Polyakov’s string classical mechanics.

Massimo Materassi

materassi@pg.infn.it, materassi@fi.infn.it

Department of Physics, University of Perugia (Italy)

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Abstract

This paper is almost an exercise in which the Hamiltonian scheme is developed for Polyakov’s classical string, by following the usual framework suggested by Dirac and Bergman for the reduction of gauge theories to their essential physical degrees of freedom. The results collected here will be useful in some forthcoming papers, where strings will be studied in the unusual context of Wigner-covariant rest frame theory.

After a short introduction outlining the work, the Lagrangean scheme is presented in Section II, where the classical equivalence between Polyakov and Nambu-Goto string is rederived. In Section III the Hamiltonian framework is worked out, primary and secondary constraints are deduced. Then Lagrange multipliers are introduced; finally the Hamiltonian equations of motion are presented.

In Section IV gauge symmetries are treated, by constructing their canonical generators; then some gauge degrees of freedom are eliminated by Dirac-Bergmann fixing procedure. In this paper only the gauge-freedom coming from primary constraints is fixed, while secondary constraints will need a deeper analysis. At the end of
Section IV the classical version of Virasoro algebra is singled out as that of the secondary constraints, surviving to the first gauge fixing.
I. INTRODUCTION.

In order to construct a good theory of relativistic insulated systems, formed by interacting subsystems, we have chosen the framework of Wigner-covariant rest frame scheme (see [3], [4], [5], [6], [7], [8], [9], [10], [11]): in this context it is possible to handle relativistic covariance in such a way that extended relativistic systems (like many body systems and fields) can be treated in a self consistent one-time formulation; many advantages of this approach are known: preservation of covariance in an easily controlled way, an easier form of the cluster decomposition property, the prospect to use almost newtonian technologies. This line of research follows the steps suggested many years ago by Dirac [12], when he introduced a space-like foliation of Minkowski spacetime giving an invariant definition of simultaneity.

Along this line of research many progresses have been achieved, some Wigner-covariant rest frame theories have already been formulated: that of $N$ relativistic particles interacting with an electromagnetic field [3], the one of charged particles interacting via Liendard-Wiechert potentials in the abelian as well as Yang-Mills case [6], that of a free Dirac field. The next step we’ve been working on is the definition of canonical center–of-mass vs relative variables for the fields [7], [8], [9], [11], so that particles and radiation will be treated in the same way.

An interesting experiment will be the application of the same framework to relativistic stringy objects, starting from classical Nambu-Goto string [2].

The motivation of this paper is essentially to review briefly the classical mechanics of Polyakov’s string, deriving its main features, by using Dirac-Bergman approach for singular Lagrangean systems [1]. The results here presented (which are not new, but are worked out explicitly in a didactic feature) will be very useful in those forthcoming works about Nambu-Goto string, whose dynamics is "included" in Polyakov’s theory.

Polyakov’s string classical theory is presented both in Lagrangean as well as canonical terms; primary and secondary constraints are singled out, their effects as canonical generators of classical gauge transformations are studied.
The main aim in this paper is to stress the canonical mechanism of generating Virasoro symmetries, in order to underline the background in which one has to move when discussing Dirac-Bergman constraint reduction for the classical string [4].

With the simplest form of Polyakov’s action (omitting the Liouville terms suggested in [13] for sake of simplicity) the worldsheet metric $h_{ab} (\sigma, \tau)$ is pure gauge and it is possible to gauge it away by using a suitable transformation generated by $\pi^{ab} (\sigma, \tau)$, the worldsheet canonical momenta that are the primary constraints of this theory. Stability conditions for $\pi^{ab} (\sigma, \tau)$ leads to two secondary constraints, which are the Virasoro constraints: here they are introduced as those symmetries surviving after conformal gauge-fixing the metric on the worldsheet.

II. LAGRANGEAN FRAMEWORK.

We’ll work with a bosonic string described by Polyakov’s action and embedded in a Minkowskian flat spacetime $M_D$ of signature $(1, D - 1)$. The action is:

$$S = -\frac{T}{2} \eta_{\mu\nu} \int d\tau \int d\sigma \sqrt{h^{ab}} \partial_{a} X^{\mu} \partial_{b} X^{\nu}.$$  \hspace{1cm} (2.1)

String motion is the evolution of the following Lagrangean coordinates

$$X^{\mu} (\sigma), \ h^{ab} (\sigma)$$

with respect to the scalar parameter $\tau$.

Our point of view is to consider $X^{\mu}$ and $h^{ab}$ as classical fields in the curved 1 + 1 background spanned by $(\sigma, \tau)$, referred to as world sheet $V_2$: the freely chosen $V_2$ coordinates will be scalars under spacetime Poincaré transformation.

There exists a worldsheet tensor calculus derived from the 1 + 1 metric $h^{ab}$, with metric-adapted connections, covariant derivatives, parallel transport and so on, while tensor calculus is trivial for the spacetime (which is flat).

The only relationship between worldsheet geometry and spacetime is that in order for
From Binet theorem

$$h = - \det \| h_{ab} \| .$$

(2.8)
\[
\det \| h^{ab} \| = \frac{1}{\det \| h_{ab} \|} = -\frac{1}{h} \Rightarrow h = \frac{1}{(h^{\tau \sigma})^2 - h^{\tau \tau} h^{\sigma \sigma}},
\] (2.9)

and so:
\[
\mathcal{L} = -\frac{T \eta_{\mu \nu}}{2 \sqrt{(h^{\tau \sigma})^2 - h^{\tau \tau} h^{\sigma \sigma}}} \left( h^{\tau \tau} \dot{X}^\mu \dot{X}^\nu + 2 h^{\tau \sigma} \dot{X}^\mu \dot{X}^\nu + h^{\sigma \sigma} X^\mu X^\nu \right).
\] (2.10)

We will need this form for \( \mathcal{L} \) later. We shall work out the Lagrangean framework treating \( X \) and \( h \) as independent variables.

The metric will obey the following Lagrangean equations
\[
\partial_a \frac{\partial \mathcal{L}}{\partial (\partial_a X^\mu)} - \partial \mathcal{L} = 0, \quad \partial_a \frac{\partial \mathcal{L}}{\partial (\partial_a h^{bc})} - \partial \mathcal{L} = 0,
\] (2.11)

and from (2.5) we get:
\[
\frac{\partial \mathcal{L}}{\partial (\partial_a X^\mu)} = -T \sqrt{h} \partial^a X_\mu, \quad \frac{\partial \mathcal{L}}{\partial X^\mu} = 0,
\] (2.12)

and even:
\[
\partial_a \frac{\partial \mathcal{L}}{\partial (\partial_a h^{bc})} = 0, \quad \frac{\partial \mathcal{L}}{\partial h^{bc}} = -\frac{T \sqrt{h}}{2} \left( \partial_b X^\mu \partial_c X_\mu - \frac{h_{bc}}{2} h^{\mu \nu} \partial_v X^\mu \partial_t X_\mu \right);
\] (2.13)

So we will write:
\[
\partial_a \left( \sqrt{h} \partial^a X_\mu \right) = 0, \quad \partial_b X^\mu \partial_c X_\mu - \frac{h_{bc}}{2} h^{\mu \nu} \partial_v X^\mu \partial_t X_\mu = 0.
\]

Let us explicit the first equation to get\[1\]:
\[
\partial_a \partial^a X_\mu = \frac{1}{2} h_{cd} \partial_a h^{cd} \partial^a X_\mu, \quad \partial_b X^\mu \partial_c X_\mu - \frac{h_{bc}}{2} h^{\mu \nu} \partial_v X^\mu \partial_t X_\mu = 0.
\] (2.15)

First let us focus the relationship

\[1\]Here the relationship
\[
\frac{\partial}{\partial h_{cd}} \sqrt{h} = -\frac{1}{2} \sqrt{h} h_{cd},
\] (2.14)

is usually employed.
\[ \partial_b X^\mu \partial_c X_\mu - \frac{h_{bc}}{2} h^{ef} \partial_e X^\mu \partial_f X_\mu = 0, \]

which is simply Euler-Lagrange equation for the metric: with the position

\[ G_{bc} = \partial_b X^\mu \partial_c X_\mu \]

it reads:

\[ G_{bc} = \frac{h_{bc}}{2} h^{ef} G_{ef} = \frac{h_{bc}}{2} \text{tr} \| G \|, \]

and from the equation of motion

\[ G_{bc} = \frac{h_{bc}}{2} \text{tr} \| G \| \]

we get:

\[ \det \| G \| = \frac{(\text{tr} \| G \|)^2}{4} \det \| h \|. \]

The absolute value of (2.19) is

\[ G = \frac{(\text{tr} \| G \|)^2}{4} h, \]

which becomes

\[ \sqrt{G} = \frac{\text{tr} \| G \|}{2} \sqrt{h}, \]

and writing \( \text{tr} \| G \| \) in more explicit form as in (2.17)

\[ \text{tr} \| G \| = h^{ab} \partial_a X^\mu \partial_b X_\mu \]

one recognizes:

\[ \sqrt{G} = \frac{\sqrt{h}}{2} h^{ab} \partial_a X^\mu \partial_b X_\mu. \]

So Euler-Lagrange equation for the independent variable metric \( h^{ab} \) reads:

\[ \frac{\sqrt{h}}{2} h^{ab} \partial_a X^\mu \partial_b X_\mu = \sqrt{G}, \quad G_{bc} = \partial_b X^\mu \partial_c X_\mu, \]
which translates (2.1) into plain Nambu-Goto action:

$$S = -T \int d\tau \int d\sigma \sqrt{G}. \quad (2.25)$$

_Nambu-Goto action is obtained from Polyakov’s (2.1) using Euler-Lagrange equations for the worldsheet metric._

Action (2.25) does really be Nambu-Goto action, i.e. $-T$ times the measure of $V_2$, considering

$$\text{mis } V_2 = \int d\tau \int d\sigma \sqrt{G}, \quad (2.26)$$

that is just considering sheet-tensor $G_{bc} = \partial_b X^\mu \partial_c X_\mu$ as the worldsheet metric. So we have to choose the option:

$$h_{bc} = G_{bc} \Rightarrow h_{bc} = \eta_{\mu\nu} \partial_b X^\mu \partial_c X^\nu. \quad (2.27)$$

Equation (2.27) simply embeds $V_2$ into $M_D$. One can find [14] very simply the generalization to General relativistic free falling string, moving on a rigid background $g_{\mu\nu} (X)$.

