THE CORRESPONDENCE BETWEEN
GEOMETRIC QUANTIZATION AND
FORMAL DEFORMATION QUANTIZATION

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ABSTRACT. Using the classification of formal deformation quantizations, and the
formal, algebraic index theorem, I give a simple proof as to which formal deformation
quantization (modulo isomorphism) is derived from a given geometric quan-
tization.

1. INTRODUCTION

There are two principal mathematical notions of “quantization”. Both share as a
starting point the idea (from physics) that the product of functions on a manifold
is deformed with a parameter $\hbar$ in such a way that the commutator is given to
leading order by the Poisson bracket as

$$[f, g] = -i\hbar \{f, g\} + O(\hbar).$$

One theory, geometric quantization, gives concrete procedures for constructing
a $C^*$-algebra for each (allowed) value of $\hbar$. In the limit as $\hbar \to 0$, each of these alge-
bras can be linearly identified with the ordinary algebra of continuous functions.
It is in this approximate sense that the elements of the algebra can be thought of as
being fixed while the product changes and satisfies Eq. (1).

In the other theory, (formal) deformation quantization (see [1, 10]), Eq. (1) is
taken to suggest an expansion in powers of $\hbar$. The $\hbar$-dependent product is ex-
pressed as a power series in $\hbar$. This power series does not, however, converge
for most smooth functions; hence, $\hbar$ can only be taken as a formal parameter and
cannot be given a specific, nonzero value.

Both these theories were originally intended to address the physical problem of
quantizing the phase space of a physical system. Physically the value of $\hbar$ is not
variable (in fact $\hbar \approx 10^{-27} \text{ g cm}^2/\text{sec}$), so deformation quantization can never be
used to fully describe what it was originally intended to. However, deformation
quantization has proven fruitful as a mathematical subject. For instance, interesting
classification results have been achieved in this abstract setting (see [2]).

As far as using deformation quantization for something like its intended pur-
pose, I believe that it should be interpreted as describing the asymptotic behavior
of a more concrete structure, such as that produced by geometric quantization.

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Any result concerning deformation quantization should then have implications for this concrete version of quantization.

Essential to exploiting this is an understanding of the relationship between deformation quantization and geometric quantization. In principle, any geometric quantization can be viewed in terms of an \( \hbar \)-dependent product which can then be asymptotically expanded to yield a deformation quantization. Since this procedure laborious at best, a shortcut to understanding what deformation quantization this gives is desirable. A comparison of my results on “quantization of vector bundles” \([3]\) with an index theorem in deformation quantization theory achieves this.

2. GEOMETRIC QUANTIZATION

My paper \([5]\) is a reference for everything in this section. Let \( M \) be a compact, connected, Kähler manifold. Let \( L \) be a Hermitian line bundle with curvature equal to the symplectic form \( \omega \) on \( M \) (which implies that \( \frac{\omega}{2\pi} \) is integral). Let \( L_0 \) a holomorphic line bundle with an inner product on sections (making \( \Gamma(M, L_0) \) a pre-Hilbert space). From these we can construct a sequence of holomorphic line bundles \( L_N := L_0 \otimes L^\otimes N \) which also have an inner product on sections.

For any \( N \) the space \( \mathcal{H}_N := \Gamma_{\text{hol}}(M, L_N) \) of holomorphic sections of \( L_N \) is a finite-dimensional Hilbert subspace of \( L^2(M, L_N) \). For \( N \) sufficiently negative, \( L_N \) is a negative line bundle and so \( \mathcal{H}_N = 0 \). Using these Hilbert spaces, we can define the matrix algebras \( A_N := \text{End}(\mathcal{H}_N) \).

The Töplitz quantization maps are completely positive maps \( T_N : \mathcal{C}(M) \to A_N \).

For any continuous function \( f \in \mathcal{C}(M) \), the action of the operator \( T_N(f) \) is defined by multiplying a section in \( \mathcal{H}_N \) by \( f \) and then orthogonally projecting back to \( \mathcal{H}_N \).

The geometric quantization maps \( Q_N : \mathcal{C}(M) \to A_N \), are also completely positive. They can be expressed as \( Q_N(f) = T_N(f + \Delta f/2N) \), where \( \Delta \) is the Laplacian.

