Robust Estimation under Heavy Contamination
using
Enlarged Models

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Abstract

In data analysis, contamination caused by outliers is inevitable, and robust statistical methods are strongly demanded. In this paper, our concern is to develop a new approach for robust data analysis based on scoring rules. The scoring rule is a discrepancy measure to assess the quality of probabilistic forecasts. We propose a simple way of estimating not only the parameter in the statistical model but also the contamination ratio of outliers. Estimating the contamination ratio is important, since one can detect outliers out of the training samples based on the estimated contamination ratio. For this purpose, we use scoring rules with an extended statistical models, that is called the enlarged models. Also, the regression problems are considered. We study a complex heterogeneous contamination, in which the contamination ratio of outliers in the dependent variable may depend on the independent variable. We propose a simple method to obtain a robust regression estimator under heterogeneous contamination. In addition, we show that our method provides also an estimator of the expected contamination ratio that is available to detect the outliers out of training samples. Numerical experiments demonstrate the effectiveness of our methods compared to the conventional estimators.

1 Introduction

In the big data era, robust data analysis is becoming more important than before. Nowadays, collecting a large dataset such as the data on the web is an easily task, while the quality of data may not be properly controlled. In such dataset, contamination caused by outliers such as incorrectly measured, or mis-recorded samples will be inevitable. Hence, robust statistical methods are demanded to extract valuable information from ubiquitous contaminated
data. The concept of outliers is elusive, and it will be difficult to establish a reliable statistical model for outliers. Hence, robust statistical methods are expected to automatically reduce the effect of outliers.

Robust statistics has a long history, and a lot of promising estimators were proposed. It is well known that the maximum likelihood estimator (MLE) suffers from a detrimental effect from outliers. The MLE can have a large bias even under a single erroneous observation. Many robust estimators were developed to reduce the bias induced by outliers. The statistical properties of robust estimators were deeply investigated by developing useful concepts such as the influence function, gross-error sensitivity, break-down point, and so forth; see \[9, 10, 13\] for details.

A way to reduce the effect of outliers is to employ weighted estimators, in which weight is introduced on each training sample. In the estimation procedure, the weight on the outlier is automatically reduced to make the estimator stable. The weighted estimators are regarded as an extension of the MLE that has a constant weight. Basu et al. \[1, 2, 11\] proposed the robust estimator based on the density-power weight, and Fujisawa and Eguchi \[6\] introduced another type of weighting scheme to deal with heavily contaminated data.

The weighted estimators are closely related to the scoring rules defined on the set of probability densities. The scoring rule is a quantity to assess the quality of probabilistic forecasts \[7\]. The MLE corresponds to the Kullback-Leibler score, and the weighted estimators introduced in the above are derived from the density-power score or gamma-score. From the standpoint of scoring rules, a unified framework of weighted estimators is recently presented by Kanamori and Fujisawa \[12\], in which a new class of scoring rule called Hölder score was proposed.

In this paper, our concern is to develop a new approach for robust data analysis based on scoring rules. Usually, the scoring rule is defined as a functional on the set of probability densities. However, there are a lot of scoring rules that can be defined over a set of non-negative functions. Exploiting such scoring rules, we propose a simple way of estimating not only the parameter in the statistical model but also the contamination ratio of outliers. Estimating the contamination ratio is important to detect outliers out of the training samples. Indeed, one can identify the outliers by picking up the estimated number of training samples in ascending order of the estimated value of the target probability density. For this purpose, we use scoring rules with an enlarged extension of statistical models, that is called the enlarged models.

We apply the proposed method to regression problems. For each inde-
pendent variable $x$, the dependent variable $y$ may be contaminated. When the contamination ratio on $y$ does not depend on $x$, i.e., the situation of homogeneous contamination, the problem is almost the same as the robust estimation of the probability density. On the other hand, when the contamination ratio depends on $x$, i.e., the heterogeneous contamination, the situation is rather complex. We propose a simple method to obtain a robust regression estimator under heterogeneous contamination. In addition, our method provides the estimator of the contamination ratio to detect the outliers out of training samples. In our approach, a scoring rule is used with an enlarged location-scale models. We prove that our methods has a small bias even under complex heterogeneous contamination. Moreover, we show that our estimator efficiently works even when both independent and dependent variables are heavily contaminated.

The remainder of the article is organized as follows. In Section 2, we introduce some scoring rules for the statistical inference. In Section 3, we propose statistical methods using enlarged models. We demonstrate how our estimator works to estimate not only the model parameter but also the contamination ratio. In Section 4, the proposed method is applied to regression problems. We show that our approach efficiently works even under heterogeneous contamination. To confirm the practical efficiency of our methods, we present some numerical experiments in Section 5. In Section 6, we close this article with a discussion of the possibility of the newly introduced estimation methods. Technical calculations and proofs are found in the appendix.

Let us summarize the notations to be used throughout the paper: Let $\mathbb{R}$ be the set of all real numbers, and $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. The univariate normal distribution with the mean $\mu$ and variance $\sigma^2$ is denoted as $N(\mu, \sigma^2)$, and the $d$-dimensional multivariate normal distribution with the mean vector $\mu$ and variance-covariance matrix $\Sigma$ is expressed as $N_d(\mu, \Sigma)$. For the function $f(x)$, the integral $\int f(x)dx$ is often denoted as $\langle f \rangle$.

## 2 Scoring Rules

Scoring rule is a class of discrepancy measures between two probability distributions, and it is widely used to statistical inference [5, 7, 8, 14, 15]. In this section, we briefly introduce some scoring rules: the density-power score, pseudo-spherical score, and Hölder score, and show some statistical properties. The density-power score and pseudo-spherical score are used for
robust parameter estimation. The Hölder score is a class of extended scoring rules including these scoring rules.

2.1 Density-power score and pseudo-spherical score

First of all, we briefly review the scoring rules. See [7] for details. Let \( p(x) \) and \( q(x) \) be probability densities on the Euclidean space \( \mathbb{R}^k \), and \( \ell(x, q) \) be a real-valued function of the point \( x \in \mathbb{R}^k \) and probability density \( q \). For the probability densities \( p \) and \( q \), the scoring rule is a real-valued function \( S(p, q) \) expressed as

\[
S(p, q) = \int p(x) \ell(x, q) dx.
\]

The scoring rule is said to be proper, if the inequality \( S(p, q) \geq S(p, p) \) holds for arbitrary probability densities \( p \) and \( q \) as long as the integral exists. Moreover, if the equality \( S(p, q) = S(p, p) \) leads to \( p = q \) almost surely, \( S(p, q) \) is called the strictly proper scoring rule. The strictly proper scoring rule \( S(p, q) \) defines the divergence \( D(p, q) = S(p, q) - S(p, p) \), that is an extension of squared distance measures on the space of probability densities. One of the most popular strictly proper scoring rules is the Kullback-Leibler (KL) score, that is defined from \( \ell(x, q) = -\log q(x) \). The divergence associated with the KL score is nothing but the KL divergence.

We can use strictly proper scoring rules for statistical inference. Let \( p_\theta(x) \) be a parametrized probability density by the parameter \( \theta \), where \( \theta \) is a member of an open subset \( \Theta \) in \( \mathbb{R}^d \). When the i.i.d. samples \( x_1, \ldots, x_n \) are observed from the probability density \( p \), the statistical model \( p_\theta \) is used to estimate the density \( p \) based on the samples. We assume that \( p \) is realized by a probability density in the model. Then, the minimization of the empirical loss,

\[
\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} \ell(x_i, p_\theta),
\]

is expected to provide a good estimate of the probability density \( p \). This is because the empirical mean converges in probability to the score \( S(p, p_\theta) \), that is minimized at \( p_\theta = p \).

Let us introduce two strictly proper scoring rules; one is the density-power score \( S_{\text{power}}(p, q) \) and the other is the pseudo-spherical score \( S_{\text{sphere}}(p, q) \). Both scores have a positive real parameter \( \gamma \). Given \( \gamma > 0 \), the density-power score is defined as

\[
S_{\text{power}}(p, q) = \gamma \langle q^{1+\gamma} \rangle - (1 + \gamma) \langle pq^\gamma \rangle,
\]
and the associated loss function is given as

$$\ell(x, q) = \gamma(q^{1+\gamma}) - (1 + \gamma)q(x)^\gamma.$$ 

See [1, 2] for details of the density-power score and its applications. On the other hand, the pseudo-spherical score [8] is defined as

$$S_{\text{sphere}}(p, q) = -\frac{\langle pq \rangle}{\langle q^{1+\gamma} \rangle^{\gamma/(1+\gamma)},}$$

that is derived from the loss function,

$$\ell(x, q) = -\frac{q(x)^\gamma}{\langle q^{1+\gamma} \rangle^{\gamma/(1+\gamma)}}.$$ 

The Hölder’s inequality assures that the pseudo-spherical score is the strictly proper scoring rule. The monotone transformation $-\frac{1}{\gamma} \log(-S_{\text{sphere}}(p, q))$ is called gamma cross entropy. The statistical property of the estimator based on gamma cross entropy was investigated in [6]. As the parameter $\gamma$ tends to zero, the estimator derived from the density-power score or pseudo-spherical score gets close to the MLE; see [1, 6] for details.

For some scoring rules, their domain can be extended to a set of non-negative functions. Indeed, one can confirm that for non-negative and non-zero functions $f$ and $g$, the inequalities $S_{\text{power}}(f, g) \geq S_{\text{power}}(f, f)$ and $S_{\text{sphere}}(f, g) \geq S_{\text{sphere}}(f, f)$ hold. For the density-power score, the equality $S_{\text{power}}(f, g) = S_{\text{power}}(f, f)$ for non-negative functions leads to $f = g$. For the pseudo-spherical score, however, the equality $S_{\text{sphere}}(f, g) = S_{\text{sphere}}(f, f)$ holds if $f$ and $g$ are linearly dependent. Note that the linearly dependent probability densities should be identical. Hence, the pseudo-spherical score is strictly proper on the set of probability densities, while it is not strictly proper on the set of non-negative functions. For the reader’s convenience, we give a self-contained short proof of the above facts in Appendix A.

