Nonzero $\theta_{13}$ for Neutrino Mixing in the Context of $A_4$ Symmetry
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Nonzero $\theta_{13}$ for neutrino mixing in the context of $A_4$ symmetry

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Abstract

In the original 2004 paper which first derived tribimaximal mixing in the context of $A_4$, i.e. the non-Abelian finite symmetry group of the tetrahedron, as its simplest application, it was also pointed out how $\theta_{13} \neq 0$ may be accommodated. On the strength of the new T2K result that $0.03(0.04) \leq \sin^22\theta_{13} \leq 0.28(0.34)$ for $\delta_{CP} = 0$ and normal (inverted) neutrino mass hierarchy, we perform a more detailed analysis of how this original idea may be realized in the context of $A_4$. 
Neutrino oscillations require nonzero neutrino masses as well as nonzero neutrino mixing angles. The current combined world data imply [1]

\[ 7.05 \times 10^{-5} \text{ eV}^2 \leq \Delta m_{21}^2 \leq 8.34 \times 10^{-5} \text{ eV}^2, \]  
\[ 2.07 \times 10^{-3} \text{ eV}^2 \leq \Delta m_{32}^2 \leq 2.75 \times 10^{-3} \text{ eV}^2, \]  
\[ 0.36 \leq \sin^2 \theta_{23} \leq 0.67, \quad 0.25 \leq \sin^2 \theta_{12} \leq 0.37, \]  
\[ \sin^2 \theta_{13} \leq 0.035 \ (90\% \ CL). \]  

However, the T2K Collaboration recently announced that a new measurement [2] has yielded a nonzero \( \theta_{13} \) at 90% confidence level, i.e.

\[ 0.03(0.04) \leq \sin^2 2\theta_{13} \leq 0.28(0.34) \]  

for \( \delta_{CP} = 0 \) and normal (inverted) neutrino mass hierarchy.

For several years now, the mixing matrix \( U_{\nu} \) linking the charged leptons \( (e, \mu, \tau) \) to the neutrino mass eigenstates \( (\nu_1, \nu_2, \nu_3) \) has often been assumed to be of tribimaximal form [3], i.e.

\[ U_{TB} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}, \]  

which predicts \( \theta_{13} = 0 \). This is particularly appealing because it was derived in 2004 [4] from the simple application of the symmetry group \( A_4 \), first used for understanding maximal \( \nu_\mu - \nu_\tau \) mixing in 2001 [5]. However, even in that original 2004 paper [4], the possibility of \( \theta_{13} \neq 0 \) was already anticipated. Although the new T2K result [2] is only 2.5\( \sigma \) away from zero, it is the most solid experimental indication to date of this possibility. Here we offer a more detailed analysis of how \( \theta_{13} \neq 0 \) may be realized in the context of \( A_4 \).

As is well-known, \( A_4 \) is the group of the even permutation of 4 objects. It is also the symmetry of the perfect three-dimensional tetrahedron [6]. It has 12 elements and 4 irreducible
representations: 1, 1', 1'', 3, with the multiplication rule

\[ 2 \times 2 = 1 + 1' + 1'' + 3 + 3. \quad (7) \]

The first step in understanding neutrino mixing is to show that \( A_4 \) allows the charged-lepton mass matrix to be diagonalized by the Cabibbo-Wolfenstein matrix \([7, 8]\)

\[ U_{CW} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad (8) \]

where \( \omega = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2 \), with three independent eigenvalues, i.e. \( m_e, m_\mu, m_\tau \).

This has been achieved in two ways. One is the original proposal of 2001 \([5]\). The other was discovered later in 2006 \([9]\). In the former, the lepton assignments are \( L_i = (\nu_i, l_i) \sim 3, \ l'_1 \sim 1, \ l'_2 \sim 1', \ l'_3 \sim 1'' \), with 3 Higgs doublets \( \Phi_i = (\phi^0_i, \phi^-_i) \sim 3 \). In the latter, they are \( L_i = (\nu_i, l_i) \sim 3, \ l''_i \sim 3 \), with 4 Higgs doublets \( \Phi_i = (\phi^0_i, \phi^-_i) \sim 3, \ \Phi_0 \sim 1 \). Assuming \( v_1 = v_2 = v_3 \) for the vacuum expectation values of \( \Phi_i \), which correspond to a \( Z_3 \) residual symmetry (lepton triality) \([10, 11, 12, 13]\), the seemingly impossible result of a diagonal charged-lepton matrix is always obtained from \( U_{CW} \) of Eq.(8), independent of the values of \( m_e, m_\mu, m_\tau \). This is a highly nontrivial result, which motivates how the otherwise arbitrary \( 3 \times 3 \) neutrino mass should be organized. It argues strongly for an underlying non-Abelian symmetry with a three-dimensional irreducible representation, the smallest of which is \( A_4 \).

