Can Low Mass Scalar Meson Nonet Survive in Large $N_c$ Limit?

Masayuki Uehara*
Takagise-Nishi 2-10-17, Saga 840-0921, Japan

March 25, 2022

Abstract

We study, within an approximate Inverse Amplitude Method to unitarize Chiral Perturbation Theory, whether low mass scalar mesons can survive in large $N_c$ regime, and show that vector mesons such as $\rho$ and $K^*$ survive as narrow width resonances, but all of the scalar meson nonet below 1GeV fade out as $N_c$ becomes large.

Recently Peláez [11] has obtained an interesting result that the complex poles on the second Riemann sheet, corresponding to the $\sigma$ and $\kappa$ states move away from the real axis. The study is performed within the Inverse Amplitude Method (IAM) to unitarize one-loop Chiral Perturbation Theory (ChPT) amplitudes $f_\pi$ [3] [5] [6]. It is well known [9, 10] that a planar amplitude of $q\bar{q}$ meson-meson scattering is of $O(N_c^{-1})$ and that widths of intermediate $q\bar{q}$ mesons are also of $O(N_c^{-1})$. Thus, the result is consistent with the common understanding that the members of the vector meson nonet including $\rho$ and $K^*$ are typical $q\bar{q}$ mesons. On the other hand the behavior of scalar mesons such as $\sigma$ and $\kappa$ is completely at variance with the nature expected for $q\bar{q}$ mesons.

Inspired by Peláez’s observation we study how amplitudes of two-NG boson scattering behave on the physical axis as $N_c$ increases within the approximate Oller-Oset-Peláez (OOP) version of IAM [7,8], and how complex poles of $f_0(980)$ and $a_0(980)$ move in an approximate manner. In the OOP version only polynomial terms with the low energy constants $L_n$’s (LEC’s) and $s$-channel loop terms are taken into account out of full $O(p^4)$ amplitudes.

In order to perform the study we have to find the explicit $N_c$ dependence in the scattering amplitudes. Since the pion decay constant $f_\pi$ is of $O(N_c^{-1/2})$ and the LEC’s are to be of $O(N_c^0)$, except for $L_4$ and $L_6$, both of which are of $O(1)$ [11] [12] [13], we fix the values of $L_n/f_\pi^2$ to those at $N_c = 3$ and put $f_\pi(N_c) = \sqrt{N_c/3} \times f_\pi(3)$ with being $f_\pi(3) = 93$ MeV. Indeed, the explicit large $N_c$ model calculations give the result that $L_n/f_\pi^2$ is $N_c$ independent for $n = 1$ to 8, including $L_7$ [12] [13]. $L_7$ is not of leading order, if we consider the $\eta' - \eta$ mixing [11], but we discard the $\eta' - \eta$ mixing and regard $L_7$ as $O(N_c)$ [12], though we use the empirical value for the $\eta$ mass.

The ingredients of IAM consist of the partial wave amplitudes with a definite isospin $I$ calculated in terms of the amplitudes of ChPT of $O(p^2)$ and $O(p^4)$. The former amplitude $T^{(2)}(s, t, u)$ is written $O(p^2/f_\pi^2)$, and then of $O(N_c^{-1})$. The latter $O(p^4)$ amplitudes have polynomial terms with LEC’s, which are written

$$T^{(4)}_{\text{poly}}(s, t, u) = \sum_{n=1,8} \frac{1}{f_\pi^2} \left( \frac{L_n}{f_\pi^2} \right) P_n(s, t, u),$$

