WORTH property, García-Falset coefficient and Opial property of infinite sums

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Abstract. We prove some results concerning the WORTH property and the García-Falset coefficient of absolute sums of infinitely many Banach spaces. The Opial property/uniform Opial property of infinite $\ell^p$-sums is also studied and some properties analogous to the Opial property/uniform Opial property for Lebesgue-Bochner spaces $L^p(\mu, X)$ are discussed.

1 Introduction

For a real Banach space $X$, denote by $X^*$ its dual space, by $B_X$ its closed unit ball and by $S_X$ its unit sphere.

We begin by recalling the important notion of fixed point property: $X$ is said to have the fixed point property (resp. weak fixed point property) if for every closed and bounded (resp. weakly compact) convex subset $C \subseteq X$, every nonexpansive mapping $F : C \to C$ has a fixed point (where $F$ is called nonexpansive if $\|F(x) - F(y)\| \leq \|x - y\|$ for all $x, y \in C$, in other words, if $F$ is 1-Lipschitz continuous).

A bounded closed convex subset $C \subseteq X$ is said to have normal structure if for each subset $B \subseteq C$ which contains at least two elements there exists a point $x \in B$ such that

$$\sup_{y \in B} \|x - y\| < \text{diam } B.$$ 

It is well known that if $C$ is weakly compact and has normal structure, then every nonexpansive mapping $F : C \to C$ has a fixed point (see for example [10, Theorem 2.1]).

The space $X$ is said to have the Opial property provided that

$$\limsup_{n \to \infty} \|x_n\| < \limsup_{n \to \infty} \|x_n - x\|$$

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holds for every weakly nullsequence \((x_n)_{n \in \mathbb{N}}\) in \(X\) and every \(x \in X \setminus \{0\}\) (one could as well use \(\liminf\) instead of \(\limsup\) or assume from the beginning that both limits exist). This property was first considered by Opial in [20] (starting from the Hilbert spaces as canonical example) to provide a result on iterative approximations of fixed points of nonexpansive mappings. It is shown in [20] that the spaces \(\ell^p\) for \(1 \leq p < \infty\) enjoy the Opial property, whereas \(L^p[0,1]\) for \(1 < p < \infty, p \neq 2\) fails to have it. Note further that every Banach space with the Schur property (i.e. weak and norm convergence of sequences coincide) trivially has the Opial property. Also, \(X\) is said to have the nonstrict Opial property if it fulfills the definition of the Opial property with \(\leq\) instead of \(<\) ([25], in [7] it is called weak Opial property). It is known that every weakly compact convex set in a Banach space with the Opial property has normal structure (see for instance [22, Theorem 5.4]).

Prus introduced the notion of uniform Opial property in [21]: a Banach space \(X\) has the uniform Opial property if for every \(c > 0\) there is some \(r > 0\) such that

\[
1 + r \leq \liminf_{n \to \infty} \|x_n - x\|
\]

holds for every \(x \in X\) with \(\|x\| \geq c\) and every weakly nullsequence \((x_n)_{n \in \mathbb{N}}\) in \(X\) with \(\liminf_{n \to \infty} \|x_n\| \geq 1\). In [21] it was proved that a Banach space is reflexive and has the uniform Opial property if and only if it has the so called property \((L)\) (see [21] for the definition), and that \(X\) has the fixed point property whenever \(X^*\) enjoys said property \((L)\).

A modulus corresponding to the uniform Opial property was defined in [17]:

\[
r_X(c) := \inf \left\{ \liminf_{n \to \infty} \|x_n - x\| - 1 \right\} \quad \forall c > 0,
\]

where the infimum is taken over all \(x \in X\) with \(\|x\| \geq c\) and all weakly nullsequences \((x_n)_{n \in \mathbb{N}}\) in \(X\) with \(\liminf_{n \to \infty} \|x_n\| \geq 1\) (if \(X\) has the Schur property, we agree to set \(r_X(c) := 1\) for all \(c > 0\)). Then \(X\) has the uniform Opial property iff \(r_X(c) > 0\) for every \(c > 0\).

In this paper, we will mostly use the following equivalent formulation of the uniform Opial property ([13, Definition 3.1]): \(X\) has the uniform Opial property iff for every \(\varepsilon > 0\) and every \(R > 0\) there is some \(\eta > 0\) such that

\[
\eta + \liminf_{n \to \infty} \|x_n\| \leq \liminf_{n \to \infty} \|x_n - x\|
\]

holds for all \(x \in X\) with \(\|x\| \geq \varepsilon\) and every weakly nullsequence \((x_n)_{n \in \mathbb{N}}\) in \(X\) with \(\limsup_{n \to \infty} \|x_n\| \leq R\).

We can also associate a modulus to this formulation in the following way:

\[
\eta_X(\varepsilon, R) := \inf \left\{ \liminf_{n \to \infty} \|x_n - x\| - \liminf_{n \to \infty} \|x_n\| \right\} \quad \forall \varepsilon, R > 0,
\]

where the infimum is taken over all \(x \in X\) with \(\|x\| \geq \varepsilon\) and all weakly nullsequences \((x_n)_{n \in \mathbb{N}}\) in \(X\) with \(\limsup_{n \to \infty} \|x_n\| \leq R\). So \(X\) has the uniform...
Opial property iff $\eta_X(\varepsilon, R) > 0$ for all $\varepsilon, R > 0$. Actually, it is enough that for every $\varepsilon > 0$ there exists some $R > 2$ with $\eta_X(\varepsilon, R) > 0$. More precisely, we have the following connection between the two moduli $r_X$ and $\eta_X$.

**Lemma 1.1.** Let $X$ be a Banach space which does not have the Schur property.

(i) For every $c > 0$ and every $R > 2$ we have

$$\min \left\{ \eta_X(c, R), \frac{R}{2} - 1 \right\} \leq r_X(c).$$

(ii) For all $\varepsilon, R > 0$ with $r_X(\varepsilon/R) > 0$ we have

$$\frac{\varepsilon r_X(\varepsilon/R)}{2 + r_X(\varepsilon/R)} = \max_{\beta \in [0, \varepsilon/2]} \min \left\{ \beta r_X(\varepsilon/R), \varepsilon - 2\beta \right\} \leq \eta_X(\varepsilon, R).$$

**Proof.** (i) Let $c > 0$ and $R > 2$. Put $\tau := \min \{ \eta_X(c, R), \frac{R}{2} - 1 \}$. Let $(x_n)_{n \in \mathbb{N}}$ be any weakly nullsequence in $X$ with $\liminf \|x_n\| \geq 1$ and let $x \in X$ with $\|x\| \geq c$. By passing to a subsequence, we may assume that $\lim_{n \to \infty} \|x_n - x\|$ and $s := \lim_{n \to \infty} \|x_n\|$ exist. If $s \leq R$ then $1 + \tau \leq s + \eta_X(c, R) \leq \lim_{n \to \infty} \|x_n - x\|$. If $s > R$ and $\|x\| > R/2$, then $\lim_{n \to \infty} \|x_n - x\| \geq \|x\| > R/2 \geq 1 + \tau$ by the weak lower semicontinuity of the norm. Finally, if $s > R$ and $\|x\| \leq R/2$, then $\lim_{n \to \infty} \|x_n - x\| \geq s - \|x\| > R/2 \geq 1 + \tau$.

(ii) The first equality is easily verified. Now chose any $\beta \in (0, \varepsilon/2)$ and put $\nu := \min \{ \beta r_X(\varepsilon/R), \varepsilon - 2\beta \}$. Let $(x_n)_{n \in \mathbb{N}}$ be a weakly nullsequence in $X$ with $\limsup \|x_n\| \leq R$ and let $x \in X$ with $\|x\| \geq \varepsilon$. Again we may assume that $\lim_{n \to \infty} \|x_n - x\|$ and $s := \lim_{n \to \infty} \|x_n\|$ exist. By the definition of $r_X$ we get $s(1 + r_X(\varepsilon/R)) \leq \lim_{n \to \infty} \|x_n - x\|$, which implies $s + \nu \leq \lim_{n \to \infty} \|x_n - x\|$ if $s > \beta$. But if $s \leq \beta$ then $\lim_{n \to \infty} \|x_n - x\| \geq \|x\| - s \geq \varepsilon - \beta \geq \nu + \beta \geq \nu + s$ and the proof is finished. \qed

In [7] J. García-Falset introduced the following coefficient of a Banach space $X$:

$$R(X) := \sup \left\{ \liminf_{n \to \infty} \|x_n + x\| : x \in B_X, (x_n)_{n \in \mathbb{N}} \in \WN(B_X) \right\},$$

where we denote by $\WN(B_X)$ the set of all weakly nullsequences in $B_X$. Obviously, $1 \leq R(X) \leq 2$ and $R(X) = 1$ if $X$ has the Schur property (in particular if $X$ is finite-dimensional or $X = \ell^1$). One has $R(c_0) = 1$ and $R(\ell^p) = 2^{1/p}$ for $1 < p < \infty$ (see [7, Corollary 3.2]). In [8, Theorem 3] it was proved that the condition $R(X) < 2$ implies that $X$ has the weak fixed point property. The reflexive spaces with $R(X) < 2$ are precisely the so called weakly nearly uniformly smooth spaces ([7, Corollary 4.4]), which were introduced in [14] and include in particular all uniformly smooth spaces.
1. Introduction

We will denote by $\delta_X$ the modulus of convexity of $X$, i.e. for $0 < \varepsilon \leq 2$

$$\delta_X(\varepsilon) := \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in B_X \text{ with } \|x - y\| \geq \varepsilon \right\}.$$ 

$X$ is uniformly rotund iff $\delta_X(\varepsilon) > 0$ for each $0 < \varepsilon \leq 2$. It is well-known that all spaces $L^p(\mu)$ for any measure $\mu$ and any $1 < p < \infty$ (in particular the spaces $\ell^p(I)$ for any index set $I$) are uniformly rotund.

