ON AN ERROR TERM FOR THE FIRST MOMENT OF TWISTED $L$-FUNCTIONS

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ABSTRACT. Let $f$ be a Hecke-Maass cusp form for the full modular group and let $\chi$ be a primitive Dirichlet character modulo a prime $q$. Let $s_0 = \sigma_0 + it_0$ with $\frac{1}{2} \leq \sigma_0 < 1$. We improve the error term for the first moment of $L(s_0, f \otimes \chi)L(s_0, \chi)$ over the family of even primitive Dirichlet characters. As an application, we show that for any $t \in \mathbb{R}$, there exists a primitive Dirichlet character $\chi$ modulo $q$ for which $L(1/2 + it, f \otimes \chi)L(1/2 + it, \chi) \neq 0$ if the prime $q$ satisfies $q \gg (1 + |t|)^{\frac{3}{54} + \varepsilon}$.

1. INTRODUCTION

It is a problem of intensive study in analytic number theory to investigate the values of automorphic $L$-functions. Among others one is particularly interested in establishing asymptotic formula for moments of $L$-functions over families, since there are many applications such as non-vanishing problem of $L$-functions, the subconvexity problem and the problem of non-existence of Landau-Siegel zeros (see Duke [3], Kowalski and Michel [10], Conrey and Iwaniec [1], and Iwaniec and Sarnak [8]).

In [2], Das and Khan proved the following asymptotic formula

$$\sum_{\chi \mod q \atop \chi(-1)=1}^\dagger L\left(\frac{1}{2}, f \otimes \chi\right)L\left(\frac{1}{2}, \chi\right) = \frac{q-2}{2} L(1, f) + O_{f, \varepsilon} \left( q^{\frac{3}{5} + \theta + \varepsilon} \right),$$

where $L(s, f \otimes \chi)$ is the $L$-function associated to a Hecke-Maass cusp form $f$ for the full modular group and a primitive Dirichlet character modulo a prime $q$, $L(s, \chi)$ is the Dirichlet $L$-function, and $L(1, f)$ denotes the $L$-function associated to $f$. Here and throughout the paper, the $\dagger$ means that the summation is over primitive characters and $\theta$ denotes the exponent towards the

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Ramanujan-Petersson conjecture for $f$, which can be taken as $\theta = \frac{7}{64}$ due to Kim and Sarnak [9] (see also Liu [11] for an interesting related result). Recently, Sono [13] generalized Das and Khan’s results to any complex number $s_0 = \sigma_0 + it_0$ with $1/2 \leq \sigma_0 < 1$. More precisely, he proved the following asymptotic formula

$$\sum_{\chi(\mod q), \chi(-1)=1} L(s_0, f \otimes \chi) \overline{L(s_0, \chi)}$$

$$= \frac{q}{2} L(2\sigma_0, f) + O_{f, \sigma_0, \varepsilon}(q^{1-\theta-2\sigma_0+\varepsilon}(q\tau)^{\theta+\varepsilon} + q^{-\frac{a}{4} + \frac{a_0 + \varepsilon}{2}}(q\tau)^{\frac{3}{2} - \frac{a}{2} \sigma_0}$$

$$+ (q\tau)^{1-\frac{a}{2} \sigma_0 + \theta + \varepsilon} + q^{-\frac{a}{2}}(q\tau)^{\frac{3}{2} - \frac{a}{2} \sigma_0 + \varepsilon} + q^{\frac{1}{2}}(q\tau)^{\frac{3}{2} - \frac{a}{2} \sigma_0 + \varepsilon} + q^{-1}(q\tau)^{\frac{3}{2} - \frac{a}{2} \sigma_0 + \varepsilon} + q^{-1}(q\tau)^{\frac{3}{2} - \frac{a}{2} \sigma_0 + \varepsilon} + q^{-1}(q\tau)^{\frac{3}{2} - \frac{a}{2} \sigma_0 + \varepsilon} + q^{-1}(q\tau)^{\frac{3}{2} - \frac{a}{2} \sigma_0 + \varepsilon} + q^{-1}(q\tau)^{\frac{3}{2} - \frac{a}{2} \sigma_0 + \varepsilon})$$,

(1.2)

where here and throughout the paper, $\varepsilon > 0$ is an arbitrarily small constant and $\tau := |t_0| + 3$.

The main aim of our paper is to improve the error term in (1.2). More precisely, we prove the following theorem.

**Theorem 1.** Let $f$ be a Hecke-Maass cusp form for $SL(2, \mathbb{Z})$. For prime values of $q$ and $s_0 = \sigma_0 + it_0$ with $1/2 \leq \sigma_0 < 1$, we have

$$\sum_{\chi(\mod q), \chi(-1)=1} L(s_0, f \otimes \chi) \overline{L(s_0, \chi)} = \frac{q}{2} L(2\sigma_0, f) + O\left(\sum_{i=1}^{4} R_i\right),$$

(1.3)

where the error terms $R_1, ..., R_4$ are given by

$$R_1 = q^{\frac{1}{4} + \varepsilon} \tau^{\frac{3}{2} (1-\sigma_0)}, \quad R_2 = (q\tau)^{\frac{3}{2} (1-\sigma_0) + \theta + \varepsilon}, \quad R_3 = q^{\frac{1}{2}} (q\tau)^{\frac{3}{2} (1-\sigma_0) + \varepsilon},$$

$$R_4 = \tau (q\tau)^{\frac{3}{2} - \left(\frac{1}{2} + \frac{\sigma_0 (1-2\theta)}{2(1+\theta)}\right) \sigma_0 + \varepsilon}.$$

The implied constant may depend on $f$, but is independent of $q$ and $\tau$.

Note that the first error term in (1.2) can be in fact dominated by the third one and for $\theta = \frac{7}{64}$, the fifth error term in (1.2) is absorbed by the last error term in (1.2). So we can ignore the first and fifth error terms in (1.2). $R_1$ is the second error term in (1.2). For the remaining terms, $R_2$ is superior to the sum of the third and the sixth error terms in (1.2) for $\theta = \frac{7}{64}$.
Obviously, $R_3$ is superior to the fourth error term in (1.2). Moreover, to compare $R_4$ and the last error term in (1.2), we let

$$g(\sigma_0) = \frac{3}{2} - \left(1 + \frac{\sigma_0(1 - 2\theta)}{2(1 + \theta)}\right)\sigma_0.$$  

We compute

$$g(\sigma_0) - \left(\frac{87}{52} - \frac{87}{52}\sigma_0 + \frac{37}{26}\right)$$

$$= -\frac{1 - 2\theta}{2(1 + \theta)}\sigma_0^2 + \frac{35}{52}\sigma_0 - \frac{9}{52} - \frac{37}{26}\theta$$

$$= -\frac{25}{71}\sigma_0^2 + \frac{35}{52}\sigma_0 - \frac{547}{1664}$$

$$= -\frac{25}{71}\left(\sigma_0 - \frac{497}{520}\right)^2 + \frac{25}{71}\left(\frac{497}{520}\right)^2 - \frac{547}{1664}.$$  

Therefore, for $1/2 \leq \sigma_0 < 1$, $g(\sigma_0) - \left(\frac{87}{52} - \frac{87}{52}\sigma_0 + \frac{37}{26}\right) \leq \frac{25}{71}\left(\frac{497}{520}\right)^2 - \frac{547}{1664} < 0$. Thus $R_4$ is superior to the last error term in (1.2) for $\theta = \frac{7}{64}$. Assembling the above argument, we find that Theorem 1 improves (1.2). In particular, for $\sigma_0 = \frac{1}{2}$, we have

$$\sum_{\chi \equiv 1 \pmod{q}} L\left(\frac{1}{2} + it_0, f \otimes \chi\right) L\left(\frac{1}{2} + it_0, \chi\right) = \frac{q}{2} L(1, f) + O\left(q^{\frac{7}{8} + \frac{3\theta}{8(1+\theta)} + \varepsilon} \tau^{\frac{15}{8} + \frac{3\theta}{8(1+\theta)} + \varepsilon}\right).$$  

(1.4)

To prove Theorem 1, we follow closely Sun [14], where the case $q$ is a product of two primes is considered. In [14], Sun also claimed that the error term in (1.1) can be improved to $O(q^{\frac{7}{8} + \frac{3\theta}{8(1+\theta)} + \varepsilon})$. Notice that (1.4) implies Sun’s result.

