Precision Neutrino Counting

Gary Steigman

Departments of Physics and Astronomy, The Ohio State University, Columbus, OH 43210

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In the framework of the standard, hot big bang cosmological model the dynamics of the early evolution of the universe is controlled by the energy density of relativistic particles, among which neutrinos play an important role. In equilibrium, the energy density contributed by one flavor of relativistic neutrinos is $7/8$ of that of the cosmic background radiation (CBR) photons. As the universe expands and cools, neutrinos decouple and their subsequent contribution to the energy density is modified by the relative heating of the CBR photons when electron-positron pairs annihilate. The small corrections to the post-$e^\pm$ annihilation energy density of the standard model neutrinos due to incomplete decoupling and finite-temperature QED effects are reviewed (correcting an error in the literature) and extended to account for possible additional relativistic degrees of freedom whose presence might modify the predictions of primordial nucleosynthesis and of the predicted CBR anisotropies.

I. INTRODUCTION

During the early evolution of the universe electroweak processes establish and maintain equilibrium between the standard model neutrinos ($\nu_e$, $\nu_\mu$, and $\nu_\tau$) and the cosmic background radiation (CBR) [1] - [3]. In equilibrium the phase space distributions are Fermi-Dirac for the neutrinos and Bose-Einstein distributions for the CBR photons and the temperatures are equal ($T_\gamma = T_\nu \equiv T_{\nu_e} = T_{\nu_\mu} = T_{\nu_\tau}$). The energy densities are related by

$$\rho_\nu \equiv \rho_{\nu_e} = \rho_{\nu_\mu} = \rho_{\nu_\tau} = \frac{7}{8} \rho_\gamma. \quad (1)$$

During early, radiation dominated (RD) epochs the universal expansion rate is uniquely determined by the energy density in relativistic particles. Prior to $e^\pm$ annihilation, at a temperature of a few MeV, the energy density receives its contributions from CBR photons, $e^\pm$ pairs, and three flavors of neutrinos,

$$\rho = \rho_\gamma + \rho_e + 3 \rho_\nu = \frac{43}{8} \rho_\gamma. \quad (2)$$

At this time ($T \sim$ few MeV) the neutrinos are beginning to decouple from the photon -- $e^\pm$ plasma and the neutron to proton ratio, which is key to the primordial abundance of $^4$He, is decreasing. As a result, the predictions of primordial nucleosynthesis depend sensitively on the early expansion rate. During these RD epochs the age and the energy density are related by $\frac{32}{3} G \rho t^2 = 1$, so that the age of the universe is known once the temperature is specified,

$$t \ T_\gamma^2 = 0.738 \ \text{MeV}^2 \ \text{s}. \quad (3)$$

In many extensions of the standard models of cosmology and of particle physics there can be “extra” energy density contained in new particles or fields, $\rho_X$. When $X$ behaves like radiation, that is when its pressure and energy density are related by $p_X = \frac{1}{3} \rho_X$, it is convenient to account for this extra energy density by normalizing it to that of an equivalent neutrino flavor [4].

$$\rho_X = \Delta N_\nu \rho_\nu = \frac{7}{8} \Delta N_\nu \rho_\gamma. \quad (4)$$

In the following it is assumed that $X$ has decoupled prior to $e^\pm$ annihilation and its temperature (or, equivalently, its number density) is used to define a “comoving volume”. For each such “neutrino-like” particle (i.e., a two-component fermion), if $T_X = T_\nu$, then $\Delta N_\nu = 1$; if $X$ is a scalar, $\Delta N_\nu = 4/7$. However, it may well be that $X$ has decoupled earlier in the evolution of the universe and has failed to profit from the heating when various particle-antiparticle pairs annihilated (or unstable particles decayed). In this case, the contribution to $\Delta N_\nu$, from each such particle will be $< 1 (< 4/7)$. Since we are interested in the deviations of $T_\gamma$ and $T_\nu$ from $T_X$ when $e^\pm$ pairs annihilate, for convenience and without loss of generality, we may define $T_X \equiv T_\nu$ prior to $e^\pm$ annihilation. In the presence of this extra component, the pre-$e^\pm$ annihilation energy density in eq. (2) is modified to,

$$\rho = \frac{43}{8} (1 + \frac{7\Delta N_\nu}{43}) \rho_\gamma. \quad (5)$$

The extra energy density speeds up the expansion of the universe so that the right hand side of the time-temperature relation in eq. (3) is smaller by the square root of the factor in parentheses in eq. (5).

