Necessary criterion for extracting thermodynamical work from qudit-entangled state

SUMIT NANDI

S.N. Bose National Centre for Basic Sciences, Block JD, Sector III, Salt Lake, Kolkata 700 106, India
E-mail: sumit.enandi@gmail.com

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Abstract. A novel criterion of extracting thermodynamical work from a bipartite pure qudit-entangled state using local operation and classical communication (LOCC) has been presented. We have shown that non-vanishing $G$-concurrence is a necessary condition for extracting work from a higher-dimensional entangled state in LOCC paradigm.

Keywords. Thermodynamical work; $G$-concurrence; qudit-entangled state.

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1. Introduction

Phenomenal inclusion of quantum information theory within the framework of thermodynamics opens up a scope to devise microengines [1] which can ‘lawfully’ draw work from a single heat bath. The important consequence of reading the microscopic degrees of freedom, in particular the orientation of spin of qubits, leads to the extraction of work from a single heat bath [2]. Needless to say, it does not violate the second law of thermodynamics since entropy of the system increases as accumulated information is diminished with the extraction of work. However, the given system may be classical or quantum by nature, but information originating out of quantum correlation can be declassified by a suitable work extraction protocol which outperforms the system containing classical information [3] only. Non-local aspects of quantum correlation is the main reason behind this, and non-locality in this case acts as a resource. Information stemming out of non-locality is regarded as the fuel as bits of information can be used to extract work from a correlated system attached to a single heat bath [4]. Thus, entanglement of the system can be exploited to perform a particular thermodynamical work extracting protocol as it is done in information processing protocols. Some novel work extraction protocols from the entangled states are presented in [5,6]. Francica et al [7] proposed a protocol, known as ergotropy, to extract work from a quantum state with respect to some reference Hamiltonian, under cyclic unitaries, and showed that quantum correlation, in particular discord, in the system can enhance work that can be gained in the given paradigm. In the light of this discussion, the role of quantum correlation has been found effective for extracting maximum ergotropy. However, in our present work, we shall consider the novel work extraction protocol presented in [8]. Maruyama et al [8] formulated a suitable criterion of separability by deriving a Bell-type equality in terms of the maximum work that can be extracted from a bipartite state. Later, this work had been carried forward for tripartite states [9]. We will generalise the protocol using qudits – higher-dimensional quantum system. Notably, information processing protocols are often advantageous with qudits [10,11].

But the fact is that the scope of qudit entanglement is much broader – it is significantly different from qubit entanglement quantitatively as well as qualitatively. For instance, a qubit-entangled state satisfies the famous positive partial transpose (PPT) criterion [12]. Thus, all entangled qubit states are negative partial transpose (NPT) which no longer remains sufficient for non-separability in qudit entanglement [13]. There exists a class of states which are PPT and entangled as well, which are known as bound entangled states. Entanglement cannot be distilled from those states [14] that raise fundamental difficulties in detecting qudit entanglement. Similarly, quantification of a single-copy pure bipartite qubit entanglement can be fully described by
a single measure whereas a bipartite qudit state needs more such ad-hoc quantities. Vidal [15] introduced a family of \(d\) entanglement monotones to describe the entanglement of a bipartite pure qudit state when only a single copy of the given state is provided. In a similar spirit, Gour [16] introduced concurrence monotones which are constructed with the Schmidt coefficients of the state. The last one, namely \(G\)-concurrence, had been the central attraction due to its extensive implication of the full dimensionality of entanglement. It seems reasonable to think of the entangled states in a \(d\)-dimensional Hilbert space \(C^d \otimes C^d\) which have \(d^2 (\leq d - 1)\) non-vanishing entanglement monotones. So, some of the novel aspects of entanglement would be missing from those states. Meanwhile, it also implies that the performance of certain protocols, if carried out by such states, would reflect the underlying structure of the state. To delve into more details, we have considered work extraction protocol by local operation and classical communication (LOCC) with bipartite qudit (\(d = 3\)) states. We have formulated a suitable criterion that ensures the success of the protocol. We have shown for the first time that extractable work is directly related to a computational measure of entanglement, namely \(G\)-concurrence. In doing so, we have also presented a physical interpretation of \(G\)-concurrence. We shall show entangled states with non-vanishing concurrence and \(G\)-concurrence are more suitable to extract work in non-local regime.

