Integrals over classical Groups, Random permutations, Toda and Toeplitz lattices

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In recent times, there has been a considerable interest in matrix Fourier-like integrals over the classical groups $O_\pm(\ell), O(\ell), Sp(\ell)$ and $U(\ell)$, due to their connection with the distribution of the length of the longest increasing sequence in random permutations and random involutions and also with the spectrum of random matrices. This connection first appeared in I. Gessel’s work [13], who showed that some generating function for the distribution of the length of the longest increasing sequence can be represented as a Toeplitz matrix. One of the purposes of this paper is to show that all those expressions are unique solutions to the Painlevé V equation, with certain initial condition. In this work, we present both, new results, concerning $O(\ell)$ and known ones, concerning $U(\ell)$; all cases are done in a same unified way.

Our method consists of appropriately adding one set of time variables $t = (t_1, t_2, \ldots)$ to the integrals for the real compact groups and two sets of times $(t, s) = (t_1, t_2, \ldots, s_1, s_2, \ldots)$ for the unitary group. The point is that these new time-dependent integrals satisfy integrable hierarchies:

(i) $O_\pm(\ell)$ and $Sp(\ell)$ correspond to the standard Toda lattice; the associated moment matrices are Hänkel, whose determinants provide the Toda $\tau$-functions.

(ii) $U(\ell)$ corresponds to a very special case of the discrete sinh-Gordon equation, leading to a new lattice, the Toeplitz lattice. This lattice involves a dual pair of infinite variables $x_i$ and $y_i$, themselves matrix integrals. Its $\tau$-functions are determinants of moment matrices, which are Toeplitz.
Both systems, the standard Toda lattice and the Toeplitz lattice are peculiar reductions of the 2-Toda lattice. Each reduction has a natural vertex operator, and so, a natural Virasoro algebra, a subalgebra of which annihilates the $\tau$-functions. Combining these equations and, in the end, evaluating the result along appropriate $(t, s)$-loci all lead, in a unifying and quick way, to different versions of the Painlevé V equation for the integrals. More details about the precise nature of the Painlevé equations will be given in propositions 3.3, 4.1 and 4.2. After this paper had been written, we found out the Toeplitz lattice coincides with the so-called Ablowitz-Ladik system; see Suris [18]. However, our approach to that system is novel.

Let $S_n$ be the group of $n!$ permutations $\pi_n$ and $S_{2n}^0$ the subset of $(2n - 1)!! = \frac{(2n)!}{2^{n!}}$ fixed-point free involutions $\pi^0$ (i.e., $(\pi^0)^2 = I$ and $\pi^0(k) \neq k$ for $1 \leq k \leq 2n$). $\pi_n$ refers to a permutation in $S_n$ and $\pi_{2n}^0$ to an involution in $S_{2n}^0$. Also consider $S_{n,k} = \{\text{words of length } n \text{ from an alphabet of } k \text{ letters}\}$.

An increasing subsequence of $\pi \in S_n$ or $S_{2n}^0$ is a sequence $1 \leq j_1 < ... < j_k \leq n$, such that $\pi(j_1) < ... < \pi(j_k)$. Define

$$\sigma(\pi_n) = \text{ length of the longest increasing subsequence of } \pi_n.$$  

In the case of $S_{n,k}$, the definition of $\sigma$ is the same, except that the subsequences must be increasing, without being necessarily strictly increasing.

Notation: The expectations $E_{O(\ell)}$, $E_{U(\ell)}$, ... refer to integration with regard to Haar measure, normalized so that $E_{O(\ell)}(1) = 1$, $E_{U(\ell)}(1) = 1$, ..., as it should. Sometimes, it will be more convenient to use integrals $\int_{O(\ell)}$, $\int_{U(\ell)}$, ..., which refer to integration with respect to Haar measure, normalized as in Proposition 1.1 below. For $U(\ell)$, the two normalizations happen to agree.

**Theorem 0.1** For every $\ell \geq 0$, the generating functions below have the following expression in terms of specific solutions of the Painlevé V equation:
(i) \[
2 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \# \{ \pi_{2n}^0 \in S_{2n}^0 \mid \sigma(\pi_{2n}^0) \leq \ell + 1 \}
= E_{O(\ell+1)} e^{x \text{ tr} M} dM + E_{O(\ell+1)} e^{x \text{ tr} \bar{M}} dM
= \exp \left( \int_0^x \frac{f_{\ell}^-(u)}{u} du \right) + \exp \left( \int_0^x \frac{f_{\ell}^+(u)}{u} du \right)
\]

(ii) \[
\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \# \{ \pi_n \in S_n \mid \sigma(\pi_n) \leq \ell \} = E_{U(\ell)} e^{\sqrt{x} \text{ tr}(M+\bar{M})} dM
= \exp \int_0^x \log \left( \frac{x}{u} \right) g_\ell(u) du,
\]

(iii) \[
\sum_{n=0}^{\infty} \frac{x^n}{n!} \# \{ \pi_n \in S_{n,k} \mid \sigma(\pi_n) \leq \ell \} = E_{U(\ell)} \det(I + M)^k e^{-x \text{ tr} \bar{M}} dM
= \exp \left( x \ell + (\ell + k) \int_0^x \frac{h_\ell(u)}{u} du \right)
\]

where \( f_\ell, g_\ell \) and \( h_\ell \) are unique solutions to three different versions of the Painlevé V equation, with the initial condition indicated below; to be precise

\[
\begin{align*}
(i) \quad & \left\{ f''' + \frac{1}{u} f'' + \frac{6}{u^2} f' - \frac{4}{u^2} f f' - \frac{16u^2 + \ell^2}{u^2} f' + \frac{16}{u} f + \frac{2(\ell^2 - 1)}{u} = 0 \\
& \text{with } f_\ell^\pm(u) = u^2 \pm \frac{u^{\ell+1}}{\ell!} + O(u^{\ell+2}), \text{ near } u = 0.
\right.
\end{align*}
\]

\[
(ii) \quad \left\{ g'' - \frac{g^2}{2} \left( \frac{1}{g-1} + \frac{1}{g} \right) + \frac{g'}{u} + \frac{2}{u} g(g-1) - \frac{\ell^2}{2u^2} g - \frac{1}{g} = 0 \\
& \text{with } g_\ell(u) = 1 - \frac{u^\ell}{(\ell!)^2} + O(u^{\ell+1}), \text{ near } u = 0.
\right.
\]
\[
\begin{aligned}
\left\{ \begin{array}{l}
h'' - \frac{h'''}{2} \left( \frac{1}{h' + 1} + \frac{1}{h'} \right) + \frac{h''}{u} + \frac{2(\ell + k)}{u} h'(h' + 1) \\
- \frac{1}{2u^2 h'(h' + 1)} \left( (u - \ell) h' - h - \ell \right) \left( (2h + u + \ell) h' + h + \ell \right) = 0
\end{array} \right.
\end{aligned}
\]

with \( h_\ell(u) = u \frac{k - \ell}{k + \ell} - \frac{u^{\ell+1}}{(\ell + 1)!} \left( \frac{k + \ell - 1}{\ell} \right) + O(u^{\ell+2}), \) near \( u = 0. \)

That the orthogonal matrix integrals (i) satisfy Painlevé V is new. The identity (i) involving orthogonal matrix integrals and random involutions is due to Rains [17]. That the \( U(\ell) \)-integral (ii) satisfies Painlevé was first established by Hisakado [12], using our methods (see [1]) and then reestablished by Tracy and Widom [19], using methods of functional analysis. The identity between random permutations and unitary matrix integrals, via Toeplitz determinants, goes back to Gessel [13]. Similarly, the \( U(\ell) \)-integral (iii) was first established by Tracy-Widom [20], again using methods of functional analysis. The relation of the combinatorics to integrals over the groups was extensively studied by Diaconis and Shashahani [11], Rains [17], Baik and Rains [7]; see also Johansson [14], Baik, Deift and Johansson [8], Aldous and Diaconis [4], Tracy and Widom [13][20].

Our methods have the benefit of providing a unifying (and also quick) way of establishing these results, new and known ones. The relationship with integrable systems can be summarized by Theorems 0.2 and 0.3:

**Theorem 0.2** Defining the integrals

(i) \( I^{\pm}_\ell(x) = \int_{O^{\pm}_\ell(\ell)} e^{x \text{tr} M} dM \)

(ii) \( I_{\ell}(x, y) = \int_{U(\ell)} e^{\text{tr}(xM - y\bar{M})} dM, \)
the expressions¹

(i) \[ q_\ell(x) = \log e^\pm \frac{I_{\ell+2}^\pm}{I_\ell^\pm}, \quad \text{with} \quad e_\ell^\pm = \frac{2}{[\ell + 2]_{\text{even}}} \quad \text{and} \quad e_\ell^- = \frac{2}{[\ell + 1]_{\text{even}}} \]

(ii) \[ q_\ell(x, y) = \log \frac{I_{\ell+1}}{I_\ell} \]

satisfy respectively

(i) \[ \frac{1}{4} \frac{\partial^2 q_\ell}{\partial x^2} = -e^{q_\ell - q_{\ell-1}} + e^{q_{\ell+1} - q_{\ell}} \quad \text{(standard Toda lattice)} \]

(ii) \[ \frac{\partial^2 q_\ell}{\partial x \partial y} = e^{q_\ell - q_{\ell-1}} - e^{q_{\ell+1} - q_{\ell}}. \quad \text{(discrete sinh-Gordon equation)} \]

Remark: Note, if the lattice is 2-periodic, i.e., \( q_\ell = q_{\ell+2k} \), then (ii) becomes the sinh-Gordon equation for \( r = q_\ell - q_{\ell-1} \):

\[ \frac{\partial^2 r}{\partial x \partial y} = 4 \sinh r. \]

Define the following probability measure on the unitary group \( U(n) \):

\[ P_{U(n)}^{t,s}(M \in dM) := \tau_n(t, s)^{-1} e^{\sum_{i=1}^\infty \text{Tr}(t_i M^i - s_i \bar{M}^i)} dM, \]

and \( h = \text{diag}(h_0, h_1, ...) \), \( h_n = \tau_{n+1}/\tau_n \), with

\[ \tau_n(t, s) := \int_{U(n)} e^{\sum_{i=1}^\infty \text{Tr}(t_i M^i - s_i \bar{M}^i)} dM. \]

Also, let \( p_i^{(1)}(t, s; z) \) and \( p_i^{(2)}(t, s; z) \) be bi-orthogonal monic polynomials in \( z \), depending on \( t \) and \( s \), satisfying \( \langle p_i^{(1)}(t, s; z), p_j^{(2)}(t, s; z) \rangle_{t,s} = \delta_{ij} h_i \), with

\[ \langle f(z), g(z) \rangle_{t,s} := \oint_{S^1} \frac{dz}{2\pi i z} f(z) g(z^{-1}) e^{\sum_{i=1}^\infty (t_i z^i - s_i z^{-i})}, \quad t, s \in \mathbb{C}^\infty. \]

¹In this statement, we use the following notation:

\[ [n]_{\text{even}} := \max \{ \text{even } x, \text{ such that } x \leq n \}. \]
The statement of Theorem 0.3 contains the elementary Schur polynomial \( p_n \), defined by
\[
e^{\sum_{i=0}^{\infty} t_i z^i} := \sum_{i \geq 0} p_i(t_1, t_2, ...) z^i
\]
and applied to the spectrum \( x_k = e^{i\theta_k} \) of the unitary matrix \( M \in U(n) \):

**Theorem 0.3** Consider the following variables, expressed in terms of the expectation for the distribution above, or expressed in terms of the bi-orthogonal polynomials evaluated at \( z = 0 \),
\[
x_n(t, s) = E_{U(n)} t s p_n(-\text{Tr } M, -\frac{1}{2} \text{Tr } M^2, -\frac{1}{3} \text{Tr } M^3, ...)
\]
\[
= \frac{p_n(-\partial_s) \tau_n(t, s)}{\tau_n(t, s)} = p_n^{(1)}(t, s; 0)
\]
\[
y_n(t, s) = E_{U(n)} t s p_n(-\text{Tr } \bar{M}, -\frac{1}{2} \text{Tr } \bar{M}^2, -\frac{1}{3} \text{Tr } \bar{M}^3, ...)
\]
\[
= \frac{p_n(\partial_s) \tau_n(t, s)}{\tau_n(t, s)} = p_n^{(2)}(t, s; 0).
\]

The \( x_n \) and \( y_n \)'s satisfy the following integrable Hamiltonian system
\[
\frac{\partial x_n}{\partial t_i} = (1 - x_n y_n) \frac{\partial H_i^{(1)}}{\partial y_n} \quad \frac{\partial y_n}{\partial t_i} = -(1 - x_n y_n) \frac{\partial H_i^{(1)}}{\partial x_n}
\]
\[
\frac{\partial x_n}{\partial s_i} = (1 - x_n y_n) \frac{\partial H_i^{(2)}}{\partial y_n} \quad \frac{\partial y_n}{\partial s_i} = -(1 - x_n y_n) \frac{\partial H_i^{(2)}}{\partial x_n},
\]

(Toeplitz lattice)

with initial condition \( x_n(0, 0) = y_n(0, 0) = 0 \) for \( n \geq 1 \) and boundary condition \( x_0(t, s) = y_0(t, s) = 1 \). The traces
\[
H_i^{(k)} = -\frac{1}{i} \text{Tr } L_{k}^i, \quad i = 1, 2, 3, ..., \quad k = 1, 2.
\]
of the matrices \( L_i \) below are integrals in involution with regard to the symplectic structure
\[
\omega := \sum_{1}^{\infty} \frac{dx_k \wedge dy_k}{1 - x_k y_k},
\]

\(^2\)They should not be confused with the bi-orthogonal polynomials \( p_i^{(k)}(t, s; z) \).
where $L_1$ and $L_2$ are given by the “rank 2” semi-infinite matrices

\[
h^{-1}L_1h := \begin{pmatrix}
-x_1y_0 & 1 - x_1y_1 & 0 & 0 \\
-x_2y_0 & -x_2y_1 & 1 - x_2y_2 & 0 \\
-x_3y_0 & -x_3y_1 & -x_3y_2 & 1 - x_3y_3 \\
-x_4y_0 & -x_4y_1 & -x_4y_2 & -x_4y_3 \\
& & & \ddots
\end{pmatrix}
\]

and

\[
L_2 := \begin{pmatrix}
-x_0y_1 & -x_0y_2 & -x_0y_3 & -x_0y_4 \\
1 - x_1y_1 & -x_1y_2 & -x_1y_3 & -x_1y_4 \\
0 & 1 - x_2y_2 & -x_2y_3 & -x_2y_4 \\
0 & 0 & 1 - x_3y_3 & -x_3y_4 \\
& & & \ddots
\end{pmatrix}
\]

Moreover, the precise “rank 2”-structure of $L_1$ and $L_2$ is preserved by the equations

\[
\frac{\partial L_i}{\partial t_n} = [(L^n_1)_+, L_i] \quad \text{and} \quad \frac{\partial L_i}{\partial s_n} = [(L^n_2)_-, L_i] \quad i = 1, 2 \quad \text{and} \quad n = 1, 2, \ldots
\]

(Two-Toda Lattice)

**Remark:** The first equation in the hierarchy above, corresponding to the Hamiltonians

\[
H_1^{(1)} = -\text{Tr} \ L_1 = \sum_{0}^{\infty} x_{i+1}y_i, \quad H_1^{(2)} = -\text{Tr} \ L_2 = \sum_{0}^{\infty} x_iy_{i+1},
\]

reads:

\[
\frac{\partial x_n}{\partial t_1} = x_{n+1}(1 - x_ny_n) \quad \frac{\partial y_n}{\partial t_1} = -y_{n-1}(1 - x_ny_n) \\
\frac{\partial x_n}{\partial s_1} = x_{n-1}(1 - x_ny_n) \quad \frac{\partial y_n}{\partial s_1} = -y_{n+1}(1 - x_ny_n).
\]
Here, we outline the ideas and the results in the paper. Throughout, consider a weight \( \rho(x)dx \) on an interval \( F \subset \mathbb{R} \), satisfying

\[
-\frac{\rho'(x)}{\rho(x)} = \sum_{i \geq 0} b_i x^i = \frac{g(x)}{f(x)}, \quad \text{with } \rho(x) \text{ decaying rapidly at } \partial F. \tag{0.0.6}
\]

We now define two time-dependent inner-products, one given by a weight \( \rho(x)dx \) on the real line \( \mathbb{R} \) and another given by a contour integration about the unit circle \( S^1 \subset \mathbb{C} \),

\[
\langle f(x), g(x) \rangle_t := \int_{\mathbb{R}} f(x)g(x)e^{\sum_{i=1}^{\infty} t_i x^i} \rho(x)dx, \quad t \in \mathbb{C}^\infty
\]

\[
\langle f(z), g(z) \rangle_{t,s} := \oint_{S^1} \frac{dz}{2\pi i} f(z)g(z^{-1})e^{\sum_{i=1}^{\infty} (t_i z^i - s_i z^{-i})}, \quad t, s \in \mathbb{C}^\infty.
\tag{0.0.7}
\]

These inner-products lead to Hänkel and Toeplitz moment matrices, respectively,

\[
\begin{align*}
\{ m_n(t) := (\langle x^i, x^j \rangle_t)_{0 \leq i,j \leq n-1} \quad & (\text{Hänkel}) \\
\{ m_n(t, s) := (\langle z^i, z^j \rangle_{t,s})_{0 \leq i,j \leq n-1} \quad & (\text{Toeplitz}).
\end{align*}
\]

The determinants \( \tau_n \) of the \( m_n \)'s have different representations: on the one hand, as multiple integrals, involving Vandermonde’s \( \Delta_n(z) \), and, on the other hand, as inductive expressions in term of \( \tau_{n-1} \), involving a vertex operator\(^4\)

\[
\chi(z) := (1, z, z^2, \ldots)
\tag{0.0.8}
\]

to be explained in (0.0.9). The \( \tau_n(t) \) and \( \tau_n(t, s) \) are respectively solutions to the standard Toda lattice, and the so-called Toeplitz lattice, both reductions

\(^3\)Decaying rapidly means: \( \rho(x)f(x) = 0 \) at finite boundary points of \( F \), or \( \rho(x)f(x)x^k \to 0 \), when \( x \to \{ \text{an infinite boundary point} \} \), for all \( k = 0, 1, 2, \ldots \).

