Radiative corrections to the background of $\mu \rightarrow e \gamma$ decay

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Abstract

Radiative muon decay in the kinematics similar to the neutrinoless decay $\mu \rightarrow e \gamma$ is considered. Radiative corrections due to one-loop virtual photons and emission of additional soft or hard photons are taken into account. Analytical expressions and numerical estimations are presented.

$$\mu \rightarrow e \gamma$$
1 Introduction

Since the discovery of the muon in 1936 its relation to electron is a puzzle. Really, the only difference between these two elementary particles is in their masses. The lepton number conservation law has no deep sources in space–time properties or gauge theories. Moreover many extensions of the Standard Model predict processes with violation of this law ($\mu \rightarrow e\gamma$, $e\gamma\gamma$, $e\bar{e}e$ etc.). Intensive search of these extensions was performed in 1977 [1]. The modern state of the subject is elucidated in papers [2, 3] and references therein. Indeed, if there is a unification of quarks and leptons, then the existence of $b \rightarrow s\gamma$ decay leads to that of $\mu \rightarrow e\gamma$. Different models give a wide range of predictions for the branching ratio of this neutrinoless muon decay. The present experimental upper limit [2] on the branching ratio is

$$B = \frac{\Gamma(\mu \rightarrow e\gamma)}{\Gamma^\mu} < 4.9 \cdot 10^{-11}.$$  

(1)

This value imposed already strong restrictions on parameters of supersymmetric [4, 5] and other models [6]. In the model independent approach [7] one gets boundaries on parameters of possible structures in the matrix element of the muon decay. Several new experiments are planned to improve the precision. They will either find the decay or put much more stronger restrictions and even discriminate some models. The forthcoming experiment at PSI (if doesn’t find the decay) will put the limit on the $\mu \rightarrow e\gamma$ decay branching ratio of about $5 \cdot 10^{-14}$. Another experiment is proposed at BNL, where they are going to reach the level of $10^{-16}$. These experiments are very important, since they have rather wide possibilities for the search of new physics comparable with those of high energy colliders. In this paper we consider the important background process

$$\mu(p) \rightarrow e(p_2) + \gamma(k_1) + (\nu_\mu + \bar{\nu}_e)(q)$$  

(2)

in the kinematical situation, imitating the neutrinoless decay. Namely, we suppose

$$n = \frac{2p_2q}{M^2} \sim l = \frac{2k_1q}{M^2} \sim \sqrt{Q^2} = \sqrt{q^2/M^2} \ll 1,$$

where $q$ is the 4–momentum carried by neutrinos, and $M$ is the muon mass. The width in the lowest order of perturbation theory was calculated many years ago [8]. The expression for the width reads:

$$d\Gamma^\mu_{\nu e\nu\gamma} = \frac{2\alpha G_F^2}{6(2\pi)^6 M} \frac{d^3p_2d^3k_1}{\varepsilon_2\omega_1} \left[-\left(\frac{M^4}{2} - q^2 \left(q^2 - \frac{M^2}{2}\right)\right) \left(\frac{p}{pk_1} - \frac{p_2}{p_2k_1}\right)^2 + 4q^2 + \frac{(k_1q)^2}{(p_2k_1)(pk_1)}(2q^2 + M^2)\right], \quad q = p - p_2 - k_1, \quad \omega_1 = k_1^0. \quad (3)$$

The validity of this formula may be confirmed in the limiting case of soft photon. A multiplier 2 was lost on right hand side (rhs) of expression for the width in [8].
The polarized muon radiative decay was considered in [9], and as a background to the neutrinoless decay it was extensively discussed in Ref. [3].

## 2 Radiative corrections

In the *imitating kinematics* (IK) we introduce the relative energy deviations of the hard electron and photon from $M/2$ and the acollinearity angle $\theta$:

\[
\sigma_1 = 1 - \frac{2\omega_1}{M}, \quad \sigma_2 = 1 - \frac{2\varepsilon_2}{M}, \quad \theta = \mathbf{p_2}, -\mathbf{k_1}.
\]  

Here we suggest

\[
\sigma_1 \sim \sigma_2 \sim \theta \ll 1.
\]  

