GENERIC INITIAL IDEALS OF POINTS AND CURVES

ALDO CONCA AND JESSICA SIDMAN

Abstract. Let \( I \) be the defining ideal of a smooth complete intersection space curve \( C \) with defining equations of degrees \( a \) and \( b \). We use the partial elimination ideals introduced by Mark Green to show that the lexicographic generic initial ideal of \( I \) has Castelnuovo-Mumford regularity \( 1 + ab(a - 1)(b - 1)/2 \) with the exception of the case \( a = b = 2 \), where the regularity is 4. Note that \( ab(a - 1)(b - 1)/2 \) is exactly the number of singular points of a general projection of \( C \) to the plane. Additionally, we show that for any term ordering \( \tau \), the generic initial ideal of a generic set of points in \( \mathbb{P}^r \) is a \( \tau \)-segment ideal.

1. Introduction

Let \( S = k[x_0, \ldots, x_r] \) where \( k \) is an algebraically closed field of characteristic zero and let \( \tau \) be a term ordering on \( S \). Let \( I \subset S \) be a homogeneous ideal. There is a monomial ideal canonically associated with \( I \), its generic initial ideal with respect to \( \tau \) denoted by \( \text{gin}_\tau(I) \) or simply \( \text{gin}_\tau I \). In this paper we study lexicographic generic initial ideals of curves and points via Green’s partial elimination ideals.

For a smooth complete intersection curve \( C \) in \( \mathbb{P}^3 \), we show that the complexity of its lexicographic generic initial ideal, as measured by Castelnuovo-Mumford regularity, is governed by the geometry of a generic projection of \( C \) to \( \mathbb{P}^2 \).

Theorem 1.1. Let \( C \) be a smooth complete intersection of hypersurfaces of degrees \( a, b > 1 \) in \( \mathbb{P}^3 \). The regularity of the lexicographic generic initial ideal of \( C \) is equal to

\[
\begin{cases}
1 + \frac{a(a - 1)b(b - 1)}{2} & \text{if } (a, b) \neq (2, 2) \\
4 & \text{if } (a, b) = (2, 2)
\end{cases}
\]

Note that, apart for the exceptional case \( a = b = 2 \), the regularity of the lexicographic generic initial ideal is \( 1 + \) the number of nodes of the curve.

The second author was supported by an NSF postdoctoral fellowship during 2002-2003.
The statement of Theorem 1.1 generalizes Example 6.10 in [15] which treats the special case where \(a = b = 3\).

Macaulay's characterization of Hilbert functions, see for instance Theorem 4.2.10 in [5], implies that any ideal \(J\) is generated in degrees bounded by the largest degree of a generator of the corresponding lex-segment \(\text{Lex}(J)\). Much more is true – Bigatti [4], Hullett [18] and Pardue [21] showed the Betti numbers of \(J\) are bounded by those of \(\text{Lex}(J)\). Let \(I\) be the ideal of \(C\) in Theorem 1.1. For such an ideal \(I\) one can compute the largest degree of a generator of \(\text{Lex}(I)\). This has been done, for instance, by D. Bayer in his Ph.D. thesis (Proposition in II.10.4, [1]) and by Chardin and Moreno -Socias [8], and it turns out to be \(\frac{a(a-1)b(b-1)}{2} + ab\). So the lexicographic generic initial ideal in Theorem 1.1 is not equal to the lex-segment ideal but nearly achieves the worst-case regularity for its Hilbert function. Moreover, as Bermejo and Lejeune-Jalabert have shown in [3], the extremal bound can only be achieved if \(C\) lies in a plane.

We also study the generic initial ideals of finite sets of points. Surprisingly, when \(X\) is a set of generic points its generic initial ideal is an initial segment.

**Theorem 1.2.** Let \(I\) be the ideal of \(s\) generic points of \(\mathbb{P}^n\). Then \(\text{gin}_\tau I\) is equal to the \(\tau\)-segment ideal \(\text{Seg}_\tau(I)\) for all term orders \(\tau\). In particular, \(\text{gin}_\text{lex} I\) is a lex-segment ideal.

The genericity required in Theorem 1.2 is quite explicit: the conclusion holds for a set \(X\) of \(s\) points if there is a system of coordinates such that the defining ideal of \(X\) does not contain non-zero forms supported on \(\leq s\) monomials. A special case of the result when \(\tau = \text{revlex}\) is proved by Marinari and Ramella in [20].

For an introduction to generic initial ideals see §15.9 in [12]. Here we just recall:

**Theorem 1.3** (Galligo, Bayer-Stillman). Given a homogeneous ideal \(I\) and a term ordering \(\tau\) on the monomials of \(S\), there exists a dense open subset \(U \subseteq \text{GL}_{r+1}(k)\) such that \(\text{gin}_{\tau} I := \text{in}_{\tau}(g \cdot I)\) is constant over all \(g \in U\) and \(\text{gin}_{\tau} I\) is Borel-fixed.

Recall also that, in characteristic 0, an ideal \(J\) is Borel-fixed if it is monomial and satisfies:

\[\text{if } m \text{ is a monomial, } x_i m \in J \implies x_j m \in J, \forall j \leq i.\]
From this property one easily shows that the regularity of a Borel-fixed ideal $J$ is the maximum degree of a minimal generators. A minimal resolution of such an ideal was constructed by Eliahou and Kervaire in [13].

As $\tau$ varies over all term orderings, both the regularity and the minimal number of generators of $\text{gin}_\tau I$ may vary greatly. The generic initial ideals with respect to the reverse lexicographic (revlex) term ordering have the minimum level of complexity possible.

**Theorem 1.4** (Bayer-Stillman [2]). If $I$ is a homogeneous ideal of $S$, and $J = \text{gin}_{\text{revlex}} I$ then

$$\text{reg } I = \text{reg } J = \max \text{ degree of a minimal generator of } J.$$

The paper is organized as follows. In §2, we set up notation and review terminology. We introduce partial elimination ideals, their basic properties, and algorithms for their computation in §3. We focus on the case of complete intersection curves in §4 and on the case of points in §5.

**Acknowledgements** We are deeply indebted to D. Speyer as much of this work originated jointly with him. We thank B. Sturmfels for his support and encouragement. We would also like to thank D. Eisenbud, R. Thomas, and M. Stillman for helpful discussions regarding partial elimination ideals and their computation and A. Dickenstein, L. Busé, and C. D’Andrea for many stimulating conversations about subresultants. We have benefitted substantially from experimentation with the computer algebra packages Macaulay 2 [14] and CoCoA [9]. We would like to thank R. Lazarsfeld and anonymous referees for helpful comments. We are also grateful for the support of MSRI, and the second author thanks the Clare Boothe Luce Program.

## 2. Notation and terminology

Let $S = k[x_0, \ldots, x_r]$ where $k$ is an algebraically closed field of characteristic zero. Denote by $m$ the irrelevant maximal ideal of $S$. For an element $\alpha = (\alpha_0, \ldots, \alpha_r) \in \mathbb{N}^{r+1}$ we let $x^\alpha$ denote $x_0^{\alpha_0} \cdots x_r^{\alpha_r}$. In this section we briefly recall notions related to term orderings and Castelnuovo-Mumford regularity. For a comprehensive introduction to general notions related to Gröbner bases see [11] and [10].

