Riesz products and spectral decompositions for rank 1 measure preserving transformations

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We consider rank one measure preserving transformations $g$ and the corresponding unitary operators $U(g)$. It is known that a generic (in the sense of Baire category) measure preserving transformation has rank one, spectral type of $U(g)$ is purely singular and is given by a Riesz product. We write explicitly spectral decompositions of $U(g)$.

1 The statement

1.1. Purpose of the paper. Consider the group $\text{Ams}(M, \mu)$ of all measure preserving transformations of a space $(M, \mu)$ with finite or $\sigma$-finite non-atomic measure. It is a topological group equipped with a topology of Polish space, in particular Baire category is well-defined on this group. Ergodic transformations form a comeagre set.

Consider $g \in \text{Ams}(M, \mu)$ and the corresponding unitary operator $U(g)$ in $L^2(M)$ (in the case of the space with a finite measure here we consider action in the space of functions with zero mean). It is known that spectral measures of generic (in the sense of Baire category) measure preserving transformations are quite strange. For a dense $G_\delta$-set in $\text{Ams}(M, \mu)$ spectrum of $U(g)$ is multiplicity free, spectral measures $\nu$ are purely singular \cite{14}, and all convolution powers of $\nu$, i.e., $\nu, \nu*\nu, \nu*\nu*\nu, \ldots$ are pairwise singular (see \cite{9}, \cite{12}). Notice that a similar statement holds for unitary operators (see \cite{4}, \cite{12}), singular spectra are usual for Schrödinger operators, see \cite{5}, \cite{10}. On some other strange properties of generic measure preserving transformations, see \cite{15}. On the other hand, singular spectra are related with strange behavior of powers $U(g)^n$, see \cite{13}, \cite{8}.

A generic measure preserving transformation has rank one. For a definition of rank of a transformation, see, e.g., \cite{11}. Any rank one transformation can be represented in a transparent form using cutting and stacking model, see e.g., \cite{7}, \cite{12} (for an equivalence of the abstract and constructive definitions, see \cite{1}). However, a verification of conjugacy/non-conjugacy of transformations having such 'normal forms' is a non-obvious question, see an interesting discussion in \cite{6}.

Spectral measures for rank one transformations admit wonderful explicit expressions (see, e.g., \cite{2}, \cite{3}, \cite{10}, \cite{12}) in terms of (generalized) Riesz products

\[ \prod_{j=1}^{\infty} \frac{1}{m_j} P_j(z) \overline{P_j(z)}, \]

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\footnote{Of course, they differs from the original F. Riesz products (1916) see \cite{17}, however the convergence has the same reason.}
where \( P_j(z) \) are 'sparse' polynomials of the form

\[
P_j(z) = 1 + \sum_{k=1}^{m_j-1} z^{r_{jk}}.
\]

Under some conditions of 'sparseness', such products converge in the sense of weak convergence of measures.

The purpose of this paper is to obtain explicit spectral decompositions of operators \( U(g) \) for rank one transformations \( g \), see Theorem 1.1.

1.2. Ring \( \mathcal{O} \). For natural \( k, l \) we have a canonical map of residue rings

\[
\mathbb{Z}/l\mathbb{Z} \leftarrow \mathbb{Z}/kl\mathbb{Z}.
\]

Fix a sequence \( m_1, m_2, \ldots \) of integers, \( m_j \geq 2 \). Consider the inverse limit

\[
\mathbb{Z}/m_1\mathbb{Z} \leftarrow \mathbb{Z}/m_1m_2\mathbb{Z} \leftarrow \mathbb{Z}/m_1m_2m_3\mathbb{Z} \leftarrow \ldots
\]

We write elements of \( \mathcal{O} \) as sequences infinite to the left

\[
s := \ldots s_3s_2s_1; \quad \text{where } s_j \text{ ranges in the set } \{0, 1, \ldots, m_j - 1\}.
\]

