Moments of Partition Functions of 2d Gaussian Polymers in the Weak Disorder Regime-I

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Abstract: Let $W_N(\beta) = E_0 \left[ e^{\sum_{n=1}^N \beta \omega(n, S_n) - N\beta^2/2} \right]$ be the partition function of a two-dimensional directed polymer in a random environment, where $\omega(i,x) \sim N(0,1)$, $i \in \mathbb{N}$, $x \in \mathbb{Z}^2$ are i.i.d. standard normal and $\{S_n\}$ is the path of a random walk. With $\beta = \hat{\beta} = \hat{\beta} \sqrt{\pi / \log N}$ and $\hat{\beta} \in (0, 1)$ (the subcritical window), $\log W_N(\beta_N)$ is known to converge in distribution to a Gaussian law of mean $-\lambda^2/2$ and variance $\lambda^2$, with $\lambda^2 = \log(1/(1 - \hat{\beta}^2))$ (Caravenna et al. in Ann Appl Probab 27(5):3050–3112, 2017). We study in this paper the moments $E[W_N(\beta_N)^q]$ in the subcritical window, for $q = O(\sqrt{\log N})$. The analysis is based on ruling out triple intersections.

1. Introduction and Statement of Results

We consider in this paper the partition function of two dimensional directed polymers in Gaussian environment, and begin by introducing the model. Set

$$W_N(\beta, x) = E_x \left[ e^{\sum_{n=1}^N \beta \omega(n, S_n) - N\beta^2/2} \right], \quad x \in \mathbb{Z}^d. \quad (1)$$

Here, $\{\omega_n, x\}_{n \in \mathbb{Z}_+, x \in \mathbb{Z}^d}$ are i.i.d. standard centered Gaussian random variables of law $\mathbb{P}$, $\{S_n\}_{n \in \mathbb{Z}_+}$ is simple random walk, and $E_x$ denotes the law of simple random walk started at $x \in \mathbb{Z}^2$. Thus, $W_N(\beta, x)$ is a random variable measurable on the $\sigma$-algebra $\mathcal{G}_N := \sigma(\{\omega(i, x) : i \leq N, x \in \mathbb{Z}^d\})$. For background, motivation and results on the rich theory surrounding this topic, we refer the reader to [15]. In particular, we mention the relation with the $d$ dimensional stochastic heat equation (SHE).

The random variables $W_N(\beta, 0)$ form a $\mathcal{G}_N$ positive martingale, and therefore converge almost surely to a limit $W_\infty(\beta, 0)$. It is well known that in dimensions $d = 1, 2,$
for any $\beta > 0$ we have $W_\infty(\beta, 0) = 0$, a.s., while for $d \geq 3$, there exists $\beta_c > 0$ so that $W_\infty(\beta_c, 0) > 0$ a.s. for $\beta < \beta_c$ and $W_\infty(\beta, 0) = 0$ for $\beta > \beta_c$. We refer to these as the weak and strong disorder regimes, respectively. In particular, for $d = 2$, which is our focus in this paper, for any $\beta > 0$, we are in the strong disorder regime.

A meaningful rescaling in dimension 2 was discovered in the context of the SHE by Bertini and Cancrini [2] and was later generalized by Caravenna et al. [5], in both the SHE and polymer setups, to a wider range of parameters for which a phase transition occurs. See also [8–10,22,28]. Introduce the mean intersection local time for random walk

$$ R_N = \mathbb{E} \log \left[ \sum_{n=1}^{N} 1_{S_n^1 = S_n^2} \right] \sim \frac{\log N}{\pi}. \quad (2) $$

The asymptotic behavior of $R_N$ follows from the local limit theorem [25, Sec. 1.2]. Further, the Erdős-Taylor theorem [21] states that

$$ \pi \log \sum_{n=1}^{N} 1_{S_n^1 = S_n^2} \Rightarrow 1. $$

Set

$$ \beta_N = \frac{\hat{\beta}}{\sqrt{R_N}}, \quad \hat{\beta} \geq 0. \quad (3) $$

We will use the short-notation $W_N = W_N(\beta_N, 0)$. With it, see [5], one has

$$ \forall \hat{\beta} < 1 : \quad \log W_N \xrightarrow{(d)} \mathcal{N} \left( -\frac{\lambda^2}{2}, \lambda^2 \right), \quad \text{with} \quad \lambda^2 = \lambda^2(\hat{\beta}) = \log \frac{1}{1 - \hat{\beta}^2}. \quad (4) $$

The convergence in (4) has recently been extended in [26] to the convergence of $W_N$ to the exponential of a Gaussian, in all $L^p$. (The critical case $\hat{\beta} = 1$, which we will not study in this paper, has received considerable attention, see [2,7,9,23].)

The spatial behavior of $W_N(\beta_N, x)$ is also of interest. Indeed, one has, see [8],

$$ G_N(x) := \sqrt{R_N} \left( \log W_N(\beta_N, x \sqrt{N}) - \mathbb{E} \log W_N(\beta_N, x \sqrt{N}) \right) \xrightarrow{(d)} \sqrt{\frac{\hat{\beta}^2}{1 - \hat{\beta}^2}} G(x), \quad (5) $$

with $G(x)$ a log-correlated Gaussian field on $\mathbb{R}^2$. The convergence in (5) is in the weak sense, i.e. for any smooth, compactly supported function $\phi$, $\int \phi(x) G_N(x) dx$ converges to a centered Gaussian random variable of variance $\hat{\beta}^2 \sigma^2(\phi) / (1 - \hat{\beta}^2)$, where

$$ \sigma^2(\phi) = \frac{1}{2\pi} \int \int \phi(x) \phi(y) \int_0^\infty z^{-1} e^{-z} dz. \quad (6) $$

One recognizes $\sigma^2(\phi)$ in (6) as the variance of the integral of $\phi$ against the solution of the Edwards–Wilkinson equation. For a related result in the KPZ/SHE setup, see [8,22,28].

Logarithmically correlated fields, and in particular their extremes and large values, have played an important recent role in the study of various models of probability theory at the critical dimension, ranging from their own study [3,4,18,29], random walk and Brownian motion [1,17], random matrices [12–14], Liouville quantum gravity [19,24], turbulence [11], and more. In particular, exponentiating Gaussian logarithmically
correlated fields yields Gaussian multiplicative chaoses, with the ensuing question of convergence towards them.

In the context of polymers, (5) opens the door to the study of such questions. A natural role is played by the random measure

\[ \mu_N^y(x) = \frac{e^{yG_N(x)}}{\mathbb{E}e^{yG_N(x)}}, \]

and it is natural to ask about its convergence towards a GMC, and about extremes of \( G_N(x) \) for \( x \) in some compact subsets of \( \mathbb{R}^2 \).

A preliminary step toward any such analysis involves evaluating exponential moments of \( G_N(0) \). This is our goal in this paper. In the following, \( q = q(N) \) denotes an integer \( q \geq 2 \) that may depend on \( N \). Our main result is the following.

**Theorem 1.1.** There exists \( \hat{\beta}_0 \leq 1 \) so that if \( \hat{\beta} < \hat{\beta}_0 \) and

\[
\limsup_{N \to \infty} \frac{3\hat{\beta}^2}{(1 - \hat{\beta}^2)} \frac{1}{\log N} \binom{q}{2} < 1,
\]

then,

\[
\mathbb{E}[W_N^q] \leq e^{(q/2)\lambda^2(1+\varepsilon_N)},
\]

where \( \varepsilon_N = \varepsilon(N, \hat{\beta}) \searrow 0 \) as \( N \to \infty \).

The proof will show that in Theorem 1.1, \( \hat{\beta}_0 \) can be taken as \( 1/96 \), but we do not expect this to be optimal.

**Remark 1.2.** With a similar method, we can also prove that the estimate (8) holds for all \( \hat{\beta} < 1 \) at the cost of choosing \( q^2 = o(\log N / \log \log N) \), see Sect. 2.4 for details. In particular, we obtain that the partition function possesses all (fixed) moments in the region \( \hat{\beta} < 1 \):

\[
\forall q \in \mathbb{N}, \sup_N \mathbb{E}[W_N^q] < \infty.
\]

As mentioned above, (9) was independently proved in [26]. (See also [27] for further precision and a multivariate generalization of the Erdős–Taylor theorem.) They also observed that together with the convergence in distribution (4), the estimate (9) implies that for all fixed \( q \in \mathbb{N}, \)

\[
\mathbb{E}[W_N^q]e^{-(q/2)\lambda^2(\hat{\beta})} \underset{N \to \infty}{\longrightarrow} 1.
\]

Note however that the estimate (8) does not yield (10) when \( q \to \infty \) with \( N \to \infty \).

**Remark 1.3.** Theorem 1.1 is of course not enough to prove convergence toward a GMC. For that, one would need to improve the error in the exponent from \( O(q^2\varepsilon_N) \) to \( O(1) \), to obtain a complementary lower bound and, more important, to derive similar multi-point estimates. We hope to return to these issues in future work.
The structure of the paper is as follows. In the next Sect. 2, we use a well-worn argument to reduce the computation of moments to certain estimates concerning the intersection of (many) random walks. After some standard preliminaries, we state there our main technical estimate, Theorem 2.1, which provides intersection estimates under the extra assumptions that all intersections are in pairs, i.e. that no triple (or more) points exist. The rest of the section provides the proof of Theorem 1.1. Section 3 then develops the induction scheme that is used to prove Theorem 2.1. Since we assume that there are no triple (or more) intersections, we may consider particles as matched in pairs at intersection times. The induction is essentially on the number of instances in which “matched particles” break the match and create a different matching. Section 4 provides a discussion of our results, their limitations, and possible extensions. In particular we explain there why the constraint on \( q \) in Theorem 1.1 limits our ability to obtain the expected sharp upper bounds on the maximum of \( \log W_N(\hat{\beta}_N, x\sqrt{N}) \). The appendices collect several auxiliary results and a sharpening of one of our estimates, see Proposition B.1.

2. Intersection Representation, Reduced Moments, and Proof of Theorem 1.1

Throughout the rest of the paper, we let \( p(n, x) = p_n(x) = P_0(S_n = x) \). There is a nice formula for the \( q \)-th moment of the partition function whose importance is apparent in previous work on directed polymers, for example in [6, 7]. Indeed,

\[
\mathbb{E}[W^q_N] = E_X^{\otimes q} e^{\sum_{i=1}^q \sum_{n=1}^N (\beta_N \omega(n, S^n_i) - \beta_N^2/2)},
\]

where \( S^1, \ldots, S^q \) are \( q \) independent copies of the simple random walk and \( E_X^{\otimes q} \) denotes the expectation for the product measure started at \( X = (x^1, \ldots, x^q) \). (If the starting point \( X \) is not specified, we assume \( X = 0 \).) Since the \( \omega(i, x) \) are Gaussian and the variance of \( \sum_{i=1}^q \beta_N \omega(n, S^n_i) \) is equal to \( \beta_N^2 \sum_{i=1}^q 1_S^n_i \), we have the following formula for the moment in terms of intersections of \( q \) independent random walks:

\[
\mathbb{E}[W^q_N] = E^{\otimes q} \left[ e^{\beta_N^2 \sum_{1 \leq i < j \leq q} \sum_{n=1}^N 1_{S^n_i = S^n_j}} \right]. \tag{11}
\]

2.1. No triple estimate. The key step in upper bounding the right-hand side of (11) is to restrict the summation to subsets where there are no triple (or more) intersections. More precisely, denote by

\[
F_n = \left\{ (\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in [1, q]^3 : \bar{\alpha} < \bar{\beta} < \bar{\gamma}, S^n_{\bar{\alpha}} = S^n_{\bar{\beta}} = S^n_{\bar{\gamma}} \right\}, \tag{12}
\]

\[
K_n = \left\{ (\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}) \in [1, q]^4 : \bar{\alpha} < \bar{\beta}, \bar{\gamma} < \bar{\delta}, \{\bar{\alpha}, \bar{\beta}\} \cap \{\bar{\gamma}, \bar{\delta}\} = \emptyset, S^n_{\bar{\alpha}} = S^n_{\bar{\beta}}, S^n_{\bar{\gamma}} = S^n_{\bar{\delta}} \right\}, \tag{13}
\]

and let

\[
G_T = \bigcap_{n \in [1, T]} (F_n \cup K_n)^C
\]

be the event that there is no triple (or more) intersection, i.e. that at each given time no more than a pair of particles are involved in an intersection.
The following theorem is the technically involved part of this paper. Its proof will be presented in Sect. 3.

**Theorem 2.1.** Fix $\hat{\beta} \in (0, 1)$. Then there exists $c = c(\hat{\beta}) > 0$ so that if (7) holds then uniformly in $T \in [1, N]$ as $N \to \infty$,

$$\sup_{\mathcal{X} \subseteq (\mathbb{Z}^2)^q} \mathbb{E}^\otimes q \left[ e^{\beta_N^2 \sum_{n=1}^T \sum_{1 \leq j < \ell \leq q} 1_{\mathcal{S}_n = \mathcal{S}_\ell}^j \mathbf{1}_{GT} } \right] \leq e^{2 \lambda_{T,N}(\frac{q}{2}) (1 + cq^{-1/2} + o(1))},$$

(14)

where $\lambda_{T,N}$ is defined as

$$\lambda_{T,N}(\hat{\beta}) = \lambda_{T,N} = \log \frac{1}{1 - \hat{\beta}^2 T \log N}.$$  

(15)

Note that as soon as $q > 9$, the expression in the left side of (14) trivially vanishes if $X = 0$. The $X$’s of interest are those that allow for non-existence of triple or more intersections.

Assuming Theorem 2.1, the proof of Theorem 1.1 is relatively straightforward. We will need the preliminary results collected in the next subsection.