Rearranging the phase volume

\[
d\Phi = \frac{d^3p_2d^3k_1}{\omega_1\varepsilon_2} = 8\pi^2 \left(\frac{M}{2}\right)^4 (1 - \sigma_1)(1 - \sigma_2)d\sigma_1d\sigma_2d\theta d\phi,
\]

and expanding the expression for the width in the Born approximation [9], we obtain:

\[
\frac{d\Gamma_{\text{Born}}}{d\sigma_1d\sigma_2d\theta d\phi} = \frac{d\Gamma_0}{d\sigma_1d\sigma_2d\theta d\phi}(1 + \delta_1),
\]

\[
\frac{d\Gamma_0}{d\sigma_1d\sigma_2d\theta d\phi} = \frac{\alpha G_F^2M^5}{3 \cdot 2^7\pi^4 R}, \quad R = \sigma_2^2(1 + \xi) + \left(4\sigma_1\sigma_2 - \frac{\theta^2}{2}\right)(1 - \xi) - \sigma_2\theta\eta,
\]

\[
\xi = s \cos(\mathbf{s}, \mathbf{p_2}), \quad \eta = s \sin(\mathbf{s}, \mathbf{p_2}) \cos \varphi, \quad \mathbf{k_1}s = \omega_1(-\xi \cos \theta - \eta \sin \theta),
\]

\[
\delta_1 = \frac{1}{R}\left[-5 + 3\xi\sigma_2^2 - 4(1 - \xi)\sigma_2^2\sigma_1 + 2(1 - \xi)\sigma_1\theta^2 + \frac{1}{2}(3 - \xi)\sigma_2\theta^2 + 4\eta\sigma_1\sigma_2\theta - \frac{5}{4}\eta\theta^3\right].
\]

Here $s$ denotes the spin of the muon, and $\varphi$ is the azimuthal angle between planes formed by $(\mathbf{s}, \mathbf{p_2})$ and $(\mathbf{s}, \mathbf{k_1})$ in the rest reference frame of the muon. Note that averaging the above expression over the angle $\varphi$ leads immediately to the result presented in [3]. We shall name higher than second order contributions on the rhs of (3) (and $\delta_1$ in rhs of (6)) as *relativistic* corrections. In this paper we will consider the radiative corrections to this width bearing in mind virtual corrections described by the Feynman diagrams drawn in Fig. 1 together with those arising from the emission of additional soft and hard photons.

For the measurement of an additional hard photon emission, two cases have to be considered: with and without external magnetic field. In the case without magnetic field the additional hard photon, moving along the final electron trajectory within a small angle, which is equal to the detector angular resolution, is registered together...
with the electron. In the opposite case (with magnetic field) those events will be rejected from statistics due to criterion: the energy of the electron is less than the maximum energy within some accuracy. The standard calculation of one–loop virtual corrections can be considerably simplified by using the IK features. Some details of our calculations (traces, vertices and the Tables of relevant integrals) are given in Appendices.

Ultraviolet divergences of loop integrals are eliminated in a standard way using the renormalization constants of the wave functions of electron and muon:

\[ Z_{1e} = 1 - \frac{\alpha}{2\pi} \left\{ \frac{1}{2} \ln \frac{\Lambda^2}{m^2} + \ln \frac{\lambda^2}{m^2} + \frac{9}{4} \right\}, \]

\[ Z_{1\mu} = 1 - \frac{\alpha}{2\pi} \left\{ \frac{1}{2} \ln \frac{\Lambda^2}{M^2} + \ln \frac{\lambda^2}{M^2} + \frac{9}{4} \right\}, \]

where \( m, \lambda, \Lambda \) are the electron mass, infrared and ultraviolet cut–off momentum parameters, \( (\lambda \ll m, \Lambda \gg M) \). The final result for the one–loop virtual corrections reads:

\[
\frac{d\Gamma^\text{virt}}{d\sigma_1 d\sigma_2 d\theta d\theta} = \frac{d\Gamma_0}{d\sigma_1 d\sigma_2 d\theta d\theta} \delta_V, \\
\delta_V = \frac{\alpha}{\pi R} \left\{ R \left( \frac{3}{2} L - (L - 1) \ln \frac{M m}{\lambda^2} + \frac{\pi^2}{6} \right) + \frac{\sigma_1^2}{4} (1 - \xi) \\
+ \sigma_2^2 \left( -3 - 2L + \frac{\pi^2}{6} \right) (1 + \xi) + \sigma_1 \sigma_2 \left[ -\frac{23}{4} + \xi \left( \frac{9}{4} + 4L + \frac{2\pi^2}{3} \right) \right] \\
+ \theta^2 \left[ \frac{13}{16} - \xi \left( \frac{3}{16} + \frac{1}{2} L + \frac{\pi^2}{12} \right) \right] + \sigma_2 \theta \eta \left( \frac{7}{4} + 2L \right) - \frac{\pi^2}{12} \sigma_1 \theta \eta \right\}, \\
L = \ln \frac{M}{m},
\]

