MODULE CHECKING OF PUSHDOWN MULTI-AGENT SYSTEMS

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ABSTRACT. In this paper, we investigate the module-checking problem of pushdown multiagent systems (PMS) against ATL and ATL* specifications. We establish that for ATL, module checking of PMS is 2EXPTIME-complete, which is the same complexity as pushdown module-checking for CTL. On the other hand, we show that ATL* module-checking of PMS turns out to be 4EXPTIME-complete, hence exponentially harder than both CTL* pushdown module-checking and ATL* model-checking of PMS. Our result for ATL* provides a rare example of a natural decision problem that is elementary yet but with a complexity that is higher than triply exponential-time.

1. Introduction

Model checking is a well-established formal-method technique to automatically check for global correctness of systems [CE81, QS82]. Early use of model checking mainly considered finite-state closed systems, modelled as labelled state-transition graphs (Kripke structures) equipped with some internal degree of nondeterminism, and specifications given in terms of standard temporal logics such as the linear-time temporal logic LTL [Pnu77] and the branching-time temporal logics CTL and CTL* [EH86]. In the last two decades, model-checking techniques have been extended to the analysis of reactive and distributed component-based systems, where the behavior of a component depends on assumptions on its environment (the other components). One of the first approaches to model check finite-state open systems is module checking [KV96], a framework for handling the interaction between a system and an external unpredictable environment. In this setting, the system is modeled as a module that is a finite-state Kripke structure whose states are partitioned into those controlled by the system and those controlled by the environment. The latter ones intrinsically carry an additional source of nondeterminism describing the possibility

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* The paper is an extended version completed with proofs of the conference paper [BMP20].
Table 1: Complexity results on finite-state model checking and finite-state module checking

|        | Model Checking | Model Checking | Module Checking | Module Checking |
|--------|----------------|----------------|-----------------|-----------------|
|        | (fixed formula) | (fixed formula) |                 |                 |
| CTL    | PTIME [EH86]   | NLOGSPACE [BVW94] | EXPTIME [KV96] | PTIME [KV96]   |
| CTL*   | PSPACE [EH86]  | NLOGSPACE [BVW94] | 2EXPTIME [KV96] | PTIME [KV96]   |
| ATL    | PTIME [AHK02]  | PTIME [AHK02]   | EXPTIME [BM17] | PTIME [BM17]   |
| ATL*   | 2EXPTIME [AHK02]| PTIME [AHK02]| 3EXPTIME [BM17]| PTIME [BM17]   |

Table 1: Complexity results on finite-state model checking and finite-state module checking

that the computation, from these states, can continue with any subset of its possible successor states. This means that while in model checking, we have only one computation tree representing the possible evolution of the system, in module checking we have an infinite number of trees to handle, one for each possible behavior of the environment. Deciding whether a system satisfies a property amounts to check that all such trees satisfy the property. This makes module checking harder to deal with. Classically, module checking has been investigated with respect to CTL and CTL* [KV96, KV97, BRS07] specifications and for $\mu$-calculus specifications [FMP08]. An extension of module checking has been also used to reason about three-valued abstractions in [dAGJ04, God03]. More recent approaches to the verification of multi-component finite-state systems (multi-agent systems) are based on the game paradigm: the system is modeled by a multi-player finite-state concurrent game, where at each step, the next state is determined by considering the “intersection” between the choices made simultaneously and independently by all the players (the agents). In this setting, properties are specified in logics for strategic reasoning such as the alternating-time temporal logics ATL and ATL* [AHK02], the latter ones being well-known extensions of CTL and CTL*, respectively, which allow to express cooperation and competition among agents in order to achieve certain goals. In particular, they can express selective quantification over those paths that are the result of the infinite game between a given coalition and the rest of the agents.

For a long time, there has been a common belief that module checking of CTL/CTL* is a special case of model checking of ATL/ATL*. The belief has been recently refuted in [JM14] where it is proved that module checking includes two features inherently absent in the semantics of ATL/ATL*, namely irrevocability and nondeterminism of strategies. On the other hand, branching-time temporal logics like CTL and CTL* do not accommodate strategic reasoning. These facts have motivated the extension of module checking to a finite-state multi-agent setting for handling specifications in ATL* [JM15, BM17], which turns out to be more expressive than both CTL* module checking and ATL* model checking [JM14, JM15]. Table 1 summarizes known results about the complexity of finite-state model checking and finite-state module checking. All the complexities in Table 1 denote tight bounds.

Verification of pushdown systems. An active field of research is model checking of pushdown systems. These represent an infinite-state formalism suitable to capture the control
flow of procedure calls and returns in programs. Model checking of (closed) pushdown systems against standard regular temporal logics (such as LTL, CTL, CTL*, and the modal μ-calculus) is decidable and it has been intensively studied in recent years leading to efficient verification algorithms and tools (see [Wal96, BEM97, BR00, AKM12, AMM14]). The verification of open pushdown systems in a two-player turn-based setting has been investigated in many works (e.g. see [LMS04, HO09]). Open pushdown systems along with the module-checking paradigm have been considered in [BMP10]. As in the case of finite-state systems, for the logic CTL (resp., CTL*), pushdown module-checking is singly exponentially harder than pushdown model-checking, being precisely 2Exptime-complete (resp., 3Exptime-complete), although with the same program complexity as pushdown model-checking (that is Exptime-complete). Pushdown module-checking has been investigated under several restrictions [ALM13, Boz11, MNP08], including the imperfect-information setting case, where the latter variant is in general undecidable [ALM13]. More recently in [MP15, CSW16], the verification of open pushdown systems has been extended to a concurrent game setting (pushdown multi-agent systems) by considering specifications in ATL* and the alternating-time modal μ-calculus. In particular, model checking of PMS against ATL* has the same complexity as pushdown module-checking against CTL* [CSW16].

Our contribution. In this paper, we extend the module-checking framework to the verification of multi-agent pushdown systems (PMS) by addressing the module-checking problem of PMS against ATL and ATL* specifications. By [JM14], the considered setting for ATL (reps., ATL*) is strictly more expressive than both pushdown module checking e.g. for CTL (reps., CTL*) and ATL (reps., ATL*) model-checking of PMS. We establish that ATL module-checking for PMS has the same complexity as pushdown module-checking for CTL, that is 2Exptime-complete. On the other hand, we show that ATL* module-checking of PMS has a very high complexity: it turns out to be exponentially harder than ATL* model-checking of PMS and pushdown module-checking for CTL*, being, precisely, 4Exptime-complete with an Exptime-complete complexity for a fixed-size formula. The upper bounds are obtained by an automata-theoretic approach. The matching lower bound for ATL* is shown by a technically non-trivial reduction from the acceptance problem for 3ExpSpace-bounded alternating Turing Machines. Our result for ATL* provides a rare example of a natural decision problem that is elementary yet but with a complexity that is higher than triply

|          | Pushdown Model Checking | Pushdown Model Checking (fixed formula) | Pushdown Module Checking | Pushdown Module Checking (fixed formula) |
|----------|-------------------------|------------------------------------------|--------------------------|------------------------------------------|
| CTL      | EXPTIME                 | EXPTIME                                  | 2EXPTIME                 | EXPTIME                                  |
|          | [Wal00]                 | [Boz06]                                  | [BMP10]                  | [BMP10]                                  |
| CTL*     | 2EXPTIME                | EXPTIME                                  | 3EXPTIME                 | EXPTIME                                  |
|          | [Boz06]                 | [Boz06]                                  | [BMP10]                  | [BMP10]                                  |
| ATL      | EXPTIME                 | EXPTIME                                  | 2EXPTIME                 | EXPTIME                                  |
|          | [CSW16]                 | [CSW16]                                  | Corollary 3.4            | Corollary 3.4                            |
| ATL*     | 3EXPTIME                | EXPTIME                                  | 4EXPTIME                 | EXPTIME                                  |
|          | [CSW16]                 | [CSW16]                                  | Cor. 3.4 & Theorem 4.1   | Corollary 3.4                            |

Table 2: Complexity results on pushdown model checking and pushdown module checking
exponential-time. To the best of our knowledge, the unique known characterization of the class 4Exptime concerns validity of $\text{CTL}^*$ on alternating automata with bounded cooperative concurrency [HRV90].

Our results confirm that pushdown module checking is exponentially harder than finite-state module checking. Indeed, like the logics $\text{CTL}$ and $\text{CTL}^*$, pushdown module checking against $\text{ATL}$ (resp., $\text{ATL}^*$) turns out to be exponentially harder that finite-state module checking against $\text{ATL}$ (resp., $\text{ATL}^*$) even for a fixed formula. This is illustrated in Tables 1 and 2, where all the complexities denote tight bounds.

The rest of the paper is organized as follows. In Section 2, we recall the concurrent game setting, the class of multi-agent pushdown systems (PMS), and the logics $\text{ATL}$ and $\text{ATL}^*$. Moreover, we introduce the PMS module-checking framework for $\text{ATL}$ and $\text{ATL}^*$ specifications. In Section 3, we describe the proposed automata-theoretic approach for solving the module-checking problem of PMS against $\text{ATL}$ and $\text{ATL}^*$, and in Section 4, we show that for the logic $\text{ATL}^*$, the considered problem is 4Exptime-hard. Finally Section 5 provides an assessment of the work done, and outlines future research directions.

2. Preliminaries

We fix the following notations. Let $AP$ be a finite nonempty set of atomic propositions, $Ag$ be a finite nonempty set of agents, and $Ac$ be a finite nonempty set of actions that can be made by agents. For a set $A \subseteq Ag$ of agents, an $A$-decision $d_A$ is an element in $Ac^A$ assigning to each agent $a \in A$ an action $d_A(a)$. For $A, A' \subseteq Ag$ with $A \cap A' = \emptyset$, an $A$-decision $d_A$ and $A'$-decision $d_{A'}$, $d_A \cup d_{A'}$ denotes the $(A \cup A')$-decision defined in the obvious way. Let $Dc = Ac^Ag$ be the set of full decisions of all the agents in $Ag$.

Let $\mathbb{N}$ be the set of natural numbers. For an infinite word $w$ over an alphabet $\Sigma$ and $i \geq 0$, $w(i)$ denotes the $(i+1)^{th}$ letter of $w$ and $w_{\geq i}$ the suffix of $w$ starting from the $(i+1)^{th}$ letter of $w$, i.e., the infinite word $w(i)w(i+1)\ldots$. For a finite word $w$ over $\Sigma$, $|w|$ is the length of $w$.

Given a set $\Upsilon$ of directions, an (infinite) $\Upsilon$-tree $T$ is a prefix closed subset of $\Upsilon^*$ such that for all $\nu \in T$, $\nu \cdot \gamma \in T$ for some $\gamma \in \Upsilon$. Elements of $T$ are called nodes and $\varepsilon$ is the root of $T$. For $\nu \in T$, a child of $\nu$ in $T$ is a node of the form $\nu \cdot \gamma$ for some $\gamma \in \Upsilon$. An (infinite) path of $T$ is an infinite sequence $\pi$ of nodes such that $\pi(i+1)$ is a child in $T$ of $\pi(i)$ for all $i \geq 0$. For an alphabet $\Sigma$, a $\Sigma$-labeled $\Upsilon$-tree is a pair $\langle T, \text{Lab} \rangle$ consisting of a $\Upsilon$-tree and a labelling $\text{Lab} : T \mapsto \Sigma$ assigning to each node in $T$ a symbol in $\Sigma$. We extend the labeling $\text{Lab}$ to paths $\pi$ in the obvious way, i.e. $\text{Lab}(\pi)$ is the infinite word over $\Sigma$ given by $\text{Lab}(\pi(0))\text{Lab}(\pi(1))\ldots$. The labeled tree $\langle T, \text{Lab} \rangle$ is complete if $T = \Upsilon^*$. Given $k \in \mathbb{N} \setminus \{0\}$, a $k$-ary tree is a $\{1, \ldots, k\}$-tree.

**Concurrent game structures (CGS).** CGS [AHK02] extend Kripke structures to a setting involving multiple agents. They can be viewed as multi-player games in which players perform concurrent actions, chosen strategically as a function of the history of the game.

**Definition 2.1 (CGS).** A CGS (over $AP$, $Ag$, and $Ac$) is a tuple $\mathcal{G} = \langle S, s_0, \text{Lab}, \tau \rangle$, where $S$ is a set of states, $s_0 \in S$ is the initial state, $\text{Lab} : S \mapsto 2^{AP}$ maps each state to a set of atomic propositions, and $\tau : S \times Dc \mapsto S \cup \{\bot\}$ is a transition function that maps a state and a full decision either to a state or to the special symbol $\bot$ ($\bot$ is for `undefined') such that for all states $s$, there exists $d \in Dc$ so that $\tau(s, d) \neq \bot$. Given a set $A \subseteq Ag$ of agents,
an $A$-decision $d_A$, and a state $s$, we say that $d_A$ is available at state $s$ if there exists an $(Ag \setminus A)$-decision $d_{Ag \setminus A}$ such that $\tau(s, d_A \cup d_{Ag \setminus A}) \in S$.

For a state $s$ and an agent $a$, state $s$ is controlled by $a$ if there is a unique $(Ag \setminus \{a\})$-decision available at state $s$. Agent $a$ is passive in $s$ if there is a unique $\{a\}$-decision available at state $s$. A multi-agent turn-based game is a CGS where each state is controlled by an agent.

Note that in modelling independent agents, usually one assumes that at each state $s$, each agent $a$ has a set $Ac_a(s) \subseteq Ac$ of actions which are enabled at the state $s$. This is reflected in the transition function $\tau$ by requiring that the set of full decisions $d$ such that $\tau(s, d) \neq \bot$ corresponds to $(Ac_a(s))_{a \in Ag}$.

We now recall the notion of strategy in a CGS $G = \langle S, s_0, Lab, \tau \rangle$. Here, we consider perfect recall strategies where an agent decides the next action by using all the available information up to the current round. A play is an infinite sequence of states $s_1s_2\ldots$ such that for all $i \geq 1$, $s_{i+1}$ is a successor of $s_i$, i.e. $s_{i+1} = \tau(s_i, d)$ for some full decision $d$. A track (or history) $\nu$ is a nonempty prefix of some play. Given a set $A \subseteq Ag$ of agents, a strategy for $A$ is a mapping $f_A$ assigning to each track $\nu$ (representing the history the agents saw so far) an $A$-decision available at the last state, denoted $\text{lst}(\nu)$, of $\nu$. The outcome function $\text{out}(s, f_A)$ for a state $s$ and the strategy $f_A$ returns the set of all the plays starting at state $s$ that can occur when agents $A$ execute strategy $f_A$ from state $s$ on. Formally, $\text{out}(s, f_A)$ is the set of plays $\pi = s_1s_2\ldots$ such that $s_1 = s$ and for all $i \geq 1$, there is $d \in Ac^{Ag \setminus A}$ such that $s_{i+1} = \tau(s_i, f_A(s_1\ldots s_i) \cup d)$.

