The fundamental Laplacian eigenvalue of the ellipse with Dirichlet boundary conditions

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Abstract

In this project, I examine the lowest Dirichlet eigenvalue of the Laplacian within the ellipse as a function of eccentricity. Two existing analytic expansions of the eigenvalue are extended: Close to the circle (eccentricity near zero) nine terms are added to the Maclaurin series; and near the infinite strip (eccentricity near unity) four terms are added to the asymptotic expansion. In the past, other methods, such as boundary variation techniques, have been used to work on this problem, but I use a different approach – which not only offers independent confirmation of existing results, but may be used to extend them. My starting point is a high precision computation of the eigenvalue for selected values of eccentricity. These data are then fit to polynomials in appropriate parameters yielding high-precision coefficients that are fed into an LLL integer-relation algorithm with forms guided by prior results.

Introduction

Let $\Omega$ be an elliptical region with boundary $\partial \Omega$, as shown in Fig. 1, and where some elementary but relevant properties are given in the Appendix.

The Dirichlet Laplacian eigenvalue problem within this region is defined by

$$\Delta \psi + \lambda \psi = 0 \quad \text{in} \quad \Omega \quad \text{with} \quad \psi = 0 \quad \text{on} \quad \partial \Omega$$

(1)

where, in general, there exists a non-accumulating, infinite tower of real eigenvalues

$$0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

(2)

and corresponding eigenfunctions, $\psi_n \in L^2$, orthonormalizable as $\int_{\Omega} \psi_m \psi_n = \delta_{mn}$. This classic problem has a long history with many results, but only relevant techniques and results are reported here.

This project is limited to examining how the lowest (fundamental) eigenvalue, $\lambda_0$, behaves as a function of ellipse parameters. Specifically, it confirms and extends two series: A Maclaurin series for small values of eccentricity ($e \approx 0$) near the circle, and an asymptotic series for large values of eccentricity ($e \leq 1$) as the highly-elongated ellipse degenerates into the infinite strip.

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The (un-normalized) eigen-solutions for these two extremes are very well known and elementary,
\[
\{ \lambda_0, \psi_0 \} = \begin{cases} 
\left\{ \left( \frac{j_0}{R} \right)^2, J_0 \left( \frac{j_0 r}{R} \right) \right\} & \text{if } e = 0 \quad \text{R-radius circle, } r = \sqrt{x^2 + y^2} \leq R \quad (3a) \\
\left\{ \left( \frac{\pi}{2e} \right)^2, \cos \left( \frac{\pi y}{2e} \right) \right\} & \text{if } e = 1 \quad \text{infinite strip, } |y| \leq \varepsilon \quad (this \ \psi_0 \not\in L^2) \quad (3b)
\end{cases}
\]
where \( j_0 \) is the Bessel function of the first kind of order zero, and \( j_{10} \approx 2.4048 \) its first root.

As noted, in the limit \( e \to 1^- \), the eigenfunction loses its \( L^2 \) property. For us, that is not a problem: We shall be interested in the eigenvalue, \( \frac{\pi}{(2\varepsilon)^2} \), since this becomes the lowest order term in the asymptotic expansion, Eq. (5), where \( \varepsilon \) of (3b) shall correspond to the stretch factor of the ellipse per Eq. (10).

**Results**

First, consider the Maclaurin series of \( \lambda_0 \) in powers of eccentricity \( e \),
\[
A = \frac{\lambda_0}{\rho} = \sum_{\nu=0}^{\infty} C_\nu(p) e^{2\nu} = 1 + \left[ \frac{\rho - 2}{32} \left( e^4 + e^6 \right) + \left\{ \frac{-7\rho^3 + 58\rho^2 + 832\rho - 1792}{32768} \right\} e^8 \right. \\
+ \left\{ \frac{-7\rho^3 + 58\rho^2 + 320\rho - 768}{16384} \right\} e^{10} + [\text{See Table 1}] + \cdots 
\]  
where \( \rho = j_{10} = 5.7831 \) is the fundamental eigenvalue of the unit-radius circle per Eq. (3a), and where the next eight terms, from \( C_6(p) e^{12} \) to \( C_{13}(p) e^{26} \), are listed in Table 1. The coefficients – expressed as rational polynomials in powers of \( \rho \) – are exact, but, for reference, rounded numerical values of the first thirty non-trivial coefficients appear in Table 2.