In [24] Sims introduced the notion of WORTH (weak orthogonality) property: $X$ is said to have the WORTH property provided that for all weakly nullsequences $(x_n)_{n \in \mathbb{N}}$ in $X$ and every $x \in X$ one has $\|x_n + x\| - \|x_n - x\| \to 0$.

Again spaces with the Schur property obviously enjoy the WORTH property. Hilbert spaces are easily seen to have the WORTH property as well. Also, the class of spaces with the WORTH property includes all so called weakly orthogonal Banach lattices (a notion introduced earlier by Borwein and Sims in [2]), which in turn includes in particular all spaces $\ell^p(I)$ for $1 \leq p < \infty$ and $c_0(I)$. However, the spaces $L^p[0,1]$ with $1 \leq p \leq \infty, p \neq 2$ do not have the WORTH property (see the remark at the end of [25]). In [24] it was proved that the WORTH property implies the nonstrict Opial property, and in [25] it was shown that a space with the WORTH property which is $\varepsilon$-inquadrat in every direction for some $0 < \varepsilon < 2$ (see [25] for the definition) has the weak fixed point property (even more, every weakly compact convex subset of such a space has normal structure). By [7, Proposition 3.6], a uniformly non-square\footnote{Recall that $X$ is said to be uniformly non-square if there is some $\delta > 0$ such that whenever $x, y \in B_X$ one has $\|x + y\| < 2(1 - \delta)$ or $\|x - y\| < 2(1 - \delta)$.} Banach space $X$ with the WORTH property satisfies $R(X) < 2$.

The degree $w(X)$ of WORTHness of $X$ was also introduced in [25] as the supremum of all $r \geq 0$ such that

$$r \liminf_{n \to \infty} \|x_n + x\| \leq \liminf_{n \to \infty} \|x_n - x\|$$

holds for all $x \in X$ and all weakly nullsequences $(x_n)_{n \in \mathbb{N}}$ in $X$. Then $1/3 \leq w(X) \leq 1$ and $X$ has the WORTH property if and only if $w(X) = 1$.

In this paper, we will study the WORTH property and the García-Falset coefficient for infinite absolute sums, and the different Opial properties specifically for infinite $\ell^p$-sums of Banach spaces (for normal structure in (finite and infinite) direct sums of Banach spaces see [5] and references therein). The next section contains the necessary preliminaries on absolute sums.
2 Preliminaries on absolute sums

Throughout this paper, if not otherwise stated, $I$ denotes a (mostly infinite) index set and $E$ a subspace of the space of all real-valued functions on $I$ which contains all functions with finite support and is endowed with an absolute, normalised norm $\|\cdot\|_E$. The latter means that $\|\cdot\|_E$ is a complete norm on $E$ such that the following conditions are satisfied:

(i) If $(a_i)_{i \in I} \in E$ and $(b_i)_{i \in I} \in \mathbb{R}^I$ such that $|a_i| = |b_i|$ for all $i \in I$ then

$$(b_i)_{i \in I} \in E \quad \text{and} \quad \|(b_i)_{i \in I}\|_E = \|(a_i)_{i \in I}\|_E.$$

(ii) $\|e_i\|_E = 1$ for all $i \in I$, where $e_i = (e_{ij})_{j \in I}$ with $e_{ij} = 0$ for $j \neq i$ and $e_{ii} = 1$.

It is important to note that such norms are automatically monotone, i.e. we actually have

$$(a_i)_{i \in I} \in E, (b_i)_{i \in I} \in \mathbb{R}^I \quad \text{with} \quad |b_i| \leq |a_i| \forall i \in I \implies (b_i)_{i \in I} \in E \quad \text{and} \quad \|(b_i)_{i \in I}\|_E \leq \|(a_i)_{i \in I}\|_E.$$

For a proof see for instance [16, Remark 2.1]. Standard examples of spaces with absolute, normalised norm are of course the spaces $\ell^p(I)$ (with $1 \leq p \leq \infty$) and $c_0(I)$.

If we put

$$E' := \left\{ (a_i)_{i \in I} \in \mathbb{R}^I : \|(a_i)_{i \in I}\|_{E'} := \sup_{(b_i)_{i \in I} \in B_E} \sum_{i \in I} |a_i b_i| < \infty \right\},$$

then $(E', \|\cdot\|_{E'})$ is again a space with absolute, normalised norm and the map $T : E' \to E^*$ defined by

$$T((a_i)_{i \in I})(b_i)_{i \in I}) := \sum_{i \in I} a_i b_i \quad \forall (a_i)_{i \in I} \in E', \forall (b_i)_{i \in I} \in E$$

is an isometric embedding. $T$ is onto if span$\{e_i : i \in I\}$ is dense in $E$, so in this case $E^* = E'$.

Now given a family $(X_i)_{i \in I}$ of Banach spaces, the absolute sum of $(X_i)_{i \in I}$ with respect to $E$ is defined as the space

$$\left[ \bigoplus_{i \in I} X_i \right]_E := \left\{ (x_i)_{i \in I} \in \prod_{i \in I} X_i : \|(x_i)\|_{i \in I} \in E \right\}$$

endowed with the norm $\|(x_i)_{i \in I}\|_E := \|(\|x_i\|)_{i \in I}\|_E$. It is not hard to see that this sum is indeed a Banach space. For $E = \ell^p(I)$ one obtains the usual $p$-sum.
3. Results on absolute sums

As regards the dual space of an absolute sum, the map

\[ S : \left( \bigoplus_{i \in I} X_i^* \right)_{E'} \to \left( \bigoplus_{i \in I} X_i^* \right)_{E} \]

\[ S((x_i^*)_{i \in I})((x_i)_{i \in I}) := \sum_{i \in I} x_i^*(x_i) \]

is an isometric embedding and it is onto if \( \text{span}\{e_i : i \in I\} \) is dense in \( E \).

3 Results on absolute sums

3.1 WORTH property of absolute sums

By [12, Theorem 4.7], \( w(X \oplus_E Y) = \min\{w(X), w(Y)\} \) holds for all Banach spaces \( X \) and \( Y \) and every absolute, normalised norm \( \|\cdot\|_E \) on \( \mathbb{R}^2 \) (actually, the notion of \( \psi \)-direct sums is used in [12], but it is an equivalent formulation (see section 2 in [12])). In particular, \( X \oplus_E Y \) has the WORTH property if and only if \( X \) and \( Y \) have the WORTH property (for this, see also [12, Theorem 4.2]). It is possible to generalise [12, Theorem 4.7] to sums of arbitrarily many Banach spaces under a mild condition on \( E \).

**Proposition 3.1.** If \( \text{span}\{e_i : i \in I\} \) is dense in \( E \) and \( (X_i)_{i \in I} \) is any family of Banach spaces then

\[ w\left( \left( \bigoplus_{i \in I} X_i \right)_E \right) = \inf\{w(X_i) : i \in I\}. \]

In particular, \( \left( \bigoplus_{i \in I} X_i \right)_E \) has the WORTH property if and only if \( X_i \) has the WORTH property for every \( i \in I \).

**Proof.** Let us write \( X = \left( \bigoplus_{i \in I} X_i \right)_E \) and \( s = \inf\{w(X_i) : i \in I\} \). We clearly have \( w(X) \leq s \). Now let \( x_n = (x_{n,i})_{i \in I} \in X \) for every \( n \in \mathbb{N} \) such that \( (x_n)_{n \in \mathbb{N}} \) converges weakly to zero and let \( x = (x_i)_{i \in I} \in X \). Without loss of generality, we may assume that the limits \( a := \lim_{n \to \infty} \|x_n + x\|_E \) and \( b := \lim_{n \to \infty} \|x_n - x\|_E \) exist.

