The families of orthogonal, unitary and quaternionic unitary Cayley–Klein algebras and their central extensions  

Francisco J. Herranz† and Mariano Santander‡

† Departamento de Física, E.U. Politécnica, Universidad de Burgos  
E–09006 Burgos, Spain

‡ Departamento de Física Teórica, Universidad de Valladolid  
E–47011 Valladolid, Spain

Abstract

The families of quasi-simple or Cayley–Klein algebras associated to anti-hermitian matrices over \( \mathbb{R} \), \( \mathbb{C} \) and \( \mathbb{H} \) are described in a unified framework. These three families include simple and non-simple real Lie algebras which can be obtained by contracting the pseudo-orthogonal algebras \( so(p,q) \) of the Cartan series \( B_l \) and \( D_l \), the special pseudo-unitary algebras \( su(p,q) \) in the series \( A_l \), and the quaternionic pseudo-unitary algebras \( sq(p,q) \) in the series \( C_l \). This approach allows to study many properties for all these Lie algebras simultaneously. In particular their non-trivial central extensions are completely determined in arbitrary dimension.

1 The three main families of CK algebras

The aim of this contribution is twofold. First, we present the structure of the three main series of Cayley–Klein (CK) algebras (orthogonal, unitary and quaternionic unitary) as associated to anti-hermitian matrices over \( \mathbb{R} \), \( \mathbb{C} \) and \( \mathbb{H} \). These families include real simple Lie algebras in the four Cartan series as well as many non-simple Lie algebras which can be obtained from the simple ones by a sequence of contractions. The CK approach is quite useful in the study of many structures associated to the algebras in a given CK family, such as their symmetric

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homogeneous spaces, Casimir invariants, quantum deformations, etc. This formalism allows a clear view of the behaviour of such properties under contraction.

And second, as an example of this approach, we focus on the complete and general results concerning the non-trivial central extensions for all the algebras in the three main CK families.

We consider $N$ real coefficients $\omega = (\omega_1, \omega_2, \ldots, \omega_N)$ and two-index coefficients $\omega_{ab}$ defined in terms of the former by
\[
\omega_{ab} := \omega_{a+1} \omega_{a+2} \cdots \omega_b \quad a, b = 0, 1, \ldots, N \quad a < b \quad \omega_{aa} := 1. \tag{1.1}
\]
Let $\mathbb{K}$ denote an associative division algebra; this can be either the reals $\mathbb{R}$, complex $\mathbb{C}$ or quaternions $\mathbb{H}$. The space $\mathbb{K}^{N+1}$ can be endowed with a hermitian (sesqui)linear form $\langle \cdot | \cdot \rangle_\omega : \mathbb{K}^{N+1} \times \mathbb{K}^{N+1} \to \mathbb{K}$ defined by
\[
\langle a | b \rangle_\omega := \bar{a}^0 b^0 + \bar{a}^1 \omega_1 b^1 + \bar{a}^2 \omega_2 b^2 + \ldots + \bar{a}^N \omega_1 \cdots \omega_N b^N = \sum_{i=0}^{N} \bar{a}^i \omega_{0i} b^i \tag{1.2}
\]
where $a, b \in \mathbb{K}^{N+1}$ and $\bar{a}^i$ means the conjugation in $\mathbb{K}$ of the component $a^i$. The underlying metric is provided by the matrix
\[
\mathcal{I}_\omega = \text{diag} (1, \omega_0, \omega_0, \ldots, \omega_0) = \text{diag} (1, \omega_1, \omega_2, \ldots, \omega_N). \tag{1.3}
\]
The antihermitian CK family over $\mathbb{K}$ is defined as the family of Lie algebras of the groups of linear isometries of this hermitian metric over the space $\mathbb{K}^{N+1}$. Thus the isometry condition for an element $U$ of the corresponding Lie group
\[
\langle U a | U b \rangle_\omega = \langle a | b \rangle_\omega \quad \forall a, b \in \mathbb{K}^{N+1}, \tag{1.4}
\]
leads to the following relations for the group element $U$ and generator $X$:
\[
U^\dagger \mathcal{I}_\omega U = \mathcal{I}_\omega \quad X^\dagger \mathcal{I}_\omega + \mathcal{I}_\omega X = 0. \tag{1.5}
\]
For $\mathbb{K} = \mathbb{R}$, $\mathbb{H}$ these matrices span a simple Lie algebra. For $\mathbb{K} = \mathbb{C}$ the algebra so determined is not simple, but becomes simple if we add the unimodularity condition:
\[
\det(U) = 1 \quad \text{trace}(X) = 0, \tag{1.6}
\]
which is known as the special condition for the Lie algebra/group. Hence from condition (1.5) (and also (1.6) when $\mathbb{K} = \mathbb{C}$) we obtain $\mathcal{I}_\omega$-antihermitian $(N+1) \times (N+1)$ matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ which generate the three families of quasi-simple or CK algebras displayed in the following table:

