, the subgroup of coboundaries $B^1(V, M) \subset Z^1(V, M)$ is equal to $\{0\}$ and therefore $H^1(V, M) = \text{Hom}(V, M)$. In this case we can reformulate the Theorem \[1.4\] in terms of a computation of the map $a^1_F$.

**Corollary 1.8.** Let $k$ be a prime finite field of characteristic $p$, $V$ be a countable vector space over $k$ acting trivially on $(M, \mathcal{F}) = (M^d, A_d)$ and assume that $p > d + 1$. Then the map $a^1 : H^1(V, M) = \text{Hom}(V, M) \rightarrow H^1_F(V, M)$ is surjective, and $\text{Ker}(a^1) = \text{Hom}_f(V, M_d)$.

**Question 1.9.** How to describe $H^n_F(V, M)$ for $n > 1$ ?

**Corollary 1.10.** Let $k$ be a prime finite field of characteristic $p$, and let $V$ be a countable vector space over $k$. Consider the filtration $(\mathcal{P}^d, B_d)$ with $W = V$ and with $V$-acting by translations $(v.P)(x) = P(x + v)$ and assume that $p > d + 1$. Then the map $a^1_B$ is surjective.

**Proof.** Let $P : V \rightarrow \mathcal{P}^d$ where $\mathcal{P}^d$ is the space of polynomial of degree $\leq d$. We assume

$$\partial P(v, v')(x) = P(v + v')(x) - P(v)(x + v') - P(v')(x)$$

is of rank $\leq i$ for any $v, v' \in V$. Let $Q(v)$ be the homogeneous degree $d$ term of $P(v)$. Then since $P(v)(x + v') - P(v)(x)$ is of degree $< d$ we have

$$Q(v + v')(x) - Q(v)(x) - Q(v')(x)$$

is of rank $\leq i + 1$. 

$\square$

**Remark 1.11.** The proof of Theorem \[1.4\] uses the inverse theorem for the Gowers norms \[10\]. One can use also prove the reverse implication modifying the arguments in \[7\], and thus an independent proof of Theorem \[1.4\] could lead to a new proof of the inverse conjecture for the Gowers norms.

2. **Proof of Theorem \[1.4\]**

For a function $f$ on a finite set $X$ we define

$$\mathbb{E}_{x \in X} f(x) = \frac{1}{|X|} \sum_{x \in X} f(x).$$

We use $X \ll L Y$ to denote the estimate $|X| \leq C(L)|Y|$, where the constant $C$ depends only on $L$. We fix a prime finite field $k$ of order $p$ and degree $d < p - 1$ and suppress the dependence of
all bounds on \(k, d\). We also fix a non-trivial additive character \(\psi\) on \(k\).

For a function \(f : G \to H\) a function between abelian groups we denote \(\Delta_h f(x) = f(x + h) - f(x)\). if \(f : G_1 \times G_2 \to H\) then for \(g_1 \in G_1\) we write \(\Delta_{g_1} f\) shorthand for \(\Delta_{(g_1, 0)} f\).

Let \(V\) be a finite vector space over \(k\). Let \(F : V \to k\). The \(m\)-th Gowers norm of \(F\) is defined by

\[
\|\psi(F)\|_{U^m}^2 = \mathbb{E}_{v_1, \ldots, v_m \in V} \psi(\Delta_{v_m} \ldots \Delta_{v_1} F(v)).
\]

These were introduced by Gowers in \([2]\), and were shown to be norms for \(m > 1\).

For a homogeneous polynomial \(P\) on \(V\) of degree \(d\) we define

\[
(2.1) \quad \tilde{P}(x_1, \ldots, x_d) = \Delta_{x_d} \ldots \Delta_{x_1} P(x).
\]

This is a multilinear homogeneous form in \(x_1, \ldots, x_d \in V\) such that

\[
P(x) = \frac{1}{d!} \tilde{P}(x, \ldots, x).
\]

**Proposition 2.1.** There exists a function \(R(L)\) such that for any finite dimensional \(k\)-vector space \(V\) and a map \(P : V \to M = M_d\) such that \(r(P(v + v') - P(v) - P(v')) \leq L\) for \(v, v' \in V\) there exists a linear map \(Q : V \to M\) such that \(r(P(v) - Q(v)) \leq R(L)\).

