Certain invariant spaces of bounded measurable functions on a sphere

Samuel A. Hokamp

Received: 13 July 2021 / Accepted: 12 August 2021 / Published online: 26 August 2021
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2021

Abstract
In their 1976 paper, Nagel and Rudin characterize the closed unitarily and Möbius invariant spaces of continuous and $L^p$-functions on a sphere, for $1 \leq p < \infty$. In this paper we provide an analogous characterization for the weak*-closed unitarily and Möbius invariant spaces of $L^\infty$-functions on a sphere. We also investigate the weak*-closed unitarily and Möbius invariant algebras of $L^\infty$-functions on a sphere.

Keywords Spaces of measurable functions · Several complex variables · Functional analysis

Mathematics Subject Classification Primary 46E30; Secondary 32A70

1 Introduction
Let $n$ be a positive integer and $S$ the unit sphere of $\mathbb{C}^n$. We say a space of complex functions defined on $S$ is \textit{unitarily invariant} if the pre-composition of any function in the set with a unitary transformation remains in the set. Similarly, we say a space of complex functions defined on $S$ is \textit{Möbius invariant} if the pre-composition of any function in the set with a biholomorphic map of the complex unit ball onto itself remains in the set. It is clear that Möbius invariance implies unitary invariance.

In [1], Nagel and Rudin determine the closed unitarily and Möbius invariant spaces of continuous and $L^p$-functions on $S$, for $1 \leq p < \infty$. The closed Möbius invariant algebras of continuous and $L^p$-functions on $S$, for $1 \leq p < \infty$, are also characterized in [1], and the closed unitarily invariant algebras of continuous functions on $S$ are considered in [2].
In this paper, we consider $L^\infty(S)$ with the weak*-topology, thereby obtaining results for $L^\infty(S)$ which are analogous to Nagel and Rudin’s results for continuous functions.

In Sect. 4.1, we determine the weak*-closed unitarily invariant subspaces of $L^\infty(S)$ (Theorem 4.1), and in Sect. 4.2, we show that those weak*-closed unitarily invariant subspaces of $L^\infty(S)$ which are algebras correspond in a certain way to the closed unitarily invariant subalgebras of continuous functions (Theorem 4.7). This correspondence leads to several results for weak*-closed unitarily invariant subalgebras of $L^\infty(S)$ (Theorems 4.10, 4.11, and 4.12) that are analogous to results for closed unitarily invariant algebras of continuous functions from [2]. These results are given in Sect. 4.3.

In Sect. 5.1, we determine the weak*-closed Möbius invariant subspaces of $L^\infty(S)$ (Theorem 5.1), and in Sect. 5.2, we determine which of the weak*-closed Möbius invariant subspaces of $L^\infty(S)$ are algebras (Theorem 5.4).

2 Preliminaries

2.1 Notation and definitions

We let $n$ be a positive integer and $S$ and $B$ be the unit sphere and unit ball of $\mathbb{C}^n$, respectively, and $\overline{B}$ the closed unit ball.

We let $\sigma$ denote the unique rotation-invariant positive Borel measure on $S$ for which $\sigma(S) = 1$. The phrase “rotation-invariant” refers to the orthogonal group $O(2n)$, the group of isometries on $\mathbb{R}^{2n}$ that fix the origin. The notation $L^p(S)$ then denotes the usual Lebesgue spaces, for $1 \leq p \leq \infty$, in reference to the measure $\sigma$, with the usual norms $|| \cdot ||_p$.

For $Y \subset C(S)$, the uniform closure of $Y$ is denoted $\overline{Y}$, and for $Y \subset L^p(S)$, $1 \leq p \leq \infty$, the norm-closure of $Y$ in $L^p(S)$ is denoted $\overline{Y}^p$. Additionally, for $Y \subset L^\infty(S)$, the weak*-closure of $Y$ in $L^\infty(S)$ is denoted $\overline{Y}^*$.

Occasionally, we let $X$ denote either $C(S)$ or one of the spaces $L^p(S)$ for $1 \leq p \leq \infty$. Then, if $Y \subset X$, the norm-closure of $Y$ in $X$ is denoted $\overline{Y}^X$.

Remark 2.1 For $Y \subset L^\infty(S)$ convex and $1 \leq p < \infty$, we have

$$\overline{Y}^* \subset \overline{Y}^p \cap L^\infty(S).$$

This follows from the fact that the topology $L^\infty(S)$ inherits from the weak topology on each $L^p(S)$ is weaker than the weak*-topology, and from the local convexity of $L^p(S)$.

We let $\mathcal{U} = \mathcal{U}(n)$ denote the group of unitary operators on the Hilbert space $\mathbb{C}^n$. Then $\mathcal{U}$ is a compact subgroup of $O(2n)$, and we use $dU$ to denote the Haar measure on $\mathcal{U}$.

We let $\mathcal{M} = \mathcal{M}(n)$ denote the Möbius group in dimension $n$. This is the group of injective holomorphic maps of $B$ onto $B$. Important to note is that each element of $\mathcal{M}$ extends to a homeomorphism of $\overline{B}$ onto $\overline{B}$, and thus maps $S$ onto $S$.
We let $Q$ denote the set of pairs of nonnegative integers; that is,

$$Q = \{(p, q) : p, q \in \mathbb{Z} \text{ and } p \geq 0, q \geq 0\}.$$  

On the sets $S$ and $B$ we define the following spaces of complex functions:

1. $C(S)$ is the space of continuous functions on $S$.
2. $A(S)$ is the space of functions which are restrictions to $S$ of holomorphic functions on $B$ that are continuous on $\overline{B}$.
3. $H^\infty(B)$ is the space of bounded holomorphic functions on $B$.
4. $H^\infty(S)$ is the space of functions in $L^\infty(S)$ which are radial limits almost everywhere of functions in $H^\infty(B)$.

If $X$ is a space of complex functions, then $\text{Conj}(X)$ denotes the space of conjugates of members of $X$.

**Definition 2.2** (Section IV.) We say a space of complex functions $Y$ with domain $S$ is **unitarily invariant** ($\mathcal{U}$-invariant) if $f \circ U \in Y$ for every $f \in Y$ and every $U \in \mathcal{U}$.

**Remark 2.3** The space $L^p(S)$ is $\mathcal{U}$-invariant, for $1 \leq p \leq \infty$.

**Definition 2.4** (Section I.) We say a space of complex functions $Y$ with domain $S$ is **Möbius invariant** ($\mathcal{M}$-invariant) if $f \circ \phi \in Y$ for every $f \in Y$ and every $\phi \in \mathcal{M}$.

### 2.2 The spaces $H(p, q)$ and the maps $\pi_{pq}$

For each $(p, q) \in Q$, we define $H(p, q)$ to be the space of harmonic, homogeneous polynomials on $S$ with total degree $p$ in the variables $z_1, z_2, \ldots, z_n$ and total degree $q$ in the variables $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n$. These spaces $H(p, q)$ are finite-dimensional (hence closed) subspaces of $C(S)$ and $L^2(S)$.

