Phase Space Reduction and Vortex Statistics

An Anyon Quantization Ambiguity

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Abstract

We examine the quantization of the motion of two charged vortices in a Ginzburg–Landau theory for the fractional quantum Hall effect recently proposed by the first two authors. The system has two second-class constraints which can be implemented either in the reduced phase space or Dirac–Gupta–Bleuler formalism. Using the intrinsic formulation of statistics, we show that these two ways of implementing the constraints are inequivalent unless the vortices are quantized with conventional statistics; either fermionic or bosonic.
1. Introduction

The fractional quantum Hall effect is an intriguing example of the effects of a highly ordered ground state much like superconductivity or superfluidity. In analogy to the description of these macroscopic quantum states by an effective field theory, there is a description of the fractional quantum Hall state by an effective scalar field theory. The former field theories describe a Bose condensed state, while the interpretation of the effective field theory for the fractional quantum Hall state is less straightforward. One can account for the fractional conductivity of the state through topological vortex excitations of the vacuum of the effective field theory which are effectively fractionally charged. These vortex excitations correspond exactly to Laughlin’s fractionally charged quasiparticles\(^1\) in the many-body theory. It has been argued\(^2\) that Laughlin’s quasiparticles are anyons, particles having statistics intermediate between fermions and bosons. This is a situation which, for point particles at least, is unique to two spatial dimensions. An interesting related question is the statistics of vortex excitations in two-dimensional superfluid films\(^3\), which is usually considered in the reduced phase space formalism.

We consider here the quantum statistics of Ginzburg-Landau vortex excitations within the intrinsic, or topological, formulation.\(^4\textit{–}6\) In the intrinsic formulation, the configuration space of identical particles is smaller than the classical configuration space because configurations which are quantum mechanically indistinguishable are identified. Quantum statistics in the intrinsic view generally arise from the nontrivial topology of the intrinsic configuration space. Because the intrinsic configuration space is not a manifold in general, one encounters problems which do not occur in the usual treatment of quantum statistics through the permutation of particle labels. The overlap of two or more particles becomes a boundary of the configuration space making it necessary to choose the boundary conditions at these points which preserve the self-adjointness of the Hamiltonian. Implementing the intrinsic formulation is reasonably straightforward when the particles are described by a non-singular quadratic Hamiltonian. When constraints are present, however, the configuration space is not uniquely defined. In the present case, there are two different configuration spaces which are relevant. One is the actual configuration space before constraints are imposed and the second is the configuration space resulting from the reduced phase space quantization. We find that only in the case of Fermi or Bose statistics do the reduced phase space and Dirac–Gupta–Bleuler methods agree.
2. The Intrinsic Formulation of Statistics

The classical configuration space for a system of $N$ identical particles moving in Euclidean $m$-space is $\mathbb{R}^{Nm}$, identical particles being classically distinguishable. Quantum mechanically, however, identical particles are not distinguishable and the true or intrinsic configuration space,

$$Q_{\text{int}} = \mathbb{R}^{Nm}\backslash \Delta,$$

is not generally a smooth manifold as it has conical singularities where two or more particles coincide. In general, the configuration space $Q_{\text{int}}$ does not have the same topology as $\mathbb{R}^{Nm}$.

The case which interests us here, that of $N$ particles moving in two-dimensional space, has an intrinsic configuration space, $Q_{\text{int}}$, which is not simply connected. When $Q_{\text{int}}$ is not simply connected, the wave functions need not be single-valued and in general will belong to a unitary irreducible representation of $\pi_1(Q_{\text{int}})$, the first homotopy group.\[4,5,7\] This is the origin of exotic statistics in two dimensions.

A technical complication of the intrinsic formulation is that the space $Q_{\text{int}}$ has singularities; boundaries from which probability might leave the system. If unitary time evolution is desired, as it is when particle number is not allowed to change, Stone’s theorem\[8\] assures us that the Hamiltonian must be self-adjoint.

