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Galois actions on Q-curves and Winding Quotients

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Abstract

We prove two “large images” results for the Galois representations attached to a degree d Q-curve E over a quadratic field K: if K is arbitrary, we prove maximality of the image for every prime p > 13 not dividing d, provided that d is divisible by q (but d ≠ q) with q = 2 or 3 or 5 or 7 or 13. If K is real we prove maximality of the image for every odd prime p not dividing dD, where D = disc(K), provided that E is a semistable Q-curve. In both cases we make the (standard) assumptions that E does not have potentially good reduction at all primes p ☐ 6 and that d is square-free.

1 Semistable Q-curves over real quadratic fields

Let K be a quadratic field, and let E be a degree d Q-curve defined over K. Let D = disc(K). Assume that E is semistable, i.e., that E has good or semistable reduction at every finite place β of K. Recall that we can attach to E a compatible family of Galois representations {σλ} of the absolute Galois group of Q: these representations can be seen as those attached to the Weil restriction A of E to Q, which is an abelian surface with real multiplication by F := Q(√±d) (cf. [E]). Let us call U the set of primes dividing D. For primes not in U, it is clear that A is also semistable, so in particular for

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every prime $\lambda$ of $F$ dividing a prime $\ell$ not in $U$ the residual representation $\overline{\sigma}_\lambda$ will be a representation “semistable outside $U$”, i.e., it will be semistable (in the sense of [Ri 97]) at $\ell$ and locally at every prime $q \neq \ell, q \notin U$. This is equivalent to say that its Serre’s weight will be either 2 or $\ell + 1$ and that the restriction to the inertia groups $I_q$ will be unipotent, for every $q \neq \ell, q \notin U$ (cf. [Ri97]).

Imitating the argument of [Ri97], we want to show that in this situation, if the image of $\overline{\sigma}_\ell$ is (irreducible and) contained in the normalizer of a Cartan subgroup, then this Cartan subgroup must correspond to the image of the Galois group of $K$, i.e., the restriction to $K$ of $\overline{\sigma}_\ell$ must be reducible. More precisely:

**Theorem 1.1.** Let $E$ be a semistable $\mathbb{Q}$-curve over a quadratic field $K$ as above. If $\ell \nmid 2dD, \lambda \mid \ell$, and $\overline{\sigma}_\lambda$ is irreducible with image contained in the normalizer of a Cartan subgroup of $\text{GL}(2, \overline{\mathbb{F}}_\ell)$, then the restriction of this residual representation to the Galois group of $K$ is reducible.

**Proof.** For any number field $X$, let us denote by $G_X$ its absolute Galois group. We know that if we take $\ell \notin U$ the residual representation $\overline{\sigma}_\lambda$ is semistable outside $U$. If this representation is irreducible and its image is contained in the normalizer $N$ of a Cartan subgroup, then there is a quadratic field $L$ such that the restriction of $\overline{\sigma}_\lambda$ to $G_L$ is reducible and the quadratic character $\psi$ corresponding to $L$ is a quotient of $\overline{\sigma}_\lambda$ (cf. [Ri 97]).

Using the description of the restriction of $\overline{\sigma}_\lambda$ to the inertia group $I_\ell$ in terms of fundamental characters, and the fact that the restriction of $\overline{\sigma}_\lambda$ to the inertia groups $I_q$, for every $q \neq \ell, q \notin U$, is unipotent, we conclude as in [Ri 97] that the quadratic character $\psi$ can only ramify at primes in $U$, and therefore that the quadratic field $L$ is unramified outside $U$, the ramification set of $K$.

