Quantum deformations of Schwarzschild and Schwarzschild–de Sitter spacetimes

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Abstract
A quantum Schwarzschild spacetime and a quantum Schwarzschild–de Sitter spacetime with a cosmological constant $\Lambda$ are constructed within the framework of a noncommutative Riemannian geometry developed in an earlier publication. The metrics and curvatures of the quantum Schwarzschild spacetime and the quantum Schwarzschild–de Sitter spacetime are computed. It is shown that up to the second order in the deformation parameter, the quantum spacetimes are solutions of a noncommutative Einstein equation.

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1. Introduction
There have been intensive activities studying noncommutative analogs of general relativity. The studies are largely motivated by the widely held belief that the usual notion of spacetime as a pseudo-Riemannian manifold needs to be modified at the Planck scale. Indeed it was demonstrated in [1] that at the Planck scale, the coordinates describing spacetime points satisfied certain nontrivial commutation relations analogous to those appearing in quantum mechanics. This suggests the relevance of noncommutative geometry to Planck scale physics.

A large volume of work has been done on noncommutative analogs of black holes [2–5] by using a variety of physically motivated methods and incorporating different physical intuitions. The papers in [3] assumed a Gaussian distribution for matter in Einstein gravity and analyzed the integrated effect of noncommutativity as corrections to the classical Schwarzschild black hole. Analysis in a similar vein was carried out in [4]. In [5], Chaichian and collaborators investigated corrections to physical quantities of black holes arising from the noncommutativity of spacetime itself by using a gauge theoretical formulation. General
relativity on a noncommutative spacetime is regarded in [5] as a noncommutative gauge theory of a deformed Lorentzian algebra analogous to the classical picture of Utiyama and Kibble. A methodology akin to this is also adopted in [6, 7, 8]. Particularly noteworthy is the study in [8], which showed that a deformation of general relativity arising from [7] differed qualitatively from the low-energy limit of string theory. Very recently, Buric and Madore in [11] (see also [10]) explored a possible moving frame formalism for a noncommutative geometry on the Moyal space as the first step toward setting up a framework for studying Schwarzschild or Schwarzschild Sitter black holes.

In [9], a theory of noncommutative Riemannian geometry over the Moyal algebra was developed by studying noncommutative surfaces embedded in higher dimensions. This theory makes use of Nash’s isometric embedding theorem [14] and its generalizations [15]. It retains key notions of usual Riemannian geometry, such as the metric and curvature, which are essential for describing gravity. Other features of the theory of [9] are its simplicity and transparent consistency, which render the theory particularly amenable to explicit computations. As we shall see in this paper, computations within the theory are no more difficult than that in usual Riemannian geometry. Furthermore, it is possible to apply the formalism of [9] to obtain a theory of $\kappa$-deformed [12] noncommutative geometry. The $\kappa$-deformed spacetimes (see [13] for references) are another class of noncommutative spacetimes much studied in the literature.

This paper applies the theory of noncommutative Riemannian geometry developed in [9] to investigate quantum aspects of gravity from a mathematical point of view. Specifically, we construct quantum deformations of the Schwarzschild spacetime and the Schwarzschild–de Sitter spacetime in the framework of [9] and investigate the physical properties of such noncommutative spacetimes. The key results on the quantum Schwarzschild spacetime are equations (3.6) and (3.8), which respectively give the metric and the Ricci curvature. The metric and the Ricci curvature of the quantum Schwarzschild–de Sitter spacetime are respectively given by equations (4.2) and (4.3). The Hawking temperature and entropy of the quantum Schwarzschild black hole are investigated, and a quantum correction to the entropy-area law is observed (see (3.7)).

The noncommutative analog of the Einstein equation in the vacuum proposed in [9] is generalized to include matter (equation (2.13)). We show that the quantum Schwarzschild spacetime and quantum Schwarzschild–de Sitter spacetime are solutions of (2.13) in the vacuum exact to the first order in the deformation parameter. However, higher order terms appear to require matters sources. It will be interesting to investigate the physical origin and implications of the source terms.

The organization of this paper is as follows. In section 2, we briefly review the noncommutative Riemannian geometry developed in [9] in the light of Nash’s isometric embedding theorem [14] and its generalization to pseudo-Riemannian manifolds [15]. In section 3, we present two constructions of a quantum Schwarzschild spacetime and study its noncommutative geometry. The Hawking temperature and entropy of the quantum Schwarzschild black hole are also analyzed. Section 4 constructs the quantum Schwarzschild–de Sitter spacetime and studies its noncommutative geometry. Section 5 concludes the paper with some brief comments on the results obtained.

2. Local quantum deformation of spacetimes

Let $(N^{1,n-1}, g)$ be an $n$-dimensional Lorentzian manifold whose metric $g$ has signature $(-1, 1, \ldots, 1)$. By results of [15], which extends Nash’s isometric embedding to pesudo-
Riemannian manifolds, there exist positive integers \( p, q \) and a set of smooth functions \( X^1, \ldots, X^p, X^{p+1}, \ldots, X^{p+q} \) on \( N^{1,n-1} \) such that
\[
g = -(dX^1)^2 - \cdots - (dX^p)^2 + (dX^{p+1})^2 + \cdots + (dX^{p+q})^2.
\] (2.1)

Let \( U \) be a coordinate chart of \( N^{1,n-1} \) with natural coordinates \( \{x^0, x^1, \ldots, x^n\} \). Following the mathematical convention, we allow the possibility that some of the coordinates are dimensionless instead of having the dimension of length or time. Let \( \hbar \) be a real indeterminate, and denote by \( \mathbb{R}[[\hbar]] \) the ring of formal power series in \( \hbar \). Here \( \hbar \) is taken to be dimensionless. Physically one may regard \( \hbar \) as the ratio of the standard model mass scale and the Planck mass.

