COHOMOLOGICAL DIMENSION WITH RESPECT TO THE LINKED IDEALS

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Abstract. Let $R$ be a commutative Noetherian ring. Using the new concept of linkage of ideals over a module, we show that if $a$ is an ideal of $R$ which is linked by the ideal $I$, then $\text{cd} (a, R) \in \{ \text{grade } a, \text{cd} (a, H^\text{grade}_I a(R)) + \text{grade } a \}$, where $\mathfrak{c} := \bigcap_{p \in \text{Ass } R} \mathfrak{p} - V(a)$ p. Also, it is shown that for every ideal $b$ which is geometrically linked with $a$, $\text{cd} (a, H^\text{grade}_b b(R))$ does not depend on $b$.

1. Introduction

Let $R$ be a commutative Noetherian ring, $a$ be an ideal of $R$ and $M$ be an $R$-module. For each $i \in \mathbb{Z}$, $H^i_a(M)$ denotes the $i$-th local cohomology module of $M$ with respect to $a$ (our terminology on local cohomology modules comes from [1]). Vanishing of these modules is an important problem in this topic and it attracts lots of interest, see for example [7] and [12]. One of the most various invariants in local cohomology theory is the cohomological dimension of $M$ with respect to the ideal $a$, i.e.

$$\text{cd} (a, M) := \text{Sup } \{ i \in \mathbb{N}_0 | H^i_a(M) \neq 0 \}.$$

In this paper, we consider the cohomological dimension of $M$ with respect to the "linked ideals" over it.

Following [13], two proper ideals $a$ and $b$ in a Cohen-Macaulay local ring $R$ is said to be linked if there is a regular sequence $\underline{x}$ in their intersection such that $a = (\underline{x}) :_R b$ and $b = (\underline{x}) :_R a$. In a recent paper, [5], the authors introduced the concept of linkage of ideals over a module and studied some of its basic properties. Let $a$ and $b$ be two non-zero ideals of $R$ and $M$ denotes a non-zero finitely generated $R$-module. Assume that $aM \neq M \neq bM$ and let $I \subseteq a \cap b$ be an ideal generating by an $M$-regular sequence. Then the

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ideals $a$ and $b$ are said to be linked by $I$ over $M$, denoted by $a \sim_{(I;M)} b$, if $bM = IM :_M a$ and $aM = IM :_M b$. This concept is the classical concept of linkage of ideals in [13], where $M = R$. Note that these two concepts do not coincide [5, 2.6] although, in some cases they do (e.g. Example [5, 2.4]). We can also characterize linked ideals over $R$, see [6, 2.7].

As an application of this generalization, one may characterize Cohen-Macaulay modules in terms of the type of linked ideals over it, see [6, 3.5].

In this paper, we consider the above generalization of linkage of ideals over a module and, among other things, study the cohomological dimension of an $R$-module $M$ with respect to the ideals which are linked over $M$. In particular, in Theorem 2.7 we show that if $a$ is an ideal of $R$ which is linked by $I$ over $M$, then

$$\text{cd} (a, M) \in \{ \text{grade } M a, \text{cd} (a, H^{\text{grade } M a}_c (M)) + \text{grade } M a \},$$

where $c := \bigcap_{p \in \text{Ass } M} M / V(a)_p$.

And in Corollary 2.11 it is shown that for every ideal $b$ which is geometrically linked with $a$ over $M$, $\text{cd} (a, H^{\text{grade } M b}_b (M))$ is constant and does not depend on $b$.

Also, we show that if $\text{cd} (b, R) < \dim (R)$ for any linked ideal $b$ over $R$, then $\text{cd} (a, R) < \dim (R)$ for any ideal $a$ (Corollary 2.17).

Throughout the paper, $R$ denotes a commutative Noetherian ring with $1 \neq 0$, $a$ and $b$ are two non-zero proper ideals of $R$ and $M$ denotes a non-zero finitely generated $R$-module.

