Morphology of Gamma-Ray Halos around Middle-aged Pulsars: Influence of the Pulsar Proper Motion

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Abstract

Recently, gamma-ray halos of a few degree extension have been detected around two middle-aged pulsars, namely, Geminga and PSR B0656+14, by the High Altitude Water Cherenkov observatory (HAWC). The gamma-ray radiation arises from relativistic electrons that escape the pulsar wind nebula and diffuse in the surrounding medium. The diffusion coefficient is found to be significantly lower than the average value in the Galactic disk. If so, given a typical transverse velocity of 300–500 km s\(^{-1}\) for a pulsar, its displacement could be important in shaping the morphology of its gamma-ray halos. Motivated by this, we study the morphology of pulsar halos considering the proper motion of pulsar. We define three evolutionary phases of the pulsar halo to categorize its morphological features. The morphology of pulsar halos below 10 TeV is double peaked or single peaked with an extended tail, which depends on the electron injection history. Above 10 TeV, the morphology of pulsar halos is nearly spherical, due to the short cooling timescale (<50 kyr) for tens of teraelectronvolt electrons. We also quantitatively evaluate the separation between the pulsar and the center of the gamma-ray halo, as well as the influence of different assumptions on the pulsar characteristics and the injected electrons. Our results suggest that the separation between the center of the gamma-ray halo above 10 TeV and the associated pulsar is usually too small to be observable by HAWC or the Large High Altitude Air Shower Observatory. Hence, our results provide a useful approach to constrain the origin of extended sources at very high energies.

Unified Astronomy Thesaurus concepts: Pulsars (1306); Gamma-ray sources (633); Extended radiation sources (504); Cosmic rays (329); Non-thermal radiation sources (1119)

1. Introduction

Pulsar wind nebulae (PWNe) are bubbles of relativistic electrons and positrons,\(^5\) accelerated when a pulsar’s relativistic wind interacts with its environment, either the supernova remnant (SNR) or the interstellar medium (ISM; e.g., Gaensler & Slane 2006). Pulsar is formed in a supernova (SN) explosion that drives a strong blast wave with \(\sim10^{51}\) erg kinetic energy expanding into the ambient medium. The pulsar and its PWN are initially surrounded by the SNR. The SNR blast wave at first moves outward freely at a speed \(\sim(5–10) \times 10^3\) km s\(^{-1}\), while the asymmetry in the SN explosion would give the pulsar a natal velocity. Observationally, we expect to see a rapidly expanding SN with a size of \(\sim1–10\) pc, a reasonably symmetric PWN near its center with a typical size at sub-parsec to parsec level, and a young pulsar at the center of the PWN, at the early evolutionary epoch of the SNR-PWN system.

The expanding SN shell starts to slow down as it sweeps up comparable mass of the surrounding ISM to that of the SN ejecta at a time \(t_{\text{sd}} \approx 1400(M_{\odot}/10M_{\odot})^{3/2}(E_{\text{SN}}/10^{51}\text{ erg})^{-1/2}(n_{\text{ISM}}/1\text{ cm}^{-3})^{-1/3}\) yr after the SN explosion (Draine 2011). Because the SNR is decelerating, the pulsar ultimately penetrates and then escapes the shell at a time \(t_{\text{cross}} \approx 45(E_{\text{SN}}/10^{51}\text{ erg})^{1/2}(n_{\text{ISM}}/1\text{ cm}^{-3})^{1/3}(v_{p}/400\text{ km s}^{-1})^{-5/3}\) kyr (van der Swaluw et al. 2003). After that, the pulsar proceeds to move through the ambient ISM with proper velocity \(v_{p}\), while the PWN at this stage is compact, smaller than 1 pc, and filled with recently injected particles (Kargaltsev et al. 2013).

The High Altitude Water Cherenkov Observatory (HAWC) recently reported the discovery of spatially extended teraelectronvolt sources surrounding two middle-aged \(t_{\text{age}} = 100–400\) kyr pulsars, namely, Geminga and PSR B0656+14 (Abeysekara et al. 2017; see also Abdo et al. 2009). The intensity profile of the observed extended teraelectronvolt sources can be explained by the inverse Compton (IC) scatterings of diffusing electrons, which are injected from the pulsars, on the cosmic microwave background (CMB) and the interstellar radiation field (ISR; Abeysekara et al. 2017; López-Coto & Giacinti 2018; Di Mauro et al. 2019; Jóhannesson et al. 2019; Tang & Piran 2019; Liu et al. 2019a, 2019b). The physical extension of the sources is at least 30 pc, which is much larger than the size of the PWN of the corresponding pulsars, likely indicating that accelerated electrons escape the PWN and produce the halo-like emissions in the ambient ISM of the pulsars.

In addition to Geminga and PSR B0656+14, many more such pulsar halos\(^6\) could have already been detected by instruments such as HAWC, the High Energy Stereoscopic System (H.E.S.S.), Fermi-Large Area Telescope (Fermi-LAT), and Large High Altitude Air Shower Observatory (LHAASO). The HAWC Collaboration recently released the 2HWC catalog (Abeysekara et al. 2018), which contains 39 sources detected close to the Galactic plane. Some of them have an extended

\(^5\) Hereafter we do not distinguish positrons from electrons for simplicity.

\(^6\) In some of the literature, they are also called teraelectronvolt halos since they were initially discovered at the teraelectronvolt band.
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morphology, and are spatially close to powerful Galactic pulsars of middle age. More recently, in the newly released 3HWC catalog (Albert et al. 2020) they highlight 12 extended teraelectronvolt sources as potential pulsar halos. In the reported PWNe and PWNe candidates by the H.E.S.S. Collaboration et al. (2018a, 2018b), the spatial extensions of some sources are significantly larger than 10 pc, which are beyond the prediction of the dynamical evolution model for PWNe (Reynolds & Chevalier 1984; van der Swaluw et al. 2001; Bucciantini et al. 2003). Some of these sources could also be pulsar halos in nature, or a mixture of PWN and halo (e.g., Liu & Yan 2020). LHAASO is a new generation instrument used to study the teraelectronvolt–petaelectronvolt gamma-ray sky (Bai et al. 2019). LHAASO contains two major gamma-ray astronomical devices: the Water Cherenkov Detector Array (WCDA) and the 1 km² Array (KM2A). Recently, LHAASO reported the discovery of 12 gamma-ray sources above 100 TeV, and some of them are pulsar halo candidates (LHAASO Collaboration et al. 2021). With its great sensitivity and large-area sky monitoring capability, LHAASO will play a key role in the detection of pulsar halos at the teraelectronvolt–petaelectronvolt band.

The morphology of gamma-ray halos is an important property for identifying pulsar halos. A pulsar’s proper motion may lead to a displacement of about $v_p \tau_{\text{age}} = 80(v_p \text{ km s}^{-1})(\tau_{\text{age}} \text{ 200 kyr}^{-1})$ pc from its birthplace. Such a displacement could be larger than the spatial extension of the teraelectronvolt halo for middle-aged pulsars, so it may largely affect the gamma-ray morphology. For example, Geminga has a transverse velocity of about $211(d/250 \text{pc})$ km s$^{-1}$ (Faherty et al. 2007) and has traveled about 70 pc in the plane of the sky. Some research groups have discussed the influence of Geminga’s proper motion on the morphology of its gamma-ray halo (Di Mauro et al. 2019; Jóhannesson et al. 2019; Tang & Piran 2019). It was found that the proper motion of Geminga can induce considerable asymmetry to its pulsar halo around 10 GeV (Di Mauro et al. 2019), while the deviation of teraelectronvolt halo from symmetry is only of the order of 5%–10% (Jóhannesson et al. 2019). At the same time, spatial offsets between the centroids of candidates of pulsar halos/ PWNe and the associated pulsars are commonly observed (Abeysekara et al. 2018; H.E.S.S. Collaboration et al. 2018a; Albert et al. 2020). Although the offset may be explained with the proper motion of the pulsar (H.E.S.S. Collaboration et al. 2018a; Di Mauro et al. 2020), it is not straightforward to connect a gamma-ray source and the associated pulsar in the case of a large offset being detected, especially without knowledge of the pulsar’s proper velocity. In addition, various factors, such as continuous injection of electrons, diffusion and cooling of electrons, magnetic and radiation field, and the limit of the angular resolution of instruments (i.e., the point-spread function (PSF)) could influence the expected offset. This could complicate the identification of pulsar halos while a detailed evaluation on the influence of these factors has not been performed.