\(^4\)For \( v = (v_0, v_1, \ldots)^\top \), \( (Av)_n = v_{n+1} \), \( (A^\top v)_n = v_{n-1} \), and \( \chi(z) := (1, z, z^2, \ldots) \).
of the semi-infinite 2d-Toda lattice\footnote{The expression $X_{12}(\frac{s+t}{2}, \frac{s-t}{2}; u, u)$ is actually independent of $s$! The expressions in (0.0.9) are inductive in $n$, because of the presence of the downwards shift $\Lambda^\top$ in $X_{12}$.}

$$I_n = n! \det \tau_n = n! \det m_n$$

$$= \left\{ \begin{array}{l}
\int_{\mathbb{R}^n} \Delta_n^2 \prod_{k=1}^n e^{\sum_{i=1}^n t_i z_k} \rho(z_k) dz_k = \int_{\mathbb{R}} d\rho(u) \left( X_{12} \left( \frac{s+t}{2}, \frac{s-t}{2}; u, u \right) I \right)_n \\
\oint_{(S^1)^n} |\Delta_n|^2 \prod_{k=1}^n e^{\sum_{i=1}^n (t_i z_k - s_i z_k)} \frac{dz_k}{2\pi i z_k} = \int_{S^1} \frac{du}{2\pi i u} \left( X_{12}(t, s; u, u^{-1}) I \right)_n, \\
\end{array} \right.$$  

(standard Toda $\tau$-functions)

(two-Toda $\tau$-functions)  

(0.0.9)

where $\tau_n(t)$ and $\tau_n(t, s)$ satisfy the following differential equations, (the second one is new)

$$\left\{ \begin{array}{l}
\frac{\partial^4}{\partial t_1^4} \log \tau_n + 6 \left( \frac{\partial^2}{\partial t_1^2} \log \tau_n \right)^2 + 3 \frac{\partial^2}{\partial t_1^2} \log \tau_n - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \log \tau_n = 0, \\
\frac{\partial^2}{\partial s_2 \partial t_1} \log \tau_n = -2 \frac{\partial}{\partial s_1} \log \frac{\tau_n}{\tau_{n-1}} - \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n - \frac{\partial^3}{\partial s_1^2 \partial t_1} \log \tau_n. \\
\end{array} \right.$$  

(KP - equation)

(two-Toda-equation)

The unique factorization of the time-dependent semi-infinite moment matrices $m_\infty$ (defined just under (0.0.7)) into lower- times upper-triangular matrices

$$\left\{ \begin{array}{l}
m_\infty(t) = S(t)^{-1} S^\top(t) \\
m_\infty(t, s) = S_1(t, s)^{-1} S_2(t, s) \\
\end{array} \right.$$  

(0.0.10)

leads to matrices $S(t), S_1(t, s), S_2(t, s)$ of the form

$$S(t) = \sum_{i \leq 0} a_i \Lambda^i, \quad S_1(t, s) = \sum_{i \leq 0} b_i \Lambda^i, \quad S_2(t, s) = \sum_{i \geq 0} b_i' \Lambda^i$$

with $b_0 = I, a_i, b_i, b'_i$ diagonal matrices. By “dressing up” the shift $\Lambda$ (defined in footnote 4), they evolve according to the following integrable systems:
\[
L(t) = S(t) \Lambda S^{-1}(t) : \text{symmetric and tridiagonal,} \quad \text{(Toda lattice)}
\]
\[
\begin{cases}
L_1(t, s) = S_1(t, s) \Lambda S_1^{-1}(t, s) \\
L_2(t, s) = S_2(t, s) \Lambda^T S_2^{-1}(t, s).
\end{cases} \quad \text{(Toeplitz lattice)}
\]

As already pointed out in (0.0.9), that expression involves the following reduction of the two-Toda vertex operator \(X_{12}\):

\[
\begin{cases}
X_{12}\left(\frac{s + t}{2}, \frac{s - t}{2}; u, u\right) =: X(t; u) = \Lambda^\top \chi(u^2) e^{\sum_{i=1}^{\infty} t_i u^i} e^{-2 \sum_{i=1}^{\infty} \frac{u^i}{i - \lambda_i}} \\
X_{12}(t, s; u, u^{-1}) = \Lambda^\top e^{\sum_{i=1}^{\infty} (t_i u^i - s_i u^{-i})} e^{-\sum_{i=1}^{\infty} \left(\frac{u^i}{i - \lambda_i} - \frac{u^{-i}}{i - \lambda_i}\right)}
\end{cases}
\]

Each of these vertex operators leads to Virasoro algebras \(J^{(2)}_m\) and \(V^{(2)}_m\) of central charge \(c = 1\) and \(c = 0\) respectively, defined by

\[
\begin{align*}
\frac{\partial}{\partial u} u^{m+1} f(u) X(t, u) \rho(u) &= \left[ J^{(2)}_m(t), X(t, u) \rho(u) \right] \\
\frac{\partial}{\partial u} u^{m+1} \frac{X_{12}(t, s; u, u^{-1})}{u} &= \left[ V^{(2)}_m(t, s), \frac{X_{12}(t, s; u, u^{-1})}{u} \right],
\end{align*}
\]

and having the explicit expressions (see notation (0.0.6))

\[
J^{(2)}_m(t) := \sum_{i \geq 0} \left( a_i \beta \gamma^{(2)}_{i+m}(t) - b_i \beta \gamma^{(1)}_{i+m+1}(t) \right) \Big|_{\beta = 2}
\]

\[
V^{(2)}_m(t, s) := \beta \gamma^{(2)}_m(t) - \beta \gamma^{(2)}_{-m}(-s) - m \left( \theta \beta \gamma^{(1)}_m(t) + (1 - \theta) \beta \gamma^{(1)}_{-m}(-s) \right) \Big|_{\beta = 1}
\]

in terms of generators \(\beta \gamma^{(2)}_m\) defined in (5.0.3) below, and arbitrary \(\theta\).

The point is that a big subalgebra of \(J^{(2)}_m\)'s and a small one of \(V^{(2)}_m\)'s annihilate \(\tau_n(t)\) and \(\tau_n(t, s)\) respectively, for appropriate \(\theta\) and for all \(n \geq 0\),

\[
\begin{cases}
J^{(2)}_m \tau_n(t) = 0 \quad \text{for} \quad m \geq -1, \\
V^{(2)}_m \tau_n(t, s) = 0 \quad \text{for} \quad m = -1, 0, 1 \quad (SL(2, \mathbb{Z})\text{- algebra}).
\end{cases}
\]
To summarize, we have that combining these equations and restricting to the three different loci $\mathcal{L}$ below, always leads to Painlevé $V$: 

\[
\begin{cases}
\text{KP} (\tau_n) = 0 \\
J^{(2)}_m \tau_n(t) = 0,
\end{cases}
\]

for $m = -1, 0$ 

\[
\mathcal{L} = \begin{cases}
t_1 = x, \text{ all other } t_i = 0
\end{cases}
\]

\[\Rightarrow\]

\[
\text{Painlevé } V \text{ for } O_{\pm}(n)-\text{integral}
\]

\[
\begin{cases}
2 - \text{Toda PDE} \\
V^{(2)}_m \tau_n(t, s) = 0,
\end{cases}
\]

for $m = -1, 0, 1$ 

\[
\mathcal{L} = \begin{cases}
t_1, s_1 \neq 0, \text{ all other } t_i, s_i = 0
\end{cases}
\]

\[\Rightarrow\]

\[
\text{Painlevé } V \text{ for } U(n)-\text{integral}
\]

\[
\begin{cases}
\text{Toeplitz relation} \\
\text{for all } t_i = -k(-1)^i \\
s_i = 0, \text{ except } s_1 = x
\end{cases}
\]

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1 Integrals over classical groups and combinatorics

This section contains a number of useful facts about integrals over groups, its relation with combinatorics and finally the behavior of some of the integrals near $x = 0$.

The situation is quite different, according to whether one integrates over the real ($O_{\pm}, Sp(\ell)$) or the complex ($U(\ell)$). The real group integrals involve the Jacobi weight, 

\[
\rho_{\alpha \beta}(z)dz := (1 - z)^\alpha (1 + z)^\beta dz,
\]

for $\alpha, \beta = \pm 1/2$ and the Tchebychev polynomials $T_n(z)$, defined by $T_n(cos \theta) := cos n \theta$. In particular, we have $T_1(z) = z$. We now have the following theorem (see Johansson [14]):

\[12\]
Proposition 1.1 (Weyl) Defining
\[ g(z) := 2 \sum_{1}^{\infty} t_i T_i(z), \]
the following holds:
\[
\int_{U(n)} e^{\sum_{1}^{\infty} \text{tr}(t_i M_i^1 - s_i M_i^1)} dM = \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} e^{\sum_{i}^{\infty} (t_i z_k^i - s_i z_k^{-i})} \frac{dz_k}{2\pi i z_k} \\
\int_{O(2n+1)_+} e^{\sum_{1}^{\infty} t_i \text{Tr} M_i^1} dM = e^{\sum_{1}^{\infty} t_i} \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^{n} e^{g(z_k)} \rho\left(\frac{1}{2}, -\frac{1}{2}\right)(z_k) dz_k \\
\int_{O(2n+1)_-} e^{\sum_{1}^{\infty} t_i \text{Tr} M_i^1} dM = e^{\sum_{1}^{\infty} (-1)^{t_i}} \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^{n} e^{g(z_k)} \rho\left(-\frac{1}{2}, \frac{1}{2}\right)(z_k) dz_k \\
\int_{O(2n)_+} e^{\sum_{1}^{\infty} t_i \text{Tr} M_i^1} dM = \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^{n} e^{g(z_k)} \rho\left(-\frac{1}{2}, -\frac{1}{2}\right)(z_k) dz_k \\
\int_{O(2n)_-} e^{\sum_{1}^{\infty} t_i \text{Tr} M_i^1} dM = \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^{n-1} e^{g(z_k)} \rho\left(\frac{1}{2}, \frac{1}{2}\right)(z_k) dz_k \\
\int_{S_p(n)} e^{\sum_{1}^{\infty} t_i \text{Tr} M_i^1} dM = \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^{n} e^{g(z_k)} \rho\left(\frac{1}{2}, \frac{1}{2}\right)(z_k) dz_k.
\]

With this normalization, we have (see Appendix 3)
\[
\int_{U(n)} dM = 1 \\
\int_{O(2n+1)_\pm} dM = 2^n \prod_{j=1}^{n} \frac{j!\Gamma^2(j-1/2)}{(n+j-1)!} \\
\int_{O(2n)_+} dM = 2^{n-1} \prod_{j=1}^{n} \frac{j!\Gamma^2(j-1/2)}{(n+j-2)!} \\
\int_{O(2n)_-} dM = 2^{n-1} \prod_{j=1}^{n-1} \frac{j!\Gamma^2(j+1/2)}{(n+j-1)!}.
\]

Letting \( t \in S_n \) denote the permutation \( k \to n+1-k \), we also state:
Proposition 1.2 The combinatorial quantities below have an expression in terms of integrals over groups:

\[
\begin{align*}
\sum_{n \geq 0} \frac{x^{2n}}{(n!)^2} \# \{ \pi \in S_n, \, \sigma_n(\pi) \leq \ell \} &= E_U(\ell) e^{x \text{Tr}(M+\bar{M})} \\
\sum_{n \geq 0} \frac{x^{2n}}{(2n)!} \# \left\{ \begin{array}{l}
\pi \in S_{2n}, \, \pi^2 = 1, \\
\pi(y) \neq \iota y, \, \sigma_{2n}(\pi) \leq 2\ell
\end{array} \right\} &= E_U(\ell) e^{x \text{Tr}(M+\bar{M})} \\
2 \sum_{n \geq 0} \frac{x^{2n}}{(2n)!} \# \left\{ \begin{array}{l}
\pi \in S_{2n}, \, \pi^2 = 1, \\
\pi(y) \neq \iota y, \, \sigma_{2n}(\pi) \leq \ell
\end{array} \right\} &= E_{O_+}(\ell) e^{x \text{Tr}M} + E_{O_+}(\ell) e^{x \text{Tr}M} \\
\sum_{n \geq 0} \frac{x^n}{n!} \# \{ \pi \in S_n, \, \pi^2 = 1, \, \sigma_n(\pi) \leq \ell \} &= e^x E_{O_+}(\ell+1) e^{x \text{Tr}M} \\
\sum_{n \geq 0} \frac{x^n}{n!} \# \{ \pi \in S_n, \, (\iota \pi)^2 = 1, \, \sigma_n(\pi) \leq \ell \} &= e^x E_{O_+}(\ell+1) e^{x \text{Tr}M} \\
\sum_{n \geq 0} \frac{x^{2n}}{(2n)!} \# \left\{ \begin{array}{l}
\pi \in S_{2n}, \, (\iota \pi)^2 = 1, \\
\pi(y) \neq \iota y, \, \sigma_{2n}(\pi) \leq 2\ell
\end{array} \right\} &= E_{O_+}(2\ell+2) e^{x \text{Tr}M} \\
\sum_{n \geq 0} \frac{x^{2n}}{(n!)^2} \# \{ \pi \in S_{2n}, \, \pi \iota = \iota \pi, \, \sigma_{2n}(\pi) \leq 2\ell \} &= E_U(\ell) e^{x \text{Tr}(M+\bar{M})} \\
\sum_{n \geq 0} \frac{x^{2n}}{(n!)^2} \# \left\{ \begin{array}{l}
\pi \in S_{2n}, \, \pi \iota = \iota \pi, \\
\sigma_{2n}(\pi) \leq 2\ell + 1
\end{array} \right\} &= E_U(\ell) e^{x \text{Tr}(M+\bar{M})} \\
\sum_{n \geq 0} \frac{x^n}{n!} \# \{ \pi \in S_{n,k}, \, \sigma_n(\pi) \leq \ell \} &= E_U(\ell) \det(I+M)^k e^{x \text{Tr}M}.
\end{align*}
\]

The first identity goes back to Gessel [13] and in this precise form to Rains [17]. The third, seventh and eight equalities between the combinatorics and the integral expression above are due to Rains [17], the next ones are due to Baik and Rains [7] and the last one is due to Tracy and Widom [20].

We now state the following elementary lemma:
Lemma 1.3

\[ e^{x^2/2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \# \{ \pi \in S_{2n} \mid \pi^2 = 1, \text{ fixed-point free} \} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \frac{(2n)!}{2^n n!} \]

\[ e^{x^2/2+x} = \sum_{n=0}^{\infty} \frac{x^n}{(n)!} \# \{ \pi \in S_n \mid \pi^2 = 1 \} = \sum_{n=0}^{\infty} \frac{x^n}{(n)!} \sum_{0 \leq m \leq \lfloor n/2 \rfloor} \left( \frac{n}{2m} \right) \frac{(2m)!}{2^m m!} \]

We now estimate the following integrals over \( O(\ell+1) \) and \( U(\ell) \) near \( x = 0 \). In all cases, one notices a big gap in the expansion, -roughly speaking- of the order \( \ell \). This would be hard to obtain at the level of the integrals, but easy to obtain via combinatorics.

Proposition 1.4 The following estimates hold, near \( x = 0 \),

\[ E_{O_{\pm}(\ell+1)} e^{x \text{Tr} M} = \exp \left( \frac{x^2}{2} \pm \frac{x^{\ell+1}}{\ell+1)!} + O(x^{\ell+2}) \right), \]

\[ E_{O_{+}(\ell+1)} e^{x \text{Tr} M} + E_{O_{-}(\ell+1)} e^{x \text{Tr} M} = 2 \exp \left( \frac{x^2}{2} + O(x^{\ell+2}) \right). \quad (1.0.3) \]

Proof: From the second relation in Lemma 1.3, it follows that

\[ \# \{ \pi \in S_n \mid \pi^2 = 1, \sigma_n(\pi) \leq \ell \} \]

\[ = \# \{ \pi \in S_n \mid \pi^2 = 1 \} = \sum_{0 \leq m \leq \lfloor n/2 \rfloor} \left( \frac{n}{2m} \right) \frac{(2m)!}{2^m m!}, \text{ for } n \leq \ell \]

\[ = \# \{ \pi \in S_{\ell+1} \mid \pi^2 = 1 \} - 1, \text{ for } n = \ell + 1. \]

Hence we have from the fourth identity of proposition 1.2, and Lemma 1.3,

\[ e^x E_{O_{-}(\ell+1)} e^{x \text{Tr} M} = \sum_{n \geq 0} \frac{x^n}{n!} \# \{ \pi \in S_n, \pi^2 = 1, \sigma_n(\pi) \leq \ell \} \]

\[ = \exp \left( \frac{x^2}{2} + x - \frac{x^{\ell+1}}{\ell+1)!} + O(x^{\ell+2}) \right), \]
and so

\[ E_{O_{-}(\ell+1)} e^{x \text{Tr} M} dM = \exp \left( \frac{x^2}{2} - \frac{x^{\ell+1}}{\ell + 1)!} + O(x^{\ell+2}) \right). \]  

(1.0.4)

But, for \(2n \leq \ell + 1\),

\[ \# \{ \pi \in S_{2n} \mid \pi^2 = 1, \sigma_{2n}(\pi) \leq \ell + 1, \text{fixed-point free} \} \]

\[ = \{ \pi \in S_{2n}, \pi^2 = 1, \text{fixed-point free} \} \]

\[ = \frac{(2n)!}{2^n n!}, \]

and so, from the third identity of Proposition 1.2,

\[ E_{O_{-}(\ell+1)} e^{x \text{Tr} M} + E_{O_{+}(\ell+1)} e^{x \text{Tr} M} \]

\[ = 2 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \# \{ \pi \in S_{2n} \mid \pi^2 = 1, \sigma_{2n}(\pi) \leq \ell + 1, \text{fixed-point free} \} \]

\[ = 2 \exp \left( \frac{x^2}{2} + O(x^{\ell+2}) \right), \]

establishing the second relation (1.0.3). Combining this formula with (1.0.4), leads to the following estimate, near \(x = 0\)

\[ E_{O_{+}(\ell+1)} e^{x \text{Tr} M} = \exp \left( \frac{x^2}{2} + \frac{x^{\ell+1}}{(\ell + 1)!} + O(x^{\ell+2}) \right), \]

establishing the first relation (1.0.3).

\[ \blacksquare \]

**Proposition 1.5** The following estimates hold, near \(x = 0\),

\[ E_{U(\ell)} e^{\sqrt{x} \text{Tr} (M + \bar{M})} dM = \exp \left( x - \frac{x^{\ell+1}}{(((\ell + 1)!)^2} + O(x^{\ell+2}) \right) \]

\[ E_{U(\ell)} \det (I + M)^k e^{-x \text{Tr} M} dM = \exp \left( kx - \frac{x^{\ell+1}}{(\ell + 1)!} \binom{k + \ell}{\ell + 1} + O(x^{\ell+2}) \right) . \]
Proof: Using the first identity of proposition 1.2, we have
\[ I_\ell := E U(\ell) e^{\sqrt{x} \operatorname{Tr}(M+\bar{M})} dM \]
\[ = \sum_0^\infty \frac{x^n}{(n!)^2} \#\{\pi \in S_n \mid \sigma_n(\pi) \leq \ell\} \]
\[ = \sum_0^\ell \frac{x^n}{n!} + \frac{x^{\ell+1}}{(\ell + 1)!} \left( (\ell + 1)! - 1 \right) + O(x^{\ell+2}) \]
\[ = \exp \left( x - \frac{x^{\ell+1}}{(\ell + 1)!} + O(x^{\ell+2}) \right). \]

Since the number of (increasing) sequences \((1,1,...,2,2,2,...,k,k,k)\) of length \(\ell+1\) and consisting of \(k\) symbols, is given by \(\binom{k+\ell}{\ell+1}\), one computes for \(S_{\ell,k} = \{\text{words of length } \ell \text{ from an alphabet of } k \text{ letters}\}\):
\[ \#\{\pi \in S_{n,k} \mid \sigma_n(\pi) \leq \ell\} = k^n, \text{ for } n \leq \ell, \]
\[ \#\{\pi \in S_{\ell+1,k} \mid \sigma_{\ell+1}(\pi) \leq \ell\} = k^{\ell+1} - \binom{k + \ell}{\ell + 1}, \text{ for } n = \ell + 1. \]

Therefore, using the last identity of Proposition 1.2, one finds
\[ I_\ell := E U(\ell) \det(I + M)^k e^{-x \operatorname{Tr} \bar{M}} dM \]
\[ = \sum_0^\infty \frac{x^n}{n!} \#\{\pi \in S_{n,k} \mid \sigma_n(\pi) \leq \ell\} \]
\[ = \sum_0^\ell \frac{x^n}{n!} k^n + \frac{x^{\ell+1}}{(\ell + 1)!} \left( k^{\ell+1} - \binom{k + \ell}{\ell + 1} \right) + O(x^{\ell+2}) \]
\[ = \exp \left( kx - \frac{x^{\ell+1}}{(\ell + 1)!} \left( k^{\ell+1} - \binom{k + \ell}{\ell + 1} \right) + O(x^{\ell+2}) \right). \]

\[ \blacksquare \]

2 Two-Toda lattice and reductions (Hänkel and Toeplitz)
2.1 Two-Toda on Moment Matrices and Identities for \(\tau\)-Functions

Two-Toda \(\tau\)-functions \(\tau_n(t, s), \ n \in \mathbb{Z}\) depend on two sets of time-variables \(t, s \in \mathbb{C}^\infty\) and are defined by the following bilinear identities, for all \(m, n \in \mathbb{Z}\):

\[
\oint_{z=\infty} \tau_n(t - [z^{-1}], s) \tau_{m+1}(t', s') e^{\sum_{i=1}^\infty (t_i - t'_i) z^i} z^{n-m-1} dz = \oint_{z=0} \tau_{n+1}(t, s - [z]) \tau_m(t', s' + [z]) e^{\sum_{i=1}^\infty (s_i - s'_i) z^{-i}} z^{n-m-1} dz,
\]

or, specified in terms of the Hirota symbol\(\footnote{for the customary Hirota symbol \(p(\partial_t)f \circ g := p(\partial_y)f(t + y)g(t - y)\big|_{y=0}\).}\), by

\[
\sum_{j=0}^\infty p_{m-n+j}(-2a)p_j(\partial_t)e^{\sum_{k=1}^\infty (a_k \frac{\partial}{\partial t_k} + b_k \frac{\partial}{\partial s_k})} \tau_{m+1} \circ \tau_n = \sum_{j=0}^\infty p_{-m-n+j}(-2b)p_j(\partial_s)e^{\sum_{k=1}^\infty (a_k \frac{\partial}{\partial t_k} + b_k \frac{\partial}{\partial s_k})} \tau_m \circ \tau_{n+1}.
\] (2.1.2)

For the semi-infinite case, the same definitions hold, but for \(n, m \geq 0\).