When written explicitly, Nambu-Goto action reads

$$S_{NG} = -T \int d\tau \int d\sigma \sqrt{|\dot{X}^\mu \dot{X}_\mu X^\alpha X'_\alpha - (\dot{X}^\mu X'_\mu)|}, \quad (2.28)$$

and so Lagrangean linear density is:

$$L_{NG} = -T \sqrt{|\dot{X}^\mu \dot{X}_\mu X^\alpha X'_\alpha - (\dot{X}^\mu X'_\mu)|}. \quad (2.29)$$

The fact that Nambu-Goto theory is included in Polyakov’s one will let us use the content of this paper in [2].

Finally, let’s consider Euler-Lagrange equations of motion for the string variables:

$$\partial_a \partial^a X_\mu = \frac{1}{2} h_{cd} \partial_a h^{cd} \partial^a X_\mu. \quad (2.30)$$
III. HAMILTONIAN FRAMEWORK.

Let’s organize an *Hamiltonian scheme for the classical free string*, described by (2.1), whose Lagrangean linear density is:

\[ \mathcal{L} = -\frac{T \eta_{\mu\nu}}{2\sqrt{(h^{\tau\sigma})^2 - h^{\tau\tau}h^{\sigma\sigma}}} \left( h^{\tau\tau} \dot{X}^\mu \dot{X}^\nu + 2h^{\tau\sigma} \dot{X}^\mu X'^\nu + h^{\sigma\sigma} X'^\mu X'^\nu \right). \] (3.1)

The \( X \) and the \( h \) variables will be thought of as independent.

A. Primary constraints and ”constraintless” Hamiltonian.

To realize a canonical version of the theory described by (3.1) we’ll have to evaluate canonical momenta of string variables

\[ P_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}_\mu} \] (3.2)

as well as of sheet variables

\[ \pi_{ab} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ab}}, \] (3.3)

assuming equal time Poisson brackets:

\[ \begin{align*}
\{X^\mu (\sigma, \tau), P_\nu (\sigma', \tau)\} &= \eta^\mu_\nu \delta (\sigma - \sigma'), \\
\{h^{ab} (\sigma, \tau), \pi_{cd} (\sigma', \tau)\} &= \frac{1}{2} \left( \delta_a^c \delta_b^d + \delta_a^d \delta_b^c \right) \delta (\sigma - \sigma'), \\
\{X^\mu (\sigma, \tau), X'^\nu (\sigma, \tau)\} &= \{P_\mu (\sigma, \tau), P_\nu (\sigma', \tau)\} = 0, \\
\{h^{ab} (\sigma, \tau), h_{cd} (\sigma', \tau)\} &= \{\pi_{ab} (\sigma, \tau), \pi_{cd} (\sigma', \tau)\} = 0.
\end{align*} \] (3.4)

Since

\[ \frac{\partial \mathcal{L}}{\partial (\partial_ah^{bc})} = 0 \] (3.5)
canonical momenta $\pi_{ab}$ vanish identically, so the theory from (3.1) is a constrained one, with primary constraints:

$$\pi_{ab} \approx 0.$$  \hspace{1cm} (3.6)

Due to the symmetry

$$h^{bc} = h^{cb}$$

the independent primary constraints are of course only three, for example the following ones:

$$\pi_{\tau\tau} \approx 0, \quad \pi_{\tau\sigma} \approx 0, \quad \pi_{\sigma\sigma} \approx 0,$$  \hspace{1cm} (3.7)

and form a first class set, as prescribed by (3.4). These constraints are strongly first class, because of assumptions (3.4). We’ll first build up the ”constraintless” part of the Hamiltonian, then we’ll include constraints with Lagrange multipliers.

String variables $X$ have nonvanishing momenta:

$$P_\mu = -T \sqrt{h} \left( h^{\tau\tau} \dot{X}_\mu + h^{\tau\sigma} X'_\mu \right).$$  \hspace{1cm} (3.8)

From (3.8) one can read back $\dot{X}_\mu$ in terms of $P_\mu$, obtaining:

$$\dot{X}_\mu = -\frac{1}{T h^{\tau\tau} \sqrt{h}} P_\mu - \frac{h^{\tau\sigma}}{h^{\tau\tau}} X'_\mu.$$  \hspace{1cm} (3.9)

With (3.8) and (3.9) we’re ready to write down the ”constraintless” Hamiltonian density:

$$\mathcal{H}_0 = -\frac{\sqrt{(h^{\tau\sigma})^2 - h^{\sigma\sigma} h^{\tau\tau}}}{2 T h^{\tau\tau}} P_\mu P_\mu - \frac{h^{\tau\sigma}}{h^{\tau\tau}} P_\mu X'_\mu - \frac{T}{2} \sqrt{(h^{\tau\sigma})^2 - h^{\sigma\sigma} h^{\tau\tau}} \frac{X'^\mu X'^\mu}{h^{\tau\tau}}.$$  \hspace{1cm} (3.10)

The whole ”constraintless” Hamiltonian is obtained integrating $\mathcal{H}_0$ along the string, and it reads:

$$H_0 = -\int_0^\pi d\sigma \left( \frac{\sqrt{(h^{\tau\sigma})^2 - h^{\sigma\sigma} h^{\tau\tau}}}{2 T h^{\tau\tau}} P_\mu P_\mu + \frac{h^{\tau\sigma}}{h^{\tau\tau}} P_\mu X'_\mu + \frac{T}{2} \sqrt{(h^{\tau\sigma})^2 - h^{\sigma\sigma} h^{\tau\tau}} \frac{X'^\mu X'^\mu}{h^{\tau\tau}} \right)$$  \hspace{1cm} (3.11)

(let’s remember Equation (2.9)).
B. Secondary constraints.

Let’s work out stability conditions for our primary constraints \( (3.6) \). We could use the equations

\[
\{ \pi_{ab}, H_C \} \approx 0
\]

(where \( H_C \) is the canonical Hamiltonian obtained from \( H_0 \) adding a linear combination of primary constraints \( (3.7) \)), but there exist an \textit{easier method} which can be employed when some constraints are canonical momenta: from Euler-Lagrange equations

\[
\frac{\partial}{\partial t} \frac{\partial L}{\partial (\partial h_{bc})} - \frac{\partial L}{\partial h_{bc}} = 0
\]

one can see that canonical momenta obey

\[
\dot{\pi}_{ab} = \frac{\partial L}{\partial h_{ab}} - \frac{\partial L}{\partial (\partial \sigma h_{ab})},
\]

(3.12)

which is

\[
\dot{\pi}_{ab} = \frac{\partial L}{\partial h_{ab}}
\]

(3.13)

for us, due to Equation \( (3.5) \). This is why we read stability conditions of \( (3.7) \) from the equations:

\[
\frac{\partial L}{\partial h^{\tau \tau}} \approx 0, \quad \frac{\partial L}{\partial h^{\tau \sigma}} \approx 0, \quad \frac{\partial L}{\partial h^{\sigma \sigma}} \approx 0.
\]

(3.14)

Remembering Equation \( (2.13) \)

\[
\frac{\partial L}{\partial h^{bc}} = -\frac{T \sqrt{\mu}}{2} \left( \partial_\mu X^\mu \partial_\nu X_\nu - \frac{h_{bc}}{2} h^{ef} \partial_\sigma X^\mu \partial_\lambda X_\mu \right),
\]

we translate conditions \( (3.7) \) into the following system:

\[\text{2This method is reliable only if one assumes lagrangian and hamiltonian motions completely equivalent.}\]
We have to use the expressions for $\dot{X}^\mu$ in terms of $P^\mu$ and those of inverted metric $h_{ab}$ in terms of $h^{ef}$.

The inversion of the matrix

$$h^{ab} = \begin{pmatrix} h^{\tau\tau} & h^{\tau\sigma} \\ h^{\tau\sigma} & h^{\sigma\sigma} \end{pmatrix}$$

leads to

$$h_{ab} = \begin{pmatrix} hh^{\sigma\sigma} & hh^{\tau\sigma} \\ hh^{\tau\sigma} & -hh^{\tau\tau} \end{pmatrix}, \quad (3.16)$$

i.e.:

$$h_{\tau\tau} = -hh^{\sigma\sigma}, \quad h_{\tau\sigma} = hh^{\tau\sigma}, \quad h_{\sigma\sigma} = -hh^{\tau\tau}. \quad (3.17)$$

So we get:

$$\begin{cases} 
\dot{X}^\mu \dot{X}_\mu - \frac{1}{2} h^{\sigma\sigma} h^{ef} \partial_\sigma X^\mu \partial_\tau X_\mu \approx 0, \\
\dot{X}^\mu X'_\mu - \frac{1}{2} h^{\tau\sigma} h^{ef} \partial_\tau X^\mu \partial_\tau X_\mu \approx 0, \\
X^\mu X'_\mu - \frac{1}{2} h^{\tau\tau} h^{ef} \partial_\tau X^\mu \partial_\tau X_\mu \approx 0. 
\end{cases} \quad (3.18)$$

