A conformally invariant gauge fixing equation and a field strength for the symmetric traceless field over four dimensional conformally flat Einstein space-time

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ABSTRACT: The conformally invariant symmetric traceless field $A$ is considered on conformally flat Einstein space-time. If $d = 4$ this field possess a scalar gauge invariance. In that case, we provide a conformally invariant gauge condition which generalizes in a simple manner, on those space-time, the Eastwood-Singer gauge condition. A field strength $F$ is built upon the potential $A$, its properties are worked out in details, giving raise to a new set of conformally invariant equations.

KEYWORDS: Conformal and W Symmetry, Gauge Symmetry

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1 Introduction

Conformal symmetry occurs in many places throughout physics and has been providing important tools to theoretical and mathematical physics. It is a symmetry enjoyed in Nature by the sourceless electromagnetism (Maxwell’s equations) and the standard model of particle physics, before the spontaneous symmetry breaking and quantum corrections are taken into account. It can also arise as an approximate symmetry, e.g. for high-energies process in particle physics or in the vicinity of a critical point. Conformal invariance might even appear as an “hidden” symmetry to non-relativistic models, such as the well-know example of the hydrogen atom, and is key to a fine-grained understanding of the spectrum of such models.

Frequently conformal symmetry and gauge symmetries occurs simultaneously. In such cases, the gauge symmetry has to be fixed which, in doing so, often spoils the conformal invariance. This needs not to happen as sometimes conformal invariance can remain while fixing the gauge symmetry. This article investigates such a case.

This article provides a conformally invariant gauge fixing equation and a field strength F to a field A solution of a restriction to conformally flat Einstein space-time (CFES) of a conformally invariant equation \( E_s(A) = 0 \), namely:

\[
(E_s(A))^{\mu_1 \ldots \mu_s} = \left( \Box + c_s R \right) A^{\mu_1 \ldots \mu_s} + a_s s \nabla^{(\mu_1} \nabla_\rho A^{\rho \mu_2 \ldots \mu_s)} + b_s (s-1) g^{(\mu_1 \mu_2} \nabla_\rho \nabla_\sigma A^{\rho \mu_3 \ldots \mu_s)} \sigma = 0,
\]

with \( A(x) \in S^s T_x M \equiv S^s \) a symmetric traceless field of rank \( s \), \( \Box A^{\mu_1 \ldots \mu_s} = g^{\alpha \beta} \nabla_\alpha \nabla_\beta A^{\mu_1 \ldots \mu_s} \), \( R \) the scalar curvature of the underlying lorentzian space-time \((M, g)\) of dimension \( d \geq 3 \).

In eq. (1.1) the coefficients are given by:

\[
a_s = -\frac{4}{d + 2s - 2}, \quad b_s = \frac{4}{(d + 2s - 4)(d + 2s - 2)}, \quad c_s = -\frac{d^2 - 2d + 4s}{4d(d - 1)} \quad (1.2)
\]

and \((\mu_1 \ldots \mu_s)\) is the, normalized, symmetrized part of the enclosed indices.

Equation (1.1) admits gauge solutions for tensors of rank \( s \geq 1 \) on CFES of dimension \( d = 4 \). We demonstrate that the gauge solutions of (1.1) at \( d = 4 \) might be restricted, while keeping the conformal invariance, thanks to a gauge condition which generalizes (to higher ranks) the Eastwood-Singer gauge fixing equation on CFES. To be precise we show that the set:

\[
\begin{align*}
E_s(A) &= 0, \\
E_0(\phi) &= 0
\end{align*}
\]

with \( \phi = \nabla_{\mu_1} \ldots \nabla_{\mu_s} A^{\mu_1 \ldots \mu_s} \), is conformally invariant and fixes the gauge.
This article is organized as follows.

In section 2 we will recall elementary, yet needed, facts on conformal invariance. Section 3 illustrates the content of the article by studying fields from rank 0 to rank 2. In such cases one sees the emergence of gauge freedom in dimension \( d = 4 \), the underlying structure between equations of different ranks when formulated on CFES. Section 4 shows that these properties are indeed found for any field \( A \) of rank \( s \geq 2 \) satisfying eq. (1.1). That is: invariance under Weyl transformations between CFES and that there are gauge solutions for \( d = 4 \). A, generally, conformally invariant equation is recovered in eq. (4.6), its restriction to CFES coincides with eq. (1.1). In section 5 we exhibit a conformally invariant gauge fixing equation. We show how it arises from the field equation in arbitrary dimensions and how it generalizes, to CFES, the Eastwood-Singer equation. These equations are precisely the set (1.3). We then show how by plugging a pure gauge field in the gauge fixing equation, thus inspecting the residual gauge freedom, one gets, as a by-product, a conformally invariant equation of order \( 2(s + 1) \) acting on scalars of weight \( s - 1 \), for \( d = 4 \). This equation is recognized to be a GJMS operator \([2]\) fulfilling Branson’s factorization formula on Einstein spaces \([3, 4]\). Our study shows a new way to establish its conformal invariance. Section 6 investigates the properties of the field strength \( F \) associated to the field \( A \). The gauge invariance of \( F \) is explicitly shown and a Lagrangian giving rise to eq. (1.1) is retro-engineered on those \( F \). Then, a set of first order equations corresponding to Maxwell’s equations on the observable fields is uncovered, a decomposition of \( F \) on \( E-B \) fields is performed and its duality is shown. Appendix A comments on the (lack of) content of eq. (1.1) in the case \( d = 2 \).

These results set the ground for the quantization of fields satisfying eq. (1.1) when \( d = 4 \) with the help of the conformally invariant gauge fixing equation (1.3) and the scalar product given in eq. (6.22). A task which will be addressed elsewhere.

# 2 A reminder on conformal invariance

Let us recall the known fact that conformal invariance recovers two different notions, namely the invariance under Weyl rescalings and invariance of a space of solutions under the action of the conformal group. See, say, the review of Kastrup \([5]\) for further references on the application of conformal invariance in theoretical physics.

## 2.1 Conformal invariance w.r.t. Weyl rescalings

On the one hand \([6, \text{ App.D}]\), starting from a manifold \( M \) equipped with a metric \( g \) a Weyl rescaling might, roughly, be defined as the application:

\[
(M, g) \mapsto (M, \bar{g}) \quad \text{s.t.} \quad \bar{g}_{\mu\nu}(x) = \omega^2(x)g_{\mu\nu}(x), \quad \omega \in C^\infty(M). \tag{2.1}
\]
Owing to this definition quantities derived from the metric, such as the curvature scalar, change according to:

\[
\begin{align*}
\Gamma^{\rho}_{\mu\nu} &= \Gamma^{\rho}_{\mu\nu} + \omega^{-1} (\delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} \omega^{\rho}_{\sigma} - g^{\rho\sigma} g_{\mu
u}) \omega_{\rho}, \\
\mathcal{R}^{\rho}_{\mu\nu\sigma} &= R^{\rho}_{\mu\nu\sigma} - \omega^{-1} (\delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} \omega^{\mu}_{\nu} - g^{\rho\sigma} g_{\mu\nu}) \omega^{\nu}_{\rho}
\end{align*}
\] (2.2)

In the above equations the semi-colon refers to the covariant derivation with respect to \( g \), \( \Gamma \) is the Levi-Civita connection, and \( [\alpha\beta] = \alpha\beta - \beta\alpha \) is the antisymmetric part of the enclosed indices. Notice that, in practice, \( M \) often will be a common subset of the space-time we are interested in.

Then, an equation, depending on the metric and symbolically written, \( E(A) = 0 \) is said to be Weyl invariant if and only if there exists a conformal weight \( h(A) \in \mathbb{R} \) such that:

\[
\mathcal{E}(A) = \omega^{h} E(A), \quad \mathcal{A} = \omega^{h} A.
\]

That is, for a given solution \( A \) of the equation \( E(A) = 0 \), the rescaled field \( \mathcal{A} = \omega^{h} A \) is a solution of \( \mathcal{E}(\mathcal{A}) = 0 \) on the Weyl related space-time \( (M, \mathcal{g}) \). Conditions such as indices symmetry and tracelessness are Weyl invariant.

### 2.2 Conformal invariance w.r.t. the conformal group \( C_{[g]} \)

On the other hand, the conformal group \( C_{g} \) of \( (M, g) \) is, by definition, the set of transformations letting the causality invariant. A convenient characterization of its infinitesimal generators \( \{X\} \) is that they fulfil the conformal Killing equation:

\[
(\mathcal{L}_{X}g)(x) = 2f_{X}(x)g(x),
\] (2.6)

with \( \mathcal{L}_{X} \) the Lie derivative along \( X \). Notice that, according to eq. (2.2), the characterization of \( X \) through eq. (2.6) implies that the conformal groups of \( (M, g) \) and \( (M, \mathcal{g}) = \omega^{2}g \) are, at least, locally isomorphic: \( C_{g} \simeq C_{\mathcal{g}} \simeq C_{[g]} \), where \([g] \) stands for the equivalence class \( g \sim \omega^{2}g \). For the purpose of this article, since conformally flat space-time are (locally) Weyl related to the Minkowski space-time, we will exploit the minkowskian conformal group. The latter is obtained by completing the Poincaré group by dilations and special conformal transformations which, in the usual rectangular coordinates, respectively read:

\[
\begin{align*}
  x^{\mu} &\rightarrow \lambda x^{\mu}, & \lambda \in \mathbb{R}^{+}_{\ast}, \\
  x^{\mu} &\rightarrow \frac{x^{\mu} + b^{\mu}x^{2}}{1 + 2b \cdot x + b^{2}x^{2}}.
\end{align*}
\] (2.7)

These two transformations fulfil eq. (2.6) with: \( f_{D}(x) = 1 \), \( f_{K_{\mu}}(x) = -2x_{\mu} \), where \( D \) and \( K_{\mu} \) are the generators of dilations and special conformal transformations respectively.
It is often convenient to write the special conformal transformation (2.7) as the product $I \circ T_b \circ I$, with $I$ the inversion $x^\mu \mapsto x^\mu/x^2$ and $T_b$ the translation $x^\mu \mapsto x^\mu + b^\mu$. Finally, for the sake of completeness, let us recall the commutations relations of the minkowskian conformal group:

\[
\begin{align*}
[X_{\mu\nu}, X_{\rho\sigma}] &= \eta_{\sigma[\mu} X_{\nu]\rho] - \eta_{\rho[\mu} X_{\nu]\sigma], \\
[P_\rho, D] &= P_\rho, \\
[K_\rho, D] &= -K_\rho, \\
[P_\mu, K_\nu] &= 2(X_{\mu\nu} - \eta_{\mu\nu}D), \\
[P_\rho, X_{\mu\nu}] &= \eta_{\rho[\mu} P_{\nu]}, \\
[K_\rho, X_{\mu\nu}] &= \eta_{\rho[\mu} K_{\nu]}, \\
[X_{\mu\nu}, D] &= 0, \\
[P_\mu, P_\nu] &= 0, \\
[K_\mu, K_\nu] &= 0,
\end{align*}
\]  

(2.8)

with $X_{\mu\nu} = -X_{\nu\mu}$ the generators of Lorentz transformations, $P_\mu$ those of translations and $\eta_{\mu\nu} = \text{diag}(+1, -1, \cdots, -1)$. Then, if one sets:

\[
X_{dd+1} = D, \quad X_{d\mu} = cK_\mu + \frac{1}{4c} P_\mu, \quad X_{\mu d+1} = cK_\mu - \frac{1}{4c} P_\mu,
\]

with $c \in \mathbb{R}^+$, the whole commutation algebra (2.8) is recast as that of $\mathfrak{o}(2, d)$:

\[
[X_{AB}, X_{CD}] = \eta_{D[A} X_{B]C} - \eta_{C[A} X_{B]D},
\]

with $A, B, C, D, \cdots = 0, 1, \cdots, d, d+1, X_{AB} = -X_{BA}$ and $\eta_{AB} = \text{diag}(-1, +1, \cdots, +1, -1, +1)$.

