Asymptotically-Optimal Motion Planning using Lower Bounds on Cost

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Abstract—Many path-finding algorithms on graphs such as A* are sped up by using a heuristic function that gives lower bounds on the cost to reach the goal. Aiming to apply similar techniques to speed up sampling-based motion-planning algorithms, we use effective lower bounds on the cost between configurations to tightly estimate the cost-to-go. We then use these estimates in an anytime asymptotically-optimal algorithm which we call Motion Planning using Lower Bounds (MPLB). MPLB is based on the Fast Marching Trees (FMT*) algorithm [1] recently presented by Janson and Pavone. An advantage of our approach is that in many cases (especially as the number of samples grows) the weight of collision detection in the computation is almost negligible with respect to nearest-neighbor calls. We prove that MPLB performs no more collision-detection calls than an anytime version of FMT*. Additionally, we demonstrate in simulations that for certain scenarios, the algorithmic tools presented here enable efficiently producing low-cost paths while spending only a small fraction of the running time on collision detection.

I. INTRODUCTION

Motion-planning algorithms aim to find a collision-free path for a robot moving amidst obstacles. The most prevalent approach in practice is to use sampling-based techniques [2]. These algorithms sample points in the robot’s configuration-space (C-space) and connect close-by configurations to construct a graph called a roadmap. Often, a cost-path is desired, where cost can be measured in terms of, for example, path length or energy consumption along the path.

Sampling-based algorithms rely on two central primitive operations: Collision Detection (CD) and Nearest Neighbors (NN) search. CD determines whether a configuration is collision-free or not and is often used to assess if a path connecting close-by configurations is collision-free. This latter operation is referred to as Local Planning (LP). An NN data structure preprocesses a set of points to efficiently answer queries such as “which are the points within radius $r$ of a given query point?” In practice, the cost of CD, primarily due to LP calls, often dominates the running time of sampling-based algorithms, and is typically regarded as the computational bottleneck for such algorithms. For a summary of the computational complexity of NN, CD and LP in sampling-based motion-planning algorithms see [3].

In their influential work, Karaman and Frazzoli [4] analyzed existing sampling-based algorithms (namely PRM [5] and RRT [6]) and introduced the notion of asymptotic optimality (AO); an algorithm is said to be AO if the cost of the solution produced by it converges to the cost of the optimal solution if the algorithm is run for sufficiently long time. They proposed AO variants of PRM and RRT called PRM* and RRT*, respectively. However, the AO of PRM* and RRT* comes at the cost of increased running time and memory consumption when compared to their non-optimal counterparts. To reduce this cost, several improvements were proposed which modify the sampling scheme [7], [8], the CD [3], or relax the optimality to asymptotic near-optimality (ANO) [9], [10], [11]. An algorithm is said to be ANO if, given an approximation factor $\varepsilon$, the cost of a solution returned by the algorithm is guaranteed to converge to within a factor of $1 + \varepsilon$ of the cost of the optimal solution.

Following the introduction of PRM* and RRT*, other AO algorithms have been suggested. RRT# [12], extends its roadmap in a similar fashion to RRT*. However, in contrast to RRT* which only performs local rewiring, RRT# efficiently propagates changes to all the relevant parts of the roadmap. Another AO algorithm, proposed by Janson and Pavone, is the Fast Marching Trees (FMT*) [1] algorithm. FMT*, reviewed in detail in Section II was shown to converge to an optimal solution faster than PRM* or RRT*.

Contribution and paper organization We show how by looking at the roadmap induced by a set of samples, we can compute effective lower bounds on the cost-to-go of nodes. This is done without performing expensive LP calls and allows to efficiently guide the search performed by the algorithm. We call our scheme Motion Planning using Lower Bounds or MPLB for short.

An interesting and useful implication of our approach is that the weight of CD and LP becomes negligible when compared to that of the NN call. Bialkowski et al. [3] introduced a technique which replaces CD calls by NN calls. Their scheme relies on additional data produced by the CD algorithm used, namely, a bound on the clearance of a configuration (if it is collision free) or on its penetration depth (if it is not collision free). Alas, such a bound is not trivial to compute for some prevalent C-spaces. In contrast, our algorithmic framework can be paired with existing off-the-shelf NN, CD and LP procedures. Thus, our results, which are more general (as they are applicable to general C-spaces), strengthen the conjecture of Bialkowski et al. [3] that NN computation and not CD may be the bottleneck of sampling-based motion-planning algorithms.

