Evolution of collective $N$ atom states in single photon superradiance

Anatoly A. Svidzinsky and Marlan O. Scully

Institute for Quantum Studies and Dept. of Physics, Texas A&M Univ., College Station TX 77843
Applied Physics and Materials Science Group, Engineering Quad, Princeton University, Princeton NJ 08544

(Dated: March 16, 2009)

We present analytical solutions for the evolution of collective states of $N$ atoms. On the one hand is a (timed) Dicke state prepared by absorption of a single photon and exhibiting superradiant decay. This is in strong contrast to evolution of a symmetric Dicke state which is trapped for large atomic clouds. We show that virtual processes yield a small effect on the evolution of the rapidly decaying timed Dicke state. However, they change the long time dynamics from exponential decay into a power-law behavior which can be observed experimentally. For trapped states virtual processes are much more important and provide new decay channels resulting in a slow decay of the otherwise trapped state.

The long standing problem of collective spontaneous emission from $N$ atoms is a subject of growing recent attention and debate. Effects of virtual processes are of particular current interest. Here we consider a system of two level atoms, $E_a - E_b = \hbar \omega$. Initially there are no photons and one of the atoms is in the excited state $a$, but we don’t know which one. That is the system is prepared in a collective $N$-atom state. The question then is how such a collective state evolves with time.

Atoms interact with common electromagnetic field and the interaction Hamiltonian is given by

$$\hat{H}_{\text{int}} = \sum_{k} \sum_{j=1}^{N} g_k (\hat{\sigma}_j e^{-i\omega t} + \hat{\sigma}_j^\dagger e^{i\omega t}) \times \left( \hat{a}_k e^{i\nu_k t - i k \cdot \mathbf{r}_j} + \hat{a}_k^\dagger e^{-i\nu_k t + ik \cdot \mathbf{r}_j} \right), \quad (1)$$

where $\hat{\sigma}_j$ is the lowering operator for atom $j$, $\hat{a}_k$ is the operator of photon with wave vector $k$, $g_k$ is the atom-photon coupling constant and $\mathbf{r}_j$ is the radius vector of the atom $j$. Evolution of the atomic system is described by the state vector

$$|\Psi\rangle = \sum_{j=1}^{N} \beta(t, \mathbf{r}_j) |b_1 b_2 \ldots a_j \ldots b_N\rangle \quad (2)$$

where $|b_1 b_2 \ldots a_j \ldots b_N\rangle$ is a Fock state in which atom $j$ is in the excited state $a$ and all other atoms being in the ground state $b$. We disregard polarization effects, that is treat photons as scalar and assume that initial state evolves slowly compared to the time of photon flight through the atomic cloud (the opposite limit has been studied in [6]).

Decay of an initial state occurs via real and virtual processes in which a virtual photon is emitted and then reabsorbed. In particular, due to counter-rotating terms in Hamiltonian virtual processes couple the single-atom excited states with those in which two atoms are excited. If all virtual processes are taken into account then for a dense atomic cloud evolution of the system is described by an integral equation with an exponential kernel [7, 8]

$$\frac{\partial \beta(t, \mathbf{r})}{\partial t} = i\frac{N}{V} \int d\mathbf{r} \frac{\exp(ik_0 |\mathbf{r} - \mathbf{r}'|)}{k_0 |\mathbf{r} - \mathbf{r}'|} \beta(t, \mathbf{r}'), \quad (3)$$

where $V = 4\pi R^3/3$ is the volume of the spherical atomic cloud, $k_0 = \omega/c$ and $\gamma$ is the single atom decay rate. We assume that atoms are uniformly distributed with density $N/V$ in a sphere of radius $R$.

