Curvature independence of statistical entropy

Judy Kupferman
Ben-Gurion University, Beer-Sheva, 84105 Israel

Abstract

We examine the statistical number of states, from which statistical entropy can be derived, and we show that it is an explicit function of the metric and thus observer dependent. We find a constraint on a transformation of the metric that preserves the number of states but does not preserve curvature. In showing exactly how curvature independence arises in the conventional definition of statistical entropy, we gain a precise understanding of the direction in which it needs to be redefined in the treatment of black hole entropy.

1 Introduction

Black hole entropy is discussed in a number of contexts: thermodynamics and statistical mechanics [1, 2, 3, 4, 6], quantum entanglement [7, 8, 9], spacetime symmetries [10, 11, 12, 13] and more. It is not clear whether entropy in all these contexts refers to the same entity, although they are frequently taken to be equivalent. The fact that black holes obey thermodynamic type laws is still not understood; for example we do not know what degrees of freedom the entropy represents. If the different versions of entropy do not coincide, the confusion increases. In order to shed some light on the matter, we examine statistical entropy in curved space, and we ask whether statistical entropy can be related to the curvature of spacetime. Our main motivation is an attempt to introduce a little more clarity into the discussion of entropy in the black hole context, by providing an unequivocal description of the curvature independence of statistical entropy as it is conventionally defined in the literature.

The Unruh effect shows that an accelerated observer sees a thermal bath of particles, while an observer in Minkowski space sees vacuum [14] so clearly the statistical entropy in the two cases will be different. Both Minkowski and Rindler metric have the same (vanishing) curvature, so it appears that the entropy must be observer dependent and not a function of curvature. On the other hand Wald’s Noether charge entropy [10, 11] is defined in terms of the curvature tensor,

$$S_{Wald} = 2\pi \int_{\Sigma} \frac{\delta L}{\delta R_{abcd}} \epsilon_{ab} \epsilon_{cd}$$

(1)
where the functional derivative is taken viewing the Riemann tensor as a field independent of $g_{ab}$, and $\epsilon_{ab}$ is the binormal to the bifurcation surface. Thus it appears to be different from the entropy defined in statistical mechanics. The question is whether for any curvature (not just in the absence of curvature) the number of states is observer dependent, or whether there is a possible dependence on curvature. The textbook definition of statistical entropy originally assumed flat space, where the choice of vacuum is unambiguous. In curved space this is not the case. Thus it would appear that this definition is not suitable for treatment of black hole entropy. We will show technically where and how curvature independence arises in the conventional definition. This will give a precise understanding of the direction in which it needs to be redefined.

In this paper we examine the statistical number of states of matter in a general curved space metric. Our motivation is understanding of black hole entropy; however, specialisation to the black hole metric includes further issues to be dealt with in future work. The relationship of the entropy of matter outside the horizon to the entropy of the black hole is not clear. ’t Hooft [4] calculated the statistical entropy of a scalar field in the black hole metric, where his motivation was to reconcile black hole physics with quantum mechanics. At the time of writing, black holes were understood to be in a quantum mechanically mixed state, and ’t Hooft attempted to describe them as pure states resembling ordinary particles. Thus black holes inhabit an extension of Hilbert space with an according Hamiltonian. This system is sensitive to observer dependence: the free falling observer perceives matter, and ’t Hooft writes that it is this matter which he considers in this paper. The distinction between presence and absence of matter is assumed to be observer dependent when considering coordinate transformations with a horizon. Another view on the relationship between the statistical entropy of a field outside the horizon and the black hole is the idea that the horizon entropy arises from the microscopic structure of spacetime and that the matter fields inherit the entropy as material kept in a hot oven inherits the temperature [15, 16]. Yet another view takes into account the fact that the entanglement entropy of a bipartite system, which expresses the quantum correlations between its subsystems, is equal to the statistical entropy of a subsystem if it is in a thermal state [17-19], thus statistical entropy of fields outside the horizon may equal entanglement entropy of the black hole system. In this paper we do not discuss these issues. We focus only on the curvature independence of statistical entropy in a general curved space, as a first small step in clarifying the relation of the different concepts of black hole entropy.

We will show that the number of states from which statistical entropy is derived is an explicit function of the metric. Since the curvature derives from the metric, it would seem that the number of states is related to curvature. However we find that for certain transformations of the metric, the number of states is preserved. These transformations do not preserve curvature. Therefore the number of states does not depend on curvature. This is shown only for a diagonal metric, but it serves as a counter example showing that in the most general case the number of states is not a

1The statistical entropy, as discussed in this paper, is fundamentally different than the usual notion of quantum entanglement entropy. This distinction is further clarified in the concluding section.
function of the spacetime curvature scalar.

