LINEAR TRANSFORMATIONS PRESERVING PROJECTIONS OF FIXED FINITE RANK

MARK PANKOV

Abstract. Let $H$ be a complex Hilbert space whose dimension is not less than 3 and let $\mathcal{F}_s(H)$ be the real vector space formed by all self-adjoint operators of finite rank. For every non-zero natural $k < \dim H$ we denote by $\mathcal{P}_k(H)$ the set of all rank $k$ projections. We describe all linear transformations $L$ of $\mathcal{F}_s(H)$ such that $L(\mathcal{P}_k(H)) \subseteq \mathcal{P}_k(H)$ for a certain natural $k$ and the restriction of $L$ to $\mathcal{P}_k(H)$ is injective. Such transformations are induced by linear or conjugate-linear isometries (except the case when $\dim H = 2k$).

1. Introduction

Let $H$ be a complex Hilbert space whose dimension is assumed to be not less than 3. For every non-zero natural $k < \dim H$ we write $\mathcal{P}_k(H)$ for the set of all rank $k$ projections (self-adjoint idempotents whose images are $k$-dimensional subspaces). All finite rank self-adjoint operators on $H$ form a real vector space which will be denoted by $\mathcal{F}_s(H)$. By the spectral theorem, every self-adjoint operator of finite rank is a real linear combination of rank one projections, and it is not difficult to show that every rank one projection can be presented as a real linear combination of rank $k$ projections for every natural $k \geq 2$. Therefore, the vector space $\mathcal{F}_s(H)$ is spanned by each $\mathcal{P}_k(H)$.

Linear transformations preserving projections of fixed finite rank were investigated in [1, 8, 10]. Aniello and Chruściński [1] obtained the following result. Let $L$ be a linear transformation of $\mathcal{F}_s(H)$ satisfying the following conditions:

- $L(\mathcal{P}_k(H)) = \mathcal{P}_k(H)$ for certain natural $k$,
- $L$ is injective.

If $\dim H \neq 2k$, then $L$ is induced by a unitary or anti-unitary operator $U$ on $H$, i.e. $L(A) = UAU^*$ for every $A \in \mathcal{F}_s(H)$. The proof is based on Győry–Šemrl’s theorem on orthogonality preserving transformations of Hilbert Grassmannians [3, 9] (see also [1]) which explain the assumption that $\dim H \neq 2k$.

Using Gehér’s arguments [4] and results from [7], we describe all linear transformations of $\mathcal{F}_s(H)$ such that for certain natural $k$ the following conditions hold:

- $L(\mathcal{P}_k(H)) \subseteq \mathcal{P}_k(H)$,
- the restriction of $L$ to $\mathcal{P}_k(H)$ is injective

(without the assumption that $\dim H \neq 2k$). In almost all cases (except the case when $\dim H = 2k$), every such transformation is induced by a linear or conjugate-linear isometry of $H$ to itself.

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2. Result

Let $U$ be a linear or conjugate-linear isometry of $H$ to itself. Then $U^*U$ is identity. For every $A \in \mathcal{F}_s(H)$ we define

$$L_U(A) = UA^*U.$$  

Then $L_U$ is a linear injective transformation of $\mathcal{F}_s(H)$. It maps the projection on a finite-dimensional subspace $X$ to the projection on $U(X)$ and

$$L_U(\mathcal{P}_k(H)) \subset \mathcal{P}_k(H)$$

for every non-zero natural $k < \dim H$. This is a linear automorphism of $\mathcal{F}_s(H)$ only in the case when $U$ is a unitary or anti-unitary operator. If $H$ is finite-dimensional, then every linear or conjugate-linear isometry of $H$ to itself is a unitary or anti-unitary operator.

Suppose that $\dim H = n$ is finite and fix non-zero natural $k < n$. Consider the linear transformation $L_k^\perp$ of $\mathcal{F}_s(H)$ defined as follows

$$L_k^\perp(A) = k^{-1}\text{tr}(A)\text{Id}_H - A,$$

where $A \in \mathcal{F}_s(H)$ and $\text{tr}(A)$ is the trace of $A$. This is a linear automorphism of $\mathcal{F}_s(H)$ which maps the projection on a $k$-dimensional subspace $X$ to the projection on the orthogonal complement $X^\perp$ and we have

$$L_k^\perp(\mathcal{P}_k(H)) = \mathcal{P}_{n-k}(H).$$

If $n = 2k$, then $L_k^\perp$ preserves $\mathcal{P}_k(H)$.

