Sharp spectral bounds for the vertex-connectivity of regular graphs

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Abstract
Let $G$ be a connected $d$-regular graph and $\lambda_2(G)$ be the second largest eigenvalue of its adjacency matrix. Mohar and O (private communication) asked a challenging problem: what is the best upper bound for $\lambda_2(G)$ which guarantees that $\kappa(G) \geq t + 1$, where $1 \leq t \leq d - 1$ and $\kappa(G)$ is the vertex-connectivity of $G$, which was also mentioned by Cioabă. As a starting point, we determine a sharp bound for $\lambda_2(G)$ to guarantee $\kappa(G) \geq 2$ (i.e., the case that $t = 1$ in this problem), and characterize all families of extremal graphs.

Keywords Adjacency matrix · Second largest eigenvalue · Connectivity · Regular graphs

Mathematics Subject Classification 05C50

1 Introduction
In this paper, we focus on the eigenvalues $d > \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ of adjacency matrix $A(G)$ of $d$-regular graphs with order $n$. Proverbially, the second largest eigenvalue $\lambda_2(G)$ and the second eigenvalue $\lambda = \max\{|\lambda_2(G)|, |\lambda_n(G)|\}$ of graphs have been paid much attention. One of the best-known cases is that it could describe the expander graphs which are useful in the design and analysis of communication networks; see the papers due to Alon (1986) for more details. Moreover, it has always been a hot topic describing the connections between the second largest eigenvalue of a regular graph and other combinatorial parameters, such as the toughness (Alon 1995; Brouwer 1995; Cioabă and Gu 2016; Cioabă and Wong 2014; Liu and Chen 2010, eg.),
the spanning trees (Cioabă and Gu 2016; Cioabă and Wong 2012; Gu et al. 2016, eg.), the (perfect) matchings (Cioabă et al. 2009; Suil and Cioabă 2010, eg.), the regular factors (Lu 2012, eg.), the matching extendability (Zhang 2020), the edge-connectivity (Abiad et al. 2018; Chandran 2004; Cioabă 2010; Krivelevich and Sudakov 2006; Suil 2016; Suil and Cioabă 2010, eg.) and so on. See the survey (Cvetković and Simić 1995) and the monograph (Brouwer and Haemers 2012, Chapter 2) for more details about the second (largest) eigenvalues of graphs.

For a graph $G$, its vertex-connectivity $\kappa(G)$ is the least number of vertices whose deletion induces a non-connected graph or a single vertex. We here pay attention to the second largest eigenvalue and vertex-connectivity of regular graphs $G$. Unlike more fruitful results about the edge-connectivity $\lambda'(G)$ of a $d$-regular graph based on $\lambda_2(G)$, there are few ones concerning the vertex-connectivity $\kappa(G)$ built on $\lambda_2(G)$, the earliest two of which are respectively Krivelevich and Sudakov’s Theorem 4.1 in Krivelevich and Sudakov (2006) and Fiedler’s Theorem 4.1 (Fiedler 1973) (note, $\lambda_2(G) = d - \mu(G)$ with $\mu(G)$ being the algebraic connectivity). Recently, Abiad et al. (2018) and Suil (2020) obtained some relations between $\lambda_2(G)$ and $\kappa(G)$ for the regular multigraph (Abiad et al. 2018; Suil 2020). For the regular simple graphs with fixed degree, Cioabă and Gu (2016) provided the first upper bound for $\lambda_2(G)$ to guarantee $\kappa(G) \geq 2$; while for given order and degree, Abiad et al. (2018), Liu et al. (2019), Hong et al. (2019) and Zhang (2020) respectively obtained the upper bounds for $\lambda_2(G)$ to guarantee $\kappa(G) \geq k (k \geq 2)$.

In this paper, we will find a sharp upper bound on the second largest eigenvalues $\lambda_2(G)$ of regular graphs with fixed degree guaranteeing the vertex-connectivity $\kappa(G) \geq 2$. In other words, we investigate the challenging problem asked by Mohar and O (private communication) and alluded to briefly by Cioabă (2010); Cioabă and Gu (2016) and raised formally by Abiad et al. (2018).

**Problem 1** For a $d$-regular simple graph or multigraph $G$ and for $1 \leq t \leq d-1$, what is the best upper bound for $\lambda_2(G)$ which guarantees that $\kappa(G) \geq t+1$ or that $\lambda'(G) \geq t+1$?

Remark, for the edge-connectivity $\lambda'(G)$, that Cioabă (2010) proved the cases $t = 1, 2$, and Suil et al. (2018) settled down for all $t \geq 3$. However, it seems unlikely to solve this problem for vertex-connectivity once and for all. As a starting point, we capture this problem in the case $t = 1$, and characterize all families of extremal graphs.

To describe our results, we introduce some notations and terminology. As usual, let $C_n$ and $K_n$ be the cycle and the complete graph with order $n$, respectively. For even $n$, let $M_n$ be the perfect matching with order $n$. The sequential join $G_1 \vee \cdots \vee G_k$ of graphs $G_1, \ldots, G_k$ is the graph formed by taking one copy of each graph and adding additional edges from each vertex of $G_i$ to all vertices of $G_{i+1}$, for $1 \leq i \leq k-1$.