If we ignore virtual contributions then Eq. (3) reduces to an equation with sinusoidal kernel

$$\frac{\partial \beta(t, \mathbf{r})}{\partial t} = -\gamma \frac{N}{V} \int d\mathbf{r} \frac{\sin(k_0 |\mathbf{r} - \mathbf{r}'|)}{k_0 |\mathbf{r} - \mathbf{r}'|} \beta(t, \mathbf{r}'), \quad (4)$$

Here we solve Eqs. (3) and (4) analytically for two initial conditions, namely the $|+\rangle$ “timed” Dicke state

$$\beta(0, \mathbf{r}) = e^{ik_0 \cdot \mathbf{r}}, \quad (5)$$

which is prepared by absorption of a single photon with wave vector $k_0$ ($k_0 = \omega/c$) [3, 4], and the symmetric Dicke state [1]

$$\beta(0, \mathbf{r}) = 1. \quad (6)$$

For a large atomic sample $R \gg \lambda$ ($\lambda = 2\pi c/\omega$ is the wavelength of the emitted photon) the $|+\rangle$ state (5) is superradiant, while (6) is a trapped state undergoing very slow decay. As we show below, virtual processes yield a small (yet interesting) effect on evolution of the rapidly decaying $|+\rangle$ state. Such states decay mainly via real Weisskopf-Wigner spontaneous emission processes. However, virtual processes can substantially modify the dynamics of trapped states and provide a main channel of decay.

Figs. 1-4 summarize our main findings. For a small atomic cloud $R \ll \lambda$ symmetric state exponentially decays according to Eq. (4) with rate $\Gamma = N\gamma$ without coupling to other states. This result has been obtained by Dicke [1]. Our Figs. 1 and 2 show, however, that virtual processes excite other states with a few % probability even in the small sample (Dicke) limit. For a
large cloud Eq. (1) predicts that symmetric state (6) is trapped, however, virtual processes lead to its slow decay as shown in Fig. 3. On the other hand, evolution of the rapidly decaying $\ket{+}$ state is only slightly affected by virtual processes (see Figs. 1 and 2). For a large sample such processes excite other states with less than about 10% probability. Thus the timed Dicke state (5) is, to a good approximation, described by Eq. (4) which ignores virtual transitions. However the symmetric state (6) is strongly affected by virtual processes as per Fig. 3.

Next we discuss the evolution of $\ket{+}$ state in detail. For $R \gg \lambda$ equation with sin kernel (11) gives

$$
\beta(t, r) = e^{ik_0 \cdot r} e^{-\Gamma t},
$$

where

$$
\Gamma = \frac{3N\gamma}{2(k_0 R)^2}.
$$

Here we find that equation with exp kernel (11) yields

$$
\beta(t, r) = e^{ik_0 \cdot r} \left[ f(t, r) + ig(t, r) \cos \theta \right],
$$

where $\theta$ is the angle between $k_0$ and $r$,

$$
f(t, r) = \frac{1}{2} \left[ J_0 \left( 2 \sqrt{1 - \frac{r}{R} \sqrt{\Gamma t}} \right) + J_0 \left( 2 \sqrt{1 + \frac{r}{R} \sqrt{\Gamma t}} \right) \right],
$$

$$
g(t, r) = \frac{1}{2} \left[ J_0 \left( 2 \sqrt{1 + \frac{r}{R} \sqrt{\Gamma t}} \right) - J_0 \left( 2 \sqrt{1 - \frac{r}{R} \sqrt{\Gamma t}} \right) \right],
$$

and $J_0(z)$ is the Bessel function. Answer (11) is universal in the sense that state evolution is determined by the dimensionless time $\Gamma t$ and change of the sample size simply results in time rescaling.
Next we calculate the probability that atoms are excited as a function of time

\[ P(t) = \frac{1}{V} \int dr |\beta(t, r)|^2. \] (17)

For the integral equation with sin kernel

\[ P_{\sin}(t) = e^{-2\Gamma t}. \] (18)

For \( \beta(t, r) \) given by Eq. (9) one can calculate the integral in Eq. (17) numerically for any \( t \), while for \( t \gg 1/\Gamma \) we find

\[ P_{\exp}(t) \approx \frac{7\sqrt{2}}{15\pi \sqrt{\Gamma t}} = 0.21 \frac{1}{\sqrt{\Gamma t}}. \] (19)

