A Balanced Truncation Primer

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Balanced truncation, a technique from robust control theory, is a systematic method for producing simple approximate models of complex linear systems. This technique may have significant applications in physics, particularly in the study of large classical and quantum systems. These notes summarize the concepts and results necessary to apply balanced truncation.

I. INTRODUCTION

Theoretical physics endeavors to produce models that predict observed physical phenomena. Though sometimes the challenge is to develop a mathematical language describing the system of interest, often (especially in the study of complex systems) one can write down the exact dynamics and gain little insight — the resulting expressions are too cumbersome, too messy, or too ill-conditioned to be useful. By exploiting symmetries and other characteristics of the particular system, one may find a simpler equivalent description of the dynamics. This description may be further simplified by using approximation techniques, e.g., asymptotic limits and small-parameter expansions. Much of the art of the field lies in finding and choosing ad hoc methods for deriving these simpler models; however, more systematic methods are clearly desirable.

Control theorists have developed a variety of model reduction techniques that systematically produce simple models of complex systems. These notes will describe balanced truncation, a model reduction technique for linear systems which is readily available in a variety of formats (e.g. MATLAB). Balanced truncation has recently been applied in physics contexts, and the results suggest it will prove a useful tool for treating large systems in both classical and quantum settings.

Much of this discussion comes directly from Dullerud and Paganini. We will present the important concepts and results necessary to apply balanced truncation, omitting both the proofs and the algorithms. We refer the reader to [1] and [2] for a more complete mathematical discussion, and to MATLAB toolboxes and their documentation for the computational methods.

In section II we will describe the input-output paradigm of control theory, and introduce state-space models, the class of systems treatable by balanced truncation. Given an arbitrary state-space model, we will characterize the smallest state-space model with identical input-output characteristics in section III, and in section IV we will show how balanced truncation is used to find smaller models with controlled approximation errors.

II. INPUT-OUTPUT MAPS AND STATE-SPACE MODELS

In many physics settings one is more concerned with the macroscopic behavior of a large system, and less concerned with the system’s microscopic details. As an admittedly contrived example, consider a pendulum in a plane at whose free end is a tank partially filled with a classical fluid (see Fig. 1). Suppose that at time $t = 0$ the system is at rest in its stable equilibrium, and our only method of disturbing the system is to exert a time-varying torque $\tau(t)$ at the pivot. Suppose further that we are concerned only with the time evolution of pendulum’s angle $\theta(t)$ — not with the distribution of the fluid, given by some high-dimensional variable $\Phi(t)$.

An exact model including the full fluid state $\Phi$ would give the system’s exact response to a driving torque, but would be quite impractical. As we only wish to describe the angular response to the driving torque (a mapping from one degree of freedom to another) one suspects a lower-dimensional model might suffice. For example, one might try treating the fluid as a point mass attached to the pendulum by a non-linear spring.

Systems of this sort are naturally phrased in a control theory language. In the typical control scenario, a time-varying input $u(t)$ drives a system with state $x(t)$ giving output signal $y(t)$, and the system dynamics have the
form

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) \\
y(t) &= h(x(t), u(t)).
\end{align*}
\] (1)

Such systems are depicted by a block diagram as shown in Fig. 2. In the example of the pendulum, the system’s state is given by \( x = (\theta, \dot{\theta}, \Phi, \dot{\Phi}) \) so as to describe the evolution with the first-order dynamics (1). The input to the system is \( u(t) = \tau(t) \), and the system’s output is \( y(t) = \theta(t) \).

Together with some initial condition (typically \( x(0) = 0 \)), the functions \( f \) and \( h \) in (1) define an input-output map \( \Psi \) taking \( u \) to \( y \). Since it is this relation with which we are concerned, rather than the system’s internal dynamics, a theoretical model for the system will suffice if it describes \( \Psi \). Given some system of the form (1), model reduction aims to produce simpler models (i.e. models with a lower-dimensional state \( x \)) that approximate the original input-output map.

