Majorization bounds for Ritz values of Rayleigh quotients of self-adjoint matrices

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Abstract

In this work we obtain a priori, a posteriori and mixed type upper bounds for the absolute change in Ritz values of Rayleigh quotients of self-adjoint matrices in terms of submajorization relations. Some of our results solve recent conjectures by Knayzev, Argentati and Zhu, that extend several known results for one dimensional subspaces to arbitrary subspaces. In particular, we revisit Nakatsukasa’s version of the tan Θ theorem of Davies and Kahan and obtain an improved version of this result. As a consequence, we obtain improved quadratic a posteriori bounds for the absolute change in Ritz values of Rayleigh quotients.

AMS subject classification: 42C15, 15A60.
Keywords: principal angles, Ritz values, Rayleigh quotients, majorization.

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1 Introduction

The study of sensitivity of Ritz values of Rayleigh quotients of self-adjoint matrices (i.e. the changes in the eigenvalues of compressions of a self-adjoint matrix) is a well established and active research field in applied mathematics [1, 3, 8, 9, 10, 11, 13, 15, 18, 19, 20, 21]. Explicitly, given a $d \times d$ complex self-adjoint matrix $A$ and isometries $X, Y$ of size $d \times k$, with ranges $X$ and $Y$ respectively, we are interested in computing upper and lower bounds for

$$|\lambda(\rho(X)) - \lambda(\rho(Y))| = (|\lambda_i(\rho(X)) - \lambda_i(\rho(Y))|)_{i \in I_k} \in \mathbb{R}_{\geq 0}$$

where $\rho(X) = X^*AX$, $\rho(Y) = Y^*AY$ are $k \times k$ complex self-adjoint matrices known as Rayleigh quotients (RQ) of $A$, and $\lambda(X), \lambda(Y) \in \mathbb{R}^k$ are the eigenvalues (counting multiplicities and arranged in non-increasing order) also known as Ritz values of the corresponding RQ.

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Typically, the bounds for the absolute change in the Ritz values of RQ are obtained in terms of the residuals $R_X = AX - X \rho(X)$ and $R_Y = AY - Y \rho(Y)$ or in terms of the principal angles between subspaces (PABS) denoted by $\Theta(X, Y) \in [0, \pi/2]^k$. Upper bounds are classified according to which parameters are used to bound the change in Ritz values (see [19]). Indeed, the a priori bounds are those obtained in terms of PABS; the a posteriori bounds are those obtained in terms of (singular values of) residuals while the mixed type bounds are obtained in terms of both PABS and residuals. It is worth pointing out that the PABS appearing in a priori bounds may not be readily available in practice. On the other hand, a posteriori bounds are based on computable singular values of residual matrices. Moreover, bounds based on residuals (i.e. both a posteriori and mixed type) are particularly convenient in case one of the spaces, say $X$, is $A$-invariant (as in this case $R_X = 0$), as opposed to (autonomous) a priori bounds.

The abstract matrix analysis formulation of the sensitivity problem stated above makes it possible to apply this theory in a variety of similarly different research areas such as: graph matching [9] in terms of spectral analysis of the graphs; signal distinction in signal processing, where Ritz values serve as harmonic signature to differentiate subspaces; finite element methods (FEM) [8], for approximation of subspaces corresponding to fundamental modes; of course, matrix analysis, e.g. for bounds for eigenvalues after matrix additive perturbations. Also, bounds for changes in Ritz values of RQ play a central role in the analysis of algorithms for simultaneous approximation of eigenvalues based on Rayleigh-Ritz methods (see [16, 17] and the references therein). By now, the role of submajorization in obtaining bounds for the change of Ritz values of RQ (recognized in the seminal paper [9]) is well known; this partial pre-order relation is a powerful tool in this context, as bounds in terms of submajorization imply a whole family of inequalities with respect to unitarily invariant norms and with respect to the class of non-decreasing convex functions ([12]).

In this work we obtain a priori, a posteriori and mixed type upper bounds for the absolute change in Ritz values of RQ of self-adjoint matrices in terms of submajorization. Some of our results solve recent conjectures from [8, 19, 20] that extend several known results for one dimensional subspaces to arbitrary subspaces. In particular, we revisit Nakatsukasa’s version of the tan $\Theta$ theorem [14] of Davies and Kahan [4] and obtain an improved version of this result. We have included some (rather simple) examples to establish comparisons with previous work (for a detailed exposition of the context, previous work, our results and some applications, see Section 3). We will consider further applications of the results herein elsewhere.

The paper is organized as follows. In Section 2 we introduce preliminary results in majorization theory and principal angles between subspaces. In Section 3 we develop our main results; our approach to obtain these results is based on methods from abstract matrix analysis, so we delay the proofs of some technical results until an appendix section. Section 3 is divided in three subsections: in Section 3.1 we prove a mixed type upper bound for the change of the Ritz values of RQ that is conjectured in [20] and show that this bound is sharp. We have also included some comments with a comparison of our results with previous works and with future applications of the results of this subsection. In Section 3.2 we establish a link between the results from Section 3.1 and an a priori upper bound for Ritz values of RQ conjectured from [8]. Although the results in this section are not sharp, they can be applied in quite general situations and they capture the order of approximation conjectured in [8]. In Section 3.3 we revisit Nakatsukasa’s version of the tan $\Theta$ theorem of Davies and Kahan and obtain an improved version of this result; we include an example that shows that this new version of the tan $\Theta$ theorem is sharp in cases in which the classical result is not. As an application, we obtain improved quadratic a posteriori error bounds for Ritz values of RQ. The paper ends with an Appendix (Section 4) in which we include a detailed background on majorization theory and present the proofs of some technical results needed in Section 3.
2 Preliminaries

Throughout our work we use the following

**Notation and terminology.** We let $\mathcal{M}_{k,d}(\mathbb{C})$ be the space of complex $k \times d$ matrices and write $\mathcal{M}_{d,d}(\mathbb{C}) = \mathcal{M}_d(\mathbb{C})$ for the algebra of $d \times d$ complex matrices. We denote by $\mathcal{H}(d) \subset \mathcal{M}_d(\mathbb{C})$ the real subspace of self-adjoint matrices and by $\mathcal{M}_d(\mathbb{C})^+$, the cone of positive semi-definite matrices. Also, $\mathcal{G}(d) \subset \mathcal{M}_d(\mathbb{C})$ and $\mathcal{U}(d)$ denote the groups of invertible and unitary matrices respectively, and $\mathcal{G}(d)^+ = \mathcal{G}(d) \cap \mathcal{M}_d(\mathbb{C})^+$.

For $d \in \mathbb{N}$, let $I_d = \{1, \ldots, d\}$. Given a vector $x \in \mathbb{C}^d$ we denote by $D_x$ the diagonal matrix in $\mathcal{M}_d(\mathbb{C})$ whose main diagonal is $x$. Given $x = (x_i)_{i \in I_d} \in \mathbb{R}^d$ we denote by $x^+ = (x_i^+)_{i \in I_d}$ the vector obtained by rearranging the entries of $x$ in non-increasing order. Moreover, if we assume further that $x_i \neq 0$, for $i \in I_k$, then $x/y = (x_i/y_i)_{i \in I_d}$, where these vectors all lie in $\mathbb{C}^k$. We also use the notation $(\mathbb{R}^d)^+ = \{x \in \mathbb{R}^d : x = x^+\}$ and $(\mathbb{R}^d_{\geq 0})^+ = \{x \in \mathbb{R}^d_{\geq 0} : x = x^+\}$. For $r \in \mathbb{N}$, we let $I_r = \{1, \ldots, r\} \subset \mathbb{R}$.

Given a matrix $A \in \mathcal{H}(d)$ we denote by $\lambda(A) = (\lambda_i(A))_{i \in I_d} \in (\mathbb{R}^d)^+$ the eigenvalues of $A$ counting multiplicities and arranged in non-increasing order. For $B \in \mathcal{M}_d(\mathbb{C})$ we let $s(B) = \lambda(|B|)$ denote the singular values of $B$, i.e. the eigenvalues of $|B| = (B^*B)^{1/2} \in \mathcal{M}_d(\mathbb{C})^+$.

Arithmetic operations with vectors are performed entry-wise i.e., in case $x = (x_i)_{i \in I_k}$, $y = (y_i)_{i \in I_k} \in \mathbb{C}^k$ then $x + y = (x_i + y_i)_{i \in I_k}$, $xy = (x_iy_i)_{i \in I_k}$, and (assuming that $y_i \neq 0$, for $i \in I_k$) $x/y = (x_i/y_i)_{i \in I_k}$, where these vectors all lie in $\mathbb{C}^k$. Moreover, if we assume further that $x, y \in \mathbb{R}^k$ then we write $x \leq y$ whenever $x_i \leq y_i$, for $i \in I_k$.

Next we recall the notion of majorization between vectors, that will play a central role throughout our work.

**Definition 2.1.** Let $x, y \in \mathbb{R}^k$. We say that $x$ is submajorized by $y$, and write $x \prec_w y$, if

$$\sum_{i=1}^j x_i^+ \leq \sum_{i=1}^j y_i^+ \quad \text{for} \quad j \in I_k.$$ 

If $x \prec_w y$ and $\text{tr} x \overset{\text{def}}{=} \sum_{i=1}^k x_i = \text{tr} y$, then we say that $x$ is majorized by $y$, and write $x \prec y$. $\triangle$

There are many fundamental results in matrix theory that are stated in terms of submajorization relations (see for example [2, 6, 12]). In what follows, we mention some elementary properties of submajorization that we will need in Section 3. We will consider some further properties and results on majorization theory in Section 4. Given $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, and $z = (z_i)_{i \in I_k} \in I^k$ we denote $f(z) = (f(z_i))_{i \in I_k} \in \mathbb{R}^k$.

**Remark 2.2.** Let $I \subset \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a convex function. Then,

1. if $x, y \in I^n$ satisfy $x \prec y$ then $f(x) \prec_w f(y)$.

