Ultrarelativistic generalized Lorentzian thermodynamics and the differential cosmic ray energy flux

R. A. Treumann 1,2 • W. Baumjohann 3

Abstract We apply the ultrarelativistic generalized Lorentzian quasi-equilibrium thermodynamic energy distribution to the energy spectrum of galactic cosmic ray fluxes. The inferred power law slopes contain a component which evolves with cosmic ray energy in steps of thirds, resembling the sequence of structure functions in fully developed Kolmogorov turbulence. Within the generalized thermodynamics the chemical potential can be estimated from the deviation of the fluxes at decreasing energy. Both may throw some light on the cosmic ray acceleration mechanism to very high energies.

Keywords Cosmic Rays; energy spectra;

1 Introduction

Half a century ago Stan Olbert 4 introduced an ad hoc model distribution function which since became known as κ-distribution. It was successfully applied to formally fitting moderate-energy electron (first by Vasyliunas 1968 and ion fluxes (cf., Christon et al. 1988, 1991 as for examples) in space. As a statistical probability distribution it ultimately found various applications (cf., Livadiotis and McComas 2013 for a review). In physics, the appearance of κ distributions as substitutes for the Maxwell-Boltzmann distribution in the stationary long-term limit (Hasegawa et al. 1985, Yoon 2012a, Yoon et al. 2012, Yoon 2012b) when particles are accelerated in interaction with fluctuating electromagnetic fields or turbulence led to suspect that they obey a deeper physical meaning in quasi-equilibrium statistical mechanics. Such a statistical mechanics, dubbed quasi-equilibrium generalized-Lorentzian statistical mechanics or thermodynamics (Treumann 1999b, Treumann and Jaroschek 2008), was based on the grand-canonical Gibbs-Lorentzian probability distribution (Treumann and Baumjohann 2014). The κ-probability distribution appeared as a version of the generalized Lorentzian phase-space quasi-equilibrium (in the above long-term sense). Its application to relativistic bosons in quantum field theory was given in (Treumann and Baumjohann 2016).

Generalized energy/momentum space Lorentzians are non-stochastic thermodynamic quasi-equilibrium momentum space distributions in the presence of correlations (Treumann 1999b, Treumann and Baumjohann 2014) provided, for instance, by interaction with a bath of plasma waves and parameterized by the power index κ. Their non-stochastic properties have so far prevented construction of a counting statistics. Their consistent correlationless, i.e. the purely stochastic limit $\kappa \to \infty$, is the Boltzmann-Maxwell phase space distribution.

The generalized-Lorentzian statistical mechanics strictly satisfies the two thermodynamic equilibrium requirements: particle number and energy conservation. Any higher order moments have no physical meaning and thus are of no physical interest. Application to non-equilibrium conditions like conservation laws in fluid mechanics including, for instance, heat flux, require adjusting the power in the energy/momentum distribution. Non-physical higher order moments can be renormalized by adding an exponential truncation factor which cuts the distribution off above some maximum energy.

In the present note we apply the generalized Lorentzian equilibrium thermodynamics to observations of ul-
2 Ultrarelativistic generalized Lorentzians

Generalized Lorentzian thermodynamics, for application to cosmic ray fluxes, must be given in ultrarelativistic form, with \( \epsilon_p \approx pc \gg mc^2 \) the particle energy. The ultrarelativistic generalized Gibbs-Lorentzian partition function (Treumann and Baumjohann 2014) yields the grand-canonical Gibbs-Lorentzian phase-space energy distribution (for our purposes suppressing the exponential cut-off which in cosmic rays comes into play only above “ankle” energies)

\[
w_{\kappa}(\epsilon_i) = A \{1 + (\epsilon_i - \mu)/\kappa T\}^{-\kappa - r}
\]

of finding a particle at thermodynamic temperature \( T \) (in energy units) in state \( i \). The transition is made from \( w_{\kappa}(\epsilon_i) \to f_{\kappa}(p) \) to the momentum distribution function \( f_{\kappa}(p) \), normalizing it to the number density \( N/V \), with volume \( V \), according to \( N/V = (2\pi\hbar)^3 \int d^3p f_{\kappa}(p) \). In the ultrarelativistic case this yields the generalized Lorentzian momentum space distribution

