Sudakov effects in $B \to \pi \ell \nu_\ell$ form factors

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Abstract

In order to obtain fundamental information about the Standard Model of particle physics from experimental measurements of exclusive hadronic two-body $B$-decays we have to be able to quantify the non-perturbative QCD effects. Although approaches based on the factorization of mass singularities into hadronic distribution amplitudes and form factors provide a rigorous theoretical framework for the evaluation of these effects in the heavy quark limit, it is not possible to calculate the $O(\Lambda_{\text{QCD}}/m_b)$ corrections in a model-independent way, because of the presence of non-factorizing long-distance contributions. It has been argued that Sudakov effects suppress these contributions and render the corresponding corrections perturbatively calculable in terms of the distribution amplitudes. In this paper we examine this claim for the simple and related example of semileptonic $B \to \pi$ decays (which have similar long-distance contributions) and conclude that it is not justified. The uncertainties in our knowledge of the mesons’ distribution amplitudes imply that the calculations of the form factors are not sufficiently precise to be useful phenomenologically. Moreover, it appears that a significant fraction of the contribution comes from the non-perturbative region of large impact parameters, and is therefore uncalculable. We also raise a number of theoretical issues in the derivation of the underlying formalism. Our conclusion is therefore a disappointing one. For $B$-decays it is not possible to invoke Sudakov effects to calculate amplitudes for decays which have long-distance divergences (end-point singularities) in the standard hard-scattering approach.

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1 Introduction

The study of $B$-decays is central to the development of our understanding of the standard model of particle physics and to the determination of its properties, parameters and limitations. The BaBar [1] and Belle [2] $B$-factories as well as other experiments will continue to provide a large amount of experimental data on $B$-decays in general and on two-body hadronic decays in particular. A measurement of the mixing-induced CP-asymmetry in the golden mode $B \to J/\Psi K_S$ (and related decays) for which there is a single weak phase, allows one to determine $\sin(2\beta)$ without any hadronic uncertainties (where $\beta$ is one of the angles of the unitarity triangle). For other decay modes however, our inability to quantify the strong interaction effects prevents us from being able to determine fundamental information about the standard model from the measured branching ratios and asymmetries with the desired precision.

Recently it was discovered that, in the $m_b \to \infty$ limit (where $m_b$ is the mass of the $b$-quark), the mass-singularities in two body hadronic $B$-decays factorize, so that the corresponding amplitudes can be written as convolutions of universal non-perturbative quantities (the light-cone distribution amplitudes of the mesons and semi-leptonic form factors) and perturbatively calculable hard-scattering kernels [3]. This provides a theoretically rigorous framework for the evaluation of decays $B \to M_1 M_2$ (where $M_1$ and $M_2$ are two mesons, at least one of which is light) in the heavy quark limit.

The physical mass of the $b$-quark is only about 4.5 GeV however, and so $O(\Lambda_{QCD}/m_b)$ corrections are significant (we refer to contributions which are suppressed by powers of $1/m_b$ as power corrections). Indeed there are "chirally enhanced" power corrections for which the scale is not $\Lambda_{QCD}$ but the much larger $m_K^2/(m_s + m_d)$. In other cases, contributions which are suppressed by powers of $1/m_b$ may be enhanced by CKM factors or for other reasons. We would therefore very much like to be able to compute the power corrections reliably. In general, mass singularities do not factorize for the power corrections, and one is therefore reduced to using model or phenomenological estimates or exploiting flavour symmetries, hence losing precision and predictive power. There is a school of thought, however, which claims that Sudakov effects regulate these mass-singularities in such a way that they are calculable in perturbation theory [4]. Indeed, calculations of $B \to M_1 M_2$ decay amplitudes which include power corrections and which use Sudakov effects to suppress the long distance effects are being presented [5]–[8]. This approach is frequently referred to as the pQCD formalism.

The possibility that the power corrections may also be calculable in perturbation theory is an exciting one of course, although one may have doubts whether Sudakov effects are effective at scales as low as $m_b$. In this paper we reexamine the calculations of the amplitudes for a related but simpler process, $B \to \pi$ semileptonic decays. In each order of perturbation theory this process is singular at long-distances. It is for this reason that the values of the form factors are used as a non-perturbative input in the factorization formulae [3]. On the other hand, in the pQCD approach, the Sudakov logarithms are resummed and the semileptonic form factors are claimed to be calculable [4] [11]. In this context we address two important questions:
1. Can Sudakov effects be used to evaluate the form-factors in perturbation theory \textit{in principle}? In particular we examine whether all (or almost all) of the contribution to the amplitudes comes from the perturbative region.

2. Are the form-factors calculable \textit{in practice}? Specifically we ask whether one can obtain phenomenologically useful results, given the uncertainties in the mesons’ wave functions.

In addition we examine some of the theoretical steps in the derivation of the pQCD formula for the semileptonic amplitudes in terms of the Sudakov form factors.

The results of our study lead to disappointing answers to both questions. When we apply the formulae used in the pQCD approach we find that our ignorance of the mesons’ wave functions leads to very large uncertainties in the predictions for the form factors. Moreover a significant fraction of the result comes from regions of phase space where perturbation theory (and hence the formulae which are being used) do not apply. We should stress that our numerical results are not in disagreement with those in refs. [5]–[9] and particularly in ref. [10]. The conclusions which we draw from our calculations however, are profoundly different from those of these authors. We believe that our calculations show that the pQCD approach cannot be used to make reliable predictions for the form factors, and hence also in the evaluation of the $B \rightarrow M_1 M_2$ decay amplitudes and in the subsequent determination of the parameters of the unitarity triangle and other studies of CP-violation. This is the main conclusion of our study and our motivation in writing this paper is to try to open a debate of this very important issue.

We also have some reservations about the theoretical foundations of the pQCD formula for semileptonic decays of $B$-mesons. In particular we do not understand the framework for the derivation of the Sudakov factor for the B-meson. We expect the Sudakov suppression in the $B$-meson’s distribution amplitude to be weak, and soft configurations with large transverse separations to contribute significantly to the $B \rightarrow \pi \ell \nu_\ell$ form factors. We will explain our concerns in some detail in sections 5 and 6.

The plan of the remainder of this paper is as follows. In section 2 we recall why the $B \rightarrow \pi$ form factors cannot be computed using the standard factorization approach. In section 3 we present an outline of the framework used in the pQCD approach. The use of Sudakov suppression of long distance contributions has been introduced into hard exclusive processes involving light hadrons in order to improve the precision of the calculations, and is now being used also in $B$-physics. In the following three sections we investigate the ingredients of the pQCD formalism for semileptonic $B$-decays in detail and highlight our concerns. Section 4 contains a discussion of the distribution amplitude of the pion and section 5 is devoted to a critical examination of the derivation of the Sudakov factor for the distribution amplitude(s) of the $B$-meson. We also investigate whether the approximation of using a single distribution amplitude for the $B$-meson is a valid one. In section 6 the pQCD expressions for the $B \rightarrow \pi$ form factors are shown to be very sensitive to the shape of the mesons’ distribution amplitudes, and to contain significant uncalculable long-distance contributions. Section 7 contains a summary of our principal conclusions. In the appendix we introduce a model distribution amplitude for the $B$-meson, which satisfies
the constraints due to the equations of motion in longitudinal and transverse momentum space.

2 \( B \rightarrow \pi \) form factors in the factorization approach

In this section we consider the semi-leptonic decay \( B(p) \rightarrow \pi(p')\ell \nu_\ell \) in the standard factorization approach, introduced by Brodsky and Lepage and by Efremov and Radyushkin (BLER) \[11, 12\]. We introduce the main ingredients of this approach, and in particular the light-cone distribution amplitudes which describe the parton content of the mesons relevant for this process. We then recall why long-distance singularities prevent the computation of \( B \rightarrow \pi \) form factors in the standard approach \[13\].

We start by defining the kinematics and our notation. The amplitude for this decay can be written in the form:

\[
A(p, p') = \frac{G_F}{\sqrt{2}} V_{ub} \left( \bar{u}_\ell \gamma_\mu (1 + \gamma_5) \ell \right) \left( \pi(p') | \bar{u} \gamma_\mu b | \bar{B}(p) \right),
\]

(1)

where \( l \) and \( \bar{\nu}_l \) represent the wave functions of the charged lepton and neutrino respectively and \( \bar{u} \) and \( b \) are quark fields. The non-perturbative QCD effects are contained in the hadronic matrix element \( \left( \pi(p') | \bar{u} \gamma_\mu b | \bar{B}(p) \right) \), which can be parametrized in terms of two invariant form factors. We choose to use a conventional definition of the form factors (in a helicity basis):

\[
\left( \pi(p') | \bar{u} \gamma_\mu b | \bar{B}(p) \right) = F_+(q^2)(p + p')_\mu + (M_B^2 - M_\pi^2) F_0(q^2) - F_+(q^2) q_\mu / q^2,
\]

(2)

where \( q_\mu = p_\mu - p'_\mu \). At \( q^2 = 0 \) we have \( F_0(0) = F_+(0) \). The notation is illustrated in Fig. 1.

We work in the rest frame of the \( \bar{B} \)-meson, and neglect the mass of the pion. We introduce light-cone coordinates, writing \( k = (k_+/\sqrt{2}, k_-/\sqrt{2}, k_\perp) \) with \( k_\pm \equiv (k_0 \pm k_3) \),
and with the metric for the transverse components chosen such that \((k_\perp)\mu(k_\perp)_\mu = -\vec{k}_\perp \cdot \vec{r}_\perp\). It will also be convenient to introduce two light-like vectors \(n_+ \equiv (\sqrt{2}, 0, 0, 0)\) and \(n_- \equiv (0, \sqrt{2}, 0, 0)\).

Using light-cone coordinates, \(p = (M_B/\sqrt{2}, M_B/\sqrt{2}, 0, 0)\) and we chose the momentum of the pion to be in the \(-\) direction, \(p' = (0, \eta M_B/\sqrt{2}, 0, 0)\). In terms of \(\eta\), the energy of the pion is given by \(p'^0 = \eta M_B/2\) and the invariant mass of the lepton pair by \(m^2_{\ell\nu} = (1-\eta)M_B^2\).

The physical range for \(\eta\) is therefore given by \(0 \leq \eta \leq 1\).

In the standard hard-scattering approach, as represented in Fig. 1, the form factors are obtained from convolutions of the mesons’ distribution amplitudes (\(\phi_\pi\) and \(\phi^B_\pi\)) and the perturbative kernel \(T\). In the following subsection we discuss the distribution amplitudes before returning to the convolution itself in section 2.2.

### 2.1 Meson distribution amplitudes

In order to evaluate the decay amplitude, we have to perform the convolution illustrated in fig. 1. In this subsection we discuss one of the principal ingredients of this convolution, namely the distribution amplitudes (or DAs) of the incoming and outgoing mesons. We will consider only the leading-twist amplitudes here.

The (leading-twist) distribution amplitude of the pion is defined as \([11]\):

\[
\langle \pi(p')|\bar{q}_a(y)q'_\beta(z)|0\rangle = i\frac{f_\pi}{4}(p'_5\gamma_5)_{\beta\alpha} \int_0^1 dx e^{i(xp'_\mu + zp'_\nu)}\phi_\pi(x; \mu) ,
\]

where \((y - z)^2 = 0, \bar{x} = 1 - x\), and we choose a normalization such that \(f_\pi \simeq 131\text{ MeV}\). A path-ordered exponential is implicit on the left-hand side, in order to maintain gauge invariance. The parameter \(\mu\) is a renormalization scale of the light-cone operators on the left-hand side.

The distribution amplitude is symmetric with respect to the interchange \(x \leftrightarrow \bar{x}\), and is normalized such that \(\int_0^1 dx \phi_\pi(x; \mu) = 1\). As the renormalization scale \(\mu\) tends to infinity, \(\phi_\pi\) tends to the asymptotic limit \(\phi_\pi^{as}(x) = 6x(1-x)\). For finite values of the renormalization scale, \(\phi_\pi\) can be expanded on the basis of Gegenbauer polynomials:

\[
\phi_\pi(x; \mu) = 6x(1-x) \left[ 1 + \sum_{p=1}^{\infty} \alpha_{2p}(\mu) C_{2p}^{(3/2)}(2x - 1) \right] .
\]

Only even polynomials contribute because of the symmetry \(x \leftrightarrow \bar{x}\). The lowest Gegenbauer polynomials contributing to the sum are \(C_{2p}^{(3/2)}(u) = 3/2 \cdot (5u^2 - 1)\) and \(C_{4}^{(3/2)}(u) = 15/8 \cdot (21u^4 - 14u^2 + 1)\). The moments \(\alpha_n\) are multiplicatively renormalized, with the anomalous dimensions increasing as \(n\) increases \([14]\) (so that the corresponding \(\alpha_{2p}\) decreases more quickly with increasing \(p\) as \(\mu \to \infty\)).