2. if $x, y \in I^n$ only satisfy $x \prec_w y$ but $f$ is further non-decreasing in $I$, then $f(x) \prec_w f(y)$. $\triangle$

**Definition 2.3.** A norm $N$ in $\mathcal{M}_d(\mathbb{C})$ is unitarily invariant (briefly u.i.n.) if $N(UAV) = N(A)$, for every $A \in \mathcal{M}_d(\mathbb{C})$ and $U, V \in \mathcal{U}(d)$. $\triangle$

Well known examples of u.i.n. are the spectral norm $\| \cdot \|_S$ and the $p$-norms $\| \cdot \|_p$, for $p \geq 1$.

**Remark 2.4.** It is well known that (sub)majorization relations between singular values of matrices are intimately related with inequalities with respect to u.i.n.’s. Indeed, given $A, B \in \mathcal{M}_d(\mathbb{C})$ the following statements are equivalent:

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1. For every u.i.n. $N$ in $\mathcal{M}_d(\mathbb{C})$ we have that $N(A) \leq N(B)$.

2. $s(A) \prec_w s(B)$. \hfill $\triangle$

**Principal Angles Between Subspaces.** Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{C}^d$ denote subspaces, with $\dim \mathcal{X} = h$ and $\dim \mathcal{Y} = k$. Let $X \in \mathcal{M}_{d,h}$ and $Y \in \mathcal{M}_{d,k}$ be such that their columns form orthonormal bases of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then, the principal angles between $\mathcal{X}$ and $\mathcal{Y}$, denoted $\pi/2 \geq \Theta_1(\mathcal{X}, \mathcal{Y}) \geq \ldots \geq \Theta_m(\mathcal{X}, \mathcal{Y}) \geq 0$ where $m = \min\{h, k\}$ - are determined by

$$\cos(\Theta_{m-i+1}(\mathcal{X}, \mathcal{Y})) = s_i(X^*Y) \quad \text{for} \quad i \in \mathbb{I}_m.$$ 

We further write $\Theta(\mathcal{X}, \mathcal{Y}) = (\Theta_i(\mathcal{X}, \mathcal{Y}))_{i \in \mathbb{I}_m} \subseteq (\mathbb{R}^m)^\perp$ for the vector of principal angles between $\mathcal{X}$ and $\mathcal{Y}$. Principal angles are a useful tool in describing the relative position and several geometric and metric aspects related with the subspaces $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{C}^d$ (see [4, 5] and the references therein).

### 3 Main results

In this section we develop our main results. The section is divided in three parts; first we prove [20, Conjecture 2.1] which establishes a mixed type bound for the error in the (absolute) change of the Ritz values of Rayleigh quotients (RQ). In the second part, we establish connections between the mixed type bounds of the first section and some a priori bounds for the change of Ritz values conjectured in [8, 10]. Finally we take a closer look at Nakatsuoka’s tan $\Theta$ theorem under relaxed the mixed type bounds of the first section and some a priori bounds for the change of Ritz values of Rayleigh quotients (RQ). In the second part, we establish connections between the mixed type bounds of the first section and some a priori bounds for the change of Ritz values conjectured in [8, 10]. Finally we take a closer look at Nakatsuoka’s tan $\Theta$ theorem under relaxed conditions from [14] and obtain an improved version of this result. As a consequence we obtain quadratic a posteriori error bounds for the change of the Ritz values of RQ that improve several known bounds. Our approach to obtain these results is based on methods from abstract matrix analysis, so we delay the proofs of some technical results until Section [4] where we have also included several classical results of this area that we will refer to in this section.

We begin by introducing the following

**Notation 3.1.** Throughout this section we consider the following notation and terminology:

1. $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{C}^d$ denote two subspaces of dimension $k$. We fix $X, Y \in \mathcal{M}_{d,k}(\mathbb{C})$ such that their columns form orthonormal bases of $\mathcal{X}$ and $\mathcal{Y}$, respectively.

2. $\Theta(\mathcal{X}, \mathcal{Y}) \subseteq (\mathbb{R}^k_{\geq 0})^\perp$ denotes the vector of principal angles between the subspaces $\mathcal{X}$ and $\mathcal{Y}$; in this case, $\cos(\Theta^\top(\mathcal{X}, \mathcal{Y})) = s(X^*Y) = (s_1(X^*Y), \ldots, s_k(X^*Y)) \subseteq (\mathbb{R}^k_{\geq 0})^\perp$.

3. For a (fixed) self-adjoint $A \in \mathcal{H}(d)$ we set $\rho(X) = X^*AX \in \mathcal{M}_k(\mathbb{C})$, $R_X = AX - X\rho(X) \in \mathcal{M}_{d,k}(\mathbb{C})$ and similarly $\rho(Y)$ and $R_Y$ for $Y$. Notice that $R_X = AX - XX^*AX = AX - P_\mathcal{X}AX = (I - P_\mathcal{X})AX = P_{\mathcal{X}^\perp}AX \in \mathcal{M}_{d,k}(\mathbb{C})$, where $P_\mathcal{X} \in \mathcal{M}_d(\mathbb{C})$ denotes the orthogonal projection onto $\mathcal{X}$ and $\mathcal{X}^\perp$ denotes the orthogonal complement of $\mathcal{X}$. We consider similar notation and identities for $\mathcal{Y}$.

4. Let $X \subseteq \mathcal{M}_{d,d-k}(\mathbb{C})$ such that their columns form an o.n.b. of $\mathcal{X}^\perp$. Then the matrix representation of $A$ in the o.n.b. given by the columns of $X$ and $X^\perp$ has the form $A = \begin{bmatrix} \rho(X) & R_{X^\perp} \cr R_X X^\perp & \rho(X^\perp) \end{bmatrix} \begin{bmatrix} \mathcal{X}^\perp \cr \mathcal{X} \end{bmatrix}$. Note that, since $R_X = (I - P_\mathcal{X})R_X$, then $s(R_X) = s(X^\perp R_X)$, so that we can think of $R_X$ as the $(2,1)$-block in the block matrix representation of $A$ as above. \hfill $\triangle$
3.1 Rayleigh-Ritz majorization error bounds of the mixed type

We consider Notations 3.1 moreover, in this subsection we further assume that $\mathcal{X}$ and $\mathcal{Y}$ are such that $\Theta_1(\mathcal{X}, \mathcal{Y}) < \frac{\pi}{2}$ that is, that $X^*Y \in \mathcal{G}(k)$ is invertible.

Our first result concerns a submajorization error bound for the distance of eigenvalue lists of self-adjoint matrices, within the context of matrix analysis theory.

**Theorem 3.2.** Let $C, D \in \mathcal{H}(k)$ and let $T \in \mathcal{G}(k)$. Then,

$$|\lambda(C) - \lambda(D)| \prec_w s(T^{-1}) s(CT - TD).$$

**Proof.** See the Appendix (Section 4).

The following result is [20] Conjecture 2.1] (see also Corollary 3.4 below).

**Theorem 3.3.** Consider Notations 3.1 and assume that $\Theta_1(\mathcal{X}, \mathcal{Y}) < \frac{\pi}{2}$. We have that

$$|\lambda(\rho(X)) - \lambda(\rho(Y))| \prec_w \frac{s(P_Y R_X) + s(P_X R_Y)}{\cos(\Theta(\mathcal{X}, \mathcal{Y}))}$$

and

$$|\lambda(\rho(X)) - \lambda(\rho(Y))| \prec_w [s(P_{X+Y} R_X) + s(P_{X+Y} R_Y)] \tan(\Theta(\mathcal{X}, \mathcal{Y})).$$

**Proof.** Set $T = X^*Y$ and notice that, since $\Theta_1(\mathcal{X}, \mathcal{Y}) < \frac{\pi}{2}$, $T \in \mathcal{M}_k(\mathbb{C})$ is invertible. Using Theorem 3.2 we get that

$$|\lambda(\rho(X)) - \lambda(\rho(Y))| \prec_w s(T^{-1}) s(\rho(X)T - T\rho(Y)),$$

where $\rho(X) = X^*AX$, $\rho(Y) = Y^*AY \in \mathcal{H}(k)$. By construction we have that

$$s(T^{-1}) = \frac{1}{\cos(\Theta(\mathcal{X}, \mathcal{Y}))} \in (\mathbb{R}_{>0})^\frac{1}{2}.$$

Arguing as in [20] Thm 4.1] we notice that

$$\rho(X)T - T\rho(Y) = X^*A X^*X^*Y - X^*YY^*AY = X^*A P_X Y - X^*P_YAY$$

$$= X^*A(I - P_{X^*})Y - X^*(I - P_{Y^*})AY$$

$$= X^*AY - X^*AP_{X^*}Y - X^*AY + X^*P_{Y^*}AY = -X^*AP_{X^*}Y + X^*P_{Y^*}AY.$$

Using that $s(C) = s(C^*)$ for $C \in \mathcal{M}_k(\mathbb{C})$, we see that

$$s(X^*A P_{X^*}Y) = s(Y^*P_{X^*}AX) = s(P_Y P_{X^*}AX) = s(P_Y R_X) \in (\mathbb{R}_{>0})^\frac{1}{2}.$$

Analogously $s(X^*P_{Y^*}AY) = s(P_X R_Y)$. The previous facts together with the sub-additivity property of taking singular values (item 1. in Theorem 4.1) imply that

$$s(\rho(X)T - T\rho(Y)) = s(-X^*A P_{X^*}Y + X^*P_{Y^*}AY) \prec_w s(P_X R_Y) + s(P_Y R_X).$$

Now, if we apply Eqs. 6 and 7 in Eq. 11, together with item 4 in Lemma 4.3 (about majorization of entrywise products), we get the relation in Eq. 12.