About fifty years ago, in 1967, Joseph [7] was first to publish up to \( C_3(p) \) and noted that only even powers of \( e \) appear. A decade or two went by (date unknown) when Henry [4] picked up the problem again, corrected a minor mistake in the Joseph result, and also reported that result to the same order. At least another decade elapsed when, in 2014, \( C_4(p) \) was determined by Boady, Grinfeld, and Johnson [1]. All of those prior efforts relied on boundary variation techniques by parametrically deforming the circle into the ellipse.

One of my contributions to this problem is the computation up to \( C_{13}(p) \), including independent verification of the coefficients up to \( C_4(p) \).

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1To line up their result with Eq. (4), first realize they used yet a different area, \( A' = \pi/\varepsilon \), so multiply Eq. (4) by \( \varepsilon = \sqrt{1-e^2} \) and re-expand in powers of \( e \). The coefficient multiplying \( e^8 \) becomes the same as theirs, \( \left[ -7\rho^3 + 58\rho^2 + 192\rho - 1792 \right] / 32768 \), for example.
where, as anticipated, the first term corresponds to the infinite strip eigenvalue per Eq. (3b). The

\[ \varepsilon \\sum_{n=1}^{\infty} c_n \varepsilon^n = \frac{\pi^2}{4e^2} + \frac{\pi}{2e} + \frac{3}{4} \left( \frac{11}{8\pi} + \frac{\pi}{12} \right) \varepsilon \]

+ \left( \frac{61}{16\pi^2} + \frac{1}{12} \right) \varepsilon^2 + \left( \frac{1971}{128\pi^3} - \frac{9}{16\pi} + \frac{3\pi}{80} \right) \varepsilon^3 + \left( \frac{20851}{256\pi^4} - \frac{271}{48\pi^2} + \frac{2}{45} \right) \varepsilon^4

+ \left( \frac{537219}{1024\pi^5} - \frac{11667}{256\pi^3} - \frac{7}{64\pi} + \frac{5\pi}{224} \right) \varepsilon^5 + O(\varepsilon^6)

(5)

Second, at the other extreme where \( \varepsilon \leq 1 \), it becomes more convenient to expand in powers of stretch factor, \( \varepsilon = \sqrt{1 - e^2} \). The so-called asymptotic expansion, valid for \( \varepsilon \geq 0 \), is now,

\[ A' = \pi e : \quad \lambda_n' = \sum_{\nu=2}^{\infty} c_{\nu} e^\nu = \frac{\pi^2}{4e^2} + \frac{\pi}{2e} + \frac{3}{4} \left( \frac{11}{8\pi} + \frac{\pi}{12} \right) e \]

\[ + \left( \frac{61}{16\pi^2} + \frac{1}{12} \right) e^2 + \left( \frac{1971}{128\pi^3} - \frac{9}{16\pi} + \frac{3\pi}{80} \right) e^3 + \left( \frac{20851}{256\pi^4} - \frac{271}{48\pi^2} + \frac{2}{45} \right) e^4 \]

\[ + \left( \frac{537219}{1024\pi^5} - \frac{11667}{256\pi^3} - \frac{7}{64\pi} + \frac{5\pi}{224} \right) e^5 + O(e^6) \]

where, as anticipated, the first term corresponds to the infinite strip eigenvalue per Eq. (3b). The

Table 1: Higher-order terms of the Maclaurin series for the ellipse per Eq. (4).
Table 2: Numerical values of the leading thirty coefficients of the Maclaurin series for the eigenvalue within the ellipse per Eq. (4), and rounded to twenty decimal places. Also indicated is \( D_\nu \), the number of digits in agreement between the numerical coefficient (via this linear regression of data) and the respective coefficient displayed in Eq. (4) with Table 1. This truncated list of coefficients is based on a fit using fifty coefficients and sixty 500-digit eigenvalues with eccentricity \( e = 0.000001 \) to 0.000060, spaced equally; and it incorporates the trivial \( C_0 = 1 \) and \( C_1 = 0 \).

