An Extension of Entanglement Measures for Pure States

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To quantify the entanglement is one of the most important topics in quantum entanglement theory. An entanglement measure is built from measures for pure states. Conditions when the entanglement measure is entanglement monotone and convex are presented, as well as the interpretation of smoothed one-shot entanglement cost. Next, a difference between the measure under the local operation and classical communication and the separability-preserving operations is presented. Then, the relation between the convex roof extended method and the way here for the entanglement measures built from the geometric entanglement measure for pure states, as well as the concurrence for pure states in two-qubit systems are considered. It is also shown that the measure is monogamous for $2 \otimes 2 \otimes d$ system.

1. Introduction

Entanglement is one of the essential features in quantum mechanics when comparing with the classical physics\cite{1,2}. It also plays key roles in quantum information processing, such as, quantum cryptography\cite{3}, teleportation\cite{4}, and superdense coding\cite{5}.

One of the most important problems is to quantify the entanglement in a composite quantum system. In 1996, Bennett et al. proposed the distillable entanglement and entanglement cost, as well as their operational interpretations\cite{6}. Vedral et al. in ref. [7] presented three necessary conditions that an entanglement measure should satisfy in 1997, and one of them is that the quantum entanglement cannot increase under local operation and classical communication (LOCC). In 2000, Vidal proposed a general mathematical framework for entanglement measures\cite{8}. There the author also presented a convex roof extended method to build entanglement monotone for bipartite entangled systems from some functions on bipartite pure states. The quantification of entanglement can also be based on the distance to the closest separable state. The most important examples are the geometric measures\cite{9,10} and the quantum relative entropy\cite{11}. Due to the monotonicity of the inner product and quantum relative entropy under LOCC, it is clear that the above two are entanglement measures. Another important method to build an entanglement measure of a bipartite state is the minimum quantum conditional mutual information of all the extensions of the state. There the authors named the measure the squashed entanglement\cite{12}. Compared with the distillable entanglement, the squashed entanglement is additive on tensor products and superadditive in general. Recently, Gour and Tomamichel have proposed an approach to quantify the general resources\cite{13}. The approach extends the resource measures from the original domain to a larger one.

In this paper, we mainly apply the approach to construct an entanglement measure for mixed states from the measures for pure states. Given an entanglement measure for pure states in bipartite systems, we first present a sufficient condition when an entanglement measure by our construction is an entanglement monotone. Then we consider the relation between an entanglement measure built from our approach and the convex roof extended method\cite{8}. As an application, we present a difference between the LOCC and separability-preserving (SEPP) operations by the Schmidt number\cite{13,14}. Then we consider the relation between the convex roof extended method and the way here for the entanglement measures built from the geometric entanglement measure for pure states, as well as the concurrence for pure states in two-qubit systems. Based on the result, we show that the concurrence satisfies the monogamy of entanglement proposed in ref. [15] for $2 \otimes 2 \otimes d$ systems.

This paper is organized as follows. In Section 2, we present the preliminary knowledge needed here, then we present a sufficient condition when the entanglement measure built here is entanglement monotone. And we also present a condition when the entanglement measure built from the way here is convex. In Section 3, we present an interpretation of the smoothed entanglement cost under the approach. In Section 4, we present some applications of the entanglement measure built here. We present a difference between LOCC and SEPP. Then we present the relationship between the geometric entanglement measure for pure states together with concurrence for pure states in two-qubit systems built from the convex roof extended method and the approach. At last, we present the entanglement measure is monogamous for $2 \otimes 2 \otimes d$ systems. In Section 5, we end with a conclusion.

2. An Entanglement Measure from Pure States

In this section, first we recall some preliminary knowledge on the entanglement measures and operations of entanglement theory. Based on the method proposed in ref. [13], we will propose some entanglement measures built from entanglement measure for...
pure states, and we present some sufficient conditions when entanglement measures are entanglement monotone, convex and subadditivity.

In the following, we denote $\mathcal{L}_{AB}$ and $D_{AB}$ as the set of operators and states respectively on $\mathcal{H}_{AB}$ with finite dimensions. If a state $\rho_{AB}$ can be written as $\rho_{AB} = \sum p_i \sigma_i \otimes \sigma'_i$, then $\rho_{AB}$ is separable, otherwise, $\rho_{AB}$ is entangled. In the following, we denote $S_{AB}$ as the set of separable states on $\mathcal{H}_{AB}$ with finite dimensions.

Assume $\Lambda : \mathcal{L}_A \rightarrow \mathcal{L}_A$ is a linear map. $\Lambda()$ is completely positive (CP) if $(I \otimes \Lambda)(\rho_{B\Lambda}) \geq 0$ for any $\rho_{B\Lambda} \geq 0$, $\Lambda()$ is trace-preserving (TP) if and only if $\text{Tr} \Lambda(\sigma) = \text{Tr} \sigma$. Next let $\Lambda : \mathcal{L}_A \rightarrow \mathcal{L}_A$, its Choi matrix is $J_\Lambda = d(I \otimes \Lambda)(|\Psi\rangle\langle\Psi|)$, here $|\Psi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum |i\rangle_A |i\rangle_B$, the sets $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ are the orthonormal bases of the systems $A$ and $B$ respectively, $d$ is the dimension of $\mathcal{H}_A$. And $\Lambda$ is CP if and only if $J_\Lambda \geq 0$.

In the distant laboratory scenario, a multipartite quantum system is distributed to several separated parties. The different parties can perform local quantum operations only on their systems, and they are permitted to coordinate their actions by classical communications. The above scenario characterizes a quantum channel which is known as LOCC.[14] LOCC are free operations in the quantum entanglement theory,[15] but they are hard to characterize mathematically.[17] One way to overcome it is to extend the set of LOCC to operations that are simpler to characterize mathematically.[18–22] Here we mainly present the definition of the separability-preserving (SEPP) operations in the following.

