Getting Wahlquist’s metric from a perturbation theory solution

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Abstract. Perturbation theory is a very convenient and powerful tool to study solutions of Einstein’s equations, and its use is extensive nowadays. Nevertheless from the formal point of view, the fact of working with an infinite dimensional manifold results in that it can not be guaranteed in general that a particular perturbation solution corresponds to the Taylor expansion of a supposed exact solution with the desired properties. Here we will present a result obtained from a solution of the perturbed Einstein’s system following the CMMR scheme for a perfect fluid region with equation of state \(\mu + (1 - n)p = \mu_0\), being \(\mu\) density and \(p\) pressure. Then we will show how it can be reached from a Post-Minkowskian and slow-rotation approximation of the Wahlquist metric. Additionally we show how conditions we got for it to be Petrov type D are automatically satisfied when we try to mimic Wahlquist’s solution, and also identify the Wahlquist parameter \(r_0\) in terms of physical quantities.

1. Previous work

We work within the analytical approximation scheme developed in [1]. Let \((\mathcal{M}, g)\) be a stationary and axisymmetric spacetime, corresponding to a compact source interior. Let also \(r\) be the time-like Killing and \(\theta\) the closed-orbits space-like one.

Adapting \(t\) and \(\varphi\) to the Killings and \(r\) and \(\theta\) to the two-dimensional surfaces orthogonal to the Killing ones, [2] we can write the inner metric with the structure

\[
g = \gamma_{tt} \omega^t \otimes \omega^t + \gamma_{t\varphi} (\omega^t \otimes \omega^\varphi + \omega^\varphi \otimes \omega^t) + \gamma_{\varphi\varphi} \omega^\varphi \otimes \omega^\varphi + \gamma_{rr} \omega^r \otimes \omega^r + \gamma_{r\theta} (\omega^r \otimes \omega^\theta + \omega^\theta \otimes \omega^r) + \gamma_{\theta\theta} \omega^\theta \otimes \omega^\theta
\]

in the cobasis \(\omega^t = dt, \omega^\varphi = dr, \omega^\theta = r d\theta, \omega^r = r \sin \theta d\varphi\). We require these coordinates to be spherical-like associated to harmonic Cartesian-like ones (in the sense that they are Cartesian in the Minkowskian limit).

\((\mathcal{M}, g)\) is filled with a compact perfect fluid without convective motions, so that its 4-velocity can be written as \(u = \psi(\xi + \omega\xi)\) with \(\psi\) a normalization factor and \(\omega\) the module of the rotation speed of the fluid. Its equation of state (EOS) is \(\mu + (1 - n)p = \mu_0\), where \(\mu\) is energy density, \(p\) pressure, \(n\) a non zero real and \(\mu_0\) a positive constant. Expressions for \(\mu\) and \(p\) in the source can be found integrating Euler’s equation \(dp = (\mu + p) d\ln \psi\) and obtaining \(\psi\) from the normalization condition \(u^mu_a = -1\). We expand the \(p = 0\) surface embedding in Legendre polynomials.

As already mentioned we follow an analytical two-parameter perturbation method. Thus we have a two-parameter family of spacetimes \((\mathcal{M}_{\lambda,\Omega}, g_{\lambda,\Omega})\). The post-Minkowskian parameter is
chosen to be $\lambda \propto \mu_0$ and the slow rotation one as $\Omega^2 = r_*^2 \omega^2 / m$, with $r_*$ the radius of the sphere with density $\mu_0$ and mass $m$. Having said that, we can proceed to solve iteratively in $\lambda$ the system of Einstein’s equations plus harmonic condition

$$\begin{bmatrix}
\Delta h_{\alpha\beta}^{(n)} &= -16\pi (T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T)^{(n)} + 2 N_{\alpha\beta}^{(n)} - \partial_\alpha F_\beta^{(n)} - \partial_\beta F_\alpha^{(n)}, \\
\partial^k \left[ h_{\alpha\beta}^{(n)} - \frac{1}{2} h^{(n)} \eta_{\alpha\beta} \right] &= - P_\alpha^{(n)},
\end{bmatrix}$$

where $\Delta$ is the flat Laplacian, $T_{\alpha\beta}$ the energy–momentum tensor $N_{\alpha\beta}$ and $P_{\alpha\beta}$ are the second and higher order terms of the Ricci tensor and the harmonic condition respectively. Finally, we make use of the slow rotation approximation to expand the shape of the surface as a power series in Legendre polynomials

$$r_\Sigma (\cos \theta) \rightarrow r_\Sigma = r_* (1 + S \Omega^2 P_2 (\cos \theta)) + O(\Omega^4)$$

with $S$ a constant (that could be determined through the matching with a certain exterior metric) and $r_*$ the radius of the fluid in the static limit. The full structure of the general solution can be found in [1] and contains the following free constants: $m_0$, $m_2$, $j_1$, $j_3$, $b_0$, $a_0$, $a_2$.

