DUALITY IN LINEAR PROGRAMMING: FROM TRICHOTOMY TO QUADRICHOTOMY

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Abstract. In this paper, we present a new approach to the duality of linear programming. We extend the boundedness to the so called inclusiveness, and show that inclusiveness and feasibility are a pair of coexisting and mutually dual properties in linear programming: one of them is possessed by a primal problem if and only if the other is possessed by the dual problem. This duality relation is consistent with the symmetry between the primal and dual problems and leads to a duality result that is considered a completion of the classical strong duality theorem. From this result, complete solvability information of the primal (or dual) problem can be derived solely from dual (or primal) information. This is demonstrated by applying the new duality results to a recent linear programming method.

1. Introduction. The duality theory of linear programming plays a very important role in the theory and applications of linear programming and in the development of computational methods for solving linear programming problems. The strong duality theorem, also known as the trichotomy theorem, is the most important theorem in the duality theory of linear programming ([1] and [5]). It tries to relate the solution status (having optimal solution, being unbounded, or being infeasible) of a primal to that of its dual. For a linear programming problem, its boundedness and feasibility properties determine its solution status as having optimal solution is equivalent to being feasible and bounded. Therefore, boundedness and feasibility are the most basic information that we need to determine in solving a linear programming problem. For this reason, we refer to the boundedness and feasibility properties of a linear programming problem as the solvability of the problem.

Solving a linear programming problem is mainly to find an optimal solution if optimal solutions exist or otherwise to identify whether the problem is unbounded or infeasible. Even though the most interesting case in solving a linear programming problem is when the problem has an optimal solution, the unbounded and infeasible cases may be important as well. In practice, a linear programming problem may become unbounded or infeasible due to modeling errors or data inaccuracy (see [2], [3] and [6], for example). In order to diagnose the cause for having no optimal solution, it is necessary to know whether the problem is infeasible (over constrained)

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or unbounded (under constrained). Certain algorithms try to solve a given linear programming problem by solving its corresponding dual problem. For this kind of algorithms, difficulty occurs when the dual problem is found to be infeasible. In this situation, the strong duality theorem tells us that the primal problem has no optimal solution, but it doesn’t tell whether the primal problem is infeasible or unbounded. In general, if the dual (or primal) problem is known to be infeasible, the strong duality theorem doesn’t tell whether the primal (or dual) problem is infeasible or unbounded. This means that the strong duality theorem doesn’t provide a ‘complete relation’ between the primal solvability and the dual solvability. On the other hand, since a primal problem and its dual problem share exactly the same set of known data, we naturally expect that their solvability should be equivalent or, in other words, mutually derivable. The above observations lead to the question: can the strong duality theorem be improved so that it represents a complete relation between the primal solvability and the dual solvability? The key to this question is to notice the ‘unbalanced’ effects of the boundedness and feasibility on linear programming problems: feasibility has a meaning for all problems, while boundedness makes sense only for feasible problems.

In this short paper, we extend the boundedness concept to the so called inclusiveness and present a basic duality relation between inclusiveness and feasibility. This leads to an improved strong duality theorem, called quadrichotomy theorem, in which the primal solvability (i.e. inclusiveness and feasibility) and the dual solvability are mutually derivable. The effect of the inclusiveness is shown by Figure 1. We note that the division of all linear programming problems into four classes using inclusiveness and feasibility, as shown in Figure 1, can be obtained by existing linear programming theory, but that can only be achieved by using both primal and dual solvability properties.

To facilitate further discussion, we consider a general linear programming problem in canonical form:

\[
\begin{align*}
\text{P: min } &\quad c^T x \\
\text{s. t. } &\quad Ax \leq b
\end{align*}
\]

where \( x = [x_1; x_2; \cdots; x_n] \in \mathbb{R}^n \) is the decision vector, \( c = [c_1; c_2; \cdots; c_n] \in \mathbb{R}^n \), \( c \neq 0 \) and \( b = [b_1; b_2; \cdots; b_m] \in \mathbb{R}^m \) are given vectors, and \( A \in \mathbb{R}^{m \times n} \) is a given \( m \times n \) matrix with \( i \)-th row \( a_i = [a_{i1}; a_{i2}; \cdots; a_{in}] \in \mathbb{R}^n \) for \( i = 1, 2, \cdots, m \) and \( m \geq n \).

Here, and in the remaining of this paper, we list the elements of a column vector in a pair of square brackets with entries separated by semicolons, and the elements of a row vector in a pair of square brackets with entries separated by commas. The \( m \times n \) matrix \( A \) with rows \( a_i, i = 1, 2, \cdots, m \), is denoted by \( A = [a_1; a_2; \cdots; a_m] \).