For the distribution amplitudes of the \(B\)-meson, we will use the notation and formalism developed in refs. [13] and [14] which we now briefly summarize \([1]\). In the heavy-quark limit,
the most general decomposition of the light-cone matrix element for the B-meson is [15, 16]:

\[
\langle 0|\bar{q}_\beta(z) b_\alpha(0)|\bar{B}(p)\rangle = -\frac{if_B}{4} \left[ \frac{\hat{p} + m}{2} \left\{ 2\tilde{\phi}_B^\alpha(t) + \frac{\phi_B^\alpha(t) - \tilde{\phi}_B^\alpha(t)}{t} \right\} \gamma_5 \right]_{\alpha\beta},
\]

(5)

where \( z^2 = 0, \ v = p/m, \ t = v \cdot z \) and a path-ordered exponential is implicit on the left-hand side again. We are working only to leading twist order and therefore have set \( M_B = m_b \equiv m \). Recalling that \( \langle 0|\bar{q}(0)\gamma_\mu\gamma_5 b(0)|\bar{B}(p)\rangle = if_Bp^\mu \), the choice of prefactor in eq. (5) implies that

\[
\tilde{\phi}_B^\alpha(t = 0) = \phi_B^\alpha(t = 0) = 1.
\]

(6)

Thus for the B-meson we have two distribution amplitudes, \( \tilde{\phi}_B^\alpha \), to consider.

In order to evaluate the amplitudes for \( B \to \pi\ell\nu_\ell \) decays we perform a convolution of the distribution amplitudes of the B-meson with the hard-scattering amplitude, as illustrated in Fig. [1]. Let \( l \) denote the momentum of the light quark in the B-meson and recall that we have chosen the momentum of the outgoing pion to be in the \( - \) direction. The \( O(\alpha_s) \) hard-scattering amplitude \( T \) is then independent of \( l_+ \). We are thus allowed to set \( z_+ = 0 \) in the decomposition in eq. (5). Following ref. [16] it is possible to write the convolution in the form

\[
\int_0^\infty dl_+ \{ P_B^T(l) \},
\]

with \( l \) set to \( (\omega/\sqrt{2}, 0, \vec{0}_\perp) \) once the momentum-space projection operator \( P_B^T \) has been applied. The projection operator is equal to:

\[
P_{\beta\alpha}^B = -\frac{if_B}{4} \left[ \frac{\hat{p} + m}{2} \left\{ \phi_B^\alpha(\omega)\gamma_\mu - \Delta(\omega)\gamma_\mu \frac{\partial}{\partial l_+^\mu} \right\} \gamma_5 \right]_{\alpha\beta},
\]

(7)

with \( \Delta(\omega) \equiv \int_0^\omega d\ell \left( \phi_B^\alpha(\ell) - \tilde{\phi}_B^\alpha(\ell) \right) \). The distribution amplitudes on the right-hand side of eq. (7) are written in momentum space,

\[
\phi_B^\pm(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, e^{i\omega t} \tilde{\phi}_B^\pm(t).
\]

(8)

Since the components of the light quark in the B-meson are expected to remain of order \( \Lambda_{QCD} \) or less, the distribution amplitudes will be suppressed for \( \omega \) larger than \( \Lambda_{QCD} \). Note also that eq. (6) implies that the distribution amplitudes in momentum space are normalized such that \( \int d\omega \, \phi_B^\pm(\omega) = \int d\omega \, \tilde{\phi}_B^\pm(\omega) = 1 \).

For a precise calculation of the form factors it is clearly important to have as much information about the mesons’ distribution amplitudes as possible. Of particular importance is the behaviour of the distribution amplitudes at the end-point \( l_+ \to 0 \). To gain some insight into this behaviour, we follow ref. [17] and note that the leading-twist decomposition in eq. (6) leads to expressions for the distribution amplitudes in terms of matrix elements of the pseudoscalar density and axial current:

\[
\langle 0|\bar{q}(z)\gamma_5 b(0)|\bar{B}(p)\rangle = -if_B M_B \tilde{\phi}_P^B,
\]

(9)

\[
\langle 0|\bar{q}(z)\gamma_\mu\gamma_5 b(0)|\bar{B}(p)\rangle = i f_B \left( \bar{\phi}_A^B A_{\mu}^\mu - M_B \tilde{\phi}_A^B A_{2\mu}^\mu \right),
\]

(10)

to-leading order in the perturbative expansion expressed in refs. [17, 18]. At this order, which is beyond the scope of our study, it is likely that the formalism will have to be modified, at least to include the dependence on the transverse momentum explicitly.
where \( \tilde{\phi}_P^B, \tilde{\phi}_{A1}^B \) and \( \tilde{\phi}_{A2}^B \) are given in terms of the distribution amplitudes \( \tilde{\phi}_\pm^B \) by

\[
\tilde{\phi}_P^B \equiv \frac{\tilde{\phi}_1^B + \tilde{\phi}_2^B}{2}, \quad \tilde{\phi}_{A1}^B \equiv \tilde{\phi}_+^B \quad \text{and} \quad \tilde{\phi}_{A2}^B \equiv \frac{i}{2} \frac{\tilde{\phi}_+^B - \tilde{\phi}_-^B}{\lambda}.
\]

(11)

For a light pseudoscalar meson, the end-point behaviour of the eigenfunctions of the evolution equations, together with sum-rule inspired arguments, suggest that \( \phi_{B1}^\pm (\omega) \sim 1 \) for \( \omega \to 0 \) \[12\]. If this behaviour is also valid for the \( B \)-meson, we might expect that \( \phi_{B}^+(l_+) = O(l_+) \) and \( \phi_{B}^-(l_+) = O(1) \) as \( l_+ \to 0 \) \[13\]. However, in spite of this argument, when studying the validity of perturbative estimates of the form factors in terms of the distribution amplitudes it must be remembered that such an end-point behaviour is a conjecture.

Another theoretically motivated constraint is a consequence of the equations of motion. Since we neglect the 3-particle distribution amplitudes (and contributions from higher Fock states), the equations of motion lead to the constraint \[14\]:

\[
\phi_{B}^\pm (l_+) = -l_+ \frac{d\phi_{B}^\pm}{dl_+} (l_+).
\]

(12)

Eq. (12) is useful in constraining models for the distribution amplitudes, and supports the conjectured behaviour of the distribution amplitudes as \( l_+ \to 0 \). We can indeed derive the expected behaviour, that \( \phi_{B}^+(l_+) = O(1) \) as \( l_+ \to 0 \) and that \( \phi_{B}^+(0) = 0 \), if we assume that the equation of motion (12) is fulfilled and that \( \lambda_{B}^{-1} \equiv \int_{0}^{\infty} dl_+ \frac{\phi_{B}^+(l_+)}{l_+} \) is finite and nonvanishing \[15\]. We can use the equation of motion to see that \( \lambda_{B}^{-1} = \phi_{B}^-(0) \) \[19\], which leads to \( \phi_{B}^+(l_+) = O(1) \) for small \( l_+ \). Eq. (12) implies then that \( \phi_{B}^+(l_+) \) vanishes at the end-point.

We end this subsection by introducing two models for the distribution amplitudes which satisfy the above constraints, and which will be useful in our investigation of the validity and reliability of the frameworks used to evaluate the form factors. The first model was proposed in ref. \[15\] on the basis of a QCD sum rule analysis:

\[
\phi_{B}^+(l_+) = \frac{l_+}{\lambda^2} \exp \left[-\frac{l_+}{\lambda}\right], \quad \phi_{B}^-(l_+) = \frac{1}{\lambda} \exp \left[-\frac{l_+}{\lambda}\right].
\]

(13)

\( \lambda \) is a parameter of order \( \Lambda_{QCD} \). Another model satisfying the above constraints is discussed in Appendix A.1:

\[
\phi_{B}^+(l_+) = \sqrt{\frac{2}{\pi \lambda^2 \lambda^2}} \exp \left[-\frac{l_+^2}{2 \lambda^2}\right], \quad \phi_{B}^-(l_+) = \sqrt{\frac{2}{\pi \lambda^2}} \exp \left[-\frac{l_+^2}{2 \lambda^2}\right].
\]

(14)

In this model, as discussed in the appendix, \( \lambda \) is related to the distribution of transverse momenta.
2.2 Expressions for the $B \to \pi$ form factors

We are now ready to discuss the semileptonic $B \to \pi$ form-factors in the standard hard-scattering approach. The leading-order $O(\alpha_s)$ contribution to the $B \to \pi$ form factors is given by the two diagrams in Fig. 2 and leads to the expressions:

\begin{align}
F_0 &= f_B \bar{f}_\pi \frac{\alpha_s \pi C_F}{N_c} \int_0^1 dx \int dl_+ \frac{\phi_\pi[(1 + \bar{x})\phi_B + \bar{x}\phi_B^\perp]}{\eta ml_+ \bar{x}^2}, \\
F_+ &= f_B \bar{f}_\pi \frac{\alpha_s \pi C_F}{N_c} \int_0^1 dx \int dl_+ \frac{\phi_\pi[(1 + \bar{x})\phi_B + \bar{x}(2\eta - 1)\phi_B^\perp]}{\eta^2 ml_+ \bar{x}^2}. \tag{16}
\end{align}

The expression for $F_+$ appears in ref. [16]. In deriving these expressions we have used the constraint from the equation of motion eq. (12) to write $\Delta(\omega) \equiv \int_0^\omega d\ell (\phi_B^\perp(\ell) - \phi_+^B(\ell)) = \omega \phi_B^B(\omega)$.

From equations (15) and (16) we see that the $B \to \pi$ form factors are very sensitive to the end-point behaviour of the distribution amplitudes for the pion and the $B$ meson. If we suppose, for example, that $\phi_\pi(x) \sim \bar{x}$ for small $\bar{x}$ and $\phi_B^\perp(l+) \sim 1$ for small $l+$, the integrations over $x$ and $l+$ are both divergent. It is therefore not possible to compute $B \to \pi$ form factors in the standard BLER approach. It is for this reason that the authors

\footnote{We thank Martin Beneke for raising this point.}
of ref. [3] proposed to consider these form factors as non-perturbative inputs (obtained for example from experiment or lattice QCD) when evaluating two-body hadronic $B$-Decays. In the pQCD approach on the other hand, Sudakov effects are invoked to regulate the integrals at the end-points. We now proceed to discuss this approach.

3 Outline of the pQCD approach

A different approach (modified factorization or pQCD) has been introduced with the aim of solving the problem discussed in section 2.2 above. In refs. [5]–[10] it is claimed that the introduction of transverse momenta and the subsequent resummation of large logarithms into Sudakov form factors regulates the end-point singularities sufficiently to enable the $B \rightarrow \pi$ form-factors to be computable. In this approach, the form-factors are expressed as

$$F_{0,\pi} = \int_0^1 dx \int_0^\infty dl_+ \int d^2\vec{b}_\perp d^2\vec{c}_\perp \tilde{\Psi}^B(l_+, \vec{c}_\perp; \mu) \tilde{H}_{0,+}(x, l_+, \vec{b}_\perp, \vec{c}_\perp, \eta, m; \mu) \tilde{\Psi}_\pi(x, \vec{b}_\perp; \mu) \ , \ (17)$$

where:

- the meson distribution amplitudes $\tilde{\Psi}_\pi$ and $\tilde{\Psi}^B$ contain the resummed Sudakov effects;
- $\vec{b}_\perp$ and $\vec{c}_\perp$ are the variables conjugate to the transverse momenta of the valence quarks and correspond to their transverse separations;
- $\tilde{H}_{0,+}$ are the Fourier transforms of the corresponding hard-scattering kernels.

In the following sections we will examine the ingredients in eq. (17) in some detail. For presentational purposes however, we hope that it will be instructive if we begin with a brief discussion of the philosophy of the pQCD approach in the simpler and much-analyzed case of the pion electromagnetic form factor [20, 21, 22]. This form factor can be computed in the standard approach [11, 23], leading to a convolution of a hard-scattering kernel $T$ with the DAs of the two pions:

$$F_{\pi \rightarrow \pi\gamma^*}(Q) = \int_0^1 du dv \phi_\pi(u; \mu) T(u, v, Q; \mu) \phi_\pi(v; \mu) \ , \ (18)$$

where $Q$ is the momentum transfer. Although the convolution in eq. (18) is finite, it has been argued [24, 25] that substantial contributions may come from end-point configurations ($u, v \rightarrow 0, 1$), where one or more propagators of the hard-scattering kernel $T$ gets close to the mass-shell, enhancing the soft contributions. It is likely that very large momentum transfers may have to be reached in order to ensure a sufficient suppression of these soft contributions.

The authors of ref. [20] proposed to exploit the ideas developed in refs. [24, 27] in order to extend the domain of applicability of eq. (18). The essential idea is to keep the transverse momenta of the valence quarks writing:

$$F_{\pi \rightarrow \pi\gamma^*} = \int_0^1 du dv \int d^2\vec{k}_\perp d^2\vec{l}_\perp \Psi_\pi(u, \vec{k}_\perp; \mu) H(u, v, \vec{k}_\perp, \vec{l}_\perp, Q; \mu) \Psi_\pi(v, \vec{l}_\perp; \mu) \ . \ (19)$$
Eq. (19) is analogous to the Fourier transform of eq. (17). The tree-level contribution to the hard-scattering kernel is now modified. A typical modification is $1/(uvQ^2) \rightarrow 1/[uvQ^2 + (\vec{k}_\perp + \vec{l}_\perp)^2]$, thus using the transverse momenta to regulate the end-point singularities. Since the transverse momenta are soft, further arguments have to be invoked in order to demonstrate that the procedure is a consistent one.

The modified hard-scattering kernel is now convoluted with a light-cone wave-functions which depend on both longitudinal and transverse components of momentum. Such wave-functions are defined by [27]:

$$i f_\pi p'_\mu \Psi_\pi(x, \vec{k}_\perp) = \int \frac{dz_- d^2 \vec{z}_\perp}{(2\pi)^3} e^{i(xp' \cdot z_- - \vec{k}_\perp \cdot \vec{z}_\perp)} \langle 0 | \bar{q}(z) \gamma_\mu q(0) | \pi(p') \rangle |_{z_+ = 0} \tag{20}$$

in an axial gauge $n \cdot A = 0 \ [n^2 \neq 0]$. We emphasize that no path-ordered exponential is present in the matrix element of eq. (20); $\Psi_\pi$ is a gauge-dependent quantity (indeed, as explained below, the gauge dependence is exploited to simplify the calculation).