Then [7], on $(M, g)$, an equation $E(A) = 0$ is said to be invariant under $C_{[g]}$, or conformally invariant, if one can realize the Lie algebra $\mathfrak{g}$ of $C_{[g]}$ such that:

\[
[E, \mathfrak{g}](A) = \xi E(A),
\]

for some function $\xi$. That is, a solution of $E(A) = 0$ is mapped to another solution of $E(A) = 0$ under the action of the conformal group $C_{[g]}$.

Notice that, while being different, invariance w.r.t. Weyl rescalings and invariance w.r.t. to the conformal group are not unrelated. Namely invariance in the former implies invariance in the latter, this is shown by considering the composition:

\[
(\omega_h)^{-1} \circ h : (M, g) \to (M, h_* g = (\omega_h)^2 g) \to (M, \overline{g} = (\omega_h)^{-2} h_* g = g)
\]

resulting in an isometry, with $h \in C_{[g]}$, $(\omega_h)^2$ the scaling of $g$ induced by $h$ which is latter compensated by the Weyl rescaling $(\omega_h)^{-1}$. The theory being invariant w.r.t. Weyl rescalings and isometries yields the claim. From the point of view of the space of solutions of the wave equation $E(A) = 0$, elements of $C_{[g]}$ map a space of solutions on itself while Weyl rescalings embed a space of solutions into another. In that respect Weyl rescalings might be thought of as intertwining operators for the conformal group $C_{[g]}$ between two spaces of solutions.

3 A guided tour from rank 0 to 2

This section studies fields from rank 0 to 2. Progressing in this manner reveals the properties of eq. (1.1) one after another and hints at what is to be expected for an arbitrary rank $s \geq 2$ on CFES. Moreover, it exposes in their simplest form the manner in which the computations will be carried in section 4 and section 5.
3.1 Rank 0: the scalar field

A textbook [8] conformally invariant equation is the scalar massless conformally coupled equation:

\[ \left( \Box - \frac{1}{4} \frac{d - 2}{d - 1} R \right) \varphi = 0. \]  

(3.1)

This equation might also be referred to as the conformal laplacian or the Yamabe operator. The above equation is Weyl invariant and, when realized on Minkowski space-time, has its space of solutions left invariant under the action of \( \text{SO}(2, d) \) once one has taken into account the scaling of the field:

\[ [T_g \varphi](x) = (\alpha(g, g^{-1}.x)) h \varphi(g^{-1}.x), \quad \forall g \in \text{SO}(2, d), \]  

(3.2)

with \( h(\varphi) = 1 - d/2 \) the conformal weight of the scalar field. The multiplier \( \alpha \) appearing in eq. (3.2) is defined through

\[ dg.s^2 = (\alpha(g, x))^2 ds^2, \]  

(3.3)

with \( ds^2 \) the squared line element. The factor \( \alpha \) then fulfils, by construction, the 1-cocycle equation:

\[ \alpha(gg', x) = \alpha(g, g'.x) \alpha(g', x), \quad \forall g, g' \in \text{SO}(2, d). \]

For \( g \) an isometry of the underlying space-time \( \alpha = 1 \) and, from eq. (3.3), eq. (3.2) agrees with the natural action on a scalar field [9].

3.2 Rank 1: the vector field

For the vector field \( A^\mu \), the rank 1 field, one can find that the equation:

\[ \Box A^\mu - \frac{4}{d} \nabla^\mu \nabla \cdot A - \frac{2}{d - 2} R^\mu_\nu A^\nu - \frac{1}{4} \frac{d(d - 4)}{(d - 1)(d - 2)} RA^\mu = 0 \]  

(3.4)

is invariant under Weyl rescalings, with \( \overline{A}^\mu = \omega^{-d/2} A^\mu \), and that its space of solutions is left invariant under the action of \( \text{SO}(2, d) \). Notice that eq. (3.4) might be rewritten as:

\[ \Box A^\mu - \frac{4}{d} \nabla^\nu \nabla^\mu A^\nu + \frac{2}{d} \frac{d - 4}{d - 2} R^\mu_\nu A^\nu - \frac{1}{4} \frac{d(d - 4)}{(d - 1)(d - 2)} RA^\mu = 0, \]  

(3.5)

to make it obvious that at \( d = 4 \) it reduces to the free Maxwell’s equations on the vector potential \( A \). Such an equation, for arbitrary \( d \), might already be found in many instances, say in [10].

For \( d = 4 \) eq. (3.4) admits \( A^\mu = \nabla^\mu \varphi \) as a gauge solution with the scalar \( \varphi \) unconstrained. That is, a solution \( A \) of eq. (3.4) is determined up to (the gradient of) a scalar \( \varphi \):

\[ A^\mu \mapsto \varphi A^\mu = A^\mu + \nabla^\mu \varphi. \]  

(3.6)

The gauge freedom showed in eq. (3.6) allowed by eq. (1.1) also means that \( A \), the vector potential, is not an observable. The gauge invariant quantities, the electric and magnetic fields, are components of the field strength \( F \) given by:

\[ (F(A))^{\alpha \mu} = \nabla^\alpha A^\mu - \nabla^\mu A^\alpha, \]  

(3.7)
which is indeed gauge invariant since for \( A^\mu = \nabla^\mu \varphi \) one has:

\[
(F(A))^{\alpha \mu} = \nabla^\alpha \nabla^\mu \varphi - \nabla^\mu \nabla^\alpha \varphi = [\nabla^\alpha, \nabla^\mu] \varphi = 0.
\] (3.8)

That being said, the vector potential \( A \) remains relevant since it is the \( U(1) \) gauge field through which the interaction is introduced in the lagrangian. In addition, to simplify as much as possible scattering computations in curved space-time, a simple expression of its two-point function would be welcome. As it has already been shown on flat and (A)dS space-time [11–14] keeping the conformal covariance by using a conformally invariant gauge fixing condition while quantizing \( A \) leads to simpler, more compact, results. This is the point of view embraced in this article.

An interesting property of eq. (3.4), in arbitrary dimensions, is that it provides a conformally invariant gauge fixing equation for the four dimensional case. Namely, taking its divergence and using the usual commutation relations of covariant derivatives leads to:

\[
\left( \frac{d-4}{d} \right) [\Box \nabla \cdot A + \frac{d}{d-2} \left( \nabla^\mu R_{\mu \nu} A^\nu - \frac{1}{4} \frac{d}{d-1} \nabla_\mu R A^\mu \right)] = 0.
\]

For \( d \neq 4 \) the term within the brackets necessarily vanishes on the space of solutions of eq. (3.4), then is conformally invariant on this space of solutions. For \( d = 4 \) the divergence of eq. (3.4) vanishes and \( \nabla \cdot A \) is unconstrained. However, one would strongly suspect

\[
G(A) = \Box \nabla \cdot A + 2 \nabla^\mu R_{\mu \nu} A^\nu - \frac{2}{3} \nabla_\mu R A^\mu = 0
\] (3.9)

to be conformally invariant on the space of solutions of eq. (3.4). It is indeed the case since:

\[
G(A) = \omega^{-4} G(A) + 2 \omega^{-5} (\Box A^\mu - \nabla^\mu \nabla \cdot A - R_{\mu \nu} A^\nu) (\nabla_\mu \omega)
\] (3.10)

where eq. (3.4) appears, with \( d = 4 \), in the right hand side. Equation (3.9) is the Eastwood-Singer gauge fixing equation [15]. Using eq. (3.9) indeed restricts the gauge freedom as now the field \( \varphi \) has to fulfil the equation

\[
\left( \Box^2 + 2 \nabla^\mu R_{\mu \nu} \nabla^\nu - \frac{2}{3} \nabla_\mu R \nabla^\mu \right) \varphi = 0.
\] (3.11)

Equation (3.11) is known as the \( d = 4 \) Paneitz operator [16] (a summary might be found in [17]) which is a fourth order conformally invariant operator on scalars of null conformal weight. This operator has also, been found by other means by Riegert [18] and Fradkin and Tseytlin [19].

### 3.3 The rank 2 field

One can try to generalize the previous scheme to the symmetric traceless rank 2 tensor \( A^{\mu \nu} \). Then one can check that, say, the equation

\[
\Box A^{\mu \nu} - \frac{4}{d+2} (\nabla^\mu \nabla_\rho A^{\nu \rho} + \nabla^\nu \nabla_\rho A^{\mu \rho}) + \frac{8}{d(d+2)} g^{\mu \nu} \nabla_\rho A^{\rho \sigma} - \frac{(d^2 - 2d + 8)}{4d(d-1)} R A^{\mu \nu}
\]

\[
+ \frac{2}{d} (R_{\mu \nu} A^{\rho \rho} + R_{\nu \rho} A^{\mu \rho}) - \frac{4(d-1)}{d} R^{\mu \nu}_{\quad \rho \sigma} A^{\rho \sigma} - \frac{4}{d} g^{\mu \nu} R_{\rho \sigma} A^{\rho \sigma} = 0
\] (3.12)
is Weyl invariant with $A^{\mu\nu} = A^{\nu\mu}$, $g_{\mu\nu}A^{\mu\nu} = 0$ and $\mathbf{A}^{\mu\nu} = \omega^{-1-d/2}A^{\mu\nu}$.

One such conformally invariant equation pops out from time to time either on flat [10, 20–23] or curved space-time (see [24] and references therein).

When trying to find gauge solutions to eq. (3.12) the computation does not seem to yield anything special. Also, taking the divergence, or twice the divergence, of eq. (3.12) does not shed much light. The blurriness appearing in the study of the rank two tensors marks the emergence of the invariant, under Weyl rescalings, Weyl tensor:

$$C^{\mu}_{\quad \rho\sigma} = R^{\mu}_{\quad \rho\sigma} - \frac{1}{d-2} \left( \delta^{\mu}_{[\nu} R_{\sigma]\rho} - g_{\rho[\nu} R^{\mu}_{\sigma]} \right) + \frac{1}{(d-1)(d-2)} R \delta^{\mu}_{[\rho} g_{\sigma]\rho},$$

which is the totally traceless part of the Riemann tensor. Thanks to this tensor eq. (3.12) is then far from being unique as one can add a term such as

$$+ \lambda C^{\mu}_{\quad \rho\sigma} A^{\rho\sigma},$$

where $\lambda$ is unconstrained by the requirement that the resulting equation has to be Weyl invariant.

To avoid such terms in the field equation one can restrict his study to conformally flat space-time. This choice, however, does not seem to make things much more simpler. This is the reason why we will narrow the scope of this article to conformally flat Einstein space-time (CFES), for which the Riemann and Ricci tensors are expressed as:

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}), \quad (3.13a)$$
$$R_{\mu\nu} = \frac{R}{d} g_{\mu\nu}, \quad (3.13b)$$
$$R = \text{Const}. \quad (3.13c)$$

Examples of CFES are Minkowski and (A)dS space-time. Then, using eq. (3.13), eq. (3.12) reduces to

$$(E_2(A))^{\mu\nu} = \Box A^{\mu\nu} - \frac{4}{d-2} (\nabla^{\mu} \nabla_{\rho} A^{\nu\rho} + \nabla^{\nu} \nabla_{\rho} A^{\mu\rho})$$
$$+ \frac{8}{d(d+2)} g^{\mu\nu} \nabla_{\rho} A^{\rho\sigma} - \frac{1}{4 d(d-1)} RA^{\mu\nu} = 0. \quad (3.14)$$

One can then follow the same steps as in the vectorial case. If one considers the field $A$ written as the derivation of a vector field $V$ such as

$$A^{\mu\nu} = \nabla^{\mu} V^{\nu} + \nabla^{\nu} V^{\mu} - \frac{2}{d} g^{\mu\nu} \nabla \cdot V, \quad (3.15)$$

then, after a computation involving eqs. (3.13), one gets:

$$\begin{align*}
(E_2(A))^{\mu\nu} &= \left( \frac{d-2}{d+2} \right) \left[ \nabla^{\mu} (E_1(V))^\nu + \nabla^{\nu} (E_1(V))^\mu - \frac{2}{d} g^{\mu\nu} \nabla \cdot (E_1(V))^{\rho} \right], \quad (3.16)
\end{align*}$$

in which the appearance of $E_1$ is worth noticing. Similarly, suppose in eq. (3.15) that $V^{\mu} = \nabla^{\mu} \varphi$, then simplifying eq. (3.16) leads to

$$\begin{align*}
(E_2(A))^{\mu\nu} &= \left( \frac{d-2}{d+2} \right) \left[ \nabla^{\mu} \nabla^{\nu} + \nabla^{\nu} \nabla^{\mu} - \frac{2}{d} g^{\mu\nu} \Box \right] E_0(\varphi), \quad (3.17)
\end{align*}$$
meaning that equation (3.14) for \( d = 4 \) allows the gauge freedom
\[
A_{\mu\nu} \mapsto \varphi A_{\mu\nu} = A_{\mu\nu} + \left( \nabla_\mu \nabla_\nu - \frac{1}{d} g_{\mu\nu} \Box \right) \varphi
\]
and \( A \) is determined up to a scalar.