For $S, T \subseteq V(G)$, let $E(S, T) = \{(u, v) | u \in S, v \in T, (u, v) \in E(G)\}$ be the set of edges from $S$ to $T$. The graph $G - S$ is derived from $G$ by deleting the vertices of $S$ and the edges incident with the vertices in $S$. If $S = \{v\}$, we denote $G - S$ by $G - v$ for convenience. For two integers $d \geq 3$ and $1 \leq c \leq d-1$, we define the following set $\mathcal{G}_{d,c}$ and the graph $G_{d,c} \in \mathcal{G}_{d,c}$ (see Fig. 1 for example) relating to our main results.
Let $d$ be odd. Define

$$G_{d,c} = \begin{cases} K_2 \lor \overline{M_{d-1}} \lor K_1 \lor K_1 \lor \overline{M_{d-1}} \lor K_2 & \text{if } c = 1 \text{ or } c = d - 1; \\ M_{d+2-c} \lor \overline{C_c} \lor K_1 \lor \overline{M_{d-c}} \lor K_{c+1} & \text{if } c \in (2, d-2) \text{ is odd}; \\ K_{d+1-c} \lor \overline{M_c} \lor K_1 \lor \overline{C_{d-c}} \lor \overline{M_{c+2}} & \text{if } c \in [2, d-2] \text{ is even}, \end{cases}$$

where $C_c = C_{c_1} \cup \cdots \cup C_{c_s}$ is the union of disjoint cycles $C_{c_i}$ and $\sum_{i=1}^s c_i = c$.

- $\mathcal{G}_{d,c}$ is the set of connected $d$-regular graphs $G$ with a cut vertex (say $u$) such that $G - u$ has a component $G_1$ with $|E(u, V(G_1))| = c$.
- Let $d \geq 3$ be odd. Define

$$G_{d,c} = K_{d+1-c} \lor \overline{M_c} \lor K_1 \lor \overline{M_{d-c}} \lor K_{c+1}.$$

We are now in the stage to present the main result of this paper.

**Theorem 1.1** Let $d \geq 3$ and $G$ be a connected $d$-regular graph.

(i) Let $d = 3$ and let $\theta_3$ be the largest root of $x^3 - 7x - 2 = 0$. If $\lambda_2(G) \leq \theta_3$ and $G \neq G_{d,1}$, then $\kappa(G) \geq 2$. Moreover, $\theta_3$ is between 2.7784 and 2.7785.

(ii) Let $d \geq 4$ be even. For $4|d$, if $\lambda_2(G) \leq \frac{1}{2}(\sqrt{d^2 + 2d + 4} + d - 2)$ and

$$G \neq G_{d,\frac{d}{2}} = K_{\frac{d}{2}+1} \lor \overline{M_{\frac{d}{2}}} \lor K_1 \lor \overline{M_{\frac{d}{2}}} \lor K_{\frac{d}{2}+1},$$

then $\kappa(G) \geq 2$.

For $4 \nmid d$, let $\theta_d$ be the largest root of $x^4 - (d-4)x^3 - (4d-4)x^2 + (\frac{d^2}{2} - 4d - 2)x + \frac{3}{4}d^2 - 3 = 0$. If $\lambda_2(G) \leq \theta_d$ and

$$G \neq G_{d,\frac{d}{2}-1} = K_{d+2} \lor \overline{M_{d-1}} \lor K_1 \lor \overline{M_{\frac{d}{2}+1}} \lor K_{\frac{d}{2}},$$

then $\kappa(G) \geq 2$. Moreover,

$$d - \frac{1}{2} + \frac{3}{4d} + \frac{5}{4d^2} - \frac{2}{d^3} < \theta_d < d - \frac{1}{2} + \frac{3}{4d} + \frac{5}{4d^2} - \frac{3}{16d^3}.$$

(iii) Let $d \geq 5$ be odd and $C_c$ be defined as above. Let $\theta_d$ be the largest root of $x^4 - (d-5)x^3 - (5d-6)x^2 + (\frac{d^2}{2} - 6d - \frac{1}{2})x + d^2 - 1 = 0$. 

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(a) For odd $d - \frac{1}{2}$, if $\lambda_2(G) \leq \theta_d$ and

$$G \neq G_{d, d-\frac{1}{2}} = M_{d+\frac{5}{2}} \lor C_{d-\frac{1}{2}} \lor K_1 \lor M_{d+1} \lor K_{d+1},$$

then $\kappa(G) \geq 2$.

(b) For even $d - \frac{1}{2}$, if $\lambda_2(G) \leq \theta_d$ and

$$G \neq G_{d, d-\frac{1}{2}} = K_{d+\frac{3}{2}} \lor M_{d-\frac{1}{2}} \lor C_{d-\frac{1}{2}} \lor M_{d+\frac{1}{2}},$$

then $\kappa(G) \geq 2$.

Moreover, $d - \frac{1}{2} + \frac{1}{d} - \frac{9}{8d^2} < \lambda_2(G_{d, d-\frac{1}{2}}) < d - \frac{1}{2} + \frac{1}{d} - \frac{9}{8d^2} + \frac{2}{d^3}$.

