QUASI-MAXWELL INTERPRETATION OF THE
SPIN-CURVATURE COUPLING

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Abstract. We write the Mathisson-Papapetrou equations of motion for a spinning particle in a stationary spacetime using the quasi-Maxwell formalism and give an interpretation of the coupling between spin and curvature. The formalism is then used to compute equilibrium positions for spinning particles in the NUT spacetime.

Introduction

The purpose of this paper is to use the quasi-Maxwell form of Einstein’s field equations for a stationary spacetime to give an interpretation of the force that acts on a spinning particle according to the Mathisson-Papapetrou equations.

The paper is divided into five sections. In the first section, we briefly review the quasi-Maxwell formalism for stationary spacetimes. The Mathisson-Papapetrou equations of motion for a spinning particle, as well as the various spin supplementary conditions, are discussed in the second section. The third section recalls the main formulae for a magnetic dipole on a magnetic field, which are compared with their gravitational counterparts in the fourth section. It turns out that there is a remarkable analogy between the two sets of formulae, shedding light into the nature of the spin-curvature coupling. In the fifth section, as an example, we use the quasi-Maxwell formalism to compute equilibrium positions for spinning particles in the NUT spacetime.

We use the notation and sign conventions in [Wal84], where latin indices represent abstract indices, except for the latin indices $i, j, \ldots$, which will be numerical indices referring to the space manifold (i.e. $i, j, \ldots = 1, 2, 3$). As is usual, we will not worry about the vertical position of space indices on orthonormal frames.

1. Quasi-Maxwell formalism for stationary spacetimes

We start by briefly reviewing the quasi-Maxwell formalism for stationary spacetimes. For more details, see [Oli02, CN05, Emb84, LL97, LBNZ98, NZ97, BS00].

Let $(M, g)$ be a chronological stationary spacetime with a complete timelike Killing vector field $T$. Then the action of $\mathbb{R}$ on $M$ by the flow of $T$ makes it a trivial principal $\mathbb{R}$-bundle [And00, BS00], and the quotient space $M/\mathbb{R}$ can be identified with a 3-dimensional submanifold $\Sigma \subset M$. One can think of this quotient as the set of all stationary observers. If $\pi : M \rightarrow \Sigma$ is the quotient map, $p \in \Sigma$ and $v \in T_p\Sigma$, we define $v^\uparrow$ to be the vector field along the integral line of $T$ through $p$ which is orthogonal to $T$ and such that $\pi_* v^\uparrow = v$. We can then define the Riemannian

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metric \( h \) on \( \Sigma \) through

\[
h(v, v) = g(v^\dagger, v^\dagger).
\]

This metric has the physical interpretation of yielding the distance measured between nearby stationary observers through Einstein’s light signaling procedure [LL97]. The **space manifold** is the Riemannian manifold \((\Sigma, h)\).

There exists a function \( \phi : \Sigma \to \mathbb{R} \), called the **gravitational potential**, such that

\[
g(T, T) = -e^{2(\phi \circ \pi)}.
\]

The **gravitational field** is then the vector field \( \mathbf{G} \) defined on \( \Sigma \) by

\[
\mathbf{G} = -\text{grad} \, \phi.
\]

It is easy to show that \(-\mathbf{G}^\dagger\) is simply the acceleration of the stationary observers.

For a particular choice of the submanifold \( \Sigma \subset M \), one can define the one-form \( A \) through

\[
A(v) = -e^{-2\phi}g(T, v)
\]

for all \( v \in T\Sigma \subset TM \). It turns out that although \( A \) depends on the choice of \( \Sigma \), the 2-form

\[
H = -e^{\phi}dA
\]

does not, and hence is well defined on the quotient. Assuming that \( \Sigma \) is orientable, we define the **gravitomagnetic field** to be the vector field

\[
\mathbf{H} = (\ast H)^\sharp,
\]

where \( \ast \) is the Hodge star and \( \sharp \) is the usual map between vectors and covectors determined by \( h \). It is not hard to show that \( \frac{1}{2} H^\dagger \) is simply the vorticity of the stationary observers.

