On normed products of operator ideals which contain $\mathcal{L}_2$ as a factor

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Abstract
We investigate quasi-Banach operator ideal products $(\mathfrak{A} \circ \mathfrak{B}, A \circ B)$ which contain $(\mathcal{L}_2, \mathcal{L}_2)$ as a factor. In particular, we ask for conditions which guarantee that $A \circ B$ is even a norm if each factor of the product is a 1-Banach ideal. In doing so, we reveal the strong influence of the existence of such a norm in relation to the accessibility of the product ideal and the structure of its factors.

Key words and phrases: Accessibility, Banach spaces, cotype 2, finite rank operators, Hilbert space factorization, operator ideals, principle of local reflexivity

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1 Introduction
This paper is devoted to an investigation of the normability of operator ideal products which contain $(\mathcal{L}_2, \mathcal{L}_2)$ as a factor (where $\mathcal{L}_2$ denotes the class of all operators which factor through a Hilbert space). It seems that 1-Banach operator ideal products play a fundamental role in the search for maximal Banach ideals which do not satisfy a transfer of the norm estimation in the classical principle of local reflexivity to their ideal norm (cf. [14]). This problem (which still is open) originated from the objective to facilitate the search for a non–accessible maximal normed Banach ideal (which is the same as a non–accessible finitely generated tensor norm in the sense of Grothendieck) (cf. [10]). Later, in 1993, Pisier constructed a counterexample (cf. [2, 31.6.]). Since each right–accessible maximal Banach ideal $(\mathfrak{A}, A)$ even satisfies such a principle of local reflexivity for operator ideals, Pisier’s counterexample of a non–accessible maximal Banach ideal naturally lead to the search for counterexamples of maximal Banach ideals $(\mathfrak{A}_0, A_0)$ for which the conjugate $(\mathfrak{A}_0^\Delta, A_0^\Delta)$ is not left-accessible, implying surprising relations between the existence of a norm on product operator ideals of type $\mathfrak{B} \circ \mathcal{L}_2$, the extension of finite rank operators with respect to a suitable operator ideal norm and the principle of local reflexivity for operator ideals (cf. [14]). The basic objects, connecting these different aspects, are product operator ideals with the property (I) and the property
and the property (S), if
\((\mathcal{A} \circ \mathcal{B}) \cap \mathfrak{F} = (\mathfrak{F} \cap \mathcal{A}) \circ \mathcal{B}\)
and the property (S), if
\((\mathcal{A} \circ \mathcal{B}) \cap \mathfrak{F} = \mathcal{A} \circ (\mathcal{B} \cap \mathfrak{F})\)
(\mathcal{F}\) denotes the class of all finite rank operators) so that each finite rank operator in \(\mathcal{A} \circ \mathcal{B}\) is the composition of two operators, one of which is of finite rank. Since each operator ideal which contains \(\mathcal{L}_2\) as a factor, has both, the property (I) and the property (S), Hilbert space factorization crystallized out as a fundamental key in these investigations.

## 2 The framework

In this section, we introduce the basic notation and terminology which we will use throughout in this paper. We only deal with Banach spaces and most of our notations and definitions concerning Banach spaces and operator ideals are standard. We refer the reader to the monographs [4], [3] and [15] for the necessary background in operator ideal theory and the related terminology. Infinite dimensional Banach spaces over the field \(\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}\) are denoted throughout by \(W, X, Y\) and \(Z\) in contrast to the letters \(E, F\) and \(G\) which are used for finite dimensional Banach spaces only. Denote for given Banach spaces \(X\) and \(Y\)

\[
\text{FIN}(X) := \{E \subseteq X \mid E \in \text{FIN}\} \quad \text{and} \quad \text{COFIN}(X) := \{L \subseteq X \mid X/L \in \text{FIN}\},
\]

where FIN stands for the class of all finite dimensional Banach spaces. The space of all operators (continuous linear maps) from \(X\) to \(Y\) is denoted by \(\mathcal{L}(X,Y)\), and for the identity operator on \(X\), we write \(Id_X\). The collection of all finite rank (resp. approximable) operators from \(X\) to \(Y\) is denoted by \(\mathfrak{F}(X,Y)\) (resp. \(\mathfrak{F}(X,Y)\)), and \(\mathcal{E}(X,Y)\) indicates the collection of all operators, acting between finite dimensional Banach spaces \(X\) and \(Y\) (elementary operators). The dual of a Banach space \(X\) is denoted by \(X'\), and \(X''\) denotes its bidual \((X')'\). If \(T \in \mathcal{L}(X,Y)\) is an operator, we indicate that it is a metric injection by writing \(T : X \hookrightarrow Y\), and if it is a metric surjection, we write \(T : X \twoheadrightarrow Y\). If \(X\) is a Banach space, \(E\) a finite dimensional subspace of \(X\) and \(K\) a finite codimensional subspace of \(X\), then \(B_X := \{x \in X : \|x\| \leq 1\}\) denotes the closed unit ball, \(J_X^E : E \hookrightarrow X\) the canonical metric injection and \(Q_X^K : X \twoheadrightarrow X/K\) the canonical metric surjection. Finally, \(T' \in \mathcal{L}(Y',X')\) denotes the dual operator of \(T \in \mathcal{L}(X,Y)\).