**Definition 1.** Assume $\Lambda : D_{AB} \rightarrow D_{AB}$ is completely positive, then $\Lambda$ is SEPP if and only if $\Lambda(\sigma_{AB})$ is separable for any separable state $\sigma \in S_{AB}$.

Next we recall $E : D(\mathcal{H}_{AB}) \rightarrow \mathbb{R}^+$ is an entanglement measure,[23] if it satisfies:

I) $E(\rho_{AB}) = 0$ if $\rho_{AB} \in S(\mathcal{H}_{AB})$

II) $E$ does not increase under the LOCC operation.

$$E(\Psi(\rho_{AB})) \leq E(\rho_{AB})$$

here $\Psi$ is LOCC.

In ref. [23], the author presented that when $E$ satisfies the following two conditions, $E$ is an entanglement monotone,

III) $E(\rho) \geq \sum_i p_i E(\sigma_i)$, here $\sigma_i = \frac{E_i(\rho_{AB})}{\rho_{AB}}$, $p_i = \text{Tr} E_i(\rho_{AB})$, $E_i$ is any unilocal quantum operation performed by any party $A$ or $B$.

IV) For any decomposition $\{p_i, \rho_i\}$ of $\rho_{AB}$

$$E(\rho) \leq \sum_i p_i E(\rho_i)$$

Condition (IV) can also be regarded as the convexity of an entanglement measure. Then we recall that if an entanglement measure $E$ for a pure state $|\psi\rangle$ is the same entanglement ordering[24] with $E$ we mean that if for any two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, $E(|\psi_1\rangle) \geq E(|\psi_2\rangle)$, then $E(|\psi_1\rangle) \geq E(|\psi_2\rangle)$.

Recently, Gour and Tomamichel proposed a new method to extend the resource measures from one domain to a larger one.[13] Yu et al. considered the coherence measures in terms of the method and presented operational interpretations for some coherence measures.[25] Here we apply this approach to the entanglement theory to present new entanglement measures, as well as the properties of the entanglement measures.

Assume $|\psi\rangle_{AB}$ is a pure state in $\mathcal{H}_{AB}$, $E$ is an entanglement measure for pure states in $\mathcal{H}_{AB}$. Then we extend the above measure for pure states to a corresponding function for the mixed states.

$$\overline{E}(\rho_{AB}) = \inf_{|\psi\rangle_{AB} \in \mathcal{R}(\rho_{AB})} E(|\psi\rangle_{AB})$$

Here the infimum takes over all the pure states in the set

$$\mathcal{R}(\rho_{AB}) = \{\psi_{AB} \in \mathcal{H}_{AB} | \rho_{AB} = \Lambda(\psi_{AB}), \Lambda \in \mathcal{T}\}$$

$\mathcal{T} \in \{\text{LOCC}, \text{SEPP}\}$

Next, we recall specific entanglement measures. Assume $|\psi\rangle_{AB} = \sum_{i=0}^{d-1} \sqrt{\lambda_i} |i\rangle_A |i\rangle_B$ is a bipartite pure state in $\mathcal{H}_{AB}$, and let $\mu^{(i)}(|\psi\rangle_{AB})$ be the vector $(\lambda_0, \lambda_1, \ldots, \lambda_{d-1})$ in decreasing order. We recall entanglement measures $E_i$ for pure states $|\psi\rangle_{AB}$, $E_i(\psi_{AB}) = \sum_{\psi \in \mathcal{R}_{\psi_{AB}}} |\langle \psi | \psi \rangle|^2$.

**Theorem 2.** Assume $\rho_{AB}$ is a bipartite state, and $\mathcal{T}$ consists of the LOCC operations.

i) If $\mathcal{T}$ is the same entanglement ordering with $E_i$, $k = 2, 3, \ldots, d - 1$, for pure states, then $\overline{E}$ satisfies the condition (III).

ii) Assume

$$E(\|\psi\rangle_{AB}) = \{\text{Tr}(\|\psi\rangle_{AB}(\|))\}$$

$$f : D(\mathcal{H}_{AB}) \rightarrow \mathbb{R}^+$$

When the function $f$ in (6) corresponding to $E$ satisfies $f(\mathcal{L}(\lambda_i \Lambda_1 + \lambda_i \Lambda_2) \leq \lambda_1 f(\Lambda_1) + \lambda_2 f(\Lambda_2)$, here $\Lambda_i, i = 1, 2$ are diagonal matrices on the space $\mathcal{H}_{AB}$, then $\overline{E}$ satisfies condition (IV). We present the proof of Theorem 2 in Appendix A.

When computing the entanglement measure in (3) and $\mathcal{T}$ is LOCC, we could decrease the size of the set $\mathcal{R}(\rho_{AB})$ of $\rho_{AB}$.

**Theorem 3.** Assume that $\rho_{AB}$ is a mixed state. $E$ is entanglement monotone for pure states, then we have that

$$\overline{E}(\rho_{AB}) = \inf_{|\psi\rangle_{AB} \in \mathcal{R}(\rho_{AB})} E(|\psi\rangle_{AB})$$

where we denote

$$\mathcal{R}(\rho_{AB}) = \{\psi \in \mathcal{R}(\rho_{AB}) | \mu^{(i)}(|\psi\rangle_{AB}) = \sum_i p_i \mu^{(i)}(|\phi_i\rangle_{AB})\}$$

We present the proof of the Theorem 3 in Appendix A.