In order to show Eq. 13 we point out that by [20] Lemma 4.1] we get that

$$s(P_X R_Y) \prec_w s(P_{X+Y} R_Y) s(\Theta(\mathcal{X}, \mathcal{Y})).$$

Since the entries of these vectors are ordered downwards, by Lemma 4.3 we can deduce that

$$s(P_X R_Y) + s(P_Y R_X) \prec_w s(P_{X+Y} R_Y) s(\Theta(\mathcal{X}, \mathcal{Y})) + s(P_{X+Y} R_X) s(\Theta(\mathcal{X}, \mathcal{Y})).$$

Hence, using Eqs. 12 and 13 together with Lemma 4.3 we see that Eq. 3 holds. 

\[\square\]
Hence, again simple computations show that
\[ |\lambda(\rho(X)) - \lambda(\rho(Y))| \leq \frac{s(P_X R_Y)}{\cos(\Theta(X, Y))} \]
and
\[ |\lambda(\rho(X)) - \lambda(\rho(Y))| \leq \frac{s(P_{\lambda} R_Y)}{\cos(\Theta(X, Y))} \tan(\Theta(X, Y)) . \]

**Corollary 3.4.** Consider Notations \ref{notations} and assume that \( \Theta_1(X, Y) < \frac{\pi}{2} \). If we further assume that \( X \) is \( A \)-invariant then

\[ |\lambda(\rho(X)) - \lambda(\rho(Y))| \leq \frac{s(P_X R_Y)}{\cos(\Theta(X, Y))} \]

\[ |\lambda(\rho(X)) - \lambda(\rho(Y))| \leq \frac{s(P_{\lambda} R_Y)}{\cos(\Theta(X, Y))} \tan(\Theta(X, Y)) . \]

**Proof.** In case \( X \) is \( A \)-invariant notice that \( R_X = 0 \). The result now follows from Theorem \ref{thm:main}. \( \square \)

It is natural to wonder whether we can improve the bounds in the previous results. As shown in the following example, the submajorization bounds in Theorem \ref{thm:main} and Corollary \ref{cor:main} are sharp.

**Example 3.5.** Let \( \lambda = (a, b, c, d) \in \mathbb{R}^4 \), where \( a < b < c < d \), and consider \( A \in \mathcal{H}(4) \) given by \( A = D_\lambda \), i.e. \( A \) is the diagonal matrix with main diagonal \( \lambda \).

Let \( X \) be the \( A \)-invariant subspace \( X = \text{span}\{e_1, e_2\} \) spanned by the first two elements of the canonical basis of \( \mathbb{C}^4 \). For \( \theta \in (0, \pi/2) \) let \( f_\theta = \cos \theta e_2 + \sin \theta e_3 \) and set \( Y_\theta = \text{span}\{e_1, f_\theta\} \). Then, the principal angles are given by \( \Theta(X, Y_\theta) = (\theta, 0) \). Let

\[
X = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \quad X_\perp = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix} \quad \text{and} \quad Y_\theta = \begin{pmatrix}
1 & 0 \\
0 & \cos \theta \\
0 & \sin \theta \\
0 & 0
\end{pmatrix}.
\]

It is straightforward to check that \( \lambda(X^*AX) = (b, a) \) and that \( \lambda(Y_\theta^*AY_\theta) = (b \cos^2 \theta + c \sin^2(\theta), a) \). Again, simple computations show that

\[
R_{Y_\theta} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & (b - c) \cos \theta \sin^2 \theta & 0 & 0 \\
0 & (c - b) \cos^2 \theta \sin \theta & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad P_X R_{Y_\theta} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & (b - c) \cos \theta \sin^2 \theta & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Hence, \( s(P_X R_{Y_\theta}) = ((b - c) \cos \theta \sin^2 \theta, 0) \). Now,

\[ |\lambda(X^*AX) - \lambda(Y_\theta^*AY_\theta)| = ((b - c) \sin^2 \theta, 0) \quad \text{and} \quad \frac{s(P_X R_{Y_\theta})}{\cos(\Theta(X, Y_\theta))} = ((b - c) \sin^2 \theta, 0) . \]

That is, Eq. \ref{eq:corollary} in Corollary \ref{cor:main} becomes an equality in this case. This also shows that Eq. \ref{eq:main} is sharp, since Eq. \ref{eq:thm} above is a particular case (when \( X \) is \( A \)-invariant).

Notice that \( X + Y_\theta = \text{span}\{e_1, e_2, e_3\} \). Since \( P_{X+Y_\theta} R_{Y_\theta} = R_{Y_\theta} \) and \( s(R_{Y_\theta}) = ((b - c) \cos \theta \sin \theta, 0) \) then

\[ s(P_{X+Y_\theta} R_{Y_\theta}) \tan(\Theta(X, Y_\theta)) = ((c - b) \sin^2 \theta, 0) . \]

By Eqs. \ref{eq:corollary} and \ref{eq:thm} we now see that Eq. \ref{eq:main} in Corollary \ref{cor:main} becomes an equality in this case. This also shows that Eq. \ref{eq:main} is sharp, since Eq. \ref{eq:thm} above is a particular case (when \( X \) is \( A \)-invariant). \( \triangle \)

**Remark 3.6** (Relations between our work and previous results). In the vector case, that is when \( X \) and \( Y \) are one dimensional spaces, Theorem \ref{thm:main} implies the upper bounds in \cite[Theorem 3.7]{[21]}, which is one the main results of that work (see also Corollary \ref{cor:main} and Remark \ref{rem:main}).
In [20] Knyazev and Zhu obtain several bounds for the absolute change of the Ritz values of Rayleigh ratios. Using Notations 3.1, the authors show (see [20, Theorem 4.2 and Corollary 4.4]) that

\[ |\lambda(\rho(X)) - \lambda(\rho(Y))|^2 \lesssim_w \frac{(s(P_X R_X) + s(P_Y R_Y))^2}{\cos^2(\Theta(X', Y))} \quad \text{and} \quad \frac{(s(P_{X+Y} R_X) + s(P_{X+Y} R_Y))^2 \tan^2(\Theta(X', Y))}. \]

(13)

Using the fact that \( f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) given by \( f(x) = x^2 \) is an increasing and convex function, then Remark 2.2 shows that Eqs. (13) and (14) follow from Eqs. (2) and (3) from Theorem 3.3. Similarly, using that \( \cos \Theta_i(X, Y) = \cos \Theta_{\max}(X, Y) \leq \cos \Theta_i(X', Y'), \) for \( i \in I_k, \) we get that Theorem 3.3 implies [20, Theorems 4.1, 4.3].

In [20] the authors show that their results can be applied in several situations such as: first order and quadratic a posteriori majorization bounds; bounds for eigenvalues after matrix additive perturbations. The previous remarks show that our bounds can also be applied in these settings. Moreover, Theorem 3.3 allows to formalize the arguments related with bounds for eigenvalues after matrix additive perturbations, and in particular with bounds for eigenvalues after discarding off-diagonal blocks from [20, Section 5] (see the detailed discussion there).

\[ \Delta \]

The bounds in Theorem 3.3 can be used to perform a detailed analysis and obtain better convergence rates for iterative algorithms related with the Rayleigh-Ritz method (see [16, 17, 21]). We will consider such applications elsewhere.

3.2 Applications: a priori majorization error bounds for Ritz values

In this section we establish a link between the majorization error bounds of the mixed type obtained in the previous section and some a priori majorization error bounds considered in [8, 10].

**Definition 3.7.** Let \( A \in \mathcal{H}(d) \) and let \( Z \subset \mathbb{C}^d \) be a subspace with \( \dim Z = p \). We consider the (spectral) spread of \( A \) relative to \( Z \), denoted \( \operatorname{Spr}(A, Z) \), given by

\[ \operatorname{Spr}(A, Z) = \lambda(A_Z) - \lambda^\uparrow(A_Z) = (\lambda_i(A_Z) - \lambda_{p-i+1}(A_Z))_{i \in I_p} \in (\mathbb{R}^p)^\uparrow, \]

where \( A_Z = P_Z A|_Z \in L(Z) \) is a self-adjoint operator. In case \( Z = \mathbb{C}^d \) then we write \( \operatorname{Spr}(A, \mathbb{C}^d) = \operatorname{Spr}(A) \).

**Remark 3.8.** Let \( A \in \mathcal{H}(d) \) and let \( X, Y \subset \mathbb{C}^d \) with \( \dim(X) = \dim(Y) = k \). Denote by \( p = \dim X + Y \). In what follows we consider the vector

\[ \operatorname{Spr}(A, X + Y) \sin(\Theta(X, Y)) = ((\lambda_i(A_{X+Y}) - \lambda_{p-i+1}(A_{X+Y})) \sin(\Theta_i(X, Y)))_{i \in I_k}. \]

Notice that, by construction, \( \operatorname{Spr}(A, X + Y) \sin(\Theta(X, Y)) \in (\mathbb{R}_{\geq 0})^\downarrow \) (see [20]).

**Remark 3.9.** (A priori error bounds for changes of Ritz values: conjectures and previous work). Let \( A \in \mathcal{H}(d) \) and let \( X, Y \subset \mathbb{C}^d \) with \( \dim(X) = \dim(Y) = k \). In [8] the authors conjectured that, in general, the following submajorization bound for the Ritz values of Rayleigh quotients holds:

\[ |\lambda(\rho(X)) - \lambda(\rho(Y))| \lesssim_w \operatorname{Spr}(A, X + Y) \sin(\Theta(X, Y)). \]

(15)

Moreover, in case \( X \) is \( A \)-invariant, the authors conjectured that

\[ |\lambda(\rho(X)) - \lambda(\rho(Y))| \lesssim_w \operatorname{Spr}(A, X + Y) \sin(\Theta(X, Y))^2. \]

(16)

These conjectures are natural extensions of results from [10] (that were obtained for \( k = 1 \)). Although [8, Conjecture 2.1] claims the validity of Eqs. (15) and (16) for arbitrary subspaces \( X \) and
\( \mathcal{Y} \) such that \( \dim \mathcal{X} = \dim \mathcal{Y} \), such bounds would become relevant in the particular case when the subspace \( \mathcal{Y} \) is a (small) perturbation of the subspace \( \mathcal{X} \). In this case, the validity of Eqs. (15) and (16) would reveal the different orders of approximation of \( \rho(X) \) by \( \rho(Y) \) in terms of PABS as well as in terms of the spectral spread of \( A \) (i.e. when considering \( A \) as well as \( \mathcal{X} \) and \( \mathcal{Y} \) as variables). Notice that these results would have immediate applications in the study of numerical stability and convergence of iterative methods related with the Rayleigh-Ritz type algorithms.