\[
f_{\kappa}^{ur}(p) = \frac{N\lambda^3 \{1 + (pc - \mu)/(\kappa T)\}^{-\kappa - r}}{V\kappa^3} \frac{B(3, \kappa + r - 3)}{(\kappa + 3)! (\kappa - 1)! (\kappa - 2)!}
\]

which is a grand canonical thermodynamic equilibrium distribution, the momentum space particle density. As required in equilibrium it conserves particle number and energy. \( \mu \) is the chemical potential related to particle number, \( \lambda = 2\pi\hbar c/T \) is the ultrarelativistic thermal de Broglie length, and \( B(\alpha, \beta) \) is the Beta function. Here \( T \) is not a so-called ‘equivalent’ temperature, it is the exact physical thermodynamic temperature (Treumann and Baumjohann 2014), the inverse of which is the partial derivative of the thermodynamic energy with respect to the entropy.

The constant \( r \) in the exponent is fixed from the thermodynamic requirement (Treumann and Baumjohann 2014) that the mean ultrarelativistic energy \( U_{ur} \) of an ideal gas is related to particle number \( N \) and thermodynamic temperature \( T \) via the ultrarelativistic ideal gas equation \( U_{ur} = 3NT \), or written in terms of the ultrarelativistic pressure \( P_{ur}V = 3NT \). Averaging the particle energy \( \epsilon_p \) yields straightforwardly

\[
U_{ur} = \frac{3\kappa}{\kappa + r - 4} NT
\]

which identifies \( r = 4 \), with ultrarelativistic Lorentzian

\[
f_{\kappa}^{ur}(p) = \frac{N\lambda^3}{2V\kappa^3} \frac{(\kappa + 3)!}{\kappa! (1 + (pc - \mu)/(\kappa T))^{\kappa + 4}}
\]

Transition to field theory, if necessary, is achieved by putting \( p = \hbar k \). The ultrarelativistic generalized-Lorentzian energy space density becomes

\[
f_{\kappa}^{ur}(\epsilon) = \frac{\epsilon^2}{\epsilon^3} f_{\kappa}^{ur}(p) \bigg|_{pc=\epsilon}
\]

These distributions have some interesting properties (Treumann and Jaroschek 2008). Firstly, the chemical potential must be negative: \( \mu < 0 \). For \( pc < -\mu = |\mu| \) the momentum space density flattens out (Treumann et al. 2004) towards low \( p \) to become \( f_{\kappa}^{ur}(p) = \text{const} \), while the energy distribution \( f_{\kappa}^{ur}(\epsilon) \propto \epsilon^2 \) assumes a parabolic dependence. It maximizes at energy

\[
\epsilon_m = \frac{\kappa T}{1 + \kappa/2} \left(1 + \frac{|\mu|}{\kappa T}\right)
\]
This is an important point, because inspection of measured momentum fluxes at small momenta $mc \ll p < -\mu / c$, if available and flattening out, allows for the determination of $|\mu|$, while measuring $\epsilon_m$ provides the ratio of chemical potential to temperature. On the other hand, for $pc \gg -\mu$ the phase space distributions become simple power laws in either case

$$f^{ur}_\kappa(p) \propto \left( \frac{pc}{\kappa T} \right)^{-(\kappa+4)}, \quad f^{ur}_\kappa(\epsilon) \propto \left( \frac{\epsilon}{\kappa T} \right)^{-(\kappa+2)} \quad (7)$$

Measuring the power law slope of the distribution at high momentum or energy immediately determines the power $\kappa$ of the generalized Lorentzian. In the following we will make use of these ways in examining the observed cosmic ray energy spectrum.

3 The cosmic ray energy spectrum

Such observations usually measure number fluxes in dependence on energy:

$$\frac{cN}{V} = c \int d\epsilon d\Omega f^{ur}_\kappa(\epsilon) \implies \frac{c}{V} \frac{dN}{d\epsilon d\Omega} = cf^{ur}_\kappa(\epsilon) \quad (8)$$

directly yielding the slope of the energy distribution from the observations. Once the slope has been determined experimentally, the ratio of chemical potential to temperature is inferred from the spectral maximum of the fluxes $f^{ur}_\kappa(\epsilon_m)$ in Eq. (8). With known $\kappa$ and $|\mu|$ known from the momentum distribution, the temperature $T$ can then be obtained. It is simple matter to show that with $f_0 \equiv [V/N(2\pi\hbar c)^3]^2 f^{ur}_\kappa(\epsilon_m) \ll |\mu|/c$, the temperature follows as

$$\kappa T = \left( \frac{(\kappa + 3)!}{(\epsilon_m f_0 \kappa!)^3} \right)^{(\kappa+2)/(3\kappa+5)} \left[ 1 + \frac{\kappa}{2} \right]^{-1/(3\kappa+5)} \quad (9)$$

