Exponential convergence in Wasserstein metric for distribution dependent SDEs

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Abstract

The existence and uniqueness of stationary distributions and the exponential convergence in $L^p$-Wasserstein distance are derived for distribution dependent SDEs from associated decoupled equations. To establish the exponential convergence, we introduce a twinned Talagrand inequality of the original SDE and the associated decoupled equation, and explicit convergence rate is obtained. Our results can be applied to SDEs without uniformly dissipative drift and distribution dependent diffusion term, which cover the Curie-Weiss model and the granular media model in double-well landscape with quadratic interaction as examples.

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1 Introduction

Stochastic differential equations with distribution dependent drifts were introduced by McKean [17] to investigate Vlasov-Poisson-Fokker-Planck systems. These type SDEs have attracted great attention since then, see e.g. [5, 16, 21] and recent works [1, 2, 3, 11, 14, 24, 27] with references therein. Let $\mathcal{P}$ denote the space of probability measures on $\mathbb{R}^d$ equipped with the weak topology. Consider the following distribution dependent SDE on $\mathbb{R}^d$

$$dX_t = b(X_t, \mathcal{L}_{X_t})dt + \sigma(X_t, \mathcal{L}_{X_t})dB_t, \quad t \geq 0,$$

where $\{B_t\}_{t \geq 0}$ is a $d$-dimensional Brownian motion on a complete filtration probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, $\mathcal{L}_{X_t}$ is the law of $X_t$ under $\mathbb{P}$, and

$$b: \mathbb{R}^d \times \mathcal{P} \to \mathbb{R}^d, \quad \sigma: \mathbb{R}^d \times \mathcal{P} \to \mathbb{R}^d \otimes \mathbb{R}^d$$
are measurable. If $\sigma(x, \mu) = \sigma(x)$ is independent of $\mu$, (1.1) is also called the McKean-Vlasov SDE. If $b(x, \mu) = b(x)$ moreover, (1.1) becomes the classical time homogenous Itô-type SDE. Distribution dependent SDEs can be derived from the associated interacting particles system by passing to the mean field limit, and the distribution dependent part of coefficients reflects the interaction of the particles system, see [21] for example. The well-posedness for (1.1) in the weak and strong sense has been intensively investigated, see e.g. [11, 19, 24, 27, 30] and references within.

The convergence to the equilibrium of the solution to McKean-Vlasov SDEs has been widely studied. In the case that $\sigma(x, \mu) = \sqrt{2I}$ with $I$ the identity matrix on $\mathbb{R}^d$ and $b(x, \mu) = -\nabla V(x) - \nabla F \ast \mu(x)$ where $V, F \in C^2(\mathbb{R}^d)$ with $F(-x) = F(x)$, $\nabla$ is the gradient operator and $\ast$ stands for the convolution on $\mathbb{R}^d$: $f \ast \mu(x) = \int_{\mathbb{R}^d} f(x-y) \mu(dy), f \in \mathcal{B}(\mathbb{R}^d), [4]$ obtained the explicit exponential convergence in mean field entropy for (1.1) in a variety of convexity conditions on confining potential $V$ and interaction potential $F$. Recently, [10] generalized results in [4] by using functional inequalities and establishing detail estimates on the associated interaction particles system. [24] obtained existence and uniqueness of stationary probability measures and the exponential convergence in Wasserstein distance for (1.1) with dissipative drifts and distribution dependent $\sigma$ satisfying

$$2(b(x, \mu) - b(y, \nu), x-y) + \|\sigma(x, \mu) - \sigma(y, \nu)\|^2_{HS} \leq C_1 W_2(\mu, \nu)^2 - C_2 |x-y|^2, x, y \in \mathbb{R}^d, \mu, \nu \in \mathcal{P}^2. \quad (1.2)$$

By using the log-Harnack inequality and the Talagrand inequality, [18] established the exponential convergence in classical entropy and Wasserstein distance under general setting of $b$ but distribution-free $\sigma$, which extended researches of [3, 10, 24]. For the general non-convex case, drift term $b$ is not uniformly dissipative w.r.t. the first variable, i.e. (1.2) holds only for large $|x-y|$. A quantitative method that combines Lyapunov functions with reflection coupling and concave distance functions is developed to investigate the longtime behavior of McKean-Vlasov SDEs without uniformly dissipative drifts, see e.g. [8, 14, 26]. In this paper, we consider (1.1) with general distribution dependent $\sigma$ and without uniformly dissipative drifts. Many methods mentioned previously fail in this case.

In our previous paper [29], existence results on stationary probability measures and criteria on phase transition (the existence of multi-stationary states) have been investigated for (1.1). The phase transition can occur for the general non-convex case with strong interaction in particular, see [29] or [3, 5, 11, 22] as well as references within. In this paper, the existence and uniqueness of stationary probability measures and the exponential convergence are established under estimates of the weakness of the interaction, see $\delta_0, \delta_1$ and $\delta_2$ in Theorem 2.2, Theorem 2.3 and Theorem 2.4 below.

Given $\mu \in \mathcal{P}$, there is a decoupled equation associated with (1.1)

$$dX^\mu_t = b(X^\mu_t, \mu)dt + \sigma(X^\mu_t, \mu)dB_t. \quad (1.3)$$

Since $\mu$ is fixed, (1.3) is a classical time homogenous Itô-type SDE. When (1.3) is well-posed and $X^\mu_0 = x \in \mathbb{R}^d$, we denote by $P^\mu_t f(x) = \mathbb{E}_x(X^\mu_t)$, where $f \in \mathcal{B}_b(\mathbb{R}^d)$.
and \( B_b(\mathbb{R}^d) \) consists of all bounded Borel measurable functions on \( \mathbb{R}^d \). \( P_t^\mu \) is the Markov semigroup associated with (1.3). There are rich researches on classical Itô-type SDEs to establish the exponential convergence for the solution to (1.3), see e.g. [7, 8, 15, 25] and references therein. We derive existence and uniqueness of stationary distributions for (1.1) from the ergodicity of (1.3). For any \( p \geq 1 \), let \( P_p = \{ \mu \in \mathcal{P} \mid \| \mu \|_p := (\mu(| \cdot |^p))^{\frac{1}{p}} < \infty \} \).

We denote by \( W_p \) the \( L^p \)-Wasserstein distance on \( \mathcal{P}_p \):

\[
W_p(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(dx, dy) \right)^{\frac{1}{p}}, \mu, \nu \in \mathcal{P}_p,
\]

where \( \mathcal{C}(\mu, \nu) \) consists of all couplings of \( \mu \) and \( \nu \). \( \mathcal{P}_p \) becomes a complete metric space under the distance \( W_p \). We assume that \( (H) \) There is \( p \geq 1 \) so that for every \( \mu \in \mathcal{P}_p \), \( P_t^\mu \) has a unique invariant probability measure \( T_\mu \in \mathcal{P}_p \), and there are \( \bar{C} > 0, \bar{\lambda} > 0 \) independent of \( \mu \) such that

\[
W_p((P_t^\mu)^* \nu, T_\mu) \leq \bar{C} e^{-\bar{\lambda} t} W_p(\nu, T_\mu), \nu \in \mathcal{P}_p. \tag{1.4} \]

Combining this with the Talagrand inequality for (1.3), we prove the mapping \( \mu \to T_\mu \) is contractive on \( \mathcal{P}_p \) when \( b, \sigma \) depend weakly on the distribution, see Theorem 2.2 below. For concrete conditions that ensure \( (H) \), one can see Remark 2.2 or \( (A2) \) with Corollary 2.5 or \( (A2') \) with Corollary 2.7. To establish exponential convergence, besides using the Talagrand inequality of the stationary distribution (Theorem 2.3), we introduce a twinned Talagrand inequality of (1.1) and (1.3), see (Ta) and Theorem 2.4 for details.

This paper is structured as follows. Our main results and corollaries are stated in Section 2. Proofs of main results are given in Section 3, and proofs of corollaries are given in Section 4.

## 2 Main results and corollaries

### 2.1 Existence and uniqueness

We first present the existence and uniqueness of stationary distributions to (1.1), which is also equivalent to that there is a unique \( \mu \) such that (1.3) with initial distribution \( \mu \) has a unique weak solution. Hence, we only need to concern with the weak well-posedness of (1.3). We assume that \( b, \sigma \) satisfy the following assumption, which also implies the strong well-posedness of (1.3) indeed.

\( (A1) \) \( b \) is continuous in the first variable, \( \sigma \) is bounded and Lipschitz in the first variable, and there exist \( K_0 \in \mathbb{R} \) and \( \delta \geq 0 \) such that

\[
2(b(x, \mu) - b(y, \nu), x - y) + (1 + (p - 2)^+) \| \sigma(x, \mu) - \sigma(y, \nu) \|_{HS}^2 \leq K_0 |x - y|^2 + \delta^2 W_p(\mu, \nu)^2, \mu, \nu \in \mathcal{P}_p. \tag{2.1} \]
To see the strong well-posedness of (1.3) with fixed \( \mu \), we can set \( \nu = \mu \) and \( y = 0 \) in (2.1). Then, taking into account that \( \sigma(x, \mu) \) is bounded in \( x \), there is \( C > 0 \) such that

\[
2\langle b(x, \mu), x \rangle + \|\sigma(x, \mu)\|_{HS}^2 \leq K_0|x|^2 + 2\langle b(0, \mu), x \rangle + \|\sigma(0, \mu)\|_{HS}^2 \\
+ \|\sigma(x, \mu)\|_{HS}\|\sigma(0, \mu)\|_{HS} \\
\leq C(1 + |x|^2).
\]

Combining this with (2.1) (setting \( \nu = \mu \)), it follows from Krylov’s criterion, see e.g. [13, Theorem 3.1.1], that (1.3) has a unique solution.