In [8, Theorem 2.1.] the authors showed that, in general,

\[
|\lambda(\rho(X)) - \lambda(\rho(Y))| \sim_w (\lambda_{\max}(A_{\mathcal{X} + \mathcal{Y}}) - \lambda_{\min}(A_{\mathcal{X} + \mathcal{Y}})) \sin(\Theta(\mathcal{X}, \mathcal{Y})), \tag{17}
\]

while, in case \( \mathcal{X} \) is \( A \)-invariant,

\[
|\lambda(\rho(X)) - \lambda(\rho(Y))| \sim_w (\lambda_{\max}(A_{\mathcal{X} + \mathcal{Y}}) - \lambda_{\min}(A_{\mathcal{X} + \mathcal{Y}})) \sin(\Theta(\mathcal{X}, \mathcal{Y}))^2, \tag{18}
\]

where \( A_{\mathcal{X} + \mathcal{Y}} = P_{\mathcal{X} + \mathcal{Y}} A|_{\mathcal{X} + \mathcal{Y}} \in L(\mathcal{X} + \mathcal{Y}) \); moreover, in [8, Theorem 2.2.] they showed that in the particular case in which \( \mathcal{X} \) is the \( A \)-invariant subspace corresponding to the \( k \) largest eigenvalues of \( A \), then

\[
0 \leq \lambda(\rho(X)) - \lambda(\rho(Y)) \sim_w (\lambda_i(A_{\mathcal{X} + \mathcal{Y}}) - \lambda_{\min}(A_{\mathcal{X} + \mathcal{Y}}))_{i \in \mathbb{I}_k} \sin(\Theta(\mathcal{X}, \mathcal{Y}))^2. \tag{19}
\]

Notice that, Eq. (19) is a stronger bound than that in Eq. (18); yet, it is weaker than the bound conjectured in Eq. (16), since \( \text{Spr}(A, \mathcal{X} + \mathcal{Y}) \leq \lambda_i(A_{\mathcal{X} + \mathcal{Y}}) - \lambda_{\min}(A_{\mathcal{X} + \mathcal{Y}}) \), for \( i \in \mathbb{I}_k \).

In what follows we apply Theorem 3.3 and obtain some results related with the conjectures from [8] described in Eqs. (15) and (16). In order to obtain these results, we take a closer look at the quantity \( s(P_{\mathcal{X}} R_Y) \) for arbitrary \( \mathcal{X} \) and \( \mathcal{Y} \), as well as in the case where \( \mathcal{X} \) is \( A \)-invariant.

**Proposition 3.10.** Let \( A \in \mathcal{H}(d) \) and let \( \mathcal{X}, \mathcal{Y} \subset \mathbb{C}^d \) with \( \dim(\mathcal{X}) = \dim(\mathcal{Y}) = k \). Then

\[
s(P_{\mathcal{X}} R_Y) \sim_w \text{Spr}(A, \mathcal{X} + \mathcal{Y}) \sin(\Theta(\mathcal{X}, \mathcal{Y})). \tag{20}
\]

**Proof.** See the Appendix (Section 4).

**Theorem 3.11.** Let \( A \in \mathcal{H}(d) \), \( \mathcal{X}, \mathcal{Y} \subset \mathbb{C}^d \) subspaces, \( \dim(\mathcal{X}) = \dim(\mathcal{Y}) = p \). If \( \Theta_1(\mathcal{X}, \mathcal{Y}) < \frac{\pi}{2} \), then

\[
|\lambda(\rho(X)) - \lambda(\rho(Y))| \sim_w \frac{2 \text{Spr}(A, \mathcal{X}, \mathcal{Y}) \sin(\Theta(\mathcal{X}, \mathcal{Y}))}{\cos(\Theta(\mathcal{X}, \mathcal{Y}))} = 2 \text{Spr}(A, \mathcal{X}, \mathcal{Y}) \tan(\Theta(\mathcal{X}, \mathcal{Y})). \tag{21}
\]

**Proof.** Theorem 3.3 establishes that

\[
|\lambda(\rho(X)) - \lambda(\rho(Y))| \sim_w \frac{s(P_{\mathcal{X}} R_Y) + s(P_Y R_X)}{\cos(\Theta(\mathcal{X}, \mathcal{Y}))}.
\]

Proposition 3.10 together with Lemma 4.3 imply that

\[
\frac{s(P_{\mathcal{X}} R_Y) + s(P_Y R_X)}{\cos(\Theta(\mathcal{X}, \mathcal{Y}))} \sim_w \frac{2 \text{Spr}(A, \mathcal{X}, \mathcal{Y}) \sin(\Theta(\mathcal{X}, \mathcal{Y}))}{\cos(\Theta(\mathcal{X}, \mathcal{Y}))}.
\]

The result follows from combining these two last inequalities. □

The next result illustrates the quadratic dependance of \( s(P_{\mathcal{X}} R_Y) \) from \( \sin(\Theta(\mathcal{X}, \mathcal{Y})) \) in case \( \mathcal{X} \) is \( A \)-invariant.
Proposition 3.12. Let $A \in \mathcal{H}(d)$, $\mathcal{X}, \mathcal{Y} \subset \mathbb{C}^d$ subspaces with $\dim(\mathcal{X}) = \dim(\mathcal{Y}) = k$. Assume that $\mathcal{X}$ is $A$-invariant. Then,

$$s(P_X R_Y) \prec_w 2 (\lambda_i(A_{X+Y}) - \lambda_{\min}(A_{X+Y}))_{i \in \mathbb{I}_k} \sin^2(\Theta(\mathcal{X}, \mathcal{Y})).$$

(22)

Proof. See the Appendix (Section 4).

Theorem 3.13. Let $A \in \mathcal{H}(d)$, $\mathcal{X}, \mathcal{Y} \subset \mathbb{C}^d$ subspaces, $\dim(\mathcal{X}) = \dim(\mathcal{Y}) = k$, and assume that $\mathcal{X}$ is $A$-invariant. If $\Theta_1(\mathcal{X}, \mathcal{Y}) < \frac{\pi}{2}$, then

$$|\lambda(\rho(\mathcal{X})) - \lambda(\rho(\mathcal{Y}))| \prec_w \frac{2}{\cos(\Theta_1(\mathcal{X}, \mathcal{Y}))} \frac{\lambda_i(A_{X+Y}) - \lambda_{\min}(A_{X+Y})_{i \in \mathbb{I}_k}}{\sin^2(\Theta(\mathcal{X}, \mathcal{Y}))}.$$ 

(23)

Proof. The result follows from Corollary 3.14 and Proposition 3.12 with an argument similar to that in the proof of Theorem 3.11 above.

Corollary 3.14. Let $A \in \mathcal{H}(d)$, $\mathcal{X}, \mathcal{Y} \subset \mathbb{C}^d$ subspaces, $\dim(\mathcal{X}) = \dim(\mathcal{Y}) = k$. If $\Theta_1(\mathcal{X}, \mathcal{Y}) < \frac{\pi}{2}$, then

$$|\lambda(\rho(\mathcal{X})) - \lambda(\rho(\mathcal{Y}))| \prec_w \frac{2}{\cos(\Theta_1(\mathcal{X}, \mathcal{Y}))} \frac{\lambda_i(A_{X+Y}) - \lambda_{\min}(A_{X+Y})_{i \in \mathbb{I}_k}}{\sin^2(\Theta(\mathcal{X}, \mathcal{Y}))}.$$ 

(24)

If we assume further that $\mathcal{X}$ is $A$-invariant, then

$$|\lambda(\rho(\mathcal{X})) - \lambda(\rho(\mathcal{Y}))| \prec_w \frac{2}{\cos(\Theta_1(\mathcal{X}, \mathcal{Y}))} \frac{\lambda_i(A_{X+Y}) - \lambda_{\min}(A_{X+Y})_{i \in \mathbb{I}_k}}{\sin^2(\Theta(\mathcal{X}, \mathcal{Y}))}.$$ 

(25)

We end this section with some remarks concerning the relation between Theorems 3.11 and 3.13, Corollary 3.14 and the conjectured bounds in Eqs. (15) and (16). As already mentioned in Remark 3.9, the bounds in Eqs. (15) and (16) would be particularly relevant in case $\mathcal{Y}$ is a (small) perturbation of $\mathcal{X}$ or, in other terms, in case that $\mathcal{X}$ and $\mathcal{Y}$ are close subspaces (e.g. $\Theta_1(\mathcal{X}, \mathcal{Y})$ is small). In order to simplify the discussion, let us assume that $\Theta_1(\mathcal{X}, \mathcal{Y}) \leq \pi/4$. We point out that this assumption holds in a number of significant situations (see for example [20, Section 5.2]). In this case, if $A \in \mathcal{H}(d)$ then Corollary 3.14 implies that

$$|\lambda(\rho(\mathcal{X})) - \lambda(\rho(\mathcal{Y}))| \prec_w (2 \sqrt{2}) \text{Spr}(A, \mathcal{X}, \mathcal{Y}) \sin(\Theta(\mathcal{X}, \mathcal{Y})).$$

(26)

Hence, under the present assumptions ($\Theta_1(\mathcal{X}, \mathcal{Y}) \leq \pi/4$), the upper bound in Eq. (26) has the conjectured order of approximation (when considering $A$ as well as the subspaces $\mathcal{X}$ and $\mathcal{Y}$ as variables), up to the constant factor $2 \sqrt{2}$.

If we further assume that $\mathcal{X}$ is $A$-invariant then by the same result we get that

$$|\lambda(\rho(\mathcal{X})) - \lambda(\rho(\mathcal{Y}))| \prec_w (2 \sqrt{2}) \frac{1}{\cos(\Theta_1(\mathcal{X}, \mathcal{Y}))} \frac{\lambda_i(A_{X+Y}) - \lambda_{\min}(A_{X+Y})_{i \in \mathbb{I}_k}}{\sin^2(\Theta(\mathcal{X}, \mathcal{Y}))}.$$ 

(27)

Again, the upper bound in Eq. (27) has the conjectured order of approximation (when considering $A$ as well as the subspaces $\mathcal{X}$ and $\mathcal{Y}$ as variables), up to the constant factor $2 \sqrt{2}$. Moreover, notice that this bound holds for an arbitrary $A$-invariant subspace $\mathcal{X}$ (as opposed the bound in Eq. (19) from [5] that is shown to hold for special choices of $A$-invariant subspaces $\mathcal{X}$).
3.3 The tan\(\Theta\) theorem revisited: improved quadratic a posteriori error bounds

In this section we revisit Nakatsukasa’s extension of Davies-Kahan’s tan(\(\Theta\)) theorem. Our motivation is the study of an improved version of this result conjectured in [20] (see Corollary 3.22 below). We first recall the separation hypothesis for Nakatsukasa’s result. As before, in this section we adopt Notation 3.1.

