FLIPS OF SEMI-STABLE 4-FOLDS WHOSE DEGENERATE FIBERS ARE UNIONS OF CARTIER DIVISORS WHICH ARE TERMINAL FACTORIAL 3-FOLDS

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Abstract. We shall investigate flipping contractions from a semi-stable 4-fold \( X \) whose degenerate fiber is a union of Cartier divisors which are terminal factorial 3-folds. Especially we shall prove that \( X \) is smooth along the flipping locus, and that the flip exists for such contractions.

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We will work over \( \mathbb{C} \), the complex number field.

§0. Introduction.

The Minimal Model Conjecture asserts that each algebraic variety has either a minimal model, or a model which admits a structure of Mori fiber space. This was classical in dimension 2, while has been considered to be extremely difficult in dimension greater than or equal to 3. By virtue of the works of Reid [R1,3], Mori [Mo2], Kawamata [Kaw1,2] and Shokurov [Sh1], however, this was reduced to the existence of flips (see [R2] Conjecture 3.7, [Kaw2] Problem 5.6). In the case of semi-stable 3-folds, Kawamata [Kaw3] solved this affirmatively. His idea is to take double
coverings to reduce the problem to the existence of flops, much simpler than that of flips (cf. Kollár [Ko1]). By applying [Kaw3], Mori [Mo4] then proved the existence of flips of 3-folds in the general case, and thus the Minimal Model Conjecture has been found to be true also in dimension 3. There are also further developments on 3-dimensional flips, such as [Utah], [KoMo], [Sh2], and [Kaw5] (see also [R5]).

Thus it is worth trying to investigate flips in dimension greater than or equal to 4 as well. In this direction, little was known, except Kawamata’s structure theorem [Kaw4] in the case of smooth 4-folds:

**Theorem 0.1. (Kawamata [Kaw4])**

Let $X$ be a smooth projective 4-fold and $g : X \to Y$ a flipping contraction. Then $\text{Exc} \ g$ is a disjoint union of $\mathbb{P}^2$’s, and those normal bundles in $X$ are all isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.

Moreover, the flip of $g$ exists.

This, together with the Termination Theorem [KaMaMa] after [Sh2], should be considered as the first step of generalizing the Minimal Model Conjecture to dimension 4.

In this paper, we shall investigate the structures of flipping contractions from semi-stable 4-folds, with a certain additional assumption (Main Theorem 0.5). This, together with Kawamata’s Theorem [Kaw4] (see also Theorem 0.1 above), shows that the flip exists for such contractions.

**Assumption 0.2. (Semi-stable degenerations) (cf. [Kaw3,5])**

Let $f : X \to \Delta$ be a projective morphism from a 4-dimensional analytic space $X$ with at most terminal singularities to the disc $\Delta := \{ z \in \mathbb{C} ||z| < 1 \}$ such that each fiber of $f$ over $\Delta - \{0\}$ is a projective 3-fold with at most terminal singularities. Moreover, assume the following conditions:

1. The central fiber $D := f^{-1}(0)$ is reduced,
2. $D$ is normal crossing outside $\text{Sing} \ X$,
3. Let $D = \bigcup_{k=1}^{r} D_k$ be the irreducible decomposition, then each $D_k$ is a normal 3-fold which is a $\mathbb{Q}$-Cartier divisor on $X$, and
4. The pair $(X, D)$ is log-terminal.

**Definition 0.3.** Let $f : X \to \Delta$ be as in Assumption 0.1. Then the contraction morphism $g : X \to Y$ associated to an extremal ray $R$ of $\overline{NE}(X/\Delta)$ is said to be a flipping contraction if $\dim \text{Exc} \ g \leq 2$, and $\text{Exc} \ g$ is called the flipping locus. If there is a projective bimeromorphic morphism $g^+ : X^+ \to Y$ over $\Delta$ from another 4-dimensional analytic space $X^+$ such that

1. The composite $X^+ \xrightarrow{g^+} Y \to \Delta$ satisfies the Assumption 0.2, except possibly (0.2.1),
2. $\rho(X^+/Y) = 1$, 

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(0.3.3) $K_{X^+}$ is $g^+$-ample, and
(0.3.4) $\dim \text{Exc } g^+ \leq 2$,
then $g^+$ is called the flip of $g$. We sometimes call the composite bimeromorphic map $g^+ \circ g : X \to X^+$ also the flip of $g$. (As for the condition (0.3.1), see Remark 5.4.)

**Remark 0.4.** Note that the flip $g^+$ of $g$ is unique if exists.

**Main Theorem 0.5.** In addition to the Assumption 0.2 and Definition 0.3, we assume furthermore the followings:

(0.5.1) Each $D_i$ is a Cartier divisor which is a terminal factorial 3-fold.