Motivated by this, we study the effect of pulsar’s proper motion on the morphology of pulsar halos at the gigaelectronvolt and teraelectronvolt bands. We focus on pulsars with an age of $\tau_{\text{age}} \geq 100$ kyr, so that the pulsar had escaped out of the SNR and we can assume electrons that escape the PWN are diffusing in the ISM (Giacinti et al. 2020). For comparison with observations, we simulate the measured morphology of the halo by convolving the model-predicted intrinsic morphology with the PSF of some instruments, such as Fermi-LAT, H.E.S.S., and LHAASO. The rest of the paper is organized as follows. In Section 2, we introduce our model for the gamma-ray emission of the pulsar halo. In Section 3, we show and analyze the resulting gamma-ray intensity map (morphology) of the pulsar halo. In Section 4, we look into the offset between the center of the pulsar halo and the pulsar’s location caused by the proper motion. We discuss the results in Section 5 and provide our conclusions in Section 6.

2. Model for the Gamma-Ray Emission of a Pulsar Halo

In the model of the pulsar halo considered in this work, relativistic electrons are continuously injected into the ambient ISM by a pulsar. We assume that the evolution of the total electron injection rate follows the pulsar’s spin-down history with a braking index $n$ (Gaensler & Slane 2006)

$$L_n(t) = \eta_e L_i(1 + t/\tau_0)^{-(n+1)/(n-1)},$$

where $t$ is the time after the pulsar’s birth, $\eta_e$ is the fraction of the spin-down energy converted into relativistic electrons, $L_i$ is the initial spin-down luminosity of the pulsar, $\tau_0 = 2\tau_e/(n - 1)$ is the initial spin-down timescale with $\tau_e$ being the characteristic age. Given the present rotation period $P$ of the pulsar, the first derivative of the period $\dot{P}$ and the initial rotation period $P_0$, we have $\tau_e = P/2\dot{P}$ and $\tau_{\text{age}} = [2\tau_e/(n - 1)](1 - (P_0/P)^{n-1})$.

The observed and derived values of the above pulsar parameters scan a large range, e.g., $1 < n < 3$; their influence will be discussed in Sections 4 and 5. For clarity, we define two benchmark pulsars for which the main results are calculated, unless specified otherwise. The parameters of the benchmark pulsars are listed in Table 1. We assume the dipole magnetic model and take $n = 3$. We refer to the value of $P, \dot{P}$ of Geminga and assume $P_0 = 50$ ms, then we get $\tau_0 = 15$ kyr. We take $L_i = 1.7 \times 10^{37}$ erg s$^{-1}$, and notice that as a normalization, $\eta_e$ or $L_i$ will not influence the morphology of the pulsar halo. As for the age of the benchmark pulsar, we consider two cases, one is $\tau_{\text{age}} = 327$ kyr, corresponding to the age of Geminga, and the other is $\tau_{\text{age}} = 100$ kyr, for which the most energetic electrons have not cooled to several teraelectronvolts.

| Parameter | Value |
|-----------|-------|
| $L_i$ erg s$^{-1}$ | $1.7 \times 10^{37}$ |
| $\tau_0$ kyr$^{-1}$ | 15 |
| $n$ | 3 |
| $\tau_{\text{age}}$ kyr$^{-1}$ | 100.327 |
| $v_p$ km s$^{-1}$ | 400 |
| $d$ kpc | 2 |

Note. The initial spin-down luminosity ($L_i$), initial spin-down timescale ($\tau_0$), braking index ($n$), age ($\tau_{\text{age}}$), proper velocity ($v_p$), and distance ($d$) of the benchmark pulsar that is defined to calculate the morphology of the pulsar halo.

Table 1

Properties of the Benchmark Pulsar
We assume the injection electron spectrum to be a power-law function with an exponential cutoff, i.e.,

\[ Q_0(E_e, t) = Q_0(t) E_e^{-\gamma_e} \exp\left(-\frac{E_e}{E_0}\right), \]

(2)

where \( Q_0(t) \) is the normalization factor, which can be determined by \( \int E_eQ_0(E_e, t)dE_e = E_x(t) \). We consider \( \gamma_e = 1 \), \( E_0 = 400 \) TeV as reference parameters. For simplicity, here we do not consider any possible influence of the PWN and the related SNR on the historically injected electrons throughout the calculation. This might affect the resulting radiation at the gigaelectronvolt band but is not supposed to influence the halo at multi-teraelectronvolt energy or beyond, since the emitting electrons of the latter cool within hundreds of years and hence we do not expect to see the contribution of electrons injected at an early epoch. The shape of the assumed injection electron spectrum may influence the morphology of the pulsar halo, and is discussed in Section 4.

The transport equation of electrons injected from a point source located at \( r_s \) is

\[ \frac{\partial n_e}{\partial t} = D(E_e) \nabla^2 n_e + \frac{\partial [b(r, E_e, t)n_e]}{\partial E_e} + Q_e(E_e, t) \delta^3(r - r_s), \]

(3)

where \( n_e(r, E_e, t) \) is the differential electron density at time \( t \) and position \( r \), and \( D(E_e) \) is the diffusion coefficient, assuming isotropic diffusion, which is assumed to be spatially homogeneous and scales with the energy as \( D(E) = D_0(E/1\text{GeV})^{1/3} \) following a Kolmogorov-type turbulence. We note that, based on HAWC’s observation of Geminga and PSR B0656+14, it has been suggested that there may be a slow diffusion zone that \( D_0 \approx 10^{24} \text{cm}^2 \text{s}^{-1} \) in the vicinity of the pulsar within a radius of 30–100 pc from the pulsar while the diffusion coefficient beyond the radius is the standard one in the ISM measured from the primary-to-secondary cosmic-ray ratio (Fang et al. 2018; Profumo et al. 2018). In this work, we do not consider the two-zone scenario but will discuss its influence on our result qualitatively.

The term \( b(r, E_e, t) \) is the energy loss rate of electrons during the propagation, due to the synchrotron and IC radiation. Here, we consider a homogeneous and constant magnetic field and radiation field. The energy loss rate, \( b \), is given by

\[ b(E_e) = - \frac{dE_e}{dt} \approx - \frac{4}{3} \sigma_T c \left( \frac{E_e}{m_e c^2} \right)^2 \times \left[ U_B + \frac{U_{ph}}{\left(1 + \frac{E_{ph}}{m_e c^2} \right)^{3/2}} \right], \]

(4)

where \( \sigma_T \) is the Thomson cross section, \( U_B = B^2/8\pi \) is the magnetic field energy density, and \( U_{ph} \) is the radiation field energy density. \( c_0 = 2.82 \) K is the typical photon energy of the blackbody/graybody target radiation field. We assume a homogeneous magnetic field of \( B = 3 \mu G \) and consider four blackbody/graybody components around the pulsar as CMB \( (T = 2.73 \) K and \( U = 0.25 \) eV cm\(^{-3}\)), far-infrared radiation (FIR) field \( (T = 40 \) K and \( U = 1 \) eV cm\(^{-3}\)), near-infrared radiation field \( (T = 500 \) K and \( U = 0.4 \) eV cm\(^{-3}\)), and visible light radiation field \( (T = 3500 \) K and \( U = 1.9 \) eV cm\(^{-3}\)). The employed ISRF refers to the model by Popescu et al. (2017) at a smaller galactocentric radius (i.e., 3–5 kpc) given more teraelectronvolt PWN and PWN candidates appear at such radius (H.E.S.S. Collaboration et al. 2018a). The influence of magnetic field and radiation field intensities on the morphology of the pulsar halo will be discussed in Section 4 and Appendix.

