Finite-Length Bounds for Joint Source-Channel Coding with Markovian Source and Additive Channel Noise to Achieve Large and Moderate Deviation Bounds

Ryo Yaguchi\textsuperscript{a} and Masahito Hayashi\textsuperscript{a,b}

\textsuperscript{a}Graduate School of Mathematics, Nagoya University
\textsuperscript{b}Centre for Quantum Technologies, National University of Singapore

Email: yaguchi.riyou@c.mbox.nagoya-u.ac.jp & masahito@math.nagoya-u.ac.jp

Abstract

We derive novel upper and lower finite-length bounds of the error probability in joint source-channel coding when the source obeys the ergodic Markov process and the channel is a Markovian additive channel or a Markovian conditional additive channel. These bounds achieve the tight bounds in the large and moderate deviation regimes.

Index Terms

Markov chain, joint source-channel coding, finite-length analysis, large deviation, moderate deviation

I. INTRODUCTION

Shannon theoretic information theory originally focuses on the asymptotic performance. Since the block length of any real code is finite, analysis with finite-blocklength is more important in a practical setting. Although the tight analysis is possible in the asymptotic regime, it is almost impossible in the finite-length regime. Hence, we usually take a strategy to find a good upper and lower bounds of the decoding error probability in the finite-length regime. Since lower and upper bounds are not unique, we need several requirements for the bounds to clarify their goodness. One is the asymptotic tightness. That is, we impose the first condition that the limit of the bound attains one of the following regimes; (1) Second order, (2) Moderate deviation, and (3) Large deviation.

To satisfy the above requirement, one may use the minimum value with respect to so many parameters. If the calculation complexity for the bound is too huge, it cannot be used in a practical use because we cannot calculate the bound. To estimate the optimal performance for a given blocklength \( n \), we need to impose the second condition that its calculation complexity is not so large, e.g., \( O(1) \), \( O(n) \), or \( O(n \log n) \).

Usually, the channel coding is discussed with the message subject to the uniform distribution. However, in the real communication, the message is not necessarily subject to the uniform distribution. To resolve this problem, we often consider the channel coding with the message subject to the non-uniform distribution. Such a problem is called source-channel joint coding and has been actively studied by several researchers \cite{7}, \cite{5}, \cite{6}, \cite{2}, \cite{4}, \cite{3}.

As a simple case, we often assume that the message is subject to the independent and identical distribution. In this case, the capacity is given as the ratio of the conventional channel capacity to the entropy of the message. Recently, Wang-Ingber-Kochman \cite{2} and Kostina-Verdú \cite{4} discussed the second-order coefficient in this problem. In the same setting, the papers \cite{7}, \cite{5} derived the exponential decreasing rate of the minimum decoding error probability when the information source is subject to an independent and identical distribution and the channel is a discrete memoryless channel. When the information source obeys a Markovian process and the channel is additive noisy channel whose additive noise simply obeys Markovian process, the paper \cite{6} derived the exponential decreasing rate of the minimum decoding error probability, and the paper \cite{3} derived the moderate deviation of the same error probability. The recent paper \cite{1} discussed the channel coding when the distribution of the additive noise in the channel is decided by the channel state, and the channel state is observed by the receiver and is subject to Markovian process. For example, Gilbert-Elliot channel with state-information available at the receiver is written as a special case of the former setting, but cannot be written as a special case of the latter setting. Hence, it is
needed to treat such a general situation to adopt a more realistic situation. In this paper, we focus on two kinds of assumptions (Assumptions 1 and 2) for such generalized additive noise channels. Under these assumptions for channels, we address joint source-channel coding with Markovian source.

The contribution of this paper is the following two points. One is to derive large and moderate deviation bounds under the above general setting, which are the generalizations of the results by the papers [6], [3]. The other is to derive upper and lower bounds of the decoding error probability that match in the large deviation regime in the above general setting. The results of our paper summarized as follows

| Channel | Finite | LD | MD | Complexity |
|---------|--------|----|----|------------|
| Direct  | Ass. 1 | Theorem 1 | Theorem 5 | Theorem 9 |
|         | Ass. 2 | Theorem 3 | Theorem 7 (Tight) | (Tight) |
| Converse| Ass. 1 | Theorem 2 | Theorem 6 | Theorem 9 |
|         | Ass. 2 | Theorem 4 | Theorem 8 (Tight) | (Tight) |

Assumption 1 contains Assumption 2. “Finite”, “LD”, and “MD” express the finite-length bound, the large deviation bound, and the moderate deviation bound, respectively.

The remaining part of this paper is organized as follows. In Section II we prepare several information quantities for Markovian process. Section III prepares several useful functions for finite-length analysis. Section IV explains several useful lemmas under the single shot setting. Section V shows our main results, i.e., our finite-length bounds and large and moderate deviation bounds. Section VI gives our numerical analysis based on our finite-length bounds.

II. INFORMATION MEASURES FOR TWO TERMINALS

In this section, we introduce some information measures and their properties will be used in latter sections.

A. Information measures for single-shot setting

Since this paper addresses finite-length setting and the large deviation analysis, we need the conditional Rényi entropy. When the joint distribution is given to be \( P_{XY} \) the conditional Rényi entropy relative to \( Q_Y \) is given as

\[
H_{1-\theta}(P_{XY}|Q_Y) := \frac{1}{\theta} \log \sum_{x,y} P_{XY}(x,y)^{1-\theta} Q_Y(y)^{\theta}.
\]  

(1)

Dependently of the choice for the distribution \( Q_Y \), we have the upper and lower types of conditional Rényi entropy.

\[
H_{1-\theta}^u(X|Y) := H_{1-\theta}(P_{XY}|P_Y),
\]

(2)

\[
H_{1-\theta}^l(X|Y) := H_{1-\theta}(P_{XY}|P_Y^{1-\theta}),
\]

(3)

where

\[
P_{Y^{1-\theta}}(y) := \frac{\left[\sum_x P_{XY}(x,y)^{1-\theta}\right]^{\frac{1}{1-\theta}}}{\sum_y \left[\sum_x P_{XY}(x,y')^{1-\theta}\right]^{\frac{1}{1-\theta}}}.
\]

(4)

To connect these two types of conditional Rényi entropy, we often focus on the following type of conditional Rényi entropy

\[
H_{1-\theta,1-\theta}(X|Y) := H_{1-\theta}(P_{XY}|P_Y^{1-\theta}').
\]

(5)

For \( P, Q \in \mathcal{P}(\mathcal{X}) \), we define Rényi divergence

\[
D_{1+s}(P||Q) := \frac{1}{s} \log \sum_z P(z)^{1+s} Q(z)^{-s}.
\]

(6)
Using Rényi divergence, we introduce two types of Rényi mutual informations
\begin{align}
I_{1-s}^\uparrow(X;Y|P_{XY}) := & D_{1-s}(P_{XY}||P_X \times P_Y), \\
I_{1-s}^\downarrow(X;Y|P_{XY}) := & -\frac{1-s}{s} \log \sum_y (\sum_x P_X(x)P_{Y|X}(y|x)^{1-s})^{\frac{s}{1-s}}.
\end{align}

B. Information measures for transition matrix

Since this paper address the Markovian information source, we prepare several information measures given in [1] for an ergodic and irreducible transition matrix $W = \{W(x, y|x', y')\}_{(x,y), (x',y') \in (\mathcal{X} \times \mathcal{Y})^2}$ on $(\mathcal{X} \times \mathcal{Y})$. For this purpose, we employ two assumptions on transition matrices, which were introduced by the paper [1].

**Definition 1** (Assumption 1 (non-hidden)). We assume the following condition for a transition matrix $W$:
\begin{equation}
\sum_x W(x, y|x', y') = W(y|y'),
\end{equation}
for every $x' \in \mathcal{X}$ and $y, y' \in \mathcal{Y}$.
When this condition holds, a transition matrix $W$ is called non-hidden (with respect to $\mathcal{Y}$).

**Definition 2** (Assumption 2). We assume one of the following conditions for a transition matrix $W$:
\begin{enumerate}
\item for every $\theta \in (-\infty, 0)$ and $(y, y') \in \mathcal{Y} \times \mathcal{Y}$,
\begin{equation}
W_\theta(y|y') = \sum_x W(x, y|x', y')^{1-\theta}.
\end{equation}
is well defined, i.e., the right hand side of (10) is independent of $x'$.
When this condition holds, a transition matrix $W$ is called strongly non-hidden (with respect to $\mathcal{Y}$).
\item $|\mathcal{Y}| = 1$.
When this condition holds, a transition matrix $W$ is called singleton.
\end{enumerate}

Assumption 1 is acquired from (10) by substituting $\theta = 0$, so Assumption 2 implies Assumption 1. When a transition matrix on $W$ satisfies Assumption 1, we define the marginal $W_\mathcal{Y}$ by $W_\mathcal{Y}(y|y') := \sum_x W(x, y|x', y')$. For the transition matrix $T$ on $\mathcal{Y}$, we also define $\mathcal{Y}_T^2 := \{(y, y') : T(y|y') > 0\}$. Then, when another transition matrix $V$ on $\mathcal{Y}$ satisfies $\mathcal{Y}_V^2 \subset \mathcal{Y}_T^2$, we define
\begin{equation}
H_{1-\theta}^{W|V}(X|Y) := \frac{1}{\theta} \log \lambda_\theta^{W|V},
\end{equation}
where $\lambda_\theta^{W|V}$ is the Perron-Frobenius eigenvalue of
\begin{equation}
W(x, y|x', y')^{1-\theta}V(y|y')^\theta.
\end{equation}

Then, the lower type of conditional Rényi entropy for the transition matrix [1] is given as
\begin{equation}
H_{1-\theta}^{W,\downarrow}(X|Y) := H_{1-\theta}^{W|V}(X|Y).
\end{equation}

Also, when $W$ satisfies Assumption 2, the upper type of conditional Rényi entropy for the transition matrix [1] is given as
\begin{equation}
H_{1-\theta}^{W,\uparrow}(X|Y) := \max_V H_{1-\theta}^{W|V}(X|Y).
\end{equation}

Furthermore, we define the information measure which is counterpart of (5). For this purpose, we introduce the following $|\mathcal{Y}| \times |\mathcal{Y}|$ matrix:
\begin{equation}
N_{\theta, \theta'}(y|y') := W_\theta(y|y')W_{\theta'}(y|y')^{1-s}. 
\end{equation}
Let \( \nu_{0,\theta} \) be the Perron-Frobenius eigenvalue of \( N_{\theta,\theta'} \). Then, we define the two-parameter conditional Rényi entropy \( [1] \) by

\[
H^{W}_{1-\theta,1-\theta'}(X|Y) := \frac{1}{\theta} \log \nu_{0,\theta'} - \frac{\theta'}{1 - \theta'} H^{W}_{1-\theta'}(X|Y).
\]

For \( \theta = 0 \), we define the conditional Rényi entropy for \( W \) by

\[
H^W(X|Y) := \lim_{\theta \to 0} H^{W}_{1-\theta}(X|Y).
\]

Also, we define following quantity.

\[
V^W(X|Y) := \lim_{\theta \to 0} \frac{2[H^{W}_{1-\theta}(X|Y) - H^W(X|Y)]}{\theta}.
\]

According to \([1]\), using (17) and (18), we obtain the following two expansions.

\[
H^{W}_{1-\theta}(X|Y) = H^W(X|Y) + \frac{\theta}{2} V^W(X|Y) + o(\theta),
\]

\[
H^{W}_{1-\theta'}(X|Y) = H^W(X|Y) + \frac{\theta'}{2} V^W(X|Y) + o(\theta)
\]

around \( \theta = 0 \).