| $\mathbb{K}$ | CK family | Generators | Dimension |
|-------------|------------|------------|-----------|
| $\mathbb{R}$ | Orthogonal $so(N+1)$ | $J_{ab} = -\omega_{ab} e_{ab} + e_{ba}$ | $\frac{1}{2}N(N+1)$ |
| $\mathbb{C}$ | Complex special unitary $su(N+1)$ | $J_{ab} = -\omega_{ab} e_{ab} + e_{ba}$ $M_{ab} = i(\omega_{ab} e_{ab} + e_{ba})$ $B_l = i(e_l e_{l-1} - e_{l-1})$ | $(N+1)^2 - 1$ |
| $\mathbb{H}$ | Quaternionic unitary $sU(N+1)$ | $J_{ab} = -\omega_{ab} e_{ab} + e_{ba}$ $M_{ab} = i_a(\omega_{ab} e_{ab} + e_{ba})$ $E_{a} = i_a e_{aa}$ | $2(N+1)^2 + (N+1)$ |
Hereafter it is assumed that \(a, b = 0, 1, \ldots, N\) and \(a < b\); \(l = 1, \ldots, N\); \(i\) is the complex unit; \(\alpha = 1, 2, 3\) and \(i_1 = i, i_2 = j, i_3 = k\) are the usual quaternionic units; and \(e_{ab}\) is the \((N + 1) \times (N + 1)\) matrix with a single 1 entry in row \(a\), column \(b\).

When all the coefficients \(\omega_a \neq 0\) these CK families include real simple algebras in the four Cartan series:

- The orthogonal CK family \(so_\omega(N + 1)\) embraces the pseudo-orthogonal algebras \(so(p, q)\) with \(p + q = N + 1\) in the Cartan series \(B_{\frac{N}{2}}\) for even \(N\), and \(D_{\frac{N+1}{2}}\) for odd \(N\).
- The special unitary CK family \(su_\omega(N + 1)\) comprises the special pseudo-unitary algebras \(su(p, q)\) with \(p + q = N + 1\) in the Cartan series \(A_N\).
- The quaternionic unitary CK family \(sq_\omega(N + 1)\) includes the quaternionic pseudo-unitary algebras \(sq(p, q)\) with \(p + q = N + 1\) (the usual name for these algebras is \(sp(p, q)\)) in the Cartan series \(C_{N+1}\).

In all cases, \(p\) and \(q\) are the number of positive and negative terms in the matrix \(I_\omega\) (1.3), so when all \(\omega_a > 0\) we recover the compact real forms \(so(N+1)\), \(su(N+1)\) and \(sq(N + 1)\). If we set one or several coefficients \(\omega_a\) equal to zero, we obtain a non-simple Lie algebra. This process is equivalent to perform the limit \(\omega_a \to 0\) which corresponds to carry out an Inönü–Wigner contraction \([11]\). Therefore each CK family includes simple as well as non-simple members and the latter appear by means of contractions. These non-simple Lie algebras preserve properties related to simplicity, and are also called quasi-simple \([12]\). For instance, it has been shown in \([11]\) that all the algebras in the orthogonal CK family \(so_\omega(N + 1)\) share the same number of functionally independent Casimir operators no matter how contracted is the Lie algebra. This is not longer true if we go beyond the CK family as the abelian algebra clearly shows: all its generators are invariants.