We will consider models of the form

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\] (2)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), and \( y \in \mathbb{R}^p \), and \( A, B, C \) and \( D \) are time-independent real matrices of sizes \( n \times n \), \( n \times m \), \( p \times n \) and \( p \times m \) respectively. (This entire discussion also holds for complex-valued systems.) Such linear models are called state-space models, and the order of a model is \( n \), the dimension of the state \( x \). For compactness, the model with matrices \( A, B, C \) and \( D \) is denoted by

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}.
\] (3)

(Note that \( D \) is unchanged.) We will call such maps similarity transformations. Because these transformations are merely a rewriting of the system dynamics, the input-output map remains the same.

As a given state-space model is not a unique description of an input-output map \( \Psi \), in the next section we ask: what is the lowest-order model with the same \( \Psi \) as the given model? After finding a lowest-order exact model, we will use balanced truncation to find lower-order models approximating \( \Psi \). This procedure is summarized in Fig. 3.

III. LOWEST-ORDER EXACT MODELS

To find state-space models of lowest order with the exact \( \Psi \) of a given model, we will split the problem into two parts: controllability and observability. We will then combine these ideas to find minimal realizations of the system.

A. Controllability

Assume the system is initially in the state \( x(0) = 0 \); given a time \( \tau > 0 \), the controllable states \( x_f \) are those for which there is an input signal \( u(t) \) yielding \( x(\tau) = x_f \). The dynamics (3) can be integrated to yield

\[
x(\tau) = \int_0^\tau e^{A(\tau-t)}Bu(t)dt,
\] (6)

which gives a linear map \( \Psi^{(\tau)}_c : u \to x(\tau) \). The controllable states form the image of this map. Since the map is linear, the image is a subspace of the state-space \( \mathbb{R}^n \), called the controllable subspace. We denote this subspace

![Fig. 2: A block diagram.](image-url)

![Fig. 3: Given an order n model, we first reduce the system to the lowest order n' ≤ n with the same input-output map, and then reduce to an approximate model of order n'' < n'.](image-url)
by $C_{AB}$, as controllability only depends on the matrices $A$ and $B$.

Using properties of the matrix exponential it can be shown that the controllable subspace is the image of the controllability matrix:

$$C_{AB} = \text{Image} \left[ B \ AB \ A^2 B \ldots A^{n-1} B \right]. \quad (7)$$

Thus we see that $C_{AB}$ is independent of $\tau$. It can also be shown that the controllable subspace is the image of the controllability gramian, an $n \times n$ matrix given by

$$X_C = \int_0^T e^{A t} B B^t e^{A^t t} dt, \quad (8)$$

and the orthogonal subspace of uncontrollable states is given by the controllability gramian's kernel. If $C_{AB} = \mathbb{R}^n$ (i.e., there exists an input signal $u$ to prepare any state $x(\tau)$) then we say that $(A, B)$ is a controllable pair.

It can be shown that given any $A$ and $B$, we can find a similarity transformation such that the transformed matrices have the block structure

$$\tilde{A} = T A T^{-1} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = T B = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} \quad (9)$$

with $(\tilde{A}_{11}, \tilde{B}_1)$ a controllable pair. Writing the state vector as $x = (x_1, x_2)$ corresponding to this block structure, we have

$$\dot{x}_1 = \tilde{A}_{11} x_1 + \tilde{A}_{12} x_2 + \tilde{B}_1 u$$
$$\dot{x}_2 = \tilde{A}_{22} x_2. \quad (10)$$

Because $x(0) = 0$, these dynamics yield $x_2(t) = 0$ for all time. Thus the dynamics for $x_1$ reduce to

$$\dot{x}_1 = \tilde{A}_{11} x_1 + \tilde{B}_1 u. \quad (11)$$

Because $(\tilde{A}_{11}, \tilde{B}_1)$ is a controllable pair, we may choose an input $u$ to prepare any state $x_1(\tau)$, and thus the transformed controllable subspace $\tilde{C}_{AB}$ is given by the states of the form $(x_1, 0)$. The orthogonal subspace, given by states of the form $(0, x_2)$, is irrelevant to the input-output map since no input can affect these states.