**Theorem 1.** Let $L$ be a linear transformation of $\mathcal{F}_s(H)$. Suppose that there is natural $k < \dim H$ such that

$$L(\mathcal{P}_k(H)) \subset \mathcal{P}_k(H)$$

and the restriction of $L$ to $\mathcal{P}_k(H)$ is injective. Then one of the following possibilities is realized:

- $L = L_U$ for a certain linear or conjugate-linear isometry $U$,
- $\dim H = 2k$ and $L = L_k^\perp L_U$, where $U$ is a unitary or anti-unitary operator.

**Remark 1.** Let $P$ be a projection of rank $k$. Consider the linear transformation of $\mathcal{F}_s(H)$ which maps every $A \in \mathcal{F}_s(H)$ to $k^{-1}\text{tr}(A)P$. It sends every projection of rank $k$ to $P$. Therefore, the second assumption in Theorem 1 (the restriction of $L$ to $\mathcal{P}_k(H)$ is injective) cannot be omitted.

3. Preliminary

3.1. Geometric characterizations of linear and conjugate-linear isometries. Let $V$ be a left vector space over a division ring $R$. The dimension of $V$ is assumed to be not less than 3. Denote by $\mathcal{G}_k(V)$ the Grassmannian formed by $k$-dimensional subspaces of $V$. Recall that a line of $\mathcal{G}_1(V)$ is the set of all 1-dimensional subspaces in a certain 2-dimensional subspaces of $V$. A map $L : V \to V$ is called *semilinear* if

$$L(x + y) = L(x) + L(y)$$

for all $x, y \in V$ and there is an endomorphism $\sigma$ of the division ring $R$ such that

$$L(ax) = \sigma(a)L(x)$$

for all $a \in R$.
for all \( x \in V \) and \( a \in R \). Every semilinear injection \( L : V \rightarrow V \) induces a transformation of \( G_1(V) \) (not necessarily injective) which maps lines to subsets of lines. We will need the following version of the Fundamental Theorem of Projective Geometry.

**Theorem 2** (Faure and Frölicher [2], Havlicek [6]). Let \( f \) be an injective transformation of \( G_1(V) \) satisfying the following conditions:

- \( f \) maps lines to subsets of lines,
- the image \( f(G_1(V)) \) is not contained in a line.

Then \( f \) is induced by a semilinear injective transformation of \( V \).

Let \( X \) be a set and let \( R \subset X \times X \) be a symmetric relation on \( X \). We write \( xRy \) if \((x, y) \in R\). A transformation \( f : X \rightarrow X \) is said to be \( R \)-preserving if

\[
xRx \implies f(x)Rf(y);
\]

in the case when

\[
xRx \iff f(x)Rf(y),
\]

we say that \( f \) is \( R \)-preserving in both directions.

**Lemma 1.** Every semilinear injective transformation of \( H \) preserving the orthogonality relation is a linear or conjugate-linear isometry.

Two elements of \( G_k(H) \) are called adjacent if their intersection is \((k - 1)\)-dimensional. Every linear or conjugate-linear isometry of \( H \) to itself induces an injective transformation of \( G_k(H) \) preserving the adjacency and orthogonality relations in both directions (note that pairs of orthogonal \( k \)-dimensional subspaces exist only in the case when \( \dim H \geq 2k \)).

**Theorem 3** (Pankov [7]). If \( \dim H > 2k > 2 \), then every adjacency and orthogonality preserving transformation of \( G_k(H) \) is induced by a linear or conjugate-linear isometry.

For the case when \( \dim H = 2k \), there is the following weak version of the above result.

