Superconformal operators in Yang-Mills theories on the light-cone

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Abstract

We employ the light-cone superspace formalism to develop an efficient approach to constructing superconformal operators of twist two in Yang-Mills theories with $\mathcal{N} = 1, 2, 4$ supercharges. These operators have an autonomous scale dependence to one-loop order and determine the eigenfunctions of the dilatation operator in the underlying gauge theory. We demonstrate that for arbitrary $\mathcal{N}$ the superconformal operators are given by remarkably simple, universal expressions involving the light-cone superfields. When written in components field, they coincide with the known results obtained by conventional techniques.

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1. Introduction

The Operator Product Expansion (OPE) is a powerful tool in quantum field theories whose applications range from the second order phase transitions in condensed matter physics to deep-inelastic scattering in QCD. It allows one to expand a product of local operators over a set of composite Wilson operators of increasing twist (= canonical dimension minus Lorentz spin) and express correlation functions in terms of the structure constants of the corresponding operator algebra. In an interacting theory, the Wilson operators mix under renormalization and acquire nontrivial anomalous dimensions. Construction of the Wilson operators possessing an autonomous scale dependence and finding the spectrum of their anomalous dimensions both at weak and strong coupling is an important problem in four-dimensional gauge theories. At weak coupling, it can be solved by calculating the corresponding mixing matrices order by order in the coupling constant and diagonalizing them perturbatively. At strong coupling, the problem awaits its solution. The maximally supersymmetric $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory takes an exceptional place among all gauge theories from this standpoint, since the gauge/string correspondence—a strong/weak duality between gauge and string theories—allows one to map Wilson operators into certain states of the string on the $AdS_5 \times S^5$ background and identify the anomalous dimension of the former as the energy of the latter (for a comprehensive review, see [3]).

In this paper we address the problem of constructing Wilson operators having an autonomous scale dependence in (super) Yang-Mills theory with an arbitrary number of supercharges $0 \leq \mathcal{N} \leq 4$. We restrict the analysis to the leading, twist-two operators. These operators possess a two-particle structure and can be obtained by expanding a nonlocal light-cone operator in the Taylor series, symbolically

$$\phi_1(z_1 n_\mu) \phi_2(z_2 n_\mu) = \sum_{k \geq 0} \frac{z_{21}^k}{k!} O_k(z_1 n_\mu),$$

where $O_k(0) = \phi_1(0) \partial_+^k \phi_2(0)$ is a local composite operator, $z_{21} \equiv z_2 - z_1$ and $\partial_+ \equiv n \cdot \partial$ is the light-cone derivative. Here $n_\mu$ is a light-like vector such that $n_\mu^2 = 0$ and the scalar variables $z_j$ (with $j = 1, 2$) define the position of a generic field $\phi_j(z_j n_\mu)$ on the light-cone. In Eq. (1.1), it is tacitly assumed that the gauge invariance is restored by inserting a gauge link between the two fields in the left-hand side and replacing the ordinary derivatives by the covariant ones in the right-hand side. The operators $O_k(0)$ mix under renormalization with operators of the same canonical dimension involving total derivatives $\partial_+^\ell O_{k-\ell}(0)$ as well as with other twist-two operators having a different “partonic” content.

It is well-known that to one-loop order the form of multiplicatively renormalizable operators is constrained by conformal invariance of the classical (super) Yang-Mills theory (for a review, see [4]). Although the conformal symmetry is broken on the quantum level (except for $\mathcal{N} = 4$ SYM) this affects the mixing matrix of Wilson operators starting from two loops only [5]. As a result, the one-loop mixing matrix inherits the symmetry of the classical Lagrangian and its eigenstates $O_N(0)$ can be classified according to representations of the (super)conformal $SU(2, 2|\mathcal{N})$ group and, more precisely, to its collinear subgroup $SL(2|\mathcal{N})$ acting on the fields “living” on the light-cone.

For instance, in the $\mathcal{N} = 0$ theory (that is, pure gluodynamics), the twist-two (parity-even)
operators are built from gauge fields only and have the form \[ 1.2 \]

\[ \mathcal{O}_N(x) = (i\partial_+)^N \text{tr} \left[ F_{+\mu}(x) C_N^{5/2} \left( \frac{\overrightarrow{D}_+ - \overleftarrow{D}_+}{\partial_+ + \partial_+} \right) F_{\mu}^+(x) \right], \]

where \( C_N^{5/2}(\xi) \) is the Gegenbauer polynomial and \( \partial_+ = \overrightarrow{\partial}_+ + \overleftarrow{\partial}_+ \) is the total derivative. Also, the arrows indicate the fields to which the derivatives are applied. The subscripts ‘+’ on symbols exhibit the projection of the corresponding Lorentz indices onto the light-cone, e.g., for the fields strength \( F_{+\mu}(x) = n^\nu F^a_{\mu\nu}(x) t^a \), and the covariant derivative, \( D_+ = n^\mu D_\mu \). The composite operator (1.2) is transformed under the conformal \( SL(2) \) transformations as a primary field with the conformal spin \( j = N + 3 \). It can be written as \( \mathcal{O}_N(x) \sim O_N + \sum_{k \geq 1} c_k \partial_+^k O_{N-k} \) with \( O_N = F_{+\mu}(x) \partial_+^N F_{+\mu}^+(x) \) and the coefficients \( c_k \) uniquely fixed by the conformal symmetry. Conformal invariance ensures that the operator (1.2) diagonalizes the one-loop dilatation operator of the \( \mathcal{N} = 0 \) theory and, therefore, \( \mathcal{O}_N(x) \) has an autonomous scale dependence. The corresponding anomalous dimension depends on the conformal \( SL(2) \) spin of the operator (1.2), but the explicit form of this dependence is not fixed by the conformal symmetry.

In supersymmetric Yang-Mills theories with \( \mathcal{N} \geq 1 \) supercharges, the conformal operators (1.2) do not have an autonomous scale dependence since they mix under renormalization with similar twist-two operators of the same conformal spin but built from the gauge field superpartners—gauginos and scalars. It is a linear combination of the \( SL(2) \) conformal operators that diagonalizes the mixing matrix and has the property of multiplicative renormalizability. To one-loop order, the resulting twist-two operators belong to an irreducible representation of the \( SL(2|\mathcal{N}) \) group and carry a definite value of the superconformal spin. We shall refer to them as superconformal operators. As before, the \( SL(2|\mathcal{N}) \) invariance allows one to determine the explicit form of these operators without going through diagrammatical calculation of the mixing matrix but it does not fix the dependence of their anomalous dimensions on the superconformal spin.

One approach to constructing the superconformal operators in the SYM theory proposed in Ref. [7] consists in examining the properties of conformal operators, like (1.2), under supersymmetric transformations belonging to the \( SL(2|\mathcal{N}) \) group. Repeatedly applying supersymmetric variations to (1.2), one can construct a supermultiplet of twist-two operators defining an irreducible representation of the \( SL(2|\mathcal{N}) \) group. The operators entering the supermultiplet diagonalize the one-loop dilatation operator of SYM theory and have the same anomalous dimension. Although the above procedure is straightforward, its implementation in SYM theory with \( \mathcal{N} > 1 \) supercharges becomes extremely cumbersome due to a large number of operators involved and growing size of supermultiplets. The resulting expressions for the superconformal operators look differently for different \( \mathcal{N} \) and do not exhibit any universal structure.

In this paper we propose another approach to constructing the superconformal operators which allows one to treat simultaneously SYM theories with an arbitrary number of supercharges \( \mathcal{N} \). It relies on the light-cone formulation of SYM theory [8, 9] and takes full advantage of the \( SL(2|\mathcal{N}) \) superconformal group. A detailed account on this formulation can be found in Ref. [10] and we summarize here its main features. Quantizing the gauge theory in the light-cone gauge \( n \cdot A^\mu(x) = A^\mu_\perp(x) = 0 \), one can use the equations of motion in order to eliminate dynamically dependent fields and reformulate the SYM theory on the light-cone in terms of propagating fields only. In case of the maximally supersymmetric, \( \mathcal{N} = 4 \) SYM they include transverse components of the gauge field \( A_\perp(x) = (A(x), \bar{A}(x)) \) of helicity \( \pm 1 \), eight gaugino fields \( \lambda^A(x) \) and \( \bar{\lambda}_A(x) \) of
helicity $\pm \frac{1}{2}$ and six scalars $\tilde{\phi}_{AB}(x)$ (with $A,B = 1,\ldots,4$). Introducing the fermionic coordinates $\theta^A$ one can rewrite the Lagrangian of the theory in terms of two distinct chiral superfields $\Phi(x^\mu, \theta^A)$ and $\Psi(x^\mu, \theta^A)$ (with $A = 1,\ldots,N$) which comprise all dynamically independent propagating fields. The latter are the coefficients in the Taylor expansion of the superfields in powers of the Grassmann (or odd) coordinates $\theta^A$. The explicit expressions for the chiral superfields are

$$\Phi(x) = \partial_+^{-1} A(x), \quad \Psi(x) = -\partial_+ \bar{A}(x) \quad (1.3)$$

for $N = 0$, \n
$$\Phi(x, \theta) = \partial_+^{-1} A(x) + \theta \partial_+^{-1} \bar{\lambda}(x), \quad \Psi(x, \theta) = -\lambda(x) + \theta \partial_+ \bar{A}(x) \quad (1.4)$$

for $N = 1$, \n
$$\Phi(x, \theta^A) = \partial_+^{-1} A(x) + \theta^A \partial_+^{-1} \bar{\lambda}_A(x) + \frac{i}{2!} \epsilon_{AB} \theta^B \phi(x),$$

$$\Psi(x, \theta^A) = i \phi(x) - \epsilon_{AB} \theta^B \bar{\lambda}(x) + \frac{1}{2!} \epsilon_{AB} \theta^B \partial_+ \bar{A}(x) \quad (1.5)$$

for $N = 2$, and \n
$$\Phi(x, \theta^A) = \Psi(x, \theta^A) = \partial_+^{-1} A(x) + \theta^A \partial_+^{-1} \bar{\lambda}_A(x) + \frac{i}{2!} \theta^A \theta^B \phi_{AB}(x)$$

$$+ \frac{1}{3!} \epsilon_{ABCD} \theta^A \theta^B \theta^C \theta^D \bar{\lambda}(x) - \frac{1}{4!} \epsilon_{ABCD} \theta^A \theta^B \theta^C \theta^D \partial_+ \bar{A}(x) \quad (1.6)$$

for $N = 4$. In the latter case, the two superfields coincide since each of them comprises all propagating modes. This is contrasted to $0 \leq N \leq 2$ models where only half of the propagating fields can be accommodated into either of the two superfields.

Similar to (1.1), all possible twist-two operators in the $N-$extended SYM theories can be obtained by expanding the $\Psi \Psi$, $\Phi \Phi$ and $\Phi \Psi$—products of two superfields located on the light-cone $x^\mu = zn^\mu$. Let us define $Z = (z, \theta^A)$ (with $A = 1,\ldots,N$) as a point in the $(N + 1)-$dimensional light-cone superspace and use a shorthand notation $\mathcal{O}(Z_1, Z_2)$ for the superfield bilinears $\text{tr}[\Psi(Z_1)\Psi(Z_2)], \text{tr}[\Phi(Z_1)\Phi(Z_2)]$ and $\text{tr}[\Phi(Z_1)\Psi(Z_2)]$. Then, the generating function for twist-two operators in $N-$extended SYM theories looks like

$$\mathcal{O}(Z_1, Z_2) = \sum_{N \geq 0} \frac{z_1^N}{N!} \sum_{n,m=0}^N \theta_1^{A_1} \ldots \theta_1^{A_n} \theta_2^{B_1} \ldots \theta_2^{B_m} \mathcal{O}_{N;A_1,\ldots,A_n;B_1,\ldots,B_n}(z_1 n_\mu). \quad (1.7)$$

Here the expansion in the right-hand side involves all possible powers of odd variables $\theta_1^{A_1}$ and the expansion coefficients define the composite twist-two operators $\mathcal{O}_{N;\{A_1,\ldots,B_2\}}(x)$.

The twist-two operators in the right-hand side of (1.7) mix under renormalization and their anomalous dimensions can be extracted from the renormalization group equation for the nonlocal light-cone superfield operator $\mathcal{O}(Z_1, Z_2)$ [10]

$$\left\{ \mu \frac{\partial}{\partial \mu} + \beta_N(g) \frac{\partial}{\partial g} + 2\gamma_N(g) \right\} \mathcal{O}(Z_1, Z_2) = -\frac{g^2 N_c}{8\pi^2} [\mathcal{H} \cdot \mathcal{O}] (Z_1, Z_2) + \mathcal{O}(g^4). \quad (1.8)$$

Here $\beta_N(g)$ is the beta-function in the SYM theory and $\gamma_N(g)$ is the anomalous dimension of the superfields. In the light-like axial gauge $A_+(x) = 0$ one has $\gamma_N(g) = \beta_N(g)/g$. The integral
operator $\mathbb{H}$ defines a representation of the one-loop dilatation operator of SYM theory on the space spanned by the operators $O(Z_1, Z_2)$. Its explicit form has been found in Ref. [10] for arbitrary number of supercharges $0 \leq \mathcal{N} \leq 4$. To solve the evolution equation (1.8) it suffices to solve the spectral problem for the operator $\mathbb{H}$. Its eigenvalues, $E(j)$, provide the spectrum of one-loop anomalous dimensions of multiplicatively renormalizable Wilson operators in the $\Psi\Psi-$, $\Phi\Phi-$ and $\Phi\Psi-$sectors, while the corresponding eigenstates determine their explicit form.

