Exact solutions, symmetries and quantization
of two-dimensional higher-derivative gravity with dynamical torsion

S. Mignemi†

Dipartimento di Scienze Fisiche, Università di Cagliari
via Ospedale 72, 09100 Cagliari, Italy

ABSTRACT
We investigate two-dimensional higher derivative gravitational theories in a Riemann-Cartan framework. We obtain the most general static black hole solutions in conformal coordinates and discuss their geometry. We also consider the hamiltonian formulation of the theory and discuss its symmetries, showing that it can be considered as a gauge theory of a non-linear generalizations of the 2-dimensional Poincaré algebra. We also show that the models can be exactly quantized in the Dirac formalism.

† e-mail: mignemi@cagliari.infn.it
1. Introduction

In this paper, we study a general class of two-dimensional higher-derivative gravitational theories in a Riemann-Cartan spacetime. As is well known, in two dimensions the Einstein-Hilbert action is a total derivative and hence cannot be used to construct a two-dimensional version of general relativity. It is however possible to construct actions whose lagrangian density is given by an arbitrary power of the Ricci scalar [1]. These models turn out to be equivalent to the ordinary Einstein-Hilbert action non-minimally coupled to a scalar field with a power-law potential [2-4]. In such formalism the field equations become second order and it is possible to obtain exact solutions and to perform the Dirac quantization of the theory [5]. Some well-known special cases of gravity-scalar theories in two dimensions include the Jackiw-Teitelboim [6] and the ”string” models [7-8].

Another useful generalization of two-dimensional gravity is given by the consideration of Riemann-Cartan geometries with non-trivial torsion. Several authors have studied various aspects of a model with action quadratic in the curvature and the torsion [9-15]. It has been proved that this model is completely integrable [9-10] and, exploiting the hamiltonian formalism, it has been shown that its symmetries are a realization of a specific non-linear algebra [11-12] and that its quantization can be performed exactly [13].

Finally, a further interesting aspect of some two-dimensional models is that they can be interpreted as gauge theories of the Poincaré or de Sitter group in two dimensions [16]. In ref. [14] it has been shown that this interpretation can be extended to theories with non-trivial torsion, provided that one generalizes the notion of gauge invariance to groups generated by non-linear algebras.

In this paper we try to extend the results obtained so far to the case of actions containing arbitrary powers of the curvature and quadratic torsion. We show that the static solutions of the field equations can be found exactly and discuss their geometry. We also investigate the hamiltonian formulation of the models and find the constraint algebra, which is a non-linear deformation of the two-dimensional Poincaré algebra and therefore permits their interpretation as a non-linear gauge theory of such algebra. Finally, we perform the Dirac quantization of the model and define the space of the physical states.

2. The action and the field equations

We consider a 2-dimensional lorentzian manifold with signature \((-,+))\) endowed with a Riemann-Cartan geometry. The geometry can be described by the zweibein field \(e_\mu^a\) and the Lorentz connection \(\omega_{ab}^\mu\), where \(a,b,..\) are tangent space indices which can take the values \(0, 1\) and \(\mu, \nu,\).. are world indices, whose values will be denoted by \(t, x\).

The curvature and torsion are defined as

\[
R_{\mu\nu}^{ab} = \partial_\mu \omega_{\nu}^{ab} - \partial_\nu \omega_{\mu}^{ab} \\
T_{\mu\nu}^a = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega_{b\mu}^a e_\nu^b - \omega_{b\nu}^a e_\mu^b
\]

and the Ricci scalar as \(R = e_\mu^a e_\nu^b R_{\mu\nu}^{ab}\). In two dimensions the Ricci scalar determines uniquely the Riemann tensor by the relation \(R_{abcd} = -\frac{1}{2} \epsilon_{abcd} R\). Moreover, the connection can be written as

\[
\omega_{ab}^\mu = \epsilon_{a}^{ab} \omega_{\mu}
\]
where $\epsilon^{ab}$ is the antisymmetric tensor $\epsilon_{ab} = -\epsilon^{ab}$, $\epsilon_{01} = 1$.

