Multicell Coordinated Beamforming with Rate Outage Constraint—Part I: Complexity Analysis

Wei-Chiang Li∗, Tsung-Hui Chang, and Chong-Yung Chi

Abstract—This paper studies the coordinated beamforming (CoBF) design in the multiple-input single-output interference channel, assuming only channel distribution information given a priori at the transmitters. The CoBF design is formulated as an optimization problem that maximizes a predefined system utility, e.g., the weighted sum rate or the weighted max-min-fairness (MMF) rate, subject to constraints on the individual probability of rate outage and power budget. While the problem is non-convex and appears difficult to handle due to the intricate outage probability constraints, so far it is still unknown if this outage constrained problem is computationally tractable. To answer this, we conduct computational complexity analysis of the outage constrained CoBF problem. Specifically, we show that the outage constrained CoBF problem with the weighted sum rate utility is intrinsically difficult, i.e., NP-hard. Moreover, the outage constrained CoBF problem with the weighted MMF rate utility is also NP-hard except the case when all the transmitters are equipped with single antenna. The presented analysis results confirm that efficient approximation methods are indispensable to the outage constrained CoBF problem.

Index terms—Interference channel, coordinated beamforming, outage probability, complexity analysis.

I. INTRODUCTION

Coordinated transmission has been recognized as a promising approach to improving the system performance of wireless cellular networks [2]. According to the level of cooperation, the coordinated transmission can be roughly classified into two categories, i.e., MIMO cooperation and interference coordination [3]. In MIMO cooperation, the transmitters, e.g., base stations (BSs), cooperate in data transmission by sharing all the channel state information (CSI) and data signals. In interference coordination, the BSs only coordinate in the transmission strategies for mitigating the inter-cell interference. Compared with MIMO cooperation, interference coordination requires the CSI to be shared only, and hence induces less overhead on the backhaul network [4]. A common model for studying interference coordination is the interference channel (IFC) [5], where multiple transmitters simultaneously communicate with their respective receivers over a common frequency band, and thus interfere with each other.

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In this paper, we consider a multiple-input single-output (MISO) IFC, wherein the transmitters are equipped with multiple antennas while the receivers are equipped with single antenna. Moreover, we are interested in the coordinated beamforming (CoBF) design problem; that is, the transmitters coordinate with each other to optimize their transmit beamforming vectors. A typical formulation of the CoBF design problem is to maximize a system utility function, e.g., the sum rate, proportional fairness rate, harmonic mean rate or the max-min-fairness (MMF) rate, assuming that the transmitters have perfect CSI. It turns out that the CoBF problems are in general difficult optimization problems. Specifically, it has been shown in [6] that, except for the MMF rate [7], the CoBF problem for the sum rate, proportional fairness rate and harmonic mean rate are NP-hard in general, implying that they cannot be efficiently solved in general. Due to this fact, a significant amount of research efforts has been devoted to the development of reliable and efficient methods for handling the CoBF problems. For example, the works [8]–[11] characterize the optimal beamforming structure in order to reduce the dimension of exhaustive search. Global optimization algorithms were also developed in [12]–[15] but are only efficient when the number of users is small. Another branch of works focus on suboptimal but computationally efficient approximation algorithms; see [6], [10], [16]–[22].

The works mentioned above all have assumed that the transmitters have perfect CSI. However, in practical wireless scenarios, especially in mobile channels, it is difficult for the transmitters to acquire accurate CSI due to time-varying channels. In contrast, the statistical distribution of the channel, i.e., the channel distribution information (CDI), can remain static in a relatively long period of time and thus is easier to obtain compared to the CSI. However, with only CDI at the transmitters, the transmission would suffer from rate outage; that is, the instantaneous channel may not reliably support low, have attracted extensive attention recently; see, e.g., [23]–[25] for the outage constrained utility maximization problem, [26], [27] for the outage constrained power minimization problem and [26], [28], [29] for the outage balancing problem. One essential fact that is commonly observed in these works is that the outage constrained CoBF problems are usually nonconvex and appear much more difficult to handle than their perfect CSI counterparts [6], [7], due to the complicated probabilistic outage constraints. Therefore, most of the works [24], [25], [29] have concentrated on developing efficient
approximation algorithms. However, unlike the perfect CSI case where the complexity status of various CoBF problems has been identified [6], [7], it is still not clear if the outage constrained CoBF problems are computational tractable or, instead, are intrinsically difficult, i.e., NP-hard [30].

Our interest in this paper lies in characterizing the computational complexity status of the outage constrained CoBF problem. Specifically, we consider the outage constrained (weighted) sum rate maximization (SRM) CoBF problem and the outage constrained (weighted) MMF CoBF problem, respectively maximize the (weighted) sum rate and (weighted) MMF rate under rate outage constraints and individual power constraints. We analytically show that the outage constrained SRM problem is NP-hard in general, and that the outage constrained MMF problem is polynomial-time solvable when each of the transmitters is equipped with only one antenna but is NP-hard when each of the transmitters is equipped with at least two antennas. The NP-hardness of the outage constrained SRM problem is established by a polynomial-time reduction from the NP-hard Max-Cut problem [30], i.e., each problem instance of the Max-Cut problem can be transformed to a problem instance of the outage constrained SRM problem in polynomial time; while the NP-hardness of the outage constrained MMF problem in the MISO scenario is established by a polynomial-time reduction from the 3-satisfiability (3-SAT) problem, which is known to be NP-complete [30]. The proposed analysis about the NP-hardness of MMF problem can also be analogously applied to prove that the outage balancing problem studied in [28] is NP-hard when each of the transmitters has at least two antennas. The complexity analysis results further motivate the development of efficient approximation algorithms for handling the outage constrained CoBF problem; see [31].

Synopsis: In Section II, we present the system model and problem formulations. The complexity analyses for the outage constrained SRM problem and outage constrained MMF problem are presented in Section III and Section IV, respectively. Finally, we draw the conclusions in Section V.

Notation: The set of n-dimensional real vectors, complex vectors and complex Hermitian matrices are denoted by \( \mathbb{R}^n \), \( \mathbb{C}^n \) and \( \mathbb{H}^n \), respectively. The set of non-negative real vectors and the set of positive-semidefinite Hermitian matrices are denoted by \( \mathbb{R}_{+}^n \) and \( \mathbb{H}_{+}^n \), respectively. The superscripts ‘T’ and ‘H’ represent the matrix transpose and conjugate transpose, respectively. We denote \( \| \cdot \| \) and [\( \cdot \)] as the vector Euclidean norm and ceiling function, respectively. \( \mathbf{A} \geq 0 \) means that matrix \( \mathbf{A} \) is positive semidefinite. We use the expression \( x \sim \mathcal{CN}(\mu, \mathbf{Q}) \) if \( x \) is circularly symmetric complex Gaussian distributed with mean \( \mu \) and covariance matrix \( \mathbf{Q} \). We denote \( \exp(\cdot) \) (or simply \( e^{(\cdot)} \)) as the exponential function, while \( \ln(\cdot) \) and \( \Pr\{\cdot\} \) represent the natural logarithm function and the probability function, respectively. For a variable \( a_{ik} \), where the domain of subscripts pair \( i, k \) are clear from context, \( \{a_{ik}\} \) denotes the set of all \( a_{ik} \) with the subscript pair \( i, k \) covering all the possible values over its domain, and \( \{a_{ik}\}_{k \neq j} \) denotes the set of all \( a_{ik} \) with the first subscript equal to \( i \). The sets \( \{a_{ik}\}_{k \neq j} \) and \( \{a_{ik}\}_{i(k) \neq (j, k)} \) are defined by the set \( \{a_{ik}\}_{k} \) excluding \( a_{ij} \) and the set \( \{a_{ik}\} \) excluding \( a_{ji} \), respectively.