Before presenting our main result, we shall revisit some prerequisites in §2 which are very much relevant for our discussion. Then, we present the framework of a work extraction protocol by LOCC in §3. Subsequently, we provide the main result in §4 with some illustrative examples. Then we will make some important remarks in §5.

### 2. Prerequisites

#### 2.1 Concurrence monotone

Concurrence monotones [16] were introduced to characterise entanglement properties of a single-copy pure entangled state of dimensionality beyond 2. These are simply functions of Schmidt coefficients of the given state. Thus, one can realise all non-local aspects of a pure state by knowing all these monotones. Concurrence monotones are computable and can be extended for mixed states by convex roof extension. For a pure state \(|\psi\rangle\) in \(C^d \otimes C^d\) having Schmidt numbers \(\lambda_0, \ldots, \lambda_{d-1}\) respectively, there exist \(d - 1\) non-trivial concurrence monotones. These monotones can be expressed in terms of a function of the Schmidt coefficients as

\[
C_1(|\psi\rangle) = \sum_{i=0}^{d-1} \lambda_i \\
C_2(|\psi\rangle) = \sum_{i<j}^{d-1} \lambda_i \lambda_j \\
\vdots \\
C_d(|\psi\rangle) = \prod_i^{d-1} \lambda_i.
\]

(1)

The first expression entails the fact that the first monotone is trivial and it is simply ‘one’, the sum of all the Schmidt coefficients. Amongst the others, the last member of the monotone family is of particular interest and known as \(G\)-concurrence which can be recast as \(C_d = d(\lambda_1 \lambda_2 \cdots \lambda_d)^{\frac{2}{d}}\). For 2 \(\otimes\) 2 pure states, it is simply the concurrence as presented in [17]. For a \(d\)-dimensional state, \(G\)-concurrence is the determinant of the reduced density matrix of the subsystem. \(G\)-concurrence reveals the dimensionality of the entanglement. It may be possible that a \(d\)-dimensional state is entangled but its \(C_d\) vanishes. By virtue of this fact, the state would be less effective for extracting information processing protocols.

#### 2.2 Thermodynamic work gain using the entanglement as a resource

In this subsection, we elaborate the protocol [8] to gain an insight into the role of entanglement in the context of thermodynamical work extraction by LOCC. Suppose Alice and Bob share a large number of bipartite states in a distant lab scenario. Both of them have a number of measurement settings \({A_\theta, A_\theta^+}\) and \({B_\theta, B_\theta^+}\), respectively. Alice measures her subsystem and let Bob know the outcome and measurement basis by classical communication. Knowing Alice’s outcome, Bob can extract \(1 - H(B_\theta | A_\theta)\) bits of work from his subsystem. Here, \(H(X|Y)\) is the Shannon entropy of \(X\) conditional on the entropy of \(Y\). We can easily verify that 

\[1 - H(B_\theta | A_\theta)\]

is maximised when the state shared between Alice and Bob is maximally entangled \(|\phi\rangle_{ME}, |\phi\rangle_{ME} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\). Quantitatively, Maruyama et al have shown that the amount of extractable work from a given state \(\rho\) is equivalent to

\[
\zeta(A(\theta), B(\theta)) = \frac{1}{2}(2 - 2H(A(\theta), B(\theta)) + H(A(\theta)) + H(B(\theta))),
\]

(2)
where $H(A(\theta))$ and $H(B(\theta))$ are the Shannon entropies corresponding to the outcome of $A(\theta)(B(\theta))$, respectively. Now, one can maximise $\xi(A(\theta), B(\theta))$ by varying $\theta$ over the great circle of the Bloch sphere. Thus, extractable work is given by

$$\mathcal{W}(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \xi(A(\theta), B(\theta))d\theta.$$  \hspace{1cm} (3)