Then, $\text{Exc } g$ is a disjoint union of $\mathbb{P}^2$'s, and $X$ is smooth along $\text{Exc } g$. Let us fix a connected component $E \simeq \mathbb{P}^2$ of $\text{Exc } g$ arbitrarily. Then

$$N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$$

and after a suitable renumbering of $\{D_1, \ldots, D_r\}$, exactly one of the followings holds:

(A-I) $E$ is a connected component of $D_1 \cap D_2$, $N_{E/D_k} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$ ($k = 1, 2$), $D_3 \cap E$, $D_4 \cap E$ are both lines in $E$ which are distinct to each other, and $D_k \cap E = \emptyset$ ($\forall k \geq 5$),

(A-II) $E$ is a connected component of $D_1 \cap D_2$, $N_{E/D_k} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$ ($k = 1, 2$), $D_3 \cap E$ is a smooth conic in $E$, and $D_k \cap E = \emptyset$ ($\forall k \geq 4$).

or

(B) $E \subset D_1$, $N_{E/D_1} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$, $D_2 \cap E$ is a line in $E$, and $D_k \cap E = \emptyset$ ($\forall k \geq 3$).

Moreover, the flip $g^+$ of $g$ exists.

As for the description of the flip $g^+$, see §5.

**Remark 0.6.**

(1) There are no flipping contractions from terminal Gorenstein 3-folds [Mo4].

(2) In particular, we may assume that $E \subset D$, in Main Theorem 0.5.

**Notation 0.7.** Let $R$ be the extremal ray of $\overline{NE}(X/\Delta)$ determining the flipping contraction $g : X \to Y$. Then for $D' = \sum_{k=1}^{r} \alpha_k D_k$ we simply write $(D'. R) > 0$ (resp. $\geq 0$, $= 0$, $\leq 0$, $< 0$) when for an irreducible curve $C \subset E$, $(D'. C) > 0$ (resp. $\geq 0$, $= 0$, $\leq 0$, $< 0$). In particular for $D = \sum_{k=1}^{r} D_k$,

(0.7.1) $(D . R) = 0$.

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§1. Preliminaries (Dividing into cases).

Lemma 1.1. \( E \) is purely 2-dimensional.

Proof. Assume that \( E \) has an irreducible component \( C \) of dimension 1 to derive a contradiction. First we claim:

(1.1.1) There is at least one \( D_k \) which contains \( C \) such that \((-K_{D_k} \cdot C) > 0\).

Actually, assume \( D_1 \) say, satisfies \( D_1 \supset C \) and \((-K_{D_1} \cdot C) \leq 0\). Since \((D \cdot C) = 0\),

\[
0 \geq (-K_{D_1} \cdot C)_{D_1} = (-K_X \cdot C)_X + (-D_1 \cdot C)_X \geq 1 + (-D_1 \cdot C)_X = 1 + \left( \sum_{k \geq 2} D_k \cdot C \right)_X.
\]

In particular

(1.1.3) \((D_k \cdot C)_X < 0\) for some \( i \geq 2\).

For such \( k \), \( C \subset D_k \) and in the same way as in (1.1.2), we have

\[
(-K_{D_k} \cdot C)_{D_k} = (-K_X \cdot C)_X + (-D_k \cdot C)_X > 0 \quad \text{(by (1.1.3))}
\]

and hence (1.1.1).

Let \( V \) be an analytic neighborhood of \( g(D_k \cap E) \) in \( g(D_k) \), \( U := g^{-1}(V) \), and \( \tilde{V} \) the normalization of \( V \). Let \((g|_U)^\sim : U \to \tilde{V}\) be the morphism induced from \( g|_U : U \to V\). (Recall that \( D_k \) is assumed to be normal (0.2.2).) This is a bimeromorphic morphism with \( \text{Exc} (g|_U)^\sim = D_k \cap E \), and \( C \) forms an irreducible component of \( \text{Exc} (g|_U)^\sim \). Let \( C^- \) be the union of all the other irreducible components of \( \text{Exc} (g|_U)^\sim \). Let \( L + L^- \) be a \((g|_U)^\sim\)-very ample divisor of \( U \) such that \( L \cap C = \emptyset \) and \( L^- \cap C^- = \emptyset \). Then \(|L|\) determines the morphism \( h : U \to U' \) over \( \tilde{V} \) which is also a bimeromorphic morphism such that

\[
\text{Exc} h = C.
\]
Since \((-K_{D_k} \cdot C) > 0\) (1.1.1), this is a flipping contraction from \(U\). Since \(D_k\) is assumed to be a terminal Gorenstein 3-fold (0.1.3), this contradicts Remark 0.6. Hence the Lemma. □