This work continues and expands our recent work [11].

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This work has been supported in part by the Israel Science Foundation (grant no. 1102/11), by the German-Israeli Foundation (grant no. 1150-82.6f2011), and by the Hermann Minkowski–Minerva Center for Geometry at Tel Aviv University.

1 Throughout this paper, when we say an NN call we mean a call to find all the nodes within radius $r(n)$ of a given node. $r(n)$ is a radius depending on the number of samples used and will be formally defined in Section II.
where we relaxed the AO of RRT* to ANO using lower bounds. The novel component here is that multiple
nodes are processed simultaneously while in RRT* the nodes are processed one at a time. This allows to efficiently
compute for all nodes an estimation of the cost-to-go which in turn can be used to speed up motion-planning
algorithms.

Our framework is demonstrated for the case where distance
is the cost function via the FMT* algorithm which is reviewed in Section [11]. As we wish to work in an
anytime setting, we introduce in Section [11] a straightforward adaptation of FMT* for anytime planning which we call
aFMT. We then proceed to present MPLB in Section [IV] We analyze aFMT* and MPLB with respect to the amount
of calls to the NN and LP procedures in Section [V] and report on experimental results in Section [VI]. Specifically, we
demonstrate in simulations that for certain scenarios, MPLB produces lower-cost paths faster (by a factor of between two
and three) than aFMT*. We conclude with a discussion and suggestions for future work in Section [VII].

II. TERMINOLOGY AND ALGORITHMIC BACKGROUND

We begin by formally stating the motion-planning problem
and introducing several procedures used by the algorithms
we present. We continue by reviewing the FMT* algorithm.

A. Problem definition and terminology

Let \( X, X_{\text{free}} \) denote the Euclidean C-space and free
space, respectively, and \( d \) the dimension of the C-space. Let
\((X_{\text{free}}, x_{\text{init}}, X_{\text{goal}})\) be the motion-planning problem where:
\( x_{\text{init}} \in X_{\text{free}} \) is the initial free configuration of the robot
and \( X_{\text{goal}} \subseteq X_{\text{free}} \) is the goal region. We will make
use of the following procedures: \( \text{sample}\_\text{free}(n) \), a
procedure returning \( n \) random configurations from \( X_{\text{free}} \);
\( \text{nearest}\_\text{neighbors}(x, V, r) \) is a procedure that returns
all neighbors of \( x \) with distance smaller than \( r \) within the
set \( V \); \( \text{collision}\_\text{free}(x, y) \) tests whether the straight-
line segment connecting \( x \) and \( y \) is contained in \( X_{\text{free}} \);
\( \text{cost}(x, y) \) returns the cost of the straight-line path
connecting \( x \) and \( y \), namely, in our case, the distance.
We consider weighted graphs \( G = (V, E) \), where the weight
of an edge \((x, y) \in E \) is \( \text{cost}(x, y) \). Given such a
graph \( G \), we denote by \( \text{cost}_G(x, y) \) the cost of the weighted
shortest path from \( x \) to \( y \). Let \( \text{cost-to-come}_G(x) \) be
\( \text{cost}_G(x_{\text{init}}, x) \) and \( \text{cost-to-go}_G(x) \) be the minimal
\( \text{cost}_G(x, x_{\text{goal}}) \) for \( x_{\text{goal}} \in X_{\text{goal}} \). Namely for every node \( x \),
\( \text{cost-to-come}_G(x) \) is the minimal cost to reach \( x \) from
\( x_{\text{init}} \) and \( \text{cost-to-go}_G(x) \) is the minimal cost to reach
\( X_{\text{goal}} \) from \( x \). Additionally, let \( B_G(x_{\text{init}}, r) \), \( B_G(X_{\text{goal}}, r) \) be the set of all nodes whose cost-to-come (respectively,
cost-to-go) value on \( G \) is smaller than \( r \). Finally, we denote
by \( \text{Dijkstra}(G, x, c) \) an implementation of Dijkstra’s algo-

