A class of entanglement monotones for general pure multipartite states based on complex projective variety

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We construct a measure of entanglement for general pure multipartite states based on Segre variety. We also construct a class of entanglement monotones based on the Plücker coordinate equations of the Grassmann variety. Moreover, we discuss and compare these measures of entanglement.

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I. INTRODUCTION

A very interesting measure of entanglement for multipartite states is usually called entanglement monotones, e.g., a measure of entanglement which is invariant under stochastic local quantum operation and classical communication (SLOCC) [1]. For example, F. Verstraete et al. [2] have presented a general mathematical framework to describe local equivalence classes of multipartite quantum states under the action of local unitary and local filtering operations. Their analysis has lead to the introduction of entanglement measures for the multipartite states, and the optimal local filtering operations maximizing these entanglement monotones were obtained. E. Briand, [3] et al. have studied the invariant theory of trilinear forms over a three-dimensional complex vector space, and apply it to investigate the behavior of pure entangled three-partite qutrit states and their normal forms SLOCC operations. They described the orbit space of the SLOCC group \( SL(3, \mathbb{C})^x3 \) both in its affine and projective versions in terms of a very symmetric normal form parameterized by three complex numbers. They have also shown that the structure of the sets of equivalent normal forms is related to the geometry of certain regular complex polytopes. A. Miyake and M. Wadati [4] have explored quantum search from the geometric viewpoint of a complex projective space. Recently, Péter Lévay [5] have constructed a class of multi-qubit entanglement monotones which was based on construction of C. Emary [6]. His construction is based on bipartite partitions of the Hilbert space and the invariants are expressed in terms of the Plücker coordinates of the Grassmannian. We have also constructed entanglement monotones for multi-qubit states based on Plücker coordinate equations of Grassmann variety, which are central notion in geometric invariant theory [7]. In this paper, we will construct a class of entanglement monotones for

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multipartite states based on some complex projective algebraic variety. In particular, in section III we will derive an entanglement measure for general multipartite states based on Segre variety and in section III we will construct a class of entanglement monotones for multipartite states, which coincides with the measure of entanglement based on the Segre variety by construction. Now, let us start by denoting a general, pure, composite quantum system with \( m \) subsystems \( \mathcal{Q} = \mathcal{Q}_1^p (N_1, N_2, \ldots, N_m) = \mathcal{Q}_1 \otimes \mathcal{Q}_2 \otimes \cdots \mathcal{Q}_m \), consisting of the pure state \( |\Psi\rangle = \sum_{k_1=1}^{N_1} \sum_{k_2=1}^{N_2} \cdots \sum_{k_m=1}^{N_m} \alpha_{k_1,k_2,\ldots,i_m} |k_1,k_2,\ldots,k_m\rangle \) and corresponding the Hilbert space \( \mathcal{H}_\mathcal{Q} = \mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2} \otimes \cdots \otimes \mathcal{H}_{\mathcal{Q}_m} \), where the dimension of the \( j \)th Hilbert space is given by \( N_j = \dim(\mathcal{H}_{\mathcal{Q}_j}) \). We are going to use this notation throughout this paper. In particular, we denote a pure two-qubit state by \( \mathcal{Q}_2^p (2,2) \). Next, let \( \rho_\mathcal{Q} \) denote a density operator acting on \( \mathcal{H}_\mathcal{Q} \). The density operator \( \rho_\mathcal{Q} \) is said to be fully separable, which we will denote by \( \rho_\mathcal{Q}^{sep} \), with respect to the Hilbert space decomposition, if it can be written as \( \rho_\mathcal{Q}^{sep} = \sum_{k=1}^{N} p_k \otimes_{j=1}^{m} \rho_{\mathcal{Q}_j}^k \), \( \sum_{k=1}^{N} p_k = 1 \) for some positive integer \( N \), where \( p_k \) are positive real numbers and \( \rho_{\mathcal{Q}_j}^k \) denotes a density operator on Hilbert space \( \mathcal{H}_{\mathcal{Q}_j} \). If \( \rho_\mathcal{Q}^p \) represents a pure state, then the quantum system is fully separable if \( \rho_\mathcal{Q}^p \) can be written as \( \rho_\mathcal{Q}^{sep} = \otimes_{j=1}^{m} \rho_{\mathcal{Q}_j} \), where \( \rho_{\mathcal{Q}_j} \) is the density operator on \( \mathcal{H}_{\mathcal{Q}_j} \). If a state is not separable, then it is said to be an entangled state. In the next section we will construct the Segre variety for multipartite states, which is a complex projective variety. However, we will not give any introduction to this complex space. The reader unfamiliar with this topic could look at the standard references for the complex projective variety, namely [10].