If \((\mathfrak{A}, \mathfrak{A})\) and \((\mathfrak{B}, \mathfrak{B})\) are given quasi–Banach ideals, we will use throughout the shorter notation \((\mathfrak{A}^d, \mathfrak{A}^d)\) for the dual ideal and the abbreviation \(\mathfrak{A}^\perp = \mathfrak{B}\) for the isometric equality \((\mathfrak{A}, \mathfrak{A}) = (\mathfrak{B}, \mathfrak{B})\). We write \(\mathfrak{A} \subseteq \mathfrak{B}\) if, regardless of the Banach spaces \(X\) and \(Y\), we have \(\mathfrak{A}(X,Y) \subseteq \mathfrak{B}(X,Y)\). The metric inclusion \((\mathfrak{A}, \mathfrak{A}) \subseteq (\mathfrak{B}, \mathfrak{B})\) is often shortened by \(\mathfrak{A}^\perp \subseteq \mathfrak{B}\).

Given quasi–Banach ideals \((\mathfrak{A}, \mathfrak{A})\) and \((\mathfrak{B}, \mathfrak{B})\), let \((\mathfrak{A} \circ \mathfrak{B}, \mathfrak{A} \circ \mathfrak{B})\) be the corresponding product ideal and \((\mathfrak{A} \circ \mathfrak{B}^{-1}, \mathfrak{A} \circ \mathfrak{B}^{-1})\) (resp. \((\mathfrak{A}^{-1} \circ \mathfrak{B}, \mathfrak{A}^{-1} \circ \mathfrak{B})\)) the corresponding ”right–quotient” (resp. ”left–quotient”). Important examples are \((\mathfrak{A}^{\text{min}}, \mathfrak{A}^{\text{min}}) := (\overline{\mathfrak{F}} \circ \mathfrak{A} \circ \overline{\mathfrak{F}}, \| \cdot \circ \mathfrak{A} \circ \| \cdot \|)\) (the minimal kernel of \((\mathfrak{A}, \mathfrak{A})\)) and \((\mathfrak{A}^{\text{max}}, \mathfrak{A}^{\text{max}}) := (\overline{\mathfrak{F}}^{-1} \circ \mathfrak{A} \circ \overline{\mathfrak{F}}^{-1}, \| \cdot \|^{-1} \circ \mathfrak{A} \circ \| \cdot \|^{-1})\) (the maximal hull of \((\mathfrak{A}, \mathfrak{A})\)). \((\mathfrak{A}^{\text{inj}}, \mathfrak{A}^{\text{inj}})\) denotes the injective hull of \(\mathfrak{A}\), the unique smallest
injective quasi–Banach ideal which contains \((\mathfrak{A}, \mathfrak{A})\), and \((\mathfrak{A}^{\text{sur}}, \mathfrak{A}^{\text{sur}})\), the surjective hull of \(\mathfrak{A}\), is the unique smallest surjective quasi–Banach ideal which contains \((\mathfrak{A}, \mathfrak{A})\).

In addition to the maximal Banach ideal \((\mathcal{L}, \| \cdot \|)\) we mainly will be concerned with the maximal Banach ideals \((\mathfrak{J}, \mathfrak{I})\) (integral operators), \((\mathcal{L}_2, \mathcal{L}_2)\) (Hilbertian operators), \((\mathcal{D}_2, \mathcal{D}_2)\) \(\overset{1}{\leq} (\mathfrak{L}^{\Delta}, \mathfrak{L}^{\Delta})\) \(\overset{1}{=} \mathfrak{P}^d_2 \circ \mathfrak{P}_2\) (2–dominated operators), \((\mathfrak{P}_p, \mathfrak{P}_p)\) (absolutely \(p\)–summing operators), \(1 \leq p \leq \infty, \frac{1}{p} + \frac{1}{q} = 1, (\mathcal{L}_\infty, \mathcal{L}_\infty) \overset{1}{=} (\mathfrak{P}^*_1, \mathfrak{P}^*_1)\) and \((\mathfrak{L}_1, \mathfrak{L}_1) \overset{1}{=} (\mathfrak{P}_{1}^* d, \mathfrak{P}_{1}^* d)\).