**Proposition 1** (Pankov [7]). Suppose that \( \dim H = 2k > 2 \) and \( f \) is an orthogonality preserving transformation of \( G_k(H) \) which also preserves the adjacency relation in both directions. Then one of the following possibility is realized:

- \( f \) is induced by a unitary or anti-unitary operator,
- \( f \) is the composition of the orthocomplementation and the transformation induced by a unitary or anti-unitary operator.

### 3.2. Some properties of projections

Denote by \( P_X \) the projection on a closed subspace \( X \subset H \).

**Lemma 2.** If \( X \) and \( Y \) are closed subspaces of \( H \), then

\[
\text{Im}(P_X + P_Y) = X \oplus Y.
\]

**Proof.** The operator \( P_X + P_Y \) is self-adjoint and we have

\[
\text{Im}(P_X + P_Y) = \text{Ker}(P_X + P_Y)^\perp = (\text{Ker}(P_X) \cap \text{Ker}(P_Y))^\perp = (X^\perp \cap Y^\perp)^\perp = X \oplus Y.
\]

\( \square \)
Two closed subspaces of $H$ are called compatible if there is an orthonormal basis of $H$ such that these subspaces are spanned by subsets of this basis. It is well-know that $P_X$ and $P_Y$ commute if and only if $X$ and $Y$ are compatible.

For any $X, Y \in \mathcal{G}_k(H)$ we denote by $\mathcal{X}_k(X, Y)$ the set of all $Z \in \mathcal{G}_k(H)$ such that $P_X + P_Y - P_Z$ is a rank $k$ projection.

**Lemma 3.** Suppose that $\dim H \geq 2k$. Then for any $X, Y \in \mathcal{G}_k(H)$ the following two conditions are equivalent:

1. $X$ and $Y$ are orthogonal.
2. $\dim(X + Y) = 2k$ and $\mathcal{X}_k(X, Y) = \mathcal{G}_k(X + Y)$.

**Proof.** (1) $\Rightarrow$ (2). It is clear that $P_X + P_Y = P_{X+Y}$ is a projection of rank $2k$. If $P_Z + P_{Z'} = P_{X+Y}$ for some $k$-dimensional subspaces $Z, Z' \subset H$, then Lemma 2 implies that $Z, Z' \subset X + Y$. Therefore, every element of $\mathcal{X}_k(X, Y)$ is contained in $X + Y$. For every $k$-dimensional subspace $Z \subset X + Y$ we have $P_Z + P_{Z'} = P_{X+Y}$, where $Z'$ is the $k$-dimensional subspace of $X + Y$ orthogonal to $Z$, i.e. $Z$ belongs to $\mathcal{X}_k(X + Y)$.

(2) $\Rightarrow$ (1). Consider a $k$-dimensional subspace $X' \subset X + Y$ spanned by some eigenvectors of the self-adjoint operator $P_X + P_Y$. Then $P_{X'}$ and $P_X + P_Y$ commute. The equality $\mathcal{X}_k(X, Y) = \mathcal{G}_k(X + Y)$ guarantees that $X'$ belongs to $\mathcal{X}_k(X, Y)$, i.e. there exists a $k$-dimensional subspace $Y' \subset H$ such that

$$P_X + P_Y = P_{X'} + P_{Y'},$$

Since $P_{X'}$ and $P_X + P_Y$ commute, the latter equality shows that $P_{X'}$ and $P_{Y'}$ commute. Therefore, $X'$ and $Y'$ are compatible and

$$P_X + P_Y = P_{X'} + P_{Y'} = P_{X''} + P_{Y''} + 2P_{X'\cap Y'},$$

where

$$X'' = X' \cap (X' \cap Y')^\perp \text{ and } Y'' = Y' \cap (X' \cap Y')^\perp.$$

By Lemma 2

$$X + Y = X' + Y' = X'' + Y'' + X' \cap Y'.$$

The dimension of this subspace is equal to $2k$ only in the case when $X' \cap Y' = 0$ which means that $X'$ and $Y'$ are orthogonal. Then $P_X + P_Y = P_{X+Y}$ is a projection of rank $2k$. We have

$$P_X + P_Y = P_{X+Y} = P_X + P_Z,$$

where $Z$ is the $k$-dimensional subspace of $X + Y$ orthogonal to $X$. Clearly, $Y = Z$, i.e. $X$ and $Y$ are orthogonal. $\Box$

**Lemma 4** (Gehér [4]). The set $\mathcal{X}_k(X, Y)$ is a one-dimensional real manifold if and only if $X$ and $Y$ are non-compatible and adjacent.