The dual of problem \( \text{P} \) can be written in the following form.

\[
\begin{align*}
\text{D: min } &\quad b^T \lambda \\
\text{s. t. } &\quad A^T \lambda = -c \\
&\quad \lambda \geq 0
\end{align*}
\]

The outline of this paper is as follows. In Section 2, the inclusiveness concept is introduced and a number of relevant results from \cite{4} is summarized. Results describing the relation between inclusiveness and boundedness are given in Section 2. Section 3 is devoted to a duality principle leading to a revised strong duality theorem. Before the final section where a brief summary of this paper is given, an application example is presented in Section 4.
2. Inclusiveness and feasibility. We start with the definition of inclusive normal cone which was first introduced in [4] for linear programming problems in the form of problem P (with only inequality constraints).

Definition 2.1. ([4]) Consider problem P as specified in Section 1. A cone generated by \( n \) linearly independent vectors \( a_{i1}^T, a_{i2}^T, \ldots, a_{in}^T \), where \( a_{ij} \) are rows of \( A \), is said to be an inclusive normal cone (or simply inclusive cone) if it contains the vector \(-c\) where \( c \in \mathbb{R}^n \) is the objective coefficient vector. The unique solution of the linear system

\[
a_{ij} x = b_{ij}, \quad j = 1, 2, \ldots, n
\]

is called the base of the inclusive cone.

The above definition of inclusive cone can be extended to problems allowing equality and/or inequality constraints. Consider a linear programming problem of \( n \) decision variables, \( k (\leq n) \) equality constraints, and \( m (\geq n - k) \) inequality constraints given in the following form:

G: \[
\begin{align*}
\text{min} & \quad \alpha^T x \\
\text{s. t.} & \quad Bx = \beta \\
& \quad Cx \leq \gamma
\end{align*}
\]
where $B$ is supposed to have full row rank $k$. Problem $G$ has an equivalent form given by

$$G': \begin{align*}
\min & \quad \alpha^T x \\
\text{s. t.} & \quad Bx \leq \beta \\
\text{s. t.} & \quad -Bx \leq -\beta \\
& \quad Cx \leq \gamma.
\end{align*}$$

Definition 2.2. An inclusive normal cone of problem $G'$ in the sense of Definition 2.1 is called an inclusive normal cone (or simply inclusive cone) of problem $G$.

Definition 2.3. A linear programming problem is said to be inclusive if it has an inclusive cone. We call a linear programming problem noninclusive if it is not inclusive.

We note that inclusive cone and inclusiveness can be defined for any linear programming problem. According to the above definitions, the dual problem $D$ is inclusive if and only if its equivalent problem $DP$ given below is inclusive.

$$DP: \begin{align*}
\min & \quad b^T \lambda \\
\text{s. t.} & \quad A^T \lambda \leq -c \\
& \quad -A^T \lambda \leq c \\
& \quad -I_{m \times m} \lambda \leq 0
\end{align*}$$

Theorems 2.4-2.6 in the following are from [4]. These results were initially given for problem $P$ but they have their trivial extensions to problems having both equality and inequality constraints. These results are helpful in understanding the duality between problems $P$ and $D$, and fundamental to the duality results of this paper.

Theorem 2.4. (optimality condition, [4]) A point $x^* \in \mathbb{R}^n$ is an optimal vertex solution of problem $P$ (or problem $G$) if and only if $x^*$ is the feasible base of some inclusive cone of problem $P$ (or problem $G$).

Theorem 2.5. ([4], the inclusive cone version of the fundamental theorem of linear programming) For an arbitrary linear programming problem in the form of problem $P$ or $G$, we have the following results:

1) If the problem has no optimal solution, then it is either infeasible or unbounded.

2) If the problem has an optimal solution, it must have an optimal solution at the base of some inclusive cone.

We note that this theorem was given for problem $P$ in [4] and therefore applies to problem $G'$. The theorem is then obviously true for problem $G$ due to the equivalence between problems $G$ and $G'$.

Theorem 2.6. The following properties hold true for problem $P$ and problem $G$.

a) Let $\bar{x}$ be the base of an inclusive cone. The objective value at $\bar{x}$ serves as a lower bound for the optimal value if the feasible region is non-empty. If $\bar{x}$ violates no constraint, then $\bar{x}$ is optimal.

b) If the feasible region of the problem is non-empty but there is no optimal solution, then the problem is inclusive.

c) If the problem is noninclusive, then it is either infeasible or unbounded.

d) If the problem is inclusive, then the following are true.
According to their inclusiveness and feasibility, both problem P and problem D can be divided into four cases: inclusive-feasible, inclusive-infeasible, noninclusive-feasible, and noninclusive-infeasible. Here, for example, the term inclusive-feasible means inclusive and feasible.

