Edge fluctuations for random normal matrix ensembles

David García-Zelada

Abstract

A famous result going back to Eric Kostlan states that the moduli of the eigenvalues of random normal matrices with radial potential are independent yet non identically distributed. This phenomenon is at the heart of the asymptotic analysis of the edge, and leads in particular to the Gumbel fluctuation of the spectral radius when the potential is quadratic. In the present work, we show that a wide variety of laws of fluctuation are possible, beyond the already known cases, including for instance Gumbel and exponential laws at unusual speeds. We study the convergence in law of the spectral radius as well as the limiting point process at the edge. Our work can also be seen as the asymptotic analysis of the edge of two-dimensional determinantal Coulomb gases and the identification of the limiting kernels.

2010 MSC: 60G55; 60F05; 60K35; 60G70

Keywords: random normal matrix; interacting particle system; extreme values; determinantal point process; Gibbs measure; Coulomb gas.

1 Introduction

We will be interested in the fluctuations of the maxima of the moduli of the eigenvalues of random normal matrices confined by a radial potential. Equivalently, these eigenvalues may be thought of as a two-dimensional radial determinantal Coulomb gas as defined in (2) below. The purpose of this article is two-fold. On the one hand, we want to present different classes of universality by describing the subtleties there may appear. In this way, we show a variety of possibilities emerging, each possibility having different interesting aspects. On the other hand, we want to convey the simplicity of the methods, which extend far beyond the examples mentioned in this article. In particular, the local limiting point process of any two-dimensional radial determinantal Coulomb gas can be easily understood as well as the local limiting point process of the norms. Furthermore, the same methods also help one study the local limiting covariance kernel for random polynomials such as the ones considered in [4]. I would like to also mention that the regularities required for the potential $V$ are minimal and that this makes the proofs even clearer by not looking at the unimportant aspects of $V$.

Let us mention some of the previous work on the fluctuations of the maxima. The first result we are aware of is the Gumbel fluctuations at speed $\sqrt{n \log n}$ obtained by Rider [21] for the farthest particle of the Ginibre ensemble. A generalization of this result can be found in the work of Chafai and Péché [5] as well as in the recent work of Ameur, Kang and Seo [1] with the same speed $\sqrt{n \log n}$. The hard-edge version has been considered by Seo [24] who showed exponential fluctuations at speed $n$. On a series of articles, [17], [14], [7], Qi and his collaborators have studied different cases related to matrix models which includes truncated circular unitary matrices and products of matrices from the spherical and from the Ginibre ensemble. A class of potentials generated by probability measures has been recently considered by Butez and the author in [4]. Fluctuations of the maxima for Coulomb gases have also attracted physicists attention as we can see, for instance, in the work of Lacroix-A-Chez-Toine, Grabsch, Majumdar and Schehr [20] where even an intermediate deviation regime is explored. The farthest particle has also been of interest for fermionic systems as in the work of Dean, Le Doussal, Majumdar and Schehr [9].
Despite these efforts, the variety of behaviors has not been really explored and the methods used up to this point have not been simple enough.

We now proceed to describe the model. Let $\mathcal{N}_n$ denote the set of $n$ by $n$ normal matrices endowed with the measure $dM$ induced by the restriction (to the regular part of $\mathcal{N}_n$) of the euclidean inner product $\langle M, N \rangle = \text{Tr}(MN^*)$ on the space of $n$ by $n$ matrices. We will be interested in the measure

$$e^{-Q(M)}dM,$$

(1)

for some continuous function $Q : \mathbb{C} \to \mathbb{R}$. By $[\mathcal{S}, \mathcal{I}, \mathcal{I}]$, if (1) has finite measure and if $M$ follows a law proportional to (1), the eigenvalues of $Q$ we are interested in the case where $C$ as a system of particles in $\mathcal{R}$.

consider a system of $n$ particles in $\mathbb{C}$ of the same charge $q$ and confined by the potential $V$. This system, known as a Coulomb gas, has the energy

$$H(x_1, \ldots, x_n) = -q^2 \sum_{i<j}^n \log |x_i - x_j| + q \sum_{i=1}^n V(|x_i|)$$

and follows the Gibbs probability measure at inverse temperature $\beta > 0$ and energy $H$, i.e. the probability measure proportional to $e^{-\beta H}d\ell_{\mathbb{C}^n}$. The case related to random normal matrices is the one where $\beta q^2 = 2$ since in this case the Gibbs probability measure is given by

$$d\mathbb{P}_n(x_1, \ldots, x_n) = \frac{1}{Z} \prod_{i<j}^n |x_i - x_j|^{2} e^{-2\kappa \sum_{i=1}^n V(|x_i|)}d\ell_{\mathbb{C}^n}(x_1, \ldots, x_n)$$

(2)

where $\kappa = 1/q$ and where $Z$ is a normalization constant. Then, the eigenvalues of a random normal matrix that follows a law proportional to (1) with $Q(z) = 2\kappa |z|^2$ can be thought of as a system of particles in $\mathbb{C}$ of charge $1/\kappa$ and at inverse temperature $\beta = 2\kappa^2$. For $\mathbb{P}_n$ to be well-defined (i.e., $Z < \infty$) we shall assume that $\kappa > n$ and that

$$\liminf_{r \to \infty} \{V(r) - \log r\} > -\infty.$$ 

To relax the condition $\kappa > n$ we can assume that the potential is strongly confining, i.e.,

$$\liminf_{r \to \infty} \frac{V(r)}{\log r} > 1$$

(3)

or, equivalently, $\liminf_{r \to \infty} \{V(r) - Q \log r\} > -\infty$ for some $Q > 1$. If (3) is not satisfied we say that the potential is weakly confining. We shall also consider the degenerate case where $V = \infty$ outside a disk which, for simplicity, we take it to be the closed unit disk centered at zero $\overline{D}$. These are examples of hard-edge systems and they are determined by a continuous function $V : [0, 1] \to \mathbb{R}$ and a positive number $\kappa$. The Gibbs probability measure would be

$$\frac{1}{Z} \prod_{i<j}^n |x_i - x_j|^{2} e^{-2\kappa \sum_{i=1}^n V(|x_i|)}d\ell_D(x_1) \ldots d\ell_D(x_n),$$

(4)

where $\ell_D$ denotes the Lebesgue measure restricted to $\overline{D}$. It may be thought of as a particular case of (2) where we let $V(r) = \infty$ for $r > 1$.

By well-known large deviation principles (see, for instance, $[\mathcal{S}$, Theorem 1.2] for an idea of the proof), if $(x_1^{(n)}, \ldots, x_n^{(n)}) \sim \mathbb{P}_n$ and $\kappa_n/n \to \infty$ (and under some conditions on $\kappa_n$ in the weakly confining case) we have that

$$\frac{1}{n} \sum_{i=1}^n \delta_{x_i^{(n)}} \underset{n \to \infty}{\overset{a.s.}{\longrightarrow}} \mu_V$$

where $\mu_V$ is a measure on $\overline{D}$.

Edge fluctuations for random normal matrices

2
where \( \mu_V \) is the probability measure that minimizes the functional

\[
\nu \mapsto \int_{\mathbb{C} \times \mathbb{C}} \left( \log \frac{1}{|x - y|} + V(|x|) + V(|y|) \right) \, d\nu(x) \, d\nu(y).
\]

Finally, let us say some words about the laws (2) and (4). The exponent 2 in \( \prod_{i<j} |x_i - x_j|^2 \) makes the system quite feasible to study. It enjoys the property of being a determinantal point process, which will be further explained in Section 3. This structure allows us to find limits of point processes by studying the limit of a function of two variables. Some examples of limits are given in Subsection 7.2 and in Proposition 2.8. Furthermore, in the study of the moduli (\(|x_1^{(n)}|, \ldots, |x_n^{(n)}|\)), an idea due to Kostlan [19, Lemma 1.4] shows that, if \( Y_0^{(n)}, \ldots, Y_{n-1}^{(n)} \) are independent random variables such that \( Y_k^{(n)} \) has a density proportional to \( r^{2k+1} e^{-2\kappa V(r)} \), then

\[
\{|x_1^{(n)}|, \ldots, |x_n^{(n)}|\} \sim \{Y_0^{(n)}, \ldots, Y_{n-1}^{(n)}\}
\]
as point processes on \([0, \infty)\). A more general statement will be recalled in Subsection 5.3. Interestingly, the asymptotic analysis of the moduli then becomes an asymptotic analysis of independent and non-identically distributed random variables.

Let us describe the content of each section.

In Section 2 we present the main results. We consider potentials whose equilibrium measures are compactly supported and we show that different behaviors for the farthest particles may arise. In Subsection 2.1 we consider potentials for which the farthest particle keeps the Coulombian behavior. This means that the limit point process at the scale considered is not Poissonian and is still related to a determinantal point process. The potentials involved may be either strongly confining or weakly confining, i.e., either (3) is true or not. In Subsection 2.2 we consider potentials for which a Gumbel fluctuation appears. One of the cases shown is a class of strongly confining potentials and the other one is a class of weakly confining potentials with the parameter \( \kappa \) growing fast to infinity. In Subsection 2.3 we consider a hard-edge potential for which exponential fluctuations at speed \( n^2 \) are obtained.

In Section 3 we give a short introduction to the determinantal structure we need together with the property that allows us to describe the cumulative distribution function of the maxima of the moduli in an explicit way. It is the generalization of a property discovered by Kostlan and rediscovered, for instance, by Fyodorov and Mehlig [12, Equation 16].

The proofs are divided in three sections. Section 4, 5 and 6 contain the proofs stated in Subsection 2.1, 2.2 and 2.3 respectively.

Finally, Section 7 contains four appendices with further information the reader might find interesting. Subsection 7.1 contains a version of Theorem 2.4 that is longer to state and where a random limiting number of particles appears. Subsection 7.2 contains the \( \alpha > 1 \) version of Theorem 2.7. Subsection 7.2 contains the limiting kernel at the edge of some systems described in Subsection 2.2 together with a remark about the relation between the limiting kernel and the Gumbel fluctuations. Subsection 7.3 explains an interesting connection between the Gumbel distribution and the limit of the maxima from Theorem 2.3.

### 2 Behaviors of the farthest particles

In this article, we will be interested in the case where \( \mu_V \) is compactly supported. Without loss of generality, we can assume that

\[
\partial \mathbb{D} \subset \operatorname{supp} \mu_V \subset \overline{\mathbb{D}},
\]

where \( \mathbb{D} \) is the open unit disk centered at zero. Recall we suppose that \( V : [0, \infty) \to \mathbb{R} \), or \( V : [0, 1] \to \mathbb{R} \) in the hard-edge case, is continuous. Using Frostman’s conditions (see [23]), the condition in (5) traduces, by adding a constant to \( V \), in the following properties for the potential.
**Definition 2.1** (Standard properties). A continuous function \( V : [0, \infty) \to \mathbb{R} \) or \( V : [0, 1] \to \mathbb{R} \) is said to satisfy the standard properties if

- \( V(r) > \log r \) when \( r < 1 \),
- \( V(r) \geq \log r \) when \( r > 1 \), and
- \( V(1) = 0 \).

Every \( V \) is assumed to satisfy these properties throughout the article. Notice that the second condition is vacuous for \( V : [0, 1] \to \mathbb{R} \), or we may say it is always true if we define \( V(r) = \infty \) for \( r > 1 \). Although this second condition will always be implied by the other conditions in the theorems, it is important to keep it in mind.

Given a sequence \( \{\kappa_n\}_{n \geq 1} \), we consider a random element \((x_1^{(n)}, \ldots, x_n^{(n)}) \sim \mathbb{P}_{\kappa_n} \), where \( \mathbb{P}_{\kappa_n} \) is defined by (2) or (4), and study

\[
M_n = \max \left( |x_1^{(n)}|, \ldots, |x_n^{(n)}| \right).
\]

More precisely, we look for two sequences of real numbers \( \{\alpha_n\}_{n \geq 1} \) and \( \{\beta_n\}_{n \geq 1} \) such that

\[\alpha_n M_n + \beta_n \]

converges in law to a non-deterministic random variable. We will see that the behavior of \( M_n \) depends on the behavior of \( V \) with respect to \( \log r \). It will also be seen to depend on the asymptotic behavior of \( \kappa_n \) or, equivalently, on the behavior of the charge of a single particle \( 1/\kappa_n \). This dependence will be explained by five examples.

- In Theorem 2.3, \( M_n \) converges to a random variable supported in a closed interval inside \([1, \infty)\). This interval depends on the behavior of \( V \) in \([1, \infty)\) but is otherwise arbitrary. For instance, the interval could be bounded away from 1 in which case an infinite number of particles stay far from the support of \( \mu_V \). Moreover, the potential can be strongly confining, in the sense of (3), while the farthest particles stay living in an annulus.

- In Theorem 2.4, \( M_n \) converges to infinity. After a proper rescaling, \( M_n \) is seen to converge to a non-deterministic random variable. Interestingly, at the same scale there are only a finite number of particles that accompany the one of modulus \( M_n \).

- Theorem 2.5 shows Gumbel fluctuations arising when the potential is strongly confining. This is stated for a family of behaviors of \( V(r) \) near \( r = 1 \). There is a compatibility, stated in Remark 7.4, between the Gumbel fluctuations and the limiting point process at the edge described in Proposition 7.2.

- Theorem 2.6 shows Gumbel fluctuations when the potential is still weakly confining but \( \kappa_n - n \to \infty \). In this case, there is no longer a compatibility between the Gumbel fluctuations and the limiting point process at the edge, described in Proposition 7.3.

- Theorem 2.7 treats the hard-edge case. The parameter for the exponential fluctuations are shown to depend on the potential near the edge but, in the particular family of potentials described, it also depends on what happens inside of the unit disk. These exponential fluctuations are obtained by using the limiting kernel at the edge from Proposition 2.8.

There are two further complementary classes of potentials described in Theorem 7.1 and Theorem 7.5 in the appendix. To complement the previous explanation, we would like to give some examples implied by the theorems. Unless otherwise stated, \( V(r) > \log r \) for \( r \neq 1 \).

We begin by giving some non-Poissonian examples. We call them Coulombian examples because they are related to some limits of determinantal point processes. These limiting point processes may be thought of as instances of Coulomb gases with an infinite number of particles.
In the left-most column of Table 1, some conditions on the potential are stated. In the second column, speed means the coefficient $\alpha_n$ necessary to obtain the convergence in law of $\alpha_nM_n$. Finally, the right-most column tells us the cumulative distribution function of the limit of $\alpha_nM_n$. For simplicity, we have chosen $\kappa_n = n + \chi$ as the inverse charge for $\chi > 0$.

| Conditions | Speed | Cumulative distribution function |
|------------|-------|---------------------------------|
| $V(r) = \log r$ only on $[R, \infty)$ | 1 | $\prod_{k=0}^{\infty} \left( 1 - \frac{R^{2k+2\chi}}{t} \right)$ |
| $V(r) = \log r$ only on $[1, R]$ | 1 | $\prod_{k=0}^{\infty} \left( 1 - \frac{t^{-2k-2\chi}}{1 - R^{-2k-2\chi}} \right)$ |
| $V(r) = \log r + \frac{1}{2} r^{-\alpha} + o(r^{-\alpha})$, $V(r) = \log r + |1 - r| + o(|1 - r|)$, $\ell := \alpha/2 - \chi \notin \mathbb{Z}$ | $n^{-1/\alpha}$ | $\prod_{k=0}^{\ell} \frac{\Gamma \left( \frac{2(k+\chi)}{\alpha}, t^{-\alpha} \right)}{\Gamma \left( \frac{2(k+\chi)}{\alpha} \right)}$ |
| $V(r) = \log r + \frac{1}{2} r^{-\alpha} + o(r^{-\alpha})$, $V(r) = \log r + |1 - r| + o(|1 - r|)$, $\ell := \alpha/2 - \chi \in \mathbb{Z}$ | $n^{-1/\alpha}$ | $\left( e^{-t^{-\alpha}} + (1 - e^{-t^{-\alpha}}) \right)^{\ell-1} \frac{\alpha}{1 + \alpha} \prod_{k=0}^{\ell-1} \frac{\Gamma \left( \frac{2(k+\chi)}{\alpha}, t^{-\alpha} \right)}{\Gamma \left( \frac{2(k+\chi)}{\alpha} \right)}$ |

The first and second examples are particular cases of Theorem 2.3. The third example is a consequence of Theorem 2.4 while the fourth example is a consequence of the random number of particles version stated in Theorem 7.1 in the appendix.

Next, we give two examples of Gumbel fluctuations. More precisely, in those cases there exist $\alpha_n$ and $\beta_n$ such that $\alpha_nM_n + \beta_n$ converges to a random variable whose cumulative distribution function is $t \in \mathbb{R} \mapsto e^{-e^{-t}}$. The left-most column of Table 2 contains the conditions on the potential, the second column contains the conditions on the inverse charge $\kappa_n$, and the right-most column contains the speed $\alpha_n$.

| Conditions | Inverse charge | Speed |
|------------|----------------|-------|
| $V(r) = \log r + \frac{1}{2\alpha} |1 - r|^{\alpha} + o(|1 - r|^{\alpha})$ | $\kappa_n = n + o \left( \frac{n}{\log n} \right)^{1/\alpha}$ | $n^{1/\alpha}(\log n)^{1-1/\alpha}$ |
| $V(r)$ as $r \to 1^-$ and $V(r) = \log r$ if $r \geq 1$ | $\kappa_n - n \to \infty$ | $2(\kappa_n - n)$ |

The first example is a particular case of Theorem 2.5 while the second example is implied by Theorem 2.6.

Finally, we give two examples of exponential fluctuations. More precisely, in those cases there exists $\alpha_n$ such that $\alpha_n(1 - M_n)$ converges to a random variable whose cumulative distribution function is $t \in [0, \infty) \mapsto 1 - e^{-at}$ for some $a > 0$. The left-most column of Table 3 contains the conditions on the potential, the second column contains the conditions on the inverse charge $\kappa_n$, and the right-most column tells us if the parameter $a$ of the exponential depends only on the conditions of the potential stated on the first column or not.

| Conditions | Inverse charge | Speed |
|------------|----------------|-------|
| $V(r) = \log r + \frac{1}{2\alpha} |1 - r|^{\alpha} + o(|1 - r|^{\alpha})$ | $\kappa_n = n + o \left( \frac{n}{\log n} \right)^{1/\alpha}$ | $n^{1/\alpha}(\log n)^{1-1/\alpha}$ |
| $V(r)$ as $r \to 1^-$ and $V(r) = \log r$ if $r \geq 1$ | $\kappa_n - n \to \infty$ | $2(\kappa_n - n)$ |
Table 3: Exponential examples

| Conditions                                                                 | Inverse charge     | Speed           | Parameter | Conditions                                                                 |
|----------------------------------------------------------------------------|--------------------|-----------------|-----------|----------------------------------------------------------------------------|
| $V(r) = \log r + (1 - r) + o(1 - r)$ as $r \to 1^-$ and $V(r) = \infty$ if $r > 1$ | No condition       | $n^2$           | Depends on $V$ inside $[0, 1]$                                             |
| $V(r) = \log r + \frac{1}{2r^\alpha}(1 - r)^\alpha + o(1 - r)$ as $r \to 1^-$ and $V(r) = \infty$ if $r > 1$ | $\kappa_n = n + o(n^{1/\alpha})$ | $n^{2/\alpha}$ | Universal                                                                 |

Recall that $V(r) = \infty$ if $r > 1$ means we are dealing with the hard-edge case and $V$ is actually a continuous function on $[0, 1]$. The first example of Table 3 can be found in Theorem 2.7 while the second example is found in Theorem 7.3 in the appendix.

**Remark 2.2 (Equivalent statement).** In every theorem we will have some $R \in [-\infty, \infty)$ and some $\alpha_n$ and $\beta_n$ such that $\{\alpha_n|x_1^{(n)}| + \beta_n, \ldots, \alpha_n|x_n^{(n)}| + \beta_n\} \cap (R, \infty)$ converges to some point process $X$ together with the convergence of $\alpha_nM_n + \beta_n$ towards the maximum of $X$. Nevertheless, both together imply and are implied by the following statement. For every $f : \mathbb{R} \to \mathbb{R}$ whose support is contained in $(R, \infty)$,

$$\sum_{i=1}^n f(\alpha_n|x_i^{(n)}| + \beta_n) \xrightarrow{\text{law}} \sum_{x \in X} f(x).$$

### 2.1 Coulombian behavior

In both of the theorems presented here $V(r) - \log r$ touches zero away from $r = 1$ and in a non trivial way. Their proofs, given in Section 4, are motivated by the methods used in [4]. Even though the proofs will give the behavior of the point process $\{x_1^{(n)}, \ldots, x_n^{(n)}\}$, the theorems will only be stated for the point process of the moduli $\{|x_1^{(n)}|, \ldots, |x_n^{(n)}|\}$. The behavior of the point process $\{x_1^{(n)}, \ldots, x_n^{(n)}\}$ may be obtained from Proposition 3.2 and Proposition 3.3.

