A vortex in a trapped Bose-Einstein condensate can experience at least two types of instabilities. (1). Macroscopic hydrodynamic motion of the vortex core relative to the center of mass of the condensate requires some process to dissipate energy. (2). Microscopic small-amplitude normal modes can also induce an instability. In one specific example, the vortex core again moves relative to the overall center of mass, suggesting that there may be only a single physical mechanism.

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1. INTRODUCTION

The remarkable recent experimental creation of Bose-Einstein condensates in trapped low-temperature alkali gases has generated great interest in the possibility of vortex states. Such vortices have been widely studied in superfluid $^4$He but no clear experimental evidence demonstrates the existence of a vortex in a trapped Bose condensate. Theoretical work has concentrated on the critical angular velocity $\Omega_{c1}$ for vortex creation, the normal modes of a condensate containing a vortex, and general considerations of stability.

The present work studies two specific types of instabilities of a vortex in a trapped Bose condensate. The first (Sec. 2) is a hydrodynamic instability involving the macroscopic motion of a vortex relative to the background condensate, including the effect of the nonuniform condensate density, which has not previously been incorporated. The second instability (Sec. 3) arises from the microscopic internal oscillations of the vortex, which has been studied in a particular geometry by Dodd et al. This latter behavior is especially clear in the weak-coupling limit, where the normal modes of the
vortex differ qualitatively from those of a vortex-free condensate.

2. HYDRODYNAMIC INSTABILITY OF A VORTEX

The behavior of a trapped condensate depends crucially on the number \( N \) of atoms in the condensate. Consider an axisymmetric trap with a potential

\[
V_{tr} = \frac{1}{2}M\left(\omega_\perp^2 \rho^2 + \omega_z^2 z^2\right),
\]

where \( M \) is the atomic mass and \((\rho, \phi, z)\) are the familiar cylindrical polar coordinates. The radial and axial oscillator lengths \( d_\perp = \sqrt{\hbar/M \omega_\perp} \) and \( d_z = \sqrt{\hbar/M \omega_z} \) characterize the condensate’s dimensions for an ideal trapped Bose gas (the corresponding volume is of order \( d_\perp^3 \equiv d_\perp^2 d_z \)).

The short-range two-body interaction potential may be written as

\[
V(r) \approx g\delta^{(3)}(r),
\]

where \( g \approx 4\pi a \hbar^2/M \) relates the coupling strength to the \( s \)-wave scattering length \( a \) (here assumed positive). The nonuniform condensate wave function \( \psi(r) \) obeys the Gross-Pitaevskii (GP) equation

\[
(T + V_{tr} + V_H) \psi = \mu \psi,
\]

where \( T = -\hbar^2 \nabla^2 / 2M \) is the kinetic-energy operator, \( V_H(r) = gN|\psi(r)|^2 = 4\pi a N \hbar^2 |\psi(r)|^2 / M \) is the Hartree potential energy of one particle with the remaining particles, and \( \mu \) is the chemical potential.

Typically, the dimensionless ratio \( a/d_0 \) is small (of order \( 10^{-3} \)), but the relevant dimensionless parameter \( Na/d_0 \) varies linearly with the condensate number \( N \). If \( Na/d_0 \) is small, then the interactions are weak, and the condensate acts like a nearly ideal Bose gas with characteristic radial and axial dimensions \( d_\perp \) and \( d_z \). In the opposite limit \( Na/d_0 \gg 1 \), the repulsive interactions predominate, and the condensate expands well beyond the harmonic-oscillator lengths. In this Thomas-Fermi (TF) limit, the kinetic energy is negligible, and the condensate density follows from the GP equation

\[
N|\psi|^2 = g^{-1} (\mu - V_{tr}) = n(0) \left( 1 - \frac{\rho^2}{R^2_\perp} - \frac{z^2}{R^2_z} \right),
\]

where \( n(0) = \mu/g \) is the central density and \( R_\perp \) and \( R_z \) are the radial and axial condensate dimensions with \( R^2_\alpha / d^2_\alpha = 2\mu / \hbar \omega_\alpha \gg 1 \) for \( \alpha = \perp \) and \( z \). In the TF limit, the normalization condition \( \int dV |\psi|^2 = 1 \) yields the dimensionless parameter

\[
\frac{Na}{d_0} = \frac{1}{15} \frac{R^5_\perp}{d^5_0} \gg 1,
\]
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where $R_0^3 = R_z^2 R_\perp$ characterizes the TF condensate volume. The repulsive interactions also introduce yet another important length [the coherence length $\xi \equiv 1/\sqrt{8\pi n(0)a}$], which here determines the vortex-core radius; in the TF limit, it obeys the relation $\xi R_0 = d_0^2$, implying the set of inequalities $\xi \ll d_0 \ll R_0$.

If the trap rotates with angular velocity $\Omega$, the relevant “free energy” is $F = E - \Omega L_z$, where $E$ is the energy of the condensate and $L_z$ is its angular momentum. The phase $S$ of the condensate wave function determines the velocity $v = (\hbar/M) \nabla S$, and the GP equation (3) provides an explicit expression for the TF free energy

$$F \approx \int dV \left( \frac{1}{2} M n v^2 + n V_{tr} + \frac{1}{2} g n^2 - M n \Omega \hat{z} \cdot r \times v \right),$$

where spatial derivatives of the density are neglected.

