Numerical method for recovering the piecewise constant right-hand side function of an elliptic equation from a boundary overdetermination data

D Kh Ivanov¹, A E Kolesov¹ and P N Vabishchevich¹,²
¹North-Eastern Federal University, 48 Kulakovskogo str., Yakutsk 677000, Russia
²Nuclear Safety Institute of RAS, 52 B. Tulskaia str., Moscow 115191, Russia
E-mail: djulus.ivanov@yandex.ru

Abstract. An elliptic problem in an open set Ω is considered with Dirichlet boundary condition on piecewise smooth boundary ∂Ω. The inverse problem is to recover piecewise-constant source term, which means an identification function of an unknown subset D ⊂ Ω. An additional information is taken as Neumann boundary condition, which leads us to consider the inverse problem under Cauchy boundary data. In this work a new computational algorithm for recovery the unknown source term is proposed. The main idea is that the identification function of subset D is replaced by a Heaviside function over the solution of an auxiliary elliptic equation. Then we formulate a minimization problem of a residual for overdetermination data, when a control is taken as the right-hand side of the auxiliary equation. Numerical implementation is based on the finite element method applying the open-source computing platform FEniCS and its package dolfin-adjoint. The capabilities of the given computational algorithm are shown by results of numerical solutions of 2D test problems.

Keywords: coefficient inverse problem, right-hand side function, elliptic equation, finite element method, FEniCS, dolfin-adjoint

1. Introduction
Inverse problems for partial differential equations are known as non-classical and often belong to the class of ill-posed problems. When considering such problems we are focusing on investigations of issues of the existence, the uniqueness and the stability of the solution over an input data, and also to developments of numerical algorithms.

Among inverse problems for elliptic equations one can derive a problem of identification of the right-hand side (RHS) of an elliptic equation by given boundary observations. This problem relates with the potential theory, for which a lot of interesting results have been obtained. The earliest work is thought to be classical Novikov’s work [1] about an uniqueness of the solution of the inverse problem of potential theory in a star shaped domain, when a constant density of mass distribution is known. In later works, e.g. [2], existence and uniqueness problems were considered in the case, where the RHS does not depend on one of the spatial variables. In [3], a wider class of uniqueness was established, when the RHS can include two functions with fewer variables.

General methods of solving inverse problems of reconstruction of the RHS function in an elliptic equation one can find in following works [4, 5, 6] and references therein. For
considered class of inverse problems the popular method is Tikhonov’s regularization in various modifications. For instance, in [7], an optimal control problem was considered for an identification of the RHS of an elliptic equation from known additional boundary measurements with representing the solution (RHS function) by a finite system of basis functions.

In this work an inverse problem of an identification of the piecewise-constant RHS function of a second order elliptic equation. An unknown RHS function is considered as the identification function of a set of non-overlapping bounded star-shaped domains. A new approach is given to approximate the solution of the considered inverse problem, when objective RHS function is sought by the Heaviside function over the solution of an auxiliary elliptic equation. An optimal control problem is posed to find the minimum of the residual functional for the boundary overdetermination, and the RHS of the auxiliary elliptic problem is taken as the control. For discretization in space the finite element method is used. The gradient iterative method is used for solving the optimal control problem.

2. Problem statement
Let’s consider a bounded open region $\Omega \subset \mathbb{R}^2$ with sufficiently smooth boundary $\partial \Omega$, and $x = (x_1, x_2) \in \Omega$. A direct problem is stated as follows: find a function $u(x)$ s.t. it satisfies the solution of a Poisson problem with homogeneous Dirichlet boundary condition

$$
-\Delta u = \rho(x), \quad x \in \Omega,
$$

$$
u(x) = 0, \quad x \in \partial \Omega,
$$

where $\rho(x) \geq 0$.

We consider an inverse problem of finding the RHS, $\rho(x)$, from a priori assumption that it has next form:

$$
\rho(x) = \begin{cases} 
1, & x \in D, \\
0, & x \in \Omega \setminus D,
\end{cases}
$$

where $D \subset \Omega$, $\partial D \cap \partial \Omega = \emptyset$. Under an assumption that $D$ consists of a finite number of star-shaped subdomains an additional information is taken as a Neumann condition on the whole boundary $\partial \Omega$:

$$
\frac{\partial u}{\partial n} = g(x), \quad x \in \partial \Omega,
$$

where $n$ is outer normal of the boundary $\partial \Omega$. So, the inverse problem can be uniquely solved, and under such conditions it was investigated, for example, in [6], see section 4.1.

