PSEUDO-GALOIS EXTENSIONS AND HOPF ALGEBROIDS

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Abstract. A pseudo-Galois extension is shown to be a depth two extension. Studying its left bialgebroid, we construct an enveloping Hopf algebroid for the semi-direct product of groups, or more generally involutive Hopf algebras, and their module algebras. It is a type of cofibered sum of two inclusions of the Hopf algebra into the semi-direct product and its derived right crossed product. Van Oystaeyen and Panaite observe that this Hopf algebroid is non-trivially isomorphic to a Connes-Moscovici Hopf algebroid, which raises interesting comparative questions.

1. Introduction

The analytic notion of finite depth for subfactors was widened to the algebraic setting of Frobenius extensions in [10], the main theorem of which states that a certain depth two Frobenius extension \( A \mid B \) with trivial centralizer is a Hopf-Galois extension. The theorem and its proof is essentially a reconstruction theorem, which uses an Ocneanu-Szymanski pairing of two centralizers on the tower of algebras above \( A \mid B \), isomorphic to the two main players in this paper \( \text{End}_B A_B \) and \( \text{End}_A A \otimes_B A_A \); then shows that the resulting algebra-coalgebra is a Hopf algebra. In the paper [12] the notion of depth two was widened to arbitrary algebra extensions whereby it was shown that the bimodule endomorphism ring \( \text{End}_B A_B \) of a depth two extension \( A \mid B \) has a bialgebroid structure as in Lu [14]. Interesting classes of examples were noted such as finite dimensional algebras, weak Hopf-Galois extensions, \( H \)-separable extensions and various normal subobjects in quantum algebra. Later in [9], the underlying fact emerged that any (including infinite index) algebra extension \( A \mid B \) is right Galois w.r.t. a bialgebroid action if and only if it is right depth two and the natural module \( A_B \) is balanced. This is in several respects analogous to the characterization of a Galois field extension as normal and separable (the splitting field of separable polynomials).

A bialgebroid is in simplest terms a bialgebra over a noncommutative base, which are of interest from the point of view of tensor categories [6] and mathematical physics [2]. A depth two extension \( A \mid B \) has a Galois action machinery consisting of a bialgebroid with base algebra the centralizer \( C_A(B) \) of the extension. For example, if \( A \) is a commutative ring, then this Galois bialgebroid specializes to the Galois biring in Winter [24]. If \( A \mid B \) is Frobenius and \( C_A(B) \) is semisimple in all base field extensions, as with reducible subfactors, the extension is a weak Hopf-Galois extension, where the Galois bialgebroid is a weak Hopf algebra [10] [4]. Bialgebroids equipped with antipodes are Hopf algebroids, a recent object of study with competing definitions of what constitutes an antipode [3] [14] [13]. In Section 3,
we investigate the Galois bialgebroid of a new type of depth two extension called a pseudo-Galois extension, which is a notion generalizing the notions of H-separable extension (e.g. an Azumaya algebra) and group-Galois extension (e.g. a Galois algebra) \[17\]. Its Galois bialgebroid is shown in Theorem 2.3 to be closely related to a certain Hopf algebroid (in the sense of Böhm and Szlachányi \[3\]) we obtain from a cofibered sum of a semi-direct product of an algebra with a group of automorphisms and its opposite right crossed product. This Hopf algebroid extends Lu’s basic Hopf algebroid on the enveloping algebra over an algebra \[14\], to the group action setting, and in Theorem 3.1 to the involutive Hopf algebra action setting. In the last section of this paper we discuss Van Oystaeyen and Panaite’s isomorphism of this “enveloping” Hopf algebroid given in the preprint QA/0508411 to this paper with a Hopf algebroid in Connes-Moscovici \[5\], which is known as a para-Hopf algebroid in \[13\]. The isomorphism may be derived from a universal condition in Proposition 3.4 or from the unit representation of a bialgebroid on its base algebra. The isomorphism of bialgebroid invariants naturally raises the possibility of a relation between certain pseudo-Galois extensions, or a generalization we propose, with Rankin-Cohen brackets \[5\].

2. Depth two and pseudo-Galois extensions

All algebras in this paper are unital associative algebras over a commutative ground ring \(K\). An algebra extension \(A \mid B\) is a unit-preserving algebra homomorphism \(B \to A\), a proper extension if this mapping is monic. The induced bimodule \(B \otimes_A B\) plays the main role below. Unadorned tensors, hom-groups and endomorphism-groups (in a category of modules) between algebras are over the ground ring unless otherwise stated. For example, \(\text{End}_A\) denotes the linear endomorphisms of an algebra \(A\), but not the algebra endomorphisms of \(A\); and \(\text{Hom}(A \otimes_A B, B)\) denotes the \(B\)-\(A\)-bimodule of right \(B\)-bimodule homomorphisms from \(A \otimes_A B\) into \(B\). The default setting is the natural module structure unless otherwise specified.

An algebra extension \(A \mid B\) is left depth two (D2) if its tensor-square \(A \otimes_B A\) as a natural \(B\)-\(A\)-bimodule is isomorphic to a direct summand of a finite direct sum of the natural \(B\)-\(A\)-bimodule \(A\): for some positive integer \(N\), we have
\[
A \otimes_B A \oplus \ast \cong A^N
\]
An extension \(A \mid B\) is right D2 if eq. (1) holds instead as natural \(A\)-\(B\)-bimodules.

Since condition (1) implies maps in two hom-groups satisfying \(\sum_{i=1}^N g_i \circ f_i = \text{id}_{A \otimes_B A}\), where \(g_i \in \text{Hom}(B \otimes_A B A, A) \cong (A \otimes_B A)^B\) (via \(g \mapsto g(1)\)) and \(f_i \in \text{Hom}(B A \otimes_A B A, B A) \cong \text{End}_B A_B := S\) via \(f \mapsto (a \mapsto f(a \otimes_B 1))\), we obtain an equivalent condition for extension \(A \mid B\) to be left D2: there is a positive integer \(N\), \(\beta_1, \ldots, \beta_N \in S\) and \(t_1, \ldots, t_N \in (A \otimes_B A)^B\) (i.e., satisfying for each \(i = 1, \ldots, n\), \(bt_i = t_i b\) for every \(b \in B\)) such that
\[
\sum_{i=1}^N t_i \beta_i(x) y = x \otimes_B y
\]
for all \(x, y \in A\).