For completeness, we review here facts and definitions about self-adjoint operators, which are explained in greater detail in Ref. [8]. Let $\hat{\mathcal{O}}$ be a densely defined operator on a separable Hilbert space $\mathcal{H}$. The operator $\hat{\mathcal{O}}$ is symmetric if and only if $\hat{\mathcal{O}} \subset \hat{\mathcal{O}}^\dagger$. That is, the domain of $\hat{\mathcal{O}}^\dagger$ is not smaller than the domain of $\hat{\mathcal{O}}$, $D(\hat{\mathcal{O}}) \subset D(\hat{\mathcal{O}}^\dagger)$, and the operators agree on the domain of $\hat{\mathcal{O}}$, $\hat{\mathcal{O}}\psi = \hat{\mathcal{O}}^\dagger\psi$ for all $\psi \in D(\hat{\mathcal{O}})$.

An operator $\hat{\mathcal{O}}$ is self-adjoint if and only if $\hat{\mathcal{O}} = \hat{\mathcal{O}}^\dagger$. That is, if and only if $\hat{\mathcal{O}}$ is symmetric and $D(\hat{\mathcal{O}}) = D(\hat{\mathcal{O}}^\dagger)$. It is easy to show that a self-adjoint operator has only real eigenvalues. To determine whether a symmetric operator $\hat{\mathcal{O}}$ is self-adjoint, we examine the spectrum of its adjoint, $\hat{\mathcal{O}}^\dagger$. If there are no solutions to $\hat{\mathcal{O}}^\dagger\psi = \pm i\psi$, then $\hat{\mathcal{O}}$ is essentially self-adjoint, that is, its closure $\hat{\mathcal{O}}^{\dagger\dagger}$ is self-adjoint. More generally, if $\hat{\mathcal{O}}$ is a symmetric operator, let $\mathcal{K}_\pm = \text{Ker}(\hat{\mathcal{O}}^\dagger \mp i)$, $n_\pm = \dim(\mathcal{K}_\pm)$. The kernels $\mathcal{K}_\pm$ are the deficiency subspaces of $\hat{\mathcal{O}}$ and
their dimensions, \( n_\pm \), are the deficiency indices of \( \hat{O} \). If \( n_+ = n_- = n \), then there is an \( n^2 \)-parameter family of self-adjoint extensions of \( \hat{O} \). The domain of the adjoint of \( \hat{O} \) is given by

\[
D(\hat{O}^\dagger) = D(\hat{O}) \oplus \mathcal{K}_+ \oplus \mathcal{K}_-.
\]

The self-adjoint extensions are characterized by the unitary maps \( U : \mathcal{K}_+ \to \mathcal{K}_- \). For each such unitary map \( U \), one finds a domain on which \( \hat{O} \) is self-adjoint:

\[
D_U(\hat{O}) = \{ \phi + \psi_+ + U\psi_+ | \phi \in D(\hat{O}), \psi_+ \in \mathcal{K}_+ \}.
\]

If \( n_+ = n_- = 0 \), then \( \hat{O} \) is essentially self-adjoint. If \( n_+ \neq n_- \), then \( \hat{O} \) is not self-adjoint and possesses no self-adjoint extensions.

### 3. Collective Quantum Mechanics of Two Vortices

Recently, two of us constructed an effective field theory for the fractional quantum Hall effect which has charged vortex excitations.\[^9\] We found that the collective coordinate action for \( N \) vortex centers \( X^A \) with integral vorticities \( n_A, A = 1, \ldots, N \), in the approximation of point vortices is

\[
S_V = \int dt \left( \sum_A \pi n_A \rho_0 \epsilon_{ab} X^A_a \dot{X}^A_b - \sum_{A<B} 2\pi \alpha n_A n_B \ln |X^A - X^B|^2 - \sum_A \pi \alpha n_A e B |X^A|^2 \right).
\]