The other method to build an entanglement monotone for a mixed state from a function for a pure state is proposed in ref. [23]. Assume

$$E(|\psi\rangle_{AB}) = \{\text{Tr}(\|\psi\rangle_{AB}(\|))\}$$

$$f : D(\mathcal{H}_{AB}) \rightarrow \mathbb{R}^+$$

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$$f : D(\mathcal{H}_{AB}) \rightarrow \mathbb{R}^+$$
If $f$ in (10) satisfies the following conditions:

1. $U$-invariant: $f(U \sigma U^\dagger) = f(\sigma), \forall \sigma \in D_{\mathcal{A}}$, $U$ is a unitary matrix on $\mathcal{H}_{\mathcal{A}}$.
2. Concave: $f(\lambda \sigma_1 + (1 - \lambda) \sigma_2) \geq \lambda f(\sigma_1) + (1 - \lambda) f(\sigma_2)$, here $\sigma_i \in D(\mathcal{H}_i), i = 1, 2, \lambda \in (0, 1)$.

then $E$ is an entanglement monotone for mixed states by the convex roof extended method,[21]

$$E_f(\rho_{AB}) = \min_{\{p, |\psi_j\rangle\}_{AB}} \sum_j p_j E(|\psi_j\rangle)$$

where the minimum takes over all the decomposition of $\{p, |\psi_j\rangle\}_{AB}$ such that $\rho_{AB} = \sum_j p_j |\psi_j\rangle \langle \psi_j|$. Then we present a condition when $E$ is convex and only if $E_f = E_f$.

**Theorem 4.** Assume that $E$ is an entanglement measure for pure states in $\mathcal{H}_{AB}$, and $\overline{E}$ is an entanglement measure for a mixed state defined in (3), then $\overline{E}$ is convex if and only if $E = E_f$.

**Proof.** As when $\rho_{AB} = |\psi\rangle\langle \psi|$ is a pure state in $\mathcal{H}_{AB}$, $E_f(|\psi\rangle\langle \psi|) = \overline{E}(|\psi\rangle\langle \psi|) = \overline{E}(\rho_{AB})$, then by the result in ref. [13], we have that $\overline{E}(\rho_{AB}) \geq E_f(\rho_{AB})$. On the other hand, from the definition of the $E_f$, $E_f$ is convex. Assume that $\{q_i, |\theta_i\rangle\}$ is the optimal decomposition of $\rho_{AB}$ in terms of $E_f$, then we have that

$$\overline{E}(\rho_{AB}) \leq \sum_j q_j E_f(|\theta_j\rangle)$$

$$= E_f(\rho_{AB})$$

The inequality is due to the convexity of $\overline{E}$, then we finish the proof.

Frequently, another property for entanglement measures is imposed, additivity. Additivity means that $\forall \sigma \in D_{\mathcal{AB}}, E(\sigma^{\otimes n}) = nE(\sigma)$. And if $E(\sigma^{\otimes n}) \leq nE(\sigma)$, we say $E$ is subadditivity. Unfortunately, this property is not always valid for many prominent entanglement measures, such as, entanglement of formation,[6,8] robustness of entanglement of formation,[29] relative entropy of entanglement,[29,30] Moreover, the relative entropy of entanglement is additive for pure states, while it is subadditivity for mixed states. Subadditivity means that $\forall \sigma \in D_{\mathcal{AB}}, E(\sigma^{\otimes n}) \leq nE(\sigma)$. Here we present a condition when $\overline{E}$ is weak subadditivity.

**Theorem 5.** Assume an entanglement measure $E$ is subadditive for a pure state $|\psi\rangle_{AB}$ in $\mathcal{H}_{AB}$. When $\rho$ is a bipartite mixed state on $\mathcal{H}_{AB}$.

$$\overline{E}(\rho^{\otimes n}) \leq n\overline{E}(\rho)$$

**Proof.** Assume $|\psi\rangle$ is the optimal pure state for a mixed state $\rho$ in terms of $\overline{E}$, then $\rho^{\otimes n}$ can be transformed into $|\psi\rangle^{\otimes n}$ by LOCC.

Due to the definition of $\overline{E}$, we have

$$\overline{E}(\rho^{\otimes n}) \leq \overline{E}(|\psi\rangle^{\otimes n})$$

$$\leq n\overline{E}(|\psi\rangle) = n\overline{E}(\rho)$$

The first inequality is due to the definition of $\overline{E}$ in (3). The second inequality is due to the subadditivity of $\overline{E}$ for pure states, the first equality is due to the assumption that $|\psi\rangle$ is the optimal for $\rho$ in terms of $\overline{E}$. \qed

In this section, we have presented some properties of a generic entanglement measure based on the formula (3).[13] Hereinafter, we will study some specific entanglement measures in terms of our way.

### 3. An Operational Interpretation of the Smoothed $\overline{E}$

In this section, we will consider the relation between the entanglement cost with the smoothed entanglement cost under our approach in the one-shot regime and asymptotic regime. In the following, we denote $\Psi_r = \frac{1}{r} \sum_{i=0}^{r-1} \{|ii\rangle\langle ii|\}$. First we present the knowledge needed.

**Definition 6.**[31] Assume $\rho \in D_{\mathcal{AB}}$, its one-shot entanglement cost is defined as

$$E_{\lambda,1}(\rho) = \log \inf_{\lambda \in \Lambda(\rho)} \{r|\lambda(\Psi_{r}) = \rho \}$$

The smoothed entanglement cost is defined as

$$E_{\lambda,1}^s(\rho) = \inf_{\lambda \in \Lambda(\rho)} E_{\lambda,1}(\rho)$$

here $B_r(\rho) = \{r^{\frac{1}{n}} \rho - \rho \leq \epsilon\}, ||\lambda|| = \text{Tr} \sqrt{\lambda^\dagger \lambda}$. Its one-shot smoothed entanglement distillation is defined as

$$E_{\lambda,1}^s(\rho) = \log \sup_{\lambda \in \Lambda(\rho)} \{r^{\frac{1}{n}} ||\lambda(\rho) - \Psi_{r}|| \leq \epsilon\}$$

Next we recall the definition of entanglement of formation and entanglement of cost.

