A PRYM HYPERGEOMETRIC

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Abstract. We study a hypergeometric local system that arises from the quantum Chen–Ruan cohomology of a family of weighted del Pezzo hypersurfaces. We prove that it is the anti-invariant variation of a pencil of genus-7 curves with respect to an involution having 4 fixed points.

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1. Introduction

Consider the hypergeometric function defined by the complex power series:

\[ F(\alpha) = \sum_{j=0}^{\infty} \frac{(18j)!j!}{(2j)!(3j)!(5j)!(9j)!} \alpha^j \]

This function satisfies an irreducible order-8 hypergeometric operator \( H \) on \( \mathbb{C}^\times \) singular at \( \alpha_0 = \frac{5^5}{3^{10}2^{20}} \). 

The local system \( \mathbb{H} \) of its solutions is a complex irreducible local system of rank 8 on \( U = \mathbb{C}^\times \setminus \{\alpha_0\} \). By [Kat90, Theorem 5.4.4] \( \mathbb{H} \) supports a canonical rational variation of (pure) Hodge structures (VHS) of weight one [CG11, Section 1.2].

Our main result is the following.

Theorem 1. Consider the pencil of genus-7 curves \( \tilde{N}_\alpha \) defined by the equation:

\[ (4y^3 - \alpha x_0 x_1^2 y^2 + 4\alpha^3 x_1^6 y + \alpha x_0^9 = 0) \subset \mathbb{P}(1, 1, 3) \]

where \( x_0, x_1, y \) are homogeneous coordinates of weights 1, 1, 3, and let \( P_\alpha : \tilde{N}_\alpha \to N_\alpha \) be the quotient by the involution \( x_1 \mapsto -x_1 \). Then, for all \( \alpha \in \mathbb{C}^\times \setminus \{\alpha_0\} \):

\[ H_\alpha = H^1(\tilde{N}_\alpha, \mathbb{Z})^- \otimes_{\mathbb{Z}} \mathbb{Q} \]

where \( H^1(\tilde{N}_\alpha, \mathbb{Z})^- \) are the antiinvariants under the involution. In particular this endows \( \mathbb{H} \) with an integral structure.

Remark 2. The curves \( \tilde{N}_\alpha, N_\alpha \) are nonsingular for all \( \alpha \in \mathbb{C}^\times \setminus \{\alpha_0\} \), and the involution \( x_1 \mapsto -x_1 \) on \( \tilde{N}_\alpha \) has four fixed points, hence \( H^1(\tilde{N}_\alpha, \mathbb{Q})^- \) has rank 8 — see Remark 14.

Theorem 1 follows from a general result on conic bundles over surfaces which is of independent interest (we set out our terminology for conic bundles in Section 3.1).
Theorem 3. Let $\pi : X \to S$ be a conic bundle over a surface $S$ with ramification data $p : \Delta \to \Delta$, and let $W$ be the subsheaf of $\mathfrak{Z}_\Delta$ of anti-invariants with respect to the involution on $\Delta$. Let $i : \Delta \hookrightarrow S$ be the inclusion.

There is a short exact sequence of mixed sheaves on $S$:

$$0 \to i_* W \to R^2\pi_* \mathcal{Z}_X(1) \to \mathcal{Z}_S \to 0$$  \hspace{1cm} (4)$$

We prove Theorem 3 in Section 3.2.

Remark 4. If $X \to S$ is a conic bundle over a rational projective surface $S$, the works [Bea77, Bel85, Bea89] describe $H^3(X, \mathbb{Z})$ in terms of the ramification data $\Delta \to \Delta$. Our Theorem 3 is a sheaf-theoretic generalisation of these results which applies to conic bundles $X \to S$ where $X$ is not necessarily proper.

1.1. Motivation and context. Our motivation to study the hypergeometric local system $\mathbb{H}$ is twofold. On the one hand $\mathbb{H}$ supports a VHS of weight one, hence it is natural to ask if $\mathbb{H}$ is related to the variation of $H^1$ of a pencil of curves. Our Theorem 1 proves that this is indeed the case.

On the other hand, the function $F$ in Equation (1) is, up to a shift in $\alpha$, a specialisation of the regularised quantum period of the family of log del Pezzo surfaces $X_{18} \subset \mathbb{P}(2, 3, 5, 9)$. This is an almost immediate consequence of [CG21, Proposition 37]. Thus our Theorem 1 implies that the pencil of curves $\tilde{N}_\alpha$ is a Landau–Ginzburg mirror of this family.

The family $X = X_{18} \subset \mathbb{P}(2, 3, 5, 9)$ appears as one of the sporadic cases of the classification of anticanonical quasismooth and wellformed log del Pezzo surfaces, by Johnson and Kollár [JK01, Theorem 8]. The mirror construction we present here is the only one that we know for the surfaces $X$. Indeed, since the anticanonical linear system of $X$ is empty, none of the standard methods apply, see [CG21, Remark 2.7].

We refer to [RRV22, Section 2–9] for an introduction to the geometry of hypergeometric motives. We refer to [CG21, Sections 1,2] for our notion of mirror symmetry for Fano varieties with quotient singularities.

1.2. Hypergeometric GKZ systems. In Section 2.1 we consider an explicit affine threefold $Z_\alpha$ depending on the parameter $\alpha \in \mathbb{C}^\times$, see for details Equation (6), and we state that the VHS supported by $\mathbb{H}$ is the variation $\mathrm{gr}^W_3 H^3_c(Z_\alpha, \mathbb{Q})(1)$ over $\alpha \neq \alpha_0$ (Proposition 7).

This identity of VHS follows from a general result by Stienstra [Sti98], which shows that certain $\mathcal{D}$-modules associated to Gel’fand–Kapranov–Zelevinsky hypergeometric systems (GKZ systems) are related to relative cohomology modules $H^* (\mathbb{T}, Z_\alpha, \mathbb{Q})$, where $Z_\alpha = (f_v = 0) \subset \mathbb{T}$ is the hypersurface cut out of the torus $\mathbb{T}$ by a parametric Laurent polynomial $f_v$ built out of the GKZ data. The hypergeometric $\mathcal{D}$-module associated to our operator $H$ arises from the restriction of such a $\mathcal{D}$-module to a hyperplane in the parameter space, and the pencil $Z_\alpha$ is obtained from the restriction of the corresponding parametric family $Z_v$ to the same hyperplane.

1.3. Sketch of the proof of Theorem 1. The starting point for the proof of Theorem 1 is the pencil of threefolds $Z_\alpha$ of Section 2.1 where $\mathbb{H} = \mathrm{gr}^W_3 H^3_c(Z_\alpha, \mathbb{Q})(1)$ over $\alpha \neq \alpha_0$.

In Section 2.2, we construct a partial compactification $Z_\alpha \subset X_\alpha \subset (\mathbb{C}^\times)^2 \times \mathbb{P}^2$ such that the projection $\pi : X_\alpha \to (\mathbb{C}^\times)^2$ to the first factor is a conic bundle.

Once we have the conic bundle, the rest of the proof follows from an application of Theorem 3 to the case at hand, together with manipulations in mixed Hodge theory. The quotient map $P : \tilde{N}_\alpha \to N_\alpha$ of Theorem 1 arises from the ramification data of the conic bundle $\pi : X_\alpha \to (\mathbb{C}^\times)^2$. We believe that Theorem 3 is the easiest way to study the Hodge structure $\mathrm{gr}^W_3 H^3_c(X_\alpha, \mathbb{Q})$.

The map $P : \tilde{N}_\alpha \to N_\alpha$ is a double cover of a genus 3-curve with branch divisor $B_\alpha$ of degree 4, see Remark 14. By [NR95, Theorem 9.14] the Prym map $\mathcal{R}_{3,4} \to A^4_\delta$, $\delta = (1, 2, 2, 2)$, is a dominant morphism generically of degree 3, but note that the construction [NR95, 9.1] does not apply here since the linear system $|B_\alpha|$ has a base point, see Remark 14.
1.4. Structure of the paper. Section 2 contains the initial setup. In Section 2.1 we construct the pencil of threefolds $Z_\alpha$ and we state Proposition 7; in Section 2.2 we build the partial compactification $Z_\alpha \subset X_\alpha$. In Section 3 we prove Theorem 3. In Section 4 we prove Theorem 1. In the Appendix we prove Proposition 7.

