The scalar field and gravity combined variable in ADM theory

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April 10, 2019

Abstract

The canonical theory of gravity together with the scalar field is written as a combined variable theory in ADM form, in the classical and quantum theory. FLRW $\kappa = 0$ cosmology is rewritten in the combined variable theory.

1 Introduction

ADM theory of space-time and gravity is extended to include the scalar field. A combined variable representation is defined in both classical and quantum theory. Equations of field theory and Hamiltonian are derived. The role of a lapse function and shift vector in the combined variable theory is studied. Limits of no gravity and no scalar field are found. Quanta of the combined variable are obtained. In FLRW $\kappa = 0$ metric of cosmology, the combined variable theory gives the evolution of space-time. Results about the energy spectrum and singularity are obtained.

Classical theory is developed in the first section. Time-dependent and space-dependent parts of both fields are separated. Then, the full Hamiltonian constraint equation is re-interpreted as a classical field equation for the combined variable field. Hamiltonian of the combined variables allow to quantize the theory in a standard way. This is done in the third section. In section four combined variable ADM theory is applied to FLRW $\kappa = 0$ cosmology and section five concludes earlier three sections. Throughout the paper $16\pi G = 1$, $\hbar = 1$ and $c = 1$ is set. All calculations are done in MATHEMATICA.

2 Classical Theory

The action for a scalar field in presence of ADM gravity is given as

$$\mathcal{A} = \int_{\mathcal{M}} dt \int_{\mathcal{M}} d^3x \left( \sqrt{|N|} \sqrt{\left( R^{(3)} + K_{ab}K_{ab} - K^2 \right) + \left( \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} |\nabla \phi|^2 - V(\phi) \right)} \right)(t,x)$$

ADM formulation of general relativity developed by Richard Arnowitt, Stanley Deser, and Charles W. Misner is a canonical formulation which is being studied over more than five decades. Refer (2, 3) for detail analysis of ADM
formulation. In ADM formulation, 4-dimensional spacetime manifold is foliated into a family of spacelike hypersurfaces. Shift vector together with lapse function decides the foliation of spacetime manifold. Canonical conjugate momenta corresponding to \( q_{ab} \), shift vector \( N^a \), lapse function \( N \), and scalar field \( \phi \) are respectively given (derived in \([3]\) section 1.2, equation 1.2.1) as

\[
P^{ab}(t, \vec{x}) := \left[ \frac{N}{|N|} \sqrt{\eta} \left( K^{ab} - K q^{ab} \right) \right](t, \vec{x})
\]  

\[C_a(t, \vec{x}) := \Pi_a(t, \vec{x}) = 0 \]

\[C(t, \vec{x}) := \Pi(t, \vec{x}) = 0 \]

\[P_\phi(t, \vec{x}) := \left[ |N| \sqrt{\eta} \phi \right](t, \vec{x})
\]

As mentioned in \([1]\) momentum \( P^{ab} \) refers to motion in the time leading out of the original \( t = constant \) surface. Extrinsic curvature \( K_{ab} \) describes how the normal to the surface converge or diverge. Only non-trivial Poisson brackets are given (borrowed from \([2]\) equation 1.2.1.9) as

\[
\{ q_{ab}(t, x), P^{cd}(t, x') \} = \delta^c_a \delta^d_b \delta^{(3)}(x, x')
\]

\[
\{ \phi(t, x), P_\phi(t, x') \} = \delta^{(3)}(x, x')
\]

There are no equations to determine lapse function \( N \) and shift vector \( N^a \). Therefore action reduces to

\[
\mathcal{A} = \int_{\mathbb{R}} dt \int_M d^3x \left\{ P^{ab} q_{ab} + P_\phi \phi - \left( N^a \mathcal{H}_a + |N| \mathcal{H}_{scalar} + |N| \mathcal{H}_\phi \right) \right\}(t, \vec{x})
\]

Where

\[
\mathcal{H}_{scalar} := \left( \frac{1}{\sqrt{\eta}} \left( q_{ac} q_{bd} - \frac{1}{2} q_{ab} q_{cd} \right) P^{ac} P^{bd} - \sqrt{\eta} R^{(3)} \right)(t, \vec{x})
\]

\[
\mathcal{H}_\phi = \sqrt{\eta} \left( \frac{P_\phi^2}{2N^2 \det(q)} + \frac{1}{2} |\nabla \phi|^2 + V(\phi) \right)(t, \vec{x})
\]

\[
\mathcal{H}_a := \left( -2q_{au} D_u P^{bc} \right)(t, \vec{x})
\]

The total Hamiltonian is a sum of Hamiltonian scalar constraints of gravity, scalar field Hamiltonian and Diffeomorphism constraints (also called vector constraints). Refer equation 1.2.6 of \([3]\) or section I.2.1 of \([2]\) for detail discussion.