4. **Proof of Theorem 1**

Let $k$ be a non-zero natural number satisfying $k < \dim H$ and let $L$ be a linear transformation of $\mathcal{F}_k(H)$ such that

$$L(P_k(H)) \subset P_k(H)$$

and the restriction of $L$ to $P_k(H)$ is injective. Since $\mathcal{F}_k(H)$ is spanned by $P_k(H)$, it is sufficient to show that the restriction of $L$ to $P_k(H)$ coincides with $L_U$ or $L^*_U$.
for a certain linear or conjugate-linear isometry $U$. Let $f$ be the transformation of $\mathcal{G}_k(H)$ induced by $L$, i.e.

$$L(P_X) = P_{f(X)}$$

for every $X \in \mathcal{G}_k(H)$. We need to prove the following: $f$ is induced by a linear or conjugate-linear isometry or $\dim H = 2k$ and $f$ is the composition of the orthocomplementation and the transformation induced by a unitary or anti-unitary operator.

First of all, we show that the general case can be reduced to the case when $\dim H \geq 2k$. Suppose that $\dim H = n$ is finite and $k > n - k$. Consider the transformation $g$ of $\mathcal{G}_{n-k}(H)$ satisfying

$$g(X) = f(X^\perp)^\perp$$

for every $X \in \mathcal{G}_{n-k}(H)$. The associated transformation of $\mathcal{P}_{n-k}(H)$ is

$$P_X \to I - L(I - P_X).$$

This transformation can be extended to the linear transformation of $\mathcal{F}_s(H)$ which sends every $A \in \mathcal{F}_s(H)$ to

$$L(A) - (n - k)^{-1}\text{tr}(A)[L(I) - I].$$

If $g$ is induced by a unitary or anti-unitary operator, then $f$ is induced by the same operator.

From this moment we suppose that $\dim H \geq 2k$.

Let $W$ be a subspace of $H$ whose dimension is finite and greater than $k$. The subspace of $\mathcal{F}_s(H)$ consisting of all self-adjoint operators whose images are contained in $W$ can be naturally identified with the real vector space $\mathcal{F}_s(W)$. Also, we observe that $\mathcal{G}_k(W)$ is a compact topological space. Using these facts, we prove the following.

Lemma 5. For every $(2k)$-dimensional subspace $W \subset H$ there is a $(2k)$-dimensional subspace $W' \subset H$ such that the restriction of $f$ to $\mathcal{G}_k(W)$ is a homeomorphism to $\mathcal{G}_k(W')$.

Proof. For every $k$-dimensional subspace $X \subset W$ we have $P_W = P_X + P_Y$, where $Y$ is the $k$-dimensional subspace of $W$ orthogonal to $X$. The image of

$$L(P_W) = L(P_X) + L(P_Y) = P_{f(X)} + P_{f(Y)}$$

coincides with $f(X) + f(Y)$ (Lemma 2). Therefore,

$$f(\mathcal{G}_k(W')) \subset \mathcal{G}_k(W'),$$

where $Z'$ is the image of $L(P_W)$. Since this image coincides with $f(X) + f(Y)$, we have $\dim W' \leq 2k$. The latter inclusion shows that the restriction of $L$ to $\mathcal{F}_s(W)$ is a linear map to $\mathcal{F}_s(W')$. The vector space $\mathcal{F}_s(W)$ is finite-dimensional and this restriction is continuous. This means that the restriction of $f$ to $\mathcal{G}_k(W)$ is continuous and, by our assumption, it is injective. Since $\mathcal{G}_k(W)$ is compact, this restriction is a homeomorphism to a subspace of $\mathcal{G}_k(W')$. Then the inequality $\dim W' \leq 2k$ implies that $W'$ is $(2k)$-dimensional. The image $f(\mathcal{G}_k(W))$ is an open-closed subset of $\mathcal{G}_k(W')$, i.e. it coincides with $\mathcal{G}_k(W')$.