It is argued that gluon exchanges suppress configurations where the quark-antiquark pair has a large transverse separation. Soft and collinear gluon exchanges between the valence quarks lead to large double-logarithmic effects, which need to be resummed. This resummation is performed in impact parameter space and so we define the Fourier conjugate of the wave-function,

$$\tilde{\Psi}_\pi(x, \vec{b}_\perp) = \int d^2 \vec{k}_\perp e^{i\vec{k}_\perp \cdot \vec{b}_\perp} \Psi(x, \vec{k}_\perp). \tag{21}$$

Expression (19) for the form factor can be re-expressed as an integral over the impact parameters,

$$F^{\pi \rightarrow \pi \gamma \gamma} = \int_0^1 du \ dv \ \int d^2 b_\perp d^2 c_\perp \tilde{\Psi}_\pi(u, b_\perp; \mu) \tilde{H}(u, v, \vec{b}_\perp, \vec{c}_\perp, Q; \mu) \tilde{\Psi}_\pi(v, \vec{c}_\perp; \mu). \tag{22}$$

An analysis of the double logarithms from higher-orders of perturbation theory shows that they can be resummed and included by multiplying the wave-function in impact parameter space by a Sudakov factor [27]. In the leading-logarithm approximation

$$\tilde{\Psi}_\pi(x, \vec{b}_\perp) \rightarrow \tilde{\Psi}_\pi^{LL}(x, b_\perp, Q; \mu) = \exp[-S^{LL}(b_\perp, Q)] \phi_\pi(x; \mu = 1/b_\perp), \tag{23}$$

where

$$S^{LL}(b_\perp, Q) = \frac{\alpha_s C_F}{4\pi} \ln^2(b_\perp Q). \tag{24}$$

When subleading logarithms are included, $S$ becomes a function of $\log[xQ/\Lambda_{QCD}]$ and $\log[1/(b_\perp \Lambda_{QCD})]$ and diverges when either of these logarithms vanish (see eq. (14) in the following section). This factor suppresses two regions: large transverse separations $b_\perp$ ($b_\perp \sim O(1/\Lambda_{QCD})$) and small fractions of longitudinal momentum $x$ ($x \sim O(\Lambda_{QCD}/Q)$). For the electromagnetic form factor of the pion [20], the Sudakov factor is considered only when it suppresses the configurations with a large transverse separation, and it is set to 1 outside of the region of large $b_\perp$. Potential suppression for end-point configurations (small $x$), and (small) enhancements for intermediate $x$ and $b_\perp$ have been neglected.
What is generally taken for the scale of the coupling constant in the hard-scattering kernel $\tilde{H}$ in eq. (22) is the largest available virtuality for the hard gluon, i.e. the maximum of $\sqrt{uvQ}$ (longitudinal), $1/b_\perp$ and $1/c_\perp$ (transverse). It is then argued that the coupling constant remains small and the perturbative approach is consistent, unless $b_\perp$ and $c_\perp$ are both $O(1/\Lambda_{\text{QCD}})$ and $u$ or $v \to 0$. Moreover, although the coupling constant is large in this soft region, the corresponding contribution is suppressed by Sudakov factors of the distribution amplitudes. The contribution from the soft region is therefore argued to be small.

Before returning to the semileptonic $B \to \pi$ decay we briefly summarize the preceding paragraphs about the possibility that the modified factorization approach might improve the BLER approach. Its principal aim is to treat the end-point regions of phase space, in order to improve the accuracy of perturbative calculations. The singularities of the propagators are smoothed by considering transverse components of the momenta. The scale of the hard-scattering kernel is taken to be the maximal (longitudinal or transverse) virtuality of the exchanged gluons. The only region of large $\alpha_s$ corresponds to small longitudinal momenta and large transverse separations. This region is expected to be strongly suppressed by the Sudakov exponential and it is therefore argued that a consistent perturbative computation is possible.

The authors of refs. [9, 10] claim that a similar procedure can be followed to compute $B \to \pi$ form factors within the pQCD approach. Let us first notice that this process is qualitatively different from the transitions $\pi \to \pi \gamma^*$ [20]–[22] and $\gamma \gamma^* \to \pi^0$ [28]. For the latter form factors, the BLER approach yields finite answers in the large $Q^2$ limit, which might however be affected by significant corrections, even at large energies. The pQCD methods are then designed to improve the accuracy of the computation, and to investigate lower momentum transfers. On the other hand, we have seen that $B \to \pi$ form factors cannot be computed in the usual factorization approach, since singular long-distance effects arise. The pQCD approach would therefore not only improve the precision of the BLER framework, but more importantly it would regulate the divergences. In refs. [3]–[10] it is claimed that indeed accurate and reliable results can be derived with no breakdown of the perturbative framework.

The main motivation for this paper is to examine the basis for such a strong claim. In the following sections, we examine the various elements involved in eq. (17), and emphasize our concerns about the validity and reliability of the pQCD approach.\footnote{The pQCD approach that we have outlined is referred to as “$k_\perp$-resummation”. Recently, it has been proposed to combine it with a second (“threshold”) resummation to study $B$-decays [6, 7, 29]. The concerns we express in sections 4 and 5 about the pQCD approach do not depend on whether the threshold resummation has been performed or not, and we have checked that our numerical analysis in section 6 is almost unchanged when both resummations are considered. We therefore focus on the $k_\perp$-resummation in the remainder of this article.}
4 Sudakov effects in the distribution amplitude of the pion

In this section we consider the first ingredient of eq. (17), the distribution amplitude of the pion $\Psi_\pi$, including Sudakov effects. The pion’s momentum is taken to be in the $-$ direction, $p' \equiv (0, Q/\sqrt{2}, \vec{0}_\perp)$ (where the components are in the $+$, $-$ and transverse directions respectively), and we consider a configuration in which the valence quarks have longitudinal momenta $x p'$ and $\bar{x} p'$ and a transverse separation $b$. The authors of ref. [27] obtain the following expression for the distribution amplitude $\Psi_\pi$ in an axial gauge $[n \cdot A = 0, n^2 \neq 0]$:

$$
\Psi_\pi(x, b, Q; \mu) = \exp \left[ -S(n, Q, b) \right] \exp \left[ -E(\mu, b) \right] \phi_\pi(x; \mu = 1/b) + O(\alpha_s(1/b)),
$$

(25)

where:

- $S$ is the Sudakov factor, which contains the resummed double logarithmic contributions, as well as some non-leading logarithms which are expected to be significant. It depends on the transverse separation $b$, on the pion’s momentum $p'$ and on the gauge-fixing vector $n$;

- $E$ is an evolution factor, relating the renormalization scale $\mu$ and $1/b$;

- $\phi_\pi$ is the standard BLER distribution amplitude.

Eq. (25) exhibits two important features. The first one is the gauge dependence of the Sudakov term $S$. Since the convolution in eq. (17) yields physical (gauge-independent) form factors, the gauge dependence of the distribution amplitude $\Psi_\pi$ has to be cancelled by that in the hard-scattering kernel. The second issue is the presence of $O(\alpha_s)$ corrections in eq. (25). The modified distribution amplitude $\Psi_\pi$ can only be replaced by the standard one $\phi_\pi$ (as in eq. (25)) for small transverse separations $b$. Specifically, there are relative $O(\alpha_s(1/b))$ corrections to eq. (25). Although the Sudakov factor should suppress the full distribution amplitude at large $b$, if this suppression is weak (or negligible), the expression (25) can not be used reliably in this region. We investigate this numerically in section 6 below.

We will return to these issues later, when we consider $B \rightarrow \pi$ form factors. We now continue with the discussion of Sudakov effects in the pion’s distribution amplitude and summarize the analysis performed in ref. [27], and then discuss the gauge-dependence of the Sudakov factor.

4.1 Derivation of the Sudakov factor for the pion

Sudakov effects are expected to suppress configurations with large transverse separations. The explicit expression arises from the resummation and subsequent exponentiation of double logarithms. Such logarithms are related to the exchange of soft and collinear gluons and double logarithmic contributions are due to an overlap of both types of divergence.
Axial gauges \( n \cdot A = 0 \) (where \( n^\mu \) is a fixed vector in the \((+,−)\) plane and \( n^2 \neq 0 \)) are particularly suitable for use in studies of Sudakov effects. In these gauges, the overlap arises when a gluon is exchanged between the valence quarks of the same meson [two-particle reducible diagrams]. Sudakov effects can therefore be analyzed independently of the physical process, and are included fully in the distribution amplitude as can be seen in eq. (25). It would however be reassuring to have an explicit demonstration of the gauge invariance of the form-factor in which the gauge-dependence of the distribution amplitudes is shown to cancel that of the hard scattering kernel.

In spite of the technicalities it is instructive to recall the derivation of eq. (25) [10, 20, 22, 23, 27]. As mentioned above, the distribution amplitude in \( x \) and impact parameter space, \( \Psi_\pi \), is defined in an axial gauge, and is gauge dependent. The only large invariant parameter that can be constructed in the axial gauge is \( \nu^2 = (p' \cdot n)^2/|n^2| \). The distribution amplitude is therefore a function \( \Psi_\pi(x,b;Q;\mu) \), where the \( Q \)-dependence is linked to the gauge dependence, and \( \mu \) is a renormalization scale. As we shall see below, the gauge dependence can be used to determine the \( Q \)-dependence.

Eq. (25) is obtained by studying the \( Q \)-dependence of \( \Psi_\pi(x,b;Q;\mu) \). This distribution amplitude is expected to exhibit the structure \( \Psi_\pi(x,b;Q;\mu) = \exp[-S(x,b,Q)] \times \Psi_\pi^0(x,b;\mu) \), where \( \Psi_\pi^0 \) is independent of \( Q \), and \( S \) is found to be a RG-invariant function containing double logarithms of \( Q \). Taking such an exponential form as an ansatz, we have

\[
Q \frac{d\Psi_\pi}{dQ} = -Q \frac{dS}{dQ} \Psi_\pi .
\]  

(26)

\( S \) contains at most double logarithmic terms in \( Q \). \( Q dS/dQ \) therefore contains at most single logarithms of \( Q \), which can be treated by usual renormalization-group methods.

To obtain the pion’s Sudakov factor, we have therefore to find a differential equation of the form, \( Q d\Psi_\pi/dQ = \mathcal{C} \cdot \Psi_\pi \) (where \( \mathcal{C} \) contains single logarithms of \( Q \)) and to integrate this equation. The procedure can be summarized in the following steps:

- \( \Psi_\pi \) depends on \( Q \) only through \( \nu^2 = (p' \cdot n)^2/|n^2| \), so that

\[
Q \frac{d\Psi_\pi}{dQ} = -\frac{n^2}{p' \cdot n} p'^\alpha \frac{d\Psi_\pi}{dn^\alpha} .
\]  

(27)

- The \( n \)-dependence of Feynman diagrams is contained only in the gluon propagators. In the axial gauge, the propagator of a gluon with momentum \( q \) is

\[
N^{\mu\nu}(q) = \frac{-i}{q^2} \left( g^{\mu\nu} - \frac{n^\mu q^\nu + q^\mu n^\nu}{n \cdot q} + n^2 \frac{q^\mu q^\nu}{(n \cdot q)^2} \right) ,
\]  

(28)

and the derivative with respect to \( n^\alpha \) can be rewritten in the form:

\[
\frac{d}{dn^\alpha} N^{\mu\nu}(q) = -\frac{1}{q \cdot n} (N^{\mu\alpha} q^\nu + N^{\nu\alpha} q^\mu) .
\]  

(29)
Consider a Feynman diagram contributing to $\Psi_\pi$. The derivative with respect to $\log Q$ is equivalent to modifying one of the gluon propagators according to eqs. (27) and (29). This is represented diagrammatically in Fig. 3 for the case of a gluon propagator attached to quark lines. We need to compute the effect of $Qd/dQ$ on all the diagrams of order $O(\alpha_s^n)$, and so we have to consider all the diagrams of order $O(\alpha_s^{n-1})$, inserting in all possible ways two elements: an external gluon propagator carrying momentum $q$ (wavy line in Fig. 3) and an injection of momentum $q$ (dashed arrow in Fig. 3). An integral over the momentum $q$ has to be performed.

When we sum over all the possible momentum injections (represented by the dashed arrows in Fig. 3), Ward identities lead to cancellations. The only surviving graphs correspond to a momentum injection at the end of one of the valence quark lines. Thus, applying $Qd/dQ$ means inserting a gluon line, with one leg attached to a valence quark line with the vertex (denoted by a square in Fig. 4):

$$- igT^A \frac{n^2}{(p' \cdot n)(q \cdot n)} p'^\alpha.$$  (30)

Figure 3: Diagrammatic representation of the application of eq. (29) for the derivative of a gluon propagator with respect to $n^\alpha$. The dashed arrow represents the injection of momentum $q$, using the standard vertex $-ig\gamma_\rho T^A$ multiplied by $q^\rho$ ($\rho = \mu$ or $\nu$).