Let \( \mathcal{A} \) be the set of formal power series in \( \hbar \) with coefficients being real smooth functions on \( U \). Namely, every element of \( \mathcal{A} \) is of form \( \sum_{i \geq 0} f_i \hbar^i \) where \( f_i \) are smooth functions on \( U \).

Then \( \mathcal{A} \) is an \( \mathbb{R}[[\hbar]] \)-module in the obvious way.

Given any two smooth functions \( u \) and \( v \) on \( N^{1,n-1} \), we denote by \( u*v \) the usual pointwise product of the two functions. We also define their star product (or more precisely, Moyal product) \( u * v \) on \( U \) by
\[
(u * v)(x) = \lim_{x' \to x} \exp \left( \hbar \sum_{i,j} \theta_{ij} \partial_i \partial_j \right) u(x)v(x'),
\] (2.2)
where \( \partial_i = \frac{\partial}{\partial x^i} \), and \((\theta_{ij})\) is a constant skew symmetric \( n \times n \) matrix. In order for the exponential to be dimensional less, the operators \( \partial_i \) of the matrix may need to have different dimensions. It is well known that such a multiplication is associative. Since \( \theta \) is constant, the Leibniz rule remains valid in the present case:
\[
\partial_i (u * v) = \partial_i u * v + u * \partial_i v.
\]

For positive integer \( m = p + q \), we define a dot product
\[
\bullet : \mathcal{A}^m \otimes \mathbb{R}[[\hbar]] \mathcal{A}^m \to \mathcal{A}^m
\] (2.3)
over \( U \) by
\[
A \bullet B = - \sum_{i=1}^p a_i \bullet b_i + \sum_{j=p+1}^{p+q} a_i \bullet b_j
\]
for all \( A = (a_1, \ldots, a_m) \) and \( B = (b_1, \ldots, b_m) \) in \( \mathcal{A}^m \). The dot product is a map of two-sided \( \mathcal{A} \)-modules.

For a given \( X \in \mathcal{A}^m \), we let \( E_i = \partial_i X \), and define
\[
g_{ij} = E_i \bullet E_j.
\]
Denote by \( \mathbf{g} = (g_{ij}) \) the \( n \times n \) matrix with entries \( g_{ij} \in \mathcal{A} \). If \( \mathbf{g} \mathrm{mod} \hbar \) is invertible over \( U \), we shall call \( \mathbf{g} \) the \textit{local quantum deformation of spacetime metric} \( g \) over \( U \).

The discussion on the metric in [9] carries over to the present situation; in particular, the invertibility of \( \mathbf{g} \) mod \( \hbar \) implies that there exists a unique inverse \( \mathbf{g}^{-1} \) such that
\[
g_{ij} g^{jk} = g^{kj} g_{ji} = \delta_i^j.
\]

Now as in [9], we define the left tangent bundle \( TX \) (respectively right tangent bundle \( \tilde{T}X \)) of the local noncommutative spacetime \((U, \mathbf{g})\) as the left (respectively right) \( \mathcal{A} \)-submodule of \( \mathcal{A}^m \) generated by the elements \( E_i \). The fact that the metric \( \mathbf{g} \) belongs to \( GL_n(\mathcal{A}) \) enables us to show that the left and right tangent bundles are projective \( \mathcal{A} \)-modules. Let
\[
E^i = g^{ij} \bullet E_j, \quad \bar{E}^i = E_j \bullet g_{ij}.
\]
which belong to \( T X \) and \( \tilde{T} X \) respectively. Then the metric gives rise to a \( \mathcal{A}\)-bimodule map 
\[ g : T X \otimes_{\mathbb{R}[h]} \tilde{T} X \rightarrow \mathcal{A} \]
with the property
\[ g(E_i, E_j) = g_j^i, \quad g(E^i, \tilde{E}^j) = g^{ij}. \]
\[ g(E^i, E_j) = \delta^i_j = g(E_j, \tilde{E}^i). \]

The connection \( \nabla_i \) on the left tangent bundle will be defined in the same way as [9], namely, by the composition of the derivative \( \partial_i \) with the projection from the free left \( \mathcal{A} \)-module \( \mathcal{A}^m \) onto the left tangent bundle. The connection \( \tilde{\nabla}_i \) on the right tangent bundle is defined similarly. In order to describe the connections more explicitly, we note that there exist \( \Gamma^k_{ij} \) and \( \tilde{\Gamma}^k_{ij} \) in \( \mathcal{A} \) such that
\[ \nabla_i E_j = \Gamma^k_{ij} \ast E_k, \quad \tilde{\nabla}_i E_j = E_k \ast \tilde{\Gamma}^k_{ij}. \]

Because the metric is invertible, the elements \( \Gamma^k_{ij} \) and \( \tilde{\Gamma}^k_{ij} \) are uniquely defined by equation (2.4). We have
\[ \Gamma^k_{ij} = \partial_i E_j \ast \tilde{E}^k \quad \tilde{\Gamma}^k_{ij} = E_k \ast \partial_i E_j. \]

It is evident that \( \Gamma^k_{ij} \) and \( \tilde{\Gamma}^k_{ij} \) are symmetric in the indices \( i \) and \( j \). The following closely related objects will also be useful later:
\[ \Gamma_{ijk} = \partial_i E_j \ast E_k, \quad \tilde{\Gamma}_{ijk} = E_k \ast \partial_i E_j. \]