II. SYSTEM MODEL AND PROBLEM STATEMENT

We consider a MISO IFC with \( K \) pairs of transmitters and receivers (i.e., \( K \) users). Each transmitter is equipped with \( N_t \) antennas and each receiver is equipped with a single antenna. Transmit beamforming is assumed for the communication between each transmitter and its intended receiver. Let \( s_i \sim \mathcal{CN}(0, 1) \) denote the message for the \( i \)-th user, and \( w_i \in \mathbb{C}^{N_t} \) denote the corresponding beamforming vector. We assume a frequency flat channel model, and the channel vector between transmitter \( i \) and receiver \( k \) is modeled as \( h_{ik} \sim \mathcal{CN}(0, Q_{ik}) \), where \( Q_{ik} \succeq 0 \) is the channel covariance matrix. The received signal for receiver \( i \) is then given by

\[
x_i = h_{ii}^H w_i s_i + \sum_{k=1, k \neq i}^{K} h_{ki}^H w_k s_k + n_i,
\]

where \( n_i \sim \mathcal{CN}(0, \sigma_i^2) \) is the additive noise at receiver \( i \) with variance \( \sigma_i^2 > 0 \). Under the assumption that single-user detection is used by each receiver, the instantaneous achievable rate (in bits/sec/Hz) of the \( i \)-th user can be expressed as

\[
r_i(\{h_{ki}\}_k, \{w_k\}_K) = \log_2 \left( 1 + \frac{|h_{ii}^H w_i|^2}{\sum_{k \neq i} |h_{ki}^H w_k|^2 + \sigma_i^2} \right).
\]

To enhance the overall system performance, a typical formulation of the CoBF design is to optimize the beamforming vectors \( \{w_k\}_K \) of all the \( K \) users so as to maximize a performance measuring system utility function, e.g., the information sum rate, under power constraints [6], which can be mathematically expressed as

\[
\max_{w_i \in \mathbb{C}^{N_t}, R_i \geq 0} U(R_1, \ldots, R_K)
\]

subject to (s.l) \( R_i \leq r_i(\{h_{ki}\}_k, \{w_k\}_K), \) \( i = 1, \ldots, K \), (3b)

\[
\|w_i\|^2 \leq P_i, \quad i = 1, \ldots, K.
\]

Here, \( R_1, \ldots, R_K \) are the respective transmission rates of the \( K \) users, and \( P_1, \ldots, P_K \) are the associated power budgets. The function \( U(R_1, \ldots, R_K) \) denotes the system utility. In this paper, we are interested in two system utilities in particular, i.e., the weighted sum rate, where \( U(R_1, \ldots, R_K) = \sum_{i=1}^{K} \alpha_i R_i \) and the weighted min rate, where \( U(R_1, \ldots, R_K) = \min_{i=1, \ldots, K} R_i/\alpha_i \). The corresponding utility optimization problems are known as the SRM problem and the MMF problem, respectively. These two problem formulations represent two extremes of the tradeoff between system throughput and the user fairness. For the SRM problem, one aims to maximize the system throughput, but the transmission may be dominated by a few of users. For the MMF problem, one places the highest emphasis on user fairness, but the achieved system throughput may not be as high. Interestingly, the complexity status of solving these two problems are also very different. Specifically, solving the SRM problem is NP-hard in general [32] which means that the problem is unlikely to be solved in a polynomial-time complexity; by contrast, the MMF problem is polynomial-time solvable [6]. Efficient approximation algorithms for handling the SRM problem have been extensively studied (see [6], [15] and references therein).
According to [25], [33], the outage constraint (4b) can be expressed as

\[
\max_{w_i \in \mathbb{C}^{N_t}, R_i \geq 0, i = 1, \ldots, K} U(R_1, \ldots, R_K) \tag{4a}
\]

subject to \(\|r_i(h_{ki})k, \{w_k\} < R_i\) \(\leq \epsilon_i, \tag{4b}\)

\[
\|w_i\|^2 \leq P_i, \quad i = 1, \ldots, K. \tag{4c}
\]

According to [25], [33], the outage constraint (4b) can be explicitly expressed as

\[
\rho_i \exp \left(\frac{(2^{R_i} - 1)\sigma^2}{\|w_i\|^2} \prod_{k \neq i} \left(1 + \frac{(2^{R_i} - 1)w_i^H Q_{ki} w_k}{\|w_i\|^2} \right)\right) \leq 1, \tag{5}
\]

where \(\rho_i = 1 - \epsilon_i \) for \(i = 1, \ldots, K\) are the satisfaction probabilities required in the downlink transmission.

Due to the complicated constraint (5), solving the outage constrained problem (4) seems more difficult than solving its perfect CSI counterpart, i.e., problem (3). However, this intuitive observation is not mathematically precise. It is hence of interest to investigate the complexity status of problem (4). In the ensuing sections, we study the complexity of solving problem (4) with weighted sum rate and minimum rate utilities, which correspond to the SRM and MMF formulations, respectively. Our complexity analysis will demonstrate that problem (4) is indeed more challenging. Specifically, problem (4) is NP-hard not only for the SRM formulation but also for the MMF formulation, while problem (3) is at least polynomial-time solvable for the MMF formulation [6]. The obtained results about the complexity of problem (4), together with the corresponding results in the literature about problem (3), are summarized in Table I on the top of the next page.

III. COMPLEXITY ANALYSIS FOR OUTAGE CONSTRAINED SRM PROBLEM

In this section, we analyze the complexity status of the outage constrained SRM problem, which can be written as

\[
\max_{w_i \in \mathbb{C}^{N_t}, R_i \geq 0, i = 1, \ldots, K} \sum_{i=1}^{K} \alpha_i R_i \tag{6a}
\]

subject to \(\rho_i \exp \left(\frac{(2^{R_i} - 1)\sigma^2}{\|w_i\|^2} \prod_{k \neq i} \left(1 + \frac{(2^{R_i} - 1)w_i^H Q_{ki} w_k}{\|w_i\|^2} \right)\right) \leq 1, \tag{6b}\)

\[
\|w_i\|^2 \leq P_i, \quad i = 1, \ldots, K. \tag{6c}
\]

Specifically, we demonstrate that problem (6) is NP-hard in general. The following theorem makes our statement precise.

**Theorem 1** The outage constrained SRM problem (6) is NP-hard in the number of users \(K\), for all \(N_t \geq 1\).

Theorem 1 indicates that problem (6) is computationally intractable, like its perfect CSI counterpart (3) with the weighted sum rate utility [32]. While both of these two problems are NP-hard, one should note that the techniques used for proving Theorem 1 are quite different than those used in [32]. In [32], it was shown that problem (3) with the weighted sum rate utility is at least as difficult as the maximum independent set problem (which is known NP-complete) [30]. However, the same idea is not applicable to the complexity analysis for problem (6), due to the much more involved constraints (6b). Instead, we show in the next subsection that problem (6) is at least as difficult as the Max-Cut problem.

**A. Proof of Theorem 1**

Here, we show that problem (6) is NP-hard even when \(N_t = 1\), which implies that problem (6) is NP-hard for the general case of \(N_t \geq 1\). For \(N_t = 1\), the CoBF design problem (6) reduces to a coordinated power control problem. Specifically, the MISO channel \(h_{ki} \in \mathbb{C}^{N_t}\) reduces to the single-input single-output (SISO) channel \(h_{ki} \in \mathbb{C}\), the channel covariance matrix \(Q_{ki} \in \mathbb{H}_{N_t}^N\) reduces to \(Q_{ki} \in \mathbb{R}_{N_t}\), and the beamformer \(w_i\) reduces to the square root of the transmit power \(p_i\); thus, problem (6) reduces to

\[
\max_{p_i \in \mathbb{R}_+, R_i \geq 0, i = 1, \ldots, K} \sum_{i=1}^{K} \alpha_i R_i \tag{7a}
\]

subject to \(\rho_i \exp \left(\frac{(2^{R_i} - 1)\sigma^2}{Q_{ki} p_i} \prod_{k \neq i} \left(1 + \frac{(2^{R_i} - 1)Q_{ki} p_k}{Q_{ki} p_i} \right)\right) \leq 1, \tag{7b}\)

\[
0 \leq p_i \leq P_i, \quad i = 1, \ldots, K. \tag{7c}
\]

We will show that the weighted Max-Cut problem, which is known to be NP-hard [30], is polynomial-time reducible to problem (7), i.e., the Max-Cut problem is a special instance of (7). For ease of the ensuing presentation, the definition of the weighted Max-Cut problem is repeated as follows.

**Definition 1** Consider an undirected and connected graph \(G = (V, E)\), where \(V = \{1, \ldots, |V|\}\) denotes the set of vertices in \(G\), and \(E = \{(i, j) \mid i \in V, j \in V, i < j\}\) denotes the set of edges in \(G\). Each edge \((i, j) \in E\) is assigned with a weight \(w_{ij} > 0\). A cut \(E(S) \subseteq E\), consists of the set of edges crossing a subset \(S \subseteq V\) and its complement \(\bar{S} = V \setminus S\). The weighted Max-Cut problem is formulated as

\[
\max_{S \subseteq V} \sum_{(i,j) \in E(S)} w_{ij}. \tag{8}
\]

To build the connection between the Max-Cut problem and problem (7), we consider an alternative formulation of (7):
Lemma 1 Problem (7) can be equivalently expressed as

\[
\max_{p_i \in \mathbb{R}, \ i = 1, \ldots, K} \sum_{i=1}^{K} \alpha_i \log_2 (1 + \zeta_i (\{p_k\}_{k \neq i}) Q_i p_i) \quad (9a)
\]

s.t. \( 0 \leq p_i \leq P_i, \ i = 1, \ldots, K, \) (9b)

where \( \zeta_i (\{p_k\}_{k \neq i}), \) which is continuously differentiable in \( \{p_k\}_{k \neq i}, \) is the unique solution that satisfies

\[
\Psi_i (x, \{p_k\}_{k \neq i}) \triangleq \rho_i \exp (\sigma_i^2 x) \prod_{k \neq i} (1 + Q_{ki} p_k \cdot x) = 1, \quad (10)
\]

for all \( i, k = 1, \ldots, K. \)

Proof: See Appendix A for details.