On the other hand, we know (by Cebotarev) that the restriction to $G_K$ of $\overline{\sigma}_\lambda$ is isomorphic to $\overline{\sigma}_{E,\ell}$. Let us assume that $\overline{\sigma}_{E,\ell}$ is irreducible (*). Its image is contained in $N$, and since the restriction of $\overline{\sigma}_\lambda$ to $G_L$ is reducible, it follows that the restriction of $\overline{\sigma}_{E,\ell}$ to $G_{L,K}$ is reducible. We are again in the case of “image contained in the normalizer of a Cartan subgroup” but now for a representation of $G_K$. Once again, the quadratic character $\psi'$ corresponding to the extension $L \cdot K/K$ is a quotient of the residual representation $\overline{\sigma}_{E,\ell}$. Using the fact that the curve $E$ is semistable we know that the restriction of this residual representation to all inertia subgroups at places relatively primes to $\ell$ give unipotent groups, and this implies as in [Ri97] that $\psi'$ is unramified outside (places above) $\ell$. But $\psi'$ corresponds to the extension $L \cdot K/K$, and
$L$ is unramified outside $U$, thus $\psi'$ is also unramified outside (places above primes in) $U$. This two facts entrain that $\ell \in U$, which is contrary to our hypothesis.

This proves that the assumption (*) contradicts the hypothesis of the theorem, i.e., that the restriction to $G_K$ of $\sigma_\lambda$ is reducible, as we wanted. □

Keep the hypothesis of the theorem above, and assume furthermore that the field $K$ is real. Then, the conclusion of the theorem together with a standard trick show that the image of $\sigma_\lambda$ can not be (irreducible and) contained in the normalizer of a non-split Cartan subgroup: the reason is simply that the representation $\sigma_\lambda$ is odd, thus the image of $c$, the complex conjugation, has eigenvalues $1$ and $-1$. In odd residual characteristic, this gives an elements which is not contained in a non-split Cartan, but if we assume that $K$ is real, we have $c$ contained in the group $G_K$, and we obtain a contradiction because as a consequence of theorem 1.1 the restriction of $\sigma_\lambda$ to $G_K$ must be contained in the Cartan subgroup. This, combined with Ellenberg’s generalizations of the results of Mazur and Momose (cf. [E]), shows that the image has to be large except for very particular primes. In fact, we have the following:

**Corollary 1.2.** Let $E$ be a semistable $\mathbb{Q}$-curve over a real quadratic field $K$ of square-free degree $d$. Assume that $E$ does not have potentially good reduction at all primes not dividing 6. Then, if $D$ is the discriminant of $K$, for every $\ell \nmid dD$, $\ell > 13$ and $\lambda | \ell$, the image of the projective representation $P(\sigma_\lambda)$ is the full $\text{PGL}(2, \mathbb{F}_\ell)$.

2. **Q-curves of composite degree over quadratic fields**

Let $E$ be a $\mathbb{Q}$-curve over a quadratic field $K$ of square-free degree $d$. Let $\lambda$ be a prime of $K$ and let us consider the projective representation $P(\overline{\sigma}_\lambda)$ coming from $E$. We can characterize the image in a subgroup of $\text{PGL}_2(\mathbb{F}_l)$ with $\lambda | l$ of the projective representation $P(\overline{\sigma}_\lambda)$ by points of modular curves as follows (proposition 2.2 [E]):

1. $P(\overline{\sigma}_\lambda)$ lies in a Borel subgroup, then $E$ is a point of $X_0(dl)^K(\mathbb{Q})$,

2. $P(\overline{\sigma}_\lambda)$ lies in the normalizer of a split Cartan subgroup then $E$ is a point of $X^*_0(d;l)^K(\mathbb{Q})$. 

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3. \( P(\overline{\sigma}_\lambda) \) lies in the normalizer of a non-split Cartan subgroup, then \( E \) is a point of \( X_{0}^{ns}(d;l)K(\mathbb{Q}) \);

where \( X^K(\mathbb{Q}) \) is the subset of \( P \in X(K) \) such that \( P^\sigma = w_dP \) for \( \sigma \) a generator of \( \text{Gal}(K/\mathbb{Q}) \) where \( w_d \) is the Frickel or Atkin-Lehner involution.

We have the following results ([E], propositions 3.2, 3.4):

**Proposition 2.1.** Let \( E \) be a \( \mathbb{Q} \)-curve of square-free degree \( d \) over \( K \) a quadratic field. We have:

1. Suppose \( P(\overline{\sigma}_\lambda) \) is reducible for some \( l = 11 \) or \( l > 13 \) with \( (p,d) = 1 \) where \( \lambda | l \). Then \( E \) has potentially good reduction at all primes of \( K \) of characteristic greater than 3.