Before our first theorem, we give a simple proposition which indicates that under (H), (1.1) has a unique stationary probability measure for small \( \delta \).

**Proposition 2.1.** Assume that (H) and (A1) hold. If \( \delta < \delta_0 \) with

\[
\delta_0 := \sup_{t > \lambda^{-1}\log \hat{C}} \left\{ \left( \frac{2t(1 - \exp\{-\frac{(p^2/2)K_0 + (p-2)^+}{(p \vee 2)K_0 + (p-2)^+}\})^{\frac{1}{p^\vee 2}}}{(1 - \hat{C}e^{-\lambda t})} \right) \right\},
\]

then there is a unique stationary probability measure for (1.1).

Let \( H(\nu|\mu) \) be the relative entropy of \( \nu \) with respect to \( \mu \):

\[
H(\nu|\mu) = \begin{cases} 
\int_{\mathbb{R}^d} \log \frac{d\nu}{d\mu} d\nu, & \text{if } \nu \ll \mu, \\
\infty, & \text{otherwise.}
\end{cases}
\]

If the invariant probability measure of \( P^\mu_t \) satisfies the Talagrand inequality, then we have the following theorem. Let

\[
K(m, p) = 2^{-\frac{1}{p^\vee 2}} ((p \vee 2)(2m - K_0) - (p - 2)^+) \frac{1}{p^\vee 2}.
\]

**Theorem 2.2.** Assume (H), (A1), and there is \( \sigma_0 > 0 \) such that

\[
\sigma(x, \mu)\sigma^*(x, \mu) \geq \sigma_0^2, \quad x \in \mathbb{R}^d, \mu \in \mathcal{P}_p.
\]

Suppose that there is \( \kappa > 0 \) such that for all \( \mu \in \mathcal{P}_p \), the invariant probability measure \( T_\mu \) of \( P^\mu_t \) satisfies

\[
W_p(\nu, T_\mu) \leq \sqrt{2\kappa H(\nu|T_\mu)}, \quad \nu \in \mathcal{P}_p.
\]

If \( \delta < \delta_0 \) with

\[
\delta_0 = \sup_{t > t_0, m > m_0} \frac{\sigma_0(1 - \hat{C}e^{-\lambda t})K(m, 2)[K(m, p) \vee t^{-\frac{1}{p^\vee 2}}]}{\sigma_0K(m, 2) + m\sqrt{\kappa t}[K(m, p) \vee t^{-\frac{1}{p^\vee 2}}]}, \\
t_0 = \lambda^{-1}\log \hat{C}, \quad m_0 = \left( \frac{(p - 2)^+}{2(p \vee 2)} + \frac{K_0}{2} \right)^+, \n\]

then there is a unique stationary probability measure for (1.1).
2.2 Exponential convergence

To investigate the exponential convergence in Wasserstein distance, we assume that the weak well-posedness of (1.1) holds. Let \( P^\ast_t \mu = \mathcal{L}^\mu_{X_t} \) be the law of weak solution with initial distribution \( \mu \). Since (2.1) implies the pathwise uniqueness of the following equation

\[
dX_t = b(X_t, P^\ast_t \mu)dt + \sigma(X_t, P^\ast_t \mu)dB_t,
\]

this equation has a unique strong solution due to the Yamada-Watanabe principle [12] and the weak well-posedness of (1.1). As a consequence, (1.1) is strong well-posed.

We first present a result without the assumption (H).

**Theorem 2.3.** Let \( p \geq 2 \), and let \( \mu \in \mathcal{P}^q \) with some \( q \geq p \). Assume (A1) and that \(|b(0, \cdot)|\) is locally bounded on \( \mathcal{P}^q \) and (2.2) holds for some \( \sigma_0 > 0 \). We also assume that (1.1) is weak well-posed for \( \mu \) and the mapping \( t \rightarrow P^\ast_t \mu \) is locally bounded in \( \mathcal{P}^q \). Suppose there is a unique stationary distribution \( \bar{\mu} \) for (1.1), and there is \( \kappa > 0 \) such that (2.3) holds with \( T_\mu \) replaced by \( \bar{\mu} \). If

\[
K_0 < \frac{\sigma_0^2}{2^{3-\frac{4}{p}}} - \left( \sqrt{\frac{p-2}{p}} - \sqrt{\frac{\sigma_0^2}{2^{3-\frac{4}{p}}}} \right)^2,
\]

then there is \( \delta_1 > 0 \) such that for (1.1) with \( \delta < \delta_1 \), there exist \( \bar{C} > 0, \bar{\lambda} > 0 \) so that

\[
W_p(P^\ast_t \mu, \bar{\mu}) \leq \bar{C}e^{-\bar{\lambda}t}W_p(\mu, \bar{\mu}).
\]

In particular, if \( p = 2 \), we have

\[
\delta_1 \geq \sqrt{(2\hat{m} - K_0)\hat{\beta}^{-1}(\Phi(2) \wedge \frac{1}{2})},
\]

\[
\bar{\lambda} \geq \frac{\delta^2\hat{\beta}}{2} \left( u - \frac{(1 + u) \log(2u)}{\log(\beta + (\beta - 2)u)} \right),
\]

where \( u = \frac{2\hat{m} - K_0}{\hat{\beta}^2} \), \( \hat{m} = \frac{\sigma_0^2}{2\hat{\beta}} \), \( \hat{\beta} = 2(1 + \frac{\hat{m}^2}{\sigma_0^2(2\hat{m} - K_0)}) \) and

\[
\Phi(x) = \inf\{ v > 0 \mid v(\nu\beta + \frac{\beta - 2}{\nu})^{-\frac{1}{\nu}} \leq x \}, \ x > 0.
\]

Instead of the Talagrand inequality for \( \bar{\mu} \), we can also use the twinned Talagrand inequality of \( (P^\ast_t)^* \) and \( P_t^\ast \) to obtain the exponential convergence:

**(Ta)** There exist nonempty \( C \subset \mathcal{P} \) and \( \kappa_t > 0 \) such that \( P_t^\ast C \subset C \) and

\[
W_p(\nu, (P_t^\ast)^\ast \mu) \leq \sqrt{2\kappa_t H(\nu)(P_t^\ast)^\ast \mu), \ \nu \in \mathcal{P}^p, t > 0, \mu \in C.
\]

The nonempty \( C \) can not contain all probability measures in \( \mathcal{P}^p \) usually. See (A2), Example 2.6 and Lemma 4.1 for concrete conditions that ensure (Ta).
Theorem 2.4. Let $\mu \in \mathcal{P}^q$ with some $q \geq p$. The assumption of Theorem 2.3 hold except the Talagrand inequality. Assume that (H) holds and there is a unique stationary distribution $\bar{\mu}$ for $(1.1)$ satisfying (Ta). Let

$$\gamma(\delta, t, \theta) = \hat{C} \left( \frac{1 + \theta}{\theta} \right)^{1 - \frac{1}{2vp}} \gamma_1(\delta, t, \theta) \frac{1}{\sqrt{v}p} e^{-\lambda t}, \delta, \theta, t > 0,$$

with

$$\gamma_1(\delta, t, \theta) = 1 + \frac{\delta^{2vp}}{(1 + \theta)^{1 - 2vp}} \left[ \int_0^t \exp \left\{ \int_s^t \left( \frac{C_1(r)\delta^{2vp}}{(1 + \theta)^{1 - 2vp}} + (2 \vee p)\lambda \right) dr \right\} ds \right],$$

$$C_1(t) = \left( 1 + \frac{\sqrt{\frac{p - 2}{p}}}{2\sigma_0 \sqrt{|K_0|} \sqrt{\frac{p - 2}{p^2}}} \right).$$

Let

$$\delta_2 = \inf \left\{ \delta > 0 \mid \inf_{t, \theta > 0} \gamma(\delta, t, \theta) \geq 1 \right\}.$$ 

Then $\delta_2 > 0$, and for $(1.1)$ with $\delta < \delta_2 \wedge \delta_0$, there are $\hat{C} > 0, \bar{\lambda} > 0$ such that $(2.5)$ holds for every $\mu \in \mathcal{P}^q \cap \mathcal{C}$. Let $t_1 > 0, \theta_1 > 0$ such that $\gamma(\delta, t_1, \theta_1) < 1$. Then $\bar{\lambda} \geq t_1^{-1} \log \frac{1}{\gamma(\delta, t_1, \theta_1)}$.

Remark 2.1. Recently, exponential convergence in the total variation distance for (reflecting) McKean-Vlasov SDEs has been investigated in [28, Theorem 2.4]. The following condition are used to character the dependence of $b(x, \mu)$ on $\mu$

$$|b(x, \mu) - b(x, \nu)| \leq c\|\mu - \nu\|_{\text{var}}, \mu, \nu \in \mathcal{P}, x \in \mathbb{R}^d,$$

where $\| \cdot \|_{\text{var}}$ denotes the total variation norm. When the constant $c$ is small enough, the existence and uniqueness and the exponential convergence can be established. We adapted a similar argument as in [28] but investigate the exponential convergence in Wasserstein distance for equations with distribution dependent $\sigma$. Since $\sigma$ here can be distribution dependent, the coupling used in [28] can not be applied.