By reformulating problem (7) as problem (9), one can compactly characterize the relation between the achievable rates and the transmit powers, e.g., the monotonicity and convexity, using the implicit functions \( \zeta_i (\cdot), \ i = 1, \ldots, K. \) As a result, it is much easier to analyze the optimal power allocation pattern based on the alternative formulation (9) than based on the original formulation (7). In the subsequent analysis, we thus focus on proving that the Max-Cut problem is polynomial-time reducible to problem (9). The idea of this proof is that, given any undirected and connected graph \( G = (\mathcal{V}, \mathcal{E}) \) and the weights \( w_{ij} > 0 \) for all \( (i, j) \in \mathcal{E}, \) we can construct a particular instance of (9) that is equivalent to the weighted Max-Cut problem (8) associated with the graph \( G \) and the weights \( \{w_{ij}\}. \) The construction is detailed as follows.

We associate each node \( i \in \mathcal{V} \) with two distinct transmitter-receiver pairs (users) in the coordinated power control problem (9), denoted by \( v_{i0}, v_{i1}. \) Moreover, each edge \( (i, j) \in \mathcal{E} \) is associated with two other users, denoted by \( e_{ij}, e_{ji}. \) The resulting set of users is the union of the user set \( \mathcal{U}_o \) associated with nodes and the user set \( \mathcal{U}_e \) associated with edges, i.e.,

\[
\mathcal{U} = \mathcal{U}_o \cup \mathcal{U}_e
\]

\[
\triangleq \{v_{10}, v_{11}, \ldots, v_{|\mathcal{V}|0}, v_{|\mathcal{V}|1}\} \cup \{e_{ij}, e_{ji}, (i, j) \in \mathcal{E}\}, \quad (11)
\]

which contains \( K = |\mathcal{U}| = 2(|\mathcal{V}| + |\mathcal{E}|) \) users in total. For these \( K \) users, we consider a particular instance of problem (9) with the following specified system parameters:

\[
\sigma_i^2 = 0.1, \quad \rho_i = 0.95, \quad P_u = \begin{cases} 1, & \text{if } u \in \mathcal{U}_o, \forall u \in \mathcal{U}, \\ 0.7, & \text{if } u \in \mathcal{U}_e, \end{cases} \quad (12a)
\]

\[
Q_{v_{ia}v_{jb}} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise}, \end{cases} \quad Q_{e_{ij}e_{ji}} = \begin{cases} 1, & \text{if } u = e_{ij}, \\ 0, & \text{otherwise}, \end{cases} \quad (12b)
\]

\[
Q_{v_{ia}e_{ik}} = \begin{cases} 1, & \text{if } (i, a) = (k, 0) \text{ or } (i, a) = (l, 1), \\ 0, & \text{otherwise}, \end{cases} \quad (12c)
\]

\[
\begin{align*}
\alpha_{v_{ia}} &= 1, \\
\alpha_{e_{ij}} &= \alpha_{e_{ji}} = \frac{w_{ij}}{2 \sum_{(k, \ell) \in \mathcal{E}} w_{k, \ell}}, & \forall v_{ia} \in \mathcal{U}_o, \\
\end{align*} \quad (12d)
\]

Based on the construction described by (11) and (12), let us consider a simple example illustrated by Fig. 1. Here, we demonstrate how a simple graph \( G = \{1, 2, 3\} \) with two edges \( \mathcal{E} = \{(1, 2), (2, 3)\} \) can be mapped to an IFC through (11) and (12). As illustrated in Fig. 1, each vertex \( i \) corresponds to two users, \( v_{i0} \) and \( v_{i1}, \) for all \( i \in \{1, 2, 3\}; \) and each edge \( (i, j) \) also corresponds to two users, \( e_{ij} \) and \( e_{ji}, \) for all \( (i, j) \in \{(1, 2), (2, 3)\}. \) For the IFC on the right-hand side in Fig. 1, user \( u \) and user \( u' \) would interfere with each other if \( Q_{uu'} = 1 \) and do not interfere if \( Q_{uu'} = 0. \) Thus, according to the parameter set in (12b), any two users associated with a common vertex in \( G \) will interfere with each other, while users associated with different vertices would not interfere with each other. Besides, the users associated with the edges only communicate with their intended receivers and do not interfere with the other users. By (12c), user \( v_{i0} \) and \( v_{i1} \) would interfere with user \( e_{ij}, \) while users \( v_{i0} \) and \( v_{i1} \) would interfere with user \( e_{ji}, \) for all edges \( (i, j). \) All the resulting interference patterns are shown in Fig. 1.

Based on the construction described by (11) and (12), the resulting problem instance of the coordinated power control problem (9) is given by

\[
\max_{p_{ia} \in \mathbb{R}, \forall v_{ia} \in \mathcal{U}_o, \forall e_{ij} \in \mathcal{U}_e} \sum_{v_{ia} \in \mathcal{U}_o} \log_2 (1 + p_{ia}^v v^v (p_{ia}^v)) \quad (13a)
\]

\[
+ \sum_{e_{ij} \in \mathcal{U}_e} \alpha_{v_{ia}} \log_2 (1 + p_{ia}^v e^e (p_{ia}^v)) \quad (13b)
\]

s.t. \( 0 \leq p_{ia}^v \leq 1, \) \( v_{ia} \in \mathcal{U}_o, \) \( e_{ij} \in \mathcal{U}_e, \)

\[
0 < p_{ij}^e \leq 0.7, \quad e_{ij} \in \mathcal{U}_e, \quad (13c)
\]

where \( p_{ia}^v \) and \( p_{ij}^e \) denote the transmission powers of user \( v_{ia} \) and \( e_{ij}, \) respectively; \( a = 1 - a \) for \( a \in \{0, 1\}; \) and by (10),
Furthermore, there exists at most one $\zeta(p_1, p_2) = \zeta(p_2, p_1)$ which is proved in Appendix B.

It is easy to see from (13) that the optimal solution to problem (13) lies in a discrete set, as stated in the following lemma.

**Lemma 2** The optimal solution of (13) must satisfy

\[
\begin{aligned}
p_{i}^* &\in \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\}, \quad \text{for } i = 1, \ldots, |\mathcal{V}|, \\
\hat{p}_{ij} &> 0, \quad \forall e_{ij} \in \mathcal{U}_e.
\end{aligned}
\]

**Proof:** Suppose that $\{\hat{p}_{i}^*, \hat{p}_{ij}^*\}_{v_{ia} \in \mathcal{U}_e, e_{ij} \in \mathcal{U}_e}$ is an optimal solution of problem (13). It is easy to see from (13a) that $\hat{p}_{ij} = 0.7$ for all $e_{ij} \in \mathcal{U}_e$, since user $e_{ij}$ does not interfere with any other user.

To complete the proof, we first show that $\hat{p}_{i}^* \in \{0, 1\}$ for all $v_{ia} \in \mathcal{U}_e$. For notational simplicity, let us focus on proving the case of $a = 0$. For any given $i \in \mathcal{V}$, let us assume that $\{\hat{p}_{ij}^*\}_{j \neq i(0)}$ are known and fixed. In this case, it is clear that $\hat{p}_{i}^*$ must also be optimal to the following problem:

\[
\begin{aligned}
\max_p \quad &F(p) \triangleq \log_2(1 + p \zeta(p_{i})) + \log_2(1 + \hat{p}_{ij} \zeta(p_{j})) \\
\text{s.t.} \quad &0 \leq p \leq 1,
\end{aligned}
\]

which is obtained from (13) by fixing $p_{i}^* = \hat{p}_{ij}$ for all $(j, b) \neq (i, 0)$ and excluding the terms irrelevant to user $v_{i0}$. Hence, we focus on showing that the optimal solution to (16) is either $p = 0$ or $p = 1$. To this end, we need the following lemma, which is proved in Appendix B.