Let

$$M(\sigma_0) = \max \left\{2(1 - \sigma_0), \frac{3 - 3\sigma_0 + 2\theta}{3\sigma_0 - 1 - 2\theta}, \frac{1 - \sigma_0}{\sigma_0}, \frac{5 + 5\theta - \sigma_0(2 + 2\theta + \sigma_0 - 2\sigma_0\theta)}{1 - \theta + \sigma_0(2 + 2\theta + \sigma_0 - 2\sigma_0\theta)}\right\}.$$  

Here we call that $\theta = 7/64$. It can be easily confirmed that the main term in (1.3) dominates the error terms if $q \gg f^{-M(\sigma_0) + \varepsilon}$. Moreover, we have

$$M\left(\frac{1}{2}\right) = \max \left\{1, \frac{3 + 4\theta}{1 - 4\theta}, \frac{15 + 18\theta}{1 - 2\theta}\right\} = \frac{543}{25}.$$  

Hence we have the following result.
Corollary 1. For a Hecke-Maass cusp form $f$ of $SL(2, \mathbb{Z})$, there exists a primitive Dirichlet character $\chi$ modulo $q$ for which $L(s_0, f \otimes \chi)$ and $L(s_0, \chi)$ do not vanish if the prime $q$ satisfies $q \gg f \tau_{M(\sigma_0)}^{+}\varepsilon$. In particular, for any $t \in \mathbb{R}$, there exists a primitive Dirichlet character $\chi$ modulo $q$ for which $L(1/2 + it, f \otimes \chi)$ and $L(1/2 + it, \chi)$ do not vanish if the prime $q$ satisfies $q \gg f (1 + |t|)^{543/25 + \varepsilon}$.

Corollary 1 improves the result of Sono [13] who proved that $L(1/2 + it, f \otimes \chi)L(1/2 + it, \chi) \neq 0$ for some primitive Dirichlet character modulo $q$ for prime values $q$ satisfying $q \gg f (1 + |t|)^{255 + \varepsilon}$.

2. Preliminaries

Let $\chi$ be an even primitive Dirichlet character modulo $q$. For $\text{Re}(s) > 1$, we define the Dirichlet $L$-function

$$L(s, \chi) = \sum_{n \geq 1} \chi(n)n^{-s},$$

which has analytic continuation to all $s \in \mathbb{C}$ and satisfies the functional equation

$$\Lambda(s, \chi) = \frac{\tau(\chi)}{\sqrt{q}} \Lambda(1 - s, \overline{\chi}),$$

where

$$\Lambda(s, \chi) = \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{s}{2}\right)L(s, \chi),$$

and $\tau(\chi) = \sum_{a \pmod{q}}^* \chi(a)e(a/q)$ is the Gauss sum. Hereafter the notation $\sum_{a \pmod{q}}^*$ means $\sum_{a \pmod{q} \ (a,q)=1}$. Let $f$ be an even Hecke-Maass cusp form for $SL(2, \mathbb{Z})$ with Laplace eigenvalue $\frac{1}{4} + T_f^2$, $T_f \in \mathbb{R}$. Let $\lambda_f(n)$ be the $n$-th Fourier coefficient of $f$. For $\text{Re}(s) > 1$, we define the Dirichlet twist of Hecke-Maass $L$-function

$$L(s, f \otimes \chi) = \sum_{n \geq 1} \lambda_f(n)\chi(n)n^{-s},$$

which has analytic continuation to the whole complex plane and satisfies the function equation

$$\Lambda(s, f \otimes \chi) = \frac{\tau(\chi)^2}{q} \Lambda(1 - s, f \otimes \overline{\chi}),$$
where
\[
\Lambda(s, f \otimes \chi) = \left( \frac{2}{\pi} \right)^s \Gamma \left( \frac{s + iT_f}{2} \right) \Gamma \left( \frac{s - iT_f}{2} \right) L(s, f \otimes \chi).
\]

To prove Theorem 1, we need the approximate functional equations for \( L(s, \chi) \) and \( L(s, f \otimes \varphi) \). We quote the follow results of Sono [13] (see Lemmas 2.6 and 2.7 in [13]).

**Lemma 1.** Let \( G(u) = e^{u^2} \). For \( s_0 = \sigma_0 + it_0 \) with \( 1/2 \leq \sigma_0 < 1 \) and for \( \chi \) an even primitive Dirichlet character of modulus \( q \), we obtain
\[
L(s_0, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^{s_0}} V_{s_0} \left( \frac{n}{\sqrt{q}} \right) + \tau(\chi) q^{-s_0} \frac{\gamma(1-s_0)}{\gamma(s_0)} \sum_{n \geq 1} \frac{\chi(n)}{n^{1-s_0}} V_{1-s_0} \left( \frac{n}{\sqrt{q}} \right),
\]
(2.1)
where \( \gamma(s) := \pi^{-s/2} \Gamma(s/2) \) and
\[
V_{s_0}(x) = \frac{1}{2\pi i} \int_{(2)} (\pi x)^{-u} \frac{\Gamma(s_0 + u)}{\Gamma(s_0)} G(u) \frac{du}{u}.
\]
The function \( V_{s_0}(x) \) satisfies
\[
V_{s_0}(x) \ll_A \min \{1, (x/\sqrt{\tau})^{-A} \},
\]
\[
V_{s_0}^{(l)}(x) \ll x^{-l}.
\]
The implied constants depend only on \( \sigma_0, A, \) and \( l \).

**Lemma 2.** Let \( f \) be an even Hecke-Maass form for \( SL(2, \mathbb{Z}) \). For \( s_0 = \sigma_0 + it_0 \) with \( 1/2 \leq \sigma_0 < 1 \) and for \( \chi \) an even primitive Dirichlet character of modulus \( q \), we obtain
\[
L(s_0, f \otimes \chi) = \sum_{n \geq 1} \frac{\lambda_f(n) \chi(n)}{n^{s_0}} W_{s_0} \left( \frac{n}{q} \right) + \tau(\chi) q^{-s_0} \frac{\bar{\gamma}(1-s_0)}{\bar{\gamma}(s_0)} \sum_{n \geq 1} \frac{\lambda_f(n) \chi(n)}{n^{1-s_0}} W_{1-s_0} \left( \frac{n}{q} \right),
\]
(2.2)
where \( \bar{\gamma}(s) := \pi^{-s} \Gamma \left( \frac{s+iT_f}{2} \right) \Gamma \left( \frac{s-iT_f}{2} \right) \) and
\[
W_{s_0}(x) = \frac{1}{2\pi i} \int_{(2)} (\pi x)^{-u} \frac{\Gamma(s_0 + u + iT_f)}{\Gamma(s_0 + iT_f)} \Gamma(s_0 + u - iT_f) \frac{du}{u} G(u).
\]
The function $W_{s_0}(x)$ satisfies

$$W_{s_0}(x) \ll_A \min\{1, (x/\tau)^{-A}\},$$

$$W_{s_0}^{(l)}(x) \ll x^{-l}.$$  

The implied constants depend only on $\sigma_0$, $A$, and $l$.

We need the Rankin-Selberg estimate (see Proposition 19.6 in Duke, Friedlander and Iwaniec [4]).

**Lemma 3.** For any $\varepsilon > 0$, we have

$$\sum_{n \leq x} |\lambda_f(n)|^2 \ll_{f, \varepsilon} x^{1+\varepsilon}.$$  

Moreover, we also need the Wilton-type bound (see Iwaniec [6], Theorem 8.1).

**Lemma 4.** For any $\alpha \in \mathbb{R}$, we have

$$\sum_{n \leq N} \lambda_f(n)e(\alpha n) \ll_{f, \varepsilon} N^{1/2+\varepsilon},$$

uniformly in $\alpha$.

The following Voronoi formula can be found in Miller and Schmid [12] (see also Godber [5], Theorem 3.2).

**Lemma 5.** Let $\psi$ be a fixed smooth function with compact support on $\mathbb{R}^+$, $c$ be a positive integer and $d, \overline{d} \in \mathbb{Z}$ with $(c, d) = 1$ and $d\overline{d} \equiv 1 (\text{mod} c)$. Then we have

$$\sum_{n \geq 1} \lambda_f(n)e\left(\frac{nd}{c}\right) \psi\left(\frac{n}{N}\right) = c \sum_{\pm} \sum_{n \geq 1} \frac{\lambda_f(n)}{n} e\left(\pm \frac{nd}{c}\right) \Psi^\pm\left(\frac{nN}{c^2}\right),$$

where for $\sigma > -1$,

$$\Psi^\pm(x) = \frac{1}{2\pi i} \int_{(\sigma)} (\pi^2 x)^{-s} G^\pm(s) \tilde{\psi}(s) ds.$$  

Here $\tilde{\psi}(s) = \int_0^\infty \psi(x)x^{s-1}dx$ is the Mellin transform of $\psi(x)$ and

$$2\pi G^\pm(s) = \frac{\Gamma\left(\frac{1+s+i\tau}{2}\right)\Gamma\left(\frac{1+s-i\tau}{2}\right)}{\Gamma\left(-s+i\tau\right)\Gamma\left(-s-i\tau\right)} \pm \frac{\Gamma\left(\frac{1+s+i\tau+1}{2}\right)\Gamma\left(\frac{1+s-i\tau+1}{2}\right)}{\Gamma\left(-s+i\tau+1\right)\Gamma\left(-s-i\tau+1\right)}.  \quad (2.3)$$
Lemma 6. If $\chi \pmod{q}$ is an even primitive character, then

$$\tau(\overline{\chi})\tau(\chi) = q.$$  \hfill (2.4)

