Direct Hamiltonization
for Nambu Systems

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Abstract
The direct hamiltonization procedure applied to Nambu mechanical systems proves that the Nambu mechanics is an usual mechanics described by only one Hamiltonian. Thus a particular case of Hamiltonian mechanics. It is also proved that any dynamical system described by the equation $\dot{\mathbf{r}} = \mathbf{A}(\mathbf{r})$ is a Nambu system.
1 Introduction

The direct Hamiltonization procedure developed in a previous paper [1] do not depend of the number of equations of motion that describes the mechanical system and so it can be applied to systems with an even number of motion equations. The Nambu mechanics [2, 3] leads with these systems and so this procedure can be applied to Nambu systems.

In section 2 it will be set two theorems. The first one proves that the Nambu Mechanics is an usual mechanics described by only one Hamiltonian and so it cannot be a generalization of the Hamiltonian mechanics. The second one sets that any dynamical system described by the motion equation \( \dot{\vec{r}} = \vec{A}(\vec{r}) \) is a Nambu system.

Section 3 gives some examples.

2 Direct Hamiltonization Procedure Applied to Nambu Systems

The equations of motion in Nambu mechanics [2] for the tridimensional problem are

\[
\dot{\vec{r}} = \vec{\nabla} h \times \vec{\nabla} g,
\]

(1)

where \( \vec{r} = (x_1, x_2, x_3) \) is the position vector and \( h \ e \ g \) constants of motion.

Nambu defined as Hamiltonians of the system the constants of motion \( h \ e \ g \) [2].

The direct Hamiltonization procedure applied to Nambu systems results in a description of this system by only one Hamiltonian, thus resulting in an usual mechanics.

Theorem 1:

The Nambu mechanics is an usual mechanics meanwhile singular.

Proof:

Let the Hamiltonian assume the following shape:

\[
H = \vec{p} \cdot (\vec{\nabla} h \times \vec{\nabla} g) + V(\vec{r})
\]

(2)
where \( \vec{p} = (p_1, p_2, p_3) \) is the momentum of the system and \( V \) an arbitrary function of space coordinates. Then from the first set of canonical equations of motion (Hamilton equations):

\[
\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} = \vec{\nabla} h \times \vec{\nabla} g, \tag{3}
\]

where the following notation is used

\[
\frac{\partial}{\partial f} = \left( \frac{\partial}{\partial f_1}, \frac{\partial}{\partial f_2}, \frac{\partial}{\partial f_3} \right).
\]

Then the Eq. (3) recovers motion equations given by Eq. (1).

To complete the direct Hamiltonization procedure it must be defined the momentum \( \vec{p} \) that is obtained from the second set of Hamilton equations as:

\[
\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} = \vec{p} \cdot \left[ \frac{\partial}{\partial \vec{r}} \left( \vec{\nabla} h \times \vec{\nabla} g \right) \right] + \frac{\partial V}{\partial \vec{r}}. \tag{4}
\]

Therefore the Hamiltonian function \( H \), given by Eq. (2), provides an Hamiltonian description to Nambu mechanics. This Hamiltonian function is a singular one since it is linear in the momentum \( \vec{p} \).

This theorem is founded in the direct Hamiltonization procedure developed in a previous paper [1].

The Hamiltonian in the direct Hamiltonization procedure [1] is given by

\[
H = A_i(q, t) \ p_i \quad (i = 1, 2, 3), \tag{2'}
\]

where the functions \( A_i \) must satisfy the Nambu motion equations.

The \( A_i \)'s are obtained directly from Eq. (1) and the Hamilton equations

\[
\dot{x}_i = \frac{\partial H}{\partial p_i} = \left( \vec{\nabla} h \times \vec{\nabla} g \right)_i = A_i,
\]

or

\[
\vec{\dot{r}} = \vec{A} = \vec{\nabla} h \times \vec{\nabla} g. \tag{3'}
\]

\(^1\)Repeated indexes means sum.
Then the Hamiltonian for the Nambu systems is obtained from Eq. (3’) and Eq. (2’)

\[ H = \left( \nabla h \times \nabla g \right) \cdot \vec{p}. \]

To this Hamiltonian can be added an arbitrary function \( V(\vec{r}) \) without changing the direct Hamiltonization procedure. This addition only modifies the definition of the momentum \( \vec{p} \) as stated in the direct Hamiltonization procedure [1]. Therefore the Hamiltonian for the Nambu systems is given by Eq. (2).