Like dual bases for projective modules, this equation is useful. For example, to show \(S\) finite projective as a left \(C_A(B)\)-module (module action given by \(r \cdot \alpha = \))
\[ \rho_r \circ \lambda_s = \lambda_r \circ \rho_s, \quad r, s \in R := C_A(B), \]

whence we may define a bimodule by composing strictly from the left:

\[ r \cdot \alpha \cdot s = \lambda_r \circ \rho_s \circ \alpha = r \alpha(?)s, \quad \alpha \in \mathcal{S}, \quad r, s \in R. \]

Next we equip \( \mathcal{S} \) with an \( R \)-coring structure \( (\mathcal{S}, R, \Delta, \varepsilon) \) as follows. The comultiplication \( \Delta : \mathcal{S} \to \mathcal{S} \otimes_s \mathcal{S} \) is an \( R \)-\( R \)-homomorphism given by

\[ \Delta(\alpha) := \sum_i \alpha(?)t_i^1 \otimes_R \beta_i = \sum_j \gamma_j \otimes_R u_j^2 \alpha(?) \]

in terms of left \( D_2 \) quasibases in the first equation or right \( D_2 \) quasibases in the second equation. There is a simplification that shows this comultiplication is a
generalization of the one in [14, Lu] (for the linear endomorphisms of an algebra):

(6) \( \phi : S \otimes_R S \xrightarrow{\cong} \text{Hom}(\text{End}_B A, B \text{End}_B R) \)

\[ \phi(\alpha \otimes_R \beta)(x \otimes_R y) = \alpha(x)\beta(y) \]

for all \( \alpha, \beta \in S \) and \( x, y \in A \). From a variant of eq. (5) we obtain:

(7) \( \phi(\Delta(\alpha))(x \otimes_R y) = \alpha(xy) \), \((x, y \in A, \alpha \in \text{End}_B A_B)\).

The counit \( \varepsilon : S \to R \) is given by evaluation at the unity element, \( \varepsilon(\alpha) = \alpha(1_A) \), again an \( R-R \)-homomorphism. It is then apparent from eq. (7) that

\[ (\varepsilon \otimes_R \text{id}_S)\Delta(\alpha) = \varepsilon(\alpha(1)) \cdot \alpha(2) = \lambda_{\alpha(1)}(1) \alpha(2) = \alpha, \quad (\alpha \in S) \]

using a reduced Sweedler notation for the coproduct of an element, and a similar equation corresponding to \((\text{id}_S \otimes_R \varepsilon)\Delta = \text{id}_S\).

Finally, the comultiplication and counit satisfy additional bialgebra-like axioms that make \((S, R, \lambda, \rho, \Delta, \varepsilon)\) a left \( R \)-bialgebroid [12, p. 80]. These are:

(8) \[ \varepsilon(1_S) = 1_R, \]

which is obvious,

(9) \[ \Delta(1_S) = 1_S \otimes_R 1_S, \]

which follows from eq. (7),

(10) \[ \forall \alpha \in S, \quad \alpha(1) \circ \rho_r \otimes_R \alpha(2) = \alpha(1) \otimes_R \alpha(2) \circ \lambda_r \]

which follows from the equation defining \( \phi \) (since both sides yield \( \alpha(xy) \)),

(11) \[ \Delta(\alpha \circ \beta) = \Delta(\alpha)\Delta(\beta) = \alpha(1) \circ \beta(1) \otimes_R \alpha(2) \circ \beta(2) \]

where eq. (10) justifies the use of a tensor algebra product in \( \text{Im} \Delta \subseteq S \otimes_R S \) and the equation follows again from equation defining \( \phi \) (as both sides equal \( \alpha(\beta(xy)) \)), and at last the easy

(12) \[ \varepsilon(\alpha \circ \beta) = \varepsilon(\alpha \circ \lambda_{\varepsilon(\beta)}) = \varepsilon(\alpha \circ \rho_{\varepsilon(\beta)}). \]

On occasion the bialgebroid \( S \) is a Hopf algebroid [3], i.e. possesses an antipode \( \tau : S \to S \). This is anti-automorphism of the algebra \( S \) which satisfies:

(1) \[ \tau \circ \rho = \lambda, \]

(2) \[ \tau^{-1}(\alpha(2)(1)) \otimes_R \tau^{-1}(\alpha(2)(2))\alpha(1) = \tau^{-1}(\alpha) \otimes 1_S \quad (\forall \alpha \in S); \]

(3) \[ \tau(\alpha(1)(1))\alpha(2) \otimes_R \tau(\alpha(1)(2)) = 1_S \otimes_R \tau(\alpha) \quad (\forall \alpha \in S). \]

Examples of Hopf algebroids are weak Hopf algebras [6] and Hopf algebras, including group algebras and enveloping algebras of Lie algebras [20]. Lu [14] provides the example \( A \otimes A^\text{op} \) of a Hopf algebroid over any algebra \( A \) with twist being the antipode, and another bialgebroid the linear endomorphisms \( \text{End}_R A \), which is a particular case of the construction \( S \).

A homomorphism of \( R \)-bialgebroids \( S' \to S \) is an algebra homomorphism which commutes with the source and target mappings, which additionally is an \( R \)-coring homomorphism (so there are three commutative triangles and a commutative square for such an algebra homomorphism to satisfy) [3]. If \( S' \) and \( S \) additionally come equipped with antipodes \( \tau' \) and \( \tau \), respectively, then the homomorphism \( S' \to S \) is additionally a homomorphism of Hopf algebroids if it commutes with the antipodes in an obvious square diagram.
2.1. Pseudo-Galois extensions. If \( \sigma \) is automorphism of the algebra \( A \), we let \( A_\sigma \) denote the bimodule \( A \) twisted on the right by \( \sigma \), with module actions defined by \( x \cdot a \cdot y = x a \sigma(y) \) for \( x, y, a \in A \). Two such bimodules \( A_\sigma \) and \( A_\tau \) twisted by automorphisms \( \sigma, \tau : A \rightarrow A \) are \( A \)-\( A \)-bimodule isomorphic if and only if there is an invertible element \( u \in A \) such that \( \tau \circ \sigma^{-1} \) is the inner automorphism by \( u \) (for send \( 1 \mapsto u \)).

Let \( B \) be a subalgebra of \( A \). Recall the characterization of a group-Galois extension \( A \mid B \), where only two conditions need be met. First, there is a finite group \( G \) of automorphisms of \( A \) such that \( B = A^G \), i.e. the elements of \( B \) are fixed under each automorphism of \( G \) and each element of \( A \) in the complement of \( B \) is moved by some automorphism of \( G \). Second, there are elements \( a_i, b_i \in A, i = 1, \ldots, n \) such that \( \sum a_i b_i = 1 \) and \( \sum a_i \sigma(b_i) = 0 \) if \( \sigma \neq \text{id}_A \).