## 2. Cohomological dimension

The cohomological dimension of an $R$-module $X$ with respect to $a$ is defined by

$$\text{cd} (a, X) := \text{Sup } \{ i \in \mathbb{N}_0 | H^i_a (X) \neq 0 \}.$$

It is a significant invariant in local cohomology theory and attracts lots of interest, see for example [7] and [12]. In this section, we study this invariant via "linkage". We begin by the definition of our main tool.

**Definition 2.1.** Assume that $aM \neq M \neq bM$ and let $I \subseteq a \cap b$ be an ideal generated by an $M$-regular sequence. Then we say that the ideals $a$
and \( b \) are linked by \( I \) over \( M \), denoted \( a \sim_{(I; M)} b \), if \( bM = IM :_{M} a \) and \( aM = IM :_{M} b \). The ideals \( a \) and \( b \) are said to be geometrically linked by \( I \) over \( M \) if \( aM \cap bM = IM \). Also, we say that the ideal \( a \) is linked over \( M \) if there exist ideals \( b \) and \( I \) of \( R \) such that \( a \sim_{(I; M)} b \). A is \( M \)-selflinked by \( I \) if \( a \sim_{(I; M)} a \). Note that in the case where \( M = R \), this concept is the classical concept of linkage of ideals in [13].

The following lemma, which will be used in the next proposition, finds some relations between local cohomology modules of \( M \) with respect to ideals which are linked over \( M \).

**Lemma 2.2.** Assume that \( I \) is an ideal of \( R \) such that \( a \sim_{(I; M)} b \). Then

1. \( \sqrt{I + \text{Ann } M} = \sqrt{(a \cap b) + \text{Ann } M} \). In particular, \( H_{a \cap b}^{i}(M) \cong H_{I}^{1}(M), \) for all \( i \).
2. Let \( I = 0 \). Then, \( \sqrt{0 :_{\text{Ann } M} a} = \sqrt{\text{Ann } aM} = \sqrt{\text{Ann } bM} \). Therefore, \( H_{\text{Ann } bM :_{M} a}^{i}(M) \cong H_{\text{Ann } aM}^{i}(M) \cong H_{b}^{i}(M) \). In other words, if \( M \) is faithful, then \( H_{a}^{i}(M) \cong H_{0 :_{M} a}^{i}(M) \).

**Proof.** (i) Let \( r \in ab \). By the assumption, \( rM \subseteq IM \). Therefore, in view of [8, 2.1], there exist an integer \( n \) and \( b_{1}, ..., b_{n} \in I \) such that \( (r^{n} + r^{n-1}b_{1} + ... + b_{n})M = 0 \). This implies that \( r + \text{Ann } M \subseteq \sqrt{I + \text{Ann } M} \), as desired. Now, the result follows using [1, 4.2.1].

(ii) Let \( r \in \text{Ann } aM \). Then, by the assumption, \( rM \subseteq bM \) and using similar argument in part (i) one can see that \( r + \text{Ann } M \subseteq \sqrt{\text{Ann } bM} \). Also, via the fact that \( ab \subseteq \text{Ann } M \), \( \sqrt{\text{Ann } aM} \subseteq 0 :_{\text{Ann } M} a \). This proves the desired equalities. Now, the isomorphisms between local cohomology modules follows using [1, 4.2.1].

**Proposition 2.3.** Let \( I \) be an ideal of \( R \) such that \( a \sim_{(I; M)} b \) and set \( t := \text{grade }_{M} I \). Then \( \text{cd } (a + b, M) \leq \text{Max } \{ \text{cd } (a, M), \text{cd } (b, M), t + 1 \} \). Moreover, if \( \text{cd } (a + b, M) \geq t + 1 \), e.g. \( a \) and \( b \) are geometrically linked over \( M \), then the equality holds.

**Proof.** Note that \( H_{I}^{i}(M) = 0 \) for all \( i \neq t \), by [1, 1.3.9 and 3.3.1]. Now, the result follows from 2.2 and using the Mayer-Vietoris sequence.
\[ ... \rightarrow H_{a \cap b}^1(M) \rightarrow H_{a+b}^{i+1}(M) \rightarrow H_a^{i+1}(M) \oplus H_b^{i+1}(M) \rightarrow H_{a \cap b}^{i+1}(M) \rightarrow ... \]

The following corollary, which is immediate by the above proposition, shows that, in spite of [3, 21.22], parts of an \( R \)-regular sequence can not be linked over \( R \).

