WEIGHTED ENDPOINT BOUNDS FOR THE BERGMAN AND CAUCHY-SZEGÖ PROJECTIONS ON DOMAINS WITH NEAR MINIMAL SMOOTHNESS

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Abstract. We study the Bergman projection, \( B \), and the Cauchy-Szegö projection, \( S \), on bounded domains with near minimal smoothness. We prove that \( B \) has the weak-type \((1, 1)\) property with respect to weighted measures assuming that the underlying domain is strongly pseudoconvex with \( C^4 \) boundary and the weight satisfies the \( B_1 \) condition, and the same property for \( S \) on domains with \( C^3 \) boundaries and weights satisfying the \( A_1 \) condition. We also obtain weighted Kolmogorov and weighted Zygmund inequalities for \( B \) and \( S \) in their respective settings as corollaries.

1. Introduction

Let \( D \subseteq \mathbb{C}^n \) be a domain. The Bergman space, \( A^2(D) \), is defined by
\[
A^2(D) := L^2(D) \cap \text{Hol}(D),
\]
where \( L^2(D) \) is the set of square integrable functions on \( D \) (with respect to Lebesgue measure) and \( \text{Hol}(D) \) is the set of holomorphic functions on \( D \). Since \( A^2(D) \) is a closed subspace of \( L^2(D) \), there exists an orthogonal projection from \( L^2(D) \) to \( A^2(D) \) which is called the Bergman projection and denoted by \( B \). The Bergman projection can be viewed as an integral operator by
\[
B f(z) = \int_D K(z, w) f(w) dV(w),
\]
where \( K \) is the reproducing kernel for \( A^2(D) \) and \( V \) represents the Lebesgue measure on \( \mathbb{C}^n = \mathbb{R}^{2n} \). We are interested in investigating the boundedness properties of \( B \).

By definition, it is clear that \( B \) acts as a bounded operator on \( L^2(D) \). However, the \( L^p(D) \) boundedness of \( B \) for \( p \neq 2 \) is more complicated. It was first shown by Zaharjuta and Judović in [19] that \( B \) has a bounded extension on \( L^p(\mathbb{D}) \) for all \( 1 < p < \infty \), where \( \mathbb{D} \) is the unit disk in \( \mathbb{C} \). In [5], Forelli and Rudin proved the \( L^p(\mathbb{B}_n) \) estimates, where \( \mathbb{B}_n \) is the unit ball in \( \mathbb{C}^n \), using Schur’s test. Later, in [15], Phong and Stein generalized this result to strongly pseudoconvex domains \( D \) with smooth boundary. More recently, in [11], Lanzani and Stein relaxed the smoothness condition on \( D \) and proved the \( L^p(D) \) bounds when \( D \subseteq \mathbb{C}^n \) is strongly pseudoconvex with \( C^2 \) boundary. Their methods rely on techniques from complex analysis, operator theory, and harmonic analysis.

Notice that \( B \) is not bounded on \( L^1(D) \) in general. This fact can be seen by taking \( D = \mathbb{B}_n \) and noting that boundedness on \( L^1(\mathbb{B}_n) \) would imply that \( B \) is bounded on \( L^\infty(\mathbb{B}_n) \) by duality. This would then contradict the well-known Rudin-Forelli estimate
\[
\sup_{z \in \mathbb{B}_n} \int_{\mathbb{B}_n} \frac{1}{|1 - \langle z, w \rangle|^{n+1}} dV(w) = \infty,
\]
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see \cite{20}.

In certain settings, the failed $L^1(D)$ bound may be replaced with the following weak-type \((1,1)\) property: there exists $C > 0$ such that

$$
\|B f\|_{L^{1,\infty}(D)} := \sup_{\lambda > 0} \lambda V(\{z \in D : |B f(z)| > \lambda\}) \leq C \|f\|_{L^1(D)}
$$

for all $f \in L^1(D)$. In other words, $B$ acts as a bounded operator from $L^1(D)$ to $L^{1,\infty}(D)$, where $L^{1,\infty}(D)$ is the space of functions $f$ on $D$ for which $\|f\|_{L^{1,\infty}(D)} < \infty$. In particular, McNeal proved the weak-type \((1,1)\) estimate for three different classes of domains, all with smooth boundary, in \cite{14}: finite type domains in $\mathbb{C}^2$, decoupled, finite type domains in $\mathbb{C}^n$, and convex, finite type domains in $\mathbb{C}^n$. The same methods with little modification also apply to strongly pseudoconvex domains with smooth boundary. The proof involves defining an appropriate quasi-metric on $D$ and using real variable techniques. See also \cite{3} for a direct proof of the weak-type \((1,1)\) estimate in the case $D = \mathbb{D} \subseteq \mathbb{C}$.

Recent attention has been given to understanding weighted bounds for the Bergman projection. We call a locally integrable function that is positive almost everywhere a weight. For a weight $\sigma$, we denote

$$
L^p_{\sigma}(D) := \left\{ f : D \to \mathbb{C} : \int_D |f|^p \sigma \, dV < \infty \right\}.
$$

For the case $D = \mathbb{B}_n$, Bekollé proved in \cite{1} that $B$ extends boundedly on $L^p_{\sigma}(\mathbb{B}_n)$ for $1 < p < \infty$ if and only if $\sigma$ satisfies the $B_p$ condition:

$$
[\sigma]_{B_p} := \sup_{B(z,r)} \left( \frac{1}{V(B(z,r))} \int_{B(z,r)} \sigma \, dV \right) \left( \frac{1}{V(B(z,r))} \int_{B(z,r)} \sigma^{- \frac{1}{p-1}} \, dV \right)^{p-1} < \infty.
$$

Here we use the notation $B(z,r)$ to represent a quasi-ball centered at $z$ with radius $r$ and $d(z,bD)$ to denote the quasi-distance from the point $z$ to the boundary of $D$ with respect to a certain quasi-metric defined on $\mathbb{B}_n \times \mathbb{B}_n$. Bekollé also addressed the case $p = 1$ for $D = \mathbb{B}_n$ by proving that $B$ extends boundedly from $L^1_{\sigma}(\mathbb{B}_n)$ to $L^{1,\infty}_{\sigma}(\mathbb{B}_n)$ if and only if $\sigma$ satisfies the $B_1$ condition:

$$
[\sigma]_{B_1} := \sup_{B(z,r)} \left( \frac{1}{V(B(z,r))} \int_{B(z,r)} \sigma \, dV \right) \left\| \sigma^{-1} \right\|_{L^1(B(z,r))} < \infty.
$$

The above result for $1 < p < \infty$ was recently extended to the near minimal smoothness case where $D$ is a strongly pseudoconvex bounded domain with $C^4$ boundary by the second author and Wick in \cite{18}. In Section \ref{sec:2}, we use the same condition for $B_1$ weights with respect to the quasi-metric defined therein.

The first main result of this paper is the weighted weak-type \((1,1)\) estimate for the Bergman projection on domains with near minimal smoothness.

**Theorem 1.1.** If $D \subseteq \mathbb{C}^n$ is a strongly pseudoconvex bounded domain with $C^4$ boundary and $\sigma$ is a $B_1$ weight on $D$, then the Bergman projection $B$ extends boundedly from $L^1_{\sigma}(D)$ to $L^{1,\infty}_{\sigma}(D)$. That is, there exists $C > 0$ such that

$$
\|B f\|_{L^{1,\infty}_{\sigma}(D)} := \sup_{\lambda > 0} \lambda \sigma(\{z \in D : |B f(z)| > \lambda\}) \leq C \|f\|_{L^1_{\sigma}(D)}
$$

for all $f \in L^1_{\sigma}(D)$.
We remark that Theorem 1.1 is new even in the unweighted setting ($\sigma = 1$). In this case, Theorem 1.1 can be viewed as an extension of McNeal’s results of [14] to domains with near minimal smoothness and also of the work of Lanzani and Stein in [11] to address the behavior at the $p = 1$ endpoint. In fact, Theorem 1.1 and an interpolation argument imply the $L^p(D)$, $1 < p < \infty$, boundedness result of [11] in the case of $D$ having $C^4$ boundary. With $B_1$ weights, Theorem 1.1 generalizes Bekollé’s endpoint weak-type result of [1] to domains with near minimal smoothness and extends the work in [18] to address the $p = 1$ endpoint.

The weak-type estimate of Theorem 1.1 implies some other useful endpoint bounds, generalizing results in [3]. In particular, one has the following weighted Kolmogorov inequality:

**Corollary 1.2.** If $D \subseteq \mathbb{C}^n$ is a strongly pseudoconvex bounded domain with $C^4$ boundary, $\sigma \in B_1$, and $0 < p < 1$, then the Bergman projection $B$ extends boundedly from $L^1_\sigma(D)$ to $L^p_\sigma(D)$. That is, there exists $C > 0$ such that

$$\|Bf\|_{L^p_\sigma(D)} \leq C\|f\|_{L^1_\sigma(D)}$$

for all $f \in L^1_\sigma(D)$.