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2. Initial setup

In this section we introduce the main objects involved in the proof of Theorem 1.

2.1. A family of threefolds. Consider the series $F$ in Equation (1). Following [Rod19] and [RRV22, Section 2], we will encode the data defining the coefficients of $F$ in the gamma list $\gamma = (-18, -1, 2, 3, 5, 9)$. The function $F$ is a solution to the order-8 irreducible hypergeometric operator on $\mathbb{C}^\times$:

$$
H = \alpha_0 \cdot D \left( D - \frac{1}{3} \right) \left( D - \frac{2}{3} \right) \prod_{n=0}^{4} \left( D - \frac{n}{5} \right) - \alpha \prod_{n=0}^{8} \left( D + \frac{2n + 1}{18} \right)
$$

where $\alpha_0$ is as in the Introduction and $D = \alpha \frac{d}{d\alpha}$. The local system $\mathbb{H}$ of the Introduction is the local system of solutions of $H$.

Remark 5. The connection between the hypergeometric parameters in (5) and the list $\gamma$ is given by the monodromy representation $\rho: \pi_1(\mathbb{P}^1 \setminus \{0, \alpha_0, \infty\}) \to \text{GL}(\mathbb{C}^8)$ of $H$. Let $\gamma_0, \gamma_\alpha, \gamma_\infty$ be three generators of $\pi_1(\mathbb{P}^1 \setminus \{0, \alpha_0, \infty\})$ such that $\gamma_\infty \gamma_\alpha \gamma_0 = 1$, and let $q_\infty$ and $q_0$ be the characteristic polynomials of $\rho(\gamma_\infty)$ and $\rho(\gamma_0)^{-1}$. Then by [BH89, Section 3]

$$
\frac{q_\infty}{q_0}(\alpha) = \frac{\phi_{18}\phi_1}{\phi_{2}\phi_3\phi_5\phi_9}(\alpha) = \frac{(\alpha^{18} - 1)(\alpha - 1)}{(\alpha^2 - 1)(\alpha^3 - 1)(\alpha^5 - 1)(\alpha^9 - 1)}
$$

where $\phi_n$ denotes the $n$-th cyclotomic polynomial.

Now consider vectors $m_1, \ldots, m_6 \in \mathbb{Z}^4$ whose affine span is primitive and such that $\gamma$ spans their affine relations, that is, $\sum_{i=1}^{6} \gamma_i m_i = 0$. Let $u_1, \ldots, u_4$ be coordinates on the torus $T = \text{Hom}(\mathbb{Z}^4, \mathbb{C}^\times) \simeq (\mathbb{C}^\times)^4$. Pick integers $k_1, \ldots, k_6$ such that $\sum_{i=1}^{6} k_i \gamma_i = 1$, and consider the Laurent polynomial

$$
f_\alpha(u) = \sum_{i=1}^{6} (-\alpha)^{k_i} u^{m_i}
$$

where $u = (u_1, \ldots, u_4)$, $u^m := u_1^{m_1}u_2^{m_2}u_3^{m_3}u_4^{m_4}$ for $m = (m_1, \ldots, m_4) \in \mathbb{Z}^4$ and $\alpha$ is a parameter in $\mathbb{C}^\times$. Denote by $Z_\alpha$ the threefold $Z_\alpha = \{f_\alpha = 0\} \subset T$. It is easy to check that $Z_\alpha$ is singular if and only if $\alpha = \alpha_0$. The threefold $Z_\alpha$ is referred to as the toric model associated to $\gamma$ in [RRV22, Section 3].

Remark 6. The vectors $m_i$ are uniquely determined up to invertible affine linear transformations of $\mathbb{Z}^4$. Therefore, the associated hypersurface $Z_\alpha \subset T^4$ is uniquely determined up to isomorphism.

In the Appendix we prove that:

Proposition 7. The local system $\mathbb{H}$ is the variation $\text{gr}_W^3 H^3_\epsilon(Z_\alpha, \mathbb{Q})(1)$, $\alpha \in \mathbb{C}^\times \setminus \{\alpha_0\}$. 


2.2. A conic bundle structure. In the rest of the paper we will choose the vectors $m_i$ as the columns of the matrix:

$$
\begin{pmatrix}
1 & 0 & 0 & 3 & 0 & 1 \\
1 & 0 & 0 & 1 & 3 & 0 \\
1 & 0 & 0 & 0 & 0 & 2 \\
0 & 2 & 1 & 0 & 0 & 0
\end{pmatrix}
$$

and we will set $(k_1, \ldots, k_6) = (0, -1, 0, 0, 0, 0)$. Then the threefold $Z_\alpha \subset \mathbb{T}$ is defined by the equation:

$$
u_1 u_2 u_3 + u_4 + u_1^3 u_2 + u_2^3 + u_1 u_3^2 - \frac{1}{\alpha} u_4^2 = 0
$$

It is clear that $Z_\alpha$ is the intersection of the threefold $X_\alpha \subset (\mathbb{C}^*)^2 \times \mathbb{P}^2$ defined by the equation:

$$
(u_1^3 u_2 + u_2^3) x_0^2 + u_1 u_2 x_0 x_1 + u_1 x_1^2 + x_1 x_2 - \frac{1}{\alpha} x_2^2 = 0
$$

with the torus $(\mathbb{C}^*)^4 \subset (\mathbb{C}^*)^2 \times \mathbb{P}^2$, where $u_1, u_2$ are coordinates on $(\mathbb{C}^*)^2$, $x_0, x_1, x_2$ are homogeneous coordinates on $\mathbb{P}^2$, and, for $x_0 \neq 0$, $u_3 = x_1/x_0$, $u_4 = x_2/x_0$. The projection $\pi: X_\alpha \rightarrow (\mathbb{C}^*)^2$ to the first factor is a conic bundle over $(\mathbb{C}^*)^2$.

Remark 8. Write $\{e_i\}$, $i = 1, 2, 3, 4$, for the standard basis of $\mathbb{Z}^4$. With the choice $m_1 = 0$, $m_2 = (2, 3, 5, 9)$, $(m_3, m_4, m_5, m_6) = (e_1, e_2, e_3, e_4)$, and $(k_1, \ldots, k_6) = (0, 1, 0, 0, 0, 0)$, one obtains the equation:

$$
1 + u_1 + u_2 + u_3 + u_4 - \frac{1}{\alpha} u_1^2 u_2^3 u_3^5 u_4^9 = 0
$$

This is the same equation associated to the list $\gamma$ in [BCM15, Equation (1.1)]. After the change of coordinates

$$
\begin{align*}
u_1 & \mapsto \frac{u_4}{u_1 u_2 u_3} & u_2 & \mapsto \frac{u_1^7}{u_3} & u_3 & \mapsto \frac{u_2}{u_1 u_3} & u_4 & \mapsto \frac{u_3}{u_2}
\end{align*}
$$

Equation (10), multiplied by $u_1 u_2 u_3$, becomes Equation (8). The point of our choice (7) is to make the conic bundle structure (9) manifest.

3. Conic bundles over surfaces

3.1. Basic notions. We recall some basic notions on conic bundles, see [Pro18] for an extensive survey.

Definition 9. A conic bundle is a projective morphism $\pi: X \rightarrow S$ where $X$ is a nonsingular 3-fold, $S$ is a surface, $\pi_* \mathcal{O}_X = \mathcal{O}_S$ and $-K_X$ is $\pi$-ample.

It follows from this that $\pi$ is flat, $S$ is nonsingular, $E = \pi_* \mathcal{O}(-K_X)$ is a rank 3 vector bundle on $S$, and every fibre is a conic.