These families of quasi-simple algebras can be obtained by applying the graded contraction theory \([13, 14]\) to any simple member in each family (e.g., \(so_\omega(N + 1)\) is provided by \(\mathbb{Z}_2^{\otimes N}\) graded contractions starting from \(so(N + 1)\) \([3, 4]\)). An alternative approach to these algebras can be found in \([13]\).

In the sequel we write down the Lie brackets for these three families of Lie algebras. We remark that the values of each real coefficient \(\omega_a\) can be reduced to \(+1, -1\) or 0 (by means of a rescaling), so that for a given \(N\) each CK family contains \(3^N\) Lie algebras (some of them are isomorphic as abstract Lie algebras).

The commutation relations of the orthogonal CK family \(so_\omega(N + 1)\) can be obtained from the matrix generators \(J_{ab}\) and read \([1, 8]\):

\[
\begin{align*}
[J_{ab}, J_{ac}] &= \omega_{ab} J_{bc}, & [J_{ab}, J_{bc}] &= -J_{ac}, & [J_{ac}, J_{bc}] &= \omega_{bc} J_{ab}, & [J_{ab}, J_{de}] &= 0.
\end{align*}
\]

(1.7)

Hereafter whenever three indices \(a, b, c\) appear they are always assumed to verify \(a < b < c\); whenever four indices \(a, b, d, e\) appear, \(a < b, d < e\) and all of them are assumed to be different; and there is no any implied sum over repeated indices.
The Lie brackets of the special unitary CK family $su_\omega(N + 1)$ are given by \([1, 9]\):

$$
\begin{align*}
[J_{ab}, J_{ac}] &= \omega_{ab}J_{bc} & [J_{ab}, J_{bc}] &= -J_{ac} & [J_{ac}, J_{bc}] &= \omega_{bc}J_{ab} \\
[M_{ab}, M_{ac}] &= \omega_{ab}J_{bc} & [M_{ab}, M_{bc}] &= J_{ac} & [M_{ac}, M_{bc}] &= \omega_{bc}M_{ab} \\
[J_{ab}, M_{ac}] &= \omega_{ab}M_{bc} & [J_{ab}, M_{bc}] &= -M_{ac} & [J_{ac}, M_{bc}] &= -\omega_{bc}M_{ab} \\
[M_{ab}, J_{ac}] &= -\omega_{ab}M_{bc} & [M_{ab}, J_{bc}] &= -M_{ac} & [M_{ac}, J_{bc}] &= \omega_{bc}M_{ab} \\
[J_{ab}, J_{de}] &= 0 & [M_{ab}, M_{de}] &= 0 & \quad [J_{de}, B_t] &= (\delta_{a,t-1} - \delta_{b,t-1} + \delta_{bl} - \delta_{al})M_{ab} \\
[M_{ab}, B_t] &= -\omega_{ab}M_{de} & \quad [J_{de}, M_{bc}] &= \omega_{bc}M_{ab} \\
\end{align*}
$$

If we discard the condition (1.6) and consequently, further to the generators of $su_\omega(N + 1)$ we add the matrix $I = i \sum_{a=0}^{N} e_{aa}$, we obtain the unitary CK family $u_\omega(N + 1)$ of dimension $(N + 1)^2$ and with commutators given by (1.8), (1.9) and

$$
\begin{align*}
[J_{ab}, I] &= 0 & [M_{ab}, I] &= 0 & [B_t, I] &= 0. \\
\end{align*}
$$

When all $\omega_a \neq 0$ we find the pseudo-unitary algebras $u(p, q)$ which are not simple (they are in the semisimple series $A_N \oplus D_1$) but their pattern is similar to that of $su(p, q)$ with respect to $\mathcal{L}_\omega$.