**Theorem 2:**

Any dynamical system described by the equations of motion

\[ \dot{\vec{r}} = \vec{A}(\vec{r}) \] (5)

is a Nambu system. Or, equivalently, it is always possible find functions \( h \), \( g \) such that

\[ \vec{A} = \nabla h \times \nabla g, \] (6)

where \( h \) and \( g \) are constants of motion of the system.

**Proof:**

If the functions \( h \) and \( g \) exists, then

\[ \vec{A} \cdot \nabla h = \left( \nabla h \times \nabla g \right) \cdot \nabla h = 0; \] (7)

\[ \vec{A} \cdot \nabla g = \left( \nabla h \times \nabla g \right) \cdot \nabla g = 0. \] (8)

The functions \( h \) and \( g \) must be solutions of this system of partial differential equations, or

\[ A_1 \frac{\partial h}{\partial x_1} + A_2 \frac{\partial h}{\partial x_2} + A_3 \frac{\partial h}{\partial x_3} = 0; \]

\[ A_1 \frac{\partial g}{\partial x_1} + A_2 \frac{\partial g}{\partial x_2} + A_3 \frac{\partial g}{\partial x_3} = 0, \]

whose auxiliar system is

\[ \frac{dx_1}{A_1} = \frac{dx_2}{A_2} = \frac{dx_3}{A_3} = \frac{dh}{0} = \frac{dg}{0}, \]
with the intermediary integrals:

\[ u_1 = C_1; \]
\[ u_2 = C_2; \]
\[ h = C_3; \]
\[ g = C_4. \]

Hence the general solution of the system of partial differential equations is:

\[ h = F_1(u_1, u_2); \]
\[ g = F_2(u_1, u_2), \] (9)

where \( u_1 \) and \( u_2 \) are constants of motion of the mechanical system described by (5), as proved below. From (5)

\[ \frac{d u_m}{dt} = \frac{\partial u_m}{\partial x_i} \frac{\partial x_i}{\partial t} = A_i \frac{\partial u_m}{\partial x_i}, \]

with \( m = 1, 2 \) and \( i = 1, 2, 3 \). As \( u_1 \) and \( u_2 \) are intermediary integrals of (7) and (8) then

\[ \frac{d u_m}{dt} = 0. \]

From the Eqs. (9) and Eq. (6)

\[ \vec{A} = \vec{\nabla} h \times \vec{\nabla} g = \vec{\nabla} F_1 \times \vec{\nabla} F_2. \] (6')

As

\[ \vec{\nabla} F_m = \frac{\partial F_m}{\partial x_i} \vec{e}_i = \frac{\partial F_m}{\partial u_n} \frac{\partial u_n}{\partial x_i} \vec{e}_i = \frac{\partial F_m}{\partial u_n} \vec{\nabla} u_n, \]

then

\[ \vec{A} = \left( \frac{\partial F_1}{\partial u_m} \vec{\nabla} u_m \right) \times \left( \frac{\partial F_2}{\partial u_n} \vec{\nabla} u_n \right) = \]
\[ = \left[ \frac{\partial F_1}{\partial u_1} \frac{\partial F_2}{\partial u_2} - \frac{\partial F_1}{\partial u_2} \frac{\partial F_2}{\partial u_1} \right] \left( \vec{\nabla} u_1 \times \vec{\nabla} u_2 \right) \]
therefore

\[ \vec{A} = [ F_1, F_2]_{u_1, u_2} \left( \vec{\nabla} u_1 \times \vec{\nabla} u_2 \right), \quad (10) \]

where

\[ [ F_1, F_2]_{u_1, u_2} = \left[ \frac{\partial F_1}{\partial u_1} \frac{\partial F_2}{\partial u_2} - \frac{\partial F_1}{\partial u_2} \frac{\partial F_2}{\partial u_1} \right]. \]

Since the most common shape of the Nambu system is:

\[ F_1 \equiv u_1; \]
\[ F_2 \equiv u_2, \]

then

\[ [ F_1, F_2]_{u_1, u_2} = [ u_1, u_2]_{u_1, u_2} = 1, \]

resulting in

\[ \vec{A} = \vec{\nabla} u_1 \times \vec{\nabla} u_2. \]

As \( u_1 \) and \( u_2 \) are constants of motion then they can be identified with \( h \) and \( g \) in Eq. (6), completing the prove of the theorem.  

