RADIUS PROBLEMS CONCERNING THE MA-MINDA TYPE STARLIKE CLASS ASSOCIATED WITH A NEPHROID DOMAIN

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ABSTRACT. Let $\mathcal{A}$ be the class of analytic functions $f(z)$ defined on the open unit disk $\mathbb{D}$ and satisfying $f(0) = f'(0) - 1 = 0$. Let $S^*_N$ be the collection of $f \in \mathcal{A}$ such that the quantity $zf'(z)/f(z)$ assumes values from the range of the Carathéodory function $\varphi_N(z) = 1 + z - z^3/3 (z \in \mathbb{D})$, which is the interior of the nephroid given by

$$\left((u - 1)^2 + v^2 - \frac{4}{9}\right)^3 - \frac{4v^2}{3} = 0.$$ 

In this paper, we find sharp $S^*_N$-radii for several geometrically defined function classes introduced in the recent past. In particular, $S^*_N$-radius for the starlike class $S^*$ is found to be $1/4$. Moreover, radii problems related to the families defined in terms of ratio of functions are also discussed.

1. Introduction

Let $\mathbb{C}$ be the complex plane and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. Let $\mathcal{H} := \mathcal{H}(\mathbb{D})$ be the totality of analytic functions defined on $\mathbb{D}$, and let $\mathcal{A}$ be the set of those $f \in \mathcal{H}$ which satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Let $S \subset \mathcal{A}$ be the family of univalent functions and $S^*(\alpha) \subset S$ be the class of starlike functions of order $\alpha (0 \leq \alpha < 1)$ defined as

$$S^*(\alpha) := \left\{ f \in \mathcal{A} : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha, \ z \in \mathbb{D} \right\}.$$ 

Further, let us define the class $C(\alpha)$ by the relation: $f \in C(\alpha)$ if and only if $zf' \in S^*(\alpha)$. The classes $S^* := S^*(0)$ and $C := C(0)$ consist of functions that are, respectively, starlike and convex in $\mathbb{D}$. For $f, g \in \mathcal{H}$, we say that $f$ is subordinate to $g$, written $f \prec g$, if there exists a function $w \in \mathcal{H}$ satisfying $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Indeed, if $f \prec g$ implies that $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Furthermore, if the function $g(z)$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Using subordination, Ma and Minda [12] defined the function class $S^*(\varphi)$ as

$$S^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}, \quad (1.1)$$

where the analytic function $\varphi : \mathbb{D} \to \mathbb{C}$ satisfies (i) $\varphi(z)$ is univalent with positive real part, (ii) $\varphi(z)$ maps $\mathbb{D}$ onto a region that is starlike with respect to $\varphi(0) = 1$, (iii) $\varphi(\mathbb{D})$ is symmetric about the real axis and (iv) $\varphi'(0) > 0$. Clearly, from the properties of the function $\varphi(z)$ and the definition (1.1), it follows that $S^*(\varphi)$ is always a subclass of the class of starlike functions $S^*$ and $S^*(\varphi) = S^*$ whenever $\varphi(z) = (1 + z)/(1 - z)$, the function which maps $\mathbb{D}$ onto the right-half plane $\text{Re} \ w > 0$. Specializing the function $\varphi(z)$ in (1.1) gives us numerous classes of functions available in the literature. We call the classes having

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such a nice representation as Ma-Minda type starlike classes. For instance, the Ma-Minda type representation of the starlike class of order $\alpha$ is $S^*(\alpha) := S^*(1 + (1 - 2\alpha)z)/(1 - z)$, the Janowski starlike class is $S^*[A, B] := S^*(1 + Az)/(1 + Bz) \ (-1 \leq B < A \leq 1)$, the class associated with the lemniscate of Bernoulli (see [22]) is $S_\alpha^* := S^*(\sqrt{1 + z})$. Below we mention a few of the recently introduced Ma-Minda type classes which will be later used in our discussion.

(a) $S_{RL}^* := S^*(\varphi_{RL})$ with
\[
\varphi_{RL}(z) = \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1 - z}{1 + 2(\sqrt{2} - 1)z}},
\]
was considered by Mendiratta et al. [14]. The function $\varphi_{RL}(z)$ maps $\mathbb{D}$ onto the region enclosed by the left-half of the shifted lemniscate of Bernoulli $((u - \sqrt{2})^2 + v^2)^{\frac{1}{2}} - 2((u - \sqrt{2})^2 + v^2) = 0$.

(b) $S_{RL}^* := S^*(z + \sqrt{1 + z^2})$ was introduced by Raina and Sokol [17]. The function $\varphi_{RL}(z) = z + \sqrt{1 + z^2}$ maps $\mathbb{D}$ onto the crescent shaped region $\{w \in \mathbb{C} : |w^2 - 1| < 2|w|, \ Re\ w > 0\}$.

(c) $S_\beta^* := S^*(e^z)$ was introduced and discussed by Mendiratta et al. [15].

(d) $S_\alpha^* := S^*(1 + 4z/3 + 2z^2/3)$ is the class associated with the cardioid $(9u^2 + 9v^2 - 18u + 5)^2 - 16(9u^2 + 9v^2 - 6u + 1) = 0$, a heart-shaped curve. It was introduced and investigated by Sharma et al. [21].

(e) $S_R^* := S^*(\varphi_0)$, where $\varphi_0(z)$ is the rational function
\[
\varphi_0(z) = 1 + \frac{z}{k - z} = 1 + \frac{1}{k}z + \frac{2}{k^2}z^2 + \frac{2}{k^3}z^3 + \cdots, \quad k = \sqrt{2} + 1,
\]
was discussed by Kumar and Ravichandran [11].

(f) $S_{lim}^* := S^*(1 + \sqrt{2}z + z^2/2)$ is associated with the limacon $(4u^2 + 4v^2 - 8u - 5)^2 + 8(4u^2 + 4v^2 - 12u - 3) = 0$ and was considered by Yunus et al. [25].

(g) Kargar et al. [9] discussed the following starlike class associated with the Booth lemniscate:
\[
BS(\alpha) := S^*(1 + \frac{z}{1 - \alpha z^2}), \quad 0 \leq \alpha < 1.
\]

(h) $S_\alpha^* := S^*(1 + \sin z)$ was introduced by Cho et al. [4].