**B. Observability**

We now consider another problem with a similar structure. Suppose the system is in some initial state $x(0) = x_0$ and $u = 0$ for all time. Based on the output $y(t)$ for $0 \leq t \leq \tau$, can we uniquely identify $x_0$? Integrating the dynamics yields

$$y(t) = C e^{A t} x_0, \quad (12)$$

which gives a linear map $\Psi^{(\tau)}_0 : x_0 \rightarrow y$ (where by $y$ we mean the output signal $y(t)$ for $0 \leq t \leq \tau$). Suppose two initial states $x_0$ and $x_1$ give the same $y$. As $\Psi^{(\tau)}_0$ is linear, the initial state $x_0 - x_1$ must give $y = 0$. We call initial states giving output $y = 0$ unobservable, since any unobservable state may be added to any other initial state without changing the output.

The unobservable states form the kernel of $\Psi^{(\tau)}_0$; as the map is linear these states form a subspace, called the unobservable subspace and denoted by $N_{CA}$. It can be shown that the unobservable subspace is given by the kernel of the observability matrix:

$$N_{CA} = \text{ker} \begin{bmatrix} C \\ C A \\ \vdots \\ C A^{n-1} \end{bmatrix}. \quad (13)$$

Thus $N_{CA}$ is independent of $\tau$. It can also be shown that $N_{CA}$ is given by the kernel of the observability gramian

$$Y_O = \int_0^T e^{A t} C^t C e^{A^t t} dt, \quad (14)$$

and the orthogonal subspace of observable states is given by the observability gramian’s image. If $N_{CA} = \{0\}$, then the entire space is observable, and we say that $(C, A)$ is an observable pair. As the observable states are given by the image of $Y_O$, and the controllable states are given by the image of $X_C$, $(C, A)$ is an observable pair if and only if $(A^t, C^t)$ is a controllable pair.

It can be shown that given any $C$ and $A$, we can find a similarity transformation such that the transformed matrices have the block structure

$$\tilde{A} = T A T^{-1} = \begin{bmatrix} \tilde{A}_{11} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{C} = C T^{-1} = \begin{bmatrix} \tilde{C}_1 \\ 0 \end{bmatrix} \quad (15)$$

with $(\tilde{C}_1, \tilde{A}_{11})$ an observable pair. Writing the state vector as $x = (x_1, x_2)$ corresponding to this block structure, we have

$$\dot{x}_1 = \tilde{A}_{11} x_1 + \tilde{A}_{12} x_2$$
$$\dot{x}_2 = \tilde{A}_{21} x_1 + \tilde{A}_{22} x_2$$
$$y = \tilde{C}_1 x_1. \quad (16)$$

Thus the time evolution of $x_1$ is never affected by $x_2$, and the output signal $y$ only depends on $x_1$. Because $(\tilde{C}_1, \tilde{A}_{11})$ is an observable pair, we can uniquely identify an initial state $x(0) = (x_1, 0)$ based on $y$. The transformed unobservable subspace $N_{\tilde{C}_{\tilde{A}}} \tilde{A}$ is given by the states of the form $(0, x_2)$, and is irrelevant to the input-output map since no output can be affected by these states.

**C. Minimal Realizations**

The notions of controllability and observability give us a means of deciding whether a state affects the system’s
input-output map: if a state is unobservable, it does not affect the output, and if a state is uncontrollable, it is unaffected by the input. Only those states that are both controllable and observable are of relevance.

We say that a state-space model given by matrices \((A, B, C, D)\) is a \textit{minimal realization} if no lower-order model gives the same input-output map \(\Psi\). The intuition above can be made precise as follows: it can be shown that a model is a minimal realization if and only if all states are both controllable and observable, i.e. \((A, B)\) is a controllable pair and \((C, A)\) is an observable pair.