### 2.2 Robustness of Estimators based on Scoring Rules

Let us introduce the robustness property of estimators based on the above scoring rules. Suppose that our target is to estimate the probability density $p_0(x)$ from the observed samples. We use the parametric statistical model $p_\theta(x), \theta \in \Theta$ for the estimation of the target density $p_0(x)$. Assume that $p_\theta(x) = p_{\theta_0}(x)$ holds for $\theta_0 \in \Theta$, i.e., the target is realized by the model. Let $w(x)$ be a probability density of contamination. Suppose that the observations $x_1, \ldots, x_n$ are drawn from the contaminated probability density,

$$p(x) = c_0p_0(x) + (1 - c_0)w(x), \quad (1)$$
in which $1 - c_0$ is the contamination ratio that typically lies in the interval $[0, 1/2)$. Here, we do not assume that $1 - c_0$ is infinitesimal, i.e., we deal with the situation of heavy contamination. Instead, we assume that for a positive constant $\gamma$, the quantity

$$\varepsilon_\theta = \langle wp_\theta^\gamma \rangle = \int w(x)p_\theta(x)^\gamma dx$$

is sufficiently small around $\theta = \theta_0$. This assumption indicates that the contamination density $w(x)$ mostly lies on the tail of the target density $p_0(x)$.

Let us consider the estimation of the target density under heavy contamination. The empirical probability density is denoted as $\hat{p}(x)$, that is expressed by the sum of Dirac’s delta function. The empirical pseudo-spherical score on the model, $S_{sphere}(\hat{p}, p_\theta)$, converges in probability to $S_{sphere}(p, p_\theta)$, and we have

$$S_{sphere}(p, p_\theta) = c_0 S_{sphere}(p_0, p_\theta) + (1 - c_0)(p_\theta^{1+\gamma})^{-\gamma/(1+\gamma)}\varepsilon_\theta.$$  

Since $\varepsilon_\theta$ is assumed to be sufficiently small around $\theta = \theta_0$, the optimal solution of $\min_\theta S_{sphere}(p, p_\theta)$ will be close to that of $\min_\theta S_{sphere}(p_0, p_\theta)$. Hence, even under heavy contamination, the pseudo-spherical score produces approximately consistent estimator of the target density $p_0$. The argument above was presented in [6].

For the density-power score, the same argument does not hold. Indeed, we have

$$S_{power}(p, p_\theta) = S_{power}(c_0 p_0, p_\theta) - (1 + \gamma)(1 - c_0)\varepsilon_\theta.$$  

(2)

Even if $\varepsilon_\theta$ is exactly zero, the minimizer of $\min_\theta S_{power}(c_0 p_0, p_\theta)$ will not be equal to $\theta = \theta_0$. Hence, the density-power score does not produce the approximately consistent estimator under heavy contamination.

### 2.3 Hölder score

As an extension of the scoring rules, Kanamori and Fujisawa proposed the Hölder score that is derived from the invariance under data transformations [12]. The Hölder score includes the density-power score and pseudo-spherical score as special cases.

Let us define the Hölder score. For a real-valued function $\phi(z)$ defined for $z \geq 0$, suppose that $\phi(1) = -1$ and $\phi(z) \geq -z^{1+\gamma}$, where $\gamma$ is a positive
real constant. Given $\gamma > 0$, the Hölder score $S_\phi$ based on the function $\phi$ is defined as

$$S_\phi(f, g) = \phi \left( \frac{\langle fg^\gamma \rangle}{\langle g^{1+\gamma} \rangle} \right) \langle g^{1+\gamma} \rangle$$

(3)

for the non-negative functions $f$ and $g$. The Hölder inequality assures that $S_\phi(f, g) \geq S_\phi(f, f)$ holds, and the equality $S_\phi(f, g) = S_\phi(f, f)$ leads to the linear dependence of $f$ and $g$. More involved argument yields that for probability densities $p$ and $q$, the equality $S_\phi(p, q) = S_\phi(p, p)$ leads to $p = q$; see [12] for details. We give a self-contained short proof of the above facts in Appendix A. Generally, the Hölder score $S_\phi(p, q)$ for the probability densities $p$ and $q$ is not expressed as the expectation with respect to $p$. However, one can substitute the empirical distribution of training samples into $p$, since $S_\phi(p, q)$ depends on $p$ through the integral $\langle pq^\gamma \rangle$. The Hölder score with $\phi(z) = \gamma - (1 + \gamma)z$ is reduced to the density-power score, and the lower bound $\phi(z) = -z^{1+\gamma}$ yields that $S_\phi(f, g) = -(S_{\text{sphere}}(f, g))^{1+\gamma}$.

The Hölder score is derived from the invariance property of the data transformation. Suppose that the probability density $p(x)$ is transformed to $\tilde{p}(z)$, when the data $x$ is changed to $z$ by an affine transformation. Then, the divergence $S_\phi(\tilde{p}, \tilde{q}) - S_\phi(\tilde{p}, \tilde{p})$ is converted into $h\{S_\phi(p, q) - S_\phi(p, p)\}$, where $h$ is a positive constant depending only on the affine transformation of data. This implies that the data transformation does not essentially change the distance structure on the set of probability densities. In addition, the affine invariance implies that the estimator defined from the Hölder score is equivariant [3]. In other words, the estimator does not essentially depend on the choice of the system of units in the measurement. This is a desirable property for statistical data analysis.

3 Robust Estimation using Enlarged Models

Detecting outliers out of training samples is an important task in data analysis. To deal with this issue, we introduce estimators of the contamination ratio based on scoring rules with enlarged models. We present some theoretical properties of the proposed estimators.

3.1 Contamination Ratio Estimation using Enlarged Models

As shown in the previous section, the estimator based on the pseudo-spherical score produces an approximately consistent estimator of the target density even under heavy contamination. However, the ratio $c_0$ in the contaminated
distribution is not estimated. Estimating the contamination ratio $1 - c_0$ is available to detect outliers out of the training samples. Using the estimated contamination ratio, one can identify the outliers out of the training samples by picking up the estimated number of training samples in ascending order of the estimated value of the target probability density. This is because the outliers are assumed to mostly lie on the tail of the underlying target density.

In order to estimate not only the target density but also the contamination ratio, we use the enlarged model $m_\xi(x)$ defined as

$$
m_\xi(x) = cp_\theta(x), \quad \xi = (c, \theta), \quad c > 0, \quad \theta \in \Theta,
$$

where $p_\theta(x)$ is a parametrized probability density and $c$ is a one-dimensional positive real parameter to estimate the ratio $c_0$.

Let us consider the estimator based on the density-power score with the enlarged model. Suppose that the samples are drawn from the contaminated probability density (1), and that $p_0 = p_{\theta_0}$ holds. In the same way as (2), we have

$$
S_{\text{power}}(p, cp_\theta) = S_{\text{power}}(c_0p_0, cp_\theta) - (1 + \gamma)(1 - c_0)c^\gamma \varepsilon_\theta
$$

(4)

for the enlarged model $cp_\theta$. If $\varepsilon_\theta$ is sufficiently small around $\theta = \theta_0$, the optimal solution of the problem $\min_{c, \theta} S_{\text{power}}(p, cp_\theta)$ will be close to that of the problem $\min_{c, \theta} S_{\text{power}}(c_0p_0, cp_\theta)$. Remember that the density-power score is strictly proper on the set of non-negative functions. Therefore, the density-power score with the enlarged model enables us to estimate both the target density $\theta_0$ and the ratio $c_0$. On the other hand, the argument in the above is not valid for the pseudo-spherical score, because $S_{\text{sphere}}(p, cp_\theta) = S_{\text{sphere}}(p, p_\theta)$ holds for all $c > 0$.

Hölder scores with some regularity conditions are also available to estimate the target density and contamination ratio. Indeed, when $\varepsilon_\theta$ is sufficiently small around $\theta = \theta_0$, the Hölder score defined from a smooth function $\phi$ satisfies

$$
S_\phi(p, cp_\theta) = \phi \left( \frac{\langle c_0p_0(cp_\theta)^\gamma \rangle}{\langle (cp_\theta)^{1+\gamma} \rangle} + \frac{1 - c_0}{c_{\langle p_\theta^{1+\gamma} \rangle} \varepsilon_\theta} \langle (cp_\theta)^{1+\gamma} \rangle \right)
$$

$$
= S_\phi(c_0p_0, cp_\theta) + O(\varepsilon_\theta).
$$

Suppose that the Hölder score is strictly proper on the set of non-negative functions. Then, it is expected that the minimizer of $S_\phi(p, cp_\theta)$ is close to $(c, \theta) = (c_0, \theta_0)$. 