From (2) we can notice that $\rho(x) = \chi_D(x)$, which is an identification function of region $D$. Therefore the inverse problem is basically to find unknown region $D$ or more precisely the boundary of $D$. There are a lot of works related to such kind of inverse interface problems based on the level set method, e.g. [8, 9]. In this paper we introduce a new method to solve the inverse problem (1)–(3) that differs from the level set method, and it is based on minimization of a residual functional:

$$
\min_{\rho} J(\rho) = \int_{\partial \Omega} \left( \frac{\partial u}{\partial n}(x; \rho) - g(x) \right)^2 \, ds.
$$

3. Computational algorithm
An approximate solution, $\rho$, of the inverse problem (1)–(4) is sought using the Heaviside function over some auxiliary smooth function $c(x)$:

$$
\chi_D(x) \rightarrow H(c(x)), \quad x \in \Omega.
$$
The auxiliary function $c(x)$ is defined from a solution of next boundary-value problem:

$$-\gamma \Delta c + c = f(x), \quad x \in \Omega, \quad c = 0, \quad x \in \partial \Omega,$$

(5)

with parameter $\gamma = \text{const} > 0$. Thus the objective RHS function (2) has the following form

$$\rho(x) = H(c(x)) = \begin{cases} 
1, & c(x) > 0, \ x \in \Omega \\
0, & c(x) \leq 0, \ x \in \Omega.
\end{cases}$$

(6)

Notice that $D$ can be defined from the function $c(x)$, where it takes positive values.

The introduction of the auxiliary problem is justified by the fact that the piecewise-constant RHS function of (1) is approximated by using the smooth function $c(x)$ over all computational domain. The auxiliary function smoothness is controlled by a term $-\gamma \Delta c$, whereas $\gamma$ has a meaning of regularization parameter. From (1)–(3) and (5)–(6) it’s seen that one can minimize the functional (4) over the RHS function $f(x)$ of (5):

$$J(f) = \int_{\partial \Omega} \left( \frac{\partial u}{\partial n}(x; f) - g(x) \right)^2 \ ds \longrightarrow \min.$$

(7)

The latter can be found by using gradient based method. In this case, one need to find derivative of the residual w.r.t. the control function. It can be made by deriving tangent linear and adjoint models of the inverse problem.

Computational implementation is performed by using the finite element method. Let’s introduce a Hilbert space $L_2(\Omega)$ with standard scalar product $(\cdot, \cdot)$ and norm $\| \cdot \|$. Let $V^h \subset H^1(\Omega)$ and $\tilde{V}^h = \{ v \in V^h : v(x) = 0, \ x \in \partial \Omega \}$. Elements of $V^h$ are represented by a system of pyramid-like piecewise-linear basis functions $\xi_i(x)$:

$$v(x) \approx \sum_{i=1}^{N_h} v_i \xi_i(x), \quad v_i = v(x_i),$$

where $\{x_i\}_{i=1}^{N_h}$ are triangulation points of the region $\Omega$. Now we introduce finite element analogues for aforesaid functions

$$u_h(x), \ c_h(x), \ \rho_h(x) \in \tilde{V}^h \quad \text{and} \quad f_h(x) \in V^h.$$  

Taking into account finite element approximations and notations we introduce an Algorithm 1 for the solution of the inverse problem (1)–(3), (5)–(7).

4. Numerical experiments

Let computational region be a unit square with $64 \times 64$ uniform grid on it. For exact values of observation data we use more refined grid as $256 \times 256$. The smoothing parameter $\gamma$ is set to 0.02 as computation results not included in this work had given us that it should be not so large — an extreme smoothness case, and not so small — an almost discrete case.

Numerical solutions of direct and inverse problems and corresponding PDEs are evaluated in python3 by using an open source library FEniCS (version 2018.1.0) [10]. For finding the minimum of the residual we use the package dolfin-adjoint (version 2018.1.0) [11] with SciPy minimization (L-BFGS-B method [12]).
Algorithm 1 Solution of the inverse problem.

| input $g$, $f_h^0$, $\gamma$, $M$, $\varepsilon_1$, $\varepsilon_2$ \{$M$ is a large number\} |
| output $\rho_h$ |
| for $k = 0$ to $M$ do |
| $c_h^k$ ← the solution of the variational problem of (5) |
| $\gamma(\nabla c_h^k, \nabla v) + (c_h^k, v) = (f_h^k, v), \forall v \in \tilde{V}^h$ |
| $\rho_h^k$ ← $H(c_h^k)$ |
| $u_h^k$ ← the solution of the variational problem of the direct problem (1) |
| $(\nabla u_h^k, \nabla v) = (\rho_h^k, v), \forall v \in \tilde{V}^h$ |
| $J_k$ ← $\int_{\partial\Omega} (\nabla u_h^k \cdot n - g)^2 \, ds$ |
| $J_{f,k}$ ← $\frac{dJ_k}{df_h^k}$ \{by solving adjoint problem, dolfin-adjoint\} |
| if $|J_{k-1} - J_k| < \varepsilon_1$ or $\|J_{f,k}\|\infty < \varepsilon_2$ then |
| $\rho_h$ ← $\rho_h^k$ |
| stop |
| end if |
| $f_h^{k+1}$ ← $f_h^k + d^k$ \{step $d^k$ is calculated by SciPy minimizer\} |
| end for |
| $\rho_h$ ← $\rho_h^k$ |

Initial guess for the unknown region $D$ is taken as a resulting circle $D_0$ with radius $r$ and center $x_0 = (X_1, X_2)$, which are found from observation data:

$$r = \left(\frac{1}{\pi} \int_{\partial\Omega} g \, ds\right)^{\frac{1}{2}}, \quad X_1 = \frac{\int_{\partial\Omega} x_1 g \, ds}{\int_{\partial\Omega} g \, ds}, \quad X_2 = \frac{\int_{\partial\Omega} x_2 g \, ds}{\int_{\partial\Omega} g \, ds}.$$

Then the initial guess for control function, RHS of (5), is taken as a smooth function:

$$f_0(x) = \begin{cases} \cos^2\left(\frac{\pi}{2r} \int_{\Omega} |x - x_0| \, dx\right), & x \in D_0, \\ 0, & x \in \Omega/D_0. \end{cases}$$

Figure 1. Approximation results of the ellipse shape at some iterations.
Figure 2. Approximation results of the rectangle shape at some iterations.