The following property shows how the classical solution status (bounded, unbounded or infeasible) are related to inclusiveness.

**Lemma 2.7.** Consider linear programming problem P or its dual problem D.

a) The problem allows an optimal solution if and only if it is inclusive-feasible.

b) It is unbounded if and only if it is noninclusive-feasible.

c) It is infeasible if and only if it is either inclusive-infeasible or noninclusive-infeasible.

**Proof.**

a). The necessity is immediately from Theorem 2.5. The sufficiency is directly from Theorem 2.6.e.1.

b). Let the problem be unbounded. Then it is feasible by definition. If the problem is also inclusive, then by a) it has an optimal solution, which contradicts the fact that the problem is unbounded. Thus, the problem is noninclusive-feasible.

Now let the problem be noninclusive-feasible. Being feasible means that the problem either is unbounded or has an optimal solution. However, having an optimal solution means that the problem is inclusive according to a). Therefore, the problem must be unbounded.

c). This is self-evident.

We can see from the above lemma that inclusiveness is a more general concept than boundedness. We also see that infeasible problems are further divided into inclusive-infeasible and noninclusive-infeasible problems.

**Example 1.** The problem given by

\[
\begin{align*}
\text{min} & \quad y \\
\text{s. t.} & \quad x - y \leq 0 \\
& \quad -x + y \leq -1 \\
& \quad -x - y \leq 0
\end{align*}
\]

is inclusive-infeasible. If its last constraint is replaced by \(x+y \leq 0\), then the problem becomes noninclusive-infeasible.

Regarding boundedness and inclusiveness, we have the following results.

**Lemma 2.8.** Problem P is inclusive if and only if at least one of its feasible subproblems obtained by dropping some of its constraints is bounded (or in other words, has an optimal solution).

**Proof.** Let the problem has a bounded subproblem. This means that one of the subproblems has an optimal solution. From Lemma 2.7, this subproblem is inclusive and hence has an inclusive cone. Since an inclusive cone for a subproblem obtained by dropping some constraints is also an inclusive cone for the original problem, it is clear that the original problem is inclusive. This proves the sufficiency. Now let the problem be inclusive. Then it has an inclusive cone according to the definition.
Dropping all constraints whose outward normals are not involved in the inclusive cone we obtain an inclusive and feasible subproblem, which according to Lemma 2.7 is bounded.

3. Duality. The following theorem shows that the feasibility and inclusiveness is pair of dual concepts.

**Theorem 3.1.** (A duality principle) Consider problem $P$ and its dual problem $D$ given in Section 1. The following statements are true.

a) Problem $P$ is inclusive if and only if problem $D$ is feasible.

b) Problem $P$ is feasible if and only if problem $D$ is inclusive.

**Proof.** a). Let problem $P$ be inclusive. Then there are $n$ independent row vectors of $A$, $a_{i1}, a_{i2}, \cdots, a_{in}$, and non-negative numbers $\lambda_{ij}$, $1 \leq j \leq n$ such that

$$\sum_{j=1}^{n} \lambda_{ij} a_{ij}^T = -c$$

Expend the $\lambda_{ij}$’s into an $m$-vector $\lambda = [\lambda_1, \lambda_2; \cdots, \lambda_m]$ such that $\lambda_k = 0$ for $1 \leq k \leq m$ and $k \neq i_j$ for $1 \leq j \leq n$. Then $\lambda$ is a feasible solution of problem $D$, and hence problem $D$ is feasible. On the other hand, if problem $D$ is feasible, then it has a basic feasible solution which clearly gives an inclusive cone for problem $P$. Hence problem $P$ is inclusive.

b) Since problem DP is in the form of problem $P$, apply a) to problem DP we see that problem DP is inclusive if and only if the standard form dual of problem DP is feasible. It is easy to find out that the dual of problem DP is

$$\min \begin{bmatrix} -c^T, c^T, 0_{1 \times m} \end{bmatrix} \begin{bmatrix} y_1; y_2; y_3 \end{bmatrix}$$

s. t. \[
\begin{bmatrix} A, -A, -I_{m \times m} \end{bmatrix} y = -b \\
y \leq 0
\]

which is equivalent to problem $P$ by letting $x = y_2 - y_1$ and removing $y_3$ by changing the equality constraints to inequality constraints. Especially, this equivalence ensures that the standard form dual of problem DP is feasible if and only if problem $P$ is feasible. Therefore, problem DP is inclusive if and only if problem $P$ is feasible. At the same time, by definition, problem DP is inclusive if and only if problem $D$ is inclusive. We see that b) follows.

