Fibonacci Numbers

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Fibonacci’s Rabbit Problem

“A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?”

—A problem from the third section of Liber abaci (1202).  

(https://www-history.mcs.st-andrews.ac.uk/Biographies/Fibonacci.html)

Recursive Definition

\[ F(\text{Next}) = F(\text{Current}) + F(\text{Productive}), \quad F(0) = F(1) = 1 \]

Rabbit Population

| Month | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|---|----|
| Rabbits | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 |
Solve one of these problems:

- A composition of $n$ is a way to write $n$ as the sum of positive integers (order matters). How many compositions are there of 8 that don’t use 1?

- How many compositions are there of 7 into odd parts?

- How many subsets are there of $\{1, 2, 3, 4, 5\}$ that include no two consecutive numbers?

- In how many ways can you tile a $2 \times 6$ rectangle with $2 \times 1$ dominoes?

- How many increasing paths are there through the honeycomb from 1 to 7?

- How many ways are there to climb a set of 6 stairs, one or two steps at a time?

- How many binary sequences of length 5 are there, with no consecutive 0’s?

- Find 6 positive integer solutions $(x, y)$ of $y^2 - xy - x^2 = \pm 1$. 

- Some problems to solve

- Solutions

- Binet’s Formula
How many *increasing* paths are there through the honeycomb from 1 to 7?

\[ \cdots \quad \text{Path}(5) + \text{Path}(6) = \text{Path}(7) \]
\[ \text{Path}(n - 2) + \text{Path}(n - 1) = \text{Path}(n) \]

| \(n\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| \(P(n)\) | 1 | 1 | 2 | 3 | 5 | 8 | 13 |

Paths ending 5-7

Paths ending 6-7
### The Fibonacci Sequence

The Fibonacci Sequence is defined by the recurrence relation:

\[ F(n) = F(n-1) + F(n-2) \] (for \( n > 2 \)), \( F(1) = 1 \), \( F(2) = 1 \)

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | \( \ldots \) |
|---|---|---|---|---|---|---|---|---|
| \( F(n) \) | 0 | 1 | 1 | 2 | 3 | 5 | 8 | \( \ldots \) |

### A Fibonacci-ish Sequence (Gibbonacci?)

A Fibonacci-ish Sequence is defined by the recurrence relation:

\[ G(n) = G(n-1) + G(n-2) \]

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | \( \ldots \) |
|---|---|---|---|---|---|---|---|---|
| \( G(n) \) | 4 | -2 | 2 | 0 | 2 | 2 | 4 | \( \ldots \) |
**Fact 1: Scaling a Fibonacci-ish Sequence yields a ...**

| $n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | ... |
|-----|----|----|----|----|----|----|----|-----|
| $G(n)$ | 4 | -3 | 1 | -2 | -1 | -3 | -4 | ... |
| $4G(n)$ | 16 | -12 | 4 | -8 | -4 | -12 | -16 | ... |

**Fact 2: If $G(0)$ is 0 then ...**

| $n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | ... |
|-----|----|----|----|----|----|----|----|-----|
| $G(n)$ | 0 | 3  | 3  | 6  | 9  | 15 | 24 | ... |
| $F(n)$ | 0 | 1  | 1  | 2  | 3  | 5  | 8  | ... |

**Fact 3: Subtracting Fibonacci-ish Sequences yields a ...**

| $n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | ... |
|-----|----|----|----|----|----|----|----|-----|
| $G(n)$ | 4 | 3  | 7  | 10 | 17 | 27 | ... |
| $H(n)$ | 2 | 1  | 3  | 4  | 7  | 11 | ... |
| $G(n) - H(n)$ | 2 | 2  | 4  | 6  | 10 | 16 | ... |
Two interesting Fibonacci-ish sequences

| n | 0   | 1   | 2   | 3   | 4   | ⋯  |
|---|-----|-----|-----|-----|-----|----|
| R(n) | 1   | r   | r²  | r³  | r⁴  | ⋯  |
| S(n) | 1   | s   | s²  | s³  | s⁴  | ⋯  |
| R(n) - S(n) | 0   | r - s | r² - s² | r³ - s³ | r⁴ - s⁴ | ⋯  |
| F(n) | 0   | 1   | \[\frac{r^2 - s^2}{r - s}\] | \[\frac{r^3 - s^3}{r - s}\] | \[\frac{r^4 - s^4}{r - s}\] | ⋯  |

Necessary (and Sufficient) Conditions

\[R(0) + R(1) = R(2)\] or \[1 + r = r^2\] and also \[1 + s = s^2\]

Both \(r\) and \(s\) are solutions of \(1 + x = x^2\).
\[r = \frac{1 + \sqrt{5}}{2} \approx 1.618\] and \[s = \frac{1 - \sqrt{5}}{2} \approx -0.618\]

Binet’s Formula

\[F(n) = \frac{r^n - s^n}{r - s} = \left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n \div \sqrt{5}\]
The End

Thank You!

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