**Definition 2.1.** We say that a total ordering $\tau$ on the monomials of $S$ is a *term ordering* if it is a well-ordering satisfying

$$x^\alpha >_\tau x^\beta \Rightarrow x^\gamma \cdot x^\alpha >_\tau x^\gamma \cdot x^\beta \ \forall \alpha, \beta, \gamma \in \mathbb{N}^{r+1}.$$
A term ordering $\tau$ on $S$ allows us to assign to each non-zero element $f \in S$ an initial term $\text{in}_\tau(f)$ and to any ideal $I$ an initial ideal $\text{in}_\tau(I)$.

In what follows we will work exclusively with homogeneous ideals and we will always require that the term ordering is degree compatible: $m > n$ if $\deg(m) > \deg(n)$.

The lexicographic and (degree) reverse lexicographic term orderings feature prominently in the literature. If $x^\alpha$ and $x^\beta$ are two monomials of the same degree, then $x^\alpha > \text{lex} x^\beta$ if the left-most non-zero entry of $\alpha - \beta$ is positive and $x^\alpha > \text{revlex} x^\beta$ if the right-most non-zero entry of $\alpha - \beta$ is negative.

Although our primary motivation for studying partial elimination ideals is to understand lexicographic initial ideals, partial elimination ideals also provide a mechanism for studying initial ideals with respect to any elimination order.

**Definition 2.2.** An elimination order for the first $t$ variables of $S$ is a term order $\tau$ such that if $f$ is a polynomial whose initial term in $\tau(f)$ does not involve variables $x_0, \ldots, x_{t-1}$, then $f$ itself does not involve variables $x_0, \ldots, x_{t-1}$.

As we shall see in Proposition 3.4, one may use an elimination order for the variable $x_0$ to compute partial elimination ideals. If $\tau$ is an elimination order for $x_0$, then it is equivalent to a $(1, r)$ product order which first sorts monomials by powers of $x_0$ and then sorts the remaining variables by an arbitrary term ordering $\tau_0$.

We will use the notion of Castelnuovo-Mumford regularity as a rough measure of the complexity of our computations.

**Definition 2.3.** Let $M$ be a finitely generated graded $S$-module, and let

$$0 \to \oplus_j S(-a_{ij}) \to \cdots \oplus_j S(-a_{1j}) \to \oplus_j S(-a_{0j}) \to M \to 0$$

be a minimal graded free resolution of $M$. We say that $M$ is $d$-regular if $a_{ij} \leq d + i$ for all $i$ and $j$, and that the regularity of $M$, denoted $\text{reg } M$, is the least $d$ such that $M$ is $d$-regular.

One may also formulate the definition of regularity in terms of vanishings of local cohomology with respect to $m$. The vanishing of the zero-th local cohomology group is related to the notion of saturation which plays an important role in the study of regularity.

**Definition 2.4.** Let $I \subseteq S$ be a homogeneous ideal. The saturation of $I$, denoted $I^{\text{sat}}$, is defined to be $I :_S m^\infty$. Note that $I_d = I_d^{\text{sat}}$ for all $d \gg 0$. We say that $I$ is $d$-saturated if $I$ agrees with its saturation in
degrees $d$ and higher. The minimum degree for which $I$ is $d$-saturated is the saturation degree (also the satiety index in [15]) of $I$.

### 3. Partial elimination ideals

Let $S = k[x_0, \ldots, x_r]$, and let $\overline{S} = k[x_1, \ldots, x_r]$. Let $\tau$ be an arbitrary elimination order on $S$ that eliminates the variable $x_0$ and hence induces a term order, denoted by $\tau_0$, on $\overline{S}$. In this section we set up the theory of partial elimination ideals over a polynomial ring in $r + 1$ variables as introduced in [15]. Much of the material in §3.1 and §3.2 appears either explicitly or implicitly in [15], but we give proofs here both to keep the presentation self-contained and to present a more algebraic point of view.

We represent any non-zero polynomial $f$ in $S$ as

$$f = f_0 x_0^p + f_1 x_0^{p-1} + \cdots + f_p$$

with $f_i \in \overline{S}$ and $f_0 \neq 0$. The polynomial $f_0$ is called the initial coefficient of $f$ with respect to $x_0$ and is denoted by $\text{incof}_{x_0}(f)$. The integer $p$ is called the $x_0$-degree of $f$ and is denoted by $\text{deg}_{x_0}(f)$.

#### 3.1. Definitions and basic facts.

In this section we define the partial elimination ideals and describe their basic algebraic and geometric properties. We begin with the definition:

**Definition 3.1** (Definition 6.1 in [15]). Let $I$ be a homogeneous ideal in $S$. The $p$-th partial elimination ideal of $I$ is defined to be the ideal

$$K_p(I) := \{\text{incof}(f) \mid f \in I \text{ and } \text{deg}_{x_0} f = p\} \cup \{0\}$$

in the polynomial ring $\overline{S} = k[x_1, \ldots, x_r]$.

It is easy to see that if $I$ is homogeneous then $K_p(I)$ is also homogeneous.

In Lemma 3.2 we gather together some elementary algebraic facts about the partial elimination ideals. We leave the proof to the reader. The decomposition of $\text{in}_\tau I$ given in part (1) is one of the motivations for the definition.

**Lemma 3.2.** Let $I$ be a homogeneous ideal.

1. $\text{in}_\tau I = \sum_p x_0^p \text{in}_{x_0} K_p(I)$.
2. Taking $K_p$ commutes with taking initial ideals: $K_p(\text{in}_\tau I) = \text{in}_{x_0} K_p(I)$
3. The partial elimination ideals are an ascending chain of ideals, i.e., $K_i(I) \subseteq K_{i+1}(I)$ for all $i$. 
One expects that if \( I \) is in generic coordinates, then the partial elimination ideals \( K_p(I) \) are already in generic coordinates. Proposition 3.3 shows that this is indeed the case.

**Proposition 3.3.** Let \( I \subset S \) be a homogeneous ideal. If \( I \) is in generic coordinates then \( \text{gin}_{x_0} K_p(I) = \text{gin}_{x_0} K_p(I) \).

**Proof.** Let \( \text{GL}_r(k) \) act on \( S \) in the usual way and extend this to an action on \( S \) in the trivial fashion by letting elements of \( \text{GL}_r(k) \) fix \( x_0 \).

We know that the ideal \( I \) determines a dense open subset \( U \subset \text{GL}_{r+1}(k) \) with the property that \( g \in U \) implies that \( \text{in}_r(gI) = \text{gin}_r I \). We show that for each \( g \in U \) there is a dense open subset \( U' \subset \text{GL}_r(k) \) so that for all \( h \in U' \)

1. \( \text{gin}_{x_0}(K_p(gI)) = \text{in}_{x_0}(hK_p(gI)) \)
2. \( hg \) is again a generic change of coordinates for \( I \).

Consider the space \( \text{GL}_r(k) \times \text{GL}_{r+1}(k) \) with projection maps \( \pi_1 \) and \( \pi_2 \) onto the first and second factors, respectively. The map \( \phi : \text{GL}_r(k) \times \text{GL}_{r+1}(k) \to \text{GL}_{r+1}(k) \) given by \( \phi(h, g) = hg \) is regular. The inverse image of \( U \) under the map \( \phi \) is a dense open subset of \( \text{GL}_r(k) \times \text{GL}_{r+1}(k) \). For each \( g \in U \) the set \( W := \pi_1(\pi_2^{-1}(g) \cap \phi^{-1}(U)) \) is a dense open subset of \( \text{GL}_r(k) \). The element \( g \) determines a dense open set \( V \subset \text{GL}_r(k) \) such that \( h \in V \) satisfies (2), i.e., each \( h \in V \) is a set of generic coordinates for \( K_p(gI) \).