The commutation rules of the quaternionic unitary CK family $sq_\omega(N + 1)$ turn out to be \([1, 11]\):

$$
\begin{align*}
[J_{ab}, J_{ac}] &= \omega_{ab}J_{bc} & [J_{ab}, J_{bc}] &= -J_{ac} & [J_{ac}, J_{bc}] &= \omega_{bc}J_{ab} \\
[M_{ab}^\alpha, M_{ac}^\alpha] &= \omega_{ab}J_{bc} & [M_{ab}^\alpha, M_{bc}^\alpha] &= J_{ac} & [M_{ac}^\alpha, M_{bc}^\alpha] &= \omega_{bc}M_{ab} \\
[J_{ab}^\alpha, M_{ac}^\alpha] &= \omega_{ab}M_{bc} & [J_{ab}^\alpha, M_{bc}^\alpha] &= -M_{ac} & [J_{ac}^\alpha, M_{bc}^\alpha] &= -\omega_{bc}M_{ab} \\
[M_{ab}^\alpha, J_{ac}^\alpha] &= -\omega_{ab}M_{bc} & [M_{ab}^\alpha, J_{bc}^\alpha] &= -M_{ac} & [M_{ac}^\alpha, J_{bc}^\alpha] &= \omega_{bc}M_{ab} \\
[J_{ab}, J_{de}] &= 0 & [M_{ab}, M_{de}] &= 0 & [J_{de}, B_t] &= (\delta_{a,d-1} - \delta_{b,d-1} + \delta_{bl} - \delta_{al})J_{ab} \\
[M_{ab}^\alpha, E_d^\alpha] &= (\delta_{ad} - \delta_{bd})M_{ab}^\alpha & [M_{ab}^\alpha, E_d^\alpha] &= -J_{ac} & [M_{ab}^\alpha, E_d^\alpha] &= (\delta_{ad} - \delta_{bd})J_{ab} \\
[J_{ab}, M_{de}^\alpha] &= 2\omega_{ab}(E_b^\beta - E_a^\beta) & [E_a^\alpha, E_b^\beta] &= 0 & [E_a^\alpha, E_b^\beta] &= 0 \\
\end{align*}
$$

$$
\begin{align*}
[M_{ab}^\beta, M_{bc}^\gamma] &= \omega_{ab}\varepsilon_{\alpha\beta\gamma}M_{bc}^\alpha & [M_{ab}^\alpha, M_{bc}^\beta] &= \varepsilon_{\alpha\beta\gamma}M_{ac}^\gamma & [M_{ac}^\alpha, M_{bc}^\beta] &= \omega_{bc}\varepsilon_{\alpha\beta\gamma}M_{ab}^\gamma \\
[M_{ab}^\alpha, M_{de}^\beta] &= 0 & [M_{ab}^\alpha, M_{de}^\beta] &= 2\omega_{ab}\varepsilon_{\alpha\beta\gamma}(E_a^\gamma + E_b^\gamma) \\
[M_{ab}^\alpha, E_d^\beta] &= (\delta_{ad} + \delta_{bd})\varepsilon_{\alpha\beta\gamma}M_{ab}^\gamma & [E_a^\alpha, E_b^\beta] &= 2\delta_{ab}\varepsilon_{\alpha\beta\gamma}E_a^\gamma \\
\end{align*}
$$

where $\varepsilon_{\alpha\beta\gamma}$ is the completely antisymmetric unit tensor with $\varepsilon_{123} = 1$, and whenever three quaternionic indices $\alpha, \beta, \gamma$ appear, they are assumed to be different.

Finally, we would like to remark that in addition to these three main ‘signature’ families of CK algebras, whose simple members $so(p, q)$, $su(p, q)$, $sq(p, q)$ can be realised as antihermitian matrices over either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, there are other CK families \([1, 9]\). In the $C_{N+1}$ Cartan series, the remaining real Lie algebra is the symplectic $sp(2(N + 1), \mathbb{R})$, which can be interpreted in terms of CK families either as the
single simple member of its own CK family \( sp_\omega(2(N + 1), \mathbb{R}) \), or alternatively, as the unitary family \( u_\omega((N + 1), \mathbb{H}') \) over the algebra of the split quaternions \( \mathbb{H}' \) (a pseudo-orthogonal variant of quaternions, where \( i_1, i_2, i_3 \) still anticommute, but their squares are \( i_1^2 = -1, i_2^2 = 1, i_3^2 = 1 \); this is not a division algebra). Likewise there are other CK families associated to the remaining real simple Lie algebras, namely: \( su^*(2(N + 1)) \approx sl(N + 1, \mathbb{H}) \) in the Cartan series \( A_{2N+1} \), \( sl(N + 1, \mathbb{R}) \approx su(N + 1, \mathbb{C}') \) also in \( A_N \), and \( so^*(2N) \) in \( D_N \).