Taking into account the emission of additional soft photon requires some care. The reason is that the energy–momentum carried by soft photons as well as by neutrinos cannot in principle be distinguished in the experiment. We introduce some small energy fraction parameter \( \Delta_1 = 2\omega_{\text{soft}}/M \ll \sigma_1, \sigma_2, \theta \) which should not affect on
observable quantities and actually cancels out in the final result. Emission of an additional soft photon, having energy lesser than $M\Delta_1/2$, can be taken into account in a usual way \[10\]. The corresponding expression looks as follows:

$$
\frac{d\Gamma_{\text{soft}}}{d^3p_2d^3k_1} = \frac{d\Gamma_0}{d^3p_2d^3k_1} \delta_s, \quad \delta_s = \frac{\alpha}{\pi} \left[ 2(L-1) \ln \frac{2\omega_{\text{soft}}}{\lambda} - L^2 + L + 1 - \frac{\pi^2}{6} \right]. \tag{9}
$$

Let us suppose, that a photon with momentum $k_2$, having energy more than $\omega_{\text{soft}}$, is emitted in such a way that we have still allowed values of the final electron and hard photon momenta. In this case the additional photon cannot be called soft, because it changes the kinematics of the process. We have to consider the corresponding contribution applying complete set of kinematical restrictions. The main condition is that the missing momentum squared must be positive:

$$
\tilde{q}^2 = (p - p_2 - k_1)^2 > 0, \quad \tilde{q} = q + k_2. \tag{10}
$$

Having in mind that the matrix element squared is proportional to the second power of small neutrino momenta, we can write down the contribution under consideration in the factorized form

$$
\frac{d\Gamma_{\gamma}}{d^3p_2d^3k_1} = \frac{d\Gamma_0}{d^3p_2d^3k_1} \frac{1}{R} \left( -\frac{\alpha}{4\pi^2} \right) \int_{\omega_2 > \omega_{\text{soft}}} \frac{d^3k_2}{\omega_2} \left( \frac{p}{pk_2} - \frac{p_2}{pk_2} \right)^2 R \Theta(\tilde{q}^2), \tag{11}
$$

$$
\tilde{R} = 2\tilde{Q}^2 + 2\tilde{l}\tilde{n} + \tilde{l}^2 + \xi(-2\tilde{Q}^2 - 2\tilde{l}\tilde{n} + \tilde{l}^2),
$$

$$
\tilde{Q}^2 = \sigma_1\sigma_2 - \frac{\theta^2}{4} - x\sigma_1, \quad \tilde{l} = \sigma_2 - x, \quad \tilde{n} = \sigma_1,
$$

$$
\tilde{q}^2 = \frac{2}{M\tilde{l}}^2, \quad \tilde{l} = \frac{2\tilde{q}p_2}{M}, \quad \tilde{n} = \frac{2\tilde{q}k_1}{M}, \quad x = \frac{2\omega}{M}, \quad \omega_2 = k_2^0.
$$

The difference in respect to the case of pure soft photon emission is that we have the shifted quantity $\tilde{R}$ instead of the Born one ($R$) under the integral sign. The above expression guarantees that the energies and angles of the observed electron and photons are the same as defined in \[4\]. Transforming the above formula we get

$$
\delta_\gamma = \frac{\alpha}{2\pi} \left\{ \int_{\Delta_1} \frac{dx}{x} R(x, c_2 = 1) \left[ -2 + 4 \ln \left( \frac{M\theta_0}{2m} \right) \right] \Theta \left( \sigma_1\sigma_2 - \frac{\theta^2}{4} - x\sigma_1 \right) \right. \right.
$$

$$
+ \int_{0}^{2\pi} \frac{d\varphi_2}{2\pi} \int_{-1}^{1} dc_2 \int_{\Delta_1} \frac{dx}{x} R(x, c_2, \varphi_2) \left( -1 + \frac{2}{1-c_2} \right) \Theta \left( \sigma_1\sigma_2 - \frac{\theta^2}{4} \right)
$$