**Lemma 1.2.** Let \(C\) be a smooth rational curve which is contained in \(E\). Assume that \(C \cap \text{Sing} \ X = \emptyset\). Let \(\text{Hilb}_{X,[C]}\) be the connected component of the Hilbert scheme \(\text{Hilb}_X\) containing the point \([C]\). Then

\[
\dim \text{Hilb}_{X,[C]} \geq 2.
\]

**Proof.** Since \(\dim X = 4\) and \((-K_X \cdot C) > 0\), this is a direct consequence of [Mo1]:

\[
\dim \text{Hilb}_{X,[C]} \geq \text{codim}_X C + (-K_X \cdot C) + \deg K_C \geq 2.
\]

□

**Corollary 1.3.** Each \(E_i\) has no open subset which admits a \(\mathbb{P}^1\)-bundle structure. In particular, \(g(E)\) is a point. □

**Proposition 1.4.** (Dividing into cases)

After a suitable renumbering of \(\{D_1, \ldots, D_r\}\), exactly one of the followings holds:

(A) \(E_i \subset D_1 \cap D_2\), and \(E_i \not\subset \bigcup_{k \geq 3} D_k\) for some \(E_i\),

(B) \(E_i \subset D_1\), \(E_i \not\subset \bigcup_{k \geq 2} D_k\), and \(E_i \cap \bigcup_{k \geq 2} D_k \neq \emptyset\) for all \(E_i\), or

(C) \(E \subset D_1\), and \(E \cap \bigcup_{k \geq 2} D_k = \emptyset\).

**Proof.** Assume not the Case (A). Then by the assumption (0.2.1) with Lemma 1.1, for any \(E_i\) there exists some \(D_{k(i)}\) such that

\[
E_i \subset D_{k(i)}, \text{ and } E_i \not\subset \bigcup_{k \neq k(i)} D_k.
\]

In particular \(\left( \sum_{k \neq k(i)} D_k \cdot R \right) \geq 0\), or equivalently

\[(D_{k(i)} \cdot R) \leq 0.\]

If \((D_{k(i)} \cdot R) < 0\), then

\[E \subset D_{k(i)} \text{ and } E \cap \bigcup_{k \neq k(i)} D_{k(i)} \neq \emptyset,\]

thus we have the Case (B) by letting \(D_1 = D_{k(i)}\). Otherwise we have the Case (C). □

The following is due to Cutkosky [C] which is a generalization of Mori [Mo2], and is an application of Fujita [F]:

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Theorem 1.5.  (Cutkosky [C])

Let $Z$ be a terminal factorial 3-fold and $h: Z \to W$ the contraction of an extremal ray of $Z$ which is birational. Let $F := \text{Exc } h$ and assume that $h(F)$ is a point. Then $(F, \mathcal{O}_F(F))$ is isomorphic either to 

$$(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)), \ (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2)), \ (\mathbb{P} \times \mathbb{P}, \mathcal{O}_{\mathbb{P} \times \mathbb{P}}(-1,-1)),$$

or $$(S_2, \mathcal{O}_{S_2}(-1)),$$

where $S_2$ is the singular quadric surface in $\mathbb{P}^3$. $h$ is the blow-up with the reduced center $P := h(F)$.

Furthermore, in the case $(F, \mathcal{O}_F(F)) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1))$ (resp. $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2))$, we have

1. $Z$ is smooth along $F$, and
2. $W$ is smooth at $P$ (resp. $W$ has a quotient singularity of type $\frac{1}{2}(1,1,1)$ at $P$).

§2. The case $E_i \subset D_1 \cap D_2$ for some $E_i$.

The aim of this section is to prove the following:

Theorem 2.1.  In the Case (A) in Proposition 1.4, the followings hold:

1. $E$ is irreducible: $E = E_i$, and is isomorphic to $\mathbb{P}^2$.
2. $X$ is smooth along $E$.
3. $N_{E/D_1} \simeq N_{E/D_2} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$. In particular $N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$.
4. After a suitable renumbering of $\{D_3, \ldots, D_r\}$, either one of the followings holds:
   (A-I) $D_3 \cap E, D_4 \cap E$ are distinct lines in $E$, and $D_k \cap E = \emptyset \ (\forall k \geq 5)$;
   or
   (A-II) $D_3 \cap E$ is a smooth conic in $E$, and $D_k \cap E = \emptyset \ (\forall k \geq 4)$,
5. The flip $g^+$ of $g$ exists.

First we shall prove the following, which is told us by Y.Kawamata and J.Kollár:

Proposition 2.2.  (Kawamata - Kollár)

In the Case (A), $E$ is irreducible, and hence is a connected component of $D_1 \cap D_2$.