Since \( \text{span}\{e_i : i \in I\} \) is dense in \( E \), it is not hard to see that we actually have \( (\|x_i\|)_{i \in I} = \sum_{i \in I} \|x_i\|e_i \). So if \( \varepsilon > 0 \) is given, we find a finite set \( J \subseteq I \) such that

\[ \left\| \sum_{i \in I \setminus J} \|x_i\|e_i \right\|_E \leq \sum_{i \in J} \|x_i\|e_i \leq \varepsilon. \]  

(3.1)

By passing to an appropriate subsequence we may assume that the limits \( a_i := \lim_{n \to \infty} \|x_{n,i} + x_i\| \) and \( b_i := \lim_{n \to \infty} \|x_{n,i} - x_i\| \) exist for each \( i \in J \).
Since \((x_{n,i})_{n \in \mathbb{N}}\) is weakly convergent to zero in \(X_i\) for every \(i \in I\) it follows that \(s a_i \leq b_i \leq s^{-1}a_i\) and consequently

\[|a_i - b_i| \leq \frac{1 - s}{s} b_i \quad \forall i \in J. \quad (3.2)\]

For every \(n \in \mathbb{N}\) we have, because of (3.1),

\[||x_n + x||_E - ||x_n - x||_E| \leq \left| \left( ||x_{n,i} + x_i|| - ||x_{n,i} - x_i|| \right)_{i \in I} \right|_E\]

\[ \leq \left| \sum_{i \in J} \left( ||x_{n,i} + x_i|| - ||x_{n,i} - x_i|| \right) e_i \right|_E + 2 \left| \sum_{i \notin J} ||x_i|| e_i \right|_E\]

\[ \leq \left| \sum_{i \in J} \left( ||x_{n,i} + x_i|| - ||x_{n,i} - x_i|| \right) e_i \right|_E + 2 \varepsilon.\]

So for \(n \to \infty\) we obtain

\[|a - b| \leq \left| \sum_{i \in J} (a_i - b_i) e_i \right|_E + 2 \varepsilon.\]

Taking (3.2) into account we arrive at

\[|a - b| \leq \frac{1 - s}{s} \left| \sum_{i \in J} b_i e_i \right|_E + 2 \varepsilon.\]

But \(\left| \sum_{i \in J} ||x_{n,i} - x_i|| e_i \right|_E \leq ||x_n - x||_E\) for each \(n\), thus \(\left| \sum_{i \in J} b_i e_i \right|_E \leq b\) and hence

\[|a - b| \leq \frac{1 - s}{s} b + 2 \varepsilon.\]

Letting \(\varepsilon \to 0\) leaves us with \(|a - b| \leq (1 - s)b/s\) which implies \(sa \leq b\) and we are done. \(\square\)

### 3.2 García-Falset coefficient of absolute sums

In [4, Theorem 7] it was proved that \(R((X_1 \oplus X_2 \oplus \cdots \oplus X_n)_E) < 2\) whenever \(R(X_i) < 2\) for \(i = 1, \ldots, n\) and \(||\cdot||_E\) is any strictly convex, absolute, normalised norm on \(\mathbb{R}^n\). For absolute sums of two Banach spaces a stronger result was obtained in [12, Theorem 3.6]: \(R(X \oplus_Y Y) < 2\) provided that \(R(X), R(Y) < 2\) and \(||\cdot||_E\) is any absolute, normalised norm on \(\mathbb{R}^2\) with \(||\cdot||_E \neq ||\cdot||_1\). For infinite sums we have the following theorem (for \(J \subseteq I\) we denote by \(\bigoplus_{i \in J} X_i\) the sum of the family whose \(i\)-th member is \(X_i\) for \(i \in J\) and \(\{0\}\) for \(i \in I \setminus J\)).
Theorem 3.2. If $I$ is an infinite index set, $E$ a subspace of $\mathbb{R}^I$ with absolute, normalised norm such that $\text{span}\{e_i : i \in I\}$ is dense in $E$ and $(X_i)_{i \in I}$ is a family of Banach spaces with

$$\alpha := \sup \left\{ R \left( \bigoplus_{i \in J} X_i \right)_E : J \subseteq I \text{ finite} \right\} < 2 \quad (3.3)$$

and $\delta_E((1 - \alpha/2)^2) > 0$, then $R(\bigoplus_{i \in I} X_i)_E < 2$.

Proof. Let us write $X = (\bigoplus_{i \in I} X_i)_E$ for short. It is well known that $\delta_E$ is continuous on $(0, 2)$ (see for example [9, Lemma 5.1]), so we can find $0 < \tau < (1 - \alpha/2)^2$ with $\delta_E(\tau) > 0$. Let $\gamma := \sqrt{\tau}$ and choose $0 < \eta < \min\{\delta_E(\tau), 1/2 - \gamma\}$.

Suppose that $R(X) = 2$. Then there would be a weakly null sequence $(x_n)_{n \in \mathbb{N}} = ((x_{n,i})_{i \in I})_{n \in \mathbb{N}}$ in $B_X$ and an element $x = (x_i)_{i \in I} \in B_X$ such that $\lim_{n \to \infty} \|x_n + x\|_E > 2 - \eta$. We may assume $\|x_n + x\|_E < 2 - 2\eta$ for all $n \in \mathbb{N}$. Since $\|x_n + x\|_E \geq \|x_n + x\|_E$ and $\eta < \delta_E(\tau)$ it follows that

$$\|x_n + x\|_E < \tau \quad \forall n \in \mathbb{N}. \quad (3.4)$$

Similarly,

$$4(1 - \eta) < 2\|x_n + x\|_E \leq \|x_n + x\| + \|x_n + x\| \leq \|x_n + x\|_E \leq 4$$

and hence

$$\|x_n + x\|_E < 2\tau \quad \forall n \in \mathbb{N}. \quad (3.5)$$

We further have $\|x\|_E \geq \|x_n + x\| - 1 > 1 - 2\eta > 2\gamma$. Since $(\|x_i\|)_{i \in I} = \sum_{i \in I} \|x_i\|e_i$ we can find a finite set $J \subseteq I$ such that

$$\left\| \sum_{i \in J} \|x_i\|e_i \right\|_E > 2\gamma. \quad (3.6)$$

Put $y := (x_i)_{i \in J}, y_n = (x_{n,i})_{i \in J} \in (\bigoplus_{i \in J} X_i)_E$ as well as $a := \sum_{i \in J} \|x_i\|e_i$ and $a_n := \sum_{i \in J} \|x_{n,i}\|e_i$. By (3.4) we have

$$\|a_n - a\|_E \leq \|x_n + x\|_E < \tau \quad \forall n \in \mathbb{N}, \quad (3.7)$$

which implies in particular $\|y\|_E - \|y_n\|_E = \|a\|_E - \|a_n\|_E < \tau$, hence $\|y_n\|_E > \|y\|_E - \tau > 2\gamma - \tau > 0$, by (3.6).

Furthermore, for every $n \in \mathbb{N}$,

$$\|a_n + a\|_E - \|y_n + y\|_E \leq \left\| \sum_{i \in J} \|x_n\| + \|x_i\| - \|x_{n,i} + x_i\| e_i \right\|_E.$$
so because of (3.5) it follows that

\[ \|a_n + a\|_E - \|y_n + y\|_E < 2\tau \quad \forall n \in \mathbb{N}. \]

Also, by (3.7), we have \[ \|a_n + a\|_E - 2\|y\|_E = \|a_n + a\|_E - 2\|a\|_E \leq \|a_n - a\| < \tau \quad \text{for each } n. \]

Consequently,

\[ \|y_n + y\|_E - 2\|y\|_E < 3\tau \quad \forall n \in \mathbb{N}. \]

Since \[ \|y_n\|_E - \|y\|_E = |1 - \|y_n\|_E/\|y\|_E| < \tau/\|y\|_E \]
we get

\[ 2 - \frac{y}{\|y\|_E} + \frac{y_n}{\|y_n\|_E} < \frac{4\tau}{\|y\|_E} < \frac{2\tau}{\gamma} \quad \forall n \in \mathbb{N}, \tag{3.8} \]

where the last inequality holds because of (3.6). Note that \((x_{n,i})_{n \in \mathbb{N}}\) converges weakly to zero in \(X_i\) for each \(i \in I\) and thus, by the representation of the dual of \(\bigoplus_{i \in J} X_i\) as \(\bigoplus_{i \in J} X_i^*\) and finiteness of \(J\), the sequence \((y_n/\|y_n\|_E)_{n \in \mathbb{N}}\) is also a weakly null sequence (as noted above, \((\|y_n\|)_{n \in \mathbb{N}}\) is bounded away from zero).

So from (3.8) and the definition of \(\alpha\) it follows that \(\alpha \geq 2(1 - \tau/\gamma)\). But \(\gamma = \sqrt{\tau}\) and \(\tau < (1 - \alpha/2)^2\), thus \(2(1 - \tau/\gamma) > \alpha\) and with this contradiction the proof is finished. \(\square\)

The above theorem reduces the case of infinite sums to the one of finite sums. The condition \(\alpha < 2\) is clearly necessary for \(R(X) < 2\).