This paper is organized as follows. First we establish the definition of the number of states, and methods of calculating the volume of phase space. We then ask under what conditions transformation of the metric will leave this volume invariant. We obtain a general transformation of any diagonal metric which displays a clear constraint on the preservation of the number of states. We examine characteristics of this transformation and look for a possible relationship to curvature. We find that in general it need not preserve curvature. That is, the number of states and thus the entropy will remain the same for systems with different curvature. This is shown for a diagonal metric in a static spacetime, but serves as a counter example showing that in general entropy is not dependent on curvature.

2 Definition of number of states

In classical thermodynamics the number of states of a nonrelativistic system is defined as follows: Take an integral over the volume of phase space \((d^3x d^3p)\), restrict it to values of momenta which fit the energy eigenvalues of the system and the number of states is

\[
N = g \int d^3x \int \frac{d^3p}{(2\pi\hbar)^3} 
\]

where \(g\) is a numerical factor related to the degeneracy. Dividing by a unit of volume in momentum space, \(2\pi\hbar\), gives the number of states in phase space with the given energy, per unit volume of phase space. Since we do not limit ourselves to nonrelativistic systems, nor to three space dimensions, a more general definition is necessary [16, 20]. The number of states is then defined as

\[
N = \int d^d x \frac{d^dp}{(2\pi\hbar)^d} dE \delta(E - E(p)). 
\]

(3)

where \(d\) denotes the number of space dimensions. Without loss of generality we are taking a constant time hypersurface. The number of states is Lorentz invariant. For a proof see [21].

Remarks on notation: for simplicity of notation in this paper, \(g_{00}\) refers to the positive value of the time coordinate of the metric, except where explicitly stated otherwise. The minus sign appears in the form of the equation. An explicitly covariant derivation which parallels ours can be found in [20]. We here keep the vector notation because it clarifies our proof in what follows.

In order to apply this definition to curved space, we need to clarify what momentum and energy refer to for a matter field or gas of particles in curved space. There are (at least) two possible

\[\text{This integral is actually } \int d^d x \frac{d^dp}{(2\pi\hbar)^d} dE [2\pi\hbar \delta(E - E(p))]\]
ways to approach the issue. One is that of [4], who took \( \psi(x) \) a scalar wave function for a light spinless particle of mass \( m \) in the Schwarzschild metric, \( m \ll 1 \ll M \) where \( M \) is the BH mass, used a WKB approximation, wrote the wave equation and defined the spatial momentum \( k(r) \) in terms of the eigenvalues of the Laplacian operator while taking energy as the eigenvalue of the time component of the Laplacian. He obtained the number of states by calculating \( \int k(r)dr \) and then summing over angular degrees of freedom. Another possibility is that of [16, 20] who treated a relativistic gas of particles, and rather than the wave equation, used the scalar invariance of the squared momentum four-vector of the particle, while the covariant energy of a particle is the projection of the timelike Killing vector on the four momentum. Both approaches give the same relationship between energy and momentum, which for a general static metric is

\[ g^{00}E^2 = \sum_i g^{ii}(p_i)^2 \]  

(4)

taking a massless particle for simplicity.

The number of states given by the volume of phase space is the product of the volume of position and momentum space. The momentum component of the number of states belongs to a constrained region in the cotangent space of the region of configuration space in question. For example, in Cartesian coordinates in flat space eq.(4) gives

\[ E^2 - p_x^2 - p_y^2 - p_z^2 = 0 \]  

(5)

and this defines a sphere of radius \( E \):

\[ 1 = \frac{p_x^2}{E^2} + \frac{p_y^2}{E^2} + \frac{p_z^2}{E^2} \]  

(6)

In statistical physics we take all energies up to a given energy, and so we look for the volume enclosed by this sphere, \( \frac{4}{3}\pi E^3 \). If the metric is not flat, the volume will be an ellipsoid. Since our proof of curvature independence rests on a counter example, we are free to take a static metric with timelike Killing vector and can define the energy accordingly.

