Nonpointlike Particles in Harmonic Oscillators

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Abstract
Quantum mechanics ordinarily describes particles as being pointlike, in the sense that the uncertainty $\Delta x$ can, in principle, be made arbitrarily small. It has been shown that suitable correction terms to the canonical commutation relations induce a finite lower bound to spatial localisation. Here, we perturbatively calculate the corrections to the energy levels of an in this sense nonpointlike particle in isotropic harmonic oscillators. Apart from a special case the degeneracy of the energy levels is removed.

1. Introduction
It has been shown that certain small corrections to the canonical commutation relations yield an interesting new short distance structure, characterised by a finite minimal uncertainty $\Delta x_0$. The algebraic and functional analytic structure underlying minimal uncertainties in positions (and/or in momenta) first appeared in [1, 2], and recent studies are e.g. [3]-[9]. The approach originated [10, 11] in the field of quantum groups [12], which is related to noncommutative geometry [13]. Part of the motivation to investigate a possible minimal uncertainty $\Delta x_0$ is related to string theory and quantum gravity where this type of short distance behaviour has been suggested to arise at the Planck scale, see e.g. [14]-[16]. For a review see [17]. On the other hand, examples of quanta which cannot be localised to a point are also e.g. nucleons, quasiparticles or various collective excitations. Our aim here is to investigate whether the new ansatz, with suitably adjusted scales, may also serve for an effective low energy description of such nonpointlike particles. To this end we consider a nonpointlike particle in a $d$-dimensional isotropic harmonic oscillator. The special case of the 1-dimensional oscillator was solved in [8]. Here, the application of different methods allows us to perturbatively calculate the spectra for the general case where we find, as a new effect, a characteristic splitting of the ordinarily degenerate energy levels.

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2. Heisenberg algebra

The associative Heisenberg algebra generated by \( x \) and \( p \) with commutation relations \( (\beta > 0 \text{ assumed small}) \) \([x, p] = i\hbar (1 + \beta p^2)\) yields the uncertainty relation \( \Delta x \Delta p \geq \hbar / 2(1 + \beta (\Delta p)^2 + \beta \langle p^2 \rangle)\) which is readily checked to imply a minimal uncertainty in positions \( \Delta x_0 = \hbar \sqrt{\beta} \). The functional analytic structure, and the properties of the states of then maximal localisation are discussed in detail in [1, 2, 8, 9].

In \( d \) dimensions the ansatz
\[
[x_i, p_j] = i\hbar \Theta_{ij}(p)
\]
yields a minimal uncertainty \( \Delta x_0 > 0 \) for appropriate choices of symmetric \( \Theta \). We will leave momentum space ‘classical’, i.e. \([p_i, p_j] = 0\). The Jacobi identity and the requirements \( x_i = x_i^*, p_i = p_i^* \) then uniquely determine the commutation relations for the now in the generic case noncommutative position operators:
\[
[x_i, x_j] = i\hbar \{x_a, \Theta^{-1}_{a \alpha} \Theta_{\alpha [j} \Theta_{\beta r, s]}\}
\]
For simplicity we adopt a geometric notation with \( f, s \) standing for \( \partial p_s f \) and where repeated indices are summed over. The associative Heisenberg algebra \( \mathcal{A} \) finds a Hilbert space representation e.g. on momentum space through:
\[
p_i \psi(p) = p_i \psi(p)
\]
\[
x_i \psi(p) = i\hbar (1/2\Theta_{ai,a} + \Theta_{ai} \partial p_a) \psi(p)
\]
For more details on the geometric structure see [5]. Assuming a rotationally isotropic situation and the minimal uncertainty \( \Delta x_0 \) to be small, we here only consider the lowest order correction terms to the canonical commutation relations
\[
\Theta_{ij}(p) = \delta_{ij} + \beta \delta_{ij} p^2 + \beta' p_i p_j
\]
where \( p^2 := \sum_{i=1}^d p_i p_i \), and where \( \beta, \beta' > 0 \) are assumed small of the first order. The same mechanism as in the one-dimensional case yields from the corresponding uncertainty relations \( \Delta x \Delta p_i \geq |\langle [x_i, p_j] \rangle|/2 \) an isotropic (i.e. \( \Delta x_{0i} = \Delta x_{0j}, \forall i, j \)) minimal uncertainty \( \Delta x_0 \) (dropping the index \( i \)):
\[
\Delta x_0 = \hbar \sqrt{\beta d + \beta'}
\]
The commutation relations in the Heisenberg algebra \( \mathcal{A} \) then read \([p_i, p_j] = 0\), and:
\[
[x_i, p_j] = i\hbar (\delta_{ij} + \beta \delta_{ij} p^2 + \beta' p_i p_j) + \mathcal{O}^2(\beta, \beta')
\]
\[
[x_i, x_j] = i\hbar (\beta'/2 - \beta)(\{x_i, p_j\} - \{x_j, p_i\}) + \mathcal{O}^2(\beta, \beta')
\]
In the momentum representation, where
\[
x_i \psi(p) = i\hbar \left[ (\beta + \beta' \frac{d + 1}{2}) p_i + (\delta_{ia} + \beta \delta_{ia} p^2 + \beta' p_i p_a) \partial p_a \right] \psi(p)
\]
we have to this order:

\[ [\mathbf{x}_i, \mathbf{x}_j]\psi(p) = \hbar^2(\beta' - 2\beta)(p_i\partial_{p_j} - p_j\partial_{p_i})\psi(p) + \mathcal{O}^2(\beta, \beta') \]  

(10)

In the case \( \beta = \beta'/2 \) the \( \mathbf{x}_i \) are commutative, which may therefore be considered a preferred choice of parameters. The framework is then translation invariant in the sense that \( \mathbf{p}_i \to \mathbf{p}_i, \mathbf{x}_i \to \mathbf{x}_i + \alpha_i \) defines an algebra homomorphism of \( \mathcal{A} \). More generally, this feature holds for any \( \Theta \) that obeys (from Eq.2) \( \Theta_{ai}\partial_{p_i}\Theta_{bc} = \Theta_{ib}\partial_{p_i}\Theta_{ac} \). In the symmetric case \( \Theta_{ij} := \delta_{ij} f(p^2) + g(p^2)p_ip_j \) this condition is \( g = 2ff'(f - 2p^2f')^{-1} \), where we may choose e.g. \( f := e^{\beta p^2} \) to recover to first order Eq.5 with \( \beta' = 2\beta \).

Recall, that generally \( Q|\lambda\rangle = \lambda|\lambda\rangle \Rightarrow (\Delta Q)|\lambda\rangle = 0 \). Indeed, due to \( \Delta x_0 > 0 \), for \( \mathcal{A} \) there no longer exists a spectral representation of the \( \mathbf{x}_i \), see [1, 2]. Therefore, in [3] the concept of quasi-position representation has been introduced. The quasi position wave function of a state \( |\psi\rangle \) is \( \psi(\xi) := \langle \phi^{\xi}\psi|\psi\rangle \) where the \( |\phi^{\xi}\rangle \) are states of now maximal spatial localisation around positions \( \xi \) (replacing position eigenstates). In [4] the concept has been extended to include a possible \( \Delta p_0 > 0 \).

We stress that corrections to the commutation relations necessarily induce new physical features, which could not alternatively be described by keeping the ordinary commutation relations and adding corrections to Hamiltonians. Although a conventional ‘point particle’ system can have the same energy spectrum, for a suitably chosen Hamiltonian, the full set of physical predictions will generally differ. This is because for all predictions, such as energy spectra, transition amplitudes and expectation values to match, systems must be related by a unitary transformation. However, unitary transformations are commutation relations preserving.

Nevertheless, in Hilbert space representations of the generalised commutation relations the usual perturbative techniques for the calculation of eigenvalues of hermitean operators, such as Hamiltonians, are still applicable.