Given a state-space model given by \(A, B, C\) and \(D\) we may find a minimal realization by isolating only those dimensions which are both controllable and observable. To do so, we perform a \textit{Kalman decomposition}, which simultaneously performs the transformations \((17)\) and \((13)\) as follows. For any state-space model there exists a similarity transformation such that the matrices of transformed model have the block structure

\[
\begin{pmatrix}
  TAT^{-1} & TB \\
  CT^{-1} & D
\end{pmatrix}
\]

and, writing the state vector as \(x = (x_1, x_2, x_3, x_4)\) corresponding to the block structure, the controllable states are of the form \((x_1, x_2, 0, 0)\) and the observable states are of the form \((x_1, 0, x_3, 0)\). Thus only states of the form \((x_1, 0, 0, 0)\) are both controllable and observable.

Eliminating all but the states \((x_1, 0, 0, 0)\) yields the state-space model

\[
\begin{pmatrix}
  \tilde{A}_{11} & \tilde{B}_1 \\
  C_1 & D
\end{pmatrix}
\]

(18)

Since we have only eliminated states irrelevant to the input-output map, this reduced system has the exact same \(\Psi\) as the original system. Further, since states of the form \((x_1, 0, 0, 0)\) were both controllable and observable, \((\tilde{A}_{11}, \tilde{B}_1)\) is a controllable pair and \((\tilde{C}_1, \tilde{A}_{11})\) is an observable pair; thus this model is a minimal realization.

Sometimes the uncontrollable and unobservable states are obvious from the form of a physical system’s dynamics, e.g. some degrees of freedom are uncoupled, or some symmetry can be exploited. In other circumstances, the physics may not make it clear which states can be eliminated without affecting the input-output map. Especially in the latter situation, an algorithmic method such as the Kalman decomposition (available in MATLAB) can be quite advantageous. Nonetheless, one feels intuitively that one should always be able to find a minimal realization analytically if one is sufficiently clever. However, we do not expect in general that standard analytic methods will be useful in seeking lower-order approximations, as we will do in the next section.

IV. LOWER-ORDER APPROXIMATE MODELS

To apply balanced truncation, we will first assume that we have reduced the model to a minimal realization. We make the further assumption that the resulting system is exponentially stable, i.e. all eigenvalues of \(A\) have strictly negative real part. (Various methods exist to extend these methods to unstable systems, e.g., \(\tilde{\beta}\), but we assume stability to prove the standard result.)

A. Quantifying Observability and Controllability

In the previous section we distinguished between states that were observable and unobservable; we now wish to quantify the observability of the observable states. To do so, consider the output signal for \(t \geq 0\) that results from the initial state \(x(0) = x_0\) when there is no input \(u\). This signal is given by \((12)\). Using the \(L_2\) norm for the signal \(y\), we have

\[
\|y\|^2 = \int_0^\infty y^\dagger(t)y(t)\,dt = x_0^\dagger \left( \int_0^\infty e^{At}C^\dagger C e^{At} \, dt \right) x_0 \]

(19)

where \(Y_O\) is the observability gramian given by \((14)\) with \(\tau \to \infty\). Recall that the kernel of \(Y_O\) is independent of \(\tau\), so choosing \(\tau \to \infty\) will not change which states are observable and which states are unobservable. (System stability is required to ensure convergence of the integral in this limit.) Scaling the norm-squared of the output by the norm-squared of the initial state yields

\[
\frac{\|y\|^2}{\|x_0\|^2} = \frac{x_0^\dagger Y_O x_0}{x_0^\dagger x_0},
\]

(20)

which quantifies the observability of the states in the direction of \(x_0\).

From \((19)\) we see that the observability gramian is Hermitian and positive semidefinite. As we have a minimal realization, all states \(x_0\) are observable \((x_0^\dagger Y_O x_0 > 0)\), and so \(Y_O\) is strictly positive definite. Thus in geometric terms analogous to the moment-of-inertia tensor, \(Y_O\) defines an “observability ellipsoid” in the state space, with the longest principal axes along the most observable directions (see Fig. 4). A similarity transformation given by \(T\) transforms the observability gramian by \(Y_O \to \left(T^{-1}\right)^\dagger Y_O T^{-1}\). As \(T\) need not be unitary, this transformation may rescale the ellipsoid’s axes as well as rotate them.

