Extended KdV equation for the case of uneven bottom

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(Dated: October 30, 2018)

We derived consistently, according to the second order perturbation approach, the extended KdV equation for an uneven bottom for the case of $\alpha = O(\beta)$ and $\delta = O(\beta^2)$. This is the only case when second order KdV type nonlinear wave equation can be derived for arbitrary bounded bottom function. Moreover, we proved that for the case of $\alpha = O(\beta)$ and $\delta = O(\beta)$ a unidirectional wave equation can be obtained neither in first order nor in higher orders when $\alpha \neq \beta$.

PACS numbers: 02.30.Jr, 05.45.-a, 47.35.Bb, 47.35.Fg
Keywords: Shallow water waves, nonlinear equations, second order perturbation approach, uneven bottom

I. INTRODUCTION

In 2014, with our co-workers, we derived the nonlinear second order wave equation for shallow water problem with uneven bottom [1, 2]. In these papers, besides standard small parameters $\alpha = \frac{a}{h}$ and $\beta = \frac{(l)}{h}$ we introduced the third one defined as $\delta = \frac{a_h}{h}$. In these definitions $a$ denotes the wave amplitude, $l$ the average wavelength and $a_h$ the amplitude of the bottom variations. We considered the case of $\alpha = O(\beta)$ and $\delta = O(\beta)$, that is, when all three small parameters are of the same order. Then, with standard assumptions for incompressible, inviscid fluid and irrotational motion, we applied the second-order perturbation approach to the set of Eulerian equations. This set, written in nondimensional variables has the following form (see, e.g., Eqs. (2)-(5) in [2])

\[ \beta \phi_{xx} + \phi_{zz} = 0, \tag{1} \]
\[ \eta_t + \alpha \phi_x \eta_x - \frac{1}{\beta} \phi_z = 0, \quad \text{for } z = 1 + \alpha \eta \tag{2} \]
\[ \phi_x + \frac{1}{2} \alpha \phi_x^2 + \frac{1}{2} \alpha \phi_z^2 + \eta = 0, \quad \text{for } z = 1 + \alpha \eta \tag{3} \]
\[ \phi_z - \beta \delta (h_x \phi_x) = 0, \quad \text{for } z = \delta h(x). \tag{4} \]

Equation (1) is the Laplace equation valid for the whole volume of the fluid. Equations (2) and (3) are so-called kinematic and dynamic boundary conditions at the surface, respectively. The equation (3) represents the boundary condition at the non-flat bottom. For abbreviation all subscripts denote the partial derivatives with respect to particular variables, i.e. $\phi_x \equiv \frac{\partial \phi}{\partial x}$, $\eta_{2x} \equiv \frac{\partial^2 \eta}{\partial x^2}$ and so on.

For the flat bottom, the boundary condition at the bottom is $\phi_z = 0$. In this case, the perturbation approach of the first order with respect to small parameters leads to the famous Korteweg-de Vries equation [3]

\[ \eta_t + \eta_x + \frac{3}{2} \eta_{xx} + \beta \frac{1}{6} \eta_{3x} = 0. \tag{5} \]

In second order, Marchant and Smyth obtained in 1990 the extended KdV equation (called also KdV2) of the form [4]

\[ \eta_t + \eta_x + \alpha \frac{3}{2} \eta_{xx} + \beta \frac{1}{6} \eta_{3x} + \alpha^2 \left( \frac{3}{8} \eta^2 \eta_x \right) \]
\[ + \alpha \beta \left( \frac{23}{24} \eta_x \eta_{2x} + \frac{5}{12} \eta_{3x} \right) + \beta^2 \frac{19}{360} \eta_{5x} = 0. \tag{6} \]

In [1, 2] we tried to extend the second-order approach to the case $\delta \neq 0$ of the non-flat bottom. Then the equation [1], limited to the second-order, allows us to express the velocity potential in the form [2, Eq. (7)]

\[ \phi = \phi^{(0)} + z \beta \delta \left( h \phi_x^{(0)} \right)_x - \frac{1}{2} \beta^2 \delta^2 \phi^{(0)}_2 x - \frac{1}{6} \beta^3 \delta \left( h \phi_x^{(0)} \right)_3 x + \frac{1}{24} \beta^2 \phi^{(0)}_4 x + \frac{1}{120} \beta^3 \delta \left( h \phi_x^{(0)} \right)_5 x - \frac{1}{720} \beta^5 \phi^{(0)}_6 x + \ldots \tag{7} \]