## 2 Central extensions

Let \( G \) an \( r \)-dimensional Lie algebra with generators \( \{X_1, \ldots, X_r\} \) and structure constants \( C^k_{ij} \). A generic central extension \( \tilde{G} \) of \( G \) by the one-dimensional algebra generated by \( \Xi \) is a Lie algebra with \((r + 1)\) generators \( \{X_1, \ldots, X_r, \Xi\} \) and commutation relations:

\[
[X_i, X_j] = \sum_{k=1}^r C^k_{ij} X_k + \xi_{ij} \Xi \quad [\Xi, X_i] = 0.
\]

Hence we consider an initial extension coefficient \( \xi_{ij} \) associated to each Lie bracket \([X_i, X_j]\). As \( \tilde{G} \) must be a Lie algebra, the extension coefficients \( \xi_{ij} \) must be antisymmetric in their indices, \( \xi_{ji} = -\xi_{ij} \), and must fulfill the following conditions coming from the Jacobi identities for the generators \( X_i, X_j, X_l \) in \( G \):

\[
\sum_{k=1}^r \left( C^k_{ij} \xi_{kl} + C^k_{lj} \xi_{ki} + C^k_{li} \xi_{kj} \right) = 0.
\]

Therefore as a first step in the study of the central extensions of a given Lie algebra \( G \) we have to solve the set of linear equations (2.2) involving all initially possible extension coefficients \( \xi_{ij} \) (2.1). Afterwards we have to find out which of these coefficients are trivial, that is, which can be removed from (2.1) by means of a change of basis in \( \tilde{G} \). Explicitly, if for a set of arbitrary real numbers \( \mu_k \) we perform a change of generators

\[
X_k \rightarrow X'_k = X_k + \mu_k \Xi,
\]

then the commutators for the generators \( \{X'_1, \ldots, X'_r, \Xi\} \) are given by (2.1) with a new set of extension coefficients

\[
\xi'_{ij} = \xi_{ij} - \sum_{k=1}^r C^k_{ij} \mu_k,
\]

where \( \delta \mu(X_i, X_j) = \sum_{k=1}^r C^k_{ij} \mu_k \) is the two-coboundary generated by \( \mu \). Extension coefficients differing by a two-coboundary correspond to equivalent extensions, and those extension coefficients which are a two-coboundary, \( \xi_{ij} = -\sum_{k=1}^r C^k_{ij} \mu_k \), correspond to trivial extensions. The classes of equivalence of non-trivial two-cocycles determine the second cohomology group of the Lie algebra, \( \mathbb{H}^2(G, \mathbb{R}) \).
The procedure we have just described can be carried out for each family of CK algebras in a global way, so that this unified approach avoids at once and for all the need of a case-by-case study for any given algebra in the family. In general, the extension coefficients arising for each CK family can be casted into three types according to their behaviour under contraction (when some $\omega_a$ vanish):

- **Type I extension coefficients**: they correspond to central extensions which are trivial for all the CK algebras belonging to a given family simultaneously, no matter of how many coefficients $\omega_a$ are equal to zero. Therefore they can be removed at once by means of redefinitions as (2.3).

- **Type II extension coefficients**: they give rise to non-trivial extensions when some $\omega_a = 0$, and to trivial ones otherwise. Therefore these extensions become non-trivial through contractions and are called pseudoextensions [16, 17].

- **Type III extension coefficients**: they must fulfill some additional conditions in such manner that whenever they are non-zero they are always non-trivial. Therefore these extensions cannot appear through the pseudoextension process.

