The parabolic Anderson model on Riemann surfaces

Antoine Dahlqvist∗ Joscha Diehl† Bruce Driver‡

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We show well-posedness for the parabolic Anderson model on 2-dimensional closed Riemannian manifolds. To this end we extend the notion of regularity structures to curved space, and explicitly construct the minimal structure required for this equation. A central ingredient is the appropriate re-interpretation of the polynomial model, which we build up to any order.

1 Introduction

The last few years have seen an explosion of literature on singular stochastic partial differential equations (singular SPDEs). The simplest instance of such an equation is the parabolic Anderson model in two dimensions, formally written as

\[ \partial_t u = \Delta u + u\xi. \]  

(PAM)

Here \( u : [0, T] \times D \to \mathbb{R} \) is looked for, where \( D \) is some 2 dimensional domain, and \( \xi \) is (time-independent) white noise on the domain \( D \). This equation is formally ill-posed (or “singular”), since \( u \) is not expected to be regular enough for the product \( u\xi \) to be well-defined analytically. The standard tool of stochastic calculus, the Itô integral, is also of no use here, since the white-noise is constant in time.

With the breakthrough results of Hairer [Hai14] and Gubinelli, Imkeller and Perkowski [GIP15] a large class of such equations has become amenable to analysis. Let us sketch the approach of [Hai14], since this is the one we shall use in this work.

- assume that \( u \) “looks like“ the solution \( \nu \) to the additive-noise equation

  \[ \partial_t \nu = \Delta \nu + \xi, \]  

which is classically well-defined via convolution with the heat semigroup \( P_t \)

∗University of Cambridge; the author is responsible for the first part of the Appendix
†Max-Planck Institute Leipzig
‡University of California San Diego; the author is responsible for the second part of the Appendix
under this assumption, if we somehow define $\nu \cdot \xi$, then the framework defines $u \cdot \xi$ automatically

- close the fixpoint argument, i.e.
  1. $u$ "looks like" $\nu$
  2. $w := P_t u_0 + \int_0^t P_{t-s} [u_s \xi] ds$
  3. then $w$ "looks like" $\nu$

It then only remains to define the missing ingredient "$\nu \cdot \xi$". This can be done probabilistically and is actually the only place in this theory that is not deterministic. Using this procedure, it is shown in [Hai14] that (PAM) possesses a unique solution for $D = T^2$, the two dimensional torus.

In this work we show that the theory can be adapted to work for $D = M$, a 2-dimensional closed Riemannian manifold. The theory of regularity structures is intrinsically a local theory (as opposed to the theory of paracontrolled distributions, which, at least at first sight is global in spirit). It is hence natural to expect that it can be applied to general geometries. It turns out that the implementation of this heuristic is not straightforward.

At least two hurdles need indeed to be bypassed. On the one hand, at the core of Euclidean regularity structures stands the space of polynomials, encoding classical Taylor expansions at any point. The operation of re-expansion from a point to another leads to a morphism from $(\mathbb{R}^d, +)$ to a space of unipotent matrices. On a manifold, one would need to look for such a space, encoding Taylor expansion and enjoying a similar structure. On the other hand, as usual for fixpoint arguments of (S)PDEs, one needs to estimate the improvement of the heat kernel in adequate spaces, which is a global operation (Schauder estimates).

To solve the first issue, we show that the space of polynomials on the tangent space of the manifolds is a suitable candidate for a canonical regularity structure, that allows to encode Hölder functions. This choice enforces a modified definition of a regularity structure. In particular one has to abandon the idea of one fixed vector space and work with vector bundles instead. For our definition of a model, there is no unipotent structure anymore and re-expansions are only approximately compatible. Within this new framework, when considering the parabolic Anderson model on a surface, we give a weak version of a Schauder estimate with elementary tools and heat kernel estimates.

This exposition does not demand any previous knowledge of regularity structures on the Reader. In this sense it is self-contained, apart from a reference to the reconstruction theorem of Hairer in our Theorem 20 and in the construction of the Gaussian model in Section 8. Its proof using wavelet analysis is of no use reproducing here. We believe that the validity of that reconstruction theorem, which we use in coordinates, is easily believed.

We follow a very hands-on approach. Instead of trying to set up a general theory of regularity structures on manifolds, we work with the smallest structure that is necessary to solve PAM. We show the Schauder estimates explicitly. Apart from introducing for the first time regularity structures on manifolds, we believe our work also has a pedagogical value. Since everything is laid out explicitly and covers the flat case $M = T^2$, it can serve as a gentle introduction to the general theory.

In future work we will investigate the algebraic foundation necessary for studying general equations, without having to build the regularity structure "by hand". For general equations a new
proof of the Schauder estimates has also to be found.  

During the writing of the present article, a different approached has been put forward in [IB2016a], where the notion of paracontrolled products using semi-groups is developed on general metric spaces. The advantage of the paracontrolled approach is that it requires less machinery. On the downside, the class of equations that can be covered is currently strictly smaller than in the setting of regularity structures. Let us point to [IB2016b] though, which pushes the framework to more general equations.

The outline of this paper is as follows. After presenting notational conventions, we give in Section 2 the notion of distributions on manifold we shall use in this work. Moreover we introduce Hölder spaces on manifolds. In Section 3 we introduce the notion of regularity structure, model and modelled distribution on a manifold. We show how these objects behave nicely under diffeomorphisms and use this fact to show the reconstruction theorem. In Section 4 we give the simplest non-trivial example of a regularity structure on a manifold; the regularity structure for “linear polynomials”. This forms the basis for the regularity structure for PAM, which is constructed in Section 5. As input it takes the product \( \nu \xi \) alluded to before. This is constructed in Section 8 via renormalization. Section 6 gives the Schauder estimate for modelled distributions in the setting of PAM and finally Section 7 solves the corresponding fixpoint equation. In Section 9 we show how the construction of Section 4 can be extended to “polynomials” of arbitrary order.

1.1 Notation

In all what follows \( M \) will be a \( d \)-dimensional closed Riemannian manifold. When we specialize to the parabolic Anderson model (PAM), the dimension will be \( d = 2 \). Denote by \( \delta > 0 \) the radius of injectivity of \( M \).

For a function \( \varphi \) supported in \( B_\delta(0) \subset \mathbb{R}^d \), we define for \( \lambda \in (0,1], p \in M \), the “scaled test function”

\[
\varphi_p^\lambda(\cdot) := \lambda^{-d} \varphi(\lambda^{-1} \exp^{-1}_p(\cdot)),
\]

extended to all of \( M \) by setting it to zero outside of \( \exp_p(B_\delta(0)) \).

For \( \tau \in \mathcal{G}, \mathcal{G} \) a graded normed vector bundle with grading \( A \) we denote by \( ||\tau||_a \) the size of component in the \( a \)-th level, \( a \in A \).

The differential of a smooth enough function \( f : M \to \mathbb{R} \) at a point \( p \) will be denoted \( d|_p f \in T_p^* M \).

Similar for higher order derivatives (see Section 9) \( \nabla^\ell|_p f \in (T_p^* M)^{\otimes \ell} \). For the action on vectors \( W \in (T_p M)^\ell \), we shall write either \( \langle \nabla^\ell|_p f, W \rangle \) or \( \nabla_W f \).

For \( \eta, r > 0 \) denote

\[
\mathcal{B}^{r,\eta} := \{ \varphi \in C^r(\mathbb{R}^d) : \text{supp } \varphi \subset B_\eta(0), ||\varphi||_{C^r(\mathbb{R}^d)} \leq 1 \},
\]

where \( B_\eta(0) := \{ x \in \mathbb{R}^d : |x| < \eta \} \). Here \( r \) will be depend the situation, and will always be large enough so that the distributions under consideration can act on \( \varphi \).

We shall use \( p, q \) for points in \( M \) and \( x, y, z \) to denote points in \( \mathbb{R}^d \). For \( x \in \mathbb{R}^d \), \( \varphi : \mathbb{R}^d \to \mathbb{R} \) we write

\[
\varphi^\lambda_x := \lambda^{-d} \varphi(\lambda^{-1}(\cdot - x)),
\]
which is consistent with the notation introduced above when considering \( \mathbb{R}^d \) as Riemannian manifold with the standard metric.

For \( \gamma \in \mathbb{R} \) we denote by \([\gamma]\) the smallest integer strictly larger than \( \gamma \).

For a pairing of a distribution \( T \) with a test function we write \( \langle T, \varphi \rangle \).

For two quantities \( f,g \) we write \( f \lesssim g \) if there exists a constant \( C > 0 \) such that \( f \leq Cg \). To make explicit the dependence of \( C \) on a quantity \( h \), we sometimes write \( f \lesssim_h g \).

2 Hölder spaces

**Definition 1.** A distribution on \( M \) is a bounded, linear functional on \( C_c^\infty(M) \) (= \( C^\infty(M) \), if \( M \) is compact).

Given a density \( \lambda \) on \( M \), \( \langle T_\lambda, \varphi \rangle := \int_M \varphi d\lambda \) defines a distribution. Distributions are hence “generalized densities”. Compare [Fri75, Section 2.8] and [Wal12, Section 1.3].

There is another definition of distributions as “generalized functions”, see [Hor13, Section 1.8]. They are equivalent when there is a canonical way to turn a function into a density and vice versa. This is the case when there is a reference density, like on a Riemannian manifold.

**Remark 2.** On a Riemannian manifold \( M \), denote the standard density by \( d|\text{Vol}| \). We can lift a function \( f \in C(M) \) to a density \( fd|\text{Vol}| \). Then, for \( f \in C_c^\infty(M) \), \( T_f \) defined as
\[
\langle T_f, \varphi \rangle := \int_M f(z)\varphi(z)d|\text{Vol}|,
\]
is a distribution.

**Definition 3** (Push-forward). Let \((\Psi, U)\) be a coordinate chart on \( M \). If \( \varphi \in C_c^\infty(\Psi(U)) \) and \( T \) is a distribution on \( M \) we can define the push-forward \( \Psi_*T \in D'(\Psi(U)) \) via
\[
\langle \Psi_*T, \varphi \rangle := \langle T, \Psi^*\varphi \rangle := \langle T, \varphi \circ \Psi \rangle.
\]

**Remark 4.** This push-forward is compatible with the pull-back of densities. Indeed, for \( f \in C(M) \) we get the distribution \( T_f := fd|\text{Vol}| \), by Remark 2. This density pulls back under \( \Psi^{-1} \) as (compare [Lee03, Proposition 16.38])
\[
fd|\text{Vol}| \mapsto f \circ \Psi^{-1} \sqrt{\det g} dy^1 \wedge \cdots \wedge dy^d,
\]
where \( y^i \) are standard coordinates on \( \mathbb{R}^d \), and \( g \) is the Riemannian metric in the coordinates \( \Psi \). Hence
\[
\langle \Psi_*T_f, \varphi \rangle = \langle T_f, \varphi \circ \Psi \rangle
= \int_M \varphi(\Psi(x))f(x)d|\text{Vol}|
= \int_{\mathbb{R}^d} \varphi(z)f(\Psi^{-1}(z))\sqrt{\det g(z)}dz
= \langle T_{(\Psi^{-1})_*f}, \varphi \rangle,
\]
where the last line is the pairing of a distribution with a test function on \( \mathbb{R}^d \) and \( T_h \) is the canonical identification of a locally integrable density \( h \) on \( \mathbb{R}^d \) with a distribution.
Recall the following definition of Hölder spaces in Euclidean space.

**Definition 5.** For \( \gamma \leq 0 \) denote by \( C^\gamma(\mathbb{R}^d) \) the space of distributions \( T \in \mathcal{D}'(\mathbb{R}^d) \) with

\[
||T||_{C^\gamma(\mathbb{R}^d)} := \sup_{x \in \mathbb{R}^d} \sup_{\lambda \in (0,1]} \sup_{\varphi \in \mathcal{B}_{r,1}} \lambda^{-\gamma} |<T, \varphi_x^\lambda>| < \infty,
\]

here \( r := [\gamma] \), \( \varphi_x^\lambda \) is defined in (4) and the set of test functions \( \mathcal{B}_{r,1} \) is defined in (3).

For \( \gamma > 0 \) we keep the classical definition, i.e.

\[
||T||_{C^\gamma(\mathbb{R}^d)} := \sum_{|\ell| \leq n} ||D^\ell T||_{\mathcal{C}^n;\mathbb{R}^d} + \sup_{|\ell|=n; x,y \in \mathbb{R}^d} \frac{|D^\ell T(x) - D^\ell T(y)|}{|x-y|^s},
\]

where \( \gamma = n + s, n \in \mathbb{N}, s \in (0,1] \).

**Remark 6.** For \( \gamma < 0 \) the norm is independent of the arbitrary upper bound \( 1 \) for the supremum over \( \lambda \) as well as the support of \( \varphi \). For every \( \lambda_0, \varepsilon_0 > 0 \)

\[
||T||_{C^\gamma(\mathbb{R}^d)} \lesssim_{\lambda_0, \varepsilon_0} \sup_{x \in \mathbb{R}^d} \sup_{\lambda \in (0,\lambda_0]} \sup_{\varphi \in \mathcal{B}_{r,0}} \lambda^{-\gamma} |<T, \varphi_x^\lambda>|,
\]

where \( r := [\gamma] \).

**Remark 7.** Every time that a condition like

\[
|<T, \varphi_x^\lambda>| \lesssim \lambda^\gamma,
\]

appears, uniformly over \( \text{supp} \varphi \subset B_{\varepsilon}(0) \), with \( ||\varphi||_{C^\gamma(B_{\varepsilon}(0))} \leq 1 \), one can equivalently demand

\[
|<T, \varphi>| \lesssim \lambda^\gamma,
\]

uniformly over \( \text{supp} \varphi \subset B_{\lambda_0}(x) \), with \( ||D^k \varphi||_{\mathcal{C}^n} \lesssim \lambda^{-d-k} \), for \( k = 0, \ldots, r \).

We need a reformulation similar to this remark, but for Schwartz test functions.

**Lemma 8.** Let \( \gamma \leq 0 \) and \( T \in C^\gamma(\mathbb{R}^d) \). Then \( T \in \mathcal{D}'(\mathbb{R}^d) \) (and not just \( T \in \mathcal{D}'(\mathbb{R}^d) \)). Define for \( r := [\gamma] \), \( \varphi \in S(\mathbb{R}^d), x_0 \in \mathbb{R}^d, \lambda \in (0,1] \)

\[
C(\varphi, \lambda, x_0, N, r) := \sup_{|k| \leq r} \sup_{x \in \mathbb{R}^d} |D^k \varphi(x)| \lambda^{d+k} \left(1 + \lambda^{-N}|x - x_0|^N\right).
\]

Then, for \( N > d \),

\[
|<T, \varphi>| \lesssim_N C(\varphi, \lambda, x_0, N)||T||_{C^\gamma(\mathbb{R}^d)} \lambda^\gamma.
\]

**Remark 9.** Note that if \( \varphi \in S(\mathbb{R}^d) \), then \( \varphi_{x_0}^\lambda := \lambda^{-d}(\varphi(\lambda^{-1}(\cdot - x_0))) \) satisfies for \( \lambda \in (0,1], r > 0, N \in \mathbb{N}, x_0 \in \mathbb{R}^d \)

\[
C(\varphi_{x_0}^\lambda, \lambda, x_0, N, r) \leq \sup_{|k| \leq r} \sup_{x \in \mathbb{R}^d} |D^k \varphi(x)| \left(1 + |x|^N\right).
\]
Proof. Let $\phi_z$, $z \in \mathbb{Z}^d$, be a partition of unity of $\mathbb{R}^d$ such that $\text{supp} \phi_z \subset B_1(z)$ and $\sup_{z \in \mathbb{Z}^d} \|\phi_z\|_{C^r} < \infty$.

Define

$$\varphi_{z,\lambda}() := \phi_z(\lambda^{-1} \cdot) \varphi().$$

Then $\sum_{z \in \mathbb{Z}^d} \varphi_{z,\lambda} = \varphi$. Write for short $C_\varphi := C(\varphi, \lambda, x_0, N, r)$. We have $\varphi_{z,\lambda} \subset B_\lambda(\lambda z)$ and

$$\|D^k \varphi_{z,\lambda}\|_\infty \lesssim C_\varphi \lambda^{-d-k} \frac{1}{1 + \lambda^{-N} |\lambda z - x_0|^N}$$

$$= C_\varphi \lambda^{-d-k} \frac{1}{1 + |z - \lambda^{-1} x_0|^N}.$$  

Then

$$|\langle T, \varphi \rangle| \leq \sum_{z \in \mathbb{Z}^d} |\langle T, \varphi_{z,\lambda} \rangle|$$

$$\leq C_\varphi \|T\|_{C^\gamma(\mathbb{R}^d)}^\gamma \sum_{z \in \mathbb{Z}^d} \frac{1}{1 + |z - \lambda^{-1} x_0|^N}$$

$$\lesssim_N C_\varphi \|T\|_{C^\gamma(\mathbb{R}^d)}^\gamma \lambda^N,$$

as desired. We used the fact that $\sum_{z \in \mathbb{Z}^d} \frac{1}{1 + |z - \lambda^{-1} x_0|^N}$ is upper bounded by $\int_{\mathbb{R}^d} \frac{1}{1 + |z - \lambda^{-1} x_0|^N} dz = \int_{\mathbb{R}^d} \frac{1}{1 + |z|^N} dz$, which is finite, since $N > d$, and independent of $\lambda$. \hfill \qed

Definition 10. Let $M$ be a closed Riemannian manifold. Let a finite partition of unity $(\phi_i)_{i \in I}$ be given on $M$, subordinate to a finite atlas $(\Psi_i, U_i)_{i \in I}$. For $\gamma \in \mathbb{R}$, define

$$C^\gamma(M) := C^\gamma(M; (\Psi_i, U_i), \phi_i) := \{ f : M \rightarrow \mathbb{R} : (\Psi_i)_* (\phi_i f) \in C^\gamma(\mathbb{R}^d), \forall i \in I \},$$

and

$$\|f\|_\gamma := \sup_{i \in I} \|(\Psi_i)_* (\phi_i f)\|_{C^\gamma(\mathbb{R}^d)}.$$ 

For $\gamma > 0$, an equivalent characterization of $C^\gamma(M)$ will be shown in Theorem 90. We now give one in the case $\gamma \leq 0$.

Lemma 11. For $\gamma \leq 0$, $M$ a closed Riemannian manifold, an equivalent norm on $C^\gamma(M)$ is given by

$$\sup_{p \in M, \lambda \in (0,1], \varphi \in B^{r,\delta}} \frac{|\langle T, \varphi^\lambda_p \rangle|}{\lambda^\gamma},$$

where we recall that $\varphi^\lambda_p$ is defined in (2).

Proof. Fix an atlas $(\Psi_i, U_i)$ with subordinate partition of unity $\phi_i$. Denote

$$C_1 := \|T\|_{C^\gamma(M; (\Psi_i, U_i), \phi_i)}$$

$$C_2 := \sup_{p \in M, \lambda \in (0,1], \varphi \in B^{r,\delta}} \frac{|\langle T, \varphi^\lambda_p \rangle|}{\lambda^\gamma},$$

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(C₁ ≥ C₂): Let φ ∈ ℂ, p ∈ M. Then
\[ \langle T, \varphi_p \rangle = \sum_i \langle T \phi_i, \varphi_p \rangle \]
\[ = \sum_i \langle (\Psi_i)_* (T \phi_i), \varphi_p \circ \Psi_i^{-1} \rangle \]
\[ = \sum_i \langle (\Psi_i)_* (T \phi_i), \tilde{\phi}_i \circ \Psi_i^{-1} \varphi_p \circ \Psi_i^{-1} \rangle \]
\[ = \sum_i \langle (\Psi_i)_* (T \phi_i), \eta_i \rangle. \]

Here \( \tilde{\phi}_i \) is such that supp \( \tilde{\phi}_i \subset U_i \) and \( \phi_i \tilde{\phi}_i = \phi_i \). Now
\[ |\text{supp } \eta_i| \leq |\Psi_i (B_{c\lambda}(p) \cap \tilde{\phi}_i)| \lesssim \lambda. \]

Indeed, this follows from
\[ \frac{1}{c} I \leq D \left[ \exp^{-1}_p \circ \Psi_i^{-1} \right] (x), \]
for some constant \( c > 0 \), for all \( i \), and \( x \in \text{supp } \tilde{\phi}_i \circ \Psi_i^{-1} \). Moreover
\[ |D^k \left[ \exp^{-1}_p \Psi_i^{-1} \right] (x)| \lesssim_k 1, \]
for all \( i \). Hence
\[ |D^k \eta_i(x)| \lesssim C_1 \lambda^{-d-k}. \]

The result then follows from Remark 7.

(C₁ ≤ C₂): We have to show
\[ |\langle (\Psi_i)_* (T \phi_i), \varphi_p \rangle| \lesssim C_2 \lambda^q, \]
for all \( \varphi \subset B_1(0), ||\varphi||_{C^q} \leq 1 \). Now for \( x \in \mathbb{R}^d \)
\[ \langle (\Psi_i)_* (T \phi_i), \varphi_p \rangle = \langle T, \phi_i \varphi_p \circ \Psi_i \rangle \]
\[ = \langle T, (\phi_i \varphi_p \circ \Psi_i) \circ \exp_{\Psi_i^{-1}(x)} \circ \exp_{\Psi_i^{-1}(x)} \rangle, \]
where the last equality holds if \( \text{supp } h_{i;\lambda;x} \subset B_{\delta/2}(\Psi_i^{-1}(x)) \), with \( h_{i;\lambda;x} := (\phi_i \varphi_p \circ \Psi_i) \).

Claim: there exists \( \lambda_0 > 0 \) such that for all \( \lambda \leq \lambda_0, i \in I, x \in \mathbb{R}^d \) either \( h_{i;\lambda;x} = 0 \) or \( \text{supp } h_{i;\lambda;x} \subset B_{\delta/2}(\Psi_i^{-1}(x)) \). Indeed, since \( I \) is finite and for all \( i \in I, \phi_i \) is compactly supported in \( U_i \) there exists \( \varepsilon_0, \varepsilon_1 > 0 \) such that if \( \lambda < \varepsilon_0 \) and if \( h_{i;\lambda;x} \neq 0 \) then \( d(x, \partial \Psi_i(U_i)) > \varepsilon_1 \). Away from the boundary, the differential of \( \Psi_i \) is bounded, and then for \( z \in \text{supp } h_i \) one has \( d(z, \Psi_i^{-1}(x)) = O(\lambda) \). This proves the claim.

Now one checks that \( h_{i;\lambda} \circ \exp_{\Psi_i^{-1}(x)} \) falls under Remark 7, and applies Remark 6. \( \square \)

As immediate consequence we get the following statement.

**Corollary 12.** Let \( (\tilde{\Psi}_j, \tilde{U}_j)_{j \in J} \) be another finite atlas with subordinate partition of unity \( (\tilde{\phi}_j)_{j \in J} \). Then for \( \gamma \leq 0 \)
\[ C^\gamma (M; (\Psi_i, U_i), \phi_i) = C^\gamma (M; (\tilde{\Psi}_j, \tilde{U}_j), \tilde{\phi}_j) \]
with equivalent norms.
3 Regularity structures on manifolds

Let $M$ be a $d$-dimensional Riemannian manifold without boundary. The two cases we are most interested in are

- $M$ is compact without boundary (i.e. closed)
- $M$ is an open bounded subset of $\mathbb{R}^d$ with induced Euclidean metric

We now give our definition of a regularity structure and a model on a manifold $M$. For concrete incarnations of these abstract definitions we refer the reader to Section 4 for the implementation of a first order “polynomial” structure; to Section 9 for a structure implementing “polynomials” of any order and right before Lemma 33 for the structure used for the parabolic Anderson model.

**Definition 13** (Regularity structure). A *regularity structure* is a graded vector bundle $G$ on $M$, with a finite grading $A = A(G) \subset \mathbb{R}$. For $\alpha \in A$, $G_{\alpha}$ denotes the vector bundle of homogeneity $\alpha$. It is assumed to be finite dimensional. We denote the fiber at $p \in M$ by $G|_p$ and the fiber of homogeneity $\alpha$ at $p$ by $G_{\alpha}|_p$. For $p \in M$, $\tau \in G|_p$, $\alpha \in A$ we write $\text{proj}_{G_{\alpha}|_p}$ for the projection of $\tau$ onto $G_{\alpha}|_p$.

**Definition 14** (Model). Let a collection of open sets $U_q \subset M$, $q \in M$, and maps

$$\Pi_q : G|_q \rightarrow D'(U_q)$$
$$\Gamma_{p\rightarrow q} : G|_q \rightarrow G|_p,$$

be given. We assume there is for every compactum $K \subset M$ a constant $\delta_K = \delta_K(\Pi, \Gamma, \{U_q\}_q) > 0$, such that $\Gamma_{p\rightarrow q}$ is defined for $p, q \in K$, $d(p, q) < \delta_K$ and for $q \in K$, $\exp_q|_{B(\delta_K(0))}$ is a diffeomorphism and $\exp_q|_{B(\delta_K(0))} \subset U_q$.

Given $\beta \in \mathbb{R}$, we say that $(\Pi, \Gamma)$ is a *model with transport precision* $\beta$ if the following entity is finite for every compactum $K \subset M$

$$||\Pi, \Gamma||_{\beta; K} := \sup_{p \in K, \ell \in A(G), \tau \in G|_p, \lambda \in (0, 1], \varphi \in \mathcal{B}r, \delta_K} \frac{||\langle \Pi_p \tau, \varphi_p^{\lambda} \rangle||}{\lambda^\ell ||\tau||} + \sup_{p, q \in K, d(p, q) < \delta_K/2, \tau \in G|_q, \lambda \in (0, 1], \varphi \in \mathcal{B}r, \delta_K/2} \frac{||\langle \Pi_q \tau - \Pi_p \Gamma_{p\rightarrow q} \tau, \varphi_p^{\lambda} \rangle||}{\lambda^\ell ||\tau||} + \sup_{\ell \in A(G), m \in \mathbb{R}, p, q \in K, d(p, q) < \delta_K, \tau \in G|_q} \frac{|\Gamma_{p\rightarrow q} \tau|_m}{d(p, q)^{(\ell-m)\vee 0} ||\tau||},$$

where we recall that the set of test functions $\mathcal{B}^{r, \delta}$ was defined in (3).

**Remark 15.** Note that the conditions on a model do not pin down the global regularity of $\Pi_q \tau$. Without loss of generality we will assume that $\Pi_q \tau \in C^\alpha(U_q)$ for all $q \in M, \tau \in G|_q$ and $\alpha := \min A(G)$.

Our definition of a regularity structure and a corresponding model are slightly more general than the original formulation by Hairer [Hai14]. This extension is necessary to accomodate the “polynomial regularity structure”, which will be constructed up to first order in Section 4 and up to any order in Section 9. Let us point out the key differences.
Derivatives of functions on a general manifold \( M \) can only be stored in a fibered space. Hence the regularity structure has to be a vector bundle and not a fixed vector space.

For this reason there cannot be a fixed structure group \( G \) in which the transport maps \( \Gamma_{p \leftarrow q} \) take value.