The quantity $\mathcal{W}$ carries a great deal of information about the non-locality of the state $\rho$. The maximal work $\mathcal{W}$ is bounded for separable states and entangled states can violate this upper bound. Now we extend this protocol for the qutrit system, i.e. the shared state between Alice and Bob is a two-qutrit pure state. A measurement setting [18] can be constructed with the projectors $M_0, M_1$ and $M_2 = I - M_0 - M_1$ where $M_i = |m_i\rangle\langle m_i|$ and

$$|m_0\rangle = e^{ix_1}\sin \theta \cos \phi |0\rangle + e^{ix_2}\sin \theta \sin \phi |1\rangle + \cos \theta |2\rangle,$$

$$|m_1\rangle = e^{ix_1}\cos \theta \cos \phi |0\rangle + e^{ix_2}\cos \theta \sin \phi |1\rangle - \sin \theta |2\rangle,$$

where $0 \leq \theta, \phi \leq \frac{\pi}{2}$ and $0 \leq x_1, x_2 \leq 2\pi$. For the sake of simplicity, we have set $x_1, x_2 = 0$. It just takes a simple step to find the upper bound of work which is ($\lesssim$) 0.65 for a separable state. Now, we evaluate the quantity $\mathcal{W}$ for the following two-qutrit states:

$$|\tilde{\Omega}\rangle = \sqrt{r}|00\rangle + \sqrt{s}|11\rangle + \sqrt{1-r-s}|22\rangle,$$

$$|\tilde{\omega}\rangle = \sqrt{r}|00\rangle + \sqrt{s}|11\rangle + \sqrt{1-r-s}|12\rangle,$$  \hspace{1cm} (4, 5)

where $r, s \in [0, 1]$. Evidently, $|\tilde{\omega}\rangle$ is entangled and its concurrence is given by $\sqrt{2(r - r^2)}$. Since its reduced density matrix has rank 2, it is straightforward that $G$-concurrence vanishes for the given state. We calculate LOCC work that can be extracted from $|\tilde{\omega}\rangle$ when the given concurrence is 0.9 and it turns out to be $\mathcal{W}(|\tilde{\omega}\rangle) = 0.50$. Although the state is highly entangled, as quantified by its concurrence, maximal work that can be gained from the state is no better than a separable one. Now, for the former state, we obtain $\mathcal{W} \simeq 0.8$ when the given concurrence is 0.9. Therefore, mere specification of concurrence would not suffice to find the quantity $\mathcal{W}$ for a qudit state. It seems reasonable as entanglement property of higher-dimensional quantum system is fully characterised by the entanglement monotones.

3. Thermodynamic work extraction protocol in the LOCC paradigm

Our approach of work extraction is pedagogical; the holistic protocol requires an entangled state being shared by two observers, namely Alice and Bob, and the former is performing a measurement in a pre-determined basis. Since the shared state is entangled, the subsystem at Bob’s end also changes accordingly. Bob is provided with several filters which can transform the microscopic degree, such as orientation of spin of a qubit, into a specific orientation. As Bob is aware of the basis deployed by Alice, he picks up the suitable filter, in each turn Alice confirms her successful operation. Now, we consider Bob’s subsystem after several such rounds of the protocol (figure 1a): after each turn of the process, Bob’s subsystem resembles a specific state. So, Bob can extract work by allowing an isothermal expansion of his subsystem. If, on the other hand Bob cannot retrieve that specific state, we shall refer the work extraction protocol as an unsuccessful one. Let us describe the protocol more vividly with the maximally entangled two-qubit state $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Alice measures her subsystem in computational basis, and as a result, Bob’s subsystem collapses into $|0\rangle$ and $|1\rangle$ corresponding to the measurement outcome $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, respectively. So, Bob does nothing for the former outcome while he applies $\sigma_x$ for the latter outcome. Here the unitary operation $\sigma_x$ can be regarded as the filter chosen by Bob. So, in each turn Bob’s state corresponds to a known state $|0\rangle$ which can be further exploited to extract thermodynamical work. In fact, this machinery would run for all measurement basis provided the shared state is maximally correlated. It is understood that the protocol does not work if a state like $|00\rangle$ is used instead, and only half of the time Bob’s subsystem resembles the pre-decided state.

We assume that Alice has performed the operation on $N_0$ particles and each operation at Alice’s end yields a particle in Bob’s lab. So, at the end of the protocol, we might expect Bob’s subsystem to consist of $N_1 \leq N_0$ objects with a specific orientation. The ratio $\frac{N_1}{N_0}$ would determine the amount of work to be extracted. Qualitatively, it seems that the ratio would tend to ‘one’ when the resource state is maximally entangled and the ratio is less than unity for partially entangled states. Thus, work extraction can be quantified from the entanglement of the resource state. We can consider ‘concurrence’ as a measure of entanglement for our present discussion.