Insert of Fig. 1 shows \( P(t) \) obtained using the exp kernel (solid line) and Eq. (8) (dash line). At \( t \lesssim 1/\Gamma \) the function \( P_{\exp}(t) \) decays as \( e^{-2\Gamma t} \), while for \( t > 1/\Gamma \) it becomes closer to its asymptotic expression (19). During the major part of the decay curve \( P_{\exp}(t) \) exhibits exponential behavior (18) and, thus, virtual processes have essentially no effect. However, virtual processes modify \( P_{\exp}(t) \) at large time yielding the power-law decay (19). Such an interesting, although small, effect can be observed experimentally.

For solution (9) the probability that atoms are in the \(+ > \) state is given by

\[ P_{\exp}^+(t) = \frac{9}{4} \left( \frac{4(\Gamma t - 2)}{(\Gamma t)^2} J_0(2\sqrt{2\Gamma t}) + \frac{\sqrt{2}}{(\Gamma t)^{3/2}} [4 - 6\Gamma t + (\Gamma t)^2 J_1(2\sqrt{2\Gamma t})]^2 \right) \] (20)

which for \( t \gg 1/\Gamma \) yields

\[ P_{\exp}^+(t) \approx \frac{9\sqrt{2}}{4\pi (\Gamma t)^{3/2}} \cos^2 \left( 2\sqrt{2\Gamma t} + \frac{\pi}{4} \right). \] (21)

In Fig. 4 we plot \( P_{\exp}^+(t) \) obtained from Eq. (20) (solid line) and compare it with those found from equation with sin kernel \( P_{\sin}^+(t) = e^{-2\Gamma t} \) (dash line). The two curves are very close to each other. This means that virtual processes practically do not change evolution of \(+ > \) state if it is considered separately. Without virtual processes the \(+ > \) state directly decays into the ground state by emitting a photon. Virtual processes yield an extra decay channel in which energy is partially transferred into other atomic states. However, as one can see from Fig. 4 the net decay rate of the \(+ > \) state into all channels remains practically the same with or without virtual processes. Insert shows probability that atoms are in any other state but \(+ > \). This curve demonstrates that during the system evolution the other states are excited with probability less than about 10% and, therefore, the effect of virtual processes is quite small for fast decaying states.

To obtain solution (9) we used the identities

\[ \exp(i k_0 |r - r'|) = i \sum_{m=0}^{\infty} (2m + 1) P_m(\hat{r} \cdot \hat{r}') \times \]

\[ \left\{ \begin{array}{ll} j_m(k_0 r) h_n^{(1)}(k_0 r'), & r > r' \\ j_m(k_0 r) h_n^{(1)}(k_0 r'), & r \leq r' \end{array} \right., \] (12)

\[ \exp(i k_0 \cdot r) = \sum_{n=0}^{\infty} i^n (2n + 1) j_n(k_0 r) P_n(\hat{k}_0 \cdot \hat{r}), \] (13)

\[ \int d\Omega_r P_m(\hat{r} \cdot \hat{r}') P_n(\hat{k}_0 \cdot \hat{r}') = \delta_{mn} \frac{4\pi}{2n + 1} P_n(\hat{k}_0 \cdot \hat{r}), \] (14)

where \( \hat{r} \) and \( \hat{k}_0 \) are unit vectors in the directions of \( r \) and \( k_0 \) respectively, \( P_n \) are the Legendre polynomials and \( j_n(z), h_n^{(1)}(z) \) are the spherical Bessel functions. In the large sample limit the ansatz (9) yields the following equations for the slowly varying functions \( f \) and \( g \)

\[ \frac{\partial f(t, r)}{\partial t} = -\frac{\Gamma}{R} \int_0^R dr' f(t, r') - \frac{\Gamma}{R} \int_r^R dr' g(t, r'), \] (15)

\[ \frac{\partial g(t, r)}{\partial t} = \frac{\Gamma}{R} \int_0^r dr' f(t, r'), \] (16)