The transport maps \( \Gamma_{p \leftarrow q} \) can also act “upwards”, see Remark 81.

The distributions \( \Pi_{p \tau} \) as well as the transports \( \Gamma_{p \leftarrow q} \) only make sense locally.

It turns out that the theory can handle these slight extensions. In particular the reconstruction theorem still holds, Theorem 22. Finally, we remark that our regularity structure does not include time and that the parabolic Anderson model will be treated by considering functions in time, valued in modelled distributions (Definition 17) on a manifold.

As in Lemma 8 we know how \( \Pi_{p \tau} \) acts on a more general class of functions:

**Lemma 16.** For a regularity structure \( \mathcal{G} \) let be given a model \( (\Pi, \Gamma) \) of transport precision \( \beta \) with \( \beta \geq \sup_{a \in A(\mathcal{G})} |a| \). Let \( p \in \mathcal{K} \), a compactum in \( M \). Let \( \phi \) satisfy the assumptions of Lemma 8 with the additional condition \( \text{supp} \phi \subset B_{\delta/4}(0) \subset \mathbb{R}^d \). Assume moreover that \( B_{\delta/2}(p) \subset \mathcal{K} \) (which can always be achieved by making \( \delta \) smaller.) Then for \( \tau \in \mathcal{G}_p \)

\[
|\langle \Pi_{p \tau}, \phi \circ \exp^{-1}_p \rangle| \lesssim C_{\phi} ||\Pi, \Gamma||_{\beta, \mathcal{K}} \lambda^\ell,
\]

where \( C_{\phi} := C(\phi, \lambda, 0, N, r) \) is defined in Lemma 8.

**Proof.** Let \( \phi_z, z \in \mathbb{Z}^d \), be a partition of unity of \( \mathbb{R}^d \) such that \( \text{supp} \phi_z \subset B_1(z) \) and \( \sup_{z \in \mathbb{Z}^d} ||\phi_z||_{C^r} < \infty \). Let \( \lambda_K := \lambda \delta_K/4 \). Define

\[
\phi_{z, \lambda_K} := \phi_z(\lambda_K^{-1} \cdot ) \phi \quad z \in \mathbb{Z}^d,
\]

so that

\[
\sum_{z \in \mathbb{Z}^d} \phi_{z, \lambda_K} = \phi.
\]

Then \( \text{supp} \phi_{z, \lambda_K} \subset B_{\lambda_K}(\lambda_K z) \cap B_{\delta K/4}(0) \). Hence \( \phi_{z, \lambda_K} \equiv 0 \) for \( |\lambda_K z| \geq \delta K/2 \). Moreover

\[
||D^k \phi_{z, \lambda_K}||_{\infty} \lesssim C_{\phi} \lambda^{-d-k} \frac{1}{1 + |z|^\nu}.
\]

Then

\[
\langle \Pi_{p \tau}, \phi \circ \exp^{-1}_p \rangle = \sum_{z \in \mathbb{Z}^d} \langle \Pi_{p \tau}, \phi_{z, \lambda_K} \circ \exp^{-1}_p \rangle = \sum_{z \in \mathbb{Z}^d, |\lambda_K z| < \delta K/2} \langle \Pi_{p \tau}, \phi_{z, \lambda_K} \circ \exp^{-1}_p \rangle = \sum_{z \in \mathbb{Z}^d, |\lambda_K z| < \delta K/2} \langle \Pi_{\exp_p(\lambda_K z)} \Gamma_{\exp_p(\lambda_K z) \leftarrow p \tau}, \phi_{z, \lambda_K} \circ \exp^{-1}_p \rangle + \langle \Pi_{p \tau - \Pi_{\exp_p(\lambda_K z)} \Gamma_{\exp_p(\lambda_K z) \leftarrow p \tau}}, \phi_{z, \lambda_K} \circ \exp^{-1}_p \rangle.
\]
Note that in the sum $|\lambda K z| < \delta_K / 2$. Hence, by assumption $q := \exp_p (\lambda K z) \in K$. Hence by definition of a model, $|\Gamma_{p \to q^\tau}|_m \leq \Pi, \Gamma\|_{\rho \cdot d(p, q)^{\beta(m)v}}$ for $\tau \in \mathcal{G}_\ell | q$. Then for those $z$

$$\langle \Pi_{\text{exp}_p (\lambda K z) \mapsto x \tau, \varphi z, \lambda K \circ \exp_p} \rangle \leq \sum_{n \leq \ell} |\Pi_{\text{exp}_p (\lambda K z) \mapsto x \tau, \varphi z, \lambda K \circ \exp_p} \rangle|
\quad + \sum_{n \succ \ell} |\Pi_{\text{exp}_p (\lambda K z) \mapsto x \tau, \varphi z, \lambda K \circ \exp_p} \rangle|
\quad \leq C_\varphi \Pi, \Gamma\|_{\beta, \lambda} \left( \sum_{n \leq \ell} \lambda^n \left( \sum_{n \leq \ell} \frac{1}{1 + |z|^m} \sum_{n \leq \ell} \frac{1}{1 + |z|^m} \right) \right).

Moreover

$$\langle \Pi_{p \tau - \Pi_{\text{exp}_p (\lambda K z) \mapsto x \tau, \varphi z, \lambda K \circ \exp_p} \rangle \leq C_\varphi \Pi, \Gamma\|_{\beta, \lambda} \frac{1}{1 + |z|^m}.

Combining,

$$\langle \Pi_{p \tau, \varphi \circ \exp_p} \rangle \leq C_\varphi \Pi, \Gamma\|_{\beta, \lambda} \left( \sum_{z \in \mathbb{Z}^d} \lambda^\ell \lambda^\ell \sum_{n \leq \ell} \frac{1}{1 + |z|^m} \sum_{n \leq \ell} \frac{1}{1 + |z|^m} \right)
\quad \leq C_\varphi \Pi, \Gamma\|_{\beta, \lambda} \lambda^\ell.

\square

**Definition 17.** Let $\mathcal{G}$ be a regularity structure and $(\Pi, \Gamma)$ a model of precision $\beta \in \mathbb{R}$. Define for $\gamma > \sup_{\alpha \in A(\mathcal{G})} |\alpha|$ the space of *modelled distributions*

$$\mathcal{G}^\gamma (M, \mathcal{G}) := \{ f : M \to \mathcal{G} : f \text{ is a section of } \mathcal{G}, ||f||_{\mathcal{G}^\gamma (\mathcal{K}, \mathcal{G})} < \infty \text{ for all compacta } \mathcal{K} \subset M \}.

with

$$||f||_{\mathcal{G}^\gamma (\mathcal{K}, \mathcal{G})} := \sum_{\ell < \gamma} \sup_{p, q \in \mathcal{K}} |f(p)|_\ell + \sup_{\ell < \gamma} \frac{\sup_{p, q \in \mathcal{K}, d(p, q) < \delta_K} |f(p) - \Gamma_{p \to q} f(q)|_\ell}{d(p, q)^{\gamma - \ell}}.

Here $\delta_K$ is the distance of points in $\mathcal{K}$ for which $\Gamma$ makes sense, see Definition 14. Note that the precision of transport $\beta$ plays no role here.

**Remark 18.** As usual for Hölder norms, for every compactum $\mathcal{K}$ an equivalent norm is obtained by replacing in the supremum, for any $\delta' \in (0, \delta_K]$, the condition $d(p, q) < \delta_K$ with the condition $d(p, q) < \delta'$.  

**Lemma 19** (Push-forward). Let $M, N$ be Riemannian manifolds. Let $\Psi : M \to N$ a diffeomorphism.

Let $\mathcal{G}$ be a regularity structure on $M$ with model $(\Pi, \Gamma)$ with transport precision $\beta \in \mathbb{R}$. Define

$$\tilde{U}_q := \Psi(U_{\Psi^{-1}(q)})$$
$$\tilde{\mathcal{G}}_q := \mathcal{G}_{\Psi^{-1}(q)}$$
$$\tilde{\Gamma}_{p \to q} := \Gamma_{\Psi^{-1}(p) \to \Psi^{-1}(q)}$$
$$\tilde{\Pi}_q \tau := \Psi_* \Pi_{\Psi^{-1}(q) \tau}$$

$q \in N, p, q \in N, q \in N, \tau \in \tilde{\mathcal{G}}_q$.
Then, $\tilde{\mathcal{G}}$ is a regularity structure on $N$ with grading $A = A$ and $(\tilde{\Pi}, \tilde{\Gamma})$ is a model with transport precision $\beta$. Moreover

1.

$$||\tilde{\Pi}, \tilde{\Gamma}||_{\beta, \mathcal{K}} \lesssim ||\Pi, \Gamma||_{\beta, \Psi^{-1}(\mathcal{K})}$$

2. Let $f, f' \in \mathcal{D}^\gamma(M, \mathcal{G})$ and define $\tilde{f}(x) := f(\Psi^{-1}(x))$, $\tilde{f}'(x) := f'(\Psi^{-1}(x))$. Then, $\tilde{f}, \tilde{f}' \in \mathcal{D}^\gamma(\Psi(U), \mathcal{G})$ and

$$||\tilde{f}||_{\mathcal{D}^\gamma(\mathcal{K}, \mathcal{G})} \lesssim ||f||_{\mathcal{D}^\gamma(\Psi^{-1}(\mathcal{K}), \mathcal{G})}$$

$$||\tilde{f} - \tilde{f}'||_{\mathcal{D}^\gamma(\mathcal{K}, \mathcal{G})} \lesssim ||f - f'||_{\mathcal{D}^\gamma(\Psi^{-1}(\mathcal{K}), \mathcal{G})}. $$

**Proof.** Since $\Psi$ has derivatives bounded below and above for every compactum, one can choose for every compactum $\mathcal{K}$ a constant $\bar{\delta}_\mathcal{K}$ as in the definition of a model, such that $\tilde{\Pi}_{p,q}$ is well-defined for $p, q \in \mathcal{K}$ and $d(p, q) < \delta_\mathcal{K}$ as well as $\exp_q^N(B_{\delta_\mathcal{K}}(0)) \subset U_q$. Here $\exp^N$ denotes the exponential map on $N$.

1. Let $q \in \mathcal{K} \subset N$ and $\tau \in \tilde{\mathcal{G}}_a|_q$ and $\varphi \in \mathcal{B}^\tau, \delta_\mathcal{K}$

$$|\langle \tilde{\Pi}_q \tau, \varphi_q^\lambda \rangle| = |\langle \Psi_{\ast} (\Pi_{\Psi^{-1}(q)} \tau), \varphi_q^\lambda \rangle|$$

$$= |\langle \Pi_{\Psi^{-1}(q)} \tau, \varphi_q^\lambda \circ \Psi \rangle|$$

$$= |\langle \Pi_{\Psi^{-1}(q)} \tau, \varphi_q^\lambda \circ \Psi \circ \exp_q \circ \exp_q^{-1} \rangle|$$

$$\lesssim ||\Pi, \Gamma||_{\beta, \Psi^{-1}(\mathcal{K})} \lambda^\alpha,$$

since $\varphi_q^\lambda \circ \Psi \circ \exp_q$ falls under Remark 7. For $p, q \in \mathcal{K} \subset N$ with $d(p, q) < \delta_\mathcal{K}$ and $\tau \in \tilde{\mathcal{G}}|_q$, we have

$$|\langle \tilde{\Pi}_q \tau - \tilde{\Pi}_p \Gamma_{x \leftarrow y} \tau, \varphi_p^\lambda \rangle| = |\langle \Psi_{\ast} (\Pi_{\Psi^{-1}(q)} \tau - \Pi_{\Psi^{-1}(p)} \Gamma_{\Psi^{-1}(p) \leftarrow \Psi^{-1}(q)} \tau), \varphi_p^\lambda \rangle|$$

$$= |\langle \Pi_{\Psi^{-1}(q)} \tau - \Pi_{\Psi^{-1}(p)} \Gamma_{\Psi^{-1}(p) \leftarrow \Psi^{-1}(q)} \tau, \varphi_p^\lambda \circ \Psi \rangle|$$

$$= |\langle \Pi_{\Psi^{-1}(q)} \tau - \Pi_{\Psi^{-1}(p)} \Gamma_{\Psi^{-1}(p) \leftarrow \Psi^{-1}(q)} \tau, \varphi_p^\lambda \circ \Psi \circ \exp_p \circ \exp_p^{-1} \rangle|$$

$$\lesssim ||\Pi, \Gamma||_{\beta, \Psi^{-1}(\mathcal{K})} \lambda^\beta,$$

again by Remark 7. Finally for $p, q \in \mathcal{K} \subset N$ with $d(p, q) < \delta_\mathcal{K}$ and $\tau \in \tilde{\mathcal{G}}|_q$, we have

$$|\tilde{\Gamma}_{p \leftarrow q} \tau| = |\Gamma_{\Psi^{-1}(p) \leftarrow \Psi^{-1}(q)} \tau|_m \lesssim ||\Pi, \Gamma||_{\beta, \Psi^{-1}(\mathcal{K})} d(p, q)^{(\ell - m)\nu_0}.$$

2. Let $p, q \in \mathcal{K} \subset N$ then

$$||\tilde{f}(q) - \tilde{\Gamma}_{p \leftarrow q} \tilde{f}(q)||_m = ||f(\Psi^{-1}(q)) - \Gamma_{\Psi^{-1}(p) \leftarrow \Psi^{-1}(q)} f(\Psi^{-1}(q))||_m$$

$$\lesssim ||f||_{\mathcal{D}^\gamma(\Psi^{-1}(\mathcal{K}), \mathcal{G})} d(\Psi^{-1}(p), \Psi^{-1}(q))^{\gamma - m}$$

$$\lesssim ||f||_{\mathcal{D}^\gamma(\Psi^{-1}(\mathcal{K}), \mathcal{G})} d(p, q)^{\gamma - m},$$

and similarly for the distance of two modelled distributions.
**Lemma 20** (Reconstruction for $M \subset \mathbb{R}^d$). Let $G$ be a regularity structure on $M$, an open connected subset of $\mathbb{R}^d$. Let $(\Pi, \Gamma)$ be a model with precision $\beta \in \mathbb{R}$. Let $\gamma > 0$ and assume $\beta \geq \gamma$. Denote $\alpha := \inf A$. Assume either that $\alpha < 0$, or that $\alpha = 0$ and that the lowest homogeneity in $G$ is given by the constant distribution (of the polynomial regularity structure of Section 4).

For every $f \in \mathcal{D}(M,G)$ there exists a unique $Rf \in C^\alpha(M)$ such that for every compactum $K \subset M$

$$\langle Rf - \Pi_x f(x), \varphi^n_x \rangle \lesssim \lambda^\gamma \|\Pi, \Gamma\|_{\beta, \overline{K}} \|f\|_{\gamma, \overline{K}}$$

Here $\varphi \in B^{r, \delta} K$, $r > |\alpha|$, (so that the action of $\Pi_x f(x)$ is well-defined) and $\overline{K} := B_{\delta, \kappa}(K)$.

**Remark 21.** Uniqueness actually holds in the class of operators $R$ that satisfy (5) with $\gamma$ replaced by any $\theta > 0$.

**Proof. Existence**

We will apply [Hai14, Proposition 3.25]. This Proposition is formulated for $\mathbb{R}^d$, but the statement is local and also holds for $M \subset \mathbb{R}^d$. So we have to verify for $\zeta_x := \Pi_x f(x)$

$$\langle \varphi^n_x, \zeta_x - \zeta_y \rangle \leq C_1 |x - y|^\gamma 2^{-nd/2 - \alpha n}$$

$$\langle \varphi^n_x, \zeta_x \rangle \leq C_2 2^{-\alpha n - nd/2}$$

uniformly over $x, y \in \overline{K}$, $n \geq n_0$, $n_0 = \log_2(\delta_\overline{K}) > 0$ and $2^{-n} \leq |x - y| \leq \delta_\overline{K}$. In [Hai14, Proposition 3.25] the upper bound 1 is chosen on $|x - y|$, but any upper bound works, so we chose $\delta_\overline{K}$, since we need $\Gamma_{x\leftarrow y}$ to be well-defined.

Here

$$\varphi^n_x := 2^{nd/2} \varphi(2^n \cdot - x),$$

and $\varphi$ is a scaling function for a wavelet basis of regularity $r > |\alpha|$. We have chosen $n_0$ also such that for $n \geq n_0$ and $x \in \overline{K}, \tau \in G|x$ the expression $\langle \Pi_x \tau, \varphi^n_x \rangle$ is well-defined. First, (7) follows from the fact that $\alpha$ is the lowest homogeneity in $A(G)$ (note that $\varphi^n_x$ is scaled to preserve the $L^2$-norm, whereas the scaling in the definition of a model preserves the $L^1$-norm).

Now

$$\langle \varphi^n_x, \zeta_x - \zeta_y \rangle = \langle \varphi^n_x, \Pi_x f(x) - \Pi_y f(y) \rangle$$

$$= \langle \varphi^n_x, \Pi_x [f(x) - \Gamma_{x\leftarrow y} f(y)] \rangle + \langle \varphi^n_x, \Pi_x \Gamma_{x\leftarrow y} f(y) - \Pi_y f(y) \rangle.$$ 

We bound the first term as

$$\lesssim \|\Pi, \Gamma\|_{\beta, \overline{K}} \sum_a 2^{-na - nd/2} |x - y|^\gamma - \alpha$$

$$\lesssim \|\Pi, \Gamma\|_{\beta, \overline{K}} 2^{-na - nd/2} |x - y|^\gamma - \alpha.$$
since $2^{-n} \leq |x - y|$. The second term is bounded as

$$
|\langle \varphi^n_x, \Pi_x \gamma y f(y) - \Pi_y f(y) \rangle| \lesssim ||\Pi, \Gamma||_{\beta, M} 2^{-n\beta - nd/2}
$$

$$
= ||\Pi, \Gamma||_{\beta, M} 2^{-n\alpha - nd/2 - n(\beta - \alpha)}
$$

$$
\lesssim ||\Pi, \Gamma||_{\beta, M} 2^{-n\alpha - nd/2} |x - y|^{\beta - \alpha}
$$

$$
\lesssim ||\Pi, \Gamma||_{\beta, M} 2^{-n\alpha - nd/2} |x - y|^{\gamma - \alpha}.
$$

This proves (6) and an application of [Hai14, Proposition 3.25] gives the existence of $Rf$ satisfying the bound (5).

The preceding argument is valid for $\alpha < 0$. For $\alpha = 0$, one can run the argument for some $\alpha' < 0$ and get unique existence of $Rf \in C^{\alpha'}$ with the claimed properties. In Corollary 23 below it is shown that actually $Rf \in C^{0}$.

**Uniqueness**

Uniqueness follows exactly as in [Hai14, Section 3].

\[\square\]

**Lemma 22** (Reconstruction for $M$ a closed Riemannian manifold). Let $M$ be a closed Riemannian manifold with regularity structure $G$ and $(\Pi, \Gamma)$ a model with transport precision $\beta \in \mathbb{R}$. Let $\gamma > 0$, and $f \in \mathcal{D}^{\gamma}(M, G)$ and assume $\beta \geq \gamma$.

Denote $\alpha := \inf A$. Assume either that $\alpha < 0$ or that $\alpha = 0$ and that the lowest homogeneity in $G$ is given by the constant distribution (of the polynomial regularity structure).

Then, there exists a unique distribution $Rf \in C^{\alpha}(M)$ such that

$$
|\langle Rf - \Pi_x f(x), \varphi^n_x \rangle| \lesssim \lambda^\gamma ||\Pi, \Gamma||_{\beta, M} ||f||_{\mathcal{D}^{\gamma}(M, G)},
$$

for $\varphi \in \mathcal{B}^{r, \delta}_M$, $r > |\alpha|$.

**Proof.** By a cutting up procedure, it is enough to show (8) for $\varphi \in \mathcal{B}^{r, \delta'}$, with $\delta' \in (0, \delta_M]$ to be chosen.

Let $(\Psi_i, \mathcal{U}_i)_{i \in I}$ an atlas with subordinate partition of unity $(\phi_i)_{i \in I}$, with $I$ finite. On each chart, we push-forward the regularity structure, model and $f$ to $\Psi_i(\mathcal{U}_i)$, with corresponding reconstruction operation $\tilde{R}_i$, model $\tilde{\Pi}_i$ and modelled distribution $\tilde{f}_i$. For each $i \in I$, fix a compactum $\mathcal{K}_i \subset \mathcal{U}_i$ such that $\text{supp} \phi_i$ is strictly contained in $\mathcal{K}_i$. By Lemma 19,

$$
||\tilde{f}_i||_{\mathcal{D}^{\gamma}(\Psi_i(\mathcal{K}_i), G)} \lesssim ||f||_{\mathcal{D}^{\gamma}(M, G)}
$$

$$
||\tilde{f}_i - f_i||_{\mathcal{D}^{\gamma}(\Psi_i(\mathcal{K}_i), G)} \lesssim ||f - f'||_{\mathcal{D}^{\gamma}(M, G)}
$$

$$
||\tilde{\Pi}_i, \Gamma \tilde{\Pi}_i||_{\beta, \Psi_i(\mathcal{K}_i)} \lesssim ||\Pi, \Gamma||_{\beta, M}.
$$

Now reconstruct in each coordinate chart as $\tilde{T}_i := \tilde{R}_i \tilde{f}_i$ using Theorem 20. Define $Rf :=$
\[
\sum_{i \in I} \phi_i(\Psi_i^{-1})_* T_i, \quad \text{Then}
\begin{align*}
\langle \mathcal{R}f - \Pi_x f(x), \varphi_x^\lambda \rangle &= \sum_i \langle \phi_i ((\Psi_i^{-1})_* T_i - \Pi_x f(x)), \varphi_x^\lambda \rangle \\
&= \sum_i \langle (\Psi_i^{-1})_* T_i - \Pi_x f(x), \phi_i \varphi_x^\lambda \rangle \\
&= \sum_i \langle T_i - (\Psi_i)_* (\Pi_x f(x)), (\phi_i \varphi_x^\lambda) \circ \Psi_i^{-1} \rangle.
\end{align*}
\]

If \(x \not\in \mathcal{U}_i\), we want the summand to vanish. So let \(\delta' := \min_i d(\text{supp} \phi_i, \partial \mathcal{U}_i)\). Then for \(\varphi \in \mathcal{B}^{r,\delta'}\), \(\lambda \in (0,1]\), we have \(\phi_i \varphi_x^\lambda \neq 0\) implies \(x \in \mathcal{U}_i\). Hence, if \(x \not\in \mathcal{U}_i\), we have \(\phi_i \varphi_x^\lambda = 0\), so the summand vanishes.

Otherwise, with \(z := \Psi_i(x)\)
\[
|\langle T_i - (\Psi_i)_* (\Pi_x f(x)), (\phi_i \varphi_x^\lambda) \circ \Psi_i^{-1} \rangle| = |\langle T_i - (\Psi_i)_* (\Pi_x f(x)), (\phi_i \lambda^{-d} \varphi(\lambda^{-1} \exp_x^{-1}(\cdot)) \circ \Psi_i^{-1} \rangle|
\]
\[
= |\langle T_i - \Pi_x f(z), (\phi_i \lambda^{-d} \varphi(\lambda^{-1} \exp_x^{-1}(\cdot)) \circ \Psi_i^{-1} \rangle|
\]
\[
\lesssim \|\tilde{T}_i, \tilde{\Gamma}_i\|_{\beta, \Psi_i(\mathcal{U}_i)} \|\tilde{f}\|_{D^\gamma(\Psi_i(\mathcal{U}_i), \tilde{G}_i)} \lambda^\gamma
\]
\[
\lesssim \|\Pi_i, \Gamma_i\|_{\beta, \mathcal{M}} \|f\|_{D^\gamma(M, \mathcal{G})} \lambda^\gamma,
\]

since \((\phi_i \lambda^{-d} \varphi(\lambda^{-1} \exp_x^{-1}(\cdot)) \circ \Psi_i^{-1}\) falls under Remark 7 around \(z\). Summing over \(i\) gives (8).

Corollary 23. In setting of the previous theorem, assume that the lowest homogeneity in \(\mathcal{G}\) is 0 and that it is given by the constant (as in the polynomial regularity structure of Section 4). Then \(\mathcal{R}f\) is given by projection onto that homogeneity, i.e.

\[
(\mathcal{R}f)(p) = f_0(p).
\]

Proof. Define \(\tilde{\mathcal{R}}f(p) := f_0(p)\), then
\[
|\langle \tilde{\mathcal{R}}f(\cdot) - \Pi_p f(p)(\cdot), \varphi_p^\lambda \rangle| = |\langle f_0(\cdot) - f_0(p), \varphi_p^\lambda \rangle| + |\sum_{\ell > 0} \Pi_p \text{proj}_{\mathcal{G}_{\ell \ell}^p} f(p), \varphi_p^\lambda \rangle|.
\]

Recall that the projection \(\text{proj}\) is defined in Definition 13. The last term is of bounded by a constant times \(\lambda^\eta\), where \(\eta\) is the smallest homogeneity strictly larger than 0.

For the second to last term we first write
\[
f_0(\cdot) - f_0(p) = \left( f(\cdot) - \Gamma_{\cdot \cdot} f(p) + \Gamma_{\cdot \cdot} \text{proj}_{\mathcal{G}^p_{\geq 0} p} f(p) \right)_0.
\]

Now, since \(f \in D^\gamma\),
\[
(f(\cdot) - \Gamma_{\cdot \cdot} f(p))_0 \lesssim d(\cdot, p)^\gamma.
\]

By the properties of a model
\[
|\Gamma_{\cdot \cdot} \text{proj}_{\mathcal{G}^p_{\geq 0} p} f(p)|_0 \lesssim d(\cdot, p)^\eta.
\]
Hence \(|f_0(\cdot) - f_0(p)| \lesssim d(\cdot, p)^{n+\gamma}\) and then
\[|\langle \tilde{f}_0(\cdot) - f_0(p), \varphi^\lambda_p \rangle| \lesssim \lambda^{n+\gamma}\]

Hence, by Remark 21, \(\tilde{R} = R\). \(\square\)

We want to apply the Lemma 22 to the terms in the heat kernel asymptotics (Theorem 40). The problem is that their support will be of order 1 (and not of order \(\lambda\) as for \(\varphi^\lambda_p\)). Hence we need the following refinement which is similar to Lemma 8.

**Lemma 24.** In the setting of Lemma 22, let \(\varphi\) satisfy the assumptions of Lemma 8 with the additional condition \(\text{supp } \varphi \subset B_{\delta^2/4}(0) \subset \mathbb{R}^d\). Then
\[|\langle Rf - \Pi_p f(p), \varphi \circ \exp^{-1}_p \rangle| \lesssim C_{\varphi} \lambda^{\gamma} \|\Pi\|_{\beta, M} \|f\|_\gamma.\]

where \(C_{\varphi} := C(\varphi, \lambda, 0, N, r)\) is defined in Lemma 8.