Definition 2.2. For a set $\Upsilon$ of directions, a Concurrent Game $\Upsilon$-Tree ($\Upsilon$-CGT) is a CGS $(T, \varepsilon, Lab, \tau)$, where $(T, Lab)$ is a $2^{AP}$-labeled $\Upsilon$-tree, and for each node $x \in T$, the successors of $x$ correspond to the children of $x$ in $T$. Every CGS $G = \langle S, s_0, Lab, \tau \rangle$ induces a $S$-CGT, denoted by $\text{Unw}(G)$, obtained by unwinding $G$ from the initial state in the usual way. Formally, $\text{Unw}(G) = \langle T, \varepsilon, Lab', \tau' \rangle$, where $\nu \in T$ iff $s_0 \cdot \nu$ is a track of $G$, and for all $\nu \in T$ and $d \in Dc$, $Lab'(\nu) = Lab(\text{lst}(\nu))$ and $\tau'(\nu, d) = \nu \cdot \tau(\text{lst}(\nu), d)$, with $\text{lst}(\varepsilon) = s_0$.

Pushdown multi-agent systems (PMS). PMS, introduced in [MP15], generalize standard pushdown systems to a concurrent multi-player setting.

Definition 2.3. A PMS (over $AP$, $Ag$, and $Ac$) is a tuple $S = \langle Q, \Gamma \cup \{\gamma_0\}, q_0, Lab, \Delta \rangle$, where $Q$ is a finite set of (control) states, $\Gamma \cup \{\gamma_0\}$ is a finite stack alphabet ($\gamma_0$ is the special stack bottom symbol), $q_0 \in Q$ is the initial state, $Lab : Q \mapsto 2^{AP}$ maps each state to a set of atomic propositions, and $\Delta : Q \times (\Gamma \cup \{\gamma_0\}) \times Dc \mapsto (Q \times \Gamma^*) \cup \{\bot\}$ is a transition function (\bot is for ‘undefined’) such that for all pairs $(q, \gamma) \in Q \times (\Gamma \cup \{\gamma_0\})$, there is $d \in Dc$ so that $\Delta(q, \gamma, d) \neq \bot$.

The size $|\Delta|$ of the transition function $\Delta$ is given by $|\Delta| = \sum_{(q', \beta) \in \text{Ran}(\Delta)} |\beta|$, where $\text{Ran}(\Delta)$ is the set of pairs $(q', \beta) \in Q \times \Gamma^*$ such that $(q', \beta) = \Delta(q, \gamma, d)$ for some $(q, \gamma, d) \in Q \times (\Gamma \cup \{\gamma_0\}) \times Dc$. A configuration of the PMS $S$ is a pair $(q, \beta)$ where $q$ is a (control) state and $\beta \in \Gamma^* \cdot \gamma_0$ is a stack content. Intuitively, when the PMS $S$ is in state $q$, the stack top symbol is $\gamma$ and the agents take a full decision $d$ available at the current configuration, i.e. such that $\Delta(q, \gamma, d) = (q', \beta)$ for some $(q', \beta) \in Q \times \Gamma^*$, then $S$ moves to the configuration with state $q'$ and stack content obtained by removing $\gamma$ and pushing $\beta$ if $\gamma = \gamma_0$ then $\gamma$ is not removed. Formally, the PMS $S = \langle Q, \Gamma \cup \{\gamma_0\}, q_0, Lab, \Delta \rangle$ induces the infinite-state CGS $G(S) = \langle S, s_0, Lab', \tau \rangle$, where $S$ is the set of configurations of $S$, $s_0 = (q_0, \gamma_0)$ (initially,
the stack contains just the bottom symbol $\gamma_0$), $Lab((q,\beta)) = Lab(q)$ for each configuration $(q,\beta)$, and the transition function $\tau$ is defined as follows for all $((q,\gamma \cdot \beta),d) \in S \times Dc$, where $\gamma \in \Gamma \cup \{\gamma_0\}$:

- either $\Delta(q,\gamma,d) = \bot$ and $\tau((q,\gamma \cdot \beta),d) = \bot$,
- or $\gamma \in \Gamma$, $\Delta(q,\gamma,d) = (q',\beta')$, and $\tau((q,\gamma \cdot \beta),d) = (q',\beta' \cdot \beta)$,
- or $\gamma = \gamma_0$ (hence, $\beta = \varepsilon$), $\Delta(q,\gamma,d) = (q',\beta')$, and $\tau((q,\gamma \cdot \beta),d) = (q',\beta' \cdot \gamma_0)$.

2.1. The logics ATL* and ATL. We recall the alternating-temporal logics ATL* and ATL proposed by Alur et al. [AHK02] as extensions of the standard branching-time temporal logics CTL* and CTL (respectively) [EH86], where the path quantifiers are replaced by more general parameterized quantifiers which allow for reasoning about the strategic capability of groups of agents. For the given sets $AP$ and $Ag$ of atomic propositions and agents, ATL* formulas $\varphi$ are defined by the following grammar:

$$\varphi ::= true \mid p \mid \neg \varphi \mid \varphi \lor \varphi \mid X\varphi \mid \varphi U \varphi \mid \langle\langle A\rangle\rangle \varphi$$

where $p \in AP$, $A \subseteq Ag$, $X$ and $U$ are the standard “next” and “until” temporal modalities, and $\langle\langle A\rangle\rangle$ is the “existential strategic quantifier” parameterized by a set $A$ of agents. Formula $\langle\langle A\rangle\rangle \varphi$ expresses the property that the group of agents $A$ has a collective strategy to enforce property $\varphi$. In addition, we use standard shorthands: the “eventually” temporal modality $F\varphi := true U \varphi$ and the “always” temporal modality $G\varphi := \neg F \neg \varphi$.

A state formula is a formula where each temporal modality is in the scope of a strategic quantifier. The logic ATL is the fragment of ATL* where each temporal modality is immediately preceded by a strategic quantifier. Note that CTL* (resp., CTL) corresponds to the fragment of ATL* (resp., ATL), where only the strategic modalities $\langle\langle Ag\rangle\rangle$ and $\langle\langle \emptyset\rangle\rangle$ (equivalent to the existential and universal path quantifiers $E$ and $A$, respectively) are allowed.

Given a CGS $G$ with labeling $Lab$ and a play $\pi$ of $G$, the satisfaction relation $G,\pi \models \varphi$ for ATL* is defined as follows (Boolean connectives are treated as usual):

- $G,\pi \models p \iff p \in Lab(\pi(0))$
- $G,\pi \models X\varphi \iff G,\pi_{\geq 1} \models \varphi$
- $G,\pi \models \varphi U \varphi_2 \iff$ there is $j \geq 0 \colon G,\pi_{\geq j} \models \varphi_2$ and $G,\pi_{\geq k} \models \varphi_1$ for all $0 \leq k < j$
- $G,\pi \models \langle\langle A\rangle\rangle \varphi \iff$ for some strategy $f_A$ for $A$, $G,\pi' \models \varphi$ for all $\pi' \in out(\pi(0),f_A)$.

For a state $s$ of $G$, $G,s \models \varphi$ if there is a play $\pi$ starting from $s$ such that $G,\pi \models \varphi$. Note that if $\varphi$ is a state formula, then for all plays $\pi$ and $\pi'$ from $s$, $G,\pi \models \varphi$ iff $G,\pi' \models \varphi$. $G$ is a model of $\varphi$, denoted $G \models \varphi$, if for the initial state $s_0$, $G,s_0 \models \varphi$. Note that $G \models \varphi$ iff $Unw(G) \models \varphi$.

2.2. ATL* and ATL Pushdown Module-checking. The module-checking framework was proposed in [KV96] for the verification of finite open systems, that is systems that interact with an environment whose behavior cannot be determined in advance. In such a framework, the system is modeled by a module corresponding to a two-player turn-based game between the system and the environment. Thus, in a module, the set of states is partitioned into a set of system states (controlled by the system) and a set of environment states (controlled by the environment).

The module-checking problem takes two inputs: a module $M$ and a branching-time temporal formula $\psi$. The idea is that the open system should satisfy the specification $\psi$ no matter how the environment behaves. Let us consider the unwinding $Unw(M)$ of $M$ into
an infinite tree. Checking whether \( Unw(M) \) satisfies \( \psi \) is the usual model-checking problem. On the other hand, for an open system, \( Unw(M) \) describes the interaction of the system with a maximal environment, i.e. an environment that enables all the external nondeterministic choices. In order to take into account all the possible behaviors of the environment, we have to consider all the trees \( T \) obtained from \( Unw(M) \) by pruning subtrees whose root is a successor of an environment state (pruning these subtrees corresponds to disabling possible environment choices). Therefore, a module \( M \) satisfies \( \psi \) if all these trees \( T \) satisfy \( \psi \).

It has been recently proved [JM14] that module checking of CTL/CTL* includes two features inherently absent in the semantics of ATL/ATL*, namely irrevocability of strategies and nondeterminism of strategies. On the other hand, temporal logics like CTL and CTL* do not accommodate strategic reasoning. These facts have motivated the extension of module checking to a multi-agent setting for handling specifications in ATL* [JM15], which turns out to be more expressive than both CTL* module checking and ATL* model checking [JM14, JM15].

In this section, we first recall the ATL* module-checking framework. Then, we generalize this setting to pushdown multi-agent systems. In the multi-agent module-checking setting, one considers CGS with a distinguished agent (the environment).

**Definition 2.4** (Open CGS). An open CGS is a CGS \( \mathcal{G} = \langle S, s_0, \text{Lab}, \tau \rangle \) containing a special agent called “the environment” \( (\text{env} \in A_g) \). Moreover, for every state \( s \), either \( s \) is controlled by the environment \( (\text{environment state}) \) or the environment is passive in \( s \) \( (\text{system state}) \).

For an open CGS \( \mathcal{G} = \langle S, s_0, \text{Lab}, \tau \rangle \), the set of environment strategy trees of \( \mathcal{G} \), denoted \( \text{exec}(\mathcal{G}) \), is the set of \( S\text{-CGT} \) obtained from \( Unw(\mathcal{G}) \) by possibly pruning some environment transitions. Formally, \( \text{exec}(\mathcal{G}) \) is the set of \( S\text{-CGT} \) \( T = \langle T, \varepsilon, \text{Lab}', \tau' \rangle \) such that \( T \) is a prefix closed subset of the set of \( Unw(\mathcal{G}) \)-nodes and for all \( \nu \in T \) and \( d \in D_c \), \( \text{Lab}'(\nu) = \text{Lab}(\text{lst}(\nu)) \), and \( \tau'(\nu, d) = \nu \cdot \tau(\text{lst}(\nu), d) \) if \( \nu \cdot \tau(\text{lst}(\nu), d) \in T \), and \( \tau'(\nu, d) = \bot \) otherwise, where \( \text{lst}(\varepsilon) = s_0 \). Moreover, for all \( \nu \in T \), the following holds:

- if \( \text{lst}(\nu) \) is a system state, then for each successor \( s \) of \( \text{lst}(\nu) \) in \( \mathcal{G} \), \( \nu \cdot s \in T \);
- if \( \text{lst}(\nu) \) is an environment state, then there is a nonempty subset \( \{ s_1, \ldots, s_n \} \) of the set of \( \text{lst}(\nu) \)-successors such that the set of children of \( \nu \) in \( T \) is \( \{ \nu \cdot s_1, \ldots, \nu \cdot s_n \} \).

Intuitively, when \( \mathcal{G} \) is in a system state \( s \), then all the transitions from \( s \) are enabled. When \( \mathcal{G} \) is instead in an environment state, the set of enabled transitions from \( s \) depend on the current environment. Since the behavior of the environment is nondeterministic, we have to consider all the possible subsets of the set of \( s \)-successors. The only constraint, since we consider environments that cannot block the system, is that not all the transitions from \( s \) can be disabled. Note that \( Unw(\mathcal{G}) \in \text{exec}(\mathcal{G}) \) \( (Unw(\mathcal{G}) \) corresponds to the maximal environment that never restricts the set of its next states).

It is worth noting that the choices made by the environment along an environment strategy tree describe a strategy of the environment which is nondeterministic. This is in contrast with the given notion of strategy for a coalition \( A \) of agents which is instead deterministic (at each round, the coalition \( A \) selects exactly one \( A \)-decision available at the current state).

For an open CGS \( \mathcal{G} \) and an ATL* formula \( \varphi \), \( \mathcal{G} \) **reactively satisfies** \( \varphi \), denoted \( \mathcal{G} \models^r \varphi \), if for all environment strategy trees \( T \in \text{exec}(\mathcal{G}) \), \( T \models \varphi \). Note that \( \mathcal{G} \models^r \varphi \) implies \( \mathcal{G} \models \varphi \) (since \( Unw(\mathcal{G}) \in \text{exec}(\mathcal{G}) \)), but the converse in general does not hold. Moreover, \( \mathcal{G} \not\models^r \varphi \) is not equivalent to \( \mathcal{G} \not\models^r \neg \varphi \). Indeed, \( \mathcal{G} \not\models^r \varphi \) just states that there is some \( T \in \text{exec}(\mathcal{G}) \) satisfying \( \neg \varphi \).
Pushdown Module-checking. An open PMS is a PMS $S$ such that the induced CGS $G(S)$ is open. Note that for an open PMS, the property of a configuration of being an environment or system configuration depends only on the control state and the symbol on the top of the stack. The pushdown module-checking problem against ATL (resp., ATL$^*$) is checking for a given open PMS $S$ and an ATL formula (resp., ATL$^*$ state formula) $\varphi$ whether $G(S) \models_r \varphi$.

Example 2.5. Consider a coffee machine that allows customers (acting the role of the environment) to choose between the following actions:

- ordering and paying a black or white coffee (actions $b$ or $w$);
- the same as in the previous point but, additionally, paying a “suspended” coffee (a prepaid coffee) for the benefit of any unknown needy customer claiming it in the future (actions $b_+$ or $w_+$);
- asking for an available prepaid (black or white) coffee (actions $b_-$ or $w_-$).

The coffee machine is modeled by a turn-based open PMS $S_{cof}$ with three agents: the environment, the brewer $br$ whose function is to pour coffee into the cup (action $pour$), and the milk provider who can add milk (action $milk$). The two system agents can be faulty and ignore the request from the environment (action $ign$). The stack is exploited for keeping track of the number of prepaid coffees: a request for a prepaid coffee can be accepted only if the stack is not empty. After the completion of a request, the machine waits for further selections. The PMS $S_{cof}$ is represented as a graph in Figure 1 where each node (control state) is labeled by the propositions holding at it: the state labeled by $choice$ is controlled by the environment, the states labeled by $reqb$ or $reqw$ are controlled by the brewer $br$, while the state labeled by $milk$ is controlled by the milk provider. The notation $push(\gamma)$ denotes a push stack operation (pushing the symbol $\gamma \neq \gamma_0$), while $pop(\gamma)$ (resp., $pop(\gamma_0)$) denotes a pop operation onto a non-empty (resp., empty) stack. The set of propositions is $\{reqw, reqb, rej, black, white\}$.

