Restricted Weyl Symmetry and Spontaneous Symmetry Breakdown of Conformal Symmetry

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Abstract

We elucidate the relation between the restricted Weyl symmetry and spontaneous symmetry breakdown of conformal symmetry. Using a scalar-tensor gravity, we show that the restricted Weyl symmetry leads to spontaneous symmetry breakdown of a global scale symmetry when the vacuum expectation value of a scalar field takes a non-zero value. It is then shown that this spontaneous symmetry breakdown induces spontaneous symmetry breakdown of special conformal symmetry in a flat Minkowski space-time, but the resultant Nambu-Goldstone boson is not an independent physical mode but expressed in terms of the derivative of the dilaton which is the Nambu-Goldstone boson of the global scale symmetry. In other words, the theories which are invariant under the general coordinate transformation and the restricted Weyl transformation exhibit a Nambu-Goldstone phase where both special conformal transformation and dilatation are spontaneously broken while preserving the Poincaré symmetry.

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1 Introduction

It is nowadays widely believed among particle physicists that global and local scale symmetries have conformal (or trace) anomaly at the quantum level [1]. However, it has been also known since a long time ago that when a scalar field called the “dilaton” \(^2\) is present and takes a non-zero vacuum expectation value in a theory, there is an ingenious way of perturbatively quantizing the theory which preserves the local scale (or Weyl) invariance [3]. Thus far, this has been several times rediscovered in Refs. [4]-[12]. The price we have to pay here is that the theory becomes non-renormalizable, but this issue is not so serious at least in the presence of the Einstein-Hilbert term of general relativity since the Einstein-Hilbert term is non-renormalizable in itself.

With the scale invariant regularization scheme [3]-[12], scale symmetries are not explicitly violated by the conformal anomaly but it is expected that they might be spontaneously broken as in gauge symmetries in quantum field theories, and as a result we could make use of scale symmetries even at the quantum level. In this article, we would like to discuss spontaneous symmetry breakdown of conformal symmetry in a flat Minkowski space-time.

In our previous work [13], we have already clarified a spontaneous symmetry breakdown of a global scale symmetry on the basis of a general scalar-tensor gravity with a complex scalar field, a gauge field and higher-derivative terms which is invariant under the restricted Weyl transformation [14]-[16]. The main purpose of this article is to generalize the case of the global scale symmetry to that of special conformal symmetry. We will see later that the special conformal symmetry and the global scale symmetry are spontaneously broken while preserving the Poincaré symmetry when the dilaton takes a non-zero vacuum expectation value.

The outline of this paper is as follows: In Section 2, we show that with the nonvanishing dilaton the global scale invariance is necessarily spontaneously broken by using the simplest scalar-tensor gravity which is invariant under the restricted Weyl transformation in addition to the global scale transformation and the general coordinate transformation. In Section 3, we discuss that the restricted Weyl transformation generates conformal transformation

\(^2\)Precisely speaking, the dilaton is a Nambu-Goldstone boson of the broken scale invariance [2]. Here we loosely use this name.
in a flat Minkowski space-time. This is a nontrivial generalization of the Zunmino’s theorem in that the full Weyl symmetry is replaced with the restricted Weyl symmetry. In Section 4, in the conformal symmetry obtained in Section 3, the special conformal symmetry as well as the global scale symmetry are spontaneously broken while the Poincaré symmetry is kept.

2 A scalar-tensor gravity

In this section, we consider the simplest scalar-tensor gravity [2] whose Lagrangian is given by

\[ \mathcal{L} = \sqrt{-g} \left( \frac{1}{2} \xi \phi^2 R - \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \right), \]

where \( \xi \) is a constant called the non-minimal coupling constant, \( \phi \) a real scalar field with a normal kinetic term (i.e., not a ghost), and \( R \) the scalar curvature. In addition to the general coordinate transformation and a global scale transformation with \( \Omega = \text{constant} \), this Lagrangian is invariant under the following restricted Weyl transformation [14]-[18], [13]

\[ g_{\mu \nu} \rightarrow g'_{\mu \nu} = \Omega^2(x) g_{\mu \nu}, \quad \Phi \rightarrow \Phi' = \Omega^{-1}(x) \Phi, \]

where the gauge transformation parameter \( \Omega(x) \), which we will call a scale factor, obeys a constraint \( \Box_g \Omega = 0 \). In order to prove the invariance, we need to use the following transformation of the scalar curvature under (2):

\[ R \rightarrow R' = \Omega^{-2}(R - 6 \Omega^{-1} \Box_g \Omega). \]

