COMPONENTS OF THE SPRINGER FIBER AND DOMINO TABLEAUX

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ABSTRACT. Consider a complex classical semi-simple Lie group along with the set of its nilpotent coadjoint orbits. When the group is of type $A$, the set of orbital varieties contained in a given nilpotent orbit is described as a set of standard Young tableaux. We parameterize both, the orbital varieties and the irreducible components of unipotent varieties in the other classical groups by sets of standard domino tableaux. The main tools are Spaltenstein’s results on signed domino tableaux together with Garfinkle’s operations on standard domino tableaux.

1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with adjoint group $G$ and write $O_f = G \cdot f$ for the coadjoint orbit of $G$ through $f$ in $\mathfrak{g}^*$. Fix a Borel subgroup $B$ of $G$ and let $\mathcal{F}$ be the flag variety $G/B$. For a unipotent element $u \in G$, $\mathcal{F}_u$ is the variety of flags in $\mathcal{F}$ fixed by the action of $u$. The orbit $O_f$ has a natural $G$-invariant symplectic structure and the Kostant-Kirillov method seeks to attach representations of $G$ to certain Lagrangian subvarieties of $O_f$ (see [GV98], [Mih96], and [Pie01]). Of particular importance is the set of orbital varieties, Lagrangian subvarieties of $O_f$ that are fixed by a given Borel subgroup of $G$.

A result of Spaltenstein identifies the set of orbital varieties for a given nilpotent orbit with the orbits of a finite group on the irreducible components of the corresponding unipotent variety [Spa77]. The main purpose of this paper is to provide new parameterizations of both, the orbital varieties contained in a given nilpotent orbit, as well as the irreducible components of the unipotent variety $\text{Irr}(\mathcal{F}_u)$.

In the case of classical groups, nilpotent coadjoint orbits are classified by partitions. Because the number of orbital varieties contained in a given orbit is finite, one expects that both orbital varieties and the components of the unipotent variety should also admit combinatorial descriptions [Ger61]. This is most apparent when $G$ is of type $A$.

Theorem ([Spa82]). Suppose that $G = GL_n(\mathbb{C})$ and the nilpotent orbit $O_f$ corresponds to the partition $\lambda$ of $n$. Then the orbital varieties contained in $O_f$ as well as the set of components $\text{Irr}(\mathcal{F}_u)$ are both parameterized by the family of standard Young tableaux of shape $\lambda$.

In the setting of other classical groups, a method similar to the one used to obtain the above can also be employed to describe both families of objects. However, the resulting parametrization by subsets of signed domino tableaux is somewhat
cumbersome (see [Spa82] and [vL89]). The following argument suggests a more appealing parameter set.

Let $S$ be the set of partitions indexing the unitary dual of $W$, the Weyl group of $G$ and write $\lambda$ for a partition lying in $S$. The dimension of the representation given by $\lambda$ is precisely the number of standard domino tableaux of shape $\lambda$. If we choose a unipotent representative $u_\lambda \in G$ in the conjugacy class corresponding to $\lambda$, then Springer’s characterization of the representations $\hat{W}$ in the top degree cohomology of $F_u$ indicates that

$$\#SDT(n) = \sum_{\lambda \in S} \dim H^{\text{top}}(F_{u_\lambda}, \mathbb{C}) = \#\{\text{Irr}(F_{u_\lambda}) \mid \lambda \in S\}$$

This suggests that $\text{Irr}(F_u)$ should correspond to a set of standard domino tableaux in a natural way. Indeed, this is the case. The precise relationship between van Leeuwen’s parameter set for $\text{Irr}(F_u)$ and the set of domino tableaux can be described in terms of Garfinkle’s notions of cycles and moving-through maps [Gar90]. After defining the notion of a distinguished cycle for a cluster of dominos, we show that moving through sets of distinguished cycles of open and closed clusters in van Leeuwen’s parameter set defines a bijection with the set of all domino tableaux of a given size.

**Theorem 1.1.** Suppose that $G$ is a complex classical simple Lie group not of type $A$. Then the collection of irreducible components of the unipotent varieties for $G$ as the unipotent element ranges over all conjugacy classes is parameterized by $SDT(n)$, the set of standard domino tableaux of size $n$.

The action of the finite group $A_u$ on the irreducible components $\text{Irr}(F_u)$ is described by [vL89]. In the signed domino parametrization, it acts by changing the signs of open clusters. We exploit this to obtain a parametrization of orbital varieties by standard domino tableaux. This time, moving through the distinguished cycles of just the closed clusters in van Leeuwen’s parameter set defines the required bijection. The result is a little simpler to state if we consider nilpotent orbits of the isometry group of a nondegenerate bilinear form, $G_e$.

**Theorem 1.2.** Suppose that $G$ is a complex classical simple Lie group not of type $A$ and $O$ is the nilpotent orbit of $G_e$ that corresponds to the partition $\lambda$. Then the set of orbital varieties contained in $O$ is parameterized by the set of standard domino tableaux of shape $\lambda$.

Parameterizations of orbital varieties by domino tableaux have been obtained in [McG99], by describing equivalence classes in the Weyl group of $G$, as well as in [Ira]. We will address the compatibility of these parameterizations with the one above in another paper.

## 2. Preliminaries

We first describe unipotent and orbital varieties, the relationship between them, and the combinatorial objects we will use in the rest of the paper.

### 2.1. Unipotent and Orbital Varieties

Let $G$ be a connected complex semisimple algebraic group, $B$ a Borel subgroup fixed once and for all, and $F = G/B$ the flag manifold of $G$. We consider the fixed point set $F_u$ of a unipotent transformation $u$ on $F$. It has a natural structure of a projective algebraic variety, called
the unipotent variety. We write \( \text{Irr}(\mathcal{F}_u) \) for the set of its irreducible components. The stabilizer \( G_u \) of \( u \) in \( G \) acts on \( \mathcal{F}_u \) and gives an action of its component group \( A_u = G_u/G_u^0 \) on \( \text{Irr}(\mathcal{F}_u) \).