In module checking, we can condition the property to be achieved on the behavior of the environment. For instance, users who never order white coffee and whose request is never rejected can be served by the brewer alone: $G(S_{cof}) \models_r AG(\neg reqw \land \neg rej) \rightarrow \langle\langle br\rangle\rangle F black$. In model checking, the same formula does not express any interesting property since $G(S_{cof}) \not\models AG(\neg reqw \land \neg rej)$. Likewise $G(S_{cof}) \models AG \neg reqw \rightarrow \langle\langle br\rangle\rangle F black$, whereas module checking gives a different and more intuitive answer: $G(S_{cof}) \not\models^r AG \neg reqw \rightarrow \langle\langle br\rangle\rangle F black$ (there are environments where requests for a prepaid coffee are always rejected).
3. Decision procedures

In this section, we provide an automata-theoretic framework for solving the pushdown module-checking problem against ATL and ATL* which is based on the use of parity alternating automata for CGS (parity ACG) [SF06] and parity Nondeterministic Pushdown Tree Automata (parity NPTA) [KPV02]. The proposed approach (which is proved to be asymptotically optimal in Section 4) consists of two steps. For the given open PMS $S$ and ATL formula (resp., ATL* state formula) $\varphi$, by exploiting known results, we first build in linear-time (resp., double exponential time) a parity ACG $A_{\neg \varphi}$ accepting the set of CGT which satisfy $\neg \varphi$. Then in the second step, we show how to construct a parity NPTA $P$ accepting suitable encodings of the environment strategy trees of $G(S)$ accepted by $A_{\neg \varphi}$. Hence, $G(S) \models^* \varphi$ iff the language accepted by $P$ is empty.

In the following, we first recall the frameworks of parity NPTA and parity ACG, and the known translations of ATL* and ATL formulas into equivalent parity ACG. Then, in Subsection 3.1, by exploiting parity NPTA, we show that given an open PMS $S$ and a parity ACG $A$, checking that no environment strategy tree of $G(S)$ is accepted by $A$ can be done in time double exponential in the size of $A$ and singly exponential in the size of $S$.

Parity NPTA [KPV02]. Here, we describe parity NPTA (without $\varepsilon$-transitions) over labeled complete $k$-ary trees for a given $k \geq 1$, which are tuples $P = (\Sigma, Q, \Gamma \cup \{\gamma_0\}, q_0, \rho, \Omega)$, where $\Sigma$ is a finite input alphabet, $Q$ is a finite set of (control) states, $\Gamma \cup \{\gamma_0\}$ is a finite stack alphabet ($\gamma_0$ is the special bottom symbol), $q_0 \in Q$ is an initial state, $\rho : Q \times \Sigma \times (\Gamma \cup \{\gamma_0\}) \to 2^{(Q \times \Gamma)^k}$ is a transition function, and $\Omega : Q \to \mathbb{N}$ is a parity acceptance condition over $Q$ assigning to each state a natural number called color. The index of $P$ is the number of colors in $\Omega$, i.e., the cardinality of $\Omega(Q)$.

Intuitively, when the automaton is in state $q$, reading an input node $x$ labeled by $\sigma \in \Sigma$, and the stack contains a word $\gamma \cdot \beta$ in $\Gamma^* \cdot \gamma_0$, then the automaton chooses a tuple $((q_1, \beta_1), \ldots, (q_k, \beta_k)) \in \rho(q, \sigma, \gamma)$ and splits in $k$ copies such that for each $1 \leq i \leq k$, a copy in state $q_i$, and stack content obtained by removing $\gamma$ and pushing $\beta_i$, is sent to the node $x \cdot i$ in the input tree.

Formally, a run of the NPTA $P$ on a $\Sigma$-labeled complete $k$-ary tree $(T, \text{Lab})$ (with $T = \{1, \ldots, k\}^*$) is a $(Q \times \Gamma^* \cdot \gamma_0)$-labeled tree $r = (T, \text{Lab}_r)$ such that $\text{Lab}_r(\varepsilon) = (q_0, \gamma_0)$ (initially, the stack contains just the bottom symbol $\gamma_0$) and for each $x \in T$ with $\text{Lab}_r(x) = (q, \gamma \cdot \beta)$, there is $((q_1, \beta_1), \ldots, (q_k, \beta_k)) \in \rho(q, \text{Lab}(x), \gamma)$ such that for all $1 \leq i \leq k$, $\text{Lab}_r(x \cdot i) = (q_i, \beta_i \cdot \beta)$ if $\gamma \neq \gamma_0$, and $\text{Lab}_r(x \cdot i) = (q_i, \beta_i \cdot \gamma_0)$ otherwise (note that in this case $\beta = \varepsilon$). The run $r = (T, \text{Lab}_r)$ is accepting if for all infinite paths $\pi$ starting from the root, the highest color $\Omega(q)$ of the states $q$ appearing infinitely often along $\text{Lab}_r(\pi)$ is even. The language $L(P)$ accepted by $P$ consists of the $\Sigma$-labeled complete $k$-ary trees $(T, \text{Lab})$ such that there is an accepting run of $P$ over $(T, \text{Lab})$.

For complexity analysis, we consider the following two parameters: the size $|\rho|$ of $\rho$ given by $|\rho| = \sum_{((q_1, \beta_1), \ldots, (q_k, \beta_k)) \in \rho(q, \sigma, \gamma)} |\beta_1| + \ldots + |\beta_k|$ and the smaller parameter $||\rho||$ given by $||\rho|| = \sum_{\beta \in \rho_0} |\beta|$ where $\rho_0$ is the set of words $\beta \in \Gamma^* \cdot \gamma_0$ occurring in $\rho$. It is well-known [KPV02] that emptiness of parity NPTA can be solved in single exponential time by a polynomial time reduction to emptiness of standard two-way alternating tree automata [Var98]. In particular, the following holds (see [KPV02, BMP10]).

Proposition 3.1. [KPV02, BMP10] The emptiness problem for a parity NPTA of index $m$ with $n$ states and transition function $\rho$ can be solved in time $O(|\rho| \cdot 2^{O(|\rho|^2 \cdot n^2 \cdot m^2 \cdot \log m)})$. 
The size $|A|$ of $A$ is $|Q| + |\text{Atoms}(A)|$, where $\text{Atoms}(A)$ is the set of atoms of $A$, i.e. the set of tuples in $Q \times \{\Box, \Diamond\} \times 2^Ag$ occurring in the transition function $\delta$.

We interpret the parity ACG $A$ over CGT. Given a CGT $T = (T, \varepsilon, Lab, \tau)$ over $AP$ and $Ag$, a run of $A$ over the input $T$ is a $(Q \times T)$-labeled $\mathbb{N}$-tree $r = (T_r, Lab_r)$, where each node of $T_r$ labelled by $(q, \nu)$ describes a copy of the automaton that is in state $q$ and reads the node $\nu$ of $T$. Moreover, we require that $Lab_r(\varepsilon) = (q_0, \varepsilon)$ (initially, the automaton is in state $q_0$ reading the root node of the input $T$), and for each $y \in T_r$ with $Lab_r(y) = (q, \nu)$, there is a set $H \subseteq Q \times \{\Box, \Diamond\} \times 2^Ag$ such that $H$ is a model of $\delta(q, Lab(\nu))$ and the set $L$ of labels associated with the children of $y$ in $T_r$ satisfies the following conditions:

- for all universal atoms $(q', \Box, A) \in H$, there is an available $A$-decision $d_A$ in the node $\nu$ of $T$ such that for all the children $\nu'$ of $\nu$ which are consistent with $d_A$, $(q', \nu') \in L$;
- for all existential atoms $(q', \Diamond, A) \in H$ and for all available $A$-decisions $d_A$ in the node $\nu$ of $T$, there is some child $\nu'$ of $\nu$ which is consistent with $d_A$ such that $(q', \nu') \in L$.

The run $r$ is accepting if for all infinite paths $\pi$ starting from the root, the highest color of the states appearing infinitely often along $Lab_r(\pi)$ is even. The language $L(A)$ accepted by $A$ consists of the CGT $T$ on $AP$ and $Ag$ such that there is an accepting run of $A$ over $T$.

**From ATL* and ATL to parity ACG.** In the following we shall exploit a known translation of ATL* state formulas (resp., ATL formulas) into equivalent parity ACG which has been provided in [BM17]. To this end we recall that, for a finite set $B$ disjunct from $AP$ and a CGT $T = (T, \varepsilon, Lab, \tau)$ over $AP$, a $B$-labeling extension of $T$ is a CGT over $AP \cup B$ of the form $(T, \varepsilon, Lab^B(\cdot), \tau)$, where $Lab^B(\nu) \cap AP = Lab(\nu)$ for all $\nu \in T$. A basic formula of ATL* is a state formula of ATL* having the form $\langle (A) \rangle \varphi$. The result exploited in the following is summarized as follows.

**Theorem 3.2.** [BM17] For an ATL* state formula (resp., ATL formula) $\varphi$ over $AP$, one can construct in doubly exponential time (resp., linear time) a parity ACG $A^\varphi$ over $2AP \cup B^\varphi$, where $B^\varphi$ is the set of basic subformulas of $\varphi$, such that for all CGT $T$ over $AP$, $T$ is a model of $\varphi$ iff there exists a $B^\varphi$-labeling extension of $T$ which is accepted by $A^\varphi$. Moreover, $A^\varphi$ has size $O(2^{|\varphi| \log(|\varphi|)})$ and index $2^{|\varphi|}$ (resp., size $O(|\varphi|)$ and index 2).
Note that while the well-known translation of CTL* formulas into alternating automata involves just a single exponential blow-up, by Theorem 3.2, the translation of ATL* formulas in alternating automata for CGS entails a double exponential blow-up. This seems in contrast with the automata-theoretic approach used in [Sch08] for solving satisfiability of ATL* (recall that ATL* satisfiability has the same complexity as CTL* satisfiability, i.e., it is $2\text{EXPTIME}$-complete [Sch08]). In particular, given an ATL* state formula $\varphi$, one can construct in singly exponential time a parity ACG accepting the set of CGT satisfying some special requirements which provide a necessary and sufficient condition for ensuring the existence of some model of $\varphi$ [Sch08]. These requirements are based on an equivalent representation of the models of a formula obtained by a sort of widening operation. However, when applied to the environment strategy trees of a CGS, such an encoding is not regular since one has to require that for all nodes in the encoding which are copies of the same environment node in the given environment strategy tree, the associated subtrees are isomorphic. Hence, the approach used in [Sch08] cannot be applied to the module-checking setting.

### 3.1. Upper bounds for ATL and ATL* pushdown module-checking

Let $S$ be an open PMS, $\varphi$ an ATL* (resp., ATL) formula, and $A_{\neg \varphi}$ the parity ACG over $2^{AP\cup B_\varphi}$ ($B_\varphi$ is the set of basic subformulas of $\varphi$) of Theorem 3.2 associated with the negation of $\varphi$. By Theorem 3.2, checking that $G(S) \models^r \varphi$ reduces to checking that there are no $B_\varphi$-labeling extensions of the environment strategy trees of $G(S)$ accepted by $A_{\neg \varphi}$. In this section, we provide an algorithm for checking this last condition. In particular, we establish the following result.

**Theorem 3.3.** Given an open PMS $S$ over $AP$, a finite set $B$ disjoint from $AP$, and a parity ACG $A$ over $2^{AP\cup B}$, checking that there are no $B$-labeling extensions of the environment strategy trees of $G(S)$ accepted by $A$ can be done in time doubly exponential in the size of $A$ and singly exponential in the size of $S$.

Thus, by Theorem 3.2 and Theorem 3.3, and since the pushdown module-checking problem against CTL is already $2\text{EXPTIME}$-complete, and $\text{EXPTIME}$-complete for a fixed CTL formula [BMP10], we obtain the following corollary.

**Corollary 3.4.** Pushdown module-checking for ATL* is in $4\text{EXPTIME}$ while pushdown module-checking for ATL is $2\text{EXPTIME}$-complete. Moreover, for a fixed ATL* state formula (resp., ATL formula), the pushdown module-checking problem is $\text{EXPTIME}$-complete.

In Section 4, we provide a lower bound for ATL* matching the upper bound in the corollary above. We present now the proof of Theorem 3.3 which is based on a reduction to the emptiness problem of parity NPTA. Given an open PMS $S$ over $AP$ and a parity ACG $A$ over $2^{AP\cup B}$, we construct in single exponential time a parity NPTA $P$ over $2^{AP\cup B}$ accepting the $B$-labeling extensions of suitable encodings of the environment strategy trees of $G(S)$ which are accepted by $A$. Since the set $B$ just occurs in the input alphabet $2^{AP\cup B}$ and the behaviour of $G(S)$ does not depend on $B$, for simplicity and without loss of generality, we assume that the set $B$ in the statement of Theorem 3.3 is empty.
Encoding of environment strategy trees of open PMS. Let us fix an open PMS $\mathcal{S} = (Q, \Gamma \cup \{\gamma_0\}, q_0, Lab, \Delta)$ over $AP$, and let $G(\mathcal{S}) = (S, s_0, Lab_S, \tau)$. For all pairs $(q, \gamma) \in Q \times (\Gamma \cup \{\gamma_0\})$, we denote by $next_S(q, \gamma)$ the finite set of pairs $(q', \beta) \in Q \times \Gamma^*$ such that there is a full decision $d$ so that $\Delta(q, \gamma, d) = (q', \beta)$. We fix an ordering on the set $next_S(q, \gamma)$ which induces an ordering on the finite set of successors of all the configurations of the form $(q, \gamma, \alpha)$. Moreover, we consider the parameter $k_S = \max\{|next_S(q, \gamma)| \mid (q, \gamma) \in Q \times (\Gamma \cup \{\gamma_0\})\}$ which represents the finite branching degree of $Unw(G(\mathcal{S}))$. Thus, we can encode each track $\nu = s_0, s_1, \ldots, s_n$ of $G(S)$ starting from the initial state, by the finite word $i_1, \ldots, i_n$ over $\{1, \ldots, k_S\}$ of length $n$ where for all $1 \leq h \leq n$, $i_h$ represents the index of state $s_h$ in the ordered set of successors of state $s_{h-1}$. Now, we observe that the transition function $\tau'$ of an environment strategy tree $T = (T, \epsilon, Lab', \tau')$ of $G(S)$ is completely determined by $T$ and the transition function $\tau$ of $G(S)$. Hence, for the fixed open CGS $G(\mathcal{S})$, $T$ can be simply specified by the underlying $2^{AP}$-labeled $T$-tree $(T, Lab')$.

We consider an equivalent representation of $(T, Lab')$ by a $(2^{AP} \cup \{\bot\})$-labeled complete $k_S$-tree $\langle\{1, \ldots, k_S\}^*, Lab_\bot\rangle$, called the $\bot$-completion encoding of $T$ ($\bot$ is a fresh proposition), where the labeling $Lab_\bot$ is defined as follows for each node $x \in \{1, \ldots, k_S\}$:

- if $x$ encodes a track $s_0 \cdot \nu$ such that $\nu$ is a node of $T$, then $Lab_\bot(x) = Lab(\nu)$ (concrete nodes);
- otherwise, $Lab_\bot(x) = \{\bot\}$ (completion nodes).

In this way, all the labeled trees encoding environment strategy trees $T$ of $G(S)$ have the same structure (they all coincide with $\{1, \ldots, k_S\}^*$), and they differ only in their labeling. Thus, the proposition $\bot$ is used to denote both “completion” nodes and nodes in $Unw(G(S))$ which are absent in $T$ (corresponding to possible disabling of environment choices).

Proof of Theorem 3.3. We show the following result which, together with Proposition 3.1, provides a proof of Theorem 3.3 (for the case $B = \emptyset$).