A field strength \( F \), which is gauge independent, for the field \( A \) can be found in section 6 in which the field strength for arbitrary ranks are worked out.

Now that the gauge freedom has been shown one can search for a gauge fixing equation similar to the Eastwood-Singer equation. Taking the divergence of eq. (3.14) and using eqs. (3.13) yields:
\[
\nabla_\mu (E_2(A))^{\mu\nu} = \left( \frac{d - 2}{d + 2} \right) (E_1(\nabla \cdot A))^\nu,
\]
where \( (\nabla \cdot A)^\nu = \nabla_\mu A^{\mu\nu} \). In the same manner, taking the divergence of eq. (3.14) twice produces:
\[
\nabla_\nu \nabla_\mu (E_2(A))^{\mu\nu} = \frac{(d - 2)(d - 4)}{d(d + 2)} E_0(\phi),
\]
where \( \phi = \nabla_\mu \nabla_\nu A^{\mu\nu} \). Those two results then suggest that the set
\[
\begin{align*}
E_2(A) &= 0, \\
E_0(\phi) &= 0, \quad (d = 4)
\end{align*}
\]
is conformally invariant, while restricting the gauge freedom allowed by \( E_2(A) = 0 \). The conformal invariance is indeed preserved, see section 5, and the scalar field \( \varphi \) now has to fulfil
\[
\left( \Box - \frac{1}{6} R \right) \Box \left( \Box + \frac{1}{3} R \right) \varphi = 0.
\]
Such a conformally, i.e. \( \text{SO}(2,4) \), gauge fixing equation was found, already, on flat space-time in \([25]\) for the tracefull field. It then reads: \( (\Box \partial_\mu \partial_\nu - \frac{1}{4} \eta_{\mu\nu} \Box^2) A^{\mu\nu} = 0 \).

4 Working out the properties of eq. (1.1)

In the previous section the prominent properties of the field equation (1.1) have been exposed: conformal invariance and gauge freedom up to a scalar for \( s \geq 1 \) and \( d = 4 \). In addition, the relations (3.16)-(3.19) suggest that, on CFES, equations of various ranks are related one to another through, some, derivations.

This section exhibits that indeed all of these properties are found in the higher rank realization of eq. (1.1). First, since we are mainly concerned with CFES, we derive two identities which reflect the mapping of a CFES to another CFES. Secondly, it is shown that, under such Weyl rescalings, the equation (1.1) is Weyl invariant. Finally, gauge solutions of eq. (1.1) are found for \( d = 4 \).

4.1 The restricted Weyl transformations

This work is mostly concerned with CFES and therefore only matter Weyl rescalings mapping a CFES on another (see, say, \([26]\) in a Riemannian setting). This means that a smaller
class of $\omega$’s has to be considered. Let us call those Weyl rescalings using such $\omega$’s restricted Weyl transformations (note to be confused with those of [27]). Without getting into the details of such transformations we can derive two identities which are fulfilled by the $\omega$’s and which will soon be needed.

Asking for eq. (2.1) to map a CFES to another CFES is tantamount to require the relations (3.13) to be preserved. First, consider eq. (3.13b) on $(M, \overline{g})$, that is:

$$\overline{R}_{\mu\nu} = \frac{1}{d} \overline{\nabla}_\mu \overline{g}_{\nu\rho}. \quad (4.1)$$

Then, plugging eq. (2.4) in the left-hand side of eq. (4.1) and setting $\overline{g}_{\mu\rho} = 0$, since $\overline{R}$ is also constant one gets:

$$\square \overline{\nabla}_{\mu} \omega = (\overline{\nabla}_\mu \omega) \left(3 \omega^{-1}(\square \omega) - \frac{R}{d(d-1)}\right) - (d-4) \omega^3 \left(\frac{1}{\omega} \overline{\nabla}_\rho \overline{\nabla}_\sigma \frac{1}{\omega}\right), \quad (4.2)$$

Equation (4.2) merely preserves eq. (3.13a) and eq. (3.13b) while the Weyl rescaled curvature scalar $\overline{R}$, at this point, is not necessarily constant. This is taken care of using eq. (3.13c) and setting $\overline{\nabla}_\mu \overline{R} = 0$, since $\overline{R}$ is also constant one gets:

$$\square \overline{\nabla}_{\mu} \omega = (\overline{\nabla}_\mu \omega) \left(3 \omega^{-1}(\square \omega) - \frac{R}{d(d-1)}\right), \quad (4.3)$$

tweaked for later convenience. Notice that the factor $\omega_I$ which arises from an inversion fulfill both eq. (4.2) and eq. (4.3), thus the SO$_0(2, d)$ invariance remains implied on conformally flat space by this weaker Weyl invariance.

### 4.2 Restricted Weyl invariance of eq. (1.1)

Performing a Weyl transformation, using eq. (2.2) and eq. (2.5), on eq. (1.1) yields:

$$(E_s(A))^{\mu_1 \ldots \mu_s} = \omega^{h-2}(E_s(A))^{\mu_1 \ldots \mu_s} + \frac{2s}{d} \omega^{h-4} \left[ \omega(\omega_{\rho\sigma} - 2\omega_{\rho} \omega_{\sigma}) A^{\mu_2 \ldots \mu_s} \right]^{\mu_1 \ldots \mu_s}
- 2s \omega^{h-4} \left[ \omega(\omega_{\rho;\sigma} - 2\omega_{\rho} \omega_{;\sigma}) A^{\mu_2 \ldots \mu_s} \sigma \right]^{\mu_1 \ldots \mu_s}
+ \frac{2s(s-1)}{d + 2s - 4} \omega^{h-4} \left[ \omega(\omega_{\rho;\sigma} - 2\omega_{\rho} \omega_{;\sigma}) g^{\mu_1 \mu_2 A^{\mu_3 \ldots \mu_s}} \right]^{\rho \sigma}, \quad (4.4)$$

with the conformal weight $h(A^{\mu_1 \ldots \mu_s}) = 1 - s - d/2$. By setting $\rho = \omega^{-1}$ eq. (4.4) might be written as:

$$(E_s(A))^{\mu_1 \ldots \mu_s} = \rho^{2-h} (E_s(A))^{\mu_1 \ldots \mu_s} + 2s \rho^{2-h} \left( (\overline{\nabla}_{\mu_1} \overline{\nabla}_{\sigma} - \frac{1}{d} (\delta_{\mu_1} \square) \rho) A^{\mu_2 \ldots \mu_s} \right) \sigma
- \frac{2s(s-1)}{d + 2s - 4} \rho^{2-h} (\overline{\nabla}_\rho \overline{\nabla}_\sigma \rho) g^{(\mu_1 \mu_2 A^{\mu_3 \ldots \mu_s})} \sigma. \quad (4.5)$$

Then, from the tracelessness of $A$ and eq. (4.2) the remaining terms vanish and yields the (restricted) Weyl invariance of eq. (1.1) between two CFES:

$$E_s(A) = \omega^{h-2} E_s(A).$$

Remark that the only property used to prove eq. (4.5) is that eq. (4.2) is fulfilled. Later, for the gauge fixing equation, the constant curvature of $(M, g)$ will come into play through the use of eq. (4.3).
4.2.1 The conformal invariance on arbitrary space-time

Equation (1.1) has been shown to be Weyl invariant under the restricted transformation (4.2). However, with eq. (4.4) and the relations (2.3) and (2.4) one would find that the equation

\[
(\Box + c_s R)A^{\mu_1 \cdots \mu_s} + a_s s \nabla^{(\mu_1} \nabla_{\mu_2} A^{\mu_2 \cdots \mu_s)} + b_s s(s-1) g^{(\mu_1 \mu_2} \nabla_{\mu_3} A^{\mu_3 \cdots \mu_s)} + c_s s(s-1) R^{(\mu_1 \rho \sigma} A_{\mu_2 \mu_3 \cdots \mu_s)} + d_s s R^{(\mu_1} \sigma A^{\mu_2 \cdots \mu_s)} + \mathcal{O}(s) s(s-1) R_{\rho \sigma} g^{(\mu_1 \mu_2} A^{\mu_3 \cdots \mu_s)} = 0,
\]

(4.6)

where the coefficients \(a_s, b_s\) and \(c_s\) are given in eq. (1.2) and

\[
d_s = \frac{2}{d-1}, \quad e_s = -\frac{2}{d} \frac{(d-1)(s-1)}{s-1}, \quad f_s = -\frac{2}{d(s-1)} \frac{(d+s-2)}{(d+2s-4)},
\]

is invariant under arbitrary Weyl rescalings. Equation (1.1) is then viewed as a restriction of eq. (4.6) to CFES.

As noticed in section 3.3 eq. (4.6) is far from unique as one can add a term such as

\[
\frac{\lambda s(s-1)}{C^{(\mu_1 \rho \sigma} A^{\mu_3 \cdots \mu_s)\rho \sigma},
\]

that is, changing in eq. (4.6) the coefficients according to

\[
c_s \rightarrow c_s + \lambda \frac{s(s-1)}{(d-1)(d-2)}, \quad e_s \rightarrow e_s + \lambda, \quad d_s \rightarrow d_s - 2\lambda \frac{s-1}{d-2}, \quad f_s \rightarrow f_s + \frac{\lambda}{d-2}.
\]

while keeping the Weyl invariance of the resulting equation intact.

4.3 Gauge invariance at \(d = 4\)

In section 3 it has been established that for \(d = 4\) and for tensors of rank 1 and 2 that the solutions of eq. (1.1) are determined up to a scalar \(\varphi\). This subsection inspects the gauge invariance of eq. (1.1) for tensors of arbitrary rank and shows explicitly that they, too, remain determined up to a scalar.

First let us, for our purpose, introduce the symmetric traceless gradient (STG), defined as

\[
(STG(f))^{\mu_1 \cdots \mu_s} = s \nabla^{(\mu_1} f^{\mu_2 \cdots \mu_s)} - \frac{s(s-1)}{(d+2s-4)} g^{(\mu_1 \mu_2} \nabla_{\mu_3} f^{\mu_3 \cdots \mu_s)}
\]

(4.8)

with \(f \in S^s_{s-1}\). In addition, let us commit the abuse of language \((STG(\varphi))^{\mu} = \nabla^\mu \varphi\).

Secondly, notice that eq. (1.1) might then be rewritten as:

\[
(\Box + c_s R)A + a_s (STG(\nabla \cdot A)) = 0.
\]

(4.9)

Now let us consider the field \(A = STG(f), f \in S^s_{s-1}\) and \(s \geq 2\). Since the coefficients \((a_s, c_s)\) fulfil the recurrence relations:

\[
a_{s-1} = \frac{a_s}{1 + a_s} \left( \frac{d + 2s - 6}{d + 2s - 4} \right); \quad c_{s-1} = \frac{1}{1 + a_s} \left( \frac{d + 2s - 3}{d - 1} \right) + \frac{(s-1)(d+s-3)}{d(d-1)} a_s + c_s,
\]

(4.10a, 4.10b)
one gets the following identity:

\[ E_s(\text{STG}(f)) = (1 + a_s) \text{STG}(E_{s-1}(f)), \]  

(4.11)

the higher rank form of eq. (3.16) where \( \text{STG} \) was given by eq. (3.15).

