Performance of Johnson–Lindenstrauss Transform for
k-Means and k-Medians Clustering

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Abstract

Consider an instance of Euclidean $k$-means or $k$-medians clustering. We show that the cost of the optimal solution is preserved up to a factor of $(1 + \varepsilon)$ under a projection onto a random $O(\log(k/\varepsilon)/\varepsilon^2)$-dimensional subspace. Further, the cost of every clustering is preserved within $(1 + \varepsilon)$. More generally, our result applies to any dimension reduction map satisfying a mild sub-Gaussian-tail condition. Our bound on the dimension is nearly optimal. Additionally, our result applies to Euclidean $k$-clustering with the distances raised to the $p$-th power for any constant $p$.

For $k$-means, our result resolves an open problem posed by Cohen, Elder, Musco, Musco, and Persu (STOC 2015); for $k$-medians, it answers a question raised by Kannan.

1 Introduction

The Euclidean $k$-clustering problem with the $\ell_p$-objective is defined as follows. Given a dataset $X \subset \mathbb{R}^m$ of $n$ points, the goal is to find a partition $C = \{C_1, C_2, \ldots, C_k\}$ of $X$ into $k$ parts (clusters) that minimizes the following cost function:

$$\text{cost}_p C = \sum_{i=1}^k \min_{u_i \in \mathbb{R}^m} \sum_{x \in C_i} \|x - u_i\|^p,$$

where $\| \cdot \|$ from now on denotes the Euclidean ($\ell_2$) norm, and the optimal points $u_i$ are called centers of clusters $C_i$. This problem is a generalization of the $k$-median ($p = 1$) and the $k$-means ($p = 2$) clustering. Algorithms for $k$-clustering (especially the Lloyd’s heuristic [Llo82] for $k$-means) are used in virtually every area of data science (see [Jai10] for a survey), for data compression, quantization, and transmission over noisy channels [Far90], and for hashing, sketching and similarity search [JDS11]. In this paper we study data-oblivious dimension reduction for $k$-clustering, which can be used to speed up clustering algorithms. This line of work has been initiated by Boutsidis, Zouzias, and Drineas [BZD10] and prior to this work, the best bounds were due to Cohen, Elder, Musco, Musco and Persu [CEM*15]. Before stating our results, let us briefly recall the notion of Euclidean dimension reduction (see [Nao18] for a broad overview of the area).

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Dimension reduction. The cornerstone dimension reduction statement for the Euclidean distance is the Johnson–Lindenstrauss Lemma [JL84]. For positive reals $p, q, \varepsilon$, we write $p \approx_{1+\varepsilon} q$ if $\frac{1}{1+\varepsilon} \cdot p \leq q \leq (1+\varepsilon) \cdot p$.

**Theorem 1.1 ([JL84]).** There exists a family of random linear maps $\pi_{m,d}: \mathbb{R}^m \to \mathbb{R}^d$ with the following properties. For every $m \geq 1$, $\varepsilon, \delta \in (0; 1/2)$ and all $x \in \mathbb{R}^m$, we have

$$\Pr_{\pi \sim \pi_{m,d}} (\|\pi x\| \approx_{1+\varepsilon} \|x\|) \geq 1 - \delta,$$

where $d = O\left(\frac{\log(n/\delta)}{\varepsilon^2}\right)$.

A straightforward corollary is that one is able to embed any $n$-point subset of a Euclidean space into an $O\left(\frac{\log n}{\varepsilon^2}\right)$-dimensional space, while preserving all of the pairwise distances up to $(1 + \varepsilon)$. This bound is known to be tight [Alo03, LN17]. The attractive feature of the dimension reduction procedure given by Theorem 1.1 is that it is data-oblivious i.e., the distribution over linear maps is independent of the set of points we apply it to.

There are several constructions of families of random maps $\pi_{m,d}$ that satisfy Theorem 1.1: projections on a random subspace [JL84, DG03] and maps given by matrices with i.i.d. Gaussian and sub-Gaussian entries [IM98, Ach03, KM+05]. All of these constructions satisfy a certain additional condition, which we will need later.

**Definition 1.2.** A family of random linear maps $\pi_{m,d}: \mathbb{R}^m \to \mathbb{R}^d$ is called sub-Gaussian-tailed if for every unit vector $x \in \mathbb{R}^m$ and every $t > 0$, one has:

$$\Pr_{\pi \sim \pi_{m,d}} (\|\pi x\| \geq 1 + t) \leq e^{-\Omega(t^2d)}.$$

Our result. Consider an instance of Euclidean $k$-clustering. We show that its cost is preserved up to a factor of $(1 + \varepsilon)$ under dimension-reduction projection into an $O(\log(k/\varepsilon)/\varepsilon^2)$-dimensional space. Further, the cost of every clustering is preserved within a factor of $(1 + \varepsilon)$. Our result applies to dimension reductions based on orthogonal and Gaussian projections and, more generally, any dimension reductions satisfying the sub-Gaussian-tail condition (in the sense of Definition 1.2). Our bound on the dimension is nearly optimal.

For $k$-means, our result resolves an open problem posed by Cohen, Elder, Musco, Musco, and Persu [CEM+15]. For a partition $C = (C_1, C_2, \ldots, C_k)$ of $X \subset \mathbb{R}^m$ and a linear map $\pi: \mathbb{R}^m \to \mathbb{R}^d$, we denote by $\pi(C)$ the respective partition of the image of $X$ under $\pi$. We now state our main result formally.

**Theorem 1.3.** Consider any family of random maps $\pi_{m,d}: \mathbb{R}^m \to \mathbb{R}^d$ that satisfies Theorem 1.1 and is sub-Gaussian-tailed (satisfies Definition 1.2). Then for every $m \geq 1$, $\varepsilon, \delta \in (0; 1/4)$ and $p \geq 1$, the following holds. For every finite $X \subset \mathbb{R}^m$ we have

$$\Pr_{\pi \sim \pi_{m,d}} (\text{cost}_p C \approx_{1+\varepsilon} \text{cost}_p \pi(C) \text{ for all partitions } C = (C_1, C_2, \ldots, C_k) \text{ of } X) \geq 1 - \delta,$$

where

$$d = O\left(\frac{p^4 \cdot \log \frac{k}{\varepsilon \delta}}{\varepsilon^2}\right).$$
In fact, we show that Theorem 1.3 holds under a slightly more general definition of a standard dimension reduction map (Definition 2.1), which is implied by sub-Gaussian tails.

Theorem 1.3 readily implies that if one solves the \( k \)-clustering problem after dimension reduction with approximation \( \lambda \geq 1 \), then the same solution yields approximation \((1+O(\epsilon))\cdot \lambda\) for the original instance. It is almost immediate to show the guarantee given by Theorem 1.3 for a fixed partition \( C \), however the total number of partitions is exponential, and we cannot afford to take the union bound over them. In fact, the actual proof of Theorem 1.3 is much more delicate.

For \( k \)-means (\( p = 2 \)), the dimension bound \( O(\log n/\epsilon^2) \) easily follows from the Johnson–Lindenstrauss lemma (Theorem 1.1) and the fact that one can express the cost function of \( k \)-means via pairwise distances. However, in many applications of clustering, \( k \ll n \), in which case the bound we obtain is much stronger. For \( p \neq 2 \), even the weaker \( O(\log n) \) bound was not known before.

There is a number of dimension reduction maps that essentially satisfy Theorem 1.1, but not quite, since they depend on \( \epsilon \) and \( \delta \) in addition to \( m \) and \( d \): most notably, sparse constructions [DKS10, KN14] as well as “fast” constructions based on subsampled randomized Fourier transform [AC06, AL09, KW11, AL13, NPW14]. For such cases the conclusion of Theorem 1.3 holds under a certain condition on the moments (see Definition 2.1).

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Finally, let us note that the bound (1) is essentially optimal. Indeed, suppose that \( X \) is a set of \( t \) pairs of points, such that the distance within each pair is 1, while distances between the pairs are very large. Suppose that \( k = 2t - 1 \). Then, preserving the cost of the optimal \( k \)-clustering is no easier than non-contracting all the distances within pairs. But the latter requires \( \Omega_p(\log k/\epsilon^2) \) dimensions if one uses a Gaussian matrix for dimension reduction.

**Related work.** There is a large body of literature on dimension reduction for \( k \)-means (which corresponds to \( p = 2 \)). Within this line of work, there are two kinds of results: data-oblivious and data-dependent. Data-oblivious results provide guarantees qualitatively similar to our Theorem 1.3 and are summarized in Figure 1. Let us now give a brief overview.

As mentioned previously, the bound on the dimension \( O(\log n/\epsilon^2) \) is a simple application of Theorem 1.1. The first bound independent of \( n \) was obtained by Boutsidis, Zouzias and Drineas [BZD10], who showed that \( O(k/\epsilon^2) \) dimensions are enough for distortion \( 2 + \epsilon \). The best bounds prior to the present work are due to Cohen, Elder, Musco, Musco and Persu [CEM+15]. They showed two incomparable bounds: \( O(k/\epsilon^2) \) dimensions with distortion \( 1 + \epsilon \), and \( O(\log k/\epsilon^2) \) dimensions with distortion \( 9 + \epsilon \). The \( 9 + \epsilon \) bound on the distortion follows from all pairwise distance between \( k \) optimal centers being approximately preserved. Our Theorem 1.3 improves upon both of the bounds shown in [CEM+15]: we get \( O(\log(k/\epsilon)/\epsilon^2) \) dimensions with distortion \( 1 + \epsilon \), thus resolving an open problem posed in [CEM+15].
The literature on data-dependent dimension reduction for \( k \)-means is ample [DFK+99, Sar06, BDM09, BZD10, FSS13, BMI13, BZMD15, CEM+15] and we refer the reader to [CEM+15] for a comprehensive overview. Let us note that none of these results obtain dimension better than \( k \).

For \( p \neq 2 \), even the \( \log n \) bound was not known previously. The question of obtaining the \( \log n \) bound for \( p = 1 \) (\( k \)-median) was explicitly posed recently by Kannan [Kan18].

A notion related to dimension reduction is that of a coreset, which is a small subsample of a dataset that approximately preserves the cost of \( k \)-clustering. A good overview of this rich line of work appears in [SW18].

1.1 Proof Overview

As we discussed earlier, it is easy to show that for any fixed clustering \( C = (C_1, \ldots, C_k) \), we have \( \text{cost}_p(\pi C) \approx_{1+\varepsilon} \text{cost}_p(C) \) w.h.p. In particular, for the optimal clustering \( C^\ast \), \( \text{cost}_p(\pi C^\ast) \approx_{1+\varepsilon} \text{cost}_p(C^\ast) \), and, consequently, the cost of the optimal clustering for \( \pi X \) is upper bounded by the cost of the optimal clustering for \( X \) up to a factor of \((1 + \varepsilon)\) w.h.p. However, it is not at all obvious how to obtain a lower bound on the cost of the optimal solution for \( \pi X \) since there may exist a clustering \( \pi C' \) of \( \pi X \) which is better than \( \pi C^\ast \). Note that we cannot use the union bound to prove Theorem 1.3 as the number of possible clusterings of \( X \) is exponential in \( n \), but the dimension \( d \) does not depend on \( n \).