Proof. By (3.15) of Iwaniec and Kowalski [7], $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)q$. Then (2.4) follows since $\chi$ is even. \hfill $\square$

3. Proof of Theorem 1

Applying the approximate functional equations in (2.1) and (2.2), we have

$$\sum_{\chi \pmod{q}} \sum_{\chi(1) = 1} \frac{L(s_0, f \otimes \chi)\overline{L(s_0, \chi)}}{L(s_0, \chi)}$$

$$= \sum_{\chi \pmod{q}} \sum_{\chi(1) = 1} \left[ \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^{s_0}} W_{s_0} \left( \frac{n}{q} \right) + \tau(\chi)^2 q^{-\frac{2s_0}{q}} \frac{\chi(1-s_0)}{\chi(s_0)} \sum_{n=1}^{\infty} \frac{\lambda_f(n)\overline{\chi}(n)}{n^{1-s_0}} W_{1-s_0} \left( \frac{n}{q} \right) \right]$$

$$\times \left[ \sum_{m=1}^{\infty} \frac{\chi(m)\gamma^{s_0}(\chi)}{m^{s_0}} \left( \frac{m}{\sqrt{q}} \right) + \tau(\chi)^2 q^{-\frac{2s_0}{q}} \frac{\gamma(1-s_0)}{\gamma(s_0)} \sum_{m=1}^{\infty} \frac{\chi(m)\overline{\gamma}(\chi)}{m^{1-s_0}} V_{1-s_0} \left( \frac{m}{\sqrt{q}} \right) \right]$$

$$:= S_1 + S_2 + S_3 + S_4,$$ \hfill (3.1)

where

$$S_1 = \sum_{m=1}^{\infty} \frac{1}{m^{s_0}} V_{s_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{s_0}} W_{s_0} \left( \frac{n}{q} \right) \sum_{\chi \pmod{q}} \sum_{\chi(1) = 1} \overline{\chi}(m)\chi(n),$$

$$S_2 = q^{-s_0} \frac{\gamma(1-s_0)}{\gamma(s_0)} \sum_{n=1}^{\infty} \frac{1}{n^{1-s_0}} V_{1-s_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{s_0}} W_{s_0} \left( \frac{n}{q} \right) \sum_{\chi \pmod{q}} \sum_{\chi(1) = 1} \chi(m)\chi(n)\tau(\chi),$$

$$S_3 = q^{-2s_0} \frac{\gamma(1-s_0)}{\gamma(s_0)} \sum_{n=1}^{\infty} \frac{1}{n^{1-s_0}} V_{1-s_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1-s_0}} W_{1-s_0} \left( \frac{n}{q} \right) \sum_{\chi \pmod{q}} \sum_{\chi(1) = 1} \chi(m)\overline{\chi}(n)\tau(\chi)^2,$$

$$S_4 = q^{-2s_0} \frac{\gamma(1-s_0)}{\gamma(s_0)} \sum_{m=1}^{\infty} \frac{1}{m^{s_0}} V_{s_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1-s_0}} W_{1-s_0} \left( \frac{n}{q} \right) \sum_{\chi \pmod{q}} \sum_{\chi(1) = 1} \overline{\chi}(m)\overline{\chi}(n)\tau(\chi)^2.$$

In the subsequent sections, we will evaluate $S_i, \ i = 1, 2, 3, 4.$
4. Estimation of $S_1$

Lemma 7. For any $\varepsilon > 0$, we have

$$S_1 = \frac{q}{2} L(2\sigma_0, f) + O\left(q^{\frac{4}{3}+\varepsilon} \tau^{\frac{3}{2}(1-\sigma_0)} + (q\tau)^{\frac{2}{3}(1-\sigma_0)+\theta+\varepsilon}\right).$$

Proof. We write

$$S_1 = \sum_{m=1}^{\infty} \frac{1}{m^{\sigma_0}} V_{\pi}(\frac{m}{q}) \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} W_{s_0}(\frac{n}{q}) \sum_{\chi(\text{mod } q)} \chi(n).$$

By the orthogonality property of Dirichlet characters, we have for $(mn, q) = 1$,

$$\sum_{\chi(\text{mod } q)} \chi(\text{mod } q) \chi^{-1} = \frac{1}{2} \sum_{\chi(\text{mod } q)} \sum_{\chi(\text{mod } q)} \chi(\pm n) \chi(m) = \frac{1}{2} \sum_{\chi(\text{mod } q)} \chi(\pm n) \chi(m) = \frac{1}{2} \sum_{\chi(\text{mod } q)} [\phi(q) 1_{m\equiv n}\text{mod } q - 1].$$

Therefore,

$$S_1 = \frac{\phi(q)}{2} \sum_{\pm} \sum_{m \geq 1} \frac{1}{m^{\sigma_0}} V_{\pi}(\frac{m}{q}) \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} W_{s_0}(\frac{n}{q})$$

$$- \sum_{m \geq 1} \frac{1}{m^{\sigma_0}} V_{\pi}(\frac{m}{q}) \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} W_{s_0}(\frac{n}{q})$$

$$= \frac{\phi(q)}{2} S_{11} - S_{12},$$

(4.1)

say. Trivially, we have

$$S_{12} = \sum_{\pm} \sum_{m \geq 1} \frac{1}{m^{\sigma_0}} V_{\pi}(\frac{m}{q}) \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} W_{s_0}(\frac{n}{q})$$

$$\ll \sum_{m \leq (q\tau)^{\frac{1}{2}+\varepsilon}} \frac{1}{m^{\sigma_0}} \sum_{n \leq (q\tau)^{1+\varepsilon}} |\lambda_f(n)| \ll (q\tau)^{\frac{3}{2}-\frac{3}{2}\sigma_0+\theta+\varepsilon}.$$  

(4.2)

Next, we evaluate $S_{11}$ which contributes the main term. We write

$$S_{11} = S_{11}^* + S_{11}^{**},$$

where...
where
\[
S_{11}^* = \sum_{m \geq 1} \sum_{\substack{m \geq 1 \atop (m,q) = 1}} \frac{1}{m^{\sigma_0}} V_m \left( \frac{m}{\sqrt{q}} \right) \sum_{n \geq 1} \frac{\lambda_f(n)}{n^{\sigma_0}} W_{s_0} \left( \frac{n}{q} \right),
\]
\[
S_{11}^{**} = \sum_{m \geq 1} \sum_{\substack{m \geq 1 \atop (m,q) = 1}} \frac{1}{m^{\sigma_0}} V_m \left( \frac{m}{\sqrt{q}} \right) \sum_{n \geq 1} \frac{\lambda_f(n)}{n^{\sigma_0}} W_{s_0} \left( \frac{n}{q} \right).
\]

We have
\[
S_{11}^* \ll (q\tau)^{\theta + \varepsilon} \sum_{m \leq (q\tau)^{\frac{1}{2} + \varepsilon}} \frac{1}{m^{\sigma_0}} \sum_{n \leq (q\tau)^{1 + \varepsilon}} \frac{1}{n^{\sigma_0}} \sum_{m \neq n \equiv \pm m \pmod{q}} \frac{1}{n^{\sigma_0}},
\]
where the contribution from \( n < m \) is at most
\[
(q\tau)^{\theta + \varepsilon} \sum_{m \leq (q\tau)^{\frac{1}{2} + \varepsilon}} \frac{1}{m^{\sigma_0}} \sum_{n \leq (q\tau)^{1 + \varepsilon}} \frac{1}{n^{\sigma_0}} \sum_{m \neq n \equiv \pm m \pmod{q}} \frac{1}{n^{\sigma_0}} \ll \frac{1}{(q\tau)^{\sigma_0}} \sum_{1 \leq k \leq q^{-1}(q\tau)^{1/2 + \varepsilon}} \frac{1}{(qk)^{\sigma_0}} \ll q^{-1}(q\tau)^{1 - \sigma_0 + \theta + \varepsilon},
\]
and the contribution from \( n > m \) is bounded by
\[
(q\tau)^{\theta + \varepsilon} \sum_{m \leq (q\tau)^{\frac{1}{2} + \varepsilon}} \frac{1}{m^{\sigma_0}} \sum_{n \leq (q\tau)^{1 + \varepsilon}} \frac{1}{n^{\sigma_0}} \sum_{m \neq n \equiv \pm m \pmod{q}} \frac{1}{n^{\sigma_0}} \ll (q\tau)^{\frac{3}{2} - \frac{3}{2}\sigma_0 + \theta + \varepsilon}.
\]

Hence
\[
S_{11}^* \ll q^{-1}(q\tau)^{\frac{3}{2}(1 - \sigma_0) + \theta + \varepsilon}. \tag{4.3}
\]
Note that
\[ S_{11}^{**} = \sum_{(n,q)=1} \frac{\lambda_f(n)}{n^{2\sigma_0}} W_{s_0} \left( \frac{n}{q} \right) V_s \left( \frac{n}{\sqrt{q}} \right), \]
which has been evaluated in Sono [13] (see Pages 1130-1131 in [13]). By Sono [13], we have
\[ S_{11}^{**} = L(2\sigma_0, f) + O(q^{-2\sigma_0 + \varepsilon} r^{\theta + \varepsilon} + q^{-\frac{3}{2} + \varepsilon} r^{\frac{3}{2}(1-\sigma_0)}). \] (4.4)