The field equations obtained from the Lagrangian (1) read

\[ \xi \phi^2 G_{\mu \nu} = T_{\mu \nu} - \xi (g_{\mu \nu} \Box_g - \nabla_\mu \nabla_\nu) (\phi^2), \]

\[ \xi \phi R + \Box_g \phi = 0, \]

\[ ^3 \text{We follow the conventions and notation of the MTW textbook [17].} \]

\[ ^4 \text{To distinguish the difference of the d’Alembertian operators, we will henceforth use } \Box_g \text{ for the Minkowski space-time and } \Box_g \text{ for the Riemannian space-time, respectively.} \]
where the Einstein tensor $G_{\mu\nu}$ and the energy-momentum tensor for the scalar field $\phi$ are defined by

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \]
\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial_\rho \phi)^2. \] (5)

Taking the trace of the former equation in (4) yields

\[ \xi \phi^2 R = (\partial_\mu \phi)^2 + 3\xi \Box_g (\phi^2). \] (6)

Multiplying the latter equation in (4) by $\phi$ leads to

\[ \xi \phi^2 R + \phi \Box_g \phi = 0. \] (7)

By eliminating $R$ from Eqs. (6) and (7), we obtain

\[ (6\xi + 1) \Box_g (\phi^2) = 0. \] (8)

In this article, we confine ourselves to the case $\xi \neq -\frac{1}{6}$ since in the specific case $\xi = -\frac{1}{6}$ we have a local scale symmetry (or equivalently, the Weyl symmetry).

Now, in order to understand the physical implications of the constraint $\Box_g \Omega = 0$ of the restricted Weyl transformation, we are interested in zero-mode solutions to the constraint. It is then obvious that the only zero-mode solution is given by $\Omega(x) = \text{const.}$ which corresponds to a global scale invariance. Thus, let us pay our attention to the scale invariance. In an infinitesimal form of the scale factor $\Omega = e^\Lambda$ with $|\Lambda| \ll 1$, the infinitesimal gauge transformation parameter $\Lambda$ must obey a constraint $\Box_g \Lambda = 0$ as well, so the zero-mode solution is given by

\[ \Lambda(x) = c, \] (9)

where $c$ is a constant.

Since there is a global invariance associated with the parameter $c$, we can construct a conserved Noether charge $Q$. Following the calculation [2, 19], it turns out that the conserved current for the global scale invariance reads

\[ J^\mu = \frac{6\xi + 1}{2} \sqrt{-g} g^{\mu\nu} \partial_\nu (\phi^2). \] (10)
We can easily verify that this current is conserved, $\partial_\mu J^\mu = 0$, by using the field equation (8). One might wonder why no derivatives of the metric appear in the expression of $J^\mu$. This is because the derivatives of $\phi$ are mixed with the metric, thus making $J^0$ serve as a generator of the metric transformation. Moreover, in case of a conformal coupling $\xi = -\frac{1}{6}$, the conserved current is identically vanishing [20, 21].

Using the corresponding Noether charge defined as $Q = \int d^3x J^0$, we find that

$$\delta g_{\mu\nu} = [iQ, g_{\mu\nu}] = 2cg_{\mu\nu}, \quad \delta \phi = [iQ, \phi] = -c\phi,$$

from which we have

$$[iQ, g_{\mu\nu}] = 2g_{\mu\nu}, \quad [iQ, \phi] = -\phi.$$  \hspace{1cm} (12)

Assuming $\langle 0|g_{\mu\nu}|0 \rangle = \eta_{\mu\nu}$\footnote{In particular, in the presence of the cosmological constant, we would need to take a more general fixed background $\bar{g}_{\mu\nu}$ which satisfies $\langle 0|g_{\mu\nu}|0 \rangle = \bar{g}_{\mu\nu}$.} and $\langle 0|\phi|0 \rangle = \phi_0$ ($\phi_0$ is a constant), and then taking the vacuum expectation value of Eq. (12) leads to

$$\langle 0|[iQ, g_{\mu\nu}]|0 \rangle = 2\eta_{\mu\nu}, \quad \langle 0|[iQ, \phi]|0 \rangle = -\phi_0.$$  \hspace{1cm} (13)

Eq. (13) implies that the global scale invariance must be broken spontaneously for $\phi_0 \neq 0$ at the quantum level [13].

