Stability properties of the Starobinsky cosmological model

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Abstract

We discuss the instabilities appearing in the cosmological model with a quasi de Sitter phase following from a fourth-order gravity theory. Both the classical equation as well as the quantization in form of a Wheeler - De Witt equation are conformally related to the analogous model with Einstein’s theory of gravity with a minimally coupled scalar field. Results are: 1. In the non-tachyonic case, classical fourth-order gravity is not more unstable than Einstein’s theory itself. 2. The well-known classically valid conformal relation is also (at least for some typical cases) valid on the level of the corresponding Wheeler - De Witt equations, which turns out to be a non-trivial statement.

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1 Introduction

The inflationary cosmological model proposed in [1] following from fourth-order gravity got the name ”Starobinsky model” in 1985, cf. [2]. Its stability properties have been discussed e.g. in [3], [4], and [5]. The results do not fully coincide, cf. the successive papers [6] and [7]. We shall not repeat those points that were caused by simple mathematical errors. However, it seems valuable to sum up that part of the discussion which originated from different notions of stability, see sect. 2.
The generalization to gravitational field equations of order higher than fourth, see [8-12], is discussed in sc. 3. When quantized, a Wheeler - De Witt equation for the Starobinsky model appears. From the first glance, one could conjecture, that the conformal relation between Einstein’s theory with a minimally coupled self-interacting scalar field and fourth-order gravity can simply be carried over, but Duff [13] showed that conformal equivalence of classical theories need not survive in the quantum theory. (For clarity, we mention that Duff considers a conformally invariant theory, i.e., a theory which is non-trivially conformally equivalent to itself whereas we consider two different but conformally equivalent theories; but the argument remains the same.) So one must look into the details of the theory, see e.g. Refs. [11, 14-18] and sc. 4.

2 Different notions of stability

We consider a Lagrangian $L = L(R)$ with curvature scalar $R$ in a region where

$$\frac{dL}{dR} \frac{d^2L}{dR^2} \neq 0.$$ (1)

Then one can perform a conformal rescaling of this fourth-order gravity theory to Einstein’s theory with a minimally coupled scalar field. In this picture, instabilities in form of ghosts (wrong sign of the kinetic term) and tachyons (wrong sign of the mass term) could appear. It holds: ghost never appear, tachyons sometimes do, but for the Starobinsky model tachyons are also excluded:

$$L = \frac{R}{2} - \frac{l^2}{12} R^2, \quad R < \frac{3}{l^2}.$$ (2)

($l \approx 10^{-28} cm$, so the inequality in (2) is not a real restriction, but one should mention that in the limit $R \rightarrow \frac{3}{l^2}$ a new type of instability occurs.) It turns out, that singularities appearing in the Starobinsky model are restricted to big bang-type singularities already known from General Relativity. So one can conclude [7]: Fourth-order gravity defined by (2) is not more unstable than General Relativity.

A second point of view is the following: General Relativity is not a renormalizable theory, but it can get this property if one adds curvature squared terms to the action. However it does not suffice to add $R^2$, one also needs the square of the Weyl tensor. The result is: the absence of both ghosts and tachyons requires the corresponding coefficient to be artificially fine-tuned to zero.
A third point of view is outlined in [5]: One could interpret the $R^2$-term in (2) to mimic some quantum gravitational effects. But then the interpretation of the solution changes: one considers the spatially flat Friedman model, solves the classical vacuum equation following from (2) and gets expanding solutions having the property that the Hubble parameter remains in a strip arbitrarily close to zero for an arbitrarily long amount of time. So, by the usual quantum fluctuations one should expect that there is a large probability for the universe to have a moment in time, where the Hubble parameter is negative. This would cause a positive probability for having a recollapse, which is classically excluded for the spatially flat Friedman model. Up to this point we agree with Suen [5]. Here we want to clarify that this argument has nothing to do with the fourth-order terms: it analogously applies to the Einstein theory and the corresponding Einstein-de Sitter model of the universe.

3 Sixth and higher order equations

From the Lagrangian $R\Box R$ a sixth-order field equation appears. It was considered in [8]. More general, a term with $R\Box^k R$ in the Lagrangian yields field equations of order $2k + 4$. It turns out [9-12] that the conformal transformation to Einstein’s theory with scalar fields is possible, but it always yields ghosts. In this sense, higher order theories give additional instabilities. Nevertheless, it makes sense to ask whether the de Sitter space-time is an attractor solution in gravity theories of the above mentioned type. For $k = 2$ see [10], for general $k$ the result is as follows: let $k \geq 1$ and

$$L = \sum_{i=0}^{k} a_i R\Box^i R, \quad a_k \neq 0 \quad (3)$$

In the set of spatially flat Friedman models the de Sitter space-time represents an attractor solution if the coefficients $a_i$ fulfil suitable (complicated) inequalities, but here it suffices to mention:

1. For every value $k$, these inequalities can be fulfilled,

2. If these inequalities are fulfilled, then $a_0 \neq 0$, i.e., if the $R^2$-term is absent, then inflation is no longer typical.

3. If the Einstein-Hilbert term is added as $+\epsilon R$ with the ‘correct’ sign (non-tachyonic case), then the de Sitter space-time becomes a transient attractor with a typical mean life time (german: ‘Verweildauer’) of the transient phase $<\tau> \sim |\epsilon|^{-\gamma}$ where $\gamma = \frac{1}{2}$. 

3
4 The Wheeler - De Witt equation

Deriving the Wheeler - De Witt equation for higher order theories different problems occur. One of the main difficulties is the question how to express higher order theories in the Hamiltonian formalism, which is of first order. Here we want to discuss two approaches, one with and the other without additional constraints. In the first one the first derivative of the cosmic scale factor with respect to the time is introduced as a new coordinate [14]. This definition has to be preserved as a constraint to the system. One considers gauge transformations and first class constraints. It is possible that there exist more first class constraints than gauge symmetries in the Lagrange-function. We expect that total and extended Hamiltonians (in the sense of the Dirac formalism) become equivalent, supposed, the Lagrangian is a geometrical one, [16].

The second approach [11, 17, 18] introduces the second derivative of the cosmic scale factor with respect to the time as a new coordinate. With this definition it is possible, at least for the Friedman models (except $k = -1$, open universe) and Lagrangian $L = f(R)$, where $f$ is an arbitrary function of $R$, to avoid an additional constraint. It is intrinsically given by one of the canonical equations. The other canonical equation is equivalent to the trace of the field equation. Classically, higher order theories are conformally related to Einstein’s theory with a minimally coupled scalar field. Now one can ask whether those theories are still conformally related to each other if they were quantized. This is not a trivial question as Duff [13] has shown that conformal equivalence does not always survive in quantum theory. It does, at least under the following conditions for the Friedman model:

1. $L = f(R)$, and $f(R)$ arbitrarily chosen function of $R$,
2. intrinsical definition for the introduced canonical coordinate,
3. the factor-ordering-problem is solved in that way, that the discussed Wheeler - De Witt equation is covariant in superspace.

If one is going to study more systematically the dynamics not only of the isotropic Friedman models, but also of arbitrary homogeneous models, the non-Hausdorff topology in the set of corresponding Lie algebras (which are related to the isometry groups of the corresponding cosmological model) might play a crucial role [19].
Here, we presented the results of the Cosmology group of Potsdam University concerning the Starobinsky model; to get a more balanced reference list one should also look at the papers cited in Refs. [1 - 20], especially, let us mention the review article [20], where the status of the Starobinsky model up to 1992 has been outlined, and an extensive reference list was given.

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