Regularized ERM on random subspaces

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Abstract

We study a natural extension of classical empirical risk minimization, where the hypothesis space is a random subspace of a given space. In particular, we consider possibly data dependent subspaces spanned by a random subset of the data. This approach naturally leads to computational savings, but the question is whether the corresponding learning accuracy is degraded. These statistical-computational tradeoffs have been recently explored for the least squares loss and self-concordant loss functions, such as the logistic loss. Here, we work to extend these results to convex Lipschitz loss functions, that might not be smooth, such as the hinge loss used in support vector machines. Our main results show the existence of different regimes, depending on how hard the learning problem is, for which computational efficiency can be improved with no loss in performance. Theoretical results are complemented with numerical experiments on large scale benchmark data sets.

1 Introduction

Despite excellent practical performances, state of the art machine learning (ML) methods often require huge computational resources, motivating the search for more efficient solutions. This has led to a number of new results in optimization [22, 41], as well as the development of approaches mixing linear algebra and randomized algorithms [30, 18, 56, 11]. While these techniques are applied to empirical objectives, in the context of learning it is natural to study how different numerical solutions affect statistical accuracy. Interestingly, it is now clear that there is a whole set of problems and approaches where computational savings do not lead to any degradation in terms of learning performance [39, 3, 31].

Here, we follow this line of research and study an instance of regularized empirical risk minimization where, given a fixed high– possibly infinite– dimensional hypothesis space, the search for a solution is restricted to a smaller– possibly random– subspace. This is equivalent to considering sketching operators [27], or equivalently regularization with random projections [56]. For infinite dimensional hypothesis spaces, it includes Nyström methods used for kernel methods [46] and Gaussian processes [55]. Recent works in supervised statistical learning has focused on smooth loss functions [39, 3, 31], whereas here we consider convex, Lipschitz but possibly non smooth losses. The lack of smoothness requires using different technical tools for the analysis. While for smooth losses linear algebra and matrix concentration can be used [51], here we need machinery from empirical processes [8]. In particular, fast rates require considering localized complexity measures [48, 68, 20].

Our main interest is characterizing the relation between computational efficiency and statistical accuracy. We do so studying the interplay between regularization, subspace size and different parameters describing how hard or easy is the considered problem. Indeed, our analysis starts from basic assumption, that eventually we first strengthen to get faster rates, and then weaken to consider more general scenarios. Our results show...
that also for convex, Lipschitz losses there are settings in which the best known statistical bounds can be obtained while substantially reducing computational requirements. Interestingly, these effects are relevant but also less marked than for smooth losses. In particular, some form of adaptive sampling seems needed to ensure no loss of accuracy and achieve sharp learning bounds. In contrast, uniform sampling suffices to achieve similar results for smooth loss functions. It is an open question whether this is a byproduct of our analysis, or a fundamental limitation. Some preliminary numerical results complemented with numerical experiments are given considering benchmark datasets.

The rest of the paper is organized as follow. In Section 2 we introduce the setting. In Section 3 we introduce the ERM approach we consider. In Section 4 we present and discuss the main results and defer the proofs to the appendix. In Section 5 we collect some numerical results.

2 Statistical learning with ERM

Let \((X, Y)\) be random variables in \(\mathcal{H} \times \mathcal{Y}\), with distribution \(P\) satisfying the following conditions.

**Assumption 1.** The space \(\mathcal{H}\) is a real separable Hilbert space with scalar product \(\langle \cdot, \cdot \rangle\), \(\mathcal{Y}\) is a Polish space, and there exists \(\kappa > 0\) such that \(\|X\| \leq \kappa\) almost surely.

Since \(X\) is bounded, the covariance operator \(\Sigma : \mathcal{H} \rightarrow \mathcal{H}\) given by \(\Sigma = \mathbb{E}[X \otimes X]\) can be shown to be self-adjoint, positive and trace class with \(\text{Tr}(\Sigma) \leq \kappa\). We can think of \(\mathcal{H}\) and \(\mathcal{Y}\) as input and output spaces, respectively, and some examples are relevant.

**Example 1.** An example is linear estimation, where \(\mathcal{H}\) is \(\mathbb{R}^d\) and \(\mathcal{Y} \subset \mathbb{R}\). Another example is kernel methods, where \(\mathcal{H}\) is a separable reproducing kernel Hilbert space on a measurable space \(\mathcal{X}\). The data are then mapped from \(\mathcal{X}\) to \(\mathcal{H}\) through the feature map \(x \mapsto K(\cdot, x)\) where \(K : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}\) is the (measurable) reproducing kernel of \(\mathcal{H}\) [48].

We denote by \(\ell : \mathcal{Y} \times \mathbb{R} \rightarrow [0, \infty)\) the loss function. Given a function \(f\) on \(\mathcal{H}\) with values in \(\mathbb{R}\), we view \(\ell(y, f(x))\) as the error made predicting \(y\) by \(f(x)\). We make the following assumption.