Figure 4: Diagrammatic representation of the derivative of $\Psi_\pi$ with respect to $\log Q$. The square vertex corresponds to $-igT^A \frac{n^2}{(p' \cdot n)(q \cdot n)} p'^\alpha$. 
• For the modified diagrams, the leading contribution for large $Q$ arises when the inserted gluon is either soft or ultraviolet. The corresponding terms have been computed at the leading order in $\alpha_s$ in ref. [27], yielding the differential equation:

\[
Q \frac{d}{dQ} \Psi_\pi(x, b, Q; \mu) = \left[ K(b\mu, g(\mu)) + \frac{1}{2} G(x\nu/\mu) + \frac{1}{2} \tilde{G}(\bar{x}\nu/\mu, g(\mu)) \right] \Psi_\pi(x, b, Q; \mu),
\]

where $K$ and $G$ collect respectively soft and ultraviolet contributions, and have opposite anomalous dimensions $\gamma_K = -\gamma_G$. The bracketed factor in eq. (31) is therefore scale-independent (in ref. [26] it is shown that this independence of the scale is also true at higher orders). The explicit expressions (at order $O(\alpha_s)$) for $K$ and $G$ in the \underline{MS} scheme are [27]:

\[
K(b\mu, g(\mu)) = -\frac{4}{3}\frac{\alpha_s}{\pi} \log\left(\frac{b^2 \mu^2 e^{2\gamma}}{4}\right),
\]

and [26, 30]

\[
\gamma_K = -\gamma_G = \frac{8}{3}\frac{\alpha_s}{\pi} + 4 \left[ \frac{67}{18} - \frac{\pi^2}{6} - \frac{5}{27} N_f \right] \left(\frac{\alpha_s}{\pi}\right)^2 + O(\alpha_s^3).
\]

• The differential equation eq. (31) can be solved directly, since the $Q$-dependence of $K(b\mu, g(\mu)) + G(\bar{x}\nu/\mu, g(\mu))$ comes only from the $\nu$-dependence of $G$. A different method was followed in refs. [26, 27]. Using RG-equations, all the $Q$-dependence can be absorbed into the scale of the coupling constant:

\[
K(b\mu, g(\mu)) + G(x\nu/\mu, g(\mu)) = K(C_1, g(C_2x\nu)) + G(1/C_2, g(C_2x\nu)) - 2 \int_{C_1/\bar{\mu}}^{C_2x\nu} \frac{d\bar{\mu}}{\bar{\mu}} A[C_1, g(\bar{\mu})],
\]

where $C_1$ and $C_2$ are free parameters, which should be tuned to avoid large logarithms in the perturbative expansion of $K$ and $G$ on the right hand-side of eq. (37). The function $A$ is given by

\[
A[C_1, g] = \frac{1}{2} \left[ \gamma_K(g) + \beta(g) \frac{\partial}{\partial g} K(C_1, g) \right].
\]

Combining eqs. (31) and (35) yields the general solution:

\[
\Psi_\pi(x, b, Q; \mu) = \exp[-\mathcal{S}(x, b, \nu)] \Psi_\pi^0(x, b; \mu),
\]

\[
\mathcal{S}(x, b, \nu) = \tilde{s}(x, b, \nu) + \tilde{s}(\bar{x}, b, \nu),
\]

\[
\tilde{s}(x, b, \nu) = \frac{1}{2} \int_{C_1/\bar{\mu}}^{C_2x\nu} \frac{d\bar{\mu}}{\bar{\mu}} \left[ 2 \log\left(\frac{C_2x\nu}{\bar{\mu}}\right) A(C_1, g(\bar{\mu})) - K(C_1, g(\bar{\mu})) - G(1/C_2, g(\bar{\mu})) \right].
\]
We can rewrite $S$ more conveniently using eqs. (32), (33), (36), and the two-loop expression for the strong coupling constant:

$$\frac{\alpha_s(\mu)}{\pi} = \frac{1}{\beta_1 \log(\mu^2/\Lambda^2_{\text{QCD}})} - \frac{\beta_2 \log \log(\mu^2/\Lambda^2_{\text{QCD}})}{\beta_1^3 \log(\mu^2/\Lambda^2_{\text{QCD}})} , \quad (40)$$

$$\beta_1 = \frac{33 - 2N_f}{12}, \quad \beta_2 = \frac{153 - 19N_f}{24}. \quad (41)$$

In this way we obtain

$$S(x, b, \nu) = s(x, b, r \cdot Q) + \bar{s}(\bar{x}, b, r \cdot Q), \quad (42)$$

where $r = \sqrt{|\lambda/\rho|}$ depends on the components of the gauge-fixing vector $n$ which we write as $n = (\lambda/\sqrt{2}, \rho/\sqrt{2}, 0)$. The gauge dependence now manifests itself as a dependence on $r$ rather than $\nu$. $s(u, b, R)$ is a function of:

$$\hat{q} = \log \left[ \frac{C_2 u R}{2\Lambda_{\text{QCD}}} \right], \quad \text{and} \quad \hat{b} = \log \left[ \frac{C_1}{b \Lambda_{\text{QCD}}} \right], \quad (43)$$

and is given explicitly by:

$$s(u, b, R) = \frac{A^{(1)}}{2\beta_1} \frac{\hat{q}}{\hat{b}} \log \left( \frac{\hat{q}}{\hat{b}} \right) + \frac{A^{(2)}}{4\beta_1^2} \left( \frac{\hat{q}}{\hat{b}} - 1 \right) - \frac{A^{(1)}}{2\beta_1} (\hat{q} - \hat{b})$$

$$- \frac{A^{(1)} \beta_2}{4\beta_1^3} \left[ \log(2\hat{b}) + 1 - \log(2\hat{q}) + 1 \right]$$

$$- \left[ \frac{A^{(2)}}{4\beta_1} - \frac{A^{(1)}}{4\beta_1} \log \left( \frac{C_1^2 e^{2\gamma - 1}}{C_2^2} \right) \right] \log \left( \frac{\hat{q}}{\hat{b}} \right) + \frac{A^{(1)} \beta_2}{8\beta_1^3} \left[ \log^2(2\hat{q}) - \log^2(2\hat{b}) \right]. \quad (44)$$

The coefficients $\beta_i$ are defined in eq. (41) and the $A^{(i)}$s in eq. (44) are given by

$$A^{(1)} = \frac{4}{3}, \quad A^{(2)} = \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{27}N_f + \frac{8}{3} \beta_1 \log \left( \frac{C_1 e^{\gamma}}{2} \right). \quad (45)$$

These coefficients come from the two-loop expression of $A$ defined in eq. (30):

$$A = \frac{\alpha_s}{\pi} A^{(1)} + \left( \frac{\alpha_s}{\pi} \right)^2 A^{(2)} + O(\alpha_s^3), \quad (46)$$

The first three terms of $A^{(2)}$ come from the two-loop anomalous dimension of $K$ [80], and the last one from the partial derivative of $K$ with respect to $g$.

Eq. (44) has been derived using the two-loop level expression of the coupling constant and of $\gamma_K$, but the one-loop expression of the functions $K$ and $G$. It is not necessary to compute their two-loop expressions if $C_1$ and $C_2$ are tuned in eq. (33) in order to avoid large contributions of $O(\alpha_s^2)$. A conventional choice is $C_1 = 1$ and $C_2 = \sqrt{2}$ [20]–[22], although we are not aware of an argument that this choice is optimal (or even good enough for the precision claimed for $B$-decay form factors [3]–[10]).
Finally we need to consider the residual function $\Psi_\pi^0(x, b; \mu)$ in eq. (37). Since the Sudakov factor suppresses the modified DA for large $b$, we have, in particular, to determine $\Psi_\pi^0(x, b; \mu)$ for small $b$. It is argued in ref. [27] that the factor $\exp(i\vec{k}_\perp \cdot \vec{b})$ in the integrand of the Fourier transform in transverse space ensures that the integral is dominated by the region $|k_\perp| < 1/b$, and hence that

$$
\Psi_\pi(x, b; \mu = 1/b) = \phi_\pi(x; \mu = 1/b) + O(\alpha_s(1/b)) .
$$

(47)

The logarithmic corrections of $O(\alpha_s(1/b))$ arise because the way that the integral is regulated at large transverse momenta is different from that in the standard definition of the BLER distribution amplitude. A second type of correction is due to the different definition of the BLER and modified DAs. Both quantities are defined from the non-local, and gauge dependent matrix element $\langle 0|\bar{q}(z)\gamma_\mu\gamma_5 q^\prime(0)|\pi(p)\rangle$ in a gauge $n \cdot A = 0$. But $\phi_\pi$ is defined in the light-cone gauge [$n^2 = 0$], whereas $\Psi_\pi$ is defined in an axial gauge [$n^2 \neq 0$]. The identification of $\phi_\pi$ and $\Psi_\pi$ holds only in the leading logarithmic (LL) approximation. Thus there are next-to-leading logarithmic (NLL) corrections in Eq. (47).

The presence of these corrections has important implications for the phenomenological applications of this formalism, as will be discussed in detail in section 6. In particular, for the calculations to be valid, the integrals have to be dominated by the region of small $b$, so that $\alpha_s(1/b)$ is in the perturbative regime. The identification of $\Psi_\pi^0(x, b; \mu)$ with $\phi_\pi(x, \mu)$ is valid for $\mu = 1/b$ and so we have to perform the RG-evolution of $\Psi_\pi$ to the scale $\mu = 1/b$. We finally arrive at:

$$
\Psi_\pi(x, b; \mu) = \exp \left[ 2 \int^{1/b}_\mu \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(g(\bar{\mu})) \right] \Psi_\pi(x, b; 1/b) ,
$$

(48)

where $\gamma_q = -\alpha_s/\pi + O(\alpha_s^2)$ is the anomalous dimension of the quark’s wave function in the axial gauges [26].

Combining eqs. (37), (47) and (48), the distribution amplitude $\Psi_\pi$ is the product of three factors: a Sudakov exponential, an evolution-related term and the standard distribution amplitude $\phi_\pi$. The combined expression is:

$$
\Psi_\pi(x, b, Q; \mu) = e^{-S} \exp \left[ 2 \int^{1/b}_\mu \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(g(\bar{\mu})) \right] \phi_\pi(x; \mu = 1/b) + O(\alpha_s(1/b)) .
$$

(49)

### 4.2 Dependence of the Sudakov form factors on the gauge and on the integration constants.

In this subsection we add some comments on the dependence of the Sudakov factor eq. (44) on $C_1$, $C_2$ and $n$. The general expression was introduced in ref. [27], but in the subsequent papers [10, 20, 22], only the particular expression with $C_1 = 1$, $C_2 = \sqrt{2}$ and $n \propto (1, -1, \vec{0}_\perp)$ was considered.
\( C_1 \) and \( C_2 \) are arbitrary parameters, introduced in order to solve the differential equation eq. (31). The full expression for \( s(u, b, R) \) is therefore independent of these parameters. However, eq. (44) is a truncated expression, derived from the one-loop expression of \( K \) and \( G \) and the two-loop expression of \( \gamma_K \). It is therefore weakly dependent on \( C_1 \) and \( C_2 \). A redefinition of these two coefficients does not affect the double logarithms (i.e. the terms proportional to \( \log(\hat{q}/\hat{b}) \)) but will have an effect on the single logarithms. Ideally \( C_1 \) and \( C_2 \) should be chosen so as to minimise the effects of the unknown higher-order terms. This requires some insights into the important regions in \( x \) and \( b \). The arguments of the logarithms in eq. (44) are positive for the regions

\[
\begin{align*}
b &\leq \frac{C_1}{\Lambda_{\text{QCD}}} \quad \text{and} \quad \frac{2\Lambda_{\text{QCD}}}{C_2 r Q} &\leq x &\leq 1 - \frac{2\Lambda_{\text{QCD}}}{C_2 r Q},
\end{align*}
\]

and Sudakov suppression arises as \( b \) reaches \( C_1/\Lambda_{\text{QCD}} \). Let us choose a specific gauge \((r)\) and a momentum transfer \((Q)\). If we consider a soft configuration for the valence quarks of the pion, the estimated Sudakov suppression varies with the choice of \( C_1 \) and \( C_2 \). Of course this dependence on \( C_1 \) and \( C_2 \) is due to the fact that we have truncated the series in \( \alpha_s \) to obtain eq. (44).

\( \Psi_\pi \) was explicitly defined in a gauge-dependent way in eq. (20). As we have seen above, this gauge dependence can be used to determine the functional dependence of the distribution amplitude on \( Q \). The gauge dependence is present in eq. (49) through the Sudakov term \( S \). To exhibit this gauge dependence more transparently we use our freedom to redefine the constants of integration \( C_1 \) and \( C_2 \), writing \( C_1 = C'_1 \) and \( C_2 = C'_2/r \), so that

\[
s(x, b, r \cdot Q) = s(x, b, Q) + \frac{A^{(1)}}{2\beta_1} \log r \log \left( \frac{\log[C'_2 x Q/(2\Lambda_{\text{QCD}})]}{\log[C'_1/(b \Lambda_{\text{QCD}})]} \right) + \ldots
\]

where the ellipsis denotes single logarithms. In eq. (51) we see explicitly the gauge dependence of the Sudakov factor.

Of course the convolutions which yield physical form factors, such as eq. (19), are gauge independent. The gauge dependence of the distribution amplitudes has to be compensated by a similar term in the hard-scattering kernel. For the case of the hard elastic scattering of two mesons, the authors of ref. [27] explain clearly how the resummation of soft gluon interactions in the hard-scattering kernel leads to a gauge dependence cancelling the one in the meson distribution amplitudes. We are not aware of similar explicit demonstrations for the electromagnetic form factor of the pion or for the \( B \to \pi \) semileptonic decay form factors. Such demonstrations would add confidence that all Sudakov effects are correctly included in the mesons’ distribution amplitudes.