It turns out that the anomalous dimensions of twist-two operators $O_n(0)$ in $\mathcal{N}$-extended SYM theories have a universal form

$$\gamma = \frac{g^2 N_c}{8\pi^2} \left( E + 2\gamma^{(0)}_{\mathcal{N}} \right) + \mathcal{O}(g^4), \quad (1.9)$$

where $\gamma^{(0)}_{\mathcal{N}}$ is the one-loop correction to the anomalous dimension of the superfield, $\gamma^{(0)}_{\mathcal{N}} = -11/6, -3/2, -1, 0$ for $\mathcal{N} = 0, 1, 2, 4$, respectively, and $E$ is the eigenvalue of the one-loop dilatation operator $\mathbb{H}$. Depending on the sector to which the Wilson operator belongs it is given by

- $\Psi\Psi-$ and $\Phi\Phi-$sectors
  $$E_{\Psi\Psi}(j) = E_{\Phi\Phi}(j) = 2 \left[ \psi(j) - \psi(1) \right], \quad (1.10)$$

- $\Phi\Psi-$sector
  $$E_{\Phi\Psi}(j) = \psi(j + 2 - \mathcal{N}/2) + \psi(j - 2 + \mathcal{N}/2) - 2\psi(1)$$
  $$+ (-1)^{j+\mathcal{N}/2} \frac{\Gamma(j - 2 + \mathcal{N}/2)}{\Gamma(j + 2 - \mathcal{N}/2)} \Gamma(4 - \mathcal{N}), \quad (1.11)$$

where $j$ is the superconformal $SL(2|\mathcal{N})$ spin of the corresponding Wilson operator and $\psi(x) = d \ln \Gamma(x)/dx$ is the Euler dilogarithm. In the $\mathcal{N} = 4$ SYM all twist-two operators belong to the $\Phi\Phi-$sector and their anomalous dimension is given by (1.10). The corresponding Wilson operators have been constructed in Ref. [10] through the diagonalization of the one-loop mixing matrix. They are given by linear combinations of two-particle operators built from propagating fields $\phi_k(x) = \{ \partial_+ A, \partial_+ \bar{A}, \lambda_A, \bar{\lambda}_A, \phi_{AB} \}$ and have the following general form

$$O_n(0) = \sum_{k,j} \phi_k(0) P_n^{(k,j)}(\partial_+^+, \partial_+^-) \phi_j(0), \quad (1.12)$$

where $P_n^{(k,j)}(x_1, x_2)$ are some polynomials in light-cone derivatives with arrows indicating, as usual, where they are applied.

In the light-cone formalism, the fields $\phi_k(x)$ are the components of the light-cone superfields, Eqs. (1.3) – (1.6). This suggests that the Wilson operators (1.12) can be constructed as

$$O_n(0) = P_n(\partial_{Z_1}; \partial_{Z_2}) O(Z_1, Z_2) \bigg|_{Z_1 = Z_2 = 0}, \quad (1.13)$$

where $O(Z_1, Z_2)$ is given by the product of two superfields, Eq. (1.7), and $P_n(\partial_{Z_1}; \partial_{Z_2})$ is a polynomial in superspace derivatives $\partial_Z = (\partial_z, \partial_{\theta^A})$, cf. Refs. [11] [12]. Equation (1.13) generalizes the
expression for the conformal operators in the $\mathcal{N} = 0$ theory, Eq. (1.2), in which case the polynomial $P_n(x_1; x_2)$, depending only on the bosonic variables, is given in terms of the Gegenbauer polynomials

$$P_n(x_1; x_2)\bigg|_{\mathcal{N}=0} = (x_1 + x_2)^n C_n^{5/2} \left(\frac{x_1 - x_2}{x_1 + x_2}\right).$$  \hspace{1cm} (1.14)

In SYM theories with $\mathcal{N} \geq 1$ supercharges, the polynomial $P_n(X_1; X_2)$—depending on two bosonic and $2\mathcal{N}$ fermionic variables via $X_i = (x_i, \vartheta_{i,A})$—can be expanded in powers of Grassmann $\vartheta_{1,A_1}$ and $\vartheta_{2,A_2}$ so that the corresponding coefficients are given by the polynomials $P_n^{(k,j)}(x_1, x_2)$, Eq. (1.12). We shall argue that the explicit form of these polynomials is uniquely determined by the superconformal $SL(2|\mathcal{N})$ symmetry. Namely, the superfields $\Phi(Z)$ and $\Psi(Z)$ are transformed under superconformal $SL(2|\mathcal{N})$ transformations according to irreducible representations of the $SL(2|\mathcal{N})$ group specified by their superconformal spins $j_\Phi = -1/2$ and $j_\Psi = (3 - \mathcal{N})/2$, respectively. As a result, the nonlocal light-cone operator $\mathcal{O}(Z_1, Z_2)$, Eq. (1.7), belongs to the tensor product of two $SL(2|\mathcal{N})$ representations. The polynomial $P_n(X_1; X_2)$ is fixed by the condition that the operator $P_n(\partial Z_1; \partial Z_2)$ has to project this tensor product onto one of its irreducible components. Then, the mixing of the local twist-two operators (1.13) with other twist-two operators is protected (to one-loop order at least) by the superconformal symmetry.

The outline of the paper is the following. In Sect. 2 we illustrate the formalism of constructing the superconformal operators on a simple example of conventional $SL(2)$ operators. We briefly review representations of the underlying collinear conformal group and establish the relation between the lowest weights in the tensor product of two $SL(2)$ modules and the twist-two conformal polynomials in the $\mathcal{N} = 0$ theory. In Sect. 3 we employ the light-cone superspace formalism and extend our consideration to supersymmetric Yang-Mills theories. We demonstrate that the above mentioned relation between the superconformal operators and the lowest weights of the $SL(2|\mathcal{N})$ group is universal and it holds true for an arbitrary number of supercharges $\mathcal{N}$. Next, in Sect. 4 we use this construction to work out the explicit expressions for the superconformal operators in the $\mathcal{N} = 1$, $\mathcal{N} = 2$, and $\mathcal{N} = 4$ SYM theories. Sect. 5 contains concluding remarks. Two appendices give details on Jacobi polynomials and the $N ≥ 1$ spinors. Then, the mixing of the local twist-two operators (1.13) with other twist-two operators is protected (to one-loop order at least) by the superconformal symmetry.

2. Conformal operators

To illustrate our approach, we first revisit the construction of the $SL(2)$ conformal operators, Eq. (1.2). We recall that twist-two operators $\mathcal{O}_{\mu_1,\ldots,\mu_6}(0)$ are composite gauge-invariant operators built from covariant derivatives $D_\mu$ and fundamental fields in (super) Yang-Mills theory $\phi_k(x) = \phi_k^a(x)\epsilon^a$, that are symmetric and traceless in any pair of Lorentz indices. They possess the Lorentz spin $N$ and the canonical dimension $2 + n$. It is convenient to contract all Lorentz indices with a light-like vector $n_\mu$ (such that $n^2 = 0$) and introduce

$$\mathcal{O}_n(0) \equiv n^{\mu_1} \ldots n^{\mu_6} \mathcal{O}_{\mu_1,\ldots,\mu_6}(0) = \sum_{k_1+k_2=n} c_{k_1k_2} \text{tr} \left[ D_+^{k_1} \phi_1(0) D_+^{k_2} \phi_2(0) \right].$$  \hspace{1cm} (2.1)

Here the local composite operators $\text{tr} \left[ D_+^{k_1} \phi_1(0) D_+^{k_2} \phi_2(0) \right]$ define a basis in the space of twist-two operators. Throughout the paper we adopt the light-cone gauge $n \cdot A(x) = A_+(x) = 0$ so that all covariant derivatives are reduced to the ordinary ones, $D_+ \equiv n \cdot D = \partial_+$. The elementary fields entering (2.1) are $\phi_k = \{\partial_+ A, \partial_+ \bar{A}, \lambda^A, \bar{\lambda}_A, \phi^{AB}\}$. 

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The twist-two operators \( \mathcal{O}_n(0) \) are uniquely specified by the set of coefficients \( c_{k_1k_2} \). They are fixed from the condition that the operators \( \mathcal{O}_n(0) \) enjoy an autonomous scale dependence, to one-loop order at least,

\[
\mu \frac{d}{d\mu} \mathcal{O}_n(0) = -\frac{g^2 N_c}{8\pi^2} \gamma^{(0)}(n) \mathcal{O}_n(0), \quad (2.2)
\]

with \( \gamma^{(0)}(n) \) being the one-loop anomalous dimension of the Wilson operator. Here we assumed for simplicity that the set of the twist-two operators \( \text{tr} \left[ D_+^{k_1} \phi_1(0) D_+^{k_2} \phi_2(0) \right] \) is closed under renormalization.

### 2.1. Conformal polynomials

Let us put into correspondence to the operator \( \mathcal{O}_n(0) \) the following homogeneous polynomial in two variables

\[
P_n(x_1, x_2) = \sum_{k_1+k_2=n} c_{k_1k_2} x_1^{k_1} x_2^{k_2}. \quad (2.3)
\]

Then, the local operator \( \mathcal{O}_n(0) \) can be obtained from \( \mathcal{O}(z_1, z_2) = \text{tr} \left[ \phi_1(z_1n_\mu) \phi_2(z_2n_\mu) \right] \) as a projection

\[
\mathcal{O}_n(0) = P_n(\partial_{z_1}, \partial_{z_2}) \mathcal{O}(z_1, z_2) \bigg|_{z_1=z_2=0}, \quad (2.4)
\]

which establishes a correspondence between local and nonlocal operators.

To determine the polynomials \( P_n(x_1, x_2) \) we require that the operators \( \mathcal{O}_n(0) \) have to satisfy \( (2.2) \). In general, for a given \( n \) one expects to find a few such operators. To enumerate them we introduce a superscript \( (\ell) \) and denote the corresponding operators and polynomials as \( \mathcal{O}_n^{(\ell)}(0) \) and \( P_n^{(\ell)}(x_1, x_2) \), respectively. Then, assuming that the set of the twist-two operators \( \mathcal{O}_n^{(\ell)}(0) \) is complete, the relation \( (2.4) \) can be inverted as

\[
\mathcal{O}(z_1, z_2) = \sum_{n,\ell} \Psi^{(\ell)}_n(z_1, z_2) \mathcal{O}_n^{(\ell)}(0), \quad (2.5)
\]

where \( \Psi^{(\ell)}_n(z_1, z_2) \) is yet another set of homogeneous polynomials of degree \( n \). Substituting \( (2.5) \) into \( (2.3) \) one finds that the two sets of polynomials are related to each other as

\[
P_n^{(\ell)}(\partial_{z_1}, \partial_{z_2}) \Psi^{(\ell)}_m(z_1, z_2) \bigg|_{z_1=z_2=0} = \delta_{nm} \delta_{\ell\ell'}. \quad (2.6)
\]

The polynomials \( \Psi^{(\ell)}_m(z_1, z_2) \) admit a power series representation similar to \( (2.3) \) with the expansion coefficients related to \( c_{k_1k_2} \) through the orthogonality condition \( (2.4) \).

The Wilson operators \( \mathcal{O}_n(0) \) are uniquely determined by the polynomials \( P_n^{(\ell)}(x_1, x_2) \) and \( \Psi^{(\ell)}_n(z_1, z_2) \). The former polynomial projects the local twist-two operator out from the nonlocal light-cone operator, Eq. \( (2.1) \), while the latter defines the coefficient function of \( \mathcal{O}_n^{(\ell)}(0) \) in the OPE expansion of the nonlocal light-cone operator, Eq. \( (2.5) \). It is easy to demonstrate that the renormalization group equation for the local operators \( (2.2) \) leads to a Schrödinger-like equation for the polynomials \( \Psi^{(\ell)}_n(z_1, z_2) \). To one-loop order, the nonlocal operator \( \mathcal{O}(z_1, z_2) \) satisfies the Callan-Symanzik equation \( (1.8) \)

\[
\mu \frac{d}{d\mu} \mathcal{O}(z_1, z_2) = -\frac{g^2 N_c}{8\pi^2} \left[ (H + 2\gamma^{(0)}_N) \cdot \mathcal{O} \right](z_1, z_2) + \mathcal{O}(g^4), \quad (2.7)
\]
with \( \mathbb{H} \) being an integral operator acting on the light-cone coordinates of fields \([13, 10]\). Substituting (2.5) into (2.7) and taking into account (2.2) one obtains

\[
[H \cdot \Psi_n](z_1, z_2) = E(n)\Psi_n(z_1, z_2),
\]

(2.8)

with \( \gamma^{(0)}(n) = E(n) + 2\gamma^{(0)}_N \). As was already mentioned, the one-loop evolution operator \( \mathbb{H} \) inherits the conformal symmetry of the classical Lagrangian of the gauge theory and, therefore, its eigenfunctions \( \Psi_n(z_1, z_2) \) can be classified according to representations of the \( SL(2) \) conformal group.

### 2.2. \( SL(2) \) conformal symmetry

It is well-known that for nonlocal light-cone operators \( O(z_1, z_2) = \text{tr} [\phi_1(z_1n_\mu)\phi_2(z_2n_\mu)] \) the full \( SO(4,2) \) conformal group is reduced to its “collinear” \( SL(2) \) subgroup. The latter acts on the light-ray \( x_\mu = zn_\mu \) as follows,

\[
z \rightarrow z' = \frac{az + b}{cz + d}, \quad \phi_k(zn_\mu) \rightarrow \phi'_k(zn_\mu) = (cz + d)^{-2j_k} \phi_k \left( \frac{az + b}{cz + d} n_\mu \right),
\]

(2.9)

with \( ad - bc = 1 \) and \( j_k \) being the conformal spin of the field \( \phi_k(zn_\mu) \). For the gauge fields \((\partial_+ A, \partial_+ \bar{A})\), gauginos \((\lambda^i, \bar{\lambda}_A)\) and scalars \((\phi^{AB})\) it takes the following values

\[
j_{\text{gauge}} = \frac{3}{2}, \quad j_{\text{gaugino}} = 1, \quad j_{\text{scalar}} = \frac{1}{2}.
\]

(2.10)

The field \( \phi_k(zn_\mu) \) defines an \( SL(2) \) representation that we shall denote as \( V_{j_k} \). The representation space is spanned by polynomials \( \{1, z_k, z_k^2, \ldots\} \in V_{j_k} \) which define the coefficient functions in the expansion of the field \( \phi_k(z_kn_\mu) \) around the origin

\[
\phi_k(z_kn_\mu) = \phi_k(0) + z_k \partial_+ \phi_k(0) + \frac{z_k^2}{2} \partial_+^2 \phi_k(0) + \ldots.
\]

(2.11)

The generators of the \( SL(2) \) transformations (2.9) can be realized as differential operators acting on the light-cone coordinates of the fields \( \phi_k(z_kn_\mu) \)

\[
L_{+ (k)} = z_k^2 \partial_{z_k} + 2j_k z_k, \quad L_{- (k)} = -\partial_{z_k}, \quad L_{0 (k)} = z_k \partial_{z_k} + j_k,
\]

(2.12)

with \( j_k \) defined in (2.10).