For \( s \in \mathcal{O} \) we assign elements \( s_j \ldots s_1 \) of \( \mathbb{Z}/m_1 \ldots m_j \) by

\[
s_j \ldots s_1 := s_1 + s_2m_1 + s_3m_1m_2 + \cdots + s_jm_1 \ldots m_{j-1}.
\]

We use a notation

\[
\mathbf{g}_j := m_j - 1, \quad \mathbf{b}_j := m_j - 2.
\]

Also, we write

\[
0_j, \quad 1_j
\]

if we want to show a position of 0 or 1 in the 'digital' notation (1.1).

Denote by \( \zeta(s) \) number of the first digit different from \( \mathbf{g} \) at the end of \( s \), by \( \iota(s) \) this digit. Namely, for

\[
s := \ldots k \mathbf{g}_j \mathbf{g}_{j-1} \ldots \mathbf{g}_1; \quad k \neq \mathbf{g}_{j+1}, \quad (1.2)
\]

we set

\[
\zeta(s) := j + 1, \quad \iota(s) := k \quad (1.3)
\]

For the number \( \ldots \mathbf{g}_2 \mathbf{g}_1; \) these functions are not defined.

By the definition \( \mathcal{O} \) is a (compact) ring, addition and product can be easily described in terms of columnar addition and multiplication. Below we need only the operation of addition of 1, it can be defined in terms of columnar addition as follows:

\[
+ \ldots s_3s_2s_1; 1;
\]
If \( s_1 \neq \frac{9}{BL} \), then we simply replace \( s_1 \mapsto s_1 + 1 \). If \( s \) has the form \( (1.2) \) then
\[ s + 1 = \ldots (k + 1)0, \ldots 0, \ldots 0. \]

The set \( \mathbb{O} \) has a natural structure of a compact metric space. Let \( s, t \in \mathbb{O} \) have the form
\[ s := \ldots k \sigma_j \ldots \sigma_1; \quad t := \ldots l \tau_j \ldots \tau_1; \]
and \( l \neq k \). Then
\[ d(s, t) = \frac{1}{m_1m_2 \ldots m_j}. \]

We get an ultrametric space. In particular, for any pair \( B_1, B_2 \) of closed balls we have \( B_1 \cap B_2 = \emptyset \), or \( B_1 \supset B_2 \), or \( B_2 \supset B_1 \). We say that a ball of level \( k \) is a ball of radius \( (m_1m_2 \ldots m_k)^{-1} \). Such balls \( B[\sigma] \) are enumerated by finite 'numbers'
\[ \sigma := \sigma_k \sigma_{k-1} \ldots \sigma_1; \]
the corresponding ball consists of all sequences
\[ \ldots s_{k+2}s_{k+1}\sigma_k \sigma_{k-1} \ldots \sigma_1; \]
where \( s_{k+1}, s_{k+2}, \ldots \) are arbitrary, \( s_i \in \{0, 1, \ldots, \frac{9}{BL} \} \).

We define a canonical probabilistic measure \( \mu \) on \( \mathbb{O} \) assuming that measure of any ball of level \( k \) is \( (m_1 \ldots m_k)^{-1} \).

1.3. Rank one transformations. Here we present a convenient for our purposes description of rank one transformations.\(^3\)

Parameters of a rank one automorphisms are following:
- We fix integers \( m_j \geq 2 \) as above.
- For each \( j = 1, 2, \ldots \), we fix positive integers \( a^0_j, a^1_j, \ldots, a^{m_j-2}_j \).

For such data we define a locally compact totally disconnected topological space \( \mathbb{V} = \mathbb{V}(m, a) \) and a transformation \( Q = Q(m, a) \) of \( \mathbb{V}(m, a) \) as follows.