The present-day \((t = t_{age})\) density of electrons with energy \( E_e \) at a radius \( r \) away from the pulsar can be calculated by

\[ n_e(r, E_e) = \int_0^{t_{age}} dt' Q_e(E_e(t'), t') \times \exp\left[-(r - r(t'))^2/4\lambda(E_e, t')\right] \frac{dE_e'}{(4\pi \lambda(E_e, t'))^{3/2}} \]

(5)

where \( \lambda(E_e, t') = \int_0^{t_{age}} D(E_e(t'')) dt'' \) and \( E_e' \) is the electron energy at injection. \( E_e(t') \) represents the trajectory of the energy evolution of an electron, whose energy is \( E_e \) at present. The relation between \( E_e(t') \) and \( E_e' \) as well as \( dE_e'/dE_e \) can be found by tracing the energy evolution of the electron via Equation (4). We take the cylindrical coordinate system and set the direction of the proper motion as the \( x \)-axis. The present pulsar position is set to \((0, 0, 0)\) and then the historical position of the pulsar can be given by \( r_s = (v_p(t_{age} - t), 0, 0) \). The measurement of the pulsar velocity along the line of sight (LOS) is generally not possible, except for several binaries (Hobbs et al. 2005). The mean transverse velocity of pulsars with \( \tau_c < 10^5 \) kyr is 307 km s\(^{-1}\), if modeling the pulsar velocity distribution with a Maxwellian distribution (Hobbs et al. 2005). By modeling the velocity distribution with two Maxвелlians, Verbunt et al. (2017) found that 32% of the pulsars with \( \tau_c < 10^5 \) kyr have an average transverse velocity of 130 km s\(^{-1}\) and 68% with 520 km s\(^{-1}\). For our benchmark pulsar, we take the angle between the pulsar proper motion direction and the LOS as \( \psi = 90^\circ \) and take \( v_p = 400 \) km s\(^{-1}\), namely, the transverse velocity of pulsar \( v_p \), to explore the morphology of the pulsar halo at different energies. We will discuss the influence of the value and direction of the pulsar proper velocity on the morphology of the pulsar halo in Section 4.

We then calculate the gamma-ray emission produced by the IC process of electrons and integrate them over the LOS at different viewing angles, following the method given by Liu et al. (2019b). We assume the distance of the benchmark pulsar to be 2 kpc, and will discuss its influence in Section 4. By taking \( \psi = 90^\circ \), the emission of the halo is projected onto the celestial plane and we obtain the gamma-ray intensity at a polar angle \( \theta \) from the pulsar and an azimuth angle \( \phi \) \((\phi = 0 \) points to the opposite direction of the proper motion) \( I'(E_e, \theta, \phi) \), which describes the intrinsic morphology of the gamma-ray halo.

We convolve \( I' \) with the PSF of various instruments at different energies (summarized in Table 2) to simulate the observed intensity map or the morphology of the gamma-ray halo by the instruments. The gamma-ray intensity map after the
convolution with the PSF is then given by

\[
I_{\gamma}(E_{\gamma}, \theta, \phi) = \int \frac{1}{2\pi\sigma^2} \times \exp\left(-\frac{l'^2}{2\sigma^2}\right) I_{\gamma}(E_{\gamma}, \theta', \phi') \sin \theta' d\theta' d\phi',
\]

where \(l' = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')\) is the angular distance between the point \((\theta, \phi)\) and the point \((\theta', \phi')\) in the celestial plane, and \(\sigma(E_{\gamma})\) is the size of the PSF as a function of gamma-ray energy.

3. Morphology of the Pulsar Halo

We first simulate and compare the morphology of the gamma-ray halo of the benchmark pulsar \((t_{age} = 327 \text{ kyr})\) with the results of Tang & Piran (2019) and Di Mauro et al. (2019), which considered continuous injection of electrons and a one-zone diffusion model. Our result is consistent with theirs qualitatively, though we assume different properties of the benchmark pulsar. Specifically, the intrinsic intensity maps of the pulsar halo at 10 GeV, 1, 10, and 100 TeV, as well as the corresponding intensity profile along the axis of the pulsar’s proper motion (x-axis) are shown in Figure 1. The influence of the pulsar proper motion on the morphology of its gamma-ray halo is energy dependent. The morphology of the pulsar halo at 10 GeV and 1 TeV is highly asymmetrical, with extended emission toward the right half part where the pulsar has passed. At 10 GeV, the pulsar halo is double peaked. The strongest emission is around the pulsar’s initial position, where the relic electrons injected at the early epoch are still emitting gamma-rays. This is because the cooling time of gigaelectronvolt-emitting electrons is longer than the age of the pulsar and the spin-down luminosity of the pulsar at the early epoch is hundreds of times larger than that at present. On the other hand, electrons injected recently form a comparatively high electron density region around the pulsar since they have not diffused far away, thus yielding another peak in the intensity profile. For teraelectronvolt-emitting electrons, their cooling time is shorter than, but still comparable to, the age of the pulsar. Therefore, the intensity map of the pulsar halo at 1 TeV extends toward \(x > 0\) but peaks around the current position of the pulsar. At higher energy (e.g., 10 and 100 TeV), the relic electrons are significantly cooled so the morphology of the halo shows a single peak centered at the current position of the pulsar without a significant extension and the emission of the halo is dominated by recently injected electrons.

In order to compare with observations, we also provide the intensity map and the profile after the convolution with the PSFs of four gamma-ray detectors in Figure 2, i.e., the PSF of Fermi-LAT for 10 GeV emission, H.E.S.S. for 1 TeV emission, LHAASO (WCDA) for 1 and 10 TeV emission, and LHAASO (KM2A) for 100 TeV emission (see Table 2). The PSF of HAWC is similar to that of LHAASO (WCDA). After the convolution, the profile becomes smoother and more extended. Such a change is more pronounced for larger PSF as can be seen by comparing the 1 TeV profile convolved with WCDA’s PSF and that convolved with H.E.S.S.’ PSF.

To understand more quantitatively the energy-dependent morphology of the pulsar halo and the influences of different model parameters, let us first define two critical timescale. The first one is \(t_{pd} = 80(E_e/1 \text{ TeV})^{1/2}(D_0/10^{26} \text{ cm}^2 \text{s}^{-1})\) \((v_0/400 \text{ km s}^{-1})\text{ yr}^{-1}\text{ kyr}\), the time when the electron’s diffusion distance \(\sqrt{\Delta r}\) is equal to the distance that the pulsar has moved \((v_0t)\). The other one, denoted by \(t_c\), is the time in which an electron cools from energy \(E_e\) (the cutoff energy in the injection spectrum) to certain energy \(E_c\) in the considered magnetic field and the radiation field. \(t_c\) can be calculated via Equation (4). For a fixed magnetic field and radiation field, the value of \(t_c\) is larger than the standard definition of the cooling timescale \(t_c(E_e)\) of electrons (i.e., \(t_c = E_e/\dot{E}_e(E_e)\)). Its value is mainly determined by \(E_e\) instead of \(E_c\). Then, we can divide the morphological evolution of the pulsar halo into three phases based on these two timescales. This definition of phases can help generalize different features and the origin of different pulsar halo morphologies. This simple physical definition works well for pulsars with \(n = 2.5\) and \(\tau_0 = 5\) kyr. The three phases of pulsar halo evolution can be given as follows:

PHASE I: \(t_{age} < t_{pd}, t_{age} < t_c\). The electrons injected at the early epoch with a high luminosity have not cooled and the pulsar’s displacement, due to the proper motion is still within the diffusion length of these relic electrons. The relic electrons produce bright radiation around the original position of the pulsar. Since relic electrons have diffused to a larger distance than the pulsar has traveled, the emissions of electrons injected at different epoch overlap with each other and yield a single and broad peak in the intensity map.

PHASE II: \(t_{pd} < t_{age} < t_c\). Since the displacement of the pulsar from its original position is proportional to \(t\), while the electron’s diffusion distance is proportional to \(t^{1/2}\), the pulsar eventually goes beyond the diffusion length of electrons injected at the early epoch, and the fresh electrons yield elongated bright regions along the pulsar trajectory. The pulsar halo is highly asymmetrical, being double peaked or single peaked with an elongated tail toward the direction \(x > 0\), such as shown in the 10 GeV intensity maps in Figure 2. The relative brightness at the original position and the current position of the pulsar depends on \(L(t)\).

PHASE III: \(t_{age} > t_c\). Electrons injected at the early epoch lost most of their energy and only recently injected electrons can produce gamma-ray emission. The morphology of the pulsar halo is again single peaked and compact, with a rough circular symmetry with respect to the current position of the pulsar, such as the 10 and 100 TeV intensity maps shown in Figure 2.

Note that in the transition stage from one phase to another phase (i.e., \(t_{age} \sim t_{pd}, t_{pd} \sim t_c\)), the feature of a certain phase mentioned above is not distinct and relies on the relative contribution from relic and newly injected electrons, i.e., \(L(t)\).

