Isotropic Cosmological Singularities: 
Other matter models

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November 1, 2018

Abstract

Isotropic cosmological singularities are singularities which can be removed by rescaling the metric. In some cases already studied ([2], [1]) existence and uniqueness of cosmological models with data at the singularity has been established. These were cosmologies with, as source, either perfect fluids with linear equations of state or massless, collisionless particles. In this article we consider how to extend these results to a variety of other matter models. These are scalar fields, massive collisionless matter, the Yang-Mills plasma of [3] and matter satisfying the Einstein-Boltzmann equation.

1 Introduction

An isotropic cosmological singularity is one which can be removed by rescaling the spacetime metric with a single function which becomes singular on a smooth spacelike hypersurface, say $\Sigma$, in the rescaled space-time ([1], [2]). In the rescaled but unphysical space-time the curvature is finite at $\Sigma$. The Weyl curvature with indices suitably arranged is conformally-invariant, so that it is also finite at the curvature singularity in the unrescaled physical space-time. The motivation for studying this class of singularities comes from Penrose’s Weyl Curvature Hypothesis ([3], [4]) which can be interpreted as the hypothesis that the Weyl curvature must be finite at any initial singularity.

In order to make mathematical progress with the Weyl Curvature Hypothesis one must first say exactly what is meant by a ‘finite Weyl curvature' singularity, that is to say a singularity at which the Weyl curvature is finite
while, of necessity, the Ricci curvature is not. There may be more than one plausible way to do this. The strategy adopted here is first to make the following definition (after [7]): A spacetime \((\tilde{M}, \tilde{g}_{ab})\) is said to admit an isotropic singularity if there exists a manifold \(M \supset \tilde{M}\), a regular Lorentz metric \(g_{ab}\) on \(M\), and a function \(\Omega\) defined on \(M\), such that

\[
\tilde{g}_{ab} = \Omega^2 g_{ab} \quad \text{for} \quad \Omega > 0
\]

\[
\Omega \to 0 \quad \text{on} \quad \Sigma
\]

where \(\Sigma\) is a smooth spacelike hypersurface in \(M\), called the singularity surface (note that tilded quantities are defined in the physical space-time, while untilded quantities are in the unphysical space-time; also \(\Sigma\) is smooth but there need be no assumption that \(\Omega\) is smooth at \(\Sigma\)). Since the Weyl tensor with its indices arranged as \(C_a^{bcd}\) is conformally invariant and is finite in \(M\), it must also be finite in \(\tilde{M}\), so that isotropic singularities form a well-defined class of cosmological singularities with finite Weyl tensor.

Consideration of the conformal Einstein equations near an isotropic singularity, for various matter models, leads naturally to a class of singular Cauchy problems for the unphysical metric \(g_{ab}\) with data given on \(\Sigma\), for which one may seek to prove suitable existence and uniqueness theorems. This program was begun in [11] and [12] where the problem was treated in the case of a perfect fluid with polytropic equation of state as matter source. There the conformal field equations were written as an evolution system for \(g_{ab}\) and it was shown that solutions which are power-series in a time-coordinate could be constructed with data just the 3-metric of the singularity surface. The first existence and uniqueness theorem for the class of singular Cauchy problems which arise was given by Claudel and Newman [4] for the particular case when the source is a radiation fluid. This theorem was adapted to solve the problem for other polytropic fluids in [5], and extended to deal with the massless Einstein-Vlasov equations, (i.e. the Einstein equations when the source is massless, collisionless matter satisfying the Vlasov or Liouville equation) in [1]. There are striking differences in the freely-specifiable data in the different cases of perfect fluid and collisionless matter. For a perfect fluid, the initial 3-metric alone is the free data while for the collisionless matter the free data is the initial distribution function subject only to a ‘vanishing dipole’ condition (the initial first and second fundamental forms are determined by the initial distribution function). It is therefore a natural question to consider other matter models, and in particular to see how the gap between these two earlier cases may be bridged.
In this article, we shall consider a range of other matter models and treat them in the style of [11] and [12]. That is, we shall seek to formulate an initial value problem with data at the singularity, to identify the data, and to show that power-series solutions may be found. We do not seek to manipulate the problems into the form in which the Claudel-Newman theorem can be applied as this is generally an extremely laborious task. For certainty, this would need to be done but in the past the study of power-series solutions has been a reliable guide to existence and uniqueness in general.

The matter models we consider are three: scalar fields, a coupled system of Einstein-Yang-Mills-Vlasov studied by Choquet-Bruhat and collaborators [3] as a model of a ‘quark-gluon plasma’, and kinetic theory with either the massive Vlasov equation or the massless Boltzmann equations. What we find can be summarised as follows:

For scalar fields subject to a particular condition on the growth of the potential with large values of the field it is possible to pose an initial value problem for which the data is just the 3-metric of the singularity surface. There is no free data for the scalar field, which diverges at the singularity but is asymptotically constant in spatial directions, and the second fundamental form of the singularity surface is automatically zero. This case is very much like the ‘stiff’ perfect fluid [11].

The Einstein-Yang-Mills-Vlasov case is similar to the (massless) Vlasov case [3]. There are data for the initial distribution function and the initial Yang-Mills electric field and magnetic potential. There is a Gauss-law type constraint, and a dipole-condition connecting all the data. There are constraints relating the first and second fundamental forms of the singularity surface to the matter variables which seem, and in some cases definitely are, strong enough to determine them uniquely in terms of the matter variables. The massive Vlasov case is essentially the same as the massless case: at the singularity, rest-mass of the particles is unimportant. When we introduce a collision term into the Vlasov (or Liouville) equation to obtain the Boltzmann equation however there is a new phenomenon. The collision term is defined as usual by a collision integral over products of phase space with a kernel determining the differential cross-section. Behaviour of the cross-section under conformal rescaling, or equivalently in the limit of large energies, now determines the problem. If the cross-section does not grow too fast then the initial value problem is like the Vlasov case and the data is the initial distribution function, which then determines the initial metric. If the cross-section grows fast enough then there is a constraint on the initial distribution function, that the collision integral must vanish initially. In this case the matter is initially in thermal equilibrium and we are back to the...
case of a perfect fluid. The data reduces to the initial 3-metric.