Here two remarks are in order. One remark is that in the scale invariant regularization method [3]-[12], by definition the “dilaton” cannot vanish anywhere to make all coupling constants be dimensionless. In the typical models with spontaneous symmetry breakdown, such a configuration is usually picked up from a potential which induces the symmetry breaking, but the existence of the scale invariance in our theory does not allow us to have such a nontrivial potential at the classical level. At the quantum level, however, we have a nontrivial effective potential inducing the spontaneous symmetry breakdown [11, 12], which is harmony with our assumption $\langle 0|\phi|0 \rangle = \phi_0 \neq 0$ in hand. As another remark, since the flat Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$ is invariant under a combination of the general coordinate transformation and the restricted Weyl transformation as will be shown in the next section,
the former equation in Eq. (13) is irrelevant to the spontaneous symmetry breakdown of the global scale invariance.

Next, let us verify explicitly that the spontaneous symmetry breakdown of the global symmetry occurs by moving from the Jordan frame (J-frame) to the Einstein frame (E-frame) [2, 22]. To do so, we will move to the Einstein frame by implementing a local scale transformation only for the metric tensor except for the scalar field

$$ g_{\mu \nu} \rightarrow g^{*}_{\mu \nu} = \Omega^2(x) g_{\mu \nu}. $$

(14)

Under this scale transformation we have [2]

$$ g^{\mu \nu} = \Omega^2(x) g^{*\mu \nu}, \quad \sqrt{-g} = \Omega^{-4} \sqrt{-g^*}, $$

$$ R = \Omega^2 (R^* + 6 \Box^* f - 6 g^{*\mu \nu} f_{\mu} f_{\nu}), $$

(15)

where we have defined

$$ f \equiv \log \Omega, \quad \Box^* f \equiv \frac{1}{\sqrt{-g^*}} \partial_\mu (\sqrt{-g^*} g^{*\mu \nu} \partial_\nu f), \quad f_{\mu} \equiv \partial_\mu f = \frac{\partial_\mu \Omega}{\Omega}. $$

(16)

Then, the Lagrangian density (1) is cast to the form

$$ \mathcal{L} = \sqrt{-g^*} \left[ \frac{1}{2} \xi \phi^2 \Omega^{-2} (R^* + 6 \Box^* f - 6 g^{*\mu \nu} f_{\mu} f_{\nu}) - \frac{1}{2} \Omega^{-2} g^{*\mu \nu} \partial_\mu \phi \partial_\nu \phi \right]. $$

(17)

To reach the Einstein frame, we have to choose the scale factor $\Omega(x)$ to satisfy the relation

$$ \Omega^2 = \frac{1}{M^2_{Pl}} \xi \phi^2, $$

(18)

where $M_{Pl}$ is the reduced Planck mass. Note that Eq. (18) shows that the “dilaton” $\phi$ cannot vanish in this case either since we cannot move to the E-frame from the J-frame in the case of the vanishing dilaton. As a result, we obtain a Lagrangian in the E-frame:

$$ \mathcal{L} = \sqrt{-g^*} \left( \frac{1}{2} M^2_{Pl} R^* - \frac{1}{2} g^{*\mu \nu} \partial_\mu \phi \partial_\nu \phi \right). $$

(19)
Here we have defined a scalar field $\sigma(x)$ and a constant $\zeta$ as

$$\phi = \xi^{-\frac{1}{2}} M_{Pl} e^{\frac{\xi}{2|\phi|}}, \quad \zeta = \sqrt{\frac{\xi}{6\xi + 1}}. \quad (20)$$