Since \( E : A \rightarrow B \) defined by \( E(a) = \sum_{\sigma \in G} \sigma(a) \) is a Frobenius homomorphism with dual bases \( a_i, b_i \), it follows that there is an \( A \)-\( A \)-bimodule isomorphism between the tensor-square and the semi-direct product of \( A \) and \( G \):

\[
(13) \quad h : A \otimes_B A \rightarrow A \times G, \quad h(x \otimes y) := \sum_{\sigma \in G} x \sigma y = \sum_{\sigma \in G} x \sigma(y) \sigma
\]

(For the inverse is given by \( h^{-1}(ar) = \sum_i ar(a_i) \otimes_B b_i \).) Thus \( A \otimes_B A \) is isomorphic to \( \oplus_{\sigma \in G} A_\sigma \) as \( A \)-\( A \)-bimodules. Mewborn and McMahon [17] relax this condition as follows:

**Definition 2.1.** The algebra \( A \) is a pseudo-Galois extension of a subalgebra \( B \) if there is a finite set \( G \) of \( B \)-automorphisms (i.e. fixing elements of \( B \)) and a positive integer \( N \) such that \( A \otimes_B A \) is isomorphic to a direct summand of \( \oplus_{\sigma \in G} A_\sigma^N \); in symbols this becomes

\[
(14) \quad A \otimes_B A \oplus \ast \cong \oplus_{\sigma \in G} A_\sigma^N,
\]

in terms of \( A \)-\( A \)-bimodules. Assume with no loss of generality that \( G \) is minimal in the group of \( B \)-automorphisms \( \text{Aut}_B(A) \) with respect to this property and \( A_\sigma \neq A_\tau \) if \( \sigma \neq \tau \) in \( G \).

It is clear that Galois extensions are pseudo-Galois. For example, let \( A \) be a simple ring with finite group \( G \) of outer automorphisms of \( A \), then \( A \) is Galois over its fixed subring \( B = A^G \), cf. [19] 2.4. Another example of a pseudo-Galois extension is an \( H \)-separable extension \( A \mid B \), which by definition satisfies \( A \otimes_B A \oplus \ast \cong A^N \) as \( A \)-\( A \)-bimodules, so we let \( G = \{ \text{id}_A \} \) in the definition above [7, 15, 17]. For example, if \( A \) is a simple ring, \( G \) a finite group of outer automorphisms of \( A \) such that each nonidentity automorphism moves an element of the center of \( A \), then the skew group ring \( A \times G \) is \( H \)-separable over \( A \) [15].

Note that the definition of pseudo-Galois extension \( A \mid B \) leaves open the possibility that \( B \) is a proper subset of the invariant subalgebra \( A^G \): if \( B \subset C \subset A^G \) and \( C \) is a separable extension of \( B \), then \( A \mid C \) is also a pseudo-Galois extension, since one may show that \( A \otimes_C A \oplus \ast \cong A \otimes_B A \) as natural \( A \)-\( A \)-bimodules via the separability element. Conversely, if \( A \mid C \) is a pseudo-Galois extension and \( C \supset B \) is \( H \)-separable, then \( A \mid B \) is pseudo-Galois by noting that \( A \otimes_B A \cong A \otimes_C C \otimes_B C \otimes_C A \). For example, if \( E \mid F \) is a finite Galois extension of fields where \( F \) is the quotient field of a domain \( R \), then the ring extension \( E \mid R \) is pseudo-Galois.
In the next proposition, we note that pseudo-Galois extensions are depth two by means of a characterization of pseudo-Galois extensions using pseudo-Galois elements.

**Proposition 2.2.** An algebra extension $A | B$ is pseudo-Galois iff there is a finite set $G$ of $B$-automorphisms and $N$ elements $r_{i, \sigma} \in C_A(B)$ and $N$ elements $e_{i, \sigma} \in (\sigma A \otimes_B A)^A$ for each element $\sigma \in G$ satisfying

$$1 \otimes_B 1 = \sum_{\sigma \in G} \sum_{i=1}^N r_{i, \sigma} e_{i, \sigma} \quad (15)$$

As a consequence, $A | B$ is left and right D2 with left and right D2 quasibases derived from the elements $r_{i, \sigma}$ and $e_{i, \sigma} \in (A \otimes_B A)^B$.

**Proof.** $(\Rightarrow)$ We note that the condition $[14]$ implies the existence of $N$ pairs of mappings $f_{i, \sigma}$ and $g_{i, \sigma}$ for each $B$-automorphism $\sigma \in G$ satisfying

$$\sum_{i=1}^N \sum_{\sigma \in G} g_{i, \sigma} \circ f_{i, \sigma} = \text{id}_{A \otimes_B A} \quad (16)$$

The mappings simplify as

$$f_{i, \sigma} \in \text{Hom}(A A \otimes_B A A, A (A_{\sigma})_A) \cong C_A(B),$$

via $F \mapsto F(1 \otimes 1)$ with inverse $r \mapsto (a \otimes a' \mapsto ar \sigma(a'))$, as well as mappings

$$g_{i, \sigma} \in \text{Hom}(A (A_{\sigma})_A, A A \otimes_B A A) \cong (\sigma A \otimes_B A)^A,$$

via $f \mapsto f(1)$ with inverse $e \mapsto (a \mapsto ae)$ where $e \in (\sigma A \otimes_B A)^A$ iff $ea = \sigma(a)e$ for each $a \in A$.

If $r_{i, \sigma}$ corresponds via the isomorphism above with $f_{i, \sigma}$, then $f_{i, \sigma}(x \otimes_B y) = xr_{i, \sigma} \sigma(y)$ for each $x, y \in A$. If $e_{i, \sigma}$ corresponds via the other isomorphism above with $g_{i, \sigma}$, then $g_{i, \sigma}(a) = ae_{i, \sigma}$. We compute:

$$x \otimes_B y = \sum_{i, \sigma \in G} (g_{i, \sigma} \circ f_{i, \sigma})(x \otimes y) = \sum_{\sigma \in G} \sum_{i=1}^N x r_{i, \sigma} \sigma(y) e_{i, \sigma}, \quad (17)$$

which shows that

$$\lambda_{r_{i, \sigma}} \circ \sigma \in \text{End}_{B} A_B, \quad e_{i, \sigma} \in (A \otimes_B A)^B \quad (18)$$

are right D2 quasibases. Setting $x = y = 1$ we obtain the eq. $[15]$. Finally use the twisted centralizer property $ea = \sigma(a)e$ for $a \in A$ and $e \in (\sigma A \otimes_B A)^A$ to obtain

$$x \otimes_B y = \sum_{\sigma \in G} \sum_{i=1}^N e_{i, \sigma} \sigma^{-1}(x) \sigma^{-1}(r_{i, \sigma}) y, \quad (19)$$