**Corollary 2.4.** Let \( (R, m) \) be local and \( x_1, ..., x_n \in m \) be an \( R \)-regular sequence, where \( n \geq 4 \). Then \( (x_{i_1}, ..., x_{i_j}) \sim (x_{i_{j+1}}, ..., x_{i_{2j}}) \), for all \( 1 < j \leq \left\lfloor \frac{n}{2} \right\rfloor \) and any permutation \( (i_1, ..., i_{2j}) \) of \( \{1, ..., 2j\} \).

Let \( M \neq aM \). It is well-known, by [1, 1.3.9], that \( \text{grade}_M a \leq \text{cd}_M (a, M) \). Then \( M \) is said to be relative Cohen-Macaulay with respect to \( a \) if

\[ \text{cd}_M (a, M) = \text{grade}_M a. \]

In the following proposition we compute the cohomological dimension of an \( R \)-module \( M \) with respect to \( a \) in two cases.

**Proposition 2.5.** Let \( I \) be an ideal of \( R \) generating by an \( M \)-regular sequence of length \( t \) and \( a \sim (I; M) b \).

(i) If \( M \) is relative Cohen-Macaulay with respect to \( a + b \), then \( H_a^i(M) = 0 \) for all \( i \notin \{\text{grade}_M a, \text{grade}_M a + b\} \).

(ii) If \( I = 0 \), then \( \text{cd}_M (a, M) = \text{cd}_M (a, \frac{M}{bM}) \).

**Proof.** (i) Using the assumption, \( H_{a \cap b}^i(M) = 0 \) for all \( i \neq t \). Now, the result follows from the isomorphism \( H_{a+b}^i(M) \cong H_a^i(M) \oplus H_b^i(M) \), for all \( i > t + 1 \), and the surjective map

\[ H_{a+b}^{t+1}(M) \rightarrow H_a^{t+1}(M) \oplus H_b^{t+1}(M) \rightarrow 0, \]

which are deduced by the Mayer-Vietoris sequence.

(ii) It follows from the fact that \( bM \) is \( a \)-torsion.

\[ \square \]

The following lemma will be used in the rest of the paper.
Lemma 2.6. Let \( I \) be a proper ideal of \( R \) such that \( a \sim_{(I;M)} b \). Then, \( \frac{M}{aM} \) can be embedded in finite copies of \( \frac{M}{IM} \).

Proof. Assume that \( F \to \frac{R}{I} \to \frac{R}{b} \to 0 \) is a free resolution of \( \frac{R}{b} \) as \( \frac{R}{I} \)-module. Then, using \( \ast := \text{Hom}_{\frac{R}{I}}(-, \frac{M}{IM}) \), we get the exact sequence \( 0 \to (\frac{R}{b})^\ast \to (\frac{R}{I})^\ast \xrightarrow{f} F^\ast \), where \( \frac{M}{aM} \cong \text{Im}(f) \subseteq F^\ast \cong \bigoplus \frac{M}{IM} \).

\[ \square \]

The next theorem, which is our main result, provides a formula for \( \text{cd} (a, M) \) in the case where \( a \) is linked over \( M \).

Theorem 2.7. Let \( I \) be an ideal of \( R \) generating by an \( M \)-regular sequence such that \( \text{Ass} \frac{M}{IM} = \text{Min} \text{Ass} \frac{M}{IM} \) and \( a \) is linked by \( I \) over \( M \). Then

\[ \text{cd} (a, M) \in \{ \text{grade} \frac{M}{M} a, \text{cd} (a, H_{c}^{\text{grade} \frac{M}{M} a}(M)) + \text{grade} \frac{M}{M} a \}, \]

where \( c := \bigcap_{p \in \text{Ass} \frac{M}{aM} - V(a)} p \).