The main interesting aspect of the following theorem is that, despite the fact that the support of $\mu \nu$ is contained in the closed unit disk, the limit of the particle farthest from the origin may live in an annulus at a finite distance from the unit disk. Moreover, in the limit, there will be an infinite number of particles living in such complement and they will only accumulate at the inner boundary of the annulus.

Notice that the following result involves a potential that does not need to be weakly confining. It is an example where the fact of being strongly confining, in our sense, does not immediately imply a Gumbel fluctuation of the farthest particle.

**Theorem 2.3 (Particles in an annulus).** Suppose $V$ satisfies the standard properties and

- $V(r) > \log r$ for every $r \in (1, R) \cup (\tilde{R}, \infty)$ and
- $V(r) = \log r$ for every $r \in [R, \tilde{R})$,

for some $R \in [1, \infty)$ and $\tilde{R} \in (1, \infty]$ such that $R < \tilde{R}$. Suppose that, for some $\chi > 0$,

$$\kappa_n = n + \chi + o(1).$$

Consider a sequence $\{Y_k\}_{k \geq 0}$ of independent random variables taking values in $[R, \tilde{R})$ such that $Y_k$ has a density proportional to $r \in [R, \tilde{R}) \mapsto r^{-2(k+\chi)-1}$. Then, as point processes on $(R, \infty)$,

$$\{|x_i^{(n)}| : 1 \leq i \leq n\} \cap (R, \infty) \xrightarrow{\text{law}} \{Y_k : k \geq 0\}.$$
Furthermore, the maximum of $|x_1^{(n)}|, \ldots, |x_n^{(n)}|$ converges in law to the maximum of $\{Y_k\}_{k \geq 0}$. More explicitly, for every $t \in (R, \tilde{R})$,

$$
\lim_{n \to \infty} \mathbb{P} (M_n \leq t) = \prod_{k=0}^{\infty} \left( \frac{R^{2(1+\chi)} - t^{-2(1+\chi)}}{R^{2(1+\chi)} - \tilde{R}^{2(1+\chi)}} \right) .
$$

The limiting point process for $\tilde{R} < \infty$ can be thought of as a conditioned version of the limiting point process for $R = \infty$ and, by a scaling, it can be taken to the limiting point process for $R = 1$. For $\chi \in (0, 1]$ the latter is related to a weighted Bergman kernel of $\mathbb{C} \setminus \overline{D}$ while for $\chi > 1$ it would be related to a truncated version of it as can be seen from the results of Proposition 4.2. We may see [15] for more information on Bergman kernels. In Proposition 7.6 from the appendix we will describe the behavior as $\chi$ goes to infinity of the limit of the maxima obtained in Theorem 2.3.

The main aspect of the next theorem is that we will only see a finite number of particles at the same scale of the particle farthest from the origin. There is a version where the number of particles is random and is stated in Theorem 7.1 for convenience of the reader. The conditions satisfied by $V$ imply that $\mu_V(\mathbb{D}) < 1$ so that $\mu_V$ must give non-zero charge to $\partial \mathbb{D}$ but it may still be charged near $\partial \mathbb{D}$.

**Theorem 2.4** (Finite limiting point process). *Suppose $V$ satisfies the standard properties and

- $V(r) > \log r$ for every $r > 1$,
- $\lim_{r \to \infty} r^\alpha (V(r) - \log r) = \gamma$,
- $\lim_{r \to 1^+} \frac{V(r)}{r - 1} > 1$ and $\lim_{r \to 1^-} \frac{V(r)}{r - 1} < 1$,

for some positive numbers $\alpha, \gamma > 0$. Suppose that, for some $\chi \in (0, \alpha/2)$ such that $\alpha/2 - \chi \notin \mathbb{Z}$,

$$
\kappa_n = n + \chi + o(1).
$$

Consider a sequence $\{Y_k\}_{k \geq 0}$ of positive independent random variables such that $Y_k$ has a density proportional to $r \in (0, \infty) \mapsto r^{-2(1+\chi)}e^{-r^{1-\alpha}}$. Then, as point processes on $(0, \infty)$,

$$
\{n^{-1/\alpha}|x_i^{(n)}| : 1 \leq i \leq n\} \overset{\text{law}}{\to} \{(2\gamma)^{1/\alpha}Y_k : 0 \leq k \leq \lfloor \alpha/2 - \chi \rfloor\}.
$$

Furthermore, $n^{-1/\alpha} \max\{|x_1^{(n)}|, \ldots, |x_n^{(n)}|\}$ converges in law to the maximum of $\{Y_k\}_{0 \leq k \leq \lfloor \alpha/2 - \chi \rfloor}$. More explicitly, for every $t > 0$,

$$
\lim_{n \to \infty} \mathbb{P} \left(n^{-1/\alpha} M_n \leq t\right) = \prod_{k=0}^{\lfloor \alpha/2 - \chi \rfloor} \frac{\Gamma \left( \frac{2(1+\chi)}{\alpha}, 2\gamma t^{-\alpha} \right)}{\Gamma \left( \frac{2(1+\chi)}{\alpha} \right)} ,
$$

where $\Gamma$ with two parameters is the upper incomplete gamma function.

If $\chi = 1$, the limit point process obtained in Theorem 2.4 is related to a system of particles in $\mathbb{C}$ with density proportional to $\prod_{i<j} |x_i - x_j|e^{-\sum_{i=1}^n |x_i|^\alpha}$. For general $\chi > 0$ the potential has an additional term. See Proposition 4.4.

**2.2 Gumbel behavior**

Here we consider two theorems where Gumbel fluctuations of the maxima of the moduli appear. They are proved in Section 5. The first theorem treats potentials that are strongly confining including the quadratic potential treated in [21] as a particular case. The second theorem treats
what we have called weakly confining potentials but where \( \kappa_n - n \to \infty \) so that Theorem 2.3 no longer applies. At the end of this section, we give some intuition on the conditions of the theorems by using potentials generated by radial positive measures.

There is an interesting connection between the Gumbel fluctuations and the local limit of these Coulomb gases at the unit circle that only holds for \( \xi = 0 \) from Theorem 2.5. Since it is not essential to obtain the Gumbel fluctuations, we have stated this in Subsection 7.2 in the appendix. By somewhat simpler calculations than the ones used to obtain the Gumbel fluctuations, we obtain the limiting kernel at the edge (Proposition 7.2 and Proposition 7.3) and in Remark 7.4 we mention its connection with the Gumbel fluctuations when there is one.

**Theorem 2.5** (Gumbel fluctuations for strongly confining potentials). Suppose \( V \) satisfies the standard properties and

- \( V(r) = \log r + \frac{\lambda_1}{\alpha}(r - 1) + o(r - 1)^{\alpha+\varepsilon} \) as \( r \to 1^+ \),
- \( V(r) = \log r + \frac{\lambda_1}{\alpha}(1 - r) + o(1 - r)^\alpha \) as \( r \to 1^- \),
- \( V(r) > \log r \) for every \( r > 1 \) and
- \( \liminf_{r \to \infty} \frac{V(r)}{\log r} > 1 \) (strongly confining),

for some positive numbers \( \varepsilon, \lambda_1, \lambda_\infty > 0 \) and \( \alpha \geq 1 \). Suppose that, for some \( \xi \in \mathbb{R} \),

\[
\kappa_n = n + \xi \left( \frac{2\lambda_1 n}{\log n} \right)^{1/\alpha} + o \left( \frac{n}{\log n} \right)^{1/\alpha}.
\]

Let \( \Delta_n \) be the unique solution to \( e^{\Delta_n/\alpha} \Delta_n = n^{1/\alpha} \) and define

\[
\delta_n = \left( \frac{\Delta_n}{2\lambda_1 \kappa_n} \right)^{1/\alpha} \quad \text{and} \quad A = \left( \frac{2}{\alpha} \right)^{1-1/\alpha} \Gamma \left( \frac{1}{\alpha} \right) \left( \frac{1}{\lambda_1^{1/\alpha}} + \frac{1}{\lambda_\infty^{1/\alpha}} \right).
\]

Then, as point processes on \( \mathbb{R} \),

\[
\{ \delta_n^{-1} \Delta_n \left( x^{(n)}_i - 1 - \delta_n \right) : 1 \leq i \leq n \} \overset{\text{law}}{\to} \mathcal{P},
\]

where \( \mathcal{P} \) is a Poisson point process on \( \mathbb{R} \) with intensity \( A^{-1} e^{-2\xi} e^{-s} \). Furthermore,

\[
\lim_{n \to \infty} \mathbb{P} \left( \delta_n^{-1} \Delta_n (M_n - 1 - \delta_n) \leq t \right) = \exp \left( -A^{-1} e^{-t-2\xi} \right)
\]

for every \( t \in \mathbb{R} \).

The \( \varepsilon \) in condition (6) is essential, i.e., the theorem would not hold if we take \( \varepsilon = 0 \) in (6). In that case, we would have to look at more details about the behavior of \( V \) near 1.

Let us say a few words about the coefficients. We can replace \( \Delta_n \) by any sequence \( \{ \Delta_n \}_{n \geq 1} \) that satisfies

\[
e^{\Delta_n/\alpha} \Delta_n \sim n^{1/\alpha}.
\]

A simpler version of \( \Delta_n \) would be

\[
\Delta_n = \log n - \alpha \log \log n
\]

but we preferred to use the solution of \( e^{\Delta_n/\alpha} \Delta_n = n^{1/\alpha} \) to emphasize the importance of \( \alpha \). Moreover, replacing \( \delta_n^{-1} \Delta_n \) by \( (2\lambda_1)^{1/\alpha} n^{1/\alpha} (\log n)^{1-1/\alpha} \) in the multiplicative coefficient we have, for instance,

\[
\lim_{n \to \infty} \mathbb{P} \left( (2\lambda_1)^{1/\alpha} n^{1/\alpha} (\log n)^{1-1/\alpha} (M_n - 1 - \delta_n) \leq t \right) = \exp \left( -A^{-1} e^{-t-2\xi} \right).
\]
Nevertheless, we have chosen to write \( \delta_n^{-1} \Delta_n \) for simplicity of notation. Finally, if \( \alpha > 1 \) we could have defined \( \delta_n = \left( \frac{\alpha}{2\pi n} \right)^{1/\alpha} \) instead of \( \left( \frac{\alpha}{2\pi s_{\nu}} \right)^{1/\alpha} \) due to the conditions on \( \kappa_n \).

The interest of the following theorem is that it involves very weakly confining potentials so that the coefficients cannot be guessed from the limiting kernel at the edge. See Proposition 7.3 and Remark 7.4 in the appendix.

**Theorem 2.6** (Gumbel fluctuations for weakly confining potentials). Suppose \( V \) satisfies the standard properties and

- \( V(r) = \log r + \frac{\lambda}{\alpha}(1 - r)^\alpha + o(1 - r)^\alpha \) as \( r \to 1^- \) and
- \( V(r) = \log r \) for every \( r \geq 1 \),

for some \( \lambda > 0 \) and \( \alpha \geq 1 \). Suppose that \( \kappa_n > n \) satisfies \( \kappa_n/n \to 1 \) and that

\[
\lim_{n \to \infty} (\kappa_n - n) = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{\kappa_n - n}{n^{1/\alpha}} = c \in [0, \infty).
\]

Let \( \Delta_n \) be the unique solution to \( e^{\Delta_n} \Delta_n = \kappa_n - n \) and define

\[
A = c \left( \frac{2}{\alpha} \right)^{1-1/\alpha} \Gamma \left( \frac{1}{\alpha} \right) \left( \frac{1}{\lambda^{1/\alpha}} \right).
\]

Then, as point processes on \( \mathbb{R} \),

\[
\left\{ 2(\kappa_n - n) \left( |x_i^{(n)}| - 1 - \frac{\Delta_n}{2(\kappa_n - n)} \right) : 1 \leq i \leq n \right\} \xrightarrow{\text{law}} P,
\]

where \( P \) is a Poisson point process on \( \mathbb{R} \) with intensity \( (1 + A)^{-1} e^{-s} ds \). Furthermore,

\[
\lim_{n \to \infty} P \left( 2(\kappa_n - n) \left( M_n - 1 - \frac{\Delta_n}{2(\kappa_n - n)} \right) \leq t \right) = \exp \left( -(1 + A)^{-1} e^{-t} \right)
\]

for every \( t \in \mathbb{R} \).

Notice that if \( \alpha = 1 \) we must have \( c = 0 \). As in the case of Theorem 2.5 we could have chosen \( \Delta_n = \log(\kappa_n - n) - \log \log(\kappa_n - n) \) but we preferred to emphasize the property \( e^{\Delta_n} \Delta_n \sim \kappa_n - n \) by using \( \Delta_n \) that satisfies exactly \( e^{\Delta_n} \Delta_n = \kappa_n - n \).

To make the conditions on Theorem 2.5 and on Theorem 2.6 more intuitive we compare them to the following setting. Let \( \nu \) be a radial locally finite positive measure on \( \mathbb{C} \) and define

\[
V^\nu(r) = \int_0^1 \frac{\nu(D_s)}{s} ds,
\]

where \( D_s \) denotes the open disk centered at zero and of radius \( s \). Notice that \( \Delta (V^\nu(\cdot \cdot | \\cdot )) = 2\pi \nu \) which tells us that \( V^\nu \) has the right of being called an electrostatic potential of \(-\nu\). Suppose that \( \nu(\overline{D}) = 1 \) and \( \nu(D_r) < 1 \) for every \( r < 1 \). Then (10) implies that \( V^\nu \) satisfies the standard properties from Definition 2.1. Furthermore, up to an additive constant, we have the well-known relation in physics \( V^\nu(|x|) = \int_{\overline{D}} \log |x - y| d\nu(y) \) for every \( x \in D \). This can be obtained either by noticing that both functions have the same Laplacians or by Furber’s theorem. See, for instance, [4] Lemma 4.1. Assume \( V^\nu(0) > -\infty \) for \( V^\nu \) to be continuous. Then, Frostman’s conditions implies that the equilibrium measure \( \mu_{V^\nu} \) is \( \nu(\cdot) \). Using (10) we get that

\[
\lim_{r \to 1^-} \frac{1 - \nu(D_r)}{(1 - r)^{\alpha - 1}} = \lambda > 0 \quad \text{implies} \quad V^\nu(r) = \log r + \frac{\lambda}{\alpha}(1 - r)^\alpha + o(1 - r)^\alpha \text{ as } r \to 1^-.
\]
so that the regularity of $\mu V$ at the unit circle would tell us some behavior of $V^\nu(r)$ near $r = 1$. If, in addition, $\nu(C) = 1$ then (10) implies that $V(r) = \log r$ for $r \geq 1$. These are precisely the conditions of Theorem 2.6. Moreover, by using (10) we see that
\[ \nu(D_r) = 1 + \lambda (r - 1)^{\alpha - 1} + o(r - 1)^{\alpha - 1 + \varepsilon} \quad \text{implies} \quad V^\nu(r) = \log r + \frac{\lambda}{\alpha}(r - 1)^\alpha + o(r - 1)^{\alpha + \varepsilon} \]
so that the conditions of Theorem 2.3 appear from the regularity of $\nu$ at the unit circle. A famous example is the quadratic potential that appears also in the Ginibre ensemble. We may obtain it by choosing $\nu = t \Sigma / \pi$ so that $V^\nu(r) = \int_1^r \frac{\nu(D_s)}{s} ds = (r^2 - 1)/2$ and $\alpha = \lambda = 2$.

### 2.3 Exponential behavior

Here the system of particles is restricted to live in the unit disk. The fluctuations obtained will be exponential and the methods used are motivated by [24]. A peculiarity of the following case is that the parameter of the exponential not only depends on what happens at the edge but it also depends on what happens inside the unit disk. Moreover, the conditions on $V$ imply that $\mu V(\partial D) > 0$ so that the scale $n^2$ needed to obtain the fluctuations of the maxima is no longer guessable from $\mu V$. This is in contrast to the case where the potential satisfies (7) for $\alpha > 1$. In this case, the scale $n^{2/\alpha}$ may be guessed from the behavior of $\mu V$ near the unit circle. We have stated the $\alpha > 1$ version as an appendix in Subsection 7.3 for convenience of the reader.

In the following theorem, $V$ will be a continuous function on $[0, 1]$.

**Theorem 2.7 (Hard-edge potential).** Suppose $V$ satisfies the standard properties and
\[ \lim_{r \to 1^-} \frac{V(r)}{r - 1} = 1 - q, \]
for some $q > 0$. Suppose that $\kappa_n/n \to 1$. Define
\[ Q = \max\{p \geq 0 : V(s) \geq (1 - p) \log s \text{ for all } s \leq 1\} \]
and $A = q^2 - (q - Q)^2$. Then, as point processes on $[0, \infty)$,
\[ \left\{ n^2 (1 - |x_i^{(n)}|) : 1 \leq i \leq n \right\} \xrightarrow{\text{law}} \mathcal{P}, \]
where $\mathcal{P}$ is a Poisson point process on $[0, \infty)$ with intensity $A$. In particular, for every $t \geq 0$,
\[ \mathbb{P}\left(n^2 (1 - M_n) \leq t\right) \to 1 - e^{-At}. \]

Following the ideas used in [24], the proof will involve a limit kernel calculation at the edge, which is is stated in the following proposition. A version for $\alpha > 1$ can be obtained by the same methods which can be seen from the proof of Theorem 7.7 in the appendix. By defining
\[ a_k^{(n)} = \left(2\pi \int_0^1 s^{2k+1} e^{-2\kappa_n V(s)} ds\right)^{-1} \]
for $k \in \{0, \ldots, n - 1\}$, we have the following.

**Proposition 2.8 (Hard-edge kernel).** Under the conditions of Theorem 2.7,
\[ \lim_{n \to \infty} \frac{1}{n^2} \sum_{k=0}^{n-1} a_k^{(n)} \left(1 + \frac{z}{n}\right)^k \left(1 + \frac{\bar{w}}{n}\right)^k = e^{z + \bar{w}} \int_0^Q \frac{q - t}{\pi} e^{-t(z + \bar{w})} dt \]
for $(z, w)$ uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}$. 

Notice that $0 < Q \leq q$. There is a nice, physical, interpretation for $q$ and $Q$. If there exists some signed measure $\mu$ that has $V$ as its potential, i.e., such that

$$V(r) = \int_1^r \frac{\mu(D_s)}{s} ds$$

then we can see that

$$1 - q = \mu(\mathbb{D}) \quad \text{while} \quad 1 - Q = \mu_V(\mathbb{D}).$$

The latter equality is a consequence of Frostman’s conditions.

**Remark 2.9** (Distance to the unit circle). By the same method used to prove Theorem 2.7, the limiting kernel of Proposition 7.2 or Proposition 7.3 allows us to find a Poissonian local limit for the process of moduli at $r = 1$ in cases that are not hard-edge systems. Nevertheless, since the point processes obtained in those cases are Poissonian on $\mathbb{R}$ with constant intensity, they give us no information on the maxima of the moduli.

## 3 Determinantal structure

The main important structure is the one of a determinantal point process that we recall in this section. We begin in Subsection 3.1 by explaining the case of two-dimensional radial determinantal Coulomb gases. Then, in Subsection 3.2 we recall the definition of point process and recall the definition and some properties of general determinantal determinantal point process. Finally, in Subsection 3.3 we state Kostlan’s idea about the description of the process of moduli which will be essential in the proofs.