Hence, the following are left D2 quasibases for $A | B$:

$$e_{i, \sigma} \in (A \otimes_B A)^B, \quad \sigma^{-1} \circ r_{i, \sigma} \in \text{End}_{B} A_B \quad (\Leftarrow)$$

Conversely, suppose we are given a finite set $G$ of $B$-automorphisms, and for each $\sigma \in G$, $N$ centralizer elements $r_{i, \sigma} \in C_A(B)$, and $N$ twisted $A$-central elements $e_{i, \sigma} \in (\sigma A \otimes_B A)^A$ for $i = 1, \ldots, N$ such that eq. $[14]$ holds. By multiplying the equation from the left by $x \in A$ and from the right by $y \in A$, we obtain eq. $[17]$ and then eq. $[18]$ by defining $f_{i, \sigma}$ and $g_{i, \sigma}$ as before, which is of course equivalent to the condition $[14]$ for pseudo-Galois extension. \qed
We note that pseudo-Galois elements in eq. (16) specialize to H-separability elements in case $G = \{ id_A \} \mathbb{I} 2.5$.

Recall that in homological algebra the enveloping algebra of an algebra $A$ is denoted by $A^e := A \otimes A^{\text{op}}$.

**Theorem 2.3.** Suppose $A|B$ is pseudo-Galois extension satisfying condition (12) with $G$ the subgroup generated by $G$ within $\text{Aut}_R A$ and $R$ the centralizer $C_A(B)$. Then there is a Hopf algebroid denoted by $R^e \rtimes G$ which maps epimorphically as $R$-bialgebroids onto the left bialgebroid $S = \text{End}_B A^e$.

**Proof.** Denote the identity in $G$ by $e$ and the canonical anti-isomorphism $R \to R^{\text{op}}$ by $r \mapsto \overline{r}$ satisfying $r s = \overline{s} \overline{r}$ for every $r, s \in R$. Note that the $B$-automorphisms of $G$ restrict to automorphisms of the centralizer $R$. The notation $C$ for $R^e \rtimes G$ is adopted at times. The Hopf algebroid structure of $R^e \rtimes G$ is given by

1. as a $K$-module (over the ground ring $K$) $R^e \rtimes G = R \otimes R^{\text{op}} \otimes K[G]$ where $K[G]$ is the group $K$-algebra of $G$;
2. multiplication given by

$$
(r \otimes \overline{s} \otimes \sigma)(u \otimes \overline{t} \otimes \tau) = r \sigma(u) \otimes \overline{v} r^{-1}(s) \otimes \sigma \tau
$$

with unity element $1_C = 1_R \otimes \overline{1_R} \otimes e$,
3. source map $s_L : R \to R^e \rtimes G$ given by $s_L(r) = r \otimes \overline{1_R} \otimes e$,
4. target map $t_L : R^{\text{op}} \to R^e \rtimes G$ given by $t_L(r) = 1 \otimes \overline{r} \otimes e$,
5. counit $\varepsilon_C : R^e \rtimes G \to R$ given by $\varepsilon_C(r \otimes \overline{s} \otimes \sigma) = r \sigma(s)$,
6. comultiplication $\Delta_C : C \to C \otimes_R C$ is given by

$$
\Delta(r \otimes \overline{s} \otimes \sigma) = (r \otimes \overline{1_R} \otimes \sigma) \otimes_R (1 \otimes \overline{s} \otimes \sigma),
$$

(7) and the antipode $\tau : C \to C$ by $\tau(r \otimes \overline{s} \otimes \sigma) = s \otimes \overline{r} \otimes \sigma^{-1}$

We will postpone the proof that this defines a Hopf algebroid over $R$ until the next section where it is shown more generally for an involutive Hopf algebra $H$ and its $H$-module algebras.

The epimorphism of left $R$-bialgebroids $\Psi : R^e \rtimes G \to S$ is given by

$$
\Psi(r \otimes \overline{s} \otimes \sigma) = \lambda_r \circ \sigma \circ \rho_s
$$

We note that $\Psi$ is an algebra homomorphism by comparing eq. (20) with

$$
\lambda_r \circ \sigma \circ \rho_s \circ \lambda_u \circ \sigma \circ \rho_v = \lambda_{r \sigma(u)} \circ \sigma \circ \rho_{v r^{-1}(s)},
$$

and $\Psi(1_C) = 1_S$. The mapping $\Psi$ is epimorphic since each $\beta \in S$ may be expressed as a sum of mappings of the form $\lambda_r \circ \sigma \circ \rho_s$ where $\sigma \in G$ and $r, s \in R$. To see this, apply $\mu(id_A \otimes \beta)$ to eq. (17) with $x = 1$, which yields

$$
\beta(y) = \sum_{i, \sigma \in G} r_{i, \sigma}(y) e_{i, \sigma}^1 e_{i, \sigma}^2 \beta (e_{i, \sigma}^1)
$$

where $e_{i, \sigma}^1 \beta (e_{i, \sigma}^2) \in R$ for each $i$ and $\sigma$.

Note next that $\Psi$ commutes with source, target and counit maps. For $\Psi(s_L(r)) = \Psi(r \otimes \overline{1_R} \otimes e) = \lambda_r$ and $\Psi(t_L(r)) = \Psi(1 \otimes \overline{r} \otimes e) = \rho_r$ for $r \in R$ (so $\Psi : C \to S$ is an $R$-$R$-bimodule map). The map $\Psi$ is counital since

$$
\varepsilon(\Psi(r \otimes \overline{s} \otimes \sigma)) = \lambda_r(\sigma(\rho_s(1))) = r \sigma(s) = \varepsilon_C(r \otimes \overline{s} \otimes \sigma).
$$
Using the isomorphism $\phi : S \otimes_R S \to \text{Hom}(B_A \otimes B_B B_A)$ for a depth two extension $A | B$ defined as above by $\phi(\alpha \otimes R \beta)(x \otimes_B y) = \alpha(x)\beta(y)$, note from eq. \[21\] that $\Psi$ is comultiplicative:

$$
\phi((\Psi \otimes R \Psi)(\Delta_{C}(r \otimes \varpi \otimes \sigma)))(x \otimes_B y) = \lambda_r(\sigma(x))\sigma(\rho_s(y)) = \\
\phi(\Delta(\lambda_r \circ \sigma \circ \rho_s))(x \otimes_B y) = \phi(\Delta(\Psi(r \otimes \varpi \otimes \sigma)))(x \otimes_B y)
$$

since $\sigma \in G$ is a group-like element satisfying $\sigma(1) \otimes_R \sigma(2) = \sigma \otimes_R \sigma$ (corresponding to the automorphism condition).

\[\square\]

Corollary 2.4. If the algebra extension $A | B$ is $H$-separable, then $G = \{\text{id}_A\}$ and $\Psi : R^c \to S$ is an isomorphism of bialgebroids, whence $S$ has the antipode $\Psi \circ \tau \circ \Psi^{-1}$.