These constraints are not completely independent, they must be reduced to a set of two independent ones only. In order to do this, we’ll employ the trace $h^{ef} \partial_\tau X^\mu \partial_\tau X_\mu$ in the rôle of a parameter, obtaining a system which is equivalent to (3.18) but simpler than it, coupling together the equations two by two. From the first equation we have $h^{ef} \partial_\sigma X^\mu \partial_\tau X_\mu$ as
\[ h^{\epsilon f} \partial_\epsilon X^\mu \partial_f X_\mu \approx -\frac{2}{h \tau \tau} \dot{X}^\mu \dot{X}_\mu \] (3.19)

and putting it into the second one\footnote{We work like this in order to never divide by the off-diagonal component \( h^{\tau \sigma} \), since by its symmetry the metric tensor is point-by-point diagonalizable, and it must be possible to put \( h^{\tau \sigma} = 0 \) as gauge fixing without any unpleasant divergency.}

\[ \dot{X}^\mu X'_\mu + \frac{h^{\tau \sigma}}{h^{\tau \sigma}} \dot{X}^\mu \dot{X}_\mu \approx 0 \] (3.20)

Involving only coordinates and canonical momenta, it reads:

\[
\left( \frac{2 (h^{\tau \sigma})^2}{Th^{\sigma \sigma} (h^{\tau \tau})^2 \sqrt{h}} - \frac{1}{Th^{\tau \tau} \sqrt{h}} \right) P^\mu X'^\mu + \left( \frac{(h^{\tau \sigma})^3}{h^{\sigma \sigma} (h^{\tau \tau})^2} - \frac{h^{\tau \sigma}}{h^{\tau \tau}} \right) X'^\mu X'^\mu + \frac{h^{\tau \sigma}}{h^{\sigma \sigma} T^2 (h^{\tau \tau})^2} \approx 0.
\] (3.21)

Let’s put together first and third equation in (3.18): we get \( h^{\epsilon f} \partial_\epsilon X^\mu \partial_f X_\mu \) from the first (as in (3.19)) and put it into the third

\[ X'^\mu X'_\mu - \frac{h^{\tau \tau}}{h^{\sigma \sigma}} \dot{X}^\mu \dot{X}_\mu \approx 0, \] (3.22)

getting something which becomes

\[
\left( 1 - \frac{(h^{\tau \sigma})^2}{h^{\sigma \sigma} h^{\tau \tau}} \right) X'^\mu X'_\mu - \frac{1}{T^2 h^{\sigma \sigma} h^{\tau \tau} \sqrt{h}} P^\mu P_\mu - \frac{2h^{\tau \sigma}}{T h^{\sigma \sigma} h^{\tau \tau} \sqrt{h}} P^\mu X'_\mu \approx 0
\] (3.23)

considering (3.9).

Finally, let’s put together the second and the third equations in (3.18): now \( h^{\epsilon f} \partial_\epsilon X^\mu \partial_f X_\mu \) comes from the third equation\footnote{...always in order to never have \( h^{\tau \sigma} \) as a divider!}

\[ h^{\epsilon f} \partial_\epsilon X^\mu \partial_f X_\mu \approx -\frac{2}{h \tau \tau} X'^\mu X'_\mu \] (3.24)

and it is put into the second one

\[ \dot{X}^\mu X'_\mu + \frac{h^{\tau \sigma}}{h^{\tau \tau}} X'^\mu X'_\mu \approx 0, \] (3.25)
which is translated into an expression involving only coordinates and momenta:

\[- \frac{1}{Th^{\tau\tau}\sqrt{h}} P_\mu X^\mu \approx 0. \tag{3.26}\]

Now we have only to put together constraints (3.21), (3.23) and (3.26) to get secondary constraints ensuring stability of the primary ones:

\[
\begin{align*}
&\left( \frac{2(h^{\tau\sigma})^2}{(T h^{\sigma\sigma} (h^{\tau\tau})^2 \sqrt{h})} - \frac{1}{T h^{\tau\tau}\sqrt{h}} \right) P_\mu X^\mu + \left( \frac{(h^{\tau\sigma})^3}{h^{\sigma\sigma} (h^{\tau\tau})^2} - \frac{h^{\tau\sigma}}{h^{\tau\tau}} \right) X'_\mu X^\mu + \frac{h^{\tau\sigma}}{h^{\sigma\sigma} T^2 (h^{\tau\tau})^2 h} P_\mu P_\mu \approx 0, \\
&\left( 1 - \frac{(h^{\tau\sigma})^2}{h^{\sigma\sigma} h^{\tau\tau}} \right) X'^\mu X^\mu - \frac{1}{T^2 h^{\sigma\sigma} h^{\tau\tau} h} P_\mu P_\mu = 0, \\
&- \frac{1}{T h^{\tau\tau}\sqrt{h}} P_\mu X^\mu \approx 0.
\end{align*}
\tag{3.27}\]

The third one, which reads equally \( P_\mu X^\mu \approx 0 \) since \( \frac{1}{T h^{\tau\tau}\sqrt{h}} \) never vanishes, can be used in the other two of (3.27). The equivalent system of stability condition is:

\[
\begin{align*}
&\left( \frac{(h^{\tau\sigma})^3}{h^{\sigma\sigma} (h^{\tau\tau})^2} - \frac{h^{\tau\sigma}}{h^{\tau\tau}} \right) X'_\mu X^\mu + \frac{h^{\tau\sigma}}{h^{\sigma\sigma} T^2 (h^{\tau\tau})^2 h} P_\mu P_\mu \approx 0, \\
&\left( 1 - \frac{(h^{\tau\sigma})^2}{h^{\sigma\sigma} h^{\tau\tau}} \right) X'^\mu X^\mu - \frac{1}{T^2 h^{\sigma\sigma} h^{\tau\tau} h} P_\mu P_\mu \approx 0, \\
&P_\mu X^\mu \approx 0.
\end{align*}
\tag{3.28}\]

We can work on the first line, getting

\[
\frac{h^{\tau\sigma}}{h^{\tau\tau}} \left[ \left( 1 - \frac{(h^{\tau\sigma})^2}{h^{\sigma\sigma} h^{\tau\tau}} \right) X'_\mu X^\mu - \frac{1}{T^2 h^{\sigma\sigma} h^{\tau\tau} h} P_\mu P_\mu \right] \approx 0
\]

and, since \( h^{\tau\sigma} \) must be considered free, we’ll hold the constraint:

\[
\left( 1 - \frac{(h^{\tau\sigma})^2}{h^{\sigma\sigma} h^{\tau\tau}} \right) X'_\mu X^\mu - \frac{1}{T^2 h^{\sigma\sigma} h^{\tau\tau} h} P_\mu P_\mu \approx 0.
\]

The independent array of secondary constraints, allowing stability for (3.6), becomes:
\[
\left(1 - \frac{(h^{\tau\sigma})^2}{h^{\sigma\sigma}h^{\tau\tau}}\right)\dot{X}^\mu X^\mu' - \frac{1}{T^2 h^{\sigma\sigma}h^{\tau\tau}h} P^\mu P_\mu \approx 0, \quad P_\mu X'^\mu \approx 0. \quad (3.29)
\]

We’ll use the following symbols

\[
\chi_1 = P_\mu X'^\mu, \quad \Gamma = \left(1 - \frac{(h^{\tau\sigma})^2}{h^{\sigma\sigma}h^{\tau\tau}}\right)\dot{X}^\mu X^\mu' - \frac{1}{T^2 h^{\sigma\sigma}h^{\tau\tau}h} P^\mu P_\mu \quad (3.30)
\]

for sake of simplicity. In this language, it’s possible to render the stability conditions clearer, replacing \(\Gamma\) with a simpler function of \(P\) and \(X'\): in fact, from (2.9) one writes:

\[
\Gamma = \det \left[ h^{ab} \right] \left( \frac{1}{T^2} P^\mu P_\mu + X'^\mu X'_\mu \right). \quad (3.31)
\]

Since \(h^{ab}\) is nonsingular, one recognizes that \(\Gamma\) vanishes if and only if

\[
\chi_2 \approx 0, \quad (3.32)
\]

where

\[
\chi_2 (P, X') = \frac{1}{T^2} P^\mu P_\mu + X'^\mu X'_\mu \quad (3.33)
\]

and one can use the following system of primary plus secondary constraints

\[
\pi_{\tau\tau} \approx 0, \quad \pi_{\tau\sigma} \approx 0, \quad \pi_{\sigma\sigma} \approx 0, \quad \chi_1 (P, X') \approx 0, \quad \chi_2 (P, X') \approx 0. \quad (3.34)
\]

C. Stability and Lagrange multipliers.

Now one has to go on to work out the stability conditions for the whole set of constraints, checking their consistency with the motion generated by the canonical Hamiltonian linear density

\[
\mathcal{H}_C = -\frac{f [h]}{2T} P^\mu P_\mu - \frac{T}{2} f [h] X'^\mu X^\mu' - \frac{h^{\tau\sigma}}{h^{\tau\tau}} P^\mu X'_\mu + \lambda^{\tau\tau} \pi_{\tau\tau} + \lambda^{\tau\sigma} \pi_{\tau\sigma} + \lambda^{\sigma\sigma} \pi_{\sigma\sigma}, \quad (3.35)
\]

obtained by adding to the ”constraintless” Hamiltonian \(\mathcal{H}_C\) a linear combination of constraints with the Lagrange multipliers. In (3.35) we put:
\[
 f [h] = \sqrt{\frac{(h^{\tau\sigma})^2 - h^{\tau\tau} h^{\sigma\sigma}}{h^{\tau\tau}}}
\]  
(3.36)

for simplicity.

In order to look for stability conditions of constraints, it’s better to express even the "constraintless” part in terms of the very constraints. It’s easy to recognize:

\[
\mathcal{H}_0 = -\frac{h^{\tau\sigma}}{h^{\tau\tau}} \chi_1 - \frac{T}{2} f [h] \chi_2.
\]

Thus the linear density for the canonical Hamiltonian is:

\[
\mathcal{H}_C = \lambda^{\tau\tau} \pi^{\tau\tau} + \lambda^{\tau\sigma} \pi^{\tau\sigma} + \lambda^{\sigma\sigma} \pi^{\sigma\sigma} - \frac{h^{\tau\sigma}}{h^{\tau\tau}} \chi_1 - \frac{T}{2} f [h] \chi_2.
\]  
(3.37)

The scalar canonical Hamiltonian reduces to a linear combination of constraints.

