On Language Varieties Without Boolean Operations

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Abstract. Eilenberg's variety theorem marked a milestone in the algebraic theory of regular languages by establishing a formal correspondence between properties of regular languages and properties of finite monoids recognizing them. Motivated by classes of languages accepted by quantum finite automata, we introduce basic varieties of regular languages, a weakening of Eilenberg's original concept that does not require closure under any boolean operations, and prove a variety theorem for them. To do so, we investigate the algebraic recognition of languages by lattice bimodules, generalizing Klíma and Polák's lattice algebras, and we utilize the duality between algebraic completely distributive lattices and posets.

1 Introduction

The introduction of algebraic methods into the study of regular languages provides a convenient classification system that allows to study finite automata and their languages in terms of associated finite algebraic structures. A celebrated example is Schützenberger's theorem [21] stating that a language is star-free if and only if its syntactic monoid is aperiodic, thus proving the decidability of star-freeness. Eilenberg's variety theorem [9] formalizes this type of correspondence as a bijection between varieties of regular languages (i.e., classes of regular languages closed under the set-theoretic boolean operations, word derivatives and preimages of monoid homomorphisms) and pseudovarieties of monoids (i.e., classes of finite monoids closed under finite products, submonoids and quotient monoids).

Numerous extensions and generalizations of Eilenberg's theorem have been discovered over the past four decades, differing from the original one by either changing the type of languages under consideration, e.g., from regular languages to \(\omega\)-regular languages [23], or by considering notions of varieties with relaxed closure properties. On the algebraic side, such a relaxation requires to replace monoids by more complex algebraic structures. For instance, Pin [17] studied positive varieties of regular languages, where the closure under complement is dropped, and proved them to biject with pseudovarieties of ordered monoids. Subsequently, Polák [18] introduced disjunctive varieties of regular languages,

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where in addition to closure under complement also the closure under intersection is dropped, and related them to pseudovarieties of idempotent semirings.

One item is conspicuously missing from this list: a variety theorem for classes of languages that need not be closed under any boolean operations, i.e. in which only closure under word derivatives and preimages of monoid homomorphisms is required. Such basic varieties of regular languages subsume all the above notions of varieties and naturally arise in several areas of automata theory, most notably in the study of languages accepted by reversible finite automata [11] or quantum finite automata [13]. In the present paper, we close this gap by developing the theory of basic varieties. As the corresponding algebraic structure we introduce lattice bimodules, a two-sorted generalization of the lattice algebras recently studied by Klíma and Polák [12], as algebraic recognizers for regular languages. The two-sorted approach allows for a clearer and more conceptual view of the underlying categorical and universal algebraic concepts. As our main result, we establish the following algebraic classification of basic varieties:

**Basic Variety Theorem.** Basic varieties of regular languages correspond bijectively to pseudovarieties of lattice bimodules.

This answers the open problem of Klíma and Polák [12] about an Eilenberg-type correspondence. Our presentation of the theorem and its proof is inspired by the recently developed duality-theoretic perspective on algebraic language theory [1, 10, 20, 22], which provides the insight that correspondences between language varieties and pseudovarieties of algebraic structures can be understood in terms of an underlying dual equivalence of categories. In our setting, we shall demonstrate that pseudovarieties of lattice bimodules can be interpreted as theories of lattice bimodules in the category AlgCDL of algebraic completely distributive lattices, while basic varieties give rise to (basic) cotheories of regular languages in the category Pos of posets. Our Eilenberg correspondence for basic varieties then boils down to an application of the duality $\text{AlgCDL} \cong \text{Pos}^\text{op}$.

Let us note that our main result is not an instance of previous category-theoretic generalizations of Eilenberg’s theorem [1, 5, 20, 22] since the two-sorted nature of lattice bimodules requires to introduce the novel concept of reduced structures, which makes the ensuing notion of pseudovariety more intricate than the ones studied in op. cit. However, much of the general methodology developed there turns out to apply smoothly, which can be seen as further evidence of its scope and flexibility. In order to make the present paper accessible to readers not familiar with the previous work, we opted to give a self-contained presentation of our results, merely assuming some familiarity with basic category theory.

## 2 Lattice Bimodules

In this section we introduce a new algebraic structure whose aim it is to capture languages varieties that are not necessarily closed under boolean operations. Our notion is a two-sorted generalization of Klíma and Polák’s lattice algebras [12]. Intuitively, for our intended purpose the following structure should be present:
(1) a monoid action that corresponds to word derivation on the language side;
(2) lattice-like operations to compensate for the missing closure under union and intersection on the language side;
(3) equational axioms specifying the interaction of (1) and (2).

From a categorical perspective, the last point means that our algebras can be modeled by a monad. This allows us to use previous work on languages recognizable by monad algebras \([5,20,22]\) as a guide towards our results.

While Klíma and Polák considered distributive lattices with an embedded monoid acting on them, we upgrade the lattice to a completely distributive lattice (shortly, CDL), i.e. a complete lattice satisfying the infinite distributive law \(\bigvee_{i\in I} \bigwedge_{j\in J_i} x_{i,j} = \bigwedge_{j\in F} \bigvee_{i\in I} x_{i,f(i)}\) for every family \(\{x_{i,j} : i \in I, j \in J_i\}\) of elements, where \(F\) is the set of all choice functions \(f\) mapping each \(i \in I\) to some \(f(i) \in J_i\). Morphisms of CDLs are maps preserving all joins and meets.

We let CDL denote the category of CDLs and their morphisms. Even though completeness makes no difference for finite structures, completely distributive lattices admit a more convenient duality theory than general distributive lattices.

In addition, in lieu of an embedded monoid we use a two-sorted structure with a monoid in the first sort. This avoids partial operations, which are somewhat awkward from the perspective of (categorical) universal algebra.

**Definition 2.1.** (1) A lattice bimodule \((M, D, \iota, \triangleright, \triangleleft)\), abbreviated as \((M, D)\), is given by a monoid \((M, \cdot, 1)\), a CDL \((D, \vee, \wedge)\), and three operations
\[
\triangleright : M \times D \to D, \quad \triangleleft : D \times M \to D, \quad \iota : M \to D,
\]
such that \(\triangleright\) and \(\triangleleft\) form a monoid biaction of \(M\) on \(D\) that distributes over the lattice operations, and \(\iota\) translates the multiplication of \(M\) to \(\triangleleft\) and \(\triangleright\); that is, for all \(m, n \in M, d \in D\) and \(\{d_i\}_{i \in I} \subseteq D\), the following equational laws hold:
\[
\begin{align*}
(m \cdot n) \triangleright d &= m \triangleright (n \triangleright d), & (m \triangleright n) \triangleleft d &= (m \triangleleft n) \triangleright d, \\
1 \triangleright d &= d, & d \triangleleft 1 &= d, \\
(m \triangleright d) \triangleleft n &= m \triangleright (d \triangleleft n), & m \triangleright \bigvee_{i \in I} d_i &= \bigvee_{i \in I} (m \triangleright d_i), \\
(m \triangleright \bigwedge_{i \in I} d_i) \triangleleft n &= m \triangleright (\bigwedge_{i \in I} d_i \triangleleft n), & (\bigwedge_{i \in I} d_i) \triangleright m &= \bigwedge_{i \in I} (d_i \triangleright m), \\
\iota(m \cdot n) &= \iota(m \cdot n), & \iota(m) \triangleright n &= \iota(m \cdot n).
\end{align*}
\]

Note that since the least and the greatest element of \(D\) are given by \(\bot = \bigvee \emptyset\) and \(\top = \bigwedge \emptyset\), resp., we also have \(m \triangleright \bot = \bot = \bot \triangleleft m\) and \(m \triangleright \top = \top = \top \triangleleft m\).

(2) A homomorphism from a lattice bimodule \((M, D, \iota, \triangleright, \triangleleft)\) to a lattice bimodule \((M', D', \iota', \triangleright', \triangleleft')\) is given by a two-sorted map \(h = (h^*, h^\circ) : (M, D) \to (M', D')\) such that \(h^*\) is a monoid homomorphism, \(h^\circ\) is a morphism of completely distributive lattices and the following diagrams commute:
\[
\begin{align*}
M \times D &\xrightarrow{\triangleright} D & D \times M &\xrightarrow{\triangleleft} D & M &\xrightarrow{\iota} D \\
M' \times D' &\xrightarrow{\triangleright'} D' & D' \times M' &\xrightarrow{\triangleleft'} D' & M' &\xrightarrow{\iota'} D'
\end{align*}
\]
Subbimodules and quotient bimodules of lattice bimodules are represented by sortwise injective and surjective homomorphisms, respectively.

We let LBM denote the category of lattice bimodules and their homomorphisms. A free lattice bimodule over a pair \((\Sigma, \Gamma)\) of sets is given by a lattice bimodule \((\hat{\Sigma}, \hat{\Gamma})\) together with a sorted map \(\eta = (\eta^*, \eta^0)\): \((\Sigma, \Gamma) \rightarrow (\hat{\Sigma}, \hat{\Gamma})\) satisfying the universal mapping property: for every sorted map \(h_0\): \((\Sigma, \Gamma) \rightarrow (M, D)\) to a lattice bimodule \((M, D)\) there exists a unique lattice bimodule homomorphism \(h: (\hat{\Sigma}, \hat{\Gamma}) \rightarrow (M, D)\) such that \(h \cdot \eta = h_0\). In the following, we denote by \(\Sigma^*\) the free monoid on the set \(\Sigma\) with neutral element \(\varepsilon \in \Sigma^*\) and by \(\text{FCDL}(\Gamma)\) the free completely distributive lattice [15] on the set \(\Gamma\). The latter can be described as the lattice of downwards closed subsets of the power set \(\mathcal{P}(\Gamma)\), or equivalently as the lattice of all formal expressions \(\bigvee_{i \in I} \bigwedge_{j \in J} x_{i,j}\), where \(x_{i,j} \in \Gamma\), modulo the equational laws of CDLs. We view \(\Gamma\) as a subset of \(\text{FCDL}(\Gamma)\).

**Proposition 2.2.** The free lattice bimodule over \((\Sigma, \Gamma)\) is given by \(\eta: (\Sigma, \Gamma) \rightarrow (\Sigma^*, \text{FCDL}(\Sigma^*) \times \Gamma \times \Sigma^*))\) with \(\eta^*(a) = a, \eta^0(b) = (\varepsilon, b, \varepsilon)\), and operations uniquely determined by the following identities for \(u, v, w \in \Sigma^*\) and \(z \in \Gamma\):

\[
\iota(u) = u, \quad uv = u\cdot v = vw, \quad (v, z, w) = (uv, z, w), \quad u\cdot v = uv, \quad (v, z, w)\cdot u = (v, z, wu).
\]

**Notation 2.3.** We write \((\Sigma^*, \Sigma^0) = (\Sigma^*, \text{FCDL}(\Sigma^*))\) for the free lattice bimodule on \((\Sigma, \emptyset)\). Note that a homomorphism \(h: (\Sigma^*, \Sigma^0) \rightarrow (M, D)\) is completely determined by its first component \(h^*: \Sigma^* \rightarrow M\). In fact, its second component \(h^0: \Sigma^0 \rightarrow D\) is the unique CDL-morphism extending the map \(\iota \cdot h^*: \Sigma^* \rightarrow D\).

We now define three properties of lattice bimodules needed subsequently.

**Definition 2.4.** A lattice bimodule \((M, D)\) is called

1. \(*\)-generated if the complete lattice \(D\) is generated by the image \(\iota[M] \subseteq D\); For all \(d \in D\) there exist elements \(m_{i,j} \in M\) such that \(d = \bigvee_{i \in I} \bigwedge_{j \in J} \iota(m_{i,j})\);
2. \(*\)-embedded if the operation \(\iota: M \rightarrow D\) is injective;
3. reduced if for every quotient bimodule \(h: (M, D) \rightarrow (M', D')\) such that \(h^*: D \rightarrow D'\) is a CDL-isomorphism, \(h\) is an LBM-isomorphism.

We note that finite \(*\)-embedded lattice bimodules are precisely the finite lattice algebras of Klíma and Polák [12]. The following lemma links the above concepts:

**Lemma 2.5.** (1) A lattice bimodule \((M, D)\) is \(*\)-generated if and only if there exists a surjective homomorphism from \((\Sigma^*, \Sigma^0)\) to \((M, D)\) for some set \(\Sigma\).

(2) Every \(*\)-embedded lattice bimodule is reduced.

(3) Every \(*\)-generated reduced lattice bimodule is \(*\)-embedded.

In Section 3 we will study lattice bimodules that are \(*\)-generated and reduced (equivalently, \(*\)-generated and \(*\)-embedded). Intuitively, these properties capture lattice bimodules whose monoid component generates the lattice component and is “minimal” with that property. This allows us to relate the two-sorted notion of lattice bimodules to the single-sorted notion of languages recognized by them.
In the categorical approach to variety theorems [22] it was shown that the key to understanding language derivatives lies in the concept of a *unary presentation* of an algebraic structure. Informally, such a presentation expresses the structure of an algebra in terms of suitable unary operations in the underlying category, which then dualize to the derivative operations on the set of languages recognized by that algebra. The heterogeneous nature of our present setting, which regards lattice bimodules as algebraic structures over the product category $\text{Set} \times \text{CDL}$, requires a slight adaptation of the concepts from op. cit.