The quantities \( \rho_0 \) and \( \alpha \) are constants arising from the original effective field theory, \( e \) is the electron charge and \( B \) is the constant background magnetic field. In what follows, we set \( \pi \rho_0 = 1 \), and \( 2\pi \alpha = 1 \), to make the manipulations more transparent. It is important to note that the action (1) is first order in time derivatives, and is thus quite different from an ordinary nonrelativistic point particle action. There are two very different methods for constructing the quantum mechanics of such an action. The first and most straightforward of these is simply to observe that the two components of each vortex center are canonically conjugate because the action (1) already has Hamiltonian form: \( S = \int [p \dot{q} - H(p, q)] dt \).
is the reduced phase space quantization, which can also be arrived at through the Dirac constraint analysis by replacing the Poisson brackets of the dynamical variables by Dirac brackets. The action (1) describes one-dimensional motion for each vortex in the reduced phase space quantization. We are free to choose, say, the $X_1$ coordinates of each vortex center to be the configuration variables and then the $X_2$ coordinates (after an integration by parts) become the momenta conjugate to the $X_1$’s,

$$X_2^A = -\frac{1}{2n_A} P_1^A.$$  \hspace{1cm} (2)

The last term in the Hamiltonian,

$$H = \sum_{A<B} 2\pi \alpha n_A n_B \ln |X^A - X^B|^2 + \sum_A \pi \alpha n_A eB |X^A|^2,$$ \hspace{1cm} (3)

is a function of the $|X^A|^2$ which become one-dimensional harmonic oscillator Hamiltonians

$$|X^A|^2 = (X_1^A)^2 + \frac{1}{(2n_A)^2}(P_1^A)^2.$$ \hspace{1cm} (4)

The states are functions only of the $X_1^A$’s. In the analysis of two vortices, the operator representing $|X^{(1)} - X^{(2)}|^2$ will be of central importance. In the reduced phase space quantization it is a harmonic oscillator, while in the Dirac–Gupta–Bleuler quantization, it is an angular momentum.

The second, more sophisticated, quantization procedure is due to Dirac. This is the procedure employed in Ref. [9]. In the Dirac method, one takes the original configuration variables $X^A$ and introduces their canonical conjugates $P^A = \partial L / \partial \dot{X}^A$. In this case we find directly the second-class phase-space constraints

$$\varphi^A_a(P^A, X^A) = P^A_a + n_A \epsilon_{ab} X^A_b \approx 0.$$ \hspace{1cm} (5)

The naive canonical Hamiltonian must be modified, by adding to it quantities which vanish when the constraints do, so that the constraints are preserved under time evolution. When second-class constraints are present, the correct Hamiltonian is often the “starred”
Hamiltonian,\(^{[11]}\)
\[
H^* = H - \{H, \varphi_a\}(\Delta^{-1})^{ab}\varphi_b, \quad (6)
\]
\[
\Delta_{ab} \equiv \{\varphi_a, \varphi_b\},
\]
although there is no unique prescription for constructing the correct Hamiltonian for a given problem. We will use the notation \(H^*\) for any modified Hamiltonian which preserves the constraints.

In the Gupta-Bleuler quantization prescription the physical quantum states are constructed so that all matrix elements of the above constraints vanish in the physical basis. The Dirac–Gupta–Bleuler prescription is more restrictive. In it, we must be able to divide the second-class constraints into “creation” and “annihilation” constraints \(\varphi_a\) and \(\varphi_b\) such that the only weakly non-vanishing Poisson brackets are those between a creation and an annihilation constraint. Upon quantization, the physical states are chosen to be those annihilated by the annihilation constraint operators
\[
\hat{\varphi}_a \left| \psi_{\text{phys}} \right> = 0. \quad (7)
\]

In order that physical states evolve into physical states, it is necessary that the annihilation constraints evolve into themselves,
\[
[H, \hat{\varphi}_a] = \Lambda_{ab} \hat{\varphi}_b, \quad (8)
\]
guaranteeing that (7) at time \(t = 0\) implies the same at all later times,
\[
\hat{\varphi}_a \left| \psi(t) \right> = \hat{\varphi}_a e^{-itH} \left| \psi(0) \right> = e^{-itH} e^{itH} \varphi_a e^{-itH} \left| \psi(0) \right> = e^{-itH} (e^{itA})_{ab} \hat{\varphi}_b \left| \psi(0) \right> = 0. \quad (9)
\]