Since we will use them throughout in this paper, let us recall the important notions of the conjugate operator ideal (cf. \([3], [8] \text{ and } [11]\)) and the adjoint operator ideal (all details can be found in the standard references \([3] \text{ and } [13]\)). Let \((\mathfrak{A}, \mathfrak{A})\) be a quasi–Banach ideal.

- Let \(\mathfrak{A}^{\Delta}(X, Y)\) be the set of all \(T \in \mathcal{L}(X, Y)\) which satisfy
  \[\mathfrak{A}^{\Delta}(T) := \sup \{| \text{tr}(TL) | \mid L \in \mathfrak{F}(Y, X), \mathfrak{A}(L) \leq 1\} < \infty.\]

  Then a Banach ideal \((\mathfrak{A}^{\Delta}, \mathfrak{A}^{\Delta})\) is obtained (here, \(\text{tr}(\cdot)\) denotes the usual trace for finite rank operators). It is called the conjugate ideal of \((\mathfrak{A}, \mathfrak{A})\).

- Let \(\mathfrak{A}^{\ast}(X, Y)\) be the set of all \(T \in \mathcal{L}(X, Y)\) which satisfy
  \[\mathfrak{A}^{\ast}(T) := \sup \{| \text{tr}(TJ_E SQ^X_K) | \mid E \in \text{FIN}(X), K \in \text{COFIN}(Y), \mathfrak{A}(S) \leq 1\} < \infty.\]

  Then a Banach ideal \((\mathfrak{A}^{\ast}, \mathfrak{A}^{\ast})\) is obtained. It is called the adjoint operator ideal of \((\mathfrak{A}, \mathfrak{A})\).

A deeper investigation of relations between the Banach ideals \((\mathfrak{A}^{\Delta}, \mathfrak{A}^{\Delta})\) and \((\mathfrak{A}^{\ast}, \mathfrak{A}^{\ast})\) needs the help of an important local property, known as accessibility, which can be viewed as a local version of injectivity and surjectivity. All necessary details about accessibility of operator ideals and its applications can be found in \([2], [11], [12], \text{ and } [13]\). So let us recall:

- A quasi–Banach ideal \((\mathfrak{A}, \mathfrak{A})\) is called right–accessible, if for all \((E, Y) \in \text{FIN} \times \text{BAN},\) operators \(T \in \mathcal{L}(E, Y)\) and \(\varepsilon > 0\) there are \(F \in \text{FIN}(Y)\) and \(S \in \mathcal{L}(E, F)\) so that \(T = J^Y_F S\) and \(\mathfrak{A}(S) \leq (1 + \varepsilon)\mathfrak{A}(T)\).

- \((\mathfrak{A}, \mathfrak{A})\) is called left–accessible, if for all \((X, F) \in \text{BAN} \times \text{FIN},\) operators \(T \in \mathcal{L}(X, F)\) and \(\varepsilon > 0\) there are \(L \in \text{COFIN}(X)\) and \(S \in \mathcal{L}(X/L, F)\) so that \(T = SQ^X_L\) and \(\mathfrak{A}(S) \leq (1 + \varepsilon)\mathfrak{A}(T)\).

- A left–accessible and right–accessible quasi–Banach ideal is called accessible.

- \((\mathfrak{A}, \mathfrak{A})\) is totally accessible, if for every finite rank operator \(T \in \mathfrak{F}(X, Y)\) acting between Banach spaces \(X, Y\) and \(\varepsilon > 0\) there are \((L, F) \in \text{COFIN}(X) \times \text{FIN}(Y)\) and \(S \in \mathcal{L}(X/L, F)\) so that \(T = J^Y_F SQ^X_L\) and \(\mathfrak{A}(S) \leq (1 + \varepsilon)\mathfrak{A}(T)\).

Let us recall the following important results on accessibility (for a detailed proof cf. \([2], [10], \text{ and } [13]\)):

**Theorem 2.1** Let \((\mathfrak{A}, \mathfrak{A})\) be a Banach ideal. Then \((\mathfrak{A}^{\text{min}}, \mathfrak{A}^{\text{min}})\) always is accessible and \((\mathfrak{A}^{\Delta}, \mathfrak{A}^{\Delta})\) is right-accessible. If in addition \((\mathfrak{A}, \mathfrak{A})\) is maximal, then \((\mathfrak{A}, \mathfrak{A})\) is right-accessible if and only if \((\mathfrak{A}^{\ast}, \mathfrak{A}^{\ast})\) is left-accessible.
\textbf{Theorem 2.2} Let \((\mathcal{A}, A)\) be a maximal Banach ideal.