**Definition 2.6.** Let $(M, D)$ be a lattice bimodule. A *unary operation* on $(M, D)$ is either a map of type $M \rightarrow M$ or $M \rightarrow D$, or a CDL-morphism $D \rightarrow D$. A set $U$ of unary operations forms a *unary presentation* of $(M, D)$ if for every pair $e = (e^*, e^\circ)$ of a surjective map $e^*: M \rightarrow M'$ and a surjective CDL-morphism $e^\circ: D \rightarrow D'$, the following statements are equivalent:

1. There exists a lattice bimodule structure on $(M', D')$ making $e$ a homomorphism of lattice bimodules.
2. For every $u \in U$, there exists $\bar{u}$ making the respective square below commute:

   $\begin{array}{ccc}
   M & \xrightarrow{u} & M \\
   e^* & \downarrow & e^* \\
   M' & \xrightarrow{u} & M'
   \end{array}$ \quad $\begin{array}{ccc}
   M & \xrightarrow{u} & D \\
   e^* & \downarrow & e^\circ \\
   M' & \xrightarrow{\bar{u}} & D'
   \end{array}$ \quad $\begin{array}{ccc}
   D & \xrightarrow{u} & D \\
   e^\circ & \downarrow & e^\circ \\
   D' & \xrightarrow{\bar{u}} & D'
   \end{array}$

**Lemma 2.7.** Every lattice bimodule $(M, D)$ admits a unary presentation composed of the following unary operations ranging over $m \in M$ and $d \in D$:

$(m \cdot), (\cdot m): M \rightarrow M, \ \iota: M \rightarrow D, \ (m \triangleright), (\triangleleft m): D \rightarrow D, \ (\triangleright d), (d \triangleleft): M \rightarrow D$.

Note that the maps $(m \triangleright)$ and $(\triangleleft m)$ are indeed CDL-morphisms, as required.

### 3 Pseudovarieties of Reduced Lattice Bimodules

In this section, we introduce pseudovarieties and theories of (reduced) lattice bimodules and show them to be in one-to-one correspondence. The concept of a pseudovariety originates in Eilenberg’s classical variety theorem [9] where a *pseudovariety of monoids* is a class of finite monoids closed under finite products, submonoids, and quotient monoids. In our setting of lattice bimodules, we shall consider pseudovarieties of $*$-generated reduced lattice bimodules. Their definition is slightly more involved than in the case of monoids because subbimodules of $*$-generated lattice bimodules are not necessarily $*$-generated and quotient bimodules of reduced lattice bimodules are not necessarily reduced.

**Definition 3.1.** A *pseudovariety of lattice bimodules* is a class $\mathcal{V}$ of $*$-generated reduced finite lattice bimodules such that

1. $\mathcal{V}$ is closed under reduced quotients: for every surjective homomorphism $e: (M, D) \twoheadrightarrow (M', D')$ of lattice bimodules, if $(M, D) \in \mathcal{V}$ and $(M', D')$ is reduced then $(M', D') \in \mathcal{V}$. 
(2) \( V \) is closed under \( \ast \)-generated subbimodules of finite products: for every injective homomorphism \( (M, D) \rightarrow \prod_{i=1}^{n} (M_i, D_i) \) of lattice bimodules, if \( (M_i, D_i) \in V \) for \( i = 1, \ldots, n \) and \( (M, D) \) is \( \ast \)-generated then \( (M, D) \in V \).

We shall also consider the notion of a local pseudovariety. It is local in the sense that it involves only quotient bimodules of a fixed free lattice bimodule \((\Sigma^*, \Sigma^\circ)\). The set of all such quotients carries a partial order: given \( e_i : (\Sigma^*, \Sigma^\circ) \rightarrow (M_i, D_i) \), \( i = 0, 1 \), put \( e_0 \leq e_1 \) iff \( e_0 \circ h \circ e_1 \) for some LBM-morphism \( h \).

**Definition 3.2.** A local pseudovariety of lattice bimodules over the finite set \( \Sigma \) is a set \( T_\Sigma \) of quotient bimodules of \((\Sigma^*, \Sigma^\circ)\) such that

1. The codomain of every \( e \in T_\Sigma \) is finite and reduced. (Note that it is also \( \ast \)-generated by Lemma 2.5(1).)
2. \( T_\Sigma \) is downwards closed: if \( e \in T_\Sigma \) and \( e' : (\Sigma^*, \Sigma^\circ) \rightarrow (M, D) \) is a quotient bimodule with reduced codomain, then \( e' \leq e \) implies \( e' \in T_\Sigma \).
3. \( T_\Sigma \) is directed: if \( e_0, e_1 \in T_\Sigma \), then there exists \( e \in T_\Sigma \) with \( e_0, e_1 \leq e \).

In order-theoretic terminology, a local pseudovariety is thus precisely an ideal in the poset of finite reduced quotient bimodules of \((\Sigma^*, \Sigma^\circ)\).

**Definition 3.3.** A theory of lattice bimodules is a family \( T = (T_\Sigma)_{\Sigma \in \text{Set}_f} \) of local pseudovarieties, with \( \Sigma \) ranging over the class \( \text{Set}_f \) of finite sets, such that for each homomorphism \( h : (\Delta^*, \Delta^\circ) \rightarrow (\Sigma^*, \Sigma^\circ) \) and \( e_\Sigma \in T_\Sigma \) their composite \( e_\Sigma \cdot h \) lifts through \( T_\Delta \), that is, there exist \( e_\Delta \in T_\Delta \) and \( h \) such that \( e_\Sigma \cdot h = h \cdot e_\Delta \).

\[
\begin{array}{ccc}
(\Delta^*, \Delta^\circ) & \xrightarrow{h} & (\Sigma^*, \Sigma^\circ) \\
\uparrow{e_\Delta} & & \downarrow{e_\Sigma} \\
(M', D') & \xrightarrow{\bar{h}} & (M, D)
\end{array}
\]

(3.1)

This notion resembles the concept of an equational theory from universal algebra: equations can be identified with quotients of free algebras, and the closure of a theory under substitution is expressed by a commutative diagram like (3.1).

**Notation 3.4.** (1) Given a theory \( T \), let \( \mathcal{V}^T \) be the class of all lattice bimodules \((M, D)\) such that some \( T_\Sigma \) contains a quotient with codomain \((M, D)\).

(2) Given a pseudovariety \( \mathcal{V} \), form the family \( \mathcal{T}^\mathcal{V} = (T_\Sigma^\mathcal{V})_{\Sigma \in \text{Set}_f} \) where \( T_\Sigma^\mathcal{V} \) consists of all quotient bimodules of \((\Sigma^*, \Sigma^\circ)\) with codomain in \( \mathcal{V} \).

The class of all pseudovarieties of lattice bimodules forms a lattice ordered by inclusion. Similarly, the class of all theories of lattice bimodules forms a lattice ordered by pointwise inclusion: \( T \leq T' \) iff \( T_\Sigma \subseteq T'_\Sigma \) for each \( \Sigma \).

**Theorem 3.5.** The maps \( \mathcal{V} \mapsto \mathcal{T}^\mathcal{V} \) and \( T \mapsto \mathcal{V}^T \) give rise to an isomorphism between the lattice of pseudovarieties of lattice bimodules and the lattice of theories of lattice bimodules.
We conclude this section with another characterization of theories, linking them to the concept of a unary presentation. For any set $\Sigma$, let $U_\Sigma$ be the canonical unary presentation of the free lattice bimodule $(\Sigma^*, \Sigma^\circ)$ given by Lemma 2.7, and denote by $\overline{U}_\Sigma$ its closure under composition. Then $\overline{U}_\Sigma$ also forms a unary presentation of $(\Sigma^*, \Sigma^\circ)$. We write $\overline{U}_\Sigma(S, T) \subseteq U_\Sigma$ for the set of unary operations in $U_\Sigma$ with domain $S$ and codomain $T$, where $S, T \in \{\Sigma^*, \Sigma^\circ\}$. In particular, $\overline{U}_\Sigma(\Sigma^\circ, \Sigma^\circ) = \{x \mapsto vwx \mid v, w \in \Sigma^*\}$.

**Definition 3.6.** (1) A quotient $e: \Sigma^\circ \rightarrow D$ in CDL is called a $U$-quotient if for every unary operation $u \in \overline{U}_\Sigma(\Sigma^\circ, \Sigma^\circ)$ there exists a CDL-morphism $\bar{u}: D \rightarrow D$ such that $e \cdot u = \bar{u} \cdot e$. We call such a $\bar{u}$ a lifting of $u$ along $e$.

(2) A local pseudovariety of $U$-quotients over the finite set $\Sigma$ is an ideal in the poset of finite $U$-quotients of $\Sigma^\circ$.

(3) A theory of $U$-quotients is a family $T = (T_\Sigma)_{\Sigma \in \text{Set}_f}$ of local pseudovarieties of $U$-quotients such that for each lattice bimodule homomorphism $h: (\Delta^*, \Delta^\circ) \rightarrow (\Sigma^*, \Sigma^\circ)$ and $e_\Sigma \in T_\Sigma$ their composite $e_\Sigma \cdot h^\circ$ lifts through $T_\Delta$: there exist morphisms $e_\Delta \in T_\Delta$ and $\bar{h}$ such that $e_\Sigma \cdot h^\circ = \bar{h} \cdot e_\Delta$.

\[
\begin{array}{ccc}
\Sigma^\circ & \xrightarrow{u} & \Sigma^\circ \\
\downarrow e & & \downarrow e \\
D & \xrightarrow{\Delta^\circ} & D
\end{array}
\quad
\begin{array}{ccc}
\Delta^\circ & \xrightarrow{h^\circ} & \Sigma^\circ \\
\downarrow e_{\Delta} & & \downarrow e_{\Sigma} \\
D' & \xrightarrow{\bar{h}} & D
\end{array}
\]

(3.2)

**Proposition 3.7.** The lattice of theories of lattice bimodules is isomorphic to the lattice of theories of $U$-quotients. The isomorphism is given by $T \mapsto T^\circ$, where $T^\circ$ consists of all quotients in $T$ restricted to their $\circ$-component.

The advantage in using theories of $U$-quotients is that they are easier to dualize but still carry as much information as theories of lattice bimodules.

## 4 Basic Varieties of Regular Languages

In this section, we study lattice bimodules as recognizers for regular languages. Their purpose is to capture classes of regular languages with no boolean closure at all, which we thus call basic varieties. Observe that since the set $2 = \{0, 1\}$ with $0 \leq 1$ forms a CDL and the set $\Sigma^*$ generates the free completely distributive lattice $\Sigma^\circ$, we get the correspondence $\mathcal{P}(\Sigma^*) \cong \text{Set}(\Sigma^*, 2) \cong \text{CDL}(\Sigma^\circ, 2)$. We use the term “language” for elements of any of these sets, identifying elements that correspond to each other via the bijections. Thus, we use the same symbol for a subset $L \subseteq \Sigma^*$ and for its characteristic function. We denote the extension of $L: \Sigma^* \rightarrow 2$ to a lattice morphism by $L^\circ: \Sigma^\circ \rightarrow 2$, and in turn denote the restriction of a lattice morphism $L: \Sigma^\circ \rightarrow 2$ to $\Sigma^*$ by $L^* = L \cdot 1: \Sigma^* \rightarrow \Sigma^\circ \rightarrow 2$.

**Definition 4.1.** A language $L: \Sigma^* \rightarrow 2$ is recognized by a finite lattice bimodule $(M, D)$ if there exists a lattice bimodule homomorphism $h: (\Sigma^*, \Sigma^\circ) \rightarrow (M, D)$ and a CDL-morphism $p: D \rightarrow 2$ with $L^\circ = p \cdot h^\circ$. 

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**Lemma 4.2.** The languages recognizable by finite lattice bimodules are precisely the regular languages.

We now introduce our concept of a language variety that we will show to correspond to pseudovarieties of lattice bimodules. It subsumes Eilenberg’s original concept [9], as well as its variants due to Pin [17] and Polák [18], by dropping the requirement of being closed under any set-theoretic boolean operations. Recall that the derivatives of a language \( L \subseteq \Sigma^* \) are the languages \( v^{-1}Lw^{-1} = \{ u \in \Sigma^* \mid uvw \in L \} \) for \( v, w \in \Sigma^* \). The preimage of \( L \) w.r.t. a monoid homomorphism \( g: \Delta^* \rightarrow \Sigma^* \) is given by \( g^{-1}L = \{ w \in \Delta^* \mid g(w) \in L \} \). In the following we write \( \text{Reg}_\Sigma \) for the set of all regular languages over \( \Sigma \).

**Definition 4.3.** (1) A basic local variety of languages over \( \Sigma \) is a set \( V_\Sigma \subseteq \text{Reg}_\Sigma \) closed under derivatives: If \( L \in V_\Sigma \) then \( v^{-1}Lw^{-1} \in V_\Sigma \) for all \( v, w \in \Sigma^* \).

(2) A basic variety of languages is a family \( (V_\Sigma \subseteq \text{Reg}_\Sigma)_{\Sigma \in \text{Set}_t} \) of local varieties closed under preimages of monoid homomorphisms: If \( L \in V_\Sigma \) then \( g^{-1}L \in V_\Delta \) for each monoid homomorphism \( g: \Delta^* \rightarrow \Sigma^* \).