### 3.1 Determinantal radial Coulomb gases

Let $n$ be a positive integer and let $\sigma$ be a finite radial positive measure on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} |z|^{2k} d\sigma(z) < \infty \quad \text{for} \quad k \leq n - 1.$$  

Then, $Z = \int_{\mathbb{C}^n} \prod_{i<j} |x_i - x_j|^2 d\sigma^\otimes(n)(x_1, \ldots, x_n) < \infty$. Moreover, if we define

$$K(z, w) = \sum_{k=0}^{n-1} a_k z^k w^k \quad \text{where} \quad a_k = \left( \int_{\mathbb{C}} |z|^{2k} d\sigma(z) \right)^{-1},$$

we have that

$$\frac{1}{n!} \det (K(x_i, x_j)_{i, j \in \{1, \ldots, n\}}) = \frac{1}{Z} \prod_{i<j} |x_i - x_j|^2.$$

Furthermore, since

$$\int_{\mathbb{C}} K(x, y) K(y, z) d\sigma(y) = K(x, z) \quad \text{and} \quad \int_{\mathbb{C}} K(x, x) d\sigma(x) = n,$$

we can see that

$$\int_{\mathbb{C}} \det (K(x_i, x_j)_{i, j \in \{1, \ldots, k\}}) d\sigma(x_k) = (n - k + 1) \det (K(x_i, x_j)_{i, j \in \{1, \ldots, k-1\}})$$

which implies the following property. If $(X_1, \ldots, X_n) \sim \frac{1}{Z} \prod_{i<j} |x_i - x_j|^2 d\sigma^\otimes(n)(x_1, \ldots, x_n)$ then, for every bounded measurable function $f : \mathbb{C} \to \mathbb{R}$,

$$\mathbb{E} \left[ \prod_{i=1}^n (1 + f(X_i)) \right] = \sum_{k=0}^n \frac{1}{k!} \int_{\mathbb{C}^k} \det (K(x_i, x_j)_{i, j \in \{1, \ldots, k\}}) \prod_{i=1}^k f(x_i) d\sigma^\otimes_k(x_1, \ldots, x_k),$$

where the sum is actually finite since $\det (K(x_i, x_j)_{i, j \in \{1, \ldots, k\}}) = 0$ for $k > n$. In this case, we say that $\{X_1, \ldots, X_n\}$ is a determinantal point process with kernel $K$ with respect to $\sigma$. 
3.2 General determinantal point processes

Let us recall what we mean by a point process or a random configuration of points. Let $M$ be a locally compact Polish space and let $C_M$ be the space of positive measures that take integer values on compact sets. Every $\mathcal{X} \in C_M$ can be written as

$$\mathcal{X} = \sum_{\lambda \in \Lambda} \delta_{x_{\lambda}}$$

for some locally finite family $\{x_{\lambda}\}_{\lambda \in \Lambda}$ of elements of $M$. We shall usually think $\mathcal{X}$ as a multiset and write $\mathcal{X} = \{x_{\lambda} : \lambda \in \Lambda\}$. For any compactly supported measurable function $f : M \to \mathbb{R}$ we may define $\hat{f} : C_M \to \mathbb{R}$ by

$$\hat{f}(\mathcal{X}) = \int_M f \, d\mathcal{X} = \sum_{\lambda \in \Lambda} f(x_{\lambda}).$$

Then, $C_M$ will be endowed with the smallest topology that makes $\hat{f}$ continuous for every compactly supported continuous function $f : M \to \mathbb{R}$. A point process will be a random element of $C_M$ and the weak convergence or convergence in law of point processes will be well-defined.

Now, to define the notion of determinantal point process, let us endow $M$ with a locally finite measure $\sigma$ and let us consider a continuous Hermitian function $K : M \times M \to \mathbb{C}$, where Hermitian means that $K(x,y) = \overline{K(y,x)}$ for every $x, y \in M$. A point process $\mathcal{X}$ of $M$ is called a determinantal point process with kernel $K$ with respect to the reference measure $\sigma$ if, for every compactly supported continuous function $f : M \to \mathbb{R}$,

$$\mathbb{E} \left[ \prod_{x \in \mathcal{X}} (1 + f(x)) \right] = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{M^k} \det(K(x_i, x_j)_{i,j \in \{1, \ldots, k\}}) \prod_{i=1}^{k} f(x_i) \, d\sigma^\otimes k(x_1, \ldots, x_k). \quad (11)$$

In fact, we may relax the continuity and the Hermiticity condition of $K$ but it will not be really necessary for our purposes. We refer to [2] and [26] for more information on determinantal point processes.

There are three aspects we wish to remark.

- The following freedom on $K$ and $\sigma$ will be useful to consider. Notice that if $\mathcal{X}$ is a determinantal point process with kernel $K$ with respect to $\sigma$ and if $H : M \to \mathbb{C} \setminus \{0\}$ is continuous, then $\mathcal{X}$ is a determinantal point process with kernel $\tilde{K}$ and reference measure $\tilde{\sigma}$ given by

$$\tilde{K}(x, y) = H(x)K(x, y)\overline{H(y)} \quad \text{and} \quad d\tilde{\sigma}(x) = |H(x)|^{-2} d\sigma(x).$$

This will be used in the case of radial determinantal Coulomb gases to move the potential term from $\sigma$ to $K$ so that we may choose the Lebesgue measure or any other convenient measure as a reference measure.

- The defining property (11) allows us to perform many calculations. In particular, it allows us to write

$$\mathbb{E} \left[ \sum_{x \in \mathcal{X}} f(x) \right] = \int_M K(x, x) \, d\sigma(x).$$

The function $\rho(x) = K(x, x)$ is known as the intensity function or the first correlation function. The determinants $(x_1, \ldots, x_k) \mapsto \det(K(x_i, x_j)_{i,j \in \{1, \ldots, k\}})$ give the so-called $k$-correlation functions. Since they will not be used here, we refer to [2] for more information.

- One of the most interesting aspects of determinantal point process is the continuity of the map that takes a kernel $K$ to the law of a point process. More precisely, if for each $n$ we are given a determinantal point process $\mathcal{X}_n$ with kernel $K_n$ with respect to the reference
measure $\sigma$ and if $\mathcal{X}$ is a determinantal point process with kernel $K_n$ with respect to the same reference measure $\sigma$, \cite[Proposition 3.10]{20} tells us that

$$K_n \xrightarrow{\text{compact-open}} K \quad \Rightarrow \quad \mathcal{X}_n \xrightarrow{\text{law}} \mathcal{X},$$

where compact – open means that the convergence is in the compact-open topology or, equivalently, that it is uniform on compact sets of $M \times M$.

### 3.3 Process of the moduli and Kostlan’s idea

Now, suppose that $M$ is a subset of $\mathbb{C}$ and suppose that $M$ and $\sigma$ are invariant under rotations. Given $I \subset \mathbb{Z}$, let $K$ be the kernel of the orthogonal projection from $L^2(M, \sigma)$ onto the closure of the subspace generated by $\{z^k\}_{k \in I}$, where we have assumed that $z^k \in L^2(M, \sigma)$ for every $k \in I$. To be more precise,

$$K(z, w) = \sum_{k \in I} a_k z^k \bar{w}^k, \quad \text{where } a_k = \left(\int_{\mathbb{C}} |z|^{2k} d\sigma(z)\right)^{-1}.$$ 

Suppose that $\mathcal{X}$ is a determinantal point process with kernel $K$ with respect to $\sigma$. Then, an explicit calculation such as the one in \cite[Theorem 4.7.1]{2} or \cite[Theorem 1.2]{5} shows that, if $f : M \to \mathbb{R}$ is invariant under rotations,

$$\mathbb{E} \left[ \prod_{x \in \mathcal{X}} (1 + f(x)) \right] = \prod_{k \in I} \left(1 + a_k \int_{\mathbb{C}} f(z)|z|^{2k} d\sigma(z)\right).$$

In particular, if $\{\tilde{Y}_k\}_{k \in I}$ are independent random variables on $M$ such that $\tilde{Y}_k \sim a_k |z|^{2k} d\sigma(z)$, we have the equality in law

$$\{|x| : x \in \mathcal{X}\} \sim \{|	ilde{Y}_k| : k \in I\},$$

where we are thinking both as point processes in $[0, \infty)$. This is the main property used in this article and it was initially considered by Kostlan in \cite{19}. The extension where $K$ is not an orthogonal projection is described in \cite[Theorem 4.7.1]{2} and will be particularly useful in the proof of Theorem \cite[3.1]{7} in the appendix.

For a more concrete example, we may consider the case from Subsection 3.1 where $\sigma$ is defined by $d\sigma(z) = e^{-V(|z|)} d\sigma_C(z)$ for some continuous function $V : [0, \infty) \to \mathbb{R}$. If $(X_1, \ldots, X_n)$ has a density proportional to $\prod_{i<j} |x_i - x_j|^2 e^{-2 \sum_{i=1}^n V(|x_i|)}$, then $(X_1, \ldots, X_n)$ is a determinantal point process with kernel

$$K(z, w) = \frac{1}{2\pi} \sum_{k=0}^{n-1} b_k z^k \bar{w}^k e^{-V(z)} e^{-V(w)}, \quad \text{where } b_k = \left(\int_0^\infty r^{2k+1} e^{-2V(r)} dr\right)^{-1},$$

with respect to Lebesgue measure. By taking $n$ positive random variables $\{Y_k\}_{0 \leq k \leq n-1}$ such that $Y_k \sim b_k r^{2k+1} e^{-2V(r)} dr$, we have that

$$\{|X_1|, \ldots, |X_n|\} \sim \{Y_k : 0 \leq k \leq n-1\}.$$ 

We may quickly see, for instance, that both point processes have the same intensity

$$\bar{\rho}(r) = \sum_{k=0}^{n-1} b_k r^{2k+1} e^{-2V(r)}$$

with respect to Lebesgue measure on $[0, \infty)$, i.e., that $\mathbb{E}[\sum_{k=0}^{n-1} f(Y_k)]$ and $\mathbb{E}[\sum_{i=1}^n f(X_i)]$ equal $\int_0^\infty f(r)\bar{\rho}(r) dr$ for every bounded measurable function $f : [0, \infty) \to \mathbb{R}$. 

4 Proofs of the Coulombian behavior

To prove Theorem 2.3 and Theorem 2.4 we will take advantage of the continuity property stated in [12]. It involves an understanding of the limiting point processes at the correct scale and in the correct region of \( \mathbb{C} \). Nevertheless, using the notation of Subsection 3.2 since the map \( \sup: \mathcal{C}_{(0,\infty)} \to [0,\infty] \) that takes a configuration of points to its supremum is not continuous, a point process convergence will not imply the convergence in law of the maxima. For this reason, it is more convenient to consider the inverted model and to use the map inf : \( \mathcal{C}_{(0,\infty)} \to [0,\infty] \) that takes a configuration of points to its infimum, which is continuous. Behind this simple reasoning is the idea that the maximum and the minimum are indistinguishable on the sphere: one can take one to the other by a rotation. The rotation used will be precisely the map \( z \mapsto 1/z \) in the complex plane. This idea was used in [4] and it is motivated by an equivariant property of the Coulomb gas model which in turn is motivated by the regular case where the Laplacian of \( V \) is thought of as a \((1,1)\)-form and \( e^{-2V} \) is thought of as a metric on the line bundle of degree one on the sphere. These objects can be found in the work of Berman [3] who consider analogous processes on complex manifolds. We emphasize that no complex geometry is needed in this article but that the ideas fit nicely in that context.

Let me begin by stating the following equivariant property of Coulomb gases.

**Lemma 4.1** (Inversion of Coulomb gases). Let \( V : \mathbb{C} \to (-\infty,\infty] \) be a measurable function and let \( \chi > 0 \). Define \( \tilde{V} : \mathbb{C} \setminus \{0\} \to (-\infty,\infty] \) by

\[
\tilde{V}(x) = V\left(\frac{1}{x}\right) + \log |x|.
\]

Then, the image of the measure

\[
\prod_{i<j}^{n} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{n} V(x_i)} d\ell_{\mathbb{C}}(x_1) \ldots d\ell_{\mathbb{C}}(x_n)
\]

under the application \((x_1, \ldots, x_n) \mapsto (1/x_1, \ldots, 1/x_n)\) is the measure

\[
\prod_{i<j}^{n} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{n} \tilde{V}(x_i)} d\Lambda_{\chi}(x_1) \ldots d\Lambda_{\chi}(x_n)
\]

where

\[
d\Lambda_{\chi}(x) = |x|^{2(\chi-1)} d\ell_{\mathbb{C}}(x)
\]

**Proof.** To avoid possible mistakes, we divide the change of variables in two steps. Consider the function \( G^{V} : \mathbb{C} \setminus \{0\} \times \mathbb{C} \setminus \{0\} \to (-\infty,\infty] \) and the positive measure \( \pi \) defined by

\[
G^{V}(x,y) = -\log |x-y| + V(x) + V(y) \quad \text{and} \quad d\pi = e^{-2(\chi+1)V} d\ell_{\mathbb{C}}.
\]

Then we may write

\[
\prod_{i<j}^{n} |x_i - x_j|^2 e^{-2(n+\chi) \sum_{i=1}^{n} V(x_i)} d\ell_{\mathbb{C}}(x_1) \ldots d\ell_{\mathbb{C}}(x_n)
\]

\[
= \exp \left( -2 \left[ - \sum_{i<j}^{n} \log |x_i - x_j| + (n + \chi) \sum_{i=1}^{n} V(x_i) \right] \right) d\ell_{\mathbb{C}}(x_1) \ldots d\ell_{\mathbb{C}}(x_n)
\]

\[
= e^{-2\sum_{i<j}^{n} G^{V}(x_i,x_j)} d\pi(x_1) \ldots d\pi(x_n).
\]

It is enough, then, to notice that the image of \( G^{V} \) and \( \pi \) under the inversion are \( \tilde{G}^{V} \) and \( \tilde{\pi} \), respectively, defined by

\[
G^{\tilde{V}}(x,y) = -\log |x-y| + \tilde{V}(x) + \tilde{V}(y) \quad \text{and} \quad d\tilde{\pi} = e^{-2(\chi+1)V} d\Lambda_{\chi}.
\]

\[\Box\]
Lemma 4.1 tells us that the inverted model we will be interested in will have the following form. Let us consider a sequence \( \{\kappa_n\}_{n \geq 1} \) of positive numbers, a lower semicontinuous function \( U : [0, \infty) \to \mathbb{R} \) such that
\[
\lim_{r \to \infty} \{U(r) - \log r\} > -\infty
\]
and a system of particles \( (x_1^{(n)}, \ldots, x_n^{(n)}) \) distributed according to the law proportional to
\[
\prod_{i < j} |x_i - x_j|^2 e^{-2\kappa_n \sum_{i=1}^n U(|x_i|)} d\Lambda_{\kappa_n-n}(x_1) \ldots d\Lambda_{\kappa_n-n}(x_n),
\]
where \( \Lambda_\chi \) is defined by
\[
d\Lambda_\chi(x) = |x|^{2(\chi-1)} d\ell_{\mathbb{C}}(x)
\]
for every \( \chi > 0 \). This can be thought of as an electrostatic system confined by \( U \) plus a potential \( \left(1 - \frac{n+1}{\kappa_n}\right) \log |\cdot| \) generated by a point charge at the origin. The following proposition will be the limiting point process convergence used to prove Theorem 4.2. It is stated in a slightly more general form since there will be no essential difference in the proof.

**Proposition 4.2 (Particles at zero potential).** Suppose \( U : [0, \infty) \to [0, \infty] \) is non-negative and lower semicontinuous. Denote
\[
\mathcal{A} = \{r \geq 0 : U(r) = 0\} \quad \text{and} \quad R = \text{ess sup } \mathcal{A},
\]
i.e., \( R \) is such that the Lebesgue measure of \( \mathcal{A} \cap (R, \infty) \) is zero but the Lebesgue measure of \( \mathcal{A} \cap (r, \infty) \) is different from zero for every \( r < R \). Assume \( R > 0 \) and that, for some \( \chi > 0 \),
\[
\kappa_n = n + \chi + o(1).
\]
Then,
\[
\{x_k^{(n)} : k \in \{1, \ldots, n\} \text{ and } |x_k^{(n)}| < R\} \xrightarrow{\text{law}} M,
\]
where \( M \) is the (inclusion into the open unit disk of radius \( R \) of a) determinantal point process on \( \{x \in \mathbb{C} : |x| < R \text{ and } U(|x|) = 0\} \) with kernel
\[
K(z, w) = \sum_{k=0}^{\infty} a_k z^k w^k, \quad a_k = \left(2\pi \int_{\mathcal{A}} r^{2k+2\chi-1} dr\right)^{-1},
\]
with respect to the reference measure \( \Lambda_\chi \).

**Proof.** For simplicity, we shall assume that \( R \leq 1 \). The general case can be obtained by a scaling. Denote \( Z = \{x \in \mathbb{C} : |x| < R \text{ and } U(|x|) = 0\} \). We will prove that
\[
\{x_k^{(n)} : k \in \{1, \ldots, n\} \text{ and } x_k^{(n)} \in Z\} \xrightarrow{\text{law}} M
\]
and that
\[
\# \{x_k^{(n)} : k \in \{1, \ldots, n\}, |x_k^{(n)}| \leq \tilde{R} \text{ and } x_k^{(n)} \notin Z\} \xrightarrow{\text{law}} 0
\]
for every \( \tilde{R} < R \). Then we conclude by the following lemma.

**Lemma 4.3 (Union with an empty point process).** Let \( X \) be a Polish space and let \( C \subset X \) be a closed subset of \( X \). Suppose we have a sequence of point processes \( \{P_n\}_{n \in \mathbb{N}} \) on \( X \) and a point process \( P \) on \( C \) such that
\[
P_n \cap C \xrightarrow{\text{law}} P \quad \text{and} \quad \#(P_n \cap K \cap C^c) \xrightarrow{\text{law}} 0 \text{ for every compact set } K \subset X.
\]
Then,
\[
P_n \xrightarrow{\text{law}} P,
\]
where \( P \) is seen as a point process in \( X \) by the natural inclusion.
Proof of the lemma. By [18, Theorem 4.11] we have to prove that

$$\sum_{x \in P_n} f(x) \xrightarrow{\text{law}} \sum_{x \in P} f(x)$$

for every continuous function $f : X \to \mathbb{R}$ with compact support. We already know that

$$\sum_{x \in P_n \cap C} f(x) \xrightarrow{\text{law}} \sum_{x \in P} f(x)$$

so that it is enough, by Slutsky’s theorem, to prove that

$$\sum_{x \in P_n \cap C^c} f(x) \xrightarrow{\text{law}} 0.$$

Let $K = \text{supp } f$. Then, by hypothesis, $\# (P_n \cap K \cap C^c) \to 0$. We can use that

$$\left| \sum_{x \in P_n \cap C^c} f(x) \right| \leq \# (P_n \cap K \cap C^c) \| f \|_{\infty}$$

to conclude.