2 Preparation

Let $M_n(F)$ be the set of all $n$ by $n$ matrices over a field $F$. A matrix $M = [m_{ij}] \in M_n(F)$ is tridiagonal if $m_{ij} = 0$, whenever $|i - j| > 1$. The following lemma is (a corrected version) from Brouwer et al. (1989, p. 130).

**Lemma 2.1** (Brouwer et al. 1989) Let $M$ be a non-negative tridiagonal matrix as follows:

$$A = \begin{pmatrix}
    a_0 & b_0 & 0 \\
    c_1 & a_1 & b_1 \\
    & \ddots & \ddots \\
    0 & c_{n-1} & b_{n-1} \\
    & a_n &
\end{pmatrix},$$

Assume each row sum of $M$ equals $d$. If $M$ has eigenvalues $\lambda_1 = d, \lambda_2, \ldots, \lambda_{n+1}$ indexed in non-increasing order, then the $n \times n$ matrix

$$\tilde{A} = \begin{pmatrix}
    d - b_0 - c_1 & b_1 & 0 \\
    c_1 & d - b_1 - c_2 & b_2 \\
    & \ddots & \ddots \\
    0 & c_{n-1} & d - b_{n-1} - c_n \\
\end{pmatrix},$$

has eigenvalues $\lambda_2, \lambda_3, \ldots, \lambda_{n+1}$.

Let $G$ be a graph on $n$ vertices. For a partition $\{V_1, V_2, \ldots, V_m\}$ of $V(G)$, the corresponding quotient matrix of the partition is an $m \times m$ matrix $(b_{ij})$, where $b_{ij} = \frac{e(V_i, V_j)}{|V_i|}$ for any $1 \leq i, j \leq m$ (note $e(V_i, V_i) = 2|E(G[V_i])|$ for $1 \leq i \leq m$). The partition
is equitable if each vertex in $V_i$ has the same number $b_{i,j}$ of neighbors in $V_j$ for any $1 \leq i, j \leq m$.

The following lemma is from Corollaries 2.5.3 and 2.5.4 in Brouwer and Haemers (2012).

**Lemma 2.2** (Brouwer and Haemers 2012) Let $B$ be the quotient matrix of a graph $G$ w.r.t. the partition \{ $V_1, \ldots, V_m$ \}. Then the eigenvalues of $B$ interlace the eigenvalues of $G$. Moreover, if this partition is equitable, then each eigenvalue (with multiplicity) of $B$ is an eigenvalue of $G$.

Look back to the graph $G_{d,c} \in \mathcal{G}_{d,c}$. Obviously, $G_{d,c}$ contains $2d + 4$ vertices for odd $d \geq 3$ and contains $2d + 3$ vertices for even $d \geq 4$. Moreover, $G_{d,c} = G_{d,d-c}$. Note that $G_{d,1} = G_{d,d-1}$ is the graph $X_d$ defined in Cioabă (2010, 4) and used in Cioabă (2010, Theorem 1.4), the proof of which implies the following conclusion.

**Lemma 2.3** (Cioabă 2010) Let $d \geq 3$ be an odd integer. Then $\lambda_2(G_{d,1}) = \lambda_2(G_{d,d-1})$ is the largest root of $f_0(x) = 0$ with $f_0(x) = x^3 - (d-3)x^2 - (3d-2)x - 2$. Moreover, if $G$ is a $d$-regular graph with cut edges, then $\lambda_2(G) \geq \lambda_2(G_{d,1})$ with equality if and only if $G = G_{d,1}$.

**Lemma 2.4** Let $G_{d,c}$ be the graph defined as above.

(i) Let $d \geq 4$ and $2 \leq c \leq d - 2$ be even integers. Then $\lambda_2(G_{d,c})$ is the largest root of equation $f_1(x) = 0$, where $f_1(x) = x^4 - (d-4)x^3 - (4d - 4)x^2 + (2cd - 2c^2 - 4d)x + 3c(d - c)$.

(ii) Let $d \geq 5$ be an odd integer and $2 \leq c \leq d - 2$. Then $\lambda_2(G_{d,c})$ is the largest root of equation $f_2(x) = 0$, where $f_2(x) = x^4 - (d-5)x^3 - (5d - 6)x^2 + (2cd - 2c^2 - 6d)x + 4c(d - c)$.

**Proof** For (i), $G_{d,c} = K_{d+1-c} \cup \overline{M}_c \cup K_1 \cup \overline{M}_{d-c} \cup K_{c+1}$ has an equitable partition $V(G_{d,c}) = V(K_{d+1-c}) \cup V(\overline{M}_c) \cup V(K_1) \cup V(\overline{M}_{d-c}) \cup V(K_{c+1})$. Then the corresponding quotient matrix is

$$B_1 = \begin{pmatrix}
    d - c & c & 0 & 0 & 0 \\
    d + 1 - c & c - 2 & 1 & 0 & 0 \\
    0 & c & 0 & d - c & 0 \\
    0 & 0 & 1 & d - c - 2 & c + 1 \\
    0 & 0 & 0 & d - c & c
\end{pmatrix},$$

which, along with Lemma 2.2, shows that the eigenvalues of $B_1$ are the eigenvalues of $G_{d,c}$. By Lemma 2.1 we have $\lambda_2(B_1) = \lambda_1(\tilde{B}_1)$, where

$$\tilde{B}_1 = \begin{pmatrix}
    -1 & 1 & 0 & 0 \\
    d + 1 - c & d - c - 1 & d - c & 0 \\
    0 & c & c - 1 & c + 1 \\
    0 & 0 & 1 & -1
\end{pmatrix}.$$ 