(i) \((\mathcal{A}, A)\) is right-accessible if and only if \(\mathcal{A}^* \circ \mathcal{A} \subseteq \mathcal{I}\).

(ii) \((\mathcal{A}, A)\) is totally accessible if and only if \(\mathcal{A}^* \equiv \mathcal{A}^\Delta\).

Pisier’s counterexample \((\mathcal{A}_F, A_F)\) shows the existence of maximal Banach ideals which neither are left nor right-accessible (cf. [2], 31.6). However, accessibility of a quasi-Banach ideal at least can be transmitted to its regular hull:

\textbf{Proposition 2.1} Let \((\mathcal{A}, A)\) be an arbitrary quasi-Banach ideal. If \((\mathcal{A}, A)\) is right-accessible (resp. totally-accessible), then the regular hull \((\mathcal{A}^{\text{reg}}, A^{\text{reg}})\) is also right-accessible (resp. totally-accessible).

\textsc{Proof:} Let \(\epsilon > 0\), \(X, Y\) be Banach spaces and \(T \in \mathcal{F}(X,Y)\) an arbitrary finite rank operator. Assume that \(\mathcal{A}\) is totally accessible or that \(X \in \text{FIN}\) and \(\mathcal{A}\) is right-accessible. In both cases, there exists a finite dimensional Banach space \(F \in \text{FIN}(Y^m)\) and an operator \(S \in \mathcal{L}(X,F)\), so that \(j_Y T = J^Y_F S\) and

\[ A(S) < (1 + \epsilon) \cdot A(j_Y T) = (1 + \epsilon) \cdot A^{\text{reg}}(T). \]

Due to the classical principle of local reflexivity for linear operators, there exists an operator \(W \in \mathcal{L}(F,Y)\) so that \(\|W\| < 1 + \epsilon\) and \(j_Y Wz = J^Y_F z\) for all \(z \in F\) which satisfy \(J^Y_F z \in j_Y(Y)\). Let \(x \in X\) and put \(z := Sx\). Then \(J^Y_F z = j_Y Tx \in j_Y(Y)\), which therefore implies that \(j_Y W S x = J^Y_F z = j_Y T x\). Now, factor \(W\) canonically through a finite dimensional subspace \(G\) of \(Y\) so that \(W = J^Y_G U\) and \(\|U\| < 1 + \epsilon\). Consequently, \(T = W S = J^Y_G(U S)\), and

\[ A^{\text{reg}}(US) < (1 + \epsilon)^2 \cdot A^{\text{reg}}(T). \]

Hence, \(A^{\text{reg}}\) is right-accessible (in each of the both cases). In the case of \(\mathcal{A}\) being totally accessible, the operator \(S\) even can be chosen as \(S = S_0 Q^X_K\), where \(K \in \text{COFIN}(X)\) and \(S_0 \in \mathcal{L}(X/K, F)\) so that

\[ A(S_0) < (1 + \epsilon) \cdot A^{\text{reg}}(T), \]

and the proof is finished. \(\blacksquare\)

\section{3 Normed operator ideal products}

Let \((\mathcal{A}, A)\) be a \(p\)-normed Banach ideal \((0 < p \leq 1)\) and \((\mathcal{B}, B)\) be a \(q\)-normed Banach ideal \((0 < q \leq 1)\). Then the product \((\mathcal{A} \circ \mathcal{B}, A \circ B)\) is a \(r\)-normed Banach ideal, where \(1/r := 1/p + 1/q\) (see \[13\], 7.1.2). Even if \(p = 1\) and \(q = 1\), \((\mathcal{A} \circ \mathcal{B}, A \circ B)\) in general is a \(1/2\)-Banach ideal only; \(A \circ B\) need not to be a norm. However, if one of the operator ideals is closed (such as e.g., \((\mathcal{F}, \| \cdot \|)\), \((\mathcal{R}, \| \cdot \|)\) or \((\mathcal{M}, \| \cdot \|)\)), then we may formulate a positive result (cf. \[1\]):

\textbf{Proposition 3.1} Let \((\mathcal{A}, A)\) and \((\mathcal{B}, B)\) be two quasi-Banach ideals. Let \(0 < p \leq 1\). Then, in each of the following cases, \((\mathcal{A} \circ \mathcal{B}, A \circ B)\) is a \(p\)-Banach ideal:

(i) \((\mathcal{A}, A)\) is a \(p\)-Banach ideal and \((\mathcal{B}, B) = (\mathcal{B}, \| \cdot \|)\) is closed,
(ii) \((\mathfrak{A}, \mathfrak{A}) = (\mathfrak{A}, \| \cdot \|)\) is closed and \((\mathfrak{B}, \mathfrak{B})\) is a \(p\)-Banach ideal.