Now the identity (4.11) enables us to look for gauge invariance for a field \( A \) obtained from a field \( g \) of rank \( r, r < s \), as one would obtain:

\[ E_s(\text{STG}^{s-r}(g)) = \prod_{i=r+1}^{s} (1 + a_i) \text{STG}^{s-r}(E_r(g)). \]  

(4.12)

Then, from eq. (4.12), the question of the existence of gauge solutions amounts to look if there is a rank \( r, r < s \), and a dimension \( d \) such that \( 1 + a_r = 0 \). From the values of the \( a_i \)'s, given in eq. (1.2), there is only one (physical) solution given by \( (a_1, d = 4) \) corresponding to the gauge freedom up to a scalar:

\[ A \mapsto \varphi A = A + \text{STG}^s(\varphi). \]  

(4.13)

It is this gauge freedom that we would like to restrict while keeping the conformal invariance.

5 Uncovering and discussing the gauge fixing equation

In the previous section we found that for \( d = 4 \) the solutions \( A \) of eq. (1.1) are determined up to a scalar \( \varphi \), see eq. (4.13). The purpose of this section is to exhibit a conformally invariant gauge fixing equation which will restrict this gauge freedom. First, we will show how that equation can be obtained from eq. (1.1) in arbitrary dimensions. Then, we prove that fixing the gauge in that manner is indeed conformally invariant. Inspecting the residual gauge freedom left to the scalar \( \varphi \) shows that \( \varphi \) is indeed constrained to belong to a certain space of solutions, as is needed. Finally, it is shown that those pure gauge solutions themselves are conformally invariant. This is demonstrated by, incidentally, providing a new way to derive Branson’s factorization formula of GJMS operators on CFES.

5.1 The derivation of the gauge fixing equation

Let us first recall that for the vector field we found, in 3.2, that the Eastwood-Singer gauge fixing equation (3.9) appears when one takes the divergence of the (generally) conformally invariant equation (3.4). Then, in 3.3, for the rank 2 tensor the presence of gauge invariance wasn’t so clear any more. After restricting ourselves to CFES we found: on the one hand explicit gauge solutions and on the other hand that by taking the divergence of the field equation, now eq. (3.14), one recovered the lower rank conformally invariant field equations, as seen on eqs. (3.18)-(3.19). Now, for an arbitrary rank \( s \) the previous section exhibited gauge solutions. Here we seek a gauge fixing equation which will constrain those solutions.

Taking the divergence of eq. (1.1) yields:

\[ \nabla_{\mu_s}(E_s(A))^{\mu_1...\mu_s} = (1 + a_s)(E_{s-1}(\nabla \cdot A))^{\mu_1...\mu_{s-1}}, \]  

(5.1)
through a direct computation, involving the commutation of covariant derivatives and using the fact that the Riemann tensor is given by eq. (3.13a) with $R = \text{Const.}$ and that the coefficients $(a_s, b_s, c_s)$ are solutions of the recurrence relations:

\begin{align}
    a_{s-1} &= \frac{a_s + 2b_s}{1 + a_s}, \quad (5.2a) \\
    b_{s-1} &= \frac{b_s}{1 + a_s}, \quad (5.2b) \\
    c_{s-1} &= \frac{1}{1 + a_s} \left( \frac{d + 2s - 3}{d(d-1)} + \frac{(s-1)(d+s-3)}{d(d-1)} a_s + c_s \right). \quad (5.2c)
\end{align}

That is, the divergence of $A$ satisfies the equation of a rank $s-1$ symmetric traceless field. By induction each divergence has to fulfil the equation $E_i$ of the appropriate rank and finally:

$$
\nabla_{\mu_1} \ldots \nabla_{\mu_s} (E_s(A))^{\mu_1 \ldots \mu_s} = \left[ \prod_{i=1}^{s} (1 + a_i) \right] E_0(\phi). \quad (5.3)
$$

Therefore, the behaviour of the divergences of $A$ is completely determined by eq. (1.1). That is, if none of the pre-factor $(1 + a_i)$, $1 \leq i \leq s$, vanish. Similarly to 4.3 for $d = 4$ the pre-factor $(1 + a_1)$ vanishes. This is the confirmation of the gauge freedom shown before for which:

$$
\phi = \nabla_{\mu_1} \ldots \nabla_{\mu_s} A^{\mu_1 \ldots \mu_s}, \quad (d = 4)
$$

is left free by eq. (1.1). That is, up to a scalar degree of freedom. Notice that this should not come as a surprise. Indeed, if one sets $b_s = -a_s/(d + 2s - 4)$, to make obvious the symmetric traceless gradient in eq. (1.1), as in eq. (4.9), then the recurrence relations in eq. (5.2) become those of eq. (4.10).

To conclude, for $d = 4$, we have shown that the $s$-fold divergence $\phi$ is left free by eq. (1.1). To correct this we choose the following gauge fixing equation:

$$
E_0(\phi) = \left( \Box - \frac{1}{6} R \right) \phi = 0, \quad (d = 4). \quad (5.4)
$$

This gauge fixing equation appears legitimate with respect to conformal invariance as on the one hand the solutions of eq. (1.1) which fulfil eq. (5.4) are left invariant by conformal transformations (see hereafter), and on the other hand in arbitrary dimensions $d \neq 4$ the corresponding equation is always satisfied by the solutions of eq. (1.1), cf. eq. (5.3). For $d = 4$ this discrepancy is rectified by enforcing eq. (5.4) as a gauge fixing equation.

### 5.2 Restricted Weyl invariance of the gauge fixing equation

Similarly to the Eastwood-Singer gauge fixing equation for the vector potential, eq. (5.3) hints at a gauge fixing equation likely to be conformally invariant on the space of solutions of $E_s(A) = 0$ between two CFES. In order to prove this property for an arbitrary rank $s$ we will first consider, again, the vectorial case in order to exemplify our strategy in its simplest case. Then the case $s \geq 2$ will be addressed.
5.2.1 The vector field, Eastwood-Singer gauge revisited

Consider the system

\[
\begin{aligned}
\square A^\mu - \nabla^\mu \nabla \cdot A - \frac{1}{4} R A^\mu &= 0, \\
\left( - \frac{1}{6} \right) \nabla \cdot A &= 0,
\end{aligned}
\]

(5.5)

with \( d = 4 \) and \((M, g)\) a CFES. Performing a Weyl rescaling yields:

\[
\bar{\phi} = \nabla \cdot \bar{A} = \omega^{-2} \nabla \cdot A + 2\omega^{-3} (\nabla \mu \omega) A^\mu.
\]

(5.6)

which might be fed to the (rescaled) conformal laplacian:

\[
\left( \square - \frac{1}{6} R \right) \bar{\phi} = \left[ \omega^{-2} \left( \square - \frac{1}{6} R \right) + 2\omega^{-2} (\nabla \alpha \omega) \nabla^\alpha + \omega^{-3} (\square \omega) \right] \bar{\phi}
= \omega^{-5} (\square \omega) (\nabla \mu \omega) A^\mu + 4\omega^{-5} (\nabla \alpha \mu \omega) \nabla^\alpha A^\mu + 2\omega^{-5} (\square \nabla \mu \omega) A^\mu
+ 8\omega^{-6} (\nabla \alpha \omega) (\nabla \mu \omega) \nabla \alpha A^\mu
- 4\omega^{-6} (\nabla \mu \omega) A^\mu - 8\omega^{-6} (\nabla \alpha \omega) (\nabla \alpha \nabla \mu \omega) A^\mu
+ 12\omega^{-7} (\nabla \alpha \omega) (\nabla \alpha \omega) (\nabla \mu \omega) A^\mu.
\]

(5.7)

As it stands the result is far from being conformally invariant on the space of solution of eq. (5.5). Let us then use eq. (4.2) to express, say, \((\nabla \alpha \nabla \mu \omega)\) in terms of other derivatives of \(\omega\), that is:

\[
(\nabla \alpha \nabla \mu \omega) = 2\omega^{-1} (\nabla \alpha \omega) (\nabla \mu \omega) + \frac{1}{4} g_{\alpha \mu} [(\square \omega) + 2\omega^{-1} (\nabla \beta \omega) (\nabla \beta \omega)].
\]

(5.8)

Plugging eq. (5.8) in eq. (5.7) simplifies greatly the right-hand side, as one then gets:

\[
\left( \square - \frac{1}{6} R \right) \bar{\phi} = \omega^{-4} \left( \square - \frac{1}{6} R \right) \phi + 2\omega^{-5} (\nabla \mu \omega) \left( \square A^\mu - \nabla^\mu \nabla \cdot A - \frac{1}{6} R A^\mu \right)
+ 2\omega^{-5} (\square \nabla \mu \omega) A^\mu - 6\omega^{-6} (\square \omega) (\nabla \mu \omega) A^\mu.
\]

Now one can use the second identity, eq. (4.3), fulfilled by the scale factor \(\omega\) of a restricted Weyl transformation. For \( d = 4 \) it reads:

\[
(\square \nabla \mu \omega) = (\nabla \mu \omega) \left( 3\omega^{-1} (\square \omega) - \frac{1}{12} R \right).
\]

(5.9)

With this last identity finally one gets:

\[
\left( \square - \frac{1}{6} R \right) \bar{\phi} = \omega^{-4} \left( \square - \frac{1}{6} R \right) \phi + 2\omega^{-5} (\nabla \mu \omega) \left( \square A^\mu - \nabla^\mu \nabla \cdot A - \frac{1}{4} R A^\mu \right),
\]

and thus shows the conformal invariance of eq. (5.5) between two CFES. A result known to be true as this is just the restriction of eq. (3.10) to our choice of space-time.
5.2.2 The general case: $s \geq 2$

To prove the invariance, under restricted Weyl rescalings, of the set given in eq. (1.3) one needs both eq. (5.8) and eq. (5.9) to be fulfilled and to take into account the properties arising from the symmetry and tracelessness of $A$. Each formula appearing in the vectorial case has to be generalized.

Let us begin with the generalization of eq. (5.6):

$$
\bar{\phi} = \nabla_{\mu_1} \ldots \nabla_{\mu_s} \omega^h A^{\mu_1 \ldots \mu_s},
$$
in which $h(A^{\mu_1 \ldots \mu_s}) = 1 - s - d/2$. Equation (4.2) can be rewritten as:

$$
\left( \nabla_{\mu} \nabla_{\nu} - \frac{1}{d} g_{\mu \nu} \Box \right) \rho = -\omega^{-2} \left( \nabla_{\mu} \nabla_{\nu} - \frac{1}{d} g_{\mu \nu} \Box \right) \omega = 0, \quad \rho = \frac{1}{\omega}. \tag{5.10}
$$

Thanks to the tracelessness of $A$ one then realizes that there cannot be derivatives of $\omega$ of degree greater or equal to 2 contracted with $A$ since

$$
(\nabla_{\mu_1} \nabla_{\mu_j} \omega) A^{\mu_1 \ldots \mu_s} = \left[ \left( \nabla_{\mu_1} \nabla_{\mu_j} - \frac{1}{d} g_{\mu_1 \mu_j} \Box \right) \omega \right] A^{\mu_1 \ldots \mu_s} = 0,
$$
in which one uses the tracelessness of $A$ and then that eq. (5.10) is fulfilled by $\omega$. This simplifies greatly the expansion of $\bar{\phi}$ as one then gets:

$$
\bar{\phi} = \sum_{i=0}^{s} \frac{\Gamma(h+1)}{\Gamma(h+1-i)} \left( \begin{array}{c} s \\ i \end{array} \right) \omega^{h-i} (\nabla \omega)^{i} \nabla^{s-i} A
$$

$$
\quad = \sum_{i=0}^{s} \frac{\Gamma(h+1)}{\Gamma(h+1-i)} \left( \begin{array}{c} s \\ i \end{array} \right) \omega^{h-i} (\nabla \omega)^{i} \nabla^{s-i} A, \tag{5.11}
$$

using the notation in which an index contracted with $A$ is not written. For instance a generic term in eq. (5.11) reads as:

$$
(\nabla \omega)^{i} \nabla^{s-i} A = (\nabla_{\mu_1} \omega) \ldots (\nabla_{\mu_i} \omega) (\nabla_{\mu_{i+1}} \ldots \nabla_{\mu_s} A^{\mu_1 \ldots \mu_s}). \tag{5.12}
$$