5 \( B \)-meson distribution amplitude(s)

In section 2.1 we have seen that there are two distribution amplitudes for the \( B \)-meson at leading twist. In the applications of the pQCD framework to \( B \)-decays [5]–[10], one linear combination of the two distribution amplitudes is neglected; specifically the two
distribution amplitudes are set equal to each other, \( \Psi_B^+ \equiv \Psi_B^- = \Psi_B^\alpha \). We consider the validity of equating the two distribution amplitudes in section 5.1. The remaining \( B^- \) meson distribution amplitude is analyzed along similar lines to that of the pion. Consider a \( B^- \) meson of momentum \( p = (m/\sqrt{2}, m/\sqrt{2}, 0_\perp) \) containing a light quark of momentum \( l \). The modified distribution amplitude used in pQCD analyses is:

\[
\Psi^B(l_+, b, p; \mu) = \exp[-S'(n, p, b)] \exp[-E(\mu, b)] \phi^B(l_+; \mu = 1/b) + O(\alpha_s(1/b)), \tag{52}
\]

where:

- \( S' \) is the Sudakov factor containing the resummed double logarithmic contributions as well as selected nonleading logarithms. It depends on the transverse separation \( b \), on the momentum of the \( B^- \) meson \( p \) and on the gauge-fixing vector \( n \).

- \( E \) is an evolution factor, relating the renormalization scale \( \mu \) and \( 1/b \).

- \( \phi^B \equiv \phi^B_+ = \phi^B_- \) is the “common” distribution amplitude of the \( B^- \) meson.

The presence of the heavy \( b \) quark significantly modifies the analysis of Sudakov effects. The exponentiated double logarithms of the Sudakov factor sum the overlapping contributions of soft and collinear gluons. We work however, in the rest frame of the \( B^- \) meson so that the term “collinear” is not defined until the convolution for the form factor is introduced (with a hard scattering kernel which is independent of \( l_- \)). The definition of the distribution amplitude therefore cannot be completely separated from the process being studied. Moreover, all the components of the momentum of the light quark are \( O(\Lambda_{\text{QCD}}) \), so that the kinematics is very different from that of an energetic pion with a light-like four-momentum. In view of these questions we examine the derivation of eq. (52) in section 5.2 below. We start however, by considering whether setting \( \Psi^B_+ = \Psi^B_- \) is theoretically motivated.

5.1 Two distribution amplitudes or one?

In refs. [1]–[10] the phenomenological analysis is performed with a single distribution amplitude for the \( B^- \) meson, setting \( \phi^B \equiv \phi^B_+ = \phi^B_- \) and \( \Psi^B \equiv \Psi^B_+ = \Psi^B_- \). Such an identification simplifies considerably the calculation of the form factors and other physical quantities, since the momentum-space projection operator \( P_B^\alpha \) in eq. (4) now reduces to:

\[
P_B^\beta\alpha = -\frac{i f_B}{4} \left\langle (p + m)\gamma_5 \right\rangle_{\alpha\beta} \phi^B(\omega). \tag{53}
\]

In view of the arguments presented in sec. 2.1 above, we find this identification of the two distribution amplitudes rather surprising. Indeed the authors of ref. [15] argue that the two DAs are likely to have a different end-point behaviour. It would be surprising to us if the pseudoscalar matrix element in eq. (3) and the axial one in eq. (10) could be described in the heavy-quark limit by the same distribution amplitude. In this section we examine
the weaker proposition in ref. [9]; that although $\phi^B_+$ and $\phi^B_-$ are different, they have the same contribution to the convolution (17), so that that their difference can be neglected when studying $B \to \pi$ form factors.

In order to assess this claim, we need the $B \to \pi$ form factors (15) and (16) written as convolutions over both longitudinal and transverse momenta:

$$F_{0,+} = f_f f_B m^2 \frac{\alpha_s \pi C_F}{N_c} \int_0^1 dx \int dl_+ \int d^2 k_\perp \int d^2 l_\perp \mathcal{I}_{0,+} ,$$

where

$$\mathcal{I}_0 = \frac{1}{\bar{x}l_+ \eta m + (\bar{k}_+ + \bar{l}_+)^2} \eta \Psi_\pi$$

$$\times \left\{ \frac{1}{\bar{x} \eta m^2 + \bar{k}_+^2} \left[ \Psi^B_+ - \bar{x} \eta \Psi^B_+ - \frac{\Delta(l_+, \bar{l}_+)}{m} \frac{\bar{k}_+ \cdot (\bar{k}_+ + \bar{l}_+)}{\bar{x}l_+ \eta m + (\bar{k}_+ + \bar{l}_+)^2} \right] \right\} ,$$

and

$$\mathcal{I}_+ = \frac{1}{\bar{x}l_+ \eta m + (\bar{k}_+ + \bar{l}_+)^2} \Psi_\pi$$

$$\times \left\{ \frac{1}{\bar{x} \eta m^2 + \bar{k}_+^2} \left[ \Psi^B_+ - \bar{x} \eta \Psi^B_+ - \frac{\Delta(l_+, \bar{l}_+)}{m} \frac{\bar{k}_+ \cdot (\bar{k}_+ + \bar{l}_+)}{\bar{x}l_+ \eta m + (\bar{k}_+ + \bar{l}_+)^2} \right] \right\} ,$$

with $\Delta(\omega, \bar{l}_+) = \int_0^\omega d\ell [\Psi^B_+(\ell, \bar{l}_+) - \Psi^B_-(\ell, \bar{l}_+)]$. $\Psi^B_\pm$ are functions of $l_+$ and $\bar{l}_+$, and $\Psi_\pi$ of $x$ and $\bar{k}_+$. In the formula for $\mathcal{I}_0$, the contribution in the second line comes from the upper diagram of Fig. 3, and the third line from the lower diagram. The second term in the formula for $\mathcal{I}_+$ comes from the lower diagram.

The authors of ref. [9] suggest that the value of $F_{0,+}$ should remain stable if we replace $\Psi^B_+$ by $\Psi^B_-$ (or vice-versa) in eqs. (55) and (59). To support this idea, they introduce the following “reasonable parametrization” for the distribution amplitudes:

$$\phi^B_+(l_+) = \delta(l_+ - \Lambda) - \Lambda \delta'(l_+ - \Lambda) , \quad \phi^B_-(l_+) = \delta(l_+ - \Lambda) ,$$

where $\Lambda$ is a parameter of order $\Lambda_{QCD}$. After integrating $\mathcal{I}_0$ and $\mathcal{I}_+$ over the transverse momentum $\bar{l}_+$, the hard-scattering kernels behave like $\ln(l_+/m)$ for small $l_+$. Convolving them with $\phi^B_-$ and $\phi^B_+$, we obtain two equal contributions proportional to $\ln(m/\Lambda)$, but with different constant terms. It is therefore argued that the two $B$ meson DAs can be set equal to each other in $B \to \pi$ form factors with reasonable precision.
The above argument however, raises a number of important questions. First of all we note that the two \( B \)-meson DAs in eq. (57) do not satisfy the equation of motion eq. (12). We also note that this argument implicitly assumes that the dependence on the transverse momentum of the modified DAs \( \Psi_\pm \) is approximately flat. Of course we accept that the model in eq. (57) was introduced only for purposes of illustration, nevertheless we do not consider it to be sufficiently general to support the case that we can identify \( \phi_{B,+} \) and \( \phi_{B,-} \), and use a single DA in the analysis. To underline this point we introduce an alternative (and equally unrealistic) model to that in eq. (57), but one which satisfies the equation of motion (12):

\[
\phi_{B,+}^B(l_+) = \delta(l_+ - \Lambda) \quad \text{and} \quad \phi_{B,-}^B(l_+) = \frac{1}{\Lambda} \Theta(l_+),
\]

(58)
where \( \Lambda = O(\Lambda_{\text{QCD}}) \), and we define \( \Theta(l_+) = 1 \) when \( 0 \leq l_+ \leq \Lambda \), and zero outside this range. We now perform the following simple exercise. We take the first term in eq. (55) (proportional to \( \Psi_{B,-} \)) and assume (as above) that its dependence on the transverse momentum is approximately flat over \([0, \Lambda]\). We then study the contributions to \( I_0 \) with \( \phi_{B,-} \) given by each of the two functions in eq. (58). In this way we investigate whether it matters which of \( \phi_{B,+} \) or \( \phi_{B,-} \) is used in the calculation (at least for this one term).

If we use \( \phi_{B,-}^B(l_+) = \Theta(l_+) / \Lambda \), we can perform the integration over \( l_+ \) and \( \vec{l}_\perp \):

\[
f_{\pi f_B} m^2 \frac{\alpha_s \pi C_F}{N_C} \int_0^1 dx \int d^2 \vec{k}_\perp \Psi_{\pi} \frac{1}{\bar{x} m^2 + \vec{k}_\perp^2} c_-(\bar{x}, K),
\]

(59)
where \( c_- \) is a function of \( \bar{x} \) and \( K = k_\perp / \Lambda \):

\[
c_- = \frac{1}{2 \Lambda^2 \bar{x}} \left[ L |1 - K^2| - S - 2L \log[1 + K^2 + |1 - K^2|] + \bar{x} - 2\bar{x} \log[2\bar{x}] + 2\bar{x} \log[\bar{x} + S + L(1 - K^2)] + 2L \log[\bar{x} + S + L(1 - K^2)] \right],
\]

(60)
and \( L = \Lambda / m \) and \( S = \sqrt{[\bar{x} + L^2(1 + K^2)]^2 - 4K^2L^2} \).

If we use \( \phi_{B,-}^B(l_+) = \delta(l_+ - \Lambda) \), we obtain a similar expression, where \( c_- \) is replaced by:

\[
c_+ = \frac{1}{\Lambda^2} \left[ \log[\bar{x} + S + L(1 - K^2)] - \log[2\bar{x}] \right].
\]

(61)
From eq. (59) we see that \( c_- \) and \( c_+ \) have to be convoluted with \( \Psi_{\pi} / (\bar{x} m^2 + \vec{k}_\perp^2) \) in order to compute the contribution of \( \Psi_{B,-} \) to the form factor \( F_0(0) \). When the integration over \( x \) and \( \vec{k}_\perp \) is performed, the largest contribution comes from the region of small \( \bar{x} \) and \( k_\perp \).

For small \( L = \Lambda / m \), the behaviour of \( c_- \) and \( c_+ \) is given by:

\[
c_- \sim - \frac{\log L}{m^2 L \bar{x}}, \quad c_+ \sim \frac{1}{m^2 L \bar{x}}
\]

(62)
which will be convoluted with the pion DA in eq. (59). We see therefore that the model in eq. (57) was artificially designed in order to yield the same leading contribution \( \ln(\Lambda_{\text{QCD}} / m) \).
In general, the contributions from $\phi_+^B$ and $\phi_-^B$ have different leading logarithmic behaviour and are significantly different in size. In the model defined by eq. (58), the expressions in eq. (62) suggest that $\phi_-^B$ yields a much larger contribution than $c_+$ and we now check this numerically.

We introduce the ratio $R = \frac{c_- - c_+}{c_- + c_+}$ which depends on $\bar{x}$ and $K$. The identification $\phi_-^B = \phi_+^B = \phi_-^B$ is justified if $R$ is small in the region which gives the dominant contribution to the convolution eq. (59). In fig. 5 we plot the ratio $R$ as a function of $\bar{x}$ for different values of $K = k_\perp / \Lambda$. For purposes of illustration we take $L = \Lambda / m = 0.25/5.28$. We see that $R$ can be as large as 40% for $k_\perp \lesssim \Lambda$, even for very small $\bar{x}$. The identification of the two $B$-meson DAs would only be valid for $\bar{x} < 0.05$, or $k_\perp \gg \Lambda$, i.e. in regions which do not contribute very much to the convolution eq. (54).

The analysis of a different, and perhaps slightly more realistic model, introduced in Appendix A.2, confirms the possibility of a large error. Of course, we have not estimated the actual error on $F_{0,+}$ when $\Psi_+^B$ and $\Psi_-^B$ are set equal to each other. In this presentation we have considered only a part of the contribution to the form factor $F_0(0)$, the scale of the coupling constant has not been defined, and Sudakov effects have not been included. Nevertheless, lessons can be learned from this brief analysis. The hard-scattering kernel enhances the small $l_+$ region, where the $B$-meson DAs are concentrated. If $\phi_+^B$ and $\phi_-^B$ have different end-point behaviours, the difference may be strongly enhanced by the hard-scattering kernel. There is therefore no convincing motivation for setting the two DAs equal to each other and such a procedure is likely to lead to unreliable results.

A related issue is the contribution from $\Delta(\omega) = \int_0^\omega d\ell \ (\phi_B^B(\ell) - \phi_+^B(\ell))$. The results of
the standard approach, eqs. (15) and (16), can be obtained by neglecting the transverse momenta in eqs. (55) and (56). We see therefore that a term proportional to $\Delta$ is contained in eqs. (15) and (16). The contribution of $\Delta$ to the standard formulae eqs. (15) and (16) is divergent when $l_+ \to 0$ or $\bar{x} \to 0$ and a large contribution from this region is therefore also expected in the pQCD approach. This contribution is dropped however, if one sets $\Psi_+^B = \Psi_-^B$ (and hence $\Delta = 0$).

We conclude that the model approximation in which the two distribution amplitudes are set equal to each other is not a generic one and in general will lead to wrong results for physical quantities.