Proof. Assume that $E$ is reducible, and let $E_1$ be an irreducible component of $E$ such that

$$E_1 \subset D_1 \cap D_2. \tag{2.2.1}$$

Since $E$ is connected, there exists another irreducible component, say $E_2$, of $E$ such that

$$E_2 \not\subset D_1 \cap D_2 \text{ and } E_1 \cap E_2 \neq \emptyset. \tag{2.2.2}$$

In particular, $(D_k \cdot R) > 0$ for either $k = 1$ or 2. Let us assume

$$D_2 \cdot R > 0. \tag{2.2.3}$$
On the other hand, by (2.2.1) with the assumption (0.2.1) and Lemma 1.1, \( E_1 \not\subset D_k \) for all \( k \geq 3 \), thus

\[ (D_k \cdot R) \geq 0 \quad (\forall k \geq 3). \]

From (2.2.3) and (2.2.4), we necessarily have

\[ (D_1 \cdot R) < 0, \]

and in particular

\[ E \subset D_1. \]

Let \( g|_{D_1} : D_1 \to g(D_1) \) be the restriction of \( g \) to \( D_1 \), let \( g(D_1)^\sim \) be the normalization of \( g(D_1) \), and \( h : D_1 \to g(D_1)^\sim \) the morphism induced from \( g|_{D_1} \).

Then \( h \) is a birational morphism such that

\[ \text{Exc } h = E. \]

Thus for a general irreducible curve \( C \) in \( E_1 \),

\[ (E_1 \cdot C)_{D_1} < 0. \]

This contradicts (2.2.3), since \( (E_1 \cdot C)_{D_1} = (D_2 \cdot C)_X \) (2.2.1). Hence \( E \) must be irreducible. \( \square \)

**Proposition 2.3.** \( E \approx \mathbb{P}^2 \).

**Proof.**

Let \( g(D_k)^\sim \) be the normalization of \( g(D_k) \) (\( k = 1, 2 \)), and let \( h_k : D_k \to g(D_k)^\sim \) be the morphism induced from \( g|_{D_k} \).

First we claim that

\[ (D_k \cdot R) < 0 \]

and \( -K_{D_k} \) is \( h_k \)-ample for both \( k = 1, 2 \).

In fact, let \( C \) be a general irreducible curve in \( E \). Then as in (2.2.8)

\[ (D_2 \cdot C)_X = (E_1 \cdot C)_{D_1} < 0, \]

and similarly \( (D_1 \cdot C)_X < 0 \). Thus

\[ (-K_{D_k} \cdot C)_{D_k} = (-K_C \cdot C)_X - (D_k \cdot C)_X > 0 \quad (k = 1, 2), \]

and we get (2.3.1).

By \( \text{Exc } h_k = E \), Corollary 1.3 and Proposition 2.2,

\[ \rho(D_k/g(D_k)^\sim) = 1 \]

and it follows from \([C]\), together with Corollary 1.3, that

\[ E \approx \mathbb{P}^2 \text{ or } S_2. \]

Assume \( E \approx S_2 \) to get a contradiction.
Let $B$ be a general (+2)-section of $E \simeq S_2$. Since $D_1$ is a Cartier divisor of $X$ and since $E$ is a Cartier divisor of $D_1$ (0.5.1), $X$ is smooth along $E$ outside the vertex. Hence

\[(2.3.3)\quad B \cap \text{Sing } X = \emptyset.\]

Consider the exact sequence:

\[0 \to N_{B/E} \to N_{B/X} \to N_{E/X} \otimes \mathcal{O}_B \to 0\]

Since $E$ is a connected component of $D_1 \cap D_2$,

\[N_{E/X} \otimes \mathcal{O}_B \simeq \mathcal{O}_B(D_1) \oplus \mathcal{O}_B(D_2),\]

and we have

\[(2.3.4)\quad c_1(N_{B/X}) = c_1(N_{B/E}) + (D_1 \cdot B) + (D_2 \cdot B)
\quad = 2 + (D_1 \cdot B) + (D_2 \cdot B).\]

On the other hand, since $B \cap \text{Sing } X = \emptyset$ (2.3.3), we can consider the deformation of $B$ inside $X$ [Mo1] (cf. [I], [Wis], [Ko2]). Let $T := \text{Hilb}_{X,[B]}$ be the connected component of the Hilbert scheme Hilb$_X$ containing the point corresponding to $B$. Then

\[(2.3.5)\quad 3 = \dim T \geq \dim X + (-K_X \cdot B) - 3
\quad = (-K_X \cdot B) + 1.\]

Moreover, since $-K_X$ is Cartier (0.5.1), and since the numerical class of $B$ is the double of the class of a ruling of $E$,

\[(-K_X \cdot B) \geq 2.\]

Hence the inequality (2.3.5) must be the equality:

\[(2.3.6)\quad (-K_X \cdot B) = 2,\]

that is,

\[(2.3.7)\quad c_1(N_{B/X}) = 0.\]

