Phase transition for accessibility percolation on hypercubes

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Abstract

In this paper, we consider accessibility percolation on hypercubes, i.e., we place i.i.d. uniform random variables on vertices of a hypercube, and study whether there is a path (possibly with back steps) connecting two vertices such that the values of these random variables increase along the path. We establish a sharp phase transition depending on the difference of the values at the two endpoints, and determine the critical window of the phase transition. Our result completely resolves a conjecture of Berestycki, Brunet, and Shi (2014).

Our work on accessibility percolation is motivated by the NK fitness model in biological evolution. Indeed, placing i.i.d. random variables on the hypercube amounts to the special case for NK fitness model for $N = K$. A second result of our work concerns the global maximum for NK fitness model, and proves that in the case when fitness variables are Gaussian it is asymptotically equivalent to the maximum of i.i.d. variables if and only if $K \to \infty$ as $N \to \infty$.

1 Introduction

For $N \in \mathbb{N}$, let $H_N = \{0, 1\}^N$ be a hypercube where two vertices are neighboring each other if their Hamming distance is precisely 1. Let $\{X_v : v \in H_N\}$ be i.i.d. random variables uniformly in $[0, 1]$. For $u, w \in H_N$, we say that $w$ is accessible from $u$ if there exists a path in $H_N$ started at $u$ and ended at $w$ such that the associated random variables $(X_v)$ are increasing along the path. For a typical choice of $u$ and $w$, the accessible probability is uniformly bounded away from 0 and 1, due to the fluctuation of $X_u$ and $X_w$. In this paper, however, we show that the conditional accessible probability given that $X_u = a$ and $X_w = b$ ($0 \leq a < b \leq 1$) admits a sharp phase transition. By symmetry, the conditional accessible probability with fixed $a$ and $b$ depends only on the Hamming distance between $u$ and $v$. Therefore, we fix $0 < \beta \leq 1$ and without loss of generality consider the case when $u = (0, 0, \ldots, 0)$, $w = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$ (here the number of 1’s in $w$ is $\lfloor \beta N \rfloor$). Furthermore, since subtracting $a$ from all $X_v$’s does not change the accessibility between any pairs, we can assume without loss of generality that $a = 0$ and $x = b - a$. Our main result is summarized in the following theorem.

**Theorem 1.1.** Let $f(x) = (\sinh x)^\beta (\cosh x)^{1-\beta}$, and let $x_0$ be a number such that $f(x_0) = 1$. Define $x_c(N) = x_0 - \frac{1}{f'(x_0)} \frac{\ln N}{N}$. For a sequence $\varepsilon_N$ with $N \varepsilon_N \to \infty$, we have

$$
\lim_{N \to \infty} \mathbb{P}(w \text{ is accessible from } u \mid X_u = 0, X_w = x_c - \varepsilon_N) = 0, \quad (1)
$$

$$
\lim_{N \to \infty} \mathbb{P}(w \text{ is accessible from } u \mid X_u = 0, X_w = x_c + \varepsilon_N) = 1. \quad (2)
$$

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In addition, for all $\Delta > 0$, there exists $0 < c_1 < c_2 < 1$ (where $c_1, c_2$ depends only on $\Delta$) such that for all $N \in \mathbb{N}$

$$c_1 \leq P(\text{w is accessible from } u \mid X_w = 0, X_u = x_c + \varepsilon_N) \leq c_2, \text{ if } |N\varepsilon_N| \leq \Delta.$$  \hspace{1cm} (3)

**Remark.** A few days before the post of this article, we noted that a paper [15] was posted in January 2015, which proved the version of (2) (without analyzing the critical window for the phase transition) for the case of $\beta \geq 0$.002. While we acknowledge the priority of [15], we emphasize that our work was carried out independently; our method is rather different and allows to derive the result for all $0 < \beta \leq 1$.

The accessibility percolation was studied by [1], where they proved (1) and conjectured (2) (both in a slightly weaker form). Our Theorem 1.1 completes the picture and describes a sharp phase transition for the accessibility percolation on hypercubes.

An analogue of Theorem 1.1 on accessibility percolation for hypercubes without back steps (i.e., one is restricted to paths of minimal length between the starting and ending vertices) was established by [9], which improves previous results [2, 19] on the same question. The accessibility percolation has also been studied on regular trees [2, 16, 18, 5] and on spherically symmetric trees [6]. In addition, the Hamiltonian increasing path on complete graph was studied in [13].

Our study on accessibility percolation is motivated by the NK-fitness landscapes, which was introduced in [11, 12] as a class of models for biological evolution. In the NK fitness model, we consider $H_N$ corresponding to, e.g., nucleobases in a DNA sequence. Let $F$ be a distribution. Given $K \leq N$, let $Y_{i, \tau}$ be i.i.d. random variables with distribution $F$ for all $1 \leq i \leq N$ and $\tau \in H_K$. For $\sigma \in H_N$, the fitness of $\sigma$ is then defined to be $X_\sigma = \sum_{i=1}^N Y_{i,(\sigma_i, \ldots, \sigma_{i+K-1})}$ (where the addition in the subscript is understood as modulo of $N$). Since the gene favors better fitness, it is natural to consider an adaptive walk on space $H_N$ such that the corresponding fitness increases until the walk is frozen at a local maximum. Theorem 1.1 is a preliminary step toward understanding the adaptive walk on NK fitness model. Indeed, our model (with i.i.d. fitness for each vertex in $H_N$) corresponds to the case when $K = N$ (the distribution $F$ does not play a role when considering increasing paths as long as $F$ is continuous).

We note that despite intensive research in theoretical biology as well as physics, there were few mathematical results [8, 7, 14, 4] on NK fitness models. In [8], some asymptotic features of NK fitness landscapes are reduced to questions about eigenvalues and Lyapunov exponents; in [7, 14], estimates on the cardinality of local maxima was provided; in [4], certain structural properties of the maxima for NK fitness model was given. In this paper, we establish the asymptotics for the maximum of NK fitness model, by proving that the maximum is asymptotically equivalent to i.i.d. Gaussian variables if and only if $K \to \infty$. While it is plausible that our proof method can extend to more general distributions, we present the proof only for the Gaussian case.

**Theorem 1.2.** Let $M_N := \max_{\sigma \in H_N} X_\sigma$ be the maximum of NK fitness model, where $F$ is a standard Gaussian distribution.

(a) If $K \to \infty$ as $N \to \infty$ then for any $\varepsilon > 0$, we have

$$P((1 - \varepsilon)\sqrt{2\ln 2} \leq \frac{M_N}{N} \leq (1 + \varepsilon)\sqrt{2\ln 2}) \geq 1 - e^{-o(\varepsilon)N}.$$  

In particular $\lim_{N \to \infty} \frac{E M_N}{N} = \sqrt{2\ln 2}$.

(b) If $K \leq K_0$ for all $N$, there exists $\varepsilon = \varepsilon(K_0)$ such that $\lim_{N \to \infty} P(M_N \geq (1 - \varepsilon)\sqrt{2\ln 2}N) = 0$. 

2
2 Accessibility percolation: antipodal case

In the current section as well as the next one, the probability measure always stands for the conditional probability given $X(u) = 0$ and $X(w) = x$. Recall that a path from $u$ to $w$ is said to be accessible (or open) if the $X(v)$’s (including $X(u)$ and $X(w)$) along the path are monotone increasing. Denote by $Z_{N,x}$ the number of such accessible paths. For clarity of presentation, in this current section we give a proof for the antipodal case (i.e., $u = \vec{0} = (0, 0, \cdots, 0)$, $X(u) = X(\vec{0}) = 0$ and $w = \vec{1} = (1, 1, \cdots, 1)$), $X(w) = X(\vec{1}) = x$. In Section 3 we modify the arguments and give a proof in general case. Throughout the paper, we write with high probability for with probability tending to 1 as $N \to \infty$.

2.1 Proof of the upper bound

This subsection is devoted to the proof of (1). Note that (1) was proved in [1] in the case when $\epsilon_N$ is a fixed positive number. Our proof relates the original model to a more tractable one and such connection will also be useful in later proofs. We start with a number of definitions.

Definition 2.1. We say that a path has length $\ell$ if it passes through $(\ell - 1)$ inner vertices (excluding the beginning and ending points). For $n, \ell \in \mathbb{N}$, let $\mathcal{M}(n, \ell)$ be the collection of paths (not necessarily self avoiding) of length $\ell$ that goes from $\vec{0}_N = (0, 0, \cdots, 0)$ to $(\vec{1}_n, \vec{0}_{N-n}) = (1, \cdots, 1, 0, \cdots, 0)$ (where there are $n$ 1’s in $(\vec{1}_n, \vec{0}_{N-n})$). Write $M(n, \ell) = |\mathcal{M}(n, \ell)|$.

Definition 2.2. For $n, \ell \in \mathbb{N}$, let $\mathcal{S}(n, \ell)$ be the collection of sequences $(a_1, \ldots, a_\ell) \in \{1, \ldots, N\}^\ell$ such that $|\{1 \leq i \leq \ell : a_i = k\}|$ is odd for $1 \leq k \leq n$ and even for $n+1 \leq k \leq N$.

For each path $v_0, v_1, \ldots, v_\ell$ on $H_N$ of length $\ell$, we associate a sequence of integers $a_1, \ldots, a_\ell$ where $a_i$ is the coordinate at which $v_{i-1}$ and $v_i$ differ. We observe that the association is a bijection between $\mathcal{M}(n, \ell)$ and $\mathcal{S}(n, \ell)$.

For $1 \leq k \leq N$, let $\mathcal{S}_k(n, \ell) \subseteq \mathcal{S}(n, \ell)$ containing all sequences such that the last number is $k$ and let $\mathcal{S}_k(n) = \bigcup_{\ell \in \mathbb{N}} \mathcal{S}_k(n, \ell)$. Let $F_1$ be a distribution supported on odd integers such that $F_1(2j + 1) = \frac{x^{2j+1}}{(2j+1)!}\sinh x$, and let $F_2$ be a distribution supported on even integers such that $F_2(2j) = \frac{x^{2j}}{(2j)!}\cosh x$ for all $j \geq 0$. Let $U_i$ be i.i.d. random variables distributed as $F_1$ for $i \in \{1, \ldots, n\} \setminus \{k\}$ and $U_k$ be i.i.d. random variables distributed as $F_2$ for $i \in \{n+1, \ldots, N\} \setminus k$, and let $U_k$ be another independent random variable with distribution $F_2$ if $1 \leq k \leq n$ and with distribution $F_1$ if $n+1 \leq k \leq N$. Given the values of $U_1, \ldots, U_N$, we let $(A_1, \ldots, A_{\ell-1}, k)$ (where $L = \sum_{i=1}^N U_i$) be a sequence uniformly at random subject to $|\{1 \leq j \leq \ell - 1 : A_j = i\}| = U_i$. We denote by $\mu_{k,n}$ the probability measure of the random sequence $A_1, \ldots, A_U, k$.

Lemma 2.3. For any sequence $(a_1, \ldots, a_{\ell-1}, k) \in \mathcal{S}_k(n, \ell)$ and $1 \leq k \leq n \leq \ell$, we have

\[
\mu_{k,n}((a_1, \ldots, a_{\ell-1}, k)) = \frac{x^{\ell-1}}{(\ell-1)!} \frac{1}{(\sinh x)^{n-1}} \frac{1}{(\cosh x)^{\ell-n+1}}.
\]

Similarly, when $n+1 \leq k \leq N$ and $\ell \geq n+2$, for any sequence $(a_1, \ldots, a_{\ell-1}, k) \in \mathcal{S}_k(n, \ell)$, we have

\[
\mu_{k,n}((a_1, \ldots, a_{\ell-1}, k)) = \frac{x^{\ell-1}}{(\ell-1)!} \frac{1}{(\sinh x)^{n-1}} \frac{1}{(\cosh x)^{\ell-n+1}}.
\]

Proof. We only prove the first case. Let $n_i = \{1 \leq j \leq \ell - 1 : a_j = i\}$. Then, we have

\[
\mu_{k,n}((a_1, \ldots, a_{\ell-1}, k)) = \mathbb{P}(U_i = n_i \text{ for all } 1 \leq i \leq N) \cdot \prod_{i=1}^{N-1} \frac{n_i!}{(\ell-1)!},
\]

(4)
where the second term counts the conditional probability of sampling \((a_1, \ldots, a_{\ell-1}, k)\) given \(U_i = n_i\) for \(1 \leq i \leq N\). By independence, we see that

\[
\mathbb{P}(U_i = n_i \text{ for all } 1 \leq i \leq N) = \prod_{i=1}^{N} \mathbb{P}(U_i = n_i) = \prod_{1 \leq i \neq k \leq n} F_1(n_i) \cdot F_2(n_k) \cdot \prod_{n+1 \leq i \leq N} F_2(n_i)
\]

\[
= \prod_{1 \leq i \neq k \leq n} \frac{n_j! \sinh x}{n_j! \sinh x} \cdot \frac{x^{n_k}}{n_k! \cosh x} \cdot \prod_{n+1 \leq i \leq N} \frac{1}{n_k! \cosh x}
\]

\[
= x^{\ell-1} \frac{1}{\prod_{i=1}^{n} (\sinh x)^{n_i} \cdot (\cosh x)^{N-n+1}}.
\]

Combined with (4), it completes the proof of the lemma.

\[\square\]

**Lemma 2.4.** We have

\[
\sum_{\ell=0}^{\infty} M(n, \ell) \frac{x^\ell}{\ell!} = (\sinh x)^n (\cosh x)^{N-n}.
\]

In addition, we have

\[
\sum_{\ell=0}^{\infty} M(n, \ell) \frac{x^{\ell-1}}{(\ell-1)!} = ((\sinh x)^n (\cosh x)^{N-n})',
\]

\[
= (\sinh x)^{n-1} (\cosh x)^{N-n-1} (n (\cosh x)^2 + (N-n)(\sinh x)^2).
\]

**Proof.** We give a proof for the second equality. The first equality can be proved in the same manner, and thus we omit the details.

Write \(\mathcal{M}_k(n, \ell) \subseteq \mathcal{M}(n, \ell)\) be the paths that are associated to \(S_k(n, \ell)\). Since \(\mu_{k,n}\) is a probability measure on \(S_k(n)\), we see that \(\sum_{\bar{a} \in S_k(n)} \mu_{k,n}(\bar{a}) = 1\). Combined with Lemma 2.3 it yields that when \(1 \leq k \leq n\)

\[
\sum_{\ell=n}^{\infty} \sum_{\bar{a} \in S_k(n, \ell)} \mu_{k,n}(\bar{a}) = \sum_{\ell=n}^{\infty} |S_k(n, \ell)| \frac{x^{\ell-1}}{(\ell-1)!} \frac{1}{(\sinh x)^{n_i}} \frac{1}{(\cosh x)^{N-n+1}} = 1.
\]

and when \(n+1 \leq k \leq N\)

\[
\sum_{\ell=n+2}^{\infty} \sum_{\bar{a} \in S_k(n, \ell)} \mu_{k,n}(\bar{a}) = \sum_{\ell=n+2}^{\infty} |S_k(n, \ell)| \frac{x^{\ell-1}}{(\ell-1)!} \frac{1}{(\sinh x)^{n_i}} \frac{1}{(\cosh x)^{N-n+1}} = 1.
\]

By the fact that \(M_k(n, \ell) = |S_k(n, \ell)|\) for \(1 \leq k \leq n, \ell \geq n\) and \(n+1 \leq k \leq N, \ell \geq n+2\) and \(M_k(n, \ell) = 0\) otherwise, we get that when \(1 \leq k \leq n\)

\[
\sum_{\ell=n}^{\infty} M_k(n, \ell) \frac{x^{\ell-1}}{(\ell-1)!} = (\sinh x)^{n-1} (\cosh x)^{N-n+1},
\]

and when \(n+1 \leq k \leq N\)

\[
\sum_{\ell=n+2}^{\infty} M_k(n, \ell) \frac{x^{\ell-1}}{(\ell-1)!} = (\sinh x)^{n+1} (\cosh x)^{N-n-1}.
\]

Combined with the fact that \(M(n, \ell) = \sum_{1 \leq k \leq N} M_k(n, \ell)\), it completes the proof of the lemma. \([\square]\)
Corollary 2.5. $\mathbb{E}Z_{N,x} \leq N(\sinh x)^{N-1} \cosh x$.