\[
H_{total} = \int d^3x \left( \frac{N}{\sqrt{\eta}} \left( q_{ac} q_{bd} - \frac{1}{2} q_{ab} q_{cd} \right) p^{ac} p^{bd} - N \sqrt{\eta} R^{(3)} \right)
\]

\[
+ \int d^3x N \sqrt{\eta} \left( \frac{p_\phi^2}{2N^2 \det(q)} + \frac{1}{2} |\nabla \phi|^2 + V(\phi) \right)
\]

\[-2 \int d^3x N^a q_{au} D_u P^{bc} \]

Separate gravitational field variables \( q_{ab} := q_{ab}(t) \tilde{q}_{ab}(\vec{x}) \) and \( P^{ab} := P^{ab}(t) \tilde{P}^{ab}(\vec{x}) \).

\[
- \left[ \frac{q_{ab} P^{ac} P^{bd}}{\sqrt{\det(q)}} \right](t) \int d^3x \left( N F_{abcd} \bar{P}^{ab} P^{cd} \right)(\vec{x})
\]
with \( \tilde{F}_{abcd} = \tilde{G}_{abcd} := \frac{1}{\sqrt{|q_a|}} (\tilde{q}_{ac} \tilde{q}_{bd} - \frac{1}{2} \tilde{q}_{ab} \tilde{q}_{cd}) \). \( \tilde{G}_{abcd} \) is a spatial part of inverse DeWitt metric. DeWitt metric has signature \((-+++++)\) which reduces to overall negative signature. Therefore \( \tilde{F}_{abcd} \) is defined to be negative of inverse DeWitt metric. Properties of DeWitt metric are discussed in detail in [4] appendix A. Choose lapse function as
\[
N(t, \tilde{x}) := N(t) \tilde{N}(\tilde{x})
\]
Then the first term becomes
\[
- \left( NF_{ab} P^a P^b \right) (t) \int d^3 x \left( \tilde{N}(\tilde{x}) F_{abcd} P^a P^d \right)(\tilde{x})
\]
with \( F_{ab} := \frac{\tilde{q}_{ab}}{\sqrt{\det q}} \). The second term in the gravitational part of Hamiltonian
\[
- N \sqrt{\det q_a(t)} \int d^3 x \tilde{N}(\tilde{x}) \sqrt{\tilde{q}_{ab}} R^3(t, \tilde{x})
\]
Separate scalar field variables into \( \phi(t, \tilde{x}) := \phi(t) \tilde{\phi}(\tilde{x}) \) and \( P_\phi(t, \tilde{x}) := P_\phi(t) \tilde{P}_\phi(\tilde{x}) \). The first term in the scalar field part of the Hamiltonian becomes
\[
\frac{p_\phi^2}{N \sqrt{\det q_a}(t)} \int d^3 x \tilde{N}(\tilde{x}) \frac{\tilde{p}_\phi^2}{2N(\tilde{x}) \sqrt{\tilde{q}_{ab}}}
\]
The second term as well as massive coupling term can be absorbed into single term as
\[
\frac{1}{2} \mu^2(\tilde{x}) N \sqrt{\det q_a(t)} \phi^2(t)
\]
with
\[
\mu^2(\tilde{x}) := \int d^3 x' \tilde{N}(x') \sqrt{\tilde{q}_{ab}} \left( |\nabla \tilde{\phi}|^2 + m^2 \tilde{\phi}^2 \right)(\tilde{x}')
\]
Combining all terms in the scalar Hamiltonian constraints as well as scalar field Hamiltonian,
\[
H_\phi + H_{\text{scalar}} = \tilde{T}_\phi \frac{p_\phi^2}{2N \sqrt{\det q_a}} - \tilde{T}_{\text{grav}} \left( NF_{ab} P^a P^b \right) + NV_{\text{CF}}
\]
With
\[
\tilde{T}_\phi := \int \tilde{N} \frac{\tilde{p}_\phi^2}{\sqrt{\tilde{q}_{ab}}} \quad \tilde{T}_{\text{grav}} := \int d^3 x \left( \tilde{N} F_{abcd} \tilde{P}^a \tilde{P}^d \right)(\tilde{x})
\]
\[
V_{\text{CF}} := - \sqrt{\det q_a} \int d^3 x \tilde{N} \sqrt{\tilde{q}_{ab}} R^{(3)} + \frac{1}{2} \mu^2 \sqrt{\det q_a} \phi^2 + \sqrt{\det q_a} \int d^3 x \tilde{N} \sqrt{\tilde{q}_{ab}} V(\phi)
\]
\( \tilde{T}_\phi \) can be thought of kinetic energy of the spatial part of scalar field, \( \tilde{T}_{\text{grav}} \) is kinetic energy due to the spatial part of gravity and \( V_{\text{CF}} \) is combined potential of both fields.
Symmetrizing the second term in terms of \( q_a \) and \( P^a \) in (20)
\[
H_\phi + H_{\text{scalar}} = \tilde{T}_\phi \frac{p_\phi^2}{2N \sqrt{\det q_a}} - \tilde{T}_{\text{grav}} \left( NF_{ab} P^a P^b \right) + NV_{\text{CF}}
\]
In classical ADM theory, there is no issue of ordering gravitational field variables. But in case of combined variable theory, different symmetric combinations may have different combined variable field theoretic extensions. Which may or may not be equivalent. Other symmetric combinations such as \( q_a P_a q_b \) or linear combination of both, these combinations may not have simple combined variable field theoretic extensions. Such extension may not even be allowed. It will be addressed in the next work. In this paper, only (23) combination is chosen.