To prove [18] we will use [26, Proposition 3.10] (recalled in [12]). A small problem appears since $\Lambda_{\kappa_n - n}$ is not a fixed reference measure. We choose any $\varepsilon > 0$ and fix instead another reference measure $\Lambda_{\chi-\varepsilon}$. The point process $\{x_k^{(n)} : k \in \{1, \ldots, n\} \text{ and } x_k^{(n)} \in Z\}$ is determinantal with kernel $K_n : Z \times Z \to \mathbb{C}$ given by

$$K_n(z, w) = \sum_{k=0}^{n-1} a_k^{(n)} z^k w^k \chi_{\kappa_n - n - \chi + \varepsilon}, \quad a_k^{(n)} = \left( 2\pi \int_0^\infty r^{2k+2\kappa_n-2n-1-2\kappa_n U(r)} dr \right)^{-1}, \quad (17)$$

with respect to $\Lambda_{\chi-\varepsilon}$. Since $\kappa_n - n - \chi + \varepsilon$ is positive for $n$ large enough, $K_n$ is continuous and we are in the context described in Subsection 3.2. We first notice that $|z\bar{w}|^{\kappa_n - n - \chi + \varepsilon}$ converges to $|z\bar{w}|^{\chi}$ for $(z, w)$ uniformly on compact sets of $\mathbb{C} \times \mathbb{C}$. We will now show that

$$\sum_{k=0}^{n-1} a_k^{(n)} z^k w^k \xrightarrow{n \to \infty} \sum_{k=0}^{\infty} a_k z^k w^k$$

for $(z, w)$ uniformly on compact sets of $D_R \times D_R$. This will be done by showing the convergence of the coefficients and by bounding each term of the series to apply Lebesgue’s dominated convergence theorem. The bound $r^{2k+2\kappa_n-2n-1-2\kappa_n U(r)} \leq (r^{\varepsilon} + r^{-\varepsilon}) r^{2k+2\chi-1-2(k+1)(\chi-\varepsilon)U(r)}$, for $\varepsilon > 0$ small enough, allows us to use Lebesgue’s dominated convergence theorem and obtain

$$\int_0^\infty r^{2k+2\kappa_n-2n-1-2\kappa_n U(r)} dr \xrightarrow{n \to \infty} \int_A r^{2k+2\chi-1} dr.$$

This gives us the convergence of $a_k^{(n)}$ towards $a_k$. Now, we need to bound the terms in the series. By definition of $A$ and since we are assuming that $R = \text{ess sup } A \leq 1$, we may choose $\varepsilon > 0$ small enough such that

$$\int_0^\infty r^{2k+2\kappa_n-2n-1-2\kappa_n U(r)} dr \geq \int_A r^{2k+2\kappa_n-2n-1-2\kappa_n U(r)} dr \geq \int_A r^{2k+2\chi+\varepsilon-1} dr$$

for $n$ large enough and for every $k \in \{0, \ldots, n-1\}$. So, we define

$$A_k = \left( 2\pi \int_A r^{2k+2\chi+\varepsilon-1} dr \right)^{-1}$$
which satisfies $a_k^{(n)} \leq A_k$ and notice that, by Laplace’s method,

$$\frac{1}{k} \log \int_\mathcal{A} r^{2k+2\chi+\varepsilon-1} dr = \frac{1}{k} \log \int_\mathcal{A} e^{kr^2 r^{2\chi+\varepsilon-1} dr} \to \sup_{k \to \infty} \{ \log r^2 \}$$

where the supremum is taken over the support of the Lebesgue measure on $\mathcal{A}$. By the definition of $R$ this supremum is log $R^2$ and the radius of convergence of $\sum_{k=0}^\infty A_k r^k$ is $R^2$. Take $r \in [0, R)$ and suppose that $|z|, |w| \leq r$. Then

$$\left| \sum_{k=0}^{n-1} a_k^{(n)} z^k \tilde{w}^k - \sum_{k=0}^\infty a_k z^k \tilde{w}^k \right| \leq \sum_{k=0}^\infty |a_k^{(n)} - a_k| |z|^k |\tilde{w}|^k \leq \sum_{k=0}^\infty |a_k^{(n)} - a_k| r^{2k}$$

where we have defined $a_k^{(n)} = 0$ for $k \geq n$. Since $|a_k^{(n)} - a_k|r^{2k}$ is bounded by $2A_k r^{2k}$ we can use Lebesgue’s dominated convergence theorem to conclude.

Now, we need to show (16). As explained in Section 3, $\{x_1^{(n)}, \ldots, x_n^{(n)}\}$ is a determinantal process with respect to $e^{-2\kappa_n U(|z|)} d\Lambda_{\chi-\varepsilon}$ and with kernel $K_n : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by the same formula (17). In particular, we have

$$\mathbb{E} \left[ \# \{ x_k^{(n)} : k \in \{1, \ldots, n\}, |x_k^{(n)}| \leq \tilde{R} \text{ and } x_k^{(n)} \notin Z \} \right] = \int_{Z^c \cap \overline{D}_{\tilde{R}}} K_n(z, z) e^{-2\kappa_n U(|z|)} d\Lambda_{\chi-\varepsilon}(z),$$

where $\overline{D}_{\tilde{R}}$ denotes the closed unit disk of radius $\tilde{R}$ and centered at zero. Since $K_n(z, z)$ converges uniformly on $\overline{D}_{\tilde{R}}$ and since $U(|z|) > 0$ for $z \in Z^c$, we can use Lebesgue’s dominated convergence theorem to conclude that

$$\mathbb{E} \left[ \# \{ x_k^{(n)} : k \in \{1, \ldots, n\}, |x_k^{(n)}| \leq \tilde{R} \text{ and } x_k^{(n)} \notin Z \} \right] \to 0$$

and then

$$\# \{ x_k^{(n)} : k \in \{1, \ldots, n\}, |x_k^{(n)}| \leq \tilde{R} \text{ and } x_k^{(n)} \notin Z \} \xrightarrow{n \to \infty} 0.$$

Recall that $(x_1^{(n)}, \ldots, x_n^{(n)})$ is distributed according to the law proportional to (14). For Theorem 2.3 and also for Theorem 7.1 in the appendix, the following will be used.

**Proposition 4.4** (Finite limiting process at zero). Suppose that $U : [0, \infty) \to [0, \infty)$ is continuous and that

- $U(r) > 0$ for $r \neq 1$,
- $\lim_{r \to 0} \frac{1}{r^\alpha} U(r) = \lambda$,
- $\lim_{r \to 1^+} \frac{U(r)}{r - 1} = l_+$ and
- $\lim_{r \to 1^-} \frac{U(r)}{1 - r} = l_-$

for some positive numbers $\lambda, \alpha, l_+, l_- > 0$. Suppose that, for some $\chi \in (0, \alpha/2]$ and $\xi \in \mathbb{R}$,

$$\kappa_n = n + \chi + \frac{\xi}{\log n} + o \left( \frac{1}{\log n} \right).$$

Then,

$$\{ n^{1/\alpha} x_k^{(n)} : k \in \{1, \ldots, n\} \} \xrightarrow{n \to \infty} \mathbb{G},$$

(18)
where $G$ is a determinantal point process on $\mathbb{C}$ with respect to $\Lambda_\chi$ and with kernel

$$K(z, w) = \sum_{k=0}^{\infty} a_k z^k \bar{w}^k e^{-\lambda|z|^\alpha} e^{-\lambda|w|^\alpha},$$

where

$$(a_k)^{-1} = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} r^{2k+2\chi-1} e^{-2\lambda r^\alpha} dr & \text{if } 2k + 2\chi > \alpha \\ \pi \left( \frac{1}{\alpha} + e^{2\chi/\alpha} \left( \frac{1}{t^\alpha} + \frac{1}{t} \right) \right) & \text{if } 2k + 2\chi < \alpha. \end{cases}$$

The number of particles $\#G$ of $G$ belongs to the interval $[\alpha/2 - \chi, \alpha/2 - \chi + 1]$. More precisely,

$$\#G = \left\lfloor \frac{\alpha}{2} - \chi \right\rfloor \quad \text{if } \alpha/2 - \chi \notin \mathbb{Z},$$

while $\#G \in \{\alpha/2 - \chi, \alpha/2 - \chi + 1\}$ with

$$\mathbb{P}(\#G = \alpha/2 - \chi + 1) = \left(1 + \frac{\alpha \lambda e^{2\chi/\alpha}}{\alpha - \chi} \left( \frac{1}{e^\alpha} + \frac{1}{e} \right) \right)^{-1} \quad \text{if } \alpha/2 - \chi \in \mathbb{Z}.$$

**Remark 4.5** (Finite Coulomb gas). Notice that $G$, conditioned to having a fixed number of particles, can be thought of as a finite Coulomb gas confined by a power-law potential plus a potential multiple of $\log |\cdot|$.

**Proof.** Notice that $\{n^{1/\alpha} x_k^{(n)} : k \in \{1, \ldots, n\}\}$ is a determinantal point process on $\mathbb{C}$ associated to the measure $\Lambda_{\kappa_n-n}$ and to the kernel

$$K_n(z, w) = \sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k e^{-\kappa_n U\left(\frac{|z|}{n^{1/\alpha}}\right)} e^{-\kappa_n U\left(\frac{|w|}{n^{1/\alpha}}\right)},$$

where

$$(a_k^{(n)})^{-1} = \int_{\mathbb{C}} |z|^{2k} e^{-2\kappa_n U\left(\frac{|z|}{n^{1/\alpha}}\right)} d\Lambda_{\kappa_n-n}(z) = 2\pi \int_{0}^{\infty} r^{2k+2\kappa_n-2n-1} e^{-2\kappa_n U\left(\frac{r}{n^{1/\alpha}}\right)} dr.$$

Since the reference measure is different for each $n$ we will consider, instead, the fixed reference measure $\Lambda_{\chi-\varepsilon}$, for some (choose any) $\varepsilon \in (0, \chi)$. The kernel will be

$$K_n(z, w) = \sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k e^{-\kappa_n U\left(\frac{|z|}{n^{1/\alpha}}\right)} e^{-\kappa_n U\left(\frac{|w|}{n^{1/\alpha}}\right)} |z|^\chi |w|^{\kappa_n-n-\chi+\varepsilon},$$

By [26], Proposition 3.10], to obtain (15) we should understand the limit of the sequence of continuous (for $n$ large enough) functions $\{K_n\}_{n \geq 1}$. Since $\lim_{r \to 0} \frac{1}{r^\alpha} U(r) = \lambda \in (0, \infty)$ we already have that, uniformly on compact sets,

$$\kappa_n U\left(\frac{|z|}{n^{1/\alpha}}\right) \to \lambda |z|^\alpha.$$

Since $\kappa_n - n \to \chi$ we know that $|z|^{\kappa_n-n-\chi+\varepsilon} \to |z|^\chi$ uniformly on compact sets. Then what is left to prove is that, uniformly on compact sets,

$$\sum_{k=0}^{n-1} a_k^{(n)} z^k \bar{w}^k \to \sum_{k=0}^{\infty} a_k z^k \bar{w}^k.$$

This will be done by showing the convergence of the coefficients, by properly bounding them and then by applying Lebesgue’s dominated convergence theorem. We want to find the limit, as $n$ goes to infinity, of

$$(2\pi a_k^{(n)})^{-1} = \int_{0}^{\infty} r^{2k+2\kappa_n-2n-1} e^{-2\kappa_n U\left(\frac{r}{n^{1/\alpha}}\right)} dr = n^{(2k+2\chi)/\alpha} \int_{0}^{\infty} r^{2k+2\kappa_n-2n-1} e^{-2\kappa_n U(r)} dr.$$

(19)
By Laplace’s method the dominant part will be around the minima of \( U \) (i.e., \( r = 0 \) and \( r = 1 \)). This suggests us to split the right-hand side integral in \( \int_0^\infty \) as

\[
\int_0^\infty = \int_0^{1/2} + \int_{1/2}^1 + \int_1^2 + \int_2^\infty.
\]

We begin by understanding the integral from 0 to 1/2. The main contribution comes from \( r = 0 \) so that we perform a change of variables and apply Lebesgue’s dominated convergence theorem (by controlling the potential \( U(r) \) from below by a multiple of \( r^\alpha \)),

\[
n(2k+2\kappa_n-2n)/\alpha \int_0^{1/2} r^{2k+2\kappa_n-2n-1}e^{-2\kappa_n U(r)} dr = \int_0^{\pi/2} r^{2k+2\kappa_n-2n-1}e^{-2\kappa_n U\left(\frac{\pi}{2\alpha}\right)} dr \rightarrow n \rightarrow \infty \int_0^\infty r^{2k+2\kappa_n-2n-1}e^{-2\lambda r^\alpha} dr.
\]

For the integral from 1/2 to 1 we notice that, by applying Lebesgue’s dominated convergence theorem (by controlling the potential from below by a multiple of \( 1-r \)),

\[
n\int_{1/2}^1 r^{2k+2\kappa_n-2n-1}e^{-2\kappa_n U(r)} dr = \int_0^{n/2} \left(1 - \frac{r}{n}\right)^{2k+2\kappa_n-2n-1}e^{-2\kappa_n U\left(\frac{r}{n}\right)} dr \rightarrow n \rightarrow \infty \int_0^\infty e^{-2\lambda r} dr = \frac{1}{2\lambda}.
\]

which implies that

\[
n(2k+2\kappa_n-2n)/\alpha \int_{1/2}^1 r^{2k+2\kappa_n-2n-1}e^{-2\kappa_n U(r)} dr \rightarrow n \rightarrow \infty \begin{cases} \infty & \text{if } 2k + 2\chi > \alpha \\ e^{2\lambda/\alpha} \frac{1}{2\lambda} & \text{if } 2k + 2\chi = \alpha \\ 0 & \text{if } 2k + 2\chi < \alpha \end{cases}.
\]

Similarly, for the integral from 1 to 2 we have

\[
n(2k+2\kappa_n-2n)/\alpha \int_1^2 r^{2k+2\kappa_n-2n-1}e^{-2\kappa_n U(r)} dr \rightarrow n \rightarrow \infty \begin{cases} \infty & \text{if } 2k + 2\chi > \alpha \\ e^{2\lambda/\alpha} \frac{1}{2\lambda} & \text{if } 2k + 2\chi = \alpha \\ 0 & \text{if } 2k + 2\chi < \alpha \end{cases}.
\]

The last integral, the one from 2 to \( \infty \), will go to zero since there is \( \varepsilon > 0 \) such that \( \varepsilon \log r \leq U(r) \) for \( r \geq 2 \) which implies that

\[
n(2k+2\kappa_n-2n)/\alpha \int_2^\infty r^{2k+2\kappa_n-2n-1}e^{-2\kappa_n U(r)} dr \leq n(2k+2\kappa_n-2n)/\alpha \int_2^\infty r^{2k+2\kappa_n-2n-1}e^{-2\kappa_n \varepsilon \log r} dr \rightarrow n \rightarrow \infty 0.
\]

In summary, we have obtained that

\[
\lim_{n \rightarrow \infty} (a_k^{(n)})^{-1} = \begin{cases} \infty & \text{if } 2k + 2\chi > \alpha \\ 2\pi \int_0^{\pi} r^{2k+2\chi-1}e^{-2\lambda r^\alpha} dr & \text{if } 2k + 2\chi = \alpha \\ 2\pi \int_0^{\pi} r^{2k+2\chi-1}e^{-2\lambda r^\alpha} dr + \pi e^{2\xi/\alpha} \left(\frac{1}{r^\alpha} + \frac{1}{\varepsilon}\right) & \text{if } 2k + 2\chi < \alpha \end{cases}.
\]

To control the coefficients we notice that by the behavior of \( U \) near the origin and since \( U \) is bounded in \([0,1]\), there exists \( C > 0 \) and \( \bar{\chi} > 0 \) such that if we define

\[
A_k = \left(2\pi \int_0^{\varepsilon^{1/\alpha}} r^{2k+2\bar{\chi}-1}e^{-Cr^\alpha} dr\right)^{-1}.
\]
we have

$$A_k \leq a^{(n)}_k$$ for $k \in \{0, \ldots, n - 1\}$.

By Laplace’s method, $\lim_{k \to \infty} \frac{1}{k} \log \left( (A_k)^{-1} \right) = \infty$ so that $\sum_{k=0}^{\infty} A_k x^k$ has an infinite radius of convergence. Finally, if $r > 0$ and $|z|, |w| \leq r$ we have

$$\left| \sum_{k=0}^{n-1} a^{(n)}_k z^k \tilde{w}^k - \sum_{k=0}^{\infty} a_k z^k \tilde{w}^k \right| \leq \sum_{k=0}^{\infty} |a^{(n)}_k - a_k| |z|^k |\tilde{w}|^k \leq \sum_{k=0}^{\infty} |a^{(n)}_k - a_k| r^{2k}$$

where $a^{(n)}_k$ is zero if $k \geq n$. By noticing that $|a^{(n)}_k - a_k| r^{2k} \leq 2A_k r^{2k}$ we apply Lebesgue’s dominated convergence theorem to conclude.

**Number of particles.** The assertion about the number of particles is an immediate consequence of [2] Theorem 4.5.3. More precisely, when $\alpha/2 - \chi \notin \mathbb{Z}$, the operator defined by $K$ is a projection onto a space of dimension $\lceil \alpha/2 - \chi \rceil$, while if $\alpha/2 - \chi \in \mathbb{Z}$, the unique non-zero eigenvalue less than one is the inverse of $1 + \alpha \lambda e^{2\xi/\alpha} (\frac{1}{r^2} + \frac{1}{\ell^2})$.

□

Having all the ingredients, we proceed to the proof of Theorem 2.3 and Theorem 2.4

**Proof of Theorem 2.3.** We shall use Proposition 4.2 with $U : [0, \infty) \to \mathbb{R}$ defined by

$$U(r) = \begin{cases} V(1/r) + \log r & \text{if } r > 0 \\ 0 & \text{if } r = 0 \end{cases}.$$

Notice that $U$ is a non-negative lower semicontinuous function and that $A = [\tilde{R}^{-1}, R^{-1}]$. By Lemma 4.1, $(1/x_1^{(n)}, \ldots, 1/x_n^{(n)})$ has a law proportional to

$$\prod_{i<j}^{n} |x_i - x_j|^2 e^{-2\kappa n \sum_{i=1}^{n} U(|x_i|)} d\Lambda_{\kappa} - n(x_1) \cdots d\Lambda_{\kappa} - n(x_n),$$

where

$$d\Lambda_{\kappa} - n(x) = |x|^{2(\kappa - n - 1)} d\ell_C(x).$$

Proposition 4.2 tells us that

$$\{1/x_k^{(n)} : k \in \{1, \ldots, n\} \text{ and } |1/x_k^{(n)}| < R^{-1} \} \xrightarrow{\text{law}} n \to \infty \mathbb{M}$$

where $\mathbb{M}$ is the (inclusion into the open unit disk of radius $R^{-1}$ of a) determinantal point process on $\{x \in \mathbb{C} : \tilde{R}^{-1} \leq |x| < R^{-1} \}$ with kernel

$$K(z, w) = \sum_{k=0}^{\infty} a_k z^k \tilde{w}^k, \quad (a_k)^{-1} = 2\pi \int_{R^{-1}}^{R^{-1}} r^{2k+2\chi-1} dr,$$

with respect to the reference measure $\Lambda_{\chi}$. In particular,

$$\{ |1/x_k^{(n)}| : k \in \{1, \ldots, n\} \text{ and } |1/x_k^{(n)}| < R^{-1} \} \xrightarrow{\text{law}} n \to \infty \{ |x| : x \in \mathbb{M} \}.$$

Suppose that $\{T_k\}_{k \geq 0}$ is a family of independent random variables in $[\tilde{R}^{-1}, R^{-1})$ such that the density of $T_k$ is given by $r \in [\tilde{R}^{-1}, R^{-1}) \mapsto 2\pi a_k r^{2k+2\chi-1}$. Then, by (13), we obtain that $\{ |x| : x \in \mathbb{M} \} \sim \{ T_k : k \geq 0 \}$. By defining $Y_k = 1/T_k$ and taking the image by $x \mapsto 1/x$ we get

$$\{ |x_k^{(n)}| : k \in \{1, \ldots, n\} \text{ and } |x_k^{(n)}| > R \} \xrightarrow{\text{law}} n \to \infty \{ Y_k : k \geq 0 \},$$
where the minimum is $R^{-1}$ if the set is empty. Since
\[ \inf \{ |1/x_k^{(n)}| : k \in \{1, \ldots, n\} \} \leq \inf \{ 1/x_k^{(n)} : k \in \{1, \ldots, n\} \} \wedge R^{-1} \]
and since $\inf \{ T_k : k \geq 0 \} < R^{-1}$ almost surely, we obtain that
\[ \inf \{ |1/x_k^{(n)}| : k \in \{1, \ldots, n\} \} \xrightarrow[\mathcal{L}] {n \to \infty} \inf \{ T_k : k \geq 0 \}, \]
and the proof may be completed by taking the image by $x \mapsto 1/x$. \hfill \Box

**Proof of Theorem 2.4.** It follows the same steps as the proof of Theorem 2.3 above by using Proposition 4.4 instead of Proposition 4.2. \hfill \Box

## 5 Proofs of the Gumbel behavior

In this section we will give the proofs of the statements from Subsection 2.2. Namely, we give the proofs of Theorem 2.5, Theorem 2.6, Proposition 7.2 and Proposition 7.3.

We begin by describing, in Subsection 5.1, the proofs of Theorem 2.5 and Theorem 2.6. In Subsection 5.2, we describe some lemmas that allow us to properly justify the steps. The convergence of the intensities, which is the main step of the proofs, is provided in Subsection 5.3. Finally, in Subsection 5.4, we complete the proofs of the Gumbel fluctuations.