We quantify controllability in a similar fashion. Suppose that at a time \(\tau\) in the past \((\tau \to -\infty)\) the system is in the state \(x = 0\), and some input \(u(t)\) drives the system for \(t \leq 0\), yielding a final state \(x(0) = x_0\). As we have a minimal realization, all states \(x_0\) are controllable and therefore can be prepared in this fashion. For each state \(x_0\) there is a minimum signal size

\[
\|u_{\text{opt}}\|^2 = \int_{-\infty}^0 u(t)^\dagger u(t)\,dt
\]

required to yield \(x(0) = x_0\).
The smaller this minimum signal, the more sensitive this state is to the input signal. Thus states with a smaller \( ||u_{\text{opt}}||^2 \) are said to be more controllable.

Consider the controllability gramian \( X_C \) given by (8) with \( \tau \to \infty \). Just as \( Y_O \), \( X_C \) is Hermitian and positive semidefinite, and because we have assumed a minimal realization, \( X_C \) is strictly positive definite and therefore invertible. It can be shown that

\[
||u_{\text{opt}}||^2 = x_0^\dagger X_C^{-1} x_0. \tag{21}
\]

It follows that

\[
\left( \frac{||u_{\text{opt}}||^2}{||x_0||^2} \right)^{-1} = \frac{x_0^\dagger X_C x_0}{x_0^\dagger x_0}, \tag{22}
\]

which quantifies the controllability of the states in the direction of \( x_0 \). Just as with \( Y_O \), \( X_C \) defines a “controllability ellipsoid” in state space, with the longest principal axes along the most controllable directions. A similarity transformation given by \( T \) transforms the controllability gramian by \( X_C \to T X_C T^\dagger \).

We have thus found that the gramians give us a useful measure of a state’s observability and controllability. One might ask whether the observability and controllability matrices of (13) and (18) could serve a similar purpose, but in fact they are only useful for determining whether a given state is observable/controllable. One might also ask if \( \tau \to \infty \) is necessary — we could choose finite \( \tau \), and quantify controllability and observability on a finite time horizon. However, finite \( \tau \) does not lead to the error bound in the approximations of the next section, which is the main result of these notes.

### B. Balanced Truncation

With the above quantification of observability and controllability, one might be tempted to prescribe some algorithm like eliminating the least observable or least controllable dimensions in the state space to yield a lower-order approximate model. However, such an approach would not necessarily be successful. Suppose, for example, that the least observable states were in the direction of the unit vector \( \hat{x} \), but that states in this direction were extremely controllable. Thus a small signal \( u \) might lead to the internal state \( x = \lambda \hat{x} \) with \( \lambda \) large. Though this state is the least observable, \( \lambda \) might be sufficiently large that the resulting output signal is non-negligible.

Instead, we wish to use observability and controllability to yield a single measure of a state’s importance to the input-output map \( \Psi \). It can be shown that given two positive definite square matrices \( X_C \) and \( Y_O \) of the same size, there exists an invertible \( T \) such that

\[
TX_C T^\dagger = (T^{-1})^\dagger Y_O T^{-1} = \Sigma \tag{23}
\]

where \( \Sigma \) is diagonal with positive real diagonal entries. Such a similarity transformation is called a balancing transformation. Geometrically, balancing transforms the observability and controllability ellipsoids so that they are identical and their principal axes lie on the coordinate axes of the state space (see Fig. 5). The resulting \( \Sigma \) is unique up to permutation of the diagonal elements, so we may choose \( T \) yielding the transformed gramians

\[
\tilde{X}_C = \tilde{Y}_O = \Sigma = \begin{bmatrix} h_1 & \cdots & h_n \end{bmatrix}
\]

with \( h_1 \geq h_2 \geq \ldots \geq h_n > 0 \). The \( h_i \) are called Hankel Singular Values (HSVs); in this transformed system \( h_i \) is the quantitative measure of both the observability and controllability of the unit basis vector \( \hat{e}_i \). Thus the basis vectors have been sorted in order of relevance to the input-output map.
Once the system is balanced, we may truncate the state-space dimensions with low HSVs to yield lower-order approximate models. Intuitively, the smaller the HSVs corresponding to truncated dimensions, the better the approximation. We will now make this idea precise.