Now consider a nilpotent element \( f \) of the dual of the Lie algebra \( g^* \) of \( G \). Write \( O_f^{ad} \) for the orbit of \( f \) under the coadjoint action of \( G \) on \( g^* \). Using the non-degeneracy of the Killing form, we can identify \( O_f^{ad} \) with a subset of \( g \). If \( b \) is the Lie algebra of \( B \) and \( n \) its unipotent radical, then the set \( O_f^{ad} \cap n \) inherits the structure of a locally closed algebraic variety from the orbit \( O_f^{ad} \). Its components are Lagrangian submanifolds of \( O_f^{ad} \) and are known as orbital varieties [Gin87]. There is a simple relationship between the set of orbital varieties contained in a given nilpotent orbit and the irreducible components of the corresponding unipotent variety. Suppose that the unipotent element \( u \) of \( G \) and the nilpotent element \( f \) of \( g^* \) correspond to the same partition.

\[ \text{Theorem 2.1 (Spa77).} \quad \text{There is a natural bijection} \]
\[ \text{Irr}(O_f^{ad} \cap n) \rightarrow \text{Irr}(\mathcal{F}_u)/A_u \]

\[ \text{between the orbital varieties contained in the nilpotent orbit } O_f^{ad} \text{ and the orbits of the finite group } A_u \text{ on } \text{Irr}(\mathcal{F}_u) \].

The set of nilpotent orbits for a classical \( G \) admits a combinatorial description by partitions. Write \( P(n) \) for the set of partitions \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k] \) of \( n \), ordered so that \( \lambda_i \geq \lambda_{i+1} \).

\[ \text{Theorem 2.2.} \quad \text{Nilpotent orbits in } g_l n \text{ are in one-to-one correspondence with the set } P(n). \]

The corresponding statement for the other classical groups is not much more difficult. To obtain slightly cleaner statements, we will state it in terms of the nilpotent orbits of the slightly larger isometry groups of nondegenerate bilinear forms. Let \( \epsilon = \pm 1 \) and consider a nondegenerate bilinear form on \( \mathbb{C}^m \) satisfying \( (x, y)_{\epsilon} = \epsilon(y, x) \) for all \( x \) and \( y \). Let \( G_\epsilon \) be the isometry group of this form and \( g_\epsilon \) be its Lie algebra. Define a subset \( P_\epsilon(m) \) of \( P(m) \) as the partitions \( \lambda \) satisfying \( \# \{ j | \lambda_j = i \} \) is even for all \( i \) with \( (-1)^i = \epsilon \). The classification of nilpotent orbits now takes the form:

\[ \text{Theorem 2.3 (Ger01).} \quad \text{Let } m \text{ be the dimension of the standard representation of } G_\epsilon. \text{ Nilpotent } G_\epsilon \text{-orbits in } g_\epsilon \text{ are in one to one correspondence with the partitions of } m \text{ contained in } P_\epsilon(m). \]

The nilpotent \( G_\epsilon \) orbits in \( g_\epsilon \) can be identified with the nilpotent orbits of the corresponding adjoint group with one exception. In type \( D \), precisely two nilpotent orbits of the adjoint group correspond to every even partition. We will write \( \mathcal{O}_f \) for the \( G_\epsilon \)-orbit through the nilpotent element \( f \) and \( \mathcal{O}_\lambda \) for the \( G_\epsilon \)-orbit that corresponds to the partition \( \lambda \) in this manner.

The group \( A_u \) is always finite, and in the setting of classical groups, it is always a two-group. More precisely:

\[ \text{Theorem 2.4 (GL89(2.4.1)).} \quad \text{The group } A_u \text{ is always trivial when } G \text{ is of type } A. \text{ In the other classical types, let } B_\lambda \text{ be the set of the distinct parts of } \lambda \text{ satisfying } (-1)^{\lambda_1} = -\epsilon. \text{ Then } A_u \text{ is a 2-group with } |B_\lambda| \text{ components}. \]
2.2. Standard Tableaux. A partition of an integer \( m \) corresponds naturally to a Young diagram consisting of \( m \) squares. We call the partition underlying a Young diagram its \emph{shape}. Recall the definitions of the sets of standard Young tableaux and standard domino tableaux from, for instance, \cite{Gar90}. We will write \( SYT(\lambda) \) and \( SDT(\lambda) \) respectively for the sets of Young and domino tableaux of shape \( \lambda \). We refer to both objects generically as \emph{standard tableaux} of shape \( \lambda \), or \( ST(\lambda) \), hoping that the precise meaning will be clear from the context. Also, we will write \( ST(n) \) for the set of all standard tableaux with largest label \( n \).

We view each standard tableau \( T \) as a set of ordered pairs \( (k, S_{ij}) \), denoting that the square in row \( i \) and column \( j \) of \( T \) is labelled by the integer \( k \). When \( T \) is a domino tableau, the domino with label \( k \), or \( D(k, T) \), is a subset of \( T \) of the form \( \{(k, S_{ij}), (k, S_{i+1,j})\} \) or \( \{(k, S_{ij}), (k, S_{i,j+1})\} \). We call these vertical and horizontal dominos, respectively. For convenience, we will refer to the set \( \{(0, S_{11})\} \) as the zero domino when in type \( B \). Whenever possible, we will omit labels of the squares and write \( S_{ij} \) for \( (k, S_{ij}) \). In that case, define \emph{label} \( S_{ij} = k \).

Definition 2.5. For a standard tableau \( T \), let \( T(i) \) denote the tableau formed by the squares of \( T \) with labels less than or equal to \( i \). A domino tableau \( T \) is \emph{admissible} of type \( X = B, C, \) or \( D \), if the shape of each \( T(i) \) is a partition of a nilpotent orbit of type \( X \).

The dominos that appear within admissible tableaux fall into three categories. Following \cite{vL89}, we call these types \( I^+, I^- \), and \( N \).

Example 2.6. Suppose that \( G \) is of type \( C \) and consider the tableaux

\[
T = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & & & \\
\end{array} \quad T' = \begin{array}{ccc}
1 & 2 & 3 \\
4 & & 5 \\
\end{array}
\]

Then \( T \) is admissible of type \( C \) but \( T' \) is not, since \( \text{shape} \ T'(2) = [3,1] \) is not the partition of a nilpotent orbit of type \( C \). The dominos \( D(1, T) \) and \( D(3, T) \) are of type \( I^- \), \( D(2, T) \) and \( D(4, T) \) are of type \( I^+ \), and \( D(5, T) \) is of type \( N \).

We also recall the notions of a cycle in a domino tableau and moving through such a cycle, as defined in \cite{Gar90}. We will think of cycles as both, subsets of dominos of \( T \), as well as just sets of their labels. Write \( MT(D(k, T), T) \) for the image of the domino \( D(k, T) \) under the moving through map and \( MT(k, T) \) for the image of \( T \) under moving through the cycle containing the label \( k \). If \( U \) is a set of cycles of \( T \) that can be moved through independent of one another, we will further abuse notation by writing \( MT(U, T) \) for the tableau obtained by moving through all the cycles in \( U \). Recall the definition of \( X \)-fixed and \( X \)-variable squares for \( X = B, C, D, \) or \( D' \) \cite{Gar90}. Under the moving through map, the labels of the fixed squares are preserved while those of variable ones may change. We will call a cycle whose fixed squares are \( X \)-fixed an \( X \)-cycle. Note also that the \( B^- \) and \( C \)-cycles as well as the \( D^- \) and \( D' \)-cycles in a given tableau \( T \) coincide.