### 5.1 Idea of the proofs of Theorem 2.5 and Theorem 2.6

For each $n \geq 1$, let us consider $n$ independent positive random variables $Y_0^{(n)}, \ldots, Y_{n-1}^{(n)}$ such that
\[ Y_k^{(n)} \sim b_k^{(n)} r^{2k+1} e^{-2\kappa_n V(r)} 1_{(0,\infty)}(r) dr, \quad \text{where} \quad b_k^{(n)} = \left( \int_0^\infty s^{2k+1} e^{-2\kappa_n V(s)} ds \right)^{-1}. \]

By Kostlan’s idea stated in Subsection 3.3, we have that
\[ \{|x_1^{(n)}|, \ldots, |x_n^{(n)}|\} \sim \{Y_0^{(n)}, \ldots, Y_{n-1}^{(n)}\}. \quad \text{(20)} \]

Let us begin by studying $M_n = \max \{|x_1^{(n)}|, \ldots, |x_n^{(n)}|\}$. By (20), we have
\[ \mathbb{P}(M_n \leq m) = \prod_{k=0}^{n-1} \left( 1 - b_k^{(n)} \int_m^\infty r^{2k+1} e^{-2\kappa_n V(r)} dr \right). \]

By using that $\log(1 - x) = -x + O(x^2)$, we can see that if $m_n$ goes to 1 slowly enough such that
\[ \sup_{k \in \{0, \ldots, n-1\}} \left( b_k^{(n)} \int_{m_n}^\infty r^{2k+1} e^{-2\kappa_n V(r)} dr \right) \xrightarrow[n \to \infty]{} 0, \quad \text{(21)} \]
we have that
\[ \log \mathbb{P}(M_n \leq m_n) \text{ converges if and only if } - \int_{m_n}^\infty \sum_{k=0}^{n-1} b_k^{(n)} r^{2k+1} e^{-2\kappa_n V(r)} dr \text{ converges}. \quad \text{(22)} \]
and their limit is the same. We recognize the first intensity of the process of moduli

\[\tilde{\rho}_n(r) = \sum_{k=0}^{n-1} \frac{b^{(n)}_k}{r^{2k+1}} e^{-2\kappa_n V(r)}\]

as minus the integrand. If \( m_n = c_n + d_n x \) for some \( \{c_n\}_{n \geq 1} \) and \( \{d_n\}_{n \geq 1} \), we would write

\[\int_{m_n}^{\infty} \tilde{\rho}_n(r) dr = \int_x^{\infty} d_n \tilde{\rho}_n (c_n + d_n s) ds.\]

So, we only need to find

\[\lim_{n \to \infty} d_n \tilde{\rho}_n (c_n + d_n s) = \rho(s)\]

and control the integrand to obtain that

\[\lim_{n \to \infty} P(M_n \leq c_n + d_n x) = \exp \left(- \int_x^{\infty} \rho(s) ds\right).\]

Notice that \( s \to d_n \tilde{\rho}_n (c_n + d_n s) \) is the intensity function of the point process

\[\{d_n^{-1}(|x^{(n)}_1| - c_n), \ldots, d_n^{-1}(|x^{(n)}_n| - c_n)\}\]

and that \( \exp \left(- \int_x^{\infty} \rho(s) ds\right) \) is the cumulative distribution function of the maximum of a Poisson point process with intensity \( \rho \). This suggests how to proceed for the study of the point process formed by \( W_k^{(n)} = d_n^{-1}(Y_k^{(n)} - c_n) \). It is motivated by \([22, 24]\). Let us consider \( \ell \) mutually disjoint bounded measurable subsets \( I_1, \ldots, I_\ell \) of \( \mathbb{R} \) and define the random variables

\[N_i^{(n)} = \sum_{k=0}^{n-1} 1_{I_i}(W_k^{(n)}).\]

We shall study \( P(N_1^{(n)} = m_1, \ldots, N_\ell^{(n)} = m_\ell) \). For this, by using that

\[1_{N_i^{(n)} = m_i} = \frac{1}{m_i!} \left( \frac{d}{d\lambda_i} \right)^{m_i} \left[ \prod_{k=0}^{n-1} \left( 1 + \lambda_i 1_{I_i}(W_k^{(n)}) \right) \right]_{\lambda_i = -1}\]

we can notice that

\[1_{N_i^{(n)} = m_1, \ldots, N_\ell^{(n)} = m_\ell} = \frac{1}{m_1!} \cdots \frac{1}{m_\ell!} \left( \frac{d}{d\lambda_1} \right)^{m_1} \cdots \left( \frac{d}{d\lambda_\ell} \right)^{m_\ell} \left[ \prod_{k=0}^{n-1} \left( 1 + \sum_{i=1}^\ell \lambda_i 1_{I_i}(W_k^{(n)}) \right) \right]_{\lambda_1 = \ldots = \lambda_\ell = -1}.\]

Then, we can write

\[P(N_1^{(n)} = m_1, \ldots, N_\ell^{(n)} = m_\ell) = \frac{1}{m_1!} \cdots \frac{1}{m_\ell!} \left( \frac{d}{d\lambda_1} \right)^{m_1} \cdots \left( \frac{d}{d\lambda_\ell} \right)^{m_\ell} \left[ \prod_{k=0}^{n-1} \left( 1 + \sum_{i=1}^\ell \lambda_i P(W_k^{(n)} \in I_i) \right) \right]_{\lambda_1 = \ldots = \lambda_\ell = -1}.\]

For each \( n \), define the polynomial function \( F_n : \mathbb{C}^\ell \to \mathbb{C}^\ell \) by

\[F_n(\lambda_1, \ldots, \lambda_\ell) = \prod_{k=0}^{n-1} \left( 1 + \sum_{i=1}^\ell \lambda_i P(W_k^{(n)} \in I_i) \right).\]
Notice that $|F_n(\lambda_1, \ldots, \lambda_\ell)| \leq F_n(r, \ldots, r)$ whenever $|\lambda_1|, \ldots, |\lambda_\ell| \leq r$. By Montel’s theorem, once we show that $F_n$ converges pointwise, it also converges uniformly on compact sets of $\mathbb{C}^\ell$. Since we are assuming (21), we have that

$$\sum_{k=0}^{n-1} \log \left(1 + \sum_{i=1}^\ell \lambda_i \mathbb{P}(W_k^{(n)} \in I_i)\right)$$

converges if and only if $\sum_{k=0}^{n-1} \sum_{i=1}^\ell \lambda_i \mathbb{P}(W_k^{(n)} \in I_i)$ converges and their limit is the same. Here the logarithm is well-defined for $n$ large enough by the usual power series. Then, we need to study the limit of

$$\sum_{k=0}^{n-1} \mathbb{P}(W_k^{(n)} \in I_i) = \int_{I_i} d_n \tilde{\rho}_n (c_n + d_ns) \, ds$$

which, since $d_n \tilde{\rho}_n (c_n + d_ns) \to \rho(s)$ and since $d_n \tilde{\rho}_n (c_n + d_ns)$ has been properly bounded to study the maxima, could be shown to converge to $\int_{I_i} \rho(s) \, ds$. We get

$$F_n(\lambda_1, \ldots, \lambda_\ell) \overset{n \to \infty}{\longrightarrow} e^{\sum_{i=1}^\ell \lambda_i \int_{I_i} \rho(s) \, ds}.$$ 

Taking the corresponding derivatives we obtain that

$$P(N_1^{(n)} = m_1, \ldots, N_\ell^{(n)} = m_\ell) \overset{n \to \infty}{\longrightarrow} \prod_{i=1}^\ell \frac{\left(\int_{I_i} \rho(s) \, ds\right)^{m_i}}{m_i!} e^{-\int_{I_i} \rho(s) \, ds}$$

which implies that

$$\{W_k^{(n)} : k \in \{0, \ldots, n-1\}\} \overset{\text{law}}{\longrightarrow} \mathcal{P},$$

where $\mathcal{P}$ is a Poisson point process with intensity $\rho$.

Now, we describe the lemmas that allow us to justify the steps described above.

### 5.2 Control of the probabilities

Next lemma, together with Lemma 5.3, imply (21) which justifies (22) for Theorem 2.5.

**Lemma 5.1** (Uniform convergence of probabilities for a power-law behavior). Let $\alpha \geq 1$. Suppose that there exists $\lambda, \bar{\lambda}, \ell > 0$ such that

$$|r - 1|^\alpha \leq V(r) - \log r \leq \lambda |r - 1|^\alpha$$

when $|r - 1| \leq \ell$ and suppose that

$$\lim_{n \to \infty} \frac{\kappa_n}{n} = 1 \quad \text{and} \quad n - \kappa_n = o\left(\kappa_n^{1/\alpha}\right).$$

Define

$$b_k^{(n)} = \left(\int_0^{\infty} s^{2k+1} e^{-2\kappa_n V(s)} \, ds\right)^{-1}.$$ 

If $\{m_n\}_{n \geq 1}$ is a sequence that satisfies

$$n^{1/\alpha} (m_n - 1) \overset{n \to \infty}{\longrightarrow} \infty,$$

then

$$\sup_{k \in \{0, \ldots, n-1\}} \left( b_k^{(n)} \int_{m_n}^{1+\ell} s^{2k+1} e^{-2\kappa_n V(s)} \, ds \right) \overset{n \to \infty}{\longrightarrow} 0.$$
Proof of Lemma 5.1. To bound the coefficients $b_k^{(n)}$, we use that

$$
\int_0^\infty s^{2k+1-2\kappa_n} e^{-2\kappa_n s} ds \geq \int_0^{1+\ell} s^{2k-2\kappa_n+1} e^{-2\kappa_n s} ds
$$

$$
= \int_{-\ell}^\ell (1+s)^{2k-2\kappa_n+1} e^{-2\kappa_n s} ds
$$

$$
= \int_0^\ell ((1+s)^{2k-2\kappa_n+1} + (1-s)^{2k-2\kappa_n+1}) e^{-2\kappa_n s} ds
$$

$$
\geq \int_0^\ell e^{-2\kappa_n s} ds
$$

$$
= \frac{1}{\kappa_n^{1/\alpha}} \int_0^{\kappa_n^{1/\alpha} \ell} e^{-2\lambda s} ds.
$$

For the integral part we can see that

$$
\int_{m_n}^{1+\ell} r^{2k+1} e^{-2\kappa_n V(r)} dr \leq \int_{m_n}^{1+\ell} r^{2k-2\kappa_n+1} e^{-2\kappa_n \lambda |r-1|^{\alpha}} dr
$$

$$
= \frac{1}{\kappa_n^{1/\alpha}} \int_{\kappa_n^{1/\alpha} (m_n-1)}^{\kappa_n^{1/\alpha} (m_n+1)} \left(1 + \frac{r}{\kappa_n^{1/\alpha}}\right)^{2k-2\kappa_n+1} e^{-2\lambda r} dr
$$

$$
\leq \frac{1}{\kappa_n^{1/\alpha}} \int_{\kappa_n^{1/\alpha} (m_n-1)}^{\kappa_n^{1/\alpha} (m_n+1)} \left(1 + \frac{r}{\kappa_n^{1/\alpha}}\right)^{2k-2\kappa_n-1} e^{-2\lambda r} dr
$$

$$
\leq \frac{1}{\kappa_n^{1/\alpha}} \int_{\kappa_n^{1/\alpha} (m_n-1)}^{\kappa_n^{1/\alpha} (m_n+1)} \exp\left(\frac{2n-2\kappa_n-1}{\kappa_n^{1/\alpha}} r\right) e^{-2\lambda r} dr
$$

$$
= o\left(\frac{1}{\kappa_n^{1/\alpha}}\right) \text{ uniformly on } k
$$

since $(2n-2\kappa_n-1)/\kappa_n^{1/\alpha} \to 0$ and $\kappa_n^{1/\alpha} (m_n-1) \to \infty$. This completes the proof.

Next lemma is the step in (21) which justifies (22) for Theorem 2.6.

Lemma 5.2 (Uniform convergence of probabilities for a ‘logarithm’ potential ). Suppose that

$$V(r) = \log r \text{ for } r \geq 1.$$ 

and that

$$\lim_{n \to \infty} \frac{\kappa_n}{n} = 1 \text{ and } \lim_{n \to \infty} (\kappa_n - n) = \infty.$$ 

Define

$$b_k^{(n)} = \left(\int_0^\infty s^{2k+1-2\kappa_n} e^{-2\kappa_n s} ds\right)^{-1}.$$ 

If $\{m_n\}_{n \geq 1}$ is a sequence that satisfies

$$(\kappa_n - n)(m_n - 1) \to \infty,$$

then

$$\sup_{k \in \{0, \ldots, n-1\}} \left( b_k^{(n)} \int_{m_n}^{1+\ell} r^{2k+1} e^{-2\kappa_n V(r)} dr \right) \to 0.$$
Proof. We use that
\[ b_k^{(n)} \leq \left( \int_1^\infty s^{2k+1} e^{-2\kappa_n V(s)} ds \right)^{-1} = 2(\kappa_n - k - 1). \]
Then, we may to notice that
\[ b_k^{(n)} \int_m^n r^{2k+1} e^{-2\kappa_n V(r)} dr \leq 2(\kappa_n - k - 1) \int_m^n r^{2k+1} e^{-2\kappa_n V(r)} dr \leq m^{n(\kappa_n - n)} \]
which goes to zero as \( n \to \infty \) since \( (\kappa_n - n)(m_n - 1) \to \infty \).

We state now a lemma that helps us neglect the intensity outside a neighborhood of 1. It will be useful in the proof of Theorem 2.5.

**Lemma 5.3 (Intensity far from 1).** Let \( V : [0, \infty) \to \mathbb{R} \) be a continuous function such that
\[ V(1) = 0, \quad \forall r > 1, \quad V(r) > \log r, \quad \text{and} \quad \liminf_{r \to \infty} \frac{V(r)}{\log r} > 1, \]
and suppose that
\[ \lim_{n \to \infty} \frac{\kappa_n}{n} = 1. \]
Define
\[ b_k^{(n)} = \left( \int_0^\infty s^{2k+1} e^{-2\kappa_n V(s)} ds \right)^{-1}. \]
Then, for every \( \varepsilon > 0 \),
\[ \sum_{k=0}^{n-1} b_k^{(n)} \int_{1+\varepsilon}^\infty r^{2k+1} e^{-2\kappa_n V(r)} dr \to 0. \]

**Proof.** Choose \( C > 1 \) such that
\[ \forall r \geq 1 + \varepsilon : V(r) \geq C \log r \]
and choose \( \delta > 0 \) such that
\[ \forall r \in [1, 1 + \delta] : V(r) \leq \log(1 + \varepsilon) \frac{C - 1}{2}. \]
Since,
\[ \int_0^\infty s^{2k+1} e^{-2\kappa_n V(s)} ds \geq \int_1^{1+\delta} s^{2k+1} e^{-\kappa_n \log(1+\varepsilon)(C-1)} ds \geq \delta(1 + \varepsilon)^{-\kappa_n(C-1)} \]
we have
\[ b_k^{(n)} r^{2k+1} e^{-2\kappa_n V(r)} \leq \delta^{-1} (1 + \varepsilon)^{\kappa_n(C-1)} r^{2k+1} e^{-2\kappa_n C} \leq \delta^{-1} r^{2k+1} e^{-\kappa_n(C+1)} \leq \delta^{-1} r^{2n-1} e^{-\kappa_n(C+1) }. \]
By integrating, taking the sum and taking the limit, the proof is completed. \( \square \)

Next lemma complements the previous one by allowing us to neglect the intensity outside a shrinking neighborhood of 1. It works when \( V \) behaves like a power of \( \alpha \) near both sides of 1 and it will be used in the proof of Theorem 2.5.
Lemma 5.4 (Intensity near 1). Let \( \alpha \geq 1 \). Suppose that there exists \( \lambda, \bar{\lambda}, \ell > 0 \) such that
\[
\lambda|1 - r|^{\alpha} \leq V(r) - \log r \leq \lambda|1 - r|^{\alpha}
\]
when \(|r - 1| \leq \ell\) and suppose that
\[
\lim_{n \to \infty} \frac{\kappa_n}{n} = 1 \quad \text{and} \quad n - \kappa_n = o\left(\kappa_n^{1/\alpha}\right).
\]
Define
\[
b_k^{(n)} = \left(\int_0^\infty s^{2k+1}e^{-2\kappa_n V(s)}ds\right)^{-1}.
\]
Then, for every \( \gamma > 0 \),
\[
\sum_{k=0}^{n-1} b_k^{(n)} \int \frac{1}{1+\frac{1}{\log n}} r^{2k+1}e^{-2\kappa_n V(r)}dr \to 0.
\]
Proof. By the proof of Lemma 5.1 we know that
\[
b_k^{(n)} \leq \bar{C}\kappa_n^{1/\alpha}
\]
for some constant \( \bar{C} > 0 \). Then, we need to study
\[
\sum_{k=0}^{n-1} \int_{1+\frac{1}{\log n}}^{1+\frac{1}{\log n}+\gamma} r^{2k+1}e^{-2\kappa_n V(r)}dr = \int_{1+\frac{1}{\log n}}^{1+\frac{1}{\log n}+\gamma} \frac{r^{2n} - 1}{r^2 - 1} e^{-2\kappa_n V(r)}dr
\]
\[
\leq (\log n)^{\gamma} \int_{1+\frac{1}{\log n}}^{1+\frac{1}{\log n}+\gamma} r^{2n+1} e^{-2\kappa_n V(r)}dr
\]
\[
\leq (\log n)^{\gamma} \int_{1+\frac{1}{\log n}}^{1+\frac{1}{\log n}+\gamma} r^{2n+1} e^{-2\kappa_n} e^{-\bar{\lambda}(r-1)^{\alpha}}dr
\]
\[
\leq (\log n)^{\gamma} \int_{1+\frac{1}{\log n}}^{1+\frac{1}{\log n}+\gamma} e^{r^{2n+1} - 2\kappa_n|\kappa_n|^{{-1}/\alpha}} e^{-2\kappa_n^{1/\alpha}}dr.
\]
Since \( (n - \kappa_n)/\kappa_n^{1/\alpha} \to 0 \) we can conclude that there is \( C > 0 \) such that
\[
\sum_{k=0}^{n-1} \int_{1+\frac{1}{\log n}}^{1+\frac{1}{\log n}+\gamma} r^{2k+1} e^{-2\kappa_n V(r)}dr \leq (\log n)^{\gamma} \int_{\kappa_n^{1/\alpha}}^{1/\gamma} e^{-Cr^{\alpha}}dr,
\]
for \( n \) large enough. Since
\[
\int_{\kappa_n^{1/\alpha}}^{1/\gamma} e^{-Cr^{\alpha}}dr = e^{-C\frac{\kappa_n^{1/\alpha}}{(\log n)^{\gamma}}(1+o(1))},
\]
we can complete the proof by using that \( (\log n)^{\gamma} e^{-C\frac{\kappa_n^{1/\alpha}}{(\log n)^{\gamma}}(1+o(1))} \to 0 \).

