Quantum energy teleportation between spin particles in a Gibbs state

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Abstract
Energy in a multipartite quantum system appears from an operational perspective to be distributed to some extent non-locally because of correlations extant among the system’s components. This non-locality allows users to transfer, in effect, locally accessible energy between sites of different system components by local operations and classical communication (LOCC). Quantum energy teleportation is a three-step LOCC protocol, accomplished without an external energy carrier, for effectively transferring energy between two physically separated, but correlated, sites. We apply this LOCC teleportation protocol to a model Heisenberg spin particle pair initially in a quantum thermal Gibbs state, making temperature an explicit parameter. We find in this setting that energy teleportation is possible at any temperature, even at temperatures above the threshold where the particles’ entanglement vanishes. This shows for Gibbs spin states that entanglement is not fundamentally necessary for energy teleportation; correlation other than entanglement can suffice. Dissonance—quantum correlation in separable states—is in this regard shown to be a quantum resource for energy teleportation, more dissonance being consistently associated with greater energy yield. We compare energy teleportation from particle A to B in Gibbs states with direct local energy extraction by a general quantum operation on B and find a temperature threshold below which energy extraction by a local operation is impossible. This threshold delineates essentially two regimes: a high temperature regime where entanglement vanishes and the teleportation generated by other quantum correlations yields only vanishingly little energy relative to local extraction and a second low-temperature teleportation regime where energy is available at B only by teleportation.

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(Some figures may appear in colour only in the online journal)
1. Introduction

Standard quantum teleportation is the transfer of a system’s unknown quantum state to a ‘blank’ second system [1]. This is tantamount to a transfer of the system itself since a system is identified by its quantum state. Quantum teleportation is accomplished using only local operations and classical communication (LOCC), but it requires quantum correlation—entanglement—between the two systems. In contrast to state teleportation, quantum energy teleportation (QET) is effectively a transfer by LOCC of energy between two components of a multipartite system [2–4]. More specifically, QET is a transfer of local energy, where local (or locally available) energy is energy that can be extracted from a system component by a local operation. Like state teleportation, QET relies on correlation between the two components, and because of these correlations, to users sited at the two components, energy appears from an operational perspective at least partially non-localized. For example, zero-point energy, while existing everywhere in a multi-body system in the ground state, cannot be extracted by local operation at any single site, and any attempt to do so would only inject additional energy into the system. Thus zero-point energy is not locally available. On the other hand, a LOCC protocol executed by two users at the sites of different subsystems can draw zero-point energy from one subsystem by injecting additional energy into the second subsystem. This is the basis of QET.

A QET protocol was theoretically demonstrated first for spin chains and quantum fields [2, 3] and subsequently for an elementary ‘minimal’ physical model [5]. This minimal model involves just a single maximally entangled spin-$1/2$ particle pair and a nondemolition measurement of the interaction Hamiltonian. These various demonstrations show that, by injecting energy at site A, energy can be extracted at site B, with no external energy carrier. The speed of extraction is limited by that of classical communication between the sites, consistent with causality. And, because QET only increases the ratio of locally available energy to the total energy in the energy extraction region, with no change to the total amount of energy, QET strictly conserves energy [2–4]. A proposal has been made to use edge channel currents in a quantum Hall system to experimentally verify QET [6].

Energy teleportation has a range of implications for fundamental physics. For example, it suggests local energy density fluctuation as a way to address entanglement in condensed matter systems. QET may in this regard constitute a new tool for a quantum Maxwell’s demon, allowing the demon to observe and react to local quantum fluctuations of an interacting many-body system at zero temperature. Past works on quantum demons assume that interactions among observed subsystems are negligibly small, eliminating a direct role for ground-state entanglement [7–9]. A demon equipped to perform QET can by indirect measurement exploit ground-state entanglement to extract work, potentially opening the way to a new paradigm for quantum information thermodynamics. Also, QET bears on local cooling in quantum many-body systems. Local measurement of zero-point fluctuation on a subsystem generally injects some energy, resulting in an excited state. We then naturally ask whether all the injected energy can be retrieved using only local operations on the measured subsystem. With the perspective of QET, the answer is no; some residual energy unavoidably remains in the system from any local-cooling procedure [2]. This is because the local measurement breaks a part of the ground-state entanglement and the broken entanglement cannot be restored by local operations. In fact, the residual energy is lower bounded by the total amount of energy that can be teleported by use of the information from the local measurement [10]. QET considered in the setting of black hole physics provides a new method [11] analogous to Hawking radiation [12] for reducing the area of the event horizon. Consider a quantum field measurement outside a massive black hole that provides information about quantum fluctuations. Positive-energy
wave packets of the field are generated during the measurement (based on approximating the quantum field’s pre-measurement state by a Minkowski vacuum state and making a passivity argument). Suppose that the black hole absorbs these wave packets. Then, significantly, part of the absorbed energy outside the horizon can be retrieved by QET. Using the measurement information, negative energy wave packets can be generated outside the horizon by extracting positive energy out of the zero-point fluctuation of the fields. The negative energy of the wave packets propagates across the event horizon and may pair-annihilate with positive energy of matter falling inside the black hole. This process is akin to spontaneous emission of Hawking radiation or, as it often called, black hole tunneling [12, 13]. The net effect of this process is to decrease the horizon area, which is proportional to the black hole entropy. This result may from an information theory viewpoint clarify the origin of black hole entropy. QET appears by these examples to be a fundamental physical process relevant to different branches of physics.