This scheme has already been applied to a pack of EOS. In [1], $\mu = \mu_0$ was used and in [3] the fluid was a polytrope. In both cases, as well as in the present one [4], the results pointed out that such source configurations could not create a Kerr exterior already in the linear solution. Here we are interested in comparing with the Wahlquist metric, so we will make us of a theorem stating that for a stationary perfect–fluid solution with rigid rotation, the metric is given by the Wahlquist family iff the solution is type D, axisymmetric and with equation of state $\mu + 3p =$ const. [5]. This translates into the conditions

$$\left\{ \begin{array}{c}
18 + 5 m_2 (n - 1) = 0 \\
m_2 \left( \frac{23}{70} - \frac{n}{5} \right) + j_3 (n - 1) + \frac{3(n - 1)}{35} = 0.
\end{array} \right.$$  

The free parameters in $g$ can be chosen to satisfy them, and in that case our perturbed metric corresponds to Wahlquist.

**2. Wahlquist metric and its coordinates**

Our aim now is to proceed in the opposite direction: take the singularity free Wahlquist metric, expand it in the appropriate parameters and try to use every freedom available to recover a particular case of the CMMR metric.

The singularity free Wahlquist metric is

$$ds^2 = -f (dt + A d\varphi)^2 + r_0^2 (\xi^2 + \eta^2) \left[ \frac{c^2 h_1 h_2}{h_1 - h_2} d\varphi^2 + \frac{d\xi^2}{(1 - k^2 \xi^2) h_1} + \frac{d\eta^2}{(1 + k^2 \eta^2) h_2} \right]$$

with $u = f^{-1/2} \partial_t$, so it is written in comoving coordinates, and the energy density and pressure are given in terms of the free parameters $\mu_0$ and $b$ as

$$\mu = \frac{1}{2} \mu_0 (3b^2 f - 1), \quad p = \frac{1}{2} b_0 (1 - b^2 f)$$

and where

$$f (\xi, \eta) = \frac{h_1 - h_2}{\xi^2 + \eta^2}, \quad A = c r_0 \left( \frac{\xi^2 h_2 + \eta^2 h_1}{h_1 - h_2} - \eta^2 \right),$$

$$h_1 (\xi) = 1 + \xi^2 + \frac{\xi}{b^2} \left[ \xi - \frac{1}{k} (1 - k^2 \xi^2)^{1/2} \text{arcsinh}(k \xi) \right], \quad \text{and} \quad k^2 \equiv \frac{1}{2} \mu_0 r_0^2 b^2,$$

$$h_2 (\eta) = 1 - \eta^2 - \frac{\eta}{b^2} \left[ \eta - \frac{1}{k} (1 + k^2 \eta^2)^{1/2} \text{arcsinh}(k \eta) \right].$$
The parameter $\mu_0$ is the energy density in the static limit and $r_0$ is related with the rotation speed of the fluid although, to the best of our knowledge, there is no proved explicit expression for it. This will be the natural choice of parameters to expand Wahlquist for us, but first we must find the change to spherical-like coordinates. If we study the static limit[6], making $\eta = \cos \chi$, $\xi = \frac{R_0}{r_0}$ and letting $r_0 \to 0$, one obtains the Whittaker metric. In Minkoskian limit, the same change gives us the flat metric in Boyer-Lindquist coordinates. This is interesting because the coordinate change from these to spherical coordinates is known, and then, what we should look for seems to be two new functions such that in the mentioned limits $\eta = \cos \chi$, $\xi = \frac{R_0}{r_0}$ holds. Once written in these coordinates, the change from Boyer-Lindquist to spherical should give us Wahlquist in the coordinates we need. The functions we choose are

$$\{\xi, \eta\} \to \{R_1, R_2\} : \begin{cases} h_1(\xi) = 1 + \frac{R_1^2}{r_0^2} \\ 1 - h_2(\eta) = \cos^2 \chi = \frac{R_2^2}{r_0^2} \end{cases}$$