Then any \( h \in U' := W \cap V \) has the property that \( hg \) is a set of generic coordinates for \( I \).

For \( h \) and \( g \) chosen as above, we have \( hK_p(gI) = K_p(hgI) \). Thus, \( \text{gin}_{x_0}(K_p(gI)) = \text{gin}_{x_0} K_p(hgI) \). By Lemma 3.2 (2), \( \text{in}_{x_0} K_p(I) = K_p(\text{in}_r I) \), which implies that \( \text{gin}_{x_0}(K_p(gI)) = K_p \text{in}_r(hgI) \).

Since \( hg \) is again generic, \( \text{gin}_r I = \text{in}_r gI = \text{in}_r hgI \). So we have \( \text{gin}_{x_0}(K_p(gI)) = K_p(\text{in}_r gI) \). Using Lemma 3.2 (2) again, we obtain \( \text{gin}_{x_0}(K_p(gI)) = \text{in}_{x_0} K_p(gI) \) and this proves the assertion. \( \Box \)

The partial elimination ideals of an arbitrary homogeneous ideal \( I \) can be recovered in an easy way from a Gröbner basis for \( I \). In practice one may want to take a \((1, r)\) product order with the reverse lexicographic ordering on the last \( r \) variables in order to minimize computations.

**Proposition 3.4.** Let \( G \) be a Gröbner basis for \( I \) with respect to an elimination ordering \( \tau \). Then the set

\[
G_p = \{ \text{incoef}_{x_0}(g) \mid g \in G \text{ and } \text{deg}_{x_0}(g) \leq p \}
\]

is a Gröbner basis for \( K_p(I) \).
Proof. Note that if \( g \in I \) and \( \deg x_0(g) = p \) then \( \text{incoef}_{x_0}(g) \in K_p(I) \) by definition. By Lemma 3.2 (3) we have that the elements of \( G_p \) are in \( K_p(I) \). We will show that their initial terms generate \( \text{in}_{x_0} K_p(I) \). Suppose that \( m \) is a monomial in the ideal \( \text{in}_{x_0} K_p(I) \). This implies that there exists \( f \in I \) such that \( \text{in}_r(f) = mx_0^p \) and hence there exists \( g \in G \) such that \( \text{in}_r(g) \mid \text{in}_r(f) \). Set \( h = \text{incoef}_{x_0}(g) \). It follows that \( \deg x_0 g \leq p \), so that \( h \in G_p \), and \( \text{in}_{x_0} h \mid m \).

By part (3) of Lemma 6.2 we know that the subscheme cut out by the \( p \)-th partial elimination ideal is contained in the subscheme defined by the \((p - 1)\)-st partial elimination ideal. The following result gives the precise relationship between the partial elimination ideals and the geometry of the projection map from \( \mathbb{P}^r \) to \( \mathbb{P}^{r-1} \).

**Theorem 3.5** (Proposition 6.2 in [15]). Let \( Z \) be a reduced subscheme of \( \mathbb{P}^r \) not containing \([1 : 0 : \cdots : 0]\) and let \( I = I(Z) \) be the homogeneous ideal of \( Z \). Let

\[
\pi : \mathbb{P}^r \to \mathbb{P}^{r-1}
\]

be the projection from the point \([1 : 0 : \cdots : 0]\). Set-theoretically, \( K_p(I) \) is the ideal of

\[
\{ z \in \pi(Z) \mid |\pi^{-1}(z)| > p \},
\]

where \(|\pi^{-1}(z)|\) denotes the length of the scheme-theoretic fiber above \( p \).

**Proof.** We prove the theorem by reducing to the affine case. We begin by introducing some notation. If \( J \subseteq S \) is a homogeneous ideal, let \( J_{(x_i)} \) denote its dehomogenization in \( k[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_r}{x_i}] \).

To show that \( K_p(I) \) cuts out the \((p+1)\)-fold points set-theoretically it suffices to show that \( K_p(I_{(x_i)}) \) cuts out the \((p+1)\)-fold points in each of the standard affine open patches of \( \mathbb{P}^{r-1} \) for \( i = 1, \ldots, r \). If we consider the ideal \( I_{(x_i)} \subseteq k[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_r}{x_i}] \) with the term ordering induced by \( \tau \) in the natural way on the monomials in \( \frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_r}{x_i} \), then by Lemma 4.8.3 in [16], \( K_p(I_{(x_i)}) \) is set-theoretically the ideal of the \((p+1)\)-fold points lying in this affine patch.

It remains for us to show that for any \( i = 1, \ldots, r \),

\[
K_p(I_{(x_i)}) = K_p(I_{(x_i)}).
\]

It is clear that \( K_p(I_{(x_i)}) \subseteq K_p(I_{(x_i)}) \). For the opposite inclusion, note that \( \frac{x_0}{x_i} \) appears in the dehomogenization of a monomial \( m \) precisely as many times as \( x_0 \) appears in \( m \), and apply the definitions. \( \square \)

In the situation of Theorem 3.5, we can see that \( K_0(I) \) is in fact radical. The ideal \( K_0(I) \) is just equal to \( I \cap \overline{S} \). On the other hand, the higher \( K_p(I) \) need not be radical even if \( I \) is a prime complete intersection of codimension 2 in generic coordinates; see Example 4.3.
3.2. Partial elimination ideals for codimension 2 complete intersection. Let

\[ f = x_0^a + f_1 x_0^{a-1} + \cdots + f_{a-1} x_0 + f_a \]

and

\[ g = x_0^b + g_1 x_0^{b-1} + \cdots + g_{b-1} x_0 + g_b \]

where \( f_1, \ldots, f_a \) and \( g_1, \ldots, g_a \) are indeterminates.

We wish to describe the partial elimination ideals of the ideal \( I_{a,b} \) generated by \( f \) and \( g \) in \( S = k[x_0, \ldots, x_r] \) after specializing the \( f_i \) and the \( g_j \) to homogeneous elements of \( \overline{S} = k[x_1, \ldots, x_r] \) of degree \( i \).

As we saw in §3.1, the partial elimination ideals of an arbitrary homogeneous ideal \( I \) can be recovered from a Gröbner basis for \( I \) and, vice versa, give information on that Gröbner basis. In this section we discuss a result of Eisenbud and Green showing that \( K_p(I_{a,b}) \) is generated by the minors of a truncation of the Sylvester matrix as long as the forms \( f \) and \( g \) are generic enough. Both Theorem 3.6 and Lemma 3.7 are well-known to experts, but we give proofs for completeness.

**Theorem 3.6** (Proposition 6.8 (3) in [15]). Assume that the \( f_i \) and the \( g_j \) are independent indeterminates and that \( p < a \leq b \). Let

\[ R = k[f_1, \ldots, f_a, g_1, \ldots, g_b, x_0], \]

where \( k \) is an arbitrary field. Define \( \text{Syl}_p(f, g) \) to be the matrix consisting of the first \( a + b - p \) rows of the Sylvester matrix of \( f \) and \( g \), i.e.

\[
\text{Syl}_p(f, g) = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & \vdots & \vdots & 1 & 0 & 0 \\
1 & 0 & g_1 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
f_a & f_{a-1} & \cdots & 1 & \cdots & 1 \\
0 & f_a & \cdots & 0 & g_b & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & f_{a-p} & \cdots & 0 & g_{b-p} \\
0 & 0 & 0 & f_{a-p} & 0 & g_{b-p}
\end{pmatrix}
\]

Then the ideal \( K_p(f, g) \subset R \) is generated by the maximal minors of the matrix \( \text{Syl}_p(f, g) \).