Indeed, any term in the expansion of $\bar{\phi}$ might be brought into the form of eq. (5.12) since $A$ is fully contracted and symmetric. Now one can express $\nabla^n A$ in terms of $\nabla$ and $\omega$. By induction on $n$ one would get:

$$
\nabla^n A = \sum_{i=0}^{n} \frac{\Gamma(d+2s-n-1+i)}{\Gamma(d+2s-n-1)} \left( \begin{array}{c} n \\ i \end{array} \right) (-1)^i \rho^{-i} (\nabla \rho)^i \nabla^{n-i} A
$$

$$
\quad = \sum_{i=0}^{n} \frac{\Gamma(d+2s-n-1+i)}{\Gamma(d+2s-n-1)} \left( \begin{array}{c} n \\ i \end{array} \right) \omega^{-i} (\nabla \omega)^i \nabla^{n-i} A, \tag{5.13}
$$

using the same argument about the derivatives of $\rho$, with eq. (5.10), and finally $\omega^{-1}(\nabla \omega) = -\rho^{-1}(\nabla \rho)$ to recast the result in terms of $\omega$ solely. Then plugging eq. (5.13) in eq. (5.11)
and inverting the order of summation yields:

\[
\bar{\phi} = \sum_{i=0}^{s} \frac{s!}{s - i!} \frac{\Gamma(h + 1)}{(i)} \frac{\Gamma(d + s - 1 + i)}{\Gamma(h + 1 - j)} \frac{\Gamma(d + s - 1 + j)}{\Gamma(d - s + 1 + j)} \left( \frac{s}{i} \right) \omega^{h-j} (\nabla \omega)^i \nabla^{s-i} A
\]

\[
= \sum_{i=0}^{s} \Gamma(d + s - 1 + i) \Gamma(d + s - 1) 2F_1 (-i, -h; d + s - 1; 1) \left( \frac{s}{i} \right) \omega^{h-j} (\nabla \omega)^i \nabla^{s-i} A
\]

\[
= \sum_{i=0}^{s} \frac{s!}{s - i!} (i + 1) \omega^{-1-s-i} (\nabla \omega)^i \nabla^{s-i} A,
\]

in which \( h \) has been set to its value and \( d = 4 \), \( 2F_1 \) is a hypergeometric function. Replacing \( \bar{\phi} \) by eq. (5.14) and applying the conformal laplacian to it yields:

\[
(\Box - \frac{1}{6} \bar{R}) \phi = \sum_{i=0}^{s} \frac{s!}{s - i!} (i + 1) \left[ \omega^{-s-i-3} \left( (\nabla \omega)^i \left( \Box - \frac{1}{6} \bar{R} \right) + i(\nabla \omega)^{i-1}(\nabla \omega) \right) + (s + i)(\nabla \omega)^{i-1} \left( \nabla \omega \right)(\nabla \omega)^i \right] + 2(\nabla \omega)^i (\nabla \omega)^{i-1} (\nabla \omega) (\nabla \omega)^i
\]

Using eq. (5.8) simplifies the above equation to:

\[
(\Box - \frac{1}{6} \bar{R}) \phi = \sum_{i=0}^{s} \frac{s!}{s - i!} (i + 1) \omega^{-s-i-3} (\nabla \omega)^i \left( \Box - \frac{1}{6} \bar{R} \right) \nabla^{s-i} A
\]

\[
- \sum_{i=1}^{s} \frac{s!}{s - i!} 2i \omega^{-s-i-3} (\nabla \omega)^{i-1} (\nabla \omega) (\nabla \omega) \nabla^{s-i} A
\]

\[
+ \sum_{i=2}^{s} \frac{s!}{s - i!} (i - 1) \omega^{-s-i-3} (\nabla \omega)^{i-2} (\nabla \omega) (\nabla \omega) \nabla^{2} \nabla^{s-i} A
\]

\[
- \sum_{i=1}^{s} \frac{s!}{s - i!} i(i + 1) \omega^{-s-i-4} (\nabla \omega)^i \nabla^{s-i} A
\]

\[
- 3s \frac{s!}{s - i!} \omega^{-2s-4} (\nabla \omega)^{s} A
\]

\[
+ \sum_{i=1}^{s} \frac{s!}{s - i!} i(i + 1) \omega^{-s-i-3} (\nabla \omega)^{i-1} (\nabla \omega) \nabla^{s-i} A.
\]

In the latter one can recognize, abusing a bit the notation, the contraction of \( (\nabla \omega)^i \) with STG \( \nabla \nabla^{s-i} A \) in the second and third line. Substituting eq. (5.9) in the sixth line cancels the fourth and fifth line and changes the coupling to the curvature to: \( 1/6 + i/12 = \)
\((i + 2)/12\). Hence, one gets:

\[
\left( \Box - \frac{1}{6} R \right) \phi = \sum_{i=0}^{s} \frac{s!}{s-i!} (i+1) \omega^{-s-i-3} (\nabla \omega)^i \left( \Box - \frac{2}{i+1} \right) \text{STG} \nabla - \frac{2+i}{12} R \right) \nabla^{s-i} A
\]

\[
= \sum_{i=0}^{s} \frac{s!}{s-i!} (i+1) \omega^{-s-i-3} (\nabla \omega)^i E_i (\nabla^{s-i} A), \quad (d = 4).
\]

On the space of solutions of \(E_s(A) = 0\), according to eq. (4.12), each term with \(1 \leq i \leq s\) vanishes thus leaving:

\[
\left( \Box - \frac{1}{6} R \right) \phi = \omega^{-s-3} \left( \Box - \frac{1}{6} R \right) \phi.
\]

This identity concludes the proof of the invariance of the gauge fixed set given in eq. (1.3) with respect to Weyl rescalings between two CFES.

### 5.3 The residual gauge freedom

Equation (5.4) provides a conformally invariant gauge fixing equation of eq. (1.1). Here it is shown that \(\varphi\) in eq. (4.13) is no longer arbitrary and its remaining gauge freedom, allowed by eq. (1.3), is found.

To begin let us consider a pure gauge field:

\[ A = \text{STG}^s(\varphi). \quad (5.15) \]

Then, plugging eq. (5.15) into eq. (5.4) yields:

\[
\left( \Box - \frac{1}{6} R \right) \nabla^s \text{STG}^s(\varphi) = \left( \Box - \frac{1}{6} R \right) \nabla^{s-1} (\nabla \cdot \text{STG}) (\text{STG}^{s-1}(\varphi))
\]

\[
= \left( \Box - \frac{1}{6} R \right) \nabla^{s-1} U^{s-1} (\text{STG}^{s-1}(\varphi))
\]

\[
= \alpha_s \left( \Box - \frac{1}{6} R \right) U^{s-1}(\nabla^{s-1} \text{STG}^{s-1}(\varphi))
\]

with

\[ U^s(A) = \Box A + \left( \frac{d+2s-4}{d+2s-2} \right) \text{STG}(\nabla \cdot A) + \frac{s(d+s-2)}{d(d-1)} RA \quad (5.19) \]

for \(A \in S^s_\circ\) and for \(\phi\) a scalar field:

\[ U^{s'}(\phi) = \left( \Box + \frac{s(d-1+s)}{d(d-1)} R \right) \phi, \quad (5.20) \]

and \(\alpha_s\) a non-vanishing numerical factor (namely \(\alpha_s = (1+s)(d+s-2)(d+2s-2)^{-1}\)). Going from eq. (5.17) to eq. (5.18), that is obtaining eq. (5.20) from eq. (5.19), is performed in the same vein as the computation already carried in 5.1, for which the commutation relations between divergences and a second order equation akin to eq. (5.19) were obtained.

Then, carrying this scheme in eq. (5.16) till its end with eq. (5.20) and minding that \(d = 4\), one gets that the scalar gauge field \(\varphi\) fulfills:

\[ \left( \Box - \frac{1}{6} R \right) \left( \Box + \frac{1}{3} R \right) \times \cdots \times \left( \Box + \frac{2}{12} R \right) \varphi = 0. \quad (5.21) \]

Therefore in the gauge transformation (4.13) the scalar field \(\varphi\) is no longer unconstrained.
5.3.1 Remark on the residual gauge freedom on de Sitter

If the underlying CFES happens to be the de Sitter space-time the equations governing the residual gauge invariance might be linked to peculiar scalar representations of the de Sitter group $SO_0(1,d)$. Indeed, in such a case, for which $R = -d(d-1)H^2$ with $H^2$ the constant Hubble radius, the first order Casimir operator $Q_1$ in the scalar representation of $SO_0(1,d)$ is related to the Laplace-Beltrami operator through $Q_1 = -H^{-2}\Box$. Then eq. (5.20) can be recast as

$$[Q_1 + j(d - 1 + j)]\phi = 0, \quad j \in \mathbb{N}.$$ 

This, precisely, is the equation fulfilled by a scalar field in the discrete series of $SO_0(1,d)$.

As a consequence, if the gauge scalar $\varphi$ transforms covariantly under the de Sitter group, it decomposes as

$$\varphi = \varphi_{cc} \oplus \bigoplus_{j=0}^{s-1} \varphi_{ds(j)},$$

in which $\varphi_{cc}$ stands for the massless conformally coupled field [28], which lies in the complementary series of $SO_0(1,d)$, and $\varphi_{ds(j)}$ is the $j$'th term in the discrete series of $SO_0(1,d)$ [29, 30]. The 0'th term of the discrete series is the, infamous, massless minimally coupled field [31–35], and field further in the discrete series are the, so-called, scalar tachyons [36].

5.4 The residual gauge freedom and its relation with GJMS operators

Let us write eq. (5.21), fulfilled by $\varphi$ once the gauge fixing equation has been imposed, as:

$$P_{2n}\varphi = \left[ \prod_{\ell=1}^{n} \left( \Box + \frac{(\ell + 1)(\ell - 2)}{12} R \right) \right] \varphi = 0, \quad n = s + 1, \quad d = 4. \quad (5.22)$$

This equation is known as Branson’s factorization formula of GJMS operators and is conformally invariant.

5.5 Conformally invariant powers of the laplacian

Let us recall that Graham-Jenne-Mason-Sparling (GJMS) [37–39] results come from the question whether or not there is a curved analogue of the conformally invariant $(SO_0(2,d))$ flat operator $\Box^n$, $n \in \mathbb{N}$. That is, does there exists an operator $P_{2n}$ from densities of weight $n - d/2$ to densities of weight $-n - d/2$ on a conformal manifold of dimension $d \geq 3$ whose leading symbol is $\Box^n$? Their result is the following. For $d$ odd there exists one such operator. For $d$ even $P_{2n}$ exists provided that the bounds $1 \leq n \leq d/2$ are satisfied. In that respect for $d = 4$ the Paneitz operator, eq. (3.11), is the critical GJMS operator. Later Branson [3, 40], through a harmonic analysis argument, proved that on $S^d$ equipped with its standard metric that $P_{2n}$ reduces to:

$$P_{2n}\varphi = \left[ \prod_{\ell=1}^{n} \left( \Box + \frac{(2\ell - 2 + d)(2\ell - d)}{4d(d - 1)} R \right) \right] \varphi.$$ 

On conformally flat Einstein spaces, since the Fefferman-Graham obstruction tensor $O_{\mu\nu}$ vanishes and the ambient metric can be recovered at arbitrary order [41], $P_{2n}$ exists
to arbitrary order. However, for $n > d/2$, $d$ even, these operators are no longer natural conformally invariant differential operators \cite{4}. Branson’s factorization formula has been extended to Einstein metrics in \cite{2,4}. For a study of the (higher) symmetries of such operators see \cite{42–44}.

5.6 Restricted invariance of eq. (5.22) and SO(2,4) invariance of the pure gauge solutions

In this subsection, we supply direct proof of the conformal invariance of eq. (5.22) by mimicking the calculus of the conformal invariance of eq. (3.11) through that of eq. (3.9). That is, this computation relies completely on the fact that for $d = 4$ both conformal invariance and gauge invariance are present. In doing so it is shown that the space of pure gauge solutions to eq. (1.1) remains invariant under the action of SO(2,4).