Under these preparations, we have three lemmas as follows. For these lemmas, we define the random variable \( X^n := (X_1, X_2, \cdots, X_n) \) where \( \{X_i\}_{i=1}^n \) are i. i. d. random variables and each random variable \( X_i \) obeys the same distribution as the random variable \( X \). Also we define the random variable \( Y^n \) in the same way.

**Proposition 3.** \([1] \) **Lemma 9** Suppose that a transition matrix \( W \) satisfies Assumption 1. Let \( W_\theta(x, y) := W(x, y|x', y')^{1-\theta} W(y|y')^\theta \) and \( v_\theta \) be the eigenvector of \( W_\theta^{T} \) with respect to the Perron-Frobenius eigenvalue \( \lambda_\theta \) such that \( \min_{x,y} v_\theta(x, y) = 1 \). Let \( w_\theta(x, y) = P_{X_i,Y_i}(x, y)^{1-\theta} P_{Y_i}(y)^{\theta} \). Then, we have

\[
(n - 1)\theta H^{W}_{1-\theta}(X|Y) + \delta_W(\theta) \leq \theta H^W(X^n|Y^n) \leq (n - 1)\theta H^{W}_{1-\theta}(X|Y) + \overline{\delta}_W(\theta),
\]

where

\[
\overline{\delta}_W(\theta) := \log v_\theta \cdot w_\theta,
\]

\[
\delta_W(\theta) := \log v_\theta \cdot w_\theta - \log \max_{x,y} v_\theta(x, y).
\]

**Proposition 4.** \([1] \) **Lemma 10** Suppose that a transition matrix \( W \) satisfies Assumption 2. Then, we have

\[
(n - 1)\frac{\theta}{1 - \theta} H^{W}_{1-\theta}(X|Y) + \xi_W(\theta) \leq \frac{\theta}{1 - \theta} H^W(X^n|Y^n) \leq (n - 1)\frac{\theta}{1 - \theta} H^{W}_{1-\theta}(X|Y) + \overline{\xi}_W(\theta),
\]

where \( \overline{\xi}_W(\theta) \) and \( \xi_W(\theta) \) is defined as follows:

For the non-hidden case, we define the \( |Y| \times |Y| \) matrix \( K_\theta \) so that

\[
K_\theta(y|y') := [\sum_x W(x, y|x', y')^{1-\theta}]^{\frac{1}{1-\theta}},
\]

and \( v_\theta \) be the eigenvector of \( K_\theta^T \) with respect to the Perron-Frobenius eigenvalue \( \kappa_\theta \) such that \( \min_y v_\theta(y) = 1 \). Let \( w_\theta \) be the \( |Y| \)-dimensional vector defined by

\[
w_\theta(y) = \left[ \sum_x P_{X,Y}(x, y)^{1-\theta} \right]^{\frac{1}{1-\theta}}.
\]
Then, $\xi_W(\theta)$ and $\xi_W^{\prime}(\theta)$ are defined as:

$$\xi_W(\theta) := \log v_\theta \cdot w_\theta, \quad (27)$$

$$\xi_W(\theta) := \log v_\theta \cdot w_\theta - \log \max_y v_\theta(y). \quad (28)$$

For the singleton case, let $W_\theta(x) := W(x|x^\prime)^{1-\theta}$ and $v_\theta$ be the eigenvector of $W_\theta^T$ with respect to the Perron-Frobenius eigenvalue $\lambda_\theta$ such that $\min_x v_\theta(x) = 1$. Let $w_\theta(x) = P_{X,Y}(x)^{1-\theta}$. Then, $\xi_W(\theta)$ and $\xi_W(\theta)$ are defined as:

$$\xi_W(\theta) := \log v_\theta \cdot w_\theta, \quad (29)$$

$$\xi_W(\theta) := \log v_\theta \cdot w_\theta - \log \max_x v_\theta(x). \quad (30)$$

**Proposition 5.** [1] lemmas 9 and 11] Suppose that a transition matrix $W$ satisfies Assumption 2. Then, we have

$$(n - 1)\lambda H_{1-\theta,1-\theta}^{\prime}(X|Y) + \xi_W(\theta, \theta') \leq \theta H_{1-\theta,1-\theta}(X^n|Y^n) \leq (n - 1)\lambda H_{1-\theta,1-\theta}^{\prime}(X|Y) + \xi_W(\theta, \theta') \quad (31)$$

where $\xi_W(\theta, \theta')$ and $\xi_W(\theta, \theta')$ are defined as follows:

For the non-hidden case with respect to $Y$, let $v_{\theta,\theta'}$ be the eigenvector of $N_{\theta,\theta'}^T$ with respect to the Perron-Frobenius eigenvalue $v_{\theta,\theta'}$ such that $\min_y v_{\theta,\theta'}(y) = 1$. Let $w_{\theta,\theta'}$ be the $|Y|$-dimensional vector defined by

$$w_{\theta,\theta'}(y) := \left[ \sum_x P_{X,Y}(x,y)^{1-\theta} \right] \left[ \sum_x P_{X,Y}(x,y)^{1-\theta'} \right]^{\frac{\theta}{1-\theta}}. \quad (32)$$

Then, $\xi_W(\theta, \theta')$ and $\xi_W(\theta, \theta')$ are defined as:

$$\xi_W(\theta, \theta') := \log v_{\theta,\theta'} \cdot w_{\theta,\theta'} - \theta \xi_W(\theta'), \quad (33)$$

$$\xi_W(\theta, \theta') := \log v_{\theta,\theta'} \cdot w_{\theta,\theta'} - \log \max_y v_{\theta,\theta'}(y) - \theta \xi_W(\theta'), \quad (34)$$

for $\theta < 0$ and

$$\xi_W(\theta, \theta') := \log v_{\theta,\theta'} \cdot w_{\theta,\theta'} - \theta \xi_W(\theta'), \quad (35)$$

$$\xi_W(\theta, \theta') := \log v_{\theta,\theta'} \cdot w_{\theta,\theta'} - \log \max_y v_{\theta,\theta'}(y) - \theta \xi_W(\theta'), \quad (36)$$

for $\theta > 0$.

For the singleton case, we define $\xi_W(\theta, \theta')$ and $\xi_W(\theta, \theta')$ by (29) and (30) independently of $\theta'$.

### III. Functions with three terminals

#### A. Functions for single shot setting

Now, to deal with joint source and channel coding, we newly introduce some functions related with three random variables $M, X, Z$. For $r > 0$ and $\theta \in (-\infty, 1)$, we define the following function.

$$U[P_{XZ}, Q_Y; r](\theta) := r \theta H_{1-\theta}(M) + \theta H_{1-\theta}(P_{XZ}|Q_Y). \quad (37)$$

Also we define its derivative

$$u[P_{XZ}, Q_Y; r](\theta) := \frac{d}{d\theta} U[P_{XZ}, Q_Y; r](\theta). \quad (38)$$

Since $U[P_{XZ}, Q_Y; r](\theta)$ is convex function, $u[P_{XZ}, Q_Y; r](\theta)$ is monotonically increasing function. Hence, we can define its inverse function $\theta[P_{XZ}, Q_Y; r](a)$ by

$$u[P_{XZ}, Q_Y; r](\theta[P_{XZ}, Q_Y; r](a)) = a, \quad (39)$$

for $\underline{a} \leq a \leq \overline{a}$, where $\underline{a} := \lim_{\theta \to -\infty} u[P_{XZ}, Q_Y; r](\theta)$ and $\overline{a} := \lim_{\theta \to 1} u[P_{XZ}, Q_Y; r](\theta)$.

When we define

$$R[P_{XZ}, Q_Y; r](a) := (1 - \theta[P_{XZ}, Q_Y; r](a))a + U[P_{XZ}, Q_Y; r](\theta[P_{XZ}, Q_Y; r](a)) \quad (40)$$

for $a \leq a \leq \overline{a}$.
for $a \leq a \leq \pi$, the derivative is calculated to be
\[
\frac{dR[P_{XZ}, Q_Y; r]}{da} = (1 - \theta(a)).
\] (41)

Hence, $R[P_{XZ}, Q_Y; r](a)$ is monotonically increasing function of $a \leq a \leq \pi$. Thus, we can define the inverse function $a[P_{XZ}, Q_Y; r](R)$ by
\[
R[P_{XZ}, Q_Y; r](a[P_{XZ}, Q_Y; r](R)) = R,
\] (42)
for $R[P_{XZ}, Q_Y; r](a) < R \leq r H_0(M) + H_0(X|Z)$.

B. Functions for two transition matrices

We define similar functions for two transition matrices $W_s$ on $\mathcal{M}$ and $W_c$ on $\mathcal{X} \times \mathcal{Z}$. Suppose that $W_c$ is non-hidden with respect to $Z$, i.e., satisfies Assumption 1.

For $r > 0$ and $\theta \in (-\infty, 1)$, we define
\[
U[W_s, W_c, \downarrow; r](\theta) := r \theta H_{1-\theta}(M) + \theta H_{1-\theta, c}(X|Z),
\] (43)
\[
u[W_s, W_c, \downarrow; r](\theta) := \frac{d}{d\theta} U[W_s, W_c, \downarrow; r](\theta).
\] (44)

Using above two functions, we define
\[
\theta[W_s, W_c, \downarrow; r](a) := (u[W_s, W_c, \downarrow; r])^{-1}(a),
\] (45)
\[
R[W_s, W_c, \downarrow; r](a) := (1 - \theta[W_s, W_c, \downarrow; r](a)) a + U[W_s, W_c, \downarrow; r](\theta[W_s, W_c, \downarrow; r](a)),
\] (46)
for $a \leq a \leq \bar{a}$, where $a := \lim_{\theta \to -\infty} u[W_s, W_c, \downarrow; r](\theta)$ and $\bar{a} := \lim_{\theta \to 1} u[W_s, W_c, \downarrow; r](\theta)$. Moreover, we define
\[
a[W_s, W_c, \downarrow; r](R) := (R[W_s, W_c, \downarrow; r])^{-1}(R),
\] (47)
for $R[W_s, W_c, \downarrow; r](a) < R \leq r H_0^W(M) + H_0^{W, c}(X|Z)$.