The idea that a study of the Boltzmann equation would lead to an understanding of the differences between the perfect-fluid case and the Vlasov case is due to Alan Rendall and Keith Anguige [10], who made some calculations in this area, and I am grateful to them for the suggestion.

The plan of the paper is as follows: we shall give a section to each of the three models described above, after ending this section by reviewing our conventions for curvature and conformal rescaling.

Throughout the paper we take the spacetime metric to have signature (+ − − −). For a metric $g_{ab}$ our definition of the Riemann tensor is

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)V^c = R^c_{dab}V^d$$

while for the Ricci tensor we take

$$R_{ab} = R^c_{acdb}.$$  

Whenever we consider a rescaling according to (1), tilded quantities will always refer to the singular, physical spacetime $(\tilde{M}, \tilde{g}_{ab})$, while un-tilded quantities will refer to the regular, unphysical spacetime $(\mathcal{M}, g_{ab})$. For metrics $g_{ab}$ and $\tilde{g}_{ab}$ related by (1), the Ricci tensors are related by:

$$\tilde{R}_{ab} = R_{ab} - 2\nabla_a \nabla_b \log \Omega + \nabla_a \log \Omega \nabla_b \log \Omega - g_{ab}(\Box \log \Omega + 2\nabla_c \log \Omega \nabla^c \log \Omega)$$  

and the Weyl tensors by:

$$\tilde{C}^a_{bcd} = C^a_{bcd}.$$  

If $\tilde{t}^a$ is a unit vector in $\tilde{M}$ then $t^a = \Omega \tilde{t}^a$ is one in $\mathcal{M}$, and if $\tilde{w}_a$ is a unit covector in $\tilde{M}$, then $w_a = \Omega^{-1}\tilde{w}_a$ is one in $\mathcal{M}$.

We will usually have a cosmic time coordinate $Z$ at our disposal. This is a smooth function in the unphysical space-time which is constant on spacelike hypersurfaces and vanishes at $\Sigma$. Its gradient is proportional to the unit normal $N_a$ to the constant-time hypersurfaces:

$$Z_a = \nabla_a Z = VN_a$$  

where $V^2 = g^{ab}Z_aZ_b$.

The intrinsic metric $h_{ab}$ on the surfaces of constant $Z$, which is also the projection orthogonal to $N^a$, is defined by

$$h_{ab} = g_{ab} - N_aN_b$$
so that $h_{ab}$ is negative-definite. We shall use $D_a$ for the intrinsic 3-dimensional metric covariant derivative on surfaces of constant $Z$.

The second-derivative of $Z$ defines the second fundamental form $K_{ab}$ of the constant-time hypersurfaces by the equation

$$\nabla_a \nabla_b Z = V(K_{ab} + N_a A_b + A_a N_b + V_Z N_a N_b)$$  \hspace{1cm} (5)

where $A_a = h_b^b \nabla_b \log V = D_a \log V$ is the acceleration of the normal congruence defined by $N_a$, and $V_Z = V^{-1} N^a \nabla_a V$ (since in general $N^a \nabla_a = V \partial / \partial Z$).

For the Cauchy problem with data on surfaces of constant $Z$ various identities relating 3 and 4-dimensional curvatures are useful. These are

$$G_{ab} N^a N^b = -\frac{1}{2} (3R - K^2 + K^{ab} K_{ab})$$ \hspace{1cm} (6)

$$R_{ac} N^a h_b^c = D_c K_b^c - D_b K^c$$ \hspace{1cm} (7)

$$R_{cd} h_a^c h_b^d = -\mathcal{L}_N K_{ab} + 3R_{ab} + 2K_{ac} K_b^c - K K_{ab} + D_a A_b - A_a A_b$$ \hspace{1cm} (8)

where $G_{ab}, R_{ab}$ are the 4-dimensional Einstein and Ricci tensors, $3R, 3R_{ab}$ are the 3-dimensional Ricci scalar and tensor, $K = h^{ab} K_{ab}$ is the trace of the extrinsic curvature, and $\mathcal{L}_N$ is the Lie-derivative along $N^a$. When we impose the Einstein equations, these identities give the Hamiltonian and momentum constraints and the evolution equation for $K_{ab}$, respectively.

The field equations in [1] and [2] were obtained in a first-order form as:

$$A^0(X) \frac{\partial X}{\partial Z} = A^i(X) \frac{\partial X}{\partial x^i} + \frac{1}{Z} B(X) X + C(X, Z).$$  \hspace{1cm} (9)

Here $X$ is a multi-component vector of unknowns including the first and second fundamental forms of the constant-$Z$ surfaces and matter variables, $Z$ is the cosmic time coordinate, $x^i$ are comoving spatial coordinates, $A^0$ and $A^i$ are symmetric matrices with $A^0$ positive definite, $C$ is analytic in $Z$ and $B$ is subject to conditions which will emerge. There is a singularity in the time at $Z = 0$ but, by assumption, nowhere else.

Suppose we are given data $X_0$ at $Z = 0$. If we want (9) to hold at $Z = 0$ we at once have a condition, that the term $B(X_0) X_0$ must vanish. We call this the Fuchsian condition for this system. If we now substitute a power series in positive powers of $Z$ we can determine the coefficients of the series provided none of the eigenvalues of $(A^0)^{-1} B(X)$ is a positive integer. With essentially these assumptions, Claudel-Newman [4] prove existence and uniqueness of solution.
Putting field equations into precisely this first-order form is therefore the acid test of existence and uniqueness. However, this is difficult to do. One can recognise the Fuchsian conditions and impose the eigenvalue condition without going that far and this is what we shall do below.