The halo of the benchmark pulsar \((t_{age} = 327 \text{ kyr})\) is already in PHASE II at the gigaelectronvolt band and PHASE III at the...
teraelectronvolt band. In Figure 3, we show the intensity profiles of the pulsar halo of the benchmark pulsar ($t_{\text{age}} = 327$ kyr) at 2 kpc. The direction of the proper motion is marked with the arrow. The stars and triangles mark the present and initial positions of the pulsar. The proper velocity is $v_p = 400$ km s$^{-1}$, $\psi = 90^\circ$, and the diffusion coefficient is $D_0(1$ GeV$) = 10^{26}$ cm$^2$ s$^{-1}$.

Figure 1. Gamma-ray intensity map of the pulsar halo and intensity profile along the axis of the pulsar’s proper motion at 10 GeV and 1, 10, and 100 TeV, considering the proper motion of the benchmark pulsar ($t_{\text{age}} = 327$ kyr) at 2 kpc. The direction of the proper motion is marked with the arrow. The stars and triangles mark the present and initial positions of the pulsar. The proper velocity is $v_p = 400$ km s$^{-1}$, $\psi = 90^\circ$, and the diffusion coefficient is $D_0(1$ GeV$) = 10^{26}$ cm$^2$ s$^{-1}$.

4. Separation Angle Estimation

We now look into the expected offset of the pulsar and the gamma-ray halo at different energies when measured by different instruments. We first produce the PSF-convolved...
intensity maps at different energies by convolving $1$–$100$ GeV emission with the PSF of Fermi-LAT, $1$–$20$ TeV emission with the PSFs of LHAASO (WCDA) and H.E.S.S., respectively, and $10$–$100$ TeV emission with the PSF of KM2A. Then, we define two kinds of offsets: one is the separation between the pulsar’ current position and the centroid of the halo (denoted by $\Theta$), and the other is the separation between the pulsar and the position of the brightest point or the intensity peak of the halo (denoted by $\Theta'$). The latter is straightforward to obtain. To get the former, we use a 2D Gaussian template as $I_\nu (r) = \frac{N_0}{2\pi \sigma_0^2} \exp \left[ - (r - r_c)^2 / 2 \sigma_0^2 \right]$, where $r_c$ is its center, to model the PSF-convolved intensity map. For each fitting, we
adjust the size of the region of interest to include the whole pulsar halo (defined as the gamma-ray intensity at the edge as 1% of the brightest point) and at the same time, maintain a higher resolution. We set \( N_0, \sigma_0, \) and \( r_c \) as free parameters, and search for the centroid of the halo \( (r_c) \) by minimizing the chi-square \( \chi^2 = \sum (I_G - I)^2 / I_c. \) This procedure can mimic the real analysis of the observation, except there is no background emission in our simulation.

Convolving the same intrinsic intensity map with the PSF of H.E.S.S. and the PSF of WCDA, respectively (for example, comparing the solid red and the solid blue lines in Figure 3), the peak of the obtained intensity profile can be different. This is also reflected in the separation angle, which can be seen in the middle panel of Figure 4. Both \( \Theta \) and \( \Theta' \) estimated with the PSF of H.E.S.S. (blue markers) are smaller than those with the PSF of WCDA (black markers). Such a tendency is particularly pronounced if the offset is evaluated by \( \Theta' \). For halo beyond teraelectronvolt energy, the position of the intensity peak in the intrinsic intensity map is basically the position of the pulsar’s current position. However, due to the asymmetry of the halo, the peak would shift to the more elongated side of the halo after convolving with the PSF of the detector. The smaller the PSF was, the less distance the peak would shift. For a detector of good angular resolution, such as H.E.S.S., the induced \( \Theta' \) could be even smaller than the resolution of our simulation if the intrinsic morphology of the halo is not sufficiently asymmetric.

The intriguing dependence of the resulting \( \Theta \) and \( \Theta' \) on various parameters, in addition to the PSF, will be discussed in the following subsections. When discussing the influence of parameters like the diffusion coefficient, magnetic field, and other parameters, we prefer to take the benchmark pulsar \( (t_{\text{age}} = 100 \text{ kyr}) \), thus the high-energy electrons producing teraelectronvolt gamma-ray emission have not cooled and the influence of the parameters can show up.

4.1. Dependence on the Gamma-Ray Energy

The obtained separation angles \( \Theta \) and \( \Theta' \) at different energies for benchmark pulsar \( (t_{\text{age}} = 327 \text{ kyr}) \) are plotted in Figure 4. To better interpret the results, we mark the evolutionary phases in each panel. The pulsar halo phase evolves following \( \text{II} \rightarrow \text{I} \rightarrow \text{III} \) or directly \( \text{II} \rightarrow \text{III} \) as the energy increases from 1 GeV to 10 TeV. The separation angle here is calculated based on a nominal distance of \( d = 2 \text{ kpc} \) for the pulsar, and the result approximately scales with \( d^{-1} \).

At the gigaelectronvolt band \( \Theta \) is energy independent and is close to the displacement of the pulsar, due to the proper motion in a time of \( t_{\text{age}} \), which is \( \Theta = 3.8^\circ \) for a nominal distance of 2 kpc from the pulsar. It reflects the emission of the huge amount of relic electrons injected at the early epoch (PHASE II and transition to PHASE III). \( \Theta' \) is constant at
PHASE II but jumps to almost zero after the pulsar halo enters PHASE III. The reason for such behavior is as follows: As the pulsar halo transits from PHASE II to PHASE III, the relic electrons start to cool so the radiation around the initial position of the pulsar becomes dimmer. At ~100 GeV, the relic electrons after cooling still make an important contribution to the entire emission of the pulsar so the decrease in $\Theta$ is limited, but the peak intensity is already lower than that of the electrons freshly injected, so $\Theta'$ is nearly zero. As energy further goes up, the cooling of the emitting electrons becomes increasingly important (PHASE III), the contribution of relic electrons reduces, leading to a more symmetric morphology of the halo at higher energy. The separation, either defined by $\Theta$ or $\Theta'$, becomes smaller. Above 10 TeV, the separation is smaller than 0.1° and is difficult to be resolved for a pulsar at 2 kpc by current gamma-ray detectors, especially for those with large PSFs such as HAWC and LHAASO.

Di Mauro et al. (2020) suggested that the offset between the pulsar and the pulsar halo induced by the pulsar proper motion can be analytically depicted by $\Theta'(E_c) = \tan(v_p \cdot t_c(E_c)/d)$ in the cooling-dominated stage (i.e., deep in PHASE III), where $t_c(E_c)$ is the cooling timescale of electrons, which dominates the gamma-ray flux at $E_c$. We overlaid the result given by this formula in Figure 4 for reference. The formula generally reproduces the trend of $\Theta$ versus gamma-ray energy, but it overestimates the separation angle. This is due to ignoring the influence of the continuous injection of electrons, which would produce gamma-ray emission along the pulsar trajectory and make the center of the pulsar halo deviate away from the position defined in the analytical formula. The influence of the continuous injection will be further discussed in Section 4.6.

4.2. Dependence on the Distance, Proper Velocity, and Diffusion Coefficient

For the same pulsar, the separation angle of the pulsar halo is supposed to scale with the distance of the pulsar $d_{\text{pcc}}$, as $\Theta(d_{\text{pcc}}) = \Theta(1 \text{ kpc})/d_{\text{pcc}}$. In Figure 5, we plot the calculated separation angle for the pulsar at different distances and the theoretical relation (dashed line). The relation fits the separation angles well except for $\Theta'$ of the benchmark pulsar at $t_{\text{age}} = 100$ kyr because these $\Theta'$ are dominated by the convolution with the PSF of the instrument, when the intrinsic separation is very small. The direction of the proper velocity $v_p$ of all pulsars in the galaxy is supposed to be randomly distributed, i.e., the cosine of the angle between $v_p$ and LOS, $\psi$, is equally distributed in the range (0, 1). Considering the observed gamma-ray flux is proportional to $1/d^2$, the gamma-ray luminosity could be larger at the initial position if the pulsar has a radial velocity and moves away from us, i.e., $\psi < 90^\circ$, and vice versa. Therefore, the LOS velocity, which is equal to $v_p \cos \psi$, may affect the pulsar halo morphology and separation angle we observed. We consider two cases for the pulsar’s proper motion: one is $v_p = 200 \text{ km s}^{-1}$ with $\psi = 90^\circ$, and the other is $v_p = 400 \text{ km s}^{-1}$ with $\psi = 30^\circ$, i.e., the transverse velocity in the plane of the sky $v_x = v_p \sin \psi$ of the two cases are the same. The separation angles of pulsars with these two velocities at different distances are plotted in Figure 5. We find that the direction of $v_p$ only causes an observable influence to the separation angle if the length that pulsar moved along the LOS is comparable to the distance of the pulsar. Thus, for the pulsar at a larger distance, we can ignore its velocity along the LOS and simply take its observed $v_p$ to calculate the theoretical gamma-ray halo morphology.