Now, we suppose that $W_c$ satisfies Assumption 2. For $r > 0$ and $\theta, \theta' \in (-\infty, 1)$, we define
\[
U[W_s, W_c, \theta'; r](\theta) := r \theta H_{1-\theta}^W(M) + \theta H_{1-\theta, 1-\theta'}(X|Z),
\] (48)
\[
u[W_s, W_c, \theta'; r](\theta) := \frac{d}{d\theta} U[W_s, W_c, \theta'; r](\theta).
\] (49)

When $\theta = \theta'$ we also define for $r > 0$ and $\theta \in (-\infty, 1)$,
\[
U[W_s, W_c, \uparrow; r](\theta) := r \theta H_{1-\theta}^W(M) + \theta H_{1-\theta, c}(X|Z),
\] (50)
\[
u[W_s, W_c, \uparrow; r](\theta) := \frac{d}{d\theta} U[W_s, W_c, \uparrow; r](\theta).
\] (51)

Using above two functions, we define
\[
\theta[W_s, W_c, \uparrow; r](a) := (u[W_s, W_c, \uparrow; r])^{-1}(a),
\] (52)
\[
R[W_s, W_c, \uparrow; r](a) := (1 - \theta[W_s, W_c, \uparrow; r](a)) a + U[W_s, W_c, \uparrow; r](\theta[W_s, W_c, \uparrow; r](a)),
\] (53)
for $a \leq a \leq \bar{a}$, where $a := \lim_{\theta \to -\infty} u[W_s, W_c, \uparrow; r](\theta)$ and $\bar{a} := \lim_{\theta \to 1} u[W_s, W_c, \uparrow; r](\theta)$. Moreover, we define
\[
a[W_s, W_c, \uparrow; r](R) := (R[W_s, W_c, \uparrow; r])^{-1}(R),
\] (54)
for $R[W_s, W_c, \uparrow; r](a) < R \leq r H_0^W(M) + H_0^{W, c}(X|Z)$. 

IV. SINGLE SHOT SETTING

A. Problem formulation

We first present the problem formulation by the single shot setting. Assume that the message $M$ takes values in $\mathcal{M}$ and is subject to the distribution $P_M$. For a channel $W_{Y|X}(y|x)$ with input alphabet $\mathcal{X}$ and output alphabet $\mathcal{Y}$, a channel code $\phi = (e, d)$ consists of one encoder $e : \mathcal{M} \to \mathcal{X}$ and one decoder $d : \mathcal{Y} \to \mathcal{M}$. The average decoding error probability is defined by

$$P_{js}[\phi|P_M, W_{Y|X}] := \sum_{m \in \mathcal{M}} P_M(m)W_{Y|X}(\{b : d(b) \neq m\}|e(m)).$$

(55)

For notational convenience, we introduce the minimum error probability under the above condition:

$$P_{js}(P_M, W_{Y|X}) := \inf_{\phi} P_{js}[\phi|P_M, W_{Y|X}].$$

(56)

B. Direct part

1) General case: We introduce several lemmas for the case when $\mathcal{M}$ is the set of messages to be sent, $P_M$ is the distribution of the messages, and $W_{Y|X}$ is the channel from $\mathcal{X}$ to $\mathcal{Y}$.

We have the following single-shot lemma for the direct part.

**Proposition 6.** [Lemma 3.8.1] For any constant $c > 0$ and for any $P_X \in \mathcal{P}(\mathcal{X})$, there exists a code $\phi = (e, d)$ such that

$$P_{js}[\phi|P_M, W_{Y|X}] \leq (P_M \times P_X \times W_{Y|X})(M, X, Y) \leq c(P_X \times \bar{W}_Y)(X, Y) + \frac{1}{c},$$

(57)

where $\bar{W}_Y(y) := \sum_x P_X(x)W_{Y|X}(y|x)$ and $P_X \times W_{Y|X}(y, x) := P_X(x)W_{Y|X}(y|x)$.

From above Proposition, we obviously have following corollary.

**Corollary 1.**

$$P_{js}(P_M, W_{Y|X}) \leq (P_M \times P_X \times W_{Y|X})(M, X, Y) \leq c(P_X \times \bar{W}_Y)(X, Y) + \frac{1}{c}.$$  

(58)

**Proof:** Since the proof of this lemma is crucial for our proof of the next novel lemma, we give a proof of this lemma as follows. We prove this lemma by using the random coding method. For the code $\phi = (e, d)$, we independently choose $e(m) \in \mathcal{X}$ subject to $P_X$. Define $D_m := \{y|P_M(m)W_{Y|X}(y|e(m)) \geq c\bar{W}_Y(y)\}$ and define decoding region of message $m$ as $D'_m := D_m \setminus (\cup_{m' \neq m} D_{m'})$. The error probability of this code can be evaluated as:

$$P_{js}[\phi|P_M, W_{Y|X}] \leq \sum_m P_M(m)(W_{Y|X}=e(m)\{P_M(m)W_{Y|X}=e(m)(Y) < c\bar{W}_Y(y)\})$$

$$+ \sum_{m' \neq m} W_{Y|X}=e(m)\{P_M(m')W_{Y|X}=e(m')(Y) \geq c\bar{W}_Y(y)\}).$$

(59)

Taking the average for the random choice, the first term is

$$E_{\phi} \sum_m P_M(m)W_{Y|X}=e(m)\{P_M(m)W_{Y|X}=e(m)(Y) < c\bar{W}_Y(y)\}$$

$$= \sum_m P_M(m) \sum_x P_X(x)W_{Y|X}=x\{P_M(m)W_{Y|X}=x(Y) < c\bar{W}_Y(y)\}$$

$$= (P_M \times P_X \times W_{Y|X}((P_M \times P_X \times W_{Y|X})(M, X, Y) < cP_X \times \bar{W}_Y(X, Y),$$

(60)
and the second term is
\[ E_P \sum_m P_M(m) \sum_{m' \neq m} W_{Y|X = e(m)} \{ P_M(m') W_{Y|X = e(m')}(Y) \geq c \bar{W}_Y(Y) \} \]
\[ = \sum_{m,m':m \neq m} P_M(m) E_{e(m')} \{ P_M(m') W_{Y|X = e(m')}(Y) \geq c \bar{W}_Y(Y) \} \]
\[ = \sum_{m,m':m \neq m} P_M(m) E_{e(m')} \bar{W}_Y \{ P_M(m') W_{Y|X = e(m')}(Y) \geq c \bar{W}_Y(Y) \} \]
\[ \leq \sum_{m,m':m \neq m} P_M(m) \frac{P_M(m')}{c} W_{Y|X = e(m')} \{ P_M(m') W_{Y|X = e(m')}(Y) \geq c \bar{W}_Y(Y) \} \]
\[ \leq \sum_{m,m':m \neq m} P_M(m) \frac{P_M(m')}{c} \leq \frac{1}{c}. \] (62)

Combining (59), (60) and (62), we have
\[ E_P P_{js}[\phi|P_M, W_{Y|X}] \leq (P_M \times P_X \times W_{Y|X}) \{ (P_M \times P_X \times W_{Y|X})(M, X, Y) \leq c (P_X \times \bar{W}_Y)(X, Y) \} + \frac{1}{c}. \] (63)
Consequently, there must exist at least one deterministic code \( \phi \) satisfying
\[ P_{js}[\phi|P_M, W_{Y|X}] \leq (P_M \times P_X \times W_{Y|X}) \{ (P_M \times P_X \times W_{Y|X})(M, X, Y) \leq c (P_X \times \bar{W}_Y)(X, Y) \} + \frac{1}{c}. \] (64)

From the above proof, we also find the following single-shot lemma for the direct part.

**Lemma 1.** For any constant \( c > 0 \) and for any distribution \( P_X \in \mathcal{P}(X) \), we have
\[ P_{js}(P_M, W_{Y|X}) \leq (P_M \times P_X \times W_{Y|X}) \{ (P_M \times P_X \times W_{Y|X})(M, X, Y) < c P_X \times \bar{W}_Y(X, Y) \} \]
\[ + (1_M \times P_X \times \bar{W}_Y) \{ (P_M \times P_X \times W_{Y|X})(M, X, Y) \geq c P_X \times \bar{W}_Y(X, Y) \}, \] (65)
where \( 1_M \) is a counting measure on \( M \). The choice \( c = 1 \) gives the minimum upper bound.

We also have following lemma.

**Lemma 2.**
\[ P_{js}(P_M, W_{Y|X}) \leq e^{sH_{1-s}(M) - sH_{1-s}(X|Y)}. \] (66)

**Proof of Lemma 2** From (61) in the proof of Proposition 6 we can evaluate the second term of (60) as
\[ \sum_{m,m':m \neq m} P_M(m) E_{e(m')} \bar{W}_Y \{ P_M(m') W_{Y|X = e(m')}(Y) \geq c \bar{W}_Y(Y) \} \]
\[ = \sum_{m,m':m \neq m} P_M(m) \sum_{x \in X} P_X(x) \bar{W}_Y \{ P_M(m') W_{Y|X = e(m')}(Y) \geq c \bar{W}_Y(Y) \} \]
\[ = \sum_{m,m':m \neq m} P_M(m) \cdot (P_X(x) \times \bar{W}_Y) \{ P_M(m') \cdot (P_X \times W_{Y|X = e(m')})(X, Y) \geq c \bar{W}_Y(Y) \} \]
\[ \leq \sum_{m,m':m \neq m} P_M(m) \cdot I_M \times P_X(x) \times \bar{W}_Y \{ (P_M(m') \times P_X \times W_{Y|X = e(m')})(X, Y) \geq c \bar{W}_Y(Y) \} \]
\[ = I_M \times P_X(x) \times \bar{W}_Y \{ (P_M(m') \times P_X \times W_{Y|X = e(m')})(X, Y) \geq c \bar{W}_Y(Y) \}. \]

So, we obtain (65).
Next, we prove that the right hand side of (65) is minimized when \( c = 1 \). For any \( c > 0 \), we can evaluate the right hand side of (65) as:

\[
(P_M \times P_X \times W_{Y|X}) \{(P_M \times P_X \times W_{Y|X})(M, X, Y) < c P_X \times W_Y(X, Y)\}
+ (1_M \times P_X \times \bar{W}_Y) \{(P_M \times P_X \times W_{Y|X})(M, X, Y) \geq c P_X \times W_Y(X, Y)\}
= 1 - \sum_{(m, x, y):(P_M \times P_X \times W_{Y|X})(m, x, y) \geq c P_X \times \bar{W}_Y(x, y)} (P_M \times P_X \times W_{Y|X})(m, x, y) - (1_M \times P_X \times \bar{W}_Y)(m, x, y)
\geq 1 - \sum_{(m, x, y):(P_M \times P_X \times W_{Y|X})(m, x, y) \geq (1_M \times P_X \times \bar{W}_Y)(m, x, y)} (P_M \times P_X \times W_{Y|X})(m, x, y) - (1_M \times P_X \times \bar{W}_Y)(m, x, y)
= (P_M \times P_X \times W_{Y|X}) \{(P_M \times P_X \times W_{Y|X})(M, X, Y) < P_X \times \bar{W}_Y(X, Y)\}
+ (1_M \times P_X \times \bar{W}_Y) \{(P_M \times P_X \times W_{Y|X})(M, X, Y) \geq P_X \times \bar{W}_Y(X, Y)\}.
\]

**Proof of Lemma 2** For any \( s \in (0, 1) \), we have

\[
(P_M \times P_X \times W_{Y|X}) \{(P_M \times P_X \times W_{Y|X})(M, X, Y) < P_X \times \bar{W}_Y(X, Y)\}
+ (1_M \times P_X \times \bar{W}_Y) \{(P_M \times P_X \times W_{Y|X})(M, X, Y) \geq P_X \times \bar{W}_Y(X, Y)\}
\leq \sum_{(P_M \times P_X \times W_{Y|X})(m, x, y) < 1_M \times P_X \times \bar{W}_Y(m, x, y)} (P_M \times P_X \times W_{Y|X})(m, x, y) \left( \frac{(1_M \times P_X \times \bar{W}_Y)(m, x, y)}{(P_M \times P_X \times W_{Y|X})(m, x, y)} \right)^s
+ \sum_{(P_M \times P_X \times W_{Y|X})(m, x, y) \geq 1_M \times P_X \times \bar{W}_Y(m, x, y)} (1_M \times P_X \times \bar{W}_Y)(m, x, y) \left( \frac{(P_M \times P_X \times W_{Y|X})(m, x, y)}{(1_M \times P_X \times \bar{W}_Y)(m, x, y)} \right)^{1-s}
= \sum_{m}(P_M \times P_X \times W_{Y|X})(m, x, y)^{1-s} \left( 1_M \times P_X \times \bar{W}_Y \right)(m, x, y)^s
= \sum_{m} P_M(m) \{ 1-M \leq s \} \int_{X,Y} \bar{W}_Y dP_X dP_Y \leq e^{s H_{1-s}(M) - s H_{1-s}(X;Y|P_X \times W_{Y|X})},
\]

However, even when \( M \) is subject to the uniform distribution, the upper bound (66) is not so tight. In the uniform case, the Gallager bound is tighter than the upper bound (66). So, modifying the derivation of the Gallager bound, we derive joint source and channel coding version of the Gallager bound as follows.