Both of these systems of maps can be assembled into a direct-product map \( T \) (respectively \( Q \)) : \( \mathcal{C}(M) \to \prod_N A_N \), where \( \prod_N A_N \) is the \( \mathcal{C}^\ast \)-algebraic direct product of the \( A_N \)'s. Define \( \mathfrak{A} \) to be the \( \mathcal{C}^\ast \)-algebra spanned by the image of \( T \) (or \( Q \), the result is the same) and the \( \mathcal{C}^\ast \)-algebraic direct sum \( \bigoplus_N A_N \). This algebra \( \mathfrak{A} \) is in fact the algebra of continuous sections of \( A_{\hat{N}} \), a continuous field of \( \mathcal{C}^\ast \)-algebras.

The base space of \( A_{\hat{N}} \) is \( \hat{N} := \{1, 2, \ldots, \infty\} \), the one-point compactification of the natural numbers (or, better, the set of \( N \) such that \( \mathcal{H}_N \neq 0 \)). The fiber over any finite \( N \in \mathbb{N} \) is \( A_N \); the fiber over \( \infty \in \hat{N} \) is \( \mathcal{C}(M) \).

Define \( \mathcal{P} : \mathfrak{A} \to \mathcal{C}(M) \) to be the evaluation of sections at \( \infty \in \hat{N} \). Define the partial traces \( \text{tr}_N : \mathfrak{A} \to \mathbb{C} \) by letting \( \text{tr}_N(a) \) be the trace of the section \( a \) evaluated at \( N \). In \([2] \) I proved:

**Theorem 1.** Let \( e = e^2 \in M_m[\mathcal{C}(M)] \) and \( \tilde{e} = \tilde{e}^2 \in M_m[\mathfrak{A}] \) be idempotent matrices such that \( \mathcal{P}(\tilde{e}) = e \). For \( N \) sufficiently large,

\[
\text{tr}_N \tilde{e} = \int_M \text{ch} \ e \wedge \text{td} \ T M \wedge e^{c_1(L_0) + N \omega/2\pi}.
\]
Here ch e is the Chern character of the bundle determined by e, and td TM is the Todd class of the holomorphic tangent bundle of M.

3. Formal Deformation Quantization

Let M again be a symplectic manifold. A (formal) deformation quantization of M (see [10]) is an algebra $A h$ which (as a vector space) is identified with $C^\infty(M) [[h]]$, the space of formal power series in $\bar{h}$ with coefficients in the smooth functions over M. Denote the $A h$-product by $* h$ and the $C^\infty(M) [[h]]$-product by apposition (e. g., $fg$). The product $* h$ is given by a formal power series

$$ f * h g = fg + \sum_{k=1}^\infty (-i h)^k \varphi_k(f, g). \quad (3) $$

This is required to be associative and $C([[h]])$-linear. It is required to satisfy $f * h 1 = f$ and $f^* * h g^* = (g * h f)^*$ where the complex conjugate $\bar{h}$ is $\bar{h}$. The only condition involving the symplectic form is the restatement of Eq. (1),

$$ f * h g - g * h f \equiv -i h \{ f, g \} \mod h^2, $$

or equivalently $\varphi_1(f, g) - \varphi_1(g, f) = \{ f, g \}$, where $\{ f, g \}$ is the Poisson bracket. Finally there is a (perhaps unnecessary) locality condition that each $\varphi_k$ is a bidifferential operator.

The archetypal example of a deformation quantization is the Moyal-Weyl deformation on a symplectic vector space $M = \mathbb{R}^{2n}$. Let $m : C^\infty(\mathbb{R}^{2n}) \otimes C^\infty(\mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n})$ be the (ordinary) multiplication map, and $\pi$ the Poisson bivector, regarded as a differential operator on $C^\infty(\mathbb{R}^{2n}) \otimes C^\infty(\mathbb{R}^{2n})$ (so that $m \circ \pi(f \otimes g) = \{ f, g \}$). The Weyl product is

$$ f * h g := m \circ \exp[-i h \pi](f \otimes g) \quad (4) $$

$$ = fg - \frac{ih}{2}(f, g) + \ldots. $$

The formal Weyl algebra $W h$ is related to this, essentially by taking germs of functions about $0 \in \mathbb{R}^{2n}$. It is constructed using formal power series $C[[\mathbb{R}^{2n}]]$ in place of smooth functions; in other words, $W h$ is $C[[\mathbb{R}^{2n}, h]]$ with the product (4).