II. COMPLETELY DECOUPLED NEUTRINOS

For standard model neutrinos the electroweak interaction rates drop below the universal expansion rate when the universe is less than one second old and the temperature is a few MeV. The “standard”, zeroth-order approximation that the neutrinos are completely decoupled prior to $e^\pm$ annihilation is
sufficiently accurate for most cosmological applications. In this approximation the CBR photons get the full benefit of the energy/entropy from the annihilating $e^\pm$ pairs, maximizing their heating relative to the decoupled neutrinos (and to any other decoupled particles, such as $X$). After $e^\pm$ annihilation, 

$$T_\nu = T_X = \left(\frac{4}{11}\right)^{1/3}T_\gamma = 0.7138T_\gamma. \quad (6)$$

In this approximation the Fermi-Dirac phase space distributions of the decoupled neutrinos is preserved as the universe expands and $\rho_{\nu_e} = \rho_{\nu_\mu} = \rho_{\nu_\tau} \equiv \rho_{\nu} \quad (\text{and} \quad \rho_X = \Delta N_\nu \rho_\gamma)$, where 

$$\frac{\rho_\nu}{\rho_\gamma} = \frac{7}{8} \left(\frac{T_\nu}{T_\gamma}\right)^4 = \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} = 0.2271, \quad (7)$$

and 

$$\frac{\rho_X}{\rho_\gamma} = \frac{7}{8} \left(\frac{T_X}{T_\gamma}\right)^4 \Delta N_\nu = \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} \Delta N_\nu. \quad (8)$$

In this zeroth-order, full decoupling approximation, the post-$e^\pm$ annihilation relativistic ($R$) energy density is, 

$$\frac{\rho_R}{\rho_\gamma} = 1 + \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} \Delta N_\nu = 1.6813(1 + 0.1351 \Delta N_\nu), \quad (9)$$

where $\Delta N_\nu \equiv 3 + \Delta N_\nu$. As long as the universe remains radiation dominated, the age and the photon temperature are simply related by, 

$$t^0 T_\gamma^2 = 1.32(1 + 0.1351 \Delta N_\nu)^{-1/2} \text{MeV}^2 \text{s}. \quad (10)$$

Both $\rho_R$ and the time–temperature relation play important roles in establishing the amplitudes and angular scales of the CBR anisotropies and any corrections to these zeroth-order results may bias the interpretation of the precise data from current ground-based and future space-based telescopes.

### III. PARTIALLY COUPLED NEUTRINOS

It has long been known that the neutrinos are not entirely decoupled from the electron-positron-photon plasma during $e^\pm$ annihilation [5]. As a result, the neutrinos get to share some of the annihilation energy with the photons and the relative heating of the photons is reduced from the zeroth-order approximation estimate. Furthermore, since the weak interactions coupling the neutrinos and the electrons are energy dependent, the neutrino phase space distributions are distorted by preferential heating of the higher energy/momentum neutrinos and the resulting distributions are no longer Fermi-Dirac. In a series of increasingly detailed calculations, many authors have tracked this evolution [5] - [12]. Although there are small differences in the quantitative results which may be traced to the differing approximations, the overall agreement among them is excellent. The results of the detailed and extensive calculations of Gnedin & Gnedin [12], which cover some seven orders of magnitude in the neutrino momentum, are adopted here.

Recall from eq. 7 that in the fully decoupled approximation, $\rho_{\nu_e}/\rho_{\gamma} = 0.2271$. Since the electron neutrinos participate in charged-current as well as in neutral-current weak interactions, they remain coupled longer than do the $\mu$ or $\tau$ neutrinos and, thereby, are heated more. In following the decoupling carefully, Gnedin & Gnedin find, 

$$\frac{\rho_{\nu_e}}{\rho_{\gamma}} = 0.2293 = \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} \times 1.0097, \quad (11)$$

while 

$$\frac{\rho_{\nu_\mu}}{\rho_{\gamma}} = 0.2285 = \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} \times 1.0061. \quad (12)$$