The names above are justified by the fact that the projection of any timelike geodesic on the space manifold, parameterized by the proper time \( \tau \), satisfies the equation

\[
\frac{Du}{d\tau} = \gamma^2 \mathbf{G} + \gamma \mathbf{u} \times \mathbf{H},
\]

where \( \mathbf{u} \) is the tangent vector and \( \gamma = (1 + u^2)^{\frac{1}{2}} \) is the energy per unit mass of the corresponding free-falling particle as measured by stationary observers. As a consequence of this equation, the particle’s total energy per unit mass

\[
E = e^{\phi} \gamma
\]

is conserved.

If \( \{ E_1, E_2, E_3 \} \) is a right-handed orthonormal local frame on the space manifold, then \( \{ E_0, E_1^\dagger, E_2^\dagger, E_3^\dagger \} \) is an orthonormal local frame on \( M \), where \( E_0 = e^{-\phi}T \). The nonvanishing Christoffel symbols associated to this frame are [Oli02]

\[
\begin{align*}
(4) \Gamma_{00}^i &= (4) \Gamma_0^0 = -G_i; \\
(4) \Gamma_{0j}^0 &= -(4) \Gamma_{0j}^i = -(4) \Gamma_{j0}^i = \frac{1}{2} H_{ij}; \\
(4) \Gamma_{jk}^i &= \Gamma_{jk}^i.
\end{align*}
\]
where $\Gamma^i_{jk}$ are the Christoffel symbols associated to $\{E_1, E_2, E_3\}$. The components of the Riemann curvature tensor are therefore given by

$$(4) \quad R_{0i0j} = -\nabla_j G_i + G_i G_j - \frac{1}{4} H_{ik} H_{kj};$$

$$(4) \quad R_{0ijk} = \frac{1}{2} (\nabla_j H_{ik} - \nabla_k H_{ij} - 2 G_i H_{jk});$$

$$(4) \quad R_{ijkl} = R_{ijkl} + \frac{1}{4} (2 H_{ij} H_{kl} + H_{ik} H_{jl} - H_{il} H_{jk}),$$

and the components of the Ricci curvature tensor are

$$(4) \quad R_{00} = -\text{div} G + G^2 + \frac{1}{2} H^2;$$

$$(4) \quad R_{0i} = \frac{1}{2} (\text{curl} H)_i - (G \times H)_i;$$

$$(4) \quad R_{ij} = R_{ij} + \nabla_i G_j - G_i G_j - \frac{1}{2} H_i H_j + \frac{1}{2} H^2 h_{ij}.$$

In particular, one can write Einstein’s equations in the quasi-Maxwell form

$\text{div} G = G^2 + \frac{1}{2} H^2 - 4\pi (\rho + \text{tr} \sigma);$  

$\text{curl} H = 2G \times H - 16\pi j;$  

$Ric + \nabla G^2 = G^2 \otimes G^2 + \frac{1}{2} H^2 \otimes H^2 - \frac{1}{2} H^2 h + 8\pi \sigma + 4\pi (\rho - \text{tr} \sigma) h,$

where $Ric$ is the Ricci curvature of the space manifold and $\rho = T^{00}, j = T^{0i} E_i$ and $\sigma = T_{ij} E_i \otimes E_j$ are the energy density, the energy current and the spatial stress tensor measured by the stationary observers. These equations imply that the scalar curvature of the space manifold is

$$R = 16\pi \rho - \frac{3}{2} H^2.$$

2. Spinning particle in General Relativity

The Mathisson-Papapetrou equations \cite{Mat37, Pap51} for the motion of a spinning particle, obtained by neglecting all multipoles higher than mass monopole and spin dipole \cite{Dix70, ADS03}, are

$$\nabla U P^a + \frac{1}{2} R^{bcd} U^b S_{cd} = 0$$  

$$\nabla U S - P \wedge U = 0$$

where $U$ is the unit tangent vector to the particle’s history, $P$ is its energy-momentum vector (not necessarily parallel to $U$) and $S$ is a rank-2 antisymmetric tensor representing the particle’s angular momentum. This system of equations is underdetermined, and must be closed by means of a spin supplementary condition. The usual choice, justified in part by the uniqueness results in \cite{Bei67, Sch79a, Sch79b}, is the Tulczyjew-Dixon condition \cite{Tul59, Dix70}