For a general static diagonal metric eq. (4) gives

\[ g^{00}E^2 = \sum_i g^{ii}(p_i)^2 \]

\[ 1 = \sum_i \frac{g^{ii}(p_i)^2}{g^{00}E^2} = \sum_i \frac{p_i^2}{g_{ii}g^{00}E^2} \]  

(7)

where \( p_i \) the spatial momenta are summed in all space directions. This is the formula for an ellipsoid with axes \( \sqrt{g_{ii}g^{00}}E \), which encloses a region whose volume in three space dimensions
would be $\frac{4}{3}\pi \sqrt{g_{xx} g_{yy} g_{zz}} \left( \sqrt{g^{00} E} \right)^3$. In $d + 1$ spacetime dimensions this becomes

$$C_d \sqrt{g_d} \left( \sqrt{g^{00} E} \right)^d$$

where $g_d$ denotes the determinant of the spatial part of the metric and $C_d$ is the volume enclosed by the $d$-dimensional unit ball. One then integrates over all momentum space. Since the measure in the momentum integral includes the root of inverse metric $g^d$, that is, the integral is given by

$$\int \frac{d^d p}{\sqrt{g_d}}$$

then the space determinant in eq.(8) cancels out, and the integral over momentum space gives the volume of a ball with radius $\sqrt{g^{00} E}$.

Therefore the number of states in $d + 1$ dimensions ($d$ space dimensions) for a diagonal metric is

$$N = C_d E^d \int_V d^d x \sqrt{g_d} \left( g^{00} \right)^{d/2}$$

$$C_d = \frac{\pi^{d/2}}{\Gamma \left( \frac{d}{2} + 1 \right)}.$$ (10)

An explicit proof for $3 + 1$ and $4 + 1$ dimensions appears in the Appendix.

3 Invariance of number of states under transformation of metrics

We wish to examine a general transformation which changes the metric while leaving the number of states invariant. We find that such a transformation exists, but does not preserve curvature. We give details of the transformation, followed by examples of the relation to curvature.

We begin with conformal rescaling. If a $d$–dimensional metric changes by $\tilde{g}_{\mu\nu} = a(x) g_{\mu\nu}$, then the number of states is

$$N_0 = \int \sqrt{g_3} d^3 x d^3 p = \int \sqrt{g_3} \frac{4\pi E^3}{3} \left( g^{00} \right)^{3/2} d^3 x.$$ $\tilde{N}$ = $\int \sqrt{\tilde{g}_3} d^3 x d^3 p$

$$= \int a^{3/2} \sqrt{g} \frac{4\pi E^3}{3} \left( \frac{1}{a} g^{00} \right)^{3/2} d^3 x$$

$$= \int \sqrt{g} \frac{4\pi E^3}{3} \left( g^{00} \right)^{3/2} d^3 x = N.$$ (11)
since \( \tilde{g}_{00} = a(x)g_{00} \) and so \( \tilde{g}^{00} = \frac{1}{a(x)}g^{00} \). This only works if the metric is uniformly rescaled, so that \( a_0 = a_i \). Thus conformal rescaling preserves the number of states. We conclude that preservation of the number of states requires a constraint on the relationship between the time and space components of the metric.

In search of a general transformation we take a general diagonal metric in \( 1 + 3 \) dimensions. Generalization to more space dimensions will be simple.

\[
\begin{pmatrix}
g_{00} & & \\
g_{xx} & g_{yy} & \\
& & g_{zz}
\end{pmatrix}
\]  

(12)

The volume of space in this metric:

\[
\int_{V} \sqrt{g_{xx}g_{yy}g_{zz}} \, d^3x
\]

(13)

where the integral is over a given volume \( V \). The volume of momentum space is

\[
\int_{V_p} \frac{d^3p}{\sqrt{g_{xx}g_{yy}g_{zz}}}
\]

(14)

where \( V_p \) is the volume in momentum space. As explained above, from eq. (11)

\[
1 = \frac{1}{g_{xx}g^{00}E^2p_x^2} + \frac{1}{g_{yy}g^{00}E^2p_y^2} + \frac{1}{g_{zz}g^{00}E^2p_z^2}
\]

(15)

which is the equation for volume of ellipsoid with axes \( \sqrt{g_{xx}g^{00}E}, \sqrt{g_{yy}g^{00}E}, \sqrt{g_{zz}g^{00}E} \). The momentum volume is obtained by integration, or more simply by just plugging in the formula for volume of ellipsoid in 3 dimensions: \( \frac{4}{3}\pi abc = \frac{4}{3}\pi \sqrt{g_{xx}g_{yy}g_{zz}}(g^{00}E^2)^{3/2} \).