In this paper, we study higher derivative actions of the form:

$$ S = \int e^2 d^2 x (R^k - \frac{\gamma}{2} T^2) \tag{2} $$

where $e = \det e^a_\mu$, $T^2 = T_{abc} T^{abc}$. Here, $k$ is any real number except 0, 1 and $\gamma$ a coupling constant. These actions generalize to Riemann-Cartan geometry the higher-derivative actions introduced in [1] and further investigated in [2-5]. The special case $k = 2$ has already been studied by several authors [9-15].

By defining a scalar field $\eta = k R^{k-1}$, it is possible, by a standard argument [2-4] to reduce the action (2) to a form which is linear in the curvature:

$$ S' = \int e^2 d^2 x \left( \eta R - \frac{\gamma}{2} T^2 + \Lambda \eta h \right) \tag{3} $$

where $\Lambda = (1 - k)k^{-k/(k-1)}$ and $h = \frac{k}{k-1}$.

The action can be further reduced to a fully first order form, by introducing a doublet of scalar fields $\eta_a$ [13]:

$$ S'' = \int e^2 d^2 x \left( \eta R + \eta_a * T^a - \frac{1}{\gamma} \eta_a \eta^a + \Lambda \eta h \right) \tag{4} $$

where $* T^a = \epsilon^{bc} T_{abc}$.

The field equations obtained by varying (3) with respect to the fields $\eta$, $e^a$ and $\omega$ can be written as:

\[
\begin{align*}
R + h \Lambda \eta^{b-1} &= 0 \\
- \nabla^b T_{cd} + T_{ac} T_{bd} - \frac{1}{4} g_{cd} (T^2 - \frac{2 \Lambda}{\gamma} \eta h) &= 0 \\
\epsilon^a_c e^\mu_a \partial_\mu \eta - \gamma \epsilon^{ab} T_{abc} &= 0
\end{align*}
\tag{5}
\]

3. The static solutions

In the following, we shall look for the static solutions of these equations. According to the discussion of the special case $k = 2$ performed in ref. [9], it results convenient to seek for the solutions in a conformal gauge. We therefore adopt the ansatz:

$$ e^0_t = e^1_x = e^{2\rho(x)} \\
e^0_x = e^1_t = 0 $$

We also assume that $\omega_\mu = \epsilon_{\mu \nu} \partial^\nu \chi$, with $\chi = \chi(x)$, which yields

$$ \omega_x = 0, \quad \omega_t = \chi' $$

where a prime denotes derivative with respect to $x$. In terms of these variables, one has:

\[
\begin{align*}
R &= 2 e^{-2\rho} \chi'' \\
T_{001} &= e^{-\rho} (\rho' - \chi')
\end{align*}
\tag{6}
\]
and the field equations become:

\[\chi'' + h \frac{\Lambda}{2} \eta h^{-1} e^{2\rho} = 0 \quad (7.a)\]

\[\eta' - \gamma (\rho' - \chi') = 0 \quad (7.b)\]

\[
\rho'' - \chi'' - \frac{1}{2}(\rho' - \chi')(\rho' + \chi') + \frac{\Lambda}{2\gamma} \eta h e^{2\rho} = 0 \quad (7.c)
\]

\[
\frac{1}{2}(\rho' - \chi')(\rho' + \chi') + \frac{\Lambda}{2\gamma} \eta h e^{2\rho} = 0 \quad (7.d)
\]

These equations admit a special solution \(\chi = \rho = \eta = 0\), corresponding to a manifold with vanishing curvature and torsion (we are considering the case of vanishing cosmological constant). The zero torsion solutions do not therefore reduce to the extremals of the action (3) with \(\gamma = 0\), which have been discussed in [5]. This fact has been observed in [9] for the special case \(k = 2\).

One can however obtain more general solutions to (7). Combining (7.c) and (7.d) one gets

\[
\rho'' - \chi'' = (\rho' - \chi')(\rho' + \chi') = -\frac{\Lambda}{\gamma} \eta h e^{2\rho} \quad (8)
\]

A first integral of the first equation (8) is

\[
\rho = \frac{1}{2}(f + \ln(Ef')) \quad (9)
\]

where \(f \equiv \rho - \chi\) and \(E\) is an integration constant.