When (3.34) are fulfilled, one has:

\[
\mathcal{H}_C \approx 0.
\]  
(3.38)

This is exactly what happens in the relativistic free classical particle, where \( H_C = \lambda (p^\mu p_\mu - m^2) \approx 0. \)

With the Hamiltonian

\[
H_C = \int_0^\pi d\sigma \left( \lambda^{\tau\tau} \pi^{\tau\tau} + \lambda^{\tau\sigma} \pi^{\tau\sigma} + \lambda^{\sigma\sigma} \pi^{\sigma\sigma} - \frac{h^{\tau\sigma}}{h^{\tau\tau}} \chi_1 - \frac{T}{2} f [h] \chi_2 \right)
\]  
(3.39)

we’re ready to work out stability conditions using:

\[
\{\pi_{ab}, H_C\} \approx 0, \quad \{\chi_1, H_C\} \approx 0, \quad \{\chi_2, H_C\} \approx 0.
\]  
(3.40)

First of all, it’s necessary to produce the Poisson bracket algebra of the constraints, where (3.4) are supposed to be fulfilled a priori.

The formal definition of equal time Poisson bracket between to quantities \( \mathcal{A} [X, P, h, \pi] \) and \( \mathcal{B} [X, P, h, \pi] \) is:
\[ \{ A(\tau), B(\tau) \} = \]
\[ = \int_0^\pi d\sigma \left[ \frac{\delta A(\tau)}{\delta X^\mu(\sigma, \tau)} \frac{\delta B(\tau)}{\delta P_\mu(\sigma, \tau)} - \frac{\delta A(\tau)}{\delta P_\mu(\sigma, \tau)} \frac{\delta B(\tau)}{\delta X^\mu(\sigma, \tau)} \right] + \]
\[ + \int_0^\pi d\sigma \left[ \frac{\delta A(\tau)}{\delta h^{ab}(\sigma, \tau)} \frac{\delta B(\tau)}{\delta \pi_{ab}(\sigma, \tau)} - \frac{\delta A(\tau)}{\delta \pi_{ab}(\sigma, \tau)} \frac{\delta B(\tau)}{\delta h^{ab}(\sigma, \tau)} \right], \]

where functional derivatives are defined à la Frechet [11]. In particular one gets:
\[ \{ \pi_{ab}(\sigma, \tau), F \} = -\frac{\delta F}{\delta h^{ab}(\sigma, \tau)} \] (3.42)

from (3.41), for any \( F[X, P, h, \pi] \).

The first three constraints are directly postulated to be strongly in involution with each other
\[ \{ \pi_{ab}(\sigma, \tau), \pi_{cd}(\sigma', \tau) \} = 0. \] (3.43)

Poisson bracketing the momenta \( \pi_{ab} \) with \( \chi_k \) we simply get zero, since \( \chi_k \) are string-dependent only:
\[ \{ \pi_{ab}(\sigma, \tau), \chi_1(\sigma', \tau) \} = 0, \{ \pi_{ab}(\sigma, \tau), \chi_2(\sigma', \tau) \} = 0. \] (3.44)

Poisson brackets between \( \chi_k \)'s are evaluated by using the relation
\[ \frac{\partial}{\partial \sigma'} \delta(\sigma' - \sigma) = -\frac{\partial}{\partial \sigma} \delta(\sigma - \sigma') \] (3.45)

(that's a distributional equality, i.e.
\[ \int f(\sigma') d\sigma' \frac{\partial}{\partial \sigma'} \delta(\sigma' - \sigma) = -\int f(\sigma') d\sigma' \frac{\partial}{\partial \sigma} \delta(\sigma - \sigma') \forall f, \]

and read:
with the product conditions (3.40) fulfilled. In fact, if we evaluate Poisson bracket of a constraint, say as it was to be shown.

They mean that those Poisson brackets are weakly zero, since from Dirac function properties one discovers:

\[
F (\sigma') \frac{\partial}{\partial \sigma} \delta (\sigma - \sigma') = \frac{\partial}{\partial \sigma} [F (\sigma) \delta (\sigma - \sigma')] - \delta (\sigma - \sigma') \frac{\partial}{\partial \sigma} F (\sigma) \quad (3.46)
\]

which leads to

\[
\begin{cases}
\{ \chi_1 (\sigma, \tau), \chi_2 (\sigma', \tau) \} = 2 \frac{\partial}{\partial \sigma} [\chi_2 (\sigma) \delta (\sigma - \sigma')] - 2 \delta (\sigma - \sigma') \frac{\partial}{\partial \sigma} \chi_2 (\sigma), \\
\{ \chi_1 (\sigma, \tau), \chi_1 (\sigma', \tau) \} = 2 \frac{\partial}{\partial \sigma} [\chi_1 (\sigma) \delta (\sigma - \sigma')] - 2 \delta (\sigma - \sigma') \frac{\partial}{\partial \sigma} \chi_1 (\sigma), \\
\{ \chi_2 (\sigma, \tau), \chi_2 (\sigma', \tau) \} = \frac{8}{T^2} \frac{\partial}{\partial \sigma} [\chi_1 (\sigma) \delta (\sigma - \sigma')] - \frac{8}{T^2} \delta (\sigma - \sigma') \frac{\partial}{\partial \sigma} \chi_1 (\sigma)
\end{cases}
\quad (3.47)
\]

in our specific case. So one can recognize:

\[
\begin{cases}
\{ \pi_{ab} (\sigma, \tau), \pi_{cd} (\sigma', \tau) \} = 0, & \{ \pi_{ab} (\sigma, \tau), \chi_i (\sigma', \tau) \} \approx 0, \\
\{ \chi_1 (\sigma, \tau), \chi_2 (\sigma', \tau) \} \approx 0, \\
\{ \chi_1 (\sigma, \tau), \chi_1 (\sigma', \tau) \} \approx 0, & \{ \chi_2 (\sigma, \tau), \chi_2 (\sigma', \tau) \} \approx 0.
\end{cases}
\quad (3.48)
\]

From the nature of \( H_C \) (that’s made of constraints only) one can consider stability conditions (3.40) fulfilled. In fact, if we evaluate Poisson bracket of a constraint, say \( \psi_\alpha \), with the product \( F [\psi] \psi_\beta \) (which \( H_C \) is made of), we get:

\[
\{ \psi_\alpha, F [\psi] \psi_\beta \} = F [\psi] \{ \psi_\alpha, \psi_\beta \} + \psi_\beta \{ \psi_\alpha, F [\psi] \} \approx 0,
\]

as it was to be shown.
D. Hamiltonian equations of motion.

Here we work out, for sake of completeness, the Hamiltonian equations of motion for sheet variables as well as string variables, and this could be done simply by using the canonical Hamiltonian linear density

\[ H_C = \lambda^{ab} \pi_{ba} - \frac{h^{\tau \sigma}}{h^{\tau \tau}} \chi_1 - \frac{T}{2} f[h] \chi_2, \]  

(3.49)

which includes primary constraints only. Secondary constraints can be added as well (as Hennaux and Teitelboim suggest in [1]), getting the extended Hamiltonian

\[ H_E = \lambda^{ab} \pi_{ba} + \left( \lambda_1 - \frac{h^{\tau \sigma}}{h^{\tau \tau}} \right) \chi_1 + \left( \lambda_2 - \frac{T}{2} f[h] \right) \chi_2 : \]  

(3.50)

generating the motion with this \( H_E \) it’s more evident how the presence of secondary constraints renders ambiguous the motion of \( P \) and \( X \) within the coordinate-momenta manifold.

In terms of Poisson brackets, Hamiltonian equations of motion read:

\[
\begin{align*}
\dot{h}^{ab} & = \{ h^{ab}, H \}, \quad \dot{\pi}_{ab} = \{ \pi_{ab}, H \}, \\
\dot{X}^{\mu} & = \{ X^{\mu}, H \}, \quad \dot{P}^{\mu} = \{ P^{\mu}, H \},
\end{align*}
\]  

(3.51)

where \( H \) is obtained by integrating (3.49) along the string,

\[
\begin{align*}
\dot{h}^{ab} & = \{ h^{ab}, H_C \}, \quad \dot{\pi}_{ab} = \{ \pi_{ab}, H_C \}, \\
\dot{X}^{\mu} & = \{ X^{\mu}, H_C \}, \quad \dot{P}^{\mu} = \{ P^{\mu}, H_C \},
\end{align*}
\]

or (3.50)

\[
\begin{align*}
\dot{h}^{ab} & = \{ h^{ab}, H_E \}, \quad \dot{\pi}_{ab} = \{ \pi_{ab}, H_E \}, \\
\dot{X}^{\mu} & = \{ X^{\mu}, H_E \}, \quad \dot{P}^{\mu} = \{ P^{\mu}, H_E \}.
\end{align*}
\]

We’ll use directly the extended Hamiltonian.
\[
H_E = \int d\sigma \left[ \lambda^{ab} \pi_{ba} + \left( \lambda_1 - \frac{h^{\sigma \tau}}{h^{\tau \tau}} \right) \chi_1 + \left( \lambda_2 - \frac{T}{2} f [h] \right) \chi_2 \right]
\]

and we’ll consider the rules (3.4) and (4.38).

Sheet variables obey the following equations

\[
\dot{h}^{ab}(\sigma, \tau) = \frac{1}{2} \left[ \lambda^{ab}(\sigma, \tau) + \lambda^{ba}(\sigma, \tau) \right],
\]

but the symmetry \( \pi_{ba} = \pi_{ab} \) allows only the symmetric part of \( \lambda^{ab} \) take part to the play, so that:

\[
\dot{h}^{ab}(\sigma, \tau) = \lambda^{ab}(\sigma, \tau).
\]

The former expresses physical emptiness of \( h^{ab} \): it can be used to invert equation (3.3), and gives us an idea of arbitrariness of \( h^{ab} \) motion.