**Proof.** The proof is based on an argument of \([4], [6]\). Our aim is to show that if \(\partial P(v, v')(x)\) is of rank \(\leq L\) for all \(v, v' \in V\) then there exists a homogeneous polynomial \(Q(v)(x)\) of degree \(\leq d\) such that \(\partial P(v, v')(0) = 0\) and \(P(v) - Q(v) \ll_{L} 1\).

**Lemma 2.2.** Let \(P : V^d \to k\) be a multilinear homogeneous polynomial of degree \(d \geq 2\) and rank \(L\). Then \(\mathbb{E}_{x \in V} \psi(P(x)) \geq C_{L,d}\) for some positive constant \(C_{L,d}\) depending only on \(L, d\).

**Proof.** We prove this by induction on \(d\). For quadratics: \(P(x_1, x_2) = \sum_{i=1}^{L} l_i^1(x_1) l_i^2(x_2)\). If \(x_1 \in \bigcap_i \text{Ker}(l_i^1)\) then \(\psi(P(\bar{x})) \equiv 1\), thus on a subspace \(W\) of codimension at most \(L\) we have \(\psi(P(\bar{x})) \equiv 1\), so that

\[
\sum_{x_1 \in W} \mathbb{E}_{x_2 \in V} \psi(P(\bar{x})) = |W|.
\]

If \(x_1\) is outside \(W\) then the inner sum is nonnegative.

Suppose now that \(d > 2\). Let \(P(x_1, \ldots, x_d)\) be multilinear homogeneous polynomial of degree \(d\) and rank \(L\). Write

\[
P(x_1, \ldots, x_d) = \sum_{j \leq d} \sum_{i \leq L_j} l_i^j(x_j) Q_i^j(x_1, \ldots, \hat{x}_j, \ldots, x_d) + \sum_{k \leq M} T_k(\bar{x}) R_k(\bar{x})
\]

with \(\bar{x} = (x_1, \ldots, x_d)\), \(l_i^j\) are linear and \(T_k(\bar{x}) R_k(\bar{x})\) is homogenous multilinear in \(x_1, \ldots, x_d\) such that the degrees of \(T_k(\bar{x})\) and \(R_k(\bar{x})\) are \(\geq 2\), and \(\sum_j L_j + M = L\).

If for all \(j\) we have \(L_j = 0\), then \(P(\bar{x}) = \sum_{k \leq L} T_k(\bar{x}) R_k(\bar{x})\) with the degree of \(T_k(\bar{x})\), \(R_k(\bar{x}) \geq 2\).

For \(x_1 \in V\) write \(P_{x_1}(x_2, \ldots, x_d) = P(x_1, x_2, \ldots, x_d)\). Then \(P_{x_1}\) is of rank \(L\) and degree \(d - 1\) for all \(x_1\) and we obtain the claim by induction.
Otherwise there is a \( j \), such that \( L_j > 0 \), without loss of generality \( j = 1 \). Let \( W = \bigcap_i \text{Ker}(l_i^1) \), then \( W \) is of codimension at most \( L_1 \). For \( x_1 \in W \) let \( P_{x_1}(x_2, \ldots, x_d) = P(x_1, x_2, \ldots, x_d) \). Consider the sum

\[
E_{x_2, \ldots, x_d \in V} \psi(P_{x_1}(x_2, \ldots, x_d)).
\]

For \( x_1 \in W \) we have \( P_{x_1} \) is of rank \( \leq L - L_1 \) and homogeneous of degree \( d - 1 \). By the induction hypothesis the above sum is \( \geq C_L d - 1 \), so that

\[
\sum_{x_1 \in W} E_{x_2, \ldots, x_d \in V} \psi(P_{x_1}(x_2, \ldots, x_d)) \geq C_{L,d-1}|W|.
\]