The conformal invariance implies that the evolution operator \( \mathbb{H} \) commutes with the two-particle \( SL(2) \) generators, \( [\mathbb{H}, L_\alpha] = 0 \) with \( \alpha = \pm, 0 \) and \( L_\alpha = L_{\alpha(1)} + L_{\alpha(2)} \). Therefore, \( \mathbb{H} \) is a function of the two-particle Casimir operator

\[
L^2 = L_{0}^2 + (L_+ L_- + L_- L_+)/2
\]

(2.13)

and its eigenstates belong to the irreducible components in the tensor product of two \( SL(2) \) modules

\[
V_{j_1} \otimes V_{j_2} = \sum_{n \geq 0} V_{j_1 + j_2 + n},
\]

(2.14)
with the space $\mathbb{V}_{j_1+j_2+n}$ spanned by the states

$$
\Psi_n^{(0)}(z_1, z_2) = (z_1 - z_2)^n, \\
\Psi_n^{(\ell)}(z_1, z_2) = (L_+)^{\ell}\Psi_n^{(0)}(z_1, z_2),
$$

(2.15)

with $\ell \geq 1$. Here $\Psi_n^{(0)}(z_1, z_2)$ is the lowest weight, $L_-\Psi_n^{(0)}(z_1, z_2) = 0$, and $\Psi_n^{(\ell)}(z_1, z_2)$ being homogeneous polynomials in $z_1$ and $z_2$ of degree $n + \ell$. \ The polynomials (2.15) diagonalize the Casimir operator

$$
L^2 \Psi_n^{(\ell)}(z_1, z_2) = (n + j_1 + j_2)(n + j_1 + j_2 - 1)\Psi_n^{(\ell)}(z_1, z_2)
$$

(2.16)

and define the eigenstates of the evolution kernel $\mathbb{H}$, Eq. (2.8). Its eigenvalues do not depend on $\ell$ and determine the one-loop anomalous dimension $\gamma^{(0)}(n)$.

Let us return to (2.5) and substitute the polynomials $\Psi_n^{(\ell)}(z_1, z_2)$ by their expressions (2.15). By construction, the corresponding operators $\mathcal{O}_n^{(\ell)}(0)$ are multiplicatively renormalizable to one-loop order and satisfy the evolution equation (2.2). To determine their explicit form we shall make use of the fact that the polynomials $\Psi_n^{(\ell)}(z_1, z_2)$, Eq. (2.15), are orthogonal with respect to the $SL(2)$ invariant scalar product.

Using the isomorphism $SL(2) \sim SU(1, 1)$, the scalar product between two “single particle” polynomials $\psi(z)$ and $\varphi(z)$ belonging to $\mathbb{V}_j$ can be defined as

$$
\langle \psi | \varphi \rangle = \int [Dz]_j \overline{\psi(z)} \varphi(z),
$$

(2.17)

where the integration goes over a unit disk in the complex $z-$plane with the measure

$$
\int [Dz]_j = \frac{1}{\pi \Gamma(2j - 1)} \int_{|z| \leq 1} d^2z (1 - |z|^2)^{2j-2}.
$$

(2.18)

The scalar product (2.17) is invariant under the $SL(2)$ transformations (2.9) with $c = b^*$ and $d = a^*$ so that $|a|^2 - |b|^2 = 1$. A simple calculation shows that

$$
\langle z^k | z^n \rangle = \frac{\delta_{kn} k!}{\Gamma(2j + k)}.
$$

(2.19)

For states belonging to the tensor product (2.14) the scalar product is defined as

$$
\langle \Psi | \Phi \rangle = \int [Dz_1]_{j_1} \int [Dz_2]_{j_2} \overline{\Psi(z_1, z_2)} \Phi(z_1, z_2).
$$

(2.20)

It is straightforward to verify that the $SL(2)$ generators (2.12) satisfy the relations $L_0^\dagger = L_0$ and $(L_-)^\dagger = -L_+$, so that the Casimir operator $L^2$, Eq. (2.13), is a self-adjoint operator with respect to the scalar product (2.20). Together with (2.16) this property ensures that the states (2.15) satisfy the orthogonality condition

$$
\langle \Psi_n^{(\ell)}(z_1, z_2) | \Psi_m^{(\ell')} (z_1, z_2) \rangle \sim \delta_{nm} \delta_{\ell \ell'}.
$$

(2.21)

\textsuperscript{1}Our definition of the integration measure differs from the conventional one by the factor $1/\Gamma(2j)$ which was introduced for the later convenience (see Sect. 4.2).
Let us now return to (2.5) and evaluate the scalar product of both sides of (2.5) with the same vector \( \Psi_n(z_1, z_2) \).

Taking into account (2.21), one gets the following expression for the \( SL(2) \) conformal operator

\[
O_n^{(\ell)}(0) \sim \langle \Psi_n^{(\ell)}(z_1, z_2) | O(z_1, z_2) \rangle = \int [Dz_1]_{j_1} \int [Dz_2]_{j_2} \overline{\Psi_n^{(\ell)}(z_1, z_2)} O(z_1, z_2). \tag{2.22}
\]

Substituting \( O(z_1, z_2) = e^{z_1 \partial w_1 + z_2 \partial w_2} O(w_1, w_2) \big|_{w_1, w_2 = 0} \), one can bring this relation to the form (2.4) with the \( P \)-polynomial determined by the scalar product

\[
P_n^{(\ell)}(x_1, x_2) = \langle \Psi_n^{(\ell)}(z_1, z_2) | e^{z_1 x_1 + z_2 x_2} \rangle = \langle (L_+)^{\ell} \Psi_n^{(0)}(z_1, z_2) | e^{z_1 x_1 + z_2 x_2} \rangle. \tag{2.23}
\]

Since \( (L_+)^{\dagger} = -L_- = \partial_{z_1} + \partial_{z_2} \), the polynomials with \( \ell \geq 1 \) can be expressed in terms of the one with \( \ell = 0 \),

\[
P_n^{(0)}(x_1, x_2) = \langle (z_1 - z_2)^n | e^{z_1 x_1 + z_2 x_2} \rangle = \int [Dz_1]_{j_1} \int [Dz_2]_{j_2} (z_1 - z_2)^n e^{z_1 x_1 + z_2 x_2}. \tag{2.25}
\]

Using the properties of the \( SL(2) \) scalar product (see Appendix B), the relations (2.23) – (2.25) can be inverted as

\[
\Psi_n^{(\ell)}(z_1, z_2) = P_n^{(\ell)}(\partial w_1, \partial w_2) \prod_{k=1,2} \Gamma(2j_k)(1 - w_k \bar{z}_k)^{-2j_k} |_{w_k = 0} = \int_0^\infty \prod_{k=1,2} dt_k t_k^{2j_k - 1} e^{-t_1 - t_2} P_n^{(\ell)}(t_1 \bar{z}_1, t_2 \bar{z}_2). \tag{2.26}
\]

Eqs. (2.23) – (2.26) establish the correspondence between the conformal polynomials defining the twist-two operators (2.4) and the lowest weights in the tensor product of two \( SL(2) \) modules, Eqs. (2.14) and (2.15).

According to (2.25), the polynomial \( P_n^{(0)}(x_1, x_2) \) is given by the lowest weight \( \Psi_n^{(0)}(z_1, z_2) \) transformed from the “coordinate” \( z \)-representation to the “momentum” \( x \)-representation. Making use of (2.19), the scalar product in (2.25) can be evaluated and yields

\[
P_n^{(0)}(x_1, x_2) = \sum_{k=0}^n (-1)^{n-k} {n \choose k} \langle \bar{z}_1^{n-k} \bar{z}_2^{n-k} | e^{z_1 x_1 + z_2 x_2} \rangle = \sum_{k=0}^n \frac{x_1^k (-x_2)^{n-k} {n \choose k}}{\Gamma(2j_1 + k) \Gamma(2j_2 + n - k)}. \tag{2.27}
\]

In turn, the sum can be expressed in terms of the Jacobi polynomials (see Eq. (A.11))

\[
P_n^{(0)}(x_1, x_2) = a_n^{(2j_1, 2j_2)} \cdot (x_1 + x_2)^n P_n^{(2j_1 - 1, 2j_2 - 1)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right), \tag{2.28}
\]

with \( a_n^{(2j_1, 2j_2)} = (-1)^n n! / [\Gamma(n + 2j_1) \Gamma(n + 2j_2)] \). Together with (2.4), this relation leads to the well-known expression [6] for the twist-two conformal \( SL(2) \) operator, \( O_n^{(0)}(0) \equiv O_n^{j_1, j_2}(0) \), built from fields of unequal conformal spins \( j_1 \) and \( j_2 \)

\[
O_n^{j_1, j_2}(x) = \partial^n_+ \text{tr} \left[ \phi_{j_1}(x) P_n^{(2j_1 - 1, 2j_2 - 1)} \left( \frac{-\partial_+}{\partial_+} \right) \phi_{j_2}(x) \right], \tag{2.29}
\]
which transforms under the $SL(2)$ transformations according to ${{\mathbf 2}, \mathbf 9}$ with the conformal spin $j = j_1 + j_2 + n$.

The following comments are in order.

The operator $O_N^{j_1, j_2}(x)$ was constructed from the lowest weight $P_N^{(0)}(x_1, x_2)$. The twist-two operators corresponding to the polynomials $P_N^{(\ell)}(x_1, x_2)$ with $\ell \geq 1$ differ from $O_N^{j_1, j_2}(x)$ by total derivatives $O_N^{(\ell)}(0) = \partial_+^\ell O_N^{j_1, j_2}(0)$ and, therefore, have the same anomalous dimension.

For $j = j_1 = j_2$, the Jacobi polynomials in $O_N^{j_1, j_2}(x)$ are reduced to the Gegenbauer polynomials, $P_N^{(2j-1,2j-1)}(x) \sim C_N^{2j-1/2}(x)$ (see Eq. (A.1)). In particular, in the $\mathcal{N} = 0$ theory, the twist-two operators are built from the gauge fields $\partial_+A$ and $\partial_+\bar{A}$ of helicity $\pm 1$ which carry the same conformal spin $j = 3/2$. The corresponding Wilson operators are given by $O_N^{3/2, 3/2}(x)$, Eq. (2.29), and involve the same conformal (Gegenbauer) polynomial $C_N^{3/2}(x)$ (see Eq. (1.2)). Still, the anomalous dimensions of these operators depend on the helicity alignment of gauge fields, Eqs. (1.10) and (1.11).

So far we have implicitly assumed that the operators $O_N^{j_1, j_2}(x)$ can not admix to other twist-two operators having a different “partonic” content. Indeed, in the $\mathcal{N} = 0$ theory (pure gluodynamics) the twist-two operators are built from the gauge field strengths only. They have an autonomous scale dependence simply because there are no other operators they could mix with. For $\mathcal{N} \geq 1$ this property is lost due to a growing number of constituent fields. In supersymmetric Yang-Mills theory the conformal operators $O_N^{j_1, j_2}(x)$ can be constructed from the gauge field, $\phi_{j=3/2} = (\partial_+A, \partial_+\bar{A})$, gaugino, $\phi_{j=1} = (\lambda^A, \bar{\lambda}_A)$, and scalars, $\phi_{j=1/2} = \phi^{AB}$. As a result, these operators mix under renormalization and the conformal symmetry alone does not allow one to resolve their mixing. Let us address this case next.

### 3. Superconformal operators

In SYM theory with $\mathcal{N}$ supercharges, the full $SU(2, 2|\mathcal{N})$ superconformal group reduces to its “collinear” $SL(2|\mathcal{N})$ subgroup when projected on the light-cone. This subgroup includes the $SL(2)$ transformations, Eq. (2.29), as well as supersymmetric transformations of fields and their superconformal generalizations (see Eqs. (1.2) – (1.3)). Then, the Wilson operators in SYM theory can be classified according to representations of the $SL(2|\mathcal{N})$ group. The operators defined in this way have autonomous scale dependence since their mixing with other operators is protected to one-loop order by the superconformal symmetry.

#### 3.1. Superconformal polynomials

In the light-cone superspace formalism, the elementary component fields of SYM theory arise as coefficients in the Taylor expansion of the light-cone superfields $\Phi(Z) = \Phi(zn_\mu, \theta^A)$ and $\Psi(Z) = \Psi(zn_\mu, \theta^A)$ in powers of Grassmann coordinates $\theta^A$ (with $A = 1, \ldots, \mathcal{N}$). This suggests to generalize the nonlinear light-cone operators $\mathbf{11}$ in the way of considering the products of two superfields $\text{tr} [\Phi(Z_1)\Phi(Z_2)]$, $\text{tr} [\Psi(Z_1)\Psi(Z_2)]$ and $\text{tr} [\Phi(Z_1)\Psi(Z_2)]$. Their expansion in $\theta^A_{1,2}$ and $z_{1,2}$ generates all possible twist-two operators in underlying SYM theory, Eq. (1.7), built from different constituent fields

$$\mathbf{\bigodot}(Z_1, Z_2) = \sum_{n, \ell} \Psi_n^{(\ell)}(Z_1, Z_2) O_n^{(\ell)}(0),$$

(3.1)
where the coefficient functions $\Psi_n^{(\ell)}(Z_1, Z_2)$ are polynomials in $Z_k = (z_k, \theta^\dagger_k)$ with $k = 1, 2$. Similarly to Eq. (2.3), the local operator $O_n^{(\ell)}(0)$ can be extracted from the nonlocal operator $O(Z_1, Z_2)$ with a help of a projection polynomial $P_n(X_1, X_2)$, Eq. (3.3), with $X_k = (x_k, \vartheta_{k,A})$. In comparison to the $\mathcal{N} = 0$ case, this polynomial depends on additional $2\mathcal{N}$ odd variables $\vartheta_{1,2,A}$. The operator $O_n^{(\ell)}(0)$ is given by (1.13) with the corresponding polynomial carrying two indices $n$ and $\ell$. As we will argue below, $n$ enumerates irreducible components in the tensor product of two $SL(2|\mathcal{N})$ representations and $\ell$ parameterizes the states within a given $SL(2|\mathcal{N})$ module. For $\mathcal{N} = 0$ the relation (3.1) coincides with (2.5).

As before, let us associate two sets of polynomials $\Psi_n^{(\ell)}(Z_1, Z_2)$ and $P_n^{(\ell)}(X_1, X_2)$ with a help of a projection polynomial $P_n(0)$ and $O_n^{(\ell)}(0)$, respectively. According to their definition, Eqs. (1.13) and (3.1), they satisfy the orthogonality condition analogous to Eq. (3.2).

\[ P_j^{(\ell)}(\partial Z_1; \partial Z_2) \Psi_{j'}^{(\ell)}(Z_1, Z_2) \big|_{Z_1=Z_2=0} = \delta_{jj'} \delta_{\ell\ell'} . \] (3.2)

Remarkably enough, the nonlocal operator (3.1) satisfies the renormalization group equation which is superficially identical to (2.7).

\[ \mu \frac{d}{d\mu} O(Z_1, Z_2) = -g^2 \mathcal{N} \frac{[ (\mathbb{H} + 2\gamma_n^{(0)}) \cdot O ] (Z_1, Z_2) + O(g^4) }{8\pi^2} . \] (3.3)

The important difference with the $\mathcal{N} = 0$ case is that the evolution kernel $\mathbb{H}$ acts both on the bosonic $z_{1,2}$ and fermionic $\theta^\dagger_{1,2}$ coordinates $[10]$. Substituting (3.1) into (3.3), one finds that the polynomials $\Psi_n^{(\ell)}(Z_1, Z_2)$ corresponding to the operators $O_n^{(\ell)}(0)$ have to satisfy the stationary Schrödinger equation

\[ \left[ \mathbb{H} \cdot \Psi_n^{(\ell)} \right] (Z_1, Z_2) = E(n) \Psi_n^{(\ell)}(Z_1, Z_2), \] (3.4)

with $\gamma^{(0)}(n) = E(n) + 2\gamma_n^{(0)}$. Notice that the functional dependence of anomalous dimension $\gamma^{(0)}(n)$ on $n$ differs for the $\Phi \bar{\Phi}$, $\Psi \bar{\Psi}$ and $\Phi \Psi$-sectors, Eqs. (2.6) and (1.11), respectively.