In this section, we discuss the main ideas we use in the proof of Theorem 1.3. We show that with high probability the following two statements hold (a) for all \( C \) we have \( \text{cost}_p(C) \leq (1 + \varepsilon) \text{cost}_p(\pi C) \) and (b) for all \( C \) we have \( \text{cost}_p(\pi X) \leq (1 + \varepsilon) \text{cost}_p(C) \). The proofs of these statements are similar.

To simplify the exposition, we focus on the former inequality in this proof overview.

To illustrate our approach, first consider an easy case when \( X \) is embedded into a \( d = O(\log n/\varepsilon^2) \) dimensional space. In this case, all distances between points in \( X \) are approximately preserved w.h.p. That is, \( \|x' - x''\| \approx_{1+\varepsilon} \|\pi x' - \pi x''\| \) for all \( x', x'' \in X \). We prove that if all distances in \( X \) are approximately preserved then for every clustering \( C = (C_1, \ldots, C_k) \) we have \( \text{cost}_p(C) \approx_{1+\varepsilon} \text{cost}_p(\pi C) \). As we shall see in a moment, the proof is immediate for \( k \)-means but requires some work for \( k \)-median and other \( \ell_p \) objectives.

For the \( k \)-means objective \((p = 2)\), we can use the following well known formula:

\[
\text{cost}_2(C) = \sum_{C \in \mathcal{C}} \frac{1}{|C|} \sum_{(x', x'') \in C \times C} \|x' - x''\|^2, \tag{2}
\]

and, similarly,

\[
\text{cost}_2(\pi C) = \sum_{C \in \mathcal{C}} \frac{1}{|C|} \sum_{(x', x'') \in C \times C} \|\pi x' - \pi x''\|^2. \tag{3}
\]

Since each term \( \|x' - x''\|^2 \) in (2) approximately equals the corresponding term \( \|\pi x' - \pi x''\|^2 \) in (3), we have \( \text{cost}_2(C) \approx_{1+O(\varepsilon)} \text{cost}_2(\pi C) \).

**One-point robust extension.** The above proof does not generalize to \( \ell_p \) objectives with \( p \neq 2 \). So our proof relies on the Kirszbraun theorem [Kir34].

**Theorem 1.4** (Kirszbraun theorem). For every subset \( X \subset \mathbb{R}^d \) and \( L \)-Lipschitz map\(^1\) \( \varphi : X \to \mathbb{R}^m \), there exists an \( L \)-Lipschitz extension \( \tilde{\varphi} \) of \( \varphi \) from \( X \) to the entire space \( \mathbb{R}^d \).

\(^1\)Recall that \( \varphi \) is an \( L \)-Lipschitz map if for all \( x', x'' \in X \), we have \( \|\varphi(x') - \varphi(x'')\| \leq L \|x' - x''\| \).
Let $Y = \pi(X)$. Observe that the map $\pi : X \to \mathbb{R}^d$ and inverse map $\pi^{-1} : Y \to \mathbb{R}^m$ are $(1 + \varepsilon)$-Lipschitz. Let $u_1, \ldots, u_k$ be the optimal centers for clusters $\pi C_1, \ldots, \pi C_k$. Using the Kirszbraun theorem, we extend the map $\pi^{-1} : Y \to \mathbb{R}^m$ to $\tilde{\varphi} : \mathbb{R}^d \to \mathbb{R}^m$ and then lift points $u_1, \ldots, u_k$ from $\mathbb{R}^d$ to $\mathbb{R}^m$ by letting $v_i = \tilde{\varphi}(u_i)$. Then, for all $y \in Y$ we have $\|v_i - \pi^{-1}y\| \leq (1 + \varepsilon)\|u_i - y\|$ or, equivalently, for all $x \in X$ we have $\|v_i - x\| \leq (1 + \varepsilon)\|u_i - \pi x\|$. We pick points $v_1, \ldots, v_k$ as centers for the clusters $C_1, \ldots, C_k$ and obtain the following bound:

$$\text{cost}_p(C) \leq \sum_{i=1}^{k} \sum_{x \in C_i} \|x - v_i\|^p \leq \sum_{i=1}^{k} \sum_{x \in C_i} (1 + \varepsilon)^p \|\pi x - u_i\|^p = (1 + \varepsilon)^p \text{cost}_p(\pi C).$$

We now return to the case when $d = O_p((\log(k/(\delta \varepsilon))/\varepsilon^2)$. Observe that when we reduce the dimension of $X$ to $d$, while most pairwise distances in $X$ are approximately preserved, some are distorted. The Kirszbraun theorem does not hold in this setting. So we prove a robust 1-point extension theorem (Theorem 4.3). Loosely speaking, this theorem states the following. Consider a finite set $C \subset \mathbb{R}^d$ and map $\varphi : C \to \mathbb{R}^m$, satisfying the following condition:

- for every $x \in C$, the distance from $x$ to all but a $\theta$ fraction of $x' \in C$ is $(1 + \varepsilon)$-preserved under $\varphi$.

Then, for every point $u \in \mathbb{R}^d$, there exists a point $v \in \mathbb{R}^m$ such that for all but $\theta'$ fraction of points $x \in C$, we have $\|x - u\| \leq (1 + \varepsilon')\|\varphi(x) - v\|$ where $\varepsilon' = O(\varepsilon)$ and $\theta' = O(\theta/\varepsilon)$.

**Non-distorted cores.** Consider a random dimension reduction map $\pi : \mathbb{R}^m \to \mathbb{R}^d$. For each realization of $\pi$, we pick the clustering $C = (C_1, \ldots, C_k)$ for which the gap between $\text{cost}_p(\pi C)$ and $\text{cost}_p(C)$ is the largest. We would like to use our new extension theorem to lift the optimal centers of $\pi C$ from $\mathbb{R}^d$ to the original space $\mathbb{R}^m$. However, we cannot do this directly, since in some clusters $C_i$ most pairwise distances can be distorted (since clusters $C_1, \ldots, C_k$ depend on $\pi$). To deal with this issue, we introduce the main technical tool of the paper—a notion of the non-distorted core.

Loosely speaking, random sets $C_1', \ldots, C_k'$ (that depend on the clustering $C$) are non-distorted cores of clusters $C_1, \ldots, C_k$ if the following four conditions are satisfied: (a) $C_i' \subset C_i$, (b) the distance from every $x$ in $C_i'$ to all but a $\theta$-fraction of other points in $C_i'$ is approximately preserved under $\pi$; (c) the distance from every $x$ in $C_i'$ to all optimal centers $c^*_1, \ldots, c^*_k$ of $C'$ is approximately preserved under $\pi$; (d) for every fixed $x \in X$, the probability that $x \in \cup_i(C_i \setminus C_i')$ is very small. We say that the random set $X' = \cup_i C_i'$ is a non-distorted core of $X$ (see Definition 2.3 for details). We show that we can remove some outliers from $X$ to obtain non-distorted cores. Note that the set of outlier points depends on $\pi$ and $C_1, \ldots, C_k$.

The proof of the existence of the non-distorted core is fairly simple for the special case when all clusters $C_i$ are of the same size. Let $D(x)$ be the set of points $x' \in X$ for which the distance between $x$ and $x'$ is distorted under $\pi$. Mark a point $x \in X$ as bad if $|D(x)| \geq \theta n/k$ or the distance from $x$ to one of the optimal centers $c^*_i$ is distorted under $\pi$. Denote the set of all bad points by $B$ and let $C_i' = C_i \setminus B$. It is clear that conditions (a) and (c) are satisfied for $C_1', \ldots, C_k'$. We need to verify conditions (b) and (d). Observe that $\mathbb{E}|D(x)| \ll \theta n/k^2$ for all $x \in X$, since the probability that the distance between two given points is distorted is much less than $\theta/k^2$ (if $d \gg \log(k/\theta)/\varepsilon^2$). Hence,

$$\Pr(x \in \cup_i(C_i \setminus C_i')) \leq \Pr(x \in B) \ll 1/k$$

for any $x \in X$ and, therefore, condition (d) is satisfied. Also, by linearity of expectation and Markov’s inequality, $|B| \leq \mathbb{E}|B| \ll n/k$ w.h.p. Consider a point $x \in C_i'$. The set of points $x' \in C_i'$
to which the distance from $x$ is distorted is $D(x) \cap C'_i = D(x) \cap C_i \setminus B$. Hence, the fraction of distorted distances is

$$\frac{|D(x) \cap C'_i \setminus B|}{|C'_i \setminus B|} \leq \frac{|D(x)|}{|C_i| - |B|} \lesssim \frac{\theta n/k}{n/k} = \theta.$$  \hfill (4)

Here we used the following bounds: (1) $|D(x)| \lesssim \theta n/k$ for $x \notin B$; (2) $|C_i| \gtrsim n/k$, and (3) $|B| \ll n/k$. This proves item (b).

The proof for the general case when the sizes of the clusters may be arbitrary is much more involved (see Lemma 3.5 and Lemma 3.7). Roughly speaking, we reduce the general case to the case when all clusters are of the same size. To this end, we introduce a carefully chosen measure $\mu$ on $X$ that we use as a proxy for the set sizes. Specifically, we show that there exists a deterministic measure $\mu$ and a random set $R$ (depending on the clustering) such that: first, $\mu(X) \leq \text{poly}(k)$; second, for every set $S \subset X$,

$$|S \cap C'_i \setminus R| \leq \mu(S) \cdot |C'_i \setminus R|;$$

and third $\Pr(x \in R) \ll 1$ (see Lemma 3.1 and Observation 3.6). We remove the random set $R$ from $X$. Then we define sets $B$ and $D(x)$ as before but use the measures $B$ and $D(x)$ instead of their sizes. We show that $\mu(B) \ll 1$ (w.h.p) and $\mu(D(x)) \ll 1$ for all $x \notin B$. Using a formula similar to (4), we get

$$\frac{|D(x) \cap C'_i \setminus B \setminus R|}{|C'_i \setminus B \setminus R|} \leq \frac{|D(x) \setminus B \setminus R|}{|C'_i \setminus B \setminus R|} \leq \frac{\mu(D(x)) |C'_i \setminus R|}{|C'_i \setminus R| - |B \setminus R|} = \frac{\mu(D(x))}{1 - \mu(B)} \ll 1.$$

**Outliers.** We are now almost done. We consider non-distorted cores $C'_1, \ldots, C'_k$ of the clusters $C_1, \ldots, C_k$, separately apply the robust 1-point extension theorem to each of them, and lift the optimal centers of $C'_1, \ldots, C'_k$ to $\mathbb{R}^m$. It only remains to take care of two sets of outliers.

The first set is the set of points we removed from clusters $X$ to obtain the non-distorted cores $C'_1, \ldots, C'_k$. In our proof sketch this set is $B \cup R$. We slightly modify our data set. We move each outlier $x$ to the optimal center of the cluster $C^*(x)$ that the point $x$ is assigned to in the optimal solution and then return $x$ back to its cluster $C_i$. Since the probability that $x$ is an outlier is very small, the cost of moving $x$ is also small (see Lemma 5.1).