Proof. We will derive an upper bound for $\mathbb{E}Z_{N,x}$ in the general (not necessarily antipodal) case. Denote by $n$ the Hamming distance of $X(u)$ and $X(w)$. Let $M'(n, \ell)$ be the subset of self-avoiding paths in $M(n, \ell)$. Write $M'(n, \ell) = |M'(n, \ell)|$. Note that for each $P \in M'(n, \ell)$, the probability that $P$ is accessible is $\frac{x^{\ell-1}}{(\ell-1)!}$. Therefore,

$$
\mathbb{E}Z_{N,x} = \mathbb{E} \sum_{\ell=0}^{\infty} \sum_{P \in M'(n, \ell)} 1_{P \text{ is accessible}} = \sum_{\ell=0}^{\infty} M'(n, \ell) \frac{x^{\ell-1}}{(\ell-1)!} \leq \sum_{\ell=0}^{\infty} M(n, \ell) \frac{x^{\ell-1}}{(\ell-1)!}
$$

$$
= (\sinh x)^{n-1} (\cosh x)^{N-n-1} [(n(\cosh x)^2 + (N-n)(\sinh x)^2),
$$

where the last inequality follows from Lemma 2.4. In the antipodal case, substituting $n = N$ in the preceding inequality gives the desired bound. \hfill \blacksquare

Proof of (1): antipodal case. In this case, $\beta = 1$ so we have $x_0 = \sinh^{-1}(1) = \ln(\sqrt{2} + 1)$, $\sinh x_0 = 1$ and $\cosh x_0 = \sqrt{2}$. We can without loss of generality assume that $\epsilon_N \leq N^{-2/3}$ since $\mathbb{P}(Z_{N,x} > 0)$ is increasing in $x$. By Corollary 2.5, we get that

$$
\mathbb{P}(Z_{N,x_0-\epsilon_N} > 0) \leq \mathbb{E}Z_{N,x_0-\epsilon_N} \leq N \sinh^{N-1}(x_0 - \epsilon_N) \cosh(x_0 - \epsilon_N)
$$

$$
= N(\sinh(x_0) - \cosh(x_0)(\frac{\sqrt{2} \ln N}{N} + o(1/N))^{N-1} \cosh(x_0 - \epsilon_N)
$$

$$
\leq N(1 - \frac{\ln N}{N} - \sqrt{2}\epsilon_N + o(1/N))^{N-1} \sqrt{2} \rightarrow 0 \text{ as } N \rightarrow \infty.
$$

\hfill \blacksquare

2.2 Proof of the lower bound

In order to prove the lower bound, we restrict to certain good paths, i.e., those with desirable properties on the growth of Hamming distances (in particular, a good path needs to be self-avoiding). Denote by $Z_{N,x,*}$ the number of good accessible path. Crucially, we demonstrate a choice of “good” path such that $\mathbb{E}Z_{N,x,*} \asymp \mathbb{E}Z_{N,x}$ and $\mathbb{E}Z_{N,x,*}^2 \asymp (\mathbb{E}Z_{N,x,*})^2$ (where $\asymp$ means that the left and right hand sides are within a constant multiplicative factor). Thus, an application of second moment method yields the existence of an accessible path with probability bounded away from 0. Finally, we use the augmenting method as employed in [9] to deduce the existence of accessible path with probability tending to 1 as $N \rightarrow \infty$.

Recall that $x_0 = \sinh^{-1}(1) = \ln(\sqrt{2} + 1) \approx 0.88137$. Let $\alpha = x_0 \coth x_0 \approx 1.24645$.

For any $\epsilon > 0$, we set $\epsilon_1, \epsilon_2, \epsilon_3$ throughout the rest of the paper as

$$
\epsilon_1 = \epsilon^{1/2}, \epsilon_2 = \epsilon^{1/4}, \text{ and } \epsilon_3 = \epsilon^{1/8}.
$$

(8)

For convenience, we also assume $N \geq 10^4$. For $u, v \in H_N$, we denote by $H(u, v)$ the Hamming distance between $u$ and $v$, i.e., the number of coordinates at which $u$ differ from $v$.

Definition 2.6. Let $\epsilon > 0$ be a sufficiently small fixed number to be selected. We say a path (or the associated sequence) $v_0, \ldots, v_L$ is good if $L \in [\alpha(1-\epsilon)N, \alpha(1+\epsilon)N]$, and the following holds:

$H(v_i, v_j) = |i - j|$, if $|i - j| = 1, 2, 3$;

$H(v_i, v_j) = |i - j|$ or $|i - j| - 2$, if $4 \leq |i - j| \leq N^\frac{1}{4}$;

$$
\frac{|i - j|}{\alpha + \epsilon_3} \leq H(v_i, v_j) \leq (1/2 + \epsilon_1)N, \text{ if } N^\frac{1}{4} \leq |i - j| \leq \alpha(1/2 + \epsilon)N;
$$
\( H(v_i, v_j) \geq \frac{|i-j|}{n+1} \), if \( \alpha(1/2 + \varepsilon)N < |i-j| \leq \alpha(1/2 + \varepsilon_2)N \);\n\( H(v_i, v_j) \geq (1/2 + \varepsilon_1)N \), if \( |i-j| > \alpha(1/2 + \varepsilon_2)N \).

It is clear from the definition that a good path is self-avoiding.

**Lemma 2.7.** There exists an \( \epsilon > 0 \) such that for all \( |x-x_0| \leq \epsilon \) and for all fixed \( \varepsilon \in (0, \epsilon) \) we have \( \mathbb{E} \mathcal{Z}_{N,x,*} \geq C_1 N \sinh N^{-1} x \cosh x \), where \( C_1 > 0 \) depends only on \( \varepsilon \).

**Proof.** In light of the bijection between the path and its associated sequence, we first consider good associated sequences. Let \( \mathcal{S}_{k,*}(n) \subseteq \mathcal{S}_k(n) \) containing all the good sequences. By Lemmas 2.3 and 2.4 it suffices to show that for each \( 1 \leq k \leq N \)

\[ \mu_{k,n}(\mathcal{S}_{k,*}(n)) \geq C_1 \text{ for an absolute constant } C_1 > 0. \] (9)

For ease of elaboration we make a slight modification to 2.3, that is, we show instead that \( \tilde{\mu}_n - 1, N - 1(\mathcal{S}_n(n - 1, N - 1)) \geq C_1 \) for \( 1 \leq k \leq n = \beta N \), or \( \tilde{\mu}_n, N - 1(\mathcal{S}_n(n - 1)) \geq C_1 \) for \( n + 1 \leq k \leq N \).

For ease of elaboration we make a slight modification to \( \mu_{k,n} \) that we also let coordinate \( k \) choose \( U_k \) according to \( F_1 \) (instead of \( F_2 \)), and delete the last update which is now constrained to be \( k \). In other words, for each \( 1 \leq i \leq N \), let \( U_i \) be i.i.d. random variables distributed as \( F_1 \). We denote by \( \tilde{\mu}_N \) the modified probability measure. Given the values of \( U_1, \ldots, U_N \), let \( (A_1, \ldots, A_L) \) (where \( L = \sum_{i=1}^N U_i \)) be a sequence uniformly at random subject to \(|\{1 \leq j \leq L : A_j = i\}| = U_i \). In the following \( \mathbb{P} \) and \( \mathbb{E} \) are with respect to this probability space of update sequence \( A_1, \ldots, A_L \) (according to measure \( \tilde{\mu}_N \)).

We argue that it suffices to prove (9) for \( \tilde{\mu}_N \). There are a number of ways to see this. For example, one may argue that if \( \tilde{\mu}_N(\mathcal{S}_n(N)) \geq C_1 \) holds, then \( \mu_{k,N}(\mathcal{S}_{k,*}(N)) \geq \frac{1}{\cosh z} C_1 \) holds, since any update sequence \( (A_1, \ldots, A_{L-1}, k) \) in \( \mathcal{S}_k(N) \) that satisfies \( U_k = 0 \) (note that \( \tilde{U}(k) \sim F_2 \) under \( \mu_{k,N} \)) has measure equal to \( \frac{1}{\cosh z} \tilde{\mu}_{N-1}(A_1, \ldots, A_{L-1} \cdot k) \), and if \( (A_1, \ldots, A_{L-1}) \) is a “good” update sequence under the space of \( \tilde{\mu}_{N-1} \) (i.e. \( \mathcal{S}(N-1) \)), then so is \( (A_1, \ldots, A_{L-1}, k) \) under the space of \( \mu_{k,N} \) (i.e. \( \mathcal{S}_k(N) \)).

It is immediate from concentration of sum of i.i.d. random variables that with probability tending to 1 as \( N \to \infty \) we have \( \alpha N - N^{2/3} \leq L \leq \alpha N + N^{2/3} \) (recall that \( \mathbb{E} U_{F_1} = x \coth x \)). It remains to consider the profile on Hamming distances, for which we split into three cases as follows. **Case 1:** \( |i-j| = 1, 2, 3 \). We prove this for general \( \beta \). We continue to denote by \( (A_1, \ldots, A_L) \) (where \( A_L = k \)) the random sequence and by \( U_i \) the number of occurrences of \( i \) in the sequence. For \( j \in \mathbb{N} \), let

\[ D_j = |\{1 \leq i \leq N : U_i = j\}|. \]

In addition, define

\[ I_i = \begin{cases} 1_{\{A_i = A_{i+1}\}} & \text{if } i = 1, 2, \ldots, L; \\ 1_{\{A_{i-L} = A_{i+2-L}\}} & \text{if } i = L + 1, L + 2, \ldots, 2L. \end{cases} \]

It is clear that with probability tending to 1 as \( N \to 1 \), we have

\[ \Lambda = (L - 1)^{-1} \sum_{i=2}^\infty D_i(i - 1) = (1 + o(1)) \frac{x^2}{\gamma} \text{ and } \max_{1 \leq i \leq N} U_i \leq 10 \log N. \] (10)
We first bound the term \( J \) where \( \sum_{i=1}^{2L} I_i = 0 \). We can write
\[
\sum_{i=1}^{2L} I_i = 0 \geq c^* - o(1).
\]
To this end, we write \( F = \sigma(U_1, U_2, U_3, \ldots) \). Clearly, there exists a finite odd number \( K \) such that
\[
\sum_{k=1}^{K} (-1)^{k+1} \left( \frac{2 \cdot x^2}{k!} \right)^k < 1.
\]
By Bonferroni inequality, we have
\[
\mathbb{P} \left( \sum_{i=1}^{2L} I_i \geq 1 \mid F \right) \leq \sum_{k=1}^{K} (-1)^{k+1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2L} \mathbb{P}(I_{i_1} = 1, I_{i_2} = 1, \ldots, I_{i_k} = 1 \mid F),
\]
where \( K \) is an odd number. In order to prove (11), it suffices to show that the left hand side of (12) is asymptotic to the right hand side of (13). For this purpose, we will split it into two parts according to whether or not any \( A_i \) is involved in more than one \( I_{i_j} \)'s (1 \( \leq j \leq k \)). More precisely we say a pair of integers \((i_j, i_{j'})\) (or equivalently \((I_{i_j}, I_{i_{j'}})\)) is intersecting if some \( A_i \) is involved in the definition of both \( I_{i_j} \) and \( I_{i_{j'}} \) where \( 1 \leq i \leq L \) and \( 1 \leq j \neq j' \leq k \). Let \( \mathcal{I}^{k,1}(\mathcal{I}^{k,2}) \) denote the set of all sequences \((i_1, i_2, \ldots, i_k)\) such that \( 1 \leq i_1 < i_2 < \cdots < i_k \leq 2L \) and it contains no (at least 1) intersecting pair, respectively. We can write
\[
\mathbb{P}(I_{i_1} = 1, I_{i_2} = 1, \ldots, I_{i_k} = 1 \mid F) = \mathcal{J}_1 + \mathcal{J}_2,
\]
where
\[
\mathcal{J}_1 = \sum_{\mathcal{I}^{k,1}} \mathbb{P}(I_{i_1} = 1, I_{i_2} = 1, \ldots, I_{i_k} = 1 \mid F), \quad \text{and} \quad \mathcal{J}_2 = \sum_{\mathcal{I}^{k,2}} \mathbb{P}(I_{i_1} = 1, I_{i_2} = 1, \ldots, I_{i_k} = 1 \mid F).
\]
We first bound the term \( \mathcal{J}_1 \). No that when there is no intersecting pair in \( I_{i_1}, I_{i_2}, \ldots, I_{i_k} \), we have
\[
\mathbb{P}(I_{i_1} = 1, I_{i_2} = 1, \ldots, I_{i_k} = 1 \mid F) \leq \frac{(L - 2k)!}{L!} \left( \sum_{i=2}^{\infty} D_i \cdot i \cdot (i - 1) \right)^k \\
\leq \left( \frac{1}{L} \right)^k (1 + o(1))((1 + o(1)) \frac{2 \cdot x^2}{\gamma})^k.
\]
Combined with the simple fact that \( |\mathcal{I}^{k,1}| \leq (2L)^k/k! \), it gives that \( \mathcal{J}_1 \leq (2x^2/\gamma)^k(1 + o(1))/k! \). In addition,
\[
\mathbb{P}(I_{i_1} = 1, I_{i_2} = 1, \ldots, I_{i_k} = 1 \mid F) \geq \frac{(L - 2k)!}{L!} \prod_{0 \leq j < k - 1} \left( \sum_{i=2}^{10 \log N} (D_i - j) \cdot i \cdot (i - 1) \right) \\
\geq \left( \frac{1}{L} \right)^k (1 + o(1))((1 + o(1)) \frac{2 \cdot x^2}{\gamma})^k.
\]
Note that \( |\mathcal{I}^{k,1}| \geq (1 + o(1))(2L)^k/k! \) since \( |\mathcal{I}^{k,1}| \geq \prod_{0 \leq j < k - 1}(2L - 100j)/k! \). Hence, we obtain that \( \mathcal{J}_1 \geq (2x^2/\gamma)^k(1 + o(1))/k! \). Altogether, we get
\[
\mathcal{J}_1 = (2x^2/\gamma)^k(1 + o(1))/k!.
\]
It remains to control $J_2$. Denote by $E_{i_1,\ldots,i_k} = \{I_{i_1} = 1, I_{i_2} = 1, \ldots, I_{i_k} = 1\}$. Note that $E_{i_1,\ldots,i_k}$ can be rewritten (or simplified) uniquely as a set of equalities

$$
A_{j_1} = A_{j_1+n_1,1} = A_{j_1+n_1,1+n_1,2} = \cdots = A_{j_1+n_1,1+n_1,2+\cdots+n_1,\alpha_1-1},
$$

$$
A_{j_2} = A_{j_2+n_2,1} = A_{j_2+n_2,1+n_2,2} = \cdots = A_{j_2+n_2,1+n_2,2+\cdots+n_2,\alpha_2-1},
$$

$$
\cdots
$$

$$
A_{j_\ell} = A_{j_\ell+n_\ell,1} = A_{j_\ell+n_\ell,1+n_\ell,2} = \cdots = A_{j_\ell+n_\ell,1+n_\ell,2+\cdots+n_\ell,\alpha_\ell-1},
$$

where $n_1,1,\ldots,n_1,\alpha_1-1, n_2,1,\ldots,n_2,\alpha_2-1, \ldots, n_\ell,1,\ldots,n_\ell,\alpha_\ell-1$ are either 1 or 2. In addition, here $a_1, a_2, \ldots, a_\ell$ are integers $\geq 2$ and $a_1 + a_2 + \cdots + a_\ell \leq 2k$ (in particular each $a_i$ is $\leq 2k$). Note that if there is any intersecting pair in $I_{i_1}, \ldots, I_{i_k}$, at least one of the $a_1, a_2, \ldots, a_\ell$ must be strictly larger than 2. Denote by $A$ the preceding set of equalities (so $A$ can also be viewed as an event), and we note that $|\{(i_1, \ldots, i_k) : E_{i_1,\ldots,i_k} = A\}| \leq (a_1 + a_2 + \cdots + a_\ell)^2 \leq (2k)^2$. Therefore we have

$$
\sum_{T^{k,2}} \mathbb{P}(E_{i_1,\ldots,i_k} \mid F) \leq (2k)^2 \mathbb{E} \left[ \sum_{D_1} \sum_{D_2} \sum_{D_3} \mathbb{P}(A \mid F) \right],
$$

(16)

where $D_1, D_2, D_3$ respectively denote the collections of all valid choices of $(a_1, a_2, \ldots, a_\ell)$, $(n_1,1,\ldots,n_1,\alpha_1-1)$ and $(j_1, j_2, \ldots, j_\ell)$. We can compute that

$$
\mathbb{P}(A \mid F) \leq \frac{(L - (a_1 + a_2 + \cdots + a_\ell))!}{L!} \prod_{j=1}^\ell \left( \sum_{i=0}^\infty D_i \cdot i \cdot (i - 1) \cdots (i - a_j + 1) \right).
$$

By concentration of sums of i.i.d. variables, it is clear that there exists $C_K$ depending only on $K$ such that with probability tending to 1 as $N \to \infty$ we have

$$
\sum_{i=1}^N U_i^{2k} \leq C_K N, \text{ for all } 1 \leq k \leq K.
$$

(17)

Thus, we can assume without loss that (17) holds. Therefore, we get that

$$
\mathbb{P}(A \mid F) \leq C'_K N^{\ell - (a_1 + a_2 + \cdots + a_\ell)} \leq C'_K / N,
$$

(18)

where $C'_K$ is another constant depending on $K$, and the second inequality follows from the fact that $a_1 + a_2 + \cdots + a_\ell > 2\ell$ (assuming there is at least one intersecting pair). Since $|D_1|, |D_2|, |D_3|$ and $\ell$ are all bounded by a number that depends only on $K$, we combine (16) and (18) and obtain

$$
\sum_{T^{k,2}} \mathbb{P}(E_{i_1,\ldots,i_k} \mid F) \leq C_K^* / N,
$$

where $C_K^* > 0$ depends only on $K$. Combined with (12), (13) and (15), this yields (11).

