The relations among two transversal submanifolds and global manifold

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In Riemann geometry, the relations among two transversal submanifolds and global manifold are discussed. By replacing the normal vector of a submanifold with the tangent vector of another submanifold, the metric tensors, Christoffel symbols and curvature tensors of the three manifolds are linked together. When the inner product of the two tangent vectors vanishes, some corollaries of these relations give the most important second fundamental form and Gauss-Codazzi equation in the conventional submanifold theory. As a special case, the global manifold is Euclidean is considered. It is pointed out that, in order to obtain the nonzero energy-momentum tensor of matter field in a submanifold, there must be the contributions of the above inner product and the other submanifold. In general speaking, a submanifold is closely related to the matter fields of the other submanifold through the above inner product. This conclusion is in agreement with the Kaluza-Klein theory and it can be applied to generalize the models of direct product of manifolds in string and D-brane theories to the more general cases — nondirect product manifolds.

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I. INTRODUCTION

Submanifold theory has reached a fruitful stage. It has been used extensively in supersymmetry and supergravity\textsuperscript{1,2} and, in particular, in strings, D-branes and M-theory.\textsuperscript{3–6} In one of our recent papers,\textsuperscript{7} using the fourth-order topological tensor current and two transversal submanifolds, we successfully obtain a new coordinate condition in general relativity, which includes the Fock’s coordinate condition as a special case. Then, with the help of the Gauss-Bonnet-Chern theorem and the transversal submanifold theory, we investigate the inner structure of the Euler-Poincaré characteristic in differential geometry in Ref. 8, which gives the global, or, topological relationship of the global manifold and the two transversal submanifolds. However, in our follow-up studies of the local, or, geometrical relations of the three manifolds, we find that thought many important formulas (such as the induced metric tensor, the second fundamental form and the Gauss-Codazzi equation) are obtained in terms of the tangent and normal vectors of submanifold, there are still two problems remained in the conventional submanifold theory of Riemann geometry. (i) The conventional submanifold theory of Riemann geometry deals with the relationship mainly between the global manifold and a single submanifold, and it always derives the geometry of submanifold from the global manifold. Can we reverse the direction, or in other words, can we build up the geometric quantity of global manifold by the correspondents of submanifolds? If we can, we will have the relations among submanifolds and the global manifold, and know how the submanifolds affect each other when the geometry of the global manifold is given. (ii) For the lack of the concrete expression of normal vector, some material calculations can not go forward deeply but be changed into a formal deduction. Can we replace the normal vector by some other vectors that play the role of normal vector under some conditions? If the answer is yes, we can extend the conventional submanifold theory of Riemann geometry. In this paper, we will discuss these two problems with the two transversal submanifolds. The results can be obviously used to generalize the models of direct product of manifolds in string
and D-brane theories to the more general cases — nondirect product manifolds.

This paper is organized as follows. In section II, as a brief review, we introduce the conventional submanifold theory of Riemann geometry. Some useful notations are also prepared. In section III, we study the relations of metric tensors, Christoffel symbols and curvature tensors of the two transversal submanifolds and global manifold. When the inner product of the tangent vectors of the two transversal submanifolds vanishes, these new relations give the conventional submanifold theory. As a special case, the global manifold is Euclidean is considered in section IV. Using the Einstein equation, the energy-momentum tensor of a submanifold is investigated. The conclusion of this paper is summarized in section V.

II. THE CONVENTIONAL SUBMANIFOLD THEORY OF RIEMANN GEOMETRY

Let \( X \) be a \( k \)-dimensional Riemann manifold with metric tensor \( g_{\mu\nu} \) and local coordinates \( x^{\mu} (\mu, \nu = 1, \cdots, k) \), and \( M \) a \( m \)-dimensional submanifold of \( X \) with local coordinates \( u^{a} (a = 1, \cdots, m; m < k) \). Then, on \( M \), one has

\[
x^{\mu} = x^{\mu}(u^{1}, \cdots, u^{m}), \quad \mu = 1, \cdots, k.
\]

The tangent vector basis of \( M \) can be expressed in terms of that of \( X \) as

\[
\frac{\partial}{\partial u^{a}} = B_{a}^{\mu} \frac{\partial}{\partial x^{\mu}}, \quad B_{a}^{\mu} = \frac{\partial x^{\mu}}{\partial u^{a}}.
\]

and the induced metric tensor \( g_{ab} \) on \( M \) is determined by

\[
g_{ab} = g_{\mu\nu} B_{a}^{\mu} B_{b}^{\nu}, \quad a, b = 1, \cdots, m.
\]

With the definition of the normal vector \( L_{A}^{\mu} \) of \( M \)

\[
g_{\mu\nu} B_{a}^{\mu} L_{A}^{\nu} = 0, \quad g_{\mu\nu} L_{A}^{\mu} L_{B}^{\nu} = \delta_{AB}, \quad A, B = m + 1, \cdots, k
\]
and the inverse matrix \([B^a, L^A] \) of the matrix \([B_a, L^A] \)

\[
B^a_b B^\mu = \delta^a_b, \quad L^A_B L^\mu = \delta^A_B, \tag{5}
\]

\[
B^a_\mu L^\mu = 0, \quad L^A_\mu B^\mu = 0, \tag{6}
\]

\[
B^a_\mu B^a_\nu + L^\mu_\nu L^\lambda = \delta^\mu_\nu, \tag{7}
\]

one can prove the completeness relations

\[
g_{\mu\nu} = g_{ab} B^a_\mu B^b_\nu + \delta_{AB} L^A_\mu L^B_\nu \tag{8}
\]

\[
g^{\mu\nu} = g^{ab} B^a_\mu B^b_\nu + \delta^{AB} L^A_\mu L^B_\nu \tag{9}
\]

and the following formulas

\[
g_{ab} B^b_\mu = g_{\mu\nu} B^\nu_a, \quad \delta_{AB} L^B_\mu = g_{\mu\nu} L^\nu_A, \tag{10}
\]

\[
g^{ab} B^a_\mu = g^{\mu\nu} B^\nu_b, \quad \delta^{AB} L^A_\mu = g^{\mu\nu} L^B_\nu, \tag{11}
\]

\[
g^{\mu\nu} B^a_\mu L^A_\nu = 0, \tag{12}
\]

where \(g^{\mu\nu}\) and \(g^{ab}\) are the inverses of \(g_{\mu\nu}\) and \(g_{ab}\), respectively. The expressions of \(B^a_\mu\) and \(L^A_\mu\) can be obtained simply by raising and lowering the indices \(\mu, a\) and \(A\) of \(B^a_\mu\) and \(L^A_\mu\), i.e.