The space \( \mathbb{V} \). We consider 'numbers' having one digit after the semicolon,
\[ s := \ldots s_3 s_2 s_1; s_0, \]
where \( \ldots s_3 s_2 s_1 \in \mathbb{O} \) are the same as above.

\[ \ldots s_3 s_2 s_1; \neq \ldots \frac{9}{BL} \frac{9}{BL} \frac{9}{BL}; \]

\(^3\)To get the standard model with cutting and stacking intervals as in [11] from considerations below, we must identify 'balls of level \( k \)' with intervals of the same level, the order of balls corresponds to the order of intervals. Our model gives kind of 'coordinates' on a measure space, which allow us to write formulas.

\(^4\)This space is metrizable (as a union of a countable number of disjoint clopen compact sets), but we do not define a metric.
Let us repeat this less formally. Take \( \ldots s_3 s_2 s_1 \in \mathcal{O} \). We look to the end of this 'number',
\[
\ldots \varphi_{\zeta-1} \ldots \varphi_{1};
\]
where
\[
\zeta = \zeta(\ldots s_3 s_2 s_1;), \quad t := t(\ldots s_3 s_2 s_1;)
\]
Then the digit after the semicolon ranges in the set \( \{1.4\} \).

Denote by \( \varphi := \varphi(m, a) \) the set of all such sequences. We denote \( a^*_\zeta - 1 \) after the semicolon by \( \varphi_{\zeta, t}^* \).

**Balls.** Next, we define \( \textit{balls } B_k[\sigma] \textit{ of level } k \). We fix a finite sequence
\[
\sigma := \sigma_k \sigma_{k-1} \ldots \sigma_1; \sigma_0,
\]
where \( \sigma_j \in \{0, 1, \ldots, m_j - 1\} \) are as above, we only require
\[
\sigma_k \sigma_{k-1} \ldots \sigma_1; \neq \varphi_{k-1} \varphi_k \ldots \varphi_{1};
\]
As above \( \sigma_0 \in \{0, \ldots, a^*_\zeta - 1\} \). Now we are ready to define the ball \( B_k[\sigma] \). It consists of all 'numbers' \( \ldots s_{k+2} s_{k+1} \sigma_k \sigma_{k-1} \ldots \sigma_1; \sigma_0 \)

**Remark.** A unique ball of level zero is \( B_0[0] = \mathcal{O} \).

Denote by \( h_k \) the number of balls of level \( k \), these numbers satisfy the recurrence relation
\[
h_k = m_k h_{k-1} + \sum_{i=0}^{m_k-2} a_k^i, \quad h_0 = 1.
\]

**Remark.** Due to the exclusion \( \{1.5\} \) balls of level \( k \) do not cover the whole space \( \mathcal{V} \).

**The measure on \( \mathcal{V} \).** We define a measure \( \mu \) on \( \mathcal{V} \) assign measure \( (m_1 \ldots m_k)^{-1} \) to each ball of level \( k \). The total measure of \( \mathcal{V} \) is
\[
\sum_{k=1}^{\infty} \frac{1}{m_1 \ldots m_k} \sum_{i=0}^{m_k-2} a_k^i.
\]
We admit both finite and infinite measures.

**The rank one transformation \( Q \).** We have a well-defined shift
\[
Q : s \mapsto s + 0;1.
\]
Namely, if the last digit $\sigma_0$ of $s \in \mathcal{V}$ is not $\frac{j}{j}$, we simply replace $\sigma_0 \mapsto \sigma_0 + 1$.

If $s$ has the form

$$s = \ldots k \frac{j}{j} \ldots \frac{j}{j} \frac{j}{j},$$

where $k \neq \frac{j}{j}$,

then

$$Qs = \ldots (k + 1) \frac{j}{j} \ldots \frac{j}{j} \frac{1}{j}.$$  

Thus we get a measure preserving map.

### 1.4. The statement.

Our purpose is to write an explicit spectral decomposition of the operator

$$Uf(s) := f(Q^{-1}s)$$

in $L^2(\mathcal{V})$.