Now we are in a position to demonstrate that the eigenfunctions $\Psi_n^{(\ell)}(Z_1, Z_2)$ are uniquely determined by the superconformal $SL(2|\mathcal{N})$ symmetry of the classical SYM Lagrangian in the light-cone formalism.

### 3.2. $SL(2|\mathcal{N})$ superconformal symmetry

The light-cone superfields $\Phi(Z)$ and $\Psi(Z)$ define representations of the $SL(2|\mathcal{N})$ group that we shall denote as $\Psi_{j\dot{A}}$ and $\Psi_{j\dot{A}}$, respectively. In the light-cone formalism, the superfields are transformed linearly under the superconformal $SL(2|\mathcal{N})$ transformations

\[ \delta_G \Phi_j(Z) = G \Phi_j(Z) , \] (3.5)

with the $SL(2|\mathcal{N})$ generators $G = \{ L^\pm, L^0, W^A, V_A^+, B, T_B^A \}$ being the differential operators acting on the coordinates of superfields as

\[ L^- = -\partial_z , \quad L^+ = 2jz + z^2 \partial_z + z (\theta \cdot \partial_\theta) , \quad L^0 = jz \partial_z + \frac{1}{2} (\theta \cdot \partial_\theta) , \]

\[ W^A = \theta^\dagger \partial_z , \quad W^A = \theta^A [ 2jz + z \partial_z + (\theta \cdot \partial_\theta) ] , \quad V_A^- = \partial_\theta^A , \quad V_A^+ = z \theta_\theta^A , \]

\[ T_B^A = \theta^A \partial_\theta^B - \frac{1}{N} \delta_B^A (\theta \cdot \partial_\theta) , \quad B = -j - \frac{1}{2} \left( 1 - \frac{2}{N} \right) (\theta \cdot \partial_\theta) , \] (3.6)
with \( \partial_z \equiv \partial/\partial z \) and \( \theta \cdot \partial_\theta \equiv \theta^A \partial/\partial \theta^A \). Here the parameter \( j \) is the superconformal spin of the superfield. In SYM theory with \( \mathcal{N} \) supercharges, the \( \Phi^- \) and \( \Psi^- \)-superfields, Eqs. (1.3) - (1.6), correspond to \( j = -1/2 \) and \( j = (3 - \mathcal{N})/2 \), respectively. A global form of the transformations (3.3) can be found in Appendix B, Eqs. (B.2) - (B.3).

The representation \( \mathcal{V}_j \) is spanned by polynomials in both \( z \) and \( \theta^A \) which arise from the Taylor expansion of the superfields \( \Phi_j(zn, \theta^4) \) around \( z = \theta^A = 0 \). It possesses the lowest weight 1 which is annihilated by the lowering operators \( L^- \), \( W^{A,-} \), \( V^-_A \) and satisfies the chirality condition \((L^0 + B) \cdot 1 = 0 \). For \( j \geq 1/2 \) the representation \( \mathcal{V}_j \) is irreducible and it is known in the literature as atypical, or chiral representation [16] [17].

The product of two light-cone superfields \( \mathcal{O}(Z_1, Z_2) \) belongs to the tensor product of two chiral representations, \( \mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2} \). The \( SL(2|\mathcal{N}) \) generators on this tensor product are given by the sum of differential operators acting on coordinates of two superfields, \( G = G_{(1)} + G_{(2)} \). The superconformal invariance implies that the one-loop evolution kernel \( \mathcal{H} \) entering (3.3) commutes with the \( SL(2|\mathcal{N}) \) generators, \( [\mathcal{H}, G] = 0 \). Therefore, \( \mathcal{H} \) depends on the two-particle superconformal spin \( \mathbb{J}_{12} \) defined as

\[
\mathbb{J}_{12}^2 = \mathbb{J}_{12}(\mathbb{J}_{12} - 1) + C_{12},
\]

with \( C_{12} = \mathcal{N}(j_1 + j_2)[1 + (j_1 + j_2)/(\mathcal{N} - 2)] \) and \( \mathbb{J}_{12}^2 \) being the two-particle quadratic Casimir operator of the \( SL(2|\mathcal{N}) \) group

\[
\mathbb{J}_{12}^2 = (L^0)^2 + L^+ L^- + (\mathcal{N} - 1)L^0 + \frac{\mathcal{N}}{\mathcal{N} - 2} B^2 - V^-_A W^{A,-} - W^{A,+} V^-_A - \frac{1}{2} T_B A^A T_A^B .
\]

To construct the eigenfunctions of the \( SL(2|\mathcal{N}) \) invariant operator \( \mathcal{H} \), Eq. (3.4), it suffices to decompose the tensor product \( \mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2} \) over the irreducible \( SL(2|\mathcal{N}) \) components. The decomposition takes the form [16] [17] [18]

\[
\mathcal{V}_{j_1} \otimes \mathcal{V}_{j_2} = \sum_{n \geq 0} \mathcal{V}_{j_1+j_2+n},
\]

where the sum in the right-hand side goes over the \( SL(2|\mathcal{N}) \) representations with the superconformal spin \( \mathbb{J}_{12} = j_1 + j_2 + n \).

The eigenfunctions \( \Psi^{(0)}_n(Z_1, Z_2) \), Eq. (3.4), belong to the \( SL(2|\mathcal{N}) \) module \( \mathcal{V}_{j_1+j_2+j} \) and can be classified as follows. For \( \ell = 0 \), the polynomial \( \Psi^{(0)}_n(Z_1, Z_2) \) is the lowest weight of \( \mathcal{V}_{j_1+j_2+n} \). By definition, it is annihilated by the lowering \( SL(2|\mathcal{N}) \) generators

\[
L^- \Psi^{(0)}_n(Z_1, Z_2) = W^{A,-} \Psi^{(0)}_n(Z_1, Z_2) = V^-_A \Psi^{(0)}_n(Z_1, Z_2) = 0
\]

and carries the superconformal spin (3.7) equal to \( j_1 + j_2 + n \),

\[
\mathbb{J}_{12} \Psi^{(0)}_n(Z_1, Z_2) = (j_1 + j_2 + n) \Psi^{(0)}_n(Z_1, Z_2).
\]

Depending on the value of the nonnegative integer \( n \), the lowest weight is given by the following expressions

\[
\Psi^{(0)}_n(Z_1, Z_2) = \begin{cases} 
1, & n = 0 \\
\varepsilon_{A_1...A_N} A_{n+1}...A_{N} \theta_{12}^{A_1} \cdots \theta_{12}^{A_n}, & 1 \leq n < \mathcal{N} \\
\varepsilon_{A_1...A_N} \theta_{12}^{A_1} \cdots \theta_{12}^{A_N} (z_1 - z_2)^{n-\mathcal{N}}, & n \geq \mathcal{N}
\end{cases}
\]
where $\theta^{1A} = \theta^{1A} - \theta^{2A}$. For $\ell \geq 1$, the polynomials $\Psi_{n}^{(\ell)}(Z_1, Z_2)$ are obtained from the lowest weight $\Psi_{0}^{(\ell)}(Z_1, Z_2)$ by applying the raising $S(L(2|\mathcal{N})$ generators $L^+, W^{A+}, V_A^+$ and $T_B^A$. To save space we do not present their explicit form here.

The lowest weight $\Psi_{0}^{(0)}(Z_1, Z_2)$, Eq. (3.12), uniquely specifies the representation $\mathcal{V}_{j_1+j_2+n}$ in the right-hand side of (3.9). One can verify that the polynomial $\Psi_{n}^{(0)}(Z_1, Z_2)$ diagonalizes the generators $B$ and $L^0$

$$B \Psi_{n}^{(0)}(Z_1, Z_2) = - \left( j_1 + j_2 + \frac{\mathcal{N} - 2}{2\mathcal{N}} n_\prec \right) \Psi_{n}^{(0)}(Z_1, Z_2),$$

$$L^0 \Psi_{n}^{(0)}(Z_1, Z_2) = \left( j_1 + j_2 + n - \frac{1}{2} n_\prec \right) \Psi_{n}^{(0)}(Z_1, Z_2),$$

with $n_\prec = \min(n, \mathcal{N})$ and carries (for $\mathcal{N} \geq 2$) a nontrivial $SU(\mathcal{N})$ charge

$$\left( T_B A T_A B \right) \Psi_{n}^{(0)}(Z_1, Z_2) = t_n \Psi_{n}^{(0)}(Z_1, Z_2),$$

with $t_n = n(\mathcal{N} - n)(\mathcal{N} + 1)/\mathcal{N}$ for $0 \leq n \leq \mathcal{N}$ and $t_n = 0$ for $n > \mathcal{N}$. For $n = 0$ the lowest weight $\Psi_{n=0}^{(0)}(Z_1, Z_2) = 1$ satisfies the chirality condition

$$(B + L^0) \Psi_{0}^{(0)}(Z_1, Z_2) = 0$$

and, as a consequence the corresponding $SL(2|\mathcal{N})$ representation $\mathcal{V}_{j_1+j_2}$, Eq. (3.9) is atypical, or chiral. For $n \geq 1$ the representation $\mathcal{V}_{j_1+j_2+n}$ is typical.

The polynomials $\Psi_{n}^{(\ell)}(Z_1, Z_2)$ defined in this way diagonalize the one-loop Hamiltonian $\mathbb{H}$ (3.4). They determine the coefficient functions accompanying the twist-two operators in the OPE expansion (2.5). By virtue of the $SL(2|\mathcal{N})$ invariance, the anomalous dimension of these operators is a function of the superconformal spin $j_1 + j_2 + n$ and does not depend on $\ell$.

### 3.3. Invariant scalar product

Let us establish a relation between the eigenstates $\Psi_{n}^{(\ell)}(Z_1, Z_2)$, determined in the previous section, and the polynomials $P_n(X_1; X_2)$ in $X = (x, \vartheta_A)$ defining the superconformal operators, Eq. (1.13). The consideration goes along the same lines as in Sect. 2.2 for $\mathcal{N} = 0$. Namely, we shall introduce an $SL(2|\mathcal{N})$ invariant scalar product on the space spanned by the polynomials in $Z_1$ and $Z_2$ and, then, project both sides of (3.1) onto the lowest weight $\Psi_{0}^{(0)}(Z_1, Z_2)$.

To begin with let us consider the $\mathcal{N} = 1$ case. An arbitrary polynomial $\Psi_j(Z)$ belonging to the chiral $SL(2|1)$ module $\mathbb{V}_j$ can be expanded in powers of $\vartheta$ as

$$\Psi_j(z, \vartheta) = \psi(z) + \vartheta \chi(z),$$

with $\psi(z)$ and $\chi(z)$ being some polynomials in $z$. Since the $SL(2|1)$ group contains the $SL(2)$ subgroup generated by the bosonic operators $L^\pm$ and $L^0$, Eq. (3.6), the polynomials $\psi(z)$ and $\chi(z)$ are transformed under the $SL(2)$ transformations according to Eq. (2.3) and carry a definite value of the $SL(2)$ spins, $j$ and $j + 1/2$, respectively. We remind that for an arbitrary (half-)integer spin $j \geq 1/2$, the $SL(2)$ scalar product on the space spanned by polynomials in $z$ is given by (2.17). This suggests to define the scalar product for the states (3.17) as

$$\langle \Psi | \Psi' \rangle_{SL(2|1)} = \langle \psi | \psi' \rangle_j + c \langle \chi | \chi' \rangle_{j + \frac{1}{2}},$$

with $c$ being a constant.
where \( c \) is an arbitrary constant and the subscript in the right-hand side indicates the value of the \( SL(2) \) conformal spin in Eq. (2.17).

By construction, Eq. (3.18) is invariant under the \( SL(2) \) transformations (2.9) for arbitrary \( c \). The value of \( c \) has to be fixed by imposing the condition of invariance of (3.18) under the \( SU(1,1|1) \) transformations, Eqs. (B.2) – (B.3). To this end, it proves convenient to rewrite (3.18) in terms of the \( \mathcal{N} = 1 \) superfields (3.17). It is straightforward to verify that for the states \( \Psi(Z) \) and \( \Psi'(Z) \) belonging to the \( SL(2|1) \) module \( V_j \) the following integral is invariant under the superconformal transformations (Eqs. (B.2) – (B.3))

\[
\langle \Psi|\Psi' \rangle_{SL(2|1)} = \frac{1}{\pi \Gamma(2j)} \int d^2 z \int d\theta d\bar{\theta} (1 - z\bar{z} - \theta \bar{\theta})^{2j - 1} \Psi(Z) \bar{\Psi}(Z).
\]

Here the integration over odd coordinates is performed using the identities

\[
\int d\theta = \int d\bar{\theta} = 0, \quad \int d\theta \theta = -\int d\bar{\theta} \bar{\theta} = 1,
\]

which is consistent with the complex conjugation assumed for odd variables \((\bar{\chi}\theta) = \bar{\theta}\bar{\chi}\) and \(\bar{Z} = (\bar{z}, \bar{\theta})\). Expanding the integrand in Eq. (3.19) in powers of \(\theta^A\) and \(\bar{\theta}^A\) one arrives at (3.18) with \(c = 1\). Eq. (3.19) defines the \( SL(2|1) \) scalar product on the space \( V_j \) spanned by the \( \mathcal{N} = 1 \) superfields (3.17).