We study QET in this paper within the framework of a coupled pair of spin-$\frac{1}{2}$ particles, focusing on the quantum Gibbs states of the particle pair. To locally interact with a single particle in the pair for the purpose of QET, we adopt the same nondemolition measurement of the interaction Hamiltonian used in [5]. To physically motivate the Gibbs states, we assume that the particles are coupled to bosonic environments (thermal baths) with interactions that are very weak relative to the interaction between the two particles. This assumption (with others) allows Gibbs states to be interpreted as a physical thermal state with equilibrium temperature as a parameter. In general, the Gibbs state approximation for multi-body systems with small degrees of freedom in contact with an ‘environment’ must be treated with care. If the interaction between the system and the thermal bath is of an order comparable to that among the system’s parts, Gibbs states poorly approximate the boundary interaction effect. However, when the interaction between the system and the thermal bath is sufficiently weak, the Gibbs state approximation is valid for ensemble averages of the experiment. This is supported by, for example, a recent analysis of canonical thermal pure quantum states [14]. Our study of these states introduces their (equilibrium) temperature as an explicit parameter, allowing us to investigate for this model (1) the extent to which temperature restricts QET, (2) the role of different forms of quantum correlation in QET and (3) the performance of QET relative to direct local energy extraction. In this investigation we show that energy teleportation is possible for any Gibbs state of the particle pair, establishing in principle that QET can be accomplished under suitable conditions at any temperature. We observe that in our Gibbs states the particles’ spins are quantum correlated (in the sense of quantum discord) at all temperatures, though significantly they are entangled only at temperatures below a certain threshold. We conclude from this that in Gibbs states the correlation essential for energy teleportation need not fundamentally be entanglement—correlation beyond entanglement can support energy teleportation. This adds to a growing number of applications in which quantum dissonance (discord without entanglement) is demonstrably a quantum resource [15–19]. Finally in this study, to better understand QET and underscore its unique capability, we compare QET from particle A to B with direct local energy extraction by a general quantum operation on B. Concerning local extraction of energy, we obtain two interesting results: (1) no energy can be extracted at B by a local unitary operation at any temperature but (2) local energy extraction at B is possible by a general (Kraus operator-sum) quantum operation, provided the temperature is above a threshold. This parametric threshold marks two temperature regimes: a high temperature regime where teleportation yields only vanishingly small amounts of energy relative to local extraction and a low-temperature teleportation regime where energy is available at B only by teleportation. These regimes indicate by their nature that some quantum correlation—entanglement or otherwise—is effectively required for Gibbs state QET.
The remainder of the paper is organized as follows. Section 2 introduces our quantum model of two spin-$\frac{1}{2}$ particles. In this section we focus on the Gibbs states associated with this model, identifying the type and degree of the quantum correlation within these states as a function of temperature. In section 3 we specify the QET protocol for teleporting energy from particle A to B and establish the central result that the protocol yields a positive amount of energy at the site of B, doing so at any temperature. In section 4 we study energy extraction at B by local operations and contrast this with QET. We conclude in section 5 with some last remarks, including a discussion of dissonance as a quantum resource for Gibbs state QET.

2. Two-particle system

We consider a model Heisenberg spin-$\frac{1}{2}$ particle pair, focusing on the Gibbs states associated with this model. These Gibbs states include as a special case the maximally entangled ground state in the QET study in [10], they have a ready physical motivation and they are a frequent vehicle for studies of entanglement and discord in spin systems [20–25]. In this section we quantify the type and degree of the quantum correlation within these states. In particular, we derive an expression for the Gibbs states’ quantum discord, such discord being called conventionally thermal discord. Quantum discord quantifies the presence of quantum correlation broadly defined, and we find positive thermal discord at all finite temperatures, across the whole class of Gibbs states, even in those Gibbs states without entanglement.

Model: Consider two spin-$\frac{1}{2}$ particles, A and B, with Hamiltonian

$$H = H_A \otimes I + I \otimes H_B + V$$

(1)

where

$$H_A = H_B = \frac{1}{m} I + \sigma_z, \quad V = 2\kappa \sigma_x \otimes \sigma_x + \frac{2\kappa^2}{m} I \otimes I$$

(2)

with $m = \sqrt{1 + \kappa^2}$ and Pauli operators $\sigma_x, \sigma_y, \sigma_z$. The particle pair model (1) and (2) is Hotta’s minimal model [10] with one independent parameter $\kappa \geq 0$ and dimensionless energy. In the components (2) of $H$, the constants—those terms involving identity operators $I$—do not change the relative magnitudes of the eigenenergies of $H$; they just serve to set the ground eigenenergy to $E_0 = 0$ and we include them here for consistency with [10]. The Hamiltonian (1) is equivalently that of a two-qubit Ising spin chain in a transverse magnetic field; viewed so, $\kappa$ is the strength of the spin coupling relative to that of the magnetic field. The two-qubit system with Hamiltonian (1) has eigenenergies

$$E_0 = 0, \quad E_1 = 2m - 2\kappa, \quad E_2 = 2m + 2\kappa, \quad E_3 = 4m$$

(3)

and corresponding eigenstates

$$|E_0\rangle = -\sqrt{\frac{m-1}{2m}} |00\rangle + \sqrt{\frac{m+1}{2m}} |11\rangle, \quad |E_1\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}},$$

$$|E_2\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |E_3\rangle = \sqrt{\frac{m+1}{2m}} |00\rangle + \sqrt{\frac{m-1}{2m}} |11\rangle,$$

in the uncoupled basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Note that $E_0 < E_1 \leq E_2 < E_3$ since $m > \kappa$.

Gibbs states: Suppose each particle in the model pair is weakly coupled (Born approximation) with its own bosonic heat bath at temperature $T$. Then the eigenstates’ canonical occupation (Gibbs) probabilities for the particle pair are, for $i = 0, 1, 2, 3$,

$$p_i(T) = \frac{1}{Z} \exp\left(-\frac{E_i}{kT}\right)$$

(4)
where $k$ is Boltzmann’s constant, and $Z$ is the partition function

$$
Z = \sum_{j=0}^{3} \exp \left( -\frac{E_j}{kT} \right)
= 2 \exp \left( -\frac{2m}{kT} \right) \left( \cosh \frac{2m}{kT} + \cosh \frac{2\kappa}{kT} \right).
$$

The quantum state $\rho(T)$ of the particle pair in thermal equilibrium at temperature $T$ is therefore

$$
\rho(T) = \sum_{i=0}^{3} p_i(T) |E_i\rangle\langle E_i| = \frac{1 \otimes I - c_1 \sigma_x \otimes \sigma_x + c_2 \sigma_y \otimes \sigma_y + c_3 \sigma_z \otimes \sigma_z - r[\sigma_z \otimes \sigma_z]}{4}.
$$

where

$$
c_1 = \frac{2}{mZ} \exp \left( -\frac{2m}{kT} \right) \left( m \sinh \frac{2\kappa}{kT} + \kappa \sinh \frac{2m}{kT} \right),
$$

$$
c_2 = \frac{2}{mZ} \exp \left( -\frac{2m}{kT} \right) \left( -m \sinh \frac{2\kappa}{kT} + \kappa \sinh \frac{2m}{kT} \right),
$$

$$
c_3 = \frac{4}{Z} \exp \left( -\frac{2m}{kT} \right) \sinh \frac{m + \kappa}{kT} \sinh \frac{m - \kappa}{kT},
$$

$$
r = \frac{2}{mZ} \exp \left( -\frac{2m}{kT} \right) \sinh \frac{2m}{kT}.
$$

In particular, we recover from (5) that $\rho(0)$ and $\rho(\infty)$ are, respectively, the ground state $|E_0\rangle\langle E_0|$ and the completely mixed state $\frac{1}{2} I \otimes I$.