Thus, a direct calculation shows that $\lambda_1(\tilde{B}_1)$ is the largest root of equation $f_1(x) = 0$. Note $\lambda_1(\tilde{B}_1) > d - \frac{1}{2}$. Let $W$ be the subspace spanned by the characteristic vectors...
of the five parts of the equitable partition of $G_{d,c}$. Clearly, the dimension of $W$ is five. Note that each eigenvector $x$ associated to an eigenvalue of $B_1$ can be extended to be an eigenvector $x'$ of $G_{d,c}$, and that the components of $x'$ corresponding to each part of the equitable partition are the same. Hence, the five independent eigenvectors extended by those of $B_1$ span $W$. Except eigenvalues of $B_1$, note that the eigenvectors of other eigenvalues of $G_{d,c}$ are orthogonal to those in $W$. So, other eigenvalues of $G_{d,c}$ are also the eigenvalues of graph $K_{d+1-c} \cup M_c \cup K_1 \cup M_{d-c} \cup K_{c+1}$, whose maximum degree is not larger than $d - 1$. Hence, the other eigenvalues of $G_{d,c}$ except the ones of $B_1$ are less than $d - 1$. Thereby, $\lambda_2(G_{d,c}) = \lambda_2(B_1) = \lambda_1(\tilde{B}_1)$, which is the largest root of equation $f_1(x) = 0$.

For (ii), firstly consider $c \in [2, d - 2]$ to be an odd integer. Clearly, $G_{d,c} = M_{d+2-c} \cup C_c \cup K_1 \cup M_{d-c} \cup K_{c+1}$ has an equitable partition $V(M_{d+2-c}) \cup V(C_c) \cup V(K_1) \cup V(M_{d-c}) \cup V(K_{c+1})$. Then, the related quotient matrix is

$$B_2 = \begin{pmatrix}
d - c & c & 0 & 0 & 0 \\
d + 2 - c & c - 3 & 1 & 0 & 0 \\
0 & c & 0 & d - c & 0 \\
0 & 0 & 1 & d - c - 2 & c + 1 \\
0 & 0 & 0 & d - c & c
\end{pmatrix}.$$ 

From Lemma 2.2 the eigenvalues of $B_2$ are also those of $G_{d,c}$. By Lemma 2.1 again we get $\lambda_2(B_2) = \lambda_1(\tilde{B}_2)$, where

$$\tilde{B}_2 = \begin{pmatrix}
-2 & 1 & 0 & 0 \\
d + 2 - c & d - c - 1 & d - c & 0 \\
0 & c & c - 1 & c + 1 \\
0 & 0 & 1 & -1
\end{pmatrix}. \quad (1)$$

By a routine computing, $\lambda_1(\tilde{B}_2)$ is the largest root of equation $f_2(x) = 0$. Note that $\lambda_1(\tilde{B}_2) > d - \frac{1}{2}$. Similarly to (i), we can verify that $\lambda_2(G_{d,c}) = \lambda_2(B_2) = \lambda_1(\tilde{B}_2)$ is the largest root of equation $f_2(x) = 0$. If $2 \leq c \leq d - 2$ is even, then $2 \leq d - c \leq d - 2$ is odd. Due to $G_{d,c} = G_{d,d-c}$, applying the above discussion to $G_{d,d-c}$ we obtain the conclusion. \hfill \Box

**Lemma 2.5** For $d \geq 3$ and $1 \leq c \leq d - 1$, let $G_{d,c}$ be the graph defined above.

(i) If $d \geq 4$ is even, then $\lambda_2(G_{d,2}) > \lambda_2(G_{d,4}) > \cdots > \lambda_2(G_{d,2\lceil \frac{d}{4}\rceil})$.

(ii) If $d \geq 3$ is odd, then $\lambda_2(G_{d,1}) > \lambda_2(G_{d,2}) > \cdots > \lambda_2(G_{d,d-1})$.

**Proof** Let $d \geq 4$ be even and $2 \leq c \leq 2\lceil \frac{d}{4}\rceil$. By Lemma 2.4(i), $\lambda_2(G_{d,c})$ is the largest root of equation $f_1(x) = 0$. Since

$$f_1(x) = x^4 - (d - 4)x^3 - (4d - 4)x^2 + (2cd - 2d^2 - 4d)x + 3c(d - c) = x^4 - (d - 4)x^3 - (4d - 4)x^2 - 4dx + c(d - c)(2x + 3),$$

\hfill \Box
then for $x > 0$ it is easy to see that $c(d - c)(2x + 3)$ is strictly increasing w.r.t. $c$. Thereby, the largest root of equation $f_1(x) = 0$ is strictly decreasing w.r.t. $c$, which implies $\lambda_2(G_{d,2}) > \lambda_2(G_{d,4}) > \cdots > \lambda_2(G_{d,\frac{d-1}{2}})$.