The above consideration can be easily generalized to \( \mathcal{N} \geq 2 \). In that case, the expansion in the right-hand side of (3.17) goes in powers of \(\theta^A\) and involves \(2^\mathcal{N}\) different polynomials which carry both the \( SU(\mathcal{N}) \) isotopic charge and the \( SL(2) \) conformal spin. Similar to (3.18), the \( SL(2|\mathcal{N}) \) scalar product is given by the sum over the \( SL(2) \) scalar products of the conformal spins varying between \( j \) and \( j + \mathcal{N}/2 \). It takes a particularly simple form when written as an integral over the superspace

\[
\langle \Psi|\Psi' \rangle_{SL(2|\mathcal{N})} = \int [DZ]_j \bar{\Psi}(Z) \Psi'(Z),
\]

where the notation was introduced for the integration measure

\[
\int [DZ]_j = \frac{1}{\pi \Gamma(2j - 1 + \mathcal{N})} \int_{|z| \leq 1} d^2 z \int \prod_{A=1}^{\mathcal{N}} (d\bar{\theta}_A d\theta^A) (1 - z\bar{z} - \theta \bar{\theta})^{2j + \mathcal{N} - 2},
\]

with \( Z = (z, \theta^A) \), \( \bar{Z} = (\bar{z}, \bar{\theta}_A) \) and \( \theta \cdot \bar{\theta} \equiv \sum_{A=1}^{\mathcal{N}} \theta^A \bar{\theta}_A \). For \( \mathcal{N} = 0 \) and \( \mathcal{N} = 1 \) these relations match Eqs. (2.17) and (3.19), respectively. For \( \mathcal{N} \geq 2 \), the integration measure over Grassmann variables is normalized via

\[
\int \prod_{A=1}^{\mathcal{N}} d\theta^A \cdot \theta^A \cdots \theta^A\mathcal{N} = \varepsilon^{A_1 \cdots A\mathcal{N}},
\]

with \( \varepsilon^{12 \cdots \mathcal{N}} = 1 \). One can verify that the scalar product (3.21) is invariant under the superconformal \( SU(1,1|\mathcal{N}) \) transformations, Eqs. (B.2) – (B.3).

The \( SL(2|\mathcal{N}) \) representation space \( V_j \) is spanned by the states \( \Psi(Z) \sim z^n \theta^{A_1} \cdots \theta^{A_L} \) given by polynomials in both even and odd variables. They are orthogonal with respect to the scalar product (3.21), namely,

\[
\langle z^n \theta^{A_1} \cdots \theta^{A_L} | z^k \theta^{B_1} \cdots \theta^{B_M} \rangle = \delta_{nk} \delta_{LM} \frac{n!}{\Gamma(2j + n + L)} \left( \delta_{A_1}^{B_1} \cdots \delta_{A_L}^{B_L} + \cdots \right),
\]

(3.24)
with ellipses standing for the terms which ensure antisymmetry of the right-hand side with respect to the permutation of any pair of lower/upper indices. For the polynomials $\Psi(Z_1, Z_2)$ belonging to the tensor product $\mathbb{V}_j \otimes \mathbb{V}_j$, the scalar product is given by the integral over $Z_1$-- and $Z_2$--coordinates in the superspace with the same measure as in (3.21).

The $SL(2|N)$ generators, $L^0, B$ and $T^A_B$, Eqs. (3.23), are self-adjoint operators with respect to the scalar product (3.21), that is $\langle \Psi L^0 | \Psi' \rangle = \langle \Psi | L^0 \Psi' \rangle$. For the remaining $SL(2|N)$ generators one finds

$$ (L^+)^\dagger = -L^- , \quad (W^A, \pm)^\dagger = V_A^\mp , \quad (3.25) $$

and, as a consequence, the Casimir operator (3.8) is a self-adjoint operator on the tensor product (3.9), $(L_{12}^2)^\dagger = L_{12}^2$. Therefore, its eigenfunctions $\Psi_n^{(\ell)}(Z_1, Z_2)$, Eq. (3.12), are mutually orthogonal with respect to (3.21),

$$ \langle \Psi_n^{(\ell)}(Z_1, Z_2) | \Psi_k^{(\ell)}(Z_1, Z_2) \rangle \sim \delta_{nk} \delta_{\ell\ell'} . \quad (3.26) $$

This relation allows one to determine the explicit form of superconformal operators in SYM theories.

Let us project both sides of Eq. (3.11) onto $\Psi_n^{(\ell)}(Z_1, Z_2)$. Taking into account (3.26), one obtains the following representation for the twist-two operators in the $N$--extended SYM

$$ \mathcal{O}_n^{(\ell)}(0) \sim \langle \Psi_n^{(\ell)}(Z_1, Z_2) | \mathcal{O}(Z_1, Z_2) \rangle = \int [DZ_1]_{j_1} [DZ_1]_{j_2} \overline{\Psi_n^{(\ell)}(Z_1, Z_2)} \mathcal{O}(Z_1, Z_2) \mathcal{O}(Z_1, Z_2) . \quad (3.27) $$

This relation is a natural generalization of a similar relation for $N = 0$, i.e., Eq. (2.22). Expanding $\mathcal{O}(Z_1, Z_2)$ in the Taylor series around $Z_1 = Z_2 = 0$ and matching (3.27) into (1.13), we identify the corresponding superpolynomials $P_n^{(\ell)}$ as

$$ P_n^{(\ell)}(X_1, X_2) = \langle \Psi_n^{(\ell)}(Z_1, Z_2) | e^{Z_1 \cdot X_1 + Z_2 \cdot X_2} \rangle = \int [DZ_1]_{j_1} [DZ_1]_{j_2} \overline{\Psi_n^{(\ell)}(Z_1, Z_2)} e^{Z_1 \cdot X_1 + Z_2 \cdot X_2} . \quad (3.28) $$

Here the notation was introduced for the momentum variables in the superspace $X_k = (x_k, \vartheta_{kA})$

$$ Z_k \cdot X_k \equiv z_k x_k + \sum_{A=1}^N \theta_k^A \vartheta_{kA} . \quad (3.29) $$

In full analogy with Eq. (2.26), the inverse transformation looks like (see Eqs. (3.1) and (3.3))

$$ \overline{\Psi_n^{(\ell)}(Z_1, Z_2)} = P_n^{(\ell)}(\partial_{w_1}, \partial_{\bar{\vartheta}_1^A}; \partial_{w_2}, \partial_{\bar{\vartheta}_2^A}) \prod_{k=1,2} \Gamma(2j_k)(1 - w_k \bar{z}_k - \vartheta \cdot \bar{\vartheta})^{-2j_k}|_{w_k = \theta_k^A = 0} = \int_0^\infty \prod_{k=1,2} dt_k t_k^{2j_k - 1} e^{-t_1 - t_2} P_n^{(\ell)}(t_1 \bar{z}_1, t_1 \bar{\vartheta}_1, t_2 \bar{z}_2, t_2 \bar{\vartheta}_2) . \quad (3.30) $$

It is instructive to compare this relation with (2.26). One notices that the only difference is that going over from $N = 0$ to $N \geq 1$ one has to enlarge the number of odd directions in the superspace.

Similar to (2.23), the polynomials $P_n^{(\ell)}(X_1, X_2)$ for $\ell \geq 1$ can be expressed in terms of the one with $\ell = 0$. The descendant eigenstates $\Psi_n^{(\ell)}(Z_1, Z_2)$ are obtained from the lowest weight $\Psi_n^{(0)}(Z_1, Z_2)$ by applying the raising operators $L^+, W^A, V_A^+, T_B^A$. Taking into account (3.28)
and (3.28), these operators can be realized as differential operators \( \hat{L}^+, \hat{W}^{A+}, \hat{V}^+_A \) and \( \hat{T}^{BA} \) acting on the polynomial \( P^{(0)}_n(X_1, X_2) \). For example, the operator \( \hat{W}^{A+} \) is defined as

\[
\langle V^- A \rangle^n \Psi_n^{(0)}(Z_1, Z_2) | e^{Z_1 \cdot X_1 + Z_2 \cdot X_2} = \langle \Psi_n^{(0)}(Z_1, Z_2) | e^{Z_1 \cdot X_1 + Z_2 \cdot X_2} \rangle (3.31)
\]

\[
= (-1)^n \sum_{k=1,2} (-x_k \partial_{\theta_{k,A}}) \langle \Psi_n^{(0)}(Z_1, Z_2) | e^{Z_1 \cdot X_1 + Z_2 \cdot X_2} \rangle = (-1)^n \hat{V}^+_A P^{(0)}_n(X_1, X_2),
\]

with the grading factor \((-1)^n\) equal to 1 (or \(-1\)) for polynomials involving even (or odd) number of grassman variables. In this way, one finds

\[
\hat{V}^+_A = \sum_{k=1,2} (-x_k \partial_{\theta_{k,A}}), \quad \hat{W}^{A+} = \bar{\vartheta}_{1,A} + \vartheta_{2,A}, \quad (3.32)
\]

\[
\hat{L}^+ = x_1 + x_2, \quad \hat{T}^{BA} = \sum_{k=1,2} \vartheta_{k,B} \partial_{\theta_{k,A}} - \frac{1}{N} \delta^A_B (\vartheta_{k,B} \cdot \partial_{\theta_{k,A}}).
\]

To evaluate the polynomials \( P^{(4)}_n(X_1, X_2) \), it suffices to apply the raising operators \( \hat{L}^+, \hat{W}^{A+}, \hat{V}^+_A \) and \( \hat{T}^{BA} \) to the lowest weight \( P^{(0)}_n(X_1, X_2) \). This allows us to restrict the consideration to the lowest weights \( P^{(0)}_n(X_1, X_2) \) only.

Eqs. (3.28) and (3.30) represent the main result of the paper. They relate to each other the \( SL(2|N) \) lowest weights, Eq. (3.12), and the polynomials defining the superconformal twist-two operators in the SYM theory with an arbitrary number of supercharges \( N \). As we will show in Sect. 4, the resulting Wilson operators coincide with known expressions obtained through diagonalization of the one-loop mixing matrix for \( N = 1, 2, 4 \).

4. Twist-two operators in SYM

Let us apply (3.28) and (3.30) to reconstruct the superconformal operators out of the lowest weights (3.12) in the \( \Psi \Psi^-, \Phi \Phi^- \) and \( \Phi \Phi^- \) sectors. We remind that for \( N = 0 \) a similar integral transformation, Eq. (2.23), relates the Jacobi polynomials, Eq. (2.28), to the \( SL(2) \) lowest weights, Eq. (2.13).

The lowest weights (3.12) are factorized into the product of two translation invariant polynomials depending separately on even and odd coordinates. Substituting (3.12) into (3.28) one can perform \( Z_1 \) and \( Z_2 \) integrations by expanding the \( SL(2|N) \) integration measure (3.21) in powers of \((\theta \cdot \bar{\theta})\)

\[
\int [DZ]_j = \sum_{k=0}^N \int [Dz]_{j+k/2} \prod_{A=1}^N (d\bar{\theta}_A d\theta^A)(\theta \cdot \bar{\theta})^{N-k}(N-k)!,
\]

with the \( SL(2) \) measure \([Dz]_{j+k/2}\) defined in (2.18). Then, for a test function of the same form as the lowest weight (3.12),

\[
\Psi(Z_1, Z_2) = (z_1 - z_2)^N \varphi_M(\theta_1 - \theta_2),
\]

with \( \varphi_M(\theta) \) being a homogenous polynomial in \( \theta^A \) of degree \( M \) (such that \( M \leq N \)), the integral in the right-hand side of (3.28) can be factorized into the product of \( z^- \) and \( \theta^- \) integrals. The
The $z$–integral is the same as for $\mathcal{N} = 0$, Eq. (2.25), and is given by the Jacobi polynomial (2.28). The $\theta$–integral can be easily evaluated with a help of (3.20) leading to (see Eq. (B.6))

$$P(X_1, X_2) = (x_1 + x_2)^N \sum_{k_1, k_2 \geq 0} c_N^{(2j_1 + k_1, 2j_2 + k_2)} \cdot P_N^{(2j_1 + k_1 - 1, 2j_2 + k_2 - 1)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) \times (\vartheta_1 \cdot \vartheta_{\bar{1}})^{k_1} (\vartheta_2 \cdot \vartheta_{\bar{2}})^{k_2} \varphi_M(\theta_1 - \theta_2) \bigg|_{\theta_{\bar{1}, \bar{2}} = 0},$$

(4.3)

with the expansion coefficients

$$c_N^{(2j_1 + k_1, 2j_2 + k_2)} = (-1)^N N!/\left[ \Gamma(N + 2j_1 + k_1)\Gamma(N + 2j_2 + k_2) k_1! k_2! \right].$$

(4.4)

It follows from (4.3) that the superconformal polynomials corresponding to the lowest weights (3.12) are given by the sum of the $SL(2)$ conformal polynomials multiplied by the product of $\vartheta$–variables.

In Sect. 3, it was tacitly assumed that the superfield $\Psi_j(Z)$ belongs to an irreducible representation of the superconformal group. This implies that its superconformal spin has to satisfy the condition $j \geq 1/2$. We remind that in the light-cone formalism the superconformal spins of the superfields $\Phi(Z)$ and $\Psi(Z)$, Eqs. (1.3) – (1.6), are $j_{\Phi} = -1/2$ and $j_{\Psi} = (3 - \mathcal{N})/2$, respectively. For $\mathcal{N} < 4$ the condition $j \geq 1/2$ is satisfied only for the $\Psi$–superfield and, therefore, Eqs. (3.28) and (3.30) can be safely applied in the $\Psi \Psi$–sector. In order to construct superconformal polynomials in $\Phi \Phi$– and $\Phi \Phi$–sectors, we have to extend our consideration to reducible representations of the $SL(2|\mathcal{N})$ group. This will be done in Sect. 4.2.

### 4.1. Superconformal operators in the $\Psi \Psi$–sector

In SYM theories with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supercharges, the twist-two operators in this sector are defined as

$$O_n^{(0)}(0) = P_n^{(0)}(\partial_{Z_1}; \partial_{Z_2}) \text{tr} [\Psi(Z_1)\Psi(Z_2)] \bigg|_{Z_1 = Z_2 = 0},$$

(4.5)

in terms of the light-cone superfield $\Psi(Z)$ given by Eqs. (1.3) and (1.5), respectively. By construction, the operators (4.5) are the lowest weights in the tensor product of two $SL(2|\mathcal{N})$ representations $\mathbb{V}_{j_{\Phi}} \otimes \mathbb{V}_{j_{\Phi}}$.