For each core $C'_i$, the second type of outliers are the points $x \in C'_i$ such that $\|x - v_i\| > (1 + \varepsilon)\|px - u_i\|$, where $u_i$ is the optimal center of $C'_i$ and $v_i$ is the optimal center lifted to $\mathbb{R}^m$. The fraction of such outliers among all points of the core is at most $\theta'$. We charge the connection costs of the outliers (i.e., the costs of assigning outliers to the centers of their clusters) to the connection costs of other vertices.

This concludes the proof. In the rest of the paper, we give a detailed proof of Theorem 1.3.

### 2 Preliminaries

Let $X \subset \mathbb{R}^d$ be an instance of Euclidean $k$-clustering with the $\ell_p$ objective. We denote the optimal $k$-clustering of $X$ by $C^* = \{C_1^*, \ldots, C_k^*\}$. Given a cluster $C$, let $\text{cost}_p C = \min_{c \in \mathbb{R}^d} \sum_{u \in \mathbb{R}^d} \|u - c\|^p$ be its cost. Given a clustering $\mathcal{C} = \{C_1, \ldots, C_k\}$, let $\text{cost}_p \mathcal{C} = \sum_{i=1}^k \text{cost}_p C_i$ be its cost. In particular, $\text{cost} C^*$ is the cost of the optimal clustering of $X$. Given a map $\pi$ and clustering $\mathcal{C} = \{C_1, \ldots, C_k\}$, we denote by $\pi(\mathcal{C})$ the clustering $\pi(C_1), \ldots, \pi(C_k)$ of $\pi(X)$.

We denote the indicator of an event $E$ by $1\{E\}$. 


Definition 2.1. Map $\pi : \mathbb{R}^m \to \mathbb{R}^d$ is an $(\varepsilon, \delta)$-random dimension reduction if

$$\frac{1}{1 + \varepsilon} \|x - y\| \leq \|\pi(x) - \pi(y)\| \leq (1 + \varepsilon) \|x - y\|$$

with probability at least $1 - \delta$ for every $x, y \in \mathbb{R}^m$. Given $p \in [1, \infty)$, map $\pi$ is an $(\varepsilon, \delta, \rho)$-dimension reduction if it additionally satisfies the following property

$$\mathbb{E}\left[\mathbb{1}\{\|\pi(x) - \pi(y)\| > (1 + \varepsilon)\|x - y\|\} \left(\frac{\|\pi(x) - \pi(y)\|^p}{\|x - y\|^p} - (1 + \varepsilon)^p\right)\right] \leq \rho. \quad (5)$$

Given $\varepsilon > 0$, we say that a random dimension reduction $\pi : \mathbb{R}^m \to \mathbb{R}^d$ is standard if it has parameters $(\varepsilon, \delta, \rho)$ and $\delta \leq \exp(-c\varepsilon^2d)$, $\rho \leq \exp(-c\varepsilon^2d)$ when $d > c'p/\varepsilon^2$ for some constants $c, c' > 0$.

We say that $\pi$ $(1 + \varepsilon)$-preserves the distance between $x$ and $y$ if $\frac{1}{1 + \varepsilon} \|x - y\| \leq \|\pi(x) - \pi(y)\| \leq (1 + \varepsilon)\|x - y\|$. Otherwise, we say that $\pi$ distorts the distance between $x$ and $y$. For brevity, we also say that $(x, y)$ is $(1 + \varepsilon)$-preserved in the former case and distorted in the latter case.

As we prove in Lemma C.1, every dimension reduction that (1) satisfies Theorem 1.1 and (2) is sub-Gaussian tailed is standard. We note that all dimension reduction constructions (satisfying Theorem 1.1) that we are aware of are sub-Gaussian tailed and thus standard. In particular, we prove in Claim C.2 that the Gaussian dimension reduction is sub-Gaussian tailed.

Definition 2.2. Consider an instance $X \subset \mathbb{R}^m$ of Euclidean $k$-clustering with the $\ell_p$ objective. We denote the optimal clustering by $C^* = (C_1, \ldots, C_k)$; we denote the optimal centers by $c_1^*, \ldots, c_k^*$.

Given a clustering $C = (C_1, \ldots, C_k)$ and a point $x \in X$, we denote the cluster that contains $x$ by $C(x)$; we denote the center of $C(x)$ by $c(x)$. In particular, $C^*(x)$ is the cluster in the optimal clustering that contains $x$, and $c(x)$ is its center.

We now introduce the notion of a $(1 - \theta)$-non-distorted core, which plays a central role in the proof of our result.

Definition 2.3. Consider an instance $X \subset \mathbb{R}^m$ of Euclidean $k$-clustering with the $\ell_p$ objective, a random dimension reduction $\pi$, and a random clustering $C = \{C_1, \ldots, C_k\}$ of $X$. We say that $X'$ is a $(1 - \theta)$-non-distorted core of $X$ if

- for every $x \in X'$, the distance from $x$ to at least a $(1 - \theta)$ fraction of points in $C(x) \cap X'$ is $(1 + \varepsilon)$-preserved by $\pi$ (always)

- for every $x \in X'$, the distances from $x$ to all centers $c_1^*, \ldots, c_k^*$ are $(1 + \varepsilon)$-preserved by $\pi$ (always).

- for every $x \in X$, $\Pr(x \notin X') \leq \theta$

3 Non-distorted core

In this section, we prove that there exists a $(1 - \theta)$-non-distorted core $X'$. 
3.1 Main Combinatorial Lemma

Lemma 3.1. Let $X$ be a finite set and $C \subset X$ be a random subset of $X$. Let $\theta \in (0, 1/2)$. Assume that $\Pr(x \in C) \geq 2\theta$ for every $x$. Then there exist a random set $R \subset C$ and deterministic measure $\mu$ on $X$ such that

1. $\mu(x) \geq \frac{1}{|C \setminus R|}$ for every $x \in C \setminus R$ (always);
2. $\Pr(x \in R) \leq \theta$ for every $x \in X$;
3. $\mu(X) = \sum_{x \in X} \mu(x) \leq \frac{\Pr(C \neq \emptyset)}{\theta}$. 

Proof. We prove this lemma by induction on the size of the set $X$. If $|X| = 0$ i.e. $X$ is empty, then properties 1–3 trivially hold. Assume that the statement holds for all sets $X'$ of size $|X'| < |X|$ and prove it for $X$.

Let $l = \theta|X|$. Define a (deterministic) set $X'$ and random subset $C' \subset X'$ as follows:

$$X' = \{ x : \Pr(x \in C \text{ and } |C| < l) \geq 2\theta \} \quad (6)$$

$$C' = \begin{cases} C \cap X', & \text{if } |C| < l \\ \emptyset, & \text{otherwise} \end{cases} \quad (7)$$

First, we prove that for some $x_0 \in X$, $\Pr(x_0 \in C \text{ and } |C| < l) \leq \theta$ and, consequently, $|X'| < |X|$. To this end, we show that the average value of $\Pr(x \in C \text{ and } |C| < l)$ for $x \in X$ is at most $\theta$:

$$\frac{1}{|X|} \sum_{x \in X} \Pr(x \in C \text{ and } |C| < l) = \frac{1}{|X|} \sum_{x \in X} \mathbb{E}[\mathbb{1}\{x \in C \text{ and } |C| < l\}] = \frac{1}{|X|} \mathbb{E}[|C| \cdot \mathbb{1}\{|C| < l\}] \leq \frac{l}{|X|} = \theta. \quad \text{Here we used that the random variable } |C| \cdot \mathbb{1}\{|C| < l\} \text{ is always less than } l.$$

Since $|X'| < |X|$ and $\Pr(x \in C') \geq 2\theta$ for every $x \in X'$, we can apply the induction hypothesis to $X'$ and $C'$. By the inductive hypothesis, there exist a random set $R' \subset X'$ and measure $\mu'$ on $X'$ satisfying properties 1–3 for the set $X'$ and random subset $C'$. Define a measure $\mu$ on $X$ and random subset $R \subset C$ as follows:

$$\mu(x) = \begin{cases} \mu'(x) + 1/l, & \text{if } x \in X' \\ 1/l, & \text{otherwise} \end{cases} \quad (8)$$

$$R = \begin{cases} R' \cup (C \setminus X'), & \text{if } |C| < l \\ R', & \text{otherwise} \end{cases} \quad (9)$$

We verify that $R$ and $\mu$ satisfy the desired conditions.

Condition 1: for every $x \in C \setminus R$, $\mu(x) \geq 1/|C \setminus R|$ (always). Fix $x \in C \setminus R$ and consider three cases. If $x \in X'$ and $|C| < l$, we have $C \setminus R = C' \setminus R'$ by (9). Thus, $\mu(x) > \mu'(x) \geq \frac{1}{|C' \setminus R'|} = \frac{1}{|C \setminus R|}$ by the induction hypothesis. If $x \in X'$ and $|C| \geq l$, then $\mu(x) \geq 1/l \geq 1/|C|$. Note that $C' = \emptyset$, since $|C| \geq l$. Hence, $R$ by (9) $R' \subset C' = \emptyset$. In particular, $1/|X'| = 1/|C \setminus R|$ and thus $\mu(x) \geq 1/|C \setminus R|$. Finally, if $x \not\in X'$, then $x \in C \setminus X' \subset R' \cup (C \setminus X')$ and $x \not\in R$. Thus, $R \neq R' \cup (C \setminus X')$. From (9), we get that $|C| \geq l$ and hence $\mu(x) = 1/l \geq 1/|C|$. Again, since $|C| \geq l$, we have $\mu(x) \geq 1/|C| = 1/|C \setminus R|$.
Condition 2: \( \Pr(x \in R) \leq 2\theta \). If \( x \in X' \), then \( \Pr(x \in R) = \Pr(x \in R') \leq 2\theta \) by the induction hypothesis. If \( x \notin X' \), then

\[
\Pr(x \in R) \overset{\text{by (9)}}{=} \Pr(x \in C \text{ and } |C| < l) \overset{\text{by (6)}}{\leq} 2\theta.
\]

Condition 3: \( \mu(X) = \sum_{x \in X} \mu(x) \leq \Pr(C \neq \emptyset)/\theta^2 \). By the inductive hypothesis, we have \( \mu'(X') \leq \Pr(C' \neq \emptyset)/\theta^2 \). Thus,

\[
\mu(X) = \mu'(X') + \frac{|X|}{l} \leq \frac{\Pr(C' \neq \emptyset)}{\theta^2} + \frac{1}{\theta}.
\]

To finish the proof, we need to show that

\[
\frac{\Pr(C' \neq \emptyset)}{\theta^2} + \frac{1}{\theta} \leq \frac{\Pr(C \neq \emptyset)}{\theta^2},
\]

or, equivalently, \( \Pr(C \neq \emptyset) - \Pr(C' \neq \emptyset) \geq \theta \). Note that if \( |C| \geq l \), then \( C \neq \emptyset \) and \( C' = \emptyset \). Thus,

\[
\Pr(C \neq \emptyset) - \Pr(C' \neq \emptyset) = \Pr(C \neq \emptyset \text{ and } C' = \emptyset) \geq \Pr(|C| \geq l).
\]

Recall that for some \( x_0 \in X \), we have \( \Pr(x_0 \in C \text{ and } |C| \leq l) \leq \theta \). Since for all \( x \in X \), \( \Pr(x \in X) \geq 2\theta \), we have

\[
\Pr(|C| \geq l) \geq \Pr(x_0 \in C \text{ and } |C| \geq l) \geq \Pr(x_0 \in C) - \Pr(x_0 \in C \text{ and } |C| \leq l) \geq \theta.
\]

This completes the proof. \(\square\)

Corollary 3.2. Let \( X \) be a finite set and \( C \subset X \) be a random subset of \( X \). Let \( \theta \in (0, 1/2) \). Then there exist a random set \( R \subset C \) and measure \( \mu \) on \( X \) such that

1. \( \mu(x) \geq \frac{1}{|C \setminus R|} \) for every \( x \in C \setminus R \) (always)
2. \( \Pr(x \in R) \leq 2\theta \) for every \( x \in X \)
3. \( \mu(X) = \sum_{x \in X} \mu(x) \leq \frac{1}{\theta^2} \)

Proof. Let \( X' = \{ x : \Pr(x \in C) \geq 2\theta \} \). We apply the lemma to \( X' \) and then let

\[
R = (R' \cup (X \setminus X')) \cap C.
\]

\(\square\)

Definition 3.3. \( \mathcal{C} = (C_1, \ldots, C_k) \) is a partial clustering of \( X \) if sets \( C_i \) are disjoint subsets of \( X \); that is, \( \mathcal{C} = (C_1, \ldots, C_k) \) is a clustering of a subset of \( X \) (possibly, the subset is equal to \( X \)).