**Lemma 3.** For any distribution \( P_X \in \mathcal{P}(\mathcal{X}) \), we have

\[
P_{\text{as}}(P_M, W_{Y|X}) \leq e^{\frac{s}{1-s} (H_{1-s}(M) - H_{1-s}(X;Y|P_X \times W_{Y|X}))},
\]

for any \( s \in [0, 1/2] \).

**Proof:** For encoder, we independently choose \( e(i) \in \mathcal{X} \) subject to \( P_X \), and for decoder, we define decoding region of the message \( i \) as

\[
D(i) := \{ y \in \mathcal{Y} | \max_{i' \neq i} P_M(i') W_{Y|X=e(i')(y)} < P_M(i) W_{Y|X=e(i)}(y) \}.
\]
And we also define

\[
\Delta_{i,j}(y) = \begin{cases} 
0 & P_M(i)W_{Y|X=e(i)}(y) < P_M(i)W_{Y|X=e(i)}(y) \\
1 & P_M(i)W_{Y|X=e(i)}(y) \geq P_M(i)W_{Y|X=e(i)}(y),
\end{cases}
\]

(69)

\[
\Delta_{i,MP}(y) = \begin{cases} 
0 & y \in D(i) \\
1 & y \notin D(i).
\end{cases}
\]

(70)

Then, for any \(0 \leq s \leq 1\) and \(0 \leq t \leq 1\),

\[
\Delta_{i,MP}(y) \leq \left( \sum_j \Delta_{i,j}(y) \right)^t \leq \left( \sum_j \frac{(P_M(j)W_{Y|X=e(i)}(y))^{1-s}}{P_M(i)W_{Y|X=e(i)}(y)^{1-s}} \right)^t,
\]

(71)

and error probability can be represented by

\[
P_{js}[\phi|P_M, W_{Y|X}] = \sum_{i,y} P_M(i)W_{Y|X=e(i)}(y) \Delta_{i,MP}(y).
\]

(72)

So that,

\[
P_{js}[\phi|P_M, W_{Y|X}] = \sum_{i,y} P_M(i)W_{Y|X=e(i)}(y) \Delta_{i,MP}(y)
\]

\[
\leq \sum_{i,y} P_M(i)W_{Y|X=e(i)}(y) \left( \sum_j \frac{(P_M(j)W_{Y|X=e(i)}(y))^{1-s}}{P_M(i)W_{Y|X=e(i)}(y)^{1-s}} \right)^t
\]

\[
\leq \sum_{i,y} P_M(i)^{1-t(1-s)}W_{Y|X=e(i)}(y)^{1-t(1-s)} \left( \sum_j (P_M(j)W_{Y|X=e(i)}(y))^{1-s} \right)^t.
\]

Taking the average for the random choice, we have

\[
E_{\phi}P_{js}[\phi|P_M, W_{Y|X}]
\]

\[
\leq \sum_{i,y} P_M(i)^{1-t(1-s)} E_{\phi}W_{Y|X=e(i)}(y)^{1-t(1-s)} \left( \sum_j P_M(j)^{1-s} E_{\phi}W_{Y|X=e(i)}(y)^{1-s} \right)^t
\]

\[
\leq \sum_{i,y} P_M(i)^{1-t(1-s)} \sum_x P_X(x)W_{Y|X}(y|x)^{1-t(1-s)} \left( \sum_j P_M(j)^{1-s} \sum_x P_X(x)W_{Y|X}(y|x)^{1-s} \right)^t
\]

\[
= \sum_{i} P_M(i)^{1-t(1-s)} \sum_{y} \left( \sum_x P_X(x)W_{Y|X}(y|x)^{1-t(1-s)} \right) \left( \sum_j P_M(j)^{1-s} \right)^t \left( \sum_x P_X(x)W_{Y|X}(y|x)^{1-s} \right)^t.
\]

(73)

By setting \(t = \frac{s}{1-s}\) in (73), we have

\[
\sum_{i} P_M(i)^{1-s} \sum_{y} \left( \sum_x P_X(x)W_{Y|X}(y|x)^{1-s} \right) \left( \sum_j P_M(j)^{1-s} \right)^{\frac{s}{1-s}} \left( \sum_x P_X(x)W_{Y|X}(y|x)^{1-s} \right)^{\frac{s}{1-s}}
\]

\[
= \left( \sum_{i} P_M(i)^{1-s} \right)^{\frac{s}{1-s}} \sum_{y} \left( \sum_x P_X(x)W_{Y|X}(y|x)^{1-s} \right)^{\frac{s}{1-s}}
\]

\[
= e^{\frac{s}{1-s}(H_{1-s}(M) - I_{1-s}(X;Y|P_X \times W_{Y|X}))}.
\]

(74)
Then we can simplify (58). We have following lemma.

\[ P_{\phi|PM, WY|X} \geq e^{-\sum_{H_{1:s}(M)-H_{1:s}(X;Y)}(X;Y|X)} \]  

(75) means that there must exist at least one deterministic code \( \phi \) satisfying

\[ P_{\phi|PM, WY|X} \geq e^{-\sum_{H_{1:s}(M)-H_{1:s}(X;Y)}(X;Y|X)} \]  

(76)

Since \( 0 \leq t \leq 1 \), \( s \) is restricted to \( 0 \leq s \leq \frac{1}{2} \). So we obtain (77).

2) Conditional additive case: Now, we proceed to the case when the channel is conditional additive. Assume that \( X \) is a module and \( Y \) is given as \( X \times Z \). Then, the channel \( W \) is called conditional additive when there exists a joint distribution \( P_{XZ} \)

\[ W_{XZ|X}(x, z|x') = P_{XZ}(x - x', z). \]  

Then we can simplify (58). We have following lemma.

**Lemma 4.** When the channel is conditional additive channel, it follows that

\[ P_{\phi|PM, W_{XZ}|X} \leq P_M \times P_{XZ}\{P_{M}(M)P_{X|Z}(X|Z) \leq c \frac{1}{|X|}\} + \frac{1}{c}, \]  

(78)

**Proof:** By setting that \( P_X \) is the uniform distribution and choosing the random variables \( X = X' \) and \( Y = XZ \) to the right hand side of (58), we have

\[ (P_M \times P_{X'}, X_{W_{XZ}|X})((P_M \times P_{X'}, X_{W_{XZ}|X})(M, X', XZ) \leq c P_{X'} \times W_{XZ}(X', X, Z)) \]

\[ = (P_M \times P_{X'}, X_{W_{XZ}|X})\{P_{M}(m)\frac{1}{|X|}P_{XZ}(x - x', z) \leq c \frac{1}{|X|}P_Z(z)\} \]

\[ = (P_M \times P_{X}, X_{W_{XZ}|X'})\{P_{M}(m)P_{X|Z}(x - x'|z) \leq c \frac{1}{|X|}\} \]

\[ = P_M \times P_{XZ}\{P_{M}(M)P_{X|Z}(X|Z) \leq c \frac{1}{|X|}\}, \]

where \( P_Z(z) := \sum_x P_{XZ}(x, z) \). Hence, (58) can be simplified to

\[ P_{\phi|PM, WY|X} \leq P_M \times P_{XZ}\{P_{M}(M)P_{X|Z}(X|Z) \leq c \frac{1}{|X|}\} + \frac{1}{c}, \]  

(79)

Also we can simplify (66) and (67). We have following lemma.

**Lemma 5.** When the channel is conditional additive channel, it follows that

\[ P_{\phi|PM, W_{XZ}|X} \leq (\frac{e^{H_{1:s}(M)+H_{1:s}(X|Z)}}{|X|})^s, \]  

(80)

and

\[ P_{\phi|PM, W_{XZ}|X} \leq (\frac{e^{H_{1:s}(M)+H_{1:s}(X|Z)}}{|X|})^{1-s}. \]  

(81)

**Proof:** Firstly, we prove (80). \( e^{sH_{1-s}(X|Y)} \) is represented as:

\[ e^{sH_{1-s}(X|Y)} = \sum_{x,y} P_{XY}(x, y)^{1-s} P_Y(y)^s. \]  

(82)

Assume that \( Y = X \times Z \) and its random variable is \( Y = XZ \). Setting \( P_{XY} = P_X \times W_{XZ|X} \), \( P_Y(y) = P_Z(z) := \sum_x P_{XZ}(x, z) \) and \( P_X \) is uniform distribution, we have
\[ e^{sH^c_{1-s}(X|Y)} \]
\[ = \sum_{x',x,z} (P_X(x')W_{XZ|X}(x, z|x'))^{1-s} P_Z(z) \]
\[ = \sum_{x',x,z} \frac{1}{|X|^{1-s}} P_{XZ}(x - x', z)^{1-s} P_Z(z) \]
\[ = \left( \frac{1}{|X|} \right)^{1-s} \sum_x e^{sH^c_{1-s}(X|Z)} \]
\[ = \frac{e^{sH^c_{1-s}(X|Z)}}{|X|^s}. \] \hspace{1cm} (83)

Substituting (83) to (66), we have (80).

And also we have
\[ e^{-\frac{s}{1-s}I^c_{1-s}(X;Y|P_X \times W_{Y|X})} \]
\[ = e^{-\frac{s}{1-s}I^c_{1-s}(X;XZ|P_X \times W_{XZ|X})} \]
\[ = \sum_{x,z} \sum_{x'} P_X(x')W_{XZ|X}(x, y|x')^{1-s} \frac{1}{x'} \]
\[ = \sum_{x,z} \sum_{x'} \frac{1}{|X|} P_{XZ}(x - x', z)^{1-s} \frac{1}{x'} \]
\[ = \sum_{x} \frac{1}{|X|^{1-s}} \sum_{x'} \sum_{z} P_Z(z)(P_{X|Z}(x - x'|z)^{1-s}) \frac{1}{x'} \]
\[ = |X|^{1-s} \sum_{x} e^{\frac{s}{1-s}H^c_{1-s}(X|Z)} \]
\[ = |X|^{1-s} e^{\frac{s}{1-s}H^c_{1-s}(X|Z)}. \] \hspace{1cm} (84)

Substituting (84) to (67), we have (81).