### $\mathcal{N} = 1$ theory

The superfield $\Psi(Z)$, Eq. (1.4), describes the negative helicity components of the gauge $(\partial + \bar{A})$ and gaugino $(\lambda)$ fields. It carries the superconformal spin $j_{\Psi} = 1$ and defines the irreducible representation of the $SL(2|1)$ group, $\Psi(Z) \in \mathbb{V}_1$. The nonlocal light-cone operator $\mathcal{O}(Z_1, Z_2) = \text{tr} [\Psi(Z_1)\Psi(Z_2)]$ belongs to the tensor product $\mathbb{V}_1 \otimes \mathbb{V}_1$ and its expansion around $Z_1 = Z_2 = 0$ produces an infinite set of twist-two operators. According to (3.12), these operators are in the one-to-one correspondence with the $SL(2|1)$ lowest weights

$$\Psi_0^{(0)}(Z_1, Z_2) = 1,$$

$$\Psi_n^{(0)}(Z_1, Z_2) = (\theta_1 - \theta_2)(z_1 - z_2)^{n-1},$$

(4.6)
where $n \geq 1$ and $Z = (z, \theta)$. Matching these expressions into (4.2) and (4.3) one finds that the corresponding superconformal polynomials are given by

\begin{align}
P_0^{(0)}(X_1; X_2) &= 1, \\
P_n^{(0)}(X_1; X_2) &= c_n \cdot (x_1 + x_2)^{n-1} \left\{ \partial_1 \mathbf{P}_{n-1}^{(2,1)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) - \partial_2 \mathbf{P}_{n-1}^{(1,2)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) \right\},
\end{align}

where $X_k = (x_k, \vartheta_k)$ and $c_n = (-1)^{n-1}/[n(1+n)!]$. It is straightforward to verify that these polynomials are related to the lowest weights (4.6) through the integral transformation (3.30).

Let us substitute (4.7) into (4.5) and take into account that $\Psi(z, \theta) = -\lambda(x) + \theta \partial_+ \tilde{A}(x)$, Eq. (1.4):

• For $n = 0$ the twist-two operator vanishes

\[ \mathcal{O}_0^{(0)}(0) = \text{tr} \left[ \Psi(0) \Psi(0) \right] = \text{tr} \left[ \lambda(0) \lambda(0) \right] = 0. \] (4.8)

• For $n \geq 1$, one finds (up to an overall normalization factor)

\[ \mathcal{O}_n^{(0)}(0) = \text{tr} \left[ \partial_+ \tilde{A}(0) \mathbf{P}_{n-1}^{(2,1)}(\partial_+, \partial_+) \lambda(0) + \lambda(0) \mathbf{P}_{n-1}^{(1,2)}(\partial_+, \partial_+) \partial_+ \tilde{A}(0) \right] = 2\sigma_{n-1} \text{tr} \left[ \partial_+ \tilde{A}(0) \mathbf{P}_{n-1}^{(2,1)}(\partial_+, \partial_+) \lambda(0) \right], \] (4.9)

with $\sigma_{n-1} = [1 + (-1)^{n-1}]/2$. Here the notation was introduced for the differential operator

\[ \mathbf{P}_{n}^{(a,b)}(\partial_+, \partial_+) = \mathbf{P}_{n}^{(a,b)} \left( \partial_+ - \partial_+ \right) \left( \partial_+ + \partial_+ \right)^n. \] (4.10)

The operators $\mathcal{O}_n^{(0)}(0)$ carry the superconformal spin $j = 2j_\Psi + n = 2 + n$ and have autonomous scale dependence to one-loop order. Their anomalous dimension is given by $E_{\Psi \Psi}(j) = 2[\psi(n+2) - \psi(1)]$, Eqs. (1.9) and (1.10). It is instructive to examine (4.9) in the forward limit (fw), that is, neglecting operators containing total derivatives. Setting $\partial_+ + \partial_+ = 0$, one finds from (4.9) with a help of (4.3)

\[ \mathcal{O}_n^{(0)}(0) \overset{\text{fw}}{=} \sigma_{n-1} \text{tr} \left[ \partial_+ \tilde{A}(0) \partial_+^{n-1} \lambda(0) \right], \] (4.11)

which is only different from zero for odd $n$. This relation gives one of the operators found in Ref. [10] by direct diagonalization of the mixing matrix.

The operator (4.9) belongs to the $\mathcal{N} = 1$ supermultiplet of twist-two operators with the aligned helicity. The remaining components of this supermultiplet are determined by descendants of the lowest weights (4.6). They are obtained by acting with the (two-particle) raising operators $L^+, W^+$ and $V^+$, Eq. (3.6), on the lowest weight $\Psi_n^{(0)}(Z_1, Z_2)$. For example, $L^+ \Psi_n^{(0)}(Z_1, Z_2)$ yields a linear combination of $(z_1 + z_2)(\theta_1 - \theta_2)(z_1 - z_2)^{n-1}$ and $(\theta_1 + \theta_2)(z_1 - z_2)^n$. When transformed into superconformal operators, the former gives a total derivative of the operator (4.9), while the latter produces the other parity component of the supermultiplet in question

\[ \sigma_{n-1} \text{tr} \left[ \partial_+ \tilde{A}(0) \mathbf{P}_{n}^{(2,1)}(\partial_+, \partial_+) \lambda(0) \right], \]
which possesses the same anomalous dimension as the operator (4.9).

We remind that the above expressions are valid in the axial gauge \((n \cdot A(x)) = A_+(x) = 0\). To write down the same expression in a covariant form, it is sufficient to substitute the light-cone by covariant derivatives \(\partial_+ = (n \cdot D)\), the antiholomorphic gauge field by the field strength tensors, \(\partial_+ A = (F^+_{+x} - i F^+_{+y})/\sqrt{2}\), and the Grassmann fermion \(\lambda\) by the two-component Weyl spinor (see Appendix A of Ref. [10]).

\(\mathcal{N} = 2\) theory

The light-cone superfield \(\Psi(Z)\), Eq. (1.5), comprises the scalar field \(\phi\), the gauge field \(\partial_+ A\) of helicity \(-1\) and the gaugino field \(\lambda^A\) (with \(A = 1, 2\)) of helicity \(-1/2\). It carries the superconformal spin \(j = 1/2\) and belongs to the irreducible \(SL(2|2)\) representation, \(\Psi(Z) \in \mathbb{V}_{1/2}\).

As before, to construct the Wilson operators in the \(\Psi\Psi\) sector, one identifies the lowest weights in the tensor product \(\mathbb{V}_{1/2} \otimes \mathbb{V}_{1/2}\), Eq. (3.12). They are

\[
\Psi_0^{(0)}(Z_1, Z_2) = 1,
\]

\[
\Psi_1^{(0)}(Z_1, Z_2) = (\theta_1 - \theta_2)^A,
\]

\[
\Psi_n^{(0)}(Z_1, Z_2) = \varepsilon_{AB}(\theta_1 - \theta_2)^A(\theta_1 - \theta_2)^B(z_1 - z_2)^{n-2},
\]

where \(n \geq 2\) and \(Z = (z, \theta^A)\) (with \(A = 1, 2\)). To determine the corresponding superconformal polynomials one matches these expressions into (4.2) and (4.3) and obtains

\[
P_0^{(0)}(X_1; X_2) = 1,
\]

\[
P_1^{(0)}(X_1; X_2) = (\partial_1 - \partial_2)^A
\]

\[
P_n^{(0)}(X_1; X_2) = c_n(x_1 + x_2)^{n-2} \left\{ - \frac{2n}{n-1} p_{n-2}^{(1, 1)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) (\partial_1 \cdot \partial_2) + p_{n-2}^{(2, 0)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) (\partial_1 \cdot \partial_1) + p_{n-2}^{(0, 2)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) (\partial_2 \cdot \partial_2) \right\},
\]

where \(c_n = (-1)^n/n!\), \(X = (x, \theta^A)\) and the notation was introduced for \((\partial_k \cdot \partial_n) \equiv \varepsilon_{kA} \partial_k A \partial_n B\). Then, one substitutes (4.13) into (4.14), takes into account that \(\Psi(Z) = i \phi(x) - \varepsilon_{AB} \lambda^A(x) + \frac{1}{2} \varepsilon_{AB} \theta^A \theta^B \partial_+ \bar{A}(x)\) and obtains the following expressions:

- For \(n = 0\), the twist-two operator is built from the complex scalar

\[
\mathcal{O}_0^{(0)}(0) = \text{tr} [\Psi(0)\Psi(0)] = - \text{tr} [\phi(0)\phi(0)].
\]

- For \(n = 1\), the operator vanishes

\[
\mathcal{O}_1^{(0)}(0) = \text{tr} [\partial_+ \Psi(0)\Psi(0) - \Psi(0)\partial_+ \Psi(0)] = 0.
\]

- For \(n \geq 2\), the twist-two operators take the form (up to an overall normalization factor)

\[
\mathcal{O}_n^{(0)}(0) = \text{tr} \left\{ - \frac{in}{(n-1)} \varepsilon_{AB} \lambda^A(0) \bar{p}_{n-2}^{(1, 1)}(\partial_+ \partial_+) \lambda^B(0)
\]

\[
+ \phi(0) \bar{p}_{n-2}^{(0, 2)}(\partial_+ \partial_+) \partial_+ \bar{A}(0) + \partial_+ \bar{A}(0) \bar{p}_{n-2}^{(2, 0)}(\partial_+ \partial_+) \phi(0) \right\}.
\]
The operators $O_n^{(0)}(0)$ carry the superconformal spin $j = 2j_\Psi + n = 1 + n$ and their one-loop anomalous dimension equals $E_{\Psi\Psi}(n + 1) = 2[\psi(n + 1) - \psi(1)]$, Eqs. (1.9) and (1.10). For $n = 0$, the operator (1.14) has a vanishing anomalous dimension and this value is protected to all orders by supersymmetry [19]. In the forward limit, for $\partial_+ + \partial_+ = 0$, the operator (1.16) is given by

$$O_n^{(0)}(0) \sim \sigma_{n-2} \exp \left\{ - \frac{in}{n-1} \varepsilon_{AB} \lambda^A(0) \bar{\partial}^{n-2} \lambda^B(0) + 2\phi(0) \bar{\partial}^{n-1} A(0) \right\},$$

(4.17)

which coincides with the results of Ref. [10]. Other components of the supermultiplet are found from descendants of the lowest weights (4.13) in the same vein as we explained above in $\mathcal{N} = 1$ SYM theory.

4.2. Superconformal operators in $\Phi\Psi$ and $\Phi\Phi$ sectors

Let us extend the analysis to twist-two operators in the $\Psi\Phi$ and $\Phi\Phi$ sectors. The important difference between the light-cone superfields $\Psi(Z)$ and $\Phi(Z)$, Eqs. (1.3) – (1.6), is that the latter involves nonlocal field operators, $\partial_+^{-1} A(x)$ and $\partial_+^{-1} \lambda(x)$. As a consequence, the superfield $\Phi(Z)$ carries a negative superconformal spin $j_\Phi = -1/2$, independent on the number of supercharges $\mathcal{N}$, and the corresponding $SL(2|\mathcal{N})$ representation $\mathbb{V}_{j_\Phi}$ is reducible.

It is easy to see that some of the formulae obtained in Sect. 3 are not well defined for negative spin $j$. For instance, integrals entering (2.26) and (3.30) are divergent for $j = -1/2$. Still, as we shall argue below, the relations between the superconformal polynomials $P_n^{(0)}(X_1, X_2)$ and the lowest weights $\Psi_n^{(0)}(Z_1, Z_2)$, Eqs. (3.28) and (3.30), are still at work when analytically continued to $j_{1,2} = -1/2$.

Let us start with the $\mathcal{N} = 0$ case and examine the Taylor expansion of the field $\Phi(zn_\mu)$, Eq. (1.3), around the origin

$$\Phi(z) = (\partial_+^{-1} A(0) + z A(0)) + \sum_{k=2}^{\infty} \frac{z^k}{k!} \partial_+^{k-1} A(0).$$

(4.18)

Here the first two terms involve the operators $\partial_+^{-1} A(0)$ and $A(0)$. Their appearance is an artefact of the light-cone formulation of the $\mathcal{N} = 0$ theory and in the covariant formulation they correspond to nonlocal, “spurious” gauge field operators.

The field $\Phi(zn_\mu)$ defines a representation of the conformal $SL(2)$ group of spin $j = -1/2$. As before, it is spanned by the coefficient functions $\mathbb{V}_{-1/2} = \{1, z, z^2, \ldots\}$ entering the Taylor expansion (4.18) and the $SL(2)$ generators are given by (2.12) with $j = -1/2$. The coefficient functions $\mathbb{V}_{sp} = \{1, z\}$ accompanying the spurious operators in (4.18) define the $SL(2)$ invariant two-dimensional subspace. For “physical”, Wilson operators the corresponding coefficient functions belong to the quotient of two spaces, $\mathbb{V}_{phys} = \mathbb{V}_{-1/2}/\mathbb{V}_{sp}$.

We remind that for positive conformal spins $j$ the vectors $z_k \in \mathbb{V}_j$ have the $SL(2)$ norm $\langle z^k | z^n \rangle \sim \delta_{kn} / \Gamma(k + 2j)$, Eq. (2.19). Analytically continuing this relation to $j = -1/2$ one finds that the vectors belonging to $\mathbb{V}_{sp} = \{1, z\}$ have a zero norm and, in addition, they are orthogonal to the vectors from $\mathbb{V}_{phys}$. Therefore, for $j = -1/2$ the $SL(2)$ scalar product (2.17) projects the field $\Phi(zn_\mu)$, Eq. (4.18), onto the subspace of physical operators

$$\langle z^k | \Phi(z) \rangle = \left\{ \begin{array}{ll} 0, & k = 0, 1 \\ \frac{1}{(k-2)!} \partial_+^{k-1} A(0), & k \geq 2 \end{array} \right.$$

(4.19)
Let us now consider the product of two $\mathcal{N} = 0$ fields $\text{tr}[\Psi(z_1)\Phi(z_2)]$ and expand it in powers of $z_1$ and $z_2$ over the set of multiplicatively renormalized operators. Depending on whether these operators involve spurious gauge field operators, they can be split into two groups

$$
\text{tr}[\Psi(z_1)\Phi(z_2)] = \sum_{n,\ell} \Psi_n^{(\ell)}(z_1, z_2) \mathcal{O}_n^{(\ell)}(0) + \sum_{n,\ell} \tilde{\Psi}_n^{(\ell)}(z_1, z_2) \tilde{\mathcal{O}}_n^{(\ell)}(0),
$$

(4.20)

where $\mathcal{O}_n^{(\ell)}(0)$ denote “physical” twist-two operators. The “spurious” operators $\tilde{\mathcal{O}}_n^{(\ell)}(0)$ involve $\partial_+ A(0)$ and $A(0)$ and, as a consequence, $\tilde{\Psi}_n^{(\ell)}(z_1, z_2)$ is linear in $z_2$. By virtue of (4.19), $\tilde{\Psi}_n^{(\ell)}(z_1, z_2)$ has zero projection onto all states in $\mathcal{V}_{3/2} \otimes \mathcal{V}_{-1/2}$. Therefore, taking the scalar product of both sides of (4.20) with the lowest weights $\Psi_n^{(0)}(Z_1, Z_2) \in \mathcal{V}_{3/2} \otimes \mathcal{V}_{-1/2}$ one finds that “spurious” operators do not contribute and only “physical” operators survive.