Further, because the energy/entropy from the $e^\pm$ is now being shared by the neutrinos, the heating of the photons relative to the $X$ (assumed to be fully decoupled prior to annihilation) is reduced, so that $T_X = 0.7144T_\gamma$ (compared to the zeroth-order estimate in eq. 6) and, 

$$\frac{\rho_X}{\rho_\gamma} = \frac{7}{8} \left(\frac{T_X}{T_\gamma}\right)^4 \Delta N_\nu = \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} \Delta N_\nu \times 1.0036. \quad (13)$$

We are now in a position to combine the post-$e^\pm$ annihilation energy densities of the photons along with that of the neutrinos (and possible $X$s) to find the relativistic energy density in the incompletely decoupled (ID) neutrino approximation. This result may be written in analogy with eq. 9 by replacing $N_\nu$ with $N_\nu^{ID}$, 

$$\frac{\rho_R^{ID}}{\rho_\gamma} \equiv 1 + \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} N_\nu^{ID}, \quad (14)$$

where 

$$N_\nu^{ID} = 3.022 + 1.0036 \Delta N_\nu, \quad (15)$$

or 

$$N_\nu^{ID} = N_\nu + (0.022 + 0.0036 \Delta N_\nu). \quad (16)$$

Thus, for the standard model case of $N_\nu = 3$, the post-$e^\pm$ annihilation energy density corresponds to 3.02 “equivalent” neutrinos. Although all authors ([5] - [12]) agree quantitatively with this $\Delta N_\nu = 0$ correction, a different value (3.03) appears in the Lopez et al. paper [13]. It is difficult to identify the source of the Lopez et al. correction, $\delta N_\nu^{ID} = 0.03$, which differs from that found earlier by some of the same authors [8][14]. The Lopez et al. value, modified by the QED effect to be addressed next ($\delta N_\nu^{QED} = 0.01$), seems to have propagated in the recent literature (see, e.g., [15][16]) and, indeed, 3.04 is the recommended default value for $N_\nu$ in the CMBFAST code of Seljak & Zaldarriaga [17], widely utilized in analyses of the CBR fluctuation spectra. The second term on the right hand side of eq. 16, the difference between $N_\nu^{ID}$ and $N_\nu$, is the incomplete decoupling correction to the standard model result, including the effect of extra relativistic energy. However, there is another effect which, while small, is the same order of magnitude as this correction for incomplete decoupling.
IV. QED CORRECTION

In the zeroth-order approximation, the $\gamma - e^\pm$ plasma is treated as a gas of free, non-interacting particles prior to $e^\pm$ annihilation. When finite temperature QED corrections are included ([18][19]), the energy density and pressure of the $\gamma - e^\pm$ plasma are reduced. As a result, when $e^\pm$ pairs annihilate they actually have less entropy to share with the photons (and the incompletely decoupled neutrinos), than would be estimated neglecting this QED correction. Accounting for this reduction leads to fewer CBR photons (and neutrinos) in the post-$e^\pm$ annihilation comoving volume, corresponding to a lower photon temperature. To quantify this effect ([18,19]) it is convenient to compare the post-$e^\pm$ annihilation photon temperature to that of the fully decoupled $X$ particles.

$$(T_X/T_\gamma)_{QED} = (4/11)^{1/3}(1 + 9.6 \times 10^{-4}).$$

(17)

Thus, in comparing the total energy density in relativistic particles to that in CBR photons alone, the relative contribution of the neutrinos (and $X$s) is *enhanced* with respect to the contributions in the zeroth-order approximation (eq. 9) and in the incompletely decoupled approximation (eq. 14). To account for the finite temperature QED correction, the $(4/11)^{1/3}$ factors on the right hand sides of eqs. 9 and 14 should be replaced by $(T_X/T_\gamma)^{1/3}_{QED}$, leading to a new, “effective” number of equivalent neutrinos $N_\nu^{eff}$,

$$N_\nu^{eff} = (11/4)^{1/3}(T_X/T_\gamma)^{1/3}_{QED}N_\nu^{ID} = 1.0038N_\nu^{ID},$$

(18)

so that

$$N_\nu^{eff} = 3.034 + 1.0074\Delta N_\nu,$$

(19)

or

$$N_\nu^{eff} - N_\nu = 0.034 + 0.0074\Delta N_\nu.$$  

(20)

This total correction to the standard model, zeroth-order approximation, $N_\nu = 3 + \Delta N_\nu$, is shown in Figure 1 as a function of any extra energy density, measured by $\Delta N_\nu$.