C. Converse part

1) General case: Firstly, combining the idea of meta converse \cite{11} and \cite[Lemma 4]{12} and the general converse lemma for the joint source and channel coding \cite[Lemma 3.8.2]{8}, we obtain the following lemma for the single shot setting. The following lemma is the same as \cite[Lemma 3.8.2]{8} when \( Q_Y = \overline{W_Y} \).

\textbf{Lemma 6.} For any constant \( c > 0 \), any code \( \phi = (e, d) \) and any distribution \( Q_Y \) on \( \mathcal{Y} \), we have
\[ P_j\phi(P_M, W_{Y|X}) \geq \sum_m P_M(m)W_{Y|X=e(m)} \{ P_M(m)W_{Y|X=e(m)}(Y) \leq cQ_Y(Y) \} - c. \] \hspace{1cm} (85)

\textbf{Proof:} First, we set
\[ \mathcal{L} := \{(m, x, y) \in (\mathcal{M}, \mathcal{X}, \mathcal{Y})| P_M(m)W_{Y|X=x}(y) \leq cQ_Y(y) \}, \] \hspace{1cm} (86)
and for each \( (m, x) \in (\mathcal{M}, \mathcal{X}) \), define
\[ \mathcal{B}(m, x) := \{ y \in \mathcal{Y}| (m, x, y) \in \mathcal{L} \}. \] \hspace{1cm} (87)
Also, for decoder \( \psi \) and each \( m \in \mathcal{M} \), we define
\[ \mathcal{D}(m) := \{ y \in \mathcal{Y}| \psi(y) = m \}. \] \hspace{1cm} (88)
In addition, we define $P_{X|M}$ so that

$$P_{X|M}(x|m) = \begin{cases} 0 & x \neq e(m) \\ 1 & x = e(m). \end{cases} \quad (89)$$

Using this, we define

$$P_{MX}(m,x) := P_M(m)P_{X|M}(x|m), \quad (90)$$
$$P_{MXY}(m,x,y) := P_M(m)P_{X|M}(x|m)W_{Y|X=e(m)}(y). \quad (91)$$

Then,

$$\sum_m P_M(m)W_{Y|X=e(m)}\{P_M(m)W_{Y|X=e(m)}(Y) \leq cQ_Y(Y)\} \quad \sum_{m} P_{MXY}(m,x,y) \quad (92)$$

$$= \sum_{(m,x,y) \in \mathcal{L}} P_{MXY}(m,x,y) \quad \sum_{(m,x) \in M, X \in B(m,x)} P_{MX}(m,x)W_{Y|X}(y|x) \quad (93)$$

$$\leq \sum_{(m,x) \in M, X \in B(m,x) \cap D(m)} P_{MX}(m,x)W_{Y|X}(y|x) + \sum_{(m,x) \in M, X \in D^c(m)} P_{MX}(m,x)W_{Y|X}(y|x) \quad (94)$$
$$= \sum_{(m,x) \in M, X \in B(m,x) \cap D(m)} P_{MX}(m,x)W_{Y|X}(y|x) + \sum_{(m,x) \in M, X \in D^c(m)} P_{MX}(m,x)W_{Y|X}(y|x) \quad (95)$$
$$= \sum_{(m,x) \in M, X \in B(m,x) \cap D(m)} P_{MX}(m,x)W_{Y|X}(y|x) + P_{js}[\phi|P_M, W_{Y|X}] \quad (96)$$

The last equality follows since the error probability can be written as

$$P_{js}[\phi|P_M, W_{Y|X}] = \sum_{(m,x) \in M, X \in D^c(m)} P_{MX}(m,x)W_{Y|X}(y|x).$$

We notice here that

$$P_M(m)W_{Y|X=e(m)}(Y) \leq cQ_Y(Y)$$

for $y \in B(m,x)$. By substituting this into (92), the first term of (92) is

$$\sum_{(m,x) \in M, X \in D^c(m)} cP_{X|M}(x|m)Q_Y(y) \quad \sum_{(m,x) \in M, X \in D^c(m)} cP_{X|M}(x|m)Q_Y(y) \quad (97)$$

$$\leq \sum_{(m,x) \in M, X \in D^c(m)} cP_{X|M}(x|m)Q_Y(y) \quad \sum_{m \in M, y \in D(m)} cP_{X|M}(x|m)Q_Y(y) \quad (98)$$

$$= c \sum_{m \in M, y \in D(m)} cP_{X|M}(x|m)Q_Y(y) \quad \sum_{m \in M} cQ_Y(D(m)) = c, \quad (99)$$

which implies (85).

2) Conditional additive case: Now, we proceed to the conditional additive case given in (77). Applying (85) to the conditional additive case, we obtain following lemma.

**Lemma 7.** For arbitrary distribution $Q_Z \in \mathcal{P}(\mathcal{Z})$, we have

$$P_{js}(P_M, W_{X,Z|X}) \geq P_M \times P_{XZ}(P_M(M) \frac{P_{XZ}(X,Z)}{Q_Z(Z)} \leq c \frac{1}{|\mathcal{X}|}) - c. \quad (100)$$

**Proof:** For some $Q_Z \in \mathcal{P}(\mathcal{Z})$, we substitute

$$Q_Y(y) = Q_{XZ}(x,z) = \frac{1}{|\mathcal{X}|} Q_Z(z) \quad (101)$$
to (85). Then, the first term of the right hand side of (93) is
\[ \sum_m P_M(m)W_{Y|X=e(m)}\{P_M(m)W_{Y|X=e(m)}(Y) \leq cQ_Y(Y) \} \]
\[ = \sum_m P_M(m)W_{X|Z}e(m)\{P_M(m)W_{X|Z}(x, z|e(m)) \leq c\frac{1}{|X|}Q_Z(z) \} \]
\[ = \sum_m P_M(m)P_{XZ}\{P_M(m)P_{XZ}(x - e(m), y) \leq c\frac{1}{|X|}Q_Z(z) \} \]
\[ = P_M \times P_{XZ}\{P_M(M)P_{XZ}(X, Z) \leq c\frac{1}{|X|}Q_Z(z) \} \].

So, we obtain (93).

Similar to [11] Theorem 5], using the monotonicity of Rényi divergence, we obtain another type of converse lemma.

**Lemma 8.** We set \( R := \log |X| \). Then, it holds that
\[ \log P_{js}[\phi|P_M, W_{Y|X}] \geq \sup_{s>0, \rho \in \mathbb{R}, \sigma \geq 0} \left[ 1 + \frac{s}{s} \left[ -U(\rho(1+s)) \right] + U(\rho) + \log \left( 1 - 2e^{U(\rho-\rho^*(\sigma))} \right) \right] \]
\[ \geq \sup_{s>0, \theta(\rho(a(R)))<\rho<1} \left[ 1 + \frac{s}{s} \left[ -U(\rho(1+s)) \right] + U(\rho) + \log \left( 1 - 2e^{U(\rho-\rho^*(\sigma))} \right) \right], \] (95)

where
\[ U(\cdot) := U[XZ, QZ; 1](\cdot), \]
\[ \theta(\cdot) := \theta[XZ, QZ; 1](\cdot), \]
\[ a(\cdot) := a[XZ, QZ; 1](\cdot). \] (98)

**Proof:** In this proof, we use the notation defined in (96)–(98).

For arbitrary \( \rho \in \mathbb{R} \), we define following new distributions.
\[ P_{M,\rho}(m) := P_M(m)^{1-\rho}e^{-\rho H_1(\cdot)}, \] (99)
\[ P_{XZ,\rho}(x, z) := P_{XZ,\rho}(x, z)^{1-\rho}e^{-\rho H_1(\cdot)}P_{XZ}(z). \] (100)

Using these, we define following joint distribution.
\[ (P_M \times P_{XZ,\rho})(m, x, z) := P_{M,\rho}(m)P_{XZ,\rho}(x, z) \]
\[ = (P_M \times P_{XZ})(m, x, z)^{1-\rho}Q_Z(z)^{1-\rho}e^{-U(\rho)}. \] (101)

For arbitrary code \( \phi = (e, d) \), we define
\[ \alpha := P_{js}[\phi|P_M, W_{XZ}|X]. \] (102)

And also, when the source distribution is \( P_{M,\rho} \) and the channel is conditional additive channel \( W_{XZ|X,\rho} \) defined by
\[ W_{XZ|X,\rho}(x, z|x') := P_{XZ,\rho}(x - x', z), \] (103)
we define
\[ \beta := P_{js}[\phi|P_{M,\rho}, W_{XZ}|X,\rho]. \] (104)

Then, for any \( s > 0 \), by the monotonicity of the Rényi divergence, we have
\[ sD_{1+s}(P_M \times P_{XZ,\rho}||P_M \times P_{XZ}) \geq \log[\beta^{1+s}\alpha^{-s} + (1 - \beta)^{1+s}(1 - \alpha)^{-s}] \geq \log \beta^{1+s}\alpha^{-s}. \] (105)
Thus, we have

$$\log \alpha \geq \frac{-sD_{1+s}(P_M \times P_{XZ},\rho||P_M \times P_{XZ}) + (1 + s) \log \beta}{s}. \quad (106)$$

For the Rényi divergence, we have

$$sD_{1+s}(P_M \times P_{XZ},\rho||P_M \times P_{XZ})$$

$$= \log \sum (P_M \times P_{XZ},\rho)^{1+s}(P_M \times P_{XZ})^{-s}$$

$$= \log \sum (P_M \times P_{XZ})^{1-(1+s)\rho}(Q_{Z})^{1+s)\rho e^{(1+s)(U(\rho))}$$

$$= U((1 + s)\rho) - (1 + s)U(\rho). \quad (107)$$

In addition, substituting $P_M = P_M(m)^{1-\rho e^{-\rho H_{1-\rho}(M)}}$ and $P_{XZ} = P_{XZ,\rho}(x, z)^{1-\rho e^{-\rho H_{1-\rho}(P_{XZ}|Q_{Z})}}$ into (93), we have

$$1 - \beta \leq (P_M \times P_{XZ,\rho})\{P_M \times P_{XZ,\rho}(m, x, z) > e^{\frac{1}{|X|}Q_{Z}(z)}\} + c. \quad (108)$$

For any $\sigma \geq 0$, the first term of right hand side of (108) can be evaluated as:

$$P_M \times P_{XZ,\rho}(m, x, z) \geq e^{\frac{1}{|X|}Q_{Z}(z)} \Rightarrow \sum_{m,x,z} (P_M \times P_{XZ,\rho}(m, x, z)$$

$$\leq \left(P_M \times P_{XZ,\rho}(m, x, z) \left(\frac{P_M \times P_{XZ,\rho}(m, x, z)}{e^{\frac{1}{|X|}Q_{Z}(z)}}\right)^{\sigma}\right)$$

$$= e^{\sigma D(P_M \times P_{XZ,\rho}||Q_{Z}(z)) + \sigma (\log |X| - \log c).}$$

Thus, by setting $c$ so that

$$\sigma D(P_M \times P_{XZ,\rho}||Q_{Z}(z)) + \sigma (\log |X| - \log c) = \log c, \quad (109)$$

we have

$$1 - \beta \leq 2e^{\frac{\sigma D(P_M \times P_{XZ,\rho}||Q_{Z}(z)) + \sigma \log |X|}{1 + \sigma}}. \quad (110)$$

For the Rényi divergence in (110), we have

$$\sigma D(P_M \times P_{XZ,\rho}||Q_{Z}(z))$$

$$= \log \sum (P_M \times P_{XZ,\rho})(m, x, z)^{1+\sigma}Q_{Z}(z)^{-\sigma}$$

$$= \log \sum \left((P_M \times P_{XZ})(m, x, z)^{1-\rho Q_{Z}(z)^{\rho e^{-U(\rho)}}}Q_{Z}(z)^{-\sigma}\right)$$

$$= \log \sum (P_M \times P_{XZ})(m, x, z)^{1-(1-\rho)^{\sigma}Q_{Z}(z)^{\rho(1-\rho)^{\sigma}}} - (1 + \sigma)U(\rho)$$

So, we have

$$\log \beta \geq \log \left(1 - 2e^{\frac{\sigma D(P_M \times P_{XZ,\rho}||Q_{Z}(z)) + \sigma \log |X|}{1 + \sigma}}\right). \quad (111)$$

Combining (106), (107) and (111), we obtain (94).