From (7.b), one has \(\eta = \gamma f\) and thus (8) yields

\[
f'' = \Lambda E \gamma h^{-1} e^f f^h f' \quad (10)
\]

which can be integrated to give

\[
f' = \Lambda E \gamma h^{-1} \int_0^f g^h e^g dg + A \quad (11)
\]

where the integral on the r.h.s. is proportional to the incomplete gamma function \(\Gamma(h + 1, -f)\) and \(A\) is an integration constant. The equation (11) can then be integrated numerically to give \(f\). This solution generalizes that obtained in ref. [9] for \(h = 2\).

One can now express the curvature and the torsion in terms of the function \(f\):

\[R = -\Lambda(\gamma f)^{h-1} \quad T^2 = e^{-f} f' \quad (12)\]

It is then easy to study the qualitative behaviour of the solutions of (11). We shall impose the positivity of \(f\) in order to avoid problems when \(h\) is not integer. From the asymptotics of (11), follows that \(f \to \infty\) for a finite value of \(x\). If \(h > 1\), a curvature singularity is
therefore always present at finite \( x \) in these coordinates, while the scalar \( T^2 \) is singular if \( h > 0 \) as \( f \to \infty \).

A numerical study permits to distinguish three different possible behaviours for \( f \) (see fig. 1-3):

a) If \( h > -1 \) and \( A > 0 \), \( f \) grows monotonically from 0 to \( \infty \) between two finite values \( x_2 \) and \( x_1 \) of \( x \).

b) If \( h > -1 \) and \( A < 0 \), the solution has two branches: one of them decreases monotonically between a constant value \( f_0 \) at \( x = -\infty \) and 0 at \( x = x_1 \), while the other grows monotonically between \( f_0 \) at \( -\infty \) and infinity at \( x = x_2 \).

c) If \( h < -1 \), for any \( A \), the behaviour of \( f \) is similar to the case b), but \( f'(x_2) \to -\infty \).

In order to investigate the properties of the solutions near the critical point, one must study the behavior of the functions \( R, T^2 \) and \( e^{2\rho} \) near \( x_1 \) and \( x_2 \). It is easy to check that for \( f \to \infty \), \( R \sim f^{h-1} \), \( T^2 \sim f^h \), \( e^{2\rho} \sim e^{2f} \), while for \( f \to 0 \), \( R \sim f^{h-1} \), \( T^2 \sim \text{const} + f^{h+1} \), \( e^{2\rho} \sim \text{const} + f^{h+1} \).

Depending on the value of \( h \), one can then distinguish several cases which are summarized in the tables 1-3. The following general results can be stated: If \( h > -1 \) and \( A > 0 \), a naked singularity is always present, either at \( x_1 \) or at \( x_2 \). More interesting are the cases where \( h > -1 \) and \( A < 0 \) or \( h > -1 \). In these cases, the two branches of the solution describe the interior and the exterior of an asymptotically flat black hole with the horizon located at \( x = -\infty \). For \( h < 1 \) (resp. \( h > 1 \)), the curvature singularity is at \( x_1 \) (resp. \( x_2 \)), while spatial infinity is at \( x_2 \) (resp. \( x_1 \)). For \( 0 < h < 1 \), however, the torsion diverges at spatial infinity.

Of course, a detailed study of the spacetime structure would require a more explicit form of the solutions. For the special cases \( h = 0 \) and \( h = 1 \), this will be afforded in a future paper.

4. First order formalism

In terms of differential forms, \( e^a = e^a_\mu dx^\mu \), \( \omega = \omega_\mu dx^\mu \), the first order lagrangian in (4) can be written as:

\[
\frac{1}{2} L = \eta_2 d\omega + \eta_a T^a + \left( \frac{\Lambda}{4} \eta_2^b + \frac{1}{2\gamma} \eta_c \eta^c \right) \epsilon_{ab} e^a e^b
\]  

(13)

where we have renamed \( \eta \) as \( \eta_2 \).

This form of the lagrangian is especially convenient because it permits to evidentiate the connection of our models with the formulation of 2-dimensional gravity as a gauge theory of the Poincaré group ISO(1,1) or one of its generalizations [16,8]. In this formalism, \( e^a \) and \( \omega \) play the role of gauge connections, while the \( \eta \) are considered as auxiliary fields. Local Poincaré transformations with parameters \( \xi^2 \) and \( \xi^a \), corresponding to Lorentz rotations and to translations, act infinitesimally on the fields according to:

\[
\delta e^a = d\xi^a + \epsilon_{ab}(\xi^b \omega - \xi^2 e^b) \quad \delta \omega = d\xi^2
\]

\[
\delta \eta_a = \epsilon_a^b \xi^2 \eta_b \quad \delta \eta_2 = \epsilon_a^b \xi^a \eta_b
\]
and \( R = d\omega \) and \( T^a \) are the field strengths corresponding to Lorentz rotations and translations respectively. The first two terms in the lagrangian (13) are invariant under these transformations, while the potential terms are not. As we shall see, the full action is in fact invariant under a non-linear generalization of the Poincaré group.