Over any manifold, we can construct a bundle $C[[TM]]$ of formal power series over each fiber of the tangent bundle. A Leibniz connection over a bundle of algebras, such as this, is one satisfying the Leibniz rule with respect to the product of sections. The constant sections of $C[[TM]]$ with respect to a flat Leibniz connection are naturally identified with the smooth functions on M. The value of a section at some $x \in M$ is a Taylor expansion about $x$ of the corresponding function.

Every fiber of the tangent bundle of a symplectic manifold is a symplectic vector space. From this, we can construct a bundle $W h M$ of formal Weyl algebras such that the fiber over $x \in M$ is the formal Weyl algebra constructed on $T_x M$. Note that the order $h^0$ part is just $W h M / h = C[[TM]]$. The structure Lie algebra for a Leibniz connection on $W h M$ is $der W h$, the derivations of the typical fiber.
A flat, Leibniz connection $\nabla$ on $\mathbb{W}^h M$ is known as a Fedosov connection. The algebra $A^h$ of $\nabla$-constant sections of $\mathbb{W}^h M$ is a deformation quantization of $\mathcal{C}^\infty(M)$. Fedosov connections always exist [2], and, in fact, any deformation quantization can be constructed in this way [3].

The space $g := \mathbb{h}^{-1} \mathbb{W}^h$ (series with an order $\mathbb{h}^{-1}$ term allowed) is a Lie algebra with the commutator as a Lie bracket. Indeed, $g$ acts on $\mathbb{W}^h$ by derivations and in fact gives all derivations of $\mathbb{W}^h$. It is thus a central extension

$$0 \to \mathbb{h}^{-1} \mathbb{C}[[\mathbb{h}]] \to g \to \text{der} \mathbb{W}^h \to 0. \quad (5)$$

Since the Fedosov connection $\nabla$ is a $\text{der} \mathbb{W}^h$-connection, it can be lifted to a $g$-connection $\tilde{\nabla}$ using Eq. (5). The flatness of $\nabla$ implies that the curvature of $\tilde{\nabla}$ is central, that is $\tilde{\nabla}^2 \in \mathbb{h}^{-1} \Omega^2(M)$. However, the lifting of $\nabla$ is not unique, so the curvature of $\tilde{\nabla}$ is not uniquely determined by $\nabla$. Fortunately, the ambiguity is only modulo exact forms, so we can define $\theta := [\tilde{\nabla}^2]/2\pi i \in \mathbb{h}^{-1} H^2_{\text{dR}}(M)$, where brackets denote the deRham cohomology class. To leading order in $\mathbb{h}$, this is given by the symplectic form as

$$\theta = \frac{[\omega]}{2\pi \mathbb{h}} + \ldots$$

The group of arbitrary automorphisms of $A^h$ decomposes as the direct product of the subgroup of automorphisms preserving $\mathbb{h}$, with the group of formal $\mathbb{h}$ reparameterizations. The group of $\mathbb{h}$-preserving automorphisms is itself an extension of the group of symplectomorphisms ($\omega$-preserving diffeomorphisms) by the group of internal automorphisms.

The cohomology class $\theta$ turns out [7] to classify deformation quantizations modulo inner automorphisms and small (connected component) symplectomorphisms. The class $\theta$, modulo “large” symplectomorphisms and formal $\mathbb{h}$ reparameterization, therefore classifies $A^h$ modulo isomorphisms.

For a given deformation quantization $A^h$ of $M$, there exists a natural trace (see [3, 4, 9]) $\text{Tr}: A^h \to \mathbb{h}^{-n} \mathbb{C}[[\mathbb{h}]]$. This is given to leading order in $\mathbb{h}$ by

$$\text{Tr} f = \int_M \frac{f \omega^n}{(2\pi \mathbb{h})^n n!} + \ldots \quad (6)$$

Using $\theta$ and this trace, a formal index theorem can be formulated (see [3] for the original, [4, 9] for clarity, and also [3, 7]). In the case of compact $M$, this reads:

**Theorem 2.** If $e = e^2 \in M_m[\mathcal{C}^\infty(M)]$ and $e = e^2 \in M_m[A^h]$ are idempotents such that $e \equiv e \mod \mathbb{h}$, then

$$\text{Tr} e = \int_M \text{ch} e \wedge \mathbb{A}(TM) \wedge e^\theta. \quad (7)$$
4. COMPARISON