2.2. A short time a priori estimate. The following lemma is a variation on Khas’minskii’s lemma [30, p. 8, Lemma 2.1].

**Lemma 2.2.** Let $Z$ be the set of all nearest-neighbor walks on $\mathbb{Z}^2$, that is $Z \in Z$ if $Z = (Z_i)_{i \in \mathbb{N}}$ where $Z_i \in \mathbb{Z}^2$ and $Z_{i+1} = Z_i \in \{ \pm e_j, j \leq d \}$ where $e_j$ are the canonical vectors of $\mathbb{Z}^2$. If for some $k \in \mathbb{N}$ and $\kappa \in \mathbb{R}$, one has

$$\eta = \sup_{Z \in \mathbb{Z}} \sup_{x \in \mathbb{Z}^2} \left( e^{k^2} - 1 \right) \mathbb{E}_x \left[ \sum_{n=1}^k 1_{S_n = Z_n} \right] < 1,$$

(16)

then

$$\sup_{Z \in \mathbb{Z}} \sup_{x \in \mathbb{Z}^2} \mathbb{E}_x \left[ e^{k^2 \sum_{n=0}^k 1_{S_n = Z_n}} \right] \leq \frac{1}{1 - \eta}. $$

(17)

**Proof.** Let $\Lambda_2 = e^{k^2} - 1$. We have:

$$\mathbb{E}_x \left[ e^{k^2 \sum_{n=1}^k 1_{S_n = Z_n}} \right] = \mathbb{E}_x \left[ \prod_{n=1}^k \left( 1 + \Lambda_2 1_{S_n = Z_n} \right) \right]$$

$$= \sum_{p=0}^{\infty} \Lambda_2^p \sum_{1 \leq n_1 < \cdots < n_p \leq k} \mathbb{E}_x \left[ \prod_{i=1}^p 1_{S_{n_i} = Z_{n_i}} \right]$$

$$= \sum_{p=0}^{\infty} \Lambda_2^p \sum_{1 \leq n_1 < \cdots < n_{p-1} \leq k} \mathbb{E}_x \left[ \prod_{i=1}^{p-1} 1_{S_{n_i} = Z_{n_i}}, \mathbb{E}_{S_{n_{p-1}}} \left[ \sum_{n=1}^{k-n_{p-1}} 1_{S_n = Z_{n+p-1}} \right] \right]$$

$$\leq \sum_{p=0}^{\infty} \Lambda_2^{p-1} \eta \sum_{1 \leq n_1 < \cdots < n_{p-1} \leq k} \mathbb{E}_x \left[ \prod_{i=1}^{p-1} 1_{S_{n_i} = Z_{n_i}} \right] \leq \cdots < \sum_{p=0}^{\infty} \eta^p = \frac{1}{1 - \eta}. $$

$\Box$
The next lemma gives an a-priori rough estimate on the moments of $W_k(\beta_N) = W_k(\beta_N, 0)$ when $k$ is small.

**Lemma 2.3.** Let $\hat{\beta} > 0$. Let $b_N > 0$ be a deterministic sequence such that $b_N = o(\sqrt{\log N})$ as $N \to \infty$. Assume that $q = O(\sqrt{\log N}) > 1$. Then, for all $k \leq e^{b_N}$,

$$
\mathbb{E}[W_k(\beta_N)^q] = \mathbb{E}^{\otimes q} \left[ e^{\beta_N^2 \sum_{1 \leq i < j \leq q} \sum_{n=1}^{k} 1_{S_n = S_n'}}^{q} \right] \leq e^{\frac{1}{\pi} (1 + \varepsilon_N) q^2 \beta_N^2 \log(k+1)}, \tag{18}
$$

for $\varepsilon_N = \varepsilon_N(\hat{\beta}) \to 0$ as $N \to \infty$.

The proof will show that the inequality in (18) remains true if $\mathbb{E}^{\otimes q}$ is replaced by $\mathbb{E}^{\otimes q_X}$ for arbitrary $X$.

**Proof.** Let $N_{i,j} = \sum_{n=1}^{k} 1_{S_n = S_n'}$. By Hölder’s inequality, we find that

$$
\mathbb{E}[W_k(\beta_N)^q] \leq \mathbb{E}^{\otimes q} \left[ e^{\frac{q \beta_N^2}{2} \sum_{1 \leq i < j \leq q} N_{i,j}^{1/2}} \right] = \mathbb{E}\left[ \mathbb{E}^{\otimes 2} \left[ e^{\frac{q \beta_N^2}{2} N_k^{1/2}} \left| S^1 \right| \right]^{q-1} \right],
$$

by independence of the $(N_{1,j})_{1 < j}$ conditioned on $S^1$. We now estimate the above conditional expectation using Lemma 2.2. Let $\kappa^2 = q \beta_N^2 / 2 \to 0$ and $\eta$ be as in (16). For any $Z \in \mathcal{Z}$ and $y \in \mathbb{Z}^2$,

$$
\mathbb{E}_y \left[ \sum_{n=1}^{k} 1_{S_n = Z_n} \right] \leq \sum_{n=1}^{k} \sup_x p_n(x),
$$

where, see Appendix A for an elementary proof,

$$
\forall n \geq 1 : \sup_x p_n(x) =: p_n^* \leq \frac{2}{\pi n}. \tag{19}
$$

Thus, $\eta \leq \frac{1}{\pi} (1 + o(1)) q \beta_N^2 \log(k+1) \to 0$, uniformly for $k \leq e^{b_N}$ as $N \to \infty$. Lemma 2.2 then yields that for such $k$’s,

$$
\mathbb{E}[W_k(\beta_N)^q] \leq \left( \frac{1}{1 - \frac{1}{\pi} (1 + o(1)) q \beta_N^2 \log(k+1)} \right)^{q-1} = e^{\frac{1}{\pi} (1 + o(1)) q^2 \beta_N^2 \log(k+1)}.
$$

$\square$

### 2.3. Proof of Theorem 1.1.

As a first step, we will prove that

$$
\mathbb{E}[W_N^q] \leq e^{(q/2)(1+cq^{-1/2}+\varepsilon_N)}, \tag{20}
$$

where $c = c(\hat{\beta}) > 0$ and $\varepsilon_N = \varepsilon_N(\hat{\beta}) \to 0$ as $N \to \infty$.

As a second step, we improve the bound in case $q$ is bounded and thus complete the proof for general $q(N)$ assuming only condition (7), by a diagonalization argument.
Recall the definitions of $\lambda_{k,N}$ in (15) and that $\lambda = \lambda_{N,N}(\hat{\beta})$. By standard convexity arguments, we note that $x \leq \log\left(\frac{1}{1-x}\right) \leq \frac{x}{1-x}$ for all $x \in [0, 1)$; hence for all $a > 1$ and $\hat{\beta} < 1$ such that $a\hat{\beta}^2 < 1$,
\begin{equation}
\forall k \leq N : \ a\hat{\beta}^2 \frac{\log k}{\log N} \leq \lambda_{k,N}(\sqrt{a}\hat{\beta})^2 \leq \frac{a\hat{\beta}^2}{1-a\hat{\beta}^2} \frac{\log k}{\log N}.
\end{equation}

Now, let
\[ I_{s,t} = \beta_N^2 \sum_{n=s+1}^{t} \sum_{i<j=q} \mathbf{1}_{S_n^i = S_n^j} \quad \text{and} \quad I_k = I_{0,k}, \]
and define
\[ M(X) := \mathbb{E}_X^{\otimes q} \left[ e^{I_N} \right] \quad \text{and} \quad M = \sup_{X \in (\mathbb{Z}^2)^q} M(X). \]

By (11), it is enough to have a bound on $M(0)$. In fact what we will give is a bound on $M$. To do so, we let $T = T_N > 0$ such that $\log T = o(\sqrt{\log N})$ and introduce the event
\[ \tau_T := \inf \{ n > T : F_n \cup K_n \text{ occurs} \}. \]

We then decompose $M(X)$ as follows:
\[ M(X) = \mathbb{E}_X^{\otimes q} \left[ e^{I_N} \mathbf{1}_{\tau_T \leq N} \right] + \mathbb{E}_X^{\otimes q} \left[ e^{I_N} \mathbf{1}_{\tau_T > N} \right] =: A(X) + B(X). \]

We start by bounding $B(X)$ from above. Let $c$ be as in Theorem 2.1. By Markov’s property,
\begin{align*}
\sup_{X \in (\mathbb{Z}^2)^q} B(X) & \leq \sup_{X \in (\mathbb{Z}^2)^q} \mathbb{E}_X^{\otimes q} \left[ e^{I_T} \right] \sup_{Y \in (\mathbb{Z}^2)^q} \mathbb{E}_Y^{\otimes q} \left[ e^{I_{N-T} \mathbf{1}_{\tau_0 > N-T}} \right] \\
& \leq e^\frac{1}{2} (1+\epsilon_N) q^2 \beta_N^2 \log T \left( \frac{q}{2} \right)^{\frac{1}{2}} \lambda_{N-T,N}(1+cq^{-1/2}+o(1)) \\
& \leq e^\left( \frac{q}{2} \right) \lambda^2 (1+cq^{-1/2}+o(1)),
\end{align*}
where in the second inequality, we used Lemma 2.3 and Theorem 2.1 and in the last inequality, we used that $\beta_N^2 \log T$ vanishes as $N \to \infty$ and that $\lambda_{N-T,N}^2 < \lambda^2$.

We will now deal with $A(X)$ and show that
\[ \sup_{X \in (\mathbb{Z}^2)^q} A(X) \leq M \epsilon_N, \quad \text{(25)} \]
with $\epsilon_N \to 0$. This, together with (23) and (24) implies that
\[ M(1-\epsilon_N) \leq e \left( \frac{q}{2} \right) \lambda^2 (1+cq^{-1/2}+o(1)), \]
which entails (20).

Toward the proof of (25), we first use Markov’s property to obtain that
\[ A(X) = \sum_{k=T}^{N} \mathbb{E}_X^{\otimes q} \left[ e^{I_k} \mathbf{1}_{\tau_T = k} \right] \leq M \sum_{k=T}^{N} \mathbb{E}_X^{\otimes q} \left[ e^{I_k} \mathbf{1}_{\tau_T = k} \right]. \]
In what follows, for $\mathcal{T} \subset \mathbb{N}$ we use the phrase "no triple+ in $\mathcal{T}$" to denote the event $(\cup_{n \in \mathcal{T}} (F_n \cup K_n))^\complement$. Similarly, for $\mathcal{I} \subset [1, q]$ we use the phrase "no triple+ in $\mathcal{I}$" to denote the event $(\cup_{n \in \mathcal{I}} (F_n^T \cup K_n^T))^\complement$ where $F_n^T$ and $G_n^T$ are defined as $F_n$ and $K_n$ but with $[1, q]$ replaced by $\mathcal{I}$. We then decompose over which event, $F_n$ or $K_n$, occurred at $\tau_T$, and then over which particles participated in the event:

$$A(X) \leq M \sum_{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \leq k} \sum_{T} N \mathbb{E}_{X}^{q} \left[ e^{l_k} \mathbf{1} \text{no triple+ in } [T, k-1] \mathbb{1}_{S_{k}^\tilde{\alpha} = S_{k}^\tilde{\beta} = S_{k}^\tilde{\gamma}} \right]$$

$$+ M \sum_{(\tilde{\alpha} < \tilde{\beta}) \neq (\tilde{\gamma} < \delta) k \in T} N \mathbb{E}_{X}^{q} \left[ e^{l_k} \mathbf{1} \text{no triple+ in } [T, k-1] \mathbb{1}_{S_{k}^\tilde{\alpha} = S_{k}^\tilde{\beta} = S_{k}^\tilde{\gamma}} \right]$$

$$= M (A_1(X) + A_2(X)).$$

By (26), it is enough to prove that $A_1(X)$ and $A_2(X)$ vanish uniformly in $X$ as $N \to \infty$ in order to obtain (25). We next show that $A_1(X)$ vanishes, the argument for $A_2(X)$ is similar. We bound $J_k$ by

$$I_k \leq J_k + J_k^\tilde{\alpha} + J_k^\tilde{\beta} + J_k^\tilde{\gamma},$$

where

$$J_k = \beta_N^2 \sum_{n=1}^{k} \sum_{1 \leq j \leq q_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma}}} \mathbf{1}_{S_{k}^{\tilde{\alpha}} = S_{k}^{\tilde{\beta}} = S_{k}^{\tilde{\gamma}}} \quad \text{and} \quad J_k^{l_0} = \beta_N^2 \sum_{n=1}^{k} \sum_{j \in [1, q] \setminus \{l_0\}} \mathbf{1}_{S_{k}^{l_0} = S_{k}^{\tilde{\alpha}}}. $$

If we let $\frac{1}{a} + \frac{3}{b} = 1$ with $1 < a \leq 2$ and $1 < b$, we have

$$\mathbb{E}_{X}^{q} \left[ e^{l_k} \mathbf{1} \text{no triple+ in } [T, k-1] \mathbb{1}_{S_{k}^\tilde{\alpha} = S_{k}^\tilde{\beta} = S_{k}^\tilde{\gamma}} \right] \leq \mathbb{E}_{X}^{q} \left[ e^{aJ_k} \mathbf{1} \text{no triple+ in } [T, k-1] \text{for particles of } [1, q] \setminus \{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\} \mathbb{1}_{S_{k}^\tilde{\alpha} = S_{k}^\tilde{\beta} = S_{k}^\tilde{\gamma}} \right]^{1/a}$$

$$\times \prod_{i_0 \in \{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\}} \mathbb{E}_{X}^{q} \left[ e^{bJ_k^{l_0}} \mathbf{1}_{S_{k}^{l_0} = S_{k}^\tilde{\alpha} = S_{k}^\tilde{\beta} = S_{k}^\tilde{\gamma}} \right]^{1/b}. $$

We treat separately the two quantities (27) and (28). Before doing so, we specify our choice of $a, b$ and $\hat{\beta}$. We assume that $\hat{\beta}^2 < 1/72$ and $a < 3/2$, with $a$ close enough to 3/2 (and $b$ close to 9) in such a way that