The Lagrangian (19) indicates that the scalar field $\sigma(x)$ is a massless Nambu-Goldstone field associated with the spontaneous symmetry breakdown of the global scale invariance. Indeed, the current $J^\mu$ in Eq. (10) can be rewritten in terms of $\sigma(x)$ as

$$J^\mu = \frac{M_{Pl}}{\zeta} \sqrt{-g} g^{\mu\nu}_\sigma \partial_\nu \sigma. \quad (21)$$

The corresponding charge is given by

$$Q = \int d^3x J^0 = \frac{M_{Pl}}{\zeta} \int d^3x \sqrt{-g} g^{0\nu}_\sigma \partial_\nu \sigma. \quad (22)$$

Since $Q$ has a linear term in $\sigma(x)$, it is obvious that the charge cannot annihilate the vacuum $|0\rangle$

$$Q|0\rangle \neq 0, \quad (23)$$

which implies the spontaneous symmetry breakdown of the scale symmetry.

This fact can be also verified by evaluating the vacuum expectation value of the commutator between $Q$ and $\sigma(x)$. Actually, from the Lagrangian (19) the canonical conjugate momentum for the scalar field $\sigma(x)$ reads

$$\pi_\sigma \equiv \frac{\partial L}{\partial \partial_0 \sigma} = -\sqrt{-g} g^{0\nu}_\sigma \partial_\nu \sigma. \quad (24)$$

Then, the Noether charge $Q$ in Eq. (22) can be rewritten as

$$Q = -\frac{M_{Pl}}{\zeta} \int d^3x \pi_\sigma. \quad (25)$$

Using the equal-time commutation relation

$$[\sigma(t, \vec{x}), \pi_\sigma(t, \vec{y})] = i\delta^3(x - y), \quad (26)$$
we obtain

\[ [iQ, \sigma(x)] = -\frac{M_{Pl}}{\zeta}. \]  

(27)

Taking the vacuum expectation value of this equation yields

\[ \langle 0 | [iQ, \sigma(x)] | 0 \rangle = -\frac{M_{Pl}}{\zeta} \neq 0, \]  

(28)

which clearly means that the scalar field \( \sigma(x) \) is the Nambu-Goldstone boson for the scale symmetry. Note that because of the definition (20), the spontaneous symmetry breakdown of the scale symmetry in the J-frame can be interpreted as that of the shift symmetry in the E-frame.

3 Conformal symmetry from the restricted Weyl symmetry

Let us recall that conformal transformation [23] can be defined as the general coordinate transformation which can be undone by the Weyl transformation when the space-time metric is the flat Minkowski metric. In the scalar-tensor gravity (1) there is no Weyl invariance, but instead we have the restricted Weyl invariance, so we could define the conformal transformation by replacing the Weyl transformation with the restricted Weyl one. With this definition, the conformal transformation is described by the equation

\[ \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2\Lambda(x)\eta_{\mu\nu}, \]  

(29)

where the infinitesimal scale factor \( \Lambda(x) \) obeys the constraint \( \Box_\eta \Lambda = 0 \).

Taking the trace of Eq. (29) enables us to determine \( \Lambda(x) \) to be

\[ \Lambda = \frac{1}{4} \partial^\rho \epsilon_\rho. \]  

(30)

Inserting this \( \Lambda \) to Eq. (29) yields

\[ \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{1}{2} \partial^\rho \epsilon_\rho \eta_{\mu\nu}, \]  

(31)
which is often called the “conformal Killing equation” in the Minkowski space-time. It is worth stressing that Eq. (31) implies the following fact: The flat Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$ is invariant in the space of the metric functions under a suitable combination of the general coordinate transformation and the restricted Weyl transformation such that

$$\delta(\epsilon_{\mu}) = \delta_{\text{GCT}}(\epsilon_{\mu}) - \delta_{\text{RW}}(\Lambda = \frac{1}{4} \partial^\rho \epsilon_{\rho}),$$

(32)

when the vector field $\epsilon_{\mu}(x)$ obeys the conformal Killing equation (31). To put it differently, the characteristic feature of the theory under consideration is that the Lagrangian (1) possesses the conformal symmetry with 15 global parameters which is a subgroup of the general coordinate transformation and the restricted Weyl transformation.