Algorithm 1 FMT* \((x_{\text{init}}, X_{\text{goal}}, n)\)

1: \( V \leftarrow \{x_{\text{init}}\} \cup \text{sample}\_\text{free}(n) \); \( E \leftarrow \emptyset \); \( T \leftarrow (V, E) \)
2: \( \text{PATH} \leftarrow \text{search}\((T, X_{\text{goal}}, 0, \infty)\) \quad \text{// See Alg. 2}\)
3: \( \text{return} \ \text{PATH} \)

\( \text{cost} \) of \( c \) has been reached. The algorithm’s implementation
updates the cost to reach each node from \( x \) and outputs the
set of nodes traversed.

Given a set of samples \( V \) and a radius \( r \), we denote by
\( G(V, r) \) the disk graph \( G \) which is the graph whose set of
vertices is \( V \) and two vertices \( x, y \in V \) are connected by an
edge if the distance between \( x \) and \( y \) is less than \( r \).

B. Fast Marching Trees (FMT*)

FMT*, outlined in Alg. [1], performs a “lazy” dynamic
programming recursion on a set of sampled configurations
to grow a tree rooted at \( x_{\text{init}} \) [14]. The algorithm samples \( n \)
collision-free nodes \( V \) (line 1). It searches for a path to \( x_{\text{goal}} \)
by building a minimum-cost spanning tree growing in cost-
to-come space (line 2 and detailed in Alg. [2]). As we explain
in Section [IV] the algorithm may benefit from using a heuris-
tic function estimating the cost-to-go of a node and a bound
on the maximal length of the path that should be found. As
these are not part of the original formulation of FMT*, we
describe the search procedure of FMT* using a cost-to-go
estimation of zero for each node and an unbounded maximal
path length (marked in red in Alg. [1] and [2]). This will allow
us to use the same pseudo-code of Alg. [2] to explain the
MPLB algorithm in Section [IV].

The search-tree is built by maintaining two sets of nodes
\( H, W \) such that \( H \) is the set of nodes added to the tree
that may be expanded and \( W \) is the set of nodes that have
not yet been added to the tree (Alg. [2] line 1). It then
computes for each node the set of nearest neighbors \( \delta \)
of radius \( r(n) \) (line 3). The algorithm repeats the following
process: the node \( z \) with the lowest cost-to-come value is
chosen from \( H \) (line 4 and 16). For each neighbor \( x \) of \( z \) that
is not already in \( H \), the algorithm finds its neighbor \( y \in H \)
such that the cost-to-come of \( y \) added to the distance between
\( x \) and \( y \) is minimal (lines 7-9). If the local path between \( y \) and
\( x \) is free, \( x \) is added to \( H \) with \( y \) as its parent (lines 10-12).
At the end of each iteration \( z \) is removed from \( H \) (line 13).
The algorithm runs until a solution is found or there are no
more nodes to process.

To ensure AO, the radius \( r(n) \) used by the algorithm is

\[
r(n) = (1 + \eta) \cdot 2 \left( \frac{1}{d} \right)^\frac{1}{4} \left( \frac{\mu(X_{\text{free}})}{\xi_d} \right)^{\frac{1}{4}} \left( \log \frac{n}{r} \right)^{\frac{1}{4}},
\]

where \( \eta > 0 \) is some small constant, \( \mu(\cdot) \) denotes the \( d \)-
dimensional Lebesgue measure and \( \xi_d \) is the volume of the
unit ball in the \( d \)-dimensional Euclidean space.

\( \text{The disk graph is sometimes referred to as the neighborhood graph.} \)

\( \text{The nearest-neighbor computation can be delayed and performed only when a node is processed but we present the batched mode of computation to}
\text{simplify the exposition.} \)
III. ANYTIME FMT* (aFMT*)

An algorithm is said to be anytime if it yields meaningful results even after a short time and it improves the quality of the solution as more computation time is available. We outline a straightforward enhancement to FMT* to make it anytime. As noted in previous work (see, e.g., [15]) one can turn a batch algorithm into an anytime one by the following general approach: choose an initial small number of samples and repeat the process. The total running time is less than twice that of the running time for the largest $n$. Note that as FMT* is AO, aFMT* is also AO.