The following lemma helps us neglect the intensity outside a shrinking neighborhood of 1 for \( V \) that equals the logarithm at the right of 1. It will be used in the proof of Theorem 2.6.
**Lemma 5.5** (Intensity near 1 for a ‘logarithm’ potential). Suppose that

$$ V(r) = \log r \quad \text{for } r \geq 1 $$

and that

$$ \lim_{n \to \infty} \frac{\kappa_n}{n} = 1 \quad \text{and} \quad \lim_{n \to \infty} (\kappa_n - n) = \infty. $$

Define

$$ b_k^{(n)} = \left( \int_0^\infty s^{2k+1} e^{-2\kappa_n V(s)} ds \right)^{-1}. $$

Then, for every $\gamma > 0$,

$$ \sum_{k=0}^{n-1} b_k^{(n)} \int_{1+\sqrt{n} - n}^{\infty} r^{2k+1} e^{-2\kappa_n V(r)} dr \to 0. $$

**Proof.** We use that

$$ b_k^{(n)} \leq \left( \int_1^\infty s^{2k+1} e^{-2\kappa_n V(s)} ds \right)^{-1} = 2(\kappa_n - k - 1). $$

Then, we need to study

$$ \sum_{k=0}^{n-1} 2(\kappa_n - k - 1) \int_{1+\sqrt{n} - n}^{\infty} r^{2k+1} e^{-2\kappa_n V(r)} dr = \sum_{k=0}^{n-1} \left( 1 + \frac{\gamma}{\sqrt{\kappa_n - n}} \right)^2(2-k+1) $$

$$ \leq \sqrt{\kappa_n - n} \left( 1 + \frac{\gamma}{\sqrt{\kappa_n - n}} \right)^2(n-k+1) $$

which goes to zero as $n \to \infty$. \hfill \Box

### 5.3 Convergence of the intensities

**Proposition 5.6** (Intensity convergence for a power-law behavior). Under the conditions and notation of Theorem 2.5, if we define

$$ b_k^{(n)} = \left( \int_0^\infty x^{2k+1} e^{-2\kappa_n V(x)} dx \right)^{-1}, $$

we have, for every $s \in \mathbb{R}$,

$$ \frac{\Delta_n}{\Lambda_n} \sum_{k=0}^{n-1} b_k^{(n)} \left( 1 + \delta_n + \frac{\delta_n}{\Delta_n} s \right)^{2k+1} e^{-2\kappa_n V(1+\delta_n + \frac{\delta_n}{\Delta_n} s)} \xrightarrow{n \to \infty} e^{-2\xi} e^{-s} \frac{A}{A}. $$

**Proof.** We begin by noticing that, since $e^{\Delta_n/\alpha} \Delta_n = n^{1/\alpha}$, we have $\Delta_n \to \infty$. Moreover, by using that $\frac{\Delta_n}{\alpha} \sim \frac{\Delta_n}{\alpha} + \log \Delta_n = \frac{1}{\alpha} \log n$ we obtain

$$ \Delta_n \sim \log n. $$

Define

$$ c_n = 1 + \left( \frac{\Delta_n}{2\lambda_n \kappa_n} \right)^{1/\alpha} \quad \text{and} \quad d_n = \frac{1}{\Delta_n} \left( \frac{\Delta_n}{2\lambda_n \kappa_n} \right)^{1/\alpha}. $$

so that we want to find the limit of

$$ d_n \sum_{k=0}^{n-1} b_k^{(n)} (c_n + d_n s)^{2k+1} e^{-2\kappa_n V(c_n + d_n s)} $$

(23)
For simplicity of notation, we do not write the subscripts and superscripts $n$ when there is no possibility of confusion. For the potential term we have

\[
V(c + ds) = \log(c + ds) + \frac{\lambda_s}{\alpha}(c + ds - 1)^\alpha + o(c + ds - 1)^{\alpha + \epsilon} \\
= \log(c + ds) + \frac{\Delta}{2\alpha, \kappa} \left(1 + \frac{s}{\Delta}\right)^\alpha + o\left(\frac{\Delta}{\kappa}\right)^{1 + \frac{\epsilon}{\alpha}} \\
= \log(c + ds) + \frac{\Delta}{2\alpha, \kappa} \left(1 + \frac{\alpha s}{\Delta} + o\left(\frac{1}{\Delta}\right)\right) + o\left(\frac{\Delta}{\kappa}\right)^{1 + \frac{\epsilon}{\alpha}} \\
= \log(c + ds) + \frac{\Delta}{2\alpha, \kappa} + \frac{s}{2\kappa} + o\left(\frac{1}{\kappa}\right).
\]

So, we can write

\[
\exp\left(-2\kappa [V(c + ds) - \log(c + ds)]\right) = \exp\left(-\frac{\Delta}{\alpha} - s + o(1)\right) \sim \frac{\Delta}{n^{1/\alpha}}e^{-s}.
\]

Then, the limit of (23) will be the same as the limit of

\[
e^{-s} \left(\frac{\Delta}{2\lambda_\kappa}\right)^{1/\alpha} \sum_{k=0}^{n-1} \frac{b_k}{n^{1/\alpha}} (c + ds)^{2(k-\lfloor \kappa \rfloor)}.
\]

where $\lfloor \kappa \rfloor$ denotes the integer part of $\kappa$. Then, by recalling that $\delta = \left(\frac{\Delta}{2\lambda_\kappa}\right)^{1/\alpha}$, it is enough to prove that $R_n$, defined by

\[
R_n = \delta \sum_{k=0}^{n-1} \frac{b_k}{n^{1/\alpha}} (1 + \frac{s}{\Delta} + \delta)^{2(k-\lfloor \kappa \rfloor)} = \delta \sum_{k=-\lfloor \kappa \rfloor}^{n-1-\lfloor \kappa \rfloor} \frac{b_k^{\lfloor \kappa \rfloor} + k}{n^{1/\alpha}} (1 + \frac{s}{\Delta} + \delta)^{2k},
\]

converges to the constant $\frac{\Delta}{n^{1/\alpha}}e^{-s}$. If $t^* = \delta [t/\delta]$, we may rewrite $R_n$ as

\[
R_n = \int_{-\delta[\kappa]}^{\delta(n-\lfloor \kappa \rfloor)} \left(\frac{b_{\lfloor \kappa \rfloor} + t^*/\delta}{n^{1/\alpha}}\right) (1 + \frac{s}{\Delta} + \delta)^{2t^*/\delta} dt.
\]

We look at each term in the integrand. We have

\[
\lim_{n \to \infty} \left(1 + \frac{s}{\Delta} + \delta\right)^{2t^*/\delta} = e^{2t}
\]

and, for

\[
\left(\frac{b_{\lfloor \kappa \rfloor} + t^*/\delta}{n^{1/\alpha}}\right)^{-1} = n^{1/\alpha} \int_0^\infty x^{2\lfloor \kappa \rfloor + t^*/\delta} + 1 e^{-2\alpha V(x)} dx,
\]

we can split the integral in two regions. We can use that $V(x) > \log x$ for $x \neq 1$, that $V$ is bounded from below and the strong confinement condition (8), to see that

\[
\lim_{n \to \infty} n^{1/\alpha} \int_{[0, 1-\varepsilon] \cup [1+\varepsilon, \infty)} x^{2\lfloor \kappa \rfloor + t^*/\delta} + 1 e^{-2\alpha V(x)} dx = 0
\]

for any $\varepsilon > 0$. We have then
\[ n^{1/\alpha} \int_{1-\varepsilon}^{1+\varepsilon} x^{2([\kappa]+t^*/\delta)+1} e^{-2\kappa V(x)} dx = n^{1/\alpha} \int_{1-\varepsilon}^{1+\varepsilon} x^{2t^*/\delta+1+2([\kappa]-\kappa)} e^{-2\kappa (V(x)-\log x)} dx \]

where we have bounded the integrand in \([-\varepsilon n^{1/\alpha}, \varepsilon n^{1/\alpha}]\) by bounding \(V(1+x) - \log(1+x)\) from below by a multiple of \(|x|^\alpha\) for \(x \in [-\varepsilon, \varepsilon]\) and applied Lebesgue’s dominated convergence theorem. We may notice that there exists \(C > 0\) and \(\varepsilon > 0\) such that by defining

\[ E_C(x) = \begin{cases} 
\exp(Cx) & \text{if } x \leq 0 \\
\exp(C^{-1}x) & \text{if } x > 0
\end{cases} \]

we have, for \(n\) large enough,

\[
\left( \frac{b_{[\kappa]+t^*/\delta}}{n^{1/\alpha}} \right) \left( 1 + \frac{n}{\Delta} \delta + \delta \right)^{2t^*/\delta} \frac{1}{1-\delta([\kappa],[\delta])}(t)
\]

\[
\leq C \left( \int_{-\varepsilon n^{1/\alpha}}^{0} E_C \left( \frac{tx}{\delta n^{1/\alpha}} \right) e^{-C|x|^\alpha} dx \right)^{-1} E_{C-1} \left( \left( \frac{s}{\Delta} + 1 \right) t \right) \frac{1}{1-\delta([\kappa],[\delta])}(t)
\]

\[
\leq C \left( \int_{-\varepsilon n^{1/\alpha}}^{0} E_C \left( \frac{(n-|\kappa|)x}{n^{1/\alpha}} \right) e^{-C|x|^\alpha} dx \right)^{-1} E_{C-1} \left( \left( \frac{s}{\Delta} + 1 \right) t \right) \frac{1}{1-\delta([\kappa],[\delta])}(t)
\]

\[
\leq \tilde{C} E_{C-1} \left( \left( \frac{s}{\Delta} + 1 \right) t \right) \frac{1}{1-\delta([\kappa],[\delta])}(t),
\]

where we have used that \((n-|\kappa|)/n^{1/\alpha} \to 0\) to find the bound \(\tilde{C}\). Since \(s\) is fixed, we may bound the previous expression to apply Lebesgue’s dominated convergence theorem and conclude that

\[
\int_{-\delta([\kappa])}^{\delta([\kappa])} \left( \frac{b_{[\kappa]+t^*/\delta}}{n^{1/\alpha}} \right) \left( 1 + \frac{n}{\Delta} \delta + \delta \right)^{2t^*/\delta} dt \to \frac{2}{A} \int_{-\infty}^{\xi} e^{2t} dt = \frac{e^{-2\xi}}{A}
\]

where

\[
\xi = \lim_{n \to \infty} \delta([\kappa] - n) = \lim_{n \to \infty} \left( \frac{\Delta}{2\lambda_+ \kappa} \right)^{1/\alpha} (\kappa - n) = \lim_{n \to \infty} \left( \frac{\log n}{2\lambda_+ n} \right)^{1/\alpha} (\kappa - n).
\]

**Proposition 5.7** (Intensity convergence for a ‘logarithm’ potential). Under the conditions and notation of Theorem 2.6, if we define

\[ b^{(n)}_k = \left( \int_0^{\infty} x^{2k+1} e^{-2\kappa_n V(x)} dx \right)^{-1}, \]

we have, for every \(s \in \mathbb{R}\),

\[
\frac{1}{2(\kappa_n - n)} \sum_{k=0}^{n-1} b^{(n)}_k \left( 1 + \frac{\Delta_n}{2(\kappa_n - n)} + \frac{s}{2(\kappa_n - n)} \right)^{2k+1} e^{-2\kappa_n V(1+\frac{\Delta_n}{2(\kappa_n - n)} + \frac{s}{2(\kappa_n - n)})} \to \frac{e^{-s}}{1+A}.
\]
Proof. First we notice that, since \( \kappa_n - n \) goes to infinity and since \( e^{\Delta_n \Delta_n} = \kappa_n - n \), we obtain that \( \Delta_n \to \infty \). Then, since \( \Delta_n \sim \Delta_n + \log \Delta_n = \log(\kappa_n - n) \), we get that

\[
\Delta_n \sim \log(\kappa_n - n).
\]

Let

\[
c_n = 1 + \frac{\Delta_n}{2(\kappa_n - n)} \quad \text{and} \quad d_n = \frac{1}{2(\kappa_n - n)}.
\]

We want to find the limit of

\[
d_n \sum_{k=0}^{n-1} b_k^{(n)} (c_n + d_n s)^k e^{-2\kappa_n V(c_n + d_n s)}.
\]  \hfill (25)

For the potential term we have

\[
\exp \left( -2\kappa_n V(c_n + d_n s) + 2n \log (c_n + d_n s) \right)
\]

\[
= \exp \left( 2(n - \kappa_n) \log \left( 1 + \frac{\Delta_n}{2(\kappa_n - n)} + \frac{s}{2(\kappa_n - n)} \right) \right)
\]

\[
\sim e^{-\Delta_n} e^{-s} = \frac{\Delta_n}{\kappa_n - n} e^{-s}.
\]

If we define \( \delta_n = \frac{\Delta_n}{2(\kappa_n - n)} \), the limit of (25) will be the same as the limit of

\[
e^{-s} \left( \delta_n \sum_{k=0}^{n-1} b_k^{(n)} \frac{1}{\kappa_n - n} \left( 1 + \frac{s}{\Delta_n} \delta_n + \delta_n \right)^{2(k-n)} \right).
\]

We want to prove that \( R_n \), defined by

\[
R_n = \delta_n \sum_{k=0}^{n-1} b_k^{(n)} \frac{1}{\kappa_n - n} \left( 1 + \frac{s}{\Delta_n} \delta_n + \delta_n \right)^{2(k-n)} = \delta_n \sum_{k=-n}^{-1} b_k^{(n)} \frac{1}{\kappa_n - n} \left( 1 + \frac{s}{\Delta_n} \delta_n + \delta_n \right)^{2k},
\]

converges to the constant \( \frac{1}{1+e^A} \). Define \( t^* = \delta_n \lfloor t/\delta_n \rfloor \) so that

\[
R_n = \int_{-n\delta_n}^{0} \frac{b_k^{(n)} t^*/\delta_n}{\kappa_n - n} \left( 1 + \frac{s}{\Delta_n} \delta_n + \delta_n \right)^{2t^*/\delta_n} dt.
\]

We look at each term in the integrand. We have

\[
\lim_{n \to \infty} \left( 1 + \frac{s}{\Delta_n} \delta_n + \delta_n \right)^{2t^*/\delta_n} = e^{2t}
\]

and, for

\[
\left( \frac{b_k^{(n)} t^*/\delta_n + n}{\kappa_n - n} \right)^{-1} = (\kappa_n - n) \int_0^\infty x^{2t^*/\delta_n + 2n + 1} e^{-2\kappa_n V(x)} dx
\]

we can split the integral in three regions. The integral from 1 to \( \infty \) can be explicitly calculated and we obtain

\[
(\kappa_n - n) \int_1^\infty x^{2t^*/\delta_n + 2n + 1} e^{-2\kappa_n V(x)} dx \to \frac{1}{2}
\]
For the integral from 0 to 1, for any \(\varepsilon > 0\), we may use that \(V(x) > \log x\) for \(x < 1\) and that \(V\) is bounded from below to obtain

\[
(\kappa_n - n) \int_0^{1-\varepsilon} x^{2t/\delta_n + 2n+1} e^{-2\kappa_n V(x)} dx \xrightarrow{n \to \infty} 0.
\]

As in the proof of Proposition 5.6, the integral from 1 to \(1 - \varepsilon\) can be understood by controlling \(V(1 + x) - \log(1 + x)\) from below by a multiple of \(|x|^\alpha\) and by using Lebesgue’s dominated convergence theorem. We obtain

\[
(\kappa_n - n) \int_1^{1-\varepsilon} x^{2t/\delta_n + 2n+1} e^{-2\kappa_n V(x)} dx
\]

\[
= \kappa_n - n \int_{-\varepsilon^{1/\alpha}}^{0} \left(1 + \frac{x}{\kappa_n^{1/\alpha}}\right) \left(2\kappa_n^{1/\alpha} + 2(n-\kappa_n)\right) e^{-2\kappa_n \left[V\left(1 + \frac{x}{n^{1/\alpha}}\right) - \log\left(1 + \frac{x}{n^{1/\alpha}}\right)\right]} dx
\]

\[
\xrightarrow{c \int_{-\infty}^{0} e^{-2\frac{x}{\alpha}|x|^\alpha} dx}
\]

\[
= c \Gamma\left(\frac{1}{\alpha}\right) \alpha^{-1/2}(2\lambda)^{-1/\alpha} = \frac{A}{2}.
\]

We can see that there exists \(C > 0\) such that, for \(n\) large enough,

\[
\left(\frac{b^{(n)}_{t^{1/\delta_n}}}{\kappa_n - n}\right) \left(1 + \frac{s}{\Delta_n^{1/\alpha}}\right) \left(2t/\delta_n + \frac{\kappa_n - n}{\kappa_n - n - 1}\right) \exp\left(\frac{s}{\Delta_n + 1}\right) \left(1 - \log\left(1 + \frac{x}{n^{1/\alpha}}\right)\right) \xrightarrow{n \to \infty} 0.
\]

By Lebesgue’s dominated convergence theorem, justified by the bound (26), we get

\[
R_n = \int_{-n\delta_n}^0 \frac{b^{(n)}_{t^{1/\delta_n + n}}}{\kappa_n - n} \left(1 + \frac{s}{\Delta_n^{1/\alpha}}\right) \left(2t/\delta_n + \frac{\kappa_n - n}{\kappa_n - n - 1}\right) \left(1 - \log\left(1 + \frac{x}{n^{1/\alpha}}\right)\right) dt \xrightarrow{n \to \infty} \frac{2}{1 + A} \int_{-\infty}^0 e^{2t} dt = \frac{1}{1 + A},
\]

where we are also using that \(n\delta_n \to \infty\), consequence of the fact that \(\frac{s}{n^{1/\alpha}}\) is bounded and that \(\Delta_n \sim \log(\kappa_n - n)\).

5.4 Proofs of Theorem 2.5 and Theorem 2.6

Proof of Theorem 2.5. Recall that \(\delta_n = \left(\frac{\Delta_n}{\Delta_n + \varepsilon_n}\right)^{1/\alpha}\). If we define \(c_n = 1 + \delta_n\) and \(d_n = \delta_n\Delta_n^{-1}\), we want to prove that, for every \(t \in \mathbb{R}\),

\[
\lim_{n \to \infty} \mathbb{P}(M_n \leq c_n + d_n t) = \exp\left(-\frac{1}{A} e^{-t - 2\delta}\right).
\]

Consider the intensity of the process of moduli \(\tilde{\rho}_n\) given by

\[
\tilde{\rho}_n(r) = r \sum_{k=0}^{n-1} b^{(n)}_k r^{2k} e^{-2\kappa_n V(r)} \quad \text{where} \quad b^{(n)}_k = \left(\int_0^\infty s^{2k+1} e^{-2\kappa_n V(s)} ds\right)^{-1}
\]

By Lemma 5.3 and Lemma 5.1, we only need to prove that, for every \(x \in \mathbb{R}\),

\[
\lim_{n \to \infty} \int_x^\infty d_n \tilde{\rho}_n(c_n + d_n s) ds = \exp\left(-\frac{1}{A} e^{-x - 2\delta}\right)
\]
as explained in Subsection 5.1. By Proposition 5.6 we have that
\[ d_n \tilde{\rho}_n (c_n + d_n s) \xrightarrow{n \to \infty} e^{-2\xi e^{-s}} A \]
for every \( s \in \mathbb{R} \). Now, we need to bound \( d_n \tilde{\rho}_n (c_n + d_n s) \) and apply Lebesgue’s dominated convergence theorem. By Lemma 5.3 and Lemma 5.4 we know that
\[ \int_{1+\frac{1}{(\log n)^r}}^{\infty} \tilde{\rho}_n (r) dr \xrightarrow{n \to \infty} 0 \]
so that we only need to control \( d_n \tilde{\rho}_n (c_n + d_n s) \) when \( c_n + d_n s \leq 1 + 1/(\log n)^\gamma \). Recall that in the proof of Proposition 5.6, it was convenient to write \( d_n \tilde{\rho}_n (c_n + d_n s) = d_n \sum_{k=0}^{n-1} b_k^{(n)} (c_n + d_n s)^{2(k-\kappa_n)+1} e^{-2\xi_n (V(c_n + d_n s) - \log(c_n + d_n s))} \).

If we integrate \( \int_1^{\infty} \) in \( t \) we may see that the term \( d_n \sum_{k=0}^{n-1} b_k^{(n)} (c_n + d_n s)^{2(k-\kappa_n)+1} \) can be bounded for \( 1 - \varepsilon < c_n + d_n s < 1 + \varepsilon \) by a constant. Meanwhile, by taking \( \varepsilon > 0 \) from condition \( (6) \) and choosing any \( \gamma > 1/\varepsilon \), we can find \( C > 0 \) such that \( e^{-2\xi_n (V(c_n + d_n s) - \log(c_n + d_n s))} \) is bounded by a multiple of \( e^{-Cs} \) in the region where \( s \geq x \) and \( c_n + d_n s \leq 1 + 1/(\log n)^\gamma \). This completes the study of the maxima.

The convergence of the point process of moduli is explained at the end of Subsection 5.1.

**Proof of Theorem 2.6.** It follows the same steps as the proof of Theorem 2.5 where now we define \( c_n = 1 + \frac{\Delta_n}{2(\kappa_n - \gamma)} \) and \( d_n = \frac{1}{2(\kappa_n - \gamma)} \). Lemma 5.2 justifies that we only need to prove
\[ \lim_{n \to \infty} \int_{1+\varepsilon}^{\infty} d_n \tilde{\rho}_n (c_n + d_n s) ds = \exp\left(-(1+A)^{-1}e^{-x}\right) \]
for every \( x \in \mathbb{R} \). Lemma 5.3 allows us to focus on \( c_n + d_n s \leq 1 + \frac{1}{\sqrt{\kappa_n - \gamma}} \) while \( (20) \) together with \( V(r) = \log r \) for \( r \geq 1 \) allows us to bound \( d_n \tilde{\rho}_n (c_n + d_n s) \) by a multiple of \( e^{-s} \) in the region where \( s \geq x \) and \( c_n + d_n s \leq 1 + \frac{1}{\sqrt{\kappa_n - \gamma}} \). The proof may be completed by using Proposition 5.7.

### 6 Proofs of the exponential behavior

We begin by the proof of Theorem 2.7, which is motivated by (24). It follows almost the same steps as the one of Theorem 2.5 and Theorem 2.6 with the main difference that we can now use the behavior of the kernel at the edge stated in Proposition 2.8.