**Case 2.** $4 \leq |i - j| \leq N^{1.2}$. Denote by $W_k$ the event that in some $k$ consecutive updates there are at least two coordinates such that all of them occur at least twice. Note that given $F$, the conditional probability that the numbers 1 and 2 both occur at least twice in the first $k$ updates is less than $\binom{k}{2} (\frac{k}{L})^2 \binom{k}{2} (\frac{k}{L})^2$. Therefore,

$$
\mathbb{P}(W_k) = \mathbb{E}(\mathbb{P}(W_k \mid F)) \leq \sum_{1 \leq i < j \leq N} \mathbb{E}\left( \binom{U_i}{2} \binom{U_j}{2} \frac{k}{L} \binom{k}{2} L \right) \leq \frac{C'^* k^4}{N} = o(1),
$$

(19)
for all $k \leq N^{1/5}$ (here $C'$ is an absolute constant).

**Case 3.** $|i - j| \geq N^{1/5}$. In this regime, we consider the following continuous version of (modified) $\mu_{k,n}$. After $U_i$'s are chosen as i.i.d. random variables according to $F_1$, we put $U_i$ copies of $i$'s independently and uniformly on $[0,1]$. Clearly the natural ordering of their positions would give us the same update sequence (in distribution) as under the modified $\mu_{k,n}$. In other words, under the probability space for the continuous model, any event that only concerns the update sequence (but not the positions of the copies of integers) would have the same probability as under the modified $\mu_{k,n}$. As such, in the rest of the proof we let $P$ and $E$ correspond to the probability measure of the continuous model.

Our strategy is to show that for $t \geq N^{-5/6}$, both the total number of updates in a time interval $I$ of length $t$ (call it $T_I$) and the number of coordinates being updated odd number of times in $I$ (call it $O_I$) are concentrated around their means respectively. Note that $T_I$ corresponds to $|i - j|$ while $O_I$ corresponds to $H(v_i, v_j)$.

Conditioning on the total number of updates (in $[0,1]$) to be $L$, the total number of updates in $I$ is the sum of $L$ i.i.d. Bernoulli random variables with mean $t$, thus by Chernoff bound,

$$P(|T_I - Lt| \geq \varepsilon Lt|L) \leq \exp(-\varepsilon^2 Lt/3). \quad (20)$$

For $O_I$ we can proceed similarly. It is the sum of $N$ i.i.d. Bernoulli random variables with mean $p_I$ where $p_I$ is the probability that a coordinate appears odd number of times in $I$. We can compute $p_I$ as follows:

$$p_I = \sum_{i=0}^{\infty} \frac{2^{2i+1}}{(2i+1)! \sinh x} \sum_{j=0}^{i} \left( \frac{2i+1}{2j+1} \right) t^{2j+1} (1-t)^{2i-2j}$$

$$= \frac{1}{\sinh x} \left( \sum_{j=0}^{\infty} \frac{(xt)^{2j+1}}{(2j+1)!} \left( \sum_{i=j}^{\infty} \frac{(x(1-t))^{2(i-j)}}{(2i-2j)!} \right) \right)$$

$$= \frac{\sinh(xt) \cosh(x(1-t))}{\sinh x}. \quad (21)$$

Notice that since $\sinh x \geq x$ for $x \geq 0$, $E(O_I) \geq cN^{1/6}$ for a constant $c > 0$ when $t \geq N^{-5/6}$. By Chernoff bound we get,

$$P(|O_I - EO_I| \geq 3\varepsilon EO_I) \leq \exp \left( -3\varepsilon^2 N \frac{\sinh(xt) \cosh(x(1-t))}{\sinh x} \right). \quad (22)$$

We will now show that with high probability $T_I$ and $O_I$ are both within $[1 - 4\varepsilon, 1 + 4\varepsilon]$ times their respective means for any interval $I$ of length $t$ when $t \geq N^{-5/6}$. To this end first divide the interval $[0,1]$ into $N$ non-overlapping intervals of equal length $1/N$ and denote by $E_L$ the event $\{L/(x \coth xN) \in [1 - \varepsilon, 1 + \varepsilon]\}$. Since $E(T_I) \geq c'N^{1/6}$ for a constant $c' > 0$, we can apply (20) and an union bound over all intervals of the form $[n_1/N, n_2/N]$ such that $(n_2 - n_1) \geq N^{1/6}$ to obtain that

$$P\left( \max_{0 \leq n_1 < n_2 \leq N, n_2 - n_1 \geq N^{1/6}} |T_{[n_1/N, n_2/N]} - Lt| \geq \varepsilon Lt \mid E_L \right) \rightarrow 0 \text{ as } N \rightarrow \infty.$$ 

Since $P(E_L) \rightarrow 1$ as $N \rightarrow \infty$, we get that $P(\mathcal{E}_T) \rightarrow 1$ as $N \rightarrow \infty$, where

$$\mathcal{E}_T = \bigcap_{I = [n_1/N, n_2/N]: n_1 - n_2 \geq N^{1/6}} \{T_I \in [\varepsilon(E(T_I)(1 - 3\varepsilon), \varepsilon(E(T_I)(1 + 3\varepsilon)) \}.$$
From (22), we can deduce in the same manner that \( \Pr(\mathcal{E}_O) \to 1 \) as \( N \to \infty \), where

\[
\mathcal{E}_O = \cap_{I=[n_1/N, n_2/N]:|n_1-n_2| \geq 100N^1/6} \{ O_I \in [\mathbb{E}(O_I)(1-3\varepsilon), \mathbb{E}(O_I)(1+3\varepsilon)] \} .
\]

So we may assume without loss that \( \mathcal{E}_T \) and \( \mathcal{E}_O \) occur. Now the probability that there are at least 100 log \( N \) points in an interval \([i/N, (i+1)/N]\) is bounded by \( \mathbb{E} \left( \frac{L}{100 \log N} \right) / N^{100 \log N} \) which is at most \( 1/N^2 \) for all large \( N \). So by applying a union bound over all the \( N \) intervals we get that the probability that any such interval contains more than 100 log \( N \) points is \( o(1) \). Without loss we assume this event does not occur in what follows. On the complement of this event (i.e., any small interval has less than 100 log \( N \) points), we can approximate any interval \( I \) of length \( t \) by intervals of form \([n_1/N, n_2/N]\) with an error of at most two small intervals. But since \( \mathbb{E}T_I \) and \( \mathbb{E}O_I \) are both of magnitude \( N^{1/6} \), a change of one or two small intervals of length \( 1/N \) in \( I \) would change \( T_I, O_I, \mathbb{E}T_I \) or \( \mathbb{E}O_I \) by at most factor of \( (1+o(1)) \) and consequently \( T_I \) and \( O_I \) would still remain within say, \([1-4\varepsilon, 1+4\varepsilon]\) times their respective means.

Now if \(|I| > (1/2 + 6\varepsilon)\), by concentration of \( T_I \) discussed above we get with probability tending to 1 as \( N \to \infty \)

\[
T_I \geq (1 - 4\varepsilon) \mathbb{E}(T_I) = (1 - 4\varepsilon) N |I| x \coth x \geq \alpha(1/2 + \varepsilon) N
\]

for all sufficiently small but fixed \( \varepsilon \). On the other hand if \(|I| \leq (1/2 + 6\varepsilon)\) then by concentration of \( O_I \), we have \( O_I \leq (1/2 + \varepsilon_1) N \) for \( \varepsilon_1 = \varepsilon^{1/2} \). Therefore, we get

\[
H(v_i, v_j) \leq (1/2 + \varepsilon_1) N, \text{ if } N^{5/8} \leq |i-j| \leq \alpha(1/2 + \varepsilon) N . \tag{23}
\]

Similar argument shows that for \( \varepsilon_2 = \varepsilon^{1/4} \)

\[
H(v_i, v_j) \geq (1/2 + \varepsilon_1) N \text{ for } |i-j| \geq \alpha(1/2 + \varepsilon_2) N . \tag{24}
\]

Finally notice that if \(|I| \not\in \left[ N^{-5/6}, (1/2 + 6\varepsilon_2) \right] \), we have \( T_I \not\in \left[ N^{1/5}, \alpha(1/2 + \varepsilon_2)N \right] \). But for \(|I| \in \left[ N^{-5/6}, (1/2 + 6\varepsilon_2) \right] \) we have \( \frac{\sinh(x|I|)\cosh(x(1-|I|))}{\sinh x} \geq x \coth x |I| \frac{1}{\alpha + \varepsilon_3} \) for \( \varepsilon_3 = 0.1 \varepsilon^{1/8} \), i.e.,

\[
\mathbb{E}O_I \geq 1/(\alpha + \varepsilon_3) \mathbb{E}T_I .
\]

By our assumptions on the concentration of \( O_I \) and \( T_I \) again, we deduce that \( O_I \geq 1/(\alpha + \varepsilon_3)T_I \) for \( \varepsilon_3 = \varepsilon^{1/8} \). This says that

\[
H(v_i, v_j) \geq |i-j|/(\alpha + \varepsilon_3) \text{ for } |i-j| \in \left[ N^{1/5}, \alpha(1/2 + \varepsilon_2)N \right] . \tag{25}
\]

Combining (11), (19), (23), (24), (25) altogether, we completed the proof of (3), and thus the proof of the lemma.

Let \( \mathcal{P} \) be the collection of good paths. For any path \( P \in \mathcal{P} \), let \( A_P \) be the event that \( P \) is accessible. So we have \( Z_{N,x,\text{good}} = \sum_{P \in \mathcal{P}} 1_{A_P} \). Notice that

\[
\mathbb{E}Z_{N,x,\text{good}}^2 = \sum_{P \in \mathcal{P}} \sum_{P' \in \mathcal{P}} \Pr(\mathbb{E}(A_P \cap A_{P'}))
\]

\[
= \sum_{P \in \mathcal{P}} \Pr(A_P) \sum_{P' \in \mathcal{P}} \Pr(A_{P'} \mid A_P)
\]

\[
= \sum_{P \in \mathcal{P}} \Pr(A_P) \mathbb{E}(Z_{N,x,\text{good}} \mid A_P) . \tag{26}
\]
So in order to estimate \( EZ_{N,x,\text{good}} \), a key step is to estimate \( E(Z_{N,x,\text{good}} | A_P) \). For any good path \( P \) of length \( L \), let \( v_0, v_1, v_2, \ldots, v_L = \overline{1} \) be the \((L+1)\) vertices it passes through. Let \( X_i \) be the (random) value at \( v_i \) (recall that \( X_0 = 0 \) and \( X_L = x \)). It’s clear that conditioning on \( P \) to be open, the \( X_i \)’s are distributed as \((L-1)\) order statistics of a sequence of i.i.d. uniform variables on \([0,x]\). We denote the successive differences of \( X_i \)’s by \( \delta_1 = X_1, \delta_2 = X_2 - X_1, \ldots, \delta_L = x - X_{L-1} \). It is well-known that \( \{\delta_1/x, \delta_2/x, \ldots, \delta_L/x\} \) has a Dirichlet distribution \( \text{Dir}(1,1,\ldots,1) \). Recall that a Dirichlet distribution \( \text{Dir}(\alpha_1, \alpha_2, \ldots, \alpha_K) \) is supported on \((x_1, x_2, \ldots, x_K)\) where \( x_i \in [0,1] \) and \( \sum_{i=1}^K x_i = 1 \), and has a density \( \frac{\Gamma(\sum_{i=1}^K \alpha_i)}{\prod_{i=1}^K \Gamma(\alpha_i)} \prod_{i=1}^K x_i^{\alpha_i-1} \).

We first state some properties of these spacings.

**Proposition 2.8.** For \( 0 < i_1 < i_2 < \cdots < i_k < L \) and nonnegative integers \( \beta_1, \beta_2, \cdots, \beta_k+1 \),

\[
(i) \quad \frac{1}{x} (X_{i_1} - X_0, X_{i_2} - X_{i_1}, \ldots, X_L - X_{i_k}) = \frac{1}{x} \left( \sum_{i=1}^{i_1} \delta_i, \sum_{i=i_1+1}^{i_2} \delta_i, \ldots, \sum_{i=i_k+1}^{L} \delta_i \right) \text{ has distribution } \text{Dir}(i_1, i_2 - i_1, \ldots, L - i_k).
\]

\[
(ii) \quad \mathbb{E}(\prod_{j=1}^{k+1} (X_{i_j} - X_{i_j-1})^{\beta_j}) \leq \prod_{j=1}^{k+1} \mathbb{E}((X_{i_j} - X_{i_j-1})^{\beta_j}).
\]

\[
(iii) \quad \mathbb{E}(X_{i_1} - X_0)^{\beta_1} \leq C \sqrt{1 + t (x^{1+1/(1+t)})^\beta_1} \text{ for } \beta_1 \leq t(i_1-1), \text{ where } C > 0 \text{ is an absolute constant.}
\]

**Proof.** (i) This follows from the aggregation property of Dirichlet distribution.

(ii) This follows from the moments of Dirichlet-distributed random variables. That is, for \( Y \sim \text{Dir}(\alpha_1, \alpha_2, \ldots, \alpha_K) \), we have

\[
\mathbb{E}(\prod_{j=1}^k Y_j^{\beta_j}) = \frac{\Gamma(\sum_{i=1}^k \alpha_i)}{\Gamma(\sum_{i=1}^k \alpha_i + \beta_i)} \prod_{i=1}^k \frac{\Gamma(\alpha_i + \beta_i)}{\Gamma(\alpha_i)} \leq \prod_{i=1}^k \frac{\Gamma(\sum_{i=1}^k \alpha_i)}{\Gamma(\beta_i + \sum_{i=1}^k \alpha_i)} \prod_{i=1}^k \frac{\Gamma(\alpha_i + \beta_i)}{\Gamma(\alpha_i)} = \prod_{j=1}^k \mathbb{E}(x_j^{\beta_j}).
\]

(iii) By standard computation, we have

\[
\mathbb{E}(X_{i_1} - X_0)^{\beta_1} = x^{\beta_1} \frac{\Gamma(L)}{\Gamma(L + \beta_1)} \frac{\Gamma(i_1 + \beta_1)}{\Gamma(i_1)} = x^{\beta_1} \frac{(L-1)!}{(L + \beta_1 - 1)! \Gamma(i_1 - 1)! \Gamma(i_1 + \beta_1 - 1)!}.
\]