This integral for $F$ provides a variational expression in the rotating frame, and the classical velocity potential $\Phi$ provides a convenient approximation for the phase $S$. The simplest case is a uniform fluid with density $n$ per unit length in a long circular cylinder with radius $R$. When a vortex with circulation $\kappa = h/M$ is added to the system at a distance $x_0 R < R$ from the symmetry axis, the change in the free energy $\Delta F$ is

$$\Delta F = \frac{M \kappa^2 \pi}{4\pi} \left[ \ln \left( \frac{R}{\xi} \right) + \ln \left( 1 - x_0^2 \right) - \frac{\Omega}{\Omega_0} \left( 1 - x_0^2 \right) \right],$$

where $\Omega_0 = \kappa/2\pi R^2 = \hbar/M R^2$ is a characteristic angular speed. In the absence of dissipation, the vortex executes a circular orbit at fixed radius under the influence of its opposite image at a distance $R/x_0 > R$ from the axis. In the presence of dissipation, however, the vortex moves to reduce its free energy. For $\Omega < \Omega_0$, the free energy $\Delta F$ decreases monotonically with increasing $x_0$, so the vortex simply spirals outward and annihilates with its image. In contrast, if $\Omega > \Omega_0$, the free energy near the axis increases with increasing $x_0$, so that a vortex near the axis tends to return to the center of the container. This situation is merely metastable for $\Omega \leq \Omega_{c1} \equiv \Omega_0 \ln(R/\xi)$, since the free energy at the center is higher than that at the wall (with a barrier at some intermediate distance), but the vortex becomes a true equilibrium state when $\Omega \geq \Omega_{c1}$.

The TF radial profile density $n = n(0)(1 - \rho^2/R^2)$ provides a more realistic description of a nonuniform rotating fluid in a long circular container. The mean density per unit length $\bar{n}$ is half the central density $n(0)$, and the corresponding vortex-induced change in the free energy is

$$\Delta F = \frac{M \kappa^2 \pi}{4\pi} (1 - x_0^2) \left[ 2 \ln \left( \frac{R}{\xi} \right) + \frac{1 + x_0^2}{x_0^2} \ln \left( 1 - x_0^2 \right) - \frac{\Omega}{\Omega_0} \left( 1 - x_0^2 \right) \right].$$
An expansion for small $x_0$ shows that a central vortex is stable for $\Omega > \Omega_{c1} \equiv \Omega_0[2\ln(R/\xi) - 1]$, unstable for $\Omega < \Omega_m \equiv \Omega_0[\ln(R/\xi) + \frac{1}{4}] = \frac{1}{2}\Omega_{c1} + \frac{3}{4}$, and metastable for $\Omega_m < \Omega < \Omega_{c1}$. Thus, inclusion of a realistic density profile affects the detailed form of the free energy (the metastable region is considerably reduced), but the qualitative picture remains unchanged.

3. STABLE AND UNSTABLE MODES OF A VORTEX

Rokhsar\cite{11} has argued that a vortex in a nearly ideal trapped condensate is unstable because the relatively large core (of order $d_\perp$) implies a bound state in the core. In particular, he noted that Dodd et al.\cite{8} found an unstable solution of the Bogoliubov equations for a condensate containing a singly quantized vortex (this unstable mode is distinct from the rigid dipole modes that necessarily oscillate at the trap frequency $\omega_\perp$ independent of the interaction parameter $Na/d_0$).

To clarify the physical meaning of this instability, it is helpful to study the relevant solutions of the Bogoliubov equations

\begin{align}
(T + V_{tr} - \mu + 2gN|\psi|^2)u_j - gN(\psi)^2v_j &= \hbar \omega_j u_j; \\
-gN(\psi^*)^2u_j + (T + V_{tr} - \mu + 2gN|\psi|^2)v_j &= -\hbar \omega_j v_j,
\end{align}

where $u_j$ and $v_j$ are the normal-mode amplitudes and $\omega_j$ is the corresponding frequency. A state $\psi$ with positive norm $\int dV (|u_j|^2 - |v_j|^2) = 1$ is potentially unstable if $\omega_j < 0$ since the creation of a quasiparticle in the $j$th mode lowers the energy relative to that of the condensate.\cite{15} For any solution of the Bogoliubov equations, the perturbations in the particle density and velocity potential are

\begin{align}
n_j' = (\psi^* u_j - \psi v_j) e^{-i\omega_j t}, \\
\Phi_j' = \frac{\hbar}{2M|\psi|^2} (\psi^* u_j + \psi v_j) e^{-i\omega_j t}.
\end{align}