with the initial conditions \( f(0, r) = 1 \) and \( g(0, r) = 0 \). One can solve Eqs. (15) and (16) using the method of Laplace transform which yields the answer (10) and (11).
Next we discuss evolution of the symmetric state \( |6\rangle \). For such initial condition Eq. (4) with sin kernel can be solved analytically for any size of the atomic sample and yields

\[
\beta(t, r) = 1 + 2F \frac{\sin(k_0 r)}{k_0 r} [1 - e^{-\Gamma t}], \tag{22}
\]

where

\[
F = \frac{k_0 R \cos(k_0 R) - \sin(k_0 R)}{k_0 R - \sin(k_0 R) \cos(k_0 R)} \tag{23}
\]

and

\[
\Gamma = \frac{3\gamma N}{2(k_0 R)^2} \left[ 1 - \frac{\sin(2k_0 R)}{2k_0 R} \right]. \tag{24}
\]

Eq. (22) shows that at the beginning the atomic system decays with the superradiant rate \( \Gamma \) but quickly ends up in a trapped state

\[
\beta(r) = 1 + 2F \frac{\sin(k_0 r)}{k_0 r}. \tag{25}
\]

Function (25) vanishes in the small sample limit \( k_0 R \ll 1 \), however, for large sample \( \beta(r) \approx 1 \) and state \( |6\rangle \) is completely trapped. Probability that atoms are excited is given by

\[
P(t) = 1 - 6 \frac{[k_0 R \cos(k_0 R) - \sin(k_0 R)]^2 [1 - e^{-2\Gamma t}]}{k_0 R - \sin(k_0 R) \cos(k_0 R)} \tag{26}
\]

For a large atomic cloud \( R \gg \lambda \) the evolution Eq. (3) with initial condition \( |6\rangle \) can be also solved analytically and the answer is expressed in terms of the Bessel functions. In Fig. 3 we plot probability that atoms are excited \( P(t) \) obtained from equation with exp (solid line) and sin (dash line) kernels. Initially atoms are prepared in the state \( |6\rangle \). Size of the atomic sample is \( R = 5\lambda \). Insert shows behavior of \( P(t) \) for exp kernel on a large time scale which exhibits interesting plateaus and oscillations. For \( t \) less then a few \( 1/\Gamma \) two curves are identical. For such time the real processes dominate and the initial state evolves into the state \( |6\rangle \) which is trapped if we omit virtual processes. Virtual processes, however, result in state decay as shown by the solid curve. State \( |6\rangle \) overlaps with many eigenstates of Eq. (3). Eigenstates which decay faster contribute to evolution at small time. As time increases \( P(t) \) decays more slowly. However, eigenfunctions of Eq. (3) are not orthogonal and, in addition, have different collective Lamb shifts. This makes state evolution richer.

In the small sample limit \( R \ll \lambda \) the initial states \( |5\rangle \) and \( |6\rangle \) are the same. Equation with sin kernel \( |4\rangle \) gives

\[
\beta(t, r) = e^{-\Gamma t}, \tag{27}
\]

where \( \Gamma = N\gamma \). For equation with exp kernel the state evolution can be obtained by noting that in the small sample limit

\[
\beta_n(t, r) = \frac{R}{r} \sin \left[ \left( \frac{\pi n + \pi}{2} \right) \frac{r}{R} \right] e^{-\lambda_n t} \tag{28}
\]

are eigenfunctions of Eq. (3) with eigenvalues \( \lambda_n \)

\[
\lambda_n = -\frac{12\pi\gamma}{\pi^2 + 96N\gamma R} + \frac{96N\gamma}{\pi^4 + 96\pi\gamma R^2}, \quad n = 0, 1, 2, \ldots \tag{29}
\]