**Lemma 3** The objective function of problem (16), i.e., $F(p) : \mathbb{R}_+ \to \mathbb{R}_+$, is a differentiable quasiconvex function. Furthermore, there exists at most one $p \geq 0$ such that $dF(p)/dp = 0$.

Suppose that $p^* \in (0, 1)$ is optimal to (16), which implies $dF(p^*)/dp = 0$. Then, by Lemma 3, we have $dF(p)/dp \leq 0$ for all $p \in [0, p^*]$ and $dF(p)/dp > 0$ for all $p \in (p^* , +\infty)$, which contradicts the fact that $p^*$ maximizes $F(p)$ over the interval $0 \leq p \leq 1$, implying that the optimal solution to problem (16) is either $p = 0$ or $p = 1$. Hence, we have proved $\hat{p}_{i}^* \in \{0, 1\}$. Similarly, we can show that $\hat{p}_{i}^* \in \{0, 1\}$. As a result, the optimal solution of problem (13) must satisfy

\[
\begin{aligned}
\left[\begin{array}{c}
\hat{p}_{i}^* \\
\hat{p}_{ij}^*
\end{array}\right] &\in \left\{ \begin{array}{c} \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \\
\left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \\
\left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \\
\left[ \begin{array}{c} 1 \\ 1 \end{array} \right]
\end{array} \right\}, \quad i = 1, \ldots, |\mathcal{V}|,
\end{aligned}
\]

and $\hat{p}_{ij}^* = 0.7$ for all $e_{ij} \in \mathcal{U}_e$.

What remains to be proved is that $\left[\hat{p}_{i0}^* \hat{p}_{ij}^*\right]^T = [0 \ 0]^T$ or $\left[\hat{p}_{i0}^* \hat{p}_{ij}^*\right]^T = [1 \ 1]^T$ for some $i \in \mathcal{V}$ is strictly suboptimal to problem (13). Let $\hat{R}$ denote the objective value of problem (13) achieved by $\{\hat{p}_{i0}^*, \hat{p}_{ij}^*\}_{v_{i0} \in \mathcal{U}_e, e_{ij} \in \mathcal{U}_e}$, and define

\[
\begin{aligned}
\mathcal{I}_0 &= \{ i \in \mathcal{V} \mid \left[\hat{p}_{i0}^* \hat{p}_{ij}^*\right]^T = [0 \ 0]^T \}, \\
\mathcal{I}_1 &= \{ i \in \mathcal{V} \mid \left[\hat{p}_{i0}^* \hat{p}_{ij}^*\right]^T = [1 \ 1]^T \}.
\end{aligned}
\]

We can bound $\hat{R}$ from below and above as follows:

\[
\begin{aligned}
\hat{R} &\geq \sum_{v_{ia} \in \mathcal{U}_e} \log_2(1 + \hat{p}_{i0} \zeta(p_{i})) \\
&= \sum_{i=1}^{\mathcal{V}} \left( \log_2(1 + \hat{p}_{i0}^* \zeta(p_{i})) + \log_2(1 + \hat{p}_{i0}^* \zeta(p_{i})) \right) \\
&= (|\mathcal{V}| - |\mathcal{I}_0| - |\mathcal{I}_1|) \times \log_2(1 + 1 \cdot \zeta(0)) \\
&\quad + 2|\mathcal{I}_1| \log_2(1 + 1 \cdot \zeta(1)) \\
&= (|\mathcal{V}| - |\mathcal{I}_0| - |\mathcal{I}_1|) \times 0.5973 + 2|\mathcal{I}_1| \times 0.0671 \\
&= |\mathcal{V}| \times 0.5973 - (|\mathcal{I}_0| \times 0.5973 + |\mathcal{I}_1| \times 0.4631),
\end{aligned}
\]

where the above numerical values are computed from the parameters set in (12). From (17a), one can see that $\hat{R}$ is no less than $|\mathcal{V}| \times 0.5973$ when $|\mathcal{I}_0| = |\mathcal{I}_1| = 0$. However, from (17b), it is clear that $\hat{R}$ is smaller than $|\mathcal{V}| \times 0.5973$, when either $|\mathcal{I}_0| \neq 0$ or $|\mathcal{I}_1| \neq 0$. Thus, the optimal power pattern $\left[\hat{p}_{i0}^* \hat{p}_{ij}^*\right]^T$ must be either $[0 \ 1]^T$ or $[1 \ 0]^T$. This completes the proof of Lemma 2.

Based on Lemma 2, we can restrict the feasible set of (13) to the subset defined in (15) without loss of optimality. Let

\[
\mathcal{S} = \{ i \mid \left[\hat{p}_{i0}^* \hat{p}_{ij}^*\right]^T = [0 \ 1]^T \} \subseteq \mathcal{V},
\]

and denote $1_{\{i \in \mathcal{S}\}}$ as the indicator function which is equal to one if $i \in \mathcal{S}$ and zero otherwise. Then, by (15), we have $\hat{p}_{i0}^* = 1_{\{i \in \mathcal{S}\}}, \hat{p}_{ij}^* = 1_{\{i \in \mathcal{S}\}}$ and $\hat{p}_{ij}^* = 0.7$. With these substitutions in problem (13), problem (13) can be equivalently reformulated as the following problem where $\mathcal{S}$ is now the optimization variable:

\[
\begin{aligned}
\max_{\mathcal{S} \subseteq \mathcal{V}} &\sum_{i=1}^{\mathcal{V}} \left[ \log_2(1 + 1 \cdot \zeta(0)) \right] \\
&\quad + \sum_{(i,j) \in \mathcal{E}(\mathcal{S})} \left[ \alpha_{e_{ij}} \log_2(1 + 0.7 \cdot \zeta(0)) \right],
\end{aligned}
\]

subject to

\[
\begin{aligned}
\sum_{(i,j) \in \mathcal{E}(\mathcal{S})} \alpha_{e_{ij}} (c_{00} + c_{11}) &\geq \sum_{(i,j) \in \mathcal{E}(\mathcal{S})} \alpha_{e_{ij}} (c_{01} + c_{10}), \\
\sum_{(i,j) \in \mathcal{E}(\mathcal{S})} \alpha_{e_{ij}} (c_{00} + c_{11}) &\geq \sum_{(i,j) \in \mathcal{E}(\mathcal{S})} \alpha_{e_{ij}} (c_{01} + c_{10}).
\end{aligned}
\]
\[ R = \max_{\mathcal{S} \subseteq \mathcal{V}} \left| \mathcal{V} \right| \cdot \log_2 \left( 1 + \frac{c_0 + c_1 \rho_i}{2} \right), \]

where \( c_0 = \log_2 \left( 1 + 0.7^c(0, 0) \right), c_1 = \log_2 \left( 1 + 0.7^c(1, 1) \right), c_{01} = \log_2 \left( 1 + 0.7^c(0, 1) \right), \) and \( c_0 = \log_2 \left( 1 + 0.7^c(1, 0) \right) \) are constants. Note that, when \((i, j) \in \mathcal{E}(S), \) vertex \( i \) and vertex \( j \) belong to different subsets \( S \) and \( \bar{S}, \) so it holds true that \( 1_{(i \in S)} = 1_{(j \in \bar{S})} \) and \( 1_{(i \in \bar{S})} = 1_{(j \in S)}; \) on the other hand, when \((i, j) \in \mathcal{E}(\mathcal{E}(S)), \) it holds true that \( 1_{(i \in S)} \neq 1_{(j \in \bar{S})} \) and \( 1_{(i \in \bar{S})} \neq 1_{(j \in S)}, \) by which we obtain (19). By (14b), one can show that \( c_0 + c_1 - c_0 > 0, \) so solving (13) is equivalent to solving the Max-Cut problem (8). Thus, we have presented a polynomial-time reduction that converts all the problem instances of the Max-Cut problem (8) to instances of the outage constrained SRM CoBF problem (6), which completes the proof of Theorem 1.