For a nearly ideal gas, the terms proportional to $gN$ in Eqs. (9), (10), and (11) can be treated in perturbation theory. If the condensate has no vortex, then the leading approximation to the condensate wave function is $\psi \approx \chi_{00}$, where I ignore the $z$-dependent part of the wave function and $\chi_{n_+,n_-}(\rho,\phi)$ is a normalized two-dimensional oscillator wave function containing $n_+$ and $n_-$ right and left circular quanta created by the raising operators $a_{\pm}^\dagger \equiv a_{\pm x}^\dagger \pm ia_{\pm y}^\dagger$.\cite{16} The lowest excited solutions of the Bogoliubov equations for this ground-state vortex-free condensate are the rigid dipole modes with frequency $\omega_\pm = \omega_\perp$ for all interaction strengths. Apart from corrections of order $g$, their detailed form

\begin{align}
u_+ &\approx \chi_{10}, \\
u_- &\approx \chi_{01}, \\
v_\pm &\approx 0, \quad \text{with} \quad \omega_\pm = \omega_\perp.
\end{align}
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follows either by direct construction or from the general explicit dipole solutions of the Bogoliubov equations. \(^{17}\)

\[ u_\pm = a_\pm^\dagger \psi, \quad v_\pm = a_\mp \psi^*, \]  

where \( \psi \) is any solution of the GP equation (3). The corresponding density and velocity-potential perturbations follow from Eqs. (11)

\[ n'_\pm \approx n_0 \rho e^{i\phi} e^{-i\omega_{\perp} t}; \quad \Phi'_\pm \approx \frac{1}{2i} \rho e^{i\phi} e^{-i\omega_{\perp} t}, \]  

where \( n_0 = |\chi_{00}|^2 \) is the unperturbed condensate density. They represent a circular motion of the rigidly displaced condensate in the positive and negative sense, respectively.

The situation is much more interesting for a singly quantized vortex, with condensate wave function \( \psi \approx \chi_{10} \propto e^{i\phi} \rho e^{-\rho^2/2} \) in the noninteracting limit. The general construction in Eq. (13) now yields

\[ \begin{pmatrix} u_+ \\ v_+ \end{pmatrix} = \left( \sqrt{2} \chi_{20} \chi_{00} \right), \quad \begin{pmatrix} u_- \\ v_- \end{pmatrix} = \left( \chi_{11} \begin{array}{c} 0 \end{array} \right), \quad \text{with} \quad \omega_\pm = \omega_{\perp} \]  

so that the + state involves a coherent superposition of a particle and a hole, differing from that for the − state. Nevertheless, they both oscillate with frequency \( \omega_{\perp} \), and the density and velocity-potential perturbations [from Eqs. (11)] have the same form

\[ n'_\pm \approx n_v \left( \rho - \frac{1}{\rho} \right) e^{i\phi} e^{-i\omega_{\perp} t}; \quad \Phi'_\pm \approx \frac{\hbar}{2M} \left( \rho \pm \frac{1}{\rho} \right) e^{i\phi} e^{-i\omega_{\perp} t} \]  

where \( n_v \approx |\chi_{10}|^2 \) is the condensate density for the vortex state (this expression also follows by an expansion of the condensate density for a small rigid displacement of both the center of mass and the position of the vortex core).

The present case of a singly quantized vortex leads to an additional anomalous mode with frequency \( \omega_a \approx -\omega_{\perp} \), reflecting the single-particle transition from the vortex state to the (lower) Gaussian ground state \( \chi_{00} \). \(^{11}\)

To zero order in the interaction parameter \( aN/d_z \), the associated Bogoliubov amplitudes are \( u_a \approx \cosh \theta \chi_{00} \) and \( v_a \approx \sinh \theta \chi_{02} \), which are properly normalized for any value of the parameter \( \theta \). To determine the actual value of \( \theta \), it is necessary to use first-order perturbation theory, yielding the expressions

\[ \begin{pmatrix} u_a \\ v_a \end{pmatrix} = \left( \sqrt{2} \chi_{00} \chi_{02} \right) \quad \text{and} \quad \omega_a \approx \omega_{\perp} \left( -1 + \frac{aN}{4d_z} \sqrt{2} / \pi \right). \]  

Like the + dipole mode of the vortex, this anomalous mode is a coherent superposition of a particle and a hole. Note that the frequency here differs from the trap frequency (and hence from the rigid dipole-oscillation
frequency) for any nonzero interaction strength. The corresponding density and velocity-potential perturbation are

\[
n'_a \approx -\frac{n_v}{\sqrt{2}} \left( \rho - \frac{2}{\rho} \right) e^{-i\phi} e^{-i\omega_at} \quad \text{and} \quad \Phi'_a \approx \frac{\hbar}{2\sqrt{2}Mi} \left( \rho + \frac{2}{\rho} \right) e^{-i\phi} e^{-i\omega_at}.
\]

A detailed study for this anomalous mode shows that the position of the vortex core is shifted by twice that of the condensate’s center of mass. In addition, the overlap integral \( \int d^2\rho \, n'_a(x \pm iy) \) vanishes identically, so this mode is not excited by the dipole oscillation of the center of mass.

It is striking that both types of instability (hydrodynamic in Sec. 2 and microscopic in Sec. 3) involve the relative motion of the vortex core and the center of mass; they may well be two descriptions of the same physics. It will be particularly interesting to study the character and role of this anomalous mode for increasing coupling strength.

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