Proof. Note that, by 2.6, \( \text{Ass} \frac{M}{aM} \subseteq \text{Ass} \frac{M}{IM} \). Set \( t := \text{grade} \frac{M}{M} a \). Without loss of generality, we may assume that \( \text{cd} (a, M) \neq t \). Hence, there exists \( p \in \text{Ass} \frac{M}{IM} - V(a) \), else, \( \sqrt{I} + \text{Ann} \frac{M}{M} = \sqrt{a} + \text{Ann} \frac{M}{M} \) which implies that \( \text{cd} (a, M) = t \). We claim that

\[ \text{(2.1)} \quad \text{grade} \frac{M}{M} (a + c) > t. \]

Suppose the contrary. So, there exist \( p \in \text{Ass} \frac{M}{IM} \) and \( q \in \text{Ass} \frac{R}{c} \) such that \( a + q \subseteq p \). By the assumption, \( p = q \) which is a contradiction to the structure of \( c \).

Let \( A := \{ p | p \in \text{Ass} \frac{M}{IM} \cap V(a) \} \). Then, in view of 2.6,

\[ \sqrt{a + \text{Ann} \frac{M}{M}} = \bigcap_{p \in \text{Min} \text{Ass} \frac{M}{aM}} p \geq \bigcap_{p \in A} p. \]

On the other hand, let \( p \in \text{Min} A \). Then, there exists \( q \in \text{Min} \text{Ass} \frac{M}{aM} \) such that \( q \subseteq p \). Hence, again by 2.6, \( q \in A \) and, by the structure of \( p, q = p \). Therefore,

\[ \text{(2.2)} \quad \sqrt{a + \text{Ann} \frac{M}{M}} = \bigcap_{p \in A} p. \]
Whence, using (2.2), it follows that
\[ \sqrt{I} + \text{Ann } M = \bigcap_{p \in \text{Ass } \frac{M}{IM}} p = \bigcap_{p \in \text{Ass } \frac{M}{IM}} p \cap c = \sqrt{a} \cap c + \text{Ann } M. \]

Now, in view of (2.1), we have the following Mayer-Vietoris sequence
\[ (2.3) \quad 0 \rightarrow H^t_a(M) \oplus H^t_c(M) \rightarrow H^t_I(M) \rightarrow N \rightarrow 0 \]
for some \( a \)-torsion \( R \)-module \( N \). Applying \( \Gamma_a(\cdot) \) on (2.3), we get the exact sequence
\[ 0 \rightarrow H^t_a(M) \oplus \Gamma_a(H^t_c(M)) \rightarrow \Gamma_a(H^t_I(M)) \rightarrow N \overset{f}{\rightarrow} H^1_a(H^t_c(M)) \rightarrow H^1_a(H^t_I(M)) \rightarrow 0 \]
and the isomorphism
\[ H^i_a(H^t_I(M)) \cong H^i_a(H^t_c(M)), \text{ for all } i > 1. \]

Also, using [10, 3.4], we have \( H^{i+t}_a(M) \cong H^i_a(H^t_I(M)) \), for all \( i \in \mathbb{N}_0 \). This implies that
\[
H^i_a(M) \begin{cases} 
\cong H^{i-t}_a(H^t_I(M)) & \text{if } i > t + 1, \\
\cong H^i_a(H^t_I(M)) & \text{if } i = t + 1, \\
\neq 0 & \text{if } i = t, \\
0 & \text{otherwise.}
\end{cases}
\]

Now, the result follows from the above isomorphisms.

The following corollary, which follows from the above theorem, provides a precise formula for \( \text{cd } (a, M) \) in the case where \( a \) is geometrically linked over \( M \) and shows how far \( \text{cd } (a, M) \) is from grade \( M^a \). Note that by [1, 1.3.9], \( \text{grade } M^a \leq \text{cd } (a, M) \).