In the sequel we present the complete results on the problem of finding all non-trivial central extensions for the three main families of CK algebras. We will directly discard type I extensions which are trivial for all CK algebras in each family.

### 2.1 Central extensions of the orthogonal CK algebras

The non-zero commutators of any central extension $\tilde{\sigma}_\omega(N+1)$ of the orthogonal CK algebra $so(\omega)(N+1)$ are given [8] by:

$$
\begin{align*}
[J_{ab}, J_{bc}] &= -J_{ac} \\
[J_{ab}, J_{a+b+1}] &= \omega_{ab}J_{b+b+1} + \omega_{a-b-1}a_{b+1}^F \Xi \\
[J_{ac}, J_{a+b+1}] &= \omega_{a+1,c}a_{a+1}^F \Xi \\
[J_{a+a+1}, J_{c+c+1}] &= \beta_{a+1,c+1}^F \Xi \\
[J_{a+a+2}, J_{a+1+a+3}] &= -\omega_{a+2}^2\beta_{a+1,a+2}^{a+3} \Xi.
\end{align*}
$$

The extension is completely characterized by the following extension coefficients:

- Two type II coefficients, $\alpha_{01}^F, \alpha_{N-1}^F$. The extension $\alpha_{01}^F$ (resp. $\alpha_{N-1}^F$) is non-trivial if $\omega_2 = 0$ (resp. $\omega_{N-1} = 0$) and is trivial otherwise.

- $(N-2)$ type II pairs, $\alpha_{a+1+a+2}^F, \alpha_{a+1+a+2}^L$ ($a = 0, 1, \ldots, N-3$), fulfilling

$$
\omega_{a+3}^2\alpha_{a+1+a+2}^F = \omega_{a+1}^2\alpha_{a+1+a+2}^L.
$$

The two extensions $\alpha_{a+1+a+2}^F, \alpha_{a+1+a+2}^L$ are both non-trivial when $\omega_{a+1} = 0$ and $\omega_{a+3} = 0$, and both are simultaneously trivial otherwise.

- $(N-1)(N-2)/2$ type III coefficients $\beta_{a+1+d+1}$ ($b = 0, 1, \ldots, N-3$ and $d = b+2, \ldots, N-1$) which must satisfy the following additional relations:

$$
\begin{align*}
\text{If } d &= b+2 : \quad \omega\beta_{b+1+b+3} = 0 \quad &\text{for } \omega = \omega_b, \omega_{b+1}\omega_{b+2}, \omega_{b+2}\omega_{b+3}, \omega_{b+4} \\
\text{If } d &> b+2 : \quad \omega\beta_{b+1+d+1} = 0 \quad &\text{for } \omega = \omega_b, \omega_{b+1}, \omega_d, \omega_{d+2}.
\end{align*}
$$

(2.7)
where it is understood that conditions as \( \omega_0 \beta = 0 \) or \( \omega_{N+1} \beta = 0 \) are not present. Whenever the extension \( \beta_{b+1,d+1} \) is non-zero it is always non-trivial.

The pseudoextension character of type II coefficients \( \alpha^{F}_{a+1,a+2} \), \( \alpha^{L}_{a+1,a+2} \) means that if \( \omega_{a+1} \neq 0 \) and \( \omega_{a+3} \neq 0 \) both can be simultaneously removed by applying a redefinition of the generator \( J_{a+1,a+2} \):

\[
J_{a+1,a+2} \rightarrow J'_{a+1,a+2} = J_{a+1,a+2} + \frac{\alpha^{F}_{a+1,a+2} \omega_{a+3}}{\omega_{a+1}} = J_{a+1,a+2} + \frac{\alpha^{F}_{a+1,a+2}}{\omega_{a+3}} \Xi. \tag{2.8}
\]

The equality is guaranteed by the condition (2.6). Thus in this case both extensions are trivial. If we perform, for instance, the contraction \( \omega_{a+1} \rightarrow 0 \), the relation (2.6) implies that \( \omega_{a+3} \alpha^{F}_{a+1,a+2} = 0 \), so if \( \omega_{a+3} \neq 0 \) the extension \( \alpha^{F}_{a+1,a+2} \) vanishes, while \( \alpha^{L}_{a+1,a+2} \) can be eliminated by using (2.8). Therefore both extensions become non-trivial only through the two contractions \( \omega_{a+1} = 0 \) and \( \omega_{a+3} = 0 \). The same happens for the remaining type II extensions \( \alpha^{L}_{01} \) and \( \alpha^{F}_{N-1,N} \) with respect to \( \omega_2 \) and \( \omega_{N-1} \), respectively. We remark that type III extensions do not appear under such process.