Example 2.7. Consider the domino tableaux \( T \) and \( T' \) from the previous example. The \( C \)-cycles in \( T \) are \{1\}, \{2,3\}, and \{4,5\} while those in \( T' \) are \{1\} and \{2,3,4,5\}. We have

\[
MT(2, T) = \begin{array}{ccc}
1 & 2 & 4 \\
3 & & 5 \\
\end{array} \quad MT(4, T) = \begin{array}{ccc}
1 & 2 & 4 \\
3 & & 5 \\
\end{array}
\]
The $D$-cycles in $T$ are $\{1,2\}$, $\{3,4\}$, and $\{5\}$, while there is only one in $T'$, mainly $\{1,2,3,4,5\}$.

3. Signed Domino Tableaux Parameterizations

The irreducible components of the unipotent variety $F_u$ for classical $G$ were described by N. Spaltenstein in [Spa82]. We summarize this parametrization as interpreted by M.A. van Leeuwen [vL89]. Its advantage lies in a particularly translucent realization of the action of $A_u$ on $\text{Irr}(F_u)$.

Fix a unipotent element $u \in G$ and let $\lambda_u$ be the partition of the corresponding nilpotent orbit. Define a map

$$\Gamma_u : F_u \longrightarrow ST(\lambda_u)$$

by the following procedure. Fix a flag $F \in F_u$. Adopting notation of [vL89], let $\lambda^{(i)}$ be the shape of the Jordan form of the unipotent operator induced by $u$ on $F^{(i)}$. The difference between the Young diagrams $\lambda^{(i)}$ and $\lambda^{(i+1)}$ is one square in type $A$ and a domino in the other classical types [Spa82]. By assigning the label $i+1$ to the set $\lambda^{(i+1)} \setminus \lambda^{(i)}$ for each $i$, we obtain a standard tableau of shape $\lambda_u$.

**Theorem 3.1.** When $G$ is of type $A$, the map $\Gamma_u$ defines a surjection onto $\text{SYT}(\lambda_u)$ that separates points of $\text{Irr}(F_u)$. That is, it defines a bijection

$$\Gamma_u : \text{Irr}(F_u) \longrightarrow \text{SYT}(\lambda_u).$$

**Corollary 3.2.** When $G$ is of type $A$, the orbital varieties $\text{Irr}(O_\lambda \cap n)$ are parameterized by the set $\text{SYT}(\lambda)$.

In the other classical types, any domino tableau in the image of $\Gamma_u$ is admissible. Admissible tableaux, however, do not fully separate the components of $F_u$. If two flags give rise to different domino tableaux in this way, they lie in different components of $F_u$. However, the converse is not true. The inverse image $F_{u,T}$ of a given admissible tableau $T$ under this identification is in general not connected. Nevertheless, the irreducible components of $F_{u,T}$ are precisely its connected components [vL89] (3.2.3). Accounting for this disconnectedness yields a parametrization of $\text{Irr}(F_u)$.

**Definition 3.3.** A signed domino tableau $T$ of shape $\lambda$ is an admissible domino of shape $\lambda$ with a choice of sign for each domino of type $I^+$. The set of signed domino tableaux is denoted $\Sigma DT(\lambda_u)$.

The set $\Sigma DT(\lambda_u)$ is too large to parameterize $\text{Irr}(F_u)$ and we follow [vL89] in defining equivalence classes. We recall the notion of a cluster of dominos.

**Definition 3.4.** Write $cl(0)$ for the cluster containing $D(1,T)$ in types $B$ and $C$. A cluster is open if it contains an $I^+$ or $N$ domino along its right edge and is not $cl(0)$. A cluster that is neither $cl(0)$ nor open is closed. Denote the set of open clusters of $T$ by $OC(T)$ and the set of closed clusters as $CC(T)$. For a cluster $C$, let $I_C$ be the domino in $C$ with the smallest label and take $S_{ij}$ as its left and uppermost square. For $X$ equal to $B$ or $C$, we say that $C$ is an $X$-cluster iff $i+j$ is odd. For $X$ equal to $D$ or $D'$, we say that $C$ is an $X$-cluster iff $i+j$ is even.

The definition of open and closed differs from [vL89] as we do not call $cl(0)$ an open cluster. The open clusters of $T$ correspond to the parts of $\lambda$ contained in $B_\lambda$, the set parameterizing the $\mathbb{Z}_2$ factors of $A_\lambda$. 


**Definition 3.5.** If \( T, T' \in \Sigma DT(\lambda) \), let \( T \sim_{op, cl} T' \) iff the underlying tableaux are the same and the products of signs in all corresponding open and closed clusters of \( T \) and \( T' \) agree. Denote the equivalence classes by \( \Sigma DT_{op, cl}(\lambda) \). Define the set \( \Sigma DT_{cl}(\lambda) \) similarly. We represent the elements of \( \Sigma DT_{op, cl}(\lambda) \) and \( \Sigma DT_{cl}(\lambda) \) as admissible tableaux with a choice of sign for each of the appropriate clusters.

There is a considerable amount of freedom in how a flag of \( F_u, T \) can be assigned an equivalence class of signed admissible domino tableaux. A particular choice is presented in \([vL89](3.4)\), defining a map \( \tilde{\Gamma}_u : F_u \rightarrow \Sigma DT_{op, cl}(\lambda_u) \).

We describe an action of \( A_u \) on \( \Sigma DT_{op, cl}(\lambda_u) \).

For \( r \in B_\lambda \), let \( b_T(r) \) be the cluster that contains a domino ending a row of length \( r \) in \( T \). Let \( \xi_r \) act trivially if \( b_T(r) = cl(0) \) and by changing the sign of the open cluster \( b_T(r) \) otherwise. For each \( r \in B_\lambda \), let \( g_r \) denote the generator of the corresponding \( \mathbb{Z}_2 \) factor of \( A_u \). One can now define the action of \( g_r \) on \( \Sigma DT_{op, cl}(\lambda_u) \) by \( g_r[T] = \xi_r[T] \).

**Theorem 3.6 ([vL89]).** Suppose that \( G \) is a classical group not of type \( A \) and \( u \) is a unipotent element of \( G \) corresponding to the partition \( \lambda \). The map \( \tilde{\Gamma}_u \) defines an \( A_u \)-equivariant bijection between the components \( \text{Irr}(F_u) \) and \( \Sigma DT_{op, cl}(\lambda) \).