**Proof.** Let \(\varphi_{z, \lambda}\) be given as in the proof of Lemma 16 with \(K := M\). Recall \(\lambda_M = \lambda \delta_M/4\) that \(\text{supp } \varphi_{z, \lambda_M} \subset B_{\lambda_M}(\lambda_M z) \cap B_{\delta_M/4}(0)\). Hence \(\varphi_{z, \lambda_M} \equiv 0\) for \(|\lambda_M z| \geq \delta_M/2\). Then with \(\zeta_r := Rf - \Pi_r f(r)\)
\[\langle \zeta_r, \varphi \circ \exp^{-1}_p \rangle = \sum_{z \in \mathbb{Z}^d} \langle \zeta_r, \varphi_{z, \lambda} \circ \exp^{-1}_p \rangle = \sum_{z \in \mathbb{Z}^d, |\lambda_M z| < \delta_M/2} \langle \zeta_r, \varphi_{z, \lambda} \circ \exp^{-1}_p \rangle + \sum_{z \in \mathbb{Z}^d, |\lambda_M z| < \delta_M/2} \left[\langle \exp_{\lambda_M}(\lambda_M z), \varphi_{z, \lambda} \circ \exp^{-1}_p \rangle + \langle \zeta_r - \exp_{\lambda_M}(\lambda_M z), \varphi_{z, \lambda} \circ \exp^{-1}_p \rangle\right].\]

Note that in the sum \(|\lambda_M z| < \delta_M/2\). Hence \(\exp_{\lambda_M}(\lambda_M z) \in M\) is well-defined. Now the first summand can be written as
\[\langle \exp_{\lambda_M}(\lambda_M z), \varphi_{z, \lambda} \circ \exp^{-1}_p \rangle = \langle \exp_{\lambda_M}(\lambda_M z), \varphi_{z, \lambda} \circ \exp^{-1}_p \circ \exp_{\lambda_M}(\lambda_M z) \circ \exp^{-1}_{\exp_{\lambda_M}(\lambda_M z)}\rangle.\]

Applying Remark 7 to \(\varphi_{z, \lambda} \circ \exp^{-1}_p \circ \exp_{\lambda_M}(\lambda_M z)\) and (8), this is bounded by a constant times \(C_{\varphi} \|\Pi\|_{\beta, M} \|f\|_{D^\gamma(M, \mathcal{G})} \lambda^{\gamma \frac{1}{1+|z|_N}}\).

The second summand is bounded as
\[\left|\langle \zeta_r - \exp_{\lambda_M}(\lambda_M z), \varphi_{z, \lambda} \circ \exp^{-1}_p \rangle\right| = \left|\langle \exp_{\lambda_M}(\lambda_M z) f(\exp_{\lambda_M}(\lambda_M z)) - \Pi_p f(x), \varphi_{z, \lambda} \circ \exp^{-1}_p \rangle\right| \lesssim C_{\varphi} \|\Pi\|_{\beta, M} \|f\|_{D^\gamma(M, \mathcal{G})} \left(\sum_{\ell} \|\lambda^\ell \lambda_M z^{\gamma - \ell} + \lambda^\beta\|^{1+(1+|z|_N)}\right).\]
Hence
\[ |\langle \zeta_q, \varphi \circ \exp_p^{-1} \rangle| \lesssim C_\varphi \| \Pi, \Gamma \|_{\beta, M} \| f \|_{D^\gamma(M, G)} \left( \sum_{z \in \mathbb{Z}^d} \lambda^\gamma \frac{1}{1 + |z|^N} + \sum_{z \in \mathbb{Z}^d} \left( \sum_{a} \lambda^a |\lambda_M z|^{\gamma-a} + \lambda^\beta \right) \frac{1}{1 + |z|^N} \right) \]
\[ \lesssim C_\varphi \| \Pi, \Gamma \|_{\beta, M} \| f \|_{D^\gamma(M, G)} \left( \lambda^\gamma + \lambda^\beta + \lambda^\gamma \sum_{z \in \mathbb{Z}^d} \frac{1}{1 + |z|^N} |z|^\gamma + |\inf_{a \in A} a| \right) \]
\[ \lesssim C_\varphi \| \Pi, \Gamma \|_{\beta, M} \| f \|_{D^\gamma(M, G)} \lambda^\gamma, \]
for \( N > d + \gamma + |\inf_{a \in A} a| \).

\[ \Box \]

### 4 Linear “polynomials” on a Riemannian manifold

The regularity structure for linear “polynomials” on the Riemannian manifold \( M \) will be built on the vector bundle \((M \times \mathbb{R}) \oplus T^* M\). For readability introduce the symbol \( \mathbf{1} \) and decree that it forms a basis for \( \mathbb{R} \). Define the graded vector bundle
\[ \mathcal{T} := (M \times \mathbb{R}1) \oplus T^* M, \]
with grading \( A(\mathcal{T}) = \{0, 1\} \). For \( q \in M \) let \( \mathcal{T}_q = \text{span}\{1\} \oplus T^*_q M \) be the fiber at \( q \). A generic element of \( \mathcal{T}_q \) will be written as
\[ 1a + \omega, \]
with \( a \in \mathbb{R}, \omega \in T^*_q M \). Let \( \mathcal{U}_q := B_\delta(q) \), where \( \delta \) is the radius of injectivity of \( M \). Define the linear map \( \Pi_q : \mathcal{T}_q \to D'(\mathcal{U}_q) \) as
\[ (\Pi_q 1)(z) = 1 \]
\[ (\Pi_q \omega)(\cdot) = \omega \exp^{-1}_q(\cdot), \quad \omega \in T^*_q M. \]

Note that since \( \mathbb{R}1 \) is a trivial fiber bundle, it is enough to specify it on the basis element 1. This is not possible on \( T^* M \). Note also that \( \Pi_q \omega \) is chosen to have value 0 and differential \( \omega \) at \( q \).

Finally define the re-expansion maps \( \Gamma_{p\leftarrow q} : \mathcal{T}_q \to \mathcal{T}_p \) as
\[ \Gamma_{p\leftarrow q} 1 = 1 \]
\[ \Gamma_{p\leftarrow q} \omega = \omega \exp^{-1}_q(p) 1 + d[\omega \exp^{-1}_q](p), \]
which is well-defined for \( d(p, q) < \delta; \delta \) the radius of injectivity of \( M \). \( \Pi \) and \( \Gamma \) together form the polynomial model, where we take \( \delta_M = \delta \) in Definition 14.

The transport of \( \omega \in T^*_q M \) is chosen such that \( \Pi_q \omega \) and \( \Pi_p \Gamma_{p\leftarrow q} \omega \) have, at \( p \), the same value and the same first derivative. Our re-expansion is not exact, i.e. we do not have \( \Pi_q \tau = \Pi_p \Gamma_{p\leftarrow q} \tau \), but we have the following.

**Lemma 25.** For \( \omega \in T^*_y M \), uniformly for \( d(p, y) \) bounded,
\[ |(\Pi_q \omega)(z) - (\Pi_p \Gamma_{p\leftarrow q} \omega)(z)| \lesssim d(z, p)^2 \]
\[ |D(\Pi_q \omega)(z) - D(\Pi_p \Gamma_{p\leftarrow q} \omega)(z)| \lesssim d(z, p) \]
\[ |D^2(\Pi_q \omega)(z) - D^2(\Pi_p \Gamma_{p\leftarrow q} \omega)(z)| \lesssim 1. \]

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Proof. Let

\[
\begin{align*}
  f(z) & := (\Pi_q \omega)(z) \\
  g(z) & := (\Pi_q \Gamma_{p \rightarrow q} \omega)(z).
\end{align*}
\]

By construction \( f(p) = g(p), df(p) = dg(p) \) and hence the statement follows from Taylor’s theorem.

\[\square\]

Remark 26. In the setting of the previous Lemma, not only \( f(p) = g(p) \) but also \( f(q) = g(q) \). Indeed, for two points \( p, q \in M \), at distance smaller than the cut locus and \( \omega_q \in T^*_y M \),

\[
\Gamma_{p \rightarrow q}(\omega_q) = \omega_q(\exp^{-1}_q(p))1 + d|_p [\omega_q(\exp^{-1}_q(\cdot))] = \omega_q(\exp^{-1}_q(p))1 + \omega_q \circ d|_p(\exp^{-1}),
\]

where the tangent map satisfies indeed \( d|_p(\exp^{-1}) : T_p M \to T_q M \). By definition,

\[
\Pi_p(\Gamma_{p \rightarrow q}(\omega_q)) = \omega_q(\exp^{-1}_q(p)) + \omega_q \circ d|_p(\exp^{-1}) \circ \exp^{-1}_q
\]
does a priori disagree with \( \Pi_q(\omega_q) = \omega_q \circ \exp^{-1}_q \), but at \( p \). Let us set \( v_q = \exp^{-1}_q(p) \) and \( v_p = \exp^{-1}_p(q) \). The path \( \gamma = (\exp_q((1 - t)v_q))_{0 \leq t \leq 1} \) is the unique path from \( p \) to \( q \), with length and speed \( d(p, q) \), staying within the cut-locus from \( y \), that is \( (\exp_p(tv_p))_{0 \leq t \leq 1} \) : in other words, for any \( 0 \leq t \leq 1 \),

\[
\exp_p(t \exp^{-1}_p(q)) = \exp_q((1 - t) \exp^{-1}_q(p)).
\]

Hence,

\[
\begin{align*}
  d|_p(\exp^{-1}_q)(v_p) & = \left. \frac{d}{dt} \right|_{t=0} \exp^{-1}_q(\exp_p(tv_p)) = \left. \frac{d}{dt} \right|_{t=0} \exp^{-1}_q(\exp_q((1 - t)v_q)) \\
  & = -v_q
\end{align*}
\]

and

\[
\Pi_p(\Gamma_{p \rightarrow q}(\omega_q))(q) = \omega_q(v_q) + \omega_q \circ d|_p(\exp^{-1}_q)(v_p) = 0 = \Pi_q(\omega_q)(q).
\]

The next lemma follows from Lemma 25 and is shown in more generality in Theorem 89.

Lemma 27. The above is a model of transport precision \( \beta = 2 \).

As a sanity check for our construction, we mention the following lemma, which is almost immediate in the flat case (see [Hai14, Lemma 2.12]). We will prove it in Section 9 in a more general setting.

Lemma 28. For \( \gamma \in (1, 2) \), a function \( f : M \to \mathbb{R} \) is in \( C^\gamma(M) \) if and only if there exists a function \( f(x) = f_0(x)1 + f_1(x) \in \mathcal{D}^\gamma(M, T) \) with \( f_0(x) = f(x) \) and \( f_1(x) \in T^*M \).

In that case: \( f_1(x) = df(x) \).
5 The regularity structure for PAM on a manifold

In the next four sections $M$ is a 2-dimensional closed manifold.

The regularity structure for PAM will be built on two copies of the vector bundle, $(M \times \mathbb{R}^2) \oplus T^*M$. We denote these two copies by $V$ and $W$. In order to distinguish the different elements of these bundles we introduce the symbols \{1, Ξ, I[Ξ], I[Ξ]Ξ\} and decree that they form a basis for $\mathbb{R}^4$. We then write

\[ W = (M \times [\mathbb{R}1 \oplus \mathcal{I}[Ξ]]) \oplus T^*M \quad \text{and} \quad V = (M \times [\mathbb{R}Ξ \oplus \mathcal{I}[Ξ]Ξ]) \oplus (Ξ T^*M), \]

where $Ξ T^*M$ is simply another copy of $T^*M$. Formally we have, $V \oplus W$. As usual we will let $\mathcal{T}|_p, V|_p, \text{and } W|_p$ denote the fibers of these bundles over $p \in M$.

The vector bundles $V$ and $W$ are graded, with gradings

\[ A(V) := \{\alpha, 2\alpha + 2, \alpha + 1\} \]
\[ A(W) := \{0, 1, \alpha + 2\}, \]

for some $\alpha \in (-3/2, -1)$ corresponding to the regularity of the driving white noise $ξ$.

For $β \in A(V)$ (or $β \in A(W)$) recall (Definition 13) that $\text{proj}_β : V \to V$ ($\text{proj}_β : W \to W$) is the projection taking an element to its $β$ - component. To be concrete, generic elements $τ \in V|_p, τ' \in W|_p$ are of the form

\[ τ = Ξa + I[Ξ]b + Ξc \]
\[ τ' = 1d + I[Ξ]e + f, \]

with $a, b, d, e \in \mathbb{R}, c, f \in T^*_pM$. And then for example

\[ \text{proj}_α τ = Ξa ∈ V_α|_p \]
\[ \text{proj}_{α+2} τ' = I[Ξ]e ∈ W_{α+2}|_p. \]

All the graded fibers have a canonical norm, where on the cotangent space we use the norm induced by the Riemannian metric. For $β \in A, τ \in V|_p$ (or $τ \in W|_p$) we write, in a slight abuse of notation, $|τ|_β := |\text{proj}_β τ|$.

The model we shall use for the parabolic Anderson model will be time dependent, so we need slight extensions of our definitions.

**Definition 29.** For $G = V, W$, assume we are given a family of models $(Π^t, Γ^t)$ on $M$ parametrized by $t \in [0, T]$. Define

\[ ||Π, Γ||_{β,M,T} := \sup_{t \leq T} ||Π^t, Γ^t||_{β,M}, \]

where $||Π^t, Γ^t||_{β,M}$ is defined in Definition 14. Note that for fixed $t$, the model comes with a reconstruction operator (Theorem 22), which we shall denote $R_t$. 

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**Definition 30** (Time-dependent modelled distributions). For $\mathcal{G} = \mathcal{V}, \mathcal{W}$, given a family of models $(\Pi^t, \Gamma^t)$ parametrized by $t \in [0, T]$, denote by $\mathcal{D}^{t, \gamma}(\mathcal{G}) = \mathcal{D}^{t, \gamma}(M, \mathcal{G})$ the corresponding spaces of modelled distributions. That is, as defined in Definition 17,

$$\|g\|_{\mathcal{D}^{t, \gamma}(M, \mathcal{G})} := \sup_{p \in M} \sup_{t \leq \gamma} \|g(p)\|_\ell + \sup_{p \neq q \in M, t \leq \gamma} \frac{\|g(q) - \Gamma^t_{p \rightarrow q} g(q)\|_\ell}{d(p, q)^{\gamma - \ell}} < \infty,$$

For $\mathcal{R} > 0$, define the modified norm

$$\|g\|_{\mathcal{D}^{t, \gamma, \mathcal{R}}(M, \mathcal{G})} := \sup_{p \in M} \sup_{t \leq \gamma} \|g(p)\|_\ell + \sup_{p \neq q \in M, t \leq \gamma} \frac{\|g(q) - \Gamma^t_{p \rightarrow q} g(q)\|_\ell}{d(p, q)^{\gamma - \ell}} + \mathcal{R} \sup_{p \neq q \in M} \|g(q) - \Gamma^t_{p \rightarrow q} g(q)\|_\mu.$$ 

Here $\mu = \alpha$, if $\mathcal{G} = \mathcal{V}$ and $\mu = 0$, if $\mathcal{G} = \mathcal{W}$.

Define $\mathcal{D}^{t, \gamma, \mathcal{R}}(M, \mathcal{G})$ to be the space of functions $f : [0, T] \rightarrow C(M, \mathcal{G})$ with $f(t) \in \mathcal{D}^{t, \gamma}(M, \mathcal{G})$ and

$$\|f\|_{\mathcal{D}^{t, \gamma, \mathcal{R}}(M, \mathcal{G})} := \sup_{t \leq T} \|f(t)\|_{\mathcal{D}^{t, \gamma}(M, \mathcal{G})} + \sup_{p \neq q \in M} \frac{\|f(t, p) - f(s, p)\|_\nu}{|t - s|^{\gamma_0}} < \infty,$$

where $\nu = \alpha$, if $\mathcal{G} = \mathcal{V}$ and $\nu = 0$, if $\mathcal{G} = \mathcal{W}$. For $\mathcal{R} > 0$, define the modified norm

$$\|f\|_{\mathcal{D}^{t, \gamma, \mathcal{R}, \mathcal{M}}(M, \mathcal{G})} := \sup_{t \leq T} \|f(t)\|_{\mathcal{D}^{t, \gamma, \mathcal{M}}(M, \mathcal{G})} + \sup_{p \neq q \in M} \frac{\|f(t, p) - f(s, p)\|_\nu}{|t - s|^{\gamma_0}}.$$ 

**Remark 31.** The modified norms with scaling parameter $\mathcal{R}$ are necessary for the fixpoint argument, see Remark 36. As usual with Hölder-type spaces on compact domains, these spaces are complete Banach spaces.

We now build the model for the structures $\mathcal{V}, \mathcal{W}$. As input we need realization of $\Xi$ and $\mathcal{T}[\Xi] \Xi$.

**Definition 32.** Assume for $T > 0$ we are given $\xi \in C^\alpha(M)$ and a family of distributions $Z^t_p \in C^\alpha(M)$, $t \in [0, T], p \in M$, satisfying

$$\langle Z^t_p, \varphi^\lambda_p \rangle \leq \lambda^{2\alpha + 2}$$

$$Z^t_q(r) = Z^t_p(r) + \int_0^t \langle p_t - s(p, \cdot) - p_{t-s}(q, \cdot), \xi \rangle ds \xi(r),$$

where the action of the heat kernel $p$ on $\xi$ is well-defined by Theorem 34. Define

$$\|\xi, Z\|_{\alpha, 2\alpha + 2, T} := \|\xi\|_{C^\alpha(M)} + \sup_{t \leq T, q \in M, \lambda \in (0, 1], \varphi \in B^{\nu, \delta}} \frac{\|Z^t_q, \varphi^\lambda_q\|}{\lambda^{2\alpha + 2}},$$

where $\nu := \|\alpha\|$ and $\delta$ is the radius of injectivity of $M$.

In our application to white-noise forcing, $\xi$ will be the white noise on $M$ and $Z$ will be constructed via Gaussian renormalization in Section 8.
Now define the models for $V$ and $W$ as

\begin{align*}
(\Pi_p^V \Xi)(z) &= \xi(z) \\
(\Pi_p^V \mathcal{T}\Xi)(z) &= \Lambda_p \\
(\Pi_p^V \Omega \Xi)(z) &= (\Pi_p^V \omega)(z)(\Pi_p^V \Xi)(z) \\
(\Pi_p^V \Gamma)(z) &= 1 \\
(\Pi_p^V \mathcal{T}\Xi)(z) &= \int_0^t \langle p_{t-r}(z, \cdot) - p_{t-r}(p, \cdot), \xi \rangle \, dr \\
(\Pi_p^V \omega)(z) &= \omega \exp^{-1}(z),
\end{align*}

with transports

\begin{align*}
\Gamma_{p\rightarrow q}^{t,V} \Xi &= \Xi \\
\Gamma_{p\rightarrow q}^{t,V} \mathcal{T}\Xi &= \Xi + \left[ \int_0^t \langle p_{t-r}(p, \cdot) - p_{t-r}(q, \cdot), \xi \rangle \, dr \right] \Xi \\
\Gamma_{p\rightarrow q}^{t,V} \omega &= \omega \exp^{-1}(p) + d_p[\omega \exp_q^{-1}] \\
\Gamma_{p\rightarrow q}^{t,W} 1 &= 1 \\
\Gamma_{p\rightarrow q}^{t,W} \mathcal{T}\Xi &= \Xi + \left[ \int_0^t \langle p_{t-r}(p, \cdot) - p_{t-r}(q, \cdot), \xi \rangle \, dr \right] 1 \\
\Gamma_{p\rightarrow q}^{t,W} \omega &= \omega \exp^{-1}(p) + d_p[\omega \exp_q^{-1}].
\end{align*}

**Lemma 33.** These are in fact models with $\delta_M = \delta$ the radius of injectivity of $M$ and the distances/norms of the model only depend on $\xi, Z$. Indeed for $G = V, W, \gamma \in \mathbb{R}$

$$||\Pi^G||_{\beta, \gamma; M} \lesssim 1 + ||\xi, Z||_{\alpha, 2\alpha + 2, T},$$

with $\beta = 2$ for $G = W$ and $\beta = 2 + \alpha$ for $G = V$.

**Proof.** By Lemma 11

$$|\langle \Pi_p^V \Xi, \varphi_p^\lambda \rangle| = |\langle \xi, \varphi_p^\lambda \rangle| \lesssim \lambda^\alpha ||\xi||_{C^\alpha(M)}.$$

By definition

$$|\langle \Pi_p^V \mathcal{T}\Xi, \varphi_p^\lambda \rangle| = |\langle \zeta_p^\lambda \varphi_p^\lambda \rangle| \lesssim \lambda^{2\alpha + 2} ||\xi, Z||_{\alpha, 2\alpha + 2, T}.$$

Moreover

$$|\langle \Pi_p^V \omega \Xi, \varphi_p^\lambda \rangle| = |\langle \xi, \omega_p \exp^{-1}(\cdot) \varphi_p^\lambda(\cdot) \rangle| \lesssim \lambda^{\alpha + 1} ||\xi||_{C^\alpha(M)},$$

since $\omega_p \exp^{-1} \varphi_p^\lambda = \lambda \psi \varphi_p^\lambda$, with $\psi(\cdot) = \omega_p(\cdot) \phi(\cdot)$.

Regarding transport, both the transport of $\Xi$ and $\mathcal{T}\Xi$ are exact by definition and

$$|\langle \Pi_q^V \omega_q \Xi - \Pi_p^V \Gamma_{p\rightarrow q} \omega_q \Xi, \varphi_p^\lambda \rangle| = |\langle \xi, \left( \Pi_q^V \omega_q - \Pi_p^V \Gamma_{p\rightarrow q} \omega_q \right) \varphi_p^\lambda \rangle| \lesssim \lambda^{\alpha + 2} ||\xi||_{C^\alpha(M)},$$
where we used Lemma 25 for the last step.

Finally

$$|\Gamma_{t+q}^{t+q}I[\Xi]\alpha| = |\int_0^t \langle p_s(p,\cdot) - p_s(q,\cdot), \xi \rangle ds \rangle \lesssim d(p,q)^{\alpha+2},$$

by the Schauder estimate Theorem 34, and

$$|\Gamma_{t+q}^{t+q}(\omega q\Xi)| \lesssim d(p,q)$$

by Lemma 27. Hence

$$||\Pi_{t+q}; \Gamma_{t+q}||_{\beta+2,\gamma,M} \lesssim 1 + ||\xi, Z||_{\alpha+2,2,\gamma,M},$$

when $\beta := 2 + \alpha$.

Analogously, one gets the bounds for $W$ with $\beta = 2$.

\[ \square \]

## 6 Schauder estimates

Let $p$ be the heat kernel on $M$. We start with a Schauder estimate for distributions. Since its proof follows the same idea as the upcoming Schauder estimate for modelled distributions, we omit the proof of the next theorem.

**Theorem 34.** Let $T > 0$, and $F \in L^\infty([0,T], C^\alpha(M))$, for $\alpha \in (-2,-1)$. Then for $t \in [0,T]$

$$|\int_0^t \langle p_{t-s}(p,\cdot), F_r \rangle dr - \int_0^t \langle p_{t-s}(q,\cdot), F_r \rangle dr | \lesssim \sup_{t \leq T} ||F_t||_{C^\alpha(M)} d(p,q)^{2+\alpha}. $$

We now prove an extension of this classical result to the space of modelled distributions. For

$$f(t,p) = f_{\alpha}(t,p)\Xi + f_{2+2\alpha}(t,p)I[\Xi] + f_{1+\alpha}(t,p)\Xi,$$

an element of $D^T\gamma^\alpha(V)\otimes^2$, define

$$(K_t f)(p) := h := h_0(t,p)1 + h_{2+\alpha}(t,p)I[\Xi] + h_1(t,p),$$

with

$$h_0(t,p) = \int_0^t \langle p_{t-s}(p,\cdot), R_s f(s) \rangle ds$$

$$h_{2+\alpha}(t,p) = f_{\alpha}(t,p)$$

$$h_1(t,p) = \int_0^t \langle p_{t-s}(z,\cdot), R_s f(s) - f_{\alpha}(t,p)P_{p}^t\Xi \rangle ds$$

The well-definedness of these terms is part of the following theorem.

\footnote{Recall from the beginning of this section that $f_{\alpha}, f_{2+2\alpha}$ are real-valued and $f_{1\alpha}$ is a section of $T^*M$.}
Theorem 35 (Schauder estimate). For \( \alpha \in (-4/3, -1) \), with \( \gamma \in (0, 2\alpha + 8/3) \), set \( \varepsilon := (2\alpha + 8/3 - \gamma)/4 \) and \( \gamma_0 = \alpha/2 + 1 - \varepsilon \). Let \( T > 0 \) and \( f \in \mathcal{D}_T^{\gamma, \gamma_0}(\mathcal{V}) \). Then, for all \( t \in [0, T] \),

\[
R_t K f = \int_0^t \langle p_{t-s}, R_s f(s) \rangle ds.
\]

Moreover, \( K f \in \mathcal{D}_T^{\tilde{\gamma}, \tilde{\gamma}_0}(\mathcal{W}) \), with \( \tilde{\gamma} = \gamma + 4/3, \tilde{\gamma}_0 = \gamma_0 \) and

\[
||K f||_{\mathcal{D}_T^{\tilde{\gamma}, \tilde{\gamma}_0}(\mathcal{W})} \lesssim ||f||_{\mathcal{D}_T^{\gamma, \gamma_0}(\mathcal{V})} \left( T^\varepsilon + T^\varepsilon N T + \frac{1}{N T} \right).
\]

Remark 36. Here we can see why we introduced the modified norm \( ||.||_{\mathcal{D}_{\tilde{\gamma}, \tilde{\gamma}_0}(\mathcal{W})} \). Without it, i.e. with \( N \equiv 1 \), the factor on the right hand side cannot be made small, which is necessary for the fixpoint argument.

Remark 37. Contrary to classical Schauder estimates, we only get an “improvement of 4/3 derivatives”. In order to get an “improvement of 2 derivatives” one has to include quadratic polynomials in the regularity structure. This is also the reason why we have to choose \( \gamma, \gamma_0 \) in such a specific way. Note that an improvement by 4/3 will be enough to set up the fix-point argument.

To be specific, in order to get an “improvement of 2 derivatives” the complete list of symbols necessary is, ordered by homogeneity,

\[
\Xi, \Xi I[\Xi], \Xi X_1, 1, \Xi I[\Xi X_1], \Xi I[\Xi], \Xi X_1, X_j, \Xi X_1, X_i, \Xi I[\Xi I[\Xi I[\Xi I]], \Xi I[\Xi I[\Xi I]], \\
\Xi I[\Xi I[\Xi]], \Xi X_1 X_j X_k, I[\Xi I[\Xi]], X_i, X_j, X_k,
\]

where \( i, j = 2, 3 \) stand for the space-directions.\(^3\) These symbols would be the building blocks for the regularity structure on flat space. On a manifold the polynomials would represent the respective symmetric covariant tensor bundles, as laid out in Section 4. The Schauder estimate has to be shown on the level of each of these symbols, and hence a treatment “by hand” as we do here would be cumbersome.

Remark 38. The following proof based on the heat kernel (almost) being a scaled test function goes back, in the flat case, to [CM2016]. A proof splitting up the heat kernel into a sum of smooth, compactly supported kernels (following the strategy of [Hai14]) is also possible, but more cumbersome.