By (4.3) and (4.4), we obtain
\[ S_{11} = L(2\sigma_0, f) + O(q^{-\frac{3}{2} + \varepsilon} r^{\frac{3}{2}(1-\sigma_0)} + q^{-1}(q^\tau)^{\frac{3}{2}(1-\sigma_0) + \theta + \varepsilon} + q^{-2\sigma_0 + \varepsilon} r^{\theta + \varepsilon}). \] (4.5)

Therefore, by (4.1), (4.2) and (4.5),
\[ S_1 = \frac{q}{2} L(2\sigma_0, f) + O \left( q^{-2\sigma_0 - \theta + \varepsilon} (q^\tau)^{\theta + \varepsilon} + q^{\frac{1}{2} + \varepsilon} r^{\frac{3}{2}(1-\sigma_0)} + (q^\tau)^{\frac{3}{2}(1-\sigma_0) + \theta + \varepsilon} \right). \]

Note that the first term is dominated by the third term. Then Lemma 7 follows. \[ \Box \]

5. Estimation of \( S_2 \)

Recall that
\[ S_2 = q^{-\sigma_0} \gamma(1 - \sigma_0) \sum_{m=1}^{\infty} \frac{1}{m^{1-\sigma_0}} V_{1 - \sigma_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{\sigma_0}} W_{s_0} \left( \frac{n}{q} \right) \sum_{\chi(\text{mod } q)} \chi(m) \chi(n) \tau(\chi). \] (5.1)

Lemma 8. For any \( \varepsilon > 0 \), we have
\[ S_2 \ll q^{\frac{1}{2}} (q^\tau)^{-\frac{3}{2}\sigma_0 + \frac{1}{2} + \varepsilon} + q^{-\frac{1}{2}} (q^\tau)^{\frac{3}{2} - \frac{3}{2}\sigma_0 + \theta + \varepsilon}. \]

Proof. By (5.1), we have
\[ S_2 = \frac{1}{2} q^{-\sigma_0} \gamma(1 - \sigma_0) \sum_{m=1}^{\infty} \frac{1}{m^{1-\sigma_0}} V_{1 - \sigma_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{\sigma_0}} W_{s_0} \left( \frac{n}{q} \right) \sum_{\chi(\text{mod } q)} \left[ 1 + \chi(-1) \right] \chi(mn) \tau(\chi) \]
\[ = \frac{1}{2} q^{-\sigma_0} \gamma(1 - \sigma_0) \sum_{m=1}^{\infty} \frac{1}{m^{1-\sigma_0}} V_{1 - \sigma_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{\sigma_0}} W_{s_0} \left( \frac{n}{q} \right) \sum_{\chi(\text{mod } q)} \chi(\pm mn) \tau(\chi). \] (5.2)
By the orthogonality of Dirichlet characters, we have
\[
\sum_{\chi \pmod{q}}^\dagger \chi(\pm mn) \tau(\chi) = \sum_{\chi \pmod{q}}^\dagger \sum_{a \pmod{q}}^\ast \overline{\chi}(a) e\left(\frac{a}{q}\right) \chi(\pm mn)
\]
\[
= \sum_{a \pmod{q}}^\ast e\left(\frac{a}{q}\right) \sum_{\chi \pmod{q}}^\dagger \overline{\chi}(a) \chi(\pm mn)
\]
\[
= \sum_{a \pmod{q}}^\ast e\left(\frac{a}{q}\right) \left[\phi(q)1_{a \equiv \pm mn \pmod{q}} - 1\right]
\]
\[
= \phi(q)e\left(\pm \frac{nm}{q}\right) + 1. \tag{5.3}
\]
Plugging (5.3) into (5.2), one has
\[
S_2 = \frac{\phi(q)}{2} S_{21} + S_{22}, \tag{5.4}
\]
where
\[
S_{21} = q^{-\sigma_0} \frac{\gamma(1 - \sigma_0)}{\gamma(s_0)} \sum_{m \geq 1} \sum_{(m, q) = 1} \frac{1}{m^{1-s_0}} V_{1-s_0} \left(\frac{m}{\sqrt{q}}\right) \sum_{n \geq 1} \frac{\lambda_f(n)}{n^{s_0}} W_{s_0} \left(\frac{n}{q}\right) e\left(\pm \frac{nm}{q}\right),
\]
\[
S_{22} = q^{-\sigma_0} \frac{\gamma(1 - \sigma_0)}{\gamma(s_0)} \sum_{m \geq 1} \sum_{(m, q) = 1} \frac{1}{m^{1-s_0}} V_{1-s_0} \left(\frac{m}{\sqrt{q}}\right) \sum_{n \geq 1} \frac{\lambda_f(n)}{n^{s_0}} W_{s_0} \left(\frac{n}{q}\right).
\]
Trivially,
\[
S_{22} \ll q^{-\sigma_0} \frac{1}{\tau^{2-\sigma_0}} \sum_{m \leq (q\tau)^{1+\epsilon}} \sum_{n \leq (q\tau)^{1+\epsilon}} \frac{|\lambda_f(n)|}{n^{\sigma_0}}
\]
\[
\ll q^{-\sigma_0} \frac{1}{\tau^{2-\sigma_0}} (q\tau)^{2\sigma_0+\epsilon} (q\tau)^{1-\sigma_0+\theta+\epsilon}
\]
\[
\ll q^{-\frac{1}{2}} (q\tau)^{\frac{3}{2} - \frac{1}{2} \sigma_0 + \theta + \epsilon}. \tag{5.5}
\]
As for $S_{21}$, removing the condition $(n, q) = 1$, we have
\[
S_{21} = q^{-\sigma_0} \frac{\gamma(1 - \sigma_0)}{\gamma(s_0)} \sum_{m \geq 1} \sum_{(m, q) = 1} \frac{1}{m^{1-s_0}} V_{1-s_0} \left(\frac{m}{\sqrt{q}}\right) \sum_{n \geq 1} \frac{\lambda_f(n)}{n^{s_0}} W_{s_0} \left(\frac{n}{q}\right) e\left(\pm \frac{nm}{q}\right)
\]
\[
+ O \left(q^{-\frac{1}{2}} (q\tau)^{\frac{3}{2} - \frac{1}{2} \sigma_0 + \theta + \epsilon}\right). \tag{5.6}
\]
By Lemma 4 and partial summation, we have
\[
\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{\sigma_0}} W_{s_0} \left( \frac{n}{q} \right) e \left( \frac{\pm nm}{q} \right) = \sum_{n \leq (q \tau)^{1+\epsilon}} \frac{\lambda_f(n)}{n^{\sigma_0}} W_{s_0} \left( \frac{n}{q} \right) e \left( \frac{\pm nm}{q} \right) + O((q \tau)^{-K})
\]
\[
= \sum_{\alpha \geq N \ll (q \tau)^{1+\epsilon}} \sum_{n=1}^{\infty} \lambda_f(n) e \left( \frac{\pm nm}{q} \right) \Phi \left( \frac{n}{N} \right) n^{-\sigma_0} W_{s_0} \left( \frac{n}{q} \right) + O((q \tau)^{-K})
\]
\[
= \sum_{\alpha \geq N \ll (q \tau)^{1+\epsilon}} \int_{1}^{\infty} \Phi \left( \frac{t}{N} \right) t^{-\sigma_0} W_{s_0} \left( \frac{t}{q} \right) d \left( \sum_{n \leq t} e \left( \frac{\pm nm}{q} \right) \lambda_f(n) \right) + O((q \tau)^{-K})
\]
\[
= \sum_{\alpha \geq N \ll (q \tau)^{1+\epsilon}} \int_{1}^{\infty} \int_{1}^{\infty} \sum_{n \leq t} \lambda_f(n) e \left( \frac{\pm nm}{q} \right) \left( \Phi \left( \frac{t}{N} \right) t^{-\sigma_0} W_{s_0} \left( \frac{t}{q} \right) \right) \frac{d}{dt} + O((q \tau)^{-K})
\]
where $K$ denotes arbitrary large number, $\Phi(x)$ is a smooth function compactly supported on $[1, 2]$ and
\[
\frac{d}{dt} \left( \Phi \left( \frac{t}{N} \right) t^{-\sigma_0} W_{s_0} \left( \frac{t}{q} \right) \right) \ll N^{-\sigma_0-1}.
\]
Hence, for $\sigma_0 \geq \frac{1}{2}$,
\[
\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{\sigma_0}} W_{s_0} \left( \frac{n}{q} \right) e \left( \frac{\pm nm}{q} \right) \ll \sum_{\alpha \geq N \ll (q \tau)^{1+\epsilon}} N \cdot N^{\frac{1}{2}+\epsilon} \cdot N^{-\sigma_0-1} \ll (q \tau)^{\epsilon}.
\]
Thus
\[
S_{21} \ll q^{-\sigma_0 \tau^{\frac{1}{2}-\sigma_0}} (q \tau)^{\frac{1}{2} \sigma_0+\epsilon} (q \tau)^{\epsilon} \ll q^{-\frac{1}{2}} (q \tau)^{\frac{1}{2}-\frac{1}{2} \sigma_0+\epsilon}. \quad (5.7)
\]
By (5.4)-(5.7),
\[
S_2 \ll q^{\frac{1}{2}} (q \tau)^{-\frac{1}{2} \sigma_0+\frac{1}{2}+\epsilon} + q^{\frac{1}{2}} (q \tau)^{\frac{1}{2}-\frac{1}{2} \sigma_0+\theta+\epsilon}.
\]
\[
\Box
\]
6. Estimation of $S_3$