By Stirling’s formula, we get that for an absolute constant \( C > 0 \)

\[
\mathbb{E}(X_{i_1} - X_0)^{\beta_1} \leq C x^{\beta_1} \frac{\sqrt{2\pi (L-1)} \frac{(L-1)}{e}^{L-1}}{\sqrt{2\pi (L + \beta_1 - 1)} \frac{(L + \beta_1 - 1)}{e}^{L+\beta_1-1}} \frac{\sqrt{2\pi (i_1 + \beta_1 - 1)} \frac{(i_1 + \beta_1 - 1)}{e}^{i_1 + \beta_1 - 1}}{\sqrt{2\pi (i_1 - 1)} \frac{(i_1 - 1)}{e}^{i_1 - 1}}
\]

\[
= C (x^{\frac{i_1 - 1}{L - 1}})^{\beta_1} \frac{\sqrt{L-1} \frac{(i_1 + \beta_1 - 1)}{e}^{i_1 + \beta_1 - 1}}{\sqrt{L + \beta_1 - 1} \frac{(i_1 - 1)}{e}^{i_1 - 1}} \frac{(1 + \frac{\beta_1}{i_1 - 1})^{1+\frac{i_1 - 1}{\beta_1}}}{(1 + \frac{\beta_1}{L-1})^{1+\frac{L-1}{\beta_1}}}. \]

Note that \( \frac{(L-1)(i_1 + \beta_1 - 1)}{(L + \beta_1 - 1)(i_1 - 1)} \leq \frac{i_1 + \beta_1 - 1}{i_1 - 1} \leq 1 + t \). In addition, the function \((1 + z)^{1+1/z}\) is increasing in \( z \) and tends to \( e \) as \( z \to 0 \), so that \( \frac{(1 + \frac{\beta_1}{i_1 - 1})^{1+\frac{i_1 - 1}{\beta_1}}}{(1 + \frac{\beta_1}{L-1})^{1+\frac{L-1}{\beta_1}}} \leq \frac{(1+t)^{1+1/t}}{e} \). Plugging these bounds into the preceding display completes the proof. \( \square \)
In order to compute $\mathbb{E}(Z_{N,x,\text{good}} \mid A_P)$, we first calculate $\mathbb{E}(Z_{N,x,\text{good}}(\vec{0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}, \vec{1}) \mid A_P)$, where $\vec{0}$, $v_{i_1}$, $v_{i_2}$, $\ldots$, $v_{i_k}$, $\vec{1}$ are vertices on path $P$ and $Z_{N,x,\text{good}}(\vec{0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}, \vec{1})$ counts the number of good accessible path $P'$ that intersects $P$ (vertex wise) at $\vec{0}$, $v_{i_1}$, $v_{i_2}$, $\ldots$, $v_{i_k}$, $\vec{1}$. Naturally these $(k + 2)$ common vertices divide both $P$ and $P'$ into $(k + 1)$ segments. The lengths of these segments on $P$ are $i_1$, $i_2 - i_1$, $\ldots$, $L - i_k$. Suppose that $P'$ visits these $(k + 2)$ common vertices at its $j_0$-th, $j_1$-th, $\ldots$, $j_{k+1}$-th steps (here we suppress the implicit dependence of $j_{k'}$ on $P'$ for each $0 \leq k' \leq k$). Then on $A_P$ we have

$$\mathbb{P}(A_{P'} \mid X_0, X_1, \ldots, X_L) = \frac{X_{i_1}^{j_1-1} \cdot (X_{i_2} - X_{i_1})^{j_2-j_1-1} \cdot \cdots \cdot (X - X_{i_k})^{j_{k+1} - j_k-1}}{(j_1 - 1)! \cdot (j_2 - j_1 - 1)! \cdot \cdots \cdot (j_{k+1} - j_k - 1)!}.$$ 

By Part (ii) of Proposition 2.8 we have

$$\mathbb{P}(A_{P'} \mid A_P) = \mathbb{E}(\mathbb{P}(A_{P'} \mid X_0, X_1, \ldots, X_L) \mid A_P) \leq \mathbb{E} \frac{Y_{i_1}^{j_1-1} \cdot (Y_{i_2} - Y_{i_1})^{j_2-j_1-1} \cdot \cdots \cdot (x - Y_{i_k})^{j_{k+1} - j_k-1}}{(j_1 - 1)! \cdot (j_2 - j_1 - 1)! \cdot \cdots \cdot (j_{k+1} - j_k - 1)!},$$

where $(Y_0, \ldots, Y_L)$ is distributed as order statistics of $(L + 1)$ i.i.d. uniform variables on $[0, x]$. Therefore, we get that

$$\mathbb{E}(Z_{N,x,\text{good}}(\vec{0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}, \vec{1}) \mid A_P) = \sum_{P' \in \mathcal{P}} \mathbb{P}(A_{P'} \mid A_P) \leq \sum_{P' \in \mathcal{P}} \prod_{\ell=1}^{k+1} \mathbb{E} \frac{(Y_{i_{\ell-1}} - Y_{i_\ell})^{j_{\ell} - j_{\ell-1} - 1}}{(j_{\ell} - j_{\ell-1} - 1)!},$$

where $F(v_{i_{\ell-1}}, v_{i_{\ell}}) = \mathbb{E}F(v_{i_{\ell-1}}, v_{i_{\ell}}, Y_{i_{\ell-1}}, Y_{i_{\ell}})$ and $F(v_{i_{\ell-1}}, v_{i_{\ell}}, y_{i_{\ell-1}}, y_{i_{\ell}})$ is the conditional expectation of the number of good open segments (given that $X_{i_{\ell-1}} = y_{i_{\ell-1}}$ and $X_{i_{\ell}} = y_{i_{\ell}}$) that connect $v_{i_{\ell-1}}$ and $v_{i_{\ell}}$. Here a good segment connecting $v$ and $u$ is a path $P^*$ that connects $v$ and $u$ such that there exists at least one good path between $\vec{0}$ and $\vec{1}$ whose subpath between $v$ and $u$ is $P^*$. Summing over $i_1, i_2, \ldots, i_k$ (and $k$) we get

$$\mathbb{E}(Z_{N,x,\text{good}} \mid A_P) \leq \sum_{k, i_1, i_2, \ldots, i_k} \prod_{\ell=1}^{k+1} F(v_{i_{\ell-1}}, v_{i_{\ell}}).$$

We can further split the sum on the right hand side into two parts, according to whether $\max\{i_1, (i_2-$
Lemma 2.10. There exists for all \( |E| \) the following two lemmas are useful for bounding (here the last inequality follows from Corollary 2.5), we complete the proof of the corollary.

Lemma 2.12. There exists \( \varepsilon \) fixed.

Proof. Plugging bounds in Lemmas 2.9 and 2.10 into (29) and then (28), we get that

\[
\sum_{k,i_1,i_2,\ldots,i_k} \prod_{\ell=1}^{k+1} F(v_{i_{\ell-1}}, v_{i_{\ell}}) = \sum_{k,i_1,i_2,\ldots,i_k} \prod_{\ell=1}^{k+1} F(v_{i_{\ell-1}}, v_{i_{\ell}}) + \sum_{k,i_1,i_2,\ldots,i_k} \prod_{\ell=1}^{k+1} F(v_{i_{\ell-1}}, v_{i_{\ell}}) \leq \left( \sum_{d=0}^{L/2} \sum_{d_1, d_2 = d} F(v_{d_1}, v_{L-d_2}) \right) \prod_{j=0}^{L} (1 + \sum_{i=1}^{\frac{L}{2}} m F(v_j, v_{j+i})) = \prod_{j=0}^{L} (1 + \sum_{i=1}^{\frac{L}{2}} m F(v_j, v_{j+i})). \quad (29)
\]

The following two lemmas are useful for bounding \( \mathbb{E}(Z_{N,x,good} \mid A_P) \).

**Lemma 2.9.** There exists \( C_2, \varepsilon > 0 \) such that \( \sum_{d=0}^{L/2} \sum_{d_1, d_2 = d} F(v_{d_1}, v_{L-d_2}) \leq C_2 N (\sinh x)^{N-1} \cosh x \) for all \( |x - x_0| < \varepsilon, \) all fixed \( \varepsilon \in (0, \varepsilon) \) and any good path \( P \).

**Lemma 2.10.** There exists \( C_3, \varepsilon > 0 \) such that \( \sum_{i=1}^{\frac{L}{2}} m F(v_j, v_{j+i}) \leq 1 + \frac{C_3}{N} \). for all \( |x - x_0| < \varepsilon, \) all fixed \( \varepsilon \in (0, \varepsilon) \), any good path \( P \), and any \( j \).

**Corollary 2.11.** There exists \( C_4, \varepsilon > 0 \) such that for all \( |x - x_0| < \varepsilon \) and all fixed \( \varepsilon \in (0, \varepsilon) \)

\[
\mathbb{E}Z_{N,x,good}^2 \leq (C_4 N \sinh^{N-1} x \cosh x + C_4) N \sinh^{N-1} x \cosh x.
\]

**Proof.** Plugging bounds in Lemmas 2.9 and 2.10 into (29) and then (28), we get that

\[
\mathbb{E}(Z_{N,x,good} \mid A_P) \leq \sum_{k,i_1,i_2,\ldots,i_k} \prod_{\ell=1}^{k+1} F(v_{i_{\ell-1}}, v_{i_{\ell}}) \leq (C_2 N (\sinh x)^{N-1} \cosh x + 1) + \frac{C_3}{N} = \frac{1}{N} e^{C_4(1+\varepsilon)\alpha}.
\]

Plugging the preceding inequality into (26) and applying the inequality

\[
\sum_{P \subset P} \mathbb{P}(A_P) = \mathbb{E}Z_{N,x,good} \leq N (\sinh x)^{N-1} \cosh x
\]

(here the last inequality follows from Corollary 2.5), we complete the proof of the corollary.

In order to prove Proposition 2.9 and Proposition 2.10 we need the following lemma.

**Lemma 2.12.** Suppose that \( N \geq 7, s \geq 1 \). Let \( g(y,s) = (\sinh y)^s (\cosh y)^{N-s} \). Then \( \frac{\partial g}{\partial y}(y,s) \) is decreasing in \( s \) for all fixed \( y > 0 \).

**Proof.** Note that

\[
\frac{\partial g}{\partial y}(y,s) = (\sinh y)^s (\cosh y)^{N-s} (s \coth y + (N-s) \tanh y)
\]

\[
= (\sinh y)^{s+1} (\cosh y)^{N-1} (\tanh y)^s (s + N (\sinh y)^2).
\]

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Therefore it suffices to show that \((\tanh y)^s(s + N(\sinh y)^2)\) is decreasing in \(s\). Taking the partial derivative with respect to \(s\) we get

\[
\frac{\partial}{\partial s} [(\tanh y)^s(s + N(\sinh y)^2)] = (\tanh y)^s + (\ln \tanh y)(\tanh y)^s(s + N(\sinh y)^2),
\]

so we only need to show that \((\coth y)^{(s+N(\sinh y)^2)} \geq e\). If \(\coth y \geq e\), then plainly we have \((\coth y)^{(s+N(\sinh y)^2)} \geq \coth y \geq e\). On the other hand, if \(\coth y < e\), then \(y > \arccot e := y_0\). Since \((\coth y)^{(\sinh y)^2}\) is increasing in \(y\), we have \((\coth y)^{(\sinh y)^2} \geq (\coth y_0)^{(\sinh y_0)^2} = e^{\frac{1}{e^2-1}} \approx 1.17\). Therefore we get \((\coth y)^{(s+N(\sinh y)^2)} \geq (\coth y)^7(\sinh y)^2 \geq (\coth y_0)^7(\sinh y_0)^2 > e\) in this case.

**Proof of Lemma 2.7.** Note that the Hamming distance between \(v_{d_1}\) and \(v_{L-d_2}\) is greater than or equal to \(N - d\), and that the distribution of \(Y_{L-d_2} - Y_{d_1}\) does not depend on \((d_1, d_2)\) provided the value of \(d = d_1 + d_2\). Therefore by (7) and Lemma 2.12 we get that

\[
F(v_{d_1}, v_{L-d_2}) \leq \mathbb{E}((\sinh y)^{N-d}(\cosh y)^d)'|y = Y_{L-d_2} - Y_{d_1} = \mathbb{E}((\sinh y)^{N-d}(\cosh y)^d)'|y = x - y_d.
\]

Computing the derivative in the right hand side of the proceeding inequality and using the fact that \(\sinh y \leq c y\), we get that

\[
F(v_{d_1}, v_{L-d_2}) \leq \mathbb{E}((\sinh y)^{N-d-1}(\cosh y)^{d-1}(N - d)(\cosh y)^2 + d(\sinh y)^2))|y = x - y_d
\]

\[
\leq \mathbb{E}((\sinh y)^{N-d-1}(\cosh y)^{d-1} N(\cosh y)^2)|y = x - y_d
\]

\[
\leq N(\cosh x)^2 \mathbb{E}(\sinh(x - y_d))^{N-d-1}(\cosh x)^{d-1}.
\]

Since \(\sinh(x - y) \leq \sinh x - \frac{\sinh x}{x} y\) for \(0 \leq y \leq x\), we get that

\[
F(v_{d_1}, v_{L-d_2}) \leq N(\cosh x)^2 \mathbb{E}(\sinh x - \frac{\sinh x}{x} y_{d})^{N-d-1}(\cosh x)^{d-1}
\]

\[
= N(\cosh x)^2(\sinh x)^{N-d-1}(\cosh x)^{d-1}(1 - \frac{Y_d}{x})^{N-d-1}. \tag{31}
\]

It remains to bound \(\mathbb{E}(1 - \frac{Y_d}{x})^{N-d-1}\). Since \(1 - \frac{Y_d}{x}\) is the \((L - d)\)th order statistic of \((L - 1)\) i.i.d. uniform variables in \([0, 1]\), it has a \(\text{Beta}(L - d, d)\) distribution. Thus, applying (27) for \(x = 1\) and \(i_1 = L - d\) we get that

\[
\mathbb{E}(1 - \frac{Y_d}{x})^{N-d-1} = \prod_{r=0}^{N-d-2} \frac{L - d + r}{L + r} \leq (1 - d/(L + N - d - 2))^{N-d-1}
\]

\[
\leq (e^{-\frac{N-d-1}{L+N-d-2}})^d \leq (\frac{0.995}{\coth x})^d, \tag{32}
\]

for \(d \leq 0.32 N\) and \(N\) sufficiently large (recall that \(L \in [\alpha N - N^2/3, \alpha N + N^2/3]\)). Here we used the inequality \(e^{-\frac{1}{\alpha N - 0.32}} \leq \frac{0.994}{\coth x_0}\) (by brutal force calculus).

For \(0.32 N \leq d \leq \alpha (1/2 + \varepsilon) N\). Set \(t = d/N\) and \(s = L/N\). Then by Stirling’s formula and calculus, we get that

\[
\prod_{r=0}^{N-d-2} \frac{L - d + r}{L + r} \leq C_8 \prod_{r=1}^{N-d} \frac{L - d + r}{L + r} \leq C_9 \frac{(L + N - 2d)L^{N-d}2^L}{(L - d)^{L-d}(L + N - d)2^L} = C_9 \left(\frac{(1 + s - 2t)^{1+s-2t}s^s}{(s-t)^{s-t}(1 + s - t)^{1+s-1}}\right)^d.
\]
Another brutal force calculus gives that

\[
\left( \frac{(1 + \alpha - 2t)^{1+\alpha-2t} \alpha^\alpha}{(\alpha - t)^{\alpha-t}(1 + \alpha - t)^{1+\alpha-t}} \right)^{\frac{1}{t}} \leq \frac{0.999}{\coth x_0}
\]

for \( t \leq \alpha (1/2 + \varepsilon) \) and \( \varepsilon \) sufficiently small. Since the function \( f(y, t) \) given by

\[
f(y, t) = \left( \frac{(1 + y - 2t)^{1+y-2t} y^y}{(y - t)^{y-t}(1 + y - t)^{1+y-t}} \right)^{t},
\]

is jointly continuous with respect to \((y, t)\) on \([1, 0.1, 1.5] \times [0.2, 0.8]\), we get that for \( N \) sufficiently large (so \( s \) is sufficiently close to \( \alpha \)) and \( 0.32 \leq t \leq \alpha (1/2 + \varepsilon) \)

\[
\left( \frac{(1 + s - 2t)^{1+s-2t} s^s}{(s - t)^{s-t}(1 + s - t)^{1+s-t}} \right)^{\frac{1}{t}} \leq \frac{0.99999}{\coth x_0}.
\]

In addition, for \( t \) sufficiently small, the right hand side in the above inequality is at most \( 0.99999/\coth x \). So we get \( \prod_{j=0}^{N-d} \frac{L-d+j}{L+t} \leq C_9 \frac{0.99999}{\coth x} d \) in this case. Combined with \( (31) \) and \( (32) \), this completes the proof of the lemma.