5.2 Derivation of the Sudakov factor for the $B$-meson

In spite of the strong reservations expressed in the previous subsection, let us suppose that we can consider a single $B$-meson distribution amplitude. In this subsection we then examine the derivation of the Sudakov factor for the DA of the $B$-meson given in ref. [10] and used in subsequent phenomenological analyses. According to the authors of ref. [10], there can be an overlap of soft and collinear divergences in gluon exchanges, when $l_+$ is much larger than $\Lambda_{QCD}$, i.e. of order $M_B$. These divergences can be resummed into a Sudakov factor, which is therefore “half” of the Sudakov factor for the pion, since there is one light valence quark instead of two. This factor would then lead to a Sudakov suppression of configurations with large transverse separations.

We work in the rest frame of the $B$-meson in which the light valence quark has components of momentum of the same order in all directions $[O(\Lambda_{QCD})]$. The term “collinear” therefore is not meaningful at this stage and no double logarithms are expected to arise, as noted by the authors of ref. [10]. If one of the light-quark components, $l_+$ say, is much larger than $\Lambda_{QCD}$, the possible Sudakov suppression happens for configurations that already highly suppressed. The impact of Sudakov effects is therefore likely to be rather weak. On the other hand, in the most likely configuration where all the light-quark components are small $[O(\Lambda_{QCD})]$, no Sudakov suppression is expected. There is therefore no perturbative QCD mechanism to suppress configurations with a large transverse separation but a small

\footnote{This contribution is however not explicit in eqs. (15) and (16) because we have used the equation of motion eq. (12) to write $\Delta(l_+) = l_+ \phi^B_{l_+}(l_+)$.}
longitudinal momentum. The consequence of this fact will be investigated in section 6.

The argument in ref. [10] attempts to proceed along the same lines as were outlined for the pion in section 4.1, by studying the dependence of the $B$-meson DA on $l_+$. We now briefly note, without comment, some of the relevant features of this argument, stressing in particular the differences with the corresponding derivation for the pion. We then present our criticisms of this derivation.

- The authors of ref. [10] aim to identify the invariant parameters on which the distribution amplitude $\Psi^B = \Psi^B_+ = \Psi^B_-$ can depend. Because of the presence of the heavy quark, many such invariant parameters are available: $p^2$, $p \cdot l$, $p \cdot n$, $l \cdot n$ and $n^2$. It is still true that the structure of the gluon propagator in the axial gauge leads to a scale invariance of $\Psi^B$ with respect to $n$, but this argument is not sufficient to deduce that $\Psi^B$ depends on a single parameter.

- It is then argued that the eikonal approximation can be used to overcome the difficulties caused by the presence of the heavy quark. Consider a gluon of momentum $q$ attached to the heavy quark line of momentum $p - l$. When $q$ is soft, or collinear to $l$, we can use the approximation:

$$\frac{p - l + \phi + m}{(p - l + q)^2 - m^2} = \frac{p^\alpha}{p \cdot q} + R . \tag{63}$$

$R$ contains terms that are suppressed by powers of $1/m$ or that vanish once contracted with $(p/ + m)$.

- The eikonal approximation leads to a scale invariance of $\Psi^B$ with respect to $p$. $p$ would only appear in ratios such as $(n \cdot p)^2/(p^2 n^2)$, which are disregarded because they cannot lead to a “large scale”. The only “large scale” would be $l$, and the relevant invariants would be $(l \cdot n)^2/n^2$ and $(l \cdot p)^2/p^2$.

The $O(\alpha_s)$ corrections are examined in the particular axial gauge $n = (1, -1, \vec{0}_\perp)$, in which $\Psi^B$ (Fig. 3) is found to exhibit a scale invariance with respect to $l$ in the same eikonal approximation. The authors of ref. [10] conclude that $(l \cdot p)^2/p^2$ has to be discarded and $\Psi^B$ depends only on the scale $\nu'^2 = (l \cdot n)^2/n^2$.

- Since $\Psi^B$ depends on a single scale $\nu'$, it is possible to replace $l_+ d/l_+$ by $d/dn^\alpha$, and to derive a differential equation for the $B$-meson distribution amplitude:

$$l_+ \frac{d}{dl_+} \Psi^B(l_+, b; \mu) = \frac{1}{2} [K(b \mu, g(\mu)) + G(\nu'/\mu)] \Psi^B(l_+, b; \mu) . \tag{64}$$

- This equation can be solved using the same techniques used for the pion in section 4.1 leading to:

$$\Psi^B(l_+, b; \mu) = e^{-\tilde{s}(1,b,\nu')} \exp \left[ 2 \int_{\mu}^{1/b} \frac{d\tilde{\mu}}{\tilde{\mu}} \gamma_q(g(\tilde{\mu})) \right] \phi^B(l_+; \mu = 1/b) + O(\alpha_s(1/b)) , \tag{65}$$

where $\tilde{s}$ is defined in eq. (39). We can rewrite the Sudakov factor in terms of $l_+$:

$$\tilde{s}(1,b,\nu') = s(l_+/m,b,m/r) ,$$

where $s$ is defined in eq. (44).
We now express our concerns about this derivation, which include the identification of the two $B$-meson distribution amplitudes, the use of an eikonal approximation for the heavy quark and the determination of the scales on which the $B$-meson distribution amplitude depends.

- It is not clear to us that multiplicative, RG-invariant Sudakov factors arise separately for $\Psi^B = (\Psi^B_+ + \Psi^B_-)/2$ and $\bar{\Psi}^B = (\Psi^B_+ - \Psi^B_-)/2$ (the latter is neglected in ref. [10]).

- The eikonal approximation eq. (63) is relevant when $l_+$ is soft. But the resummation of double logarithms is performed in the region where $l_+$ is large. The relevance of eq. (63) in this case is not clear to us.

- A demonstration of the $l$-scale invariance of the $O(\alpha_s)$ corrections to $\Psi^B$, in a particular gauge, does not ensure the scale invariance of the function $\Psi^B$. And even if this property was satisfied by $\Psi^B$, $(l \cdot p)^2/p^2$ and $(l \cdot n)^2/n^2$ are both $l$-scale dependent and therefore could not be relevant variables for $\Psi^B$. If $\Psi^B$ is scale independent with respect to $n$, $p$ and $l$, it cannot depend on any “large scale”. The invariants with this property are

$$\frac{(p \cdot n)^2}{[p^2 n^2]} = \frac{\rho}{4\lambda} \left(1 + \frac{\lambda}{\rho}\right)^2, \quad \frac{(l \cdot p)^2 n^2}{[p^2 (l \cdot n)^2]} = \frac{\lambda}{\rho}, \quad (66)$$

where $n = (\lambda/\sqrt{2}, \rho/\sqrt{2}, \vec{0}_\perp)$ and only the $l_+$ component of $l$ has been retained.

Moreover, the questions raised by the derivation of the pion Sudakov factor remain to be answered:

- How is the gauge dependence of $\Psi^B$ cancelled by that in the hard-scattering kernel?

- Is it legitimate and sufficiently precise to identify $\phi_B$ and $\Psi^B$?

We have tried and failed to understand the derivation of the Sudakov factor for the distribution amplitudes of the $B$-meson. In our view it is based on a number of unjustified assumptions and steps. Given that the dominant contributions come from the region of small $l_+$, it seems to us that Sudakov effects are unlikely to be significant for the $B$-meson. If this is the case however, the integrals in eqs. (15) and (16) may give significant contributions (or even diverge) at small $l_+$.

6 Numerical study

In the previous sections we have explained our reservations about the pQCD formalism used to evaluate the semileptonic form factors and other physical quantities. We now temporarily set aside these reservations and perform a numerical study of the pQCD evaluation of the $B \rightarrow \pi$ semileptonic form factors. We are particularly interested in two questions:
1. Are the results sensitive to the choice of distribution amplitudes?
We will see that the answer to the first question is yes, so that the form factors are not calculable in practice.

2. For “reasonable” choices of distribution amplitudes, does all of the contribution come from the perturbative region?
The answer to this question is in general no, so that the form factors are not calculable also in principle. This conclusion is in addition to our theoretical reservations.

6.1 pQCD expression of $B \to \pi$ form factors
The formulae for the form factors in eqs. (54), (55) and (56) can be rewritten in impact parameter space:

$$F_{0,+} = \int_0^1 dx \int dl_+ \int b_+ db_+ \int c_+ dc_+ \tilde{\Psi}(x, b; \mu) \tilde{H}_{0,+}(x, l_+, b_+, c_+, \eta; \mu) \tilde{\Psi}(l_+, c_+; \mu),$$

where

$$\tilde{H}_0 = f_{\pi f_B} m^2 \frac{\pi C_F}{N_c} \alpha_s(\mu) \eta \times \left( (1 + \bar{x} \eta) G(\bar{x} l_+ \eta m, b_+, \bar{x} \eta m^2, c_+) + \frac{l_+ \bar{\eta}}{m} G(l_+ \eta m, c_+, \bar{x} l_+ \eta m, b_+) \right),$$

and

$$\tilde{H}_+ = f_{\pi f_B} m^2 \frac{\pi C_F}{N_c} \alpha_s(\mu) \times \left( (1 + \bar{x} \eta) G(\bar{x} l_+ \eta m, b_+, \bar{x} \eta m^2, c_+) - \frac{l_+ \bar{\eta}}{m} G(l_+ \eta m, c_+, \bar{x} l_+ \eta m, b_+) \right),$$

with the function $G$ given in terms of Bessel functions by:

$$G(A, b_+, B, c_+) = K_0(\sqrt{A} b_+) \left[ \theta(b_+ - c_+) K_0(\sqrt{B} b_+) I_0(\sqrt{B} c_+) \right. + \theta(c_+ - b_+) K_0(\sqrt{B} c_+) I_0(\sqrt{B} b_+) \left. \right] .$$

The angular integrations in the transverse plane have been performed, since the distribution amplitudes are independent of the orientation of the transverse momenta.

There are higher order corrections to the hard-scattering amplitude of order $\log(t/\mu)$, where $t$ is the large scale associated with the hard gluon. In ref. [10] this was taken as $t = \max(\sqrt{l_+ \eta m}, 1/b_+, 1/c_+)$. In order to avoid large corrections, we can perform the evolution of the hard-scattering kernels $H_{0,+}$ from $\mu$ to $t$ using the RG equation:

$$\left( \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right) H_{0,+} = 4\gamma_q H_{0,+},$$

25
where \( \gamma_q = -\alpha_s/\pi \) is the anomalous dimension of the quark’s wave function in the axial gauge. For \( H_{0,+}(t) \) we take its expression at lowest order in \( \alpha_s \). At this order \( H \) does not contain a gauge dependence able to cancel the gauge dependence of \( \Psi_\pi \) and \( \Psi_B \). In particular, there is no resummation of soft gluon contributions in the hard-scattering kernel in a similar way to the work of ref. \[27\] for hard elastic scattering of mesons. We obtain:

\[
F_0 = f_\pi f_B m^2 \frac{\pi C_F}{N_c} \int_{0}^{1} dx \int dl_+ \int b_1 db_\perp \int c_1 dc_\perp \\
\times \phi_\pi(x; 1/b_\perp) \phi^B(l_+; 1/c_\perp) \exp[-S(x, l_+, \eta, m, b_\perp, c_\perp)] \alpha_s(t) \\
\times \eta \left[ \left(1 + \bar{\eta}\right) G(\bar{x} l_+ \eta m, b_\perp, \bar{x} \eta m^2, c_\perp) + \frac{l_+ \bar{n}}{m} G(l_+ \eta m, c_\perp, \bar{x} l_+ \eta m, b_\perp) \right],
\]

(72)

and

\[
F_+ = f_\pi f_B m^2 \frac{\pi C_F}{N_c} \int_{0}^{1} dx \int dl_+ \int b_1 db_\perp \int c_1 dc_\perp \\
\times \phi_\pi(x; 1/b_\perp) \phi^B(l_+; 1/c_\perp) \exp[-S(x, l_+, \eta, m, b_\perp, c_\perp)] \alpha_s(t) \\
\times \left[ \left(1 + \bar{\eta}\right) G(\bar{x} l_+ \eta m, b_\perp, \bar{x} \eta m^2, c_\perp) - \frac{l_+ \bar{n}}{m} G(l_+ \eta m, c_\perp, \bar{x} l_+ \eta m, b_\perp) \right],
\]

(73)

where the function \( S \) combines Sudakov factors and evolution-related terms:

\[
S = s(x, b_\perp, \eta m r) + s(\bar{x}, b_\perp, \eta m r) + s(l_+/m, c_\perp, m r) \\
+ 2 \int_{1/b_\perp}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(g(\bar{\mu})) + 2 \int_{1/c_\perp}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(g(\bar{\mu})).
\]

(74)

The function \( s \) is defined in eq. \[44\]. We follow refs. \[10, 20\] and take the particular values \( C_1 = 1, C_2 = \sqrt{2} \) and \( r = 1 \), so that

\[
\hat{q} = \log \left[ u Q / (\sqrt{2} \Lambda_{\text{QCD}}) \right], \quad \text{and} \quad \hat{b} = \log \left[ 1 / (b \Lambda_{\text{QCD}}) \right].
\]

(75)

In eqs. \[72\] and \[73\], the transverse separations are integrated from 0 to \( 1/\Lambda_{\text{QCD}} \), because the Sudakov factors suppress any configuration with larger transverse separations. We can accept this statement in the case of the pion \( (b_\perp) \). But the Sudakov factor for the \( B \)-meson is weak for the usual configurations where \( l_+ \) is of order \( \Lambda_{\text{QCD}} \) or less. We may therefore worry that the integration with respect to \( c_\perp \) is artificially cut-off, and omits regions which might yield significant contributions to the form factors (such as small \( l_+ \) but large \( c_\perp \)).