$$
- \frac{x}{2} \left( \sigma_1 + \sigma_2 + c_2(\sigma_1 - \sigma_2) - \theta \sqrt{1-c_2^2 \cos \varphi_2} \right), \right\} \right\}, \tag{12}
$$

$$
c_2 = \cos(\hat{k}_2, \hat{p}_2), \quad x_{\text{max}} = \frac{1}{2} (\sigma_1 + \sigma_2 + \sqrt{(\sigma_1 - \sigma_2)^2 + \theta^2}).
$$
In this expression we introduced an auxiliary parameter $\theta_0$ in order to separate the contribution, when the additional photon is emitted collinear to the electron momentum; $\theta_0 \ll 1$. So, the first term of Eq. (12) can be integrated analytically in order to keep track of the leading logarithmic part. We checked that the final expression does not depend on $\theta_0$.

Then we arrive to the total answer, that has the form

$$\frac{d\Gamma}{d\Gamma_0} = 1 + \delta_1 + \delta_V + \delta_S + \delta_\gamma.$$  (13)

The dependence on the soft photon parameter $\Delta_1$ cancels out in the sum $\delta_S + \delta_\gamma$ whereas the fictitious photon mass $\lambda$ disappears in the sum $\delta_S + \delta_\gamma$.

If the experimental set–up does not distinguish in the detector an electron with a collinear photon, we have to modify our results in the following way. Let $\theta_0$ define the aperture of the narrow cone, within which the two particles would be detected as a unique one. Then we should take the non–shifted value for $R$ in the first integral of Eq. (12). We have to add also the rest contribution of hard photon emission within the same cone. It can be obtained using the quasireal electron method [11]:

$$\frac{d\Gamma^{\text{hard}}}{d\sigma_1 d\sigma_2 \theta d\theta} = \frac{d\Gamma_0}{d\sigma_1 d\sigma_2 \theta d\theta} \frac{\alpha}{\pi} \int_1^x dx \frac{1}{x} \frac{1}{x_\text{max}} \ln \left( \frac{M\theta_0}{2m} \right),$$  (14)

$$x_\text{max}' = \sigma_2 - \frac{\theta^2}{4\sigma_1}.
$$

The lower limit comes here from the $\Theta$-function in the first integral of Eq. (12).

In the presence of a magnetic field, when the electron trajectory is curve, the above expression will give a part of the background to the process $\mu \rightarrow e\gamma\gamma$, considered in paper [12]. Really, the final electron will have the energy $M(1 - x)/2$, whereas the quantities $\sigma_1, \sigma_2, \theta$ are the same as in the case of single photon emission.

### 3 Conclusions

In Table 1 we give numerical values for $\delta_1$, $\delta_{SV\gamma} = \delta_S + \delta_V + 2\delta_\gamma$ versus $\sigma_1, \sigma_2, \theta$, which characterize the missing energy and momentum (see Eq. (4)), and the degree of muon polarization $\xi$. For typical expected values of $\sigma_1 \sim \sigma_2 \sim \theta \sim 10^{-2}$ (we give 6 points) one can see, that the relativistic and QED corrections should be taken into account on the same footing. The resulting correction to the Born–level decay width $d\Gamma_0$ (see Eq. (3)) is given by the sum $\delta_{SV\gamma} + \delta_1$.

A measurement of the radiative muon decay in the kinematics close to that of neutrinoless decay is required to get an independent normalization for the search of the decay $\mu \rightarrow e\gamma$. For this aim our results, we hope, would be important.
10^2\sigma_1 & 10^2\sigma_2 & 10^2\theta & 10^2\delta_1 & 10^2\delta_{SV\gamma} \\
\xi = 0  & \xi = 0.5 & \xi = -0.5 & \xi = 0  & \xi = 0.5 & \xi = -0.5 \\
1.0  & 1.0  & 1.0  & -1.2 & -1.0 & -1.3 & -10.5 & -10.7 & -10.3 \\
3.0  & 3.0  & 3.0  & -3.7 & -3.0 & -4.0 & -8.3 & -8.5 & -8.1 \\
5.0  & 5.0  & 5.0  & -6.1 & -5.0 & -6.7 & -7.2 & -7.5 & -7.1 \\
6.0  & 6.0  & 3.0  & -10.0 & -8.6 & -10.8 & -6.6 & -6.9 & -6.5 \\
3.0  & 3.0  & 5.9  & 4.4  & 3.8  & 4.9  & -13.1 & -13.2 & -13.0 \\
4.0  & 4.0  & 3.0  & -6.0 & -5.0 & -6.5 & -7.5 & -7.8 & -7.4 \\