**Proof:** It is sufficient to prove the case (i); (ii) follows similarly. So let \((\mathfrak{A}, \mathfrak{A})\) be \(p\)-normed and \((\mathfrak{B}, \| \cdot \|)\) be closed. Let \(X\) and \(Y\) be arbitrary Banach spaces and \(T_1, T_2 \in (\mathfrak{A} \circ \mathfrak{B})(X,Y)\). It remains to show that

\[
(A \circ \| \cdot \|)^p(T_1 + T_2) \leq (A \circ \| \cdot \|)^p(T_1) + (A \circ \| \cdot \|)^p(T_2).
\]

Let \(\epsilon > 0\). Then there exist Banach spaces \(Z_1, Z_2\) and operators \(R_1 \in \mathfrak{A}(Z_1, Y), R_2 \in \mathfrak{A}(Z_2, Y), S_1 \in \mathfrak{B}(X, Z_1), S_2 \in \mathfrak{B}(X, Z_2)\) so that \(T_1 = R_1S_1, T_2 = R_2S_2, \|S_1\| \leq 1, \|S_2\| \leq 1, A(R_1) \leq (1 + \epsilon) \cdot (A \circ \| \cdot \|)(T_1)\) and \(A(R_2) \leq (1 + \epsilon) \cdot (A \circ \| \cdot \|)(T_2)\). We now consider the Banach space \(W := l_\infty(Z_1, Z_2)\) consisting of all elements \((z_1, z_2) \in Z_1 \times Z_2\) so that \(\|(z_1, z_2)\|_\infty := \max(\|z_1\|, \|z_2\|) < \infty\). Let \(J_i : Z_i \to W\) be the canonical injections and \(Q_i : W \to Z_i\) the corresponding canonical surjections \((i = 1, 2)\). Then \(S := J_1S_1 + J_2S_2 \in \mathfrak{B}(X, W)\) and \(\|Sx\|_\infty = \|(S_1x, S_2x)\|_\infty = \max(\|S_1x\|, \|S_2x\|) \leq \|x\|\) for all \(x \in X\). Hence, \(\|S\| \leq 1\). Put \(R := R_1Q_1 + R_2Q_2\). Then \(R \in \mathfrak{A}(W, Y)\) and

\[
(A(R))^p \leq (A(R_1))^p + (A(R_2))^p.
\]

The construction therefore implies that \(T_1 + T_2 = RS\) and

\[
(A \circ \| \cdot \|)^p(T_1 + T_2) = (A \circ \| \cdot \|)^p(RS) \leq (A(R) \cdot \|S\|)^p \leq (A(R))^p \leq (A(R_1))^p + (A(R_2))^p \leq (1 + \epsilon)^p \cdot ((A \circ \| \cdot \|)^p(T_1) + (A \circ \| \cdot \|)^p(T_2)),
\]

and the proof is finished. \(\blacksquare\)

An immediate (non-trivial) consequence is the

**Corollary 3.1** Let \((\mathfrak{A}, \mathfrak{A})\) be a \(p\)-Banach ideal \((0 < p \leq 1)\). Then \((\mathfrak{A}^{\text{min}}, \mathfrak{A}^{\text{min}})\) also is a \(p\)-Banach ideal.

Unfortunately, we still cannot present explicate sufficient criteria which show the existence of (an equivalent) ideal norm on product ideals in the general case. It seems to be much more easier to show that a certain product ideal cannot be a normed one by using arguments which involve trace ideals and the ideal of nuclear operators (the smallest Banach ideal). Even more holds: if \(\mathfrak{A} \circ \mathfrak{L}_2\) is a \(1\)-Banach ideal for certain operator ideals \(\mathfrak{A}\), then \(\mathfrak{A} \circ \mathfrak{L}_2\) is not right–accessible (cf. theorem 3.4)! To investigate more carefully the general case, we recall an important factorization property for finite rank operators which had been introduced by Jarchow and Ott in their paper \([3]\). It not only turns out to be very useful for an investigation of local structures in (product) operator ideals; this factorization property was used as the main tool to show that \(\mathfrak{L}_\infty\) and \(\mathfrak{L}_1\) are not totally accessible – answering an open question of Defant and Floret (see \([2]\), 21.12 and \([14]\)). So, let us recall the definition of this factorization property and its implications:

**Definition 3.1 (Jarchow/Ott)** Let \((\mathfrak{A}, \mathfrak{A})\) and \((\mathfrak{B}, \mathfrak{B})\) be arbitrary quasi–Banach ideals. Let \(L \in \mathfrak{A}(X,Y)\) an arbitrary finite rank operator between two Banach spaces \(X\) and \(Y\). Given \(\epsilon > 0\), we can find a Banach space \(Z\) and operators \(A \in \mathfrak{A}(Z,Y), B \in \mathfrak{B}(X,Z)\) so that \(L = AB\) and

\[
A(A) \cdot B(B) \leq (1 + \epsilon) \cdot (A \circ B)(L).
\]

(i) If the operator \(A\) is of finite rank, we say that \(\mathfrak{A} \circ \mathfrak{B}\) has the property \((I)\).