If $\Omega$ is any subset of $Q$, we define the set $E_\Omega$ to be the algebraic sum of the spaces $H(p, q)$ such that $(p, q) \in \Omega$. Each $E_\Omega$ is consequently $\mathcal{U}$-invariant. Our notation for the spaces $H(p, q)$ and $E_\Omega$ is the same used by Rudin in [3]. For the maps $\pi_{pq}$ (defined in Theorem 2.5(c)), the notation in [1,3] is the same, so we use it as well.

**Theorem 2.5** (Theorem 4.6) Let $(p, q) \in Q$.

(a) For every $z \in S$, $H(p, q)$ contains a unique function $K_z$ that satisfies

(i) $K_z \circ U = K_z$ for every $U \in \mathcal{U}$ with $Uz = z$, and

(ii) $\int_S |K_z|^2 d\sigma = K_z(z) > 0$.

(b) $H(p, q)$ is $\mathcal{U}$-minimal; that is, $H(p, q)$ is $\mathcal{U}$-invariant with no proper $\mathcal{U}$-invariant subspaces.

(c) If $X$ is a function space with domain $S$ and $\pi_{pq}$ is defined for $f \in X$ by

$$\pi_{pq} f(z) = \int_S f \overline{K_z} d\sigma,$$

then $\pi_{pq}$ is a projection of $X$ onto $H(p, q)$.  

\[\mathcal{C} \text{ Springer} \]
Each $\pi_{pq}$ is an orthogonal projection in $L^2(S)$. Further, the spaces $H(p, q)$ are pairwise orthogonal in $L^2(S)$, and $L^2(S)$ is the direct sum of the spaces $H(p, q)$. That is,

$$L^2(S) = \bigoplus_{Q} E^2_{\Omega}$$

Observe that each $E_{\Omega}$ is $U$-invariant since each $H(p, q)$ is $U$-invariant.

**Remark 2.6** Explicitly, Theorem 2.5(d) says that each $f \in L^2(S)$ has a unique expansion $f = \sum f_{pq}$, with each $f_{pq} \in H(p, q)$, which converges unconditionally to $f$ in the $L^2$-norm. Since $\pi_{pq}$ is the identity map on $H(p, q)$ and the spaces $H(p, q)$ are pairwise orthogonal, we have $f_{pq} = \pi_{pq} f$ for $(p, q) \in Q$. Thus,

$$f = \sum \pi_{pq} f.$$ 

Each $\pi_{pq}$ is continuous as the orthogonal projection of $L^2(S)$ onto the finite-dimensional subspace $H(p, q)$. Thus, $\pi_{pq}$ annihilates a subset of $L^2(S)$ if and only if it annihilates its closure. The following is a consequence of this and Remark 2.6:

**Remark 2.7** For each set $\Omega \subset Q$, we have

$$\overline{E^2_{\Omega}} = \{ f \in L^2(S) : \pi_{pq} f = 0 \text{ when } (p, q) \notin \Omega \}.$$ 

### 2.3 $U$- and $M$-invariant subspaces and subalgebras of $C(S)$

In this section we summarize the results of Nagel and Rudin concerning the unitarily and Möbius invariant subspaces and subalgebras of $C(S)$. The results we draw on for this paper are formulated in [1] and [2], but the reader can also find them presented in [3].

In [1], Nagel and Rudin establish that each closed $U$-invariant subspace of $C(S)$ or $L^p(S)$ for $1 \leq p < \infty$ corresponds to some $\Omega \subset Q$ (Theorem 2.8). This $\Omega$ is the set of all $(p, q) \in Q$ such that $\pi_{pq}$ does not annihilate the space.

**Theorem 2.8** ([1] Theorem 4.4) Let $X$ be any of the spaces $C(S)$ or $L^p(S)$ for $1 \leq p < \infty$. If $Y$ is a closed $U$-invariant subspace of $X$, then $Y = \overline{E^X_{\Omega}}$ for some $\Omega \subset Q$.

The characterization of the closed $U$-invariant subalgebras of $C(S)$ is dependent on the dimension $n$. The case when $n = 1$ is simple and is covered in [1]: Let $\Omega$ be any additive semigroup of integers. The set of continuous functions on $S$ whose Fourier coefficients vanish on the complement of $\Omega$ is a closed $U$-invariant algebra, and there are no others.

In [2], Rudin develops a combinatorial criterion that describes the sets $\Omega \subset Q$ (called algebra patterns) which induce closed $U$-invariant subalgebras of $C(S)$ when $n \geq 3$. 

@ Springer
Theorem 2.9  ([2] Theorem 1) The following property of a set $\Omega \subset Q$ implies $\Omega$ is an algebra pattern: If $(p, q), (r, s) \in \Omega$ and

$$\mu = \min\{p + q, r + s, p + r, q + s\},$$

then $\Omega$ contains all points $(p + r + j, q + s + j)$ with $0 \leq j \leq \mu$. Conversely, every algebra pattern has this property when $n \geq 3$.

The case $n = 2$ has some exceptions which Rudin also addresses in [2]. Theorem 2.9 can also be formulated in terms of the spaces in the following definition.

Definition 2.10  ([2] Definition 1.5) Given $(p, q), (r, s) \in Q$, we define the space $H(p, q) \cdot H(r, s)$ to be the linear span of products $fg$, where $f \in H(p, q)$ and $g \in H(r, s)$.

Theorem 2.9 then yields several examples of algebra patterns, namely the sets described in Definition 2.11. As in [2], we define $\Delta_k$ for each integer $k$ to be the set $\{(p, q) \in Q : p - q = k\}$.

Definition 2.11  ([2] Definition 1.6) Let $d$ be any integer, $\Sigma$ any (possibly empty) additive semigroup of positive integers, and $(p, q) \in Q$ such that $0 \leq q < p$. We define

1. $G(d)$ to be the union of all $\Delta_{kd}$, for $-\infty < k < \infty$,
2. $G(\Sigma)$ to be the union of $\Delta_0$ and all $\Delta_k$ for $k \in \Sigma$, and
3. $G(p, q)$ to be the set consisting of $(p, q)$ and points

$$(mp - j, mq - j),$$

where $m = 2, 3, 4, \ldots$, and $0 \leq j \leq mq$.

The following theorems of [2] show the importance of these algebra patterns. The first two results are also in [1] and describe every self-adjoint closed $\mathcal{U}$-invariant algebra of continuous functions on $S$ (with $\Sigma$ empty).

Theorem 2.12  ([2] Theorem 2) If $\Omega$ is an algebra pattern that contains some $(p, q)$ with $p > q$ and some $(r, s)$ with $r < s$, then $\Omega = G(d)$ for some $d > 0$.

Theorem 2.13  ([2] Theorem 3) Suppose $\Omega$ is an algebra pattern such that $p \geq q$ for every $(p, q) \in \Omega$ and $(a, a) \in \Omega$ for some $a > 0$.

1. If $n \geq 3$, then $\Omega = G(\Sigma)$.
2. If $n = 2$ and if $G^*(\Sigma)$ is obtained from $G(\Sigma)$ by deleting all $(m, m)$ with $m$ odd, then $G^*(\Sigma)$ is also an algebra pattern, and $\Omega$ is either $G(\Sigma)$ or $G^*(\Sigma)$.