Inserting (7) into (2) and (3) and retaining only terms up to second-order one obtains the second-order Boussinesq's system

\[ \eta_t + w_x + \alpha (\eta w)_x - \frac{1}{6} \beta w_{3x} - \frac{1}{2} \alpha \beta (\eta w_{2x})_x \]
\[ - \frac{1}{120} \beta^2 w_{5x} - \delta (h w)_x + \frac{1}{2} \beta \delta (h w)_3 x = 0, \tag{8} \]
\[ w_t + \eta_x + \alpha w_{2x} - \frac{1}{2} \beta w_{2xt} + \frac{1}{24} \beta^2 w_{4xt} + \beta \delta (h w)_x \]
\[ + \frac{1}{2} \alpha \beta [-2 (\eta w)_x + w_x w_{2x} - w w_{3x}] = 0. \tag{9} \]

This set of Boussinesq’s equations is correct.
Recently, it was pointed out in [1] that our next steps, performed in [1, 2] and leading to the KdV2 equation for uneven bottom were inconsistent, and therefore the derived equation [2, Eq. (18)] bears no relevant solution to the problem considered.

We agree with this criticism. We derived our equation [2, Eq. (18)] in good faith. However, using different notations for small parameters \( \alpha, \beta, \delta \) we did not recognize the proper order of terms related to the bottom function.

The next parts of this article contain the following results.

- The Boussinesq’s system \([8, 9]\) cannot be reduced (for arbitrary shape of the bottom function) to a single KdV-type equation even in the first order. In consequence, the same is true for any higher order equations for the case of \( \alpha = O(\beta) \) and \( \delta = O(\beta) \).

- In Section IV we test motion of solitons over uneven bottom of the trapezoidal shape. This bottom function is piecewise linear. Two cases, a bump, and a well are tested. Initial conditions are taken as the KdV solitons for the case of \( \alpha = O(\beta) \) and \( \delta = O(\beta) \) and as the KdV2 solitons for the case of \( \alpha = O(\beta) \) and \( \delta = O(\beta^2) \).

II. (NON)EXISTENCE OF WAVE EQUATION FOR THE CASE OF \( \alpha = O(\beta) \) AND \( \delta = O(\beta) \)

In his Comment [5], the author points out that the consistent second order perturbation approach can be achieved when all small parameters are related to only one, assuming for instance

\[
\alpha = A\beta, \quad \delta = q\beta,
\]

where the constants \( A, q \) are of the order of 1. The presence of the factors \( A \) and \( q \) in the following steps eases to recognize the origin of particular terms.

In standard approach the velocity potential is assumed in the form of the series \( \phi(x, z, t) = \sum_{m=0}^{\infty} z^m \phi^{(m)}(x, t) \). For flat bottom case \( \delta = q = 0 \) equations (13) and (4) allow us to express all \( \phi^{(m)}(x, t) \) with even \( m \) only, by \( f(x, t) := \phi^{(0)}(x, t) \) and its even \( x \)-derivatives. For the uneven bottom case, to satisfy the equation \([6]\), the velocity potential has to contain also odd \( m \) terms. In general the velocity potential fulfilling Laplace equation can be expressed in the following form

\[
\phi(x, z, t) = \sum_{m=0}^{\infty} \frac{(-1)^m \beta^m f z^m}{(2m)!} \frac{\partial^{2m} f}{\partial x^{2m}} + \sum_{m=0}^{\infty} \frac{(-1)^m \beta^{m+1} \delta^{2m+1} F}{(2m + 1)!} \frac{\partial^{2m+1} F}{\partial x^{2m+1}} z^{2m+1},
\]

where \( F = F(x, t) \). Explicit form of this velocity potential is

\[
\phi = f - \frac{1}{2} \beta z^2 f_{xx} + \frac{1}{24} \beta^2 z^4 f_{4x} - \frac{1}{120} \beta^3 z^6 f_{6x} + \cdots + \beta z G - \frac{1}{6} \beta^2 z^3 G_{2x} + \frac{1}{120} \beta^3 z^5 G_{4x} + \cdots,
\]

where \( G = F_x \). Substituting (12) into (11) gives (with \( z = q\beta h \)) nontrivial relation between the functions \( G \) and \( f \)

\[
G - q\beta(hf_x)_x = \frac{1}{2} q^2 \beta^3(h^2 G_x)_x + \frac{1}{6} q^3 \beta^4(h^3 f_{3x})_x + \cdots
\]