Table 1: Numerical estimations for the corrections \(\delta_1\) and \(\delta_{SV\gamma}\) versus \(\sigma_1, \sigma_2, \theta, \xi\)

We would like to mention here result obtained in [13] on the background to the three lepton neutrinoless decay \(\mu^+ \to e^+e^-e^-\). For an experimental set–up when the electron and positron energies \(\epsilon^\pm\) are measured, it reads

\[
d\Gamma = \frac{\alpha^2}{\pi^2} \frac{13}{36} (2 - w)^2 \ln \frac{M^2}{m^2}, \quad w = \frac{2}{M} (\epsilon_1^+ + \epsilon_2^+ + \epsilon^-), \quad w \to 2. \tag{15}
\]

We have to discuss some features of the results presented. At first we note, that the large logarithm \(L\) does not factorize before the Born–like structure \((R)\), as one may expect. We claim that the factorization theorem, which was proved for high energy processes, should not work here. Another problem is that if one integrated out over the whole phase volume of the second photon, he would still have in the answer the logarithm of the mass ratio. Formally, this violates the Kinoshita–Lee–Nauenberg theorem [14]; the formula is infinite in the limit \(m \to 0\). But again, the conditions of the theorem allow us to say, that the process of radiative muon decay is a legal exception. One can see the same situation in radiative muon decay at the Born level [10, 13].

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**Appendix A. Tables of integrals**

Here we put the tables of relevant integrals appearing in the loop momentum integration. The denominators of amplitudes, which correspond to Feynman diagrams
drawn in Fig.1, have the following form:

\[(1) = (P - k)^2 - M^2, \quad (2) = (p_2 - k)^2 - m^2,\]
\[(1) = (p - k_1 - k)^2 - M^2 \approx k^2 - 2kp_2 - M^2, \quad (2) = (p_2 + k_1 - k)^2 - m^2 \approx k^2 - 2kp + M^2, \quad (0) = k^2 - \lambda^2.\] (A.1)

We use a symbol \(\approx\) to underline the peculiarity of imitating kinematics. Namely, working out traces we use

\[p_2^2 = k_1^2 = 0, \quad 2p_2k_1 = M^2 = 1, \quad q = 0.\] (A.2)

The scalar integrals considered have a form

\[\int \frac{dk}{(i)(j)}, \quad \int \frac{dk}{(i)(j)(k)}, \quad \int \frac{dk}{(i)(j)(k)(l)}, \quad dk = \frac{d^4k}{i\pi^2}.\] (A.3)

Vector and tensor integrals are parametrized as follows:

\[\int \frac{k^\mu dk}{N} = ck_1^\mu + dp_2^\mu,\]
\[\int \frac{k^\mu k^\nu dk}{N} = gg^\mu\nu + \alpha k_1^\mu k_1^\nu + \beta p_2^\mu p_2^\nu + \gamma (k_1, p_2)^\mu\nu,\]
\[\int \frac{k^\mu k^\nu k^\sigma dk}{N} = (G^{(1)}(g, k_1) + G^{(2)}(g, p_2) + \kappa (k_1)^3 + \tau (p_2)^3 + \psi(p_2, k_1, k_1) + \rho(p_2, p_2, k_1))^\mu\nu\sigma,\] (A.4)

where we denote different symmetrical combinations, for instance:

\[(g, a)^{ijk} = g^{ij}a^k + g^{ik}a^j + g^{jk}a^i, \quad (a, b)^{ij} = a^ia^j + a^ja^i, \ldots\] (A.5)

Below we put the values of the coefficients and the scalar integrals. In the tables 2 \(\div\) 7 we used \(Y = \ln \frac{\Lambda^2}{M^2}, \quad L = \ln \frac{M}{m}, \quad X = \frac{\pi^2}{6}, \quad Z = \ln \frac{Mm}{\lambda^2}.\) All the integrals we put in dimensionless form by setting \(M = 1.\)