In constructing the states for two identical unit vortices \(n_1(n_2) = 1\), it is useful to take center-of-mass and relative coordinates
\[
X^{\text{CM}} = \frac{1}{2}(X^{(1)} + X^{(2)}),
\]
\[
X^{\text{rel}} = X^{(1)} - X^{(2)}, \quad (10)
\]
because the Hamiltonian and the constraints separate;

\[
H = H_{\text{CM}} + H_{\text{rel}} = eB|X_{\text{CM}}|^2 + \ln|X_{\text{rel}}|^2 + \frac{1}{4}eB|X_{\text{rel}}|^2, \tag{11}
\]

\[
\varphi^\text{CM}_a = P^\text{CM}_a + 2\epsilon_{ab}X^\text{CM}_b \approx 0, \\
\varphi^\text{rel}_a = P^\text{rel}_a + \frac{1}{2}\epsilon_{ab}X^\text{rel}_b \approx 0.
\]

We find that the complex combinations of constraints

\[
\varphi^\text{CM} = \varphi^\text{CM}_1 + i\varphi^\text{CM}_2 \approx 0, \quad \varphi^\text{rel} = \varphi^\text{rel}_1 + i\varphi^\text{rel}_2 \approx 0, \tag{12}
\]

\[
\bar{\varphi}^\text{CM} = \varphi^\text{CM}_1 - i\varphi^\text{CM}_2 \approx 0, \quad \bar{\varphi}^\text{rel} = \varphi^\text{rel}_1 - i\varphi^\text{rel}_2 \approx 0,
\]

can be written simply in terms of 

\[
Z = X^\text{CM}_1 + iX^\text{CM}_2, \quad \xi = X^\text{rel}_1 + iX^\text{rel}_2
\]

and the combinations of their momenta 

\[
P = \frac{1}{2}(P_1 - iP_2), \quad \bar{P} = \frac{1}{2}(P_1 + iP_2) \text{ as follows.}
\]

\[
\varphi^\text{CM} = -2i(iP_Z + Z), \quad \varphi^\text{rel} = -2i(i\bar{P}_\xi + \frac{1}{4}\xi), \\
\bar{\varphi}^\text{CM} = -2i(iP_Z - \bar{Z}), \quad \bar{\varphi}^\text{rel} = -2i(i\bar{P}_\xi - \frac{1}{4}\bar{\xi}). \tag{13}
\]

The Hamiltonian operator must have Poisson brackets with the constraints again yielding a combination of constraints. Because the center-of-mass and relative Hamiltonians are functions of $|X_{\text{CM}}|^2$ and $|X_{\text{rel}}|^2$ respectively, we may write the correct Hamiltonians as functions of the starred variables $|X_{\text{CM}}|^2*$ and $|X_{\text{rel}}|^2*$, which have Poisson brackets with each constraint in the basis (13) proportional to that constraint.

\[
|X_{\text{CM}}|^2* = |X_{\text{CM}}|^2 + \frac{1}{4}\{|X_{\text{CM}}|^2, \varphi^\text{CM}_a\} \epsilon_{ab} \varphi^\text{CM}_b, \\
|X_{\text{rel}}|^2* = |X_{\text{rel}}|^2 + \{|X_{\text{rel}}|^2, \varphi^\text{rel}_a\} \epsilon_{ab} \varphi^\text{rel}_b. \tag{14}
\]

The Hamiltonian as an explicit function of the starred variables is

\[
H^*_\text{CM} = H_{\text{CM}}(|X_{\text{CM}}|^2*) = \frac{eB}{2}(iZP_Z - i\bar{Z}\bar{P}_Z), \\
H^*_\text{rel} = H_{\text{rel}}(|X_{\text{rel}}|^2*) = \ln[2(i\xi P_\xi - i\bar{\xi}\bar{P}_\xi)] + \frac{eB}{2}(i\xi P_\xi - i\bar{\xi}\bar{P}_\xi). \tag{15}
\]

To go to quantum operators, we simply replace the momenta by their derivative forms

\[
\hat{P}_Z = -i\partial_Z, \quad \hat{P}_\xi = -i\partial_\xi, \\
\hat{\bar{P}}_Z = -i\partial_Z, \quad \hat{\bar{P}}_\xi = -i\partial_\xi. \tag{16}
\]

The physical states are those which are annihilated by the constraints $\varphi^\text{rel} = -2i(\partial_\xi + \frac{1}{4}\xi)$.
and $\varphi_{\text{CM}} = -2i(\partial_Z + Z)$. These states have the form

$$
\Psi(\xi, Z) = F(\xi)e^{-\frac{1}{4}|\xi|^2} G(Z)e^{-|Z|^2},
$$

(17)

where $F$ and $G$ are holomorphic functions which do not grow too fast at infinity so that the states are normalizable.