(i) Recently, Khattar et al. [10] introduced and discussed in detail the classes $S_{\alpha, e}^* := S^*(\alpha + (1 - \alpha)e^z)$ and $S_{\alpha, L}^* := S^*(\alpha + (1 - \alpha)\sqrt{1 + z})$, $0 \leq \alpha < 1$.

Clearly, for $\alpha = 0$, these classes reduce to $S_e^*$ and $S_L^*$, respectively.

(j) Very recently, Wani and Swaminathan [24] introduced the function class
\[
S_{Ne}^* := S^*(1 + z - z^3/3),
\]
and proved that the function $\varphi_{Ne}(z) := 1 + z - z^3/3$ maps $\mathbb{D}$ onto the interior of a 2-cusped curve called nephroid given by
\[
\left((u - 1)^2 + v^2 - \frac{4}{9}\right)^3 - \frac{4u^2}{3} = 0.
\]

Apart from studying several characteristic properties of the domain bounded by the nephroid (1.2), the authors in [24] discussed certain inclusion results, subordination results, Fekete-Szegö problem etc. related to the class $S_{Ne}^*$. In this paper, we consider the function class $S_{Ne}^*$ and solve several radii problems.
Radius Problem. Let $F$ and $G$ be two subclasses of $A$. Then the $F$-radius for the class $G$ is the largest number $\rho \in (0, 1)$ such that $r^{-1}f(rz) \in F$ for all $f \in G$, where $0 < r \leq \rho$. The problem of finding the number $\rho$ is called a radius problem in geometric function theory. For brevity, we write $R_F(G) = \rho$, if $\rho$ is the $F$-radius of $G$. For example, see [7, Chapter 13], $R_{S^*}(S) = \tanh(\pi/4) \approx 0.655$, $R_C(S) = R_C(S^*) = 2 - \sqrt{3} \approx 0.267$, $R_C(\alpha)(S^*) = (2 - \sqrt{3 + \alpha^2})/(1 + \alpha)$ and $R_C(\alpha)(C) = (1 - \alpha)/(1 + \alpha)$. Solving the radius problems is continuing to be an active area of research. For recent results in this direction, see [1, 2, 3, 4, 10, 11, 14, 15, 18, 21] and the references therein.

For our convenience, we give a simple geometric interpretation to the above definition in terms of $S_{Ne}^*$ as:

Definition 1.1. Let $D_r := \{z : |z| < r\}$ and $\Omega_{Ne} := \varphi_{Ne}(D)$. By $S_{Ne}^*$-radius for the class $G \subset A$, denoted by $R_{S_{Ne}^*}(G)$, we mean the largest number $\rho \in (0, 1)$ such that each $f \in G$ satisfies

$$L(D_r) \subset \Omega_{Ne} \text{ for every } r \leq \rho, \text{ where } L(z) = \frac{zf'(z)}{f(z)}.$$ 

2. A Preliminary Lemma

Lemma 2.1. Let $1/3 < a < 5/3$, and let $r_a$ be given by

$$r_a = \begin{cases} a - \frac{1}{3}, & \frac{1}{3} < a \leq 1 \\ \frac{5}{3} - a, & 1 \leq a < \frac{5}{3}. \end{cases}$$
Then
\[ \{ w \in \mathbb{C} : |w - a| < r_a \} \subseteq \Omega_N. \]

**Proof.** For \( z = e^{it} \), the parametric equations of the nephroid \( \varphi_N(z) = 1 + z - z^3/3 \) are
\[ u(t) = 1 + \cos t - \frac{1}{3} \cos 3t, \quad v(t) = \sin t - \frac{1}{3} \sin 3t, \quad -\pi < t \leq \pi. \]

Therefore, the square of the distance from the point \((a, 0)\) to the points on the curve (1.2) is given by
\[ z(t) = (a - u(t))^2 + (v(t))^2 \]
\[ = \frac{16}{9} + (a - 1)^2 - 4(a - 1) \cos t - \frac{4}{3} \cos^2 t + \frac{8}{3} (a - 1) \cos^3 t \]
\[ = \frac{16}{9} + (a - 1)^2 - 4(a - 1)x - \frac{4}{3} x^2 + \frac{8}{3} (a - 1)x^3 =: H(x), \]
where \( x = \cos t, -\pi < t \leq \pi \). Since the nephroid is symmetric about the real axis, it is sufficient to consider \( 0 \leq t \leq \pi \). A simple computation shows that \( H'(x) = 0 \) if and only if
\[ x = x_0 = \frac{1 - \sqrt{1 + 18(a - 1)^2}}{6(a - 1)} \quad \text{and} \quad x = x_1 = \frac{1 + \sqrt{1 + 18(a - 1)^2}}{6(a - 1)}. \]

![Figure 2. (i) Graph of \( x_0 = x_0(a) \), (ii) Graph of \( x_1 = x_1(a) \), where \( a \in (1/3, 5/3) \).](image)

It is clear that, for \( 1/3 < a < 5/3 \), only the number \( x_0 = \left( 1 - \sqrt{1 + 18(a - 1)^2} \right) / 6(a - 1) \) lies between \(-1\) and \(1\). Moreover,
\[ H''(x_0) = -\frac{8}{3} \sqrt{1 + 18(a - 1)^2} < 0 \quad \text{for each } a. \]

This shows that \( x_0 \) is the point of maxima for the function \( H(x) \) and hence, \( H(x) \) is increasing in the interval \([ -1, x_0 ]\) and decreasing in \([x_0, 1]\). Therefore, for \( 1/3 < a < 5/3 \), we have
\[ \min_{0 \leq t \leq \pi} z(t) = \min \{ H(-1), H(1) \}. \]
Since $H(1) - H(-1) = -\frac{8(a-1)}{3}$, so that $H(-1) \leq H(1)$ whenever $a \leq 1$ and $H(1) \leq H(-1)$ whenever $a \geq 1$. Thus, we conclude that
\[
r_a = \min_{0 \leq t \leq \pi} \sqrt{z(t)} = \begin{cases} \sqrt{H(-1)} = a - \frac{1}{3}, & \text{whenever } \frac{1}{3} < a \leq 1 \\ \sqrt{H(1)} = \frac{5}{3} - a, & \text{whenever } 1 \leq a < \frac{5}{3}. \end{cases}
\]
This further implies that the disk $\{w \in \mathbb{C} : |w - a| < r_a\}$ completely lies inside the region $\Omega_{\text{Ne}} = \varphi_{\text{Ne}}(\Delta)$. This completes the proof of the lemma. \hfill $\square$