8
3.2 Theoretical Properties of Estimators

Let us consider the optimization of the empirical Hölder score,

$$\min_{c, \theta} S_{\phi}(\tilde{p}, cp_{\theta}), \quad \text{s. t. } 0 < c \leq 1, \quad \theta \in \Theta,$$

(5)

where $\tilde{p}$ is the empirical probability density of training samples, $x_1, \ldots, x_n$. We show the relation between the problem (5) and the minimization of the pseudo-spherical score

$$\min_{\theta} S_{\text{sphere}}(\tilde{p}, p_{\theta}), \quad \text{s. t. } \theta \in \Theta.$$

(6)

Let $c(\theta)$ be the function

$$c(\theta) = \langle \tilde{p}p_{\theta} \gamma \rangle \langle p^{1+\gamma}_{\theta} \rangle,$$

(7)

where $\langle \tilde{p}p_{\theta} \gamma \rangle = \frac{1}{n} \sum_{i=1}^{n} p_{\theta}(x_i)^{\gamma}$. The function $c(\theta)$ connects (5) and (6). Indeed, we have

$$S_{\phi}(\tilde{p}, cp_{\theta}) = \phi \left( \frac{\langle \tilde{p}p_{\theta} \gamma \rangle}{c\langle p^{1+\gamma}_{\theta} \rangle} \right) c^{1+\gamma} \langle p^{1+\gamma}_{\theta} \rangle \geq - \frac{(\tilde{p}p_{\theta} \gamma)^{1+\gamma}}{\langle p^{1+\gamma}_{\theta} \rangle^{\gamma}} = -(-S_{\text{sphere}}(\tilde{p}, p_{\theta}))^{1+\gamma},$$

and the equality holds for $c = c(\theta)$. Details are presented in the following lemma and theorem. The proof is found in Appendix B.

**Lemma 1.** For the function $\phi$ in the Hölder score, suppose $\phi(1) = -1$ and $\phi(z) > -z^{1+\gamma}$ for $z \neq 1$. For arbitrary positive real number $u$, let us define $\psi_u(z)$ as $\psi_u(z) = z^{1+\gamma} \phi(u/z)$ for $z > 0$. Suppose that the function $\psi_u(z)$ is strictly decreasing on the open interval $(0, u)$. Then, for any fixed parameter $\theta \in \Theta$, the optimal solution of the problem

$$\min_{c} S_{\phi}(\tilde{p}, cp_{\theta}), \quad \text{s. t. } 0 < c \leq 1$$

(8)

is uniquely given as $c = \min\{1, c(\theta)\}$, in which the function $c(\theta)$ is defined by (7).

**Remark 1.** The function $\phi(z) = \gamma - (1 + \gamma)z$ that produces the density-power score satisfies the conditions in Lemma 1. Let us confirm the condition concerning the function $\psi_u(z)$. For the density-power score, we have $\psi_u(z) = z^{1+\gamma}(\gamma - (1 + \gamma)u/z)$, and the derivative is $\psi_u'(z) = \gamma(1 + \gamma)z^{\gamma-1}(z - u)$. Hence, $\psi_u'(z) < 0$ holds for $z \in (0, u)$.
Theorem 1. Let \((\hat{c}, \hat{\theta})\) be an optimal solution of (5). In addition to the assumptions in Lemma 1, we assume that \(c(\theta)\) in (7) is continuous in the vicinity of \(\hat{\theta}\). If \(0 < \hat{c} < 1\) holds, the parameter \(\hat{\theta}\) is a local optimal solution of (6). Otherwise, the parameter \(\hat{\theta}\) is an optimal solution of the problem

\[
\min_{\theta} S_\phi(p, p_\theta), \quad \text{s.t. } \theta \in \Theta. \tag{9}
\]

A simple optimization procedure of the problem (5) is constructed based on the above theorem. Suppose that the assumptions in Theorem 1 holds. Moreover, we assume that the problem (6) has the unique local optimal solution, \(\tilde{\theta} \in \Theta\). If \(0 < c(\tilde{\theta}) \leq 1\), the parameter \((c(\tilde{\theta}), \tilde{\theta})\) is an optimal solution of (5). Otherwise, solve the problem (9), and let \(\bar{\theta}\) be the optimal solution. Then, the point \((c, \theta) = (1, \bar{\theta})\) is an optimal solution of (5). Iterative algorithms are available to solve (6) and (9); see [6, 1] for details. When some assumptions in the above argument are violated, we use the standard non-linear constrained optimization methods such as active set methods. Since the constrained inequality \(0 < c \leq 1\) is easy to deal with, the non-linear optimization methods will also efficiently work to solve (5).

We evaluate the bias of the estimator. Let us define \(\xi_1 = (c_1, \theta_1)\) as an optimal solution of

\[
\min_{c, \theta} S_\phi(p, cp_\theta), \quad \text{s.t. } c > 0, \; \theta \in \Theta, \tag{10}
\]

where \(p(x)\) is defined by (1), and define \(\varepsilon_1 = \langle wp_\gamma \rangle\). Similarly to Lemma 1 and Theorem 1, the optimal parameter \(\xi_1\) does not depend on the function \(\phi\) under a mild assumption.

Theorem 2. Suppose that \(\xi_1 = (c_1, \theta_1)\) is the unique optimal solution of (10). Define \(f_0(\xi) = S_{\text{power}}(c_0p_0, cp_\theta)\) and \(f_1(\xi) = S_{\text{power}}(p, cp_\theta)\) as the function of \(\xi = (c, \theta)\). For \(\xi_1 = (c_1, \theta_1)\) and \(\varepsilon_1 = \langle wp_\gamma \rangle\), let \(\mathcal{N}\) be a convex set satisfying

\[
\{\xi \in (0, 1] \times \Theta | f_1(\xi) \leq f_1(\xi_1) + (1 + \gamma)\varepsilon_1\} \subset \mathcal{N}.
\]

Suppose that \(f_0(\xi)\) is second order differentiable on \(\mathcal{N}\). Let \(H_\xi\) be the Hessian matrix of \(f_0(\xi)\), and suppose that there exists a positive real number \(\delta\) such that all eigenvalues of \(H_\xi, \xi \in \mathcal{N}\) are greater than \(\delta\). Then, \(\|\xi_1 - \xi_0\| = O(\varepsilon_1^{1/2})\) holds.

The proof is found in Appendix C.

The asymptotic distribution of the estimator based on (5) depends on the parameter \(c_0\) of the sample distribution (1). For the ratio \(c_0\) such that
0 < c_0 < 1, the standard asymptotic expansion is available to derive the asymptotic distribution. When c_0 = 1, the asymptotic normality will not hold because of the singularity of the statistical model. The asymptotic distribution is, however, obtained by using the asymptotic expansion under nonstandard conditions [17]. The following theorem presents the expression of the asymptotic distribution. The matrices Σ_ξ and Λ_θ, and a small quantity ¯ε that appear in the theorem are defined in the proof in Appendix D.

**Theorem 3.** Let \( \xi_0 = (c_0, \theta_0) \in (0, 1] \times \Theta \) be the target parameter in (1), where \( p_0(x) = p_{\theta_0}(x) \) is assumed. Suppose that the conclusion of Theorem 2, i.e., \( \|\xi_1 - \xi_0\| = O(\bar{\varepsilon}^{1/2}) \) holds for \( \xi_1 = (c_1, \theta_1) \) that is an optimal solution of (10). An optimal solution of (5) is denoted as \( \hat{\xi} = (\hat{c}, \hat{\theta}) \in \mathbb{R}^{1+d} \). Let \( \tilde{\theta} \) and \( \bar{\theta} \) be the d-dimensional optimal solutions of (6) and (9), respectively. Then, the following asymptotic properties hold:

1. Suppose \( 0 < c_0 < 1 \). In addition to the assumptions in Theorem 2, suppose the regularity conditions such that the random vector \( \sqrt{n}(c_1 - c_0, \tilde{\theta} - \theta_0) \) converges in distribution to a \( (1+d) \)-dimensional multivariate normal distribution with the mean zero. Then, the asymptotic distribution of the estimator \( \hat{\xi} = (\hat{c}, \hat{\theta}) \) is given as the \( d+1 \) dimensional normal distribution, i.e.,

\[
\sqrt{n}(\hat{\xi} - \xi_1) \xrightarrow{d} N_{1+d}(0, \Sigma_\xi_0 + O(\bar{\varepsilon}^{1/2})),
\]

and \( \xi_1 = \xi_0 + O(\bar{\varepsilon}^{1/2}) \).

2. Suppose \( \xi_0 = (1, \theta_0) \). In addition to the assumptions in Theorem 1, suppose the regularity conditions such that \( \sqrt{n}(c_1 - 1, \tilde{\theta} - \theta_0) \) and \( \sqrt{n}(c_0 - 1, \bar{\theta} - \theta_0) \) converge in distribution to \( (1+d) \)-dimensional multivariate normal distributions with the mean zero. Then, the asymptotic distribution of the estimator is expressed as

\[
\sqrt{n}(\hat{\xi} - \xi_0) \xrightarrow{d} Z,
\]

in which \( Z = (Z_0, Z_1, \ldots, Z_d) \) is the random variable having the probability density

\[
\phi_{d+1}(z_0, z_1; \Sigma_{\xi_0})1[z_0 \leq 0] + \frac{1}{2} \delta(z_0)\phi_d(z_1; \Lambda_{\theta_0}),
\]

where \( z_0 \in \mathbb{R} \) corresponds to \( Z_0 \) and \( z_1 = (z_1, \ldots, z_d) \in \mathbb{R}^d \) corresponds to \( (Z_1, \ldots, Z_d) \). Here, \( \phi_d(z; \Sigma) \) denotes the probability density of the distribution \( N_d(0, \Sigma) \), and \( \delta(z) \) is the Dirac’s delta function. The indicator function \( 1[A] \) takes 1 if \( A \) is true, and 0 otherwise.
Remark 2. Some calculation yields that for $\xi = (c, \theta)$, the dependency of $\Sigma_\xi$ on $c$ and $\theta$ is given as

$$\Sigma_\xi = \begin{pmatrix} ca_\theta - c^2 & b^T_\theta \\ b_\theta & \frac{1}{c} D_\theta \end{pmatrix},$$

where $a_\theta \in \mathbb{R}$, $b_\theta \in \mathbb{R}^d$, $D_\theta \in \mathbb{R}^{d \times d}$ are quantities that depend only on the parameter $\theta$. When $\gamma$ tends to zero, the vector $b_\theta$ goes to the zero vector. We omit the concrete expression of the quantities above, since they are somewhat complex. The matrix $\frac{1}{c} D_\theta$ is the asymptotic variance of the estimator $\hat{\theta}$ based on the pseudo-spherical score. This is proportional to the reciprocal of $c$ that indicates the ratio of samples from the target distribution. The same result about the matrix $\frac{1}{c} D_\theta$ is presented in [6].