We need some notation. For each ball $B_k[\sigma]$ of level $k$ we define its indicator function

$$I_k[\sigma](s) := \begin{cases} 1, & \text{if } s \in B[\sigma]; \\ 0, & \text{if } s \notin B[\sigma]. \end{cases}$$

Next, for each ball $B_k[\sigma] = B_k[\sigma_k \ldots \sigma_1; \sigma_0]$ of level $k$ we define integers

$$\sigma_k \ldots \sigma_\beta := \sigma_\beta + \sigma_{\beta+1} m_{\beta+1} + \cdots + \sigma_0 m_{\beta+1} \ldots m_k, \quad \beta \geq 1,$$

and the function $\Upsilon_k$ on the set of balls of level $k$ by

$$\Upsilon_k(\sigma) := \sum_{\alpha=1}^{k} (\sigma_k \ldots \sigma_{\alpha+1} \sum_{i} a_{\alpha}^i + \sum_{i<\sigma_\alpha} a_{\alpha}^i) + \sigma_0. \quad (1.7)$$

Denote by $S^1$ the unit circle $|z| = 1$ on the complex plane. Denote by $d\hat{z}$ the standard probabilistic Lebesgue measure on $S^1$. For each $k$ we define a polynomial function $\Theta_k(z)$ on $S^1$ by

$$\Theta_k(z) := \sum_{p=0}^{m_k-1} z^{p h_{k-1} + \sum_{i<p} a_{k-1}^i} \quad (1.8)$$

Consider the sequence of measures

$$d\kappa_n := \prod_{k=1}^{n} \frac{1}{m_k} \Theta_k(z) \overline{\Theta_k(z)} \frac{d\hat{z}}{d\hat{z}} \quad (1.9)$$

and the weak limit (Riesz product)

$$d\kappa(z) = \prod_{k=1}^{\infty} \frac{1}{m_k} \Theta_k(z) \overline{\Theta_k(z)} \frac{d\hat{z}}{d\hat{z}} := \lim_{n \to \infty} d\kappa_n. \quad (1.10)$$
Denote by $\mathcal{F} = \mathcal{F}(V)$ the space of locally constant compactly supported functions on $V$. Consider the map $\mathcal{R}$ from $\mathcal{F}$ to $C^\infty(S^1)$ that sends each indicator function $I_k[\sigma]$ to a function $\Phi_k[\sigma]$ on $S^1$ given by

$$\Phi_k[\sigma] := \frac{\Psi_k(\sigma)}{\prod_{j=1}^k \Theta_j(z)}.$$  

(1.11)

In particular, $\Psi_0[0] = 1$.

**Theorem 1.1** If functions $\Theta_k(z)$ have no zeros on the circle, then the map $\mathcal{R}$ extends to a unitary operator $L^2(V) \to L^2(S^1, d\kappa)$ and

$$\mathcal{R}(Q^{-1}s) = z \mathcal{R}(s).$$

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## 2 Proof

**2.1. The function $\Upsilon_k$.** The map $Q$ sends any ball of level $k$ to a ball of level $k$ except the ball

$$B_k[9_k \ldots 9_2 9_1; 9_0].$$

(2.1)

The image of the last ball under the map $Q$ is a disjoint union of balls of level $(k+1)$,

$$Q B_k[9_k \ldots 9_2 9_1; 9_0] = Q \left( B_{k+1}[0_k 9_k \ldots 9_2 9_1; 9_0] \cup B_{k+1}[1_k 9_k \ldots 9_2 9_1; 9_0] \cup \ldots \cup B_{k+1}[8_k 9_k \ldots 9_2 9_1; 9_0] \cup \cdots \cup B_{k+1}[8_k 9_k \ldots 9_2 9_1; 0] \right).$$

**Lemma 2.1**

$$B_k[\sigma_k \ldots \sigma_1; \sigma_0] = Q^{\Upsilon(\sigma_k \ldots \sigma_1; \sigma_0)} B_k[0_k \ldots 0_k; 0] .$$