To specify this relation let us express \( G \) as a series

\[
G = G_0 + \beta G_1 + \beta^2 G_2 + \beta^3 G_3 + \beta^4 G_4 + \cdots
\]

Substituting (14) into (13) and collecting powers of \( \beta \) we get \( G_0 = 0 \), \( G_2 = 0 \) and

\[
G_1 = q(hf_x)_x, \quad G_3 = q^2(h^2 G_x)_x, \quad G_4 = \frac{1}{6} q^3(h^3 f_{3x})_x \quad \cdots
\]

So, the function \( G \) is given by

\[
G = \beta q(hf_x)_x + \beta^3 q^2(h^2 G_x)_x + \cdots
\]

Since we are interested in second order equations, we can safely reject all terms except the first one in (16) since after substitution of (16) to the velocity potential (12) they contribute in at least the fourth order in \( \beta \). This approximation allows us to express the \( x \)-dependence of the velocity potential through \( f, h \) and their \( x \)-derivatives.

Remark: The form of (17) indicates that attempts to derive a wave equation of the higher order than second are practically unfeasible.

Then we obtain velocity potential in the following form

\[
\phi = f - \frac{1}{2} \beta z^2 f_{xx} + \frac{1}{24} \beta^2 z^4 f_{4x} - \frac{1}{120} \beta^3 z^6 f_{6x} + \cdots + \beta^2 q(hf_x)_x - \frac{1}{6} \beta^3 z^3 q(hf_x)_{3x} + \frac{1}{120} \beta^4 z^5 q(hf_x)_{5x} + \cdots
\]

Inserting (17) into (2) and (3) and retaining terms up to second order yields the set of the Boussinesq equations
in the following form (as usual $w = f_x$)

$$
\eta_t + w_x + \beta \left( A(\eta w)_x - \frac{1}{6} w_{3x} - q(hw)_x \right) = 0
$$

(18)

$$
+ \beta^2 \left( -A \frac{1}{2} (\eta w_2)_x + \frac{1}{120} w_{5x} - q(hw)_x \right) = 0
$$

(19)

$$
w_t + \eta_x + \beta \left( w w_x - \frac{1}{2} w_{2xt} \right) + \beta^2 \left[ A \left( -\eta (w_x)_x + \frac{1}{2} w_x w_{2x} - \frac{1}{6} w_{6x} \right) + \frac{1}{24} w_{4xt} + q(hw)_x \right] = 0.
$$

Inserting $A\beta = \alpha$ and $q\beta = \delta$ into (18)-(19) we regain Eqs. (8)-(9) from [2], as well as Eqs. (8)-(9) in Section II.

Below we prove that the Boussinesq set (18)-(19) cannot be reduced to the KdV - type wave equation even in the first order. It is well known that in the lowest (zero) order the Boussinesq set reduces to

$$
\eta_t + w_x = 0, \quad w_t + \eta_x = 0.
$$

(20)

In the first order the Boussinesq set reduces to

$$
\eta_t + w_x + \alpha (\eta w)_x - \frac{1}{6} \beta w_{3x} - \delta (hw)_x = 0,
$$

(21)

$$
w_t + \eta_x + \alpha w w_x - \frac{1}{2} \beta w_{2xt} = 0.
$$

(22)

Assume that in the first order

$$
w = \eta + \alpha Q^{(\alpha)} + \beta Q^{(\beta)} + \delta Q^{(\delta)},
$$

(23)

where all correction functions have to fulfill conditions $Q_x = -Q_x$ necessary for right moving wave. Then we substitute (23) into equations (21)-(22), express time derivatives in terms of $x$-derivatives and retain term only to the first order. This yields

$$
\alpha(Q_x^{(\alpha)} + 2\eta\eta_x) + \beta(Q_x^{(\beta)} - \frac{1}{6} \eta_{3x}) + \delta(Q_x^{(\delta)} - (h\eta)_x) = 0
$$