Since electrons are continuously injected into the ambient medium of the pulsar, a larger $v_p$ would induce a more elongated tail of the halo toward $x > 0$, as shown in Figure 3. This is correspondingly reflected in the $\Theta - \Theta'$ relation, as a monotonous increase of the separation angle with the proper velocity. The transition of the slope of the relation can be explained by the transition of the morphology of the halo. For example, the jump in $\Theta'$ with increasing $v_p$ for the 10 GeV halo (the top-left panel of Figure 6) is due to the pulsar halo transiting from single peak to double peak (i.e., PHASE I to PHASE II). The reason is the same for the jump in $\Theta'$ with increasing $D_0$ for the 10 GeV halo (top-right panel of Figure 6). The other interesting transition occurs when the Gaussian

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7 The formula given by Di Mauro et al. (2020) is an approximation and does not take into account the Klein–Nishina effect, which is important at high energy. Note that the Klein–Nishina effect is considered in their numerical simulations.

8 An exception is the $\Theta'$ for H.E.S.S. as shown with open blue circles. The offset between the position of the peak and the position of the pulsar can hardly be identified in our simulation, due to its small PSF. See also the discussion at the beginning of Section 4.
template is used to estimate the offset and the intensity profile is convolved with the H.E.S.S. PSF; as shown with blue stars in the middle-left panel of Figure 6, the $\Theta - v_{tr}$ relation becomes flat when $v_{tr} > 400$ km s$^{-1}$. The reason is that the morphology of the intrinsic halo at the teraelectronvolt band has a single broad peak in PHASE I, which can be described with a Gaussian template. In PHASE II, the morphology of the halo is single peaked with an elongated tail at the teraelectronvolt band. Due to the good resolution of H.E.S.S. (i.e., a small PSF), such a feature is reserved in the PSF-convolved intensity profile and deviates from the Gaussian template. Therefore, when using the Gaussian template to fit the intensity map measured by H.E.S.S., the peak is overweighted, while the elongated tail barely influences the determination of the centroid. By contrast, when the intensity profile in PHASE II is convolved with the PSF of WCDA, the latter’s large PSF smears out the fine structure, leaving only a single broad peak (see the solid blue curve in Figure 3), which is similar to that in PHASE I. As a result, we do not expect to see such a prominent transition in the $\Theta - v_{tr}$ relation if the halo is measured by WCDA, as shown with the black crosses in the middle-left panel of Figure 6.

With decreasing diffusion coefficient, electrons injected at the early epoch diffuse more slowly and yield a brighter region at $x > 0$, so we expect a more asymmetric pulsar halo and subsequently a larger separation angle with a smaller $D_0$, as shown in the right panels of Figure 6. Specifically, if the emitting electrons at the beginning stage are not cooled, a smaller $D_0$ results in a smaller $t_{pd}$, leading to the earlier transition of the halo from PHASE I to PHASE II.
radiation from the relic electrons and the fresh electrons becomes distinct as electrons are better confined around the position where they are injected, so the separation angle becomes larger. Similarly, although no relic electrons survived in PHASE III, a smaller $D_0$ can nevertheless enhance the radiation of electrons injected at earlier times, so the halo would appear more elongated toward $x > 0$. However, we note that the influence of $D_0$ on $\Theta$ or $\Theta'$ is limited by the age of the pulsar in PHASE II or by the cooling timescale of electrons in PHASE III because the separation angle would be at most $\nu_t r_{\text{age}} / d$ in PHASE I/II or $\nu_t r_{\text{age}} / d$ in PHASE III. This explains why the slope of $\Theta(\Theta')$ against the diffusion coefficient becomes flat at the low $D_0$ end, as shown in the right panels of Figure 6.

4.3. Dependence on the Injection Spectrum of Electrons

When the early injected high-energy electrons cool to lower energies, they can produce lower-energy gamma-ray emission. Therefore, the electron injection spectrum may affect the separation angle of the pulsar halo. Here, we compare two additional cases of power-law electron injection spectrum, one softer ($\gamma_e = 2.5$) and the other harder ($\gamma_e = 1.5$), with that in the benchmark case ($\gamma_e = 2.0$).

The gamma-ray emission at the initial position of the pulsar is produced by electrons cooled from higher energy, and a harder spectrum in general leads to more high-energy relic electrons. As such, the emission at the initial position is enhanced given a harder spectrum with respect to that at the pulsar’s current position. This is consistent with the results for $E > 1$ TeV, as shown in Figure 7, where we see the separation angles with $\gamma_e = 1.5$ is larger than that with $\gamma_e = 2.0$, and the difference can be resolved by the instruments. On the other hand, comparing the case of $\gamma_e = 2.0$ with the case of $\gamma_e = 2.5$, we find that the influence of a softer spectrum is not significant because the majority of the halo’s tereaelectronvolt emission is already emitted by recently injected electrons and the centroid is around the current position of the pulsar. Although a softer injection spectrum would further reduce the contribution of relic electrons, it cannot make $\Theta$ significantly smaller.

Note that the situation is different at the gigaelectronvolt band (shown in the top-left panel of Figure 7) because a harder injection spectrum would lead to fewer electrons that radiate at such low energies, provided the same bolometric injection luminosity. Therefore, opposite to the case at the tereaelectronvolt band, the separation angle at the gigaelectronvolt band would be reduced given a harder injection spectrum. However, since the cooling is not important at low energy, the influence of the injection spectrum is not significant, except at 20 GeV for $\Theta'$. This is because $\Theta'$ measures the separation between the brightest point from the pulsar in the intensity map. At the gigaelectronvolt band, the morphology is double peaked, due to the slow diffusion and the inefficient cooling of electrons, as discussed earlier. The intensity ratio between the peak at the pulsar’s position and the peak at the original position increases with energy, whereas the position of the brightest point

Figure 7. The separation angle $\theta$ and $\theta'$ of the benchmark pulsar ($t_{\text{age}} = 100$ kyr) with different electron injection spectra at different energies. The blue markers are the results of the power-law spectrum ($\gamma_e = 2.0$) with an exponential cutoff. The gray markers are the results of softer power-law spectrum ($\gamma_e = 2.5$). The black markers are the results of harder power-law spectrum ($\gamma_e = 1.5$). Where no blue points appear is because the blue points are overlaid with gray points.
transfers from the original position at $E \lesssim 10$ GeV to the pulsar’s current position at $E \gtrsim 10$ GeV. Around 20 GeV, the intensities between the two peaks are comparable to each other and hence the position of the brightest point in the intensity map becomes sensitive to model parameters even if the influence is small.

We note that the injection electron spectrum was assumed to be a broken power law in some of the literature (e.g., Bucciantini et al. 2011; Ishizaki et al. 2017), usually with a spectral break around 0.1–1 TeV. Since the spectrum after the break is soft, we do not expect that assuming a broken power-law spectrum has a significant influence on the separation angle in the benchmark case.

4.4. Dependence on the Magnetic Field Strength

The strengths of the radiation field and the magnetic field affect the cooling of electrons, and consequently, the separation angle of the pulsar halo. The gamma-ray emission we are concerned with is mainly produced by electrons upscattering the FIR and CMB radiation. We find the influence of the radiation field on the separation angle is finite and the discussion is presented in Appendix. On the other hand, the energy loss timescale through the synchrotron process is comparable to the IC process and is the main energy loss process for $>10$ TeV electrons when $B = 3 \mu$G. Thus, the influence of $B$ is more important. We compare the separation angles of $B = 1$, $B = 3 \mu$G, and $B_0 = 6 \mu$G, as shown in Figure 8. The separation angle of the stronger magnetic field is smaller since the faster cooling of relic electrons and correspondingly less dominant gamma-ray emission, and vice versa. The difference in $\Theta$ is observable by LHAASO (WCDA) and H.E.S.S. and reaches two times above 4 TeV. In the interpretation of the observed offset of the pulsar halo, the magnetic field strength can affect the theoretical calculation and conclusion.