Just as pseudovarieties of reduced lattice bimodules can be presented as theories, basic varieties of languages correspond uniquely to cotheories. As suggested by the terminology, theories and cotheories form dual concepts, see Section 5. In the following, \( \mathcal{P}(X) \) denotes the poset of subsets of a set \( X \). Recall that an ideal of \( \mathcal{P}(X) \) is a subset \( I \subseteq \mathcal{P}(X) \) that is downwards closed and upwards directed.

**Definition 4.4.** A basic cotheory of regular languages is a family

\[
T = (I_\Sigma \subseteq \mathcal{P}(\text{Reg}_\Sigma))_{\Sigma \in \text{Set}_t}
\]

of ideals with the following properties:

(1) Every element \( F_\Sigma \in I_\Sigma \) is a finite basic local variety.

(2) \( T \) is closed under preimages of monoid homomorphisms: If \( F_\Sigma \in I_\Sigma \), then \( g^{-1}[F_\Sigma] = \{ g^{-1}L \mid L \in F_\Sigma \} \in I_\Delta \) for each monoid homomorphism \( g: \Delta^* \rightarrow \Sigma^* \).

In diagrammatic terms, (1) means that for every \( u \in U_\Sigma(\Sigma^\circ, \Sigma^\circ) \), viewed as a map \( u: \Sigma^* \rightarrow \Sigma^* \) by restricting its domain and codomain, the preimage map \( u^{-1}: \mathcal{P}(\Sigma^*) \rightarrow \mathcal{P}(\Sigma^*) \) restricts to \( F_\Sigma \). Indeed, since \( U_\Sigma(\Sigma^\circ, \Sigma^\circ) \) consists of all unary operations of the form \( x \mapsto vxw \) for \( v, w \in \Sigma^* \), the map \( u^{-1} \) is given by \( L \mapsto v^{-1}Lw^{-1} \). Similarly, (2) means that for every \( F_\Sigma \in I_\Sigma \) and \( g: \Delta^* \rightarrow \Sigma^* \), the map \( g^{-1}: \mathcal{P}(\Sigma^*) \rightarrow \mathcal{P}(\Delta^*) \) restricts to one between \( F_\Sigma \) and some \( F_\Delta \in I_\Delta \).

\[
\begin{align*}
\mathcal{P}(\Sigma^*) & \xrightarrow{u^{-1}} \mathcal{P}(\Sigma^*) & \mathcal{P}(\Sigma^*) & \xrightarrow{g^{-1}} \mathcal{P}(\Delta^*) \\
\subseteq & \uparrow \quad \quad \uparrow \subseteq & \subseteq & \uparrow \subseteq \\
F_\Sigma & \xrightarrow{} F_\Sigma & F_\Sigma & \xrightarrow{} F_\Delta \\
(4.1)
\end{align*}
\]

**Theorem 4.5.** The lattice of basic varieties of regular languages (ordered by inclusion) is isomorphic to the lattice of basic cotheories of regular languages. The isomorphism and its inverse are given pointwise for \( \Sigma \in \text{Set}_t \) by the maps \( I_\Sigma \mapsto \bigcup I_\Sigma \) and \( V_\Sigma \mapsto \{ F \subseteq V_\Sigma \mid F \text{ is a finite basic local subvariety of } V_\Sigma \} \).
5 Duality and the Basic Variety Theorem

The glue between the algebraic concepts of Section 3 and the language-theoretic ones of Section 4 is provided by duality, more precisely, the dual equivalence $\text{AlgCDL} \cong^\text{op} \text{Pos}$ between the full subcategory $\text{AlgCDL}$ of $\text{CDL}$ given by algebraic completely distributive lattices and the category $\text{Pos}$ of posets and monotone maps [8]. Recall that a complete lattice $D$ is algebraic if every element is a join of compact elements, where $c \in D$ is compact if for every subset $S \subseteq D$ with $c \leq \bigvee S$ one has $c \leq \bigvee F$ for some finite $F \subseteq S$. Since all free CDLs and finite CDLs are algebraic, a theory of $U$-quotients (Definition 3.6) lives in the category $\text{AlgCDL}$. Similarly, a basic cotheory of regular languages (Definition 4.4) lives in $\text{Pos}$, viewing the set $\mathcal{P}(\Sigma^*)$ of languages as a poset ordered by inclusion.

Let us now make the key observation that, under the above duality, theories of $U$-quotients dualize to basic cotheories of regular languages: One can show that, up to isomorphism, the duals of the commutative squares (3.2) in $\text{AlgCDL}$ are precisely the commutative squares (4.1) in $\text{Pos}$ where $g = h^*$ and $F_\Sigma$ and $F_\Delta$ are the posets of languages recognized by $e_\Sigma$ and $e_\Delta$, respectively. We can therefore bring the results of the previous sections together to establish our main result:

**Theorem 5.1 (Basic Variety Theorem).** The lattice of basic varieties of regular languages is isomorphic to the lattice of pseudovarieties of lattice bimodules.

**Proof.** We simply compose all the previously established lattice isomorphisms:

Pseudovarieties of lattice bimodules

$\cong$ Theories of lattice bimodules (Theorem 3.5)

$\cong$ Theories of $U$-quotients (Proposition 3.7)

$\cong$ Basic cotheories of regular languages (Duality)

$\cong$ Basic varieties of regular languages (Theorem 4.5) $\Box$

Spelling out the four isomorphisms in the proof, from top to bottom we transform between the following collections:

$$\mathcal{V} = \{(\Sigma^*, \Sigma^\circ) \to (M, D) \mid (M, D) \in \mathcal{V}) \}_{\Sigma \in \text{Set}}$$

$\cong$

$$\mathcal{T}^\mathcal{V} = \{(\Sigma^*, \Sigma^\circ) \to (M, D) \mid (M, D) \in \mathcal{V}) \}_{\Sigma \in \text{Set}}$$

$\cong$

$\{\Sigma^\circ \to D \mid e \in \mathcal{T}^\mathcal{V})\}_{\Sigma \in \text{Set}}$

$\cong^\text{op}$

$\{I_\Sigma \hookrightarrow \mathcal{P}(\text{Reg}_\Sigma)\}_{\Sigma \in \text{Set}}$

$\cong$

$\{V_\Sigma \hookrightarrow \text{Reg}_\Sigma\}_{\Sigma \in \text{Set}}$

Thus, starting from the top, a pseudovariety $\mathcal{V}$ of lattice bimodules is sent to the basic variety of all regular languages recognized by some lattice bimodule in $\mathcal{V}$. 
Conversely, starting from the bottom, a basic variety \((V_\Sigma)_{\Sigma \in \text{Set}_f}\) of languages is sent to the pseudovariety of all \(*\)-generated reduced finite lattice bimodules \((M, D)\) such that every language \(L \subseteq \Sigma^*\) recognized by \((M, D)\) lies in \(V_\Sigma\).

### 6 Quantum Finite Automata

In this section we present a natural example of a basic variety of regular languages that is not closed under union and intersection and therefore not captured by any previously known Eilenberg-type correspondence. It is concerned with languages accepted by *quantum finite automata* (QFA). Several different notions of QFA have been proposed and studied, varying in their expressive power; see e.g. the recent survey paper by Ambainis and Yakaryılmaz [2]. Here, we focus on the model of *Kondacs-Watrous quantum finite automata* (KWQFA) [13], also known in the literature as *measure-many quantum finite automata*.

A KWQFA \(M = (Q, \Sigma, T, q_0, Q_{\text{acc}}, Q_{\text{rej}}, Q_{\text{non}})\) is given by a finite set \(Q\) of *basis states*, an input alphabet \(\Sigma\) not containing the end markers \(\kappa\) and \(\$\), an initial state \(q_0 \in Q\) and a partition \(Q_{\text{acc}} \cup Q_{\text{rej}} \cup Q_{\text{non}}\) of \(Q\) into accepting, rejecting and non-halting states. The transitions are specified by a family of unitary linear maps \(T_\sigma : \mathcal{H}_Q \rightarrow \mathcal{H}_Q\) \((\sigma \in \Sigma \cup \{\kappa, \}$\)) on the complex Hilbert space \(\mathcal{H}_Q\) with orthonormal basis \(\Sigma\). Thus, denoting the basis vectors by \(|q\rangle\) \((q \in Q)\), every element \(|\psi\rangle\) of \(\mathcal{H}_Q\) can be uniquely expressed as a linear combination \(|\psi\rangle = \sum_{q \in Q} \alpha_q |q\rangle\) with \(\alpha_q \in \mathbb{C}\). The *states* of \(M\) are those \(|\psi\rangle \in \mathcal{H}_Q\) with norm \(\sum_{q \in Q} |\alpha_q|^2 = 1\). Note that a unitary transformation \(T_\sigma\) maps states to states. A *measurement* collapses the state \(|\psi\rangle\) to the basis state \(|q\rangle\) with probability \(|\alpha_q|^2\).

Initially, the automaton is in the basis state \(|q_0\rangle\). An input \(w \in \Sigma^*\) is processed by first adding the left \((\kappa)\) and right \((\$)\) end markers. Then, for every successive symbol \(\sigma\) in \(w = \kappa w \$\) the corresponding transformation \(T_\sigma\) is applied and a measurement is performed. The automaton halts and accepts if the resulting basis state lies in \(Q_{\text{acc}}\), halts and rejects if it lies in \(Q_{\text{rej}}\), and continues with processing the next input letter if it lies in \(Q_{\text{non}}\). Thus, if the QFA is in the state \(|\psi\rangle = \sum_{q \in Q_{\text{acc}}} \alpha_q |q\rangle + \sum_{q \in Q_{\text{rej}}} \beta_q |q\rangle + \sum_{q \in Q_{\text{non}}} \gamma_q |q\rangle\) after reading the current input symbol but before making the measurement, it accepts with probability \(\sum_{q \in Q_{\text{acc}}} |\alpha_q|^2\), rejects with probability \(\sum_{q \in Q_{\text{rej}}} |\beta_q|^2\) and continues processing the input with probability \(\sum_{q \in Q_{\text{non}}} |\gamma_q|^2\). This yields an overall probability \(p \in [0, 1]\) that the input word \(w\) is accepted, i.e. that at any stage of the computation the automaton reaches a state in \(Q_{\text{acc}}\).

We say that \(M\) *accepts* the language \(L \subseteq \Sigma^*\) (with bounded error) if there exists a real number \(p > 1/2\) such that \(M\) accepts every word in \(L\) with probability \(\geq p\) and rejects every word not in \(L\) with probability \(\geq p\). The class of languages accepted by KWQFA is denoted by \(\text{RMM}\). It is known to be a proper subclass of the class of all regular languages; for instance, \(\{a, b\}^*a \notin \text{RMM}\) [13, Proposition 7]. Subsequent work has identified certain “forbidden configurations” in the minimal deterministic finite automaton of a regular language making it unrecognizable by a KWQFA [4, 6]. In this way, it was shown that \(\text{RMM}\) is not closed under union and intersection [4, Corollary 3.2]. However, \(\text{RMM}\) is closed
under preimages of monoid homomorphisms and derivatives [6, Theorem 4.1] and thus forms a basic variety of regular languages.

The questions whether $\text{RMM}$ is decidable and whether it has an algebraic characterization remain open problems in the theory of quantum automata [3]. Our Basic Variety Theorem provides strong evidence that such a characterization must exist: it asserts that $\text{RMM}$ corresponds to a pseudovariety of reduced lattice bimodules, which by Theorem 3.5 admits an (abstract form of) equational presentation. We expect that the latter can be turned into a more concrete form using profinite equations over free lattice bimodules $(\Sigma^*, \Sigma^\diamond)$, analogous to Reiterman’s [19] description of pseudovarieties of finite monoids in terms of profinite equations over free monoids $\Sigma^*$. A concrete profinite axiomatization of the pseudovariety induced by $\text{RMM}$ might pave the way towards the decidability of that class: deciding whether a given regular language lies in $\text{RMM}$ reduces to checking whether its syntactic lattice bimodule satisfies the equational axioms.

7 Conclusion and Future Work

We have introduced a new two-sorted algebraic structure, lattice bimodules, for the recognition of regular languages. Our main result is a new Eilenberg-type correspondence between basic varieties of regular languages, which need not be closed under set-theoretic boolean operations, and pseudovarieties of reduced lattice bimodules. The proof is guided by the recent category-theoretic approach to algebraic language theory and makes use of the duality between algebraic completely distributive lattices and posets.

An immediate next step to unleash the full power of our new variety theorem is to establish a Reiterman-type theorem for lattice bimodules leading to a description of pseudovarieties of lattice bimodules in terms of profinite equations. The recent categorical account of (profinite) equational theories [7,16] should provide inspiration in this direction. This may lead to new results on the decidability of basic varieties of regular languages, e.g. language classes recognized by different models of reversible automata [11] or quantum automata (cf. Section 6).