Recall that

$$S_3 = q^{-2s_0 - \frac{1}{2}} \sum_{m=1}^{\infty} \frac{1}{m^{1-s_0}} V_{1-s_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1-s_0}} W_{1-s_0} \left( \frac{n}{q} \right) \times \sum_{\chi \pmod{q}, \chi(-1)=1} \chi(m) \bar{\chi}(n) \tau(\bar{\chi}) \tau(\chi)^2. \quad (6.1)$$

**Lemma 9.** For any $\varepsilon > 0$, we have

$$S_3 \ll q^{\frac{1}{2}} (q \tau)^{1 - \frac{1}{2} \sigma_0 + \varepsilon} + q^{-\frac{1}{2}} (q \tau)^{\frac{3}{2} - \frac{1}{2} \sigma_0 + \theta + \varepsilon}.$$

**Proof.** By $(6.1)$ and $(2.4)$ in Lemma 9,

$$S_3 = \frac{1}{2} q^{1-2s_0 - \frac{3}{2}} \sum_{m=1}^{\infty} \frac{1}{m^{1-s_0}} V_{1-s_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1-s_0}} W_{1-s_0} \left( \frac{n}{q} \right) \times \sum_{\chi \pmod{q}} \chi(\pm m \bar{m}) \tau(\chi).$$

By the orthogonality of Dirichlet characters, we have

$$\sum_{\chi \pmod{q}} \chi(\pm m \bar{m}) \tau(\chi) = \sum_{\chi \pmod{q}} e\left( \frac{a}{q} \right) \sum_{\chi \pmod{q}} \chi(\pm m \bar{m})$$

$$= \sum_{\chi \pmod{q}} e\left( \frac{a}{q} \right) \left[ \phi(q) \varepsilon_{a \equiv \pm m \bar{m} \pmod{q}} - 1 \right]$$

$$= \phi(q) e\left( \frac{\pm m \bar{m}}{q} \right) + 1.$$

Thus

$$S_3 = \frac{\phi(q)}{2} q^{1-2s_0 - \frac{3}{2}} \sum_{m \geq 1 \pmod{1}} \frac{1}{m^{1-s_0}} V_{1-s_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n \geq 1 \pmod{1}} \frac{\lambda_f(n)}{n^{1-s_0}} W_{1-s_0} \left( \frac{n}{q} \right) \varepsilon\left( \frac{\pm m \bar{m}}{q} \right)$$

$$+ q^{1-2s_0 - \frac{3}{2}} \sum_{m \geq 1 \pmod{1}} \frac{1}{m^{1-s_0}} V_{1-s_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n \geq 1 \pmod{1}} \frac{\lambda_f(n)}{n^{1-s_0}} W_{1-s_0} \left( \frac{n}{q} \right)$$

$$:= \frac{\phi(q)}{2} q^{1-2s_0 - \frac{3}{2}} \gamma(1-s_0) \gamma(1-s_0) S_{31} + q^{1-2s_0 - \frac{3}{2}} \gamma(1-s_0) \gamma(1-s_0) S_{32}, \quad (6.2)$$
say. Trivially, we have

\[ S_{32} \ll \sum_{m \leq (q \tau)^{\frac{1}{2} + \varepsilon}} \sum_{n \leq (q \tau)^{1 + \varepsilon}} \frac{|\lambda_f(n)|}{n^{1 - \sigma_0}} \ll (q \tau)^{\frac{3}{2} \sigma_0 + \theta + \varepsilon}. \quad (6.3) \]

Removing the condition \((n, q) = 1\), we have

\[ S_{31} = \sum_{\pm} \sum_{m \geq 1} \sum_{(m, q) = 1} \frac{1}{m^{1 - \sigma_0}} \left( \frac{m}{\sqrt{q}} \right) \sum_{n \geq 1} \frac{\lambda_f(n)}{n^{1 - \sigma_0}} W_{1 - \sigma_0} \left( \frac{n}{q} \right) e \left( \pm \frac{nm}{q} \right) + O(q^{-1}(q \tau)^{\frac{3}{2} \sigma_0 + \theta + \varepsilon}) \quad (6.4) \]

By partial integration once and Lemma \[ \square \], we obtain

\[ \ll \sum_{\pm} \sum_{m \geq 1} \sum_{(m, q) = 1} \frac{1}{m^{1 - \sigma_0}} \left( \frac{m}{\sqrt{q}} \right) \sum_{n \geq 1} \frac{\lambda_f(n)}{n^{1 - \sigma_0}} W_{1 - \sigma_0} \left( \frac{n}{q} \right) e \left( \pm \frac{nm}{q} \right) \phi \left( \frac{n}{N} \right) \]

\[ \ll \sum_{\pm} \sum_{m \geq 1} \sum_{(m, q) = 1} \frac{m^{\sigma_0 - 1}}{m^{1 - \sigma_0}} \left( \frac{m}{\sqrt{q}} \right) \int_0^\infty t^{s_0 - 1} W_{1 - \sigma_0} \left( \frac{t}{q} \right) \phi \left( \frac{t}{N} \right) d \left( \sum_{n \leq t} \lambda_f(n) e \left( \pm \frac{nm}{q} \right) \right) \]

\[ = \sum_{\pm} \sum_{m \geq 1} \sum_{(m, q) = 1} \frac{m^{\sigma_0 - 1}}{m^{1 - \sigma_0}} \left( \frac{m}{\sqrt{q}} \right) \int_0^\infty \sum_{n \leq t} \lambda_f(n) e \left( \pm \frac{nm}{q} \right) t^{s_0 - 1} W_{1 - \sigma_0} \left( \frac{t}{q} \right) \phi \left( \frac{t}{N} \right) dt \]

\[ \ll (q \tau)^{\frac{3}{2} \sigma_0 + \varepsilon} \sum_{\pm} \sum_{m \geq 1} \sum_{(m, q) = 1} \frac{m^{\sigma_0 - 1}}{m^{1 - \sigma_0}} N \cdot N^{\frac{1}{2} + \varepsilon} N^{\sigma_0 - 2} \]

\[ \ll (q \tau)^{\frac{3}{2} \sigma_0 - \frac{1}{2} + \varepsilon}. \quad (6.5) \]

By applying \((5.113)\) in Iwaniec and Kowalski \[ \square \], we have

\[ \frac{\gamma(1 - \sigma_0)}{\gamma(s_0)} = O(\tau^{\frac{3}{2} - 3\sigma_0}). \quad (6.6) \]

By \((6.2)-(6.6)\), we conclude that

\[ S_3 \ll q^{2 - 3\sigma_0} \tau^{\frac{3}{2} - 3\sigma_0} \left( (q \tau)^{\frac{3}{2} \sigma_0 - \frac{1}{2} + \varepsilon} + q^{-1}(q \tau)^{\frac{3}{2} \sigma_0 + \theta + \varepsilon} \right) + q^{1 - 3\sigma_0} \tau^{\frac{3}{2} - 3\sigma_0} (q \tau)^{\frac{3}{2} \sigma_0 + \theta + \varepsilon} \]

\[ \ll q^{\frac{1}{2}} (q \tau)^{1 - \frac{3}{2} \sigma_0 + \varepsilon} + q^{-\frac{1}{2}} (q \tau)^{\frac{3}{2} - \frac{3}{2} \sigma_0 + \theta + \varepsilon}. \]

\[ \square \]
7. Estimation of \( S_4 \)

Recall that

\[
S_4 = q^{-2s_0} \frac{\gamma(1 - s_0)}{\gamma(s_0)} \sum_{m = 1}^{\infty} \frac{1}{m^{s_0}} V_{s_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n = 1}^{\infty} \frac{\lambda_f(n)}{n^{1-s_0}} W_{1-s_0} \left( \frac{n}{q} \right) \sum_{\chi \pmod{q}}^{\dagger} \chi(m) \overline{\chi}(n) \tau(\chi)^2.
\]

By the orthogonality of Dirichlet characters, we have

\[
\sum_{\chi \pmod{q}}^{\dagger} \frac{\chi(m) \overline{\chi}(n) \tau(\chi)^2}{\chi(-1) = 1} = \sum_{\chi \pmod{q}}^{\dagger} (\chi(-1) + 1) \frac{\chi(mn) \tau(\chi)^2}{\chi(-1) = 1} = \frac{1}{2} \sum_{\chi \pmod{q}}^{\dagger} \left( \sum_{q' \pmod{q}}^{*} \chi(a) e \left( \frac{a}{q'} \right) \right)^2.
\]