We now consider the neutrino mass matrix in the original \( A_4 \) basis. Let there be 6 heavy Higgs triplets \([14]\):

\[ \xi_1 \sim 1, \ \xi_2 \sim 1', \ \xi_3 \sim 1'', \ \xi_i \sim 3 \ (i = 4, 5, 6), \quad (9) \]

where \( \xi_i = (\xi_i^+, \xi_i^-, \xi_i^0) \). Then

\[ M_\nu = \begin{pmatrix} a + b + c & f & e \\ f & a + \omega b + \omega^2 c & d \\ e & d & a + \omega^2 b + \omega c \end{pmatrix}, \quad (10) \]
where \(a\) comes from \(\langle \xi_1^0 \rangle\), \(b\) from \(\langle \xi_2^0 \rangle\), \(c\) from \(\langle \xi_3^0 \rangle\), \(d\) from \(\langle \xi_4^0 \rangle\), \(e\) from \(\langle \xi_5^0 \rangle\), \(f\) from \(\langle \xi_6^0 \rangle\). As it stands, there is of course no prediction at all. For a pattern to emerge, the way \(A_4\) breaks into its subgroups must be considered. For \(b = c\) and \(e = f = 0\), which breaks \(A_4\) to \(Z_2\), the neutrino mass matrix, written in the basis where the charged-lepton mass matrix is diagonal, is given by

\[
M_{\nu}^{(e, \mu, \tau)} = U_{\nu}^\dagger W U_{\nu} = \begin{pmatrix}
a + (2d/3) & b - (d/3) & b - (d/3) \\
b - (d/3) & b + (2d/3) & a - (d/3) \\
b - (d/3) & a - (d/3) & b + (2d/3)
\end{pmatrix},
\]

which is indeed diagonalized by \(U_T\) of Eq.(6), with eigenvalues \(m_1 = a - b + d\), \(m_2 = a + 2b\), and \(m_3 = -a + b + d\). It has been shown [15] how this pattern is obtained from \(A_4\) alone with the help of lepton number.

Deviations from tribimaximal mixing may be obtained for \(b \neq c\). This will allow \(\nu_1\) to mix with \(\nu_3\) and \(\theta_{13}\) becomes nonzero. However, this same mixing will move \(\theta_{12}\) to a larger value [4] so that \(\tan^2 \theta_{12} > 0.5\) which is not favored by current data. To allow \(\tan^2 \theta_{12} < 0.5\), it was proposed [4] that \(e = -f \neq 0\) in Eq.(10). This is maintained by an assumed residual symmetry of the \(\xi \Phi \Phi\) soft terms of the Higgs potential under which \(\xi_5 \leftrightarrow -\xi_6\) and \(\Phi_2 \leftrightarrow \Phi_3\). As a result, the neutrino mass matrix under \(U_T\) is no longer diagonal, but is given by [4]

\[
M_{\nu}^{(1, 2, 3)} = \begin{pmatrix}
m_1 & 0 & m_4 \\
0 & m_2 & m_5 \\
m_4 & m_5 & m_3
\end{pmatrix},
\]

where \(m_1 = a - (b + c)/2 + d\), \(m_2 = a + b + c\), \(m_3 = -a + (b + c)/2 + d\), \(m_4 = \sqrt{3}/2(c - b)\) and \(m_5 = -i\sqrt{2}e\). If \(m_4 = 0\), then \(\nu_2\) mixes with \(\nu_3\) and it can be shown that the allowed range of \(\theta_{23}\) from Eq.(3) implies \(\sin^2 2\theta_{13} \leq 0.04\) which lies on the outer edge of the allowed region of Eq.(5). In the following, we consider both \(m_4\) and \(m_5\) to be nonzero and study various numerical solutions to the T2K data.

The atmospheric neutrino mixing is assumed to be maximal, i.e. \(\sin^2 2\theta_{23} = 1\), which is also the assumption of T2K in obtaining their new result. The solar neutrino mixing is
taken to be $\sin^2 2\theta_{12} = 0.87 \pm 0.3$ [1]. We also use $\Delta m_{32}^2 = 2.40 \times 10^{-3}$ eV$^2$ which is the value used by T2K, and $\Delta m_{21}^2 = 7.65 \times 10^{-5}$ eV$^2$. For the central value of $\theta_{12} = 34.43^\circ$, we have $\tan^2 \theta_{12} = 0.47$ which is rather close to the tribimaximal prediction of 0.5. Using this and assuming the central value of $\sin \theta_{13} = 0.168$ ($\sin^2 \theta_{13} = 0.11$), the zero entry of the neutrino mass matrix of Eq.(12) implies the condition

$$0.007655m'_1 - 0.020990m'_2 + 0.013342m'_3 = 0,$$

(13)

where $m'_{1,2,3}$ are the mass eigenvalues of Eq.(12). Hence they are related to the measured $\Delta m_{32}^2$ and $\Delta m_{21}^2$ by

$$m'_2 = \pm \sqrt{m'_1^2 + \Delta m_{21}^2},$$

(14)

$$m'_3 = \pm \sqrt{m'_1^2 + \Delta m_{21}^2/2 \pm \Delta m_{32}^2}.$$

(15)