Furthermore, several generalizations of our work are conceivable. The most obvious one is to replace the duality $\text{AlgCDL} \cong \text{Pos}^\text{op}$ by an abstract dual equivalence $\mathcal{A} \cong \mathcal{B}^\text{op}$ between suitable categories $\mathcal{A}$ and $\mathcal{B}$, and to consider the recognition of languages by $\mathcal{A}$-bimodules. We anticipate that this minor generalization already recovers results closely related to the original Eilenberg theorem for $\mathcal{A}$ being the category of sets, and to Polák’s variety theorem for idempotent semirings for $\mathcal{A}$ being the category of complete semilattices. In an orthogonal direction, the monoid action on the algebra may be generalized to the action of a monad $T$ on the category of sets, but the dependence between the monad $T$ and the category $\mathcal{A}$ is not obvious and remains to be investigated.

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Appendix

This appendix provides full proofs and additional details for all our results.

A Details for Section 2

Lemma A.1. The category $\text{LBM}$ has the factorization system of (sortwise) surjective and injective morphisms. More precisely, every lattice bimodule homomorphism $h$ factorizes as $h = m \cdot e$, where $e$ is a surjective and $m$ is an injective homomorphism, and for every commutative square

$$
\begin{array}{ccc}
(M_1, D_1) & \xrightarrow{e} & (M_2, D_2) \\
\downarrow & & \downarrow \\
(M_3, D_3) & \xrightarrow{m} & (M_4, D_4)
\end{array}
$$

with $e$ surjective and $m$ injective, there exists a unique diagonal fill-in $d$ making both triangles commute.

While this lemma is not difficult to prove directly, it also follows immediately from the fact that the category $\text{LBM}$ forms a variety of (infinitary) algebras and is thus monadic over the product category $\text{Set} \times \text{Set}$ [14]. This implies that $\text{LBM}$ inherits the (surjective, injective) factorization system of $\text{Set} \times \text{Set}$.

Notation A.2. Let $\text{Quo}_{\mathcal{C}}(X)$ denote the set of all finite quotients (represented by surjective morphisms) of an object $X$ in the category $\mathcal{C} \in \{\text{Set}, \text{CDL}, \text{LBM}\}$. If $\mathcal{C}$ is clear from context, we may omit the superscript. We equip $\text{Quo}_{\mathcal{C}}(X)$ with the order $e \leq e'$ if $e$ factorizes through $e'$, i.e. $e = q \cdot e'$ for some $q$. This makes $\text{Quo}_{\mathcal{C}}(X)$ a poset if we identify isomorphic quotients.

Remark A.3. Quotients in $\mathcal{C} \in \{\text{Set}, \text{CDL}, \text{LBM}\}$ satisfy the homomorphism theorem: given two quotients $e, e'$ of the same object, we have $e \leq e'$ if and only if the kernel of $e'$ of contained in the kernel of $e$; that is, for each $x, y$ in the domain of $e'$,

$$e'(x) = e'(y) \text{ implies } e(x) = e(y).$$

Proof of Proposition 2.2

Let $h_0 = (h_0^0, h_0^1): (\Sigma, \Gamma) \to (M, D)$ be a two-sorted map into a lattice bimodule $(M, D)$. We need to show that there exists a unique $\text{LBM}$-morphism $h: (\Sigma, \text{FCDL}(\Sigma^* + \Sigma^* \times \Gamma \times \Sigma^*)) \to (M, D)$ satisfying $h \cdot \eta = h_0$.

1. Existence. Let $h^*: \Sigma^* \to M$ be the unique monoid morphism with $h^*(a) = h_0^0(a)$ for each $a \in \Sigma$. Moreover, let $h^\circ: \text{FCDL}(\Sigma^* + \Sigma^* \times \Gamma \times \Sigma^*) \to D$ be the unique $\text{CDL}$-morphism with

$$h^\circ(w) = h^*(w) \text{ and } h^\circ(v, z, w) = h^*(v) \triangleright h^\circ(z) \triangleright h^*(w) \quad (A.1)$$

for $v, w \in \Sigma^*$ and $z \in \Gamma$. An easy verification shows that $h = (h^*, h^\circ)$ is an $\text{LBM}$-morphism from $(\Sigma^*, \text{FCDL}(\Sigma^* + \Sigma^* \times \Gamma \times \Sigma^*))$ into $(M, D)$ extending $h_0$. 
Thus, $\iota(a)$ is a monoid congruence for every $h = W$. Lemma A.5.

For every lattice bimodule $b$ since this holds when precomposed with the universal map $m$ surjective. By the universal property of $\eta$, $\eta$ hold for all $v, w \in \Sigma^*$ and $z \in \Gamma^*$, which proves that $h^\circ$ is uniquely determined by $h_0^\circ$ and $h^*$. \hfill $\Box$

Lemma A.4. Free lattice bimodules are projective: for every LBM-morphism $h: (\Sigma^*, \Sigma^\circ) \to (M', D')$ and every surjective LBM-morphism $g: (M, D) \to (M', D')$ there exists an LBM-morphism $f: (\Sigma^*, \Sigma^\circ) \to (M, D)$ with $h = g \cdot f$.

Proof. For each $a \in \Sigma$ choose $m_a \in M$ with $g^*(m_a) = h^*(a)$, using that $g$ is surjective. By the universal property of $(\Sigma^*, \Sigma^\circ)$, there exists a unique homomorphism $f: (\Sigma^*, \Sigma^\circ) \to (M, D)$ with $f^*(a) = m_a$ for all $a \in \Sigma$. Then $h = g \cdot f$ since this holds when precomposed with the universal map $\eta: (\Sigma, \emptyset) \to (\Sigma^*, \Sigma^\circ)$.

Lemma A.5. For every lattice bimodule $(M, D)$ the operation $\iota: M \to D$ induces a monoid congruence $\equiv_i$ on $M$ given by

$$m \equiv_i n \iff \iota(m) = \iota(n).$$

Thus, $\iota[M]$ carries a monoid structure with $\iota[M] \cong M/\equiv_i$.

Proof. Clearly $\equiv_i$ is an equivalence relation. For $m_1 \equiv_i, m_2, n_1 \equiv_i n_2$ we have

$$\iota(m_1n_1) = \iota(m_1) \cdot \iota(n_1) = \iota(m_2) \cdot \iota(n_1) = \iota(m_2n_1) = m_2 \cdot \iota(n_1) = m_2 \cdot \iota(n_2) = \iota(m_2n_2)$$

and thus $m_1n_1 \equiv_i m_2n_2$, showing that $\equiv_i$ is a monoid congruence. \hfill $\Box$

Proof of Lemma 2.5

(1) Let $h: (\Sigma^*, \Sigma^\circ) \to (M, D)$ be surjective. Given $d \in D$, choose an element $W = \bigvee_{j \in J} \bigwedge_{k \in K_j} \iota(w_{jk}) \in \Sigma^\circ$, where $w_{jk} \in \Sigma^*$, such that $h(W) = d$. Then

$$d = h(W) = h\left( \bigvee_{j \in J} \bigwedge_{k \in K_j} \iota(w_{jk}) \right) = \bigvee_{j \in J} \bigwedge_{k \in K_j} h(\iota(w_{jk})) = \bigvee_{j \in J} \bigwedge_{k \in K_j} \iota(h(w_{jk})),$$

proving that $(M, D)$ is $*$-generated.

Conversely, suppose that $(M, D)$ is $*$-generated. Choose $\Sigma = M$ and let $h: (\Sigma^*, \Sigma^\circ) \to (M, D)$ be the unique lattice bimodule morphism with $h^*(a) = a$ for every $a \in \Sigma$. Since $D$ is $*$-generated, for each $d \in D$ we have

$$d = \bigvee_{j \in J} \bigwedge_{k \in K_j} \iota(m_{jk}).$$
for some \( m_{jk} \in M \). Since \( h^* \) is surjective, there exist \( w_{jk} \in \Sigma^* \) with \( h^*(w_{jk}) = m_{jk} \). It follows that

\[
d = \bigvee_{j \in J} \bigwedge_{k \in K_j} \imath(h^*(w_{jk}))
= \bigvee_{j \in J} \bigwedge_{k \in K_j} \imath(w_{jk})
= \imath \bigvee_{j \in J} \bigwedge_{k \in K_j} (w_{jk})
\]

and so both components of \( h \) are surjective.

(2) Let \((M, D)\) be an \( \ast \)-embedded lattice bimodule and let \( h:\ (M, D) \to (M', D')\) be a quotient with \( h^* \) a CDL-isomorphism. Then \( h^* \) is injective and thus \( h^* \cdot \imath = \imath' \cdot h^* \) is injective as well. Hence \( h^* \) is injective, and so \( h \) is an isomorphism.

(3) Let \((M, D)\) be a \( \ast \)-generated reduced lattice bimodule. Since \((M, D)\) is \( \ast \)-generated, we can define a left action of the monoid \( \imath[M] \) on \( D \) by

\[
[m], \triangleright \bigvee_{j \in J} \bigwedge_{k \in K_j} \imath(m_{jk}) := \bigvee_{j \in J} \bigwedge_{k \in K_j} m \triangleright \imath(m_{jk})
\]

where \([m]_\imath\) is the equivalence class of \( m \) under \( \equiv \), see Lemma A.5. It is well defined since for \( m \equiv n \), we have

\[
[m]_\imath \triangleright \bigvee_{j \in J} \bigwedge_{k \in K_j} \imath(m_{jk}) = \bigvee_{j \in J} \bigwedge_{k \in K_j} m \triangleright \imath(m_{jk})
= \bigvee_{j \in J} \bigwedge_{k \in K_j} \imath(m \cdot m_{jk})
= \bigvee_{j \in J} \bigwedge_{k \in K_j} \imath(m) \triangleleft m_{jk}
= \bigvee_{j \in J} \bigwedge_{k \in K_j} \imath(n) \triangleleft m_{jk}
= [n]_\imath \triangleright \bigvee_{j \in J} \bigwedge_{k \in K_j} \imath(m_{jk}).
\]

Thus \((\imath[M], D)\) carries the structure of a lattice bimodule with \( \triangleright \) defined as above, \( \triangleleft \) defined symmetrically, and the unary operation \( \imath[M] \to D \) given by \( [m]_\imath \to \imath(m) \). Moreover, letting \( \imath' : M \to \imath[M] \) denote the codomain restriction of \( \imath \), we see that \((\imath', id) : (M, D) \to (\imath[M], D)\) is a lattice bimodule homomorphism. Since \((M, D)\) is reduced and \( id \) is an isomorphism, we conclude that \( \imath' \) is an isomorphism. This implies that \( \imath \) is injective, so \((M, D)\) is \( \ast \)-embedded.

\( \square \)

**Lemma A.6.** Finite products and quotients of finite \( \ast \)-generated lattice bimodules are again \( \ast \)-generated.
Proof. The statement for quotients follows immediately from Lemma 2.5(1). As for products, it suffices to prove the statement for binary products since the \(n\)-ary case follows by iteration and the empty product is the trivial lattice bialgebra \((1, 1)\), which is obviously \(*\)-generated. Let \((M, D), (M', D')\) be \(*\)-generated. To show that \((M, D) \times (M', D') = (M \times M', D \times D')\) is \(*\)-generated, let \((d, d') \in D \times D'\); we need to show \((d, d')\) to be generated by elements of \(M \times M'\). By hypothesis, there exist elements \(m_{ik} \in M\) and \(m'_{rk} \in M'\) such that

\[
d = \bigvee_{l=1}^{n} \bigwedge_{k=1}^{n_l} m_{ik} \quad \text{and} \quad d' = \bigvee_{r=1}^{m} \bigwedge_{k=1}^{m_r} m'_{rk},
\]

for some natural numbers \(n, m, n_l, m_r\). (For notational convencience, we identify elements \(m \in M\) and \(m' \in M'\) with their images \(\iota(m) \in D\) and \(\iota(m') \in D'\). Let

\[
o = \max\{n_l, m_r \mid l = 1 \ldots n, r = 1 \ldots m\}.
\]

Using idempotence, extend the conjunctions by the first factor so each conjunction is the same size.

\[
\bigwedge_{k=1}^{n_l} m_{ik} \mapsto \left( \bigwedge_{k=1}^{n_l} m_{ik} \right) \land \left( \bigwedge_{i=1}^{o-n_l} m_{i1} \right) =: T_l
\]

\[
\bigwedge_{k=1}^{m_r} m'_{rk} \mapsto \left( \bigwedge_{k=1}^{m_r} m'_{rk} \right) \land \left( \bigwedge_{i=1}^{o-m_r} m'_{r1} \right) =: T'_r,
\]

so \(d = \bigvee_{l=1}^{n} T_l =: L\) and \(d' = \bigvee_{r=1}^{m} T'_r\). Without loss of generality we can assume \(m \leq n\). Then, using idempotence laws again, extend \(d'\) to having as many disjuncts as \(d\) to get

\[
d = \bigvee_{r=1}^{m} T'_r = \left( \bigvee_{r=1}^{m} T'_r \right) \lor \left( \bigvee_{r=1}^{n-m} T_1 \right) =: R.
\]

Now the terms \(L\) and \(R\) have an equal number of conjunctions and disjuncts so there exist \(n_{\alpha\beta} \in M, n'_{\alpha\beta} \in M'\) with

\[
(d, d') = (\bigwedge_{\alpha} n_{\alpha\beta}, \bigwedge_{\beta} n'_{\alpha\beta}) = \bigvee_{\alpha} \bigwedge_{\beta} (n_{\alpha\beta}, n'_{\alpha\beta})
\]

which proves \(D \times D'\) to be \(*\)-generated. \(\square\)

Lemma A.7. Subbimodules and products of \(*\)-embedded lattice bimodules are \(*\)-embedded.

\begin{proof}
Restrictions and products of injective functions are injective. \(\square\)
\end{proof}

Example A.8. In general, subbimodules of \(*\)-generated lattice bimodules are not \(*\)-generated, and quotient bimodules of reduced lattice bimodules are not reduced. To see this, consider the lattice bimodule \((\mathbb{Z}/2\mathbb{Z}, D)\) where \(\mathbb{Z}/2\mathbb{Z}\) is the additive group of integers modulo 2, and \(D = \{\bot, \top, 0, 1\}\) is the diamond lattice induced by the order \(\bot \leq i \leq \top\), \(i \in \mathbb{Z}/2\mathbb{Z}\). The operation \(\iota: \mathbb{Z}/2\mathbb{Z} \to D\) is the obvious injection; this determines \(\triangleright\) and \(<\) uniquely. Then \((\mathbb{Z}/2\mathbb{Z}, D)\) is \(*\)-generated and \(*\)-embedded and thus reduced by Lemma 2.5. However:
(1) The subbimodule \( (\{0\}, D) \hookrightarrow (\mathbb{Z}/2\mathbb{Z}, D) \) is not \( \ast \)-generated.