In figure 1, we have depicted the work extraction protocol: initially two observers are provided with two identical systems composed of $N_0$ randomly polarised objects (represented by the arrowheads in the upper left and right boxes). Then, the observer A measures its subsystem and communicates to B the used measurement basis, according to which B applies a suitable filter and consequently, its subsystem is filled with $N_1 \leq N_0$ objects with a known polarisation state.

Now, we substitute the working substance with the following qutrit-entangled states given as
The state $|\omega\rangle$, corresponding to each measurement outcome $|\omega\rangle$ of Alice, the latter can find a suitable unitary transformation, Bob’s subspace corresponds to a definite state, say $|n\rangle\langle m|$, corresponding to Alice’s each measurement outcome, proving that our theorem is sufficient for accomplishing a successful work extraction protocol.

Now, we will proceed to prove the necessity of the theorem. As prescribed in the previous section, work extraction protocol begins with Alice who measures her subsystem with a general measurement setting consisting of the projectors $m_0, m_1, \ldots, m_d$ which satisfy \( \sum_{i=0}^{d-1} |m_i\rangle\langle m_i| = \frac{1}{d} \). Corresponding to each measurement outcome $i$ pertaining probability $p_i$ of Alice, Bob’s

4. Condition for maximal work extraction by LOCC

We proceed now to expose a criterion for extracting work in our paradigmatic situation.

**Theorem.** A pure bipartite qudit entangled state can be used as a resource state for extracting thermodynamic work by means of LOCC if it has non-vanishing G-concurrence.

To prove the above theorem, let us start with a general bipartite two-qudit state

\[
|\Psi\rangle = \sum_{i,j=0}^{d-1} a_{ij} |i\rangle_A |j\rangle_B,
\]  

where the complex coefficients $a_{ij}$ satisfy the normalisation condition $\sum_{i,j} a_{ij}^* a_{ij} = 1$. The subscripts $A(B)$ denote that the subsystems are distributed between Alice and Bob, respectively. The state written in the above form can always be expressed as follows:

\[
|\Psi\rangle = \sum_{i} \lambda_i |\tilde{i}\rangle_A |\tilde{i}\rangle_B,
\]  

where $\lambda_i$ are the Schmidt coefficients of $|\Psi\rangle$. If Alice makes a measurement and communicates the outcome to Bob, then corresponding to each measurement outcome of Alice, the latter can find a suitable unitary $V$ to transform his subsystem into a specific state. While elaborating our work extraction protocol in the previous section, we found that a state would be useful for extracting thermodynamic work. Thus, it is an indicative consequence of the fact that entanglement of a qudit state quantified by concurrence is not sufficient to ensure the success of the protocol. In this spirit, we shall proceed to find the necessary criterion so that a qudit state can be used to extract work successfully in a thermodynamical system.
subsystem is collapsed into the given one-qudit state $\sum_j \tilde{a}_{ij} |j\rangle$, where $\tilde{a}_{ij} = \frac{a_{ij}}{d^2}$. The protocol would be successful if Bob finds a suitable unitary operation $U$ such that the collapsed state is identified as a fixed reference state $\rho$ which can be expressed as $\rho = \sum_{n=0}^{d-1} |n\rangle\langle n|$ where $\sum_n |n\rangle\langle n| = 1$. Here, we emphasise that $n$ has the same dimension as the index $j$. To accomplish this, we need a one-qutrit unitary operation $U$ such that

$$|j\rangle = \sum_n U_{jn} |n\rangle,$$  

where $U_{jn}$ are the elements of $U$. Condition (11) gives interesting properties of the complex elements $a_{ij}$. It would be possible if $a_{ij}$ satisfy the following condition:

$$\sum_i a_{ij}^{*} a_{ij} = \delta_{jj'}.$$  

To check this, let us consider Bob’s subsystem $\rho_B$ by tracing over Alice’s subsystem as

$$\rho_B = \sum_{i,j,j'} a_{ij}^{*} a_{ij} |j\rangle\langle j'|.$$

Now, invoking eqs (11) and (12), we find

$$\rho_B = \sum_{j,j',n,n'} \delta_{jj'} U_{jn'}^{*} U_{jn} |n\rangle\langle n'|$$