II. SEGRE VARIETY AND MULTIPARTITE ENTANGLEMENT MEASURE

In this section, we will define the Segre variety for a multi-projective space [10] and then based on this variety, we will construct an entanglement measure for general pure multipartite states. For example, we can construct a projective variety of \( \mathbb{CP}^{N_1-1} \times \mathbb{CP}^{N_2-1} \times \cdots \times \mathbb{CP}^{N_m-1} \) by its Segre embedding. Let \((\alpha_1, \alpha_2, \ldots, \alpha_{N_i})\) be points defined on \( \mathbb{CP}^{N_i-1} \). Then the Segre map

\[
S_{N_1,\ldots,N_m} : \mathbb{CP}^{N_1-1} \times \mathbb{CP}^{N_2-1} \times \cdots \times \mathbb{CP}^{N_m-1} \rightarrow \mathbb{CP}^{N_1 N_2 \cdots N_m - 1}
\]

is well defined for \( \alpha_{i_1 i_2 \cdots i_m}, 1 \leq i_1 \leq N_1, 1 \leq i_2 \leq N_2, \ldots, 1 \leq i_m \leq N_m \) as a homogeneous coordinate-function on \( \mathbb{CP}^{N_1 N_2 \cdots N_m - 1} \). Moreover, the image of the Segre map is a complex projective variety, which is called Segre variety. Now, let us consider the composite quantum system \( \mathcal{Q}_m^p (N_1, N_2, \ldots, N_m) \). Then, the Segre ideal \( \mathcal{I}_{\mathcal{Q}_1=\mathcal{Q}_2=\mathcal{Q}_3=\cdots=\mathcal{Q}_m}^1 \) representing a subsystem \( \mathcal{Q}_1 \) that is unentangled with \( \mathcal{Q}_2 \mathcal{Q}_3 \cdots \mathcal{Q}_m \) is generated by the six 2-by-2 minors of

\[
\text{Mat}_{N_1, N_2, \ldots, N_m}^1 = \begin{pmatrix}
\alpha_{1,1,\ldots,1} & \alpha_{1,1,\ldots,2} & \cdots & \alpha_{1, N_2, \ldots, N_m} \\
\alpha_{2,1,\ldots,1} & \alpha_{2,1,\ldots,2} & \cdots & \alpha_{2, N_2, \ldots, N_m} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{N_1,1,\ldots,1} & \alpha_{N_1,1,\ldots,N_m} & \cdots & \alpha_{N_1, N_2, \ldots, N_m}
\end{pmatrix},
\]
The other \( m-1 \) Segre ideal of multi-projective space, that is, \( \mathcal{I}^2_{Q_2} = \mathcal{Q}_2 \mathcal{Q}_3 \mathcal{Q}_m, \ldots, \mathcal{I}^m_{Q_m} = \mathcal{Q}_1 \mathcal{Q}_2 \mathcal{Q}_{m-1} \) can be defined by permutation indices of elements of matrix \( \text{Mat}_{N_1, N_2 \ldots N_m}^{1} \). For example, \( \mathcal{I}^j_{Q_j} = \mathcal{Q}_1 \mathcal{Q}_2 \mathcal{Q}_{j-1} \mathcal{Q}_{j+1} \mathcal{Q}_{m-1} \) is generated by 2-by-2 minors of \( \text{Mat}_{N_j, N_1 N_2 \ldots N_{j-1} N_{j+1} N_{m}}^{j} \), where the hat over \( j \)th-subsystem \( \hat{Q}_j \) indicate that we should delete this subsystem from right hand side of the Segre ideal. Now, based on 2-by-2 minors of these ideals we can construct an entanglement measure for general pure multipartite states. So, let \( \mathcal{M}_{\nu, \mu}(\text{Mat}_{N_j, N_1 N_2 \ldots N_{j-1} N_{j+1} N_{m}}^{j}) \) denotes the 2-by-2 minors of matrix \( \text{Mat}_{N_j, N_1 N_2 \ldots N_{j-1} N_{j+1} N_{m}}^{j} \), which is generated by rows \( \nu \) and \( \mu \), where \( \nu < \mu < N_j \). Then an entanglement measure for a general pure multipartite states is give by