**Definition 7.**[6] Assume $|\psi\rangle_{AB} \in \mathcal{H}_{AB}$, its entanglement of formation is defined as

$$E_f(|\psi\rangle) = S(\text{Tr} |\psi\rangle_{AB} |\psi\rangle)$$

here $S(\rho) = - \text{Tr} \rho \log \rho$.

When $\rho \in D_{\mathcal{AB}}$ is a mixed state, its entanglement of formation is defined by the convex roof extended method,

$$E_f(\rho) = \min_{\{p, |\psi_j\rangle\}_{AB}} \sum_j p_j E_f(|\psi_j\rangle)$$

where the minimization takes over all the decompositions $\{p, |\psi_j\rangle\}_{AB}$ of $\rho$ such that $\rho = \sum_j p_j |\psi_j\rangle_{AB} \langle \psi_j|$. Assume $\rho \in D_{\mathcal{AB}}$, its entanglement cost is

$$E_{\lambda}(\rho) = \inf \{r \lim_{n \to \infty} \inf_{\Lambda(\rho)} \text{Tr} [\rho^{\otimes n} - \Lambda(\Psi_{r}^{2n})] = 0\}$$

Next we present a similar smoothed entanglement measure defined in (3).
Definition 8. [13] Assume $\rho_{AB} \in D_{AB}$, $E$ is an entanglement measure defined in (3), the smoothed extension of $E$, $\overline{E}$ is defined as

$$\overline{E}(\rho) = \inf_{\rho \in D_{AB}} E(\rho)$$

$$= \inf_{\rho \in D_{AB}} \inf \{ E(\ket{\psi}) | \Lambda(\ket{\psi}) = \rho, \Lambda \in \text{LOCC} \}$$

(21)

Here we restrict $\mathcal{T}$ defined in (3) to be LOCC, and the second inf takes over all the pure states $|\psi\rangle$ such that $\Lambda(|\psi\rangle) = |\psi\rangle$, $\Lambda \in \text{LOCC}$.

When we take $E$ for pure states as entanglement cost, we have the following theorem in the one-shot regime.

Theorem 9. Assume $\rho_{AB} \in D_{AB}$, $\epsilon \in [0, 1)$, then $\overline{E}_{\epsilon, 1}(\rho) = E_{\epsilon, 1}(\rho)$.

Proof. Assume $|\psi_\epsilon\rangle$ is the optimal pure state for $\epsilon$ in terms of $E_{\epsilon, 1}$, and let

$$\overline{E}_{\epsilon, 1}(\rho) = E_{\epsilon, 1}(|\psi_\epsilon\rangle) = r_\epsilon$$

(22)

then let $\Omega(|\psi_\epsilon\rangle) = r_\epsilon$, here $\epsilon \in B_\epsilon(\rho)$, we have

$$|\Omega \Lambda(|\psi_\epsilon\rangle) - \Omega(|\psi_\epsilon\rangle)|$$

$$\leq |\Lambda(|\psi_\epsilon\rangle) - |\psi_\epsilon\rangle|$$

(23)

Here $\Lambda$ is the optimal LOCC operations in terms of entanglement of cost for $|\psi_\epsilon\rangle$. The first inequality is due to that the trace-preserving quantum operations are contractive, [32] the second inequality is due to the assumption above. Then we have $\overline{E}_{\epsilon, 1}(\rho) \geq E_{\epsilon, 1}(\rho)$.

Next we prove the other side. Assume $r_\epsilon$ is the optimal of $\rho$ in terms of $E_{\epsilon, 1}(\rho)$ and $E_{\epsilon, 1}(\rho) = r_\epsilon$. Let $|\phi_i\rangle$ be the optimal for $\rho$, in terms of $E_{\epsilon, 1}$. Then we would show that $E_{\epsilon, 1}(|\phi_i\rangle) = r_\epsilon$.

First $E_{\epsilon, 1}(|\phi_i\rangle) > r_\epsilon$ is impossible, as by the definition of $E_{\epsilon, 1}$, $|\phi_i\rangle$ can be the optimal pure state for $\rho$, in terms of $E_{\epsilon, 1}$. Next if $E_{\epsilon, 1}(|\phi_i\rangle) < r_\epsilon$, then by the similar thought of (23), we see it is impossible. Then we have

$$E_{\epsilon, 1}(|\phi_i\rangle) = \overline{E}_{\epsilon, 1}(\rho) = r_\epsilon$$

(24)

Due to the definition of $\overline{E}_{\epsilon, 1}$, we have $\overline{E}_{\epsilon, 1}(\rho) \leq E_{\epsilon, 1}(\rho)$. Then we finish the proof.

Next we present a result in the asymptotic regime, which may imply the counterpart of entanglement of formation.

Theorem 10. Assume $\rho \in D_{AB}$, then we have

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \overline{E}_{\epsilon, 1}(\rho^n) = E_1(\rho)$$

(25)

The proof of Theorem 10 is placed in Appendix A.

4. Applications

This section is organized as follows. In the Section 4.1, we present a difference between the quantity defined in (3) between LOCC and SEPP. In the Section 4.2, we present the relation between the entanglement generated from the geometric entanglement measure for pure states by the convex roof extended method and the approach proposed in (3). In Section 4.3, we present that for the two-qubit system, the entanglement generated from concurrence for pure states by the convex roof extended method and the way proposed here are equal, and we present that the entanglement measure is monogamous under the definition of monogamy proposed in ref. [15].