Additionally, one also gets the following Zygmund inequality as a corollary:

**Corollary 1.3.** If $D \subseteq \mathbb{C}^n$ is a strongly pseudoconvex bounded domain with $C^4$ boundary and $\sigma \in B_1$, then the Bergman projection $B$ extends boundedly from $(L \log^+ L)_\sigma(D)$ to $L^1_\sigma(D)$. That is, there exists $C > 0$ such that

$$\|Bf\|_{L^1_\sigma(D)} \leq C\|f\|_{(L \log^+ L)_\sigma(D)}$$

for all $f \in (L \log^+ L)_\sigma(D)$.

Refer to Section 4 for a precise definition of the Zygmund spaces $L \log^+ L$ and their norms.

We also study the Cauchy-Szegő projection on domains with near minimal smoothness. The Hardy space, $H^2(bD)$, is defined to be the following closure in $L^2(bD)$:

$$H^2(bD) := \{f \in L^2(bD) : f = F |_{bD}, \ F \in \text{Hol}(D), \ \text{and} \ F \in C^0(\overline{D})\}.$$ 

Since $H^2(bD)$ is a closed subspace of $L^2(bD)$, there is an orthogonal projection from $L^2(bD)$ to $H^2(bD)$ which we call the Cauchy-Szegő projection and denote by $S$. We can view $S$ as an integral operator via

$$Sf(z) = \int_{bD} K(z, w)f(w)\,dS(w),$$

where $K$ is the reproducing kernel for $H^2(bD)$ and $S$ denotes the induced Lebesgue measure on $bD$.

Again, it is clear that $S$ is a bounded operator on $L^2(bD)$. The bounds of $S$ on $L^p(bD)$ for $1 < p < \infty$ have a long history beginning in the case $D = \mathbb{D}$, where $S$ is the Cauchy transform. In this case, the classical theorem of M. Riesz asserts that $S$ acts as a bounded operator on $L^p(\mathbb{D})$ for $1 < p < \infty$. Recently, the $L^p(bD)$ bounds for $S$ on domains with minimal smoothness were proved by Lanzani and Stein in [12]. In particular, they showed that if $D \subseteq \mathbb{C}^n$ is strongly pseudoconvex and bounded with $C^2$ boundary, then $S$ extends as a bounded operator on $L^p(bD)$ for $1 < p < \infty$.

The characterization of weighted bounds for $S$ in the case $D = \mathbb{D}$ is given by the $A_p$ condition of Hunt, Muckenhoupt, and Wheeden from [7]. For $1 < p < \infty$, a weight $\sigma$
satisfies the $A_p$ condition if
\[
[\sigma]_{A_p} := \sup_B \left( \frac{1}{S(B)} \int_B \sigma \, dS \right) \left( \frac{1}{S(B)} \int_B \sigma^{-\frac{1}{p-1}} \, dS \right)^{p-1} < \infty,
\]
where supremum is taken over all quasi-balls (with respect to the quasi-metric defined in Section 3) $B \subseteq bD$. When $p = 1$, we say $\sigma$ is an $A_1$ weight if
\[
[\sigma]_{A_1} := \sup_B \left( \frac{1}{S(B)} \int_B \sigma \, dS \right) \|\sigma^{-1}\|_{L^\infty(B)} < \infty.
\]

The bounds of $S$ on $L_p^\sigma(bD)$ for $1 < p < \infty$ and $\sigma \in A_p$ were recently established in the near minimal smoothness case where $D$ is a strongly pseudoconvex bounded domain with $C^3$ boundary by the second author and Wick in [18].

The second main result of this paper is the weighted weak-type $(1, 1)$ estimate for the Cauchy-Szegő projection on domains with near minimal smoothness.

**Theorem 1.4.** If $D \subseteq \mathbb{C}^n$ is a strongly pseudoconvex bounded domain with $C^3$ boundary and $\sigma$ is an $A_1$ weight on $bD$, then the Cauchy-Szegő projection $S$ extends boundedly from $L^1_\sigma(bD)$ to $L^1_{\sigma, \infty}(bD)$. That is, there exists $C > 0$ such that
\[
\|Sf\|_{L^1_{\sigma, \infty}(bD)} := \sup_{\lambda > 0} \lambda \sigma(\{z \in bD : |Sf(z)| > \lambda\}) \leq C\|f\|_{L^1_\sigma(bD)}
\]
for all $f \in L^1_\sigma(bD)$.

We remark that Theorem 1.4 is new even in the unweighted setting. Theorem 1.4 can be viewed as a weighted extension of the work of Lanzani and Stein in [14] and of the second author and Wick in [18] to address the behavior at the $p = 1$ endpoint in the case of near minimal smoothness.

Similar to the case of the Bergman projection, we obtain a weighted Kolmogorov inequality and a weighted Zygmund inequality for the Cauchy-Szegő projection.

**Corollary 1.5.** If $D \subseteq \mathbb{C}^n$ is a strongly pseudoconvex bounded domain with $C^3$ boundary, $\sigma \in A_1$, and $0 < p < 1$, then the Cauchy-Szegő projection $S$ extends boundedly from $L^1_\sigma(bD)$ to $L^p_\sigma(bD)$. That is, there exists $C > 0$ such that
\[
\|Sf\|_{L^p_\sigma(bD)} \leq C\|f\|_{L^1_\sigma(bD)}
\]
for all $f \in L^1_\sigma(bD)$.

**Corollary 1.6.** If $D \subseteq \mathbb{C}^n$ is a strongly pseudoconvex bounded domain with $C^3$ boundary and $\sigma \in A_1$, then the Cauchy-Szegő projection $S$ extends boundedly from $(L \log L)_\sigma(bD)$ to $L^1_\sigma(bD)$. That is, there exists $C > 0$ such that
\[
\|Sf\|_{L^1_\sigma(bD)} \leq C\|f\|_{(L \log L)_\sigma(bD)}
\]
for all $f \in (L \log L)_\sigma(bD)$.

Throughout this paper, we use the notation $A \lesssim B$ to mean $A \leq CB$ for some $C > 0$ that could possibly depend on $n$, anything intrinsic to $D$, or $A_1, B_1$ weight characteristics. Although we will not keep track of constants depending on the weights, we will explicitly mention whenever their conditions are used. We say $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$. We use the notation $\langle f \rangle_{E, \mu}$ to denote the average $\frac{1}{\mu(E)} \int_E f \, d\mu$. We just write $\langle f \rangle_E$ to represent this average when $\mu$ is Lebesgue measure in Section 2 and induced Lebesgue measure on
bD in Section 3. For a weight \( \sigma \), we write \( \sigma(E) \) to represent \( \int_E \sigma \, dV \) in Section 2 and to represent \( \int_E \sigma \, dS \) in Section 3.

The paper is organized as follows. In Section 2, we prove the weighted weak-type \((1,1)\) inequality for the Bergman projection, Theorem 1.1. In Section 3, we prove the weighted weak-type \((1,1)\) estimate for the Cauchy-Szegő projection, Theorem 1.4. Finally in Section 4, we obtain Corollaries 1.2, 1.3, 1.5, and 1.6 via general principles.

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## 2. The Bergman Projection

### 2.1. Setup

Let \( D \subseteq \mathbb{C}^n \) be a strongly pseudoconvex bounded domain with \( C^4 \) boundary. This means that there exists a strictly plurisubharmonic, \( C^4 \) defining function \( \rho \) with \( D = \{ z \in \mathbb{C}^n : \rho(z) < 0 \} \) and \( \nabla \rho(z) \neq 0 \) for \( z \in \partial D \).

Our general approach is to construct an auxiliary operator \( T \) that produces and reproduces holomorphic functions. We follow the same construction as in [11, 16], first constructing an operator \( T_1 \) that reproduces (but does not produce) holomorphic functions, and then introducing an operator \( T_2 \) to correct it. The operator \( T \) is taken to be \( T_1 + T_2 \).

To construct \( T_1 \), we use the holomorphic integral representations known as Cauchy-Fantappié integrals. For \( w \in D \), we define the Levi polynomial at \( w \) as follows:

\[
P_w(z) := \sum_{j=1}^{n} \frac{\partial \rho}{\partial w_j}(w)(z_j - w_j) + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial w_j \partial w_k}(w)(z_j - w_j)(z_k - w_k).
\]

We have to modify the Levi polynomial slightly to make it usable for our purposes. In particular, using the strict pseudoconvexity of \( D \), it is possible to choose a smooth cutoff function \( \chi = \chi(z,w) \) with \( \chi \equiv 1 \) when \( |z-w| < \delta/2 \) and \( \chi \equiv 0 \) when \( |z-w| > \delta \) for a small constant \( \delta > 0 \) such that the function

\[
g(z,w) := -\chi P_w(z) + (1 - \chi)|z-w|^2
\]

satisfies

\[
\Re g(z,w) \gtrsim -\rho(w) - \rho(z) + |z-w|^2.
\]

Define the \((1,0)\) form in \( w \), \( G(z,w) \), as follows:

\[
G(z,w) := \chi \left( \sum_{j=1}^{n} \frac{\partial \rho}{\partial w_j}(w) \, dw_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial w_j \partial w_k}(w)(z_k - w_k) \, dw_j \right) + (1-\chi) \sum_{j=1}^{n} (\bar{w}_j - \bar{z}_j) \, dw_j.
\]

Notice that

\[
\langle G(z,w), w-z \rangle = g(z,w) + \rho(w),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the action of a \((1,0)\) form on a vector in \( \mathbb{C}^n \). Now we take

\[
\eta(z,w) := \frac{G(z,w)}{g(z,w)}
\]

and define

\[
T_1 f(z) := \frac{1}{(2\pi i)^n} \int_D (\overline{\partial}_w \eta)^n f(w),
\]

where the exponent \( n \) denotes the wedge product taken \( n \) times.