2. Suppose \( P(\overline{\sigma}_\lambda) \) lies in the normalizer of a split Cartan subgroup of \( PGL_2(\mathbb{F}_l) \) where \( \lambda | l \) for \( l = 11 \) or \( l > 13 \) with \( (l,d) = 1 \). Then \( E \) has good reduction at all primes of \( K \) not dividing 6.

After this result we need to study what happens when the image lies in the non-split Cartan situation. For this case, Ellenberg obtains for the situation of \( K \) an imaginary quadratic field, that there is a constant depending of the degree \( d \) and the quadratic imaginary field \( K \) such that if the image of \( P(\overline{\sigma}_\lambda) \) lies in a non-split Cartan and \( l > M_{d,K} \) then \( E \) has potentially good reduction at all primes of \( K \), see proposition 3.6 [E]. He centers in the twisted version for \( X^K \) to obtain this result. We obtain a similar result in a non-twisted situation for \( X^K \), and with \( K \) non necessarily imaginary.

We impose once for all that \( d \), the degree, is even. We denote \( d = 2\tilde{d} \). First, let us construct an abelian variety quotient of the Jacobian of \( X_{0}^{ns}(2\tilde{d};l) \) on which \( w_{2\tilde{d}} \) acts as 1 and having \( \mathbb{Q} \)-rang zero. Then using “standard” arguments, that we will reproduce here for reader’s convenience, we obtain our result on the non-split Cartan situation.

By the Chen-Edixhoven theorem, we have an isogeny between \( J_{0}^{ns}(2;l) \) and \( J_{0}(2l^2)/w_{l^2} \). Darmon and Merel [DM, prop.7.1] construct an optimal quotient \( A_f \) with \( \mathbb{Q} \)-rang zero. They construct \( A_f \) as the associated abelian variety to a form \( f \in S_2(\Gamma_0(2l^2)) \) with \( w_{l^2}f = f \) and \( w_{2}f = -f \).

Then, in this situation, we construct now a quotient morphism

\[
\pi_f : J_{0}(2\tilde{d}l^2) \to A'_f
\]
such that the actions of \( w_{2d} \) and \( w_{l2} \) on \( J_0(2d\tilde{l}^2) \) give both the identity on \( A'_f \) if \( \tilde{d} \neq 1 \). Moreover, we can see that \( A'_f \) is preserved by the whole group \( W \) of Atkin-Lehner involutions. We construct \( A'_f \) from \( f \in S_2(\Gamma_0(2l^2)) \) and we go to increase the level.

We denote by \( B_n \) the operator on modular forms of weight 2 that acts as:

\[
f|B_n(\tau) = f(n\tau) = n^{-1}f|_{A_n}, \text{ where } A_n = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}
\]

from level \( M \) to level \( Mk \) with \( n \mid k \). We denote by

\[
B_n : J_0(M) \to J_0(Mk)
\]

the induced map on jacobians.

**Lemma 2.2.** With the above notation and supposing that \( (\tilde{n}, k) = 1 \) and \( g \) is a modular form which is an eigenform for the Atkin-Lehner involution \( w_{\tilde{n}} \) in \( J_0(M) \), then \( g|_{B_n} \) is also an eigenform for the Atkin-Lehner involution \( w_{\tilde{n}} \) in \( J_0(Mk) \) with the same eigenvalue.

**Proof.** We only need to show that there exist \( w_{\tilde{n},M} \) and \( w_{\tilde{n},Mk} \), the Atkin-Lehner involution of \( \tilde{n} \) at level \( M \) and \( Mk \) respectively, such that:

\[
A_n w_{\tilde{n},Mk} = w_{\tilde{n},M} A_n
\]

which is easy to check. \( \Box \)

With the above lemma we can rewrite lemma 26 in [AL] as follows

**Lemma 2.3 (Atkin-Lehner).** Let \( g \) a form in \( \Gamma_0(M) \), eigenform for all the Atkin-Lehner involutions \( w_l \) at this level. Let \( q \) be a prime. Then the form

\[
g|_{B_q^{\alpha}} \pm q^{(\delta-2\alpha)}g|_{B_1=Id}
\]

is a form in \( \Gamma_0(Mq^\alpha) \) which is an eigenform for all Atkin-Lehner involutions at level \( Mq^\alpha \) where \( \delta \) is defined by \( q^{1-\delta}||M \) and \( q^\alpha||Mq^\alpha \). Moreover, let us impose that \( \delta \neq 2\alpha \). Then the eigenvalue of this form for \( w_{q^\alpha(Mq^\alpha)} \) is \( \pm \) the eigenvalue of \( w_{q^\alpha(M)} \) on \( g \).