\[
\mathcal{E}(\mathcal{Q}_m^p(N_1, \ldots, N_m)) = \left( N_m \sum_{j=1}^{m} \sum_{\nu > \mu = 1} |\mathcal{M}_{\nu, \mu}(\text{Mat}_{N_j, N_1 N_2 \ldots N_{j-1} N_{j+1} N_{m}}^{j})|^2 \right)^{1/2}.
\]

By construction, that is, the definition of the Segre ideals, this measure vanish on separable set of a general pure multipartite state. However, we need to show that this measure is an entanglement monotones. This can be done using the construction of entanglement monotones based on the Plücker coordinates of the Grassmann variety, which we will discuss in the following section.

### III. GRASSMANN VARIETY

In this section, we will define the Grassmann variety [11] and then we will construct a measure of entanglement based on Plücker coordinate equations of Grassmann variety. Let \( \text{Gr}(r, d) \) be the Grassmann variety of the \( r-1 \)-dimensional linear projective subspaces of \( \mathbb{C}P^{d-1} \). Now, we can embed \( \text{Gr}(r, d) \) into \( \mathbb{P}^{r-1}(\Lambda^r(\mathbb{C}^d)) = \mathbb{C}P^{N^r} \), \( N^r = \binom{d}{r} - 1 \), by using the Plücker map \( L \rightarrow \Lambda^r(L) \), where the exterior product \( \Lambda^r(\mathbb{C}^d) \) for \( 1 \leq r \leq d \) is a subspace of \( \mathbb{C}^{N_1} \otimes \cdots \otimes \mathbb{C}^{N_m} \), spanned by the anti-symmetric tensors. The Plücker coordinates \( P_{i_1, i_2, \ldots, i_r}, 1 \leq i_1 < \cdots < i_r \leq d \) are the projective coordinates in this projective space. Next, let \( \mathbb{C}[\Lambda(r, d)] \) be a polynomial ring with the Plücker coordinates \( P_I \) indexed by elements of the set \( \Lambda(r, d) \) of ordered \( r \)-tuples in \( \{1, 2, \ldots, d\} \) as its variables. Then the image of the map \( \kappa : \mathbb{C}[\Lambda(r, d)] \rightarrow \text{Pol}(\text{Mat}_{r,d}) \), which assigns \( P_{i_1, i_2, \ldots, i_r} \) the bracket polynomial \( [i_1, i_2, \ldots, i_r] \) (the bracket function on the \( \text{Mat}_{r,d} \)), whose values on a given matrix is equal to the maximal minor formed by the columns from a set of \( \{1, 2, \ldots, d\} \) is equal to the sub-ring of the invariant of the polynomials. Moreover, the kernel \( \mathcal{I}_{r,d} \) of the map \( \kappa \) is equal to the homogeneous ideal of the Grassmann in its Plücker embedding. Furthermore, the homogeneous ideal \( \mathcal{I}_{r,d} \) defining \( \text{Gr}(r, d) \) in its Plücker embedding is generated by the quadratic polynomials