(1) $8b\hat{\beta}^2 < 1$,  \hspace{1cm} (ii) $\limsup_{N \to \infty} \frac{1}{N} q^2 \beta_N^2 =: \rho_0 < 1/a$ \hspace{1cm} and

(3) $\limsup_{N \to \infty} \frac{\hat{\beta}^2}{1 - a\hat{\beta}^2 \log N} (1 + cq^{-1/2}) =: \rho_1 < 1/a$.}

Note that (ii) and (iii) are assured to hold for $a$ close enough to 3/2 thanks to the assumption (7) which implies that $\limsup_{N} \pi^{-1} q^2 \beta_N^2 \leq \frac{2}{3}$ since $\beta_N \sim \pi \hat{\beta}^2 / \log N$. We chose $\hat{\beta}^2 < 1/72$ to allow (i).
We first bound (27). If \( k \leq e^{(\log N)^{1/3}} \), then, using that \( J_k \) does not depend on the \( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \) particles, (27) is bounded by

\[
E_X^{\otimes q} \left[ e^{a J_k} \right]^{1/a} P^{\otimes 3}_{(x^a, x^\beta, x^\gamma)} \left( S_k^\tilde{\alpha} = S_k^\tilde{\beta} = S_k^\tilde{\gamma} \right)^{1/a} \leq C e^{\frac{1}{3} (1 + \epsilon_N) \beta^2 \gamma^2 (\log (k+1)) / a k^{-2/a}},
\]

for some \( c > 0 \) and uniformly in \( X \in (\mathbb{Z}^2)^q \), where we have used in the inequality Lemma 2.3 and that \( \sum_x p_k(x)^3 \leq (p_k^*)^2 \leq k^{-2} \) by (19). For \( k \geq e^{(\log N)^{1/3}} \), we rely on (14) to bound (27) by

\[
P^{\otimes 3}_{(x^a, x^\beta, x^\gamma)} \left( S_k^\tilde{\alpha} = S_k^\tilde{\beta} = S_k^\tilde{\gamma} \right)^{1/a} \leq C k^{-2/a} \left( E_X^{\otimes q} \left[ e^{a J_k} \right] \sup_{y} E_Y^{\otimes (q-3)} \left[ e^{a J_k - T} \right] \right)^{1/a} \leq C e^{\frac{1}{3} (1 + \epsilon_N) \beta^2 \gamma^2 (\log T) / a} e^{(\frac{3}{2}) (1 + \epsilon_N) (\sqrt{\alpha \beta} / a k^{-2/a})},
\]

where we have used that \( J_{k-T} \leq C + J_{k-T-1} \) by (7). For (28), we apply the Cauchy-Schwarz inequality to find that

\[
E_X^{\otimes q} \left[ e^{b J_k^0} \right]^{1/b} \leq E_X^{\otimes q} \left[ e^{2b J_k^0} \right]^{1/2b} k^{-1/b},
\]

where we again used that \( \sum_x p_k(x)^3 \leq (p_k^*)^2 \leq k^{-2} \) by (19). Now observe that by conditioning on \( S^T \), we have

\[
E_X^{\otimes q} \left[ e^{2b J_k^0} \right] \leq \sup_{y \in \mathbb{Z}^2} E_Y^{S^T} \left[ \sup_{x \in \mathbb{Z}^2} E_X^{S^2} \left[ e^{2b \beta^2 N \sum_{n=1}^k 1_{S_n = S_k}^2} \right] \right]^{q-1},
\]

where uniformly on all nearest neighbor walks \( Z \in \mathbb{Z} \),

\[
(e^{2b \beta^2 N} - 1) \sup_{x \in \mathbb{Z}^2} E_x \sum_{n=1}^k 1_{S_n = Z_n} \leq 4(1 + o(1)) b \beta^2 \log (k+1) \log N \leq 8 b \beta^2 \log (k+1) \log N
\]

for all \( N \) large, because \( \beta^2 N \sim \pi \beta^2 / \log N \) and \( \sup_x p_n(x) \leq 2 / (\pi n) \), see (2) and (19). Hence by Lemma 2.2 with (29)-(i),

\[
\sup_{X \in (\mathbb{Z}^2)^q} E_X^{\otimes q} \left[ e^{b J_k^0} \right]^{1/b} \leq \left( \frac{1}{1 - 8 b \beta^2 \log (k+1) \log N} \right)^{(q-1)/2b} k^{-1/b} \leq e^{c (\log (k+1) / \log N)} k^{-1/b},
\]

for some universal constant \( c > 0 \), using (7).
We thus find that

\[
\sup_{X \in (\mathbb{Z}^2)^q} A_1(X) \leq q^3 \sum_{k = T}^{[e^{(\log N)^{1/3}}]} e^{\frac{1}{2} (1 + \varepsilon N) q^2 \beta_N^2 \log(k+1)} \left( e^{\frac{3}{4} \log(k+1)} \beta_N^{3/2} \right) k^{-2/a} e^{\frac{3}{4} \log(k+1) \sqrt{2 N}} k^{-3/b} + C q^3 e^{\frac{1}{4} (1 + \varepsilon N) q^2 \beta_N^2 \log T} \times
\]

\[
\sum_{k = [e^{(\log N)^{1/3}}]}^{N} e^{\frac{1}{2} (1 + \varepsilon N) q^2 \beta_N^2 \log(k+1)} \left( e^{\frac{3}{4} \log(k+1)} \beta_N^{3/2} \right) k^{-2/a} e^{\frac{3}{4} \log(k+1) \sqrt{2 N}} k^{-3/b}.
\]

This is bounded by

\[
q^3 \sum_{k = T}^{[e^{(\log N)^{1/3}}]} k^{-1 - \frac{\varepsilon}{2}} \leq C q^3 T^{-\delta},
\]

for \( N \) large enough. Hence, we can set \( T = [e^{(\log N)^{1/4}}] \) (which satisfies \( \log T = o(\sqrt{\log N}) \)), so that \( q^3 T^{-\delta} \to 0 \) as \( N \to \infty \). Relying on (21), the second sum in (31) is bounded by

\[
C q^3 e^{\varepsilon \log T} \sum_{k = [e^{(\log N)^{1/3}}]}^{N} e^{\frac{1}{2} (1 + \varepsilon N) q^2 \beta_N^2 \log(k+1)} \left( e^{\frac{3}{4} \log(k+1)} \beta_N^{3/2} \right) k^{-1 - \frac{\varepsilon}{2}} \leq C q^3 e^{-\frac{\delta}{2} (\log N)^{1/3} + c \log N}\]

for some \( \varepsilon > 0 \). Then the quantity in the last line vanishes as \( N \to \infty \). (Note that we decomposed the sum for \( k \geq e^{(\log N)^{1/3}} \) and let \( \log T = (\log N)^{1/4} \) to ensure that \( (\log N)^{1/3} \) \( \gg \) \( (\log N)^{1/4} \) in the last display). By (31) we have thus proven that \( \lim_{N \to \infty} \sup_{X \in (\mathbb{Z}^2)^q} A_1(X) = 0 \).

When dealing with \( A_2 \), we have to use Hölder’s inequality as in (27), (28) with 4 particles instead of 3, so in this case we can choose \( a \sim 3/2 \) and \( b \sim 12 \), and the condition (i) in (29) is satisfied with the restriction \( \beta^2 < 1/96 \). The rest of the argument follows the same line as for \( A_1 \) and we get that \( \lim_{N \to \infty} \sup_{X \in (\mathbb{Z}^2)^q} A_2(X) = 0 \).

As a result, we have shown that (20) holds. This proves (8) when \( q \to \infty \). When \( q = q_0 \), (20) yields that \( W_N \) is bounded in any \( L^p \), \( p > 1 \). This fact combined with (4) implies the convergence (10) for all fixed \( q \), which implies that (8) holds in the case \( q = q_0 \) as well.

We now turn to the general case, where we only assume that \( q(N) \) satisfies (7). Suppose that (8) does not hold, so that we can find \( \varepsilon_0 > 0 \) and a subsequence \( q' = q(\varphi(N)) \) such that

\[
\forall N \in \mathbb{N}, \quad \mathbb{E} W_N^{q' N} > e^{\frac{1}{2} \varepsilon_0^2 (q' N)^{(1 + \varepsilon_0)}}.
\]
One can distinguish two cases. If \( q'_{N} \) is bounded, then up to extracting a sub-sequence, we can suppose that \( q'_{N} \) converges to some \( q_{0} \geq 2 \). Then, one can check that by (4), we must have \( \mathbb{E} W_{N}^{q_{N}} \rightarrow e^{\lambda^{2}(q_{0})/2} \) (for example, using Skorokhod’s representation theorem and Vitali’s convergence theorem with the fact that \( W_{N} \) is bounded in any \( L^{p} \)). But this is impossible by (32). On the other hand, if \( q'_{N} \) is not bounded, up to extracting a sub-sequence we can suppose that \( q'_{N} \rightarrow \infty \). But then (32) cannot be true because (8) holds with \( q = q'_{N} \rightarrow \infty \). Therefore (8) must hold for any sequence \( q(N) \) that satisfies (7).

\[ \square \]

2.4. On Remark 1.2. We describe the changes needed for obtaining the claim in Remark 1.2. Recall the definitions of \( F_{n} \) and \( K_{n} \), see (12) and (13), and (11). Set

\[
A_{N} = \sum_{n=1}^{N} 1 \{ F_{n} \cup K_{n} \} \sum_{1 \leq i < j \leq q} 1_{S_{n}^{i} = S_{n}^{j}},
\]

\[
B_{N} = \sum_{n=1}^{N} 1_{F_{n}} \sum_{1 \leq i < j \leq q} 1_{S_{n}^{i} = S_{n}^{j}}, \quad C_{N} = \sum_{n=1}^{N} 1_{K_{n}} \sum_{1 \leq i < j \leq q} 1_{S_{n}^{i} = S_{n}^{j}}.
\]

Note that for any \( u_{N} \geq 1 \), we can check that \( E_{X}^{\otimes q} \left[ e^{u_{N} \beta_{N}^{2} A_{N}} \right] \) is bounded above by \( \Psi_{N,q}(X) \) of (40) with \( T = N \) and \( \beta_{N} \) replaced by \( \hat{\beta} \). Using Hölder’s inequality it is enough to show (together with the proof of Theorem 2.1, which actually controls \( \sup_{X} \Psi_{N,q}(X) \)) that for any \( \hat{\beta} < 1 \) and \( q_{N} = o(\log N / \log \log N) \), there exist \( u_{N} \rightarrow \infty \) so that

\[
\sup_{X} E_{X}^{\otimes q} \left[ e^{u_{N} \beta_{N}^{2} B_{N}} \right]^{1/u_{N}} \rightarrow N \rightarrow \infty 1, \quad \sup_{X} E_{X}^{\otimes q} \left[ e^{u_{N} \beta_{N}^{2} C_{N}} \right]^{1/u_{N}} \rightarrow N \rightarrow \infty 1. \tag{33}
\]

We sketch the proof of the first limit in (33), the proof of the second is similar. By Corollary C.2 (applied on the space of \( q \)-tuples of path, with \( f(Y_{n}) = v_{N} \beta_{N}^{2} \sum_{1 \leq i < j \leq q} 1_{S_{n}^{i} = S_{n}^{j}} 1_{F_{n}} \)), it suffices to show that

\[
\lim_{N \rightarrow \infty} \sup_{X \in \mathcal{Z}_{n}} E_{X}^{\otimes q} \left[ v_{N} \beta_{N}^{2} B_{N} \right] = 0. \tag{34}
\]

To see (34), fix \( K \in [1, N] \). By (19), we have that

\[
E_{X}^{\otimes q} \sum_{n=1}^{N} 1_{F_{n}} \sum_{1 \leq i < j \leq q} 1_{S_{n}^{i} = S_{n}^{j}} \leq \sum_{n=1}^{K} \frac{q(q-1)}{2} C_{n} + \sum_{n=K+1}^{N} \sum_{1 \leq i < j \leq q} E_{X}^{\otimes q} 1_{F_{n}} 1_{S_{n}^{i} = S_{n}^{j}}. \tag{35}
\]

For \( i < j \leq q \) and \( r \in \{0, 1, 2\} \), further denote

\[ F_{n}^{i,j,r} = \{ (\bar{\alpha}, \bar{\beta}, \bar{\gamma}) : \bar{\alpha} < \bar{\beta} < \bar{\gamma} \leq q, S_{n}^{i} = S_{n}^{j} = S_{n}^{\bar{\gamma}}, |\{\bar{\alpha}, \bar{\beta}, \bar{\gamma}\} \cap \{i, j\}| = r \}. \]

We have that

\[
\sum_{1 \leq i < j \leq q} E_{X}^{\otimes q} 1_{F_{n}} 1_{S_{n}^{i} = S_{n}^{j}} = \sum_{1 \leq i < j \leq q} E_{X}^{\otimes q} 1_{F_{n}^{i,j,2}} 1_{S_{n}^{i} = S_{n}^{j}} \sum_{r=0}^{2} E_{X}^{\otimes q} 1_{F_{n}^{i,j,r}} 1_{S_{n}^{i} = S_{n}^{j}}.
\]
We first focus on the term $r = 0$. By independence, (19) and the union bound,
\[
\sum_{1 \leq i < j \leq q} E_X^{\otimes q} 1_{F_i^*} 1_{S_i^* = S_j^*} \leq \sum_{1 \leq i < j \leq q} C \frac{n}{a_i} \sum_{a_i < a_i' < q} \sup_{x_i \in \mathbb{Z}^2} \prod_{y_i = 1}^{3} P_{x_i}(S_n = y) \leq \frac{Cq^5}{n^3}.
\]
When $r = 1$, the condition in the indicator function becomes that there exist $\bar{a} < \bar{b} \leq q$ such that $S_n^i = S_n^i = S_n^\beta$. Hence, the term for $r = 1$ is bounded by
\[
\sum_{1 \leq i < j \leq q} \sum_{a_i < a_i' < q} \sup_{x_i \in \mathbb{Z}^2} \prod_{y_i = 1}^{4} P_{x_i}(S_n = y) \leq \frac{Cq^4}{n^3}.
\]
Similarly, we can bound the term for $r = 2$ by a constant times $q^3/n^2$. Using (35), we find that for all $K \in [1, N]$,
\[
\sup_{x \in \mathbb{Z}^q} E_X^{\otimes q} [v_N \beta_N^2 B_N] \leq \frac{Cv_N \beta_N^2}{\log N} \left( \frac{q(q - 1)}{2} \log K + \frac{q^5}{K^2} + \frac{q^4}{K^2} + \frac{q^3}{K} \right). \tag{36}
\]
For $K = \lceil (\log N)^{3/4} \rceil$, and $q^2 = o(\log N / \log \log N)$, we find that (34) holds with a well-chosen $v_N \to \infty$.