4. Vortex Statistics

The fact that vortices are indistinguishable quantum objects must be built into the quantum mechanics of several vortices. When two identical vortices are present there is no restriction on the center-of-mass coordinate but, from the intrinsic viewpoint, we must identify relative configurations differing only by the exchange of the vortices.

In the reduced phase space quantization the relative coordinate $X_{1}^{\text{rel}}$ is restricted to non-negative values. It is important to note that the condition $X_{1}^{\text{rel}} = 0$ does not imply that the vortices are actually coincident. Coincidence also requires that $X_{2}^{\text{rel}} = 0$, so it is reasonable to expect that in the reduced phase space formalism there should be no probability loss at $X_{1}^{\text{rel}} = 0$. Self-adjointness of the Hamiltonian, which guarantees this, will require that a boundary condition be put on the wave functions at $X_{1}^{\text{rel}} = 0$.

The relative configuration space of two vortices in the reduced phase space quantization is the half-line,

$$
Q_{\text{RPS}} = \mathbb{R}^+, \quad (1)
$$

and the Hamiltonian,

$$
H = \ln(x^2 + p^2) + \frac{eB}{4}(x^2 + p^2),
$$

(2)

is a function of the positive operator

$$
\tilde{O} = x^2 + p^2. \quad (3)
$$

Here we use $x$ as the relative coordinate $X_{1}^{\text{rel}}$ and $p$ for its canonical momentum, $-X_{2}^{\text{rel}}$. At the configuration space boundary, $x = 0$, we can use the von Neumann theory$^{[8,12]}$ outlined in section 2 to find the conditions required to guarantee that no probability leaks away.
Because it is a positive operator, it is sufficient to require that $\hat{O}$ be self-adjoint. We start from the domain

$$D^{(0)}(\hat{O}) = \{\psi \in L^2(\mathbb{R}^+) \mid \psi(0) = \psi'(0) = 0\}. \quad (4)$$

Let us denote the eigenvalue of $\hat{O}$ as $2\lambda + 1$. The normalizable eigenfunctions of $\hat{O}^\dagger$ on the half line are $\psi_\lambda(x) = e^{-x^2/2}H_\lambda(x)$, where

$$H_\lambda(x) = \frac{2^\lambda}{\sqrt{\pi}} \left[ \cos\left(\frac{\pi \lambda}{2}\right) \Gamma\left(\frac{\lambda}{2} + \frac{1}{2}\right) \mathbf{1}F_1\left(-\frac{\lambda}{2}, \frac{1}{2}; x^2\right) + 2x \sin\left(\frac{\pi \lambda}{2}\right) \Gamma\left(\frac{\lambda}{2} + 1\right) \mathbf{1}F_1\left(\frac{1}{2} - \frac{\lambda}{2}, \frac{3}{2}; x^2\right) \right] \quad (5)$$

is a combination of confluent hypergeometric functions $\mathbf{1}F_1$ which reduces to the ordinary Hermite polynomials when $\lambda$ is a non-negative integer. For the eigenfunctions $\psi_\lambda(x)$ to be normalizable on the whole line, it is necessary that $\lambda$ be a non-negative integer. There is no such restriction if normalizability on the positive half line is the only requirement. We can see directly from (5) that the deficiency indices of $\hat{O}$ on the domain (4) are $n_+ = n_- = 1$ and thus, that there is a one-parameter family of self-adjoint extensions to $\hat{O}$. These self-adjoint domains are parametrized by a single real number, $\Theta$.

$$D_\Theta(\hat{O}) = \{\psi \in L^2(\mathbb{R}^+) \mid \psi'(0) = \tan\left(\frac{1}{2}\Theta\right) \psi(0)\}, \quad -\pi < \Theta \leq \pi. \quad (6)$$

For each value of $\Theta$ we can solve for the spectrum of $\hat{O}$.