Now, \( \hat{H} |\Phi \rangle = 0 \) is interpreted as classical field equation for classical combined variable field \( \Phi \). This definition is unique up to chosen symmetric combination of gravitational field variables.

\[
- \left( \frac{\hat{T}_\phi}{2N \sqrt{\det q_a}} \frac{\partial^2}{\partial \phi^2} - \hat{T}_{\text{grav}} \frac{\partial}{\partial q_a} N F_{ab} \frac{\partial}{\partial q_b} - N V_{\text{CF}} \right) \Phi(\phi, q_a) = 0 \tag{24}
\]

OR

\[
(\partial^\mu \eta_{\mu \nu} \partial^\nu - V_{\text{CF}}) \Phi(\phi, \vec{q}) = 0
\]

Where the metric for superspace \((\phi, \vec{q})\) is defined as

\[
\eta_{\mu \nu} := \left( \frac{1}{N \sqrt{\det q_a}}(t) T_\phi(\vec{x}), -\hat{T}_{\text{grav}}(\vec{x}) N(t) F_{ab}(t) \right)
\tag{25}
\]

and \( \partial^\mu = \left( \frac{\partial}{\partial \phi}, \frac{\partial}{\partial q_a} \right) \). This allows us to write an action for combined variable field as,

\[
\mathcal{A} = \int d\phi \int d^3 q \frac{1}{2} \left( \frac{\hat{T}_\phi}{2N \sqrt{\det q_a}} \left( \frac{\partial \Phi}{\partial \phi} \right)^2 - \hat{T}_{\text{grav}} \frac{\partial}{\partial q_a} \left( \frac{\partial \Phi}{\partial q_a} \right) F_{ab} \frac{\partial}{\partial q_b} - N V_{\text{CF}} \Phi^2 \right)
\tag{26}
\]

Invariance of action under \( \Phi \rightarrow \Phi + \delta \Phi \) gives

\[
- \left( \frac{\hat{T}_\phi}{2N \sqrt{\det q_a}} \frac{\partial^2}{\partial \phi^2} - \hat{T}_{\text{grav}} \frac{\partial}{\partial q_a} \left( N F_{ab} \frac{\partial}{\partial q_b} \right) - N V_{\text{CF}} \right) \Phi(\phi, q_a) = 0 \tag{27}
\]

\( D \) in (26) represents number of independent components of 3-metric \( q_a(t) \). For example in case of \( q_1(t) = q_2(t) = q_3(t) = q \), \( D = 1 \). The action chosen is not unique but the simplest form of action which satisfies above field equation.

Vector constraints

\[
H_{\text{vector}} = -2 \int d^3 x N^a q_{ac} D_b P^{bc} = -2 (q_a P_a) \left( \int d^3 x N^a q_{ac} D_b P^{bc} \right) = \alpha_{\text{Diff}}(\vec{x}) q_a P_a
\tag{28}
\]

translate into combined variable theory as

\[
H_{\text{vector}} \Phi = -i \alpha_{\text{Diff}}(\vec{x}) q_a \frac{\partial \Phi}{\partial q_a}
\tag{29}
\]

There are two possibilities

\[
0 = \begin{cases} \alpha_{\text{Diff}}(\vec{x}) \\ q_a \frac{\partial \Phi}{\partial q_a} \end{cases}
\]
$\partial_{\nu}^{\phi}$ is directional derivative of combined variable field along $q_{\nu}$. Assuming that to be zero would also make the second term in (25) zero because $F_{ab} := \frac{q_{\nu}^{a}q_{\nu}^{b}}{\sqrt{\det q_{\nu}}}$. Therefore, this assumption is not valid. This implies

$$\alpha_{\text{Diff}}(\vec{x}) := -2 \int d^{3}x N^{\mu} \bar{q}_{\nu} D_{\nu} \bar{P}^{\mu} = 0$$

(30)

That means diffeomorphism constraints do not play any role in the dynamics of the combined variable field. It puts restrictions on the spatial parts of gravitational field variables. Unlike other fields, gravity being a dynamical theory of space-time itself, it cannot evolve with respect to external time. It evolves with respect to matter field. Therefore, momentum conjugate to the combined variable field is

$$\Pi := \frac{\partial \mathcal{L}}{\partial \left( \frac{\partial \Phi}{\partial \phi} \right)} = \frac{T_{\phi}}{2N \sqrt{\det q_{\nu}}} \left( \frac{\partial \Phi}{\partial \phi} \right)$$