\[
B^a_\mu = g^{ab} g_{\mu\nu} B^b_\nu, \quad L^A_\mu = \delta^{AB} g_{\mu\nu} L^B_\nu. \tag{14}
\]

After some simple calculations, the relationship between the Christoffel symbols of \(M\) and \(X\) can be written as

\[
\Gamma^c_{ab} = B^c_a B^b_\nu B^\mu_\lambda \Gamma^\lambda_{\mu\nu} + B^c_\mu \frac{\partial B^\mu_a}{\partial u^b}. \tag{15}
\]

The covariant derivative of \(B^\mu_a\) is defined by

\[
H^\mu_{ab} \equiv \nabla_a B^\mu_b = \frac{\partial B^\mu_b}{\partial u^a} - \Gamma^c_{ab} B^\mu_c + B^c_\mu \Gamma^\lambda_{\nu\lambda} B^\lambda_b \tag{16}
\]
which is called the Euler-Schouten curvature tensor. Substituting (15) into (16) and using (7), the Euler-Schouten curvature $H^\mu_{\ ab}$ is changed into

$$H^\mu_{\ ab} = (B^\lambda_a B^\rho_b \Gamma^\nu_{\lambda \rho}) L^\lambda_A L^\mu_A$$

which shows $H^\mu_{\ ab}$ can be expanded by the normal vector $L^\mu_A$. The expansion coefficients are defined as the second fundamental form $\Omega^A_{\ ab}$, i.e.

$$\Omega^A_{\ ab} \equiv H^\mu_{\ ab} L^A_\mu = (B^\lambda_a B^\rho_b \Gamma^\nu_{\lambda \rho}) L^A_\nu.$$  (18)

Therefore, the covariant derivative of $B^a_\mu$ can be rewritten as

$$H^\mu_{\ ab} = \nabla_a B^\mu_b = \Omega^A_{\ ab} L^A_\mu,$$  (19)

which leads to

$$H^\mu_{\ ab} B^c_\mu = B^c_\mu \nabla_a B^\mu_b = \Omega^A_{\ ab} L^A_\mu B^c_\mu = 0.$$  (20)

Making use of (20) and the generalized Ricci formula

$$R^d_{\ abc} = B^\mu_a B^\nu_b B^\lambda_c B^\rho_d R_{\ mu \lambda \rho} - B^d_\mu \nabla_a B^\mu_b + B^d_\rho \nabla_b B^\mu_c,$$  (21)

one obtains the most important Gauss-Codazzi equation in the conventional submanifold theory of Riemann geometry

$$R^d_{\ abc} = B^\mu_a B^\nu_b B^\lambda_c B^\rho_d R_{\ mu \lambda \rho} + \nabla_a B^d_\mu \nabla_b B^\mu_c - \nabla_b B^d_\mu \nabla_a B^\mu_c$$  (22)

where $R^d_{\ abc}$ and $R_{\ mu \lambda \rho}$ are the curvature tensors of $M$ and $X$, respectively.

So far, one can see that all of the above consequences are based on the normal vector $L^\mu_A$, i.e. the definition (4). Though the normal vector takes great success in the conventional submanifold theory, it can not give the relationship between two submanifolds. In our recent papers,\textsuperscript{7,8} we have obtained two transversal submanifolds and, then, we can replace the normal vector by the tangent vector of another submanifold. However, in this case, the orthogonal and orthonormal conditions in (4) are not held in general.
III. THE RELATIONS AMONG TWO TRANSVERSAL SUBMANIFOLDS AND GLOBAL MANIFOLD

Besides of the above-discussed submanifold $M$, let $N$ be another $n$-dimensional submanifold of $X$ with $n = k - m$ and local coordinates $v^A$ ($A = 1, \ldots, n$). The parametric equation of $N$ is

$$x^\mu = x^\mu(v^1, \ldots, v^n).$$  \hspace{1cm} (23)

Similarly, the tangent vector basis of $N$ can be expressed as

$$\frac{\partial}{\partial v^A} = C^\mu_A \frac{\partial}{\partial x^\mu}, \quad C^\mu_A = \frac{\partial x^\mu}{\partial v^A},$$  \hspace{1cm} (24)

and the induced metric tensor $g_{AB}$ is given by

$$g_{AB} = g_{\mu\nu} C^\mu_A C^\nu_B, \quad A, B = 1, \ldots, n.$$  \hspace{1cm} (25)

In the following, in order to investigate the relations among $X$, $M$ and $N$, we will consider the case that $M$ and $N$ are transversal at a point $p \in X$. Under this condition, we have

$$T_p(X) = T_p(M) + T_p(N),$$  \hspace{1cm} (26)

where $T_p(X)$, $T_p(M)$ and $T_p(N)$ are the tangent spaces of $X$, $M$ and $N$ at $p$, respectively. The expression (26) is to say

$$\frac{\partial}{\partial x^\mu} = B^a_\mu \frac{\partial}{\partial u^a} + C^A_\mu \frac{\partial}{\partial v^A}, \quad k = m + n,$$

in which $B^a_\mu$ and $C^A_\mu$ are the expansion coefficients of $\partial/\partial x^\mu$ in terms of $\partial/\partial u^a$ and $\partial/\partial v^A$, respectively. Then, substituting (27) into (2) and (24), we obtain

$$B^a_\mu B^b_\mu = \delta^b_a, \quad B^A_\mu C^A_\mu = 0,$$  \hspace{1cm} (28)

$$C^A_\mu B^a_\mu = 0, \quad C^A_\mu C^B_\mu = \delta^B_A.$$  \hspace{1cm} (29)

While inserting (2) and (24) into (27) gives

$$B^a_\mu B^\nu_a + C^A_\mu C^\nu_A = \delta^\nu_\mu.$$  \hspace{1cm} (30)
The formulas (28) — (30) tell us that the expansion coefficients $B^a_\mu$ and $C^A_\mu$ are just determined by the inverse matrix of the matrix $(B^a_\mu, C^A_\mu)$.