By (7.1), we have

\[
\sum_{\chi \pmod{q}}^{\dagger} \chi(mn) \tau(\chi)^2 = \frac{1}{2} \sum_{a \pmod{q}}^{*} \sum_{b \pmod{q}}^{*} e \left( \frac{a+b}{q} \right) \left[ \phi(q) 1_{ab = \pm mn \pmod{q}} - 1 \right] + \frac{1}{2} \sum_{a \pmod{q}}^{*} \sum_{b \pmod{q}}^{*} e \left( \frac{a+b}{q} \right).
\]

By (7.1), we have

\[
S_4 = \frac{1}{2} \phi(q) q^{-2s_0} \frac{\gamma(1 - s_0)}{\gamma(s_0)} S_{41} - q^{-2s_0} \frac{\gamma(1 - s_0)}{\gamma(s_0)} S_{42}.
\]

where

\[
S_{41} = \sum_{\pm} \sum_{\substack{m \geq 1 \cr (m, q) = 1}}^{\dagger} \frac{1}{m^{s_0}} V_{s_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n \geq 1}^{\dagger} \frac{\lambda_f(n)}{n^{1-s_0}} W_{1-s_0} \left( \frac{n}{q} \right) S(1, \pm mn; q),
\]

\[
S_{42} = \sum_{\pm} \sum_{\substack{m \geq 1 \cr (m, q) = 1}}^{\dagger} \frac{1}{m^{s_0}} V_{s_0} \left( \frac{m}{\sqrt{q}} \right) \sum_{n \geq 1}^{\dagger} \frac{\lambda_f(n)}{n^{1-s_0}} W_{1-s_0} \left( \frac{n}{q} \right).
\]

say. By Weil’s bound for Kloosterman sums and Lemma \( \mathfrak{L} \) we have

\[
S_{42} \ll \sum_{m \leq (q\tau)^{1/2+\varepsilon}} m^{-\sigma_0} \sum_{n \leq (q\tau)^{1+\varepsilon}} \frac{|\lambda_f(n)|}{n^{1-\sigma_0}} \ll (q\tau)^{1/2+\varepsilon} (q\tau)^{\sigma_0+\theta+\varepsilon} \ll (q\tau)^{1/2+\sigma_0+\theta+\varepsilon}. \quad (7.3)
\]
Making smooth partitions of unity into dyadic segments to the sums over \( m \) and \( n \), we arrive at

\[
S_{41} = \sum_{\pm} \sum_{2^{\alpha_1} = M \ll (q\tau)^{1/2+\varepsilon}} \sum_{2^{\alpha_2} = N \ll (q\tau)^{1+\varepsilon}} T(\pm, M, N),
\]

where

\[
T(\pm, M, N) = \sum_{(m, q) = 1} m^{-\sigma_0} V_{m_0} \left( \frac{m}{\sqrt{q}} \right) W_1 \left( \frac{m}{M} \right) \sum_{(n, q) = 1} \lambda_f(n) n^{s_0} W_1(n) W_2 \left( \frac{n}{N} \right) S(1, \pm mn; q) \quad (7.4)
\]

Here \( W_j(x) \in C^\infty_c(1, 2) \) \((j = 1, 2)\) satisfying \( W_j(l) \ll l^{\varepsilon} \) for \( l \geq 0 \).

**Lemma 10.** For any \( \varepsilon > 0 \), we have

\[
T(\pm, M, N) \ll \tau^{-\frac{1}{2}} (q\tau)^{\frac{1}{2} + \beta_1 - (\sigma_0 + \varepsilon)} + \tau(q\tau)(1 - \beta_1) + \varepsilon + \tau(q\tau)^{\frac{1}{2} + \beta_2 - \sigma_0} + \varepsilon + \tau(q\tau)^{\frac{1}{2} + \beta_2 - (1 - \sigma_0) + (2 - \beta_2) \theta + \varepsilon}. \quad (7.5)
\]

**Proof.** We distinguish three cases according to the ranges of \( M \) and \( N \). Let \( \beta_1 \) and \( \beta_2 \) be positive parameters to be chosen later. By (7.4), we assume \( 0 < \beta_1 < \frac{1}{2} \) and \( 0 < \beta_2 < 1 \).

**Case I:** \( M < (q\tau)^{\beta_1}, N < (q\tau)^{\beta_2} \).

By Lemma 3 and Weil’s bound for Kloosterman sums, we have

\[
T(\pm, M, N) \ll q^{\frac{1}{2}} \sum_{m \geq M} m^{-\sigma_0} \sum_{n \leq N} n^{\sigma_0-1} \ll q^{\frac{1}{2}} N^{\sigma_0+\varepsilon} M^{1-\sigma_0} \ll q^{\frac{1}{2}} (q\tau)^{\beta_1 - (\sigma_0 + \varepsilon)}. \quad (7.6)
\]

**Case II:** \( M \geq (q\tau)^{\beta_1} \).

In this case, we apply Poisson summation formula to the \( m \)-sum to get

\[
\sum_{(m, q) = 1} \frac{1}{m^{\sigma_0}} V_{m_0} \left( \frac{m}{\sqrt{q}} \right) W_1 \left( \frac{m}{M} \right) e \left( \frac{\pm amn}{q} \right) \]

\[
= \frac{1}{M^{\sigma_0}} \sum_\beta \sum_{\beta \mod q} e \left( \frac{\pm \beta m}{q} \right) \left( \frac{M}{m} \right)^{\sigma_0} V_{m_0} \left( \frac{m}{\sqrt{q}} \right) W_1 \left( \frac{m}{M} \right) \]

\[
= \frac{1}{M^{\sigma_0}} \sum_\beta \sum_{\beta \mod q} e \left( \frac{\pm \beta m}{q} \right) \frac{1}{q} \sum_{m \in \mathbb{Z}} e \left( \frac{\beta m}{q} \right) \int_\mathbb{R} \left( \frac{M}{t} \right)^{\sigma_0} V_{m_0} \left( \frac{t}{\sqrt{q}} \right) W_1 \left( \frac{t}{M} \right) e \left( \frac{-mt}{q} \right) dt \]

\[
= q^{-1} M^{1-\sigma_0} \sum_{m \in \mathbb{Z}} \sum_\beta e \left( \frac{(m \pm \alpha m)\beta}{q} \right) J(m, q), \quad (7.7)
\]
where

\[ J(m, q) = \int_{\mathbb{R}} t^{-s_0} V_{\overline{\sigma}} \left( \frac{Mt}{q} \right) W_1(t) e \left( \frac{-Mmt}{q} \right) dt. \]

By repeated partial integrations,

\[ J(m, q) \ll \left( 1 + \frac{|m|M}{q} \right)^{-A} \]

for any \( A > 0 \). Thus the contribution from \( |m| \geq (q\tau)^{1+\varepsilon}/M \) can be arbitrarily small.

Plugging (7.7) into (7.5), one has

\[
T(\pm, M, N) = q^{-1} M^{1-s_0} \sum_{|m| \leq (q\tau)^{1+\varepsilon}/M} J(m, q) \sum_{(n,q)=1} \frac{\lambda_f(n)}{n^{1-s_0}} W_1 \left( \frac{n}{q} \right) W_2 \left( \frac{n}{N} \right) \]

\[
\times \sum_{a \mod q} \sum_{\beta \mod q} e \left( \frac{a}{q} \right) e \left( \frac{(m \pm \bar{a} n)\beta}{q} \right) + O \left( (q\tau)^{-A} \right) \]

\[
:= q^{-1} M^{1-s_0} \sum_{|m| \leq (q\tau)^{1+\varepsilon}/M} J(m, q) \sum_{(n,q)=1} \frac{\lambda_f(n)}{n^{1-s_0}} W_1 \left( \frac{n}{q} \right) W_2 \left( \frac{n}{N} \right) c(m, n; q) \]

\[
+ O \left( (q\tau)^{-A} \right), \tag{7.8}
\]

where

\[
c(m, n; q) = \sum_{a \mod q} \sum_{\beta \mod q} e \left( \frac{a}{q} \right) e \left( \frac{(m \pm \bar{a} n)\beta}{q} \right). \]

1° If \( m = 0 \), then

\[
c(0, n; q) = \sum_{a \mod q} \sum_{\beta \mod q} e \left( \frac{a}{q} \right) e \left( \frac{\pm \bar{a} n\beta}{q} \right) = \begin{cases} -\phi(q), & \text{if } q \mid n, \\ 1, & \text{if } q \nmid n. \end{cases}
\]