In first order formalism, the field equations are given by:

\[
\begin{align*}
T^a + \frac{1}{\gamma} \eta^a \epsilon_{bce} e^b e^c &= 0 \\
d\omega + \frac{\Lambda}{4} \eta_2^{h-1} \epsilon_{ab} e^a e^b &= 0 \\
d\eta_a + \eta_b \epsilon_a^b \omega + \left( \frac{\Lambda}{2} \eta_2^h + \frac{1}{\gamma} \eta_c \eta^c \right) \epsilon_{ab} e^b &= 0 \\
d\eta_2 + \eta_a e^a \eta = 0
\end{align*}
\]  

whose static solutions can be written in terms of the function \( f \) defined above as:

\[
\begin{align*}
\epsilon_t^0 &= e_x^1 = \sqrt{E f' e^f} \\
\omega_t &= \frac{\Lambda}{2} E \gamma^{h-1} f^h e^f - \frac{1}{2} f' \quad \omega_x = 0 \\
\eta_0 &= \gamma \sqrt{\frac{f'}{E e^f}} \quad \eta_1 = 0 \quad \eta_2 = \gamma f
\end{align*}
\]

5. The hamiltonian formalism

Another advantage of the first order formalism is that it leads naturally to a hamiltonian formulation of the model, and hence permits a straightforward discussion of its symmetries and quantization. In fact, after integration by parts, the lagrangian density can be written as:

\[
\frac{1}{2} \mathcal{L} = \eta_a \epsilon_x^a + \eta_2 \omega_x \\
+ e_t^a (\eta_t^a + \epsilon_a^b \eta_b \omega_x) + \frac{\Lambda}{2} \eta_2^h \epsilon_{ab} e_x^b + \frac{1}{\gamma} \eta_c \eta^c \epsilon_{ab} e_x^b + \omega_t (\eta_2^o + \eta_a e^a \eta_x^b)
\]

where a dot denotes time derivative and a prime spatial derivative.

The lagrangian (15) has a canonical structure, with coordinates \((e^a_x, \omega_x)\), conjugate momenta \((\eta_a, \eta_2)\) and Lagrange multipliers \((e_t^a, \omega_t)\) enforcing the constraints:

\[
\begin{align*}
G_a &= \eta_a^o + \epsilon_a^b \eta_b \omega_x + \left( \frac{\Lambda}{2} \eta_2^h + \frac{1}{\gamma} \eta_c \eta^c \right) \epsilon_{ab} e_x^b = 0 \\
G_2 &= \eta_2^o + \eta_a \epsilon_{ab} e_x^b = 0
\end{align*}
\]
Combining the two constraints (16), one can deduce that
\[
\frac{1}{2}(\eta_a^a)' - \frac{\Lambda}{2}\eta^a\eta^a' - \frac{1}{\gamma}\eta_a^a\eta_a^a' = 0
\] (17)
which implies the existence of the conserved quantity
\[
Q \equiv \eta_a^a e^{-2\eta_2^2/\gamma} - \Lambda \left(\frac{\gamma}{2}\right)^{h+1} \Gamma \left(h + 1, \frac{2\eta_2^2}{\gamma}\right)
\] (18)
with \(\Gamma\) the incomplete gamma function.