For any \( x_1 \notin W \), \( P_{x_1} \) is of degree \( d - 1 \), and of rank \( < \infty \) thus by the induction hypothesis,

\[
E_{x_2, \ldots, x_d \in V} \psi(P_{x_1}(x_2, \ldots, x_d)) \geq 0.
\]

Thus

\[
E_{x_1 \in V} E_{x_2, \ldots, x_d \in V} \psi(P_{x_1}(x_2, \ldots, x_d)) \geq C_{L,d-1}|W|/|V|,
\]

and we obtain the claim. \( \square \)

Let \( P : V \to M_d \) be a map such that for all \( u, v \):

\[
\text{rk}(P(v) + P(u) - P(v + u)) < L.
\]

We define a function on \( V \times W^d \) by

\[
f(v, x_1, \ldots, x_d) = \psi(\tilde{P}(v)(x_1, \ldots, x_d)) = \psi(\tilde{P}(v)(\bar{x})).
\]

**Lemma 2.3.** \( \|f\|_{U^{d+2}} \geq c_L \).

**Proof.** We expand

\[
\Delta_{(v_{d+2}, \bar{h}_{d+2})} \cdots \Delta_{(v_1, \bar{h}_1)} \tilde{P}(v)(\bar{x}) = \sum_{k=0}^{d+2} \Delta_{\bar{h}_{d+2}} \cdots \Delta_{\bar{h}_{k+1}} \Delta_{v_k} \cdots \Delta_{v_1} \tilde{P}(v)(\bar{x} + \bar{h}_1 + \ldots + \bar{h}_k)
\]

with \( v_i \in V \) and \( \bar{h}_i \in V^d \). Since \( P_v \) is of degree \( d \) the above is equal

\[
\sum_{k=2}^{d+2} \Delta_{\bar{h}_{d+2}} \cdots \Delta_{\bar{h}_{k+1}} \Delta_{v_k} \cdots \Delta_{v_1} \tilde{P}(v)(\bar{x} + \bar{h}_1 + \ldots + \bar{h}_k)
\]

Since \( \text{rk}(P(v) + P(u) - P(v + u)) < L \), for any \( v_1 + u_1 = v_2 + u_2 \) we have

\[
\text{rk}(P(v_1) + P(u_1) - P(v_2) - P(u_2)) < 2L.
\]

that for \( k \geq 2 \) we have \( \Delta_{v_k} \cdots \Delta_{v_1} \tilde{P}(v) \) is of rank \( \ll_L 1 \). For fixed \( v, v_1, \ldots, v_k \), the above polynomial can be expresses as a multilinear homogeneous polynomial of degree \( d \) in \( y_1, \ldots, y_d \) with

\[
y_j = (x_j, h_{1j}^1, \ldots, h_{d+2}^j).
\]

which is of rank that is bounded in terms of \( L, d \). Now apply previous lemma. \( \square \)
By the inverse theorem for the Gowers norm [11] there is a polynomial $Q$ on $V^{d+1}$ of degree $d+1$ on with $Q : V \times V^d \to k$ s.t. such that
\[ |E_{v,x_1,\ldots,x_d} \psi(\tilde{P}(v)(x_1,\ldots,x_d) - Q(v,x_1,\ldots,x_d))| \gg_L 1.\]

By an application of the triangle and Cauchy-Schwarz inequalities we obtain
\[ |E_{v,x_1,\ldots,x_d} \psi(\tilde{P}(v)(x_1,\ldots,x_d) - Q(v,x_1,\ldots,x_d))|^2 \]
\[ \leq |E_{v,x_1,\ldots,x_{d-1}} E_{x_d} \psi(\tilde{P}(v)(x_1,\ldots,x_d) - Q(v,x_1,\ldots,x_d))|^2 \]
\[ = E_{v,x_1,\ldots,x_{d-1}} E_{x_d} (\tilde{P}(v)(x_1,\ldots,x_{d-1},x'_d) - \tilde{P}(v)(x_1,\ldots,x_d) - Q(v,x_1,\ldots,x_{d-1},x'_d) + Q(v,x_1,\ldots,x_d)). \]

Where the last equality follows from the fact that $\tilde{P}$ is homogeneous multilinear form and thus
$\tilde{P}(v)(x_1,\ldots,x_{d-1},x'_d) - \tilde{P}(v)(x_1,\ldots,x_d) = \tilde{P}(v)(x_1,\ldots,x_{d-1},x'_d)$.