Theorem 3.5. Given an open PMS $\mathcal{S} = (Q, \Gamma \cup \{\gamma_0\}, q_0, Lab, \Delta)$ over $AP$ and a parity ACG $A = (Q_A, \delta_A, \delta, \Omega)$ over $2^{AP}$ with index $h$, one can build in single exponential time, a parity NPTA $\mathcal{P}$ accepting the set of $2^{AP} \cup \{\bot\}$-labeled complete $k_S$-trees which are the $\bot$-completion encodings of the environment strategy trees of $G(S)$ which are accepted by $A$. Moreover, $\mathcal{P}$ has index $O(h|A|^2)$, number of states $O(|Q| \cdot (h|A|^2)^{O(h|A|^2)})$, and transition function $\rho$ such that $||\rho|| = O(|\Delta| \cdot (h|A|^2)^{O(h|A|^2)})$.

Proof. First, we observe that for the given parity ACG $A$ and an input CGT $T$, we can associate in a standard way to $A$ and $T$ an infinite-state two player parity game, where player 0 plays for acceptance, while player 1 plays for rejection. Winning strategies of player 0 correspond to accepting runs of $A$ over $T$. Thus, since the existence of a winning strategy in parity games implies the existence of a memoryless one, we can restrict ourselves to consider only memoryless runs of $A$, i.e. runs $r = \langle T_r, Lab_r \rangle$ where the behavior of $A$ along $r$ depends only on the current input node and current state. Formally, $r$ is memoryless if for all nodes $y$ and $y'$ of $r$ having the same label, the subtrees rooted at the nodes $y$ and $y'$ of $r$ are isomorphic. We now provide a representation of the memoryless runs of $A$ over the environment strategy trees of the open CGS $G(S)$ induced by the given open PMS $\mathcal{S}$.

Fix an environment strategy tree $T = (T, \epsilon, Lab_T, \tau)$ of $G(S)$ and let $\langle\{1, \ldots, k_S\}^*, Lab_\bot\rangle$ be the $\bot$-completion encoding of $T$. Recall that $Atoms(A)$ is the set of atoms of $A$, i.e., the set of tuples in $Q_A \times \{\square, \Diamond\} \times 2^{AP}$ occurring in the transition function $\delta$ of $A$. 

Let $Ann = 2^{QA \times \text{Atoms}(A)}$ be the finite set of annotations and $\Upsilon = (2^{AP} \times Ann \times Ann) \cup \{\bot\}$. For an annotation $an \in Ann$, we denote by $\text{Dom}(an)$ the set of $A$-states $q$ such that $(q, \text{atom}) \in an$ for some atom $\text{atom} \in \text{Atoms}(A)$. Moreover, we denote by $\text{Cod}(an)$ the set of $A$-states occurring in the atoms of $an$. For example, if $an = \{(q_1, (q_1', \Diamond, A_1)), (q_2, (q_2', \Box, A_2))\}$, then $\text{Dom}(an) = \{q_1, q_2\}$ and $\text{Cod}(an) = \{q_1', q_2'\}$.

We represent memoryless runs $r$ of $A$ over $T$ as annotated extensions of the $\bot$-completion encoding $\langle\{1, \ldots, k_S\}^*, \text{Lab}_\bot\rangle$ of $T$, i.e., $\Upsilon$-labeled complete $k_S$-trees $\langle\{1, \ldots, k_S\}^*, \text{Lab}_\Upsilon\rangle$, where for every concrete node $x \in \{1, \ldots, k_S\}$ encoding a node $\nu_x$ of $T$, $\text{Lab}_\Upsilon(x)$ is of the form $\langle\text{Lab}_\bot(x), an, an'\rangle$ (recall that $\text{Lab}_\bot(x) = \text{Lab}_\Upsilon(\nu_x)$), and for every completion node $x$, $\text{Lab}_\Upsilon(x) = \text{Lab}_\bot(x) = \{\bot\}$. Intuitively, the meaning of the first annotation $an$ and the second annotation $an'$ in the label of a concrete node $x$ is as follows:

- $\text{Dom}(an)$ represents the set of $A$-states $q$ associated with the copies of $A$ in the run $r$ which read the input node $\nu_x$ of $T$, while for each $q \in \text{Dom}(an)$, the set of atoms $\text{atom}$ such that $(q, \text{atom}) \in an$ represents the model of $\delta(q, \text{Lab}_\Upsilon(\nu_x))$ selected by $A$ in $r$ on reading node $\nu_x$ in state $q$. Note that $\text{Cod}(an)$ represents the set of target states of the moves in $an$.

- Additionally, the second annotation $an'$ in the labeling of node $x$ keeps tracks, in case $x$ is not the root, of the subset of the moves in the first annotation of the parent $\nu'$ of $\nu_x$ in $T$ for which, starting from $\nu'$, a copy of $A$ is sent to the current node $\nu_x$ along $r$.

Moreover, we require that the two annotations $an$ and $an'$ are consistent, i.e., $an' = \emptyset$ if $x$ is the root and $\text{Cod}(an') = \text{Dom}(an)$ otherwise. An annotated extension $\langle\{1, \ldots, k_S\}^*, \text{Lab}_\Upsilon\rangle$ of $\langle\{1, \ldots, k_S\}^*, \text{Lab}_\bot\rangle$ is well-formed if it satisfies the local requirements informally expressed above. We deduce the following result.

Claim 1. One can construct in singly exponential time a parity NPTA $P_{wf}$ over $\Upsilon$-labeled complete $k_S$-trees accepting the set of well-formed annotated extensions of the $\bot$-completion encodings of the environment strategy trees of $G(S)$. Moreover, $P_{wf}$ has number of states $O(|Q| \cdot 2^{O(|QA| \cdot |\text{Atoms}(A)|)})$, index 1, and transition function $\rho$ such that $||\rho|| = O(|\Delta|)$.

The proof of Claim 1 is postponed at the end of the proof of Theorem 3.5.

Note that the well-formedness requirement just ensures that the annotated extension $\langle\{1, \ldots, k_S\}^*, \text{Lab}_\Upsilon\rangle$ of $\langle\{1, \ldots, k_S\}^*, \text{Lab}_\bot\rangle$ encodes a memoryless run $r$ of the ACG $A$ over the input $T$. In order to ensure that $\langle\{1, \ldots, k_S\}^*, \text{Lab}_\Upsilon\rangle$ encodes a run $r$ which is also accepting, we need to enforce additional global requirements on the annotated extension $\langle\{1, \ldots, k_S\}^*, \text{Lab}_\Upsilon\rangle$.

Let $\pi$ be an infinite path of $\langle\{1, \ldots, k_S\}^*, \text{Lab}_\Upsilon\rangle$ from the root which does not visit $\bot$-labeled nodes. Then, $\text{Lab}_\Upsilon(\pi)$ “collects” all the infinite sequences $\nu$ of states in $Q_A$ along the run $r$ associated with the input path of the environment strategy tree $T$ encoded by $\pi$. In order to check the acceptance condition on the individual parallel paths $\nu$, the infinite sequence of annotations $\text{Lab}_\Upsilon(\pi)$ must allow to distinguish the individual infinite paths over $Q_A$ grouped by $\text{Lab}_\Upsilon(\pi)$. This is because we exploit the second annotation $an'$ in the labeling $(\text{Lab}_\bot(x), an, an')$ of a concrete node $x$. In particular, the individual paths over $Q_A$ grouped by $\text{Lab}_\Upsilon(\pi)$ correspond to the so-called $Q_A$-paths of $\text{Lab}_\Upsilon(\pi)$ which are defined as follows.

For all $i \geq 0$, let $\text{Lab}_\Upsilon(\pi(i)) = (\sigma_i, an_i, an'_i)$. Then, a $Q_A$-path of $\text{Lab}_\Upsilon(\pi)$ is an infinite sequence $q_0q_1\ldots$ of $Q_A$-states such that for all $i \geq 0$, $q_i \in \text{Dom}(an_i)$ and $(q_i, (q_{i+1}, m, A)) \in an_i \cap an'_{i+1}$ for some $m \in \{\Box, \Diamond\}$ and set $A$ of agents.
We need to check that all these $Q_\mathcal{A}$-paths satisfy the acceptance parity condition of $\mathcal{A}$. To this end, we construct a standard parity nondeterministic tree automaton (parity NTA) $\mathcal{A}_{acc}$ over $\Upsilon$-labeled complete $k_S$-trees which accepts an input tree if all the $Q_\mathcal{A}$-paths associated with the infinite paths of the input tree starting at the root satisfy the acceptance parity condition of $\mathcal{A}$. In order to construct $\mathcal{A}_{acc}$, we proceed as follows.

We first easily construct a co-parity nondeterministic word automaton $\mathcal{B}$ over $\Upsilon$ with $O(|Q_\mathcal{A}| \cdot |\text{Atoms}(\mathcal{A})|)$ states and index $h$ (the index of $\mathcal{A}$) which accepts an infinite word over $\Upsilon$ if it contains a $Q_\mathcal{A}$-path that does not satisfy the parity acceptance condition of $\mathcal{A}$. We now co-determinize $\mathcal{B}$, i.e., determine it and complement it in a singly-exponential construction [Saf88] to obtain a deterministic parity word automaton $\mathcal{B}'$ that rejects violating $Q_\mathcal{A}$-paths. By [Saf88], $\mathcal{B}'$ has $(nh)^O(nh)$ states and index $O(nh)$, where $n = |Q_\mathcal{A}| \cdot |\text{Atoms}(\mathcal{A})|$. Then the parity NTA $\mathcal{A}_{acc}$ is obtained from $\mathcal{B}'$ by simply running $\mathcal{B}'$ in parallel over all the branches of the input which do not visit a $\bot$-labeled node. Note that $\mathcal{A}_{acc}$ has $(nh)^O(nh)$ states and index $O(nh)$.

Then, the parity NPTA $\mathcal{P}$ satisfying Theorem 3.5 is obtained by projecting out the annotation components of the input trees accepted by the intersection of the NPTA $\mathcal{P}_{wf}$ of index 1 in Claim 1 with the parity NTA $\mathcal{A}_{acc}$ (recall that parity NPTA are effectively and polynomial-time closed under projection and intersection with nondeterministic tree automata [KPV02]).

In order to conclude the proof of Theorem 3.5 it remains to prove Claim 1.

**Proof of Claim 1.** In order to define the NPTA $\mathcal{P}_{wf}$ satisfying Claim 1, we need additional definitions. Recall that for an annotation $an \in \text{Ann}$, $\text{Dom}(an)$ denotes the set of $\mathcal{A}$-states $q$ such that $(q, \text{atom}) \in an$ for some atom $\text{atom} \in \text{Atoms}(\mathcal{A})$, while $\text{Cod}(an)$ denotes the set of states occurring in the atoms of $an$. Moreover, for each state $q \in Q_\mathcal{A}$, we denote by $\text{Atoms}(q, an)$ the set of atoms $\text{atom}$ such that $(q, \text{atom}) \in an$.

Let $(q, \gamma) \in Q \times (\Gamma \cup \{\gamma_0\})$ with $\text{next}_S(q, \gamma) = \{(q_1, \beta_1), \ldots, (q_k, \beta_k)\}$ for some $1 \leq k \leq k_S$. For a move $\eta = (p, (p', m, A)) \in Q_\mathcal{A} \times \text{Atoms}(\mathcal{A})$ and a non-empty subset $X$ of $\text{next}_S(q, \gamma)$, we say that $X$ is consistent with the move $\eta$ if the following holds:

- case $m = \square$: there is an $\mathcal{A}$-decision $d_A$ such that $X$ coincides with the set of pairs $\Delta(q, \gamma, d)$ where $d$ is a full decision consistent with $d_A$;
- case $m = \triangledown$: there is a surjective function $f : \mathcal{A}^U \mapsto X$ such that for each $\mathcal{A}$-decision $d_A$, $f(d_A) = \Delta(q, \gamma, d)$ for some full decision $d$ consistent with $d_A$.

For an annotation $an$ and a tuple $(an_1, \ldots, an_k)$ of $k$ annotations, we say that $(an_1, \ldots, an_k)$ is consistent with annotation $an$ and the pair $(q, \gamma)$, with $\text{next}_S(q, \gamma) = \{(q_1, \beta_1), \ldots, (q_k, \beta_k)\}$, if the following holds:

- $an = \bigcup_{i=1}^k an_i$;
- for each move $\eta = (p, (p', m, A)) \in an$, let $X_\eta$ be the subset of $\{(q_1, \beta_1), \ldots, (q_k, \beta_k)\}$ consisting of the pairs $(q_i, \beta_i)$ such that $\eta \in an_i$. Then, $X_\eta$ is consistent with the move $\eta$.

We denote by $\text{Cons}(q, \gamma, an)$ the set of tuples $(an_1, \ldots, an_k)$ of $k$ annotations which are consistent with the annotation $an$ and the pair $(q, \gamma)$.

We now define the parity NPTA $\mathcal{P}_{wf}$ of index 1 satisfying Claim 1. Essentially, given a $\Upsilon$-labeled complete $k_S$-tree $\{1, \ldots, k_S\}^*, \text{Lab}_\Upsilon$, the automaton $\mathcal{P}_{wf}$, by simulating the behaviour of the open PMS $\mathcal{S}$ and by exploiting the transition function of the parity ACG $\mathcal{A}$, checks that the input is a well-formed annotated extension of the $\bot$-completion encoding of
some environment strategy tree of \( G(S) \). Formally, the NPTA \( P_{\text{wf}} = \langle \Upsilon, P, \Gamma \cup \{ \gamma_0 \}, p_0, \rho, \Omega : p \in P \mapsto \{ 0 \} \rangle \) is defined as follows.

The set \( P \) of states consists of the triples \((q, an, m)\) where \( q \in Q \) is a state of the PMS \( S \), \( an \in \text{Ann} \) is an annotation, and \( m \in \{ \bot, \top, \vdash \} \) is a state marker such that \( an = \emptyset \) if \( m = \bot \). When the state marker \( m \) is \( \bot \), the NPTA \( P_{\text{wf}} \) can read only the letter \( \bot \), while when the state marker is \( \top \), \( P_{\text{wf}} \) can read only letters in \( \Upsilon \setminus \{ \bot \} \). Finally, when \( P_{\text{wf}} \) is in states of the form \((q, an, \vdash)\), then it can read both letters in \( \Upsilon \setminus \{ \bot \} \) and the letter \( \bot \). In this case, it is left to the environment to decide whether the transition to a configuration of the simulated PMS \( S \) of the form \((q, \beta)\) is enabled. Intuitively, the three types of states are used to ensure that the environment enables all transitions from enabled system configurations, enables at least one transition from each enabled environment configuration, and disables transitions from disabled configurations. Moreover, the annotation \( an \) in a control state \((q, an, m)\) of \( P_{\text{wf}} \) represents the guessed subset of the moves in the first annotation of the parent \( x' \) (if any) of the current concrete input node for which, starting from \( x' \), a copy of \( A \) is sent to the current input node (in the transition function, we require that in case the current input symbol \( \sigma \) is not \( \bot \), \( an \) coincides with the second annotation of \( \sigma \)).