3. No Triple Intersections: Proof of Theorem 2.1

Recall that $T \in [1, N]$. For compactness of notation in the rest of the paper, set
\[
\sigma_N^2 = \sigma_N^2(\beta) = e^{\beta^2_N} - 1. \tag{37}
\]
By (2), there exist $\delta_N = \delta(N, \beta)$ and $\delta'_N = \delta'(N, \beta)$ that vanish as $N \to \infty$ such that
\[
\sigma_N^2 = \beta^2_N (1 + \delta_N) = \frac{\pi \beta^2}{\log N} (1 + \delta'_N). \tag{38}
\]

3.1. Expansion in chaos. In this section, we show that the moment without triple intersections can be bounded by a rather simple expansion. Introduce the following notation: for $n = (n_0, n_1, \ldots, n_k)$ and $x = (x_0, x_1, \ldots, x_k)$, let $p_{n,x} = \prod_{i=1}^{k} p(n_i - n_{i-1}, x_i - x_{i-1})$. Let $C_q = \{(i, j) \in [1, q]^2 : i < j \}$.

**Proposition 3.1.** For all $X = (x^1_0, \ldots, x^q_0) \in (\mathbb{Z}^2)^q$, we have
\[
E_X^{\otimes q} \left[ e^{\sigma_N^2 \sum_{n=1}^{T} \sum_{1 \leq i < j \leq q} 1_{S_i^* = S_j^*} 1_{G_T}} \right] \leq \Psi_{N,q}(X), \tag{39}
\]
where
\[
\Psi_{N,q}(X) = \sum_{k=0}^{\infty} \sigma_N^{2k} \sum_{1 \leq n_1 < \cdots < n_k \leq T} \prod_{r=1}^{k} 1_{x^r_i = x^r_j} \prod_{i=1}^{q} p_{(0, n_1, \ldots, n_k), (x^i_0, x^i)} \tag{40}
\]
where we recall (37) for the definition of $\sigma_N$. 
Moments of partition functions of 2d polymers

(By convention, here and throughout the paper, the term $k = 0$ in sums as (40) equals 1.)

**Proof.** For brevity, we write $G$ for $G_T$. For $X = (x_0^1, \ldots, x_0^q) \in (\mathbb{Z}^2)^q$, using the identity $e^{\beta N} 1_{s_n = s_0^j} - 1 = \sigma_N^2 1_{s_n = s_0^j}$,

$$M_{N,q}^{\text{no triple}}(X) := E_X^{\otimes q} \left[ e^{\beta_N \sum_{n=1}^T \sum_{1 \leq i < j \leq q} 1_{s_n = s_0^j} 1_{G}} \right]^{\otimes q} \prod_{n \in [1,T], (i,j) \in C_q} \left( 1 + \sigma_N^2 1_{s_n = s_0^j} 1_G \right) .$$

Expand the last product to obtain that:

$$M_{N,q}^{\text{no triple}}(X) = \sum_{k=0}^{\infty} \sigma_N^{2k} \sum_{(n_r, i_r, j_r) \in [1,T] \times C_q, r=1,\ldots,k} \prod_{r=1}^k 1_{s_{n_r} = s_0^{i_r}} 1_{G} ,$$

where we have used the lexicographic ordering on 3-tuples $(n, i, j)$. Since there are no triple or more particle intersections on the event $G$, the sum above can be restricted to 3-tuples $(n_r, i_r, j_r)_{r \leq k}$ such that $n_r < n_{r+1}$ for all $r < k$. Hence,

$$M_{N,q}^{\text{no triple}}(X) = \sum_{k=0}^{\infty} \sigma_N^{2k} \sum_{1 \leq n_1 < \cdots < n_k \leq T, (i_r, j_r)_{r \leq k} \in C_q^k} \prod_{r=1}^k 1_{s_{n_r} = s_0^{i_r}} 1_{G} \leq \Psi_{N,q}(X) ,$$

where $\Psi$ is defined in (40), and where we have bounded $1_G$ by 1 in the inequality. \qed

### 3.2. Decomposition in two-particle intersections.

In this section, we rewrite $\Psi_{N,q}$ in terms of successive two-particle interactions. We generalize a decomposition used in [7, Section 5.1] that was restricted to a third moment computation ($q = 3$). The following notation is borrowed from their paper. Let

$$U_N(n, x) := \begin{cases} \sigma_N^2 E_0^{\otimes 2} \left[ e^{\beta_N^2 \sum_{l=1}^{n-1} 1_{s_l^1 = s_l^2} 1_{s_{n-1}^1 = s_{n-1}^2 = x}} \right] & \text{if } n \geq 1, \\ 1 & \text{if } n = 0, \end{cases}$$

and

$$U_N(n) := \sum_{z \in \mathbb{Z}^2} U_N(n, z) = \begin{cases} \sigma_N^2 E_0^{\otimes 2} \left[ e^{\beta_N^2 \sum_{l=1}^{n-1} 1_{s_l^1 = s_l^2} 1_{s_{n-1}^1 = s_{n-1}^2}} \right] & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$
Observe that, by the identity $e^{\frac{\beta^2}{N} \sum_{i=1}^{\infty} 1_{S_i^1=S_i^2}} - 1 = \sigma_N^2 \sum_{i=1}^{\infty} 1_{S_i^1=S_i^2}$, one has for all $n \geq 1$,

$$
\mathbb{E}_{0}^{\otimes 2} \left[ e^{\frac{\beta^2}{N} \sum_{i=1}^{n-1} 1_{S_i^1=S_i^2}} \right] = \mathbb{E}_{0}^{\otimes 2} \left[ \prod_{l=1}^{n-1} \left( 1 + \sigma_N^2 1_{S_l^1=S_l^2} \right) 1_{S_n^1=S_n^2} = x \right]
$$

$$
= \sum_{k=0}^{\infty} \sigma_N^{2k} \sum_{n_0=0<n_1<\cdots<n_k<n} \mathbb{E}_{0}^{\otimes 2} \left[ \prod_{r=1}^{k} 1_{S_{n_r}^1=S_{n_r}^2} 1_{S_{n_r}^1=S_{n_r}^2} = x \right].
$$

Hence for all $n \geq 1$:

$$
U_N(n, x) = \sigma_N^2 \sum_{k=0}^{\infty} \sigma_N^{2k} \sum_{n_0=0<n_1<\cdots<n_k<n} \prod_{r=1}^{k+1} p_{n_r-n_{r-1}} (x_r - x_{r-1})^2.
$$

Now, in the sum in (40), we observe that (only) two particles interact at given times $(n_1, \ldots, n_k)$. So we define $a_1 = n_1$ and $b_1 = n_2$ such that $(n_1, n_2, \ldots, n_k)$ are the successive times that verify $(i_1, j_1) = (i_2, j_2) = \cdots = (i_r, j_r)$ before a new couple of particles $(i_{r+1}, j_{r+1}) \neq (i_1, j_1)$ is considered, and we let $k_1 = r$ be the number of times the couple is repeated. Define then $a_2 \leq b_2, a_3 \leq b_3, \ldots, a_m \leq b_m$ similarly for the next interacting couples, with $m$ denoting the number of alternating couples and $k_2, \ldots, k_m$ the numbers of times the couples are repeated successively.

Further let $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_m)$ with $X_r = (x_1^r, \ldots, x_q^r)$ and $Y_r = (y_1^r, \ldots, y_q^r)$ denote respectively the positions of the particles at time $a_r$ and $b_r$. We also write $X = (x_0^p)_{p \leq q}$, for the initial positions of the particles at time 0. We call a diagram $I$ of size $m \in \mathbb{N}$ any collection of $m$ couples $I = (i_r, j_r)_{r \leq m} \in \mathbb{C}^m$ such that $(i_r, j_r) \neq (i_{r+1}, j_{r+1})$. We denote by $\mathcal{D}(m, q)$ the set of all diagrams of size $m$.

If we re-write $\Psi_{N,q}(X)$ according to the decomposition that we just described, we find that:

$$
\Psi_{N,q}(X) = \sum_{m=0}^{\infty} \sum_{1 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \cdots \leq a_m \leq b_m \leq T} \sum_{k_1 \in [1, b_1-a_1+1], \ldots, k_m \in [1, b_m-a_m+1]} \sigma_N^{2k_1+\cdots+2k_m}
$$

$$
\times \prod_{p \leq q} p_{a_1} (x_1^p - x_0^p) \left( \prod_{r=1}^{m-1} \prod_{p \leq q} p_{a_{r+1} - b_r} (x_{r+1}^p - y_r^p) \right) \prod_{r=1}^{m} 1_{x_r^i = x_r^j} 1_{y_r^i = y_r^j}
$$

$$
\times \sum_{a_r < n_1 < \cdots < n_{k_r-2} < b_r} \prod_{s=1}^{k_r-2} 1_{z_r^s = z_r^s} \prod_{p \leq q} p_{(a_r, n_1, \ldots, n_{k_r-2}; b_r)} (x_{k_r-1}^p, z_1^p, \ldots, z_{k_r-2}^p, y_r^p). \quad \text{See Fig. 1 for a pictorial description of the intersections associated with a diagram.}
$$

Summing over all the configurations between time $a_r$ and $b_r$ gives a contribution of $\sigma_N^2 1_{x_r^i = y_r^i}$ when $a_r = b_r$, and
\[
\sum_{k=2}^{\infty} \sigma_N^{2k} \sum_{n_0 = a_r < n_1 < \cdots < n_{k-2} < b_r = n_{k-1}} p_{n_i - n_{i-1}} (x_i - x_{i-1})^2 \\
= \sigma_N^2 U_N (b_r - a_r, y_r^lr - x_r^lr) 
\]
when \(a_r < b_r\) (in this case \(k_r \geq 2\) by definition). It directly follows that:

\[
\Psi_{N,q}(X) = \sum_{m=0}^{\infty} \sigma_N^{2m} \sum_{1 \leq a_1 < a_2 < b_2 < \cdots < a_m \leq \overline{b} \leq T} A_{X,a,b,X,Y,I},
\]
where

\[
A_{X,a,b,X,Y,I} = \prod_{p \leq q} p_{a_1}(x_1^p - x_0^p) \prod_{r=1}^{m} U_N(b_r - a_r, y_r^lr - x_r^lr) 1_{x_r^lr = y_r^lr} 1_{y_r^lr = y_r^lr} \\
\times \prod_{p \notin \{i_r, j_r\}} p(b_r - a_r, y_r^p - x_r^p) \prod_{r=1}^{m-1} p(a_{r+1} - b_r, x_r^p - y_r^p).
\]

We can further simplify the expression (45). Let \(I = (i_r, j_r)_{r \leq m} \in D(m, q)\) be any diagram. For all \(r \leq m\), denote by \(k_r^l\) the last index \(l < r\) such that \(i_r \in \{i_l, j_l\}\), i.e. \(k_r^l = \text{sup}\{l \in [1, r-1] : i_r \in \{i_l, j_l\}\}\). When the set is empty we set \(k_r^l = 0\). Define \(k_r^2\) similarly for \(j_r\) instead of \(i_r\) and let \(k_r = k_r^1 \lor k_r^2\). See Fig. 1.

**Proposition 3.2.** For all \(X \in \mathbb{Z}^2\),

\[
\Psi_{N,q}(X) = \sum_{m=0}^{\infty} \sigma_N^{2m} \sum_{1 \leq a_1 < a_2 < b_2 < \cdots < a_m \leq \overline{b} \leq T} A_{X,a,b,x,y,I},
\]
where

\[
A_{X,a,b,x,y,I} = \prod_{p \in \{i_1, j_1\}} p_{a_1}(x_1^p - x_0^p) \prod_{r=1}^{m} U_N(b_r - a_r, y_r - x_r) \\
\times \prod_{r=1}^{m-1} p(a_{r+1} - b_{k_r^1}, x_{r+1}^1 - y_{k_r^1}^1) p(a_{r+1} - b_{k_r^2}, x_{r+1}^2 - y_{k_r^2}^2).
\]

**Proof.** Denote \(x_r = x_r^lr\) and \(y_r = y_r^lr\). We obtain (48) from (46) by using the semi group property of the random walk transition probabilities and summing, at intersection times, over the location of particles not involved in the intersection. \(\square\)

**Proposition 3.3.** We have that

\[
\sup_{X \in \mathbb{Z}^2} \Psi_{N,q}(X) \leq \sum_{m=0}^{\infty} \sum_{I \in D(m, q)} \sigma_N^{2m} A_{m,N,I}.
\]
where

\[ A_{m,N,t} = \sum_{u_i \in [1, T], v_i \in [0, T], 1 \leq i \leq m} p_{2m_1}^* U_N(v_m) \prod_{r=1}^{m-1} U_N(v_r) p_{v_r+2u_{r+1}+2\tilde{u}_{r+1}}^* \]  \hspace{1cm} (50)

with

\[ \tilde{u}_r = \begin{cases} \sum_{i=\bar{k}_r+1}^{r-1} u_i & \text{if } \bar{k}_r < r - 1, \\ \frac{u_r - 1}{2} & \text{if } \bar{k}_r = r - 1, \end{cases} \]  \hspace{1cm} (51)

and \( p_k^* = \sup_{x \in \mathbb{Z}^2} p_k(x) \).