**Corollary 2.8.** Let \( I \) be an ideal of \( R \) generating by an \( M \)-regular sequence and \( a \) and \( b \) be geometrically linked by \( I \) over \( M \). Also, assume that \( M \) is not relative Cohen-Macaulay with respect to \( a \). Then
\[ \text{cd } (a, M) = \text{cd } (a, H^{\text{grade } M^a}(M)) + \text{grade } M^a. \]
Proof. First, we show that

\[(2.4) \quad \{p | p \in \text{Ass} \frac{M}{IM} - V(a)\} = \{p | p \in \text{Ass} \frac{M}{IM} \cap V(b)\}.
\]

Let \(p \in \text{Ass} \frac{M}{IM} - V(a)\). Then, by 2.2(i), \(p \in V(b)\). On the other hand, if \(p \in \text{Ass} \frac{M}{IM} \cap V(b)\) then \(p \not\in a\), else,

\[0 : \frac{M}{IM} a \cap 0 : \frac{M}{IM} b = 0 : \frac{M}{IM} a + b \neq 0,
\]

which is a contradiction.

Now, in view of [5, 2.8(iii)],

\[\sqrt{b + \text{Ann} M} = \bigcap_{p \in \text{Ass} \frac{M}{IM} \cap V(b)} p.
\]

This, in conjunction with (2.4), implies that

\[\sqrt{b + \text{Ann} M} = \bigcap_{p \in \text{Ass} \frac{M}{IM} - V(a)} p.
\]

Now, the result follows using similar argument as used in the proof of theorem 2.7. \(\square\)

Proposition 2.9. Let \(I\) be an ideal of \(R\) generating by an \(M\)-regular sequence and \(a\) and \(b\) be geometrically linked by \(I\) over \(M\). Then \(\text{grade} M a + b = \text{grade} M I + 1\).

Proof. Assume that \(a\) and \(b\) are geometrically linked by some \(M\)-regular sequence \(I\) of length \(t\) over \(M\). Then \(a\) and \(b\) are geometrically linked by zero over \(\frac{M}{IM}\). In view of the fact that \(\text{grade} M a + b = \text{grade} \frac{M}{IM} (a + b) + t\), one can replace \(M\) by \(\frac{M}{IM}\) and assume that \(I = 0\). Hence, it is enough to show that \(\text{grade} M a + b = 1\).

Let \(p \in V(a + b)\). By [5, 2.12], \(aR_p\) and \(bR_p\) are geometrically linked by zero over \(M_p\). In conjunction with the facts that \(\text{grade} M a + b \leq \text{grade} M_p (a + b) R_p\) and \(\text{grade} M a + b \geq 1\), it is enough to show that \(\text{grade} M_p (a + b) R_p = 1\). Therefore, one may assume that \(R\) is local.

On the contrary, assume that \(\text{grade} M a + b > 1\). Using the long exact sequence

\[0 \to \text{Hom} \left( \frac{R}{a + b}, M \right) \to \text{Hom} \left( \frac{R}{a}, M \right) \oplus \text{Hom} \left( \frac{R}{b}, M \right) \to \text{Hom} \left( \frac{R}{a \cap b}, M \right) \to \text{Ext}^1_R \left( \frac{R}{a + b}, M \right)
\]
and the assumption \((a \cap b)M \subseteq aM \cap bM = 0\), we get
\[ M \cong aM \oplus bM. \]

By the fact that \(abM = 0\) we get, for any \(i > 0\), \(b^iM = bM\). Then, by krull theorem, \(bM = 0\) and so \(aM = M\), which is a contradiction. □

**Remark 2.10.**

(i) An ideal can be linked with more than one ideal. As an example, let \(R\) be local and \(x, y, z\) be an \(M\)-regular sequence. Then, \(Rx\) is geometrically linked with \(Ry\) and \(Rz\) over \(M\).

(ii) Let \(a\) and \(b\) be geometrically linked by \(I\) over \(R\). Then, by [6, 2.4], \(\sqrt{a}\) is linked by \(I\). In particular, \(\sqrt{a}\) is linked by every ideal \(I' \subset \sqrt{a}\) which is generated by a maximal \(R\)-sequence in \(a\). Indeed, by [5, 2.8] and [4, Theorem 1],

\[ \text{Ass} \frac{R}{a} = V(a) \cap \text{Ass} \frac{R}{I} = \text{Ass} \text{Hom}_R(\frac{R}{a}, \frac{R}{I}) = V(a) \cap \text{Ass} \frac{R}{I}, \]

which, in view of [6, 2.4], implies that \(\sqrt{a}\) is linked by \(I'\).