In contrast to the commutative case, \( \Gamma_{ijk} \) and \( \tilde{\Gamma}_{ijk} \) do not coincide in general. We have
\[ \Gamma^k_{ij} = c_{ijl} g^l + \Upsilon_{ijl} g^k, \quad \tilde{\Gamma}^k_{ij} = g^{kij} - g^{kij} \ast \Upsilon_{ijl}, \]

where
\[ c_{ijl} = \frac{1}{2} (\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}), \]
\[ \Upsilon_{ijl} = \frac{1}{2} (\partial_i E_l \ast E_j - \partial_j E_l \ast E_i). \]

Therefore, the left and right connections involve two parts. The part \( c \Gamma_{ijl} \) depends on the metric only, while the noncommutative torsion \( \Upsilon_{ijl} \) embodies extra information. In the present case, the noncommutative torsion explicitly depends on the embedding. In the classical limit with \( \hbar = 0 \), \( \Upsilon_{ijl} \) vanishes and both \( \Gamma_{ijk} \) and \( \tilde{\Gamma}_{ijk} \) reduce to the standard Levi-Civita connection.

The connections are metric compatible in the following sense [9, proposition 2.7]
\[ \partial_i g(Z, \tilde{Z}) = g(\nabla_i Z, \tilde{Z}) + g(Z, \nabla_i \tilde{Z}), \quad \forall Z \in T X, \quad \tilde{Z} \in \tilde{T} X. \]

This is equivalent to the fact that
\[ \partial_i g_{jk} - \Gamma_{ijk} - \tilde{\Gamma}_{ijk} = 0. \]

In contrast to the commutative case, equation (2.7) by itself is not sufficient to uniquely determine the connections \( \Gamma_{ijk} \) and \( \tilde{\Gamma}_{ijk} \); the noncommutative torsion needs to be specified independently.

Let \([\nabla_i, \nabla_j] := \nabla_i \nabla_j - \nabla_j \nabla_i \) and \([\tilde{\nabla}_i, \tilde{\nabla}_j] := \tilde{\nabla}_i \tilde{\nabla}_j - \tilde{\nabla}_j \tilde{\nabla}_i \). Straightforward calculations show that for all \( f, Z \in \mathcal{A} \),
\[ [\nabla_i, \nabla_j](f \ast Z) = f \ast [\nabla_i, \nabla_j]Z, \quad Z \in T X, \]
\[ [\tilde{\nabla}_i, \tilde{\nabla}_j](W \ast f) = [\tilde{\nabla}_i, \tilde{\nabla}_j]W \ast f, \quad W \in \tilde{T} X. \]

Clearly the right-hand side of the first equation belongs to \( T X \), while that of the second equation belongs to \( \tilde{T} X \). Thus the maps \([\nabla_i, \nabla_j]\) and \([\tilde{\nabla}_i, \tilde{\nabla}_j]\) are left and right \( \mathcal{A}\)-module homomorphisms, respectively. So we can always write
\[ [\nabla_i, \nabla_j]E_k = R_{kij}^l \ast E_l, \quad [\tilde{\nabla}_i, \tilde{\nabla}_j]E_k = E_l \ast \tilde{R}_{kij}^l. \]
for some $R^l_{kij}, \tilde{R}^l_{kij} \in A$. We refer to $R^l_{kij}$ and $\tilde{R}^l_{kij}$ respectively as the Riemann curvatures of the left and right tangent bundles of the noncommutative spacetime $(U, g)$. We have

$$R^l_{kij} = -\partial_j \Gamma^l_{ik} - \Gamma^p_{jk} \Gamma^l_{ip} + \partial_i \Gamma^l_{jk} - \Gamma^p_{jp} \Gamma^l_{ip},$$

$$\tilde{R}^l_{kij} = -\partial_j \tilde{\Gamma}^l_{ik} - \tilde{\Gamma}^p_{jk} \Gamma^l_{ip} + \partial_i \tilde{\Gamma}^l_{jk} - \tilde{\Gamma}^p_{jp} \tilde{\Gamma}^l_{ip}.$$

Let us define

$$R_{klij} = R^p_{kij} \ast g_{pl}, \quad \tilde{R}_{klij} = -g_{kp} \ast \tilde{R}^p_{lji}.$$  

Then these Riemannian curvatures of the left and right tangent bundles coincide [9, lemma 2.12]:

$$R_{klij} = \tilde{R}_{klij}.$$ 

Therefore we only need to study the Riemannian curvature on one of the tangent bundles.

Another important property of the Riemannian curvature is that it satisfies the noncommutative analogs of the first and second Bianchi identities [9, theorem 4.3]. Note also that $R_{klij} = -R_{klji}$, but there is no simple rule to relate $R_{klij}$ to $R_{klij}$ in contrast to the commutative case.

Let $R_{ij} = g_{ip} R^p_{ij}, R = g^{ij} \ast R_{ij},$ (2.9) and call them the Ricci curvature and scalar curvature, respectively. Let $R^l_j = g^{ik} \ast R_{klj},$ (2.10) then the scalar curvature is $R = R^l_j$. Let us also introduce the following object,

$$\Theta^l_p := g^{ik} \ast R^l_{kpi},$$ (2.11)

which we will refer to as the $\Theta$ curvature. In the commutative case, $\Theta^l_p$ coincides with $R^l_p$, but it is no longer true in the present setting. However, note that

$$\Theta^l_j = g^{ik} \ast R^l_{kli} = g^{ik} \ast R_{ki} = R.$$ (2.12)

The analysis of the noncommutative second Bianchi identity in [9, section 4.2] showed that $R^l_j + \Theta^l_j - \delta^l_j R$ was an analog of the usual Einstein tensor (more precisely, 2 times the Einstein tensor) in the case of vanishing cosmological constant. Based on the analysis, a noncommutative Einstein equation in the vacuum without a cosmological constant was proposed in [9, section 4.3].