Theorem 2.14  ([2] Theorem 4) Fix $(p, q) \in Q$ with $p > q$. Let $\Omega(p, q)$ be the smallest algebra pattern that contains $(p, q)$.

1. If $n \geq 3$, then $\Omega(p, q) = G(p, q)$.
2. If $n = 2$, then $\Omega(p, q)$ is obtained from $G(p, q)$ by deleting all $(mp - 1, mq - 1)$ for $m = 2, 3, 4, \ldots$, and all $(2p - j, 2q - j)$ for $j$ odd.
Lastly, Nagel and Rudin show in [1] (Theorem B) that there are only six closed $\mathcal{M}$-invariant subspaces of $C(S)$:

1. The null space $\{0\}$,
2. The space of constant functions $\mathbb{C}$,
3. The space $A(S)$,
4. The space $\text{Conj}(A(S))$,
5. The closure of the space $A(S) + \text{Conj}(A(S))$, and
6. The space $C(S)$.

This is clear from Theorem 2.8 and the following lemma.

**Lemma 2.15** ([1] Lemma 5.2) Suppose $Y$ is a closed $\mathcal{M}$-invariant subspace of $C(S)$, $p \geq 1$ an integer, and $H(p, q) \subset Y$ for some $q \geq 0$. Then $H(p - 1, q) \subset Y$ and $H(p + 1, q) \subset Y$.

The closed $\mathcal{M}$-invariant subalgebras of $C(S)$ are those spaces above that are also algebras. Observe that in general all are algebras except for the closure of $A(S) + \text{Conj}(A(S))$. This is an algebra only when $n = 1$.

### 3 Closures of $\mathcal{U}$-invariant sets

In this section we verify that $\mathcal{U}$-invariance is preserved by closures in the spaces $C(S)$ and $L^p(S)$ (Corollaries 3.2 and 3.4). We prove this by showing that $\mathcal{U}$ induces a class of $L^p$-isometries on $L^p(S)$ and a class of isometries on $C(S)$ (Theorem 3.1), as well as a class of weak*-homeomorphisms on $L^\infty(S)$ (Theorem 3.3).

**Theorem 3.1** Suppose $X$ is any of the spaces $C(S)$ or $L^p(S)$ for $1 \leq p \leq \infty$ and $U \in \mathcal{U}$. If $\Phi_U : X \to X$ is the map given by $\Phi_U(f) = f \circ U$, then $\Phi_U$ is a bijective linear isometry.

**Proof** If $X$ is any of the spaces $L^p(S)$ for $1 \leq p \leq \infty$, the rotation-invariance of $\sigma$ yields that the map $\Phi_U$ is an $L^p$-isometry (hence injective). The linearity and surjectivity of $\Phi_U$ is clear. The case when $X$ is the space $C(S)$ follows from the previous case when $p = \infty$, since the $L^\infty$-norm coincides with the uniform norm on $C(S)$. \qed

**Corollary 3.2** Suppose $X$ is any of the spaces $C(S)$ or $L^p(S)$ for $1 \leq p \leq \infty$. If $Y \subset X$ is $\mathcal{U}$-invariant, then $\overline{Y}^X$ is $\mathcal{U}$-invariant.

**Theorem 3.3** Let $U \in \mathcal{U}$. If $\Phi_U : L^\infty(S) \to L^\infty(S)$ is the map given by $\Phi_U(f) = f \circ U$, then $\Phi_U$ is a weak*-homeomorphism.

**Proof** Recall the weak*-topology on $L^\infty(S)$ is a weak topology induced by the maps on $L^\infty(S)$ of the form

$$\Lambda_g f = \int_S fg \, d\sigma.$$
for some \( g \in L^1(S) \). Thus, \( \Phi_U \) is continuous with respect to the weak*-topology if and only if \( \Lambda_g \circ \Phi_U \) is continuous for all maps \( \Lambda_g \).

Fix \( g \in L^1(S) \). We observe that

\[
(\Lambda_g \circ \Phi_U)(f) = \Lambda_g(f \circ U) = \int_S (f \circ U) \cdot g \, d\sigma,
\]

for every \( f \in L^\infty(S) \).

By Remark 2.3, \((f \circ U) \cdot g\) is an element of \( L^1(S) \). Application of the map \( U^{-1} \) yields

\[
\int_S (f \circ U) \cdot g \, d\sigma = \int_S [(f \circ U) \cdot g] \circ U^{-1} \, d\sigma = \int_S (f \circ U \circ U^{-1}) \cdot (g \circ U^{-1}) \, d\sigma = \int_S f \cdot (g \circ U^{-1}) \, d\sigma.
\]

Since \((g \circ U^{-1}) \in L^1(S)\), for every \( f \in L^\infty(S) \) we have

\[
(\Lambda_g \circ \Phi_U)(f) = \int_S f \cdot (g \circ U^{-1}) \, d\sigma = \Lambda_{g \circ U^{-1}} f.
\]

We conclude \( \Phi_U \) is continuous on \( L^\infty(S) \) with respect to the weak*-topology.

Finally, the map \( \Phi_{U^{-1}} : L^\infty(S) \to L^\infty(S) \) given by \( \Phi_{U^{-1}}(f) = f \circ U^{-1} \) is clearly the inverse of \( \Phi_U \). By a similar argument, \( \Phi_{U^{-1}} \) is continuous with respect to the weak*-topology, and thus \( \Phi_U \) is a weak*-homeomorphism.

\( \square \)

**Corollary 3.4** Suppose \( Y \) is a \( \mathcal{U} \)-invariant subset of \( L^\infty(S) \). Then \( Y^* \) is \( \mathcal{U} \)-invariant.

We lastly generalize (3.1) for later use.

**Theorem 3.5** Let \( 1 \leq p \leq \infty \) and let \( p' \) be its conjugate exponent. Then

\[
\int_S (f \circ U) \cdot g \, d\sigma = \int_S f \cdot (g \circ U^{-1}) \, d\sigma,
\]

for \( f \in L^p(S) \), \( g \in L^{p'}(S) \), and \( U \in \mathcal{U} \).

\section*{4 Unitarily invariant spaces}

### 4.1 The weak*-closed \( \mathcal{U} \)-invariant subspaces of \( L^\infty(S) \)

Recall that each \( E_\Omega \) is \( \mathcal{U} \)-invariant, and consequently, each \( \overline{E_\Omega}^* \) is a weak*-closed \( \mathcal{U} \)-invariant subspace of \( L^\infty(S) \) by Corollary 3.4. The main result of this section (Theorem 4.1) is that the spaces \( \overline{E_\Omega}^* \) are the only weak*-closed \( \mathcal{U} \)-invariant subspaces of \( L^\infty(S) \), and thus each weak*-closed \( \mathcal{U} \)-invariant subspace of \( L^\infty(S) \) is characterized by the corresponding set \( \Omega \). As was the case for Theorem 2.8, \( \Omega \) is the set of all \((p, q) \in \mathbb{Q}\) such that \( \pi_{pq} \) does not annihilate the space.
Theorem 4.1 If $Y$ is a weak*-closed $\mathcal{U}$-invariant subspace of $L^\infty(S)$, then $Y = \overline{E}_\Omega^*$ for some $\Omega \subset Q$.