Phase space is given as:

\[
N = \int_{V} d^3x \int_{V_p} d^3p.
\]

(16)

We now transform the metric in arbitrary way but keeping it diagonal:

\[
\begin{pmatrix}
a(\vec{x})g_{00} & & \\
& b(\vec{x})g_{xx} & \\
& c(\vec{x})g_{yy} & \\
& d(\vec{x})g_{zz}
\end{pmatrix}
\]

(17)

We plug this into the term for phase space. First we calculate the volume of momentum space for

\footnote{This can have a prefactor of (2\(\pi\))\(^{-3}\) when calculating the density of modes per unit of phase space}
the transformed metric. We obtain
\[ \frac{1}{a(\vec{x})} g^{00} E^2 = \frac{1}{b(\vec{x})} g^{xx} p_x^2 + \frac{1}{c(\vec{x})} g^{yy} p_y^2 + \frac{1}{d(\vec{x})} g^{zz} p_z^2 \] (18)

and using eq. (15)
\[ 1 = \frac{1}{a(\vec{x})} b(\vec{x}) g_{xx} g^{00} E^2 p_x^2 + \frac{1}{a(\vec{x})} c(\vec{x}) g_{yy} g^{00} E^2 p_y^2 + \frac{1}{a(\vec{x})} d(\vec{x}) g_{zz} g^{00} E^2 p_z^2 \] (19)

so that the volume becomes
\[ V_p = 4 \pi^{3/2} \sqrt{b(\vec{x}) c(\vec{x}) d(\vec{x}) g_{xx} g_{yy} g_{zz} \left( \frac{g^{00}}{a(\vec{x})} E^2 \right)^{3/2}}. \] (20)

This will equal the volume before the transformation if
\[ b(\vec{x}) c(\vec{x}) d(\vec{x}) = a(\vec{x})^3. \] (21)

Thus we have identified the constraint for an arbitrary transformation to preserve the volume of phase space.

We looked for some kind of general algebraic characterization for this kind of matrix, but found none. It belongs to \( GL(n, R) \) but does not represent a particular symmetry. Conformal transformations are a subgroup of our transformation. Certain non conformal transformations also preserve the number of states. This holds if the determinants cancel out: That is, for \( d \) space dimensions, the time part \( a(x) \) when raised to the \( d^{th} \) power, has to equal the determinant of the space part. Take
\[ A = \begin{pmatrix} a(x) & 0 & 0 & 0 \\ 0 & a(x)^2 & 0 & 0 \\ 0 & 0 & a(x) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \] (22)

As with the conformal transformation, we still have \( \sqrt{g_3} = a^{3/2} \sqrt{g_3} \) and and so \( \tilde{N} = N_0 \).

So in general our constraint is:
\[ (g_{00})^d = det g_{\text{space}} \] (23)

where \( d \) is the number of space dimensions and \( g_{\text{space}} \) is the determinant of the spatial part of the metric.

We can regard the transformation matrix, labeled \( A \), as two blocks, separating the time and space components:
\[ A = \begin{pmatrix} T \\ S \end{pmatrix} \] (24)
where $T$ is a 1x1 matrix, and $S$ is a diagonal matrix of rank $d$ where $d$ is the dimension of space (rank of $A$ is the dimension of space-time, $d+1$). Then the constraint requires

$$
det(S) = det(T)^d
$$

$$
det(A) = det(T)^{2d}.
$$

(25)

4 Relation to curvature

In $d+1$ dimensions

$$
N \sim \int d^3x \sqrt{\frac{g_d}{(g_{00})^d}}
$$

(26)

where $g_d$ denotes the determinant of the space part of metric. To preserve the number of states we have to preserve the ratio $g_d/g_{00}^d$, which entails the constraint on the determinant as detailed above. The question becomes: given a change of metric for which this constraint holds, will such a constraint ensure preservation of scalar curvature? If so preservation of the number of states would entail preservation of curvature, which is an observer independent characteristic.

We take two matrices representing two possible transformations of a given 3-dimensional metric:

$$
A = \frac{1}{L} \begin{pmatrix} x & x & x \\ \sqrt{\frac{g_{xx}}{L}} & 2 & \frac{x^2}{L^2} \\
\end{pmatrix}
$$

$$
B = \begin{pmatrix} \frac{x}{L} & \sqrt{\frac{g_{xx}}{L}} & 2 & \frac{x^2}{L^2} \\
\end{pmatrix}
$$

(27)

where $L$ is a constant with dimension of length. Both transformation matrices preserve the constraint given above, while their curvature differs. That is, taking a flat metric for example, after undergoing each of these transformations it would have the same number of states as previously, but different curvature. The first transformation would give $R = \frac{3L}{2x}$, the second gives $R = \frac{L}{\sqrt{2x^3}}$. This is because the second one has fewer Christoffel signs, since the derivative must be $\partial_x$, and $\partial_x g_{yy} = 0$. Therefore clearly imposing the constraint on a metric transformation will not necessarily preserve the curvature of the original metric.