Thus the contribution from the terms with \( m = 0 \) is at most

\[
\frac{M^{1-s_0}}{q} |J(0, q)| \sum_{q \mid n} \frac{\lambda_f(n)}{n^{1-s_0}} \phi(q) + \frac{M^{1-s_0}}{q} |J(0, q)| \sum_{q \nmid n} \frac{\lambda_f(n)}{n^{1-s_0}} \]

\[
\ll q^{-1} M^{1-s_0} N^{\sigma_0+\theta+\varepsilon}. \tag{7.9}
\]
2° If \( m \neq 0 \), then 

\[
c(m, n; q) = \sum_{\beta \mod q}^* e \left( \frac{a}{q} \right) \left( \sum_{\beta \mod q} e \left( \frac{(m \pm an)\beta}{q} \right) - 1 \right) = qe \left( \frac{\mp nm}{q} \right) + 1.
\]

Thus the contribution from the terms with \( m \neq 0 \) is

\[
M^{1-\sigma_0} \sum_{(n, q)=1} \frac{\lambda_f(n)}{n^{1-\sigma_0}} W_{1-\sigma_0} \left( \frac{n}{q} \right) W_2 \left( \frac{n}{N} \right) \sum_{0<|m|\leq(q\tau)^{1+\varepsilon}/M} e \left( \frac{\mp nm}{q} \right) J(m, q)
\]

\[
+ q^{-1} M^{1-\sigma_0} \sum_{(n, q)=1} \frac{\lambda_f(n)}{n^{1-\sigma_0}} W_{1-\sigma_0} \left( \frac{n}{q} \right) W_2 \left( \frac{n}{N} \right) \sum_{0<|m|\leq(q\tau)^{1+\varepsilon}/M} J(m, q). \tag{7.10}
\]

Note that the second term in (7.10) is bounded by

\[
q^{-1} M^{1-\sigma_0} N^{\sigma_0 + \theta + \varepsilon} (q\tau)^{1+\varepsilon}/M \ll \tau^{1+\varepsilon} N^{\sigma_0 + \theta + \varepsilon} M^{-\sigma_0}. \tag{7.11}
\]

Removing the condition \((n, q) = 1\) in the first term in (7.10) at a cost of \( O(\tau^{1+\varepsilon} N^{\sigma_0 + \theta + \varepsilon} M^{-\sigma_0}) \) and combining (7.8)-(7.11), we obtain

\[
T(\pm, M, N) = M^{1-\sigma_0} \sum_{n \geq 1} \frac{\lambda_f(n)}{n^{1-\sigma_0}} W_{1-\sigma_0} \left( \frac{n}{q} \right) W_2 \left( \frac{n}{N} \right) \sum_{0<|m|\leq(q\tau)^{1+\varepsilon}/M} J(m, q) e \left( \frac{\mp nm}{q} \right)
\]

\[
+ O \left( \tau^{1+\varepsilon} N^{\sigma_0 + \theta + \varepsilon} M^{-\sigma_0} + q^{-1} M^{1-\sigma_0} N^{\sigma_0 + \theta + \varepsilon} \right). \tag{7.12}
\]

By Lemma 4 and partial summation, we have

\[
\sum_{n \geq 1} \frac{\lambda_f(n)}{n^{1-\sigma_0}} W_{1-\sigma_0} \left( \frac{n}{q} \right) W_2 \left( \frac{n}{N} \right) e \left( \frac{\mp nm}{q} \right)
\]

\[
= \int_N^{2N} t^{\sigma_0 - 1} W_{1-\sigma_0} \left( \frac{t}{q} \right) W_2 \left( \frac{t}{N} \right) d \left( \sum_{n \leq t} \lambda_f(n) e \left( \frac{\mp nm}{q} \right) \right)
\]

\[
= \int_N^{2N} \sum_{n \leq t} \lambda_f(n) e \left( \frac{\mp nm}{q} \right) t^{\sigma_0 - 1} W_{1-\sigma_0} \left( \frac{t}{q} \right) W_2 \left( \frac{t}{N} \right) \, dt
\]

\[
\ll \tau N^{\sigma_0 - 1/2 + \varepsilon} \ll \tau (q\tau)^{\sigma_0 - 1/2 + \varepsilon}. \tag{7.13}
\]
Thus by (7.12) and (7.13),

\[
T(\pm, M, N) \ll M^{1-\sigma_0} \left( \frac{q^{\tau}}{M} \right)^{1+\varepsilon} + \tau^{1+\varepsilon} N^{\sigma_0+\theta+\varepsilon} M^{-\sigma_0} + q^{-1} M^{1-\sigma_0} N^{\sigma_0+\theta+\varepsilon}
\]

\[
\ll \tau(q\tau)^{(1-\beta_1)\sigma_0+\frac{1}{2}+\varepsilon} + \tau(q\tau)^{(1-\beta_1)\sigma_0+\theta+\varepsilon} + q^{-1}(q\tau)^{(1-\beta_1)\sigma_0+\beta_1+\theta+\varepsilon}
\]

\[
\ll \tau(q\tau)^{(1-\beta_1)\sigma_0+\frac{1}{2}+\varepsilon} + q^{-1}(q\tau)^{(1-\beta_1)\sigma_0+\beta_1+\theta+\varepsilon}.
\]

(7.14)

Here we recall that \( \theta = \frac{7}{64} \).

Case III: \( M < (q\tau)^{\beta_1}, N \geq (q\tau)^{\beta_2} \).

Note that

\[
\sum_{(m,q)=1} \frac{1}{m^{s_0} V_{s_0}} \left( \frac{m}{\sqrt{q}} \right) W_1 \left( \frac{m}{M} \right) = \lambda_f(n) \sum_{q|\lambda} \frac{\lambda_f(n)}{n^{1-s_0}} W_1 \left( \frac{n}{q} \right) W_2 \left( \frac{n}{N} \right) S(1, \pm mn; q)
\]

\[
\ll M^{1-\sigma_0} q^{-1} N^{\sigma_0+\theta} \sqrt{q}
\]

\[
\ll q^{-\frac{1}{2}} (q\tau)^{\beta_1(1-\sigma_0)} (q\tau)^{\sigma_0+\theta+\varepsilon}
\]

\[
\ll q^{-\frac{1}{2}} (q\tau)^{\beta_1(1-\sigma_0)+\sigma_0+\theta+\varepsilon},
\]

where we have used Weil’s bound for Kloosterman sums. Thus we can write (7.5) as

\[
T(\pm, M, N) = T^*(\pm, M, N) + O \left( q^{-\frac{1}{2}} (q\tau)^{\beta_1(1-\sigma_0)+\sigma_0+\theta+\varepsilon} \right),
\]

(7.15)

where

\[
T^*(\pm, M, N) = \sum_{(m,q)=1} \frac{1}{m^{s_0} V_{s_0}} \left( \frac{m}{\sqrt{q}} \right) W_1 \left( \frac{m}{M} \right)
\]

\[
\times \sum_{n} \frac{\lambda_f(n)}{n^{1-s_0}} W_1 \left( \frac{n}{q} \right) W_2 \left( \frac{n}{N} \right) S(1, \pm mn; q).
\]

(7.16)

Opening the Kloosterman sum and applying Voronoi summation formula in Lemma 5 to the sum over \( n \), we get

\[
\sum_{n} \frac{\lambda_f(n)}{n^{1-s_0}} W_1 \left( \frac{n}{q} \right) W_2 \left( \frac{n}{N} \right) S(1, \pm mn; q)
\]

\[
= q N^{s_0-1} \sum_{a \mod q}^\ast e \left( \frac{a}{q} \right) \sum_{n \geq 1} \frac{\lambda_f(n)}{n} e \left( \frac{\pm anm}{q} \right) \Psi \left( \frac{nN}{q^2} \right),
\]

(7.17)
where for $\sigma > -1$,

$$
\Psi^\pm(y) = \frac{1}{2\pi i} \int_{(\sigma)} (\pi^2 y)^{-s} G^\pm(s) \left( \int_0^\infty u^{s_0-1} W_1(u) W_2(u) u^{-s-1} du \right) ds. \quad (7.18)
$$

Note that the sum over $a$ equals

$$
\sum_{a \equiv q \mod q} e \left( \frac{(1 \pm nm)a}{q} \right) = q \sum_{n \equiv \pm m \mod q} 1.
$$

Thus (7.14) is equal to

$$
q N^{s_0-1} \sum_{\pm} \sum_{n \geq 1} \frac{\lambda_f(n)}{n} \Psi^\pm \left( \frac{nN}{q^2} \right) \left( q \sum_{n \equiv \pm m \mod q} 1 - 1 \right). \quad (7.19)
$$