Hamilton’s equations for sheet momenta are:

\[
\dot{\pi}_{ab}(\sigma, \tau) \approx 0,
\]

and represent the stability of \( \pi_{ab} \) as constraints.

Let’s deal with string variables; the Lagrangean coordinates obey to:

\[
\dot{X}_\mu(\sigma, \tau) = \left( \lambda_1(\sigma, \tau) - \frac{h^{\sigma \tau}(\sigma, \tau)}{h^{\tau \tau}(\sigma, \tau)} \right) X'_\mu(\sigma, \tau) + \left( \frac{2\lambda_2(\sigma, \tau)}{T^2} - \frac{f[h(\sigma, \tau)]}{T} \right) P_\mu(\sigma, \tau).
\]

This represents the inversion formula for (3.2), in which Lagrange multipliers \( \lambda_1(\sigma, \tau) \) and \( \lambda_2(\sigma, \tau) \) appear: equation (3.55) does coincide with (3.9) when multipliers are chosen to vanish.

The real physically meaningful equation of motion for the string variables constructed with extended Hamiltonian is anyway that of \( P_\mu \):

\[
\dot{P}_\mu(\sigma, \tau) =
\]

\[
= \left[ \lambda'_1(\sigma, \tau) - \frac{\partial}{\partial \sigma} \left( \frac{h^{\sigma \tau}(\sigma, \tau)}{h^{\tau \tau}(\sigma, \tau)} \right) \right] P_\mu(\sigma, \tau) + \left( \lambda_1(\sigma, \tau) - \frac{h^{\sigma \tau}(\sigma, \tau)}{h^{\tau \tau}(\sigma, \tau)} \right) P'_\mu(\sigma, \tau) +
\]

\[
(2\lambda'_2(\sigma, \tau) - T f'[h(\sigma, \tau)]) X'_\mu(\sigma, \tau) + (2\lambda_2(\sigma, \tau) - T f[h(\sigma, \tau)]) X''_\mu(\sigma, \tau).
\]

Lagrange multipliers appear here too, with their degree of arbitrariness.
IV. SYMMETRIES AND CONSTRAINTS.

Let’s now analyze the meaning of constraints shown by Polyakov’s string, treating them as canonical generators of gauge transformations.

A. Gauge transformations for sheet variables.

The physical system with action

\[ S = -\frac{T}{2} \int d\tau \int d\sigma \sqrt{h} \partial_a X^\mu \partial_b X_\mu \]  

(4.1)

shows five first class constraints

\[
\begin{align*}
\pi_{\tau\tau} & \approx 0, \\
\pi_{\tau\sigma} & \approx 0, \quad P^\mu X'_\mu \approx 0, \quad \frac{P^\alpha P_\alpha}{T^2} + X'^\alpha X'_\alpha \approx 0, \\
\pi_{\sigma\sigma} & \approx 0,
\end{align*}
\]

three primary and two secondary.

Since the canonical momenta of variables \( h^{ab} \) representing the worldsheet metric are constraints, these variables are functions of \( \sigma \) and \( \tau \) which can be arbitrarily changed at each instant of the motion, without changing the physical state of the system. In particular, we’ve already shown in (3.53) that Legendre transformations from \( \dot{h}^{ab} \) to \( \pi_{ab} \) must be inverted inserting arbitrary scalar functions of \( \tau \).

It’s possible to use this arbitrariness affecting \( h^{ab} \) by operating some gauge fixing in order to make simpler the string description: we’ll now perform this gauge-fixing.

This operation needs some mathematical conditions: gauge fixing is the position of some conditions:\footnote{...which must fulfill \[ \text{[1]} \]:}
Let’s start with a particular metric configuration

\[ h_{ab} = \tilde{h}_{ab} \]  \hspace{1cm} (4.4)

being \( \tilde{h}_{ab} \) assigned, and let’s look for the \( \pi_{ef} \)-generated transformation leading to the wanted form \( h_{0}^{ab} \).

Let’s define the functional derivative differential operator \( \Pi_{ef} \), acting on quantities depending from sheet variables, such that:

\[ \Pi_{ef} (\sigma) F[h] = \{ F[h], \pi_{ef} (\sigma) \}, \]  \hspace{1cm} (4.5)

and let’s appreciate:

\[ \Pi_{ef} (\sigma) F[h] = \frac{\delta F[h]}{\delta h_{ef} (\sigma)}. \]

We can thus use the operator identification:

\[ \Pi_{ef} (\sigma) = \frac{\delta}{\delta h_{ef} (\sigma)}. \]  \hspace{1cm} (4.6)

- **Accessibility**: from the original form of \( h_{ab} \) it must be possible to reach a new form \( h_{0}^{ab} \) identically satisfying \((4.3)\). This must be done with a sequence of transformations generated by suitable second class constraints, \( \pi_{ab} \) in our present case.

- **Completeness**: relationships \((4.3)\) must completely fix the gauge, i.e. the form of \( h_{0}^{ab} \) we’ve given the sheet variables must be uninvariant under those transformations of the gauge we wanted to fix; in particular this condition is formulated as

\[ \{ C_n \left[ h^{ab} \right], \pi_{ef} \} \neq 0, \]  \hspace{1cm} (4.2)

in terms of Poisson brackets.
Any infinitesimal element of $\pi_{ef}$-algebra

$$\int \epsilon^{ef} (\sigma) \pi_{ef} (\sigma) \, d\sigma$$

will cause a transformation

$$\mathcal{F}[h] \rightarrow \mathcal{F}[h] + \delta \mathcal{F}[h]$$

on any functional $\mathcal{F}[h]$, such that:

$$\delta \mathcal{F}[h] = \int \epsilon^{ef} (\sigma) \{ \mathcal{F}[h] , \pi_{ef} (\sigma) \} \, d\sigma. \quad (4.7)$$

This will be applied in the form (4.6), getting:

$$\delta \mathcal{F}[h] = \int d\sigma \epsilon^{ef} (\sigma) \frac{\delta \mathcal{F}[h]}{\delta h^{ef} (\sigma)}. \quad (4.8)$$

Infinitesimal variation on the metric is then:

$$\delta h^{ab} (\sigma) = \epsilon^{ab} (\sigma). \quad (4.9)$$

Equations (4.7) and (4.8) can be extended to finite forms, simply by exponentiating the infinitesimal version:

$$\mathcal{F}'[h] = \mathcal{F}[h] + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \int d\sigma \epsilon^{ef} (\sigma) \frac{\delta}{\delta h^{ef} (\sigma)} \right)^n \mathcal{F}[h], \quad (4.10)$$

i.e.:

$$\mathcal{F}' = \left[ \exp \left( \int d\sigma \epsilon^{ef} (\sigma) \frac{\delta}{\delta h^{ef} (\sigma)} \right) \right] \mathcal{F}. \quad (4.11)$$

These canonical generators can be very easily exponentiated, since their bracket algebra is trivial. The fact that their algebra is abelian will be a key point all over our discussion.

When we work with functions from $\mathbb{R}$ into $\mathbb{R}$ we have

$^6$Of course one should be careful about the convergence of the integrals involved; we will not be so careful here, this paper is simply intended to be a solved exercise!
(D_a f) (y) = f (y + a).

(4.12)

where:

\[ D_a = \sum_{n=0}^{+\infty} \frac{1}{n!} a^n \partial_x^n, \]

(4.13)

and \( a \) is a fixed real number. The following mathematical object

\[ D_\epsilon = \exp \left( \int d\sigma \epsilon^e f(\sigma) \frac{\delta}{\delta h^e (\sigma)} \right) \]

(4.14)

is a thing much more sophisticated than \( D_a \), here we make functional derivatives, and have to deal with an infinite number of degrees of freedom... Anyway, we still exponentiate abelian operators \( \frac{\delta}{\delta h^e (\sigma)} \), with coefficients \( \epsilon^e f(\sigma) \) which are functionally constant with respect to the variables \( h^e (\sigma) \). We arrive to the naïve conclusion:

\[ F'[h] = D_\epsilon F[h] \Rightarrow F'[h] = F[h + \epsilon]. \]

(4.15)

Metric sheet tensor changes from \( h \) into \( h + \epsilon \), so that if we want to go from \( \tilde{h}^{ab} \) to \( h^{ab}_0 \) fulfilling (4.3) it will be possible to do it by that \( D_\epsilon \) with

\[ \epsilon^e f(\sigma) = h^e_0(\sigma) - \tilde{h}^e(\sigma). \]

(4.16)

B. Some interesting functionals.

So our system is described by a linear canonical Hamiltonian density

\[ \mathcal{H}_C = \lambda^{\tau \tau} \pi_{\tau \tau} + \lambda^{\tau \sigma} \pi_{\tau \sigma} + \lambda^{\sigma \sigma} \pi_{\sigma \sigma} - \frac{h_{\tau \sigma}}{h^{\tau \tau}} \chi_1 - \frac{T}{2} f[h] \chi_2, \]

(4.17)

where \( f[h] \) is defined in (3.36). There are five first class constraints, three involving only sheet variables

\[ \pi_{\tau \tau} \approx 0, \quad \pi_{\tau \sigma} \approx 0, \quad \pi_{\sigma \sigma} \approx 0, \]

(4.18)

and two involving only string variables:
\[ P^\mu X'_\mu \approx 0, \quad \frac{P^\alpha P_\alpha}{T^2} + X'^\alpha X'_\alpha \approx 0. \]  

(4.19)

From singular Lagrangean system theory \[1\], one knows that first class constraints are gauge transformation generators, which change in a \( \tau \)-dependent way the formal motion through the phase space, without changing the physical state of the system. Here we’ll underline which transformation is generated by each constraint.