**Proof.** Let \( R_{\leq t} \) denote the vector space of polynomials in \( R \) with \( \deg_{x_0} \leq t \). To compute \( K_p(f, g) \) we want to find all \( A, B \in R \) such that

\[ Af +Bg = c_0 x_0^p + c_1 x_0^{p-1} + \cdots + c_{p-1} x_0 + c_p. \]
with \( c_i \in k[f_1, \ldots, f_a, g_1, \ldots, g_b] \). Note that it suffices to find all \( A, B \) satisfying the equation (1) where \( \deg x_0 A \leq b - 1 \) and \( \deg x_0 B \leq a - 1 \).

The matrix \( \text{Syl}_p(f, g) \) gives a linear map
\[
R_{\leq b-1} \oplus R_{\leq a-1} \to R_{\leq a+b-1}/(1, x_0, \ldots, x_0^{p-1}).
\]
The kernel of \( \text{Syl}_{p+1}(f, g) \) consists of the set of all \( (A, B) \in R_{\leq b-1} \oplus R_{\leq a-1} \) that satisfy equation (1). The image of \( \ker \text{Syl}_{p+1}(f, g) \) under \( \text{Syl}_p(f, g) \) is exactly the set
\[
\{ c_0 x_0^p \mid c_0 x_0^p + c_1 x_0^{p-1} + \cdots + c_{p-1} x_0 + c_p \in (f, g) \}.
\]

We will show that the maximal minors of \( \text{Syl}_p(f, g) \) generate the image of \( \ker \text{Syl}_{p+1}(f, g) \) under the map \( \text{Syl}_p(f, g) \) as long as \( \text{Syl}_{p+1}(f, g) \) drops rank in the expected codimension. The proof that \( \text{Syl}_{p+1}(f, g) \) does indeed drop rank in codimension \( p + 2 \) will be given in Lemma 3.7.

If \( \text{Syl}_{p+1}(f, g) \) drops rank in the expected codimension, then since \( R \) is Cohen-Macaulay we can conclude that the Buchsbaum-Rim complex resolves the cokernel of \( \text{Syl}_{p+1}(f, g) \). (See Eisenbud [12] A2.6 for details.)

Using the Buchsbaum-Rim complex we can give explicit formulas for elements of \( \ker \text{Syl}_{p+1}(f, g) \) indexed by \( T \subseteq \{1, \ldots, a + b\} \) with \( |T| = a + b - p \). Define \( \text{Syl}_{p+1}(f, g)^T \) to be the \((a+b-p-1) \times (a+b-p)\) matrix consisting of all of the columns of \( \text{Syl}_{p+1}(f, g) \) indexed by elements of \( T \). Define \( W_T \) to be the vector of length \( a + b \) whose \( i \)-th entry is 0 if \( i \notin T \) and \( \text{sign}(i) \det \text{Syl}_{p+1}(f, g)^{T-(i)} \) if \( i \in T \) where \( \text{sign}(i) = 1 \) if the number of elements of \( T \) less than \( i \) is even, and -1 if the number of elements of \( T \) less than \( i \) is odd. The Buchsbaum-Rim complex is a resolution precisely when the vectors \( W_T \) generate the kernel of \( \text{Syl}_{p+1}(f, g) \).

Finally, we apply \( \text{Syl}_p(f, g) \) to the elements \( W_T \) constructed above. The dot product of \( W_T \) with each of the first \( a + b - p - 1 \) rows of \( \text{Syl}_p(f, g) \) is zero since \( W_T \) is in the kernel of \( \text{Syl}_{p+1}(f, g) \). The dot product of \( W_T \) with the last row is just the expansion of the maximal minor of \( \text{Syl}_p(f, g) \) corresponding to the columns indexed by \( T \) by this final row. Therefore,
\[
\text{Syl}_p(f, g) \cdot W_T = \det \text{Syl}_p(f, g)^T x_0^p.
\]

\( \square \)

**Lemma 3.7.** If \( f_i \) and \( g_j \) are independent indeterminates and \( p < a \leq b \) the matrix \( \text{Syl}_p(f, g) \) drops rank in the expected codimension \( p + 1 \).

**Proof.** We will show that the set where \( \text{Syl}_p(f, g) \) fails to have maximal rank, that is, where \( \dim_k \ker \text{Syl}_p(f, g) \geq p + 1 \), has codimension \( p + 1 \) in
the space of all \( f \) and \( g \) where the \( f_i \) and \( g_i \) take values in \( k \). The result follows if we can show that for any specialization of the indeterminates \( f_i \) and \( g_j \) to values in \( k \), \( \dim_k \ker \text{Syl}_p(f, g) \geq p + 1 \) if and only if \( f \) and \( g \) have a common factor of degree \( p + 1 \).

It is clear that if \( f \) and \( g \) have a common factor of degree \( p + 1 \) then \( \dim_k \ker \text{Syl}_p(f, g) \geq p + 1 \), since we can use the \( (p+1) \) common factors to construct \( (p+1) \) syzygies on \( f \) and \( g \) with distinct degrees.

To prove the other direction, we will use induction on \( p \).

Suppose that \( p = 0 \). Then
\[
\dim_k \ker \text{Syl}_0(f, g) > 0
\]
if and only if \( \text{Res}(f, g) = 0 \). It is well-known (see [11]) that \( \text{Res}(f, g) = 0 \) if and only if \( f \) and \( g \) have a common factor of degree at least one.

We treat the case where \( p > 0 \). Our assumption implies that we can find \( p + 1 \) linearly independent elements \((A_0, B_0), \ldots, (A_p, B_p)\) of the kernel of \( \text{Syl}_p(f, g) \).

Since
\[
A_0f + B_0g, \ldots, A_pf + B_pg \in \text{span}(1, \ldots, x_0^{p-1}),
\]
there is a nontrivial linear relation \( \sum \lambda_i(A_i f + B_i g) = 0 \). Hence, \( f \) and \( g \) must have a common factor so that \( f = (x - \alpha)f' \) and \( g = (x - \alpha)g' \).

By induction, we will be done if we can show that the dimension of
\[
\{(A, B) \in k[x_0]_{\leq b-2} \oplus k[x_0]_{\leq a-2} \mid A f' + B g' \in \text{span}(1, x_0, \ldots, x_0^{p-2})\}
\]
is \( \geq p \). But, we can assume (after reordering and cancelling leading terms) that for \( i \geq 1 \), \( \deg A_i \leq b - 2 \) and \( \deg B_i \leq a - 2 \). Consequently, for \( i \geq 1 \),
\[
A_i f' + B_i g' \in \text{span}(1, x_0, \ldots, x_0^{p-2}).
\]

\[\square\]

Note that the first \( b \) rows of \( \text{Syl}_p(f, g) \) contain constants. Recall the following fact:

**Lemma 3.8** (pg. 10 in [4]). Suppose that \( M = (m_{i,j}) \) is a \( p \times q \) matrix with entries in a commutative ring. If \( m_{p,q} \) is a unit, then the ideal generated by the maximal minors of \( M \) is the same as the ideal generated by the maximal minors of the \((p-1) \times (q-1) \) matrix \( N \) with entries
\[
n_{i,j} = m_{i,j} - m_{p,j}m_{i,q}m_{p,q}^{-1} \quad 1 \leq i \leq p-1, \quad 1 \leq j \leq q-1.
\]

We have the following:

**Corollary 3.9** (See the remark following Proposition 6.9 in [15]). Let \( a \leq b \) and assume that the \( f_i \) and \( g_i \) are sufficiently general homogeneous polynomials of degree \( i \) in variables \( x_1, \ldots, x_r \). Assume also that \( p < a \).
(1) The ideal of maximal minors of $\text{Syl}_p(f, g)$ is always contained in $K_p(f, g)$. It has the expected codimension, $p+1$, if $p \leq r-1$.