4.1. An Example on an Entanglement Measure Under SEPP

Definition 11. [24] Assume $|\psi\rangle_{AB}$ is a pure state, its Schmidt number

$$\text{Sch}(|\psi\rangle_{AB}) = \text{Rank}(\rho_{AB}).$$

(26)

here $\rho_{AB} = \text{Tr}_B |\psi\rangle_{AB}$. When $\rho_{AB}$ is a mixed state, then its Schmidt number $\text{Sch}(\rho)$ is $k$, if (i) there exists a decomposition of $\{|\psi_i\rangle\}$ such that the Schmidt number of all the pure states $|\psi_i\rangle$ are at most $k$, (ii) for any decomposition $\{|\psi_i\rangle\}$ of $\rho_{AB}$, there exists at least one pure state $|\psi_i\rangle$ in the set $\{|\psi_i\rangle\}$ with its Schmidt number at least $k$.

In ref. [13], the authors showed that when the entanglement measure $E$ is the Schmidt number, $T = \text{LOCC}$, $\text{Sch}(\rho_{AB}) = \text{Sch}(\rho_{AB})$. In ref. [35], the authors presented the following interesting result.

Lemma 12. [35] For every bipartite state $\rho$ and any positive integer $k$, there exists a SEPP $\Lambda$ such that $\Lambda(|\psi_i\rangle) = \rho_{AB}$ if and only if $R(\rho) \leq R(|\psi_i\rangle)$, here $|\psi_i\rangle = \frac{1}{\sqrt{N}} \sum_j |j\rangle$ is a maximally entangled state, and $R(\rho)$ is its robustness of entanglement which is defined as follows

$$R(\rho) = \min_{\epsilon \in \mathcal{N}(\rho_{AB})} \min_{\frac{s}{1 + s} \in S_{AB}}\{s^{\frac{1}{s}}\}^2$$

(27)

Then we present that when $T \in \text{SEPP}$, the entanglement measure built from the method here can be decreased.

Example 13. Assume that $|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$, let

$$\rho = \frac{1}{4} |\phi_1\rangle\langle\phi_1| + \frac{3}{4} |\phi_2\rangle\langle\phi_2|.$$  

(28)

$$|\phi_1\rangle = \frac{1}{2} |00\rangle + \frac{1}{6} |11\rangle + \frac{1}{22} |22\rangle + \frac{5}{6} |33\rangle.$$  

(29)

$$|\phi_2\rangle = \frac{1}{2} |00\rangle + \frac{1}{6} |11\rangle + \frac{1}{22} |22\rangle + \frac{\sqrt{46}}{13} |33\rangle.$$  

(30)

In ref. [28], the authors showed that when $|\phi\rangle = \sum_i \sqrt{T_i}|i\rangle$, $R(|\phi\rangle) = \sum_i T_i^2 - 1$, then

$$R(|\psi\rangle) = 2.$$  

(31)

$$R(|\rho_1\rangle) = 1.7778, \quad R(|\rho_2\rangle) = 1.5529.$$  

(32)

Next as $R$ is convex, we have $R(\rho) \leq 1.7778 < 2$, due to the Lemma 12, we have that there exists a SEPP $\Lambda$ such that $\Lambda(|\psi\rangle) = \rho$. However, it is clear to see that $\text{Sch}(|\psi\rangle) = 3$, $\text{Sch}(\rho) = 4$, that is, when
$$T$$ stands for the LOCC in (3), $$\overline{\text{Sch}}(\rho) = 4$$, when $$T$$ is SEPP in (3), $$\overline{\text{Sch}}(\rho) \leq 3$$.

4.2. The Extension of Geometric Entanglement Measure

Here we discuss the connection between the extension of geometric entanglement measure by the approach of (3) and the original definition defined in ref. [9]. The latter for the pure state is defined as the maximum overlap between $$\rho$$ with any fully product states $$\{|a_1, a_2\rangle\}$$, and extended to the mixed state by the convex roof extended method characterized above. That is,

$$G(\rho) = \frac{1}{2} - \max_{\{|\psi\rangle\in\{|a_1, a_2\rangle\}} \langle \psi | \rho | \psi \rangle$$

where the minimum is taken over all the pure state decompositions of $$\rho_{AB}$$. The GME is a fundamental multipartite entanglement measure in the past decades.[36–38] The GME can quantify the entanglement of experimentally realizable states, GHZ states, W states and graph states for one-way quantum computing,[39] topological quantum computing,[40] and six-photons Dicke states,[41] respectively.

Here we first recall the definition of concurrence for bipartite quantum states. Assume $$|\psi\rangle_{AB} \in H_{AB}$$, its concurrence[6] is defined as

$$C(|\psi\rangle_{AB}) = \sqrt{2(1 - \text{Tr} \rho_1^2)}$$

Here $$\rho_1 = \text{Tr}_B |\psi\rangle_{AB} \langle \psi|$$. When $$\rho_{AB}$$ is a bipartite mixed state, its concurrence is defined as

$$C(\rho_{AB}) = \min_{\{p, |\varphi\rangle\}} \sum_i p_i C(|\varphi_i\rangle)$$

where the minimum takes over all the decompositions $$\{p, |\varphi_i\rangle\}$$ of $$\rho_{AB}$$ such that $$\rho_{AB} = \sum_i p_i |\varphi_i\rangle \langle \varphi_i|$$.

Moreover, when $$|\psi\rangle_{AB}$$ is a bipartite qubit pure state, By the Schmidt decomposition, $$|\psi\rangle_{AB}$$ can be written as

$$|\psi\rangle_{AB} = \sqrt{\lambda_0} |00\rangle + \sqrt{\lambda_1} |11\rangle$$

Next we present the relation between $$C$$ and $$\overline{C}$$ of a bipartite state $$\rho_{AB}$$.