3. $S_{\text{Ne}}^*$-Radius of Several Important Families

For $-1 \leq B < A \leq 1$, let $\mathcal{P}[A, B]$ be the collection of analytic functions $p : \Delta \to \mathbb{C}$ that are of the form $p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ and satisfy the subordination $p(z) \prec (1 + A z)/(1 + B z)$. Further, we set $\mathcal{P}[1 - 2\alpha, -1] = \mathcal{P}(\alpha)$ ($0 \leq \alpha < 1$) and $\mathcal{P}(0) = \mathcal{P}$. Also, recall [8] that the Janowski starlike class $S^*[A, B]$ consists of functions $f \in S$ satisfying $zf'(z)/f(z) \in \mathcal{P}[A, B]$.

Lemma 3.1 ([13, Lemma 1, p. 514]). If $p \in \mathcal{P}$, then for $|z| = r < 1$
\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2r}{1 - r^2}.
\]

Lemma 3.2 ([18, Lemma 2.1, p. 267]). If $p \in \mathcal{P}[A, B]$, then for $|z| = r < 1$
\[
\left| p(z) - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leq \frac{(A - B)r}{1 - B^2r^2}.
\]

For the particular case $p \in \mathcal{P}(\alpha)$, we get
\[
\left| p(z) - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{2(1 - \alpha)r}{1 - r^2}.
\]

Theorem 3.1. Let $0 \leq B < A \leq 1$. Then the sharp $S_{\text{Ne}}^*$-radius for the class $S^*[A, B]$ is given by
\[
\mathcal{R}_{S_{\text{Ne}}^*} (S^*[A, B]) = \min \left\{ 1, \frac{2}{3A - B} \right\}.
\]
In particular, if $1 - B \leq 3(1 - A)$, then $S^*[A, B] \subset S_{\text{Ne}}^*$.

Proof. Let $f \in S^*[A, B]$. Then $zf'(z)/f(z) \in \mathcal{P}[A, B]$ and Lemma 3.2 gives
\[
\frac{|zf'(z)/f(z) - 1 - ABr^2|}{1 - B^2r^2} \leq \frac{(A - B)r}{1 - B^2r^2}.
\]
(3.1)
The inequality (3.1) represents a disk with center $\frac{1 - ABr^2}{B^2r^2}$ and radius $\frac{(A - B)r}{B^2r^2}$. Since $B \geq 0$, we have $(1 - ABr^2)/(1 - B^2r^2) \leq 1$. In view of Lemma 2.1, the disk (3.1) lies inside the region $\Omega_{\text{Ne}}$ bounded by the nephroid (1.2) if
\[
\frac{(A - B)r}{1 - B^2r^2} \leq \frac{1 - ABr^2}{1 - B^2r^2} - \frac{1}{3}.
\]
Simplifying the above inequality, we obtain $r \leq 2/(3A - B)$. For sharpness of the estimate, consider the function
\[
f(z) = \begin{cases} \frac{z (1 + Bz)^{\frac{4B}{3}}}{ze^{Az}}, & \text{if } B \neq 0 \\ 0, & \text{if } B = 0. \end{cases}
\]
The function \( \tilde{f}(z) \) satisfies
\[
\frac{zf'(z)}{f(z)} = \frac{1+Az}{1+Bz},
\]
implying that \( \tilde{f} \in S^*[A, B] \). Also
\[
\frac{zf'(z)}{f(z)} \bigg|_{z=z_0} = \frac{1}{3}, \quad z_0 = -\frac{2}{3A-B}
\]
shows that the result is sharp for the function \( \tilde{f} \in S^*[A, B] \). In particular, if \( 1-B \leq 3(1-A) \), then \( 2/(3A-B) \geq 1 \) and we have \( R_{S^*_{Ne}}(S^*[A,B]) = 1 \). Thus, \( S^*[A,B] \subset S^*_{Ne} \). \( \square \)

**Remark 1.** For the case \( 0 \leq B < A \leq 1 \), the inequality \( 1-B \leq 3(1-A) \) is sufficient to conclude that \( S^*[A,B] \subset S^*_{Ne} \). In fact, \( 1-B \leq 3(1-A) \) implies \( 1-B^2 \leq 3(1-AB+(B-A)) < 3(1-AB) \) and, \( 0 \leq B < A \leq 1 \) gives \( 3(1-AB) \leq 3(1-B^2) \). Combining, we obtain \( 1-B^2 < 3(1-AB) \leq 3(1-B^2) \). Now the inclusion relation \( S^*[A,B] \subset S^*_{Ne} \) follows from [24, Theorem 3.3].

**Theorem 3.2.** Let \(-1 \leq B < A \leq 1 \) with \( B \leq 0 \). Then the sharp \( S^*_{Ne} \)-radius for the class \( S^*[A,B] \) is given by
\[
R_{S^*_{Ne}}(S^*[A,B]) = \min\left\{1, \frac{2}{3A-5B}\right\}.
\]
In particular, if \( 3(1+A) \leq 5(1+B) \), then \( S^*[A,B] \subset S^*_{Ne} \).

**Proof.** As in Theorem 3.1, \( f \in S^*[A,B] \) implies
\[
\frac{zf'(z)}{f(z)} - \frac{1-ABr^2}{1-B^2r^2} \leq \frac{(A-B)r}{1-B^2r^2}.
\]
Since \( B \leq 0 \), the number \( (1-ABr^2)/(1-B^2r^2) \geq 1 \). Hence, in view of Lemma 2.1, the above disk lies in \( \Omega_{Ne} \) if
\[
\frac{(A-B)r}{1-B^2r^2} \leq \frac{5}{3} - \frac{1-ABr^2}{1-B^2r^2}.
\]
Solving, we get \( r \leq 2/(3A-5B) \). The result is sharp for the function \( \tilde{f}(z) \in S^*[A,B] \) defined in Theorem 3.1, as \( zf'(z)/\tilde{f}(z) \) assumes the value \( 5/3 \) at the point \( z_0 = 2/(3A-5B) \). \( \square \)

**Corollary 3.1.** The sharp \( S^*_{Ne} \)-radius for the class \( S^*([\alpha]) = S^*[1-2\alpha,-1] \) is \( \frac{2}{3(1-2\alpha)+\alpha} \). The estimate is sharp for the function \( k_\alpha(z) = z(1-z)^{2\alpha-2} \).