From this and (2.3.4),

\[(D_1 \cdot B) + (D_2 \cdot B) = -2.\]

Again since $(D_k \cdot B)$ is an even integer ($k = 1, 2$),

\[(2.3.8)\quad (D_k \cdot B) \geq 0 \quad \text{for either } k = 1 \text{ or } 2,\]

which contradicts (2.3.1). Hence we must have $E \simeq \mathbb{P}^2$ (2.3.2). $\square$
Corollary 2.4. $X$ is smooth along $E$, and

$$N_{E/D_1} \simeq N_{E/D_2} \simeq \mathcal{O}_{\mathbb{P}^2}(-1).$$

Proof. First by Proposition 2.3 and Lemma 1.5, $D_1, D_2$ are smooth along $E$, and hence so is $X$, since $D_1$ is a Cartier divisor of $X$.

Next, let $h_k: D_k \to g(D_k)$ be as in (2.3.0). Since $\text{Exc } h_k = E$ for both $k = 1, 2$,

$$N_{E/D_k} \simeq \mathcal{O}_{\mathbb{P}^2}(a_k) \text{ with } a_k < 0 \ (k = 1, 2).$$

In particular $N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2}(a_1) \oplus \mathcal{O}_{\mathbb{P}^2}(a_2)$. Moreover since $-K_X$ is $g$-ample, we must have

$$a_1 = a_2 = -1. \quad \square$$

Corollary 2.5. The flip of $g$ exists.

Proof. Since $X$ is smooth along $E$, this is a direct consequence of Kawamata [Kaw4]. $\square$

The rest thing we have to prove is the following:

Proposition 2.6. After a suitable renumbering of $\{D_3, \ldots, D_r\}$, either one of the followings holds:

(I) $D_3 \cap E, D_4 \cap E$ are distinct lines in $E$, and $D_k \cap E = \emptyset \ (\forall k \geq 5)$, or

(II) $D_3 \cap E$ is a smooth conic in $E$, and $D_k \cap E = \emptyset \ (\forall k \geq 4)$.

Proof. Let $l$ be any line in $E$. Since $N_{E/D_k} \simeq \mathcal{O}_{\mathbb{P}^2}(-1) \ (k = 1, 2)$,

$$(D_k \cdot l)_X = (-K_X \cdot l)_X + (K_{D_k} \cdot l)_{D_k}$$

$$= 1 - 2 = -1 \ (k = 1, 2).$$

Thus either

(2.6.1) $$(D_3 \cdot l) = 2, \ (D_k \cdot l) = 0 \ (\forall k \geq 4), \text{ or}$$

(2.6.2) $$(D_3 \cdot l) = (D_4 \cdot l) = 1, \ (D_k \cdot l) = 0 \ (\forall k \geq 5).$$

In the case (2.6.1), $D_3 \cap E$ must be a smooth conic $C$ in $E \simeq \mathbb{P}^2$, by the assumption (0.2.1). Hence we have the case (II).

On the other hand, in the case (2.6.2), obviously we have (I). $\square$

Now the proof of Theorem 2.1 is completed.
§3. The case $E_i \subset D_1$, $E_i \not\subset \bigcup_{k \geq 2} D_k$ and $E_i \cap \bigcup_{k \geq 2} D_k \neq \emptyset$ for any $E_i$.

The aim of this section is:

**Theorem 3.1.** In the Case (B) in Proposition 1.4, the followings hold:

1. $E$ is irreducible and is isomorphic to $\mathbb{P}^2$,
2. $X$ is smooth along $E$,
3. $N_{E/D_1} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$ and $N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$,
4. After a suitable renumbering of $\{D_2, \ldots, D_r\}$, $D_2 \cap E$ is a line in $E$, and $D_k \cap E = \emptyset$ ($\forall k \geq 3$), and
5. The flip $g^+$ of $g$ exists.

First we shall prove:

**Lemma 3.2.** In the Case (B),

1. $E \subset D_1$ and $(D_1 \cdot R) < 0$.
2. For any irreducible curve $C$ in $E$,
   
   $$(-K_{D_1} \cdot C) \geq 2.$$

**Proof.** (1) By the condition (B), $E \subset D_1$ and $\left( \sum_{k \geq 2} D_k \cdot R \right) > 0$, that is,

$$ (D_1 \cdot R) < 0. $$

(2) Let $C \subset E$ be any irreducible curve. Then

$$ (-K_{D_1} \cdot C) = -(D_1 \cdot C) + (-K_X \cdot C) \geq 2,$$

since $D_1$ and $-K_X$ are both Cartier. \(\square\)

**Lemma 3.3.** There exists an irreducible component $E_1$, say, of $E$ which is isomorphic to $\mathbb{P}^2$ such that

$$ N_{E_1/D_1} \simeq \mathcal{O}_{\mathbb{P}^2}(-1), \quad N_{E_1/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}, \quad \text{and} \quad \mathcal{O}_{E_1}(D_1) \simeq \mathcal{O}_{\mathbb{P}^2}(-1).$$

**Proof.**

(3.3.0) Let $g(D_1)^\sim$ be the normalization of $g(D_1)$, and consider the morphism $h : D_1 \to g(D_1)^\sim$ induced from $g|_{D_1}$, as in (2.2.7), (2.3.0).