The action (2.1) shows lots of symmetries. It is invariant under the worldsheet reparametrizations

\[ \tilde{\sigma} = \tilde{\sigma} (\sigma, \tau), \quad \tilde{\tau} = \tilde{\tau} (\sigma, \tau) \]  

(4.20)

diffeomorphic maps from \( V_2 \) into \( V'_2 \), as well as under Weyl transformations

\[ \tilde{h}^{ab} (\sigma, \tau) = \Lambda (\sigma, \tau) h^{ab} (\sigma, \tau) . \]  

(4.21)

The term \( h^{ab} \partial_a X^\mu \partial_b X^\nu \) is diff-invariant since the sheet-tensor indices are all correctly saturated; moreover the measure

\[ d^2 V = d\tau d\sigma \sqrt{h} \]  

(4.22)

is notoriously a diff-invariant one, so \( S \) does be.

We have

\[ \sqrt{\tilde{h}' (h')^{ab}} = \sqrt{h} h^{ab} \]  

(4.23)

changing the metric as suggested in (4.21), while \( -\frac{T}{2} \eta_{\mu\nu} d\tau d\sigma \partial_a X^\mu \partial_b X^\nu \) is really unaffected by Weyl transformation (4.21), so that \( S \) is invariant under local rescaling too.

We can try to understand these symmetries in terms of transformations generated by the first class constraints (4.18) and (4.19). We have to be particularly careful with diffeomorphisms, because the constraints act only on the fields \( h, \pi, X \) and \( P \), not on the \( V_2 \)-coordinates directly; we can still map reparametrizations (4.20) into the corresponding transformations which affect the fields as a consequence of those coordinate changes. For
example, letting $\sigma^a$ be any "old" worldsheet variable and $\sigma'^a$ any "new" one, we can still recognize:

$$h'^{ab} = \frac{\partial \sigma'^a}{\partial \sigma^c} \frac{\partial \sigma'^b}{\partial \sigma^d} h_{cd}, \quad \pi'_c = \frac{\partial \sigma'^a}{\partial \sigma^c} \frac{\partial \sigma'^b}{\partial \sigma^d} \pi_{ab}, \quad X' = X, \quad P' = P. \quad (4.24)$$

Let’s deal with sheet constraints (4.18); from:

$$\{\pi_{ab} (\sigma, \tau), F\} = -\frac{\delta F}{\delta h^{ab} (\sigma, \tau)} \quad (4.25)$$

it’s easy to regard $\pi_{ab}$ as the canonical generators of "translations along $h^{ab} (\sigma, \tau)$". Let’s use these sheet constraints to realize Weyl rescaling generators.

One has to get

$$\tilde{h}^{ab} (\sigma, \tau) = e^{\Lambda (\sigma, \tau)} h^{ab} (\sigma, \tau) \quad (4.26)$$

on sheet variables, while nothing has to happen to the string variables.

Let’s consider an $O (\Lambda)$ version of (4.26)

$$\tilde{h}^{ab} (\sigma, \tau) = \left[ 1 + \Lambda (\sigma, \tau) + \ldots \right] h^{ab} (\sigma, \tau), \quad (4.27)$$

so that the sheet metric tensor changes as:

$$\delta h^{ab} = \Lambda h^{ab}. \quad (4.27)$$

The infinitesimal generator of these transformations is a functional $W_\Lambda$ such that:

$$\delta h^{ab} = \{ h^{ab}, W_\Lambda \} . \quad (4.28)$$

Let’s try the function

$$w_\Lambda (\sigma, \tau) = \Lambda (\sigma, \tau) \ h^{ab} (\sigma, \tau) \pi_{ba} (\sigma, \tau) \quad (4.29)$$

with $\Lambda \in C^\infty (V_2, \mathbb{R})$; one has:

$$\{ h^{ef} (\sigma, \tau), w_\Lambda (\sigma', \tau) \} = \Lambda (\sigma', \tau) h^{ef} (\sigma, \tau) \delta (\sigma - \sigma'). \quad (4.30)$$

We get the right functional if we define
\[ \mathcal{W}_\Lambda [h, \pi] = \int_{0}^{\pi} \Lambda(\sigma, \tau) h^{ab}(\sigma, \tau) \pi_{ba}(\sigma, \tau) \, d\sigma \]  

(4.31)

and obtain:

\[ \{ h^{\text{ef}}(\sigma, \tau), \mathcal{W}_\Lambda \} = \Lambda(\sigma, \tau) h^{\text{ef}}(\sigma, \tau), \]  

(4.32)

which authorizes us to state:

- \( \mathcal{W}_\Lambda [h, \pi] \) is the canonical generator of Weyl rescalings.

It’s interesting to find that Weyl invariance of the classical theory is related to the energy-stress sheet tensor of the string. Let us define that tensor as in [15]

\[ T_{ab} = -\frac{2}{T \sqrt{h}} \frac{\partial \mathcal{L}}{\partial h^{ab}}; \]  

(4.33)

than let us assume the primary constraints

\[ \pi_{ab} \approx 0 : \]  

(4.34)

the derivative

\[ \dot{\mathcal{W}}_\Lambda = \dot{\Lambda} h^{ab} \pi_{ba} + \Lambda \dot{h}^{ab} \pi_{ba} + \Lambda h^{ab} \dot{\pi}_{ba}, \]

thus becomes:

\[ \dot{\mathcal{W}}_\Lambda \approx \Lambda h^{ab} \dot{\pi}_{ba}. \]  

(4.35)

The condition in order for \( \dot{\mathcal{W}}_\Lambda \approx 0 \) to be fulfilled (that is: in order for the Weyl generating charge \( \mathcal{W}_\Lambda \) to be conserved, and so for the theory to be consistently Weyl-invariant) is

\[ \dot{\pi}_{ba} = 0, \]

which becomes

\[ \frac{\partial \mathcal{L}}{\partial h_{ba}} = 0 \]
due to the Lagrange equations (2.11) and Lagrangean singularity (3.5). From (4.33) one gets:

$$\dot{\omega}_\Lambda (\sigma, \tau) \approx -\frac{T}{2} \sqrt{h(\sigma, \tau)} \Lambda (\sigma, \tau) h^{ab}(\sigma, \tau) T_{ba}(\sigma, \tau) \quad \forall \Lambda \in C^\infty(\mathbb{V}_2, \mathbb{R}),$$

(4.36)

which allows the affirmation: *the vanishing of stress-energy tensor trace is the condition for the local rescaling Weyl transformations to be symmetries of the theory, because it’s the condition for $\mathcal{W}_\Lambda$ to be constant.*

Let’s now deal with string variable constraints, those $\chi_k$’s defined as follows:

$$\chi_1 = P^\mu X'_\mu, \quad \chi_2 = \frac{P^\alpha P_\alpha}{T^2} + X'^\alpha X'_\alpha, \quad (4.37)$$

First of all, regarding them as canonical generators of gauge transformations acting on string variables, we obtain:

$$\begin{align*}
\{X_\mu(\sigma, \tau), \chi_1(\sigma', \tau)\} &= \partial_\sigma X_\mu(\sigma, \tau) \delta(\sigma' - \sigma), \\
\{P_\mu(\sigma, \tau), \chi_1(\sigma', \tau)\} &= -P_\mu(\sigma, \tau) \partial_{\sigma'} \delta(\sigma' - \sigma), \\
\{X_\mu(\sigma, \tau), \chi_2(\sigma', \tau)\} &= \frac{2}{T^2} P_\mu(\sigma, \tau) \delta(\sigma' - \sigma), \\
\{P_\mu(\sigma, \tau), \chi_2(\sigma', \tau)\} &= -2 \partial_\sigma X_\mu(\sigma, \tau) \partial_{\sigma'} \delta(\sigma' - \sigma).
\end{align*}$$

(4.38)

The only very understandable formula in (4.38) is the first one

$$\{X_\mu(\sigma, \tau), \chi_1(\sigma', \tau)\} = \partial_\sigma X_\mu(\sigma, \tau) \delta(\sigma' - \sigma),$$

\footnote{Since $P$ and $X$ haven’t zero Poisson brackets with these constraints $\chi_1$ and $\chi_2$, they aren’t gauge-invariant at all: if one wanted to get really gauge-invariant variables for the string, one should make one more canonical transformation, that is one more Dirac-Bergman reduction. This will be tried in \textcircled{2}.}
which regards $\chi_1$ as a canonical generator of translations along $\sigma$. In fact, defining the quantity

$$D_f(\tau) = \int f(\sigma, \tau) \chi_1(\sigma, \tau) \, d\sigma$$

(4.39)

and then using it to transform canonically string variable $X_\mu(\sigma, \tau)$, we deduce

$$\{X_\mu(\sigma, \tau), D_f(\tau)\} = f(\sigma, \tau) \partial_\sigma X_\mu(\sigma, \tau):$$

(4.40)

this is typically the action of the canonical generator of the transformation

$$\tilde{\tau} = \tau, \quad \tilde{\sigma} = \sigma + f(\sigma, \tau),$$

always thinking of $f(\sigma, \tau)$ as an "infinitesimal" function.

Such a polite relationship is rather difficult to single out for the other equations (4.38): less cumbersome, more encouraging results are obtained by combining functionally the $\chi_k$’s. For example, using the following functional

$$M_f(\tau) = -\int f(\sigma, \tau) \left[ \frac{h^{\tau\sigma}(\sigma, \tau)}{h^{\tau\tau}(\sigma, \tau)} \chi_1(\sigma, \tau) + \frac{T}{2h^{\tau\tau}(\sigma, \tau) \sqrt{h(\sigma, \tau)}} \chi_2(\sigma, \tau) \right] \, d\sigma$$

(4.41)

as a canonical generator, the following result is obtained

$$\{X_\mu(\sigma, \tau), M_f(\tau)\} = f(\sigma, \tau) \dot{X}_\mu(\sigma, \tau),$$

(4.42)

while when $M_f(\tau)$ acts on the canonical momentum $P_\mu(\sigma, \tau)$ it yields:

$$\{P_\mu(\sigma, \tau), M_f(\tau)\} = -\partial_\sigma \left[ f(\sigma, \tau) \left( \frac{h^{\tau\sigma}(\sigma, \tau)}{h^{\tau\tau}(\sigma, \tau)} P_\mu(\sigma, \tau) + \frac{T}{2h^{\tau\tau}(\sigma, \tau) \sqrt{h(\sigma, \tau)}} \partial_\sigma X_\mu(\sigma, \tau) \right) \right]$$

(4.43)

(these are both worked out using (3.8) and (3.9) equations).