Now, we restrict the range of $\rho$ so that $\theta(a(R)) < \rho < 1$, and take

$$\sigma = \frac{\rho - \theta(a(R))}{1 - \rho}, \quad (112)$$

we obtain the second inequality.
V. \textit{n}-fold Markovian conditional additive channel

A. Formulation for general case

Firstly, we give general notations for channel coding when the message obeys Markovian process. We assume that the set of messages is $\mathcal{M}^k$. Then, we assume that the message $M^k = (M_1, \ldots, M_k) \in \mathcal{M}^k$ is subject to the Markov process with the transition matrix $\{W_s(m|m')\}_{m,m' \in \mathcal{M}}$. We denote the distribution for $M^k$ by $P_{M^k}$.

Now, we consider very general sequence of channels with the input alphabet $\mathcal{X}^n$ and the output alphabet $\mathcal{Y}^n$. In this case, the transition matrix is $\{W_{Y^n|X^n}(y^n|x^n)\}_{x^n \in \mathcal{X}^n, y^n \in \mathcal{Y}^n}$. Then, a channel code $\phi = (e, d)$ consists of one encoder $e : \mathcal{M}^k \to \mathcal{X}^n$ and one decoder $d : \mathcal{Y}^n \to \mathcal{M}^k$. Then, the average decoding error probability is defined by

$$P_j[\phi, k, n|W_s, W_{Y^n|X^n}] := \sum_{m^k \in \mathcal{M}^k} P_{M^k}(m^k) W_{Y^n|X^n}(\{y^n : d(y^n) \neq m^k\}|e(m^k)).$$

(113)

For notational convenience, we introduce the error probability under the above condition:

$$P_j(k, n|W_s, W_{Y^n|X^n}) := \inf_{\phi} P_j[\phi, k, n|W_s, W_{Y^n|X^n}].$$

(114)

When there is no possibility for confusion, we simplify it to $P_j(k, n)$. Instead of evaluating the error probability $P_j(n, k)$ for given $n, k$, we are also interested in evaluating

$$K(n, \varepsilon|W_s, W_{Y^n|X^n}) := \sup \{k : P_j(n, k|W_s, W_{Y^n|X^n}) \leq \varepsilon\}$$

(115)

for given $0 \leq \varepsilon \leq 1$.

B. Formulation for Markovian conditional additive channel

In this section, we address an \textit{n}-fold Markovian conditional additive channel [1]. That is, we consider the case when the joint distribution for the additive noise obeys the Markov process. To formulate our channel, we prepare notations. Consider the joint Markovian process on $\mathcal{X} \times \mathcal{Z}$. That is, the random variables $X^n = (X_1, \ldots, X_n) \in \mathcal{X}^n$ and $Z^n = (Z_1, \ldots, Z_n) \in \mathcal{Z}^n$ are assumed to be subject to the joint Markovian process defined by the transition matrix $\{W_c(x, z|x', z')\}_{x, z \in \mathcal{X}, z' \in \mathcal{Z}}$. We denote the joint distribution for $X^n$ and $Z^n$ by $P_{X^n, Z^n}$. Now, we assume that $\mathcal{X}$ is a module, and consider the channel with the input alphabet $\mathcal{X}^n$ and the output alphabet $(\mathcal{X} \times \mathcal{Z})^n$. The transition matrix for the channel $W_{X^n, Z^n|X^n, Z^n}$ is given as

$$W_{X^n, Z^n|X^n, Z^n}(x^n, z^n|x'^n, z'^n) = P_{X^n, Z^n}(x^n - x'^n, z^n)$$

(116)

for $z^n \in \mathcal{Z}^n$ and $x^n, x'^n \in \mathcal{X}^n$. Also, we denote $\log |\mathcal{X}|$ by $R$. In the following discussion, we use the channel capacity $C := \log |\mathcal{X}| - H(W|X|Z)$, which is shown in [1]. In this case, we denote the average error probability $P_j[\phi, k, n|W_s, W_{X^n, Z^n|X^n}]$ and the minimum average error probability $P_j(k, n|W_s, W_{X^n, Z^n|X^n})$ by $P_{jca}[\phi, k, n|W_s, W_c]$ and $P_{jca}(k, n|W_s, W_c)$, respectively. Then, we denote the maximum size $K(n, \varepsilon|W_s, W_{Y^n|X^n})$ by $K_{ca}(n, \varepsilon|W_s, W_c)$. When we have no possibility for confusion, we simplify them to $P_{jca}[\phi, k, n]$, $P_{jca}(k, n)$, and $K_{ca}(n, \varepsilon)$, respectively.

In the following discussion, we assume Assumption 1 or 2 for the joint Markovian process described by the transition matrix $\{W_c(x, z|x', z')\}_{x, z \in \mathcal{X}, z' \in \mathcal{Z}}$. The paper [1] derives the single-letterized channel capacity under Assumption 1. Among author’s knowledge, the class of channels satisfying Assumption 1 is the largest class of channels whose channel capacity is known. When $\mathcal{Z}$ is singleton and the channel is the noiseless channel given by identity transition matrix $I$, our problem is the source coding with Markovian source. In this case, the memory size is equal to the cardinality $|\mathcal{X}|^k$, we denote the minimum error probability $P_{jca}(k, n|W_s, I)\mathcal{X}|$ by $P_s(k, n|W_s)$.

C. Finite-length bound

1) Assumption 1: Now, we assume Assumption 1. Combining Proposition $5$ and $8[0]$ of Lemma $5$, we have an upper bound of the minimum error probability as follows.

**Theorem 1** (Direct Bound). When Assumption 1 holds, setting $R = \log |\mathcal{X}|$, we have

$$\log P_j(k, n) \leq \inf_{s \in (0, 1)} \left[-nsR + (n - 1)U|W_s, W_c, \frac{k - 1}{n - 1}(s) + \delta(s)\right],$$

(117)
where

$$\delta(s) := \overline{\omega}_W(s) + \overline{\omega}_\ell(s).$$  \hspace{1cm} (118)

Combining Proposition 3 and (94) of Lemma 8 we have a lower bound of the minimum error probability as follows.

**Theorem 2** (Converse bound). When Assumption 1 holds, setting \( R = \log |\mathcal{X}| \), we have

$$\log P_j(k,n) \geq \sup_{s>0,\theta(a(R))<\rho<1} \frac{1+s}{s} \left[ - (n-1) \frac{U(\rho(1+s))}{1+s} + (n-1)U(\rho) + \delta_1(s,\rho) \right]$$

$$+ \log \left( 1 - 2e^{(n-1)((\rho-\theta(a(R)))a(R)+U(\theta(a(R))-U(\rho)))+\delta_2(\rho)} \right),$$  \hspace{1cm} (119)

where

$$U(\cdot) := U[W_k,W_{<\ell} ; \frac{\theta - 1}{n-1} ](\cdot),$$

$$\theta(\cdot) := \theta[W_k,W_{<\ell} ; \frac{\theta - 1}{n-1} ](\cdot),$$

$$a(\cdot) := a[W_k,W_{<\ell} ; \frac{\theta - 1}{n-1} ](\cdot),$$

and where

$$\delta_1(s,\rho) := -\frac{\overline{\omega}_W((1+s)\rho) + \overline{\omega}_\ell((1+s)\rho)}{1+s} + \tilde{\omega}_W(\rho) + \tilde{\omega}_\ell(\rho),$$

$$\delta_2(\rho) := \frac{(1-\rho)(\overline{\omega}_W((\rho(a(R))) + \overline{\omega}_\ell((\rho(a(R))) - (1-\rho(a(R))))(\tilde{\omega}_W(\rho) - \tilde{\omega}_\ell(\rho)) - (\rho(a(R)) - \rho)R}{1-\rho(a(R))}.$$  \hspace{1cm} (123)

**Proof:** We first substitute \( P_{XZ} = P_{X^n Z^n} Q_Z = P_{Z^n} \) to (94) of Lemma 8 and use Proposition 3. Then, we restrict the range of \( \rho \) as \( \theta(a(R)) < \rho < 1 \) and set \( \sigma = \frac{\rho - \theta(a(R))}{1-\rho(a(R))} \). Then, we have the claim of the Theorem. \( \blacksquare \)

2) Assumption 2: Next, we assume Assumption 2. Combining Proposition 4 and (81) of Lemma 5 we have an upper bound of the minimum error probability as follows.

**Theorem 3** (Direct Bound). When Assumption 2 holds, setting \( R = \log |\mathcal{X}| \), we have

$$\log P_j(k,n) \leq \inf_{s \in [0,\frac{1}{2}]} -nsR + \frac{(n-1)U[W_k,W_{<\ell} ; \frac{\theta - 1}{n-1} ](s)}{1-s} + \xi(s),$$  \hspace{1cm} (125)

where

$$\xi(s) := \frac{\overline{\omega}_W(s) + \overline{\omega}_\ell(s)}{1-s}.$$  \hspace{1cm} (126)

Combining Proposition 5 and (94), we have a lower bound of the minimum error probability as follows.

**Theorem 4** (Converse Bound). When Assumption 2 holds, setting \( R = \log |\mathcal{X}| \), we have

$$\log P_j(k,n) \geq \sup_{s>0,\theta(a(R))<\rho<1} \frac{1+s}{s} \left[ - (n-1) \frac{U_\theta(a(R))(\rho(1+s))}{1+s} + (n-1)U_\theta(a(R))(\rho) + \delta_1(s,\rho) \right]$$

$$+ \log \left( 1 - 2e^{(n-1)((\rho-\theta(a(R)))a(R)+U(\theta(a(R)))-U(\rho)))+\delta_2(\rho)} \right).$$  \hspace{1cm} (127)
where
\begin{align}
\delta_1 & := \zeta_{W_c}(\rho, \theta(a(R))) - \zeta_{W_c}((1 + s)\rho, \theta(a(R))), \\
\delta_2 & := \\
(1 - \rho)\{\zeta_{W_c}(\theta(a(R))) + \zeta_{W_c}(\theta(a(R)), \theta(a(R)))\} - (1 - \theta(a(R)))\{\delta_{W_c}(\rho) - \zeta_{W_c}(\rho, \theta(a(R)))\} - (\theta(a(R)) - \rho)R
\end{align}
and where
\begin{align}
\theta(\cdot) & := [W_s, W_c, \uparrow; \frac{k - 1}{n - 1}]\{\cdot\}, \\
a(\cdot) & := a[W_s, W_c, \uparrow; \frac{k - 1}{n - 1}]\{\cdot\}, \\
U^\uparrow(\cdot) & := U[W_s, W_c, \uparrow; \frac{k - 1}{n - 1}]\{\cdot\}, \\
U_{\theta(a(R))}(\cdot) & := U[W_s, W_c, \theta(a(R)); \frac{k - 1}{n - 1}]\{\cdot\}.
\end{align}

Proof: We first substitute \(P_{X|Z} = P_{X^n|Z^n} Q_Z = P_{Z^n}^{1-\theta(a(R))}\) to (114) of Lemma 8 and use Proposition 4 and 5. Then, we restrict the range of \(\rho\) as \(\theta(a(R)) < \rho < 1\) and set \(\sigma = \frac{\rho - \theta(a(R))}{1 - \rho}\). Then, we have the claim of the Theorem.

