Modeling and Optimization of Layer-by-Layer Structures

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Abstract. In the present paper a differential-geometric approach is developed to modeling of residual stresses in layered (LbL) structures obtained as a result of successive curing of thin layers of material. The objects of modeling are the structures obtained by sequential adsorption of a large number of thin layers. During this assembling, the internal (residual) stresses appear in the multilayered structure while local deformations turn out to be incompatible. This leads to accumulation of residual stresses and distortion of the final shape of the LbL structure. To reduce these factors, geometric compensation is used, the calculation of which is extremely laborious with a large number of layers. Geometric methods allow us to implement fast algorithms for obtaining the compensation. They are based on the “smoothing” of the multilayer LbL structure and its representation by introducing a smooth body-manifold with non-Euclidean connection that characterizes the incompatibility of deformations. Connection is determined from the solution of the evolutionary problem, which formalizes the course of the technological process. The classical fields of the mechanics of a continuous medium are put in correspondence with their non-Euclidean counterparts. As an example, model problems for cylindrical LbL elastic structures are considered. A model problem of structural optimization is solved to determine the optimal strategy of the technological process.

1. Introduction

Recently, the technique of obtaining thin films by layer-by-layer assembling (LbL) has been intensively developing. It was initially aimed at improving the quality of coatings and other nanostructural objects. Lately great advance in understanding the internal interaction between layers and particles have been made. This gave impetus to the development of LbL technologies that nowadays have reached substantial sophistication in structure and variety of components [1, 2]. At present one should expect significant research efforts to be directed at fine-tuning and optimization of complex multilayered structure as well as at design of specific stack architecture for improving their functionalities.

One can describe LbL assembly as the sequential adsorption of positively and negatively charged species or layers that are held in the compound by a wide range of intermolecular interactions, such as ion–ion, Van der Waals, hydrophobic interactions, etc. [2]. In this regard the LbL structures as a rule are self-stressed. This feature, from one hand, can be qualified as undesirable, and the efforts of developers should be directed towards reducing internal stresses. From the other hand, one is able use this property of LbL structures for the benefit and design them with a predetermined distribution of internal stresses for improving their specific features. All this leads to the formulation of problems of structural optimization [3, 4]. Note that the approach of small deformations seems to be too rough for the majority of LbL structures and comprehensive modeling can be provided in the framework of...
nonlinear mechanics only. Moreover one has to have an appropriate tool for modeling incompatible local deformations, similar to them in nonlinear dislocation mechanics. Such a tool is available, and this is the Geometrical Mechanics [5,6]. The effectiveness of differential-geometrical approach was demonstrated in classical works [7–12] as well as in a number of recent investigations [13–16]. The present work is aimed at the development of methods of Geometrical Mechanics in the application to the modeling of LbL structures.

Some words should be said about major ideas of Geometrical Mechanics. In the classical elasticity theory, deformations are defined relatively to a reference shape which is assumed to be free from stresses [17]. As it was already mentioned, LbL structures do not possess such shape, so classical approach cannot be applied for their analysis. It is possible however to find stress-free shape in more general sense. To do this one can define reference shape as an image of considered LbL structure with respect to an embedding into non-Euclidean space with specific, material connection [18–21]. For simple solids this connection becomes affine. This fact has simple geometrical explanation. Indeed it is easy to see formal correspondence between the mathematical description of incompatible deformations and the geometry of spaces endowed with affine connection. In simple materials the transformation of each elementary volume into a uniform state is carried out by a linear transformation. By assumption these transformations are determined by a set of continuous functions. The moving frame method proposed by E. Cartan for constructing of the space with affine connection also uses a smooth field of linear transformations that acts on coordinate frame [22–24]. Thus, these two approaches have much in common and the geometric point of view may be applied for modeling of solids whose local deformations do not satisfy the compatibility conditions, particularly, LbL solids. Connection that is determined in such a way is called material connection [20]. There are two basic ways to introduce a material connection. One can use directly the correspondence between local deformations that brings infinitesimally part of the body into stress-free state and linear transformations that acts on coordinate frame. This approach results in flat space with non-trivial torsion, called Weitzenböck space or teleparallel space [22]. Another way is based on the correspondence between a local metric defined by local deformation and the construction of a Riemannian space by given distribution for the metric. Below we use both approaches. For simple bodies these approaches give equivalent state description because the state of simple material is defined by only symmetrical part of local deformation (in virtue of frame indifference principle [19]).

Since the present work affects the theory of parallelizable manifolds we will briefly discuss here the purely geometric aspects [23, 24]. We have pointed out seven basic concepts and give below brief comments to them.

1. **Global frame fields.** Let \( \mathcal{M} \) be a \( n \)-dimensional smooth manifold. Recall that a global frame field is a \( n \)-tuple \((e_i)_{i=1,...,n}\) of vector fields on \( \mathcal{M} \) that are linearly independent at each point \( x \in \mathcal{M} \). It follows from this assumption that they must be globally non-zero. A \( n \)-dimensional smooth manifold is said to be parallelizable if one can define a global frame field on it. One can regard global frame field as a global section of the principal fiber bundle \( GL(\mathcal{M}) \to \mathcal{M} \) and, consequently, treat it as a set of \( n \) smooth maps \( e_i : \mathcal{M} \to GL(\mathcal{M}) \) such that \((e_i)\) defines a basis for the tangent vector space \( T_x\mathcal{M} \) for every \( x \). Another way for characterizing the frame \((e_i)\) is to say that it represents a linear isomorphism \((e_i) : \mathbb{R}^n \to T_x\mathcal{M}, (v^1,...,v^n) \mapsto v^i e_i\). The inverse isomorphism

\[
(\theta^i) : T_x\mathcal{M} \to \mathbb{R}^n, v \mapsto (\theta^1(x)v,...,\theta^n(x)v)
\]

defines the coframe field at \( x \) and, consequently, a global section of the principal fiber bundle \( G^L(\mathcal{M}) \to \mathcal{M} \). The above mentioned linear isomorphisms are inverse to each other. Thus, every frame field has a unique reciprocal coframe field \((\theta^i)\), which can be defined by the reciprocal property \(\theta^i(e_j) = \delta^i_j\). At this stage we can already establish a correspondence between geometrical and
3. Nonholonomic frames. If \( \{e_i, e_j\} \) and \( d\theta^i \) are non-vanishing for some \((i, j)\) then \( (e_i) \) represents a nonholonomic frame field. It cannot be directly integrable into the natural frame field of any coordinate chart. However the local frame fields are integrable in more general sense that a vector field is parallel iff its components are constant functions. Locally, \( d\theta^i = 0, \) \( d\theta^j = 0, \) where \([\cdot, \cdot]\) are Lie brackets [23]. This property is equivalent to the integrability conditions for the local frame field. The vanishing of 1-forms \( d\theta^i \) means that there are \( n \) functions \( \chi_i \) on \( U \) such that \( \theta^i = d\chi_i \). Moreover, linearity independence of 1-forms \( \theta^i \) leads to the fact that the map \( x: U \rightarrow \mathbb{R}^n \) is a diffeomorphism. From the point of view adopted in continuum mechanics it means the existence of a displacement field that transforms the stress-free shape into some deformed (actual) shape.

4. Parallelism. The pure geometric idea of parallel vector fields plays an important role in the theory being developed, since it allows one to formalize the idea of a material parallel transfer. Recall that a vector field \( v \) on \( \mathcal{M} \) is said to be parallel iff its components \( v^i \) are constant functions. Locally, a vector \( v \in T_x\mathcal{M} \) is parallel to a vector \( v' \in T_y\mathcal{M}, \) \( x \neq y \) iff \( v^i = v'^i. \) Suchwise a vector field is parallel iff all of its values, i.e. vectors from tangent spaces, are pair-wise parallel to each other. In the framework of continuum mechanics one can describe deformation by observing the distortion of the
material fiber during its transfer along the curve, provided that the preimage of this fiber in the reference form remains parallel to itself. All that is required to generalize classical representations is to assume that the reference form is embedded in a space with a non-Euclidean rule of parallel transport.

5'. Connection. To introduce a connection that facilitates such parallel translation we assume that the frame field itself is parallel. Let \( v \) be a vector field:

\[
v = v^i \partial_i = \tilde{v}^i e_i, \quad \tilde{v}^i = T^{ij}_k v^j.
\]

Some calculations give

\[
d\tilde{v}^i = dT^{ij}_k v^j + T^{ij}_k dv^j = \tilde{T}^{ij}_k \nabla v^j, \quad \nabla v^j = dv^j + \Gamma^j_i v^i, \quad \Gamma^j_i = \Omega^j_i dx^i, \quad \Gamma^j_i = \frac{\tilde{\Omega}^{ij}_k}{\partial x^k}.
\]

Note that since \( d(\tilde{T}^{ij}_k) = d\delta^j_i = 0 \) then \( \Gamma^j_i = -d\Omega^j_i \tilde{T}^{i}_k \) also. Suppose that \( v \) is parallel. According to aforementioned definition it means that \( v^i \) are constant functions and hence \( d\tilde{v}^i = 0 \) and \( \nabla v^i = 0 \). In the case of general vector field \( v \)

\[
\nabla v = \Omega^j_i \nabla v^j \otimes \partial_i = \nabla \tilde{v}^i \otimes e_i.
\]

Thus, \( \Gamma^j_i = 0 \) relative to the nonholonomic frame field \( e_i \) and one can say that geometry as observed in that frame field is “nonholonomic Euclidean” [22]. The \( \Gamma^j_i \) is called 1-form teleparallelism connection. In the framework of continuum mechanics this kind of connection formalizes local transformations of representative (infinitesimal) part to the uniform state, so-called crystal state [27]. This question will be discussed in section 5.

6'. Torsion and curvature. Special properties of so defined material connection can be characterized by torsion and curvature of it. Recall that torsion can be defined by 2-form [28]

\[
\mathcal{T}^i = d\theta + \Gamma^i_k \wedge \theta^k,
\]

that in general does not vanish. Note that if one has two parallel vector fields \( v \) and \( w \) then

\[
\mathcal{T}^i(v, w)e_i = -[v, w].
\]

It can be shown that curvature 2-form [28]

\[
\mathcal{R}^i_j = d\Gamma^i_j + \Gamma^i_k \wedge \Gamma^k_j
\]

that corresponds to teleparallel connection vanishes. Indeed, direct calculations shows that [12]

\[
\mathcal{R} = d\tilde{T} \wedge d\Omega + (d\tilde{T} \Omega) \wedge (d\tilde{T} \Omega) = d\tilde{T} \wedge d\Omega + (d\tilde{T}) \wedge (\Omega d\tilde{T} \Omega) = 0.
\]

7'. Metric. Finally let us pay some attention to the second, metrical way for introducing the material connection. Let \( \mathcal{M} \) be a parallelizable manifold. We can define on \( \mathcal{M} \) the volume element \( V \) and the metric \( g \) according to

\[
V = \theta^1 \wedge \theta^2 \wedge \theta^3, \quad g = g_{\theta^i \theta^j} = g_{\theta^i} \otimes \theta^j.
\]

This allows us to determine Levi-Civita connection [28]

\[
\tilde{\Gamma}^{\mu}_{\nu \kappa} = \frac{1}{2} g^{\lambda \mu} \left( \partial_{\nu} g_{k \lambda} + \partial_{\kappa} g_{\nu \lambda} - \partial_{\nu} g_{\kappa \lambda} \right).
\]
However the Levi-Civita connection does not have vanishing curvature. If the metric $g$ is identified with local deformation measures then so defined connection can be treated as material. This question will be discussed in section 4.

Some words about notation. In the paper we use mathematical terminology and definitions that are consistent with those in [24, 30]. In particular, if $\mathcal{M}$ is a smooth manifold, then symbols $T_p\mathcal{M}$ and $T^*_p\mathcal{M}$ for each point denote respectively tangent and cotangent spaces to $\mathcal{M}$ at a point $p \in \mathcal{M}$. The tangent and cotangent bundles are denoted by $T\mathcal{M}$ and $T^*\mathcal{M}$ respectively. The symbol $\otimes$ stays for the abstract tensor product. We denote by $\langle v, u \rangle$ the value of the following bilinear operation

$$\langle \cdot, \cdot \rangle : T^*_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}, \quad \langle v, u \rangle := v(u).$$

Some words should be mentioned about vector bundles and sections. If $(\mathcal{C}, \mathcal{M}, \pi)$ is a vector bundle [24] with base $\mathcal{M}$, total space $\mathcal{C}$, and projection $\pi : \mathcal{C} \to \mathcal{M}$, then by a section [24] of such the bundle we mean a mapping $s : \mathcal{M} \to \mathcal{C}$ that satisfies the following condition: $\pi \circ s = 1$, in which $1 : \mathcal{M} \to \mathcal{M}$ is the identity map. We use the term “field” as a synonym for “section”. Note that the construction of vector bundle allows one to define the notion of smooth section similarly to the notion of smooth mapping between manifolds. It is convenient instrument that is used in the modern physics [28].

The following notations are used for sets of smooth sections:

a) $\mathcal{C}(\mathcal{M})$, the set of all smooth scalar functions $f : \mathcal{M} \to \mathbb{R}$;

b) $\text{Vec}(\mathcal{M})$, the set of all smooth vector fields $u : \mathcal{M} \to T\mathcal{M}$;

c) $\text{CVec}(\mathcal{M})$, the set of all smooth covector fields $v : \mathcal{M} \to T^*\mathcal{M}$.

The mapping $\langle \cdot, \cdot \rangle$ induces bilinear operations $\cdot \cdot$ and $\partial \cdot$, which are acting on dyads as follows:

$$\cdot \cdot : (T \otimes \mathcal{O}, v) \mapsto T \otimes \mathcal{O} \cdot = \langle \mathcal{O}, v \rangle T, \quad \partial \cdot (v, \mathcal{O} \otimes T) \mapsto v \partial \mathcal{O} \otimes T = \langle \partial, v \rangle T.$$ 

Here $T$ is a smooth section of a vector bundle, and $v \in \text{Vec}(\mathcal{M})$, $\mathcal{O} \in \text{CVec}(\mathcal{M})$ are vector and covector fields.

If $\mathcal{M}$ and $\mathcal{N}$ are smooth manifolds, $\pi : \mathcal{M} \to \mathcal{N}$ is a smooth mapping, and $T, Q$ are respectively smooth sections of vector bundles over $\mathcal{M}$ and $\mathcal{N}$, then the symbols $\pi^*T$ and $\pi'^*Q$ denote pushforward and pullback of these fields [30]. For a given Riemannian metric musical isomorphisms are denoted by $(\cdot) : T\mathcal{M} \to T^*\mathcal{M}$ and $(\cdot)^\sharp : T^*\mathcal{M} \to T\mathcal{M}$ [24].

Symbol $[a_{ij}]$ denotes a matrix with elements $a_{ij}$.

2. Body and Physical Space

A body $\mathcal{B}$ is a set with continuum cardinality. Its elements $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$, are referred to as material points. This set is endowed with smooth $n$-dimensional manifold structure [24]. That is, $\mathcal{B}$ is a Hausdorff second-countable topological space with a fixed maximal atlas $A_{\text{max}}$. Each chart from this atlas is represented by an ordered pair $(U, \varphi)$, in which $U \subset \mathcal{B}$ is an open set, and $\varphi : U \to O$ is a homeomorphism between $U$ and some open set $O \subset \mathbb{R}^n$. We refer to $\varphi$ as coordinate mapping. For each two charts $(U, \varphi)$ and $(V, \psi)$ from $A_{\text{max}}$ one has that either i) $U \cap V = \emptyset$, or ii) $U \cap V \neq \emptyset$ and the mapping $\psi \circ \varphi^{-1}\big|_{\varphi(U \cap V)} : \varphi(U \cap V) \to \psi(U \cap V)$ is a $C^\infty$-diffeomorphism.

A physical space is a formalization of the ideal “laboratory”. In the present paper we consider the physical space as a 3-dimensional Euclidean point space $\mathcal{E}$ with the associated translation vector.
space \( \mathcal{V} \). That is, the following Weyl axioms hold \([31]\): there exists a mapping \( w: \mathcal{E} \times \mathcal{E} \to \mathcal{V} \), \((x, y) \mapsto xy\), such that

a) for each \( x, y, z \in \mathcal{E} \) Chasles’ identity holds: \( xy + yz + zx = 0 \);

b) for each \( x \in \mathcal{E} \) the mapping \( w(x, \cdot): \mathcal{E} \to \mathcal{V} \), \( y \mapsto xy \) is bijective.