The following corollary shows that for all ideals \(b\) which are geometrically linked with \(a\) over \(M\), \(\text{cd} (a, H^\text{grade}_{Mb}(M))\) is constant.

**Corollary 2.11.** Let \(a\) be linked over \(M\). Then, for every ideal \(b\) which is geometrically linked with \(a\) over \(M\), \(\text{cd} (a, H^\text{grade}_{Mb}(M))\) is constant. In particular,

\[ \text{cd} (a, H^\text{grade}_{Mb}(M)) = \begin{cases} 1, & M \text{ is relative Cohen-Macaulay with respect to } a, \\ \text{cd} (a, M) - \text{grade}_M a, & \text{otherwise.} \end{cases} \]

**Proof.** Assume that \(M\) is relative Cohen-Macaulay with respect to \(a\) and \(a\) and \(b\) are geometrically linked by some \(M\)-regular sequence \(I\) of length \(t\) over \(M\). Then, by [5, 2.8], we have the following Mayer-Vietoris sequence

\[ 0 \rightarrow H^i_a(M) \oplus H^i_b(M) \rightarrow H^i_M(M) \rightarrow N \rightarrow 0 \]
for some $a$-torsion $R$-module $N$. Applying $\Gamma_a(-)$ on (2.5), we get the exact sequence

$$H_a^{i-1}(N) \rightarrow H_a^i(H_b^i(M)) \rightarrow H_a^i(H_t^i(M)),$$

for $i > 1$. Now, by [10, 3.4] and the assumption, we get $H_a^i(H_b^i(M)) = 0$, for $i > 1$. On the other hand, again by [5, 2.8], $\Gamma_a(H_b^i(M)) = 0$.

Therefore, using the convergence of spectral sequences

$$H_a^i(H_b^i(M)) \Rightarrow H_a^{i+j}(M)$$

and the assumption, we get $H_a^1(H_b^i(M)) \cong H_a^{i+1}(M)$. Now, by 2.9, $H_a^1(H_b^i(M)) \neq 0$ and $\text{cd} (a, H_b^\text{grade}_a(M)) = 1$.

In the case where $M$ is not relative Cohen-Macaulay with respect to $a$, the result follows from 2.8.

\[\Box\]

Convention 2.12. Assume that $I$ is an ideal of $R$ which is generated by an $M$-regular sequence. We define the set

$$S_{(I:M)} := \{a \subset R| I \subsetneq a, a = IM :_R IM :_M a\}.$$

$S_{(I:R)}$ actually contains all linked ideals by $I$.

The following proposition, which is needed in the next two items, shows that any ideal $a$ with $aM \neq M$ can be embedded in a radical ideal $a'$ of $S_{(I:M)}$ for some $I$.

Proposition 2.13. Assume that $aM \neq M$. Then,

(i) There exists an ideal $I$, generating by an $M$-regular sequence, such that $a$ can be embedded in a radical element $a'$ of $S_{(I:M)}$ with $\text{grade}_M a' = \text{grade}_M a =: t$. Also, $a'$ can be chosen to be the smallest radical ideal with this property.

(ii) Let $a'$ be as in (i). Then $\text{Ass} H_a^1(M) = \text{Ass} \frac{R}{a'}$. In particular, $a' = \bigcap_{p \in \text{Ass} H_a^1(M)} p$ and it is independent of the choice of the ideal $I$.

Proof. (i) Let $x_1, \ldots, x_t \in a$ be an $M$-regular sequence such that $a \subset Z_R(\underbrace{M}_{(x_1, \ldots, x_t)_M})$. Replacing $(x_1, \ldots, x_t)$ with $(x_1, \ldots, x_t)$, we may assume, in addition, that $x_1, \ldots, x_t \notin a$ and it is not a prime ideal. Set $\Lambda :=
\{p \mid p \in \text{Ass}(\frac{M}{(x_1, \ldots, x_t)M}), a \subseteq p\} \text{ and } a' = \cap_{p \in \Lambda} p. \text{ Then, setting } I := (x_1, \ldots, x_t) \text{ and using [6, 2.4], } a \subseteq a' \text{ is a radical ideal of } S(I; M).