Remark 2.2. If (A1) holds with $K_0 < 0$ and $p \leq 2$, then $m_0 = 0$ and $(1.4)$ holds with $\hat{C} = 1$, $\lambda = -K_0$. If $\sigma$ is bounded in addition, then $(2.3)$ holds with $\kappa = \frac{\|\sigma\|_{K_0}}{K_0}$, see e.g. [6, Theorem 5.6]. In this situation,

$$\delta_0 = \sup_{t > t_0} \frac{(1 - \hat{C}e^{-\bar{\lambda}t})(K(0, 2))^2}{K(0, 2)} = \sqrt{-K_0}.$$ 

Then $\delta < \delta_0$ if and only if $\delta^2 + K_0 < 0$, which is the condition “$C_2 > C_1$” used in [24, Theorem 3.1 (2)] for $(1.2)$. Due to $K_0 < 0$ and $p \leq 2$, $C_1(t) \equiv 1$ and

$$\inf_{t, \theta > 0} \gamma(\delta, t, \theta)^2 = \inf_{t, \theta > 0} \left\{ \frac{(1 + \theta)\left( (1 + \theta)\delta^2 e^{(1 + \theta)\delta t} + 2\bar{\lambda}e^{-2\bar{\lambda}t} \right)}{\theta((1 + \theta)\delta^2 + 2\bar{\lambda})} \right\}$$

$$= \inf_{\theta > 0} \left\{ \frac{1 + \theta}{\theta} \left( \frac{2\bar{\lambda}e^{-2\theta} \frac{\lambda^2}{(1 + \bar{\theta})^{2\lambda}} + 1 \wedge 1}{1 + \theta} \right) \right\}.$$
In this case, we can derive exponential convergence rate $\bar{\lambda}$ by [15, 25]. We use the following condition which is a modified form of [25, (2.23)].

Our results can be applied to SDEs without uniformly dissipative drifts investigated

2.3 Corollaries and examples

Our results can be applied to SDEs without uniformly dissipative drifts investigated by [15] [25]. We use the following condition which is a modified form of [25, (2.23)]. In this case, we can derive exponential convergence rate $\bar{\lambda}$ involving $\hat{C}$ and $\lambda$.

(A2) $b$ are continuous in the first variable; $\sigma$ is Lipschitz in the first variable and satisfies

$$\|\sigma\|_\infty := \sup_{x \in \mathbb{R}^d, \mu \in \mathcal{P}^1} \|\sigma(x, \mu)\|_{HS} < \infty, \quad (2.9)$$

there exist constant $r_0 > 0, K_0 \geq 0, K_1 > 0$ and $\delta \geq 0$ such that $\mu, \nu \in \mathcal{P}^1$

$$2(b(x, \mu) - b(y, \nu), x - y) + \left(2\frac{\|\sigma\|^2}{\sigma_0^2} - 1_{[d=1]}\right) \|\sigma(x, \mu) - \sigma(y, \nu)\|_{HS}^2$$

$$\leq ((K_0 + K_1)1_{|x-y| \leq r_0} - K_1) |x - y|^2 + \delta^2 W_1(\mu, \nu)^2.$$  

Then we have the following corollary.

**Corollary 2.5.** Assume that (A2) holds. Then there are $\lambda > 0$ and $\hat{C} \geq 1$ such that for $\delta < \delta_0$ with

$$\delta_0 = \sup_{t > \lambda^{-1} \log \hat{C}} \frac{\sigma_0(1 - \hat{C}e^{-\lambda t})}{\sqrt{2\sigma_0\|\sigma\|_{HS}} \sqrt{K_1 t + K_0\|\sigma\|^2_{HS}}}$$

there is a unique stationary probability measure for (1.1).

Suppose $|b(0, \cdot)|$ is locally bounded on $\mathcal{P}^1$. Let $\mu \in \mathcal{P}$ satisfy

$$\int_{\mathbb{R}^d} e^{\theta|x-y|^2} \mu(dx)\mu(dy) < \infty, \quad \theta < \frac{K_1}{4\|\sigma\|^2_{\infty}} \wedge \frac{K_3}{4\|\sigma\|^2_{\infty}} \quad (2.10)$$

for some $K_3 > 0$, and let

$$\alpha = \left(1 + \frac{\|\sigma\|_{\infty}}{\sigma_0} \sqrt{\frac{K_0}{K_1 \wedge K_3}} \right)^2, \quad \delta_2 = \sqrt{\lambda^{-1} \Phi(2\hat{C}^2)}$$

where

$$\Phi(u) = \inf \left\{ v > 0 \mid \frac{\log(u)}{u} \leq v \right\}, \quad u > 0.$$  

Then for (1.1) with $\delta < \delta_2 \wedge \delta_0$, there are $C > 0, \bar{\lambda} > 0$ such that

$$W_1(P_t^\mu, \bar{\mu}) \leq \hat{C} e^{-\bar{\lambda} t} W_1(\mu, \bar{\mu}).$$

with

$$\bar{\lambda} = \left(\frac{\alpha \delta^2 + \bar{\lambda}}{2}\right) \left(\frac{\log(2\hat{C}^2)}{\log(\lambda/\alpha \delta^2)} + \frac{\alpha \delta^2 - \bar{\lambda}}{\alpha \delta^2 + \bar{\lambda}}\right).$$
Example 2.6. Let $\sigma$ satisfy (2.7), (2.9) and there $C_1, \delta > 0$ so that
\[
\left(2\frac{\|\sigma\|_{\infty}^2}{\sigma_0^2} - \mathds{1}_{[d=1]}\right) \|\sigma(x, \mu) - \sigma(y, \nu)\|_{H^1}^2 \leq C_1|x - y|^2 + \frac{\delta^2}{2}W_1(\mu, \nu)^2.
\]
Let $b$ be of the following form
\[
b(x, \mu) = b_1(x) + \int_{\mathbb{R}^d} b_2(x, z)\mu(dz), \quad x \in \mathbb{R}^d,
\]
where $b_1(x) = -2\theta_1|x|^2 + \frac{\theta_2}{2}x^2$ with some $\theta_1 > 0, \theta_2 \geq 0$, $b_2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and
\[
C_2 := \sup_{(x, z) \in \mathbb{R}^{2d}} \left|\partial_1 b_2(x, z)\right| < \infty, \quad \sup_{(x, z) \in \mathbb{R}^{2d}} \left|\partial_2 b_2(x, z)\right| \leq \frac{\delta}{\sqrt{2}}.
\]
Then
\[
2\langle b_1(x) - b_1(y), x - y \rangle \leq -\theta_1|x - y|^4 + \theta_2|x - y|^2.
\]
and
\[
2\langle \mu(b_2(x, \cdot)) - \nu(b_2(y, \cdot)), x - y \rangle
\leq 2\langle \mu(b_2(x, \cdot) - b_2(y, \cdot)) + \mu(b_2(y, \cdot)) - \nu(b_2(y, \cdot)), x - y \rangle
\leq 2C_2|x - y|^2 + 2\int_{\mathbb{R}^d} (b_2(y, z_1) - b_2(y, z_2), x - y)\pi(dz_1, dz_2)
\leq 2C_2|x - y|^2 + \sqrt{2}\delta W_1(\mu, \nu)|x - y|
\leq (2C_2 + 1)|x - y|^2 + \frac{\delta^2}{2}W_1(\mu, \nu)^2.
\]
Therefore, (A2) holds.

If $\sigma(x, \mu)$ is independent of $x$, then we can assume
(A2') $b$ is continuously differentiable in the first variable with $\sup_{\mu \in \mathcal{P}_2} |b(0, \mu)| < \infty$, and there exist $r_0 > 0, K_0 \geq 0, K_1 > 0$ and $\delta \geq 0$ such that
\[
2\langle b(x, \mu) - b(y, \nu), x - y \rangle + \|\sigma(\mu) - \sigma(\nu)\|_{H^1}^2
\leq \left((K_0 + K_1)\mathds{1}_{[|x - y| \leq r_0]} - K_1\right)|x - y|^2 + \delta^2 W_2(\mu, \nu)^2, \quad x, y \in \mathbb{R}^d, \mu, \nu \in \mathcal{P}_2.
\]
Suppose that $\sigma$ also satisfies (2.2) and (2.9). Then (1.1) is well-posed by [24]. It follows from [25, Theorem 2.1(2)] that there are $\hat{\lambda} > 0$ and $\hat{C} \geq 1$ such that (1.4) holds and there is $\kappa > 0$ such that (2.3) holds with $p = 2$. Hence, we have the following corollary.