Now, we can discuss the relations among $g_{\mu\nu}$, $g_{ab}$ and $g_{AB}$. Using $g_{\mu\nu} = g_{\lambda\rho}\delta^\lambda_\mu \delta^\rho_\nu$, from (3), (25) and (30) we can prove

$$
g_{\mu\nu} = g_{ab} B^a_\mu B^b_\nu + g_{\lambda\rho} B^\lambda_a C^\rho_A B^a_\mu C^\nu_A + g_{\lambda\rho} C^\lambda_A B^\rho_a C^\mu_A B^a_\nu + g_{AB} C^A_\mu C^B_\nu.
$$

(31)

Corresponding to the orthogonal relation in (4), the inner product of $\partial/\partial u^a$ and $\partial/\partial v^A$ is defined as

$$
g_{aA} \equiv \left< \frac{\partial}{\partial u^a}, \frac{\partial}{\partial v^A} \right> = g_{\mu\nu} B^\mu_a C^\nu_A
$$

(32)

satisfying

$$
g_{aA} = g_{Aa}.
$$

(33)

When $C^a_A$ is orthogonal to $B^a_\mu$, we have $g_{aA} = 0$. So, (31) can be further read as

$$
g_{\mu\nu} = g_{ab} B^a_\mu B^b_\nu + g_{aA} (B^a_\mu C^A_\nu + C^A_\mu B^a_\nu) + g_{AB} C^A_\mu C^B_\nu
$$

(34)

which is the generalization of the completeness relation (8). When $g_{aA} = 0$ and $g_{AB} = \delta_{AB}$, (34) goes back to (8). (34) is nothing but what we seek to show the relations among $g_{\mu\nu}$, $g_{ab}$ and $g_{AB}$. From (34) we see that, besides of the contributions of $g_{ab}$ and $g_{AB}$, there are still the mixed terms of $g_{aA}$ in $g_{\mu\nu}$, which will lead to very important new results in our later calculations. The generalizations of the formulas (10) — (12) are

$$
g_{\mu\nu} B^\nu_a = g_{ab} B^b_\mu + g_{aA} C^A_{\mu},
$$

(35)

$$
g_{\mu\nu} C^\nu_A = g_{AB} C^B_\mu + g_{aA} B^a_\mu,
$$

(36)

$$
g_{\mu\nu} B^\nu_a = g^{ab} B^b_\mu - g^{\mu\nu} g^{ab} g_{bA} C^A_{\nu},
$$

(37)

$$
g_{\mu\nu} C^\nu_A = g^{AB} C^B_\mu - g^{\mu\nu} g^{AB} g_{ab} B^\nu_a,
$$

(38)

$$
g^{ab} = g^{\mu\nu} B^a_\mu B^b_\nu + g^{\mu\nu} g^{ac} g_{cA} C^A_{\nu} B^b_\mu,
$$

(39)

$$
g^{AB} = g^{\mu\nu} C^A_{\mu} C^B_{\nu} + g^{\mu\nu} g^{AD} g_{aD} B^a_\nu C^B_{\mu},
$$

(40)
which will return to (10) — (12) when $g_{aA} = 0$ and $g_{AB} = \delta_{AB}$, where $g^{AB}$ is the inverse of $g_{AB}$. Comparing (35) — (40) with (10) — (12), we stress that, besides of the differences between the conventional terms, the most important are the additional terms of $g_{aA}$ because they provide another way of linking the indices $\mu$ and $a$ or $\mu$ and $A$. Due to the existence of $g_{aA}$, the expressions of $B^a_{\mu}$ and $C^A_{\mu}$ are no longer as in (14) but become the equations of $B^a_{\mu}$ and $C^A_{\mu}$

\[ B^a_{\mu} = g_{\mu\nu}g^{ab}B^b_{\nu} - g^{ab}g_{bA}C^A_{\mu} \]  

(41)

\[ C^A_{\mu} = g_{\mu\nu}g^{AB}C^B_{\nu} - g^{AB}g_{aB}B^a_{\mu}. \]  

(42)

Eliminating $C^A_{\mu}$ in (41) and $B^a_{\mu}$ in (42), we get

\[ (g_{ab} - g_{aAg}^{\mu\nu}g_{Bb})B^a_{\mu} = g_{\mu\nu}(B^\nu_{\nu} - g_{\nuAg}g^{AB}C^\nu_{B}) \]  

(43)

\[ (g_{AB} - g_{aAg}^{\mu\nu}g_{Bb})C^A_{\mu} = g_{\mu\nu}(C^\nu_{B} - g_{\nuBg}g^{ab}B^\nu_{b}) \]  

(44)

which can be looked upon as another definition of $B^a_{\mu}$ and $C^A_{\mu}$. It is obvious that (41) — (44) can go back to (14) when $g_{aA} = 0$ and $g_{AB} = \delta_{AB}$.

In the following, we will study the relations among the Christoffel symbols of $X$, $M$ and $N$. From (34), the relationship among $\Gamma^\lambda_{\mu\nu}$, $\Gamma^c_{ab}$ and $\Gamma^D_{AB}$ can be calculated out to be

\[ \Gamma^\lambda_{\mu\nu} = B^a_{\mu} B^b_{\nu} g^{\lambda \rho} B^d_{\rho} g_{dc} \Gamma^c_{ab} + C^A_{\mu} C^B_{\nu} g^{\lambda \rho} C^E_{\rho} g_{ED} \Gamma^D_{AB} \]  

\[ + B^A_{\mu} \frac{\partial B^a_{\nu}}{\partial x^\mu} + C^A_{\mu} \frac{\partial C^A_{\nu}}{\partial x^\mu} + \frac{1}{2} \gamma^\lambda_{\mu\nu} \]  

(45)

where

\[ \gamma^\lambda_{\mu\nu} = g^{\lambda \rho} B^a_{\rho} C^A_{\nu} \frac{\partial g_{aA}}{\partial x^\mu} + g^{\lambda \rho} C^A_{\rho} B^a_{\nu} \frac{\partial g_{aA}}{\partial x^\mu} + g^{\lambda \rho} B^a_{\rho} C^A_{\mu} \frac{\partial g_{aA}}{\partial x^\nu} \]  

\[ + g^{\lambda \rho} C^A_{\rho} B^a_{\mu} \frac{\partial g_{aA}}{\partial x^\nu} - g^{\lambda \rho} B^a_{\mu} C^A_{\nu} \frac{\partial g_{aA}}{\partial x^\rho} - g^{\lambda \rho} C^A_{\mu} B^a_{\nu} \frac{\partial g_{aA}}{\partial x^\rho} \]  