C. Conformal gauge fixing.

In order to render
simpler, we’d like to get a diagonal sheet metric.

Let’s use the following gauge fixing

\[ h^{ab} - \eta^{ab} = 0, \]  

(4.44)

where \( \eta^{ab} \) is simply the Minkowskian \( 1 + 1 \) metric. Since we want to get the configuration

\[ \|h^{ab}_0\| = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]  

(4.45)

we’ll have to choose a transformation (4.14) with coefficients:

\[ \|\epsilon^{ab}\| = \begin{pmatrix} -1 - \tilde{h}^{\tau\tau} & -\tilde{h}^{\tau\sigma} \\ -\tilde{h}^{\tau\sigma} & 1 - \tilde{h}^{\sigma\sigma} \end{pmatrix}. \]  

(4.46)

This gauge fixing is possible, since (4.46) is always an admitted choice; moreover, gauge fixing (4.44) is complete, since it’s unstable under further \( \pi_{\epsilon \tau} \)-generated transformation because:

\[ \{ h^{ab}(\sigma, \tau) - \eta^{ab}, \pi_{\epsilon \tau}(\sigma', \tau) \} = \frac{1}{2} (\delta^a_c \delta^b_d + \delta^a_d \delta^b_c) \delta(\sigma' - \sigma) \neq 0. \]  

(4.47)

From now on we’ll always work in the conformal gauge, as the condition (4.44) is referred to. Polyakov’s action becomes:

\[ S = -\frac{T}{2} \int d\tau \int d\sigma \sqrt{h} \left( h^{\tau\tau} \dot{X}^\mu \dot{X}_\mu + 2h^{\tau\sigma} \dot{X}^\mu X'_\mu + h^{\sigma\sigma} X'^\mu X'_\mu \right). \]  

(4.48)

We’ll gauge out the gauge degrees of freedom related to the constraints

\[ \pi_{\tau\tau} \approx 0, \quad \pi_{\tau\sigma} \approx 0, \quad \pi_{\sigma\sigma} \approx 0, \]  

(4.49)

since by the conformal fixing

\[ h^{\tau\tau} + 1 = 0, \quad h^{\tau\sigma} = 0, \quad h^{\sigma\sigma} - 1 = 0 \]  

(4.50)
first class set (4.49) will be changed into a second class set, adding the \( (h^{ab} - \eta^{ab}) \)'s:

\[
\pi_{\tau\tau} \approx 0, \quad \pi_{\tau\sigma} \approx 0, \quad \pi_{\sigma\sigma} \approx 0, \quad h^{\tau\tau} + 1 \approx 0, \quad h^{\tau\sigma} \approx 0, \quad h^{\sigma\sigma} - 1 \approx 0. \quad (4.51)
\]

It’s possible to read as strong equations these (4.51), using suitable Dirac brackets instead of usual symplectic product (3.41).

The symplectic matrix of the second class constraints is defined as:

\[
C(\sigma, \sigma') = \begin{pmatrix}
\{\pi_{\tau\tau}(\sigma), h^{\tau\tau}(\sigma') + 1\} & \{\pi_{\tau\tau}(\sigma), h^{\tau\sigma}(\sigma')\} & \{\pi_{\tau\tau}(\sigma), h^{\sigma\sigma}(\sigma') - 1\} \\
\{\pi_{\tau\sigma}(\sigma), h^{\tau\sigma}(\sigma') + 1\} & \{\pi_{\tau\sigma}(\sigma), h^{\tau\tau}(\sigma')\} & \{\pi_{\tau\sigma}(\sigma), h^{\sigma\sigma}(\sigma') - 1\} \\
\{\pi_{\sigma\sigma}(\sigma), h^{\tau\tau}(\sigma') + 1\} & \{\pi_{\sigma\sigma}(\sigma), h^{\tau\sigma}(\sigma')\} & \{\pi_{\sigma\sigma}(\sigma), h^{\sigma\sigma}(\sigma') - 1\}
\end{pmatrix}, \quad (4.52)
\]

and reads:

\[
C(\sigma, \sigma') = -\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \delta(\sigma - \sigma'). \quad (4.53)
\]

Its inverse matrix has the same form:

\[
C^{-1}(\sigma, \sigma') = -\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \delta(\sigma - \sigma'). \quad (4.54)
\]

Dirac brackets are defined as follows:

\[
\{\mathcal{F}, \mathcal{G}\}^* = \{\mathcal{F}, \mathcal{G}\} - \int d\sigma \int d\sigma' \left\{\mathcal{F}, \Psi_A(\sigma)\right\} \left(C^{-1}(\sigma, \sigma')\right)^{AB} \left\{\Psi_B(\sigma'), \mathcal{G}\right\}, \quad (4.55)
\]

where \( \Psi_A \) are (1.54) constraints, with \( A \) from 1 to 6; from the form

\[
(C^{-1}(\sigma, \sigma'))^{AB} = -\delta^{AB} \delta(\sigma - \sigma'), \quad (4.56)
\]

we immediately get:

\[
\{\mathcal{F}, \mathcal{G}\}^* = \{\mathcal{F}, \mathcal{G}\} + \sum_{A=1}^{6} \int d\sigma \left\{\mathcal{F}, \Psi_A(\sigma)\right\} \left\{\Psi_A(\sigma), \mathcal{G}\right\}. \quad (4.57)
\]

Explicitly:
\{F, G\}^* = \{F, G\} +
\int d\sigma \{F, \pi_{\tau \tau}(\sigma)\}\{\pi_{\tau \tau}(\sigma), G\} + \int d\sigma \{F, \pi_{\tau \sigma}(\sigma)\}\{\pi_{\tau \sigma}(\sigma), G\} +
\int d\sigma \{F, \pi_{\sigma \sigma}(\sigma)\}\{\pi_{\sigma \sigma}(\sigma), G\} +
\int d\sigma \{F, h^{\tau \tau}(\sigma)\}\{h^{\tau \tau}(\sigma), G\} +
\int d\sigma \{F, h^{\tau \sigma}(\sigma)\}\{h^{\tau \sigma}(\sigma), G\} + \int d\sigma \{F, h^{\sigma \sigma}(\sigma)\}\{h^{\sigma \sigma}(\sigma), G\}.

(4.58)

Since Dirac bracketing allows us to write
\[ \Psi_A(\sigma, \tau) = 0 \]
strongly, we can directly modify quantities involving the gauged out constraints. From string velocity to string momentum now we go by:
\[ P_\mu = T \dot{X}_\mu \]
and come back by:
\[ \dot{X}_\mu = \frac{P_\mu}{T}, \]
which replace (3.8) and (3.9) respectively.

Linear canonical Hamiltonian density becomes:
\[ H_C = \frac{P^\alpha P_\alpha}{2T} + \frac{T}{2} X'^{\alpha} X'^{\alpha}, \]
and the extended correspondent:
\[ H_E = \lambda_1 P^\alpha X'^{\alpha} + \left( \frac{\lambda_2 + T}{2} \right) \left( \frac{P^\alpha P_\alpha}{T^2} + X'^{\alpha} X'^{\alpha} \right). \]

(4.62)

Surviving constraints involving $P$ and $X$ are left formally invariant, while their expressions in terms of $\dot{X}$ and $X$ change as:
\[ \dot{X}^{\alpha} X'^{\alpha} \approx 0, \quad \dot{X}^{\alpha} \dot{X}^{\alpha} + X'^{\alpha} X'^{\alpha} \approx 0. \]

(4.63)
Dirac symplectic product among string variables reads:

\[
\{ X^\mu(\sigma, \tau), \dot{X}_\nu(\sigma', \tau) \}^* = T\eta^\mu_\nu \delta(\sigma - \sigma').
\] (4.64)

From now on, we’ll write this relationship without the asterisk * (which was there to remind us that a gauge reduction took place).

The equation of motion (3.56) due to the canonical Hamiltonian is modified as

\[
\dot{P}_\mu = TX''_\mu,
\] (4.65)

that is:

\[
\ddot{X}_\mu - X''_\mu = 0.
\] (4.66)

Working with the extended Hamiltonian (4.62) we would get

\[
\ddot{X}_\mu - \lambda_1' \dot{X}_\mu - \lambda_1 \dot{X}'_\mu - 2\frac{\lambda_2'}{T} X'_\mu - \left(2\frac{\lambda_2}{T} + 1\right) X''_\mu = 0.
\] (4.67)

After deciding the conformal gauge (4.44) to be used from here on, let’s show that its conclusions are coherent with what’s been discovered within Lagrangean framework. There (in (2.13)) it was stressed that Euler-Lagrange equations for \(h^{ab}\) led to:

\[
h_{bc} = \eta_{\mu\nu} \partial_b X^\mu \partial_c X^\nu,
\] (4.68)

i.e. the sheet metric from Lagrangean equations was shown to be that of \(\mathbb{V}_2\) when embedded into \(M_D\). This spacetime is flat, anyway, and has \(\eta_{\mu\nu}\) as metric tensor, so we have:

\[
h_{\tau\tau} = \dot{X}^2, \ h_{\sigma\sigma} = (X')^2, \ h_{\tau\sigma} = \dot{X} \cdot X'.
\] (4.69)

Constraints (4.63) simply tell us that the choice

\[
h_{\tau\tau} = -1, \ h_{\sigma\sigma} = +1, \ h_{\tau\sigma} = 0
\]

is admitted, consistently with (4.68) and (4.69).
D. Surviving gauge symmetries.