**Remark 2.4 (AL).** In the case \( \delta = 2\alpha \) let us take the form \( g|_{B_q^{2\alpha}} \). Then it satisfies the following: it is an eigenform for the Atkin-Lehner involutions at level \( Mq^\alpha \) with eigenvalue for the Atkin-Lehner involution at \( q \) equal to that of the Atkin-Lehner involution at \( q \) on \( g \) (\( g \) of level \( M \)).
Let us remark that if the condition $\delta \neq 2\alpha$ is satisfied we can choose a form in level $Mq^\alpha$ with eigenvalue of the Atkin-Lehner involution at $q$ as one wishes: 1 or -1. This condition is always satisfied if $(M, q) = 1$, situation that we will use in this article. With this remarks the following lemma is clear by induction:

**Lemma 2.5.** Let $g$ be a modular form of level $M$ which is an eigenvector for all the Atkin-Lehner involutions at level $M$. Then we can construct by the above lemma a modular form $f$ of level $Mk$ $(k \in \mathbb{N})$ which is an eigenvector for all the Atkin-Lehner involutions at level $Mk$, and moreover the eigenvalue at the primes $q|M$ with $(q, k) = 1$ is the same that the one for the Atkin-involution of this prime at $g$ at level $M$, and we can choose (1 or -1) the eigenvalue for the Atkin-Lehner involution at the primes $q$ with $(q, k) \neq 1$ if this prime satisfies the condition $\delta \neq 2\alpha$ of the above lemma.

Let us write a result in the form that will be useful for our exposition, noting here that the even level condition can be removed.

**Corollary 2.6.** Let us write $\tilde{d} = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ with $(\tilde{d}, 2p^2) = 1$. We have a map

$$I_{\chi_{p_1}, \ldots, \chi_{p_r}} : J_0(2p^2) \to J_0(2\tilde{d}p^2)$$

whose image is stable under the action of $W$, and we can choose the action of $w_{2\tilde{d}}$ on the quotient as $\pm$ the action of $w_2$ for an initial form $g \in S_2(\Gamma_0(2p^2))$ eigenform for the Atkin-Lehner involutions at level $2p^2$.

**Proof.** From lemma 27 in [AL], we have a base for the modular forms which are eigenforms for the Atkin-Lehner involutions. Applying the lemma of Atkin-Lehner above we have the result for $\tilde{d} = p_1^{\alpha_1}$, we have to consider

$$I_{\chi_{p_1}} = |B_{p_1}^{\alpha_1} + \chi(p_1)p_1^{-\alpha_1}|_{B_{p_1} = Id},$$

where we can choose $\chi(p_1)$ as 1 or -1 depending on how we want the Atkin-Lehner involution at the prime $p_1$ to act on the quotient. Inductively we obtain the result. \qed

Applying the above corollary with $\tilde{d}$ square-free ($\alpha_i = 1$) in our situation ($\tilde{d} \neq 1$) and choosing $w_{2\tilde{d}} = 1$, we take

$$A_f' := I_{\chi_{p_1}, \ldots, \chi_{p_r}}(A_f),$$

which is by construction a subvariety of $J_0(2\tilde{d}l^2)$ isogenous to $A_f$ which is stable under $W$ (at level $2\tilde{d}l^2$) on which $w_{2\tilde{d}}$ and $w_{l^2}$ acts as identity. In particular the $\mathbb{Q}$-rank of $A_f'$ is zero (recall that we started with an $A_f$ of $\mathbb{Q}$-rank
By the Chen-Edixhoven isomorphism, we obtain a quotient map
\[ \pi'_f : J_0^{ns}(2\tilde{d}; l) \to A'_f. \]
\( \pi'_f \) is compatible with the Hecke operators \( T_n \) with \( (n, 2\tilde{d}) = 1 \) (see for example lemma 17 [AL]) and moreover \( \pi'_f \circ w_{2\tilde{d}} = \pi'_f \). Let us recall that we are interested in points on \( X_0^{ns}(2\tilde{d}; l)^K(\mathbb{Q}) \) (we want to study the non-split Cartan situation). We have the following commutative diagram:

\[
\begin{array}{ccc}
J_0^{ns}(2\tilde{d}; l) & \rightarrow & A'_f \\
\downarrow i & & \downarrow id \\
J := J_0^{ns}(2\tilde{d}; l)^K & \rightarrow & A'_f \\
\end{array}
\]

where \( i \) is an isomorphism such that \( i^\sigma = w_{2\tilde{d}} \circ i \) with \( \sigma \) the non-trivial element of \( Gal(K/\mathbb{Q}) \). Observe that \( \psi_f := \pi'_f \circ i^{-1} : J \to A'_f \) is defined over \( \mathbb{Q} \) because,

\[
\psi_f^\sigma = (\pi'_f)^\sigma \circ (i^{-1})^\sigma = \pi'_f \circ w_{2\tilde{d}} \circ i^{-1} = \pi'_f \circ i^{-1} = \psi_f.
\]

Let \( R_0 \) be the ring of integers of \( K(\zeta_l + \zeta_l^{-1}) \) and \( R = R_0[1/2\tilde{d}] \), then \( X_0^{ns}(2\tilde{d}; l) \) has a smooth model over \( R \) and the cusp \( \infty \) of \( X_0^{ns}(2\tilde{d}; l) \) is defined over \( R \) [DM]. We define

\[
h : X_0^{ns}(2\tilde{d}; l)/R \to J_0^{ns}(2\tilde{d}; l)/R
\]

by \( h(P) = [P] - [\infty] \). Then it follows by lemma 3.8 [E]

**Lemma 2.7.** Let \( \beta \) be a prime of \( R \). Then the map,

\[
\pi'_f \circ h : X_0^{ns}(2\tilde{d}; l)/R \to A'_f/R
\]

is a formal immersion at the point \( \infty \) of \( X_0^{ns}(2\tilde{d}; l)(\mathbb{F}_\beta) \).

We can prove a result for the non-split Cartan situation with a constant independent of the quadratic field.
Proposition 2.8. Let $K$ be a quadratic field, and $E/K$ be a $Q$-curve of square-free degree $d = 2\tilde{d}$, with $\tilde{d} > 1$. Suppose that the image of $P(\sigma)$ lies in the normalizer of a non-split Cartan subgroup of $PGL_2(\mathbb{F}_l)$ with $\lambda | l$ for $l > 13$ with $(2d,l) = 1$. Then $E$ has potentially good reduction at all primes of $K$.

Proof. We can follow closely the proof of prop.3.6 in [E], let us reproduce it here for reader’s convenience. Take $\beta$ a prime of $K$ where $E$ has potentially multiplicative reduction, if $\beta | l$ then the image of the decomposition group $G_\beta$ under $P(\sigma)$ lies in a Borel subgroup. By hypothesis this image lies in the normalizer of a non-split Cartan subgroup. We conclude that the size of this image has order at most 2, which means that $K_\beta$ contains $\mathbb{Q}(\zeta_l + \zeta_l^{-1})$, which is impossible once $l \geq 7$.