where $P_n$’s are the second order polynomial terms of $s$, $t$, or $u$ and meson mass squared. The polynomial term $T^{(4)}$ is of $O(1/N_c)$, because we fix $L_n/f_\pi^2$’s to the values at $N_c = 3$. The $s$-channel loop terms given as $t^{(2)}(s)J(s)t^{(2)}(s)$ are proportional to $O(s^2/f_\pi^2)$, where $J(s)$ is the one-loop function regularized as the $\overline{MS}$ - 1 scheme at the renormalization scale $\mu$ [11], and $t^{(2)}$ is partial wave projected from $T^{(2)}(s, t, u)$. Since the $t$- and $u$-channel loop terms and tadpole terms are also of $O(p^4/f_\pi^2)$, they are of $O(1/N_c^2)$. Thus, the OOP version is expected to be more valid as $N_c$ becomes large, because while the polynomial terms with LEC’s are of $O(N_c^{-1})$, the discarded terms are of $O(N_c^{-2})$. The $s$-channel terms are crucial to

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*E-mail: ueharam@cc.saga-u.ac.jp
realize unitarity, though they are of $O(N_c^{-2})$. This different $N_c$ dependence produces different behavior of the amplitudes in the large $N_c$ regime. We note that IAM gives a complex pole at a reasonable point in meson-meson scattering amplitudes, but this does not mean the IAM can give a complete prediction on the existence or non-existence of a resonance, because the amplitude depends on the values of LEC’s, which are determined so as for the amplitudes to reproduce extensive low energy data including resonant behaviors or prominent structures. In this sense the study of the large $N_c$ behavior could provide a good laboratory to test the nature of hypothesized resonances.

The values of LEC’s, which reproduce phase shifts reasonably consistent with the experimental data, are summarized in Table I. These phenomenological values deviate somewhat from those of the large $N_c$ model calculations, but we use the above formulation of the $N_c$ dependence of the IAM amplitudes. We point out that Our set is not chosen as the best solution to the overall fitting with the existing data. (Ours set in this work is slightly different from that of Ref. 8.)

### Vector channels

At first, we discuss the behaviors of vector mesons in the single channel formalism. The simple reason why the phase shift of the $\rho$ or $K^*$ channels increases across $\pi/2$ is due to the fact that the real part of the denominator, $t^{(2)}(s) - t^{(4)}_{\text{poly}}(s) - t^{(2)}\text{Re}[J(s)]t^{(2)}$, develops a zero at the resonance position. We note that while the first and second terms behave as $O(1/f^2)$, the third term coming from the loop term does as $O(1/f^2)$. This implies that the zero almost does not vary depending on the value of $N_c$. The zero depends dominantly on the combination of LEC’s $2L_3 - L_2 + L_3$ as pointed out in Ref. 2. On the other hand the imaginary part $t^{(2)}\rho(s)t^{(2)}$ having a form $1/f^4$ becomes smaller and smaller as $N_c$ increases, where the phase space factor $\rho(s) = \text{Im} J(s) = k/(8\pi\sqrt{s})$ with $k$ being the CM momentum of two scattering mesons.

| $N_c$ | $L_1$ | $L_2$ | $L_3$ | $L_5$ | $L_7$ | $L_8$ |
|-------|-------|-------|-------|-------|-------|-------|
| Large $N_c$ | 0.79  | 1.58  | -3.17 | 0.43  | -0.42 | 0.46  |
| IAM I  | 0.56  | 1.21  | -2.79 | 1.4   | -0.44 | 0.78  |
| IAM III| 0.60  | 1.22  | -3.02 | 1.9   | -0.25 | 0.84  |
| Ours   | 0.55  | 0.98  | -2.95 | 0.81  | -0.25 | 0.49  |
| ChPT   | 0.52 ± 0.23 | 0.72 ± 0.24 | -2.70 ± 0.99 | 0.65 ± 0.12 | -0.26 ± 0.15 | 0.47 ± 0.18 |

Table 1: $L_n \times 10^3$: The set Large $N_c$ is taken from Ref. 12, the sets IAM I and III for IAM with the full $O(p^4)$ amplitudes are taken from Ref. 11, where $L_4 = -0.36$ and $L_9 = 0.07$ in I and $L_4 = 0$ and $L_9 = 0.07$ in III, the set Ours is used in this work and the set ChPT is the 2000 version of LEC’s taken from Ref. 14. Used are $L_4 = L_6 = 0$ in Large $N_c$, Ours and ChPT sets. The renormalization scale is $\mu = M_\rho$ except for Ours, which uses $\mu = 1.07$ GeV.