(24)

and

$$
\alpha(-Q_x^{(\alpha)} + \eta\eta_x) + \beta(-Q_x^{(\beta)} + \frac{1}{2} \eta_{2x}) + \delta(-Q_x^{(\delta)}) = 0.
$$

(25)

Subtracting (24) from (25) and using the argument that small parameters are arbitrary within physically relevant intervals we arrive into three independent equations which can be integrated

$$
Q_x^{(\alpha)} = -\frac{1}{2} \eta\eta_x \implies Q^{(\alpha)} = -\frac{1}{4} \eta^2,
$$

(26)

$$
Q_x^{(\beta)} = \frac{3}{2} \eta\eta_x \implies Q^{(\beta)} = \frac{1}{3} \eta_{2x},
$$

(27)

$$
Q_x^{(\delta)} = \frac{1}{2} (h\eta)_x \implies Q^{(\delta)} = \frac{1}{2} (h\eta).
$$

(28)

It is well known that solutions $Q^{(\alpha)}$ and $Q^{(\beta)}$ of equations (26)-(27) supply the appropriate first order corrections for $\delta = 0$, that is for the flat bottom case, and supply the KdV equation. However, $Q^{(\delta)}$ correction, derived under the necessary condition $Q_t^{(\delta)} = -Q_x^{(\delta)}$ violates this condition since

$$
(h\eta)_x = h_x \eta + h\eta_x \quad \text{whereas} \quad (h\eta)_t = h\eta_x.
$$

(29)

This contradiction shows that it is not possible to make the Boussinesq set for the case of $\alpha = O(\beta)$ and $\delta = O(\beta)$ compatible within functions which can supply KdV - type wave equation even in the first order approach. Therefore for the case of $\alpha = O(\beta)$ and $\delta = O(\beta)$, there are no higher order wave equations possible, as well.

In [3], the author claimed that for the first order Boussinesq’s equations (18)-(19) the appropriate correction $Q^{(\delta)}$ can be found for the specific case $h(x) = kx$. He proposed

$$
w = \eta - \frac{1}{4} \eta^2 + \beta \left( \frac{1}{3} \eta_{2x} + \frac{1}{4} (2kx\eta + k \int \eta dx) \right).
$$

(30)

Insertion (30) into (18) yields

$$
\eta_t + \eta_x + \alpha(2\eta\eta_x) + \beta \left( \frac{1}{2} \eta\eta_x + \frac{1}{6} \eta_{3x} - \frac{1}{4} qk(\eta + 2x\eta_x) \right) = 0,
$$

(31)

whereas insertion (30) into (18) gives (after replacing $t$-derivatives by $x$-derivatives)

$$
\eta_t + \eta_x + \beta \left( \frac{3}{2} \eta\eta_x + \frac{1}{6} \eta_{2x} - \frac{1}{4} qk(\eta + 2x\eta_x) \right) = 0.
$$

(32)

The equations (31) and (32) are compatible only for $\alpha = \beta$ (or equivalently for $A = 1$ in notation used in [3]).

For a general case $\alpha \neq \beta$ the Boussinesq equations when $\alpha = O(\beta)$ and $\delta = O(\beta)$ cannot be made compatible even in the first order.

The bottom function $h(x) = kx$ is unbound on $x \in \mathbb{R}$ which contradicts the definition of the parameter $\delta = \frac{q_k}{k}$, where $q_k$ is the amplitude of the bottom function. Also from a physics standpoint, the bottom function can not grow infinitely, because for some values of $x$ the bottom would be above the water surface. Perhaps a meaningful use of this equation would be the case when $h(x)$ is a piecewise linear and bounded. In Section IV we examine this case in numerical simulation.

III. DERIVATION OF THE NONLINEAR WAVE EQUATION FOR THE CASE OF \( \alpha = O(\beta) \) AND \( \delta = O(\beta^2) \)

In this case we set

$$
\alpha = A\beta, \quad \delta = q\beta^2.
$$

(33)
Now, we insert the general form of velocity potential \( \phi \) into the bottom boundary condition which in this case is
\[
\phi_z - q\beta^3(h_x \phi_x) = 0, \quad \text{for} \quad z = q\beta^2 h(x)
\]
(34)

obtaining relation similar to \( (13) \)
\[
G - q\beta^2(h_x f_x) = -\frac{1}{2} q^2 \beta^3(h^2 G_x)_x + \frac{1}{6} q^3 \beta^7(h^3 f_x)_x
\]
(35)
\[
+ \frac{1}{24} q^4 \beta^{10}(h^4 G_{3x})_x - \frac{1}{120} q^5 \beta^{12}(h^5 f_{5x})_x + \cdots = 0.
\]