The 3-dimensional vector space \( \mathcal{V} \) is endowed with an inner product \( \cdot: \mathcal{V} \times \mathcal{V} \to \mathbb{R} \).

A Cartesian frame \((o, (i^k)_{k=1}^3)\) is chosen in \( \mathcal{E} \). Here \( o \in \mathcal{E} \) is the origin and \((i^k)_{k=1}^3\) is an orthonormal basis of \( \mathcal{V} \), i.e., \( i_p \cdot i_k = \delta_{pk} \), for each \( p, k = 1, 2, 3 \). The basis which is dual to \((i^k)_{k=1}^3\), i.e., \( i^k \) are elements of the dual vector space \( \mathcal{V}^* \) and \( \langle i^k, i_p \rangle = \delta_{ip} \), for each \( p, k = 1, 2, 3 \).

The space \( \mathcal{E} \) has a trivial smooth manifold structure. Its minimal atlas \( A_{\text{min}} \) consists of one chart \((x, c)\), where \( c: \mathcal{E} \to \mathbb{R}^3 \) is Cartesian arithmetization:

\[
\forall x \in \mathcal{E}: \quad c(x) := ((x-o)i_1, (x-o)i_2, (x-o)i_3).
\]

In what follows, the tangent space \( T_x \mathcal{E} \) to \( \mathcal{E} \) at any point \( x \) is identified with the translation space \( \mathcal{V} \).

The physical space \( \mathcal{E} \) contains all shapes of every body \( \mathcal{B} \). These are images \( \varkappa(\mathcal{B}) \) of smooth embeddings \( \varkappa: \mathcal{B} \to \mathcal{E} \). Such a mapping is referred to as configuration. The set of all configurations is denoted by \( \mathcal{C}(\mathcal{B}) \).

**Remark 1.** A body \( \mathcal{B} \) is observable in the physical space \( \mathcal{E} \) only through its shapes. This fact was allegorically described by M. Epstein in \([6]\): "The body itself dwells in the Platonic world of differentiable manifolds, of which we can only see the manifestations in the world of phenomena, amenable to perception, in this case, in the guise of configurations."

Since each configuration \( \varkappa \in \mathcal{C}(\mathcal{B}) \) is a homeomorphism onto its image, \( \varkappa(\mathcal{B}) \), it is convenient to introduce a new mapping \( \hat{\varkappa} \), which is obtained by the restriction of the codomain of \( \varkappa \) into \( \varkappa(\mathcal{B}) \). That is\(^\dagger\),

\[
\hat{\varkappa}: \mathcal{B} \to \varkappa(\mathcal{B}), \quad \hat{\varkappa}(x) = \varkappa(x).
\]

This mapping is a homeomorphism. Relations between the body, the physical space and configurations are illustrated in figure 1.

**Remark 2.** Formalization of the body as a three-dimensional smooth manifold was provided by W. Noll and C.-C. Wang in \([20, 21]\). In their considerations the smooth structure was generated by configurations that are mappings from the body into the physical space. M. Gurtin and A. Murdoch in \([32]\) developed theory of elastic material surfaces (that are two-dimensional smooth manifolds) by using Noll’s approach. Since that time the mathematical theory of smooth manifolds was deeply developed, but key ideas in mechanical interpretation are the same. \(^\dagger\)

\(^\dagger\) It is assumed that \( \dim \mathcal{B} \leqslant \dim \mathcal{E} \).

\(^\dagger\) In the present paper we use the following notation. Let \( f: X \to Y \) be a mapping. Then by \( \hat{f} \) we denote the mapping \( \hat{f}: X \to f(X) \), that assigns to each \( x \in X \) a value \( \hat{f}(x) := f(x) \).
Figure 1. A body $\mathcal{B}$, the physical space $\mathcal{E}$, a configuration $\mathcal{X}$.

Since the body $\mathcal{B}$ is not observable by itself, neither are material fibers. If some configuration $\mathcal{X} \in C(\mathcal{B})$ is chosen, then for each material fiber $c: \mathcal{I} \to \mathcal{B}$ and its infinitesimal element $u \in T\mathcal{B}$, the images $\mathcal{X} \circ c(\mathcal{I})$ and $T\mathcal{X}(u)$ of the composition $\mathcal{X} \circ c: \mathcal{I} \to \mathcal{E}$ and the tangent map $T\mathcal{X}: T\mathcal{B} \to T\mathcal{E}$ are observable. Thus, the tangent map $T\mathcal{X}$ plays the same role as the conventional deformation gradient. Thereby, we refer to $T\mathcal{X}$ as configuration gradient.

3. Material Uniformity and Inhomogeneity

Let $x$ be a material point of a body $\mathcal{B}$. Local configuration at $x$ formalizes the idea of infinitesimal localization: if the “observer” is placed into infinitesimal neighborhood of the point $x$, then it can distinguish configurations only within this neighborhood. Assume that this localization has the first order: the “observer” cannot distinguish configurations that have the same values at $x$ and the same linear approaches. Mathematical formalization of this idea may be provided as follows. Let $\sim_x$ be the following equivalence relation on the set $C(\mathcal{B})$:

$$\forall \mathcal{X}_1, \mathcal{X}_2 \in C(\mathcal{B}) : (\sim_x) \Leftrightarrow ((\mathcal{X}_1(x) = \mathcal{X}_2(x)) \wedge (T_x\mathcal{X}_1 = T_x\mathcal{X}_2)).$$

Following W. Noll [20] we refer to each equivalence class $\text{cl}_x$ as local configuration at $x \in \mathcal{B}$. A local configuration $\text{cl}_x$ contains such configurations that are not distinguishable from each other by the imaginary “observer”.

Denote the set of all local configurations at $x$ by $\mathcal{L}_x$. Each local configuration $\text{cl}_x$ generates linear mapping $K_x: T_x\mathcal{B} \to V$ as follows. Choose some configuration $\mathcal{X} \in \text{cl}_x$ and define $K_x$ as $K_x = T_x\mathcal{X}$. According to the definition of the relation $\sim_x$, the mapping $K_x$ does not depend on the choice of a representative $\mathcal{X} \in \text{cl}_x$ and is fully determined by the class $\text{cl}_x$. It maps infinitesimal neighborhood of the point $x$, i.e., tangent space $T_x\mathcal{B}$, into infinitesimal neighborhood of the point $x = \mathcal{X}(x)$, $\mathcal{X} \in \text{cl}_x$. By definition, there is a one-to-one correspondence between such mappings and elements from $\mathcal{L}_x$. In the paper we identify a local configuration $\text{cl}_x$ with the corresponding linear mapping $K_x$.

Let $TE \otimes T^*\mathcal{B} = \bigcup_{(X,x) \in \mathcal{B} \times \mathcal{E}} T_xE \otimes T^*_x\mathcal{B}$. Introduce a section $K: \mathcal{B} \to TE \otimes T^*\mathcal{B}$ as follows:

$$\forall x \in \mathcal{B} : (K_x \in \mathcal{L}_x).$$

We refer to the section $K$ as reference [20]. By its definition, the reference $K$ is a two-point tensor field with dyadic decomposition $K = K^k_{ij} \otimes E^i$, in which $(E^i)$ is a coframe field on the body $\mathcal{B}$.
The material at any point \( \mathcal{X} \in \mathfrak{B} \) is characterized by its response. Let \( R \) be a set of response descriptors, i.e., a set of sets of objects that are determined by \(^3\) “measurers”. The body \( \mathfrak{B} \) is simple [20], if there is a mapping \( \mathcal{R}_\mathcal{X} : \mathcal{L}_\mathcal{X} \to R \) for any material point \( \mathcal{X} \). In the paper we consider only simple bodies. Each of them is materially uniform, i.e., the material at any two of its points is the same. The mathematical formalization of such an idea is based on the notion of material isomorphism [20]. That is, let \( \mathcal{X}, \mathcal{Y} \in \mathfrak{B} \) be material points. An invertible linear mapping \( \Phi_{xY} : T_\mathcal{Y} \mathfrak{B} \to T_\mathcal{X} \mathfrak{B} \) is called as material isomorphism from \( T_\mathcal{Y} \mathfrak{B} \) onto \( T_\mathcal{X} \mathfrak{B} \), if the equality
\[
\mathcal{R}_\mathcal{X}(K_\mathcal{X}) = \mathcal{R}_\mathcal{Y}(K_\mathcal{Y} \Phi_{xY}),
\]
holds for each \( K_\mathcal{X} \) from \( \mathcal{L}_\mathcal{X} \). Let the set of all material isomorphisms from \( T_\mathcal{Y} \mathfrak{B} \) onto \( T_\mathcal{X} \mathfrak{B} \) denote by \( g_{xY} \). Thus, the simple body is materially uniform if \( g_{xY} \neq \emptyset \) for all material points \( \mathcal{X}, \mathcal{Y} \in \mathfrak{B} \).

It follows from the material uniformity that there exists a uniform reference \( K^R \) [20], i.e., such a reference that the linear mapping
\[
\Phi_{xY} := [K_\mathcal{X}]^{-1} K^R,
\]
is a material isomorphism from \( T_\mathcal{Y} \mathfrak{B} \) onto \( T_\mathcal{X} \mathfrak{B} \) for all material points \( \mathcal{X}, \mathcal{Y} \in \mathfrak{B} \). According to the definition, the following equality holds for a uniform reference \( K^R : \mathcal{R}_\mathcal{X}(K^R) = \mathcal{R}_\mathcal{Y}(K^R) \). This means that the response of the body is the same at each material point. Another consequence is that images of the linear mappings \( K^R_\mathcal{X} \) coincide and are represented by a vector subspace of \( \gamma \) of the dimension \( \dim B \). Thus, each infinitesimal neighborhood \( T_\mathcal{X} \mathfrak{B} \) may be obtained from such a space by means of \( [K^R_\mathcal{X}]^{-1} \).

A uniform reference \( K^R \) generates a family \( \text{Ref}_\mathfrak{B} = \{K^R_\mathcal{X}\}_{\mathcal{X} \in \mathfrak{B}} \) of local configurations, that are values of \( K^R \). If a representative \( \mathcal{X}^R_\mathcal{X} \) is chosen from each \( K^R_\mathcal{X} \), then one arrives at the family \( \text{Ref}_\mathfrak{B} = \{\mathcal{X}^R_\mathcal{X}\}_{\mathcal{X} \in \mathfrak{B}} \) of global configurations
\[
\mathcal{X}^R_\mathcal{X} : \mathfrak{B} \to \mathcal{E}, \quad \mathcal{X}^R_\mathcal{X} : \mathcal{Y} \to \mathcal{X}^R_\mathcal{X}(\mathcal{Y}),
\]
such that each of them transforms an infinitesimal neighborhood of a point \( \mathcal{X} \) into a uniform shape [25].

Physically, a materially uniform simple body consists of infinitesimal representative volumes with the following property. Each of them, if being separated from others and placed in “testing machine”, gives the same response on the same deformation. It leads to the idea that each representative volume of such a body may be obtained from one fixed subject to the corresponding local deformation. This fixed infinitesimal object is referred to as archetyp e and the corresponding mapping as implant. Let \( K^R \) be a uniform reference. Fix a material point \( \mathcal{X}_0 \in \mathfrak{B} \). Thus, the material isomorphism \( \Phi_{xX_0} = [K^R_\mathcal{X}_0]^{-1} K^R_\mathcal{X}_0 \) from \( T_\mathcal{X}_0 \mathfrak{B} \) onto \( T_\mathcal{X} \mathfrak{B} \) allows us to obtain \( T_\mathcal{X} \mathfrak{B} \) from \( T_\mathcal{X}_0 \mathfrak{B} \) for each \( \mathcal{X} \in \mathfrak{B} \). The tangent space \( T_\mathcal{X}_0 \mathfrak{B} \) and the linear mapping \( P_\mathcal{X} = \Phi_{xX_0} \) are viewed respectively as an archetyp e and an implant. It is M. Epsteins’ standpoint [5], in which the archetyp e is considered as a material object. By W. Nolls’ standpoint [20] the archetyp e is a spatial object. Thus, it is represented by \( K^R_{\mathcal{X}_0} [T_\mathcal{X}_0 \mathfrak{B}] \). In such consideration the implant is represented by the composition \( \Phi_{xX_0} [K^R_\mathcal{X}_0]^{-1} \), i.e., \( P_\mathcal{X} = [K^R_\mathcal{X}]^{-1} \).

\(^3\) In particular, \( R \) may consist of second order tensors that represent values of the Cauchy stress tensor.
If $K^R$ is a uniform reference and there exists a configuration $x \in C(\mathcal{B})$, such that $K^R_x = T^x x^R$ for each material point $x \in \mathcal{B}$, then the body $\mathcal{B}$ is homogeneous. In this case one can apply $x^R_x := x$ and consider the global uniform shape $x(\mathcal{B})$ as a reference shape. In the contrary, the body $\mathcal{B}$ is inhomogeneous if such the configuration does not exist. It means that there does not exist a global uniform shape for the body $\mathcal{B}$ in the Euclidean physical space $\mathcal{E}$. Figure 2 illustrates the case of an inhomogeneous body. A red cube represents an infinitesimal neighborhood in an uniform state.

**Figure 2.** Materially uniform inhomogeneous body.

**Remark 3.** Instantaneous material composition of LbL structure may be represented by a materially uniform body $\mathcal{B}$. In this case, one can divide the body $\mathcal{B}$ into a collection of “joined” layers. Each layer is homogeneous, but as a whole, the body is inhomogeneous. 

The key idea of the geometric approach is to give the body $\mathcal{B}$ a role of the fictional reference shape and to derive equations in material description relatively to it. Since these equations use the notion of parallel translation, one needs to define the affine connection on the body $\mathcal{B}$, which should also reflect material properties of the body. We refer to such a connection as material connection.

### 4. Synthesizing of the Material Metric

Using elements of the family $\overline{Ref}_{\mathcal{B}}$ one can obtain the collection $\{(\mathcal{B}, G^{(x)})\}_{x \in \mathcal{B}}$ of Riemannian spaces with metrics $G^{(x)}$ that are defined relatively to the embeddings $x^R_x$. Thus,

$$\forall x \in \mathcal{B} \ \forall u, v \in T_x \mathcal{B}: \ G^{(x)}_{|_x} (u, v) := [T_x x^R_x (u)] - [T_x x^R_x (v)].$$

The corresponding Levi-Civita connection is Euclidean. Let us synthesize Riemannian metric, non-Euclidean in general case. Assume that the section $G: \mathcal{B} \to T^* \mathcal{B} \otimes T^* \mathcal{B}$, defined by a formula $G_x := G^{(x)}_{|_x}$, is smooth. Each of values $G^{(x)}_{|_x}$ represents a symmetric positive definite bilinear form. Hence, $G$ is a Riemannian metric. Since it is defined on the body $\mathcal{B}$, we refer to $G$ as material metric. By the definition, the material metric is combined from the family of Riemannian metrics $\{G^{(x)}\}_{x \in \mathcal{B}}$, i.e.,

$$\forall x \in \mathcal{B} \ \forall u, v \in T_x \mathcal{B}: \ G_x (u, v) := [T_x x^R_x (u)] - [T_x x^R_x (v)].$$

In terms of the uniform reference $K^R$ this relation takes the form [20]

$$\forall x \in \mathcal{B} \ \forall u, v \in T_x \mathcal{B}: \ G_x (u, v) := [K^R_x (u)] - [K^R_x (v)].$$

(3)

The material metric $G$ defines Levi-Civita connection. The coefficients $\Gamma^i_{\alpha\beta}$ of the latter in a coordinate frame $(\partial_\alpha)$ may be calculated by using the following expressions
\[
\Gamma_{\alpha\beta}^{\gamma} = \frac{G_{\alpha\beta}}{2} \left( \partial_{\alpha} G_{\beta\gamma} + \partial_{\beta} G_{\gamma\alpha} - \partial_{\gamma} G_{\alpha\beta} \right),
\]

where \( G_{\alpha\beta} = g_{\alpha}^{\epsilon} K_{\alpha\epsilon} K_{\beta}^{\epsilon} \), \([G_{\alpha\beta}] = [G_{\alpha\beta}]^{-1}\) and \( g_{\alpha} = e_{\alpha} e_{\alpha} \). Here \((e_{\alpha})_{k=1}^{3}\) is a local frame field for some curvilinear coordinate system in \( E \).