## 2 Scalar fields

We begin by considering cosmological models whose source is a scalar field. We shall find that these may have isotropic singularities provided the potential satisfies a condition and that there is then an initial value problem for which the data is the 3-metric of the singularity surface Σ. As one might expect, this case is similar to the perfect fluid with pressure equal to density (see [11]).

If the source of the gravitational field is a scalar field \( \phi \) with potential \( U(\phi) \) then the energy-momentum tensor in the physical space-time is

\[
\tilde{T}_{ab} = \phi_a \phi_b - \tilde{g}_{ab} \left( \frac{1}{2} \tilde{g}^{cd} \phi_c \phi_d + U(\phi) \right)
\]

writing \( \phi_a \) for \( \nabla_a \phi \). The conservation equation for this \( \tilde{T}_{ab} \) is satisfied by virtue of the field equation for the scalar field which is

\[
\Box \phi = U'(\phi).
\]  

(10)

The tensor which we want to substitute into (2) is the trace-reversed energy-momentum tensor:

\[
\tilde{T}_{ab} - \frac{1}{2} \tilde{T} \tilde{g}_{ab} = \phi_a \phi_b + \tilde{g}_{ab} U(\phi).
\]  

(11)

Now from (2), (11) and the Einstein equations we find

\[
R_{ab} = \frac{2}{\Omega} \nabla_a \nabla_b \Omega - \frac{4}{\Omega^2} \nabla_a \Omega \nabla_b \Omega + g_{ab} \left( \frac{1}{\Omega} \Box \Omega + \frac{1}{\Omega^2} \nabla_c \Omega \nabla^c \Omega \right)
\]

\[
+ \frac{8\pi G}{c^2} (\phi_a \phi_b + \Omega^2 g_{ab} U(\phi)).
\]  

(12)

where we don’t yet know what to take for \( \Omega \), but if there is to be an isotropic cosmological singularity then this must hold for some \( \Omega \), and \( \tilde{T}_{ab} \) must be singular at \( \Omega = 0 \). In particular \( R_{ab} \) is finite at \( \Sigma \), where \( \Omega \) vanishes, and by (12) at \( O(\Omega^{-2}) \) this requires

\[
\phi_a \sim \frac{\Omega_a}{\Omega}
\]  

(13)
in this limit. Looking carefully at (12), we see that we cannot require \( \Omega \) to be smooth at \( \Sigma \) but we can require \( \Omega^2 \) to be smooth (just as for the perfect fluid with \( p = \rho \)). Now to satisfy (13) we need \( \phi \) to be a function of \( Z \) at least as \( Z \) tends to zero and \( \phi \) to infinity. Thus near the singularity, \( \phi \) tends to infinity and the surfaces of constant \( \phi \) are space-like. Consequently we can take a function of \( \phi \) to be the cosmic time coordinate \( Z \) and use this to define \( \Omega \). The correct function to take is determined by (12) as

\[
Z = \Omega^2 = \exp\left(\frac{\phi}{\alpha}\right)
\]

(14)

where \( \alpha \) is a constant to be determined (if \( \alpha > 0 \) then \( \phi \to -\infty \) as \( Z \to 0 \) while if \( \alpha < 0 \) then \( \phi \to +\infty \)). We choose units with \( \frac{8\pi G}{c^2} = 1 \) and substitute (14) into (12) to find

\[
ZR_{ab} = \nabla_a \nabla_b Z - \frac{1}{Z} (\alpha^2 - \frac{3}{2}) \nabla_a Z \nabla_b Z + \left(\frac{1}{2} \square Z + Z^2 U(\phi)\right) g_{ab}
\]

(15)

where we have also multiplied through by \( Z \). Now we can see that if (13) is to hold at \( \Sigma \), where \( Z = 0 \), then necessarily \( \alpha^2 = \frac{3}{2} \). Furthermore, with this choice we have a regular field equation in the unphysical space-time provided the term with the potential \( U \) is finite at \( \Sigma \). This requires \( Z^2 U(\phi) \) to be finite as \( Z \) tends to zero, or equivalently \( \exp\left(\frac{2\phi}{\alpha}\right) U(\phi) \) to be finite as \( \phi \to \pm \infty \) where the correct choice of sign depends on the sign of \( \alpha \).

If we calculate the contracted Bianchi identity in the unphysical space-time using (13) then we find that this is equivalent to the equation

\[
\square Z = \frac{1}{\alpha} Z^2 U'(\phi)
\]

(16)

which can also be obtained from (13) by conformally transforming and changing the dependent variable.

We next need to formulate a Cauchy problem for the Einstein equations in the form (15) with the matter equation (13). We perform a 3+1-splitting with respect to the hypersurfaces of constant \( Z \) and use \( Z \) as a time-coordinate. The variables in the Cauchy problem will be \( V, h_{ab} \), and \( K_{ab} \) at each value of \( Z \) and we need evolution equations for each of these, with the possibility of constraints between them.