$$= \sum_{j,n,n'} U_{jn'}^{*} U_{jn} |n\rangle\langle n'|.$$  

We put the condition of unitary $\sum_j U_{jn'}^{*} U_{jn} = \delta_{nn'}$ and plug it into the last equation to obtain

$$\rho_B = \sum_{n,n'} \delta_{nn'} |n\rangle\langle n'|$$

$$= \sum_n |n\rangle\langle n| = \rho.$$  

Thus, we obtain a suitable criterion of a successful work extracting protocol of the complex coefficients $a_{ij}$ of state (9). The condition implies that Bob’s subspace constitutes an orthonormal basis. Now, it suffices to evaluate $G$-concurrence of the given state (9). We know that $G$-concurrence of a state of the form (9) can be expressed as $G \propto \prod_{j=0}^{d-1} |\langle j|j\rangle|$. Thus, invoking condition (12), we see that the resource state has non-vanishing $G$-condition which turns out to be necessary to carry out the prescribed task with the state (9). This completes the proof of the theorem.

The theorem implies that the given state would be useful for the task if all of its Schmidt coefficients exist.

Figure 2. A comparative behaviour of the quantity $W(\rho)$ as a monotonic function of the state parameter $x$.

Figure 3. A comparative behaviour of the quantity $W(\rho)$ as a monotonic function of $\alpha$.

For a two-level quantum system, it would simply imply that $|\Psi\rangle$ is entangled with non-vanishing concurrence. Beyond the two-level system, concurrence seems to be insufficient to ensure work extraction protocol. The state with vanishing $G$-concurrence would not be a potential resource state to extract thermodynamical work in the given paradigm as it was shown in the last section: the state $|\Omega\rangle$ was found to be useful for the protocol even if $|\omega\rangle$ was not as suitable.

To assert our theorem, we shall extend the protocol [8] by two well-known classes of mixed states to show that the quantity $W$ is a monotone of $G$-concurrence.

**Example.** Here we consider the state $\rho = x|\omega\rangle\langle\omega| + (1 - x)|\Omega\rangle\langle\Omega|$. We mention that $G$-concurrence of $\rho$, for this particular decomposition, can be given as $1 - x$. A quantitative behaviour of the quantity $W(\rho)$ for the given state is depicted in figure 2. The plot shows that $W(\rho)$ in the non-local regime decreases as $x$ increases.

**Example.** Lastly, we shall produce an example to show that our criterion (12) is robust against white noise. We
Consider the mixed state 
\[ \rho_{\text{mix}} = \alpha |\Omega\rangle\langle \Omega| + (1 - \alpha) |01\rangle\langle 01|. \] (16)

Although no general prescription exists to compute concurrence monotones of arbitrary mixed states, the \( G \)-concurrence of the given state \( \rho_{\text{mix}} \) has been found in [19] and it is given by \( \alpha \). We plot the quantity \( \mathcal{W}(\rho_{\text{mix}}) \) for the given state \( \rho_{\text{mix}} \) in figure 3. It follows that the extractable work \( \mathcal{W}(\rho_{\text{mix}}) \) monotonically increases with its \( G \)-concurrence.

5. Conclusion

We have found the necessary criterion to extract thermodynamical work from a qutrit-entangled state. It turns out that the non-vanishing \( G \)-concurrence is necessary for a state so that it can be used as a resource for the prescribed protocols. It is to be noted that the result can also be generalised straightforwardly for any qudit state. The result is not so surprising in itself as we know that specification of a family of entanglement monotones is necessary to characterise the entanglement property of a qudit state. So a state with high concurrence value but vanishing \( G \)-concurrence is not suitable for extracting work in the thermodynamic regime. We have explicitly found the necessary condition to extract work successfully. Apart from a proof in full generality, suitable examples have been constructed to strengthen our result. We have provided a few classes of mixed states to show that the quantity \( \mathcal{W} \) monotonically increases with \( G \)-concurrence. Thus, we have also obtained a physical interpretation of \( G \)-concurrence of qudit-entangled states. We mention that extensive characterisation of entanglement in qudit regime needs \( G \)-concurrence, perhaps other monotones also which may occur beyond the qutrit states. In our discussion, we have seen that entanglement might be present in the subspaces of a qutrit state which is quantified by its concurrence. Indeed, it appears to be insufficient to accomplish certain LOCC protocol while entanglement arising out of full dimensionality of the state space is very much required.

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