**Theorem 2.1** Two-Toda \(\tau\)-functions satisfy

(i) the KP-hierarchy in \(t\) and \(s\) separately, of which the first equation reads:

\[
\left(\frac{\partial}{\partial t_1}\right)^4 \log \tau + 6 \left(\left(\frac{\partial}{\partial t_1}\right)^2 \log \tau\right)^2 + 3 \left(\frac{\partial}{\partial t_2}\right)^2 \log \tau - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \log \tau = 0;
\]

(ii) an identity, involving \(t, s\) and nearest neighbors \(\tau_{n-1}, \tau_n\):

\[
\frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n = -\frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2};
\]

(iii) a (new) **Identity** involving \(t, s\) and nearest neighbors \(\tau_{n-1}, \tau_n\):

\[
\frac{\partial^2}{\partial s_2 \partial t_1} \log \tau_n = -2 \frac{\partial}{\partial s_1} \log \frac{\tau_n}{\tau_{n-1}} - \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n - 3 \frac{\partial^3}{\partial s_1^2 \partial t_1} \log \tau_n.\] (2.1.3)
The proof of this theorem will be given later in this section.

The “wave vectors” defined in terms of \( \tau_n(t, s) \),

\[
\Psi_1(t, s, z) = \left( \frac{\tau_n(t - [z^{-1}], s)}{\tau_n(t, s)} e^{\sum_{i=1}^{\infty} t_i z_i z^{-n}} \right)_{n \in \mathbb{Z}} = e^{\sum_{i=1}^{\infty} t_i z_i} S_1 \chi(z)
\]

\[
\Psi_2^*(t, s, z) = \left( \frac{\tau_n(t, s + [z])}{\tau_n(t, s)} e^{-\sum_{i=1}^{\infty} s_i z_i^{-i} z^{-n}} \right)_{n \in \mathbb{Z}} = e^{-\sum_{i=1}^{\infty} s_i z_i^{-i}} (S_2^{-1})^\top \chi(z^{-1}),
\]

specify lower- and upper-triangular wave matrices \( S_1 \) and \( S_2 \) respectively. They, in turn, define a pair of matrices \( L_1 \) and \( L_2 \)

\[
L_1 := S_1 \Lambda S_1^{-1} = \sum_{-\infty < i \leq 0} a_i^{(1)} \Lambda_i + \Lambda, \quad L_2 := S_2 \Lambda^\top S_2^{-1} = \sum_{-1 \leq i < \infty} a_i^{(2)} \Lambda_i,
\]

where \( \Lambda = (\delta_{j-i,1})_{i,j \in \mathbb{Z}} \), and \( a_i^{(1)} \) and \( a_i^{(2)} \) are diagonal matrices depending on \( t = (t_1, t_2, \ldots) \) and \( s = (s_1, s_2, \ldots) \). Then

\[
z \Psi_1 = L_1 \Psi_1 \quad \text{and} \quad z^{-1} \Psi_2^* = L_2^\top \Psi_2^*,
\]

and the matrices \( L_i \) satisfy the 2-Toda lattice equations:

\[
\frac{\partial L_i}{\partial t_n} = [(L_1^n)_+, L_i] \quad \text{and} \quad \frac{\partial L_i}{\partial s_n} = [(L_2^n)_-, L_i] \quad i = 1, 2 \quad \text{and} \quad n = 1, 2, \ldots
\]

with \( \Psi_1 \) and \( \Psi_2^* \) satisfying the differential equations:

\[
\frac{\partial \Psi_1}{\partial t_n} = (L_1^n)_+ \Psi_1 \quad \frac{\partial \Psi_1}{\partial s_n} = (L_2^n)_- \Psi_1 \\
\frac{\partial \Psi_2^*}{\partial t_n} = -((L_1^n)_+)^\top \Psi_2^* \quad \frac{\partial \Psi_2^*}{\partial s_n} = -((L_2^n)_-)^\top \Psi_2^*.
\]

\[\chi(z) = \ldots, z^{-1}, 1, z^1, \ldots \] in the bi-infinite case
\[\chi(z) = (1, z, z^2, \ldots) \] in the semi-infinite case.

Also \( \Lambda^{-1} \) should always be interpreted as \( \Lambda^\top \) in the semi-infinite case.
For future use, define the diagonal matrix:

\[ h := (\ldots, h_{-1}, h_0, h_1, \ldots), \text{ where } h_k(t, s) := \frac{\tau_{k+1}(t, s)}{\tau_k(t, s)}. \]  

(2.1.8)

In [4], we have shown that \( L^k_1 \) has the following expression in terms of \( \tau \)-functions:

\[
L^k_1 = \sum_{\ell=0}^{\infty} \text{diag} \left( p_\ell(\tilde{\partial}_t) \frac{\tau_{n+k-\ell+1} \circ \tau_n}{\tau_{n+k-\ell+1} \tau_n} \right) \Lambda^{k-\ell} 
\]

\[
hL^\top_2 h^{-1} = \sum_{\ell=0}^{\infty} \text{diag} \left( p_\ell(-\tilde{\partial}_s) \frac{\tau_{n+k-\ell+1} \circ \tau_n}{\tau_{n+k-\ell+1} \tau_n} \right) \Lambda^{k-\ell}.
\]

(2.1.9)

There is a general involution in the equation, which we shall frequently use, namely \( t \leftrightarrow -s, L_1 \leftrightarrow hL^\top_2 h^{-1} \).

Finally, we define the 2-Toda vertex operator, which is the generating function for the algebra of symmetries, acting on \( \tau \)-functions (it will play a role later!):

\[
X_{12}(t, s; u, v) = \Lambda^{-1} e^{\sum_{i=1}^{\infty} (u_i w_i - s_i v_i)} e^{-\sum_{i=1}^{\infty} \left( -\frac{\partial}{\partial t_i} - \frac{\partial}{\partial s_i} \right) \chi(uv)},
\]

(2.1.10)

leading to a Virasoro algebra (see Appendix 1) with \( \beta = 1 \) and thus with central charge \( c = -2 \),

\[
\frac{\partial}{\partial u} u^{k+1} X_{12}(t, s; u, v) = \left[ \beta \mathcal{J}_k^{(2)}(t) \right]_{\beta=1} X_{12}(t, s; u, v),
\]

\[
u \mathcal{J}_k X_{12}(t, s; u, v) = \left[ \beta \mathcal{J}_k^{(1)}(t) \right]_{\beta=1} X_{12}(t, s; u, v),
\]

(2.1.11)

with generators (in \( t \)), explicitly given by (5.0.3). Similarly, the involution \( u \leftrightarrow v, t \leftrightarrow -s \) leads to the same Virasoro algebra in \( s \), with same central charge.

-----

8 \( p_\ell(f \circ g) \) refers to the Hirota operation, defined before. Here the \( p_\ell \) are the elementary Schur polynomials \( e^{\sum_{i=0}^{\infty} t_i z^i} := \sum_{i \geq 0} p_i(t) z^i \). Also \( p_\ell(\tilde{\partial}_t) := p_\ell(\frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \frac{1}{3} \frac{\partial}{\partial t_3}, \ldots) \) and \( p_\ell(-\tilde{\partial}_s) := p_\ell(-\frac{\partial}{\partial s_1}, -\frac{1}{2} \frac{\partial}{\partial s_2}, -\frac{1}{3} \frac{\partial}{\partial s_3}, \ldots) \).
Proposition 2.2

\[
(L^1_k)_{n,n} = \frac{p_k(\tilde{\partial}_t)\tau_{n+1} \circ \tau_n}{\tau_{n+1}\tau_n} = \frac{\partial}{\partial t_k} \log \frac{\tau_{n+1}}{\tau_n}
\]

\[
(hL^1_kh^{-1})_{n,n} = \frac{p_k(-\tilde{\partial}_s)\tau_{n+1} \circ \tau_n}{\tau_{n+1}\tau_n} = -\frac{\partial}{\partial s_k} \log \frac{\tau_{n+1}}{\tau_n} \quad (2.1.12)
\]

and

\[
(L^1_k)_{n,n+1} = \frac{p_{k-1}(\tilde{\partial}_t)\tau_{n+2} \circ \tau_n}{\tau_{n+2}\tau_n} = \frac{\partial^2}{\partial s_1\partial t_k} \log \frac{\tau_{n+1}}{\tau_n}
\]

\[
(hL^1_kh^{-1})_{n,n+1} = \frac{p_{k-1}(-\tilde{\partial}_s)\tau_{n+2} \circ \tau_n}{\tau_{n+2}\tau_n} = -\frac{\partial^2}{\partial s_1\partial t_{j+1}} \log \frac{\tau_{n+1}}{\tau_n} \quad (2.1.13)
\]

Proof: Relations (2.1.12) follows from (2.1.7) and a standard argument; see [4], Theorem 0.1, formula (0.15).

To prove (2.1.13), set \(m = n + 1\), all \(b_k\) and \(a_k = 0\), except for one \(a_{j+1}\), in the Hirota bilinear relation (2.1.2). The first nonzero term in the sum on the left hand side of that relation, which is also the only one containing \(a_{j+1}\) linearly, reads

\[
p_{j+1}(-2a)p_{j+1}(\tilde{\partial}_t)e^{a_{j+1}\partial_{\tau_{n+1}}} \tau_{n+2} \circ \tau_n + \ldots = -2a_{j+1}p_{j+1}(\tilde{\partial}_t)\tau_{n+2} \circ \tau_n + O(a^2_{j+1}),
\]

whereas the right hand side equals

\[
p_0(0)p_1(\tilde{\partial}_s)e^{a_{j+1}\partial_{\tau_{n+1}}} \tau_{n+1} \circ \tau_{n+1} = \frac{\partial}{\partial s_1} (1 + a_{j+1}\frac{\partial}{\partial t_{j+1}} + \ldots) \tau_{n+1} \circ \tau_{n+1}.
\]

Comparing the coefficients of \(a_{j+1}\) in (2.1.14) and (2.1.15) yields

\[-2 p_{j+1}(\tilde{\partial}_t)n \circ \tau_n = \frac{\partial^2}{\partial s_1\partial t_{j+1}} \tau_{n+1} \circ \tau_{n+1};\]

in particular, we have

\[
p_{k-1}(\tilde{\partial}_t)\tau_{n+1} \circ \tau_n = -\frac{\partial^2}{\partial s_1\partial t_k} \log \tau_{n+1},
\]

(2.1.16)
and so, for $k = 1,$

$$\frac{\tau_n \tau_{n+2}}{\tau_{n+1}^2} = - \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_{n+1}. \quad (2.1.17)$$

Dividing (2.1.16) and (2.1.17) leads to the first equality in (2.1.13), since according to (2.1.9), the $(n, n+1)$-entry of $L_1^k$ is precisely given by (2.1.16).

The similar result for $L_2^k$ is given by the involution

$$t \longleftrightarrow -s \text{ and } L_1 \longleftrightarrow hL_2^\top h^{-1}. \quad \blacksquare$$

**Lemma 2.3** The first upper-subdiagonal of $L_1^2$ and $hL_2^\top h^{-1}$ reads:

$$
\begin{align*}
(L_1^2)_{n,n+1} & = \frac{\partial}{\partial t_1} \log \frac{\tau_{n+2}}{\tau_n} \\
& = \frac{\partial^2}{\partial s_1 \partial t_2} \log \tau_{n+1} \\
& = \frac{\partial}{\partial s_1} \log \left( \frac{\tau_{n+1}}{\tau_n} \right)^2 \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_{n+1} \\
& = \frac{\partial}{\partial t_1} \log \left( \frac{\tau_{n+1}}{\tau_n} \right)^2 \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_{n+1} \\
\end{align*}
$$

$$
\begin{align*}
(hL_2^\top h^{-1})_{n,n+1} & = -\frac{\partial}{\partial s_1} \log \frac{\tau_{n+2}}{\tau_n} \\
& = \frac{\partial^2}{\partial t_1 \partial s_2} \log \tau_{n+1} \\
& = \frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{n+1} \\
& = -\frac{\partial}{\partial s_1} \log \left( \frac{\tau_{n+1}}{\tau_n} \right)^2 \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_{n+1} \right) \quad . \quad (2.1.18)
\end{align*}
$$
Proof: From Proposition 2.2 ($k = 2$), we have the first two identities in (2.1.18); it also follows from these identities that

$$\frac{\partial^2 \log \tau_{n+1}}{\partial s_1 \partial t_2} = \frac{\partial^2 \log \tau_{n+1}}{\partial s_1 \partial t_1} \frac{\partial}{\partial t_1} \frac{\partial^2 \ln \tau_{n+1}}{\partial s_1 \partial t_1} \log \tau_{n+1}$$

which establishes the first equation (2.1.18). The second equation (2.1.18) is simply the dual of the first one by $t_i \mapsto -s_i$.

Proof of Theorem 2.1: The first statement concerning the KP hierarchy is standard. The proof of the second identity follows immediately from (2.1.16), for $k = 1$, and the third identity from the last identity in the proof of Lemma 2.3 and the duality.

A prominent example of the semi-infinite 2-Toda lattice is given by an (arbitrary) $(t, s)$-dependent semi-infinite matrix

$$m_\infty(t, s) = (\mu_{ij}(t, s))_{0 \leq i, j < \infty}, \text{ with } m_n(t, s) = (\mu_{ij}(t, s))_{0 \leq i, j \leq n-1}, \quad (2.1.19)$$

evolving according to the equations

$$\frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty \quad \text{and} \quad \frac{\partial m_\infty}{\partial s_k} = -m_\infty(\Lambda^\top)^k. \quad (2.1.20)$$

According to [2], the formal solution to this problem is given by

$$m_\infty(t, s) = e^{\sum_{i=1}^{\infty} \tau_i \lambda_i} m_\infty(0, 0) e^{-\sum_{i=1}^{\infty} s_i \lambda_i} = S_1^{-1}(t, s) S_2(t, s), \quad (2.1.21)$$

where the associated unique factorization into lower-times upper-triangular matrices actually lead to the wave matrices $S_1$ and $S_2$, as defined in the
general Toda theory \((2.1.4)\). The expression \((2.1.21)\) contains the matrix of Schur polynomials\(^9\)

\[
e^{\sum_{i=1}^{\infty} t_i \Lambda_i} = \sum_{0}^{\infty} \Lambda^i p_i(t) = \begin{pmatrix} p_j(t) \end{pmatrix}_{1 \leq i, j < \infty}
\]

of which a truncated version is given by the following \(n \times \infty\) submatrix \(p_i(t)\):

\[
E_n(t) = \begin{pmatrix}
1 & p_1(t) & p_2(t) & \ldots & p_{n-1}(t) \\
0 & 1 & p_1(t) & \ldots & p_{n-2}(t) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & p_1(t) \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix} p_{j-i}(t) \end{pmatrix}_{1 \leq i \leq n, 1 \leq j < \infty}
\] \((2.1.22)\)

So, for a semi-infinite initial condition \(m_{\infty}(0,0)\), the \(\tau\)-functions of the 2-Toda problem are given by

\[
\tau_n(t,s) := \det m_n(t,s) = \det \left( E_n(t) m_{\infty}(0,0) E_n^T(-s) \right).
\] \((2.1.23)\)

Incidentally, the wave vectors \(\Psi_1\) and \(\Psi_2\) define monic polynomials \(p^{(1)}(x)\) and \(p^{(2)}(y)\),

\[
\Psi_1 := e^{\sum_{k=1}^{\infty} t_k z^k} p^{(1)}(z) \quad \text{and} \quad \Psi_2^* := e^{-\sum_{k=1}^{\infty} s_k z^{-k} h^{-1}} p^{(2)}(z^{-1}) = e^{-\sum_{k=1}^{\infty} s_k z^{-k} \left(S^{-1}_2\right)^\dagger \chi(z^{-1})},
\] \((2.1.24)\)

which are bi-orthogonal with regard to the original matrix \(m_{\infty}\); that is, for all \(t, s\):

\[
\langle p^{(1)}_m, p^{(2)}_l \rangle = \delta_{n,m} h_n \quad \text{for the inner-product defined by} \quad \langle x^i, y^j \rangle := \mu_{ij},
\] \((2.1.25)\)

with \(h_n\) as in \((2.1.8)\).

\(^9\)The Schur polynomials \(p_i\), defined by \(e^{\sum_{i=1}^{\infty} t_i z^i} = \sum_{k=0}^{\infty} p_k(t) z^k\) and \(p_k(t) = 0\) for \(k < 0\), are not to be confused with the bi-orthogonal polynomials \(p^{(k)}_i, k = 1,2\).
Proposition Given the semi-infinite initial condition \( m_\infty(0,0) \), the 2-Toda \( \tau \)-function has the following expansion in Schur polynomials:

\[
\tau_n(t, s) = \sum_{\lambda, \nu} \det(m_{\lambda, \nu}) s_\lambda(t) s_\nu(-s), \quad \text{for } n > 0, \quad (2.1.26)
\]

where the sum is taken over all Young diagrams \( \lambda \) and \( \nu \), with first columns \( \hat{\lambda}_1 \) and \( \hat{\nu}_1 \leq n \) and where

\[
m_{\lambda, \nu} := (\mu_{\lambda_i-i+n, \nu_j-j+n})_{1 \leq i,j \leq n}. \quad (2.1.27)
\]

Proof: Note that every increasing sequence \( 1 \leq k_1 < \ldots < k_n < \infty \) can be mapped into a Young diagram \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0 \), by setting \( k_j = j + \lambda_{n+1-j} \). Relabeling the indices \( i, j \) with \( 1 \leq i, j \leq n \), by setting \( j' := n - j + 1, \quad i' := n - i + 1 \), we have \( 1 \leq i', j' \leq n \) and \( k_j - i = \lambda_{j'} - j' + i' \) and \( k_i - 1 = \lambda_{i'} - i' + n \). The same can be done for the sequence \( 1 \leq \ell_1 < \ldots < \ell_n < \infty \), leading to the Young diagram \( \nu \), using the same relabeling.