Curvature in these examples is affected by the number of terms with an $x$ derivative, while the determinant is not. Thus the constraint on the determinant does NOT preserve curvature. This is intuitively understandable: the determinant indicates volume but gives no information as to the spatial distribution of the volume.
4.1 Examples in various dimensions

In 1 + 1 dimensions a transformation that preserves N must be conformal: \( g_{00} = g_{xx} \) since \( \det g_{\text{space}} = g_{xx} \). In 2+1 dimensions we give two examples of transformations that preserve N. One is conformal, the other non conformal but symmetric:

Conformal:

\[
A = \frac{1}{L} \begin{pmatrix} x & x \\ x & x \end{pmatrix}, \quad R = \frac{3L}{2x^3}
\] (28)

Symmetric:

\[
B = \frac{1}{L} \begin{pmatrix} \sqrt{xy} & x \\ x & y \end{pmatrix}, \quad R = L \left( \frac{5x^2 - 6y\sqrt{xy}}{8x^3y^2} \right).
\] (29)

An asymmetric example is like the one given in the previous section for a Euclidean metric.

Note that plugging in the value \( x = y \) after deriving \( R \) for matrix \( B \) does not give the curvature of matrix \( A \). This is because the derivation of \( R \) takes into account the direction of each component as well as its numerical value. If one plugs in \( y = x \) before deriving \( R \) all the derivatives \( \partial_y \) vanish, giving the different result.

We next look at 3+1 dimensions. The constraint requires \( |(g_{00})^3| = \det g_3 \). Comparing several matrices that obey this constraint and inspecting their curvature:

\[
A = \frac{1}{L} \begin{pmatrix} \frac{x}{L} & \frac{x^3}{L^3} \\ \frac{x^3}{L^3} & 1 \end{pmatrix}, \quad R = \frac{2L^3}{x^3}
\]

\[
B = \frac{1}{L} \begin{pmatrix} (xyz)^{\frac{1}{3}} \\ x \\ y \\ z \end{pmatrix}, \quad R = \frac{4L}{9} \left( \frac{1}{x^3} + \frac{1}{y^3} + \frac{1}{z^3} \right)
\]

\[
C = \frac{1}{L} \begin{pmatrix} x \\ x \\ x \end{pmatrix}, \quad R = \frac{3L}{2x^3}
\] (30)

A few comments: 1) The curvature for the third transformation is the same as for the conformal matrix in 1+2 dimensions. 2) As before, setting \( x = y = z \) after calculating the curvature for matrix \( B \) does not give the same result as the curvature for matrix \( C \). Again, this is because the direction of the variable contributes in calculating \( R \), and not just its numeric value. This sheds
light on the fact that the number of states, which is proportional to the volume of phase space, is different from curvature, which incorporates information on the distribution of that volume. A constraint on the determinant, representing Euclidean volume, is not the same as that on Ricci curvature, which in fact represents the amount by which the volume of a geodesic ball in a curved Riemannian manifold deviates from that of the standard ball in Euclidean space.

4.1.1 Rindler vs Schwarzschild:

The transformation from Minkowsky to Rindler space is not diagonal. It mixes time and space coordinates and that is why N is different from flat space. We cannot conclude from this that curvature is irrelevant to statistical entropy. That conclusion can only be drawn from the general proof given above.

The Schwarzschild metric diverges at the boundary and it was found that the number of states (and thus the entropy) is different from that of Minkowski space \[4,19\]. This is not the same as the difference between the number of states in Rindler and Minkowski spaces. In the Schwarzschild case the argument above does apply, since the transformation metric from Minkowski to Schwarzschild metric is diagonal. The Schwarzschild number of states differs from that of Minkowski because of the redshift on energy: \(g_{00}(r)\).

4.2 Discussion

Our transformation leaves N invariant because it preserves the relationship between the volume of momentum space and of position space. \((g^{00})^{3/2}\) is the variable part of momentum space, and \(\sqrt{g_{d}}\) is the variable part of position space. \(N\) is invariant so long as the relation between the two is preserved, so that if position space shrinks, momentum space grows and vice versa: \(a(x)^d\) multiplying \(\sqrt{g_{\text{space}}}\) equals \(1/a(x)^d\) multiplying momentum space.

We examined the question whether in curved space the number of states, and the statistical entropy derived from this, is observer dependent or is related to a physical quantity such as curvature. We found that it is observer dependent and not related to the intrinsic geometry.

One might argue that a proof of curvature independence must show that there are no cases at all where curvature is preserved under a transformation that preserved the number of states. In fact it is quite possible that in some case curvature might be preserved. We claim that this must be seen as a coincidence because the constraint on preservation of the number of states relates to the determinant. By definition, there is a difference between the determinant, which represents Euclidean volume and does not depend on directions in space, and curvature which does depend on directions in space. The number of states does not depend on directions in space and so it can be preserved even if the directional characteristics and thus the curvature are changed. There will be a subgroup where transformation of the number of states will indeed preserve curvature. But
one cannot assume that any given number of states, and the entropy derived from it, relate to a spacetime with a given curvature.