(31)

where $\mathcal{L}$ is the Lagrangian density for combined variable theory. Hamiltonian for combined variable theory is obtained using Legendre transformation,

$$H = \int d^{3}q \frac{N}{2} \left( \frac{2 \sqrt{\det q_{\nu}}}{T_{\phi}} \right) \Pi^{2} + \bar{T}_{\text{grav}} \left( \frac{\partial \Phi}{\partial q_{\nu}} \right) F_{ab} \left( \frac{\partial \Phi}{\partial q_{b}} \right) + V_{\text{CF}} \Phi^{2}$$

(32)

Although $H$ is called as Hamiltonian, it should rather be considered as an observable which gives $\phi$ evolution. This Hamiltonian is different for different lapse function. Therefore choosing a particular lapse function is equivalent of selecting a particular scalar field. Evolution of $\Pi$ can be solved using

$$\frac{\partial \Pi}{\partial \phi} = [\Pi, H]_{\text{PB}}$$

(33)

Invariance of an action of the combined variable field under infinitesimal change in the scalar field as well as 3-metric gives us stress-energy tensor. The procedure for obtaining this tensor (can be found in section 1.5 of [6]). Identify $x^{\mu}$ with $q^{\mu}$ and $\phi$ as $\Phi$ is quite generic.

$$T_{\nu}^{\mu} := \frac{\partial \mathcal{L}}{\partial \left( \partial_{\nu} \Phi \right)} \partial_{\nu} \Phi - \delta_{\nu}^{\mu} \mathcal{L}$$

(34)

Here, $\mu, \nu = 0$ is scalar field dependent component and $\mu, \nu = 1, 2, 3$ are gravitational components. $T_{0}^{\phi}$ is identified as the energy density of the combined variable field and $-T_{j}^{i}$ ($i, j = 1, 2, 3$) as pressure.

**Flat space-time:**

If we take flat space-time limit of a complex combined variable field, the Hamiltonian reduces to

$$H = \frac{1}{2} \left( \frac{2}{T_{\phi}} \right) |\Gamma_{\nu}^{\mu} + V_{\text{CF}} |\Phi_{\nu}^{2} |$$

(35)

Taking $\Phi = e^{ik\nu \phi}$ we recover the Hamiltonian of the scalar field in flat space-time with the difference of overall factor half.

$$H = \frac{1}{2} \left( \frac{T_{\phi}}{2} \right) P_{\phi}^{2} + V_{\text{CF}}$$

(36)
with

\[
V_{CF} = \frac{1}{2} \left( \int d^3x \left( |\nabla \tilde{\phi}|^2 + m^2 \tilde{\phi}^2 \right)(\tilde{x}) \right) \tilde{\phi}^2(t) + \int d^3x V(\tilde{\phi})
\]  

(37)

The Hamiltonian of combined variable theory gives physical time evolution in the flat space-time limit. This is because if we take \( \Phi \approx e^{\pm iP_k \phi} e^{\pm i\bar{p}_k \bar{\phi}} \), combined variable theory reduces to ADM theory. But in absence of gravity, the Hamiltonian gives physical time evolution.

**Absence of scalar field:**
In absence of scalar field \( \phi \), the Hamiltonian evolution is a gauge transformation. We take a complex combined variable field.

\[
\tilde{T}_{grav} F^{ab} \left| \frac{\partial \Phi}{\partial \bar{q}_b} \right|^2 + V_{grav} |\Phi|^2 \approx 0
\]  

(38)

Taking \( \Phi \approx e^{\pm i\bar{p}_b} \bar{q}^b \) we recover vector (28) as well as scalar Hamiltonian constraints.

\[
\tilde{T}_{grav} F^{ab} p^a p^b + V_{grav} \approx 0
\]  

(39)

**Lapse function, shift vector and combined variable dynamics:**
In order to see the role of lapse function in the Hamiltonian dynamics, find it’s conjugate momentum

\[
\Pi_N := \frac{\partial L}{\partial \left( \frac{\partial N}{\partial \phi} \right)} = 0
\]  

(40)

Hamiltonian equations for lapse function and it’s conjugate momentum are

\[
\frac{\partial N}{\partial \phi} = \{N, H\} = 0 \quad \frac{\partial \Pi_N}{\partial \phi} = \{\Pi_N, H\}
\]  

(41)

Since the conjugate momentum of lapse function itself is a primary constraint, the second equation gives secondary constraints. This shows that the lapse function does not evolve in the combined variable dynamics.

As discussed in (30), the shift vector does not play any role in the combined variable dynamics. It puts restrictions on the spatial part of metric \( q_{ab}(t, \tilde{x}) \) and it’s conjugate momentum.