**Proof.** We have a natural linear order on the set of balls of level $k$:

$$B_k[\tau_k \ldots \tau_1; \tau_0] < B_k[\sigma_k \ldots \sigma_1; \sigma_0]$$

(2.2)

if for some $j$ we have $\sigma_i = \tau_i$ for all $i > j$ and $\sigma_j > \tau_j$. If we have $Q \Upsilon$, then

$$B_k[\sigma_k \ldots \sigma_1; \sigma_0] = Q^{\Upsilon} B_k[\tau_k \ldots \tau_1; \tau_0]$$
for some positive $L$. In particular, 

$$B_k[\sigma_k \ldots \sigma_1; \sigma_0] = Q^N B_k[0_k \ldots 0_1; 0]$$

for some positive $N$. Clearly, $N$ is the cardinality of the set of all balls $\prec B_k[\sigma_k \ldots \sigma_1; \sigma_0]$. Each ball of this type has a form 

$$B[\tau_k \ldots \tau_{\alpha+1} \{g_{\alpha-1} \ldots g_1, \tau_0\}], \quad \text{where } l \neq g_{m_{\alpha}} \tag{2.3}$$

for some $\alpha$.

Thus, we fix $\alpha$ and evaluate number of balls (2.3), which are $\prec B_k[\tau_k \ldots \tau_{\alpha}; \tau_0]$. This can happened in the following cases:

1) If 

$$\tau_k \ldots \tau_{\alpha+1} \prec \sigma_k \ldots \sigma_{\alpha+1}.$$ 

For each $\tau_k \ldots \tau_{\alpha+1}$ we have $\sum_{i} a_{m_{\alpha}}$ balls of this type.

2) If 

$$\tau_k \ldots \tau_{\alpha+1} = \sigma_k \ldots \sigma_{\alpha+1}, \quad l \prec \sigma_{\alpha}.$$ 

Number of such balls is $\sum_{i \prec \sigma_{\alpha}} a_i$.

3) Exceptional case $\alpha = 0$,

$$\tau_k \ldots \tau_{\alpha} = \sigma_k \ldots \sigma_{\alpha}, \quad \tau_0 = \sigma_{\alpha-1} \ldots \sigma_1 = g_{\alpha-2} \ldots g_1, \quad \tau_0 \prec \sigma_0$$

Then we get $\sigma_0$ balls.

This gives us formula (1.7) for the exponent $N$. \hfill \Box

**Corollary 2.2** Let $B_k[\sigma]$ be a ball of level $k$ different from (1.2). Then 

$$U I_k[\sigma] = I_k[Q \sigma].$$

We also get the following expression for number $h_k$ of balls of level $k$:

$$h_k = \Upsilon_k(g_{k} \ldots g_2 \{g_1 \ldots g_{m_{k+1}-2}\}).$$

**Lemma 2.3** The numbers $h_k$ satisfy the recurrence relation (1.6).

**Proof.** Each ball of level $k$ is a union of $m_{k+1}$ subballs of level $(k+1)$. Additionally, we have balls of the type 

$$B_{k+1}[j \{g_k \ldots g_1, \tau\}].$$

Number of such balls is $\sum_{j=0}^{m_{k+1}-2} a_{k+1}^j$. \hfill \Box

**2.2. Self-consistency of the definition of the map $\mathcal{R}$.** Let us represent a ball $B_k[\sigma]$ of level $k$ as a union of $m_{k+1}$ disjoint balls of level $(k+1)$,

$$B_k[\sigma_k \ldots \sigma_1; \sigma_0] = \prod_{i=0}^{m_{k+1}-1} B_{k+1}[i \sigma_k \ldots \sigma_1; \sigma_0].$$
We must show that
\[ R I_k[\sigma_k \ldots \sigma_1; \sigma_0] = \sum_{i=0}^{m_{k+1}-1} R I_{k+1}[i \sigma_k \ldots \sigma_1; \sigma_0]. \]