Definition 10. The discriminant of the conic bundle is the curve

$$
\Delta = \{ s \in S \mid \pi^{-1}(s) \text{ is a singular conic} \} \subset S
$$

It is well-known that $\Delta \subset S$ is a nodal curve, and the set

$$
\Delta_s = \{ s \in S \mid \pi^{-1}(s) \text{ is a a double line} \} \subset \Delta
$$

is the singular locus of $\Delta$, see Equation (13). Below we write

$$
\Delta_0 := \Delta \setminus \Delta_s
$$

Definition 11. The ramification data of the conic bundle is the double cover $p: \tilde{\Delta} \rightarrow \Delta$, where $\tilde{\Delta}$ is the curve parametrizing irreducible components of the singular conics over $\Delta$. We denote by $\tau: \tilde{\Delta} \rightarrow \tilde{\Delta}$ the involution of the cover.
It is well-known [Bea77, Bel85, Bea89] that if \( \pi : X \to S \) is a conic bundle and \( X \) is proper, then \( H^3(X, \mathbb{Z}) \) is the \( \tau \)-noninvariant part of \( H^1(\Delta, \mathbb{Z}) \).

If \( X \) is not proper, then \( H^3(X, \mathbb{Z}) \) — and \( H^3_c(X, \mathbb{Z}) \) — carries a mixed Hodge structure, and our Theorem 3 states that there is a surjection \( R^2\pi_*\mathbb{Z}_X \to \mathbb{Z}_S \) with kernel the subsheaf of \( p_*\mathbb{Z}_\Delta \) of anti-invariants under the involution. This description of \( R^2\pi_*\mathbb{Z}_X \) provides a tool to study \( H^3(X, \mathbb{Z}) \) — and \( H^3_c(X, \mathbb{Z}) \) — when \( X \) is not proper.

In Section 4 we will use Theorem 3 to study \( H^3_c(X_\alpha, \mathbb{Z}) \), where \( X_\alpha \) is the threefold of Equation 9.

Let \( s \in S \). In a small affine neighbourhood \( s \in U \), one can define \( X_U := \pi^{-1}(U) \subset U \times \mathbb{P}^2 \) by an equation of the form:

\[
\sum_{0 \leq i,j \leq 2} a_{ij}x_ix_j
\]

where \( a_{ij} \in \mathbb{C}[U] \). Then \( \Delta \cap U = (\det(a_{ij}) = 0) \). Moreover, by a change of coordinates, one can rewrite (12) as:

\[
\begin{align*}
    b_0x_0^2 + b_1x_1^2 + b_2x_2^2 &= 0 \iff \text{rank}(a_{ij}) = 3 \iff s \notin \Delta \\
    b_0x_0^2 + b_1x_1^2 + c_2x_2^2 &= 0 \iff \text{rank}(a_{ij}) = 2 \iff s \in \Delta_0 \\
    b_0x_0^2 + c_1x_1^2 + c_2x_2^2 + c_3x_1x_2 &= 0 \iff \text{rank}(a_{ij}) = 1 \iff s \in \Delta_s
\end{align*}
\]

where \( b_i \neq 0, c_i = 0, \text{mult}_s(c_1) = \text{mult}_s(c_2) = 1 \).

Let \( \delta_k, k = 1, 2, 3 \), be the \( 2 \times 2 \)-minors of the matrix \( (a_{ij}) \) obtained by deleting the \( k \)th row and column.

By (13), on each irreducible component \( \Delta_i \) of \( \Delta \), the double cover \( p : \tilde{\Delta} \to \Delta \) is specified by a minor \( \delta = \delta_k \) which does not vanish identically along \( \Delta_i \).

3.2. Proof of Theorem 3. We prove Theorem 3, following the setup and notation of the previous section.

**Lemma 12.** Let \( \pi : X \to S \) be a conic bundle with ramification data \( p : \tilde{\Delta} \to \Delta \). The involution \( \tau \) on \( \tilde{\Delta} \) induces an involution, which for simplicity we still denote by \( \tau \), on \( p_*\mathbb{Z}_{\tilde{\Delta}} \). Let \( \mathcal{W} \subset p_*\mathbb{Z}_{\tilde{\Delta}} \) be the subsheaf of anti-invariants with respect to \( \tau \), equivalently defined by the short exact sequence:

\[
0 \to \mathcal{W} \to p_*\mathbb{Z}_{\tilde{\Delta}} \to \mathbb{Z}_\Delta \to 0
\]

where \( p_* : \mathbb{Z}_{\tilde{\Delta}} \to \mathbb{Z}_\Delta \) is the trace map. Consider the diagram:

\[
\begin{array}{ccc}
X_U & \longrightarrow & X \\
\pi_U \downarrow & & \downarrow \pi \\
U & \underset{j}{\rightarrow} & S \leftarrow \pi \Delta
\end{array}
\]

where \( U = S \setminus \Delta, X_U = \pi^{-1}(U), X_\Delta = \pi^{-1}(\Delta) \) and \( \pi_U : X_U \to U, \pi_\Delta : X_\Delta \to \Delta \) are the restrictions.

We have:

(i) \( R^0\pi_*\mathbb{Z}_X = \mathbb{Z}_S \) and \( R^1\pi_*\mathbb{Z}_X = (0) \);

(ii) The sheaf \( R^2\pi_*\mathbb{Z}_X \) has stalks:

\[
(R^2\pi_*\mathbb{Z}_X)_s = \begin{cases} \mathbb{Z} & \text{if } s \in U \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } s \in \Delta_0 \\ \mathbb{Z} & \text{if } s \in \Delta_s \end{cases}
\]

(iii) the restriction \( j^*R^2\pi_*\mathbb{Z}_X = R^2\pi_U_*\mathbb{Z}_{X_U} \) is isomorphic to \( \mathbb{Z}_U(-1) \);

(iv) the restriction \( i^*R^2\pi_*\mathbb{Z}_X = R^2\pi_\Delta_*\mathbb{Z}_{X_\Delta} \) is isomorphic to \( p_*\mathbb{Z}_{\tilde{\Delta}}(-1) \).

**Proof.** To prove (i), note that for all \( s \in S \), \( (R^0\pi_*\mathbb{Z}_X)_s = H^0(X_s, \mathbb{Z}) = \mathbb{Z} \), and \( \Gamma_S(\pi_*\mathbb{Z}_X) = H^0(X, \mathbb{Z}) = \mathbb{Z} \). On the other hand, for all \( s \in S \), \( (R^1\pi_*\mathbb{Z}_X)_s = H^1(X_s, \mathbb{Z}) = (0) \).

The statement in (ii) is immediate.
To prove (iii), it is enough to notice that \( c_1 \mathcal{O}(1)|_{X_U} \in \text{Hom}(\mathcal{O}_U, R^2\pi_{U*}\mathcal{Z}_X) = H^0(U, R^2\pi_{U*}\mathcal{Z}_X) \) is a global section of \( R^2\pi_{U*}\mathcal{Z}_X \).

To prove (iv), consider the diagram:

\[
\begin{array}{ccc}
\tilde{\Delta}_0 \times_{\Delta_0} X_{\Delta_0} & \rightarrow & X_{\Delta_0} \\
\downarrow & & \downarrow \\
\tilde{\Delta}_0 & \rightarrow & X_{\Delta_0} \\
\downarrow p_0 & & \downarrow \pi_0 \\
\Delta_0 & \rightarrow & \Delta_0
\end{array}
\]

where \( \tilde{\Delta}_0 = p^{-1}(\Delta_0) \), \( X_{\Delta_0} = \pi^{-1}(\Delta_0) \) and \( p_0: \tilde{\Delta}_0 \rightarrow \Delta_0 \) and \( \pi_0: X_{\Delta_0} \rightarrow \Delta_0 \) are the restrictions. The fiber product \( \tilde{\Delta}_0 \times_{\Delta_0} X_{\Delta_0} = Z_1 + Z_2 \) is the sum of two irreducible components. Denoting by \( Z \) one of these components, \( Z \) induces an isomorphism \( z: p_0_*Z_{\tilde{\Delta}_0} \rightarrow R^2\pi_{0*}Z_{X_{\Delta_0}}(1) \). One concludes by pushing forward both sides of this isomorphism. \( \square \)