Corollary 3.4. Let \( X \) be a finite set and \( \mathcal{C} = (C_1, \ldots, C_k) \) be a random partial clustering of \( X \). Let \( \theta \in (0, 1/2) \). Then there exist a random set \( R \) and measure \( \mu \) on \( X \) such that

1. \( \mu(x) \geq \frac{1}{|C_i \setminus R|} \) for every \( i \in [k] \) and \( x \in C_i \setminus R \) (always)
2. \( \Pr(x \in R) \leq \theta \) for every \( x \in X \)

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3. \( \mu(X) = \sum_{x \in X} \mu(x) \leq \frac{4k^3}{\theta^2} \)

Proof. We separately apply Corollary 3.2 with \( \theta' = \theta/(2k) \) to every cluster \( C_1, \ldots, C_k \) and obtain sets \( R_1, \ldots, R_k \) and measures \( \mu_1, \ldots, \mu_k \). Then we let

\[
R = \bigcup_{i=1}^k R_i \quad \text{and} \quad \mu(x) = \sum_{i=1}^k \mu_i(x).
\]

\[\square\]

3.2 Pruning lemma

Lemma 3.5. Consider a set of points \( X \). Let \( \pi \) be a \((\varepsilon, \delta)\)-random dimension-reduction map. Let \( \theta \in (0, 1/2) \). Assume that with \( \varepsilon \in (0, 1) \) and \( \delta \leq \frac{\theta^7}{10k^3} \). Let \( C = (C_1, \ldots, C_k) \) be a random partial clustering of \( X \). Denote by \( C(x) \) the cluster that contains \( x \), if \( x \) is clustered, and \( C(x) = \emptyset \), otherwise. There exists a random subset \( X' \subset X \) such that

- for every \( x \in X' \), the distance from \( x \) to at least a \((1 - \theta)\) fraction of points in \( C(x) \cap X' \) is \((1 + \varepsilon)\)-preserved (always);
- for every \( x \in X \), \( \Pr(x \notin X' \text{ and } x \in \bigcup_{i=1}^k C_i) \leq \theta \);
- \( X' \subset \bigcup_{i=1}^k C_i \).

Proof. Let \( \alpha = \theta/(1 + \theta) \). We apply Corollary 3.4 to random partial clustering \( C_1, \ldots, C_k \), with \( \theta' = \theta/3 \), and get a measure \( \mu \) on \( X \) and random subset \( R \subset X \). Denote \( C'(x) = C(x) \setminus R \). Let \( M = \mu(X) \leq \frac{4k^3}{\theta^2} \). We will need the following observation.

Observation 3.6. For every \( S \subset C'(x) \), we have \( |S| \leq \mu(S)|C'(x)| \).

Proof. By Corollary 3.4, \( \mu(x) \geq 1/|C'(x)| \) for every \( x \in S \). Therefore, \( \mu(S) \geq |S|/|C'(x)| \) and \( |S| \leq \mu(S)|C'(x)| \). \[\square\]

Consider the product measure \( \mu^{\otimes 2} \) on \( X \times X \), defined by \( \mu^{\otimes 2}((x, y)) = \mu(x) \cdot \mu(y) \). Then \( \mu(X \times X) = M^2 \). Let \( D = \{(x, y) : (x, y) \text{ is distorted}\} \) be the set of distorted pairs. Note that for every pair \( (x, y) \), \( (x, y) \) belongs to \( D \) with probability at most \( \delta \). Thus, \( \operatorname{E}[\mu^{\otimes 2}(D)] \leq \delta M^2 \). By Markov’s inequality,

\[
\Pr(\mu^{\otimes 2}(D) \geq \alpha^2) \leq \frac{\delta M^2}{\alpha^2} \leq \frac{\theta}{3}.
\]

If \( \mu^{\otimes 2}(D) \geq \alpha^2 \), we let \( X' = \emptyset \). In this case, all distances in \( X' \) are preserved as \( X' \) is an empty set. We now consider the main case when \( \mu^{\otimes 2}(D) < \alpha^2 \). We say that \( x \in X \) is bad if

\[
\mu(\{y \in C'(x) : (x, y) \in D\}) \geq \alpha.
\]

(10)

Denote the set of bad points by \( B \) (note that \( B \) is a random set) and define \( X' \) as

\[
X' = \left( \bigcup_{i=1}^k C_i \right) \setminus (R \cup B).
\]
Let us verify that $X'$ satisfies the desired properties. First, we check that the distance from every $x \in X'$ to at least a $1 - \theta$ fraction of points in $C(x) \cap X'$ is $(1 + \varepsilon)$-preserved. That is, $| \{ y \in C'(x) : (x, y) \in D \} | \leq \theta | C(x) \cap X'|$. To this end, we estimate the measures of the sets $B$, $\{ y \in C'(x) : (x, y) \in D \}$, and $C(x) \cap X'$. For every $x \in B$, 

$$
\mu^2(\{ (x, y) \in D \}) = \mu(x) \cdot \mu(\{ y \in X : (x, y) \in D \}) \geq \mu(x) \cdot \mu(\{ y \in C'(x) : (x, y) \in D \}) \geq \mu(x) \cdot \alpha.
$$

Consequently, 

$$
\mu^2(D) \geq \sum_{x \in B} \mu^2(\{ (x, y) \in D \}) \geq \alpha \mu(B)
$$

and $\mu(B) \leq \mu^2(D)/\alpha < \alpha$.

Consider now $x \in X'$. Since $x \notin B$, 

$$
\mu(\{ y \in C(x) \cap X' : (x, y) \in D \}) \leq \mu(\{ y \in C'(x) : (x, y) \in D \}) \leq \alpha.
$$

Also, $\mu(C'(x) \cap B) \leq \mu(B) \leq \alpha$. By Observation 3.6, $| \{ y \in C(x) \cap X' : (x, y) \in D \} | \leq \alpha| C'(x) |$ and $| C'(x) \cap B | \leq \alpha| C'(x) |$. From the latter inequality, we get $| C(x) \cap X' | = | C'(x) \setminus B | \geq (1 - \alpha)| C'(x) |$. We conclude that 

$$
| \{ y \in C(x) \cap X' : (x, y) \in D \} | \leq \frac{\alpha}{1 - \alpha}| C(x) \cap X' | = \theta| C(x) \cap X' |,
$$

as required.

To finish the proof, we need to bound $\Pr( x \notin X' \text{ and } x \in \bigcup_{i=1}^k C_i ) \leq \theta$. Note that $x$ is in $\bigcup_{i=1}^k C_i$ but not in $X'$ only if one of the following events happens:

- $\mu^2(D) \geq \alpha^2$. As we showed above, the probability of this event is at most $\theta/3$.
- $\mu^2(D) < \alpha^2$ and $x \in R$. The probability of this event is at most $\theta' \leq \theta/3$.
- $\mu^2(D) < \alpha^2$ and $x \in B$. By Markov’s inequality, the probability of this event is at most 

$$
\Pr(x \in B) \leq \mathbb{E}[\mu(\{ y \in C'(x) : (x, y) \in D \})]/\alpha \leq \delta M/\alpha \leq \theta/3,
$$

here we used that first for $x \in B$, $\mu(\{ y \in C'(x) : (x, y) \in D \})/\alpha \geq 1$ see (10), and second, 

$$
\mathbb{E}[\mu(\{ y \in C'(x) : (x, y) \in D \})] \leq \sum_{y \in X} \Pr((x, y) \in D) \mu(y) \leq \sum_{y \in X} \delta \mu(y) = \delta M.
$$

Thus, $\Pr( x \notin X' \text{ and } x \in \bigcup_{i=1}^k C_i ) \leq \theta$. 

\[\square\]

### 3.3 Existence of a non-distorted core

**Lemma 3.7.** Let $\pi$ be a $(\varepsilon, \delta)$-random dimension-reduction map with $\varepsilon > 0$ and $\theta \in (0, 1/2)$, and $\delta = \frac{\theta^2}{10k\varepsilon^2}$. Then there exists a $(1 - \theta)$-non-distorted core $X'$.

**Proof.** Let $Y$ be the set of points $y$ in $X$ whose distances to all centers $c_1^*, \ldots, c_k^*$ are $(1 + \varepsilon)$-preserved. Note that $\Pr(x \notin X) \leq k\delta \leq \theta/640$. We apply Lemma 3.5 to partial clustering $C_i \cap Y$ and obtain a random subset $X'$. We have, $\Pr(x \notin X') = \Pr(x \notin Y) + \Pr(x \in Y \setminus X') \leq \theta$. It follows that $X'$ is a $(1 - \theta)$-non-distorted core. 

\[\square\]
4 One point extension

Consider two sets of points $X$ and $Y$ in Euclidean space and a one-to-one map $\varphi : X \rightarrow Y$. Suppose that for every point $x$ in $X$, the distances from $x$ to all but a $\theta$ fraction of $x'$ in $X$ do not increase under the map $\varphi$. In this section, we show that in this case, we have $\text{cost}_p(Y) \leq (1 + O(\theta^{1/(p+1)})) \text{cost}_p(X)$. We prove this statement (Lemma 4.4) using a robust version of the classic Kirszbraun Theorem (Theorem 4.3). To state our results, we need several definitions.

**Definition 4.1** ($\theta$ everywhere-sparse graphs). We say that a graph $G = (V, E)$ is $\theta$ everywhere-sparse if the degree of every vertex $v$ in $V$ is at most $\theta |V|$. 