(2) The quotient bimodule \((id, !): (\mathbb{Z}/2\mathbb{Z}, D) \rightarrow (\mathbb{Z}/2\mathbb{Z}, 1)\) is not reduced since it is not \( \ast \)-embedded even though it is \( \ast \)-generated.

**Definition A.9.** A lattice bimodule congruence \( \equiv \) on a lattice bimodule \((M, D)\) is a pair \((\equiv_M \subseteq M \times M, \equiv_D \subseteq D \times D)\) such that \( \equiv_M \) is a monoid congruence on \( M \), \( \equiv_D \) is a complete lattice congruence on \( D \) and the operations between the two sets preserve the congruences, that is, for all \( m, m' \in M \) and \( d, d' \in D \),

\[- m \equiv_M m' \text{ implies } \iota(m) \equiv_D \iota(m');
- m \equiv_M m' \text{ and } d \equiv_D d' \text{ implies } m \triangleright d \equiv_D m' \triangleright d' \text{ and } d \triangleleft m \equiv_D d' \triangleleft m'.\]

**Remark A.10.** Lattice bimodule congruences on \((M, D)\) correspond uniquely to quotient lattice bimodules of \((M, D)\). More precisely, a quotient

\[e = (e^*, e^\circ): (M, D) \rightarrow (M', D')\]

in \( \text{Set} \times \text{CDL} \) carries a quotient lattice bimodule (i.e. there exists a lattice bimodule structure on \((M', D')\) making \( e \) a homomorphism of lattice bimodules) if and only if its kernel relation \((\equiv_M, \equiv_D)\), defined by

\[m \equiv_M m \text{ iff } e^*(m) = e^*(m') \quad \text{and} \quad d \equiv_D d' \text{ iff } e^\circ(d) = e^\circ(d'),\]

forms a lattice bimodule congruence. This follows immediately from the homomorphism theorem (see Remark A.3).

**Proof of Lemma 2.7**

Let \( \equiv = (\equiv_M, \equiv_D) \) be a pair of an equivalence relation on \( M \) and a CDL congruence on \( D \). By the equivalence of quotients and congruences, see Remark A.10, and the homomorphism theorem, it suffices to show that \( \equiv \) is a lattice bimodule congruence iff it is stable under the unary operations in \( U \). The latter means that for all \( u: S \rightarrow T \) in \( U \), where \( S, T \in \{M, D\} \), and all \( a, b \in S \) with \( a \equiv_S b \) we have \( u(a) \equiv_T u(b) \).

Clearly every \text{LBM}-congruence is stable under \( U \). Conversely, suppose that \( \equiv \) is stable under \( U \) and that \( m \equiv_M m', n \equiv_M n', d \equiv_D d' \). Then

\[- m \cdot n = (m \cdot)(n) \equiv_M (m')(n') = m \cdot n' = (\cdot n')(m) \equiv_M (\cdot n')(m') = m' \cdot n' \text{ because } (m \cdot), (\cdot n') \in U. \]

Thus, \( \equiv_M \) is a monoid congruence.

\[- \iota(m) \equiv_D \iota(m') \text{ because } \iota \in U.
- m \triangleright d = (m \triangleright)(d) \equiv_D (m \triangleright)(d') = m \triangleright d' = (\triangleright d')(m) \equiv_D (\triangleright d')(m') = m' \triangleright d' \text{ because } (m \triangleright), (\triangleright d') \in U.
- d \triangleleft m \equiv_D d' \triangleleft m', \text{ analogously.}\]

Thus, \( \equiv \) is an \text{LBM}-congruence. \(\square\)
B Details for Section 3

Remark on Definition 3.1
Notice that in (1) the bimodule \((M', D')\) is necessarily \(*\)-generated by Lemma A.6. Similarly, in (2) the bimodule \((M, D)\) is necessarily reduced by Lemma A.7 and 2.5.

**Lemma B.1.** Let \(\mathcal{T}\) be a theory of lattice bimodules and let \((M, D)\) be a \(*\)-generated reduced finite lattice bimodule. Then the following are equivalent:

1. There exists \(h \in \mathcal{T}_\Sigma\) with codomain \((M, D)\) for some \(\Sigma \in \text{Set}_\iota\).
2. Every lattice bimodule homomorphism \(f: (\Delta^*, \Delta^\circ) \to (M, D)\) with \(\Delta \in \text{Set}_\iota\) factors through some element of \(\mathcal{T}_\Delta\).

**Proof.** For (1) \(\Rightarrow\) (2), let \(f: (\Delta^*, \Delta^\circ) \to (M, D)\). By hypothesis there exists an alphabet \(\Sigma\) and a quotient \(h: (\Sigma^*, \Sigma^\circ) \to (M, D)\) in \(\mathcal{T}_\Sigma\). Using Lemma A.4 we can choose a morphism of lattice bimodules \(g: (\Delta^*, \Delta^\circ) \to (\Sigma^*, \Sigma^\circ)\) with \(f = h \cdot g\). Since \(\mathcal{T}\) is a theory, \(f = h \cdot g\) factorizes through some \(\bar{h} \in \mathcal{T}_\Delta\).

\[
\begin{array}{ccc}
(\Delta^*, \Delta^\circ) & \xrightarrow{g} & (\Sigma^*, \Sigma^\circ) \\
\downarrow{\bar{h}} & & \downarrow{\bar{h}} \\
(M', D') & \xrightarrow{\bar{h}} & (M, D)
\end{array}
\]

For (2) \(\Rightarrow\) (1), suppose that \((M, D)\) satisfies (2). Since \((M, D)\) is \(*\)-generated, there exists a surjective homomorphism \(f: (\Delta^*, \Delta^\circ) \to (M, D)\) for some \(\Delta \in \text{Set}_\iota\) by Lemma 2.5(1); in fact, the proof of that lemma shows that one can choose \(\Delta = M\). By assumption, \(f\) factors through some \(h \in \mathcal{T}_\Delta\), i.e. \(f \leq h\). Since \(\mathcal{T}_\Delta\) is downwards closed, we conclude \(f \in \mathcal{T}_\Delta\). \(\square\)

Recall from Notation 3.4 the class \(\mathcal{V}^T\) associated to a theory \(\mathcal{T}\). The elements of \(\mathcal{V}^T\) are those \(*\)-generated reduced finite lattice bimodules satisfying the equivalent conditions of Lemma B.1.

**Lemma B.2.** If \(\mathcal{T}\) is a theory of lattice bimodules, then \(\mathcal{V}^T\) is a pseudovariety of lattice bimodules.

**Proof.** Let \(\mathcal{T}\) be a theory. The class \(\mathcal{V}^T\) is closed under reduced quotients because all \(\mathcal{T}_\Sigma\) are downwards closed. To show that \(\mathcal{V}^T\) is closed under \(*\)-generated subbimodules of finite products, suppose that \(m: (M, D) \to \prod_{i \in I} (M_i, D_i)\) is such a subbimodule with \((M_i, D_i) \in \mathcal{V}^T\), \(I\) finite. We prove \((M, D) \in \mathcal{V}^T\) by condition (2) of Lemma B.1, viz. that any \(h: (\Delta^*, \Delta^\circ) \to (M, D)\) factors through \(\mathcal{T}_\Delta\). Let \(p_i: \prod_{i \in I} (M_i, D_i) \to (M_i, D_i)\) denote the product projections. Then, since \((M_i, D_i) \in \mathcal{V}^T\), the homomorphisms \(p_i \cdot m \cdot h\) each factor through some \(f_i\) in \(\mathcal{T}_\Delta\) via \(k_i: (M_i, D_i) \to (M_i, D_i)\). Since \(\mathcal{T}_\Delta\) is a local pseudovariety and \(I\) is finite, the \(f_i\) have an upper bound \(f \in \mathcal{T}_\Delta\), i.e. \(f_i = l_i \cdot f\) for some \(l_i\). Then the
diagonal fill-in property, applied to the left square in the commutative diagram below, yields \( g: (M, D) \to (M, D) \) with \( h = g \cdot f \). This proves \((M, D) \in \mathcal{V}^T\). □

\[
\begin{array}{c}
\begin{array}{ccc}
(\Delta^*, \Delta^o) & \xrightarrow{f} & (M, D) \\
\downarrow h & & \downarrow k_i \\
(M, D) & \xrightarrow{m} & \prod_{i \in I} (M_i, D_i) \\
\end{array}
\end{array}
\]

**Lemma B.3.** For any pseudovariety of reduced lattice bimodules \( \mathcal{V} \), the family \( \mathcal{T}^\mathcal{V} \) (see Notation 3.4) is a theory of lattice bimodules.

**Proof.** We first show each \( \mathcal{T}^\mathcal{V}_2 \) to be a local pseudovariety. It is downwards closed since \( \mathcal{V} \) is closed under reduced quotients. To show directness let \( e_i: (\Sigma^*, \Sigma^o) \to (M_i, D_i), i = 1, 2 \), be two quotients in \( \mathcal{T}^\mathcal{V}_2 \). The image \( \langle e_1, e_2 \rangle[\langle \Sigma^*, \Sigma^o \rangle] \subseteq (M_1, D_1) \times (M_2, D_2) \) of \( \langle \Sigma^*, \Sigma^o \rangle \) under \( \langle e_1, e_2 \rangle \) is \( \ast \)-generated by Lemma 2.5(1) and reduced by Lemma A.7. Since \( \mathcal{V} \) is a pseudovariety, is follows that the lattice bimodule \( \langle e_1, e_2 \rangle[\langle \Sigma^*, \Sigma^o \rangle] \) lies in \( \mathcal{V} \). Therefore, the codomain restriction of \( \langle e_1, e_2 \rangle \) to its image \( \langle e_1, e_2 \rangle[\langle \Sigma^*, \Sigma^o \rangle] \) is an element of \( \mathcal{T}^\mathcal{V}_2 \). It is an upper bound for both \( e_1 \) and \( e_2 \), so \( \mathcal{T}^\mathcal{V} \) is directed.

To confirm that \( \mathcal{T}^\mathcal{V} \) is a theory, let \( e \in \mathcal{T}^\mathcal{V}_2 \) with codomain \( (M, D) \), and \( h: (\Delta^*, \Delta^o) \to (\Sigma^*, \Sigma^o) \). Factorize \( e \cdot h \) into a surjective lattice bimodule homomorphism \( \bar{e}: (\Delta^*, \Delta^o) \to (M', D') \) followed by an injective homomorphism \( h: (M', D') \to (M, D) \). Then \( (M', D') \) is a \( \ast \)-generated subbimodule of \( (M, D) \) and therefore itself in \( \mathcal{V} \), since \( \mathcal{V} \) is a pseudovariety. Thus \( \bar{e} \in \mathcal{T}^\mathcal{V}_{\Sigma} \), as required.

\[
\begin{array}{ccc}
(\Delta^*, \Delta^o) & \xrightarrow{h} & (\Sigma^*, \Sigma^o) \\
\downarrow \bar{e} & & \downarrow e \\
(M', D') & \xrightarrow{h} & (M, D) \\
\end{array}
\]

**Proof of Theorem 3.5**

1. For any pseudovariety \( \mathcal{V} \) of lattice bimodules it holds that \( \mathcal{V} = \mathcal{V}^T \) where \( \mathcal{T} := \mathcal{T}^\mathcal{V} \). Indeed, to show \( \mathcal{V} \subseteq \mathcal{V}^T \), let \( (M, D) \in \mathcal{V} \). Since \( (M, D) \) is \( \ast \)-generated, there exists a surjective homomorphism \( h: (\Sigma^*, \Sigma^o) \to (M, D) \) for some \( \Sigma \in \mathcal{S}_t \). Then \( h \in \mathcal{T}_\Sigma \) by the definition of \( \mathcal{T} \), and so \( (M, D) \in \mathcal{V}^T \). Conversely, to show \( \mathcal{V}^T \subseteq \mathcal{V} \) suppose that \( (M, D) \in \mathcal{V}^T \). Then there exists a \( h \in \mathcal{T}^\mathcal{V}_2 \) with codomain \( (M, D) \). But then, by definition of \( \mathcal{T}^\mathcal{V} \), \( (M, D) \) must have been in \( \mathcal{V} \).