If $A | B$ is $G$-Galois, then $\Psi : R^c \bowtie G \to S$ is a split epimorphism of bialgebroids.

Proof. Note that $R^c$ is isomorphic as algebras to the subalgebra $R^c \bowtie \{e\}$. The first statement follows from \[S\], since $\Psi(r \otimes \varpi) = \lambda_r \circ \rho_s$ is shown there to be an isomorphism of bialgebroids.

If $A | B$ is $G$-Galois, then $A \bowtie B \cong A \times G$ via $h$ above. Since each $\sigma \in G$ fixes elements of $B$, it follows that $(A \otimes_B A)^B \cong R \times G$. Since $A | B$ is a Frobenius extension, $\text{End}_B A \cong A \otimes_B A$ via $f \mapsto \sum_i f(a_i) \otimes b_i$ with inverse $x \otimes_B y \mapsto \lambda_r \circ \rho_s \circ \lambda_y$. This restricts to $\text{End}_B A \cong (A \otimes_B A)^B$. Putting the two together yields

$$
\Phi : S \xrightarrow{\cong} R \times G, \quad \Phi(\alpha):= \sum_{\sigma \in G} \sum_{i=1}^n \alpha(a_i)\sigma(b_i)\sigma
$$

with inverse $r \times \tau \mapsto \lambda_r \circ \tau$. Then the algebra epimorphism $\Phi \circ \Psi : R^c \bowtie G \to R \times G$ simplifies to

$$
(\Phi \circ \Psi)(r \otimes \varpi \bowtie \sigma) = \Phi(\lambda_r \circ \sigma \circ \rho_s) = \sum_{\tau \in G} \sum_{i=1}^n \lambda_{r\sigma(\tau(a_i))} \circ \sigma \circ \tau.
$$

which is split by the monomorphism $R \times G \to R^c \bowtie G$ given by $r \times \sigma \mapsto r \otimes \varpi \bowtie \sigma$, an algebra homomorphism by an application of eq. \[21\].

\[\square\]

3. AN ENVELOPING HOPF ALGEBROID OVER ALGEBRAS IN CERTAIN TENSOR CATEGORIES

Let $H$ be a Hopf algebra with bijective antipode $S$ and $A$ a left $H$-module algebra, i.e. an algebra in the tensor category of $H$-modules. Motivated by the left bialgebroid of a pseudo-Galois extension as studied in section 3, we define a type of enveloping algebra $A^c \bowtie H$ for the smash product algebra $A \times H$. It is a left bialgebroid over $A$, and a Hopf algebroid in case $H$ is involutive such as a group algebra or the enveloping algebra of Lie algebra. In terms of noncommutative algebra, it is the minimal algebra which contains subalgebras isomorphic to the Hopf algebra $H$, the standard enveloping algebra $A^c$ of an algebra $A$, and the semi-direct or crossproduct algebra $A \times H$ as well as its derived right crossproduct algebra $H \times A\text{_{op}}$. In terms of category theory, it is derived from the pushout construction \[13\] of the inclusion $H \hookrightarrow A \times H$ and its opposite via the isomorphism $S : H \to H\text{_{cop,op}}$.

Theorem 3.1. Suppose $B := A^c \bowtie H$ is the vector space $A \otimes A\text{_{op}} \otimes H$ with multiplication

\[23\] $(a \bowtie h \bowtie h)(c \bowtie d \bowtie k) := a(h_{(1)} \cdot c) \otimes d(S(k_{(2)}) \cdot b) \bowtie h_{(2)}k_{(1)}$. 

Then $B$ is a left bialgebroid over $A$ with structure given in eqs. (24) through (28). If $S^2 = \text{id}_H$, then $B$ is a Hopf algebroid with antipode eq. (29).

Proof. Clearly the unity element $1_B = 1_A \otimes 1_A \bowtie 1_H$. The multiplication is associative, since
\[
[(a \otimes b) \bowtie h](c \otimes d \bowtie k)|c \otimes f j) = (a(h_1) \cdot c) \otimes d(S(k_2) \cdot b) \bowtie h_2 \cdot k_1 = \epsilon h_1 \cdot c \otimes S(k_2) \cdot b \bowtie h_2 \cdot k_1 = (a \otimes b \bowtie h) \cdot (c \otimes d \bowtie k) \cdot (e \otimes f j) = \epsilon h_1 \cdot c \otimes S(k_2) \cdot b \bowtie h_2 \cdot k_1,
\]
It follows that $B$ is an algebra.

Define a source map $s_L : A \to B$ and target map $t_L : A \to B$ by
\[
s_L(a) = a \otimes 1_A \bowtie 1_H
\]
\[
t_L(a) = 1_A \otimes \pi \bowtie 1_H,
\]
an algebra homomorphism and anti-homomorphism, respectively. It is evident that $t_L(x)s_L(y) = s_L(y)t_L(x)$ for all $x, y \in A$. The $A$-$A$-bimodule structure induced from $x \cdot b \cdot y = s_L(x)t_L(y)b$ for $b \in B$ is then given by $(a, c \in A, h \in H)$
\[
x \cdot (a \otimes \pi \bowtie h) \cdot y = xa \otimes c(S(h_2) \cdot y) \bowtie h_1.
\]

The counit $\epsilon : B \to A$ is defined by
\[
\epsilon(a \otimes \pi \bowtie h) := a(h \cdot c).
\]
Note that $\epsilon$ is an $A$-$A$-bimodule homomorphism via its application to the RHS of eq. (28):
\[
\epsilon(xa \otimes c(S(h_2) \cdot y) \bowtie h_1) = xa \cdot h_1 \cdot (c(S(h_2) \cdot y)) = xa(h \cdot c) = xa \cdot \epsilon(a \otimes \pi \bowtie h) \cdot y,
\]
since $h \cdot (xy) = h_1 \cdot x(h_2) \cdot y$, the measuring axiom on $A$.