**Thermal discord**: The total correlation, both quantum and classical, in a bipartite system in a quantum state $\omega$ is quantified [28, 29] by the quantum mutual information, which is given by

$$
I[\omega] = S(\omega_A) + S(\omega_B) - S(\omega)
$$

where $\omega_A = \text{tr}_B[\omega]$ and $\omega_B = \text{tr}_A[\omega]$ are the marginal states of parts A and B of the system and $S(\cdot)$ is von Neumann entropy [30]. For a qubit pair in the Gibbs state (5), the joint and marginal entropies (in bits) of the pair are

$$
S(\rho(T)) = -\sum_{i=0}^{3} p_i(T) \log_2 p_i(T)
$$

and

$$
S(\rho_A(T)) = S(\rho_B(T)) = h(r)
$$

where in (10) the $p_i(T)$ are the Gibbs probabilities (4), and $r$ in (11) is given by (8) with $h(\cdot)$ defined by

$$
h(x) = \frac{1 + x}{2} \log_2 \frac{2}{1 + x} + \frac{1 - x}{2} \log_2 \frac{2}{1 - x}.
$$

The quantum mutual information in a qubit pair in state $\rho(T)$ is, from (10) and (11) and after some calculation,

$$
I[\rho(T)] = 2h(r) - \log_2 Z - \frac{\langle H \rangle}{kT} \log_2 e
$$

where $\langle H \rangle = \text{tr}[H \rho(T)]$ is the average energy of the particle pair in the thermal state $\rho(T)$. 5
The classical part of the total correlation (9) in parts A and B of a bipartite quantum system is defined to be the reduced uncertainty about the state of, say, A by measurement of B [28, 29]. Suppose we make a von Neumann measurement \( \{M_k\} \) of B with one-dimensional projectors \( M_k \) such that \( \sum_k M_k = I \). This measurement casts the bipartite system, originally in state \( \omega \), into the state

\[
\omega_k = \frac{1}{q_k} (I \otimes M_k) \omega (I \otimes M_k)
\]

with probability \( q_k = \text{tr}[(I \otimes M_k) \omega (I \otimes M_k)] \). Depending on the measurement outcome, the reduction in uncertainty about the state of A is \( S(\omega_A) - S(\omega_k) \), with average reduction \( S(\omega_A) - \sum_k q_k S(\omega_k) \). The supremum of this average reduction through measuring B is defined to be the classical part

\[
C[\omega] = \sup_{\{M_k\}} \left( S(\omega_A) - \sum_k q_k S(\omega_k) \right)
\]

(14)

of the total correlation in \( \omega \). The optimization in (14) is more generally taken over quantum measurements described by positive operator-valued measures, but for two-qubit states the optimal measurement is known to be projective [31]. Definition (14), involving as it does measurement of subsystem B of the bipartite system, is not symmetrical in A and B and, in fact, the two possible versions of \( C[\omega] \) are generally not equal [28, 32]. This is not a present concern, though, because \( \rho(T) \) is qubit exchange symmetric.

The supremum in the definition (14) of classical correlation is readily found analytically for two-qubit Bell-diagonal states [33]. Beyond the Bell-diagonals, though, the supremum presents a much greater challenge, and there are some erred results in the literature [34]. The Gibbs states \( \rho(T) \) in (5) are not Bell-diagonal. Instead, they are a subset of the broader class of states studied in [23, 35, 36], where in each of these studies the classical correlation was found by numerical search. Here, expressions (6), (7), and (8) for \( c_1, c_3, \) and \( r \) constrain the form of \( \rho(T) \) in (5) sufficiently to allow us to determine the supremum in (14) and obtain an analytical expression for the classical correlation. We find

\[
C[\rho(T)] = h(r) - h(\sqrt{r^2 + c_1^2}).
\]

(15)

Details of the calculation of (15) are given in appendix A.

The difference between the total correlation in \( \rho(T) \) in (13) and its classical correlation in (15) is the quantum discord \( D[\rho(T)] \). Quantum discord quantifies the quantum correlation, entanglement and otherwise, in a bipartite state [28, 29], and it has different important operational interpretations [37–40] to support its use for this purpose. When the state is separable, any non-zero discord is due strictly to quantum correlation other than entanglement. Discord in Gibbs states is commonly called thermal discord. Also, following Modi et al [41], we call positive discord in the absence of entanglement dissonance and say that a separable state with positive discord is dissonant. From (9) and (15) the discord in \( \rho(T) \) is

\[
D[\rho(T)] = h(r) + h(\sqrt{r^2 + c_1^2}) \log_2 Z - \frac{H}{kT} \log_2 e.
\]

(16)

The ground thermal state \( \rho(0) = |E_0\rangle |E_0\rangle \) is pure so its quantum correlation is solely entanglement, and its discord (16) is the entropy of entanglement \( D[\rho(0)] = S(\rho_A(T)) = h(r) \). Figure 1 shows for chosen values of \( \kappa \) that \( D[\rho(T)] \) is positive and decreasing with temperature \( T \) for all \( T \).

**Entanglement.** Entanglement is a particular form of quantum correlation, readily detected in the Gibbs states \( \rho(T) \) by the PPT criterion [26]. The PPT criterion yields (see appendix B) that \( \rho(T) \) is separable if and only if

\[
m \cosh \frac{2\kappa}{kT} \geq \kappa \sinh \frac{2m}{kT}.
\]

(17)
Figure 1. Log-log plots of thermal discord in the state $\rho(T)$. The left panel shows cases of the particle pair model with $\kappa \geq 1$, and the right panel shows cases with $\kappa \leq 1$. The state $\rho(T)$ is separable for temperatures above $T_c$, to the right of $\bullet$ on each curve.

This condition is saturated by a critical temperature $T = T_c$, which increases with the coupling $\kappa$. Below $T_c$ (in the shaded region in figure 5) the particles are entangled to some degree, while at and above $T_c$ there is zero entanglement and $\rho(T)$ is separable. A temperature threshold for entanglement is usual for thermal spin-$\frac{1}{2}$ systems [27].

3. Energy teleportation

We now apply the QET protocol to our model particle pair in the thermal state $\rho(T)$ given by (5), to teleport energy from the site of particle A to that of B. The QET protocol is known [10] to accomplish this in the zero-temperature case $\rho(0) = |E_0\rangle\langle E_0|$. We show that the QET protocol succeeds with any Gibbs state $\rho(T)$, even those states without entanglement.