(2) Assume $p \leq r-2$. Then we have:
   (a) $K_p(f, g)$ is equal to the ideal of maximal minors of $\text{Syl}_p(f, g)$.
   (b) $K_p(f, g)$ is also the ideal generated by the maximal minors of a matrix of size $(a-p) \times a$ whose $(i, j)$-th entry is either 0 or has degree $b+i-j$.
   (c) $\text{reg} K_p(f, g) = ab + \binom{a-p+1}{2} - \binom{a+1}{2} + p(a-p-1)$.

Proof. Let $R = k[f_1, \ldots, f_a, g_1, \ldots, g_b, x_0]$ where the $f_i$ and $g_j$ are indeterminates as in Theorem 3.6. Generators for $K_p(f, g)$ as an ideal in $R$ also generate the $p$-th partial elimination ideal of the ideal generated by $f$ and $g$ in the ring $R \otimes_k k[x_1, \ldots, x_r]$, which we will denote by $K_p(f, g) \otimes k[x_1, \ldots, x_r]$. An elementary argument shows that if $p+1 \leq r$, then for sufficiently general forms $f_i, g_j \in k[x_1, \ldots, x_r]$, the specialization of the matrix $\text{Syl}_p(f, g)$ still drops rank in the expected codimension.

Thus, (1) and (2,a) follows from the proof of Theorem 3.6 and from Lemma 3.7. Part (b) of (2) follows from (2, a) and from iterated use of Lemma 3.8. Finally (2, c) follows from (2, b) and from Lemma 3.11. □

Remark 3.10. The above corollary is sharp in the sense that, in general, $K_{r-1}(f, g)$ strictly contains the ideal of maximal minors of $\text{Syl}_{r-1}(f, g)$. For instance, one can check with CoCoA that this happens if $r = 3$ and $a = b = 4$.

Lemma 3.11. Let $X = (h_{ij})$ be an $m \times n$ matrix of forms with $m \leq n$. Assume $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ are integers such that $\deg(h_{ij}) = a_i + b_j > 0$ whenever $h_{ij} \neq 0$. Assume that the ideal $I_m$ of maximal minors of $X$ has the expected codimension $n - m + 1$. Then

$$\text{reg } I_m = \sum_i a_i + \sum_j b_j + (\max(a_i) - 1)(n - m)$$

Proof. The Eagon-Northcott complex gives a resolution of $I_m$ which is minimal since the entries of the matrices in the resolution are (up to sign) the entries of $X$ and 0. Keeping track of the shifts one obtains the formula above. The same formula can be derived from the result [6, Cor.1.5] of Bruns and Herzog. Another formula for the regularity appears in [10]. □

In particular we have:
Corollary 3.12. Let $I$ be the ideal of a smooth complete intersection $C$ in $\mathbb{P}^3$ defined by two forms $f$ and $g$ of degrees $a, b > 1$. Assume that $I$ is in generic coordinates. We have:

(a) $K_1(f, g)$ is equal to the ideal of maximal minors of $\text{Syl}_1(f, g)$ and has codimension 2 in $k[x_1, x_2, x_3]$.

(b) $K_2(f, g)$ contains the ideal of maximal minors of $\text{Syl}_2(f, g)$ and both ideals have codimension 3 in $k[x_1, x_2, x_3]$.

Proof. We will use a geometric argument to show that if $f$ and $g$ are in sufficiently general coordinates, then $\text{Syl}_1(f, g)$ has codimension 2 and $\text{Syl}_2(f, g)$ has codimension 3 in $k[x_1, x_2, x_3]$. Since these codimensions are the expected values for those determinantal ideals, the conclusion will follow by Corollary 3.9.

Recall the classical fact that a generic projection of a smooth irreducible curve in $\mathbb{P}^3$ has only nodes as singularities. (See Theorem IV.3.10 in [17].) It follows that after a generic change of coordinates, the image of the projection from the point $[1 : 0 : 0 : 0]$ will have only nodes as singularities. As a consequence, we see that for each point $q \in \mathbb{P}^2$, the fiber of the projection of the curve $C$ will contain at most two points, and the set of $q$ with $\pi^{-1}(q) = 2$ is finite. In other words, $\deg \gcd(f(x_0, q), g(x_0, q)) \leq 2$ and equality holds for only finitely many $q$. From the proof of Lemma 3.7, we can see $\text{Syl}_p(f, g)$ drops rank at $q$ if and only if $f(x_0, q)$ and $g(x_0, q)$ have a common factor of degree $\geq p + 1$. Therefore, we see that $\text{Syl}_1(f, g)$ drops rank at a finite set of points and and $\text{Syl}_2(f, g)$ does not drop rank at any point in $\mathbb{P}^2$. □

4. The lexicographic gin of a complete intersection curve in $\mathbb{P}^3$

Let $I_{a,b}$ be a codimension 2 complete intersection ideal in the polynomial ring $S = k[x_0, x_1, x_2, x_3]$ defined by two forms of degrees $a, b > 1$. Let $C = V(I_{a,b})$ be the curve in $\mathbb{P}^3$ defined by $I_{a,b}$. We will assume that $C$ is smooth and in generic coordinates. In other words, we assume that $I_{a,b}$ is prime, that the singular locus of $S/I_{a,b}$ consists solely of the homogeneous maximal ideal and that $I_{a,b}$ is in generic coordinates. We have that $C$ has degree $ab$ and genus $ab(a + b - 4)/2 + 1$. From Theorem 3.5 we know that $K_0(I_{a,b})$ is the radical ideal of the projection $\pi : C \to \mathbb{P}^2$ from the point $[1 : 0 : 0 : 0]$. Since $C$ is in generic coordinates by assumption, the projection $\pi$ is generic. Proposition 4.1 describes additional numerical data associated to $\pi(C)$. 

Proposition 4.1. The ideal $K_0(I_{a,b})$ is generated by a single polynomial of degree $ab$. It cuts out a degree $ab$ curve with $ab(a - 1)(b - 1)/2$ nodes.

Proof. We already know that $K_0(I_{a,b})$ is the radical ideal of $\pi(C)$ which has degree $ab$. So it remains to show that $\pi(C)$ has $ab(a - 1)(b - 1)/2$ nodes.

Since a general projection of any space curve has only nodes as singularities, we have that $\pi(C)$ is a plane curve with only nodes as singularities. Since $C$ is the normalization of $\pi(C)$ and $C$ has genus $ab(a + b - 4)/2 + 1$, $\pi(C)$ has

$$\frac{(ab - 1)(ab - 2)}{2} - \left(\frac{ab(a + b - 4)}{2} + 1\right) = \frac{a(a - 1)(b)(b - 1)}{2}$$

nodes (see Remark 3.11.1 in [17]). \qed

Already, we can begin to describe the generators of $\text{gin}_{\text{lex}} I_{a,b}$:

Corollary 4.2. The ideal $\text{gin}_{\text{lex}} I_{a,b}$ contains $x_1^{ab}$ and this is the only generator that is not divisible by $x_0$.