IV. COMPLEXITY ANALYSIS FOR OUTAGE CONSTRAINED MMF PROBLEM

In this section, we turn our attention to the outage constrained MMF problem, i.e.,

\[ \max_{w_i \in \mathcal{C}^N, \ R_i \geq 0, \ i = 1, \ldots, K} R \triangleq \min_{i = 1, \ldots, K} R_i / \alpha_i \quad (21a) \]

\[ \text{s.t.} \quad \rho_i \exp \left( \frac{2R_i - 1}{\sigma_i^2} \right) \sum_{k \neq i} \left( 1 + \frac{2R_i - 1}{\sigma_i^2} w_{k}^H Q_{ki} w_i \right) \leq 1, \quad (21b) \]

\[ \|w_i\|^2 \leq P_i, \ i = 1, \ldots, K. \quad (21c) \]

In contrast to its perfect CSI counterpart, which can be transformed into a quasiconvex problem for multiple antennas and multiple users [6], we will show that problem (21) is polynomial-time solvable for any single antenna case, i.e., \( N_t = 1, \) but NP-hard in the number of users \( K \) when \( N_t \geq 2. \)

To proceed with the complexity analysis, let us first introduce a feasibility problem. That is, given a target rate \( \bar{R} \geq 0, \)

Find \( w_1, \ldots, w_K \)

\[ \text{s.t.} \quad \rho_i \exp \left( \frac{2\alpha_i R - 1}{\sigma_i^2} \right) \sum_{k \neq i} \left( 1 + \frac{2\alpha_i R - 1}{\sigma_i^2} w_k^H Q_{ki} w_i \right) \leq 1, \quad (22b) \]

\[ \|w_i\|^2 \leq P_i, \ i = 1, \ldots, K. \quad (22c) \]

Note that problem (22) is closely related to problem (21). Their relation is specified in the following lemma.

**Lemma 4** Let \( R^* \) denote the optimal value of problem (21). It holds true that, for any \( \bar{R} \geq 0, \) problem (22) is feasible if and only if \( \bar{R} \leq R^* \). Furthermore, the set of optimal beamformers to problem (21) is a subset of the feasible set of problem (22) when \( \bar{R} < R^* \), and these two sets coincide when \( \bar{R} = R^* \).

**Proof:** Let \( \{w_i^*\} \) denote an optimal beamforming to problem (21). Since the left-hand side of constraint (22b) is strictly increasing in \( \bar{R}, \) we know that \( \{w_i^*\} \) is feasible to problem (22) if \( \bar{R} \leq R^* \). Hence, problem (22) is feasible if \( \bar{R} \leq R^* \), and all the optimal beamformers to problem (21) are feasible to problem (22). On the other hand, suppose that problem (22) is feasible and \( \{w_i\} \) is a feasible point. Then, \( \{w_i\} \) is clearly a feasible beamformer of (21) that achieves objective value \( \bar{R} \), implying \( R^* \geq \bar{R} \). Furthermore, when \( \bar{R} = R^* \), all the feasible beamformers to problem (22) achieve the optimum objective value of problem (21), and hence are optimal beamformers to (21). \( \blacksquare \)

Lemma 4 infers that problem (21) is polynomial-time solvable if and only if problem (22) is polynomial-time solvable. In particular, if problem (22) can be efficiently solved, then problem (21) can be efficiently solved by a bisection algorithm which involves solving a series of problem (22) [34. §4.2.5]. On the other hand, if one can solve problem (21), then one can correctly determine whether problem (22) is feasible and, if yes, obtain a set of feasible beamformers. Therefore, these two problems belong to the same complexity class. In the next two subsections, we investigate the complexity of problem (21) for the SISO case \( (N_i = 1), \) and MISO case \( (N_t = 1) \) based on this observation.

**A. Single-Antenna Case**

In this subsection, we consider the SISO case, and present a polynomial-time bisection algorithm, which involves solving a finite number of the feasibility problem (22), for solving problem (21) to the global optimum. When \( N_i = 1, \) problem (21) degenerates to the following power control problem

\[ \max_{p_i \geq 0, \ R_i \geq 0, \ i = 1, \ldots, K} R \quad (23a) \]

\[ \text{s.t.} \quad \rho_i \exp \left( \frac{2R_i - 1}{\sigma_i^2} \right) \sum_{k \neq i} \left( 1 + \frac{2R_i - 1}{\sigma_i^2} p_k Q_{ki} \right) \leq 1, \quad (23b) \]

\[ p_i \leq P_i, \ i = 1, \ldots, K. \quad (23c) \]

where \( p_i \) is the transmit power of transmitter \( i, \) and \( Q_{ki} \) is the variance of the SISO channel \( h_{ki}, \) for all \( i, k = 1, \ldots, K. \) Similarly, problem (22) reduces to

Find \( (p_1, \ldots, p_K) \)

\[ \text{s.t.} \quad \rho_i \exp \left( \frac{2\alpha_i R - 1}{\sigma_i^2} \right) \sum_{k \neq i} \left( 1 + \frac{2\alpha_i R - 1}{\sigma_i^2} p_k Q_{ki} \right) \leq 1, \quad (24b) \]

\[ p_i \leq P_i, \ i = 1, \ldots, K. \quad (24c) \]

which is feasible if and only if the optimal value of (23) is no less than \( \bar{R}. \) First of all, we show that problem (24) is polynomial-time solvable.

Consider the change of variables \( \tilde{p}_i = \ln p_i, \) for \( i = 1, \ldots, K. \) Then, we can solve problem (24) by solving the**
following convex problem
\[
\begin{align*}
\min_{\hat{\alpha},\rho_i, i=1,\ldots,K} & \quad \hat{\alpha} \\
\text{s.t.} & \quad \ln \rho_i + \frac{(2^\alpha \bar{R} - 1)\sigma_i^2}{Q_{ii}} e^{-\bar{p}_i} \\
& \quad + \sum_{k \neq i} \ln \left(1 + \frac{(2^\alpha \bar{R} - 1)Q_{ki}}{Q_{ii}} e^{-\bar{p}_i + \bar{p}_k}\right) \leq \hat{\alpha}, \\
& \quad \bar{p}_i \leq \ln P_i, \ i = 1,\ldots,K.
\end{align*}
\]

Specifically, it is not difficult to verify by using an argument similar to Lemma 4 that problem (24) is feasible if and only if the optimal \(\hat{\alpha}\) of problem (25) is less than or equal to zero, and that every optimal solution of problem (25) directly serves as a feasible point of problem (24) provided that problem (24) is feasible. Therefore, based on Lemma 4, one can solve problem (23) in a bisection manner by solving a sequence of convex problems (25). The bisection algorithm is described in Algorithm 1.

\textbf{Algorithm 1 Bisection algorithm for solving problem (23)}

1. Set \(\bar{R}_0 := 0\), \(\bar{R}_u := \min_i \frac{1}{\alpha_i} \log_2 \left(1 + \frac{P_i Q_{ii} \ln(1/\rho_i)}{\sigma_i^2}\right)\), and set the solution accuracy to \(\delta > 0\);
2. repeat
3. \(\hat{R} := (\bar{R}_0 + \bar{R}_u)/2\);
4. Solve problem (25), and denote the solution as \(\left(\bar{p}_i^{\hat{R}}, \bar{R}\right)\); and set
5. \(\tilde{R} := \hat{R}\) if \(\hat{\alpha}^* \leq 0\); otherwise, set \(\bar{R}_u := \bar{R}\);
6. until \(\bar{R}_u - \bar{R}_0 < \delta\);
7. Output \(p_i = e^{\bar{p}_i}, R = \bar{R}, i = 1,\ldots,K\), as a solution to problem (23).

Problem (25) is in fact equivalent to the outage balancing power control problem studied in [28], which can be efficiently solved by a nonlinear Proner-Frondenius theory-based algorithm with overall complexity of \(O(K \ln(K/\epsilon))\) (see [28, Algorithm 1]), where \(\epsilon\) specifies the solution accuracy. Therefore, the overall complexity of Algorithm 1 is \(\kappa \cdot O(K \ln(K/\epsilon))\), where

\[
\kappa \triangleq \left\lfloor \log_2 \left(\delta^{-1} \bar{R}_u \right) \right\rfloor
\]

is the number of bisection iterations required for Algorithm 1 to achieve a solution accuracy \(\delta\). Algorithm 1 has a polynomial-time complexity since \(\kappa\) is finite in practical situations. Hence, we have the complexity of the MMF CoBF problem for the case of \(N_t = 1\) as stated in the following theorem.

\textbf{Theorem 2} When \(N_t = 1\), the outage constrained MMF CoBF problem (21) (i.e., problem (23)) is polynomial-time solvable.

\textbf{B. Multiple-Antenna Case}

In contrast to the single transmit antenna case, in this subsection, we show that the outage constrained MMF CoBF problem (21) is NP-hard when each of the transmitters is equipped with multiple antennas.

\textbf{Theorem 3} When \(N_t \geq 2\), the outage constrained MMF CoBF problem (21) is NP-hard in the number of users \(K\).

\textbf{Proof:} As inferred from Lemma 4, it suffices to show that solving the feasibility problem (22) is NP-hard when \(N_t \geq 2\). The main idea of the proof is to show that the 3-satisfiability (3-SAT) problem, which is known to be NP-complete [30], is reducible to problem (22). The 3-SAT problem is defined as follows.