**Discussion:** In ADM theory, 4-dimensional curvature scalar splits into intrinsic curvature and extrinsic curvature parts. In combined variable theory, intrinsic curvature \( R^{(3)} \) plays a role of potential energy. It is similar to that of a role of \( m^2 \) in the scalar field theory. Temporal parts of 3-metric \( q_{ab} \) along with the scalar field forms a 4-dimensional superspace. Combined variables spread over the space of \( q_{ab}(t) \) and evolve with \( \phi(t) \). The extrinsic curvature part decides the distribution of combined variables in this space of \( q_{ab}(t) \). The scalar field energy plays a role of kinetic energy.

**3 Quantum Theory**

The Hamiltonian of combined variable theory allow us to interpret it as a collection of infinitely many quantum harmonic oscillators. This is achieved by defining creation and annihilation operators with coupling part of the last term replaced with unsettled function \( \omega(\phi, q_{ab}) \). The extra quadratic coupling term
appears due to the second term in the right hand side of (47). These two terms together are the third term in the Hamiltonian function. Therefore, $\omega$ is chosen in such a way that satisfies (48). Non-trivial commutation relation between $\Phi$ and $\Pi$ is given as

$$[\Phi(\phi, \tilde{q}), \Pi(\phi, \tilde{q})] = i\delta(\tilde{q}, \tilde{q}')$$ (42)

Combined variables $\Phi$, $\Pi$, creation and annihilation operators, Hamiltonian are all operators is to be understood. Notation $\hat{\cdot}$ is avoided for simplicity and introduced in the end. Creation and annihilation operators are defined as

$$a := \frac{\sqrt{N}}{\sqrt{2}} \bigg( \sqrt{2 \sqrt{\det q_a}} \frac{\sqrt{T}{\varphi}}{T} \Pi + i \int \sqrt{\det q_a} \left( \frac{1}{\det q_a} \right)^{\frac{1}{4}} q_a \frac{\partial \Phi}{\partial q_a} + i \omega \Phi \bigg)$$ (43)

$$a^* := \frac{\sqrt{N}}{\sqrt{2}} \bigg( \sqrt{2 \sqrt{\det q_a}} \frac{\sqrt{T}{\varphi}}{T} \Pi - i \int \sqrt{\det q_a} \left( \frac{1}{\det q_a} \right)^{\frac{1}{4}} q_a \frac{\partial \Phi}{\partial q_a} - i \omega \Phi \bigg)$$ (44)

Here $\omega(\phi, q_a)$ is unsettled function which will be fixed later.

$$a^* a = \frac{N}{2} \left( \frac{2 \sqrt{\det q_a}}{T} \frac{\sqrt{T}{\varphi}}{T} \Pi^2 + T_{\varphi} \left( \frac{\partial \Phi}{\partial q_a} \right) \left( \frac{\partial \Phi}{\partial q_a} \right)^{\frac{1}{2}} + \omega^2 \Phi^2 \right)$$

$$+ \frac{N}{2} \left( i \int \sqrt{\det q_a} \left( \frac{\partial \Phi}{\partial q_a} \right) \left( \frac{\partial \Phi}{\partial q_a} \right)^{\frac{1}{2}} + i \int \sqrt{\det q_a} \left( \frac{\partial \Phi}{\partial q_a} \right)^{\frac{1}{2}} \omega [\Pi, \Phi] \right)$$

Since,

$$[\Pi(\tilde{q}), q_a^* \frac{\partial \Phi(\tilde{q}^*)}{\partial q_a^*}] = D \delta(\tilde{q}, \tilde{q}')$$ (46)

$D$ is the number of independent components of 3-metric $q_{ab}$. e.g. for isotropic case $q_1 = q_2 = q_3 = q$, $D = 1$. For $q_1 \neq q_2 \neq q_3$, $D = 3$. Notice that the last term in the (45) can be written as

$$N \left( \frac{1}{\det q_a} \right)^{\frac{1}{4}} q_a \omega \frac{\partial \Phi}{\partial q_a} = \frac{\partial}{\partial q_a} \left( N \left( \frac{1}{\det q_a} \right)^{\frac{1}{4}} q_a \omega \Phi^2 \right) - \left( \frac{\partial}{\partial q_a} \left( N \left( \frac{1}{\det q_a} \right)^{\frac{1}{4}} q_a \omega \Phi^2 \right) \right)$$

(47)

When we take integral over the metric space, the first term becomes zero if combined variable $\Phi$ is chosen such that $N \left( \frac{1}{\det q_a} \right)^{\frac{1}{4}} q_a \omega \Phi^2$ remains constant on the surface. Now, first three terms along with the last term in integral of (45) give Hamiltonian if we set

$$\frac{N}{2} \omega^2 - \int \sqrt{\det q_a} \left( \frac{\partial}{\partial q_a} \left( N \left( \frac{1}{\det q_a} \right)^{\frac{1}{4}} q_a \omega \Phi^2 \right) \right) = \frac{1}{2} NV_{CF}$$ (48)