**Proof of Lemma 2.11** Recall that \( P = v_0, v_1, \ldots, v_L \) is a good path of length \( L \). For an arbitrary \( j \), we will bound \( F(v_j, v_{j+1}) \) in a number of regimes depending on the value of \( i \), as follows.

**Case (a):** \( i = 1 \). Since for any good path (or good segment), the Hamming distance between a pair of vertices on the path is 1 if and only if these two vertices are neighboring each other in the path, we get that \( F(v_j, v_{j+1}) \leq 1 \).

**Case (b):** \( i = 2 \). The Hamming distance between \( v_j \) and \( v_{j+2} \) (since \( P \) is good) is precisely 2, and thus the length of the subpath of any good segment between \( v_j \) and \( v_{j+2} \) is either 2 or 4. Therefore, \( F(v_j, v_{j+2}, y_j, y_{j+2}) \leq 2(y_{j+2} - y_j) + (N(4)2!) \frac{(y_{j+2} - y_j)^3}{3!} = 2(y_{j+2} - y_j) + 2N(y_{j+2} - y_j)^3 \). Combined with \( (31) \) of Proposition 2.8 it gives that

\[
F(v_j, v_{j+2}) \leq 20/N.
\]

**Case (c):** \( i = 3 \). The Hamming distance between \( v_j \) and \( v_{j+3} \) (since \( P \) is good) is precisely 3, and thus the length of the subpath of any good segment between \( v_j \) and \( v_{j+3} \) is either 3 or 5. Therefore, \( F(v_j, v_{j+3}, y_j, y_{j+3}) \leq 3(y_{j+3} - y_j)^2 + (N(5)3!) \frac{(y_{j+2} - y_j)^4}{4!} = 3(y_{j+3} - y_j)^2 + (\frac{3}{2}N)(y_{j+2} - y_j)^4 \). Combined with \( (31) \) of Proposition 2.8 it yields that

\[
F(v_j, v_{j+3}) \leq 1000 \cdot N^{-2}.
\]

**Case (d):** \( 4 \leq i \leq N^\frac{1}{4} \). By definition of good path and good segment again, we see that the possible values of \((H(v_j, v_{j+i}), L(v_j, v_{j+i}))\) are \((i, i)\), \((i, i+2)\), \((i-2, i-2)\) and \((i-2, i)\). Therefore \( F(v_j, v_{j+i}, y_j, y_{j+i}) \) is at most

\[
i(y_{j+i} - y_j)^{i-1} + \frac{N(i+2)!}{(i+1)!}(y_{j+i} - y_j)^{i+1} + (i-2)(y_{j+i} - y_j)^{i+3} + \frac{N(i)!}{(i-1)!}(y_{j+i} - y_j)^{i-1}.
\]

Combined with \( (31) \) of Proposition 2.8 it yields that

\[
F(v_j, v_{j+i}) \leq 10^4 i N^{i-3} \text{ for sufficiently large } N.
\]
Case (e): \( N^{\frac{1}{5}} \leq i \leq L/2 \). Recall definitions of \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) in (8). By properties of good paths, we have \( \frac{1}{\alpha + \varepsilon_3} \leq H(v_j, v_{j+i}) \leq (1/2 + \varepsilon_1)N \). Therefore (by properties of good paths again) all good segment that joins \( v_j \) and \( v_{j+i} \) must have length \( L(v_j, v_{j+i}) \leq \alpha (1/2 + \varepsilon_2)N \), so that \( L(v_j, v_{j+i}) \) also satisfies \( L(v_j, v_{j+i}) \leq (\alpha + \varepsilon_3)H(v_j, v_{j+i}) \leq (\alpha + \varepsilon_3)N \). Since \( i \) is large enough, by (111) of Proposition 2.8 we get that

\[
\mathbb{E}(X_{j+i} - X_j)^\ell \leq C \left( \frac{1}{L - 1} \right)^{\ell} \left( 1 + (\alpha + \varepsilon_3) \right)^{1+1/(\alpha + \varepsilon_3)} e, \text{ for } \ell \leq (\alpha + \varepsilon_3)i.
\]

Therefore by (7) and Lemma 2.12, we have

\[
F(v_j, v_{j+i}) \leq C((\sinh y)^{\frac{1}{\alpha + \varepsilon_3}} (\cosh y)^{N - \frac{1}{\alpha + \varepsilon_3}})^{\frac{1}{\alpha + \varepsilon_3}} \leq C_1 10^2 (\sinh y)^{\frac{1}{\alpha + \varepsilon_3}} (\cosh y)^{N - \frac{1}{\alpha + \varepsilon_3}}.
\]

Set \( a = \frac{N(\alpha + \varepsilon_3)}{L - 1} \), \( c = \frac{x}{\left( 1 + (\alpha + \varepsilon_3) \right)^{1+1/(\alpha + \varepsilon_3)} e} \), and \( c_0 = \frac{x_0 (1 + \alpha)^{1+1/\alpha}}{e} \approx 1.39 \). Clearly \( c \) would be sufficiently close to \( c_0 \) if \( t \) (and therefore \( \varepsilon \)) is sufficiently small.

Let \( t = \frac{3}{L - 1} \) and \( f(t) = (\sinh(ct))^\frac{1}{\alpha + \varepsilon_3} (\cosh(ct))^\frac{N}{\alpha + \varepsilon_3} \). Then the preceding display can be rewritten as \( F(v_j, v_{j+i}) \leq C_1 10^2 (f(t))^{L-1} \). In order to estimate \( F(v_j, v_{j+i}) \), we analyze the behavior of the function (as well as the derivatives) of \( f(t) \). By straightforward computation, we get that

\[
((\alpha + \varepsilon_3) \ln f(t))' = t \ln \sinh(ct) + (a - t) \ln \cosh(ct),
\]

\[
((\alpha + \varepsilon_3) \ln f(t))'' = \ln \sinh(ct) - \ln \cosh(ct) + c(a - t) \tanh(ct) + c(a - t) \tanh(ct).
\]

Further, we can compute

\[
((\alpha + \varepsilon_3) \ln f(t))'' = c \coth(ct) - c \tanh(ct) + c \coth(ct) - c \tanh(ct) - \frac{c^2 t}{(\sinh(ct))^2} + \frac{c^2 (a - t)}{(\cosh(ct))^2} + \frac{c^2}{(\sinh(ct))^2 (\cosh(ct))^2} \sinh(2ct) - c \cosh(2ct) > 0.
\]

for \( t \leq 1/2 \) (since \( c t < c/2 < 0.8 \)).

Therefore \( (\alpha + \varepsilon_3) \ln f(t) \), and consequently \( f(t) \) is convex up to \( t = 1/2 \). Thus we have \( f(t) \leq \max(f(\frac{N^{\frac{1}{5}}}{2}), f(\frac{1}{2})) \), and so \( F(v_j, v_{j+i}) \leq \max(C_1 10^2 (f(\frac{N^{\frac{1}{5}}}{2}))^{L-1}, C_1 10^2 (f(\frac{1}{2}))^{L-1}) \).

However, since \( (f(\frac{1}{2}))^{2(\alpha + \varepsilon_3)} = \sinh(c/2) \cos(c/2)^{2(\alpha + \varepsilon_3)} N^{\frac{1}{5} - \frac{1}{\alpha + \varepsilon_3}} \) which is sufficiently close to \( \sinh(c_0/2) \cos(c_0/2) = \sinh(c_0)/2 < 1 \) if \( t \) is sufficiently small and \( N \) is sufficiently large, we have in this case \( f(\frac{1}{2}) \leq p \) where \( p \) is a constant strictly less than 1. Thus, \( (f(\frac{1}{2}))^{L-1} \leq p^{L-1} \). In addition, \( (f(\frac{N^{\frac{1}{5}}}{2}))^{L-1} \leq (N^{-\frac{1}{5}}) N^{\frac{1}{5}} (1 + N^{-\frac{1}{5}})^N \) for sufficiently large \( N \). Thus in this case we have for \( N \) sufficiently large

\[
F(v_j, v_{j+i}) \leq C_1 10^2 \max(p^{L-1}, (N^{-\frac{1}{5}}) N^{\frac{1}{5}} (1 + N^{-\frac{1}{5}})^N).
\]

Conclusion. Summing \( F(v_j, v_{j+i}) \) over \( 1 \leq i \leq L/2 \), and applying the bounds we obtained in Cases (a), (b), (c), (d), (e), we get that \( \sum_{i=1}^{\frac{L}{2}} F(v_j, v_{j+i}) \leq 1 + \frac{C_6}{N} \) for some \( C_6 > 0 \), completing the proof of the lemma. 

\( \square \)
Proposition 2.13. There exists $0 \leq K < 1$ such that, if \( \liminf_{N \to \infty} \mathbb{P}(Z_{N,x_c+\varepsilon_N} > 0) \geq C \) for some constant $C \geq 0$, then whenever $N \varepsilon_N \to \infty$ we have
\[
\liminf_{N \to \infty} \mathbb{P}(Z_{N,x_c+\varepsilon_N} > 0) \geq 1 - (1 - C)K.
\]

Proof. Our strategy basically follows that of \cite{9}. First we pick four vertices $a_1, a_2, b_1, b_2$ satisfying: $a_1$ and $a_2$ are neighbors of $\tilde{0}$ and have a value in $[0, \varepsilon_N/3]$, $b_1$ and $b_2$ are neighbors of $\tilde{1}$ and have a value in $[x - \varepsilon_N/3, x]$, and none of the four pairs $(a_i, b_j)$ are antipodal. Since $N \varepsilon_N \to \infty$, this can be achieved with probability $1 - o_N(1)$.

Without loss of generality assume that the only coordinates of $a_1, a_2, b_1$ and $b_2$ that are different from $\tilde{0}$ or $\tilde{1}$ are $1, 2, 3$ and $4$, respectively. Let $\tilde{H}_1$ and $\tilde{H}_2$ be the $(N-2)$ dimensional sub-hypercubes of $\{0,1\}^N$ formed by $a_1, b_1$ and $a_2, b_2$, respectively. That is, $\tilde{H}_1$ is the sub-hypercube with the first coordinate being $1$ and the third coordinate being $0$, and $\tilde{H}_2$ is the sub-hypercube with the second coordinate being $1$ and the fourth coordinate being $0$. Let $H'_2$ be $\tilde{H}_2 \setminus \tilde{H}_1$. Denote by $p_{\tilde{H}_1}$ and $p_{H'_2}$ the probabilities that there is an accessible path in $\tilde{H}_1$ (from $a_1$ to $b_1$) and $H'_2$ (from $a_2$ to $b_2$) respectively. From independence of $\tilde{H}_1$ and $\tilde{H}_2$ we get $\mathbb{P}(Z_{N,x_c+\varepsilon_N} > 0) \geq 1 - (1 - p_{\tilde{H}_1})(1 - p_{H'_2}) - o_N(1)$. Clearly $p_{\tilde{H}_1} \geq \mathbb{P}(Z_{N-2,x_c+\varepsilon_N/3} > 0) \geq C - o_N(1)$.

It remains to show that $p_{H'_2}$ is bounded from below by a positive constant $1 - K$. To this end, we note that if we only consider the good path in $\tilde{H}_2$ (from $a_2$ to $b_2$) which only updates Coordinate $1$ and Coordinate $3$ once and Coordinate $3$ is updated before Coordinate $1$ (that is, in the associated sequence the numbers $1$ and $3$ occur precisely once each and $3$ occurs ahead of $1$), such path must be contained in $H'_2$. Clearly, the number of such open paths has the second moment less than $\mathbb{E}Z^2_{N-2,x_c+\varepsilon_N/3,\text{good}}$ and the first moment within an absolute multiplicative constant of $\mathbb{E}Z_{N-2,x_c+\varepsilon_N/3}$ (i.e., the first moment is at least $C_{11}(N-2)\sinh^{N-3}(x)\cosh x \cdot x^2/2$ for a constant $C_{11} > 0$). Combined with Lemma 2.7 and Corollary 2.11 this yields that $p_{H'_2} \geq 1 - K$ for some constant $K < 1$. This completes the proof of the proposition. \( \square \)

Proof of \eqref{eq:2}: antipodal case. Applying Proposition 2.13 recursively completes the proof of \eqref{eq:2}. \( \square \)

At the end of this section, we provide

Proof of \eqref{eq:3}: antipodal case. For the lower bound, it suffices to consider $x = x_c - \Delta/N$. It is clear that in this case $N(\sinh x)^{N-1} \cosh x \geq c(\Delta)$ where $c(\Delta) > 0$ depends only on $\Delta$. Say $\liminf_N N(\sinh x)^{N-1} \cosh x \geq C_{15}$ for $C_{15} > 0$. Applying second moment method and using Corollary 2.11 and Lemma 2.7 we obtain that
\[
\mathbb{P}(Z_{N,x} > 0) \geq \mathbb{P}(Z_{N,x,\text{good}} > 0) \geq \frac{(\mathbb{E}Z_{N,x,\text{good}})^2}{\mathbb{E}Z^2_{N,x,\text{good}}} \geq c_1(\Delta),
\]
where $c_1(\Delta) > 0$ depends only on $\Delta$.

For the upper bound, it suffices to consider $x = x_c + \Delta/N$. Let $K > 0$ be number depending on $\Delta$ that we specify later. The idea is to condition on the values of the neighbors of $\tilde{0}$. Let $v_1, v_2, \ldots, v_N$ be neighbors of $\tilde{0}$, and let $Y_1, Y_2, \ldots, Y_N$ be the corresponding fitness values. Let $\mathcal{E} = \{Y_i \geq K/N \text{ for all } 1 \leq i \leq K\}$. It is clear that
\[
\liminf_{N \to \infty} \mathbb{P}(\mathcal{E}) > e^{-K}.
\]

\( (33) \)
Let $Z_{N,x,i}$ be the number of open paths joining $\vec{0}$ and $\vec{i}$ which visit $v_i$ in the first step. Then by (7), we get that

$$E(Z_{N,x,i} \mid Y_i = y_i) \leq ((\sinh t)^{N-1} \cosh t)'|_{t=x-y_i} \leq 2N(\sinh(x - y_i))^{N-2}.$$  

Therefore we get that (recall that $x_0$ is a number such that $\sinh x_0 = 1$ and $x_c = x_0 - \frac{\sqrt{2} \ln N}{2}$)

$$E(Z_{N,x,i}; \mathcal{E}) \leq \int_{K/N}^{1} 2N(\sinh(x - y_i))^{N-2} dy_i \leq \int_{x-1}^{x_0 - \frac{\sqrt{2} \ln N}{2} + \frac{K}{N}} 2N(\sinh y)^{N-2} dy

= N^{-1}(\sqrt{2}e^{\sqrt{2}(\Delta - K)} + o_N(1)),$$

where the last transition follows from [17] problem 213 (in Part Two Chapter 5 section 2) (the solutions is contained in the book [17]). Therefore, we get that

$$\mathbb{P}(Z_{N,x} \geq 1 \mid \mathcal{E}) \leq \frac{N E(Z_{N,x,i}; \mathcal{E})}{\mathbb{P}(\mathcal{E})} \leq \sqrt{2}e^{\sqrt{2}(1-\sqrt{2})K} + o_N(1).$$

Choosing $K = 100(\Delta + 1)$, we get that $\mathbb{P}(Z_{N,x} = 0) \mid \mathcal{E} \geq 1/2 + o_N(1)$. Combined with (35) completes the proof.