Imposing the self-adjointness condition (6) on the eigenfunctions $\psi_\lambda(x) = e^{-x^2/2}H_\lambda(x)$, we find a relation determining the eigenvalues, $2\lambda + 1$, of $\hat{O}$,

$$2 \tan\left(\frac{\pi \lambda}{2}\right) \frac{\Gamma\left(\frac{\lambda}{2} + 1\right)}{\Gamma\left(\frac{\lambda}{2} + \frac{1}{2}\right)} = \tan\left(\frac{1}{2}\Theta\right). \quad (7)$$

For $\Theta = 0$ or $\pi$, the spectrum of $\hat{O}$ can be determined immediately from eq. (7). The spectrum is $2\lambda + 1$ with $\lambda = 2n, 2n+1$ respectively, $n = 0, 1, 2, \ldots$. Other values of the parameter $\Theta$ yield spectra which are not evenly spaced, although they become so asymptotically in $\lambda$,

$$\lambda \simeq 2n + \frac{\tan\left(\frac{\Theta}{2}\right)}{\pi \sqrt{n}}, \quad n \gg \frac{\tan^2\left(\frac{\Theta}{2}\right)}{\pi^2}, \quad \Theta \neq \pi. \quad (8)$$

We note that only in the two special situations, $\Theta = 0$ or $\pi$, when the spectrum is a subset of the harmonic oscillator spectrum on the whole line, can the wave functions be extended


smoothly to normalizable functions on the whole line. In the case \( \Theta = 0 \) the eigenstates can be extended to even functions while in the case \( \Theta = \pi \) they can be extended to odd functions on the line. These are the cases of bosonic or fermionic vortices, respectively. This is clear because the wave functions are symmetric or antisymmetric under the exchange \( x \to -x \).

There is no such easy characterization of the states for general \( \Theta \), although, for a related problem, Hansson, Leinaas and Myrheim have argued that such states be interpreted as anyonic states.\(^{3,14}\) If we try to extend the general \( \Theta \) states smoothly to the whole line, we find that they are not normalizable.

In the Dirac–Gupta–Bleuler quantization, the relative configuration space is the cone

\[
Q_{\text{DGB}} = \mathbb{R}^2 \backslash \{0\} \sim (X_1^{\text{rel}}, X_2^{\text{rel}}) \sim (-X_1^{\text{rel}}, -X_2^{\text{rel}}), \tag{9}
\]

which we can parametrize by polar coordinates

\[
Q_{\text{DGB}} = \{(r, \phi) \mid 0 < r < \infty, 0 \leq \phi < \pi, \xi = re^{i\phi}\}. \tag{10}
\]

Because the configuration space \( Q_{\text{DGB}} \) is not simply connected, it is not necessary that the state function be single-valued on it, but only that its modulus \( |\Psi(r, \phi)|^2 = |\Psi(r, \phi + \pi)|^2 \) be single-valued. Technically speaking, we require that the state function be a section of a \( U(1) \) bundle over the configuration space \( Q_{\text{DGB}} \). That is,

\[
\Psi(r, \phi + \pi) = e^{i\theta} \Psi(r, \phi), \quad 0 \leq \theta < 2\pi, \tag{11}
\]

or, equivalently, that as a function of complex variables \( \xi \) and \( \bar{\xi} \), it have the monodromy

\[
\Psi(e^{i\pi} \xi, e^{-i\pi} \bar{\xi}) = e^{i\theta} \Psi(\xi, \bar{\xi}). \tag{12}
\]

According to eq. (15), the Hamiltonian is again the same function of an operator \( \hat{\mathcal{O}} \),

\[
\hat{\mathcal{O}} = \hat{\mathcal{O}}_{\text{DGB}} = 2(\xi \partial_\xi - \bar{\xi} \partial_{\bar{\xi}}) = -2i \frac{\partial}{\partial \phi}, \tag{13}
\]

whose self-adjoint extensions we wish to find. We note that, compared to its classical expression, the operator (13) has an ordering ambiguity and that in order for there to be any
possibility of agreement with the reduced phase space method we must add the normal-ordering constant 1 to $\hat{O}_{\text{DGB}}$. The Hilbert space of states now is quite a bit smaller than all normalizable states. Instead of reducing the configuration space, the constraints now determine the physical Hilbert space

$$\mathcal{H}_\text{phys} = \text{Ker}(\hat{\varphi}^{\text{rel}}) = \{\Psi \in L^2(Q_{\text{DGB}}), \Psi(r, \phi + \pi) = e^{i\theta}\Psi(r, \phi) \mid \Psi(0) < \infty, \hat{\varphi}^{\text{rel}}\Psi = 0\}. \quad (14)$$