3.2.1. Proof of Theorem 3. Consider the relative Poincaré morphism:

\[
P: Z_X[2](1) \rightarrow D_{X/S}
\]

where \( D_{X/S} = \pi^!Z_S \) is the relative dualizing complex of \( X \) over \( S \). The functor \( R^0\pi_* \) applied to the morphism (16) induces the morphism:

\[
P_2: R^2\pi_*Z_X(1) \rightarrow R^0\pi_*\pi^!Z_S
\]

which, on each stalk, is the usual Poincaré map of \( X_s \):

(18) \( P_{2,s}: H^2(X_s, \mathbb{Z})(1) \rightarrow H_0(X_s, \mathbb{Z}) \)

Note that \( R^0\pi_*\pi^!Z_S \simeq Z_S \) by local Verdier duality

\[
R\text{Hom}(R\pi_!Z_X, Z_S) = R\pi_*R\text{Hom}(Z_X, \pi^!Z_S)
\]

and that \( P_2 \) is a surjective morphism. In fact, for \( s \notin \Delta_0 \), the map \( P_{2,s} \) is an isomorphism. For \( s \in \Delta_0 \), since \( X_s \) is the union of two distinct lines, \( H^2(X_s, \mathbb{Z})(1) \simeq \mathbb{Z}^2 \), and map \( P_{2,s} \) sends \( \mathbb{Z}^2 \ni (k_1, k_2) \rightarrow k_1 + k_2 \in \mathbb{Z} \)

see Figure 1, thus \( \ker(P_{2,s}) = \{(k, -k) \in \mathbb{Z}^2\} \). It follows that over \( \Delta \) the morphism \( P_2 \) is the same as the morphism \( p_*Z_{\Delta_0} \rightarrow Z_{\Delta} \), whose kernel is the sheaf \( \mathcal{W} \) of anti-invariants with respect to \( \tau \). This proves the statement of the theorem. \( \square \)

\[\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{image.png}
\caption{A section of \( R^2\pi_*Z \) about a degenerate fiber \( X_s = L_1 \cup L_2 \) looks like the union of two horizontal disks meeting \( X_s \) at \( k_1p_1 + k_2p_2 \), where \( p_i \in L_i \). The map \( P_{2,s} \) sends \( k_1p_1 + k_2p_2 \) to \( (k_2 + k_2)p \), where \( p \) is the class of the point.}
\end{figure}\]
4. Proof of Theorem 1

In this section we prove Theorem 1 of the Introduction. For convenience in what follows we fix $\alpha \neq \alpha_0$ and we omit all reference to $\alpha$.

The partial compactification $Z \subset X$. In Section 2.2 we built a partial compactification $X \subset (\mathbb{C}^\times)^2 \times \mathbb{P}^2$ of $Z$. Lemma (19) below states that $\text{gr}_3^W H^3_c(Z, \mathbb{Q}) = \text{gr}_3^W H^3_c(X, \mathbb{Q})$, thus one might replace $Z$ with $X$ in Proposition 7. The advantage of working with $X$ is that the first projection $\pi: X \to (\mathbb{C}^\times)^2$ is a conic bundle.

Lemma 13. There is an identity of pure Hodge structures:

$$\text{gr}_3^W H^3_c(Z, \mathbb{Z}) = \text{gr}_3^W H^3_c(X, \mathbb{Z})$$  \hspace{1cm} (19)

**Proof.** Consider the divisor $D = X \setminus Z$. There is a long exact sequence of mixed Hodge structures:

$$\cdots \to H^3_c(D, \mathbb{Z}) \to H^3_c(Z, \mathbb{Z}) \to H^3_c(X, \mathbb{Z}) \to H^3(D, \mathbb{Z}) \to \cdots$$  \hspace{1cm} (20)

To prove (19), we show that $\text{gr}_3^W H^2_c(D, \mathbb{Z}) = \text{gr}_3^W H^3_c(D, \mathbb{Z}) = (0)$. We have that $D = D_0 \cup D_1 \cup D_2$, where $D_i = X \cap \{ x_i = 0 \}$, $i = 0, 1, 2$, and

$$D_0 \cap D_1 = D_0 \cap D_2 = \varnothing \quad D_1 \cap D_2 = (u_1^2 + u_2^2 = 0) \subset (\mathbb{C}^\times)^2$$

This implies that, for all $i$, $H^i_c(D, \mathbb{Z}) = H^i_c(D_0, \mathbb{Z}) \oplus H^i_c(D_1 \cup D_2, \mathbb{Z})$.

The surface $D_0$ is given by

$$(u_1^2 x_1^2 - x_2^2 = 0) \subset (\mathbb{C}^\times)^2 \times \mathbb{P}^1$$

Hence $D_0 \simeq (\mathbb{C}^\times)^2$, thus $H^2_c(D_0, \mathbb{Z})$ is a pure Hodge structure of weight 0, and $H^3_c(D_0, \mathbb{Z})$ is a pure Hodge structure of weight 2. It follows that

$$\text{gr}_3^W H^i_c(D, \mathbb{Z}) = \text{gr}_3^W H^i_c(D_1 \cup D_2, \mathbb{Z}) \quad i = 2, 3$$  \hspace{1cm} (21)

We have a long exact sequence of mixed Hodge structures:

$$\cdots \to H^{i-1}_c(D_1, \mathbb{Z}) \to H^i_c(D_2 \setminus D_1, \mathbb{Z}) \to H^i_c(D_1 \cup D_2, \mathbb{Z}) \to H^i_c(D_1, \mathbb{Z}) \to \cdots$$  \hspace{1cm} (22)

The surface $D_1$ is the smooth surface given by:

$$\left( (u_1^3 u_2 + u_2^3)x_0^2 + x_0 x_2 - \frac{1}{\alpha} x_2^2 = 0 \right) \subset (\mathbb{C}^\times)^2 \times \mathbb{P}^1$$

Then a natural compactification of $D_1$ is the smooth degree two del Pezzo surface $\overline{D}_1$ given by

$$\left( (u_1^3 u_2 + u_0 u_2^3) + u_0^2 x_2 - \frac{1}{\alpha} x_2^2 = 0 \right) \subset \mathbb{P}(1, 1, 1, 2)$$

Hence $\text{gr}_3^W H^2_c(D_1, \mathbb{Z}) = \text{gr}_3^W H^3_c(D_1, \mathbb{Z}) = 0$. The surface $D_2$ is the smooth surface given by:

$$\left( (u_1^3 u_2 + u_2^3)x_0^2 + u_1 u_2 x_0 x_1 + u_1 x_2^2 = 0 \right) \subset (\mathbb{C}^\times)^2 \times \mathbb{P}^1$$

Since $D_2 \setminus D_1$ is nonsingular but noncompact, we have $\text{gr}_3^W H^2_c(D_1 \setminus D_2, \mathbb{Z}) = (0)$. Then it follows from (22) that $\text{gr}_3^W H^2_c(D_1 \cup D_2, \mathbb{Z}) = (0)$. On the other hand, since $D_1 \cap D_2$ is a nonsingular and noncompact curve, we have $\text{gr}_3^W H^3_c(D_2 \setminus D_1, \mathbb{Z}) = \text{gr}_3^W H^3_c(D_2, \mathbb{Z})$. Now, since the projection $\phi: D_2 \to (\mathbb{C}^\times)^2$ to the first factor is a double (branched) cover of $(\mathbb{C}^\times)^2$, we have that $\text{gr}_3^W H^3_c(D_2, \mathbb{Z}) = \text{gr}_3^W H^3_c((\mathbb{C}^\times)^2, \phi_* \mathbb{Z}) = (0)$. It follows from (22) that $\text{gr}_3^W H^3_c(D_1 \cup D_2, \mathbb{Z}) = 0$. This concludes the proof.  \hspace{1cm} $\Box$
Ramification data of $\pi: X \to (\mathbb{C}^\times)^2$. By means of the substitutions $x_2 \mapsto x_2 + \alpha x_0/2$, $x_1 \mapsto x_1 - u_2 x_0/2$, we rewrite Equation (9) as

$$
\left( u_1^3 u_2 + u_2^3 + \frac{\alpha}{4} - \frac{u_1 u_2^2}{4} \right) x_0^2 + u_1 x_1^2 - \frac{1}{\alpha} x_2^2 = 0
$$

(23)

In matrix notation, we represent (23) by the diagonal matrix:

$$
\begin{bmatrix}
  u_1^3 u_2 + u_2^3 + \frac{\alpha}{4} - \frac{u_1 u_2^2}{4} & 0 & 0 \\
  0 & u_1 & 0 \\
  0 & 0 & -\frac{1}{\alpha}
\end{bmatrix}
$$

(24)

Then the discriminant of $\pi: X \to (\mathbb{C}^\times)^2$ is the smooth curve $\Delta \subset (\mathbb{C}^\times)^2$ defined as:

$$
\Delta = \left( 4 u_1^3 u_2 + 4 u_2^3 + \alpha - u_1 u_2^2 = 0 \right) \subset (\mathbb{C}^\times)^2
$$

(25)

The $2 : 1$ étale cover $p: \widetilde{\Delta} \to \Delta$ associated to $\pi: X \to (\mathbb{C}^\times)^2$ is specified by the minor $\delta_1 = -\frac{1}{\alpha} u_1$ of (24).