\[
\mathcal{P}_{I,J} = \sum_{t=1}^{r+2} (-)^t P_{i_1, \ldots, i_{r-1}, i_t} P_{j_1, \ldots, j_{r-1}, j_t, j_{t+1}, \ldots, j_{r+1}},
\]

where \( I = (i_1 \ldots i_{r-1}), 1 \leq i_1 < \cdots < i_{r-1} < j_t \), and \( J = (j_1, \ldots, j_{r+1}), 1 \leq j_1 < \cdots < j_{r+1} \leq d \) are two increasing sequences of numbers from the set \( \{1, 2, \ldots, d\} \). Note that the equations \( \mathcal{P}_{I,J} = 0 \) define the Grassmannian \( \text{Gr}(r, d) \).
are called the Plücker coordinate equations. For example, for \( \text{Gr}(2, d) \) and \( r = 2 \), we have

\[
\mathcal{P}_{I, J} = \sum_{t=1}^{4} (-1)^t P_{i_1, j_t} P_{j_1 \ldots j_{t-1} j_{t+1} \ldots j_3} = -P_{i_1, j_1} P_{j_2 \ldots j_3} + P_{i_1, j_2} P_{j_1 \ldots j_3} - P_{i_1, j_3} P_{j_1, j_2},
\]

where \( I = (i_1) \), and \( J = (j_1, j_2, j_3) \). Note that, by its construction, the Grassmannian \( \text{Gr}(2, d) \) is invariant under \( \text{SL}(2, \mathbb{C}) \). Now, let us consider a quantum system \( \mathcal{Q}_m^p(2, 2, \ldots, 2) \), where in this case we have \( r = N_j = 2 \) and \( d = N_1 N_2 \cdots \hat{N}_j \cdots N_m = 2^{m-1} \). Then, we define

\[
\mathcal{E}_{I, J}(\text{Mat}^j_{N_j, N_1 N_2 \cdots \hat{N}_j \cdots N_m}) = \sum_{t=1}^{4} (P^t_{i_1, j_t} \mathcal{P}^t_{1, j_t} + P^t_{j_1 \ldots j_{t-1} j_{t+1} \ldots j_3} \mathcal{P}^t_{j_1 \ldots j_{t-1} j_{t+1} \ldots j_3})
\]

\[
= P^t_{i_1, j_1} \mathcal{P}^t_{1, j_1} + P^t_{j_1 \ldots j_3} \mathcal{P}^t_{j_1 \ldots j_3} + P^t_{j_1, j_2} \mathcal{P}^t_{j_1, j_2} + P^t_{j_1 \ldots j_2} \mathcal{P}^t_{j_1 \ldots j_2},
\]

where \( \text{Mat}^j_{N_j, N_1 N_2 \cdots \hat{N}_j \cdots N_m} \) is given by matrix \( \mathcal{P} \) for all \( N_j = 2 \) and \( j = 1, 2, \ldots, m \). Next, if we assume that the sequences \( I, J \) denote the columns of the \( \text{Mat}^j_{N_j, N_1 N_2 \cdots \hat{N}_j \cdots N_m} \), then we can define a class of entanglement monotones for the multi-qubit states by

\[
\mathcal{E}(\mathcal{Q}_m^p(2, 2, \ldots, 2)) = \left( N \sum_{j=1}^{m} \mathcal{E}_{I, J}(\text{Mat}^j_{N_j, N_1 N_2 \cdots \hat{N}_j \cdots N_m}(1, 2)) \right)^{1/2}.
\]