**Definition 3.15.** Let \(A \in \mathcal{H}(d)\) and let \(\mathcal{X}, \mathcal{Y} \subset \mathbb{C}^d\) be subspaces with \(\dim \mathcal{X} = \dim \mathcal{Y} = k\), such that \(\mathcal{X}\) is \(A\)-invariant. Let \([X, X^\perp], [Y, Y^\perp] \in \mathcal{U}(d)\) be unitary matrices such that the columns of (the \(d \times k\) matrices) \(X\) and \(Y\) form ONB’s of \(\mathcal{X}\) and \(\mathcal{Y}\) respectively. Given \(\delta > 0\) we say that \((A, \mathcal{X}, \mathcal{Y}, \delta)\) satisfies the Davies-Kahan-Nakatsukasa (DKN) separation property if there exist \(a \leq b\) such that

1. \(\lambda_i(X^\perp A X) = \lambda_i(P_{X^\perp} A P_{X^\perp}) \in [a, b], \quad \text{for } i \in \mathbb{I}_{d-k};\)

2. \(\lambda_i(Y^* A Y) = \lambda_i(P_Y A P_Y) \in (\infty, a - \delta] \cup [b + \delta, \infty), \quad \text{for } i \in \mathbb{I}_k.\)

Next we state Nakatsukasa’s tan\(\Theta\) theorem under relaxed conditions.

**Theorem 3.16** ([14]). Let \(A \in \mathcal{H}(d)\), \(\mathcal{X}, \mathcal{Y} \subset \mathbb{C}^d\) and \(\delta > 0\) be such that \((A, \mathcal{X}, \mathcal{Y}, \delta)\) satisfies the DKN separation property. Then, \(\Theta_1(\mathcal{X}, \mathcal{Y}) < \pi/2\) and we have that

\[\delta \|\tan(\Theta(\mathcal{X}, \mathcal{Y}))\| \leq \|R_\mathcal{Y}\|,\]

for every unitarily invariant norm \(\|\cdot\|\). Equivalently, \(\delta \tan(\Theta(\mathcal{X}, \mathcal{Y})) \prec_{\text{w}} s(R_\mathcal{Y}).\)

**Remark 3.17.** Let \(A \in \mathcal{H}(d)\), \(\mathcal{X}, \mathcal{Y} \subset \mathbb{C}^d\) and \(\delta > 0\) be such that \((A, \mathcal{X}, \mathcal{Y}, \delta)\) satisfies the DKN separation property. Then, Theorem 3.16 requires the knowledge of the full matrix \(A\) in order to bound the (norm of the) vector \(\tan(\Theta(\mathcal{X}, \mathcal{Y}))\) from above. Instead, it would be interesting to bound the vector \(\tan(\Theta(\mathcal{X}, \mathcal{Y}))\) from above (only) in terms of the self-adjoint operator \(A_{\mathcal{X} + \mathcal{Y}} = P_{\mathcal{X} + \mathcal{Y}} A P_{\mathcal{X} + \mathcal{Y}} \in \mathcal{L}(\mathcal{X} + \mathcal{Y}).\) In the next result we show that the tan\(\Theta\) theorem mentioned above allow to obtain such a result. Moreover, we will also see that it is possible to describe separation hypothesis for \((A_{\mathcal{X} + \mathcal{Y}}, \mathcal{X} + \mathcal{Y})\), that are more general than the DKN separation hypothesis for \((A, \mathcal{X}, \mathcal{Y})\), for which the tan\(\Theta\) theorem holds; arguing in terms of interlacing inequalities, we can show that these separation hypothesis on \(A_{\mathcal{X} + \mathcal{Y}}\) provide better separation constants than the DKN conditions on the complete matrix \(A.\)

We formalize the content of the previous remark - with a small variation on the notation - in the following result. First, we recall some facts related with the relative position of two subspaces.

**Remark 3.18.** Let \(\mathcal{X}, \mathcal{Y} \subset \mathbb{C}^d\) be two subspaces with \(\dim \mathcal{X} = \dim \mathcal{Y} = k\). Consider the mutually orthogonal subspaces

\[\mathcal{H}_{00} = \mathcal{X}^\perp \cap \mathcal{Y}^\perp, \quad \mathcal{H}_{10} = \mathcal{X} \cap \mathcal{Y}^\perp, \quad \mathcal{H}_{01} = \mathcal{X}^\perp \cap \mathcal{Y}, \quad \mathcal{H}_{11} = \mathcal{X} \cap \mathcal{Y},\]

and \(\mathcal{H}_g = \mathbb{C}^d \ominus (\mathcal{H}_{00} \oplus \mathcal{H}_{10} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_{11})\) which is called the generic part of the pair \((\mathcal{X}, \mathcal{Y})\). Each of these five (possible zero) subspaces reduces each projection \(P_X\) and \(P_Y\). Moreover, the subspaces \(\mathcal{X}_g = \mathcal{X} \cap \mathcal{H}_g\) and \(\mathcal{Y}_g = \mathcal{Y} \cap \mathcal{H}_g\) are in generic position so that \(\mathcal{H}_g = \mathcal{X}_g + \mathcal{Y}_g.\) For details of this well known construction and several fundamental results see [5].

**Theorem 3.19.** Let \(A \in \mathcal{H}(d)\), \(\mathcal{X}, \mathcal{Y} \subset \mathbb{C}^d\) be such that \(\dim \mathcal{X} = \dim \mathcal{Y} = k\). Let \(A_{\mathcal{X} + \mathcal{Y}} = S^* A S \in \mathcal{H}(p)\), where \(S \in \mathcal{M}_{d,p}(\mathbb{C})\) is such that its columns form and ONB for \(\mathcal{X} + \mathcal{Y}\). Then,

1. If \(\delta > 0\) is such that \((A, \mathcal{X}, \mathcal{Y}, \delta)\) satisfies the DKN separation property then there exists \(\delta' \geq \delta\) such that \((A_{\mathcal{X} + \mathcal{Y}}, S^* \mathcal{X}, S^* \mathcal{Y}, \delta')\) satisfies the DKN separation property.
2. If \( \delta' > 0 \) is such that \((A_{X+Y}, S^*X, S^*Y, \delta')\) satisfies the DKN separation property, then
\[
\delta' \| \tan(\Theta(X, Y)) \| \leq \| A_{X+Y} Y_S - Y_S (Y_S^* A_{X+Y} Y_S) \| = \| P_{X+Y} R_Y \| ,
\]
for every unitarily invariant norm \( \| \cdot \| \), where \( Y_S = S^*Y \in M_{p,k}(C) \).

**Proof.** We first show item 1. Let \( X, Y \in M_{d,k}(C) \) be such that their columns form orthonormal bases of \( X \) and \( Y \), respectively. By hypothesis, there exist \( a \leq b \) such that
\[
\lambda_i(X_i^*AX_i) \in [a, b] \quad \text{for} \quad i \in \mathbb{I}_{d-k} \quad \text{and} \quad \lambda_i(Y_i^*AY) \in (\infty, a - \delta] \cup [b + \delta, \infty) \quad \text{for} \quad i \in \mathbb{I}_k ,
\]
where \( X_i \in \mathcal{M}_{d,d-k}(C) \) is such that its columns for an ONB for \( X \). Let \( Z = X + Y \) and notice that \( S \in \mathcal{M}_{d,p}(C) \) is an isometry from \( C^p \) onto \( Z \). Moreover, the matrix \( S^*AS \in \mathcal{H}(p) \) is such that \( S^*X, S^*Y \in \mathcal{M}_{p,k} \) are isometries from \( C^k \) onto \( S^*X, S^*Y \subseteq C^p \), respectively. Consider the mutually orthogonal subspaces
\[
H_{X_11} = X \cap Y , \quad X_g = H_g \cap X \quad \text{and} \quad X_{g_2} = H_g \cap X_g ,
\]
where \( H_g \) is the subspace of \( C^d \) corresponding to the generic part of the pair \((X, Y)\) (see Remark 4.18). By Theorem 4.10 we have that \( \Theta_1(X, Y) < \pi/2 \) so then, \( X^+ \cap Y = \{0\} = X \cap Y \). Thus,
\[
X = H_{X_11} \oplus X_g , \quad Z = H_{X_11} \oplus X_g \oplus X_{g_2} \quad \text{and} \quad X_{g_2} = Z \ominus X.
\]

Let \( X' \in \mathcal{M}_{d,(p-k)}(C) \) be such that its columns form an orthonormal basis of \( X_{g_2} \subseteq X \). Then, \( X_S = S^*X \in \mathcal{M}_{p,(p-k)}(C) \) is an isometry from \( C^{p-k} \) onto \( S^*X_{g_2} = (S^*X)^{\perp} \subseteq C^p \). In order to check the DKN separation property for \((A_{X+Y}, S^*X, S^*Y)\) we consider the eigenvalues of
\[
(X^*_S)^*(S^*AS)X_S = (X')^* S S^*AS S^*X' = (X')^* AX' \in \mathcal{H}(p-k) ,
\]
since \( SS^* = P_Z \in \mathcal{M}_{d}(C), P_Z X' = X' \) and \((X')^* P_Z = (X')^* \). Hence, we now see that
\[
\lambda_i((X^*_S)^*(S^*AS)X_S) = \lambda_i(P_{X_{g_2}} A P_{X_{g_2}}) \quad \text{for} \quad i \in \mathbb{I}_{p-k} .
\]

Since \( X_{g_2} \subseteq X \) we have that \( P_{X_{g_2}} A P_{X_{g_2}} \) is a compression of \( P_{X^{\perp}} A P_{X^{\perp}} \). Using the interlacing inequalities for compressions of self-adjoint matrices (see [2]), we get that
\[
\lambda_i((P_{X^{\perp}} A P_{X^{\perp}})) \in [a, b] \quad \text{for} \quad i \in \mathbb{I}_{d-k} \implies \lambda_i(P_{X_{g_2}} A P_{X_{g_2}}) \in [a, b] \quad \text{for} \quad i \in \mathbb{I}_{p-k} .
\]  

(29)

On the other hand, notice that
\[
Y_S^* (S^*AS) Y_S = Y^* P_Z A P_Z Y = Y^* A Y
\]
since, as before, \( SS^* = P_Z, P_Z Y = Y \) and \( Y^* P_Z = Y^* \). Therefore, we get that
\[
\lambda_i(Y_S^* (S^*AS) Y_S) = \lambda_i(Y^* A Y) \in (\infty, a - \delta] \cup [b + \delta, \infty) \quad \text{for} \quad i \in \mathbb{I}_k .
\]  

(30)

Item 1. now follows from Eqs. (29) and (30) and the fact that \( S^*X \subseteq C^p \) is, by construction, an \( A_{X+Y} \)-invariant subspace.