By Corollary 2.2, it is sufficient to consider the case
\[ \sigma = 0_k \ldots 0_1; 0, \]
i.e., we must verify the following identity:
\[ R I_k[0_k \ldots 0_1; 0] = \sum_{j=0}^{m_{k+1}-1} R I_{k+1}[j_{k+1} 0_k \ldots 0_1; 0]. \]

In the right-hand side we have
\[ m_{k+1} - 1 \sum_{j=0}^{m_{k+1}-1} \sum_{i<j} a_i k+1 I_{k+1}[j_{k+1} 0_k \ldots 0_1; 0] = \sum_{j=0}^{m_{k+1}-1} z^{h_k + \sum_{i<j} a_i k+1} \Theta_k+1(z) \prod_{p=1}^{j-1} \Theta_p(z) = R I_k[0_k \ldots 0_1; 0]. \]

2.3. Riesz products. Decompose
\[ P_k := \prod_{j=1}^k \frac{1}{m_j} \Theta_j(z) \overline{\Theta_j(z)} = \sum_{\alpha=-N}^N u^{(k)}_\alpha z^\alpha. \]

**Lemma 2.4**

a) Coefficients \( u^{(k)}_\alpha \) satisfy
\[ 0 \leq u^{(1)}_\alpha \leq u^{(2)}_\alpha \leq \ldots \leq 1. \]

b) \( u^{(k)}_0 = 1. \)

c) In the product
\[ \prod_{j=l+1}^k \frac{1}{m_j} \Theta_j(z) \overline{\Theta_j(z)} = \sum_{\alpha=-N}^N u^{(l;k)}_\alpha z^\alpha \quad (2.4) \]
we have
\[ u^{(l;k)}_\alpha = 0 \quad \text{for all } \alpha \text{ satisfying } 0 < |\alpha| < h_l. \]

**Proof.** We have
\[ T_j(z) := \frac{1}{m_j} \Theta_j(z) \overline{\Theta_j(z)} = \]
\[ = 1 + \frac{1}{m_j} \sum_{0 \leq p < q < m_j} \left( z^{(q-p)h_{j-1} + \sum_{i \leq r < q} a_i_{j-1} + \sum_{i \leq r < q} a_i_{j-1}} \right). \]

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Coefficients are nonnegative and this implies non-negativity and increasing of \( u^{(k)}_n \) (this is a part of the statement a).

The maximal degree of a monomial in (2.5) is

\[
\nu_j := (m_j - 1)h_{j-1} + \sum_{i=0}^{m_j-2} a_j^i,
\]

This value corresponds for \( q = m_j - 1, \ p = 0 \). We have

\[
\nu_j = h_j - h_{j-1}
\]

Therefore the maximal degree of a monomial in \( P_k \) is

\[
\nu_1 + \nu_2 + \cdots + \nu_k = (h_1 - 1) + (h_2 - h_1) + \cdots + (h_k - h_{k-1}) = h_k - 1,
\]

and minimal (negative) degree is \(-h_k + 1\).

On the other hand, minimal positive degree in \( T_{k+1} \) is \( h_k + a^0_{k+1} \), it is achieved for \( q = 1, \ p = 0 \). Therefore the term \( z^0 \) in \( P_k T_{k+1} \) can arise only as a product of \( z^0 \cdot z^0 \). This implies statement b.

Thus \( \int P_k(z)\,dz = 1 \), the expression \( P_k(z) \) is positive, therefore other Fourier coefficients are \( \leq 1 \). This completes a proof of the statement a.