The Proof of Theorem 4.1 requires Lemma 4.2 (the analogue to Lemma 12.3.5 of [3]), which we prove in Sect. 4.4.

Lemma 4.2 Let $Y \subset L^\infty(S)$ be a $\mathcal{U}$-invariant space. Then for $g \in L^\infty(S)$, we have that $g \notin \overline{Y}^2$ whenever $g \notin \overline{Y}^*$.

We use Lemma 4.2 to establish some final observations.

Remark 4.3 From Remark 2.1 and Lemma 4.2, we have for any $\mathcal{U}$-invariant subspace $Y \subset L^\infty(S)$,

$$
\overline{Y}^* = \overline{Y}^2 \cap L^\infty(S).
$$

Remark 4.4 Remarks 4.3 and 2.7 give a description of the sets $\overline{E}_\Omega^*$:

$$
\overline{E}_\Omega^* = \overline{E}_\Omega^2 \cap L^\infty(S) = \{ f \in L^\infty(S) : \pi_{pq} f = 0 \text{ when } (p, q) \notin \Omega \}.
$$

Proof of Theorem 4.1 Let $Y$ be a weak*-closed $\mathcal{U}$-invariant subspace of $L^\infty(S)$. From Remark 4.3,

$$
Y = \overline{Y}^* = \overline{Y}^2 \cap L^\infty(S).
$$

Since $Y$ is $\mathcal{U}$-invariant, so is $\overline{Y}^2$ from Corollary 3.2. By Theorem 2.8,

$$
\overline{Y}^2 = \overline{E}_\Omega^2,
$$

where $\Omega' = \{ (p, q) \in Q : \pi_{pq} \overline{Y}^2 \neq 0 \}$.

We define $\Omega = \{ (p, q) \in Q : \pi_{pq} \overline{Y} \neq 0 \}$. Then, Remark 4.4 yields

$$
\overline{E}_\Omega^2 \cap L^\infty(S) = \overline{E}_\Omega^*.
$$

We have $\Omega = \Omega'$ by the continuity of each $\pi_{pq}$, and thus $\overline{E}_\Omega^2 = \overline{E}_\Omega^2$. \qed

4.2 The weak*-closed $\mathcal{U}$-invariant subalgebras of $L^\infty(S)$

We now determine those sets $\Omega \subset Q$ whose corresponding weak*-closed $\mathcal{U}$-invariant spaces $\overline{E}_\Omega^*$ are algebras. The main result is that the subsets of $Q$ which induce closed $\mathcal{U}$-invariant subalgebras of $C(S)$ and those which induce weak*-closed $\mathcal{U}$-invariant subalgebras of $L^\infty(S)$ are exactly the same (Theorem 4.7). We can then utilize results in [2] to characterize certain weak*-closed $\mathcal{U}$-invariant subalgebras of $L^\infty(S)$ (Sect. 4.3).

Definition 4.5 A set $\Omega \subset Q$ is a $C(S)$-algebra pattern if $\overline{E}_\Omega$ is an algebra in $C(S)$. 

$\square$ Springer
Definition 4.6 A set $\Omega \subset Q$ is a $L^\infty(S)$-algebra pattern if $\overline{E}_\Omega^*$ is an algebra in $L^\infty(S)$.

Theorem 4.7 A set $\Omega \subset Q$ is a $C(S)$-algebra pattern if and only if $\Omega$ is a $L^\infty(S)$-algebra pattern.

The proof of Theorem 4.7 requires the following necessary and sufficient condition for the set $\overline{E}_\Omega^*$ to be an algebra. Recall the space $H(p, q) \cdot H(r, s)$ from Definition 2.10.

Theorem 4.8 Let $\Omega \subset Q$. Then $\overline{E}_\Omega^*$ is an algebra if and only if $H(p, q) \cdot H(r, s) \subset \overline{E}_\Omega^*$ for $(p, q), (r, s) \in \Omega$.

The proof of Theorem 4.8 is given in Sect. 4.5. The following lemma plays a role in the proofs of both Theorems 4.7 and 4.8.

Lemma 4.9 Suppose $\Omega \subset Q$ and $(p, q), (r, s) \in Q$. The following are all true or all false:

1. $H(p, q) \cdot H(r, s) \subset E_\Omega$.
2. $H(p, q) \cdot H(r, s) \subset \overline{E}_\Omega$.
3. $H(p, q) \cdot H(r, s) \subset \overline{E}_\Omega^*$.

Proof Since $E_\Omega \subset \overline{E}_\Omega \subset \overline{E}_\Omega^*$, we have that (1) implies (2) and (2) implies (3). To show (3) implies (1), observe $H(p, q) \cdot H(r, s)$ is finite-dimensional $\mathcal{U}$-invariant subspace of $C(S)$. Thus, by Theorem 2.8 there exists some set $\Omega' \subset Q$ such that

$$H(p, q) \cdot H(r, s) = E_{\Omega'}.$$

Suppose $(a, b) \in \Omega'$. Assuming (3) is true, we get that

$$H(a, b) \subset H(p, q) \cdot H(r, s) \subset \overline{E}_\Omega^*.$$

A consequence of Remark 4.3 is that for any $\mathcal{U}$-invariant subspace $Y$ of $L^\infty(S)$, $\pi_{pq} Y = 0$ if and only if $\pi_{pq} \overline{Y} = 0$. As a result, we get that $H(a, b) \subset E_\Omega$. Since $E_\Omega$ must also contain the algebraic sum $E_{\Omega'}$ of the sets $H(a, b)$ for $(a, b) \in \Omega'$, we have that (1) holds.

Proof of Theorem 4.7 The proof follows from applications of Lemma 4.9 and Theorem 4.8.

Observe that $\overline{E}_\Omega$ is an algebra if and only if $H(p, q) \cdot H(r, s) \subset \overline{E}_\Omega$ for $(p, q), (r, s) \in \Omega$. This holds if and only if $H(p, q) \cdot H(r, s) \subset \overline{E}_\Omega^*$ for $(p, q), (r, s) \in \Omega$ (Lemma 4.9), which holds if and only if $\overline{E}_\Omega^*$ is an algebra (Theorem 4.8).

Thus, $\overline{E}_\Omega$ is an algebra if and only if $\overline{E}_\Omega^*$ is an algebra. Hence, $\Omega$ is a $C(S)$-algebra pattern if and only if $\Omega$ is a $L^\infty(S)$-algebra pattern.

Having established Theorem 4.7, we thus have that when $n \geq 3$, the weak*-closed $\mathcal{U}$-invariant subalgebras of $L^\infty(S)$ are characterized by those sets $\Omega \subset Q$ described in Theorem 2.9, and that the same exceptions exist when $n = 2$.  

 Springer
4.3 Applications of Theorem 4.7

In this section we state analogues to theorems from [2] which Theorem 4.7 allows us to establish. We omit the proofs, since each is a simple matter of applying the stated result of [2] followed by Theorem 4.7. Recall the sets \(G(d), G(\Sigma), \) and \(G(p, q)\) from Definition 2.11; by Theorem 4.7, each is an \(L^\infty(S)\)-algebra pattern.