![Figure 1: $N_c$ dependence of phase shift (left) and cross section (right) of the $\rho$ channel. Lines correspond to $N_c = 3, 5, 10$ and 30 from the top to the bottom. The inclined dotted line (right) shows the kinematical limit of the cross section.](image-url)
Thus, the width decreases as like as $O(1/N_c)$, but the resonance position stays almost at the same point. This is the behavior on the physical real axis corresponding just to the behavior seen on the complex II sheet \cite{1}.

![Graphs showing phase shift and cross section]

Figure 2: $N_c$ dependence of phase shift (left) and cross section (right) of the $K^*$ channel. Lines correspond to $N_c = 3, 5, 10$ and $30$ from the top to the bottom. The inclined dotted line (right) shows the kinematical limit of the cross section.

If we consider the meson-meson scattering in the two channel formalism with the same LEC’s, the combination of LEC’s $2L_1 - L_2 + L_3$ develops the zero at almost the same point for $r_{11}$, $r_{12}$ and $r_{22}$, where $r_{ij} = r_{ij}^{(2)} - Re[r_{ij}^{(4)}]$, as pointed out in Ref. \cite{8}. Our LEC’s set gives unfortunately a little bit larger mass value for $\rho$, but smaller one for $K^*$. The octet component of the isoscalar vector meson $|V_8\rangle = 1/\sqrt{3} |\omega\rangle + 2/\sqrt{3} |\phi\rangle$ has a mass below the $K\bar{K}$ threshold both in our model and IAM with the full $T^{(4)}$ amplitudes \cite{5} as if it is a bound state with the decreasing residue as $O(1/N_c)$, so that if the pole shifts above the $K\bar{K}$ threshold, the shrinking width should occur.

Thus, we can conclude that the vector mesons explained by IAM have the nature consistent with the $q\bar{q}$ mesons.

**Scalar channels**

Now, we proceed to the studies on the scalar channels. 

$(I, J) = (0, 0)$

This channel contains the enigmatic $\sigma$ and the $f_0(980)$ states. Using the two-channel IAM formalism with the ($\pi\pi$) and ($K\bar{K}$) channels, we calculate the $N_c$ dependence of the phase shift and the cross section as shown in Fig. 3. We remember that the $f_0(980)$ state is generated as a bound state in the $K\bar{K}$ channel, where the kaon loop contribution is crucial \cite{5}. If $N_c$ increases, however, the real part of the $s$-channel loop term becomes small and the bound state pole shifts to the $K\bar{K}$ threshold and gets into the unphysical sheet through the cut starting with the $K\bar{K}$ threshold. In contrast to the vector channels the $N_c$ dependence of the phase shift and cross section shows drastic changes, therefore; the sharp rise of the phase shift near the $K\bar{K}$ threshold seen at $N_c = 3$ and $4$ disappears even at $N_c = 5$, and shows a cusp-like behavior, and then the phase shift and the cross section shrink to the null structure after the cusp behavior disappears.

Where does the $f_0(980)$ pole go to? We approximately calculate the pole position by expanding the amplitude in powers of $k_2$, the momentum of the $K\bar{K}$ channel, up to the first order, and observe that the pole moves into the upper (lower) half plane of the IV sheet from the lower (upper) half plane of the II sheet, winding around the branch point at $K\bar{K}$ threshold, and goes away from the real axis as shown in the left side of Fig. 4. Of course, positions at large $N_c$ are not so reliable owing to the rough approximation. No matter where the pole moves to, it holds valid that the physical trace of $f_0$ vanishes in the large $N_c$ limit.
Figure 3: $N_c$ dependence of phase shift (left) and cross section (right) of the (0,0) channel. Solid, dotted, dot-dot-dashed and dashed lines are for $N_c = 3, 5, 8$ and 12 respectively.