Let us illustrate now these results with some interesting algebras [8]:

1. \( \text{so}(p,q) \) with \( p + q = N + 1 \) (all \( \omega_a \neq 0 \)).

All type III coefficients vanish due to (2.7), and all type II are trivial as they can be removed by means of redefinitions (2.8). Therefore, as it is well known (Whitehead’s lemma) these algebras have no non-trivial extensions: \( \dim (H^2(\text{so}(p,q), \mathbb{R}) = 0. \)

2. \( \text{iso}(p,q) \) with \( p + q = N \) (\( \omega_1 = 0 \) and the others are non-zero).

These algebras include the Euclidean \( \text{iso}(N) \) and Poincaré \( \text{iso}(N-1,1) \) ones. The case \( N = 2 \) is special since \( \omega_1 = \omega_{N-1} = 0 \), so that there is single non-trivial extension \( \alpha^{L}_{N-1,N} = \alpha^{F}_{12} \) and \( \dim (H^2(\text{iso}(p,q), \mathbb{R})) = 1 \); this extension corresponds to a uniform and constant force field in the \( 1 + 1 \) Minkowskian free kinematics for the Poincaré algebra \( \text{iso}(1,1) \). However this extension is no longer possible if \( N > 2 \) (\( 0 = \omega_1 \neq \omega_{N-1} \)) and there are no non-trivial type II extensions; furthermore it can be checked that all type III ones vanish so that \( \dim (H^2(\text{iso}(p,q), \mathbb{R})) = 0. \)

3. \( \text{iiiso}(p,q) \) with \( p + q = N - 1 \) (\( \omega_1 = \omega_2 = 0 \) and the others are non-zero).

We have to distinguish three cases according to the values of \( N \):

| \( N \) | Non-trivial extensions | \( \dim (H^2(\text{iiiso}(p,q), \mathbb{R})) \) |
|-------|------------------------|---------------------------------|
| 2     | \( \alpha^{L}_{01}, \alpha^{L}_{12} \) | 2                              |
| 3     | \( \alpha^{L}_{01}, \alpha^{F}_{23}, \beta_{13} \) | 3                              |
| \( > 3 \) | \( \alpha^{L}_{01} \) | 1                              |

The physical role of these extensions for the Galilean algebra \( \text{iiiso}(N-1) \) is as follows: the only non-trivial coefficient which appears for any \( N \), \( \alpha^{L}_{01} \), is the mass; \( \alpha^{F}_{12} \) is a constant force for \( \text{iiiso}(1) \); \( \alpha^{F}_{23} \) is a sort of non-relativistic remainder of the Thomas precession for \( \text{iiiso}(2) \) [8] and \( \beta_{13} \) has no physical meaning and disappears once we move from the Galilean algebra to the corresponding Lie group.

7
This is the most contracted CK algebra: the orthogonal flag algebra. All the conditions (2.6) and (2.7) are fulfilled and all possible extensions are non-trivial, that is, there are \(2(N - 1)\) type II and \((N - 1)(N - 2)/2\) type III non-trivial extensions.