**Definition 4.2** (Distance expansion graph). Consider two finite metric spaces $(X, d_X)$ and $(Y, d_Y)$ and a map $\varphi : X \rightarrow Y$. Define the distance expansion graph for $\varphi$ on elements of the space $X$ as follows. A pair of vertices $(x', x'')$ is an edge in the graph if and only if

$$d_Y(\varphi(x'), \varphi(x'')) > d_X(x', x'').$$

**Theorem 4.3** (Robust one point extension theorem for $L_2$ spaces). Consider two finite (multi)sets of points $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^{d''}$ and a map $\varphi : X \rightarrow Y$. Let $G = (X, E)$ be the distance expansion graph for $\varphi$ with respect to the Euclidean distance. Suppose that $G$ is $\theta$ everywhere-sparse. Then, for every $u \in \mathbb{R}^d$ and positive $\varepsilon$, there exists $v \in \mathbb{R}^{d''}$ such that for all but possibly a $\theta'(\varepsilon)$ fraction of points $x$ in $X$ we have

$$\|\varphi(x) - v\| \leq (1 + \varepsilon)\|x - u\|,$$

where $\theta'(\varepsilon) = 2(1 + \varepsilon)^2 \cdot \theta / \varepsilon$.

First, we show how to derive the main result of this section (Lemma 4.4) from Theorem 4.3 and then prove Theorem 4.3 itself.

**Lemma 4.4.** Consider two finite multisets of points $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^{d''}$ and a one-to-one map $\varphi : X \rightarrow Y$. Let $G = (X, E)$ be the distance expansion graph for $\varphi$ with respect to the Euclidean distance. Suppose that $G$ is $\theta$ everywhere-sparse with $\theta \leq 1/4^{p+1}$. Then, for every $p \geq 1$, we have the following inequality on the cost of the clusters $X$ and $Y$:

$$\text{cost}_p(X) \leq (1 + 3^{p+2}\theta^{1/(p+1)}) \text{cost}_p(Y).$$

**Proof.** Fix a parameter $\varepsilon = \theta^{1/(p+1)} \leq 1/4$. Let $u^*$ be the optimal center for the cluster $X$. By Theorem 4.3, there exists a point $v^* \in \mathbb{R}^{d''}$ such that for all but possibly a $\theta' = 2(1 + \varepsilon)^2 \cdot \theta / \varepsilon$ fraction of points $x$ in $X$ we have

$$\|\varphi(x) - v^*\| \leq (1 + \varepsilon)\|x - u^*\|.$$

Let $\tilde{X}$ be the set of points $x$ for which the inequality above holds. Then, $|\tilde{X}| \geq (1 - \theta')|X|$. Let us place the center of the cluster $Y$ to $v^*$. This will give an upper bound on the cost $\text{cost}_p(Y)$:

$$\text{cost}_p(Y) \leq \sum_{y \in Y} \|y - v^*\|^p = \sum_{x \in X} \|\varphi(x) - v^*\|^p.$$  \hspace{1cm} (11)

We now need to estimate the right hand side of (11). For $x \in \tilde{X}$, we already have a bound: $\|\varphi(x) - v^*\|^p \leq (1 + \varepsilon)^p \|x - u^*\|^p$. We use the following claim to bound $\|\varphi(x) - v^*\|^p$ for $x \in X \setminus \tilde{X}$.
Claim 4.5. For all $x \in X$, we have

$$
\|\varphi(x) - v^*\|^p \leq (1 + \varepsilon)^p \|x - u^*\|^p + \frac{3^p}{\varepsilon^{p-1}} \cdot \frac{2}{|X|} \sum_{x' \in \bar{X}} \|x' - u^*\|^p.
$$

Proof. Fix $x \in X$. Let $I_x$ be the set of its non-neighbors in the distance expansion graph. I.e., $x' \in I_x$ if $\|\varphi(x) - \varphi(x')\| \leq \|x - x'\|$. Since the distance expansion graph is $\theta$ everywhere sparse, the set $I_x$ contains at least $(1 - \theta)|X|$ points and $I_x \cap \bar{X}$ contains at least $(1 - \theta - \varepsilon)|X|$ points. Consider an arbitrary $x' \in I_x \cap \bar{X}$. By the triangle inequality, we have

$$
\|\varphi(x) - v^*\| \leq \|\varphi(x) - \varphi(x')\| + \|\varphi(x') - v^*\|.
$$

Now, $\|\varphi(x) - \varphi(x')\| \leq \|x - x'\|$ because $x' \in I_x$ and $\|\varphi(x') - u^*\| \leq (1 + \varepsilon)\|x' - u^*\|$ because $x' \in \bar{X}$. Thus,

$$
\|\varphi(x) - v^*\| \leq \|x - x'\| + (1 + \varepsilon)\|x' - u^*\|.
$$

Using the triangle inequality once again, we get

$$
\|\varphi(x) - v^*\| \leq (\|x - u^*\| + \|x' - u^*\|) + (1 + \varepsilon)\|x' - u^*\| = \|x - u^*\| + (2 + \varepsilon)\|x' - u^*\|.
$$

By Lemma A.1 applied to the inequality above,

$$
\|\varphi(x) - v^*\|^p \leq (1 + \varepsilon)^p \|x - u^*\|^p + \left(\frac{1 + \varepsilon}{\varepsilon}\right)^{p-1} (2 + \varepsilon)^p \|x' - u^*\|^p
\leq (1 + \varepsilon)^p \|x - u^*\|^p + \frac{3^p}{\varepsilon^{p-1}} \|x' - u^*\|^p,
$$

(12)

here we used that $(1 + \varepsilon)(2 + \varepsilon) < 3$ for $\varepsilon \leq 1/4$.

We now average (12) over all $x' \in I_x \cap \bar{X}$ and use the bound $|I_x \cap \bar{X}| \geq (1 - \varepsilon - \theta)|X| \geq |X|/2$:

$$
\|\varphi(x) - v^*\|^p \leq (1 + \varepsilon)^p \|x - u^*\|^p + \frac{3^p}{\varepsilon^{p-1}} \frac{1}{|I_x \cap \bar{X}|} \sum_{x' \in I_x \cap \bar{X}} \|x' - u^*\|^p
\leq (1 + \varepsilon)^p \|x - u^*\|^p + \frac{3^p}{\varepsilon^{p-1}} \cdot \frac{2}{|X|} \sum_{x' \in \bar{X}} \|x' - u^*\|^p.
$$

This concludes the proof of Claim 4.5.\hfill\Box

We now split the right hand side of (11) into two sums:

$$
\text{cost}_p(Y) \leq \sum_{x \in X} \|\varphi(x) - v^*\|^p = \sum_{x \in \bar{X}} \|\varphi(x) - v^*\|^p + \sum_{x \in X \setminus \bar{X}} \|\varphi(x) - v^*\|^p.
$$

(13)

Write

$$
\sum_{x \in \bar{X}} \|\varphi(x) - v^*\|^p \leq (1 + \varepsilon)^p \sum_{x \in X} \|x - u^*\|^p,
$$

13
and, using Claim 4.5,

$$\sum_{x \in X \setminus \tilde{X}} \| \varphi(x) - v^* \|^p \leq (1 + \varepsilon)^p \sum_{x \in X \setminus \tilde{X}} \| x - u^* \|^p + \frac{3\varepsilon^p}{\varepsilon^{p-1}} \cdot \frac{2}{|X|} \sum_{x \in X \setminus \tilde{X}} \| x' - u^* \|^p$$

$$= (1 + \varepsilon)^p \sum_{x \in X \setminus \tilde{X}} \| x - u^* \|^p + \frac{3\varepsilon^p}{\varepsilon^{p-1}} \cdot \frac{2|X \setminus \tilde{X}|}{|X|} \sum_{x' \in \tilde{X}} \| x' - u^* \|^p$$

$$\leq (1 + \varepsilon)^p \sum_{x \in X \setminus \tilde{X}} \| x - u^* \|^p + \frac{3\varepsilon^p}{\varepsilon^{p-1}} \cdot 2\theta' \sum_{x' \in \tilde{X}} \| x' - u^* \|^p.$$

Here we used that $|X \setminus \tilde{X}| \leq \theta'|X|$. We now have bounds for the both terms in the right hand side of inequality (13). We plug them in and obtain the following upper bound on $\text{cost}_p(Y)$:

$$\text{cost}_p(Y) \leq (1 + \varepsilon)^p \sum_{x \in X} \| x - u^* \|^p + \frac{3\varepsilon^p}{\varepsilon^{p-1}} \cdot 2\theta' \sum_{x' \in \tilde{X}} \| x' - u^* \|^p$$

$$\leq \left( (1 + \varepsilon)^p + \frac{2\theta' \cdot 3\varepsilon^p}{\varepsilon^{p-1}} \right) \text{cost}_p(X).$$

Observe that $(1 + \varepsilon)^p \leq 1 + p(1 + \varepsilon)^{p-1} \varepsilon < 1 + 3\varepsilon^p$ and

$$\frac{2\theta' \cdot 3\varepsilon^p}{\varepsilon^{p-1}} = \frac{4\theta(1 + \varepsilon)^2 \cdot 3\varepsilon^p}{\varepsilon^p} = 4(1 + \varepsilon)^2 \cdot 3\varepsilon \leq 7 \cdot 3\varepsilon^p.$$

Therefore,

$$\text{cost}_p(Y) \leq (1 + 3\varepsilon^p \cdot 2) \text{cost}_p(X).$$

\[\square\]

### 4.1 Proof of Theorem 4.3

We prove Theorem 4.3 using a duality argument. Fix a positive $\varepsilon$; and denote the size of $X$ by $n$. Let $\eta = (\theta'(\varepsilon)n)^{-1}$. Consider the following convex polytope:

$$\Lambda_\eta = \{ \lambda \in \mathbb{R}^X : \sum_{x \in X} \lambda_x = 1; \lambda_{x'} \leq \eta \text{ for all } x' \in X \}.$$

For every $\lambda \in \Lambda_\eta$, $u' \in \mathbb{R}^d$, and $v' \in \mathbb{R}^{d'}$, let

$$f(X, \lambda, u') = \sum_{x \in X} \lambda_x \| u' - x \|^2 \quad \text{and} \quad f(\varphi(X), \lambda, v') = \sum_{x \in X} \lambda_x \| v' - \varphi(x) \|^2. \quad (14)$$

That is, $f(X, \lambda, u')$ is the cost of a single cluster $X$ with a center in $u'$ according to the weighted $k$-means objective. The weight of a point $x \in X$ is $\lambda_x$. Similarly, $f(\varphi(X), \lambda, v')$ is the cost of the cluster $\varphi(X)$ with a center in $v'$. The optimal centers for clusters $X$ and $\varphi(X)$ are located at the centers of mass of $X$ and $\varphi(X)$ respectively. Thus, for a given $\lambda \in \Lambda_\eta$, $X$, and $\varphi$, the objective
functions \( f(X, \lambda, u') \) and \( f(\varphi(X), \lambda, v') \) are minimized when \( u' = \sum x \lambda_x x \) and \( v' = \sum x \lambda_x \varphi(x) \). Consequently (see Section B for details),

\[
\begin{align*}
\min_{u' \in \mathbb{R}^d} f(X, \lambda, u') &= \sum_{(x', x'')} \lambda_{x'} \lambda_{x''} \|x' - x''\|^2; \quad \text{and} \\
\min_{v' \in \mathbb{R}^d} f(\varphi(X), \lambda, v') &= \sum_{(x', x'')} \lambda_{x'} \lambda_{x''} \|\varphi(x') - \varphi(x'')\|^2,
\end{align*}
\]

where \( P \) is the set of all unordered pairs \((x', x'')\) with \( x', x'' \in X \).