(ii) If the operator \(B\) is of finite rank, we say that \(\mathfrak{A} \circ \mathfrak{B}\) has the property \((S)\).

Important examples are the following (see \([3]\), lemma 2.4):
• If \( \mathfrak{B} \) is injective, or if \( \mathfrak{A} \) contains \( L^2 \) as a factor, then \( \mathfrak{A} \circ \mathfrak{B} \) has the property (I).

• If \( \mathfrak{A} \) is surjective, or if \( \mathfrak{B} \) contains \( L^2 \) as a factor, then \( \mathfrak{A} \circ \mathfrak{B} \) has the property (S).

Since \( L^2 \circ \mathfrak{A} \) is injective for every quasi–Banach ideal \( (\mathfrak{A}, \mathfrak{A}) \) (see [13], lemma 5.1.), \( \mathfrak{B} \circ L^2 \circ \mathfrak{A} \) therefore has the property (I) as well as the property (S), for all quasi–Banach ideals \( (\mathfrak{A}, \mathfrak{A}) \) and \( (\mathfrak{B}, \mathfrak{B}) \). Such ideals are exactly those which contain \( L^2 \) as factor – in the sense of [8].

**Theorem 3.1** Let \( (\mathfrak{A}, \mathfrak{A}) \) be a maximal Banach ideal. Then both, the maximal \( \frac{1}{2} \)-Banach ideal \( \mathfrak{A}^{\text{inj}} \circ L^2 \) and the injective hull of the maximal \( \frac{1}{2} \)-Banach ideal \( \mathfrak{A} \circ L^2 \) are totally accessible.

**Proof:** Since every Hilbert space has the metric approximation property and since \( \mathfrak{A}^{\text{inj}} \perp (\mathfrak{A}^{\text{inj}})^{**} \) is right–accessible, an easy calculation shows that

\[
\mathfrak{A}^{\text{inj}} \circ L^2 \perp (\mathfrak{A}^{\text{inj}})^{\Delta} \circ L^2.
\]

(1)

Since \( (\mathfrak{A}^{\text{inj}})^{\Delta} \) is right–accessible, the total accessibility of \( L^2 \) and the property (S) of the product ideal \( (\mathfrak{A}^{\text{inj}})^{\Delta} \circ L^2 \) even imply that \( (\mathfrak{A}^{\text{inj}})^{\Delta} \circ L^2 \) is totally accessible (cf. [14, proposition 4.1]). Hence, \( \mathfrak{A}^{\text{inj}} \circ L^2 \) is totally accessible (due to (1)), and in particular we obtain that \( (\mathfrak{A} \circ L^2)^{\text{inj}} \perp (\mathfrak{A}^{\text{inj}} \circ L^2)^{\text{inj}} \) is totally accessible.\[\square\]

Now, let \( (\mathfrak{A}, \mathfrak{A}) \) be a maximal Banach ideal so that \( L^2 \circ \mathfrak{A} \) is even a norm on the (maximal) product ideal \( (L^2 \circ \mathfrak{A}, L^2 \circ \mathfrak{A}) \). Then \( \mathfrak{A}^* \perp (L^2 \circ \mathfrak{A})^* \perp L^\infty \) (cf. [13, proposition 5.1.]) and \( L^\infty \perp \mathfrak{M}^\Delta \perp L^\Delta \). Given Banach spaces \( X \) and \( Y \) so that both, \( X' \) and \( Y \) have cotype 2, [16, theorem 4.9.] tells us, that any finite rank operator \( L \in \mathfrak{F}(Y, X) \) satisfies

\[
N(L) \leq (2C_2(X') \cdot C_2(Y))^{\frac{3}{2}} \cdot D_2(L).
\]

Hence,

\[
\mathfrak{M}^\Delta(X, Y) \subseteq D_2^\Delta(X, Y) \perp L^\Delta(X, Y),
\]

and we have proven a rather surprising fact (revealing the strong influence of a norm on an operator ideal product):