**Theorem 3.2.** (A quadrichotomy theorem) For any linear programming problem $P$ with dual problem $D$, exactly one of the following is true.

a) Both problem $P$ and problem $D$ have optimal solutions, which is equivalent to each of the following.

1. Problem $P$ is inclusive-feasible.
2. Problem $D$ is inclusive-feasible.
3. Both problem $P$ and problem $D$ are inclusive-feasible.
4. Problem $P$ has an optimal solution.
5. Problem $D$ has an optimal solution.
6. Both problem $P$ and problem $D$ are inclusive.
7. Both problem $P$ and problem $D$ are feasible.

When both problem $P$ and problem $D$ have optimal solutions, their optimal values, denoted by $v^*(P)$ and $v^*(D)$ respectively, satisfy $v^*(P) = -v^*(D)$.

b) Problem $P$ is unbounded, which is equivalent to each of the following.
1. Problem P is noninclusive-feasible.
2. Problem D is inclusive-feasible.
3. Problem P is noninclusive and problem D is inclusive.
4. Problem P is feasible and problem D is infeasible.

1. Problem D is unbounded.
2. Problem D is noninclusive-feasible.
3. Problem P is inclusive and problem D is noninclusive.
4. Problem P is infeasible and problem D is feasible.

c) Problem P is inclusive-infeasible, which is equivalent to each of the following.
1. Problem D is unbounded.
2. Problem D is noninclusive-feasible.
3. Problem P is inclusive and problem D is noninclusive.
4. Problem P is infeasible and problem D is feasible.

d) Problem P is noninclusive-infeasible, which is equivalent to each of the following.
1. Problem D is noninclusive-infeasible.
2. Both problem P and problem D are noninclusive.
3. Both problem P and problem D are infeasible.
4. Both problem P and problem D are noninclusive-infeasible.

Proof. The results of the theorem can be directly obtained from Theorems 2.5–3.2 and Lemma 1. Here, as an example, we only prove a). In fact, the equivalence to a.3 is from Lemma 2.7. The equivalence of a.1, a.2, a.3, a.6, and a.7 is directly from Theorem 3.2. Lemma 2.7 implies the equivalence between a.1 and a.4 and between a.2 and a.5. To prove \( v^*(P) = -v^*(D) \), let \( x^* \) be an optimal solution of problem P achieved at the feasible base of an inclusive cone generated by \( n \) linearly independent vectors \( a_{i_1}, a_{i_2}, \ldots, a_{i_n} \). Let the dual feasible solution corresponding to this inclusive cone be denoted by \( \lambda^* \), then the components of \( \lambda^* \) satisfy

\[
\lambda^*_k = \begin{cases} 
\geq 0, & k = i_j, j = 1, 2, \ldots, n \\
0, & \text{otherwise}
\end{cases}
\]

Since

\[
[a_{i_1}; a_{i_2}; \ldots; a_{i_n}]x^* = [b_{i_1}; b_{i_2}; \ldots; b_{i_n}]
\]

and

\[
[a_{i_1}; a_{i_2}; \ldots; a_{i_n}]^T[\lambda^*_1; \lambda^*_2; \ldots; \lambda^*_n] = -c,
\]

we have

\[
v^*(P) = c^T x^* = -[\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_n][a_{i_1}; a_{i_2}; \ldots; a_{i_n}]x^* = \cdots = -b^T \lambda^* = -v^*(D).
\]

Now, the classical duality theorem of linear programming is a direct consequence of Theorem 3.2.

4. Application. As an application of the duality results of this paper, we consider the numerical solution of a linear programming problem. At the termination of any linear programming solver, the output should include one of the following three possible outcomes regarding the problem’s solvability: optimal solution found; the problem is unbounded; the problem is infeasible. However, even the most popular linear programming methods, including the dual simplex method (DSM), the interior point method (IPM), and the primal-dual method (PDM), are not always able
to determine whether a linear programming problem is infeasible or unbounded if it has no optimal solution. We demonstrate this by the following examples.