The results in this paper apply to a diagonal metric only, but this is sufficient as it serves as a counter example. The question arises: what of Wald’s entropy? Since statistical entropy is derived from the number of states, whereas Wald’s entropy is an explicit function of curvature, this indicates a difference between these two concepts of entropy. Calculation for Einstein gravity gives the same result in both cases, but for generalized theories of gravity Wald entropy could contain terms derived from the curvature, so the concepts themselves do not coincide.

However the issue may not be so simple. Phase space is defined as the product of the volume of position and momentum spaces. The definition arose in the context where momentum refers to kinetic momentum which is also the canonical conjugate to position. However even in spherical coordinates, kinetic and conjugate momentum do not coincide [22], and a treatment in curved space should generalize the definition to conjugate momentum. If this is done, one then notes that gravitational Lagrangian includes the Ricci scalar, and Ricci tensors as well in the generalized theories of gravity with which Wald dealt. The Lagrangian of a particle in a gravitational background will include at least two terms, the matter Lagrangian and the gravitational term. Each will have a generalized momentum conjugate to the dynamical variable in the Lagrangian. Therefore it may be necessary to redefine statistical entropy to take into account a more general formulation of phase space. This cannot be done simply by adding gravitational degrees of freedom; for example, adding gravitational degrees of freedom to the statistical calculation would not give one fourth the area but one half, and thus differ from the other derivations of entropy. A clue to suitable redefinition of the term may be found in the example discussed at the start of the paper: Minkowski and Rindler space. Statistical entropy was originally defined for flat space, where choice of vacuum is unambiguous. In treatment of curved space there should be a way to incorporate the choice of vacuum into the concept of phase space.

Another issue is that of divergence on the horizon. While this may be an artifact of quantum uncertainty[19] still a more thorough investigation is necessary before drawing conclusions on the relationship of the two entropies.

Note: There is a claim that entanglement entropy and statistical entropy are one and the same. In [23] it was shown for explicit examples that entanglement entropy does not depend on curvature. For a discretized region in curved space it was found that even when the space is large enough for the effects of curvature to be noticeable, entropy remains proportional to area and is not affected by the curvature of the background. This qualitative similarity to our result reinforces the idea that entanglement and statistical entropy may be the same, and that they differ from Noether charge entropy.

In conclusion, we have shown that the number of states is a function of the metric and is preserved under specific transformations of the metric, which do not necessarily preserve curvature. Therefore the number of states calculated with the accepted definition of phase space does
not depend on curvature, and neither does the statistical entropy derived from it. For general theories of gravity it may be necessary to redefine statistical entropy taking into account a more general concept of phase space in some subtle manner, but as the definitions stand, it appears that statistical entropy and Wald entropy differ.

This research was supported by the Israel Science Foundation Grant No. 239/10. We thank Ramy Brustein, Merav Hadad and Frol Zapolsky for helpful discussions, and Joey Medved for comments on the manuscript.

A Phase space in 3+1 and in 4+1 dimensions

The number of states in $d+1$ dimensions (d space dimensions) for a diagonal metric works out to be

$$ N = CE^d \int \sqrt{g_d} \left( g^{00} \right)^{\frac{d}{2}} $$

$$ C = \frac{\pi^{\frac{d}{2}}}{\Gamma \left( \frac{d}{2} + 1 \right)} \quad (31) $$

and $g_d$ is the determinant of the spatial components of the metric.

Proof in 3+1 dimensions:

Eq.(4) gives:

$$ g^{00} E^2 - g^{xx} p_x^2 - g^{yy} p_y^2 - g^{zz} p_z^2 = 0 $$

$$ p_x = \sqrt{g_{xx}} \sqrt{g^{00} E^2 - g^{yy} p_y^2 - g^{zz} p_z^2} $$

$$ \int d^3p = \int dp_y \int dp_z \int \frac{\sqrt{g_{yy} g^{00} E}}{\sqrt{g_{xx} \sqrt{g^{00} E^2 - g^{yy} p_y^2} \sqrt{g_{zz} \sqrt{g^{00} E^2 - g^{zz} p_z^2}}}} $$

$$ = 2 \sqrt{g_{xx}} \int dp_y \int dp_z \sqrt{g^{00} E^2 - g^{yy} p_y^2 - g^{zz} p_z^2} $$
We label $g^{00}E^2 - g^{yy}p_y^2 \equiv A^2$. Then the integral over $p_z$ becomes

$$
\int_{-\sqrt{g_{zz}A}}^{\sqrt{g_{zz}A}} dp_z \sqrt{A^2 - g^{zz}p_z^2} = A \int_{-\sqrt{g_{zz}A}}^{\sqrt{g_{zz}A}} dp_z \sqrt{1 - \frac{p_z^2}{g_{zz}A^2}}
= A^2 \sqrt{g_{zz}} \int_{-1}^{1} du \sqrt{1 - u^2} = A^2 \sqrt{g_{zz}} \frac{\pi}{2}.
\tag{32}
$$