**Theorem 3.2** Let \( (\mathfrak{A}, \mathfrak{A}) \) be a maximal Banach ideal so that the product ideal \( (L^2 \circ \mathfrak{A}, L^2 \circ \mathfrak{A}) \) is normed. Let \( X \) and \( Y \) be arbitrary Banach spaces so that both, \( X' \) and \( Y \) have cotype 2. Then

\[
\mathfrak{A}^*(X, Y) \perp (L^2 \circ \mathfrak{A})^*(X, Y) \subseteq L^\Delta(X, Y)
\]

and

\[
L^\Delta(T) \leq (2C_2(X') \cdot C_2(Y))^{\frac{3}{2}} \cdot (L^2 \circ \mathfrak{A})^*(T) \leq (2C_2(X') \cdot C_2(Y))^{\frac{3}{2}} \cdot \mathfrak{A}^*(T)
\]

for all operators \( T \in \mathfrak{A}^*(X, Y) \).

To maintain the previous statement, even a permutation of the factors \( \mathfrak{A} \) and \( L^2 \) in the product \( L^2 \circ \mathfrak{A} \) is allowed:
Theorem 3.3 Let \((\mathfrak{A}, A)\) be a maximal Banach ideal so that the product ideal \((\mathfrak{A} \circ L_2, A \circ L_2)\) is normed. Let \(X\) and \(Y\) be arbitrary Banach spaces so that both, \(X'\) and \(Y\) have cotype 2. Then
\[
\mathfrak{A}^*(X, Y) \subseteq (\mathfrak{A} \circ L_2)^*(X, Y) \subseteq L_2(X, Y)
\]
and
\[
L_2(T) \leq (2C_2(X') \cdot C_2(Y))^{\frac{1}{2}} \cdot (A \circ L_2)^*(T) \leq (2C_2(X') \cdot C_2(Y))^{\frac{1}{2}} \cdot A^*(T)
\]
for all operators \(T \in \mathfrak{A}^*(X, Y)\).

Proof: Let \((\mathfrak{A}, A)\) and \(X, Y\) be as before and let \(\mathfrak{A} \circ L_2\) be normed. Then \(\mathfrak{A} \not\subseteq \mathfrak{A}^{dd}\), and \(\mathfrak{A} \circ L_2\) is a maximal (and therefore a regular) Banach ideal (cf. [14, lemma 4.3]). Since the injective \(\frac{1}{2}\)–Banach ideal \(L_2 \circ \mathfrak{A}^{dd}\) is also regular (cf. [13 lemma 5.1]), an easy calculation shows that
\[
(\mathfrak{A} \circ L_2)^d \subseteq L_2 \circ \mathfrak{A}^{dd}
\]
and
\[
\mathfrak{A} \circ L_2 \subseteq (L_2 \circ \mathfrak{A}^{dd})^d.
\]

Since \(A \circ L_2\) is a norm, \((A \circ L_2)^d\) obviously is a norm too. Hence, if we apply the previous theorem to the normed product ideal \((\mathfrak{A} \circ L_2)^d \subseteq L_2 \circ \mathfrak{A}^{dd}\), we obtain
\[
(\mathfrak{A} \circ L_2)^d \subseteq (L_2 \circ \mathfrak{A}^{dd})^d \subseteq (A \circ L_2)^d(X, Y) \subseteq L_2(X, Y),
\]
and
\[
L_2(T) \leq C \cdot (A \circ L_2)^*(T') \leq C \cdot A^*(T')
\]
for all operators \(T \in \mathfrak{A}^{*d}(X, Y)\) (where \(C := (2C_2(X') \cdot C_2(Y))^{\frac{1}{2}}\)). Now, since \(Y\) has the same cootype as its bidual \((Y')'\) with identical cotype constants (cf. [3, corollary 11.9]), the proof is finished.

Let \((\mathfrak{A}, A)\) be a given ultrastable quasi–Banach ideal so that \((\mathfrak{A} \circ L_2, A \circ L_2)\) is right–accessible. Our aim is to show that in this case \((\mathfrak{A} \circ L_2, A \circ L_2)\) and \((L_2 \circ \mathfrak{A}^*, L_2 \circ \mathfrak{A}^*)\) both together cannot be normed. To this end, we need a lemma which is of its own interest:

Lemma 3.1 Let \((\mathfrak{A}_0, A_0)\) be a maximal Banach ideal so that space(\(\mathfrak{A}_0\)) contains a Banach space without the approximation property. Then there does not exist a maximal Banach ideal \((\mathfrak{C}, C)\) so that \(\mathfrak{C} \circ L_\infty\) has the property (I) and \(\mathfrak{C} \subseteq \mathfrak{A}_0^{-1} \circ \mathfrak{P}_1\).