We now show item 2. Fix an unitarily invariant norm \( \| \cdot \| \). Using that \( X, Y \subseteq Z \) and the fact that \( S^* \) is an isometry from \( Z \) onto \( C^p \), we see that \( \Theta(X, Y) = \Theta(S^*X, S^*Y) \). Then, an application of Nakatsukasa’s \( \tan \Theta \) theorem (Theorem 4.16) to the self-adjoint matrix \( S^*AS \in \mathcal{H}(p) \) and subspaces \( S^*X, S^*Y \subseteq C^p \) shows that
\[
\delta' \| \tan(\Theta(X, Y)) \| \leq \| A_{X+Y} Y_S - Y_S (Y_S^* A_{X+Y} Y_S) \| ,
\]
Then, we have that
\[
\lambda \text{ we let above by } X
\]
It is clear that
\[
\Theta(X) = \theta
\]
Remark 3.20. With the notation of Theorem 3.19 and using Remark 2.4 then Eq. (28) is equivalent to the majorization relation
\[
\delta' \tan(\Theta(X,Y)) \preceq s(A_{X+Y} Y_S - Y_S (Y_S^* A_{X+Y} Y_S)) = s(P_{X+Y} R_Y)
\]
in terms of the separation constant \(\delta'\) for \(A_{X+Y} = S^* A S, S^* X\) and \(S^* Y\).

\[\triangle\]

Consider the notation in Theorem 3.19. Let \(\delta > 0\) be such that \((A, X, Y, \delta)\) satisfies the DKN. Given a unitarily invariant norm \(\| \cdot \|\) then, Theorem 3.19 allows to bound \(\| \tan(\Theta(X,Y))\|\) from above by
\[
\| \tan(\Theta(X,Y))\| \leq \|R_Y\| / \delta. \tag{31}
\]
On the other hand, by item 2 in Theorem 3.19 there exists \(\delta' \geq \delta > 0\) such that \((A_{X+Y}, S^* X, S^* Y, \delta')\) satisfies the DKN separation property, so that we get the upper bound
\[
\| \tan(\Theta(X,Y))\| \leq \|P_{X+Y} R_Y\| / \delta'. \tag{32}
\]
Since \(\|P_{X+Y} R_Y\| \leq \|R_Y\|\) and \(\delta \leq \delta'\), we immediately see that the upper bound in Eq. (32) improves the classical bound in Eq. (31). In order to compare these two bounds in some more detail, let us consider the following Example 3.21.

**Example 3.21.** Let \(\tilde{\lambda} = (a, b, d, c) \in \mathbb{R}^4\), where \(a < b < c < d\), and let \(\tilde{A} \in \mathcal{H}(4)\) be given by \(\tilde{A} = D_3\). For the purposes of this example, we consider the real parameter \(c \in (b, d)\) as variable (while \(a, b, d\) are fixed).

Let \(X, Y_\theta \subset \mathbb{C}^4\) be as in Example 3.3 i.e. \(X = \text{span}\{e_1, e_2\}\) and \(Y_\theta = \text{span}\{e_1, f_\theta\}\). Recall that \(\Theta(X, Y_\theta) = (\theta, 0)\). In particular, \(\tan(\Theta(X, Y_\theta)) = (\tan \theta, 0)\) in this case.

It is clear that \(X + Y_\theta = \text{span}\{e_1, e_2, e_3\}\). Let
\[
X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad X_\perp = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Y_\theta = \begin{pmatrix} 1 & 0 \\ 0 & \cos \theta \\ 0 & \sin \theta \\ 0 & 0 \end{pmatrix}.
\]

Then, we have that \(\lambda(Y_\theta^* \tilde{A} Y_\theta) = (b \cos^2 \theta + d \sin^2 \theta, a)\), while \(\lambda(X_\perp^* \tilde{A} X_\perp) = (d, c)\). Therefore, if we let \(\theta_0(c) = \theta_0 = \arcsin \left( \sqrt{\frac{c-b}{d-b}} \right) \) and we set
\[
\delta_\theta = c - (b \cos^2 \theta + d \sin^2 \theta) > 0 \quad \text{for} \quad 0 < \theta < \theta_0,
\]
then \((\tilde{A}, X, Y_\theta, \delta_\theta)\) satisfies the DKN-separation property, and \(\delta_\theta\) is the optimal (largest) separation constant and the separation property holds only for \(0 < \theta < \theta_0\) in this case. Again, simple computations show that \(s(R_{Y_\theta}) = ((d - b) \cos \theta \sin \theta, 0)\).
Now, Eq. (31) obtained from Theorem 3.16 becomes
\[ \tan \theta \leq \frac{(d - b) \cos \theta \sin \theta}{c - b \cos^2 \theta + d \sin^2 \theta} \quad \text{for} \quad 0 < \theta < \theta_0. \] (33)

Notice that \( \lim_{\theta \to \theta_0^+} \theta_0 = 0 \) i.e., the range of \( \theta \) for which we can apply the bound in Eq. (33) tend to become small. In the limit case in which \( b = c \) (i.e. multiple eigenvalues) then we can not apply the bound (33) (the separation constant in this case is \( \delta_0 = 0 \)). Finally, if we consider the limit case in which \( \theta \) becomes small, then the upper bound is comparable with the upper bound \( \left( \frac{d - b}{c - b} \right) \tan \theta \) (\( > \tan \theta \)).

On the other hand, \( \mathcal{X} + \mathcal{Y} \subseteq \mathcal{C} e_3 \), the subspace spanned by \( e_3 \). In this case, if we let \( X' = (0, 0, 1, 0)^t \), it is clear that \( \lambda((X_0')^\ast \tilde{A} X_0') = d \). Therefore, if we let \( \delta' = d - (b \cos^2 \theta + d \sin^2 \theta) > 0 \), for \( \theta \in (0, \pi/2) \), we get that \( (\tilde{A}_{X + \mathcal{Y}}, S^* \mathcal{X}, S^* \mathcal{Y}, \delta'_0) \) satisfies the DKN-separation property, where \( S \in \mathcal{M}_{4,3}(\mathbb{C}) \) is the matrix whose columns are the first three elements in the canonical basis. In this case we have that
\[ s_1(P_{\mathcal{X} + \mathcal{Y}_0} R_{\mathcal{Y}_0}) = \frac{(d - b) \cos \theta \sin \theta}{(d - b) \cos^2 \theta} = \tan \theta, \]
and hence, the upper bound in Eq. (32) coincides with \( \tan \theta \) (where \( \tan(\Theta(\mathcal{X}, \mathcal{Y})) = (\tan \theta, 0) \)) i.e. the upper bound is sharp. Notice that the bound is applicable for every \( \theta \in (0, \pi/2) \).

The following result was conjectured in [20].

**Corollary 3.22.** Let \( A \in \mathcal{H}(d) \), let \( \mathcal{X}, \mathcal{Y} \subseteq \mathbb{C}^d \) and \( \delta > 0 \) be such that \( (A, \mathcal{X}, \mathcal{Y}, \delta) \) satisfies the DKN separation property. Then,
\[ \delta \| \tan(\Theta(\mathcal{X}, \mathcal{Y})) \| \leq \| P_{\mathcal{X} + \mathcal{Y}} R_{\mathcal{Y}} \|. \]
for every unitarily invariant norm \( \| \cdot \| \).

**Proof.** Let \( S \in \mathcal{M}_{d,p}(\mathbb{C}) \) be such that its columns form and ONB for \( \mathcal{X} + \mathcal{Y} \). By item 1. in Theorem 3.19 there exists \( \delta' \geq \delta \) such that \( (S^* A S, S^* \mathcal{X}, S^* \mathcal{Y}, \delta') \) satisfies the DKN separation property. By item 2. of the same result, we have that
\[ \delta \| \tan(\Theta(\mathcal{X}, \mathcal{Y})) \| \leq \delta' \| \tan(\Theta(\mathcal{X}, \mathcal{Y})) \| \leq \| P_{\mathcal{X} + \mathcal{Y}} R_{\mathcal{Y}} \|. \]

Finally, we get the following quadratic a posteriori error bound for the simultaneous approximation of eigenvalues of \( A \) by the Ritz values corresponding to Rayleigh quotients for which a DKN separation property holds.

**Theorem 3.23.** Let \( A \in \mathcal{H}(d) \), let \( \mathcal{X}, \mathcal{Y} \subseteq \mathbb{C}^d \) and \( \delta > 0 \) be such that \( (A, \mathcal{X}, \mathcal{Y}, \delta) \) satisfies the DKN separation property. Then, for every unitarily invariant norm \( \| \cdot \| \) we have that
\[ \| \lambda(\rho(\mathcal{X})) - \lambda(\rho(\mathcal{Y})) \| \leq \frac{\| P_{\mathcal{X} + \mathcal{Y}} R_{\mathcal{Y}} \|^2}{\delta}. \]

**Proof.** This is a consequence of Corollary 3.4 and Theorem 3.19.

Theorem 3.23 allows to obtain the following extension of [19] Theorem 5.3 (see Remark 3.25 below) which is a quadratic a posteriori majorization error bound for simultaneous approximation of consecutive eigenvalues.
Corollary 3.24. Let $A \in \mathcal{H}(d)$ and let $\mathcal{Y} \subset \mathbb{C}^d$ be such that:

1. $\lambda_1(Y^*AY) < \lambda_i(A)$, where $j \in \mathbb{I}_{d-k}$ is the smallest such index;

2. $\lambda_i(Y^*AY) \geq \lambda_{i+j}(A)$, for $i \in \mathbb{I}_k$.