The QET protocol proceeds in three steps. Step I is a measurement of particle A’s observable $\sigma_x$. This local measurement on A, independent of B, is a nondemolition measurement in as much as the observable $\sigma_x$ commutes with the interaction component (2) of the system Hamiltonian, $[\sigma_x \otimes I, V] = 0$. This measurement has the key effect of moving the particle pair to a new state with changed average local and total energies. Step II of the protocol is to classically communicate the outcome $\alpha = \pm 1$ of the step I measurement to the site of particle B. In step III, at the site of B, the communicated outcome $\alpha$ is used to choose a unitary operation $U(\alpha)$ for local application. This local unitary operation changes, again, the average energy of the particle pair. A gain $E_A > 0$ in system energy in step I indicates that the measurement device at site A has deposited energy into the particle pair, while a system energy loss $E_B > 0$ in step III indicates that energy is extracted at site B from the particle pair. These energy changes combine to achieve the effect of energy transport from site A to site B. This effect is termed energy teleportation because (1) being limited only by the communication speed in step II, it can be accomplished faster than the energy diffusion velocity within the system, (2) A and B can be a physical distance apart, and (3) no external energy carrier is involved. To show that the protocol succeeds for any Gibbs state $\rho(T)$ of our qubit pair, we check that the particle pair’s energy gain $E_A$ in step I and energy loss $E_B$ in step III are both positive. This QET protocol is superficially similar to remote state preparation (RSP). In each of QET and RSP the aim is to prepare subsystem B by operation on subsystem A, and each relies on correlation between the two subsystems. The particular aim of RSP, though, is to impose a known state on the target subsystem, measuring success by the fidelity of the
achieved state [42]. And to accomplish this, the RSP protocol depends on the specific quantum state to be imposed. By contrast, rather than aim to establish a particular state, QET seeks to maximize the locally available energy in subsystem B, and to this end, the QET protocol is designed in accord with the interaction component of the Hamiltonian, and not directly with any initial or desired quantum state.

Prior to initiating the QET protocol, the average energy in the particle pair in the Gibbs state $\rho(T)$ is

$$\langle H \rangle = \text{tr}[H \rho(T)] = \sum_{i=0}^{3} p_i(T) E_i$$

where the $p_i(T)$ are the Gibbs probabilities (4) and the $E_i$ are the system energies (3). Simple calculation yields

$$\langle H \rangle = 2m - 2\kappa c_1 - 2r \quad (18)$$

in terms of (6) and (8).

Consider measuring particle A as specified by the QET protocol. The projectors associated with the observable $\sigma_x$ are

$$\Pi_{1}(\alpha) = \frac{1}{2} (I + \alpha \sigma_x)$$

for $\alpha = \pm 1$, and the post-measurement system state is, depending on $\alpha$,

$$\rho_{\Pi}(T, \alpha) = \frac{[\Pi_{1}(\alpha) \otimes I] \rho(T) [\Pi_{1}(\alpha) \otimes I]}{q(\alpha)} \quad (19)$$

where $q(\alpha) = \text{tr}[\{\Pi_{1}(\alpha) \otimes I\} \rho(T) \{\Pi_{1}(\alpha) \otimes I\}]$ is the probability of the outcome $\alpha$. The energy in the post-measurement state (19) is

$$\langle H_{\Pi}(\alpha) \rangle = \text{tr}[H \rho_{\Pi}(T, \alpha)] = \frac{1}{q(\alpha)} \sum_{i=0}^{3} p_i(T) \langle E_i | H_{\Pi}(\alpha) | E_i \rangle \quad (20)$$

where $H_{\Pi}(\alpha) = [\Pi_{1}(\alpha) \otimes I] H [\Pi_{1}(\alpha) \otimes I]$. Averaging the energies (20) over the two measurement outcomes $\alpha = \pm 1$, we have

$$\langle H_{\Pi} \rangle = \sum_{i=0}^{3} p_i(T) \langle E_i | (H_{\Pi}(1) + H_{\Pi}(-1)) | E_i \rangle,$$

which after some calculation is

$$\langle H_{\Pi} \rangle = 2m - 2\kappa c_1 - r. \quad (21)$$

According to (18) and (21), the average gain $E_A = \langle H_{\Pi} \rangle - \langle H \rangle$ in system energy that results from the measurement of particle A is $E_A = r$. The quantity $r$ can be seen from (8) to be a positive decreasing function of $T$ for all $\kappa$. Therefore, the QET protocol’s measurement of particle A injects energy into the system on average, injecting more energy for lower temperature.

Now we consider the extraction of energy at the site of particle B. Suppose that the outcome $\alpha$ of the measurement of particle A has been communicated to the site of B, and suppose the local unitary operation

$$U(\alpha) = I \cos \theta - i \alpha \sigma_y \sin \theta \quad (22)$$

specified by step III of the QET protocol is applied to B, where the angle $\theta$ in (22) is an adjustable real parameter. The state of the particle pair at the completion of step III is, depending on $\alpha$,

$$\rho_{III}(T, \alpha) = \frac{[\Pi(\alpha) \otimes U(\alpha)] \rho(T) [\Pi(\alpha) \otimes U(\alpha)]}{q(\alpha)} \quad (23)$$
The energy in the state (23) is
\[
\langle H_{\text{III}}(\alpha) \rangle = \frac{1}{q(\alpha)} \sum_{i=0}^{3} \langle E_{i}|H_{\text{III}}(\alpha)|E_{i} \rangle
\]  
(24)
where \( H_{\text{III}}(\alpha) = [\Pi(\alpha) \otimes U(\alpha)] H [\Pi(\alpha) \otimes U(\alpha)] \). Averaging the energies (24) over the two measurement outcomes \( \alpha = \pm 1 \), we have
\[
\langle H_{\text{III}} \rangle = \sum_{i=0}^{3} p_{i}(T) \langle E_{i}|(H_{\text{III}}(1) + H_{\text{III}}(-1))|E_{i} \rangle.
\]
We then calculate that
\[
\langle H_{\text{III}} \rangle = 2m + \frac{c_2 - c_1}{2} (2\kappa \cos 2\theta - \sin 2\theta) - r(\kappa \sin 2\theta + (m^2 + \kappa^2) \cos 2\theta).
\]  
(25)
Comparing (21) and (25), we find that the average loss of energy in the particle pair due to the local unitary operation \( U(\alpha) \) is
\[
E_{B}(\theta) = \langle H_{I} \rangle - \langle H_{\text{III}} \rangle = a(\kappa, T) \sin 2\theta - b(\kappa, T)(1 - \cos 2\theta)
\]  
(26)
where the coefficients \( a(\kappa, T), b(\kappa, T) \) are
\[
a(\kappa, T) = \kappa r + \frac{c_2 - c_1}{2} \frac{4\kappa}{ZkT} \exp \left( -\frac{2m}{kT} \right) \left( s \left( \frac{2m}{kT} \right) - s \left( \frac{2\kappa}{kT} \right) \right),
\]
\[
b(\kappa, T) = (\kappa^2 + m^2)r - \kappa(c_2 - c_1) \frac{4}{ZkT} \exp \left( -\frac{2m}{kT} \right) \left( 2\kappa^2 s \left( \frac{2\kappa}{kT} \right) + (\kappa^2 + m^2)s \left( \frac{2m}{kT} \right) \right)
\]
with \( s(x) = \sinh(x)/x \). The energy \( E_{B}(\theta) \) is the average energy extracted at site B by the QET protocol, as a function of the angle \( \theta \) used in (22) in step III. The optimal choice \( \theta = \theta_{o} \) to maximize \( E_{B}(\theta) \) is, from (26), given by
\[
\tan 2\theta_{o} = \frac{a(\kappa, T)}{b(\kappa, T)}.
\]
Substituting \( \theta_{o} \) into (26), we find that the maximum extracted energy \( E_{B} = E_{B}(\theta_{o}) \) at site B with the QET protocol is
\[
E_{B} = \sqrt{a(\kappa, T)^2 + b(\kappa, T)^2} - b(\kappa, T).
\]  
(27)
Plots of the maximum extracted energy \( E_{B} \) for different \( \kappa \) in figure 2 show that \( E_{B} \) is a decreasing function of temperature, with a temperature threshold for the decrease for \( \kappa < 1 \) (weak spin coupling). In the regime \( \kappa < 1 \) where this threshold exists, the temperature of the particle pair can be increased up to the threshold with almost no decrease in teleported energy. Figure 2 also indicates that maximum energy teleportation occurs with \( \kappa \approx 1 \); that is, when the strength of the particles’ coupling is comparable to that of the external magnetic field.