Corollary 2.7. Assume that (2.2), (2.9) and (A2'). Then there are $\hat{\lambda} > 0$, $\hat{C} \geq 1$ and $\kappa > 0$ such that for $\delta < \delta_0$ with
\[
\delta_0 = \sup_{t > \lambda^{-1} \log \hat{C}} \frac{\sigma_0(1 - \hat{C} e^{-\lambda t})}{\sqrt{\kappa(2\sigma_0 \sqrt{t + K_0} \kappa t)}},
\]
there is a unique stationary probability measure for (1.1). If $K_0 < \frac{\sigma_0^2}{2\kappa}$ in addition, we obtain the exponential convergence for all $\mu \in \mathcal{P}_2$ with $\delta_1$ and the convergence rate $\hat{\lambda}$ given by (2.6) and (2.7).
3 Proofs of Proposition 2.1 and theorems

Proof of Proposition 2.1

We prove that the mapping \( \mu \to T_\mu \) is contractive on \( \mathcal{P}^p \). Let \( \nu, \mu \in \mathcal{P}^p \). It follows from (1.3) that

\[
W_p(T_\mu, T_\nu) \leq W_p(T_\mu, (P^\mu)^* T_\nu) + W_p((P^\mu)^* T_\nu, T_\nu)
\]

Substituting this into (3.1), we arrive at

\[
\nu \leq e^{-\lambda t}W_p(T_\mu, T_\nu) + W_p((P^\mu)^* T_\nu, (P^\nu)^* T_\nu).
\]

This implies that

\[
\frac{d}{dt}|X^\mu_t - X^\nu_t|^p \leq \frac{p}{2} |X^\mu_t - X^\nu_t|^{p-2} \left\{ 2(b(X^\mu_t, \mu) - b(X^\nu_t, \nu), X^\mu_t - X^\nu_t) 
+ \|\sigma^\ast(X^\mu_t, \mu) - \sigma^\ast(X^\nu_t, \nu)\|_{HS}^2 + (p - 2)\|\sigma^\ast(X^\mu_t, \mu) - \sigma^\ast(X^\nu_t, \nu)\|^2 \right\} dt 
+ p|X^\mu_t - X^\nu_t|^{p-2}(X^\mu_t - X^\nu_t, (\sigma(X^\mu_t, \mu) - \sigma(X^\nu_t, \nu))dB_t)
\]

Substituting this into (3.1), we arrive at

\[
W_p((P^\mu)^* T_\nu, (P^\nu)^* T_\nu)^p \leq \mathbb{E}|X^\mu_t - X^\nu_t|^p
\]

This implies that

\[
W_p(T_\mu, T_\nu) \leq \hat{C}e^{-\hat{\lambda} t}W_p(T_\mu, T_\nu) + \delta \left( \frac{2t(1 - e^{-\frac{p(K_0+1)-2}{2}t})}{p(K_0 + 1) - 2} \right)^{\frac{1}{p}} W_p(\mu, \nu).
\]

Consequently, for \( t > \frac{1}{\hat{\lambda}} \log \hat{C} \),

\[
W_p(T_\mu, T_\nu) \leq \delta \left( \frac{2t(1 - e^{-\frac{pK_0+p-2}{2}t})}{pK_0 + (p - 2)} \right)^{\frac{1}{p}} (1 - \hat{C}e^{-\hat{\lambda} t})^{-1} W_p(\mu, \nu)
\]

Substituting this into (3.1), we arrive at

\[
W_p(T_\mu, T_\nu) \leq \delta \left( \frac{2t(1 - e^{-\frac{pK_0+p-2}{2}t})}{pK_0 + (p - 2)} \right)^{\frac{1}{p}} (1 - \hat{C}e^{-\hat{\lambda} t})^{-1} W_p(\mu, \nu).
\]
If \( p \in [1, 2) \), then it follows from (3.2) with \( p = 2 \) that
\[
\mathbb{E}|X_t^\mu - X_t^\nu|^2 \leq \int_0^t e^{-K_0(t-s)} \delta^2 W_p^2(\mu, \nu) ds = \frac{\delta^2 t(1 - e^{-K_0 t})}{K_0} W_p^2(\mu, \nu).
\]
This together with (3.1) and \( W_p \leq W_2 \) implies (3.3) for \( p < 2 \).

Therefore, by the definition of \( \delta_0 \),
\[
W_p(T_{\mu}, T_{\nu}) \leq \frac{\delta}{\delta_0} W_p(\mu, \nu), \quad \mu, \nu \in \mathcal{P}^p. \tag{3.4}
\]

The assertion follows by applying the Banach fixed point theorem to \( T \) on \( \mathcal{P}^p \).

In (3.1), we use the synchronous couplings for \( ((P^\mu_t)^* T_{\nu}, (P^\nu_t)^* T_{\nu}) \) to estimate \( W_p((P^\mu_t)^* T_{\nu}, T_{\nu}) \) under (A1). We can use the coupling by change of measure and the Talagrand inequality to estimate \( W_p((P^\mu_t)^* T_{\nu}, T_{\nu}) \).

**Proof of Theorem 2.2**

By (3.1), we focus on the estimate of \( W_p((P^\mu_t)^* T_{\nu}, T_{\nu}) \). To this aim, we first construct a coupling \((\tilde{X}_t^\mu, \tilde{X}_t^\nu)\) as follows
\[
\begin{align*}
d\tilde{X}_t^\mu &= b(\tilde{X}_t^\mu, \mu)dt + \sigma(\tilde{X}_t^\mu, \mu)dB_t \\
&\quad - m\sigma(\tilde{X}_t^\mu, \mu)\sigma^{-1}(X_t^\nu, \nu)(\tilde{X}_t^\mu - X_t^\nu)dt, \quad \tilde{X}_0^\mu = T_{\nu}, \\
d\tilde{X}_t^\nu &= b(\tilde{X}_t^\nu, \nu)dt + \sigma(\tilde{X}_t^\nu, \nu)dB_t, \quad \tilde{X}_0^\nu = X_0^\nu,
\end{align*}
\]
where \( m > m_0 \) is a constant and \( \tilde{X}_0^\nu = T_{\nu} \) means that \( \tilde{X}_0^\nu \) is a random variable with law \( T_{\nu} \). By (2.1) and that \( \sigma \) is bounded and Lipschitz in the first variable, this coupling is well-posed for any \( t > 0 \). Let
\[
\tilde{B}_t = B_t - m \int_0^t \sigma^{-1}(X_s^\nu, \nu)(\tilde{X}_s^\mu - X_s^\nu)ds.
\]

Then \((\tilde{X}_t^\mu, \tilde{X}_t^\nu)\) satisfies
\[
\begin{align*}
d\tilde{X}_t^\mu &= b(\tilde{X}_t^\mu, \mu)dt + \sigma(\tilde{X}_t^\mu, \mu)dB_t, \quad \tilde{X}_0^\mu = T_{\nu}, \\
d\tilde{X}_t^\nu &= b(\tilde{X}_t^\nu, \nu)dt + \sigma(\tilde{X}_t^\nu, \nu)dB_t + m(\tilde{X}_t^\mu - X_t^\nu)dt, \quad X_0^\nu = T_{\nu}.
\end{align*}
\]

Let
\[
R^{\mu, \nu}_t = \exp \left\{ \int_0^t \left< m\sigma^{-1}(X_s^\nu, \nu)(\tilde{X}_s^\mu - X_s^\nu), dB_s \right> + \frac{m^2}{2} \int_0^t \left| \sigma^{-1}(X_s^\nu, \nu)(\tilde{X}_s^\mu - X_s^\nu) \right|^2 ds \right\}. \tag{3.5}
\]

As in e.g. [23], by combining the Girsanov theorem with the stopping time technique, the Fatou lemma and the martingale convergence theorem, we can derive that for any \( t > 0 \), \( \{B_s\}_{0 \leq s \leq t} \) is a Brownian motion under \( Q = R^{\mu, \nu}_t \mathbb{P} \) and
\[
\sup_{s \in [0, t]} \mathbb{E} R^{\mu, \nu}_s \log R^{\mu, \nu}_s \leq \frac{tm^2 \delta^2 W_p^2(\mu, \nu)^2}{2\sigma_0^2(2m - K_0)}, \quad t > 0. \tag{3.6}
\]
It follows from the Itô formula that
\[ d|\tilde{X}_t^\mu - X_t^\nu|^2 \leq \left( (K_0 - 2m)|\tilde{X}_t^\mu - X_t^\nu|^2 + \delta^2 W_p(\mu, \nu)^2 \right) dt \\
+ 2\langle \tilde{X}_t^\mu - X_t^\nu, (\sigma(\tilde{X}_t^\mu, \mu) - \sigma(X_t^\nu, \nu))d\tilde{B}_t \rangle. \tag{3.7} \]

If \( p \geq 2 \), then it follows from the Itô formula that
\[ d|\tilde{X}_t^\mu - X_t^\nu|^p \leq \left( \frac{p(K_0 - 2m) + p - 2}{2} |\tilde{X}_t^\mu - X_t^\nu|^p + \delta^p W_p(\mu, \nu)^p \right) dt \\
+ p|\tilde{X}_t^\mu - X_t^\nu|^{p-2}\langle \tilde{X}_t^\mu - X_t^\nu, (\sigma(\tilde{X}_t^\mu, \mu) - \sigma(X_t^\nu, \nu))d\tilde{B}_t \rangle. \]

Combining this with (3.8), we have that
\[ \mathbb{E}_Q |\tilde{X}_t^\mu - X_t^\nu|^{2p} \leq \int_0^t e^{(\frac{p(2mK_0 - 2m)}{2} + (p-2)^+)(t-s)\delta^2 W_p(\mu, \nu)^{p/2}} ds \]
\[ \leq \frac{2\delta^2 W_p(\mu, \nu)^{p/2}(1 - e^{-\frac{(p-2)(2m - K_0) - (p-2)^+}{2}})}{(p \vee 2)(2m - K_0) - (p-2)^+} \]
\[ = \left( \frac{\delta W_p(\mu, \nu)}{K(m, p) \vee t^{1/p^2}} \right)^{p/2}. \tag{3.8} \]

Denote by \( \mathcal{L}_X^Q \) and \( \mathcal{L}_{\tilde{X}}^Q \) the law of \( X_t^\nu \) and \( \tilde{X}_t^\mu \) under \( Q \). Due to the uniqueness of (3.3), \( \mathcal{L}_X^Q = (\mathcal{P}^\mu)_* \tau^\nu \). Then
\[ \mathcal{L}_X^Q(f) = \mathbb{E}_Q f(X_t^\nu) = \mathbb{E} R_{\mu, \nu} f(X_t^\nu) \]
\[ = \mathbb{E} (\mathbb{E}[R_{\mu, \nu}|X_t^\nu]) f(X_t^\nu) \]
\[ = \int_{\mathbb{R}^d} (\mathbb{E}[R_{\mu, \nu}|X_t^\nu = x]) f(x) \mathcal{L}_X^Q(dx), \ f \in \mathcal{B}_b(\mathbb{R}^d). \]