Now let us suppose that $E$ has potentially multiplicative reduction over $\beta$ with $\beta \nmid l$, denote by $l'$ the prime of $\mathbb{Q}$ such that $\beta | l'$. It corresponds to a cusp on $X_0^{ns}(2\tilde{d}; l)$ where we will take reduction modulo $\beta$. The cusps of $X_0^{ns}(2\tilde{d}; l)$ have minimal field of definition $\mathbb{Q}(\zeta_l + \zeta_l^{-1})$ [DM, §5], and $K$ is linearly disjoint from $\mathbb{Q}(\zeta_l + \zeta_l^{-1})$; it follows that the cusps of $X_0^{ns}(2\tilde{d}; l)$ which lie over $\infty \in X_0(2d)$ form a single orbit under $G_K$. If $\tilde{\beta}$ is a prime of $L = K(\zeta_l + \zeta_l^{-1})$ over $\beta$, then the point $P \in X_0^{ns}(2\tilde{d}; l)(K)$ parametrizing $E$ reduces mod $\beta$ to some cusp $c$. By applying Atkin-Lehner involutions at the primes dividing $2\tilde{d}$, we can ensure that $P$ reduces to a cusp which lies over $\infty$ in $X_0(2\tilde{d})$. By the transitivity of the Galois action, we can choose $\tilde{\beta}$ so that $P$ actually reduces to the cusp $\infty \bmod \tilde{\beta}$. Using the condition that a $K$-point of $X_0^{ns}(2\tilde{d}; l)$ reduces to $\infty$, we have then that the residue field $\mathcal{O}_K/\beta$ contains $\zeta_l + \zeta_l^{-1}$, and this implies that $(l')^4 \equiv 1 \bmod l$, in particular $l' \neq 2,3$ when $l \geq 7$.

We have constructed a form $f$ and an abelian variety $A_f$ isogenous to the one associated to $f$ with $\mathbb{Q}$-rank zero and $w_{2\tilde{d}}$ acting as 1 on it, and we have a formal immersion $\phi = \pi_f \circ h$ at $\infty$

\[ X_0^{ns}(d; l) K/R \rightarrow A_f'/R. \]

Let us consider $y = P$ our point from the $Q$-curve and $x = \infty$ at the curve $X = X_0^{ns}(2\tilde{d}; l)/R_\beta$, we know that they reduce at $\beta$ to the same cusp if $P$ has multiplicative reduction. Let us consider then $\phi(P)$ the point in $A_f'(L)$ with $L = K(\zeta_l + \zeta_l^{-1})$. Let $n$ be an integer which kills the subgroup of $J_0^{ns}(2\tilde{d}; l)$ generated by cusps, it exists by Drinfeld-Manin, then $nh(P) \in J_0^{ns}(2\tilde{d}; l)$ and let $\tau \in \text{Gal}(L/\mathbb{Q})$ and not fixing $K$, then $P^\tau = w_{2\tilde{d}}P$ and we obtain that
$n\phi(P)^{\tau} = n\phi(P)$ then lies in $A'_f(\mathbb{Q})$ which is a finite group and then torsion, concluding that $\phi(P)$ is torsion (this is getting a standard argument [DM, lemma 8.3]).

Since $l' > 3$ the absolute ramification index of $R_\beta$ at $l'$ is at most 2. Then it follows from known properties of integer models (see for example [E, prop.3.1]) that $x$ and $y$ reduce to distinct point of $X$ at $\beta$, in contradiction with our hypothesis on $E$.

Putting together propositions 2.1 and 2.8, we obtain:

**Corollary 2.9.** Let $E$ be a $\mathbb{Q}$-curve over a quadratic field $K$ of square-free composite degree $d = 2\tilde{d}$, with $\tilde{d} > 1$. Assume that $E$ does not have potentially good reduction at all primes not dividing 6. Then, for every $\ell \nmid 2\tilde{d}$, $\ell > 13$ and $\lambda | \ell$, the image of the projective representation $P(\bar{\sigma}_\lambda)$ is the full $\text{PGL}(2, \mathbb{F}_\ell)$.

To conclude, observe that if we take a $\mathbb{Q}$-curve over a quadratic field whose degree $d$ is odd and composite (and square-free), there are more cases where the above result still holds: for example if $3 \mid d$ the result holds because all the required results from [DM] (in particular, the existence of a non-trivial Winding Quotient in $S_2(3p^2)$) are also proved in this case. Moreover, since the only property of the small primes $q = 2$ or 3 required for all the results we need from [DM] to hold is the fact that the modular curve $X_0(q)$ has genus 0, we can apply them to any of $q = 2, 3, 5, 7, 13$, and so we conclude that the above result applies whenever $d$ is composite (and square-free) and divisible by one such prime $q$.

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