\section{D. Large deviation bounds}

In this section, for some constant \(r > 0\), we fix the coding rate \(\frac{k}{n}\) to be \(r\) by using the real number \(R := \log |\mathcal{X}|\).

1) Assumption 1: Now, we assume Assumption 1. Using Theorem 11, we can upper bound the exponent of the minimum error probability as follows. By setting \(k = nr\), taking logarithm and normalizing the both side of (117), we obtain following theorem.

**Theorem 5** (Direct Bound). Assume that Assumption 1 holds and set \(R = \log |\mathcal{X}|\). When the rate \(r\) satisfies \(rH^{W_c}(M) + H^{W_c}(X|Z) < R\), we have
\begin{equation}
\liminf_{n \to \infty} - \frac{1}{n} \log P_j(nr, n) \geq E_{1,j}(r),
\end{equation}
where \(E_{1,j}(r)\) is error exponent function defined as
\begin{equation}
E_{1,j}(r) := \sup_{s \in (0,1)} [sR - U[W_s, W_c, \downarrow; r]\{s\}].
\end{equation}

**Remark 7.** This theorem is a conditional additive version of [6] Proposition 1.

Using Theorem 2, we can lower bound exponent of the minimum error probability as follows. By setting \(k = nr\), we obtain following theorem.

**Theorem 6** (Converse Bound). Assume that Assumption 1 holds and set \(R = \log |\mathcal{X}|\). When the rate \(r\) satisfies \(rH^{W_c}(M) + H^{W_c}(X|Z) < R < rH^{W_c}(M) + H^{W_c}(X|Z)\), we have
\begin{equation}
\limsup_{n \to \infty} - \frac{1}{n} \log P_j(rn, n) \leq \overline{E}_{1,j}(r),
\end{equation}
where \(\overline{E}_{1,j}(r)\) is error exponent function defined as
\begin{equation}
\overline{E}_{1,j}(r) := \theta(a(R))a(R) - U[W_s, W_c, \downarrow; r]\{\theta(a(R))\} = \sup_{\theta \leq 1} \frac{\theta R - U(\theta)}{1 - \theta},
\end{equation}
\begin{align}
\zeta_{W_c}(\rho) & := \sup_{\theta \leq 1} \frac{\theta R - U(\theta)}{1 - \theta}, \\
\zeta_{W_c}(\theta(a(R))) & := \sup_{\theta \leq 1} \frac{\theta R - U(\theta)}{1 - \theta}. 
\end{align}
where

\[ U(\cdot) := U[W_s, W_c, \uparrow; r](\cdot), \]
\[ \theta(\cdot) := \theta[W_s, W_c, \uparrow; r](\cdot), \]
\[ a(\cdot) := a[W_s, W_c, \uparrow; r](\cdot). \]

**Remark 8.** This theorem is a conditional additive version of [6, Theorem 2].

**Proof:** From Theorem 2 we have

\[ \limsup_{n \to \infty} \frac{1}{n} \log P_j(k, n) \leq \frac{1 + s}{s} \left[ \frac{U(\rho(1 + s))}{1 + s} - U(\rho) + \delta_1(s, \rho) \right] \]
\[ = \rho s \left[ \frac{U(\rho(1 + s)) - U(\rho)}{s \rho} \right] - U(\rho) \]
\[ \to \rho u(\rho) - U(\rho) \quad \text{(as } s \to 0) \]
\[ \to \theta(a(R))u(\theta(a(R))) - U(\theta(a(R))) \quad \text{(as } \rho \to \theta(a(R))) \]
\[ = \theta(a(R))a(R) - U(\theta(a(R))), \]

where \( u(\cdot) := u[W_s, W_c, \uparrow; r](\cdot). \)

This part will be done similar to [11, Theorem 21]. In this case, the direct part bound does not coincide with the converse part bound, in general. To derive the exact value of the exponent, we need a stronger assumption.

2) **Assumption 2:** Next, we assume Assumption 2, which is stronger than Assumption 1. Using Theorem 3, we can upper bound the exponent of the minimum error probability as follows. By setting \( k = nr \), taking logarithm and normalizing the both side of (125), we obtain following theorem.

**Theorem 7** (Direct Bound). Assume that Assumption 2 holds and set \( R = \log |X| \). When the rate \( r \) satisfies \( rH_{M}(W_s) + H_{M}(X|Z) < R \), we have

\[ \liminf_{n \to \infty} \frac{1}{n} \log P_j(rn, n) \geq E_{2,j}(r), \]

where \( E_{2,j} \) is an error exponent function defined as

\[ E_{2,j}(r) := \sup_{s \in [0, \frac{1}{2}]} \frac{sR - U[W_s, W_c, \uparrow; r](s)}{1 - s}. \]

Using Theorem 4 we can lower bound the exponent of the minimum error probability as follows. By setting \( k = nr \), we obtain following theorem.

**Theorem 8** (Converse Bound). Assume that Assumption 2 holds and set \( R = \log |X| \). When the rate \( r \) satisfies \( rH_{M}(W_s) + H_{M}(X|Z) < R < rH_{0}^{W}(M) + H_{0}^{W}(X|Z) \), we have

\[ \limsup_{n \to \infty} \frac{1}{n} \log P_j(rn, n) \leq \overline{E}_{2,j}(r), \]

where \( \overline{E}_{2,j}(r) \) is an error exponent function defined as

\[ \overline{E}_{2,j}(r) := \theta(a(R))a(R) - U(\theta(a(R))) \]
\[ = \sup_{0 \leq \theta \leq 1} \frac{\theta R - U(\theta)}{1 - \theta}, \]

where

\[ U(\cdot) := U[W_s, W_c, \uparrow; r](\cdot), \]
\[ \theta(\cdot) := \theta[W_s, W_c, \uparrow; r](\cdot), \]
\[ a(\cdot) := a[W_s, W_c, \uparrow; r](\cdot). \]
Thus, the lower bound in Theorem 7 coincides with the upper bound in Theorem 8. So we have

**Remark 10.** Theorem 9 is conditional additive channel version of [3, Theorem 1].

Assume that Assumption 1 holds. Then, for arbitrary

\[ \lim_{n \to \infty} \frac{1}{n} \log P_j(k, n) \leq \frac{1 + s}{s} \left[ \frac{U_{\theta(a(R))}(\rho(1+s))}{1+s} - U_{\theta(a(R))}(\rho) \right] \]

\[ = \rho \frac{U_{\theta(a(R))}(\rho(1+s)) - U_{\theta(a(R))}(\rho)}{s\rho} - U(\rho) \]

\[ \to \rho u_{\theta(a(R))}(\rho) - U_{\theta(a(R))}(\rho) \quad (\text{as } s \to 0) \]

\[ \to \theta(a(R)) u(\theta(a(R))) - U(\theta(a(R))) \quad (\text{as } \rho \to \theta(a(R))) \]

\[ = \theta(a(R)) a(R) - U(\theta(a(R))), \quad (149) \]

where \( u_{\theta(a(R))}(\cdot) := u(W_s, W_c, \theta(a(R)); r)(\cdot) \) and \( u^*(\cdot) := u(W_s, W_c, \uparrow; r)(\cdot). \)

**Corollary 2.** Combining the above theorems, we obtain the exact expression of the exponent of the minimum error probability when we define the critical rate \( R_{cr} \) as

\[ R_{cr} := R(W_s, W_c, \uparrow; r) \left( u(W_s, W_s, \uparrow; r) \left( \frac{1}{2} \right) \right). \quad (150) \]

For \( R \leq R_{cr} \), we can rewrite the upper bound in Theorem 8 as

\[ \sup_{s \in [0, 1]} \frac{\theta R - U(\theta)}{1 - \theta} = \theta(a(R)) a(R) - U(\theta(a(R))). \quad (151) \]

Thus, the lower bound in Theorem 7 coincides with the upper bound in Theorem 8. So we have

\[ \lim_{n \to \infty} \frac{1}{n} \log P_j(rn, n) = \sup_{s \in [0, 1]} \frac{\theta R - U(\theta)}{1 - \theta} = \theta(a(R)) a(R) - U(\theta(a(R))). \quad (152) \]

**Remark 9.** Now, we consider the case when \( Z \) is singleton and the transition matrix \( W_c \) of the additive noise is the identity matrix \( I \), which is the same as the data compression with Markovian source. Since \( C = \log |\mathcal{X}| \), we have

\[ \lim_{n \to \infty} \frac{1}{n} \log P_s(nr, n|W_s) \leq \sup_{s \in (0, 1)} [sR - rsH_{1-s}^W(M)] \quad (153) \]

\[ \lim_{n \to \infty} \frac{1}{n} \log P_s(nr, n|W_s) \geq \sup_{\theta \leq 1} \frac{\theta R - r\theta H_{1-s}^W(M)}{1 - \theta}, \quad (154) \]

which is the same as the result of [1] Theorem 12.

**E. Moderate deviation bound**

Next, we proceed to the rate is in the moderate deviation regime, in which, the coding rate \( r_n \) behaves as \( r_n := \frac{k}{n} = \frac{C}{H_{W_s}} - \delta n^{-t} \) with \( t \in (0, \frac{1}{2}) \). Then, the minimum error probability can be evaluated as follows.