\section{Accessibility percolation: general case}

Since most of our proof in the antipodal case carries over to the general case, in the following proof for the general case we will emphasize the parts that require non-trivial modification.

Fix $0 < \beta < 1$ throughout the rest of the section. Recall that $f(x) = (\sinh x)\beta(\cosh x)^{1-\beta}$, and that $x_0$ be the unique root of $f(x) = 1$. We have

$$f'(x) = (\beta \coth x + (1 - \beta) \tanh x)(\sinh x)\beta(\cosh x)^{1-\beta},$$

so that $f'(x_0) = \beta \coth x_0 + (1 - \beta) \tanh x_0$. In addition, it is straightforward to check that $0 < f''(x_0) < \infty$.

\textbf{Proof of (1): general case.} In light of (5) we denote by

$$M_{N,\beta,x} = ((\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N})' = ((f(x))^{N})' = N(f(x))^{N-1}f'(x).$$

Recall that (from the statement of Theorem (1.1))

$$x_c = x_0 - \frac{1}{N} \ln N = x_0 - \frac{1}{\beta \coth x_0 + (1 - \beta) \tanh x_0} \frac{\ln N}{N}.$$

We see that $M_{N,\beta,x}$ is of constant order. Note that for $|x-x_0| \leq 1/10$, we have $M_{N,\beta,x} \geq N(f(x))^N$.

Since $\mathbb{P}(Z_{N,x} > 0)$ is monotone in $x$ we can assume without loss of generality that $\varepsilon_N \leq N^{-2/3}$.

With this assumption, we have for $x = x_c \pm \varepsilon_N$,

$$(x - x_0)^2 = (x_c \pm \varepsilon_N - x_0)^2 = o(1/N)$$

and thus

$$f(x) = f(x_0) - f'(x_0)(x - x_0) + o(1/N).$$

Therefore, $M_{N,\beta,x_c,\varepsilon_N} \to 0$. Combined with (7), it gives that $E Z_{N,x_c,\varepsilon_N} \to 0$, yielding (1). \hfill \Box
We next turn to prove \([22]\). To this end, we need to revise the definition of a good path. Let us first have a brief discussion in order to motivate the modification of the notation of good. Since there would be odd and even number of updates at \(\beta N\) and \((1 - \beta)N\) many positions respectively, the “typical” length of a path is \(\gamma N\) where \(\gamma\) is given by

\[
\gamma = \beta x_0 \coth x_0 + (1 - \beta)x_0 \tanh x_0 = x_0 f'(x_0).
\]

It will be useful to extend the continuous model introduced in the proof of (Case 3) Lemma 2.7 to the general case. For \(1 \leq i \leq \beta N\), let \(U_i\)’s be i.i.d. random variables according to \(F_1\), and for \(\beta N + 1 \leq i \leq N\) let \(U_i\)’s be i.i.d. random variables according to \(F_2\). Given the values of \(U_i\)’s, we put \(U_i\) copies of ‘\(i\)’s independently and uniformly on \([0,1]\). Clearly the natural ordering of their positions would give us the same update sequence (in distribution) as under the modified \(\mu_{k,\beta N}\). In other words, under the probability space for the continuous model, any event that only concerns the update sequence (but not the positions of the copies of integers) would have the same probability as under the modified \(\mu_{k,\beta N}\). Let us also define a function \(g(t)\) as

\[
g(t) = \beta \frac{\sinh(x_0 t) \cosh(x_0 (1 - t))}{\sinh x_0} + (1 - \beta) \frac{\sinh(x_0 (1 - t)) \sinh(x_0 t)}{\cosh x_0}.
\]

It follows from a straightforward calculation (analogous to \([21]\)) that \(g(t)N\) is the expected Hamming distance travelled in time \(t\) in the aforementioned continuous model (i.e., \(g(t)N\) is the expected number of coordinates that has appeared an odd number of times up to \(t\)). For a pair of vertices \(u\) and \(v\), it is useful to introduce the notion \(H'(u,v)\), which is defined to be the Hamming distance restricted to the first \(\beta N\) coordinates (i.e., the number of the first \(\beta N\) coordinates at which \(u\) differs from \(v\)).

**Definition 3.1** (general case). Let \(\varepsilon > 0\) be a small number to be specified and set \(\varepsilon_4 = \varepsilon^{1/8}\). We call a path \(v_0, v_1, \ldots, v_L\) (or the corresponding associated sequence) [good](#) if it satisfies the following:

(a) The number of occurrences of the first \(\beta N\) coordinates lies within

\[
[\beta x_0 \coth x_0 (1 - \varepsilon), \beta x_0 \coth x_0 (1 + \varepsilon)]N
\]

and the number of occurrences of the last \((1 - \beta)N\) coordinates lies within

\[
[(1 - \beta)x_0 \tanh x_0 (1 - \varepsilon), (1 - \beta)x_0 \tanh x_0 (1 + \varepsilon)]N.
\]

(b) For \(|i - j| = 1, 2, 3\) we have

\[
H(v_i, v_j) = |i - j|, \text{ if } |i - j| = 1, 2, 3;
\]

(c) For \(|i - j| > 3\) we have

\[
H'(v_i, v_j) = |i - j| \text{ or } |i - j| - 2, \text{ if } 4 \leq |i - j| \leq N^{1/4};
\]

\[
H'(v_i, v_j) \leq (1/2 + \varepsilon_1)\beta N, \text{ if } |i - j| \leq |\gamma(1/2 + \varepsilon)|N;
\]

\[
H'(v_i, v_j) \geq (1/2 + \varepsilon_1)\beta N, \text{ if } |i - j| \geq \gamma(1/2 + \varepsilon_2)N;
\]

\[
H(v_i, v_j) \geq \frac{2g(1/2)|i - j|}{\gamma + \varepsilon_4}, \text{ if } N^{1/4} < |i - j| \leq \gamma(1/2 + \varepsilon_2)N.
\]

(d) Let \(D(v_0, v_i)\) be the number of updates of the first \(\beta N\) coordinates in the first \(i\) steps, and \(D(v_L - i, v_L)\) be the number of updates of the first \(\beta N\) coordinates in the last \(i\) steps. Then both \(D(v_0, v_i)\) and \(D(v_L - i, v_L)\) are \(\leq \left(\frac{\beta \coth x_0}{\beta \coth x_0 (1 - \beta) \tanh x_0} + \varepsilon_4\right)i \leq \delta i\) for any \(i \leq L/2\).
As in the antipodal case, it is clear that a good path is self-avoiding. In addition, we have \( L \in [\gamma(1 - \varepsilon)N, \gamma(1 + \varepsilon)N] \).

**Lemma 3.2.** There exists an \( \iota > 0 \) such that for all \( |x - x_0| \leq \iota \) and for all fixed \( \varepsilon \in (0, \iota) \) we have \( \mathbb{E}Z_{N,x,*} \geq C'_1 M_{N,\beta,x} \), where \( C'_1 > 0 \) depends only on \( \varepsilon \).

**Proof.** The proof is highly similar to that of Lemma [2.7] and thus we only give a brief sketch of the arguments.

Recall the continuous model introduced preceding to Definition [3.1] (and recall that we also refer to a sequence of numbers to a sequence of updates, where each number is interpreted as a coordinate and corresponds to flipping the value at the coordinate). We will show that the set of good sequences is bounded below by a constant under this (modified) continuous model.

We first observe that Properties (a) and (c) can be satisfied by a random sequence with probability tending to 1 as \( N \to \infty \). This can be derived from the concentration of \( T_I \) (which corresponds to \( |i - j| \), \( O_I \) (which corresponds to \( H(v_i, v_j) \)) and \( O'_I \) (the number of coordinates in the first \( \beta N \) coordinates that are updated odd number of times in a time interval \( I \), corresponds to \( H'(v_i, v_j) \)) for all intervals \( I \) with \( |I| \geq N^{-5/6} \) in the continuous model. The proof is highly similar to that of Lemma [2.7] and thus details are omitted. In addition, we claim that Properties (b) and (d) can be satisfied simultaneously with probability bounded from below. Altogether, this would imply the desired bound in the lemma.

Therefore, it remains to verify the aforementioned claim. To this end, we show that the update sequence \((A_1, \ldots, A_L)\) can be obtained as a two-step procedure, in each of which one property can be satisfied. For convenience, write \( L_1 = \sum_{i=1}^{\beta N} U_i \) and \( L_2 = \sum_{i=\beta N+1}^N U_i \).

First, conditioned on the values of \( U_i \)’s for each \( i \) and Property (a), we choose \( L_1 \) indices \( i_1 < i_2 < \cdots < i_{L_1} \) uniformly from \( 1, 2, \cdots, L \) and call them type 1. For convenience, denote by \( I \) the collection of type 1 vertices. Let \( j_1 < j_2 < \cdots < j_{L_2} \) be the rest of the indices and call them type 2. Denote by \( \mathcal{E} \) the following event:

\[
|\{1, \ldots, i\} \cap I|, |\{L - i + 1, \cdots, L\} \cap I| \leq \left( \frac{\beta \coth x_0}{\beta \coth x_0 + (1 - \beta) \tanh x_0} + \varepsilon \right) i \text{ for all } 1 \leq i \leq L/2.
\]

We wish to lower bound \( \mathbb{P}(\mathcal{E}) \). To this end, for each \( 1 \leq i \leq L \) let \( T_i = 1_{\{i \text{ is of type 1}\}} \). Note that \( T_1, T_2, \cdots, T_L \) can be viewed as a sample without replacement from \( L_1 \) 1’s and \( L_2 \) 0’s. By [10, Theorem 4], we have for any \( n \),

\[
\mathbb{P}\left( \sum_{i=1}^n T_i \geq \frac{L_1}{L_1 + L_2} + \varepsilon \right) \leq \exp(-2n \varepsilon^2),
\]

and

\[
\mathbb{P}\left( \sum_{i=L-n+1}^L T_i \geq \frac{L_1}{L_1 + L_2} + \varepsilon \right) \leq \exp(-2n \varepsilon^2).
\]

By a union bound over \( M \leq n \leq \frac{L}{2} \) (where \( M \) depending only on \( \varepsilon \) is chosen later), we get

\[
\mathbb{P}(\mathcal{E}_1) \geq 1 - \frac{2 \exp(-2\varepsilon^2 M)}{1 - \exp(-2\varepsilon^2)},
\]

where

\[
\mathcal{E}_1 = \left\{ \frac{\sum_{i=1}^n T_i}{n} \leq \frac{L_1}{L_1 + L_2} + \varepsilon \text{ and } \frac{\sum_{i=L-n+1}^L T_i}{n} \leq \frac{L_1}{L_1 + L_2} + \varepsilon \text{ for any } M \leq n \leq \frac{L}{2} \right\}.
\]
Let $K$ be all pairs $(k_1, k_2)$ such that $\frac{M-k_1}{M} \leq \frac{L_1}{L_1+L_2} + \varepsilon$ and $\frac{M-k_2}{M} \leq \frac{L_1}{L_1+L_2} + \varepsilon$. For $(k_1, k_2) \in E$, define events

$$\mathcal{E}_2(k_1) = \{T_i = 0 \text{ for } 1 \leq i \leq k_1\} \cap \{T_i = 1 \text{ for } k_1 + 1 \leq i \leq M\},$$

$$\mathcal{E}_3(k_2) = \{T_i = 0 \text{ for } L - k_2 + 1 \leq i \leq L\} \cap \{T_i = 1 \text{ for } L - M + 1 \leq i \leq L - k_2\}.$$ 

We get that

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2(k_1) \cap \mathcal{E}_3(k_2)) = \binom{M}{k_1}^{-1} \binom{M}{k_2}^{-1} \mathbb{P}(\sum_{i=1}^{M} T_i = M - k_1, \sum_{i=L-M+1}^{L} T_i = M - k_2, \mathcal{E}_1)$$

$$\geq 2^{-2M} \mathbb{P}(\sum_{i=1}^{M} T_i = M - k_1, \sum_{i=L-M+1}^{L} T_i = M - k_2, \mathcal{E}_1). \quad (34)$$

Note that for all $(k_1, k_2) \in K$, on the event $\mathcal{E}_2(k_1) \cap \mathcal{E}_3(k_2)$ we have that $\frac{\sum_{i=1}^{n} T_i}{n} \leq \frac{L_1}{L_1+L_2} + \varepsilon$ and $\frac{\sum_{i=L-n+1}^{L} T_i}{n} \leq \frac{L_1}{L_1+L_2} + \varepsilon$ for all $1 \leq n \leq M$. Therefore, we have

$$\mathcal{E} \supseteq \bigcup_{(k_1, k_2) \in K} \mathcal{E}_1 \cap \mathcal{E}_2(k_1) \cap \mathcal{E}_3(k_2).$$

In addition, we observe that

$$\mathcal{E}_1 = \bigcup_{(k_1, k_2) \in K} \mathcal{E}_1 \cap \{\sum_{i=1}^{M} T_i = M - k_1\} \cap \{\sum_{i=L-M+1}^{L} T_i = M - k_2\}.$$ 

Summing (34) over all $(k_1, k_2) \in K$ and using the preceding two displays, we deduce that

$$\mathbb{P}(\mathcal{E}) \geq 2^{-2M} \left(1 - \frac{2 \exp(-2\varepsilon^2 M)}{1 - \exp(-2\varepsilon^2)}\right). \quad (35)$$

Provided Property (a) we have

$$\frac{L_1}{L_1+L_2} + \varepsilon \leq \frac{\beta x_0 \coth x_0 (1+\varepsilon) N}{\beta x_0 \coth x_0 (1-\varepsilon) N + (1-\beta) x_0 \tanh x_0 (1-\varepsilon) N} + \varepsilon \leq \frac{\beta \coth x_0}{\beta \coth x_0 + (1-\beta) \tanh x_0} + \varepsilon_4$$

for sufficiently small $\varepsilon$. By (35) and choosing $M$ depending on $\varepsilon$, we have proved that Property (d) is satisfied with probability bounded by a number depending only on $\varepsilon$ from below.

Now, conditioned on the previous step, let $(B_1, B_2, \ldots, B_{L_1})$ be a sequence uniformly at random subject to $\{|1 \leq j \leq L_1 : B_j = i| = U_i$ for $i = 1, 2, \ldots, \beta N$, and independently let $(C_1, C_2, \ldots, C_{L_2})$ be a sequence uniformly at random subject to $\{|1 \leq j \leq L_2 : C_j = i| = U_i$ for $i = \beta N + 1, \beta N + 2, \ldots, N$. Let $A_{ik} = B_k$ for $1 \leq k \leq L_1$ and $A_{jk} = C_k$ for $1 \leq k \leq L_2$ (recall that $i_k$’s and $j_k$’s are sampled in the previous step). Similar to the proof of Case 1 for Lemma 2.7, we have that with high probability (with respect to $U_i$’s), we have that $B_i \neq B_{i+1}$ and $B_i \neq B_{i+2}$ hold for all $1 \leq i \leq L_1$ with at least constant probability; and with high probability (with respect to $U_i$’s), we have $C_i \neq C_{i+1}$ and $C_i \neq C_{i+2}$ hold for all $1 \leq i \leq L_2$ with at least constant probability. In addition, note that $B_i \neq B_{i+1}, B_i \neq B_{i+2}$ for all $1 \leq i \leq L_1$ and $C_i \neq C_{i+1}, C_i \neq C_{i+2}$ for all $1 \leq i \leq L_2$ together would imply that $A_i \neq A_{i+1}$ and $A_i \neq A_{i+2}$ for all $1 \leq i \leq L,$
which corresponds to Property (b). By the (conditional) independence of \((B_1, B_2, \ldots, B_L)\) and 
\((C_1, C_2, \ldots, C_{L_2})\), we see that Property (b) can be satisfied with probability bounded by a constant 
from below.

Finally, it is clear that the update sequence \((A_1, \ldots, A_L)\) obtained from the aforementioned two-
step procedure has the same distribution as our original definition. This completes the verification 
of our claim.

\[\text{Lemma 3.3.} \quad \text{There exists } C'_2, \iota > 0 \text{ such that for all } |x - x_0| < \iota, \text{ all fixed } \varepsilon \in (0, \iota) \text{ and any good path } P = v_0, v_1, \ldots, v_L \text{ we have} \]

\[\sum_{d=0}^{L/2} \sum_{d_1 + d_2 = d} F(v_{d_1}, v_{L-d_2}) \leq C'_2 N f(x)^N \sim N(f(x))^{N-1} f'(x).\]

\[\text{Proof.} \quad \text{Within the proof, the number } C \text{ denote for a numerical constant whose value may depend on} \]
\((C'_2, \iota, \varepsilon)\) and could vary from line by line. We continue to use notations specified in Definition 3.1.