The following is the analogue to Theorem 2.12.

**Theorem 4.10**  If \(\Omega\) is an \(L^\infty(S)\)-algebra pattern that contains some \((p, q)\) with \(p > q\) and some \((r, s)\) with \(r < s\), then \(\Omega = G(d)\) for some \(d > 0\).

The following is the analogue to Theorem 2.13.

**Theorem 4.11**  Suppose \(\Omega\) is an \(L^\infty(S)\)-algebra pattern such that \(p \geq q\) for every \((p, q) \in \Omega\) and \((a, a) \in \Omega\) for some \(a > 0\).

1. If \(n \geq 3\), then \(\Omega = G(\Sigma)\).
2. If \(n = 2\) and if \(G^*(\Sigma)\) is obtained from \(G(\Sigma)\) by deleting all \((m, m)\) with \(m\) odd, then \(G^*(\Sigma)\) is also an algebra pattern, and \(\Omega\) is either \(G(\Sigma)\) or \(G^*(\Sigma)\).

The following is the analogue to Theorem 2.14.

**Theorem 4.12**  Fix \((p, q) \in Q\) with \(p > q\). Let \(\Omega^\infty(p, q)\) be the smallest \(L^\infty(S)\)-algebra pattern containing \((p, q)\).

1. If \(n \geq 3\), then \(\Omega^\infty(p, q) = G(p, q)\).
2. If \(n = 2\), then \(\Omega^\infty(p, q)\) is obtained from \(G(p, q)\) by deleting points \((mp - 1, mq - 1)\) for \(m = 2, 3, 4, \ldots\), and points \((2p-j, 2q-j)\) for \(j\) odd.

As in [2], each self-adjoint algebra \(E^*_\Omega\) is described by Theorems 4.10 or 4.11 (with \(\Sigma\) empty).

4.4 The Proof of Lemma 4.2

The following is the analogue to Lemma 12.3.3 of [3].

**Lemma 4.13**  Let \(g \in L^\infty(S)\). Then the map \(\varphi : \mathcal{U} \to L^\infty(S)\) given by \(\varphi(U) = g \circ U\) is weak*-continuous.

**Proof**  To show that \(\varphi\) is weak*-continuous, we verify each \(\Lambda_h \circ \varphi\) is continuous, where \(\Lambda_h\) is the map \(L^\infty(S) \to \mathbb{C}\) given by integration against the function \(h \in L^1(S)\).

Observe the map \(\Lambda_h \circ \varphi\) is given by

\[
(\Lambda_h \circ \varphi)(U) = \int_S (g \circ U) \cdot h \, d\sigma.
\]

From Lemma 12.3.3 of [3], the map \(U \mapsto h \circ U\) is continuous from \(\mathcal{U}\) into \(L^1(S)\). Thus, the map \(U \mapsto g \cdot (h \circ U^{-1})\) is continuous. We apply Theorem 3.5 to get that

\[
U \mapsto \int_S g \cdot (h \circ U^{-1}) \, d\sigma = \int_S (g \circ U) \cdot h \, d\sigma
\]
is continuous, as desired.

\[\text{Proof of Lemma 4.2} \] Suppose \(g \in L^\infty(S)\) and \(g \notin \overline{Y}^*\). There exists some weak*-continuous linear functional \(\Gamma\) on \(L^\infty(S)\) such that \(\Gamma f = 0\) for \(f \in Y\), and \(\Gamma g = 1\).

Since each weak*-continuous linear functional on \(L^\infty(S)\) is induced by an element of \(L^1(S)\), there exists some \(h \in L^1(S)\) such that \(\Gamma f = \int_S Fh \, d\sigma\) for \(F \in L^\infty(S)\).

From Lemma 4.13, there exists a neighborhood \(N\) of the identity in \(U\) such that

\[\text{Re} \int_S (g \circ U) \cdot h \, d\sigma > \frac{1}{2}\]

for \(U \in N\). We choose a continuous map \(\psi : \mathcal{U} \to [0, \infty)\) such that \(\int \psi \, dU = 1\) and the support of \(\psi\) is contained in \(N\) (recall \(dU\) denotes the Haar measure on \(\mathcal{U}\)).

We now define a map \(\Lambda\) on \(L^\infty(S)\) by

\[\Lambda F = \int_S h(z) \int_{\mathcal{U}} \psi(U) \cdot F(Uz) \, dU \, d\sigma(z), \quad \text{for } F \in L^\infty(S).\]

We fix \(F \in L^\infty(S)\) and \(z \in S\) and define the map \(\mathcal{F}_z : \mathcal{U} \to \mathbb{C}\) by \(U \mapsto F(Uz)\).

Since

\[\int_{\mathcal{U}} |\mathcal{F}_z|^2 \, dU = \int_{\mathcal{U}} |F(Uz)|^2 \, dU = \int_S |F|^2 \, d\sigma = ||F||_2^2 < \infty,\]

we get \(\mathcal{F}_z \in L^2(\mathcal{U})\), and since \(\psi\) is continuous, we have that \(\psi \in L^2(\mathcal{U})\). Further,

\[\left| \int_{\mathcal{U}} \psi \mathcal{F}_z \, dU \right| \leq \left( \int_{\mathcal{U}} |\psi|^2 \, dU \right)^{\frac{1}{2}} \left( \int_{\mathcal{U}} |\mathcal{F}_z|^2 \, dU \right)^{\frac{1}{2}} = ||\psi||_2 \cdot ||F||_2,\]

so that for \(F \in L^\infty(S)\),

\[||\Lambda F|| \leq ||\psi||_2 \cdot ||F||_2 \int_S |h| \, d\sigma = ||\psi||_2 \cdot ||F||_2 \cdot ||h||_1.\]

The linearity of \(\Lambda\) on \(L^\infty(S)\) is clear. Thus, \(\Lambda\) defines an \(L^2\)-continuous linear functional on \(L^\infty(S)\), and hence extends to an \(L^2\)-continuous linear functional \(\Lambda_1\) on \(L^2(S)\). By interchanging the integrals in the definition of \(\Lambda\), we see that \(\Lambda_1\) annihilates \(Y\), since \(Y\) is \(\mathcal{U}\)-invariant. Further,

\[\text{Re} \Lambda_1 g = \int_{\mathcal{U}} \psi(U) \left( \text{Re} \int_S g(Uz) \cdot h(z) \, d\sigma(z) \right) \, dU > \int_N \psi(U) \cdot \frac{1}{2} \, dU = \frac{1}{2}.\]

We conclude that \(g \notin \overline{Y}^2\). \(\square\)
4.5 The Proof of Theorem 4.8

Observe that Theorem 4.8 is not easily established due to the fact that multiplication of $L^\infty$-functions is not continuous with respect to the weak*-topology. However, this multiplication is separately continuous (Definition 4.14), which is sufficient to prove Theorem 4.8.

Definition 4.14 Let $A$ be a topological space and $m$ a map $A \times A \to A$. For each element $a \in A$, we define the maps

$m_a : A \to A$ given by $m_a(x) = m(a, x)$, and

$m^a : A \to A$ given by $m^a(x) = m(x, a)$.