With this definition, one can show that \( T_1 \) is majorized by a positive operator \( \Gamma \) which can be interpreted as a Calderón-Zygmund operator. This approach is taken in the proof of
Proposition 2.4 below. Notice also that the kernel of $\mathcal{T}_1$ is continuous on $\overline{D} \times \overline{D}$ away from the boundary diagonal $\{(z,z) : z \in bD\}$. Moreover, the following is proven in [11]:

**Lemma 2.1.** If $f \in L^1(D)$ is holomorphic on $D$, then for all $z \in D$,

$$T_1 f(z) = f(z).$$

Lemma 2.1 says that $T_1$ reproduces holomorphic functions. However, in general $T_1$ does not produce holomorphic functions from $L^1$ data since its kernel is not holomorphic in the variable $z$. We introduce a correction operator, $T_2$, to overcome this difficulty. The operator $T_2$ is constructed by solving a $\overline{\partial}$ problem on a smoothly bounded, strongly pseudoconvex domain $\Omega$ that contains $D$. These details are unimportant for our purposes; we only need the following from [11]:

**Lemma 2.2.** There exists an integral operator

$$T_2 f(z) := \int_D K_2(z,w) f(w) dV(w)$$

with continuous kernel $K_2(z,w)$ on $\overline{D} \times \overline{D}$ so that the operator $T := T_1 + T_2$ satisfies the following properties:

1. If $f \in L^1(D)$, then $T f$ is holomorphic on $D$.
2. If $f \in L^1(D)$ and $f$ is holomorphic on $D$, then $T f(z) = f(z)$ for $z \in D$.

It is important to note that Lemma 2.2 together with the definition of $T_1$ implies that the kernel of $T$ is continuous on $\overline{D} \times \overline{D}$ away from the boundary diagonal.

We have constructed an auxiliary operator $T$ that produces and reproduces holomorphic functions. Since $B$ also produces and reproduces holomorphic functions, we arrive at the following operator equations that hold on $L^2(D)$:

$$TB = B \quad \text{and} \quad BT = T.$$  

Taking adjoints in the first identity, subtracting from the second, and some rearrangement yields the familiar Kerzman-Stein equation:

$$(2.1) \quad B(I - (T^* - T)) = T.$$  

The proof of Theorem 1.1 follows easily from the following two facts.

**Proposition 2.3.** If $\sigma$ is a $B_1$ weight, then the operator $I - (T^* - T)$ is invertible on $L^1_{\sigma}(D)$.

**Proposition 2.4.** If $\sigma$ is a $B_1$ weight, then $T$ maps $L^1_{\sigma}(D)$ to $L^1_{\sigma,\infty}(D)$ boundedly.

**Proof of Theorem 1.1.** Using Proposition 2.3, we may rewrite (2.1) as

$$B = T(I - (T^* - T))^{-1}.$$  

The bound of $B$ from $L^1_{\sigma}(D)$ to $L^1_{\sigma,\infty}(D)$ follows from Proposition 2.3 and Proposition 2.4.  

The remainder of this section is devoted to proving Proposition 2.3 and Proposition 2.4. Proposition 2.3 will follow from the spectral theorem for compact operators on a Banach space. In particular, we will show that $T^* - T$ is compact on $L^1_{\sigma}(D)$ and also that 1 is not an eigenvalue of $T^* - T$ on $L^1_{\sigma}(D)$. Proposition 2.4 relies on methods from Calderón-Zygmund theory reminiscent of the ideas in [1].

The arguments in [6, 13, 14, 18] make use of an appropriately constructed quasi-metric $d$ that reflects the geometry of the boundary. Technically, the quasi-metric $D$ is only defined

**Proposition 2.4.** If $\sigma$ is a $B_1$ weight, then $T$ maps $L^1_{\sigma}(D)$ to $L^1_{\sigma,\infty}(D)$ boundedly.
for points \( z, w \) sufficiently close to the boundary, but we will abuse notation and define objects as if \( d \) were defined globally. This reduction is possible because the kernels of all the relevant operators are uniformly continuous on compact subsets of \( \overline{D} \times \overline{D} \) off the boundary diagonal and all the necessary properties will hold for trivial reasons.

The quasi-metric \( d \) locally satisfies:

\[
d(z, w) \approx |z_1 - w_1| + \sum_{j=2}^{n} |z_j - w_j|^2,
\]

where the coordinates \( z_j \) and \( w_j \) are taken in a special holomorphic coordinate system centered at \( w \). The coordinate function \( z_1 \) corresponds to the radial direction, while \( z_2, \ldots, z_n \) describe the complex tangential directions. In [13], these coordinates were used to obtain favorable estimates on the Bergman kernel for smoothly bounded domains \( D \), and in [18] they were used in the case when \( D \) has less boundary regularity.

We use the constant \( c > 0 \) to denote the implicit constant in the triangle inequality for \( d \):

\[
d(z, w) \leq cd(z, \zeta) + cd(\zeta, w).
\]

We denote balls with respect to this quasi-metric by \( B(z, r) := \{ w \in D : d(z, w) < r \} \). If \( B \) is a quasi-ball, then its center and radius are represented by \( c(B) \) and \( r(B) \) respectively, meaning \( B = \{ w \in D : d(c(B), w) < r(B) \} \). We also write \( kB \) to denote the \( k \)-fold dilate of \( B \), that is \( kB := \{ w \in D : d(c(B), w) < kr(B) \} \).

Importantly, the triple \( (D, d, V) \) forms a space of homogeneous type in the sense of Coifman and Weiss introduced in [2]. In particular, \( V \) satisfies the following growth condition with respect to quasi-balls induced by \( d \):

\[
(2.2) \quad V(B(z, r)) \approx r^{n+1}
\]

for all \( z \in D \) and \( r > 0 \). Moreover, the distance function \( d \) can be extended to \( \overline{D} \times \overline{D} \) and we may also define \( d(z, bD) := \inf_{w \in bD} d(z, w) \), see [6]. Notice that for a \( B_1 \) weight \( \sigma \), \( \sigma dV \) also satisfies a particular doubling property for quasi-balls close to the boundary:

\[
\sigma(B(z, 2r)) \lesssim \left( \inf_{w \in B(z, 2r)} \sigma(w) \right) V(B(z, 2r)) \lesssim \sigma(B(z, r))
\]

for any \( z \in D \) and \( r > 0 \) such that \( r > kd(z, bD) \) for some absolute \( k > 0 \) (the first inequality above depends on \( [\sigma]_{B_1} \)). For sets \( E, F \subseteq \overline{D} \), we write \( d(E, F) := \inf_{z \in F} d(z, w) \).

We work with a maximal operator \( \mathcal{M} \) adapted to our setting. For locally integrable \( f \), define

\[
\mathcal{M}f(z) := \sup_{B(w, r) \ni z \atop r > d(w, bD)} \langle |f| \rangle_{B(w, r)}.
\]

Note that a weight \( \sigma \) is in \( B_1 \) if and only if \( \mathcal{M} \sigma (z) \lesssim \sigma(z) \) for almost every \( z \in D \).

2.2. Inversion of the “mild” operator. To deduce the compactness of \( T^* - T \), we use a more general result which follows from [4, Corollary 4.1]. In the following lemma, \( \mathcal{K} \) is an integral operator given by

\[
\mathcal{K}f(x) = \int_X k(x, y) f(y) \, d\mu(y)
\]

and \( k_y(x) = k(x, y) \).
Lemma 2.5. Let \((X, \mu)\) be a positive measure space. Suppose that \(k : X \times X \to \mathbb{R}\) is a measurable function such that \(\|\int_X k(x, \cdot) \, d\mu(x)\|_{L^\infty(X, \mu)} < \infty\). If the set \(\{k_y\}_{y \in X}\) is relatively compact in \(L^1(X, \mu)\), then \(K\) and \(K^*\) are compact operators on \(L^1(X, \mu)\) and \(L^\infty(X, \mu)\) respectively.

To justify the relative compactness of \(\{k_y\}\) in our application of Lemma 2.5, we use the following characterization for relatively compact sets, which can be viewed as a generalization of the classical Riesz-Kolmogorov theorem.

Lemma 2.6. Let \(\mu\) be a finite Borel measure on \(X\) such that \(\inf_{x \in X} \mu(B(x, r)) > 0\) for any \(r > 0\) and let \(1 \leq p < \infty\). If \(K \subseteq L^p(X, \mu)\) is a bounded set satisfying

\[
\lim_{r \to 0^+} \sup_{f \in K} \int_X |f(x) - \langle f \rangle_{B(x,r),\mu}|^p \, d\mu(x) = 0,
\]

then \(K\) is relatively compact in \(L^p(X, \mu)\).