**Proof.** By (47), it is enough to show that

\[ \sup_{X \in (\mathbb{Z}^2)^q} \sum_{1 \leq a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_m \leq b_m \leq T} \tilde{A}_{X,a,b,x,y,I} \leq A_{m,N,I}. \]  \hspace{1cm} (52)

We begin by summing on \( y_m \), which gives a contribution of

\[ \sum_{y_m} U_N(b_m - a_m, y_m - x_m) = U_N(b_m - a_m), \]
where $U_N(n)$ is defined in \((42)\). Then summing on $x_m$ gives a factor

$$\sum_{x_m} p(a_m - b_{\bar{\epsilon}_m}, x_m - y_{\bar{\epsilon}_m}) p(a_m - b_{\bar{\epsilon}_m}, x_m - y_{\bar{\epsilon}_m})$$

$$= p(2a_m - b_{\bar{\epsilon}_m} - b_{\bar{\epsilon}_m}, y_{\bar{\epsilon}_m} - y_{\bar{\epsilon}_m}) \leq p(2a_m - b_{\bar{\epsilon}_m} - b_{\bar{\epsilon}_m}).$$

By iterating this process we obtain that the sum on $x, y$ is bounded (uniformly on the starting point $X$) by

$$\sum_{r=1}^{m-1} U_N(b_r - a_r) p(2a_r + 1 - b_{\bar{\epsilon}_{r+1}}, x_{\bar{\epsilon}_{r+1}}) \leq p_N(b_0 - a_0, x_0).$$

If we introduce the change of variables $u_i = a_i - b_i$ and $v_i = b_i - a_i$ with $b_0 = 0$, then equation \((52)\) follows from combining that $2a_r + 1 - b_{\bar{\epsilon}_{r+1}} \geq v_r + 2a_{r+1} + 2\bar{u}_{r+1}$ with the monotonicity of $p_N$ in $n$, which follows from

$$p_{n+1} = \sup_{y} \sum_{x} p_{n}(x) p_{1}(y - x) \leq p_{n}.$$ \hspace{1cm} (53)

$\square$

### 3.3. Estimates on $U_N$. It is clear from Proposition 3.3 that the function $U_N$ plays a crucial role in our moment estimates, which we will obtain by an induction in the next subsection. In the current subsection, we digress and obtain a-priori estimates on $U_N$ and $\mathbb{E}[W_n(\beta_N)^2]$). Appendix B contains some improvements that are not needed in the current work but may prove useful in follow up work.

**Proposition 3.4.** There exists $N_0 = N_0(\hat{\beta})$ such that for all $N \geq N_0$ and all $n \leq N$,

$$\mathbb{E} \left[ W_n(\beta_N)^2 \right] \leq \frac{1}{1 - \sigma_N^2 R_n}. \hspace{1cm} (54)$$

Furthermore, there exists $\varepsilon_n = \varepsilon(n, \hat{\beta}) \to 0$ as $n \to \infty$, and $\varepsilon_{n,N}$ satisfying $|\varepsilon_{n,N}| \leq \varepsilon_n$, such that for all $N \geq n$,

$$\mathbb{E} \left[ W_n(\beta_N)^2 \right] = (1 + \varepsilon_{n,N}) \frac{1}{1 - \hat{\beta}_2 \log n}. \hspace{1cm} (55)$$

**Proof.** We first choose $N_0 = N_0(\hat{\beta})$ large enough such that for all $N \geq (n \lor N_0)$, we have $\sigma_N^2 R_n < 1$. That this is possible follows from \((19)\) which yields that

$$\forall n \in \mathbb{N}, \quad R_n = \sum_{s=1}^{n} p_{2s}(0) \leq \frac{1}{\pi} \sum_{s=1}^{n} \frac{1}{s} \leq \frac{1}{\pi} \log(n + 1). \hspace{1cm} (56)$$

For the rest of the proof, we continue in this setup. Similarly to \((43)\), we have (letting $n_0 = x_0 = 0$) that

$$\mathbb{E} \left[ W_n(\beta_N)^2 \right] = E_0 \left[ e^{\hat{\beta}_N \sum_{k=1}^{n} S_k^1 S_k^2} \right]$$
\[
= \sum_{k=0}^{\infty} \sigma_N^{2k} \sum_{0<n_1<\cdots<n_k \leq n} \sum_{x_1,\ldots,x_k \in \mathbb{Z}^2} \prod_{i=1}^{k} p_{n_i-n_{i-1}}(x_i - x_{i-1})^2. \tag{57}
\]

Hence, if for each \( k \in \mathbb{N} \) in (57) we let \( n_i - n_{i-1} \) run free in \([1, n]\), we obtain that
\[
\mathbb{E} \left[ W_n(\beta N)^2 \right] \leq \sum_{k=0}^{\infty} \sigma_N^{2k} \left( \sum_{m=1}^{n} \sum_{x \in \mathbb{Z}^2} p_m(x) \right)^k = \sum_{k=0}^{\infty} \sigma_N^{2k} R_n^k = \frac{1}{1 - \sigma_N^2 R_n},
\]
which gives (54). On the other hand, if for each \( k \in \mathbb{N} \) in (57) we let \( n_i - n_{i-1} \) run free in \([1, n/k]\), we have
\[
\mathbb{E} \left[ W_n(\beta N)^2 \right] \geq 1 + \sum_{k=1}^{\infty} \sigma_N^{2k} \left( \sum_{m=1}^{n/k} \sum_{x \in \mathbb{Z}^2} p_m(x) \right)^k \geq 1 + \sum_{k=1}^{\log n} \sigma_N^{2k} R_n^{k/n} = \frac{1 - (\sigma_N^2 R_n/\log n)^{\log n + 1}}{1 - \sigma_N^2 R_n/\log n}.
\]
By (38) and the fact that \( R_n \sim \frac{1}{\pi} \log n \) as \( n \to \infty \) by (2), we find that for all \( N \geq n \),
\[
\mathbb{E} \left[ W_n(\beta N)^2 \right] \geq (1 + \delta_n) \frac{1}{1 - \hat{\beta}^2 \log n/\log N},
\]
with \( \delta_n = \delta_n(\hat{\beta}) \to 0 \) as \( n \to \infty \). Combining this with (54) entails (55). \( \square \)

**Proposition 3.5.** For all \( M \geq 1 \), we have:
\[
\sum_{n=0}^{M} U_N(n) = \mathbb{E} \left[ W_M^2 \right]. \tag{58}
\]
Moreover, there is \( C(\hat{\beta}) > 0 \) and \( N_0(\hat{\beta}) > 0 \) such that, for \( N \geq N_0(\hat{\beta}) \) and for all \( n \leq N \),
\[
U_N(n) \leq C \frac{1}{(1 - \hat{\beta}^2 \log n/\log N)^2} \frac{1}{n \log N}. \tag{59}
\]

**Remark 3.6.** When \( n \to \infty \), one can take the constant \( C \) that appears in (59) arbitrarily close to one. See Appendix B.

**Proof.** By (44), we have, for \( n \geq 1 \), with \( x_0 = 0 \),
\[
U_N(n) = \sigma_N^2 \sum_{k=1}^{\infty} \sigma_N^{2(k-1)} \sum_{0<n_1<\cdots<n_k=n} \sum_{x_1,\ldots,x_k \in \mathbb{Z}^2} \prod_{i=1}^{k} p_{n_i-n_{i-1}}(x_i - x_{i-1})^2 \tag{60}
\]
\[
= \sigma_N^2 \sum_{k=1}^{\infty} \sigma_N^{2(k-1)} \sum_{0<n_1<\cdots<n_k=n} \prod_{i=1}^{k} p_{2n_i-2n_{i-1}}(0).\]
Therefore,

\[
\sum_{n=0}^{M} U_N(n) = 1 + \sum_{k=1}^{\infty} \sigma_N^{2k} \sum_{0 < n_1 < \cdots < n_k \leq M} \prod_{i=1}^{k} p_{n_i - n_{i-1}} (x_i - x_{i-1})^2
\]

\[
= \mathbb{E}[W^2_M],
\]

which yields (58).

We now prove (59) by expressing \( U_N \) as a function of a renewal process. From (60), one can see that the following representation for \( U_N(n) \) holds when \( n \geq 1 \):

\[
U_N(n) = \sum_{k=1}^{\infty} (\sigma_N^2 R_N)^k p \left( \tau_k^{(N)} = n \right),
\]

where the \( \tau_k^{(N)} \) are renewal times defined by

\[
\tau_k^{(N)} = \sum_{i \leq k} T_i^{(N)},
\]

with \( (T_i^{(N)}) \) being i.i.d. random variables with distribution

\[
P \left( T_1^{(N)} = n \right) = \frac{1}{R_N} p_{2n}(0) 1_{1 \leq n \leq N}, \quad \text{and} \quad R_N = \sum_{n=1}^{N} p_{2n}(0).
\]

The renewal formulation that is used here is due to [6]. We also refer to [21, Chapter 1] for the connection of polymer models to renewals processes in a general context. By [6, Proposition 1.5], there exists \( C > 0 \) such that for all \( n \leq N \),

\[
P \left( \tau_k^{(N)} = n \right) \leq C k p \left( T_1^{(N)} = n \right) p \left( T_1^{(N)} \leq n \right)^{k-1}.
\]

Hence, using that \( \sum_{k=1}^{\infty} k a^k = \frac{1}{(1-a)^2} \) for \( a < 1 \),

\[
U_N(n) \leq C \sum_{k=1}^{\infty} (\sigma_N^2 R_N)^k p \left( T_1^{(N)} = n \right) p \left( T_1^{(N)} \leq n \right)^{k-1}
\]

\[
= C p_{2n}(0) \frac{\sigma_N^2 R_N}{R_N \left( 1 - \sigma_N^2 R_N \frac{R_n}{R_N} \right)^2},
\]

which gives (59) by (2), (19) and (38). \( \square \)
3.4. Summing on the $v_j$'s. In the following, we denote
\[
F(u) = \frac{1}{u} \frac{1}{1 - \hat{\beta}^2 \log(u)}.
\] (62)

By differentiation with respect to $u$ one checks that $F$ is non-increasing.

Proposition 3.7. There exist $N_0(\hat{\beta}) > 0$ and $\varepsilon_N = \varepsilon(N, \hat{\beta}) \searrow 0$ as $N \to \infty$, such that for all $N \geq N_0(\hat{\beta})$,
\[
\sup_{X \in \{\mathbb{Z}\}^d} \Psi_{X,q}(X) \leq \sum_{m=0}^{\infty} \sigma_N^{2m} \sum_{i \in D(m,q)} \left( \frac{1}{\pi} \right)^{m-1} \tilde{A}_{m,N,1},
\] (63)

where, recalling (51),
\[
\tilde{A}_{m,N,1} = \frac{1}{1 - \hat{\beta}^2} \sum_{u_i \in \{1,2,3\}} (1 + \varepsilon_N)^m p_{2m_1}^* \prod_{r=2}^m F(u_r + \tilde{u}_r) \mathbf{1}_{\sum_{i=1}^r u_i \leq T}.
\] (64)

Proof. By (50), (55) and (58), summing over $v_m$ in $A_{m,N,1}$ gives a factor bounded by
\[
\frac{1}{1 - \hat{\beta}^2} (1 + o(1)).
\] We will now estimate the sum over the variable $v_{m-1}$. Let $w = u_m + \tilde{u}_m$.
(Note that by definition $3/2 \leq w \leq T \leq N$, and that $w$ might be a non-integer multiple of $1/2$.) Writing $v = v_{m-1}$, the sum over $v_{m-1}$ in (50) gives a factor
\[
\sum_{v=0}^T U_N(v) p_{v+2w}^* =: S_{\leq w} + S_{> w},
\] (65)

where $S_{\leq w}$ is the sum on the left hand side of (65) restricted to $v \leq \lfloor w \rfloor$ and $S_{> w}$ is the sum for $v \geq \lfloor w \rfloor$. Using (19) and (59), there exists a constant $C = C(\hat{\beta}) > 0$ such that
\[
S_{> w} \leq \frac{C}{\log N} \sum_{v=\lfloor w \rfloor+1}^T \frac{1}{v^2} \leq \frac{1}{C} \frac{1}{\log N \, w}.
\] (66)

Using (53) and (58),
\[
S_{\leq w} \leq p_{2w}^* \sum_{v=0}^{\lfloor w \rfloor} U_N(v) = p_{2w}^* \mathbb{E}[W_{\lfloor w \rfloor}^2] \leq p_{2w}^* \frac{1}{\frac{1}{1 - \sigma_N^2 R_{\lfloor w \rfloor}}},
\] (67)

where the second inequality holds by (54) for all $N \geq N_0(\hat{\beta})$ since $w \leq N$. Let $\delta_N = \delta(N, \hat{\beta}) \to 0$ such that (38) holds, and let $N'_0 = N'_0(\hat{\beta}) > N_0(\hat{\beta})$ be such that $\sup_{N \geq N'_0} \sup_{n \leq N} \hat{\beta}^2 \frac{1+\log n}{\log N} (1 + \delta_N) < 1$. By (19) and (56), we obtain that
\[
p_{2w}^* \frac{1}{\frac{1}{1 - \hat{\beta}^2 \frac{1+\log \log n}{\log N} (1 + \delta_N)}} \leq \frac{1}{\pi} \frac{1}{w} \frac{1}{1 - \hat{\beta}^2 \frac{1+\log \log w}{\log N} (1 + \delta_N)}.
\]

Moreover, as there is $C(\hat{\beta}) \in (0, \infty)$ such that
\[
\sup_{N \geq N'_0} \sup_{n \leq N} \frac{1}{1 - \hat{\beta}^2 \frac{1+\log n}{\log N} (1 + \delta_N)} \leq C(\hat{\beta}),
\]

By (51) and (57), summing over $v_{m-1}$ gives a factor bounded by
\[
\frac{1}{1 - \hat{\beta}^2} (1 + o(1)).
\]
we see that there exists $\varepsilon'_N = \varepsilon'(N, \tilde{\beta}) \searrow N \to \infty 0$ such that for all $n \leq N$,

$$\frac{1}{1 - \tilde{\beta}^2 \frac{4 + \log n}{\log N}} \leq \frac{\varepsilon'_N}{1 - \tilde{\beta}^2 \frac{\log n}{\log N}}.$$ 

Coming back to (67), we obtain that for all $N \geq N_{0,1}^\ast (\tilde{\beta})$,

$$S_{\leq w} \leq \frac{1}{\pi w} \frac{1 + \varepsilon'_N}{1 - \tilde{\beta}^2 \frac{\log w}{\log N}}.$$

We finally obtain from (68) and (66) that there exists $\varepsilon'_N = \varepsilon'(N, \beta) \searrow N \to \infty 0$ such that the sum in (65) is smaller than

$$(1 + \varepsilon'_N) \frac{1}{\pi w} \frac{1}{1 - \tilde{\beta}^2 \frac{\log w}{\log N}} = (1 + \varepsilon'_N) \frac{1}{\pi} F(u_m + \tilde{u}_m).$$

Repeating the same observation for $v_{m-2}, \ldots, v_1$ leads to Proposition 3.7. \qed

3.5. The induction pattern. Our next goal is to sum over $(u_r)_{r \leq m}$ that appear in (64). We will sum by induction starting from $r = m$ and going down to $r = 1$. To do so, we first need to define the notion of good and bad indices $r$. While performing the induction, encountering a bad index will add some nuisance term to the estimate. We will then show that, for typical diagrams, the bad indices are rare enough so that the nuisance can be neglected.