Multiplying it by $\partial^\mu \partial^\nu$, we obtain

$$\Box_{\eta} \partial^\rho \epsilon_{\rho} = 0.$$  

(33)

Moreover, multiplying Eq. (31) by $\partial^\mu \partial_{\lambda}$ and then symmetrizing the indices $\nu$ and $\lambda$ leads to the desired equation

$$\partial_{\lambda} \partial_{\nu} \partial^\rho \epsilon_{\rho} = 0,$$

(34)

where we have used Eqs. (31) and (33). It turns out that a general solution to Eq. (34) reads

$$\epsilon^{\mu} = a^{\mu} + \omega^{\mu\nu} x_{\nu} + \lambda x^{\mu} + b^{\mu} x^2 - 2 x^{\mu} b_{\rho} x^{\rho},$$

(35)

where $a^{\mu}, \omega^{\mu\nu} = -\omega^{\nu\mu}, \lambda$ and $b^{\mu}$ are all constant parameters and they correspond to the translation, the Lorentz transformation, the dilatation$^6$ and the special conformal transformation, respectively.

$^6$For clarity, we will call a global scale transformation in a flat Minlowski space-time “dilatation”. Dilatation is usually interpreted as a subgroup of the general coordinate transformation in such way that the space-time coordinates are transformed as $x^{\mu} \rightarrow \Omega x^{\mu}$ in the flat space-time where $\Omega$ is a constant scale factor, whereas the global scale transformation is a rescaling of all lengths by the same $\Omega$ by $g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}$. The two viewpoints are completely equivalent since all the lengths are defined via the line element $ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$.  

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At this stage, we have to verify that the infinitesimal scale factor \( \Lambda \) generated by the “conformal Killing vector” \( \epsilon^\mu \) in Eq. (35) satisfies the constraint of the restricted Weyl symmetry. Actually, substituting Eq. (35) into Eq. (30), we have

\[
\Lambda = \lambda - 2b_\mu x^\mu.
\] (36)

This is nothing but zero-mode solutions to the constraint \( \Box_\eta \Lambda = 0 \), so the restricted Weyl transformation can generate the conformal transformation in a flat Minkowski background as in the full Weyl transformation.

To summarize, we have shown that the restricted Weyl symmetry, together with the general coordinate invariance, generates the conformal symmetry in the flat Minkowski background. This result is a generalization of the well-known Zumino’s theorem [24] which insists that the theories invariant under both the general coordinate transformation and the Weyl transformation (or local scale transformation) possess conformal transformation in the flat Minkowski background. In the theory in hand, the Weyl transformation is replaced with the restricted Weyl transformation which is a subgroup of the Weyl transformation.

4 Spontaneous symmetry breakdown of special conformal symmetry

In this section, we will show that in the conformal transformation obtained in the previous section through the restricted Weyl transformation, both the special conformal transformation and dilatation are spontaneously broken.

Before doing so, let us recall the generators and algebra of conformal transformation [23]. The generators of the translation \( (P_\mu) \), the Lorentz transformation \( (M_{\mu\nu}) \), the dilatation \( (D) \) and the special conformal transformation \( (K_\mu) \) respectively take the form

\[
\begin{align*}
P_\mu & = -i\partial_\mu, \\
M_{\mu\nu} & = -i(x_\mu \partial_\nu - x_\nu \partial_\mu), \\
D & = ix^\mu \partial_\mu, \\
K_\mu & = i(2x_\mu x^\rho \partial_\rho - x^2 \partial_\mu).
\end{align*}
\] (37)

The conformal group is a simple Lie group and its algebra reads

\[
[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\rho\nu} - \eta_{\nu\sigma} M_{\rho\mu}),
\]
\[
\begin{align*}
[P_\mu, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho), \\
[P_\mu, P_\nu] &= 0, \\
[P_\mu, D] &= iP_\mu, \\
[M_{\mu\nu}, D] &= 0, \\
[K_\mu, D] &= -iK_\mu, \\
[P_\mu, K_\nu] &= -2i(\eta_{\mu\nu}D + M_{\mu\nu}), \\
[K_\mu, K_\nu] &= 0, \\
[K_\mu, M_{\rho\sigma}] &= -i(\eta_{\mu\rho}K_\sigma - \eta_{\mu\sigma}K_\rho).
\end{align*}
\] (38)

We assume that the vacuum is invariant under the Poincaré transformation

\[P_\mu|0\rangle = 0, \quad M_{\mu\nu}|0\rangle = 0.\] (39)

In Section 2, we have already proved that the global scale transformation is spontaneously broken. In particular, setting \(g_{\mu\nu} = \eta_{\mu\nu}\), the global scale transformation, i.e., the dilatation, must be spontaneously broken as well. The corresponding Nambu-Goldstone boson is the massless dilaton \(\sigma(x)\) as seen in Eq. (28), which gives us the equation

\[\langle 0| [D, \sigma(x)]|0 \rangle \equiv v \neq 0,\] (40)

where \(v\) is a non-zero constant.