**Corollary 3.2.** The sharp \( S^*_{Ne} \)-radius for the class \( S^* \) is \( \frac{1}{4} \) and the sharpness holds for the Koebe function \( k(z) = z/(1-z)^2 \).

**Corollary 3.3.** The sharp \( S^*_{Ne} \)-radius for the convex class \( C \) is \( \frac{2}{5} \).

**Proof.** From Theorem 3.2, the \( S^*_{Ne} \)-radius for the class \( S^* \left( \frac{1}{2} \right) = S^*[0,-1] \) is \( \frac{2}{5} \). Using the fact that \( C \subset S^* \left( \frac{1}{2} \right) \) [16, Theorem 2.6a, p. 57], it follows that \( R_{S^*_{Ne}}(C) \) is at least \( 2/5 \). Further, at the point \( z = 2/5 \), the function \( \ell(z) = z/(1-z) \in C \) satisfies
\[
\frac{z\ell'(z)}{\ell(z)} = \frac{1}{1-2/5} = \frac{5}{3},
\]
showing that \( R_{S^*_{Ne}}(C) \leq 2/5 \). Hence, the estimate \( R_{S^*_{Ne}}(C) = 2/5 \) is sharp. \( \square \)

For \( \beta > 1 \), consider the class of functions \( M(\beta) \) introduced by Uralegaddi et al. [23] as
\[
M(\beta) := \left\{ f \in A : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \beta, \ z \in \mathbb{D} \right\}.
\]
For this interesting class, we prove the following radius result.
Theorem 3.3. The sharp $S_{Ne}^*$-radius for the class $\mathcal{M}(\beta)$ is
\[
R_{S_{Ne}^*}(\mathcal{M}(\beta)) = \frac{1}{3\beta - 2}.
\]
Proof. Let $f \in \mathcal{M}(\beta)$. Then $zf'(z)/f(z) \in \mathcal{P}[1 - 2\beta, -1]$, and hence from Lemma 3.2 we get the following inequality
\[
\left| \frac{zf'(z)}{f(z)} - \frac{1 + (1 - 2\beta)r^2}{1 - r^2} \right| \leq \frac{2(\beta - 1)r}{1 - r^2}. \tag{3.2}
\]
For each $\beta > 1$ and $r \in (0, 1)$,
\[
\frac{1 + (1 - 2\beta)r^2}{1 - r^2} < 1.
\]
Therefore from Lemma 2.1 and the disk (3.2), it follows that the quantity $zf'(z)/f(z)$ takes values from the interior of the nephroid (1.2) if
\[
\frac{2(\beta - 1)r}{1 - r^2} \leq \frac{1 + (1 - 2\beta)r^2}{1 - r^2} - \frac{1}{3},
\]
or equivalently $(3\beta - 2)r^2 + 3(\beta - 1)r - 1 \leq 0$. The last inequality further gives $r \leq 1/(3\beta - 2)$. For sharpness, consider the function $f_0(z) = z(1 - z)^{2\beta - 2} \in \mathcal{M}(\beta)$. It is easy to verify that $zf_0'(z)/f_0(z) = 1/3$ for $z = (3\beta - 2)^{-1}$. Therefore, the radius cannot be increased further. \qed

The class of close-to-starlike functions of type $\alpha$ ($0 \leq \alpha < 1$) introduced by Reade [19] is defined as
\[
CS^*(\alpha) := \left\{ f \in \mathcal{A} : \frac{f}{g} \in \mathcal{P}, g \in S^*(\alpha) \right\}.
\]
Theorem 3.4. The sharp $S_{Ne}^*$-radius for the class $CS^*(\alpha)$ is given by
\[
R_{S_{Ne}^*}(CS^*(\alpha)) = \rho_0 = \frac{2}{6 - 3\alpha + \sqrt{52 - 48\alpha + 9\alpha^2}}.
\]
Proof. Let $f \in CS^*(\alpha)$. Then, for some $g \in S^*(\alpha)$, $h(z) = \frac{f(z)}{g(z)} \in \mathcal{P}$. Applying Lemma 3.1, we have
\[
\left| \frac{zh'(z)}{h(z)} \right| \leq \frac{2r}{1 - r^2}. \tag{3.3}
\]
Since $g \in S^*(\alpha)$ implies $\frac{g'(z)}{g(z)} \in \mathcal{P}(\alpha)$, Lemma 3.2 gives
\[
\left| \frac{zg'(z)}{g(z)} - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{2(1 - \alpha)r}{1 - r^2}. \tag{3.4}
\]
In view of the identity
\[
\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)},
\]
it follows from (3.3) and (3.4) that
\[
\left| \frac{zf'(z)}{f(z)} - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{2(2 - \alpha)r}{1 - r^2}. \tag{3.5}
\]
The inequality (3.5) represents a disk with center $a := \frac{1 + (1 - 2\alpha)r^2}{1 - r^2}$ and radius $r_0 := \frac{2(2 - \alpha)r}{1 - r^2}$. As $a > 1$, we conclude from Lemma 2.1 that the disk (3.5) lies inside the region $\Omega_{Ne}$. \qed
bounded by the nephroid (1.2) if \( r_0 \leq \frac{3}{8} - a \). That is, if \((4 - 3\alpha)r^2 + 3(2 - \alpha)\rho - 1 \leq 0 \). This on solving yields \( r \leq \rho_0 \). For sharpness, consider the functions \( f, g \in A \) defined as
\[
f(z) = \frac{z(1+z)}{(1-z)^{3-2\alpha}} \quad \text{and} \quad g(z) = \frac{z}{(1-z)^{2-2\alpha}}.
\]
Since \( f(z)/g(z) = (1+z)/(1-z) \in P \) and \( g \in S^*(\alpha) \) [6, p. 141], the function \( f \in CS^*(\alpha) \).

Also, at the point \( z = \rho_0 \), \( zf'(z)/f(z) \) assumes the value 5/3. This proves the sharpness of the result.