V. DISCUSSION

Prior to $e^\pm$ annihilation the total energy density in the standard model ($N_\nu = 3$) is given in terms of the CBR energy density by eq. 2. In the presence of extra (relativistic) energy density, $\Delta N_\nu > 0$, the $\rho_R - \rho_\gamma$ relation is modified to that in eq. 5. After $e^\pm$ annihilation the relativistic energy density is shared by photons and decoupled neutrinos (and $X$s) and the total energy density is modified by the relative heating of the photons, neutrinos, and $X$s. When careful account is taken of the incomplete decoupling of the neutrinos during $e^\pm$ annihilation, as well as of the finite temperature QED effects on the equation of state of the pre-annihilation plasma, the total energy density ($\rho_R \equiv \rho_R^{eff}$), normalized to the CBR photon energy density, is

$$\frac{\rho_R}{\rho_\gamma} = 1 + \frac{7}{8}(11)^{4/3}N_\nu^{eff},$$

(21)

where eq. 20 provides the connection between $N_\nu^{eff}$ and $N_\nu$. For the standard model case of $N_\nu = 3$, the correction to the post-$e^\pm$ annihilation energy density, 0.034, is small; but, for $\Delta N_\nu > 0$, this correction term grows. The new term, proportional to $\Delta N_\nu$, will dominate the correction for $\Delta N_\nu \gtrsim 4.5$, and should not be ignored if precision requires that the first term be included. The difference between the zeroth-order energy density (eq. 9) and the corrected one (eq. 21) is also shown in Figure 1. It is interesting to note that in a joint BBN + CBR analysis of the constraints on $N_\nu^{eff}$, Hansen et al. ([20]) recommend a standard model ($\Delta N_\nu = 0$) value of $N_\nu^{eff} = 3.034$, in excellent agreement with that presented here, but they make no mention of the additional correction when $\Delta N_\nu$ differs from zero.

![FIG. 1. The solid curve is the correction to the zeroth-order result for the equivalent number of neutrinos as a function of the “extra” equivalent number of neutrinos. The dotted curve shows the fractional change in the post-$e^\pm$ annihilation age when the universe is radiation dominated. The dashed curve is the difference between the corrected, and zeroth-order radiation energy densities in units of the CBR photon energy density.](image)

Since the post-$e^\pm$ annihilation expansion rate is determined by $\rho_R$ (as long as the universe remains RD), the age – temperature relation will be modified from its zeroth-order form (eq. 10),

$$\frac{t^0}{t} = \left(\frac{1 + 0.1351(N_\nu^{eff} - 3)}{1 + 0.1351(N_\nu - 3)}\right)^{1/2}.$$  

(22)

The fractional change in the age of the universe ($\Delta t/t \equiv (t - t^0)/t$) is shown in Figure 1. Since the CBR fluctuation...
spectrum is being painted on the microwave sky during those epochs when the universe is making the transition from radiation to matter dominated, modifications to the zeroth-order expressions for the expansion rate and the radiation energy density will effect the details of the resulting anisotropies.

As cosmology enters this new era of precision science, it may be necessary to account for even the very small changes from the zeroth-order approximation summarized here. For example, Lopez et al. (13) suggest that measurements accurate to $\delta N_{\nu} \approx 0.03$, a $\sim 1\%$ determination, will be possible. At present, such precision seems a distant dream. For example, for $\Delta N_{\nu} = 0$, the effect on the BBN-predicted helium abundance is very small (14,19), $\delta Y_P = 1.5 \times 10^{-4}$, a correction buried in the overall uncertainty in the BBN prediction ($\gtrsim 4 \times 10^{-4}$ [21]). The small speed up in the post-e+e- expansion rate will modify the BBN-predicted abundances of the other light elements, but here, too, the changes (for $\Delta N_{\nu} = 0$) are overwhelmed by the current theoretical uncertainties (see, e.g., [21]). As for the CBR angular fluctuations, there are degeneracies between $N_{\nu}^{eff}$ and many other cosmological parameters, leading at present to only very weak constraints (22,23,20). $N_{\nu}^{eff} \lesssim 7-17$. At this stage, differences between $N_{\nu}^{eff}$ and $N_{\nu}$ at the 1% level are beyond our grasp. We can only hope that our experimental colleagues will rise to the challenge and provide data of such exquisite precision that the corrections reviewed and summarized here will be important.

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