We can further speed up this method by reusing both existing samples and connections from previous iterations. Due to lack of space, we omit these details and refer the interested reader to the extended version of our paper [16].

IV. ALGORITHMIC FRAMEWORK

We are now ready to present our approach to exploiting lower bounds on cost in order to speed up sampling-based motion-planning algorithms.

Given a random infinite sequence of collision-free samples $S = s_1, s_2 \ldots$ denote by $V_i(S)$ the set of the first $2^i$ elements of $S$. Let $G_i(S) = G(V_i(S), r([V_i(S)]))$ and let $H_i(S) \subseteq G_i(S)$ be the subgraph containing collision-free edges only (here $r(n)$ is the radius defined in Eq. 1). For brevity, we omit $S$ when referring to $V_i(S), G_i(S)$ and $H_i(S)$. Moreover, when we compare our algorithm to the aFMT* algorithm, we do so for runs on the same random infinite sequence $S$. Clearly, for any two nodes $x, y \in V_i$, $cost_G(x, y) \leq cost_H(x, y)$. Thus for any node $x \in V_i$, $cost-to-go_G(x) \leq cost-to-go_H(x)$. Namely, the cost-to-go computed using the disk graph $G$ is a lower bound on the cost-to-go that may be obtained using $H_i$. We call this the lower bound property. For an illustration, see Fig. 1.

Algorithm 2 search $(T, X_{goal}, cost\_to\_go, c_{\text{max}})$

```
1: $W \leftarrow V \setminus \{x_{\text{init}}\}; \quad H \leftarrow \{x_{\text{init}}\}$
2: for all $v \in V$ do
3: $N_v \leftarrow \text{nearest_neighbors}(V \setminus \{v\}, v, r(n))$
4: $z \leftarrow x_{\text{init}}$
5: while $z \notin X_{\text{goal}}$ do
6: $H_{\text{new}} \leftarrow \emptyset; \quad X_{\text{near}} \leftarrow W \cap N_z$
7: for $x \in X_{\text{near}}$ do
8: $Y_{\text{near}} \leftarrow H \cap N_x$
9: $y_{\text{min}} \leftarrow \arg\min_{y \in Y_{\text{near}}} \{\text{cost}_T(y) + \text{dist}(y, x)\}$
10: if $\text{collision_free}(y_{\text{min}}, x)$ then
11: $T_{\text{parent}}(x) \leftarrow y_{\text{min}}$
12: $H_{\text{new}} \leftarrow H_{\text{new}} \cup \{x\}; \quad W \leftarrow W \setminus \{x\}$
13: if $H = \emptyset$ then
14: return FAILURE
15: $z \leftarrow \arg\min_{x \in H} \{\text{cost}_T(x) + \text{cost\_to\_go}(x)\}$
16: if $\text{cost}_T(z) + \text{cost\_to\_go}(z) \geq c_{\text{max}}$ then
17: return FAILURE
18: return $\text{PATH}$
```

We present Motion Planning using Lower Bounds, or MPLB (outlined in Alg. 3). Similar to aFMT*, the algorithm runs in iterations and at the $i$’th iteration, uses $V_i$ as its set of samples. Unlike aFMT*, each iteration consists of a pre-processing phase (line 4) of computing a lower bound on the cost-to-go values and a searching phase (line 5) where a modified version of FMT* is used.

Let $c_i(\text{ALG})$ denote the cost of the solution obtained by an algorithm ALG using $V_i$ as the set of samples (set $c_0(\text{ALG}) \leftarrow \infty$). We now show that only a subset of the nodes sampled in each iteration need to be considered. We then proceed to describe the two phases of MPLB.

A. Promising nodes

We use the lower bound property to consider only a subset of $V_i$ that will be used in the $i$’th iteration. Intuitively, we only wish to consider nodes that may produce a solution that is better than the solution obtained in previous iterations. This leads us to the definition of promising nodes:

Definition 1: A node $x \in V_i$ is promising (at iteration $i$) if $\text{cost\_to\_come}_{H_i}(x) + \text{cost\_to\_go}_{H_i}(x) < c_{i-1}(\text{MPLB})$.