4 Regression Problems

Let us consider the application of scoring rules to regression problems. In Section 4.1, the regression problems under homogeneous contamination is studied. In Section 4.2 we deal with heterogeneous contamination. The density-power score and pseudo-spherical score are used to derive the estimators for regression problems. In [12], it is proved that Hölder score that is available for regression problems is expressed as a mixture of the density-power score and pseudo-spherical score. For simplicity, we focus on the estimators based on the density-power score and pseudo-spherical score.

4.1 Homogeneous Contamination

Let us consider the regression problems based on the training samples, $(x_i, y_i), i = 1, \ldots, n$ that are i.i.d. samples from the joint probability density $p(y|x)q(x)$. Under the heavy contamination for the output variable $y$, the conditional density $p(y|x)$ is supposed to be expressed as

$$p(y|x) = c_0 p_0(y|x) + (1 - c_0) w(y|x), \quad (11)$$

where $p_0(y|x)$ is the target conditional density. The contamination ratio $1 - c_0$ is a constant number that typically lies in the interval $[0, 1/2)$, i.e., $1/2 < c_0 \leq 1$. In the above model, the contamination ratio is independent of $x$, and such situation is called the homogeneous contamination in this paper. The conditional density $w(y|x)$ describes the conditional density of outliers. To estimate the target conditional density, we use the parametric model $p_\theta(y|x) = p(y|x; \theta)$ or its extension, $m_\xi(y|x) = cp_\theta(y|x)$ with $\xi = (c, \theta)$. We
assume that the target density is included in the model $p_\theta(y|x)$, i.e., $p_\theta(y|x)$ is expressed as $p_\theta(y|x) = p_{\theta_0}(y|x)$ for a parameter $\theta_0 \in \Theta$.

We use the density-power score to estimate the target conditional density. Remember that the pseudo-spherical score with the enlarged model does not work to estimate the contamination ratio. Given two functions $f(y|x)$ and $g(y|x)$ having two arguments $x$ and $y$ and a probability density $q(x)$, let us define the conditional density-power score as

$$S_{\text{power}}(f, g; q) = \int S_{\text{power}}(f(\cdot|x), g(\cdot|x))q(x)dx,$$

where $S_{\text{power}}(f(\cdot|x), g(\cdot|x))$ is the density-power score between $f(y|x)$ and $g(y|x)$ as the function of $y$ for a fixed $x$. It is straightforward to confirm that the inequality $S_{\text{power}}(f, g; q) \geq S_{\text{power}}(f, f; q)$ holds and that the equality $S_{\text{power}}(f, g; q) = S_{\text{power}}(f, f; q)$ leads to $f(y|x) = g(y|x)$ almost everywhere under the measure defined from $q(x)dx dy$. By overloading the notation $\langle f \rangle$ of $f(x, y)$ to representing $\int f(x, y)dx dy$, the conditional density-power score is expressed as

$$S_{\text{power}}(f, g; q) = \gamma \langle qq^{1+\gamma} \rangle - (1 + \gamma) \langle f q g^\gamma \rangle$$

In regression problems based on the samples from $p(y|x)q(x)$, we can employ $S_{\text{power}}(p, p\theta; q)$ or $S_{\text{power}}(p, c p\theta; q)$ as the loss function for statistical inference. Let us define $\tilde{p}(y|x)\tilde{q}(x)$ as the empirical probability density of the training samples. Substituting $\tilde{p}(y|x)\tilde{q}(x)$ into $p$ and $q$ in $S_{\text{power}}(p, c p\theta; q)$, we obtain the empirical approximation,

$$S_{\text{power}}(\tilde{p}, c p\theta; \tilde{q}) = -(1 + \gamma) \langle \tilde{p} q(c p\theta)^\gamma \rangle + \gamma \langle \tilde{q} (c p\theta)^{1+\gamma} \rangle$$

$$= -\left(1 + \gamma\right) c^\gamma \sum_{i=1}^{n} p_\theta(y_i|x_i)^\gamma + \gamma c^{1+\gamma} \sum_{i=1}^{n} \int p_\theta(y|x_i)^{1+\gamma} dy.$$

As the sample size tends to infinity, the above empirical approximation converges in probability to $S_{\text{power}}(p, c p\theta; q)$ at each parameter $(c, \theta)$. Under the contamination [11], we have

$$S_{\text{power}}(p, c p\theta; q) = S_{\text{power}}(c_0 p_\theta, c p\theta; q) - (1 + \gamma)(1 - c_0)c^\gamma \tilde{\epsilon}_\theta,$$

where $\tilde{\epsilon}_\theta$ is defined as $\tilde{\epsilon}_\theta = \langle w q p^\gamma \rangle$. Let $\epsilon_\theta(x)$ be

$$\epsilon_\theta(x) = \langle w(\cdot|x)p_\theta(\cdot|x)^\gamma \rangle = \int w(y|x)p_\theta(y|x)^\gamma dy,$$

$$\theta(x) = \langle w(\cdot|x)p_\theta(\cdot|x)^\gamma \rangle = \int w(y|x)p_\theta(y|x)^\gamma dy,$$

$$\theta(x) = \langle w(\cdot|x)p_\theta(\cdot|x)^\gamma \rangle = \int w(y|x)p_\theta(y|x)^\gamma dy,$$
then, we have $\tilde{\varepsilon}_\theta = \int \varepsilon_\theta(x)q(x)dx$. In a similar manner to the argument in Section 2.2, since $\varepsilon_\theta(x)$ is expected to be sufficiently small for each $x$, so is $\tilde{\varepsilon}_\theta$ around $\theta = \theta_0$. Then, the optimal solution of $\min_{c,\theta} S_{\text{power}}(p, cp\theta; q)$ will be close to the optimal solution of $\min_{c,\theta} S_{\text{power}}(cp_0, cp\theta; q)$, implying that the minimization of the empirical approximation $\min_{c,\theta} S_{\text{power}}(\tilde{p}, cp\theta; \tilde{q})$ is expected to provide a good estimator of the target parameter $\theta_0$ and the ratio $c_0$.

As shown in Section 3.1, the minimization of the conditional density-power score is related to the minimization of the pseudo-spherical score. In the regression problems, let us define the pseudo-spherical score between two conditional probability densities, $p(y|x)$ and $p_{\theta}(y|x)$ under the base measure $q(x)$ as

$$S_{\text{sphere}}(p, p_{\theta}; q) = -\frac{\langle pqp\theta \gamma \rangle}{\langle qp_{\theta}^{1+\gamma} \rangle^{\gamma/(1+\gamma)}}.$$  

Note that the empirical probability density $\tilde{p}(y|x)\tilde{q}(x)$ is directly substituted into $S_{\text{sphere}}(p, p_{\theta}; q)$.

Given training samples, the estimator is obtained by solving the problem,

$$\min_{c,\theta} S_{\text{power}}(\tilde{p}, cp\theta; \tilde{q}), \quad \text{s. t. } 0 < c \leq 1, \; \theta \in \Theta. \tag{13}$$

Let us define $c_{\text{reg}}(\theta)$ as

$$c_{\text{reg}}(\theta) = \frac{\langle \tilde{p}q\theta \gamma \rangle}{\langle q\theta^{1+\gamma} \rangle^{\gamma/(1+\gamma)}} = \frac{1}{n} \sum_{i=1}^{n} p_{\theta}(y_i|x_i)^{\gamma} \int p_{\theta}(y|x_i)^{1+\gamma} dy.$$  

Then, for arbitrary fixed parameter $\theta \in \Theta$, we can verify that

$$\min_{c>0} S_{\text{power}}(\tilde{p}, cp\theta; \tilde{q}) = S_{\text{power}}(\tilde{p}, c_{\text{reg}}(\theta)p\theta; \tilde{q}) = -(-S_{\text{sphere}}(\tilde{p}, p\theta; \tilde{q}))^{1+\gamma}.$$  

Hence, in the same way as in Theorem 1, we obtain the theoretical property of the estimator based on (13).

**Theorem 4.** Let $(\hat{c}, \hat{\theta})$ be an optimal solution of (13). Suppose that $c_{\text{reg}}(\theta)$ is continuous around $\theta = \theta$. If $0 < \hat{c} < 1$, the parameter $\hat{\theta}$ is a local optimal solution of the problem,

$$\min_{\theta} S_{\text{sphere}}(\tilde{p}, p\theta; \tilde{q}), \quad \text{s. t. } \theta \in \Theta. \tag{14}$$

Otherwise, the parameter $\hat{\theta}$ is an optimal solution of

$$\min_{\theta} S_{\text{power}}(\tilde{p}, p\theta; \tilde{q}), \quad \text{s. t. } \theta \in \Theta. \tag{15}$$
We omit the proof, since the proof is almost the same as in Theorem 1. Based on the above theorem, we present a simple optimization procedure of the problem (13). We assume that the problem (14) has the unique local optimal solution, $\tilde{\theta} \in \Theta$. If $0 < c_{\text{reg}}(\tilde{\theta}) \leq 1$, the parameter $(c_{\text{reg}}(\tilde{\theta}), \tilde{\theta})$ is an optimal solution of (13). Otherwise, for $\theta = \bar{\theta}$ that is an optimal solution of (15), the point $(c, \bar{\theta}) = (1, \bar{\theta})$ is an optimal solution of (13). Even if the assumptions in the above argument are violated, we can exploit the standard non-linear constrained optimization methods such as active set methods. Since the constrained inequality $0 < c \leq 1$ is easy to deal with, the non-linear optimization methods will also efficiently work to solve (13).