We next show (ii). For $d = 3$, the result is trivial. Let $d \geq 5$ be odd and $2 \leq c \leq \frac{d-1}{2}$. By Lemma 2.4(ii), $\lambda_2(G_{d,c})$ is the largest root of equation $f_2(x) = 0$. Since

$$f_2(x) = x^4 - (d - 5)x^3 - (5d - 6)x^2 + (2cd - 2c^2 - 6d)x + 4c(d - c)$$

then for $x > 0$ we get that $2c(d - c)(x + 2)$ is strictly increasing w.r.t. $c$. Therefore, the largest root of equation $f_2(x) = 0$ is strictly decreasing w.r.t. $c$, and hence $\lambda_2(G_{d,2}) > \lambda_2(G_{d,3}) > \cdots > \lambda_2(G_{d,\frac{d-1}{2}})$.

For $c = 1$, from Lemma 2.3 it follows that $\lambda_2(G_{d,1})$ is the largest root of $f_0(x) = 0$. By Lemma 2.4(ii) again, $\lambda_2(G_{d,2})$ is the largest root of equation

$$f_2(x)|_{c=2} = x^4 - (d - 5)x^3 - (5d - 6)x^2 - (2d + 8)x + 8d - 16 = (x + 2)(x^3 - (d - 3)x^2 - (3d - 2)x - 2) + 2x(2d - 5 - x) + 8d - 12 = 0.$$ 

For $d \geq 5$ and $\lambda_2(G_{d,1}) \leq x < d$, we get $f_0(x) \geq 0$, and then $x^4 - (d - 5)x^3 - (5d - 6)x^2 - (2d + 8)x + 8d - 16 > 0$. Hence, $\lambda_2(G_{d,1}) > \lambda_2(G_{d,2})$, and so $\lambda_2(G_{d,1}) > \lambda_2(G_{d,2}) > \cdots > \lambda_2(G_{d,\frac{d-1}{2}})$.

This completes the proof. 

\[ \square \]

3 Proof of Theorem 1.1

Recall, for two integers $d \geq 3$ and $1 \leq c \leq d - 1$, that the family $\mathcal{G}_{d,c}$ is defined in Section 1.

Lemma 3.1 Let $G \in \mathcal{G}_{d,c}$. If $d \geq 3$ is odd and $c = 1$ or $c = d - 1$, then $\lambda_2(G) \geq \lambda_2(G_{d,c})$ with equality holds if and only if $G = G_{d,c}$.

Proof In this case, $G$ has cut edges. Then the result follows from Lemma 2.3. \[ \square \]

Lemma 3.2 Let $G \in \mathcal{G}_{d,c}$. If $d \geq 4$ and $c \in [2, d - 2]$ are even, then $\lambda_2(G) \geq \lambda_2(G_{d,c})$, where the equality holds if and only if $G = G_{d,c}$.

Proof Since $G \in \mathcal{G}_{d,c}$, then there exists a cut vertex $u$ such that $|E(u, V(G_1))| = c$, where $G_1$ is a component of $G - u$. Let $G_2$ be the union of other components of $G - u$. Then $|E(u, V(G_2))| = d - c$. If $|V(G_1)| \leq d$, then $c = |E(u, V(G_1))| \geq |V(G_1)|(d + 1 - |V(G_1)|) \geq d$, a contradiction. Hence, $|V(G_1)| \geq d + 1$, and so is $|V(G_2)|$ similarly. Let $V_1$ and $V_2$ be the sets of the neighbors of $u$ in $G_1$ and in $G_2$ respectively. Then $|V_1| = c$ and $|V_2| = d - c$. Set $G_1' = G_1 - V_1$ and $G_2' = G_2 - V_2$. Thus, $|V(G_1')| = p \geq d + 1 - c$ and $|V(G_2')| = q \geq d + 1 - (d - c) = c + 1$. After
putting $|E(V(G_1'), V_1)| = r$ and $|E(V(G_2'), V_2)| = t$, we obtain a partition of $G$ with $V(G) = V(G_1') \cup V_1 \cup \{u\} \cup V_2 \cup V(G_2')$ whose quotient matrix is

$$B_3 = \begin{pmatrix}
  d - \frac{r}{p} & \frac{r}{p} & 0 & 0 & 0 \\
  \frac{r}{c} & d - 1 - \frac{r}{c} & 1 & 0 & 0 \\
  0 & c & 0 & d - c & 0 \\
  0 & 0 & 1 & d - 1 - \frac{r}{d-c} & \frac{r}{d-c} - \frac{r}{q} \\
  0 & 0 & 0 & \frac{r}{q} & d - \frac{r}{q}
\end{pmatrix}. \quad (2)
$$

Using Lemmas 2.1 and 2.2 we get $\lambda_2(G) \geq \lambda_2(B_3) = \lambda_1(\widetilde{B}_3)$, where $\widetilde{B}_3$ is

$$\begin{pmatrix}
  d - \frac{r}{p} & \frac{r}{p} & 1 & 0 & 0 \\
  \frac{r}{c} & d - c - 1 & d - c & 0 & 0 \\
  0 & c & c - 1 & \frac{r}{d-c} & \frac{r}{d-c} - \frac{r}{q} \\
  0 & 0 & 1 & d - \frac{r}{d-c} & \frac{r}{d-c} - \frac{r}{q}
\end{pmatrix}. \quad (3)
$$

**Fact 1** The largest root $\lambda_1(\widetilde{B}_3)$ of matrix $\widetilde{B}_3$ is strictly decreasing w.r.t. $r$ and $t$.