Assume that there exists a radical ideal c \in S(I; M) \text{ such that } a \subseteq c. \text{ Hence, by [6, 2.4](iii), } \text{Ass } \frac{R}{c} \subseteq \text{Ass } \frac{M}{IM} \cap V(a) = \Lambda. \text{ Therefore, } a' \subseteq c.

(ii) By [4, Theorem 1] and [14, 1.4], we have
\[
\text{Ass } H^t_a(M) = \text{Ass } \text{Ext}^t_R(R_a, M).
\]
Also, using the above notations and in view of [2, 1.2.4 and 1.2.27],
\[
\text{Ass } \text{Ext}^t_R(R_a, M) = \text{Ass } \text{Hom}_R(R_a, M) = V(a) \cap \text{Ass } \frac{M}{IM} = \text{Ass } \frac{R}{a'}.\]

Now, by the above equalities, we have
\[
a' = \bigcap_{p \in \text{Ass } H^t_a(M)} p.
\]

\[\square\]

Corollary 2.14. Let a be a linked ideal over M. Then, \(\sqrt{a + \text{Ann } M} = \bigcap_{p \in \text{Ass } H^t_a(M)} p.\)

Proof. It follows from [6, 2.4](v) and the above proposition. \[\square\]

The following theorem provides some conditions in order to have \(cd(a, M) < \dim M.\)

Theorem 2.15. Let \((R, m)\) be local and \(x = x_1, \ldots, x_t\) be an M-regular sequence of length t. Assume that \(H^t_{p \text{ dim } M}(M) = 0\) for all \(p \in \text{Ass } R_{(x)M}^M.\) Then, \(H^t_{a \text{ dim } M}(M) = 0\) for any ideal \(a \supseteq (x)\) with \(\text{grade } M a = t.\)

Proof. Let \(n := \dim M\) and \(b \in S((x); M)\) be a radical ideal. Then, by [6, 2.4], \(b = \bigcap_{i=1}^l p_i\) for some \(l \in \mathbb{N}\) and \(p_1, \ldots, p_l \in \text{Ass } R_{(x)M}^M.\) By the assumption, \(H^n_b(M) = 0\) when \(l = 1.\) In the case \(l > 1,\) set \(c = \bigcap_{i=2}^l p_i.\) Then, using the Mayer-Vietoris sequence
\[
\ldots \rightarrow H^n_{p_1}(M) \oplus H^n_c(M) \rightarrow H^n_b(M) \rightarrow 0
\]
and the inductive hypothesis, we have \(H^n_b(M) = 0.\)

Now, let \(a \supseteq (x)\) be an ideal with \(\text{grade } M a = t.\) Then, by 2.13, there exists a radical ideal \(b \in S((x); M)\) such that \(a \subseteq b.\) Let \(b = a + (y_1, \ldots, y_m).\) Then, using induction on \(m\) and [1, 8.12], it is straightforward to see that there exists an onto homomorphism \(H^n_b(M) \rightarrow H^n_a(M) \rightarrow 0,\) and the result follows.
Remark 2.16. Let the situations be as in the above theorem and assume, in addition, that \((R, \mathfrak{m})\) is complete. Let \(\mathfrak{a} \supseteq (x)\) be an ideal with grade \(M \mathfrak{a} = t\). Then, the Lichtenbaum-Hartshorne Theorem shows that \(\mathfrak{a}\) can not be coprimary with a member of \(\text{Assh } M\), i.e. there is no \(\mathfrak{p} \in \text{Assh } M\) with \(\sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m}\).

Corollary 2.17. There is a linked ideal \(\mathfrak{b}\) over \(R\) such that \(H^\dim R(R) \neq 0\).

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