(46)

is the contribution of the first-order derivative of $g_{aA}$. The effect of $g_{aA}$ is included in the factors $g^{\lambda \rho} B^d_{\rho} g_{dc}$ and $g^{\lambda \rho} C^E_{\rho} g_{ED}$. From (45), $\Gamma^c_{ab}$ can be expressed in terms of $\Gamma^\lambda_{\mu\nu}$ as

\[ \Gamma^c_{ab} = B^a_{\mu} B^b_{\nu} g^{cd} B^d_{\rho} g_{\rho\lambda} \Gamma^\lambda_{\mu\nu} + g^{cd} B^d_{\rho} g_{\rho\lambda} \frac{\partial B^\lambda_{b}}{\partial x^\mu}. \]  

(47)
When \( g_{aA} = 0 \), from (41) we have \( g^{cd}B^\rho_d g_{\rho\lambda} = B^\lambda_x \). Then,

\[
\Gamma^c_{ab} = B^\mu_a B^\nu_b B^c_\lambda \Gamma^\lambda_{\mu\nu} + B^c_\lambda \frac{\partial B^\lambda_b}{\partial u^a} \tag{48}
\]

which is just the formula given in (15). So, we see that the relationship between \( \Gamma^c_{ab} \) and \( \Gamma^\lambda_{\mu\nu} \) in the conventional submanifold theory is a special case of (47) which is only a corollary of (45). The similar consequences are held for \( \Gamma^D_{AB} \), too. In this sense, (45) is the total relationship among \( \Gamma^\lambda_{\mu\nu} \), \( \Gamma^c_{ab} \) and \( \Gamma^D_{AB} \). Since the effect of \( g_{aA} \) has been considered in (45) and (47), the covariant derivative of \( B^\mu_a \) can be still defined by

\[
\nabla_a B^\mu_b = \frac{\partial B^\mu_b}{\partial u^a} - \Gamma^c_{ab} B^\mu_c + B^\nu_a \Gamma^\mu_{\nu\lambda} B^\lambda_b. \tag{49}
\]

However, under the present condition, from (47) \( \nabla_a B^\mu_b \) can be represented by

\[
\nabla_a B^\mu_b = (B^\lambda_a B^\nu_b \Gamma^\nu_\lambda \rho + \frac{\partial B^\nu_b}{\partial u^\alpha}) C^A_\nu C^\mu_A - (B^\lambda_a B^\nu_b \Gamma^\nu_\lambda \rho + \frac{\partial B^\nu_b}{\partial u^\alpha}) C^A_\nu g_{dA} g^{cd} B^\mu_c \tag{50}
\]

which shows \( \nabla_a B^\mu_b \) can not be expanded by \( C^A_\mu \) only but by \( B^\mu_a \) and \( C^A_\mu \) together. This extension is also due to the existence of \( g_{aA} \). Corresponding to the second fundamental form defined in (18), let us denote the expansion coefficients as

\[
\Theta^A_{ab} \equiv (\nabla_a B^\mu_b) C^A_\mu, \quad \Phi^c_{ab} \equiv (\nabla_a B^\mu_b) B^c_\mu. \tag{51}
\]

Then,

\[
\Theta^A_{ab} = (B^\lambda_a B^\nu_b \Gamma^\nu_\lambda \rho + \frac{\partial B^\nu_b}{\partial u^\alpha}) C^A_\nu, \tag{52}
\]

\[
\Phi^c_{ab} = -(B^\lambda_a B^\nu_b \Gamma^\nu_\lambda \rho + \frac{\partial B^\nu_b}{\partial u^\alpha}) C^A_\nu g_{dA} g^{dc} \tag{53}
\]

and

\[
\nabla_a B^\mu_b = \Theta^A_{ab} C^\mu_A + \Phi^c_{ab} B^\mu_c. \tag{54}
\]

One may have noticed that the expansion coefficients are not independent. They are related by

\[
\Phi^c_{ab} = -\Theta^A_{ab} g_{dA} g^{dc}. \tag{55}
\]
Though \( \nabla_a B^\mu_b \) cannot be expanded by \( C_\mu^A \) only, we find \( \nabla_a (g^{bc}g_{\mu\nu}B^\nu_c) \) does, i.e.

\[
\nabla_a (g^{bc}g_{\mu\nu}B^\nu_c) = \frac{\partial (g^{bc}g_{\mu\nu}B^\nu_c)}{\partial x^a} + \Gamma^b_{bc} g^{cd} g_{\mu\nu} B^\nu_d - B^\nu_a \Gamma^\lambda_{\nu\alpha} g_{bc} \Gamma^\alpha_{\rho\beta} B^\rho_c - B^\nu_a \Gamma^\lambda_{\nu\alpha} g_{bc} \Gamma^\alpha_{\rho\beta} B^\rho_c
\]

\[
= \frac{\partial (g^{bc}g_{\mu\nu}B^\nu_c)}{\partial x^a} + \frac{\partial (g^{bc}g_{\mu\nu}B^\nu_c)}{\partial u^a} - B^\nu_a \Gamma^\lambda_{\nu\alpha} g_{bc} \Gamma^\alpha_{\rho\beta} B^\rho_c + \Gamma^b_{bc} g^{cd} g_{\mu\nu} B^\nu_d + \Gamma^b_{bc} g^{cd} g_{\mu\nu} B^\nu_d + \Gamma^b_{bc} g^{cd} g_{\mu\nu} B^\nu_d + \Gamma^b_{bc} g^{cd} g_{\mu\nu} B^\nu_d
\]

\[
= (\nabla_a (g^{bc}g_{\mu\nu}B^\nu_c)) C^\mu_A C^A_\mu
\]

which leads to

\[
(\nabla_a (g^{bc}g_{\mu\nu}B^\nu_c)) B^\mu_d = 0.
\]

So, \( \nabla_a (g^{bc}g_{\mu\nu}B^\nu_c) \) will play the role of \( \nabla_a B^\mu_b \) in some places, such as in the Gauss-Codazzi equation.