In terms of comoving coordinates \( x^i, i = 1, 2, 3 \) the metric can be written as

\[
ds^2 = \frac{dZ^2}{V^2} + h_{ij} dx^i dx^j
\]

(17)

with \( V \) and \( h_{ij} \) as in (3) and (4).
From the trace of (5) we obtain

$$\Box Z = V(K + \frac{\partial V}{\partial Z})$$  \hspace{1cm} (18)

where $K = g^{ab}K_{ab}$, which with (11) gives an evolution equation for $V$. Substituting from (5) and (16) into (15) puts the Einstein equations into the form

$$Z R_{ab} = V(K_{ab} + N_a A_b + A_a N_b + VZ N_a N_b) + g_{ab}(\frac{1}{2\alpha}Z^2 U'(\phi) + Z^2 U(\phi)).$$  \hspace{1cm} (19)

From (19) and (6) we obtain the Hamiltonian constraint as

$$-2G_{ab}N^aN^b = 3R - K^2 + K_{ab}K^{ab} = 2(\frac{VK}{Z} + ZU(\phi))$$  \hspace{1cm} (20)

and from (7) the momentum constraint as

$$R_{ac}N^a h^c_b = D_c K^c_b - D_b K = \frac{1}{Z} D_b \log V$$  \hspace{1cm} (21)

where we have used the definition of $A_a$ from (1). For the evolution equations we have the evolution of $V$ from (16) and (18):

$$\frac{\partial V}{\partial Z} = -K + \frac{1}{\alpha V}Z^2 U'(\phi)$$  \hspace{1cm} (22)

then the evolution of $h_{ab}$, which is essentially the definition of $K_{ab}$:

$$\mathcal{L}_N h_{ab} = 2K_{ab}$$  \hspace{1cm} (23)

and finally from (8) and (19) the evolution of $K_{ab}$

$$\mathcal{L}_N K_{ab} = 3R_{ab} + 2K_{ac}K^c_b - KK_{ab} + D_a A_b - A_a A_b - \frac{V}{Z}K_{ab} + h_{ab}(\frac{1}{2\alpha}Z^2 U'(\phi) + Z^2 U(\phi)).$$  \hspace{1cm} (24)

To summarise, the evolution equations are (22), (23) and (24), with constraints (20) and (21). Because the contracted Bianchi identity for (15) is (16) which in turn is (22) and part of the system, we know that the evolution preserves the constraints. The evolution equations have a singularity in the ‘time’ at $Z = 0$. In order to put them in the form of (9) it would be necessary to introduce many more variables to make the system first-order in space. Rather than do this, we can draw similar conclusions in the present
form. Specifically we can read off the Fuchsian constraints and check the
eigenvalue conditions noted after (9).

For the Fuchsian constraints, if (21) is to hold at \( Z = 0 \) then we need
\( D_b V = 0 \) so that \( V \) is constant on \( \Sigma \). By a constant rescaling we can suppose
that \( V = 1 \) there. Now for (20) at \( Z = 0 \) we need initially
\[
K = -\lim(Z^2 U(\phi))
\]
while from (24) we need initially
\[
K_{ab} = -h_{ab} \left( \lim \left( \frac{1}{2\alpha} Z^2 U'(\phi) + Z^2 U(\phi) \right) \right).
\]
For consistency between (25) and (26) we need to impose
\[
Z^2 U(\phi) + \frac{3}{4\alpha} Z^2 U'(\phi) \to 0 \text{ as } Z \to 0.
\]
This is in addition to the condition found earlier that \( Z^2 U(\phi) \) have a finite
limit as \( Z \to 0 \). Assuming this condition on \( U(\phi) \) holds then the Fuchsian
conditions can be solved to find that the free data at \( \Sigma \) consist of just the
initial 3-metric \( h_{ab} \). The other variables are \( V \), which is one without loss of
generality, and \( K_{ab} \), which is determined by (26).

It is straightforward to construct solutions as power series in \( Z \), if one
assumes power series in \( Z \) for \( U(\phi) \), and these are unique with the given data
(this happens because the only singular evolution equation is (24) and the
singular term has a negative coefficient). While we don’t convert the system
into a form in which the theorem of Claudel-Newman [4] can be applied,
the existence of unique power series solutions has in the past been a reliable
guide to the applicability of this theorem.

It is worth noting that \( \Sigma \) is necessarily umbilic, in that its second fun-
damental form is a multiple of its metric, and this multiple is in fact a
constant. Consequently the magnetic part of the Weyl tensor (see e.g. [5])
is zero at \( \Sigma \) and the electric part of the Weyl tensor is proportional to the
trace-free part of the intrinsic Ricci tensor of \( \Sigma \). The initial Weyl tensor is
finite, as it must be, and is zero if and only if \( \Sigma \) has vanishing trace-free
Ricci tensor or equivalently has a homogeneous and isotropic metric. But
now uniqueness for the Cauchy problem would mean that the cosmological
model was a Friedman-Robertson-Walker model. Thus with scalar fields we
have the same situation as with polytropic perfect fluids, namely that if the
Weyl tensor is zero initially then it is always zero.
3 The Einstein-Yang-Mills-Liouville equations

The Einstein-Yang-Mills-Liouville equations have been considered by Choquet-Bruhat and coworkers as a model of the (early) universe where the matter content is a ‘quark-gluon plasma’ [3]. The model has a distribution function for massless, collisionless matter with colour-charges which produces a Yang-Mills field. With space-time indices $a, b, \ldots$ and Lie algebra indices $\alpha, \beta, \ldots$ the matter variables are the Yang-Mills potential $\tilde{A}_a^\alpha$ and the distribution function $\tilde{f}(x^a \tilde{p}_b, q^\alpha)$ where the vector $q^\alpha$ is a vector of colour-charges.

We retain the convention that tilded quantities relate to the physical space-time and untilded to the rescaled, unphysical space-time. In fact $A_a^\alpha, f$ and $p_b$ are all unchanged under rescaling, as is the Yang-Mills field $F_{ab}^\alpha$, which is defined by

$$F_{ab}^\alpha = \nabla_a A_b^\alpha - \nabla_b A_a^\alpha + c_{\beta\gamma}^\alpha A_a^\beta A_b^\gamma$$

where $c_{\beta\gamma}^\alpha$ are the structure constants for the Lie algebra, say $G$.