Let $\mathcal{U}$ be the $A$-invariant space spanned by the eigenvectors associated with $\lambda_i(A)$, for $1 \leq i \leq j$, and set $\mathcal{X} = (I - P_U)\mathcal{Y}$. If we let $\eta = \lambda_j(A) - \lambda_1(Y^*AY) > 0$ then, we have that

$$
\|(\lambda_{i+j}(A))_{i \in \mathbb{I}_k} - \lambda(\rho(Y))\| \leq \frac{\|P_{\mathcal{X}+\mathcal{Y}} R_Y\|^2}{\eta},
$$

for every unitarily invariant norm $\| \cdot \|$. 

Proof. Let $\mathcal{V} = \mathcal{U} + \mathcal{Y}$ and notice that $\mathcal{U} \cap \mathcal{Y} = \{0\}$; hence, $p = \dim \mathcal{V} = \dim \mathcal{U} + k$ i.e. $j = \dim \mathcal{U} = p - k$. Moreover, $\mathcal{V} \cap \mathcal{U} = (I - P_U)\mathcal{Y} = \mathcal{X}$; then, in particular, $\dim \mathcal{X} = \dim \mathcal{Y}$ and $\mathcal{V} \cap \mathcal{X} = \mathcal{U}$. Also notice that $\Theta_1(\mathcal{X}, \mathcal{Y}) < \pi/2$ or otherwise, we would have that $\mathcal{U} \cap \mathcal{Y} \neq \{0\}$, since $\mathcal{V} \cap \mathcal{X} = \mathcal{U}$.

Let $V \in \mathcal{M}_{d,p}(\mathbb{C})$ be such that its columns form a ONB of $\mathcal{V}$ and set $A_V = V^*AV \in \mathcal{H}(p)$. Similarly, let $X, Y \in \mathcal{M}_{d,k}(\mathbb{C}), U \in \mathcal{M}_{d,p-k}(\mathbb{C})$ be such that their columns form ONB’s of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{U}$ respectively; set $X_V = V^*X$, $Y_V = V^*Y \in \mathcal{M}_{p,k}(\mathbb{C})$ and $U_V = V^*U \in \mathcal{M}_{p,p-k}(\mathbb{C})$. Then, the columns of $U_V$ span $U_V \subset \mathbb{C}^p$ an $A$-invariant space of $A_V$. In particular, the columns of $X_V$ span $X_V \subset \mathbb{C}^p$ which is also an $A$-invariant space of $A_V$. In this case $X_V^\perp = U_V$ and $\Theta_1(X_V, Y_V) = \Theta_1(\mathcal{X}, \mathcal{Y}) < \pi/2$, where $\mathcal{Y}_V \subset \mathbb{C}^p$ is the space spanned by the columns of $Y_V$. Notice that, by construction $\lambda_i(Y_V^*A_V Y_V) = \lambda_i(Y^*AY)$, for $i \in \mathbb{I}_k$. Since $\mathcal{X} \subset U_V^\perp$ by the interlacing inequalities for compressions of self-adjoint matrices and Item 2 above, we get that

$$
\lambda_i(X_V^*A_V X_V) = \lambda_i(X^*A X) \leq \lambda_i(U_V) = \lambda_{j+i}(A) \leq \lambda_i(Y^*AY) = \lambda_i(Y_V^*A_V Y_V) \quad (34)
$$

for $i \in \mathbb{I}_k$, where $U_V \in \mathcal{M}_{d,d-j}(\mathbb{C})$ is such that its columns for an ONB for $U_V^\perp$. On the other hand, by hypothesis $(A_V, X_V, Y_V, \eta)$ satisfies the DKN separation property (recall that $X_V^\perp = U_V$). Hence, by Theorem 3.23 we conclude that

$$
\|\lambda(X_V^*A_V X_V) - \lambda(Y_V^*A_V Y_V)\| \leq \frac{\|P_{X_V+Y_V} (A_V Y_V - Y_V (Y_V^*A_V Y_V))\|^2}{\eta} \quad (35)
$$

By Eq. (34) we get that

$$
|\lambda_{i+j}(A))_{i \in \mathbb{I}_k} - \lambda(Y_V^*A_V Y_V)| \prec_w |\lambda(X_V^*A_V X_V) - \lambda(Y_V^*A_V Y_V)|.
$$

On the other hand, arguing as in the proof of Theorem 3.19 we see that

$$
\|P_{X_V+Y_V} (A_V Y_V - Y_V (Y_V^*A_V Y_V))\| = \|P_{X+Y} R_Y\|.
$$

The result follows from these last facts together with Eq. (35) and Remark 2.24. 

Remark 3.25. We mention that the hypothesis in item 1. in Corollary 3.24 is that there exists an eigenvalue $\beta$ of $A$ such that $\lambda_1(Y^*AY) < \beta$. Indeed, in this case we can apply the interlacing inequalities and get that $\lambda_i(Y^*AY) \geq \lambda_{d-k+i}(A)$, for $i \in \mathbb{I}_k$. Therefore, $\beta = \lambda_j(A)$ for some $1 \leq j \leq d - k$.

The hypothesis in item 2. is rather restrictive and difficult to check in general. Nevertheless, we mention two cases in which the hypotheses in Corollary 3.24 can be easily checked:

1. In case the hypothesis in item 1 holds for $j = d - k$ then, by the interlacing inequalities

$$
\lambda_i(Y^*AY) \geq \lambda_{i+d-k}(A) \quad \text{for} \quad i \in \mathbb{I}_k,
$$

so the hypothesis in item 2 automatically hold.
2. In case $k = 1$ that is, if $\mathcal{X} = \mathbb{C} y$ for a unit norm vector $y \in \mathbb{C}^d$, the hypotheses become the existence of $j \in \mathbb{N}$ such that $\lambda_{j+1} < \lambda_j(A)$; then, Corollary 3.24 implies that

$$0 \leq \langle Ay, y \rangle - \lambda_{j+1}(A) \leq \frac{\|P_{\mathcal{X}} Ay - \langle Ay, y \rangle y\|}{\lambda_j(A) - \langle Ay, y \rangle},$$

where $\mathcal{X} = \mathbb{C} x$, for $x = (I - P_{\mathcal{U}})y \in \mathbb{C}^d$; this is [19, Theorem 5.3]. As explained in [19], Corollary 3.24 encodes several known bounds related with eigenvalue estimation even when $k = 1$. △

4 Appendix

Here we collect several and well known results about majorization, used throughout our work. The first result deals with submajorization relations between singular values of arbitrary matrices in $\mathcal{M}_d(\mathbb{C})$. For detailed proofs of these results and general references in majorization theory see [2, 6, 12].

**Theorem 4.1.** Let $C, D \in \mathcal{M}_d(\mathbb{C})$. Then,

1. $s(C + D) \prec_w s(C) + s(D)$;
2. $s(\text{re}(C)) \prec_w s(C)$;
3. $s(CD) \prec_w s(C)s(D)$;
4. If we assume that $CD \in \mathcal{H}(d)$ then $s(CD) \prec_w s(\text{re}(DC))$.

For hermitian matrices we have the following majorization relations

**Theorem 4.2.** Let $C, D \in \mathcal{H}(d)$. Then,

1. $\lambda(C) - \lambda(D) \prec \lambda(C - D) \prec \lambda(C) - \lambda(D)$;
2. $|\lambda(C) - \lambda(D)| \prec_w s(C - D)$;
3. Let $\mathcal{P} = \{P_j\}_{j=1}^r$ be a system of projections (i.e. they are mutually orthogonal projections on $\mathbb{C}^d$ such that $\sum_{i=1}^r P_i = I$). If $C_{\mathcal{P}}(C) = \sum_{i=1}^r P_i C P_i$, then $\lambda(C_{\mathcal{P}}(C)) \prec \lambda(C)$.

In the next result we describe several elementary but useful properties of (sub)majorization between real vectors.

**Lemma 4.3.** Let $x, y, z \in \mathbb{R}^k$. Then,

1. $x^\dagger + y^\dagger \prec_w x + y \prec x^\dagger + y^\dagger$;
2. If $x \prec_w y$ and $y, z \in (\mathbb{R}^k)_\downarrow$ then $x + z \prec_w y + z$;

If we assume further that $x, y, z \in \mathbb{R}^k_{\geq 0}$ then,

3. $x^\dagger y^\dagger \prec_w x y \prec_w x^\dagger y^\dagger$;
4. If $x \prec_w y$ and $y, z \in (\mathbb{R}^k_{\geq 0})_\downarrow$ then $x z \prec_w y z$.

**Proposition 4.4.** Let $1 \leq k < d$ and let $E \in \mathcal{M}_{k,(d-k)}(\mathbb{C})$. Then

$$\hat{E} = \begin{pmatrix} 0 & E \\ E^* & 0 \end{pmatrix} \in \mathcal{H}(d) \quad \text{and} \quad \lambda(\hat{E}) = (s(E), -s(E^*))_\downarrow \in (\mathbb{R}^d_{\geq 0})_\downarrow.$$
Theorem 4.5. [Theorem 4.6, 3] Let \( \mathcal{X}, \mathcal{Y} \subset \mathbb{C}^d \) be such that \( \dim(\mathcal{X}) = \dim(\mathcal{Y}) = k \). Then
\[
\lambda(P_\mathcal{X} P_\mathcal{Y} \perp P_\mathcal{X}) = s(P_\mathcal{X} P_\mathcal{Y} \perp P_\mathcal{X}) = s^2(P_\mathcal{Y} P_\mathcal{X} \perp) = s^2(P_\mathcal{Y} \perp P_\mathcal{Y}) = (\sin^2(\Theta(\mathcal{X}, \mathcal{Y})), 0_{d-k}).
\]

Notice that item 2. below is Theorem 3.2 from Section 3.