Proof of Theorem 35. The first statement follows from the definition of \( h_0 \) and the fact that reconstruction of modelled distributions taking values only in positive homogeneities is given by the projection onto homogeneity 0, see Lemma 23.

Recall that \( \delta_M = \delta \), the radius of injectivity. By Remark 18 we can, and will only consider \( d(p, y) < \delta/4 \). Introduce the short notation

\[
C_f := ||f||_{\mathcal{D}_T^{\gamma, \gamma_0}(M, \mathcal{V})} \\
C_\Pi := \sup_{t \leq T} ||\Pi^{I, \mathcal{V}, \Gamma^{I, \mathcal{V}}}||_{\beta, M}.
\]

\(^3\)Assuming that one builds a regularity structure including space and time.
Note that $||\xi||_{C^\alpha(M)} \leq C_H$.

We shall need the following facts. Since

$$|f(t, p) - \Gamma_{p\rightarrow q}^t f(t, q)| \leq C_N d(p, q)^{\gamma'-\alpha},$$

we have

$$|f(t, p) - f(t, q)| \leq |f(t) - \Gamma_{p\rightarrow q}^t f(t, q)| + |f_{2\alpha+2}(t, q)| \int_0^t \langle p_t - s(p, \cdot) - p_t - s(q, \cdot), \xi \rangle ds + |f_{1+\alpha}(t, q) \exp^{-1}_q(p)|$$

$$\leq C_f \frac{d(p, q)}{2^{\alpha}} C_f \Pi_d(p, q)^{2+\alpha} + C_f d(p, q)$$

$$\leq \left(\frac{C_f}{2^\alpha} + C_f + C_f \Pi_d\right) d(p, q)^{2+\alpha}, \quad \text{(9)}$$

where we used the classical Schauder estimate Theorem 34.

Moreover for a function $\varphi$ satisfying the assumptions of Lemma 16 and Lemma 24 (recall that $R_t$ is the reconstruction operator of Theorem 22 associated to the model $(\Pi^t, \Gamma^t)$)

$$|\langle R_t f(t) - f_\alpha(t, p) \xi, \varphi \circ \exp_p^{-1}\rangle| \leq \left|\langle R_t f(t) - \Pi_{p\rightarrow q} f(t), \varphi \circ \exp_p^{-1}\rangle\right| + \left|\langle \Pi_{p \rightarrow q}^t f(t) - f_\alpha(t, p) \xi, \varphi \circ \exp_p^{-1}\rangle\right|$$

$$= \left|\langle R_t f(t) - \Pi_{p \rightarrow q}^t f(t), \varphi \circ \exp_p^{-1}\rangle\right| + \left|\langle f_{2\alpha+2}(t, p) \Pi_{p}^t (I[\Xi] \Xi) + \Pi_{p}^t (f_{\alpha+1}(t, p) \Xi), \varphi \circ \exp_p^{-1}\rangle\right|$$

$$\leq C_f C_{\Pi} \lambda^\gamma + C_f C_{\Pi} \lambda^{2\alpha+2} + C_f d(p, q)^{2+\alpha}$$

$$\leq C_f C_{\Pi} \lambda^{2\alpha+2}, \quad \text{(10)}$$

and similarly

$$|\langle R_t f(t), \varphi \circ \exp_p^{-1}\rangle| \leq \left|\langle R_t f(t) - \Pi_{p \rightarrow q}^t f(t), \varphi \circ \exp_p^{-1}\rangle\right| + \left|\langle \Pi_{p \rightarrow q}^t f(t), \varphi \circ \exp_p^{-1}\rangle\right|$$

$$\leq C_f C_{\Pi} \lambda^\gamma + C_f C_{\Pi} (\lambda^\alpha + \lambda^{2\alpha+2} + \lambda^{\alpha+1})$$

$$\leq C_f C_{\Pi} \lambda^\alpha. \quad \text{(11)}$$

We now estimate each term in the definition of the norm $||K f||_{D^{s, \gamma}_x, \gamma_0, n(W)}$.

**Space regularity**
Regarding the easier term involving $R$ where $p$

\[
\nabla \text{where Homogeneity 0}
\]

We now treat the term involving $R^N$ using heat asymptotics, Theorem 40.

Regarding the easier term involving $R^N$ we write

\[
\int_0^t \left\langle R^N_t(p,\cdot) - R^N_t(q,\cdot) - d|R^N_t(p,\cdot)\exp_q^{-1}(p), R_s f(s) - f(a(t,q)\xi) \right\rangle ds
\]

where $\nabla$ acts on the dummy variable $\cdot$ and convolution acts on $\cdot$ and $\gamma$ is the geodesic connection $q$ to $p$. Since

\[
||R_s f(s) - f(a(t,q)\xi)||_a \lesssim C_f C_{\Pi}
\]

this expression is well-defined for $N$ large enough and of order

\[
C_f C_{\Pi} \sup_{\theta \leq 1} |\gamma(\theta)|^2 = C_f C_{\Pi} d(p,q)^2.
\]

We now treat the term involving $p^N$. Denoting by $g(t,s)$ the integrand of the above integral, for $s \in [t - d(p,q)^2, t]$,

\[
|g(t,s)| \leq |\left\langle p_{t-s}^N(p,\cdot), R_s f(s) - f(a(t,q)\xi) \right\rangle| + |\left\langle p_{t-s}^N(q,\cdot), R_s f(s) - f(a(t,q)\xi) \right\rangle|
\]

The first term we bound as

\[
\left\langle p_{t-s}^N(p,\cdot), R_s f(s) - f(a(t,q)\xi) \right\rangle \leq \left| \left\langle p_{t-s}^N(p,\cdot), R_s f(s) - f(a(s,p)\xi) \right\rangle \right| + \left| \left\langle p_{t-s}^N(p,\cdot), (f(a(s,p) - f(a(t,q)))\xi) \right\rangle \right|
\]

\[
\lesssim C_f C_{\Pi} |t - s|^{(2\alpha+2)/2} + C_{\Pi}|t - s|^{\alpha/2} \left( \frac{C_f}{C_{\Pi}} + C_f + C_f C_{\Pi} \right) d(p,q)^{2+\alpha} + C_f |t - s|^{\alpha},
\]

\[
C_f |t - s|^{\alpha/2} \left( \frac{C_f}{C_{\Pi}} + C_f + C_f C_{\Pi} \right) d(p,q)^{2+\alpha} + C_f |t - s|^{\alpha}
\]
where we used (10) together with Lemma 41 (i), as well as to Hölder continuity of \( f_\alpha \) in space (9) and in time.

The second we bound as
\[
\left| \left< p_{t-s}^N(q, \cdot), R_s f(s) - f_\alpha(t, q) \xi \right> \right| \\
\leq \left| \left< p_{t-s}^N(q, \cdot), R_s f(s) - f_\alpha(s, q) \xi \right> \right| + \left| \left< p_{t-s}^N(q, \cdot), (f_\alpha(s, q) - f_\alpha(t, q)) \xi \right> \right| \\
\lesssim C_f C_H |t-s|^{(2\alpha+2)/2} + C_H C_f |t-s|^{\alpha/2} |t-s|^{\gamma_0},
\]
where we used (10) together with Lemma 41 (i) as well as to Hölder continuity of \( f_\alpha \) in time.

The last one we bound as
\[
\left| \left< d_q p_{t-s}^N(q, \cdot) \exp_q^{-1}(p), R_s f(s) - f_\alpha(t, q) \xi \right> \right| \\
\leq \left| \left< d_q p_{t-s}^N(q, \cdot) \exp_q^{-1}(p), R_s f(s) - f_\alpha(s, q) \xi \right> \right| \\
+ \left| \left< d_q p_{t-s}^N(q, \cdot) \exp_q^{-1}(p), (f_\alpha(s, q) - f_\alpha(t, q)) \xi \right> \right| \\
\lesssim C_f C_H d(p, q)|t-s|^{(2\alpha+2)/2-1/2} + C_H d(p, q)|t-s|^{\alpha/2-1/2} C_f |t-s|^{\gamma_0},
\]
where we used (10) together with Lemma 41 (ii) as well as to Hölder continuity of \( f_\alpha \) in space (9) and in time.

Hence
\[
|g(t, s)| \lesssim \left( |t-s|^{1/2} + d(p, q) \right) \\
\times \left( C_f C_H |t-s|^{(2\alpha+2)/2-1/2} + C_H \left( \frac{C_f}{\Pi} + C_f + C_f C_H \right) d(p, q)^{2+\alpha} |t-s|^{\alpha/2-1/2} \right. \\
\left. + C_H C_f |t-s|^{\alpha/2-1/2} |t-s|^{\gamma_0} \right),
\]
and then by Lemma 39
\[
\int_{t-d(p, q)^2}^{t} |g(t, s)| ds \lesssim T^\varepsilon \left[ C_f C_H d(p, q)^{2\alpha+4-2\varepsilon} + C_H \left( \frac{C_f}{\Pi} + C_f + C_f C_H \right) d(p, q)^{2\alpha+4-2\varepsilon} \right. \\
\left. + C_H C_f d(p, q)^{\alpha+2+2\gamma_0-2\varepsilon} \right],
\]
if
\[
(2\alpha + 2)/2, \ \alpha/2 + \gamma_0, \ \alpha/2, \ (\alpha + 2)/2 - 1/2, \ \alpha/2 - 1/2 + \gamma_0 > -1 + \varepsilon.
\]
Then the following are upper bounds to \( \bar{\gamma} \)
\[
2\alpha + 4 - 2\varepsilon, \ \alpha + 2\gamma_0 + 2 - 2\varepsilon.
\]
Both are satisfied under our assumptions.

Now consider \( s \in [0, t - d(p, q)^2] \). By [DS15, Theorem 6.1] we have
\[
p_{t-s}^N(p, \cdot) - p_{t-s}^N(q, \cdot) - d_q p_{t-s}^N(q, \cdot) \exp_q^{-1}(p) = \int_0^1 \nabla^2 p_{t-s}^N(\gamma(r), \cdot) (\dot{\gamma}(r) \otimes \dot{\gamma}(r)) (1 - r) dr,
\]
where $\gamma(r) := \exp_q(rv), v := \exp_q^{-1}(p)$, for any $r \in [0,1]$, and $\nabla^2$ is acting on the first variable of $p^N$. Now
\[
g(t, s) = \left\langle p_{t-s}^N(p, \cdot) - p_{t-s}^N(q, \cdot) - d|q| p_{t-s}^N(q, \cdot) \exp_q^{-1}(p), \mathcal{R}_s f(s) - f_\alpha(t, q) \right\rangle
\]
\[
= \left\langle p_{t-s}^N(p, \cdot) - p_{t-s}^N(q, \cdot) - d|q| p_{t-s}^N(q, \cdot) \exp_q^{-1}(p), \mathcal{R}_s f(s) - f_\alpha(s, q) \right\rangle
\]
\[
+ \left\langle p_{t-s}^N(p, \cdot) - p_{t-s}^N(q, \cdot) - d|q| p_{t-s}^N(q, \cdot) \exp_q^{-1}(p), (f_\alpha(s, q) - f_\alpha(t, q)) \right\rangle.
\]
The first term we bound as
\[
\left| \left\langle p_{t-s}^N(p, \cdot) - p_{t-s}^N(q, \cdot) - d|q| p_{t-s}^N(q, \cdot) \exp_q^{-1}(p), \mathcal{R}_s f(s) - f_\alpha(s, q) \right\rangle \right|
\]
\[
= \left| \int_0^1 \left\langle (\nabla^2 p_{t-s}^N(\gamma(r), \cdot) (\dot{\gamma}(r) \otimes \ddot{\gamma}(r)), \mathcal{R}_s f(s) - f_\alpha(s, q) \right\rangle (1 - r) \right| \right|
\]
\[
\lesssim \int_0^1 |v|^2 C_f C_{\Pi} |t - s|^{(2 + 2\alpha)/2 - 1} (1 - r) dr
\]
\[
\lesssim |v|^2 C_f C_{\Pi} |t - s|^{(2 + 2\alpha)/2 - 1}
\]
\[
= d(p, q)^2 C_f C_{\Pi} |t - s|^{(2 + 2\alpha)/2 - 1},
\]
where we used (10) together with Lemma 41. 4

The second term we bound as
\[
\left| \left\langle p_{t-s}^N(p, \cdot) - p_{t-s}^N(q, \cdot) - d|q| p_{t-s}^N(q, \cdot) \exp_q^{-1}(p), (f_\alpha(s, q) - f_\alpha(t, q)) \right\rangle \right|
\]
\[
= \left| \int_0^1 \left\langle (\nabla p_{t-s}^N(\gamma(r), \cdot) (\dot{\gamma}(r) \otimes \ddot{\gamma}(r)), (f_\alpha(s, q) - f_\alpha(t, q)) \right\rangle \right| dr
\]
\[
\lesssim d(p, q)^2 C_f C_{\Pi} |t - s|^{(2 + 2\alpha)/2 - 1} |t - s|^\gamma_0,
\]
where we used Lemma 41 and the Hölder continuity of $f_\alpha$ in time.

Hence by Lemma 39
\[
\int_0^{t - d(p, q)^2} g(t, s) ds \lesssim T^\varepsilon \left( C_f C_{\Pi} d(p, q)^{4 + 2\alpha - 2\varepsilon} + C_{\Pi} C_f d(p, q)^{\alpha + 2 + 2\gamma_0 - 2\varepsilon} \right),
\]
if
\[
(2\alpha + 2)/2 - 1, \quad \alpha/2 + \gamma_0 < -1 + \varepsilon.
\]

Then the following are upper bounds to $\bar{\gamma}$
\[
2\alpha + 4 - 2\varepsilon, \quad 2 + \alpha + 2\gamma_0 - 2\varepsilon.
\]

4In coordinates,
\[
\nabla^2 p_{t-s}^N(\gamma(r), \cdot) = \left( \partial_{ij} p_{t-s}^N(\gamma(r), \cdot) - \sum_k \Gamma_{ij}^k \partial_k p_{t-s}^N(\gamma(r), \cdot) \right) dx^i \otimes dx^j,
\]
where $\Gamma$ are the Christoffel symbols. This gives the quadratic factor in $|\dot{\gamma}(r)| = d(p, q)$. The blowup in $t - s$ follows from an application of Lemma 41 (i), (ii) to the components here.
Both are satisfied under our assumptions.

Hence

\[
\mathfrak{N}|f(t,p) - \Gamma_{p\rightarrow q}^t f(t,q)|_\alpha \lesssim C_f \left( T^\gamma + T^\gamma \mathfrak{N} \right) d(p,q)^\gamma.
\]

**Homogeneity** $\alpha + 2$

\[
|h(t,p) - \Gamma_{p\rightarrow q}^t h(t,q)|_{\alpha+2} = |h_{\alpha+2}(t,p) - h_{\alpha+2}(t,q)|
\]

\[
= |f_{\alpha}(t,p) - f_{\alpha}(t,q)|
\]

\[
\lesssim \frac{1}{\mathfrak{N}} \|f\|_{\mathcal{D}_p^\gamma} d(p,q)^{\gamma - \alpha}
\]

\[
= \frac{1}{\mathfrak{N}} \|f\|_{\mathcal{D}_p^\gamma} d(p,q)^{(\gamma+2) - (\alpha+2)},
\]

so we need

\[
\bar{\gamma} \leq \gamma + 2,
\]

which is satisfied under our assumptions.

**Homogeneity** 1

As on homogeneity 0, we write $p = p_N + R_N$. We only treat the term involving $p_N$.

\[
(h(t,p) - \Gamma_{p\rightarrow q}^t h(t,q))_1 = \int_0^t \left\langle d p_{t-s}^N(p,\cdot), R_s f(s) - f_\alpha(t,p) \Pi_p^s \Xi \right\rangle ds
\]

\[
- d|_p \left[ z \mapsto \int_0^t \left\langle d p_{t-s}^N(q,\cdot), R_s f(s) - f_\alpha(t,q) \Pi_q^s \Xi \right\rangle ds \exp_q^{-1}(z) \right] =: \int_0^t g(t,s) ds.
\]

It is enough to bound this expression acting on $X \in T_p M$. Write

\[
\zeta_p^s := R_s f(s) - f_\alpha(s,p) \Pi_p^s \Xi.
\]

For $s \in \lfloor t - d(p,q)^2, t \rfloor$ we bound ($\bullet$ denotes the dummy variable on which $X$ is acting, $\cdot$ denotes the dummy variable in the distribution-pairing)

\[
\left| \left\langle d|_p p_{t-s}^N(\bullet,\cdot)(X), R_s f(s) - f_\alpha(t,p) \Pi_p^s \Xi \right\rangle \right|
\]

\[
\leq \left| \left\langle d|_p p_{t-s}^N(\bullet,\cdot)(X), \zeta^s_p \right\rangle \right| + \left| \left\langle d|_p p_{t-s}^N(\bullet,\cdot)(X), (f_\alpha(s,q) - f_\alpha(t,q)) \xi \right\rangle \right|
\]

\[
= \left| \left\langle X \left( p_{t-s}^N(\bullet,\cdot) \right), \zeta^s_p \right\rangle \right| + \left| \left\langle X \left( p_{t-s}^N(\bullet,\cdot) \right), (f_\alpha(s,q) - f_\alpha(t,q)) \xi \right\rangle \right|
\]

\[
\lesssim C_f C_\Pi |t-s|^{(2+2\alpha)/2 - 1/2} + C_f C_\Pi |t-s|^{(2+2\alpha)/2 - 1/2} |t-s|^{\gamma_0},
\]

where we used (10) together with Lemma 41 (ii), as well as the Hölder continuity of $f_\alpha$ in time.

Now

\[
\left| \left\langle d|_p \left[ z \mapsto d p_{t-s}^N(q,\cdot) \exp_q^{-1}(z) \right](X), R_s f(s) - f_\alpha(t,p) \Pi_p^s \Xi \right\rangle \right|
\]

\[
\leq \left| \left\langle d|_p \left[ z \mapsto d p_{t-s}^N(q,\cdot) \exp_q^{-1}(z) \right](X), \zeta^s_p \right\rangle \right| + \left| \left\langle d|_p \left[ z \mapsto d p_{t-s}^N(q,\cdot) \exp_q^{-1}(z) \right](X), (f_\alpha(s,q) - f_\alpha(t,q)) \xi \right\rangle \right|
\]

\[
= \left| \left\langle d|_q p_{t-s}^N(q,\cdot) d|_p \exp_q^{-1}(z) \right( X \right), \zeta^s_p \right\rangle \right| + \left| \left\langle d|_q p_{t-s}^N(q,\cdot) d|_p \exp_q^{-1}(z) \right( X \right), (f_\alpha(s,q) - f_\alpha(t,q)) \xi \right\rangle \right|
\]

\[
\lesssim C_f C_\Pi |t-s|^{(2+2\alpha)/2 - 1/2} + C_f C_\Pi |t-s|^{(2+2\alpha)/2 - 1/2} |t-s|^{\gamma_0},
\]
where we used (10) together with Lemma 41 (ii) with $Y_p := d_{|p}\exp_{q}^{-1}(z)(X)$, as well as the Hölder continuity of $f_{\alpha}$ in time.

Hence by Lemma 39
\[
\int_{t-d(p,q)^2}^{t} |g(t,s)|ds \lesssim T\varepsilon \left(C_f C_{\Pi d}(p,q)^{3+2\alpha-2\varepsilon} + C_f C_{\Pi d}(p,q)^{\alpha+1+2\gamma_0-2\varepsilon}\right)
\]
if
\[
(1+2\alpha)/2, \quad \alpha/2 - 1/2 + \gamma_0 > -1 + \varepsilon.
\]
Then the following are upper bounds to $\bar{\gamma} - 1$
\[
3 + 2\alpha - 2\varepsilon, \quad \alpha + 1 + 2\gamma_0 - 2\varepsilon.
\]
Both are satisfied under our assumptions.

Consider now $s \in [0, t-d(p,q)^2]$. Again it is enough to bound the term acting on some $X \in T_p M$.
For notational simplicity let $v(z) := d_{|z}p_{t-s}^{-N}(z,\cdot) d_{|p}\exp_{q}^{-1}(X)$ and $\zeta_p^s = \mathcal{R}_s f(s) - f_{\alpha}(s,p)\xi$. We then write the term to bound as
\[
\langle dp_{t-s}^{-N}(p,\cdot), \mathcal{R}_s f(s) - f_{\alpha}(t,p)\Pi_{t-s}^t \Xi \rangle \langle X \rangle - \langle dp_{t-s}^{-N}(q,\cdot) \exp_{q}^{-1}(z) \rangle \langle X \rangle - \langle v(q), \Pi_{t-s}^t \Xi \rangle - \langle v(q), \mathcal{R}_s f(s) - f_{\alpha}(t,q)\Pi_{t}^t \Xi \rangle
\]
\[
= \langle v(p), \mathcal{R}_s f(s) - f_{\alpha}(t,p)\Pi_{t-s}^t \Xi \rangle - \langle v(q), \mathcal{R}_s f(s) - f_{\alpha}(t,q)\Pi_{t}^t \Xi \rangle
\]
\[
= \langle v(p) - v(q), \zeta_p^s \rangle + \langle v(p), (f_{\alpha}(s,p) - f_{\alpha}(t,p))\xi \rangle - \langle v(q), (f_{\alpha}(s,p) - f_{\alpha}(t,q))\xi \rangle
\]
\[
= \langle v(p) - v(q), \zeta_p^s \rangle + \langle v(p) - v(q), (f_{\alpha}(s,p) - f_{\alpha}(t,p))\xi \rangle + \langle v(q), (f_{\alpha}(t,q) - f_{\alpha}(t,p))\xi \rangle.
\]
Now with $\gamma(t) := \exp_{q}(tv), v := \exp_{q}^{-1}(p)$,
\[
\langle v(p) - v(q), \zeta_p^s \rangle = \int_{0}^{1} \langle d|_{\gamma(r)} v(\gamma(r)), \zeta_p^s \rangle \, dr 
\]
\[
\lesssim d(p,q)C_f C_{\Pi} |t - s|^{(2+2\alpha)/2-1}.
\]
where we used (10) together with Lemma 41 (iii).

Similarly
\[
\left| \langle v(p) - v(q), (f_{\alpha}(s,p) - f_{\alpha}(t,p))\xi \rangle \right| = \left| \int_{0}^{1} \langle d|_{\gamma(r)} v(\gamma(r)), (f_{\alpha}(s,p) - f_{\alpha}(t,p))\xi \rangle \, dr \right|
\]
\[
\lesssim d(p,q)C_f C_{\Pi} |t - s|^{\gamma_0} |t - s|^{\alpha/2-1},
\]
where we used Lemma 41 (iii) and the Hölder continuity of $f_{\alpha}$ in time.

Finally
\[
\left| \langle v(q), (f_{\alpha}(t,q) - f_{\alpha}(t,p))\xi \rangle \right| \lesssim \left( C_f + C_f C_{\Pi} \right) C_{\Pi d}(p,q)^{2+\alpha} |t - s|^{\alpha/2-1/2},
\]
where we used Lemma 41 (ii) and the Hölder continuity of $f_{\alpha}$ in space (9).
Hence by Lemma 39
\[
\int_0^{t-d(p,q)^2} |g(t,s)|ds \lesssim T^\varepsilon \left( C_f C_\Pi d(p,q)^{3+2\alpha-2\varepsilon} \right)
+ C_f C_\Pi d(p,q)^{\gamma_0+\alpha+1-2\varepsilon} + \left( \frac{C_f}{\Pi} + C_f + C_f C_\Pi \right) C_\Pi d(p,q)^{3+2\alpha-2\varepsilon}
\]
if
\[(2 + 2\alpha - 2)/2, \alpha/2 - 1/2, \alpha/2 - 1 + \gamma_0 < -1 + \varepsilon.\]

Then the following are upper bounds for \( \bar{\gamma} - 1 \)
\[2\alpha + 3 - 2\varepsilon, 1 + \alpha + \gamma_0 - 2\varepsilon.\]
Both are satisfied under our assumptions.

Then
\[|f(t,p) - \Gamma_t^{p\leftarrow q} f(t,q)|_1 \lesssim C_f \left( T^\varepsilon + T^\varepsilon \Pi \right) d(p,q)^{\bar{\gamma}-1}.\]

**Time regularity**

As on homogeneity 0 we write \( p = p^N + R^N \). We only treat the term involving \( p^N \).
\[
h_0(t,p) - h_0(s,p) = \int_0^t \left< p^N_t - p^N_s \right> \mathcal{R}_r f(r) dr - \int_0^t \left< p^N_s - p^N_r \right> \mathcal{R}_r f(r) dr
= \int_t^s \left< p^N_t - p^N_r \right> \mathcal{R}_r f(r) dr + \int_0^s \left< p^N_s - p^N_r \right> \mathcal{R}_r f(r) dr.
\]

Now using (11) and Lemma 41 (i)
\[
\int_t^s \left| \left< p^N_t - p^N_r \right> \mathcal{R}_r f(r) \right| dr \lesssim C_f C_\Pi \int_t^s (t-r)^{\alpha/2} dr
= C_f C_\Pi \int_t^s (t-r)^{\varepsilon(t-r)^{\alpha/2-\varepsilon}} dr
\lesssim T^\varepsilon C_f C_\Pi |t-s|^{(\alpha+2)/2-\varepsilon}.
\]
Further, again using (11) and Lemma 41 (i)

\[
\int_0^s \left| \left\langle p_{t-r}^N(p, \cdot) - p_{s-r}^N(p, \cdot), R_r f(r) \right\rangle \right| dr = C_f C_{\Pi} \int_0^s \left| \int_{s-r}^{t-r} \partial_t p_{\theta}^N(p, \cdot), R_r f(r) \right| d\theta \bigg| dr \\
\lesssim C_f C_{\Pi} \int_0^s \int_{s-r}^{t-r} \theta^{\alpha/2 - 1} d\theta dr \\
= C_f C_{\Pi} \frac{1}{(\alpha/2)} \int_0^s \left[ (t-r)^{\alpha/2} - (s-r)^{\alpha/2} \right] dr \\
\leq T^s C_f C_{\Pi} \left| \frac{1}{(\alpha/2)} \int_0^s \left[ (t-r)^{\alpha/2-\varepsilon} - (s-r)^{\alpha/2-\varepsilon} \right] dr \right| \\
= T^s C_f C_{\Pi} \left| - \frac{1}{(\alpha/2)} \frac{1}{(\alpha/2) + 1 - \varepsilon} \times \left[ |t-s|^{\alpha/2+1-\varepsilon} - t^{\alpha/2+1-\varepsilon} + s^{\alpha/2+1-\varepsilon} \right] \right| \\
\lesssim T^s C_f C_{\Pi} |t-s|^{\alpha/2+1-\varepsilon},
\]

if

\[
\alpha/2 - \varepsilon > -1.
\]

We then need

\[
\tilde{\gamma}_0 - \varepsilon \leq \alpha/2 + 1 - \varepsilon.
\]

Both are satisfied under our assumptions.

Then

\[
|h_0(t, p) - h_0(s, p)| \lesssim T^s C_f |t-s|^\tilde{\gamma}_0.
\]

We used the following lemmas.