To exemplify our scheme let us, again, consider the vectorial case. The main idea is that the gauge fixing condition (3.9) is conformally invariant on the space of solutions of eq. (3.5), for $d = 4$. Then, if one plugs in eq. (3.9) a pure gauge solution the “on the space of solutions of eq. (3.5)” is already taken care of and one is left to look if there is a Weyl rescaling of the scalar field resulting (only) in the appropriate rescaling of the pure gauge vector. That is, does there exist $w \in \mathbb{R}$ such that the equation

$$A_\mu = \nabla_\mu \omega^w \varphi = \omega^0 \nabla_\mu \varphi = A_\mu,$$

is fulfilled? The answer is positive with $w = 0$. Extended to our case the question now is, does there exist $w \in \mathbb{R}$ such that

$$\text{STG}^s(\omega^w \varphi) = \omega^h \text{STG}^s(\varphi)$$

with $h(A) = 1 - s - d/2$?

First let us study the case with $s = 2$. With no assumptions on $\omega$ one obtains:

$$\frac{\text{STG}^2(\varphi)^\mu}{\text{STG}^2(\varphi)} = \frac{\omega^w - 4}{\omega^{w-4}} \left[ (\nabla^\mu \omega)(\nabla^\nu + 2 d g^\mu \partial^\nu (\nabla_\alpha \omega)) \right] \varphi$$

$$+ 2 w (w-1) \omega^{w-5} \left[ \left( \nabla^\mu \omega \right)(\nabla^\nu + 2 d g^\mu \partial^\nu (\nabla_\alpha \omega)) \right] \varphi$$

$$+ 2 w (w-1) \omega^{w-6} \left[ \left( \nabla^\mu \omega \right)(\nabla^\nu + 2 d g^\mu \partial^\nu (\nabla_\alpha \omega)) \right] \varphi$$

$$- 2 w \omega^{w-3} \left[ \left( \nabla^\mu \nabla^\nu - \frac{1}{d} \right) \frac{1}{\omega} \right] \varphi. \quad (5.24)$$

If one sets $w = 1$ the right hand side of eq. (5.24) is greatly simplified but, still, does not produce the desired result. Then, one can notice that under the restricted Weyl transformation eq. (4.2) that the remaining term vanishes. This is the point of view that we will adopt here, but it also hints at the gauge invariance

$$A_{\mu\nu} \mapsto \varphi A_{\mu\nu} = A_{\mu\nu} + \left[ \nabla_\mu \nabla_\nu - \frac{1}{d} g_{\mu\nu} \Box + \frac{1}{d-2} \left( R_{\mu\nu} - \frac{1}{d} g_{\mu\nu} R \right) \right] \varphi$$

of eq. (4.6) on (more) generic space-time.
Setting $\rho = \omega^{-1}$, $v = -w$ and noticing the identity:

$$\overline{\text{STG}}(f) = \rho^2 \text{STG}(f), \quad \forall f \in S_0^s,$$

simplifies the right hand side of eq. (5.23) which might then be expressed as:

$$\overline{\text{STG}}^s(\omega^w \varphi) = \overline{\text{STG}}^s(\rho^v \varphi) = \rho^2 \text{STG}(\rho^2 \text{STG}(...(\rho^v \varphi)...)) \quad (s \text{ times}).$$

To get a tractable formula let us also use the following notation:

$$\times : S_0^{s_1} \times S_0^{s_2} \rightarrow S_0^{s_1+s_2},$$

$$f, g \mapsto (s_1 + s_2)! \overline{(f \cdot g)}_{ST}\frac{s_1! s_2!}{s_1^{s_2}!}$$

in which $|ST$ is the projector ($|ST|_{ST} = |ST|$) onto $S_0^{s_1+s_2}$. On scalars it reduces to the pointwise product of two functions. Then, thanks to this definition, accommodated to that of the symmetric traceless gradient (STG) given in eq. (4.8), one has a Leibniz identity:

$$\text{STG}(f \times g) = \text{STG}(f) \times g + f \times \text{STG}(g).$$

Then, beginning with $\rho^v \varphi = \rho^v \times \varphi$, using eq. (4.2) to discard any term such as $\text{STG}^i(\rho)$ with $i \geq 2$ and an induction on the degree yields:

$$\overline{\text{STG}}^s(\rho^v \varphi) = \sum_{i=0}^{s} \frac{\Gamma(v+s)}{\Gamma(v+s-i)} \left(\frac{s}{i}\right) \rho^{v+2s-i} (\text{STG}(\rho)) \times i \times \text{STG}^{s-i}(\varphi).$$

Finally, setting $w = s - 1$, that is $v = 1 - s$, to cancel the terms with $i \geq 1$, and restoring the $\omega$'s provides the identity:

$$\text{STG}(\omega^{s-1} \varphi) = \omega^{-s-1} \text{STG}^s(\varphi). \quad (5.25)$$

However, for the equation eq. (5.22) to be conformally invariant the scaling factor of the resulting field in eq. (5.25) has to agree with that of $A$, namely:

$$h(A) = 1 - s - \frac{d}{2} = -s - 1, \quad (5.26)$$

which is achieved only for $d = 4$. The SO(2,4) invariance of eq. (5.22) is inherited from that of $\Box^n$ on Minkowski space-time, a well known result [45].

Notice that the identity obtained in eq. (5.25) can be interpreted in the following manner: a pure gauge solution on $(M, g)$ is mapped on a pure gauge solution on $(M, \bar{g})$. Similarly, this implies that the space of pure gauge solutions of eq. (1.1) is left invariant under the action of SO(2,4).

6 The field strength $F$

It has been shown in section 4.3 that eq. (1.1) admits gauge solutions, through the gauge transformation given in eq. (4.13). This section is devoted to the construction, upon the potential $A$, of a field strength $F$ similar to the Faraday tensor, eq. (3.7).
To tackle this problem we proceed in close analogy with Maxwell’s case \((s = 1)\) and obtain a strength tensor for the general case \((s \geq 2)\). First, the equivalent of Maxwell’s equations, written as a divergence, provides a candidate \(F\) for the field strength, its symmetries are registered. Then, it is showed that \(F\) vanishes on gauge solutions, on CFES, hence is a field strength. The equations on \(A\) are then translated into a system of two first-order conformally invariant equations on \(F\), one of which vanishes for \(F\) derived from a potential and \((M, g)\) being conformally flat. Then, as the Faraday tensor decomposes in electric and magnetic fields \(E-B\), it is explicitly shown that \(F\) decomposes in a set of 3-dimensional traceless symmetric tensors. We then show, on flat space, how the conformally invariant system of equations is written on those \(E-B\) fields and prove their duality. Finally, a gauge invariant and conformally invariant scalar product on the space of solutions of eq. (1.1) is given.

6.1 Reconstructing the field strength \(F\)

To produce a field strength let us consider, instead of eq. (1.1), the equation:

\[
\Box A - \left(\frac{d + 2s - 4}{s(d + s - 3)}\right) \text{STG}(\nabla \cdot A) - \left(\frac{d + s - 2}{d(d - 1)}\right) RA = 0, \tag{6.1}
\]

which, in arbitrary dimension, has the gauge invariance showed in eq. (4.13). For \(s = 1\) eq. (6.1) is the restriction of Maxwell equation \(\nabla_\mu (\nabla^\mu A^\nu - \nabla^\nu A^\mu) = 0\) to a CFES. For \(d = 4\) eq. (1.1) and eq. (6.1) are the same. Under the assumption that eq. (3.13) is fulfilled eq. (6.1) can be recast as

\[
\nabla_\alpha (F(A)^\alpha_{\mu_1 \cdots \mu_s}) = 0, \tag{6.2}
\]

where \(F\) is built upon \(A\) by:

\[
(F(A))^{\alpha_{\mu_1 \cdots \mu_s}} = (DA)^{\alpha_{\mu_1 \cdots \mu_s}} \\
= \nabla^{\alpha} A^{\mu_1 \cdots \mu_s} - \nabla^{(\mu_1} A^{\mu_2 \cdots \mu_s)_{\alpha}} \\
- \frac{(s-1)}{(d+s-3)} [g^{\alpha(\mu_1} \nabla_{\sigma} A^{\mu_2 \cdots \mu_s)_{\sigma}} - g^{\mu_1 \mu_2} \nabla_\sigma A^{\mu_3 \cdots \mu_s}_{\alpha\sigma}], \tag{6.3}
\]

found already in [46] and from which we borrow the notation \(D\). For \(s = 1\) one has \(F^{\alpha \mu} = \nabla^\alpha A^\mu - \nabla^\mu A^\alpha\). We will show, on CFES, that \(F\) is a field strength as it is independent of the gauge scalar field \(\varphi\).

Notice that on a generic space-time, without the assumption that eq. (3.13) is fulfilled, the equation resulting from (6.2) is still for \(d = 4\) a conformally invariant equation as one, schematically, gets:

\[
\nabla_\alpha (F(A)^\alpha_{\mu_1 \cdots \mu_s}) = (4.6) + (4.7), \quad \lambda = \frac{s + 2}{s(s - 1)}, \quad d = 4.
\]

From eq. (6.3) one can workout the identities fulfilled by the field strength, namely:

\[
F^{(\alpha_{\mu_1 \cdots \mu_s})} = 0, \quad g_{\mu_1 \mu_2} F^{\alpha_{\mu_1 \cdots \mu_s}} = 0, \quad g_{\alpha \mu_1} F^{\alpha_{\mu_1 \cdots \mu_s}} = 0, \tag{6.4}
\]
thanks to which one can find that \( F \) possesses
\[
\text{DoF}(F) = \frac{(d+s-4)!}{(d-3)!(s+1)!} s(d+2s-2)(d+s-2)
\] (6.5)

independent components. Such as the Faraday tensor, the relations given in eq. (6.4) could serve as the defining properties of \( F \) and under the appropriate assumptions eventually there would exist a potential \( A \) such that \( F \) is given by eq. (6.3). Later, in section 6.3, it is shown that the equation \( \nabla_\alpha F^{\alpha \mu_1...\mu_s} = 0 \) can be completed by another equation of motion which vanishes as soon as there exists a potential \( A \) (similar to the sourceless Maxwell’s equations) and \( g \) is conformally flat.

6.2 On a CFES the field strength \( F \) is gauge invariant

Let \((M, g)\) be a CFES and consider the field \( A = \text{STG}(f) \) with \( f \in S^{s-1}_s \). Let us also make the extra assumption that the field \( f \) satisfies eq. (6.1) written at the rank \( s-1 \). Then, after some algebra in which the strategy is that as soon as a laplacian hits \( f \) to replace it by the remaining of eq. (6.1), one gets:
\[
(F(\text{STG}(f)))^{\alpha \mu_1...\mu_s} = (s-1) \left[ \nabla^{[\mu_1}(F(f))^{\alpha]} \mu_2...\mu_s \right] - \frac{s-1}{d+2s-4} g^{\mu_1\mu_2} \nabla_{\sigma} (F(f))^{[\alpha]} \mu_3...\mu_s
\]
\[
= \left( s-1 \right) \left( \text{STG}(F(f)) \right)^{[\alpha]} \mu_1...\mu_s,
\]
in which the \(|\alpha|\) means that this superscript \( \alpha \) is not involved in the STG process. Suppressing the indices it reads as
\[
F(\text{STG}(f)) = \left( \frac{s-1}{s} \right) \text{STG}(F(f)).
\] (6.6)

This scheme can be pushed further. Let \( g \in S^r_s \), \( r < s \), and in addition let \( g \) be a solution of eq. (6.1) written at the rank \( r \). Using eq. (4.12) adapted to eq. (6.1) it follows that \( \text{STG}^n(g) \) is a solution of eq. (6.1) written at the rank \( r + n \). Then, from eq. (6.6) one gets:
\[
F(\text{STG}^{s-r}(g)) = \frac{r}{s} \text{STG}^{s-r}(F(g)).
\] (6.7)

Finally, one can consider a scalar field \( \varphi \) and a pure gauge field obtained from that scalar \( A = \text{STG}^s(\varphi) \). Since eq. (6.1) has solutions determined up to a scalar \( \text{STG}^n(\varphi) \) is a solution whatever the rank is and the identity (6.7) reads as
\[
F(\text{STG}^s(\varphi)) = \frac{1}{s} \text{STG}^{s-1}(F(\text{STG}(\varphi))) = 0,
\]
in which the last equality is nothing but eq. (3.8). Thus, the field strength \( F \) vanishes on gauge solutions.