Other two terms in the integral of (45) are delta functions and both together is interpreted as a vacuum energy. The unsettled $\omega$ is thus a solution to above
Riccati equation.

\[
\int d^3q \ a^\dagger a = H + \int d^3q N \left( -D \sqrt{\frac{T_{\text{grav}}}{2T_\phi}} + \omega (\det q_a) \right) \delta(0) \tag{49}
\]

Or

\[
H = \int d^3q \ a^\dagger a = \int d^3q N \left( -D \sqrt{\frac{T_{\text{grav}}}{2T_\phi}} - \omega (\det q_a) \right) \delta(0) \tag{50}
\]

The second term \(\delta(0)\) corresponds to vacuum energy. Only non-trivial commutation relation between creation and annihilation operators is given as

\[
[a(\phi, \bar{q}), a^\dagger(\phi, \bar{q})] = N \sqrt{\frac{1}{2T_\phi}} \left( D \sqrt{T_{\text{grav}} - \omega (\det q_a)} \right) \delta(\bar{q}, \bar{q}') \tag{51}
\]

A role of creation and annihilation operator changes when \(D \sqrt{T_{\text{grav}}} \) is less than \(\omega (\det q_a)\). In that case, number operator \(\hat{n}\) is defined as \(n \propto a a^\dagger\) and for

\[
a a^\dagger = n \left( N \sqrt{\frac{1}{2T_\phi}} \left( D \sqrt{T_{\text{grav}} - \omega (\det q_a)} \right) \right) \quad \text{for} \quad D \sqrt{T_{\text{grav}}} \geq \omega (\det q_a) \tag{52}
\]

\[
a a^\dagger = n \left( N \sqrt{\frac{1}{2T_\phi}} \left( \omega (\det q_a) \right) - D \sqrt{T_{\text{grav}}} \right) \quad \text{for} \quad D \sqrt{T_{\text{grav}}} < \omega (\det q_a) \tag{53}
\]

Then the Hamiltonian without a vacuum energy term is written as

\[
\hat{H} = \int d^3q \ N \sqrt{\frac{1}{2T_\phi}} \left( D \sqrt{T_{\text{grav}} - \omega (\det q_a)} \right) \hat{n} \tag{54}
\]

This Hamiltonian is a collection of infinitely many quanta of the combined variable field. The spectrum is geometric. The state of single quantum is represented by \(|n, q_a, \phi\rangle\). \(n\) represents quantum number of combined variable field quantum with metric \(q_a\) and scalar field \(\phi\). Creation operator acting on the vacuum produces a single combined variable field quantum.

\[
\hat{a}^\dagger_{(n, q_a, \phi)} |0\rangle = |1, q_a, \phi\rangle \tag{55}
\]

Annihilation operator acting on the vacuum state gives 0.

Adjoint operator of creation operator \(\hat{a}\) for real combined variables is

\[
\frac{\sqrt{N}}{\sqrt{2}} \left( \sqrt{\frac{2 \det q_a}{T_\phi}} \Pi - i \sqrt{T_{\text{grav}}} \left( \frac{1}{\det q_a} \right)^{\frac{1}{4}} q_a \frac{\partial \phi}{\partial q_a} - i \omega^* \Phi \right)
\]

which is not an annihilation operator defined in the \(43\) unless \(\omega\) is real. Therefore only real solutions to the Riccati equation make Hamiltonian a self-adjoint operator. This Riccati equation arises in the process of quantization as an inevitable condition. An application of proper boundary conditions on this Riccati equation guarantees uniqueness of the quantum theory.
4 FLRW $\kappa = 0$ cosmology

Here, the standard ADM theory is re-written in terms of temporal part $q(t)$ of 3-metric and its canonical conjugate momentum. $q(t)$ is related to the scale factor $a(t)$ by $q = a^2$. FRW metric is given as

$$g_{\mu\nu} = \text{diag}(1, -q(t)^2, -q(t)^2 \sin(\theta))$$ (56)

Assume massless scalar field which is constant everywhere.

ADM theory:
Choose lapse function $N = 1$ and shift vector $N^a = 0$. Total Hamiltonian is then

$$H_{\text{total}} = H_\phi + H_{\text{scalar}} = \dot{\bar{T}}_\phi \frac{p_\phi^2}{2 \sqrt{\det q_a}} - \bar{T}_{\text{grav}} \left( \frac{p^a q_a q_\phi}{\sqrt{\det q_a}} p^b \right)$$

$$H_{\text{total}} = \frac{p_\phi^2}{2 q^2} - \frac{1}{3} q^2 p^2$$ (57)