Let $L = L_N \in \mathbb{N} \setminus \{1, 2\}$ to be determined later. Given a diagram $I \in D(m, q)$, we say that $r \in [1, m]$ is a long jump if $r - \bar{k}_r > L + 2$, which means that the last times that the two particles $i_r, j_r$ have been involved in an intersection are not too recent. We say that $r$ is a small jump if it is not a long jump. (See Fig. 1 for a pictorial description of short (top) and long (bottom) jumps). Since small jumps reduce drastically the combinatorial choice on the new couple that intersects, the diagrams that will contribute to the moments will contain mostly long jumps. Let $K = K(I)$ denote the number of small jumps and $s_1 < \cdots < s_K$ denote the indices of small jumps. We also set $s_{K+1} = m + 1$. For all $i \leq K + 1$ such that $s_i - s_{i-1} > L + 1$, we mark the following indices $\{s_i - kL - 1 : k \in \mathbb{N}, s_i - kL - 1 > s_{i-1}\}$ as stopping indices. We then call any long jump $r$ a fresh index if $r$ is stopping or if $r + 1$ is a small jump. Note that any stopping index is a fresh index. If $m$ is a long jump we also mark it as a fresh index. The idea is that if $k$ is a fresh index and $k - 1$ is a long jump, then $k - 1, k - 2, \ldots$ avoid nuisance terms until $k - i$ is a stopping index or a small jump; we remark that since our induction will be downward from $m$, these nuisance-avoiding indices occur in the induction following a fresh index. Hence we say that an index $r$ is good if it is a long jump that is not fresh. An index $r$ is bad if it is not good. For any given diagram, one can easily determine the nature of all indices via the following procedure: (i) mark all small jumps; (ii) mark every stopping index; (iii) mark all fresh indices; (iv) all the remaining indices that have not been marked are good indices.

For all $I \in D(m, q)$, we define for all $r < m$,

$$\varphi(r) = \varphi(r, I) =$$
\[
\begin{align*}
\begin{cases}
\inf \{ r' \in [r, m], r' \text{ is fresh} \} - L & \text{if } r \text{ is not a stopping index and } r + 1 \text{ is a long jump}, \\
r & \text{otherwise}.
\end{cases}
\end{align*}
\]

We also set \( \varphi(m) = m \). Here are a few immediate observations:

**Lemma 3.8.** (i) If \( r \) is good, then \( r + 1 \) is a long jump.
(ii) If \( r \in [2, m - 1] \) is good, then \( \varphi(r - 1) = \varphi(r) \).
(iii) If \( r \in [2, m] \) is fresh, then \( \varphi(r - 1) = r - L \).
(iv) If \( r \in [1, m - 1] \) is such that \( r + 1 \) is a long jump, then \( \varphi(r) \leq r \).

**Remark 3.9.** Point (iv) ensures that \( \varphi(r) \leq r \) for all \( r \leq m \). By (ii), this implies in turn that \( \varphi(r) \leq r - 1 \) when \( r \) is good.

**Proof.** Proof of (i). Suppose that \( r \) is good. It must be that \( r < m \) since by definition \( m \) is either fresh or a small jump. Now, \( r + 1 \) must be a long jump otherwise \( r \) would be fresh.

Proof of (ii). Let \( r \in [2, m - 1] \) be a good index. We distinguish two cases. First suppose that \( r - 1 \) is not a stopping index. Then \( r - 1 \) cannot be fresh because \( r \) is not a small jump. Therefore \( \varphi(r - 1) = \inf \{ r' > r - 1, r' \text{ is fresh} \} - L \). Furthermore, by (i), we have that \( \varphi(r) = \inf \{ r' \geq r, r' \text{ is fresh} \} - L \) and thus \( \varphi(r - 1) = \varphi(r) \).

Now assume that \( r - 1 \) is stopping. Then \( \varphi(r - 1) = r - 1 \). Moreover, by definition \( r, \ldots, r + L - 1 \) are long jumps and either \( r + L - 1 \) is a stopping index or \( r + L \) is a small jump. Therefore \( r + L - 1 \) is a fresh index and \( r, \ldots, r + L - 2 \) are good, so that \( \varphi(r) = \inf \{ r' \geq r, r' \text{ is fresh} \} - L \).

Proof of (iii). Let \( r \in [2, m - 1] \) be a fresh index. We first note that \( r - 1 \) cannot be a stopping index. Indeed, if \( r \) is a stopping index, then \( r - 1 \) cannot be stopping by definition; if \( r \) is not a stopping index, then as \( r \) is fresh, \( r + 1 \) must be a small jump and thus \( r - 1 \) cannot be stopping. Now, as \( r - 1 \) is not stopping and \( r \) is fresh, we obtain that \( \varphi(r - 1) = r - L \). (Note that \( r - 1 \) cannot be fresh because \( r - 1 \) is not stopping and \( r \) is a long jump.)

Proof of (iv). It is enough to show that \( r_0 := \inf \{ r' \in [r, m], r' \text{ is fresh} \} \leq r + L \). Since \( s_i = 1 \) and \( s_{K+1} = m + 1 \), we can find \( i \leq K \) such that \( r \in [s_i, s_{i+1}] \). Now first suppose that \( s_{i+1} - r \leq L + 1 \). As \( r + 1 \) is a long jump, \( s_{i+1} > r + 1 \geq s_i + 1 \) and so \( s_{i+1} - 1 \) is a long jump because it is in \( (s_i, s_{i+1}) \). Hence \( s_{i+1} - 1 \) is fresh (note that this remains true when \( s_{i+1} = m + 1 \)) and we obtain that \( r_0 \leq s_{i+1} - 1 \leq r + L \). Otherwise if \( s_{i+1} - r > L + 1 \), we can let \( r_* = s_{i+1} - k_0L - 1 \), \( k_0 \in \mathbb{N} \), be the stopping index of \( (s_i, s_{i+1}) \) that satisfies \( r_* - L < r \leq r_* \) (this is the smallest stopping time larger than \( r \)). By definition \( r_* \) is fresh, therefore \( r_0 \leq r_* \leq r + L \).

For all \( v \in [1, T] \), we further let

\[
f(v) = \frac{\log N}{\hat{\beta}^2} \log \left( \frac{1 - \hat{\beta}^2 \frac{\log v}{\log N}}{1 - \hat{\beta}^2 \frac{\log T}{\log N}} \right).
\]

Note that \( f \) is non-increasing. Recall (51) and the definition of \( F \) in (62).
Lemma 3.10. For all \( m \geq 2 \), \( I \in \mathcal{D}(m, q) \), \( k \in [1, m - 1] \) and \( \sum_{i=1}^{m-k} u_i \leq T \) with \( u_i \in [1, T] \),

\[
\sum_{u_i \in [1,T],m-k<i\leq m} \prod_{r=m-k+1}^{m} F(u_r + \tilde{u}_r) I_{\sum_{i=1}^{m-k} u_i \leq T} = \sum_{i=0}^{k} \frac{c_i^k}{(k-i)!} \left( \frac{1}{1 - \beta^2} \right)^i f \left( \sum_{i=\varphi(m-k)}^{m-k} u_i \right)^{k-i},
\]

with \( c_0^1 = 1 \), \( c_1^1 = 2 \), \( c_{i+1}^k = c_i^k + 2 \gamma_k^m \sum_{j=0}^{i-1} c_j^k \) for \( i \leq k + 1 \) with \( \gamma_k^m = 1_{m-k \text{ is bad}} \) and \( c_i^k = 0 \) for \( i > k \).

Remark 3.11. The \( c_i^k \)'s depend on \( m \) and \( I \in \mathcal{D}(m, q) \).

Before turning to the proof, we need another result that plays a key role in the proof of Lemma 3.10 and which clarifies the role of good indices.

Lemma 3.12. For all \( k \in [0, m - 2] \), \( j \leq k \) and \( \sum_{i=1}^{m-k-1} u_i \leq T \) with \( u_i \in [1, T] \),

\[
S_k f^j(u_1, \ldots, u_{m-k-1}) := \sum_{u_{m-k-1}}^{T} F(u_{m-k} + \tilde{u}_{m-k}) f \left( \sum_{i=\varphi(m-k)}^{m-k-1} u_i \right)^j I_{\sum_{i=1}^{m-k-1} u_i \leq T} \leq \frac{1}{j+1} f \left( \sum_{i=\varphi(m-k)}^{m-k-1} u_i \right)^{j+1} + \gamma_k^m \sum_{l=1}^{j+1} \frac{j!}{(j+1-l)!(1-\beta^2)^j} f \left( \sum_{i=\varphi(m-k-l)}^{m-k-1} u_i \right)^{j+1-l}.
\]

Remark 3.13. When \( m - k \) is good, the right-hand side of (70) is reduced to a single term. When \( m - k \) is bad, a nuisance term appears.

Proof. We divide the proof into three cases.

Case 1: \( m - k \) is good. Necessarily \( m - k + 1 \) is a long jump by Lemma 3.8-(i), so if we let \( r_{\text{fresh}} = \inf\{r' > m - k, r' \text{ is fresh}\} \), then \( \varphi(m - k) = r_{\text{fresh}} - L \). By Remark 3.9 and since \( r_{\text{fresh}} > m - k \), we have

\[
(m-k) - L < \varphi(m-k) \leq (m-k) - 1.
\]

Define

\[
v := \sum_{i=\varphi(m-k)}^{m-k-1} u_i \in [1, T].
\]

As \( m - k \) is a long jump, we first observe that

\[
\tilde{u}_{m-k} \geq u_{m-k-1} + \cdots + u_{m-k-L-1} \geq v.
\]
Since $F$ and $f$ are non-increasing, see (62), this implies that

$$S_k f^j \leq \sum_{u_{m-k}=1}^T F(u_{m-k} + v) f(u_{m-k} + v)^j 1_{u_{m-k}+v \leq T}$$

$$\leq \int_v^T \frac{1}{u} \frac{1}{1 - \beta^2 \log(u) \log N} f(u)^j du = \left[ - \frac{1}{j+1} f(x)^{j+1} \right]_v^T = \frac{1}{j+1} f(v)^{j+1},$$

where in the comparison to the integral, we have used that $F(x) f(x)^j$ decreases in $x \in [1, \ldots, T]$. Given that $\varphi(m-k-1) = \varphi(m-k)$ by Lemma 3.8-(ii), we have the identity $v = \sum_{i=\varphi(m-k-1)}^{m-k-1} u_i$. Hence (70) holds.

**Case 2:** $m - k$ is fresh. By Lemma 3.8-(iii), we have $\varphi(m-k-1) = m - k - L$. Note that in contrast with Case 1, we have from the definition that $\varphi(m-k) = m - k$ and therefore we cannot follow the same argument as for Case 1. Instead, we now define $v$ summing from $\varphi(m-k-1)$ and not $\varphi(m-k)$, i.e.

$$v := \sum_{i=\varphi(m-k-1)}^{m-k-1} u_i \in [1, T].$$

We then decompose

$$S_k f^j = S_k^{\leq v} f^j + S_k^{> v} f^j,$$

where $S_k^{\leq v} f^j$ is the restriction of the sum in $S_k f^j$ to $u_{m-k} \in [1, v]$. Given that $m - k$ is a long jump, the bounds (71) hold again. Hence, using that $F$ and $f$ are non-increasing, we find that

$$S_k^{\leq v} f^j \leq \sum_{u_{m-k}=1}^v \frac{1}{u_{m-k} + v} \frac{1}{1 - \beta^2 \log(u_{m-k}+v)} f(u_{m-k})^j 1_{u_{m-k}+v \leq T}$$

$$\leq \frac{1}{v} \frac{1}{1 - \beta^2} \sum_{u_{m-k}=1}^v f(u_{m-k})^j$$

(73)

$$\leq \frac{1}{v} \frac{1}{1 - \beta^2} \left( f(1)^j + \int_1^v f(x)^j dx \right),$$

by comparison to a integral (using that $f$ is non-increasing). By integrating by part and using that $f'(x) = \frac{-1}{x \log(x)}$, we see that for all $j \geq 1$,

$$f(1)^j + \int_1^v f(x)^j dx = vf(v)^j - j \int_1^v x f'(x) f(x)^{j-1} dx$$

$$\leq vf(v)^j + \frac{j}{1 - \beta^2} \int_1^v f(x)^{j-1} dx.$$

If we iterate the integration by parts, we obtain that

$$f(1)^j + \int_1^v f(x)^j dx \leq v \sum_{i=0}^j \frac{j!}{(j-i)!} \left( \frac{1}{1 - \beta^2} \right)^i f(v)^{j-i},$$
and so

\[ S_k^{\leq v} f^j \leq \sum_{i=0}^{j} \frac{j!}{(j-i)!} \left( \frac{1}{1 - \tilde{\beta}^2} \right)^{i+1} f(v)^{j-i}. \]  

(74)

On the other hand, we have

\[ S_k^{> v} f^j \leq \sum_{u_{m-k}=1}^{T} \frac{1}{u_{m-k}} \frac{1}{1 - \tilde{\beta}^2 \log(u_{m-k})} f(u_{m-k})^j \]

\[ \leq \int_v^T \frac{1}{x} \frac{1}{1 - \tilde{\beta}^2 \log(x)} f(x)^j \, dx = \left[ - \frac{1}{j+1} f(x)^{j+1} \right]_v^T \leq \frac{1}{j+1} f(v)^{j+1}. \]

Combining the two previous estimates yields (70).