We say $m$ is separately continuous if the maps $m_a$ and $m^a$ are continuous for $a \in A$. Further, if $B \subset A$, we say $B$ is invariant under $m$ if $m(B \times B) \subset B$.

Proposition 4.15 Let $f \in L^\infty(S)$. The left multiplication operator $m_f : L^\infty(S) \to L^\infty(S)$ defined by $m_f(h) = fh$ is weak*-continuous.

Proof Let $\Lambda$ be a weak*-continuous linear functional on $L^\infty(S)$ and $g \in L^1(S)$ be such that $\Lambda h = \int_S h g \, d\sigma$ for $h \in L^\infty(S)$. We observe then that

$$(\Lambda \circ m_f)(h) = \Lambda(fh) = \int_S (fh)g \, d\sigma = \int_S h(fg) \, d\sigma.$$ 

Since $fg \in L^1(S)$, $\Lambda \circ m_f$ is a weak*-continuous linear functional on $L^\infty(S)$.

We can similarly define a right multiplication operator $m^g : L^\infty(S) \to L^\infty(S)$ given by $m^g(h) = hg$. Since pointwise multiplication for functions is commutative, the right multiplication operators are also weak*-continuous.

Remark 4.16 By Proposition 4.15, the multiplication map $m : L^\infty(S) \times L^\infty(S) \to L^\infty(S)$ given by $m(f, g) = fg$ is separately continuous with respect to the weak*-topology.

Lemma 4.17 Let $A$ be a topological space and $m : A \times A \to A$. Suppose $B \subset A$ with $B$ invariant under $m$. If $m$ is separately continuous, then $\overline{B}$ is invariant under $m$.

Proof We begin by proving the following claim: If $f \in \overline{B}$, then $m_f(B) \subset \overline{B}$.

Let $g \in B$. Since $B$ is invariant under $m$, we get that $m^g(B) \subset B$. Observe since $m$ is separately continuous, the map $m^g$ is continuous, and thus we have

$$m_f(g) = m^g(f) \in m^g(\overline{B}) \subset \overline{m^g(B)} \subset \overline{B}.$$ 

Since $g \in B$ was arbitrary, this proves the claim.

We are now ready to prove the lemma.

Let $f, g \in \overline{B}$. Then, we have that

$$m(f, g) = m_f(g) \in m_f(\overline{B}) \subset \overline{m_f(B)},$$

\[\square\] Springer
since $mf$ is continuous. However, by the above claim, $mf(B) \subset B$, and thus $m(f, g) \in B$.

Since we chose $f, g$ arbitrarily from $B$, we conclude that $B$ is invariant under $m$.

\[ \square \]

**Proof of Theorem 4.8** Suppose $E^*_\Omega$ is an algebra and fix pairs $(p, q), (r, s) \in \Omega$. If we let $h \in H(p, q) \cdot H(r, s)$, then

$$h = \sum_{k=1}^{n} f_k g_k,$$

where $f_k \in H(p, q)$ and $g_k \in H(r, s)$ for all $k$. Since $H(a, b) \subset E_\Omega \subset E^*_\Omega$ for $(a, b) \in \Omega$, we have that $f_k, g_k \in E^*_\Omega$ for all $k$. Since $E^*_\Omega$ is an algebra, we have that $f_k g_k \in E^*_\Omega$ for all $k$, and further $h \in E^*_\Omega$. Thus, $H(p, q) \cdot H(r, s) \subset E^*_\Omega$, which completes the forward direction.

We now show the backward direction. Suppose $H(p, q) \cdot H(r, s) \subset E^*_\Omega$ for $(p, q), (r, s) \in \Omega$. To show that $E^*_\Omega$ is an algebra, we must verify that $E^*_\Omega$ is invariant under the multiplication map $m : L^\infty(S) \times L^\infty(S) \to L^\infty(S)$ given by $m(f, g) = fg$ for $f, g \in L^\infty(S)$. By Remark 4.16, this multiplication map is separately continuous. Thus, by showing that $E_\Omega$ is invariant under $m$, we can get that $E^*_\Omega$ is invariant under $m$ by Lemma 4.17.

Let $f, g \in E_\Omega$. Then,

$$f = \sum_{i=1}^{n} f_i, \text{ where } f_i \in H(p_i, q_i) \text{ and } (p_i, q_i) \in \Omega, \text{ for } 1 \leq i \leq n, \text{ for some } n,$$

and

$$g = \sum_{j=1}^{m} g_j, \text{ where } g_j \in H(r_j, s_j) \text{ and } (r_j, s_j) \in \Omega, \text{ for } 1 \leq j \leq m, \text{ for some } m.$$

Further, we have that

$$fg = \sum_{i=1}^{n} \sum_{j=1}^{m} f_i g_j, \text{ and } f_i g_j \in H(p_i, q_i) \cdot H(r_j, s_j) \text{ for all } i, j.$$

Since $H(p_i, q_i) \cdot H(r_j, s_j) \subset E^*_\Omega$ for all $i, j$, we have that $H(p_i, q_i) \cdot H(r_j, s_j) \subset E_\Omega$ for all $i, j$ by Lemma 4.9. Hence, $f_i g_j \in E_\Omega$ for all $i, j$. As a subspace, $E_\Omega$ is in particular closed under finite sums, and thus $fg \in E_\Omega$. Since $f$ and $g$ were chosen arbitrarily in $E_\Omega$, we conclude that $E_\Omega$ is invariant under $m$. By Lemma 4.17, $E^*_\Omega$ is invariant under $m$, and hence $E^*_\Omega$ is an algebra.

\[ \square \]

**5 Möbius invariant spaces**

**5.1 The weak*-closed $\mathcal{M}$-invariant subspaces of $L^\infty(S)$**

In this section we determine the weak*-closed $\mathcal{M}$-invariant subspaces of $L^\infty(S)$ (Theorem 5.1). We begin by noting that every $\mathcal{M}$-invariant subset of $L^\infty(S)$ is also
\(\mathcal{U}\)-invariant, since every bijective holomorphic map from \(B\) to \(B\) is a unitary transformation. Thus by Theorem 4.1, determining the weak*-closed \(\mathcal{M}\)-invariant subspaces of \(L^\infty(S)\) only requires finding those sets \(\Omega \subset Q\) such that \(E^*_\Omega\) is \(\mathcal{M}\)-invariant. Similarly to Theorem B of [1], there are only six such spaces.

**Theorem 5.1** The following are the weak*-closed \(\mathcal{M}\)-invariant subspaces of \(L^\infty(S)\):

1. The null space \(\{0\}\),
2. The space of constant functions \(\mathbb{C}\),
3. The space \(H^\infty(S)\),
4. The space \(\text{Conj}(H^\infty(S))\),
5. The weak*-closure of the space \(H^\infty(S) + \text{Conj}(H^\infty(S))\), and
6. The space \(L^\infty(S)\).

The proof of this result uses the following lemmas. The first (Lemma 5.2) is the analogue to Lemma 2.15, which it also requires in its proof. The use of the second, Lemma 5.3, is clear from the statement.