As suggested in Sec. 1, we can try to construct a deformation quantization from a geometric quantization. Let $Q_N$ represent either the geometric or Töplitz quantization maps. Suppose that we choose a sequence of maps $Q_N^{\text{inv}} : \mathcal{A}_N \rightarrow \mathcal{E}^\infty(M)$ such that $Q_N \circ Q_N^{\text{inv}} = \text{id}$ and $Q_N^{\text{inv}} \circ Q_N \rightarrow \text{id}$ as $N \rightarrow \infty$. Using such maps, we can pull the product on $\mathcal{A}_N$ back to $\mathcal{E}(M)$ and define,

$$f \ast^N g := Q_N^{\text{inv}}[Q_N(f)Q_N(g)].$$

If all goes well, for any smooth functions $f, g \in \mathcal{C}^\infty(M)$, we can define $f \ast_h g$ by taking the asymptotic expansion of $f \ast^N g$ as $N \rightarrow \infty$ and setting $N = \hbar^{-1}$. The requirement that $\mathcal{C}^\infty(M)[[\hbar]]$ be closed under $\ast_h$ is equivalent to the requirement that the image of each $\varphi_i$ in Eq. (3) is in $\mathcal{C}^\infty(M)$. Associativity of $\ast_h$ is automatic.

The necessity of making a (somewhat arbitrary) choice of $Q_N^{\text{inv}}$ in this construction is a rather unpleasant feature; fortunately, it can be eliminated. Suppose (for any $f, g$, and $k$) that we take the difference of $f \ast^N g$ with the order $N^{-k}$ partial sum of the series (3) for $f \ast_{N^{-1}} g$. The defining property of the asymptotic expansion is that the norm of this difference is of order $\hbar - 1$ goes to 0 as $N \rightarrow \infty$.

Since the quantization maps are norm-contracting, we can apply the limit before taking the norm. This cancels out the $Q_N^{\text{inv}}$ and shows that for any $f, g \in \mathcal{C}^\infty(M)$ and $k \in \mathbb{N}$,

$$\lim_{N \rightarrow \infty} \left\| N^kQ_N(f)Q_N(g) - \sum_{j=0}^{k} (-i)^j N^{k-j} Q_N[\varphi_j(f, g)] \right\| = 0. \quad (8)$$

Using the property that $\lim_{N \rightarrow \infty} \|Q_N(f)\| = \|f\|$ (see [3]), it is easy to verify that a $\ast_{\hbar}$-product satisfying (8) is unique. It is proven in [3] that a $\ast_{\hbar}$-product satisfying Eq. (8) does exist and defines a deformation quantization, although only for the (slightly) restricted case of $L_0$ trivial.

Recall from Sec. 3 that the algebra $\mathcal{A}$ is the space of continuous sections of $\mathcal{A}_0$, a continuous field of $C^*$-algebras. It is useful to refine $\mathcal{A}$ to a space of smooth sections. Let $A^S$ be the subspace of elements in $\mathcal{A}$ which have an asymptotic expansion as $N \rightarrow \infty$. To be precise,

$$A^S := \left\{ a \in \mathcal{A} \mid \forall f \in \mathcal{C}^\infty(M)[[N^{-1}]] \forall k \in \mathbb{N} : \lim_{N \rightarrow \infty} N^k\|a - Q_N(f_k)\| = 0 \right\}, \quad (9)$$

where $f_{(k)}$ is the order $N^{-k}$ partial sum of $f$. The fact that (8) can be satisfied for an algebraically closed $\ast_{\hbar}$ implies that $A^S \subset \mathcal{A}$ is a subalgebra. When the $f$ in Eq. (8) exists, it is unique, so there is a well defined asymptotic expansion homomorphism $\iota : A^S \rightarrow \mathcal{A}_h; a \mapsto f$.

The subalgebra $A^S$ is holomorphically closed. To verify this, it is sufficient to check that if $a \in A^S$ and $F : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic on the disc of radius $\|a\|$, then $F(a)$ has the asymptotic expansion $F(\iota(a))$.

Quotienting by $\mathfrak{h}^k$ is a way of only dealing with the algebras $A^S$ and $\mathcal{A}_h$ to order $\mathfrak{h}^{k-1}$. When we do this, the homomorphism $\iota$ induces an isomorphism
This leads to an invariant construction for \( A_0^S = \mathbb{A}^S / \mathbb{h}^S \mathbb{A}^S \). Note that \( A_0^S := \mathbb{h} A^S \) is simply the ideal of sections in \( A^S \) which vanish at \( \mathbb{h} = 0 \). Taking powers of this gives a nested sequence of ideals \( (A_0^S)^k = \mathbb{h}^k A^S \). We can express \( A_0^S \) as an algebraic inverse limit

\[
A_0^S = \lim_{\longrightarrow} A^S / (A_0^S)^k,
\]

using the obvious projections \( A^S / (A_0^S)^{k+1} \rightarrow A^S / (A_0^S)^k \). The homomorphism \( \iota \) is recovered canonically from this inverse limit construction. This construction only depends on the specification of \( A^S \subset A \), which is the same for both the geometric and Töplitz quantization maps.