Significantly for what we wish to establish, we see in all cases in figure 2 that \( E_{B} \) is positive. In fact, inspection of (27) shows immediately that \( E_{B} \) is positive for all finite temperatures \( T \) and non-zero particle couplings \( \kappa \). This establishes one of our central results: in the setting of our two-particle model, the QET protocol yields a positive amount of energy at site B. It does so at any temperature and across the whole family of spin particle pair systems parameterized by \( \kappa > 0 \), with maximum teleported energy \( E_{B} \) given by (27).
Figure 2. Log-log plots of teleported energy $E_B(\theta_o)$ extracted at the site of particle B. In correspondence with figure 1, the left panel shows cases of the particle pair model with coupling $\kappa \geq 1$, and the right panel shows cases with $\kappa \leq 1$.

4. Energy extraction without teleportation

We have shown that the QET protocol extracts energy from particle B. One might ask whether energy could as well be extracted from B directly by a local one-qubit operation without the exercise of the QET protocol. To answer this question and better understand what transpires in energy teleportation, suppose we execute a frustrated version of the protocol in which no measurement of particle A is made and, therefore, nothing is communicated to the site of B. In other words, suppose we skip steps I and II of the protocol and just perform a conditionally unitary operation locally on particle B, consistent with step III of the protocol. A conditionally unitary operation made locally on B takes the form $I \otimes W$ where $I$ is the identity operation (on A) and the operation $W$ on a qubit in state $\tau$ takes the form

$$W(\tau) = \sum_k p_k W_k \tau W_k^\dagger$$

where each operator $W_k$ is unitary. We allow any number of unitary operators $W_k$ in (28), with any probabilities $p_k$ that are independent of the particles’ history such that $\sum_k p_k = 1$, and we seek the operation $W$ that extracts the most energy possible when applied to particle B of the particle pair in state $\rho(T)$. A general one-qubit unitary operator is [30]

$$W = \begin{pmatrix} e^{i\frac{u}{2}} \cos \frac{w}{2} & -e^{-i\frac{v}{2}} \sin \frac{w}{2} \\ e^{i\frac{v}{2}} \sin \frac{w}{2} & e^{-i\frac{u}{2}} \cos \frac{w}{2} \end{pmatrix}$$

with real angles $u, v, w$. Consider the operation $W$ in (28) with just the single operator $W_1 = W$; i.e., $p_1 = 1$. We find after some calculation that, after application of $I \otimes W$ for this case, the average energy in the particle pair is

$$\text{tr}[H(I \otimes W)\rho(T)(I \otimes W)^\dagger] = 2m - r(1 + \cos w) + \kappa c_1 \cos v(1 - \cos w) - \kappa c_1 \cos u(1 + \cos w).$$

We seek values of the angles $u, v, w$ of $W$ to minimize (29) and thereby extract the maximum amount of energy with $W$. Because $r, c_1 > 0$ we easily see that (29) is minimum uniquely when $u = w = 0$; that is, when $W = I$, in which case (29) is exactly (18) and zero energy is extracted. This result exemplifies Gibbs states’ general property of passivity [43], whereby any non-trivial unitary operation necessarily increases the system energy and no unitary operation on B can extract energy, at any temperature. When $W$ in (28) is a non-trivial sum involving
more than one unitary operator, the particle pair energy is minimized when each $W_k = I$. Thus, no local, conditionally unitary operation made on particle B, made without a measurement of A and the knowledge therefrom, can do better than extract zero energy. In the QET protocol the measurement of particle A both ‘sets’ particle B and provides information to the site of B for exploiting that setting. This is the essence of QET.

No conditional unitary operation applied locally to particle B without preparation at and communication from A can extract energy from B. With the QET protocol, on the other hand, energy can be extracted locally from B. This comparison, energy teleportation versus a local conditionally unitary operation, seems most apt since step III in energy teleportation is one of two unitary operations, the choice depending on information sent from A to B. Conditionally unitary operations are not the most general one-qubit operations, of course. One might ask about how QET fares in contest with a general quantum operation applied locally to B. We explore this question now.

A quantum operation for a qubit in state $\tau$ is, in general operator-sum form [30],

$$\mathcal{G}(\tau) = \sum_{k=1}^{4} K_k \tau K_k^\dagger$$

(30)

with Kraus operators

$$K_k = \begin{pmatrix} s_k & t_k \\ u_k & v_k \end{pmatrix}$$

(31)

whose complex-valued elements $s_k, t_k, u_k, v_k$ satisfy the completeness condition

$$\sum_{k=1}^{4} K_k^\dagger K_k = I.$$ 

(32)

This condition can be expressed in terms of the elements of the $K_k$ as

$$s^\dagger s + u^\dagger u = 1$$

$$t^\dagger t + v^\dagger v = 1$$

$$s^\dagger t + u^\dagger v = 0$$

(33)

where $s = (s_1, s_2, s_3, s_4)^\top$, etc. We call a vector $z = (s^\top, t^\top, u^\top, v^\top)^\top$ feasible and write $z \in \mathcal{F}$ if $z$ satisfies the conditions in (33). Using these conditions, we find after some calculation that

$$\text{tr} \left[ \mathcal{H} \sum_k (I \otimes K_k) \rho(T) (I \otimes K_k)^\dagger \right] = 2m - 2r - (1-r)u^\dagger u + (1+r)t^\dagger t$$

$$- \kappa c_1 (s^\dagger v + v^\dagger s + u^\dagger t + t^\dagger u) .$$

(34)

Then, subtracting (34) from (18), we find, for any $\kappa$ and $T$, that the energy extracted by applying $\mathcal{G}$ locally to B is

$$\Omega(z) = (1-r)u^\dagger u - (1+r)t^\dagger t + \kappa c_1 (s^\dagger v + v^\dagger s + u^\dagger t + t^\dagger u - 2).$$