Applying Cauchy-Schwarz $d - 1$ more times we obtain
\[ E_{v,x_1,x'_1,\ldots,x_{d-1},x'_d} \psi(\tilde{P}(v(x'_1,\ldots,x'_d)) - \Delta x'_1 \ldots \Delta x'_d Q(v,x_1,\ldots,x_d)) \gg_L 1 \]

One more application of Cauchy-Schwarz gives
\[ E_{v',x',x'_1,\ldots,x_{d-1},x'_d} \psi(\tilde{P}(v')(x',x'_1,\ldots,x'_d) - \Delta v' \Delta x'_1 \ldots \Delta x'_d Q(v,x_1,\ldots,x_d)) \gg_L 1. \]

Since $Q$ is a polynomial of degree $d + 1$, $\Delta v' \Delta x'_1 \ldots \Delta x'_d Q$ is independent of $v, x_1, \ldots, x_d$ so we obtain
\[ |E_{v',x',x'_1,\ldots,x_{d-1},x'_d} \psi(\tilde{P}(v+v') - \tilde{P}(v')(x'_1,\ldots,x'_d) - Q'(v',x'_1,\ldots,x'_d))| \gg_L 1 \]

with $\tilde{Q}$ a multilinear homogeneous form in $v', x'_1, \ldots, x'_d$. Denote by $\tilde{Q}(v)$ the function on $W^d$ given by $\tilde{Q}(v, \bar{x}) = Q(v, \bar{x})$. Then
\[ E_{v',v'} |E_{x'_1,\ldots,x'_d} \psi(\tilde{P}(v+v') - \tilde{P}(v) - \tilde{Q}(v')(x'_1,\ldots,x'_d))| \gg_L 1. \]

By [3] (Proposition 6.1) and [13] (Lemma 4.17) it follows that for at least $\gg_L |V|^2$ values of $v, v'$ we have $P(v + v') - P(v) - Q'(v')$ is of rank $\ll_L 1$, where $Q'(v)(x) = \tilde{Q}(v)(x, \ldots, x)/d^!$ where $Q'(v)$ is linear in $v$. Recall now that $P(v + v') - P(v) - P(v')$ is of rank $\leq L$, so that we get a set $E$ of size $\gg_L |V|$ of $v$ for which
\[ P(v) = Q'(v) + R(v) \]

with $R(v)$ of rank $\leq L$. Since $E \gg_L |V|$, by the Bogolyubov lemma (see e.g. [12]) $2E$ contains a subspace $E'$ of codimension $K \ll_L 1$ in $V$. For $v \in E'$, define
\[ P'(v) = Q'(v). \]

Let $P' : V \to M_d$ be any extension of $P'$ linear in $v \in V$. Then $P'(v) - P(v)$ is of rank $\ll_L 1$, and $P'(v)$ is a cocycle. \qed
**Remark 2.4.** In the case where \( p \leq d + 2 \), by the inverse theorem for the Gowers norms over finite fields the polynomial \( Q \) in the above argument on \( V \times V^d \) would be replaced by a nonclassical polynomial see [11], and the same argument would give that the approximate cohomology obstructions lie in the nonclassical degree \( d \) polynomials - these are functions \( P : V \to \mathbb{T} \) satisfying \( \Delta_{h_{d+1}} \ldots \Delta_{h_1} P \equiv 0 \).