The constituent ‘particles’ of the collisionless matter follow the Yang-Mills version of the Lorentz force law which, for a particle with colour-charges given by $q^\alpha$ can be written as the system:

$$\frac{dx^a}{ds} = g^{ab} p_b$$
$$\frac{dp_a}{ds} = -\frac{1}{2} \frac{\partial g_{bc}}{\partial x^a} p_b p_c + q^\alpha \eta_{\alpha\beta} F_{ab}^\beta$$
$$\frac{dq^\alpha}{ds} = -c_{\beta\gamma}^\alpha A_a^\beta p^\beta q^\gamma$$

(29)

here $\eta_{\alpha\beta}$ is the metric on $G$ (assuming it is semi-simple). Again the system (29) is conformally-covariant in that it takes the same appearance if tildes are inserted throughout.

The Vlasov or Liouville equation for the distribution function follows from (29) and the chain rule:

$$\frac{\partial f}{\partial x^a} \frac{dx^a}{ds} + \frac{\partial f}{\partial p_a} \frac{dp_a}{ds} + \frac{\partial f}{\partial q^\alpha} \frac{dq^\alpha}{ds} = 0.$$ 

The Yang-Mills current is obtained as an integral over the distribution function. Now we do need to watch the conformal rescaling. In the unphysical space-time we consider the vector field:

$$J_a^\alpha = \int q^\alpha p_a f \omega p \omega q$$
where $\omega_p$ is the standard Lorentz-invariant volume-form on the null cone, which can be written with respect to a 3+1-splitting as
\begin{equation}
\omega_p = \frac{1}{p^0 \sqrt{-g}} dp_1 dp_2 dp_3
\end{equation}
and $\omega_q$ is an invariant volume-form on $G$.

Under conformal rescaling we have
\begin{equation}
\tilde{\omega}_p = \Omega^{-2} \omega_p
\end{equation}
and then the physical Yang-Mills current is
\begin{equation}
\tilde{J}^\alpha_a = \Omega^{-2} J^\alpha_a.
\end{equation}
The physical Yang-Mills equation relates the divergence of the physical Yang-Mills field to the physical Yang-Mills current. Now this equation transforms well under conformal rescaling and in the unphysical variables we find
\begin{equation}
\nabla_a F^{\alpha\beta\gamma} + c_{\beta\gamma} A_\beta^\alpha F^{\beta\gamma} = J^\alpha_a.
\end{equation}
The energy-momentum tensor is a sum of two terms. For the Yang-Mills field consider
\begin{equation}
T^{YM}_{ab} = \eta_{\alpha\beta} (F^\alpha_{bc} F^{\beta c} - \frac{1}{4} g_{ab} F^{\alpha cd} F_{cd}^{\beta})
\end{equation}
and for the collisionless matter consider
\begin{equation}
T^{L}_{ab} = \int f p_a p_b \omega_p \omega_q.
\end{equation}
Note that both these tensors are trace-free. The physical energy-momentum tensor is then
\begin{equation}
\tilde{T}_{ab} = \Omega^{-2} (T^{YM}_{ab} + T^{L}_{ab}).
\end{equation}
We write down the Einstein equations in the unphysical space-time with the aid of (2). Following the argument in [1] we may choose the cosmic time coordinate $Z$ so that $\Box Z = 0$ everywhere and $V = 1$ at $\Sigma$ and we can then take $Z = \Omega$ (there is sufficient conformal gauge freedom to allow this, provided there is an isotropic cosmological singularity, because the energy-momentum tensor is trace-free). From (2) we obtain
\begin{equation}
Z^2 R_{ab} = 2Z \nabla_a \nabla_b Z - 4 \nabla_a Z \nabla_b Z + V^2 g_{ab} + T^{YM}_{ab} + T^{L}_{ab}.
\end{equation}
The dependent variables on each surface of constant $Z$ are $V, h_{ab}$ and $K_{ab}$ as in section 2, together with the Yang-Mills potential $A_\alpha^a$ and the Yang-Mills electric field $F^\alpha_a := F_{ab} N^b$, and the distribution function $f$. There
is sufficient freedom in the conformal gauge to ensure that \( K = 0 \) at \( \Sigma \). We choose the temporal Yang-Mills gauge so that \( N^a A^\alpha_a = 0 \). The Yang-Mills equations (33) imply a Gauss-law constraint:

\[
N_b(\partial^a F_{ab\alpha} - c^\alpha_{\beta\gamma} A^\beta_a F^{\alpha\gamma}) = 0
\]

which translates as

\[
D_a E^{a\alpha} + c^\alpha_{\beta\gamma} A^\beta_a E^{a\gamma} = \frac{1}{\sqrt{-h}} \int q^\alpha f dp_1 dp_2 dp_3.
\]

(36)

This constraint holds on every surface of constant \( Z \) and in particular needs to hold on the data at \( \Sigma \).

We perform a 3+1-splitting with respect to the hypersurfaces of constant \( Z \) as before, and decompose the Einstein equations (33) into evolution and constraints. The Fuchsian constraints for data at \( \Sigma \) will follow from the condition that the right-hand-side in (33) must be \( O(Z^2) \). We write spatial indices \( i, j, k, \ldots = 1, 2, 3 \) and then the conditions arise from the vanishing of the \( O(1) \) terms in the \( (i, j) \) and \( (0, i) \) components in (33) and the vanishing of the \( O(Z) \) terms in the \( (i, j) \) components. (The \( O(Z) \) terms in the \( (0, i) \) components vanish by virtue of these conditions since the contracted Bianchi conditions hold.) The conditions are most easily written in terms of multipole moments \( \chi_i, \chi_{ij}, \chi_{ijk}, \ldots \) of the matter density and \( J^\alpha_i, J^\alpha_{ij}, J^\alpha_{ijk}, \ldots \) of the charge density which we define at \( \Sigma \) by