Theorem 15. Assume $$\rho_{AB} \in D_{AB}$$ is a two-qubit state, then we have

$$\overline{C}(\rho_{AB}) = C(\rho_{AB})$$

We place the proof of Theorem 15 in Appendix A. Next We remark that from the proof, other entanglement measures in terms of the approach proposed in (3) for the states in two-qubit systems can be obtained, such as Tsallis$$q$$-entanglement measure[43] and Renyi$$q$$-entanglement measure[44] when $$q[45]$$ and $$\alpha[46]$$ are in some regions.

Then we show that $$\overline{C}$$ is monogamous for a 2$$\otimes$$2$$\otimes$$d space in terms of a generalized monogamy relation.[15] Monogamy of entanglement (MoE) is a fundamental property that can distinguish entanglement from classical correlations. Mathematically, MoE means that it can be characterized as in terms of an entanglement measure $$E$$ for a tripartite system $$A, B$$ and $$C$$,

$$E_{ABC} \geq E_{AB} + E_{AC}$$

here $$E_{AB}$$ denotes the entanglement $$AB$$ in terms of $$E$$. Although many entanglement measures satisfy the above inequality for multi-qubit systems,[12,45–47] the above inequality is not valid in general in terms of almost all entanglement measures for multipartite higher dimensional systems.[45,50] Recently, a generalized monogamy relation for an entanglement measure $$E$$ was proposed in ref. [15]. There the authors defined that an entanglement measure $$E$$ is monogamous for a tripartite system $$A, B$$ and $$C$$ if for any $$\rho_{ABC} \in D_{ABC}$$. 

$$E_{ABC} = E_{AB} + E_{AC} = 0$$

4.3. Some Results on Two-Qubit States

Here we first recall the definition of concurrence for bipartite quantum states. Assume $$|\psi\rangle_{AB} \in H_{AB}$$, its concurrence[6] is defined as

$$C(|\psi\rangle_{AB}) = \sqrt{2(1 - \text{Tr} \rho_1^2)}$$

Next, assume $$\{p, |\varphi_i\rangle\}$$ is the optimal decomposition of $$\rho$$ in terms of $$G_j$$. Due to Theorem 3, let $$|\varphi\rangle$$ be the pure state with $$\mu(\varphi) = \sum_j p_j |\varphi_j\rangle$$, by the main result in ref. [26], $$\varphi \rightarrow \rho$$, by the definition of $$\overline{G}$$ and (35), we have

$$\overline{G}(\rho) \geq G_j(\rho)$$

Next, assume $$\{p, |\varphi_i\rangle\}$$ is the optimal decomposition of $$\rho$$ in terms of $$G_j$$. Due to Theorem 3, let $$|\varphi\rangle$$ be the pure state with $$\mu(\varphi) = \sum_j p_j |\varphi_j\rangle$$, by the main result in ref. [26], $$\varphi \rightarrow \rho$$, by the definition of $$\overline{G}$$ and (35), we have

$$\overline{G}(\rho) \geq G_j(\rho)$$

Combining with (38) and (39), we finish the proof. □
Corollary 16. Let $\rho_{ABC} \in D_{ABC}$ on a $H_A \otimes H_B \otimes H_C$ space, then if $\overline{C}(\rho_{ABC}) = \overline{C}(\rho_{AB})$, then $\overline{C}(\rho_{AC}) = 0$.

Proof. Due to the Theorem 15 and the assumption, we have

$$\overline{C}(\rho_{ABC}) = \overline{C}(\rho_{AB}) = C(\rho_{AB}) \quad (47)$$

By the Theorem 1 in ref. [13],

$$\overline{C}(\rho_{ABC}) \geq C(\rho_{AB}) \geq C(\rho_{AC}) = \overline{C}(\rho_{AC}) \quad (48)$$

then combing (47) and (48), we have that

$$\overline{C}(\rho_{ABC}) \geq C(\rho_{AB}) \geq C(\rho_{AC})$$

that is, $\overline{C}(\rho_{ABC}) = C(\rho_{AB})$. As in ref. [51], the authors presented that $C$ is monogamous, then $\overline{C}(\rho_{AC}) = 0$, that is, $\rho_{AC}$ is separable. On the other hand, a separable pure state can be transformed into a separable mixed state through LOCC, then we have

$$\overline{C}(\rho_{AC}) = 0 \quad (50)$$

From the similar analysis, we have that the entanglement measure in terms of the approach proposed in (3) for Tsallis-$q$ entanglement entropy, Rényi-$\alpha$ entanglement entropy satisfy the (46) for the space $H_A \otimes H_B \otimes H_C$.

5. Conclusion

In the paper, we have considered an approach to build an entanglement measure for mixed states based on the measure for pure states. First we have presented when a generic entanglement measure is entanglement monotone, convex and subadditivity, and we also have considered the relationship between the entanglement measure built by the convex roof extended method and the approach proposed in (3). Then we have presented some results on entanglement measures generated from entanglement cost, schmidt number, geometric entanglement measure and concurrence for pure states built from (3). At last, we have shown the entanglement measure is monogamous for the space $H_A \otimes H_B \otimes H_C$. We hope our work could shed some light on related studies.

Appendix A

Here we present the proof of Theorems 2, 3, 10 and 15.

A.1. The Proof of Theorem 2

Theorem 2: Assume $\rho_{AB}$ is a bipartite state, and $T$ consists of the LOCC operations. i) If $E$ is the same entanglement ordering with $E_2$, $k = 2, 3, \ldots, d - 1$, for pure states, then $E$ satisfies the condition (II).

ii) Assume

$$E(|\psi\rangle_{AB}) = f(\text{Tr}(|\psi\rangle_{AB}(|\psi\rangle_{AB})))$$

$$f : D(H_A) \rightarrow R^+ \quad (A1)$$

When the function $f$ in (A1) corresponding to $E$ satisfies $f(\lambda A_1 + \lambda A_2) \leq \lambda f(A_1) + \lambda f(A_2)$, here $A_i$, $i = 1, 2$ are diagonal matrices on the space $H_i$, then $E$ satisfies the condition (IV).