**Proof of Theorem 1.4** Let \( k = \mathbb{F}_q \), and let \( V \) be an countable vector space over \( k \). Denote \( V_n = k^n \), then \( V = \bigcup V_n \). Let \( M \subset k[x_1, ..., x_n, ...] \) be the subspace of homogeneous polynomials of degree \( d \). For any \( l \geq 1 \) we denote by \( N_l \subset M \) the subset of polynomials of in \( x_1, ..., x_l \) and denote by \( p_l : M \to N_l \) the projection defined by \( x_i \to 0 \) for \( i > l \). Observe that \( p_n \) does not increase the rank. By Proposition 2.4 there is a constant \( C \) depending only on \( L, d \) (and \( k \)) such that for any \( n \) there exists a linear map \( \phi_n : V_n \to M \) such that \( \text{rank} \ (P(v) - \phi_n(v)) \leq C \), for \( v \in V_n \).

We now show that the existence of such linear maps \( \phi_n \) implies the existence of a linear map \( \psi : V \to M \) such that \( \text{rank} (R(v) - \psi(v)) \leq C \), for \( v \in V \).

**Lemma 2.5.** Let \( X_n, n \geq 0 \) be finite not empty sets and \( f_n : X_{n+1} \to X_n \) be maps. Then one can find \( x_n \in X_n \) such that \( f_n(x_{n+1}) = x_n \).

**Proof.** This result is standard, but for the convenience of a reader we provide a proof. The claim is obviously true if the maps \( f_n \) are surjective. For any \( m > n \) we define the subset \( X_{m,n} \subset X_n \) as the image of

\[
\tilde{f}_n \circ \ldots \circ f_{m-1} : X_m \to X_n
\]

It is clear that for a fixed \( n \) we have

\[
X_n \supset X_{n+1,n} \supset \ldots \supset X_{m,n} \supset \ldots
\]

We define \( Y_n \) as the intersection \( \cap_{m > n} X_{m,n} \). Since the set \( X_n \) is finite, the sets \( X_{m,n} \) stabilize as \( m \) grows and hence \( Y_n \) is not empty. Let \( \tilde{f}_n \) be the restriction of \( f_n \) on \( Y_{n+1} \). Now the maps \( \tilde{f}_n : Y_{n+1} \to Y_n \) are surjective, thus the lemma follows. \( \square \)

**Lemma 2.6.** Let \( P : V \to M \) be a map such that for any \( n \) there exists a linear map \( \phi_n : V_n \to M \) such that \( \text{rank} \ (P(v) - \phi_n(v)) \leq C \), \( v \in V_n \). Then there exists a linear map \( \psi : V \to M \) such that \( \text{rank} (P(v) - \psi(v)) \leq C \), for all \( v \in V \).

**Proof.** Let \( l(n) \) be such that \( P(V_n) \subset N_{l(n)} \) and \( \psi_n = p_{l(n)} \circ \phi_n \). Since \( p_n \) does not increase the rank, and since \( (p_{l(n)} \circ P)(V_n) = R(V_n) \) we have

\[
(\star_n) \quad \text{rank}(P(v) - \psi_n(v)) \leq C, \quad v \in V_n.
\]

We apply Lemma 2.5 to the case when \( X_n \) is the set of linear maps \( \psi_n : V_n \to N_{l(n)} \) satisfying \( (\star_n) \) and \( f_n \) are the restriction from \( V_{n+1} \) onto \( V_n \) we find the existence of linear maps \( \psi_n : V_n \to N_{l(n)} \) satisfying the condition \( (\star_n) \) and such that the restriction of \( \psi_{n+1} \) onto \( V_n \) is equal to \( \psi_n \). The system \( \{\psi_n\} \) defines a linear map \( \psi : V \to M \).