\textbf{Definition 2} Given \(N\) Boolean variables and \(M\) clauses each containing exactly three literals of different Boolean variables, the 3-SAT problem is to determine whether there exists a truth assignment of the Boolean variables such that all the \(M\) clauses hold true.

For ease of exposition, we use “\(\lor\)”, “\(\land\)”, “\(\neg\)” to denote the logical disjunction (OR), negation, and formulate a 3-SAT problem as a feasibility problem. Specifically, a 3-SAT problem instance with \(N\) Boolean variables \(x_1,\ldots,x_N\) and \(M\) clauses \(d_m = y_{i_m} \lor y_{j_m} \lor y_{k_m}, i_m, j_m, k_m \in \{1,\ldots,N\}, m = 1,\ldots,M\), where \(y_{i_m}\) is either \(x_{i_m}\) or its negation \(\neg x_{i_m}\), and so are \(y_{j_m}\) and \(y_{k_m}\), can be written as the following feasibility problem:

\[
\begin{align*}
\text{Find } [x_1,\ldots,x_N]^T & \in \{0,1\}^N \\
\text{s.t. } d_m & = y_{i_m} \lor y_{j_m} \lor y_{k_m} = 1, \ m = 1,\ldots,M.
\end{align*}
\]

Given any \(N\) Boolean variables \(x_1,\ldots,x_N\) and \(M\) clauses \(d_1,\ldots,d_M\), we construct a problem instance of (22) that is equivalent to the corresponding 3-SAT problem, i.e., problem (26), as follows.

We associate each \(x_n\) with five users, denoted by the set \(V_n \triangleq \{v_{n0}, v_{n1}, v_{n2}, v_{n3}, v_{n4}\}\) for \(n = 1,\ldots,N\), and associate the \(M\) clauses \(d_1,\ldots,d_M\) with \(M\) users \(C \triangleq \{c_1,\ldots,c_M\}\). The entire set of users is thus

\[
U = \bigcup_n V_n \cup V_N \cup C,
\]

which contains a total of \(K = 5N + M\) users. For these \(K\) users, we consider a particular problem instance of (22) with the following specified system parameters:

\[
\begin{align*}
N_t = 2, \quad & \bar{R} = 1, \quad \alpha_u = 1, \quad P_u = 1, \quad \rho_u = \rho = 0.9, \quad \forall u \in U, \\
\sigma_{v_{n\ell}}^2 & = \begin{cases} \ln \rho^{-1}, & \ell = 0, \\
\ln \left(\frac{10}{\rho} \rho^{-1}\right), & \ell \neq 0, \end{cases}, \quad \sigma_{c_m}^2 = 0.01, \forall m, \\
Q_{u \ell} & = \begin{bmatrix} 1 & 0 \ 0 & 1 \end{bmatrix}, \quad \forall u \in U, \quad Q_{v_{n\ell},v_{n\ell}} = \begin{bmatrix} 1 & 0 \ 0 & 1 \end{bmatrix}, \quad \forall u, \forall \ell \neq 0, \\
Q_{v_{n0},v_{n4}} & = \begin{bmatrix} 1 & 0 \ 0 & 0 \end{bmatrix}, \quad \text{if } x_n \in \{y_{i_n}, y_{j_n}, y_{k_n}\}, \quad \forall n, \\
Q_{u_1,u_2} = 0, \quad \text{for all } u_1, u_2 \text{ not specified above},
\end{align*}
\]
where $w_{nl}^e$ and $w_{nl}^c$ denote the beamformers of users $v_{nl}$ and $c_n$, respectively. Next, we show that the given 3-SAT problem instance is satisfiable if and only if (28) is feasible.

To begin with, we show that any feasible point of problem (28) corresponds to a truth assignment satisfying the 3-SAT problem (26). Note that constraint (28b) can be written as

$$\|w_{nl}^e\|^2 \geq \sigma^2_{v_{nl}} (\ln \rho^{-1})^{-1} = 1, \ n = 1, \ldots, N,$$

where the equality comes from (27b). Combining constraints (28b) with (28e), we have $\|w_{nl}^e\|^2 = 1$ for $n = 1, \ldots, N$. Similarly, one can rewrite (28c) as

$$\|w_{nl}^c\|^2 \geq \sigma^2_{v_{nl}} (\ln \rho^{-1})^{-1} = 1,$$

$$\|w_{nl}^c\|^2 = \frac{1}{\rho} \exp \left( -\frac{\text{Trace}(w_{n\ell}^c w_{n\ell}^c))}{\|w_{n\ell}^e\|^2} \right) - 1,$$

where $n = 1, \ldots, N$, $\ell = 1, 2, 3, 4$. In addition, $w_{nl}^c$ must satisfy either $|w_{nl}^c| = 0$ or $|w_{nl}^c| = 1$. In other words, $w_{nl}^c$ must satisfy the second elements of $w_{nl}^e$, thus a feasible $w_{nl}^c$ must satisfy $|w_{nl}^c| = 0$ or $|w_{nl}^c| = 1$. In addition, constraints (28d) and (28e) imply that

$$\min_{\|w_{nl}^e\|^2 \leq 1} \min_{\|w_{nl}^c\|^2 \leq 1} \left\{ \rho e^2 \sum_{\tau_m \in \{i,m,j,m\}} \left( 1 + \frac{(w_{\tau_m}^0) H\sigma_{v_{\tau_m}\ell} c_m w_{\tau_m}^0}{\|w_{\tau_m}^c\|^2} \right) \right\} = \min_{\|w_{nl}^e\|^2 \leq 1} \min_{\|w_{nl}^c\|^2 \leq 1} \left\{ \rho e^{2c_m} \sum_{\tau_m \in \{i,m,j,m\}} \left( 1 + \frac{(w_{\tau_m}^0) H\sigma_{v_{\tau_m}\ell} c_m w_{\tau_m}^0}{\|w_{\tau_m}^c\|^2} \right) \right\} \leq 1. \ (34)$$

By combining (32), (33), and (34), we come up with the result that the feasible beamformers to problem (28) must satisfy

$$w_{nl}^e \in \left\{ \bigcup_{\theta \in [0,2\pi]} [e^{i\theta} 0]^T \right\} \bigcup \left\{ \bigcup_{\theta \in [0,2\pi]} [0 e^{i\theta}]^T \right\}, \ \forall n,$$

$$w_{nl}^c \in \left\{ \bigcup_{\theta \in [0,2\pi]} [e^{i\theta} 0]^T \right\} \bigcup \left\{ \bigcup_{\theta \in [0,2\pi]} [0 e^{i\theta}]^T \right\}, \ \forall n,$$

$$\|w_{nl}^e\|^2 = 1, \ \forall n, \ \forall \ell.$$
where \( \tau_m \in \{ i_m, j_m, k_m \}, \forall m = 1, \ldots, M \). Suppose that \( \bar{w}_w^n, \bar{w}_m^n, \bar{w}_n^n, n = 1, \ldots, N, m = 1, \ldots, M, \ell = 0, 1, \ldots, 4 \), are feasible to (28), and set
\[
x_n = \begin{cases} 1, & \text{if } \bar{w}_n^n \in \bigcup_{\theta \in [0, 2\pi)} [e^{\theta T}] \cup [0, e^{\theta T}] \\
0, & \text{if } \bar{w}_n^n \in \bigcup_{\theta \in [0, 2\pi)} [e^{\theta T}] \cup [0, e^{\theta T}] \end{cases}
\tag{37}
\]

Then, we have \((\bar{w}_{w}^\tau_{m0})^T Q_{v,t_{m0},c_{m}} \bar{w}_m^\tau_{m0} \) = 0 if and only if \( y_{m} = 1 \) according to (36). Furthermore, it can be verified by (27a), (27b) and (36) that condition (35b) is satisfied if and only if \((\bar{w}_w^\tau_{m0})^T Q_{v,t_{m0},c_{m}} \bar{w}_m^\tau_{m0} = 0 \), for some \( \tau_m \in \{ i_m, j_m, k_m \}, \forall m \), implying that all of the clauses \( d_1, \ldots, d_M \) are true. Hence, the truth assignment (37) satisfies the given 3-SAT problem instance (26).