**Appendix B. Gauge invariant subset of Feynman diagrams**

Amplitudes, describing Feynman diagrams with loop correction to the real photon emission vertex, and the ones, taking into account self– energy of fermions (typical diagrams are shown in Fig.1a,b), provide a gauge invariance in respect to the real photon polarization vector. It has a universal form and may be taken into account
by substitutions in Born amplitude of the form

$$\hat{\mathbf{p}} - \hat{k}_1 + \frac{M}{-2pk_1} \hat{\mathbf{e}}u(p) \rightarrow \frac{\alpha}{2\pi} \left[ A_1 \left( \hat{\mathbf{e}} - \frac{\hat{k}_1 p_e}{pk_1} \right) + A_2 \hat{k}_1 \hat{\mathbf{e}} \right] u(p),$$

$$A_1 = \frac{M}{2(M^2 + t)} \left( 1 - \frac{t}{M^2 + t} L_t \right), \quad t = -2pk_1, \quad L_t = \log \frac{-t}{M^2},$$

$$A_2 = \frac{N + 1}{t} + \frac{1}{2(t + M^2)} - \frac{2t^2 + 3tM^2 + 2M^4}{2t(M^2 + t)^2} L_t,$$

$$N = \frac{M^2}{t} \left[ \text{Li}_2(1) - \text{Li}_2 \left( \frac{M^2 + t}{M^2} \right) \right], \quad \text{Li}_2(z) = -\int_0^1 \frac{\ln(1-zx)}{x} dx,$$

where $\mathbf{e}$ is the polarization vector of the real photon. In the IK we have $A_1 = 1/(4M), A_2 = (\pi^2/6 - 1/4)/M^2$. In a similar fashion for the diagrams, which can be obtained from depicted in Fig.1a,b by emitting a real photon from another leg, we have

$$\bar{u}(p_2) \hat{\mathbf{p}}_2 + \hat{k}_1 + \frac{m}{2p_2k_1} \rightarrow \frac{\alpha}{2\pi} \bar{u}(p_2) \left[ B_1 \left( \hat{\mathbf{e}} - \frac{p_2 e}{p_2 k_1} \right) + B_2 \hat{k}_1 \hat{\mathbf{e}} \right].$$

In the IK, omitting the terms disappearing in the limit of zero electron mass, we have $B_1 = 0, B_2 = (1/2 - 2L)/M^2$.

### Appendix C. Averaging on neutrino states, traces

To rearrange bispinors in the matrix element we use Fierz identity:

$$\bar{u}_1 O_a u_2 \bar{u}_3 O_a u_4 = -\bar{u}_3 O_b u_2 \bar{u}_1 O_b u_4, \quad O_a = \gamma_a(1 + \gamma_5)/2.$$  \hspace{1cm} (C.1)

Summing on the neutrino spin states of the matrix element squared one obtains

$$\Sigma \bar{u}_3 O_a u_2 (\bar{u}_3 O_b u_2)^* = 2((q_1 q_2)^{ab} - q_1 q_2 g^{ab}) = L_{ab}. \hspace{1cm} (C.2)$$

Averaging over the neutrino momentum is performed using the invariant integration:

$$\int \frac{d^3q_1 d^3q_2}{q_{10}q_{20}} q_1^a q_2^b \delta^4(q_1 + q_2 - q) = \frac{\pi}{6}(q^2 g_{ab} + 2q^a q^b). \hspace{1cm} (C.3)$$

Application of this formula to the tensor $L^{ab}$ gives the result:

$$\int \frac{d^3q_1 d^3q_2}{q_{10}q_{20}} L^{ab} \delta^4(q_1 + q_2 - q) = \frac{4\pi}{3}(q^a q^b - q^2 g^{ab}) = \frac{4\pi}{3} O^{ab}. \hspace{1cm} (C.4)$$