There is only one solution to Eq.(13), i.e.

$$m'_1 = 0.0246 \text{ eV}, \quad m'_2 = -0.0261 \text{ eV}, \quad m'_3 = -0.0552 \text{ eV},$$

(16)

which exhibits normal mass hierarchy. From this solution, we then obtain $m_{1,2,3,4,5}$ and the original $A_4$ parameters $a, b, c, d, e$. The $\nu_e$ mass observed in nuclear beta decay is given by

$$\sum_i |U_{ei}|^2 |m'_i| = 0.026 \text{ eV}.$$  

The effective mass $m_{ee}$ for neutrinoless double beta decay is

$$m_{ee} = |a + (2/3)d| = |(2/3)m_1 + (1/3)m_2|.$$

(17)

We plot in Figs. 1 to 3 the solutions for $|m'_{1,2,3}|$ and $m_{ee}$ as a function of $\sin^2 2\theta_{13}$ in the range 0.03 to 0.135 [corresponding to the upper bound given in Eq.(4)] for $\sin^2 \theta_{23} = 1$ and the values $\sin^2 2\theta_{12} = 0.84, 0.87, 0.90$. Thus $m_{ee}$ is predicted to be at most 0.04 eV. As for the $\nu_e$ mass in nuclear beta decay, it can be read off as approximately given by $(2|m'_1| + |m'_2|)/3$. We also plot in Fig. 4 the values of $m_{1,2,3,4,5}$ for $\sin^2 2\theta_{12} = 0.87$. This shows that $m_4$ and $m_5$, i.e. the parameters of $A_4$ which deviate from tribimaximal mixing, are indeed small. In
terms of $A_4$ symmetry, the following breaking patterns are in effect: in the charged-lepton sector, $A_4$ breaks to $Z_3$ (which may be verified experimentally from Higgs-boson decay [13]); in the neutrino sector, $A_4$ breaks first to $Z_2$ (the tribimaximal limit), and then $Z_2$ is also broken with the pattern $b \neq c$ and $f = -e$, which may be maintained by a suitably chosen set of soft terms.

In conclusion, on the strength of the recent observation [2] of a nonzero $\theta_{13}$ for neutrino mixing, the original $A_4$ proposal of 2004 [4] is updated. We find that solutions are indeed possible with the most recent data but only in a normal hierarchy of neutrino masses, i.e. $|m_1'| < |m_2'| < |m_3'|$. We confirm that the parameters of $A_4$ which deviate from tribimaximal mixing, i.e. $m_4$ and $m_5$, are indeed small. We also make predictions on the effective $m_{ee}$ in neutrinoless double beta decay.

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Figure 1: Physical neutrino masses $|m'_{1,2,3}|$ and the effective $m_{ee}$ for neutrinoless double beta decay of this model in the range $0.03 \leq \sin^2 2\theta_{13} \leq 0.135$ for $\sin^2 2\theta_{23} = 1$ and $\sin^2 2\theta_{12} = 0.84$. 

\[ \sin^2(2\theta_{12}) = .84 \]
Figure 2: Physical neutrino masses $|m'_{1,2,3}|$ and the effective $m_{ee}$ for neutrinoless double beta decay of this model in the range $0.03 \leq \sin^2 2\theta_{13} \leq 0.135$ for $\sin^2 2\theta_{23} = 1$ and $\sin^2 2\theta_{12} = 0.87$. 

$\sin^2(2\theta_{12}) = .87$
Figure 3: Physical neutrino masses $|m'_{1,2,3}|$ and the effective $m_{ee}$ for neutrinoless double beta decay of this model in the range $0.03 \leq \sin^2 2\theta_{13} \leq 0.135$ for $\sin^2 2\theta_{23} = 1$ and $\sin^2 2\theta_{12} = 0.90$. 
Figure 4: The $A_4$ parameters $m_{1,2,3,4,5}$ of this model in the range $0.03 \leq \sin^2 2\theta_{13} \leq 0.135$ for $\sin^2 2\theta_{23} = 1$ and $\sin^2 2\theta_{12} = 0.87$. 