5. Synthesizing of Material Connection

5.1. Affine connection

Now we switch on to the question of defining material connection. Let us review some aspects from the affine connection theory. In what follows the symbol \( \mathcal{M} \) denotes a smooth \( n \)-dimensional manifold. Affine connection \( \nabla \) is a mapping [33]

\[
\nabla : \text{Vec}(\mathcal{M}) \times \text{Vec}(\mathcal{M}) \rightarrow \text{Vec}(\mathcal{M}), \quad (u, v) \mapsto \nabla_{u} v,
\]

for which the following conditions hold:

a) \( \forall u, v, w \in \text{Vec}(\mathcal{M}) : \nabla_{u+v} w = \nabla_{u} w + \nabla_{v} w \);

b) \( \forall u, v \in \text{Vec}(\mathcal{M}) : \forall f \in \mathcal{F}(\mathcal{M}) : \nabla_{u} v = f \nabla_{v} v ;

c) \( \forall u, v, w \in \text{Vec}(\mathcal{M}) : \nabla_{v} (u+w) = \nabla_{u} v + \nabla_{v} w ;

d) \( \forall u, v \in \text{Vec}(\mathcal{M}) : \forall f \in \mathcal{F}(\mathcal{M}) : \nabla_{u} (f v) = f \nabla_{u} v + (uf) v \) (Leibniz rule).

In d) \( uf \) denotes the action of a vector field \( u \) on a scalar function \( f \). Here \( u \) is identified with derivation [24]. The vector field \( \nabla_{u} v \) represents covariant derivative of \( v \) among \( u \).

In a fixed (nonholonomic) frame field \((e_{i})_{i=1}^{n} \), \( e_{i} \in \text{Vec}(\mathcal{M}) \), and the corresponding coframe field \((\Theta_{i})_{i=1}^{n} \), \( \Theta_{i} \in C\text{Vec}(\mathcal{M}) \), the mapping \( \nabla \) is represented by \( n \) scalar mappings \( \Gamma_{\mu}^{\alpha} = (\Theta_{i} , e_{\mu} , e_{\alpha}) \in \mathcal{F}(\mathcal{M}) \).

Remark 4. Since \( \nabla_{e_{\mu}} e_{\alpha} = \Gamma_{\mu}^{\alpha} e_{\alpha} \), for vector fields \( u = u^{i} e_{i} \) and \( v = v^{i} e_{i} \), one has the following coordinate representation of covariant derivative \( \nabla_{u} v \):

\[
\nabla_{u} v = u^{i} \left( \partial_{i} v^{k} + v^{j} \Gamma_{kj}^{i} \right) e_{k} = u^{i} \nabla_{e_{i}} v^{k} e_{k},
\]

where \( \nabla_{e_{i}} v^{k} = (dx^{k}, \nabla_{e_{i}} v) \). If the field \( \nabla v = \nabla_{e_{i}} v^{k} e_{k} \otimes dx^{k} \) is introduced then \( \nabla_{u} v = \nabla v \otimes u \).

Let us obtain the transformation law of the mappings \( \Gamma_{\mu}^{\alpha} \). Let the mappings \( \Omega_{i} \in \mathcal{F}(\mathcal{M}) \) define a smooth field of non-degenerated \( n \times n \) matrices \( \Omega_{i} \). There is a frame field related to this field according to

\[
\mathcal{M} \ni p \mapsto (e_{i}^{\prime})_{i=1}^{n} , \quad e_{i}^{\prime} = \Omega_{i} e_{j} .
\]

The corresponding coframe field \( \mathcal{M} \ni p \mapsto (\Theta_{i}^{\prime})_{i=1}^{n} \) is expressed through “old” coframe field via the relation \( \Theta_{i}^{\prime} = \Theta_{ij}^{\prime} \Theta_{j}^{\epsilon} \), where \( [\Theta_{ij}^{\prime}] = [\Omega_{i}]^{-1} \). Finally, let \( e_{i} = A_{i}^{j} \partial_{x^{j}} \), where \( (x^{j}) \) are local coordinates on \( \mathcal{M} \). Then

\[
\nabla_{e_{j}} e_{k}^{\prime} = \nabla_{\Theta_{ij}^{\prime} \Theta_{j}^{\epsilon}} e_{k}^{\epsilon} = (\Omega_{j}^{\gamma} \nabla_{\Theta_{j}^{\epsilon}} e_{k}^{\gamma}) = (\Omega_{j}^{\gamma} \nabla_{\Theta_{j}^{\epsilon}} e_{k}^{\gamma} + e_{\epsilon} (\Omega_{j}^{\gamma} e_{k}^{\gamma})),
\]

from which, since \( \Gamma_{\mu}^{\alpha} = (\Theta_{i}^{\prime}, \nabla_{e_{i}} e_{\alpha}^{\prime}) = (\nabla_{e_{i}} \Theta_{i}^{\prime} , \nabla_{e_{i}} e_{\alpha}^{\prime})\) and \( e_{\epsilon} (\Omega_{j}^{\gamma}) = A_{i}^{j} \partial_{x^{j}} \Omega_{i}^{\gamma} \), it follows that

\[
\Gamma_{\mu}^{\alpha} = \nabla_{\Theta_{i}^{\prime}} \Omega_{j}^{\gamma} \left( \Theta_{i}^{\prime} \Theta_{j}^{\epsilon} \nabla_{e_{i}} e_{\alpha}^{\prime} + \Theta_{i}^{\prime} e_{\epsilon} (\Omega_{j}^{\gamma} e_{k}^{\gamma}) \right).
\]
Taking into account the relations \( \Gamma_{\mu q}^m = \langle \partial^m, \nabla_q e_q \rangle \) and \( \langle \partial^m, e_q \rangle = \delta_q^m \), we arrive at the final expression:

\[
\Gamma_{jk}^m = \Gamma_{\mu q}^m \Omega_j^\mu \Omega_k^\nu + A' (\Omega_q^m \Omega_j^\nu) \partial_{\nu} \Omega_k^m.
\]

The expression (4) defines the affine connection transformation rule in the general case: when one changes an “old” nonholonomic frame field into a “new” one. If the “old” frame field is a coordinate frame, then \( A' = \delta_j^i \) and the formula (4) transforms into

\[
\Gamma_{jk}^m = \Gamma_{\mu q}^m \Omega_j^\mu \Omega_k^\nu + \Omega_q^m \partial_{\nu} \Omega_k^m.
\]

If both frame fields are coordinate frames, i.e. the affine connection transformation is generated by coordinate transformation, then \( \Omega_j^\mu = \partial(x')^j / \partial(x)^i \), where \((x')^i\) are “old” coordinates and \((x')^j\) are “new” ones. In this case the relation (5) takes the form:

\[
\Gamma_{jk}^m = \Gamma_{\mu q}^m \frac{\partial x'^i}{\partial x^m} \frac{\partial x'^j}{\partial x^m} + \frac{\partial^2 x'^m}{\partial x^i \partial x^j} + \frac{\partial x'^m}{\partial x^i} \frac{\partial x'^m}{\partial x^j} \frac{\partial x'^m}{\partial x^i}.
\]

Connection coefficients may have nonzero values in an Euclidean space when one considers curvilinear coordinates, and in more general spaces, such as Riemann and Cartan spaces. In order to distinguish the space of non-Euclidean geometry, it is convenient to use tensor fields [33] of the torsion

\( \mathfrak{T} : \text{Vec}(\mathbb{M}) \times \text{Vec}(\mathbb{M}) \to \text{Vec}(\mathbb{M}) \),

curvature

\( \mathfrak{R} : \text{Vec}(\mathbb{M}) \times \text{Vec}(\mathbb{M}) \times \text{Vec}(\mathbb{M}) \to \text{Vec}(\mathbb{M}) \),

and nonmetricity

\( \Omega : \text{Vec}(\mathbb{M}) \times \text{Vec}(\mathbb{M}) \times \text{Vec}(\mathbb{M}) \to F(\mathbb{M}) \).

For a given Riemannian metric \( g \) these fields are defined by the relations

\[
\mathfrak{T}(u, v) = \nabla_u v - \nabla_v u - [u, v],
\]

\[
\mathfrak{R}(u, v, w) = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u, v]}w,
\]

\[
\Omega(u, v, w) = g(\nabla_u v, w) + g(v, \nabla_u w) - u[g(v, w)],
\]

for \( u, v, w \in \text{Vec}(\mathbb{M}) \). Here \([\cdot, \cdot] : \text{Vec}(\mathbb{M}) \times \text{Vec}(\mathbb{M}) \to \text{Vec}(\mathbb{M}) \) denote Lie brackets.

**Remark 5.** Nonmetricity is a measure of affine connection compatibility with the Riemannian metric. If \( \Omega = 0 \) then affine connection is compatible with the metric. ♦

**Remark 6.** Torsion is a measure of affine connection non-symmetry. Let \( (\partial_i')^k\) be a coordinate frame field on \( \mathbb{M} \). Then

\[
\mathfrak{T}(u, v) = \nabla_u v - \nabla_v u - [u, v] = u'v'(\Gamma_{jk}^i - \Gamma_{ji}^k)\partial_k,
\]

and one has that \( \mathfrak{T} = (\Gamma_{jk}^i - \Gamma_{ji}^k)\partial_k \otimes dx^i \otimes dx^j \). It means that \( \mathfrak{T} = 0 \) if and only if \( \Gamma_{jk}^i = \Gamma_{ji}^k \), for all \( i, j, k \). ♦

Given fields of torsion, curvature and nonmetricity, they fully determine affine connection via the formula
\[ \Gamma_{ij} = 2\Gamma_{ik}^m g_{mk} = \partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk} + T_{ij} - T_{jk} + T_{ji} + Q_{ij} - Q_{jk} - Q_{ik}, \]

5.2. Weitzenböck connection

Now let us use Cartan's moving frame method to define a special connection. Assume that a smooth field of non-degenerated $n \times n$ matrices $[\Omega'_i]$ is given. This field is generated by some scalar mappings $\Omega'_i \in F(\mathcal{M})$. Using the field $[\Omega'_i]$ one can define the new frame field

\[ \mathcal{M} \ni p \mapsto (z_i)_{|p}^o, \quad z_i = \Omega_i^j \partial_j. \]

In particular, one has $z_i = \Omega_j^i$ for the space $\mathcal{E}$. This case corresponds to the classical moving frame method: an orthonormal frame is deformed by means of $[\Omega'_i]$ and becomes nonholonomic.

Let us obtain such an affine connection $\nabla$ that the following relations hold:

\[ \nabla z_i = 0, \quad i, j = 1, \ldots, n. \]

It is Weitzenböck connection. The corresponding space is the space with absolute parallelism. The connection coefficients relatively to the frame $(z_i)$ are equal to zero: $\Gamma_{ik}^j = 0$. According to the formula (5) we have the following system of equations relatively to the connection coefficients $\Gamma_{iq}^m$:

\[ \Gamma_{iq}^m \Omega'_m \partial_j \Omega'_q + \Omega'_m \partial_q \partial_i \Omega'_q = 0. \]

Multiply both sides of the last equation on $\Omega'_q \partial_q \Omega'_i$ and sum over $i, j, k$. Then one arrives at the expressions

\[ \Gamma_{iq}^m \delta^q_k \delta^k_i + \delta^m_k \delta^q_j \partial_q \partial_i \Omega'_q = 0, \]

from which it is possible to establish the formula

\[ \Gamma_{iq}^m = -\Omega'_q \partial_q \Omega'_i. \]

The obtained relation can be transformed as follows

\[ \Gamma_{iq}^m = -\Omega'_q \partial_q \Omega'_i = -\Omega'_q \partial_q \Omega'_q \delta^q_i = -\Omega'_q \partial_q \Omega'_q \partial_q \Omega'_q, \]

and then, using the relation $\partial q \Omega'_q = -\Omega'_q \partial_q \Omega'_q \partial_q \Omega'_q$, we take the final result:

\[ \delta^k_i = \Omega'_k \partial_q \Omega'_q. \] (6)

5.3. Material connection

Now consider a uniform reference $K^R$ of a three-dimensional materially uniform inhomogeneous body\(^4\) $\mathcal{B}$. It represents the field $X \mapsto K^R_X$ of non-degenerate linear transformations. These transformations, in turn, generate the implant field $X \mapsto P_X = [K^R_X]^{-1}$. Choose some configuration $\hat{\mathcal{C}} \in \mathcal{C}(\mathcal{B})$. The shape $\hat{\mathcal{C}}(\mathcal{B})$ represents a “model” of the body. In addition, one embeds all infinitesimal neighborhoods into $\hat{\mathcal{C}}(\mathcal{B})$ by means of $T\hat{\mathcal{C}}$. Because of inhomogeneity, the shape $\hat{\mathcal{C}}(\mathcal{B})$ is non-uniform (in particular, with internal stresses). The implant field $P$, up to embedding,

\[ \text{dim } \mathcal{B} = \text{dim } \mathcal{E}. \]

Then $K^R_X$ is a vector space isomorphism. In the general case the key idea is the same.
generates the frame field $z_i = P^i_j$ defined on the shape $\tilde{\mathcal{B}}$. This field is nonholonomic (see figure 3).

![Natural local shapes](image)

![Continuous field of frames (nonholonomic)](image)

**Figure 3.** Correspondence between local shapes and space with absolute parallelism.

Parallelize the frame field $(z_k)$ by means of Weitzenböck connection. According to (6) one gets

$$\Gamma^k_{ij} = P^k_i \delta_{ij} [P^{-1}]^m_j,$$

or, in terms of local configuration,

$$\Gamma^k_{ij} = [(K^R)^{-1}]_{ij} \delta_i [K^R]^m_j.$$

The Weitzenböck space $(\tilde{\mathcal{B}}, \nabla)$ represents the desired global uniform shape.