\( ^{10} \) For a given Young diagram \( \lambda_1 \geq \ldots \geq \lambda_n \), define \( s_\lambda(t) = \det(p_{\lambda_i-i+j}(t))_{1 \leq i,j \leq n} \).
Applying the Cauchy-Binet formula twice, expression (2.1.23) leads to:

\[
\tau_n(t, s) = \det(E_n(t) m_\infty(0,0) E_n^\top(-s)) = \sum_{1 \leq k_1 < \ldots < k_n < \infty} \det(p_{k_j-i}(t))_{1 \leq i, j \leq n} \det((m_\infty(0,0))^{k_i, i})_{1 \leq i, i \leq n} \\
= \sum_{1 \leq k_1 < \ldots < k_n < \infty} \det(p_{k_j-i}(t))_{1 \leq i, j \leq n} \sum_{1 \leq \ell_1 < \ldots < \ell_n < \infty} \det(\mu_{k_{i-1}, j-1})_{1 \leq i, j \leq n} \det(p_{\ell_i-j}(-s))_{1 \leq i, j \leq n} \\
= \sum_{\lambda, \nu \leq n} \det(p_{\lambda_{i-1}, j-1} \nu_{i-1} \nu_{i-1} + j)_{1 \leq i, j, n} \det(\mu_{\lambda_{i-1}, j-1} \nu_{i-1} \nu_{i-1} + j)_{1 \leq i, j, n} \det(p_{\nu_{i-1} \nu_{i-1} + j}(-s))_{1 \leq i, j, n} \\
= \sum_{\lambda, \nu \leq n} \lambda_{i-1} \nu_{i-1} \nu_{i-1} + j \lambda_{i-1} \nu_{i-1} \nu_{i-1} + j \ s_{\lambda}(t) s_{\nu}(-s).
\]

\[\blacksquare\]

### 2.2 Reduction to Hänkel matrices: the standard Toda lattice and a Virasoro algebra of constraints

In the notation of (2.1.7), consider the locus of \((L_1, L_2)\)'s such that \(L_1 = L_2\). This means the matrix \(L_1 = L_2\) is tridiagonal. From the equations (2.1.7), it follows that along that locus,

\[
\frac{\partial(L_1 - L_2)}{\partial t_n} = 0 \quad \frac{\partial(L_1 - L_2)}{\partial s_n} = 0.
\]

We now define new variables \(t'_n\) and \(s'_n\) by

\[
t'_n = t_n - s_n \quad \text{and} \quad s'_n = t_n + s_n,
\]

and thus

\[
\frac{\partial}{\partial t'_n} = \frac{1}{2} \left( \frac{\partial}{\partial t_n} - \frac{\partial}{\partial s_n} \right), \quad \frac{\partial}{\partial s'_n} = \frac{1}{2} \left( \frac{\partial}{\partial t_n} + \frac{\partial}{\partial s_n} \right).
\]

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Then
\[\frac{\partial L_i}{\partial s'} = \frac{1}{2} \left( \frac{\partial}{\partial t_n} + \frac{\partial}{\partial s_n} \right) L_i = [(L^n_1)_+ + (L^n_2)_-, L_i] = [L^n_i, L_i] = 0 \quad (2.2.3)\]
and
\[\frac{\partial L_i}{\partial t'_n} = \frac{1}{2} \left( \frac{\partial}{\partial t_n} - \frac{\partial}{\partial s_n} \right) L_i = \frac{1}{2} [(L^n_1)_+ - (L^n_2)_+ + L^n_i, L_i] = [(L^n_i)_+, L_i], \text{ using } L_1 = L_2. \quad (2.2.4)\]

So, equation (2.2.3) implies that \(L_1 = L_2\) is independent of \(s'\). Since \(\tau(t, s)\) is a function of \(t - s\) only, we may set \(\tau(t') := \tau(t - s)\). After noting (see (2.1.12))
\[\frac{\partial}{\partial t_k} \log h_n = (L^k_1)_{n,n},\]
this situation leads to the standard Toda lattice equations on symmetric and tridiagonal matrices \(L = h^{-1/2}L_1 h^{1/2}\) and wave vectors (expressed in terms of the 2-Toda wave vectors (2.1.4))
\[\Psi(t', z) := h^{-1/2} \Psi_1(t, s; z) e^{-\frac{1}{2} \sum t_i^+ z^i} = h^{1/2} \Psi_2(t, s; z^{-1}) e^{\frac{1}{2} \sum t_i^+ z^i} = e^{\frac{1}{2} \sum t_i^+ z^i} \left( z^{n} \frac{\tau_n(t' - [z^{-1}])}{\sqrt{\tau_n(t') \tau_{n+1}(t')}} \right)_{n \geq 0} \quad (2.2.5)\]

namely
\[L \Psi = z \Psi, \quad \frac{\partial \Psi}{\partial t'_n} = \frac{1}{2} (L^n_{sk}) \Psi \quad \text{and} \quad \frac{\partial L}{\partial t'_n} = \frac{1}{2} [(L^n_{sk}, L)] \quad (2.2.6)\]

(Standard Toda lattice)

where \((\cdot)_{sk}\) refers to the skew-part in the skew and lower-triangular Lie decomposition.

We now define the standard Toda vertex operator as the reduction of the two-Toda vertex operator \(X_{12}\), defined in (2.1.10), using (2.2.1) and (2.2.2):
\[\Xi(t'; z) := \Xi_{12}(t, s; z, z) \bigg|_{t' \rightarrow t', t' \rightarrow t'} = \Lambda^\top \chi(z^2) e^{\sum t_i z^i} e^{-2 \sum t_i} \frac{z^{-1}}{t'} \frac{\partial}{\partial t'}. \quad (2.2.7)\]
In the rest of this section, we shall omit $t'$ in $t'$ and $s'$. This vertex operator $X(t; z)$ generates a Virasoro algebra, with $\beta = 2$ and thus central charge $c = 1$ (see Appendix 1 and (5.0.5))

$$\frac{d}{du} u^{k+1}X(t, u) = \left[ \frac{\beta \gamma_k^{(2)}(t)}{\beta = 2}, X(t; u) \right], \ k \in \mathbb{Z}; \quad (2.2.8)$$

An interesting semi-infinite example of the standard Toda lattice is obtained by considering a weight $\rho(z)dz = e^{-V(z)}dz$ defined on an interval $F \subset \mathbb{R}$, satisfying (0.0.6) and an inner-product

$$\langle f, g \rangle_t := \int_{\mathbb{R}} f(z)g(z)e^{\sum_{i=1}^{\infty} t_i z_i} \rho(z)dz, \quad (2.2.9)$$

leading to a $t$-dependent moment matrix

$$m_\infty(t) = (\mu_{ij}(t))_{0 \leq i, j \leq \infty} = (\langle y^i, y^j \rangle_t)_{0 \leq i, j < \infty}. \quad (2.2.10)$$

(Hankel matrix)

**Theorem 2.4** *Adler-van Moerbeke [1]*) The vector $\tau(t) = (\tau_0, \tau_1(t), ...) \quad \tau_n(t) := \frac{1}{n!} \int_{\mathcal{H}_n} e^{\text{Tr}(-V(M)+\sum_{i=1}^{\infty} t_i M^i)} dM

$$= \frac{1}{n!} \int_{F^n} \Delta_n(z)^2 \prod_{k=1}^{n} e^{\sum_{i=1}^{\infty} t_i z_i} \rho(z_k)dz_k

= \det (\mu_{ij}(t))_{0 \leq i, j \leq n-1} \quad (2.2.11)$$

is a set of $\tau$-functions for the standard Toda lattice. Also, each $\tau_n(t)$ satisfies the KP hierarchy, of which the first equation is given in Theorem 2.1. It also satisfies (i) of theorem 0.2, with $t_1 = x$. Moreover, the $\tau_n$'s satisfy the
following Virasoro constraints for $\beta = 2$, (see (2.2.8)),

$$0 = \mathcal{J}^{(2)}_m \tau(t), \quad m \geq -1,$$

$$:= \sum_{k \geq 0} \left( -a_k \sum_{i+j=k+m} : \beta \mathcal{J}^{(1)}_i \mathcal{J}^{(1)}_j : + b_k \beta \mathcal{J}^{(1)}_{k+m+1} \right) \bigg|_{\beta=2} \tau(t)$$

$$= \left( \sum_{k \geq 0} \left( -a_k ( \beta \mathcal{J}^{(2)}_{k+m} + 2n \beta \mathcal{J}^{(1)}_{k+m} + n^2 \mathcal{J}^{(0)}_{k+m} ) ight. \right. + \left. \left. b_k ( \beta \mathcal{J}^{(1)}_{k+m+1} + n\delta_{k+m+1,0} ) \right) \right) \bigg|_{\beta=2} \tau_n(t) \bigg|_{n \geq 0}$$

(2.2.12)

where the $a_k$ and $b_k$ are the coefficients (0.0.6) of the rational function $\rho'/\rho$, and where the $\beta \mathcal{J}^{(i)}_k$ and $\beta \mathcal{J}^{(i)}_k$ for $\beta = 2$, are given by (5.0.1), (5.0.3) and (5.0.5). The relation between the vertex operator $X(t, u)$ and $\mathcal{J}^{(2)}_m$ is given by

$$\frac{\partial}{\partial u} u^{m+1} f(u) X(t, u) \rho(u) = [-\mathcal{J}^{(2)}_m, X(t, u) \rho(u)]. \quad (2.2.13)$$

**Sketch of proof:** Formula (2.2.13) is a direct consequence of (2.2.8), while (2.2.12) hinges on the fact that the vector $I = (\tau_0, \tau_1, ..., n!\tau_n, ...)$ of Toda lattice $\tau$-functions is a fixed point for a certain integrated vertex operator, in the following sense:

$$(\mathcal{Y}I)_n := I_n, \quad \text{for } n \geq 1, \quad \text{where } \mathcal{Y} := \int_F du \rho(u) X(t, u),$$

which is just an iterated integral formula. Upon integrating (2.2.13) on the full range $F$, deduce $[\mathcal{J}^{(2)}_m, \mathcal{Y}] = 0$ from the boundary conditions (0.0.6). Acting on $I$ with this relation, one deduces (2.2.12) by induction on $n$ and the fact that $\tau_0 = 1$.

2.3 Reduction to Toeplitz matrices: two-Toda Lattice and an SL(2,\mathbb{Z})-algebra of constraints

Consider the following inner product, depending on $(t, s)$,

$$\langle f(z), g(z) \rangle_{t,s} := \int_{S^1} \frac{\rho(dz)}{2\pi i z} f(z) g(z^{-1}) e^{\sum_{i=1}^{\infty} (t_i z^i - s_i z^{-i})}, \quad (2.3.1)$$
where the integral is taken over the unit circle $S^1$ around the origin in the complex plane $\mathbb{C}$. Instead of having $z^{k\top} = z^k$ in the Hänkel inner-product, we have $z^{k\top} = z^{-k}$ in this inner-product:

$$\langle z^k f(z), g(z) \rangle_{t,s} = \langle f(z), z^{-k} g(z) \rangle_{t,s}. \quad (2.3.2)$$

Thus the moment matrix $m_{\infty}$, with entries

$$\mu_{k\ell}(t, s) = \langle z^k, z^\ell \rangle_{t,s} = \oint_{S^1} \frac{\rho(z)dz}{2\pi iz} e^{\sum_{i=1}^{\infty} (t_i z_i - s_i z_i^{-1})}, \quad (2.3.3)$$

is a Toeplitz matrix for all $t, s$, satisfying the differential equations (2.1.20) of the 2-Toda lattice, i.e.,

$$\frac{\partial \mu_{k\ell}}{\partial t_i} = \mu_{k+i, \ell} \quad \text{and} \quad \frac{\partial \mu_{k\ell}}{\partial s_i} = -\mu_{k, \ell+i}.$$

**Theorem 2.5** For $\rho(dz) = dz$, the vector $\tau(t, s) = (\tau_0 = 1, \tau_1(t, s), ...)$, with

$$\tau_n(t, s) = \int_{U(n)} e^{\sum_{i=1}^{\infty} \text{Tr}(t_i M^i - s_i \bar{M}^i)} dM$$

$$= \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left( e^{\sum_{i=1}^{\infty} (t_i z_k - s_i \bar{z}_k^{-1})} \frac{dz_k}{2\pi i z_k} \right)$$

$$= \det (\mu_{k\ell}(t, s))_{0 \leq k, \ell \leq n-1}$$

$$= \sum \{ \text{Young diagrams } \lambda \text{ with first column } \leq n \} s_\lambda(t) s_\lambda(-s) \quad (2.3.4)$$

is a vector of $\tau$-functions for the two-Toda lattice. Hence, they satisfy the identity (2.1.3), and (ii) of Theorem 0.2, with $t_1 = x, s_1 = y$. Moreover, they are annihilated by the following algebra of three Virasoro partial differential operators, which form a $\text{SL}(2, \mathbb{Z})$- algebra, (note: the expression below is a semi-infinite vector),

$$0 = V_k^{(2)}(t, s) \quad \text{for } \begin{cases} k = -1, & \theta = 0 \\ k = 0, & \theta \text{ arbitrary} \\ k = 1, & \theta = 1 \end{cases} \quad \text{only}$$

$$= \left( \beta \mathbb{J}_k^{(2)}(t) - \beta \mathbb{J}_{-k}^{(2)}(-s) - k(\theta \beta \mathbb{J}_k^{(1)}(t) + (1 - \theta) \beta \mathbb{J}_{-k}^{(1)}(-s)) \right)\bigg|_{\beta=1} \tau(t, s), \quad (2.3.5)$$
where $\theta$ is an arbitrary parameter and $\beta J_k^{(2)}$ for $\beta = 1$ is given in (5.0.4). This is a subalgebra of a Virasoro algebra (generated by the 2-Toda vertex operator (2.1.10))

$$
\frac{d}{du} u^{k+1} \frac{X_{12}(t, s; u, u^{-1})}{u} = \left[ V_k^{(2)}(t, s), \frac{X_{12}(t, s; u, u^{-1})}{u} \right], \quad k \in \mathbb{Z}, \quad (2.3.6)
$$

of central charge $c = 0$:

$$
\left[ V_k^{(2)}, V_\ell^{(2)} \right] = (k - \ell)V_{k+\ell}. \quad (2.3.7)
$$

The statement hinges on the following statement about the vertex operator:

**Proposition 2.6** For general weight $\rho(dz) = e^{V(z)}dz$, the column vector of 2-Toda $\tau$-functions (slightly rescaled), $\tau(t, s) = (\tau_0, \tau_1, ...)$, with

$$
I_n(t, s) = n! \tau_n = n! \int_{U(n)} e^{\sum_{i=1}^\infty \text{Tr}(t_i M_i - s_i M_i^t)} e^{\text{Tr} V(M)} dM \quad (2.3.8)
$$

is a fixed point, in the sense,

$$
(Y(t, s; \rho) I)_n = I_n, \quad \text{for } n \geq 1, \quad (2.3.9)
$$

for the operator

$$
Y(t, s; \rho) = \int_{S^1} \frac{\rho(du)}{2\pi i u} X_{12}(t, s; u, u^{-1}). \quad (2.3.10)
$$

**Proof of Proposition 2.6 and Theorem 2.5:** On the one hand, using the fact that $\bar{z} = 1/z$ on the circle $S^1$ and a property of Vandermonde determinants\(^{11}\)

\(^{11}\)The following holds:

$$
\det(u_k^{l-1})_{1 \leq l, k \leq N} \det(v_k^{l-1})_{1 \leq l, k \leq N} = \sum_{\sigma \in S_N} \det(u_{\sigma(k)}^{l-1} v_{\sigma(k)}^{k-1})_{1 \leq l, k \leq N}.
$$
and Theorem 1.1, we have, for a general weight $\rho(dz) = e^{V(z)}dz$,

\[ n!\tau_n(t, s) = n! \int_{U(n)} e^{\sum_{i=1}^{\infty} \text{Tr}(t_i M^i - s_i M^i)} e^{\text{Tr} V(M)} dM \]

\[ = \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left( e^{\sum_{i=1}^{\infty} (t_i z_k^i - s_i z_k^i)} \frac{\rho(dz_k)}{2\pi i z_k} \right) \]

\[ = \int_{(S^1)^n} \Delta_n(z) \Delta_n(\bar{z}) \prod_{k=1}^{n} \left( e^{\sum_{i=1}^{\infty} (t_i z_k^i - s_i z_k^i)} \frac{\rho(dz_k)}{2\pi i z_k} \right) \]

\[ = \sum_{\sigma \in S_n} \det \left( \int_{S^1} z^{\ell-1} z^m \sum_{i}^{\infty} (t_i z^i - s_i z^i) \frac{\rho(dz)}{2\pi iz} \right)_{1 \leq \ell, m \leq n} \]

\[ = n! \det \left( \int_{S^1} z^{\ell-1} z^m \sum_{i}^{\infty} (t_i z^i - s_i z^i) \frac{\rho(dz)}{2\pi iz} \right)_{1 \leq \ell, m \leq n} \]

yielding the third equality of (2.3.4). On the other hand, for $n \geq 1$, we have

\[ I_n(t, s) = n!\tau_n(t, s) \]

\[ = \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left( e^{\sum_{i=1}^{\infty} (t_i z_k^i - s_i z_k^i)} \frac{\rho(dz_k)}{2\pi i z_k} \right) \]

\[ = \int_{S^1} \frac{\rho(du)}{2\pi i u} e^{\sum_{i}^{\infty} (t_i u^i - s_i u^i)} u^{n-1} u^{-n+1} \]

\[ \int_{(S^1)^{n-1}} \Delta_{n-1}(z) \Delta_{n-1}(\bar{z}) \prod_{k=1}^{n-1} \left( 1 - \frac{z_k}{u} u \right) \left( 1 - \frac{z_k}{u} \right) e^{\sum_{i=1}^{\infty} (t_i z_k^i - s_i z_k^i)} \frac{\rho(dz_k)}{2\pi i z_k} \]

\[ = \int_{S^1} \frac{\rho(du)}{2\pi i u} e^{\sum_{i}^{\infty} (t_i u^i - s_i u^i)} e^{-\sum_{i=1}^{\infty} \left( \frac{u - \partial}{\partial u} - \frac{\partial}{\partial \bar{u}} \right)} \]

\[ \int_{(S^1)^{n-1}} \Delta_{n-1}(z) \Delta_{n-1}(\bar{z}) \prod_{k=1}^{n-1} e^{\sum_{i=1}^{\infty} (t_i z_k^i - s_i z_k^i)} \frac{\rho(dz_k)}{2\pi i z_k} \]

\[ = \int_{S^1} \frac{\rho(du)}{2\pi i u} e^{\sum_{i=1}^{\infty} (t_i u^i - s_i u^i)} e^{-\sum_{i=1}^{\infty} \left( \frac{u - \partial}{\partial u} + \frac{\partial}{\partial \bar{u}} \right)} I_{n-1}(t, s) \]

\[ = \left( \mathcal{Y}(t, s; \rho) I(t, s) \right)_{n}, \quad \text{(2.3.12)} \]
from which (2.3.9) follows.