Then, in the lowest order
\[
G = q\beta^2(h_x f_x)
\]
(36)

which inserted into \( (12) \) gives the velocity potential as
\[
\phi = f - \frac{1}{2} \beta z^2 f_{2x} + \frac{1}{24} \beta^2 z^4 f_{4x} - \frac{1}{720} \beta^3 z^6 f_{6x} + \cdots
\]
(37)
\[
+ q\beta^3 z(h_x f_x) - \frac{1}{6} q\beta^3 z^3(h_x f_x)_{3x} + \frac{1}{120} q\beta^3 z^5(h_x f_x)_{5x} + \cdots.
\]

In this case the Boussinesq system has the form
\[
\eta_t + w_x + \beta \left(A(\eta w)_x - \frac{1}{6} \eta w_{3x}\right)
\]
(38)
\[
+ \beta^2 \left(-A \frac{1}{2}(\eta w_{2x})_x + \frac{1}{120} w_{5x} - q(hw)_x\right) = 0,
\]
\[
w_t + \eta_x + \beta \left(A w w_x - \frac{1}{2} w_{2x}\right)
\]
(39)
\[
+ \beta^2 \left(-A(\eta w_{xt})_x + A \frac{1}{2} w_x w_{2x} - A \frac{1}{2} w w_{3x}
\]
\[
+ \frac{1}{24} w_{4x}\right) = 0.
\]

In the first order this system reduces to the common KdV system, with
\[
w = \eta + \beta \left(-A \frac{1}{4} \eta^2 + \frac{1}{3} \eta \eta_x\right)
\]
(40)

which ensures the KdV equation
\[
\eta_t + \eta_x + \beta \left(A \frac{3}{2} \eta \eta_x + \frac{1}{6} \eta \eta_{3x}\right) = 0.
\]
(41)

Now, we aim to satisfy the Boussinesq system \( (38)-(39) \) with the terms of the second order included. Then, we set
\[
w = \eta + \beta \left(-A \frac{1}{4} \eta^2 + \frac{1}{3} \eta \eta_x\right) + \beta^2 Q.
\]
(42)

Then we insert the trial function \( (12) \) into \( (38) \) and \( (39) \) and retain terms up to second order in \( \beta \). This yields the set of two equations
\[
\eta_t + \eta_x + \beta \left(A \frac{3}{2} \eta \eta_x + \frac{1}{6} \eta \eta_{3x}\right) + \beta^2 \left(-A^2 \frac{3}{4} \eta^2 \eta_x\right)
\]
(43)
\[
+ \frac{1}{12} \eta x \eta_{2x} - \frac{1}{12} \eta \eta_{3x} + \frac{17}{360} \eta \eta_{5x} - q(h \eta)_x + Q_x = 0
\]
and
\[
\eta_t + \eta_x + \beta \left(-A \frac{1}{2} \eta \eta_x + \frac{1}{6} \eta \eta_{2x}\right)
\]
(44)
\[
+ \beta^2 \left(-A^2 \frac{3}{4} \eta^2 \eta_x - A \frac{1}{2} \eta \eta_{2x} + A \frac{3}{4} \eta \eta_{2x} + A \frac{5}{6} \eta \eta_{3x}
\]
\[
+ \frac{3}{4} \eta \eta_{2x} - \frac{1}{6} \eta \eta_{3x} - \frac{1}{8} \eta \eta_{4x} + Q_t\right) = 0.
\]

Now, we subtract the equation \( (44) \) from \( (43) \). This gives
\[
\beta \left(A \frac{1}{2} \eta \eta_x + \frac{1}{6} \eta \eta_{3x}\right)
\]
(45)
\[
+ \beta^2 \left[A \frac{1}{2} \eta \eta_x - \frac{1}{4} \eta \eta_{2x} + A \frac{3}{4} \eta \eta_{2x} + A \frac{1}{2} \eta \eta_{3x}\right]
\]
\[
+ \frac{1}{8} \eta \eta_{4x} + \frac{3}{17} \eta_{12} + Q - Q_t - q(h \eta)_x = 0.
\]