| \( \nu \) | \( C_\nu \) | \( C_\nu/C_\nu-1 \) | \( D_\nu \) | \( \nu \) | \( C_\nu \) | \( C_\nu/C_\nu-1 \) |
|---|---|---|---|---|---|---|
| 2 | 0.11822456134208701629 | 458 | 17 | 0.05770190566258202267 | 97.004 |
| 3 | 0.11822456134208701629 | 446 | 18 | 0.0560709781866375885 | 97.173 |
| 4 | 0.11003095525016373549 | 0.93069 | 34 | 0.05457135306802184693 | 97.326 |
| 5 | 0.10183734915824045469 | 0.92553 | 20 | 0.05318746732402259177 | 97.464 |
| 6 | 0.0946980942824285786691 | 0.92990 | 21 | 0.05190553831652441797 | 97.590 |
| 7 | 0.08861319050401597214 | 0.93574 | 34 | 0.05071400061380283923 | 97.704 |
| 8 | 0.08341794996558013471 | 0.94137 | 31 | 0.04960296200320839231 | 97.809 |
| 9 | 0.07894686465249571895 | 0.94641 | 38 | 0.04856391475086479136 | 97.905 |
| 10 | 0.0750629658798235035 | 0.95084 | 37 | 0.0475895045497284014 | 97.994 |
| 11 | 0.0716627779628316421 | 0.95471 | 36 | 0.046673441159790177 | 98.075 |
| 12 | 0.0686339082734667271 | 0.95810 | 35 | 0.04580986156578441533 | 98.150 |
| 13 | 0.06599134928344895227 | 0.96108 | 34 | 0.044991768028498705 | 98.219 |
| 14 | 0.063597537418726261359 | 0.96373 | 29 | 0.04422199837110163796 | 98.284 |
| 15 | 0.06143976881171471065 | 0.96607 | 30 | 0.043489536757551810 | 98.344 |
| 16 | 0.0594838738373514015 | 0.96817 | 31 | 0.0427934472789372947 | 98.399 |

coefficients – expressed as rational polynomials in powers of \( \pi \) – are exact; but, for reference, rounded numerical values of the first ten coefficients appear in Table 3.

This asymptotic expansion has a very different history. Troesch and Troesch \([9,8]\) in 1973 appear to have started the discussion\[2\] and derived the first three terms, up to zeroth order (i.e., 3/4) based on expansions of the roots of the modified Mathieu functions for large eccentricity (cf., \([9]\) paragraph 6\[3\]). Several decades later, in the mid-2000s, Borisov and Freitas \([2]\) considered the ellipse as one example using a boundary variation method. They published the first four terms, i.e., the first line of Eq. (5), up to first order in \( \varepsilon \). My second contribution to this problem is the next four terms of Eq. (5), i.e., the last two lines.

Of note is that as one moves down these series, the terms appear to become more complicated, but intriguing patterns do emerge – suggesting future work. For example, with the Maclaurin series, the ratio of successive coefficients (see Table 2) appears to approach unity (as \( \nu \to \infty \)). This fact suggests that a better representation is possible, quite analogous to the way that \( 1 + x + x^2 + \ldots \) is better represented by \( 1/(1-x) \). Despite that, the two canonical representations chosen for this report are intended to help compare with prior results.

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\[2\] Although this ellipse problem has a very long history, they appear to be the first who promoted expanding the eigenvalue in powers of \( \varepsilon \). In the second quoted article, they raise an intriguing Kac-like question: Can an elliptical drum have a spectrum similar to a stringed instrument?

\[3\] In my notation, they had (with \( m=k=0 \)), \( (\lambda_0')^{1/2} \varepsilon = \pi/2 + \varepsilon^2/(2\pi) + O(\varepsilon^3) \), and squaring, \( \lambda_0' \varepsilon^2 = \pi^2/4 + \pi\varepsilon/2 + 3\varepsilon^2/4 + O(\varepsilon^3) \).
Table 3: Numerical values of the leading ten coefficients of the asymptotic series for the eigenvalue within the ellipse per Eq. (5), and rounded to twenty decimal places. Also indicated is $D_c$, the number of digits in agreement between the numerical coefficient (via these fits of data) and the respective coefficient displayed in Eq. (5). This truncated list of coefficients is based on polynomial interpolations using thirty-six 200-digit eigenvalues with eccentricity $e = 0.999800$ to 0.999995, each one incorporating the (known) closed-form lower-order coefficients.