Consequently, \( \mathcal{L}_X^Q \ll \mathcal{L}_{\tilde{X}}^Q \) with
\[ \frac{d\mathcal{L}_X^Q}{d\mathcal{L}_{\tilde{X}}^Q} = \mathbb{E}[R_{\mu, \nu}|X_t^\nu = x], \ \mathcal{L}_{\tilde{X}}^Q \text{-a.s.} \]

Hence, it follows from the Jensen inequality and (3.8) that
\[ H(\mathcal{L}_X^Q|\mathcal{L}_{\tilde{X}}^Q) = \mathbb{E}(\mathbb{E}[R_{\mu, \nu}|X_t^\nu] \log \mathbb{E}[R_{\mu, \nu}|X_t^\nu]) \]
\[ \leq \mathbb{E}(\mathbb{E}[R_{\mu, \nu} \log R_{\tilde{X}}|X_t^\nu] = \mathbb{E}[R_{\mu, \nu} \log R_{\tilde{X}}]) \]
\[ \leq \frac{m^2 t \delta^2 W_p(\mu, \nu)^2}{2\delta^2 (2m - K_0)} = t \left( \frac{m \delta W_p(\mu, \nu)}{\sqrt{2\delta^2 K(m, 2)}} \right)^2. \tag{3.9} \]

By (3.8), we have that
\[ W_p((P^\mu)_* \tau^\nu, \tau^\nu) \leq W_p((P^\mu)_* \tau^\nu, \mathcal{L}_X^Q) + W_p(\mathcal{L}_X^Q, \tau^\nu) \]
\[ \leq \left( \mathbb{E}|X_t^\mu - X_t^\nu|^{p/2} \right)^{2/p} + W_p(\mathcal{L}_X^Q, \tau^\nu). \]
Substituting this into (3.10), we arrive at

\[ H \leq \frac{\delta W_p(\mu, \nu)}{K(m, p) \vee t^{-1/2}} + W_p(\mathcal{L}_{X_t^\mu}^q, \mathcal{T}_t). \]  

(3.10) ine-Wp-mm0

It follows from (2.3) and (3.9) that

\[ W_p(\mathcal{L}_{X_t^\mu}^q, \mathcal{T}_t) \leq \sqrt{2\kappa H(\mathcal{L}_{X_t^\mu}^q | \mathcal{T}_t)} = \sqrt{2\kappa H(\mathcal{L}_{X_t^\mu}^p | (P_t)^\ast \mathcal{T}_t)} \]

\[ = \sqrt{2\kappa H(\mathcal{L}_{X_t^\mu}^q | \mathcal{L}_{X_t^\mu}^p)} \leq \frac{m\sqrt{kl}}{\sigma_0 K(m, 2)} \delta W_p(\mu, \nu). \]

Substituting this into (3.10), we arrive at

\[ W_p((P_t)^\ast \mathcal{T}_t, \mathcal{T}_t) \leq \left( \frac{1}{K(m, p) \vee t^{-1/2}} + \frac{m\sqrt{kl}}{\sigma_0 K(m, 2)} \right) \delta W_p(\mu, \nu). \]  

(3.11) ine-W-CH

Taking into account (1.4) and (3.1), we have that for \( t > \hat{\lambda}^{-1} \log \hat{C}, m > m_0 \)

\[ W_p(\mathcal{T}_t, \mathcal{T}_t) \leq \left( 1 - \hat{\lambda} e^{-\hat{\lambda}t} \right)^{-1} \left( \frac{1}{K(m, p) \vee t^{-1/2}} + \frac{m\sqrt{kl}}{\sigma_0 K(m, 2)} \right) \delta W_p(\mu, \nu). \]

Hence, it follows from the definition of \( \delta_0 \) that (3.4) holds.

Therefore, the assertion follows from (3.4) and the Banach fixed point theorem.

\[ \square \]

**Proof of Theorem 2.3**

Consider the coupling as follows

\[ d\hat{X}_t = b(\hat{X}_t, \mu_t)dt + \sigma(\hat{X}_t, \mu_t)dB_t \]

\[ - m\sigma(\hat{X}_t, \mu_t)\sigma^{-1}(\hat{\bar{X}}_t^\mu, \hat{\bar{\mu}})(\hat{X}_t - \hat{\bar{X}}_t^\mu)dt, \hat{X}_0 = \mu, \]

(3.12) tld\hat{X}_t

where \( m > m_0 \) and

\[ \mathbb{E}[\hat{X}_0 - \hat{\bar{X}}_0^\mu] = W_p(\mu, \hat{\bar{\mu}})^p. \]

Since \( t \to P_t^\mu \mu \) is locally bounded on \( \mathcal{P}^q \), (A1) and that \( |b(0, \cdot)| \) is locally bounded on \( \mathcal{P}^q \), (3.12) is well-posed for any \( t > 0 \). Then \( \mathcal{L}_{X_t^\mu}^p \equiv \hat{\mu} \) due to (A1) which yields the well-posedness of (1.3). Let

\[ \hat{B}_t = B_t - \int_0^t m\sigma^{-1}(\hat{X}_s^\mu, \hat{\bar{\mu}})(\hat{X}_s - \hat{\bar{X}}_s^\mu)ds. \]

Then \( (\hat{X}_t, \hat{X}_t^\mu) \) satisfies

\[ d\hat{X}_t = b(\hat{X}_t, \mu_t)dt + \sigma(\hat{X}_t, \mu_t)d\hat{B}_t, \]

\[ d\hat{X}_t^\mu = b(\hat{\bar{X}}_t^\mu, \hat{\bar{\mu}})dt + \sigma(\hat{X}_t^\mu, \hat{\bar{\mu}})d\hat{B}_t + m(\hat{X}_t - \hat{\bar{X}}_t^\mu)dt. \]

Similarly, as in [23], we have that \( \{\hat{B}_s\}_{0 \leq s \leq t} \) is a Brownian motion under \( \hat{\mathbb{Q}} = \hat{R}_t \mathbb{P} \) with

\[ \hat{R}_t := \exp \left\{ \int_0^t \left( m\sigma^{-1}(\hat{X}_s^\mu, \hat{\bar{\mu}})(\hat{X}_s - \hat{\bar{X}}_s^\mu), dB_s \right) \right\} \]
Moreover, by \((p \vee 2)(K_0 - 2m) + (p - 2)^+ < 0\) since \(m > m_0\), we have as \((3.3)\) and \((3.9)\) that

\[
W_p(\mathcal{L}_X \hat{\mu}, \mathcal{L}_X^\hat{\mu}) \leq \mathbb{E}[\hat{X}_t - \hat{X}_0]^2
\]

Moreover, by \((p \vee 2)(K_0 - 2m) + (p - 2)^+ < 0\) since \(m > m_0\), we have as \((3.3)\) and \((3.9)\) that

\[
W_p(\mathcal{L}_X \hat{\mu}, \mathcal{L}_X^\hat{\mu}) \leq \mathbb{E}[\hat{X}_t - \hat{X}_0]^2
\]

Moreover, by \((p \vee 2)(K_0 - 2m) + (p - 2)^+ < 0\) since \(m > m_0\), we have as \((3.3)\) and \((3.9)\) that

\[
W_p(\mathcal{L}_X \hat{\mu}, \mathcal{L}_X^\hat{\mu}) \leq \mathbb{E}[\hat{X}_t - \hat{X}_0]^2
\]
We choose
\[ \hat{m} = \sqrt{\frac{\sigma_0^2}{2^{3-\frac{2}{p}}} \sqrt{\frac{p-2}{p}}} \sqrt{\frac{\sigma_0^2}{2^{3-\frac{2}{p} \kappa}}}. \]
Then by (2.4), we have that
\[
\left( \frac{\sigma_0^2}{2^{3-\frac{2}{p} \kappa}} - \frac{\sigma_0^2}{2^{3-\frac{2}{p} \kappa}} \sqrt{1 - \frac{2^{3-\frac{2}{p} \kappa}}{\sigma_0^2} K_0} \right) \vee \left( \frac{K_0}{2} + \frac{p-2}{2p} \right) \vee 0
\]
\[ < \hat{m} < \frac{\sigma_0^2}{2^{3-\frac{2}{p} \kappa}} + \frac{\sigma_0^2}{2^{3-\frac{2}{p} \kappa}} \sqrt{1 - \frac{2^{3-\frac{2}{p} \kappa}}{\sigma_0^2} K_0}. \]
This implies that
\[
\frac{2^{3p-2} \kappa^2 \hat{m}^p}{\sigma_0^p (2\hat{m} - K_0)^2} < 1.
\]
Then
\[
\lim_{t \to +\infty} a_1(\hat{m}, t) = \frac{2^{3p-2} \kappa^2 \hat{m}^p}{\sigma_0^p (2\hat{m} - K_0)^2} < 1.
\]
Choosing large enough \( \hat{t} \) and small \( \hat{\delta} > 0 \), one can see that for all \( 0 < \delta < \hat{\delta} \), \( \gamma(\delta, \hat{m}, \hat{t}) < 1 \). Hence
\[
\delta_1 := \inf \left\{ \delta > 0 \left| \inf_{t>0, m>m_0} \gamma(\delta, m, t) \geq 1 \right. \right\} \geq \hat{\delta} > 0. \tag{3.15}
\]
For \( \delta < \delta_1 \), there exist \( \hat{m} > 0 \) and \( \hat{t} > 0 \) so that \( \gamma(\delta, \hat{m}, \hat{t}) < 1 \) and
\[
W_p(\mu_{\hat{t}}, \bar{\mu}) \leq \gamma(\delta, \hat{m}, \hat{t}) W_p(\mu, \bar{\mu}).
\]
It follows from the Markov property \( P_{t+s}^s = P_{t}^s P_{s}^s \) that
\[
W_p(P_{t+\hat{t}}^s \mu, \bar{\mu}) \leq (\gamma(\delta, \hat{m}, \hat{t}))^{\hat{t} + 1} W_p(P_{t-\hat{t}}^s \mu, \bar{\mu}) \leq C(\delta, \hat{m}, \hat{t}) e^{-\bar{\lambda} \hat{t}} W_p(\mu, \bar{\mu}),
\]
where
\[
\bar{\lambda} = \hat{t}^{-1} \log \frac{1}{\gamma(\delta, \hat{m}, \hat{t})}, \quad C(\delta, \hat{m}, \hat{t}) = \gamma(\delta, \hat{m}, \hat{t})^{-1} \sup_{0 \leq t \leq \hat{t}} \gamma(\delta, \hat{m}, t).
\]
In particular, if \( p = 2 \), we have that \( \hat{m} = \frac{\sigma_0^2}{2\kappa} \) and
\[
\gamma(\delta, \hat{m}, t) = 2 \frac{(2\hat{m} - K_0)}{\delta^2 \beta + 2\hat{m} - K_0} e^{-2\hat{m} - K_0} t
\]
\[ + 2 \left( \frac{\delta^2 \beta}{\delta^2 \beta + 2\hat{m} - K_0} + \frac{\kappa \hat{m}^2}{\sigma_0^2 (2\hat{m} - K_0)} \right) e^{\delta^2 \beta t},
\]
where
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Then