**Lemma 5.2** Let \(Y\) be a weak*-closed \(\mathcal{M}\)-invariant subspace of \(L^\infty(S)\), \(p \geq 1\) an integer, and \(H(p, q) \subset Y\) for some \(q \geq 0\). Then \(H(p - 1, q) \subset Y\) and \(H(p + 1, q) \subset Y\).

**Proof** Since \(Y\) is \(\mathcal{M}\)-invariant, \(Y\) is in particular \(\mathcal{U}\)-invariant, and thus there exists some \(\Omega \subset Q\) such that \(Y = E^*_\Omega\) by Theorem 4.1. Consequently, we get \(H(p, q) \subset E^*_\Omega\).

We now define \(Y_C = Y \cap C(S)\). Since \(Y\) is weak*-closed in \(L^\infty(S)\), we have that \(Y\) is norm-closed in \(L^\infty(S)\), and hence \(Y_C\) is closed in \(C(S)\). Further, \(Y_C\) is \(\mathcal{M}\)-invariant as the intersection of \(\mathcal{M}\)-invariant spaces. We also observe that \(E^*_\Omega \subset Y_C\) and hence \(H(p, q) \subset Y_C\).

Thus by Lemma 2.15, we have \(H(p - 1, q) \subset Y_C\) and \(H(p + 1, q) \subset Y_C\), and the desired result follows accordingly.

**Lemma 5.3** Let \(\Omega\) be the set \(\{(p, q) : q = 0\}\). Then the space \(E^*_\Omega\) is the space \(H^\infty(S)\).

The Proof of Lemma 5.3 uses some results concerning the Poisson kernel and the Poisson integrals of \(L^1\)-functions. As such, we save the Proof of Lemma 5.3 for Sect. 5.3.

**Proof of Theorem 5.1** Since the roles of \(p\) and \(q\) can be switched in Lemma 2.15, the roles of \(p\) and \(q\) in Lemma 5.2 can also be switched. Thus, if \(Y\) is a weak*-closed \(\mathcal{M}\)-invariant subspace of \(L^\infty(S)\) and \(\Omega\) is the subset of \(Q\) such that \(E^*_\Omega = Y\), repeated application of Lemma 5.2 yields only the following six possible options for the set \(\Omega\):

1. \(\Omega = \emptyset\).
2. \(\Omega = \{(0, 0)\}\).
3. \(\Omega = \{(p, q) : q = 0\}\).
4. \(\Omega = \{(p, q) : p = 0\}\).
5. \(\Omega = \{(p, q) : \text{Either } p = 0 \text{ or } q = 0\}\).
6. \(\Omega = Q\).

\(\square\) Springer
These are the same six possible options found in [1]. We now determine the spaces they induce. This process is easiest for the sets in (1), (2), (6), and a special case for (5).

When \( \Omega = \emptyset \), the corresponding weak*-closed \( \mathcal{M} \)-invariant subspace is the algebra consisting only of the zero function: By Remark 4.4,

\[
\overline{E}_\emptyset^* = \{ f \in L^\infty(S) : \pi_{pq} f = 0 \text{ for all } (p, q) \}.
\]

Clearly, 0 ∈ \( \overline{E}_\emptyset^* \), and further, if \( g \in \overline{E}_\emptyset^* \), we have \( g = \sum \pi_{pq} g = 0 \).

When \( \Omega = \{(0, 0)\} \), the corresponding weak*-closed \( \mathcal{M} \)-invariant subspace is the algebra of constant functions: By Remark 4.4,

\[
\overline{E}_{\{(0, 0)\}}^* = \{ f \in L^\infty(S) : \pi_{pq} f = 0 \text{ for all } (p, q) \neq (0, 0) \}.
\]

Clearly, \( \mathbb{C} \subset \overline{E}_{\{(0, 0)\}}^* \), and further, if \( g \in \overline{E}_{\{(0, 0)\}}^* \), we have \( g = \sum \pi_{pq} g = \pi_{00} g \in \mathbb{C} \).

For \( \Omega = Q \), we have that \( \overline{E}_Q^* = L^\infty(S) \). This follows from Remark 4.3 and Theorem 2.5.

As a special case for the set in (5), when \( n = 1 \) we get that \( \overline{E}_Q^* = L^\infty(S) \). This is due to the fact that either \( p = 0 \) or \( q = 0 \) for each space \( H(p, q) \) when \( n = 1 \). Thus, the set \( \Omega \) consists of all possible pairs \( (p, q) \); that is, \( \Omega = Q \). Since \( \overline{E}_Q^* = L^\infty(S) \), we get that \( \overline{E}_Q^* = L^\infty(S) \).

Determining the spaces \( \overline{E}_\Omega^* \) for the remaining options for \( \Omega \) (\( n > 1 \) for the fifth option) is not as easy. Fortunately, the descriptions of these spaces from Remark 4.4 show that determining \( \overline{E}_\Omega^* \) for the set in (3) will suffice to get the remaining two spaces.

By Lemma 5.3, we have that \( \overline{E}_\Omega^* = H^\infty(S) \) when \( \Omega = \{(p, q) : q = 0 \} \). Observe then for any nonnegative integer \( p \),

\[ H(0, p) = \text{Conj}(H(p, 0)). \]

Consequently, we get \( \overline{E}_\Omega^* = \text{Conj}(H^\infty(S)) \) when \( \Omega = \{(p, q) : p = 0 \} \).

When \( \Omega = \{(p, q) : \text{Either } p = 0 \text{ or } q = 0 \} \), we similarly have that

\[
\overline{E}_\Omega^* = \{ f \in L^\infty(S) : \pi_{pq} f = 0 \text{ when } p > 0 \text{ and } q > 0 \},
\]

which is the weak*-closure of \( H^\infty(S) + \text{Conj}(H^\infty(S)) \) by the previous two cases. ⊫

### 5.2 The weak*-closed \( \mathcal{M} \)-invariant subalgebras of \( L^\infty(S) \)

**Theorem 5.4** The following are the weak*-closed \( \mathcal{M} \)-invariant subalgebras of \( L^\infty(S) \):

1. The null space \( \{0\} \),
2. The space of constant functions \( \mathbb{C} \),
3. The space \( H^\infty(S) \),
4. The space \( \text{Conj}(H^\infty(S)) \),
5. The weak*-closure of the space $H^\infty(S) + \text{Conj}(H^\infty(S))$ if and only if $n = 1$, and

6. The space $L^\infty(S)$.

**Proof** The spaces listed in the above statement are the weak*-closed $\mathcal{M}$-invariant subspaces of $L^\infty(S)$ by Theorem 5.1. All except the weak*-closure of $H^\infty(S) + \text{Conj}(H^\infty(S))$ are clearly algebras for all $n$. Recall when $n = 1$, we have that the weak*-closure of this space is $L^\infty(S)$, which is clearly an algebra. We thus need only show that for $n > 1$, the weak*-closure of $H^\infty(S) + \text{Conj}(H^\infty(S))$ is not an algebra.