It is already shown in [5] section II, (2.7) and (2.8) that the ADM theory is invariant under two rescaling [5] section II, (2.6). Therefore omitted fiducial volume part. Equations of motion are

$$\dot{\phi} = [\phi, H_{\text{total}}] = \frac{p_\phi}{q^2}$$
$$\dot{P}_\phi = [P_\phi, H_{\text{total}}] = 0$$ (58)

$$\dot{q} = [q, H_{\text{total}}] = -\frac{2}{3} q^2 p$$
$$\dot{P} = [P, H_{\text{total}}] = \frac{3 p_\phi^2}{4 q^2} - \frac{1}{6} q^2$$ (59)

$$\ddot{q} = -\frac{2}{3} \left( \frac{\dot{q}}{2 q} p + q^2 \dot{P} \right)$$ (60)

Plugging equations of $\dot{q}$ and $\dot{P}$ into above equation and using the fact that $H_{\text{total}} = 0$, we get

$$\ddot{q} = -\frac{\dot{q}^2}{2 q}$$ (61)

and $H_{\text{total}} = 0$ guarantees $\dot{q}^2 = \frac{2}{3} \rho_\phi$, where $\rho_\phi \equiv \dot{\phi}^2$ is scalar field energy density. Equation for scalar field energy density becomes

$$\dot{\rho}_\phi = -3 \left( \frac{\dot{q}}{q} \right) \dot{q}$$ (62)

Since the scalar field is massless and constant everywhere, $\rho_\phi = p_\phi$. The standard results of FLRW $\kappa = 0$ models are $\frac{\dot{a}}{a} = \frac{1}{3} \rho_\phi$ (10.73,7), $\dot{\rho} = -\frac{4}{3} \rho_\phi$ (10.80,7) with $\dot{a} = -2 \frac{\dot{a}}{a}$ and (10.82,7). Results obtained above can be easily verified by taking $q = a^2$.

Discussion:
According to the standard FLRW flat universe model in the ADM form, $\dot{q} > 0$ tells that the Universe expands in presence of scalar field and $\dot{q} < 0$ suggests
that this rate of expansion decreases with time. In absence of the scalar field, the Universe would have remained static. Refer chapter 10 of (7) for further detailed analysis of FLRW models.

**Classical combined variable field:**

Action for the combined variable field defined by (22) which has symmetrized gravitational part of a field equation is given by

\[
\mathcal{A} = \int d\phi \int dq \left( \frac{v_0}{2q^2} \left( \frac{\partial \Phi}{\partial \phi} \right)^2 - \frac{1}{3} v_0 q^2 \left( \frac{\partial \Phi}{\partial q} \right)^2 \right)
\]  

(63)

\[ D = 1 \] because gravitational part of superspace is 1 dimensional. It gives following field equation

\[
\left( \frac{v_0}{2q^2} \frac{\partial^2}{\partial \phi^2} - \frac{1}{3} v_0 q^2 \frac{\partial^2}{\partial q^2} - \frac{1}{6} v_0 q^\frac{1}{2} \frac{\partial}{\partial q} \right) \Phi(\phi, q) = 0
\]  

(64)

The superspace \((\phi, q)\) is 2 dimensional. In the limit \(q \to 0\), the first term is dominant and \(\Phi \to A\phi + B\). In the limit \(q \to \infty\), the other two terms are dominant and \(\Phi \to Aq^\frac{5}{9} + B\). Energy density and pressure for this combined variable field are given as

\[
\rho = \frac{v_0}{4q^2} \left( \frac{\partial \Phi}{\partial \phi} \right)^2 + \frac{1}{6} v_0 q^\frac{1}{2} \left( \frac{\partial \Phi}{\partial q} \right)^2
\]  

(65)

\[
p = \frac{v_0}{4q^2} \left( \frac{\partial \Phi}{\partial \phi} \right)^2 - \frac{1}{6} v_0 q^\frac{1}{2} \left( \frac{\partial \Phi}{\partial q} \right)^2
\]  

(66)

Energy density \(\rho\) as well as pressure has singularity, the ‘Big Bang singularity’ at \(q = 0\). The Big Bang singularity exists even in absence of the scalar field because \(\Phi = Aq^\frac{5}{9} + B\). Therefore the second term diverges at the origin. In the limit \(q \to \infty, \rho \to 0\) and \(p \to 0\) with \(p = -\rho\). This is the equation of state for pure gravitational field described by FLRW \(\kappa = 0\) metric in absence of the scalar field.

**Discussion:**

The combined variable field \(\Phi\) behaves more like a scalar field in early development of the Universe. As Universe expands the scalar field energy density decreases and gravitational field energy density starts dominating. In absence of the scalar field, momentum \((31)\) conjugate to combined variable field becomes primary constraint and \(\frac{\partial \rho}{\partial q} = 0\). Evolution is not a physical evolution. There is nothing with respect to which gravitational field can evolve and situation becomes static. In other words, the gravitational field evolves resulting into the expansion of the Universe. Later, scalar field energy density tends to zero. In absence of scalar field, gravitational field cannot evolve.