8
The list of $T$ and $S$ is given below:

\[
\begin{align*}
T_1 &= \hat{p}_2\gamma_a(\gamma_b - 2\hat{k}_1p_b/M^2)(\hat{p} + M)\gamma_b(\hat{p}_2 + M)\gamma_c/M^4, \\
T_2 &= \hat{p} + 2\gamma_a\hat{k}_1\gamma_b(\hat{p} + M)\gamma_b(\hat{p}_2 + M)\gamma_c/M^4, \\
T_3 &= \hat{p}_2\gamma_a\hat{k}_1\gamma_b\hat{p}_b\gamma_c\hat{p}_b\gamma_b/M^4,
\end{align*}
\]

\[
\begin{align*}
S_{31} &= (\hat{p} + 1)\gamma_c\hat{p}\gamma_b\hat{p}_2\gamma_b(\hat{p}_2 - \hat{k})\gamma_a(\hat{p}_2 - \hat{k} + 1)\gamma_b(\hat{p} - \hat{k} + 1)\gamma_{\mu}, \\
S_{12} &= (\hat{p}_2 + 1)\gamma_b(\hat{p} + 1)\gamma_c\hat{p}\gamma_b\hat{p}_2\gamma_b(\hat{p}_2 - \hat{k})\gamma_a(\hat{p}_2 - \hat{k} + 1)\gamma_{\mu}, \\
S_{21} &= (\hat{p} + 1)\gamma_c\hat{p}\gamma_b\hat{p}_2\gamma_b\gamma_a(\hat{p} - \hat{k} + 1)\gamma_{\mu}, \\
S_{41} &= (\hat{p} + 1)\gamma_c\hat{p}\gamma_b\hat{p}_2\gamma_b\gamma_a(\hat{p} - \hat{k} + 1)\gamma_{\mu}, \\
S_{32} &= (\hat{p} + 1)\gamma_b(\hat{p}_2 + 1)\gamma_c\hat{p}_2\gamma_b(\hat{p}_2 - \hat{k})\gamma_a(\hat{p}_2 - \hat{k} + 1)\gamma_b(\hat{p} - \hat{k} + 1)\gamma_{\mu}, \\
S_{22} &= (\hat{p} + 1)\gamma_b(\hat{p}_2 + 1)\gamma_c\hat{p}_2\gamma_b\gamma_c(\hat{p}_2 - \hat{k})\gamma_a(\hat{p}_2 - \hat{k} + 1)\gamma_{\mu}, \\
S_{11} &= (\hat{p}_2 + 1)\gamma_b(\hat{p} + 1)\gamma_b(\hat{p}_2 + 1)\gamma_c\hat{p}_2\gamma_b(\hat{p}_2 - \hat{k})\gamma_a(\hat{p}_2 - \hat{k} + 1)\gamma_{\mu}, \\
S_{42} &= (\hat{p} + 1)\gamma_b(\hat{p}_2 + 1)\gamma_c\hat{p}_2\gamma_b\gamma_a(\hat{p} - \hat{k} + 1)\gamma_{\mu}, \\
S_{41} &= (\hat{p} + 1)\gamma_b(\hat{p}_2 + 1)\gamma_c\hat{p}_2\gamma_b\gamma_a(\hat{p}_2 - \hat{k} + 1)\gamma_{\mu}, \\
S_{22} &= (\hat{p} + 1)\gamma_b(\hat{p}_2 + 1)\gamma_c\hat{p}_2\gamma_b\gamma_a(\hat{p} - \hat{k} + 1)\gamma_{\mu}, \\
S_1 &= S_{11} - S_{12}, \quad S_2 = S_{21} - S_{22}, \quad S_3 = S_{31} - S_{32}, \quad S_4 = S_{41} - S_{42}.
\end{align*}
\]

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Table 2: Scalar integrals with 2 denominators

|       | (01) | (01) | (02) | (02) | (12) | (12) | (21) | (22) | (11) |
|-------|------|------|------|------|------|------|------|------|------|
|       | \(Y + 1\) | \(Y\) | \(Y + 2L + 1\) | \(Y + 1\) | \(Y\) | \(Y\) | \(Y + 2L - 1\) | \(Y - 1\) |   |

\[
\begin{array}{cccccc}
(012) & (011) & (012) & (112) & (012) \\
\hline
-LZ & -X & -2L - 1 & -1 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
(022) & (122) & (0112) & (0122) \\
\hline
2L^2 - X & 1 - 2L & ZL - X & -ZL - 2L^2 + X \\
\end{array}
\]