**Proof of Fact 1** We only prove the conclusion for $r$, as it is similar for $t$. By a computation, the characteristic polynomial of $\widetilde{B}_3$ is equal to $h(x) = |xI - \widetilde{B}_3| = (x - d)h_1(x) + rh_2(x)$, where $h_2(x) = \left(\frac{r}{p} + \frac{t}{c}\right)h_1(x) + \frac{t}{c(d-c)} - \frac{r}{c}(x - c + 1)(x - d + \frac{r}{d-c} + \frac{r}{q})$, and $h_1(x)$ is the next characteristic polynomial of a principle sub-matrix of $\widetilde{B}_3$.

$$\begin{pmatrix}
  d - c - 1 & d - c & 0 \\
  c & c - 1 & \frac{r}{d-c} \\
  0 & 1 & d - \frac{r}{d-c} - \frac{r}{q}
\end{pmatrix}.$$

Clearly, for $x \geq \lambda_1(\widetilde{B}_3)$ we get $h_1(x) > 0$ and $h(x) \geq 0$. Thus, for $\lambda_1(\widetilde{B}_3) \leq x < d$ we arrive at $h_2(x) > 0$ and hence $(x - d)h_1(x) + rh_2(x) > h(x) \geq 0$ for any $r_1 > r$. Thereby, the largest root of equation $(x - d)h_1(x) + rh_2(x) = 0$ is less than $\lambda_1(\widetilde{B}_3)$.

So, the largest root of $\widetilde{B}_3$ is strictly decreasing w.r.t. $r$. \hfill $\Box$

Employing Fact 1, we can make $r$ and $t$ as large as possible. By the well-known Perron-Frobenius Theorem, the largest eigenvalue of an irreducible matrix will strictly decrease if its positive elements decrease. Then for given $r$ and $t$ we set $p$ and $q$ as small as possible in $\widetilde{B}_3$. To obtain the minimum value of $\lambda_1(\widetilde{B}_3)$, we will show that $p \leq d - 1$ and $q \leq d - 1$.

We only need to show $p \leq d - 1$, since it is similar for $q \leq d - 1$. Note $r \leq c(d-1)$.

If $p \geq d - 1$, Employing Fact 1, we can let $r = c(d-1)$. Then $\widetilde{B}_3$ is reduced to the following matrix

$$D_0 = \begin{pmatrix}
  1 - \frac{c(d-1)}{p} & 1 & 0 & 0 \\
  d - 1 & d - c - 1 & d - c & 0 \\
  0 & c & c - 1 & \frac{t}{d-c} + \frac{t}{q} \\
  0 & 0 & 1 & d - \frac{r}{d-c} - \frac{r}{q}
\end{pmatrix}.$$
By Perron–Frobenious Theorem (see (Brouwer and Haemers 2012)), \( \lambda_1(D_0) \) is minimized when \( p \) is minimized (i.e., \( p = d - 1 \)). Thus we can assume \( p \leq d - 1 \).

By the above discussion, we can let \( d + 1 - c \leq p \leq d - 1 \) and \( c + 1 \leq q \leq d - 1 \). Then we have \( r = cp \) and \( t = (d - c)q \) by Fact 1. (We can obtain such graphs just by letting \( G[V_1], G[V(G'_1)], G[V_2] \) and \( G[V(G'_2)] \) be four regular graphs of degrees \( d - 1 - p, d - c, d - 1 - q \) and \( c \), respectively. Note that for any two positive integers \( d < t \), there exists a \( d \)-regular graph on \( t \) vertices if \( td \) is even.) Then \( \tilde{B}_3 \) is reduced to the following matrix

\[
D = \begin{pmatrix}
p & d - c - 1 & 0 & 0 \\
1 & d - c - p & 0 & 0 \\
0 & c & c - 1 & q \\
0 & 0 & 1 & c - q \\
\end{pmatrix}.
\]

Fact 2 The largest root \( \lambda_1(D) \) of matrix \( D \) is a strictly increasing function w.r.t. \( p \) and \( q \).

Proof of Fact 2 We only prove the conclusion for \( p \), as it is similar for \( q \). A straightforward calculation yields the characteristic polynomial of \( D \) is equal to \( g(x) = \) \( |xI - D| = (x - d + c)g_1(x) + pg_2(x) \), where \( g_2(x) = g_1(x) - (x - c + 1)(x - c + q) + q \),

and \( g_1(x) \) is the characteristic polynomial of next principle sub-matrix of \( D \),

\[
\begin{pmatrix}
d - c - 1 & d - c \\
c & c - 1 & q \\
0 & 1 & c - q \\
\end{pmatrix}.
\]

Note that \( \lambda_1(D) > d - 1 \) by Perron–Frobenious Theorem, since \( D \) has the following principle sub-matrix whose largest eigenvalue is larger than \( d - 1 \):

\[
\begin{pmatrix}
d - c - 1 & d - c \\
c & c - 1 \\
\end{pmatrix}
\]