$$S^{ab} P_b = 0.$$

Less popular choices include the Mathisson-Pirani condition \cite{Mat37, Pir56}

$$S^{ab} U_a = 0.$$
Dixon’s supplementary condition implies that the angular momentum tensor is of the form

\[ S^{ab} = \frac{1}{m} \epsilon^{abcd} P_c s_d, \]

where \( \epsilon \) is the volume element and \( m = \sqrt{-g(P,P)} \) is the particle’s dynamical mass (which can be seen to be constant). The spin vector

\[ s^a = -\frac{1}{2m} \epsilon^{abcd} P_b S_{cd} \]

(which is the relativistic analogue of the angular momentum vector) is orthogonal to \( P \) (hence spacelike), and satisfies

\[ \nabla_U s = \frac{1}{m^2} g(\nabla_U P, s) P. \]

It is possible to show [K72, TdFC76, Sem99] that

\[ U^a = \frac{g(P, U)}{m^2} \left( P^a + \frac{1}{\Delta} S^{ab} R_{bced} P^c S_{de} \right), \tag{1} \]

where

\[ \Delta = 2m^2 + \frac{1}{2} R^{abcd} S_{ab} S_{cd}. \]

Therefore \( U \) and \( \frac{1}{m} P \) differ by a Lorentz transformation of order

\[ \frac{|s|^2}{m^2 r^2}, \]

where \( r \) is the local radius of curvature. Assuming that the particle is not rotating too fast, we can take \( P \) and \( U \) to be parallel, \( P = mU \), in which case Dixon’s and Mathisson’s spin supplementary conditions coincide. The (approximate) Mathisson-Papapetrou equations are then simply written as

\[ m \nabla_U U^a = -\frac{1}{2} R^a_{bced} P^b \epsilon^{cdef} U^e s_f; \tag{2} \]

\[ \nabla_U s = g(\nabla_U U, s) U. \tag{3} \]

These equations are consistent with the spin supplementary condition \( g(U, s) = 0 \), as \( s \) is Fermi-Walker transported along the motion.

The first equation indicates that a net force arises from the particle’s spin coupling to the spacetime curvature. The expression for this force, however, is not particularly enlightening. We will see that within the quasi-Maxwell formalism its expression is actually quite natural.

3. MAGNETIC DIPOLE ON A MAGNETIC FIELD

As is well known (see for instance [Jac98]), a magnetic dipole with moment \( m \) placed in a magnetic field \( B \) will experience a torque

\[ N = m \times B, \]

which can be thought of as arising from the potential energy

\[ U = -m \cdot B. \]
Such dipole will experience a net force
\[ \mathbf{F} = \nabla \mathbf{B} \cdot \mathbf{m} - (\text{div} \, \mathbf{B}) \mathbf{m}, \]
where the last term is not usually written because of Maxwell’s equation div\( \mathbf{B} = 0 \).

4. Spinning particle in Quasi-Maxwell formalism

We will consider the special case when the particle is at rest with respect to the stationary observers, i.e.
\[ U = E_0. \]
The spin supplementary condition then yields
\[ s^0 = 0. \]
Using the formulae for the Christoffel symbols, it is easily seen that the equation for the Fermi-Walker transport of \( s \) can be written as
\[
\frac{ds^i}{d\tau} + (4) \Gamma^i_{0j}s^j = 0 \iff \frac{ds^i}{d\tau} - \frac{1}{2} H_{ij}s^j = 0,
\]
i.e.
\[ (4) \quad Ds = -\frac{1}{2} \mathbf{H} \times \mathbf{s}. \]
This formula shows that the particle’s angular momentum changes under a gravitomagnetic field exactly like a magnetic dipole moment under a magnetic field (except for a factor \( \frac{1}{2} \)).