Unfortunately, the author does not know whether the simpler condition \(\beta := \sup_{i \in I} R(X_i) < 2\) would be already enough to ensure that \(\alpha < 2\). The proof of [4, Theorem 7] shows that for \(\beta < 2\) one has for every finite subset \(J \subseteq I\) with \(|J| = N\) that \(R\left(\bigoplus_{i \in J} X_i\right) \leq 2 - \delta\), where first \(\varepsilon > 0\) is chosen such that \(\beta(1 + N\varepsilon) < 2\) and then \(0 < \delta < \min\{2\delta_E(\varepsilon), 2 - \beta(1 + N\varepsilon)\}\), so it still might be that \(R\left(\bigoplus_{i \in J} X_i\right)\) tends to 2 for \(N \to \infty\).

Next we will discuss some applications of Theorem 3.2. First, since the Schur property is inherited by finite sums, we get the following corollary.

**Corollary 3.3.** If \((X_i)_{i \in I}\) is a family of Banach spaces with the Schur property (in particular, a family of finite-dimensional Banach spaces) and \(\text{span}\{e_i : i \in I\}\) is dense in \(E\) with \(\delta_E(1/4) > 0\), then \(R\left(\bigoplus_{i \in I} X_i\right) < 2\). In particular, \(R\left(\bigoplus_{i \in I} X_i\right)_p < 2\) for all \(1 < p < \infty\).

For another application of Theorem 3.2 consider the following example.

**Example 3.4.** If \(N \geq 2\) and \(I_1, \ldots, I_N\) are non-empty sets at least one of which is infinite, then

\[ R\left(\bigoplus_{k=1}^N c_0(I_k)\right)_p = 2^{1/p} \tag{3.9} \]
for every $1 \leq p < \infty$. Consequently, by Theorem 3.2, if $(I_k)_{k \in I}$ is any family of non-empty sets we have that

$$R \left( \bigoplus_{k \in I} c_0(I_k) \right)_p < 2 \text{ for } 1 < p < \infty.$$ 

Proof. To prove (3.9) put $X := \left[ \bigoplus_{k=1}^N c_0(I_k) \right]_p$ and suppose without loss of generality that $X_1$ is infinite. Fix a sequence $(i_n)_{n \in \mathbb{N}}$ of distinct elements of $I_1$ and any $j \in I_2$ and put $x_n := (e_{i_n}, 0, \ldots, 0) \in S_X$ as well as $x := (0, e_j, 0, \ldots, 0) \in S_X$. Then $x_n \to 0$ weakly in $X$ and $\|x_n + x\|_p = 2^{1/p}$ for each $n$, thus $2^{1/p} \leq R(X)$.

To prove the reverse inequality let $x_n = (x_{n,1}, \ldots, x_{n,N}) \in B_X$ for each $n \in \mathbb{N}$ such that $x_n \to 0$ weakly and let $x = (x_1, \ldots, x_N) \in B_X$. Without loss of generality we can suppose that $\lim_{n \to \infty} \|x_n + x\|_p$ and also $a_k := \lim_{n \to \infty} \|x_{n,k}\|_\infty$ exists for each $k \in \{1, \ldots, N\}$.

Take an arbitrary $\varepsilon > 0$. Then for each $k \in \{1, \ldots, N\}$ the set $J_k := \{i \in I_k : |x_k(i)| > \varepsilon\}$ is finite. Since $x_n \to 0$ weakly we have $x_{n,k}(i) \to 0$ for all $k \in \{1, \ldots, N\}$ and all $i \in I_k$. It follows that there exists $n_0 \in \mathbb{N}$ such that $|x_{n,k}(i)| \leq \varepsilon$ for all $k \in \{1, \ldots, N\}$, all $i \in J_k$ and all $n \geq n_0$.

But then $|x_{n,k}(i) + x_k(i)| \leq |x_{n,k}(i)| + |x_k(i)| \leq \max\{|x_{n,k}(i)|, |x_k(i)|\} + \varepsilon$ for all $k \in \{1, \ldots, N\}$, all $i \in I_k$ and all $n \geq n_0$. From this we can conclude

$$\|x_n + x\|_p^p = \sum_{k=1}^N \|x_{n,k} + x_k\|_\infty^p \leq \sum_{k=1}^N (\max\{|x_{n,k}\|_\infty, |x_k|_\infty\} + \varepsilon)^p \quad \forall n \geq n_0.$$ 

For $n \to \infty$ it follows that

$$\lim_{n \to \infty} \|x_n + x\|_p^p \leq \sum_{k=1}^N (\max\{a_k, |x_k|_\infty\} + \varepsilon)^p.$$ 

Letting $\varepsilon \to 0$ we obtain

$$\lim_{n \to \infty} \|x_n + x\|_p^p \leq \sum_{k=1}^N \max\{a_k^p, |x_k|_\infty^p\} \leq \sum_{k=1}^N (a_k^p + |x_k|_\infty^p)$$

$$= \lim_{n \to \infty} \|x_n\|_p^p + \|x\|_p^p \leq 2^{1/p} \text{ and we are done.}$$

Recall that a Banach space $X$ is said to be a $U$-space if for any two sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in $S_X$ and every sequence $(x_n^*)_{n \in \mathbb{N}}$ in $S_{X^*}$ the conditions $\|x_n + y_n\| \to 2$ and $x_n^*(x_n) = 1$ for each $n \in \mathbb{N}$ imply $x_n^*(y_n) \to 1$. $U$-spaces were introduced by Lau in [15]. Uniformly rotund and uniformly
smooth spaces are examples of $U$-spaces. Gao [6] defined the modulus of $u$-convexity of $X$ by

$$u_X(\varepsilon) := \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in S_X \exists x^* \in S_{X^*} \; x^*(x) = 1, x^*(y) \leq 1 - \varepsilon \right\}$$

for $0 < \varepsilon \leq 2$. Then $X$ is a $U$-space if and only if $u_X(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$. Obviously, $u_X \geq \delta_X$. By [19, Theorem 5] $R(X) < 2$ if $u_X(\varepsilon) > 0$ for some $0 < \varepsilon < 1$.

Putting several results together it is now possible to obtain the following corollary.

**Corollary 3.5.** Let $(X_i)_{i \in I}$ be a family of Banach spaces and $1 < p < \infty$. Suppose that there exist four pairwise disjoint (possibly empty) subsets $I_1, I_2, I_3, I_4 \subseteq I$ such that

(i) $X_i$ has the Schur property for each $i \in I_1$,

(ii) $\inf_{i \in I_2} u_{X_i}(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$,

(iii) for each $i \in I_3$ there is a set $J_i$ with $X_i = c_0(J_i)$.

(iv) $I_4$ is finite and $R(X_i) < 2$ for all $i \in I_4$.

Then $R\left( \bigoplus_{i \in I} X_i \right)_p < 2$.

**Proof.** Let us put $X := \left[ \bigoplus_{i \in I} X_i \right]_p$ and $X_k := \left[ \bigoplus_{i \in I_k} X_i \right]_p$ for $k = 1, 2, 3, 4$ (or $X_k = \{0\}$ if $I_k = \emptyset$). By Corollary 3.3 we have $R(X_1) < 2$ and by Example 3.4 we have $R(X_3) < 2$. Also, by [11, Corollary 3.17] and the remarks after [11, Definition 1.5] $X_2$ is again a $U$-space, so $R(X_2) < 2$. From the aforementioned result [4, Theorem 7] it follows that $R(X_4) < 2$ and since $X \cong X_1 \oplus_p X_2 \oplus_p X_3 \oplus_p X_4$, [4, Theorem 7] implies that $R(X) < 2$. \qed

The case of $c_0$-sums is not covered by the above results. However, it is easy to prove the following proposition directly.

**Proposition 3.6.** Let $(X_i)_{i \in I}$ be any family of Banach spaces and $X := \left[ \bigoplus_{i \in I} X_i \right]_{c_0(I)}$. Then

$$R(X) = \sup_{i \in I} R(X_i).$$

**Proof.** We clearly have $\alpha := \sup_{i \in I} R(X_i) \leq R(X)$. To prove the reverse inequality, fix any weakly null sequence $(x_n)_{n \in \mathbb{N}}$ and any $x = (x_i)_{i \in I} \in B_X$. Without loss of generality, we may assume that $\lim_{n \to \infty} \|x_n + x\|_\infty$ exists. Let $\varepsilon > 0$ be arbitrary. Then $J := \{ i \in I : \|x_i\| \geq \varepsilon \}$ is finite, so by passing to an appropriate subsequence once more we may also assume that $\lim_{n \to \infty} \|x_{n,i} + x_i\|$ exists for all $i \in J$.\[11\text{ of } 22\]
Since $x_{n,i} \to 0$ weakly for all $i \in I$ it follows that $\lim_{n \to \infty} \|x_{n,i} + x_i\| \leq R(X_i) \leq \alpha$ for all $i \in J$, so $\|x_{n,i} + x_i\| \leq \alpha + \varepsilon$ for all $i \in J$ and all sufficiently large $n$. But for $i \in I \setminus J$ we have $\|x_{n,i} + x_i\| \leq \|x_{n,i}\| + \|x_i\| \leq 1 + \varepsilon \leq \alpha + \varepsilon$. Consequently, $\|x_n + x\|_{\infty} \leq \alpha + \varepsilon$ for all sufficiently large $n$. Since $\varepsilon > 0$ was arbitrary, we are done.