**Lemma 39.** Let \( \rho_1, \rho_2 \in \mathbb{R}, g : \mathbb{R}^2 \to \mathbb{R} \) and assume

\[
g(t, s) \leq C_1 |t-s|^\rho_1, \quad s \in [t-A, t] \\
g(t, s) \leq C_2 |t-s|^\rho_2, \quad s \in [0, t-A].
\]

Let \( \varepsilon \geq 0 \) such that \( \rho_1 - \varepsilon > -1, \rho_2 - \varepsilon < -1 \). Then

\[
\int_{t-A}^{t} g(t, s) ds \lesssim C_1 T^A A^{\rho_1+1-\varepsilon} \\
\int_{t-A}^{t} g(t, s) ds \lesssim C_2 T^A A^{\rho_2+1-\varepsilon}.
\]
Proof. Indeed

\[
\begin{align*}
\int_{t-A}^{t} g(t, s) ds &\leq \int_{t-A}^{t} |t - s|^{\rho_1} ds \\
&\leq T^\varepsilon \int_{t-A}^{t} |t - s|^{\rho_1 - \varepsilon} ds \\
&= T^\varepsilon (-\frac{1}{\rho_1 + 1} |t - s|^{\rho_1 + 1 - \varepsilon} |t - A|) \\
&= T^\varepsilon (-\frac{1}{\rho_1 + 1} [0 - A^{\rho_1 + 1 - \varepsilon}]) \\
&\lesssim T^\varepsilon A^{\rho_1 + 1 - \varepsilon},
\end{align*}
\]

and

\[
\begin{align*}
\int_{0}^{t-A} g(t, s) ds &\leq \int_{0}^{t-A} |t - s|^{\rho_2} ds \\
&= T^\varepsilon \int_{0}^{t-A} |t - s|^{\rho_2 - \varepsilon} ds \\
&= T^\varepsilon (-\frac{1}{\rho_2 + 1} |t - s|^{\rho_2 + 1 - \varepsilon} |0 - A|) \\
&= T^\varepsilon (-\frac{1}{\rho_2 + 1} [A^{\rho_2 + 1 - \varepsilon} - t^{\rho_2 + 1 - \varepsilon}]) \\
&\lesssim T^\varepsilon A^{\rho_2 + 1 - \varepsilon}.
\end{align*}
\]

The following result on heat kernel asymptotics is classical and its proof can be found for example in [D, Theorem 3.10] and [BGV92, Theorem 2.30] See also [Ros97, Section 3.2]. In these references the norm \( ||\cdot||_{C^\ell(M \times M)} \) is defined via a partition of unity as in Definition 10. There is a slight difference to our notation. In the cited references, \( C^1 \) for example means “continuously differentiable”, while in our notation it only means “Lipshitz continuous”. But it is enough to know that our norm is dominated by the norm in the references.

**Theorem 40.** Let \( M \) be a \( d \)-dimensional, closed Riemannian manifold and \( p \) be the heat kernel on \( M \). Then there exist smooth functions \( (\Phi_i(p, q))_{i \geq 0} \) such that if we define for \( N \geq 1 \)

\[
p^N(t, p, q) := t^{-d/2} \exp(-d(p, q)^2/4t) \sum_{i=0}^{N} t^i \Phi_i(p, q),
\]

we have

\[
||\partial^k_t (p_t - p^N_t)||_{C^\ell(M \times M)} \lesssim t^{N - d/2 - \ell/2 - k}.
\]

Here \( \Phi_i(p, q) = 0 \), for \( d(p, q) \geq \delta/4 \).
Lemma 41. Let
\[ p^N_t(p, q) = t^{-d/2} \exp(-d(p, q)^2/4t) \sum_{i=0}^{N} t^i \Phi_i(p, q), \]
\( \psi, \Phi_i \) smooth and with \( \Phi_i(p, q) = 0 \), for \( d(p, q) \geq \delta/4 \).

Let \( p \in M \) and define for \( z \) in the range of \( \exp^{-1}_p \), \( Y_p \in T_p M \) a tangent vector and \( Z \in \Gamma(TM) \) a vector field
\[ \varphi_t(z) := p^N_t(p, \exp_p(z)) \]
\[ \varphi^Y_t(z) := X_p p^N_t(\bullet, \exp_p(z)) \]
\[ \varphi^{Y:Z}_t(z) := Y_p [\bullet \mapsto Z_{\bullet} p^N_t(\bullet, \cdot)]|_{\exp_p(z)}. \]

(Note that because of the small support of \( p^N \), these are globally well-defined smooth functions by continuation with zero outside of the range of \( \exp^{-1}_p \).)

Then for any multiindex \( k \), any \( n \geq 0 \) and \( \ell = 0, 1 \).

(i) \( |\partial_{t}^{k} \varphi_t(z)| \lesssim_{\ell, n, k} (\sqrt{t})^{-d-k-2\ell} \frac{1}{1+(|z|/\sqrt{t})^n} \),

(ii) \( |D^{k} \varphi^Y_t(z)| \lesssim_{n, k} |Y| (\sqrt{t})^{-d-1-k} \frac{1}{1+(|z|/\sqrt{t})^n} \),

(iii) \( |D^{k} \varphi^{Y:Z}_t(z)| \lesssim_{Z, n, k} |Y| (\sqrt{t})^{-d-2-k} \frac{1}{1+(|z|/\sqrt{t})^n} \).

Proof. The summands of \( p^N \) are of the same form, apart from the factors \( t^i_i = 0, \ldots, N \). Since for \( i \geq 1 \) they improve the singularity at \( t = 0 \), it is enough to treat \( N = 0 \).

Then
\[ \varphi_t(z) = t^{-d/2} \exp(-|z|^2/4t) \Phi(p, \exp_p(z)). \]

Since \( z \mapsto \Phi(p, \exp_p(z)) \) is smooth, uniformly in \( p \), with support in \( B_1(\delta/8) \), and the factor \( 1/4 \) in the exponential is irrelevant, we consider
\[ \varphi_t(z) = t^{-d/2} \exp(-|z|^2/t), \]

where we abuse notation and keep the same name. Now this is the Schwartz function \( z \mapsto \exp(-z^2) \) scaled by a factor of \( \sqrt{t} \), and so part (i) with \( \ell = 0 \) follows from by Remark 9.

Now
\[ \partial_{t} \left[ t^{-d/2} \exp(-|z|^2/t) \right] = (-d/2)t^{-d/2-1} \exp(-|z|^2/t) + t^{-d/2} \exp(-|z|^2/t)|z|^2 t^{-2}. \]

The first term is treated as above, now having the additional prefactor \( t^{-1} = (\sqrt{t})^{-2} \).

We write the second term as
\[ t^{-d/2-1} \exp(-|z|^2/t) \left( \frac{|z|}{\sqrt{t}} \right)^2 = t^{-d/2-1} \phi_0 \sqrt{t}(z), \]
where $\phi(s) := s^2 \exp(-s^2)$ is Schwartz. By Remark 9 part (i) with $\ell = 1$ is proven.

For the second statement

$$Y_p p_t(p, q) = Y_p \left[ t^{-d/2} \exp(-d(p, q)^2/4t) \Phi(p, q) \right]$$

$$= -\frac{1}{2} t^{-d/2-1} \exp(-d(p, q)^2/4t) Y_p \left[ d^2(p, q) \right] \Phi(p, q) + \frac{1}{2} t^{-1} \exp(-d(p, q)^2/4t) Y_p \left[ \Phi(p, q) \right].$$

The first term has worse blowup in $t$ and the factor $1/4$ in the exponential is irrelevant, so it is enough to consider $f(z)g(z)$ where

$$f(z) := t^{-d/2-1} \exp(-|z|^2/t)$$

$$g(z) := Y_p \left[ d^2(p, \cdot) \right] |\exp_p(z)|.$$

Now for a multiindex $k$

$$D^k [f(z)g(z)] = \sum_{\beta \leq k} c_{\beta, k} D^{k-\beta} f(z) D^\beta g(z).$$

By Lemma 42

$$|D^\beta g(z)| \lesssim |z| \quad \text{if } |\beta| = 0$$

$$|D^\beta g(z)| \lesssim 1 \quad \text{else.}$$

and by Lemma 44

$$|D^{k-\beta} f(z)| \lesssim t^{-d/2-1-|k-\beta|/2} \left( \frac{|z|}{t^{1/2}} \right)^{|k-\beta|} \exp(-|z|^2/t).$$

Hence for $|k - \beta| \leq |k| - 1$

$$|f^{k-\beta}(z)g^{(i)}(z)| \lesssim t^{-d/2-1-|k-\beta|/2} \left( \frac{|z|}{t^{1/2}} \right)^{k-i} \exp(-|z|^2/t) \lesssim t^{-d/2-1-2|k|/2} \left( \frac{|z|}{t^{1/2}} \right)^{k-i} \exp(-|z|^2/t).$$

For $|k - \beta| = |k|$ we have $|\beta| = 0$ and then

$$|D^{k-\beta} f(z) D^\beta g(z)| \lesssim t^{-d/2-1-k/2} \left( \frac{d}{t^{1/2}} \right)^k \exp(-|z|^2/t)|z|

\lesssim t^{-d/2-1-2k/2} \left( \frac{d}{t^{1/2}} \right)^{k+1} \exp(-|z|^2/t).$$

The second statement then follows, since $s \mapsto s^j \exp(-s^2)$ is a Schwartz function, for any $j \geq 0$.

The third statement follows in a similar fashion from Lemma 42 and Lemma 43.

\[\Box\]

**Lemma 42.** Let $Y_p \in T_p M$ acting on the first component of $d^2$ as follows

$$g(z) := Y_p \left[ d^2(p, \cdot) \right] |\exp_p(z)|.$$

Then

$$|g(z)| \lesssim |z||Y|$$

$$|D^\beta g(z)| \lesssim |Y|, \quad \text{for any multiindex } \beta.$$
Proof. Since \((p, q) \mapsto d^2(p, q)\) is smooth, we only need to show \(g(z) \lesssim |z||Y|\).

Let \(h(q) = Y_p [d^2(p, q)]\). Fix \(q\) and take coordinates \(\exp_q^{-1}\). Then
\[
|h(q)| = |Y^i \partial_{r_i} |d^2(\exp_q(r), q)| = |Y^i 2r_i| \lesssim |Y|d(p, q).
\]

Then \(|g(z)| = |h(\exp_p(z))| \lesssim |Y||z|\) as desired.

Lemma 43. Let
\[
g(z) := Z \left[ Z \left[ d(\cdot, \cdot)^2 \right] \right]_{\cdot = \exp_p(z)}.
\]

Then, for any multi-index \(\beta\),
\[
|D^\beta g(z)| \lesssim |Y|, \quad i \geq 0.
\]

Proof. This follows from the fact that \((p, q) \mapsto d^2(p, q)\) is smooth.

Lemma 44. For any multi-index \(k\)
\[
|D^k \exp(-|z|^2/t)| \lesssim_k t^{-|k|/2} \left( \frac{|z|}{1+t/2} \right)^{|k|} \exp(-|z|^2/t).
\]

Proof. This can be verified using the Faa di Bruno formula.

7 Fixpoint argument

The following lemma follows from a direct application of the definition of modelled distribution.

Lemma 45. Define “multiplication by \(\Xi\)” as the vector bundle morphism \(m^\Xi : W \to V\) satisfying
\[
m^\Xi(1) := \Xi
\]
\[
m^\Xi(I[\Xi]) := I[\Xi] \Xi
\]
\[
m^\Xi(\omega) := \omega \Xi \quad p \in M, \omega \in T^*_p M.
\]

If \(f \in \mathcal{D}_{T}^\gamma, \gamma_0(M, W)\) then \(m(f) \in \mathcal{D}_{T}^\gamma, \gamma_0(M, V)\) and for \(\Re > 0\)
\[
||m^\Xi(f)||_{\mathcal{D}_{T}^\gamma, \gamma_0(M, V)} = ||f||_{\mathcal{D}_{T}^\gamma, \gamma_0(M, W)}.
\]

Theorem 46. Let \(u_0 \in C^\infty(\mathbb{R}^2)\). Define \(v_t := Pt u_0\) and lift it to the regularity structure as
\[
V_t(p) := 1v_t(p) + I[\Xi] 0 + d_p v_t.
\]

Let \((\xi, Z)\) be given as in Definition 32 and let \(\Pi_p^G, \Gamma_{p^{-1}}^G\) be the corresponding models given by Lemma 33, \(\mathcal{G} = V, W\). Let \(\alpha \in (-4/3, -1), \, \gamma := \alpha/2 + 1\) and \(\gamma \in (4/3, 2\alpha + 4)\). Then there exists \(T > 0\) and a unique \(u \in \mathcal{D}_{T}^\gamma, \gamma_0(M, W)\) such that on \([0, T]\)
\[
u_t = K_t \left[ m^\Xi(u) \right] + V_t.
\]
\textbf{Proof.} We follow a standard fixpoint argument. Denote
\[ B(R, \mathfrak{A}) := \{ f \in D_T^{\gamma,\mathfrak{A}}(M, W) : ||f - V||_{D_T^{\gamma,\mathfrak{A}}(M, W)} \leq R \}. \]
Denote for \( f \in B(R, \mathfrak{A}) \)
\[ \Phi(f) := K_t [m^\Xi(f)] + V_t. \]
Claim: for any \( \mathfrak{A} > 0 \), there is \( R > 0 \) such that \( \Phi(B(R, \mathfrak{A})) \subset B(R, \mathfrak{A}) \).
Indeed, by Theorem 35 and Lemma 45, for a constant \( c > \mathfrak{A} \),
\[ ||\Phi(f) - V||_{D_T^{\gamma,\mathfrak{A}}(M, W)} = ||K_t [m^\Xi(f)]||_{D_T^{\gamma,\mathfrak{A}}(M, W)} \]
\[ \leq c ||m^\Xi(f)||_{D_T^{\gamma-4/3,\mathfrak{A}}(M, V)} \left( T^\mathfrak{A} + T^\mathfrak{A} + \frac{1}{\mathfrak{A}} \right) \]
\[ = c ||f||_{D_T^{\gamma-4/3,\mathfrak{A}}(M, V)} \left( T^\mathfrak{A} + T^\mathfrak{A} + \frac{1}{\mathfrak{A}} \right) \]
\[ \leq c ||f||_{D_T^{\gamma,\mathfrak{A}}(M, V)} \left( T^\mathfrak{A} + T^\mathfrak{A} + \frac{1}{\mathfrak{A}} \right), \]
since \( \alpha > -4/3 \). Hence for \( T \) small enough and \( \mathfrak{A} \) large enough, \( \Phi(B(R, \mathfrak{A})) \subset B(R, \mathfrak{A}) \), for any \( R > 0 \).
Let us show that \( \Phi \) is a contraction on \( B(R, \mathfrak{A}) \): for any \( f, f' \in B(R, \mathfrak{A}) \),
\[ ||\Phi(f) - \Phi(f')||_{D_T^{\gamma,\mathfrak{A}}(M, W)} = ||K_t [m^\Xi(f - f')]||_{D_T^{\gamma,\mathfrak{A}}(M, W)} \]
\[ \leq c ||m^\Xi(f - f')||_{D_T^{\gamma-4/3,\mathfrak{A}}(M, V)} \left( T^\mathfrak{A} + T^\mathfrak{A} + \frac{1}{\mathfrak{A}} \right) \]
\[ = c ||f - f'||_{D_T^{\gamma-4/3,\mathfrak{A}}(M, V)} \left( T^\mathfrak{A} + T^\mathfrak{A} + \frac{1}{\mathfrak{A}} \right) \]
\[ \leq c ||f - f'||_{D_T^{\gamma,\mathfrak{A}}(M, V)} \left( T^\mathfrak{A} + T^\mathfrak{A} + \frac{1}{\mathfrak{A}} \right). \]
Hence for \( T \) small enough and \( \mathfrak{A} \) large enough, \( \Phi \) is a contraction on \( B(R, \mathfrak{A}) \) for any \( R > 0 \).
We therefore get unique existence of a solution for small \( T > 0 \). \qed

\section{8 Appendix - The Gaussian model}

Let \( \xi \) be white noise on \( M \). We recall that \( \xi \) is a Gaussian process associated to the Hilbert space \( L^2(M, \text{vol}_M) \), on a probability space \( (\Omega, \mathcal{B}, \mathbb{P}) \).

\textbf{Lemma 47.} There exists a realization of \( \xi \) such that almost surely for any \( \alpha < -1 \), \( \xi \in C^\alpha \).

\textbf{Proof.} For any coordinate chart \( \psi \) defined on an open subset \( U \subset M \), and \( \rho \) a positive function with support in \( U, \xi_{\mathcal{U}} = \rho \circ \psi^{-1} \psi_* \xi \) is a Gaussian process associated to the Hilbert space \( L^2(\mathbb{R}^2, \rho^2 \circ \psi^{-1} \det(g \circ \psi^{-1})) \). Note that \( \xi_{\mathcal{U}} \) has the same law as \( \eta \nu \), with \( \eta := \rho \circ \psi^{-1} \sqrt{g \circ \psi^{-1}} \).
and \( \nu \) a white-noise on \( \mathbb{R}^d \). According to [Hai14, Lemma 10.2] \( \nu \) has a version which is almost surely in \( C^\alpha(\mathbb{R}^d) \) and hence \( \xi_\mathcal{U} \in C^\alpha(\mathbb{R}^2) \).

Let now \((\rho_i)_{1\leq i\leq n}\) be a partition a unity subordinated to an atlas \((\mathcal{U}_i, \psi_i)_{1\leq i\leq n}\). Then, there is a realization of \((\xi_\mathcal{U})_{1\leq i\leq n}\) such that almost surely for all \( \alpha < -1, i \in \{1, \ldots, n\}, \xi_\mathcal{U} \in C^\alpha(\mathbb{R}^2) \). Then, \( \sum_{i=1}^n \psi_i^* \xi_\mathcal{U} \) is a realization of \( \xi \) belonging almost surely to \( C^\alpha(M) \).

Thanks to this realization, we can already define the transport map used in the following Lemma (point (i)).

**Lemma 48.** Let \( \xi \) be the white noise on \( M \) and \( Z_p^t, p \in M, t \in [0, T] \) be a collection of random distributions on \( M \) such that for some \( \alpha \in (-4/3, 1) \), some \( \kappa > 0 \),

(i) \[ Z_p^t(\cdot) = Z_p^t(\cdot) + \int_0^t \left< p_{t-r}(p, \cdot) - p_{t-r}(q, \cdot), \xi \right> dr \xi(\cdot), \]

(ii) \[ \sup_{p \in M, 0 \leq t, s \leq T, \lambda > 0, \varphi \in B^{|\alpha|}} \lambda^{-(2\alpha + 2)} \lambda^{|(Z_p^0, \varphi^\lambda)|^2 + |t-s|^{-\kappa}(Z_p^t - Z_p^s, \varphi^\lambda)|^2} < \infty, \tag{12} \]

(iii) for any \( \varphi \in C^\infty(M), t \in [0, T], p \in M \), \( \langle Z_p^t, \varphi \rangle \) is in the second Wiener chaos. Then, there is a version of \( Z \) and a constant \( h > 0 \) such that a.s.

\[ \sup_{p \in M, 0 \leq t, s \leq T, \lambda > 0, \varphi \in B^{|\alpha|}} \lambda^{-(2\alpha + 2)} \left( |(Z_p^0, \varphi^\lambda)| + |t-s|^{-h}|(Z_p^t - Z_p^s, \varphi^\lambda)| \right) < \infty. \tag{13} \]

**Proof.** For \( t > s \geq 0 \), define for a chart \((\Psi, \mathcal{U})\)

\[ \tilde{Z}^{s,t}_x := \Psi^*_s(Z_{\Psi^{-1}(x)} - Z_{\Psi^{-1}(x)}^t), \quad x \in \Psi(\mathcal{U}) \]

\[ \xi := \Psi^*_s \xi. \]

Note that \( \tilde{Z}^{s,t}_x, x \in \Psi(\mathcal{U}) \) and \( \xi \) are elements of \( D'(\Psi(\mathcal{U})) \). Then

\[
\left< \tilde{Z}^{s,t}_y, \varphi \right> = \left< Z^{s,t}_{\Psi^{-1}(y)}; \varphi \circ \Psi \right>
= \left< Z^{s,t}_{\Psi^{-1}(x)} + \int_s^t \left[ p_{t-r}(\Psi^{-1}(x), \cdot) - p_{t-r}(\Psi^{-1}(y), \cdot) \right] dr, \xi \circ \Psi \right>,
= \left< \tilde{Z}^{s,t}_x + \int_s^t \left[ p_{t-r}(\Psi^{-1}(x), \cdot) - p_{t-r}(\Psi^{-1}(y), \cdot) \right] dr, \xi \circ \Psi \right>,
= \left< \tilde{Z}^{s,t}_x + S^{s,t}(x \leftarrow y) \xi, \varphi \right>, 
\tag{14}
\]

where we denote \( S^{s,t}(x \leftarrow y) := \int_s^t \left[ p_{t-r}(\Psi^{-1}(x), \cdot) - p_{t-r}(\Psi^{-1}(y), \cdot) \right] dr, \xi \).
Define the regularity structure and model (in the stronger sense of [Hai14])
\[
\mathcal{T} := \text{span}\{\Xi\} \oplus \text{span}\{I[\Xi]\Xi\} \oplus \text{span}\{1\}
\]
\[
\Pi_{x,t}\Xi := \xi
\]
\[
\Pi_{x,t}I[\Xi]\Xi := \bar{Z}_{x,t}^s
\]
\[
\Pi_{x,t}1 := 1
\]
\[
\Gamma_{x-y}\Xi := \Xi
\]
\[
\Gamma_{x-y}I[\Xi]\Xi := I[\Xi]\Xi + S_{x,t}(x \leftarrow y)\Xi
\]
and the sector (in the sense of [Hai14, Definition 2.5])
\[
V := \text{span}\{\Xi\} \oplus \text{span}\{I[\Xi]\Xi\}.
\]
One can then apply [Hai14, Proposition 3.32] to get for every compacta \(K \subset \subset K \subset \Psi(U)\), and \(\varphi \in B^{\alpha,1}\), \(r := ||\alpha||\), with \(\varphi^\perp \subset K\),
\[
|\langle \bar{Z}_{x,t}^s, \varphi_x^\lambda \rangle| \lesssim \lambda^{2\alpha+2}||\Pi_{x,t}I[\Xi]\Xi||_{V,K}
\]
\[
\lesssim \lambda^{2\alpha+2} (1 + ||\Gamma_{x,t}\Xi||_{V,K})
\]
\[
\times \sup_{a'=0,2\alpha+2} \sup_{\tau \in V_u} \sup_{n \in \mathbb{N}} \sup_{z \in \text{dyadic}\cap K} \lambda^{(2\alpha+2)n} (\Pi_{x,t}^\tau, \varphi_{z}^n)
\]
\[
\lesssim \lambda^{2\alpha+2} \left(1 + \sup_{x,y \in \mathbb{R}^d} |x-y|^{-\alpha+2} S_{s,t}(x \leftarrow y)\right)
\]
\[
\times \sup_{n \in \mathbb{N}} \sup_{z \in \text{dyadic}\cap K} \left(2^{(2\alpha+2)n} (\bar{Z}_{z,t}^s, \varphi_{z}^n) + 2^{\alpha n} \langle \bar{Z}_{z,t}^s, \varphi_{z}^n \rangle \langle \xi, \varphi_{z}^n \rangle \right).
\]  \tag{15}

Then, for \(q \in \mathbb{N}^*\) large enough, using (iii), equivalence of moments and then (i)
\[
E[\sup_{n \geq 0} \sup_{|z| < \delta} \sup_{z \in \text{dyadic}} \left(2^{(2\alpha+2)n} |\langle \bar{Z}_{z,t}^s, \varphi_{z}^{2-n} \rangle|^q + 2^{\alpha n} |\langle \bar{Z}_{z,t}^s, \varphi_{z}^{2-n} \rangle|^q \langle \xi, \varphi_{z}^{2-n} \rangle |^q |^q \right)]
\]
\[
\lesssim \sum_{n \geq 0} 2^{2n} \left(2^{(2\alpha+2)n} \sup_{|z| < \delta} E[|\langle \bar{Z}_{z,t}^s, \varphi_{z}^{2-n} \rangle|^2]^2 + 2^{\alpha n} \sup_{|z| < \delta} E[|\langle \bar{Z}_{z,t}^s, \varphi_{z}^{2-n} \rangle|^2 |\langle \xi, \varphi_{z}^{2-n} \rangle|^2]^2 \right)
\]
\[
\lesssim |t-s|^{\frac{3\alpha}{2}}.
\]  \tag{16}

Let now \((\Psi_{i,U_i})\) be a finite atlas with subordinate partition of unity \(\phi_i\). Then for \(s, t \in [0, T]\), \(p \in M\), \(\varphi \in B^{\alpha,\delta}\)
\[
\langle Z_{p,t}^s, \varphi_p^\lambda \rangle = \sum_{i : \phi_i \varphi_p^\lambda \neq 0} \langle Z_{p,t}^s, \phi_i \varphi_p^\lambda \rangle,
\]
Now for \(\lambda\) small enough, \(\phi_i \varphi_p^\lambda \neq 0\) implies that \(\text{supp} \varphi_p^\lambda \subset U_i\) and in particular \(p \in U_i\). Hence
\[
\langle Z_{p,t}^s, \varphi_p^\lambda \rangle = \sum_{i : \phi_i \varphi_p^\lambda \neq 0} \langle \bar{Z}_{\Psi_{i}(p)}^{s,t}, (\phi_i \varphi_p^\lambda) \circ \Psi_i^{-1} \rangle,
\]
\[5\text{in the notation of [Hai14, Proposition 3.32], } \varphi^n \text{ stands for } 2^{-nd/2} \varphi^{2-n} \]
where $Z_{x,t}^{s,i} := (\Psi_i)_s(Z_{x}^{-1}(x) - Z_{x}^i(x))$. We can apply Remark 7 to $(\phi_i, \varphi^\lambda_p) \circ \Psi_i^{-1}$ and can estimate, using (15),

$$
\langle Z_{p,s}^{s,t}, \varphi^\lambda_p \rangle \lesssim \lambda^{2\alpha+2} \left(1 + \sup_{x,y \in \mathbb{R}^d} |x - y|^{-(\alpha+2)} S^{s,t}(x \leftarrow y)\right) \times \sum_{\lambda \geq 0} \sup_{n \in \mathbb{N}} \sup_{\mathcal{K}_i} \left(2^{2(\alpha+2)n} \langle Z_{s,t}^{s,i}, \varphi^\lambda_2 \rangle + 2^{\alpha n} \langle Z_{s,t}^{s,i}, \varphi^\lambda_2 \rangle \langle \hat{\Phi}_i, \varphi^\lambda_2 \rangle \right),
$$

here for every $i$, $\mathcal{K}_i$ is some compactum satisfying $\Psi_i(\sup \phi_i) \subset \mathcal{K}_i \subset \Psi_i(\mathcal{U}_i)$. Then, by (16),

$$
\mathbb{E} \left[ \sup_{p \in M, \varphi \in B^{r,\delta}} |\langle Z_{p,s}^{s,t}, \varphi^\lambda_p \rangle|^q \right] \lesssim |t - s|^{\frac{q\alpha}{2}}.
$$

Let us formulate a setting where we can apply Kolmogorov’s continuity theorem in time. Endow the linear space of maps $Y : M \to \mathcal{D}'(M)$, such that for any $p \in M$, $\text{supp}(Y_p) \subset B(p, \delta)$, with the norm

$$
||Y|| := \sup_{p \in M, \lambda \in (0,1), \varphi \in B^{r,\delta}} \lambda^{-2(\alpha+2)}|\langle Y_p, \varphi^\lambda_p \rangle|,
$$

and consider the Banach space $\chi = \{Y : ||Y|| < \infty\}$. We apply this to

$$
Y^t_p := Z_{p}^t p^t r_p.
$$

Here $\rho_p := \rho \circ \exp_p^{-1}$, with $\rho$ smooth, $\text{supp} \rho \subset B_\delta(0)$ and $\rho \equiv 1$ on $B_{\delta - \epsilon}(0)$ for some $\epsilon > 0$ small enough. Then, from the argument before, for any $s, t \geq 0$ and $q$ large enough, we have

$$
\mathbb{E} ||Y^t - Y^s||^q \lesssim |t - s|^{\frac{q\alpha}{2}}.
$$

The result now follows from the Kolmogorov continuity theorem. 