In view of [9, section 4.2], the following appears to be a reasonable proposal for a noncommutative Einstein equation over $U$,

$$R^l_j + \Theta^l_j - \delta^l_j R + 2\delta^l_j \Lambda = 2T^l_j,$$ (2.13)

where $T^l_j$ is some generalized ‘energy–momentum tensor’ and $\Lambda$ is the cosmological constant. This reduces to the vacuum equation suggested in [9] when $T^l_j = 0$ and the cosmological constant vanishes. We hope to provide a mathematical justification for this proposal in future work, where the defining properties of $T^l_j$ will also be specified.

In the commutative limit, we recover the usual Einstein equation from equation (2.13). However, it is yet to be seen whether the equation in the noncommutative setting correctly describes physics.

We shall test the aspects of the validity of the noncommutative Einstein equation by examining whether it is possible to find solutions of the equation, which may be considered as noncommutative analogs of physically important spacetimes in the commutative setting, for example, the Schwarzschild spacetime and Schwarzschild–de Sitter spacetime. As we shall see presently, this is indeed possible.
3. Quantum deformation of the Schwarzschild spacetime

In this section, we investigate noncommutative analogs of the Schwarzschild spacetime using the general theory discussed in the previous section. Recall that the Schwarzschild spacetime has the following metric,

\[ g = -\left(1 - \frac{2m}{r}\right)dr^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2), \]  

(3.1)

where \( m = \frac{2GM}{c^2} \) is constant, with \( M \) interpreted as the total mass of the spacetime. In the formula for \( m \), \( G \) is the Newton constant and \( c \) is the speed of light. The Schwarzschild spacetime can be embedded into a flat space of 6 dimensions in the following two ways [16, 17]:

(i) Kasner’s embedding:

\[ X^1 = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} \sin t, \]
\[ X^2 = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} \cos t, \]
\[ X^3 = f(r), \quad (f')^2 + 1 = \left(1 - \frac{2m}{r}\right)^{-1} \left(1 + \frac{m^2}{r^4}\right), \]
\[ X^4 = r \sin \theta \cos \phi, \]
\[ X^5 = r \sin \theta \sin \phi, \]
\[ X^6 = r \cos \theta, \]

with the Schwarzschild metric given by

\[ g = -(dX^1)^2 - (dX^2)^2 + (dX^3)^2 + (dX^4)^2 + (dX^5)^2 + (dX^6)^2. \]

(ii) Fronsdal’s embedding:

\[ Y^1 = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} \sinh t, \]
\[ Y^2 = \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} \cosh t, \]
\[ Y^3 = f(r), \quad (f')^2 + 1 = \left(1 - \frac{2m}{r}\right)^{-1} \left(1 - \frac{m^2}{r^4}\right), \]
\[ Y^4 = r \sin \theta \cos \phi, \]
\[ Y^5 = r \sin \theta \sin \phi, \]
\[ Y^6 = r \cos \theta, \]

with the Schwarzschild metric given by

\[ g = -(dY^1)^2 + (dY^2)^2 + (dY^3)^2 + (dY^4)^2 + (dY^5)^2 + (dY^6)^2. \]

Let us now construct a noncommutative analog of the Schwarzschild spacetime. Denote \( x^0 = t, x^1 = r, x^2 = \theta \) and \( x^3 = \phi \). We deform the algebra of functions in these variables by imposing on it the Moyal product defined by (2.2) with the following anti-symmetric matrix:

\[ (\theta_{\mu\nu})^3_{\mu,\nu=0} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}. \]  

(3.2)
Denote the resultant noncommutative algebra by $\mathcal{A}$. Note that in the present case the nonzero components of the matrix $(\theta_{\mu\nu})$ are dimensionless.

Now we regard the functions $X^i$, $Y^i$ ($1 \leq i \leq 6$) appearing in both Kasner’s and Fronsdal’s embeddings as elements of $\mathcal{A}$. For $\mu = 0, 1, 2, 3$, and $i = 1, 2, \ldots, 6$, let

$$E^i_\mu = \frac{\partial X^i}{\partial x^\mu}, \quad \text{for Kasner’s embedding},$$

$$E^i_\mu = \frac{\partial Y^i}{\partial x^\mu}, \quad \text{for Fronsdal’s embedding}.$$  

(3.3)

Following the general theory of the last section, we define the metric and noncommutative torsion for the noncommutative Schwarzschild spacetime by

1. in the case of Kasner’s embedding

$$g_{\mu\nu} = -E^1_\mu \ast E^1_\nu - E^2_\mu \ast E^2_\nu + \sum_{j=3}^{6} E^j_\mu \ast E^j_\nu,$$

$$\Upsilon_{\mu\nu\rho} = \frac{1}{2} \left( \partial_\mu E^1_\nu \ast E^1_\rho - \partial_\nu E^2_\mu \ast E^2_\rho + \sum_{j=3}^{6} \partial_\mu E^j_\nu \ast E^j_\rho \right)$$

$$+ \frac{1}{2} \left( -E^1_\rho \ast \partial_\mu E^1_\nu - E^2_\rho \ast \partial_\nu E^2_\mu + \sum_{j=2}^{6} E^j_\rho \ast \partial_\mu E^j_\nu \right);$$