\[
\inf_{t > 0} \gamma(\delta, \hat{m}, t)^2 = \begin{cases} 
\beta, & \delta \geq \sqrt{\frac{1}{2}(2\hat{m} - K_0)\hat{\beta}^{-1}}, \\
2 \left( \frac{\hat{\beta}}{2} + \left( \frac{\hat{\beta}}{2} - 1 \right) u \right)^{\frac{n}{n+1}} u^{\frac{1-u}{u+1}}, & \delta < \sqrt{\frac{1}{2}(2\hat{m} - K_0)\hat{\beta}^{-1}},
\end{cases}
\]

where \( u = \frac{2\hat{m} - K_0}{\hat{\beta}^2} \) and for \( \delta < \sqrt{\frac{1}{2}(2\hat{m} - K_0)\hat{\beta}^{-1}} \), the optimal \( t \) is

\[
\hat{t} = \frac{1}{\delta^2 \beta + (2\hat{m} - K_0) \log \frac{u^2}{\frac{\beta}{2} + (\frac{\hat{\beta}}{2} - 1) u}}.
\]

Thus

\[
\left\{ \delta > 0 \mid \inf_{t > 0} \gamma(\delta, \hat{m}, t)^2 \geq 1 \right\}
= \left\{ 0 < \delta < \sqrt{\frac{1}{2}(2\hat{m} - K_0)\hat{\beta}^{-1}} \mid 2 \left( \frac{\hat{\beta}}{2} + \left( \frac{\hat{\beta}}{2} - 1 \right) u \right)^{\frac{n}{n+1}} u^{\frac{1-u}{u+1}} \geq 1 \right\}
= \left\{ 0 < \delta < \sqrt{\frac{1}{2}(2\hat{m} - K_0)\hat{\beta}^{-1}} \mid 2 \geq v(\hat{\beta} + \hat{\beta} - 2)^{-\frac{n}{n+1}}, v = \frac{\delta^2 \hat{\beta}}{2\hat{m} - K_0} \right\}.
\]

Hence

\[
\delta_1 \geq \inf \left\{ \delta > 0 \mid \inf_{t > 0} \gamma(\delta, \hat{m}, t)^2 \geq 1 \right\}
= \inf \left\{ \sqrt{\frac{1}{2}(2\hat{m} - K_0)\hat{\beta}^{-1} v} \mid 0 < v < \frac{1}{2}, v(\hat{\beta} + \hat{\beta} - 2)^{-\frac{n}{n+1}} \leq 2 \right\}
= \sqrt{\frac{1}{2}(2\hat{m} - K_0)\hat{\beta}^{-1}(\Phi(2) \wedge \frac{1}{2})}.
\]

Therefore, (2.6) follows, and

\[
\bar{\lambda} \geq -\frac{1}{2t} \log \left( 2 \left( \frac{\hat{\beta}}{2} + \left( \frac{\hat{\beta}}{2} - 1 \right) u \right)^{\frac{n}{n+1}} u^{\frac{1-u}{u+1}} \right)
= \frac{\delta^2 \hat{\beta}}{2} \left( u - \frac{(1 + u) \log(2u)}{\log \frac{2u^2}{\beta + (\hat{\beta} - 2)u}} \right).
\]

**Proof of Theorem 2.4**

Fix \( \mu \in \mathcal{P}_q \cap \mathcal{C} \). Since \( \bar{\mu} \) is the stationary distribution, \( T_{\bar{\mu}} = \bar{\mu} \). Since (1.1) is weak well-posed for \( \mu \in \mathcal{P}_q \), \( P_t^* \mu \) is well-defined. By (1.4), we have that

\[
W_p(P_t^* \mu, \bar{\mu}) \leq W_p(P_t^* \mu, (P_t^\bar{\mu})^* \mu) + W_p((P_t^\bar{\mu})^* \mu, \bar{\mu})
\leq W_p(P_t^* \mu, (P_t^\bar{\mu})^* \mu) + \hat{C} \epsilon^{-\lambda} W_p(\mu, \bar{\mu}).
\]

(3.16) \( \text{ine-Wmu0} \)

Denote \( \mu_t = P_t^* \mu \). We consider the following coupling

\[
d\tilde{X}_t = b(\tilde{X}_t, \mu_t)dt + \sigma(\tilde{X}_t, \mu_t)dB_t
\]
\[-m\sigma(\tilde{X}_t, \mu_t)\sigma^{-1}(\tilde{X}_t, \tilde{\mu})(\tilde{X}_t - X_t)dt, \quad \tilde{X}_0 = \mu, \tag{3.17}\]

d\tilde{X}_t = b(\tilde{X}_t, \tilde{\mu})dt + \sigma(\tilde{X}_t, \tilde{\mu})dB_t, \quad X_0 = \tilde{X}_0,

where \(m > m_0\). Let

\[
\tilde{B}_t = B_t - \int_0^t m\sigma^{-1}(X_s, \mu_s)(\tilde{X}_s - X_s)ds.
\]

Then \((\tilde{X}_t, X_t)\) satisfies

\[
d\tilde{X}_t = b(\tilde{X}_t, \mu_t)dt + \sigma(\tilde{X}_t, \mu_t)d\tilde{B}_t,
\]

\[
dX_t = b(X_t, \tilde{\mu})dt + \sigma(X_t, \tilde{\mu})dB_t + m(\tilde{X}_t - X_t)dt.
\]

We can prove that \(\{\tilde{B}_s\}_{0 \leq s \leq t}\) is a Brownian motion under \(\tilde{Q} = \tilde{R}_t\mathbb{P}\) with

\[
\tilde{R}_t := \exp\left\{ \int_0^t \left( \frac{m^2}{2} \int_0^t \left| \sigma^{-1}(X_s, \mu_s)(\tilde{X}_s - X_s) \right|^2 ds \right) \right\}, \tag{3.18}
\]

and

\[
\sup_{s \in [0, t]} \mathbb{E}_{\tilde{R}_s} \log \tilde{R}_s \leq \frac{m^2\delta^2}{2\sigma_0^2(2m - K_0)} \int_0^t W_p(\mu_s, \tilde{\mu})^2 ds, \tag{3.19}
\]

\[
\mathbb{E}_{\tilde{Q}} |X_t - \tilde{X}_t|^{2p} \leq \left( \int_0^t e^{-K_0s}2^p(t-s)2p \mathbb{E}_{\tilde{Q}} |X_t - \tilde{X}_t|^{2p} ds \right) \leq \delta^{p/2} \left( \int_0^t W_p(\mu_s, \tilde{\mu})^{4p} ds \right)^{1/2},
\]

where we have used in the last inequality that \((p/2)(K_0 - 2m) + (p/2)^+ \leq 0\). It follows from the uniqueness in law of (1.7) that \(\mathcal{L}_t^{\tilde{Q}} = \mu_t\). Then

\[
W_p(\mathcal{L}_t^{\tilde{Q}}) \leq W_p(\mathcal{L}_t^{\tilde{Q}} - \mathcal{L}_t^{\tilde{Q}} + \mathcal{L}_t^{\tilde{Q}}) \leq \delta^{p/2} \left( \int_0^t W_p(\mu_s, \tilde{\mu})^{4p} ds \right)^{1/2} = \delta \left( \int_0^t W_p(\mu_s, \tilde{\mu})^{4p} ds \right)^{1/2}.
\]

Due to \(\mathcal{L}_t^{\tilde{Q}}(f) = \mathbb{E}_{\tilde{R}_t} f(X_t)\) and (3.19), we also have by \(\mu \in \mathcal{C}\) and (2.8) that

\[
W_p(\mathcal{L}_t^{\tilde{Q}}, \mathcal{L}_t^{p}) \leq \sqrt{2\kappa_t \mathbb{E}_{\tilde{R}_t} \log \tilde{R}_t} \leq \sqrt{2\kappa_t \mathbb{E}_{\tilde{R}_t} \log \tilde{R}_t} \leq \frac{m\delta}{\sigma_0 \sqrt{2m - K_0}} \left( \int_0^t W_p(\mu_s, \tilde{\mu})^2 ds \right)^{1/2} \leq \frac{m\delta}{\sigma_0 \sqrt{2m - K_0}} \left( \int_0^t W_p(\mu_s, \tilde{\mu})^{2p} ds \right)^{1/2p}.
\]