Now, let us consider the relations of the curvature tensors of \( X, M \) and \( N \). Similar to the covariant derivatives of \( B^\mu_a \) and \( C^\mu_A \) in the conventional submanifold theory, we denote the notations \( \nabla_\mu B^\nu_a, \nabla_\mu B^a, \nabla_\mu C^\nu_\lambda \) and \( \nabla_\mu C^A_\nu \) as

\[
\nabla_\mu B^\nu_a = B^b_a \frac{\partial B^\nu_a}{\partial x^b} - B^b_a \Gamma^\nu_{ba} B^\nu_c + \Gamma^\nu_{\mu\lambda} B^\lambda_a
\]

\[
= \frac{\partial B^\nu_a}{\partial x^b} - \Gamma^\nu_{ba} B^\nu_c - \Gamma^\nu_{\mu\lambda} B^\lambda_a
\]

\[
\nabla_\mu C^\nu_\lambda = C^B_\mu \frac{\partial C^\nu_\lambda}{\partial x^B} - C^B_\mu \Gamma^\nu_{B\lambda} C^D_\mu + \Gamma^\nu_{\mu\lambda} C^\lambda_\mu
\]

\[
\nabla_\mu C^A_\nu = \frac{\partial C^A_\mu}{\partial x^\mu} + C^B_\mu \Gamma^A_{BD} C^D_\nu - \Gamma^\nu_{\mu\lambda} C^A_\lambda
\]

Then, after long and complicated calculations, the curvature tensor \( R^\rho_{\mu\nu\lambda} \) of \( X \) can be expressed by

\[
R^\rho_{\mu\nu\lambda} = B^b_a B^c_d g^{de} B^e_{gde} R^a_{abc} + C^A_\mu C^B_\nu C^D_\lambda g^{\rho\sigma} C^E_\sigma g_{EF} R_{ABD}^F
\]

\[
+ \nabla_\mu B^c_a \nabla_\nu B^\lambda_b + \nabla_\nu C^A_\lambda \nabla_\nu C^A_\lambda - \nabla_\nu B^a_\mu \nabla_\nu B^\mu_c - \nabla_\nu C^a_\rho \nabla_\nu C^A_\lambda
\]

\[
+ \Theta + \Lambda + \Phi + \Psi + \Omega + \Sigma
\]

where \( R^a_{abc} \) and \( R_{ABD}^F \) are the curvature tensors of \( M \) and \( N \), and

\[
\Theta = g_{\alpha A} \frac{\partial B^d_\alpha}{\partial x^\mu} B^b_a B^c_d g^{bc} g^{\rho\sigma} C^A_\beta \Gamma^\alpha_{\rho\beta} + \frac{\partial C^B_\alpha}{\partial x^\mu} B^b_a B^c_d g^{bc} g^{\rho\sigma} C^A_\beta \Gamma^\alpha_{\rho\beta} + \frac{\partial (g^{\rho\sigma} C^A_\alpha)}{\partial x^\mu} B^b_a B^c_d \Gamma^\alpha_{\rho\beta}
\]

10
\[
\Lambda = \sum g_{\alpha\beta} \frac{\partial C^E}{\partial x^\alpha} C^D C^E g_{\alpha^2} B_{\alpha^2} \nabla^A + \sum g_{\alpha\beta} \frac{\partial C^E}{\partial x^\alpha} C^D C^E g_{\alpha^2} B_{\alpha^2} \nabla^A
\]

(63)

\[
\Phi = -\sum g_{\alpha\beta} \frac{\partial C^E}{\partial x^\alpha} C^D C^E g_{\alpha^2} B_{\alpha^2} \nabla^A - \sum g_{\alpha\beta} \frac{\partial C^E}{\partial x^\alpha} C^D C^E g_{\alpha^2} B_{\alpha^2} \nabla^A
\]

(64)
\[
\Psi = -\frac{1}{2}g_{\alpha\lambda}(B_{\mu}^b g^{\rho\sigma} C_\sigma B_{\alpha}^c \Gamma_{bc}^\alpha \gamma_\rho - C_\mu^B g^{\rho\sigma} B_a^c C_{\alpha}^D \Gamma_{BD}^\alpha \gamma_\rho + B_{\alpha}^b g_{\beta}^{\rho\sigma} C_\sigma B_{\alpha}^c \Gamma_{bc}^\alpha \gamma_\rho - C_\mu^B g^{\rho\sigma} B_a^c C_{\alpha}^D \Gamma_{BD}^\alpha \gamma_\rho)
\]

\[
\Omega = \frac{1}{2} \gamma_\rho^{\mu\alpha} \gamma_\lambda^{\alpha} - \frac{1}{2} \gamma_\rho^{\mu\alpha} \gamma_\mu^{\alpha}
\]

\[
\Sigma = \frac{1}{2} \frac{\partial^2 g_{\alpha\lambda}}{\partial x^\mu \partial x^\nu} g^{\rho\sigma} B_a^c C_{\rm a}^D + \frac{1}{2} \frac{\partial^2 g_{\alpha\lambda}}{\partial x^\mu \partial x^\nu} g^{\rho\sigma} C_\sigma B_{\alpha}^c C_{\mu}^D - \frac{1}{2} \frac{\partial^2 g_{\alpha\lambda}}{\partial x^\mu \partial x^\sigma} g^{\rho\sigma} B_a^c C_{\mu}^D
\]

in which \(\gamma_\mu^{\mu\mu}\) includes the first-order partial derivatives of \(g_{\alpha\lambda}\) and is defined in (65). Hence, \(\Theta, \Lambda, \Phi, \Psi, \Omega\) and \(\Sigma\) are the contributions of \(g_{\alpha\lambda}, g_{\alpha\lambda} g_{\beta\lambda}, \partial g_{\alpha\lambda}, g_{\alpha\lambda} g_{\beta\lambda}, g_{\alpha\lambda} \partial g_{\beta\lambda}\) and \(\partial^2 g_{\alpha\lambda}\), respectively. Though the expressions of \(R_{\mu\nu\lambda}^\rho\) in (62) — (68) are very complicated and strongly nonlinear, the curvature tensor \(R_{abc}^d\) of \(M\) can be represented in terms of \(R_{\mu\nu\lambda}^\rho\) simply as

\[
R_{abc}^d = B_a^u B_b^v B_c^w g_{\sigma\rho} R_{\mu\nu\lambda}^\rho + \nabla_a (g^{de} B_e^u g_{\mu\nu}) \nabla_b B_c^\nu
\]

\[
-\nabla_b (g^{de} B_e^u g_{\mu\nu}) \nabla_a B_c^\nu
\]

where

\[
\nabla_a (g^{de} B_e^u g_{\mu\nu}) = \frac{\partial (g^{de} B_e^u g_{\mu\nu})}{\partial u^a} + \Gamma_{ab}^d g_e^b g_{\mu\nu} - B_a^\lambda \Gamma_{\lambda\nu}^\rho g^{de} B_e^u g_{\mu\rho}
\]

\[
\nabla_b B_c^\nu = \frac{\partial B_c^\nu}{\partial u^b} - \Gamma_{bc}^a B_a^\nu + B_b^a \Gamma_{\nu\lambda}^\rho B_c^\lambda
\]