Define the class \( W \) as
\[
W := \left\{ f \in A : \frac{f}{z} \in P \right\}.
\]

Introduced first by MacGregor [13], this class was recently studied for radius problems in [1, 4, 14, 15]. Following them, we prove the following sharp radius result related to the class \( W \).

**Theorem 3.5.** The sharp \( S^*_N \)-radius for the class \( W \) is given by
\[
R_{S^*_N}(W) = \frac{2}{3 + \sqrt{13}} \approx 0.302776.
\]

**Proof.** If \( f \in W \), then \( h(z) = \frac{f(z)}{z} \in P \) and hence Lemma 3.1 gives
\[
\left| \frac{zh'(z)}{h(z)} \right| \leq \frac{2r}{1 - r^2}.
\]
On using the above inequality in the identity
\[
\frac{zf'(z)}{f(z)} = 1 + \frac{zh'(z)}{h(z)},
\]
we obtain
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{2r}{1 - r^2}.
\]
Now using Lemma 2.1, it follows that \( f \in S^*_N \) if \( 2r/(1-r^2) \leq 2/3 \), i.e., if \( r \leq 2/(3 + \sqrt{13}) \).

For the function \( f(z) = z(1+z)/(1-z) \) satisfying \( f(z)/z \in P \), we have
\[
\left| \frac{zf'(z)}{f(z)} \right|_{z = \frac{2}{3 + \sqrt{13}}} = \frac{1}{3} \quad \text{and} \quad \left| \frac{zf'(z)}{f(z)} \right|_{z = \frac{2}{3 + \sqrt{13}}} = \frac{5}{3}.
\]
This demonstrates that the radius estimate is sharp for the function \( f(z) = z(1+z)/(1-z) \) \( \in W \).

Before proving the next result, we note that \( S^*_L(\alpha) \subset S^*_N \) for \( \alpha \geq 1/3 \) and \( S^*_L \subset S^*_N \)
for \( \alpha \geq \frac{3e-5}{3e-3} \); see Wani and Swaminathan [24, Theorem 3.1].

**Theorem 3.6.** For the function classes \( BS(\alpha) \), \( S^*_L(\alpha) \), and \( S^*_L \) introduced in Section 1, we have the following radius results:

(i) For \( 0 \leq \alpha < 1 \), \( R_{S^*_N}(BS(\alpha)) = R_{S^*_L} = \frac{1}{3 + \sqrt{9 + 16\alpha}} \).
(ii) For \( 0 \leq \alpha \leq 1/3 \), \( R_{S^*_N}(S^*_L(\alpha)) = \frac{4(2-3\alpha)}{9(1-\alpha)^2} \). In particular, \( R_{S^*_N}(S^*_L) = \frac{\pi}{3} \).
(iii) For \( 0 \leq \alpha \leq \frac{3e-5}{3e-3} \), \( R_{S^*_N}(S^*_L(\alpha, e)) = \log \left( \frac{5-3\alpha}{3-3\alpha} \right) \). In particular, \( R_{S^*_N}(S^*_L) = \log \left( \frac{\pi}{3} \right) \approx 0.510826 \).

The estimate in each part is sharp.
Proof. (i) Let \( f \in BS(\alpha) \). Then \( \frac{zf'(z)}{f(z)} \prec G_\alpha(z) = 1 + \frac{z}{1 - \alpha z^2} \) and hence, for \( |z| = r \), we have
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{z}{1 - \alpha z^2} \right| \leq \frac{r}{1 - \alpha r^2}.
\]
In view of Lemma 2.1, the disk (3.6) lies completely in \( \Omega_N \) if \( r/(1 - \alpha r^2) \leq 2/3 \). This gives \( r \leq 4/(3 + \sqrt{9 + 16\alpha}) = \mathcal{R}_\alpha \). For sharpness, consider the function
\[
f_B(z) = z \left( \frac{1 + \sqrt{\alpha z}}{1 - \sqrt{\alpha z}} \right)^{1/(2\sqrt{\alpha})}.
\]
It can be easily verified that \( \frac{zf'_B(z)}{f_B(z)} = G_\alpha(z) \) and hence \( f_B \in BS(\alpha) \). Also, a straightforward calculation shows that
\[
\left| \frac{zf'_B(z)}{f_B(z)} \right|_{z = \mathcal{R}_\alpha} = \frac{1}{3} \quad \text{and} \quad \left| \frac{zf'_B(z)}{f_B(z)} \right|_{z = \mathcal{R}_\alpha} = \frac{5}{3}.
\]
This proves that the estimate is best possible (see Figure 3).

\[
\text{(i) } \alpha = 0.9 \quad \text{(ii) } \alpha = 0.1
\]

\textbf{Figure 3.} Sharpness of \( \mathcal{R}_\alpha \). The Shaded region is \( G_\alpha(|z| < \mathcal{R}_\alpha) \) with (i) \( \alpha = 0.9 \) and (ii) \( \alpha = 0.1 \).

(ii) Let \( f \in S_1^*(\alpha) \). Then \( \frac{zf'(z)}{f(z)} \prec \alpha + (1 - \alpha)\sqrt{1 + z} \) and hence
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \alpha + (1 - \alpha)\sqrt{1 + z} - 1 \right| = (1 - \alpha)\left| 1 - \sqrt{1 + z} \right|
\]
An application of Lemma 2.1 shows that $f \in S_{N_e}$ if $(1 - \alpha)(1 - \sqrt{1 - r}) \leq 2/3$, which on simplification gives $r \leq \mathcal{R}_{S_{N_e}}(S^*_L(\alpha))$. For sharpness, consider the function

$$f(z) = z + (1 - \alpha)z^2 + \frac{1}{16}(1 - \alpha)(1 - 2\alpha)z^3 + \cdots$$

which satisfies $zf'(z)/f(z) = \alpha + (1 - \alpha)\sqrt{1 + z}$ and hence is a member of $S^*_L(\alpha)$. Verification shows that $zf'(z)/f(z)$ assumes the value 1/3 at the point $z = -4(2 - 3\alpha)/(1 - \alpha)^2$. Further, taking $\alpha = 0$ yields the sharp radius estimate for the class $S^*_L$.