Using the identity

\[
1 = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin[(2n+1)x]
\]

one can expand the initial condition \( \beta(0, r) = 1 \) in terms of \( \beta_n(0, r) \). As a result, time evolution of the symmetric state is given by

\[
\beta(t, r) = \frac{8R}{\pi^2 r} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \left( n + \frac{1}{2} \right) \frac{\pi r}{R} e^{-\lambda_n t} \tag{30}
\]

and probability to find atoms excited is

\[
P(t) = \frac{96}{\pi^4} \sum_{n=0}^{\infty} \exp \left[ -2\Re(\lambda_n) t \right] \left( \frac{1}{2n+1} \right)^4. \tag{31}
\]

Fig. 1 shows \( P(t) \) given by Eq. (31) (solid line) and compares it with the answer obtained omitting virtual processes \( P(t) = \exp(-2\Gamma t) \) (dash line). The two curves are close to each other, but Eq. (31) yields a few % of population trapped which slowly decays with time.

In Fig. 2 we plot probability that atoms are in the symmetric state \( |6\rangle \) obtained for \( R = 0.01\lambda \) from Eq. (30) (solid line) and compare it with \( P(t) = \exp(-2\Gamma t) \) (dash line). The two curves are very close meaning that the net decay rate of the symmetric state into all channels is the same with or without virtual processes. Insert shows probability to find atoms in any other state but symmetric state \( |6\rangle \) for \( R = 0.01\lambda \) (solid line) and \( R = 0.03\lambda \) (dash-dot line) obtained from Eq. (31). Dependence of the imaginary part of \( \lambda_n \) (collective Lamb shift) on \( n \) is the reason for oscillations. Period of oscillations is proportional to \( k_0 R \). The other states are excited with a few % probability. Thus, in the small sample limit, virtual photons also yield a small (but interesting) effect on evolution of fast decaying states.

In summary, we consider evolution of two collective states of \( N \) atoms, the \( |+\rangle > \) state which decays fast and the symmetric state which is trapped for \( R \gg \lambda \). We obtain analytical formulas for the atomic state vector as a function of time. We show that virtual processes yield a small effect on evolution of the rapidly decaying states. However, they change the long time dynamics from exponential decay into power-law which can be observed experimentally. For trapped states virtual processes qualitatively modify state evolution. Namely, they provide new decay channels which ultimately result in a slow decay of the otherwise trapped state.
We thank R. Friedberg and J. Manassah for stimulating discussion and gratefully acknowledge the support of the Office of Naval Research (Award No. N00014-07-1-0184 and N0001408-1-0948) and the Robert A. Welch Foundation (Award A-1261).

[1] R.H. Dicke, Phys. Rev. 93, 99 (1954).
[2] V. Ernst and P. Stehle, Phys. Rev. 176, 1456 (1968); N. E. Rehler and J. H. Eberly, Phys. Rev. A 3, 1735 (1971); R. Bonifacio et al., Phys. Rev. A 4, 302 (1971); S. Prasad and R. Glauber, Phys. Rev. A 31, 1583 (1985).
[3] M. Scully, E. Fry, C.H.R. Ooi and K. Wodkiewicz, Phys. Rev. Lett. 96, 010501 (2006).
[4] M. Scully, Laser Phys. 17, 635 (2007).
[5] J.H. Eberly, J. Phys. B 39, S599 (2006); I. Mazets and G. Kurizki, J. Phys. B 40, F105 (2007).
[6] A.A. Svidzinsky, J.T. Chang and M.O. Scully, Phys. Rev. Lett. 100, 160504 (2008).
[7] A.A. Svidzinsky and J.T. Chang, Phys. Rev. A 77, 043833 (2008).
[8] R. Friedberg and J. T. Manassah, Phys. Lett. A 372, 2514 (2008).
[9] R. Friedberg and J. Manassah, Phys. Lett. A 372, 6833 (2008); Opt. Com. 281, 4391 (2008).
[10] A.A. Svidzinsky and J.-T. Chang, Phys. Lett. A 372, 5732 (2008); R. Friedberg and J. T. Manassah, Phys. Lett. A 372, 5734 (2008); M. Scully and A.A. Svidzinsky, Phys. Lett. A 373, 1283 (2009).