| $v$ | $c_v$ | $c_v/c_{v-1}$ | $D_v$ | $v$ | $c_v$ | $c_v/c_{v-1}$ |
|-----|-------|----------------|-------|-----|-------|----------------|
| -2  | 2.4674011002723965471 | 48 | 3 | 0.43538365077995525294 | 0.9271040 | 40 |
| -1  | 1.57079632679489661923 | 0.63662474 | 4 | 0.30855816280914840552 | 0.7087038 | 38 |
| 0   | 0.75000000000000000000 | 0.4774645 | 5 | 0.27983128169766678772 | 0.9069036 | 36 |
| 1   | 0.69947548130186160990 | 0.9326343 | 6 | 0.19027912693622176700 | 0.679980 | 34 |
| 2   | 0.4696203459674608896 | 0.6713941 | 7 | 0.1598173976222463 | 0.839910 | 33 |

**Technique**

Boundary variation techniques are mathematically elegant, but as workers have discovered and acknowledged, getting simple results is quite challenging. The other method – using Mathieu functions – is also quite elegant, but it too leads to interesting computational challenges.

The method I use is quite different from those other methods. I view it more as a brute force method, however, it is guided and motivated by those prior results.

My starting point is a high-precision computation of the fundamental eigenvalue for judiciously chosen values of eccentricity. For the ellipse eigenvalues, I use the same method [5] as popularized by Fox, Henrici, and Moler [3] fifty years earlier. The key to make it work well is to use multi-precision software and pay attention to the spacing of boundary points.

Next, sets of computed eigenvalues (“data points”) are fit to an appropriate model equation and high-precision coefficients are computed. By adding terms (and data points), the precision of each coefficient can be improved, and thus estimated. It is quite typical to work with coefficients that appear to be precise to more than a few dozen digits, and often to hundreds of digits.

When the leading coefficient is of sufficient precision, it is fed into an LLL integer relation algorithm with an ansatz guided by prior results. If the precision of the coefficient is too low, the LLL output is far from unique. As the coefficient precision increases, a viable candidate for the integer relation clearly emerges, and with increasing confidence. After numerical evidence supports a result for a coefficient, it is incorporated into the dependent variable of the model equation, and the process repeated in search of the next coefficient.

A very similar method works quite well for the $1/S$ expansion of the fundamental Dirichlet Laplacian eigenvalue of the $S$-sided regular polygon [6].

To get the coefficients in the MacLaurin series, Eq. (4) with Table 1, thirty 500-digit eigenvalues, equally spaced from $e = 0.000001$ to 0.000030, were sufficient. Each eigenvalue required about five hours on my commodity hardware (i7, 6-core, desktop) using free software (pari/gp). A thirty-term interpolating polynomial in even
powers of $e$, with those thirty eigenvalues, was then determined. Numerical values of the first thirty non-trivial coefficients (using sixty eigenvalues) of the Maclaurin series are listed in Table 2.

To get the coefficients in the asymptotic expansion, Eq. (5), some twenty 200-digit eigenvalues in the interval from $e = 0.999980$ to 0.999995 were sufficient. That entire eigenvalue computation took a few days with the same setup. Like the Maclaurin series, an interpolating polynomial, this time in powers of $\varepsilon$, was used.