Now let us consider the Jacobi identity [25]

\[\left[[P_\mu, K_\nu], \sigma\right] + \left[[K_\nu, \sigma], P_\mu\right] + \left[[\sigma, P_\mu], K_\nu\right] = 0.\] (41)

Next, using the conformal algebra for \([P_\mu, K_\nu]\) in Eq. (38) and the equation

\[\left[P_\mu, \sigma\right] = -i\partial_\mu \sigma,\] (42)

we can obtain

\[-2i(\eta_{\mu\nu}D + M_{\mu\nu}, \sigma) + \left[[K_\nu, \sigma], P_\mu\right] + i[\partial_\mu \sigma, K_\nu] = 0.\] (43)

Taking the vacuum expectation value of the algebra (43) and using Eqs. (39) and (40) provides us with the final result

\[\langle 0| [K_\nu, \partial_\mu \sigma]|0 \rangle = -2v\eta_{\mu\nu}.\] (44)

Eq. (44) clearly shows that the special conformal transformation is also spontaneously broken, and the corresponding Nambu-Goldstone boson can be identified with \(\partial_\mu \sigma\) [25]. Thus, the Nambu-Goldstone boson associated with the spontaneous symmetry breakdown of the special conformal transformation is not an independent mode but expressed in terms of the derivative of the Nambu-Goldstone boson of the dilatation or the global scale transformation [26].
5 Conclusion

In this paper, based on a scalar-tensor gravity, we have elucidated the relation between the restricted Weyl symmetry and spontaneous symmetry breakdown of conformal symmetry. What we have shown is that the theories which are simultaneously invariant under the general coordinate transformation and the restricted Weyl transformation have conformal transformation when we take the flat Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$, but both special conformal transformation and dilatation are spontaneously broken. This fact should be contrasted to the Zumino's theorem which insists that the theories which are invariant under the general coordinate transformation and the Weyl transformation necessarily have conformal transformation without any spontaneous symmetry breakdown when we take the flat Minkowski metric.

An interesting point is that the Nambu-Goldstone boson of the special conformal transformation is not an independent field but the derivative of the Nambu-Goldstone boson of the dilatation. This phenomenon has been already observed in Refs. [26, 25], but for the first time we have presented a concrete gravitational model which realizes this phenomenon by starting with the restricted Weyl transformation. Although we have derived the above results in terms of a specific theory, i.e., the scalar-tensor gravity, we believe that the same phenomenon could occur even in more general and realistic theories if they have both the general coordinate invariance and the restricted Weyl symmetry. In this respect, it is worthwhile to point out that the restricted Weyl symmetry plays an important role since only the global scale symmetry, which often coexists with the restricted Weyl symmetry, cannot generate the conformal group owing to the lack of the special conformal transformation.

In series of papers related to the restricted Weyl symmetry [13, 18], we have investigated some aspects of the theories with the restricted Weyl symmetry such as the meaning of the constraint, the origin of the restricted Weyl symmetry and the spontaneous symmetry breakdown of global symmetries etc. In these papers, it was mentioned that the restricted Weyl symmetry is very similar to the restricted gauge symmetry in QED which emerges as a residual symmetry when fixing the gauge symmetry by the Lorenz gauge $\partial^\mu A_\mu = 0$ in the sense that the both symmetries have a constraint on the transformation parameter (in the restricted Weyl symmetry $\square_g \Omega = 0$ whereas
in the restricted gauge one $\Box_\theta \theta = 0$), and they play a role in spontaneous symmetry breakdown of global symmetries. In future, we wish to construct a more realistic model beyond the standard model on the basis of such the restricted symmetries.

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