Hence,

\[
W_p(\tilde{P}_t^\mu, (\tilde{P}_t^\mu)^* \mu) \leq W_p(\tilde{P}_t^\mu, \mathcal{L}_t^{\tilde{Q}}) + W_p(\mathcal{L}_t^{\tilde{Q}}, \mathcal{L}_t^{p}) \mu)
\]
Taking optimal $m$, we have that

$$
\inf_{m > m_0} \frac{m}{\sqrt{2m - K_0}} = \frac{m}{\sqrt{2m - K_0}} \bigg|_{m = \frac{1}{2} \left( \frac{(p-2)^+}{p^2} \sqrt{|K_0|} + K_0 \right)} = \frac{K_0 + \left( \frac{(p-2)^+}{p^2} \sqrt{|K_0|} \right)}{2 \sqrt{|K_0|} \sqrt{\frac{(p-2)^+}{p^2}}}.
$$

Then

$$
\inf_{t, \theta > 0, m > m_0} \gamma(\delta, t, m, \theta) = \inf_{t, \theta > 0} \gamma \left( \delta, t, \frac{1}{2} \left( \frac{(p-2)^+}{p^2} \sqrt{|K_0|} + K_0 \right), \theta \right) \equiv \inf_{t, \theta > 0} \gamma(\delta, t, \theta).
$$

Hence

$$
\delta_2 = \inf \left\{ \delta > 0 \left| \inf_{t, \theta > 0} \gamma(\delta, t, \theta) \geq 1 \right. \right\}
$$
Due to \((Ta)\) and \(\bar{\delta} < \delta\) for all \(t' > m_0, \theta > 0\) and \(\delta' > 0\) such that for all \(\delta < \delta'\) it holds that \(\gamma(\delta, t, m, \theta) < 1\). Thus \(\delta_2 \geq \delta' > 0\).

For \(\delta < \delta_2 \wedge \delta_0\), let \((t_1, \theta_1)\) so that \(\gamma(\delta, t_1, \theta_1) < 1\). Then
\[
W_p(\mu_1, \bar{\mu}) \leq \gamma(\delta, t_1, \theta_1) W_p(\mu, \bar{\mu}).
\]

Due to \((Ta)\) and \(P^s_t \mu \in \mathcal{P}^q\), we have that \(P^s_t(\mathcal{P}^q \cap C) \subset \mathcal{P}^q \cap C\). Then, it follows from the Markov property \(P^s_{t+s} = P^s_t P^s_s\) that
\[
W_p(P^s_t \mu, \bar{\mu}) \leq (\gamma(\delta, t_1, \theta_1))^{1/\bar{\lambda}_1} W_p(P^s_{t-t_1} | \bar{\nu} \mu, \bar{\mu}) \leq \bar{C}(\delta, t_1, \theta_1) e^{-\lambda t} W_p(\mu, \bar{\mu}),
\]
where \(\bar{\lambda} = t_1^{-1} \log \frac{1}{\gamma(\delta, t_1, \theta_1)}\) and
\[
\bar{C}(\delta, t_1, \theta_1) = \gamma(\delta, t_1, \theta_1)^{-1} \sup_{0 \leq t \leq t_1} \gamma(\delta, t, \theta_1).
\]

\(\square\)

4 Proofs of corollaries

**Proof of Corollary 2.5**

To prove Corollary 2.5, we first establish the \(W_1\)-transportation cost inequality under the assumption \((A2)\). Let \(\nu_0 \in \mathcal{P}^1\) and \(\{\nu_t\}_{t \geq 0} \subset \mathcal{P}^1\) such that the following SDE has a unique solution
\[
dY_t = b(Y_t, \nu_t) dt + \sigma(Y_t, \nu_t) dB_t, \quad Y_0 = \nu_0.
\]

**Lemma 4.1.** Assume that \(b, \sigma\) satisfies \((A2)\). If \(\nu_0\) satisfies (2.10), then \(\mathcal{L}^p_{\tilde{Y}_t}\) satisfies (2.10) and \(W_1\)-transportation cost inequality
\[
W_1(\nu, \mathcal{L}^p_{\tilde{Y}_t}) \leq \sqrt{2 \|\sigma\|_{L^2}^2 (K_1 \wedge K_3)^{-1} H(\nu|\mathcal{L}^p_{\tilde{Y}_t})}, \quad t \geq 0.
\]

**Proof.** We establish \(W_1\)-transportation cost inequality for (1.3) by using (1.5) and Theorem 2.3, see also [20, Theorem 3.2].

Let \(\{B_t\}_{t \geq 0}\) be a Brownian motion independent of \(\{B_t\}_{t \geq 0}\), and let \(\tilde{Y}_t\) be the solution of the following equation
\[
d\tilde{Y}_t = b(\tilde{Y}_t, \nu_t) dt + \sigma(\tilde{Y}_t, \nu_t) dB_t, \quad \tilde{Y}_0 = \mu_0,
\]
and \(\tilde{Y}_0\) is independent of \(Y_0\). It follows from the Itô formula that
\[
d|Y_t - \tilde{Y}_t|^2 = 2(Y_t - \tilde{Y}_t, b(Y_t, \nu_t) - b(\tilde{Y}_t, \nu_t)) dt + \|\sigma(Y_t, \nu_t)\|_{HS}^2 dt + 2(Y_t - \tilde{Y}_t, \sigma(Y_t, \nu_t) dB_t - \sigma(\tilde{Y}_t, \nu_t) dB_t)
\]
\[
\leq \left( (K_0 + K_1) 1_{|Y_t - \tilde{Y}_t| \leq \tau_0} - K_1 \right) |Y_t - \tilde{Y}_t|^2 dt
\]
Hence, according to \([6, (1.5)\) and Theorem 2.3\), (4.1) holds.

Thus \(\kappa\) implies that

\[\text{Theorem 2.6 and (2.23)\},}\]

For every \((A1)\) holds with \(\theta < \infty\) and \(Y\) is independent of solutions to (1.3) and that \(\sigma\) holds for any \(\theta < \infty\). Due to the uniqueness of solutions to (1.3) and that (2.10) holds for \(\mu\) with \(\kappa = \|\sigma\|_\infty^2 K_1^{-1}\). Then by Theorem 2.2

Due to the uniqueness of solutions to (1.3) and that \(Y_0\) and \(\bar{Y}_0\) are independent with the same law, \(\bar{Y}_1\) and \(\bar{Y}_\infty\) are independent with the same law. Then it follows from (4.2) that for any \(\theta < \frac{K_1}{4\|\sigma\|_\infty^2}\)

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} e^{\theta|y_1-y_2|^2} \mathcal{L}_{Y_1}(dy_1) \mathcal{L}_{\bar{Y}_1}(dy_2) = \mathbb{E} e^{\theta|Y_1-\bar{Y}_1|^2} < +\infty, \ t \geq 0.
\]

Hence, according to [6, (1.5) and Theorem 2.3], (4.1) holds.

We now turn to the proof of Corollary 2.5. According to \((2.2), (2.9)\) and \(25\) Theorem 2.6 and (2.23), \((H)\) holds with \(p = 1\). It is clear that \((A2)\) implies that \((A1)\) holds with \(p = 1\).

(1) Due to \((H)\), \((P_{\mu}^t)^* \delta_x \to \mathcal{T}_\mu\) weakly as \(t \to +\infty\), where \(\delta_x\) is the Dirac measure centred on \(x \in \mathbb{R}^d\). This, together with that \(2.10\) holds for \(\mu_0 = \delta_x\) and any \(K_3 > 0\), implies by Lemma [4.1] and [6] Lemma 2.2 that (2.3) holds for any \(\mathcal{T}_\mu\) with \(\kappa = \|\sigma\|_\infty^2 K_1^{-1}\). Then by Theorem 2.2

\[
\sup_{t > t_0, m > m_0} \frac{\sigma_0 (1 - \tilde{C} e^{-\lambda t}) K(m, 2)}{\sigma_0 K(m, 2) + m \sqrt{\kappa t} K(m, p) \vee t^{-\frac{1}{p+2}}} \leq \frac{1}{\sigma_0 K(m, 2) + m \sqrt{\kappa t} K(m, p) \vee t^{-\frac{1}{p+2}}}
\]
\[
\begin{align*}
\geq & \sup_{t > \lambda^{-1}\log \hat{C}, m > K_0/2} \frac{\sigma_0 (1 - \hat{C} e^{-\lambda t}) \sqrt{2m - K_0}}{\sigma_0 + m \|\sigma\|_\infty \sqrt{t}} \\
= & \sup_{t > \lambda^{-1}\log \hat{C}} \frac{\sigma_0 (1 - \hat{C} e^{-\lambda t}) \sqrt{K_1}}{2\sigma_0 \|\sigma\|_\infty \sqrt{K_1 t} + \|\sigma\|_\infty^2 K_0 t},
\end{align*}
\]
where in the last equality we set
\[
m = \frac{1}{2} \left\{ \left( \frac{2\sigma_0 \sqrt{K_1} + \|\sigma\|_\infty K_0 \sqrt{t}}{\|\sigma\|_\infty \sqrt{t}} \right)^2 + K_0 \right\}.
\]
Then we have proved the first assertion.