3. The Harmonic Oscillator

A low energy approximation for most kinds of oscillations is a \( d \)-dimensional harmonic oscillator, which we here for simplicity choose isotropic:

\[ H := \sum_{i=1}^{d} \left( \frac{\mathbf{p}_i^2}{2m} + \frac{m\omega^2\mathbf{x}_i^2}{2} \right) \]  

(11)

The Hamiltonian acts on momentum space as

\[ H.\psi(p) = \left[ \frac{p_i^2}{2m} - \frac{m\omega^2\hbar^2}{2} \sum_{i=1}^{d} \left( \left( \beta + \beta'\frac{d+1}{2} \right)p_i + \partial_{p_i} + \beta p_i^2\partial_{p_i} + \beta'p_ip_a\partial_{p_a} \right)^2 \right] \psi(p) \]  

(12)

which is, to first order in \( \beta, \beta' \):

\[ H.\psi(p) = \left[ \frac{p_i^2}{2m} - \frac{m\omega^2\hbar^2}{2} \left( \beta d + \beta'(d + d^2)/2 + (4\beta + (2 + 2d)\beta')p_ip_a\partial_{p_a} \right) \right] \psi(p) \]
\[ + 2\beta p^2 \partial^2 + 2\beta' p_i p_a \partial p_i \partial p_a + \partial^2 \] \] \( \psi(p) \)  

(13)

Since we are dealing with harmonic oscillators it is convenient to further transform into a Fock representation where \( |\psi\rangle = \psi(a^\dag)|0\rangle \). The multiplication and differentiation operators \( p_i \) and \( \partial_p j \) can be represented through

\[
 p_j \langle \psi \rangle = i(m\omega\hbar/2)^{1/2}(a_j^\dag - a_j)|\psi\rangle \\
 \partial_p j \langle \psi \rangle = -i(2m\omega\hbar)^{-1/2}(a_j^\dag + a_j)|\psi\rangle
\]

(14)

(15)

where \( a_i a_j^\dag - a_j^\dag a_i = \delta_{ij} \), so that

\[
 p_i \partial_i = -\frac{d}{2} + \text{NN-terms} \\
 p^2 \partial^2 = -N^2 - N(d + 1) - \frac{d(d + 2)}{4} + \frac{1}{2} \sum_{i,j=1}^d a_i^2 a_j^2 + \text{NN-terms} \\
 p_ip_j \partial_i \partial_j = N + \frac{d(d + 4)}{4} - \frac{1}{2} \sum_{i,j=1}^d a_i^2 a_j^2 + \text{NN-terms}
\]

(16)

(17)

(18)

where \( N_i := a_i a_i^\dag \), \( N := \sum_{i=1}^d N_i \), and where NN-terms are terms that contain nonequal numbers of raising and lowering operators. Substituting Eqs. 16-18 in Eq. 13 yields for the action of \( H \) on Fock space:

\[
 H \langle \psi \rangle = \left[ \hbar \omega \left( N + \frac{d}{2} \right) \\
 + m\omega^2 \hbar^2 \left( \beta N^2 + (\beta(d + 1) - \beta')N + (\beta d(d + 4) - 3\beta' d)/4 \right) \\
 - m\omega^2 \hbar^2 \beta - \beta' \frac{d}{2} \sum_{i,j=1}^d a_i^2 a_j^2 + \text{NN-terms} \right] \langle \psi \rangle
\]

(19)

The natural length scale of the harmonic oscillator is the inverse length in the exponent of the Hermite functions: \( (\hbar/m\omega)^{1/2} \). Let us replace the parameters \( \beta \) and \( \beta' \) by more intuitive dimensionless parameters \( k, k' \) which measure the minimal uncertainty length scales associated with \( \beta \) and \( \beta' \) in units of the length scale of the oscillator (see Eq.6): \( k := \hbar \sqrt{\beta}/(\hbar/m\omega)^{1/2} \), \( k' := \hbar \sqrt{\beta'}/(\hbar/m\omega)^{1/2} \), i.e. we have \( \beta = k^2/m\omega\hbar \), \( \beta' = k'^2/m\omega\hbar \) and thus, from Eq.6

\[
 \Delta x_0 = \sqrt{k^2d + k'^2} \sqrt{\frac{\hbar}{m\omega}}
\]