4.5. Evolution of the Separation Angle with Pulsar Age

The age of the pulsar directly influences the displacement of the pulsar and thus the separation angle of the pulsar halo. We calculate the separation angle at 10 GeV, and 1 and 10 TeV of the benchmark pulsar with different $t_{\text{age}}$. The result is plotted in Figure 9, with the evolutionary phases marked. At PHASE I and II, $\Theta$ increases with $t_{\text{age}}$, linearly and its value is limited by the displacement of the pulsar. This reflects that the relic electrons contribute the majority of the gamma-ray emission. When the pulsar halo enters PHASE III, $\Theta$ starts to decrease with $t_{\text{age}}$. This is due to the cooling of relic electrons, and the morphology of the pulsar halo becomes more symmetric. For $\Theta'$, there is a jump after the pulsar halo enters PHASE II, as the pulsar halo transfers to being double peaked. And similar to $\Theta$, $\Theta'$ also decreases with $t_{\text{age}}$ at PHASE III. Besides, the cooling timescale of tens of teraelectronvolt electrons is less than 50 kyr; thus, only electrons injected within this timescale can still radiate $>10$ TeV gamma-ray emission. Thus, for the middle-aged pulsar ($t_{\text{age}} > 100$ kyr), $\Theta$ and $\Theta'$ of $>10$ TeV halo are nearly constant with $t_{\text{age}}$. 
4.6. Maximum Separation Angle

The analytical estimation of the separation angles shown as black curves in Figure 4 largely exceeds the numerical results in PHASE III. This is due to ignoring the continuous injection of electrons in the analytical formula as discussed above. We note that in certain conditions the influence of the continuous injection could be suppressed. Practically, as the pulsar wind luminosity decreases with time, the maximum energy of electrons achievable in the termination shock may also decrease with time. When the maximum energy is below a certain energy, we can regard the injection of electrons of this energy as being ceased. The termination of injection would cause a larger offset because electrons are not injected at the position close to the present position of the pulsar. However, if the injection stops too early, i.e., the time experienced from the epoch of the injection termination to the present time, denoted by \( t_s \), is longer than the cooling timescale \( t_c \), we do not expect to see the emission of electrons at present. Therefore, we expect the most favorable condition for a large offset to be \( t_s \sim t_c \). We then calculate the evolution of the pulsar halo at each energy by turning off the electron injection at \( t = \text{age} - t_c \) to maximize the separation angle. The result is shown in Figure 10, assuming \( B = 3 \mu \text{G} \). The offset obtained in this condition (\( Q_{\text{max}} \)) is independent of the electron injection history. We find that the \( Q_{\text{max}} \) relation in the range of 1–100 TeV can be empirically depicted by

\[
Q_{\text{max}} = 3^5 \left( \frac{E}{1 \text{ TeV}} \right)^{-0.77} \left( \frac{v_t}{400 \text{ km s}^{-1}} \right) \left( \frac{d}{2 \text{ kpc}} \right)^{-1},
\]

except in the case that the offset is evaluated by \( Q' \) while the halo is measured by an instrument of good angular resolution such as H.E.S.S. We note that the offset above 100 TeV is quite small, due to the rapid cooling of ultrahigh-energy electrons and is difficult to be resolved by the current gamma-ray instrument, i.e., LHAASO-KM2A, unless the pulsar is located sufficiently close to Earth (e.g., at a distance of \( \sim 100 \text{ pc} \) from Earth) and has a high proper velocity (e.g., \( \gtrsim 1000 \text{ km s}^{-1} \)).

5. Discussion

5.1. The Evolution of the Spin-down Luminosity of a Pulsar

The electron luminosity injected by a pulsar is assumed to be proportional to the spin-down luminosity of a pulsar, which is determined by the braking index \( n \) and the initial spin-down timescale \( \tau_0 \). In the previous sections, we considered a typical value \( n = 3 \), assuming the pulsar to be a magnetic dipole, and \( \tau_0 = 15 \text{ kyr} \), following the case of Geminga. However, these two parameters are not necessarily universal for other middle-
aged pulsars. Thus, we discuss the uncertainty in the electron injection history \(L_e(t)\), i.e., \(n\) and \(\tau_0\), in this subsection.

The braking index \(n\) is defined as \(\dot{v}/v^2\), where \(v = 2\pi/P\) is the angular frequency of the pulsar and \(\dot{v}(v)\) is the time differential of \(v(v)\). The direct measurement of \(n\) requires the pulsar to be young enough to have a measurable \(\dot{v}\). Most of the young pulsars have \(1 < n < 3\) (Lyne et al. 1993, 1996; Livingstone et al. 2005; Weltevrede et al. 2011; Roy et al. 2012; Hamil et al. 2015), but recently it has also been found at a pulsar with \(n > 3\) (Archibald et al. 2016). On the other hand, these young pulsars may not represent the properties of older pulsars. The decay of the pulsar magnetic field and inclination angle can induce the evolution of \(n\) with time, i.e., increasing with time (Tauris & Konar 2001; Johnston & Karastergiou 2017). Recently, Cholis et al. (2018) used the measurement of electron and positron spectrum to constrain the spin-down properties of pulsars. They found that \(n < 2.5\) is disfavored for the majority of middle-aged pulsars (> 100 kyr).

The initial spin-down timescale \(\tau_0\) can be derived if the initial period of the pulsar, \(P_0\), is further given. The measurement of \(P_0\) is only possible for several pulsars’ associated SNR, so that one may estimate the true age of the pulsar and derive \(P_0\). Based on this method, Popov & Turolla (2012) found that \(P_0\) ranges from 20–300 ms, in a Gaussian distribution with an average of 100 ms and standard deviation of 100 ms. Another method to study the distribution of \(P_0\) of Galactic pulsars is to search for a distribution of \(P_0\) that can reproduce the properties of observed pulsar population, via modeling the evolution of the pulsar from birth (Faucher-Giguère & Kaspi 2006; Johnston & Karastergiou 2017). Assuming \(n\) is a constant, Faucher-Giguère & Kaspi (2006) found that the distribution of \(P_0\) has a mean value of 300 ms with a wide distribution. Combining the above two methods, the derived \(\tau_0\) of most middle-aged pulsars ranges from 2–100 kyr.

Besides, from the perspective of the selection effect, those middle-aged pulsars that can produce observable pulsar halos are less likely to have a small \(n\) and/or a small \(\tau_0\) because such parameters lead to a rapid decline in the spin-down luminosity of the pulsar and hence a lower electron injection luminosity at the present time.

From a theoretical point of view, either a smaller \(n\) or a smaller \(\tau_0\) leads to a more rapid decline of the spin-down luminosity with time, which enhances the relative contribution of the relic electrons, and vice versa. To look into the influences of the electron injection history (i.e., \(n\) and \(\tau_0\)), we compare the gamma-ray profiles and separation angles under different combinations of the two parameters, including \(\tau_0 = 2, 5, \) and 30 kyr and \(n = 3, n = 2 \) and 2.5, and \(\tau_0 = 15\) kyr, \(n = 2.5\), and \(\tau_0 = 5\) kyr, while the other parameters are the same as the benchmark parameters. We choose to show the corresponding profiles of the 1 TeV gamma-ray halo at \(t_{\text{age}} = 100\) kyr (Figure 11) because the halo is at the transition stage from PHASE I to II/III (i.e., \(t_{\text{age}} \lesssim t_c \lesssim t_{\text{psf}}, \) where \(t_c = 115\) kyr, \(t_{\text{psf}} = 126\) kyr), and the morphological feature of the halo is sensitive to the electron injection history. We normalize the electron injection luminosity to be \(L_e(t = 100\text{ kyr}) = 2.9 \times 10^{33}\text{ erg s}^{-1}\), which is the spin-down luminosity of Geminga at 100 kyr under the benchmark parameters. Note that although the luminosity at \(t_{\text{age}}\) is the same, the intensity at the current pulsar’s position (\(x = 0\)) is different. This is due to the different electron injection histories.