4.2 Location-Scale Models for Heterogeneous Contamination

We consider the regression problems under the non-constant contamination ratio. Suppose that the contaminated conditional probability density of the target $p_0(y|x)$ is expressed as

$$p(y|x) = c_0(x)p_0(y|x) + (1 - c_0(x))w(y|x),$$

where $w(y|x)$ denotes the conditional distribution of extreme outliers. The contamination ratio $1 - c_0(x)$ typically lies in $[0, 1/2)$, i.e., $1/2 < c_0(x) \leq 1$ holds at each $x$. We assume $0 < c_0(x) \leq 1$. The situation such that the ratio $c_0$ may depend on the independent variable is called heterogeneous contamination. To deal with the heterogeneous contamination, we assume that the target $p_0(y|x)$ is represented as the location scale model

$$p_\theta(y|x) = \frac{1}{\sigma} s\left(\frac{y - f_\beta(x)}{\sigma}\right), \quad \theta = (\beta, \sigma), \quad \sigma > 0,$$

where $s(y)$ is a probability density on $\mathbb{R}$ with the mean zero and the unit variance. The parameter $\sigma$ denotes the standard deviation, and $f_\beta(x)$ with the parameter $\beta$ is the regression function. Let us assume that $p_0(y|x) = p_{\theta_0}(y|x)$ holds for a parameter $\theta_0 \in \Theta$. The enlarged location scale model is defined as $m_\xi(y|x) = cp_\theta(y|x)$, $\xi = (c, \theta)$ for $0 < c \leq 1$ and $\theta \in \Theta$. We show that the constant parameter $c$ efficiently works even under heterogeneous contamination.

The conditional density-power score defined in Section 4.1 is employed. The empirical approximation $S_{\text{power}}(\tilde{p}, cp_\theta; \tilde{q})$ converges in probability to $S_{\text{power}}(p, cp_\theta; q)$. Let us consider the optimal solution of $\min_{c, \theta} S_{\text{power}}(p, cp_\theta; q)$
under heterogeneous contamination. The direct calculation yields that

$$S_{\text{power}}(p, cp; q) = S_{\text{power}}(c_0p_0, cp; q) - (1 + \gamma)c^7 \int (1 - c_0(x))q(x)\varepsilon_\theta(x)dx,$$

where $\varepsilon_\theta(x)$ is defined in (12). Suppose that $\varepsilon_\theta(x)$ is sufficiently small at each $x$ around $\theta = \theta_0$. Then, the second term of the right-side in the above expression will be negligible, and the optimal solution of $S_{\text{power}}(c_0, cp; q)$ will be close to the optimal solution of $S_{\text{power}}(c_0p, cp; q)$ in which $c_0$ may depend on $x$.

Let us consider the minimization problem

$$\min_{c, \theta} S_{\text{power}}(c_0p_0, cp; q), \quad c > 0, \quad \theta \in \Theta.$$  \hspace{1cm} (16)

We revisit the constraint $c \leq 1$ later. Using the same idea as in Theorem 4, we obtain the inequality

$$S_{\text{power}}(c_0p_0, cp; q) \geq - \left( \frac{\langle c_0p_0q_\theta^\gamma \rangle}{\langle q_\theta^{1+\gamma} \rangle^{\gamma/(1+\gamma)} \gamma/(1+\gamma)} \right)^{1+\gamma}. $$

The equality holds by setting

$$c = \frac{\langle c_0p_0q_\theta^\gamma \rangle}{\langle q_\theta^{1+\gamma} \rangle^{\gamma/(1+\gamma)}}. $$

In the integral of the location-scale model, the variable change $z = (y - f_\beta(x))/\sigma$ produces the equality,

$$\langle q_\theta(\cdot|x)^{1+\gamma} \rangle = \sigma^{-\gamma} \int s(z)^{1+\gamma}dz,$$

i.e., the integral $\int p_\theta(y|x)^{1+\gamma}dy$ does not depend on $x$, and then, $\langle q_\theta^{1+\gamma} \rangle = \langle q_\theta(\cdot|x)^{1+\gamma} \rangle$. Hence, we obtain

$$\frac{\langle c_0p_0q_\theta^\gamma \rangle}{\langle q_\theta^{1+\gamma} \rangle^{\gamma/(1+\gamma)}} = \int c_0(x)q(x)\frac{\langle p_\theta(\cdot|x)p_\theta(\cdot|x)^\gamma \rangle}{\langle q_\theta(\cdot|x)^{1+\gamma} \rangle^{\gamma/(1+\gamma)}}dx$$

$$= - \int c_0(x)q(x)S_{\text{sphere}}(p_0(\cdot|x), p_\theta(\cdot|x))dx.$$

Therefore, the optimization of the conditional density-power score is represented as

$$\min_{c, \theta} S_{\text{power}}(c_0p_0, cp; q) = \min_{\theta} - \left( - \int c_0(x)q(x)S_{\text{sphere}}(p_0(\cdot|x), p_\theta(\cdot|x))dx \right)^{1+\gamma}.$$
The optimal solution of the pseudo-spherical score at each $x$ is given as $\theta = \theta_0$. For the optimal parameter $\theta_0$, the optimal ratio is presented as

$$c = \frac{\langle c_0 q_0^{1+\gamma} \rangle}{\langle q_0^{1+\gamma} \rangle} = \int c_0(x)q(x)dx \leq 1,$$

where the property of the location-scale models is used in the integral.

In summary, the optimal solution of the problem (16) is given by the target mode parameter $\theta = \theta_0$ and the expected ratio $c = \int c_0(x)q(x)dx$. Since the expected ratio is less than or equal to 1, the problem (16) with the additional constraint $c \leq 1$ has the same optimal solution. The expected contamination ratio is obtained by $1 - \int c_0(x)q(x)dx$. Therefore, the minimization of the empirical approximate $S_{\text{power}}(\bar{p}, cpq; \bar{q})$ will produce an estimator of the target model parameter and the expected contamination ratio even under heavy heterogeneous contamination.

The minimization problem of the empirical approximate, $S_{\text{power}}(\bar{p}, cpq; \bar{q})$, is common in the homogeneous and heterogeneous situations. Hence, Theorem 4 with the location-scale model also holds in the current situation. For the location-scale model, the integral in $\langle \bar{q} \rho_0^{1+\gamma} \rangle$ is expressed as $\sigma^{-\gamma} \int s(z)^{1+\gamma}dz$. Once the integral of $s(z)^{1+\gamma}$ is computed, any additional integral is not required in the process of the optimization. This is a computational advantage of the location-scale model.

5 Numerical Experiments

We conducted numerical experiments to evaluate the statistical properties of robust estimators including the preceding technical developments. First, synthetic datasets for density estimation problems and regression problems were employed. Then, benchmark datasets were used to compare robust estimators for regression problems. We borrowed the setup of regression problems from [18].

5.1 Synthetic data

First, we show illustrative examples of robust estimation.

**Density Estimation:** The training samples $x_1, \ldots, x_n \in \mathbb{R}^2$ were drawn from the two-dimensional standard normal distribution $N_2(0, I)$, where $0$ is the zero vector and $I$ is the identity matrix. To seed the outliers, 20% of the training samples were randomly chosen and their values were replaced
with the samples each component of which was generated from the normal distribution $N(10, 10^2)$. The sample size was set to $n = 50$. Figure 1 depicts the scatter plot of the observations including outliers. The statistical model $p_\theta(x)$ is the full-model of the two-dimensional normal distribution $N_2(\mu, \Sigma)$, i.e., the five dimensional parameter $\theta$ consists of the mean vector and the variance-covariance matrix. The estimated parameter based on the maximum likelihood estimator was given as

$$\hat{\mu}_{MLE} = \begin{pmatrix} 2.70 \\ 1.86 \end{pmatrix}, \quad \hat{\Sigma}_{MLE} = \begin{pmatrix} 39.40 & 20.76 \\ 20.76 & 20.28 \end{pmatrix}.$$

As the robust estimator, we employed the density-power score $S_{\text{power}}$ with $\gamma = 0.1$ and the enlarged model $cp_\theta(x)$. Then, the estimated parameter of the target density $N_2(0, I)$ was

$$\hat{\mu} = \begin{pmatrix} 0.05 \\ 0.11 \end{pmatrix}, \quad \hat{\Sigma} = \begin{pmatrix} 0.91 & -0.03 \\ -0.03 & 0.85 \end{pmatrix}.$$

In addition, the proposed method provided the estimator of the contamination ratio $1 - \hat{c}$. By picking up $50(1 - \hat{c})$ samples in ascending order of the estimated value of the conditional probability density, $p_{\hat{\theta}}(y_i | x_i)$, one can identify the outliers. In this example, the estimated contamination ratio was 0.198, and the detected outliers are indicated as the triangle points in Figure 1.

**Regression:** Let us consider the simple linear regression problem. The independent variable $x \in \mathbb{R}$ was drawn from the standard normal distribution $N(0, 1)$, and the target density $p_0(y|x)$ was defined from the regression function $y = 1 + 10x + \varepsilon$, where the noise $\varepsilon$ is generated from $N(0, 1)$. As the outlier $(x, y)$, $x$ was drawn from $N(1, 0.8^2)$ and $y$ was the absolute value of the random variable drawn from $N(0, 70^2)$. The left panel of Figure 2 depicts the scatter plot of the observations including outliers. The sample size was 50, and the expected contamination ratio was set to $1 - c_0 = 0.3$. The right panel presents the estimated regression functions based on the least square estimator and the proposed method using the density-power score with $\gamma = 0.1$. Our approach produced a reasonable result, while the least square estimator was significantly affected by the outliers. By picking up $50(1 - \hat{c})$ samples in ascending order of the estimated value of the conditional probability density, $p_{\hat{\theta}}(y_i | x_i)$, one can identify the outliers. The estimated contamination ratio was 0.265, and the triangle points denote the detected outliers.