(20)

so that

\[
 H \langle \psi \rangle = \hbar \omega \left[ N + \frac{d}{2} + k^2 N^2 + \left( k^2(d + 1) - k'^2 \right) N + \frac{k^2 d(d + 4) - 3k'^2 d}{4} \right. \\
 - \left. \frac{k^2 - k'^2}{2} \sum_{i,j=1}^d a_i^2 a_j^2 + \text{NN-terms} \right] \langle \psi \rangle
\]

(21)
4. First order corrections to the spectra

We read off from Eq.21 that $H$ consists of a diagonal part with degenerate eigenvalues (in more than one dimension), and a nondiagonal term $\sum_{i,j=1}^{d} a_i a_j^{\dagger}$ proportional to $(k^2 - k'^2)$. As a new effect, this nondiagonal term can lead to a splitting of the normally $g(n,d)$-fold degenerate eigenvalues $E_n$ of the $d$-dimensional isotropic harmonic oscillator. We recall the degeneracy function: $g(n,d) = \frac{(n+d-1)!}{n!(d-1)!}$. As is well known, in the calculation of the eigenvalues of $A := B + C$, for $A$, $B$, $C$ hermitian, the first order perturbative corrections to degenerate eigenvalues of $B$ are the eigenvalues of the perturbing matrix $C$ when restricted to the corresponding eigenspaces.

Thus, here the $g(n,d)$-fold degenerate energy levels $E_n$ split into levels $E'_{n_r}$:

$$E'_{n_r}(k, k') = \hbar \omega \left( n + \frac{d}{2} + k^2 n^2 + k^2 (d+1)n - k'^2 n + \frac{k^2 d(d+4) - 3k'^2 d}{4} \right)$$

$$- \frac{k^2 - k'^2}{2} \times \text{r'th Eigenvalue} \left( \left( \sum_{i,j=1}^{d} a_i a_j^{\dagger} \right) \left| H_n \right) \right)$$

(22)

where $r = 1, 2, ..., g(n,d)$ and where the eigenspaces $\mathcal{H}_n$ of the diagonal part of the Hamiltonian are

$$\mathcal{H}_n := \text{span} \left\{ (r_1! \cdot ... \cdot r_d!)^{-1/2} a_1^{\dagger r_1} \cdot \cdot \cdot a_d^{\dagger r_d} | 0 \right\} \sum_{i=1}^{d} r_i = n$$

(23)

The matrix elements of NN-terms vanish in $\mathcal{H}_n$, i.e. for $m = \sum_{i=1}^{d} r_i = \sum_{i=1}^{d} s_i$ there holds $\langle 0 | a_1^{s_1} \cdot \cdot \cdot a_d^{s_d} (\text{NN-terms}) a_1^{r_1} \cdot \cdot \cdot a_d^{r_d} | 0 \rangle = 0$, so that the NN-terms of Eq.21 do not contribute in Eq.22.

For the calculation of the eigenvalues of $\sum_{i,j=1}^{d} a_i a_j^{\dagger}$ in $\mathcal{H}_n$ we can choose the ON-basis given in Eq.23 to obtain the matrix elements $(n = \sum_{i=1}^{d} r_i = \sum_{i=1}^{d} s_i)$:

$$\langle r_1, ..., r_d | \sum_{i,j}^{d} a_i^{2} a_j^{\dagger 2} | s_1, ..., s_d \rangle = \sum_{i,j=1}^{d} \sqrt{(r_i + 1)(r_i + 2)(s_j + 1)(s_j + 2)}$$

$$\times \delta_{r_1, s_1} \delta_{r_2, s_2} \delta_{r_3, s_3} \delta_{r_d, s_d}$$

(24)

We begin with the one-dimensional case. For this case the momentum space Schrödinger equation proved to be exactly solvable in terms of hypergeometric functions, yielding to first order in $\beta$, from Eqs.53,56,69 in [8]:

$$E_n' = \hbar \omega (n + 1/2) + m \omega^2 \hbar^2 \beta (n^2/2 + n/2 + 1/4)$$

(25)

Indeed, we recover this result from Eqs.[9,22] as the special case $d = 1$ (note that $a^2 a^{\dagger 2} = N^2 + 3N + 2$ and that $\beta + \beta'$, corresponds to $\beta$ in [8]).