(35)

To find the maximum energy that can be extracted locally by $\mathcal{G}$, we maximize (35) subject to (33). The solution space for this problem has 32 real parameters (corresponding to the 16 complex elements of the Kraus operators), with non-trivial constraints imposed by (33). Despite the evident challenge, a simple analytical expression for the maximum can be given. The maximum of $\Omega(z)$ subject to (33) is

$$\max_{z \in \mathcal{F}} \Omega(z) = \begin{cases} \sqrt{1 - r^2 + 4r(1-r)} - 2\kappa c_1 - r, & \kappa c_1 < \frac{1-r^2}{2r} \\ 0, & \text{otherwise} \end{cases}$$

(36)
This maximum has two branches, a zero branch obtained by the identity operation and a positive branch obtained by the local quantum operation $G$ with Kraus operators

$$K_1 = \begin{pmatrix} \cos \alpha & 0 \\ 0 & \cos \beta \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & \sin \beta \\ \sin \alpha & 0 \end{pmatrix}, \quad K_3 = K_4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(37)

with angles $\alpha$ and $\beta$ whose sum $\sigma = \alpha + \beta$ and difference $\delta = \alpha - \beta$ are given by

$$\cos \sigma = \frac{2k c_1 r}{1 - r^2}, \quad \cos \delta = \frac{2k c_1}{\sqrt{(1 - r^2)(1 - r^2 + 4k^2 c_1^2)}}.$$ 

(38)

To prove (36), let $z = (s^\top, t^\top, u^\top, v^\top)^\top$ be any feasible vector $z \in F$ and associate to $z$ the vector $z_o = (s_o^\top, t_o^\top, u_o^\top, v_o^\top)^\top$ with

$s_o = (s, 0, 0, 0)^\top, \quad t_o = (0, t, 0, 0)^\top, \quad u_o = (0, u, 0, 0)^\top, \quad v_o = (v, 0, 0, 0)^\top

where $s = \sqrt{{s}^2}, t = \sqrt{{t}^2}, u = \sqrt{{u}^2}v$ and $v = \sqrt{{v}^2}$ are the magnitudes of $s, t, u$ and $v$. This association of $z_o$ with $z$ is the proof’s key step. We have $z_o \in F$. Also, $u \cdot u = u^2 = u_o^\top u_o$, and $t \cdot t = t^2 = t_o^\top t_o$, and, by the Cauchy–Schwarz inequality, $s^2 v + v^2 s \leq 2sv = s^\top v_o + v^\top s_o$, and $u^\top t + t^\top u \leq 2ut = u_o^\top t_o + t_o^\top u_o$. Using these results with $0 \leq r \leq 1$ and $c_1 \geq 0$, we have $\Omega(z) \leq \Omega(z_o)$. Therefore, the maximum of $\Omega(z)$ subject to the conditions of (33) equals the maximum of

$$\omega(s, t, u, v) = (1 - r)u^2 - (1 + r)t^2 + 2k c_1 (sv + ut - 1)$$

subject to $s^2 + u^2 = 1$ and $t^2 + v^2 = 1$. Set

$s = \cos \alpha, \quad t = \sin \beta, \quad u = \sin \alpha, \quad v = \cos \beta,$

and let $\sigma = \alpha + \beta$ and $\delta = \alpha - \beta$. Then $\omega(s, t, u, v) = \sigma(\sigma, \delta)$ where

$$\sigma(\sigma, \delta) = \sin \alpha \sin \delta + r \cos \sigma \cos \delta + 2k c_1 \cos \delta - r - 2k c_1$$

(39)

and the maximum of $\Omega(z)$ subject to constraints (33) is the unconstrained maximum of $\sigma(\sigma, \delta)$. Solving for the stationary point(s) of $\sigma(\sigma, \delta)$ yields (38) for $2k c_1 r < 1 - r^2$ and $\sigma = \delta = 0$ otherwise. Substituting (38) into (39) gives the positive branch of (36); putting $\sigma = \delta = 0$ into (39) gives the zero branch of (36).

Two cases, $\kappa = 0$ and $T = \infty$, of (35) have particular physical interest and can be solved directly to independently check (36). In both cases $\rho(T)$ has zero discord and the QET protocol yields zero energy.

$\kappa = 0$. This is the case of uncoupled spin-$\frac{1}{2}$ particles in the product state $\rho(T) = \rho_A \otimes \rho_B$ where $\rho_A = \rho_B = \frac{1}{2} (I - r \sigma_z)$ with $r = \tanh \frac{1}{2} T$. We are in this case effectively just seeking the maximum amount of energy that can be extracted from a thermal qubit with Hamiltonian $H_B = I + \sigma_z$ as in (2). For this case (35) is just

$$\Omega(z) = (1 - r)u \cdot u - (1 + r)t \cdot t,$$

(40)

and, subject to (33), the maximum of (40) can be seen by inspection to be $1 - r$. The energy (18) initially in the two uncoupled particles is $2 - 2r$, half associated with each particle. We conclude that in the case $\kappa = 0$ the optimal local quantum operation $G$ extracts all the energy $1 - r$ associated with particle $B$. This agrees with (36); for $\kappa = 0$ we have $q = 0$ and the positive branch of (36) applies, yielding $1 - r$ for the energy locally available at $B$. There is no non-local energy in $A$ and $B$ in this example.
Maximum energy extractable by local operation at B

The maximum of \( (41) \) subject to \( (33) \) is 1 by inspection. This maximum energy extractable locally by \( \mathcal{G} \) agrees with \( (36) \); \( r = q = 0 \) for \( T = \infty \) so the positive branch of \( (36) \) applies with maximum value 1. The non-local energy in A and B is \( 2m - 2 \) in this example.

In terms of our state parameters \( \kappa \) and \( T \), the condition \( 2\kappa r c_1 < 1 - r^2 \) in \( (36) \) is equivalent to

\[
\left( \kappa \sinh \frac{2m}{kT} + m \sinh \frac{2\kappa}{kT} \right)^2 < 2m^2 \cosh \frac{2\kappa}{kT} \left( \cosh \frac{2m}{kT} + \cosh \frac{2\kappa}{kT} \right). \tag{42}
\]

A unique non-zero temperature threshold \( T_1 \) saturates condition \( (42) \) for each coupling \( \kappa \). Below temperature \( T_1 \) no energy can be extracted at site B by any local operation \( \mathcal{G} \). Figure 3 shows this threshold for different couplings \( \kappa \). The emergence of this threshold in our simple two-particle model is unexpected. Its existence points to a distinctive ability of energy teleportation. Below

\( T = \infty \). Again the particles are uncoupled, with in this case \( \rho(\infty) = \frac{1}{2}I \otimes I \) with energy \( \text{tr}[H\rho(\infty)] = 2m \). Also, \( r = c_1 = 0 \) so

\[
\Omega(z) = u^\dagger u - t^\dagger t. \tag{41}
\]

The maximum of \( (41) \) subject to \( (33) \) is 1 by inspection. This maximum energy extractable locally by \( \mathcal{G} \) agrees with \( (36) \); \( r = q = 0 \) for \( T = \infty \) so the positive branch of \( (36) \) applies with maximum value 1. The non-local energy in A and B is \( 2m - 2 \) in this example.