On the other hand, suppose that the given 3-SAT problem (26) can be satisfied and \( \hat{x}_n \in \{0, 1\}, n = 1, \ldots, N, \), is a truth assignment such that the clauses \( d_1, \ldots, d_M \) are true, i.e., constraints (26b) are satisfied. Then, analogous to the above analysis, it can be verified that
\[
\bar{w}_w^n = \begin{cases} [1 0]^T, & \text{if } \hat{x}_n = 1, \forall n, \\
[0 1]^T, & \text{if } \hat{x}_n = 0, \forall n, \end{cases}
\tag{37}
\]

is feasible to problem (28). Thus, we have proven the equivalence of the given 3-SAT problem (26) and problem (28), namely, the 3-SAT problem is reducible to problem (22). As a result, determining the feasibility of problem (22) is NP-hard when \( N_t \geq 2 \), and hence solving problem (21) is also NP-hard according to Lemma 4.

Remark 1 The preceding proof of the NP-hardness of the feasibility problem (22) can also be analogously applied to show the NP-hardness of the following CoBF design problem for outage balancing [28]:
\[
\min_{\rho, \bar{w}_{c,1} \bar{w}_{c,L} \in [0,1], i = 1, \ldots, K} \frac{1}{\rho} \tag{38a}
\]
\[
s.t. \begin{cases}
\rho_i \exp \left( \frac{\sqrt{1 + \frac{(2R_i - 1)\rho_i^2}{w_{c_i}^T Q_{c_i} w_{c_i}}} - \frac{(2R_j - 1)w_{c_i}^T Q_{c_i} w_{c_i}}{\rho_i^2}}{1 + \frac{(2R_j - 1)w_{c_i}^T Q_{c_i} w_{c_i}}{\rho_i^2}} \right) \leq 1, & (38b) \\
\| \bar{w}_{c_i} \|^2 \leq P_i, & i = 1, \ldots, K, & (38c)
\end{cases}
\]

where \( \rho = \min\{\rho_1, \ldots, \rho_K\} \) and \( R_i \) is the given data rate of the \( i \)th user, \( i = 1, \ldots, K \). Then, following a similar argument as Lemma 4, one can show that problem (38) is solvable (e.g., by bisecting over \( \rho \in [0, 1] \)) if and only if the feasibility problem (22) (with \( \rho_i = \rho, \alpha_i = \bar{R}_i = R_i, i = 1, \ldots, K \)) is solvable. Therefore, the NP-hardness of problem (22) when \( N_t \geq 2 \) also implies that problem (38) is NP-hard for outage balancing.

V. CONCLUSIONS

In this paper, we have studied the complexity status of the outage constrained SRM and MMF CoBF problems. In particular, we have established the NP-hardness of the two problems by showing that they are at least as difficult as the Max-Cut problem and the 3-SAT problem, respectively. Besides, a subclass, i.e., the SISO case of the outage constrained MMF problem is identified polynomial-time solvable. Since the MMF CoBF problem is known polynomial-time solvable under perfect CSI, our result implies that the outage constrained CoBF design problems are indeed more challenging. Motivated by our complexity analysis results, efficient algorithms for obtaining high-quality approximate solutions to the outage constrained CoBF problems are further pursued in the companion paper [31].
2) Either \( f(p) > 0 \) for all \( 0 \leq p \leq 1 \), or \( f(p) < 0 \) for all \( 0 \leq p \leq 1 \), i.e., the function \( F(p) \) is either strictly increasing or strictly decreasing.

That is, \( F(p) \) is a quasiconvex function, and there exists at most one \( p \geq 0 \) such that \( dF(p)/dp = 0 \).

Note that the two conditions in (A.2) are actually equivalent. Hence, it suffices to prove (A.2a). To this end, let us introduce the following lemma.

**Lemma 5** Given \( \bar{p} \geq 0 \) fixed, the functions \( \zeta^v(p) \), which satisfies (14a), and \( \zeta^v(p, \bar{p}) = \zeta^v(p, \bar{p}) \), which satisfies (14b), are strictly decreasing for \( p \geq 0 \). Except for the functions \( p\zeta^v(p) \) and \( p\zeta^v(p, \bar{p}) \) are strictly increasing for \( p \geq 0 \).

**Proof**: See Appendix C.

Given any \( \bar{p} \geq 0 \) and based on Lemma 5, we can derive a lower bound \( g(\bar{p}|\bar{P}) \) of \( f(\bar{p}) \) in 2 for all \( p \geq \bar{p} \), which is given in (A.3) on the top of the next page. To obtain the first inequality in (A.3), one can observe that \( \frac{1}{1+\bar{p}\zeta^v(p)} \) and \( \frac{1}{1+\bar{p}\zeta^v(p, \bar{p})} \) in the second term are increased due to the fact that \( \zeta^v(p) \leq \zeta^v(\bar{P}) \), and \( \frac{x}{1+x} \) is nondecreasing for \( x \) in the interval \( (0, 1) \). Similar reasons also apply to the summation term since \( \zeta^v(p, \bar{p}) \leq \zeta^v(\bar{P}, \bar{p}) \) and \( p\zeta^v(p, \bar{p}) \geq \zeta^v(\bar{P}, \bar{p}) \) for all \( p \geq \bar{P} \). The second inequality in (A.3) is obtained by the monotonicity of \( \frac{x}{1+x} \) and \( \zeta^v(p, \bar{p}) \). Note that the monotonic properties of \( \zeta^v(p) \), \( \zeta^v(p, \bar{p}) \), \( p\zeta^v(p) \), \( p\zeta^v(p, \bar{p}) \) and \( \frac{x}{1+x} \) are strict, so the two inequalities in (A.3) hold with equality if and only if \( p = \bar{P} \); that is,

\[
\begin{align*}
\begin{cases}
(f(p) \ln 2 > g(\bar{p}|\bar{P}), & p > \bar{P}, \\
(f(p) \ln 2 = g(\bar{p}|\bar{P}), & p = \bar{P}.
\end{cases}
\end{align*}
\]  

(A.4)

Next, we show that \( g(\bar{p}|\bar{P}) \geq 0 \) for all \( p \geq \bar{P} \) if \( f(\bar{P}) \geq 0 \), which, together with (A.4) then implies (A.2a). By defining

\[
\begin{align*}
a_0 &= \sigma^2 + \sigma^2 \bar{P} \zeta^v(\bar{P}) \\
b_0 &= \bar{p} \zeta^v(\bar{P}) \\
a_1 &= (1 + \bar{P} \bar{p} \zeta^v(\bar{P}, \bar{p})) [\bar{p} \zeta^v(\bar{P}, \bar{p}) + \sigma^2 (1 + \bar{p} \bar{p} \zeta^v(\bar{P}, \bar{p}))] \\
&\quad + \bar{p} \bar{p} \zeta^v(\bar{P}, \bar{p}) \zeta^v(\bar{P}, \bar{p}) \\
b_1 &= 0.7 \bar{p} \bar{p} \zeta^v(\bar{P}, \bar{p}) \\
c_1 &= b_1 \cdot (1 + \bar{p} \bar{p} \zeta^v(\bar{P}, \bar{p})) \\
&\geq 0, \quad \forall j \in \{k|e_{ik} \in \mathcal{U}_c\},
\end{align*}
\]

which are independent of \( p \), one can respectively express \( g_0(\bar{p}|\bar{P}) \) and \( g_1(\bar{p}|\bar{P}) \) as

\[
\begin{align*}
g_0(\bar{p}|\bar{P}) &= \frac{b_0}{(1 + b_0)(a_0 + p)}, \\
g_1(\bar{p}|\bar{P}) &= \frac{\alpha_{e_{ik}} c_j}{(1 + b_1)(a_1 + p)}, \quad \forall j \in \{k|e_{ik} \in \mathcal{U}_c\}.
\end{align*}
\]  

(A.5a)

(A.5b)