Table 3: Scalar integrals with 3 and 4 denominators

|       | (01) | (01) | (02) | (02) | (12) | (12) |
|-------|------|------|------|------|------|------|
|       | \(d\) | \(c\) | \(d\) | \(c\) | \(d\) | \(c\) |
|       | \(\frac{1}{2}Y - \frac{1}{4}\) | \(0\) | \(\frac{1}{2}Y + L - \frac{1}{4}\) | \(\frac{1}{2}Y + \frac{1}{4}\) | \(Y - \frac{1}{2}\) | \(\frac{1}{2}Y - \frac{1}{2}\) |
|       | \(\frac{1}{2}Y - \frac{1}{4}\) | \(0\) | \(\frac{1}{2}Y + L - \frac{1}{4}\) | \(\frac{1}{2}Y + \frac{1}{4}\) | \(Y - \frac{1}{2}\) | \(\frac{1}{2}Y - \frac{1}{2}\) |

Table 4: Vector integrals with 2 denominators

|       | (012) | (011) | (012) | (112) | (012) |
|-------|------|------|------|------|------|
|       | \(d\) | \(c\) | \(d\) | \(c\) | \(d\) |
|       | \(-2L\) | \(-1\) | \(-L - \frac{1}{3}\) | \(-1\) | \(-\frac{1}{2}\) |
|       | \(-1\) | \(X - 2\) | \(0\) | \(-\frac{1}{3}\) | \(-\frac{1}{2}\) |

Table 5: Vector integrals with 3 and 4 denominators

|       | (022) | (122) | (0112) | (0122) |
|-------|------|------|--------|--------|
|       | \(d\) | \(c\) | \(d\) | \(c\) |
|       | \(2L^2 - 2L - X\) | \(2L - 2\) | \(0\) | \(1\) |
|       | \(1 - 2L\) | \(X - 1\) | \(1 - 2L\) |   |

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Table 6: 2-rank tensor integrals with 3 and 4 denominators

|       | (012) | (011) | (012) | (112) | (012) |
|-------|-------|-------|-------|-------|-------|
| $g$   | $\frac{1}{2} Y + \frac{1}{3}$ | $\frac{1}{4} Y - \frac{1}{2} X + \frac{5}{3}$ | $\frac{1}{2} Y$ | $\frac{1}{2} Y - \frac{1}{2}$ | $\frac{1}{2} Y + \frac{1}{3}$ |
| $\beta$ | $- L$ | $- X + \frac{2}{3}$ | $- \frac{3}{2} L - \frac{1}{6}$ | $- 1$ | $- \frac{1}{3}$ |
| $\alpha$ | $- \frac{1}{2}$ | $- X + \frac{2}{3}$ | $0$ | $- \frac{1}{2}$ | $- \frac{1}{2}$ |
| $\gamma$ | $- \frac{1}{2}$ | $2 X - \frac{1}{2}$ | $0$ | $- \frac{1}{2}$ | $- \frac{1}{2}$ |

Table 7: 3-rank tensor integrals with 4 denominators

|       | (022) | (122) | (0112) | (0122) |
|-------|-------|-------|--------|--------|
| $g$   | $\frac{1}{2} Y + \frac{2}{3}$ | $\frac{1}{3} Y$ | $- \frac{1}{2} X + \frac{1}{3}$ | $- \frac{1}{3}$ |
| $\beta$ | $2 L^2 - 3 L - X + \frac{1}{2}$ | $- 2 L + 1$ | $L - X + \frac{3}{4}$ | $- 2 L^2 + 2 L + X - \frac{1}{2}$ |
| $\alpha$ | $L - 1$ | $- \frac{5}{2} L + \frac{1}{3}$ | $- X + \frac{3}{4}$ | $- L + \frac{5}{4}$ |
| $\gamma$ | $2 L - \frac{3}{2}$ | $- L + \frac{1}{2}$ | $2 X - 3$ | $- 2 L + 2$ |

Table 7: 3-rank tensor integrals with 4 denominators

|       | $G^{(1)}$ | $G^{(2)}$ | $\tau$ |
|-------|-----------|-----------|--------|
| (0112) | $\frac{1}{3} X - \frac{1}{3}$ | $- \frac{1}{2} X + \frac{5}{3}$ | $\frac{2}{3} L - X + \frac{29}{30}$ |
| (0122) | $- \frac{1}{3}$ | $- \frac{1}{3}$ | $- 2 L^2 + 3 L + X - 1$ |
| $\kappa$ | $\rho$ | $\psi$ |
| (0112) | $X - \frac{27}{10}$ | $3 X - \frac{14}{9}$ | $- 3 X + 5$ |
| (0122) | $- \frac{5}{3} L + \frac{11}{3}$ | $- 2 L + \frac{5}{2}$ | $- L + 1$ |