Concerning $\ell^1$-sums it was already proved in [12, Theorem 3.13] that $R(X \oplus_1 Y) < 2$ if and only if both $X$ and $Y$ have the Schur property. The proof of the “only if” part directly generalises to sums of arbitrarily many spaces and since it was proved in [26] that the $\ell^1$-sum of any family of Banach spaces has the Schur property if and only if each summand has the Schur property, we obtain the following characterisation.

**Proposition 3.7.** Let $I$ be any index set with at least two elements. Let $(X_i)_{i \in I}$ be a family of Banach spaces and $X := \bigoplus_{i \in I} X_i$. The following assertions are equivalent:

(i) $R(X) < 2$,

(ii) $X_i$ has the Schur property for each $i \in I$,

(iii) $X$ has the Schur property,

(iv) $R(X) = 1$.

### 3.3 Opial properties of finite absolute sums

In this subsection we will briefly consider Opial properties of finite sums. This is surely well-known, but we will include the results and some of their proofs here as the author was not able to find them explicitly in the literature.

Recall that an absolute, normalised norm $\|\cdot\|_E$ on $\mathbb{R}^m$ is said to be strictly monotone if for all $a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m) \in \mathbb{R}^m$ we have

$$\|a\|_E = \|b\|_E \text{ and } |a_i| \leq |b_i| \; \forall i = 1, \ldots, m \Rightarrow |a_i| = |b_i| \; \forall i = 1, \ldots, m.$$  

It is easy to see that strictly convex, absolute, normalised norms are strictly monotone.

**Proposition 3.8.** Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^m$ and $X_1, \ldots, X_m$ Banach spaces with nonstrict Opial property. Then $\bigoplus_{i=1}^m X_i$ has the nonstrict Opial property. If moreover $\|\cdot\|_E$ is strictly monotone and each $X_i$ has the Opial property, then $\bigoplus_{i=1}^m X_i$ also has the Opial property.

The proof is straightforward and will be omitted.

As is well-known, every strictly monotone, absolute, normalised norm on $\mathbb{R}^m$ is actually uniformly monotone in the following sense (the proof consists in an easy compactness argument).
Lemma 3.9. Let $\|\cdot\|_E$ be a strictly monotone, absolute, normalised norm on $\mathbb{R}^m$. Let $\varepsilon, R > 0$. The there exists $\delta > 0$ such that for all $a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m) \in \mathbb{R}^m$ with $\|b\|_E \leq R$ and $|a_i| \leq |b_i|$ for $i = 1, \ldots, m$ we have

$$\|b\|_E - \|a\|_E < \delta \Rightarrow |b_i| - |a_i| < \varepsilon \forall i = 1, \ldots, m.$$ 

Utilizing this fact, one can see the following.

Proposition 3.10. Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^m$ which is strictly monotone and $X_1, \ldots, X_m$ Banach spaces with the uniform Opial property. Then $X := \left( \bigoplus_{i=1}^m X_i \right)_E$ also has the uniform Opial property.

Proof. Let $\varepsilon, R > 0$ and put $\eta := \min\{\eta_{X_i}(\varepsilon/m, R) : i = 1, \ldots, m\}$. Choose a $0 < \delta \leq 1$ according to Lemma 3.9 corresponding to the values $\eta$ and $3R + 1$.

Now consider a weakly nullsequence $(x_n)_{n \in \mathbb{N}} = ((x_{n,1}, \ldots, x_{n,m}))_{n \in \mathbb{N}}$ in $X$ with $\limsup \|x_n\|_E \leq R$ and an element $y = (y_1, \ldots, y_m) \in X$ with $\|y\|_E \geq \varepsilon$.

Since $\|y\|_E \leq \sum_{i=1}^m \|y_i\|$ there is some $i_0 \in \{1, \ldots, m\}$ with $\|y_{i_0}\| \geq \varepsilon/m$.

There is no loss of generality in assuming that all the limits in the following calculations exist. From the definition of $\eta$ we get

$$\lim_{n \to \infty} \|x_{n,i_0}\| + \eta \leq \lim_{n \to \infty} \|x_{n,i_0} - y_{i_0}\|.$$ 

Since each $X_i$ has in particular the nonstrict Opial property, we also have

$$\lim_{n \to \infty} \|x_{n,i}\| \leq \lim_{n \to \infty} \|x_{n,i} - y_i\| \quad \forall i \in \{1, \ldots, m\} \setminus \{i_0\}.$$ 

If $\|y\|_E \leq 2R + 1$, then $\lim_{n \to \infty} \|x_n - y\|_E \leq \lim_{n \to \infty} \|x_n\|_E + 2R + 1 \leq 3R + 1$ and the choice of $\delta$ implies $\lim_{n \to \infty} \|x_n\|_E + \delta \leq \lim_{n \to \infty} \|x_n - y\|_E$.

If on the other hand $\|y\|_E > 2R + 1$, then $\lim_{n \to \infty} \|x_n - y\|_E \geq \|y\|_E - \lim_{n \to \infty} \|x_n\|_E \geq R + 1 \geq (R + 1)/2$. So $X$ has the uniform Opial property. \qed

3.4 Opial properties of some infinite sums

We will first show that the Opial and nonstrict Opial property are preserved under infinite $\ell^p$-sums.

Proposition 3.11. If $1 \leq p < \infty$, $I$ is any index set and $(X_i)_{i \in I}$ a family of Banach spaces with the Opial property (nonstrict Opial property), then $X := \left( \bigoplus_{i \in I} X_i \right)_p$ also has the Opial property (nonstrict Opial property).

Proof. We will only prove the strict case, the nonstrict case is treated analogously. Let $x_n = (x_{n,i})_{i \in I} \in X$ for every $n \in \mathbb{N}$ such that $(x_n)_{n \in \mathbb{N}}$ converges weakly to zero and let $x = (x_i)_{i \in I} \in X \setminus \{0\}$. Fix $i_0 \in I$ with $x_{i_0} \neq 0$. We may assume that $\lim_{n \to \infty} \|x_n\|_p$ and $\lim_{n \to \infty} \|x_n - x\|_p$ as well
Since \( a := \lim_{n \to \infty} \|x_{n,i_0}\| \) and \( b := \lim_{n \to \infty} \|x_{n,i_0} - x_{i_0}\| \) exist. Note also that \((x_{n,i})_{n \in \mathbb{N}}\) is a weakly nullsequence in \(X_i\) for each \(i \in I\). So since \(X_{i_0}\) has the Opial property it follows that \(\delta := b^p - a^p > 0\). Put \(K := \sup_{n \in \mathbb{N}}\|x_{n}\|_p\) and let \(0 < \varepsilon \leq 1\). We can find a finite set \(J \subseteq I\) with \(i_0 \in J\) such that

\[
\left\|\left(\|x_i\chi_{I \setminus J(i)}\|_{i \in I}\right)_{i \in I}\right\|_p < \varepsilon,
\]

where \(\chi_{I \setminus J}\) denotes the characteristic function of \(I \setminus J\). By passing to a further subsequence, we can assume that \(\lim_{n \to \infty} \|x_{n,i}\|\) and \(\lim_{n \to \infty} \|x_{n,i} - x_i\|\) exist for all \(i \in J\). Then, using the Opial property of each of the summands \(X_i\), the definition of \(\delta\) and (3.10), we obtain

\[
\lim_{n \to \infty} \left\|x_n\right\|^p = \sum_{i \in J \setminus \{i_0\}} \lim_{n \to \infty} \|x_{n,i}\|^p + \lim_{n \to \infty} \left\|\left(\|x_{n,i}\|\chi_{I \setminus J(i)}\|_{i \in I}\right)\right\|^p
\]

\[
\leq \lim_{n \to \infty} \left(\left\|\left(\|x_{n,i} - x_i\|\chi_{I \setminus J(i)}\|_{i \in I}\right)\right\| + \delta + \lim_{n \to \infty} \left\|\left(\|x_{n,i}\|\chi_{I \setminus J(i)}\|_{i \in I}\right)\right\|^p
\]

\[
\leq \lim_{n \to \infty} \left(\left\|\left(\|x_{n,i} - x_i\|\chi_{I \setminus J(i)}\|_{i \in I}\right)\right\| + \delta + \lim_{n \to \infty} \left(\left\|\left(\|x_{n,i} - x_i\|\chi_{I \setminus J(i)}\|_{i \in I}\right)\right\| + \varepsilon\right)^p
\]