(3.4)

2. in the case of Fronsdal’s embedding

$$g_{\mu\nu} = -E^1_\mu \ast E^1_\nu + \sum_{j=2}^{6} E^j_\mu \ast E^j_\nu, $$

$$\Upsilon_{\mu\nu\rho} = \frac{1}{2} \left( -\partial_\mu E^1_\nu \ast E^1_\rho + \sum_{j=2}^{6} \partial_\mu E^j_\nu \ast E^j_\rho \right)$$

$$+ \frac{1}{2} \left( -E^1_\rho \ast \partial_\mu E^1_\nu + \sum_{j=2}^{6} E^j_\rho \ast \partial_\mu E^j_\nu \right).$$

(3.5)

Some lengthy but straightforward calculations show that the metrics and the noncommutative torsions are respectively equal in the two cases. Since the noncommutative torsion will not be used in later discussions, we shall not spell it out explicitly. However, we record the metric $g = (g_{\mu\nu})$ of the quantum deformation of the Schwarzschild spacetime as follows:

$$g_{00} = -\left(1 - \frac{2m}{r}\right),$$

$$g_{01} = g_{10} = g_{02} = g_{20} = g_{03} = g_{30} = 0,$$

$$g_{11} = \left(1 - \frac{2m}{r}\right)^{-1} \left[1 + \left(1 - \frac{2m}{r}\right)(\sin^2 \theta - \cos^2 \theta) \sinh^2 \tilde{h}\right],$$

$$g_{12} = g_{21} = 2r \sin \theta \cos \theta \sinh \tilde{h},$$

$$g_{13} = -g_{31} = -2r \sin \theta \cos \theta \sinh \tilde{h} \cosh \tilde{h},$$

$$g_{22} = r^2[1 - (\sin^2 \theta - \cos^2 \theta) \sinh^2 \tilde{h}],$$

$$g_{23} = -g_{32} = r^2(\sin^2 \theta - \cos^2 \theta) \sinh \tilde{h} \cosh \tilde{h},$$

$$g_{33} = r^2[\sin^2 \theta + (\sin^2 \theta - \cos^2 \theta) \sinh^2 \tilde{h}].$$

(3.6)
It is interesting to observe that the quantum deformation of the Schwarzschild metric (3.6) still has a black hole with the event horizon at \( r = 2m \). The Hawking temperature and entropy of the black hole are respectively given by
\[
T = \frac{1}{2} \left. \frac{\partial g_{00}}{\partial r} \right|_{r=2m} = \frac{1}{4m}, \quad S_{bh} = 4\pi m^2.
\]

They coincide with the temperature and entropy of the classical Schwarzschild black hole of mass \( M \). However, the area of the event horizon of the noncommutative black hole receives corrections from the quantum deformation of the spacetime. Let \( \bar{g} = (g_{22} \, g_{23} \, g_{32} \, g_{33}) \). We have
\[
A = \iint_{r=2m} \sqrt{\det \bar{g}} \, d\theta \, d\phi
= \iint_{r=2m} r^2 \sin \theta \sqrt{1 + (\sin^2 \theta - \cos^2 \theta) \sinh^2 \bar{h}} \, d\theta \, d\phi
= 16\pi m^2 \left( 1 - \frac{\bar{h}^2}{6} + O(\bar{h}^4) \right).
\]

This leads to the following relationship between the horizon area and entropy of the noncommutative black hole:
\[
S_{bh} = A \frac{A}{4} \left( 1 + \frac{\bar{h}^2}{6} + O(\bar{h}^4) \right). \tag{3.7}
\]

Let us now consider the Ricci and the \( \Theta \) curvature of the deformed Schwarzschild metric. We have
\[
R^1_0 = R^2_0 = R^3_0 = R^0_1 = R^0_2 = R^0_3 = 0,
\]
\[
\Theta^1_0 = \Theta^2_0 = \Theta^3_0 = \Theta^0_1 = \Theta^0_2 = \Theta^0_3 = 0,
\]
\[
R^0_0 = \Theta^0_0 = - \frac{m[2m + 3r + 3(m + r) \cos 2\theta]}{r^4} \bar{h}^2 + O(\bar{h}^4),
\]
\[
R^1_1 = \Theta^1_1 = \frac{m[-14m + 3r + (-11m + r) \cos 2\theta]}{r^4} \bar{h}^2 + O(\bar{h}^4),
\]
\[
R^2_2 = \Theta^2_2 = \frac{2m \cos^2 \theta \cot \theta}{r^4} \bar{h}^2 + O(\bar{h}^4),
\]
\[
R^3_3 = - \Theta^3_3 = \frac{2m \cot \theta}{r^4} \bar{h} + O(\bar{h}^3),
\]
\[
R^1_0 = \Theta^1_0 = \frac{5m(-2m + r) \sin 2\theta}{r^3} \bar{h}^2 + O(\bar{h}^4),
\]
\[
R^2_2 = \Theta^2_2 = \frac{m[4(m + r) + (6m + 5r) \cos 2\theta]}{r^4} \bar{h}^2 + O(\bar{h}^4),
\]
\[
R^3_3 = - \Theta^3_3 = \frac{4m}{r^3} \bar{h} + O(\bar{h}^3),
\]
\[
R^1_1 = - \Theta^1_1 = \frac{m(2m - r) \sin 2\theta}{r^3} \bar{h} + O(\bar{h}^3),
\]
\[
R^2_1 = - \Theta^2_1 = \frac{4m \cos^2 \theta}{r^3} \bar{h} + O(\bar{h}^3),
\]
\[
R^3_1 = - \Theta^3_1 = \frac{m[-8m + 8r + (-6m + 9r) \cos 2\theta]}{r^4} \bar{h}^2 + O(\bar{h}^4). \tag{3.8}
\]
Note that $R^i_j = \Theta^i_j$ for all $i$, and $R^i_i = -\Theta^i_i$ if $i \neq j$. Let us write