Indeed, let us examine the expression for the $SL(2)$ conformal operator (2.29) for $j_1 = -1/2$ and $j_2 = 3/2$ corresponding to the conformal spins of $\Phi$– and $\Psi$–fields, Eq. (1.3). Using the properties of the Jacobi polynomials (see Eqs. (A.3) and (A.6)) one obtains (for $n \geq 2$)

$$
\mathcal{O}_n^{-1/2,3/2}(0) = \text{tr} \left[ \Phi(0) \mathbb{P}_n^{(-2,2)} \left( \begin{array}{c} \partial_+ \nabla_+ \\ \partial_+ \Phi \end{array} \right) \Psi(0) \right]
$$

(4.21)

\begin{align*}
&\sim \text{tr} \left[ \partial_+^2 \Phi(0) \mathbb{P}_n^{(-2,2)} \left( \begin{array}{c} \partial_+ \nabla_+ \\ \partial_+ \Phi \end{array} \right) \Psi(0) \right] = \mathcal{O}_n^{3/2,3/2}(0),
\end{align*}

where $\partial_+^2 \Phi(0) = \partial_+ A(0)$, $\Psi = -\partial_+ \tilde{A}(0)$ and the $\mathbb{P}$–operator was defined in (4.10). Eq. (4.21) defines the $SL(2)$ conformal operator in the $\mathcal{N} = 0$ theory in the $\Phi\Psi$–sector. This operator has the $SL(2)$ conformal spin $j = j_\Psi + j_\Phi + n = 1 + n$ and its one-loop anomalous dimension is given by $E_{\Phi\Psi}(n + 1)$, Eqs. (1.9) and (1.11), evaluated for $\mathcal{N} = 0$. In a similar manner, for $j_1 = j_2 = -1/2$ one finds the conformal operators in the $\Phi\Phi$–sector (for $n \geq 4$)

$$
\mathcal{O}_n^{1/2,-1/2}(0) = \text{tr} \left[ \Phi(0) \mathbb{P}_n^{(-2,-2)} \left( \begin{array}{c} \partial_+ \nabla_+ \\ \partial_+ \Phi \end{array} \right) \Phi(0) \right]
$$

(4.22)

\begin{align*}
&\sim \text{tr} \left[ \partial_+^2 \Phi(0) \mathbb{P}_n^{(-2,-2)} \left( \begin{array}{c} \partial_+ \nabla_+ \\ \partial_+ \Phi \end{array} \right) \partial_+^2 \Phi(0) \right] = \mathcal{O}_n^{3/2,-3/2}(0),
\end{align*}

where $\partial_+^2 \Phi(0) = \partial_+ A(0)$.

This operator has the conformal spin $j = 2j_\Phi + n = -1 + n$ and its one-loop anomalous dimension is given by $E_{\Phi\Phi}(n - 1)$, Eqs. (1.9) and (1.11).

Let us extend the analysis to $\mathcal{N} \geq 1$. Examining the expansion of the superfield $\Phi(z \mu, \theta^A)$, Eq. (1.4) – (1.6), around the origin $z = \theta^A = 0$ one observes that the coefficient functions accompanying nonlocal, “spurious” field components define the $SL(2|\mathcal{N})$ invariant subspace $\mathcal{V}_\text{sp} = \{1, z, \theta^A, z \theta^A\}$ (with $\lambda = 1, \ldots, \mathcal{N}$). Applying (3.24) for $j = -1/2$ one finds that the states belonging to $\mathcal{V}_\text{sp}$ have a zero norm and they are orthogonal to all states in $\mathcal{V}_{j_\Phi}$. Therefore, the scalar product of $\text{tr}[\Phi(Z_1)\Psi(Z_2)]$ and $\text{tr}[\Phi(Z_1)\Psi(Z_2)]$ with the $SL(2|\mathcal{N})$ lowest weights $\Psi_n^{(0)}(Z_1, Z_2)$ eliminates the contribution of operators involving nonlocal field components and only retains Wilson operators. This allows one to use a general expression for the superconformal polynomials, Eqs. (3.28) and (4.3), and analytically continue it to the values of superconformal spins $j_\Phi = -1/2$ and $j_\Psi = (3 - \mathcal{N})/2$ corresponding to the $\Phi$– and $\Psi$–superfields. Notice that the lowest weights, Eqs. (3.12) and (4.2), do not depend on the superconformal spins of the superfields. This dependence enters into (4.3) only through the coefficients $c_N^{(2j_1 + k_1,2j_2 + k_2)}$ and indices of the Jacobi polynomials.

\footnote{By virtue of the relation $\partial_+^2 \Phi(0) = -\Psi(0) = \partial_+ A(0)$ (see Eq. (1.3)), the twist-two operators in the $\Phi\Phi$– and $\Psi\Psi$–sectors are complex conjugated to each other.}
\( N = 1 \) theory

Let us first consider the twist-two operators in the \( \Phi \Phi \) sector for \( N = 1 \) and \( N = 2 \). According to (1.3) and (1.5), the light-cone superfields \( \Phi(Z) \) and \( \Psi(Z) \), Eqs. (1.3) – (1.6) involve two different sets of the fundamental fields which are mutually conjugated to each other. This suggests that the same relation should hold between the superconformal operators in the \( \Phi \Phi \) and \( \Psi \Psi \) sectors. Indeed, substituting \( j_1 = j_2 = -1/2 \) into (4.3) and going over through the calculation of the expansion coefficients \( c_N^{(2j_1+k_1,2j_2+k_2)} \), Eq. (1.3), one finds that for \( N = 1 \) and \( N = 2 \) the superconformal operators in the \( \Phi \Phi \) sectors are given by the same expressions as before, Eqs. (4.8), (4.9) and Eqs. (4.14) – (4.16), respectively, provided that the fields \( \partial_+ \bar{A}, \lambda^A, \phi \) are replaced by the corresponding conjugated fields \( \partial_+ A, \bar{\lambda}_A, \bar{\phi} \). Obviously, this substitution does not affect anomalous dimensions of Wilson operators.

In the \( \Phi \Psi \) sector, the superconformal operators are defined as

\[
O_n^{(0)}(0) = P_n^{(0)}(\partial Z_1; \partial Z_2) \left[ \Phi(Z_1) \Psi(Z_2) \right]_{Z_1 = Z_2 = 0},
\]

with the polynomials given by (4.3) for \( j_1 = -1/2 \) and \( j_2 = (3 - N)/2 \) and the lowest weights (1.2) of the form (3.12).

For \( N = 1 \), the superconformal polynomials corresponding to the lowest weights (1.6) are

\[
P_n^{(0)}(X_1; X_2) = (x_1 + x_2)^{n-1} \left\{ b_n^{(1)} P_{n-1}^{(-1,1)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) \partial_1 - b_n^{(2)} P_{n-1}^{(-2,2)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) \partial_2 \right\},
\]

with \( b_n^{(0)} = (-1)^{n-1} \Gamma(n)/[\Gamma(n+1) \Gamma(n-1)] \). Applying the identities (A.5) and (A.6) one obtains

\[
P_n^{(0)}(X_1; X_2) = -b_n^{(0)} (x_1 + x_2)^{n-1} \left[ P_{n-2}^{(1,1)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) \frac{x_1 \partial_1}{x_2 + x_1} + P_{n-3}^{(2,2)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) \frac{x_1 \partial_2}{(x_1 + x_2)^2} \right],
\]

for \( n \geq 2 \). It follows from (1.4) that

\[
\partial_1 \partial_2 \Phi(Z_1) \Psi(Z_2) \big|_{\theta_1,\theta_2 = 0} = -\bar{\lambda}(z_1) \lambda(z_2),
\]

\[
\partial_1^2 \partial_2 \Phi(Z_1) \Psi(Z_2) \big|_{\theta_1,\theta_2 = 0} = \partial_+ A(z_1) \partial_+ \bar{A}(z_2).
\]

Therefore, the twist-two operators defined by the superconformal polynomials (1.24) are

\[
O_n^{(0)}(0) = \text{tr} \left\{ \bar{\lambda}(0) P_{n-2}^{(1,1)} (\partial_+, \partial_+) \lambda(0) - \partial_+ A(0) P_{n-3}^{(2,2)} (\partial_+, \partial_+) \partial_+ \bar{A}(0) \right\},
\]

with \( n \geq 2 \). They carry the superconformal spin \( j = j_\Phi + j_\Psi + n = n + 1/2 \) and their one-loop anomalous dimension equals \( E_{\Phi \Psi}(n + 1/2) \), Eqs. (1.9) and (1.11) for \( N = 1 \). In the forward limit, for \( \partial_+ + \partial_+ = 0 \), the operator (4.27) is given by

\[
O_n^{(0)}(0) \overset{\text{fw}}{\sim} \text{tr} \left\{ \bar{\lambda}(0) \partial_+^{n-2} \lambda(0) - \frac{n - 2}{n + 1} \partial_+ A(0) \partial_+^{n-2} \bar{A}(0) \right\},
\]

which is the results obtained in Ref. [10].

\textsuperscript{3}Here and in what follows it is tacitly assumed that the Jacobi polynomials \( P_n^{(a,b)} \) vanish for negative \( n \).
\( \mathcal{N} = 2 \) theory

For \( \mathcal{N} = 2 \) the lowest weight are given by (4.42) and the conformal spins of the superfields are \( j_\Phi = -1/2 \) and \( j_\Psi = 1/2 \). One applies (4.3) and calculates the corresponding superconformal polynomials as \( P_k^{(0)}(X_1; X_2) = 0 \) for \( k = 0, 1 \) and

\[
P_n^{(0)}(X_1; X_2) = (x_1 + x_2)^{n-2} \left\{ b_{n-1}^{(2)} \ P_n^{(-2,2)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) (\vartheta_2 \cdot \vartheta_2) \right. - 2b_{n-1}^{(1)} \ P_n^{(-1,1)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) (\vartheta_1 \cdot \vartheta_2) + \left. b_{n-1}^{(0)} \ P_n^{(0,0)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) (\vartheta_1 \cdot \vartheta_1) \right\},
\]

for \( n \geq 2 \) and the \( b \)-coefficients defined in (4.24). Simplification of the Jacobi polynomials with a help of (A.5) and (A.6) yields

\[
P_n^{(0)}(X_1; X_2) = b_{n-1}^{(0)}(x_1 + x_2)^{n-2} \left\{ P_n^{(2,2)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) \left( \frac{\vartheta_2 \cdot \vartheta_2}{x_2 + x_1} \right) x_1^2 \right. + \left. 2P_n^{(1,1)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) \left( \vartheta_1 \cdot \vartheta_2 \right) x_1 + P_n^{(0,0)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right) (\vartheta_1 \cdot \vartheta_1) \right\}.
\]

Its substitution into (1.13) leads to the following expression for the twist-two operator (up to an overall normalization factor)

\[
\mathcal{O}_n^{(0)}(0) = \text{tr} \left\{ \tilde{\phi}(0) \ P_{n-2}^{(0,0)} \left( \partial_+, \partial_+ \right) \phi(0) + \tilde{\lambda}_A(0) \ P_{n-3}^{(1,1)} \left( \partial_+, \partial_+ \right) \lambda^A(0) - \partial_+ A(0) \ P_{n-4}^{(2,2)} \left( \partial_+, \partial_+ \right) \partial_+ \bar{A}(0) \right\}.
\]

This operator has the superconformal spin \( j = j_\Phi + j_\Psi + n = n \) and its one-loop anomalous dimension is given by \( E_{\Phi \Psi}(n) \), Eqs. (1.9) and (1.11) for \( \mathcal{N} = 2 \). In the forward limit, for \( \partial_+ + \bar{\partial}_+ = 0 \), the operator (4.31) is given by

\[
\mathcal{O}_n^{(0)}(0) \ \overset{\text{fw}}{=} \text{tr} \left\{ \partial_+ A(0) \partial_+^{n-3} \bar{A}(0) - \frac{n}{n - 3} \tilde{\lambda}_A(0) \partial_+^{n-3} \lambda^A(0) - \frac{n(n - 1)}{(n - 2)(n - 3)} \tilde{\phi}(0) \partial_+^{n-2} \phi(0) \right\},
\]

which agrees with the expression obtained in Ref. [10].

\( \mathcal{N} = 4 \) theory

A unique feature of the \( \mathcal{N} = 4 \) theory is that all twist-two operators belong to the \( \Phi \Phi \)–sector. The superconformal spin of the \( \Phi \Phi \)–superfield equals \( j_\Phi = -1/2 \) and the lowest weights in the tensor product of two \( SL(2|4) \) modules are given for \( \mathcal{N} = 4 \) by (3.12)

\[
\Psi_0^{(0)}(Z_1, Z_2) = 1,
\]

\[
\Psi_{n<4}^{(0)}(Z_1, Z_2) = \varepsilon_{A_1 A_2 A_3 A_4} \prod_{k=1}^n (\theta_1 - \theta_2)^{A_k},
\]

\[
\Psi_{n\geq4}^{(0)}(Z_1, Z_2) = (z_1 - z_2)^{n-4} \varepsilon_{A_1 A_2 A_3 A_4} \prod_{k=1}^4 (\theta_1 - \theta_2)^{A_k}.
\]
Let us match these expressions into (4.2) and (4.3). For the lowest weights \( \Psi_n^{(0)}(Z_1, Z_2) \) with \( 0 \leq n < 4 \) the expansion coefficients entering (4.3) take the form \( c_n^{(-1+k_1,-1+k_2)} = 1/[\Gamma(k_1-1)\Gamma(k_2-1)] \) and are different from zero provided that \( k_1, k_2 \geq 2 \). Since \( k_1 + k_2 = n \), one concludes that \( c_n^{(-1+k_1,-1+k_2)} = 0 \) for \( n \geq 3 \) leading to

\[
P_n^{(0)}(X_1, X_2) = 0, \quad (0 \leq n < 4)
\]

For \( n \geq 4 \), the superconformal polynomial (4.3) corresponding to the lowest weight \( \Psi_n^{(0)}(Z_1, Z_2) \) takes the form

\[
P_n^{(0)}(X_1, X_2) = 4! \sum_{k=n_{\text{min}}}^{4-n_{\text{min}}} (-1)^k b_{n-k}^{(k-2)} (x_1 + x_2)^{n-4} P_{n-4}^{(2,2-k)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right)
\times \varepsilon_{A_1A_2A_3A_4} \prod_{m_1=1}^{k} \partial_{1,A_{m_1}} \prod_{m_2=k+1}^{4} \partial_{2,A_{m_2}},
\]

where \( n_{\text{min}} \equiv \max(6-n, 0) \), \( X_k = (x_k, \partial_{k,A}) \) and the \( b \)-coefficients were defined in (4.24). Here, the ordering of \( \partial \)-variables is such that \( \prod_{m=1}^{n} \partial_{A_m} = \partial_{A_1} \ldots \partial_{A_n} \).