6. Strains and Stresses

6.1. Cauchy-Green tensors

Since material metric $G : \mathcal{B} \to T^* \mathcal{B} \otimes T^* \mathcal{B}$ has been synthesized in section 4, we can define the notion of the transposed configuration gradient. Let $\mathcal{X} \in \mathcal{B}$. We denote by a $T_{\mathcal{X}} T^\perp$ a unique linear map defined on $\mathcal{Y}$ with values in $T_{\mathcal{X}} \mathcal{B}$, transpose to the tangent map $T_{\mathcal{X}} T^\perp : T_{\mathcal{X}} \mathcal{B} \to \mathcal{Y}$, i.e. the following relation is held for $T_{\mathcal{X}} T^\perp$:

$$\forall u \in T_{\mathcal{X}} \mathcal{B} \quad \forall v \in \mathcal{Y} : \quad [T_{\mathcal{X}} T^\perp (u)] v = G(u, T_{\mathcal{X}} T^\perp (v)).$$

By using $T_{\mathcal{X}} T^\perp$ and $T_{\mathcal{X}} T^\perp$ the following tensors can be constructed:

$$C_{\mathcal{X}} := T_{\mathcal{X}} T^\perp \circ T_{\mathcal{X}} T^\perp \in T_{\mathcal{X}} \mathcal{B} \otimes T_{\mathcal{X}} \mathcal{B}, \quad \mathcal{X} \in \mathcal{B},$$

$$B_{\mathcal{X}} := T_{\mathcal{X}^{-1}(x)} T^\perp \circ T_{\mathcal{X}^{-1}(x)} T^\perp \in \mathcal{Y} \otimes \mathcal{Y}, \quad x \in \mathcal{X}(\mathcal{B}).$$

We refer to the tensors $C_{\mathcal{X}}$ and $B_{\mathcal{X}}$ as right and left Cauchy-Green tensors [17]. In components,

$$C_{\mathcal{X}} = G^\alpha{}^\beta |_{x} g^j |_{x} (x) \left( \frac{\partial x^i}{\partial \mathcal{X}^\alpha} \right) \left( \frac{\partial x^j}{\partial \mathcal{X}^\beta} \right) \left( \frac{\partial x^k}{\partial \mathcal{X}^\gamma} \right) \left( \frac{\partial x^l}{\partial \mathcal{X}^\sigma} \right) \delta_{x^k} \otimes d\mathcal{X}^\gamma |_{x}, \quad (7)$$
\[ B_z = G^{\alpha \beta} \left| \frac{\partial \chi^\gamma}{\partial \chi^\alpha} \right| g_\beta \left| \frac{\partial \chi^\lambda}{\partial \chi^\gamma} \right| e_\lambda \otimes e^\gamma \Big|_z. \] (8)

Here \( \varphi: \mathcal{X} \mapsto (\mathcal{X}', ..., \mathcal{X}''') \) is a coordinate mapping on \( \mathfrak{B} \) , \((\mathcal{X}', ..., \mathcal{X}''') \) are curvilinear coordinates on \( \mathcal{E} \) with the local frame field \((e_i)\), \((\mathcal{X}', ..., \mathcal{X}''') \mapsto (x', x'', x''')\) is a coordinate representation of \( \mathcal{X} \), \([G^{\alpha \beta}] = [G_{\alpha \beta}]^{-1} \), \( G_{\alpha \beta} = G(\partial_{\alpha}, \partial_{\beta}) \), and \( g_\beta = \varepsilon_i \cdot e_j \). Greek indices varies from 1 to \( n = \dim \mathfrak{B} \) , and Latin ones varies from 1 to 3 . Thus, one has the sections

\[ C: \mathfrak{B} \rightarrow T^* \mathfrak{B} \otimes T^* \mathfrak{B} , \quad \text{and} \quad B: \mathcal{X}(\mathfrak{B}) \rightarrow T \mathcal{E} \otimes T^* \mathcal{E}. \]

### 6.2. Covector-valued exterior stress-forms

By a part \( \mathfrak{Y} \) of the body \( \mathfrak{B} \) we mean a \( n \)-dimensional oriented smooth submanifold with the boundary in \( \mathfrak{B} \) [24]. Let smooth sections \( T: \mathcal{X}(\mathfrak{B}) \rightarrow T^* \mathcal{E} \otimes T^* \mathcal{E} \) and \( P: \mathfrak{B} \rightarrow T^* \mathcal{E} \otimes T^* \mathfrak{B} \) correspond to the Cauchy and first Piola-Kirchhoff stress tensor fields.

**Remark 7.** The following interpretation for the section \( T \) takes place. Let \( \mathfrak{Y} \) be a part of the body \( \mathfrak{B} \) and \( n: \partial_{\mathcal{X}}(\mathfrak{Y}) \rightarrow \mathcal{V} \) be a unit normal vector field that is applied to the boundary of the shape \( \mathcal{X}(\mathfrak{B}) \). Then the covector field \( x \mapsto t_n(x) := T[n] \) represents a classical stress vector field \( t_n \). If \( v \) is a spatial velocity field then the scalar field \( \langle t_n, v \rangle = t_n \cdot v \) represents a surface density of the power generated by contact forces on the boundary \( \partial_{\mathcal{X}}(\mathfrak{Y}) \). The interpretation of the section \( P \) is quite cumbersome since the notion of oriented area element of the \( \partial \mathfrak{B} \) requires clarification. This problem was considered in a variety of works [34, 35] and we follow the ideas that are given in them.

The reasoning starting point is as follows. The exterior form is an object which can be integrated over a smooth manifold. Let the classical situation hold. The body has a stress free reference shape, that is the image of a configuration \( \mathcal{X}_R \in \mathcal{E}(\mathfrak{B}) \) , and there is a motion, which may be represented by a family \( \{ \mathcal{X}_t \}_{t \in \mathbb{T}} \) of configurations \( \mathcal{X}_t \in \mathcal{E}(\mathfrak{B}) \). Here \( \mathbb{T} \) is a real axis interval. The power on the actual shape is represented by the integral

\[ p = \int_{\partial_{\mathfrak{X}}(\mathfrak{Y})} v \cdot T_n \, nds, \]

and the power on the reference shape can be written in the form

\[ p = \int_{\partial_{\mathcal{X}_R}(\mathfrak{Y})} V \cdot P_n \, Nds, \]

where \( v \), \( V \) are the spatial and material velocities and \( n \), \( N \) are normal vector fields. In the framework of these reasoning we consider the first Piola-Kirchhoff stress tensor as \( P: \mathcal{X}_R(\mathfrak{B}) \rightarrow T^* \mathcal{E} \otimes T^* \mathcal{X}_R(\mathfrak{B}) \). Note that \( nds \) and \( Nds \) are oriented area elements, and, thus, can be considered as exterior forms. The idea is to view on \( T_n nds \) and \( P_n Nds \) as to one-type objects that are covector-valued exterior forms. By returning to the general case, we introduce the sections

\[ T = *_2 T , \quad P = *_2 P , \] (9)

where \( *_2 \) is the Hodge star operation applied to the “second leg”. Thus, one gets coordinatewise

\[ T = T_{\alpha \beta} e^\alpha \otimes (*_2 e^\beta) , \quad P = P_{\alpha \beta} e^\alpha \otimes (*_2 E^\beta) , \]

where \( T = T_{\alpha \beta} e^\alpha \otimes e^\beta , \quad P = P_{\alpha \beta} e^\alpha \otimes E^\beta , \) and \( e^\alpha , E^\alpha \) are coframe fields on \( \mathcal{E} \) and \( \mathfrak{B} \). Since [30]
\[ *E^\beta = \sqrt{G} \sum_{j=1}^{\delta} (-1)^{j-1} G^{\beta j} E^j \wedge \ldots \wedge E^u , \]

\[ *e^b = \sqrt{g} \sum_{j=1}^{3} (-1)^{j-1} g^{bj} e^j \wedge e^j \wedge e^b , \]

we arrive at the following representations for (9):

\[ T = \sum_{j=1}^{3} (-1)^{j-1} g^{bj} T_{ab} \sqrt{g} e^a \wedge \left( e^j \wedge e^j \wedge e^b \right) , \quad (10) \]

\[ P = \sum_{j=1}^{3} (-1)^{j-1} G^{\beta j} P_{\alpha \beta} \sqrt{G} e^\alpha \wedge \left( E^j \wedge \ldots \wedge E^\gamma \wedge \ldots \wedge E^u \right) . \quad (11) \]

Here \([g^\beta] = [g_{\beta}]^{-1} , \quad g = \det[g_{\beta}] , \quad g_\beta = e_\beta \cdot e_j \). Similarly, \([G^{\alpha \beta}] = [G_{\alpha \beta}]^{-1} , \quad G = \det[G_{\alpha \beta}] \) and \(G_{\alpha \beta} = G(E_\alpha , E_\beta)\). Note that the power which is generated by stress forms is represented by the following expressions:

\[ p = \int_{\mathcal{H}(\mathfrak{F})} v_{\perp} T , \quad p = \int_{\mathcal{H}} V_{\perp} P . \]

The analog of the classical Piola transformation is represented in terms of pullback [35]:

\[ P = \varepsilon^* T . \quad (12) \]

Here \(\varepsilon^*\) is the pullback provided by the configuration \(\varepsilon\) on the “second leg” of \(T\), i.e.

\[ P = T_{ab} e^a \wedge \varepsilon^* (*e^b) . \]

The exterior form \(\varepsilon^*(*e^b)\) is defined on \(\mathfrak{B}\).

6.3. The particular case

Consider the particular case: \(n = 3\). Obtain the representation of the Piola-Kirchhoff stress form through the Cauchy one by using (12). We begin with the relation (10) which has in our considerations, the form

\[ T = T_{ab} \sqrt{g} \, dx^a \wedge \left( g^{bj} dx^j \wedge dx^3 \right) - g^{b2} dx^1 \wedge dx^3 + g^{b3} dx^1 \wedge dx^2 , \quad (13) \]

To use (12) we need a formula for \(\varepsilon^* (dx^i \wedge dx^j)\). It follows from the definition of pullback that

\[ \varepsilon^* (dx^i \wedge dx^j) = \frac{\partial \varepsilon^j}{\partial \mathfrak{X}^a} \frac{\partial \varepsilon^i}{\partial \mathfrak{X}^b} d\mathfrak{X}^a \wedge d\mathfrak{X}^b . \]

After substituting formula (13) into (12), taking into account the linearity of \(\varepsilon^*\) and the above formula for \(\varepsilon^* (dx^i \wedge dx^j)\), we obtain the relation

\[ P = J T_{ab} \sqrt{g} g^{b2} d\mathfrak{X}^a \wedge (F^{-1})^2_{i} d\mathfrak{X}^i \wedge d\mathfrak{X}^3 \wedge d\mathfrak{X}^3 - (F^{-1})^2_{i} d\mathfrak{X}^i \wedge d\mathfrak{X}^3 \wedge d\mathfrak{X}^3 + (F^{-1})^2_{i} d\mathfrak{X}^i \wedge d\mathfrak{X}^3 \wedge d\mathfrak{X}^3 , \quad (14) \]

where \(J = \det \left( \frac{\partial \varepsilon^j}{\partial \mathfrak{X}^a} \right)\) and \(F^{-1} = (F^{-1})^2_{i} \varepsilon^i \wedge dx^i \) is the inverse to \(F = T\mathfrak{X}\). Since components for \(F^{-1}\) and its transpose, \(F^{-T}\) are related by the expression
the relation (14) may be rewritten in terms of \((F^T)_{ij}\). Now let us note that the expression (11) in our particular case takes the form

\[
P = P_{ab} \sqrt{G} e^{\alpha} \otimes \left( G^{\alpha\beta} d\xi^\beta \wedge \xi^\gamma - G^{\alpha\beta} \xi^\beta \wedge \xi^3 + G^{\alpha\beta} \xi^\beta \wedge \xi^2 \right).
\]

Then we get the final expression

\[
P_{ab} = \frac{\sqrt{G}}{J} T_{ab} (F^T)^{\alpha}_{\beta},
\]

in which \(g = \text{det}[g_{ij}]\), and \(G = \text{det}[G_{ij}]\).

Remark 8. In the case of incompressible material the factor \(\sqrt{G}/J\) in the expression (15) may be not equal to 1, in contrast to the classical Piola transformation. The reason is of the use of different coordinate maps on the reference and actual shapes. ♦

6.4. The Eshelby energy-momentum tensor

The Eshelby energy-momentum tensor can be generalized for the body \(B\) as follows. The balance equation on the body has the form

\[
\text{Div} P^i + \rho_0 b^i_0 = 0.
\]

Here \(P: B \rightarrow T^*E \otimes T^*B\) is the first Piola-Kirchhoff stress tensor, \(P^i = P^{\alpha\beta} \partial_{\xi^\alpha} \otimes \partial_{\xi^\beta}\) and

\[
\text{Div} P^i = [\partial_{x^\alpha} P^{\alpha\beta} + P^{\alpha\beta} \Gamma_{\alpha}^{\gamma\beta} + P^{\alpha\beta} F_{\alpha}^{\gamma\beta} \gamma_{\gamma\beta}] \partial_{x^\beta},
\]

where \(\Gamma_{\alpha}^{\gamma\beta}\) and \(\gamma_{\gamma\beta}\) are Levi-Civita connection coefficients on \(B\) and \(E\), respectively; \(\rho_0\) is the mass density and \(b^i_0\) is the body force. Applying the contraction with \(F^T = (F^T)^{\alpha}_{\beta} \partial_{\xi^\alpha} \otimes dx^\beta\) to (16) gives us

\[
F^T \lrcorner \text{Div} P^i + \rho_0 F^T \lrcorner b^i_0 = 0.
\]

Further we will use the bilinear operation \(\lrcorner\), that satisfies the following property:

\[
\partial_{x^\alpha} \otimes dx^\alpha \otimes d\xi^\beta \partial_{\xi^\beta} \otimes \partial_{x^\beta} = \langle dx^\alpha, \partial_{x^\alpha} \rangle \langle d\xi^\beta, \partial_{\xi^\beta} \rangle \partial_{x^\beta}.
\]

The following formula can be verified directly:

\[
\text{Div}(F^T \lrcorner P^i) = F^T \lrcorner \text{Div} P^i + (\text{VF}^T) \lrcorner P^i.
\]

By using (18) in the equation (17) we have

\[
\text{Div}(F^T \lrcorner P^i) - (\text{VF}^T) \lrcorner P^i + \rho_0 F^T \lrcorner b^i_0 = 0.
\]

\[\text{According to [17], the covariant derivative of a two point tensor field } Q = Q^{\alpha\beta} \partial_{\xi^\alpha} \otimes \partial_{\xi^\beta} \text{ has the representation } \nabla Q = Q^{\alpha\beta} \partial_{\xi^\alpha} \otimes \partial_{\xi^\beta} \otimes d\xi^\beta, \text{ in which } Q^{\alpha\beta} = \partial_{\xi^\alpha} Q^{\alpha\beta} + Q^{\alpha\beta} \Gamma_{\alpha}^{\gamma\beta} + Q^{\alpha\beta} F_{\alpha}^{\gamma\beta} \gamma_{\gamma\beta}. \text{ It follows that } \text{Div} Q = Q^{\alpha\beta}.\]

\[\text{Let us remind the reader that, for convenience, in coordinate representations we use Greek indices for the body and Latin ones for the physical space.}\]
Let $W : \mathfrak{B} \to \mathbb{R}$ be the elastic energy density defined on $\mathfrak{B}$. Subtracting the field $^7 (\nabla_a W)^i$ from the left and right sides of the obtained equation leads to

$$\text{Div}(F^T \otimes P^i) - (\nabla_a W)^i - (\nabla F^T) \otimes P^i + \rho_0 F^T \cdot b_0^i = - (\nabla_a W)^i. \quad (19)$$

Denote $I = \partial_{x^i} \otimes d\mathcal{X}^i$ (the identity tensor on $\mathfrak{B}$). Since

$$\text{Div}(WI^i) = \{G^a{}_{ab} \partial_{x^a} W\} \partial_{x^b} = (\nabla_a W)^i,$$

the expression (19) takes the final form

$$\text{Div}(e) + (\nabla F^T) \otimes P^i - \rho_0 F^T \cdot b_0^i = (\nabla_a W)^i, \quad (20)$$

where $e = WI^i - F^T \otimes P^i$ is the Eshelby energy-momentum tensor.

Let $\bar{W}$ be the elastic energy density represented as a function of points and fields defined on $\mathfrak{B}$ according to

$$W_\alpha = \bar{W}(x; F_\alpha, \ldots).$$

By following [27] let us define $\nabla_a W|_{\text{impl}}$ as

$$\nabla_a W|_{\text{impl}} = \nabla_a W - \nabla_a W|_{\text{expl}}, \quad \text{where} \quad \nabla_a W|_{\text{expl}} = \nabla_a \bar{W}(x; F_{\alpha_0}, \ldots)|_{x_0 = x}.$$

In this case the relation (20) can be rewritten in the form [27]:

$$-\text{Div}(e) = f^i_{\text{int}} + f^i_{\text{ext}} + f^i_{\text{inh}},$$

where $f^i_{\text{int}} = (\nabla F^T) \otimes P^i - \nabla_a W|_{\text{impl}}$ is the internal force, $f^i_{\text{ext}} = -\rho_0 F^T \cdot b_0^i$ is the body force, and $f^i_{\text{inh}} = -\nabla_0 W|_{\text{expl}}$ is the Eshelby force.

7. Models of LbL structures
The object of the present paper is a solid with variable composition, i.e. deformable body composed of a set of material points, variable in time. Layer-by-Layer structures may serve as example. Such solids can be represented by the family of bodies $\mathfrak{S} = \{\mathfrak{B}_\alpha\}_{\alpha \in I}$, where $I$ is a linearly ordered set of indexes.

Without any loss of generality one may assume that $I \subseteq \mathbb{R}$. Each element of the family $\mathfrak{S}$, namely a set $\mathfrak{B}_\alpha$, characterizes an instantaneous composition of the solid. The cardinality of the set $I$ determines the type of the process. When $\text{Card} I = N < \infty$, the process may be qualified as discrete, while $\text{Card} I = \aleph_1$ corresponds to continuous process [36]. Without loss of generality it is assumed that $I = \{1, \ldots, N\}$ in the case of discrete process, and $I = [a_0, a_1] \subseteq \mathbb{R}$ in the case of continuous one.