With a sufficiently precise (numerical) coefficient in hand, the technique to determine an expression is given by example, here for the third-order term in Eq. (5), i.e., the coefficient $c_3$ multiplying $\varepsilon^3$. Guided by lower order expressions, an ansatz might look like

$$a_1 c_3 + \frac{a_2}{\pi^3} + \frac{a_3}{\pi} + \frac{a_4}{\pi} + a_5 \pi + a_6 \pi^3 = 0$$

where the six “small” integers $a_i$ are to be sought using LLL. In this particular example, if the LLL output has $a_2 = a_6 = 0$, that adds more confidence. Fitting the numerical eigenvalue data to the model equation indicates that

$$c_3 = 0.43538365077995525294060384502545762439389188 \cdots$$

where the underlined leading 36 digits appear to be correct (based only on the fit to the data). With that, the LLL routine (unambiguously) yields,

$$a_1 = 640 \quad a_2 = 0 \quad a_3 = -9855 \quad a_4 = 360 \quad a_5 = -24 \quad a_6 = 0$$

from which the coefficient is constructed. Incidentally, this constructed coefficient reproduces the numerical coefficient,

$$\left( \frac{1971}{128\pi^3} - \frac{9}{16\pi} + \frac{3\pi}{80} \right) = 0.43538365077995525294060384502545762443837546 \cdots$$

matching the above $c_3$ to 36 digits. This then becomes the convincing evidence (valid to 36 digits) that the LLL gave the correct result.

Two effects provide a practical limit to this method. Given a set of computed eigenvalues, as one progresses along a series: (1) numerical precision of the coefficients decreases and (2) the number of terms (rational polynomial in powers of $\rho$ or $\pi$) needed to represent a coefficient increases. At some point, these two effects conspire in such a way that LLL routine simply fails to offer an unambiguous solution. All of the results in this report were pushed to the limit and only unambiguous, unique solutions are presented. The simple way to extend results even further is with a more extensive eigenvalue computation.

### Conclusion

The method outlined in this report appears to be quite fruitful in confirming and extending power series expansions of the eigenvalues of the Laplacian for the ellipse. The main results of this report are summarized in Eq. (4) with Table 1, and Eq. (5). This method nicely complements other methods, such as boundary variation methods, providing independent confirmation and hints on what terms look like on down those series.

The ingredients necessary for the method to work are (1) very high precision eigenvalue computations, (2) an analytical model to give hints at what a series might look like (in terms of $\pi$ or $j_{0,1}$, for example), (3) an integer relation algorithm, and (4) a little patience and luck.

Of course, having these results raises many other questions, especially as simple patterns are exposed. Questions might include: What do higher eigenvalues look like? What are the convergence properties? How do these results relate to Mathieu functions? Are there better representations such as rational polynomials or continued fractions? What are the recurrence relations for the coefficients? Can these results help guide boundary variation techniques that solve the same problem? Indeed, there seem to be more new questions raised than answers provided.
Appendix

Refer to Fig. 1 for artwork. To fix the notation,

\[
\left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1 \quad \varepsilon = \frac{a}{b} = \sqrt{1-e^2} \quad e = \sqrt{1-e^2} \quad A = \pi ab
\]  

(10)

where \(a\) and \(b\) are, respectively, the semi-major and semi-minor axes, \(\varepsilon\) the stretch factor, \(e\) the eccentricity, and \(A\) the area. Without loss of generality, it is tacitly assumed that \(a \geq b\) so that as eccentricity varies from zero to unity, the stretch factor varies from unity to zero.

Key to sorting out the expressions in this report – and comparing to other results – is the invariant product (eigenvalue × area) for a given eccentricity. I shall use two popular conventions, distinguished using a prime, and here showing which quantities depend on eccentricity,

- **Constant area:** \(A = \pi \lambda_0(e) a(e) b(e) = 1 \) \hspace{1cm} (11a)
- **Constant semi-major axis:** \(A'(e) = \pi \varepsilon(e) \lambda'_0(e) = \lambda_0(e)/\varepsilon(e) \) \hspace{1cm} (11b)

where, in both cases, the unit-radius circle is the shape when \(e = 0\), and “constant” means independent of \(e\). The invariant product \(A\lambda_0 = A'\lambda'_0\) means \(\lambda'_0\varepsilon = \lambda_0\). The numerical values of both \(\lambda_0\) and \(\lambda'_0\) diverge as \(e \to 1^-\), but the product \(\lambda'_0\varepsilon^2 = \lambda_0\varepsilon = (\pi/2)^2\) in that limit, see Eq. (3b).

To ameliorate the inevitable confusion, the ellipse area shall always be clearly specified for each eigenvalue expression or set of data.

References

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