Lemma 2.6 was originally stated for the case of metric spaces in [9], but we will need a version from [8, Lemma 1] in the case where we only have a quasi-metric.

We next apply Lemma 2.5 and Lemma 2.6 to prove the following result.

Lemma 2.7. If \(\sigma\) is a \(B_1\) weight, then the operator \(T^* - T\) is compact on \(L^1_\sigma(D)\).

Proof. First, we note that \(\sigma dV\) is a finite Borel measure on \(D\). Using the \(B_1\) condition and the fact that \(B(z, R) = D\) for \(z \in D\) and sufficiently large \(R\), one has

\[
\sigma(D) \lesssim \left( \inf_{w \in D} \sigma(w) \right) V(D).
\]

The infimum condition on the measure \(\sigma dV\) can be verified using a compactness argument and the fact that \(B(z, r)\) contains a Euclidean ball with radius comparable to \(r^{1/2}\), which was proved in [18, Proposition 3.5]. Let \(k(z, w)\) denote the kernel of \(T^* - T\) with respect to Lebesgue measure. The following key properties of \(k(z, w)\) are proved in [18, Lemma 3.14]:

\[
|k(z, w)| \lesssim |g(z, w)|^{-\left(n + \frac{1}{2}\right)} \lesssim d(z, w)^{-\left(n + \frac{1}{2}\right)}
\]

as well as

\[
|k(z, w)| \lesssim \min \left\{ d(z, bD)^{-\left(n + \frac{1}{2}\right)}, d(w, bD)^{-\left(n + \frac{1}{2}\right)} \right\}.
\]

Here, the assumption that the boundary of \(D\) is of class \(C^4\) is in fact crucial. Let \(\tilde{k}(z, w)\) denote the kernel of \(T^* - T\) with respect to the weighted measure \(\sigma dV\) and notice \(\tilde{k}(z, w) = k(z, w)\sigma(w)^{-1}\).

We claim that there exists \(M > 0\) such that \(\sup_{w \in D} \int_D |\tilde{k}(z, w)| \sigma(z) \, dV(z) < M\). To see this, fix \(w \in D\) and integrate over dyadic annuli, choosing \(R\) so that \(B(w, R) = D\) and letting \(N\) be the largest positive integer such that \(B(w, 2^{-N} R)\) meets the boundary of \(D\).
We use the above control of $|k(z, w)|$ and (2.2) to obtain
\[
\int_D \hat{k}(z, w)|\sigma(z)\, dV(z) = \sigma(w)^{-1} \int_D |k(z, w)|\sigma(z)\, dV(z)
\]
\[
= \sigma(w)^{-1} \sum_{j=0}^N \int_{B(w, 2^{-j+1}R)} d(z, w)^{-(n+\frac{1}{2})}\sigma(z)\, dV(z)
\]
\[
+ \sigma(w)^{-1} \int_{B(w, 2^{-N+1}R)} d(w, bD)^{-(n+\frac{1}{2})}\sigma(z)\, dV(z)
\]
\[
\lesssim \sigma(w)^{-1} \sum_{j=0}^N 2^{-j/2}R^{1/2} \int_{B(w, 2^{-j}R)} \sigma(z)\, dV(z)
\]
\[
+ \sigma(w)^{-1} \frac{d(w, bD)^{1/2}}{V(B(w, d(w, bD)))} \int_{B(w, d(w, bD))} \sigma(z)\, dV(z)
\]
\[
\leq \sigma(w)^{-1} \sum_{j=0}^N 2^{-j/2}R^{1/2}M\sigma(w) + \sigma(w)^{-1}d(w, bD)^{1/2}M\sigma(w)
\]
\[
\lesssim \sigma(w)^{-1}(R^{1/2} + d(w, bD)^{1/2})M\sigma(w)
\]
\[
\lesssim R^{1/2}.
\]

Note that we used the $B_1$ condition in the last line above. All the implicit constants are independent of $w$, and $R$ is also independent of $w$ since we can just take $R$ to be the diameter of $D$ in the quasi-metric. This establishes the claim. Notice that this argument also shows that replacing the region of integration by a quasi-ball $B(w, \delta)$ yields
\[
(2.3) \quad \int_{B(w, \delta)} \hat{k}(z, w)|\sigma(z)\, dV(z) \lesssim \delta^{1/2} + d(w, bD)^{1/2},
\]
where the implicit constant is independent of $w$.

Now we must show the crucial condition
\[
\lim_{r \to 0^+} \sup_{w \in D} \sigma(w)^{-1} \int_D |k_w(z) - \langle k_w \rangle_{B(z, r), \sigma dV}|\sigma(z)\, dV(z) = 0,
\]
where $k_w(z) = k(z, w)$. Fix $\varepsilon > 0$, $w \in D$, and let $\delta > 0$ and $0 < r < \delta$ be constants to be fixed later. We emphasize all constants obtained will ultimately be independent of $w$.

Let $G := \{z \in D : d(z, w) \geq \delta \text{ or } d(z, bD) \geq \delta\}$. We will first estimate
\[
\sigma^{-1}(w) \int_G |k_w(z) - \langle k_w \rangle_{B(z, r), \sigma dV}|\sigma(z)\, dV(z).
\]
Recall that the kernel function $k(z, w)$ is uniformly continuous on compact subsets off the boundary diagonal, so in particular the function $k_w(z)$ is uniformly continuous on $G$ with a modulus of continuity independent of $w$. We can choose $r$ sufficiently small relative to $\delta$ and independent of $w$ so that we have $|k_w(z) - \langle k_w \rangle_{B(z, r), \sigma dV}| < \varepsilon$ for $z \in G$ and hence,
\[
\sigma(w)^{-1} \int_G |k_w(z) - \langle k_w \rangle_{B(z,r)}| \sigma(z) \, dV(z) \leq \varepsilon \sigma(w)^{-1} \int_D \sigma(z) \, dV(z) \\
\lesssim \varepsilon \sigma(w)^{-1} \mathcal{M} \sigma(w) \\
\lesssim \varepsilon
\]
as required. We used the \( B_1 \) condition of \( \sigma \) in the last inequality above.

Now we need to estimate the integral on \( D \setminus G \). Note \( D \setminus G = B(w, \delta) \cap A \), where \( A := \{ z : d(z, bD) < r \} \). We have

\[
\sigma(w)^{-1} \int_{D \setminus G} |k_w(z) - \langle k_w \rangle_{B(z,r)}| \sigma(z) \, dV(z) \leq \sigma(w)^{-1} \left( \int_{D \setminus G} |k_w(z)| \sigma(z) \, dV(z) \\
+ \int_{D \setminus G} |\langle k_w \rangle_{B(z,r)}| \sigma(z) \, dV(z) \right).
\]

By (2.3), it is easy to deduce

\[
\sigma(w)^{-1} \int_{D \setminus G} |k_w(z)| \sigma(z) \, dV(z) \lesssim \delta^{1/2}.
\]

We will also show

\[
\sigma(w)^{-1} \int_{D \setminus G} |\langle k_w \rangle_{B(z,r)}| \sigma(z) \, dV(z) \lesssim \delta^{1/2}
\]

using similar methods. We consider two separate regions of integration based on the relative positions of \( z \) and \( w \). First, suppose that \( cr < \frac{1}{2} d(z, w) \). One can show that if \( \zeta \in B(z, r) \), then \( d(z, w) \lesssim d(\zeta, w) \) with an implicit constant independent of \( z \) and \( w \). We then estimate

\[
\sigma(w)^{-1} \int_{(B(w, \delta) \setminus B(w, 2cr)) \cap A} |\langle k_w \rangle_{B(z,r)}| \sigma(z) \, dV(z) \\
\leq \sigma(w)^{-1} \int_{(B(w, \delta) \setminus B(w, 2cr)) \cap A} \frac{1}{\sigma(B(z, r))} \int_{B(z, r)} d(\zeta, w)^{-\left(n + \frac{1}{2}\right)} \sigma(\zeta) \, dV(\zeta) \sigma(z) \, dV(z) \\
\lesssim \sigma(w)^{-1} \int_{B(w, \delta) \cap A} d(z, w)^{-\left(n + \frac{1}{2}\right)} \sigma(z) \, dV(z) \\
\lesssim \delta^{1/2}
\]
as before. We have used the \( B_1 \) condition of \( \sigma \) in the third inequality above.