Let’s deal with the string theory in which the sheet constraints $\pi_{ab} \approx 0$ have been gauged away by the position (4.44): now we’ve not any more five constraints, but only two first class ones (the weakly vanishing $\chi_1$ and $\chi_2$).

This new theory still shows all the symmetries generated by the two surviving constraints:

$$P^\mu X'_\mu \approx 0, \quad \frac{P^\alpha P_\alpha}{T^2} + X'^\alpha X'_\alpha \approx 0.$$  \hspace{1cm} (4.70)

Constraints (4.70) generate canonically all the worldsheet transformations which leave the conformal action

$$S = \frac{T}{2} \int d\tau \int d\sigma \left[ \dot{X}^2 - (X')^2 \right]$$  \hspace{1cm} (4.71)

unchanged.

In this paragraph we want to show that those gauge transformations which survive after the gauge-fixing (4.44) can be rewritten producing the very well known Virasoro algebra: due to this fact, string theory will be interpreted as a conformal field theory, that’s one of its most pregnant property.

1. Stress-energy tensor.

The first thing we want to stress is the meaning of the quantities $\chi_1$ and $\chi_2$ in the conformal gauge we’ve chosen; they correspond to the stress-energy tensor components, defined as follows [13]:

$$T_{ab} (\eta) = - \frac{2}{T \sqrt{h}} \frac{\delta S}{\delta h_{ab}} \Big|_{h=\eta}.$$  \hspace{1cm} (4.72)

In order to evaluate this $T_{ab} (\eta)$ it’s necessary to use the still gauge-unfixed Polyakov’s action (2.1), compute the derivatives of (4.72), and finally restrict ourselves to the conformal choice for the gauge. The tensor is evaluated as

$$T_{ab} (h) = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab} \left( h^{\rho \tau} \partial_\rho X^\mu \partial_\tau X_\mu \right).$$  \hspace{1cm} (4.73)
In the conformal gauge the latter becomes:

\[ T_{ab}(\eta) = \partial_a X^\mu \partial_b X_\mu + \frac{1}{2} \eta_{ab} \left( \dot{X}^2 - (X')^2 \right). \]  

(4.74)

When conformally gauge-fixed, the stress-energy tensor elements are the very constraints \(\chi_1\) and \(\chi_2\) which survive the gauge-fixing as further symmetries:

\[ T_{\tau\tau}(\eta) = \frac{1}{2} \left( \dot{X}^2 + (X')^2 \right), \quad T_{\sigma\sigma}(\eta) = \frac{1}{2} \left( \dot{X}^2 + (X')^2 \right), \quad T_{\tau\sigma}(\eta) = \dot{X} \cdot X', \]  

(4.75)

that is

\[ T_{\tau\sigma}(\eta) = \chi_1, \quad T_{\tau\tau}(\eta) = T_{\sigma\sigma}(\eta) = \frac{1}{2} \chi_2. \]  

(4.76)

So within the conformal gauge the theory shows the following constraints:

\[ T_{ab} \approx 0, \]  

(4.77)

omitting the gauge-fixing symbol \((\eta)\).

This tensor \(T_{ab}\) is interesting, because it corresponds in general to the sheet stress-energy tensor, that is to \((1 + 1)\)-current which describes the flux of canonical \(\sigma\)- and \(\tau\)-translation generators along the worldsheet, during the motion of \(X^\mu(\sigma, \tau)\). This is deduced from the application of Noether’s theorem to Polyakov’s Lagrangean density.

The sheet densities of the canonical generators of translations along \(\tau\) and \(\sigma\) are

\[ \rho_a = -\dot{X}^\mu \partial_a X_\mu + \frac{1}{2} \eta^a \left( \ddot{X}^2 - (X')^2 \right), \]  

(4.78)

where \(a = \tau, \sigma\), and the current density of the \(a\)-th canonical generator simply is:

\[ J_a = -X'^\mu \partial_a X_\mu + \frac{1}{2} \eta^\sigma \left( \ddot{X}^2 - (X')^2 \right). \]  

(4.79)

It’s easy to show that the equation

\[ \dot{\rho}_a + J'_a = \partial_a X^\mu \left( \dot{X}_\mu - X''_\mu \right) \]  

(4.80)

is fulfilled by \(\rho_a\) and \(J_a\), which becomes a continuity equation \(\dot{\rho}_a + J'_a = 0\) when equations of motion (4.66) are considered. So one can write
\[ \dot{\rho}_a + J'_a = 0. \quad (4.81) \]

The canonical generators of sheet translations are obtained from \( \rho_\tau \) and \( \rho_\sigma \) integrating along the string at fixed \( \tau \), and are:

\[ Q_\tau = \int d\sigma T^{\tau \tau}(\sigma, \tau) \quad Q_\sigma = \int d\sigma T^{\tau \sigma}(\sigma, \tau). \quad (4.82) \]

Continuity laws (4.81) allow the relationships

\[ \dot{Q}_\tau = 0, \quad \dot{Q}_\sigma = 0 \quad (4.83) \]

to hold along the string motion.

Let’s turn for a moment to equations (4.75): from them one can derive

\[ \rho_\tau = -\frac{1}{2} \chi_2, \quad \rho_\sigma = \chi_1; \quad (4.84) \]

this makes \( \rho_\tau \) and \( \rho_\sigma \) have zero Poisson brackets. Moreover, since the canonical Hamiltonian is (see equation (4.61))

\[ H_C = \frac{T}{2} \chi_2 \quad (4.85) \]

it’s to be expected that \( Q_\tau \) and \( Q_\sigma \) are constant along the motion.

The matrix of the sheet stress-energy tensor \( T_{ab} \) contains the constraints \( \chi_k \)’s:

\[ \|T_{ab}\| = \begin{pmatrix} \frac{1}{2} \chi_2 & \chi_1 \\ \chi_1 & \frac{1}{2} \chi_2 \end{pmatrix}. \]

The weak vanishing of \( T_{ab} \) components is completely equivalent to that of \( \chi_k \)’s, while their linear combination

\[ T_{ab} h^{ba} \approx 0 \quad (4.86) \]

vanishes as a stability condition for Weyl’s rescaling invariance of Polyakov’s action, as it is shown by equation (4.36).
2. Virasoro algebra.

There’s a local conservation law

$$\partial^a T_{ab} = 0$$

(4.87)

to which tensor $T_{ab}$ undergoes, and it allows us to build up an infinite number of conserved quantities for Polyakov’s string motion. All these quantities form the Poisson algebra of the rich group of conformal transformations, the very gauge symmetries generated by $\chi_k$’s.

First of all, we have to extend string variables between $\sigma = -\pi$ and $\sigma = 0$ in order to produce those conserved quantities; an even extension is introduced for the velocity

$$\dot{X}_\mu (-\sigma, \tau) = \dot{X}_\mu (\sigma, \tau) ,$$

(4.88)

while gradients must undergo to an odd extension:

$$X'_\mu (-\sigma, \tau) = -X'_\mu (\sigma, \tau) .$$

(4.89)

Let’s then consider a smooth function

$$f \in C^\infty ([-\pi, \pi], \mathbb{C})$$

and define the following functional of secondary constraints:

$$L [f] = T^4 \int_{-\pi}^{+\pi} f (\sigma) \left[ 2\chi_1 (\sigma) \frac{\eta}{T} + \chi_2 (\sigma) \right] d\sigma ,$$

(4.90)

which reads

$$L [f] = T^4 \int_{-\pi}^{+\pi} f (\sigma) \left[ \frac{P_\mu (\sigma)}{T} + X'_\mu (\sigma) \right] \left[ \frac{P^\nu (\sigma)}{T} + X'_\nu (\sigma) \right] d\sigma$$

(4.91)

in terms of string variables.

The important fact is that these functionals have a closed Poisson bracket algebra. The linear combination of functionals expressing the Poisson bracket of two given functionals $L [f]$ and $L [g]$ is rather easy:
\[
\{ L[f], L[g] \} = \frac{T}{4} \int_{-\pi}^{+\pi} \left[ f(\sigma) g'(\sigma) - f'(\sigma) g(\sigma) \right] \left[ \frac{2\chi_1(\sigma)}{T} + \chi_2(\sigma) \right] d\sigma,
\]
i.e. by definition (4.90):
\[
\{ L[f], L[g] \} = L[f g' - f'g]. \tag{4.92}
\]

Defining:
\[
f g' - f'g = f \times g \tag{4.93}
\]
one has:
\[
\{ L[f], L[g] \} = L[f \times g]. \tag{4.94}
\]

This tells us that equation (4.90) functionals span a closed symplectic algebra \([16]\). This gives a canonical realization of the conformal group \([17]\). This canonical group is very big, so big as the set of functions \(f\) and \(g\) which is possible to construct the \(L\)’s with, so it’s worth simplifying it ordering its elements in a more transparent way. We can do this Fourier-decomposing functions \(f\) as:
\[
f(\sigma) = \sum_{n \in \mathbb{Z}} A_n \exp\left(in\sigma\right), \tag{4.95}
\]
This induces the following basis for the algebra (4.94)
\[
L_n = \frac{T}{4} \int_{-\pi}^{+\pi} d\sigma \exp\left(in\sigma\right) \left[ \frac{2\chi_1(\sigma)}{T} + \chi_2(\sigma) \right]. \tag{4.96}
\]
These \(L_n\) are the very well known Virasoro charges: it’s evident that the conservation of these theoretical charges
\[
\dot{L}_n = 0 \tag{4.97}
\]
follows without problems from secondary constraint stability. Equation (4.97) yields as well
\[
\dot{L}[f] = 0 \quad \forall \ f, \tag{4.98}
\]
by the definition of (4.90).

In terms of Z-number indices the algebra (4.94) becomes:

$$\{L_m, L_n\} = -i (m - n) L_{m+n},$$

(4.99)

which is referred to as Virasoro classical algebra [18]. By definition, Virasoro charges are thus the very constraints of conformal invariance:

$$L_m \approx 0.$$  

(4.100)

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