Since \( A_u \) acts by changing the signs of the open clusters of \( \Sigma DT_{op, cl}(\lambda) \), it is simple to parameterize the \( A_u \) orbits on \( \text{Irr}(F_u) \).

**Corollary 3.7.** Suppose that \( G \) is a classical group not of type \( A \) and \( O'_\lambda \) is the nilpotent orbit corresponding to the partition \( \lambda \). The orbital varieties \( \text{Irr}(O_\lambda \cap n) \) are parameterized by \( \Sigma DT_{cl}(\lambda) \).

### 4. Domino Tableaux Parameterizations

We show how to index the components \( \text{Irr}(F_u) \) and \( \text{Irr}(O_\lambda \cap n) \) by families of standard tableaux. In type \( A \), this is Theorem 1. For the other classical types, we define maps from domino tableaux with signed clusters to the set of standard domino tableaux by applying Garfinkle’s moving through map to certain distinguished cycles.

#### 4.1. Definition of Bijections

Consider an \( X \)-cluster \( C \) and let \( I_C \) be the domino in \( C \) with the smallest label. Let \( Y_C \) be the \( X \)-cycle through \( I_C \). We call it the *initial* cycle of the cluster \( C \).

**Proposition 4.1.** A cluster of an admissible domino tableau \( T \) that is either open or closed contains its initial cycle.

We defer the proof to another section. Armed with this fact, we can propose a map

\[
\Phi : \Sigma DT_{op, cl}(n) \rightarrow SDT(n)
\]

by moving through the distinguished cycles of all open and closed clusters with positive sign. More explicitly, for a tableau \( T \in \Sigma DT_{op, cl}(n) \), let \( C^+(T) \) denote the set of open and closed clusters of \( T \) labelled by a \((+\) and let \( \sigma(T) = \{ Y_C \mid C \in C^+(T) \} \) be the set of their distinguished cycles. Write \( |T| \) for the standard domino tableau underlying \( T \). We define

\[
\Phi(T) = MT(\sigma(T), |T|).
\]
Lemma 4.2. The map $\Phi : \Sigma DT_{op,cl}(n) \rightarrow SDT(n)$ is a bijection. We can view the set $\Sigma DT_{d}(n)$ as a subset of $\Sigma DT_{op,cl}(n)$ by assigning a negative sign to each unsigned open cluster of a domino tableau in $\Sigma DT_{d}(n)$. Restricted to $\Sigma DT_{d}(n)$, $\Phi$ preserves the shapes of tableaux and defines a bijection $\Phi : \Sigma DT_{cl}(\lambda) \rightarrow SDT(\lambda)$ for each $\lambda$ a shape of a nilpotent orbit.

Proof. We check that $\Phi$ is well-defined, that its image lies in $SDT(n)$, and then construct its inverse. We first need to know that the definition of $\Phi$ does not depend on which order we move through the cycles in $\sigma(T)$. It is enough to check that if $Y_C$ and $Y_{C'} \in \sigma(T)$, then $Y_{C'}$ is also lies in $\sigma(MT(|T|, Y_C))$. While this statement is not true for arbitrary cycles, in our setting, this is Lemma 4.4.

The image of $\Phi$ indeed lies in $SDT(n)$. That $\Phi(T)$ is itself a domino tableau follows from the fact that moving through any cycle of $|T|$ yields a domino tableau. Hence $\Phi(T) \in SDT(n)$ and if $T \in \Sigma DT_{cl}(\lambda)$ then $\Phi(T) \in SDT(\lambda)$ since in this case $\Phi$ moves through only closed cycles.

The definition of a cluster forces the initial domino $I_C$ of every closed cluster to be of type $I^+$. By the definition of moving through, the image of $MT(I_C, T)$ in $MT(Y_C, T)$ is inadmissible, i.e. it is a horizontal domino not of type $N$. In general, all the inadmissible dominos in $\Phi(T)$ appear within the image of distinguished cycles under moving through. Furthermore, the lowest-numbered domino within each cycle is the image of the initial domino of some distinguished cycle. With this observation, we can construct the inverse of $\Phi$. We define a map

$$\Psi : \Phi(\Sigma DT_{op,cl}(n)) \rightarrow \Sigma DT_{op,cl}(n)$$

that satisfies $\Psi \circ \Phi = \text{Identity}$. Let $\iota(\Phi(T))$ be the set of cycles in $\Phi(T)$ that contain inadmissible dominos. We define $\Psi(\Phi(T)) = MT(\Phi(T), \iota(\Phi(T)))$. By the above discussion, $\iota(\Phi(T))$ contains precisely the images of cycles in $\sigma(T)$. Hence

$$\Psi(\Phi(T)) = MT(\Phi(T), \iota(\Phi(T))) = MT(MT(|T|, \sigma(T))) = T$$

as desired. Thus $\Phi$ is a bijection onto its image in $SDT(n)$ and restricted to $\Sigma DT_{cl}(\lambda)$, it is a bijection with its image in $SDT(\lambda)$. As we already know that the sets $\Sigma DT_{d}(\lambda)$ and $\Sigma DT(\lambda)$ both parameterize the same set of orbital varieties, and that $\Sigma DT_{op,cl}(n)$ and $SDT(n)$ both parameterize the same set of irreducible components of unipotent varieties, $\Phi$ must provide bijections between these two sets. \hfill \Box

Theorems 1.1 and 1.2 are immediate consequences.

Example 4.3. Let $G$ be of type $D$ and suppose that both $u$ and $O_\lambda$ correspond to the partition $\lambda = [3^2]$. The van Leeuwen parameter set $\Sigma DT_{op,cl}([3^2])$ for $\text{Irr}(F_u)$ is:

$$\begin{array}{cccccc}
1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 1 & 2 & 3 \\
\end{array}$$

The image of $\Sigma DT_{op,cl}([3^2])$ under $\Phi$ is the following set of standard domino tableaux. We write the image of a given tableau in the same relative position. Note that this parameter set for $\text{Irr}(F_u)$ consists of all tableaux of shapes $[3^2]$ and $[4, 2]$.

$$\begin{array}{cccccc}
1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 1 & 2 & 3 \\
\end{array}$$
The van Leeuwen parameter set $\Sigma DT_{\chi}(\{3^2\})$ for the orbital varieties contained in $O_\lambda$ is:

$$\begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{array}$$

Its image under $\Phi$ is the set of all domino tableaux of shape $\{3^2\}$. Again, we write the image of a tableau in the same relative position.

$$\begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{array}$$

4.2. Independence of Moving Through Initial Cycles.

**Lemma 4.4.** Consider open or closed clusters $C$ and $C'$ and their initial cycles $Y_C$ and $Y_{C'}$. Then $Y_C$ is again a cycle in $MT(|T|, Y_{C'})$.