### 6.2 Models of distribution amplitudes

In order to compute the \( B \to \pi \) form factors, we have to choose models for the distribution amplitudes of the pion and the \( B \) meson. It can be seen from eq. \[19\] that the modified pion DA \( \Psi_\pi(x, b; \mu) \) is the product of three factors: a Sudakov exponential, an evolution-related term and the BLER pion DA \( \phi_\pi(x; \mu = 1/b) \). In ref. \[10\], the \( b \)-dependence due
to the renormalization scale of $\phi_\pi(x; \mu = 1/b)$ has been neglected in eqs. (72) and (73), because its numerical impact is believed to be small. We will follow this prescription here. In a similar way, the $b$-dependence of the $B$-meson DA in eqs. (72) and (73) is neglected.

We use two different models for the leading-twist pion DA [11, 12]:

$$\phi_\pi^{(a)}(x) = 6x(1-x)[1 + \alpha_2 C_2^{(3/2)}(2x - 1) + \alpha_4 C_4^{(3/2)}(2x - 1)] ,$$  \hspace{1cm} (76)  

$$\phi_\pi^{(b)}(x) = 30x(1-x)(1-2x)^2 ,$$  \hspace{1cm} (77)  

where $C_n^{(3/2)}(u)$ are the Gegenbauer polynomials. Even though we neglect evolution effects in the computation of the $B \rightarrow \pi$ form factors, the BLER distribution amplitudes do depend on the renormalization scale $\mu$. In model $\phi_\pi^{(a)}$ the $\mu$-dependence is contained in the coefficients $\alpha_2$ and $\alpha_4$; model $\phi_\pi^{(b)}$ was proposed originally at the scale $\mu_0 = 0.5$ GeV [12].

We will use three models for the “common” $B$-meson DA; in all three models the contribution from the region with $l_+ \geq m$ is suppressed. Defining $\xi = l_+/m$, the first two

Figure 7: Models $\phi_B^{(1)}$, $\phi_B^{(2)}$ and $\phi_B^{(3)}$ for the $B$ distribution amplitude, as functions of $\xi = l_+/m$. 
models are \([10]\):

\[
\phi^B_{(1)}(l_+) = N_{(1)} \sqrt{\xi(1 - \xi)} \exp \left[ -\frac{m^2}{2\omega^2\xi^2} \right], \tag{78}
\]

\[
\phi^B_{(2)}(l_+) = N_{(2)} \frac{\xi(1 - \xi)^2}{m^2 + \omega(1 - \xi)} \tag{79}
\]

and we introduce a third model, with a different end-point behaviour:

\[
\phi^B_{(3)}(l_+) = N_{(3)}(1 - \xi) \exp \left[ -\frac{m}{\omega} \xi \right]. \tag{80}
\]

The normalization constants \(N_{(i)}\) are obtained from the integral \(\int dl_+ \phi^B(l_+) = 1\). Fig. 7 shows the general shape of the three models. \(l_+\) is concentrated around \(\omega/\sqrt{2}\) in the first model and we expect \(\omega = O(\Lambda_{\text{QCD}})\). The second model has a broad distribution in \(l_+\), and may therefore be considered highly unlikely to be physical. The third model is concentrated in the small \(l_+\) region.

We stress that the three models in eqs. (78)–(80) are introduced to study the dependence on the expressions (72) and (73) on the end-point behaviour of the distribution amplitudes. Our reservations about the consistency of using a single wave function for the \(B\)-meson remain of course. In particular the question of whether \(1/\lambda_B\) is finite for each of the three models depends on whether the functions in eqs. (78)–(80) are interpreted as corresponding to \(\phi^B_+\) or \(\phi^B_-\), and cannot be answered in the approximation in which a single distribution amplitude is used for the \(B\)-meson.

### 6.3 Dependence on the shape

We now use eqs. (72) and (73) to compute the form factors \(F_+\) and \(F_0\) for the models of the \(\pi\) and \(B\)-meson DAs introduced in section 6.2. We are able to reproduce the quoted results in ref. [10] for the first two models for the \(B\) meson \(\phi^B_{(1)}\) and \(\phi^B_{(2)}\). We wish to study the model-dependence of these results. For illustration we will present our results for the favourable kinematic situation \(\eta = 1\) (i.e. \(q^2 = 0\), where \(F_0 = F_+\)). The results are presented for the following choice of parameters: \(f_B = 0.19\) GeV, \(m = 5.28\) GeV, and \(\Lambda_{\text{QCD}} = 0.25\) GeV.

For the \(B\)-meson DA we consider the three models introduced above and vary the shape parameter \(\omega\). We use the two models for the pion DA in eqs. (76) and (77). Eq. (72) is linear in \(\phi_\pi\). If we choose the expansion in Gegenbauer polynomials \(\phi_\pi = \phi_\pi(a)\), we can give separately the numerical contributions from each of the three terms in eq. (76), i.e. the contribution from the asymptotic DA and those proportional to \(\alpha_2\) and \(\alpha_4\). The corresponding values are indicated in Tab. 1. For instance, if we choose \(\phi^B = \phi^B_{(1)}\), \(\omega=0.4\) GeV and \(\phi_\pi = \phi_\pi(a)\), the form factors at \(q^2 = 0\) are \(F_{0,+}(0) = 0.12 + \alpha_2 \cdot 0.25 + \alpha_4 \cdot 0.25\).

We see that the computed values of \(F_{0,+}(0)\) depend very significantly on the distribution amplitudes of both mesons. Models concentrated around small values of \(l_+\) (\(\phi^B_{(3)}\) for small \(\omega\)) or \(\bar{x}\) (\(\phi^B_{(2)}\)) give larger values than models which have a broader spread (\(\phi^B_{(3)}\), asymptotic pion
Table 1: Dependence of $F_0(0) = F_+ (0)$ on the $B$-meson and pion DAs. The models of $B$-meson DA depend on the shape parameter $\omega$ (in GeV for models 1 and 3, GeV$^2$ for model 2). The columns “As.”, “$\alpha_2$” and “$\alpha_4$” correspond to $\phi_\pi = 6x(1-x)$, $\phi_\pi = 6x(1-x)C(3/2)_2(2x-1)$ and $\phi_\pi = 6x(1-x)C(3/2)_4(2x-1)$ respectively. $\phi^{(\alpha)}_{\pi}$ is a linear combination of these three terms. “C-Z” corresponds to the model $\phi^{(b)}_{\pi}$.

| Model  | $\omega$ | As.  | $\alpha_2$ | $\alpha_4$ | C-Z  |
|--------|----------|------|------------|------------|------|
| $\phi^B_{(1)}$ | 0.1   | 0.28 | 0.46       | 0.46       | 0.58 |
|        | 0.2   | 0.19 | 0.35       | 0.38       | 0.42 |
|        | 0.3   | 0.14 | 0.29       | 0.29       | 0.33 |
|        | 0.4   | 0.12 | 0.25       | 0.25       | 0.28 |
|        | 0.5   | 0.10 | 0.22       | 0.26       | 0.25 |
|        | 0.6   | 0.09 | 0.19       | 0.23       | 0.22 |
| $\phi^B_{(2)}$ | -27.5 | 0.05 | 0.12       | 0.15       | 0.13 |
|        | -25.5 | 0.04 | 0.10       | 0.13       | 0.10 |
|        | -22.5 | 0.03 | 0.09       | 0.11       | 0.09 |
|        | -20.5 | 0.03 | 0.08       | 0.11       | 0.09 |
|        | -17.5 | 0.03 | 0.08       | 0.10       | 0.08 |
| $\phi^B_{(3)}$ | 0.1   | 0.36 | 0.52       | 0.51       | 0.70 |
|        | 0.3   | 0.21 | 0.36       | 0.38       | 0.45 |
|        | 0.5   | 0.16 | 0.29       | 0.32       | 0.35 |
|        | 0.7   | 0.14 | 0.25       | 0.28       | 0.30 |
|        | 0.9   | 0.12 | 0.23       | 0.26       | 0.27 |

DA). This can be readily understood since the hard-scattering kernel enhances distribution end-points (without invoking Sudakov effects the contributions from the end-point regions are divergent).

The conclusion of our investigation is that very good control of the behaviour of the DA’s at the end-points is needed in order to be able to compute the form factors with a precision which would be useful for phenomenological studies. From table 1 we see that with our present knowledge of the distribution amplitudes this is not the case.

### 6.4 Dependence on the cut-off in impact parameter space

In the previous subsection we have seen that with our present knowledge of the mesons’ distribution amplitudes, it is not possible to calculate the form factors with the required precision. We now investigate whether, for model distribution amplitudes of the form used above, all (or almost all) of the contribution to the form factors comes from the perturbative region of phase space. In the preceding sections we have seen that in order to set $\Psi_{\pi}(x,b_\perp,Q;\mu = 1/b_\perp)$ equal to $\phi_\pi(x;\mu = 1/b_\perp)$ (see eq. (47)) we require $b_\perp$ to be small [27] (there is a similar requirement for the $B$-meson). We now check whether all of the contribution does indeed come from the region of small impact parameters.
Figure 8: Dependence of $F_{0,+}(0)$ on the cut-offs $b_\pi^c$ and $b_B^c$ for the integration over $b_\perp$ (pion impact parameter) and $c_\perp$ ($B$-meson impact parameter). For purposes of illustration, in this figure we choose $\phi_{(1)}^B$ as the model distribution amplitude for the $B$-meson, and the asymptotic distribution amplitude for the pion.

In fig. 8 we evaluate the integrals in eqs. (72) and (73), but with a cut-off introduced for the impact parameters for the pion or the $B$-meson. For purposes of illustration we take $\phi_{(1)}^B$ for the $B$-meson’s DA, and the asymptotic distribution amplitude for the pion, but similar plots can readily be obtained for the other models of distribution amplitudes. The figure contains plots with the integrals over the impact parameters performed over the regions $0 \leq b_\perp \leq b_\pi^c$ and $0 \leq c_\perp \leq 1/\Lambda_{QCD}$ (left-hand figure) or $0 \leq b_\perp \leq 1/\Lambda_{QCD}$ and $0 \leq c_\perp \leq b_B^c$ (right-hand figure). Fig. 8 shows the dependence of $F_{0,+}(0)$ on the cut-offs $b_\pi^c$ (left) and $b_B^c$ (right). In order for the calculations to be consistent, we require that the contribution from the regions of large impact parameters is negligible, and hence that the curves in fig. 8 reach a plateau for values of $b_\pi^c$ and $b_B^c$ in the perturbative region.

The purpose of our investigation is to check whether this is the case. In general, and for the models used in fig. 8 in particular, the answer is clearly no. Even if we optimistically take 500 MeV as the value above which perturbation theory holds, we see that the curves in the figure are far from saturating at this scale. There is a significant (and uncalculable) contribution from the nonperturbative region of phase space, where eqs (25) and (52) are affected by large corrections, impossible to estimate. Contributions from regions with larger impact parameters cannot be calculated reliably and hence we conclude that pQCD calculations for the form factors are not valid.

A similar study was reported in ref. [10], in which the authors impose that for a consistent pQCD computation, most of the contribution should come from the region where $\alpha_s(1/b_\perp)/\pi$ and $\alpha_s(1/c_\perp)/\pi$ are smaller than 0.5, i.e. the impact parameters are smaller than $b_\pi^c = b_B^c \equiv b^c \leq 0.6/\Lambda_{QCD}$. The authors concluded that pQCD approach was relatively self-consistent, since a large contribution comes from the perturbative region. Our discussion above makes it clear that we do not accept the conclusion of ref. [10].

The authors of ref. [4] considered a weaker criterion, that most of the contribution
Figure 9: Contribution to $F_{0,+}(0)$ as a function of the coupling constant $\alpha_s(t)$ in the hard-scattering kernel. For purposes of illustration, in this figure we choose $\phi_B^{(1)}$ as the model distribution amplitude for the $B$-meson (with $\omega = 0.1$ GeV), and the asymptotic distribution amplitude for the pion.

should come from the region of phase space in which the maximal available virtuality (longitudinal or transverse) of the gluon, $t \equiv \max(\sqrt{x_{\eta m}}, 1/b_\perp, 1/c_\perp)$ is in the perturbative regime. They take $t$ as the scale of the coupling constant in the hard-scattering kernel. We have argued above that this condition is insufficient for the consistency of the calculations, $1/b_\perp$ and $1/c_\perp$ must also separately be in the perturbative regime. Of course, since $t \geq 1/b_\perp$ and $t \geq 1/c_\perp$, for any set of distribution amplitudes a larger fraction of the form-factor comes from the region is which $t > t^c$ where $t^c$ is a cut-off defining the perturbative region of phase space, than from the regions in which $1/b_\perp > t^c$ or $1/c_\perp > t^c$.

We have however considered this weaker criterion, and studied the contribution to the form factors coming from the region of phase space where $a_1 \leq \alpha_s(t)/\pi \leq a_2$. Fig. 9 is the resulting histogram for $\phi_B = \phi_B^{(1)}$, $\omega=0.1$ GeV and $\phi_\pi = \phi_\pi^{(a)}$. The last bar on the right-hand side is the contribution of $\alpha_s(t)/\pi$ larger than 0.9. We find that the fraction coming from the region in which $t$ is not perturbative, is large. Numerical studies of other models for the distribution amplitudes indicate that the contribution from the region of non-perturbative $t$ can be large, but this depends very much on the choice of distribution amplitudes. It may be relatively small for some choices of the distribution amplitudes and large for others.