In \( (45) \), in order to replace \( t \)-derivatives by \( x \)-derivatives we use \( Q_t = -Q_x \) and the properties of the first order equation \( (11) \), that is, \( \eta_t = -\eta_x - \beta (A \frac{1}{2} \eta \eta_x + \frac{1}{6} \eta \eta_{3x}) \), again retaining only terms up to second order. Solving the result with respect to \( Q_x \) and integrating over \( x \) we find
\[
Q = \frac{1}{4} q(h \eta) + A^2 \frac{1}{4} \eta^3 + A^2 \frac{3}{16} \eta^2 + A \frac{1}{2} \eta \eta_{3x} + \frac{1}{10} \eta_{4x}.
\]
(46)

This form of the correction function makes the Boussinesq system \( (38)-(39) \) compatible and allows to derive explicit form for the wave equation for the case of \( \alpha = O(\beta) \) and \( \delta = O(\beta^2) \). Finally, we have
\[
w = \eta + \beta \left(-A^2 \frac{1}{4} \eta^2 + \frac{1}{3} \eta \eta_x\right) + \beta^2 \left(\frac{1}{2} q(h \eta)\right)
\]
(47)
\[
+ A^2 \frac{1}{4} \eta^3 + A^2 \frac{3}{16} \eta^2 + A \frac{1}{2} \eta \eta_{2x} + \frac{1}{10} \eta_{4x}
\]
and
\[
\eta_t + \eta_x + \beta \left(A \frac{3}{2} \eta \eta_x + \frac{1}{6} \eta \eta_{3x}\right) + \beta^2 \left(-\frac{1}{2} q(h \eta)_x\right)
\]
(48)
\[
- A^2 \frac{3}{8} \eta^2 \eta_x + A \frac{23}{24} \eta \eta_{3x} + A \frac{1}{2} \eta \eta_{2x} + A \frac{5}{12} \eta \eta_{3x} + \frac{19}{360} \eta_{5x} = 0.
\]
(49)

The equation \( (48) \) is the nonlinear wave equation, for uneven bottom, when \( \alpha = \beta, \delta = O(\beta^2) \), derived consistently within second order perturbation approach.

Since \( \delta = q\beta^2 \) we can come back to original notations for small parameters, used in \( 2 \). Then equations \( (47) \) and \( (48) \) take the following forms
\[
w = \eta + \frac{1}{4} \alpha \eta^2 + \frac{1}{3} \beta \eta_{2x} + \frac{1}{2} \frac{1}{2} \eta \eta_{2x} + \frac{1}{8} \alpha^2 \eta^3 + \alpha \beta \left(\frac{3}{16} \eta^2 + \frac{1}{2} \frac{1}{2} \eta \eta_{2x}\right)
\]
(50)
\[
+ \frac{1}{10} \beta^2 \eta_{4x}
\]
and
\[ t=(2k+1)\cdot 2.5 \]
\[ t=2k\cdot 2.5 \]
\[ t=(2k+1)\cdot 2.5 \]
\[ \eta_t + \eta_x + \frac{3}{2} \alpha \eta_{xx} + \frac{1}{6} \alpha \eta_{xxx} - \delta \frac{1}{2} (h\eta)_x - \frac{3}{8} \alpha^2 \eta^2 \eta_x = 0. \] (51)

These forms of equations (50) and (51) may be misleading, since the terms without \( \delta \), looking as first order ones, are, in fact, of second order.

The equation (51), limited to the case \( \delta = q = 0 \), is the extended KdV equation or KdV2 \([4]\). This equation is nonintegrable. Despite this fact, we found several forms of analytic solutions to KdV2: soliton solutions in \([2]\), cnoidal solutions \((\sim \text{cn}^2)\) in \([6]\) and superposition cnoidal solutions \((\sim \text{dn}^2 \pm \sqrt{\text{mcn} \text{dn}})\) in \([7,8]\).

The equation (51) is the second order wave equation directly taking into account bottom variation derived consistently for the case of \( \alpha = O(\beta) \) and \( \delta = O(\beta^2) \).

The wave equation (51) is very similar to the erroneous \([2\text{, Eq. (18)}]\). The latter contains, apart from the leading term from the bottom \(-\frac{1}{4} \delta (h\eta)_x\), two other terms which resulted from not fully consistent derivation.