**Assumption 2 (Lipschitz loss).** The loss function \(\ell : \mathcal{Y} \times \mathbb{R} \rightarrow [0, \infty)\) is convex and Lipschitz in its second argument, namely there exists \(G > 0\) such that for all \(y \in \mathcal{Y}\) and \(a, a' \in \mathbb{R}\),

\[
\ell(y, a) - \ell(y, a') \leq G|a - a'| \quad \text{and} \quad \ell_0 = \sup_{y \in \mathcal{Y}} \ell(y, 0). \tag{1}
\]

**Example 2 (Hinge loss \\& other loss functions).** The main example we have in mind is the hinge loss \(\ell(y, a) = |1 - ya|_+ = \max\{0, 1 - ya\}\), with \(\mathcal{Y} = \{-1, 1\}\), which is convex but not differentiable, and for which \(G = 1\) and \(\ell_0 = 1\). Another example is the logistic loss \(\ell(y, a) = \log(1 + e^{-ya})\), for which \(G = 1\) and \(\ell_0 = \log 2\).

Given a loss, the corresponding expected risk \(L : \mathcal{H} \rightarrow [0, \infty)\) is for all \(w \in \mathcal{H}\)

\[
L(w) = \mathbb{E} \left[ \ell(Y, \langle w, X \rangle) \right] = \int_{\mathcal{H} \times \mathcal{Y}} \ell(y, \langle w, x \rangle) dP(x, y),
\]

and can be easily shown to be convex and Lipschitz continuous.

In this setting, we are interested in the problem of solving

\[
\min_{w \in \mathcal{H}} L(w), \tag{2}
\]

when the distribution \(P\) is known only through a training set of independent samples \(D = (x_i, y_i)_{i=1}^n \sim P^n\). Since we only have the data \(D\), we cannot solve the problem exactly and given an empirical approximate solution \(\hat{w}\), a natural error measure is the the excess risk \(L(\hat{w}) - \inf_{w \in \mathcal{H}} L(w)\), which is a random variable through its dependence on \(\hat{w}\), and hence on the data. In the following we are interested in characterizing its distribution for finite sample sizes. Next we discuss how approximate solutions can be obtained from data.
2.1 Empirical risk minimization (ERM)

A natural approach to derive approximate solutions is based on replacing the expected risk with the empirical risk \( \hat{L}(w) = \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, \langle w, x_i \rangle) \).

We consider regularized empirical risk minimization (ERM) based on the solution of the problem,

\[
\min_{w \in \mathcal{H}} \hat{L}_\lambda(w), \quad \hat{L}_\lambda(w) = \hat{L}(w) + \lambda \|w\|^2.
\]

(3)

Note that \( \hat{L}_\lambda : \mathcal{H} \rightarrow \mathbb{R} \) is continuous and strongly convex, hence there exists a unique minimizer \( \hat{w}_\lambda \). If we let \( \hat{X} \) denote the data matrix, by the representer theorem [54, 42] there exists \( c \in \mathbb{R}^n \) such that

\[
\hat{w}_\lambda = \hat{X}^\top c.
\]

(4)

The expression of the coefficient \( c \) depends on the considered loss function. Next, we comment on different approaches to obtain a solution when \( \ell \) is the hinge loss. We add one remark first.

Remark 1 (Constrained ERM). A related approach is based on considering the problem

\[
\min_{\|w\| \leq R} \hat{L}(w).
\]

Minimizing (3) can be seen as a Lagrange multiplier formulation of the above problem. While these problems are equivalent [10], the exact correspondence is implicit. As a consequence their statistical analysis differ. We primarily discuss Problem (3), but also analyze Problem (5) in Appendix H.

2.2 Computations with the hinge loss

Minimizing (3) can be solved in many ways and we provide some basic considerations. If \( \mathcal{H} \) is finite dimensional, iteratively via gradient methods can be used. For example, the subgradient method [10] applied to (3) is given, for some suitable \( \hat{w}_0 \) and step-size sequence \( (\eta_t)_t \), by

\[
w_{t+1} = w_t - \eta_t \left( \frac{1}{n} \sum_{i=1}^{n} y_i x_i g_t(w_t) + 2\lambda w_t \right),
\]

(6)

where \( g_t(w) \in \partial \ell(y_t, \langle w, x_t \rangle) \) is the subgradient of the map \( a \mapsto \ell(y_t, a) \) evaluated at \( a = \langle w, x_t \rangle \), see also [37]. The corresponding iteration cost is \( O(nd) \) in time and memory. Clearly, other variants can be considered, for example adding a momentum term [35], stochastic gradients and minibatching or considering other approaches for example based on coordinate descent [45]. When \( \mathcal{H} \) is infinite dimensional a different approach is possible, provided \( \langle x, x' \rangle \) can be computed for all \( x, x' \in \mathcal{H} \). For example, it is easy to prove by induction that the iteration in (6) satisfies \( w_t = \hat{X}^\top c_{t+1} \), where

\[
c_{t+1} = c_t - \eta_t \left( \frac{1}{n} \sum_{i=1}^{n} y_i c_t g_i(\hat{X}^\top c_t) + 2\lambda c_t \right),
\]