$$R^i_j = R^i_j(0) + \hbar R^i_j(1) + \hbar^2 R^i_j(2) + \cdots,$$

$$\Theta^i_j = \Theta^i_j(0) + \hbar \Theta^i_j(1) + \hbar^2 \Theta^i_j(2) + \cdots.$$  \hfill (3.9)

Then the formulae for $R^i_j$ and $\Theta^i_j$ show that

$$R^i_j(0) = \Theta^i_j(0), \quad R^i_j(1) = -\Theta^i_j(1), \quad R^i_j(2) = \Theta^i_j(2).$$

Naively generalizing the Einstein tensor $R^i_j - \frac{1}{2} \delta^i_j R$ to the noncommutative setting, one ends up with a quantity that does not vanish at order $\hbar$, as can easily be shown using the above results. However,

$$R^i_j + \Theta^i_j - \delta^i_j R = 0 + O(\hbar^2).$$

This indicates that the proposed noncommutative Einstein equation (2.13) captures some essence of the underlying symmetries in the noncommutative world.

Now the deformed Schwarzschild metric (3.6) satisfies the vacuum noncommutative Einstein equation (2.13) with $T^i_j = 0$ and $\Lambda = 0$ to first order in the deformation parameter. However, if we take into account higher order corrections in $\hbar$, the deformed Schwarzschild metric no longer satisfies the noncommutative Einstein equation in the vacuum. Instead, $R^i_j + \Theta^i_j - \delta^i_j R = T^i_j$ with $T^i_j$ being of order $O(\hbar^2)$ and is given by

$$T^0_0 = T^0_1 = T^0_2 = T^0_3 = T^1_0 = T^1_1 = T^1_2 = T^2_0 = 0,$$

$$T^0_0 = \frac{m[8m - 9r + (4m - 9r) \cos 2\theta]}{r^4} + O(\hbar^2),$$

$$T^1_1 = \frac{-m[4m + 3r + (4m + 5r) \cos 2\theta]}{r^4} + O(\hbar^2),$$

$$T^2_1 = \frac{2m \cos^2 \theta \cot \theta}{r^4} + O(\hbar^2),$$

$$T^1_2 = \frac{5m(-2m + r) \sin 2\theta}{r^3} + O(\hbar^2),$$

$$T^2_2 = \frac{m[14m - 2r + (13m - r) \cos 2\theta]}{r^4} + O(\hbar^2),$$

$$T^3_3 = \frac{m[2m + r + (m + 3r) \cos 2\theta]}{r^4} + O(\hbar^2).$$  \hfill (3.10)

A possible physical interpretation of the results is the following. We regard the $\hbar$ and higher order terms in the metric $g_{ij}$ and associated curvature $R_{ijkl}$ as arising from quantum effects of gravity. Then $T^i_j$ obtained in (3.10) should be interpreted as quantum corrections to the classical Einstein tensor.

4. Quantum deformation of the Schwarzschild–de Sitter spacetime

In this section, we investigate a noncommutative analog of the Schwarzschild–de Sitter spacetime. Since the analysis is parallel to that on the quantum Schwarzschild spacetime, we shall only present the pertinent results.

Recall that the Schwarzschild–de Sitter spacetime has the following metric,

$$g = -\left(1 - \frac{r^2}{\ell^2} - \frac{2m}{r}\right) dr^2 + \left(1 - \frac{r^2}{\ell^2} - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$  \hfill (4.1)
where $\Lambda > 0$ is the cosmological constant and $m$ is related to the total mass of the spacetime through the same formula as in the Schwarzschild case. This spacetime can be embedded into a flat space of 6 dimensions in two different ways.

1) The generalized Kasner embedding:

$$X^1 = \left(1 - \frac{r^2}{l^2} - \frac{2m}{r}\right)^{\frac{1}{2}} \sin t,$$

$$X^2 = \left(1 - \frac{r^2}{l^2} - \frac{2m}{r}\right)^{\frac{1}{2}} \cos t,$$

$$X^3 = f(r), \quad (f')^2 + 1 = \left(1 - \frac{r^2}{l^2} - \frac{2m}{r}\right)^{-1} \left[1 + \left(\frac{m}{r^2} - \frac{r}{l^2}\right)^2\right].$$

$$X^4 = r \sin \theta \cos \phi,$$

$$X^5 = r \sin \theta \sin \phi,$$

$$X^6 = r \cos \theta,$$

with the Schwarzschild–de Sitter metric given by

$$g = -(dX^1)^2 - (dX^2)^2 + (dX^3)^2 + (dX^4)^2 + (dX^5)^2 + (dX^6)^2.$$

2) The generalized Fronsdal embedding:

$$Y^1 = \left(1 - \frac{r^2}{l^2} - \frac{2m}{r}\right)^{\frac{1}{2}} \sinh t,$$

$$Y^2 = \left(1 - \frac{r^2}{l^2} - \frac{2m}{r}\right)^{\frac{1}{2}} \cosh t,$$

$$Y^3 = f(r), \quad (f')^2 + 1 = \left(1 - \frac{r^2}{l^2} - \frac{2m}{r}\right)^{-1} \left[1 - \left(\frac{m}{r^2} - \frac{r}{l^2}\right)^2\right].$$