In the pre-processing phase, MPLB will traverse $G_i$ (and not $H$) to collect a set of nodes that contains all promising nodes.

Algorithm 3 MPLB $(x_{\text{init}}, X_{\text{goal}}, n_0)$

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1: $V \leftarrow \{x_{\text{init}}\}; \quad n \leftarrow n_0; \quad c_{\text{prev}} \leftarrow \infty$
2: while $\text{time_permits()}$ do
3: $V \leftarrow V \cup \text{Sample\_free}(n); \quad E \leftarrow \emptyset; \quad T \leftarrow (V, E)$
4: $\text{estimate\_cost\_to\_go}(V, x_{\text{init}}, X_{\text{goal}}, c_{\text{prev}})$
5: $\text{PATH} \leftarrow \text{search}(T, X_{\text{goal}}, cost\_to\_go, c_{\text{prev}})$
6: $n \leftarrow 2^n; \quad c_{\text{prev}} \leftarrow \text{cost\_PATH}$
7: return $\text{PATH}$
```

Algorithm 4 estimate\_cost\_to\_go $(V, x_{\text{init}}, X_{\text{goal}}, c)$

```
1: $V_{\text{preproc}} \leftarrow \text{Dijkstra}(G(V, r([V])), x_{\text{init}}, \frac{c}{2})$
2: $V_{\text{preproc}} \leftarrow V_{\text{preproc}} \cup \text{Dijkstra}(G(V, r([V])), X_{\text{goal}}, \frac{c}{2})$
3: for $x \in V \setminus V_{\text{preproc}}$ do
4: $\text{cost\_to\_go}(x) \leftarrow \infty$
5: $\text{Dijkstra}(G(V_{\text{preproc}}, r([V_{\text{preproc}}])), X_{\text{goal}}, c)$
```
nodes (and possibly other nodes), compute a lower bound on their cost-to-go and use this set in the searching phase.

B. Preprocessing phase: Estimating the cost-to-go

Recall that in the preprocessing phase, outlined in Alg. 4, we wish to compute a lower bound on the cost-to-go for (a subset of) nodes \( x \in V_i \). Specifically, the only nodes we wish to consider are promising nodes. This is done by collecting the set of nodes \( V_{\text{preproc}} = B_{G_i} \left( x_{\text{init}}, \frac{c_{i-1}(\text{MPLB})}{2} \right) \) and \( B_{G_i} \left( X_{\text{goal}}, \frac{c_{i-1}(\text{MPLB})}{2} \right) \). Namely, by performing one traversal from \( x_{\text{init}} \) (line 1) and one traversal from \( X_{\text{goal}} \) (line 2), all nodes such that cost-to-come \( H_i \leq \frac{c_{i-1}(\text{MPLB})}{2} \) or cost-to-go \( G_i \leq \frac{c_{i-1}(\text{MPLB})}{2} \) are found. Clearly, any node not in either set is not promising (lines 3-4).

After collecting all nodes in \( V_{\text{preproc}} \), MPLB computes the distance of every such node from \( X_{\text{goal}} \) (line 5). This is done by running a shortest paths algorithm on the graph \( G_i \) restricted to the nodes in \( V_{\text{preproc}} \). This distance is stored for each node and will be used as a lower bound on the cost-to-go. We note that this preprocessing phase only uses NN calls and does not use any CD calls (as there are no LP calls).

C. Searching phase: Using cost-to-go estimations

The lower bounds computed in the preprocessing phase allow for two algorithmic enhancements to the searching phase when compared to aFMT*: (i) incorporating the cost-to-go estimation in the ordering scheme of the nodes and (ii) discarding nodes that are found to be not promising.

Node ordering: Recall that in aFMT*, \( H \) is the set of nodes added to the tree that may be expanded and that these nodes are ordered according to their cost-to-come value (Alg. 2, line 16). Instead, we suggest using the cost-to-come added to the cost-to-estimation to order the nodes in \( H \). This follows exactly the formula of A* [17] which performs a Dijkstra-like search on a set of nodes. The nodes that were encountered but not processed yet (\( H \) in our setting) are ordered according to a cost function \( f() = g() + h() \). Here, \( g(x) \) is the (computed) cost-to-come value of \( x \) (cost-to-come \( H_i \) in our case) and \( h \) is a lower bound on the cost-to-go of \( x \) to the goal (cost-to-go \( G_i \) in our case). aFMT* essentially uses the trivial heuristic \( h = 0 \). Instead, we suggest to use a much sharper bound to speed up the search towards the goal.