(2) It follows from Lemma 4.1 and (2.10) that (2.8) holds with \(\kappa_t \equiv \|\sigma\|_\infty (K_1 \wedge K_3)^{-1}\) and \(\mathcal{C}\) consists of probability measures satisfying (2.10). The inequality (2.10) also yields that \(\mu \in \mathcal{P}_2\). Since \(W_1 \leq W_2\), (A2) implies the strong well-posedness of (1.1) with initial distribution \(\mu\) and \(t \to P_t^* \mu\) is locally bounded in \(\mathcal{P}_2\), see e.g. [24].

It follows from Theorem 2.4 with \(\theta = 1\) and Lemma 4.1 that
\[
\gamma(\delta, t, 1)^2 = \frac{2\hat{C}^2}{\alpha \delta^2 + \lambda} \left( \frac{\alpha \delta^2 e^{2\alpha \delta^2 t} + \hat{\lambda} e^{-2\lambda t}}{2\hat{C}^2 \left( \frac{\hat{\lambda}}{\alpha \delta^2} \right)^{\alpha \delta^2 + \lambda}} \right) \delta > \sqrt{\lambda \alpha^{-1}}.
\]
Then
\[
\inf_{t > 0} \gamma(\delta, t, 1)^2 = \begin{cases} 
2\hat{C}^2 \frac{2\hat{C}^2 \alpha \delta^2}{\alpha \delta^2 + \lambda} \frac{\alpha \delta^2 - \alpha \delta^2 + \lambda}{\alpha \delta^2 + \lambda} \delta > \sqrt{\lambda \alpha^{-1}} \quad & 
\delta \leq \sqrt{\lambda \alpha^{-1}} \ \text{(i)} \\
2\hat{C}^2 \left( \frac{\hat{\lambda}}{\alpha \delta^2} \right)^{\alpha \delta^2 + \lambda} \delta \leq \sqrt{\lambda \alpha^{-1}} \quad & \text{(ii)}
\end{cases}
\]
Since for \(\delta > \sqrt{\lambda \alpha^{-1}}\)
\[
\frac{2\hat{C}^2 \alpha \delta^2}{\alpha \delta^2 + \lambda} > \hat{C}^2 \geq 1,
\]
we have that
\[
\inf \left\{ \delta > 0 \bigg| \inf_{t, \theta > 0} \gamma(\delta, t, \theta) \geq 1 \right\}
\geq \inf \left\{ \delta > 0 \bigg| \inf_{t > 0} \gamma(\delta, t, 1) \geq 1 \right\}
\geq \inf \left\{ 0 < \delta \leq \sqrt{\lambda \alpha^{-1}} \bigg| 2\hat{C}^2 \geq \left( \frac{\alpha \delta^2}{\lambda} \right)^{\alpha \delta^2 + \lambda} \right\}
\geq \inf \left\{ \delta \bigg| 0 < u \leq 1, \ \frac{\alpha \delta^2}{\lambda} \leq 2\hat{C}^2, \ u = \frac{\alpha \delta^2}{\lambda} \right\}
\geq \sqrt{\lambda \alpha^{-1}} \Phi(2\hat{C}^2),
\]
where in the last equality we have used that the function \(u^{\frac{\alpha \delta^2}{\lambda}}\) decrease in \((0, 1)\) and increase in \((1, +\infty)\). Then we obtain \(\delta_2\).

For \(\delta < \delta_2 \wedge \delta_0\), then \(\delta < \sqrt{\lambda \alpha^{-1}} \Phi(2\hat{C}^2)\) which yields that \(\hat{\lambda}/(\alpha \delta^2) > 1\). Thus we can choose \(\hat{t} = \frac{\log(\lambda/\alpha \delta^2)}{\alpha \delta^2 + \lambda} \) so that
\[
\gamma(\delta, \hat{t}(\delta), 1)^2 = 2\hat{C}^2 \left( \frac{\lambda}{\alpha \delta^2} \right)^{\alpha \delta^2 + \lambda} < 1
\]
Then

\[ W_1(\mu_t, \mu_1) \leq \gamma(\delta, \hat{t}, 1)W_1(\mu, \mu_1). \]

It follows from the Markov property \( P_{t-s}^* = P_s^*P_s^* \) that

\[ W_1(P_t^*\mu, \mu_1) \leq (\gamma(\delta, \hat{t}, 1))^{\frac{1}{p}} W_1(P_t^*\mu, \mu_1) \leq C \varepsilon^{-\lambda t} W_1(\mu, \mu_1), \]

with

\[ \bar{\lambda} = \frac{1}{t} \log \frac{1}{\gamma(\delta, \hat{t}, 1)} = \left( \frac{\log(2\lambda^2)}{2} + \frac{\log(\lambda/\alpha \delta^2)}{\alpha \delta^2 + \lambda} \right), \]

\[ \bar{C} = \gamma(\delta, \hat{t}, 1)^{-1} \sup_{0 \leq t \leq T} \frac{C_1^2 (\alpha \delta^2 \lambda \varepsilon^{-\lambda t} + \bar{\lambda} e^{-2 \lambda t})}{\alpha \delta^2 + \lambda}. \]

\textbf{Proof of Corollary 2.7}

The condition (A2') yields the strong well-posedness to (1.1), see e.g. [24]. It follows from (A2') and [15] Corollary 1.8 that for any \( \mu \in \mathcal{P}_p \), there is \( T_\mu \in (0, 1/p) \) and (1.4) holds for \( p = 1 \). We use [25, Theorem 2.1 (2)] to prove that (1.4) and (2.3) holds for \( p = 2 \). To this end, we only need to prove that \( P_t^\mu \) has an invariant probability measure \( T_\mu \) and there is \( \theta > 0 \) independent of \( \mu \) such that \( T_\mu(e^{\theta \mu}) < \infty \).

For the solution to (1.3) with \( X^0_t = 0 \), it follows from (A2'), the Itô formula and the Hölder inequality that

\[ d|X^\mu_t|^2 \leq \left( (K_0 + K_1) 1_{|X^\mu_t| \leq r_0} - K_1 \right) |X^\mu_t|^2 + 2\langle X^\mu_t, \sigma(\mu)dB_t \rangle \]

\[ + \| \sigma(\mu) \|^2_{H^2} dt + 2\langle b(0, \mu), X^\mu_t \rangle dt \]

\[ \leq \left( (K_0 + K_1) 1_{|X^\mu_t| \leq r_0} - \frac{K_1}{2} \right) |X^\mu_t|^2 + 2\langle X^\mu_t, \sigma(\mu)dB_t \rangle \]

\[ + \| \sigma(\mu) \|^2_{H^2} dt + K_1^{-1} \sup_{\mu \in \mathcal{P}_p} \| b(0, \mu) \|^2 dt. \]

Then

\[ d e^{\theta \mu |X^\mu_t|^2} \leq \theta e^{\theta \mu |X^\mu_t|^2} \left( (K_0 + K_1) 1_{|X^\mu_t| \leq r_0} - \frac{K_1}{2} \right) |X^\mu_t|^2 dt \]

\[ + \left\{ \| \sigma \|^2_{L^\infty} + \sup_{\mu \in \mathcal{P}_p} \| b(0, \mu) \|^2 \right\} dt \]

\[ + 2\theta^2 e^{\theta \mu |X^\mu_t|^2} \| \sigma \|^2_{L^\infty} |X^\mu_t|^2 dt + 2\theta e^{\theta \mu |X^\mu_t|^2} \langle X^\mu_t, \sigma(\mu)dB_t \rangle. \]

This implies that there is \( \theta_0 > 0 \) independent of \( \mu \) such that for \( \theta < \theta_0 \)

\[ d e^{\theta \mu |X^\mu_t|^2} \leq \left( C_1 - C_2 e^{\theta \mu |X^\mu_t|^2} \right) dt + 2\theta e^{\theta \mu |X^\mu_t|^2} \langle X^\mu_t, \sigma(\mu)dB_t \rangle \]

with some \( C_1, C_2 > 0 \) independent of \( \mu \). Then

\[ \frac{1}{t} \int_0^t E e^{\theta \mu |X^\mu_s|^2} ds \leq \frac{C_1}{C_2}. \]
which implies \( \sup_{\mu \in \mathcal{P}^2} T_{\mu}(e^{\theta|x|^2}) < \infty \). From this and (A2'), which implies that
\[
2\langle b(x, \mu) - b(y, \mu), x - y \rangle \leq ((K_0 + K_1) \mathbf{1}_{|x-y| \leq r_0} - K_1) |x-y|^2, \ \mu \in \mathcal{P}^2,
\]
we have that (1.4) and (2.3) holds for \( p = 2 \) according to [25, Theorem 2.1 (2)].

Applying Theorem 2.2 with \( p = 2 \), we have that
\[
\sup_{t > t_0, m > m_0} \sigma_0 \left( 1 - \hat{C}e^{-\hat{\lambda}t} \right) K(m, 2) \left[ K(m, p) \vee t^{-\frac{1}{p+1}} \right] \geq \sup_{t > t_0, m > m_0} \sigma_0 \left( 1 - \hat{C}e^{-\hat{\lambda}t} \right) \sqrt{2m - K_0} \sigma_0 + m \sqrt{\kappa t} = \sup_{t > t_0} \sigma_0 \left( 1 - \hat{C}e^{-\hat{\lambda}t} \right) \sqrt{\kappa(2\sigma_0 \sqrt{t} + K_0 \kappa t)}.
\]

Hence the first assertion follows.

Applying Theorem 2.2 with \( p = 2 \), we prove the second assertion.

\[\square\]

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