For \( n = 4 \), the relation (4.35) takes a simple form

\[
P_4^{(0)}(X_1, X_2) = 4! b_{1}^{(0)} \varepsilon_{A_1A_2A_3A_4} \partial_{1,A_1} \partial_{1,A_2} \partial_{2,A_3} \partial_{2,A_4}.
\]

For \( n \geq 5 \), one of the indices of the Jacobi polynomial in (4.35) takes negative values. Applying the identities (A.5) and (A.6) one obtains after some algebra (for \( \ell = k - 2 \))

\[
P_n^{(0)}(X_1, X_2) = 4! b_{n-3}^{(0)} \sum_{\ell=0}^{2-n_{\text{min}}} \frac{\kappa_{\ell}}{(2 + \ell)!(2 - \ell)!} (x_1 + x_2)^{n-4-\ell} P_{n-4-\ell}^{(\ell,\ell)} \left( \frac{x_2 - x_1}{x_2 + x_1} \right)
\times \varepsilon_{A_1A_2A_3A_4} \left[ (-x_2)^{2+\ell} \prod_{m_1=1}^{2+\ell} \partial_{1,A_{m_1}} \prod_{m_2=3+\ell}^{4} \partial_{2,A_{m_2}} + x_1^{2-\ell} \prod_{m_1=1}^{2-\ell} \partial_{1,A_{m_1}} \prod_{m_2=3-\ell}^{4} \partial_{2,A_{m_2}} \right],
\]

with \( \kappa_{\ell=0} = 1/2 \) and \( \kappa_{\ell\neq0} = 1 \).

Let us translate these expressions into the superconformal operators

\[
\mathcal{O}_n^{(0)}(0) = P_n^{(0)}(\partial_{Z_1}; \partial_{Z_2}) \left. \text{tr} \left[ \Phi(Z_1)\Phi(Z_2) \right] \right|_{Z_1=Z_2=0},
\]

with the \( \Phi \)-superfield given by (1.6). One finds from (4.34) that \( \mathcal{O}_n^{(0)}(0) \) vanishes for \( 0 \leq n < 4 \). For \( n = 4 \), it follows from (4.36) that (up to an overall normalization factor)

\[
\mathcal{O}_4^{(0)}(0) = \frac{1}{4} \varepsilon_{A_1A_2A_3A_4} \partial_{\theta_1A_1} \partial_{\theta_1A_2} \partial_{\theta_2A_3} \partial_{\theta_2A_4} \left. \text{tr} \left[ \Phi(Z_1)\Phi(Z_2) \right] \right|_{Z_1=Z_2=0} = -\frac{1}{2} \text{tr} \left[ \tilde{\phi}_{A_1A_2}(0) \phi^{A_1A_2}(0) \right],
\]

where \( \phi^{A_1A_2} = \frac{1}{2} \varepsilon_{A_1A_2A_3A_4} \tilde{\phi}_{A_1A_4} \). For \( n \geq 5 \), the substitution of (4.37) into (4.38) yields

\[
\mathcal{O}_n^{(0)}(0) = \sigma_n \text{tr} \left\{ 2 \lambda_A(0) \mathcal{P}^{(1,1)}_{n-5} (\partial_+, \partial_+) \right. \lambda^A(0)
- 2 \partial_+ A(0) \mathcal{P}^{(2,2)}_{n-6} (\partial_+, \partial_+) \partial_+ \bar{A}(0) - \frac{1}{2} \mathcal{P}^{(0,0)}_{n-4} (\partial_+, \partial_+) \phi^{AB}(0) \right\},
\]

24
with \( \sigma_n = [1 + (-1)^n]/2 \). It reproduces the operator \( S^2 \mathbf{g}_{n+5} \) from Ref. [20]. This operator carries the \( SL(2|4) \) superconformal spin \( j = 2j_\Phi + n = n - 1 \) and its one-loop anomalous dimension is given by \( E_{\Phi \Phi} (n - 1) \), Eqs. (1.9) and (1.10). In the forward limit, i.e., for \( \overrightarrow{\partial} + \overleftarrow{\partial} = 0 \), the operator (4.40) reduces to

\[
\mathcal{O}^{(0)}_n(0) \sim \sigma_n \text{tr} \left\{ \partial_+ A(0) \partial_{\alpha}^{-5} \bar{A}(0) - \frac{n-2}{n-5} \bar{\lambda}_A(0) \partial_+^{n-5} \lambda^4(0) \right. \\
- \left. \frac{(n-2)(n-3)}{4(n-4)(n-5)} \phi_{AB}(0) \partial_+^{n-4} \phi^{AB}(0) \right\}.
\] (4.41)

Again it is in accordance with Refs. [21, 10].

We remind that the twist-two operators constructed in this section are the lowest weights of the superconformal \( SL(2|N) \) group. The remaining twist-two operators belonging to the same supermultiplet are defined by the “descendant” superconformal polynomials \( P_n^{(\ell)}(X_1, X_2) \) with \( \ell \geq 1 \). These polynomials are obtained from the lowest weight \( P_n^{(0)}(X_1, X_2) \) by applying the raising operators (3.32).

5. Conclusions

In the present work we have formulated an efficient framework for constructing supermultiplets of conformal operators in supersymmetric Yang-Mills theories with an arbitrary number of supercharges. These operators belong to irreducible representations of the “collinear” \( SL(2|N) \) group and their mixing with other operators is protected to one-loop order by superconformal symmetry. The central point in our analysis was representation for the superconformal operators in terms of the superfields within the light-cone formalism. The latter turns out to be advantageous as compared with the conventional covariant formulation as it naturally accommodates supersymmetric models with an arbitrary number of supersymmetries \( 0 \leq N \leq 4 \) and treats them in the same fashion.

Our approach is based on a map of local Wilson operators into polynomials defining the lowest weights in the tensor product of two representations of the “collinear” \( SL(2|N) \) supergroup. As an outcome of the present analysis we have found a concise expression for all superconformal operators of twist two in supersymmetric Yang-Mills theories with \( 0 \leq N \leq 4 \). It has a remarkably simple form and exhibits universal features which are not obvious or hidden if addressed by other means. When written in components, the obtained expressions coincide with the known twist-two operators derived by the conventional technique based on a brute force inspection of transformation properties of local Wilson operators under the action of supersymmetry generators and closure of their algebra [7, 20]. These operators form a basis of eigenstates of the one-loop dilation operators in the underlying gauge theories and enter into the operator product expansion of diverse correlation functions, see, e.g., [22, 23] and references therein.

Acknowledgements

This work was supported in part by the grant 03-01-00837 of the Russian Foundation for Fundamental Research (A.M. and S.D.) and by the grant VH-NG-004 of the Helmholtz Association (A.M.). One of us (G.K.) is grateful to V. Braun for useful discussions and hospitality at the Institute for Theoretical Physics, University of Regensburg.
A Appendix A: Jacobi polynomials

The $SL(2)$ conformal operator $O_{j,j}(n)$, Eq. \[ \text{(2.23)}, \] is expressed in terms of the Jacobi polynomial $P_{n}^{(2j_{1}+1,2j_{2}-1)}(x)$. This polynomial is defined as

$$P_{n}^{(\alpha,\beta)}(x) = \frac{\Gamma(n+1+\beta)}{n!\Gamma(1+\beta)} \left(\frac{x-1}{2}\right)^{n} 2F_{1}\left(-n,-n-\alpha;\beta+1;\frac{x+1}{x-1}\right) \quad \text{(A.1)}$$

and has the following properties:

- **parity**
  $$P_{n}^{(\alpha,\beta)}(x) = (-1)^{n}P_{n}^{(\beta,\alpha)}(-x) \quad \text{(A.2)}$$

- **asymptotic behaviour**
  $$P_{n}^{(\alpha,\beta)}(x) = \frac{\Gamma(2n+\alpha+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!} (x/2)^{n} + O(x^{n-1}) \quad \text{(A.3)}$$

- **relation to the Gegenbauer polynomial**
  $$P_{n}^{(\alpha,\alpha)}(x) = C_{n}^{\alpha+1/2}(x) \frac{\Gamma(2\alpha+1)\Gamma(n+\alpha+1)}{\Gamma(n+2\alpha+1)\Gamma(\alpha+1)} \quad \text{(A.4)}$$

The $SL(2)$ conformal operators $O_{j,j}(n)$, $O_{j_{1}+1,j_{2}-1}(0)$ and $O_{j_{1}+1,j_{2}-2}(0)$ carry the same conformal spin $j_{1} + j_{2} + n$ and are not independent. The relations between them can be established with a help of two identities (with $\alpha, \beta \geq 0$)

$$P_{n}^{(-\alpha,\beta)}(x) = \left(\frac{x-1}{2}\right)^{\alpha} P_{n-\alpha}^{(\alpha,\beta)}(x) \frac{\Gamma(n+1-\alpha)\Gamma(n+1+\beta)}{\Gamma(n+1-\alpha+\beta)\Gamma(n+1)}, \quad \text{(A.5)}$$

$$P_{n}^{(-\alpha,-\beta)}(x) = \left(\frac{x-1}{2}\right)^{\alpha} \left(\frac{x+1}{2}\right)^{\beta} P_{n-\alpha-\beta}^{(\alpha,\beta)}(x), \quad \text{(A.6)}$$

valid for $n - \alpha \geq 0$ and $n - \alpha - \beta \geq 0$, respectively.

B Appendix B: $SL(2|\mathcal{N})$ collinear group

The superfield $\Phi_{j}(Z) \equiv \Phi_{j}(z, \theta^{A})$ defines a representation of the $SL(2|\mathcal{N})$ group labelled by its superconformal spin, $\mathbb{V}_{j}$. The superconformal invariance implies that the evolution equation \[ \text{(1.8)} \] has to be invariant under the $SL(2|\mathcal{N})$ transformations of superfields \[ \text{(3.3)} \] generated by the operators \[ \text{(3.6)}. \]

The scalar product on $\mathbb{V}_{j}$ is given by \[ \text{(3.21)} \] and \[ \text{(3.22)}. \] It is invariant under the superconformal transformations $\Psi(Z) \rightarrow e^{iG} \Psi(Z)$

$$\langle e^{iG} \Psi | e^{iG} \Psi' \rangle = \langle e^{-iG^\dagger} e^{iG} \Psi | \Psi' \rangle = \langle \Psi | \Psi' \rangle, \quad \text{(B.1)}$$

provided that $G$, given by a linear combination of the $SL(2|\mathcal{N})$ generators \[ \text{(3.6)} \], is a self-adjoint operator, $G^\dagger = G$. In particular, the operators $L^{-}$, $L^{+}$ and $L^{0}$ generate projective, $SU(1,1)$ transformations on the light-cone

$$e^{iaL^{0}} \Phi_{j}(Z) = (e^{ia})^{j} \Phi_{j}(e^{ia} z, e^{i\alpha} \theta^{A}),$$

$$e^{iaL^{+} - iaL^{-}} \Phi_{j}(Z) = \frac{1}{(bz + a)^{2j}} \Phi_{j} \left(\frac{az + b}{bz + a}, \frac{\theta^{A}}{bz + a}\right), \quad \text{(B.2)}$$

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with \( a = \cosh |\alpha| \) and \( b = i(\alpha/\alpha)^{1/2} \sinh |\alpha| \). The odd operators \( W^{A,\pm} \) and \( V^{\pm}_A \) generate conformal transformations in the superspace and satisfy (3.25). For arbitrary \( \mathcal{N} \) these transformations have a rather complicated form whereas for \( \mathcal{N} = 1 \) they look as

\[
e^{\varepsilon W^- + \varepsilon V^+} \Phi_j(Z) = \Phi_j((z + \varepsilon \theta)\rho, (\theta + \varepsilon z)/\rho),
\]

\[
e^{\varepsilon W^+ + \varepsilon V^-} \Phi_j(Z) = \frac{\rho^{2j}}{(1 - \varepsilon \theta)^{2j}} \Phi_j \left( \frac{z \rho}{1 - \varepsilon \theta}, \theta + \varepsilon \right), \tag{B.3}
\]

where \( \rho = 1 + \varepsilon \varepsilon / 2 \).

The superfield \( \Phi_j(Z) \) admits the following integral representation

\[
\Phi_j(Z) = \Gamma(2j) \int [D\mathcal{W}]_j (1 - Z \cdot \mathcal{W})^{-2j} \Phi_j(W), \tag{B.4}
\]

where \( Z = (z, \theta^A), W = (w, \vartheta^A) \) and \( W \cdot \bar{Z} = w \bar{z} + \vartheta^A \bar{\theta}_A \). To verify (B.4), one substitutes \( \Phi_j(W) \) by its Taylor expansion around the origin \( \Phi_j(W) \sim w^n \vartheta^A_1 \ldots \vartheta^A_L \) and performs integration with the help of (3.24). Applying (B.4), the superconformal operator (1.13) can be written as

\[
\mathcal{O}^{(0)}_n(0) = \left. P^{(0)}_n(\partial Z_1, \partial Z_2) \int \prod_{k=1}^2 [D\mathcal{W}_k]_{jk} \Gamma(2j_k)(1 - Z_k \cdot \bar{W}_k)^{-2j_k} \mathcal{O}(W_1, W_2) \right|_{Z_{1,2}=0}. \tag{B.5}
\]

Matching this identity into (3.27), one arrives at the relation (3.30) which is valid for arbitrary \( \mathcal{N} \). For \( \mathcal{N} = 0 \), it leads to (2.26).

A general expression for the superconformal polynomial \( P(X_1, X_2) \) is given by (4.3). To obtain this expression, one substitutes (4.2) into (3.28) and applies (4.1)

\[
P(X_1, X_2) = \sum_{k_1, k_2=0}^{\mathcal{N}} \frac{1}{(\mathcal{N} - k_1)!(\mathcal{N} - k_2)!} \int [Dz_1]_{j+k_1/2} \int [Dz_2]_{j+k_2/2} \bar{z}^{n_{12}} e^{z_1 x_1 + z_2 x_2} \tag{B.6}
\]

\[
\times \int \prod_{A=1}^{\mathcal{N}} (d\theta_1, d\theta_1^A) \int \prod_{A=1}^{\mathcal{N}} (d\theta_2, d\theta_2^A) \langle \theta_1 \cdot \theta_1 \rangle^{\mathcal{N}-k_1} \langle \theta_2 \cdot \theta_2 \rangle^{\mathcal{N}-k_2} \varphi_M(\theta_{12}) e^{\theta_1 \cdot \theta_1 + \theta_2 \cdot \theta_2}.
\]

The \( z \)-integral is the same as in (2.25), and is given by the Jacobi polynomial (2.28). The \( \theta \)-integral can be easily evaluated with a help of (3.20).

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