Let $u_1, u_2, u_3$ be coordinates on $(\mathbb{C}^\times)^3$. Then:

$$
\widetilde{\Delta} = \left( \alpha u_3^2 - u_1 = 4 u_1^3 u_2 + 4 u_2^3 + \alpha - u_1 u_2^2 = 0 \right) \subset (\mathbb{C}^\times)^3
$$

(26)

with $p$ the projection $(u_1, u_2, u_3) \mapsto (u_1, u_2)$. Equivalently, $\widetilde{\Delta}$ is the curve:

$$
\widetilde{\Delta} = \left( 4\alpha^3 u_1^6 u_2 + 4 u_2^3 + \alpha - \alpha u_1^2 u_2^2 = 0 \right) \subset (\mathbb{C}^\times)^2
$$

(27)

and $p: \widetilde{\Delta} \to \Delta$ is the double cover defined by $u_1, u_2 \mapsto \alpha u_1^2, u_2$. Let $M = \mathbb{Z} e_1 + \mathbb{Z} e_2$ be the character lattice of the torus in (25) and let $\widetilde{M} = \mathbb{Z} \frac{e_1}{2} + \mathbb{Z} e_2$ be the character lattice of the torus in (27). In Figure 2 we draw the Newton polytopes $P, \tilde{P}$ of the polynomials defining $\Delta$ and $\widetilde{\Delta}$.

**Figure 2.** A picture of the Newton polytope $P$ of $\Delta$, as well as the Newton polytope $\tilde{P}$ of $\widetilde{\Delta}$. The two arrows in the picture represent the standard lattice basis vectors; the interior lattice points of $P$ are in red, and the remaining interior lattice points of $\tilde{P}$ in green.

Remark 14. The closure of $\Delta$ in $\mathbb{P}^2$ is the smooth quartic:

$$
N = \left( 4 u_1^3 u_2 + 4 u_0 u_2^3 + \alpha u_0^4 - u_0 u_1 u_2^2 = 0 \right) \subset \mathbb{P}^2
$$

where $u_0, u_1, u_2$ are now homogeneous coordinates on $\mathbb{P}^2$, and $\Delta = N \setminus \{ p_0, \ldots, p_4 \}$, with $p_0 = (0 : 1 : 0)$, $p_i = (u_1 = u_3^2 + \alpha u_0^3 = 0)$, $i = 1, 2, 3$, and $p_4 = (0 : 0 : 1)$, see Figure 3. Hence $\text{gr}_W^1 H_0^1(\Delta, \mathbb{Z}) = H^1(N, \mathbb{Z})$ has rank 6.

The closure of $\widetilde{\Delta}$ in $\mathbb{P}^2$ is singular at the point $(0 : 0 : 1)$. Its normalisation is manifestly the genus-7 curve $\tilde{N} \subset \mathbb{P}(1, 1, 3)$ given by Equation (2), hence $\text{gr}_W^1 H^1(\widetilde{\Delta}, \mathbb{Z}) = H^1(\tilde{N}, \mathbb{Z})$ has rank 14.

We have a commutative diagram:
Remark 15. The local system \( \text{gr}^W H^1_c(\Delta_\alpha, \mathbb{Q}) = H^1(N_\alpha, \mathbb{Q}) \) is the hypergeometric local system associated to \( \gamma = (-9, 1, 3, 5) \) — indeed, \( \gamma \) spans the affine relations of the monomials in Equations (25), (27).

Lemma 16. There is an identity of pure Hodge structures:

\[
\text{gr}^W H^3_c(X, \mathbb{Z}) = \text{gr}^W H^1_c((\mathbb{C}^\times)^2, R^2\pi_*\mathbb{Z})
\]

Proof. Consider the Leray spectral sequence of \( \pi \) with second page \( E^{p,q}_2 = H^p_c((\mathbb{C}^\times)^2, R^q\pi_*\mathbb{Z}) \Rightarrow H^{p+q}_c(X, \mathbb{Z}) \). Since the second page of the spectral sequence is:

\[
\begin{array}{cccc}
H^0_c((\mathbb{C}^\times)^2, R^2\pi_*\mathbb{Z}) & H^1_c((\mathbb{C}^\times)^2, R^2\pi_*\mathbb{Z}) & H^2_c((\mathbb{C}^\times)^2, R^2\pi_*\mathbb{Z}) & H^3_c((\mathbb{C}^\times)^2, R^2\pi_*\mathbb{Z}) \\
(0) & (0) & (0) & (0) \\
H^0_c((\mathbb{C}^\times)^2, \mathbb{Z}) & H^1_c((\mathbb{C}^\times)^2, \mathbb{Z}) & H^2_c((\mathbb{C}^\times)^2, \mathbb{Z}) & H^3_c((\mathbb{C}^\times)^2, \mathbb{Z})
\end{array}
\]

one has that \( d_2 : E^{p,q}_2 \rightarrow E^{p+2,q-1}_2 \) is zero, thus \( E^{p,q}_3 = E^{p,q}_2 \). For reasons of space, the only nonzero \( d_3 \) are:

\( d_3 : H^0_c((\mathbb{C}^\times)^2, R^2\pi_*\mathbb{Z}) \rightarrow H^3_c((\mathbb{C}^\times)^2, \mathbb{Z}) \) and \( d_3 : H^1_c((\mathbb{C}^\times)^2, R^2\pi_*\mathbb{Z}) \rightarrow H^4_c((\mathbb{C}^\times)^2, \mathbb{Z}) \)

and \( d_4 = 0 \) for all \( i \geq 4 \), thus \( E^{p,q}_\infty = E^{p,q}_2 \) where

\[
\begin{align*}
E^{0,3}_4 &= E^{2,1}_4 = 0 \\
E^{1,2}_4 &= \ker \left( d_3 : H^1_c((\mathbb{C}^\times)^2, R^2\pi_*\mathbb{Z}) \rightarrow H^4_c((\mathbb{C}^\times)^2, \mathbb{Z}) \right) \\
E^{3,0}_4 &= \coker \left( d_3 : H^0_c((\mathbb{C}^\times)^2, R^2\pi_*\mathbb{Z}) \rightarrow H^4_c((\mathbb{C}^\times)^2, \mathbb{Z}) \right)
\end{align*}
\]
and \( d_i = 0 \) for all \( i \geq 4 \), vanish, thus \( E^p,q_4 = E^p,q_3 \). Then there is a short exact sequence:

\[
0 \to E^{3,0}_4 \to H^3_c(X, \mathbb{Z}) \to E^{1,2}_4 \to 0
\]

Since \( H^3((\mathbb{C}^\times)^2, \mathbb{Z}) \) is a pure Hodge structure of weight 2 and \( H^1_c((\mathbb{C}^\times)^2, \mathbb{Z}) \) is a pure Hodge structure of weight 4, it follows that

\[
gr^W_3 H^3_c(X, \mathbb{Z}) = gr^W_3 E^{1,2}_4 = gr^W_3 H^1_c((\mathbb{C}^\times)^2, R^2 \pi_* \mathbb{Z})
\]

This proves the statement. \( \square \)

**Lemma 17.** As in Section 3.2, denote by \( \mathbb{W} \subset p_* \mathbb{Z}_\Delta \) the subsheaf of anti-invariants.