(iii) $f \in S^*_{\alpha,e}$ implies $zf'(z)/f(z) < \alpha + (1 - \alpha)e^z$. For $|z| = r$, this gives

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq (1 - \alpha)|e^z - 1| \leq (1 - \alpha)(e^r - 1) \leq \frac{2}{3}$$

if $r \leq \log\left(\frac{5 - 3\alpha}{3 - 3\alpha}\right) = \mathcal{R}_{S_{N_e}}(S^*_{\alpha,e})$. The result is sharp for the function $g(z) \in S^*_{\alpha,e}$ defined as

$$g(z) = z + (1 - \alpha)z^2 + \frac{1}{4}(1 - \alpha)(3 - 2\alpha)z^3 + \cdots$$

On taking $\alpha = 0$, we obtain the sharp radius estimate for the class $S^*_e$.

**Theorem 3.7.** For the function classes $S^*_{RL}$, $S^*_C$, $S^*_R$ and $S^*_lim$ introduced in Section 1, we have:

(i) $\mathcal{R}_{S^*_{N_e}}(S^*_{RL}) = \frac{56}{122 - 41\sqrt{2}} \approx 0.874764$.

(ii) $\mathcal{R}_{S^*_{N_e}}(S^*_C) = \sqrt{2} - 1 \approx 0.414214$.

(iii) $\mathcal{R}_{S^*_{N_e}}(S^*_R) = \frac{1}{3\sqrt{2} - 3} \approx 0.804738$.

(iv) $\mathcal{R}_{S^*_{N_e}}(S^*_lim) = \frac{2\sqrt{2}}{3 + \sqrt{15}} \approx 0.411528$.

The estimate in each part is best possible.

**Proof.** (i) Let $f \in S^*_{RL}$. Then

$$\frac{zf'(z)}{f(z)} < \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1 - z}{1 + 2(\sqrt{2} - 1)z}}.$$ 

Therefore, for $|z| = r < 1$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1 - z}{1 + 2(\sqrt{2} - 1)z}} - 1 \leq 1 - \left(\sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1 + r}{1 - 2(\sqrt{2} - 1)z}}\right)$$

In view of Lemma 2.1, the above disk lies inside $\Omega_{N_e}$ provided

$$1 - \left(\sqrt{2} - (\sqrt{2} - 1)\sqrt{\frac{1 + r}{1 - 2(\sqrt{2} - 1)z}}\right) \leq \frac{2}{3},$$

or equivalently $r \leq \frac{56}{122 - 41\sqrt{2}} = \mathcal{R}_{S^*_{N_e}}(S^*_{RL})$. The result is sharp for the function $f_0 \in S^*_{RL}$ defined as

$$f_0(z) = z \left(\sqrt{1 - z} + \frac{\sqrt{1 + 2(\sqrt{2} - 1)z}}{2}\right)^{2\sqrt{2} - 2} \times \exp(g_0(z))$$
where
\[ g_0(z) = \sqrt{2(\sqrt{2} - 1)} \times \tan^{-1}\left( \frac{\sqrt{2(\sqrt{2} - 1)} \left( \frac{2(\sqrt{2} - 1) z + 1 - \sqrt{1 - z}}{2(\sqrt{2} - 1) \sqrt{1 - z} + \sqrt{2(\sqrt{2} - 1) z + 1}} \right)}{2(\sqrt{2} - 1) \sqrt{1 - z} + \sqrt{2(\sqrt{2} - 1) z + 1}} \right). \]

(ii) \( f \in S^*_C \) implies \( \frac{zf'(z)}{f(z)} < 1 + 4z/3 + 2z^2/3 \). This subordination in turn gives
\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{4}{3}z + \frac{2}{3}z^2 \right| \leq \frac{2}{3}(r^2 + 2r), \quad |z| = r. \]

Applying Lemma 2.1, we conclude that \( f \in S^*_N \) if \( \frac{2}{3}(r^2 + 2r) \leq \frac{2}{3} \) or if \( r \leq \sqrt{2} - 1 \). To verify the sharpness of the estimate, consider the function
\[ f_C(z) := z \exp \left( \frac{4}{3}z + \frac{1}{3}z^2 \right). \]

Since \( \frac{zf'(z)}{f_C(z)} = 1 + \frac{1}{3}z + \frac{2}{3}z^2 \), the function \( f_C(z) \) is a member of \( S^*_C \). Also, at the point \( z = \sqrt{2} - 1 \), the value of \( zf'(z)/f_C(z) \) is 5/3. Hence the result is sharp.

(iii) Let \( f \in S^*_R \). Then \( \frac{zf'(z)}{f(z)} < 1 + \frac{1}{k} \left( \frac{k + z}{k - z} \right) \), where \( k = \sqrt{2} + 1 \). For \( |z| = r \), this further implies
\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{z}{k} \left( \frac{k + z}{k - z} \right) \right| \leq \frac{r^2 + 2r}{k(k - r)} \leq \frac{2}{3} \]
if \( 3r^2 + 5kr - 2k^2 \leq 0 \), or if \( r \leq 1/(3\sqrt{2} - 3) \). Now, Lemma 2.1 implies that \( \mathcal{R}_{S^*_N}(S^*_R) = 1/(3\sqrt{2} - 3) \). The estimate is sharp for the function \( f_R \in S^*_R \) defined by
\[ f_R(z) := \frac{k^2z}{(k - z)^2} e^{-z/k}, \quad k = \sqrt{2} + 1. \]

(iv) If \( f \in S^*_L \), then \( \frac{zf'(z)}{f(z)} < 1 + \sqrt{2}z + z^2/2 \) and, for \( |z| = r \),
\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{1}{2} \left( r^2 + 2\sqrt{2}r \right) \leq \frac{2}{3} \]
provided \( 3r^2 + 6\sqrt{2}r - 4 \leq 0 \) or \( r \leq 2\sqrt{2}/(3 + \sqrt{15}) \). The result is sharp for the function \( f_L(z) = z \exp \left( \sqrt{2}z + z^2/4 \right) \in S^*_L \).

Theorem 3.8. The \( S^*_N \)-radii for the classes \( S^*_Q \) and \( S^*_S \) are given by

(i) \( \mathcal{R}_{S^*_N}(S^*_Q) = \frac{1}{6} (\sqrt{17} - 1) \approx 0.520518 \),
(ii) \( \mathcal{R}_{S^*_N}(S^*_S) = \sinh^{-1}(\frac{\sqrt{2}}{3}) = \log \left( \frac{\sqrt{13} + 2}{3} \right) \approx 0.625145 \).