In light of the classification of deformation quantizations by cohomology classes, the obvious question now is: If a deformation quantization can be successfully constructed from a geometric quantization, then what is \( \Theta \)? This question is easily answered by comparing Theorems 1 and 2.

**Theorem 3.** For the deformation quantization derived from the geometric quantization of a compact, Kähler manifold \( M \), the classifying cohomology class is

\[
\Theta = \frac{[\omega]}{2\pi \mathbb{h}} + c_1(L_0) + \frac{1}{2} c_1(TM),
\]

**Proof.** Theorem 1 shows that \( \text{tr}_N 1 \) grows as a polynomial in \( N \). With the inequality \( |\text{tr}_N a| \leq \|a\|_N \text{tr}_N 1 \), this shows that for any \( a \in \ker \iota \) (that is, \( a \sim 0 \)), \( \text{tr}_N a \sim 0 \). This shows that the asymptotic expansion of \( \text{tr}_N a \) gives a well defined, \( \mathbb{C}[\mathbb{h}] \)-linear trace on \( \text{Im} \iota \subset A^\mathbb{h} \). Because of (10), this extends uniquely to all of \( A^\mathbb{h} \) and must therefore be proportional to the trace \( \text{Tr} \) (by the uniqueness of \( \text{Tr} \), see [7]). To be precise, for any \( a \in A^S \), the asymptotic expansion of \( \text{tr}_N a \) as \( N = h^{-1} \rightarrow \infty \) is

\[
\text{tr}_N a \sim \beta \text{Tr}[\iota(a)],
\]

where \( \beta \in \mathbb{C}[[\mathbb{h}]] \) is independent of \( a \). Of course, this extends to matrices over \( A^S \).

Choose any \( a \in M_m[A^S] \) such that \( \mathcal{P}(a) = e \). This is idempotent modulo \( A^S \) in the sense that \( a^2 - a \in M_m[A^S] \). Since \( A^S \) is holomorphically closed, we can use a standard contour integral trick to construct from \( a \) and idempotent \( \tilde{e} \in M_m[A^S] \) such that \( \tilde{e} - a \in M_m[A^S] \). Hence, \( \mathcal{P} \tilde{e} = 1 \), which is the hypothesis of Thm. 2. Equation (2) gives an exact polynomial expression for \( \text{tr}_N \tilde{e} \) for \( N \) sufficiently large; this polynomial is the asymptotic expansion of \( \text{tr}_N \tilde{e} \).

The idempotent \( e := \iota(a) \) satisfies the hypothesis of Thm. 2 that \( e \equiv e \mod h \), so \( \text{Tr}[\iota(a)] \) is given by Eq. (2).

Combining these results gives,

\[
\int_M \text{ch} e \wedge \hat{A}(TM) \wedge e^\theta = \int_M \text{ch} e \wedge \text{td} TM \wedge e^{c_1(L_0) + \omega / 2\pi h}.
\]

Recall that the \( \hat{A} \) and Todd classes are related by \( \text{td} TM = e^{\frac{1}{2} c_1(TM)} \wedge \hat{A}(TM) \), where \( c_1 \) is the first Chern class. Now, noting that the possible values of \( \text{ch}(e) \)
span $H^*_\text{dr}(\mathcal{M})$, and that $\hat{\mathcal{A}}(\mathcal{T}\mathcal{M})$ is invertible, this gives
\[
\theta + \ln \beta = c_1(L_0) + \frac{[\omega]}{2\pi \hbar} + \frac{1}{2}c_1(\mathcal{T}\mathcal{M}).
\]
All terms of this equation are of degree 2 except for $\ln \beta$ which is of degree 0; therefore, $\ln \beta = 0$.

**Corollary 4.** For any $\alpha \in \mathcal{A}^S$,
\[
\text{tr}_N \alpha \sim \text{Tr} [\iota(\alpha)]
\]
as $N = \hbar^{-1} \to \infty$.

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