We accept the following simplifying assumptions:

a) For each $\alpha \in I$ the set $\mathfrak{B}_\alpha$ is a materially uniform simple body with inhomogeneities.

b) Layers are sequentially join to the solid and become the inner part of it. During the process material cannot be removed. Thus, the family $\mathfrak{S}$ is ordered by the subset relation, i.e.

---

7 Note that $\nabla_a W$ is a covector field, and $(\nabla_a W)^i$ is vector field.

8 The expression $\nabla_a \bar{W}(x; F_{x_0}, \ldots)|_{x_0 = x}$ has the following sense: covariant derivative is taken from the field $\mathfrak{X} \mapsto \bar{W}(\mathfrak{X}; F_{x_0}, \ldots)$, where $F_{x_0}, \ldots$ are field values at a fixed point $x_0$. After calculation of covariant derivative all entries of $x_0$ are replaced by $x$. 

---
In addition, for any \( \alpha < \beta \) the body \( \mathcal{B}_\alpha \) is a “part” of the \( \mathcal{B}_\beta \) and, hence, \( \mathcal{B}_\alpha \) should have topology induced by\(^9\) \( \mathcal{B}_\beta \).

Thus, it is convenient to consider bodies from \( \mathcal{G} \) as smooth submanifolds\(^{[24]}\) of an ambient \( n \)-dimensional smooth manifold \( M \) that represents the total variable body composition. It means that \( M \) is sufficiently large to hold each element of the \( \mathcal{G} \). We refer to \( M \) as \textit{material manifold}. In the case of continuous process we enable to represent each element of the family \( \mathcal{G} \), which corresponds to \( \alpha \in \mathbb{I} \), as \( \mathcal{B}_\alpha := \mathcal{B} \cup \mathcal{B}_\alpha \). Here the symbol \( \mathcal{B}_\alpha \) represents an extra body, formed by extra material that inflowed to the substrate \( \mathcal{B} \). In the case of discrete process we obtain the following expression: \( \mathcal{B}_{s+1} := \mathcal{B} \cup L_s \). Here this symbol \( L_s \) corresponds to the layer that is attached to the previous assembly \( \mathcal{B}_s \). In both cases the symbol \( \cup \) denotes the operation of \textit{joining}, that assigns to each pair of bodies a new body, that contains both of them.

Although the partition of the body into layers and its assembling seems to be natural operations on the set of material points, constituting the body, their formal implementations have technical issues. In particular, the set difference of the body and its part results on a set, which is not open and therefore it cannot be a body. Moreover, after the union operation applied to two bodies with a partially coincide boundary, the intersection of boundaries would not be a part of the result (recall that we treat a body as an open set). Below we will represent our view on how to solve these problems.

7.1. Layers and their joining

Intuitive notions of extra body (or layer, in discrete case) and joining operation may be formalized as follows. Denote the topology of \( M \) by \( \mathcal{T} \). Since each three-dimensional body containing in \( M \) is an open set, and vice versa, it is sufficient to define the operation of extra body (or layer) detachment over open sets. We consider the case when one body (or “substrate”) contains in another. Let \( \mathcal{M} \) and \( \mathcal{N} \) be \( n \)-dimensional smooth submanifolds of \( M \) (that is, bodies), and \( \mathcal{N} \subset \mathcal{M} \). Then, the set \( \mathcal{N} \) is open in the topological space \( \mathcal{M} \). Therefore, the difference \( \mathcal{M} \setminus \mathcal{N} \) represents a closed set relatively to the \textit{induced topology} \( \mathcal{T}_{\mathcal{M}} \) of \( \mathcal{M} \). Hence, one has that \( \mathcal{M} \setminus \mathcal{N} = \mathcal{M} \setminus \mathcal{N} \). We arrive at the following representation\(^{[37]}\):

\[
\mathcal{M} \setminus \mathcal{N} = \text{Int}(\mathcal{M} \setminus \mathcal{N}) \cup \partial_{\mathcal{M}}(\mathcal{M} \setminus \mathcal{N}),
\]

where the topological operations of closure, interior (\( \text{Int}(\cdot) \)), and boundary (\( \partial_{\mathcal{M}}(\cdot) \)) are considered in the topology \( \mathcal{T}_{\mathcal{M}} \). The set \( \text{Int}(\mathcal{M} \setminus \mathcal{N}) \) is open in \( \mathcal{M} \). Since interior operation is defined as the union of all open sets containing in the given set, we have that

\[
\text{Int}(\mathcal{M} \setminus \mathcal{N}) = \text{Int}(\mathcal{M} \setminus \mathcal{N}),
\]

where \( \text{Int} \) is the \textit{interior operation relatively to the topology of} \( \mathcal{M} \). Denote

\[
\mathcal{I}(\mathcal{M}, \mathcal{N}) := \text{Int}(\mathcal{M} \setminus \mathcal{N}).
\]

It is the desired extra body (or, respectively, layer). Thus, if one introduce the set

\[
A = \{ (\mathcal{M}, \mathcal{N}) \in 2^M \times 2^M | \mathcal{M} \in \mathcal{I}, \mathcal{N} \in \mathcal{T}_{\mathcal{M}} \},
\]

then the following operation is defined:

\(^9\) In general case smooth manifolds \( \mathcal{B}_\alpha, \mathcal{B}_\beta \) (\( \alpha, \beta \in \mathbb{I}, \alpha \neq \beta \)) have independent topological structures, that may differ one from other\(^{[25]}\).
\[ \mathcal{I}: A \to \mathcal{I}, \quad \mathcal{I}(M, \mathcal{M}) \to \text{Int}(M \setminus \mathcal{M}). \]

Note that if \( \mathcal{M} \subset \mathcal{M}_i \subset \mathcal{M}_{i+1} \), then, by definition, \( \mathcal{I}(\mathcal{M}_i, \mathcal{M}_{i+1}) \subset \mathcal{I}(\mathcal{M}_i, \mathcal{M}) \).

The operation \( \mathcal{I} \) allows one to partition any \( n \)-dimensional body \( M \) into a body \( \mathcal{M} \in \Sigma_M \) and the body \( \mathcal{I}(\mathcal{M}, \mathcal{M}) \). The symbol \( \lor \) indicates this partition and is referred to as \emph{partitioning operation}:

\[ M = \mathcal{M} \lor \mathcal{I}(\mathcal{M}, \mathcal{M}) := \mathcal{M} \cup \text{Int}(M \setminus \mathcal{M} \cup \partial_{\mathcal{M}}(M \setminus \mathcal{M})). \]

For a continuous process one has \( \mathcal{B}_a := \mathcal{B} \lor \mathcal{B}_a \), where \( \mathcal{B}_a = \mathcal{I}(\mathcal{B}_a, \mathcal{B}) \), and for a discrete one, \( \mathcal{B}_{s+1} := \mathcal{B}_s \lor \mathcal{L}_s \), in which \( \mathcal{L}_s = \mathcal{I}(\mathcal{B}_{s+1}, \mathcal{B}_s) \). In addition, \( \mathcal{L}_0 := \mathcal{B}_1 \), where \( \mathcal{B}_1 \) represents a substrate. Thus, \( \mathcal{B}_N = (\mathcal{B}_0 \lor \mathcal{L}_0) \lor \mathcal{L}_1 \lor \mathcal{L}_2 \lor \cdots \lor \mathcal{L}_{N-1} \).

\textbf{Remark 9.} The operation \( \lor \) is not equal to the conventional set union and has the following physical sense, which we will explain in the case of discrete process. When we obtain sequentially new assembly \( \mathcal{B}_{s+1} \) from the previous assembly \( \mathcal{B}_s \) and the layer \( \mathcal{L}_s \), we have to include all interior points of \( \mathcal{B}_s \) and \( \mathcal{L}_s \) and, besides them, such points of common boundary of \( \overline{\mathcal{B}_s} \) and \( \overline{\mathcal{L}_s} \) which represent a thin film between “gluing” solids.

\subsection*{7.2. Discrete process}

Let discrete LbL structure is represented by a family \( \mathcal{S} = \{ \mathcal{B}_s \}_{s=1}^N \). Using the idea of layering developed in subsection 7.1 allows one to formulate the following assumptions:

\begin{itemize}
  \item[a)] For each \( s \in \{1, ..., N\} \) the set \( \mathcal{B}_s \) is a \emph{three-dimensional inhomogeneous materially uniform simple body}. A stress-free state is meant by the uniform state.
  \item[b)] For each \( s \in \{1, ..., N\} \) one has decomposition \( \mathcal{B}_s = (\mathcal{B}_1 \lor \mathcal{L}_1) \lor \mathcal{L}_2 \lor \mathcal{L}_3 \lor \cdots \lor \mathcal{L}_{s-1} \). Each layer \( \mathcal{L}_s \) is a \emph{three-dimensional materially uniform simple body that has stress-free shape in \( \mathcal{E} \)}.
  \item[c)] \emph{Stress-strain state of each \( \mathcal{B}_s \) depends on the previous one of \( \mathcal{B}_{s-1} \)}.
\end{itemize}

The deformation of each layer \( \mathcal{L}_s \) can be described in the framework of the classical continuum mechanics. If \( \mathbf{x}_s^L, \mathbf{x}_s \in \mathcal{C}(\mathcal{L}_s) \) are respectively the stress-free reference configuration and the actual configuration of a layer \( \mathcal{L}_s \), then the corresponding deformation is defined as follows

\[ \gamma_s = \mathbf{x}_s - [\mathbf{x}_s^L]^T : \mathcal{C}(\mathcal{L}_s) \to \mathcal{C}(\mathcal{L}_s). \]

There is a finite family \( \{ \gamma_s \}_{s=0}^{N-1} \) of deformations and a finite family of \( \{ \mathbf{x}_s^L(\mathcal{L}_s) \}_{s=0}^{N-1} \) of reference shapes. The body \( \mathcal{B}_s \) is observed in some actual configuration \( \mathbf{x}_b \in \mathcal{C}(\mathcal{B}_s) \) and the deformations of the layers can be agreed with this configuration by the requirement \( \mathbf{x}_s := \mathbf{x}_b|_{\mathcal{L}_s} \), for \( k \in \{0, ..., s-1\} \).

Thus, \( \mathcal{B}_s \) has one actual shape and a family of reference shapes. Each of this reference shapes has internal stresses, except the open subset, that corresponds to the reference shape of a layer.

The assumption c) asserts that each element of the family \( \mathcal{S} \) represents a kind of “memory”. That is, it stores the information about stress-strain state of the LbL structure at the corresponding instant. Thus, layer deformations cannot be independent: they obey some recurrent relations. In section 8 we will consider an example of such relation.

\subsection*{7.3. Continuous process}

A some class of solids with continuous inhomogeneity can be singled out in the following way. We intend to model a situation, when material continuously inflows to a substrate. Let \( \mathcal{B} \) be a three-dimensional materially uniform homogeneous simple body, that represents a substrate, and let
$\mathcal{G} = \{ \mathcal{B}_\alpha \}_{\alpha \in \mathbb{I}}$ be a family of three-dimensional materially uniform simple bodies with inhomogeneities that represents a solid with variable material composition. Here by a uniform state we mean a stress-free state. Each element of the family $\mathcal{G}$ contains $\mathcal{B}$. Assume that $\mathbb{I} = [a_0, a_1]$ and that $\mathcal{B}_{a_0} = \emptyset$.

According to subsection 7.1 for each $\alpha \in \mathbb{I}$ take $\mathcal{B}_\alpha := \text{Int}(\mathbb{B}_\alpha \setminus \mathcal{B})$. Thus, we have arrived at the relation $\mathcal{B}_\alpha := \mathcal{B} \cup \mathcal{B}_\alpha$, and constructed the new family $\mathcal{G} = \{ \mathcal{B}_\alpha \}_{\alpha \in \mathbb{I}}$ of three-dimensional materially uniform simple bodies with inhomogeneities. Now let us formulate the following assumption respectively to the family $\mathcal{G}$:

a) For each $\alpha \in \mathbb{I}$ there exists a smooth fiber bundle $(\mathcal{B}_\alpha, \mathcal{J}_\alpha, \pi_\alpha, \mathcal{F})$ with one-dimensional base $\mathcal{J}_\alpha$, total space $\mathcal{B}_\alpha$ and projection $\pi_\alpha : \mathcal{B}_\alpha \to \mathcal{J}_\alpha$. Each fiber $\mathcal{G}_{\alpha,i} := \pi_\alpha^{-1}(\{i\})$, $i \in \mathcal{J}_\alpha$, is diffeomorphic to a typical fiber $\mathcal{F}$, which is a smooth two-dimensional manifold. The smooth fiber bundle $(\mathcal{B}_\alpha, \mathcal{J}_\alpha, \pi_\alpha, \mathcal{F})$ satisfies the following property. There exists a family $\mathcal{R}_\mathcal{B}_\alpha \{ \mathcal{X}_{a,\mathcal{B}} \}_{\alpha \in \mathbb{I}}$ of configurations $\mathcal{X}_{a,\mathcal{B}} \in \mathcal{C}(\mathcal{B}_\alpha)$ with the index set $\mathcal{B}_\alpha$ and the following attribute. If $\mathcal{X} \in \mathcal{B}_\alpha$ then each point from a two-dimensional submanifold $\mathcal{X}_{a,\mathcal{B}}(\mathcal{G}_{\alpha,i,\mathcal{B}})$ in $\mathcal{X}$ has some uniform spatial neighborhood. Moreover, if $\mathcal{X}, \mathcal{Y} \in \mathcal{G}_{\alpha,i}$, for some $\alpha \in \mathcal{J}_\alpha$, then $\mathcal{X}_{a,\mathcal{B}} = \mathcal{Y}_{a,\mathcal{B}}$.

The assumption a) asserts that $\mathcal{B}_\alpha$ can be represented as a union of disjoint two-dimensional manifolds $\mathcal{G}_{\alpha,i}$. That is,

$$\forall \alpha \in \mathbb{I} : \mathcal{B}_\alpha = \bigcup_{i \in \mathcal{J}_\alpha} \mathcal{G}_{\alpha,i}.$$ 

The mapping $\pi_\alpha$, that generates each $\mathcal{G}_{\alpha,i}$, is smooth and surjective. It follows from such a representation of $\mathcal{B}_\alpha$: i) there exists a family $\mathcal{R}_\mathcal{B}_\alpha \{ \mathcal{X}_{a,\mathcal{B}} \}_{\alpha \in \mathbb{I}}$, as in section 3; ii) not only $\mathcal{X}_{a,\mathcal{B}}(\mathcal{X})$ has a uniform neighborhood, but all points of the whole set $\mathcal{X}_{a,\mathcal{B}}(\mathcal{G}_{\alpha,i,\mathcal{B}})$ are satisfying this property; iii) configurations $\mathcal{X}_{a,\mathcal{B}}$ that correspond to material points, belonging to the same $\mathcal{G}_{\alpha,i}$, coincide. The idea of fibration is illustrated in figure 4.

![Figure 4. Fibration of a body $\mathcal{M}$.](image)

Recall that the body $\mathcal{B}$, as a whole, has a uniform shape. Assume that

b) For each $\alpha \in \mathbb{I}$ there exists a configuration $\mathcal{X}_{a,\mathcal{B}} \in \mathcal{C}(\mathcal{B}_\alpha)$, such that $\mathcal{X}_{a,\mathcal{B}}(\mathcal{B})$ is a uniform shape and that the following property is satisfied. For any $\mathcal{X} \in \mathcal{B}_\alpha$ the deformation

$$\tilde{\gamma}_{a,\mathcal{B}} = \mathcal{X}_{a,\mathcal{B}} \circ \tilde{\gamma}_{a,\mathcal{B}}^{-1} : \mathcal{X}_{a,\mathcal{B}}(\mathcal{B}_\alpha) \to \mathcal{X}_{a,\mathcal{B}}(\mathcal{B}_\alpha),$$

where
from the shape $\chi_a(\mathcal{B}_a)$ into the shape $\chi^B_a(\mathcal{B}_a)$ is prescribed and depends on $k \geq 1$ free scalar parameters $\rho^a_{1,\alpha}, \ldots, \rho^a_{k,\alpha} \in \mathbb{R}$.

c) The mappings $\tilde{b}^i_a : \mathcal{B}_a \to \mathbb{R}$, that are defined by the rule $\tilde{b}^i_a : x \mapsto \rho^a_{i,\alpha}$, are smooth for all $j = 1, \ldots, k$ and $\alpha \in \mathbb{I}$.