Figure 4: $N_c$ dependence of the $f_0(980)$ pole (left) and $a_0(980)$ pole (right). Both of the poles wind aroundthe branch point at $K\bar{K}$ threshold to go to the upper half plane of the IV sheet.

Figure 5: $N_c$ dependence of phase shift (left) and cross section (right) of the (1,0) channel. $N_c = 3, 4, 6$ and 12 from the top to bottom. The vertical lines show the $K\bar{K}$ threshold.

$(I, J) = (1, 0)$
This channel contains the $a_0(980)$ state, which appears as a cusp-like sharp peak, because the complex pole sits above the $K\bar{K}$ threshold as $(1.071 - i0.0198)$ GeV in the II sheet at $N_c = 3$ in this work.
This state may be a typical example of the structure generated by the channel coupling between the $\pi\eta$ and $K\bar{K}$ channel [5]. As $N_c$ increases the rising phase shift above the cusp bends down to a flat behavior and the cross section having a harp peak shrinks to the null structure. The pole moves from the sheet II to the sheet IV and leaves from the real axis as shown in the left side of Fig. 4. The different pole movement between the isoscalar and isovector channel would come from the difference between the strong attractive $\pi\pi$ interaction in the former channel and the weak repulsive $\pi\eta$ one in the latter channel.

$$(I, J) = (1/2, 0)$$

The null behaviors of the phase shift and cross section for large $N_c$ are almost similar to the channels discussed above. The phase shift and the cross section are calculated in terms of the single $\pi K$ channel amplitude, because there appears a fictitious zero in the $\eta K$ amplitude in the OOP version. But we note that since the channel coupling between the $\pi K$ and $\eta K$ channel is weak, the results by the multichannel formalism are almost the same as those by the single channel formalism above the fictitious zero. The calculations with use of the full $T^{(4)}$ give no such an unwanted zero [5] and the results are almost the same as ours. We emphasize that the $\pi K$ scattering cross section has a broad peak at 800 MeV at $N_c = 3$ as well as the similar peak of the $\pi\pi$ cross section at 550 MeV. These peaks could be called as $\kappa$ and $\sigma$ "resonances".

Figure 6: $N_c$ dependence of phase shift (left) and cross section (right) of the $(1/2, 0)$ channel. $N_c = 3$, 5, 8 and 12 from the above to bottom.

We have calculated $N_c$ dependence of the vector and scalar meson channels though the approximate IAM under the rough assumption that $L_n/f_\pi^2$'s are $N_c$ independent and $f_\pi(N_c) = \sqrt{N_c/3} \times f_\pi(3)$. This assumption would be the more valid as $N_c$ becomes the larger, because the effects by remaining $O(1/N_c)$ terms would disappear, and then the results obtained here would remain valid in the full IAM calculations. Thus, we observe that all of the low mass scalar mesons including the $f_0(980)$ and $a_0(980)$ cannot survive in the large $N_c$ limit. This makes the sharp contrast to the vector mesons, which survive as the resonances with the extremely narrow widths. Our conclusion is that the vector meson nonet has the nature consistent with the $q\bar{q}$ meson in large $N_c$ QCD, but the low mass scalar meson nonet cannot survive in the large $N_c$ limit and then does not have the $q\bar{q}$ nature. Finally, we emphasize that the scalar meson nonet should not be understood as particles, which can propagate with a definite mass and coupling constants, but as dynamical effects generated in coupled channel meson-meson scattering covering wide mass ranges below 1 GeV, where chiral symmetry and unitarity play crucial roles. Their structures reveal themselves or not depending strongly on reactions, therefore. Chiral unitary approach provides a consistent formalism, which realizes such a picture in almost all of the scattering and production processes.

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