Let \( F(v', \lambda) = (1 + \varepsilon)^2 f(X, \lambda, u) - f(\varphi(X), \lambda, v') \). Our goal is to show that

\[
\max_{v' \in \mathbb{R}^{d'}} \min_{\lambda \in \Lambda} F(v', \lambda) \geq 0.
\]

**Lemma 4.6.** Inequality (17) holds.

**Proof.** Observe that functions \( f(X, \lambda, u) \) and \( f(\varphi(X), \lambda, v) \) are linear in \( \lambda \) for fixed \( u \) and \( v \) and convex in \( u \) and \( v \) respectively for a fixed \( \lambda \) (see (14)). Hence, \( v' \mapsto F(v', \lambda) \) is a concave function for every \( \lambda \); and \( \lambda \mapsto F(v', \lambda) \) is a linear function for every \( v' \). Therefore, by the von Neumann minimax theorem, we have

\[
\max_{v' \in \mathbb{R}^{d'}} \min_{\lambda \in \Lambda} F(v', \lambda) = \min_{\lambda \in \Lambda} \max_{v' \in \mathbb{R}^{d'}} F(v', \lambda).
\]

Thus, it suffices to prove that \( \max_{v' \in \mathbb{R}^{d'}} F(v', \lambda) \geq 0 \) for all \( \lambda \in \Lambda \). Fix \( \lambda \in \Lambda \). Then,

\[
\begin{align*}
\max_{v' \in \mathbb{R}^{d'}} F(v', \lambda) &= \max_{v' \in \mathbb{R}^{d'}} (1 + \varepsilon)^2 f(X, \lambda, u) - f(\varphi(X), \lambda, v') \\
&= (1 + \varepsilon)^2 f(X, \lambda, u) - \min_{v' \in \mathbb{R}^{d'}} f(\varphi(X), \lambda, v') \\
&\geq \min_{v' \in \mathbb{R}^{d'}} (1 + \varepsilon)^2 f(X, \lambda, u') - \min_{v' \in \mathbb{R}^{d'}} f(\varphi(X), \lambda, v').
\end{align*}
\]

Using formulae (15) and (16) for the minimum of the function \( f \), we have

\[
\begin{align*}
\max_{v' \in \mathbb{R}^{d'}} F(v', \lambda) &\geq (1 + \varepsilon)^2 \sum_{(x', x'')} \lambda_{x'} \lambda_{x''} \|x' - x''\|^2 - \sum_{(x', x'')} \lambda_{x'} \lambda_{x''} \|\varphi(x') - \varphi(x'')\|^2 \\
&= \sum_{(x', x'')} \lambda_{x'} \lambda_{x''} \left[(1 + \varepsilon)^2 \|x' - x''\|^2 - \|\varphi(x') - \varphi(x'')\|^2\right].
\end{align*}
\]

We now split the sum on the right hand side into two parts: the sum over pairs \((x', x'') \in E\) and pairs \((x', x'') \notin E\). Then, we upper bound each term in each of the sums. For \((x', x'') \in E\), we use a trivial bound

\[
(1 + \varepsilon)^2 \|x' - x''\|^2 - \|\varphi(x') - \varphi(x'')\|^2 \geq -\|\varphi(x') - \varphi(x'')\|^2.
\]

For \((x', x'') \notin E\), we have \(\|x' - x''\|^2 \geq \|\varphi(x') - \varphi(x'')\|^2\) by the definition of the distance expansion graph, and hence

\[
(1 + \varepsilon)^2 \|x' - x''\|^2 - \|\varphi(x') - \varphi(x'')\|^2 \geq ((1 + \varepsilon)^2 - 1)\|\varphi(x') - \varphi(x'')\|^2.
\]

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Denote $\varepsilon' = (1 + \varepsilon)^2 - 1$. We obtain the following bound:

\[
\max_{v' \in \mathbb{R}^d} \ F(v', \lambda) \geq \varepsilon' \sum_{(x', x'') \in P \setminus E} \lambda_{x'} \lambda_{x''} \|\varphi(x') - \varphi(x'')\|^2 - \sum_{(x', x'') \in E} \lambda_{x'} \lambda_{x''} \|\varphi(x') - \varphi(x'')\|^2 \tag{18}
\]

We estimate the second sum using the following claim.

**Claim 4.7.** For all $x', x'' \in X$ and $\lambda \in \Lambda_\eta$, we have

\[
\|\varphi(x') - \varphi(x'')\|^2 \leq 2 \sum_{x \in X} \lambda_x \|\varphi(x) - \varphi(x')\|^2 + \lambda_x \|\varphi(x) - \varphi(x'')\|^2.
\]

**Proof.** The desired inequality is the convex combination of the relaxed triangle inequalities for squared Euclidean distances (see Corollary A.2):

\[
\|\varphi(x') - \varphi(x'')\|^2 \leq 2 \left[\|\varphi(x) - \varphi(x')\|^2 + \|\varphi(x) - \varphi(x'')\|^2\right]
\]

with weights $\lambda_x$. Note that $\sum_{x \in X} \lambda_x = 1$ for each $\lambda \in \Lambda_\eta$. \qed

By Claim 4.7,

\[
\sum_{(x', x'') \in E} \lambda_{x'} \lambda_{x''} \|\varphi(x') - \varphi(x'')\|^2 \leq 2 \sum_{(x', x'') \in E} \lambda_{x'} \lambda_{x''} \sum_{x \in X} \left(\lambda_x \|\varphi(x) - \varphi(x')\|^2 + \lambda_x \|\varphi(x) - \varphi(x'')\|^2\right)
\]

\[
= 2 \sum_{x, x' \in X} \left[\sum_{x'' : (x', x'') \in E} \lambda_{x''}\right] \lambda_x \lambda_{x'} \|\varphi(x) - \varphi(x')\|^2.
\]

Since the degree of every vertex $x'$ in the distance expansion graph is at most $\theta n$ and each $\lambda_{x''} \leq \eta$, we have $\sum_{x'' : (x', x'') \in E} \lambda_{x''} \leq \theta n$. Therefore,

\[
\sum_{(x', x'') \in E} \lambda_{x'} \lambda_{x''} \|\varphi(x') - \varphi(x'')\|^2 \leq 2\theta n \sum_{x, x' \in X} \lambda_x \lambda_{x'} \|\varphi(x) - \varphi(x')\|^2 = 4\theta n \sum_{(x', x'') \in P} \lambda_x \lambda_{x''} \|\varphi(x') - \varphi(x'')\|^2.
\]

Note that $4\theta n = 2\varepsilon/(1 + \varepsilon)^2 < \varepsilon'/(1 + \varepsilon')$. We plug in the bound above in Equation (18) and obtain the desired inequality:

\[
\max_{v' \in \mathbb{R}^d} \ F(v', \lambda) \geq \varepsilon' \sum_{(x', x'') \in P} \lambda_{x'} \lambda_{x''} \|\varphi(x') - \varphi(x'')\|^2 - \frac{(1 + \varepsilon') \cdot \varepsilon'}{1 + \varepsilon'} \sum_{(x', x'') \in P} \lambda_{x'} \lambda_{x''} \|\varphi(x') - \varphi(x'')\|^2 = 0.
\]

\[\square\]
Let $v$ be the point that maximizes the functional $\min_{\lambda \in \Lambda_\eta} F(v, \lambda)$. By Lemma 4.6, $\min_{\lambda \in \Lambda_\eta} F(v, \lambda) \geq 0$ or, in other words, $F(v, \lambda) \geq 0$ for all $\lambda \in \Lambda_\eta$. Consider the set $S = \{ x \in X : \| \varphi(x) - v \| > (1 + \varepsilon)\| x - u \| \}$. If $S = \emptyset$ then we are done. Otherwise, define a vector $\lambda^*$ as follows

$$\lambda^*_x = \begin{cases} 1/|S|, & \text{if } x \in S \\ 0, & \text{otherwise.} \end{cases}$$

Note that $(1 + \varepsilon')\| x - u \|^2 - \| \varphi(x) - v \|^2$ is negative for all $x \in S$ (by the definition of $S$). Hence,

$$F(v, \lambda^*) = \frac{1}{|S|} \sum_{x \in S} (1 + \varepsilon')\| x - u \|^2 - \| \varphi(x) - v \|^2 < 0$$

and, consequently, $\lambda^* \notin \Lambda_\eta$. Therefore, $1/|S| > \eta$ (otherwise, $\lambda^*$ would belong to $\Lambda_\eta$) and $|S| \leq 1/\eta = \theta'(\varepsilon)n$. This finishes the proof of Theorem 4.3 since for all $x \notin S$, we have $\| \varphi(x) - v \| \leq (1 + \varepsilon)\| x - u \|$.

**Corollary 4.8.** Consider a finite multiset of points $X \subset \mathbb{R}^d$ and a map $\varphi : X \to \mathbb{R}^{d'}$. Assume that $\varphi$ $(1 + \varepsilon)$-preserves the distance from every point $x \in X$ to a $(1 - \theta)$ fraction of points in $X$ for $\theta \in (0, 1/4^{p+1})$. Then, for every $p \geq 1$ and $D = (1 + \varepsilon)^p(1 + 3^{p+2}\theta^{1/(p+1)})$, we have

$$D^{-1} \text{cost}_p(X) \leq \text{cost}_p(\varphi(X)) \leq D \text{cost}_p(X).$$

**Proof.** Let $Y = \varphi(X)$. We apply Lemma 4.4 to maps $(1 + \varepsilon)\varphi$ and $(1 + \varepsilon)\varphi^{-1}$ and get the desired result. \qed

## 5 Bounding the cost of $k$-clustering

In this section, we show that if there exists a $(1 - \theta)$-non-distorted core then with high probability $\text{cost}_p \mathcal{C} \approx \text{cost}_p \pi(\mathcal{C})$ for any clustering $\mathcal{C}$ of $X$.

**Lemma 5.1.** Consider an instance $X$ of Euclidean $k$-clustering with the $\ell_p$-objective. Let $\pi$ be a $(\varepsilon, \delta, \theta)$-random dimension reduction, and $\mathcal{C} = (C_1, \ldots, C_k)$ be a random clustering of $X$. Let $\alpha \in (0, 1)$ (the failure probability) and $\theta \in (0, 1/4^{p+1})$.