These estimates are not sharp.

Proof. (i) Let \( f \in S^*_Q \). Then \( zf'(z)/f(z) < z + \sqrt{1 + z^2} \) and hence
\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq z + \sqrt{1 + z^2} - 1 \leq 1 + r - \sqrt{1 - r^2}, \quad |z| = r < 1. \quad (3.7) \]

From Lemma 2.1, it follows that the disk (3.7) lies inside \( \Omega_{S^*_N} \) if \( 1 + r - \sqrt{1 - r^2} \leq 2/3 \), or if \( r^2 + \frac{r}{4} - \frac{1}{4} \leq 0 \). This inequality gives the desired radius estimate. We note that a graphical observation shows that the estimate is not sharp.
provided \( r \leq \sinh^{-1}(2/3) \). The desired result follows by an application of Lemma 2.1. \( \square \)

4. \( S_{Ne,n}^* \)-Radius For The Classes of Ratio Functions

Let \( n \in \mathbb{N} := \{1, 2, 3, \ldots\} \). Let \( \mathcal{A}_n \) be the collection of all analytic functions \( f(z) \) of the form \( f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \) and let \( \mathcal{P}_n(\alpha) \) be those \( p \in \mathcal{H} \) having the form \( p(z) = 1 + \sum_{k=n}^{\infty} p_k z^k \) and satisfying \( \Re(p(z)) > \alpha \) \((0 \leq \alpha < 1)\) for all \( z \in \mathbb{D} \). Furthermore, let

\[
S_{Ne,n}^* := S_{Ne}^* \cap \mathcal{A}_n, \quad S_{n}^* := S^* \cap \mathcal{A}_n, \quad C_n := C \cap \mathcal{A}_n \quad \text{and} \quad \mathcal{P}_n := \mathcal{P}_n(0).
\]

In this section, we will find the \( S_{Ne,n}^* \)-radius for some families defined by the ratio of analytic functions. To prove our results, we will use the following lemma.

**Lemma 4.1 ([20, Lemma 2])**. Let \( p \in \mathcal{P}_n(\alpha) \), then, for \( |z| = r \),

\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2nr^n(1 - \alpha)}{(1 - r^n)(1 + (1 - 2\alpha)r^n)}.
\]

In particular, if \( p \in \mathcal{P}_n \) then

\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2nr^n}{1 - r^{2n}}.
\]

Now let us define the following function classes:

\[
\mathcal{G}_1 := \left\{ f \in \mathcal{A}_n : \frac{f}{g} \in \mathcal{P}_n \quad \text{and} \quad \frac{g(z)}{z} \in \mathcal{P}_n, \quad g \in \mathcal{A}_n \right\},
\]

\[
\mathcal{G}_2 := \left\{ f \in \mathcal{A}_n : \frac{f}{g} \in \mathcal{P}_n \quad \text{and} \quad \frac{g(z)}{z} \in \mathcal{P}_n \left( \frac{1}{2} \right), \quad g \in \mathcal{A}_n \right\},
\]

\[
\mathcal{G}_3 := \left\{ f \in \mathcal{A}_n : \left| \frac{f(z)}{g(z)} - 1 \right| < 1 \quad \text{and} \quad \frac{g(z)}{z} \in \mathcal{P}_n, \quad g \in \mathcal{A}_n \right\},
\]

and

\[
\mathcal{G}_4 := \left\{ f \in \mathcal{A}_n : \left| \frac{f(z)}{g(z)} - 1 \right| < 1, \quad g \in \mathcal{C}_n \right\}.
\]

**Theorem 4.1.** For the function classes \( \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \) and \( \mathcal{G}_4 \), we have the following radii results.

(i) \( \mathfrak{R}_{S_{Ne,n}^*}(\mathcal{G}_1) = \rho_1 = \left( \frac{1}{3n+\sqrt{9n^2+1}} \right)^{1/n} \),

(ii) \( \mathfrak{R}_{S_{Ne,n}^*}(\mathcal{G}_2) = \rho_2 = \left( \frac{1}{3n+\sqrt{9n^2+24n+16}} \right)^{1/n} \),

(iii) \( \mathfrak{R}_{S_{Ne,n}^*}(\mathcal{G}_3) = \rho_3 = \left( \frac{1}{3n+\sqrt{9n^2+24n+16}} \right)^{1/n} \),

(iv) \( \mathfrak{R}_{S_{Ne,n}^*}(\mathcal{G}_4) = \rho_4 = \left( \frac{1}{3(n+1)+\sqrt{9n^2+42n+16}} \right)^{1/n} \).

Each estimate is sharp.
Proof. (i): Let \( f \in G_1 \), and let \( p, h : \mathbb{D} \to \mathbb{C} \) be defined as
\[
p(z) = \frac{g(z)}{z} \quad \text{and} \quad h(z) = \frac{f(z)}{g(z)}.
\]
Clearly, \( p, h \in P_n \). Thus, for \( |z| = r \), we have the following inequalities from Lemma 4.1
\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2nr^n}{1 - r^{2n}} \quad \text{and} \quad \left| \frac{zh'(z)}{h(z)} \right| \leq \frac{2nr^n}{1 - r^{2n}}.
\] (4.1)
Moreover, we have \( f(z) = g(z)h(z) = zp(z)h(z) \), which gives the identity
\[
\frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)} + \frac{zh'(z)}{h(z)}.
\] (4.2)
Using the inequalities (4.1), the above identity yields the disk
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{4nr^n}{1 - r^{2n}}.
\]
In view of Lemma 2.1, this disk lies in the region \( \Omega_{N_e} \) if \( 4nr^n/(1 - r^{2n}) \leq 2/3 \), or equivalently \( r^{2n} + 6nr^n - 1 \leq 0 \), which further gives \( r \leq r_1 \). To verify the sharpness of the estimate \( \Re S_{N_e,n}(G_1) \), consider the functions \( f_1, g_1 \in A_n \) given by
\[
f_1(z) = z \left( \frac{1 + z^n}{1 - z^n} \right)^2 \quad \text{and} \quad g_1(z) = z \left( \frac{1 + z^n}{1 - z^n} \right).
\]
It is clear that \( f_1(z)/g_1(z) = g_1(z)/z = (1 + z^n)/(1 - z^n) \in P_n \), and hence \( f_1 \in G_1 \). Also
\[
\frac{zf_1'(z)}{f_1(z)} = 1 + \frac{4nz^n}{1 - z^{2n}}.
\]
and at the point \( z = r_1 \), \( zf_1'(z)/f_1(z) = 5/3 \). This proves the sharpness of the radius estimate \( r_1 \).