On the other hand, if \( d(z, w) \leq 2cr \), then \( B(z, r) \subseteq B(w, Cr) \) and \( B(w, r) \subseteq B(z, Cr) \), where \( C = 2c^2 + c \). We first consider a further subcase where \( d(w, bD) < r \). In this case, note \( d(z, bD) \lesssim r \) on this set as well by the quasi-triangle inequality. Thus, we calculate:
\[
\sigma(w)^{-1} \int_{B(w,2cr)} \frac{1}{\sigma(B(z,r))} \int_{B(z,r)} d(\zeta, w)^{-\left(n+\frac{1}{2}\right)} \sigma(\zeta) dV(\zeta) \sigma(z) dV(z)
\]
\[
\leq \sigma(w)^{-1} \int_{B(w,2cr)} \frac{1}{\sigma(B(w,r))} \int_{B(w,r)} d(\zeta, w)^{-\left(n+\frac{1}{2}\right)} \sigma(\zeta) dV(\zeta) \sigma(z) dV(z)
\]
\[
\lesssim \delta^{1/2} \int_{B(w,\delta)} \frac{1}{\sigma(B(w,r))} \int_{B(w,r)} d(\zeta, w)^{-\left(n+\frac{1}{2}\right)} \sigma(z) dV(z)
\]
\[
\lesssim \delta^{1/2}
\]
using the \(B_1\) condition in the second inequality and the doubling property of \(\sigma\) in the third inequality. For the second subcase, suppose \(d(w, bD) \geq r\) and note that we still assume \(d(z, w) \leq 2cr\), so in fact have \(d(w, bD)^{-\left(n+1/2\right)} \lesssim d(z, w)^{-\left(n+1/2\right)}\). We estimate
\[
\sigma(w)^{-1} \int_{B(w,2cr)} |\langle k_w \rangle_{B(z,r), \sigma dV}| d\sigma(z)
\]
\[
\leq \sigma(w)^{-1} \int_{B(w,2cr)} \frac{1}{\sigma(B(z,r))} \int_{B(z,r)} d(w, bD)^{-\left(n+\frac{1}{2}\right)} \sigma(\zeta) dV(\zeta) \sigma(z) dV(z)
\]
\[
\lesssim \sigma(w)^{-1} \int_{B(w,\delta)} d(z, w)^{-\left(n+\frac{1}{2}\right)} \sigma(z) dV(z)
\]
\[
\lesssim \delta^{1/2},
\]
where we have used the \(B_1\) condition in the third inequality.

Thus, we obtain
\[
\sigma(w)^{-1} \int_{D_G} |k_w(z) - \langle k_w \rangle_{B(z,r), \sigma dV}| \sigma(z) dV(z) \lesssim \delta^{1/2}
\]
with an independent implicit constant. This can be made less than \(\varepsilon\) by making an appropriately small choice of \(\delta\), completing the proof. \(\square\)

We need the following lemma to conclude that \((I - (T^* - T))\) is invertible on \(L^1_\sigma(D)\).

**Lemma 2.8.** If \(\sigma \in B_1\), the number 1 is not an eigenvalue of \(T^* - T\) considered as an operator on \(L^1_\sigma(D)\).

**Proof.** The proof proceeds in the same way as \([18\text{ Corollary 3.17}]\). In particular, it was proved in \([18\text{ Proposition 3.16}]\) that there exists \(\varepsilon > 0\) so that \(T^* - T\) maps \(L^p(D)\) to \(L^{p+\varepsilon}(D)\) boundedly for \(p \geq 1\). Thus, if 1 were an eigenvalue for \(T^* - T\) with eigenvector \(f \in L^1_\sigma(D)\), then we would have
\[
\|f\|_{L^{1+\varepsilon}(D)} = \|(T^* - T)f\|_{L^{1+\varepsilon}(D)} \lesssim \|f\|_{L^1_\sigma(D)} \lesssim \|f\|_{L^1_\sigma(D)},
\]
noting that a weight in \(B_1\) is bounded below. If we repeat this argument a second time, we get \(f \in L^{1+2\varepsilon}(D)\). In fact, we can iterate arbitrarily many times to obtain \(f \in L^p(D)\) for all \(p \geq 1\). In particular, \(f \in L^2(D)\). This is a contradiction because 1 is not an eigenvalue of \(T^* - T\) on \(L^2(D)\), since all of these eigenvalues are purely imaginary. \(\square\)

**Proof of Proposition 2.3.** This follows immediately from Lemma 2.7 and Lemma 2.8 using the spectral theorem for compact operators. \(\square\)
2.3. Weak-type estimate for the auxiliary operator. To show the weighted weak-type (1, 1) property for $T$, we first prove the analogous bound for our maximal operator $M$.

**Lemma 2.9.** If $\sigma$ is a $B_1$ weight, then $M$ maps $L^1_{\sigma}(D)$ into $L^{1,\infty}_{\sigma}(D)$ boundedly.

**Proof.** It suffices to prove the estimate for the centered version of $M$,

$$ \widetilde{M}f(z) := \sup_{r > d(z, B)} |f|_{B(z, r)}, $$

since we have the pointwise equivalence $\widetilde{M}f \leq Mf \lesssim \widetilde{M}f$. Indeed, the first inequality is clear, and the second is justified by the fact that $|f|_{B(z, 2\rho(B))} \lesssim |f|_{B(z, 2\rho(B))}$ for any $z \in D$ and quasi-ball $B$ containing $z$.

Let $f \in L^1_{\sigma}(D)$, $\lambda > 0$, and $E_\lambda := \{\widetilde{M}f > \lambda\}$. We show that

$$ \sigma(E_\lambda) \lesssim \frac{1}{\lambda} \|f\|_{L^1_{\sigma}(D)}. $$

For each $z \in E_\lambda$, let $B_z$ be a quasi-ball centered at $z$ such that $r(B_z) > d(z, B\sigma)$ and $\langle|f|\rangle_{B_z} > \lambda$. Apply a Vitali-type lemma to obtain a subcollection $\{B_j\}_{j=1}^\infty$ of $\{B_z\}_{z \in E_\lambda}$ consisting of pairwise disjoint quasi-balls such that there exists $R \geq 1$ with $E_\lambda \subseteq \bigcup_{j=1}^\infty RB_j$.

Use the doubling property of $\sigma$, the $B_1$ property of $\sigma$, and the selection property of the $B_j$ to conclude

$$ \sigma(E_\lambda) \leq \sum_{j=1}^\infty \sigma(RB_j) \lesssim \sum_{j=1}^\infty \sigma(B_j) $$

$$ \lesssim \sum_{j=1}^\infty \left( \frac{1}{\|\sigma^{-1}\|_{L^\infty(B_j)}} \right) V(B_j) \leq \sum_{j=1}^\infty \left( \inf_{w \in B_j} \sigma(w) \right) \frac{1}{\lambda} \int_{B_j} |f| \, dV \leq \frac{1}{\lambda} \|f\|_{L^1_{\sigma}(D)}. $$

For $k \in (0, 1)$, define $B_k(z) := B(z, kd(z, B\sigma))$ and

$$ R_k f(z) := \langle f \rangle_{B_k(z)}. $$

The following was proved in [6, Lemma 3.4]. Note that in the setting of [6], $D$ had smooth boundary, but that assumption was not needed to establish the following lemma.

**Lemma 2.10.** For all $k \in (0, \frac{1}{2\epsilon})$ and all nonnegative $f, g \in L^1(D)$, we have

$$ \int_D f(R_k g) \, dV \lesssim \int_D (R_{k'} f) g \, dV, $$

where $k' := \frac{ck}{1-ck}$.

**Lemma 2.11.** There exists $k > 0$ such that $|g(z, w)| \approx |g(z, w')|$ for all $z, w, w' \in D$ satisfying $d(w, w') \leq kd(w, B\sigma)$.

**Proof.** From the proof of [18, Lemma 3.12], we know $|g(z, w)| \approx |g(z, w')|$ whenever $d(w, w') \leq Ck^\epsilon$, where $C > 0$ is an absolute constant. If $d(w, B\sigma) < \frac{C}{k} d(z, w)$, then $d(w, w') \leq kd(w, B\sigma) < Ck^\epsilon$, and hence $|g(z, w)| \approx |g(z, w')|$. 


We may now assume that $d(z, w) \leq \frac{k}{2}d(w, bD)$. In this case, we use the triangle inequality, the fact that $|g(z, w) - g(z, w')| \lesssim d(w, w')^{1/2}d(z, w)^{1/2} + d(w, w')$ (which was proven in [18 Lemma 3.12]), and the assumptions to get
\[
|g(z, w)| \leq |g(z, w) - g(z, w')| + |g(z, w')| \\
\lesssim d(w, w')^{1/2}d(z, w)^{1/2} + d(w, w') + |g(z, w')| \\
\leq \left( \frac{k}{C_{1/2}} + k \right) d(w, bD) + |g(z, w')|.
\]
Now using the triangle inequality and the hypothesis, we have
\[
d(w, bD) \leq cd(w, w') + cd(w', bD) \leq ckd(w, bD) + cd(w', bD).
\]
Choosing $k$ sufficiently small, the above line implies $d(w, bD) \lesssim d(w', bD)$, and so
\[
|g(z, w)| \lesssim d(w', bD) + |g(z, w')|.
\]
Again referring to [18 Lemma 3.12], we have $d(w', bD) \lesssim |g(z, w')|$, and we conclude
\[
|g(z, w)| \lesssim |g(z, w')|.
\]
A symmetric argument proves the reverse inequality, establishing the lemma. \qed

The following lemma is a modified version of the Calderón-Zygmund decomposition.