Proof: Assume that the statement is false. Then there exists a (maximal) Banach ideal \((\mathfrak{A}, A)\) so that \(\mathfrak{A}_0 \subseteq \mathfrak{P}_1 \circ (\mathfrak{A}^*)^{-1} \subseteq (\mathfrak{A}^*\Delta)^{inj}\). Due to the assumed property (I) of \(\mathfrak{A}^* \circ L_\infty\), the proof of theorem 3.4 in [14] shows that even \(((\mathfrak{A}^*\Delta)^{inj})^{dd} \subseteq (\mathfrak{A}^{inj})^\Delta \subseteq \mathfrak{P}_\Delta\). Since \(\mathfrak{A}_0\) was assumed to be a maximal Banach ideal, we therefore obtain \(\mathfrak{A}_0 \not\subseteq \mathfrak{A}_0^{-1} \circ \mathfrak{P}_1\) which is a contradiction.

Corollary 3.2 Let \((\mathfrak{A}_0, A_0)\) be a maximal Banach ideal so that space(\(\mathfrak{A}_0\)) contains a Banach space without the approximation property. If \((\mathfrak{A}_0^{-1} \circ \mathfrak{P}_1) \circ L_\infty\) has the property (I), \(\mathfrak{A}_0\) is not left–accessible.

\(^{1}\)In particular, it follows that \(\mathfrak{A} \circ L_2\) is surjective (cf. [14, 8.5.9.]).
**Theorem 3.4** Let \((B, \mathcal{B})\) be an ultrastable quasi–Banach ideal so that \(B \subseteq L_\infty\). If \(B \circ L_2\) is right–accessible, \(B \circ L_2\) cannot be a \(1\)–Banach ideal.

**Proof:** Assume that the statement is false and put \(B_0 := (L_\infty \circ L_2)^*\) and \(\mathfrak{A} := (B \circ L_2)^*\). Then
\[
\mathfrak{A}^* \overset{1}{\subseteq} (B \circ L_2)^{\max} \overset{1}{=} (B \circ L_2)^{\reg} \overset{1}{=} B^{\reg} \circ L_2
\]
is right–accessible (due to proposition 2.1) and contains \(L_2\) as a factor so that in particular \(\mathfrak{A}^* \circ L_\infty\) has the property (I). Since \(B \subseteq L_\infty\),
\[
B_0 \circ \mathfrak{A}^* \subseteq \mathfrak{A} \circ \mathfrak{A}^* \overset{1}{\subseteq} J \overset{1}{\subseteq} P_1,
\]
and it follows that \(\mathfrak{A}^* \subseteq B_0^{-1} \circ P_1\). Since \(Id_P \in B_0\) (cf. [14, proposition 4.4]), lemma 3.1 leads to a contradiction.\(\square\)

Now let us assume that \((B, B)\) is even a *maximal Banach ideal* so that \(B \subseteq L_\infty\). If \(B \circ L_2\) were normed, then \(B \circ L_2\) would be a maximal and surjective Banach ideal, implying that
\[
P_1^{d} \overset{1}{=} 1^{\text{sur}} \overset{1}{=} (P^\max)^{\text{sur}} \subseteq B \circ L_2.
\]
Hence,
\[
(B \circ L_2)^* \overset{1}{\subseteq} P_1^{d*} \overset{1}{=} L_1.
\]
Since \(B \subseteq L_\infty\), it even follows that \(P_1 \overset{1}{=} L_\infty \subseteq (B \circ L_2)^* \overset{1}{\subseteq} L_1\) which is a contradiction (cf. [2, 27.2.]). So, in this case we obtain a stronger result:

**Theorem 3.5** Let \((B, B)\) be a maximal Banach ideal so that \(B \subseteq L_\infty\). Then \(B \circ L_2\) cannot be a \(1\)–Banach ideal.

**Corollary 3.3** Let \((\mathfrak{A}, A)\) be a quasi-Banach ideal. If \((\mathfrak{A}, A)\) is a maximal Banach ideal or if \((\mathfrak{A}, A)\) is regular and \((\mathfrak{A} \circ L_2, A \circ L_2)\) is right-accessible, then \((A \circ L_2, A \circ L_2)\) and \((L_2 \circ \mathfrak{A}^*, L_2 \circ A^*)\) both together cannot be normed.

**Proof:** Let \((A \circ L_2, A \circ L_2)\) be a 1-Banach ideal. Due to our assumption, \(A \not\subseteq L_\infty\). If the injective quasi-Banach ideal \((L_2 \circ \mathfrak{A}^*, L_2 \circ A^*)\) were also a normed one, then we would obtain \(P_1 \overset{1}{\subseteq} L_2 \circ \mathfrak{A}^* \overset{1}{\subseteq} A^*\) (cf. [13], proposition 5.1) – a contradiction.\(\square\)

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