For $d = 2$, straightforward calculation now yields the $g(n,2) = n + 1$ eigenvalues of $\sum_{i,j=1}^{2} a_i^{2} a_j^{\dagger 2}$ in $\mathcal{H}_n$. If $n$ is odd, these can be put into the form $4s(n + 2 - s)$
for \( s = 1, \ldots, (n+1)/2 \) with all eigenvalues two-fold degenerate. For \( n \) even, \( s \) runs \( s = 1, \ldots, (n+2)/2 \) with the last eigenvalue nondegenerate. Using Eq. (22) we therefore obtain the energy levels for \( d = 2 \) (with the degeneracies given within brackets):

\[
E_{n,s} (k'k) = \hbar \omega \left[ n + 1 + k^2(n^2 + 3n + 3) - k'^2 \left( n + \frac{3}{2} \right) \right] - (k^2 - k'^2) \left\{ \begin{array}{l}
2 \cdot 1(n + 2 - 1) \\
2 \cdot 2(n + 2 - 2) \\
2 \cdot 3(n + 2 - 3) \\
\vdots \\
(n + 1)(n + 3)/2 \\
(n + 2)^2/2
\end{array} \right. (2\times) \text{ for } n \text{ odd} \\
(n + 2)^2/2 (1\times) \text{ for } n \text{ even}
\]

Of particular interest is the 3-dimensional oscillator. To illustrate the calculation, consider e.g. the splitting of the second excited energy level \( E_2 \). We may choose as an ON-basis of \( \mathcal{H}_2 \) (see Eq. (23)): \( e_1 := |2, 0, 0\rangle, e_2 := |0, 2, 0\rangle, e_3 := |0, 0, 2\rangle, e_4 := |0, 1, 1\rangle, e_5 := |1, 0, 1\rangle, e_6 := |1, 1, 0\rangle \), in which:

\[
\left( \sum_{i,j=1}^{d} a_i^2 a_j^2 \right) \mid \mathcal{H}_2 = \begin{pmatrix}
16 & 2 & 2 & 0 & 0 & 0 \\
2 & 16 & 2 & 0 & 0 & 0 \\
2 & 2 & 16 & 0 & 0 & 0 \\
0 & 0 & 0 & 14 & 0 & 0 \\
0 & 0 & 0 & 0 & 14 & 0 \\
0 & 0 & 0 & 0 & 0 & 14
\end{pmatrix}
\]

The eigenvalues are: 14, 14, 14, 14, 14, 20. Thus, the ordinarily 6-fold degenerate second excited level \( E_2 \) splits into two energy levels, one of which is 5-fold degenerate and one nondegenerate (see Eq. (30)). The calculation of the first few excited states shows the systematics in the splitting of the levels:

\[
E'_0 = \hbar \omega \left[ \frac{3}{2} + \frac{21k^2 - 9k'^2}{4} - (k^2 - k'^2) \cdot 3 (1\times) \right]
\]

\[
E'_{1s} = \hbar \omega \left[ \frac{5}{2} + \frac{41k^2 - 13k'^2}{4} - (k^2 - k'^2) \cdot 5 (3\times) \right]
\]

\[
E'_{2s} = \hbar \omega \left[ \frac{7}{2} + \frac{69k^2 - 17k'^2}{4} - (k^2 - k'^2) \cdot \left\{ \begin{array}{l}
7 (5\times) \\
10 (1\times)
\end{array} \right. \right]
\]

\[
E'_{3s} = \hbar \omega \left[ \frac{9}{2} + \frac{105k^2 - 21k'^2}{4} - (k^2 - k'^2) \cdot \left\{ \begin{array}{l}
9 (7\times) \\
14 (3\times)
\end{array} \right. \right]
\]

\[
E'_{4s} = \hbar \omega \left[ \frac{11}{2} + \frac{149k^2 - 25k'^2}{4} - (k^2 - k'^2) \cdot \left\{ \begin{array}{l}
11 (9\times) \\
18 (5\times) \\
21 (1\times)
\end{array} \right. \right]
\]

\[
E'_{5s} = \hbar \omega \left[ \frac{13}{2} + \frac{201k^2 - 29k'^2}{4} - (k^2 - k'^2) \cdot \left\{ \begin{array}{l}
13 (11\times) \\
22 (7\times) \\
27 (3\times)
\end{array} \right. \right]
\]
5. The translation invariant case