The gauge invariance of \( D_A \) was already noted in flat space in [46], for a generalization to more generic (w.r.t. CFES) curved background one would have to have a better understanding of the gauge solutions.
6.3 The equations on $F$ and their conformal invariance

Presently we have found that, at least on CFES, one can find a field strength $F$ and that the equation on the potential $A$ might be rewritten as a first order equation on $F$ (eq. (6.2)). From Maxwell’s equations point of view we do know that then there lack a second equation (the sourceless one) to get a well posed set of PDEs on $F$. Here we show that one can find such one set of equations and that all the properties found for Maxwell’s equations remain as soon as one restricts oneself to conformally flat space-time.

Let us assume that $F$ fulfils eq. (6.4), with no further assumptions, and let us define

$$ (DF)^{\alpha\beta\, \mu_1...\mu_s} = \nabla^{[\alpha\, F^\beta]}_{\mu_1...\mu_s} + \frac{s}{s+1} \nabla^{[\mu_1\, F]_{[\alpha\, \beta][\mu_2...\mu_s]}} $$

$$ - \frac{s}{(d+s-4)(s+1)} \left[ s g^{[\mu_1][\alpha\, \nabla_\sigma F^\beta]}_{\mu_2...\mu_s}\sigma ight] $$

$$ + (s-1) g^{(\mu_1\mu_2\nabla_{\sigma} F^{(\alpha\, \beta)(\mu_3...\mu_s)})} + g^{(\mu_1)[\alpha\, \nabla_{\sigma} F^{(\beta)\mu_2...\mu_s)}} \right] $$

in which the notation $D$ seems natural if one compares eq. (6.8) to eq. (6.3). Notice that for $s = 1$ one gets

$$ (DF)^{\alpha\beta\, \mu} = \nabla^\alpha F^\beta \mu - \nabla^\beta F^\alpha \mu + \frac{1}{2} \nabla^\mu F^{(\alpha\, \beta)} = \nabla^\alpha F^\beta \mu + \nabla^\beta F^\mu \alpha + \nabla^\mu F^\alpha \beta, $$

Then, if $F$ happens to derive from a potential $A$ as in eq. (6.3) one would get

$$ (DF)^{\alpha\beta\, \mu_1...\mu_s} = (D^2 A)^{\alpha\beta\, \mu_1...\mu_s} $$

$$ = (1-s) \left[ C^{\alpha\beta\, \sigma}_{\mu_1 A^\mu_2...\mu_s}\sigma + C^{(\mu_1\mu_2)[\alpha\, A^\beta][\mu_3...\mu_s)]}\sigma \right] $$

$$ + \frac{(1-s)(2-s)}{d+s-4} C_{\nu_1\nu_2 \sigma} \left[ g^{\delta_{\rho} g^{(\alpha\, \beta)}(\mu_1 \delta^\rho_{\nu_2} \delta^\mu_3 + \delta^\rho_{\nu_2} g^{(\alpha\, \beta)}(\mu_1 \delta^\rho_{\nu_2} \delta^\mu_3 $$

$$ + \delta^\rho_{\nu_2} g^{(\gamma\, \delta)(\mu_1 g^\mu_2)} \right] A^{\mu_4...\mu_s)}_{\nu_1\nu_2\nu_3} $$

which vanishes when either $s = 1$ (thanks to Bianchi identity, already used here) or if $(M, g)$ is conformally flat for which one gets:

$$ D^2 A = 0, \quad (M, g) \equiv \text{conformally flat}. $$

One then is led to consider the system

$$ \begin{cases} (DF)^{\alpha\beta\, \mu_1...\mu_s} = 0, \\ \nabla_\alpha F^\alpha_{\mu_1...\mu_s} = 0, \end{cases} \quad (M, g) \equiv \text{conformally flat}. \quad (6.9) $$

In addition one shows, taking into account the symmetries registered in eq. (6.4), that if, and only if, $d = 4$ the set (6.9) is conformally invariant with $\overline{F}^{\alpha\, \mu_1...\mu_s} = \omega^{-3-s} F^{\alpha\, \mu_1...\mu_s}$. If $F$ derives from a potential $A$ this conformal invariance agrees with that of $F$ with

$$ (\overline{F}(A))^{\alpha\, \mu_1...\mu_s} = \omega^{-3-s} (F(A))^{\alpha\, \mu_1...\mu_s}, \quad \overline{A}^{\mu_1...\mu_s} = \omega^{-1-s} A^{\mu_1...\mu_s}, \quad d = 4. \quad (6.10) $$

Finally, still for $d = 4$, instead of the set (6.9) one would rather consider the system

$$ \begin{cases} (d DF)^{\mu_1...\mu_s} = 0, \\ \nabla_\alpha F^\alpha_{\mu_1...\mu_s} = 0, \end{cases} \quad (M, g) \equiv \text{conformally flat}, \quad d = 4, \quad (6.11) $$
in which \((*DF)^{\mu_1...\mu_s} = s \varepsilon^{[\mu_1}_{\alpha\beta\gamma} (DF)^{\alpha\beta]} \mu_2...\mu_s)\gamma\), with \(\varepsilon\) the totally antisymmetric tensor.

To obtain a wave operator on the field strength \(F\) one could consider the system

\[
\begin{align*}
\nabla_\beta (DF)^{\alpha\beta \mu_1...\mu_s} &= 0, \\
\nabla_\alpha F^{\alpha \mu_1...\mu_s} &= 0.
\end{align*}
\]

Let us simply remark that this system is no longer conformally invariant.

### 6.4 An \(E-B\) decomposition of \(F\)

Notice, from eq. (6.5), that for \(d = 4\) the following occurs:

\[
\text{DoF}(F) \bigg|_{d=4} = 2s(s + 2) = 2 \sum_{j=1}^{s} (2j + 1).
\]

That is, \(F\) possesses the appropriate number of independent components to be written as the sum over \(2s\) symmetric traceless tensors with respect to \(\text{SO}(3)\). This subsection investigates one such decomposition and its consequences.

First, we propose a definition of such fields and immediately show that, indeed, \(F\) is entirely expressible in terms of those fields. Then, we write how the set (6.11) is translated on those fields, a task simplified once one has shown that (6.11) have a duality property. Finally, we register how \(\text{so}(2,4)\) acts upon those fields.

Every computation is performed on flat space from now on. The results which are obtained here can be lifted to conformally flat space-time through eq. (6.10).

#### 6.4.1 Definition of the \(E-B\) fields

On flat space let us define

\[
\begin{align*}
E^{i_1...i_j}_j &= M^{i_1...i_j}_j - \text{traces}, \\
B^{i_1...i_j}_j &= \frac{1}{s+1} N^{i_1...i_j}_j - \text{traces},
\end{align*}
\]

with \(1 \leq j \leq s\), the traces are with respect to the 3-dimensional metric \(\delta^{ij} = \delta^{ij} = -\eta_{ij}\) and we defined

\[
M^{i_1...i_j}_j = F^{0...0i_1...i_j}, \quad N^{i_1...i_j}_j = j \varepsilon^{(i_1|bc F^{bc} 0...0|j_2...i_j)}.
\]

One can decompose \(F\) in the (analogues of) \(E\) and \(B\) if an arbitrary component can, unambiguously, be written in terms of such 3-dimensional fields, that is if one can invert eqs. (6.12)-(6.13).

#### 6.4.2 The inversion of eqs. (6.12)-(6.13)

The \(M_j-N_j\)’s in terms of the \(E_j-B_j\)’s. From the properties fulfilled by \(F\), recorded in eq. (6.4), and the definition (6.13) one finds that

\[
\begin{align*}
\delta_{iaib} M^{i_1...i_j}_j &= M^{i_1...\hat{i}_a...\hat{i}_b...i_j}_j, \\
\delta_{iaib} N^{i_1...i_j}_j &= N^{i_1...\hat{i}_a...\hat{i}_b...i_j}_j.
\end{align*}
\]
in which a hat over an index means that said index is omitted. From eq. (6.12) and eq. (6.14) one can invert the $M_j$'s in terms of the $E_j$'s since

$$E_j^{i_1 \ldots i_j} = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \alpha_{jk} \delta^{(i_1 i_2 \ldots i_{2k})} \delta^{(i_{2k+1} \ldots i_j)} M_{j-2k}^{i_1 \ldots i_{2k+1} \ldots i_j}, \quad (6.15)$$

in which the coefficients $\alpha_{jk}$ are determined by asking that the right hand side of eq. (6.15) is traceless, which is well posed thanks to eq. (6.14) once one has chosen an $\alpha_{j0}$ (here $\alpha_{j0} = 1$ for $E$ and $M$ and $\alpha_{j0} = 1/s + 1$ for $B$ and $N$).

Then, eq. (6.15) might, rightly, be viewed as an upper triangular transformation between the $E_j$'s and the $M_j$'s and as such can be inverted with:

$$M_j^{i_1 \ldots i_j} = E_j^{i_1 \ldots i_j} - \alpha_{j1} \delta^{(i_1 i_2 E_j^{i_3 \ldots i_j})} - (\alpha_{j2} - \alpha_{j1}\alpha_{j-2\ 1}) \delta^{(i_1 i_2 \delta^{(i_3 i_4 E_j^{i_5 \ldots i_j})})} - \ldots$$

$$= \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \beta_{jk} \delta^{(i_1 i_2 \ldots i_{2k}} \delta^{(i_{2k+1} \ldots i_j)} E_{j-2k}^{i_1 \ldots i_{2k+1} \ldots i_j}).$$

The same arguments apply to the inversion of the $B_j$'s in terms of the $N_j$'s. One could, actually, find that:

$$\alpha_{jk} = (-1)^k \frac{j!}{j-2k!} \left( \frac{2j - 2k!!}{2j - 1!!} \right) = (-1)^k \frac{j!}{2j - k!} \left( \frac{2j - 2k!!}{2j - 1!!} \right) = \frac{j!}{2j - k!} \left( \frac{2j - 2k!!}{2j - 1!!} \right)$$

$$\beta_{jk} = \frac{j!}{j-2k!} \left( \frac{2j - 4k + 1!!}{2j - 2k + 1!!} \right) = \frac{j!}{2j - k!} \left( \frac{2j - 4k + 1!!}{2j - 2k + 1!!} \right)$$

for $k \geq 1$ and $\alpha_{j0} = \beta_{j0} = 1$ otherwise. In the above we used the identity on bifactorials of odd numbers: $2n - 1!! = 2n!/2^n n!$.

$F$ in terms of the $M_j$-$N_j$'s. We are now concerned with the second step of the inversion, that is to write an arbitrary component of $F$ in terms of the $M_j$'s and the $N_j$'s. From the definition (6.13) an arbitrary $F_{0 \ldots 0 i_1 \ldots i_j}$ might already be written in terms of the $M_j$'s. What is left to show is that $F^l_{0 \ldots 0 i_1 \ldots i_j}$ might also be written in terms of the $M_j$'s and the $N_j$'s.

From the symmetries of $F$, cf. eq. (6.4), one can write the following (sub-)identity:

$$F^l_{i_1 \ldots i_j 0 \ldots 0} = \frac{j}{j + 1} F^{ll}_{(i_1 i_2 \ldots i_j) 0 \ldots 0} - \left( \frac{s - j}{j + 1} \right) F^0_{i_1 \ldots i_j 0 \ldots 0}, \quad (6.16)$$

in which the second term of the r.h.s., from the above remark, is already inverted in terms of the $M_j$'s. Then, it suffices to show that one can express the first term of the r.h.s. of eq. (6.16) unambiguously in terms of the $M_j$'s and the $N_j$'s.