There is an identity of mixed Hodge structures:

\[
H^1_c(\mathbb{W}, \mathbb{Z}) = H^1_c((\mathbb{C}^\times)^2, R^2 \pi_* \mathbb{Z})(1)
\]

**Proof.** By Theorem 3, there is a short exact sequence of mixed sheaves:

\[
0 \to i_* \mathbb{W} \to R^2 \pi_* \mathbb{Z}_X(1) \to \mathbb{Z}_{(\mathbb{C}^\times)^2} \to 0
\]

Applying the functor \( R^* \Gamma_c(\mathbb{C}^\times)^2, - \) to this sequence, we obtain the long exact sequence of compactly supported cohomology groups:

\[
\cdots \to H^0((\mathbb{C}^\times)^2, \mathbb{Z}) \to H^1_c(\mathbb{W}, \mathbb{Z}) \to H^1_c((\mathbb{C}^\times)^2, R^2 \pi_* \mathbb{Z})(1) \to H^1_c((\mathbb{C}^\times)^2, \mathbb{Z}) \to \cdots
\]

The statement follows from the fact that \( H^0((\mathbb{C}^\times)^2, \mathbb{Z}) = H^1_c((\mathbb{C}^\times)^2, \mathbb{Z}) = 0 \). \( \square \)

**Lemma 18.** There is an an exact sequence:

\[
0 \to H^1_c(\Delta, \mathbb{W}) \to H^1_c(\Delta, \mathbb{Z}) \overset{p_*}{\to} H^1_c(\Delta, \mathbb{Z})
\]

where \( p_* \) is the Gysin morphism.

**Proof.** Apply the functor \( R^* \Gamma_c(\Delta, -) \) to the the short exact sequence (14) defining \( \mathbb{W} \) and observe that \( H^0_c(\Delta, \mathbb{Z}) \simeq H_2(\Delta, \mathbb{Z}) = 0 \). \( \square \)

**Remark 19.** Note that the Gysin morphism \( p_* \) in (33) is not surjective but has cokernel \( \mathbb{Z}/2\mathbb{Z} \). Over \( \mathbb{Q} \) one has the short exact sequence:

\[
0 \to H^1_c(\Delta, \mathbb{W}_\mathbb{Q}) \to H^1_c(\widetilde{\Delta}, \mathbb{Q}) \overset{p_*}{\to} H^1_c(\Delta, \mathbb{Q}) \to 0
\]

where \( \mathbb{W}_\mathbb{Q} = \mathbb{W} \otimes \mathbb{Z} \mathbb{Q} \).

**Proof of Theorem 1.** Applying in sequence Lemma 13, Lemma 16, Lemma 17, one sees that:

\[
gr^W_3 H^3_c(Z, \mathbb{Z}) = gr^W_3 H^3_c(X, \mathbb{Z}) = gr^W_3 H^1_c((\mathbb{C}^\times)^2, R^2 \pi_* \mathbb{Z}) = (gr^W_1 H^1_c(\Delta, \mathbb{W}) )(-1)
\]

Then the statement follows from Proposition 7 together with Lemma 18, Remark 14, and Remark 19. \( \square \)

**Appendix A. Proof of Proposition 7.**

The proof of Proposition 7 uses the theory of GKZ systems: we summarise what we need in the next section.

**A.1. GKZ systems.** Let \( N \) and \( n \) be two positive integers. Let \( A = \{a_1, \ldots, a_N \} \) be a set of \( N \) vectors in \( \mathbb{Z}^n \), and let \( \overline{A} = \{\overline{a}_1, \ldots, \overline{a}_N \} \subset \mathbb{Z}^{n+1} \) where \( \overline{a}_i = (1, a_i) \). Let \( \beta \) be a vector in \( \mathbb{C}^{n+1} \). Assume that the vectors \( \overline{a}_i \) generate a rank-(\( n+1 \)) sublattice of \( \mathbb{Z}^{n+1} \).

Let \( \mathbb{C}^N \) be the affine space with coordinates \( v_1, \ldots, v_N \). Let \( D_\mathbb{C}^N \) be the sheaf of polynomial linear partial differential operators on \( \mathbb{C}^N \). We simply write \( D \) whenever the context allows it. For all \( i \in \{1, \ldots, N \} \) we write \( \delta_i = \delta_{v_i}, D_i = v_i \delta_i \).
Definition 20. The GKZ system with parameters \((A, \beta)\) is the differential system:

\[
M(A, \beta) = \left\{ \begin{array}{l}
\sum_{j=1}^{N} a_j D_i - \beta \\
\prod_{l_i>0}(\delta_i)^{l_i} - \prod_{l_i<0}(\delta_i)^{-l_i} \quad \text{for} \quad l \in \mathbb{L}
\end{array} \right. \]

where \(\mathbb{L}\) is the lattice of integral relations among the \(a_i\):

\[
\mathbb{L} = \{ l = (l_1, \ldots, l_n) \in \mathbb{Z}^N : \sum_{i=1}^{N} l_i a_i = 0 \}
\]

Note that \(\mathbb{L}\) is a lattice of rank \(N - n - 1\).

Definition 21. The GKZ \(\mathcal{D}\)-module with parameters \((A, \beta)\) is the quotient:

\[
\mathcal{M}(A, \beta) = \mathcal{D}/(M(A, \beta))
\]

where \((M(A, \beta))\) denotes the left ideal generated by the partial differential operators of \(M(A, \beta)\).

Let \(T = \text{Spec} \mathbb{C}[Z^n] \simeq (\mathbb{C}^\times)^n\) and let \(u_1, \ldots, u_n\) be coordinates on \(T\). Write \(u = (u_1, \ldots, u_n)\). Every Laurent polynomial \(f \in \mathbb{C}[[u_1, \ldots, u_n]]\) defines an affine hypersurface

\[
Z_f = (f = 0) \subset T
\]

Observe that the complement \(T \setminus Z_f\) is isomorphic to the affine hypersurface \(Z_F = (F = 0) \subset T\) where \(T = \text{Spec} \mathbb{C}[Z^{n+1}]\) is the \((n + 1)\)-dimensional torus with coordinates \(u_0, u_1, \ldots, u_n\), and \(F(u_0, u) = x_0 f(u) - 1\).

Now let \(A = \{a_1, \ldots, a_N\}\) as above. To every point \(v = (v_1, \ldots, v_N) \in \mathbb{C}^N\) one can associate the Laurent polynomial:

\[
f_v(u) = \sum_{j=1}^{N} v_j u_j^{a_j} \in \mathbb{C}[[u_1, \ldots, u_n]]
\]

where \(u^m = u_1^{m_1} u_2^{m_2} \cdots u_n^{m_n}\) for \(m = (m_1, \ldots, m_n) \in \mathbb{Z}^n\), the affine hypersurface \(Z_{f_v} \subset T\), and the affine hypersurface \(Z_{F_v} \subset T\). We simply write \(f, Z_f, F, Z_F\) whenever the the subscript is clear from the context.

The following result relates the \(\mathcal{D}\)-module \(\mathcal{M}(A, 0)\) and the Laurent polynomials \(f_v\).

Theorem 22. [Sti98, Theorem 8] Let \(A\) be as above. Assume that the following two conditions hold:
1. the vectors \(a_i = (1, a_i)\) generate \(\mathbb{Z}^{n+1}\);
2. \(A = \Delta \cap \mathbb{Z}^n\) for some \(n\)-dimensional lattice polyhedron \(\Delta\).

Let \(E_A = E_A(v_1, \ldots, v_N)\) be the principal \(A\)-determinant and let \(U = \mathbb{C}^N \setminus \{E_A = 0\}\). Then the local system of flat sections of \(\mathcal{M}(A, 0)|_U\) is the variation of relative cohomology \(H^{n+1}(\overline{T}, Z_{F_v}, \mathbb{Q})\) on \(U\).