Next, we present numerical experiments of linear regression problems under heavy contamination. The problem setup is similar to the setup in [18].
For $x \in \mathbb{R}^d$, $d = 5$ and $y \in \mathbb{R}$, the target density $p_0(y|x)$ was defined from the regression function $y = x^T \theta_0 + \varepsilon$, where the target parameter $\theta_0$ was generated from the multivariate normal distribution $N_d(0, I)$. The distribution of the noise $\varepsilon$ was the normal distribution $N(0, 1/4)$, and the independent variable $x$ was drawn from the uniform distribution on $[0, 1]^d$. The estimation accuracy was evaluated on 1000 test points that were drawn from the joint probability of $(x, y)$ in the above.

Let us consider two setups for contamination. In the first setup, each dependent variable $y_i$ was re-sampled as the outlier from $N(0, 10^8)$ with the contamination probability $1 - c_0$, while the independent variable was not changed. In the second, both $x$ and $y$ were resampled from $N(0, 10^4)$ and $N(0, 10^8)$, respectively. The estimators using enlarged models are designed to deal with heavy contamination in the first setup. We present that the proposed methods efficiently work even in the second setup.

In the regression problems, the following methods were compared: least square method (L2), median regression estimator base on $L_1$-loss (L1), robust estimator using Huber loss (Huber) [10], least trimmed square method (LTS) [16], robust estimator using the bounded Geman-McClure loss (GemMc) [4], robust MM-estimator (MM-est) [13, Chap. 5], and the proposed method us-
Figure 2: Left panel: the scatter plot of training samples for regression estimation. Right panel: The solid line is the estimated regression function by our methods, and the broken line is the estimation result of the least square estimator. The triangle points are identified as the outliers by our methods.
ing the density-power score with enlarged model \((S_{\text{power}})\). The LTS method requires an estimate of the contamination ratio. In our experiments, the true ratio \(1 - c_0\) was fed to the LTS method. In the present setup, the linear regression model includes the intercept, while the regression model used in [18] did not have the intercept. The model \(p_\theta(y|x)\) with the parameter \(\theta = (\beta_0, \beta_1, \sigma)\) was defined from \(y = \beta_1 + \beta_0^T x + \varepsilon, \varepsilon \sim N(0, \sigma^2)\), and the enlarged model was given as \(cp_\theta(y|x)\).

For each estimator, we computed the averaged root mean square errors (RMSE) over 100 iterations. The contamination ratio estimated by using the proposed methods is also presented. The upper part of Table 1 reports the numerical results of the first setup, i.e., contamination only for the dependent variable. When the samples were not contaminated, all estimators efficiently worked as shown in the left column of the table. Indeed, the all RMSEs were close to optimal value \(1/2\), i.e., the standard deviation of the noise \(\varepsilon\). This result is almost the same as that in [18]. As shown in the middle and right columns, the least square method and Huber estimator tended to be affected by outliers. The lower part of Table 1 reports the results of the second setup. In addition to L2 and Huber, the L1-estimator was degraded by outliers. Even under heavy contamination, GemMc, MM-est and the proposed method performed well. We also found that the estimator \(S_{\text{power}}\) was useful for the estimation of the contamination ratio even under the second setup. In this experiments, the choice of \(\gamma\) in the density-power score did not significantly affect the estimation accuracy.

5.2 Benchmark data

We used four benchmark datasets taken from the StatLib repository and DELVE: cal-housing, abalone, pumadyn-32fh, and bank-8fh. These were the same as the datasets used in [18]. Cal-housing dataset has 8 features and one dependent variable (median House Value). Abalon dataset has originally 8 features and one output (rings). However, one discrete feature, “Gender or Infant”, is removed, and we use 7 features and one dependent variable. Pumadyn-32fh has 32 features and one output variable (ang acceleration of joint 6). Bank-8fh has 8 features and one output variable (rejection rate). The dependent variable of bank-8fh dataset denotes the probability, and hence, the logistic regression would be appropriate to analyze bank-8fh. However, we dealt with the rejection rate just as a real number in order to investigate the robustness property of the regression estimators.

For each dataset, 100 training samples and 1000 test samples were randomly selected. Let us consider two kinds of contamination, i.e., contamination.
tion of only dependent variable \((y\text{-contamination})\), and that of both independent and dependent variables \(((x, y)\text{-contamination})\). To seed outliers, some amount of training samples were randomly chosen, and their \(y\) values were multiplied by 10000 in the first setup. In the second setup, \(x\) values were also multiplied by 100. The contamination ratio was set to \(1 - c_0 = 0.05, 0.2\) or \(0.4\), while only the case of \(1 - c_0 = 0.05\) was examined in [15].

For the model fitting, we employed the linear regression model with the intercept. In addition, the normal distribution was assumed for the conditional probability model \(p_\theta(y|x)\). In [15], the regularization technique was used. In the numerical experiments of this article, we did not use the regularization, since the regression model used in the experiments was rather simple. Again, the true contamination ratio \(1 - c_0\) was used in the LTS estimator.

Table 2 (resp. Table 3) reports the RMSE on the test samples under the setup of \(y\)-contamination (resp. \((x, y)\)-contamination). As shown in [13, 15], any estimator based on minimizing a convex loss such as L2, L1, and Huber was sensitive to even small amount of outliers. Under the heavy contamination, also the LTS estimator was degraded by the outliers. Other estimators, GemMc, MM-est and \(S_{\text{power}}\) were not degraded even under heavy contamination. In both \(y\)-contamination and \((x, y)\)-contamination, \(S_{\text{power}}\) with \(\gamma = 0.1\) efficiently performed for the estimation of the model parameter \(\theta\) and the contamination ratio \(1 - c_0\). Also, other estimators based on non-convex losses such as GemMc, MM-est and \(S_{\text{power}}\) with \(\gamma = 0.5\) and 1.0 provided rather stable results. In Pumadyn-32fh dataset, the MM-est performed worse. In our experiments, the MM-est got sensitive for fairly high-dimensional data. When the estimator was trapped in local minima, the estimation accuracy was not high. The estimator \(S_{\text{power}}\) with a small \(\gamma\) was expected to have the unique local minima. Thus, the problematic local minima would be avoided. In practice, \(S_{\text{power}}\) with \(\gamma = 0.1\) provided an accurate estimator.

6 Conclusion

In this paper, the robust statistical inference under heavy contamination is studied. In order to estimate not only the model parameter but also the contamination ratio, scoring rules such as the density-power score or pseudospherical score are applied with enlarged models. The proposed method is used for regression problems. Even under heterogeneous contamination, the proposed method with the location-scale model provides an estimate of the
expected contamination ratio besides a robust estimator of the target model parameter. Using the estimator of the contamination ratio, one can identify the outliers out of the observed samples. Numerical experiments showed the effectiveness of our approach.

As shown in [13, 18], the convex loss function does not provide strong robustness to heavy contamination. This fact makes the optimization in the robust estimation harder. In the numerical experiments, the multi-start strategy is used as well as the other robust estimators. For the clipped loss function, Yu et al, [18] proposed the relaxation approach for efficient computation. This approach is not directly available to our methods, since the loss functions proposed in this paper are not expressed as the form of the clipped loss. A future work is to study numerical algorithms that are specialized for robust statistical inference.

A Preliminaries of Scoring Rules

The density-power score and pseudo-spherical score are described as a special case of the Hölder score (3). Indeed, the density-power score is derived from \( \phi(z) = \gamma - (1 + \gamma)z \), and the pseudo-spherical score is derived from \( \phi(z) = -z^{1+\gamma} \). First, we prove the inequality for Hölder score, \( S_\phi(f, g) \geq S_\phi(f, f) \).

Then, we show the condition the equality for each score.

Given non-negative functions \( f \) and \( g \), Hölder’s inequality leads to

\[
\langle fg^\gamma \rangle \leq \langle f^{1+\gamma} \rangle^{1/(1+\gamma)} \langle g^{1+\gamma} \rangle^{\gamma/(1+\gamma)}
\]

for \( \gamma > 0 \). The equality holds if and only if \( f \) and \( g \) are linearly dependent. From the inequality \( \phi(z) \geq -z^{1+\gamma} \) for \( z \geq 0 \), we have

\[
S_\phi(f, g) - S_\phi(f, f) = \phi \left( \frac{\langle fg^\gamma \rangle}{\langle g^{1+\gamma} \rangle} \right) \langle g^{1+\gamma} \rangle + \langle f^{1+\gamma} \rangle \\
\geq - \left( \frac{\langle fg^\gamma \rangle}{\langle g^{1+\gamma} \rangle} \right)^{1+\gamma} \langle g^{1+\gamma} \rangle + \langle f^{1+\gamma} \rangle \\
\geq 0. \quad \text{(Hölder’s inequality)}
\]

Hence, the property of pseudo-spherical score was shown. For the density-power score, suppose that \( S_{\text{power}}(f, g) = S_{\text{power}}(f, f) \) holds. Then, the inequalities in the above should become equality. The equality condition of Hölder’s inequality leads that \( f \) and \( g \) are linearly dependent. For the function \( \phi(z) \) of the density-power score, \( \phi(z) = -z^{1+\gamma} \) holds only when \( z = 1 \).
Hence, \( \langle fg^\gamma \rangle / \langle g^{1+\gamma} \rangle = 1 \) should hold. For non-negative and non-zero linearly dependent functions \( f \) and \( g \), the equality \( \langle fg^\gamma \rangle / \langle g^{1+\gamma} \rangle = 1 \) leads to \( f = g \).