\( \square \)

We now prove the result stated in the abstract by proving the equality \( R(V,r,k) = R^f(V,r,k) \). Let \( \Gamma \) be an abelian group, \( \Gamma = \bigcup \Gamma_n \) where \( \Gamma_n \) are finitely generated groups. Let \( k \) be a finite field, and let \( V, W \) be \( k \)-vector spaces with bases \( v_j, w_j \). For \( n \geq 1 \) let \( V_n, W_n \) be the spans of \( v_j, w_j, 1 \leq j \leq n \). We denote by \( i_n : V_n \to V_{n+1} \) the natural imbedding and by \( \beta_n : W_{n+1} \to W_n \) the natural projection. Denote \( \text{Hom}^f(V,W) \) the finite rank homomorphisms from \( V \to W \).
Proposition 2.7. Suppose there exists $C = C(c)$ such that for any map $a^I : \Gamma \to \text{Hom}^I(V,W)$ such that

$$r(a^I(\gamma + \gamma'') - a^I(\gamma') - a^I(\gamma'')) \leq c$$

there exists a homomorphism $\chi^I : \Gamma \to \text{Hom}^I(V,W)$ such that $r(a^I(\gamma) - \chi^I(\gamma)) \leq C$. Then for any map $a : \Gamma \to \text{Hom}(V,W)$ such that

$$r(a(\gamma + \gamma'') - a(\gamma') - a(\gamma'')) \leq c$$

there exists a homomorphism $\chi : \Gamma \to \text{Hom}(V,W)$ such that $r(a(\gamma) - \chi(\gamma)) \leq C$.

Proof. We will use the following fact that is an immediate consequence of König’s lemma: Let $X$ be a locally finite tree, $x \in X$. If for any $N$ there exists a branch starting at $x$ of length $N$ then there exists an infinite branch starting at $x$.

Let $a : \Gamma \to \text{Hom}(V,W)$ be a map such that

$$r(a(\gamma + \gamma'') - a(\gamma') - a(\gamma'')) \leq c$$

We define $F_n = \text{Hom}(V_n, W_n)$. Let $Y_n = \text{Hom}(\Gamma_n, F_n)$ and $q_n : Y_{n+1} \to Y_n$ be given by

$$q_n(\chi_{n+1}) = \beta_n \circ \chi'_{n+1} \circ i_n$$

where $\chi'_{n+1}$ is the restriction of $\chi_{n+1}$ on $\Gamma_n$.

We denote by $X_n \subset Y_n$ the subset of homomorphisms $\chi_n$ of $\Gamma_n$ such that

$$r(\beta_n \circ a(\gamma) \circ i_n - \beta_n \circ \chi_n(\gamma) \circ i_n) \leq C.$$ Let $X$ be the disjoint union of $X_n$ and we connect $\chi_n \in X_n$ with $\chi_{n+1} \in X_{n+1}$ if $\chi_n = q_n(\chi_{n+1})$.

By the assumption $X_n$ are finite not empty sets and for any $n$ there exists a branch from $X_0$ to $X_n$ (any $\chi_n \in X_n$ defines such a branch). Now the Lemma 2.5 implies the existence a character $\chi : \Gamma \to \text{Hom}(V,W)$ such that $r(a(\gamma) - \chi(\gamma)) \leq C$. □

To conclude the proof of Theorem 1.4 we calculate the kernel of the map $a^1$:

Proposition 2.8. The kernel of $a^1$ consists of maps $P$ of finite rank.

Proof. Suppose $V, W$ are of dimension $n_1, n_2$ respectively. All the bounds below are independent of $n_1, n_2$. Suppose $P : V \to M_d$ is a linear map with $r(P) \leq L$. Let $\tilde{P}$ be the multilinear version of $P$ as in (2.1). Let $f(v, \bar{x}) = \psi(P(v)(\bar{x}))$. Now $\tilde{P}(\bar{x})$ is a multilinear polynomial on $V \times W^d$ of degree $d + 1$.

For any fixed $v$ we have

$$\mathbb{E}_{\bar{x} \in W^d} \psi(\tilde{P}(v)(\bar{x})) \geq C(L),$$

and thus

$$\mathbb{E}_{v \in V} \mathbb{E}_{\bar{x} \in W^d} \psi(\tilde{P}(v)(\bar{x})) \geq C(L).$$