Given the vertex operator $X_{12}(t, s; u, u^{-1})$, we now compute the corresponding Virasoro algebra, using (2.1.11),

$$u \frac{d}{du} u^k X_{12}(t, s; u, u^{-1})$$

$$= \left( u^{k+1} \frac{d}{du} + k u^k \right) X_{12}(t, s; u, u^{-1})$$

$$= \left( u^{k+1} \frac{\partial}{\partial u} - v^{1-k} \frac{\partial}{\partial v} + k u^k \right) X_{12}(t, s; u, v) \bigg|_{v = u^{-1}}$$

$$= \left( \frac{\partial}{\partial u} u^{k+1} - \frac{\partial}{\partial v} v^{1-k} - k u^k \right) X_{12}(t, s; u, v) \bigg|_{v = u^{-1}}$$

$$= \left[ \frac{\beta J^{(2)}_k(t) - \beta J^{(2)}_{-k}(-s)}{-k} - k \left( \beta J^{(1)}_k(t) + (1 - \theta) J^{(1)}_{-k}(-s) \right) \right] X_{12}(t, s; u, u^{-1})$$

$$= \left[ \beta J_k^{(2)}, X_{12}(t, s; u, u^{-1}) \right]_{\beta = 1}, \quad (2.3.13)$$

from which (2.3.6) follows. Verifying (2.3.7) goes by explicit computation, using (5.0.2).

Since (by virtue of (2.3.6)) for Lebesgue measure on $S^1$,

$$\left[ \beta J_k^{(2)}, \mathcal{Y}(t, s, \rho = 1) \right] = \left[ \beta J_k^{(2)}, \int_{S^1} X_{12}(t, s; u, u^{-1}) \frac{du}{2\pi i u} \right]$$

$$= \int_{S^1} \left[ \beta J_k^{(2)}, X_{12}(t, s; u, u^{-1}) \frac{du}{2\pi i u} \right]$$

$$= \int_{S^1} \frac{du}{2\pi i} \frac{d}{du} u^{k+1} X_{12}(t, s; u, u^{-1})$$

$$= 0,$
we have, using the notation (2.3.10) and the fact that, for \( n \geq 0 \) and \( \rho = 1 \), the integrals \( I_n = n! \tau_n(t, s) \) are fixed points for \( Y(t, s, dz) \); hence

\[
0 = \left( [\mathcal{V}_k^{(2)}, (\mathcal{Y}(t, s; dz))]^n \right)_n \\
= \left( \mathcal{V}_k^{(2)} \mathcal{Y}(t, s; dz)^n I - \mathcal{Y}(t, s; dz)^n \mathcal{V}_k^{(2)} I \right)_n \\
= \left( \mathcal{V}_k^{(2)} I - \mathcal{Y}(t, s; dz)^n \mathcal{V}_k^{(2)} I \right)_n.
\]

Taking the \( n^\text{th} \) component and taking into account the presence of \( \Lambda^{-1} \) in \( X_{12}(t, s; u, u^{-1}) \), we find

\[
0 = \left( \mathcal{V}_k^{(2)} I - \mathcal{Y}(t, s; dz)^n \mathcal{V}_k^{(2)} I \right)_n \\
= \mathcal{V}_k^{(2)} I_n - \int_{S_1} \frac{du}{2\pi i u} \sum_{i=1}^{\infty} (t^{w_i} - s^{w_i}) e^{-\sum_{i=1}^{\infty} \left( \frac{w_i}{i} + i \frac{\partial}{\partial u} - \frac{\partial}{\partial s} \right)} \\
\cdots \int_{S_1} \frac{du}{2\pi i u} \sum_{i=1}^{\infty} (t^{w_i} - s^{w_i}) e^{-\sum_{i=1}^{\infty} \left( \frac{w_i}{i} + i \frac{\partial}{\partial u} - \frac{\partial}{\partial s} \right)} \mathcal{V}_k^{(2)} I_0.
\]

In the notation (2.3.5) and (5.0.4), \( \mathcal{V}_k^{(2)}(t, s) \) has the following form

\[
\mathcal{V}_k^{(2)}(t, s) = \frac{1}{2} \left( J_k^{(2)}(t) - J_{-k}^{(2)}(-s) + (2n + k + 1)J_k^{(1)}(t) - (2n - k + 1)J_{-k}^{(1)}(-s) \right) \\
- k \left( \theta J_k^{(1)}(t) + (1 - \theta)J_{-k}^{(1)}(-s) \right).
\]

Working out the expression above leads to the expression written out in (4.0.7) and one checks immediately that, given \( \tau_0 = 1 \),

\[
\mathcal{V}_k^{(2)}(t, s)\tau_0 = 0 \quad \text{only for} \quad \left\{ \begin{array}{l} k = -1, \quad \theta = 0 \\
 k = 0, \quad \theta \quad \text{arbitrary} \\
 k = 1, \quad \theta = 1 \end{array} \right. 
\]

ending the proof of Theorem 2.5, except for the last equality of (2.3.4), which follows easily from the last Proposition of subsection 2.1. Indeed,

\[
m_\infty(0, 0) = \left( \int_{S_1} \frac{z^{k-t} dz}{2\pi i z} \right)_{1 \leq k, t < \infty} = I_\infty,
\]

where \( I_\infty \) denotes the semi-infinite identity matrix. Finally, one uses identity (2.1.27) and the fact that all the determinants of submatrices \( (I_\infty)^{\lambda\mu} \) (in the notation (2.1.27)) are zero, except when the Young diagrams \( \lambda \) and \( \mu \) are equal. ■
Remark: According to the strong Szegő theorem, we have:
\[
\tau_n = \exp\left(-\sum_{k=1}^{\infty} k t_k s_k\right) \text{ for } n \to \infty,
\]
provided \(\sum_1^{\infty} k(|t_k|^2 + |s_k|^2) < \infty\). Therefore the Toeplitz case yields boundary conditions for \(\tau_n\) at both extremities, namely \(n = 0\) and \(n = \infty\).

2.4 Toeplitz matrices and the structure of \(L_1\) and \(L_2\)

The associated 2-Toda matrices \(L_1\) and \(L_2\) have a very peculiar structure, when the initial \(m_\infty\) matrix is Toeplitz, as we shall see in the main theorem of this section. Throughout, we shall be using the multiplication operator identity \(z^\top = z^{-1}\) with regard to the inner product (2.3.1). This characterizes the Toeplitz case. Remember, from section 2.1, the polynomials (combining (2.1.4) and (2.1.24))

\[
p^{(1)}_n(z) = z^n \frac{\tau_n(t - [z^{-1}], s)}{\tau_n(t, s)} \quad \text{and} \quad p^{(2)}_n(z) = z^n \frac{\tau_n(t, s + [z^{-1}])}{\tau_n(t, s)}
\]

are bi-orthogonal for the special inner-product (2.3.1); also consider the vector notation:

\[
p^{(i)} = (p^{(i)}_0, p^{(i)}_1, ...), \quad p^{(i)}_\Lambda = (p^{(i)}_1, p^{(i)}_2, ...) \quad \text{and} \quad h = \text{diag}\left(\tau_1, \tau_2, ...\right).
\]

Theorem 2.7 The lower-triangular parts\(^{12}\) of the matrices \(L_1\) and \(hL_2^\top h^{-1}\), arising in the context of a Toeplitz matrix \(m_\infty\), are the projection of a rank 2 matrix:

\[
L_1 = -\left(h p^{(1)}_\Lambda(0) \otimes h^{-1} p^{(2)}(0)\right)_{-0} + \Lambda
\]

\[
hL_2^\top h^{-1} = -\left(h p^{(2)}_\Lambda(0) \otimes h^{-1} p^{(1)}(0)\right)_{-0} + \Lambda.
\]

\(^{12}\)In the formulae below \(A_{-0}\) denotes the lower-triangular part of \(A\), including the diagonal.
Corollary 2.8 (Unsymmetric identities) In particular

\[ p_{n+1}^{(1)}(0)p_{n+1}^{(2)}(0) = 1 - \frac{h_{n+1}}{h_n} \]

\[ p_{n+1}^{(1)}(0)p_{n+1}(0) = 1 - \frac{h_{n+1}}{h_n} \]

\[ p_{n+1}^{(1)}(0)p_{n}^{(2)}(0) = -\frac{\partial}{\partial t_1} \log h_n \]

\[ p_{n+1}^{(2)}(0)p_{n+1}^{(1)}(0) = \frac{\partial}{\partial s_1} \log h_n \]

\[ p_{n+1}^{(1)}(0)p_{n-1}^{(2)}(0) = -\frac{h_{n-1}}{h_n} \left( \frac{\partial}{\partial t_1} \right)^2 \log \tau_n \]

\[ p_{n+1}^{(2)}(0)p_{n-1}^{(1)}(0) = -\frac{h_{n-1}}{h_n} \left( \frac{\partial}{\partial s_1} \right)^2 \log \tau_n \]

\[ \frac{(\frac{h_n}{h_{m+1}})^2 \left( 1 - \frac{h_{n+1}}{h_n} \right) \left( 1 - \frac{h_{m+1}}{h_m} \right)}{\tau_m^{2} \tau_{n+1}^{2}} \left( p_{n-m} (\tilde{\partial}_t) \tau_{m+2} \circ \tau_n \right) \cdot \left( p_{n-m} (-\tilde{\partial}_s) \tau_{m+2} \circ \tau_n \right) \]

In particular, for \( m = n - 1 \),

\[ \left( 1 - \frac{h_{n+1}}{h_n} \right) \left( 1 - \frac{h_n}{h_{n-1}} \right) = -\frac{\partial}{\partial t_1} \log h_n \frac{\partial}{\partial s_1} \log h_n. \]  

Identity (2.4.2) was already observed by Hisakado in [12]. We first need a Lemma, which explains the peculiar structure of the bi-orthogonal polynomials \( p_n^{(1)}(y) \) and \( p_n^{(2)}(z) \), associated with the inner-product (2.3.1):

Lemma 2.10 (Hisakado [12]) The following holds:

\[ p_{n+1}^{(1)}(z) - z p_n^{(1)}(z) = p_{n+1}^{(1)}(0)z^n p_n^{(2)}(z^{-1}) \]

\[ p_{n+1}^{(2)}(z) - z p_n^{(2)}(z) = p_{n+1}^{(2)}(0)z^n p_n^{(1)}(z^{-1}). \]  

\[ ^{13}\text{See footnote 8 for notation } p_k(\tilde{\partial}_t) \text{ and } p_k(-\tilde{\partial}_s). \text{ The bi-orthogonal polynomials } p_k^{(i)} \text{ should not be confused with the Schur polynomials } p_k. \]
Proof: The following orthogonality relations hold for $1 \leq i \leq n$:

$$\langle p^{(1)}_{n+1}(z) - zp^{(1)}_n(z), z^i \rangle = \langle p^{(1)}_{n+1}(z), z^i \rangle - \langle p^{(1)}_n(z), z^{i-1} \rangle = 0$$

and

$$\langle z^n p^{(2)}_n(z^{-1}), z^n \rangle = \langle z^{n-1}, p^{(2)}_n(z) \rangle = 0.$$ 

Therefore the two must be proportional and since

$$\langle \cdot, \cdot \rangle$$

are, on the other hand, the first identity (2.4.3) follows. The second one follows by duality.

Proof of Theorem 2.7 and Corollaries 2.8 and 2.9: On the one hand, for $n > m \geq -1$

$$\langle p^{(1)}_{n+1}(z) - zp^{(1)}_n(z), p^{(2)}_{m+1}(z) - zp^{(2)}_m(z) \rangle = -\langle p^{(1)}_n(z), p^{(2)}_{m+1}(z) \rangle$$

$$= -\langle p^{(1)}_n(z) + \ldots + (L_1)_{n,m+1} p^{(1)}_{m+1}(z) + \ldots , p^{(2)}_{m+1}(z) \rangle$$

$$= -(L_1)_{n,m+1} \langle p^{(1)}_{m+1}(z), p^{(2)}_{m+1}(z) \rangle$$

$$= -(L_1)_{n,m+1} h_{m+1}, \quad (2.4.4)$$

and, on the other hand, for $n \geq m \geq -1$

$$\langle p^{(1)}_{n+1}(z) - zp^{(1)}_n(z), p^{(2)}_{m+1}(z) - zp^{(2)}_m(z) \rangle$$

$$= \langle p^{(1)}_{n+1}(0) z^n p^{(2)}_n(z^{-1}), p^{(2)}_{m+1}(0) z^m p^{(1)}_m(z^{-1}) \rangle$$

$$= p^{(1)}_{n+1}(0) p^{(2)}_{m+1}(0) \langle z^n p^{(1)}_n(z), p^{(2)}_m(z) \rangle$$

$$= p^{(1)}_{n+1}(0) p^{(2)}_{m+1}(0) \langle p^{(1)}_n(z) + \ldots , p^{(2)}_m(z) \rangle$$

$$= p^{(1)}_{n+1}(0) p^{(2)}_{m+1}(0) h_n. \quad (2.4.5)$$

Comparing (2.4.4) and (2.4.5) yields

$$(L_1)_{n,m+1} = -h_{n} p^{(1)}_{n+1}(0) h_{m+1}^{-1} p^{(2)}_{m+1}(0), \quad n > m \geq -1, \quad (2.4.6)$$

proving the first expression of Theorem 2.7; the second one is obtained by the usual duality $L_1 \mapsto hL_2 h^{-1}$, $t \mapsto -s$ and so $p^{(1)} \mapsto p^{(2)}$ (see formulae in the beginning of this section). For $n = m$, we compute

$${}^{14}\text{Define } p^{(2)}_{-1}(z) = 0.$$
\[
\langle p_{n+1}^{(1)}(z) - zp_{n+1}^{(2)}(z), p_{n+1}^{(2)}(z) - zp_{n+1}^{(2)}(z) \rangle
\]
\[
= \langle p_{n+1}^{(1)}(z), p_{n+1}^{(2)}(z) \rangle + \langle zp_{n+1}^{(1)}(z), zp_{n+1}^{(2)}(z) \rangle - \langle zp_{n+1}^{(1)}(z), p_{n+1}^{(2)}(z) \rangle - \langle p_{n+1}^{(1)}(z), zp_{n+1}^{(2)}(z) \rangle
\]
\[
= h_{n+1} + h_n - h_{n+1} - h_{n+1} = h_n - h_{n+1},
\]
which upon comparison with (2.4.5) for \( n = m \) yields the first line of corollary 2.8:
\[
p_{n+1}^{(1)}(0)p_{n+1}^{(2)}(0)h_n = h_n - h_{n+1}.
\] (2.4.7)

Remember from (2.1.9) and (2.1.12), we have:
\[
L_1 = \sum_{k=-\infty}^{-2} \text{diag} \left( \frac{p_{1-k}(\tilde{\partial}_t)\tau_{n+k+1} \circ \tau_n}{\tau_{n+k+1} \tau_n} \right) \Lambda^k
\]
\[
+ \left( \frac{\partial}{\partial t_1} \right)^2 \log \tau_n \Lambda^{-1} + \frac{\partial}{\partial t_1} \log h_n \Lambda^0 + \Lambda
\]
\[
hL_2 h^{-1} = \sum_{k=-\infty}^{-2} \text{diag} \left( \frac{p_{1-k}(\tilde{\partial}_s)\tau_{n+k+1} \circ \tau_n}{\tau_{n+k+1} \tau_n} \right) \Lambda^k
\]
\[
+ \left( \frac{\partial}{\partial s_1} \right)^2 \log \tau_n \Lambda^{-1} - \frac{\partial}{\partial s_1} \log h_n \Lambda^0 + \Lambda.
\]
Together with the theorem, this yields corollary 2.8.

Finally, upon multiplying relations (2.4.1), setting \( m + 1 = n - k \), \( n > m \),
and using the relation above, one obtains, using Corollary 2.8,
\[
\left( \frac{h_{m+1}}{h_n} \right)^2 \left( \frac{p_{n-m}(\tilde{\partial}_t)\tau_{m+2} \circ \tau_n}{\tau_{m+2}^2 \tau_n^2} \right) \left( \frac{p_{n-m}(\tilde{\partial}_s)\tau_{m+2} \circ \tau_n}{\tau_{m+2}^2 \tau_n^2} \right)
\]
\[
= p_{n+1}^{(1)}(0)p_{n+1}^{(2)}(0)p_{n+1}^{(2)}(0)\left( 1 - \frac{h_{n+1}}{h_n} \right)p_{n+1}^{(1)}(0)
\]
\[
= \left( 1 - \frac{h_{n+1}}{h_n} \right) \left( 1 - \frac{h_{m+1}}{h_m} \right)
\]
\[
= \frac{1}{\tau_{n+1}^2 \tau_{m+1}^2} \left( \tau_{n+1}^2 - \tau_n \tau_{n+2} \right) \left( \tau_{m+1}^2 - \tau_m \tau_{m+2} \right),
\] (2.4.8)
which is precisely Corollary 2.9. Relation (2.4.2) is a special case of (2.4.8),
setting \( m = n - 1 \).
Proof of Theorem 0.3: The structure of $L_1$ and $L_2$ follows from theorem 2.7. The statement about the mathematical expectation follows from:

$$p^{(1)}_n(t, s; z) = z^n \frac{\tau_n(t - [z^{-1}], s)}{\tau_n(t, s)}$$

$$= \sum_{k=0}^{n} z^k \frac{p_{n-k}(-\partial_t)\tau_n(t, s)}{\tau_n(t, s)}$$

$$= \frac{1}{\tau_n} \sum_{k=0}^{n} z^k \int_{U(n)} p_{n-k}(-\operatorname{Tr} M, -\frac{1}{2} \operatorname{Tr} M^2, -\frac{1}{3} \operatorname{Tr} M^3, \ldots) e^{\sum_{i=1}^{\infty} \operatorname{Tr}(t_i M^i - s_i \bar{M}^i)} dM,$$

and similarly

$$p^{(2)}_n(t, s; z) = z^n \frac{\tau_n(t, s + [z^{-1}])}{\tau_n(t, s)}$$

$$= \sum_{k=0}^{n} z^k \frac{p_{n-k}(\partial_s)\tau_n(t, s)}{\tau_n(t, s)}$$

$$= \frac{1}{\tau_n} \sum_{k=0}^{n} z^k \int_{U(n)} p_{n-k}(-\operatorname{Tr} \bar{M}, -\frac{1}{2} \operatorname{Tr} \bar{M}^2, -\frac{1}{3} \operatorname{Tr} \bar{M}^3, \ldots) e^{\sum_{i=1}^{\infty} \operatorname{Tr}(t_i \bar{M}^i - s_i \bar{M}^i)} dM.$$

Finally, we check the Hamiltonian flow statement for the first flow. Indeed, from the equations for $\Psi$ (after (2.1.7)), (2.1.23), Theorem 2.7, and the first relation of Corollary 2.8, it follows that ($h_{-1} = 0$)

$$\frac{\partial x_n}{\partial t_1} = \left. \frac{\partial p^{(1)}_n(t, s; z)}{\partial t_1} \right|_{z=0}$$

$$= - \left. (L_1)_{-p^{(1)}} \right|_{z=0}$$

$$= h_n p^{(1)}_{n+1}(t, s; 0) \sum_{i=0}^{n-1} \frac{p^{(1)}_i(t, s; 0) p^{(2)}_i(t, s; 0)}{h_i}$$

$$= h_n x_{n+1} \sum_{i=0}^{n-1} \frac{x_i y_i}{h_i}$$

$$= h_n x_{n+1} \sum_{i=0}^{n-1} \left( \frac{1}{h_i} - \frac{1}{h_{i-1}} \right) = x_{n+1} \frac{h_n}{h_{n-1}} = x_{n+1} (1 - x_n y_n),$$
and similarly for the other coordinates ending the proof of Theorem 0.3.