Note that the Hamming distance between \(v_{d_1}\) and \(v_{L-d_2}\) is at least \(\beta N - D(v_0, v_{d_1}) - D(v_{L-d_2}, v_L)\), which is at least \(\beta N - \delta d\) (see definition of \(\delta\) in Property (d) of Definition 3.1). Continue to let 
\((Y_0, \ldots, Y_L)\) be distributed as order statistics of \((L + 1)\) i.i.d. uniform variables on \([0, x]\). Note that 
the distribution of \(Y_{L-d_2} - Y_{d_1}\) does not depend on \((d_1, d_2)\) provided the value of \(d = d_1 + d_2\).

Therefore by (7) and Lemma 2.12 we have for \(d \geq 1\)

\[F(v_{d_1}, v_{L-d_2}) \leq \mathbb{E}((\sinh y)^{\beta N - \delta d}(\cosh y)^{(1-\beta)N + \delta d}) | \gamma = Y_{L-d_2} - Y_{d_1} \]

\[= \mathbb{E}((\sinh y)^{\beta N - \delta d}(\cosh y)^{(1-\beta)N + \delta d}) | \gamma = y - y_d. \tag{36}\]

Note that \(x - Y_d\) is the \((L - d)\)th order statistic of \((L - 1)\) i.i.d. uniform variables in \([0, x]\), thus \(x - Y_d\) 
has a Beta\((L - d, d)\) distribution. Thus, the density of \(x - Y_d\) is \(\frac{1}{x} \frac{(L-1)!}{(L-d-1)!(d-1)!} \)
for \(y \in [0, x]\). Therefore

\[\mathbb{E}((\sinh y)^{\beta N - \delta d}(\cosh y)^{(1-\beta)N + \delta d}) | \gamma = x - X_d \]

\[= \int_0^x ((\sinh y)^{\beta N - \delta d}(\cosh y)^{(1-\beta)N + \delta d}) \frac{1}{x} \frac{(L-1)!}{(L-d-1)!(d-1)!} \]

\[dy. \tag{37}\]

We split the above integral into two parts according to whether \(y\) is smaller or greater than \(\frac{x}{2}\), and 
denote by \(\mathcal{J}_1(d)\) the integral over \([0, \frac{x}{2}]\) and by \(\mathcal{J}_2(d)\) the integral over \([\frac{x}{2}, x]\). On one hand, by 
Lemma 2.12 we have

\[((\sinh y)^{\beta N - \delta d}(\cosh y)^{(1-\beta)N + \delta d})' \leq ((\sinh y)^{\beta N - \delta \frac{L}{2}}(\cosh y)^{(1-\beta)N + \delta \frac{L}{2}})' \]

\[= (\sinh y)^{\beta N - \delta \frac{L}{2} - 1}(\cosh y)^{(1-\beta)N + \delta \frac{L}{2} - 1}((\beta N - \delta \frac{L}{2})\cosh y + ((1-\beta)N + \delta \frac{L}{2})\sinh y). \]

Since \(\cosh y \leq \cosh(\frac{x}{2})\) and \(\sinh y \leq \sinh(\frac{x}{2})\) for \(y \in [0, \frac{x}{2}]\), we get that

\[((\sinh y)^{\beta N - \delta d}(\cosh y)^{(1-\beta)N + \delta d})' \leq CN(\sinh(\frac{x}{2}))^{\beta N - \delta \frac{L}{2} - 1}(\cosh(\frac{x}{2}))^{(1-\beta)N + \delta \frac{L}{2} - 1} \]

\[\leq CN(\sinh(\frac{x}{2}))^{\beta N - \delta \frac{7(1+\varepsilon)N}{2}}(\cosh(\frac{x}{2}))^{(1-\beta)N + \delta \frac{3(1+\varepsilon)N}{2}}, \]

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where the last inequality follows from Property (a) of Definition 3.1. Therefore

\[
\frac{b}{d} \sum_{d=1}^{L} (d+1)J_{1}(d) \leq CN^{3}(\sinh(\frac{x}{2}))^{\beta N - \delta \gamma(1+2\epsilon)N} (\cosh(\frac{x}{2}))^{(1-\beta)N + \delta \gamma(1+\epsilon)N} \\
\quad \leq CN^3 r^N (\sinh x)^{\beta N} (\cosh x)^{(1-\beta)N},
\]

where \(0 < r < 1\) is a constant that depends only on \(\beta\). Here we used a fact (by brutal force computation) that

\[
\frac{(\sinh(\frac{x}{2}))^{\beta - \delta \gamma(1+2\epsilon)N}}{(\sinh(\frac{x}{2}))^{(1-\beta)N + \delta \gamma(1+\epsilon)N}} < 1.
\]

On the other hand, for \(y \in [\frac{x}{2}, x]\) we have \(\coth y \leq \coth(\frac{x}{2})\) and \(\tanh y \leq \tanh(x)\). Combined with the assumption that \(|x - x_0| < \epsilon\), it yields that

\[
((\sinh y)^{\beta N - \delta d}(\cosh y)^{(1-\beta)N + \delta d}) \frac{1}{x} = (\sinh y)^{\beta N - \delta d}(\cosh y)^{(1-\beta)N + \delta d}((\beta N - \delta d) \coth y + ((1-\beta)N + \delta d) \tanh y)\frac{1}{x} \\
\quad \leq CN (\sinh y)^{\beta N - \delta d}(\cosh y)^{(1-\beta)N + \delta d} \\
\quad \leq CN ((\sinh y)^{\beta N}(\cosh y)^{(1-\beta)N})(\coth y)^{\delta(d-2)}.
\]

Therefore, the integrand of \((37)\) is at most \(CN((\sinh y)^{\beta N}(\cosh y)^{(1-\beta)N})\varphi(x, y, d, \beta, N, L)\) for \(y \in [\frac{x}{2}, x]\), where

\[
\varphi(x, y, d, \beta, N, L) = (\coth y)^{\delta(d-2)} \left(\frac{y}{x}\right)^{L-d-1} (1 - \frac{y}{x})^{d-1} \frac{(L-1)!}{(L-d-1)!(d-1)!}.
\]

Therefore, we can derive that

\[
\frac{b}{d} \sum_{d=1}^{L} (d+1)J_{2}(d) \leq CN \int_{\frac{x}{2}}^{x} ((\sinh y)^{\beta N}(\cosh y)^{(1-\beta)N}) \sum_{d=1}^{L} (d+1)\varphi(x, y, d, \beta, N, L) \, dy.
\]

Note that for \(d = 1\), we have

\[
(d+1)\varphi(x, y, d, \beta, N, L) = (\tanh y)^{\delta(\frac{y}{x})} L - 2(L - 1) \leq C(\frac{y}{x})^L (L - 1).
\]

In addition, for \(d \geq 2\), we have

\[
(d+1)\varphi(x, y, d, \beta, N, L) \\
\leq (1 - \frac{y}{x} \coth y)^{\delta(d-2)} (1 - \frac{y}{x})^{d-2} \left(\frac{y}{x}\right)^{L-d-1} \frac{(L-3)!}{(L-d-1)!(d-2)!} \frac{(L-1)(L-2)(d+1)}{(d-1)!} \\
\leq 3(1 - \frac{y}{x}) L^2 \cdot [((\coth y)^{\delta})(1 - \frac{y}{x})^{d-2} \left(\frac{y}{x}\right)^{L-d-1} \frac{(L-3)!}{(L-d-1)!(d-2)!}].
\]

Observing that the second part of the right hand side of the preceding inequality is a binomial term, we get that

\[
\frac{b}{d} \sum_{d=2}^{L} L (d+1)\varphi(x, y, d, \beta, N, L) \leq 3(1 - \frac{y}{x}) L^2 \cdot ((\coth y)^{\delta}(1 - \frac{y}{x}) + \frac{y}{x}) L^{-3}.
\]
Applying (40) and (41) and using Property (a) of Definition 3.1, we get that
\[
\frac{L}{2} \sum_{d=1}^{L} (d+1) \varphi(x, y, d, \beta, N, L) \leq C \left( \frac{y}{x} \right)^{\gamma(1-2\epsilon)N} (L - 1) + 3L^2 \left( 1 - \frac{y}{x} \right) ((\coth y)^{\delta} (1 - \frac{y}{x}) + \frac{y}{x})^\gamma(1+\epsilon)N.
\]

Therefore (39) translates to
\[
\frac{L}{2} \sum_{d=1}^{L} (d+1) J_2(d)
\]
\[
\leq CN \int_{\frac{y}{x}}^{\infty} ((\sinh y)^{\beta N} (\cosh y)^{(1-\beta)N}) (C \left( \frac{y}{x} \right)^{\gamma(1-2\epsilon)N} (L - 1) + 3L^2 \left( 1 - \frac{y}{x} \right) ((\coth y)^{\delta} (1 - \frac{y}{x}) + \frac{y}{x})^\gamma(1+\epsilon)N) dy
\]
For convenience, write
\[
\psi_1(y) = \frac{((\sinh y)^{\beta} (\cosh y)^{1-\beta}) \cdot \left( \frac{y}{x} \right)^{\gamma(1-2\epsilon)}}{(\sinh x)^{\beta} (\cosh x)^{1-\beta}},
\]
\[
\psi_2(y) = \frac{((\sinh y)^{\beta} (\cosh y)^{1-\beta}) \cdot ((\coth y)^{\delta} (1 - \frac{y}{x}) + \frac{y}{x})^\gamma(1+\epsilon)}}{(\sinh x)^{\beta} (\cosh x)^{1-\beta}},
\]
\[
\psi_3(y) = \beta \ln \sinh y + (1 - \beta) \ln \cosh y.
\]
We show that both \( \psi_1(y) \) and \( \psi_2(y) \) satisfy \( \psi(y) = 1 \) and \( \ln \psi(y) \leq -K(x - y) \) for \( y \in [\frac{y}{x}, x] \), where \( K > 0 \) is a constant that only depends on \( \beta \). As a result, both \( \int_{\frac{y}{x}}^{x} N(\psi(y))^N dy \) and \( \int_{\frac{y}{x}}^{x} N^2(x - y)(\psi(y))^N dy \) would stay bounded as \( N \to \infty \), so that (42) is bounded by \( CN (\sinh x)^{\beta N} (\cosh x)^{(1-\beta)N} \).

It is relatively easy to check \( \psi_1(y) \) has the desired properties since it is increasing in \( y \), so we focus on \( \psi_2(y) \). Note that
\[
\ln \psi_2(y) = \psi_3(y) - \psi_3(x) + \gamma(1 + \epsilon) \ln((\coth y)^{\delta} (1 - \frac{y}{x}) + \frac{y}{x}).
\]
We compute the derivatives of \( \psi_3(y) \) as follows:
\[
\psi'_3(y) = \beta \coth y + (1 - \beta) \tanh y,
\]
\[
\psi''_3(y) = \beta (1 - (\coth y)^2) + (1 - \beta) (1 - (\tanh y)^2).
\]
Since \( \coth y \geq \coth x \geq \left( \frac{1-\beta}{\beta} \right)^{\frac{1}{\delta}} \) for \( y \leq x \), we have \( \psi''_3(y) \) is increasing in \( y \). By Taylor’s theorem (Lagrange form of the remainder) we find that for some \( \xi \in [y, x] \),
\[
\psi_3(y) = \psi_3(x) + \psi'_3(x)(y - x) + \frac{\psi''_3(\xi)}{2} (y - x)^2
\]
\[
\leq \psi_3(x) + \psi'_3(x)(y - x) + \frac{\psi''_3(x)}{2} (y - x)^2.
\]
Now set \( C(y) := (\coth y)^{\delta} - 1, \theta(y) := 1 - \frac{y}{x} \). Note that
\[
(\coth y)^{\delta} (1 - \frac{y}{x}) + \frac{y}{x} = 1 + (1 - \frac{y}{x}) C(y) = 1 + \theta(y) C(y).
\]

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Clearly $0 \leq \theta(y)C(y) \leq \theta(\frac{x}{y})C(\frac{x}{y}) = \frac{1}{2}((\coth \frac{x}{y})^\delta - 1) \leq 0.75$. Since $\ln(1 + t) \leq t - \frac{t^2}{2}$ for $0 \leq t \leq 0.75$, we get
\[
\ln((\coth y)^\delta (1 - \frac{y}{x} + \frac{y}{x})) \leq \theta(y)C(y) - \frac{(\theta(y))^2}{3}(C(y))^2.
\]
Therefore,
\[
\ln \psi_2(y) \leq -\psi_2'(x)(1 - \frac{y}{x}) + \frac{\psi_2''(x)x^2}{2}(1 - \frac{y}{x})^2 + \gamma(1 + \varepsilon)\theta(y)C(y) - \gamma(1 + \varepsilon)(\theta(y))^2(\frac{C(y)}{3})^2
\]
\[
= \theta(y)\gamma(1 + \varepsilon)[-\frac{(\theta(y))^2}{3}(C(y))^2 + C(y) - \frac{1}{\gamma(1 + \varepsilon)}(\psi_2'(x) - \frac{\psi_2''(x)x^2}{2}\theta(y))].
\]
Set $c := \frac{\psi_2''(x_0)x_0^2}{2\gamma}$. We wish to show that the factor in the square bracket is less than some constant $-\eta$ where $\eta > 0$ only depends on $\beta$, i.e.,
\[
-\frac{(\theta(y))^2}{3}(C(y))^2 + C(y) - 1 + c\theta(y) \leq -\eta.
\]
Since $|x - x_0| < \varepsilon$, $\varepsilon < \iota$, and $\iota$ can be made arbitrarily small, we only need to show that for some constant $\eta_1 > 0$ which only depends on $\beta$,
\[
-\frac{(\theta(y))^2}{3}(C(y))^2 + C(y) - 1 + c\theta(y) \leq -\eta.
\]
Let $q(s) := -\frac{(\theta(y))^2}{3}s^2 + s - 1 + c\theta(y)$. Solving the quadratic equation $q(s) = 0$ with respect to $s$, we get the smaller root
\[
r(y) := -1 + \sqrt{1 - \frac{4\theta(y)}{3}(1 - c\theta(y))}.
\]
Note that $q'(s) = -\frac{2\theta(y)}{3}s + 1$, and $q'(C(y)) = -\frac{2\theta(y)}{3}C(y) + 1 \geq 0.5$. So if we showed for $\frac{x}{y} \leq y \leq x$, $C(y) \leq r(y) - \eta_2$ for some constant $\eta_2 > 0$ which only depends on $\beta$, then $q'(C(y)) + \eta_2 = q'(C(y)) - \frac{2\theta(y)}{3}\eta_2 \geq 0.5 - \frac{2}{3}\eta_2$, so that $0 = q(r(y)) \geq q(C(y) + \eta_2) \geq q(C(y)) + \eta_2(0.5 - \frac{2}{3}\eta_2)$ and we can take $\eta_2 = \eta_2(0.5 - \frac{2}{3}\eta_2)$.