\[
\chi_i = \frac{1}{\sqrt{-h}} \int f p_i d^3p \omega_q
\]

\[
\chi_{ij} = \frac{1}{\sqrt{-h}} \int \frac{f p_i p_j}{p^0} d^3p \omega_q
\]

\[
\chi_{ijk} = \frac{1}{\sqrt{-h}} \int \frac{f p_i p_j p_k}{(p^0)^2} d^3p \omega_q
\]

and so on, and

\[
J^\alpha_i = \frac{1}{\sqrt{-h}} \int \frac{q^\alpha f p_i}{p^0} d^3p \omega_q
\]

\[
J^\alpha_{ij} = \frac{1}{\sqrt{-h}} \int \frac{q^\alpha f p_i p_j}{(p^0)^2} d^3p \omega_q
\]

\[
J^\alpha_{ijk} = \frac{1}{\sqrt{-h}} \int \frac{q^\alpha f p_i p_j p_k}{(p^0)^3} d^3p \omega_q
\]

and so on, where it is understood that \( f, h \) and \( p^0 \) are evaluated at \( \Sigma \). We have used (30) and in these definitions

\[
(p^0)^2 = -h^{ij} p_i p_j.
\]
Now the Fuchsian conditions are

\[
0 = h_{ij} + \chi_{ij} - \eta_{\alpha\beta}(E^\alpha_i E^\beta_j - F^\alpha_i F^\beta_j)
+ \frac{1}{4} h_{ij}(2E^\alpha_j E^k\beta + F^\alpha_{km} F^{km\beta}) \tag{37}
\]

\[
0 = \chi_i - \eta_{\alpha\beta} E^\beta_j \tag{38}
\]

and the most complicated which can be written

\[
K_{\alpha\beta} Q_{\alpha\beta} = h_{ij} D_i \chi_{ij} + c^{\alpha\beta\gamma}_\alpha A_{\beta\gamma}\chi_{ij}
- \eta_{\alpha\beta}(4E^\alpha_i E^\beta_j) + 2F^\alpha_{(i} J^\beta_{j)m} + F^{\alpha\beta} J^{\beta}_{ij}
- \tilde{D}_k E^\alpha_{(i} F^{k\beta}_{j)} - \tilde{D}_i E^\alpha_{(j} F^{k\beta}_{k)} + 2E^\alpha_{(i} \tilde{D}^k F^{k\beta}_{j)}
\]

\[
+ h_{ij}(\tilde{D}_k E^\alpha_{(k} E^{\beta}_{m)} - E^\alpha_{ij} F^{\beta}_{j}) \tag{39}
\]

where

\[
Q_{\alpha\beta\gamma} = 4h_{ij} h_{mj} - h_{tm} h_{ij} - \chi_{\alpha\beta\gamma}
- \eta_{\alpha\beta}(E^\alpha_i E^\beta_j + E^\alpha_i E^\beta_j h_{mi} + E^\alpha_i E^\beta_j h_{ij} + E^\alpha_i E^\beta_j h_{i} + E^\alpha_i E^\beta_j h_{j})
- E^\alpha_i E^\beta_j h_{tm} - E^\alpha_i E^\beta_j h_{ij}
- F^\alpha_{ij} F^{\beta}_{jm} - F^\alpha_{ij} F^{\beta}_{jm} + h_{ij} F^\alpha_{ik} F^{k\beta}_{m} + h_{tm} F^\alpha_{ik} F^{k\beta}_{j}
- h_{ij} h_{mj}(2E^\alpha_i E^{k\beta} + F^\alpha_{kn} F^{kn\beta})
+ h_{tm} h_{ij}(\frac{1}{2} F^\alpha_i E^{k\beta} - \frac{1}{4} F^\alpha_{kn} F^{kn\beta}) \tag{40}
\]

If we consider first (37), then we recall from [2] that in the case of vanishing Yang-Mills fields this actually determines the initial metric \(h_{ij}\) from the initial distribution function \(f\). It is hard to see whether this remains true for arbitrary Yang-Mills fields but it will certainly remain true for small fields.

Next note that, by virtue of (37) and (38), \(Q_{\alpha\beta\gamma}\) is symmetric and trace-free on each index pair, and symmetric under interchange of the two pairs. If we think of it as a map from trace-free symmetric tensors to trace-free symmetric tensors, it necessarily has real eigenvalues and we can solve (39) for \(K_{ij}\) provided none of these eigenvalues is zero (recall that \(K_{ij}\) is trace-free at \(\Sigma\) by choice of conformal gauge). Again this is a difficult question in general although we know from [2] that it is true if the Yang-Mills fields are zero, so that it will be true if they are small.

Finally (38) and (36) are constraints on the initial matter variables. After the Fuchsian conditions we should check the eigenvalues as in the discussion
following (9), but again we can argue that they must still satisfy the required condition, at least for small fields.

To summarise, the data at Σ consists of the Yang-Mills potential $A^\alpha_i$, from which the magnetic field $F^\alpha_{ij}$ is calculated by (28), the Yang-Mills electric field $E^\alpha_i$, the distribution function $f$ and the first and second fundamental forms $h_{ij}$ and $K_{ij}$. These are subject to the Gauss-law constraint (36) and the three Fuchsian constraints (37), (38) and (39). In the pure Vlasov case, with the Yang-Mills fields omitted, (37) suffices to determine $h_{ij}$ from $f$ and (38) then determines $K_{ij}$ from $f$ (see [2] or [1] for details of this). Thus in that case the free data is just $f$ subject to the counterpart of (38). With the Yang-Mills fields present it is not so easy to see that (37) determines a unique $h_{ij}$ for arbitrary Yang-Mills fields and that (39) determines a unique $K_{ij}$, but this must certainly be true for small fields by continuity. Once the Fuchsian constraints are satisfied, and relying on the experience of earlier cases, we expect to have existence and uniqueness of solutions.

