THE LEAST ACTION METHOD, CDM AND $\Omega$

A.M. Dunn$^1$
Institute of Astronomy
University of Cambridge, Madingley Road
Cambridge, CB3 0HA, UK

R. Laflamme$^2$
Theoretical Astrophysics, T-6, MSB288
Los Alamos National Laboratory
Los Alamos, NM87545, USA

Abstract

Peebles has suggested an interesting method to trace back in time positions of galaxies called the least action method. This method applied on the Local Group galaxies seems to indicate that we live in an $\Omega \approx 0.1$ Universe. We have studied a CDM N-body simulation with $\Omega = 0.2$ and $H = 50\text{km} s^{-1}/\text{Mpc}$ and compare trajectories traced back from the Least Action Principle and the center of mass of the particle forming CDM halos. We show that the agreement between these set of trajectories is at best qualitative. We also show that the line of sight peculiar velocities are underestimated. This discrepancy is due to orphans, CDM particles which do not end up in halos. By varying the density parameter $\Omega$ in the least action principle we show that using this method we would underestimate the density of the Universe by a factor of 4-5.
1. Introduction

The density parameter $\Omega$ is one of the most important parameters characterising our Universe. There are many methods to measure $\Omega$, but a factor of 10 uncertainty in its value remains. Recently Peebles suggested that it may be possible to estimate its value in the Local Neighborhood, by tracing Local Group galaxies back in time.

Peebles (1989, 1990, 1994) used the principle of least action to find complete trajectories for Local Group galaxies. The idea is to assume that galaxies growing out of small density perturbations in the early universe will have negligible peculiar velocities with respect to the Hubble flow. This is a reasonable assumption as we know that the microwave background has very small anisotropies. Using zero initial peculiar velocities as one boundary condition and the present positions of the galaxies as the other, trial orbits are iteratively varied so as to minimise the action. The method has been criticised since the galaxies are treated as point particles throughout their history, even though the size of the galaxies must be comparable to their separation at early times. However, the least action principle leaves the final velocities of the galaxies unconstrained, and its ability to reproduce the observed radial velocities remains a powerful test of the validity of the trajectories. For the Local Group galaxies, Peebles has obtained remarkable agreement between the observed radial velocities and those calculated from the least action principle. Obtaining reliable trajectories for nearby galaxies might shed light into the origin of their angular momentum (Dunn & Laflamme, 1993).

Although the Least Action Method provides a powerful tool for investigating galaxy orbits, its predictions are only as good as the assumptions they stand on. These are, that a) galaxies initially have negligible peculiar velocity with respect to the Hubble flow, b) galaxies can be represented as point particles throughout their history, c) mergers have little effect on a galaxy’s motion, and d) light traces mass. One way to test the validity of some of these assumptions is to apply the Least Action Method to a numerical simulation of the universe. In the simulation we have complete information about the particle trajectories, which we compare with the predictions made by the Least Action Method. In this letter we use a cold dark matter (CDM) simulation. Although CDM may not be able to reproduce all the observable features of our universe, it does at least represent a possible universe in which all the particles are governed by Hamiltonian dynamics.

In the first section we give details of the simulations we have used and comment on the groups that we have studied. Secondly we compare the trajectories obtained from the CDM simulation and the Least Action Method and compare their ‘line of sight’ velocities. Finally we comment on the origin of the discrepancy.
2. The Least Action Method.

Peebles’ use of the least action method (LAM) selects a set of classical trajectories for a group of galaxies (point masses) which are interacting through gravity, against the background of an expanding universe model. This method differs from the usual application of the least action principle in that boundary conditions are applied to the beginning and end of each trajectory. The trajectories are constrained such that

\[ \delta x_i = 0 \text{ at } t = t_0, \quad a^2 \frac{dx_i}{dt} \to 0 \text{ at } a \to 0 \]  

(2.1)

where \( a \) is the scale factor of the universe and \( x_i(a) \) is the trajectory of the \( i \)th galaxy in comoving coordinates. That is, the galaxies are fixed at their present positions at the present epoch, and their peculiar velocities vanish as we approach the Big Bang. Trial trajectories for the set of galaxies are adjusted in order to find a stationary point in the action.