By Lemma 3.2 (2), $-K_{D_1}$ is $h$-ample. Take the contraction morphism $D_1 \to V$ associated to any extremal ray of $\overline{NE}(D_1/g(D_1)^\sim)$. Then by Cutkosky [C], together
with Corollary 1.3 and Lemma 3.2 (2), the exceptional locus, say $E_1$, must be isomorphic to $\mathbb{P}^2$, and

$$N_{E_1/D_1} \simeq \mathcal{O}_{\mathbb{P}^2}(-1).$$

(3.3.1)

In particular $D_1$ and hence $X$ is smooth along $E_1$.

Let $\mathcal{O}_{E_1}(D_1) \simeq \mathcal{O}_{\mathbb{P}^2}(a)$ ($a \in \mathbb{Z}$). Then from (3.3.1) and the exact sequence:

$$0 \to N_{E_1/D_1} \to N_{E_1/X} \to \mathcal{O}_{E_1}(D_1) \to 0$$

we have

$$N_{E_1/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(a).$$

Since $E_1$ never deforms inside $X$, and since $\mathcal{O}_{E_1}(-K_X)$ is ample, we necessarily have $a = -1$, and we are done. □

3.4. Consider the local flip (Kawamata flip) \[Kaw4\]

$$\eta^{(0)}: X \dasharrow X^{(1)}$$

with respect to $E_1$, and let

$$g^{(1)'}: X^{(1)} \to Y$$

be the structure morphism. Let $E_1^{+ (1)} \simeq \mathbb{P}^1$ be the flipped curve, and let $E_i^{(1)}$ ($i \geq 2$), $D^{(1)}$ be the proper transform of $E_i$, $D_k$ in $X^{(1)}$, respectively. Then

(3.4.1)

$X^{(1)}$ is smooth along $E_1^{+ (1)}$

[loc.cit.], and

(3.4.2)

$E_1^{+ (1)} \subset D_2^{(1)}$, and $\text{Exc } g^{(1)'} = E_1^{+ (1)} \cup \bigcup_{i \geq 2} E_i^{(1)}$.

Hence $g^{(1)'}$ is factored through

$$g^{(1)}: X^{(1)} \to Y^{(1)}$$

(see the diagram (3.5.1) below) such that

(3.4.3)

$-K_{X^{(1)}}$ is $g^{(1)}$-ample and $E^{(1)} := \text{Exc } g^{(1)} = \bigcup_{i \geq 2} E_i^{(1)}$.

3.5. If $\text{Exc } g^{(1)}$ is still reducible, then return back to the situation of Lemma 3.3, with the substitutions of $X$, $D_1$ by $X^{(1)}$, $D_1^{(1)}$, respectively. Then we again find an $E_2^{(1)} \simeq \mathbb{P}^2$, say, which has the normal bundle $\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(a)$ in $X^{(1)}$. Do the same procedure as in 3.4 above for $X^{(1)}$ instead of $X$, to get $X^{(2)}$ and $g^{(2)}: X^{(2)} \to Y^{(2)}$, satisfying the similar condition to (3.4.3). If we continue the processes successively until $\text{Exc } g^{(p)}$ becomes irreducible, we get the diagram

$$\begin{array}{ccccccc}
X & \eta^{(0)} & \to & X^{(1)} & \eta^{(1)} & \to & X^{(2)} & \ldots & \eta^{(p-1)} & \to & X^{(p)} \\
g \downarrow & & & & & & g^{(2)} \downarrow & & & & g^{(p)} \downarrow \\
\Delta & \leftarrow & Y & \leftarrow & Y^{(1)} & \leftarrow & Y^{(2)} & \ldots & \leftarrow & Y^{(p)}
\end{array}$$

(3.5.1)

with

(3.5.2)

$\text{Exc } g^{(p)} = E^{(p)}_{p+1} \simeq \mathbb{P}^2$, along which $X^{(p)}$ and $D_1^{(p)}$ are smooth.
Proposition 3.6. In Lemma 3.3, $E$ is irreducible: $E = E_1 \simeq \mathbb{P}^2$. The flip of $g$ exists.