A similar expression holds for the energy distribution at maximum energy $f^{ur}_\kappa(\epsilon_m)$. If $|\mu|$ cannot be determined from either of them, $T^{-1} = (\partial U/\partial S)|_V$ as derivative of energy with respect to entropy $S$ must be calculated from the theoretical expression for the entropy [Treuermann and Baumjohann 2014]. Working out this programme completes the quasi-equilibrium generalized-Lorentzian thermodynamics of cosmic ray fluxes. In the cosmic ray energy fluxes, $\epsilon_m$ and $f_0$ are not directly observable thus inhibiting its completion without adding further information. Instead, inferring the power law index $\kappa$ from published cosmic ray fluxes Fig. 1 is of sufficient interest as it contains the hidden microscopic physics of the acceleration.

The above distribution holds if the observer is placed inside the generation region. For cosmic ray observations this is not the case, however, as cosmic rays readily escape from their source and propagate out into space where they become further accelerated. One thus expects that just the power law tail, possibly with some weak deviations from power law at low energies, becomes visible, indicating flattening of the distribution.

The theoretical distribution $f^{ur}_\kappa(\epsilon)$ at energies above $\epsilon > -\mu$ is monotonous. Deviations from monotony indicate effects not covered by one single function. Also for stationary spectra one can neglect the effect of time dispersion such that the observed long-term spectra can be taken as averaging out all contributions of time variability of the cosmic ray sources.

4 Power law index

The observed spectrum of galactic cosmic ray particle fluxes (taken from publicly available spectra reproduced in Fig. 1) is not uniform in energy. The fluxes exhibit different energy ranges of different well-distinguishable slopes $s$. Galactic cosmic rays have composite spectra, as they do not consist of one single particle species. Rather they are composed of different particle components coming from a variety of sources and being subject to primary and secondary acceleration processes [Schlickeiser 2002].

These slopes can be read from Fig. 2 (Antoni et al. 2005; Amenomori et al. 2008; Apel et al. 2011; Bertain et al. 2011; Aartsen et al. 2013, 2016; Abbasi et al. 2008; Abraham et al. 2015; Aharonian et al. 2008; Ivanov et al. 2015a; Ivanov 2015; Valino and Pierre Auger Collaboration 2015; Tiniakov et al. 2015; Aab et al. 2016) and are given in Table 1 for the different energy ranges in three forms, as slope $s$, $\kappa = s - 2$, and in the fractional form $\kappa$ closest to the inferred index $s - 2$. (One may note that $\kappa = s - 2$ is the genuine power law index of the generalized Lorentzian, with the number $2 = r - 2$ the thermodynamically required constant addition to warrant energy conservation and adjusting for the equation of state. Thus it is $\kappa$, not $s$, which contains the hidden physics.)

The indices $s$ are approximate in the sense of the precision of the data scatter on the log-log scale with an estimated uncertainty in slope $s \lesssim 0.1$. However, the three determined ranges are sufficiently far apart from each other that this uncertainty is of little importance when considering the slopes on an equal theoretical basis. The last column in Table 1 is the closest fractional approximation of the determined $\kappa$ index within this uncertainty of the measurement. These fractions are sufficiently close to the measurements and sufficiently far apart from each other to justify the fractional approximation. (For instance, $\kappa \sim 0.7$ is close to $0.66 \ldots$ and $\sim 0.4$ away from the next $\kappa \lesssim 1.1$.)
5 Discussion

The list of slopes suggests that the index \( \kappa \) of the ultrarelativistic quasi-equilibrium Lorentzian thermodynamics applied to the galactic cosmic ray spectrum (excluding the bump of the “knee”) changes gradually in steps of thirds with energy before, at the “ankle”, returning to its initial slope \( \kappa \sim \frac{2}{3} \). On excluding “knee” and “ankle”, one thus has the sequence

\[
\kappa_n = \frac{n}{3}, \quad n = 2, 3, 4
\]

which is graphically shown in Fig. 2. The changes in slope with energy indicate that the cosmic ray energy fluxes are not determined by one single process only over more than 10 orders of magnitude in energy. It is known that they are composed of particles of different chemical composition while coming from different sources. However, the evolution of \( \kappa \) in well defined quantized steps of thirds indicates some inherent process hidden in the evolution of the spectra. It consists of – at least – three different stages. This fact extracted from the observations should throw light on the acceleration process of charged particles up to the highest cosmic ray energies of \( > 10^7 \) TeV, while it is most interesting that this internal spectral structure is uncovered by interpreting the cosmic ray spectra as generalized Lorentzian thermodynamic quasi-equilibrium spectra.