Lemma 2.12. For any $\lambda > 0$, $k \in (0,1)$, and nonnegative $f \in L^1(D)$, we can write $f \approx f_1 + f_2$, where

1. $R_k f_1 \lesssim \lambda$,
2. there exists a countable collection of almost disjoint quasi-balls $\mathcal{F}$ such that $r(B) \geq \frac{1}{2}d(B, bD)$ for each $B \in \mathcal{F}$ and $f_2 \approx \sum_{B \in \mathcal{F}} f_{2, B}$ where the $f_{2, B}$ are supported in $B$ with $\|f_{2, B}\|_B \leq \lambda$, and
3. $\sum_{B \in \mathcal{F}} \sigma(B) \lesssim \frac{1}{\lambda} \|f\|_{L^1(D)}$.

Proof. Apply a Whitney decomposition to write
\[
\{Mf > \lambda\} = \bigcup_{B \in \mathcal{F}'} B,
\]
where $\mathcal{F}'$ is a countable collection of almost disjoint quasi-balls for which there exists $K > 1$ such that $KB \cap \{Mf \leq \lambda\} \neq \emptyset$ for all $B \in \mathcal{F}'$. We take
\[
\mathcal{F} := \left\{ B \in \mathcal{F}' : r(B) \geq \frac{1}{2}d(B, bD) \right\}.
\]
Put
\[
f_1 := f \chi_{\{Mf \leq \lambda\} \cup \bigcup_{B \in \mathcal{F}' \setminus \mathcal{F}} B} \quad \text{and} \quad f_2 := f \chi_{\bigcup_{B \in \mathcal{F}} B}.
\]
Clearly, $f \approx f_1 + f_2$.

To show (1), we first claim that $R_k f_1(z) \lesssim Mf_1(z)$ for any $z \in D$. Indeed, since the radius of $\frac{k+1}{k}B_k(z)$ is greater than $d(z, bD)$ and using [2.2], we have
\[
R_k f_1(z) = \langle f_1 \rangle_{B_k(z)} \lesssim \langle |f_1| \rangle_{\frac{k+1}{k}B_k(z)} \lesssim Mf_1(z).
\]
Therefore it is enough to prove $Mf_1 \lesssim \lambda$. To this end, fix $z \in D$ and let $B_0$ be a quasi-ball containing $z$ that intersects $bD$. If either $B_0 \cap \{Mf \leq \lambda\} \neq \emptyset$ or if $f \equiv 0$ on $B_0$, then $\langle |f_1| \rangle_{B_0} \leq \lambda$. Otherwise, $B_0 \cap B \neq \emptyset$ for some $B \in \mathcal{F}' \setminus \mathcal{F}$. Notice that $CB_0 \supseteq KB$ with
\[ C = c^3(K + 1) + c, \text{ since } d(c(B_0), bD) < r(B_0) \text{ and } r(B) < \frac{1}{2} d(B, bD). \] Since \( KB \cap \{ \mathcal{M} f \leq \lambda \} \neq \emptyset \) and using (2.2), we have
\[ \langle |f_1| \rangle_{B_0} \lesssim \langle |f_1| \rangle_{C_* B_0} \leq \lambda. \]
Therefore (1) holds.

For (2), note that the properties of the collection \( \mathcal{F} \) are satisfied by construction. Take \( f_{2,B} := f \chi_B \) for \( B \in \mathcal{F} \). Since \( KB \cap \{ M f \leq \lambda \} \neq \emptyset \), we have \( \langle |f_{2,B}| \rangle_B \lesssim \lambda \).

Finally, (3) follows from the almost disjointness of the quasi-balls in \( \mathcal{F} \) and Lemma 2.9
\[ \sum_{B \in \mathcal{F}} \sigma(B) \lesssim \sigma \left( \bigcup_{B \in \mathcal{F}} B \right) \leq \sigma \left( \{ M f > \lambda \} \right) \lesssim \frac{1}{\lambda} \| f \|_{L^1_b(D)}. \]

Proof of Proposition 2.4. Since \( T_2 \) has a bounded kernel and \( \sigma(D) < \infty \) (using the \( B_1 \) condition), it is immediate that \( T_2 \) is bounded on \( L^1_\sigma(D) \), and hence from \( L^1_\sigma(D) \) to \( L^{1, \infty}_\sigma(D) \). It is thus sufficient to prove the estimate for \( T_1 \).

As in [18], we define a positive comparison operator \( \Gamma \) by
\[ \Gamma f(z) := \int_D f(w) \frac{dV(w)}{|g(z, w)|^{n+1}}. \]
It can easily be shown that
\[ |T_1 f(z)| \lesssim \Gamma |f|(z), \]
so it suffices to prove that \( \Gamma \) maps \( L^1_\sigma(D) \) to \( L^{1, \infty}_\sigma(D) \).

Let \( f \) be a nonnegative and continuous function on \( D \) and let \( \lambda > 0 \). We will show that
\[ \sigma \left( \{ \Gamma f > \lambda \} \right) \lesssim \frac{1}{\lambda} \| f \|_{L^1_b(D)}. \]
A density argument and doubling the implied constant in the display above yields the result for general \( f \in L^1_\sigma(D) \).

Apply Lemma 2.12 to write
\[ f \approx f_1 + f_2 \approx f_1 + \sum_{B \in \mathcal{F}} f_{2,B}, \]
where the properties and notations from the lemma hold. Then
\[ \sigma \left( \{ \Gamma f > \lambda \} \right) \leq \sigma \left( \left\{ \Gamma f_1 > \frac{\lambda}{C} \right\} \right) + \sigma \left( \left\{ \Gamma f_2 > \frac{\lambda}{C} \right\} \right) \]
\[ \leq \sigma \left( \left\{ \Gamma f_1 > \frac{\lambda}{C} \right\} \right) + \sigma \left( \bigcup_{B \in \mathcal{F}} RB \right) + \sigma \left( \left\{ z \in D \setminus \bigcup_{B \in \mathcal{F}} RB : \Gamma f_2(z) > \frac{\lambda}{C} \right\} \right) \]
for some \( C > 0 \) and where \( R > 1 \) will be fixed later. Therefore it is enough to bound
\[ I := \sigma \left( \{ \Gamma f_1 > \lambda \} \right), \]
\[ \text{II} := \sigma \left( \bigcup_{B \in \mathcal{F}} RB \right), \quad \text{and} \]
\[ \text{III} := \sigma \left( \left\{ z \in D \setminus \bigcup_{B \in \mathcal{F}} RB : \Gamma f_2(z) > \lambda \right\} \right) \]
by constants multiplied by $\frac{1}{\lambda}\|f\|_{L^1_\sigma(D)}$.

To address I, we first claim that there exists $k > 0$ such that for all integrable and nonnegative $u$, we have

$$\Gamma u(z) \lesssim \Gamma (R^k u)(z),$$

where $k' = \frac{c_1}{1 - c_1}$ Indeed, by Lemma 2.11, we have $|g(z, w)| \approx |g(z, w')|$ for all $z \in D$ and $w' \in B_k(w)$. Using the above and Lemma 2.10, we deduce

$$\Gamma u(z) = \int_D \frac{1}{|g(z, w)'|^{n+1}} u(w) dV(w)$$

$$\approx \int_D \left( \int_{B_k(w)} \frac{1}{|g(z, w)'|^{n+1}} dV(w') \right) u(w) dV(w)$$

$$\lesssim \int_D \frac{1}{|g(z, w)'|^{n+1}} \left( \int_{B_{k'}(w)} u(w') dV(w') \right) dV(w)$$

$$= \Gamma (R^k u)(z).$$

Therefore, using Chebyshev’s inequality, the above claim, the $L^2_\sigma(D)$ bound of $\Gamma$ (see [18]), property (1) of Lemma 2.12, Lemma 2.10, and the $B_1$ condition of $\sigma$ we have

$$I \lesssim \frac{1}{\lambda^2} \int_D (\Gamma f_1)^2 \sigma dV$$

$$\lesssim \frac{1}{\lambda^2} \int_D (\Gamma (R^k f_1))^2 \sigma dV$$

$$\lesssim \frac{1}{\lambda^2} \int_D (R^k f_1)^2 \sigma dV$$

$$\lesssim \frac{1}{\lambda} \int_D (R^k f_1) \sigma dV$$

$$\lesssim \frac{1}{\lambda} \int_D f_1 (R^k \sigma) dV$$

$$\lesssim \frac{1}{\lambda} \int_D f_1 \sigma dV$$

$$\lesssim \frac{1}{\lambda} \|f\|_{L^1_\sigma(D)}.$$

The control of II follows from the doubling property of $\sigma$ and property (3) of Lemma 2.12

$$II \lesssim \sum_{B \in F} \sigma(RB) \lesssim \sum_{B \in F} \sigma(B) \lesssim \frac{1}{\lambda} \|f\|_{L^1_\sigma(D)}.$$ 