The initial state \( p_0 \) is given by \((q_0, \emptyset, \top)\). Finally, the transition function \( \rho : P \times \Upsilon \times (\Gamma \cup \{ \gamma_0 \}) \rightarrow 2^{(P \times \Gamma)^{k_S}} \) is defined as follows. According to the definition of \( P \), the automaton \( P_{\text{wf}} \) can be in a state of the form \((q, \emptyset, \bot)\), \((q, an, \top)\), or \((q, an, \vdash)\). Both in the first and the third cases, \( P_{\text{wf}} \) can read \( \bot \), which means that the automaton is reading a disabled or a completion node. Thus, independently from the fact that the actual configuration of the automaton is associated with an environment or a system configuration of the open PMS \( S \), \( \rho \) propagates states of the form \((q, \emptyset, \bot)\) to all children of the reading node. In case the automaton is in a state of the form \((q, an, \top)\) or \((q, an, \vdash)\) and reads a label different from \( \bot \), the possible successor states further depend on the particular kind of the configuration in which the automaton is. If \( P_{\text{wf}} \) is in a system configuration of \( S \), then all the children of the reading node associated with the successors of such a configuration in the CGS \( G(S) \) must not be disabled and so, \( \rho \) sends to all of them states with marker \( \top \). If \( P_{\text{wf}} \) is in an environment configuration of \( S \), then all the children of the reading node, but one, associated with the successors of such a configuration in the CGS \( G(S) \) may be disabled and so, \( \rho \) sends to all of them states with marker \( \vdash \), except one, to which \( \rho \) sends a state with marker \( \top \). Formally, let \((q, an, m) \in P\), \( \sigma \in \Upsilon \), and \( \gamma \in \Gamma \cup \{ \gamma_0 \} \) with \( \text{next}_G(q, \gamma) = ((q_1, \beta_1), \ldots, (q_k, \beta_k)) \) \((1 \leq k \leq k_S)\). Then, \( \rho((q, an, m), \sigma, \gamma) \) is defined as follows:

- Case \( m \in \{ \bot, \vdash \} \), \( \sigma = \bot \), and \( an = \emptyset \):
  \[
  \rho((q, \emptyset, m), \bot, \gamma) = \{ (\langle (q, \emptyset, \bot), \varepsilon \rangle, \ldots, \langle (q, \emptyset, \bot), \varepsilon \rangle) \}_{k_S \text{ pairs}}
  \]
  That is, \( \rho((p, m), \bot, A) \) contains exactly one \( k_S \)-tuple. In this case all the successors of the current \( S \)-configuration are disabled.

- Case \( m \in \{ \top, \vdash \} \), \( (q, \gamma) \) is associated with system \( S \)-configurations, \( \sigma = (\text{Lab}(q), an', an) \) for some annotation \( an' \) such that \( \text{Cod}(an') = \text{Dom}(an') \), and for each \( q_A \in \text{Dom}(an') \),
Atoms\((q_A, an')\) is a model of \(\delta(q_A, Lab(q))\):
\[
\rho((q, an, m), \sigma, \gamma) = \bigcup_{\langle an_1, \ldots, an_k \rangle \in Cons(q, \gamma, an')} \{ \langle ((q_1, an_1, \top), \beta_1), \ldots, ((q_k, an_k, \top), \beta_k) \rangle, ((q, \emptyset, \bot), \varepsilon), \ldots, ((q, \emptyset, \bot), \varepsilon) \} \}
\]

In this case, all the \(k\) successors of the current system \(S\)-configuration are enabled. Moreover, the automaton guesses a tuple \(\langle an_1, \ldots, an_k \rangle\) of \(k\) annotations which are consistent with the first annotation \(an'\) of the input node and the pair \((q, \gamma)\), and sends state \((q_i, an_i, \top)\) to the \(i\)th child of the current input node for all \(1 \leq i \leq k\).

- **Case** \(m \in \{\top, \bot\}\), \((q, \gamma)\) is associated with environment \(S\)-configurations, \(\sigma = (Lab(q), an', an)\) for some annotation \(an'\) such that \(Cod(an) = Dom(an')\), and for each \(q_A \in Dom(an')\), Atoms\((q_A, an')\) is a model of \(\delta(q_A, Lab(q))\): in this case \(\rho((q, an, m), \sigma, \gamma)\) is defined as follows
\[
\bigcup_{\langle an_1, \ldots, an_k \rangle \in Cons(q, \gamma, an')} \{ \langle ((q_1, an_1, \top), \beta_1), \ldots, ((q_k, an_k, \top), \beta_k) \rangle, ((q, \emptyset, \bot), \varepsilon), \ldots, ((q, \emptyset, \bot), \varepsilon) \} \}
\]

In this case, the automaton guesses a tuple \(\langle an_1, \ldots, an_k \rangle\) of \(k\) annotations which is consistent with the first annotation \(an'\) of the input node and the pair \((q, \gamma)\) and, additionally, guesses an index \(1 \leq i \leq k\). With these choices, the automaton sends state \((q_i, an_i, \top)\) to the \(i\)th child of the current input node and, additionally, ensures that the \(i\)th successor of the current environment \(S\)-configuration is enabled while all the other successors may be disabled.

- **All the other cases**: \(\rho((q, an, m), \sigma, \gamma) = \emptyset\).

Note that \(P_{\text{uf}}\) has \(O(|Q| \cdot 2^{O(|Q_A| \cdot |\text{Atoms}(A)|)})\) states, \(|\rho| = O(|\Delta|)\), and \(|\rho| = O(|\Delta| \cdot 2^{O(k_S \cdot |Q_A| \cdot |\text{Atoms}(A)|)})\). This concludes the proof of Claim 1.

\[\square\]

4. **4EXPTIME-hardness of ATL* pushdown module-checking**

In this section, we establish the following result.

**Theorem 4.1.** Pushdown module-checking against ATL* is 4EXPTIME-hard even for two-player turn-based PMS of fixed size.

Theorem 4.1 is proved by a polynomial-time reduction from the acceptance problem for 3EXPSPACE-bounded Alternating Turing Machines (ATM, for short) with a binary branching degree. Formally, such a machine is a tuple \(\mathcal{M} = (\Sigma, Q, Q_\exists, Q_\forall, q_0, \delta, F)\), where \(\Sigma\) is the input alphabet which contains the blank symbol \(\#\), \(Q\) is the finite set of states which is partitioned into \(Q = Q_\forall \cup Q_\exists\), \(Q_\exists\) (resp., \(Q_\forall\)) is the set of existential (resp., universal) states, \(q_0\) is the initial state, \(F \subseteq Q\) is the set of accepting states, and the transition function \(\delta\) is a mapping \(\delta : Q \times \Sigma \rightarrow (Q \times \Sigma \times \{\leftarrow, \rightarrow\})^2\). Note that since \(\mathcal{M}\) has a binary branching.
degree, the transition function $\delta$ nondeterministically associates to each pair state/input symbol $(q, \sigma)$ two possible moves, where each move is represented by a triple $(q', \sigma', d)$ consisting of a target state $q'$, the symbol $\sigma'$ to write in the tape cell currently pointed by the reading head, and a symbol $d \in \{\leftarrow, \rightarrow\}$ encoding the movement of the reading head: $\leftarrow$ (resp., $\rightarrow$) means that the reading head moves one cell to the left (resp., to the right) of the current cell.

Formally, configurations of $\mathcal{M}$ are words in $\Sigma^* \cdot (Q \times \Sigma) \cdot \Sigma^*$. A configuration $C = \eta \cdot (q, \sigma) \cdot \eta'$ denotes that the tape content is $\eta \cdot \sigma \cdot \eta'$, the current state (resp., current input symbol) is $q$ (resp., $\sigma$), and the reading head is at position $|\eta| + 1$. From a configuration $C$, the machine $\mathcal{M}$ nondeterministically chooses a triple $(q', \sigma', d)$ in $\delta(q, \sigma) = \{(q_1, \sigma_1, d_1), (q_r, \sigma_r, d_r)\}$, and then moves to state $q'$, writes $\sigma'$ in the current tape cell, and its reading head moves one cell to the left or to the right, according to $d$. We denote by $\text{succ}_l(C)$ and $\text{succ}_r(C)$ the successors of $C$ obtained by choosing respectively the left and the right triple in $\{(q_1, \sigma_1, d_1), (q_r, \sigma_r, d_r)\}$ (note that the terms ‘left’ and ‘right’ here should not be confused with the movement of the reading head of the ATM). The configuration $C$ is accepting (resp., universal, resp., existential) if the associated state $q$ is in $F$ (resp., in $Q_f$, resp., in $Q_\exists$).

Given an input $\alpha \in \Sigma^+$, a (finite) computation tree of $\mathcal{M}$ over $\alpha$ is a finite tree in which each node is labeled by a configuration. The root of the tree is labeled by the initial configuration associated with $\alpha$. An internal node that is labeled by a universal configuration $C$ has two children, corresponding to $\text{succ}_l(C)$ and $\text{succ}_r(C)$, while an internal node labeled by an existential configuration $C$ has a single child, corresponding to either $\text{succ}_l(C)$ or $\text{succ}_r(C)$. The tree is accepting if each leaf is labeled by an accepting configuration. An input $\alpha \in \Sigma^+$ is accepted by $\mathcal{M}$ if there is an accepting computation tree of $\mathcal{M}$ over $\alpha$.

If the ATM $\mathcal{M}$ is $3\text{EXPSPACE}$-bounded, then there is a constant $c \geq 1$ such that for each $\alpha \in \Sigma^+$, the space needed by $\mathcal{M}$ on input $\alpha$ is bounded by $\text{Tower}(\alpha^c, 3)$, where for all $n, h \in \mathbb{N}$, $\text{Tower}(n, h)$ denotes a tower of exponentials of height $h$ and argument $n$ (i.e., $\text{Tower}(n, 0) = n$ and $\text{Tower}(n, h + 1) = 2^{\text{Tower}(n, h)}$). It is well-known [CKS81] that the acceptance problem for $3\text{EXPSPACE}$–bounded ATM (with a binary branching degree) is $4\text{EXPTIME}$-complete even if the ATM is assumed to be of fixed size.

Fix a $3\text{EXPSPACE}$–bounded ATM $\mathcal{M}$ and an input $\alpha \in \Sigma^+$. Let $n = |\alpha|$. W.l.o.g. we assume that the constant $c$ is $1$ and $n > 1$. Hence, any reachable configuration of $\mathcal{M}$ over $\alpha$ can be seen as a word in $\Sigma^* \cdot (Q \times \Sigma) \cdot \Sigma^*$ of length exactly $\text{Tower}(n, 3)$, and the initial configuration is

$$(q_0, \alpha(0))\alpha(1) \ldots \alpha(n - 1) \cdot (\#)^t$$

where $t = \text{Tower}(n, 3) - n$. Note that for an ATM configuration $C = u_1 u_2 \ldots u_{\text{Tower}(n, 3)}$ and for all $i \in [1, \text{Tower}(n, 3)]$ and dir $\in \{l, r\}$, the value $u'_i$ of the $i$-th cell of $\text{succ}_{\text{dir}}(C)$ is completely determined by the values $u_{i-1}$, $u_i$ and $u_{i+1}$ (taking $u_{i+1}$ for $i = \text{Tower}(n, 3)$ and $u_{i-1}$ for $i = 1$ to be some special symbol, say $\dagger$). Thus, we denote by $next_{\text{dir}}(u_{i-1}, u_i, u_{i+1})$ the value $u'_i$ of the $i$-th cell of $\text{succ}_{\text{dir}}(C)$ (note that the function $next_{\text{dir}}$ can be trivially obtained from the transition function of $\mathcal{M}$). According to the previous observation, we use the set $\Lambda$ of triples of the form $(u_p, u, u_s)$ where $u \in \Sigma \cup (Q \times \Sigma)$, and $u_p, u_s \in \Sigma \cup (Q \times \Sigma) \cup \{\dagger\}$. We prove the following result from which Theorem 4.1 directly follows.

**Theorem 4.2.** One can construct, in time polynomial in $n$ and the size of $\mathcal{M}$, a turn-based PMS $S$ and an $\text{ATL}^*$ state formula $\varphi$ over the set of agents $Ag = \{sys, env\}$ such that $\mathcal{M}$ accepts $\alpha$ iff there is an environment strategy tree in $\text{exec}(\mathcal{G}(S))$ that satisfies $\varphi$ iff $\mathcal{G}(S) \not\vDash^{env} \neg \varphi$. Moreover, the size of $\mathcal{G}(S)$ depends only on the size of $\mathcal{M}$. 

The rest of this section is devoted to the proof of Theorem 4.2.

**Encoding of ATM configurations.** We first define an encoding of the ATM configurations by using the following set $Main$ of atomic propositions:

$$Main := \Lambda \cup \{0, 1, \forall, \exists, \ell, t, r, f\} \cup \{s_1, s_2, s_3, e_1, e_2, e_3\}.$$ 

In the encoding of an ATM configuration, for each ATM cell, we record the content of the cell, the location (cell number) of the cell on the ATM tape, and the contents of the previous and next cell (if any). In order to encode the cell number, which is a natural number in $[0, \text{Tower}(n, 3) - 1]$, for all $1 \leq h \leq 3$, we define the notions of $h$-block and well-formed $h$-block. For $h = 1, 2$, well-formed $h$-blocks encode integers in $[0, \text{Tower}(n, h) - 1]$, while well-formed 3-blocks encode the cells of ATM configurations. In particular, for $h = 2, 3$, a well-formed $h$-block encoding a natural number $m \in [0, \text{Tower}(n, h) - 1]$ is a sequence of $\text{Tower}(n, h - 1)$ well-formed $(h - 1)$-blocks, where the $i^{th}$ $(h - 1)$-block encodes both the value and (recursively) the position of the $i^{th}$-bit in the binary representation of $m$.

Formally, a 0-block is a word of length 1 of the form $\{b\}$ where $b \in \{0, 1\}$ ($b$ is the content of $\{b\}$).

For each $1 \leq h \leq 3$, an $h$-block $bl$ is a word of the form $\{s_h\} \cdot bl_0 \ldots bl_t \cdot \{\tau\} \cdot \{e_h\}$, where

- $t \geq 1$,
- $\tau \in \{0, 1\}$ if $h \neq 3$, and $\tau \in \Lambda$ otherwise ($\tau$ is the content of $bl$),
- and for all $0 \leq i \leq t$, $bl_i$ is an $(h - 1)$-block.

Note that the $h$-block $bl$ is enclosed by the start delimiter $s_h$ and the end delimiter $e_h$. We say that the $h$-block $bl$ is well-formed if the following additional condition hold:

- $t = \text{Tower}(n, h - 1) - 1$ and
- whenever $h > 1$, then the $(h - 1)$-block $bl_i$ is well-formed and has number $i$ for each $0 \leq i \leq t$.

If $bl$ is well-formed, then the number of $bl$ is the natural number in $[0, \text{Tower}(n, h) - 1]$ whose binary code is given by $b_0 \ldots b_t$ where $b_i$ is the content of the sub-block $bl_i$ for all $0 \leq i \leq t$.