In this paper a matter dominated universe with no cosmological constant is assumed, thus

\[ H dt = \frac{a^{1/2} da}{F^{1/2}}, \]  

(2.2)

where \( F = (\Omega + (1-\Omega)a) \), \( H \) is the present Hubble constant and \( \Omega \) is the density parameter. Following Peebles (1990), the action for particles moving in such a universe is,

\[ S = \int_{t_0}^{t_0} \left[ \sum_i \frac{m_i a^2}{2} \left( \frac{dx_i}{dt} \right)^2 + \frac{G}{a} \sum_{i \neq j} \frac{m_i m_j}{|x_i - x_j|} + \frac{2}{3} \pi G \rho_b a^2 \sum m_i \right] \]  

(2.3)

from which we can deduce the equation of motion,

\[ a^{1/2} \frac{d}{da} a^{3/2} \frac{dx_i}{da} + \frac{(1 - \Omega)a^2}{2F} \frac{dx_i}{da} = \frac{\Omega}{2F} [x_i + \frac{R_0^3}{M_T} \sum_j \frac{m_j (x_j - x_i)}{|x_j - x_i|^3}]. \]  

(2.4)

Here \( R_0 \) is the radius of a sphere which would enclose a homogeneous distribution of the total mass \( M_T \) of the group of galaxies considered, \( R_0^3 \equiv M_T (\frac{4}{3} \pi \rho^0) \). Note that this equation is slightly different from the one used by Peebles as we do not assume a flat \((k = 0)\) universe .

It is very hard to have exact analytic solutions for the coupled system of equations (2.4). However Peebles succeeded in obtaining approximate solutions using trial functions of the form

\[ x_i(a) = x_i^0 + \sum_n C^n_i f_n(a) \]  

(2.5)
where \( \mathbf{x}_i^p \) are the present positions of the galaxies and the \( f_n \) are linearly independent functions chosen to satisfy the boundary conditions (2.1). In this paper we take \( f_n = a^n(1 - a) \) for \( n = 0, \ldots, 4 \). The classical solutions are obtained by introducing \( \mathbf{x}_i(a) \) in the action and iteratively modifying the coefficient \( C_i^n \) to obtain a stationary action. As Peebles did, we verify that the least action solutions are good approximations to real solutions by evolving the classical equations of motion starting with the initial positions and velocities derived from the least action solutions at \( z = 60 \).

3. CDM simulation.

In order to understand the limits of the Least Action Method, it is important to compare it with some other method. We used a CDM, N-body simulation of Kauffmann and White (1992). It is a PPPM simulation with 262 144 particle, representing an \( \Omega = 0.2 \) universe. Scaled to \( H = 50 \text{km s}^{-1} \text{Mpc}^{-1} \), it encompasses a size of 100 Mpc with particles of mass \( 5.2 \times 10^{10} \text{M}_\odot \).

We have studied a few groups containing 10 or so galaxy halos. The halos are determined by a friend of a friends algorithm. They were chosen in order to match the conditions in the Local Group; two dominant galaxies with peculiar velocities towards each other, a mass ratio of roughly 4:3 and somewhat isolated from high density mass concentrations. Due to the limitations of dynamic range in the simulation, we could not find any such halos with a separation of 0.7 Mpc but had to go to approximately 2 Mpc. These halos also had masses approximately 5-10 times greater than M31 and the MW. In addition to the two central galaxies, the galaxies around them up to a distance of 20 Mpc were selected to form a group. We thus intend to study the effect of the spatial distribution of the halo, the influence of the nearby galaxies on their dynamics and also the effect of particles not linked to any halo (orphans).

We have selected 9 groups which had 2 halos of the order of 200 particles within 2 Mpc. We investigated roughly 10 galaxies around these two center halos. All the groups had similar behavior and we therefore for brevity present the results of only one of them here.
4. Comparison.