In terms of our state parameters \( \kappa \) and \( T \), the condition \( 2\kappa r c_1 < 1 - r^2 \) in \( (36) \) is equivalent to

\[
\left( \kappa \sinh \frac{2m}{kT} + m \sinh \frac{2\kappa}{kT} \right)^2 < 2m^2 \cosh \frac{2\kappa}{kT} \left( \cosh \frac{2m}{kT} + \cosh \frac{2\kappa}{kT} \right). \tag{42}
\]

A unique non-zero temperature threshold \( T_1 \) saturates condition \( (42) \) for each coupling \( \kappa \). Below temperature \( T_1 \) no energy can be extracted at site B by any local operation \( \mathcal{G} \). Figure 3 shows this threshold for different couplings \( \kappa \). The emergence of this threshold in our simple two-particle model is unexpected. Its existence points to a distinctive ability of energy teleportation.
Figure 4. Two temperature-dependent energy regimes. QET yields energy throughout the $\kappa - T$ parameter space; in the teleportation regime energy extraction is possible only by QET. In the local extraction regime a local operation yields more energy than QET. In the narrow temperature window $(T_1, T_2)$ between the two regimes, QET yields more energy than available by a local operation.

$T_1$ energy can be extracted at site B only by QET; no local operation, unitary or otherwise, at site B can accomplish this.

Above $T_1$ is a second temperature threshold $T_2$, defined by $\max_z \Omega(z) = E_B$. This is the temperature where the energies available at particle B by teleportation and by local operation are equal. In the temperature window $(T_1, T_2)$ a local operation on B can extract energy, but not as much as that yielded by energy teleportation. Above temperature $T_2$, a local operation can yield more energy than teleportation. Returning to figure 2, we can check that for any coupling $\kappa$ the amount of energy teleported at temperatures above $T_2$ is vanishingly small relative to the amount available by $\mathcal{G}$. We conclude therefrom that there are effectively two energy regimes: a teleportation regime below the window $(T_1, T_2)$ where energy can be extracted from B by teleportation but not by any local operation $\mathcal{G}$ and a local extraction regime where energy can be extracted by $\mathcal{G}$ but very little energy can be teleported. These two energy regimes are shown in figure 4.

5. Discussion and summary

Product states cannot support QET. The two cases, $\kappa = 0$ and $T = \infty$, of the previous section illustrate this—in each case $\rho(T)$ is a product state and the teleported energy is zero—and one can readily see that this is generally true. In our two-particle model QET requires some correlation between the particles at the two sites. By definition, the quantum part of this correlation is measured by the quantum discord in the particles’ state. We found for all finite temperatures and non-zero degrees of spin coupling both that quantum correlation is present in $\rho(T)$ and that, in fact, energy teleportation can be accomplished with the QET protocol. The quantum correlation is solely entanglement at zero temperature. At temperatures above
Figure 5. $\kappa$–$T$ parameter space of two-particle Gibbs states. Gibbs states in the shaded region are entangled. The contours in the unshaded region are parametric families $\rho_c(T)$ of separable Gibbs states with constant classical correlation, $C[\rho_c(T)] = .1, .3, .5, .7, .9$.

$T_e$ (marked by the points on the curves in figure 1), $\rho(T)$ is separable, the quantum correlation that exists between particles A and B is strictly dissonant, and this dissonance is acting as a quantum resource for energy teleportation. This means not just that energy teleportation occurs when dissonance is present in $\rho(T)$. It means that quantum states with more dissonance yield greater teleported energy, and that this occurs under controlled conditions in which all other correlations are held constant (and cannot, therefore, account for the greater energy.) At temperatures above $T_e$ where $\rho(T)$ is strictly dissonant, $\rho(T)$ has for fixed $\kappa$ a varying amount of classical correlation $C[\rho(T)]$ that could conceivably be the source of energy increases. To eliminate $C[\rho(T)]$ as a possible factor, we consider $\rho(T)$ as a two-parameter ($T$ and $\kappa$) family of states and choose (as in figure 5) parametric families $\rho_c(T)$ of Gibbs states with constant $C[\rho_c(T)]$—accomplished by adjusting $\kappa$ to keep $C[\rho(T)]$ constant as $T$ is increased. When this is done, we get the results in figure 6. The curves in figure 6 are parametric plots of teleported energy versus the quantum dissonance in $\rho_c(T)$ as temperature is increased starting at $T_e$ and the coupling $\kappa$ is varied to maintain constant $C[\rho(T)]$. The curves correspond to five fixed values of $C[\rho_c(T)]$, chosen to represent the 0–1 range of classical correlation present in the separable Gibbs states in figure 5. On each curve in figure 6, both entanglement and classical correlation are fixed (to zero in the case of entanglement); only the dissonance is varying. These curves show, with all other forms of correlation held constant, a direct relationship between $E_B$ and the Gibbs state dissonance: greater dissonance consistently yields greater energy. Dissonance is acting in the case of the Gibbs states studied here as a quantum resource. Other examples of dissonance as a resource are known [15–18]; more examples, we expect, will contribute to a fuller understanding of dissonance, discord and quantum correlation.

When we consider a strictly local quantum operation to extract energy from particle B of our spin pair, we find the temperature threshold $T_1$ below which no energy extraction is possible. That such a threshold emerges in even our very simple two-particle model is
remarkable. We refer to temperatures below $T_1$ as the teleportation regime because at these temperatures the QET protocol unlocks energy that is otherwise not locally accessible. We also identified a threshold $T_2 > T_1$ above which more energy can be extracted by a local operation than by energy teleportation. In fact, in the spin system we consider, energy teleportation yields only an vanishingly small amount of energy above $T_2$, while at these temperatures significant energy can be extracted by local operation on $B$. We therefore call these temperatures above $T_2$ the local extraction regime. Between these two identified regimes is a narrow transitional temperature window. It would interesting to know whether, and to what extent, these regimes exist in other spin chains. In particular, we would like to know whether a teleportation regime with an associated $T_1$ threshold similar to that identified here is generic to different classes of spin chains.