**Example 4.3.** Let $n = 2$. In this case $\text{Tower}(n, 2) = 16$ and $\text{Tower}(n, 1) = 4$. Thus, we can encode by well-formed 2-blocks all the integers in $[0, 15]$. For example, let us consider the number 14 whose binary code (using $\text{Tower}(n, 1) = 4$ bits) is given by 0111 (assuming that the first bit is the least significant one). For each $b \in \{0, 1\}$, the well-formed 2-block with content $b$ and number 14 is given by

$$\{s_2\}\{s_1\}\{0\}\{0\}\{e_1\}\{s_1\}\{1\}\{0\}\{1\}\{e_1\}\{s_1\}\{0\}\{1\}\{\tau\}\{e_1\}\{s_1\}\{1\}\{1\}\{e_1\}\{b\}\{e_2\}.$$ 

Note that the 1-sub-blocks also encode the position of each bit in the binary code of 14. Now, let us consider $\tau \in \Lambda$ and $\ell \in [0, 2^{10} - 1]$, and let $b_0 \ldots b_{15}$ be the binary code of $\ell$. Then, the well-formed 3-block with content $\tau$ and number $\ell$ is given by the word $\{s_3\}bl_0 \ldots bl_{15}\{\tau\}\{e_3\}$, where for each $i \in [0, 15]$, $bl_i$ is the well-formed 2-block having content $b_i$ and number $i$.

ATM configurations $C = u_1 u_2 \ldots u_k$ (note that here we do not require that $k = \text{Tower}(n, 3)$) are then encoded by words $w_C$ of the form

$$w_C = tag_1 \cdot bl_1 \ldots bl_k \cdot tag_2,$$ 

where
Figure 2: Subtree of the computation tree of the open PMS $S$ rooted at an $f$-node (pop-phase)

- $tag_1 \in \{\{l\}, \{r\}\}$,
- for each $i \in [1, k]$, $bl_i$ is a 3-block whose content is $(u_{i-1}, u_i, u_{i+1})$ (where $u_0 = \vdash$ and $u_{k+1} = \vdash$),
- $tag_2 = \{f\}$ if $C$ is accepting, $tag_2 = \{\exists\}$ if $C$ is non-accepting and existential, and $tag_2 = \forall$ otherwise.

The symbols $l$ and $r$ are used to mark a left and a right ATM successor, respectively. We also use the symbol $l$ to mark the initial configuration. If $k = Tower(n, 3)$ and for each $i \in [1, k]$, $bl_i$ is a well-formed 3-block having number $i - 1$, then we say that $w_C$ is a well-formed code of $C$. A sequence $w_{C_1} \cdot \ldots \cdot w_{C_p}$ of well-formed ATM configuration codes is faithful to the evolution of $M$ if for each $1 \leq i < p$, either $w_{C_{i+1}}$ is marked by symbol $l$ and $C_{i+1} = succ_l(C_i)$, or $w_{C_{i+1}}$ is marked by symbol $r$ and $C_{i+1} = succ_r(C_i)$.

**Behaviour of the PMS $S$ and encoding of accepting computation trees on $\alpha$.**

The PMS $S$ in Theorem 4.2 generates, for different environment behaviors, all the possible computation trees of $M$. External nondeterminism is used in order to produce the actual symbols of each ATM configuration code. Whenever the PMS $S$ reaches the end of an existential (resp., universal) guessed ATM configuration code $w_C$, it simulates the existential (resp., universal) choice of $M$ from $C$ by external (resp., internal) nondeterminism, and, in particular, $S$ chooses a symbol in $\{l, r\}$ and marks the next guessed ATM configuration with this symbol.\(^1\) This ensures that, once we fix the environment behavior, we really get a tree $T$ where each existential ATM configuration code is followed by (at least) one ATM configuration code marked by a symbol in $\{l, r\}$, and every universal configuration is followed (in different branches) by two ATM configurations codes, one marked by the symbol $l$ and the other one marked by the symbol $r$.

\(^1\)For external (resp., internal) nondeterminism, we mean that the choices are resolved by the environment (resp., system) player.
We have to check that the guessed computation tree $T$ (corresponding to environment choices) corresponds to a legal computation tree of $\mathcal{M}$ over $\alpha$. To that purpose, we have to check several properties about each computation path $\pi$ of $T$, in particular:

- the ATM configurations codes are well-formed (i.e., the Tower$(n,3)$-bit counter is properly updated),
- $\pi$ is faithful to the evolution of $\mathcal{M}$.

The PMS $S$ cannot guarantee by itself these requirements. Thus, these checks are performed by a suitable ATL* formula $\varphi$. However, in order to construct an ATL* formula of size polynomial in $n$ and in the size of the ATM $\mathcal{M}$, we need to ‘isolate’ the (arbitrary) selected path $\pi$ from the remaining part of the tree. This is the point where we use the stack of the PMS $S$. As the ATM configurations codes are guessed symbol by symbol, they are pushed onto the stack of the PMS $S$. This phase is called push-phase. Note that in this phase the unique nodes which are controlled by the system player are the nodes labeled by the proposition $\forall$, where the system player simulates the universal choices of the ATM $\mathcal{M}$ from a universal configuration.

Whenever the end of an accepting computation path $\pi$ (i.e., a sequence of ATM configuration codes where the last ATM configuration is accepting) is reached, the PMS moves to the so called pop-phase. Let us denote by $\langle T_\pi, Lab_\pi \rangle$ the subtree of the full computation tree of the PMS $S$ rooted at the last node of $\pi$ (note that the last node of $\pi$ is labeled by $\{f\}$).

We now describe the branching behaviour of $S$ along $\langle T_\pi, Lab_\pi \rangle$ (pop-phase). The structure of $\langle T_\pi, Lab_\pi \rangle$ is also illustrated in Figures 2 and 3. By using both internal and external nondeterminism, the PMS pop the entire computation path $\pi$ from the stack. In this way, the PMS $S$ partitions the sanity checks for $\pi$ into separate branches (corresponding to the reverse of $\pi$) and augmented with additional information by means of the extra atomic propositions $\text{check}_3, \text{check}_2, \text{check}_1$). In particular, in the pop-phase, the unique nondeterministic or branching nodes (i.e., the nodes with at least two children) are end nodes, i.e., nodes labeled by one of the propositions in $\{e_1, e_2, e_3\}$. These nodes have, in particular, a binary branching degree. Moreover:

- the branching behaviour at the branching $e_3$-nodes along $\langle T_\pi, Lab_\pi \rangle$ is subdivided in two sub-phases. In the first sub-phase, the branching $e_3$-nodes are controlled by the system player, and $S$ marks by internal nondeterminism the $\Lambda$-content of exactly one 3-block $bl_3$ of $\pi$ with the special symbol $\text{check}_3$. This means, in particular, that for each 3-block $bl_3$ of $\pi$, there is a play of $\langle T_\pi, Lab_\pi \rangle$ such that the unique $\text{check}_3$-marked 3-block corresponds to $bl_3$. This is illustrated in the left part of Figure 2. After having marked a 3-block with $\text{check}_3$, $S$ moves to the second sub-phase, where the branching $e_3$-nodes are controlled by the environment player. In particular, in case the marked 3-block $bl_3$ does not belong to the first configuration code of $\pi$, $S$ marks by external nondeterminism the $\Lambda$-content of exactly one 3-block $bl'_3$ with the special symbol $\text{check}_3$ by ensuring that $bl_3$ and $bl'_3$ belong to two consecutive configurations codes along $\pi$. Hence, for all 3-blocks $bl_3$ and $bl'_3$ of $\pi$ such that $bl_3$ and $bl'_3$ belong to adjacent configurations and $bl'_3$ follows $bl_3$ along the reverse of $\pi$, there is a play of $\langle T_\pi, Lab_\pi \rangle$ such that the unique $\text{check}_3$-marked 3-block corresponds to $bl_3$ and the unique $\text{check}_3$-marked 3-block corresponds to $bl'_3$. This is illustrated in the right part of Figure 2.

\footnote{recall that the last symbol of an accepting configuration code is $\{f\}$}
• The branching behaviour at the $e_2$-nodes and $e_1$-nodes along $\langle T_\pi, Lab_\pi \rangle$, which is illustrated in Figure 3, is as follows. The $e_2$-nodes are controlled by the system player, while the $e_1$-nodes are controlled by the environment nodes. In particular, for each 2-block $bl_2$ of $\pi$, $S$ generates by internal nondeterminism, starting at the $e_2$-node (of the reverse) of $bl_2$, a tree copy of $bl_2$ (check 2-block-tree). This tree copy is structured as follows (see Figures 3(b) and 3(c)):
- there is an infinite path $\rho$ from the $e_2$-node of (the reverse of ) $bl_2$ whose labeling consists of a marked copy (of the reverse) of $bl_2$ (the content of $bl_2$ is marked by the special symbol check$_2$) followed by the suffix $\emptyset^\omega$ (see Figures 3(c));
- there are additional branches chosen by external nondeterminism starting at the $e_1$-nodes of the infinite path $\rho$. As illustrated in Figure 3(c), these additional branches represent marked copies of the (reverse of) 1-sub-blocks $bl_1$ of $bl_2$ (the content of $bl_1$ is marked by the special symbol check$_1$).

Note that for the $e_1$-nodes of $\langle T_\pi, Lab_\pi \rangle$, only the ones belonging to check 2-block trees are branching.

Note that for each $h = 1, 2, 3$, in a marked $h$-block $bl$, only the content (i.e., the symbol preceding the end-symbol) of $bl$ is marked.

Hence, the subtree $\langle T_\pi, Lab_\pi \rangle$ of the full computation tree of $G(S)$ associated with this pop-phase and the specific accepting computation path $\pi$, satisfies the following: each main play (i.e., a play of $\langle T_\pi, Lab_\pi \rangle$ which does not get trapped into a check 2-block-tree) corresponds to the reverse of $\pi$ (followed by a suffix with label $\emptyset^\omega$) with the unique difference that exactly one 3-block $bl_3$ is marked by check$_3$ and (in case $bl_3$ does not belong to the first ATM configuration code of $\pi$) exactly one 3-block $bl'_3$ is marked by $\widehat{\text{check}}_3$. The PMS $S$ ensures that $bl_3$ and $bl'_3$ belong to two consecutive configurations codes along $\pi$ (where $bl_3$ precedes $bl'_3$ along the main play) and, independently from the environment choices, all the 3-blocks $bl_3$ of $\pi$ are checked (i.e., there is a main play whose $\overline{\text{check}}_3$-marked block corresponds to $bl_3$).

The additional check 2-block-trees are intuitively used to isolate 2-blocks for ensuring by an ATL* formula $\varphi$ that the ATM configuration codes along $\pi$ are well-formed and $\pi$ is faithful to the evolution of $M$. In particular, as detailed in the proof of Lemma 4.6, the ATL* formula $\varphi$ requires that the given environment strategy tree of $G(S)$ satisfies the following:

• all the environment choices in each 2-block check-tree are enabled,
• the environment choices from the $\{e_3\}$-nodes controlled by the environment player are deterministic. This entails that the subtree rooted at the $s_3$-node of a check$_3$-marked 3-block $bl_3$ which does not belong to the first ATM configuration code contains exactly one check$_3$-marked 3-block $bl'_3$.

Then, by exploiting the previous two requirements, the ATL* formula $\varphi$ existentially quantifies over strategies of the system player whose outcomes get trapped into a check 2-block-tree in order to ensure that for the given sequence $\nu$ of configuration codes (associated with an accepting computation path $\pi$ of the push-phase), the following holds:

• the configuration codes along $\nu$ are well-formed,
• for each check$_3$-marked 3-block $bl_3$ which does not belong to the first ATM configuration code of $\nu$, the associated check$_3$-marked 3-block $bl'_3$ satisfies the following: $bl_3$ and $bl'_3$ have the same number and the $\Lambda$-contents of $bl_3$ and $bl'_3$ are consistent with the transition function of $M$. Since $bl_3$ and $bl'_3$ belong to two adjacent configuration codes along $\nu$, the
Figure 3: Marked copies of 2-blocks in the pop-phase of the open PMS $S$

previous conditions ensure that $\nu$ is faithful to the evolution of $M$. Note that in order to enforce that $bl_3$ and $bl'_3$ have the same number, for each 2-sub-block $bl_2$ of $bl_3$, the formula $\varphi$ requires the existence of a system strategy $f$ starting at the $e_2$-node of $bl_2$ which gets trapped into the check 2-block-tree of a 2-sub-block $bl'_2$ of $bl'_3$ such that the copy of $bl'_2$ in the check 2-block-tree and $bl_2$ have the same number and the same content. Note that the additional $check_1$-branches of the check 2-block-tree of $bl'_2$ are used to check by an LTL formula, asserted at the outcomes of the system strategy $f$, that $bl_2$ and the copy of $bl'_2$ have the same number.

Let $AP = Main \cup \{check_1, check_2, check_3, \hat{check}_3\}$. We now formally define the $AP$-labeled trees associated with the accepting environment strategy trees of $G(S)$, i.e. the environment strategy trees where each play from the root visits a $\{f\}$-labeled node. In the following, a $2^{AP}$-labeled tree is minimal if the children of each node have distinct labels. A branching node of a tree is a node having at least two distinct children. A tree-code is a finite minimal $2^{AP}$-labeled tree $\langle T, Lab \rangle$ such that

1. for each maximal path $\pi$ from the root, $Lab(\pi)$ is a sequence of ATM configuration codes;
2. a node $x$ is labeled by $\{f\}$ iff $x$ is a leaf;
3. each node labeled by $\{\forall\}$ has two children, one labeled by $\{l\}$ and one labeled by $\{r\}$.

Intuitively, tree-codes correspond to the maximal portions of the accepting environment strategy trees of $G(S)$ where $S$ performs push operations (push-phase). We now extend a tree-code $\langle T, Lab \rangle$ with extra nodes in such a way that each leaf $x$ of $\langle T, Lab \rangle$ is expanded in a tree, called check-tree (pop-phase).

**Check-trees.** The definition of check-trees is based on the notion of check 2-block-tree and simple check-tree. The structure of a check 2-block-tree for a 2-block $bl_2$ is illustrated in Figure 3(c). Note that the unique branching nodes are labeled by $\{e_1\}$ (and are controlled by the environment). A partial check 2-block-tree for $bl_2$ is obtained from the check 2-block-tree for $bl_2$ by pruning some choices from the $\{e_1\}$-branching nodes. Given a sequence $\nu$ of ATM configuration codes, a simple check-tree for $\nu$ is a minimal $2^{AP}$-labeled tree $\langle T, Lab \rangle$ such that
for each path \( \pi \) from the root, \( \text{Lab}(\pi) \) corresponds to the reverse of \( \nu \) followed by \( \emptyset^\omega \) but there is exactly one 3-block \( bl_3 \) of \( \nu \) whose content is additionally marked by proposition \( \text{check}_3 \), and in case \( bl_3 \) does not belong to the first configuration code of \( \nu \), there is exactly one 3-block \( bl'_3 \) whose content is marked by proposition \( \hat{\text{check}}_3 \); moreover, \( bl'_3 \) and \( bl_3 \) belong to two consecutive configuration codes, and \( bl'_3 \) precedes \( bl_3 \) along \( \nu \);

- for each 3-block \( bl_3 \) of \( \nu \), there is a path \( \pi \) from the root such that the sequence of nodes associated with \( bl_3 \) is marked by \( \text{check}_3 \) (i.e., all the 3-blocks of \( \nu \) are checked);

- each branching node \( x \) has label \( \{e_3\} \) and two children: one labeled by \( \{\lambda\} \) and the other one labeled by \( \{\lambda, \text{tag}\} \) for some \( \lambda \in \Lambda \) and \( \text{tag} \in \{\text{check}_3, \hat{\text{check}}_3\} \). If \( \text{tag} = \text{check}_3 \) (resp., \( \text{tag} = \hat{\text{check}}_3 \)), we say that \( x \) is a \( \text{check}_3 \)-branching (resp., \( \hat{\text{check}}_3 \)-branching) node.