A simple way to define $Z_{p}^t$ is to consider the Wick product of the random variables involved. For any $t > 0$, the heat kernel and the heat operator are denoted respectively by $p_t : M^2 \to \mathbb{R}$ and $P_t$, and we write for any $p, q \in M$, $q_t(p) = p_t(p, p)$. According to Lemma 47 and Theorem 40, we can consider $P_t(\xi)$ as a function and the map $t \in \mathbb{R}_{>0} \mapsto P_t(\xi) \in C^\infty(M)$ is continuous.

We set for any $p \in M, t \in \mathbb{R}_{>0}$ and any function $\varphi \in C^\infty(M),$

$$
Z_{t}^p := \int_0^t (\xi \circ P_s(\xi) - \xi P_s(\xi)(p))ds,
$$

where for any $s > 0$ and $\varphi \in C^\infty(M),$

$$
(\xi \circ P_s(\xi), \varphi) = \langle \xi, \varphi P_s(\xi) \rangle - \mathbb{E}[\langle \xi, \varphi P_s(\xi) \rangle].
$$

Note that for any $s > 0$,

$$
\xi \circ P_s(\xi) = P_s(\xi) - q_s.
$$

For any $t \geq 0$, let us consider the operator $K_t = \int_0^t P_s ds$ and for any $p, q \in M$ with $p \neq q$, set $k_t(p, q) = \int_0^t p_s(p, q) ds$. Let us note that the operator

$$
K_t^2 = \int_{0 \leq s, s' \leq t} P_{s+s'} ds ds' = \int_0^{2t} sP_s ds,
$$

has a continuous kernel according to Theorem 40, that we shall denote $k_{2,t}.$
Proposition 49. For any $t \in \mathbb{R}_{\geq 0}$, almost surely for any $p \in M$ and $\varphi \in C^\infty(M)$, $(Z_p^t, \varphi)$ is well-defined and there exists a modification of the process given by $(\langle Z_p^t, \varphi \rangle)_{p \in M, \varphi \in C^\infty(M), t \geq 0}$ such that almost surely (13) holds true.  

Proof of Proposition 49. It is enough to prove the assumption of Lemma 48. Let us fix $p \in M, 0 < r < t$. Recall that $\delta$ is the radius of injectivity of $M$ and let $r := 1$.

Let us first check that for any $\varphi \in C^\infty(M)$, $Z^t(\varphi)$, is well defined. Therefor, let us recall – see Theorem 40 – that

$$L := \sup_{p \in M, r \in [0, t]} r q_r(p) < \infty. \quad (18)$$

The Wick formulas imply for any $s > 0$,

$$\text{Var}(\langle \xi, \varphi P_s \xi \rangle) = \int q_2(z) \varphi(z)^2 dz + \int p_4(z, z') \varphi(z) \varphi(z') dz dz' \leq 2 \text{vol}(M) L |\varphi|_\infty^2 s^{-1}.$$ 

It follows that

$$\mathbb{E}[\int_0^t (\langle P_s \xi \varphi, \xi, \varphi \rangle) ds] \leq \int_0^t \text{Var}(\langle \xi, \varphi P_s \xi \rangle)^{1/2} ds < \infty.$$ 

Besides, $\mathbb{E}((\langle \xi, \varphi \rangle^2 P_s \xi(p))^2) = q_2(p) |\varphi|_{L^2}^2 + P_s(\varphi)(p)^2 \leq s(L + 1) |\varphi|_\infty^2$, so that $Z^t_p(\varphi)$ is well defined. We shall now prove that for any $\kappa < 0$,

$$\sup_{t \in [0, T], p \in M, \varphi \in \mathcal{B}^{r, \delta}} \lambda^{-2\kappa} \mathbb{E}[\langle Z^t_p(\varphi^\lambda) \rangle^2] < \infty, \quad (19)$$

which together with Lemma 47 shall yield the claim. We fix now $\kappa < 0$. Let us first prove that the expectation of the second integrand in $\Pi^t_p(\xi, \varphi)$ is almost surely of homogeneity $\kappa$. Indeed, according to [D], for $T > 0$ fixed, there exists $C_T > 0$, such that, for all $0, t < T, p, q \in M$, with $p \neq q$,

$$|k_t(p, q)| \lesssim C_T + \int_0^T e^{-\frac{d(p, q)^2}{4s}} ds = C_T + \int_0^{+\infty} e^{-\frac{v}{4\delta^2}} \frac{dv}{v} \lesssim |\log(d(p, q))|.$$ 

Since

$$\mathbb{E}(\langle \xi P_s \xi(p), \varphi \rangle) = P_s \varphi(p),$$

it follows that for any $\kappa < 0$,

$$\sup_{p \in M, \varphi \in \mathcal{B}^{r, \delta}} |\mathbb{E}(\langle \xi K_t(p), \varphi^\lambda \rangle)| = \sup_{p \in M, \varphi \in \mathcal{B}^{r, \lambda}} |K_t(\varphi^\lambda)(p)| \leq C_T \lambda^\kappa.$$ 

Setting $I_{p, s} = \xi P_s \xi - \xi P_s \xi(p)$ and $I_{p, s} := I_{p, s} - \mathbb{E}[I_{p, s}]$, it remains to estimate

$$\langle Z^t_p, \varphi^\lambda \rangle := \langle Z^t_p, \varphi^\lambda \rangle - \mathbb{E}[\langle Z^t_p, \varphi^\lambda \rangle] = \int_0^t \langle I_{p, s}, \varphi^\lambda \rangle ds$$

\footnote{In particular, almost surely, for all $\varphi \in C^\infty(M)$ and $p \in M, t \mapsto \langle \Pi^t_p(\tau), \varphi \rangle$ is measurable and bounded.}

\footnote{Where me denote the distribution $\langle \mathbb{E}[I_{p, s}], \varphi \rangle := \mathbb{E}(\langle I_{p, s}, \varphi \rangle)$.}
For any $\varphi \in B^{r,\delta}, s, s' \geq 0$,
\[
\mathbb{E} [ : I_{p,s} ; \varphi^\lambda_p ] : I_{p,s'} ; \varphi^\lambda_p ] = \int_M (p_{s+s'}(q,q) + p_{s+s'}(p,p) - 2p_{s+s'}(q,p))\varphi_p^\lambda(q)^2 dq \\
+ \int_{M^2} (p_s(q,q' - p_s(p,q'))(p_{s'}(q,q') - p_{s'}(p,q'))\varphi_p^\lambda(q)\varphi_p^\lambda(q')dq dq'.
\]
and
\[
\mathbb{E} [ : Z^L_p ; \varphi^\lambda_p ] = \int_M (k_{2,t}(q,q) + k_{2,t}(p,p) - 2k_{2,t}(q,p))\varphi_p^\lambda(q)^2 dq \\
+ \int_{M^2} (k_t(q,q') - k_t(p,q'))(k_t(q,q') - k_t(p,q'))\varphi_p^\lambda(q)\varphi_p^\lambda(q')dq dq' \\
\leq 2 \int_M (k_{2,t}(q,q) + k_{2,t}(p,p) - 2k_{2,t}(q,p))\varphi_p^\lambda(q)^2 dq,
\]
where the second line follows from the Cauchy-Schwarz inequality. It follows from the bound (21) that there exists $C > 0$, such that for any $\varphi \in B^{r,\delta}, t \in [0, T],$
\[
\mathbb{E} [ : \Pi(\Xi_t) ; \varphi^\lambda_p ] \leq C \int d(p,q)^{2-\delta}|\varphi_p^\lambda(q)|^2 dq.
\]
Hence for any $\lambda > 0$ and $\varphi \in B^{r,\delta},$
\[
\mathbb{E} [ : Z^L_p ; \varphi^\lambda_p ] \leq C\lambda^{-\delta}.
\]
\[\square\]

**Lemma 50.** For any $\nu > \eta > 0, T \geq 0$, there exists $C > 0$, such that for any $q \in M, t \in [0, T],$
\[
|k_{2,t}(q,q) + k_{2,t}(p,p) - 2k_{2,t}(q,p)| \leq C t^n d(p,q)^{2-2\nu}. \tag{21}
\]

**Proof.** On the one hand, according to (17) and Theorem 40, the left-hand-side of (21) is uniformly bounded by $C_T t$, for all $t \in [0, T]$, for some $C_T > 0$. On the other hand, the estimate (21) would hold true, with $\eta = 0$, if $K_t$ would be replaced by $C^2$ symmetric function on $M^2$. Indeed if $K : M^2 \to \mathbb{R}$ is a $C^2$ symmetric function,
\[
|K(q,q) + K(p,p) - 2K(q,p)| \leq \int_{0 \leq r, s \leq 1} \|\nabla_2 \gamma_s \nabla_1 \gamma_r K(\gamma_s, \gamma_r)\|_{\infty} dr ds, \tag{22}
\]
where the index below the connexion symbol indicate the variable on which the latter is acting, and $\gamma$ is a geodesic from $p$ to $q$. According to Theorem 40, one can therefore consider $K_{t,N} = \int_0^t sP^N_s ds$ in place of $K^2_t$, as soon as $N$ is large enough. This same Theorem ensures that there exists a smooth function $\Phi : [0, T] \times M^2 \to \mathbb{R}_{0+}$ such that for all $t \in (0, T], p, q \in M,$
\[
p^N_t(p,q) = (2\pi \tau)^{-1} e^{-\frac{d(p,q)^2}{4\tau}} \Phi(\tau, p, q).
\]
Let us set $q_\tau(r) = \frac{1}{2\pi \tau} e^{-\frac{r^2}{4\tau}},$ for any $r, \tau > 0$. We shall apply (22) to $K_{t,\varepsilon} = \int_0^{t \varepsilon} sP^N_s ds$, for any fixed $\varepsilon > 0$. Up to a constant, the integrand of the right-hand-side of (22) is bounded by

40
\[
\int_\varepsilon^t (d(p,q)\|\nabla_{1,\gamma_s} q_\tau \circ d(\gamma_s, \gamma_r)\| + \|\nabla_{1,\gamma_s} \nabla_{2,\gamma_s} q_\tau \circ d(\gamma_s, \gamma_r)\|) \, d\tau.
\]
Let us set \( R = d(p,q) \). The first term can be bounded by
\[
R^2 \int_\varepsilon^t \tau^{-1} |s-r| e^{-\frac{p^2 (s-r)^2}{2s}} \, d\tau \leq R^2 |s-r| \int_\varepsilon^t \frac{R^2 |r-s|^2}{2s} e^{-u/2} \frac{du}{u} \leq C_T R^2 |s-r| \log \frac{T}{|s-r|R^2},
\]
and the second by
\[
R^2 \int_\varepsilon^t (\tau^{-1} + \frac{R^2 (r-s)^2}{s} e^{-\frac{\gamma^2 (s-r)^2}{2s}}) \, d\tau \\ \leq R^2 \left( 2 \log \frac{T}{|s-r|R^2} + \int_{R^2 (r-s)^2 / t}^{\infty} e^{-u/2} \frac{du}{u} \right), \]
\[
\leq C_T R^2 \log \frac{T}{|s-r|R^2},
\]
for some constant \( C_T > 0 \). These two bounds, once integrated in (22), imply that for any \( \alpha > 0 \), the left-hand-side of (21) is bounded by \( C_T d(p,q)^{2-\alpha} \), uniformly on \( p, q \in M \) and \( t \in [0,T] \). Using the bound \( \min \{a,b\} \leq a^\eta b^{1-\eta} \), for \( a, b, \eta \in (0,1) \), gives (21).

9 Appendix - Higher order “polynomials”

We recall the regularity structure of polynomial functions in flat space \( \mathbb{R}^d \) given in [Hai14]. It is used to abstractly describe functions in \( C^\gamma (\mathbb{R}^d) \), \( \gamma > 0 \), and also forms a central ingredient for general regularity structures associated with singular SPDEs. Let \( \gamma > 0 \) and \( n = \lfloor \gamma \rfloor \), that is \( n \in \mathbb{N} \) and \( \gamma \in (n,n+1) \). For simplicity of notation let \( d = 1 \). Define
\[
\mathcal{T}^{\text{flat}} := \bigoplus_{\ell=0}^n \text{span}\{X^\ell\},
\]
where \( \text{span}\{X^\ell\} \) denotes the one-dimensional vector space spanned by the abstract symbol \( X^\ell \). Hence \( \mathcal{T}^{\text{flat}} \simeq \mathbb{R}^{n+1} \).

Given \( x, y \in \mathbb{R} \) and \( \ell \in \mathbb{N} \), we define the linear maps, \( \Pi_x : \mathcal{T}^{\text{flat}} \rightarrow \mathcal{D}'(\mathbb{R}) \) and \( \Gamma_{x-y} : \mathcal{T}^{\text{flat}} \rightarrow \mathcal{T}^{\text{flat}} \), which are uniquely determined by
\[
\Pi_x^{\text{flat}} X^\ell := (\cdot - x)^\ell \\
\Gamma_{x-y}^{\text{flat}} X^\ell := \sum_{i \leq \ell} \binom{\ell}{i} (x-y)^{\ell-i} X^i.
\]
In this case one has \( \Pi_x^{\text{flat}} \Gamma_{x-y}^{\text{flat}} = \Pi_y^{\text{flat}} \tau \) for all \( \tau \in \mathcal{T}^{\text{flat}} \). One can use this regularity structure to describe regular functions.

**Lemma 51** ([Hai14, Lemma 2.12]). Let \( f : \mathbb{R} \rightarrow \mathbb{R} \). Then \( f \in C^\gamma (\mathbb{R}) \) if and only if there exists \( \hat{f} : \mathbb{R} \rightarrow \mathcal{T}^{\text{flat}} \) with \( \hat{f}_0(x) = f(x) \) and
\[
|\hat{f}(x) - \Gamma_{x-y}^{\text{flat}} \hat{f}(y)|_\ell \lesssim |x-y|^{\gamma-\ell}.
\]
In that case \( \hat{f}_\ell(x) = f^{(\ell)}(x) \), \( \ell = 0, \ldots, n \). \(^8\)

\(^8\)Here we recall the notation of \( \hat{f}_\ell(x) \) as the component of \( \hat{f}(x) \) on the \( \ell \)-th homogeneity, i.e. the coefficient in front of \( X^\ell \).
9.1 Higher order covariant derivatives

We want to mirror as best we can the flat space polynomial model described above, in the general context of a $d$ dimensional Riemannian manifold. In order to do so we need to store higher order derivatives of functions $f : M \to \mathbb{R}$ in a coordinate independent fashion. There is a canonical way to do this on a Riemannian manifold by making use of the associated Levi-Civita connection.

We recall the notion of higher order covariant derivatives of functions $f : M \to \mathbb{R}$ on a Riemannian manifold with Levi-Civita$^9$ connection (see for example [Lee06, Lemma 4.6]).

**Definition 52.** Define $\nabla^\ell|_p f \in [T_p M]^\otimes \ell \cong [T_p M^{\otimes \ell}]^*$ by,

$$\nabla^0|_p f = f(p), \quad \langle \nabla|_p f, X_1(p) \rangle = \langle d|_p f, X_1(p) \rangle,$$

and then inductively by;

$$\left\langle \nabla|_p f, X_1(p) \otimes \cdots \otimes X_\ell(p) \right\rangle = \left[ X_1 \langle \nabla^{\ell-1} f, X_2 \otimes \cdots \otimes X_\ell \rangle \right]|_p$$

$$- \sum_{m=2}^\ell \langle \nabla^{\ell-1} f, X_1 \otimes \cdots \otimes X_{m-1} \otimes \nabla X_m \otimes X_{m+1} \otimes \cdots \otimes X_\ell \rangle|_p,$$

where $X_1, \ldots, X_\ell$ are arbitrary vector fields on $M$.

A few remarks are in order.

1. As the notation suggests, $\nabla^\ell|_p f$ is indeed tensorial, i.e. the right side of the previously displayed equation really only depends on the vector fields, $\{X_i\}_{i=1}^\ell$, through their values at $p$.

2. In the literature $\nabla f$ sometimes denotes the gradient of $f$. We never use the gradient of a function in this work.

3. We shall also sometimes write $\nabla^\ell_W f = \langle \nabla^\ell|_p f, W \rangle$ for any $W \in (T_p M)^{\otimes \ell}$.

**Lemma 53.** If $f$ is an $\ell$-times continuously differentiable function in a neighborhood of $p \in M$, $v \in T_p M$, and $\gamma_v(t) := \exp_p(tv)$, then

$$\left. \frac{d^\ell}{dt^\ell} \right|_{t=0} f(\gamma_v(t)) = \nabla^\ell_v \circ t f, \quad \text{for all } v \in T_p M.$$ 

(24)

More generally, if $\ell, n \in \mathbb{N}_0$, $f$ is an $(\ell + n + 1)$-times continuously differentiable function in a neighborhood of $p \in M$, $//_t(\gamma_v) : T_{\gamma_v(0)} M \to T_{\gamma_v(t)} M$ is parallel translation along $\gamma_v$, $W_0 \in T_p M^{\otimes \ell}$, and $W_t := //_t(\gamma_v)^{\otimes \ell} W_0$, then

$$\left. \frac{d^k}{dt^k} \nabla^\ell_W f \right|_{t=0} = \nabla^\ell+k_{\gamma_v(t)^{\otimes k} \circ t W_t} f \quad \forall 0 \leq k \leq n + 1.$$ 

(25)

$^9$In general, $\nabla$ can be any affine connection.
Proof. Let \( \gamma_v (t) := \exp_p (tv) \) so that \( \gamma_v (t) \) solves the geodesic differential equation, \( \nabla \dot{\gamma}_v (t) / dt = 0 \) with \( \dot{\gamma}_v (0) = v \). The proof is completed by showing (by induction) that

\[
\frac{d^k}{dt^k} f (\gamma_v (t)) = \nabla_{\dot{\gamma}_v (t) \otimes k} f \quad \text{for} \quad 1 \leq k \leq \ell.
\] (26)

The case \( k = 1 \) amounts to the definition that \( \nabla v f = v f = df (v) \) for all \( v \in TM \). For the induction step we have by the product rule;

\[
\frac{d^{k+1}}{dt^{k+1}} f (\gamma_v (t)) = \frac{d}{dt} \nabla_{\dot{\gamma}_v (t) \otimes k} f = \nabla_{\dot{\gamma}_v (t) \otimes (k+1)} f + \nabla_{\dot{\gamma}_v (t) \otimes k} f
\]

wherein the last equality we have again used the product rule to conclude that \( \frac{\sum}{dt} [\dot{\gamma}_v (t) \otimes k] = 0 \). The result now follows by evaluating (26) at \( k = \ell \) and \( t = 0 \). The more general assertion in (25) is proved similarly. One only need observe that \( \nabla W_i / dt = 0 \) and hence the presence of \( W_i \) in the expressions in no way changes the computations. \( \square \)

Definition 54 (Symmetrizations). If \( V \) is a real vector space and \( \ell \in \mathbb{N} \), we let \( \text{Sym}_\ell : V^{\otimes \ell} \rightarrow V^{\otimes \ell} \) denote the symmetrization projection uniquely determined by

\[
\text{Sym}_\ell (v_1 \otimes \cdots \otimes v_\ell) = \frac{1}{\ell!} \sum_{\sigma \in S_\ell} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(\ell)}
\]

where \( S_\ell \) is the permutation group on \( \{1, 2, \ldots, \ell\} \). Often we will simply write \( \text{Sym} \) for \( \text{Sym}_\ell \) as it will typically be clear what \( \ell \) is from the argument put into the symmetrization function.

As usual we let \( V^* \) denote the dual space to a vector space \( V \) and let \( \langle \cdot, \cdot \rangle \) denote the pairing between a vector space and its dual. We will often identify \( (V^*)^{\otimes \ell} \) with \( [V^{\otimes \ell}]^* \) where the identification is uniquely determined by

\[
\langle \varepsilon^1 \otimes \cdots \otimes \varepsilon^\ell, v_1 \otimes \cdots \otimes v_\ell \rangle = \varepsilon^1 (v_1) \cdots \varepsilon^\ell (v_\ell) \quad \forall \quad \{\varepsilon^i\}_{i=1}^{\ell} \subset V^* \text{ and } \{v_i\}_{i=1}^{\ell} \subset V.
\]

We also identify \( (V^*)^{\otimes \ell} \) with the space of multi-linear maps from \( V^\ell \rightarrow \mathbb{R} \) using,

\[
T (v_1, \ldots, v_\ell) = \langle T, v_1 \otimes \cdots \otimes v_\ell \rangle \quad \forall \quad T \in (V^*)^{\otimes \ell} \text{ and } \{v_i\}_{i=1}^{\ell} \subset V.
\]

Under these identification we have

\[
\langle \text{Sym}[T], W \rangle = \langle T, \text{Sym}[W] \rangle \quad \forall \quad T \in (V^*)^{\otimes \ell} \text{ and } W \in V^{\otimes \ell}
\]

and

\[
\text{Sym}[T] (v_1, \ldots, v_\ell) = \langle T, \text{Sym} (v_1 \otimes \cdots \otimes v_\ell) \rangle \quad \forall \quad T \in (V^*)^{\otimes \ell} \text{ and } \{v_i\}_{i=1}^{\ell} \subset V.
\]

Remark 55. If \( T \in [V^*]^{\otimes \ell} \) and \( v_1, \ldots, v_\ell \in V \), then

\[
\frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_\ell} \bigg|_{s_1 = \cdots = s_\ell = 0} T (s_1 v_1 + \cdots + s_\ell v_\ell)^{\otimes \ell} = \sum_{\sigma \in S_\ell} T (v_{\sigma(1)}, \ldots, v_{\sigma(\ell)})
\]
and therefore,
\[
\operatorname{Sym}[T](v_1, \ldots, v_\ell) := \frac{1}{\ell!} \frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_\ell} |_{s_1=\cdots=s_\ell=0} T\left((s_1 v_1 + \cdots + s_\ell v_\ell)^{\otimes \ell}\right).
\]

This formula shows that the symmetric part \(\operatorname{Sym}[T]\) of \(T\) is completely determined by the knowledge of \(T(v, v, \ldots, v)\) for all \(v \in V\).

**Definition 56.** Let \(\Sigma^{\ell} T_p^* M\) denote the symmetric tensors in \([T_p^* M]^{\otimes \ell}\) and for \(T \in [T_p^* M]^{\otimes \ell}\), let \(\operatorname{Sym}[T] \in \Sigma^{\ell} T_p^* M\) denote the symmetrization of \(T\) as above.

**Example 57.** If \(U\) is an open subset of \(M\) and \(f\) is \(\ell\)-times continuously differentiable on \(U\), then \(\operatorname{Sym}[\nabla^\ell f]\) defines a local section (over \(U\)) of \(\Sigma^{\ell} T^* M\). Moreover since \(v^{\otimes \ell}\) is symmetric for all \(v \in T_p M\) we may write (24) as

\[
\left. \frac{d^\ell}{dt^\ell} \right|_{t=0} f(\gamma_v(t)) = \left\langle \operatorname{Sym}\left[\nabla^\ell f\right], v^{\otimes \ell} \right\rangle, \text{ for all } v \in T_p M. \tag{27}
\]

**Theorem 58** (Taylor’s Theorem on \(M\)). Let \(\ell, n, p \in \mathbb{N}_0\), \(p \in M\), \(v \in T_p M\), \(\gamma_v(t) := \exp_p(tv)\), //\(\gamma_v) : T_{\gamma_v(0)}M \to T_{\gamma_v(t)}M\), \(W_0 \in T_{\gamma_v(0)} M^{\otimes \ell}\), and \(W_t := //\(\gamma_v) W_0\). If \(f\) is \((\ell + n + 1)\)-times continuously differentiable on \(U\), where \(U\) is an open set containing \(\gamma_v([0, 1])\), then

\[
\nabla^\ell_{W_1} f = \sum_{k=0}^n \frac{1}{k!} \nabla^{\ell+k}_{v^{\otimes k} \otimes W_0} f + \frac{1}{n!} \int_0^1 \left[ \nabla^{\ell+n+1}_{\gamma_v(t)^{(n+1)} \otimes W_1} f \right] \cdot (1-t)^n \, dt. \tag{28}
\]

When \(\ell = 0\) the previous equation is to be interpreted as (also see [DS15, Theorem 6.1])

\[
f(\exp_p(v)) = \sum_{k=0}^n \frac{1}{k!} \nabla^k_{v^{\otimes k}} f + \frac{1}{n!} \int_0^1 \left[ \nabla^{n+1}_{\gamma_v(t)^{(n+1)}} f \right] \cdot (1-t)^n \, dt \tag{29}
\]

\[
= \sum_{k=0}^n \frac{1}{k!} \left\langle \operatorname{Sym}\left[\nabla^k f\right], v^{\otimes k} \right\rangle + \frac{1}{n!} \int_0^1 \left\langle \operatorname{Sym}\left[\nabla^{n+1}_{\gamma_v(t)} f\right], \dot{\gamma}_v(t)^{(n+1)} \right\rangle (1-t)^n \, dt, \tag{30}
\]

where

\[
\frac{1}{0!} \left\langle \operatorname{Sym}\nabla^0_{p} f, v^{\otimes 0} \right\rangle := f(p).
\]

**Proof.** Let \(g(t) := \nabla^\ell_{W_t} f\) and recall that the standard Taylor’s theorem with remainder states;

\[
g(1) = \sum_{k=0}^n \frac{1}{n!} g^{(k)}(0) + \frac{1}{n!} \int_0^1 g^{(n+1)}(t) (1-t)^n \, dt.
\]

The results now follow by using Lemma 53 in order to compute the \(g^{(k)}(t)\) for \(1 \leq k \leq n + 1\). \(\square\)

Theorem 58 has the following immediate corollaries.
Corollary 59. Moreover,
\[
f(\exp_p(v)) = \sum_{k=0}^{n+1} \frac{1}{k!} \langle \Sym \nabla^k f, v^{\otimes k} \rangle + o_f\left(|v|^{n+1}\right) \tag{31}
\]
where \( o_f\left(|v|^{n+1}\right) \leq \varepsilon(v) |v|^{n+1} \) and \( \varepsilon(v) \to 0 \) as \( |v| \to 0 \).