We fix $\Omega = \{(p, q) : \text{Either } p = 0 \text{ or } q = 0 \}$ and consider the coordinate functions $f(z) = z_1$ and $g(z) = \bar{z}_n$. Clearly, both $f$ and $g$ are elements of the weak*-closure of $H^\infty(S) + \text{Conj}(H^\infty(S))$, since $f \in H(1, 0) \subset E_{\Omega}$ and $g \in H(0, 1) \subset E_{\Omega}$. However, their product, given by $(fg)(z) = z_1 \bar{z}_n$, is an element of $H(1, 1)$. Hence $fg \notin E_{\Omega}$ and thus the weak*-closure of $H^\infty(S) + \text{Conj}(H^\infty(S))$ is not an algebra when $n > 1$. 

\[\square\]

5.3 The Proof of Lemma 5.3

The Proof of Lemma 5.3 uses the following lemma as well as standard results concerning the Poisson integrals of $L^1$-functions. These results can be found in many texts, one of which is [4].

**Lemma 5.5** Let $f$ be holomorphic on $B$ and $0 < r < 1$. If we define $f_r(\zeta) = f(r\zeta)$ for $\zeta \in S$, then $\pi_{pq} f_r = 0$ whenever $q > 0$.

**Proof** Since $f$ is holomorphic on $B$, $f$ has a power series representation $\sum f_\alpha \zeta^\alpha$ on $B$ which converges absolutely and uniformly to $f$ on compacta of $B$ ([3]).

Observe then that $f_r \in C(S)$ and $\sum f_\alpha r^{|\alpha|} \zeta^\alpha$ converges uniformly to $f_r$ on $S$. In particular, $\sum f_\alpha r^{|\alpha|} \zeta^\alpha$ converges to $f_r$ in $L^2(S)$, and hence is the unique $L^2$ expansion of $f_r$. Clearly then, $\pi_{pq} f_r = 0$ whenever $q > 0$. 

\[\square\]

**Proof of Lemma 5.3** Let $\Omega = \{(p, q) : q = 0 \}$ and suppose that $f \in H^\infty(S)$. To show that $f \in E^S_{\Omega}$, we verify that $\pi_{pq} f = 0$ whenever $q > 0$. Let $g \in H^\infty(B)$ be the function such that

$$
\lim_{r \to 1} g(r\zeta) = f(\zeta)
$$

for almost all $\zeta \in S$. For each positive integer $m$, we define the function $g_m$ on $S$ by

$$
g_m(\zeta) = g\left(\frac{m - 1}{m} \zeta\right)
$$

for $\zeta \in S$. Then,

$$
\lim_{m \to \infty} g_m(\zeta) = f(\zeta)
$$

for almost all $\zeta \in S$.

\[\square\] Springer
We now fix \((p, q) \in Q\) and \(z \in S\). Recall from Theorem 2.5(a), there exists a unique element \(K_z \in H(p, q)\) such that

\[
(\pi_{pq} h)(z) = \int_S h K_z \, d\sigma,
\]

for \(h \in L^2(S)\). Clearly, then, for \(z \in S\), we have that

\[
\lim_{m \to \infty} g_m(\xi) K_z(\xi) = f(\xi) K_z(\xi)
\]

for almost all \(\xi \in S\).

Each \(K_z \in C(S)\) and hence has uniform norm \(||K_z||\). Further, since \(g \in H^\infty(B)\), \(g\) has uniform norm \(||g||\). Thus, for each \(z \in S\),

\[
|g_m(\xi) K_z(\xi)| \leq ||g|| \cdot ||K_z||.
\]

for \(\xi \in S\), and by the Lebesgue Dominated Convergence Theorem,

\[
\lim_{m \to \infty} \int_S g_m(\xi) K_z(\xi) \, d\sigma(\xi) = \int_S f(\xi) K_z(\xi) \, d\sigma(\xi).
\]

Hence, \((\pi_{pq} g_m)(z) \to (\pi_{pq} f)(z)\) as \(m \to \infty\), for \(z \in S\). From Lemma 5.5, \(\pi_{pq} g_m = 0\) whenever \(q > 0\). Hence, \(\pi_{pq} f = 0\) whenever \(q > 0\), and thus \(f \in \mathcal{E}_\Omega^*\).

Suppose now \(f \in \mathcal{E}_\Omega^*\) and let \(P[f]\) denote the Poisson integral of \(f\). We observe that \(P[f]\) is bounded on \(B\) and further, \(P[f]\) has radial limits \(f(\xi)\) at almost every \(\xi \in S\). We lastly show that \(P[f]\) is holomorphic on \(B\) as the uniform limit on compacta of a sequence of holomorphic functions.

Recall from Remark 2.6, the sum \(\sum \pi_{pq} f\) converges absolutely and unconditionally to \(f\) in the \(L^2\)-norm. Let \(\pi_p\) denote the map \(\pi_{p0}\) for all \(p\). We then have

\[
f = \sum_{p=0}^\infty \pi_p f,
\]

with convergence in the \(L^2\)-norm.

Each \(\pi_p f\) is a holomorphic polynomial as an element of \(H(p, 0)\), and so is each partial sum \(s_k\), defined by

\[
s_k = \sum_{p=0}^k \pi_p f.
\]

Observe \(s_k\) converges to \(f\) in the \(L^2\)-norm, and hence in the \(L^1\)-norm. Since each \(s_k\) is holomorphic on \(B\) and continuous on \(\overline{B}\), the corresponding Poisson integral \(P[s_k]\) is holomorphic on \(B\).
Let $K$ be a compact subset of $B$. Observe there exists some $M_K > 0$ such that

$$P(z, \zeta) \leq M_K$$

for $(z, \zeta) \in K \times S$.

Suppose now that $z \in K$. We thus have

$$\left| P[s_k](z) - P[f](z) \right| \leq \int_S |s_k(\zeta) - f(\zeta)| P(z, \zeta) d\sigma(\zeta)$$

$$\leq M_K \int_S |s_k(\zeta) - f(\zeta)| d\sigma(\zeta).$$

The integral farthest to the right above goes to 0, since $s_k \to f$ in the $L^1$-norm, and hence

$$P[s_k] \to P[f]$$

uniformly on $K$. We conclude that $P[s_k]$ converges to $P[f]$ uniformly on compacta of $B$, and thus, $P[f]$ is holomorphic on $B$. Since $f$ is the radial limit almost everywhere on $S$ of a bounded holomorphic function on $B$, we conclude that $f \in H^\infty(S)$. \[\square\]

**Data Availability Statement** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**References**

1. Nagel, A., Rudin, W.: Moebius-invariant function spaces on balls and spheres. Duke Math. J. **43**(4), 841–865 (1976)
2. Rudin, W.: Unitarily invariant algebras of continuous functions on spheres. Houston J. Math. **5**(2), 253–265 (1979)
3. Rudin, W.: Function theory in the unit ball of $\mathbb{C}^n$. In: Classics in Mathematics. Springer-Verlag, Berlin (2008) *(Reprint of the 1980 edition)*
4. Axler, S., Bourdon, P., Ramey, W.: Harmonic Function Theory. Graduate Texts in Mathematics, vol. 137, 2nd edn. Springer-Verlag, New York (2001)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.