But, since \(|s^p - t^p| \leq pA^{p-1}|s - t|\) for all \(0 \leq s, t \leq A\), we also have

\[
\lim_{n \to \infty} \left(\left\|\left(\|x_{n,i} - x_i\|\chi_{I \setminus J(i)}\|_{i \in I}\right)\right\| + \varepsilon\right)^p \leq \lim_{n \to \infty} \left(\left\|\left(\|x_{n,i} - x_i\|\chi_{I \setminus J(i)}\|_{i \in I}\right)\right\| + \varepsilon\right)^p - \left(\left\|\left(\|x_{n,i} - x_i\|\chi_{I \setminus J(i)}\|_{i \in I}\right)\right\| + \varepsilon\right)^p
\]

\[
\leq \lim_{n \to \infty} \left(\left\|\left(\|x_{n,i} - x_i\|\chi_{I \setminus J(i)}\|_{i \in I}\right)\right\| + p(K + \|x\|_p + 1)^{p-1}\varepsilon.
\]

It follows that

\[
\lim_{n \to \infty} \left\|x_n\right\|^p \leq \lim_{n \to \infty} \left\|x_n - x\right\|^p - \delta + p(K + \|x\|_p + 1)^{p-1}\varepsilon.
\]

Since \(\varepsilon \in (0,1]\) was arbitrary and \(\delta\) independent of \(\varepsilon\), we conclude

\[
\lim_{n \to \infty} \left\|x_n\right\|^p \leq \lim_{n \to \infty} \left\|x_n - x\right\|^p - \delta < \lim_{n \to \infty} \left\|x_n - x\right\|^p
\]

and the proof is finished. \(\square\)

\(c_0\) is a typical example of a Banach space which has the nonstrict Opial property but not the usual (strict) Opial property. Next we will see that \(c_0\)-sums preserve the nonstrict Opial property.

**Proposition 3.12.** Let \(I\) be any index set and \((X_i)_{i \in I}\) a family of Banach spaces with the nonstrict Opial property. Then \(X := \bigoplus_{i \in I} X_i\) has the nonstrict Opial property.
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Proof. Let $x_n = (x_{n,i})_{i \in I} \in X$ for every $n \in \mathbb{N}$ such that $(x_n)_{n \in \mathbb{N}}$ converges weakly to zero and let $x = (x_i)_{i \in I} \in X$. Take $\varepsilon > 0$ to be arbitrary and find a finite subset $J \subseteq I$ such that $\|x_i\| \leq \varepsilon$ for every $i \in I \setminus J$. Again there is no loss of generality in assuming that all the limits involved in the following calculations exist. Since each $X_i$ has the nonstrict Opial property, we have

$$
\lim_{n \to \infty} \|x_{n,i}\| \leq \lim_{n \to \infty} \|x_{n,i} - x_i\| \quad \forall i \in J.
$$

Therefore we obtain

$$
\lim_{n \to \infty} \|x_n\| = \max \left\{ \max_{i \in J} \lim_{n \to \infty} \|x_{n,i}\|, \lim_{n \to \infty} \left( \|x_{n,i}\| \chi_{I \setminus J}(i) \right)_{i \in I} \right\}
$$

$$
\leq \max \left\{ \max_{i \in J} \lim_{n \to \infty} \|x_{n,i} - x_i\|, \lim_{n \to \infty} \left( \|x_{n,i} - x_i\| \chi_{I \setminus J}(i) \right)_{i \in I} \right\}
$$

$$
\leq \lim_{n \to \infty} \max \left\{ \max_{i \in J} \|x_{n,i} - x_i\|, \left( \|x_{n,i} - x_i\| \chi_{I \setminus J}(i) \right)_{i \in I} \right\} + \varepsilon
$$

$$
= \lim_{n \to \infty} \|x_n - x\| + \varepsilon
$$

and since $\varepsilon > 0$ was arbitrary we are done. \qed

Concerning the uniform Opial property, we have the following result for infinite $\ell^p$-sums, resembling in structure Theorem 3.2.

**Theorem 3.13.** Let $1 \leq p < \infty$ and let $I$ be an infinite index set. For a family $(X_i)_{i \in I}$ of Banach spaces put $X_J := \bigoplus_{i \in J} X_i$ for every finite $J \subseteq I$. Suppose that

$$
\omega(\varepsilon, R) := \inf \{ \eta_{X_J}(\varepsilon, R) : J \subseteq I \text{ finite} \} > 0 \quad \forall \varepsilon, R > 0.
$$

Then $X := \bigoplus_{i \in I} X_i$ has the uniform Opial property.

**Proof.** Let $0 < \varepsilon \leq 1$ and $R > 0$. We put $\nu := \min\{3R + 1, \omega(\varepsilon/2, R)\}$ and $\tau := \min\{1, 3R + 1 - ((3R + 1)^p - \nu^p)^{1/p}\}$. Now let us consider a weakly nullsequence $(x_n)_{n \in \mathbb{N}} = ((x_{n,i})_{i \in I})_{n \in \mathbb{N}}$ in $X$ with $\lim\sup\|x_n\|_p \leq R$ and let $x = (x_i)_{i \in I} \in X$ with $\|x\|_p \geq \varepsilon$. As before, we may assume that $\lim_{n \to \infty}\|x_n\|_p$ and $\lim_{n \to \infty}\|x_n - x\|_p$ exist. Let $K := \sup_{n \in \mathbb{N}}\|x_n\|_p$. For $0 < \alpha \leq \varepsilon/2$ we can find a finite subset $J \subseteq I$ such that

$$
\left\| \left( \|x_i\| \chi_{I \setminus J}(i) \right)_{i \in I} \right\|_p \leq \alpha.
$$

It follows that

$$
\left\| \sum_{i \in J} \|x_i|c_i|_p \right\| \geq \|x\|_p - \alpha \geq \varepsilon/2.
$$

(3.11)
We may also assume that \( \lim_{n \to \infty} \| x_{n,i} \| \) and \( \lim_{n \to \infty} \| x_{n,i} - x_i \| \) exist for all \( i \in J \). Analogous to the proof of Proposition 3.11 we can show that
\[
\lim_{n \to \infty} \| x_n \|_p^p \leq \lim_{n \to \infty} \sum_{i \in J} \| x_{n,i} \|_p^p + \lim_{n \to \infty} \| (\| x_{n,i} - x_i \| \chi_{I,J}(i))_{i \in I} \|_p^p \\
+ p(K + \| x \|_p + 1)^{p-1} \alpha. \tag{3.12}
\]
If we put \( y_n := (x_{n,i})_{i \in J} \) for each \( n \in \mathbb{N} \) and \( y := (x_i)_{i \in J} \), then \( (y_n)_{n \in \mathbb{N}} \) is a weakly nullsequence in \( X_J \) with \( \lim_{n \to \infty} \| y_n \|_p \leq \lim_{n \to \infty} \| x_n \|_p \leq R \) and \( y \in X_J \) with \( \| y \|_p \geq \varepsilon/2 \) (because of (3.11)), thus
\[
\lim_{n \to \infty} \| y_n \|_p + \eta_{X_J}(\varepsilon/2,R) \leq \lim_{n \to \infty} \| y_n - y \|_p. \tag{3.13}
\]
Since \( (a - b)^p \leq a^p - b^p \) for all \( a \geq b \geq 0 \) we can deduce from (3.12) and (3.13) that
\[
\lim_{n \to \infty} \| x_n \|_p^p \leq \lim_{n \to \infty} \| y_n - y \|_p^p - \eta_{X_J}(\varepsilon/2,R)^p + p(K + \| x \|_p + 1)^{p-1} \alpha \\
\leq \lim_{n \to \infty} \| x_n - x \|_p^p - \nu^p + p(K + \| x \|_p + 1)^{p-1} \alpha.
\]
Letting \( \alpha \to 0 \) we obtain
\[
\lim_{n \to \infty} \| x_n \|_p^p \leq \lim_{n \to \infty} \| x_n - x \|_p^p - \nu^p. \tag{3.14}
\]
If \( \| x \|_p \geq 2R + 1 \) then \( \lim_{n \to \infty} \| x_n - x \|_p \geq 2R + 1 - \lim_{n \to \infty} \| x_n \|_p \geq \lim_{n \to \infty} \| x_n \|_p + 1 \geq \lim_{n \to \infty} \| x_n \|_p + \tau. \)

Now consider the case \( \| x \|_p < 2R + 1 \). Define \( f(s) := s - (s^p - \nu^p)^{1/p} \) for all \( s \geq \nu \). It is easily checked that \( f \) is decreasing. Since \( \lim_{n \to \infty} \| x_n - x \|_p \leq \lim_{n \to \infty} \| x_n \|_p \leq 3R + 1 \) it follows from (3.14) that
\[
\lim_{n \to \infty} \| x_n \|_p \leq \lim_{n \to \infty} \| x_n - x \|_p - f( \lim_{n \to \infty} \| x_n - x \|_p ) \\
\leq \lim_{n \to \infty} \| x_n - x \|_p - f(3R + 1) \leq \lim_{n \to \infty} \| x_n - x \|_p - \tau
\]
and the proof is complete. 