The generalization to Nambu systems of greater dimension can be done easily.

3 Direct Hamiltonization Applied to a Classical Nambu System

3.1 Example of the First Theorem

As an example of direct Hamiltonization procedure applied to Nambu systems it will be considered the rigid rotator [2].

The generalized coordinates of the rigid rotator are the components of the angular momentum \( \vec{l} \).

The equations of motion can be written as:

\[ \dot{\vec{l}} = \vec{\nabla} h \times \vec{\nabla} g, \]
where $h$ is the half of the norm square of the angular momentum

$$h = \frac{1}{2} \left( l_x^2 + l_y^2 + l_z^2 \right),$$

and $g$ the kinetic energy of the system

$$g = \frac{1}{2} \left[ \frac{l_x^2}{I_x} + \frac{l_y^2}{I_y} + \frac{l_z^2}{I_z} \right],$$

$I_x$, $I_y$ and $I_z$ being the inertial moments. Hence $h$ and $g$ are motion constants.

As

$$\vec{\nabla} h = (l_x, l_y, l_z) = \vec{l}$$

and

$$\vec{\nabla} g = \left( \frac{l_x}{I_x}, \frac{l_y}{I_y}, \frac{l_z}{I_z} \right),$$

then

$$\dot{\vec{l}} = (a_{xy} l_y l_z, a_{xz} l_x l_z, a_{yz} l_y l_x),$$

where

$$a_{mn} = \frac{1}{I_m} - \frac{1}{I_n};$$

with $m, n = x, y, z$.

And the Hamiltonian for this system can be obtained from (2):

$$H = \vec{p} \cdot (\vec{\nabla} h \times \vec{\nabla} g),$$

or

$$H = a_{xy} p_x l_y l_z + a_{xz} p_y l_x l_z + a_{yz} p_z l_y l_x.$$

To this Hamiltonian it can be added an arbitrary function $V(\vec{r})$. 
3.2 Example of the Second Theorem

Consider a dynamical system described by the equations of motion:
\[
\begin{align*}
\dot{x} &= 2x(z^2 - y^2); \\
\dot{y} &= 2y(x^2 - z^2); \\
\dot{z} &= 2z(y^2 - x^2).
\end{align*}
\]

As this is a Nambu system let determine the constants of motion \( h \) and \( g \), which are solutions of the partial differential equations (7) and (8).

From Eqs. (5)
\[ A_k = \dot{x}_k \quad (k = 1, 2, 3), \]
then the auxiliary system for Eqs. (9) are
\[
\begin{align*}
\frac{dx}{2x(z^2 - y^2)} &= \frac{dy}{2y(x^2 - z^2)} &= \frac{dz}{2z(y^2 - x^2)} &= \frac{dh}{0} = \frac{dg}{0},
\end{align*}
\]
with the intermediary integrals
\[
\begin{align*}
u_1 &= x^2 + y^2 + z^2; \\
u_2 &= x y z.
\end{align*}
\]
therefore it can be chosen
\[
\begin{align*}
h &= u_2 = x y z; \\
g &= u_1 = x^2 + y^2 + z^2,
\end{align*}
\]
which are motion constants, as it was proved in theorem 2, so that the Nambu system is described by the equations of motion (1).

The development of the equations of motion of this system leads to
\[
\begin{align*}
\hat{\mathbf{r}} &= \vec{\nabla}(x y z) \times \vec{\nabla}(x^2 + y^2 + z^2) = \\
&= (y z, x z, x y) \times (2x, 2y, 2z),
\end{align*}
\]
or
\[
\hat{\mathbf{r}} = \left(2x(z^2 - y^2), 2y(x^2 - z^2), 2z(y^2 - x^2)\right),
\]
which can be identified with the initial equations of motion of the system. Therefore this is a Nambu system.
4 Final Remarks

The above theorems can be extended to Nambu system described by $N$ constant of motion (the Nambu Hamiltonians) that makes use of a greater number of Nambu Hamiltonians (or constants of motion) since the direct Hamiltonization procedure allows to determine the Hamiltonian function for any mechanical system described only by its equations of motion.

Furthermore the use of the direct hamiltonization procedure proves that the Nambu mechanics cannot be considered as a generalization of Hamiltonian mechanics as it is a particular case of this mechanics.

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