**Proof.** If $C$ and $C'$ are clusters of the same type, then so are their initial cycles and the lemma is [Gar90](1.5.29). Otherwise, without loss of generality, take $C$ to be a $C$-cluster and $C'$ to be a $D$-cluster. As the proof in the other cases is similar, we can also assume that $Y_C$ is $C$-boxed while $Y_{C'}$ is $D$-boxed.

Suppose that the dominos $D(r) \in Y_C$ and $D(s) \in Y_{C'}$ lie in relative positions compatible with the diagram

$$\begin{array}{c}
\square \\
\square
\end{array}$$

where the box labelled by $r$ is fixed. The same squares in $MT(|T|, Y_{C'})$ have the labels

$$\begin{array}{c}
\square \\
\square
\end{array}$$

for some $s'$. To prove the lemma, we need to show that $s < r$ implies $s' < r$ and $s > r$ implies $s' > r$. Since our choice of $r$ and $s$ was arbitrary, this will show that $Y_C$ remains a cycle. There are two possibilities for the domino $D(s)$. It is either horizontal or vertical and must occupy the following squares:

Case (i):$$\begin{array}{c}
\square \\
\square
\end{array}$$

Case (ii):$$\begin{array}{c}
\square \\
\square
\end{array}$$

Case (i). In this case, $s < r$ always. Garfinkle’s rules for moving through imply that $MT(|T|, D(r)) \cap C' \neq \emptyset$. This is a contradiction since we know by hypothesis that $Y_C \neq Y_{C'}$. Hence this case does not occur.

Case (ii). First suppose $s > r$. Then the our squares within $MT(|T|, Y_{C'})$ must look like

$$\begin{array}{c}
\square \\
\square
\end{array}$$

for some $s' \neq s$. Since the tableau $MT(Y_{C'}, T)$ is standard, this requires that $s' > s$ implying $s' > r$ which is what we desired. Now suppose $s < r$ and suppose the
squares in our diagram look like

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6
\end{array}
\]

As in Case (i), we find that \(D(t) \notin C'.\) Since \(D(t) \in C,\) type \(D(s) = I^+\) implies type \(D(t) = I^-\), type \(D(r) = I^-\), and type \(D(u) = I^+.\) Otherwise, the rules defining clusters would force \(s\) to lie in the cluster \(C'.\) Now \(D(u)\) lies in the initial cycle of a closed cluster of same type as \(C'.\) Since it lies on the periphery and its type is \(I^+,\) then its top square must be fixed. In particular, \(\text{MT}(D(r)) \cap D(u) \neq \emptyset.\) This is a contradiction, implying that this case does not arise.

To finish the proof, we must examine the possibility that \(D(s)\) and \(D(r)\) lie in the relative positions described by

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6
\end{array}
\]

This case is completely analogous and we omit the proof. □

This lemma shows that the image of moving though a subset of distinguished cycles is independent of the order in which these cycles are moved through. Note, however, that a similar result is not true for subsets of arbitrary cycles.

4.3. Nested Clusters and the Periphery of a Cluster. We aim to show that closed and open clusters contain their distinguished cycles. The proof has two parts. First, we show that \(Y_C\) is contained in a larger set of clusters \(\overline{C},\) defined as the union of \(C\) with all of its nested clusters. Then, we show that \(Y_C\) intersects each of the nested clusters trivially.

Let \(C\) be a cluster of a tableau \(T\) and denote by \(\text{row}_k T = \{S_{k,j} \mid j \geq 0\}\) the \(k\)th row of \(T.\) Define \(\text{col}_k T\) similarly. If \(\text{row}_k T \cap C \neq \emptyset,\) let \(\inf_k C = \inf \{j \mid S_{k,j} \in \text{row}_k T \cap C\}\) and \(\sup_k C = \sup \{j \mid S_{k,j} \in \text{row}_k T \cap C\}.\)

**Example 4.5.** Consider the following tableau of type \(D.\) It has two closed clusters given by the sets \(C = \{1, 2, 3, 4, 5, 8, 9, 10, 11, 12\}\) and \(C' = \{6, 7\}.\)

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6
\end{array}
\]

\(C\) is a \(D\)-cluster while \(C'\) is a \(B\)-cluster. \(Y_C\) is then a \(D\)-cycle and consists of the dominos in the set \(\{1, 3, 5, 11, 12, 10, 9, 2\}.\) \(T\) has two other \(D\)-cycles, \(\{4, 6\}\) and \(\{7, 8\}.\) Both intersect \(C,\) but are not contained within it. The \(B\)-cycle \(Y_C'\) equals \(\{6, 7\}\) and is contained in \(C'.\) Hence an \(X\)-cluster may not contain all the \(X\)-cycles through its dominos. However, it always contains its initial cycle. Also notice that \(C\) completely surrounds \(C'.\) We call such interior clusters *nested*.

Nested clusters complicate the description of clusters. To simplify our initial results, we would like to consider the set formed by a cluster together with all of its nested clusters. To be more precise:
Definition 4.6. Let \( C' \) be a cluster of \( T \). It is nested in \( C \) if all of the following are satisfied:

\[
\begin{align*}
\inf\{k|\text{row}_kT \cap C' \neq \emptyset\} & > \inf\{k|\text{row}_kT \cap C \neq \emptyset\} \\
\sup\{k|\text{row}_kT \cap C' \neq \emptyset\} & < \sup\{k|\text{row}_kT \cap C \neq \emptyset\} \\
\inf\{k|\text{col}_kT \cap C' \neq \emptyset\} & < \inf\{k|\text{col}_kT \cap C \neq \emptyset\} \\
\sup\{k|\text{col}_kT \cap C' \neq \emptyset\} & > \sup\{k|\text{col}_kT \cap C \neq \emptyset\}
\end{align*}
\]

Define \( \overline{C} \) to be the union of \( C \) together with all clusters nested within it. We will write \( \text{periphery}(\overline{C}) \) for the set of dominos in \( \overline{C} \) that are adjacent to some square of \( T \) that does not lie in \( \overline{C} \). Note that \( \text{periphery}(\overline{C}) \) is a subset of the original cluster \( C \).

Example 4.7. In the above tableau, \( C' \) is nested in \( C \). Furthermore, \( C \cup C' = \overline{C} = T \), and \( \text{periphery}(\overline{C}) = \mathcal{Y}C \subset C \).