4 Massive Einstein-Vlasov and Einstein-Boltzmann

In this section we do two things. First we look at massive Einstein-Vlasov in the expectation that near the singularity there should be no difference between massless collisionless matter and massive collisionless matter. Then, restricting to massless for simplicity we seek to drop the collisionless condition. This is partly to move to a more realistic situation but also to resolve a puzzle left over from earlier work and noted in the Introduction.

For an isotropic singularity with a perfect fluid (at least for linear equations of state [2]) the data are just the initial 3-metric. There are no separate data for the matter. For an isotropic singularity with massless collisionless matter the data is just the initial distribution function, which determines the initial first and second fundamental forms [2]. This time there are no separate data for the geometry. It was suggested to me by Alan Rendall [10] that by including a collision term in the Vlasov equation, in other words by turning to the Einstein-Boltzmann equations, one might find a bridge between these two extreme examples. This turns out to be the case.

We begin by considering the massive Einstein-Vlasov equations, by which we mean the Einstein equations whose source is collisionless matter where the matter is now supposed to be composed of particles of a single nonzero mass, say $m$. The presence of the mass changes the detail of the equations but we shall see that that change has no influence near to the singularity.

In this case, the stress-energy-momentum tensor in the physical variables
\[ \tilde{T}_{ab}^L = \int f p_a p_b \tilde{\omega}_p \]  

(41)

with

\[ \tilde{\omega}_p = \frac{1}{p^0 \sqrt{-g}} dp_1 dp_2 dp_3. \]  

(42)

The distribution function \( f \) is supported on the mass shell where

\[ \tilde{g}^{ab} p_a p_b = m^2 \]

which we must solve to find \( \tilde{p}^0 \) in (12). We subsequently regard \( \tilde{p}^0 \) (and later \( p^0 \)) as a function of \( p_i \) and the coordinates, and the distribution function as a function only of \( p_i \) and \( x^a \).

Under conformal rescaling according to (1) we still have (31) with \( \omega_p \) as in (30) but now with

\[ p^0 = V^2 p_0 = V(-h^{ij} p_i p_j + m^2 \Omega^2)^{\frac{1}{2}}. \]  

(43)

As one might expect, the mass becomes insignificant near the singularity, where \( \Omega \) vanishes. The unphysical energy-momentum tensor to go into (35) may be written, with the aid of the metric (17) and the expression (43) as

\[ T_{ab}^L = \frac{1}{\sqrt{-h}} \int f p_a p_b dp_1 dp_2 dp_3 \]

\[ (\tilde{-h}^{km} p_k p_m + m^2 \Omega^2)^{\frac{1}{2}}. \]

Provided \( f \) is supported away from the origin in \( p \)-space, which one usually assumes for regularity (see e.g. [1]), the energy momentum tensor can be expanded as a series in positive powers of \( \Omega \). In particular in the limit at \( \Sigma \) it is the same as in the massless case. Consequently in rescaling we take \( \Omega \) to be \( Z \) just as in the massless case.

We also have to change the Vlasov equation in the presence of mass. The Vlasov equation expresses the vanishing of the derivative of the distribution function along the geodesic flow, or equivalently the vanishing of its Poisson bracket with the Hamiltonian. In the unphysical variables this is

\[ \mathcal{L}_\tilde{X} f(p_a, x^b) = \{\frac{1}{2} \tilde{g}^{ab} p_a p_b, f\}_{PB} \]

\[ = \tilde{g}^{ab} p_a \frac{\partial f}{\partial x^b} - \frac{1}{2} \frac{\partial \tilde{g}^{ab}}{\partial x^c} p_a p_b \frac{\partial f}{\partial p_c} \]

\[ = 0. \]  

(44)
Written in this form, it is easy to see how to conformally transform the equation. With the metric as in (17) and rescaling as in (1) we obtain
\[
V^2 p_0 \frac{\partial f}{\partial Z} + h^{ij} p_i \frac{\partial f}{\partial x^j} - (V \frac{\partial V}{\partial x^i}(p_0)^2 + \frac{1}{2} \frac{\partial h^{jk}}{\partial x^i} p_j p_k) \frac{\partial f}{\partial p_i} + m^2 \frac{\partial \Omega}{\partial x^i} \frac{\partial f}{\partial p_i} = 0. \tag{45}
\]

The last term in this actually vanishes because \( \Omega \) is \( Z \), but there are still explicit appearances of \( Z \) in (45) through \( p_0 \). However these are always in the numerator and we get no new singularities from them.

We conclude that, at least at the level of looking for power series solutions in the time-variable, there should be no difference between massive and massless Vlasov.