Proof. The proof that (3) is an entanglement measure can be found in ref. [13]. If $|\psi\rangle_{AB}$ is a pure state. There exists a decomposition $\{p_i, |\phi_i\rangle\}$ such that $|\psi\rangle_{LOCC} \rightarrow \{p_i, |\phi_i\rangle\}$ of $\rho$, then by the assumption and the results in ref. [26],

$$\overline{E}(|\psi\rangle) \geq \sum_k p_k \overline{E}(|\phi_k\rangle) \quad (A2)$$

Next when $\rho$ is a mixed state, assume $|\psi\rangle$ is the optimal pure state for $\rho$ in terms of $E$, there exists a decomposition $\{r_j, |\eta_j\rangle\}$ of $\rho$ such that

$$|\psi\rangle \rightarrow \{r_j, |\eta_j\rangle\}$$

$$\mu^I(|\psi\rangle) < \sum_{j} r_j \mu^I(|\eta_j\rangle) \quad (A3)$$

here $\rho = \sum_j r_j |\eta_j\rangle\langle\eta_j|$, the second formula is due to the results in [26]. Assume $E_q$ is a unilocal operation on party $B$, then let $\rho_{jk} = \frac{t_{jk} q_k}{\sum_j t_{jk} q_k}$, $\rho_k = \frac{E_q(\rho_{jk})}{\sum_j t_{jk} q_k}$, $t_{jk} = \text{Tr} E_q(\rho_{jk})$. Due to the definition of $E_q$, it is invariant under local unitary operations, it is monotone under the actions

$$\rho \rightarrow \rho \otimes \rho_1$$

$$\rho \rightarrow \text{Tr}_Q \rho \quad (A4)$$

here $\rho_1$ is a state added by one party to its subsystem, $Q$ is held by $B$, and $\text{Tr}_Q \rho$ is the partial trace on $Q$. When $E_q$ stands for the unilocal von Neumann measurement $\{I \otimes M_k\}$, $\rho_k$ can be pure, and we write $\rho_k = |\xi_k^R\rangle\langle\xi_k^R|$. The above formula is due to the Theorem 1 in ref. [26].

Next let $|\chi_k\rangle$ be a pure state with

$$\mu^I(|\chi_k\rangle) = \sum_j \frac{r_j t_{jk}}{m_k} \mu^I(|\xi_k^R\rangle) \quad (A6)$$

here $m_k = \sum_j r_j t_{jk}$, then

$$|\chi_k\rangle \rightarrow |\xi_k^R\rangle \quad (A7)$$
A.2. The Proof of Theorem 3

We have that
\[ \mu_i(\rho) \leq \sum_k q_k \mu_i(\rho_k) \] (A8)

where we denote
\[ \mathcal{O}(\rho_{AB}) = \{ |\psi\rangle \in R(\rho_{AB}) | \mu_i(\rho_{AB}) \rangle = \sum i p_i \mu_i(\rho_i_{AB}) \} \] (A17)
\[ \rho_{AB} = \sum_i p_i |\phi_i\rangle \langle \phi_i| \] (A18)

Proof. By the definition of \( \mathcal{O}(\rho_{AB}) \), we have that \( \mathcal{O}(\rho_{AB}) \leq \inf_{|\psi\rangle \in R(\rho_{AB})} E(|\psi\rangle) \). Then we prove the other side of (A16).

Assume that \(|\psi\rangle_{AB} \) is a pure state in \( R(\rho_{AB}) \) in terms of \( E \), by the similar analysis in Theorem 2, then we have that there exists a decomposition \( \{\rho_i, |\phi_i\rangle_{AB}\} \) of \( \rho_{AB} \) such that
\[ \mu_i(|\psi\rangle_{AB}) < \sum_i p_i \mu_i(|\phi_i\rangle_{AB}) \] (A19)

the formula (A19) is due to the result in ref. [26]. Next if we take \(|\psi\rangle \) with \( \mu_i(|\psi\rangle) = \sum p_i \mu_i(|\phi_i\rangle_{AB}) \), combing with (A19), we have that \( \mu^i(|\psi\rangle_{AB}) < \mu^i(|\psi\rangle_{AB}) \). Then combing the result in ref. [27] and the definition of \( E \), we finish the proof. □

A.3. The Proof of Theorem 10

**Theorem 10:** Assume \( \rho \in \mathcal{D}_{AB} \), then we have
\[ \lim_{n \to \infty} \lim_{\epsilon \to 0} \frac{E_{\epsilon}(\rho^{\otimes n})}{n} = E_{\epsilon}(\rho) \] (A20)

Proof. From the definition of entanglement cost in (20), we have that \( \forall \epsilon \), when \( n \to \infty \), there exists a \( \Lambda \in \mathcal{LOCC} \) such that
\[ \Lambda(\Psi_{\epsilon}^{\otimes n}) = \rho_{AB} \), \( \rho_{AB} \in B(\rho^{\otimes n}) \), that is
\[ E_{\epsilon}(\Psi_{\epsilon}^{\otimes n}) \geq E_{\epsilon}(\rho_{AB}) \geq E_{\epsilon}(\rho^{\otimes n}) \] (A21)

the second inequality is due to the definition of the smoothed entanglement cost. Next denote \( \sigma \) as the state such that \( \mathcal{E}_{\epsilon}(\sigma^{\otimes n}) = \mathcal{E}_{\epsilon}(\rho^{\otimes n}) \). As when \(|\psi\rangle_{AB} \) is a pure state, \( E_{\epsilon}(|\psi\rangle_{AB}) = E_{\epsilon}(|\psi\rangle_{AB}) = -\text{Tr} \rho_{AB} \log \rho_{AB} \), and the Theorem 1 in ref. [13], we have
\[ \mathcal{E}_{\epsilon}(\sigma^{\otimes n}) \geq E_{\epsilon}(\rho^{\otimes n}) \] (A22)