As we are dealing with a perturbed calculation, we only need the approximate expressions of $\xi = \xi(R_1)$ and $\eta = \eta(R_2)$. Expanding in $\mu_0$ and $r_0$ all the constants in $h_1(\xi)$ and $h_2(\eta)$, we get

$$\xi^2 = \frac{R_1^2}{r_0^2} - \frac{1}{6} \mu_0 \frac{R_1^4}{r_0^4} \left\{ 1 - \frac{1}{15} \mu_0 R_1^2 \left[ 2 - \mu_0 \left( \sigma_1 + \frac{37}{84} R_1^2 \right) \right] \right\} + \mathcal{O}(\mu_0)^4 \quad (8)$$

$$\eta^2 = \frac{R_2^2}{r_0^2} + \frac{1}{6} \mu_0 \frac{R_2^4}{r_0^4} \left\{ 1 + \frac{1}{15} \mu_0 R_2^2 \left[ 2 - \mu_0 \left( \sigma_1 - \frac{37}{84} R_2^2 \right) \right] \right\} + \mathcal{O}(\mu_0)^4 \quad (9)$$

And now, the change

$$r^2 \sin^2 \theta = \sin^2 \chi (r_0^2 + R_1^2)$$

$$r \cos \theta = R_1 \cos \chi$$

should give us the spherical-like coordinates desired.

### 3. From Wahlquist to CMMR

Now we face the problem of the identification of the approximation parameters and making every term in Wahlquist and CMMR equal.

To get an idea of the problems arising in the general case, we analyze the static limit. Prior to make any comparison, we stop the CMMR metric making $\varphi' = \varphi - \omega t \rightarrow \mathbf{u} = \psi \partial_t$. Now, we use the clear relation between mass parameters: $\lambda = \frac{3}{2} \pi \mu_0 r_0^2$ and make $r_0 \to 0$. The $g_{rr}$ term of the expression of Whittaker we get is

$$dr^2 \left( 1 - \frac{r^2 \lambda}{r_s^2} - \frac{2r^2 \lambda^2 (7r^2 + 15\sigma)}{5r_s^4} \right)$$

and upon comparison with the corresponding static limit of CMMR

$$dr^2 \left( 1 + m_0 \lambda - \frac{r^2 \lambda}{r_s^2} - \frac{2r^4 \lambda^2}{35r_s^4} + \frac{2m_0 r^2 \lambda^2}{5r_s^2} - \frac{24r^2 S \lambda^2}{5r_s^2} \right)$$

we can see that there are discrepancies among $r^4$ terms. To some extent this was to be expected since CMMR is written in coordinates associated to harmonic ones and no such a condition has been imposed on Wahlquist yet. In this particular case, the two metrics can be rendered exactly equal with a change of the form $r \to r(1 + \lambda^2 f(r))$.
In the general case we keep the relation between mass parameters. From comparing both \( g_{\varphi \varphi} \) terms, we can guess that 
\[
r_0 = w \sqrt{\lambda} \Omega r_s,
\]
with \( w \) a proportionality factor. This leads to a new kind of discrepancy, giving rise to \( \text{const} \times \Omega^2 \) and \( \text{const} \times \Omega^4 \) terms. This issue can be solved using the freedom in the time scale. Eventually, every “bad term” is eliminated through the changes

\[
\begin{align*}
t &= t' (1 + \mu_0 F_1 + \mu_0^2 F_2) + \cdots \\
r &= r' (1 + \mu_0 G_1 [r, \theta] + \mu_0^2 G_2 [r, \theta]) + \cdots \\
\theta &= \theta' + \mu_0 \sin \theta H_1 [r, \theta] + \mu_0^2 \sin \theta H_2 [r, \theta] + \cdots
\end{align*}
\] (11)

3.1. Adjusting terms

Once the dependencies in \( r^n \) and \( \Omega^n \) are corrected, only the adjustment of the constant factors remains. Here we can still carefully make use of a change with the form (12), and finally, we find that both metrics exactly coincide up to order \((\lambda^1, \Omega^2)\) if the free constants of CMMR take the values

\[
\begin{align*}
m_0 &= 0 \\
m_2 &= \frac{6}{5} (1 + 2 \lambda S) \\
j_1 &= \frac{9}{5} \Omega^2 S \\
j_3 &= \frac{36}{175}
\end{align*}
\] (13)

These values satisfy the Petrov type D condition (4), as was to be expected. The final adjustment also fixes the proportionality factor \( w \) and the Wahlquist surface potential \( b \) to be

\[
\begin{align*}
w(r_0 = w \sqrt{\lambda} \Omega r_s) &\rightarrow w = 1 - \frac{\Omega^2}{10} \quad \text{(14)} \\
b^2 &= (1 + \Omega^2) (1 + 2 \lambda S) \quad \text{(15)}
\end{align*}
\]

4. Conclusions

In this work we have arrived at the same metric through two different ways: direct construction on perturbation theory and perturbing a known exact solution. Thus, we have checked the reliability of the method at least when applied to the singularity free Wahlquist case. Additionally, we have identified the uncertain Wahlquist parameter \( r_0 \) in terms of physical quantities.

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