For III, we claim that if $R > 1$ is sufficiently large and $u$ is supported on a quasi-ball $B$, then

$$\Gamma u(z) \lesssim \Gamma (\langle u \rangle_B \chi_B)(z)$$

for all $z \in D \setminus RB$. Indeed, as stated in the proof of Lemma 2.11, we have $|g(z, w)| \approx |g(z, w')|$ whenever $d(w, w') \leq Cd(z, w)$. For $z \in D \setminus RB$ and $w, w' \in B$, we use the triangle inequality to obtain $d(w, w') < 2c_r(B)$ and $\frac{R-c}{c}r(B) < d(z, w)$. Thus, if $R$ is chosen large enough so
that $2c \leq C^{\frac{\mu - \nu}{\nu}}$, we have $d(w, w') \leq Cd(z, w)$, and hence $|g(z, w)| \approx |g(z, w')|$. The claim follows via using Fubini’s theorem

$$\Gamma u(z) = \int_B \frac{1}{|g(z, w)|^{n+1}} u(w) \, dV(w) \approx \int_B \left( \frac{1}{V(B)} \int_B \frac{1}{|g(z, w')|^{n+1}} \, dV(w') \right) u(w) \, dV(w)$$

$$= \int_B \frac{1}{|g(z, w')|^{n+1}} \left( \frac{1}{V(B)} \int_B u(w) \, dV(w) \right) \, dV(w') \approx \Gamma (\langle u \rangle_B \chi_B)(z).$$

Using the above claim, we have

$$\Gamma f_2(z) \approx \sum_{B \in \mathcal{F}} \Gamma f_{2,B}(z) \lesssim \sum_{B \in \mathcal{F}} \Gamma(\langle f_{2,B} \rangle_B \chi_B)(z) = \Gamma \tilde{f}_2(z)$$

for $z \in D \setminus \bigcup_{B \in \mathcal{F}} RB$, where $\tilde{f}_2 := \sum_{B \in \mathcal{F}} \langle f_{2,B} \rangle_B \chi_B$. Therefore, to control III, it suffices to prove

$$\sigma(\{ \Gamma \tilde{f}_2 > \lambda \}) \lesssim \frac{1}{\lambda} \| f \|_{L^\lambda(D)}.$$  

To accomplish this, apply Chebyshev’s inequality, the bound of $\Gamma$ on $L^2_\sigma(D)$, property (2) of Lemma 2.12, the $B_1$ condition of $\sigma$, and the almost disjointness of the quasi-balls in $\mathcal{F}$

$$\sigma(\{ \Gamma \tilde{f}_2 > \lambda \}) \lesssim \frac{1}{\lambda^2} \int_D (\Gamma \tilde{f}_2)^2 \sigma \, dV$$

$$\lesssim \frac{1}{\lambda^2} \int_D \tilde{f}_2^2 \sigma \, dV$$

$$\lesssim \frac{1}{\lambda} \sum_{B \in \mathcal{F}} \int_B (f)_B \sigma \, dV$$

$$\lesssim \frac{1}{\lambda} \sum_{B \in \mathcal{F}} \int_B f \sigma \, dV$$

$$\lesssim \frac{1}{\lambda} \| f \|_{L^\lambda(D)}.$$

\[\square\]

### 3. The Cauchy-Szegő Projection

Throughout this section, we assume that the domain $D$ has class $C^3$ boundary. This means the same thing as in Section 2 except the defining function $\rho$ is only assumed to be of class $C^3$. We construct an auxiliary operator, $\mathcal{C}$, that produces and reproduces boundary values of holomorphic functions. This construction proceeds in a similar way to the construction in Section 2, and we will reuse certain notations to refer to objects playing analogous roles. For more details concerning the construction of this operator, we refer the reader to [12].

Let $P_w(z)$ denote the Levi polynomial, but this time at $w \in bD$. In this case, we set

$$g(z, w) = -\chi P_w(z) + (1 - \chi)|z - w|^2,$$
where \( \chi = \chi(z, w) \) is an appropriately chosen \( C^\infty \) cutoff function with \( \chi \equiv 1 \) when \( |z - w| < \delta/2 \) and \( \chi \equiv 0 \) when \( |z - w| > \delta \) for some \( \delta > 0 \) so that
\[
\text{Re } g(z, w) \gtrsim -\rho(z) + |z - w|^2.
\]

The \((1, 0)\) form in \( w, G(z, w) \), is defined exactly the same way as in the Bergman case. This time we have
\[
\langle G(z, w), w - z \rangle = g(z, w).
\]

As before, we set
\[
\eta(z, w) := G(z, w) g(z, w)
\]
and define similarly for \( z \in D \)
\[
\text{C}_1 f(z) := \frac{1}{(2\pi i)^n} \int_{bD} j^* ((\overline{\partial_w} \eta)^{n-1} \wedge \eta) f(w)
\]
where \( j : bD \to \mathbb{C}^n \) denotes the inclusion map. Notice now the integration takes place over the boundary rather than the interior of the domain.

We have the following lemma, which grants that \( \text{C}_1 \) reproduces holomorphic functions from their boundary values.

**Lemma 3.1.** If \( F \) is holomorphic on \( D \), continuous on \( \overline{D} \), and \( f = F|_{bD} \), then for all \( z \in D \),
\[
\text{C}_1 f(z) = F(z).
\]

The problem again is that \( \text{C}_1 \) does not generally produce holomorphic functions. This is corrected with the following lemma.

**Lemma 3.2.** There is a continuous \((n, n - 1)\) form \( \text{C}_2(z, w) \) in \( w \) that depends smoothly on the parameter \( z \in \overline{D} \) so that if we define
\[
\text{C}_2 f(z) := \int_{bD} \text{C}_2(z, w) f(w)
\]
and
\[
\text{C} := \text{C}_1 + \text{C}_2,
\]

then the operator \( \text{C} \) satisfies the following:

1. If \( f \in L^1(bD) \), then \( \text{C} f \) is holomorphic on \( D \).
2. If \( F \) is holomorphic on \( D \), continuous on \( \overline{D} \), and \( f = F|_{bD} \), then \( \text{C} f(z) = F(z) \) for all \( z \in D \).

We define the operator \( \mathcal{C} f := \text{C} f|_{bD} \) for a class of functions satisfying a certain type of Hölder continuity and refer to [12] for the details. It was proved in [12] that the operator \( \mathcal{C} \) extends boundedly on \( L^p(bD) \) for all \( 1 < p < \infty \). The Kerzman-Stein equation now takes the following form:
\[
\mathcal{S}(I - (\mathcal{C}^* - \mathcal{C})) = \mathcal{C}.
\]

To prove that \( \mathcal{S} \) is weak-type \((1, 1)\) with respect to the \( A_1 \) weight \( \sigma \), we proceed in two steps as before. In particular, we have the following two propositions:

**Proposition 3.3.** If \( \sigma \) is an \( A_1 \) weight, then the operator \( I - (\mathcal{C}^* - \mathcal{C}) \) is invertible on \( L^1_\sigma(bD) \).

**Proposition 3.4.** If \( \sigma \) is an \( A_1 \) weight, then \( \mathcal{C} \) maps \( L^1_\sigma(bD) \) to \( L^{1, \infty}_\sigma(bD) \) boundedly.

We can now prove Theorem 1.4.
Proof of Theorem 1.4. This follows directly from Proposition 3.3 and Proposition 3.4.

The proof of Proposition 3.3 proceeds as in the Bergman case. We again appeal to Lemma 2.5 and Lemma 2.6 to prove a compactness result. In this case, the underlying space is \( X = bD \) and the finite Borel measure is \( \sigma \, dS \). We can define the appropriate quasi-metric:

\[
d(z, w) := |g(z, w)|^{1/2}.
\]

It was proved in [12] that \( d \) is indeed a quasi-metric and that \( (D, d, S) \) is a space of homogeneous type. Additionally, we have \( S(B(z, r)) \approx r^{2n} \).

Lemma 3.5. If \( \sigma \) is an \( A_1 \) weight, then the operator \( C^* - C \) is compact on \( L^1_\sigma(bD) \).

Proof. Let \( k(z, w) \) denote the kernel of the operator \( C^* - C \). We consider the family of functions \( k_w(z) = k(z, w) \) for \( w \in bD \). By Lemma 2.5, it suffices to show that \( \{ k_w : w \in bD \} \) is relatively compact in \( L^1_\sigma(bD) \), which we can do by verifying the criteria of Lemma 2.6. The infimum condition can be verified as before. This set is clearly bounded in \( L^1_\sigma(bD) \); this follows by observing as in [18] that

\[
\int_{bD} |k(z, w)| \sigma(z) \, dS(z) \lessgtr \sigma(w).
\]

In particular, this is deduced from the bound \( |k(z, w)| \lessgtr d(z, w)^{-2n+1} \) (which relies on the domain having boundary of class \( C^3 \)) and a dyadic integration argument similar to the one presented in Lemma 2.7. Similarly, we obtain that

\[
\int_{B(w, \delta)} |k(z, w)| \sigma(z) \, dS(z) \lessgtr \delta \sigma(w).
\]

Notice that this bound does not involve a \( d(z, bD) \) term which highlights a key difference from the case of the Bergman projection.

The second conclusion mirrors very closely the argument in the Section 2, so we only sketch the ideas. Namely, for a fixed function \( k_w \), we excise a small ball about \( w \) and integrate the function \( |k_w(z) - \langle k_w \rangle_{B(z, r), \sigma dS}| \) over this ball and its complement. The integral on the complement of the ball can be controlled by uniform continuity, since \( k(z, w) \) is continuous off the boundary diagonal. The integral over the ball is controlled via the triangle inequality and splitting into regions as in the proof of Lemma 2.7. It should be noted that it is not necessary to split into subcases based on the distance of points \( z \) and \( w \) to \( bD \) because all the integration occurs on the boundary and \( A_1 \) weights satisfy a true doubling property.