Using the expressions for the components of the Riemann tensor, one can compute the force due to coupling between spin and curvature to have components
\[
F^i = -\frac{1}{2} (4) R^i_{\ 0jk} \varepsilon^{kml} (-1) s^l = \frac{1}{4} (\nabla_j H^i_k - \nabla_k H^i_j - 2G_i H^j_k) \varepsilon_{jkl}s^l
\]
where \( \varepsilon \) is the volume element of the space manifold. Therefore
\[
F^i = \frac{1}{2} (\nabla_j H^i_k - G_i H^j_k) \varepsilon_{jkl}s^l = \frac{1}{2} (\nabla_j (\varepsilon_{ikm} H^m) - G_i \varepsilon_{jkm} H^m) \varepsilon_{jkl}s^l
\]
\[
= \frac{1}{2} (\delta_{ij} \delta_{ml} - \delta_{il} \delta_{mj}) (\nabla_j H^m) s^l - \frac{1}{2} (\delta_{jj} \delta_{ml} - \delta_{jl} \delta_{mj}) G_i H^m s^l
\]
\[
= \frac{1}{2} (\nabla_{(j} H^i_{k)} s^l - \frac{1}{2} (\nabla_j H^i)^l s^i - \frac{1}{2} (3\delta_{ml} - \delta_{nl}) G_i H^m s^l
\]
that is,
\[
(5) \quad \mathbf{F} = \frac{1}{2} \nabla \mathbf{H} \cdot \mathbf{s} - \frac{1}{2} (\text{div} \, \mathbf{H}) \mathbf{s} - (\mathbf{s} \cdot \mathbf{H}) \mathbf{G}.
\]
Therefore the spin-curvature coupling can be interpreted as consisting of a gravitomagnetic part, which mimics the magnetic force on a magnetic dipole (apart from the factor \( \frac{1}{2} \)), plus the weight of the dipole energy.

This similarity between the spin-curvature coupling and the force on a magnetic dipole was noticed long ago in the context of the linearized theory by Wald [Wal72]. Equation (5) extends this analogy to an arbitrarily strong (albeit stationary) field. Notice that only the first term in (5) appears in [Wal72], as the other two are not linear in the fields (it is easily seen that div\( \mathbf{H} = -\mathbf{G} \cdot \mathbf{H} \)).

\[ ^1 \text{We have } \varepsilon^{0ijk} = -\varepsilon_{ijk} \text{ on any orthonormal frame.} \]
5. Spinning particle in NUT spacetime

We would like to use equation (5) to compute equilibrium positions for spinning particles in stationary spacetimes. Unfortunately, this is a situation where the approximate equations (2) and (3) are expected to break down. Indeed, since the components of the Riemann tensor are quadratic in the components of the fields $G$ and $H$, we expect these to be of order $\frac{1}{r}$, where $r$ is the local radius of curvature. Therefore the force on the spinning particle is of order $\frac{|s|}{r^2}$.

For the particle to remain at equilibrium, this force must balance the gravitational force $mG$, which is of order $\frac{m}{r}$. Therefore, we expect $\frac{|s|}{r^2} \sim \frac{m}{r} \Rightarrow \frac{|s|}{mr} \sim 1$ at any equilibrium position, whereas the approximate equations assume that this quantity is small.

Equation (7), however, implies that the relation $U = \frac{1}{m} P$ is exact if $s \times F = 0$, where $F$ is the force on the spinning particle given in (5); for an equilibrium position, this is equivalent to requiring that $s \times G = 0$. Therefore the equilibrium positions computed using equation (5) will be exact for the Tulczyjew-Dixon supplementary condition if the angular momentum is aligned with the gravitational field.

For completeness, let us remark that the Mathisson-Pirani supplementary condition leads to the motion equation

\[ m \nabla_U U^a = -\frac{1}{2} R^a_{\ bcd} U^b S^{cd} + S^{ab} \nabla_U U^b, \]

where the constant $m$ is now defined as $m = -g(P, U)$ [Mat37, Pir56, MTW73, Ste04]. For an equilibrium position, this will reduce to the approximate equation (2) when $s \times (G \times H) = 0$. Therefore the equilibrium positions computed using equation (5) will be exact for the Mathisson-Pirani supplementary condition if the angular momentum is aligned with the “Poynting vector” $G \times H$. Thus for instance the equilibrium positions along the axis of the Kerr-de Sitter solution, computed in [SK06] using the Mathisson-Pirani supplementary condition, coincide with the equilibrium positions computed using Tulczyjew-Dixon supplementary condition, and can both be obtained from (5).