(ii): Let \( f \in G_2 \), and define the functions \( p, h : \mathbb{D} \to \mathbb{C} \) by \( p(z) = g(z)/z \) and \( h(z) = f(z)/g(z) \). Then \( f(z) = zp(z)h(z) \) with \( p \in P_n(1/2) \) and \( h \in P_n \). In light of the identity (4.2), it follows from Lemma 4.1 that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{2nr^n}{1 - r^{2n}} + \frac{nr^n}{1 - r^n} = \frac{3nr^n + nr^{2n}}{1 - r^{2n}}.
\]
Now from Lemma 2.1, \( f \in S_{N_e} \) provided
\[
\frac{3nr^n + nr^{2n}}{1 - r^{2n}} \leq \frac{2}{3}.
\]
That is, if \((3n + 2)r^{2n} + 9nr^n - 2 \leq 0\), which gives the desired result \( r \leq r_2 \). To prove that \( r_2 \) is the sharp \( S_{N_e,n} \) radius for the class \( G_2 \), define
\[
f_2(z) = \frac{z(1 + z^n)}{(1 - z^n)^2} \quad \text{and} \quad g_2(z) = \frac{z}{1 - z^n}.
\]
It is easy to see that \( \frac{f_2}{g_2} \in P_n \) and \( \frac{f_2}{z} \in P_n(1/2) \), which shows that \( f_2 \) is a member of \( G_2 \). Further,
\[
\left| \frac{zf_2'(z)}{f_2(z)} \right|_{z=r_2} = \left| \frac{1 + 3nz^n + (n - 1)z^{2n}}{1 - z^{2n}} \right|_{z=r_2} = \frac{5}{3}.
\]
Hence, the estimate is sharp.
(iii): Let \( f \in G_3 \). Define the functions \( p, h : \mathbb{D} \to \mathbb{C} \) by

\[
p(z) = \frac{g(z)}{z} \quad \text{and} \quad h(z) = \frac{g(z)}{f(z)}.
\]

Obviously \( p \in \mathcal{P}_n \), and \( h \in \mathcal{P}_n(1/2) \) due to the fact that

\[
\left| \frac{f(z)}{g(z)} - 1 \right| < 1 \iff \frac{g}{f} \in \mathcal{P}_n \left( \frac{1}{2} \right).
\]

Observe that \( f(z) =zp(z)/h(z) \) and

\[
\frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)} - \frac{zh'(z)}{h(z)}.
\]

Applying Lemma 4.1 and simplifying, we obtain

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{3nr^n + nr^{2n}}{1 - r^{2n}} \leq 2
\]

if \((3n + 2)r^{2n} + 9nr^n - 2 \leq 0 \) or \( r \leq \rho_3 \). Therefore, by Lemma 2.1, the quantity \( zf'(z)/f(z) \) lies inside \( \Omega_{N_e} \) provided \( r \leq \rho_3 \). For sharpness, consider the functions

\[
f_3(z) = \frac{z(1 + z^n)^2}{1 - z^n} \quad \text{and} \quad g_3(z) = \frac{z(1 + z^n)}{1 - z^n}
\]

satisfying

\[
\left| \frac{f_3(z)}{g_3(z)} - 1 \right| = |z|^n < 1 \quad \text{and} \quad \frac{g_3(z)}{z} = \frac{1 + z^n}{1 - z^n} \in \mathcal{P}_n,
\]

so that \( f_3 \in G_3 \). Moreover, it is easy to see that \( \frac{zf_3'(z)}{f_3(z)} \) assumes the value \( 1/3 \) at the point \( z = \rho_3 e^{i\pi/n} \). This shows that the estimate is best possible.

(iv): Let \( f \in G_4 \), and let \( h : \mathbb{D} \to \mathbb{C} \) be given by \( h(z) = g(z)/f(z) \), where \( g \in \mathcal{A}_n \) is some convex function. As earlier, we have \( h \in \mathcal{P}_n(1/2) \) and hence Lemma 4.1 gives

\[
\left| \frac{zh'(z)}{h(z)} \right| \leq \frac{nr^n}{1 - r^n}, \quad (4.3)
\]

Further, as convexity of \( g \) implies \( g \in \mathcal{P}_n(1/2) \), we have from [18, Lemma 2.1]

\[
\left| \frac{zg'(z)}{g(z)} - 1 \right| \leq \frac{r^n}{1 - r^{2n}}, \quad (4.4)
\]

In view of (4.3), (4.4) and the identity

\[
\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} - \frac{zh'(z)}{h(z)},
\]

we calculate that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{(n + 1)r^n + nr^{2n}}{1 - r^{2n}}.
\]

Since for every \( r \in (0,1) \), the number \( 1 - r^{2n} < 1 \), it follows from Lemma 2.1 that the above disk lies inside \( \Omega_{N_e} \) provided

\[
\frac{(n + 1)r^n + nr^{2n}}{1 - r^{2n}} \leq \frac{1}{1 - r^{2n}} - \frac{1}{3}.
\]
or $(3n - 1)r^{2n} + 3(n + 1)r^n - 2 \leq 0$. This inequality gives $r \leq \rho_4$. Now define

$$f_4(z) = \frac{z(1 + z^n)^2}{(1 - z^n)^{1/n}} \quad \text{and} \quad g_4(z) = \frac{z}{(1 - z^n)^{1/n}}.$$ 

Evidently, $f_4 \in G_4$ as $g_3 \in C_n$ and

$$\left| \frac{f_4(z)}{g_4(z)} - 1 \right| = |z|^n < 1.$$ 

Also, at the point $z = \rho_4$, we have $zf_4'(z)/f_4(z) = 5/3$. This proves that the radius cannot be increased further. □

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