**Proof of Theorem 2.7.** For this case, the convergence of the point process of moduli implies the fluctuations of the maxima. Nevertheless, for simplicity, we shall follow the steps explained in Subsection 5.1 and show first the fluctuations of the maxima. Let us choose \( m_n = 1 - \frac{x}{n} \) and recall that
\[ \tilde{\rho}_n (r) = \sum_{k=0}^{n-1} b_k^{(n)} r^{2k+1} e^{-2\xi_n V(r)} \quad \text{where} \quad b_k^{(n)} = \left( \int_0^1 s^{2k+1} e^{-2\xi_n V(s)} ds \right)^{-1} \]

To see that \( (21) \) holds we notice that there exists \( C < 1 \) such that
\[ b_k^{(n)} \leq \left( \int_{1-\varepsilon}^1 s^{2k+1} e^{-2\xi_n C \log s} ds \right)^{-1} \leq \left( \int_{1-\varepsilon}^1 s^{2n-1} e^{-2\xi_n C \log s} ds \right)^{-1} \]
which is bounded by $\tilde{C}n$ for some $\tilde{C} > 0$ independent of $n$ and $k$. Then, we have
\[
\sup_{k \in \{0, \ldots, n-1\}} b_k^{(n)} \int_{1-\frac{\tilde{C}}{n}}^{1} e^{-2\kappa V(r)} dr \leq \tilde{C} n \int_{1-\frac{\tilde{C}}{n}}^{1} r e^{-2(1-Q) \log r} dr \xrightarrow{n \to \infty} 0.
\]
Define
\[
f_n(r) = \frac{1}{n^2} \sum_{k=0}^{n-1} b_k^{(n)} \left(1 + \frac{r}{n}\right)^{2k} e^{-2\kappa (1+\frac{r}{n})} \quad \text{and} \quad f(r) = 2 \int_0^Q (q-t) e^{2r(q-t)} dt.
\]
Notice that $2\kappa V \left(1 + \frac{r}{n}\right) \to 2r(1-q)$ uniformly on compact sets of $(-\infty, 0]$. Then, we may use Proposition 2.8 and recall that $2\pi a_k^{(n)} = b_k^{(n)}$ to obtain that
\[
f_n(r) \xrightarrow{n \to \infty} f(r)
\]
for $r$ uniformly on compact sets of $(-\infty, 0]$. By the steps described in Subsection 5.1, we are interested in the limit of
\[
\frac{1}{n^2} \rho_n \left(1 + \frac{s}{n^2}\right) = \left(1 + \frac{s}{n^2}\right) f \left(\frac{s}{n}\right).
\]
But since the convergence of $f_n$ towards $f$ is uniform in compact sets, we have that
\[
\frac{1}{n^2} \rho_n \left(1 + \frac{s}{n^2}\right) \xrightarrow{n \to \infty} f(0),
\]
for $s$ uniformly on compact sets of $\mathbb{R}$. Then
\[
\int_0^x \frac{1}{n^2} \rho_n \left(1 + \frac{s}{n^2}\right) ds \xrightarrow{n \to \infty} -xf(0) = -2x \int_0^Q (q-t) dt = x \left( q^2 - (q - Q)^2 \right),
\]
which is what we wanted to prove. For the point process convergence, we may proceed as explained at the end of Subsection 5.1.

Now, we proceed to the proof of Proposition 2.8.

**Proof of Proposition 2.8.** Define
\[
F_n(z) = \frac{1}{n} \sum_{k=0}^{n-1} a_k^{(n)} \left(1 + \frac{z}{n}\right)^k.
\]
Then, we have to prove that
\[
\lim_{n \to \infty} F_n \left( z + \bar{w} + \frac{z\bar{w}}{n} \right) = \int_0^Q \frac{q-t}{\pi} e^{-t(z+\bar{w})} dt
\]
for $(z, w)$ uniformly on compact sets of $\mathbb{C} \times \mathbb{C}$. Since $|F_n(z)| \leq F_n(r)$ whenever $|z| \leq r$, by Montel’s theorem it is enough to show that
\[
\lim_{n \to \infty} F_n(r) = \int_0^Q \frac{q-t}{\pi} e^{-rt} dt
\]
for every $r \geq 0$. Define $t^* = \frac{|tn|}{n}$ so that
\[
F_n(r) = \int_0^1 \frac{a_k^{(n)}}{n} \left(1 + \frac{r}{n}\right)^{nt^*} dt.
\]
We will study the limiting behavior of
\[
\left( \frac{a_{nt^*}}{n} \right)^{-1} = 2\pi n \int_0^1 s^{2 nt^* + 1} e^{-2\kappa_n V(s)} ds = n \int_0^\infty e^{-2\kappa_n V(s) + 2 nt^* \log s} s ds
\]
and, due to Laplace’s method, it will depend on the points where \( V(s) - t \log s \) is minimum. We know that the minimum is negative if \( t < 1 - Q \) so that the sequence converges exponentially to infinity. On the other hand, if \( t > 1 - Q \), the minimum is zero and it is attained at \( s = 1 \) so that
\[
\left( \frac{a_{nt^*}}{n} \right)^{-1} \xrightarrow{n \to \infty} 2\pi \int_0^1 \left( 1 + \frac{s}{n} \right)^{2 nt^* + 1} e^{-2\kappa_n V(1 + \frac{s}{n})} ds
\]
By bounding \( V(s) \) from above by a multiple of \( \log s \) we can see that there is \( C > 0 \) such that
\[
\frac{a_{nt^*}}{n} \left( 1 + \frac{r}{n} \right)^{nt^*} 1_{[0,1]}(t) \leq C(t + C)e^{rt} 1_{[0,1]}(t)
\]
so that we may apply Lebesgue’s dominated convergence theorem and obtain
\[
F_n(r) \xrightarrow{n \to \infty} \int_{1-Q}^{1} \frac{t - (1 - q)}{r} \pi e^{rt} dt = \int_0^Q \frac{q - s}{\pi} e^{r(1-s)} ds.
\]

7 Appendices

7.1 Random number of particles

For completeness, we state here the complementary version of Theorem 2.4. Here the precise limiting behavior will depend on some other parameters which make the statement a little more cumbersome.

**Theorem 7.1** (Finite limiting process: Random number of particles). Suppose that, for some positive numbers \( \alpha, \gamma, Q_+, Q_- > 0 \),

- \( V(r) > \log r \) for every \( r > 1 \),
- \( \lim_{r \to \infty} r^{\alpha} (V(r) - \log r) = \gamma \),
- \( \lim_{r \to 1^+} \frac{V(r)}{r - 1} = 1 + Q_+ \) and
- \( \lim_{r \to 1^-} \frac{V(r)}{r - 1} = 1 - Q_- \).

Suppose that, for some \( \chi \in (0, \alpha/2] \) such that \( \alpha/2 - \chi \in \mathbb{Z} \) and for some \( \xi \in \mathbb{R} \),
\[
\kappa_n = n + \chi + \frac{\xi}{\log n} + o \left( \frac{1}{\log n} \right).
\]
Consider a sequence \( \{Y_k\}_{k \geq 0} \) of independent random variables where \( Y_k \) follows the law
\[
\frac{r^{-2(k+\chi)} e^{-r^{-\alpha}} 1_{(0,\infty)}(r) dr}{\alpha \Gamma \left( \frac{2(k+\chi)}{\alpha} \right) r}
\]
and define
\[ p = \frac{\alpha \gamma e^{2\xi/\alpha}(Q_+ + Q_-)}{Q_+ Q_-} \]

Consider a random variable \( N \) independent of \( \{Y_k\}_{k \geq 0} \) such that
\[ \mathbb{P}\left(N = \frac{\alpha}{2} - \chi\right) = \frac{p}{1 + p} \quad \text{and} \quad \mathbb{P}\left(N = \frac{\alpha}{2} - \chi + 1\right) = \frac{1}{1 + p}. \]

Then, for every continuous function \( f : [0, \infty) \to \mathbb{R} \) whose support is contained in \((0, \infty)\),
\[ \sum_{k=0}^{n} f\left(n^{-1/\alpha}|x_k^{(n)}|\right) \xrightarrow{\text{law}} \sum_{k=0}^{N-1} f\left((2\gamma)^{1/\alpha} Y_k\right). \]

In particular, \( n^{-1/\alpha} \max\{|x_1^{(n)}|, \ldots, |x_n^{(n)}|\} \) converges in law to the maximum of \( \{Y_k\}_{0 \leq k \leq N-1} \).

More explicitly, for every \( t > 0 \),
\[ \lim_{n \to \infty} \mathbb{P}\left(n^{-1/\alpha} \max\{|x_1^{(n)}|, \ldots, |x_n^{(n)}|\} \leq t\right) = \left(\prod_{k=0}^{\alpha/2-1} \Gamma\left(\frac{2(k+1)+\alpha}{\alpha}\right) \frac{\Gamma\left(\frac{2(k+1)+\alpha}{\alpha}\right)}{\Gamma\left(2\gamma t^{-\alpha}\right)} \right)\left(e^{-2\gamma t^{-\alpha}} + (1 - e^{-2\gamma t^{-\alpha}}) \frac{p}{1 + p}\right). \]

**Proof.** As in the proof of Theorem 2.4, this theorem is a consequence of Proposition 4.4 together with [2, Theorem 4.5.3]. We could have also used [2, Theorem 4.7.1] which is specific for radial systems. \( \Box \)

Notice that in the extreme case \( \alpha = 2\chi \) the limiting distribution of the maxima is a convex combination of a Fréchet distribution and a Dirac measure at zero. The same techniques can be used to generalize this theorem to a case where \( V(r) \) has different kind of singularities at \( r = 1 \). In those generalizations actual Fréchet distributions can be obtained and the point process that accompanies the farthest particle would go to a point process with just one particle.

### 7.2 Comparison between the kernel and Gumbel fluctuations

It is interesting to compare the results from Theorem 2.5 and Theorem 2.6 with the limiting behavior of the point process at a point \( z \) of the unit circle, let us say \( z = 1 \). Since, as explained in [12], the convergence of the point process is implied by the convergence of the kernels, we shall state now the convergence of these kernels (up to a conjugation of norm one). The proofs are given at the end of this subsection. Define
\[ K_n(z, w) = \sum_{k=0}^{n-1} a^{(n)}_k z^k w^{k+1} e^{-\kappa_\alpha V(|z|)} e^{-\kappa_\alpha V(|w|)} \quad \text{where} \quad a^{(n)}_k = \left(2\pi \int_0^\infty s^{2k+1} e^{-2\kappa_\alpha V(s)} ds\right)^{-1} \]

and let
\[ \tilde{K}_n(z, w) = \left(\frac{z}{|z|}\right)^{[\kappa_\alpha]} K_n(z, w) \left(\frac{w}{|w|}\right)^{[\kappa_\alpha]}. \]

We will state the versions for \( \alpha > 1 \) since the versions for \( \alpha = 1 \) are somewhat longer to state.

**Proposition 7.2** (Limiting kernel for strongly confining potentials). Suppose \( V \) satisfies the standard properties and
- \( V(r) = \log r + \frac{\lambda}{\alpha}(r - 1)^\alpha + o(r - 1)^\alpha \) as \( r \to 1 \), \( (27) \)
- \( V(r) \geq \log r \) for every \( r > 1 \) and
for some $\alpha > 1$ and $\lambda > 0$. Suppose that $\kappa_n/n \to 1$ and that
\[
\lim_{n \to \infty} \frac{n - \kappa_n}{n^{1/\alpha}} = \zeta \in (-\infty, \infty],
\]

Then, if $W : \mathbb{C} \to \mathbb{R}$ is defined by $W(x + iy) = \frac{\lambda}{\alpha} |x|^\alpha$, we have that
\[
\lim_{n \to \infty} \frac{1}{n^{2/\alpha}} K_n \left(1 + \frac{z}{n^{1/\alpha}}, 1 + \frac{w}{n^{1/\alpha}}\right) = \frac{e^{-W(z)e^{-W(w)}}}{2\pi} \int_{-\infty}^{\zeta} e^{(z + \bar{w})t} \int_{-\infty}^{\infty} e^{2st} e^{-2W(s)} ds dt
\]
for $(z, w)$ uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}$.

Notice that in condition (27) there is no need of $\varepsilon$ as in (6) from Theorem 2.5. When $\zeta = \infty$ we obtain the weighted Bergman kernel with weight
\[
x + iy \mapsto e^{-2 \frac{|x|^\alpha}{\alpha}}
\]
which is, for $\alpha = 2$, the well-known Ginibre kernel up to a conjugation. For $\zeta < \infty$, we obtain a truncated version of a Bergman kernel. Moreover, for $\alpha = 2$ we get the well-known error function kernel if $\zeta = 0$ and a horizontally translated version of it for general finite $\zeta$. This error function kernel is known to be universal under some analyticity conditions [10].

**Proposition 7.3** (Limiting kernel for weakly confining potentials). Suppose $V$ satisfies the standard properties and
\[
V(r) = \log r + \frac{\lambda}{\alpha} (1 - r)^\alpha + o(1 - r)^\alpha \quad \text{as} \quad r \to 1^{-} \quad \text{and} \quad V(r) = \log r \quad \text{for} \quad r \geq 1,
\]
for some $\alpha > 1$ and $\lambda > 0$. Suppose that $\kappa_n > n$ satisfies
\[
\lim_{n \to \infty} \frac{n - \kappa_n}{n^{1/\alpha}} = \zeta \in (-\infty, 0].
\]

Then, if $W : \mathbb{C} \to \mathbb{R}$ is defined by $W(x + iy) = \frac{\lambda}{\alpha} |\min(x, 0)|^\alpha$, we have that
\[
\lim_{n \to \infty} \frac{1}{n^{2/\alpha}} K_n \left(1 + \frac{z}{n^{1/\alpha}}, 1 + \frac{w}{n^{1/\alpha}}\right) = \frac{e^{-W(z)e^{-W(w)}}}{2\pi} \int_{-\infty}^{\zeta} e^{(z + \bar{w})t} \int_{-\infty}^{\infty} e^{2st} e^{-2W(s)} ds dt
\]
for $(z, w)$ uniformly on compact subsets of $\mathbb{C} \times \mathbb{C}$.

The version for $\alpha = 1$, not stated here, does not only depend on the behavior of the potential at the unit circle. It can be understood by using the same methods, and results similar to Proposition 7.3 would be obtained. We may see another example in [13] Theorem 4.3].

In Proposition 7.3, when $\zeta = 0$, we obtain the weighted Bergman kernel with weight
\[
x + iy \mapsto e^{-2 \frac{|\min(x, 0)|^\alpha}{\alpha}}.
\]
For $\zeta < 0$, we obtain a truncated version of that Bergman kernel. Furthermore, defining $W$ as in Proposition 7.3 we have that for $(z, w)$ uniformly on compact sets of $\{x + iy \in \mathbb{C} : x > 0\}^2$,
\[
\frac{e^{-W(z)e^{-W(w)}}}{2\pi} \int_{-\infty}^{\zeta} e^{(z + \bar{w})t} \int_{-\infty}^{\infty} e^{2st} e^{-2W(s)} ds dt \to \frac{e^{(z + \bar{w})t}}{\pi} dt
\]
which is the unweighted Bergman kernel of $\{x + iy \in \mathbb{C} : x > 0\}$ when $\zeta = 0$. The $\alpha = 2$ version
\[
K(x + iy, a + ib) = \frac{e^{-\frac{1}{2} \min(x, 0)^2} e^{-\frac{1}{2} \min(a, 0)^2}}{2\pi} \int_{-\infty}^{\zeta} \int_{-\infty}^{\infty} e^{(x+i(x+y))t} \frac{e^{2st} e^{-\lambda^2 t} ds - \frac{1}{2\pi}}{dt}
\]
has recently appeared in [1] (at least for ζ = 0) and it satisfies the nice but expected property

\[ e^{\lambda(T-x/2)y}K(x-T + iy, a - T + ib)e^{-i\lambda(T-a/2)b} \xrightarrow{T \to \infty} \frac{\lambda}{2\pi} e^{-\frac{1}{2}\lambda((x-a)^2+(y-b)^2)}e^{i\frac{\lambda}{2}(ya-xb)}, \]

where we may recognize the Ginibre kernel at the right-hand side. In this way, the α = 2 case for ζ = 0 interpolates between the determinantal point process that has the Bergman kernel as its kernel and the Ginibre point process. Now, we state the promised relation between the Gumbel fluctuations and the limiting kernels.

**Remark 7.4** (Connection between Gumbel fluctuations and the limiting kernel). Choose α > 1 and λ > 0 and take W : R → R defined by W(r) = \(\frac{1}{\alpha}|r|^\alpha\). Let us define \(f : \mathbb{R} \to \mathbb{R}\) by

\[ f(r) = e^{-2W(r)} \int_{-\infty}^{0} e^{2rt} \int_{-\infty}^{\infty} e^{2st} e^{-2W(s)} ds dt. \tag{28} \]

Using a change of variables in the integral, we may see that

\[ f(r) \sim_{r \to \infty} \frac{e^{-2\alpha r}}{2r \int_{-\infty}^{\infty} e^{-2W(s)} ds}. \]

Then, the sought connection, which may be seen as a connection between Proposition 7.2 and Proposition 5.7, is the convergence

\[ \left( \frac{\Delta_n}{2\lambda n} \right)^{1/\alpha} \frac{n^{2/\alpha}}{\Delta_n} f \left( n^{1/\alpha} \left( \frac{\Delta_n}{2\lambda n} \right)^{1/\alpha} \left( 1 + \frac{s}{\Delta_n} \right) \right) \xrightarrow{n \to \infty} \frac{e^{-s}}{2f \int_{-\infty}^{\infty} e^{-2W(s)} ds}, \]

where we recognize A from Theorem 2.5 as the denominator. Nevertheless, the parameter ξ in Theorem 2.5 cannot be guessed by only using the behavior at the unit circle. On the other hand, if we define \(W(r) = \frac{\lambda}{\alpha} \min(r,0)^\alpha\) and f as in (28), we notice that

\[ f(r) \sim_{r \to \infty} \frac{1}{2r^2}. \]

In particular, the behavior at the unit circle would not allow us to guess Theorem 2.6.

Now, we proceed to give the proofs of Proposition 7.2 and Proposition 7.3. The ideas are somewhat simpler than the ones used to prove Proposition 5.6 and Proposition 5.7.