IV. NUMERICAL TESTS

In this section, we tentatively examine the motion of appropriate solitons entering the region where the bottom is no longer even. In these tests, we use our numerical code based on the finite difference method. The code was described in detail in \([2]\).

A. The case of \( \alpha = O(\beta) \) and \( \delta = O(\beta) \)

In this part we present evolution of the KdV solitons obtained with numerical solution of the equation (52). Since this equation is valid only for \( \alpha = \beta \) and all three parameters should be of the same order we set in these test \( \alpha = \beta = \delta = 0.25 \). As a bottom function \( h(x) \) we chose a piecewise function of trapezoid shape located at \( x_1 = 5, x_2 = 10, x_3 = 20, x_4 = 25 \). Since the equation (52) is valid only for the linear bottom function \( h(x) = kx \) the trapezoidal bottom is allowed. The size and location of the trapezoid allows us also to compare the results with those presented in \([9]\). Note that the bottom function is drawn not in scale. The initial condition is the KdV soliton with the amplitude equal to 1, that is, \( \eta(x,t = 0) = \sec^2 \left( \frac{\sqrt{2} \beta}{\sqrt{m_0}} x \right) = \sec^2 \left( \sqrt{0.75} x \right) \).

In the case presented in Fig. 1 the soliton first slows down and then accelerates with the amplitude increase and decrease, respectively. In the case presented in Fig. 2 the soliton first accelerates and then slows down with the amplitude decrease and increase, respectively. Therefore in the latter case, the distance covered by the soliton at \( t = 40 \) is larger than in the former case.

B. The case of \( \alpha = O(\beta) \) and \( \delta = O(\beta^2) \)

In this case there exists KdV2 solitons, that is solitons of the equation (51) when \( \delta = 0 \), that is, for the flat bottom (see, Sect. V in \([2]\)). The amplitude of such solitons is one for \( \alpha \approx 0.2424 \). Since we compare motion of solitons with the same amplitude (equal to 1) we present below numerical solutions when \( \alpha = \beta = 0.2424 \) and \( \delta = 2\beta^2 \approx 0.1175 \). Then the initial condition is
\[ \eta(x, t = 0) = \sec^2 \left( \sqrt{0.599} x \right). \]

that this radiation still exists, but with much smaller

Distortions of the soliton shape caused by interaction with uneven bottom observed in Figs. 3 and 4 are much smaller than those in Figs. 1 and 2.

Comparison of the numerical evolution of KdV2 solitons obtained with the equation \(51\) with that resulted from the erroneous equation \(2\) Eq. (18)] shows an important difference. The radiation of small amplitude wave-train in front of the main wave, present in evolution according to \(2\) Eq. (18)] seems to dissappear in evolution according to the equation \(51\) displayed in Figs. 3 and 4.

The thorough inspection of the calculated data reveals amplitude (in Figs. 3 and 4 this amplitude is comparable to the linewidth). In order to enhance this effects we performed additional calculations in which we set \(\alpha = \beta = 0.2424\) and \(\delta = 3\beta^2 \approx 0.176\). Several profiles of the wave obtained in the numerical evolution of KdV2 soliton according to the equation \(51\) are displayed in Fig. 5. The creation and then detachment of the small amplitude wave packet in front of the main wave is clearly exposed in the insert. This is qualitatively the same feature as observed in our previous papers \(1, 2, 9\) for wave motion according to the erroneous equation \(2\) Eq. (18)]. Quantitatively the effect has much smaller amplitude, for realistic values of parameters \(\alpha, \beta, \delta\) it is smaller than 1\% of the solitons amplitude. On the other hand, even such small effect suggests the origin of the very tiny wrinkles observed always on the water surface at the seashore.

We are sure that this is the real effect, not an artifact of numerical simulation. Since our code utilizes periodic boundary conditions we performed calculations on much wider \(x\)-interval than displayed in figures above. In such cases, when the soliton moves far from the end of the \(x\)-interval, the boundary conditions do not influence the shape of the localized wave.

**ACKNOWLEDGMENTS**

We thank the author of \(5\) for his detailed explanation of consistent perturbation approach in the case of several small parameters. His remarks allowed us to derive the correct second order wave equation explicitly containing terms from the uneven bottom, for arbitrary, bounded bottom function.
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