$$Y^4 = r \sin \theta \cos \phi,$$

$$Y^5 = r \sin \theta \sin \phi,$$

$$Y^6 = r \cos \theta,$$

with the Schwarzschild–de Sitter metric given by

$$g = -(dY^1)^2 - (dY^2)^2 + (dY^3)^2 + (dY^4)^2 + (dY^5)^2 + (dY^6)^2.$$

Let us now construct a noncommutative analog of the Schwarzschild–de Sitter spacetime. Denote $x^0 = t, x^1 = r, x^2 = \theta$ and $x^3 = \phi$. We deform the algebra of functions in these variables by imposing on it the Moyal product defined by (2.2) with the anti-symmetric matrix (3.2). Denote the resultant noncommutative algebra by $\mathcal{A}$.

Now we regard the functions $X_i, Y_i (1 \leq i \leq 6)$ appearing in both the generalized Kasner embedding and the generalized Fronsdal embedding as elements of $\mathcal{A}$. Let $E_{\mu}^i$ ($\mu = 0, 1, 2, 3$ and $i = 1, 2, \ldots, 6$) be defined by (3.3) but for the generalized Kasner and Fronsdal embeddings, respectively. We also define the metric and noncommutative torsion for the noncommutative Schwarzschild–de Sitter spacetime by equations (3.4) and (3.5) for the generalized Kasner and Fronsdal embeddings, respectively. As in the case of the noncommutative Schwarzschild spacetime, we can show that the metrics and the noncommutative torsions are respectively equal for the two embeddings. We record the metric $g = (g_{\mu\nu})$ of the quantum deformation of the Schwarzschild–de Sitter spacetime as follows:
\( g_{00} = -\left( 1 - \frac{r^2}{l^2} - \frac{2m}{r} \right) \),
\( g_{01} = g_{10} = g_{02} = g_{20} = g_{03} = g_{30} = 0 \),
\( g_{11} = \left( 1 - \frac{r^2}{l^2} - \frac{2m}{r} \right) \left[ 1 + \left( 1 - \frac{r^2}{l^2} - \frac{2m}{r} \right) (\sin^2 \theta - \cos^2 \theta) \sin^2 \vartheta \right]^{-1} \),
\( g_{12} = g_{21} = 2r \sin \theta \cos \theta \sinh \vartheta \),
\( g_{13} = -g_{31} = -2r \sin \theta \cos \theta \sinh \vartheta \cosh \vartheta \),
\( g_{22} = r^2 [1 - (\sin^2 \theta - \cos^2 \theta) \sinh^2 \vartheta] \),
\( g_{23} = -g_{32} = r^2 (\sin^2 \theta - \cos^2 \theta) \sinh \cosh \vartheta \),
\( g_{33} = r^2 [\sin^2 \theta + (\sin^2 \theta - \cos^2 \theta) \sinh^2 \vartheta] \).

Let us now consider the Ricci curvature and the \( \Theta \) curvature of the deformed Schwarzschild metric. We have

\[
R^1_0 = R^3_0 = R^1_0 = R_0^3 = 0, \\
\Theta^0_0 = \Theta^0_3 = \Theta^0_1 = \Theta^0_1 = 0, \\
R^0_0 = \Theta^0_0 = \frac{3}{l^4} + \frac{1}{l^4} \left( 10m - 3r \right) r^3 + 10r^6 - l^4 m (2m + 3r) \\
-3 \left[-2l^2 m r^3 - 2r^6 + l^4 m (m + r) \right] \cos 2\theta \hat{h}^2 + O(\hat{h}^4), \\
R^1_1 = \Theta^1_1 = \frac{3}{l^4} + \frac{1}{l^4} \left( 16m - 9r \right) r^3 + 16r^6 + l^4 m (-14m + 3r) \\
+ \{2l^2 (5m - 2r)r^3 + 10r^6 + l^4 m (-11m + r) \} \cos 2\theta \hat{h}^2 + O(\hat{h}^4), \\
R^1_2 = \Theta^1_2 = \frac{2l^2 m - 4r^3 \cos^2 \theta \cot \vartheta \cosh \vartheta^2 + O(\hat{h}^4),} \\
R^1_3 = \Theta^1_3 = \frac{-2l^2 m - 4r^3 \cot \vartheta \cosh \vartheta + O(\hat{h}^3),} \\
R^2_1 = \Theta^2_1 = -\frac{2l^2 m - 4r^3 \cos^2 \theta \cot \vartheta \cosh \vartheta + O(\hat{h}^3),} \\
R^2_2 = \Theta^2_2 = \frac{-2l^2 m + 4r^3 \cos^2 \theta \cosh \vartheta + O(\hat{h}^3),} \\
R^2_3 = \Theta^2_3 = \frac{3}{l^4} + \frac{1}{l^4} \left( 22m - 5r \right) r^3 + 10r^6 + 4l^4 m (m + r) \\
+ \{6r^6 + l^4 m (15m + 4r) + l^4 m (6m + 5r) \} \cos 2\theta \hat{h}^2 + O(\hat{h}^4), \\
R^3_1 = \Theta^3_1 = \frac{8}{l^2 + \frac{4m}{r^3}} \hat{h} + O(\hat{h}^3), \\
R^3_2 = \Theta^3_2 = \frac{\left(l^2 m - 4r^3 \right)[l^2 (2m - r) + r^3 \sin 2\vartheta \cosh \vartheta + O(\hat{h}^3),} \\
R^3_1 = \Theta^3_3 = \frac{\left(l^2 m - 4r^3 \right)[l^2 (2m - r) + r^3 \sin 2\vartheta \cosh \vartheta + O(\hat{h}^3),} \\
R^3_1 = \Theta^3_3 = \frac{3}{l^4} + \frac{1}{l^4} \left( -8l^4 m (m - r) + l^2 (28m - 5r) r^3 + 16r^6 \\
+ 3 \left\{ l^2 m r^3 + 4r^6 + l^4 m (-2m + 3r) \right\} \cosh \vartheta \hat{h}^2 + O(\hat{h}^4).} \]
Note that if we expand $R^i_j$ and $\Theta^i_j$ into power series in $\hbar$ in the form (3.9), we again have