3 Painlevé equations for $O(n)$ and $Sp(n)$ integrals

3.1 Painlevé equations associated with the Jacobi weight

**Theorem 3.1** (Painlevé equation and the Jacobi weight) The function $H_n(x) = x \frac{d}{dx} \log \tau_n(x)$, with

$$\tau_n(x) := c_n \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^{n} e^{xz_k}(1-z_k)^{\alpha}(1+z_k)^{\beta} dz_k, \quad (3.1.1)$$

satisfies the Painlevé V equation ($a := \alpha + \beta$, $b := \alpha - \beta$, and $\alpha, \beta > -1$)

$$x^2 H''' + x H'' + 6xH'^2 - (4H + 4x^2 - 4bx + (2n + a)^2) H' + (4x - 2b)H$$
$$+ 2n(n + a)x - bn(2n + a) = 0,$$

(3.1.2)

with initial condition

$$H(0) = 0 \quad \text{and} \quad H'(0) = \frac{-nb}{a + 2n}. \quad (3.1.3)$$

**Corollary 3.2**

$$\tilde{H}_n(x) = x \frac{d}{dx} \log e^{-x\tau_n(2x)} \quad (3.1.4)$$

satisfies the Painlevé V equation

$$\frac{1}{2}x^2 \tilde{H}''' + \frac{1}{2}x \tilde{H}'' + 3x(\tilde{H}')^2 - \frac{1}{2} \left( 4\tilde{H} + 16x^2 - 8(b + c)x + (2n + a)^2 \right) \tilde{H}' +$$
$$\left( 8x - 2(b + c) \right) \tilde{H} + (4n(n + a) + c(2b + c))x - \frac{1}{2}(2n + a)(2n(b + c) + ac) = 0.$$  

(3.1.5)

with

$$\tilde{H}(0) = 0 \quad \text{and} \quad \tilde{H}'(0) = -\frac{2n(b + c) + ac}{2n + a}. \quad (3.1.6)$$
Proof of Theorem 3.1: Since $\alpha, \beta > -1$, the boundary condition (0.0.6) on $\rho(z)$ is fulfilled; so, we may apply Theorem 2.4. We set

$$a_0 = 1, a_1 = 0, a_2 = -1, b_0 = \alpha - \beta =: b, b_1 = \alpha + \beta =: a$$

and all other $a_i = b_j = 0$ in (2.2.12), implying that

$$\tau_n(t_1, t_2, \ldots) := \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{\sum_{i=1}^n t_iz_i^k} (1 - z_k)^\alpha (1 + z_k)^\beta dz_k$$

satisfies the equations ($m = 1, 2, 3, \ldots$)\(^{15}\):

$$0 = \frac{J_m^{(2)}}{\tau_n} \tau_n$$

$$= \sum_{k \geq 0} \left( -a_k \sum_{i+j=k+m-2} : \beta \mathfrak{J}_i^{(1)} \beta \mathfrak{J}_j^{(1)} : + b_k \beta \mathfrak{J}_{k+m-1}^{(1)} \right) \bigg|_{\beta=2} \tau_n$$

$$= \left( J_m^{(2)} - J_{m-2}^{(2)} - 2nJ_{m-2}^{(1)} + (2n+a)J_m^{(1)} + bJ_{m-1}^{(1)} - n^2 \delta_{m,2} + nb\delta_{m,1} \right) \tau_n.$$ 

Then introducing the function $F_n := \log \tau_n(t)$, the two first Virasoro constraints for $m = 1, 2$ divided by $\tau_n$ are given by

$$\frac{J_1^{(2)}}{\tau_n} \tau_n = \left( \sum_{i \geq 1} it_i \frac{\partial}{\partial t_{i+1}} - \sum_{i \geq 2} it_i \frac{\partial}{\partial t_{i-1}} + (2n+a) \frac{\partial}{\partial t_1} \right) F_n + n(b - t_1) = 0$$

$$\frac{J_0^{(2)}}{\tau_n} \tau_n = \left( \sum_{i \geq 1} it_i \frac{\partial}{\partial t_{i+2}} - \sum_{i \geq 1} it_i \frac{\partial}{\partial t_i} + b \frac{\partial}{\partial t_1} + \frac{\partial^2}{\partial t_1^2} + (2n+a) \frac{\partial}{\partial t_2} \right) F_n$$

$$+ \left( \frac{\partial F_n}{\partial t_1} \right)^2 - n^2 = 0.$$ 

(3.1.7)

These expressions and their first $t_1$- and $t_2$- derivatives, evaluated along the locus

$$\mathcal{L} := \{ t_1 = x, \text{ all other } t_i = 0 \}$$

\(^{15}\)The $J_m^{(i)}$ below are the ones of (5.0.3), for $\beta = 2$. 

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read as follows:

\[
0 = \left. \frac{J_0(t_1)}{\tau_n} \right|_L = \left. \left( t_1 \frac{\partial}{\partial t_2} + (2n + a) \frac{\partial}{\partial t_1} \right) F_n + n (b - t_1) \right|_L
\]

\[
0 = \left. \frac{J_1(t_2)}{\tau_n} \right|_L = \left. \left( t_1 \frac{\partial}{\partial t_3} + (b - t_1) \frac{\partial}{\partial t_1} + (2n + a) \frac{\partial}{\partial t_2} + \frac{\partial^2}{\partial t_1^2} \right) F_n + \left( \frac{\partial F_n}{\partial t_1} \right)^2 - n^2 \right|_L
\]

\[
0 = \left. \frac{\partial J_0(t_1)}{\partial t_1} \frac{\tau_n}{\tau_n} \right|_L = \left. \left( \sum_{i \geq 1} i t_i \frac{\partial^2}{\partial t_{i+1} t_1} + \frac{\partial}{\partial t_2} - \sum_{i \geq 2} i t_i \frac{\partial^2}{\partial t_{i-1} t_1} \right) \right|_L - n
\]

\[
= \left. \left( t_1 \frac{\partial^2}{\partial t_2 t_1} + \frac{\partial}{\partial t_2} + (2n + a) \frac{\partial^2}{\partial t_1^2} \right) F_n \right|_L - n
\]

\[
0 = \left. \frac{\partial J_1(t_2)}{\partial t_1} \frac{\tau_n}{\tau_n} \right|_L = \left. \left( \sum_{i \geq 1} i t_i \frac{\partial^2}{\partial t_{i+2} t_1} + \frac{\partial}{\partial t_3} - \sum_{i \geq 2} i t_i \frac{\partial^2}{\partial t_{i-1} t_1} \right) \right|_L + b \frac{\partial^2}{\partial t_1^2}
\]

\[
+ \frac{\partial^3}{\partial t_1^3} + (2n + 1) \frac{\partial^2}{\partial t_2 t_1} \right) \right|_L - n
\]

\[
= \left. \left( t_1 \frac{\partial^2}{\partial t_3 t_1} + \frac{\partial}{\partial t_3} + (b - t_1) \frac{\partial^2}{\partial t_1^2} - \frac{\partial}{\partial t_1} + \frac{\partial^3}{\partial t_1^3} \right) \right|_L + (2n + a) \frac{\partial^2}{\partial t_2 t_1} \right) \right|_L - n
\]

\[
0 = \left. \frac{\partial J_2(t_3)}{\partial t_2} \frac{\tau_n}{\tau_n} \right|_L = \left. \left( \sum_{i \geq 1} i t_i \frac{\partial^2}{\partial t_{i+2} t_2} + 2 \frac{\partial}{\partial t_3} - \sum_{i \geq 2} i t_i \frac{\partial^2}{\partial t_{i-1} t_2} - 2 \frac{\partial}{\partial t_1} \right) \right|_L
\]

\[
+ (2n + a) \frac{\partial^2}{\partial t_1 t_2} \right) \right|_L - n
\]

\[
= \left. \left( t_1 \frac{\partial^2}{\partial t_2^2} + 2 \frac{\partial}{\partial t_3} - 2 \frac{\partial}{\partial t_1} + (2n + a) \frac{\partial^2}{\partial t_1 t_2} \right) \right|_L - n
\]

\[
= \left( t_1 \frac{\partial^2}{\partial t_2^2} + 2 \frac{\partial}{\partial t_3} - 2 \frac{\partial}{\partial t_1} + (2n + a) \frac{\partial^2}{\partial t_1 t_2} \right) F_n
\]
The five equations above form a (triangular) linear system in five unknowns

\[
\begin{align*}
\frac{\partial F_n}{\partial t_2} & , \quad \frac{\partial F_n}{\partial t_3} , \quad \frac{\partial^2 F_n}{\partial t_1 \partial t_2} , \quad \frac{\partial^2 F_n}{\partial t_1 \partial t_3} , \quad \frac{\partial^2 F_n}{\partial t_2^2} ,
\end{align*}
\]

Setting \( t_1 = x \) and \( F_n' = \partial F_n/\partial x \), the solution is given by the following expressions,

\[
\begin{align*}
\frac{\partial F_n}{\partial t_2} \bigg|_L = & -\frac{1}{x} \left( (2n + a)F_n' + n(b - x) \right) \\
\frac{\partial F_n}{\partial t_3} \bigg|_L = & -\frac{1}{x^2} \left( x (F_n'' + F_n'^2 + (b - x)F_n' + n(n + a)) - (2n + a)((2n + a)F_n' + bn) \right) \\
\frac{\partial^2 F_n}{\partial t_1 \partial t_2} \bigg|_L = & -\frac{1}{x^2} \left( (2n + a)(xF_n'' - F_n') - bn \right) \\
\frac{\partial^2 F_n}{\partial t_1 \partial t_3} \bigg|_L = & -\frac{1}{x^3} \left( x^2(F_n'' + 2F_n'^2) - x ((x^2 - bx + 1)F_n'' + F_n'^2 + bF_n' + (2n + a)^2 F_n'' + n(n + a)) + 2(2n + a)^2 F_n' + 2bn(2n + a) \right) \\
\frac{\partial^2 F_n}{\partial t_2^2} \bigg|_L = & \frac{1}{x^3} \left( x (2F_n'^2 + 2bF_n' + ((2n + a)^2 + 2)F_n'' + 2n(n + a)) - 3(2n + a)^2 F_n' - 3bn(2n + a) \right).
\end{align*}
\]

Putting these expressions into the KP-equation (Theorem 2.1), and setting

\[
G(x) := F_n'(x) = \frac{d}{dx} \log \tau_n(x),
\]

we find

\[
\begin{align*}
x^3 G''' + 4x^2 G'' + x \left( -4x^2 + 4bx + 2 - (2n + a)^2 \right) G' + 8x^2 GG' + 6x^3 G'^2 + 2xG^2 + \left( 2bx - (2n + a)^2 \right) G + n(2x - b)(n + a) - bn^2 &= 0.
\end{align*}
\]

(3.1.8)

Finally, the function

\[
H(x) := xG(x) = x \frac{d}{dx} \log \tau_n(x)
\]
satisfies
\[ x^2 H''' + x H'' + 6xH'^2 - (4H + 4x^2 - 4bx + (2n + a)^2) H' + (4x - 2b)H + 2n(n + a)x - bn(2n + a) = 0. \]

(3.1.9)

According to Cosgrove \[9\], this 3rd order equation can be transformed into a master Painlevé equation, which one recognizes to be Painlevé V; see Appendix 2.

From section 8, Appendix 4, identity (8.0.5), it now follows that
\[
H_n'(0) = \frac{\tau_n'(0)}{\tau_n(0)} = \frac{\sum_{i=1}^{n} \int_{[-1,1]^n} \Delta_n(z)^2 z_i \prod_{k=1}^{n} \rho(\alpha,\beta)(z_k) dz_k}{\int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^{n} \rho(\alpha,\beta)(z_k) dz_k} = n\langle y_1 \rangle = -\frac{nb}{a + 2n}.
\]

This ends the proof of Theorem 3.1.

Proof of Corollary 3.2: The differential equation for
\[
\tilde{H}(x) = x \frac{d}{dx} \log e^{-cx} \tau_n(2x) = H(2x) - cx
\]
is obtained by first setting \(x \mapsto 2x\) in the differential equation (3.1.2) and then setting \(H(2x) = \tilde{H}(x) + cx\). This leads to the differential equation (3.1.5), which is, of course, also Painlevé V. Relation (3.1.6) follows at once from (3.1.4).

3.2 Proof of Theorem 0.1 (\(O(n)\) and \(Sp(n)\))

We give here a more detailed version of Theorem 0.1 (i):

Proposition 3.3 Given the integral\[16\]
\[
I_\ell^\pm(x) = \int_{O_\pm(\ell)} e^{x \text{Tr} M} dM,
\]

\[16\]The integral over the symplectic group \(Sp(n - 1)\) can be identified with \(O(2n)\).
the expressions\footnote{In this statement, we use the following notation: \([n]_{\text{even}} := \max \{\text{even } x, \text{ such that } x \leq n\}.\)}

\[ q_{\ell}(x) = \log e_{\ell}^{\pm} \frac{I_{\ell+2}^{\pm}}{I_{\ell}^{\pm}}, \quad \text{with} \quad e_{\ell}^{+} = \frac{2}{[\ell + 2]_{\text{even}}} \quad \text{and} \quad e_{\ell}^{-} = \frac{2}{[\ell + 1]_{\text{even}}} \]

satisfy the standard Toda lattice equations:

\[ \frac{1}{4} \frac{\partial^2 q_{\ell}}{\partial x^2} = -e^{q_{\ell} - q_{\ell-1}} + e^{q_{\ell+1} - q_{\ell}}. \]

**Proposition 3.4** The function

\[ f_{\ell}^{\pm}(x) = x \frac{d}{dx} \log \int_{O(\ell+1)_{\pm} \text{ or } Sp(\ell+1)} e^{x \text{Tr} M} dM \] (3.2.1)

is the unique solution to the 3rd order equation (i) in Theorem 0.1:

\[ \begin{cases} f''' + \frac{1}{x} f'' + \frac{6}{x} f f' - \frac{16}{x^2} f f' - \frac{16}{x} f' + \frac{16}{x} f + \frac{2(\ell^2 - 1)}{x} = 0 \\ \text{with } f_{\ell}^{\pm}(x) = x^2 \pm \frac{x^{\ell+1}}{\ell!} + O(x^{\ell+2}), \text{ near } x = 0. \end{cases} \] (3.2.2)

This 3rd order equation can be transformed into the following second order equation in \( f \), quadratic in \( f'' \):

\[ \frac{x^2}{4} f''^{\prime\prime} = - \left( x f'^2 - \left( 4x^2 + \frac{\ell^2}{4} \right) f' + x(\ell^2 - 1) \right) f' + (f'^2 - 8xf' + \ell^2 - 1)f + 4f^2, \]

which in turn leads to the standard Painlevé equation (6.0.3) for

\[ \alpha = -\beta = \frac{(\ell + 1)^2}{8}, \quad \gamma = 0, \quad \delta = -8. \]
Proof of Proposition 3.3: Proposition 1.1 and identity (2.2.11) imply

\begin{itemize}
  \item $I_{2n+1}^+(x) = n! e^x \tau_n(2x, 0, \ldots)$
  \item $I_{2n}^+(x) = n! \tau_n(2x, 0, \ldots)$
  \item $I_{2n+1}^-(x) = n! e^{-x} \tau_n(2x, 0, \ldots)$
  \item $I_{2n}^-(x) = (n-1)! \tau_{n-1}(2x, 0, \ldots)$.
\end{itemize}

Note that, since the functions $\tau_n(t-s) = \tau_n(t, s)$ satisfy differential equation (ii) of Theorem 2.1, we obtain for the function $\tau_n(t)$, by subtracting two consecutive equations:

$$\frac{\partial^2}{\partial t^2} \log \frac{\tau_{n+1}}{\tau_n} = \frac{\tau_n \tau_{n+2}}{\tau_{n+1}^2} - \frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2},$$

from which the standard Toda lattice equations follow. \hfill \blacksquare

Proof of Proposition 3.4: From Corollary 3.2 it follows that

$$\tilde{H}_n(x) = x \frac{d}{dx} \log \left( e^{-cx} \int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n e^{2xzk} (1 - z_k)^\alpha (1 + z_k)^\beta \, dz_k \right)$$

(3.2.3)

satisfies the Painlevé V equation (3.1.5). Then in view of Theorem 1.1, $\tilde{H}_n(x)$ corresponds to $f_\ell(x)$ in (3.2.1), when the parameters $n, a = \alpha + \beta, b = \alpha - \beta$ and $c$ take on the following values:

- $O(\ell + 1)_-$, with $\ell$ even: $n = \ell/2, a = 0, b = -1, c = 1$
- $O(\ell + 1)_-$, with $\ell$ odd: $n = (\ell - 1)/2, a = 1, b = 0, c = 0$
- $O(\ell + 1)_+$, with $\ell$ even: $n = \ell/2, a = 0, b = 1, c = -1$
- $O(\ell + 1)_+$, with $\ell$ odd: $n = (\ell + 1)/2, a = -1, b = 0, c = 0$
- $Sp(\ell+1)$, with $\ell$ odd: $n = (\ell - 1)/2, a = 1, b = 0, c = 0$.

Setting these values into equation (3.1.5) leads at once to equation (i) of Theorem 0.1, namely

$$f''' + \frac{1}{x} f'' + \frac{6}{x} f'^2 - \frac{4}{x^2} ff' - \frac{16n^2 + \ell^2}{x^2} f' + \frac{16}{x} f + \frac{2(\ell^2 - 1)}{x} = 0.$$
Moreover, for these values, we have that $b + c = ac = 0$ and so from (3.1.6), it follows that

$$f_\ell(0) = \tilde{H}_\ell(0) = 0 \quad \text{and} \quad f'_\ell(0) = \tilde{H}'_\ell(0) = -\frac{2n(b + c) + ac}{2n + a} = 0. \quad (3.2.4)$$

According to appendix 2 (see Cosgrove [9]), this third order equation has a first integral, which is second order in $f$ and quadratic in $f''$, thus introducing a constant $c$:

$$\frac{x^2}{4} f''^2 = - \left( x f'^2 - \left( 4x^2 + \frac{\ell^2}{4} \right) f' + x(\ell^2 - 1) \right) f'$$

$$+ (f'^2 - 8xf' + \ell^2 - 1)f + 4f^2 - \frac{c}{4}.$$ 

Evaluating this differential equation at $x = 0$ leads to, since $f(0) = 0$,

$$c = \ell^2 f'(0)^2 = 0, \quad \text{using (3.2.4).}$$

Setting $f = \tilde{f} - \ell^2/4$ in order to get the equation in Cosgrove's form [10],

$$\frac{x^2}{4} \tilde{f}''^2 = - \left( x \tilde{f}'^2 - 4x^2 \tilde{f}' - x(\ell^2 + 1) \right) \tilde{f}'$$

$$+ (\tilde{f}'^2 - 8x\tilde{f}' - (\ell^2 + 1))\tilde{f} + 4\tilde{f}^2 + \frac{\ell^2}{4}.$$ 

In the notation (6.0.2), we have

$$a_1 = 16, \quad a_2 = 4(\ell^2 + 1), \quad a_3 = 0, \quad c = -\frac{\ell^2}{4}.$$ 

Solving (6.0.3) for $\alpha, \beta, \gamma, \delta$ leads to the canonical form for Painlevé V, with

$$\alpha = -\beta = \frac{(1 + \ell)^2}{8}, \quad \gamma = 0, \quad \delta = -8,$$

and according to Appendix 4, $f''_\ell(0) = 2$, ending the proof of the first half of Theorem 0.1.