Out of the definition (6.13) of the $N_j$'s, of the properties fulfilled by the antisymmetric tensor $\varepsilon_{ijk}$ and those of $F$, given in eq. (6.4), one can establish the following formula:

$$\varepsilon^{kl(i_1 N_j^{i_2 \ldots i_j)}} = j F^{ll}_{(i_1 i_2 \ldots i_j) 0 \ldots 0} - (s + 1) \delta^{(i_1 F^{00}_{i_2 \ldots i_j}) 0 \ldots 0}$$

$$- (j - 2) \delta^{(i_1 i_2 F^{ll}_{i_1 i_4 \ldots i_j}) 0 \ldots 0} + (s + 1) \delta^{(i_1 i_2 F^{00}_{i_3 \ldots i_j}) 0 \ldots 0}$$
put in a more suitable form, for our purpose, as:

\[ j F^l (i_1 i_2 ... i_j)_{0...0} = (j - 2) \delta^{(i_1 i_2} F^l [i_3 ... i_j)]_{0...0} \]

\[ + \varepsilon^{k (i_1 N^i_{j2} ... i_j)} k + (s + 1) (\delta^{(i_1 M_{j-1}^i} j - \delta^{(i_1 i_2 M_{j-1}^i j)}). \]

By recursively using the above formula one can invert \( F^l (i_1 i_2 ... i_j) \) in terms of the \( M_j \)’s and the \( N_j \)’s terminating with either

\[ F^l (i_1 i_2 ... i_j)_{0...0} = \varepsilon^{k l_1 N^k_1} \]

or

\[ F^l (i_1 i_2)_{0...0} + F^l (i_2 i_1)_{0...0} = \frac{1}{2} (\varepsilon^{k l_1 N^2_2} + \varepsilon^{k l_2 N^1_2}) \]

\[ + \frac{1}{2} (s + 1) (\delta^{i_1 M^i_1} + \delta^{i_2 M^i_2} - 2 \delta^{i_1 i_2 M^i_j}), \]

depending on the parity of \( j \).

This means that the first term in eq. (6.16) can unambiguously be written in terms of the \( N_j \)’s and the \( M_j \)’s. From eq. (6.13) it is also the case for the second term in eq. (6.16). Hence \( F^l \ 0...0i_1 ... i_j \) has a well posed decomposition over the \( M_j \)’s and the \( N_j \)’s.

**F in terms of the \( E_j\)-\( B_j \)’s.** Finally, \( F \) can be written in terms of the \( M_j\)-\( N_j \)’s, which themselves can be written in terms of the \( E_j\)-\( B_j \)’s. Hence, an arbitrary component of \( F \) might be written uniquely in terms of the \( E_j\)-\( B_j \)’s that is:

\[ F = \bigoplus_{j=1}^s (E_j \oplus B_j). \]  

(6.17)

6.4.3 The field equation written on the \( E_j\)-\( B_j \)’s

The conformally invariant equations fulfilled by \( F \) can be translated upon the \( E_j\)-\( B_j \)’s fields. Those equations are here derived, a task greatly simplified as soon as their duality have been found. First, instead of eq. (6.11), let us write the set under consideration as

\[ \left\{ \begin{array}{lcl} \partial_\alpha \tilde{F}^\alpha \mu_1 ... \mu_s = 0, \\ \partial_\alpha F^\gamma \mu_1 ... \mu_s = 0, \end{array} \right. \quad \text{on} \quad (\mathbb{R}^4, \eta), \]

(6.18)

in which we set \( *DF \propto \partial \tilde{F} \), thus getting:

\[ \tilde{F}^\alpha \mu_1 ... \mu_s = \frac{s}{s + 1} \varepsilon^{\alpha (\mu_1 \beta_1} F^{|\beta|} \mu_2 ... \mu_s) \gamma. \]

A careful inspection of \( \tilde{F} \) shows that it fulfils all the properties that \( F \) possesses, registered in eq. (6.4), and as such might be decomposed over fields \( \tilde{E}_j \) and \( \tilde{B}_j \) according to eqs. (6.12)-(6.13). Performing one such decomposition, while knowing the \( E_j \)’s and \( B_j \)’s of \( F \), one recognizes that:

\[ \tilde{E}_j = - B_j, \quad \tilde{B}_j = E_j, \]
and then $\tilde{F} = -F$, showing that eq. (6.18) remains invariant under the electric-magnetic duality

$$(E_j, B_j) \mapsto (-B_j, E_j).$$

(6.19)

This duality can be extended to infinitesimal rotations as:

$$\delta E_j = -\theta B_j, \quad \delta B_j = \theta E_j, \quad |\theta| \ll 1,$$

or integrated to the finite rotation:

$$\theta E_j = \cos \theta E_j - \sin \theta B_j,$$
$$\theta B_j = \sin \theta E_j + \cos \theta B_j,$$

(6.20)

as a symmetry of the system (6.18) and generalize to the known duality rotation of electromagnetism [47, 48]. An interesting question would be to find whether this applies also for the (on shell) action as it is the case for electromagnetism [49]? Notice however as showed recently [50], following [51, 52], that one such symmetry cannot be made local, at least not without sacrificing the gauge invariance [53]. It might be worth to stress that the duality (6.19), and then the rotation (6.20), has to be applied to all fields $(E_j, B_j)$ for all $j \leq s$.

Then, if one introduces the following notations:

$$\text{DIV}_3 E_j \equiv \partial_i E_j^{i \ldots j},$$
$$\text{STG}_3 E_{j-1} \equiv j^k (i_1 \partial_k E_j^{i_2 \ldots i_j}) - \text{traces},$$
$$\text{STC}_3 E_j \equiv \varepsilon^{kl} (i_1 \partial_k E_j^{i_2 \ldots i_j} l),$$

writing $H_j = E_j + i B_j$, with $i^2 = -1$, and thanks to the duality (6.19) the system (6.18) now reads as

$$\partial_t H_j - i \left( \frac{s + j}{j + 1} \right) \text{STC}_3 H_j - \left( \frac{s - j}{j + 1} \right) \text{DIV}_3 H_{j+1} + \frac{1}{j} \left( 1 + \frac{2(s-j)}{(j+1)(j+2)} \right) \text{STG}_3 H_{j-1} = 0,$$

with $H_j = 0$ for $j \leq 0$ and for $j > s$. The complex conjugated equation holds on the complex conjugated fields $\overline{H}_j$.

6.4.4 The action of SO(2,4) on the $E_j$-$B_j$ fields

Let us deduce the action of SO(2,4) on the $E_j$-$B_j$ fields from that on the field strength $F$ by:

$$(g E_j)(F) = E_j(g F), \quad g \in \mathfrak{so}(2, 4).$$
Then one gets:

\[
(P_\mu H_j)^{i_1...i_j} = \partial_\mu H_j^{i_1...i_j},
\]

\[
(DH_j)^{i_1...i_j} = (x \cdot \partial + 2)H_j^{i_1...i_j},
\]

\[
(X_{0m} H_j)^{i_1...i_j} = x_0[\partial_m] H_j^{i_1...i_j} + (\Sigma_{0m} H_j)^{i_1...i_j}
\]

\[
= x_0[\partial_m] H_j^{i_1...i_j} + \frac{s + 1}{j + 1} \varepsilon^{km(i_1} H_j^{i_2...i_j)k} - \frac{j(s - j)}{j + 1} H_j^{i_1...i_j} + \frac{(s + 3)j(j + 1) + 2(s + 1)}{(j + 1)(j + 2)} \left( \delta^{m(i_1} H_j^{i_2...i_j)j} - \frac{j - 1}{2j - 1} \delta^{i_1i_2} H_j^{i_3...i_j} \right),
\]

\[
(X_{mn} H_j)^{i_1...i_j} = x_{[m} \partial_{n]} H_j^{i_1...i_j} + (\Sigma_{mn} H_j)^{i_1...i_j} = x_{[m} \partial_{n]} H_j^{i_1...i_j} + j \delta_{[m}^{i_1} \delta_{n]}^{i_2} H_j^{i_3...i_j}.
\]

\[
(K_\rho H_j)^{i_1...i_j} = [x^2 \partial_\rho - 2x_\rho (x \cdot \partial + 2)] H_j^{i_1...i_j} + x^\sigma (\Sigma_{\sigma \rho} H_j)^{i_1...i_j} = x^\sigma (X_{\sigma \rho} H_j)^{i_1...i_j} - x_\rho (DH_j)^{i_1...i_j} - 2x_\rho H_j^{i_1...i_j}.
\]

Notice that the full conformal group acts independently on the fields $H_j$ and $\overline{H}_j$.

### 6.5 A Lagrangian and a scalar product out of $F$

Let us conclude this section with various remarks.

First, notice that the equation of motion, eq. (6.2), can be derived from the action:

\[
S(g, A) = \int \frac{s}{2(s + 1)} (F(A))_\alpha \mu_1...\mu_s (F(A))^\alpha \mu_1...\mu_s \ dVol_g
\]

\[
= \int \frac{s}{2(s + 1)} (F(A))^2 \ dVol_g.
\]

Notice also, in view of quantization, using eq. (6.4), that the elements on the "diagonal" of $F$ (i.e. $F^{0...0}$, $F^{1...1}$, ...) are null. Then, having chosen a system such that the 0’th coordinate is the one by which the space-time is foliated by Cauchy hypersurfaces one would get, in a conventional canonical quantization, that the 0’th momenta identically vanishes. This hints at a gauge freedom which is likely to persist on a broader class of space-time (than CFES).

Finally, again still in view of quantization, thanks to the operator $\mathcal{D}$ and eq. (6.21), one can equip the space of solutions to eq. (1.1) with a symplectic form

\[
\langle A, A' \rangle = \int \left[ A_\mu_1...\mu_s (\mathcal{D}A')^\alpha \mu_1...\mu_s - A'_\mu_1...\mu_s (\mathcal{D}A)^\alpha \mu_1...\mu_s \right] d\Sigma_\alpha,
\]

in which the integral is carried over the Cauchy hypersurface $\Sigma$. One such product, for $d = 4$, is conformally invariant and vanishes on pure gauge solutions.

### A The special case of $d = 2$

The dimension $d = 2$ is far too particular with respect to conformal invariance and cannot be examined on the same footing as the others. This section describes the (lack of) content of eq. (1.1) for $d = 2$. 

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One could consider eq. (1.1) and wonder what happens for $d = 2$. First, the conformal invariance under restricted Weyl transformations remains, the same applies to the invariance w.r.t. SO(2,2). However, following the same steps has in the main text, one would find that the equation admits solutions determined up to a vector (as then $(1 + a_2)|_{d=2} = 0$). Similarly, one could get the impression that the set:

$$\{ E_s(A) = 0, \quad E_1(V) = 0, \quad (d = 2) \}$$

where $V^{\mu_1} = \nabla_{\mu_2} \cdots \nabla_{\mu_s} A^{\mu_1 \cdots \mu_s}$, is conformally invariant while restricting the (new-found) gauge freedom. While the former happens to be true, the latter is not as $E_1(V) = 0$ is found to be automatically fulfilled on the space of solutions of $E_s(A) = 0$. In order to show this, notice first that $A$ possesses, whatever the rank $s$, only 2 independent components. In the usual cartesian coordinates let us choose those as:

$$\Phi^\pm = A^{00 \cdots 00 \pm} A^{00 \cdots 01}.$$ 

Then, turning to the chiral coordinates $x^\pm = x^0 \pm x^1$ eq. (1.1) is rewritten as:

$$\begin{cases} 
\partial_+^2 \Phi^+ = 0, \\
\partial_-^2 \Phi^- = 0.
\end{cases}$$

(A.1)

The system is conformally invariant and the rank from which $\Phi^\pm$ emanates is read off the way the field transforms. For $d = 2$ the fields admits the gauge transformation:

$$A \mapsto {}^a A = A + \text{STG}^{s-1}(a), \quad s \geq 2,$$

with $a$ an arbitrary vector field. On the independent components $\Phi^\pm$ the above reads as:

$${}^a \Phi^\pm = \Phi^\pm + s! \partial_+^{s-1} a^\pm, \quad a^\pm = a^0 \pm a^1.$$ (A.2)

Now, for a given field $A \simeq (\Phi^+, \Phi^-)$ one can find $(a^+, a^-)$ such that the gauge transformed field (A.2) is null. This means that any field $A$, $s \geq 2$, is in the (gauge) equivalence class of the trivial solution. Hence that field carries no physical content.

This can also be seen on the gauge fixing equation since:

$$\partial_{\mu_1} \partial_{\mu_2} A^{\mu_1 \cdots \mu_s} \propto (\partial_+^2 \Phi^+ + \partial_-^2 \Phi^-) = 0,$$

the $(s - 1)$-divergence of the field vanishes anyway. Therefore, the equation $E_1(V) = 0$ is no constraint on the space of solutions of (A.1).

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