**Case 3:** \( m - k \) is a small jump. We have that \( f \left( \sum_{i=\varphi(m-k)} u_i \right) \leq f(u_{m-k}) \) as \( f \) is non-increasing. Moreover, \( \tilde{u}_{m-k} \geq \frac{u_{m-k}}{2} \) always holds. Hence, if we use the same decomposition as in (72) with \( v = u_{m-k-1} \), we find using that \( F \) is non-increasing that

\[ S_k^{\leq v} f^j \leq \sum_{u_{m-k}=1}^{v} \frac{1}{u_{m-k} + v/2} \frac{1}{1 - \tilde{\beta}^2 \log(u_{m-k} + v/2)} f(u_{m-k})^j \]

\[ \leq \frac{2}{v} \frac{1}{1 - \tilde{\beta}^2} \left( f(1)^j + \int_1^v f(x)^j \, dx \right) \]

\[ \leq 2 \sum_{i=0}^{j} \frac{j!}{(j-i)!} \left( \frac{1}{1 - \tilde{\beta}^2} \right)^{i+1} f(v)^{j-i}, \]

where we have used the integration by parts from Case 2 and that \( f \) is non-increasing in the comparison to the integral. Furthermore, we have \( S_k^{> v} \leq \frac{1}{j+1} f(v)^{j+1} \) as in Case 2. Finally, since \( m - k \) is not a long jump we have \( \varphi(m-k-1) = m-k-1 \) and therefore (70) follows.

**Proof of Lemma 3.10.** We prove the lemma by induction on \( k \). The case \( k = 1 \) follows from Lemma 3.12 with \( j = k = 0 \).

Assume now that (69) holds for some \( k \in [1, m-2] \). Then by (70) we obtain that the left hand side of (69) for the index \( k+1 \) is smaller than the sum of all the entries of the following matrix, where we have set \( \mu = 1 - \tilde{\beta}^2 \) and \( f = f(v) \) with \( v = \sum_{i=\varphi(m-k-1)} u_i \):

\[
\begin{pmatrix}
\frac{c_k}{(k+1)!} f^{k+1} & \frac{2\gamma_k c_k}{k^{(k+1)!}} f^k & \frac{2\gamma_k c_k}{(k+1)!} f^{k-1} & \ldots & \frac{2\gamma_k c_k}{\mu^{k!}} f^0 & \frac{2\gamma_k c_k}{\mu^{k+1}} f^0 \\
0 & \frac{c_k}{k^{(k+1)!}} f^k & \frac{2\gamma_k c_k}{(k+1)!} f^{k-1} & \ldots & \frac{2\gamma_k c_k}{\mu^{k!}} f^0 & \frac{2\gamma_k c_k}{\mu^{k+1}} f^0 \\
0 & 0 & \frac{c_k}{(k+1)!} f^{k-1} & \ldots & \frac{2\gamma_k c_k}{\mu^{k!}} f^0 & \frac{2\gamma_k c_k}{\mu^{k+1}} f^0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \frac{c_k}{\mu^{k+1}} f^0 & \frac{2\gamma_k c_k}{\mu^{k+1}} f^0 \\
\end{pmatrix},
\]  

(75)

and collecting the coefficients of the various terms \( f^r, r \in [0, k+1] \), which amounts to summing over the columns of (75), gives (69) for \( k+1 \).
Recall (64). Lemma 3.10 yields the following.

**Proposition 3.14.** There exists $C = C(\hat{\beta}) > 0$ and $\varepsilon_N = \varepsilon(N, \hat{\beta}) \to 0$ as $N \to \infty$, such that

$$\tilde{A}_{m,N,I} \leq C(1 + |\varepsilon_N|)^m \sum_{i=0}^{m} \frac{c_i^m}{(m - i)!} \times \left( \frac{\log N}{\hat{\beta}^2} \right)^{m-i} \frac{1}{(1 - \hat{\beta}^2)^i},$$  

(76)

where $\lambda_{T,N}$ is defined in (15).

**Proof.** By Proposition 3.7 and Lemma 3.10 applied to $k = m - 1$, we have:

$$\tilde{A}_{m,N} \leq \frac{1}{1 - \hat{\beta}^2} (1 + \varepsilon_N)^m \sum_{u_1=1}^{T} \frac{C}{u_1} \sum_{i=0}^{m-1} \frac{c_i^{m-1}}{(m-1-i)!} f(u_1)^{m-1-i} \frac{1}{(1 - \hat{\beta}^2)^i},$$

where in the second inequality, we have used that

$$\sum_{u_1=1}^{T} \frac{1}{u_1} f(u_1)^{m-i} \leq \sum_{u_1=1}^{T} F(u_1) f(u_1)^{m-i} \leq f(1)^{m-i} \int_{1}^{T} F(u_1) f(u_1)^{m-i} du_1$$

$$= f(1)^{m-i} + \frac{f(1)^{m-i+1}}{m-i+1},$$

using that $F(u) f(u)$ is non-increasing in the comparison to the integral. This yields (76) since

$$c_i^{m-1} + c_{i-1}^{m-1} \mathbf{1}_{i \geq 1} \leq c_i^{m-1} + 2\gamma_{m-1} \sum_{j=0}^{i-1} c_j^{m-1} = c_i^{m},$$

as $\gamma_{m-1} = 1$ since the index 1 is always bad (it is a small jump).

**Lemma 3.15.** For all $I \in D(m, q)$, for all $k \leq m$:

$$\forall i \leq k, \quad c_i^k \leq 3^i \prod_{r=1}^{k-1} (1 + \gamma_r^m).$$  

(77)

**Proof.** We prove it by induction on $k$. The estimate holds for $k = 1$ since $c_0^1 = 1$ and $c_1^1 = 2$. Suppose that (77) holds for some $k \leq m - 1$. Then, for all $i \leq k + 1$,

$$c_i^{k+1} = c_i^k + 2\gamma_k^m \sum_{j=0}^{i-1} c_j^k \leq \prod_{r=1}^{k-1} (1 + \gamma_r^m) \left( 3^i + 2\gamma_k^m \sum_{j=0}^{i-1} 3^i \right) \leq 3^i \prod_{r=1}^{k} (1 + \gamma_r^m).$$

\[\square\]
3.6. **Proof of Theorem 2.1.** By Proposition 3.1, it is enough to show that

\[
\sup_{X \in (\mathbb{Z}^2)^q} \Psi_{N,q}(X) \leq e^{\lambda_{T,N}^2 \left( \frac{1}{2} \right) + c q^{3/2 + o(1)} q^2},
\]

for some \( c = c(\beta) \). Using Proposition 3.7, we have

\[
\sup_{X \in (\mathbb{Z}^2)^q} \Psi_{N,q}(X) \leq \sum_{m=0}^{\infty} \sigma_N^{2m} \left( \frac{1}{\pi} \right)^{m-1} \sum_{I \in D(m,q)} \tilde{A}_{m,N,I},
\]

where (76) gives an upper bound on the \( \tilde{A}_{m,N,I} \). Observe that by (77), we have

\[
\epsilon_i^m \leq 3^i 2^{\sum_{l=1}^{m-1} I_l \text{ is bad}} \leq 3^i 2^{\sum_{l=1}^{m} I_l \text{ is bad}} \leq 3^i 2^{2 n(I) + m/L + 1},
\]

where \( n(I) \) is the number of small jumps in \( I \). Indeed, an index \( r \) is bad if it is a small jump or a fresh index. The number of small jumps is \( n(I) \). A fresh index is either equal to \( m \), or a stopping index, or an index adjacent to a small jump, so the number of fresh indices is at most \( 1 + n(I) \) plus the number of stopping indices. Since stopping indices are spaced at least \( L \) steps apart, there are at most \( m/L \) stopping indices. Hence there are at most \( 2 n(I) + m/L + 1 \) bad indices. For a fixed \( n \leq m \), let us compute the number of diagrams in \( D(m, q) \) such that \( n(I) = n \). One has first to choose the location of the small jumps, which gives \( \binom{m}{n} \) possibilities. Now if \( m \) is a small jump \( (m - \bar{k}_m \leq L + 2) \), it means that at least one of the two particles \( \{i_m, j_m\} \) is the same as one of the particles \( \{i_{m-L-2}, j_{m-L-2}, \ldots, i_{m-1}, j_{m-1}\} \), therefore there are at most \( 2(L+2)q \) choices for the couple \( \{i_m, j_m\} \). On the other hand, if \( \{i_m, j_m\} \) is a long jump, there are at most \( \left( \frac{q}{2} \right)^m \) possibilities. By repeating the argument, we finally find that the number of diagrams in \( D(m, q) \) such that \( n(I) = n \) is less than \( \binom{m}{n} (2L + 2)q^n \left( \frac{q}{2} \right)^{m-n} \).

Hence, by (79), Proposition 3.14, Lemma 3.15 and (38), there exists \( \varepsilon_N \searrow 0 \) such that

\[
\sup_{X \in (\mathbb{Z}^2)^q} \Psi_{N,q}(X) \leq C(\beta) \times \\
\sum_{m=0}^{\infty} (1 + \varepsilon_N)^m \sum_{n=0}^{m} \binom{m}{n} (2L + 2q)^n \left( \frac{q}{2} \right)^{m-n} \sum_{i=0}^{m} 3^i 2^{2 n+m/L+1} \left( \frac{\beta^2}{\log N} \right)^i \lambda_{T,N}^{2(m-i)} \left( 1 - \frac{\beta^2}{\log N} \right)^i.
\]

The sum over \( n \) gives a factor of \( (8(L+2)q + \left( \frac{q}{2} \right))^m \). Exchanging the sum in \( i \) and \( m \) entails

\[
\sup_{X \in (\mathbb{Z}^2)^q} \Psi_{N,q}(X) \leq C \sum_{i=0}^{\infty} (1 + \varepsilon_N)^i 3^i 2^{i/L} \left( \frac{\beta^2}{\log N} \right)^i \left( 8(L+2)q + \left( \frac{q}{2} \right) \right)^i \frac{1}{\left( 1 - \frac{\beta^2}{\log N} \right)^i} \\
\times \sum_{m=i}^{\infty} (1 + \varepsilon_N)^{m-i} \times \left( 8(L+2)q + \left( \frac{q}{2} \right) \right)^{m-i} 2^{(m-i)/L} \left( \frac{\lambda_{T,N}^{2(m-i)}}{(m-i)!} \right)^{m-i}.
\]

So if we assume that

\[
r = 3(1 + \varepsilon_N) \frac{2^{1/L}}{(1 - \beta^2)} \left( \frac{\beta^2}{\log N} \right) \left( 8(L+2)q + \left( \frac{q}{2} \right) \right) < 1,
\]

(80)
we obtain the bound:

\[
\sup_{X \in (\mathbb{Z}^2)^q} \Psi_{N,q}(X) \leq \frac{C}{1 - r} e^{(1 + \varepsilon_N)(8(L + 2)q + (q/2)^2)} 2^{1/L} \lambda T_{T,N}^2.
\] (81)

Taking \( L = \lceil \sqrt{q} \rceil \) then gives, together with (7), the desired conclusion (78), since \( 2^{1/L} \leq 1 + 2/L \) for \( L \geq 1 \).

\[ \square \]

4. Discussion and Concluding Remarks

We collect in this section several comments concerning the results of this paper.

1. Our results allow one already to obtain some estimates on the maximum of \( Y_N(x) := \log W_N(\beta N, x) \) over subsets \( D \subset [0, 1]^2 \). Specifically, let \( \gamma > 0 \) be given and define \( Y^*_N = \sup_{x \in D} Y_N(x) \). By Chebyshev’s inequality we have that

\[
P\left(Y^*_N \geq \delta \sqrt{\log N}\right) \leq 2N^{2\gamma} P\left(Y_N(0) \geq \delta \sqrt{\log N}\right) \leq 2N^{2\gamma} \mathbb{E}[W^q_N] e^{-q \delta \sqrt{\log N}} \leq N^{2\gamma} e^{-q \delta \sqrt{\log N} + o(1)},
\]

where we used (8) in the last inequality. The optimal \( q \) (disregarding the constraint in (7)) is \( q / \sqrt{\log N} = \delta / \lambda^2 \), and for that value the right side of the last display decays to 0 if \( \delta^2 > 4\gamma \lambda^2 \). The condition on \( q \) in (7) then gives the constraint that \( \gamma < \frac{1}{6} \lambda^2 \frac{1 - \beta^2}{\beta^2} \), which for \( \hat{\beta} \) small reduces to \( \gamma < 1/6 \). We however do not expect that this Chebyshev based bound is tight, even if one uses the optimal \( q \) disregarding (7); this is due to inherent inhomogeneity in time of the model. We hope to return to this point in future work.

2. In view of the last sentence in Remark 1.3, it is of interest to obtain a lower bound on \( \mathbb{E}[W^q_N] \) that matches the upper bound, that is, \( \mathbb{E}[W^q_N] \geq e^{(q/2)^2 \lambda^2 (1 - \varepsilon_N)} \). This can be found in [16].