Sudakov suppression of configurations with large transverse separations is not efficient enough for $B \to \pi$ form factors, whereas it worked reasonably well in the case of the pion electromagnetic form factor. This should not surprise us. Recall that there is a fundamental difference between the two processes in the BLER approach: the first suffers from long-distance divergences, whereas the soft contributions are finite in the latter case. One would therefore expect that a much stronger Sudakov suppression is required, and not
achieved, for a self-consistent pQCD approach of $B$-decays.

As explained above, we have serious concerns about the derivation and the expression of the Sudakov factor for the $B$-meson. Note that the weakness of the Sudakov suppression in eq. (52) can be expected on general grounds for the $B$-system. We have seen that a Sudakov effect was argued for $B$-mesons with a large longitudinal light-quark momentum $l_+$, which is a highly unlikely configuration. On the other hand, there should be no effect for the standard situation: $l_+ \sim l_- \sim l_\perp \sim \Lambda_{QCD}$. “Large” transverse separations ($c_\perp \geq \Lambda_{QCD}$) are not suppressed by this mechanism. Therefore, even if an expression of the form (52) could be derived for the Sudakov factor in the $B$-meson, it would lead to large contributions coming from nonperturbative regions.

We conclude this section by restating that the pQCD predictions for $B \rightarrow \pi$ form factors receive a substantial (but uncalculable) contribution from configurations with large transverse separations. The Sudakov suppression appears to be too weak to cure this problem. Although the numerical details of our study depend on the choice of models for the distribution amplitudes, the qualitative conclusion is general. We are therefore forced to conclude that the pQCD predictions for $B \rightarrow \pi$ semileptonic form factors (and other physical processes which are affected by end-point singularities) are not valid.

7 Conclusion

In this paper we have studied Sudakov effects in predictions for $B \rightarrow \pi$ semileptonic form factors. In the standard approach, these form-factors are not calculable due to the presence of long-distance effects from end-point regions of (longitudinal) phase-space. We have investigated the claim that Sudakov effects suppress these long-distance contributions sufficiently for the form-factors to be calculable reliably and precisely in perturbation theory. Our conclusion is that this is not the case. The same arguments can be used to conclude that Sudakov effects cannot be invoked to make reliable predictions for other processes which have end-point singularities, such as power corrections to the amplitudes for exclusive two-body $B$-decays [3, 8, 9, 31].

Among the reasons for our conclusion are:

1. As explained in detail in section 5, it is not possible for us to accept that the formalism currently used to derive the Sudakov factor for the $B$-meson is theoretically sound.

2. Even if one accepts the Sudakov factors for both $B$ and $\pi$ mesons, the uncertainty in the mesons’ distribution amplitudes (particularly at small $\bar{x}$ and $l_+$) means that the form factors for $B \rightarrow \pi l_\nu$ decays cannot be evaluated with sufficient precision to be phenomenologically useful. This was investigated numerically in section 6.3.

3. For these $B$-decays the kinematic parameters are such that substantial contributions to the form-factors come from the nonperturbative region (see sec. 6.4), and are hence uncalculable. Sudakov effects are too weak to suppress contributions from the regions of phase space with large transverse separations. We therefore conclude that the pQCD approach is invalid for semileptonic $B$-decays.
Following the completion of this work, there appeared the interesting paper by H. Kawa-
mura et al. [32], in which the heavy quark theory and equations of motion are used to show
that, under the assumption in which the three-parton distribution amplitudes are neglected,
the two B-meson distribution amplitudes $\phi^B_+$ and $\phi^B_-$ can be determined (more generally
it is shown in this paper that the two- and three-parton distribution amplitudes can be
related). These results reinforce our conclusions, in particular it is shown in ref. [32] that
$\phi^B_-$ does not vanish at the end-point. Numerical studies with these distribution amplitudes
lead to the same conclusions as presented in section 6.

As mentioned in the introduction, our inability to evaluate the power corrections in two-
bond nonleptonic $B$-decay amplitudes ($B \to M_1 M_2$, for two mesons $M_1$ and $M_2$) severely
limits the precision with which we can deduce fundamental information from experimental
measurements of the rates and CP-asymmetries. Our conclusion, that Sudakov suppression
of long-distance effects is too weak (and too unreliable) to be useful in extending the
range of applicability of perturbative QCD in $B$-physics, is therefore a disappointing one.
Nevertheless, given the fact that many phenomenological studies of $B$-decays are being
performed which are based on the Sudakov suppression of end-point singularities, we felt
that it was important to articulate our concerns about the validity and reliability of this
approach.

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A A separable model for the $B$-meson

In this appendix we introduce a model for the two leading-twist $B$-meson distribution
amplitudes (in longitudinal and transverse momentum space). This model is introduced
for illustrative purposes and is used in section 6. It satisfies the constraints arising from
the equation of motion.

A.1 Constraints from the equations of motion.

We will generalise eq. (8) to include the effects of transverse momenta. Since the hard-
scattering kernel is independent of $L_-$, we can consider eq. (8) for $z_+ = 0$ (but $z^2 \neq 0$). The
most general decomposition is:

$$\langle 0 | \bar{q}_\beta(z) b_\alpha(0) | \bar{B}(p) \rangle = - \frac{i f_B}{4} \left[ \frac{\hat{p} + m}{2} \left\{ 2 \tilde{\Psi}_+^B(z^2, t) + \frac{\tilde{\Psi}_-^B(z^2, t)}{t} \right\} \gamma_5 \right]_{\alpha\beta}, \ (81)$$
with \( m = M_B = m_b \), \( p = mv \) and \( t = v \cdot z \) \((v = p/M_B)\). A path-ordered exponential is implicitly present in the gauge-independent matrix element. We can introduce the Fourier transforms:

\[
\Psi_{B \pm}(l_+, \vec{l}_\perp) = \int \frac{d^2 \vec{z}_\perp}{(2\pi)^2} \frac{dz_z}{2\pi} e^{i(l_z - \vec{l}_\perp \cdot \vec{z}_\perp)} \tilde{\Psi}_{B \pm}(z_-, -\vec{z}_\perp^2). \tag{82}
\]

We have defined \( l = (l_+/\sqrt{2}, 0, \vec{l}_\perp) \) and \( z = (0, z_- \sqrt{2}, \vec{z}_\perp) \), so that \( z^2 = -\vec{z}_\perp^2 \) and \( t = z_- \).

In ref. [16] constraints were derived on the leading-twist \( B \)-meson DAs, neglecting the effects of three-particle \((q\bar{q}g)\) and higher Fock states:

\[
\partial \tilde{\Psi}_B + \frac{\partial \tilde{\Psi}_B}{\partial t} + \frac{1}{4} \frac{\partial^2 \tilde{\Psi}_B}{\partial t^2} \bigg|_{z_- = 0} = 0, \tag{83}
\]

\[
\frac{\partial \tilde{\Psi}_B}{\partial z^2} + \frac{1}{4} \frac{\partial^2 \tilde{\Psi}_B}{\partial t^2} \bigg|_{z_- = 0} = 0, \tag{84}
\]

where \( \tilde{\Psi}_B \) depend on \( z^2 \) and \( t = v \cdot z \). We consider a model in which the dependence on the longitudinal and transverse momenta is factorized:

\[
\Psi_{B \pm}(l_+, \vec{l}_\perp) = \phi_{B \pm}^B(l_+) \times \tau_{B \pm}(l_\perp), \tag{85}
\]

where we chose the normalization conditions

\[
\int dl_+ \phi_{B \pm}^B(l_+) = 1 \quad \text{and} \quad \int d^2 \vec{l}_\perp \tau_{B \pm}(l_\perp) = 1. \tag{86}
\]

In this case, the two constraints become in the momentum space:

\[
\phi_{B \pm}^B(l_+) = -l_+ \frac{d\phi_{B \pm}^B(l_+)}{dl_+}, \tag{87}
\]

\[
l_+^2 \phi_{B \pm}^B(l_+) = \lambda^2 \phi_{B \pm}^B(l_+), \tag{88}
\]

where \( \lambda^2 = \int d^2 \vec{l}_\perp \tilde{r}_\perp \tau_{B \pm}(l_\perp) \). Combining the two equations, we obtain the differential equation:

\[
\frac{d\phi_{B \pm}^B(l_+)}{dl_+} = -\frac{l_+}{\lambda^2} \phi_{B \pm}^B(l_+), \tag{89}
\]

and the corresponding (normalized) solutions:

\[
\phi_{B \pm}^B(l_+) = \sqrt{\frac{2}{\pi \lambda^2}} \exp \left[ -\frac{l_+^2}{2\lambda^2} \right] \quad \text{and} \quad \phi_{B \pm}^B(l_+) = \sqrt{\frac{2}{\pi \lambda^2}} \frac{l_+}{\lambda^2} \exp \left[ -\frac{l_+^2}{2\lambda^2} \right]. \tag{90}
\]

The value of \( \lambda \) depends on the model used for the transverse momentum, and measures the dispersion of its values. For example, the step-function distribution:

\[
\tau_{B \pm}(l_\perp) = \begin{cases} 
1/(\pi \lambda_\perp^2) & \text{for } 0 \leq l_\perp \leq \lambda_\perp \\
0 & \text{otherwise}
\end{cases} \tag{91}
\]

leads to \( \lambda = \lambda_\perp/\sqrt{2} \), whereas the Gaussian distribution:

\[
\tau_{B \pm}(l_\perp) = \frac{1}{2\pi \sigma_\perp^2} \exp \left[ -\frac{l_\perp^2}{2\sigma_\perp^2} \right] \tag{92}
\]

has \( \lambda = \sqrt{2}\sigma_\perp \).
A.2 Can we set $\Psi^B_+ = \Psi^B_-$?

As an exploratory exercise we estimate the error in the calculated value of the form-factors (see eqs. (54), (55) and (56)) caused by setting $\Psi^B_+$ equal to $\Psi^B_-$. For this exercise, we take for the DA of the $B$-meson the separable model of App. A.1, with a flat transverse distribution, eqs. (85) and (91). We set $\lambda_\perp = \Lambda$, with $\Lambda = O(\Lambda_{\text{QCD}})$. For this model the first term in eq. (55) is:

$$
S_-(\eta) = \int_0^1 dx_\pi(x) \int_0^\infty dl_+ \phi^B_+(l_+) \int d^2 \vec{k}_\perp \tau_+(k_\perp) \int d^2 \vec{l}_\perp \tau_+(l_\perp) \int\frac{1}{x l_+ \eta m + (\vec{k}_\perp + \vec{l}_\perp)^2 \eta m^2 + k_\perp^2} \int\frac{1}{x l_+ \eta m + (\vec{k}_\perp + \vec{l}_\perp)^2 \eta m^2 + l_\perp^2}
$$

$$
= \frac{\eta}{\Lambda^4} \int_0^1 dx_\pi(x) \int_0^\infty dl_+ \phi^B_-(l_+) \int_0^\Lambda k_\perp dk_\perp \int_0^\Lambda l_\perp dl_\perp \int\frac{1}{x l_+ \eta m + (\vec{k}_\perp + \vec{l}_\perp)^2 \eta m^2 + k_\perp^2} \int\frac{1}{x l_+ \eta m + (\vec{k}_\perp + \vec{l}_\perp)^2 \eta m^2 + l_\perp^2}
$$

$$
= \frac{\eta}{\Lambda^4} \int_0^1 dx_\pi(x) \int_0^\infty dl_+ \phi^B_-(l_+) \int_0^\Lambda k_\perp dk_\perp \int_0^\Lambda l_\perp dl_\perp \int\frac{1}{x l_+ \eta m + \kappa \sqrt{(\vec{x} l_+ \eta m + \kappa + \lambda)^2 - 4 \kappa \lambda}}
$$

The following integral is useful for the angular integration over the transverse momenta:

$$
\int_0^{2\pi} \frac{1}{a + b \cos \theta} d\theta = \frac{2\pi}{\sqrt{a^2 - b^2}} \quad \text{for } a > b.
$$

In eq. (95), the integrals over $\lambda, \kappa$ and $l_+$ can be performed numerically and we find:

$$
I_-(\vec{x}, \eta) = \frac{\eta}{\Lambda^4} \int_0^\infty dl_+ \phi^B_-(l_+) \int_0^\Lambda k_\perp dk_\perp \int_0^\Lambda l_\perp dl_\perp \int\frac{1}{x l_+ \eta m + \kappa \sqrt{(\vec{x} l_+ \eta m + \kappa + \lambda)^2 - 4 \kappa \lambda}}
$$

The same integral can be computed with $\Psi^B_+$ instead of $\Psi^B_-$, and the result is called $I_+(\vec{x}, \eta)$. For illustration, the two functions are plotted in Fig. 10 for the case $\eta = 1$. Throughout the region in $x$, $I_-$ remains about twice as large as $I_+$.

In order to estimate the error on the form factors we need to take a model for the DA of the pion. For example, if we take the asymptotic form, $\phi_\pi(x) = 6x(1 - x)$, we obtain $S_-(\eta = 1) = 1.32 \text{ GeV}^{-4}$ and $S_+(\eta = 1) = 0.59 \text{ GeV}^{-4}$. The identification of $\Psi_+$ and $\Psi_-$ leads therefore to an error of at least 30% in this contribution to the form factors.

References
Figure 10: First term $I_-(\bar{x}, \eta = 1)$ in eq. (55), with the separable model for $\Psi_B^-$, and corresponding integral $I_+(\bar{x}, \eta = 1)$ when $\Psi_B^-$ is replaced by $\Psi_B^+$. 

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