To go from Vlasov to Boltzmann is to go from collisionless matter to the inclusion of a collision-term in (44). Symbolically we have
\[
L_\tilde{X} f = C(f, f) \tag{46}
\]
where \( C(f, f) \) is a bilinear functional in \( f \). We set up the formalism for this following Choquet-Bruhat [3]. We restrict to 2-body collisions and suppose that incoming particles with 4-momenta \( p^{(0)} \) and \( p^{(1)} \) (temporarily suppressing space-time indices) collide to give outgoing particles with 4-momenta \( p^{(2)} \) and \( p^{(3)} \), where all particles have mass \( m \) and total 4-momentum \( P \) is conserved:
\[
P = p^{(0)} + p^{(1)} = p^{(2)} + p^{(3)}
\]
If we choose \( p^{(0)} \) then there are 3 degrees of freedom in \( p^{(1)} \), two in \( p^{(2)} \) and none in \( p^{(3)} \). The collision term is written as an integral:
\[
C(f, f) = \int (f_2 f_3 - f_0 f_1) k(x^a, p^{(0)}, p^{(1)}, p^{(2)}, p^{(3)}) \tilde{\omega}_{p^{(1)}} \tilde{\xi}_2 \tag{47}
\]
where \( f_n = f(x^a, p^{(n)}_i) \), \( k \) is a kernel to be discussed below, \( \tilde{\omega}_p \) is as in (40) but with the time-direction defined by \( p^{(0)} \), and \( \tilde{\xi}_2 \) is an invariant (Leray) 2-form on the space of allowed \( p^{(2)} \) (see [3] for a precise definition). The kernel \( k \) is related to the differential cross-section and is usually assumed, for reasons of symmetry, to be a function only of the total energy involved in the collision, which is
\[
\tilde{s} = g^{ab} P_a P_b \tag{48}
\]
and the angle \( \theta \) between \( p^{(0)} - p^{(1)} \) and \( p^{(2)} - p^{(3)} \), which is the scattering angle in the centre-of-mass frame, so we may write \( k(\tilde{s}, \theta) \). For Choquet-Bruhat [3] \( k \) is the differential cross-section but for other authors, e.g. Grout et
al \[ k = \tilde{s}\sigma(\tilde{s}, \theta) \] where \( \sigma \) is the differential cross-section. The important consideration for us is the behaviour of the various terms in \( \frac{d}{d\tau} \) under conformal rescaling. We know how to transform \( \tilde{s} \) from \( \frac{d}{d\tau} \) and \( \tilde{\omega}_p \) from \( \frac{d}{d\tau} \). The Leray form \( \xi_2 \) is conformally-invariant \[ \frac{d}{d\tau} \] and so the rescaling of \( \frac{d}{d\tau} \), at least for the massless case is

\[
\mathcal{L}_X f = \Omega^2 \mathcal{L}_{\tilde{X}} f = \Omega^2 C(f, f) = \int (f_2 f_3 - f_0 f_1) k(\Omega^{-2}s, \theta)\omega(\xi_1, \xi_2). \tag{49}
\]

We want the Boltzmann equation \( \frac{d}{d\tau} \) to hold up to \( \Sigma \) and so we need to see if it gives another Fuchsian condition. This will depend on the behaviour of the collision term and specifically on the behaviour of the kernel \( k \) as \( \Omega \) goes to zero or equivalently on the behaviour of \( k(\tilde{s}, \theta) \) as \( \tilde{s} \) increases without bound.

If \( k \) is sufficiently well-behaved that the integral in \( \frac{d}{d\tau} \) exists in the limit for every \( f \) then we get no new Fuchsian condition. The situation is just like the Vlasov case dealt with in \[ \frac{d}{d\tau} \] and we should expect existence and uniqueness with \( f_0 \) as the free data (subject to the Fuchsian constraint corresponding to \( \chi_i = 0 \)). The other Fuchsian constraints determine the first and second fundamental forms of \( \Sigma \) in terms of \( f_0 \).

However if \( k \) diverges for large \( \tilde{s} \) then we do get a new Fuchsian condition: for the collision term to be finite, the term \( (f_2 f_3 - f_0 f_1) \) in the integral must vanish at \( \Sigma \). As is familiar from the literature (see e.g. \[ \frac{d}{d\tau} \]), this condition requires the initial distribution function to take the form

\[
f_0(x^i, p_j) = \exp(-\alpha(x^i) - \beta^a(x^i)p_a) \tag{50}
\]

for a function \( \alpha(x^i) \) and a vector-field \( \beta^a(x^i) \), both defined at \( \Sigma \). Next, the Fuchsian condition \( \chi_i = 0 \) when imposed on a function \( f_0 \) of the form of \( \frac{d}{d\tau} \) requires that the vector \( \beta^a \) be proportional to the normal \( N^a \) to \( \Sigma \). Now the initial distribution function takes the form

\[
f_0 = \exp(-\alpha(x^i) - \beta(x^i)p_0) \tag{51}
\]

which is formally the distribution for local equilibrium with \( \alpha \) and \( \beta \) related to chemical potential and inverse temperature (for global equilibrium these would be constants). Note that \( \beta \) is not the physical inverse temperature. This would be \( \bar{\beta} = \Omega\beta \) with the rescaling conventions in use here, and this vanishes at \( \Sigma \) as one would expect.
The quantity $p_0$ in (51) is understood as a function of $x^i$ and $p_j$ as in (43) but now

$$p_0 = (-h^{ij}p_ip_j)^{1/2}$$

since at $\Sigma$ we can take $V = 1$ (and we are supposing for simplicity that $m = 0$). With this in (51), the Fuchsian constraint corresponding to (38) is satisfied provided $\alpha$ and $\beta$ are related by

$$\beta^4 e^{-\alpha} = 256\pi^2.$$

Finally the third Fuchsian constraint, corresponding to (39), forces the initial second fundamental form to vanish, because the third moment $\chi_{ijk}$ of $f_0$ vanishes for a distribution function of the form of (51). Thus the data consist of the initial metric $h_{ij}$ and a function $\beta(x^i)$ which we can call the (unphysical) initial inverse temperature. The matter is initially in local equilibrium, so that the stress-tensor is the stress-tensor for a perfect fluid. However, there is no reason to suppose that this form for $f$ will be preserved by the evolution (49).

In conclusion, we see that, as suggested by Rendall and Anguige [10], the Boltzmann case sits between the perfect-fluid and Vlasov cases studied in [1] and [2], and depending on the behaviour of the differential cross-section at large energies it will, as regards data at the singularity, approach the Vlasov case or the perfect fluid case but with an extra function.

Acknowledgement

Part of this work was done at the Albert Einstein Institute in Golm in September 2000 and I am grateful to them for hospitality and to Alan Rendall for useful discussions.

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