### 2.2 Central extensions of the unitary CK algebras

The commutation relations of any central extension \(\bar{su}_\omega(N+1)\) of the special unitary CK algebra \(su_\omega(N+1)\) can be written [3] as the commutation relations (1.8) together with:

\[
\begin{align*}
[J_{ab}, M_{ab}] &= -2\omega_{ab} \sum_{s=a+1}^{b} B_s + \sum_{s=a+1}^{b} \omega_{s-1, s-1}\omega_{ab} \alpha_s \Xi \\
[B_k, B_l] &= \beta_{kl} \Xi \quad k < l
\end{align*}
\]

which will replace those in (1.3). The possible extension coefficients are

- \(N\) type II coefficients \(\alpha_k\) \((k = 1, \ldots, N)\). Each of them gives rise to a non-trivial extension if \(\omega_k = 0\) and to a trivial one otherwise.
- \(N(N - 1)/2\) type III coefficients \(\beta_{kl}\) \((k < l \text{ and } k, l = 1, \ldots, N)\), satisfying

\[
\omega_k \beta_{kl} = 0 \quad \omega_l \beta_{kl} = 0.
\]

Thus, \(\beta_{kl}\) vanishes when at least one of the parameters \(\omega_k, \omega_l\) is different from zero. When \(\beta_{kl}\) is non-zero it is always non-trivial.

On the other hand, the Lie brackets of any central extension \(\bar{u}_\omega(N+1)\) of the unitary CK algebra \(u_\omega(N+1)\) are given [9] by (1.8), (2.9) and:

\[
\begin{align*}
[J_{ab}, I] &= 0 \\
[M_{ab}, I] &= 0 \\
[B_k, I] &= \gamma_k \Xi
\end{align*}
\]

which will replace those in (1.10). The extension is completely characterized by the above extension coefficients \(\alpha_k\) and \(\beta_{kl}\) together with:

- \(N\) type III coefficients \(\gamma_k\) \((k = 1, \ldots, N)\) satisfying

\[
\omega_k \gamma_k = 0.
\]

When \(\gamma_k\) is non-zero, the extension that it determines is non-trivial.

Unlike the orthogonal CK family we can give now the following closed expressions for the dimension of the second cohomology group of these two unitary CK families which establish the number of non-trivial extension coefficients according to the number \(n\) of coefficients \(\omega_k\) which are equal to zero for a given unitary CK algebra:

\[
\dim (H^2(su_\omega(N+1), \mathbb{R})) = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}
\]
\[ \dim (H^2(u_\omega(N + 1), \mathbb{R})) = n + \frac{n(n - 1)}{2} + n = \frac{n(n + 3)}{2}. \] (2.14)

The first term \( n \) in the sum of (2.13) and (2.14) corresponds to the central extensions \( \alpha_k \), the second term \( \frac{n(n-1)}{2} \) to the \( \beta_{kl} \) and the third term \( n \) in (2.14) to the \( \gamma_k \).

As far as the special unitary CK algebras is concerned, we find that the second cohomology group is trivial for the simple \( su(p, q) \) algebras with all \( \omega_a \neq 0 \) \( (n = 0) \). The inhomogeneous \( iu(p, q) \) algebras with \( \omega_1 = 0 \) and all other constants \( \omega_a \neq 0 \) \( (n = 1) \) have, in any dimension, a single non-trivial extension: \( \alpha_1 \). The algebras with \( \omega_1 = \omega_2 = 0 \) and the remaining \( \omega_a \neq 0 \) \( (n = 2) \) have always three non-trivial extensions: \( \alpha_1, \alpha_2 \) and \( \beta_{12} \). The special unitary flag algebra with all \( \omega_a = 0 \) \( (n = N) \) has the maximum number of non-trivial extensions: \( N(N + 1)/2 \). A similar discussion can be performed for the CK family \( u_\omega(N + 1) \).

### 2.3 Central extensions of the quaternionic unitary CK algebras

The algebraic structure of the CK family \( sq_\omega(N + 1) \) seems to be much more complicated than those in the orthogonal and unitary CK families. However, the solution to the central extension problem comes as a surprise and is quite simple to state \[10\]: all central extensions of any Lie algebra belonging to the quaternionic unitary CK family \( sq_\omega(N + 1) \) are always trivial for any \( N \) and for any values of the set of constants \( \omega_1, \omega_2, \ldots, \omega_N \):

\[ \dim (H^2(sq_\omega(N + 1), \mathbb{R})) = 0. \] (2.15)

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