B Proofs of Lemma 1 and Theorem 1

Proof of Lemma 1. As defined in Section 2.3, the function \( \phi \) in the Hölder score satisfies \( \phi(z) \geq -z^{1+\gamma} \). Thus, we have \( \psi_u(z) \geq -u^{1+\gamma} \) for \( z > 0 \). Since the inequality \( \phi(z) > -z^{1+\gamma} \) is assumed for \( z \neq -1 \), the equality \( \psi_u(z) = -u^{1+\gamma} \) is satisfied only when \( z = u \). Hence, we have,

\[
S_{\phi}(\tilde{p}, c\theta p_\theta) = \psi_{c(\theta)}(c) \langle p_\theta^{1+\gamma} \rangle \geq -\langle p_\theta^{1+\gamma} \rangle c(\theta)^{1+\gamma},
\]

and the equality holds only for \( c = c(\theta) \). In addition, \( \psi_{c(\theta)}(c) \) is assumed to be strictly decreasing on the open interval \( c \in (0, c(\theta)) \). If \( c(\theta) \leq 1 \) holds, clearly \( c = c(\theta) \) is the optimal solution of (8). Otherwise, \( c = 1 \) is optimal, since the inequality \( 1 < c(\theta) \) assures that \( S_{\phi}(\tilde{p}, c\theta p_\theta) \) is strictly decreasing with respect to \( c \) over the interval \( (0, 1] \). In summary, the optimal solution is expressed as \( c = \min\{1, c(\theta)\} \). □

Proof of Theorem 1. When \( \hat{c} = 1 \) holds, the statement of the theorem is clear. Let us suppose that \( 0 < \hat{c} < 1 \) holds. Note that the equality

\[
S_{\phi}(\tilde{p}, c(\theta) p_\theta) = -(-S_{\text{sphere}}(\tilde{p}, p_\theta))^{1+\gamma}
\]

holds for \( \theta \in \Theta \). Due to Lemma 1, \( \hat{c} = c(\hat{\theta}) \) should hold, because under the assumptions of Lemma 1, the optimal value of \( c \) should be expressed as \( \min\{1, c(\theta)\} \) for each \( \theta \in \Theta \). Let \( N \subset \Theta \) be an open neighborhood of \( \hat{\theta} \) such that \( 0 < c(\theta) < 1 \) holds for all \( \theta \in N \). Then, we have

\[
\min_{c \in (0,1], \theta \in \Theta} S_{\phi}(\tilde{p}, c(\theta) p_\theta) = S_{\phi}(\tilde{p}, c(\hat{\theta}) p_{\hat{\theta}}) = \min_{\theta \in N} S_{\phi}(\tilde{p}, c(\theta) p_\theta)
\]

Due to the equality (17), the minimization of \( S(\tilde{p}, c(\theta) p_\theta) \) on \( N \) is identical to the minimization of the pseudo-spherical score \( S_{\text{sphere}}(\tilde{p}, p_\theta) \) on \( N \). Therefore, \( \hat{\theta} \) is a local optimal solution of (6). Generally, the set \( N \) cannot be replaced with \( \Theta \), since the optimal solution of \( \min_{\theta \in \Theta} S(\tilde{p}, c(\theta) p_\theta) \) may not satisfy the constraint \( c(\theta) \leq 1 \). □
C Proof of Theorem 2

Proof. The point \( \xi_0 = (c_0, \theta_0) \) is the unique minimizer of \( f_0(\xi) \). In the same way as the proof of Theorem 1, we can prove that the point \( \xi_1 = (c_1, \theta_1) \) is also the minimizer of \( f_1(\xi) = S_{\text{power}}(p, cp\theta) \). For \( \xi = (c, \theta) \), the equality (4) leads to

\[
f_1(\xi) \leq f_0(\xi) \leq f_1(\xi) + (1 + \gamma)\varepsilon_0,
\]

where the constraint \( c_0, c \in (0, 1] \) is used to derive the second inequality. Then, \( f_1(\xi_1) \leq f_0(\xi_0) \) should hold, since \( f_1(\xi_1) \leq f_1(\xi_0) \leq f_0(\xi_0) \). In addition, we have

\[
f_0(\xi_0) \leq f_0(\xi_1) \leq f_1(\xi_1) + (1 + \gamma)\varepsilon_1,
\]

implying that \( \xi_0, \xi_1 \in \mathcal{N} \). Moreover, we obtain

\[
f_0(\xi_0) - (1 + \gamma)\varepsilon_1 \leq f_1(\xi_1) \leq f_0(\xi_0).
\]

Taylor expansion of \( f_0(\xi_1) \) around \( \xi_0 \) and the assumption on the Hessian matrix yield that

\[
f_0(\xi_0) + \frac{\delta}{2}\|\xi_0 - \xi_1\|^2 - (1 + \gamma)\varepsilon_1 \leq f_0(\xi_0).
\]

Therefore, \( \|\xi_0 - \xi_1\| = O(\varepsilon_1^{1/2}) \) holds. \( \square \)

D Proof of Theorem 3

Proof of the first statement in Theorem 3. Suppose \( 0 < c_0 < 1 \). Since \( c(\tilde{\theta}) \) is assumed to be a \( \sqrt{n} \)-consistent estimator of \( c_1 \), the large deviation theory assures that the inequality \( c(\tilde{\theta}) < 1 \) holds with the probability more than \( 1 - e^{-\alpha n} \), where \( \alpha \) is a positive constant. Therefore, the constraint \( c \leq 1 \) in (5) does not affect the asymptotic distribution of the estimator \( \hat{\xi} = (\hat{c}, \hat{\theta}) \). When \( c(\tilde{\theta}) < 1 \), the estimator given by \( (\hat{c}, \hat{\theta}) = (c(\tilde{\theta}), \tilde{\theta}) \) does not depend on the choice of the function \( \phi \), as shown in Theorem 1. The density-power score is employed to calculate the asymptotic distribution of the estimator. Note that the function \( \phi \) of the density-power score satisfies the assumptions in the theorem. Suppose that the density-power score \( S_{\text{power}}(p, cp\theta) \) is expressed as

\[
S_{\text{power}}(p, cp\theta) = \int p(x)\ell(x, cp\theta)dx.
\]
The minimum solution is \( \xi_1 = (c_1, \theta_1) \). The asymptotic theorem of the M-estimator shows that the asymptotic distribution of \( \sqrt{n}(\hat{\xi} - \xi_1) \) is the multivariate normal distribution with the mean zero and variance-covariance matrix \( \Sigma_p = J_p^{-1}K_pJ_p^{-1} \), where the \( d + 1 \) by \( d + 1 \) matrices \( J_p \) and \( K_p \) are given as

\[
(J_p)_{ij} = \left. \left\langle p \cdot \frac{\partial^2}{\partial \xi_i \partial \xi_j} \ell(\cdot, cp_\theta) \right\rangle \right|_{\xi = \xi_1},
\]

\[
(K_p)_{ij} = \text{Cov}_p \left[ \frac{\partial}{\partial \xi_i} \ell(X, cp_\theta), \frac{\partial}{\partial \xi_j} \ell(X, cp_\theta) \right] \bigg|_{\xi = \xi_1}.
\]

Let \( s_{\theta,i} \) be \( \frac{\partial}{\partial \theta_i} \log p_\theta(x) \), and \( s_{\theta,ij} \) be \( \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log p_\theta(x) \), and let \( \Theta_0 \) be a convex subset of \( \Theta \) such that \( \theta_1, \theta_0 \in \Theta_0 \). Let us define \( \bar{\epsilon}_{\theta,ij} \) as

\[
\bar{\epsilon}_{\theta,ij} = \max\left\{ \epsilon_{\theta}, \langle wp^{2\gamma}_\theta \rangle, \langle wp^{2\gamma}_\theta | s_{\theta,i} \rangle, \langle wp^{2\gamma}_\theta | s_{\theta,i} s_{\theta,j} \rangle, \langle wp^{2\gamma}_\theta | s_{\theta,i} s_{\theta,j} \rangle \right\},
\]

and let \( \bar{\epsilon} = \sup\{ \bar{\epsilon}_{\theta,ij} \mid i, j = 1, \ldots, d, \theta \in \Theta_0 \} \). Using \( \epsilon_1 = \epsilon_{\theta_1} \leq \bar{\epsilon} \), we have \( \| \xi_0 - \xi_1 \| = O(\epsilon^{1/2}) \). Then, we obtain

\[
(J_p)_{ij} = (J_0)_{ij} + O(\epsilon^{1/2}), \quad (K_p)_{ij} = (K_0)_{ij} + O(\epsilon^{1/2}),
\]

where the \((1 + d)\) by \((1 + d)\) matrices \( J_0 \) and \( K_0 \) are defined as

\[
J_0 = \left. \left\langle c_0 p_{\theta_0} \cdot \frac{\partial^2}{\partial \xi_i \partial \xi_j} \ell(\cdot, cp_\theta) \right\rangle \right|_{\xi = \xi_0}, \quad K_0 = \text{Cov}_{c_0 p_{\theta_0}} \left[ \frac{\partial}{\partial \xi_i} \ell(X, cp_\theta), \frac{\partial}{\partial \xi_j} \ell(X, cp_\theta) \right] \bigg|_{\xi = \xi_0}.
\]

In the above, we assumed that the derivatives of \( \langle wp^{\gamma}_\theta \rangle \) and \( \langle wp^{2\gamma}_\theta \rangle \) up to the third order on \( \Theta_0 \) are uniformly bounded by an integrable function. As a result, the asymptotic variance-covariance matrix is given as \( \Sigma_p = \Sigma_{\xi_0} + O(\epsilon^{1/2}) \), where \( \Sigma_{\xi_0} \) is defined as \( \Sigma_{\xi_0} = J_0^{-1}K_0J_0^{-1} \).

\[\square\]