The study of the algebra of constraints permits to discuss the symmetries of the theory. The calculation of the Poisson brackets of the constraints yields
\[
\{G_a, G_2\} = \epsilon_a^b G_b \quad \{G_a, G_b\} = \epsilon_{ab} \frac{\Lambda}{2} h\eta_2^{h-1} G_2 + \frac{1}{\gamma} \eta_c G_c
\] (19)
with coordinate dependent structure functions. This algebra acts locally on the fields by the infinitesimal transformations:
\[
\delta e^a = d\xi^a + \epsilon_a^b (\xi^b \omega - \xi^2 e^b) - \frac{1}{\gamma} \epsilon_{bc} \xi^b e^c \eta^a \quad \delta \omega = d\xi^2 - \frac{\Lambda}{2} h\eta_2^{h-1} \epsilon_{ab} \xi^a e^b
\]
\[
\delta \eta_a = \epsilon_a^b \left[\xi^2 \eta_b + \xi_b \left(\frac{\Lambda}{2} \eta_2^{h} + \frac{1}{\gamma} \eta_a^{a} \eta_a\right)\right] \quad \delta \eta_2 = \epsilon_a^b \xi^a \eta_b
\]
as can be checked by computing the commutators \(\delta e^a = \{G_a, e^b\}\), etc.

The lagrangian (13) is invariant under these transformations up to a total derivative. Our model can therefore be considered as a gauge theory of the group generated by the non-linear algebra (19), realized by means of its action on the lagrangian (13). The generalization of the usual gauge theories to non-linear algebras has been introduced in [14], where also the special case \(h = 2\) of our model has been examined.

It must be noticed, however, that in this form the algebra fails to close. In order to construct a closed algebra one has to include in it also the fields \(\eta_i\) (\(i = 0, 1, 2\)) and to consider the family of generators \(A(\eta_i) + B(\eta_i)G_1\), with \(A, B\) analytic functions of \(\eta_i\) [11]. One has then:
\[
\{\eta_a, \eta_2\} = \{\eta_a, \eta_b\} = 0
\]
\[
\{G_2, \eta_2\} = 0 \quad \{G_a, \eta_b\} = \epsilon_{ab} \left(\frac{\Lambda}{2} \eta_2^{h} + \frac{1}{\gamma} \eta_c \eta^c\right)
\] (20)
\[
\{G_2, \eta_a\} = -\{G_a, \eta_2\} = \epsilon_a^b \eta_b
\]
The resulting algebra is a nonlinear deformation of \(iso(1, 2)\) of the kind discussed in [17].

6. Dirac quantization

The model can now be quantized in the Dirac formalism, by replacing the Poisson brackets with commutators and imposing the Gauss law on the physical states. In a
momentum representation for the wave functional, $e^a \rightarrow i \frac{d}{d\eta_a}$, $\omega \rightarrow i \frac{d}{d\eta_2}$, the constraint equations become:

$$\left[ \eta'_a + i \epsilon_a^b \eta_b \frac{\partial}{\partial \eta_2} + i \epsilon_{ab} \left( \frac{\Lambda}{4} \eta_2^b + \frac{1}{2\gamma} \eta_c \eta^c \right) \frac{\partial}{\partial \eta_b} \right] \Psi(\eta_a, \eta_2) = 0 \quad (21)$$

$$\left( \eta'_2 + i \epsilon_a^b \eta_a \frac{\partial}{\partial \eta_b} \right) \Psi(\eta_a, \eta_2) = 0 \quad (22)$$

The solution of these equations can be written as:

$$\Psi = \delta(Q')e^{i\Omega}\psi(Q) \quad (23)$$

where $Q$ is given in (18) and

$$\Omega = \int \frac{\epsilon^{ab}\eta_2 \eta_a \eta_b}{\eta^c \eta_c} \quad (24)$$

The parameters $\Lambda$ and $\gamma$ enter in (23) only through the parameter $Q$, which classifies the quantum states. Some special cases of the solution (23) have been obtained in [5,13,18].

It should be pointed out, however, that one cannot straightforwardly define a Schrödinger equation, since due to the constraints (21,22), the hamiltonian vanishes on the physical states. This is a well-known problem in the hamiltonian quantization of gravity and can be solved by fixing a gauge: in a two-dimensional context it has been treated in ref. [13].

7. Conclusions

We have shown that most of the results obtained in two-dimensional gravity with quadratic curvature and torsion can be extended to the case of an action containing arbitrary powers of the curvature scalar. A possible generalization of these results would be to consider actions containing arbitrary functions of the curvature, which can be treated essentially by the same methods used here [3,4]. Another interesting point would be to find the most general solutions of the field equations, including time-dependent ones, which we have not considered. It seems plausible that this can be achieved by means of a suitable generalization of the procedure followed in ref. [9] and [10] in the special case of quadratic curvature.

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