Of course, from combinatorics (Proposition 1.4), we have a much stronger statement:

$$E_{O_\pm(\ell+1)} e^{x \Tr M} = \exp \left( \frac{x^2}{2} \pm \frac{x^{\ell+1}}{(\ell + 1)!} + O(x^{\ell+2}) \right),$$

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and thus

\[ f^{\pm}_\ell(x) = x \frac{d}{dx} \log E_{O(\ell+1)} e^{\Tr M} dM = x^2 \pm \frac{x^{\ell+1}}{\ell!} + O(x^{\ell+2}), \text{ near } x = 0. \]

It remains to show the uniqueness of the solution to the initial value problem (3.2.2). Indeed, substituting \( f(x) = x^2 + \sum_{i \geq 3} a_i x^i \) into the third order differential equation (3.2.2) for \( f \) yields the recursive formula for the coefficients:

\[
3(4 - \ell^2)a_3 = 0
\]

\[
(i + 1)(i^2 - \ell^2)a_{i+1} - 16(i - 2)a_{i-1} + \sum_{2 \leq m, n \leq i-1 \atop n + m = i+1} n a_n (6m - 4)a_m = 0, \text{ for } i \geq 3. 
\]

Therefore, if \( \ell \geq 3 \), we have inductively \( a_3 = \ldots = a_\ell = 0 \), from (3.2.5). Setting \( i = \ell \), in the equation above shows that the coefficient \( a_{\ell+1} \) is free and can therefore be specified; it is specified by the combinatorics, namely \( a_{\ell+1} = \pm ((\ell+1)!)^{-1} \). Once \( a_{\ell+1} \) is fixed, all the subsequent \( a_i \)'s are determined by (3.2.5).

\[ \square \]

## 4 Painlevé equations for \( U(n) \) integrals

In this section, we show items (ii) and (iii) of Theorem 0.1:

**Proposition 4.1**

\[
g_n(x) = \frac{d}{dx} x \frac{d}{dx} \log \int_{U(n)} e^{\sqrt{\pi} \Tr (M+\bar{M})} dM \]  

(4.0.1)

is the unique solution to the initial value problem (Painlevé V equation):

\[
\begin{aligned}
g''_n - \frac{g'^2_n}{2} \left( \frac{1}{g_n - 1} + \frac{1}{g_n} \right) + \frac{g'^n_n}{x} - \frac{n^2}{2x^2} \frac{(g_n - 1)}{g_n} + \frac{2}{x} g_n (g_n - 1) &= 0 \\
\text{with } g_n(x) &= 1 - \frac{x^n}{(n!)^2} + O(x^{n+1}), \text{ near } x = 0.
\end{aligned}
\]
Proposition 4.2

\[ h_n(x) = \frac{E_{U(n)} \text{Tr} M \det(I + M)^k e^{-x \text{Tr} \bar{M}}}{E_{U(n)} \det(I + M)^k e^{-x \text{Tr} M}} \]

\[ = \frac{1}{n + k} x \frac{d}{dx} \log E_{U(n)} \det(I + M)^k e^{-x \text{Tr}(I + \bar{M})} dM \]  

is the unique solution to the initial value Painlevé V equation, as well:

\[
\begin{cases}
  h'''' - \frac{1}{2} \left( \frac{1}{h'} + \frac{1}{h'''} + 1 \right) h''^2 + \frac{h''}{x} + \frac{2(n + k)}{x} h'(h' + 1) \\
  - \frac{2x^2 h' (h' + 1)}{x} (x - n) h' - h - n) \left( (2h + x + n) h' + h + n \right) = 0
\end{cases}
\]

with \( h := h_n(x) = \frac{k - n}{k + n} - \frac{x^{n+1}}{(n + 1)!} \left( \frac{k + n - 1}{n} \right) + O(x^{n+2}) \), near \( x = 0 \).

Proof of Propositions 4.1: The proofs of the two propositions are almost identical, except in the end one specializes to a different locus.

Throughout, we shall be using the diagonal elements (2.1.12) of \( L_1 \) and \( hL_2^\top h^{-1} \):

\[ b_n = \frac{\partial}{\partial t_1} \log \frac{\tau_n}{\tau_{n-1}} = (L_1)_{n-1,n-1} \quad \text{and} \quad b^*_n = -\frac{\partial}{\partial s_1} \log \frac{\tau_n}{\tau_{n-1}} = (hL_2^\top h^{-1})_{n-1,n-1}. \]  

(4.0.3)

From (2.4.2), (2.1.17), (2.1.3), Theorem 2.5, and (5.0.4), the integral below, which is also the determinant of a Toeplitz matrix,

\[
\tau_n(t, s) = \int_{U(n)} e^{\sum_i \infty} \text{Tr}(t_i M_i - s_i \bar{M_i}) dM
\]

\[
= \det \left( \int_{S^2} z^{k-\ell} e^{\sum_i \infty} (t_i z^i - s_i z^{-i}) \frac{dz}{2\pi iz} \right)_{0 \leq k, \ell \leq n-1}
\]

(4.0.4)

satisfies the following three relations:
(i) Toeplitz:

$$T(\tau)_n = \frac{\partial}{\partial t_1} \log \frac{\tau_n}{\tau_{n-1}} \frac{\partial}{\partial s_1} \log \frac{\tau_n}{\tau_{n-1}}$$

$$+ \left(1 + \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n\right) \left(1 + \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n - \frac{\partial}{\partial s_1} \left(\frac{\partial}{\partial t_1} \log \frac{\tau_n}{\tau_{n-1}}\right)\right)$$

$$= -b_n b^*_n + \left(1 + \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n\right) \left(1 + \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n - \frac{\partial}{\partial s_1} b_n\right) = 0,$$

(4.0.5)

(ii) two-Toda:

$$\frac{\partial^2 \log \tau_n}{\partial s_2 \partial t_1} = -2 \frac{\partial}{\partial s_1} \log \frac{\tau_n}{\tau_{n-1}} \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n - \frac{\partial^3}{\partial s_1^2 \partial t_1} \log \tau_n$$

$$= 2b^*_n \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n - \frac{\partial^3}{\partial s_1^2 \partial t_1} \log \tau_n, \quad (4.0.6)$$

(iii) Virasoro:

$$V_{-1}\tau_n = \left(\sum_{i \geq 1} (i + 1)t_{i+1} \frac{\partial}{\partial t_i} - \sum_{i \geq 2} (i - 1)s_{i-1} \frac{\partial}{\partial s_i}\right) + n \left(t_1 + \frac{\partial}{\partial s_1}\right) \tau_n = 0$$

$$V_0\tau_n = \sum_{i \geq 1} \left(it_i \frac{\partial}{\partial t_i} - is_i \frac{\partial}{\partial s_i}\right) \tau_n = 0$$

$$V_1\tau_n = \left(-\sum_{i \geq 1} (i + 1)s_{i+1} \frac{\partial}{\partial s_i} + \sum_{i \geq 2} (i - 1)t_{i-1} \frac{\partial}{\partial t_i} + n \left(s_1 + \frac{\partial}{\partial t_1}\right)\right) \tau_n = 0.$$

(4.0.7)
Therefore we have

\[ 0 = \frac{1}{\tau_n} (\mathcal{V}_1 + \mathcal{V}_0) \tau_n \]
\[ = \left( \sum_{i \geq 1} ((i + 1)t_{i+1} + it_i) \frac{\partial}{\partial t_i} - \sum_{i \geq 2} ((i - 1)s_{i-1} + is_i) \frac{\partial}{\partial s_i} \right) \log \tau_n + nt_1 \]

\[ 0 = \frac{1}{\tau_n} (\mathcal{V}_0 + \mathcal{V}_1) \tau_n \]
\[ = \left( \sum_{i \geq 2} ((i - 1)t_{i-1} + it_i) \frac{\partial}{\partial t_i} - \sum_{i \geq 1} ((i + 1)s_{i+1} + is_i) \frac{\partial}{\partial s_i} \right) \log \tau_n + ns_1 \]

\[ 0 = \frac{\partial}{\partial t_1} \left( \frac{\mathcal{V}_1 \tau_n}{\tau_n} \right) \]
\[ = \left( \sum_{i \geq 1} (it_i \frac{\partial^2}{\partial t_1 \partial t_i} - is_i \frac{\partial^2}{\partial t_1 \partial s_i}) + \frac{\partial}{\partial t_1} \right) \log \tau_n \]

\[ 0 = \frac{\partial}{\partial s_1} \left( \frac{\mathcal{V}_1 \tau_n}{\tau_n} \right) \]
\[ = \left( -\sum_{i \geq 1} (i + 1)s_{i+1} \frac{\partial^2}{\partial s_1 \partial s_i} + \sum_{i \geq 2} (i - 1)t_{i-1} \frac{\partial^2}{\partial s_1 \partial t_i} + n \frac{\partial^2}{\partial s_1 \partial t_1} \right) \log \tau_n + n \]

\[ 0 = \frac{\partial}{\partial s_1} \left( \frac{\mathcal{V}_0 \tau_n}{\tau_n} \right) \]
\[ = \left( -\sum_{i \geq 1} (is_i \frac{\partial^2}{\partial s_1 \partial s_i} - it_i \frac{\partial^2}{\partial s_1 \partial t_i}) - \frac{\partial}{\partial s_1} \right) \log \tau_n \]

(4.0.8)
For the sake of this proof, consider the
locus \( \mathcal{L} = \{ \text{all } t_i = s_i = 0, \text{except } t_1, s_1 \neq 0 \} \).

From (4.0.7), we have on \( \mathcal{L} \),
\[
\frac{V_0 \tau_n}{\tau_n} \bigg|_{\mathcal{L}} = \left( t_1 \frac{\partial}{\partial t_1} - s_1 \frac{\partial}{\partial s_1} \right) \log \tau_n \bigg|_{\mathcal{L}} = 0,
\]
implying \( \tau_n(t, s) \bigg|_{\mathcal{L}} \) is a function of \( x := -t_1 s_1 \) only. Therefore we may write
\( \tau_n \bigg|_{\mathcal{L}} = \tau_n(x) \), and so, along \( \mathcal{L} \), we have
\[
\frac{\partial}{\partial t_1} = -s_1 \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial s_1} = -t_1 \frac{\partial}{\partial x}, \quad \frac{\partial^2}{\partial t_1 \partial s_1} = -\frac{\partial}{\partial x} \frac{\partial}{\partial x}.
\]
Setting
\[
f_n(x) = \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \log \tau_n(x) = -\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_n(t, s) \bigg|_{\mathcal{L}},
\]
and using \( x = -t_1 s_1 \), the two-Toda relation (4.0.6) takes on the form
\[
s_1 \frac{\partial^2}{\partial s_2 \partial t_1} \bigg|_{\mathcal{L}} = s_1 \left( 2 b_{n}^{*} \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n - \frac{\partial}{\partial s_1} \left( \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n \right) \right)
\]
\[
= x \left( 2 t_1 f_n + f_n' \right).
\] (4.0.9)

Setting this relation (4.0.9) into the Virasoro relations (4.0.7) and (4.0.8), we have
\[
0 = \frac{V_0 \tau_n}{\tau_n} - \frac{V_0 \tau_{n-1}}{\tau_{n-1}} \bigg|_{\mathcal{L}} = \left( t_1 \frac{\partial}{\partial t_1} - s_1 \frac{\partial}{\partial s_1} \right) \log \frac{\tau_n}{\tau_{n-1}} \bigg|_{\mathcal{L}} = t_1 b_n + s_1 b_{n}^{*}
\] (4.0.10)
\[
0 = \left( -s_1 \frac{\partial^2}{\partial s_2 \partial t_1} + n \frac{\partial^2}{\partial t_1 \partial s_1} \right) \log \tau_n \bigg|_{\mathcal{L}} + n
\]
\[
= -x \left( 2 \frac{b_{n}^{*}}{t_1} f_n(x) + f_n'(x) \right) + n(-f_n(x) + 1).
\] (4.0.11)

This is a system of two linear relations (4.0.10) and (4.0.11) in \( b_n \) and \( b_{n}^{*} \), whose solution, together with its derivatives, are given by:
\[
b_{n}^{*} = -\frac{b_n}{s_1} = -n(f_n - 1) + x f_n', \quad \frac{\partial b_n}{\partial s_1} = \frac{\partial}{\partial x} \frac{b_n}{s_1} = \frac{x(f_n f_n'' - f_n^2) + (f_n + n)f_n'}{2 f_n^2}.
\]
Setting $\partial^2 \log \tau_n / \partial s_1 \partial t_1 = -f_n$ into the Toeplitz relation (4.0.5) yields
\[ b_n b_n^* = (1 - f_n) \left( 1 - f_n - \frac{\partial}{\partial s_1} b_n \right), \]
which, using the expressions above for $b_n$, $b_n^*$ and $\partial b_n / \partial s_1$, yields the differential equation:
\[
f''_n - \frac{1}{2} f'^2_n \left( \frac{1}{f_n - 1} + \frac{1}{f_n} \right) + \frac{1}{x} f'_n + \frac{n^2 (-f_n + 1)}{2x^2 f_n} - \frac{2}{x} f_n (-f_n + 1) = 0.
\]
(4.0.12)

Note, along the locus $\mathcal{L}$, we may set $t_1 = \sqrt{x}$ and $s_1 = -\sqrt{x}$, since it respects $t_1 s_1 = -x$. Thus,
\[ f_n(x) = \frac{d}{dx} x \frac{d}{dx} \log \tau_n(x), \]
with
\[ \tau_n(t, s) \big|_{\mathcal{L}} = \int_{U(n)} e^{\text{Tr}(t_1 M - s_1 \bar{M})} dM \bigg|_{\mathcal{L}} = \int_{U(n)} e^{\sqrt{x} \text{Tr}(M + \bar{M})} dM, \]
satisfies (4.0.12). The behavior of $f_n(x)$ near $x = 0$ is given by Proposition 1.5 and the above formula, with the uniqueness established as in the orthogonal case, proving Proposition 4.1.

Remark: Setting
\[ f_n(x) = \frac{w(x)}{w(x) - 1} \]
leads to standard Painlevé V, with $\alpha = \delta = 0$, $\beta = -n^2/2$, $\gamma = -2$.

Proof of Propositions 4.2: For fixed $k \in \mathbb{R}$, $k \neq 0$, consider the locus
\[ \mathcal{L} = \{ \text{all } it_i = -k(-1)^i \text{ and } s_i = 0, \text{ except } s_1 = x \}. \]
Then, setting
\[ f_n(x) = \frac{\partial}{\partial t_1} \log \tau_n \bigg|_{\mathcal{L}}, \]
(4.0.13)
the Toda relations (4.0.6) become:

$$-x \frac{\partial^2}{\partial s_2 \partial t_1} \log \tau_n \bigg|_L = -2xb_n^* \frac{\partial^2}{\partial s_1 \partial t_1} \log \tau_n + x \frac{\partial^3}{\partial s_1^2 \partial t_1} \log \tau_n$$

$$= -2xb_n^* f'_n + xf_n''.$$  (4.0.14)

The Virasoro relations (4.0.8) become, using (4.0.3) and the locus,

$$0 = \left( \frac{(V_0 + V_1)n}{\tau_n} - \frac{(V_0 + V_1)n-1}{\tau_{n-1}} \right) \bigg|_L$$

$$= \left( -x \frac{\partial}{\partial s_1} + (n + k) \frac{\partial}{\partial t_1} \right) \log \tau_n + nx$$

$$- \left( -x \frac{\partial}{\partial s_1} + (n - 1 + k) \frac{\partial}{\partial t_1} \right) \log \tau_{n-1} - (n - 1)x$$

$$= \left( -x \frac{\partial}{\partial s_1} + (n + 1) \frac{\partial}{\partial t_1} \right) \log \frac{\tau_n}{\tau_{n-1}} + \frac{\partial}{\partial t_1} \log \tau_n + x$$

$$= -x \frac{\partial}{\partial s_1} \log \frac{\tau_n}{\tau_{n-1}} + (n + 1) \frac{\partial}{\partial t_1} \log \frac{\tau_n}{\tau_{n-1}} + f_n + x$$

$$= x(b_n^* + (n + 1)b_n^*) + f_n + x.$$  (4.0.15)

and, using (4.0.8) and (4.0.14)

$$0 = \frac{\partial}{\partial t_1} \left( \frac{(V_{n-1} + V_0)n}{\tau_n} \right) \bigg|_L$$

$$= \left( \frac{\partial}{\partial t_1} - x \frac{\partial^2}{\partial t_1 \partial s_2} + (n - x) \frac{\partial^2}{\partial s_1 \partial t_1} \right) \log \tau_n + n = 0$$

$$= f_n + (n - x)f'_n + n - 2xb_n^* f'_n + xf_n''.$$  (4.0.16)

So, as before, we have a linear system in $b_n$ and $b^*_n$ whose solution is

$$b_n = \frac{xf''_n + f'_n(2f_n + x + n) + f_n + n}{2f'_n(n + k - 1)}$$

$$b^*_n = \frac{xf''_n - f'_n(x - n) + f_n + n}{2xf'_n}.$$  

Substituting this solution into the Toeplitz relation (4.0.5):

$$b_n b^*_n = (1 + f'_n)(1 + f'_n - \frac{\partial}{\partial x} b_n)$$
yields
\[
f''''_n - \frac{1}{2} \left( \frac{1}{f''_n} + \frac{1}{f'_n + 1} \right) f'''_n + \frac{f''''_n}{x} + \frac{2(n + k)}{x} f'_n (f'_n + 1)
- \frac{1}{2x^2 f'_n (f'_n + 1)} \left( (x - n) f'_n - f''_n - n \right) \left( (2f'_n + x + n) f'_n + f'_n + n \right) = 0.
\]

It remains to compute \( f_n(x) \) as in (4.0.13). Note that
\[
\tau_n(x) := \tau_n(t, s) \bigg|_{L} = \int_{U(n)} e^{\text{Tr} \sum_{i}^{\infty} (t, M_i - s_i, \bar{M}_i)} \frac{dM}{\text{det}(I + M)^k e^{-x \text{Tr} M} dM}.
\]

Therefore
\[
f_n(x) = \frac{\partial}{\partial t_1} \log \tau_n \bigg|_{L}
= \int \text{Tr} M e^{\text{Tr} \sum_{i}^{\infty} (t, M_i - s_i, \bar{M}_i)} \frac{dM}{\text{det}(I + M)^k e^{-x \text{Tr} M} dM}
= \frac{\int \text{Tr} M \det(I + M)^k e^{-x \text{Tr} M} dM}{\int \det(I + M)^k e^{-x \text{Tr} M} dM}
= \frac{1}{n + k} x \frac{d}{dx} \log \tau_n e^{-nx}.
\]

This last equality \( \equiv \) will be shown later in Lemma 4.3. To conclude the proof of Proposition 4.2, observe from (4.0.17) and Proposition 1.5, that
\[
f_n(x) = \frac{1}{n + k} \left( \frac{x}{dx} \log \int_{U(n)} \det(I + M)^k e^{-x \text{Tr} M} dM - nx \right)
= \frac{x k - n}{k + n} - \frac{x^{n+1}}{(n + 1)!} \binom{k + n - 1}{n} + O(x^{n+2}),
\]
concluding the proof of Proposition 4.2.
Proof of Theorem 0.1: Upon integrating the expressions (4.0.1) and (4.0.2) and exponentiating, one finds the expressions (ii) and (iii) of Theorem 0.1, upon using respectively the initial conditions (0.0.3) and the first identity of Lemma 4.3.