are the covariant derivatives of \(g^{de} B_e^u g_{\mu\nu}\) and \(B_c^\nu\). When \(g_{\alpha\lambda} = 0\), the two tangent vectors \(B_a^\mu\) and \(C_A^\mu\) of \(M\) and \(N\) are orthogonal each other and \(C_A^\mu\) plays the role of the normal vector of \(B_a^\mu\) in the conventional submanifold theory. In this case, from (41) we have \(g^{de} B_e^\sigma g_{\sigma\rho} = B_d^\rho\) and then

\[
R_{abc}^d = B_a^u B_b^v B_c^w R_{\mu\nu\lambda}^\rho + \nabla_a B_d^u \nabla_b B_c^\nu - \nabla_b B_d^u \nabla_a B_c^\nu
\]
which is just the most important Gauss-Codazzi equation in the conventional submanifold
theory of Riemann geometry. So, we see that the Gauss-Codazzi equation is a special case of
(69) which is only a corollary of (62). The similar consequences are held for $R_{ABD}^E$, too. In
this sense, (62) — (68) are the total relationship among $R_{\mu\nu\lambda}^\rho$, $R_{abc}^d$ and $R_{ABD}^E$. Comparing
(69) with (72), as indicated in above, $\nabla_a (g^{de} B^\mu_c g_{\mu\nu})$ takes the place of $\nabla_a B^d_c$ in the general
case. The formulas derived in this section can be applied to generalize the models of direct
product of manifolds in string and D-brane theories to the more general cases — nondirect
product manifolds.

IV. A SPECIAL CASE

In this section, we do not pay more attention to the condition $g_{aA} = 0$ which has been
studied extensively in Riemann geometry, but change to consider the case $g_{\mu\nu} = \delta_{\mu\nu}$, i.e. the
global manifold $X$ is Euclidean. In this case, we have

$$g_{\mu\nu} = \delta_{\mu\nu}, \quad \Gamma^\lambda_{\mu\nu} = 0, \quad R_{\mu\nu\lambda}^\rho = 0. \quad (73)$$

This is the condition that we obtain a new coordinate condition in general relativity,\(^7\) which
includes the Fock’s coordinate condition as a special case. Then, in order to maintain the
Euclidean property of $X$, the induced metric tensors and the Christoffel symbols of $M$ and
$N$ must satisfy

$$\delta_{\mu\nu} = g_{ab} B^a_{\mu} B^b_{\nu} + g_{aA}(B^a_{\mu} C^A_{\nu} + C^A_{\mu} B^a_{\nu}) + g_{AB} C^A_{\mu} C^B_{\nu} \quad (74)$$

and

$$0 = B^a_{\mu} B^b_{\nu} \delta^{\lambda\rho} B^d_{\rho} g_{dc} \Gamma^c_{ab} + C^A_{\mu} C^B_{\nu} \delta^{\lambda\rho} C^E_{\rho} g_{ED} \Gamma^D_{AB} + B^a_{\mu} \frac{\partial B^b_{\nu}}{\partial x^\mu} + C^A_{\mu} \frac{\partial C^d_{\nu}}{\partial x^\mu} + \frac{1}{2} \gamma^\lambda_{\mu\nu}. \quad (75)$$

respectively. In this case, $g_{ab}$ and $\Gamma^c_{ab}$ are represented by

$$g_{ab} = \delta_{\mu\nu} B^a_{\mu} B^b_{\nu} \quad (76)$$

$$\Gamma^c_{ab} = g^{cd} B^a_{\mu} \delta_{\mu\nu} \frac{\partial B^b_{\nu}}{\partial u^d}. \quad (77)$$
Here, we point out that (76) is not the vielbein expression of $g_{ab}$ because the dimension of $\mu$ is not equal to that of $a$ and $B^\mu_a$ is not the vielbein field. And (77) is quite different from the corresponding form of $\Gamma_{ab}$ in (15) because (77) includes the contribution of $C^A_{\mu}$ in $g^{cd}B^\mu_d\delta_{\mu\nu}$ by taking account of (41). In fact, this conclusion is also held for (47) in the general case because these two transversal submanifolds must maintain the geometric property of the given global manifold. On the curvature tensors $R_{abc}^d$ and $R_{ABD}^E$ of $M$ and $N$, we have

$$0 = B^a_{\mu}B^b_{\nu}B^c_{\lambda}\delta_{\rho\sigma}B^d_{\sigma}g_{de}R_{abc}^e + C^A_{\mu}C^B_{\nu}C^C_{\lambda}\delta_{\rho\sigma}C^E_{\sigma}g_{EF}R_{ABD}^F$$

$$+ D_\mu B^\rho_a + D_\mu C^a_{\nu}C^A_{\lambda} - D_\nu B^\rho_a - D_\nu C^a_{\nu}C^A_{\lambda} - D_\nu C^a_{\nu}D_\mu C^A_{\lambda}$$

$$+ \Theta' + \Lambda' + \Phi'$$

(78)

where

$$D_\mu B^\rho_a = B^b_\mu \frac{\partial B^\rho_a}{\partial x^b} - B^b_\mu \Gamma^c_{ba} B^\rho_c, \quad D_\nu B^\rho_a = \frac{\partial B^\rho_a}{\partial x^\nu} + B^b_\nu \Gamma^a_{bc} B^\rho_c$$

(79)

$$D_\mu C^a_{\nu} = C^B_{\mu} \frac{\partial C^a_{\nu}}{\partial x^B} - C^B_{\mu} \Gamma^D_{BA} C^D_{\nu}$$

(80)

and

$$\Theta' = g_{aA}(B^b_\mu B^d_{\nu}B^e_{\lambda})\Gamma_a^\Gamma_b \Gamma_c^\Gamma_d + C^B_{\mu} C^E_{\nu} C^F_{\lambda} \delta_{\rho\sigma} B^a_{\sigma} \Gamma^A_{BD} \Gamma^D_{EF} + B^b_\mu \delta_{\rho\sigma} C^a_{\sigma} \frac{\partial B^e_{\lambda}}{\partial x^\nu} \Gamma_a^\Gamma_b$$

$$+ C^B_{\mu} \delta_{\rho\sigma} B^a_{\sigma} \frac{\partial C^a_{\nu}}{\partial x^B} \Gamma_b^\Gamma_c - \frac{\partial C^a_{\nu}}{\partial x^B} B^b_\mu \nabla^c B^\rho_c \Gamma_a^\Gamma_b$$

$$- B^b_\mu B^d_{\nu} B^e_{\lambda} \delta_{\rho\sigma} C^a_{\sigma} \Gamma_b^\Gamma_c$$

$$- B^b_\mu B^d_{\nu} B^e_{\lambda} \delta_{\rho\sigma} C^a_{\sigma} \Gamma_b^\Gamma_c - B^b_\mu \delta_{\rho\sigma} C^A_{\mu} \Gamma^A_{BD} \Gamma^D_{EF} - B^b_\mu \delta_{\rho\sigma} C^a_{\sigma} \frac{\partial B^e_{\lambda}}{\partial x^\nu} \Gamma_a^\Gamma_b$$