Proof. The generators of $\text{gin}_{\text{lex}} I_{a,b}$ are elements of $x_0^p \text{gin}_{\text{lex}} K_p(I_{a,b})$ for various $p$. So clearly, the generators of $\text{gin}_{\text{lex}} K_0(I_{a,b})$ are the only generators of $\text{gin}_{\text{lex}} I_{a,b}$ not containing a factor of $x_0$. But $K_0(I_{a,b})$ is principal, generated by a form of degree $ab$ in generic coordinates. The leading term of such a form is $x_1^{ab}$. \qed

We are ready to prove the main result of the paper:

Theorem [17]. Set $I = I_{a,b}$, $K_p = K_p(I)$. By virtue of Lemma 3.2 and since $x_0^a \in \text{gin}_{\text{lex}} I$ we have

$$\text{gin}_{\text{lex}} I = \sum_{p=0}^{a} x_0^p \text{gin}_{\text{lex}} K_p$$

From Proposition 4.1 we know that $\text{gin}_{\text{lex}} K_0 = (x_1^{ab})$. The proof consists of three steps. First, we compute the regularity of $\text{gin}_{\text{lex}} K_1$ explicitly. Then we show that the regularity of $\text{gin}_{\text{lex}} K_p - p \leq 1 + \text{reg} \text{gin}_{\text{lex}} K_1$ for $2 \leq p \leq a - 1$. Finally, we will show that $\text{gin}_{\text{lex}} I$ actually requires a generator of degree $\frac{1}{2}a(a - 1)(b(b - 1) + 1$, which will complete the proof.

By Corollary 3.12 we have that $K_1$ is the ideal of maximal minors of a matrix of size $(a - 1) \times a$ whose $ij$ entry has degree $b + i - j$. The resolution of $K_1$ is given by the Hilbert-Burch complex. It is then easy to determine the degree of $K_1$ from the numerical data of the resolution. We obtain that $K_1$ is unmixed and of degree $\frac{1}{2}a(a - 1)(b(b - 1)$.
1). We also know that the radical of $K_1$ is the ideal of definition of $\frac{1}{2}a(a - 1)b(b - 1)$ points. It follows that $K_1$ itself is the radical ideal defining $\frac{1}{2}a(a - 1)b(b - 1)$ points. We can conclude from Corollary \ref{cor:reg} that $\text{reg} \gin_{\text{lex}}(K_1) = \frac{1}{2}a(a - 1)b(b - 1)$.

We now prove that for $p > 1$, the degrees of the generators of $x_0^p \gin_{\text{lex}} K_p$ are bounded above by $1 + \text{reg} \gin_{\text{lex}} K_1$, that is, by $1 + \frac{1}{2}a(a - 1)b(b - 1)$. This will imply that $\text{reg} \gin_{\text{lex}} I$ is $(ab, 1 + \frac{1}{2}a(a - 1)b(b - 1))$.

From Corollary \ref{cor:reg} \ref{cor:reg2} we have that the ideal, say $J$, of the maximal minors of $\text{Syl}_2(f, g)$ is Artinian (i.e. $K[x_1, x_2, x_3]/J$ is Artinian) and is contained in $K_2$ and that $J$ is contained in $K_p$ for $p > 1$. The regularity of an Artinian ideal $D$ is given by the smallest $k$ such that the $k$-th power of the maximal ideal is contained in the ideal $D$ and hence does not change when passing to the initial ideal. It follows that $\text{reg} \gin_{\text{lex}} K_p \leq \text{reg} J$ for every $p > 1$. So the generators of $x_0^p \gin_{\text{lex}} K_p$ are in degrees $\leq p + \text{reg} J$. Taking into consideration that $K_a = (1)$, it is enough to show that $\text{reg} J \leq \frac{1}{2}a(a - 1)b(b - 1) + 1 - p$ for all $p = 2, \ldots, a - 1$. So we may assume $a > 2$ and we have to show that $\text{reg} J \leq \frac{1}{2}a(a - 1)b(b - 1) + 2 - a$. To compute the regularity of $J$, we first use Lemma \ref{lem:reg} \ref{lem:reg1} to get rid of the units in the matrix defining $J$ and then we use Lemma \ref{lem:reg} \ref{lem:reg2}. We get $\text{reg}(J) = ab + \left(\frac{a - 1}{2}\right) - \left(\frac{a + 1}{2}\right) + 2(a - 3)$.

So it remains to show that

$$ab + \left(\frac{a - 1}{2}\right) - \left(\frac{a + 1}{2}\right) + 2(a - 3) \leq \frac{1}{2}a(a - 1)b(b - 1) + 2 - a$$

that is

$$1/2a^2b^2 - 1/2a^2b - 1/2ab^2 - 1/2ab - a + 7 \geq 0$$

for all $3 \leq a \leq b$. This is a simple calculus exercise.

To finish the proof, we will show that if $m$ is a minimal generator of $\gin_{\text{lex}} K_1$, of degree $\frac{1}{2}a(a - 1)b(b - 1)$, then $x_0^0 m$ is a minimal generator of $\gin_{\text{lex}} I$. If $x_0^0 m$ is not a minimal generator of $\gin_{\text{lex}} I$, then it must be divisible by some monomial $n$ that is a minimal generator of $\gin_{\text{lex}} I$. This implies that $n \mid x_0 m$ and that $n$ must be in $\gin_{\text{lex}} K_0$. However, this means that $n \mid m$ and $n \in \gin_{\text{lex}} K_1$ since $K_0 \subseteq K_1$. This contradicts our choice of $m$ as a minimal generator. We conclude that $x_0 m$ must be a minimal generator of $\gin_{\text{lex}} I$. \hfill $\square$

**Example 4.3.** One can check (using CoCoA, for instance) that $I = (x^3 - y^2z, y^3 - z^2t)$ defines an irreducible complete intersection curve $C$ with just one singular point and that $K_1(gI)$ with $g$ a generic change of
coordinates is not radical. Indeed, $K_1(gI)$ has degree 18 and it defines only 11 points, namely the 11 singular points of the generic projection of $C$ to $\mathbb{P}^2$. In this case, the regularity of $\text{gin}_{\text{lex}}(I)$ is 16 and not 19 as in the smooth case.

5. The regularity of gins of points

Set $S = k[x_0, \ldots, x_r]$. We start with the following well-known lemma:

**Lemma 5.1.** Let $I$ be a homogeneous ideal of $S$ such that $S/I$ has Krull dimension 1 and $\deg(S/I) = e$. Set $c = \min\{j | \dim[S/I]_i = e \text{ for all } i \geq j\}$. Then $\text{reg}(I) \leq \max\{e, c\}$.

**Proof.** Let $J$ be the saturation of $I$. Then $S/J$ is a 1-dimensional CM (Cohen-Macaulay) algebra. It is well-known and easy to see that $\text{reg}(J) \leq \deg(S/J) = e$ and $\dim[S/J]_i = e$ for all $i \geq e - 1$. Let $p$ denote the saturation degree (satiety index) of $I$, i.e. the least $j$ such that $I_i = J_i$ for all $i \geq j$. From the characterization of regularity in terms of local cohomology it follows immediately that $\text{reg}(I) = \max\{\text{reg}(J), p\}$. To conclude, it is enough to show that $p \leq \max\{e, c\}$. If $p > e$ then $I_i = J_i$ for all $i \geq p$ and $I_{p-1} \subsetneq J_{p-1}$. Thus, $\dim[S/I]_i = e$ for all $i \geq p$ and $\dim[S/I]_{p-1} > e$. Hence $p = c$ and we are done. □

**Corollary 5.2.** Let $I$ be a homogeneous ideal of $S$ such that $S/I$ has Krull dimension 1. Assume that the Hilbert function of $I$ is equal to the Hilbert function of a 1-dimensional CM ideal. (e.g. $I$ is an initial ideal of a 1-dimensional CM ideal). Then $\text{reg}(I) \leq \deg(S/I)$.