For \( x \geq \lambda_1(D) \), clearly we get \( g_1(x) > 0 \). For any \( d + 1 - c \leq p' < p \), we now prove \( (x - d + c)g_1(x) + p g_2(x) > 0 \) whenever \( x \geq \lambda_1(D) \). Recall \( \lambda_1(D) > d - 1 \). For any \( \lambda_1(D) \leq x < d \), if \( g_2(x) \geq 0 \), then \( (x - d + c)g_1(x) + p' g_2(x) > 0 \). If \( g_2(x) < 0 \) for some \( \lambda_1(D) \leq x < d \), then \( (x - d + c)g_1(x) + p g_2(x) > (x - d + c)g_1(x) + pg_2(x) \geq 0 \). Hence, \( (x - d + c)g_1(x) + p g_2(x) > 0 \) for \( x \geq \lambda_1(D) \), which implies that the largest root of equation \( (x - d + c)g_1(x) + p' g_2(x) = 0 \) is less than \( \lambda_1(D) \). So, the largest root \( \lambda_1(D) \) of matrix \( D \) is strictly increasing w.r.t. \( p \). \( \square \)

In view of Fact 2, we can set \( p = d + 1 - c \) and \( q = c + 1 \) which together with \( r = cp \) and \( t = (d - c)q \) leads to \( \tilde{B}_3 = \tilde{B}_1 \) (defined in Lemma 2.4). Consequently, \( \lambda_2(G) \geq \lambda_2(G_{d,c}) \) with equality holds if and only if \( G = G_{d,c} \). \( \square \)

For the completeness of next lemma, we give an imitate proof with the similar as Lemma 3.2.

Lemma 3.3 Let \( G \in \mathcal{G}_{d,c} \) and \( d \geq 5 \) be an odd integer.
(i) If \( c \in [2, d - 2] \) is odd, then \( \lambda_2(G) \geq \lambda_2(G_{d,c}) \) with equality if and only if
\[
G = \overline{M}_{d+2-c} \cup \overline{C}_c \cup K_1 \cup \overline{M}_{d-c} \cup K_{c+1},
\]
where \( C_c \) is the union of disjoint cycles on \( c \) vertices.

(ii) If \( c \in [2, d - 2] \) is even, then \( \lambda_2(G) \geq \lambda_2(G_{d,c}) \) with equality if and only if
\[
G = K_{d+1-c} \cup \overline{M}_c \cup K_1 \cup \overline{C}_{d-c} \cup \overline{M}_{c+2},
\]
where \( C_{d-c} \) is the union of disjoint cycles on \( d - c \) vertices.

**Proof**  We only need to show (i). Otherwise, if \( c \in [2, d - 2] \) is even, then \( d - c \) is odd. Due to \( G_{d,c} = G_{d,d-c} \), then the proof is similar to the case in which \( c \) is odd.

Since \( G \in \mathcal{G}_{d,c} \), then there exists a cut vertex \( u \) such that \(|E(u, V(G_1))| = c\), where \( G_1 \) is a component of \( G - u \). Let \( G_2 \) be the union of other components of \( G - u \). Then \( |E(u, V(G_2))| = d - c \). Similarly to Lemma 3.2, we have \(|V(G_1)| \geq d + 1\) and \(|V(G_2)| \geq d + 1\). Since \( d|E(G_1)| = 2|E(G)| + c \) and \( d|E(G_2)| = 2|E(G_2)| + d - c \), then \(|V(G_1)| \) is odd, and thus \(|V(G_1)| \geq d + 2\) and \(|V(G_2)| \) is even. Let \( V_1 \) and \( V_2 \) be the sets of the neighbors of \( u \) in \( G_1 \) and in \( G_2 \) respectively. Then \(|V_1| = c\) and \(|V_2| = d - c\). Set \( G_1' = G_1 - V_1 \) and \( G_2' = G_2 - V_2 \). Thus, \(|V(G_1')| = p \geq d + 2 - c\) and \(|V(G_2')| = q \geq d + 1 - (d - c) = c + 1\). Set \(|E(V(G_1'), V_1)| = r\) and \(|E(V(G_2'), V_2)| = t\). We obtain a partition of \( G \) with \( V(G) = V(G_1') \cup V_1 \cup \{u\} \cup V_2 \cup V(G_2') \) whose quotient matrix is just \( B_3 \) in (2). By Lemmas 2.1 and 2.2, we have \( \lambda_2(G) \geq \lambda_2(B_3) = \lambda_1(B_3) \), where \( B_3 \) is defined in (3). As shown in Lemma 3.2, by Fact 1 we get that the largest root \( \lambda_1(B_3) \) of matrix \( B_3 \) is strictly decreasing w.r.t. \( r \) and \( t \).

By the well-known Perron-Frobenius Theorem, the largest eigenvalue of an irreducible matrix will strictly decrease if its positive elements decrease. Then we can make \( p \) and \( q \) as small as possible, and set \( r \) and \( t \) as large as possible. Note that \( r \leq c(d - 1) \) and \( t \leq (d - c)(d - 1) \). Similar to Lemma 3.2, we can let \( d + 2 - c \leq p \leq d - 1 \), \( c + 1 \leq q \leq d - 1 \), and \( r = cp \) and \( t = (d - c)q \). Thus, \( B_3 \) is equal to matrix \( D \) defined in (4).