The configuration $\chi_a$, which has been introduced in assumption b), is referred to as intermediate configuration. Deformation parameters $\rho^a_{1,\alpha}, \ldots, \rho^a_{k,\alpha}$ determine its degrees of freedom. It is convenient to introduce the mappings $b^i_a : \mathcal{B}_a \to \mathbb{R}$,

$$b^i_a(x) = \begin{cases} 0, & x \in \mathcal{B} \cup \partial \mathcal{B}_a (\mathcal{B}_a \setminus \mathcal{B}), \\ \tilde{b}^i_a(x), & x \in \mathcal{B}_a, \end{cases}$$

that are defined on the whole $\mathcal{B}_a$. The set $\partial \mathcal{B}_a (\mathcal{B}_a \setminus \mathcal{B})$, which has been introduced in subsection 7.1, represents the set of jump points, and, in the physical sense, the thin layer between the substrate and the extra body. It follows from the homogeneity of the body $\mathcal{B}$ that its material Riemannian metric is Euclidean. Thus, in general case, material metric in the whole body $\mathcal{B}_a$ would have points of discontinuity in $\partial \mathcal{B}_a (\mathcal{B}_a \setminus \mathcal{B})$. Hence, it is sufficient to provide our reasonings only for bodies $\mathcal{B}_a$.

Since the deformation $\tilde{\chi}_a$ is prescribed with accuracy up to parameters $\rho^a_{1,\alpha}$, the family $\{\hat{K}^B_a\}_{a=\mathcal{B}_a}$ of uniform references can be obtained. To this end let us introduce local coordinates $(\xi^i_a)_{i=1}$ on the intermediate shape $\chi_a(\mathcal{B}_a)$. Using the map $\chi_a$, it is possible to transfer these coordinates to the body $\mathcal{B}_a$. The frame and coframe on the body are denoted by $(\hat{\mathcal{E}}^i_a, \hat{\mathcal{E}}^i_a)$, and $(d\xi^1_a, d\xi^2_a, d\xi^3_a)$ consequently. Despite the fact that local coordinates on the body and its intermediate shape have the same notations, the corresponding frames and coframes are different. This is indicated in the table 1.

| Coordinate map | $\mathcal{B}_a$ | $\chi_a(\mathcal{B}_a)$ |
|----------------|-----------------|------------------------|
| Coordinates   | $H \circ \chi_a : \mathcal{B}_a \to (\xi^1_a, \xi^2_a, \xi^3_a)$ | $H : \chi_a \to (\xi^1_a, \xi^2_a, \xi^3_a)$ |
| Frame         | $(\hat{\mathcal{E}}^i_a, \hat{\mathcal{E}}^i_a, \hat{\mathcal{E}}^i_a)$ | $(\mathcal{E}^i_a, \mathcal{E}^i_a, \mathcal{E}^i_a)$ |
| Coframe       | $(d\xi^1_a, d\xi^2_a, d\xi^3_a)$ | $(e^i_a, e^i_a, e^i_a)$ |

**Remark 10.** Transferring local coordinates from intermediate shape to the body by means of the mapping $\chi_a$ is equivalent to the consideration of $\chi_a(\mathcal{B}_a)$ as a “model” of $\mathcal{B}_a$. The body “dresses up” coordinate net taken from the set $\chi_a(\mathcal{B}_a)$.

For any point $x \in \mathcal{B}_a$, the tangent map (configuration gradient) $T_x \chi_a : T_x \mathcal{B}_a \to \mathcal{V}$ has the following dyadic decomposition:

$$T_x \chi_a = e^i_{\mathcal{B}_a} \big|_x \otimes d\xi^i_{\mathcal{B}_a} \big|_x + e^i_{\mathcal{B}_a} \big|_x \otimes d\xi^i_{\mathcal{B}_a} \big|_x + e^i_{\mathcal{B}_a} \big|_x \otimes d\xi^i_{\mathcal{B}_a} \big|_x,$$

where $x = \chi_a(\mathcal{B}_a)$.
Introduce local coordinates \((\Xi^i_{\alpha})_{i=1}^3\) on the each shape \(\varkappa^R_{\alpha,\mathcal{X}}(\mathcal{B}_{\alpha})\). Let the coordinate representation of \(\tilde{\varkappa}_{\alpha,\mathcal{X}}\) in the pair of coordinates \((\Xi^i_{\alpha})_{i=1}^3\) and \((\Xi^j_{\alpha})_{j=1}^3\) has the form
\[
\tilde{\varkappa}_{\alpha,\mathcal{X}} \leftrightarrow (\varkappa^1_{\alpha,\mathcal{X}}, \varkappa^2_{\alpha,\mathcal{X}}, \varkappa^3_{\alpha,\mathcal{X}}).
\]
Then the deformation gradient at point \(y \in \varkappa_{\alpha}(\mathcal{B}_{\alpha})\) is represented by
\[
T_y\tilde{\varkappa}_{\alpha,\mathcal{X}} = \left. \frac{\partial \gamma_j^i(y)}{\partial \xi^j_{\alpha}} (\xi^1_{\alpha}, \xi^2_{\alpha}, \xi^3_{\alpha}; b^i_\alpha, \ldots, b^3_\alpha(\mathcal{X})) \right|_{\gamma^\varepsilon_{\alpha,\mathcal{X}}(y)} \otimes e^j_{\varepsilon_{\alpha,\mathcal{X}}(y)}.
\]
Thus, the tangent map \(T_{\gamma}\varkappa_{\alpha,\mathcal{X}} : T_{\gamma} \mathcal{B}_{\alpha} \rightarrow \mathcal{Y}\) to \(\varkappa_{\alpha,\mathcal{X}}\) at point \(\gamma \in \mathcal{B}_{\alpha}\) may be calculated as a composition \(T_{\gamma}\varkappa_{\alpha,\mathcal{X}} = T_{T_{\gamma}\tilde{\varkappa}_{\alpha,\mathcal{X}}} \circ T_{\gamma} \tilde{\varkappa}_{\alpha,\mathcal{X}}|_{\gamma^\varepsilon_{\alpha,\mathcal{X}}}:\)
\[
T_{\gamma}\varkappa_{\alpha,\mathcal{X}} = \left. \frac{\partial \gamma_j^i(y)}{\partial \xi^j_{\alpha}} (\xi^1_{\alpha}, \xi^2_{\alpha}, \xi^3_{\alpha}; b^i_\alpha, \ldots, b^3_\alpha(\mathcal{X})) \right|_{\gamma^\varepsilon_{\alpha,\mathcal{X}}(y)} \otimes d\xi^j_{\alpha} |_{\gamma},
\]
where \(y = \varkappa^R_{\alpha,\mathcal{X}}(\gamma)\). Finally, the uniform reference \(K^R_{\alpha}\) is defined at point \(\mathcal{X} \in \mathcal{B}_{\alpha}\) by the following expression:
\[
K^R_{\alpha} |_{\mathcal{X}} = \left. \frac{\partial \gamma_j^i(y)}{\partial \xi^j_{\alpha}} (\xi^1_{\alpha}, \xi^2_{\alpha}, \xi^3_{\alpha}; b^i_\alpha, \ldots, b^3_\alpha(\mathcal{X})) \right|_{\gamma^\varepsilon_{\alpha,\mathcal{X}}(y)} \otimes d\xi^j_{\alpha} |_{\mathcal{X}},
\]
where \(x = \varkappa^R_{\alpha,\mathcal{X}}(\mathcal{X})\). Thus, assumptions a) – c) are sufficient to determine uniform reference up to mappings \(b^i_\alpha\).

By using obtained formula (21), the material metric \(G_{\alpha}\) defined by (3) is derived. The section
\[
G_{\alpha}(\mathcal{X}) = \begin{cases} (\varkappa_{\alpha})^\varepsilon, & \mathcal{X} \in \mathcal{B} \cup \partial \mathcal{B}_{\alpha}(\mathcal{B}_{\alpha} \setminus \mathcal{B}); \\ G_{\alpha}(\mathcal{X}), & \mathcal{X} \in \mathcal{B}_{\alpha}, \end{cases}
\]
represents the total material metric \(G_{\alpha}\) in \(\mathcal{B}_{\alpha}\). Here \(g\) is the Riemannian metric on \(\mathcal{E}\) which is induced by the scalar product \((\cdot,\cdot)\).

The functions \(b^i_\alpha\), which appear in (21), represent distortion parameters and are defined by physical mechanisms accompanying the continuous process. Among such mechanisms are shrinkage of adhered layers and inheritance of inhomogeneities. The latter implies that material properties of adhered layers do not change after joining. Complete assumptions a) – c) by one more:

a) For each \(j \in \{1, \ldots, k\}\), if \(\alpha, \beta \in \mathbb{I}\), \(\alpha < \beta\), then
\[
\hat{b}^j_{\beta} |_{\gamma^\varepsilon_{\alpha,\mathcal{X}}} = \hat{b}^j_{\alpha}.
\]
This assumption expresses the idea of inheritance.

8. Example: Cylinder

8.1. Discrete process
Consider an example. In order to avoid heavy numerical computations, assume that this solid is an inhomogeneous hollow cylinder, which inhomogeneity appears during the process of its assembling
from a number of thin unstressed hollow cylindrical parts. Its incompatibility can be clarified from technological point of view as follows. Hollow cylindrical layers join consequently and each layer undergo shrinkage at once after joining. Apply the following assumptions:

a) Layers $\mathcal{L}_k$ have stress free reference configurations in three-dimensional Euclidean point space $\mathcal{E}$:

$$\mathcal{X}^k : \mathcal{L}_k \rightarrow \mathcal{E}, \; k = 0, \ldots, N-1.$$ 

b) Images of the layers $\mathcal{L}_k$ under configurations $\mathcal{X}^k$, $\mathcal{X}^k(\mathcal{L}_k) \subset \mathcal{E}$, are represented by hollow cylinders with the inner radii $R^k_i$ and the exterior radii $R^k_e$ and common to all height $h$.

c) All bodies $\mathcal{B}_s$, $s = 2, \ldots, N$ , result by sequential joining of layers $\mathcal{L}_k$ to $\mathcal{B}_1$. Image of any body $\mathcal{B}_s$ under the actual configuration $\mathcal{X}_s$ is a hollow cylinder. Cylinders $\mathcal{X}_s(\mathcal{L}_k)$ are coaxial. Denote by $r^k_{i,s}$ actual inner radius of $k$ -th layer, and by $r^k_{e,s}$ actual exterior radius of $k$ -th layer. The index $s$ after comma designates the number of assembly.

d) Each layer before joining to assembly undergoes axisymmetric plane deformation.

e) Inner radius of the $\mathcal{B}_s$ in reference configuration, $R^0_i$, is equal to $\rho$, and reference thicknesses $\Delta^k = R^k_i - R^k_e$, $k = 0, \ldots, N-1$, of the layers are prescribed. The other reference radii are unknown, but they can be determined from the following recurrence relation: the inner reference radius of the layer $\mathcal{L}_{s+1}$ is equal to $R^{s+1}_i = S^{s+1} r^{s+1}_{e,s}$, where $0 < S^{s+1} < 1$ is the shrinkage coefficient, and $r^{s+1}_{e,s}$ is the exterior actual radius of the layer $\mathcal{L}_{s+1}$.

These assumptions are schematically illustrated in figure 5.

![Figure 5. Cylindrical LbL structure.](image)

Consider the assembly with the number $s$ ($s \in \{1, \ldots, N\}$. Since $\mathcal{X}_s(\mathcal{B}_s)$ is a hollow cylinder, it is convenient to use the cylindrical coordinates $(r, \theta, z)$ that are related with the Cartesian coordinates via formulae:

$$x^1 = r \cos \theta, \quad x^2 = r \sin \theta, \quad x^3 = z.$$ 

The elements of the local frame $(e_r, e_\theta, e_z)$, that corresponds to cylindrical coordinates, can be derived as $e_c = (\partial x^c) / (\partial c) \mathbf{i}_c$. The index $c$ designates one of the symbols $r$, $\theta$, or $z$, and $x = x^c(r, \theta, z) \mathbf{i}_c$ is the position vector field, represented by the Cartesian frame. By $(e^r, e^\theta, e^z)$ we denote the coframe field $e^c : \mathcal{X}_s(\mathcal{B}_s) \rightarrow \mathcal{V}$, $e^c(e_j) = \delta^c_j$, $c, j \in \{r, \theta, z\}$. 

23
Let \( k \) be the layer index in the assembly with the number of layers \( s; k \in \{0,\ldots,s-1\} \). According to the assumptions, stated above, the deformation of reference shape \( \mathcal{F}_k(L_k) \) is simply “bloating”. To this end we introduce cylindrical coordinates \((R^k, \Theta^k, Z^k)\) on the set \( \mathcal{F}_k(L_k) \). The deformation \( d_k : \mathcal{F}_k(L_k) \to \mathcal{F}_k(L_k) \) belongs to the class of universal deformations [38] and can be represented in coordinate free notation as follows:

\[
x(X) = e_{r^k} \sqrt{(e_{r^k} \cdot \mathbf{X})^2} + a_r e_{r^k} \otimes e^{r^k} X_k.
\]

In coordinates \((R^k, \Theta^k, Z^k)\) and \((r, \theta, z)\) it can be written as:

\[
d_k : (R^k, \Theta^k, Z^k) \mapsto (r, \theta, z) = \left( (R^k)^2 + a_r^2, \Theta^k, Z^k \right).
\]

where \( a^k \) is the deformation parameter.

The deformation gradient \( \mathbf{F}_k \), its transpose, \( \mathbf{F}_k^T \), and the left Cauchy-Green tensor \( \mathbf{B}_k = \mathbf{F}_k \mathbf{F}_k^T \) in coordinates \((R^k, \Theta^k, Z^k)\) and \((r, \theta, z)\) are written in the forms

\[
\mathbf{F}_k = \frac{R^k}{\sqrt{(R^k)^2 + a_r^2}} e_{r^k} \otimes e^{r^k} + e_{\theta^k} \otimes e^{\theta^k} + e_z \otimes e^z, \quad (23)
\]

\[
\mathbf{F}_k^T = \frac{R^k}{\sqrt{(R^k)^2 + a_r^2}} e_{r^k} \otimes e' + \frac{(R^k)^2 + a_r^2}{(R^k)^2} e_{\theta^k} \otimes e^\theta + e_z \otimes e', \quad (24)
\]

In derivation of (23)–(24) we take into account that the metric tensors \( \mathbf{G}_k \) and \( \mathbf{g} \) on \( \mathcal{F}_k(L_k) \) and \( \mathcal{F}_k(L_k) \) have the following representations:

\[
\mathbf{G}_k = e^{r^k} \otimes e^{r^k} + (R^k)^2 e^{\theta^k} \otimes e^{\theta^k} + e^z \otimes e^z, \quad \mathbf{g} = e' \otimes e' + r^2 e^\theta \otimes e^\theta + e^z \otimes e^z.
\]

Assume that layers are made of hyperelastic material. The constitutive relation has the form: \( \mathbf{T} = -p \mathbf{I} + J_1 \mathbf{B} + J_2 \mathbf{B}^{-1} \), where \( \mathbf{I} \) is the identity tensor, \( p \) is the hydrostatic component, \( J_1 = (1 + \beta) \mu / 2 \), \( J_2 = (\beta - 1) \mu / 2 \), and \( \beta \) and \( \mu \) are the material constants. Taken into account the diagonality of \( \mathbf{B} \), one can derive the expression for Cauchy stress tensor\(^{10} \) in the layer with the number \( k \):

\[
\mathbf{T}_k = \mathbf{T}_k^{rr} e_r \otimes e_r + \mathbf{T}_k^{\theta \theta} e_\theta \otimes e_\theta + \mathbf{T}_k^{zz} e_z \otimes e_z, \\
\mathbf{T}_k^{rr} = -p + J_1 \frac{r^2 - a_r^k}{r^2} + J_2 \frac{r^2}{r^2 - a_r^k}, \quad \mathbf{T}_k^{\theta \theta} = -p + J_1 \frac{1}{r^2 - a_r^k} + J_2 \frac{r^2}{r^2 - a_r^k}, \\
\mathbf{T}_k^{zz} = -p + J_1 + J_2.
\]

It follows from the balance of linear momentum that

\(^{10} \)Here we consider Cauchy stress tensor as a symmetric field \( \mathbf{T} : \mathcal{F}(\mathbb{B}) \to \mathcal{V} \otimes \mathcal{V} \). For a normal vector field \( \mathbf{n} : \partial \mathcal{F}(\mathbb{B}) \to \mathcal{V} \) we have the following representation for the stress vector: \( \mathbf{t}_n = \mathbf{T}_n \mathbf{n} \).
\[
\frac{\partial T''_k}{\partial r} + \frac{T''_k}{r} - r T^{00}_k = 0, \quad \frac{\partial T^{00}_k}{\partial \theta} = 0, \quad \frac{\partial T^{zz}_k}{\partial z} = 0.
\]  

(25)

One can obtain components of \( T^i_k \) by using expressions (25):

\[
T''_k = \frac{\mu}{2} \left( \ln \frac{r^2 - a^2_k}{r^2} - \frac{a^2_k}{r^2} \right) + p^0_{0,s}, \quad T^{00}_k = T''_k + \frac{\mu}{r^2} \left( \ln \frac{r^2 - a^2_k}{r^2} - \frac{r^2 - a^2_k}{r^2} \right),
\]

\[
T^{zz}_k = T''_k + \mu a_k \frac{r^2 - (1 + \beta) a^2_k / 2}{r^2 (r^2 - a^2_k)}.
\]

Here \( p^k_{0,s} \) are the constants of integration.