**Proof.** This follows from Lemma 5.1 since the assumption implies that $\dim[S/I]_i = \deg(S/I)$ for all $i \geq \deg(S/I) - 1$. □

**Corollary 5.3.** Let $I$ be the ideal of a set $X$ of $s$ points of $\mathbb{P}^r$. Then $\text{reg} \text{gin}_{\text{lex}} I = s$.

**Proof.** By Corollary 5.2 we have $\text{reg} \text{gin}_{\text{lex}} I \leq s$. A general projection of $X$ to $\mathbb{P}^1$ will give $s$ distinct points. This implies that $x_{r-1}^s$ is in $\text{gin}_{\text{lex}} I$. Since we work with the lex order, $x_{r-1}^s$ is a minimal generator of $\text{gin}_{\text{lex}} I$. □

We want to show now that for a set of generic points the gin lex and indeed any gin has a very special form: it is a segment ideal. Consider the polynomial ring $S = k[x_0, \ldots, x_r]$ equipped with a term order $\tau$. Assume that $x_0 >_\tau x_1 >_\tau \cdots >_\tau x_r$.

**Definition 5.4.** A vector space $V$ of forms of degree $d$ is said to be a $\tau$-segment if it is generated by monomials and for every monomial $m$ in $V$ and every monomial $n$ of degree $d$ with $n >_\tau m$ one has $n \in V$. 
Given a non-negative integer \( u \leq \binom{r+d}{r} \) there exists exactly one \( \tau \)-segment of forms of degree \( d \) and of dimension \( u \): it is the space generated by the \( u \) largest monomials of degree \( d \) with respect to \( \tau \) and it will be denoted by \( \text{Seg}_\tau(d, u) \). Given a homogeneous ideal \( I \) for every \( d \) we consider the \( \tau \)-segment \( \text{Seg}_\tau(d, \dim I_d) \) and define

\[
\text{Seg}_\tau(I) = \bigoplus_d \text{Seg}_\tau(d, \dim I_d).
\]

By the very definition, \( \text{Seg}_\tau(I) \) is a graded monomial vector space and simple examples show that \( \text{Seg}_\tau(I) \) is not an ideal in general. But there are important exceptions: Macaulay’s numerical characterization of Hilbert functions \cite[Thm.4.2.10]{Macaulay} can be rephrased by saying that for every homogeneous ideal \( I \) the space \( \text{Seg}_{\text{lex}}(I) \) is an ideal. In the following lemma we collect a few simple facts about segments that will be used in the proof of that result.

**Lemma 5.5.** Let \( \tau \) be a term order and let \( V \subset S_a \) be a \( \tau \)-segment with \( \dim S_a/V \leq a \). Then \( S_1V \) is a \( \tau \)-segment with \( \dim S_{a+1}/VS_1 = \dim S_a/V \).

**Proof.** First observe that since \( x_{r-1}^a > x_{r-1}^{a-j}x_{r}^j \) for \( j = 1, \ldots, a \) we have that \( x_{r-1}^a \in V \) and hence \( (x_0, \ldots, x_{r-1})^a \subseteq V \). To prove that \( VS_1 \) is a \( \tau \)-segment assume that \( n \) is a monomial of degree \( a + 1 \) such that \( x_i m < n \) with \( m \) in \( V \); we have to show that \( n \in VS_1 \). Let \( k \) be the largest index such that \( x_k \) divides \( n \), so that \( n = x_k n_1 \). If \( k \geq i \) then \( x_i n_1 \geq x_k n_1 > x_i m \). It follows that \( n_1 > m \) and hence \( n_1 \in V \) so that \( n \in VS_1 \). If, instead, \( k < i \) then \( n \in (x_0, \ldots, x_{r-1})^{a+1} \) which is contained in \( VS_1 \) since we have seen already that \( (x_0, \ldots, x_{r-1})^a \) is contained in \( V \).

To conclude, it is enough to show that the map \( \phi \) induced by multiplication by \( x_r \) is an isomorphism from \( S_a/V \) to \( S_{a+1}/VS_1 \). We show first that \( \phi \) is injective. If \( m \) is a monomial in \( S_a \setminus V \), then \( mx_r \not\in VS_1 \). Otherwise, \( mx_r = nx_i \) for some \( n \in V \) and some \( i \), and then \( m > n \), a contradiction. To prove that \( \phi \) is surjective, consider a monomial \( m \) in \( S_{a+1} \setminus VS_1 \). Then \( m = x_r n \) since \( (x_0, \ldots, x_{r-1})^{a+1} \subseteq VS_1 \). Obviously, \( n \not\in V \). So \( \phi \) is surjective. \( \square \)

**Proposition 5.6.** Let \( I \) be the ideal defining \( s \) points, say \( P_1, \ldots, P_s \), of \( \mathbb{P}^r \). Assume that there exists a coordinate system \( x_0, x_1, \ldots, x_r \) such that \( I \) does not contain forms of degree \( \leq s \) supported on \( \leq s \) monomials. Then \( \text{gen}_r I = \text{Seg}_\tau(I) \) for all term orders \( \tau \). In particular \( \text{gen}_{\text{lex}} I = \text{Seg}_{\text{lex}}(I) \).

**Proof.** It is easy to see that the assumption implies that the Hilbert function of \( S/I \) is the expected one, namely \( \dim[S/I]_d = \min \{ s, \binom{r+d}{r} \} \)
for all $d$. Fix a term order $\tau$. For a given $d \leq s$ consider the set $M_d$ of the smallest (with respect to $\tau$) $\min\{s, \binom{r+d}{r}\}$ monomials of degree $d$. By assumption these monomials are a basis of $S/I$ in degree $d$. It follows immediately that $\text{in}_\tau I_d = \text{Seg}_{\tau}(I)_d$ for every $d \leq s$. From Lemma 5.1 we know that in $\tau I$ does not have generators in degree $\geq s$. Then in $\tau I_d = \text{Seg}_{\tau}(I)_d$ for all $d \geq s$. On the other hand, it follows from Lemma 5.5 that $\text{Seg}_{\tau}(I)_d = \text{Seg}_{\tau}(I)_s$ for all $d \geq s$. We have seen already that in $\tau I_s = \text{Seg}_{\tau}(I)_s$. Therefore we may conclude that in $\tau I_d = \text{Seg}_{\tau}(I)_d$ also for all $d \geq s$. We have shown that in $\tau I = \text{Seg}_{\tau}(I)$. From this it follows that $\text{gin}_\tau I = \text{Seg}_{\tau}(I)$ (see the construction/definition of $\text{gin}$ given in [12, Theorem 15.18].) $\square$

We can now prove the main result of this section:

**Theorem 1.2.** Let $P_1, \ldots, P_s$ be generic points in $\mathbb{P}^r$. Fix a coordinate system on $\mathbb{P}^r$ and let $(a_{i0}, a_{i1}, \ldots, a_{ir})$ be the coordinates of $P_i$. It is enough to show that the assumption of Proposition 5.6 holds (in the given coordinates) for a generic choice of the $a_{ij}$. For any $d \leq s$ consider the $s \times \binom{r+d}{r}$ matrix $X_d$ whose rows are indexed by the points, the columns by the monomials of degree $d$ and whose $ij$-th entry is obtained by evaluating the $j$-th monomial at the $i$-th point. The assumption of Proposition 5.6 is equivalent to the fact that any maximal minor of $X_d$ is non-zero for $d \leq s$. If we consider the $a_{ij}$ as variables over some base field then every minor of $X_d$ is a non-zero polynomial in the $a_{ij}$ since no cancellation can occur in the expansion. So these are finitely many non-trivial polynomial conditions on the coordinates of the points. $\square$

As we have already said, the genericity condition required in Theorem 1.2 implies that the Hilbert function of the ideal $I$ of $s$ points of $\mathbb{P}^r$ is given is the expected one:

$$\dim[S/I]_j = \min(s, \binom{r+j}{r}).$$

One may wonder whether it is enough to assume that the Hilbert function is generic to conclude that $\text{gin}_\tau I$ is $\text{Seg}_{\tau}(I)$ for an ideal of points. The next example answer this question.