**Quantum combined variable field:**

Quantization of the theory is straightforward once Riccati equation \((48)\) is solved. \(\omega\) which is the solution to the Riccati equation allows us to write the Hamiltonian in terms of creation and annihilation operators.

\[
\frac{1}{2} \omega^2 - \sqrt{\frac{1}{3} v_0} \frac{\partial}{\partial q} (q^\frac{4}{9} \omega) = 0
\]  

(67)
Solution to this equation is given as

\[ \omega = -\frac{1}{3} \nu_0 \frac{1}{q^{1/4} (\sqrt{q} - c_1)} \]  

(68)

The Hamiltonian for FRW \( \kappa = 0 \) is

\[ \hat{H} = \int dq \left| \sqrt{\frac{1}{6}} \left( 1 + \frac{\sqrt{q}}{\sqrt{q} - c_1} \right) \right| \hat{H} \]  

(69)

The Hamiltonian is a collection of infinitely many combined variable field quantum. A value of constant \( c_1 \) can be fixed by applying an initial condition to the Riccati equation. The energy is quantized in the units of

\[ h = \sqrt{\frac{1}{6}} \left( 1 + \frac{\sqrt{q}}{\sqrt{q} - c_1} \right) \]  

(70)

Case \( c_1 > 0 \):

The energy of quantum increase with decrease in \( q \). This quantum is a quantum of gravity and scalar field together. For large \( q \), matter density is negligible. The Universe is mostly empty. In this limit, this quantum can be interpreted as a quantum of gravity and the energy of quantum becomes independent of \( q \). As \( q \) decreases, the energy of quantum increases. Near \( q = q_0 = c_1^2 \), it hits a singularity. That means if we try to compress the quantum by increasing the energy, it's size approaches \( q_0 \) but it will always be greater than \( q_0 \).

Case \( c_1 = 0 \):

The energy of quantum in this case remains constant.

Case \( c_1 < 0 \):

In the large \( q \) limit, the energy of quantum is constant. As \( q \) decreases, the energy of quantum decreases.

Figure 1: \( h = 1 + \frac{\sqrt{q}}{\sqrt{q} - c_1} \) with \( c_1 = 1 \)
Discussion: In order to find the correct choice of quantum theory, coherent states are taken. These coherent states are eigen states of annihilation operator.

\[ \hat{a}|\alpha\rangle = \alpha|\alpha\rangle \] (71)

The uncertainty in the Hamiltonian density \( \hat{a}^\dagger \hat{a} \) is calculated using above definition and (51)

\[ \langle \Delta (\hat{a}^\dagger \hat{a}) \rangle = \langle \hat{a}^\dagger \hat{a} \hat{a} \hat{a}^\dagger \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2 = |\alpha|^2 h \] (72)

These uncertainties are proportional to \( h \). If we assume that the Universe is becoming more and more classical as it expands then, uncertainties should decrease with increase in \( q \). \( h \) is decreasing function of \( q \) only in the case of \( c_1 > 0 \). Therefore, under this assumption \( c_1 > 0 \) is the correct quantum theory. In this quantum theory, \( q := q_0 = c_1 \) is a lower bound to the size of quantum. The Universe being collection of these quanta of the combined variable field, cannot be compressed to this size. In other words, \( q = q_0 \) is the beginning of the Universe and not \( q = 0 \). In general, the quantum of combined variable field is a quantum of both scalar field and gravitational field combined together. In the limit \( q \to \infty \), scalar field energy density becomes zero as discussed in the classical theory. Therefore, in this limit the quantum can be interpreted as a quantum of gravitational field.

5 Conclusion

If the Klein-Gordon field is a result of quantizing relativistic particles then, the combined variable field is a result of quantizing gravitational field together with the scalar field. Unlike ADM theory where field equations tell us how gravitational field and scalar field evolve relative to one another, the combined combined variable theory tells, both gravitational and scalar field combined to form a combined variable field which is distributed over \( q_a(t) \) and evolve relative to \( \phi(t) \). On one particle (\( \Phi \approx e^{\pm q_0 P} e^{e^{\pm q_0 P}} \) like interpretation, the combined variable theory reduces to the ADM theory. Hamiltonians are different for observers in two different frames and are related to each other through lapse function. Shift vectors does not play any role in the combined variable dynamics. It puts restrictions on the spatial part of gravitational field variables.

In the absence of scalar field \( \phi \), Hamiltonian constraints get recovered and combined variable field becomes static. In the absence of gravity, combined variable field behaves as a standard (special relativistic) scalar field and Hamiltonian gives physical time evolution.

The theory applied to FLRW \( \kappa = 0 \) model in order to understand its implications. The classical combined variable theory agrees with the standard ADM theory. Both classical theories are different viewpoint of the same reality. The quantum of the combined variable theory is a quantum of both scalar and field and gravitational field together. For \( q \to \infty \), it can be interpreted as a quantum of gravity.
6 Acknowledgement

I am immensely grateful to Prof. Ajay Patwardhan for providing expertise. This work would not have possible without his guidance. Any errors are my own and should not tarnish his reputation.

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