In ref. [33], the author showed that when \( \rho, \sigma \in \mathcal{D}_{AB}, \epsilon ||\rho - \sigma|| \leq \epsilon \),
\[ |E_{\epsilon}(\rho) - E_{\epsilon}(\sigma)| \leq \delta \log d + (1 + \delta) h(\frac{\delta}{1 + \delta}) \] (A23)

where \( d \) is the dimension of the smaller of the two systems. Without loss of generality, we assume \( \text{Dim} H_A = \text{Dim} H_B = d \), \( \delta = \sqrt{e(2 - \epsilon)} \), \( h(e) = -e \log (1 - e) \).
\[ \mathcal{E}_{\epsilon}(\rho^{\otimes n}) \geq E_{\epsilon}(\sigma^{\otimes n}) \] (A24)

that is,
\[ \mathcal{E}_{\epsilon}(\rho^{\otimes n}) \geq E_{\epsilon}(\sigma^{\otimes n}) - n \delta \log d + (1 + \delta) h(\frac{\delta}{1 + \delta}) \] (A25)
As \( \lim h(\epsilon) = 0 \) and \( \lim \delta = 0 \), then we have
\[
E_c(\rho) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{E_c(|\psi^+_n\rangle\langle\psi^+_n|)}{n} \geq \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{E_c(\rho^\epsilon)}{n} \geq \lim_{n \to \infty} \frac{E_c(\rho)}{n} = E_c(\rho)
\]
The last equality is due to the result in ref. [34], then we finish the proof. \( \square \)

A.4. The Proof of Theorem 15

Theorem 15: Assume \( \rho_{AB} \in D_{AB} \) is a two-qubit state, then we have
\[
\overline{C}(\rho_{AB}) = C(\rho_{AB})
\]

Proof. Assume \( \{|\phi_i\rangle\} \) is the optimal decomposition of \( \rho_{AB} \) in terms of \( C \), that is, for any decomposition \( \{|\phi_i\rangle\} \) of \( \rho_{AB} \),
\[
\sum_i p_i C(|\phi_i\rangle) \leq \sum_k q_k C(|\varphi_k\rangle).
\] (A28)

Let \( |\chi\rangle \) be a pure state with \( \mu(|\chi\rangle) = \sum_i p_i \mu(|\phi_i\rangle) \), then by the theorem 1 in ref. [26], we have \( |\chi\rangle \) can be transformed into \( |\phi_i\rangle \) with probability \( p_i \), that is, \( |\chi\rangle \rightarrow |\phi_i\rangle \).

Next in ref. [42], the authors showed that for a two-qubit state \( \rho_{AB} \), there exists a decomposition \( \{|\varphi_k\rangle\} \) of \( \rho_{AB} \) such that
\[
C(\rho_{AB}) = \min_{\{|\varphi_k\rangle\}} \sum_k r_k C(|\varphi_k\rangle)
\]

Then we have
\[
\sum r_k \sqrt{1 - C^2(|\varphi_k\rangle)} = \sqrt{1 - C^2(\rho_{AB})} \geq \sqrt{1 - \sum q_k C(|\varphi_k\rangle)^2} \geq \sum q_k \sqrt{1 - C^2(|\varphi_k\rangle)}
\] (A30)

here we denote that \( \{|\varphi_k\rangle\} \) is an arbitrary decomposition of \( \rho_{AB} \). The first inequality is due to the definition of concurrence, the second inequality is due to the Cauchy-Schwarz inequality. By the equality (43), we have that
\[
\sum r_k \mu(|\varphi_k\rangle) > \sum q_k \mu(|\phi_k\rangle)
\] (A31)

Next we prove that the state \( |\varphi\rangle \) with \( \mu(|\varphi\rangle) = \sum_k r_k \mu(|\varphi_k\rangle) \) is the optimal for \( \rho_{AB} \) in terms of \( C \). If \( |\xi\rangle \in R(\rho_{AB}) \), then there exists a decomposition \( \{|m_i, |\phi_i\rangle\} \) of \( \rho_{AB} \),
\[
\mu^i(|\xi\rangle) < \sum r_l \mu^i(|\phi_l\rangle) < \sum q_k \mu^i(|\varphi_k\rangle)
\] (A32)

On the other hand, as
\[
C(|\varphi\rangle) = C(\rho), \forall \lambda,
\]
\[
\lambda_0 = \frac{1 + \sqrt{1 - C^2}}{2}, \lambda_1 = \frac{1 - \sqrt{1 - C^2}}{2},
\] (A34)

then we have that
\[
\mu^i(|\varphi\rangle) = \mu^i(|\varphi_k\rangle) > \mu^i(|\xi\rangle)
\] (A35)

that is, \( |\xi\rangle \rightarrow |\varphi\rangle \). Due to the definition of \( \overline{C} \), we have that \( |\varphi\rangle \) is the optimal pure state for \( \rho_{AB} \) in terms of \( E \). Then by the above analysis and the definition of \( \overline{C}(\rho_{AB}) \), we have that
\[
C(\rho_{AB}) = \overline{C}(\rho_{AB})
\] (A36)

Then we finish the proof. \( \square \)

Acknowledgements
Authors were supported by the NNSF of China (Grant No. 11871089), and the Fundamental Research Funds for the Central Universities (Grant Nos. KG12080401 and ZG21651902).

Conflict of Interest
The authors declare no conflict of interest.

Keywords
entanglement measure, monogamy of entanglement

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