Now if $c < \frac{1}{3}$, $r(y)$ is convex in $y$ (if $c > \frac{1}{3}$, $r(y)$ is concave in $y$). This can be seen by observing that $r = -1 + \sqrt{1 - \frac{4\theta(y)}{3}(1 - c\theta(y))}$ is the inverse function of $\theta = \frac{r - 1}{r - c}$, whose properties such as monotonicity and convexity are not hard to justify. Therefore
\[
r(y) \geq r'(\frac{3x}{4})(y - \frac{3x}{4}) + r(\frac{3x}{4}) := t(y)
\]
where $t(y)$ can be computed as
\[
t(y) = -\frac{1}{x}(\frac{120}{\sqrt{3c + 24}} - 24)(y - \frac{3x}{4}) + 6 - \sqrt{3c + 24}.
\]
Since $C(y)$ is also convex in $y$, we only need to have $t(x) \geq C(x) + \eta_2$ and $t(\frac{x}{y}) \geq C(\frac{x}{y}) + \eta_2$, i.e.,
\[
-\frac{30}{\sqrt{3c + 24}} + 12 - \sqrt{3c + 24} \geq (\coth x)^\delta - 1 + \eta_2
\]
\[\text{(43)}\]
and
\[ \frac{30}{\sqrt{3c + 24}} - \sqrt{3c + 24} \geq (\coth \frac{x}{2})^\beta - 1 + \eta_2. \] (44)

If \( c = \frac{1}{3}, \) \( r(y) \equiv 1, \) this can be incorporated in either the case for \( c < \frac{1}{3} \) or the case for \( c > \frac{1}{3}. \) If \( c > \frac{1}{3}, \) \( r(y) \) is concave in \( y, \) so we only need to have \( r(x) \geq C(x) + \eta_2 \) and \( r(\frac{y}{2}) \geq C(\frac{y}{2}) + \eta_2, \) i.e.,
\[ 1 \geq (\coth x)^\delta - 1 + \eta_2 \] (45)
and
\[ 3 - \sqrt{3(c + 1)} \geq (\coth \frac{x}{2})^\delta - 1 + \eta_2. \] (46)

All of the (43), (44), (45) and (46) boil down to comparisons of constants which only involve \( x_0 \) (note that \( |x - x_0| < \iota, \varepsilon < \iota, \) and \( \iota \) can be made arbitrarily small), so we have shown that (42) is bounded by \( CN(\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N}. \)

Combining (36), (37), (38) and (42), noting that when \( d = 0, \) \( F(v_0, v_L) \leq N(f(x))^{N-1} f'(x), \) we conclude that \( \sum_{d=0}^{L-d_0} \sum_{d_1+\cdots+d_2=d} F(v_{d_1}, v_{L-d_2}) \leq C'_{2}N(\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N}. \)

**Lemma 3.4.** There exists \( C'_3, \iota > 0 \) such that \( \sum_{i=1}^{L} F(v_j, v_{j+i}) \leq 1 + \frac{C'}{N} \) for all \( |x - x_0| < \iota, \) all fixed \( \varepsilon \in (0, \iota), \) any good path \( P, \) and any \( j. \)**

**Proof.** The proof can be carried out in the same manner as that of Lemma 2.10 except that the role of \( \alpha + \varepsilon_3 \) in Case (e) is now replaced by \( \frac{\iota + \varepsilon_3}{\varepsilon_3(1/2)}. \) We thus omit the details. \( \square \)

**Corollary 3.5.** There exists \( C'_4, \iota > 0 \) such that for all \( |x - x_0| < \iota \) and all fixed \( \varepsilon \in (0, \iota) \)
\[ \mathbb{E}Z_{N,x,good}^2 \leq (C'_4 N(\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N} + C'_3) N(\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N}. \]

**Proof.** Plugging bounds in Lemmas 3.3 and 3.4 into (29) and then (28), we get that
\[ \mathbb{E}(Z_{N,x,good}^2 \mid A_P) \leq \sum_{k, i_1, i_2, ..., i_k} \prod_{\ell=1}^{k+1} F(v_{i_{\ell-1}}, v_{i_{\ell}}) \leq (C'_2 N(\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N} + 1 + \frac{C'_3}{N})^{1+\varepsilon_3} \]
\[ \leq (C'_2 N(\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N} + 1) e^{C'_3(1+\varepsilon_3)} \] (47)

Plugging the preceding inequality into (26) and applying the inequality
\[ \sum_{P \in \mathcal{P}} \mathbb{P}(A_P) \leq \mathbb{E}Z_{N,x,good}^2 \leq N(\sinh x)^{\beta N}(\cosh x)^{(1-\beta)N}(\beta \coth x + (1 - \beta) \tanh x) \]
(here the last inequality follows from Corollary 2.5), we complete the proof of the corollary. \( \square \)

**Proposition 3.6.** There exists \( 0 \leq K' < 1 \) such that, if \( \lim_{N \to \infty} \mathbb{P}(Z_{N,x+\varepsilon_N} > 0) \geq C \) for some constant \( C \geq 0, \) then whenever \( N \varepsilon_N \to \infty \) we have
\[ \lim_{N \to \infty} \mathbb{P}(Z_{N,x+\varepsilon_N} > 0) \geq 1 - (1 - C)K'. \]
Proof. The basic idea is the same as Proposition 2. Fix a large integer $M$. We first choose vertices $A_1, \ldots, A_M, B_1, \ldots, B_M$, and $C_1, \ldots, C_M, D_1, \ldots, D_M$ such that for $1 \leq i \leq M$:

- The only coordinate that $A_{i-1}$ and $A_i$ differs is $a_i$. The only coordinate that $B_{i-1}$ and $B_i$ differs is $b_i$. The only coordinate that $C_{i-1}$ and $C_i$ differs is $c_i$. The only coordinate that $D_{i-1}$ and $D_i$ differs is $d_i$ (set $A_0 = C_0 = 0$ and $B_0 = D_0 = \beta$).
- All of the $4M$ coordinates $a_i, b_i, c_i$ and $d_i$ are different and are among the first $\beta N$ coordinates.
- $X(A_i), X(C_i) \in \left[\frac{(i-1)\varepsilon-N}{4M}, \frac{i\varepsilon-N}{4M}\right]$ and $X(B_i), X(D_i) \in \left[x - \frac{i\varepsilon-N}{4M}, x - \frac{(i-1)\varepsilon-N}{4M}\right]$.

Since $N\varepsilon_N \to \infty$, this can be achieved with probability $1 - o_N(1)$.

Now let $M_2 = \frac{(1-\beta)}{\beta}2M$, and select (arbitrary) coordinates $e_1, e_2, \cdots, e_{M_2}$ and $f_1, f_2, \cdots, f_{M_2}$ among the last $(1-\beta)N$ coordinates. Let $\tilde{H}_1$ be the $(N - 2M - M_2)$ dimensional sub-hypercube formed by $A_M$ and $B_M$ with values at coordinates $e_1, e_2, \cdots, e_{M_2}$ being 0, i.e.,

$$\tilde{H}_1 = \{ \sigma \in H_N : \sigma_{e_i} = 0 \text{ for all } 1 \leq i \leq M_2, \sigma_{a_i} = 1 \text{ for all } 1 \leq i \leq M, \sigma_{b_i} = 0 \text{ for all } 1 \leq i \leq M \}.$$ 

Similarly, let $\tilde{H}_2$ be the $(N - 2M - M_2)$ dimensional sub-hypercube formed by $C_M$ and $D_M$ with values at coordinates $f_1, f_2, \cdots, f_{M_2}$ being 0, i.e.,

$$\tilde{H}_2 = \{ \sigma \in H_N : \sigma_{f_i} = 0 \text{ for all } 1 \leq i \leq M_2, \sigma_{c_i} = 1 \text{ for all } 1 \leq i \leq M, \sigma_{d_i} = 0 \text{ for all } 1 \leq i \leq M \}.$$ 

Let $H'_2 = \tilde{H}_2 \setminus \tilde{H}_1$. Denote by $p_{\tilde{H}_1}$ and $p_{H'_2}$ the probabilities that there is an accessible path in $\tilde{H}_1$ (from $A_M$ to $B_M$) and $H'_2$ (from $C_M$ to $D_M$) respectively. Since $\tilde{H}_1$ and $H'_2$ are disjoint, by independence we get $\mathbb{P}(Z_{N,x+\varepsilon_N/2} > 0) > 1 - (1 - p_{\tilde{H}_1})(1 - p_{H'_2}) - o_N(1)$. From the construction above it is clear that we are reduced to accessibility percolation in $(N - 2M - M_2)$ dimension with $x \geq x_c + \varepsilon_N/2$, in either $\tilde{H}_1$ (from $A_M$ to $B_M$) or $H'_2$ (from $C_M$ to $D_M$). Thus,

$$p_{\tilde{H}_1} \geq \mathbb{P}(Z_{N-2M-M_2,x_c+\varepsilon_N/2} > 0) \geq C - o_N(1).$$ 

To show that $p_{H'_2}$ is bounded from below by a positive constant $1 - K'$, we only consider good path in $H_2$ (from $C_M$ to $D_M$) which updates each of coordinates $a_1$ and $b_1$ precisely once and $b_1$ is updated before $a_1$. Such paths must be contained in $H'_2$. Clearly, the number of such open paths has the second moment less than $\mathbb{E}Z^2_{N-2M-M_2,x_c+\varepsilon_N/2,good}$ and the first moment within an absolute multiplicative constant of $\mathbb{E}Z_{N-2M-M_2,x_c+\varepsilon_N/2}$. Combined with Lemma 3.2 and Corollary 3.5 it yields that $p_{H'_2} \geq 1 - K'$ for some constant $K' < 1$. This completes the proof of the proposition. □

Proof of (2): general case. Applying Proposition 3.6 recursively completes the proof of (2). □

Proof of (3): general case. For the lower bound, it suffices to consider $x = x_c - \Delta/N$. It is clear that in this case $N \sinh(x) \beta^N \cosh(x)^{1-\beta} \geq c(\Delta)$ where $c(\Delta) > 0$ depends only on $\Delta$. Applying second moment method and using Corollary 3.5 and Lemma 3.2 we obtain that

$$\mathbb{P}(Z_{N,x} > 0) \geq \mathbb{P}(Z_{N,x,good} > 0) \geq \frac{(\mathbb{E}Z_{N,x,good}^2)^2}{\mathbb{E}Z_{N,x,good}^2} \geq c_1(\Delta),$$

where $c_1(\Delta) > 0$ depends only on $\Delta$.

For the upper bound, the proof is basically the same except that the role of $\sinh(x)$ is now played by $f(x) = \sinh(x)^\beta N \cosh(x)^{1-\beta} N$. □
4 Proof of Theorem 1.2

We first provide the proof for the case that $K \to \infty$ as $N \to \infty$. The upper bound follows from a straightforward union bound, i.e.,

$$
P(M_N \geq (1+\varepsilon)N\sqrt{2\ln 2}) \leq 2^N P(X_\sigma \geq (1+\varepsilon)N\sqrt{2\ln 2}) \leq e^{-c(\varepsilon)N},
$$

for $c(\varepsilon) > 0$ depending only on $\varepsilon$. For the lower bound, note that the covariance of any two vertices, is a decreasing function in $K$. Therefore, by Slepian’s lemma [20] we have $M_{N,K}$ is stochastically dominated by $M_{N,K'}$ provided $K < K'$. So we can assume without loss of generality that $\frac{N}{K} \to \infty$.

We chop up the $N$ coordinates into $\frac{N}{K}$ intervals of length $K$. We use the following algorithm to find a $\hat{\sigma}$ such that $X_{\hat{\sigma}}$ is large. First we let $\hat{\sigma}_1 = \cdots = \hat{\sigma}_K = 0$. In what follows, we set $\sigma_i = \hat{\sigma}_i$ for $1 \leq i \leq K$.

Next, we inductively run the following procedure for $j = 1, \ldots, \lfloor N/K \rfloor - 1$:

- Choose $(\hat{\sigma}_{jK+1}, \hat{\sigma}_{jK+2}, \cdots, \hat{\sigma}_{(j+1)K})$ such that

  $$
  \max_{(\sigma_{jK+1}, \ldots, \sigma_{(j+1)K}) \in \{0,1\}^K} \sum_{i=2}^{K+1} Y_{i+(j-1)K}(\sigma_{i+(j-1)K}, \ldots, \sigma_{i+jK-1}) = \sum_{i=2}^{K+1} Y_{i+(j-1)K}(\hat{\sigma}_{i+(j-1)K}, \ldots, \hat{\sigma}_{i+jK-1})
  $$

- Set for the future $\sigma_i = \hat{\sigma}_i$ for $jK + 1 \leq i \leq (j+1)K$.

Finally, set $\hat{\sigma}_i = 0$ for $\lfloor N/K \rfloor K + 1 \leq i \leq N$. Note that for each $j = 1, \ldots, \lfloor N/K \rfloor - 1$, we have $\{\sum_{i=2}^{K+1} Y_{i+(j-1)K}(\sigma_{i+(j-1)K}, \ldots, \sigma_{i+jK-1}) : (\sigma_{jK+1}, \ldots, \sigma_{(j+1)K}) \in \{0,1\}^K\}$ behaves exactly as a binary branching random walk (BRW) of depth $K$ (see, e.g., [3] for a definition of BRW). Furthermore, this BRW is independent from all previous BRWs occurred in the iteration. By an estimate of the maximum of the BRW [3], we have that

$$
\sum_{i=2}^{K+1} Y_{i+(j-1)K}(\hat{\sigma}_{i+(j-1)K}, \ldots, \hat{\sigma}_{i+jK-1}) = \sqrt{2\ln 2} K - \frac{3}{2\sqrt{2\ln 2}} \ln K + W_j K
$$

where $W_j$’s are i.i.d. random variable with finite exponential moments. Write $W = \sum_{j=1}^{\lfloor N/K \rfloor - 1} W_j$, and write $Y = \sum_{i=2}^{\lfloor N/K \rfloor - 1} Y_{i,\hat{\sigma}_i,\ldots,\hat{\sigma}_{i+jK-1}}$. In addition, write $m_K = \sqrt{2\ln 2} K - \frac{3}{2\sqrt{2\ln 2}} \ln K$. Thus, we get that

$$
P(M_N \leq (1-\varepsilon)N\sqrt{2\ln 2}) \leq P(X_\sigma \leq (1-\varepsilon)N\sqrt{2\ln 2}) \leq P(m_K (N/K - 2) + W + Y \leq (1-\varepsilon)N\sqrt{2\ln 2}).
$$

At this point, a simple application on the concentration for $W + Y$ (which are sums of i.i.d. variables with finite exponential moments) completes the proof.

Next, we turn to the case when $K \leq K_0$ for all $N \in \mathbb{N}$. Write $m_N = (1 - \varepsilon)\sqrt{2\ln 2}N$ where $\varepsilon > 0$ is to be selected. Let $A = \{4Ki : 1 \leq i \leq N/4K\}$. For any $k \in A$, let $T_{\sigma,k} = \sum_{i=k-K+1}^{k} Y_{i,\sigma_i,\ldots,\sigma_{i+K-1}}$ and let $T_{\sigma,k}' = \sum_{i=k-K+1}^{k} Y_{i,\sigma_i',\ldots,\sigma_{i+K-1}}$ where $\sigma_i' = \sigma_i$ if and only if $i \neq k$. Thus, we have $X_\sigma - X_{\sigma'} = T_{\sigma,k} - T_{\sigma,k}'$. Define

$$
\mathcal{E}_\sigma = \{|\{k : T_{\sigma,k} \leq 10K\}| \leq \frac{N}{10K}\}.
$$
Now it is clear that we can choose $\varepsilon$ sufficiently small depending on $K_0$ such that there exists $\delta = \delta(K_0)$ satisfying
\[ \mathbb{P}(\{X_\sigma \geq \bar{m}_N\} \cap \mathcal{E}_\sigma) \leq 2^{-N-\delta N}. \]
In addition, notice that for fixed $\sigma$ we have that $\{T'_{\sigma,k} : k \in A\}$ is independent of $X_\sigma$ and of $\{T_{\sigma,k} : k \in A\}$. Denoting by $\tau = \arg \max_{\sigma \in H_N} X_\sigma$, we see that
\begin{align*}
\mathbb{P}(M_N \geq \bar{m}_N) \leq & \sum_{\sigma \in H_N} \mathbb{P}(\{X_\sigma \geq \bar{m}_N\} \cap \mathcal{E}_\sigma) + \sum_{\sigma \in H_N} \mathbb{P}(\{X_\sigma \geq \bar{m}_N\} \cap \mathcal{E}_\sigma^c \cap \{\sigma = \tau\}) \\
\leq & 2^{-\delta N} + \sum_{\sigma \in H_N} \mathbb{P}(X_\sigma \geq \bar{m}_N)(\mathbb{P}(T'_{4K} \leq 10K))^{\frac{N}{10K}} \to 0
\end{align*}
as $N \to \infty$, where we choose $\varepsilon$ a sufficiently small number depending only on $K_0$. \hfill \Box

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