The pulsar’s present position is shifted by about 40 pc from the initial position or 1.1° for \(d = 2\) kpc. Comparing the solid gray curve, which represents the profile of the benchmark pulsar \((t_{\text{age}} = 100\text{ kyr})\), but with different evolution of \(L_e(t)\), after convolving with the PSF of H.E.S.S.

![Figure 11](image-url)

The gamma-ray intensity profile at 1 TeV of the benchmark pulsar \((t_{\text{age}} = 100\text{ kyr})\), but with different evolution of \(L_e(t)\), after convolving with the PSF of H.E.S.S.
and $\tau_0$ converge at $t_{age}$ or at the deep PHASE III (e.g., at $t_{age} = 300$ kyr, comparable to the age of Geminga). Comparing the evolution of the separation angle at 1 TeV to that at 10 TeV, we can see that the higher the gamma-ray energy, the faster it converges. Indeed, when relic electrons have been severely cooled in the deep PHASE III, the injection history can barely influence the present morphology of the halo.

5.2. Application to the Observation

As we discussed above, it is increasingly difficult to produce a resolvable offset between the center of the pulsar halo and the pulsar by the proper motion at higher energy. An observable $\Theta$ requires that either the pulsar is located close by, its proper velocity is extremely high, the magnetic field is weak, or its
braking index and the $\tau_0$ of the pulsar is small. However, offsets at such a high energy between extended teraelectronvolt sources and the positions of associated middle-aged pulsars have already been observed by HAWC and LHAASO.

The 3HWC catalog gives 12 extended teraelectronvolt sources and the separation between the sources and their candidate pulsars (Albert et al. 2020). Note that the analysis of the 3HWC catalog does not optimize the sizes of the sources and most sources are identified as point sources, so the reported source position is not necessary to reveal the true centroid of the source. Nevertheless, we compare the theoretical separation angle induced by the pulsar motion and the observed offsets to roughly judge whether the association is possible. We list the 3HWC sources and potential pulsars in Table 3. If we assume the separation is caused by a pulsar proper motion, four 3HWC sources can be excluded from the association with candidate pulsars, due to the unaligned pulsar motion direction. To associate 3HWC J0540+228 with pulsar B0540+23, the magnetic field of the ISM needs to be smaller than $1 \mu G$, or the braking index of pulsar needs to be at least smaller than 2. For the other eight 3HWC sources, the association is possible with the proper combination of parameters and we need further knowledge of the proper motion velocity of their candidate pulsars to provide a robust judgment.

Very recently, the LHAASO Collaboration et al. (2021) reported the discovery of 12 gamma-ray sources above 100 TeV with more than a $7\sigma$ statistical significance. Among them, two pulsar halo candidates, i.e., LHAASO J2032+4102 and LHAASO J1929+1745, are possibly associated with the middle-aged pulsars PSR J2032+4127 and PSR J1928+1746, respectively. We may observe the separated observation with the theoretical maximum $\Theta$ induced by pulsar proper motion (Equation (7)) to quickly estimate whether the association could be true. We find that it is impossible for LHAASO J2032+4102 to be entirely associated with PSR J2032+4127, due to the small $v_{\text{rel}}$ of the pulsar, and the association of LHAASO J1929+1745 with PSR J1928+1746 requires an extremely large $v_{\text{rel}}$. The results are also shown in Table 3.

If a 3HWC source or a LHAASO source is truly a pulsar halo, but the spatial offset between the source and the related pulsar is very large, we have to resort to more complicated but probably realistic scenarios, such as the anisotropic particle diffusion scenario if the geometric configuration of the magnetic field in the surrounding ISM of the pulsars is not chaotic (Liu et al. 2019b). Let us envisage a specific scenario where the magnetic field on the east side of a pulsar is largely parallel to the periphery of the PWN while on the west side, it is largely radial. The diffusion of escaping electrons toward the east is then suppressed because the cross-field diffusion is slow, while the electrons can quickly diffuse toward the west. The pulsar halo under such a configuration of the magnetic field would be significantly extended toward the west side and a large offset is expected between the pulsar and the centroid of the halo. Of course, the source is not necessarily a pulsar halo. Some sources could have an asymmetric morphology intrinsically. With certain specific conditions, a $\sim$10 kyr aged PWN could vigorously expand (Khangulyan et al. 2018), while the asymmetric reverse shock arising from the SN ejecta could crush one side of the PWN (Blondin et al. 2001; Gaensler et al. 2003; Aharonian et al. 2006) and/or electrons preferentially escape from one side of the PWN, due to the geometry of the magnetic field in the PWN (Liu & Yan 2020). These scenarios might cause a significant offset between the centroid of the teraelectronvolt emission and the pulsar’s location. Alternatively, the source could be composed of multiple origins. There is more than one source candidate around the two LHAASO sources, as shown in Extended Data Table 2 in LHAASO Collaboration et al. (2021). Giacinti et al. (2020) suggested that

### Table 3

| 3HWC          | Pulsar     | $\tau_c$(kyr) | $d$ (kpc) | $v_{\text{rel}}$(km s$^{-1}$) | $\theta_{\text{obs}}$(°) | Comment |
|---------------|------------|---------------|-----------|-------------------------------|--------------------------|---------|
| J0540+228     | B0540+23   | 253           | 1.56      | 215                           | 0.83                     | B < 1 $\mu G$ or $n < 2$ |
| J0543+231     | B0540+23   | 253           | 1.56      | 215                           | 0.36                     | Unaligned |
| J0631+169     | J0633+1746 | 342           | 0.19      | 128                           | 0.95                     | Possible |
| J0634+180     | J0633+1746 | 342           | 0.19      | 128                           | 0.38                     | Unaligned |
| J0659+147     | B0656+14   | 111           | 0.29      | 60                            | 0.51                     | Unaligned |
| J0702+147     | B0656+14   | 111           | 0.29      | 60                            | 0.77                     | Unaligned |
| J1739+099     | J1740+1000 | 114           | 1.23      | ...                           | 0.13                     | Unclear |
| J1831+095     | J1831-0952 | 128           | 3.68      | ...                           | 0.27                     | Unclear |
| J1912+103     | J1913+1011 | 169           | 4.61      | ...                           | 0.31                     | Unclear |
| J1923+169     | J1925+1720 | 115           | 5.06      | ...                           | 0.67                     | Unclear |
| J1928+178     | J1925+1720 | 115           | 5.06      | ...                           | 0.85                     | Unclear |
| J2031+415     | J2032+4127 | 201           | 1.33      | ...                           | 0.11                     | Unclear |

| LHAASO        | Pulsar     | $\tau_c$(kyr) | $d$ (kpc) | $v_{\text{rel}}$(km s$^{-1}$) | $\theta_{\text{obs}}$(°) | Comment |
|---------------|------------|---------------|-----------|-------------------------------|--------------------------|---------|
| J2032+4102    | J2032+4127 | 201           | 1.4$^a$   | 20.4$^b$                      | 0.42                     | Unlikely |
| J1929+1745    | J1928+1746 | 82.6          | 4.6       | ...                           | 0.25                     | $v_{\text{rel}} > 2700$ km s$^{-1}$ |

Notes. Pulsar halo candidates in 3HWC and LHAASO source list (Column 1), related pulsars (Column 2), the pulsar’s characteristic age (Column 3), distance (Column 4), velocity (Column 5), and the observed separation between pulsar and source (Column 6). The properties of the pulsars are from the ATNF pulsar catalog (Manchester et al. 2005), unless otherwise specified. Our brief comment on whether the separation between the pulsar and the source can be explained by the pulsar proper motion is given in the last column: "Unaligned" means the association is impossible, due to the unaligned pulsar motion direction with the relative position of the 3HWC source and candidate pulsar; "Possible" means the offset can in principle be explained by the proper motion of the pulsar, given the measured the velocity and direction of the pulsar’s proper motion; “Unclear” means no measurement on the proper motion of the pulsar, but the offset may be explained by the proper motion if both the velocity and the direction are appropriate.

$^a$ Rygl et al. (2012).

$^b$ Jennings et al. (2018).
the energy density of the relativistic electron density inside the source being smaller than that of the ISM might be a criterion for the pulsar halo. Multimwavelength observations with high angular resolution will be crucial to reveal the true nature of the sources.