2. For any theory \( \mathcal{T} \) of lattice bimodules it holds that \( \mathcal{T} = \mathcal{T}^\mathcal{V} \), where \( \mathcal{V} = \mathcal{V}^\mathcal{V} \).

To see this, we first show \( \mathcal{T} \subseteq \mathcal{T}^\mathcal{V} \). For \( h \in \mathcal{T}_\Sigma \) with codomain \( (M, D) \) we get \( (M, D) \in \mathcal{V} \), so also \( h \in \mathcal{T}^\mathcal{V}_2 \). For the direction \( \mathcal{T}^\mathcal{V} \subseteq \mathcal{T} \) suppose that \( h \in \mathcal{T}^\mathcal{V}_2 \). Then its codomain \( (M, D) \) is a lattice bimodule in \( \mathcal{V} \), so by definition there exists some \( e \in \mathcal{T}_\Sigma \) with codomain \( (M, D) \). By Lemma A.4 we can choose a \( g: (\Sigma^*, \Sigma^o) \to (\Delta^*, \Delta^o) \) with \( h = e \cdot g \). Since \( \mathcal{T} \) is a theory, there exist \( \bar{e} \in \mathcal{T}_\Sigma \).
and \( \bar{g} \) to make the diagram below commute. Then \( h \leq \bar{e} \in T_\Sigma \) and therefore \( h \in T_\Sigma \) because \( T_\Sigma \) is downwards closed.

\[
\begin{array}{c}
(\Sigma^*, \Sigma^\circ) \xrightarrow{g} (\Delta^*, \Delta^\circ) \\
\downarrow \bar{e} \\downarrow \hline \hline \\
(M', D') \xrightarrow{\bar{g}} (M, D)
\end{array}
\]

(3) Parts (1) and (2) show that the maps \( V \mapsto T_\Sigma V \) and \( T_\Sigma \rightarrow V \) are mutually inverse bijections. Moreover, by definition both maps are clearly order-preserving, which shows that they define an isomorphism of lattices.

Our next aim is to prove Proposition 3.7. The key to this result lies in the observation that \( U \)-quotients of \( \Sigma^\circ \) and reduced quotients of \((\Sigma^*, \Sigma^\circ)\) are in one-to-one correspondence. This is based on the following construction:

**Notation B.4.**

1. For notational simplicity, for a lattice bimodule homomorphism \( h = (h^*, h^\circ) \), we sometimes omit the superscripts \( (-)^* \) and \( (-)^\circ \) and denote both components by \( h \).

2. For any quotient \( e : \Sigma^\circ \rightarrow D \) in CDL, we define a pair \( \equiv_e = (\equiv_e^*, \equiv_e^\circ) \) of equivalence relations on \((\Sigma^*, \Sigma^\circ)\) as follows: For \( x, y \in \Sigma^s \), where \( s \in \{*, \circ\} \), put \( x \equiv_e^s y \iff e(u(x)) = e(u(y)) \) for every \( u \in \overline{U}_\Sigma(\Sigma^s, \Sigma^\circ) \).

Note that \( \equiv_e^\circ \) is a CDL-congruence because all \( u \in \overline{U}_\Sigma(\Sigma^\circ, \Sigma^\circ) \) are CDL-morphisms. Moreover, \( \equiv_e^s \) is stable under all unary operations in \( U \) since \( \overline{U}_\Sigma \) is closed under composition. Thus, Lemma 2.7 (see also Remark A.10) shows that \( \equiv_e \) induces a quotient lattice bimodule of \((\Sigma^*, \Sigma^\circ)\), denoted by \( e_R = (e_R^*, e_R^\circ) : (\Sigma^*, \Sigma^\circ) \rightarrow (\Sigma^*/e, \Sigma^\circ/e) \).

Note that there is no semantic ambiguity in the term \( e_R^\circ \). If \( e \in \text{CDL} \) then it can only be read as \( (e_R)^\circ \) and conversely for \( e \in \text{LBM} \) as \( (e^\circ)_R \).

The key properties of the quotient \( e_R \) are established by the next lemma:

**Lemma B.5.** For any CDL-quotient \( e : \Sigma^\circ \rightarrow D \), the following holds true:

1. \( e_R \) is the smallest lattice bimodule quotient of \((\Sigma^*, \Sigma^\circ)\) with \( e \leq e_R^\circ \).
2. The lattice bimodule \((\Sigma^*/e, \Sigma^\circ/e)\) is reduced.
3. If \( e \) is a \( U \)-quotient, then \( e_R \) is the unique reduced lattice bimodule quotient with \( e_R^\circ = e \).

**Proof.** We first prove the following auxiliary statement (#):

For any lattice bimodule quotient \( h = (h^*, h^\circ) : (\Sigma^*, \Sigma^\circ) \rightarrow (M', D') \):

Whenever \( e \leq h^\circ \) then also \( e_R \leq h \).
To see this, let \( e \leq h^\circ \), so \( e = g \cdot h^\circ \) for some \( g \). To prove that \( e_R \leq h \), we apply the homomorphism theorem: given \( x, y \in \Sigma^e \) with \( h(x) = h(y) \) we need to show that \( e_R(x) = e_R(y) \), that is, \( (e \cdot u)(x) = (e \cdot u)(y) \) for every \( u \in U_\Sigma(\Sigma^e, \Sigma^e) \). Since \( U_\Sigma \) is a unary presentation and \( h \) is a lattice bimodule quotient there exists a lifting \( \bar{u} \) of \( u \) along \( h \). Then

\[
e(u(x)) = g(h(u(x))) = g(\bar{u}(h(x))) = g(\bar{u}(h(y))) = g(h(u(y))) = e(u(y)),
\]
as required. Now we proceed to prove the statements from the lemma.

1. First, we use the homomorphism theorem to prove \( e \leq e^R_R \). Let \( x, y \in \Sigma^e \) with \( e^R_R(x) = e^R_R(y) \), so \( e(u(x)) = e(u(y)) \) for any \( u \in U_\Sigma(\Sigma^e, \Sigma^e) \). In particular \( e(x) = e(id(x)) = e(id(y)) = e(y) \) since \( id \in U_\Sigma(\Sigma^e, \Sigma^e) \). That \( e_R \) is the smallest lattice bimodule quotient of \((\Sigma^*, \Sigma^o)\) with \( e \leq e^R_R \) follows from (\#).

2. Given a lattice bimodule quotient \( g = (g^*, g^o) : (\Sigma^e/e, \Sigma^o/e) \rightarrow (M', D') \) with \( g^o \) an isomorphism in \( \text{CDL} \). Since trivially \( g \cdot e_R \leq e_R \), the opposite \( e_R \leq g \cdot e_R \) would suffice to prove \( g \) is an isomorphism. To do so, we first show \( e \leq g^o \cdot e^R_R \) by using the homomorphism theorem:

For \( x, y \in \Sigma^e \) with \( g^o(e^R_R(x)) = g^o(e^R_R(y)) \) we get \( e^R_R(x) = e^R_R(y) \) because \( g^o \) is an isomorphism and thus \( e(x) = e(y) \) by part (1) of this lemma. Now to derive \( e_R \leq g \cdot e_R \) we apply (\#).

3. Suppose that \( e \) is a \( U \)-quotient. We prove \( e^R_R = e \) by showing they have the same kernel: For \( x, y \in \Sigma^e \), we have \( x \equiv^e y \) iff \( e(x) = e(y) \). The “only if” direction follows directly from part (1). Conversely, let \( e(x) = e(y) \). For all \( u \in U_\Sigma(\Sigma^e, \Sigma^o) \) with a lifting \( \bar{u} \) along \( e \) we compute \( e(u(x)) = \bar{u}(e(x)) = \bar{u}(e(y)) = e(u(y)) \), so \( x \equiv^e y \) and this proves the “if” direction.

For the uniqueness suppose that \( h = (h^*, h^o) : (\Sigma^*, \Sigma^o) \rightarrow (M', D) \) is reduced with \( h^o = e \). Then trivially \( e \leq h^o \) and therefore \( e_R \leq h \) by (\#). Thus, there exists a homomorphism \( g \) with \( e_R = g \cdot h \) and since \( e = e^R_R = g^o \cdot h^o = g^o \cdot e \) we see that \( g^o = id \) since \( e \) is epi. Since \( (M', D) \) is reduced we conclude that \( g \) is an isomorphism and thus \( e_R \) and \( h \) form the same quotient of \((\Sigma^*, \Sigma^o)\).

**Remark B.6.** For any lattice bimodule quotient \( h : (\Sigma^*, \Sigma^o) \rightarrow (M, D) \) the second component \( h^o \) is a \( U \)-quotient; this follows immediately from the fact that \( U_\Sigma \) is a unary presentation of \((\Sigma^*, \Sigma^o)\).

**Lemma B.7.** The maps \( h \mapsto h^o \) and \( e \mapsto e_R \) define an isomorphism between the poset of reduced lattice bimodule quotients of \((\Sigma^*, \Sigma^o)\) and the poset of \( U \)-quotients of \( \Sigma^o \).

**Proof.** We show the assignments to be mutually inverse. Let \( e : \Sigma^o \rightarrow D \) be a \( U \)-quotient. Then Lemma B.5.3 shows that \( e^R_R = e \). Conversely, if \( h : (\Sigma^*, \Sigma^o) \rightarrow (M, D) \) is reduced then \( h^o \) is a \( U \)-quotient, so it follows from Lemma B.5.3 that \((h^o)_R = h \) since they agree on the second component. The map \( h \mapsto h^o \) is clearly monotone. To show that the map \( e \mapsto e_R \) is monotone take \( U \)-quotients \( e, f : \Sigma^o \rightarrow D_i \) with \( e \leq f \). Then \( e \leq f \leq f^o_R \) by Lemma B.5.1 and hence \( e_R \leq f_R \), again by Lemma B.5.1. \( \square \)
Proof of Proposition 3.7
We show that the isomorphism of Lemma B.7 induces an isomorphism between theories of reduced lattice bimodules and theories of $U$-quotients. Explicitly, this isomorphism maps a theory $\mathcal{T}$ of reduced lattice bimodules to the theory $\mathcal{T}^\circ$ of $U$-quotients containing all quotients $e^\circ$ with $e \in \mathcal{T}$. Its inverse maps a theory $\mathcal{T}$ of $U$-quotients to the theory $\mathcal{T}_R^\circ$ of reduced lattice bimodules containing all $e_R$ with $e \in \mathcal{T}$. Since these maps are clearly mutually inverse, the only thing we need to show is that they are well-defined, i.e. they actually map theories to theories.

(1) Given a theory of reduced lattice bimodules $\mathcal{T} = (\mathcal{T}_\Sigma)_{\Sigma \in \text{Set}}$, we show that the corresponding family $\mathcal{T}^\circ = (\mathcal{T}_\Sigma^\circ)_{\Sigma \in \text{Set}}$ is a theory of $U$-quotients. Clearly, each $\mathcal{T}_\Sigma^\circ$ is an ideal since $\mathcal{T}_\Sigma$ is an ideal. Given a $U$-quotient $e^\circ$ that is the second component of some $e: (\Sigma^*, \Sigma^\circ) \rightarrow (M, D)$ in $\mathcal{T}_\Sigma$ and $g: (\Delta^*, \Delta^\circ) \rightarrow (\Sigma^*, \Sigma^\circ)$, choose a reduced lattice bimodule quotient $\bar{e}: (\Delta^*/\bar{e}, D') \rightarrow (M', D')$ in $\mathcal{T}_\Delta$ and a lifting $\bar{g}: (M', D') \rightarrow (M, D)$ with $\bar{g} \cdot \bar{e} = e \cdot g$. Dropping the $*$-component yields $\Delta^\circ := (\Delta^*/\bar{e}, D)$ and $\Delta^\circ := (\Sigma^*, \Sigma^\circ)$, and thus the desired lifting for $e^\circ \cdot g^\circ$ along $\bar{e}^\circ$ in $\mathcal{T}^\circ$. This proves that $\mathcal{T}^\circ$ is a theory of $U$-quotients.