The Cauchy stress tensor \( T^i_k \) is defined on each layer up to parameters \( a^k_i \) and \( p^k_{0,s} \). Some boundary conditions can be satisfied by choosing the corresponding values of such the parameters. We assume that on inner, \( r = r^0_i \), and on exterior, \( r = r^{i+1}_e \), boundaries of assembly, the uniform hydrostatic loading \( p_{r,s} \) and \( p_{e,s} \) are given:

\[
T^i_k \cdot e^i_{|r^0_i} = p_{r,s} e_s, \quad T^i_k \cdot e^i_{|r^{i+1}_e} = p_{e,s} e_s,
\]

and also, that layers in actual shape are in ideal contact:

\[
T^i_k \cdot e^i_{|r^{i+1}_k} = T^i_{k+1} \cdot e^i_{|r^{i+1}_k}, \quad a^k_{e,s} = a^{k+1}_{r,s}, \quad k = 0, \ldots, s - 2.
\]

Conditions (26) – (27) can be satisfied if \( a^k_i \) and \( p^k_{0,s} \), \( k = 0, \ldots, s - 1 \), are solutions of the following system of non-linear equations:

\[
\frac{\mu}{2} \left[ \ln \frac{(R^0)^2}{(R^0)^2 + a^2_0} - \frac{a^2_0}{(R^0)^2 + a^2_0} \right] + p^0_0 = p_{r,s},
\]

\[
\frac{\mu}{2} \left[ \ln \frac{(R^{i+1})^2}{(R^{i+1})^2 + a^2_i} - \frac{a^2_i}{(R^{i+1})^2 + a^2_i} \right] + p^{i+1}_{0,s} = p_{e,s},
\]

\[
\frac{\mu}{2} \left[ \ln \frac{(R^i)^2}{(R^i)^2 + a^2_i} - \frac{a^2_i}{(R^i)^2 + a^2_i} \right] + p^i_0 = \frac{\mu}{2} \left[ \ln \frac{(R^{i+1})^2}{(R^{i+1})^2 + a^2_{i+1}} - \frac{a^2_{i+1}}{(R^{i+1})^2 + a^2_{i+1}} \right] + p^{i+1}_{0,s} = p_{r,s},
\]

(28)

Solving the system (28) one can consistently exclude unknowns \( p^k_{0,s} \) and arrive at the system of \( s \) equations relatively to the parameters \( a^k_i \):

\[
\prod_{k=0}^{s-1} \frac{\alpha_k + v_k x_i}{\beta_k + v_k x_i} = W_i \exp \left[ \sum_{k=0}^{s-1} \frac{1 - \gamma_k}{(\alpha_k + v_k x_i)(\beta_k + v_k x_i)} \right],
\]

\[
\alpha^k_i = \alpha^0_i + A_k, \quad k = 1, \ldots, s - 1,
\]

(29)

where \( A_0 = 0, \quad A_k = \sum_{p=0}^{k-1} (R^p)^2 - (R^{p+1})^2, \quad k = 1, \ldots, s - 1, \quad \gamma_k = (R_i^k / R_i^0)^2, \quad \alpha_k = 1 + A_k / (R_i^0)^2, \quad \beta_k = \gamma_k + A_k / (R_i^0)^2, \quad v_k = (R_i^k / R_i^0)^2, \quad x_i = a^0_i / (R_i^0)^2, \quad W_i = \exp \left[ \frac{2 p_{r,s} - p_{e,s}}{\mu} \right]. \)
Conditions of e) with equations (28) allows one to determine the parameters $a^k_s$ and $p_{0,s}^k$ recursively. Parameters $a^k_s$ are determined from the system (29). Hereafter the parameters $p_{0,s}^k$ can be determined from the equations (28).

Since the Cauchy stress tensor $T_k$ in each layer $\Sigma_k$ is determined, the first Piola-Kirchhoff stress tensor $P_k$ is derived from the formula $P_k = J_k T_k F_k^{-\top}$, where $J_k = \det F_k$. For incompressible material $J_k = 1$. Thus $P_k$ has the following dyadic decomposition (indexes corresponded to layer numbers are suppressed for the sake of brevity):

$$P^k = P^R_k e_r \otimes e_r + P^{\omega \omega}_k e_\omega \otimes e_\omega + P^{ZZ}_k e_z \otimes e_z,$$

$$P^R_k = \frac{\sqrt{R^2 + a}}{R} T^\omega = p_0 \frac{\sqrt{R^2 + a}}{R} + \frac{\mu}{2} \frac{\sqrt{R^2 + a}}{R} \left( \ln \frac{R^2}{R^2 + a} - \frac{a}{R^2 + a} \right),$$

$$P^{\omega \omega}_k = T^{\omega \omega} = p_0 + \frac{\mu}{2} \left( \ln \frac{R^2}{R^2 + a} - \frac{a}{R^2 + a} \right) + \mu \frac{R^2 + (1-\beta)a/2}{R^2 (R^2 + a)}.$$
Let us make intermediate conclusions for the case of discrete LbL-assembling. Relations for strain and stresses were obtained in the framework of classical Euclidean continuum mechanics. Indeed, measures of strain and stresses are defined for each layer, and, in totality, form piecewise continuous mappings defined on actual shape. To satisfy boundary conditions one needs to solve the system (28). Nevertheless for a large amount of layers one may need a huge amount of time to solve the system. This must represent considerable computational problem. Thereby it may be preferable to model the appropriate continuous process. Methods for modeling such continuous processes are discussed below.

8.2. Continuous process
Consider a continuous assembling process. We do that in the framework of general reasonings given in subsection 7.3. Each set \( \mathcal{E}_{a,X}(\mathcal{B}_a, \sigma_a(X)) \) represents a cylindrical surface, the whole body \( \mathcal{B}_a \) is observed as a finite hollow cylinder in \( \mathcal{E} \), and so do \( \mathcal{E}_{a,X}(\mathcal{B}_a) \). Figure 6 illustrates the situation.

Let \( \mathcal{E}_a \in \mathcal{C}(\mathcal{B}_a) \) satisfy the assumption b) of subsection 7.3 (i.e., it is intermediate configuration), and, in addition, assume that \( \mathcal{E}_a(\mathcal{B}_a) \) is a hollow cylinder. Thus, we endow the set \( \mathcal{E}_a(\mathcal{B}_a) \) with cylindrical coordinates \( (R_a, \Theta_a, Z_a) \). The corresponding coordinate map is denoted by \( H : X \mapsto (R_a, \Theta_a, Z_a) \). The cylindrical coordinates are related with the Cartesian ones as follows:

\[
X^1 = R_a \cos \Theta_a, \quad X^2 = R_a \sin \Theta_a, \quad X^3 = Z_a.
\]

Figure 6. Actual shape \( \mathcal{E}_a(\mathcal{B}_a) \) of the body \( \mathcal{B}_a \) and reference shape \( \mathcal{E}_{a,X}(\mathcal{B}_a) \).

Transfer the cylindrical coordinates \( (R_a, \Theta_a, Z_a) \) to the body \( \mathcal{B}_a \) by means of \( \mathcal{E}_a \). The frame and coframe on the body are denoted by \( (\partial_{R_a}, \partial_{\Theta_a}, \partial_{Z_a}) \), and \( (dR_a, d\Theta_a, dZ_a) \), respectively.

Each of the cylinders \( \mathcal{E}_{a,X}(\mathcal{B}_a) \) can be endowed with own cylindrical coordinates \( (R_{a,X}, \Theta_{a,X}, Z_{a,X}) \). Assume that the deformation \( \tilde{\gamma}_{a,X} \) from subsection 7.3 has the following representation in the pair of coordinates \( (R_a, \Theta_a, Z_a) \) and \( (R_{a,X}, \Theta_{a,X}, Z_{a,X}) \):

\[
R_{a,X} = \sqrt{(R_a)^2 - \rho_{a,X}}, \quad \Theta_{a,X} = \Theta_a, \quad Z_{a,X} = Z_a, \tag{30}
\]
where \( \rho_{\alpha, x} \in \mathbb{R} \). The uniform reference \( K^{R}_{\alpha} \) is defined via expression (21):

\[
K^{R}_{\alpha} |_{x} = \frac{R_{\alpha}}{\sqrt{(R_{\alpha})^{2} - b_{\alpha}(x)}} \left| e_{\alpha, x} \right|_{x} \otimes dR_{\alpha} |_{x} + \left| e_{\alpha, x} \right|_{x} \otimes \Theta_{\alpha} d\Theta_{\alpha} + \left| e_{\alpha, x} \right|_{x} \otimes dZ_{\alpha} |_{x},
\]

where \( x = x^{R}_{\alpha, x}(X) \) and \( (e_{\alpha, x}, e_{\alpha, x}, e_{\alpha, x}) \) is the coordinate frame corresponding to the coordinates \( (R_{\alpha, x}, \Theta_{\alpha, x}, Z_{\alpha, x}) \). The mapping \( b_{\alpha} := b^{i}_{\alpha} \) represents a distortion parameter. From the property \( a \) (see subsection 7.3) it follows that the coordinate representation of the mapping \( b_{\alpha} \) depends on \( R_{\alpha} \) only.

We obtained the family \( \{b_{\alpha}\}_{\alpha \in \mathbb{I}} \) of mappings \( b_{\alpha} : R_{\alpha} \mapsto \tilde{b}_{\alpha}(R_{\alpha}) \) that are defined on finite interval \( \mathbb{I}_{\alpha} \subset \mathbb{R} \). Let us choose intermediate configurations with the following property:

\[
\forall \alpha_{1}, \alpha_{2} \in \mathbb{I} : (\alpha_{1} < \alpha_{2}) \Rightarrow (\tilde{b}_{\alpha_{1}} |_{\mathbb{I}_{\alpha_{1}}} = \tilde{b}_{\alpha_{2}} |_{\mathbb{I}_{\alpha_{2}}}),
\]

that corresponds to the \( d \) from 7.3.

**Remark 11.** If \( \alpha_{1} < \alpha_{2} \) then \( \mathcal{X}_{\alpha_{1}}(\mathcal{B}_{\alpha_{1}}) \subset \mathcal{X}_{\alpha_{2}}(\mathcal{B}_{\alpha_{2}}) \) and stress-strain state at points in \( \mathcal{X}_{\alpha_{2}}(\mathcal{B}_{\alpha_{2}}) \), which are in \( \mathcal{X}_{\alpha_{1}}(\mathcal{B}_{\alpha_{1}}) \), is fixed.

Now one can obtain the family \( \{G_{\alpha}\}_{\alpha \in \mathbb{I}} \) of material metrics:

\[
G_{\alpha} : \mathcal{B}_{\alpha} \rightarrow T^{*} \mathcal{B}_{\alpha} \otimes T^{*} \mathcal{B}_{\alpha}, \quad G_{\alpha}(u, v) = [K^{R}_{\alpha} \otimes u] \cdot [K^{R}_{\alpha} \otimes v].
\]

In the explicit form:

\[
G_{\alpha} = \frac{(R_{\alpha})^{2}}{(R_{\alpha})^{2} - b_{\alpha}(R_{\alpha})} dR_{\alpha} \otimes dR_{\alpha} + \left((R_{\alpha})^{2} - b_{\alpha}(R_{\alpha})\right) d\Theta_{\alpha} \otimes d\Theta_{\alpha} + dZ_{\alpha} \otimes dZ_{\alpha}.
\]

Thus, the family \( \{\mathcal{B}_{\alpha}, G_{\alpha}\}_{\alpha \in \mathbb{I}} \) of Riemannian spaces is obtained. Coefficients of the non-zero Levi-Civita connection \( \nabla_{\alpha} \) are represented by expressions (for the sake of brevity the indices \( \alpha \) and \( \circ \) are omitted):

\[
\Gamma_{i1}^{i} = -\frac{2 \tilde{b}(r) + rd\tilde{b}(r)}{2(r^{3} - rb(r))}, \quad \Gamma_{i2}^{i} = \frac{(r^{2} - \tilde{b}(r)(-2r + \tilde{b}(r))}{2r^{2}}, \quad \Gamma_{12}^{2} = \Gamma_{21}^{2} = -\frac{2r - \tilde{b}(r)}{2r^{2} - 2b(r)}.
\]

The curvature of so-obtained connection is non-zero in general. Its Ricci invariant has the form\(^{11}\):

\[
\text{Ric}_{\alpha} = \frac{R \tilde{b}_{\alpha}(R_{\alpha}) - \tilde{b}_{\alpha}(R_{\alpha})}{(R_{\alpha})^{3}},
\]

and reaches zero only when \( \tilde{b}_{\alpha}(R_{\alpha}) = C_{1}(R_{\alpha})^{2} + C_{2} \). This mathematical fact reflects the specific features of such bodies: the incompatibility of deformations, the presence of intrinsic stresses.

The intermediate configuration \( \mathcal{X}_{\alpha} \) is used for the material metric synthesizing. Now apply

\(^{11}\text{Here prime (') denote } d/dR_{\alpha}.\)
\[ b_a(R_a) = \begin{cases} 
0, & R_a < R_{a,c}, \\
\tilde{b}_a(R_a), & R_a > R_{a,c}. 
\end{cases} \tag{35} \]

In this formula \( R_{a,c} \), \( R_{a,c} \) are prescribed the inner and outer radii of \( \mathcal{K}_a(\mathcal{B}) \), respectively. The zero value corresponds to the substrate \( \mathcal{B} \), which is homogeneous simple body. Total material metric \( G \) in \( \mathcal{B}_a \) is defined via formula (22). Endow the set \( \mathcal{K}_a(\mathcal{B}_a) \) with cylindrical coordinates \( (r_a, \theta_a, z_a) \). In the coordinates the deformation \( d_a \) from \( \mathcal{K}_a(\mathcal{B}) \) to \( \mathcal{K}_a(\mathcal{B}_a) \) is assumed to be in the form

\[
d_a : (R_a, \Theta_a, Z_a) \mapsto (r_a, \Theta_a, z_a), \quad r_a = \sqrt{(R_a)^2 + A_a}, \quad \Theta_a = \Theta_a, \quad z_a = Z_a,
\]

where \( A_a \in \mathbb{R} \). Hence, the deformation gradient is defined as

\[
F_a = \frac{R_a}{\sqrt{(R_a)^2 + A_a}} e_a \otimes e^{R_a} + e_{\theta_a} \otimes e^{\theta_a} + e_{z_a} \otimes e^{z_a}. \tag{36}
\]

The configuration gradient \( T_x \mathcal{K}_a \) can be represented as multiplicative decomposition

\[
T_x \mathcal{K}_a = F_a \circ T_x \mathcal{K}_a. \quad \text{According to (36) it has the form:}
\]

\[
T_x \mathcal{K}_a = \frac{R_a}{\sqrt{(R_a)^2 + A_a}} e_a \bigg|_{x} \otimes dR_a \bigg|_{x} + e_{\theta_a} \bigg|_{x} \otimes d\Theta_a \bigg|_{x} + e_{z_a} \bigg|_{x} \otimes dZ_a \bigg|_{x}, \tag{37}
\]

where \( x = \mathcal{K}_a(\mathcal{X}) \).