Discarding nodes: In the preprocessing stage MPLB computes a set of nodes that \( m \)ay be promising, though for each such node, the cost-to-come value was not computed. In the searching phase, once a node is added to the tree, its cost-to-come value will not change in the current iteration. Thus, every node \( x \) added to the tree with cost-to-come \( H_i \) (\( \geq c_{i-1}(\text{MPLB}) \)) is discarded as it cannot be promising. This implies that MPLB will terminate an iteration when it is evident that the previous iteration’s solution cannot be improved (see Alg. 2, lines 17-18).

In Section VI we demonstrate through various simulations that using lower bounds has a significant effect on the running time of the algorithm in practice. Ordering the nodes using a heuristic that tightly estimates the cost-to-go allows MPLB to expand a smaller portion of the nodes \( V_i \) while discarding nodes allows to focus the search only on nodes that may potentially improve the existing solution.

V. COMPARATIVE ANALYSIS AND DISCUSSION

We compare aFMT* and MPLB with respect to the size of the tree constructed in the searching phase and with respect to the primitive procedures, namely NN and LP. This is done by quantifying the number of NN and LP calls performed by both algorithms and allows us to discuss the fundamental differences between the two algorithms.

Let \( \#_{\text{NN}}(\text{ALG}) \), \( \#_{\text{LP}}(\text{ALG}) \) denote the number of NN and LP calls performed by an algorithm ALG in iteration \( i \), respectively for a fixed sequence of samples \( S \). Recall that when comparing the two algorithms, it is done for the same sequence \( S \).

A. Search-tree size

Let \( V_i(\text{ALG}) \subseteq V_i \) denote the set of nodes in the tree in the \( i \)th iteration of an algorithm ALG.

Lemma 1: At every iteration, the set of nodes traversed in MPLB’s searching phase is not larger than that of aFMT*.

Proof: Every node \( x \) in the tree of aFMT* has cost-to-come not larger than \( c_i(\text{aFMT}) \). Thus, the size of the search-tree of aFMT* is: \( |V_i(\text{aFMT})| = |\{x \in V_i | \text{cost-to-come}_{H_i}(x) \leq c_i(\text{aFMT})\}| \).

Similar to aFMT*, each node \( x \) traversed by MPLB in the searching phase has cost-to-come \( \text{cost-to-come}_{H_i}(x) + \text{cost-to-go}_{G_i}(x) \leq c_i(\text{aFMT}) \). Additionally, due to node discarding (see Section IV), cost-to-come \( \text{cost-to-come}_{H_i}(x) + \text{cost-to-go}_{G_i}(x) \leq c_{i-1}(\text{MPLB}) \). Thus, the size of the search-tree of MPLB is: \( |V_i(\text{MPLB})| = |\{x \in V_i | \text{cost-to-come}_{H_i}(x) + \text{cost-to-go}_{G_i}(x) \leq \min(c_{i-1}(\text{MPLB}), c_i(\text{aFMT}))\}| \).

Namely, \( |V_i(\text{MPLB})| \leq |V_i(\text{aFMT})| \).

B. Nearest neighbor calls (NN)

To quantify the number of NN queries performed by each algorithm we note the following observations (explained in detail in the extended version of this paper [16] due to lack of space):

Observation 1: The number of NN calls performed by aFMT* can be bounded from below as follows: \( \#_{\text{NN}}(\text{aFMT}) \geq |V_i(\text{aFMT})| \).

Observation 2: The number of NN calls performed by MPLB is: \( \#_{\text{NN}}(\text{MPLB}) = \left| \left\{x \in V_i | x \in B_{G_i} \left( x_{\text{init}}, \frac{c_{i-1}(\text{MPLB})}{2} \right) \cup B_{G_i} \left( X_{\text{goal}}, \frac{c_{i-1}(\text{MPLB})}{2} \right) \right\} \right| \). Thus, MPLB may perform more NN queries than FMT*.

C. Local planning calls (LP)

The LP will be called whenever either algorithm (aFMT* or MPLB) attempts to insert a node to the search-tree (line 10 in Alg. 2). Thus we can state the following lemma:
Lemma 2: If MPLB performs an LP call for the edge \((x, y)\) in the \(i\)th iteration then aFMT* will perform an LP call for the edge \((x, y)\) as well.