**Example 5.7.** (a) Consider the ideal $I$ of 7 points of $\mathbb{P}^3$ with generic Hilbert function. The ideal $I$ contains 3 quadrics. If the 3 quadrics have a common linear factor, then $\text{gin}(I_2)$ is $x_0(x_0, x_1, x_2)$ no matter what the term order is. So in particular, $\text{gin}_{\text{revlex}} I$ is not $\text{Seg}_{\text{revlex}}(I)$ in degree 2. Explicitly, one can take the 7 points with coordinates $(0, 0, 0, 1), (0, 0, 1, 1), (0, 0, 2, 1), (0, 1, 0, 1), (0, 1, 1, 1), (0, 2, 0, 1), (1, 0, 0, 1).$
(b) Consider the 10 points of $P^3$ with coordinates $(a, b, c, 1)$ where $a, b, c$ are non negative integers with $a + b + c \leq 2$ and let $I$ the corresponding ideal. One can check with (and even without) the help of a computer algebra system that the 10 points have the generic Hilbert function and that any generic projection to $P^2$ gives 10 points on a cubic. This, in turn, implies that $\text{gin}_{\text{lex}} I$ contains $x_2^3$ while $\text{Seg}_{\text{lex}}(I)$ does not contain it.

The next example shows that, even for Hilbert functions of generic points in $P^2$, the segment ideals are special among the Borel-fixed ideals.

**Example 5.8.** Consider the ideal $I$ of 7 generic points in $P^2$. The Hilbert function of $S/I$ is $(1, 3, 6, 7, 7, 7, \ldots)$. There are exactly 8 Borel-fixed ideals with this Hilbert function, they are:

1. $(x^3, x^2y, x^2z, xy^3, xy^2z, xyz^3, xz^5, y^7)$, lex
2. $(x^3, x^2y, x^2z, xy^3, x^2y^2z, xyz^3, y^6)$, $(6, 2, 1)$
3. $(x^3, x^2y, x^2z, x^3y^2, x^2y^2z, y^7)$, $(4, 2, 1)$
4. $(x^3, x^2y, x^2z, xy^3, y^4)$
5. $(x^3, x^2y, x^2z, x^2y^2, xy^2z, xyz^3, xz^5, y^7)$,
6. $(x^3, x^2y, x^2z, xy^3, x^2z^2, xzy^3, y^6)$,
7. $(x^3, x^2y, x^2z, x^2y^2, x^2z^2, y^5)$,
8. $(x^3, x^2y, x^2z, y^4)$, revlex

The ideals (1),(2),(3) and (8) are segments (with respect to the term order or weight indicated on the right) while the remaining four are non-segments. Let us check, for instance, that (4) is not a segment. Suppose, by contradiction, it is a segment with respect to a term order $\tau$. Then since $x^2z$ is in and $xy^2$ is out, we have $x^2z >_\tau xy^2$ and hence $xz >_\tau y^2$. We deduce that $xy^2z >_\tau y^4$. But since $y^4$ is in then also $xy^2z$ must be in and this is a contradiction. Summing up, among the 8 Borel-fixed ideals only (1),(2),(3) and (8) are gins of $I$.

**References**

[1] D. Bayer, *The Division Algorithm and the Hilbert Scheme*, Ph.D. thesis, Harvard University, 1982.
[2] D. Bayer and M. Stillman, *A criterion for detecting m-regularity*, Invent.Math. 87 (1987), 1–11.
[3] I. Bermejo and M. Lejeune-Jalabert, *Sur la complexité du calcul des projections d’une courbe projective*, Comm. Alg. 27 (1999), no. 7, 3211–3220.
[4] A. Bigatti, *Upper bounds for the Betti numbers of a given Hilbert function*, Comm. Algebra 21 (1993), no. 7, 2317–2334.
[5] W. Bruns and J. Herzog, *Cohen Macaulay Rings*, Cambridge University Press, Cambridge (1993).
[6] W. Bruns and J. Herzog, *On the computation of a-invariants*, Manuscripta Math. 77 (1992), no. 2-3, 201–213.
[7] W. Bruns and U. Vetter, *Determinantal Rings*, Lecture Notes in Mathematics, 1327, Springer-Verlag, New York, 1980.
[8] M. Chardin and G. Moreno-Socías, *Regularity of lex-segment ideals: some closed formulas and applications*, Proc. Amer. Math. Soc. 131 (2002), no. 4, 1093–1102.
[9] A. Caponi, G. Niesi, L. Robbiano, *CoCoA, a system for doing Computations in Commutative Algebra*, Available via anonymous ftp from: cocoa.dima.unige.it.
[10] N. Budur, M. Casanellas and E. Gorla *Hilbert functions of irreducible arithmetically Gorenstein schemes*, to appear in J. Algebra.
[11] D. Cox, J. Little, and D. O’Shea, *Ideals, varieties, and algorithms*, Springer-Verlag, New York (1997).
[12] D. Eisenbud, *An Introduction to Commutative Algebra with a View Towards Algebraic Geometry*, Springer-Verlag, New York (1995).
[13] S. Eliahou and M. Kervaire, *Minimal resolutions of some monomial ideals*, J. Algebra 129 (1990), 1–25.
[14] D. Grayson and M. Stillman, *Macaulay 2 – a system for computation in algebraic geometry and commutative algebra*, [http://www.math.uiuc.edu/Macaulay2](http://www.math.uiuc.edu/Macaulay2), 1997.
[15] M. Green, *Generic initial ideals*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., 166, Birkhäuser, Basel, (1998), 119–186.
[16] M. Haiman, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. 14, 2001, 941-1006.
[17] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.
[18] H. Hulett, *Maximum Betti numbers of homogeneous ideals with a given Hilbert function*, Comm. Algebra 21 (1993), no. 7, 2335–2350.
[19] M. Kreuzer and L. Robbiano, *Computational commutative algebra*, Springer-Verlag, Berlin, 2000.
[20] M. G. Marinari and L. Ramella, *Some properties of Borel ideals*, J. Pure Appl. Algebra, 139 (1999), no. 1-3, 183–2000.
[21] K. Pardue, *Deformation classes of graded modules and maximal Betti numbers*, Illinois J. Math., 40 (1996), no. 4, 564–585.

E-mail address: conca@dima.unige.it

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI GENOVA, VIA DODECANESO 35, I-16146 GENOVA, ITALIA

E-mail address: jsidman@mtholyoke.edu

415A Clapp Lab, Department of Mathematics and Statistics, Mount Holyoke College, South Hadley, MA 01075.