$$R^i_j(0) = \Theta^i_j(0), \quad R^i_j(1) = -\Theta^i_j(1), \quad R^i_j(2) = \Theta^i_j(2).$$

By using the above results one can easily show that the deformed Schwarzschild–de Sitter metric (4.2) satisfies the vacuum noncommutative Einstein equation (2.13) (with $T_i^j = 0$) to first order in the deformation parameter:

$$R^i_j + \Theta^i_j - \delta^i_j R + \delta^i_j \frac{\hbar}{T^2} = 0 + O(\hbar^3).$$

Further analyzing the deformed Schwarzschild–de Sitter metric, we note that $R^i_j + \Theta^i_j - \delta^i_j R + \delta^i_j \frac{\hbar}{T^2} = T^j_i$ with $T^j_i$ being of order $O(\hbar^2)$ and is given by

$$T^0_0 = T^3_3 = T^0_3 = T^3_0 = T^0_0 = T^3_3 = T^0_3 = T^3_0 = 0,$$

$$T^0_1 = -2l^2 (14m - 3r)r^3 + 16n^6 + l^4 m(-8m + 9r) + \{2l^2 mr^3 + 11r^6 + l^4 m(-4m + 9r)\} \cos 2\theta \frac{\hbar^2}{l^4 r^4} + O(\hbar^4),$$

$$T^1_1 = -2l^2 mr^3 + 10r^6 + l^4 m(4m + 3r) + \{7r^6 + 4l^2 r^3(4m + r) + l^4 m(4m + 5r)\} \cos 2\theta \frac{\hbar^2}{l^4 r^4} + O(\hbar^4),$$

$$T^2_2 = \frac{2(l^2 m - 4r^3)}{l^2 r^4} \cos^2 \theta \cot \theta \hbar^2 + O(\hbar^4),$$

$$T^1_2 = -\frac{l^2 (2m - r) + r^3(5l^2 m + 4r^3)}{l^4 r^3} \sin 2\theta \hbar^2 + O(\hbar^4),$$

$$T^2_3 = -2[l^2 (2m - r) + r^3(7l^2 m + 4r^3) + 8r^6 + l^4 m(-7m + r)] \cos 2\theta \frac{\hbar^2}{l^4 r^4} + O(\hbar^4),$$

$$T^3_1 = [5l^2 m r^3 - 5r^6 + l^4 m(3m + 3r)] \cos 2\theta \frac{\hbar^2}{l^4 r^4} + O(\hbar^4).$$

Similar to the case of the quantum Schwarzschild spacetime, one may regard this as quantum corrections to the Einstein tensor.

5. Conclusion

Working within the framework of the noncommutative Riemannian geometry of [9], we have obtained in this paper quantum analogs of the Schwarzschild spacetime and Schwarzschild–de Sitter spacetime, and studied their noncommutative geometries. The quantum Schwarzschild spacetime has been constructed in two ways, respectively, mimicking the Kasner and Fronsdal embeddings of the classical Schwarzschild spacetime in 6 dimensions. The metrics and curvatures of the resultant quantum spacetimes coincide, and are shown to be solutions of a noncommutative analog of the Einstein equation in the vacuum exact to the first order in the deformation parameter. The Hawking temperature and entropy of the quantum Schwarzschild black hole have been computed and shown to coincide with the usual quantities. However, the area of the horizon has received quantum corrections, and this in turn leads to a modification of the entropy-area law. We have also constructed the quantum Schwarzschild–de Sitter spacetime using two embeddings, and shown that to the first order in the deformation parameter.
parameter, the spacetime is a solution of the vacuum noncommutative Einstein equation with a cosmological constant.

Quantum deformations of the plane-fronted waves and other spacetimes will be studied in a forthcoming paper [18].

Works on noncommutative relativity and black holes reported in the literature are largely based on physical intuitions. A fundamental theory is much desired for developing the subject to a higher level of sophistication. We hope that future work will develop the theory of [9] into a coherent mathematical framework for noncommutative gravity.

Two problems deserve particular attention. One is a possible first principle derivation of the noncommutative Einstein equation (2.13) (e.g., based on an action principle) and the other is the understanding of the symmetries governing (2.13). The latter problem is closely related to the study of a noncommutative analog of the ‘diffeomorphism group’. In [9, section 5], we introduced noncommutative general coordinate transformations and obtained the transformation rules of the metric, Riemannian curvature tensor and other quantities under such transformations. While the material of [9, section 5] is well adapted to studying the symmetries of equation (2.13), the approach of [7] based on deforming the Hopf algebraic structure of differential operators will also be worth investigating. We will return to the issues alluded to here in future publications.

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