Once we have identified the galactic halos we can use the least action method described in section 2 to trace them back in time. We can also trace back the particles making the halo in the final step of the CDM simulation. Figure 1. shows the 3 projections of the halos trajectories. We can see that there is a very rough agreement between the LAM trajectories and the CDM ones.

Assuming that these are in rough agreement we can compare the line of sight velocities of these two trajectories. It is seen from figure 2 that with the same parameter $\Omega$ and $H$, the LAM would overestimate this velocity. By dividing $\Omega$ by a factor of 4 we we would obtain reasonable line of sight velocity. It is this problem that we address now.

First let’s see the effect of modifying $\Omega$ in the LAM. By varying $\Omega$ in the LAM we can change the line of sight velocities. Here there are two factors to take into account. Changing $\Omega$ will change the time elapsed since the Big Bang, increasing $\Omega$ decrease the elapsed time and thus increased the velocity. The second factor is that with a larger $\Omega$ the radius $R_0$ of eq. (2.4), the radius of a sphere which would enclose a homogeneous distribution of the total mass, is smaller. Thus we do not have to go as far to gather the mass to make the halos, thus decreasing the velocity. These two factors conspire against each other but a simple calculation for a 2 body system show that the first one wins. Thus as shown in figure 2 as $\Omega$ decreases the velocity decreases.

We must now answer the question of why, for the same $\Omega$ and $H$ we have the LAM line of sight velocities being larger than the CDM one. One incorrect justification would be to think that the radius $R_o$ to gather the halo’s mass, should be increased in proportion to the fraction of CDM particles which do not end up in halos. However this is incorrect as can be shown by comparing the trajectories from the CDM and LAP simulations. The correct answer lies in the fact that in the CDM model there are orphans, i.e. CDM particles which are not linked to halos. In the early times the background of orphans is roughly homogeneously distributed and cancel the force due to the particles which will eventually make the halos. From the LAP point of view, this will in fact reduce the force or the effective mass of the halos and thus, in order to end up at their known final positions, they will have to start at a closer distance than the LAM has given us. This has the effect of reducing the velocities, as the CDM simulation shows.

As mentioned earlier, we can also investigate the effect of the spatial distribution of the halos. Consider equation (2.4), all the terms are linear (in $x^i$) except the last one on the right. We have compared the contribution of this term, which we call by abuse of language, the inhomogeneous component of force, when we do the sum over particles in
different ways. In the LAM we assume that the halos interact as point sources, that is, the important part of the force only acts between the centers of mass of each halo,

\[
F^1_a = \sum_b \frac{x_b - x_a}{|x_b - x_a|^3}
\]

(4.1)

where \(a\) is the target galactic halo and the sum over \(b\) is over the center of mass of the nearby halos.

The second approach is obtained by summing over all the particles in each of the halos rather than just their center of mass. We have also divided by the number of particles of the target halo \((N_a)\) to get the force on its center of mass. This will give an estimate of the effect of the higher multiple moment of the halos.

\[
F^2_a = \frac{1}{N_a} \sum_{a_i} \sum_{b_j} \frac{x_{b_j} - x_{a_i}}{|x_{b_j} - x_{a_i}|^3}
\]

(4.2)

here the sum over \(a_i\) is over all particles of the target halo and the one over \(b_j\) is over all particles of the halo \(b\) and then over all halos in our sample. This will essentially sum over everything except the orphans.

The third quantity is

\[
F^3_a = \frac{1}{N_a} \sum_{a_i} \sum_j \frac{x_j - x_{a_i}}{|x_j - x_{a_i}|^3}
\]

(4.3)

where the sum \(a_i\) is over all particles of the target halo and the sum \(j\) is over all particles within 20\,Mpc of the CM of the target group at \(r = 9.98465\) i.e., the last frame of the CDM model. This corresponds to the true force on the halo. We have modified the distance of 20\,Mpc to a shorter distance and without significant change in the results (for the force on the last frame).