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**Appendix A**

The classical correlation (14) in the Gibbs state $\rho(T)$ is

$$C(\rho(T)) = h(r) - \min_{\{M_k\}} \sum_k q_k S(\rho_k)$$

(A.1)
where, with \( h(\cdot) \) as in (12), \( h(r) = S(\rho_A(T)) \) is the von Neumann entropy of \( \rho_A(T) = \text{tr}_B[\rho(T)] \). The projective measurement \( \{M_0, M_1\} \) of qubit B has the general form \( M_k = |k\rangle \langle k'| \)

\[
|0\rangle' = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle,
\]

\[
|1\rangle' = \sin \frac{\theta}{2} |0\rangle - e^{i\phi} \cos \frac{\theta}{2} |1\rangle.
\]

The measurement \( \{M_0, M_1\} \) changes \( \rho(T) \) to one of two states

\[
\rho_k = \frac{1}{q_k} (I \otimes M_k)\rho(T)(I \otimes M_k) = \frac{1}{q_k} J_k \otimes M_k, \quad k = 0, 1
\]

where

\[
J_0 = \frac{q_0}{2} - \frac{c_1 \cos \phi \sin \theta}{4} \sigma_x + \frac{c_2 \sin \phi \sin \theta}{4} \sigma_y - \frac{r - c_3 \cos \theta}{4} \sigma_z,
\]

\[
J_1 = \frac{q_1}{2} + \frac{c_1 \cos \phi \sin \theta}{4} \sigma_x - \frac{c_2 \sin \phi \sin \theta}{4} \sigma_y - \frac{r + c_3 \cos \theta}{4} \sigma_z
\]

with corresponding probabilities

\[
q_0(\theta) = \text{tr}[(I \otimes M_0)\rho(T)(I \otimes M_0)] = \frac{1 - r \cos \theta}{2},
\]

\[
q_1(\theta) = \text{tr}[(I \otimes M_1)\rho(T)(I \otimes M_1)] = \frac{1 + r \cos \theta}{2}.
\]

The eigenvalues of \( \rho_0 \) and \( \rho_1 \) are, respectively,

\[
0, 0, \frac{1}{2} \pm \frac{\sqrt{A(\theta, \phi) - 2rc_3 \cos \theta}}{4q_0}
\]

\[
0, 0, \frac{1}{2} \pm \frac{\sqrt{A(\theta, \phi) + 2rc_3 \cos \theta}}{4q_1}
\]

where

\[
A(\theta, \phi) = (c_1^2 \cos^2 \phi + c_2^2 \sin^2 \phi) \sin^2 \theta + c_3^2 \cos^2 \theta + r^2.
\]

The minimum in (A.1) is

\[
\min_{\{M_k\}} \sum_k q_k S(\rho_k) = \min_{\theta, \phi}(q_0(\theta) S(\rho_0) + q_1(\theta) S(\rho_1))
\]

\[
\text{(A.4)}
\]

where

\[
S(\rho_0) = h \left( \frac{\sqrt{A(\theta, \phi) - 2rc_3 \cos \theta}}{2q_0(\theta)} \right)
\]

\[
S(\rho_1) = h \left( \frac{\sqrt{A(\theta, \phi) + 2rc_3 \cos \theta}}{2q_1(\theta)} \right)
\]

because the zero eigenvalues in (A.2) and (A.3) do not contribute to the entropies of \( \rho_0, \rho_1 \).

The function \( h(x) \) is decreasing for \( 0 < x < 1 \) and \( c_1 > c_2 \) so the minimum in (A.4) is found at \( \phi = 0 \) for all angles \( \theta \). We have then that the minimum in (A.1) is

\[
\min_{\{M_k\}} \sum_k q_k S(\rho_k) = \min_{\theta} \left( q_0(\theta) h \left( \frac{\sqrt{c_1^2 \sin^2 \theta + (r - c_3 \cos \theta)^2}}{2q_0(\theta)} \right) + q_1(\theta) h \left( \frac{\sqrt{c_1^2 \sin^2 \theta + (r + c_3 \cos \theta)^2}}{2q_1(\theta)} \right) \right).
\]

\[
\text{(A.5)}
\]
The expression to be minimized in (A.5) has, for arbitrary values of $c_1$, $c_3$, and $r$, at least two and sometimes more local extrema [34]. However, for the class of Gibbs states $\rho(T)$ in (5) with $c_1$, $c_3$, and $r$ restricted to values given by (6), (7), and (8), one can check numerically that the minimum in (A.5) is achieved by $\theta = \frac{\pi}{2}$ for all $\kappa$ and $T$. Since $q_0(\frac{\pi}{2}) = q_1(\frac{\pi}{2}) = \frac{1}{2}$, the minimum in (A.5) is $h(\sqrt{c_1^2 + r^2})$. Substituting this in (A.1) gives our result (15) for $C[\rho(T)]$.

Appendix B

The four eigenvalues of the partial transpose of $\rho(T)$ are

$$
\lambda_{1,\pm} = e^{\mp \frac{\pi}{mZ}} \left( m \cosh \frac{2m}{kT} \pm \kappa \sinh \frac{2m}{kT} \right),
$$

$$
\lambda_{2,\pm} = e^{\mp \frac{\pi}{mZ}} \left( m \cosh \frac{2m}{kT} \pm \sqrt{m^2 \sinh^2 \frac{2m}{kT} + \sinh^2 \frac{2m}{kT}} \right).
$$

Eigenvalues $\lambda_{1+}, \lambda_{2+}$ are always positive. Also, $\lambda_{2-}$ is always positive since

$$
\lambda_{2-} - \lambda_{2+} = e^{\mp \frac{\pi}{mZ}} \left( m^2 \cosh^2 \frac{2m}{kT} - m^2 \sinh^2 \frac{2m}{kT} - \sinh^2 \frac{2m}{kT} \right)
$$

$$
= e^{\mp \frac{\pi}{mZ}} \left( \kappa^2 \cosh^2 \frac{2m}{kT} - m^2 \sinh^2 \frac{2m}{kT} + 1 \right)
$$

$$
\geq 4\kappa^2 e^{\mp \frac{\pi}{mZ}} \left( \frac{kT}{2m} \right)^2 \cosh^2 \frac{2m}{kT} - \left( \frac{kT}{2\kappa} \right)^2 \sinh^2 \frac{2m}{kT}
$$

$$
\geq 4\kappa^2 e^{\mp \frac{\pi}{mZ}} \left( \frac{kT}{2m} \right)^2 \cosh^2 \frac{2m}{kT} - \left( \frac{kT}{2m} \right)^2 \sinh^2 \frac{2m}{kT}
$$

$$
= \frac{\kappa^2 e^{\mp \frac{\pi}{mZ}}}{m^2 Z^2} > 0,
$$

where inequality (B.1) holds because $m > \kappa$ and $\sinh x/x$ is strictly increasing for $x > 0$. Therefore, according to the PPT criterion, the thermal state $\rho(T)$ is separable if and only if $\lambda_{1-} \geq 0$. This is the source of condition (17).

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