Let the balanced state-space model with sorted HSVs be given by matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$. Choosing some $r$ such that $h_r$ is strictly greater than $h_{r+1}$, we write the state vector $x = (x_1, x_2)$ where $x_1$ gives the first $r$ coordinates, and $x_2$ gives the last $n - r$ coordinates. We then write the state-space model in the corresponding block structure

$$
\begin{pmatrix}
\tilde{A} & \tilde{B} \\
\tilde{C} & \tilde{D}
\end{pmatrix} =
\begin{pmatrix}
\tilde{A}_{11} & \tilde{A}_{21} & \tilde{B}_1 \\
\tilde{A}_{12} & \tilde{A}_{22} & \tilde{B}_2 \\
\tilde{C}_1 & \tilde{C}_2 & \tilde{D}
\end{pmatrix}
$$

(25)

and truncate the $n - r$ least significant dimensions of the model, yielding the order $r$ model

$$
\begin{pmatrix}
\tilde{A}_{11} & \tilde{B}_1 \\
\tilde{C}_2 & \tilde{D}
\end{pmatrix}.
$$

(26)

This procedure is called balanced truncation.

We now give bounds on the resulting approximation error. Let $u(t)$ be some input signal on $t \in (-\infty, \infty)$ that is finite with respect to the $L_2$ norm

$$
||z||^2 = \int_{-\infty}^{\infty} z(t)^\dagger z(t) dt.
$$

(27)

Let $y(t)$ and $\tilde{y}(t)$ be the resulting output signals of the original and truncated systems respectively. Stability of the original system [2] and the strict inequality $h_r > h_{r+1}$ guarantees stability of the truncated system [20]; since both systems are stable, the output signals $y$ and $\tilde{y}$ are also finite with respect to the norm (27), as is the error signal $\tilde{y} - y$. Now, letting $h_1^r > h_2^r > \ldots > h_k^r$ be the distinct HSVs of the truncated $n - r$ dimensions, it can be shown that

$$
h_1^r \leq \max_a \frac{||\tilde{y} - y||}{||u||} \leq 2 \sum_{j=1}^{k} h_j^r
$$

(28)

Thus even though sinusoidal inputs and outputs are unbounded in the $L_2$ norm (27), the error is controlled in this fashion.

$$
\begin{align*}
&u(t) = \sin(\omega t) \rightarrow y(t) = A_\omega \sin(\omega t + \phi_\omega).
\end{align*}
$$

(29)

V. CONCLUSION

According to [1], balanced realizations first appeared in the control literature in 1981, and the proof of the error bound on truncated models first appeared in 1984. Until recently, however, the physics community has made little use of this powerful tool. We believe that balanced truncation will be of particular value when building simulations of and theoretical models for the evolution of macroscopic quantities in large complex systems, and it is hoped that these notes will be helpful in such studies.

[1] G. E. Dullerud and F. G. Paganini, A Course in Robust Control Theory (Springer-Verlag, 2000).

[2] K. Zhou, J. C. Doyle, and K. Glover, Robust and Optimal Control (Prentice Hall, 1996).

[3] B. Rahn, A. C. Doherty, and H. Mabuchi (2001), quant-ph/0111003.

[4] M. Sznaier, A. C. Doherty, et al. (2001), submitted to the 2002 American Control Conference.

[5] Model reduction routines are available in the MATLAB Control System Toolbox, Robust Control Toolbox, and Mu-Analysis and Synthesis Toolbox.