Lemma 6. The transformation $f$ is adjacency preserving in both directions.
Proof. If \( X, Y \in \mathcal{G}_k(H) \) and \( Z \in \mathcal{X}_k(X, Y) \), then \( L \) sends the rank \( k \) projection \( P_X + P_Y - P_Z \) to the rank \( k \) projection

\[
L(P_X) + L(P_Y) - L(P_Z) = P_{f(X)} + P_{f(Y)} - P_{f(Z)}
\]

which implies that

\[ f(\mathcal{X}_k(X, Y)) \subset \mathcal{X}_k(f(X), f(Y)). \]

Let \( W \) and \( W' \) be as in the previous lemma. Lemma 2 shows that

\[ \mathcal{X}_k(X, Y) \subset \mathcal{G}_k(W) \quad \text{and} \quad \mathcal{X}_k(X', Y') \subset \mathcal{G}_k(W') \]

for any \( k \)-dimensional subspaces \( X, Y \subset W \) and \( X', Y' \subset W' \). Since \( f(\mathcal{G}_k(W)) = \mathcal{G}_k(W') \), we have \( f(\mathcal{P}_k(W)) = \mathcal{P}_k(W') \) which guarantees that

\[ f(\mathcal{X}_k(X, Y)) = \mathcal{X}_k(f(X), f(Y)) \]

for any \( k \)-dimensional subspaces \( X, Y \subset W \). The restriction of \( f \) to \( \mathcal{G}_k(W) \) is a homeomorphism to \( \mathcal{G}_k(W') \) and Lemma 3 implies that \( X, Y \in \mathcal{G}_k(W) \) are non-compatible and adjacent if and only if \( f(X), f(Y) \) are non-compatible and adjacent. The set of all elements of \( \mathcal{G}_k(W) \) adjacent to \( X \in \mathcal{G}_k(W) \) is the closure of the set of all \( Y \in \mathcal{G}_k(W) \) such that \( X, Y \) are non-compatible and adjacent. Therefore, the restriction of \( f \) to \( \mathcal{G}_k(W) \) is adjacency preserving in both directions. Since for any two \( k \)-dimensional subspaces of \( H \) there is a \( (2k) \)-dimensional subspace containing them, \( f \) is adjacency preserving in both directions. \( \square \)

Lemma 7. The transformation \( f \) is orthogonality preserving.

Proof. If \( X \) and \( Y \) are orthogonal, then \( \dim(X + Y) = 2k \) and

\[ \mathcal{X}_k(X, Y) = \mathcal{G}_k(X + Y) \]

(Lemma 3). By Lemma 5 and arguments from the proof of Lemma 4 we have

\[ \mathcal{X}_k(f(X), f(Y)) = f(\mathcal{X}_k(X, Y)) = f(\mathcal{G}_k(X + Y)) = \mathcal{G}_k(W'), \]

where \( W' \) is a certain \( (2k) \)-dimensional subspace of \( H \). Then Lemma 4 implies that \( f(X) \) and \( f(Y) \) are orthogonal. \( \square \)

Suppose that \( k = 1 \). By our assumption, \( f \) is injective. Lemma 5 shows that \( f \) is a transformation of \( \mathcal{G}_1(H) \) sending lines to subsets of lines. Also, the image \( f(\mathcal{G}_1(H)) \) is not contained in a line (this follows from the condition \( \dim H \geq 3 \) and Lemma 7). By Theorem 2 \( f \) is induced by an injective semilinear transformation of \( H \). This semilinear transformation is orthogonality preserving (since \( f \) is orthogonality preserving) and Lemma 1 implies that it is a linear or conjugate-linear isometry.

In the case when \( k \geq 2 \), the transformation \( f \) is adjacency preserving in both directions (Lemma 5) and orthogonality preserving (Lemma 7). Theorem 3 and Proposition 1 give the claim.

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Faculty of Mathematics and Computer Science, University of Warmia and Mazury, Słoneczna 54, Olsztyn, Poland
E-mail address: pankov@matman.uwm.edu.pl