The following lemma follows the exact same argument as Lemma 2.8.

Lemma 3.6. If \( \sigma \in A_1 \), the number 1 is not an eigenvalue of \( C^* - C \) considered as an operator on \( L^1_\sigma(bD) \).

Therefore, we can prove Proposition 3.3.

Proof of Proposition 3.3. This proposition follows from Lemma 3.5, Lemma 3.6, and the spectral theorem for compact operators on a Banach space.

To complete the proof of Theorem 1.4 it remains to prove Proposition 3.4.

Proof of Proposition 3.4. As in [12], write \( C = C^\sharp + R \). It is proven in [12] that \( C^\sharp \) is a Calderón-Zygmund operator with respect to the quasi-metric \( d \). Thus, by standard theory, \( C^\sharp \) maps \( L^1_\sigma(bD) \) to \( L^{1,\infty}_\sigma(bD) \) boundedly for \( \sigma \in A_1 \).
On the other hand, the operator $\mathcal{R}$ has a kernel $r(z,w)$ that satisfies
\[
\int_{bD} |r(z,w)| \sigma(w) \, dS(w) \lesssim \sigma(z)
\]
and
\[
\int_{bD} |r(z,w)| \sigma(z) \, dS(z) \lesssim \sigma(w)
\]
for $\sigma \in A_1$ (see [18]). A simple argument using Fubini’s theorem shows that $\mathcal{R}$ is bounded on $L^1_\sigma(bD)$. This completes the proof. □

4. Kolmogorov and Zygmund Inequalities

We first prove the general fact that the weak-type $(1,1)$ estimate implies the Kolmogorov inequality on finite measure spaces.

**Theorem 4.1.** Let $T$ be a linear operator and $(X,\mu)$ a finite measure space. If $T$ maps $L^1(X,\mu)$ to $L^{1,\infty}(X,\mu)$ boundedly and $0 < p < 1$, then $T$ extends boundedly from $L^1(X,\mu)$ to $L^p(X,\mu)$.

**Proof.** Using the distribution function and the weak-type $(1,1)$ assumption, we have for any $t > 0$:
\[
\|Tf\|_{L^p(X,\mu)}^p = \int_0^\infty p \lambda^{p-1} \mu(\{x \in X : |Tf(x)| > \lambda\}) \, d\lambda \\
= \int_0^t p \lambda^{p-1} \mu(\{x \in X : |Tf(x)| > \lambda\}) \, d\lambda + \int_t^\infty p \lambda^{p-1} \mu(\{x \in X : |Tf(x)| > \lambda\}) \, d\lambda \\
\leq t^p \mu(X) + \frac{p}{1-p} t^{-p-1} \|f\|_{L^1(X,\mu)}.
\]
Taking $t = \|f\|_{L^1(X,\mu)}$ completes the proof. □

**Proof of Corollary 1.2.** This follows immediately from Theorem 1.1 and Theorem 4.1. □

**Proof of Corollary 1.5.** This follows immediately from Theorem 1.4 and Theorem 4.1. □

Before proving our Zygmund inequalities, we first define the space $L \log^+ L$, which falls within the scope of Orlicz spaces. We call a function $\Phi : [0, \infty] \to [0, \infty]$ a Young function if $\Phi$ is continuous, convex, increasing, and satisfies $\Phi(0) = 0$. Given a measure space $(X,\mu)$ and a Young function $\Phi$, the associated Orlicz space, $L^\Phi(X,\mu)$, is the linear hull of all measurable functions on $X$ satisfying
\[
\int_X \Phi(|f|) \, d\mu < \infty
\]
equipped with the following Luxemburg norm:
\[
\|f\|_{L^\Phi(X,\mu)} := \inf \left\{ \lambda > 0 : \int_X \Phi \left( \frac{|f|}{\lambda} \right) \, d\mu \leq 1 \right\}.
\]

The Zygmund space $L \log^+ L(X,\mu)$ is defined to be the Orlicz space $L^\Phi(X,\mu)$ associated with the Young function $\Psi(t) = t \log^+ t$, where $\log^+(t) := \max\{\log(t), 0\}$. We use the notation $(L \log^+ L)_\sigma(D)$ to represent $L \log^+ L(D, \sigma \, dV)$ for a domain $D \subseteq \mathbb{C}^n$ and a weight $\sigma$ on $D$ and we similarly write $(L \log^+ L)_\sigma(bD)$ for $L \log^+ L(bD, \sigma \, dS)$ with $\sigma$ a weight on $bD$. We refer to [10,17] for thorough treatments of Orlicz spaces.
We next prove that the weak-type \((1,1)\) and \(L^2\) bounds imply the Zygmond inequality on general finite measure spaces.

**Theorem 4.2.** Let \(T\) be a linear operator and \((X, \mu)\) a finite measure space. If \(T\) is bounded on \(L^2(X, \mu)\) and maps \(L^1(X, \mu)\) to \(L^{1,\infty}(X, \mu)\) boundedly, then \(T\) extends boundedly from \(L \log^+ L(X, \mu)\) to \(L^1(X, \mu)\).

**Proof.** Let \(f \in L \log^+ L(X, \mu)\) be given and normalized to assume \(\|f\|_{L \log^+ L(X, \mu)} = 1\). Observe that \(L^1(X, \mu)\) is the Orlicz space \(L^\Phi(X, \mu)\) with Young function \(\Phi(t) = t\). Define \(\Phi_1\) by

\[
\Phi_1(t) = \begin{cases} 0 & \text{if } 0 \leq t < 2 \\ t - 2 & \text{if } 2 \leq t \leq \infty \end{cases}
\]

and notice that \(\Phi\) and \(\Phi_1\) are equivalent Young functions in the sense that

\[
\Phi_1(t) \leq \Phi(t) \leq \Phi_1(2t)
\]

for all \(t \geq 2\). Therefore by [10, Theorem 13.2 and Theorem 13.3], it suffices to prove

\[
\|Tf\|_{L^\Phi_1(X, \mu)} \lesssim 1.
\]

For a fixed \(\lambda > 0\), write \(f = f_0 + f_\infty\), where \(f_0 := f\chi_{\{|f| \leq \lambda\}}\) and \(f_\infty := f\chi_{\{|f| > \lambda\}}\). Using the assumed bounds of \(T\) and the distribution function, we have

\[
\mu(\{|Tf| > 2\lambda\}) \leq \mu(\{|Tf_0| > \lambda\}) + \mu(\{|Tf_\infty| > \lambda\})
\]

\[
\leq \frac{1}{\lambda^2} \|f_0\|^2_{L^2(X, \mu)} + \frac{1}{\lambda} \|f_\infty\|_{L^1(X, \mu)}
\]

\[
\approx \frac{1}{\lambda^2} \int_0^\lambda s \mu(\{|f| > s\}) \, ds + \frac{1}{\lambda} \int_\lambda^\infty \mu(\{|f| > s\}) \, ds.
\]

Use the distribution function, a change of variables, the above estimate, and Fubini’s Theorem, direct estimates, and the normalization \(\|f\|_{L \log^+ L(X, \mu)} = 1\) to deduce

\[
\int_X \Phi_1(|Tf|) \, d\mu = \int_2^\infty \mu(\{|Tf| > \lambda\}) \, d\lambda \approx \int_1^\infty \mu(\{|Tf| > 2\lambda\}) \, d\lambda
\]

\[
\leq \int_1^\infty \frac{1}{\lambda^2} \int_0^\lambda s \mu(\{|f| > s\}) \, ds \, d\lambda + \int_1^\infty \frac{1}{\lambda} \int_\lambda^\infty \mu(\{|f| > s\}) \, ds \, d\lambda
\]

\[
= \int_0^1 s \mu(\{|f| > s\}) \left( \int_1^\infty \frac{1}{\lambda^2} \, d\lambda \right) \, ds + \int_1^\infty s \mu(\{|f| > s\}) \left( \int_1^s \frac{1}{\lambda} \, d\lambda \right) \, ds
\]

\[
+ \int_1^\infty \mu(\{|f| > s\}) \left( \int_1^s \frac{1}{\lambda} \, d\lambda \right) \, ds
\]

\[
= \int_0^1 s \mu(\{|f| > s\}) \, ds + \int_1^\infty (1 + \log s) \mu(\{|f| > s\}) \, ds
\]

\[
\leq \mu(X) + \int_X \Psi(|f|) \, d\mu
\]

\[
\lesssim 1,
\]

where \(\Psi(t) = t \log^+(t)\). Thus \(\|Tf\|_{L^\Phi_1(X, \mu)} \lesssim 1\) as desired. \(\square\)

**Proof of Corollary 1.3.** This follows immediately from Theorem 1.1 and Theorem 4.2 \(\square\)

**Proof of Corollary 1.6.** This follows immediately from Theorem 1.4 and Theorem 4.2 \(\square\)
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