Assume that $X' \subset X$ is a $(1 - \theta)$-non-distorted core. Then

$$\text{cost}_p \pi(\mathcal{C}) \leq A \left( \text{cost}_p \mathcal{C} + c_{\alpha\varepsilon\theta} \text{cost}_p \mathcal{C}^* \right)$$

$$\text{cost}_p \mathcal{C} \leq A \left( \text{cost}_p \pi(\mathcal{C}) + c_{\alpha\varepsilon\theta} \text{cost}_p \mathcal{C}^* \right)$$

with probability at least $1 - \alpha - (k/\theta)\delta$, where $A = (1 + \varepsilon)^{3p-2}(1 + 3^{p+2}\varepsilon^{1/(p+1)})$ and $c_{\alpha\varepsilon\theta} = \frac{3(1+\varepsilon)^{p+1}}{\alpha\varepsilon\theta}$.

**Proof.** Let $\mathcal{C}^* = (C_1^*, \ldots, C_k^*)$ be the optimal clustering of $X$, and $c_1^*, \ldots, c_k^*$ be the optimal centers of $C_1^*, \ldots, C_k^*$, respectively. Let $c^*(x) = c_j^*$ if $x \in C_j^*$. Consider the event $\mathcal{E}$ that all the distances between the centers $c_j^*$ are $(1 + \varepsilon)$-preserved. Note that $\Pr(\mathcal{E}) \geq 1 - (k/\theta)\delta$. We assume below that $\mathcal{E}$ happens.

Let

$$g(x) = \begin{cases} x, & \text{if } x \in X' \\ c^*(x), & \text{if } x \notin X'. \end{cases}$$

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Consider multiset $\tilde{C}_i = g(C_i)$ for every $i$. Note that every points in $\tilde{C}_i$ is either in $X'$ or equal to some $c_i^*$. Now, since $X'$ is a $(1 - \theta)$-non-distorted core, the map $\pi$ is $(1 + \varepsilon)$-preserves the distances from every $x \in X'$ to at least a $(1 - \theta)$ fraction of the points in $X'$ and to all $c_i^*$. Further, since $\mathcal{E}$ happens all distances between centers $c_i^*$ are $(1 + \varepsilon)$-preserved. Therefore, we can apply Corollary 4.8 to multiset $\tilde{C}_i$ and map $\pi$. We get that

$$D^{-1} \text{cost}_p(\tilde{C}_i) \leq \text{cost}_p(\pi(\tilde{C}_i)) \leq D \text{cost}_p(\tilde{C}_i)$$

for $D = (1 + \varepsilon)^p(1 + 3^p + 2\theta^{1/(p + 1)})$. Adding up (21) over all $i$, we get

$$D^{-1} \text{cost}_p(\tilde{C}) \leq \text{cost}_p(\pi(\tilde{C})) \leq D \text{cost}_p(\tilde{C}).$$

Denote the optimal centers of clusters $\tilde{C}_i$ by $\tilde{c}_i$. Note that

$$\text{cost}_p C \leq \sum_{x \in X} \|x - \tilde{c}(g(x))\|^p \quad \text{and} \quad \text{cost}_p \tilde{C} = \sum_{x \in X} \|g(x) - \tilde{c}(g(x))\|^p.$$

Compare the summations in the upper bound for $\text{cost}_p C$ and the formula for $\text{cost}_p \tilde{C}$ term by term. For $x \in X'$, $g(x) = x$ and thus the terms in both summations are equal. For $x \notin X'$, the terms in the first and second summations are equal to $\|x - \tilde{c}(c^*(x))\|^p$ and $\|c^*(x) - \tilde{c}(c^*(x))\|^p$, respectively. Therefore,

$$\text{cost}_p C - (1 + \varepsilon)^{p-1} \text{cost}_p \tilde{C} \leq \sum_{x \in X \setminus X'} \|x - \tilde{c}(c^*(x))\|^p - (1 + \varepsilon)^{p-1} \|c^*(x) - \tilde{c}(c^*(x))\|^p.$$

Observe that $\|x - \tilde{c}(c^*(x))\| \leq \|x - c^*(x)\| + \|c^*(x) - \tilde{c}(c^*(x))\|$. Applying Lemma A.1 with $r = 1$, we get

$$\|x - \tilde{c}(c^*(x))\|^p \leq (1 + \varepsilon)^{p-1} \|c^*(x) - \tilde{c}(c^*(x))\|^p + \left(\frac{1 + \varepsilon}{\varepsilon}\right)^{p-1} \|x - c^*(x)\|^p.$$

Therefore,

$$\text{cost}_p C - (1 + \varepsilon)^{p-1} \text{cost}_p \tilde{C} \leq \left(\frac{1 + \varepsilon}{\varepsilon}\right)^{p-1} \sum_{x \in X \setminus X'} \|x - c^*(x)\|^p.$$

Similarly,

$$\text{cost}_p \tilde{C} - (1 + \varepsilon)^{p-1} \text{cost}_p C \leq \left(\frac{1 + \varepsilon}{\varepsilon}\right)^{p-1} \sum_{x \in X \setminus X'} \|x - c^*(x)\|^p$$

$$\text{cost}_p \pi(C) - (1 + \varepsilon)^{p-1} \text{cost}_p \pi(C) \leq \left(\frac{1 + \varepsilon}{\varepsilon}\right)^{p-1} \sum_{x \in X \setminus X'} \|\pi(x) - \pi(c^*(x))\|^p$$

$$\text{cost}_p \pi(\tilde{C}) - (1 + \varepsilon)^{p-1} \text{cost}_p \pi(C) \leq \left(\frac{1 + \varepsilon}{\varepsilon}\right)^{p-1} \sum_{x \in X \setminus X'} \|\pi(x) - \pi(c^*(x))\|^p$$
Using inequality (22), we obtain
\[
\text{cost}_p \pi(C) \leq A \left( \text{cost}_p C + \varepsilon^{1-p} \sum_{c \in \mathcal{X} \setminus \mathcal{X}'} R_x \right)
\]
\[
\text{cost}_p C \leq A \left( \text{cost}_p \pi(C) + \varepsilon^{1-p} \sum_{c \in \mathcal{X} \setminus \mathcal{X}'} R_x \right)
\]
where \( A = (1 + \varepsilon)^{2(p-1)} D \) and \( R_x = \|x - c^*(x)\|^p + \|\pi(c^*(x)) - \pi(x)\|^p \).

It remains to prove that \( \varepsilon^{1-p} \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} \mathbb{1} \{ \mathcal{E} \} R_x \leq c_{\alpha \theta} \text{cost}_p C^* \) with probability at least \( 1 - \alpha \).

To this end, we show that \( \mathbb{E} \left[ \mathbb{1} \{ \mathcal{E} \} \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} R_x \right] \leq 3(1 + \varepsilon)^p \theta \text{cost}_p C^* \) and then use Markov’s inequality. From property (5) in Definition 2.1 we get that for every \( x \in \mathcal{X} \),
\[
\mathbb{E} \left[ \max(\|\pi(c^*(x)) - \pi(x)\|^p - (1 + \varepsilon)^p \|c^*(x) - x\|^p, 0) \right] \leq \theta \|c^*(x) - x\|^p.
\]

Therefore,
\[
\mathbb{E} \left[ \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} \|\pi(c^*(x)) - \pi(x)\|^p \right] \leq (1 + \varepsilon)^p \mathbb{E} \left[ \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} \|c^*(x) - x\|^p \right] + \mathbb{E} \left[ \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} \max(\|\pi(c^*(x)) - \pi(x)\|^p - (1 + \varepsilon)^p \|c^*(x) - x\|^p, 0) \right]
\]

Now, we bound the second term as
\[
\sum_{x \in \mathcal{X}} \mathbb{E} \left[ \max(\|\pi(c^*(x)) - \pi(x)\|^p - (1 + \varepsilon)^p \|c^*(x) - x\|^p, 0) \right] \leq \sum_{x \in \mathcal{X}} \theta \|c^*(x) - x\|^p = \theta \text{cost}_p C^*
\]

Therefore,
\[
\mathbb{E} \left[ \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} \|\pi(c^*(x)) - \pi(x)\|^p \right] \leq (1 + \varepsilon)^p \mathbb{E} \left[ \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} \|c^*(x) - x\|^p \right] + \theta \text{cost}_p C^*.
\]

We also have,
\[
\mathbb{E} \left[ \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} \|x - c^*(x)\|^p \right] \leq \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} \|x - c^*(x)\|^p \Pr \left( x \notin \mathcal{X}' \right) \leq \theta \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} \|x - c^*(x)\|^p = \theta \text{cost}_p C^*.
\]

Therefore,
\[
\mathbb{E} \left[ \mathbb{1} \{ \mathcal{E} \} \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} R_x \right] \leq \mathbb{E} \left[ \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} \|\pi(c^*(x)) - \pi(x)\|^p + \|x - c^*(x)\|^p \right] \leq 3(1 + \varepsilon)^p \theta \text{cost}_p C^*.
\]

By Markov’s inequality,
\[
\Pr \left( \mathbb{1} \{ \mathcal{E} \} \varepsilon^{1-p} \sum_{x \in \mathcal{X} \setminus \mathcal{X}'} R_x \geq c_{\alpha \theta} \text{cost}_p C^* \right) \leq \varepsilon^{1-p} \cdot \frac{3(1 + \varepsilon)^p \theta \text{cost}_p C^*}{c_{\alpha \theta} \text{cost}_p C^*} = \alpha.
\]

We conclude that
\[
\Pr ((19) \text{ and } (20) \text{ hold}) \geq 1 - \alpha - \Pr (\mathcal{E}) \geq 1 - \alpha - \left( \frac{k}{2} \right) \delta.
\]
6 Proof of main results

In this section, we prove the main results: Theorem 6.1, Theorem 6.2 and Theorem 1.3.

**Theorem 6.1.** Let $X$ be an instance of $k$-clustering with the $\ell_p$-objective. Let $\varepsilon \in (0, 1/4)$ (the distortion parameter) and $\alpha \in (0, 1)$ (the failure probability). Let $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^d$ be a standard random dimension-reduction map with

$$d = \frac{C(p^2 \log k + p \log \frac{1}{\varepsilon \alpha})}{\varepsilon^2}$$

(where $C$ only depends on the parameters of map $\pi$ in Definition 2.1).

I. Let $\mathcal{C} = (C_1, \ldots, C_k)$ be a random clustering of $X$ (clustering $\mathcal{C}$ may depend on $\pi$). Then with probability at least $1 - \alpha$, we have

$$\text{cost}_p \pi(\mathcal{C}) \leq (1 + \varepsilon)^{3p} \text{cost}_p \mathcal{C}$$

$$\text{cost}_p \mathcal{C} \leq \frac{(1 + \varepsilon)^{3p-1}}{1 - \varepsilon} \text{cost}_p \pi(\mathcal{C})$$

(23)

(24)

(In particular, the cost of the optimal clustering of $\pi(X)$ is approximately equal to the cost of the optimal clustering of $X$.)

II. Further, with probability at least $1 - \alpha$ the following event $\mathcal{D}$ happens: for every clustering $\mathcal{C}$ of $X$ inequalities (23) and (24) hold.