**Example 2.** The problem in Example 1 was solved using the MATLAB routine ‘linprog.m’ (MATLAB Version 7.5.0.342 (R2007b), Optimization Toolbox Version 3.1.2 (R2007b), with the default algorithm ‘large scale IPM’), the output is as below:

```matlab
>>[x,v]=linprog([0;1],[1 -1 1 1; -1 1 1; -1 -1; 0 -1; 0; 0],
    [0; -1; 0; 1; 0; 0; 0; 0])
Exiting: One or more of the residuals, duality gap, or total relative error has stalled:
both the primal and the dual appear to be infeasible.
```

**Example 3.** We solved the problem in Example 1 with its last constraint replaced by $x + y \leq 0$. This time, we used the routine ‘linprog.m’ in MATLAB Version 7.9.0.529 (R2009b), Optimization Toolbox Version 4.3 (R2009b), which uses the PDM as its default algorithm. The output is shown below.

```matlab
>>[x,v]=linprog([0;1],[1 -1; -1 1; 1 1; 0 -1; 0; 0; 0; 0],
    [0; -1; 0; 1; 0; 0; 0; 0])
Exiting: One or more of the residuals, duality gap, or total relative error has stalled:
the dual appears to be infeasible and the primal unbounded since the primal objective < -1e+10 and the dual objective < 1e+6.
```

**Example 4.** In this example, we solved the dual problem of the problem in Example 1. We used the same named routine ‘linprog.m’ in TOMLAB Version 7.2 CPLEX, which uses the CPLEX Dual Simplex LP solver. The output is

```matlab
>>[x,v]=linprog([0;-1;0],[1 -1 -1; -1 1 1; -1 1 1; -1 0 0; 0 -1 0; 0 -1 0; 0 0 0; 0 0 -1],
    [0;-1;0;1;0;0;0;0])
linprog (CPLEX): The problem is infeasible
```

The outputs for all three examples above are incorrect. In fact, one can easily see that the dual problem of the problem in Example 2 is feasible, the primal problem in Example 3 is infeasible, and the primal problem in Example 4 is unbounded.

These examples show that the major linear programming solvers all have difficulties in determining unboundedness and infeasibility in certain cases. With the duality theory of this paper and the inclusive normal cone based exterior climbing algorithm developed in [4], the existing difficulties shown above can be resolved. For instance, in order to solve a given primal problem, we can use the centered climbing ladder algorithm ([4]) to solve the corresponding dual problem (with inequality only constraints) and then apply the new duality result to obtain complete solvability information for the primal problem. The logical steps involved are as follows.

1. The algorithm starts by finding an initial inclusive cone for the dual problem. If there does not exist any initial inclusive cone, the dual problem is noninclusive. According to Theorem 3.2, the primal problem is infeasible.
2. If the dual problem is inclusive, the algorithm is capable of finding an initial inclusive cone. Starting from the initial inclusive cone, the centered climbing ladder algorithm can find a dual optimal solution if dual optimal solution exists. In this case, an optimal solution for the primal is immediately available.
3. Or, otherwise, the algorithm identifies that the dual problem is infeasible (and inclusive). Thus, according to Theorem 3.2, the primal problem is noninclusive-feasible, or equivalently, unbounded.
5. **Summary.** In this paper, we introduced the inclusiveness concept and presented some new and/or revised duality theorems including a duality relation between inclusiveness and feasibility and the quadrichotomy theorem. The later can be considered as a completion of the trichotomy (strong duality) theorem. An application of the current duality theorems is given.

**REFERENCES**

[1] M. C. Ferris, O. L. Mangasarian and S. J. Wright, “Linear Programming with MATLAB,” MPS-SIAM Series on Optimization, 7, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, Mathematical Programming, Society (MPS), Philadelphia, PA, 2007.

[2] H. J. Greenberg, *How to analyze the results of linear programs—part 3: Infeasibility diagnosis*, Interface, 23 (1993), 120–139.

[3] C. Li, X. He, B. Chen, Z. Gong, B. Chen and Q. Zhang, *Infeasibility diagnosis on the linear programming model of production planning in refinery*, Chinese J. Chem. Eng., 14 (2006), 569–573.

[4] Y. Liu, *An exterior point linear programming method based on inclusive normal cones*, Journal of Industrial and Management Optimization, 6 (2010), 825–846.

[5] D. G. Luenburg and Y. Ye, “Linear and Nonlinear Programming,” 3rd edition, International Series in Operations Research & Management Science, 116, Springer, New York, 2008.

[6] G. Roodman, *Post-infeasibility analysis in linear programming*, Management Science, 25 (1979), 916–922.

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