Finally, a check-tree for \( \nu \) is a minimal \( 2^{\text{AP}} \)-labeled tree \( \langle T, \text{Lab} \rangle \) which is obtained from some simple check-tree \( \langle T', \text{Lab}' \rangle \) for \( \nu \) by adding for each node \( x \) of \( T' \) with label \( \{e_2\} \) an additional child \( y \) and a subtree rooted at \( y \) so that the subtree rooted at \( x \) obtained by removing all the descendants of \( x \) in \( T' \) is a partial check 2-block-tree for the 2-block associated with node \( x \) in \( T' \). Thus, in a check-tree, we have four types of branching nodes: \( \text{check}_3 \)-branching nodes and \( \{e_2\} \)-branching nodes which are controlled by the system, and \( \text{check}_3 \)-branching nodes and \( \{e_1\} \)-branching nodes which are controlled by the environment.

**Extended tree-codes.** An extended tree-code is a minimal \( 2^{\text{AP}} \)-labeled tree \( \langle T_e, \text{Lab}_e \rangle \) such that there is a tree-code \( \langle T, \text{Lab} \rangle \) so that \( \langle T_e, \text{Lab}_e \rangle \) is obtained from \( \langle T, \text{Lab} \rangle \) by replacing each leaf \( x \) with a check-tree for the sequence of labels associated with the path of \( \langle T, \text{Lab} \rangle \) starting at the root and leading to \( x \). By construction and the intuitions given about the PMS \( S \), we easily obtain the following result.

**Lemma 4.4.** One can build, in time polynomial in the size of the ATM \( M \), a PMS \( S \) over \( \text{AP} \) and \( \text{Ag} = \{\text{env, sys}\} \) such that the following holds:

- the set of \( 2^{\text{AP}} \)-labeled trees \( \langle T, \text{Lab} \rangle \) associated with the accepting environment strategy trees \( \langle T, \text{Lab}, \tau \rangle \) in \( \text{exec}(\mathcal{G}(S)) \) coincides with the set of extended tree-codes;

- for each accepting environment strategy tree \( \langle T, \text{Lab}, \tau \rangle \) in \( \text{exec}(\mathcal{G}(S)) \), the unique nodes controlled by the system in a check-subtree of \( \langle T, \text{Lab}, \tau \rangle \) are the \( \text{check}_3 \)-branching nodes and the \( \{e_2\} \)-branching nodes.

**Construction of the ATL* formula \( \varphi \) in Theorem 4.2.**

**Definition 4.5 (Well-formed Check-trees).** A check-tree \( \langle T, \text{Lab} \rangle \) for a sequence \( \nu \) of ATM configuration codes is well-formed if

- \( \langle T, \text{Lab} \rangle \) satisfies the goodness property, which means that:
  - there are no \( \hat{\text{check}}_3 \)-branching nodes, \( ^3 \) i.e., the unique branching \( e_3 \)-nodes are the \( \text{check}_3 \)-branching nodes (which are controlled by the system player). This entails that the subtree rooted at the \( \{s_3\} \)-node of a \( \text{check}_3 \)-marked 3-block contains at most one \( \text{check}_3 \)-marked 3-block.
  - Each \( \{e_1\} \)-node in a partial check 2-block-tree has two children (i.e., all the environment choices in the \( \{e_1\} \)-branching nodes are enabled).

- The ATM configuration codes in \( \nu \) are well-formed;

\( ^3 \)Recall that a \( \hat{\text{check}}_3 \)-branching node is a \( e_3 \)-node having two children, one marked by \( \text{check}_3 \) and one which is not marked.
• \( \nu \) starts with the code of the initial configuration for \( \alpha \);
• fairness condition: \( \nu \) is faithful to the evolution of \( M \) and for each path visiting a (well-formed) check\(_3\)-marked 3-block \( bl_3 \) and a (well-formed) check\(_3\)-marked 3-block \( bl'_3 \), \( bl_3 \) and \( bl'_3 \) have the same number.

An extended tree-code \( \langle T_e, Lab_e \rangle \) is well-formed if each check-tree in \( \langle T_e, Lab_e \rangle \) is well-formed. Evidently, there is a well-formed extended tree-code if and only if there is an accepting computation tree of \( M \) over \( \alpha \). We show the following result that together with Lemma 4.4 provides a proof of Theorem 4.2.

**Lemma 4.6.** One can construct in time polynomial in \( n \) and \(|AP|\), an ATL* state formula \( \varphi \) over \( AP \) and \( Ag = \{ \text{env}, \text{sys} \} \) such that for each environment strategy tree \( T = \langle T, Lab, \tau \rangle \) in \( \text{exec}(G(S)) \), \( T \) is a model of \( \varphi \) iff \( \langle T, Lab \rangle \) is a well-formed extended tree-code.

**Proof.** The ATL* formula \( \varphi \) is given by
\[
\varphi := AF f \land AG(f \rightarrow (\varphi_{\text{good}} \land \varphi_{\text{init}} \land \varphi_{3\text{bl}} \land \varphi_{\text{conf}} \land \varphi_{\text{fair}}))
\]
where for an environment strategy tree \( T = \langle T, Lab, \tau \rangle \) of the PMS \( S \) of Lemma 4.4, the first conjunct ensures that \( T \) is accepting (recall that \( T \) is accepting iff each play from the root visits a \( \{f\} \)-labeled node), while the subformulas \( \varphi_{\text{good}}, \varphi_{\text{init}}, \varphi_{3\text{bl}}, \varphi_{\text{conf}}, \) and \( \varphi_{\text{fair}} \) ensure the following for each check-tree \( \langle T_e, Lab_e \rangle \) of \( T \), where \( \nu \) is the sequence of ATM configuration codes associated with \( \langle T_e, Lab_e \rangle \):

• \( \varphi_{\text{good}} \) is a CTL formula requiring that \( \langle T_e, Lab_e \rangle \) satisfies the goodness property in Definition 4.5;
• \( \varphi_{\text{init}} \) is a CTL* formula guaranteeing that the first configuration code of \( \nu \) is associated with an ATM configuration of the form \( (q_0, \alpha(0))\alpha(1)\ldots\alpha(n-1) \cdot (#)^k \) for some \( k \geq 0 \);
• \( \varphi_{3\text{bl}} \) is a CTL* formula enforcing well-formedness of 3-blocks along \( \nu \);
• \( \varphi_{\text{conf}} \) is an ATL* formula requiring that the ATM configuration codes along \( \nu \) are well-formed;
• finally, \( \varphi_{\text{fair}} \) is an ATL* formula ensuring that \( \nu \) satisfies the fairness condition in Definition 4.5.

Fix a check-tree \( \langle T_e, Lab_e \rangle \) of an accepting environment strategy tree of the PMS \( S \), and let \( \nu \) be the sequence of ATM configuration codes associated with \( \langle T_e, Lab_e \rangle \).

The CTL formula \( \varphi_{\text{good}} \) ensuring the goodness property in Definition 4.5 is defined as follows:
\[
\varphi_{\text{good}} := AG(\neg(EX\check{3} \land EX\neg\check{3})) \land AG(\neg(check_2 \rightarrow AG(e_1 \rightarrow (EX\check{1} \land EX\neg\check{1}))))
\]
where

• the first conjunct ensures that there are no \( \check{3} \)-branching nodes, i.e., no \( e_3 \)-node of the check tree has both a child marked by \( \check{3} \) and a child which is not marked by \( \check{3} \);
• the second conjunct asserts that each \( e_1 \)-node associated with a marked 2-block has exactly two children. Recall that each \( e_1 \)-node associated with a marked 2-block has at most two children, one which is not marked and the other one which is marked by \( \check{1} \) (see Figure 3(c)).
The definition of the CTL* formula $\varphi_{init}$ is involved but standard.

$$\varphi_{init} := EF \left([f \lor \exists \lor \forall] \land ((l \land \neg r) \cup (l \land \neg EX \lor p)) \land \bigvee_{p \in AP} (e_3 \rightarrow X^{\psi#}U (e_3 \land X(\psi_{n-1} \land \ldots \land (e_3 \land X(\psi_1 \land XG-e_3)) \ldots))) \right)$$

where $\psi# := \bigvee_{(u_p, #, u_s), s \in \Lambda} (u_p, (g_0, \alpha(0)), u_s)$, and for all $2 \leq i \leq n$, $\psi_i := \bigvee_{(u_p, \alpha(i-1), u_s)} (u_p, \alpha(i-1), u_s)$

Recall that the paths of the check-tree $\langle T_c, \text{Lab}_c \rangle$ are associated to the reverse of $\nu$ and the first symbol (resp., the last symbol) of a configuration code is of the form $\{p\}$ where $p \in \{l, r\}$ (resp., $p \in \{f, \exists, \forall\}$). Moreover, each play of the check-tree $\langle T_c, \text{Lab}_c \rangle$ has a suffix labeled by $\Psi^*$. Thus, the previous formula asserts that the last configuration code along the reverse of $\nu$ (corresponding to the first configuration code of $\nu$) has the form $(q_0, \alpha(0))\alpha(1)\ldots\alpha(n-1) \cdot (\#^k)$ for some $k \geq 0$.

**Construction of the ATL* formula $\varphi_{3bl}$**. The CTL* formula $\varphi_{3bl}$ requires that the 3-blocks along $\nu$ are well-formed (hence, the $2^n$-bit counter in a 3-block is properly updated).

$$\varphi_{3bl} := \varphi_{2bl} \lor \varphi_{2, first} \land \varphi_{2, last} \land \varphi_{2, inc}$$

The conjunct $\varphi_{2bl}$ checks that the 2-blocks are well-formed. Again recall that the paths of the check-tree $\langle T_c, \text{Lab}_c \rangle$ are associated to the reverse of $\nu$.

$$\varphi_{2bl} := AG \left(e_1 \rightarrow (X^{n+2}s_1 \land \bigvee_{i=1}^{n+2} X^ib) \land AG \left(e_1 \land X^{n+3}s_2 \rightarrow \bigwedge_{i=2}^{n+1} X^i0 \right) \land \right.$$  

$$\left. AG \left(-s_1 \land Xe_1 \rightarrow \bigwedge_{i=3}^{n+2} X^i1 \lor AG \left(e_1 \land X^{n+3}e_1 \rightarrow \right. \right.$$  

$$\left. \bigvee_{i=2}^{n+1} X^i1 \land X^{n+3+i} \land \bigwedge_{j=2}^{n-i} X^jb \land X^{n+3+j}b \land \bigwedge_{j=i+1}^{n+1} X^j0 \land X^{n+3+j} \right) \right)$$

where:

- the first conjunct in the definition of $\varphi_{3bl}$ ensures well-formedness of 1-blocks. Recall that the reverse of a well-formed 1-block is of the form $\{e_1\}\{b\}\{b\} \ldots \{b\}\{s\}$, where $b, b_1, \ldots, b_n \in \{0, 1\}$ and $b$ is the content of the 1-block.
- The second conjunct ensures that the first 1-block $bl_1$ of a 2-block has number 0, i.e., the reverse of $bl_1$ has the form $\{e_1\}\{b\}\{0\} \ldots \{0\}\{s\}$ for some $b \in \{0, 1\}$.
- The third conjunct ensures that the last 1-block $bl_1$ of a 2-block has number 0, i.e., the reverse of $bl_1$ has the form $\{e_1\}\{b\}\{1\} \ldots \{1\}\{s\}$ for some $b \in \{0, 1\}$.
- Finally, the last conjunct ensures the for two adjacent 1-blocks $bl_1$ and $bl_1'$ along a 2-block, $bl_1$ and $bl_1'$ have consecutive numbers.

The second conjunct $\varphi_{2, first}$ in the definition of $\varphi_{3bl}$ ensures that the first 2-block $bl_2$ of a 3-block along $\nu$ has number 0, i.e., the content of each 1-sub-block of $bl_2$ is 0.

$$\varphi_{2, first} := AG \left( e_2 \land X \neg e_2 U s_3 \right) \rightarrow X \left( \neg e_2 \land (e_1 \rightarrow X0) \cup s_3 \right)$$
The second conjunct $\varphi_{2,\text{last}}$ guarantees that the last 2-block $bl_2$ of a 3-block has number $2^n - 1$, i.e., the content of each 1-sub-block of $bl_2$ is 1.

$$\varphi_{2,\text{last}} := \text{AG}
\left(\neg s_2 \land X e_2 \land F s_2 \implies X \left(\neg s_2 \land (e_1 \rightarrow X 1)\right) \lor s_2\right)$$

Finally, the last conjunct $\varphi_{2,\text{inc}}$ in the definition of $\varphi_{3bl}$ guarantees that for all adjacent 2-blocks $bl_2$ and $bl_2'$ of a 3-block along $\nu$, $bl_2$ and $bl_2'$ have consecutive numbers. For this, assuming that $bl_2'$ follows $bl_2$ along the reverse of $\nu$, we need to check that there is a 1-sub-block $\overline{bl_1}$ of $bl_2$ whose content is 1 and the following holds:

- the 1-sub-block of $bl_2'$ with the same number as $\overline{bl_1}$ has content 0;
- Let $bl_1$ be a 1-sub-block of $bl_2$ distinct from $\overline{bl_1}$, and $bl_1'$ be the 1-sub-block of $bl_2'$ having the same number as $bl_1$. Then, $bl_1$ and $bl_1'$ have the same content if $bl_1$ precedes $\overline{bl_1}$ along the reverse of $bl_2$; otherwise, the content of $bl_1$ is 0 and the content of $bl_1'$ is 1.