Plugging this in we get

$$
\int d^3p = 2 \sqrt{g_{xx}g_{zz}} \frac{\pi}{2} \int_{-\sqrt{g_{yy}g^{00}E}}^{\sqrt{g_{yy}g^{00}E}} dp_y \left( g^{00}E^2 - g^{yy}p_y^2 \right)
= \sqrt{g_{xx}g_{zz} g_{yy}} \left[ 2 \sqrt{g_{yy}} \left( g^{00}E^2 \right)^{3/2} - \frac{2}{3} g^{yy} \left( \sqrt{g_{yy}g^{00}E} \right)^3 \right]
= \sqrt{g_{xx}g_{zz} g_{yy}} \frac{4}{3} \pi \left( \sqrt{g^{00}} \right)^3.
\tag{33}
$$

One more dimension:

$$
g^{00}E^2 - g^{xx}p_x^2 - g^{yy}p_y^2 - g^{zz}p_z^2 - g^{ww}p_w^2 = 0
p_w = \sqrt{g_{ww}} \sqrt{g^{00}E^2 - g^{xx}p_x^2 - g^{yy}p_y^2 - g^{zz}p_z^2}
$$

$$
\int d^3p = \sqrt{g_{yy}g^{00}E} \sqrt{g_{xx}g^{00}E^2 - g^{yy}p_y^2} \sqrt{g_{xx}g^{00}E^2 - g^{yy}p_y^2 - g^{zz}p_z^2} \sqrt{g_{xx}g^{00}E^2 - g^{yy}p_y^2 - g^{zz}p_z^2 - g^{ww}p_w^2}
\times \int_{-\sqrt{g_{ww}g^{00}E}}^{\sqrt{g_{ww}g^{00}E}} dp_w \sqrt{g^{00}E^2 - g^{xx}p_x^2 - g^{yy}p_y^2 - g^{zz}p_z^2}
= 2 \sqrt{g_{ww}} \sqrt{g^{00}E^2 - g^{xx}p_x^2 - g^{yy}p_y^2 - g^{zz}p_z^2}
\times \int_{-\sqrt{g_{xx}g^{00}E}}^{\sqrt{g_{xx}g^{00}E}} dp_y \sqrt{g^{00}E^2 - g^{xx}p_x^2 - g^{yy}p_y^2 - g^{zz}p_z^2}
\times \int_{-\sqrt{g_{xx}g^{00}E}}^{\sqrt{g_{xx}g^{00}E}} dp_z \sqrt{g^{00}E^2 - g^{xx}p_x^2 - g^{yy}p_y^2 - g^{zz}p_z^2}
\tag{34}
$$

\[13\]
We label \( g^{00} E^2 - g^{zz} p_z^2 - g^{yy} p_y^2 \equiv A^2 \). Then the integral over \( p_x \) becomes

\[
\sqrt{g_{xx}} A \int_{-\sqrt{g_{xx}} A}^{\sqrt{g_{xx}} A} dp_x \sqrt{A^2 - g^{xx} p_x^2} = A \int_{-\sqrt{g_{xx}} A}^{\sqrt{g_{xx}} A} dp_x \sqrt{1 - \frac{p_x^2}{g_{xx} A^2}}
\]

\[
= A^2 \sqrt{g_{xx}} \int_{-1}^{1} du \sqrt{1 - u^2} = A^2 \sqrt{g_{xx} \frac{\pi}{2}}.
\]

Plugging this in we get

\[
\hat{d}^3 p = 2 \sqrt{g_{xx} g_{ww}} \frac{\pi}{2} \int_{-\sqrt{g_{ww} g^{00}}}^{\sqrt{g_{ww} g^{00}}} dp_y \int_{-\sqrt{g_{yy} g^{00}}}^{\sqrt{g_{yy} g^{00}}} dp_z (g^{00} E^2 - g^{yy} p_y^2 - g^{zz} p_z^2)
\]

Let us label \( g^{00} E^2 - g^{yy} p_y^2 \equiv B^2 \). Then the \( p_z \) integral becomes

\[
\int_{-\sqrt{g_{zz} B}}^{\sqrt{g_{zz} B}} dp_z (B^2 - g^{zz} p_z^2) = \frac{4}{3} \sqrt{g_{zz} B^3} = \frac{4}{3} \sqrt{g_{zz} (g^{00} E^2 - g^{yy} p_y^2)^{3/2}}.
\]

Integrate over \( p_y \):

\[
\int_{-\sqrt{g_{yy} g^{00}}}^{\sqrt{g_{yy} g^{00}}} dp_y (g^{00} E^2 - g^{yy} p_y^2)^{3/2} = \sqrt{g_{yy} \frac{3}{8} \pi} \left( \sqrt{g_{00} E} \right)^4.
\]

(this was done with Mathematica, you get a result containing \( Arctan[\infty] = \frac{\pi}{2} \)).