The next two propositions describe properties of dominos that occur along the left and right edges of \( \overline{C} \). Recall that our definition of the cycle \( \mathcal{Y}C \) endows \( C \) as well as \( \overline{C} \) with a choice of fixed and variable squares by defining the left and uppermost square of \( I_C \) as fixed.

Proposition 4.8. Suppose that \( C \) is a non-zero cluster of a domino tableau \( T \) and that the intersection of the \( k \)-th row of \( T \) with \( C \) is not empty. Then the dominos \( D(\text{label}(T_{k,\text{inf},c}),T) \) and \( D(\text{label}(T_{k,\text{inf},\overline{C}},T) \) are both of type \( I^+ \). If \( C \) is also closed, then the dominos \( D(\text{label}(T_{k,\text{sup},c}),T) \) and \( D(\text{label}(T_{k,\text{sup},\overline{C}},T) \) are of type \( I^- \).

Proof. The first statement is true for all non-zero clusters by \( \text{VLX} \) (3.3). The second statement is the defining property of closed clusters.

Proposition 4.9. Suppose that \( C \) is a non-zero cluster of a domino tableau \( T \). If the domino \( D \) consisting of the squares \( S_{pq} \) and \( S_{p+1,q} \) lies in \( \text{periphery}(\overline{C}) \), then

1. \( S_{pq} \) is fixed if type \( D = I^+ \) and
2. \( S_{p+1,q} \) is fixed if type \( D = I^- \)

Proof. Case (i). Assume that there is a \( D' \) in the \( \text{periphery}(\overline{C}) \) of type \( I^+ \) whose uppermost square is not fixed. Then \( \text{periphery}(\overline{C}) \) must contain two type \( I^+ \) dominos \( E = \{S_{kl},S_{k+1,t}\} \) and \( E' = \{S_{k+1,m},S_{k+2,m}\} \) with the squares \( S_{kl} \) and \( S_{k+2,m} \) fixed and \( |m-l| \) minimal.

Assume \( m < l \). The opposite case can be proved by a similar argument. Because \( E' \) is of type \( I^+ \), there is an integer \( t \) such that \( m < t < l \), \( S_{k+1,t} \in \text{periphery}(\overline{C}) \), and \( t \) is maximal with these properties. Let \( F \) be the domino containing \( S_{k+1,t} \). \( F \) has to be \( \{S_{k+1,t},S_{k+2,t}\} \) and of type \( I^- \). If its type was \( I^- \) or \( N \), \text{VLX} (3.3 (17)) would force \( S_{k+1,t+1} \) to be in \( \text{periphery}(\overline{C}) \) as well. If \( F \) on the other hand was \( \{S_{k+1,t},S_{k+2,t}\} \), this would contradict the minimality of \( |m-l| \). We now consider two cases.

(a) Assume \( t = l - 1 \). Because \( E \) and \( F \) lie in \( \text{periphery}(\overline{C}) \) and hence in \( C \), \( C \) must contain a domino of type \( N \) of the form \( \{S_{u,l-1},S_{u+1,l}\} \) with \( u > k + 2 \) and \( u \) minimal with this property. The set of squares \( \{S_{p,l-1}|k+2 < p < \)}
u) \cup \{S_p|k+1<p<u\} must be tiled by dominos, which is impossible, as its cardinality is odd.

(b) Assume \( t < l-1 \). We will contradict the maximality of \( t \). Because \( E \) and \( F \) both lie in \( C \), \( C \) must contain a sequence \( H_\alpha \) of dominos of type \( N \) satisfying

\[
H_\alpha = \{ S_{k+1+f(\alpha),t+2\alpha}, S_{k+1+f(\alpha),t+2\alpha+1} \}
\]

where \( 0 \leq \alpha \leq \frac{l-t+1}{2} \). We choose each \( H_\alpha \) such that for all \( \alpha \), \( f(\alpha) \) is minimal with this property. Because the sets \( \{S_{k+p,t}|k+1<p<k+1+f(\frac{l-t+1}{2})\} \) and \( \{S_{k+p,t}|k+2<p<k+1+f(0)\} \) have to be tiled by dominos of type \( I^{+} \) and \( I^{-} \) respectively, \( f(0) \) has to be even and \( f(\frac{l-t+1}{2}) \) has to be odd. Hence there is a \( \beta \) such that \( f(\beta) \) is even and \( f(\beta+1) \) is odd.

Assume \( f(\beta) < f(\beta+1) \), but the argument in the other case is symmetric.

Let \( G \) be the domino containing the square \( S_{k+1+f(\beta),t+2\beta+2} \). \( G \) must belong to \( C \), as \( H_\beta \) and \( G \) is either of type \( I^{-} \) or \( N \). The type of \( G \) cannot be \( N \), however, as this would contradict the condition on \( f \). Hence \( G \) must be of type \( I^{+} \). If \( G \) equals \( \{S_{k+1+f(\beta),t+2\beta+2}, S_{k+f(\beta),t+2\beta+2}\} \). Then by successive applications of [vL89] (3.3 (17)), the set of dominos

\[
\{\{S_{k+f(\beta)\gamma=1 or 2 and 0 \leq \gamma \leq f(\beta) - 2 \text{ is contained in } C \text{ as well. But this means that } t + 2\beta + \epsilon \text{ for } \epsilon = 1 \text{ or } 2 \text{ satisfies the defining property of } t, \text{ contradicting its maximality.}

Case (ii). We would like to show that the bottom square is fixed for every \( I^{-} \)domino in \( \text{periphery} (\mathcal{C}) \). It is enough to show that this is true for one such domino, as an argument similar to that in case (i) can be repeated for the others. Let 

\[
l = \inf \{k|row_k \in T \cap \mathcal{C} = \emptyset \}
\]

Then by [vL89] and the definition of fixed, we know that \( S_{l,\inf} \) is fixed. As \( \{S_{l,\sup_1}, S_{l+1,\sup_1}\} \) is a domino of type \( I^{-} \) in \( \text{periphery} (\mathcal{C}) \), we have found the desired domino.

\[
\text{Lemma 4.10. The following inclusions hold when } C \text{ is an open or closed cluster: } \text{periphery} (\mathcal{C}) \subset \mathcal{V}_C \subset \mathcal{C}.
\]

\[
\text{Proof. Recall that our choice of a fixed square in } \mathcal{I}_C \text{ defines the fixed squares in all of } \mathcal{C}. \text{ Define } \mathcal{C} \text{ as } \mathcal{C} \text{ when } C \text{ is closed and } \mathcal{C} \text{ union with all empty holes and corners of } |T| \text{ adjacent to } C \text{ when } C \text{ is open. [Gar90] (1.5.5). We show that the image } MT(D,T) \text{ of } D \text{ in } \text{periphery} (\mathcal{C}) \text{ lies in } \mathcal{C}. \text{ This shows the second inclusion, as if any domino in } \text{periphery} (\mathcal{C}) \text{ stays in } \mathcal{C} \text{ under moving through, then so must the cycle } \mathcal{V}_C. \text{ The first inclusion is a consequence of the argument and the definitions of moving through and clusters. We differentiate cases accounting for different domino positions along } \text{periphery} (\mathcal{C}).
\]

Case (i). Take \( D = \{(k,S_{ij}),(k,S_{i+1,j})\} \) and suppose type \( D = I^{+} \). Because \( D \) lies on \( \text{periphery} (\mathcal{C}) \), Proposition 4.9 implies that \( S_{ij} \) is fixed. Due to [vL89] (3.3.17(ii)) and \( \mathcal{C} \), \( S_{i,j+1} \in \mathcal{C} \).