The comultiplication $\Delta : B \to B \otimes_A B$ is defined by
\[
\Delta(a \otimes \pi \bowtie h) := (a \otimes 1_A \bowtie h_1) \otimes_A (1_A \otimes \pi \bowtie h_2).
\]
It is an $A$-$A$-homomorphism:
\[
\Delta(xa \otimes c(S(h_2) \cdot y) \bowtie h_1) = (xa \otimes 1_A \bowtie h_1) \otimes_A (1_A \otimes c(S(h_2) \cdot y) \bowtie h_2)
\]
\[
= \epsilon \cdot \Delta(a \otimes \pi \bowtie h) \cdot y
\]
by eq. (28). The left counit equation $(\epsilon \otimes_A \text{id}_B)\Delta = \text{id}_B$ follows from
\[
\epsilon(a \otimes 1_A \bowtie h_1) \cdot (1 \otimes \pi \bowtie h_2) = a(h_1) \cdot 1 \cdot (1 \otimes \pi \bowtie h_2) = a \otimes \pi \bowtie h,
\]
since $h \cdot 1_A = \epsilon(h)1_A$ in the $H$-module algebra $A$. The right counit equation $(\text{id}_B \otimes \epsilon)\Delta = \text{id}_B$ follows from
\[
(a \otimes 1_A \bowtie h_1) \cdot \epsilon(1 \otimes \pi \bowtie h_2) = (a \otimes 1 \bowtie h_1) \cdot (h_2 \cdot c) = a \otimes (S(h_2) \cdot h_2) \cdot c \bowtie h_1 = a \otimes \pi \bowtie h.
\]
Hence $(B, A, \Delta, \epsilon)$ is an $A$-$coring$.

We check the remaining bialgebroid axioms:
\[
\Delta(1_B) = 1_B \otimes 1_B, \quad \epsilon(1_A) = 1_A.
\]
are apparent from eqs. (28) and (27). The axiom corresponding to eq. (10) computes as:

\[
\Delta(a \otimes b \bowtie h)(t_L(c) \otimes_A 1_B) = (a \otimes 1 \bowtie h_{(1)}) (1 \otimes \tau \bowtie 1_H) \otimes_A (1 \otimes b \bowtie h_{(2)}) = (a \otimes \tau \bowtie h_{(1)}) \otimes_A (1 \otimes b \bowtie h_{(2)}),
\]

and on the other hand

\[
\Delta(a \otimes b \bowtie h)(1_B \otimes_A s_L(c)) = (a \otimes 1 \bowtie h_{(1)}) \otimes_A (1 \otimes b \bowtie h_{(2)}) (c \otimes 1 \bowtie 1_H) = (a \otimes 1 \bowtie h_{(1)}) \otimes_A (1 \otimes b \bowtie h_{(2)}) (c \otimes 1 \bowtie h_{(3)}) \otimes_A (1 \otimes b \bowtie h_{(4)}) = \Delta(a \otimes b \bowtie h) (t_L(c) \otimes_A 1_B).
\]

Next, the comultiplication is multiplicative:

\[
\Delta((a \otimes \overline{b} \bowtie h)(c \otimes \overline{d} \bowtie k)) = \Delta((a(h_1) \otimes \overline{d} \bowtie k) \bowtie h_{(2)}k_{(1)}) = (a(h_1) \otimes \overline{d} \bowtie k_{(1)} \bowtie c(h_2)(h_3) \cdot b) = (a(h_1) \cdot c(h_2)(h_3) \cdot b) = (a \bowtie \overline{b} \bowtie h)(c \otimes \overline{d} \bowtie k).
\]

The counit satisfies

\[
\varepsilon((a \otimes \overline{b} \bowtie h)(c \otimes \overline{d} \bowtie k)) = \varepsilon((a \bowtie h \bowtie h_{(2)})(c \otimes \overline{d} \bowtie k)) = \varepsilon((a \bowtie h \bowtie h_{(1)}) \bowtie h_{(2)}(c \otimes \overline{d} \bowtie k)) = \Delta(\overline{a} \bowtie h) \Delta(c \otimes \overline{d} \bowtie k).
\]

Similarly, \(\varepsilon((a \otimes \overline{b} \bowtie h)(c \otimes \overline{d} \bowtie k)) = \varepsilon((a \bowtie \overline{h} \bowtie h_{(1)}) \bowtie h_{(2)}(c \otimes \overline{d} \bowtie k))\) for all \(a, b, c, d \in A, h, k \in H\). Thus \(B\) is a bialgebroid over \(A\).

Suppose the antipode on \(H\) is bijective and satisfies \(S^2 = \text{id}_H\). Define an antipode on \(B\) by \(a, b \in A, h \in H\)

\[
(29) \quad \tau(a \otimes \overline{b} \bowtie h) = b \otimes \overline{a} \bowtie S(h).
\]

Denote the compositional inverse of \(S\) by \(\overline{S}\). Then \(\tau\) has inverse,

\[
\tau^{-1}(a \otimes \overline{b} \bowtie h) = b \otimes \overline{a} \bowtie \overline{S}(h).
\]

Note that \(\tau\) is an anti-automorphism of \(B\):

\[
\tau((c \otimes \overline{d} \bowtie k)) = (d \otimes \overline{S}(k_{(1)}) \bowtie \overline{S}(k_{(2)})) = S(h_{(1)})S(h_{(2)}) = \tau(a \bowtie h_{(1)} \bowtie h_{(2)}).
\]

The antipode satisfies the three axioms (11) - (13):

\[
\tau(t_L(a)) = \tau(1 \otimes \overline{a} \bowtie 1_H) = 1 \otimes \overline{a} \bowtie 1_H = s_L(a),
\]

for all \(a \in A\). Next, for \(b := a \otimes \overline{b} \bowtie h \in B\),

\[
\tau^{-1}(b_{(1)}) \otimes_A \tau^{-1}(b_{(2)}) = (c \otimes \overline{S}(h_{(1)}) \bowtie \overline{S}(h_{(2)})) \bowtie \overline{S}(h_{(3)}) \bowtie h_{(4)} = c \otimes \overline{S}(h_{(1)}) \bowtie \overline{S}(h_{(2)}) \bowtie h_{(3)} \bowtie h_{(4)} \bowtie \overline{S}(h_{(5)}) \bowtie h_{(6)} = 1_B \otimes_A 1_B.
\]