We have plotted the result in Figure 3 where the magnitude of the different ‘forces’ and the angle between the first two and \(F^3_a\) are shown. From this figure we can see that at early times the force \(F^1_a\) is overestimated by a factor of roughly 2, which is not unexpected, since galaxies make poor approximations to point particles at early times. We can also see that the force \(F^2_a\), which includes higher multipoles of halos, is not a very good approximation since there are serious discrepancies and scatter between the direction of this vector and the true force \(F^3_a\). We must therefore reject the suggestion of Branchini and Carlberg (1994), that the discrepancy between the CDM and LAM line of sight velocities might be due to neglecting the shape of the CDM halos.
We should also point out that another possible problem for the LAM is the existence of mergers. In one of our groups there was a significant merger and for this halo the force was not very well represented by the one at its center of mass. For a merger to have an important effect it must be the result of roughly equally massive halos which come in rather different directions. A detailed study of mergers in CDM model is needed to quantitatively know if this is a potentially serious problem for the LAM.

5. Conclusion.

In this letter we have shown that Peebles’ Least Action Method underestimates the value of $\Omega$ for a CDM universe. The main discrepancy is due to neglecting the effect of orphans, CDM particles which have are not members of any halos. They are scattered uniformly in the early stage of the universe and therefore reduce the force on the particles which will eventually form halos. Thus the proto-halos must start at a shorter distance than what is expected in Peebles original suggestion. This is equivalent to failure of one of the key assumptions of the Least Action Method, that is, that light traces mass (at least at kiloparsec to megaparsec scales). Of course, it is quite possible that this assumption does in fact hold for the universe we live in. However, since there is little observational or theoretical reason to suppose this, we must call in to doubt previously published results based on the LAM. We conclude that the dynamics of the Local Group and a careful examination of its line of sight velocities do not exclude a closed Universe.

Acknowledgments. We would like to thank B. Bromley, D. Lynden-Bell, S. White, M. Warren and W.H. Zurek for useful comments. We would also like to thank the NASA HPCC program for support.

5. References

Branchini, E. & Carlberg, R.G., Testing the Least Action Principle in and $\Omega_o = 1$ Universe, SISSA preprint 56-94-A.
Dunn, A.M. & Laflamme, R., 1993, M.N.R.A.S., 264, 865.
Kauffman, G & White, S.D.M., 1992, M.N.R.A.S., 258, 511.
Peebles, P.J.E., 1989, Astrophys.J., 344,L53.
Peebles, P.J.E., 1990, Astrophys.J., 362, 1.
Peebles, P.J.E., 1994, Orbits of nearby galaxies, Astrophys.J., to appear.
Figure captions.

Figure 1. Projection of the CDM and least action trajectories for galaxies in the chosen group. The broken line are the trajectories from the CDM simulation by following the center of mass of the particles forming a halo in the last frame. The plain line are the LAM trajectories. We can see that the agreement is at best qualitative. (Units in Mpc).

Figure 2. Least Action Method line of sight velocities (with respect to one target galaxy of the chosen group) as a function of the CDM line of sight velocity. The best fit corresponds to an adjusted density parameter $\Omega \approx 0.05$ a factor of 4-5 higher than the CDM simulation parameter.

Figure 3. Plot of the inhomogeneous part of the ‘force’ (eq. [4.1-3]) $F^1_a$ and $F^2_a$ with respect to the true force $F^3_a$ on a typical halo. We see that the force on a halo is not well approximated by the force due to other halos, orphans (CDM particles not bound to halos) have an important contribution.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/astro-ph/9411002v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/astro-ph/9411002v1
This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/astro-ph/9411002v1
Numerical Galaxy Orbits
Least Action velocity (km/s)

CDM velocity (km/s)

\[ x \Rightarrow \Omega = 1 \]

\[ \bullet \Rightarrow \Omega = 0.2 \]

\[ \circ \Rightarrow \Omega = 0.05 \]
Magnitude of the force \((1/\text{Mpc}^2/\text{N}_{\text{target}})\)

**Target 876**

- \(F^1_a\)
- \(F^2_a\)
- \(F^3_a\)

**Angle(°)**

- \(F^1_a \leftrightarrow F^3_a\)
- \(F^2_a \leftrightarrow F^3_a\)

**Target 888**

**Target 820**