In order to check these conditions, we exploit the branches of the check 2-block-tree in $\langle T_c, Lab_c \rangle$ associated with (a copy of) $bl_2'$ which lead to check$_1$-marked copies of the 1-sub-blocks of $bl_2'$ (see Figure 3(c)). Note that these branches consist (of the reverse) of a check$_1$-marked 1-sub-block of $bl_2'$ followed by the suffix $\theta^\omega$. Then, the formula $\varphi_{2,\text{inc}}$ is defined as follows.

$$\varphi_{2,\text{inc}} := \text{AG}
\left(\left(\left(e_2 \land X (\neg s_3 \lor e_2)\right) \implies X \left\{\neg e_2 \land \left(e_1 \rightarrow \bigvee_{b \in \{0,1\}} \theta(b, b)\right)\right\}
\right.
\left.\cup \left\{\theta(1, 0) \land e_1 \land X (\neg e_2 \land \left(e_1 \rightarrow \theta(0, 1)\right)\lor e_2)\right\}\right)$$

where for all $b, b' \in \{0, 1\}$, the auxiliary subformula $\theta(b, b')$ in the definition of $\varphi_{2,\text{inc}}$ requires that for the current 1-sub-block $bl_1$ of $bl_2$ and for the path from $bl_1$ which leads to the check$_1$-marked copy $bl_1'$ of the 1-sub-block of $bl_2'$ having the same number as $bl_1$, the following holds: the content of $bl_1$ is $b$ and the content of $bl_1'$ is $b'$.

$$\theta(b, b') := X b \land E \left[\neg e_2 \lor \left(e_2 \land X (\text{check}_2 \land F (\text{check}_1 \land b'))\right)\right] \land \bigwedge_{i=1}^n \left[X^{i+1} c \land F (\text{check}_1 \land X^i e)\right] \land \bigvee_{c \in \{0, 1\}}$$

We now illustrate the crucial part of the construction. By definition of $\varphi_{\text{good}}$ and $\varphi_{3bl}$, we can assume that the check-tree $\langle T_c, Lab_c \rangle$ is good and all the 3-blocks along $\nu$ are well-formed. For defining the remaining ATL$^\star$ formulas $\varphi_{\text{conf}}$ and $\varphi_{\text{fair}}$, we exploit the following pattern: starting from an $\{e_2\}$-node $x_{bl_2}$ related to a 2-block $bl_2$ of the good check-tree $\langle T_c, Lab_c \rangle$, we need to isolate another 2-block $bl_2'$ following $bl_2$ along the reverse of $\nu$ and checking, in particular, that $bl_2$ and $bl_2'$ have the same number. Moreover, for the case of the formula $\varphi_{\text{conf}}$, we require that the 3-block of $bl_2'$ is adjacent to the 3-block of $bl_2$ within the same ATM configuration code, while for the case of the formula $\varphi_{\text{fair}}$, we require that the 3-block of $bl_2$ (resp., $bl_2'$) is check$_3$-marked (resp., check$_3$-marked) in the considered path of $\langle T_c, Lab_c \rangle$.

Recall that in a good check-tree, the unique nodes controlled by the system are the check$_3$-branching nodes and the $\{e_2\}$-nodes, and each unmarked 2-block is associated with a check 2-block-tree (2-CBT for short). In particular, in a 2-CBT, all the nodes, but the root (which is an $\{e_2\}$-node), are controlled by the environment. Moreover, each strategy of the system selects exactly one child for each node controlled by the system. Hence, there is a strategy $I_{bl_2}$ of the player system such that
• (*) each play consistent with the strategy \( f_{bl_2} \) starting from the \( \{e_2\}\)-node \( x_{bl_2} \) “gets trapped” in the 2-\( CBT \) of \( bl'_2 \), and
• (**) each path starting from the node \( x_{bl_2} \) and leading to some marked 1-block of the 2-\( CBT \) for \( bl'_2 \) is consistent with the strategy \( f_{bl_2} \).

Thus, in order to isolate a 2-block \( bl'_2 \), an ATL* formula “guesses” the strategy \( f_{bl_0} \) and check that conditions (*) and (**) are fulfilled by simply requiring that each outcome from the current node \( x_{bl_2} \) visits a node marked by proposition \( check_2 \). Additionally, by exploiting the branches of the 2-\( CBT \) leading to marked 1-blocks, we can check by a formula of size polynomial in \( n \) and the size of \( \mathcal{M} \) that \( bl_2 \) and \( bl'_2 \) have the same number. We now proceed with the technical details about the construction of the ATL* formulas \( \varphi_{conf} \) and \( \varphi_{fair} \).

**Construction of the ATL* formula \( \varphi_{conf} \).** The ATL* formula \( \varphi_{conf} \) is defined as follows.

\[
\varphi_{conf} := \varphi_{3, first} \land \varphi_{3, last} \land \varphi_{3, inc}
\]

The conjunct \( \varphi_{3, first} \) requires that the first 3-block \( bl_3 \) of an ATM configuration code along \( \nu \) has number 0, i.e., the content of each 2-sub-block of \( bl_3 \) is 0.

\[
\varphi_{3, first} := AG\left( [e_3 \land X(\neg e_3 \cup (l \lor r))] \rightarrow X(\neg e_3 \land (e_2 \rightarrow X0)) U (l \lor r) \right)
\]

The second conjunct \( \varphi_{3, last} \) guarantees that the last 3-block \( bl_3 \) of an ATM configuration code has number \( Tower(n, 3) - 1 \), i.e., the content of each 2-sub-block of \( bl_3 \) is 1.

\[
\varphi_{3, last} := AG\left( [\neg s_3 \land Xe_3 \land Fs_3] \rightarrow X(\neg s_3 \land (e_2 \rightarrow X1)) U s_3 \right)
\]

The last conjunct \( \varphi_{3, inc} \) in the definition of \( \varphi_{conf} \) checks that for all adjacent 3-blocks \( bl_3 \) and \( bl'_3 \) of an ATM configuration code along \( \nu \), \( bl_3 \) and \( bl'_3 \) have consecutive numbers. For this, assuming that \( bl'_3 \) follows \( bl_3 \) along the reverse of \( \nu \), we need to check that there is a 2-sub-block \( \overline{bl}_2 \) of \( bl_3 \) whose content is 1 and the following holds:

- the 2-sub-block of \( bl'_3 \) with the same number as \( \overline{bl}_2 \) has content 0;
- Let \( bl_2 \) be a 2-sub-block of \( bl_3 \) distinct from \( \overline{bl}_2 \), and \( bl'_2 \) be the 2-sub-block of \( bl'_3 \) having the same number as \( bl_2 \). Then, \( bl_2 \) and \( bl'_2 \) have the same content if \( bl_2 \) precedes \( \overline{bl}_2 \) along the reverse of \( bl_3 \); otherwise, the content of \( bl_2 \) is 0 and the content of \( bl'_2 \) is 1.

Formula \( \varphi_{3, inc} \) is then defined as follows.

\[
\varphi_{3, inc} := AG\left( [e_3 \land X(\neg l \land \neg r) U e_3] \rightarrow X\left( \neg e_3 \land (e_2 \rightarrow \bigvee_{b \in \{0,1\}} \eta(b, b)) \right) U \{\eta(1, 0) \land e_2 \land X(\neg e_3 \land (e_2 \rightarrow \eta(0, 1))) U e_3\} \right)
\]

where for all \( b, b' \in \{0,1\} \), we exploit the auxiliary formula \( \eta(b, b') \) to require from the current \( e_2\)-node \( x \) of the current 2-sub-block \( bl_2 \) of \( bl_3 \) that the content of \( bl_2 \) is \( b \) and the 2-sub-block \( bl'_2 \) of \( bl'_3 \) having the same number as \( bl_2 \) has content \( b' \). In order to ensure the last condition, the formula \( \eta(b, b') \) asserts the existence of a strategy \( f_x \) of the player system such that the following two conditions hold:

1. each outcome of \( f_x \) from the node \( x \) visits a node marked by \( check_2 \) whose parent (\( e_2\)-node) belongs to a 2-block of \( bl'_3 \). This ensures that all the outcomes “get trapped” in the same check 2-block-tree associated with some 2-block \( bl'_2 \) of \( bl'_3 \). Moreover, \( bl'_2 \) has content \( b' \).
(2) For each outcome $\pi'$ of $f_x$ from $x$ which leads to a marked 1-sub-block $bl_1'$ of $bl_2'$ (hence, a marked copy of a 1-sub-block of $bl_3'$), denoting by $bl_1$ the 1-sub-block of $bl_2$ having the same number as $bl_1'$, it holds that $bl_1'$ and $bl_1$ have the same content. This ensures that $bl_2$ and $bl_2'$ have the same number.

The first (resp., second) condition is implemented by the first (resp., second) conjunct in the argument of the strategic quantifier $\langle \langle \text{sys} \rangle \rangle$ in the definition of $\eta(b, b')$ below.

$$\eta(b, b') := Xb \land \langle \langle \text{sys} \rangle \rangle \left( \left[ -e_3 \cup (e_3 \land X(-e_3 \cup \langle \langle \text{check}_2 \land b' \rangle \rangle)) \right] \land \left[ F\langle \langle \text{check}_1 \rightarrow X((-e_2 \land \langle \langle e_1 \rightarrow X\eta_1 \rangle \rangle) \cup \langle \langle s_2 \rangle \rangle) \right] \right)$$

$$\eta_1 := \bigwedge_{i=1}^{i=n} \bigvee_{b \in \{0,1\}} (X^i b) \land F(\langle \langle \text{check}_1 \land X^i b \rangle \rangle) \rightarrow \bigvee_{b \in \{0,1\}} (b \land F(\langle \langle \text{check}_1 \land b \rangle \rangle))$$

Note that for each outcome $\pi'$ of strategy $f_x$ which leads to a marked 1-sub-block $bl_1'$ of $bl_2'$, the subformula $\eta_1$ of $\eta(b, b')$ is asserted at the content node of each 1-sub-block $bl_1'$ of $bl_2'$. Thus, $\eta_1$ requires that whenever $bl_1'$ and $bl_1$ have the same number, then $bl_1'$ and $bl_1$ have the same content as well.

**Construction of the ATL* formula $\varphi_{\text{fair}}$.** We can assume that the check-tree $(T_c, Lab_c)$ is good and all the ATM configuration codes along $\nu$ are well-formed. Since $(T_c, Lab_c)$ satisfies the goodness property, for each check3-marked 3-block $bl_3$ which does not belong to the first configuration code of $\nu$, there is exactly one check3-marked 3-block $bl_3'$ in the subtree of $(T_c, Lab_c)$ rooted at the $s_3$-node of $bl_3$. Moreover, $bl_3$ and $bl_3'$ belong to two adjacent configuration codes along $\pi$. Thus, by construction, in order to ensure that $\nu$ is faithful to the evolution of $M$, it suffices to require that for each (well-formed) check3-marked 3-block $bl_3$ in $(T_c, Lab_c)$ which does not belong to the first configuration code of $\nu$, the associated (well-formed) check3-marked 3-block $bl_3'$ satisfies the following conditions, where $(u_p, u, u_s) \ (r, (u_p', u', u_s'))$ is the content of $bl_3$ (resp., $bl_3'$):

- $bl_3$ and $bl_3'$ have the same number,
- $u = next_{\nu}(u_p', u', u_s')$ if $l$ marks the ATM configuration code of $bl_3$, and $u = next_{\nu}(u_p', u', u_s')$ otherwise.

Thus, formula $\varphi_{\text{fair}}$ is defined as follows:

$$\varphi_{\text{fair}} := \bigwedge_{\text{dir} \in \{l, r\}} \text{AG} \left( \left[ \text{check}_3 \land [(-l \land -r) \cup (\text{dir} \land X(\exists \nu \lor \forall \nu))] \right] \rightarrow \left[ ((-e_3 \land (e_2 \rightarrow \psi_\pi)) \cup \langle \langle s_3 \rangle \rangle) \land \langle \langle \text{check}_3 \land (u_p, u, u_s) \cup \langle \langle s_3 \rangle \rangle \rangle \right] \right)$$

where the auxiliary formula $\psi_\pi$ in the definition of $\varphi_{\text{fair}}$ requires from the current $e_2$-node $x$ of the current 2-sub-block $bl_2$ of $bl_3$ that the 2-sub-block $bl_2'$ of $bl_3'$ having the same number as $bl_2$ has the same content as $bl_2$ too. In order to ensure the last condition, the formula $\psi_\pi$ asserts the existence of a strategy $f_x$ of the player system such that the following holds:

1. each outcome of $f_x$ from the node $x$ visits a node marked by check2 whose parent (e2-node) belongs to a check3-marked 3-block. This ensures that all the outcomes “get trapped” in the same 2-block check-tree associated with some 2-block $bl_2'$ of $bl_3'$. Moreover, $bl_2$ and $bl_2'$ have the same content.
(2) For each outcome \( \pi' \) of \( f_x \) from \( x \) which leads to a marked 1-sub-block \( b_{l1}' \) (hence, a marked copy of a 1-sub-block of \( b_{l2}' \)), denoting by \( b_{l1} \) the 1-sub-block of \( b_{l2} \) having the same number as \( b_{l1}' \), it holds that \( b_{l1} \) and \( b_{l1}' \) have the same content. This ensures that \( b_{l2} \) and \( b_{l2}' \) have the same number.

Thus, formula \( \psi \) is defined as follows.

\[
\psi := \langle (\mathrm{sys}) \rangle \left( \bigvee_{b \in \{0,1\}} (Xb \land F\{\mu_{\text{check}3} \land (\neg e_3 \cup (b \land \text{check}2))\}) \right) \land [F\text{check}1 \rightarrow X((\neg e_2 \land (e_1 \rightarrow X\eta_1) \cup s_2))] 
\]

where \( \eta_1 \) corresponds to the homonymous subformula of the auxiliary formula \( \eta(b,b') \) used in the definition of \( \varphi_{3,\mathrm{inc}} \). This concludes the proof of Lemma 4.6.

\[ \square \]

5. Conclusion

Module checking is a useful game-theoretic framework to deal with branching-time specifications. The setting is simple and powerful as it allows to capture the essence of the adversarial interaction between an open system (possibly consisting of several independent components) and its unpredictable environment. The work on module checking has brought an important contribution to the strategic reasoning field, both in computer science and AI [AHK02]. Recently, CTL/CTL\(^*\) module checking has come to the fore as it has been shown that it is incomparable with ATL/ATL\(^*\) model checking [JM14]. In particular the former can keep track of all moves made in the past, while the latter cannot. This is a severe limitation in ATL/ATL\(^*\) and has been studied under the name of irrevocability of strategies in [AGJ07]. Remarkably, this feature can be handled with more sophisticated logics such as Strategy Logics [CHP10, MMPV14], ATL with strategy contexts [LM15], and quantified CTL [LM14]. However, for such logics, the relative model checking question for finite-state multi-agent systems (modelled by finite-state concurrent game structures) turns out to be non-elementarily decidable.

In this paper, we have addressed the module-checking problem of multi-agent pushdown systems (PMS) against ATL and ATL\(^*\) specifications. PMS endow finite-state multi-agent systems with an additional expressive power, the possibility of using a stack to store unbounded information. The stack is the standard low level mechanism which allows to structure agents in modules and to implement recursive calls and returns of modules. Hence, the considered framework is suitable for formally reasoning on the behaviour of software agents with (recursive) procedural modularity. As a main contribution, we have established the exact computational complexity of pushdown module-checking against ATL and ATL\(^*\). While for ATL, the considered problem is 2EXPTIME-complete, which is the same complexity as pushdown module-checking for CTL, for ATL\(^*\), pushdown module-checking turns out to be 4EXPTIME-complete, hence exponentially harder than both CTL\(^*\) pushdown module-checking and ATL\(^*\) model-checking of PMS. As future work, we aim to investigate the considered problems in the setting of imperfect information under memoryless strategies. We recall that this setting is decidable in the finite-state case [AHK02]. However, moving to pushdown systems one has to distinguish whether the missing information relies in the control states, in the pushdown store, or both. We recall that in pushdown module-checking only the former case is decidable for specifications given in CTL and CTL\(^*\) [ALM+13].
Another interesting question to investigate is the exact computational complexity of pushdown module checking against the fragment ATL$^+$ of ATL*, where each temporal modality is immediately preceded either by a strategic quantifier or by a Boolean connective. Our results just imply that pushdown module checking against ATL$^+$ lies somewhere between $2\text{Exptime}$ and $4\text{Exptime}$.

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