5.3. Influence of Two Diffusion Zones

In our calculation of electrons diffusion, we consider a spatially homogeneous diffusion coefficient around the pulsar at a scale of ~100 pc. Some previous studies have suggested that there could exist two diffusion zones in this region, with a slow diffusion zone within a few tens of parsecs around the pulsar and a normal diffusion zone beyond the radius (Fang et al. 2018; Profumo et al. 2018; Tang & Piran 2019). Our present numerical treatment cannot deal with two-zone diffusion with the proper motion of pulsar because the diffusion would become highly anisotropic if the pulsar has moved close to the boundary between the slow diffusion zone and the normal diffusion zone. Such a situation could happen since the displacement of the pulsar, due to the proper motion \( v_{\text{pulsar}} \) could be comparable to the size of the slow diffusion zone. Nevertheless, we may infer the influence of two-zone diffusion qualitatively. If the slow diffusion zone is centered at the initial position of a pulsar, a large fraction of the huge amount of electrons injected at the early epoch would be still confined inside the slow diffusion zone if not cooled. On the other hand, the electrons injected recently would more likely diffuse in the normal diffusion zone, and lead to a comparatively low particle density around the current position of the pulsar. As a consequence, it is beneficial to yield a large offset. On the contrary, if the slow diffusion zone moves with the pulsar, or it is centered at the pulsar’s present position, electrons injected at an early epoch would largely diffuse in the normal diffusion zone and hence their radiation becomes diffuse and faint, while electrons injected recently would diffuse in the slow diffusion zone and form a comparatively high density around the pulsar. As a result, the halo would be dominated by the radiation of electrons injected recently, appearing more symmetrical with a smaller offset. We leave the detailed and quantitative discussion of different mechanisms of two-zone diffusion and their influences to future work. Note that the two-zone diffusion model mainly affects the morphology of the pulsar halo at the giga-electronvolt–tera-electronvolt bands, which can extend beyond the size of the slow diffusion zone. For the pulsar halo above 10 TeV (PHASE III), the situation is similar to the single diffusion zone scenario because the size of the halo is generally limited, due to rapid cooling.

Our analysis here is consistent with the result of Jöhnkensson et al. (2019). They considered the two-zone diffusion model to explain the pulsar halo of Geminga. They considered three scenarios about the slow diffusion zone: (1) the slow diffusion zone centers at the birthplace of the pulsar; (2) it centers at the current position of the pulsar; and (3) it moves with the pulsar. For the first scenario to be compatible with the observation, the slow diffusion zone of the second scenario needs to be large enough to include both the birthplace and the current position of Geminga, which is similar to the one-zone scenario. The other two scenarios result in a more symmetrical halo, with a single peak and a short tail, due to the reason we discussed above.

6. Conclusions

In this paper, we studied the theoretical gamma-ray morphology of a pulsar halo and the spatial offset between the halo and its associated pulsar considering the pulsar’s proper motion. We divide the evolution of the pulsar halo into three characteristic phases, based on three timescales, namely, when the displacement of the pulsar equals the diffusion length of the initially injected particle \( \tau_{\text{pd}} \), the cooling timescale of electrons \( \tau_{e} \), and the age of the pulsar \( \tau_{\text{age}} \). For a pulsar with \( \tau_{\text{age}} < \tau_{\text{pd}} \) and \( \tau_{\text{age}} < \tau_{e} \), the pulsar halo would appear as a single-peaked morphology with a broad peak (PHASE I). For \( \tau_{\text{pd}} < \tau_{\text{age}} < \tau_{e} \), the morphology of the pulsar halo would show significant asymmetry with two humps or one hump with an extended plateau/tail (PHASE II). For \( \tau_{e} < \tau_{\text{age}} \), the morphology becomes single peaked again but with a narrow peak centered at the current position of the pulsar (PHASE III). Note that in the phase-transition stage (i.e., \( \tau_{\text{pd}} \sim \tau_{\text{age}} \), or \( \tau_{e} \sim \tau_{\text{age}} \)), the morphological feature of a phase may not be distinct and is affected by the electron injection history. Our study can give a clue to the origin of the asymmetrical morphology of the pulsar halos at the giga-electronvolt–tera-electronvolt bands. If the age and the proper velocity of a pulsar are known, we can quickly figure out which phase its pulsar halo is experiencing and judge whether the observed morphology is consistent with the origin of the pulsar halo. It would be helpful to understand the origin of the extended gamma-ray sources.

We defined two kinds of offsets between the pulsar halo and the pulsar: one is the separation between the centroid of the halo and the pulsar’s position (\( \Theta \)) and the other is the separation between the brightest position in the halo and the pulsar’s position (\( \Theta' \)). The influence of various model parameters on the separation angles is studied. We found that both separation angles generally decrease with increasing gamma-ray energy. The offset becomes difficult to be resolved above 10 TeV, due to the very rapid cooling of the emitting electrons, unless the pulsar is in close proximity to Earth (e.g., \( \lesssim 100 \) pc) and/or has a high proper velocity (\( > 1000 \text{ km s}^{-1} \)). We also found that \( \Theta \) and \( \Theta' \) have a weak dependence on some parameters, i.e., the electron injection spectrum and background photon field. Other parameters, such as the magnetic field strength of ISM, affect \( \Theta \) significantly and should be treated carefully when interpreting the observation. From the dependence we obtained, we can estimate the theoretical separation angle. Then, by comparing it to the observed value, we can either exclude the possibility of some association between the candidate pulsar and the extended emission observed by HAWC or LHAASO, or put constraints on parameters that are needed to make the association possible.

Under an extreme assumption that electron injection terminates at \( t = \tau_{\text{age}} - \tau_{e} \), we gave the maximum separation angle (\( \Theta_{\text{max}} \) and \( \Theta'_{\text{max}} \)) that the pulsar’s proper motion can induce at PHASE III. We found that the maximum separation angle of the emission centroid can be given by \( \Theta_{\text{max}} = 3\left(E_{e}/1 \text{ TeV}\right)^{0.77} \left(v_{\text{pulsar}}/400 \text{ km s}^{-1}\right) \left(d/2 \text{ kpc}\right)^{-3/2} \), empirically, assuming \( B = 3 \mu \text{G} \). Even for instruments with good angular resolution, such as H.E.S.S., we do not expect to see an offset between the brightest point in the halo and the position of the pulsar for reasonable parameters. Therefore, if too large an offset is detected, it would require the consideration of a more complex configuration of the magnetic field and correspondingly anisotropic diffusion of electrons. Alternatively, it might imply that the source is not simply a pulsar halo.
Multiwavelength observation with high angular resolution would then be helpful to reveal the true origin of the gamma-ray source. So far, only a few pulsar halos have been detected. Future observation by LHAASO and HAWC should be able to bring us more samples and provide the opportunity to explore the mechanism of the pulsar halo offset and test our model.

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Appendix
Dependence of Separation Angle on Radiation Field

As we discussed in the main text, the strength of the radiation field affects the cooling of electrons and correspondingly affects the separation angle of the pulsar halo. Here, we take a weaker radiation field (the one at 13 kpc from the Galactic center) around the pulsar as CMB ($T = 2.73$ K and $U = 0.25$ eV cm$^{-3}$), FIR radiation field ($T = 30$ K and $U = 0.1$ eV cm$^{-3}$), near-infrared radiation field ($T = 500$ K and $U = 0.04$ eV cm$^{-3}$), and visible light radiation field ($T = 5000$ K and $U = 1.9$ eV cm$^{-3}$) and compare the separation angles of the benchmark model photon field (BMF) used in the main text with those of this weaker photon field (WF). The result is plotted in Figure 14. Overall, the separation angle of the WF is smaller than the one of the BMF. This is because in the BMF, the gamma-ray emission from gigaelectronvolts to about 10 TeV is produced by electrons and the FIR field. But with a 10 times weaker FIR field, the gigaelectronvolt–teraelectronvolt gamma-ray emission is produced by the IC process between electrons and the CMB field in the WF model. Thus, higher energy electrons are needed in the WF model and correspondingly a shorter cooling timescale causes the gamma-ray emission to be more focused around the present pulsar position. The difference in $\Theta$ induced by different radiation fields is observable only at around 1 TeV by H.E.S.S.; we thus conclude that the influence of a different photon field around a pulsar is finite.

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Figure 14. The separation angle $\Theta$ and $\theta'$ of the benchmark pulsar ($t_{age} = 100$ kyr) but assuming a different background radiation field at different energies. The blue points are based on the assumption that the BMF and the gray points assume the WF.