Suppose that there exists \( 0 \leq \bar{P} \leq 1 \) satisfying \( f(\bar{P}) \geq 0 \). Then, by (A.4), we have \( f(\bar{P}) \ln 2 = g(\bar{P}|\bar{P}) = \frac{\zeta^v(\bar{P})}{1 + \bar{p} \zeta^v(\bar{P})} - \frac{g_0(\bar{P}|\bar{P}) - g_0(\bar{P}|\bar{P})}{\bar{P} \zeta^v(\bar{P}), \bar{P} \zeta^v(\bar{P})} \geq 0 \). Since \( \frac{\zeta^v(\bar{P})}{1 + \bar{p} \zeta^v(\bar{P})} > 0 \), \( g_0(\bar{P}|\bar{P}) \geq 0 \) and \( g_1(\bar{P}|\bar{P}) \geq 0 \) for all \( j \in \{k|e_{ik} \in \mathcal{U}_c\} \), we can further infer from \( g(\bar{P}|\bar{P}) \geq 0 \) that, when \( p = \bar{P} \),

\[
\begin{align*}
\beta_0 \cdot \left[ \zeta^v(\bar{P}) \right] - g_0(\bar{P}|\bar{P}) &= \left[ \zeta^v(\bar{P}) a_0 b_0 (1 + b_0) - b_0 \right] \\
&\left[ (1 + \bar{p} \zeta^v(\bar{P}))(1 + b_0)(a_0 + p) \right] \\
&\left[ \beta_0 (1 + b_0) - b_0 \right] \cdot \zeta^v(\bar{P}) [1 + \bar{p} \zeta^v(\bar{P})] (1 + b_0)(a_0 + p) \\
&\geq 0, \\
\beta_j \cdot \left[ \zeta^v(\bar{P}) \right] &= \left[ \zeta^v(\bar{P}) a_j b_0 (1 + b_0) - a_0 c_j \right] \\
&\left[ (1 + \bar{p} \zeta^v(\bar{P}))(1 + b_0)(a_0 + p) \right] \\
&\left[ \beta_0 (1 + b_0) - a_0 c_j \right] \cdot \zeta^v(\bar{P}) [1 + \bar{p} \zeta^v(\bar{P})] (1 + b_0)(a_0 + p) \\
&\geq 0, \quad \forall j \in \{k|e_{ik} \in \mathcal{U}_c\},
\end{align*}
\]

(A.6a)

(A.6b)

where

\[
\begin{align*}
\beta_0 &= \frac{g_0(\bar{P}|\bar{P}) + \sum_{\{k|e_{ik} \in \mathcal{U}_c\}} g_j(\bar{P}|\bar{P})}{\bar{P} \zeta^v(\bar{P})}, \\
\beta_j &= \frac{g_j(\bar{P}|\bar{P}) + \sum_{\{k|e_{ik} \in \mathcal{U}_c\}} g_k(\bar{P}|\bar{P})}{\bar{P} \zeta^v(\bar{P})}, \quad \forall j \in \{k|e_{ik} \in \mathcal{U}_c\},
\end{align*}
\]

and \( \beta_0 + \sum_{\{k|e_{ik} \in \mathcal{U}_c\}} \beta_j = 1 \). Next, we aim to show that \( \left[ \beta_0 (1 + b_0) - b_0 \right] \geq 0 \) and \( \left[ \beta_j (1 + b_0) - a_0 c_j \right] \geq 0 \), which, when applied to (A.6), imply that, \( \frac{\beta_0 \zeta^v(\bar{P})}{1 + \bar{p} \zeta^v(\bar{P})} - g_0(\bar{P}|\bar{P}) \geq 0 \) and each \( \beta_j \zeta^v(\bar{P}) - g_j(\bar{P}|\bar{P}) \geq 0 \) for all \( p \geq \bar{P} \). To this end, consider the following inequalities

\[
\begin{align*}
\tilde{p} e^s \zeta^v(1 + \bar{p} \zeta^v) &\geq \rho (1 + \bar{p} \zeta^v)(1 + \bar{p} \bar{p} \zeta^v), \\
\tilde{p} e^s \zeta^v(1 + \bar{p} \bar{p} \zeta^v) &\geq \rho (1 + \bar{p} \bar{p} \zeta^v)(1 + \bar{p} \bar{p} \zeta^v),
\end{align*}
\]  

(A.7a)

(A.7b)
for all $\zeta^c \geq 0$, $\zeta^c \geq 0$, $p \geq 0$ and $\bar{\rho}_{j1} \geq 0$, where the inequalities are owing to $e^x \geq 1 + x$ for all $x$. Since $\rho = 0.95 < 1$, for any $p \geq 0$, there must exist $\zeta^c(p) = \zeta^c(p) > 0$ such that

$$\rho(1 + \sigma^2 \zeta^c(p))(1 + p \bar{\zeta}^c(p)) = \rho(1 + \sigma^2 \zeta^c(p))(1 + p \bar{\zeta}^c(p)) = 1.$$  

Besides, by (14a), one has

$$\rho e^{\sigma^2 \zeta^c(p)}(1 + p \bar{\zeta}^c(p)) = 1,$$

$$\rho e^{\sigma^2 \zeta^c(p)}(1 + p \bar{\zeta}^c(p)) = 1,$$  

Thus, it follows from (A.7a) that $\zeta^c(p) \geq \zeta^c(p)$ for all $p \geq 0$. Therefore, for $0 \leq \bar{P} \leq 1$, we have

$$\zeta^c(\bar{\rho}_{j1}) a_0 = \zeta^c(\bar{\rho}_{j1}) \sigma^2 (1 + \bar{P} \zeta^c(\bar{P})) \leq \zeta^c(0) \sigma^2 (1 + \zeta^c(1)) \leq \ln \rho^{-1} \times (1 + \zeta^c(1))$$

$$\approx \ln \rho^{-1} \times \left(1 - \frac{1+\sigma^2}{2\sigma^2} + \frac{1+(\sigma^2)\sigma^2}{4\sigma^4} - \frac{1}{\sigma^2}\right) \approx 0.0537 < 1,$$

where the first inequality comes from Lemma 5, $\zeta^c(0) = \ln(1/\rho) / \sigma^2$ is obtained from (14a), and the approximate value $0.0537$ is obtained by using $\rho = 0.95$ and $\sigma^2 = 0.1$ in (12a).

By applying (A.8) to (A.6a), we conclude that

$$\beta_0(1 + b_0) - b_0 \geq 0;$$  

otherwise (A.6a) does not hold true. Similarly, we have

$$\zeta^c(p) \geq \zeta^c(p, \bar{\rho}_{j1}),$$

for all $p \geq 0$, $\bar{\rho}_{j1} \geq 0$,

$$\zeta^c(\bar{\rho}_{j1}) a_j = \zeta^c(\bar{\rho}_{j1}) [(1 + \bar{P} \zeta^c(\bar{P}, \bar{\rho}_{j1})) \bar{\rho}_{j1} + \sigma^2 (1 + \bar{\rho}_{j1} \zeta^c(\bar{P}, \bar{\rho}_{j1}))] \leq \zeta^c(0) [1 + \zeta^c(1, \bar{\rho}_{j1})] + \zeta^c(1, \bar{\rho}_{j1})] \leq \frac{1}{\sigma^2} \ln \rho^{-1} [1 + \zeta^c(1)] + \zeta^c(1) \approx 0.0537 < 1,$$

and thus

$$\beta_j(1 + b_j) - \alpha_{e_1, e_j} \geq 0, \quad \forall j \in \{k|e_j \in U_c\},$$

as inferred from (A.6b). By (A.6), (A.9) and (A.11), we have

$$\frac{\beta_j(1 + b_j) - \alpha_{e_1, e_j}}{1 + \rho e^{\sigma^2 \zeta^c(p)}} - g_j(p|\bar{P}) \geq 0, \quad \forall p \geq \bar{P}, \quad j \in \{0\} \cup \{k|e_j \in U_c\}.$$

By summing up the above inequalities, we have $g(p|\bar{P}) \geq 0$ for all $p \geq \bar{P}$. Hence, we have proved (A.2a), and the proof of Lemma 5 is complete.

**APPENDIX C**

**PROOF OF LEMMA 5**

Suppose that $0 \leq p_1 < p_2$. Then, we can obtain by (14a) that

$$1 = 0.95 e^{\sigma^2 (p_2)} (1 + \zeta^c(p_2)) p_2 = 0.95 e^{\sigma^2 (p_1)} (1 + \zeta^c(p_1)) p_1 < 0.95 e^{\sigma^2 (p_1)} (1 + \zeta^c(p_1)) p_1.$$  

Note that the above inequality is strict since $\zeta^c(p) > 0$ for any $p \geq 0$. Because the function $\rho e^{\sigma^2 (p)} (1 + \zeta^c(p)) p_1$ is strictly increasing in $\zeta^c(p)$, the above inequality implies that $\zeta^c(p_2) < \zeta^c(p_1)$, i.e., the function $\zeta^c(p)$ is a strictly decreasing function for $p \geq 0$. Accordingly, we can further obtain

$$1 = 0.95 e^{\sigma^2 (p_2)} (1 + \zeta^c(p_2)) p_2 = 0.95 e^{\sigma^2 (p_1)} (1 + \zeta^c(p_1)) p_1 > 0.95 e^{\sigma^2 (p_1)} (1 + \zeta^c(p_1)) p_1,$$

which implies $\zeta^c(p_2) p_2 > \zeta^c(p_1) p_1$, namely, $p \zeta^c(p)$ is strictly increasing for $p \geq 0$. The remaining statements about $\zeta^c(\bar{p})$ and $p \zeta^c(p, \bar{p})$ in Lemma 5 can be proved similarly.

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