(a) Suppose \( S_{i-1,j+1} \) in not in \( \mathcal{C} \). Then \( r = \text{label}(S_{i-1,j+1}) < k \). Otherwise \( S_{i-1,j} \) and \( S_{ij} \) would both belong to the same cluster by [vL89] (3.3.17(ii)). Since \( S_{i-1,j} \) and \( S_{i-1,j+1} \) are in the same cluster by [vL89] (3.3.17(ii)) or (3.3.17(iii)), this contradicts our assumption. Now [Gar90] (1.5.26) forces \( MT(D,T) = \{(k,S_{ij}),(k,S_{i,j+1})\} \), and since \( S_{ij} \) and \( S_{i,j+1} \) both belong to \( \mathcal{C} \), so must \( MT(D,T) \).
We divide the problem into a few cases. Now (Gar90)(1.5.26) implies \( MT(D, T) \subset \{ S_i, S_{i-1, j}, S_{i, j+1}\} \). As all of these squares lie in \( \overline{C} \), we must also have \( MT(D, T) \subset \overline{C} \).

**Case (ii).** Suppose \( D = \{(k, S_{ij}), (k, S_{ij+1})\} \) and that the square \( S_{i, j+1} \) is fixed. By (vL89)(3.3.17(ii)) and 3.4, \( S_{i, j+1} \in \overline{C} \).

(a) Suppose \( S_{i, j+1} \) is not in \( \overline{C} \). Then \( S_{i-1, j+2} \) lies in \( |T| \) but not in \( \overline{C} \), as by (vL89)(3.3.17(i)) or (3.3.17(iii)), they both belong to the same cluster. The definition of a cluster forces \( r = \text{label}(S_{i-1, j+2}) < k \) and (Gar90)(1.5.26(ii)) implies \( MT(D, T) = \{ S_{i, j+1}, S_{i, j+2}\} \). Since the squares \( S_{i, j+2} \) as well as \( S_{i, j+1} \) are both contained in \( \overline{C} \), so is \( MT(D, T) \).

(b) Suppose \( S_{i-1, j+1} \) lies in \( \overline{C} \). Then because the domino \( MT(D, T) \) must be a subset of \( \{ S_{i+1, j+1}, S_{i, j+2}, S_{i, j+1}\} \), it must also be a subset of \( \overline{C} \).

**Case (iii).** Suppose \( D = \{(k, S_{ij}), (k, S_{i, j+1})\} \) and that the square \( S_{ij} \) is fixed. Then \( S_{ij} \in \overline{C} \) by (vL89)(3.3.17(iii)).

(a) Suppose first that \( S_{i+1, j-1} \) is not in \( \overline{C} \). Then \( r = \text{label}(S_{i+1, j-1}) > k \) by either (vL89)(3.3.17(ii)) or (3.3.17(iii)). But (Gar90)(1.5.26(iii)) forces \( MT(D, T) \) to be precisely \( \{ S_{ij}, S_{i, j-1}\} \) which is a subset of \( \overline{C} \).

(b) If \( S_{i+1, j-1} \in \overline{C} \), then \( S_{i+1, j} \in \overline{C} \) as well, by (vL89)(3.3.17(i)) or (3.3.17(iii)). They either must belong to the same cluster or \( S_{i+1, j} \) is an empty hole or corner. But by (Gar90)(1.5.26(iii)(iv)), \( MT(D, T) \) is a subset of \( \{ S_{ij}, S_{i+1, j}, S_{i, j-1}\} \), all of whose squares lie in \( \overline{C} \).

**Case (iv).** Suppose \( D = \{(k, S_{ij}), (k, S_{i, j+1})\} \) and that the domino \( D \) is of type \( I^- \). The square \( S_{i+1, j} \) is then fixed and \( S_{i+1, j-1} \in \overline{C} \).

(a) Assume that \( S_{i+2, j-1} \in \overline{C} \). Then \( S_{i+2, j} \in \overline{C} \). Since \( MT(D, T) \) is the domino \( \{ S_{i+1, j}, S_{i+1, j-1}\} \), \( MT(D, T) \in \overline{C} \) as both possibilities are contained in \( \overline{C} \).

(b) Assume \( S_{i+2, j-1} \) is not in \( \overline{C} \). We have \( r = \text{label}(S_{i+2, j-1}) > k \), otherwise \( D(r, T) \) and hence \( S_{i+2, j-1} \) would lie in \( \overline{C} \). But then \( MT(D, T) = \{ S_{i+1, j}, S_{i+1, j-1}\} \), so it is contained in \( \overline{C} \).

These cases describe all possibilities by (L9).

What remains is to see that the initial cycle \( Y_C \) is contained within the cluster \( C \). It is enough to show that its intersection with any closed cluster nested in \( C \) is empty, as open clusters cannot be nested. Our proof relies on the notion of X-boxing Gar90(1.5.2). We restate the relevant result.

**Proposition 4.11** (Gar90(1.5.9) and (1.5.22)). Suppose that the dominos \( D(k, T) \) and \( D(k', T) \) both belong to the same X-cycle. Then

1. \( D(k, T) \) is X-boxed iff \( MT(D(k, T), T) \) is not X-boxed.
2. \( D(k, T) \) and \( D(k', T) \) are both simultaneously X-boxed or not X-boxed.

**Lemma 4.12.** If \( C' \subset \overline{C} \) is a closed cluster nested in \( C \), then \( Y_C \cap C' = \emptyset \).

**Proof.** It is enough to show that \( \text{periphery}(C') \cap Y_C = \emptyset \), as this forces \( C' \cap Y_C = \emptyset \). We divide the problem into a few cases.