According to relations (7) and (8) one has formulæ

\[
C_a = \frac{(R_a)^2 - b_a(R_a)}{(R_a)^2 + A_a} \frac{\partial}{\partial R_a} \otimes dR_a + \left( \frac{(R_a)^2 + A_a}{(R_a)^2 - b_a(R_a)} \right) \frac{\partial}{\partial \theta_a} \otimes d\Theta_a + \frac{\partial}{\partial \tau_a} \otimes dZ_a,
\]

\[
B_a = \frac{r_a^2 - A_a - b_a(\sqrt{r_a^2 - A_a})}{r_a^2} e_a \otimes e^{R_a} + \frac{r_a^2}{r_a^2 - A_a - b_a(\sqrt{r_a^2 - A_a})} e_{\theta_a} \otimes e^{\theta_a} + e_{z_a} \otimes e^{z_a}. \tag{38}
\]

All is ready to calculate the stresses in the actual configuration \( \mathcal{K}_a \). Using relation \( T = -pI + J_x B + J_x B^{\top} \), as in subsection 8.1, and taking into account (38) we obtain that

\[
(T_a)^s = T_a^{\alpha'\alpha} e_{\alpha_a} \otimes e_{\alpha'} + T_a^{\theta'\theta} e_{\theta_a} \otimes e_{\theta'} + T_a^{z'z} e_{z_a} \otimes e_{z'}.
\]

\[
T_a^{\alpha'\alpha}(r) = B_a + \mu \int_{r_a}^{r_a + \mu} \left[ \frac{\tau}{\tau^2 - A_a - b_a(\sqrt{\tau^2 - A_a})} - \frac{\tau^2 - A_a - b_a(\sqrt{\tau^2 - A_a})}{\tau^2} \right] d\tau, \tag{40}
\]

\[
T_a^{\alpha'\alpha} = \frac{T_a^{\alpha'\alpha}}{r_a} + \mu \left( \frac{r_a^2}{r_a^2 - A_a - b_a(\sqrt{r_a^2 - A_a})} - \frac{r_a^2 - A_a - b_a(\sqrt{r_a^2 - A_a})}{r_a^2} \right),
\]

\[
T_a^{\theta'\theta} = T_a^{\theta'\theta} + \mu \left( A_a + b_a(\sqrt{r_a^2 - A_a}) \right) \frac{r_a^2 - (1 + \beta)(A_a + b_a(\sqrt{r_a^2 - A_a}))}{r_a^2 (r_a^2 - A_a - b_a(\sqrt{r_a^2 - A_a}))},
\]

\[
T_a^{z'z} = T_a^{z'z} + \mu \left( A_a + b_a(\sqrt{r_a^2 - A_a}) \right) \frac{r_a^2 - (1 + \beta)(A_a + b_a(\sqrt{r_a^2 - A_a}))}{r_a^2 (r_a^2 - A_a - b_a(\sqrt{r_a^2 - A_a}))}. \]
Here $B_\alpha$ is the constant of integration that has appeared after integrating the equilibrium equation, and $r_{a;i}$ is the actual inner radius. The corresponding Cauchy exterior 2-form is represented as follows

$$T_a = r_a T^b_{\alpha' \alpha} e^{\alpha'} \otimes \left(e^{\alpha'} \wedge e^{\alpha}\right) - r_a^2 T^b_{\alpha' \alpha} e^{\alpha'} \otimes \left(e^{\alpha'} \wedge e^{\alpha}\right) + r_a T_{a'\alpha} e^{\alpha} \otimes \left(e^{\alpha'} \wedge e^{\alpha}\right).$$

Since the Cauchy stress tensor (39) is known, we can obtain the first Piola-Kirchhoff tensor by the formula (15).

Let us obtain the Eshelby energy-momentum tensor. Since

$$P = \frac{R_a \sqrt{(R_a)^2 + A_a}}{(R_a)^2 - b_a(R_a)} T^a_{\alpha'} e^{\alpha'} \otimes dR_a + (R_a)^2 - b_a(R_a) T^a_{\alpha'} e^{\alpha'} \otimes d\Theta_a + T_{a'\alpha} e^{\alpha'} \otimes dZ_a,$$

we arrive at the formula

$$F_a^T = \frac{R_a \sqrt{(R_a)^2 + A_a}}{(R_a)^2 - b_a(R_a)} e^{\alpha} \otimes dR_a + (R_a)^2 - b_a(R_a) e^{\alpha} \otimes d\Theta_a + e_{a'} \otimes dZ_a.$$
deformation parameters are unknown. On the interior and exterior boundaries of \( \mathcal{X}_1(\mathcal{B}_1) \) consider the following conditions:

\[
T^{\alpha \alpha}_{11}(r_{in}) = p_i(\alpha), \quad T^{\alpha \alpha}_{11}(r_{out}) = p_e(\alpha),
\]

where \( T^{\alpha \alpha}_{11}(r) \) is defined by (40). These equalities lead to the relation

\[
\int_{\mathcal{K}_{11}(\alpha)} \left[ \frac{1}{\xi^2} - \frac{\xi^2 - b_\alpha(\xi)}{(\xi^2 + A_\alpha)^2} \right] \xi d\xi = \frac{p_i(\alpha) - p_e(\alpha)}{\mu}. \quad (41)
\]

Another condition relates \( b_\alpha \) with shrinkage \( S \):

\[
b_\alpha(R_{11}(\alpha)) + A_\alpha = S(\alpha). \quad (42)
\]

Both (41) and (42) allow us to determine \( A_\alpha \) and \( b_\alpha \). According to (35) equation (41) takes the form

\[
\int_{\mathcal{K}_{11}(\alpha)} \left[ \frac{1}{\xi^2} - \frac{\xi^2 - b_\alpha(\xi)}{(\xi^2 + A_\alpha)^2} \right] \xi d\xi + \int_{\mathcal{K}_{11}(\alpha)} \left[ \frac{1}{\xi^2} - \frac{\xi^2 - \tilde{b}_\alpha(\xi)}{(\xi^2 + A_\alpha)^2} \right] \xi d\xi = \frac{p_i(\alpha) - p_e(\alpha)}{\mu},
\]

or, since the first integral can be calculated directly,

\[
\ln \frac{R_{11}(\alpha) \sqrt{(R_{11}(\alpha))^2 + A_\alpha}}{R_{in}(\alpha) \sqrt{(R_{11}(\alpha))^2 + A_\alpha}} = A_\alpha \left( \frac{1}{(R_{11}(\alpha))^2 + A_\alpha} - \frac{1}{(R_{11}(\alpha))^2 + A_\alpha} \right) + \int_{\mathcal{K}_{11}(\alpha)} \left[ \frac{1}{\xi^2} - \frac{\xi^2 - \tilde{b}_\alpha(\xi)}{(\xi^2 + A_\alpha)^2} \right] \xi d\xi = \frac{p_i(\alpha) - p_e(\alpha)}{\mu}. \quad (43)
\]

Equation (42) should be rewritten in terms of \( \tilde{b}_\alpha \):

\[
\tilde{b}_\alpha(R_{11}(\alpha)) + A_\alpha = S(\alpha). \quad (44)
\]

Thus, for this case one should solve the evolutionary problem (43), (44) to determine \( \tilde{b}_\alpha \) and \( A_\alpha \).

8.3. Numerical results

To compare strain and stress distributions in discrete structurally inhomogeneous solids with similar distributions in solids with continuously distributed inhomogeneity consider following examples.

a) Hollow cylinder is assembled from a number of thin hollow cylindrical parts. We suppose that each part possesses a natural configuration. A sequence of assemblies with common to all total volume \( V \) and increasing number of parts \((10, 30, 200, 1000) \) is studied.

b) Hollow cylinder with volume \( V \), the same as in a), and piecewise continuous distribution for deformation parameter \( b_\alpha \) that is taken to be the continuous approximation of the sequence of deformation parameters \( a_i \) obtained for assembly from a) with maximal numbers of parts.

c) Hollow cylinder with volume \( V \), the same as in a) and b), and piecewise continuous distribution for \( b_\alpha \) that is obtained as numerical solution of evolutionary problem (43), (44). From general considerations discussed above it is expected that stress and strain distributions in solids with discrete inhomogeneity tend to the corresponded stress and strain distributions in a solid with some piecewise continuous \( b_\alpha \), that can be obtained as a solution of evolutionary problem (43), (44) or can be represented as approximation of deformation parameters \( a_i \) of sufficiently dense assembly. The following calculations numerically illustrate this fact.
For calculations we took the following values of parameters [2]. For each of the assemblies inner radius $\rho$ of the first cylindrical layer in reference configuration and its thickness $\Delta^0$ are taken equal to $\rho = \Delta^0 = 2$ mkm. The reference thicknesses $\Delta^k$, $k \geq 1$, for each of assemblies are taken equal to $\Delta^k = 22/(N-1)$ mkm, where $N$ is the number of layers. The shrinkage coefficients are taken equal to $S^k = 0.7$. Suppose that inner and outer hydrostatic loadings, $p_{i,i}$ and $p_{rr,i}$, vanish. In computations we put $\beta = 0.24$.

Strain and stress distributions are calculated by equations obtained in subsection 8.1. In figures 7 and 8 the numerical results are shown, where dotted line corresponds to 10 layers, dot-dashed line corresponds to 30 layers, dashed line corresponds to 200 layers. The case of 1000 layers is not shown on graphs because its difference from continuous distributions discussed below is graphically indistinguishable. Angle brackets in $Y_{\alpha\beta}$ denote that the corresponding component of $Y$ is considered in physical (with unit norm vectors) frame. The symbols $Y_{r\alpha r\alpha}$, $Y_{r\beta r\beta}$, $Y_{\alpha\beta r\beta}$ correspondingly denote radial, circumferential and axial physical components of a tensor $Y$. So do $Y_{r\alpha r\beta}$, $Y_{r\beta r\alpha}$, $Y_{\alpha\beta r\alpha}$, but relatively to the pair of cylindrical coordinate systems.

Results for continuously structural inhomogeneous solid that corresponds to considered above discrete cases can be obtained in the following way. Let the mapping $r \mapsto b_x(r)$ given by

$$b_x(r) = \begin{cases} 3.1651, & r < 3.5826, \\ 4.5681 + 1.3347r + 3.5255 \cdot 10^{-1} r^2 + 8.2975 \cdot 10^{-3} r^3 - 1.6201 \cdot 10^{-4} r^4, & r > 3.5826. \end{cases}$$

be the numerical approximation of the values for deformation parameters in the most dense fragmentation (1000 layers). Material metric in this case is Riemannian with nontrivial curvature. This fact is illustrated by the distribution of Ricci invariant (34) shown in figure 8f. Note that Ricci invariant is equal to zero on the part of material manifold corresponded to the initial body. Adhered material forms the part of the body with non-Euclidean material metrics.

The corresponding Cauchy stress tensor $T^{\text{approx}}$ is obtained in subsection 8.2, and similarly, the corresponding first Piola-Kirchhoff stress tensor and Eshelby energy-momentum tensor. We considered the intermediate shape as actual and put $A_{rr}$ to be equal to zero. The corresponding distributions are shown in figures 7, 8a, 8c, 8e by solid lines.

The function

$$x \mapsto \delta(x) = \frac{T_{w}^{\text{approx}}(x) - T_{w}^{\text{discr}}(x)}{\max T_{w}^{\text{discr}}},$$

estimates the difference between stress states in solids with discrete and continuous inhomogeneities. Plot of this function is shown in figure 9; label 1 designates 10 layers, label 2 designates 30 layers, label 3 designates 200 layers, and label 4 designates 1000 layers. In the right upper corner of the figure 9 the logarithms of absolute values for $\delta$, i.e. $\ln |\delta|$, are shown.

9. Optimization Problem
Since analytical solution for the cylindrical problem stated in Section 8 is known, one can derive the intensity $\tau$ of normal stresses, which is defined by the formula

$$\tau := \sqrt{-3I_z(\text{Dev}T)}.$$  \hspace{1cm} (45)

Here,

$$I_z(Q) = \frac{1}{2} (\text{tr}^2 Q - \text{tr} Q)$$

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is the second invariant of the 2-nd rank tensor $\mathbf{Q}$, and $\text{Dev} \mathbf{T}$ is the deviator of stress tensor $\mathbf{T}$. In stress tensor components,

$$2\tau^2 = (T_{<rr>} - T_{<\theta\theta>})^2 + (T_{<rr>} - T_{<zz>})^2 + (T_{<zz>} - T_{<\theta\theta>})^2,$$

where the symbols $T_{<rr>}$, $T_{<\theta\theta>}$, $T_{<zz>}$ denote respectively radial, circumferential, and axial physical components of the tensor $\mathbf{T}$, which is defined by the relations from subsection 8.1.

**Figure 7.** Relative physical components $T_{<ij>}/\mu$ and $P_{<ij>}/\mu$ of the Cauchy and first Piola-Kirchhoff stress tensors.
Figure 8. Relative physical components $E_{\langle ij \rangle} / \mu$ of Eshelby energy-momentum tensor; distributions of deformation parameter, the first and Ricci invariants.

Figure 9. The estimation of the difference between stress states in solids with discrete and continuous inhomogeneities.
Let us consider internal hydrostatic loadings as optimization parameters. Thus, we introduce a set \( \mathbb{R}^N \) of all \( N \)-tuples. Each tuple \( p = (p_1, \ldots, p_N) \in \mathbb{R}^N \) represents hydrostatic loadings for internal surfaces of the \( N \) layers constituting an assembly. Let external hydrostatic loadings, reference thicknesses, reference radii of the first layer, shrinkage and material parameters are given. Stress intensity \( \tau \) defined by the formula (45) is an objective function depending on \( p \).

The optimization problem is to find such a vector \( p \) that minimizes the maximal value of stress intensity \( \tau_{\text{max}} \) of the terminal assembly under given working inner pressure, i.e.

\[
p^* = \arg \min_{p \in \mathbb{R}^N} \max_{X \in \mathbb{B}} \tau(X, p).
\]

![Figure 10. Stress intensity \( \tau \) vs. incompatibility parameters \( \Delta R_1 \) and \( \Delta R_2 \).](image)

The approaches applied for solving the optimization problem (46) require special attention, as the resulting functional is neither convex nor smooth. These properties can be demonstrated even on the example of the simplest assembling of two layers. In figure 10 the graph of the intensity function with respect to the radii incompatibility \( \Delta R_1 \) and \( \Delta R_2 \) is presented. The parameters \( \Delta R_j \) for \( j = 1, 2 \) mean the differences between the outer radius \( R_{j-1}^o \) of the previous layer \( j - 1 \) in the reference configuration and the inner radius \( R_j^i \) of the layer \( j \). These gaps (\( \Delta R_j < 0 \)) or overlaps (\( \Delta R_j > 0 \)) define uniquely the resulting stress-strain state of the structure and can be recalculate as functions of control tuple \( p \).

As seen from the figure the surface \( \tau_{\text{max}}(\Delta R_1, \Delta R_2) \) processes some folds, convex regions, and even visible local minima near the global one.

With great confidence, we can assume that the similar or more complicated patterns are inherent in the multidimensional case \( (N > 2) \). Thus, it is unavoidable to search for an reasonable initial control vector in the attraction basin of the global minimum point in order to approach to the lowest level of stress intensity.

10. Conclusion Remarks
To summarize the work we note the following

- All basic concepts that form the basis of classical continuum mechanics can be generalized on non-Euclidean analysis.
- This approach allow to obtain continual models for multilayered LbL structures with large number of layers.
• For computations such continual models are much more effective then discrete models, based on classical (Euclidean) approach.
• Structural optimization of nonlinearly deformable LbL structures makes it possible to improve the quality of the design, but due to the substantial nonlinearity of the problem it is ambiguous and requires a delicate comprehension.

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