(2) Given a theory $\mathcal{T}$ of $U$-quotients, we show that $\mathcal{T}_R$ is a theory of reduced lattice bimodules. Clearly, each $(\mathcal{T}_R)_\Sigma$ is an ideal since $\mathcal{T}_\Sigma$ is an ideal. Now let $e: \Sigma^\circ \rightarrow D$ be a $U$-quotient in $\mathcal{T}_\Sigma$ and $f: (\Delta^*, \Delta^\circ) \rightarrow (\Sigma^*, \Sigma^\circ)$. We need to show that $e_R \cdot f$ has a lifting. Apply the lifting property of $\mathcal{T}$ to $e$ to obtain the following commutative diagram, where $\bar{e} \in \mathcal{T}_\Delta$ is a lifting of $e :: f$:

What remains is to find the first component $\hat{h}$ of the dashed arrow below:

$$
\begin{array}{c}
\begin{array}{c}
(\Delta^*, \Delta^\circ) \xrightarrow{f} (\Sigma^*, \Sigma^\circ) \\
(\cdot, e) \downarrow \hspace{1cm} \downarrow (\cdot, e) \\
(\cdot, \bar{e}) \downarrow \hspace{1cm} \downarrow (\cdot, \bar{e}) \\
(\cdot, \bar{e}) \downarrow \hspace{1cm} \downarrow (\cdot, \bar{e}) \\
(1, D') \rightarrow (1, D') \\
(\cdot, f) \rightarrow (1, D')
\end{array}
\end{array}
\right)
$$

$$
\begin{array}{c}
\begin{array}{c}
(\Delta^*, \Delta^\circ) \xrightarrow{f} (\Sigma^*, \Sigma^\circ) \\
(\cdot, e) \downarrow \hspace{1cm} \downarrow (\cdot, e) \\
(\cdot, \bar{e}) \downarrow \hspace{1cm} \downarrow (\cdot, \bar{e}) \\
(\cdot, \bar{e}) \downarrow \hspace{1cm} \downarrow (\cdot, \bar{e}) \\
(1, D') \rightarrow (1, D') \\
(\cdot, f) \rightarrow (1, D')
\end{array}
\end{array}
\right)
$$
By the homomorphism theorem, it suffices to show that \( \bar{e}_R(w') = \bar{e}_R(w) \) implies \( e_R(f(w)) = e_R(f(w')) \). From the assumption \( \bar{e}_R(w) = \bar{e}_R(w') \) it follows that \( e(\iota(w)) = e(\iota(w')) \) by the definition of \( \equiv \), and so

\[
\iota(e_R^*(f^*(w))) = e(f(\iota(w))) = \hat{f}(\bar{e}(\iota(w'))) = \iota(e_R^*(f^*(w'))).
\]

Since \( (\Sigma^*/e, D) \) is reduced and \( \star \)-generated and so \( \iota \) is injective by Lemma 2.5(3), we get \( e_R^*(f^*(w)) = e_R^*(f^*(w')) \), as required.

This proves that \( e_R \cdot f \) factors through \( \bar{e}_R \) via \( (\hat{h}, \hat{f}) \) and we are done. \( \square \)

C Details for Section 4

Proof of Lemma 4.2

Recall that a language \( L : \Sigma^* \to 2 \) is regular iff it is recognizable by a finite monoid; that is, there exists a finite monoid \( M \), a monoid homomorphism \( h : \Sigma^* \to M \) and a map \( p : M \to 2 \) such that \( L = p \cdot h \).

Suppose that \( L : \Sigma^* \to 2 \) is recognized by a finite lattice bimodule \( (M, D) \) via \( h : (\Sigma^*, \Sigma^0) \to (M, D) \) and \( p : D \to 2 \). Then \( h^\star \) is a monoid homomorphism that recognizes \( L \) via \( p \cdot \iota \). Since \( M \) is a finite monoid, this proves that \( L \) is regular.

Conversely, if \( L \) is regular, then there exists a finite monoid \( M \), a monoid homomorphism \( h : \Sigma^* \to M \) and a map \( p : M \to 2 \) with \( L = p \cdot h \). We may assume \( h \) to be surjective; if necessary, replace \( h \) by its codomain restriction \( h : \Sigma^* \to h[\Sigma^*] \). Thus, the outside in the diagram below commutes.

\[
\begin{array}{c}
\Sigma^* \\
\downarrow h \\
M \\
\downarrow \iota \\
\text{FCDL}(M)
\end{array} \quad \begin{array}{c}
\Sigma^0 \\
\downarrow h^0 \\
2 \\
\end{array}
\]

In analogy to the proof of Proposition 2.2, the pair \( (M, \text{FCDL}(M)) \) carries a canonical lattice bimodule structure and the monoid homomorphism \( h \) extends to a lattice bimodule homomorphism \( (h, h^\diamond) : (\Sigma^*, \Sigma^0) \to (M, \text{FCDL}(M)) \). Let \( p^\diamond \) to be the extension of \( p \) to a \( \text{CDL} \)-morphism. Then \( (h, h^\diamond) \) is a lattice bimodule homomorphism that recognizes \( L \) via \( p^\diamond \). Note that \( \text{FCDL}(M) \) is finite because \( M \) is finite.

Recall that pseudovarieties of lattice bimodules consist of \( \star \)-generated reduced bimodules. This restriction does not limit the recognized languages:

Lemma C.1. Every language \( L \) recognizable by a finite lattice bimodule is recognizable by a finite \( \star \)-generated reduced lattice bimodule.
Proof. The lattice bimodule \((M, \text{FCDL}(M))\) constructed in the proof of Lemma 4.2 has these properties: It is \(*\)-generated and \(*\)-embedded by definition, and thus reduced by Lemma 2.5.

Remark C.2 (Languages over \(\mathbb{U}\)-quotients). By Lemma C.1 we know that it suffices to work with \(*\)-generated reduced lattice bimodules. There is an obvious equivalent formulation of language recognition in terms of finite \(\mathbb{U}\)-quotients. We therefore use the same terminology as for lattice bimodules, i.e. say that a variety of languages \(V\) is \(\mathbb{U}\)-reduced if it is \(\mathbb{U}\)-generated and \(\mathbb{U}\)-embedded by definition. Since every finite \(\mathbb{U}\)-quotient \(h\) recognizes the languages \(L\) iff its corresponding \(\mathbb{U}\)-quotient \(h^\#\) recognizes \(L\).

Proof of Theorem 4.5

(1) We first prove an auxiliary statement (*):

If \(V_{\Sigma}\) is a basic local variety of regular languages over the alphabet \(\Sigma\), every finite subset \(S \subseteq_f V_{\Sigma}\) is contained in a finite local subvariety of \(V_{\Sigma}\).

To see this, let \(S \subseteq_f V_{\Sigma}\). For any \(L \in S\), let \(\text{Deriv}(L)\) denote the set of all derivatives of \(L\). It is finite since \(L\) is regular, and a local subvariety of \(V_{\Sigma}\) by definition. Since \(S\) is finite, the union \(F = \bigcup_{L \in S} \text{Deriv}(L)\) is again a finite local subvariety and \(S \subseteq F \subseteq_f V_{\Sigma}\).

(2) We show that the two functions given in the statement of the theorem are well-defined, i.e. map varieties to cotheories and vice versa. For any basic local variety of languages \(V_{\Sigma}\), the set

\[
I^{V_{\Sigma}} = \{ F \subseteq V_{\Sigma} \mid F \text{ is a finite basic local subvariety of } V_{\Sigma}\}
\]

is an ideal of finite local subvarieties of \(V_{\Sigma}\): it is clearly downwards closed, and it is upwards directed because unions of basic local varieties are basic local varieties. For the same reason the set \(\bigcup I_{V_{\Sigma}}\) is a basic local variety for any ideal \(I_{V_{\Sigma}}\) of finite local varieties.

If \(T = (I_{V_{\Sigma}})_{\Sigma \in \text{Set}_{\Sigma}}\) is a cotheory of regular languages then the family \(V^T = (\bigcup I_{V_{\Sigma}})_{\Sigma \in \text{Set}_{\Sigma}}\), is a basic variety of regular languages: For every monoid homomorphism \(g: \Delta^* \to \Sigma^*\) the function \(g^{-1}\) restricts locally to some \(F' \in I_{\Delta}\) for all \(F \in I_{\Sigma}\), so

\[
g^{-1}\bigcup I_{\Sigma} = g^{-1}\bigcup_{F \in I_{\Sigma}} F = \bigcup_{F \in I_{\Sigma}} g^{-1}[F] \subseteq \bigcup_{F \in I_{\Sigma}} F' \subseteq \bigcup I_{\Delta}
\]

and thus \(g^{-1}L \in \bigcup I_{\Delta}\) for all languages \(L \in \bigcup I_{\Sigma}\).

Conversely, let \(V = (V_{\Sigma})_{\Sigma \in \text{Set}_{\Sigma}}\) be a basic variety of regular languages. To show \(T = (I^{V_{\Sigma}})_{\Sigma \in \text{Set}_{\Sigma}}\) a cotheory of regular languages, let \(g: \Delta^* \to \Sigma^*\), and take some \(F \in I^{V_{\Sigma}}\). Since \(F\) is finite, we have that \(g^{-1}[F]\) is a finite basic local subvariety of \(V_{\Delta}\); it is closed under derivatives because \(v^{-1}(g^{-1}L)w^{-1} = g(v)^{-1}Lg(w)^{-1}\) for all \(v, w \in \Delta^*\). Thus, \(g^{-1}[F]\) lies in \(V_{\Delta}\).
It remains to prove that the two constructions are mutually inverse. First we show that \( V_\Sigma = \bigcup I_{\Sigma} \) for any basic local variety \( V \). By (\( \star \)), for every \( L \in V_\Sigma \) we have \( L \in F \subseteq V_\Sigma \) for some finite local subvariety \( F \), so \( L \in F \subseteq \bigcup I_{\Sigma} \). For the other direction we have \( \bigcup I_{\Sigma} \subseteq \bigcup P(V_\Sigma) = V_\Sigma \).

Next we have to show for any basic cotheory \( I = (I_{\Sigma})_{\Sigma \in \text{Set}} \) that \( I_{\Sigma} = I_{\Sigma} \) with \( V_\Sigma = \bigcup I_{\Sigma} \). This time the inclusion \( I_{\Sigma} \subseteq I_{\Sigma} \) is clear. Now let \( F \in I_{\Sigma} \). Then each \( L \in F \) is element of some \( F_L \in I_{\Sigma} \). Since \( F \subseteq \bigcup_{L \in F} F_L \in I \) and \( I_{\Sigma} \) is downwards closed, this proves \( F \in I \).

Both assignments are clearly order-preserving. \( \square \)

\[ \text{Details for Section 5} \]

We first provide some details about the duality between algebraic completely distributive lattices and posets. Let us start by recalling some standard terminology from order theory [8]. A subset \( D \subseteq P \) of a poset \( P \) is a down-set if \( p \in D \) and \( p' \leq p \) implies \( p' \in D \). For \( p \in P \) the down-set \( \downarrow p = \{ a \in P \mid a \leq p \} \) is called the principal down-set of \( p \). Since intersections and unions of down-sets are again down-sets we see that the set \( D(P) \) of down-sets of \( P \) forms a completely distributive lattice. An element \( c \) of a complete lattice \( D \) is compact if whenever \( c \leq \bigvee S \) then \( c \leq \bigvee F \) for some finite subset \( F \subseteq S \), and join-prime if \( c \leq \bigvee S \) implies \( c \leq s \) for some \( s \in S \). We denote the sets of compact and join-prime elements of a lattice \( D \) with \( K(D) \) and \( J_p(D) \), respectively. A complete lattice \( D \) is algebraic if every element is the join of all compact elements below it, that is, \( d = \bigvee (\downarrow d \cap K(D)) \) for all \( d \in D \). Algebraic CDLs form a full subcategory of CDL denoted \( \text{AlgCDL} \). For any poset \( P \) the lattice \( D(P) \) is algebraic; its join-primes are the principal downsets \( \downarrow p \) (\( p \in P \)), and its compact elements are finitely generated down-sets (i.e. finite unions of principal down-sets).

**Proposition D.1.** The category \( \text{AlgCDL} \) is dually equivalent to the category \( \text{Pos} \) of posets and monotone maps, witnessed by the equivalence functor

\[
D : \text{Pos} \xrightarrow{\cong} \text{AlgCDL}^{\text{op}}
\]

that maps a poset \( P \) to the lattice \( D(P) \) of down-sets, and a monotone map \( f : P \to Q \) to the CDL morphism \( D(f) = f^{-1} : D(Q) \to D(P) \).

We provide a proof of this well-known duality for the convenience of the reader.

**Proof.** By [8, Thm. 10.29] a CDL is algebraic iff it is isomorphic to \( D(P) \) for some poset \( P \), which implies that \( D \) is isomorphism-dense. To show that \( D \) is an equivalence functor, it remains to prove that it is full and faithful.