The splitting of the energy levels for the case $k'^2 = 2k^2$ is interesting for the framework is then translation invariant and the $x_i$ commute (Eq.10). (Note that there would be no splitting for $k'^2 = k^2$).

$$E'_0 = \frac{3}{2} \hbar \omega + \frac{3}{4} (\Delta x_0)^2 m \omega^2 \quad (1 \times) \quad (34)$$

$$E'_1 = \frac{5}{2} \hbar \omega + \frac{3}{4} (\Delta x_0)^2 m \omega^2 \quad (3 \times) \quad (35)$$

$$E'_2 = \frac{7}{2} \hbar \omega + (\Delta x_0)^2 m \omega^2 \cdot \begin{cases} 63/20 \quad (5 \times) \\ 15/4 \quad (1 \times) \end{cases} \quad (36)$$

$$E'_3 = \frac{9}{2} \hbar \omega + (\Delta x_0)^2 m \omega^2 \cdot \begin{cases} 99/20 \quad (7 \times) \\ 119/20 \quad (3 \times) \end{cases} \quad (37)$$

$$E'_4 = \frac{11}{2} \hbar \omega + (\Delta x_0)^2 m \omega^2 \cdot \begin{cases} 143/20 \quad (9 \times) \\ 171/20 \quad (5 \times) \\ 183/20 \quad (1 \times) \end{cases} \quad (38)$$

$$E'_5 = \frac{13}{2} \hbar \omega + (\Delta x_0)^2 m \omega^2 \cdot \begin{cases} 195/20 \quad (11 \times) \\ 231/20 \quad (7 \times) \\ 251/20 \quad (3 \times) \end{cases} \quad (39)$$

Eqs.34-39 give the first few levels for this case and $d = 3$, expressed in terms of the then only free parameter $\Delta x_0$. For a graph of the spectrum see Fig.1.

![Fig1: Splitting of the energy levels of the 3-dim isotropic harmonic oscillator for a nonpointlike particle with relative size $k = 1/10$.]
6. Discussion

We are considering oscillators for which a lowest order anharmonicity originates in the nonpointlike nature of the oscillating particle. Instead of describing nonpointlike particles from first principles, such as nucleons in terms of their quark content, we here aim at an effective low energy description which accounts for nonpointlikeness through the introduction of a finite $\Delta x_0$. Our calculation of the lowest order corrections to the energy levels of an in this sense nonpointlike particle in an isotropic harmonic oscillator predicts the new effect of a characteristic fine splitting of the ordinarily degenerate energy levels.

While rotation invariance of the commutation relations allows two free parameters, the additional requirement of translation invariance (the case $\beta' = 2\beta, k'^2 = 2k^2$), reduces these to one free parameter $\Delta x_0 = \hbar((d+2)\beta)^{1/2} = k((d+2)\hbar/m\omega)^{1/2}$. With only a single parameter $\Delta x_0$ to fit, it should in principle be possible to experimentally check the applicability of the approach for any given species of nonpointlike particles.

The new approach of course also allows to straightforwardly calculate arbitrary further details of the system, such as wave functions, transition amplitudes or probability amplitudes for spatially localised measurements, in arbitrary dimensions and also for non-isotropic oscillators.

Concerning the possibility of a fundamental $\Delta x_0 > 0$, which we mentioned in the beginning, it should be interesting to apply similar perturbative methods to the case of the Hydrogen atom. This should yield the relation between the scale of an assumed non-pointlikeness of the electron and the scale of the thereby caused effects on the Hydrogen spectrum. The high precision to which such effects can be experimentally excluded could then yield an interesting upper bound for a possible fundamental non-pointlikeness $\Delta x_0$ of the $e^-$. This issue is presently under investigation. Recent studies on minimal uncertainties and regularisation in field theory are [6, 7]. For related studies see e.g. [8, 9].

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