Moreover, by repeated integration by parts, the $u$-integral in (7.18) is bounded by $\left( \frac{\tau}{1+|m|} \right)^j$ for any integer $j \geq 0$. Therefore, by (2.3) and Stirling’s formula,

$$
\Psi^\pm(y) \ll y^{-\sigma} \int_{|t| \leq \tau^{1+\varepsilon} q^{\varepsilon}} (1 + |t|)^{2\sigma+1} \left( \frac{\tau}{1+|t|} \right)^j dt + (q\tau)^{-A}
$$

for any $A > 0$. Taking $\sigma = j/2 - 1 - \varepsilon$ with any fixed $j \geq 1$, one has

$$
\Psi^\pm(y) \ll y^{-j/2 + 1 + \varepsilon \tau^j} \int_{|t| \leq \tau^{1+\varepsilon} q^{\varepsilon}} (1 + |t|)^{-1-2\varepsilon} dt + (q\tau)^{-A} \ll y^{1+\varepsilon} \left( \frac{y}{\tau^2} \right)^{-j/2}
$$

for any fixed $j \geq 1$. Therefore, the contribution from $\frac{nN}{q^2} \tau^j \geq (q\tau)^\varepsilon$ in (7.19) is negligible. For smaller $y$, we move the contour of integration in $\Psi^\pm(y)$ to $\text{Re}(s) = -1 + \varepsilon$ to get

$$
\Psi^\pm(y) \ll y^{1-\varepsilon} \int_{|t| \leq \tau^{1+\varepsilon} q^{\varepsilon}} (1 + |t|)^{-1+\varepsilon} dt \ll (q\tau)^\varepsilon y^{1-\varepsilon}. \quad (7.20)
$$

By putting (7.19) into (7.16), we obtain

$$
T^\pm(\pm,M,N) = q N^{s_0-1} \sum_{\pm} \sum_{(m,q)=1} \frac{1}{m^{s_0}} \lambda_f(m) \left( \frac{m}{q} \right) W_1 \left( \frac{m}{M} \right) \times \sum_{n \leq (q\tau)^{2+\varepsilon}/N} \frac{\lambda_f(n)}{n} \Psi^\pm \left( \frac{nN}{q^2} \right) \left( q \sum_{n \equiv \pm m \mod q} 1 - 1 \right).
$$
Using (7.20) and Lemma 3, we have
\[
T^*(\pm, M, N) \ll q^2 N^{\sigma_0 - 1} \sum_{m \leq M} m^{-\sigma_0} \frac{\lambda_f(m)}{m} \frac{mN}{q^2} + q^2 N^{\sigma_0 - 1} \sum_{m \leq M} m^{-\sigma_0} \sum_{1 \leq k \leq \tau(q) \frac{1 + \varepsilon}{N}} \sum_{n = \pm m + kq} n^{-1} \cdot \frac{nN}{q^2}
\]
\[
+ qN^{\sigma_0 - 1} \sum_{m \leq M} m^{-\sigma_0} \sum_{n \leq (q\tau)^{2 + \varepsilon}/N} \frac{|\lambda_f(n)|}{n} \cdot \frac{nN}{q^2}
\]
\[
\ll N^{\sigma_0} M^{1-\sigma_0+\varepsilon} + q\tau^2 N^{\sigma_0-1} M^{1-\sigma_0+\varepsilon} (\frac{q\tau}{N})^{2\theta} + q\tau^2 N^{\sigma_0-1} M^{1-\sigma_0+\varepsilon}
\]
\[
\ll N^{\sigma_0-1} M^{1-\sigma_0+\varepsilon} q\tau^2 (\frac{q\tau}{N})^{2\theta}
\]
\[
\ll \tau (q\tau)^{1+(1-\sigma_0)\beta_1-(1-\sigma_0)\beta_2+(2-\beta_2)\theta+\varepsilon}. \tag{7.21}
\]

By (7.15) and (7.21), we get
\[
T(\pm, M, N) \ll \tau (q\tau)^{1+(1-\sigma_0)\beta_1-(1-\sigma_0)\beta_2+(2-\beta_2)\theta+\varepsilon} + q^{-\frac{1}{2}} (q\tau)^{\beta_1(1-\sigma_0)+\sigma_0+\theta+\varepsilon}. \tag{7.22}
\]

Note that \((2 - \beta_2)\theta > \theta\) for \(\beta_2 < 1\) and \(1 - (1 - \sigma_0)\beta_2 > \sigma_0\). The second term in (7.22) is dominated by the first term. Thus
\[
T(\pm, M, N) \ll \tau (q\tau)^{1+(1-\sigma_0)\beta_1-(1-\sigma_0)\beta_2+(2-\beta_2)\theta+\varepsilon}. \tag{7.23}
\]

Therefore, Lemma 10 follows from (7.6), (7.14) and (7.23). \square

By Lemma 10 and (7.4),
\[
S_{41} \ll \tau^{-\frac{1}{2}} (q\tau)^{\frac{1}{2}+(1-\sigma_0)\beta_1+\beta_2\sigma_0+\varepsilon} + \tau (q\tau)^{(1-\beta_1)\sigma_0+\frac{1}{2}+\varepsilon}
\]
\[
+ \tau (q\tau)^{1+(1-\sigma_0)\beta_1-(1-\sigma_0)\beta_2+(2-\beta_2)\theta+\varepsilon}. \tag{7.24}
\]

Solving the equations
\[
\begin{align*}
\frac{1}{2} + (1 - \sigma_0)\beta_1 + \beta_2\sigma_0 &= (1 - \beta_1)\sigma_0 + \frac{1}{2}, \\
1 + (1 - \sigma_0)\beta_1 - (1 - \sigma_0)\beta_2 + (2 - \beta_2)\theta &= (1 - \beta_1)\sigma_0 + \frac{1}{2},
\end{align*}
\]
we choose \(\beta_1\) and \(\beta_2\) as
\[
\beta_1 = \frac{\sigma_0(1-2\theta)}{2(1+\theta)}, \quad \beta_2 = \frac{1+4\theta}{2(1+\theta)}. \tag{7.25}
\]
Thus by (7.24),

\[
S_{41} \ll \tau(q\tau)^{1/2} \left( 1 - \frac{\sigma_0(1-\theta)}{2(1+\theta)} \right)^{\sigma_0+\varepsilon}.
\]  

(7.26)

By Stirling’s formula,

\[
\frac{\tilde{\gamma}(1-s)}{\gamma(s)} \ll \tau^{1-2\sigma_0}.
\]  

(7.27)

Substituting (7.3), (7.25)-(7.27) into (7.2), we conclude that

\[
S_4 \ll (q\tau)^{1-2\sigma_0} \tau(q\tau)^{1/2} \left( 1 - \frac{\sigma_0(1-\theta)}{2(1+\theta)} \right)^{\sigma_0+\varepsilon} + q^{-2\sigma_0} \tau(q\tau)^{1/2} \left( \frac{\sigma_0}{2} + \theta + \varepsilon \right)
\]

\[
\ll \tau(q\tau)^{1/2} \left( 1 + \frac{\sigma_0(1-\theta)}{2(1+\theta)} \right)^{\sigma_0+\varepsilon} + \tau(q\tau)^{1/2} \left( \frac{\sigma_0}{2} + \theta + \varepsilon \right).
\]

Note that 1 + \frac{\sigma_0(1-\theta)}{2(1+\theta)} < \frac{3}{2} and \frac{3}{2} > \frac{1}{2} + \theta for \theta \leq \frac{\pi}{64}. Therefore, the first term dominates the second term and

\[
S_4 \ll \tau(q\tau)^{1/2} \left( 1 + \frac{\sigma_0(1-\theta)}{2(1+\theta)} \right)^{\sigma_0+\varepsilon}.
\]  

(7.28)

8. Completing the proof of Theorem 1

By (8.1), Lemma 7-9 and (7.28), we have

\[
\sum_{\chi(\mod q)}^* L(s_0, f \otimes \chi) L(s_0, \chi) = \frac{q}{2} L(2\sigma_0, f) + E,
\]

(8.1)

where

\[
E \ll q^{1/2+\varepsilon} \tau^{1/2} \left( 1 - \sigma_0 \right) + (q\tau)^{1/2} \left( 1 - \sigma_0 \right)^{\sigma_0+\varepsilon} + q^{1/2} (q\tau)^{-1/2} \sigma_0^{1/2+\varepsilon}
\]

\[
+ q^{1/2} (q\tau)^{-1/2} \sigma_0^{1/2+\varepsilon} + \tau(q\tau)^{-1/2} \left( 1 + \frac{\sigma_0(1-\theta)}{2(1+\theta)} \right)^{\sigma_0+\varepsilon}.
\]

Note that for \frac{1}{2} \leq \sigma_0 < 1, the third term dominates the fourth term. Therefore,

\[
E \ll q^{1/2+\varepsilon} \tau^{1/2} \left( 1 - \sigma_0 \right) + (q\tau)^{1/2} \left( 1 - \sigma_0 \right)^{\sigma_0+\varepsilon} + q^{1/2} (q\tau)^{1/2} \left( 1 - \sigma_0 \right)^{\sigma_0+\varepsilon}
\]

\[
+ \tau(q\tau)^{-1/2} \left( 1 + \frac{\sigma_0(1-\theta)}{2(1+\theta)} \right)^{\sigma_0+\varepsilon}.
\]

(8.2)

By (8.1) and (8.2), Theorem 1 follows.
ON AN ERROR TERM FOR THE FIRST MOMENT OF TWISTED $L$-FUNCTIONS

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