To see that \( D \) is faithful, let \( f, g : P \to Q \) with \( D(f) = D(g) \). Then for all \( q \in Q \) we have we have \( D(f)(\downarrow q) = D(g)(\downarrow q) \), that is, for all \( p \in P \),

\[
f(p) \leq q \iff g(p) \leq q.
\]

Substituting \( f(p) \) and \( g(p) \) for \( q \) shows that \( f(p) \leq g(p) \) and \( g(p) \leq f(p) \) for all \( p \in P \), thus proving \( f = g \). This shows that \( D \) is faithful.
Finally, we show that $\mathcal{D}$ is full. Let $g: \mathcal{D}(Q) \to \mathcal{D}(P)$ be a CDL morphism. For each $p \in P$ let $D_p$ denote the least down-set of $Q$ whose $g$-image contains $p$:

$$D_p = \bigcap \{ D \in \mathcal{D}(Q) \mid p \in g(D) \} \in \mathcal{D}(Q).$$

Note that $p \in g(D_p)$ because $g$ preserves intersections. The set $D_p$ is join-prime since if $D_p \subseteq \bigcup_i K_i$ then $p \in g(K_j)$ for some $j$, and so $D_p \subseteq K_j$ by the minimality of $D_p$. The join-prime elements of $\mathcal{D}(Q)$ are precisely the principal down-sets of $Q$, hence $D_p = \downarrow f(p)$ for some unique $f(p) \in Q$. This defines a monotone map $f: P \to Q$ via $p \mapsto f(p)$ for which we prove $f^{-1} = g$. If $D \in \mathcal{D}(Q)$, then

$$p \in f^{-1}(D) \iff f(p) \in D \iff \downarrow f(p) \subseteq D \iff D_p \subseteq D \iff \bigcap_{p \in g(K)} K \subseteq D \iff p \in g(D)$$

which gives $f^{-1}(D) = g(D)$ and so $g = f^{-1} = \mathcal{D}(f)$, proving that $\mathcal{D}$ is full. □

**Remark D.2.** (1) We may identify $\mathcal{D}(P)$ with the lattice $\text{Pos}(P, 2)$ of monotone functions into the two-chain $2 = \{0 < 1\}$ via

$$D \mapsto \chi_{D^c},$$

$$f^{-1}[0] \leftrightarrow f,$$

where $\chi_{D^c} : P \to 2$ denotes the characteristic function of the complement of $D$. Thus, up to natural isomorphism, the equivalence $\mathcal{D}$ is given by the hom-functor

$$\text{Pos}(\cdot, 2) : \text{Pos} \xrightarrow{\cong} \text{AlgCDL}^{\text{op}}.$$

(2) It is also instructive to see how the duality operates in the other direction: From [8, Theorem 10.29] we know that for a completely distributive algebraic lattice $D$ the poset $P$ with $\mathcal{D}(P) \cong D$ is isomorphic to $J_p(D)$, the poset of join-prime elements of $D$, but equipped with the dual order of $D$: for $p, q \in J_p(D)$ we have $p \leq_J(D) q$ if and only if $q \leq_D p$. Note that we have the isomorphism $J_p(D) \cong \text{AlgCDL}(D, 2)$ by identifying an element of $d \in J_p(D)$ with the morphism $f : D \to 2$ sending $d' \in D$ to $1$ if $d \leq d'$. The order $\leq_{J_p(D)}$ is the order induced by the pointwise ordering on $\text{AlgCDL}(D, 2)$. Thus, the inverse equivalence of $\mathcal{D}$ is, up to natural isomorphism, the hom-functor

$$\text{AlgCDL}(\cdot, 2) : \text{AlgCDL}^{\text{op}} \xrightarrow{\cong} \text{Pos}.$$

**Remark D.3.** To make use of the duality $\text{AlgCDL}^{\text{op}} \cong \text{Pos}$ in our setting, let us note that the concept of a $\sqcup$-quotient $c : \Sigma^\circ \to D$ actually “lives” in the full subcategory $\text{AlgCDL}$ of $\text{CDL}$ since it involves only free or finite CDLs. Every free CDL $\Sigma^\circ$ is algebraic since the elements of the form $\bigwedge_{i \in I} w_i \ (w_i \in \Sigma^\circ)$ are join-prime. Moreover, every finite lattice $D$ is algebraic because every element $d \in D$ is compact. Furthermore, the factorization system of $\text{CDL}$ restricts to $\text{AlgCDL}$, and we can thus safely adopt our concepts of quotients into $\text{AlgCDL}$.
Lemma D.4. AlgCDL inherits the factorization system of surjective and injective morphisms from CDL.

Proof. For the proof we use that a lattice is an algebraic CDL if and only if it is isomorphic to a complete lattice of sets [8, Theorem 10.29]. So let \( h: L \to M \) be a function between complete lattices of sets that preserves arbitrary unions and intersections. In CDL the morphism \( h \) factorizes into

\[
L \xrightarrow{p} K \xrightarrow{i} M
\]

with \( p \) surjective and \( i \) injective. Since \( K \) is a complete sublattice of \( M \) and thus also isomorphic to a lattice of sets, we see that \( K \) is algebraic. Thus, \( h = i \cdot p \) is a factorization of \( h \) in AlgCDL. \( \square \)

Remark D.5. Note that quotients (i.e. surjective homomorphisms) in AlgCDL dualize to subposets (i.e. maps \( m \) satisfying \( x \leq y \) iff \( m(x) \leq m(y) \)) in Pos. This follows immediately from the definition of the dual equivalence, but also from the fact that quotients in AlgCDL correspond to strong epimorphisms and subposets correspond to strong monomorphisms in Pos. Using this duality, \( \mathbb{U} \)-quotients in AlgCDL admit a natural dual interpretation in terms of the languages recognized by them (cf. Remark C.2):

1. If we start with a finite \( \mathbb{U} \)-quotient \( e: \Sigma^\diamond \to D \) in AlgCDL, it dualizes to the embedding of a finite subposet

\[
J_p(e): J_p(D) \hookrightarrow J_p(\Sigma^\diamond).
\]

The join-primes of \( \Sigma^\diamond \) are given by \( J_p(\Sigma^\diamond) \cong AlgCDL(\Sigma^\diamond, 2) \cong \text{Set}(\Sigma^*, 2) \cong \mathcal{P}(\Sigma^*) \), so we may regard \( J_p(e) \) as a subobject \( J_p(D) \hookrightarrow \mathcal{P}(\Sigma^*) \), i.e. a set of languages. Now let \( k: J_p(D) \cong AlgCDL(D, 2) \to \text{Rec}(e) \) be the bijection given by \( p \mapsto p \cdot e \). Then, using that \( J_p(e) \) is given by precomposition with \( e \), we see that the map \( k \) makes the following triangle commute:

\[
\begin{array}{ccc}
\text{AlgCDL}(D, 2) & \xrightarrow{\cong} & \text{Rec}(e) \\
\P(\Sigma^*) & \xrightarrow{k} & \\
\end{array}
\]

Therefore \( \text{AlgCDL}(D, 2) \) and \( \text{Rec}(e) \) are isomorphic subposets of \( \mathcal{P}(\Sigma^*) \). Note that all elements of \( \text{Rec}(e) \) are regular languages by Lemma 4.2. Since \( e \) is a \( \mathbb{U} \)-quotient, for every \( u \in \mathbb{U}_{\Sigma}(\diamond, \diamond) \) there exists a lifting:

\[
\begin{array}{ccc}
\Sigma^\diamond & \xrightarrow{u} & \Sigma^\diamond \\
\downarrow e & & \downarrow e \\
D & \longrightarrow & D
\end{array}
\]
Dualizing this diagram yields

\[
\begin{array}{ccc}
\mathcal{P}(\Sigma^*) & \xrightarrow{u^{-1}} & \mathcal{P}(\Sigma^*) \\
\downarrow & & \downarrow \\
\text{Rec}(e) & \xrightarrow{-} & \text{Rec}(e)
\end{array}
\]  

(D.2)

indicating that for all \(v, w \in \Sigma^*\) the word derivation function \(L \mapsto v^{-1}Lw^{-1}\) restricts to the subset \(\text{Rec}(e)\). In other words, \(\text{Rec}(e)\) is a finite basic local variety of languages.

(2) Conversely, if we start out with a finite basic local variety of languages \(i: V_\Sigma \hookrightarrow \text{Reg}_\Sigma \hookrightarrow \mathcal{P}(\Sigma^*)\) then \(V_\Sigma\) is a subobject of \(\mathcal{P}(\Sigma^*)\) in \(\text{Pos}\). Its dual \(\mathcal{D}(i)\) is therefore a quotient of \(\Sigma^\circ\) in \(\text{AlgCDL}\). Since \(V_\Sigma\) is closed under all word derivatives \(L \mapsto v^{-1}Lw^{-1}\) represented by elements \(u \in \mathbb{U}_\Sigma(\circ, \circ)\), the map \(u^{-1}\) on \(\mathcal{P}(\Sigma^*)\) restricts to \(V_\Sigma\), i.e. we have the commutative diagrams

\[
\begin{array}{ccc}
\mathcal{P}(\Sigma^*) & \xrightarrow{u^{-1}} & \mathcal{P}(\Sigma^*) \\
\downarrow & & \downarrow \\
V_\Sigma & \xrightarrow{-} & V_\Sigma
\end{array}
\]

Thus, dually, \(\mathcal{D}(i)\) is a quotient in \(\text{AlgCDL}\) such that every \(u \in \mathbb{U}\) has a lifting:

\[
\begin{array}{ccc}
\Sigma^\circ & \xrightarrow{u} & \Sigma^\circ \\
\downarrow & & \downarrow \\
\mathcal{D}(V_\Sigma) & \xrightarrow{-} & \mathcal{D}(V_\Sigma)
\end{array}
\]

This proves that \(\mathcal{D}(i)\) is a \(U\)-quotient for any finite basic local subvariety \(i: V_\Sigma \hookrightarrow \mathcal{P}(\Sigma^*)\). More specifically, \(\mathcal{D}(i)\) is the \(U\)-quotient recognizing precisely the languages in \(V_\Sigma\); we see this since any \(L \in V_\Sigma\) is representable by the triangle

\[
\begin{array}{ccc}
\mathcal{P}(\Sigma^*) & \xrightarrow{L_i} & \Sigma^\circ \\
\downarrow & & \downarrow \\
V_\Sigma & \xrightarrow{L} & \mathcal{D}(V_\Sigma)
\end{array}
\]

that dualizes to

\[
\begin{array}{ccc}
\Sigma^\circ & \xrightarrow{L} & \Sigma^\circ \\
\downarrow & & \downarrow \\
\mathcal{D}(V_\Sigma) & \xrightarrow{-} & \mathcal{D}(V_\Sigma)
\end{array}
\]

proving that \(\mathcal{D}(V_\Sigma)\) recognizes \(L\). Conversely, any \(L\) recognized by \(\mathcal{D}(V_\Sigma)\) dualizes to some element of \(V_\Sigma\) if we start with the triangle in \(\text{AlgCDL}\).

We have thus established the following result:

**Proposition D.6.** The lattice of finite \(U\)-quotients of \(\Sigma^\circ\) is isomorphic to the lattice of finite basic local varieties over \(\Sigma\). The isomorphism is given by

\[
e \mapsto (\text{Rec}(e) \hookrightarrow \mathcal{P}(\Sigma^*))
\]
This isomorphism easily extends to the level of ideals:

**Corollary D.7 (Duality between local varieties).** For each \( \Sigma \in \text{Set} \), the lattice of local pseudovarieties of \( \mathbb{U} \)-quotients over \( \Sigma \) is isomorphic to the lattice of ideals \( I_{\Sigma} \) of finite basic local varieties over \( \Sigma \). The isomorphism is given by

\[
\mathcal{T}_\Sigma \mapsto \{ \text{Rec}(e) : e \in \mathcal{T}_\Sigma \}.
\]

**Remark D.8.** As the final step, we observe that the above local correspondence extends to a global one between theories of \( \mathbb{U} \)-quotients and basic cotheories of regular languages. Suppose that \( \mathcal{T} = (\mathcal{T}_\Sigma)_{\Sigma \in \text{Set}} \) is a theory of \( \mathbb{U} \)-quotients. Thus, for all lattice bimodule homomorphisms \( h : (\Delta^*, \Delta^o) \to (\Sigma^*, \Sigma^o) \) and every \( e \in \mathcal{T}_\Sigma \) there exists a lifting of \( e \cdot h^o \) through \( \mathcal{T}_\Delta \):

\[
\begin{array}{ccc}
\Delta^o & \xrightarrow{h^o} & \Sigma^o \\
\downarrow & & \downarrow \\
D' & \longrightarrow & D
\end{array}
\]

Thus, letting \( g = h^* \) denote the monoid morphism in the first component of \( h \), the dual diagram in \( \text{Pos} \) then precisely states that the corresponding family of ideals of finite basic local varieties is closed under preimages of \( g \), and vice versa.

\[
\begin{array}{ccc}
\mathcal{P}(\Delta^*) & \leftarrow g^{-1} & \mathcal{P}(\Sigma^*) \\
\downarrow & & \downarrow \\
\text{Rec}(\pi) & \longleftarrow & \text{Rec}(e)
\end{array}
\]

We have thus established the following result:

**Proposition D.9 (Duality between theories and cotheories).** The lattice of theories of \( \mathbb{U} \)-quotients is isomorphic to the lattice of basic cotheories of regular languages. The isomorphism is given by

\[
\mathcal{T} \mapsto \mathcal{T} = (I_{\Sigma})_{\Sigma \in \text{Set}}, \text{ with } I_{\Sigma} = \{ \text{Rec}(e) : e \in \mathcal{T}_{\Sigma} \}.
\]

**Proof of Theorem 5.1**
We simply compose all the previously established lattice isomorphisms:

\[
\begin{array}{ccc}
\text{Pseudovarieties of lattice bimodules} & \cong & \text{Theories of lattice bimodules} \quad (\text{Theorem 3.5}) \\
& \cong & \text{Theories of } \mathbb{U} \text{-quotients} \quad (\text{Proposition 3.7}) \\
& \cong & \text{Basic cotheories of regular languages} \quad (\text{Proposition D.9}) \\
& \cong & \text{Basic varieties of regular languages} \quad (\text{Theorem 4.5})
\end{array}
\]