This scaling in \( \kappa \) reminds of Kolmogorov’s (Kolmogorov 1962; 1991a) scaling of the turbulent structure functions (Monin and Yaglom 1975; Frisch et al. 1978) in fully developed inertial range turbulence whose exponents \( \zeta_n \) are given in similar sequential order

\[
\zeta_n = \frac{n}{3}, \quad n = 2, 3, \ldots
\]

In a picture where the power law spectrum of cosmic rays is progressively generated by the interaction between the charged particle component of the magnetic turbulence as the main feature of diffusive Fermi acceleration, this similarity between the quantised steps of the spectral slope and the structure functions suggests that at increasing energies stepwise higher order turbulent structure functions start contributing to or even dominating the diffusive acceleration of the charged galactic cosmic ray particles to ever higher energies. As particles get relativistically heavier with increasing energy, Fermi acceleration becomes more efficient. One might speculate that this leads to the involvement of higher order turbulent structures.

6 Remarks and conclusions

Independent of the above, in the generalized Lorentzian thermodynamic quasi-equilibrium interpretation both the “knee” and “ankle” indicate the possible working of additional processes. Flattening of the spectrum at the “knee” suggests a shift in energy space towards higher energies, which occurs over roughly half an order of magnitude before the steeper slope belonging to the next third in \( \kappa \) takes over. In this interpretation the flattening indicates the start of a new Lorentzian quasi-equilibrium that takes over at higher energies. At “knee” energy the effect of the chemical potential of the superimposed Lorentzian would be non-negligible.
The energy of the "knee" is then identified with the transition to the new higher energy Lorentzian equilibrium and thus a change in chemical potential. This implies the onset of a different process at "knee" energies, for instance the above mentioned inclusion of the next-higher order turbulent structure function.

The meaning of such a high energy chemical potential is difficult to understand. In the region far below "knee"-energy the chemical potential can be read from the deviation of the power law at energies around $\epsilon \lesssim 10$ GeV (data not included here in Fig. 1) and might thus be related to the $\epsilon = 1$ GeV rest energy of protons. In an analogous interpretation, a chemical potential at "knee" energy would suggest the presence of some unknown hypothetical particle of 'very high energy/rest mass' the order of say $m \gtrsim 10$ TeV/c$^2$, which should then have been generated in the primary cosmic ray production process in the galactic sources. Afterwards they would have passed the turbulent acceleration process to become further accelerated. Because of their large mass, diffusive acceleration would be rather efficient.

Similarly, from the sudden flattening of the cosmic ray fluxes at "ankle"-energies, a chemical potential of the order of $|\mu| \lesssim (10^4 - 10^5)$ TeV would be inferred. Tempted to assume indication of another high energy particle, the inferred rest mass should be of the order of, say, $m > (10^2 - 10^3)$ TeV/c$^2$. Nucleons of such rest mass energies are so far unknown in our accessible Universe.

The accepted interpretation of the cosmic ray flux spectra (cf., e.g., Schlickeiser 2002) and their "knee" and "ankle" sections is in terms of superposition of acceleration spectra of the various known chemical particle components generated in AGNs, stars, supernovae and shocks (Balogh and Treumann 2013, Bykov and Treumann 2011). These particles undergo many acceleration cycles in diffusive Fermi acceleration of the various heavy charged known elemental nucleons in turbulent magnetic fields (Schlickeiser 2002) being small-angle scattered and slowly pushed up in energy until reaching the quasi-equilibrium state in the generalized Lorentzian thermodynamics.

This slow though efficient diffusive acceleration process is by now quite well established. It involves distributed magnetic fields and many of such magnetic scattering centers located within a large spatial volume. It has by now been sufficiently well developed being in its ultimate completion phase while including a variety of anomalous processes, for instance astrophysical turbulence where the turbulent mechanical energy is ultimately dissipated in electric current filaments (Treumann and Baumjohann 2015) on leptonica
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