It follows that $\tilde{P}(v)(\bar{x})$ is of bounded rank $\ll L$ and thus of the form

$$\tilde{P}(v)(\bar{x}) = \sum_{j=1}^K \hat{Q}_j(v, \bar{x})\hat{R}_j(v, \bar{x})$$
with \( \tilde{Q}_j, \tilde{R}_j \) of degree \( \geq 1 \), for any fixed \( v \) also \( \tilde{Q}_j, \tilde{R}_j \) are of degree \( \geq 1 \), and \( K \ll L \). For any fixed \( x \), \( \tilde{P}(V) \) is linear and thus either \( \tilde{Q}_j, \tilde{R}_j \) are constant as a function of \( v \). Recall that \( P(v)(x) = \frac{1}{d!} \tilde{P}(v)(x, \ldots, x) \), and let \( Q_j(v, x) = \frac{1}{d!} \tilde{Q}_j(v, x, \ldots, x) \), similarly \( R_j \). Let \( J \) be the set of \( j \) in the sum \( P = \sum_j Q_j R_j \) such that \( Q_j \) is linear in \( v \) and does not depend on \( x \). Let \( V' = \bigcap_j \text{Ker} Q_j \). The restriction of \( P \) to \( V' \) has finite (that is by a constant which does not depend on \( n_1, n_2 \)) rank. Since \( \text{codim}(V') \leq |J| \) we see that \( P \) has rank that is bounded by a constant which does not depend on \( n_1 \) and \( n_2 \).

Now let \( V, W \) be infinite. Let \( V_n, l(n), p_l(n) \) be as in Lemma 2.6. Let \( P_n(V_n) = (p_l(n) \circ P)(V_n) \). Now apply Lemma 2.6 for \( X \) the collection of finite rank maps from \( V_n \to N_l(n) \), and \( f_n \) the restriction as before. This finishes a proof of Theorem 1.4.

\[ \square \]

**REFERENCES**

[BL] Bhowmick A., Lovett S. Bias vs structure of polynomials in large fields, and applications in effective algebraic geometry and coding theory.

[1] Bergelson, V., Tao, T., Ziegler, T. An inverse theorem for the uniformity seminorms associated with the action of \( F_\infty^\infty \). Geom. Funct. Anal. 19 (2010), no. 6, 1539-1596.

[2] Gowers, T. A new proof of Szemerédi’s theorem. Geom. Funct. Anal. 11, (2001) 465588.

[3] Green, B., Tao, T. The distribution of polynomials over finite fields, with applications to the Gowers norms. Contrib. Discrete Math. 4 (2009), no. 2, 1-36.

[4] Green, B., Tao, T. An equivalence between inverse sumset theorems and inverse conjectures for the \( U_3 \) norm. Math. Proc. Cambridge Philos. Soc. 149 (2010), no. 1, 1-19.

[5] Kazhdan, D. On \( \epsilon \)-representations. Israel J. Math. 43 (1982), no. 4, 315-323.

[6] Lovett, S. Equivalence of polynomial conjectures in additive combinatorics. Combinatorica 32 (2012), no. 5, 607-618.

[7] Samorodnitsky, A. Low-degree tests at large distances, STOC 07.

[8] Tao, T. https://terrytao.wordpress.com/2008/11/09/a-counterexample-to-a-strong-polynomial-freiman-ruzsa-conjecture/

[9] Schmidt, W.M. Bounds for exponential sums Acta Arith. 44 (1984), 281-297.

[10] Tao, T., Ziegler, T. The inverse conjecture for the Gowers norm over finite fields via the correspondence principle. Anal. PDE 3 (2010), no. 1, 1-20.

[11] Tao, T., Ziegler, T. The inverse conjecture for the Gowers norm over finite fields in low characteristic. Ann. Comb. 16 (2012), no. 1, 121-188.

[12] Wolf, J. Finite field models in arithmetic combinatorics ten years on. Finite Fields Appl. 32 (2015), 233-274.

EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, GIVAT RAM THE HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, 91904, ISRAEL

E-mail address: david.kazhdan@mail.huji.ac.il

EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, GIVAT RAM THE HEBREW UNIVERSITY OF JERUSALEM, JERUSALEM, 91904, ISRAEL

E-mail address: tamarz@math.huji.ac.il