(81)

$$\Lambda' = \frac{\partial g_{aA}}{\partial x^\mu} B^b_\mu B^c_{\lambda} \delta_{\rho\sigma} C^A_{\sigma} \Gamma_a^\Gamma_b$$

$$- \frac{\partial g_{aA}}{\partial x^\mu} C^B_{\mu} C^C_{\lambda} \delta_{\rho\sigma} B^a_{\sigma} \Gamma_B^\Gamma_D + \frac{1}{2} \frac{\partial g_{aA}}{\partial x^\mu} \delta_{\rho\sigma} \frac{\partial (B^a_{\sigma} C^a_{\lambda})}{\partial x^\nu}$$

$$+ \frac{1}{2} \frac{\partial g_{aA}}{\partial x^\mu} \delta_{\rho\sigma} \frac{\partial (C^A_{\mu} B^a_{\lambda})}{\partial x^\nu}$$

$$- \frac{1}{2} \frac{\partial g_{aA}}{\partial x^\mu} \delta_{\rho\sigma} \frac{\partial (B^a_{\sigma} C^a_{\nu})}{\partial x^\mu}$$

$$+ \frac{1}{2} \frac{\partial g_{aA}}{\partial x^\mu} \delta_{\rho\sigma} \frac{\partial (C^A_{\nu} B^a_{\lambda})}{\partial x^\mu}$$

$$- \frac{1}{2} \frac{\partial g_{aA}}{\partial x^\mu} \delta_{\rho\sigma} \frac{\partial (B^a_{\sigma} C^a_{\nu})}{\partial x^\mu}$$

$$- \frac{1}{2} \frac{\partial g_{aA}}{\partial x^\mu} \delta_{\rho\sigma} \frac{\partial (C^A_{\nu} B^a_{\lambda})}{\partial x^\mu}$$

(82)
\[
\Phi' = \frac{1}{2} \frac{\partial^2 g_{\alpha\beta}}{\partial x^\mu \partial x^\nu} \delta^\rho_{\sigma} B^\sigma_{\rho} C^\nu_{\alpha} + \frac{1}{2} \frac{\partial^2 g_{\alpha\beta}}{\partial x^\mu \partial x^\nu} \delta^\rho_{\sigma} C^\alpha_{\rho} B^\nu_{\sigma} - \frac{1}{2} \frac{\partial^2 g_{\alpha\beta}}{\partial x^\mu \partial x^\nu} \delta^\rho_{\sigma} B^\sigma_{\rho} C^\nu_{\alpha} + \frac{1}{2} \frac{\partial^2 g_{\alpha\beta}}{\partial x^\mu \partial x^\nu} \delta^\rho_{\sigma} C^\alpha_{\rho} B^\nu_{\sigma}
\]

are the contributions of \(g_{\alpha\beta}, \partial g_{\alpha\beta}\) and \(\partial^2 g_{\alpha\beta}\), respectively. Correspondingly, \(R_{\alpha\beta\gamma}^d\) in (69) becomes

\[
R_{\alpha\beta\gamma}^d = D_a(g^{de} B^e_{\mu} \delta_{\mu\nu}) D_b B^\nu_{c} - D_b(g^{de} B^e_{\mu} \delta_{\mu\nu}) D_a B^\nu_{c}
\]

in which

\[
D_a(g^{de} B^e_{\mu} \delta_{\mu\nu}) = \frac{\partial(g^{de} B^e_{\mu} \delta_{\mu\nu})}{\partial u^a} + \Gamma^d_{\alpha\beta} b^e_{\mu} \delta_{\mu\nu}
\]

\[
D_b B^\nu_{c} = \frac{\partial B^\nu_{c}}{\partial u^b} - \Gamma^c_{\alpha\beta} B^\nu_{a}
\]

are the covariant derivatives of \(g^{de} B^e_{\mu} \delta_{\mu\nu}\) and \(B^\nu_{c}\) in the present condition. From (77) \(R_{\alpha\beta\gamma}^d\) is further read as

\[
R_{\alpha\beta\gamma}^d = \frac{\partial(g^{de} B^e_{\mu} \delta_{\mu\nu})}{\partial u^a}(C^\rho_{A} - B^\rho_{f} g^{fe} g_{e\alpha}) C^A_{\nu} \frac{\partial B^\nu_{c}}{\partial u^b} - \frac{\partial(g^{de} B^e_{\mu} \delta_{\mu\nu})}{\partial u^b}(C^\rho_{A} - B^\rho_{f} g^{fe} g_{e\alpha}) C^A_{\nu} \frac{\partial B^\nu_{c}}{\partial u^a}.
\]

Since \(C^\mu_{A}\) has nothing to do with \(u^a\), we obtain

\[
R_{\alpha\beta\gamma}^d = \frac{\partial(g^{de} g_{e\alpha})}{\partial u^a} C^A_{\mu} \frac{\partial B^\mu_{c}}{\partial u^b} - \frac{\partial(g^{de} B^e_{\mu} \delta_{\mu\nu})}{\partial u^a} B^\nu_{f} g^{fg} g_{g\alpha} C^A_{\lambda} \frac{\partial B^\lambda_{c}}{\partial u^b} - \frac{\partial(g^{de} g_{e\alpha})}{\partial u^b} C^A_{\mu} \frac{\partial B^\mu_{c}}{\partial u^a} + \frac{\partial(g^{de} B^e_{\mu} \delta_{\mu\nu})}{\partial u^b} B^\nu_{f} g^{fg} g_{g\alpha} C^A_{\lambda} \frac{\partial B^\lambda_{c}}{\partial u^a}
\]

\[
= D_a(g^{de} g_{e\alpha}) C^A_{\mu} \frac{\partial B^\mu_{c}}{\partial u^b} - D_b(g^{de} g_{e\alpha}) C^A_{\mu} \frac{\partial B^\mu_{c}}{\partial u^a}.
\]