Figure 3. Approximation results of the two circles shape at some iterations.

Figure 4. Approximation results of the star shape at some iterations.

The choice of the initial guess for $f(x)$ should be done carefully, otherwise the algorithm may fail or terminate earlier if the auxiliary function $c(x)$ becomes strictly negative at some iteration.

The Heaviside function is approximated by a following smooth function

$$H_\eta(c) = \begin{cases} 0, & c < 0, \\ 0.5 - 0.5 \cos \left( \frac{\pi c}{\eta} \right), & 0 \leq c < \eta, \\ 1, & c \geq \eta \end{cases}$$

(8)

with a small parameter $\eta$, in our case $\eta = 0.01$. Notice, due to the statement of the auxiliary problem (5) and the assumption that $\partial D \cap \partial \Omega = \emptyset$ the Heaviside function should vanish at zero.
Table 1. Details of shapes for the unknown region $D$.

| Shape form | Description |
|------------|-------------|
| Ellipse    | $((x_1 - 0.5)/0.35)^2 + ((x_2 - 0.5)/0.15)^2 \leq 1$ |
| Rectangle  | $|x_1 - 0.35| \leq 0.1 \cap |x_2 - 0.6| \leq 0.25$ |
| Two circles| $(x_1 - 0.7)^2 + (x_2 - 0.3)^2 \leq 0.15^2 \cup (x_1 - 0.4)^2 + (x_2 - 0.75)^2 \leq 0.15^2$ |
| Star       | $((x_1 - 0.5)^2 + (x_2 - 0.5)^2)(1 + 0.8 \sin(6 \tan^{-1}((x_2 - 0.5)/(x_1 - 0.5)))) \leq 0.005$ |

In this work, true measurement data is derived from evaluation of the direct problem in the refined grid. We simulate test problems with different forms of the unknown region $D$ for RHS of the equation (1), see Figures 1–4 and Table 1 for more details. In the first pictures, an initial approximation along with an initial circle $D_0$ for control function (yellow solid line) are drawn. The exact contour of unknown region is drawn by black solid line. The second and third pictures represent approximations at some intermediate iterations, the last picture is for the exit iteration, when the algorithm terminates with $\varepsilon_1 = 10^{-14}$ and $\varepsilon_2 = 10^{-12}$.

5. Conclusion
In this work we presented new method of finding the piecewise-constant right-hand side function of the elliptic equation from total boundary observation data. Capabilities of the method are demonstrated by 2D test problems for different shapes of unknown region $D$. Notice that this method can be easily applied for the other coefficients of elliptic equations.

Acknowledgments
The work was supported by RFBR grant 17-01-00689 and Mega grant of Russian Government #14.Y26.31.0013.

References
[1] Novikov PS 1938 On the uniqueness of the solution of the inverse potential problem Dokl. AN SSSR 18(3) 165–8
[2] Prilepko AI 1967 Inverse problems of potential theory Diff. Eq. 3(1) 30–44
[3] Vabishchevich PN 1982 Uniqueness of the solution of an inverse problem for determining the right-hand side of an elliptic equation Diff. Eq. 18(8) 1450–3
[4] Samarskii AA and Vabishchevich PN 2008 Numerical methods for solving inverse problems of mathematical physics (Walter de Gruyter) 52
[5] Kabanikhin SI 2011 Inverse and ill-posed problems: theory and applications (Walter De Gruyter) 55
[6] Isakov V 2017 Inverse problems for partial differential equations (Springer) 3rd ed. 127
[7] Vabishchevich PN 1985 An inverse problem of reconstruction of the right-hand side of an elliptic equation and its numerical solution Diff. Eq. 21(2) 277–84
[8] Isakov V, Leung S and Qian J 2011 A fast local level set method for inverse gravimetry Communications in computational physics 10(4) 1044–70
[9] Ito K, Kunisch K and Li Z 2001 Level-set function approach to an inverse interface problem Inverse problems 17(5) 1225–42
[10] Logg A, Mardal K-A and Wells G 2012 Automated solution of differential equations by the finite element method: The FEniCS book (Springer Science & Business Media)
[11] Funke SW and Farrell PE 2013 A framework for automated PDE-constrained optimisation Preprint arXiv:1302.3894
[12] Zhu C, Byrd RH, Lu P and Nocedal J 1997 Algorithm 778: L-bfgs-b: Fortran subroutines for large-scale bound-constrained optimization. ACM Trans. Math. Softw. 23 550–60