Proof. First $X^{(p)}$ is smooth along $E^{(p)}_{p+1}$ (3.5.2). Hence $X^{(p-1)}$ is smooth along

$$\eta^{(p-1)-1}(E^{(p)}_{p+1} - E^{(p)}_{p+1} \cap E^{(p)}_p) = E^{(p-1)}_{p+1} - E^{(p-1)}_p \cap E^{(p-1)}_p,$$

over which the flip $\eta^{(p-1)}$ is an isomorphism.

On the other hand, $X^{(p-1)}$ is smooth also along the flipping surface $E^{(p-1)}_p \simeq \mathbb{P}^2$. Thus

(3.6.1) $X^{(p-1)}$ is smooth along $E^{(p-1)}_p \cup E^{(p-1)}_{p+1} = \text{Exc } g^{(p-1)}$.

If we go further this argument from $X^{(p-1)}$ upstream to $X^{(0)} = X$, we conclude that

(3.6.2) $X$ is smooth along $E$.

Hence by Kawamata [Kaw4], $E$ must be irreducible, $E \simeq \mathbb{P}^2$, and the flip $g^+$ of $g$ exists. □

3.7. Proof of Theorem 3.1.

Since $\mathcal{O}_E(D_1) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$ (Lemma 3.3), for any line $l$ in $E$ we have $(D_1 \cdot l) = -1$, i.e.

$$\left( \sum_{k \geq 2} D_k \cdot l \right) = 1.$$

Thus we find an unique $D_2$, say, among $\{D_2, \ldots, D_r\}$ such that $D_2 \cap E$ is a line, and $D_k \cap E = \emptyset$ (\forall $k \geq 3$).

Hence we get Theorem 3.1. □

§4. The exclusion of the case $E \subset D_1$ and $E \cap \bigcup_{k \geq 2} D_k = \emptyset$.

In this section we shall exclude the Case (C) in Proposition 1.4:

Theorem 4.1. The Case (C) in Proposition 1.4 never happens.

We adopt the simplified proof of Theorem 4.1 based on J.Kollár’s idea.

Theorem 4.2. (Kollár - Mori [KoMo] 11.4)

Let $f: U \to \Delta$ be a surjective morphism from a 4-fold $U$ with at most terminal singularities to the disc $\Delta$. Let $U_0$ be its central fiber. Assume that $U_0$ has at most terminal singularities, and there is a proper bimeromorphic morphism $\varphi_0: U_0 \to V_0$ to a germ $(V_0, Q)$ of some normal 3-fold such that

$$R^1 \varphi_0^* \mathcal{O}_{U_0} = 0.$$
Then there exists a proper surjective morphism \( \varphi : U \to V \) to some normal 4-fold \( V \) which factors through \( f \) such that

1. The central fiber of the structure morphism \( V \to \Delta \) is isomorphic to \( V_0 \), and
2. \( \varphi |_{U_0} = \varphi_0 \) under this identification.

4.3. Proof of Theorem 4.1. (Following the idea of J.Kollár)

First by the condition (C),
\[
\left( \sum_{k \geq 2} D_k \cdot R \right) = 0,
\]
*i.e.*
\[
(D_1 \cdot R) = 0.
\]
Hence for any irreducible curve \( C \subset E \), we have
\[
(-K_{D_1} \cdot C) = (-K_X \cdot C).
\]

Let \( U \) be an analytic neighborhood of \( E \) in \( X \) so that
\[
U \cap f^{-1}(t) \text{ contains no proper 1-dimensional subspaces } \forall t \in \Delta - \{0\},
\]
and consider \( g|_U : U \to g(U) \).

Let \( D_1^\circ := D_1 \cap U (= D \cap U) \), and let \( g(D_1^\circ) \sim \) be the normalization of \( g(D_1^\circ) \).

Then the induced morphism \( h^\circ : D_1^\circ \to g(D_1^\circ) \sim \) is a bimeromorphic morphism such that
\[
-K_{D_1^\circ} \text{ is } h^\circ \text{-ample.}
\]
Consider \( \overline{NE}(D_1^\circ \cap g(D_1^\circ) \sim) \). Choose any extremal ray of it, and let \( \varphi_0 : D_1^\circ \to \varphi_0(D_1^\circ) \) be the associated contraction. Since \( \text{Exc } \varphi_0 \subset E \) and \( \dim \text{Exc } \varphi_0 = 2 \) (Remark 0.6),
\[
\text{Exc } \varphi_0 = E_i \text{ for some } i.
\]
Let
\[
Q := \varphi_0(E_i).
\]

Then by \( R^1\varphi_0_*O_{D_1^\circ} = 0 \) and Theorem 4.2 we have a proper surjective morphism \( \varphi : U \to V \) to some normal 4-fold \( V \) over \( \Delta \) such that
\[
\left\{ \begin{array}{l}
(V)_0 \simeq \varphi_0(D_1^\circ), \\
\varphi|_{D_1^\circ} = \varphi_0.
\end{array} \right.
\]