Continuing our notation \(b = a \otimes \overline{b} \bowtie h \in B\), note too that

\[
\tau(b_{(1)}) \otimes \tau(b_{(2)}) = (1 \otimes \overline{S}(h_{(1)}) \bowtie \overline{S}(h_{(2)})) \bowtie \overline{S}(h_{(3)}) \bowtie h_{(4)} = 1_B \otimes \overline{S}(h_{(1)}) \bowtie \overline{S}(h_{(2)}) = 1_B \otimes \overline{S}(h_{(1)}) \bowtie \overline{S}(h_{(2)}) = 1_B \otimes \overline{S}(h_{(1)}) = 1_B \otimes 1_B.
\]

Hence, \(B\) is a Hopf algebroid. \(\square\)
Given a group $G$, its group algebra $K[G]$ over a commutative ring $K$ is an involutive Hopf algebra [24]. Moreover, if $G$ acts by automorphisms on a $K$-algebra $A$, then $A$ is a left $K[G]$-module algebra and $A \rtimes G$ is identical with the semidirect product [24]. Thus the construction $R^e \bowtie G$ (covering the left bialgebroid of a pseudo-Galois extension in Section 3) is a Hopf algebroid, and we record the following.

Corollary 3.2. Given a $K$-algebra $A$ and a group $G$ of algebra automorphisms of $A$, the algebra $A^e \bowtie K[G]$ is a Hopf algebroid over $A$.

Recall that Lu [14] defines over an algebra $A$ a Hopf algebroid $A^e$. This is a Hopf subalgebroid of the construction in the theorem above.

Corollary 3.3. Let $H$ be an involutive Hopf algebra and $A$ a left $H$-module algebra. Then the Hopf algebroid $A^e \bowtie H$ contains subalgebras isomorphic to

1. Lu’s Hopf algebroid $A \otimes A^{\text{op}}$
2. the semidirect product $A \rtimes H$
3. its derived right crossproduct $H \ltimes A^{\text{op}}$
4. the Hopf algebra $H$

Proof. It is easy to check from eq. [24] that the following mappings

1. $A^e \hookrightarrow A^e \bowtie H$ given by $a \otimes b \mapsto a \otimes b \bowtie 1_H$ is an algebra monomorphism as well as a homomorphism of Hopf algebroids over $A$ (i.e., it commutes with the source, target, counit, comultiplication and antipode maps above and those given in [14]);
2. $j_1 : A \times H \hookrightarrow A^e \bowtie H$ given by $j_1(a \# h) := a \otimes 1_H \bowtie h$ is an algebra monomorphism, where we recall that the multiplication in $A \rtimes H$ is given by
   $$(a \# h)(b \# k) = a(h_{(1)} \cdot b) \# h_{(2)} k$$
3. $j_2 : H \ltimes A^{\text{op}} \hookrightarrow A^e \bowtie H$ given by $j_2(h \# \overline{a}) := 1 \otimes \overline{a} \bowtie h$ is an algebra monomorphism, where $\overline{\sigma} \cdot h := \overline{\sigma(h) \cdot a}$ defines the derived right action of $H$ on $A^{\text{op}}$ and the multiplication in $H \ltimes A^{\text{op}}$ (cf. [16] p. 22) is given by
   $$(h \# \overline{a})(k \# \overline{b}) = h k_{(1)} \# (\overline{a} \cdot k_{(2)}) \overline{b}.$$ 
4. $H \hookrightarrow A^e \bowtie H$ given by $h \mapsto 1 \otimes \overline{1} \bowtie h$ is an algebra monomorphism as well as a Hopf algebra homomorphism (for it commutes with the counit, comultiplication and antipode mappings of $H$ and $A^e \bowtie H$ if $A$ is a faithful $K$-algebra and $K$ is identified with $K1_A$).

The construction $A^e \bowtie H$ for a Hopf algebra and a left $H$-module algebra is a type of cofibered sum [22] p. 99) of the algebra monomorphisms $i_1 : H \hookrightarrow A \rtimes H$ and $i_2 : H \hookrightarrow H \ltimes A^{\text{op}}$ defined by $i_1(h) := 1_A \# h$ and $i_2(h) := h \# 1_A$ for each $h \in H$. We note that $j_1 \circ i_1 = j_2 \circ i_2$, both sending $h \mapsto 1_H \otimes 1_A \bowtie h$. Also define the algebra monomorphism $k_1 : A \hookrightarrow A \rtimes H$ by $k_1(a) := a \# 1_H$ and anti-monomorphism $k_2 : A \hookrightarrow H \ltimes A^{\text{op}}$ by $k_2(a) := 1_H \# \overline{a}$. Note that

$$j_2(k_2(a))j_1(k_1(b)) = (1_A \otimes \overline{a} \bowtie 1_H)(b \otimes 1_A \bowtie 1_H) = j_1(k_1(b))j_2(k_2(a)),$$

for all $a, b \in A$. 

□
Then there is a uniquely defined algebra homomorphism

\[ H \otimes H \to A \times H \]

with Hopf algebroid comultiplication given on monomials by (integers \( k \geq p, q \geq 0 \))

\[ \Delta(X^n Y^m Z^k) = \sum_{p+q=k} \binom{k}{p} X^n Z^p \otimes A Y^m Z^q \]
count by
\[
\varepsilon(X^m Y^n Z^k) = \begin{cases} 
\frac{m!}{(m-k)!}X^{n+m-k} & \text{if } m \geq k \\
0 & \text{if } k > m
\end{cases}
\]
and antipode by
\[
\tau(X^m Y^n Z^k) = (-1)^k X^m Y^n Z^k.
\]

4. Discussion

Böhm and Brzeziński [24 A.1] generalize the construction \( A^c \bowtie H \) in the previous section to a certain module algebra \( A \) w.r.t. the action of a Hopf algebroid \( H \) which is twisted by an \( A \)-valued cocycle on \( H \).

Panaite and Van Oystaeyen [21] observe that the Hopf algebroid \( A^c \bowtie H \) constructed in the last section is isomorphic to the Hopf algebroid \( A \odot H \odot A \) in Connes-Moscovici [5] with antipode given in [13], which arises in a quite different context. The algebra \( A \odot H \odot A \) formed from a Hopf algebra \( H \) and a left \( H \)-module algebra \( A \) is an algebra homomorphism satisfying with \( f \), Note the algebra homomorphism
\[
(\alpha \otimes \beta \cdot x) (\gamma \otimes \delta \cdot y) = \alpha \otimes \beta \cdot (\gamma \otimes \delta \cdot x) (\gamma \otimes \delta \cdot y)
\]
leads to a mapping \( F : A^c \bowtie H \to A \odot H \odot A \) given by
\[
f_2(k \# b) := 1_A \otimes k(1) \odot k(2) \cdot b
\]
is an algebra homomorphism satisfying with \( f_1 \) the hypotheses of Prop. 5.3. This leads to a mapping \( F : A^c \bowtie H \to A \odot H \odot A \) given by
\[
a \otimes \overline{b} \bowtie h \mapsto a \otimes h \cdot h(1) \odot h(2) \cdot b,
\]
which is the isomorphism in [24 2.4].