Remark 23. Conditions 1 and 2 are necessary for Theorem 22 to hold, see [Sti98, Part IIIB]. To prove Proposition 7, we will need the conclusion of Theorem 22 to hold for a GKZ system that does not satisfy Condition 2.

Remark 24. If the Laurent polynomial \(f\) is \(\Delta\)-regular [Bat93, Definition 3.3] there is an exact sequence of mixed Hodge structures:

\[
0 \rightarrow H^n(\overline{T}, \mathbb{Q}) \rightarrow H^n(Z_F, \mathbb{Q}) \rightarrow H^{n+1}(\overline{T}, Z_F, \mathbb{Q}) \rightarrow H^{n+1}(\overline{T}, \mathbb{Q}) \rightarrow 0
\]

see [Sti98, Equation (55)]. This sequence splits into the two short exact sequences:

\[
0 \rightarrow H^n(\overline{T}, \mathbb{Q}) \rightarrow H^n(Z_F, \mathbb{Q}) \rightarrow PH^n(Z_F, \mathbb{Q}) \rightarrow 0
\]

\[
0 \rightarrow PH^n(Z_F, \mathbb{Q}) \rightarrow H^{n+1}(\overline{T}, Z_F, \mathbb{Q}) \rightarrow H^{n+1}(\overline{T}, \mathbb{Q}) \rightarrow 0
\]
where the symbol $PH^\bullet$ denotes primitive cohomology.

One has that $H^n(T, \mathbb{Q}) = \mathbb{Q}^{n+1}(-(n))$ and $H^{n+1}(T, \mathbb{Q}) \simeq \mathbb{Q}(-(n+1))$. By \cite[Proposition 5.2]{Bat93},
\[
\dim H^n(Z_F, \mathbb{Q}) = \text{vol}(\Delta) + n, \quad \text{thus } \dim PH^a(Z_F, \mathbb{Q}) = \text{vol}(\Delta) - 1 \quad \text{and } \dim H^{n+1}(T, Z_F, \mathbb{Q}) = \text{vol}(\Delta).
\]

Note that, by (41),
\[
W_i PH^a(Z_F, \mathbb{Q}) = W_i H^{n+1}(T, Z_F, \mathbb{Q}) \quad \text{for all } i \leq 2n + 1.
\]
Moreover, the residue map $\text{Res} : H^n(Z_F, \mathbb{Q}) \to H^{n-1}(Z_F, \mathbb{Q})(-1)$ induces an isomorphism $PH^a(Z_F, \mathbb{Q}) \simeq PH^{n-1}(Z_F, \mathbb{Q})(-1)$, see \cite[Section 5]{Bat93}.

A.2. Proof of Proposition 7. We follow the same setup and notation of Section A.1.

Preliminary lemmas. Consider the order-19 reducible hypergeometric operator $\tilde{H}$ on $\mathbb{C}^\times$:
\[
\tilde{H} = \alpha_0 D \left( D - \frac{1}{2} \right) \prod_{i=0}^{2} \left( D - \frac{i}{3} \right) \prod_{i=0}^{4} \left( D - \frac{i}{5} \right) \prod_{i=0}^{8} \left( D - \frac{i}{9} \right) - \alpha (D + 1) \prod_{i=1}^{18} \left( D + \frac{i}{18} \right)
\]
where $D = ad/d\alpha$ as in Section 2. Since $\alpha(D + 1 + \delta) = (D + \delta)\alpha$ for all $\delta \in \mathbb{C}$, we have $\tilde{H} = G \cdot H$, where $H$ is the operator in Equation (3) and
\[
G = D \left( D - \frac{1}{2} \right) \prod_{i=0}^{8} \left( D - \frac{i}{9} \right)
\]

Let $\Delta$ be the convex hull of the vectors $m_i, i = 1, \ldots, 6$ in Equation (8). The lattice points of $\Delta$ are given by the $m_i$ and the vector $(0, 1, 0, 1)$. Let $a_i = m_i, i = 1, \ldots, 6$, and write $A' = \{a_1, \ldots, a_6\}$.

Lemma 25. Let $k_i$ be integers such that $\sum_{i=1}^{6} k_i \gamma_i = 1$ and consider the morphism
\[
\mathbb{C}^* \overset{j}{\to} \mathbb{C}^6 \quad \text{given by } v_i \mapsto (-\alpha)^{k_i}
\]

Then there is an isomorphism of $D$-modules:
\[
D/D\tilde{H} = j^* \mathcal{M}(A', \mathbb{Q})
\]

Sketch of Proof. The lattice $\mathbb{L}$ of integral relations among $\bar{a}_1, \ldots, \bar{a}_6$ is generated by $\gamma = (-18, -1, 2, 3, 5, 9)$. Thus the GKZ system $M(A', \mathbb{Q})$ is the system of partial differential equations:
\[
M(A', \mathbb{Q}) = \left\{ \begin{array}{l}
\sum_{i=1}^{6} D_i = 0 \\
D_1 + 3D_4 + D_6 = 0 \\
D_1 + D_4 + 3D_5 = 0 \\
D_1 + 2D_6 = 0 \\
2D_2 + D_3 = 0 \\
\delta_3^2 \delta_5 \delta_6^2 \delta_9 - \delta_1^9 \delta_2 = 0
\end{array} \right.
\]

The key point is the following. A function $\Phi(v_1, \ldots, v_6)$ satisfies the first five equations of the system if and only if $\Phi(v_1, \ldots, v_6) = G(\prod_{i=1}^{6} v_i^{\gamma_i})$ for some function of one variable $G$; then the last equation of the system is satisfied if and only if $G$ satisfies the hypergeometric ODE:
\[
2D(2D - 1)3D(3D - 1)(3D - 2) 5D \cdots (5D - 4) 9D \cdots (9D - 8) - (-\alpha) \cdot 18D \cdots (18D + 17) = 0
\]
It follows that $\Phi(j(\alpha)) = G(-\alpha)$ is solution to the hypergeometric operator:
\[
L = 2D(2D - 1)3D(3D - 1)(3D - 2) 5D \cdots (5D - 4) 9D \cdots (9D - 8) - \alpha \cdot 18D \cdots (18D + 17)
\]
and by \cite[Lemma 3.3]{Kat90} the $D$-module $D/DL$ is isomorphic to the $D$-module $D/D\tilde{H}$.

Remark 26. This is why GKZ $D$-modules are called ‘hypergeometric’. 

\footnote{The hypergeometric operator $\tilde{H}$ can be computed from the coefficients of the function $F$ via the two-term recursion relation, see \cite[Section 5.3]{Kat90}.}
To \((v_1, \ldots, v_6) \in \mathbb{C}^6\) associate the Laurent polynomial \(f_{(v_1, \ldots, v_6)} = \sum_{j=1}^{6} v_j u^{a_j}\) as in (38), the hypersurface \(Z_{f_{(v_1, \ldots, v_6)}} \subset \mathbb{T}\), and the hypersurface \(Z_{F_{(v_1, \ldots, v_6)}} \subset \mathbb{T}\).

**Lemma 27.** Let \(U' = \mathbb{C}^6 \setminus \{E_A = 0\}\). The local system of flat sections of \(\mathcal{M}(A', \mathbb{Q})|_{U'}\) is isomorphic to \(H^5(\mathbb{T}, Z_{F_{(v_1, \ldots, v_6)}}, \mathbb{Q})\) on \(U'\).