From Fact 2 it follows that the largest root \( \lambda_1(D) \) is strictly increasing w.r.t \( p \) and \( q \). Thus, we can set \( p = d + 2 - c \) and \( q = c + 1 \), which implies \( B_3 = \tilde{B}_2 \) defined in (1). Consequently, \( \lambda_2(G) \geq \lambda_2(G_{d,c}) \), and equality holds if and only if \( G = \overline{M}_{d+2-c} \cup \overline{C}_c \cup K_1 \cup \overline{M}_{d-c} \cup K_{c+1}, \) where \( C_c \) is the union of disjoint cycles on \( c \) vertices.

**Proof of Theorem 1.1** For (i), from Lemma 2.3 we get that \( \lambda_2(G_{3,1}) \) is the largest root of equation \( x^3 - 7x - 2 = 0 \), which is between 2.7784 and 2.7785.

For (ii) and (iii), assume that \( G \) has a cut vertex, say \( u \). Then, there exists some component \( G_1 \) of \( G - u \) such that \(|E(u, V(G_1))| = c \leq \frac{d-1}{2}\) if \( d \) is odd, or such that \( c \leq 2\lfloor \frac{d}{4} \rfloor \) if \( d \) is even. From Lemmas 3.1–3.3 and 2.5 we get \( \lambda_2(G) \geq \lambda_2(G_{d,\frac{d-1}{2}}) \) with the equality iff \( G = G_{d,\frac{d-1}{2}} \) when \( d \) is odd, or \( \lambda_2(G) \geq \lambda_2(G_{d,2\lfloor \frac{d}{4} \rfloor}) \) with the equality iff \( G = G_{d,2\lfloor \frac{d}{4} \rfloor} \) when \( d \) is even, a contradiction. Consequently, \( \kappa(G) \geq 2 \).
If $d$ is even and $4 \mid d$, then by Lemma 2.4(i) we get that $\lambda_2(G_{d, \frac{d}{2}})$ is the largest root of equation

$$0 = x^4 - (d - 4)x^3 - (4d - 4)x^2 + \left(\frac{d^2}{2} - 4d\right)x + \frac{3}{4}d^2$$

which is just $\frac{1}{2} (d - 2 + \sqrt{d^2 + 2d + 4})$.

If $d$ is even and $4 \nmid d$, by Lemma 2.4(i) again we obtain that $\lambda_2(G_{d, \frac{d}{2}-1})$ is the largest root of equation $f(x) = 0$, where $f(x) = x^4 - (d - 4)x^3 - (4d - 4)x^2 + (\frac{d^2}{2} - 4d - 2)x + \frac{3}{4}d^2 - 3$. Set $f_1(y) = f(d - \frac{1}{2} + \frac{3}{4d} + \frac{5}{4d^2} + \frac{y}{d^3})$. A direct calculation shows, for $d \geq 6$, that

$$f_1(-2) = -\frac{19}{16} - \frac{7}{32d} + \frac{151}{64d^2} - \frac{435}{64d^3} + \frac{45}{256d^4} - \frac{9}{d^5} - \frac{2537}{128d^6} + \frac{2127}{64d^7} - \frac{1583}{256d^8} + \frac{107}{8d^9} + \frac{27}{2d^{10}} - \frac{40}{d^{11}} + \frac{16}{d^{12}} < 0,$$

which implies $\lambda_2(G_{d, \frac{d}{2}-1}) > d - \frac{1}{2} + \frac{3}{4d} + \frac{5}{4d^2} - \frac{2}{d^3}$. Similarly, for $y \geq -\frac{3}{16}$ we get $f_1(y) > 0$ that indicates $\lambda_2(G_{d, \frac{d}{2}-1}) < d - \frac{1}{2} + \frac{3}{4d} + \frac{5}{4d^2} - \frac{3}{16d^3}$.

If $d \geq 5$ is odd, by Lemma 2.4(ii) we have that $\lambda_2(G_{d, \frac{d+1}{2}})$ is the largest root of equation $h(x) = 0$ with $h(x) = x^4 - (d - 5)x^3 - (5d - 6)x^2 + (\frac{d^2}{2} - 6d - \frac{1}{2})x + d^2 - 1$. Set $h_1(y) = f(d - \frac{1}{2} + \frac{1}{d} - \frac{9}{8d^2} + \frac{y}{d^3})$. By computation we arrive at

$$h_1(0) = -\frac{11}{8} - \frac{81}{32d} - \frac{123}{64d^2} - \frac{21}{128d^3} + \frac{145}{64d^4} + \frac{1341}{512d^5} + \frac{1701}{512d^6} - \frac{729}{128d^7} + \frac{6561}{4096d^8} < 0,$$

which shows $\lambda_2(G_{d, \frac{d+1}{2}}) > d - \frac{1}{2} + \frac{1}{d} - \frac{9}{8d^2}$. Similarly, for $y \geq 2$ we get $h_1(y) > 0$ that leads to $\lambda_2(G_{d, \frac{d+1}{2}}) < d - \frac{1}{2} + \frac{1}{d} - \frac{9}{8d^2} + \frac{2}{d^3}$.

This finishes the proof. \qed

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