Plugging this back in,

\[
\int d^3 p = \frac{4}{3} \left( \frac{3}{8} \right) \pi^2 \sqrt{g_{xx} g_{ww} g_{yy} g_{zz}} (g^{00})^2 E^4
\]

\[
= \frac{\pi^2}{2} \sqrt{g_4} (g^{00})^2 E^4
\]

and so

\[
N = \frac{\pi^2}{2} E^4 \int_V d^4 x \sqrt{g_4} (g^{00})^2
\]

\[
= C_d E^4 \int_V d^4 x \sqrt{g_4} (g^{00})^\frac{4}{3}, \quad \left( C = \frac{\pi^\frac{4}{3}}{\Gamma(\frac{4}{3} + 1)} \right)
\]

(39)
just as we claimed. We would like to be able to prove by induction that if it’s true for $N_d$ it’s true for $N_{d+1}$ but (so far) we can’t generalize the integration: the integrand becomes $(g^{00} E^2 - g^{(d+1,d+1)} p_{d+1})^{d/2}$.

References

[1] J.M.Bardeen, B.Carter and S.W.Hawking, Commun.Math.Phys.31,161 (1973).
[2] J.D.Bekenstein, Phys. Rev. D 7, 2333 (1973).
[3] G.W. Gibbons and S.W. Hawking, Phys. Rev. D, 15, 2752–2756, (1977).
[4] G.’t Hooft, Nucl. Phys. B256, 727 (1985).
[5] T. Padmanabhan, Rep. Prog. Phys. 73,046901 (2010) [arXiv:0911.5004v2] and references therein.
[6] R. M. Wald, Living Rev. Relativ. 4, 6 (2001). [http://www.livingreviews.org/lrr-2001-6, arXiv:gr-qc/9912119v2].
[7] Luca Bombelli, Rabinder K. Koul, Joohan Lee, and Rafael D. Sorkin, Phys. Rev. D 34, 373 (1986).
[8] Mark Srednicki, Phys. Rev. Lett. 71, 666 (1993).
[9] M.B.Plenio, J. Eisert, J. Dreissig and M. Cramer, Phys. Rev. Lett. 94, 060503 (2005) [arXiv:quant-ph/0405142v3]. See also J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. 82, 277 (2010) [arXiv:0808.3773v4].
[10] Robert M.Wald, Phys. Rev. D 48, 3427-3431 (1993) [arXiv:gr-qc/9307038v1].
[11] V. Iyer and R.M.Wald, Phys. Rev. D 50, 846 (1994) [arXiv:gr-qc/9403028v1].
[12] S.Carlip, Class.Quant.Grav.16:3327-3348 (1999) [arXiv:gr-qc/9906126v2].
[13] S. Silva, Class.Quant.Grav. 19 (2002) 3947-3962 [hep-th/0204179].
[14] W.G. Unruh, Phys. Rev. D 14 (4), 870 (1976).
[15] T.Padmanabhan, 2010 Mod. Phys. Lett. A 25 1129 (2010) [arXiv:0912.3165]; Phys. Rev. D 81 124040 (2010) [arXiv:1003.5665];
[16] Sanved Kolekar, T.Padmanabhan, Phys. Rev.D83, 064034 (2011) [arXiv:1012.5421]
[17] D.Kabat, Nuclear Physics B 453, 281 (1995) [arXiv:hep-th/9503016]
[18] W. Israel, Phys. Lett. A57, 107 (1976).
[19] R.Brustein and J.Kupferman, Phys. Rev.D83, 124014 (2011) [arXiv:1010.4157v2].

[20] T. Padmanabhan. Physics Letters A, 136,203–205, (1989).

[21] T. Padmanabhan, *Gravitation: Foundations and Frontiers*, Cambridge University Press, (2010), p.36.

[22] Lee Ting Hsang, An Chong Shan and Zhai Tian Yi, International Journal of Theoretical Physics 29, 9 (1990)

[23] Katja Ried, quant-ph/1309.7380v1 (2013).