Recall equality (4.0.17) (\(\ast\)) still needed proof:

\[ \tau_n(0) = \int_{U(n)} \det(I+M)^k dM = 1, \quad \frac{\partial \tau_n}{\partial t_1}(0) = \int_{U(n)} \text{Tr} M \det(I+M)^k dM = 0 \]

and

\[
\frac{\int \text{Tr} M \det(I + M)^k e^{-x \text{Tr } \bar{M}} dM}{\int \det(I+M)^k e^{-x \text{Tr } \bar{M}} dM} = \frac{1}{n+k} x \frac{d}{dx} \log e^{-nx} \int_{U(n)} \det(I + M)^k e^{-x \text{Tr } \bar{M}} dM
\]

\[
= -x \left( \frac{\int \text{Tr} \bar{M} \det(I + M)^k e^{-x \text{Tr } \bar{M}} dM}{\int \det(I+M)^k e^{-x \text{Tr } \bar{M}} dM} + n \right) . \tag{4.0.18}
\]

Proof: Recall from (4.0.17) that \(f(x)\) is the left hand side of (4.0.18). At first we show, using the Toeplitz matrix representation (2.3.11) in the third identity, that \(f(0) = 0\); indeed,

\[
\begin{align*}
\tau_n(0) f(0) &= \int_{U(n)} \text{Tr} M \det(I + M)^k dM \\
&= \frac{d}{d\varepsilon} \left. \int \det(I + M)^k \det(I + \varepsilon M) dM \right|_{\varepsilon = 0} \\
&= \frac{d}{d\varepsilon} \det \left( \int_{S^1} z^{\ell-m} (1 + z)^k (1 + \varepsilon z) \frac{dz}{2\pi i z} \right)_{0 \leq \ell, m \leq n-1} \bigg|_{\varepsilon = 0} \\
&= \frac{d}{d\varepsilon} \det \begin{pmatrix} 1 & * \\ O & 1 \end{pmatrix} \bigg|_{\varepsilon = 0} \\
&= \frac{d}{d\varepsilon} (1) = 0. \tag{4.0.19}
\end{align*}
\]
The equality $\ast$ is due to the fact that
\[
\int_{S^1} z^{\ell-m}(1+z)^k(1+\varepsilon z) \frac{dz}{2\pi i z} = 0 \quad \text{for } \ell - m \geq 1
\]
\[= 1 \quad \text{for } \ell = m.
\]
The same, but even simpler argument shows $\tau_n(0) = 1$, by replacing $1 + \varepsilon z$ by 1 in (4.0.19). From (4.0.8), it also follows that
\[
0 = \left. \frac{\partial}{\partial s_1} (\cal{V}_0 + \cal{V}_1) \frac{\tau_n}{\tau_n} \right|_\mathcal{L}
\]
\[
= \left. (n+k) \frac{\partial^2}{\partial s_1 \partial t_1} - s_1 \frac{\partial^2}{\partial s_1^2} - \frac{\partial}{\partial s_1} \right|_\mathcal{L} \log \tau_n + n
\]
\[
= \left. (n+k) \frac{\partial f}{\partial x} - \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \log \tau_n \right|_\mathcal{L} + n
\]
\[
= \left. (n+k) \frac{\partial f}{\partial x} - \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \log \tau_n e^{-nx} \right|_\mathcal{L}.
\]
Integrating this expression from 0 to $x$ yields
\[
(n+k)(f(x) - f(0)) = x \frac{\partial}{\partial x} \log \tau_n e^{-nx};
\]
the fact that $f(0) = 0$ establishes the first identity of (4.0.18). The second identity of (4.0.18) follows from
\[
f(x) = \frac{1}{n+k} x \frac{d}{dx} \log e^{-nx} \int_{U(n)} \det(I + M)^k e^{-x \Tr M} dM
\]
\[
= -\frac{x}{n+k} \left( \frac{\int \Tr M \det(I + M)^k e^{-x \Tr M} dM}{\int \det(I + M)^k e^{-x \Tr M} dM} + n \right),
\]
ending the proof of Lemma 4.3.

5 Appendix 1: Virasoro algebras

In [3], we defined a Heisenberg and Virasoro algebra of vector operators $\beta \mathcal{J}_k^{(i)}$, depending on a parameter $\beta > 0$:
\[
(\beta \mathcal{J}_k^{(1)})_n = \beta J_k^{(1)} + n J_k^{(0)} \quad \text{and} \quad (\mathcal{J}_k^{(0)})_n = n J_k^{(0)} = n \delta_{0k},
\]
57
and

\[ \beta J_k^{(2)}(\beta) = \frac{\beta}{2} \sum_{i+j=k} : \beta J_i^{(1)} \beta J_j^{(1)} : + \left( 1 - \frac{\beta}{2} \right) \left( (k+1) \beta J_k^{(1)} - k \beta J_k^{(0)} \right) \]

\[ = \left( \frac{\beta}{2} J_k^{(2)} + \left( n\beta + (k+1)(1 - \frac{\beta}{2}) \right) \beta J_k^{(1)} + \frac{n((n-1)\beta + 2)}{2} J_k^{(0)} \right)_{n \in \mathbb{Z}}. \]  

(5.0.1)

The \( \beta J_k^{(2)} \)'s satisfy the commutation relations: (see [3])

\[
\begin{align*}
\left[ \beta J_k^{(1)} , \beta J_\ell^{(1)} \right] &= \frac{k}{\beta} \delta_{k,-\ell} \\
\left[ \beta J_k^{(2)} , \beta J_\ell^{(1)} \right] &= -\ell \beta J_{k+\ell}^{(1)} + k(k+1) \left( \frac{1}{\beta} - \frac{1}{2} \right) \delta_{k,-\ell} \\
\left[ \beta J_k^{(2)} , \beta J_\ell^{(2)} \right] &= (k-\ell) \beta J_{k+\ell}^{(2)} + c \left( \frac{k^3 - k}{12} \right) \delta_{k,-\ell},
\end{align*}
\]

(5.0.2)

with central charge

\[ c = 1 - 6 \left( \left( \frac{\beta}{2} \right)^{1/2} - \left( \frac{\beta}{2} \right)^{-1/2} \right)^2. \]

In the expressions above,

\[
\begin{align*}
\beta J_k^{(1)} &= \frac{\partial}{\partial t_k} \text{ for } k > 0 \\
&= \frac{1}{\beta}(-k)t_{-k} \text{ for } k < 0 \\
&= 0 \text{ for } k = 0 \\
\beta J_k^{(2)} &= \sum_{i+j=k} \frac{\partial^2}{\partial t_i \partial t_j} + \frac{2}{\beta} \sum_{-i+j=k} it_i \frac{\partial}{\partial t_j} + \frac{1}{\beta^2} \sum_{-i-j=k} it_i jt_j \text{ (5.0.3)}
\end{align*}
\]
In particular, for $\beta = 1$ and 2, the $J_{k}^{(2)}$ take on the form:

$$J_{k}^{(2)}(t)|_{\beta=1} = \frac{1}{2} \left( \beta J_{k}^{(2)} + (2n + k + 1) \beta J_{k}^{(1)} + n(n+1)J_{k}^{(0)} \right)_{n \in \mathbb{Z}},$$

$$J_{k}^{(2)}(t)|_{\beta=2} = \left( \beta J_{k}^{(2)} + 2n \beta J_{k}^{(1)} + n^2 J_{k}^{(0)} \right)_{n \in \mathbb{Z}}. \quad (5.0.4)$$

### 6 Appendix 2: Chazy classes

Given arbitrary polynomials $P(z), Q(z), R(z)$ of degree 3, 2, 1 respectively, Cosgrove [9], (A.3), shows that the following third order equation

$$f''' + \frac{P'}{P} f'' + \frac{6}{P} f'^2 - 4 \frac{P'}{P^2} f f' + \frac{P''}{P^2} f^2 - \frac{2Q'}{P^2} f + \frac{2R}{P^2} = 0 \quad (6.0.1)$$

has a first integral, which is second order in $f$ and quadratic in $f''$,

$$f'' + \frac{4}{P^2} \left( (P f'^2 + Q f' + R)f' - (P' f'^2 + Q' f' + R')f \right) + \frac{1}{2} \left( P'' f'^2 + Q'' f^2 - \frac{1}{6} P''' f^3 + c \right) = 0; \quad (6.0.2)$$

c is the integration constant. This is a master Painlevé equation, containing the 6 Painlevé equations. When the polynomials $P, Q$ and $R$ have the following form

$$P = x, \quad Q = -\frac{a_1}{4} x^2, \quad R = -\frac{1}{4} (a_2 x + a_3),$$

then equation (6.0.2) can be reduced to the Painlevé V equation [10], p. 70:

$$w''' = \left( \frac{1}{2w} + \frac{1}{w-1} \right) w'^2 - \frac{1}{x} w' + \frac{(w - 1)^2}{x^2} \left( \alpha w + \beta \right) + \frac{\gamma w}{x} + \frac{\delta w(w + 1)}{w - 1},$$

with

$$a_1 = -2\delta$$

$$a_2 = \frac{1}{4} \gamma^2 + 2\beta \delta - \delta (1 - \sqrt{2\alpha})^2$$

$$a_3 = \beta \gamma + \frac{1}{2} \gamma (1 - \sqrt{2\alpha})^2$$

$$c = -\frac{1}{32} \gamma^2 ((1 - \sqrt{2\alpha})^2 - 2\beta) + \frac{1}{32} \delta ((1 - \sqrt{2\alpha})^2 + 2\beta)^2.$$
7 Appendix 3: The volume of the orthogonal and symplectic groups

Selberg’s integral (see Mehta [15], p 340), renormalized over \([-1,1]^{n}\),

\[
\int_{[-1,1]^{n}} \Delta_{n}(x)^{2\gamma} \prod_{j=1}^{n} (1-x_{j})^{\alpha}(1+x_{j})^{\beta} dx_{j} 
\]

\[
= 2^{n(\alpha+\beta+\gamma(n-1)+1)} \prod_{j=0}^{n-1} \frac{\Gamma(\alpha+j\gamma+1)\Gamma(\beta+j\gamma+1)\Gamma(\gamma+j)}{\Gamma(\gamma+1)\Gamma(\alpha+\beta-\gamma+\gamma(n+j)+2)} 
\]

\[
= 2^{n(n+\alpha+\beta)} \prod_{j=1}^{n} \frac{j! \Gamma(j+\alpha)\Gamma(j+\beta)}{\Gamma(n+j+\alpha+\beta)}, \text{ upon setting } \gamma = 1,
\]

leads to the value of \(c_{2n}^{\pm}\) and \(c_{2n-1}^{\pm}\) in Theorem 1.1:

\[
\begin{align*}
&\bullet \alpha = -\beta = \pm \frac{1}{2} \quad \longrightarrow \quad \int_{O(2n+1)_{\pm}} dM = 2^{n^{2}} \prod_{j=1}^{n} \frac{j!(j-1/2)\Gamma^{2}(j-1/2)}{(n+j-1)!} \\
&\bullet \alpha = \beta = -\frac{1}{2} \quad \longrightarrow \quad \int_{O(2n)_{+}} dM = 2^{n(n-1)} \prod_{j=1}^{n} \frac{j!\Gamma^{2}(j-1/2)}{(n+j-2)!} \\
&\bullet \alpha = \beta = \frac{1}{2}, \quad n \mapsto n - 1 \quad \longrightarrow \quad \int_{O(2n)_{-}} dM = 2^{n(n-1)} \prod_{j=1}^{n-1} \frac{j!\Gamma^{2}(j+1/2)}{(n+j-1)!}.
\end{align*}
\]

8 Appendix 4: Direct evaluation of integrals over the orthogonal group and their derivatives at \(x = 0\)

Refering to Theorem 3.1, formulae (3.1.3) and (3.1.4), we evaluate \(d/dx \log \tau_{n}(x)\) and \(d^{2}/dx^{2} \log \tau_{n}(x)\) directly from the integral representation, not using the combinatorial interpretation of the integrals. To do this, we need the Aomoto extension [6] (see Mehta [15], p. 340) of Selberg’s integral:  

\[\text{where } Re \alpha, Re \beta > -1, Re \gamma > - \min \left(\frac{1}{n}, \frac{Re \alpha + 1}{n - 1}, \frac{Re \beta + 1}{n - 1}\right)\]
\[ \langle x_1 \ldots x_m \rangle := \frac{\int_0^1 \ldots \int_0^1 x_1 \ldots x_m |\Delta(x)|^{2\gamma} \prod_{j=1}^n x_j^\alpha (1-x_j)^\beta \, dx_1 \ldots dx_n}{\int_0^1 \ldots \int_0^1 |\Delta(x)|^{2\gamma} \prod_{j=1}^n x_j^\alpha (1-x_j)^\beta \, dx_1 \ldots dx_n} = \prod_{j=1}^m \frac{\alpha + 1 + (n-j)\gamma}{\alpha + \beta + 2 + (2n-j-1)\gamma}. \] (8.0.1)

In particular, setting \( \gamma = 1 \), formula (8.0.1) implies

\[ \langle x_1 \rangle = \frac{n + \alpha}{2n + \beta + \alpha} \quad \text{and} \quad \langle x_1 x_2 \rangle = \frac{(n + \alpha - 1)(n + \alpha)}{(2n + \beta + \alpha - 1)(2n + \beta + \alpha)}. \] (8.0.2)

and from the identity (see [13], p. 349)

\[ (2n + \beta + \alpha + 1) \langle x_1^2 \rangle = (2n + \alpha) \langle x_1 \rangle - (n - 1) \langle x_1 x_2 \rangle, \]

we derive

\[ \langle x_1^2 \rangle = \frac{(n + \alpha)(3n^2 + 2\beta n + 3\alpha n + \alpha \beta + \alpha^2 - 1)}{(2n + \beta + \alpha - 1)(2n + \beta + \alpha)(2n + \beta + \alpha + 1)}. \] (8.0.3)

We now consider the following ratio of integrals: (remember \( \rho_{\alpha\beta}(z) := (1-z)^{\alpha}(1+z)^{\beta} \))

\[ \langle y_1 \ldots y_m \rangle_{[-1,1]} := \frac{\int_{[-1,1]^n} y_1 \ldots y_m \Delta_n(y)^2 \prod_{k=1}^n \rho_{(\alpha,\beta)}(y_k) \, dy_k}{\int_{[-1,1]^n} \Delta_n(y)^2 \prod_{k=1}^n \rho_{(\alpha,\beta)}(y_k) \, dy_k}. \] (8.0.4)

The relationship between the two integrals (8.0.1) and (8.0.4) is obtained by setting \( x_j = \frac{1-y_j}{2} \); so, we have

\[ \langle x_1 \rangle = \frac{1}{2}(1-\langle y_1 \rangle), \quad \langle x_1 x_2 \rangle = \frac{1}{4}(1-2\langle y_1 \rangle + \langle y_1 y_2 \rangle), \quad \langle x_1^2 \rangle = \frac{1}{4}(1-2\langle y_1 \rangle + \langle y_1^2 \rangle). \]

So, these relations, upon using (8.0.2) and (8.0.3) and upon setting \( \alpha = \frac{a+b}{2}, \beta = \frac{a-b}{2} \), yield

\[ \langle y_1 \rangle = \frac{-b}{a+2n}, \quad \langle y_1 y_2 \rangle = \frac{b^2 - a - 2n}{(a+2n-1)(a+2n)}. \] (8.0.5)
\[ \langle y_1^2 \rangle = \frac{b^2(a + n) + n(a + 2n)^2 - (a + 2n)}{(a + 2n - 1)(a + 2n)(a + 2n + 1)}. \]

Hence, setting

\[ I_n^{(\alpha, \beta)}(x) := \int_{[-1, 1]^n} \Delta_n(z)^2 \prod_{1}^{n} e^{2\pi z_k \rho_{\alpha, \beta}(z_k)} dz_k, \]

we compute for future use:

\[ \gamma(n) := 2 \left. \frac{I''_n}{I_n} \right|_{x=0} = 8 \langle \left( \sum_{1}^{n} y_i \right)^2 \rangle = 8 \left( n \langle y_1^2 \rangle + n(n - 1) \langle y_1 y_2 \rangle \right) = 8n \left( \langle y_1^2 \rangle + (n - 1) \langle y_1 y_2 \rangle \right) = 8n \frac{(a + 2n)(b^2n + a + n) - b^2}{(a + 2n - 1)(a + 2n)(a + 2n + 1)}. \] (8.0.6)

Note this formula applies to a general \( I_n^{(\alpha, \beta)}(x) \), where a combinatorial interpretation is absent. These considerations will now be applied to the orthogonal case. Indeed, considering the special values of \( \alpha \) and \( \beta \) and thus for \( a \) and \( b \), we evaluate:

- \( a = -1, \ b = 0 : \ \gamma(n) = 2 \)
- \( a = 1, \ b = 0 : \ \gamma(n) = 2 \)
- \( a = 0, \ b = 1 : \ \gamma(n) = 4 \)
- \( a = 0, \ b = -1 : \ \gamma(n) = 4 \).

It is easily seen that

\[ \left( x \frac{d}{dx} \log \int e^{x \text{Tr} M \text{d}M} \right)' = 2 \left( \int e^{x \text{Tr} M \text{d}M} \right)' - 2 \left( \frac{\int e^{x \text{Tr} M \text{d}M}}{\int e^{x \text{Tr} M \text{d}M}} \right)^2 + O(x) \]
and so, using (3.2.4) and the fact that the volume \( \int dM \) does not vanish,

\[
\left( \frac{d}{dx} \log \int e^{x \text{Tr} M} dM \right)^{\prime\prime} \bigg|_{x=0} = 2 \left( \frac{\int e^{x \text{Tr} M} dM}{\int e^{x \text{Tr} M} dM} \right)^{\prime\prime} \bigg|_{x=0}
\]

Using \( f'(0) = 0 \) to evaluate \( I'_n(0) \) below and using (1.0.2) and (8.0.6), we now verify in each of the cases:

\[
f''_{2n-1}(0) = \left( \frac{d}{dx} \log \int_{O(2n)} e^{x \text{Tr} M} dM \right)^{\prime\prime} \bigg|_{x=0}
\]

\[
= 2 \left( \frac{I''_{n}(-\frac{1}{2},-\frac{1}{2})}{I_{n}} \right) \bigg|_{x=0} = \gamma(n) |_{a=-1,b=0} = 2
\]

\[
f''_{2n-1}(0) = \left( \frac{d}{dx} \log \int_{O(2n)} e^{x \text{Tr} M} dM \right)^{\prime\prime} \bigg|_{x=0}
\]

\[
= 2 \frac{I_{n-1}^{(\frac{1}{2},\frac{1}{2})}}{I_{n-1}} = \gamma(n-1) |_{a=1,b=0} = 2
\]

\[
f''_{2n}(0) = \left( \frac{d}{dx} \log \int_{O(2n+1)} e^{x \text{Tr} M} dM \right)^{\prime\prime} \bigg|_{x=0}
\]

\[
= 2 \left( \frac{e^{x} I_{n}^{(\frac{1}{2},-\frac{1}{2})}}{e^{x} I_{n}^{(\frac{1}{2},-\frac{1}{2})}} \right) \bigg|_{x=0}
\]

\[
= 2 \left( \frac{I_{n}'' + 2 I_{n}' + I_{n}}{I_{n}} \right) \bigg|_{x=0}
\]

\[
= 2 \left( \frac{I_{n}''}{I_{n}} - 1 \right), \text{ using } (e^{x} I_{n}(x))' \bigg|_{x=0} = I_{n}'(0) + I_{n}(0) = 0
\]

\[
= -2 + \gamma(n) |_{a=0,b=-1} = 2
\]

\[
f''_{2n}(0) = \left( \frac{d}{dx} \log \int_{O(2n+1)} e^{x \text{Tr} M} dM \right)^{\prime\prime} \bigg|_{x=0}
\]

\[
= 2 \left( \frac{e^{-x} I_{n}^{(-\frac{1}{2},\frac{1}{2})}}{e^{-x} I_{n}^{(-\frac{1}{2},\frac{1}{2})}} \right) \bigg|_{x=0}
\]

\[
= 2 \left( \frac{I_{n}'' - 2 I_{n}' + I_{n}}{I_{n}} \right) \bigg|_{x=0}, \text{ using } (e^{-x} I_{n}(x))' \bigg|_{x=0} = 0
\]

\[
= 2 \left( \frac{I_{n}''}{I_{n}} - 1 \right) = -2 + \gamma(n) \bigg|_{a=0,b=-1} = 2
\]
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