As an example, we now compute equilibrium positions for spinning particles in NUT spacetime. Recall that the NUT spacetime is a stationary solution of the vacuum Einstein field equations describing a gravitomagnetic monopole, given in local coordinates by

\[ g = -e^{2\phi}(dt + 2l \cos \theta d\varphi)^2 + e^{-2\phi} dr^2 + (r^2 + l^2)(d\theta^2 + \sin^2 \theta d\varphi^2), \]

where

\[ e^{2\phi} = 1 - 2\frac{Mr + l^2}{r^2 + l^2} \]

and $M, l$ are two parameters representing the total mass and (half) the gravitomagnetic charge [NTU63, LBNZ98, SKM03]. The metric of the space manifold

\[ \text{is interesting to note that the linear momentum of the field produced by a magnetic dipole } m \text{ on an electric field } E \text{ has the approximate expression } P_{\text{field}} = E \times m \text{ [Jac98].} \]
determined by the stationary observers is
\[ h = e^{-2\phi} dr^2 + (r^2 + l^2)(d\theta^2 + \sin^2 \theta d\varphi^2), \]
and the gravitomagnetic potential 1-form is
\[ A = 2l \cos \theta d\varphi. \]
Thus the gravitomagnetic field corresponds to the 2-form
\[ H = -e^\phi dA = 2l e^\phi \sin \theta d\theta \wedge d\varphi, \]
and hence to the 1-form
\[ H^\sharp = \star H = \frac{2l}{r^2 + l^2} dr, \]
implying that \( H \) is radial. If we choose the angular momentum \( s \) to be also radial, then the equilibrium positions will be exact for the Tulczyjew-Dixon supplementary condition. Moreover, since the 1-form associated to the gravitational field is
\[ G^\sharp = -d\phi = -\phi' dr, \]
where the prime denotes differentiation with respect to \( r \), we have \( G \times H = 0 \), and the equilibrium positions will also be exact for the Mathisson-Pirani supplementary condition.

By equation (4), the angular momentum \( s \) will be constant. We now compute the 1-forms associated to each term on the expression (5) for the force on a spinning particle (here we must keep track of the vertical position of the indices, as we are working with a non-orthonormal coordinate basis). To begin with, we have
\[ (\nabla H \cdot s)^\sharp = \nabla_r H_r s^r dr = (H'_r - (H_r + \phi' H_r) s^r dr = (H'_r + \phi' H_r) s^r dr. \]
(the relevant Christoffel symbols can be readily computed from the geodesic equations of the space manifold). Moreover,
\[ (\text{div} H)s^\sharp = -(\nabla H \cdot s)r^r dr = -G^r H_r s_r dr = -G_r H_r s^r dr = \phi' H_r s^r dr. \]
Finally,
\[ (s \cdot H)G^\sharp = -s^r H_r \phi' dr. \]
Collecting these terms together yields
\[ F^\sharp = \frac{1}{2} (H'_r + \phi' H_r) s^r dr. \]
The equation for the equilibrium positions is then
\[ mG^\sharp + F^\sharp = 0 \iff -2m\phi' + (H'_r + \phi' H_r) s^r = 0 \iff \frac{s^r}{m} = \frac{(e^{2\phi})'}{(H_r e^{2\phi})'}. \]
For \( r \gg \sqrt{M^2 + l^2} \), we have
\[ e^{2\phi} \approx 1 - \frac{2M}{r}, \quad H_r e^{2\phi} \approx \frac{2l}{r^2}, \]
and hence the equilibrium condition reduces to
\[ \frac{s^r}{m} \approx -\frac{Mr}{2l}. \]
In particular, the angular momentum must point towards the center if \( l > 0 \). This could easily be computed from the linearized formulae in [Wal72]. In the strong
field region, however, things are more complicated. For instance, near the horizon \( r_H = M + \sqrt{M^2 + l^2} \) one has
\[
e^{2\phi} \simeq 0
\]
and hence the equilibrium condition reduces to
\[
\frac{s'}{m} \simeq \frac{(e^{2\phi})'}{H_r(e^{2\phi})'} = \frac{1}{H_r} = \frac{r^2 + l^2}{2l}.
\]
In particular, the angular momentum must point away from the center in this region.

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