Case (i). Suppose \( \{ \text{type } Y_C, \text{type } Y_{C'} \} = \{ C, D' \} \). We investigate the intersection of \( \text{periphery}(C') \) with \( Y_C \). It cannot contain dominos of types \( I^+ \) and \( I^- \); because
the boxing property is constant on cycles according to Proposition 4.11(ii), such dominoes would have to be simultaneously \( C \) and \( D \)-boxed, which is impossible. If \( D(k, T) \in \text{periphery}(C') \cap \mathcal{Y}_C \) is of type \( (N) \), \( D(k, T) \) and \( MT(D(k, T), T) \) are both \( C \) and \( D' \)-boxed. This contradicts Proposition 4.11(ii), forcing \( \text{periphery}(C') \cap \mathcal{Y}_C = \emptyset \). The proof is identical when the set \( \{ \text{type } \mathcal{Y}_C, \text{type } \mathcal{Y}_{C'} \} \) equals \( \{ B, D \} \) instead.

Case (ii). Suppose \( \{ \text{type } \mathcal{Y}_C, \text{type } \mathcal{Y}_{C'} \} = \{ C, D \} \). The proof is similar to the first case, except this time, dominoes of type \( N \) cannot be simultaneously \( C \) and \( D \)-boxed. Again, the proof is identical when the set \( \{ \text{type } \mathcal{Y}_C, \text{type } \mathcal{Y}_{C'} \} \) equals \( \{ B, D' \} \) instead.

Case (iii). Suppose \( \{ \text{type } \mathcal{Y}_C, \text{type } \mathcal{Y}_{C'} \} \subset \{ B, C \} \) or \( \{ D, D' \} \). Then by the definition of cycles, \( \mathcal{Y}_C \cap \mathcal{Y}_{C'} = \emptyset \). We know \( \text{periphery}(C') \subset \mathcal{Y}_C \subset \mathcal{Y}' \) by Lemma 4.10, implying again that \( \text{periphery}(C') \cap \mathcal{Y}_C = \emptyset \).

\( \square \)

5. The \( \tau \)-Invariant for Orbital Varieties

A natural question is whether our method of describing orbital varieties by\( \text{standard tableaux} \) gives the same parametrization as \( \text{[McG99]} \). More precisely, if \( \pi : \text{Irr}(\mathcal{F}_u)/A_u \rightarrow \text{Irr}(O_u \cap \mathfrak{n}) \) is the bijection of \( \text{[Spa77]} \) does the same tableau parameterize both \( C \in \text{Irr}(\mathcal{F}_u)/A_u \) and its image \( \mathcal{V} = \pi(C) \)? Write \( \mathcal{T}(C) \) for the domino tableau corresponding to the \( A_u \)-orbit \( C \in \text{Irr}(\mathcal{F}_u)/A_u \) via the map of the previous section and \( \mathcal{T}(\mathcal{V}) \) for the domino tableau used to parameterize \( \mathcal{V} \) in \( \text{[McG99]} \).

Let \( \Pi \) be the set of simple roots in \( \mathfrak{g} \). The \( \tau \) invariant, a subset of \( \Pi \), is defined for orbital varieties in \( \text{[Jos84]} \) and for components of the Springer fiber in \( \text{[Spa82]} \). It is constant on each \( A_u \)-orbit. For a standard domino tableau \( T \), it can be defined in terms of the relative positions of the dominoes. We say that a domino \( D \) lies higher than \( D' \) in a tableau \( T \) if the rows containing squares of \( D \) have indices strictly smaller than the indices of the rows containing squares of \( D' \). Then \( \tau(T) \) consists of precisely the simple roots \( \alpha_i \) whose indices satisfy:

1. \( i = 1 \) and the domino \( D(1, T) \) is vertical,
2. \( i > 1 \) and \( D(i-1, T) \) lies higher than \( D(i, T) \) in \( T \).

According to \( \text{[Gar93]} \), there is a unique tableau of a given shape within each equivalence class of tableaux generated by the generalized \( \tau \)-invariant. We show

**Theorem 5.1.** Suppose that \( C \in \text{Irr}(\mathcal{F}_u)/A_u \) and that \( \mathcal{V} = \pi(C) \). Then 
\[
\tau(\mathcal{T}(C)) = \tau(\mathcal{T}(\mathcal{V})).
\]

**Proof.** In fact, we show that all of following sets are equal.
\[
\tau(\mathcal{T}(\mathcal{V})) = \tau(\mathcal{V}) = \tau(C) = \tau(\mathcal{T}(C)).
\]

The first equality follows from \( \text{[McG99]} \) and \( \text{[Jos84]} \). The second from the definition of \( \pi \). We verify the third.

Recall the map \( \Phi : \text{SDT}_{op,cl} \rightarrow \text{SDT} \) defined in the previous section. We prove that if \( \tilde{T} \in \text{SDT}_{op,cl} \) parameterizes the irreducible component \( C \in \text{Irr}\mathcal{F}_u \) in \( \text{[L89]} \), then its \( \tau \)-invariant \( \tau(C) \) is precisely the \( \tau \)-invariant of the standard domino tableau \( \Phi(\tilde{T}) = \mathcal{T}(C) \) as defined above. The content of the proof is a description of the effect of \( \Phi \) on the characterization of the \( \tau \)-invariant of the components of the Springer fiber given in \( \text{[Spa82]} \).

That \( \alpha_1 \in \tau(C) \) if and only if \( \alpha_1 \in \tau(\Phi(\tilde{T})) \) is clear in types \( B \) and \( C \) since \( D(1, T) \) never lies within a closed cluster and hence remains unaltered by \( \Phi \). In type \( D \), the conditions
for $\alpha_i$, when $i \leq 2$, to lie in $\tau(C)$ described by Spaltenstein translate exactly to our conditions for $\alpha_i$ to lie in $\tau(\Phi(\overline{T}))$.

For $i > 1$, suppose that either $D(i, T)$ or $D(i-1, T)$ lies in some $K \in CC^+(T)$. If $K$ contains more than two dominos, then \cite{Gar93} (III.1.4) implies that $\alpha_i \in \tau(C)$ iff $\alpha_i \in \tau(\Phi(\overline{T}))$.

So suppose that $K$ contains exactly two dominos. If, in fact, $K = \{D(i), D(i-1)\}$, the simple root $\alpha_i$ must lie in $\tau(C)$. But $D(i-1)$ is higher than $D(i)$ in $MT(C, T)$, implying by the definition of $\Phi$ that $\alpha_i \in \tau(\Phi(\overline{T}))$ as well. The remaining possibility is that only one of the dominos $D(i)$ and $D(i-1)$ lies in the two-domino cluster $K$. Then the fact that $\alpha_i \in \tau(C)$ iff $\alpha_i \in \tau(\Phi(\overline{T}))$ follows by inspection. \hfill $\square$

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