**Proof of Proposition 7.2.** Define

\[ F_n(z) = \frac{1}{n^{2/\alpha}} \sum_{k=0}^{n-1} \theta_k^{(n)} \left( 1 + \frac{z}{n^{1/\alpha}} \right)^{k-\lfloor \kappa_n \rfloor}, \quad \theta_n(z) = \left| 1 + \frac{z}{n^{1/\alpha}} \right|^{\lfloor \kappa_n \rfloor - \kappa_n} \]

and

\[ W_n(z) = \kappa_n \left( V \left( \left| 1 + \frac{z}{n^{1/\alpha}} \right| \right) - \log \left| 1 + \frac{z}{n^{1/\alpha}} \right| \right) \]

so that we have

\[ \frac{1}{n^{2/\alpha}} \tilde{K}_n \left( 1 + \frac{z}{n^{1/\alpha}}, 1 + \frac{w}{n^{1/\alpha}} \right) = F_n \left( z + \bar{w} + \frac{z\bar{w}}{n^{1/\alpha}} \right) e^{-W_n(z)} e^{-W_n(w)} \theta_n(z) \theta_n(w). \]

Then, it is enough to show that \(F_n, W_n\) and \(\theta_n\) converge uniformly on compact sets of \(\mathbb{C}\). Since \(\lfloor \kappa_n \rfloor - \kappa_n\) is bounded we have that, for \(z\) uniformly on compact sets of \(\mathbb{C}\),

\[ \theta_n(z) \xrightarrow{n \to \infty} 1. \]
By (27) we have that, for \( x + iy \) uniformly on compact sets of \( \mathbb{C} \),

\[
W_n(x + iy) \xrightarrow{n \to \infty} \frac{\lambda}{\alpha} |x|^\alpha.
\]

We only need to understand \( F_n \). But for any \( R > 0 \), sup\( |z| \leq R \) \( |F_n(z)| \leq F_n(R) + F_n(-R) \) for \( n \) large enough. So, if \( F_n|_{\mathbb{R}} \) converges pointwise, by Montel’s theorem \( F_n \) converges uniformly on compact sets of \( \mathbb{C} \). Now, take \( r \in \mathbb{R} \), define \( t^* = \frac{|\kappa_{n}|}{n^{1/\alpha}} \) and perform the change of variables

\[
t^* n^{1/\alpha} = k - |\kappa_{n}| \quad \text{so that}
\]

\[
F_n(r) = \int_{- \frac{|\kappa_{n}|}{n^{1/\alpha}}}^{\frac{R - |\kappa_{n}|}{n^{1/\alpha}}} \frac{a_{[\kappa_{n}] + t^* n^{1/\alpha}}}{n^{1/\alpha}} \left( 1 + \frac{r}{n^{1/\alpha}} \right) t^{* n^{1/\alpha}} \, dt.
\]

We have

\[
\lim_{n \to \infty} \left( 1 + \frac{r}{n^{1/\alpha}} \right) t^{* n^{1/\alpha}} = e^{rt}.
\]

For the term

\[
\left( \frac{a_{[\kappa_{n}] + t^* n^{1/\alpha}}}{n^{1/\alpha}} \right)^{-1} = 2\pi n^{1/\alpha} \int_{0}^{\infty} s^{2([\kappa_{n}] + t^* n^{1/\alpha}) + 1} e^{-2\kappa_{n} V(s)} \, ds
\]

\[
= 2\pi n^{1/\alpha} \int_{0}^{\infty} s^{2t^* n^{1/\alpha} + 2([\kappa_{n}] - \kappa_{n}) + 1} e^{-2\kappa_{n} V(s) - \log s} \, ds
\]

\[
= 2\pi \int_{-\infty}^{\infty} \left( 1 + \frac{s}{n^{1/\alpha}} \right)^{2t^* n^{1/\alpha} + 2([\kappa_{n}] - \kappa_{n}) + 1} e^{-2\kappa_{n} V(1 + \frac{s}{n^{1/\alpha}}) - \log \left( 1 + \frac{s}{n^{1/\alpha}} \right)} \, ds
\]

we can notice that it is enough to consider the integral from \(-\varepsilon n^{1/\alpha}\) to \(\varepsilon n^{1/\alpha}\). Indeed, since \( V(s) > \log s \) if \( s > 1 \) and since the potential is strongly confining, we may see that

\[
2\pi n^{1/\alpha} \int_{1 + \varepsilon}^{\infty} s^{2([\kappa_{n}] + t^* n^{1/\alpha}) + 1} e^{-2\kappa_{n} V(s)} \, ds \xrightarrow{n \to \infty} 0
\]

for every \( \varepsilon > 0 \). On the other hand, since \( V(s) > \log s \) if \( s < 1 \) and since the potential is bounded near the origin, we have that

\[
2\pi n^{1/\alpha} \int_{0}^{1 - \varepsilon} s^{2([\kappa_{n}] + t^* n^{1/\alpha}) + 1} e^{-2\kappa_{n} V(s)} \, ds \xrightarrow{n \to \infty} 0
\]

for every \( \varepsilon > 0 \). Then, for \( \varepsilon > 0 \) small enough we may use (27) to control the integrand in (29) from \(-\varepsilon n^{1/\alpha}\) to \(\varepsilon n^{1/\alpha}\) and apply Lebesgue’s dominated convergence theorem to obtain

\[
\left( \frac{a_{[\kappa_{n}] + t^* n^{1/\alpha}}}{n^{1/\alpha}} \right)^{-1} \xrightarrow{n \to \infty} 2\pi \int_{-\infty}^{\infty} e^{2st} e^{-\frac{2\kappa_{n}}{s^{1/\alpha}}} \, ds.
\]

Since there exists \( C > 0 \) and \( \varepsilon > 0 \) such that if we define

\[
E_C(x) = \begin{cases} 
\exp(Cx) & \text{if } x \leq 0 \\
\exp(C^{-1} x) & \text{if } x > 0
\end{cases}
\]

we have

\[
\frac{a_{[\kappa_{n}] + t^* n^{1/\alpha}}}{n^{1/\alpha}} \left( 1 + \frac{r}{n^{1/\alpha}} \right)^{2t^* n^{1/\alpha}} \left( 1 - \frac{|\kappa_{n}|}{n^{1/\alpha}} \right)^{s_{[\kappa_{n}]}}(s) \leq C \left( \int_{-\varepsilon n^{1/\alpha}}^{\varepsilon n^{1/\alpha}} E_C(st) e^{-C|s|^{1/\alpha}} \, ds \right)^{-1} E_{C^{-1}}(rt),
\]
where the right-hand side is integrable in $t$ for $n$ large enough, we can apply Lebesgue’s dominated convergence theorem to conclude

$$
\lim_{n \to \infty} F_n(r) = \frac{1}{2\pi} \int_{-\infty}^{\eta} \frac{e^{rt}}{\int_{-\infty}^{\infty} e^{2st} e^{-2k|s|^\alpha} ds} dt.
$$

**Proof of Proposition 7.3** As in the proof of Proposition 7.2 we only need to understand

$$
F_n(z) = \frac{1}{n^{2/\alpha}} \sum_{k=0}^{n-1} a_k^{(n)} \left( 1 + \frac{z}{n^{1/\alpha}} \right)^{k-[\kappa n]}.
$$

As before, we only need to study the pointwise convergence of $F_n|_{\mathbb{R}}$. Take $r \in \mathbb{R}$ and define $t^* = \left\lfloor \frac{tn^{1/\alpha}}{n^{1/\alpha}} \right\rfloor$ so that

$$
F_n(r) = \int_{-\frac{[\kappa n]}{n^{1/\alpha}}}^{\frac{2-[\kappa n]}{n^{1/\alpha}}} \frac{a_k^{(n)} + t^* n^{1/\alpha}}{n^{1/\alpha}} (1 + \frac{r}{n^{1/\alpha}})^{t^* n^{1/\alpha}} dt.
$$

We have

$$
\lim_{n \to \infty} \left( 1 + \frac{r}{n^{1/\alpha}} \right)^{t^* n^{1/\alpha}} = e^{rt}.
$$

The term

$$
\left( \frac{a_k^{(n)} + t^* n^{1/\alpha}}{n^{1/\alpha}} \right)^{-1}
$$

can be split in two parts. The integral from 1 to $\infty$ can be explicitly calculated and we obtain

$$
2\pi n^{1/\alpha} \int_{1}^{\infty} s^{2([\kappa n] + t^* n^{1/\alpha}) + 1} e^{-2\kappa V(s)} ds \to \frac{\pi}{t} = 2\pi \int_{0}^{\infty} e^{2st} ds
$$

(30)

For the integral from 0 to 1 there is no difference from the proof of Proposition 7.2 and we get

$$
2\pi n^{1/\alpha} \int_{0}^{1} s^{2([\kappa n] + t^* n^{1/\alpha}) + 1} e^{-2\kappa V(s)} ds \to 2\pi \int_{-\infty}^{0} e^{2st} e^{-2t W(s)} ds
$$

In summary, recalling that $W(r) = \frac{\lambda}{2} \min(r, 0)^{\alpha}$, we have obtained

$$
\lim_{n \to \infty} \left( \frac{a_k^{(n)} + t^* n^{1/\alpha}}{n^{1/\alpha}} \right)^{-1} \to 2\pi \int_{-\infty}^{\infty} e^{2st} e^{-2W(s)} ds.
$$

Since this $F_n$ is dominated by the $F_n$ from the proof of Proposition 7.2 we may apply Lebesgue’s dominated convergence theorem to obtain

$$
\lim_{n \to \infty} F_n(r) = \frac{1}{2\pi} \int_{-\infty}^{\eta} \frac{e^{rt}}{\int_{-\infty}^{\infty} e^{2st} e^{-2W(s)} ds} dt.
$$

We could have also used the inverse of the left-hand side of (30) to bound the integrand in $F_n(r)$ since it can be explicitly calculated.
7.3 Exponential fluctuations for different behaviors

Recall that
\[
\bar{\rho}_n(r) = \sum_{k=0}^{n-1} b_k^{(n)} r^{2k+1} e^{-2\kappa_n V(r)} \quad \text{where} \quad b_k^{(n)} = \left( \int_0^1 s^{2k+1} e^{-2\kappa_n V(s)} ds \right)^{-1}.
\]

The \( \alpha = 2 \) case of the following theorem is treated in [24] with additional conditions on \( V \).

In the following theorem, \( V \) will be a continuous function on \([0, 1]\).

**Theorem 7.5.** Suppose \( V \) satisfies the standard properties and
\[
V(r) = \log r + \frac{\lambda}{\alpha} (1 - r)^\alpha + o(1 - r)^\alpha \quad \text{as} \quad r \to 1^-,
\]
for some \( \alpha > 1 \) and \( \lambda > 0 \). Suppose that \( \kappa_n \) satisfies
\[
\lim_{n \to \infty} \frac{n - \kappa_n}{n^{1/\alpha}} = \zeta \in (-\infty, \infty].
\]

Define \( F : (-\infty, 0] \to \mathbb{R} \) by
\[
F(r) = e^{-2\frac{\lambda}{\alpha}|r|^\alpha} \int_{-\infty}^{\zeta} e^{2rt} e^{-2\frac{\lambda}{\alpha}|s|^\alpha} ds.
\]

Then, for \( r \) uniformly on compact sets of \((-\infty, 0]\),
\[
\frac{1}{n^{2/\alpha}} \bar{\rho}_n \left( 1 + \frac{r}{n^{1/\alpha}} \right) \xrightarrow{n \to \infty} F(r).
\]

Furthermore, as point processes on \([0, \infty)\),
\[
\left\{ n^{2/\alpha} (1 - |x_i^{(n)}|) : 1 \leq i \leq n \right\} \xrightarrow{\text{law}} \mathcal{P},
\]
where \( \mathcal{P} \) is a Poisson point process on \([0, \infty)\) with intensity \( F(0) \). In particular, for every \( t \geq 0 \),
\[
\mathbb{P} \left( n^{2/\alpha} (1 - M_n) \leq t \right) \to 1 - e^{-F(0)t}.
\]

**Proof.** Taking \( t^* = \frac{|n^{1/\alpha}|}{n^{1/\alpha}} \), we may write
\[
\frac{1}{n^{2/\alpha}} \bar{\rho}_n \left( 1 + \frac{r}{n^{1/\alpha}} \right) = \frac{1}{n^{2/\alpha}} \sum_{k=0}^{n-1} b_k^{(n)} \left( 1 + \frac{r}{n^{1/\alpha}} \right)^{2(k - |\kappa_n|) + 1} e^{-2\kappa_n \left( V\left( 1 + \frac{r}{n^{1/\alpha}} \right) - \log\left( 1 + \frac{r}{n^{1/\alpha}} \right) \right)}
\]
\[
= e^{-2\kappa_n \left( V\left( 1 + \frac{r}{n^{1/\alpha}} \right) - \log\left( 1 + \frac{r}{n^{1/\alpha}} \right) \right)} \int_{-\frac{|\kappa_n|}{n^{1/\alpha}}}^{\frac{b_{\{\kappa_n\}} + t^* n^{1/\alpha}}{n^{1/\alpha}}} b_{\{\kappa_n\} + t^* n^{1/\alpha}} \left( 1 + \frac{r}{n^{1/\alpha}} \right)^{2t^* n^{1/\alpha} + 1} dt.
\]

By noticing that
\[
\left( \frac{b_{\{\kappa_n\} + t^* n^{1/\alpha}}}{n^{1/\alpha}} \right)^{-1} = \int_{-n^{1/\alpha}}^{0} \left( 1 + \frac{s}{n^{1/\alpha}} \right)^{2(|\kappa_n| + t^* n^{1/\alpha}) + 1} e^{-2\kappa_n \left( V\left( 1 + \frac{r}{n^{1/\alpha}} \right) - \log\left( 1 + \frac{r}{n^{1/\alpha}} \right) \right)} ds
\]
\[
\xrightarrow{n \to \infty} \int_{-\infty}^{0} e^{2st} e^{-2\frac{\lambda}{\alpha}|s|^\alpha} ds
\]
we obtain
\[ \frac{1}{\nu^{2/\alpha}} \rho_n \left( 1 + \frac{r}{\nu^{1/\alpha}} \right) \xrightarrow{n \to \infty} e^{-2\lambda |r|^\alpha} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{2st} e^{-2\lambda |s|^\alpha} ds dt = F(r) \]
for \( r \) uniformly on compact sets of \( (-\infty, 0] \). The justifications to apply Lebesgue’s dominated converges theorem in the previous two integrals are as in the proof of Proposition 7.2. Finally,
\[ \frac{1}{\nu^{2/\alpha}} \rho_n \left( 1 + \frac{r}{\nu^{2/\alpha}} \right) \xrightarrow{n \to \infty} F(0) \]
also for \( r \) uniformly on compact sets of \( (-\infty, 0] \) so that we obtain the result by using the ideas described in Subsection 5.1 as in the proof of Theorem 2.7.

### 7.4 Gumbel distribution and weakly confining fluctuations

Here we establish a connection between the limit of the maxima in Theorem 2.3 for \( \tilde{R} = \infty \) and the Gumbel distribution.

**Proposition 7.6 (Weakly confining and Gumbel distribution).** For each \( \chi > 0 \) let \( X_\chi \) be a random variable with cumulative distribution function
\[ P(X_\chi \leq t) = \prod_{k=0}^{\infty} \left( 1 - t^{-2k-2\chi} \right) \]
for every \( t \geq 1 \) and let \( \varepsilon_\chi > 0 \) denote the unique solution to
\[ e^{\chi \varepsilon_\chi} \varepsilon_\chi = 1. \]
Then, as \( \chi \to \infty \), we have
\[ 2\chi(X_\chi - 1 - \varepsilon_\chi/2) \to G \]
where \( G \) has a standard Gumbel distribution, i.e.,
\[ P(G \leq a) = e^{-e^{-a}} \]
for every \( a \in \mathbb{R} \).

**Proof.** Define \( b_\chi \) by
\[ 2b_\chi = \varepsilon_\chi + a/\chi. \]
We have to prove that
\[ \lim_{\chi \to \infty} \prod_{k=0}^{\infty} \left( 1 - (1 + b_\chi)^{-2k-2\chi} \right) = e^{-e^{-a}}. \]
We notice that, as \( \chi \to \infty \),
\[ \log \prod_{k=0}^{\infty} \left( 1 - (1 + b_\chi)^{-2k-2\chi} \right) \sim - \sum_{k=0}^{\infty} (1 + b_\chi)^{-2k-2\chi} \]
which is due to the fact that \( (1 + b_\chi)^x \to \infty \) and \( \log(1 - x) \sim -x + o(x) \). Then
\[ \sum_{k=0}^{\infty} (1 + b_\chi)^{-2k-2\chi} = \frac{(1 + b_\chi)^{-2\chi}}{1 - (1 + b_\chi)^{-2}} \sim \frac{e^{-2\chi b_\chi + \chi O(b_\chi)^2}}{2b_\chi} \sim \frac{e^{-\chi \varepsilon_\chi}}{\varepsilon_\chi} e^{-a} = e^{-a} \]
which concludes the proof.

Similarly to Theorem 2.5 and Theorem 2.6, the proposition still holds if we change \( \varepsilon_\chi \) to \( \varepsilon_\chi = \chi^{-1} (\log \chi - \log \log \chi) \). Nevertheless, we preferred to use the defining relation \( e^{\chi \varepsilon \varepsilon} = 1 \) due to its importance in the proof.
References

[1] Yacin Ameur, Nam-Gyu Kang and Seong-Mi Seo. On boundary confinements for the Coulomb gas. Analysis and Mathematical Physics 10, (2020), no. 4, paper no. 68, 42 pp.

[2] John Ben Hough, Manjunath Krishnapur, Yuval Peres and Bálint Virág. Zeros of Gaussian analytic functions and determinantal point processes. Vol. 51 of University Lecture Series. American Mathematical Society, Providence, RI, 2009.

[3] Robert J. Berman. Determinantal Point Processes and Fermions on Complex Manifolds: Large Deviations and Bosonization. Communications in Mathematical Physics 327, (2014), no. 1, 1-47.

[4] Raphael Butez and David García-Zelada. Extremal particles of two-dimensional Coulomb gases and random polynomials on a positive background. Preprint arXiv:1811.12225, 2020.

[5] Djalil Chafaï and Sandrine Péché. A note on the second order universality at the edge of Coulomb gases on the plane. Journal of Statistical Physics 156, (2014), no. 2, 368–383.

[6] Djalil Chafaï, David García-Zelada and Paul Jung. Macroscopic and edge behavior of a planar jellium. Journal of Mathematical Physics 61, 033304 (2020).

[7] Shuhua Chang, Deli Li and Yongcheng Qi. Limiting Distributions of Spectral Radii for Product of Matrices from the Spherical Ensemble. Journal of Mathematical Analysis and Applications 461, (2018), no. 2, 1165–1176.

[8] Ling-Lie Chau and Oleg Zaboronsky. On the structure of correlation functions in the normal matrix model. Communications in Mathematical Physics 196 (1998), no. 1, 203–247.

[9] David Dean, Pierre Le Doussal, Satya Majumdar and Grégory Schehr. Statistics of the maximal distance momentum in a trapped Fermi gas at low temperature. Journal of Statistical Mechanics: Theory and Experiment, 063301, 2017.

[10] Paul Dupuis, Vaios Laschos and Kavita Ramanan. Large deviations for configurations generated by Gibbs distributions with energy functionals consisting of singular interaction and weakly confining potentials. Electronic Journal of Probability 25 (2020), paper no. 46, 41 pp.

[11] Peter Elbau and Giovanni Felder. Density of eigenvalues of random normal matrices. Communications in Mathematical Physics 259 (2005), issue 2, 433–450.

[12] Yan V. Fyodorov and Bernhard Mehlig. Statistics of resonances and nonorthogonal eigenfunctions in a model for single-channel chaotic scattering. Physical review E 66, 045202, 2002.

[13] David García-Zelada. Edge fluctuations for a class of two-dimensional determinantal Coulomb gases. Available at https://arxiv.org/abs/1812.11170v2 2019.

[14] Wenhao Gui and Yongcheng Qi. Spectral Radii of Truncated Circular Unitary Matrices. Journal of Mathematical Analysis and Applications 458 (2018), issue 1, 536-554.

[15] Haakan Hedenmalm, Boris Korenblum and Kehe Zhu. Theory of Bergman spaces. Graduate Texts in Mathematics 199. Springer-Verlag, New York, 2000. x+286 pp.

[16] Haakan Hedenmalm and Aron Wennman. Planar orthogonal polynomials and boundary universality in the random normal matrix model. Preprint arXiv:1710.06493 2020.

[17] Tiefeng Jiang and Yongcheng Qi. Spectral radii of large non-hermitian random matrices. Journal of Theoretical Probability 30, 2017, issue 1, 326-364.
[18] Olav Kallenberg. *Random Measures, Theory and Applications*. Volume 77 of Probability Theory and Stochastic Modelling. Springer-Verlag, New York, 2017.

[19] Eric Kostlan. *On the spectra of Gaussian matrices*. Linear Algebra and its applications 162/164, 1992, 385–388. Directions in matrix theory (Auburn, AL, 1990).

[20] Bertrand Lacroix-A-Chez-Toine, Aurélien Grabsch, Satya Majumdar and Grégory Schehr. *Extremes of 2d Coulomb gas: universal intermediate deviation regime*. Journal of Statistical Mechanics: Theory and Experiment, 013203, 2018.

[21] Brian Rider. *A limit theorem at the edge of a non-Hermitian random matrix ensemble*. Journal of Physics A: Mathematical and General 36, 2003, no. 12, 3401–3409.

[22] Brian Rider. *Order Statistics and Ginibre’s Ensembles*. Journal of Statistical Physics 114, 1139–1148 (2004).

[23] Edward Saff and Vilmos Totik. *Logarithmic potentials with external fields*. Grundlehren der Mathematischen Wissenschaften 316. Springer-Verlag, Berlin, 1997. xvi+505 pp

[24] Seong-Mi Seo. *Edge scaling limit of the spectral radius for random normal matrix ensembles at hard edge*. Journal of Statistical Physics 181, 1473–1489 (2020).

[25] Sylvia Serfaty. *Systems of points with Coulomb interactions*. Proceedings of the International Congress of Mathematicians 2018, 1, 935–978, Rio de Janeiro, 2018.

[26] Tomoyuki Shirai and Yoichiro Takahashi. *Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes*. Journal of Functional Analysis 205, 2003, issue 2, 414–463.

[27] Christopher Sinclair and Maxim Yattselev. *Universality for ensembles of matrices with potential theoretic weights on domains with smooth boundary*. Journal of Approximation Theory 164, 2012, issue 5, 682-708.

E-mail address: david.garcia-zelada@univ-amu.fr