We see that when \(g_{\alpha\beta} = 0\), \(R_{\alpha\beta\gamma}^d = 0\) and when \(g_{\alpha\beta} \neq 0\), \(R_{\alpha\beta\gamma}^d\) may not vanish. What does this mean? From (34) we know that when \(g_{\alpha\beta} = 0\), the global manifold \(X\) is the direct product of \(M\) and \(N\). Under this condition, (88) tells us when \(X\) is Euclidean, both the two submanifolds \(M\) and \(N\) are Euclidean, too, i.e. \(R^k = R^m \times R^n\). This is the trivial case either in mathematic or in physics. However, when \(g_{\alpha\beta} \neq 0\), \(X\) can not be expressed as the direct product of \(M\) and \(N\). Even when \(X\) is Euclidean, \(M\) and \(N\) are not Euclidean but affect
each other through $g_{\alpha A}$. In this sense, we say $g_{\alpha A}$ is the bridge between $M$ and $N$. After the symmetrization, the Ricci tensor of $M$ is

$$
R_{bc} = -\frac{1}{2} \left( g^{af} B^\nu_f \delta_\nu^\lambda \frac{\partial (B^\lambda_d g^{de} g_{eA})}{\partial u^a} C^A_{\mu} \frac{\partial B^c_{\mu}}{\partial u^b} + g^{af} B^\nu_f \delta_\nu^\lambda \frac{\partial (B^\lambda_d g^{de} g_{eA})}{\partial u^c} C^A_{\mu} \frac{\partial B^b_{\mu}}{\partial u^a} \right) + \frac{1}{2} \frac{\partial (B^\lambda_d g^{de} g_{eA})}{\partial u^a} C^A_{\mu} \frac{\partial B^b_{\mu}}{\partial u^b} \right),
$$

(89)

which gives the scalar curvature

$$
R = -B^a_a g^{af} B^\nu_f \delta_\nu^\lambda \frac{\partial (B^\lambda_d g^{de} g_{eA})}{\partial x^\sigma} C^A_{\mu} \frac{\partial C^A_{\mu}}{\partial x^\rho} - \frac{\partial (B^\lambda_d g^{de} g_{eA})}{\partial x^\rho} \frac{\partial C^A_{\mu}}{\partial x^\sigma} B^b_b g^{bc} B^\mu_c.
$$

(90)

Then one has

$$
R_{ij} - \frac{1}{2} g_{ij} R = -B^a_a g^{af} B^\nu_f \delta_\nu^\lambda \frac{\partial (B^\lambda_d g^{de} g_{eA})}{\partial x^\sigma} C^A_{\mu} \frac{\partial C^A_{\mu}}{\partial x^\rho} - \frac{\partial (B^\lambda_d g^{de} g_{eA})}{\partial x^\rho} \frac{\partial C^A_{\mu}}{\partial x^\sigma} B^b_b g^{bc} g_{ij}.
$$

(91)

Corresponding to the Einstein equation of $M$

$$
R_{ij} - \frac{1}{2} g_{ij} R = -8\pi GT_{ij},
$$

(92)

the energy-momentum tensor of matter field in $M$ is determined by

$$
T_{ij} = \frac{1}{16\pi G} B^a_a g^{af} B^\nu_f \delta_\nu^\lambda \frac{\partial (B^\lambda_d g^{de} g_{eA})}{\partial x^\sigma} C^A_{\mu} \frac{\partial C^A_{\mu}}{\partial x^\rho} - \frac{\partial (B^\lambda_d g^{de} g_{eA})}{\partial x^\rho} \frac{\partial C^A_{\mu}}{\partial x^\sigma} B^b_b g^{bc} g_{ij}.
$$

(93)

in which $G$ is the Newton constant. Here one can see that, in order to obtain the nonzero energy-momentum tensor $T_{ij}$ of matter field, there must be the contributions of the bridge $g_{\alpha A}$ and the transversal submanifold $N$. So we conclude that the other transversal submanifold $N$ is closely related to the matter fields of the submanifold $M$. This conclusion is the significant generalization to the conventional submanifold theory of Riemann geometry and it is in agreement with the Kaluza-Klein theory, of which the high-dimensional coordinates of space-time are linked to the gauge fields through the so-called dimensional reduction.

V. CONCLUSIONS
In this paper, by replacing the normal vector of a submanifold with the tangent vector of another submanifold, the total relations of the metric tensors, Christoffel symbols and curvature tensors of two transversal submanifolds and global manifold are discussed. The corresponding formulas are the generalizations of the conventional submanifold theory of Riemann geometry. When the inner product of the two tangent vectors vanishes, all of the above consequences will return to the conventional submanifold theory and some corollaries will give the most important second fundamental form and Gauss-Codazzi equation. As a special case, the global manifold is Euclidean is considered. It is pointed out that, in order to maintain the property of the given global manifold, the metric tensors, Christoffel symbols and curvature tensors of the two transversal submanifolds must satisfy some conditions. Furthermore, the above inner product plays the role of bridge through which the two transversal submanifolds affect each other and one submanifold is closely related to the matter fields of another submanifold. This conclusion is the significant generalization to the conventional submanifold theory of Riemann geometry and it is in agreement with the Kaluza-Klein theory, of which the high-dimensional coordinates of space-time are linked to the gauge fields through the so-called dimensional reduction. The consequences obtained in this paper can be applied to generalize the models of direct product of manifolds in string and D-brane theories to the more general cases — nondirect product manifolds. The applications of this paper will be studied in our later work.

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1. J. Erlich, A. Naqvi and L. Randall, Phys. Rev. D 58, 046002 (1998).

2. S. Hyun, Y. Kiem and H. Shin, Phys. Rev. D 57, 4856 (1998).

3. C.S. Chu, P.S. Howe and E. Sezgin, Phys. Lett. B 428, 59 (1998).

4. I. Kishimoto and N. Sasakura, Phys. Lett. B 432, 305 (1998).

5. S.W. Hawking and M.M. Taylor-Robinson, Phys. Rev. D 58, 025006 (1998).

6. S. Ferrara, A. Kehagias, H. Partouche and A. Zaffaroni, Phys. Lett. B 431, 42 (1998).

7. G.H. Yang and Y.S. Duan, Mod. Phys. Lett. A 13, 745 (1998).

8. G.H. Yang, Mod. Phys. Lett. A 13, 2123 (1998).

9. L. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, NJ, 1964).