Consider the product (2.4), i.e., \( \prod_{j=l+1}^k T_j \). We expand each \( T_j \) as (2.5). Each term of the expansion has a form

\[
z^{\xi_{j_1}} z^{\xi_{j_2}} \cdots z^{\xi_{j_p}},
\]

where \( z^{\xi_{j_s}} \) appears from the expansion of \( T_{j_s} \),

\[
l + 1 \leq j_1 < j_2 < \cdots < j_p \leq k,
\]

and \( \xi_{j_s} \neq 0 \) for all \( s \) (i.e., we exclude factors \( z^0 \) from (2.8), for the term \( \prod_{j=l+1}^k z^0 \) expression (2.8) is empty, empty product is 1). To be definite, assume that \( \xi_{j_p} > 0 \). Then \( \xi_{j_p} \geq h_{j_p-1} \). On the other hand for \( s < p \) we have \( \xi_{j_s} \geq -\nu_{j_s} \).

Therefore the total degree of

\[
\xi_{j_p} + \sum_{s<p} \xi_{j_s} \geq h_{j_p-1} - \sum_{s<p} \nu_{j_s} \geq h_{j_p-1} - \sum_{l+1<i<j_p} \nu_i = h_{j_p-1} - \sum_{l+1<i<j_p} (h_i - h_{i-1}) = h_l.
\]

This proves the last statement. \( \square \)

**Corollary 2.5** The sequence

\[
\prod_{k=1}^N \frac{1}{m_k} \Theta_k(z) \Theta_k(z) \,dz
\]

of probabilistic measures on the circle weakly converges.
This statement is well-known. To be complete, we give a proof. Expressions $P_k(z) \, dz$ are probabilistic measures, since the Fourier coefficient at $z^0$ is 1. Other Fourier coefficients are increasing and $\leq 1$. Therefore, we have a weak convergence of measures.

2.4. Proof of Theorem 1.1.

Lemma 2.6 The map $R$ is an isometry.

Proof. It is sufficient to show that for two indicator functions $I[\sigma], I[\tau]$ we have

$$\langle RI[\sigma], RI[\tau]\rangle_{L^2(S^1, d\omega)} = \langle I[\sigma], I[\tau]\rangle_{L^2(V)} = \mu(B[\sigma] \cap B[\tau]).$$

Dividing balls into smaller subballs we reduce the statement to the case of balls of the same level, say, $k$. In other words, we must prove the following identity

$$\langle RU^p I_k[0_k \cdots 0_k; 0], RU^q I_k[0_k \cdots 0_k; 0]\rangle_{L^2(S^1, d\omega)} = \begin{cases} \prod_{j=1}^k m_j^{-1}, & \text{if } p = q; \\ 0, & \text{otherwise,} \end{cases}$$

where $p, q < h_k$. The left hand side is

$$\lim_{N \to \infty} \int_{S^1} \frac{z^p}{\prod_{j=1}^k \Theta_j(z)} \frac{\tau^q}{\prod_{j=1}^k \Theta_j(z)} \prod_{j=1}^N \frac{1}{m_j} \Theta_j(z) \Theta_j(z) \, d\omega =$$

$$= \prod_{j=1}^k \frac{1}{m_j} \lim_{N \to \infty} \int_{S^1} z^{p-q} \frac{1}{\prod_{j=k+1}^N \Theta_j(z)} \, d\omega \quad (2.9)$$

By Lemma 2.4b, the prelimit integral is 1 if $p = q$. If $p \neq q$, then by Lemma 2.4c the integral is 0. □

End of proof of Theorem 1.1 The map $R$ is an isometric embedding of $L^2(V)$ to $L^2(S^1, d\omega)$ and send $U$ to the multiplication by $z$. The function $I_0[0]$ is the indicator function of $\emptyset \subset V$. Its image under $R$ is $f(z) = 1$. Next, $R$ send $U^n I_0[0]$ to $z^n$. Therefore, the image of $R$ contains all polynomials in $z, z^{-1}$. By the Weierstrass theorem the image of $R$ is dense in the space of continuous functions on $S^1$ and therefore it is dense in $L^2(S^1, d\omega)$.

□

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