**Proof of the second statement in Theorem** Suppose \( c_0 = 1 \). In this case, there is no outliers, and \( p = p_0 = p_{\theta_0} \) holds. Hence, under the regularity conditions, \( \langle c(\hat{\theta}), \hat{\theta} \rangle \) converges to \( \xi_0 = (1, \theta_0) \) almost surely, and \( \hat{\theta} \) also converges to \( \theta_0 \) almost surely. The asymptotic behaviour of \( \langle c(\hat{\theta}), \hat{\theta}, \hat{\theta} \rangle \) is obtained by using the asymptotic expansion. Let us define \( s_\theta \) as the \( \mathbb{R}^d \)-valued score.
function $\frac{\partial}{\partial \theta} \log p_\theta(x)$, and let $u$ and $v$ be

$$u = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{p_{\theta_0}(x_i)^\gamma - \langle p_{\theta_0}^{1+\gamma}\rangle\},$$

$$v = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{p_{\theta_0}(x_i)^\gamma s_{\theta_0}(x_i) - \langle p_{\theta_0}^{1+\gamma} s_{\theta_0}\rangle\}.$$

Then, the asymptotic expansion of the estimating equation for $c(\tilde{\theta}), \tilde{\theta}$ and $\tilde{\theta}$ yields that

$$\sqrt{n} \left( \frac{c(\tilde{\theta}) - 1}{\tilde{\theta} - \theta_0} \right) = \left( \frac{\langle p_{\theta_0}^{1+\gamma} s_{\theta_0}^T \rangle}{\langle p_{\theta_0}^{1+\gamma} s_{\theta_0} \rangle} \right)^{-1} \begin{pmatrix} u \\ v \end{pmatrix} + o_p(1),$$

$$\sqrt{n} (\tilde{\theta} - \theta_0) = \left( \frac{\phi''(1)}{\gamma(1+\gamma)} + \frac{\langle p_{\theta_0}^{1+\gamma} s_{\theta_0}^T \rangle}{\langle p_{\theta_0}^{1+\gamma} s_{\theta_0} \rangle} \right)^{-1} \begin{pmatrix} \phi''(1) \\ \gamma \end{pmatrix} \langle p_{\theta_0}^{1+\gamma} s_{\theta_0} \rangle + o_p(1).$$

The asymptotic expansion of $\sqrt{n}(\tilde{\theta} - \theta_0)$ is shown in the proof of Theorem 5 in [12]. The asymptotic probability densities of $\sqrt{n}(c(\tilde{\theta}) - 1, \tilde{\theta} - \theta_0)$ and $\sqrt{n}(c(\tilde{\theta}) - 1, \tilde{\theta} - \theta_0)$ are respectively denoted as $p(z_0, z_1)$ and $p(z_0, z_2)$ for $z_0 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d$, in which the notation is overloaded. Under the regularity condition, $p(z_0, z_1)$ and $p(z_0, z_2)$ are $(1+d)$-dimensional normal distributions with the mean zero. Let us define $\bar{p}$ and $\bar{\rho}$ as

$$\bar{p}(z_0, z_1) = 2 p(z_0, z_1) \mathbf{1}[z_0 \leq 0], \quad \bar{\rho}(z_2) = 2 \int p(z_0, z_2) \mathbf{1}[z_0 > 0] dz_0,$$

where the indicator function $\mathbf{1}[A]$ takes 1 if $A$ is true, and 0 otherwise. Informally, $\bar{p}(z_0, z_1)$ denotes the conditional probability density $p(z_0, z_1 | z_0 \leq 0)$, and $\bar{\rho}(z_2)$ denotes $p(z_2 | z_0 > 0)$. The symmetry of the distribution assures that the asymptotic probability such that $\sqrt{n}(\tilde{\theta} - \theta_0)$ is equal to $1/2$. Hence, the asymptotic probability density of $\sqrt{n}(\tilde{\theta} - \theta_0)$ is expressed as

$$\frac{1}{2} \bar{p}(z_0, z) + \frac{1}{2} \bar{\rho}(z_0) \bar{\rho}(z).$$

The first term is equal to $\phi_{1+d}(z_0, z; \Sigma_{\xi_0}) \mathbf{1}[z_0 \leq 0]$, that corresponds to the distribution of the estimator $(c(\tilde{\theta}), \tilde{\theta})$ in the case of $c(\tilde{\theta}) \leq 1$. The second term corresponds to the distribution of the estimator $(1, \tilde{\theta})$ in the
case of $c(\tilde{\theta}) > 1$. The density $\tilde{p}(z)$ is expressed as the $d$-dimensional normal distribution with the mean zero and the variance-covariance matrix $\Lambda_{\theta_0}$ that is determined from the asymptotic expansions and the integral in the above.

\[
\begin{align*}
\end{align*}
\]

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## Outlier Probability for variable $y$

| Methods       | $1 - c_0 = 0.0$ | $1 - c_0 = 0.2$ | $1 - c_0 = 0.4$ |
|---------------|-----------------|-----------------|-----------------|
| L2            | 0.51 ± 0.01     | 1093.9 ± 358.97 | 1528.91 ± 454.99 |
| L1            | 0.52 ± 0.01     | 0.54 ± 0.02     | 0.58 ± 0.05     |
| Huber         | 0.52 ± 0.01     | 1.40 ± 0.21     | 621.76 ± 335.89 |
| LTS           | 0.52 ± 0.01     | 0.52 ± 0.01     | 15.64 ± 19.07   |
| GemMc         | 0.52 ± 0.01     | 0.52 ± 0.02     | 0.54 ± 0.02     |
| MM-est        | 0.52 ± 0.01     | 0.52 ± 0.01     | 0.53 ± 0.02     |
| $S_{\text{power}}$ ($\gamma = 0.1$) | 0.52 ± 0.01 | 0.52 ± 0.01 | 0.53 ± 0.02 |
| $S_{\text{power}}$ ($\gamma = 0.5$) | 0.52 ± 0.01 | 0.52 ± 0.01 | 0.54 ± 0.02 |
| $S_{\text{power}}$ ($\gamma = 1.0$) | 0.53 ± 0.02 | 0.54 ± 0.03 | 0.56 ± 0.03 |

| $1 - \hat{c}$ ($\gamma = 0.1$) | 0.00 ± 0.00 | 0.17 ± 0.07 | 0.36 ± 0.13 |
| $1 - \hat{c}$ ($\gamma = 0.5$) | 0.01 ± 0.01 | 0.19 ± 0.05 | 0.40 ± 0.04 |
| $1 - \hat{c}$ ($\gamma = 1.0$) | 0.06 ± 0.10 | 0.25 ± 0.09 | 0.42 ± 0.15 |

## Outlier Probability for variables $(x, y)$

| Methods       | $1 - c_0 = 0.0$ | $1 - c_0 = 0.2$ | $1 - c_0 = 0.4$ |
|---------------|-----------------|-----------------|-----------------|
| L2            | 0.52 ± 0.01     | 313.76 ± 232.81 | 532.41 ± 353.29 |
| L1            | 0.52 ± 0.01     | 18.80 ± 6.73    | 13.71 ± 5.21    |
| Huber         | 0.52 ± 0.01     | 18.98 ± 6.96    | 60.11 ± 64.27   |
| LTS           | 0.52 ± 0.01     | 1.17 ± 0.79     | 1.23 ± 0.63     |
| GemMc         | 0.52 ± 0.01     | 0.53 ± 0.02     | 0.54 ± 0.02     |
| MM-est        | 0.52 ± 0.01     | 0.52 ± 0.01     | 0.54 ± 0.02     |
| $S_{\text{power}}$ ($\gamma = 0.1$) | 0.52 ± 0.01 | 0.52 ± 0.01 | 0.53 ± 0.02 |
| $S_{\text{power}}$ ($\gamma = 0.5$) | 0.52 ± 0.01 | 0.52 ± 0.02 | 0.54 ± 0.02 |
| $S_{\text{power}}$ ($\gamma = 1.0$) | 0.53 ± 0.02 | 0.54 ± 0.03 | 0.56 ± 0.05 |

| $1 - \hat{c}$ ($\gamma = 0.1$) | 0.00 ± 0.00 | 0.17 ± 0.07 | 0.33 ± 0.15 |
| $1 - \hat{c}$ ($\gamma = 0.5$) | 0.01 ± 0.01 | 0.20 ± 0.04 | 0.40 ± 0.04 |
| $1 - \hat{c}$ ($\gamma = 1.0$) | 0.05 ± 0.06 | 0.26 ± 0.10 | 0.41 ± 0.16 |

Table 1: Top table shows the results in the case that only the dependent variable $y$ is incurred by outliers. Bottom table shows the results in the case that both independent and dependent variables $(x, y)$ are contaminated. In the synthetic regression problems, RMSEs on 10000 clean test samples are computed. The training sample size is 100, and an observation consists of the pair of the 5-dimensional independent variable and a dependent variable. The contamination ratio $1 - c_0$ is set to 0.0 (clean training data), 0.2 or 0.4. For each estimator, we employed the linear regression model with the intercept. The averaged RMSE over 100 iterations is computed, and also the estimated contamination ratio is presented.
### Table 2: The numerical results on benchmark datasets are presented. The training samples size is 100, and the contamination ratio $1 - c_0$ is set to 0.05, 0.2 or 0.4. Only the dependent variable $y$ is contaminated by the outliers. For each estimator, we employed the linear regression model with the intercept, and computed the averaged RMSE over 100 iterations. Also the estimated contamination ratio is presented.
Table 3: The numerical results on benchmark datasets are presented. The training samples size is 100, and the contamination ratio $1 - c_0$ is set to 0.05, 0.2 or 0.4. Both the independent and dependent variables $(x, y)$ are contaminated by the outliers. For each estimator, we employed the linear regression model with the intercept, and computed the averaged RMSE over 100 iterations. Also the estimated contamination ratio is presented.