**Proof.** Let \(\alpha_7 = (0, 1, 0, 1)\) and write \(A = \{\alpha_1, \ldots, \alpha_7\}\). Since the GKZ system \(M(A, 0)\) satisfies the two conditions of Theorem 22, letting \(U = \mathbb{C}^7 \setminus \{E_A = 0\}\), the local system of flat sections of \(\mathcal{M}(A, \mathbb{Q})|_U\) is isomorphic to the variation \(H^5(\mathbb{T}, Z_{F_{(v_1, \ldots, v_7)}}, \mathbb{Q})\) on \(U\). Let

\[\iota: \mathbb{C}^6 = (v_7 = 0) \hookrightarrow \mathbb{C}^7\]

be the inclusion. It is clear that the restriction of the variation \(H^5(\mathbb{T}, Z_{F_{(v_1, \ldots, v_7)}}, \mathbb{Q})\) to the hyperplane \((v_7 = 0)\) is the variation \(H^5(\mathbb{T}, Z_{F_{(v_1, \ldots, v_6)}}, \mathbb{Q})\). Then it is enough to show that the restriction \(\iota^*(\mathcal{M}(A, 0)) = \mathcal{M}(A', 0)\). This follows from the proof of the main result of [FFW11, Theorem 2.2] and it can be shown computationally with the Dmodules .m2 package [LT] of Macaulay2 [GS].

**Proof of Proposition 7.** Lemma 25 and Lemma 27 imply that the local system of flat sections of \(D/D\tilde{H}\) is the variation \(H^5(\mathbb{T}, Z_{F_{(v_1, \ldots, v_6)}}, \mathbb{Q})\), where

\[(45) \quad v_i = (-\alpha)^{\gamma_i} i = 1, \ldots, 6 \quad \text{and} \quad \sum_{i=1}^{6} k_i \gamma_i = 1\]

Note that the hypersurface \(Z_{f_{(v_1, \ldots, v_6)}} \subset \mathbb{T}\), with the \(v_i\) as in (45) is (up to isomorphism) the hypersurface \(Z_{\alpha} \subset \mathbb{T}\) of Section 2. In what follows we omit the subscript \(-_{(v_1, \ldots, v_6)}\) from our notation.

Denote by \(\tilde{\mathbf{H}}\) the local system of solutions to \(\tilde{H}\). Since \(\tilde{H} \in D\tilde{H}\), there is an inclusion of local systems \(\mathbf{H} \hookrightarrow \tilde{\mathbf{H}}\). For all \(\alpha \neq \alpha_0 \in \mathbb{C}^\times\) we have a short exact sequence:

\[0 \to PH^1(Z_{F}, \mathbb{Q}) \to H^5(\mathbb{T}, Z_{F}, \mathbb{Q}) \to H^5(\mathbb{T}, \mathbb{Q}) \to 0\]

see Remark 24. Its dual is:

\[0 \to H^5_c(\mathbb{T}, \mathbb{Q}) \to \text{Hom}(H^5(\mathbb{T}, Z_{F}), \mathbb{Q}) \to PH^1_c(Z_{F}, \mathbb{Q})(-1) \to 0\]

where \(PH^d_c\) denotes primitive cohomology with compact support.

One has \(H^5_c(\mathbb{T}) \simeq \mathbb{Q}\). Moreover, since \(PH^1(Z_{F}, \mathbb{Q}) \simeq PH^3(Z_{f}, \mathbb{Q})(-1)\), we have \(PH^1_c(Z_{F}, \mathbb{Q})(-1) \simeq PH^3_c(Z_{f}, \mathbb{Q})\). It follows that, for all \(m \geq 1\), one has

\[W_m \text{Hom}(H^5(\mathbb{T}, Z_{F}), \mathbb{Q}) \simeq W_m PH^3_c(Z_{f}, \mathbb{Q})\]

By Batyrev–Borisov’s formula [BB96], the Hodge–Deligne numbers \(h^{p,q}\) of \(PH^3_c(Z_{f}, \mathbb{Q})\) are given by:

\[
\begin{array}{cccc}
0 & 0 & 4 \\
1 & 7 & 4 \\
1 & 1 & 0 & 0
\end{array}
\]

where we put \(h^{p,q}\) in position \((p, q) \in \mathbb{Z}^2\), with \((0, 0)\) lying in the bottom-left corner. Then, since \(\mathbf{H}\) is irreducible, the weight-one variation it supports must be the variation \(\text{gr}_3^W PH^3_c(Z_{f}, \mathbb{Q})(1)\) over \(\mathbb{C}^\times \setminus \{\alpha_0\}\).

Note that \(\text{gr}_3^W PH^3_c(Z_{f}, \mathbb{Q}) = \text{gr}_3^W H^3_c(Z_{f}, \mathbb{Q})\). This concludes the proof.
References

[Bat93] Victor V. Batyrev, Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori, Duke Math. J. 69 (1993), no. 2, 349–409. MR 1203231

[BB96] Victor V. Batyrev and Lev A. Borisov, Mirror duality and string-theoretic Hodge numbers, Invent. Math. 126 (1996), no. 1, 183–203. MR 1408560

[BCM15] Frits Beukers, Henri Cohen, and Anton Mellit, Finite hypergeometric functions, Pure Appl. Math. Q. 11 (2015), no. 4, 559–589. MR 3613122

[Bea77] Arnaud Beauville, Variétés de Prym et jacobienes intermédiaires, Scientific annals of the École Normale Supérieure (1977), no. 3, 309–391 (fr).

[Bea89], Prym varieties: a survey, Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987), Proc. Sympos. Pure Math., vol. 49, Amer. Math. Soc., Providence, RI, 1989, pp. 607–620. MR 1013156

[Bel85] Mauro C. Beltrame, On the Chow group and the intermediate Jacobian of a conic bundle, Annali di Matematica Pura ed Applicata 141 (1985), 331–351.

[BH89] Frits Beukers and Gerrit J. Heckman, Monodromy for the hypergeometric function \( _nF_{n-1} \), Invent. Math. 95 (1989), no. 2, 325–354. MR 974906

[CG11] Alessio Corti and Vasily V. Golyshev, Hypergeometric equations and weighted projective spaces, Sci. China Math. 54 (2011), no. 8, 1577–1590. MR 2824960

[CG21] Alessio Corti and Giulia Gugliatt, Hyperelliptic integrals and mirrors of the Johnson-Kollár del Pezzo surfaces, Trans. Amer. Math. Soc. 374 (2021), no. 12, 8603–8637. MR 4337923

[FFW11] Maria-Cruz Fernández-Fernández and Uli Walther, Restriction of hypergeometric \( D \)-modules with respect to coordinate subspaces, Proc. Amer. Math. Soc. 139 (2011), no. 9, 3175–3180. MR 2811272

[GS] Daniel R. Grayson and Michael E. Stillman, Macaulay2, a software system for research in algebraic geometry, Available at http://www.math.uiuc.edu/Macaulay2/.

[JK01] Jennifer M. Johnson and János Kollár, Kähler–Einstein metrics on log del Pezzo surfaces in weighted projective 3-spaces, Ann. Inst. Fourier (Grenoble) 51 (2001), no. 1, 69–79. MR 1821068

[Kat90] Nicholas M. Katz, Exponential sums and differential equations, Annals of Mathematics Studies, vol. 124, Princeton University Press, Princeton, NJ, 1990. MR 1081536

[LT] Anton Leykin and Harrison Tsai, Dmodules: A Macaulay2 package. Version 1.4.0.1, A Macaulay2 package available at https://github.com/Macaulay2/M2/tree/master/M2/Macaulay2/packages.

[NR95] Donihakkalu Shankar Nagaraj and Sundararaman Ramanan, Polarisations of type \((1, 2, \ldots, 2)\) on abelian varieties, Duke Math. J. 80 (1995), no. 1, 157–194. MR 1360615

[Pro18] Yuri G. Prokhorov and Sundararaman Ramanan, Pairs in \( (\Delta^n, m) \) and \((\Delta^n, \Delta_m)\): On the rationality problem for conic bundles, Russian Mathematical Surveys 73 (2018), no. 3, 375.

[Rod19] Fernando Rodriguez Villegas, Mixed Hodge numbers and factorial ratios, arXiv e-prints (2019).

[RRV22] David P. Roberts and Fernando Rodriguez Villegas, Hypergeometric motives, Notices Amer. Math. Soc. 69 (2022), no. 6, 914–929.

[Sti98] Jan Stienstra, Resonant hypergeometric systems and mirror symmetry, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), World Sci. Publ., River Edge, NJ, 1998, pp. 412–452. MR 1672077

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