Theorem 4.6. Let \( C, D \in \mathcal{H}(k) \). Then,
1. if \( T \in \mathcal{G}(k)^+ \), then \( s(C - D) \prec_w s(T^{-1}) s(CT - TD) \).
2. if \( T \in \mathcal{G}(k) \), then \( |\lambda(C) - \lambda(D)| \prec_w s(T^{-1}) s(CT - TD) \).

Proof. We first show item 1. Since \( T \) is positive and invertible, using Theorem 4.2 (item 3.) we get that
\[
s(C - D) = s(CT^{1/2} T^{-1/2} - T^{-1/2} T^{1/2} D)) = s(T^{-1/2} (CT^{1/2} - T^{1/2} DT^{1/2}) T^{-1/2}) \prec_w s(T^{-1} s(T^{1/2} (C - D) T^{1/2}).
\]
By Theorem 4.1 (items 2. and 4.) and the fact that \( \text{re}(DT) = \text{re}(TD) \) we obtain that
\[
s(T^{1/2} (C - D) T^{1/2}) \prec_w s([\text{re}[(C - D)|T]]) = s([\text{re}[CT - TD]]) \prec_w s(CT - TD), \tag{36}
\]
By the previous inequalities and Lemma 4.3 we see that
\[
s(C - D) \prec_w s(T^{-1}) s(CT - TD). \tag{37}
\]
In order to show item 2, consider a representation of \( T \) given by \( T = USV^* \), where \( U, V \in \mathcal{U}(k) \) are unitary matrices and \( \Sigma \in \mathcal{M}_k(\mathbb{C}) \) is the diagonal matrix with main diagonal \( s(T) \in \mathbb{R}_{\geq 0}^k \) (notice that such representation follows from the SVD decomposition of \( T \)); note that \( \Sigma \) is definite positive and invertible. Using item 2 in Theorem 4.2 and (the already proved) item 1. of the statement we get
\[
|\lambda(C) - \lambda(D)| = |\lambda(U^* CU) - \lambda(V^* DV)| \prec_w s(U^* CU - V^* DV) \prec_w s(\Sigma^{-1}) s(U^* CU \Sigma - \Sigma V^* DV) = s(T^{-1}) s(U^*(CT - TD)V) = s(T^{-1}) s(CT - TD).
\]

In what follows we re-state and prove two propositions of Section 3.2.

Proposition 3.10. Let \( A \in \mathcal{H}(d) \) and let \( \mathcal{X}, \mathcal{Y} \subset \mathbb{C}^d \) with \( \dim(\mathcal{X}) = \dim(\mathcal{Y}) = k \). Then
\[
s(P_\mathcal{X} R_\mathcal{Y}) \prec_w \text{Spr}(A, \mathcal{X} + \mathcal{Y}) \sin(\Theta(\mathcal{X}, \mathcal{Y})). \tag{38}
\]

Proof. Consider \( A \in \mathcal{H}(d) \) and \( \mathcal{X}, \mathcal{Y} \subset \mathbb{C}^d \) with \( \dim(\mathcal{X}) = \dim(\mathcal{Y}) = k \). In what follows we show that
\[
s(P_\mathcal{X} R_\mathcal{Y}) \prec_w \text{Spr}(A, \mathcal{X} + \mathcal{Y}) \sin(\Theta(\mathcal{X}, \mathcal{Y})).
\]
We begin with a simple reduction argument. A simple calculation show that \( (s(P_\mathcal{X} R_\mathcal{Y}), 0_{d-k}) = s(P_\mathcal{X} (A P_\mathcal{Y} - P_\mathcal{Y} A P_\mathcal{Y})) \in (\mathbb{R}_{\geq 0}^d)^k \). Let \( Z = \mathcal{X} + \mathcal{Y} \) with \( \dim Z = p \), and consider the matrix representations with respect to the decomposition \( \mathbb{C}^d = Z \oplus Z^\perp \):
\[
P_\mathcal{X} = \begin{pmatrix} P_\mathcal{X} & 0 \\ 0 & 0 \end{pmatrix}, \quad P_\mathcal{Y} = \begin{pmatrix} P_\mathcal{Y} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \ast & * \\ * & \ast \end{pmatrix},
\]
where $P^X$, $P^Y$, $A_Z = P_Z A|_Z \in L(Z)$ are self-adjoint operators. In this case we have

$$P^X (A P_Y - P_Y A P_Y) = \begin{pmatrix} P^X (A_Z P^Y - P^Y A_Z P^Y) & 0 \\ 0 & 0 \end{pmatrix}. $$

Hence, $(s(P^X Y), 0_{d-k}) = s(P^X (A_Z P^Y - P^Y A_Z P^Y)) = s(P^X (I_Z - P^Y) A_Z P^Y)$. Thus, we can assume further that $C^d = Z = X + Y$ and show that

$$s(P^X Y), 0_{d-k}) = s(P^X (P_Y A P_Y)) \triangleq \sin(\Theta(X, Y)), 0_{d-k}).$$

Now using multiplicative Lidskii’s

$$s(P^X P^Y \perp A P_Y) = s(P^X P^Y \perp P^Y \perp A P_Y) \triangleq s(P^X P^Y \perp A P_Y).$$

First notice that by Theorem 4.5, we have that $s(P^X P^Y \perp) = (\sin(\Theta(X, Y)), 0_{d-k})$. On the other hand, consider the matrix representation induced by the decomposition $C^d = Y \oplus Y \perp$:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \text{ and set } A_1 := \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & A_{21} \\ A_{21} & 0 \end{pmatrix}. $$

Then, we have that $A = A_1 + A_2$. Now, $A_1$ is a pinching of $A$ (associated with the system of projections $\{P_Y, P^Y \perp\}$) so $\lambda(A_1) < \lambda(A)$ so then

$$- \lambda^\dagger(A_1) < -\lambda^\dagger(A).$$

Using Lidskii’s additive inequality for $A_2 = A - A_1$ (see item 1 in Theorem 4.4)

$$\lambda(A_2) < \lambda(A) - \lambda^\dagger(A_1).$$

Combining (42) and (43), we obtain

$$\lambda(A_2) < \lambda(A) - \lambda^\dagger(A) = \text{Spr}(A) \in \mathbb{R}^d. $$

By Proposition 4.3, we get that $\lambda(A_2) = (s(A_{21}), -s(A_{21}))^\dagger$; in particular, $s(A_{21}) = (\lambda_i(A_2))_{i \in I_k}$. Now, $s(P^Y \perp A P_Y) = (s(A_{21}), 0_{d-k})$; thus, we see that

$$s(P^Y \perp A P_Y) = s(A_{21}), 0_{d-k}) = ((\lambda_i(A_2))_{i \in I_k}, 0_{d-k}) \triangleq (\text{Spr}(A))_{i \in I_k}, 0_{d-k}),$$

where $\text{Spr}(A) = (\text{Spr}(A))_{i \in I_k}$. Using Eqs. (40) and (45) together with Lemma 4.3, we finally get that

$$s(P^X P^Y \perp A P_Y) \triangleq \sin(\Theta(X, Y)), 0_{d-k}) \in \mathbb{R}_{\geq 0}^d.$$ 

Now the result follows from the last submajorization relation, by considering the first $k$ entries of both vectors.

**Proposition 3.12.** Let $A \in \mathcal{H}(d)$, $X, Y \subset \mathbb{C}^d$ subspaces with $\dim(X) = \dim(Y) = k$. Assume that $X$ is $A$-invariant. Then,

$$s(P X Y) \triangleq 2 (\lambda_i(A_{X+Y}) - \lambda_{\text{min}}(A_{X+Y})_{i \in I_k} \text{ sin}^2(\Theta(X, Y))).$$

**Proof.** Arguing as in the proof of Proposition 3.10, we can assume further that $C^d = X + Y$. With this assumption, we consider first the case where $A \in \mathcal{M}_d(\mathbb{C})^+$ and show that

$$s(P X Y) \triangleq 2 (\lambda_i(A))_{i \in I_k} \text{ sin}^2(\Theta(X, Y)).$$
Indeed, the $A$-invariance of $X$, allows us to write $A = P_XAP_X + P_{X^\perp}AP_{X^\perp}$. With this decomposition in mind and using the fact that $(s(P_{X^\perp}R_Y), 0_{d-k}) = s(P_{X^\perp}P_{Y^\perp}AP_{Y^\perp})$, we have that

\[ s(P_{X^\perp}P_{Y^\perp}AP_{Y^\perp}) = s(P_{X^\perp}P_{Y^\perp}AP_{X^\perp}A P_{X^\perp}P_{Y^\perp} + P_{X^\perp}P_{Y^\perp}AP_{X^\perp}P_{Y^\perp}) \]

\[ \preceq_w s(P_{X^\perp}P_{Y^\perp}AP_{X^\perp}P_{Y^\perp}) + s(P_{X^\perp}P_{Y^\perp}AP_{X^\perp}P_{Y^\perp}) \quad \text{def} \quad M. \]

Using Theorem 4.1 (multiplicative Lidskii’s), the fact that $0_d \leq s(P_{X^\perp}P_{Y^\perp}) \leq I_d$ and Theorem 4.5, we get

\[ M \preceq_w s(P_{X^\perp}P_{Y^\perp}AP_{X^\perp}AP_{Y^\perp}) + s(P_{X^\perp}P_{Y^\perp}AP_{X^\perp}P_{Y^\perp}) \]

\[ \preceq_w 2 \lambda(A) (\sin^2(\Theta(X, Y)), 0_{d-k}) \in (\mathbb{R}_{\geq 0})^d, \]

since $A \in M_d(\mathbb{C})^+$ is positive semi-definite. The result now follows from the previous facts.

In general, for $A \in \mathcal{H}(d)$ consider the auxiliary matrix $\tilde{A} = A - \lambda_{\min}(A) I \in M_d(\mathbb{C})^+$. Notice that

\[ R_Y(\tilde{A}) = \tilde{A}Y - Y(Y^*\tilde{A}Y) = AY - Y(Y^*AY) = R_Y \quad \text{and} \quad \lambda(\tilde{A}) = \lambda(A) - \lambda_{\min}(A) I_d. \]

The result now follows from these facts and from Eq. 17 applied to $\tilde{A}$. □

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