As a corollary we obtain again the already known result that the \( \ell^p \)-sum of any family of Banach spaces with the Schur property has the uniform Opial property (see [22, Example 4.23 (2.)] or [23, Theorem 7]).

**Corollary 3.14.** Let \( 1 \leq p < \infty \) and let \((X_i)_{i \in I}\) be a family of Banach spaces with the Schur property. Then \( \left( \bigoplus_{i \in I} X_i \right)_p \) has the uniform Opial property.

The author does not know whether the condition \( \inf_{i \in I} \eta_{X_i} > 0 \) is already enough to ensure that \( \left( \bigoplus_{i \in I} X_i \right)_p \) has the uniform Opial property (the proof of Proposition 3.10 does not give a uniform lower bound for the moduli of the finite sums).
4 Some Opial-type properties in Lebesgue-Bochner spaces

We consider a complete, finite measure space \((S, \mathcal{A}, \mu)\) and a real Banach space \(X\). First recall that for \(1 \leq p \leq \infty\) the Lebesgue-Bochner space \(L^p(\mu, X)\) is defined as the space of all Bochner-measurable functions \(f : S \to X\) (modulo equality \(\mu\)-almost everywhere) such that \(\|f(\cdot)\| \in L^p(\mu)\). Equipped with the norm \(\|f\|_p := \|\|f(\cdot)\|\|_p\), \(L^p(\mu, X)\) becomes a Banach space.

As was mentioned in the introduction, even the spaces \(L^p[0,1], 1 < p < \infty, p \neq 2\), of scalar-valued functions do not have the Opial property. However, some results which are in a certain sense analogous to the Opial property are available. For example it was shown in [3] that any bounded sequence \((f_n)_{n \in \mathbb{N}}\) in \(L^p(\mu)\) (\(0 < p < \infty\)) which converges pointwise almost everywhere to a function \(f \in L^p(\mu)\) satisfies

\[
\lim_{n \to \infty} (\|f_n\|_p^p - \|f_n - f\|_p^p) = \|f\|_p^p
\]

and hence

\[
\liminf_{n \to \infty} \|g_n - g\|_p^p = \liminf_{n \to \infty} \|g_n\|_p^p + \|g\|_p^p
\]

for any bounded sequence \((g_n)_{n \in \mathbb{N}}\) in \(L^p(\mu)\) which converges pointwise almost everywhere to zero and every \(g \in L^p(\mu)\).

In [1, Chapter 2, Lemma 3.3] it was shown that any sequence \((f_n)_{n \in \mathbb{N}}\) in \(L^1(\mu, X)\) (where \((S, \mathcal{A}, \mu)\) is a probability space and \(X\) an arbitrary Banach space) and any \(f \in L^1(\mu, X)\) such that

\[
\lim_{n \to \infty} \mu(\{t \in S : \|f_n(t) - f(t)\| \geq \varepsilon\}) = 0 \quad \forall \varepsilon > 0
\]

satisfy the equality

\[
\liminf_{n \to \infty} \|f_n - f\|_1 + \|f - g\|_1 = \liminf_{n \to \infty} \|f_n - g\|_1
\]

for every \(g \in L^1(\mu, X)\).

We are now going to consider pointwise weak convergence almost everywhere in Lebesgue-Bochner spaces and prove some results analogous to the Opial property in this setting.

**Proposition 4.1.** Let \((S, \mathcal{A}, \mu)\) be a complete, finite measure space, \(1 \leq p < \infty\) and \(X\) a Banach space with the nonstrict Opial property. Let \((f_n)_{n \in \mathbb{N}}\) be a bounded sequence in \(L^p(\mu, X)\) such that \((f_n(t))_{n \in \mathbb{N}}\) converges weakly to zero for almost every \(t \in S\). Suppose further that there is a function \(g \in L^p(\mu)\) such that \(\|f_n(t)\| \to g(t)\) for almost every \(t \in S\). Then

\[
\int_A \liminf_{n \to \infty} \|f_n(t) - f(t)\|^p d\mu(t) - \int_A g(t)^p d\mu(t) \\
geq \limsup_{n \to \infty} \|f_n - f\|^p_p - \limsup_{n \to \infty} \|f_n\|^p_p
\]
holds for every \( f \in L^p(\mu, X) \) and every \( A \in \mathcal{A} \). In particular,

\[
\limsup_{n \to \infty} \|f_n\|_p \leq \limsup_{n \to \infty} \|f_n - f\|_p \quad \forall f \in L^p(\mu, X).
\]

**Proof.** Without loss of generality we can assume that \( \lim_{n \to \infty} \|f_n(t)\| = g(t) \) and \( f_n(t) \to 0 \) weakly for every \( t \in S \) and also that \( \lim_{n \to \infty} \|f_n\|_p \) and \( \lim_{n \to \infty} \|f_n - f\|_p \) exist. Now let \( A, f \in L^p(\mu, X) \) and \( 0 < \varepsilon < 1 \). By the equi-integrability of finite subsets of \( L^1(\mu) \) there exists \( \delta > 0 \) such that

\[
B \in \mathcal{A}, \mu(B) \leq \delta \Rightarrow \int_B h(t) \, d\mu(t) \leq \varepsilon \tag{4.1}
\]

for each \( h \in \{ ||f(\cdot)||^p, g, \liminf_{n \to \infty} ||f_n(\cdot) - f(\cdot)||^p \} \).

By Egorov’s theorem there exists \( C \in \mathcal{A} \) with \( \mu(S \setminus C) \leq \delta \) such that \( \lim_{n \to \infty} ||f_n(t)||^p = g(t)^p \) uniformly in \( t \in C \), which implies

\[
\lim_{n \to \infty} \int_C ||f_n(t)||^p \, d\mu(t) = \int_C g(t)^p \, d\mu(t). \tag{4.2}
\]

By passing to a further subsequence if necessary, we may assume that the limit

\[
\lim_{n \to \infty} \int_C ||f_n(t) - f(t)||^p \, d\mu(t)
\]

exists. Now we can calculate, using (4.2) and the nonstrict Opial property of \( X \),

\[
\lim_{n \to \infty} ||f_n||_p^p = \int_C g(t)^p \, d\mu(t) + \lim_{n \to \infty} \int_{S \setminus C} ||f_n(t)||^p \, d\mu(t)
\]

\[
= \int_{C \cap A} g(t)^p \, d\mu(t) + \int_{C \setminus A} g(t)^p \, d\mu(t) + \lim_{n \to \infty} \int_{S \setminus C} ||f_n(t)||^p \, d\mu(t)
\]

\[
\leq \int_{C \cap A} g(t)^p \, d\mu(t) + \int_{C \setminus A} \liminf_{n \to \infty} ||f_n(t) - f(t)||^p \, d\mu(t)
\]

\[
+ \lim_{n \to \infty} \int_{S \setminus C} ||f_n(t)||^p \, d\mu(t) \leq \int_C \liminf_{n \to \infty} ||f_n(t) - f(t)||^p \, d\mu(t)
\]

\[
+ \int_{C \cap A} (g(t)^p - \liminf_{n \to \infty} ||f_n(t) - f(t)||^p) \, d\mu(t) + \lim_{n \to \infty} \int_{S \setminus C} ||f_n(t)||^p \, d\mu(t). \tag{4.3}
\]

But

\[
\left| \int_{C \cap A} g(t)^p \, d\mu(t) - \int_A g(t)^p \, d\mu(t) \right| = \int_{A \setminus C} g(t)^p \, d\mu(t) \leq \varepsilon \tag{4.4}
\]

because of \( \mu(S \setminus C) \leq \delta \) and (4.1). Analogously,

\[
\int_{A \setminus C} \liminf_{n \to \infty} ||f_n(t) - f(t)||^p \, d\mu(t) \leq \varepsilon. \tag{4.5}
\]
Putting (4.3), (4.4) and (4.5) together we obtain

$$\lim_{n \to \infty} \|f_n\|_p^p \leq \int_C \liminf_{n \to \infty} \|f_n(t) - f(t)\|^p \, d\mu(t) + \lim_{n \to \infty} \int_{S \setminus C} \|f_n(t)\|^p \, d\mu(t)$$

$$+ \int_A \left( g(t)^p - \liminf_{n \to \infty} \|f_n(t) - f(t)\|^p \right) \, d\mu(t) + 2\varepsilon$$

$$\leq \lim_{n \to \infty} \int_C \|f_n(t) - f(t)\|^p \, d\mu(t) + \lim_{n \to \infty} \int_{S \setminus C} \|f_n(t)\|^p \, d\mu(t)$$

$$+ \int_A \left( g(t)^p - \liminf_{n \to \infty} \|f_n(t) - f(t)\|^p \right) \, d\mu(t) + 2\varepsilon,$$

where we have used Fatou’s lemma in the second step.