Comparing the two isomorphic Hopf algebroids (see [21] for details) we note that the antipode in \( A^c \bowtie H \) is given by a simpler formula, while the \( A \)-\( A \)-bimodule structure in \( A \odot H \odot A \) is simpler. The multiplication in \( A^c \bowtie H \) is closer to the smash product of a Hopf algebra with a bimodule algebra, which is the method of proof in [21].

It should also be noted that [21 3.1, 3.2] provides an equivalent condition to that in proposition 5.3 which shows \( A^c \bowtie H \) is a certain universal bialgebroid.

Let us emphasize the picture of universals for bialgebroids over a fixed base ring \( A \). As observed in [14], for any (finite projective) algebra \( A \) there is a homomorphism of bialgebroids \( A^c \to \text{End } A \), where \( x \otimes \gamma \mapsto \lambda_x \circ \rho_\gamma \), since \( \text{End } A \) is a terminal object in a category of \( A \)-bialgebroids (existence in [14 Prop. 3.7], uniqueness: an easy argument). For similar reasons, \( A^c \) is an initial object in this category. For \( A \) a left \( H \)-module algebra, this homomorphism factors through the bialgebroids \( A^c \bowtie H \), \( A \odot H \odot A \), or any bialgebroid over \( A \) as follows.

Let \( S \) be a bialgebroid over \( A \) with source, target mappings \( s_L, t_L : A \to S \) and counit \( \varepsilon : S \to A \). In addition to Lu's mapping above, define bialgebroid arrows \( A^c \to S \), \( a \otimes \overline{b} \mapsto s_L(a)t_L(b) \) and the Xu anchor mapping \( S \to \text{End } A \) given by \( x \mapsto \varepsilon(\overline{s_L(x)}) \). P. Xu's anchor map [29] corresponds to the action of \( S \) on \( A \) via source and counit [14 3.7], for which \( A \) becomes the unit module in the tensor category of \( S \)-modules.

**Proposition 4.1.** The natural arrows defined above form a commutative triangle of bialgebroid homomorphisms.
Proof. This follows readily from bialgebroid identities such as $\varepsilon \circ s_L = \text{id}_A = \varepsilon \circ t_L$. 

The anchor mapping $A^e * H \to \text{End} A$ is given by $(a, b \in A, h \in H)$
\begin{align}
(38) \quad a \otimes b \triangleright h & \mapsto \lambda_a \circ \lambda_{h^0} \circ \rho_b
\end{align}
where $\lambda_{h^0}$ denotes the endomorphism given by left action by $h$, $x \mapsto h \triangleright x$. (Note that this is an algebra homomorphism since for $a, b, c, d \in A, h, k \in H$,
\begin{align}
\lambda_a \circ \lambda_{h^0} \circ \rho_b \circ \lambda_c \circ \lambda_{k^0} \circ \rho_d = \lambda_{a(h_{(1)}\triangleright c)} \circ \lambda_{k_{(1)}\triangleright h_{(2)}^\dagger} \circ \rho_d(S(k_{(2)}\triangleright b)),
\end{align}
which is the eq. (37) up to a simple re-writing.)

Xu’s anchor mapping for the Connes-Moscovici bialgebroid $A \otimes H \otimes A$ is the mapping $A \otimes H \otimes A \to \text{End} A$ given by sending $a \otimes h \otimes b$ into the endomorphism $\varepsilon((a \otimes h \otimes b)(x \otimes 1_H \otimes 1_A)) = \varepsilon(a(h_{(1)}\triangleright x) \otimes h_{(2)} \triangleright b) = a(h_{(1)}\triangleright x)\varepsilon(h_{(2)}b) = \lambda_a \circ \rho_b \circ \lambda_{h^0}(x)$. Note that in $\text{End} A$ we have
\begin{align}
(39) \quad \lambda_a \circ \lambda_{h^0} \circ \rho_b = \lambda_a \circ \rho_{h_{(2)}\triangleright b} \circ \lambda_{h_{(1)}^\dagger},
\end{align}
which lifts to the isomorphism $A^e * H \to A \otimes H \otimes A$.

We propose a generalization of pseudo-Galois extension to pseudo-Hopf-Galois extension as follows. Let $H$ be a finite dimensional (or finite projective) Hopf algebra acting from the left on an $H$-module algebra $A$, and $B$ be a subalgebra contained in the subalgebra of invariants $A^H = \{b \in A : \forall h \in H, h \triangleright b = \varepsilon(h)b\}$. With $R := C_A(B)$ denoting the centralizer as usual, note that $H$ restricts to an action on $R$. To be a pseudo-Hopf-Galois extension, we require the algebra extension $A \mid B$ to be D2, and we require the bialgebroid homomorphism
\begin{align}
(40) \quad R^e \bowtie H \to \text{End}_B AB, \quad r \otimes \Phi \bowtie h \mapsto \lambda_r \circ \lambda_{h^0} \circ \rho_s
\end{align}
to be surjective. For example, if $A \mid B$ is a Hopf-Galois extension (technically, right $H^*$-Galois), it is pseudo-Hopf-Galois since it is D2 and by $[12]$
\begin{align}
(41) \quad \Psi : R \times H \to \text{End}_B AB, \quad \Psi(r \# h) := \lambda_r \circ \lambda_{h^0}.
\end{align}
In addition, if $A \mid B$ is H-separable, it is pseudo-Hopf-Galois since it is D2 $[12]$ and $R^e \cong \text{End}_B AB$ via $r \otimes \Phi \mapsto \lambda_r \circ \rho_s$. These are the two examples we wish to generalize at once.

The following is a third class of example of a pseudo-Hopf-Galois extension. Let $A \mid B$ have a split injective Galois mapping $\beta : A \otimes_B A \to A \otimes H^*$ as $A-B$-bimodules and let its trace function $A \to B$ be (a non-surjective) Frobenius homomorphism $[20]$ chs. 4, 8]. Then $A \mid B$ is D2 and the mapping in eq. (41) is a split epimorphism via the commutative square below.
\[
A \otimes_B A \xrightarrow{\beta} A \otimes H^* \\
\cong \quad \cong \\
\text{End}(A_B) \xleftarrow{\Psi} A\#H
\]

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