Proof: The LP procedure will be called for every pair of nodes \(x, y\) in the search tree such that: (i) \(x, y\) are neighbors in \(G_i\) (namely their distance is less than \(r(V_i))\), (ii) \(\text{cost-to-come}_{H_i}(x) < \text{cost-to-come}_{H_i}(y)\) (namely \(x\) is inserted to the tree before \(y\)), and (iii) the edge \((z, y)\) is not collision-free for all other neighbors \(z\) of \(y\) in the tree that could potentially lead to smaller cost-to-come values of \(y\).

If MPLB performs an LP call for the edge \((x, y)\) then conditions (i),(ii) and (iii) hold for the samples \(x, y\) in MPLB. To prove the lemma we show that they hold for the samples \(x, y\) in aFMT*. Condition (i) holds trivially as it is a property of the samples. Note that the cost-to-come of any node \(z\) computed by both algorithms equals to \(\text{cost-to-come}_{H_i}(z)\). Using this observation and that \(V_{\text{preproc}} \subseteq V_i\) (namely the nodes used by MPLB is a subset those used by aFMT*), conditions (ii) and (iii) hold as well. □

D. Discussion

From the above analysis we conclude that MPLB will perform no more LP calls than aFMT*. It may perform more NN calls than aFMT*. As we demonstrate empirically in the Evaluation section, the number of NN calls that MPLB performs may actually be smaller than that of aFMT*. Moreover, as the number of iterations increases, MPLB performs only a tiny fraction of the number of LP calls performed by aFMT*.

VI. Evaluation

We present simulations evaluating the performance of MPLB as an anytime algorithm on 2, 3 and 6 dimensional C-spaces. All experiments were run on a 2.5GHz Intel Core i7 processor with 8GB of memory. The MPLB and aFMT* implementations are based on the FMT* implementation provided by Pavone’s research group using the Open Motion Planning Library (OMPL 0.10.2) [18]. Each result is averaged over one hundred different runs. Scenarios and additional material are available at [http://acg.cs.tau.ac.il/projects/MPLB](http://acg.cs.tau.ac.il/projects/MPLB).

The AO proof of FMT* (and thus of aFMT* and MPLB) relies on the fact that the C-space is Euclidean. Thus, we start by studying the motion of robots translating in the plane and in space (Fig. 2a and 2b). Next, we continue to examine the behavior of the algorithms in SE(3) (Fig. 2c). Here the radius provided for FMT* (Eq. (1)) is irrelevant due to the differences in the rotational and translational components of the C-space. Hence, for both aFMT* and MPLB, we chose to connect each node to its \(k\) NN, where \(k(n) = 9 \log n\): Karaman and Frazzoli [4] proposed a variant of RRG where each node is connected to its \(k_{\text{RRG}}(n) \geq 2e\log n\). Although this variant was analyzed for Euclidean spaces only, applying it to non-Euclidean spaces works well in practice (see, e.g. [11]).

| \(n\) | \(\text{ratio}_i(MPLB)\) | \(\text{ratio}_i(aFMT)\) | \(\text{ratio}_i(MPLB)\) | \(\text{ratio}_i(aFMT)\) |
|------|----------------|----------------|----------------|----------------|
| 1.6K | 0.71 | 0.38 |
| 3.2K | 0.53 | 0.31 |
| 6.4K | 0.68 | 0.33 |
| 12.8K | 0.68 | 0.19 |
| 25.6K | 0.69 | 0.20 |
| 51.2K | 0.99 | 0.05 |

Fig. 4: Percentage of time spent for each of the main components in each iteration for both algorithms for the Grids Scenario. Each iteration is represented by the number of samples used. The left (right) bars of each iteration represent the result of aFMT* (MPLB, respectively). Note that the time of each iteration for each algorithm is different.

A. Fast convergence to high-quality solutions

We start by comparing the cost of a solution obtained by aFMT* and MPLB as a function of time (Fig. 3). In all scenarios MPLB typically finds a solution of given cost between two to three times faster than aFMT*. In the Corridors scenario [Fig. 2a] the convergence rate can be sped up by using an approximation factor (see suggestion for future work in Section VII). Interestingly, as we will show, the speed-up achieved by MPLB is done while spending a smaller proportion of the time on LP compared to aFMT*.

B. Nearest Neighbors and Local Planning calls