By (4.3.2), (4.3.3) and (4.3.6),
\[
\varphi \text{ is a flipping contraction with } \text{Exc } \varphi = E_i.
\]
By the above construction,
\[
((V)_0, Q) \text{ is a germ of a 3-dimensional terminal singularity which is a Cartier divisor of the 4-dimensional flipping singularity } (V, Q) \text{. Note that } V-Q \text{ is Gorenstein, since so is } U.
\]

If the singularity index of \( ((V)_0, Q) \) is greater than 1, then this is never deformed to be Gorenstein [Sc], [R1,4], [Mo3,4], [KS], a contradiction to (4.3.9). So \( ((V)_0, Q) \) must be Gorenstein, and hence a hypersurface singularity [loc.cit]. Then \( (V, Q) \) is also a hypersurface singularity (4.3.9), which again contradicts (4.3.9), since flipping singularities can never be \( \mathbb{Q} \)-Gorenstein. Hence the Theorem 4.1. \( \square \)
By completely the same argument as in 4.3, we can prove the following which might be a little stronger than Theorem 4.1:

**Remark 4.4.** Let \( g : X \to Y \) be as in Definition 0.3, and let \( E \) be any connected component of \( \text{Exc} \, g \) as in Main Theorem 0.5. Instead of (0.5.1), we assume the following:

(4.4.1) \( E \subset D_1 \), and \( g(D_1) \) has a terminal singularity at \( g(E) \).

Then

\[ E \cap \bigcup_{k \geq 2} D_k \neq \emptyset. \]

\( \square \)

§5. Description of flips.

**Notation 5.0.** Let \( g : X \to Y \) be as in Main Theorem 0.5. \( g \) is of type either (A-I), (A-II) or (B). Let \( g^+ : X^+ \to Y \) be the flip of \( g \), let \( E^+ \simeq \mathbb{P}^1 \) be the flipped curve, and \( D_k^+ \) the proper transform of \( D_k \) \((k = 1, \ldots, r)\). Moreover, let

\[ \eta := g^{+ - 1} \circ g : X \dashrightarrow X^+. \]

Recall that \( N_{E^+/X^+} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3} \) [Kaw4].

In the following 5.1 through 5.3, we shall describe the flip \( g^+ : X^+ \to Y \), depending on the type of \( g \).

5.1. (For the Type (A-I))

If \( g : X \to Y \) is of Type (A-I), then

(1) \( E^+ \cap D_1^+ \cap D_2^+ = \emptyset \), and \( (D_1^+ \cdot E^+) = (D_2^+ \cdot E^+) = 1 \).

(2) \( E^+ \subset D_3^+ \cap D_4^+ \), \( D_3^+ \), \( D_4^+ \) are smooth along \( E^+ \), and

\[ N_{E^+/D_3^+} \simeq N_{E^+/D_4^+} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}. \]

(3) \( N_{D_3 \cap E/D_3} \simeq N_{D_4 \cap E/D_4} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \),

and \( \eta|_{D_k} : D_k \dashrightarrow D_k^+ \) gives the flop of \( D_k \supset D_k \cap E \) \((k = 3, 4)\).

5.2. (For the Type (A-II))

If \( g : X \to Y \) is of Type (A-II), then

(1) \( E^+ \cap D_1^+ \cap D_2^+ = \emptyset \), and \( (D_1^+ \cdot E^+) = (D_2^+ \cdot E^+) = 1 \).

(2) \( E^+ \subset D_3^+ \), and \( D_3^+ \) has the canonical singularity of type \( 1/2(1, 1, 0) \) along \( E^+ \).

(3) \( N_{D_3 \cap E/D_3} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 2} \),

and \( \eta|_{D_3} : D_3 \dashrightarrow D_3^+ \) gives the anti-flip of \( D_3 \supset D_3 \cap E \).
5.3.  *(For the Type (B))*

If \( g : X \rightarrow Y \) is of Type (B), then

1. \((D_1^+. E^+) = 1.\)
2. \(E^+ \subset D_2^+, \text{ and } D_2^+ \text{ is smooth along } E^+.
3. \(N_{D_2 \cap E / D_2} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}, \text{ and } \eta|_{D_2} : D_2 \rightarrow D_2^+ \) gives the flop of \( D_2 \supset D_2 \cap E. \)

**Remark 5.4.** If \( g \) is of Type (A-I) or (B), then the flip \( g^+ \) again satisfies all the conditions of the Assumption 0.2.

On the other hand, if \( g \) is of Type (A-II), then \( X^+ \) is still smooth while \( D_3^+ \) is singular, so \((0.2.1)\) fails for such \( g^+. \)

This is the reason why in (0.3.1) we eliminated the condition \((0.2.1)\) from the definition of flips.

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