The inner product is the usual one. We have introduced the condition that the wave function be regular at the origin, though it is not required for normalizability, in order that the operator $\hat{O}$ be positive and the Hamiltonian have real eigenvalues. The condition in eq. (14) is a very strong analyticity requirement. Because of this requirement, the deficiency indices are $n_+ = n_- = 0$. That is, $\hat{O}_{\text{DGB}}$ is essentially self-adjoint on $\mathcal{H}_\text{phys}$. The existence of the anyonic states follows directly from the condition that an exchange of the vortices leave the state vector unchanged up to a phase. Using the normal ordered operator $\hat{O}_{\text{DGB}} = -2i\frac{\partial}{\partial \phi} + 1$, and eq. (12), we find

$$e^{i\pi \hat{J}}\Psi = e^{i\pi(\hat{O}_{\text{DGB}} - 1)/2}\Psi = \Psi(e^{i\pi \xi}, e^{-i\pi \bar{\xi}}) = e^{i\theta}\Psi(\xi, \bar{\xi}), \quad (15)$$

implying directly the spectrum of $\hat{O}_{\text{DGB}}$.

$$\hat{O}_{\text{DGB}} |n\rangle_\theta = \left(2(2n + \frac{\theta}{\pi}) + 1\right) |n\rangle_\theta, \quad n = 0, 1, 2, 3, \ldots. \quad (16)$$

When $\theta = 0$, the states are bosons, and when $\theta = \pi$ the states are fermions. For all values of $\theta$ the eigenvalues are evenly spaced, while the eigenvalues in the reduced phase space quantization are only evenly spaced for bosons or fermions. Up to a normal-ordering constant, the spectrum of $\hat{O}$ is the same in both quantizations as long as the vortices are taken to be either fermions or bosons.
5. Conclusions

In the intrinsic formulation of quantum statistics, we have found that the reduced phase space and Dirac–Gupta–Bleuler quantizations of identical vortices are equivalent (in the sense that the observables have identical spectra) only when the vortices are quantized with conventional Bose or Fermi statistics. In each case the relative Hamiltonian is given by

\[ H = \ln(\hat{O}) + \frac{eB}{4} \hat{O}, \]  

but the specific operators \( \hat{O} \) are different. In the reduced phase space quantization, \( \hat{O} \) has a one parameter set of self-adjoint extensions which determine the vortex statistics, while in the Dirac–Gupta–Bleuler quantization \( \hat{O} \) is essentially self-adjoint and the quantum statistics arise from the topology of the configuration space. The statistics parameter comes in through the choice of a specific \( U(1) \) bundle.

Following Refs. [4] and [14], we might identify vortices in the reduced phase space quantization with the most general boundary conditions as anyons, although this is a delicate issue as we observe that the states in the reduced phase space quantization can be chosen real and are therefore invariant under time reversal, while the states (and the Hamiltonian) in the Dirac–Gupta–Bleuler quantization are not time-reversal invariant because they cannot be real. The time reversal invariance in the reduced phase space quantization results from the fact that the Hamiltonian is a real function of the squares of the dynamical variables and not of the variables themselves. Thus there is a loss of some phase information. In particular, it is impossible to know whether the relative coordinate \( X_{1}^{\text{rel}} \) leads or lags \( X_{2}^{\text{rel}} \) since this information is lost when the circular motion of the vortices is projected onto a single line. Besides preserving phase information, the Dirac–Gupta–Bleuler quantization is also preferable if we wish to interpret anyonic vortices as Laughlin quasiparticles in the fractional quantum Hall effect because there is an exact correspondence between states (17) and the Laughlin state.
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