**Proof.** I. Let $\theta = \min(\varepsilon^{p+1}3^{(p+1)(p+1)}, \varepsilon^{p}/(6(1 + \varepsilon)^p))$. We choose constant $C$ in the condition for $d$ so that $\pi$ is a dimension reduction with parameters $(\varepsilon, \delta, \rho)$ such that $\delta \leq \theta/((10^4k^5)(\frac{5}{2})\delta \leq \alpha/2$, and $\rho \leq \theta$ (we can do this, since $\pi$ is a standard dimension reduction). Now we apply Lemmas 3.7 and 5.1. By our choice of parameters, $A \leq (1 + \varepsilon)^{3p-1}$ and $c_{\alpha \varepsilon \theta} \leq \varepsilon$ in the statement of Lemma 5.1. Also, $\text{cost}_p \mathcal{C}^* \leq \text{cost}_p \mathcal{C}$. We have,

$$\text{cost}_p \pi(\mathcal{C}) \leq (1 + \varepsilon)^{3p} \text{cost}_p \mathcal{C}$$

$$(1 - \varepsilon) \text{cost}_p \mathcal{C} \leq (1 + \varepsilon)^{3p-1} \text{cost}_p \pi(\mathcal{C})$$

with probability at least $1 - \alpha/2 - (\frac{5}{2})\delta \geq 1 - \alpha$. The statement of the theorem follows.

II. For each realization of $\pi$, let $\tilde{\mathcal{C}}$ be a clustering that violates inequality (23) or (24), if there exists such a clustering, and $\tilde{\mathcal{C}}$ be an arbitrary clustering, otherwise. We get a random clustering $\tilde{\mathcal{C}}$. We have,

$$\Pr(\mathcal{D}) = \Pr(\tilde{\mathcal{C}} \text{ satisfies (23) or (24)}) \leq \alpha,$$

as required.

As a corollary, we get the following formulation of the theorem.

**Theorem 6.2.** Let $X$ be an instance of $k$-clustering with the $\ell_p$-objective. Let $\varepsilon \in (0, 1/4)$ (the distortion parameter) and $\alpha \in (0, 1/2)$ (the failure probability). Let $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^d$ be a standard random dimension-reduction map with

$$d = \frac{Cp^4 \log(k/(\varepsilon \alpha))}{\varepsilon^2}.$$
Then with probability at least $1 - \alpha$:

$$(1 - \varepsilon) \text{cost}_p C \leq \text{cost}_p \pi(C) \leq (1 + \varepsilon) \text{cost}_p C \quad \text{for every clustering} \ C$$

**Proof.** We apply Theorem 6.1 with $\varepsilon' = (1 + \varepsilon)^{1/(3p)} - 1 = O(\varepsilon/p)$ and obtain the desired inequality.

Now we prove Theorem 1.3.

**Proof of Theorem 1.3.** We apply Lemma C.1 to $\pi_{m,d}$. Since $\pi_{m,d}$ satisfies the condition of Theorem 1.1 and is sub-Gaussian tailed, it is a standard dimension reduction. Applying Theorem 6.2, we get the desired result.

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A Inequality for the sum of $p$-th powers

**Lemma A.1.** Let $x$ and $y_1, \ldots, y_r$ be non-negative real numbers, and $\varepsilon > 0$, $p \geq 1$. Then

$$
\left( x + \sum_{i=1}^{r} y_i \right)^{p} \leq (1 + \varepsilon)^{p-1} x^{p} + \left( \frac{(1 + \varepsilon)r}{\varepsilon} \right)^{p-1} \sum_{i=1}^{r} y_i^{p}.
$$

**Proof.** Let $t = \frac{1}{1+\varepsilon}$. Write,

$$
\left( x + \sum_{i=1}^{r} y_i \right)^{p} = \frac{1}{t^{p}} \left( t x + \sum_{i=1}^{r} \frac{1-t}{r} \left( \frac{r t y_i}{1-t} \right) \right)^{p}.
$$

The expression in the parentheses on the right is a convex combination of numbers $x, \frac{r t y_1}{1-t}, \ldots, \frac{r t y_r}{1-t}$ (which add up to 1). Applying Jensen’s inequality, we get

$$
\frac{1}{t^{p}} \left( t x + \sum_{i=1}^{r} \frac{1-t}{r} \left( \frac{r t y_i}{1-t} \right) \right)^{p} \leq \frac{t x^{p}}{t^{p}} + \frac{1-t}{r^{p}} \sum_{i=1}^{r} \left( \frac{r t y_i}{1-t} \right)^{p} = (1 + \varepsilon)^{p-1} x^{p} + \left( \frac{(1 + \varepsilon)r}{\varepsilon} \right)^{p-1} \sum_{i=1}^{r} y_i^{p}.
$$

$\square$

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Corollary A.2 (Relaxed triangle inequalities). For any vectors $u, v, w$ and numbers $\varepsilon > 0$, $p \geq 1$, we have

1. $\|u - w\|^p \leq (1 + \varepsilon)\|u - v\|^p + \left(\frac{1 + \varepsilon}{\varepsilon}\right)^{p-1}\|v - w\|^p$.

2. $\|u - w\|^2 \leq 2\|u - v\|^2 + 2\|v - w\|^2$.

Proof. Using Lemma A.1, we get

$$\|u - w\|^p \leq (\|u - v\| + 2\|v - w\|)^p \leq (1 + \varepsilon)\|u - v\|^p + \left(\frac{1 + \varepsilon}{\varepsilon}\right)^{p-1}\|v - w\|^p.$$  

Item 2 is a special case of this inequality with $\varepsilon = 1$ and $p = 2$. 

\[\square\]

B Closed-form expression for the cost of a cluster

In this section, we derive a well known formula (15) for computing the cost of a cluster with respect to the $k$-means objective. The optimal center of the cluster formed by points in set $X$ with weights $\lambda_x$ is located in the center of mass of $X$. Let $a$ and $b$ be i.i.d random variables such as $\Pr(a = x) = \Pr(b = x) = \lambda_x$. Then, the cost of the cluster $X$ equals $\mathbb{E}[(a - \mathbb{E}a)^2] = \text{Var}[a]$ and

$$\sum_{(x', x'') \in P} \lambda_{x'} \lambda_{x''} \|x' - x''\|^2 = \frac{1}{2} \sum_{(x', x'') \in X \times X} \lambda_{x'} \lambda_{x''} \|x' - x''\|^2 = \frac{1}{2} \mathbb{E}\|a - b\|^2 = \mathbb{E}a^2 - (\mathbb{E}a)^2 = \text{Var}[a].$$

C Sub-Gaussian Tailed Dimension Reduction

In this section, first we prove that every sub-Gaussian tailed dimension reduction is standard. Then we show that the Gaussian dimension reduction is sub-Gaussian tailed.

Lemma C.1. Let $\varepsilon < 1/2$. Assume that a family of random maps $\pi_{m,d} : \mathbb{R}^m \to \mathbb{R}^d$ satisfies the condition of Theorem 1.1 and is sub-Gaussian tailed (satisfies Definition 1.2). Then $\pi_{m,d}$ is a standard dimension reduction (see Definition 2.1).

Proof. Denote the parameters of dimension reduction $\pi_{m,d}$ by $(\varepsilon, \delta, \rho)$. Since $\pi_{m,d}$ satisfies the condition of Theorem 1.1, $\delta \leq \exp(-c\varepsilon^2 d)$ for some $c$.

Let $u$ be a unit vector in $\mathbb{R}^m$ and $\xi = \|\pi(u)\| - 1$. Since $\pi_{m,d}$ is sub-Gaussian tailed, $\Pr(\xi > t) \leq \exp(-ct^2 d)$ for some $c$. We assume that $d \geq c(p - 1)/\varepsilon^2$. We have,

$$\mathbb{E} \left[ \mathbb{1} \{\|\pi(u)\| > (1 + \varepsilon)\|u\|\} \left(\frac{\|\pi(u)\|^p}{\|u\|^p} - (1 + \varepsilon)^p\right) \right] = \mathbb{E} \left[ \mathbb{1} \{\xi > \varepsilon\} ((1 + \xi)^p - (1 + \varepsilon)^p) \right]$$

$$= \int_\varepsilon^\infty ((1 + t)^p - (1 + \varepsilon)^p) d\Pr(\xi \leq t) = \int_\varepsilon^\infty p(1 + t)^{p-1} \Pr(\xi > t) dt$$

$$\leq \int_\varepsilon^\infty p(1 + t)^{p-1} e^{-ct^2 d} dt = \int_\varepsilon^\infty (p(1 + t)^p e^{-ct^2 d/2}) e^{-ct^2 d/2} dt$$

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By differentiating \( g(t) = p(1 + t)^{p-1} e^{-c t^2 / 2} \) by \( t \), we get that \( g'(t) \) is decreasing when \( t(1 + t) \geq \frac{p-1}{c} \). Since \( d \geq \frac{c(p-1)}{\varepsilon^2} \), \( g(t) \) attains its maximum on \( [\varepsilon, \infty) \) when \( t = \varepsilon \). We have

\[
g(t) \leq p(1 + \varepsilon)^{p-1} e^{-c \varepsilon^2 / 2} \leq e^{-(p-1)/\varepsilon^2 + \ln p + (p-1) \ln(1+\varepsilon)} \leq 1
\]

(here, we used that \( \varepsilon \leq 1/2 \) and \( d \geq c(p-1)/\varepsilon^2 \). Therefore,

\[
\mathbb{E} \left[ (1 + \xi)^p - (1 + \varepsilon)^p \right] \leq \int_{\varepsilon}^{\infty} e^{-c t^2 / 2} dt \leq \int_{\varepsilon}^{\infty} \frac{t}{\varepsilon} e^{-c t^2 / 2} dt = \frac{1}{c d \varepsilon} \int_{\varepsilon^2 / 2}^{\infty} e^{-c s} ds = \frac{1}{c d \varepsilon} e^{-c d \varepsilon^2 / 2} < e^{-c d \varepsilon^2 / 2}.
\]

Consider a \( d \times m \) matrix \( G \) whose entries are i.i.d. Gaussian random variables \( \mathcal{N}(0,1) \). Matrix \( G \) defines a linear dimension reduction \( \pi(u) = G u / \sqrt{d} \). [IM98]

**Claim C.2.** The Gaussian dimension reduction \( \pi \) (defined above) is sub-Gaussian tailed.

**Proof.** Let \( u \) be a unit vector in \( \mathbb{R}^m \), \( \xi = \| \pi(u) \| - 1 \) and \( \eta = \sqrt{d} (\xi + 1) = \sqrt{d} \| \pi(u) \| \). Note that \( \eta^2 \) has the \( \chi^2 \)-distribution with \( d \) degrees of freedom. As was shown by Laurent and Massart [LM00, Lemma 1, Inequality (4.3)], \( \Pr(\eta^2 - d \geq 2 \sqrt{d \cdot x} + 2x) \leq e^{-x} \) for any positive \( x \). Plugging in \( x = t^2 d / 2 \) (where \( t > 0 \)), we get

\[
\Pr(\xi \geq t) = \Pr((\xi + 1)^2 \geq 1 + 2t + t^2) \leq \Pr((\xi + 1)^2 - 1 \geq \sqrt{2t} + t^2)
= \Pr(\eta^2 - d \geq 2 \sqrt{d x} + 2x) \leq e^{-x} = e^{-t^2 d / 2}.
\]

\( \square \)