We profiled aFMT* and MPLB and collected the total time spent on CD for point sampling, LP for edges, NN calls and cost computations. Results for the Grids scenario are presented in Fig. 4 (similar behavior was observed for the other scenarios as well). Clearly, CD computation time (due to sampling, not LP) is negligible for both algorithms and cost calculation plays a larger (but still small) role for MPLB. CD calls due to LP calls are the main bottleneck for aFMT* (starting at around 65% and gradually decreasing to 45%). For MPLB they start as a main time consumer but as samples are added their percentage of the overall iteration time becomes quite small (around 2% for the last iteration). NN calls play an almost complementary role to the LP and for the last iteration take 40% of the total running time for the MPLB algorithm while taking less than 20% for aFMT*.

In the table the right reports on the ratio of NN and LP calls performed by MPLB and aFMT* for the Grids scenario. The number of NN calls performed by MPLB is lower than those performed by aFMT*. As expected, MPLB performs significantly less LP calls than aFMT*.

VII. Conclusion and Outlook

In this work we show that by using effective lower bounds and with no compromise on the cost of paths produced by the algorithm, the weight of CD (via LP calls) may become almost negligible with respect to NN calls. This follows the
ideas presented by Bialkowski et al. [3] but uses different, more general methods. Looking into NN computation, one can notice that AO algorithms such as sPRM* [4], FMT* and MPLB rely on a specific type of NN computation: given a set \( P \) of \( n \) points, either compute for each point all its \( k \) nearest neighbors, or all neighbors within distance \( r \) from the point. In both cases, \( P \) is known in advance and \( k \) (or \( r \)) are parameters that do not change throughout the algorithm or throughout a single iteration of the algorithm.

This calls for using application-specific NN algorithms and not general purpose ones. For example, the recent work on randomly shifted grids by Aiger et al. [19] may be used. Indeed, we show that using this data structure allows to significantly speed up motion-planning algorithms [20].

A different possibility to enhance MPLB is to relax AO to ANO: Asymptotically-optimal motion-planning algorithms such as MPLB often, from a certain stage of their execution, invest huge computational resources at only slightly improving the cost of the current best existing solution. Similar to the approach presented by the authors in a previous work [11] one can construct a variant such that given an approximation factor \( \varepsilon \), the cost of the solution obtained is within a factor of \( 1 + \varepsilon \) from the solution that MPLB would obtain for the same set of samples. We expand on this idea in the extended version of our paper [16]. Preliminary results, presented in Fig. 3a show the potential benefit of this approach.

VIII. ACKNOWLEDGEMENTS

We wish to thank Marco Pavone and his co-workers for their advice and support regarding the FMT* algorithm.

Fig. 2: Scenarios used for the evaluation. (a) Two dimensional setting for a point robot. A low-cost path is easy to find yet in order to find a high-quality path, the robot needs to pass through two narrow passages. (b) Three-dimensional C-space for a translating robot in space. To find the shortest path the robot needs to pass through a three-dimensional grid. (c) Six-dimensional C-space for an L-shaped robot translating and rotating in space. Finding a path is relatively easy yet much time is needed to converge to the optimal path. Start and target configurations for (b) and (c) are depicted by green and red robots, respectively. The Home scenario is provided by the OMPL [18] distribution.

Fig. 3: Average cost vs. time. Cost values are normalized such that a cost of one represents the cost of an optimal path. Low and high error bars denote the twentieth and eightieth percentile, respectively.

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