**Theorem 9.** Assume that Assumption 1 holds. Then, for arbitrary \( t \in (0, \frac{1}{2}) \) and \( \delta > 0 \), it holds that

\[ \lim_{n \to \infty} \frac{1}{n^{1-2t}} \log P_j \left( \frac{nC}{HW_s} - \delta n^{-t}, n \right) = \frac{1}{2} \cdot \frac{\delta^2}{(H_{W_s}^r(M))^2} \left[ \frac{C}{H_{W_s}^r(M)} V_{W_s}(M) + V_{W_s}(X|Z) \right]. \quad (155) \]

**Remark 10.** Theorem 2 is conditional additive channel version of [3] Theorem 1.

**Proof:** From Theorem 1, we obtain

\[ -\log P_j(k, n) \geq \sup_{s \in (0, 1)} [nsR - (k - 1)sH_{1-s}^W(M) - (n - 1)sH_{1-s}^W(X|Z) - \delta(s)] \]

\[ \geq \sup_{s \in (0, 1)} [nsR - (k - 1)sH_{1-s}^W(M) - (n - 1)sH_{1-s}^W(X|Z)] + \inf_{s \in (0, 1)} [-\delta(s)] \]

\[ \geq n[sR - r_n sH_{1-s}^W(M) - s'H_{1-s'}^W(X|Z)] + o(n^{1-2t}). \quad (156) \]
By (19), Taylor expansions of $H_{1-s}(M)$ and $H_{1-s}(X|Z)$ in the neighborhood of $s = 0$ are

$$H_{1-s}(M) = H^W(M) + \frac{1}{2} s V^W(M) + o(s), \quad (157)$$

$$H_{1-s}(X|Z) = H^W(X|Z) + \frac{1}{2} s V^W(X|Z) + o(s). \quad (158)$$

Substituting these expansions into (156), we obtain

$$-\log P_j\left(\frac{nC}{H^W} - \delta, n\right) \geq \frac{-s^2}{2} \left(\frac{C}{H^W} V^W(M) + V^W(X|Z)\right) + s' \delta n^{-t} H^W(M)$$

$$- s'(C + H^W(X|Z) - \log |X'|) - \frac{\delta n^{-t} s^2}{2} + o(s^2) + o(n^{1-2t}). \quad (159)$$

Now, we set $s' := \frac{\delta n^{-1} H^W(M)}{n V^W(M) + V^W(X|Z)}$ which satisfies $s \in [0, 1]$ for enough large $n$. Then, we have

$$-\log P_j(k, n) \geq \frac{n}{s} \left(\frac{\delta n^{-2t} (H^W)^2}{2(\frac{C}{H^W}) V^W(M) + V^W(X|Z)}\right) + o(n^{-2t})$$

$$\geq -\frac{n}{s} \left(\frac{\delta^2}{2(\frac{C}{H^W}) V^W(M) + V^W(X|Z)}\right) + o(n^{-2t}), \quad (160)$$

that is,

$$\liminf_{n \to \infty} -\frac{1}{n^{1-2t}} \log P_j(k, n) \geq \frac{1}{2} \cdot \left(\frac{\delta^2}{(\frac{C}{H^W}) V^W(M) + V^W(X|Z)}\right). \quad (161)$$

On the other hands, by choosing $\rho = \frac{1}{\frac{C}{H^W}(\frac{C}{H^W})^2 + \frac{C}{H^W} V^W(M) + V^W(X|Z)}$, Theorem 2 implies that

$$\limsup_{n \to \infty} -\frac{1}{n^{1-2t}} \log P_j(k, n) \leq \lim_{n \to \infty} n^{2t} \frac{1 + s}{s} \rho [r_n (H_{1-(1+s)}(M) - H^W(M)) + (H^W(X|Z) - H^W(X|Z))]$$

$$\leq \lim_{n \to \infty} n^{2t} \frac{1 + s}{s} \rho \left(\frac{C}{H^W} V^W(M) + V^W(X|Z)\right)$$

$$= \lim_{n \to \infty} n^{2t} (1 + s) \rho \left(\frac{C}{H^W} V^W(M) + V^W(X|Z) - \delta n^{-t} V^W(M)\right)$$

$$= (1 + s) \frac{1}{2} \cdot \left(\frac{\delta^2}{(\frac{C}{H^W}) V^W(M) + V^W(X|Z)}\right)$$

$$\leq \frac{1}{2} \cdot \left(\frac{\delta^2}{(\frac{C}{H^W}) V^W(M) + V^W(X|Z)}\right) \quad (s \to 0). \quad (162)$$

Now, we consider the case when $Z$ is singleton and the transition matrix $W_c$ of the additive noise is the identity matrix $I$. When $k = \frac{C}{H^W(M)} n - \frac{C}{H^W(M)^2} (\frac{C}{H^W(M)})^{-t} \delta' n^{1-t}$, the minimum error probability $P_s(k, n|W_s)$ is characterized as follows. Setting $\delta = \frac{C}{H^W(M)^2} (\frac{C}{H^W(M)})^{-t} \delta'$ i.e., $k = \frac{C}{H^W(M)} n - \frac{C}{H^W(M)^2} (\frac{C}{H^W(M)})^{-t} \delta' n^{1-t}$, the minimum error probability $P_s(k, n|W_s)$ and using $C = \log |X'|$, we obtain
\[
\lim_{k \to \infty} - \frac{1}{k^{1-2t}} \log P_j(k, n) \\
= \lim_{k \to \infty} - \left[ \frac{C}{H^{W_s}(M)} n - C(\frac{C}{H^{W_s}(M)})^{-t\delta' n^{-t}} \right]^{1-2t} \log P_j(k, n) \\
= \lim_{k \to \infty} - \left( \frac{1}{H^{W_s}(M)} \right)^{1-2t} \left( 1 - H^{W_s}(M)(\frac{C}{H^{W_s}(M)})^{-t\delta' n^{-t}} \right)^{1-2t} \log P_j(k, n) \\
= \left( \frac{H^{W_s}(M)}{C} \right)^{1-2t} \lim_{k \to \infty} - \frac{1}{n^{1-2t}} \log P_j(k, n) \left( 1 - H^{W_s}(M)(\frac{C}{H^{W_s}(M)})^{-t\delta' n^{-t}} \right)^{1-2t} \\
= \left( \frac{H^{W_s}(M)}{C} \right)^{1-2t} \frac{1}{2} \frac{\delta^2}{(H^{W_s}(M))^2} \left[ \frac{C}{H^{W_s}(M)} V^{W_s}(M) \right] \\
= \left( \frac{C^2}{H^{W_s}(M)} \right)^{1-2t} \frac{1}{2} \frac{\delta^2}{(H^{W_s}(M))^2} \left[ \frac{C}{H^{W_s}(M)} V^{W_s}(M) \right] \\
= \frac{\delta^2}{2V^{W_s}(M)}. \tag{163}
\]

This result coincides with \[\text{I.} \text{ Theorem 11}\].

VI. NUMERICAL EXAMPLE

Finally, to demonstrate the advantage of our finite-length bounds, we numerically evaluate the achievability bound in Theorem 3 and the converse bound in Theorem 4. Due to the efficient construction of our bounds, we could calculate both bounds with huge size \( n = 1 \times 10^6 \) because the calculation complexity behaves as \( O(1) \).

We employ the following parametrization \( W(p, q) \) for the binary transition matrix:

\[
W(p, q) := \begin{bmatrix}
1 - p & q \\
p & 1 - q
\end{bmatrix}. \tag{164}
\]

We consider the case when \( W_s = W_c = W(0.1, 0.2) \). The optimal transmission rate \( \frac{C}{H^{W_s}(M)} \) and the dispersion \( \frac{1}{(H^{W_s}(M))^2} \left[ \frac{C}{H^{W_s}(M)} V^{W_s}(M) + V^{W_s}(X|Z) \right] \) are calculated to be 0.807317 and 6.12809, respectively. Also, the exponent \( E(0.75) \) is calculated to be 0.0002826, which is approximated by \( E_{md}(0.75n, n)/n = 0.0002680 \).

When \( n = 10000 \), Fig. 1 calculates the upper and lower bounds of \( - \frac{1}{n} \log P_j(k, n) \) based on Theorems 3 and 4. Also, it shows the comparison them with the approximations \( nE(k/n) \) and \( E_{md}(k, n) \) by Theorems 7 and 2. Fig. 2 addresses the quantity \( \frac{1}{n} \log P_j(0.75n, n) \) in the same way.

ACKNOWLEDGMENTS

MH is very grateful to Professor Vincent Y. F. Tan and Professor Shun Watanabe for helpful discussions and comments. The works reported here were supported in part by a MEXT Grant-in-Aid for Scientific Research (B) No. 16KT0017, the Okawa Research Grant and Kayamori Foundation of Informational Science Advancement.

REFERENCES

[1] M. Hayashi and S. Watanabe, “Finite-Length Analyses for Source and Channel Coding on Markov Chains,” arXiv:1309.7528 (2013).
[2] D. Wang, A. Inger, and Y. Kochman, “The Dispersion of Joint Source-Channel Coding,” Proc. 49th Annual Allerton Conf., Allerton House, Monticello, IL, USA, 2011, pp. 180 - 187; arXiv:1109.6310
[3] Y. Y. F. Tan, S. Watanabe, and M. Hayashi, “Moderate Deviations for Joint Source-Channel Coding of Systems With Markovian Memory”, in Proc. 2014 IEEE ISIT, Honolulu, HI, USA, June 29 - July 4, 2014, pp.1687
[4] V. Kostina and S. Verdú, “Lossy joint source-channel coding in the finite blocklength regime,” Proceedings of 2012 IEEE International Symposium on Information Theory, 1-6 July 2012, Cambridge, MA, USA, pp. 1553-1557.
[5] A. T. Campo, G. Vazquez-Vilar, A. G. i Fàbregas, T. Koch and A. Martínez, “Achieving Csiszár’s Source-Channel Coding Exponent with Product Distributions,” Proceedings of 2012 IEEE International Symposium on Information Theory, 1-6 July 2012, Cambridge, MA, USA, pp. 1548 - 1552.
Fig. 1. Graphs of the upper and lower bounds of $-\log P_j(k, n)$ based on Theorems 3 and 4 when $n = 10000$. Blue line is the upper bound of $-\log P_j(k, n)$ based on Theorem 4. Black line is the lower bound of $-\log P_j(k, n)$ based on Theorem 3. Yellow line is $nE(k/n)$. Green line is $E_{md}(k, n)$.

Fig. 2. Graphs of the upper and lower bounds of $-\frac{1}{n}\log P_j(0.75n, n)$ based on Theorems 3 and 4. Blue line is the upper bound of $-\frac{1}{n}\log P_j(0.75n, n)$ based on Theorem 4. Black line is the lower bound of $-\frac{1}{n}\log P_j(0.75n, n)$ based on Theorem 3. Yellow line is $E(0.75)$.

[6] Y. Zhong, F. Alajaji and L. Lorne Campbell, “Joint Source-Channel Coding Error Exponent for Discrete Communication Systems With Markovian Memory,” IEEE Trans. Inf. Theory, vol. 53, No. 12, 4457-4472 (2007).
[7] I. Csiszár, “Joint source-channel error exponent,” Probl. Contr. Inf. Theory, vol. 9, pp. 315–328, 1980.
[8] T. S. Han, Information-Spectrum Methods in Information Theory. Springer, 2003.
[9] R. G. Gallager, Information Theory and Reliable Communication. John Wiley & Sons, 1968.
[10] I. Csiszár and J Körner, Information Theory: Coding Theorems for Discrete Memoryless Systems. Academic Press, 1981.
[11] H. Nagaoka, Strong converse theorems in quantum information theory, Proceedings of ERATO Workshop on Quantum Information Science 2001, Univ. Tokyo, 6-8 September 2001, Tokyo, Japan, pp. 33.
[12] M. Hayashi and H. Nagaoka, “General formulas for capacity of classical-quantum channels,” IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 49, NO. 7: 1753-1768, 2003.
[13] M. Tomamichel, and M. Hayashi, “Operational Interpretation of Renyi Information Measures via Composite Hypothesis Testing Against Product and Markov Distributions,” Proceedings of 2016 IEEE International Symposium on Information Theory, 10-15 July 2016, Barcelona, Spain, pp. 585 - 589; arXiv:1511.04874
[14] A. N. Tikhomirov, “On the convergence rate in the central limit theorem for weakly dependent random variables,” Theor. Probabil. and Its Applic., 25(4), 790-809, 1980.