Proof. According to (30), (31) holds with
\[
o_f\left(|v|^{n+1}\right) = \frac{1}{n!} \int_0^1 \left\langle g(v, t), v^{\otimes (n+1)} \right\rangle (1-t)^n \, dt
\]
where
\[
g(v, t) := \Sym \left[ \nabla_{\gamma_v(t)}^{n+1} f \right] / \gamma_v^{\otimes (n+1)} - \Sym \left[ \nabla_{\gamma_v(0)}^{n+1} f \right] \to 0 \text{ as } v \to 0.
\]
Therefore \( o_f\left(|v|^{n+1}\right) \leq \varepsilon(v) |v|^{n+1} \) where
\[
\varepsilon(v) = \frac{1}{n!} \int_0^1 \|g(v, t)\|(1-t)^n \, dt \to 0 \text{ as } |v| \to 0.
\]

Remark 60. Since parallel translation is isometric it follows (continuing the notation in Theorem 58) that
\[
\left| \nabla_{\gamma_v(t)}^{\ell+n+1} f \right| \leq \left\| \nabla^{\ell+n+1} f \right\|_{[T_{\gamma_v(t)}^* M]^{\otimes (\ell+n+1)}} \|v|^{n+1} \|W_0\|
\]
and hence
\[
\left| \nabla^{\ell} f - \sum_{k=0}^{n} \frac{1}{k!} \nabla^{\ell+k} v^{\otimes k} W_0 f \right|
\leq \frac{1}{(n+1)!} \|W_0\| \cdot \max_{0 \leq \ell \leq 1} \left\| \nabla^{\ell+n+1} f \right\|_{[T_{\gamma_v(0)}^* M]^{\otimes (\ell+n+1)}} \cdot d(p, \exp_p(v))^{n+1}. \tag{32}
\]

Since \( M \) is a compact Riemannian manifold it is necessarily complete and therefore, by the Hopf–Rinow theorem, for each \( q \in M \) we may find at least one \( v \in T_p M \) such that \( q = \exp_p(v) \) and \( d(q, p) = |v| \). Using these remarks we can reformulate (29) as follows.

Corollary 61. If \( f \) is \((n+1)\)-times continuously differentiable on \( M \), \( p, q \in M \), and \( v \in T_p M \) is chosen so that \( q = \exp_p(v) \) and \( d(q, p) = |v| \), then
\[
f(q) = \sum_{k=0}^{n} \frac{1}{k!} \langle \Sym \nabla^k f, v^{\otimes k} \rangle + O_f\left(d(p, q)^{n+1}\right) \tag{33}
\]
where
\[
\left| O_f\left(d(p, q)^{n+1}\right) \right| \leq \frac{1}{(n+1)!} \max_{m \in M} \|\Sym \nabla^{n+1} f\| d(p, q)^{n+1}.
\]
Furthermore if \( f \) is \( n \)-times continuously differentiable on \( M \) then
\[
f(q) = \sum_{k=0}^{n} \frac{1}{k!} \langle \Sym \nabla^k f, v^{\otimes k} \rangle + o_f\left(d(p, q)^n\right). \]
Definition 62 (Taylor approximations). Suppose that $U \subset M$ is an open subset of $M$, $p \in U$, $f$ a $n$-times continuously differentiable function on $M$ and $\varepsilon > 0$ is sufficiently small so that $B_p(\varepsilon) \subset U$ and $\varepsilon$ is smaller than the injectivity radius of $M$. We then define, $\text{Tay}^n_p f \in C^\infty (B_p(\varepsilon))$ by

$$(\text{Tay}^n_p f)(q) := \sum_{k=0}^n \frac{1}{k!} \left< \text{Sym} \nabla^k|_pf, [\exp_{p}^{-1}(q)]^{\otimes k} \right>.$$ (34)

With this notation we have the following local version of Corollary 61.

Corollary 63. If $f$ is a $n$-times continuously differentiable function on $M$, $p, q \in M$ with $d(p, q)$ smaller than the injectivity radius of $M$, then

$$f(q) = (\text{Tay}^n_p f)(q) + o(d(p, q)^n).$$ (35)

Remark 64. In the case $M = \mathbb{R}^d$ and $f$ is a polynomial of degree at most $n$, it follows by Taylor’s theorem that $f = \text{Tay}^n_p f$ for all $p \in \mathbb{R}^d$. So in the flat case the error term in (35) is no longer present.

Lemma 65. If $f$ is a $n$-times continuously differentiable function on $M$ and $f(q) = o(d(p, q)^n)$, then $(V^f)(p) = 0$ for any $n$th - order differential operator $V$ and in particular, $\nabla^k|_pf = 0$ for all $0 \leq k \leq n$.

Proof. Let $(\Psi, U)$ be a chart on with $p \in U$ and $\Psi(p) = 0$ and define $F := f \circ \Psi^{-1} \in C^n(\tilde{U} := \Psi(U))$. Then the give assumption implies $F(x) = o(|x|^n)$ and therefore for any $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$ small we have $F(tx) = o(t^n)$ from which it easily follows that

$$0 = \frac{d^k}{dt^k} F(tx) |_{t=0} = \left< \left( D^k F \right)(0), x^\otimes k \right>$$ for all $0 \leq k \leq n$.

As $(D^k F)(0)$ is symmetric and $x \in \mathbb{R}^d$ was arbitrary we may conclude that $(D^k F)(0) = 0 \in (\mathbb{R}^d)^{\otimes k}$ for $0 \leq k \leq n$. As any $n$th - order differential operator $U$ on $C^n(M)$ may be written locally as

$$V f = \sum_{k=0}^n \left< \left( D^k F \right)(\Psi), W_k \right>$$

for some smooth functions, $W_k : U \to (\mathbb{R}^d)^{\otimes k}$ for each $0 \leq k \leq n$, it follows that

$$(V^f)(p) = \sum_{k=0}^n \left< \left( D^k F \right)(\Psi(p)), W_k(p) \right> = \sum_{k=0}^n \left< \left( D^k F \right)(0), W_k(p) \right> = 0.$$ (35)

Corollary 66. If $f$ a $n$-times continuously differentiable function on $M$ and $V$ is an $n$th - order differential operator, then

$$(V^f)(p) = \left[ V \left( \text{Tay}^n_p f \right) \right](p)$$

and in particular,

$$\nabla^n|_pf = \left[ \nabla^n|_p \left( \text{Tay}^n_p f \right) \right](p)$$

from which it follows that $\nabla^n|_pf$ is a linear combination of $\{\text{Sym} \nabla^k|_pf\}_{k=0}^n$. 

\[46\]
We will make the last assertion of Corollary 66 more explicitly in Corollary 72 and Remark 74. The upshot is that there is no loss of information in only keeping track of the symmetrizations of the covariant derivatives.

**Corollary 67.** If \( f \in C^\infty(M) \), \( p, q \in M \) with \( d(p, q) \) then for \( 0 \leq k \leq n \) we have

\[
\left\| \nabla^k_q [f - (\text{Tay}_p^n f)] \right\| \leq \frac{1}{(n + 1 - k)!} \max_{0 \leq t \leq 1} \left\| \nabla^{n+1}|_{\gamma_t}(f - \text{Tay}_p^n f) \right\| \cdot d(p, q)^{n+1-k},
\]

where \( v := \exp_p^{-1}(q) \).

**Proof.** Let us apply the estimate in (32) with \( f \) replaced by \( g := f - \text{Tay}_p^n f \) keeping in mind that \( \nabla^k|_p [f - (\text{Tay}_p^n f)] = 0 \) for \( 0 \leq k \leq n \) by Corollary 66. This allows us to conclude for \( W_0 \in T_pM^\otimes k \) that

\[
\left\| \nabla^k_{W_1} [f - \text{Tay}_p^n f] \right\| \leq \frac{1}{(n + 1 - k)!} \left\| W_0 \right\| \cdot \max_{0 \leq t \leq 1} \left\| \nabla^{n+1}g \right\| \left( T_{\gamma(t)}^* M \right)^{\otimes (n+1)} \cdot d(p, q)^{n+1-k}
\]

where \( v := \exp_p^{-1}(q) \). As the map \( W_0 \to W_1 \) is an isometry it follows that

\[
\left\| \nabla^k_q [f - (\text{Tay}_p^n f)] \right\| \leq \frac{1}{(n + 1 - k)!} \max_{0 \leq t \leq 1} \left\| \nabla^{n+1}g \right\| \left( T_{\gamma(t)}^* M \right)^{\otimes (n+1)} \cdot d(p, q)^{n+1-k}
\]

\[ \square \]

**9.1.1 Symmetric parts of covariant derivatives determine all derivatives**

We will now make Corollary 66 more precise.

**Definition 68.** If \( (x, U := \text{dom}(x)) \) is a chart on \( M \), let \( D^x \) denote the flat covariant derivative on \(TU \) determined by \( D^x \frac{\partial}{\partial x^j} = 0 \) for \( 1 \leq j \leq d \).

**Remark 69.** If \( V = \sum_{j=1}^d V_j \frac{\partial}{\partial x^j} \) is a vector field on \( U \) and \( v \in T_m M \) then \( D^x_v V = \sum_{j=1}^d (v V_j) \frac{\partial}{\partial x^j}|_m \). Using \( D^x \frac{\partial}{\partial x^j} = 0 \), it easily follows that for all \( \ell \in \mathbb{N} \) and \( f \) \( \ell \)-times continuously differentiable we have

\[
\left\langle (D^x)^\ell \left\| m f, \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_\ell}} \right\| \right\rangle = \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_\ell}} \right\| \right\rangle f
\]

and in particular \( (D^x)^\ell f \in \Sigma^\ell TU^\ast U \).

**Lemma 70.** Suppose that \( (x, U := \text{dom}(x)) \) is a chart on \( M \), \( D = D^x \) is the covariant derivative of Notation 68. Then, there exists a family of sections \( Q_\ell,n \in \Gamma \left[ \text{Hom} \left( TU^{\otimes n}, TU^{\otimes \ell} \right) \right] \) for \( 1 \leq \ell \leq n \), such that \( Q_{n,n} = I \) and for all \( n \)-times continuously differentiable functions \( f \),

\[
\left\langle \nabla^n p f, W \right\rangle = \sum_{\ell=1}^n \left\langle D^\ell p f, Q_{\ell,n} W \right\rangle \quad \forall W \in [T_p M]^{\otimes n}.
\]

(36)
Proof. Let \( D = D^x \) and \( \Gamma \) be the \( \text{End}(TU) \) – valued connection one form on \( TU \) so that \( \nabla = D + \Gamma \). It is enough to verify that (36) holds on a basis for \( T_pU \otimes^n \). To this end, let \( i_j \in \{1, 2, \ldots, d\} \), for \( 1 \leq j \leq n \) and let \( V_j = \frac{\partial}{\partial x^j} \). Then,
\[
\langle \nabla f, V_1 \rangle = V_1 f = \langle D f, V_1 \rangle ,
\]
which shows that (36) holds for \( n = 1 \). For the sake of completing the proof by induction, let us now assume that (36) holds at level \( n - 1 \) and below. In particular we assume
\[
\langle \nabla^n f, V_{n-1} \otimes \cdots \otimes V_1 \rangle = \sum_{\ell=1}^{n-1} \langle D^\ell f, Q_{\ell,n-1} V_{n-1} \otimes \cdots \otimes V_1 \rangle .
\]
On one hand,
\[
V_n \langle \nabla^{n-1} f, V_{n-1} \otimes \cdots \otimes V_1 \rangle = \langle \nabla^n f, V_n \otimes \cdots \otimes V_1 \rangle + \langle \nabla^{n-1} f, \nabla V_n [V_{n-1} \otimes \cdots \otimes V_1] \rangle = \langle \nabla^n f, \nabla [V_{n-1} \otimes \cdots \otimes V_1] \rangle + \sum_{k=1}^{n-1} \langle \nabla^{n-1} f, V_{n-1} \otimes \cdots [\nabla (V_n) V_k] \otimes \cdots \otimes V_1 \rangle ,
\]
while on the other hand (using the induction hypothesis, the product rule, and \( DV_k = 0 \) for all \( k \)),
\[
V_n \langle \nabla^{n-1} f, V_{n-1} \otimes \cdots \otimes V_1 \rangle = \sum_{\ell=1}^{n-1} V_n \langle D^\ell f, Q_{\ell,n-1} V_{n-1} \otimes \cdots \otimes V_1 \rangle = \sum_{\ell=1}^{n-1} \langle D^{\ell+1} f, V_n \otimes Q_{\ell,n-1} V_{n-1} \otimes \cdots \otimes V_1 \rangle + \sum_{\ell=1}^{n-1} \langle D^\ell f, (D V_n Q_{\ell,n-1}) V_{n-1} \otimes \cdots \otimes V_1 \rangle = \langle D^n f, V_n \otimes \cdots \otimes V_1 \rangle + \sum_{\ell=1}^{n-2} \langle D^\ell f, V_n \otimes Q_{\ell,n-1} V_{n-1} \otimes \cdots \otimes V_1 \rangle + \sum_{\ell=1}^{n-1} \langle D^\ell f, (D V_n Q_{\ell,n-1}) V_{n-1} \otimes \cdots \otimes V_1 \rangle .
\]
Comparing the last two displayed equations shows,
\[
\nabla^n_{V_n \otimes \cdots \otimes V_1} f = \langle D^n f, V_n \otimes \cdots \otimes V_1 \rangle + \langle D f, [D V_n Q_{1,n-1}] V_{n-1} \otimes \cdots \otimes V_1 \rangle + \sum_{\ell=2}^{n-1} \langle D^\ell f, V_n \otimes Q_{\ell-1,n-1} V_{n-1} \otimes \cdots \otimes V_1 \rangle + \sum_{\ell=1}^{n-1} \langle D^\ell f, (D V_n Q_{\ell,n-1}) V_{n-1} \otimes \cdots \otimes V_1 \rangle + \sum_{k=1}^{n-1} \langle \nabla^{n-1} f, V_{n-1} \otimes \cdots [\nabla (V_n) V_k] \otimes \cdots \otimes V_1 \rangle .
\]
From this expression it follows that \( \nabla^n_{V_n \otimes \cdots \otimes V_1} f \) may be expressed in the form claimed in (36). \( \square \)
Corollary 71. Let us continue the notation in Lemma 70. Then, there exists

\[ \tilde{Q}_{\ell,n} \in \Gamma \left[ \text{Hom} \left( TU^{\otimes n}, TU^{\otimes \ell} \right) \right], \quad \text{for } 1 \leq \ell \leq n, \]

such that \( \tilde{Q}_{n,n} = I \) and for all \( n \)-times continuously differentiable functions \( f \),

\[ \langle D^n f, W \rangle = \sum_{\ell=1}^{n} \left\langle \text{Sym} \nabla^\ell f, \tilde{Q}_{\ell,n} W \right\rangle, \quad \forall \ W \in TM^{\otimes n}. \] (37)

Proof. The proof is again by induction on \( n \). For \( n = 1 \), we have \( D_W f = W f = \nabla_W f \), so there is nothing to prove. For the inductive step, suppose that (37) holds at level \( n - 1 \) and below. From (36) with \( W \) replaced by \( \text{Sym}_n W \), it follows that,

\[ \langle \text{Sym} [\nabla^n f], W \rangle = \langle \nabla^n f, \text{Sym}_n W \rangle = \sum_{\ell=1}^{n} \left\langle D^\ell f, Q_{\ell,n} \text{Sym}_n W \right\rangle \]

\[ = \langle D^n f, W \rangle + \sum_{\ell=1}^{n-1} \left\langle D^\ell f, Q_{\ell,n} \text{Sym}_n W \right\rangle, \]

wherein the last equality we have used that \( D^n f \) is already symmetric. From the previous equation along with the inductive hypothesis, we conclude that \( \langle D^n f, W \rangle \) may be expressed as described in (37).

Corollary 72. If \( \nabla \) is a covariant derivative on \( TM \), then there exists

\[ Q^\nabla_{\ell,n} \in \Gamma \left[ \text{Hom} \left( TM^{\otimes n}, TM^{\otimes \ell} \right) \right], \quad \text{for } 1 \leq \ell \leq n, \]

such that \( Q^\nabla_{n,n} = I \) and for all \( n \)-times continuously differentiable functions \( f \),

\[ \langle \nabla^n f, W \rangle = \sum_{\ell=1}^{n} \left\langle \text{Sym} \nabla^\ell f, Q^\nabla_{\ell,n} W \right\rangle, \quad \forall \ W \in TM^{\otimes n}. \] (38)

Proof. First suppose that \( M = U \), as in Lemma 70. Then combining the results of Lemma 70 and Corollary 71, there exists \( Q^\ell_{\ell,n} \in \Gamma \left[ \text{Hom} \left( TU^{\otimes n}, TU^{\otimes \ell} \right) \right] \) such that (38) holds for all \( W \in TU^{\otimes M} \).

Let \( \{x_\alpha\}_{\alpha=1}^N \) be a collection of charts on \( M \) such that \( \{\text{dom}(x_\alpha)\}_{\alpha=1}^N \) is an open cover of \( M \) and \( \{\psi_\alpha\}_{\alpha=1}^N \) be a partition of unity relative to this cover. To complete the proof we define

\[ Q^\nabla_{\ell,n} := \sum_{\alpha=1}^{N} \psi_\alpha Q^\nabla_{\ell,n}. \]

We note the following corollary for completeness.
Corollary 73. If $\nabla$ is a covariant derivative on $TM$ on $L$ is a linear $n^{th}$-order differential operator on $C^\infty(M)$, then there exists smooth sections, $W_\ell \in \Gamma\left(\Sigma^\ell TM\right)$ for $0 \leq \ell \leq n$ such that
\[
Lf = \sum_{\ell=0}^{n} \nabla^\ell_W f \text{ for all } f \in C^\infty(M).  \tag{39}
\]

Proof. By definition $Lf$ is locally given by $Lf = \sum_{\ell=0}^{n} \langle D_n f, A_n \rangle$ for some $A_n \in \Gamma\left(\Sigma^\ell TU\right)$. Using Corollaries 71 and 72, we may locally express $Lf$ as in (39). The global picture may then be constructed using a partition of unity argument. \qed

Remark 74. Our proof of Corollary 72 was local in nature and hence does not give much information about how the $Q_{\ell,n}^\nabla$ depend on $\nabla$. It is possible to give a global proof of Corollary 72 which would show that $Q_{\ell,n}^\nabla$ may be constructed from certain combinations of covariant derivatives of the torsion and curvature tensor of $\nabla$. Here is a sketch of this argument. In this sketch we let $v \wedge w := v \otimes w - w \otimes v$ for any $v, w \in T_p M$.

1. If $v_1, \ldots, v_n \in T_p M$ and $1 \leq i < n$, then
\[
\nabla^n_{v_0 \otimes \cdots \otimes v_{i-1} \otimes v_i} f = \left( \nabla^{n-i-1}_{v_0 \otimes \cdots \otimes v_{i+1} - \otimes v_i} \left[ R(\cdot, \cdot) \nabla^{i-1} f \right], v_{i+1} \otimes v_1 \otimes \cdots \otimes v_1 \right) + \left( \nabla^{n-i-1}_{v_0 \otimes \cdots \otimes v_{i+2} - \otimes v_i} \left[ \nabla T(\cdot, \cdot) \nabla^{i-1} f \right], v_{i+1} \otimes v_1 \otimes \cdots \otimes v_1 \right)
\]
where $R(\cdot, \cdot) \nabla^{i-1} f$ is the appropriate action of the curvature tensor of $\nabla$ on $\nabla^{i-1} f$ and $T$ is the torsion tensor of $\nabla$.

2. As a consequence of item 1. and the fact that every permutation is a composition of transpositions, it follows that for any permutation $\sigma \in S_n$,
\[
\nabla^n_{v_{\sigma(n)} \otimes \cdots \otimes v_{\sigma(1)}} f = \nabla^n_{v_0 \otimes \cdots \otimes v_1} f + \sum_{\ell=1}^{n-1} \left\langle \nabla^\ell f, Q(\sigma)_{\ell,n} v_0 \otimes \cdots \otimes v_1 \right\rangle, \tag{40}
\]
where $Q(\sigma)_{\ell,n} \in \Gamma\left[\text{Hom}\left(TM^\otimes n, TM^\otimes \ell\right)\right]$ are constructed from certain combinations of covariant derivatives of the torsion and curvature tensor of $\nabla$.

3. Summing (40) on $\sigma$ and then dividing by $n!$ and setting
\[
Q_{\ell,n} = \frac{1}{n!} \sum_{\sigma \in S_n} Q(\sigma)_{\ell,n}
\]
shows
\[
\left\langle \text{Sym} \nabla^n f, v_n \otimes \cdots \otimes v_1 \right\rangle = \left\langle \nabla^n f, v_n \otimes \cdots \otimes v_1 \right\rangle + \sum_{\ell=1}^{n-1} \left\langle \nabla^\ell f, Q_{\ell,n} v_n \otimes \cdots \otimes v_1 \right\rangle, \tag{41}
\]
where the $Q_{\ell,n} \in \Gamma\left[\text{Hom}\left(TM^\otimes n, TM^\otimes \ell\right)\right]$ are constructed from certain combinations of covariant derivatives of the torsion and curvature tensor of $\nabla$. 50
4. Using (41) recursively then shows there exists $Q_{\ell,n}^\nabla \in \Gamma \left[ \text{Hom} \left[ TM^{\otimes n}, TM^{\otimes \ell} \right] \right]$ such that

\[
\langle \nabla^n f, v_n \otimes \cdots \otimes v_1 \rangle = \langle \text{Sym} \nabla^n f, v_n \otimes \cdots \otimes v_1 \rangle + \sum_{\ell=1}^{n-1} \left( \langle \text{Sym} \nabla^\ell f, Q_{\ell,n}^\nabla v_n \otimes \cdots \otimes v_1 \rangle \right),
\]

where each $Q_{\ell,n}^\nabla$ is constructed from certain combinations of covariant derivatives of the torsion and curvature tensor of $\nabla$.

9.2 The regularity structure and model

We are now ready to set up the regularity structure for “polynomials” up to order $n$ on a manifold.

**Definition 75.** Fix $n \geq 0$ and let $\mathcal{T} = \bigoplus_{\ell=0}^n \Sigma^\ell T^*_p M$ be the vector bundle over $M$ with fiber at $p \in M$ given by

\[
\mathcal{T}|_p := \bigoplus_{\ell=0}^n \Sigma^\ell T^*_p M.
\]

(42)

For $p \in M$ and $z$ near $p$, let

\[
G^n_p (z) := \sum_{\ell=0}^n \frac{1}{\ell!} \left[ \exp^{-1}_p(z) \right]^{\otimes \ell} \in \bigoplus_{\ell=0}^n \Sigma^\ell T^*_p M.
\]

(43)

The vector bundle $\mathcal{T}$ will be used to store higher order derivatives of functions. On flat space $\mathbb{R}^d$ such “abstract Taylor expansions” were realized as honest functions using polynomials, see (23). Polynomials are the simplest function that have specified derivatives at one point. On the manifold we instead choose polynomials in exponential coordinates.

**Definition 76** (Model). For $\tau = (\tau_0, \ldots, \tau_n) \in \mathcal{T}_p$ define

\[
(\Pi_p \tau)(z) := \langle \tau, G^n_p(z) \rangle = \left\langle \tau, \sum_{\ell=0}^n \frac{1}{\ell!} \left[ \exp^{-1}_p(z) \right]^{\otimes \ell} \right\rangle
\]

\[
= \sum_{\ell=0}^n \frac{1}{\ell!} \tau_\ell \left( \left[ \exp^{-1}_p(z) \right]^{\otimes \ell} \right).
\]

These local “Taylor polynomials” are a good substitute for the usual Taylor polynomials in the flat space theory, as Lemma 77 and Corollary 78 below demonstrate.

**Lemma 77.** Let $A = A_0 + A_1 + \cdots + A_n \in \mathcal{T}_p$, with $A_\ell \in \Sigma^\ell T^*_p M$, for $\ell = 0, \ldots, n$, and define

\[
\varphi(q) := \langle A, G^n_p(q) \rangle = \sum_{\ell=0}^n \frac{1}{\ell!} A_\ell (\exp^{-1}_p(q)^{\otimes \ell}).
\]

Then,

\[
\varphi(p) = A_0 \quad \text{and} \quad \text{Sym}[(\nabla)^\ell \varphi]|_p = A_\ell, \quad \forall \ \ell = 1, \ldots, n.
\]
Proof. Let $\gamma_v(t) = \exp_p(tv)$. Then,

$$
\varphi(\gamma_v(t)) = \sum_{\ell=0}^n \frac{1}{\ell!} A_\ell(\exp_p^{-1}(\gamma_v(t) \otimes \ell)) = \sum_{\ell=0}^n t^\ell \ell! A_\ell \left( v \otimes \ell \right)
$$

and hence by Lemma 53

$$
\nabla_{v \otimes \ell} \varphi = \left. \frac{d^\ell}{dt^\ell} \right|_{t=0} \varphi(\gamma_v(t)) = A_\ell \left( v \otimes \ell \right),
$$

which suffices to complete the proof by Remark 55.

We now have the immediate corollary of this lemma.

**Corollary 78.** Let $\tau \in \Sigma^T_p M$. Then, for $i = 0, \ldots, n$,

$$
\operatorname{Sym}[\nabla^i_p \Pi_{p} \tau] = \begin{cases} 
\tau, & i = \ell \\
0, & \text{else}
\end{cases}
$$

**Remark 79.** Let $x$ be exponential coordinates around $p \in M$, i.e. suppose that $x = (x^1, \ldots, x^d)$ where $\{x^i(q)\}^d_{i=1}$ are the coordinates of $\exp_p^{-1}(q)$ relative to some basis $\{u_i\}^d_{i=1}$ of $T_p M$. Then with $v = \sum^d_{i=1} v^i u_i \in T_p M$,

$$
\sum_{i_1 \ldots i_\ell=1}^d \partial_{i_1 \ldots i_\ell} f(p) v^{i_1} \ldots v^{i_\ell} = \frac{d^\ell}{dt^\ell} |_{t=0} f(\exp_p(tv)) = \left< \nabla^\ell_p f, v \otimes \ell \right>
$$

which makes sense for $d(p, q) < \delta$, the radius of injectivity of $M$.

**Remark 81.** For $n \geq 2$ this transport will in general also go “upwards.” That is, if $\tau \in \mathcal{T}_\alpha|y$ some $\alpha < n$, then in general $\Gamma_{x \leftarrow y} \tau$ will have components in homogeneities strictly larger than $\alpha$. This is not allowed in the original formulation of a regularity structure by Hairer [Hai14, Definition 2.1]. As we have seen in the main text, this poses no problem, since our modified definition of a model (Definition 14) allows for it. We moreover believe that any transport that wants to achieve the following lemma for a “polynomial model” is forced to do this.
The definitions have been arranged so that $\Pi_q^\tau$ and $\Pi_p^\tau \Gamma_{p-q}^\tau$ agree at $p$ to order $n$, i.e. $\Pi_q^\tau$ and $\Pi_p^\tau \Gamma_{p-q}^\tau$ along with all derivatives up to order $n$ agree at $p$.