Since $\mu(S \setminus C) \leq \delta$ we have $\int_{S \setminus C} \|f(t)\|^p \, d\mu(t) \leq \varepsilon$, by (4.1). Hence

$$\lim_{n \to \infty} \int_{S \setminus C} \|f_n(t)\|^p \, d\mu(t) \leq \lim_{n \to \infty} \left( \left( \int_{S \setminus C} \|f_n(t) - f(t)\|^p \, d\mu(t) \right)^{1/p} + \varepsilon^{1/p} \right)^p$$

Since $|s^p - t^p| \leq pA^{p-2}|s - t|$ for all $0 \leq s, t \leq A$, we obtain as in the proof of Proposition 3.11

$$\lim_{n \to \infty} \int_{S \setminus C} \|f_n(t)\|^p \, d\mu(t) \leq \lim_{n \to \infty} \int_{S \setminus C} \|f_n(t) - f(t)\|^p \, d\mu(t) + pL^{p-1}\varepsilon^{1/p},$$

where $L := \|f\|_p + 1 + \sup_{n \in \mathbb{N}} \|f_n\|_p$. From (4.6) and (4.7) it follows that

$$\lim_{n \to \infty} \|f_n\|_p^p \leq \lim_{n \to \infty} \int_S \|f_n(t) - f(t)\|^p \, d\mu(t) + pL^{p-1}\varepsilon^{1/p}$$

$$+ \int_A \left( g(t)^p - \liminf_{n \to \infty} \|f_n(t) - f(t)\|^p \right) \, d\mu(t) + 2\varepsilon.$$

Letting $\varepsilon \to 0$ now leads to the desired inequality. \(\square\)

If $X$ has the Opial property, we have the following corollary.

**Corollary 4.2.** Let $(S, \mathcal{A}, \mu)$ be a complete, finite measure space, $1 \leq p < \infty$ and $X$ a Banach space with the Opial property. Let $(f_n)_{n \in \mathbb{N}}$ be a bounded sequence in $L^p(\mu, X)$ such that $(f_n(t))_{n \in \mathbb{N}}$ converges weakly to zero for almost every $t \in S$. Suppose further that there is a function $g \in L^p(\mu)$ such that $\|f_n(t)\| \to g(t)$ for almost every $t \in S$. Then

$$\limsup_{n \to \infty} \|f_n\|_p < \limsup_{n \to \infty} \|f_n - f\|_p \forall f \in L^p(\mu, X) \setminus \{0\}.$$

**Proof.** Just put $A := \{ t \in S : f(t) \neq 0 \}$ in Proposition 4.1. Then $\mu(A) > 0$ and since $X$ has the Opial property we have $\liminf \|f_n(t) - f(t)\| < g(t)$ for every $t \in A$, so the result follows from Proposition 4.1. \(\square\)
In the case that $X$ even has the uniform Opial property, we have the following two results.

**Theorem 4.3.** Let $(S, \mathcal{A}, \mu)$ be a complete, finite measure space, $1 \leq p < \infty$ and $X$ a Banach space with the uniform Opial property. Let $M, R > 0$ and $f \in L^p(\mu, X) \setminus \{0\}$. Then there exists $\eta > 0$ such that the following holds: whenever $(f_n)_{n \in \mathbb{N}}$ is a sequence in $L^p(\mu, X)$ with $\sup_{n \in \mathbb{N}}\|f_n\|_p \leq R$ such that $(f_n(t))_{n \in \mathbb{N}}$ converges weakly to zero and $\lim_{n \to \infty}\|f_n(t)\| \leq M$ for almost every $t \in S$, then

$$\limsup_{n \to \infty}\|f_n\|_p + \eta \leq \limsup_{n \to \infty}\|f - f\|_p.$$ 

**Proof.** We define $\tau := \|f\|_p(2\mu(S))^{-1/p}$ and $A := \{t \in S : \|f(t)\| \geq \tau\}$. If $\mu(A) = 0$, then we would obtain $\|f\|_p^p \leq \mu(S \setminus A)r^p \leq \|f\|_p^p/2$, contradicting the fact that $f \in L^p(\mu, X) \setminus \{0\}$. Thus $\mu(A) > 0$.

Next we put $w := \eta_X(\tau, M), \delta := \min\{(3R + 1)^p, \mu(A)w^p\}, \omega := R + 1 - ((R + 1)^p - \delta)^{1/p}$ and finally $\eta := \min\{\omega, 1\}.$

Now let $(f_n)_{n \in \mathbb{N}}$ be as above. Without loss of generality we may assume that $g(t) := \lim_{n \to \infty}\|f_n(t)\| \leq M$ and $f_n(t) \to 0$ weakly for every $t \in S$. The definition of $\eta_X$ implies

$$\liminf_{n \to \infty}\|f_n(t) - f(t)\| - g(t) \geq \eta_X(\tau, M) = w \ \forall t \in A.$$ 

Since $(a - b)^p \leq a^p - b^p$ for all $a \geq b \geq 0$, it follows that

$$\liminf_{n \to \infty}\|f_n(t) - f(t)\|^p - g(t)^p \geq w^p \ \forall t \in A.$$ 

Combining this with Proposition 4.1 leads to

$$\limsup_{n \to \infty}\|f_n - f\|^p - \limsup_{n \to \infty}\|f_n\|^p \geq \mu(A)w^p \geq \delta.$$ 

As in the proof of Theorem 3.13, by distinguishing the two cases $\|f\|_p \geq 2R + 1$ and $\|f\|_p < 2R + 1$, we can deduce from this that

$$\limsup_{n \to \infty}\|f_n\|_p + \eta \leq \limsup_{n \to \infty}\|f_n - f\|_p.$$ 

\[\square\]

**Theorem 4.4.** Let $(S, \mathcal{A}, \mu)$ be a complete, finite measure space, $1 \leq p < \infty$ and $X$ a Banach space with the uniform Opial property. Let $p < r \leq \infty$ and $\varepsilon, M, R, K > 0$. Then there exists $\eta > 0$ such that the following holds: whenever $(f_n)_{n \in \mathbb{N}}$ is a sequence in $L^p(\mu, X)$ with $\sup_{n \in \mathbb{N}}\|f_n\|_p \leq R$ such that $(f_n(t))_{n \in \mathbb{N}}$ converges weakly to zero and $\lim_{n \to \infty}\|f_n(t)\| \leq M$ for almost every $t \in S$ and $f \in L^r(\mu, X) \subseteq L^p(\mu, X)$ such that $\|f\|_r \leq K$ and $\|f\|_p \geq \varepsilon$, then

$$\limsup_{n \to \infty}\|f_n\|_p + \eta \leq \limsup_{n \to \infty}\|f_n - f\|_p.$$
Proof. We put $s := r/p \in (1, \infty]$. Let $s' \in [1, \infty)$ such that $1/s' + 1/s = 1$. Choose $0 < \tau < \varepsilon \mu(S)^{-1/p}$ and put $Q := (\varepsilon^p - \mu(S)^p) K^{-s'}$. Let $w := \eta_X(\tau, M)$ and $\delta := \min\{Q w^p, (3R + 1)^p\}$. $w$ and $\eta$ are also defined as in the previous proof.

Now let $(f_n)_{n \in \mathbb{N}}$ and $f$ be as above. For $A := \{ t \in S : \| f(t) \| \geq \tau \}$ we have

$$
\begin{aligned}
\varepsilon^p & \leq \| f \|_p^p = \int_A \| f(t) \|^p \, d\mu(t) + \int_{S \setminus A} \| f(t) \|^p \, d\mu(t) \\
& \leq \int_A \| f(t) \|^p \, d\mu(t) + \mu(S \setminus A) \tau^p \leq \mu(A)^{1/s'} \| f \|_p^p + \mu(S) \tau^p \\
& \leq \mu(A)^{1/s'} K^p + \mu(S) \tau^p,
\end{aligned}
$$

where we have used Hölder’s inequality in the second line. It follows that $\mu(A) \geq Q$. As in the previous proof we can deduce that

$$
\limsup_{n \to \infty} \| f_n - f \|_p^p - \limsup_{n \to \infty} \| f_n \|_p^p \geq \mu(A) w^p \geq Q w^p \geq \delta
$$

and from there we get to

$$
\limsup_{n \to \infty} \| f_n \|_p^p + \eta \leq \limsup_{n \to \infty} \| f_n - f \|_p
$$

as in the proof of Theorem 3.13. \qed

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