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Appendix A: Proof of (19)

First note that \( p_{2n}^* \leq p_{2n}(0) \) since, by the Cauchy-Schwarz inequality,

\[
p_{2n}(x) = \sum_y p_n(x - y)p_n(y) \leq \sum_y p_n(y)^2 = p_{2n}(0).
\]

Let \( p_{2n}^{(d)} \) be the return probability of \( d \)-dimensional SRW to 0. A direct computation gives that \( p_{2n}^{(2)} = (p_{2n}^{(1)})^2 \) (see e.g. [20, Page 184]). We will show that \( a_n = \sqrt{2n}p_{2n}^{(1)} \) is increasing. We have,

\[
a_n = \sqrt{2n}2^{-2n} \binom{2n}{n}.
\]

Hence,

\[
\frac{a_{n+1}}{a_n} = \frac{1}{4} \sqrt{\frac{n+1}{n}} \left( \frac{2n+2}{n+1} \right)^2 = \sqrt{\frac{1}{n(n+1)(n+(n+1)/2}}.
\]

Since \( (a + b)/2 \geq \sqrt{ab} \), we conclude (using \( a = n \) and \( b = n + 1 \)) that \( a_{n+1}/a_n \geq 1 \).

Let \( p_{2n+1}^{(1)} \) be the probability of the 1-dimensional SRW to come back to 1 in \( 2n+1 \) steps. By the random walk representation [20, Remark in Pg. 185], we have that \( p_{2n+1}^* \leq (p_{2n+1}^{(1)})^2 \). A similar line of argument to the above shows that \( b_n = \sqrt{2n+1}p_{2n+1}^{(1)} \) is increasing in \( n \). Indeed,

\[
\frac{b_{n+1}}{b_n} = \frac{1}{4} \sqrt{\frac{2n+3}{2n+1}} \left( \frac{2n+3}{2n+2} \right)^2 \frac{\sqrt{2n+1}(n+1)}{\sqrt{2n+1}(n+2)}
\]

where the first fraction is bigger than 1 by the formula \( (a + b)/2 \geq \sqrt{ab} \), as well as the second fraction by expanding the products.

Now, we know from the local limit theorem that \( a_n \) and \( b_n \) converge to \( 2/\sqrt{2\pi} \), thus they are always smaller than this limit. This leads to (19).

Appendix B: Improved Estimates on \( U_N \)

When \( n \) is taken large enough, the estimate (59) can be improved as follows.

Proposition B.1. There exists \( \varepsilon_n = \varepsilon(n, \bar{\beta}) \to 0 \) such that as \( n \to \infty \), uniformly for \( N \geq n \),

\[
U_N(n) = (1 + \varepsilon_n) \frac{\bar{\beta}^2}{\left(1 - \bar{\beta}^2 \frac{\log n}{\log N}\right)^2} \frac{1}{n \log N}.
\]
Proof. Since \((S_1^n - S_2^n) \overset{(d)}{=} (S_2^n)\), we can write
\[
U_N(n + 1) = \sigma_N^2 E_0 \left[ e^{\beta_n^2 \sum_{i=1}^n 1_{S_{2i} = 0}} \right]. \tag{83}
\]
Consider \(\ell = \ell_n = n^{1-\varepsilon_n}\) with \(\varepsilon_n = \frac{1}{\log \log n}\), so that \(\ell_n = o(n)\) and \(\varepsilon_n \to 0\).

First step: As \(n \to \infty\) with \(n \leq N\),
\[
E_0 \left[ e^{\beta_n^2 \sum_{i=1}^n 1_{S_{2i} = 0}} \right] \sim E_0 \left[ e^{\beta_n^2 (\sum_{i=1}^{\ell_n} 1_{S_{2i} = 0} + \sum_{i=n-\ell_n}^n 1_{S_{2i} = 0})} \right] \tag{84}
\]
We compute the norm of the difference which, using that \(|e^{-x} - 1| \leq |x|\) for \(x \geq 0\), is less than
\[
E_0 \left[ e^{\beta_n^2 \sum_{i=1}^n 1_{S_{2i} = 0}} \times e^{\beta_n^2 \sum_{j=\ell}^{n-\ell} 1_{S_{2j} = 0}} \right] = \beta_n^2 \sum_{j=\ell}^{n-\ell} E_0 \left[ e^{\beta_n^2 \sum_{i=1}^j 1_{S_{2i} = 0}} \right] E_0 \left[ e^{\beta_n^2 \sum_{i=1}^{n-j} 1_{S_{2i} = 0}} \right],
\]
where we have used Markov’s property in the second line. By (83) and (59), the last sum is smaller than
\[
C \beta_n^2 \sum_{j=\ell}^{n-\ell} \frac{1}{j} \leq 2C \beta_n^2 \sum_{j=\ell}^{n/2} \frac{1}{j} \leq \frac{1}{n} C' \beta_n^2 \log \left( \frac{n}{\ell_n} \right) \leq \frac{1}{n} C' \varepsilon_n = o(n^{-1}).
\]
Since the left hand side of (84) is bigger than \(cn^{-1}\) for some constant \(c > 0\), this shows (84).

Second step: As \(n \to \infty\) with \(n \leq N\),
\[
E_0 \left[ e^{\beta_n^2 (\sum_{i=1}^{\ell_n} 1_{S_{2i} = 0} + \sum_{i=n-\ell_n}^n 1_{S_{2i} = 0})} \right] \sim E_0 \left[ e^{\beta_n^2 \sum_{i=1}^{\ell_n} 1_{S_{2i} = 0}} \right] E_0 \left[ e^{\beta_n^2 \sum_{i=n-\ell_n}^n 1_{S_{2i} = 0}} \right]. \tag{85}
\]
By Markov’s property, we can write the LHS of (85) as
\[
\sum_{x \in \mathbb{Z}^2} E_0 \left[ e^{\beta_n^2 \sum_{i=1}^{\ell_n} 1_{S_{2i} = 0} + \sum_{i=n-\ell_n}^n 1_{S_{2i} = 0}} \right] E_x \left[ e^{\beta_n^2 \sum_{i=n-\ell_n}^n 1_{S_{2i} = 0}} \right] = \sum_{x \in \mathbb{Z}^2} E_0 \left[ e^{\beta_n^2 \sum_{i=1}^{\ell_n} 1_{S_{2i} = 0} + \sum_{i=n-\ell_n}^n 1_{S_{2i} = 0}} \right] E_x \left[ e^{\beta_n^2 \sum_{i=n-\ell_n}^n 1_{S_{2i} = x}} \right].
\]
Therefore the difference in (85) writes \(\sum_{x \in \mathbb{Z}^2} \Delta_x\) with
\[
\Delta_x := E_0 \left[ e^{\beta_n^2 \sum_{i=1}^{\ell_n} 1_{S_{2i} = x} \left( 1_{S_{2n-\ell_n} = 0} - 1_{S_{2n-\ell_n} = x} \right)} \right].
\]
Since \(E_0 \left[ e^{\beta_n^2 \sum_{i=1}^{\ell_n} 1_{S_{2i} = 0}} \right] \leq C(\beta)\) by (54), we have
\[
\sum_{|x| > \sqrt{tn^{\varepsilon/4}}} |\Delta_x| \leq C \sum_{|x| > \sqrt{tn^{\varepsilon/4}}} E_0 \left[ e^{\beta_n^2 \sum_{i=1}^{\ell_n} 1_{S_{2i} = x}} \right].
\]
By Hölder’s inequality with $p^{-1} + q^{-1} = 1$, and $p$ small enough so that $\sqrt{p} \hat{\beta} < 1$,

$$E_0 \left[ e^{\beta N \sum_{i=1}^{\ell} 1_{S_{2i}=0} 1_{S_{2\ell}=x}} \right] \leq E_0 \left[ e^{\beta_0 N \sum_{i=1}^{\ell} 1_{S_{2i}=0}} \right]^{1/\hat{p}} p_{2\ell}(x)^{\hat{\beta}/\hat{p}} \leq C(\hat{\beta}) e^{-\frac{1}{2\hat{q}} \frac{|x|^2}{n}},$$

for $n$ large enough. Therefore,

$$\sum_{|x| > \sqrt{\ell n^{\epsilon/4}}} |\Delta_x| \leq C \sum_{|x| > \sqrt{\ell n^{\epsilon/4}}} \ell_n^{-1} e^{-\frac{1}{2\hat{q}} \frac{|x|^2}{n}}, \leq C e^{-\frac{1}{2\hat{q}} n^{\epsilon/2}} = o(n^{-1}).$$

We now estimate the sum on $\Delta_x$ for $|x| \leq \sqrt{\ell n^{\epsilon/4}}$. We start by bounding the expectation inside the definition of $\Delta_x$:

$$E_0 \left[ e^{\beta_0^{\epsilon} \sum_{i=1}^{\ell} 1_{S_{2i}=0} \left( 1_{S_{2\ell-\ell}=0} - 1_{S_{2n-\ell}=x} \right)} \right] = \sum_{y \in \mathbb{Z}^2} E_0 \left[ e^{\beta_0^{\epsilon} \sum_{i=1}^{\ell} 1_{S_{2i}=0} 1_{S_{\ell}=y}} \right] (p_{2n-2\ell}(y) - p_{2n-2\ell}(y - x)). \tag{86}$$

By the same argument as above, we can prove that the above sum restricted to $|y| \geq \sqrt{\ell n^{\epsilon/4}}$ is negligible with respect to $n^{-1}$, uniformly for $|x| \leq \sqrt{\ell n^{\epsilon/4}}$. On the other hand, by the local limit theorem we have

$$\sup_{|x| \leq \sqrt{\ell n^{\epsilon/4}}, |y| \leq \sqrt{\ell n^{\epsilon/4}}} |p_{2n-2\ell}(y) - p_{2n-2\ell}(y - x)| = o(n^{-1}),$$

since $\ell_n n^{\epsilon/2} = n^{1-\epsilon_n/2} = o(n)$. Thus, the quantity in (86) is bounded uniformly for $|x| \leq \sqrt{\ell n^{\epsilon/4}}$ by

$$E_0 \left[ e^{\beta_0^{\epsilon} \sum_{i=1}^{\ell} 1_{S_{2i}=0}} \right] \times o(n^{-1}) = o(n^{-1}).$$

This completes the proof of (85).

**Third step:** As $n \to \infty$ with $n \leq N$,

$$E_0 \left[ e^{\beta_0^{\epsilon} \sum_{i=n-\ell}^{n} 1_{S_{2i}=0} 1_{S_{2n}=0}} \right] \sim E_0 \left[ e^{\beta_0^{\epsilon} \sum_{i=1}^{\ell} 1_{S_{2i}=0}} \right] p_{2n}(0). \tag{87}$$

Equivalence (87) can be proven by following the same line of arguments as used to prove (85), hence we omit its proof.

Now, combining the three steps leads to the equivalence

$$E_0 \left[ e^{\beta_0^{\epsilon} \sum_{i=1}^{n} 1_{S_{2i}=0} 1_{S_{2n}=0}} \right] \sim E_0 \left[ e^{\beta_0^{\epsilon} \sum_{i=1}^{\ell} 1_{S_{2i}=0}} \right]^2 p_{2n}(0).$$

By (55), as $\log \ell \sim \log n$, we have

$$E_0 \left[ e^{\beta_0^{\epsilon} \sum_{i=1}^{\ell} 1_{S_{2i}=0}} \right] \sim \frac{1}{1 - \beta_0^{\epsilon} \log n \log N},$$

and so (82) follows from (83) and the last two displays.

\relax
Appendix C: Khas’minskii’s Lemma for Discrete Markov Chains

The following theorem is another discrete analogue of Khas’minskii’s lemma, compare with Lemma 2.2.

**Theorem C.1.** Let \((Y_n)_n\) be any markov chain on a discrete state-space \(E\) and let \(f : E \to \mathbb{R}_+\). Then for all \(k \in \mathbb{N}\), if

\[
\eta_0 := \sup_{x \in E} \mathbb{E}_x \left[ \sum_{n=1}^{k} (e^{f(Y_n)} - 1) \right] < 1, \tag{88}
\]

one has

\[
\sup_{x \in E} \mathbb{E}_x \left[ e^{\sum_{n=1}^{k} f(Y_n)} \right] \leq \frac{1}{1 - \eta_0}. \tag{89}
\]

**Proof.** Denote by \(D_n = e^{f(Y_n)} - 1\). We have,

\[
\begin{align*}
\mathbb{E}_x \left[ e^{\sum_{n=1}^{N} f(Y_n)} \right] &= \mathbb{E}_x \left[ \prod_{n=1}^{N} (1 + D_n) \right] = \sum_{p=0}^{\infty} \sum_{1 \leq n_1 < \cdots < n_p \leq k} \mathbb{E}_x \left[ \prod_{i=1}^{p-1} D_{n_i} \sum_{n=1}^{k-n_p} D_n \right] \\
&\leq \sum_{p=0}^{\infty} \eta_0^p \sum_{1 \leq n_1 < \cdots < n_p \leq k} \mathbb{E}_x \left[ \prod_{i=1}^{p-1} D_{n_i} \right] \leq \cdots \leq \sum_{p=0}^{\infty} \eta_0^p = \frac{1}{1 - \eta_0}.
\end{align*}
\]

\(\square\)

**Corollary C.2.** Let \((Y_n)_n\) be any markov chain on a discrete state-space \(E\) and let \(f : E \to [0, 1]\). Then for all \(k \in \mathbb{N}\), if

\[
\eta_1 := \sup_{x \in E} \mathbb{E}_x \left[ \sum_{n=1}^{k} f(Y_n) \right] < 1, \tag{90}
\]

one has

\[
\sup_{x \in E} \mathbb{E}_x \left[ e^{\sum_{n=1}^{k} f(Y_n)} \right] \leq \frac{1}{1 - \eta_1}. \tag{91}
\]

**Proof.** Simply observe that \(e^{f(x)} - 1 \leq e^c f(x)\) and apply Theorem C.1. \(\square\)
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