**Lemma 82.** Let $\tau \in \mathcal{T}|_q$ and $p, z \in U$ where $U$ is a sufficiently small neighborhood of $q$. If $V$ is a differential operator of order $k \leq n$ defined on $U$, then

$$|V \left[\Pi_q^\tau - \Pi_p^\tau \Gamma_{p-q}^\tau\right](z)\right| \lesssim_{V} |\tau| d(z, p)^{n+1-k}.$$

**Proof.** Let

$$g(z) = \langle \Pi_q^\tau \rangle(z) = \langle \tau, G_q^n(z) \rangle$$

so that

$$\langle \Pi_p^\tau \Gamma_{p-q}^\tau \rangle(z) = \left(\text{Tay}_p^n\right)(z).$$

Using Corollary 67, we have the estimate,

$$\left\|\nabla^k |_{\gamma_v(t)}(g - \text{Tay}_p^n g)\right\| \leq \frac{1}{(n+1-k)!} \max_{0 \leq t \leq 1} \left\|\nabla^{n+1}|_{\gamma_v(t)}(g - \text{Tay}_p^n g)\right\| \cdot d(p, z)^{n+1-k},$$

where $v := \exp_p^{-1}(z)$. For $d(z, p) < \varepsilon$ and $0 \leq t \leq 1$, let $[p, z]_t := \exp(t \exp_p^{-1}(z))$ so that $t \rightarrow [p, z]_t$ is the geodesic joining $p$ to $z$ parametrized by $[0, 1]$. Then we have

$$\max_{0 \leq t \leq 1} \left\|\nabla^{n+1}|_{\gamma_v(t)}(g - \text{Tay}_p^n g)\right\| \leq \max_{w : d(w, p) \leq d(p, z)} \left\|\nabla^{n+1}|_{w}(g - \text{Tay}_p^n g)\right\|$$

and so we have

$$\left\|\nabla^k |_{\gamma_v(t)}(g - \text{Tay}_p^n g)\right\| \leq \frac{1}{(n+1-k)!} \max_{w : d(w, p) \leq d(p, z)} \left\|\nabla^{n+1}|_{w}(g - \text{Tay}_p^n g)\right\| \cdot d(p, z)^{n+1-k}.$$

For the proof of the first half of Theorem 90 below, it is convenient to introduce the notion of a parallelism on a vector bundle, $E$, over $M$.

**Definition 83 (Diagonal domains).** Let $\mathcal{U}$ be an open set on $M$. An open set $\mathcal{D}_{\mathcal{U}} \subseteq M \times M$ is a $\mathcal{U}$-diagonal domain if it contains the diagonal of $\mathcal{U}$, that is $\Delta_{\mathcal{U}} := \bigcup_{p \in \mathcal{U}} (p, p) \subseteq \mathcal{D}_{\mathcal{U}}$. A local diagonal domain is a $\mathcal{V}$-diagonal domain for some nonempty open $\mathcal{V} \subseteq M$.

If $\mathcal{U} = M$, we write $\mathcal{D} := \mathcal{D}_{\mathcal{M}}$ and refer to $\mathcal{D}$ simply as a diagonal domain.

**Definition 84 (Parallelisms).** Let $E$ be a vector bundle over $M$ and $\text{Hom}(E) \rightarrow M \times M$ be the associated vector bundle over $M \times M$ with fibers, $\text{Hom}_{(q,p)}(E) := L(E_p, E_q)$ for $(q, p) \in M \times M$, where $L(E_p, E_q)$ denote the set of all linear transformations from $E_p$ to $E_q$. A smooth local section $U \in \Gamma(\text{Hom}(E))$ with domain $\mathcal{D}$ (i.e. $U(p, q) \in L(E_p, E_q)$ for all $(q, p) \in \mathcal{D}$) is called a parallelism if $U(p, p) = I_p$. If $U$ is only defined on a local diagonal domain, we refer to $U$ as a local parallelism.
Example 85 (Parallel translation and parallelisms). One natural example of a parallelism when \((M,g)\) is a Riemannian manifold and \(E\) is equipped with a covariant derivative, \(\nabla^E\), is to define

\[
U^\nabla(q,p) := \left\| \frac{d}{dt} \left( t \to \exp_p \left( t \exp_p^{-1}(q) \right) \right) \right\|,
\]

where \(p,q \in M\) are “close enough” so there is a unique vector \(v_p\) with minimum length such that \(q = \exp^\nabla_p(v_p)\) and \(\left\| \frac{d}{dt} \right\|\) denotes the parallel translation operator on \(E\) relative to \(\nabla^E\). For our purposes below \(E\) will be a bundle associated to \(TM\) and \(\nabla^E\) will be the induced connection on this bundle associated to the Levi-Civita covariant derivative on \((M,g)\).

Example 86 (Charts and parallelisms). Each chart \((\Psi, U)\) induces a local parallelism on \((T^*M)^{\otimes \ell}\) for any \(\ell \in \mathbb{N}\) as follows. If \(A \in (T^*_p M)^{\otimes \ell}\) is expressed as

\[
A = \sum_{i_1, \ldots, i_{\ell}=1}^d A_{i_1, \ldots, i_{\ell}} d\Psi^{i_1}|_p \otimes \cdots \otimes d\Psi^{i_\ell}|_p,
\]

then we define \(U^\Psi(q,p) A \in T^*_q M^{\otimes \ell}\) by

\[
U^\Psi(q,p) A = \sum_{i_1, \ldots, i_{\ell}=1}^d A_{i_1, \ldots, i_{\ell}} d\Psi^{i_1}|_q \otimes \cdots \otimes d\Psi^{i_\ell}|_q.
\]

In other words, \(U^\Psi(q,p)\) is uniquely determined by requiring

\[
\left\langle U^\Psi(q,p) A, \frac{\partial}{\partial \Psi^{i_1}}|_p \otimes \cdots \otimes \frac{\partial}{\partial \Psi^{i_\ell}}|_p \right\rangle = \left\langle A, \frac{\partial}{\partial \Psi^{i_1}}|_p \otimes \cdots \otimes \frac{\partial}{\partial \Psi^{i_\ell}}|_p \right\rangle
\]

for all \(q \in U\) and \(1 \leq i_1, i_2, \ldots, i_{\ell} \leq d\). [This example is basically a special case of Example 85 where one takes \(\nabla\) to be the flat connection, \(D^\Psi\), defined in Notation 68.]

With the aid of a parallelism, we can now define the notion of \(\gamma\) – Hölder section, \(S\), on \(E\). In what follows we assume that \(E\) is equipped with a smoothly varying inner product, \(\langle \cdot, \cdot \rangle_E\). We do not necessarily assume that \(\nabla^E\) is compatible with \(\langle \cdot, \cdot \rangle_E\) or that \(U(p,q)\) is unitary for all \((p,q) \in D\).

Lemma 87. Let \(S\) be a continuous section of a vector bundle \(E\). Let \((U, D), (U', D)\) be parallelisms on \(E\). Then for every compactum \(K \subset D\)

\[
||U(q,p)S(p) - S(q)|| \leq CK \left(||U'(q,p)S(p) - S(q)|| + d(p,q)\right), \forall p, q \in K.
\]

Proof. We work in a local trivialization. Let \(U, U' : \mathbb{R}^d \times \mathbb{R}^d \to GL(\mathbb{R}^N)\) be smooth functions such that \(U(x,x), U'(x,x) = I\), which we view to be a parallelism on the trivial bundle, \(\mathbb{R}^d \times \mathbb{R}^N\) over \(\mathbb{R}^d\). A continuous section of this bundle may be identified with a continuous function, \(S : \mathbb{R}^d \to \mathbb{R}^N\). Then

\[
||U(x,y)S(y) - S(x)|| \leq \|U(x,y) - U'(x,y)\| S(y)|| + ||U'(x,y)S(y) - S(x)||. \]

The statement then follows from smoothness of \(U, U'\), the fact that they coincide at \(x, x\) and local boundedness of \(S\). \(\Box\)
Lemma 88. Let \( f \in C(M) \), \( \gamma > 0 \) and \( n = \lfloor \gamma \rfloor \in \mathbb{N}_0 \). Then \( f \in C^\gamma(M) \) (as in Definition 10) iff \( f \) a \( n \)-times continuously differentiable function on \( M \) and for any (local) parallelisms \( (U) \) on the vector bundle \( \Sigma^n T^* M \), \( \text{Sym}[\nabla^n f] \) satisfies
\[
|U(q,p)\text{Sym}[\nabla^n|_p f] - \text{Sym}[\nabla^n|_q f]| \lesssim d(q,p)^{\gamma-n}, \tag{44}
\]
where \( \nabla \) is the Levi-Civita covariant derivative.

Proof. Recall from Definition 10, that \( f \in C(M) \) is in \( C^\gamma(M) \) iff \( f \circ \Psi^{-1} \in C^\gamma(\Psi(U)) \) for every coordinate chart \( (\Psi, U) \). These conditions are equivalent to \( f \) being \( n \)-times continuously differentiable and the \( n \)-th—derivatives of \( f \circ \Psi^{-1} \) are locally \( (\gamma-n) \)-Hölder on \( \Psi(U) \). The latter condition may be expressed as saying
\[
|U(q,p) D^n|_p f - D^n|_q f| \lesssim d(q,p)^{\gamma-n}, \tag{45}
\]
where \( D = D^\Psi \) is the flat connection defined in Notation 68. From Lemma 70 and Corollary 71 we may express
\[
D^n f = \text{Sym}[\nabla^n f] + Lf \tag{46}
\]
where \( L \) is a linear differential operator of order at most \( n-1 \). As \( Lf \) is continuously differentiable it follows that
\[
(q,p) \to U^\Psi(q,p)(Lf)_p - (Lf)_q
\]
is continuously differentiable and vanishes at \( q = p \) and therefore (by the fundamental theorem of calculus)
\[
\left|U^\Psi(q,p)(Lf)_p - (Lf)_q\right| \lesssim d(q,p). \tag{47}
\]
From (46) and (47) it follows that (45) is equivalent to
\[
|U^\Psi(q,p)\text{Sym}[\nabla^n|_p f] - \text{Sym}[\nabla^n|_q f]| \lesssim d(q,p)^{\gamma-n}. \tag{48}
\]
Lastly using Lemma 87 we conclude that the estimates in (48) and (44) are also equivalent. \( \square \)

Theorem 89. Fix \( n \in \mathbb{N} \) and construct \( T \) and \( (\Pi, \Gamma) \) as above. Then \( T \) is a regularity structure and \( (\Pi, \Gamma) \) is a model of transport precision \( n + 1 \).

Proof. The fact that \( T \) is a regularity structure is immediate. Let us now set \( \delta = \delta_M \) to be the injectivity radius of \( M \) and for \( q \in M \), let \( U_q := \exp_q(B_{\delta_M}(0)) \).

We have to check that
\[
\|\|\langle\Pi, \Gamma\rangle\|_{\beta, M} < \infty.
\]
The homogeneity estimate, \( |\langle\Pi_{p\tau}, \varphi_{p\tau}^\lambda\rangle| \lesssim \lambda^\ell \), for \( \tau \in T_p \), follows from the fact that \( \Pi_{p\tau} \) is a monomial of order \( \ell \) in \( \exp_{-p} \)-coordinates. Lemma 82 gives the transport precision, i.e.
\[
|\langle\Pi_q \tau - \Pi_p \Gamma_{p \tau - q \tau}, \varphi_{p \tau}^\lambda\rangle| \lesssim \lambda^{n+1} \text{ for all } \tau \in T_q.
\]
Let \( D \) be the covariant derivative induced by the chart \( \exp_q^{-1} \). Using Lemma 70 we get
\[
\langle \text{Sym} \left[ \nabla^m|_{\exp_q^{-1}} \right], W \rangle = \sum_{i=1}^{m} \langle D^i|_{\exp_q^{-1}} \rangle, Q_{i,n} \rangle \lesssim \sum_{i=1}^{m} d(p,q)^{\ell-i}|W| \lesssim d(p,q)^{\ell-m}|W|,
\]
and hence \( |\Gamma_{\ell}|_{\exp_q^{-1}}| \lesssim d(p,q)^{\ell-m} \), which finishes the proof.

We are finally able to characterize \( C^\gamma(M) \) in terms of the “polynomial” regularity structure.

**Theorem 90.** Let \( \gamma \in (0, \infty) \setminus \mathbb{N} \) and \( f : M \to \mathbb{R} \) continuous. Then, \( f \in C^\gamma(M) \) if and only if there is \( \hat{f} \in D^\gamma(M, T) \) with \( \hat{f}_0(p) = f(p) \). In that case,
\[
\hat{f}_\ell(p) = \text{Sym} \left[ \nabla^\ell|_p f \right].
\]

Proof. \((\implies)\) Let \( f \in C^\gamma(M) \) and define
\[
\hat{f}(p) := \sum_{\ell=0}^{\lfloor \gamma \rfloor} \text{Sym} \left[ \nabla^\ell|_p f \right],
\]
i.e. \( \hat{f}_\ell(p) := \text{Sym} \left[ \nabla^\ell|_p f \right] \) for \( 0 \leq \ell \leq \lfloor \gamma \rfloor = n \). We have to check that \( \hat{f} \in D^\gamma(M, T) \), i.e. for all \( \ell \leq \lfloor \gamma \rfloor \) and \( d(p,q) < \delta \)
\[
| (\hat{f}(q) - \Gamma_{q-p}\hat{f}(p))_\ell | \lesssim d(p,q)^{\gamma-\ell}
\]
or equivalently, using the definition of \( \Gamma_{q-p} \), if
\[
g(q) := \left( \Pi_p \hat{f}(p) \right)(q) = \left( \hat{f}(p), G^n_p(q) \right),
\]
we must show
\[
| \text{Sym} \left[ \nabla^\ell|_q (f - g) \right] | \lesssim d(p,q)^{\gamma-\ell}.
\]

\( \ell = n; \)
Recall \( \gamma - n \in (0, 1] \). Now the term to bound in (49) reads as
\[
| \text{Sym} \left[ \nabla^n|_q f - \nabla^n|_q \left( \sum_{i=0}^{n} \langle \nabla^i|_p f, \exp_p^{-1}(\cdot) \otimes^i \rangle \right) \right] | \leq | \text{Sym} \left[ \nabla^n|_q f - \nabla^n|_q \left( \langle \nabla^n|_p f, \exp_p^{-1}(\cdot) \otimes^n \rangle \right) \right] | + \sum_{i=0}^{n-1} \text{Sym} \left[ \nabla^n|_q \langle \langle \nabla^i|_p f, \exp_p^{-1}(\cdot) \otimes^i \rangle \rangle \right] |.
\]

By Lemma 78
\[
\text{Sym} \left[ \nabla^n|_q \left( \langle \nabla^i|_p f, \exp_p^{-1}(\cdot) \otimes^i \rangle \right) \right] = 0, \text{ at } q = p,
\]
\(^{10}\)The space of modelled distributions was defined in Definition 17.
and since the expression is smooth in $q$ we can focus on

$$|\text{Sym} [\nabla^n|_q f - \nabla^n|_q \{\nabla^n|_p f, \exp^{-1}(\cdot) \otimes^q\}]|$$

Define on the vector bundle $\Sigma^n T^* M$ the parallelism

$$U(q, p)S := \nabla^n|_q (S, \exp^{-1}(\cdot) \otimes^n).$$ (50)

Then by Lemma 88

$$|\text{Sym}[\nabla^n|_q f] - U(q, p) \text{Sym}[\nabla^n|_p f]| \lesssim d(q, p)^{\gamma - n},$$

so for $\ell = n$ we are done.

$\ell = 0, \ldots, n - 1$: We need to show (49). It is enough to bound for $w \in T_p M$, with $v := \exp^{-1}(q)$,

$$|\langle \nabla^\ell|_q (f - g), (// t(\gamma_0)w)^\otimes\ell \rangle| \lesssim |w|^\ell d(p, q)^{\gamma - \ell}.$$ Here $// t(\gamma_0) : T_{\gamma_0(0)} M \to T_{\gamma_0(t)} M$ denotes the parallel transport along $\gamma_0(t) := \exp_p(tv)$.

For this purpose, define

$$W_t := // t (\gamma_0) w_1 \otimes \cdots \otimes // t (\gamma_0) w_\ell,$$

and $F := f - g$. Since $W_t$ and $\dot{\gamma}_0(t)$ are parallel along $\gamma_0(t)$ it follows that

$$\frac{d^k}{dt^k} \nabla^\ell_{W_t} F = \nabla^{\ell+k}_{\dot{\gamma}_0(t) \otimes k \otimes W_t} F \forall 0 \leq k \leq n - \ell.$$ Therefore by Taylor’s theorem and the fact that $\nabla^n|_p F_p = 0$ for $0 \leq m \leq n^{11}$, we have

$$\nabla^\ell_{W_t} F = \sum_{k=0}^{n-\ell-1} \frac{1}{k!} \nabla^{\ell+k}_{v^0 \otimes k \otimes W_t} F + \frac{1}{(n - \ell - 1)!} \int_0^1 \left[ \nabla^n_{\gamma_0(t) \otimes n-\ell \otimes W_t} F \right] \cdot (1 - t)^{n-\ell-1} dt$$

$$= \frac{1}{(n - \ell - 1)!} \int_0^1 \left[ \nabla^n_{\dot{\gamma}_0(t) \otimes n-\ell \otimes W_t} F \right] \cdot (1 - t)^{n-\ell-1} dt.$$ (51)

Since $g$ is smooth we apply the fundamental theorem of calculus to find

$$\nabla^n_{\dot{\gamma}_0(t) \otimes (n-\ell) \otimes W_t} g = \nabla^n_{v^0 \otimes (n-\ell) \otimes W_0} g + \int_0^t \nabla^{n+1}_{\dot{\gamma}_0(\tau) \otimes (n-\ell) \otimes W_\tau} g d\tau$$

$$= \nabla^n_{v^0 \otimes (n-\ell) \otimes W_0} f + \int_0^t \nabla^{n+1}_{\dot{\gamma}_0(\tau) \otimes (n-\ell) \otimes W_\tau} g d\tau$$

$$= \nabla^n_{v^0 \otimes (n-\ell) \otimes W_0} f + O \left( |v|^{(n-\ell) + 1} |W_0| \right).$$

Using this estimate, it follows that

$$\nabla^n_{\dot{\gamma}_0(t) \otimes (n-\ell) \otimes W_t} F = \nabla^n_{\dot{\gamma}_0(t) \otimes (n-\ell) \otimes W_t} f - \nabla^n_{v^0 \otimes (n-\ell) \otimes W_0} f + O \left( |v|^{(n-\ell) + 1} |W_0| \right).$$

$^{11}$This follows by the very construction of $g$ along with Corollary 72 and Lemma 78.
Since
\[ \nabla^n_{\gamma_v(t) \otimes (n-\ell) \otimes w} f = \nabla^n|_{\gamma_v(t)} f \left( \, \text{Ch}_{\ell}(\gamma_v) \, v \right) \otimes \left( \, \text{Ch}_{n-\ell}(\gamma_v) \, w \right) \otimes \ell \\
= \left( \, \text{Ch}_{\ell}(\gamma_v) \, f, v \right) \otimes (n-\ell) \otimes w \otimes \ell, \]
As shown in the step \( \ell = n \), we then get
\[
\left| \nabla^n_{\gamma_v(t) \otimes (n-\ell) \otimes w} f - \nabla^n_{v \otimes (n-\ell) \otimes w_0} f \right| \leq C d \left( \gamma_v(t), p \right) \gamma^{-n} |v|^{n-\ell} |w|^{\ell} = C |v|^{\gamma-\ell} |w|^{\ell}.
\]
and hence
\[
\left| \nabla^n_{\gamma_v(t) \otimes (n-\ell) \otimes w} F \right| \leq \left[ C |v|^{\gamma-\ell} + O \left( |v|^{n-\ell+1} \right) \right] |w|^{\ell} \leq C' |v|^{\gamma-\ell} |w|^{\ell}.
\]
Plugging this estimate back into (51) shows,
\[
\left| \nabla^{\ell} F \right| \leq C |v|^{\gamma-\ell} |w|^{\ell},
\]
which completes the proof of (49).

\( \leftarrow \)

Recall that \( \gamma \in (n, n+1) \), for some \( n \in \mathbb{N} \).

**Step 1:** We will show that \( f \) is \( n \)-times differentiable and \( \text{Sym} [\nabla^\ell f] = \hat{f}_\ell \) for \( \ell = 0, \ldots, n \). This will be done by induction.

So assume for some \( \ell = 0, \ldots, n-1 \) we know that

- \( f \) is \( \ell \)-times differentiable
- \( \text{Sym}[\nabla^i f] = \hat{f}_i, \quad i = 0, \ldots, \ell \)

By Taylor’s theorem (Theorem 58)
\[
f(\exp_p(v)) = \sum_{j=0}^{\ell-1} \frac{1}{j!} \text{Sym}[\nabla^j_p f] \left( v \otimes j \right) + \frac{1}{(\ell-1)!} \int_0^1 (1-t)^{\ell-1} \text{Sym}[\nabla^\ell |_{\gamma_v(t)} f] \left( \, \text{Ch}_{\ell}(\gamma_v) \right) \right). \tag{52}
\]

Now by assumption
\[
|\text{Sym} [\nabla^\ell (f - g)]|_q \lesssim d(q, p)^{\gamma-\ell},
\]
where
\[
g(q) := \left( \Pi_p \hat{f} (p) \right) (q) = \left\langle \hat{f} (p), G^n_p (q) \right\rangle.
\]
Hence
\[
\text{Sym} [\nabla^\ell |_{\gamma_v(t)} f] = \text{Sym} [\nabla^\ell |_{\gamma_v(t)} g] + O(|tv|^{\gamma-\ell}).
\]
Plugging this into (52) and using the fact that \( |\gamma_v(t)| = |v| \) we get
\[
f(\exp_p(v)) = \sum_{j=0}^{\ell-1} \frac{1}{j!} \text{Sym}[\nabla^j_p f] \left( v \otimes j \right) + \frac{1}{(\ell-1)!} \int_0^1 (1-t)^{\ell-1} \text{Sym}[\nabla^\ell |_{\gamma_v(t)} g] \left( \, \text{Ch}_{\ell}(\gamma_v) \right) \right) + O(|v|^\gamma). \tag{53}
\]
Now, since \( g \) is smooth and \( \frac{\partial}{\partial t} \dot{\gamma}_v (t) = 0 \), we have
\[
\frac{d}{dt} \left[ (\nabla (t) g )_{\gamma_v (t)} \dot{\gamma}_v (t)^{\otimes \ell} \right] = \left[ (\nabla (t+1) g )_{\gamma_v (t)} \dot{\gamma}_v (t)^{\otimes (\ell+1)} \right] \text{ and }
\]
\[
\frac{d^2}{dt^2} \left[ (\nabla (t) g )_{\gamma_v (t)} \dot{\gamma}_v (t)^{\otimes \ell} \right] = \left[ (\nabla (t+2) g )_{\gamma_v (t)} \dot{\gamma}_v (t)^{\otimes (\ell+2)} \right]
\]
and therefore by Taylor’s theorem (in one variable) together with Lemma 53
\[
\text{Sym} \left[ \nabla^\ell g \right]_{\gamma_v (t)} \dot{\gamma}_v (t)^{\otimes \ell} = \left[ \nabla^\ell g \right]_{\gamma_v (t)} \dot{\gamma}_v (t)^{\otimes \ell}
\]
\[
= \left[ \nabla^\ell g \right]_p v^{\otimes \ell} + t \left[ \nabla^{\ell+1} g \right]_p v^{\otimes (\ell+1)} + O \left( |tv|^{\ell+2} \right)
\]
\[
= \left[ \nabla^\ell f \right]_p v^{\otimes \ell} + t \dot{f}_{\ell+1} (p) v^{\otimes (\ell+1)} + O \left( |tv|^{\ell+2} \right)
\]
\[
= \text{Sym} \left[ \nabla^\ell f \right]_p v^{\otimes \ell} + t \dot{f}_{\ell+1} (p) v^{\otimes (\ell+1)} + O \left( |tv|^{\ell+2} \right) \quad (54)
\]
A simple integration by parts argument shows
\[
\frac{1}{k!} \int_0^1 (1-t)^k t dt = \frac{1}{(k+2)!} \quad (55)
\]
Combining (53), (54) and (55), we get
\[
f(\exp_p (v)) = \sum_{j=0}^{\ell-1} \frac{1}{j!} \text{Sym} [\nabla^j]_p f \left( v^{\otimes j} \right) + \frac{1}{\ell!} \text{Sym} [\nabla^\ell]_p f \left( v^{\otimes \ell} \right) + \frac{1}{(\ell+1)!} \dot{f}_{\ell+1} (p) \left( v^{\otimes (\ell+1)} \right) + O (|v|)^{\gamma}.
\]
(56)
As \( v \mapsto \exp_p (v) \) is a local diffeomorphism, it now follows from (56) that \( f \) is \( \ell + 1 \) times differentiable at \( p \) and moreover since,
\[
f(\exp_p (tv)) = \sum_{j=0}^{\ell} \frac{k^j}{j!} \text{Sym} [\nabla^j]_p f \left( v^{\otimes j} \right) + \frac{k^{\ell+1}}{(\ell+1)!} \dot{f}_{\ell+1} (p) \left( v^{\otimes (\ell+1)} \right) + O (|t|)^{\gamma},
\]
we may conclude, using Lemma 53 that
\[
\nabla^{\ell+1} |_{p} v^{\otimes (\ell+1)} = \frac{d^{\ell+1}}{dt^{\ell+1}} |_{t=0} f(\exp_p (tv)) = \dot{f}_{\ell+1} (p) \left( v^{\otimes (\ell+1)} \right).
\]
Then by Remark 55 it follows that
\[
\text{Sym} [\nabla^{\ell+1} f]_p = \dot{f}_{\ell+1} (p).
\]

**Step 2:** So far we have shown that \( f \) is \( n \)-times continuously differentiable and that \( \text{Sym} [\nabla^\ell]_p f \) = \( \dot{f}_\ell (p) \) for \( \ell = 0, \ldots, n \). Then with \( U \) defined in (50) we have
\[
| \text{Sym} [\nabla^n]_q f - U(q,p) \text{Sym} [\nabla^n]_p f | \leq | \dot{f}_n (q) - \Gamma_{q-p} \dot{f}_n (p) | + \sum_{\ell=n-1} | \text{Sym} [\nabla^n]_q \Pi_{\ell} \dot{f}_\ell (p) |.
\]
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The second to last term is of order $d(q,p)^{\gamma-n}$ by assumption. Moreover, for $\ell \leq n-1$, by Corollary 78, we have $\nabla^n|_p \Pi_p \hat{f}_\ell(p) = 0$. Hence the last term is of order $d(q,p) \lesssim d(q,p)^{\gamma-n}$. By Lemma 88 we hence get that $f \in C^\gamma(M)$.



\begin{flushright}
$\square$
\end{flushright}

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