This measure for multi-qubit states, which correspond to rows one and two of matrix \( \mathcal{P} \) and its permutations, is invariant under action of \( \text{SL}(2, \mathbb{C}) \). However, for multi-qubit states we don’t need to used the row indexing \((1, 2)\). Now, if we exchange these rows with any other rows say \( \nu, \mu \) of matrix \( \mathcal{P} \), then it is still invariant under action of \( \text{SL}(2, \mathbb{C}) \). Thus, in this way we can construct an entanglement measure for general pure multipartite states as follows

\[
\mathcal{E}(\mathcal{Q}_m^p(N_1, \ldots, N_m)) = \left( N_\nu \sum_{j=1}^{m} \sum_{\forall \nu > \mu = 1} \mathcal{E}_{I, J}(\text{Mat}^j_{N_j, N_1 N_2 \cdots \hat{N}_j \cdots N_m}(\nu, \mu)) \right)^{1/2},
\]

where \( \text{Mat}^j_{N_j, N_1 N_2 \cdots \hat{N}_j \cdots N_m}(\nu, \mu) \) refer to rows \( \nu \) and \( \mu \) of matrix \( \mathcal{P} \). Now, we can see that entanglement measure based on the Segre ideals defined in equation \( \mathcal{E} \) coincides with entanglement measure based on the Plücker coordinate equations of the Grassmann variety defined in equation \( \mathcal{E} \). However, for this general case we have shown that these measure are invariant under repeated action of \( \text{SL}(2, \mathbb{C}) \). Thus, for multi-qubit states these measures of entanglement are entanglement monotones but for general pure multipartite states one need, in general, to show that these measure are invariant under action of \( \text{SL}(r, \mathbb{C}) \), so these measure of entanglement should be used with caution for general states. An alternative way to construct an entanglement monotones for a general pure multipartite is as follows. Let us consider a quantum system \( \mathcal{Q}_m^p(N_1, N_2, \ldots, N_m) \), where in this case we have \( r = N_j \) and \( d = N_1 N_2 \cdots \hat{N}_j \cdots N_m \) and let

\[
\mathcal{E}_{I, J}(\text{Mat}^j_{N_j, N_1 N_2 \cdots \hat{N}_j \cdots N_m}) = \sum_{t=1}^{r+2} (P^t_{i_1 \ldots i_{r-1} j_t} \mathcal{P}^t_{1, i_{r-1} j_t} + P^t_{j_1 \ldots j_{r-1} j_t} \mathcal{P}^t_{j_1 \ldots j_{r-1} j_t}).
\]
where \( I = (i_1 \ldots i_{r-1}), 1 \leq i_1 < \cdots < i_{r-1} < j_i \), and \( J = (j_1, \ldots, j_{r+1}), 1 \leq j_1 < \cdots < j_{r+1} \leq d \) are two increasing sequences of numbers from the set \( \{1, 2, \ldots, d\} \) and \( \text{Mat}_{N_j, N_i, N_2 \cdots \hat{N}_j \cdots N_m}^j \) is given by matrix (2). Then we can define a measure of entanglement for general pure multipartite states by

\[
E(Q^p_m(N_1, N_2, \ldots, N_m)) = \left( N \sum_{j=1}^m E_{I,J}(\text{Mat}_{N_j, N_i, N_2 \cdots \hat{N}_j \cdots N_m}^j) \right)^{1/2}.
\]

Thus, by construction this measure of entanglement is invariant under action of \( SL(r, \mathbb{C}) \), that is, a class of entanglement monotones for multipartite states. However, other properties of this measure needs further investigation.

**IV. CONCLUSION**

In this paper, we have constructed a class of entanglement monotones for multipartite states based on the Segre variety and the Grassmannian \( \text{Gr}(r, d) \), which was defined in terms of the Plücker coordinate equations. In particular, we have shown that entanglement monotones constructed based on Segre variety coincides with the one constructed by mean of the Plücker coordinate equations. Thus, we have used some standard but advanced mathematical tools from complex algebraic geometry to construct a class of entanglement monotones for multipartite states. However, the problem of quantifying multipartite entanglement still needs further investigation and there are many unanswered questions. We hope that our result can gives some geometrical insight to solving such interesting problem of the fundamental quantum theory with wide application in emerging field of the quantum information science.

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