(7)

and where \( e_1, \ldots, e_n \) is the canonical basis in \( \mathbb{R}^n \). The cost of the above iteration is \( O(n^2 C_K) \) for computing \( g_i(w) \in \partial \ell(y_i, \langle \hat{X}^\top c_t, x_i \rangle) = \partial \ell(y_i, \langle \hat{X}^\top c_t, x_i \rangle) \), where \( C_K \) is the cost of evaluating one inner product. Also in this case, a number of other approaches can be considered, see e.g. [48 Chap. 11] and references therein. We illustrate the above ideas for the hinge loss.

Example 3 (Hinge loss & SVM). Considering problem (3) with the hinge loss corresponds to support vector machines for classification. With this choice \( \partial \ell(y_i, \langle w, x_i \rangle) = 0 \) if \( y_i \langle w, x_i \rangle = 1 \), \( \partial \ell(y_i, \langle w, x_i \rangle) = [-1, 0] \) if \( y_i \langle w, x_i \rangle = 1 \) and \( \partial \ell(y_i, \langle w, x_i \rangle) = -1 \) if \( y_i \langle w, x_i \rangle < 1 \). In particular, in (7) we can take \( g_i(w) = -1_{[y_i \langle w, x_i \rangle \leq 1]} \).
3 ERM on random subspaces

In this paper, we consider a variant of ERM based on considering a subspace \( B \subset H \) and the corresponding regularized ERM problem,

\[
\min_{\beta \in B} \hat{L}_\lambda(\beta). 
\]

As clear from (4), choosing \( B = H_n = \text{span}\{x_1, \ldots, x_n\} \) is not a restriction and yields the same solution as considering (3). From this observation a natural choice is to consider for \( m \leq n \),

\[
B_m = \text{span}\{\tilde{x}_1, \ldots, \tilde{x}_m\} 
\]

with \( \{\tilde{x}_1, \ldots, \tilde{x}_m\} \subset \{x_1, \ldots, x_n\} \) a subset of the input points. A basic idea we consider is to sample the points uniformly at random. Another more refined choice we consider is sampling exactly or approximately (see Definition 1 in the Appendix) according to the leverages scores [17]

\[
l_i(\alpha) = \left\langle x_i, (\tilde{X} \tilde{X}^\top x + \alpha I_n)^{-1} x_i \right\rangle \quad i = 1, \ldots, n. 
\]

While leverage scores computation is costly, approximate leverage scores (ALS) computation can be done efficiently, see [38] and references therein. Following [39], other choices are possible. Indeed for any \( q \in \mathbb{N} \) and \( z_1, \ldots, z_q \in H \) we could consider \( B = \text{span}\{z_1, \ldots, z_q\} \) and derive a formulation as in (11) replacing \( \tilde{X} \) with the matrix \( Z \) with rows \( z_1, \ldots, z_q \). We leave this discussion for future work. Here, we focus on the computational benefits of considering ERM on random subspaces and analyze the corresponding statistical properties.

The choice of \( B_m \) as in (9) allows to improve computations with respect to (4). Indeed, \( \beta \in B_m \) is equivalent to the existence of \( b \in \mathbb{R}^m \) s.t. \( \beta = \tilde{X}^\top b \), so that we can replace (8) with the problem

\[
\min_{b \in \mathbb{R}^m} \frac{1}{n} \sum_{i=1}^n \ell \left( y_i, \left\langle \tilde{X} b, x_i \right\rangle \right) + \lambda \left\langle b, \tilde{X} \tilde{X}^\top b \right\rangle_m. 
\]

Further, since \( \tilde{X} \tilde{X}^\top \in \mathbb{R}^{m \times m} \) is symmetric and positive semi-definite, we can derive a formulation close to that in (3), considering the reparameterization \( a = (\tilde{X} \tilde{X}^\top)^{1/2} b \) which leads to,

\[
\min_{a \in \mathbb{R}^m} \frac{1}{n} \sum_{i=1}^n \ell \left( y_i, (a, x_i)_m \right) + \lambda \|a\|_m^2, 
\]

where for all \( i = 1, \ldots, n \), we defined the embedding \( x_i \mapsto x_i = ((\tilde{X} \tilde{X}^\top)^{1/2})^\top \tilde{X} x_i \). Note that this latter operation only involves the inner product in \( H \) and hence can be computed in \( O(m^3 + nm^2 C_K) \) time. The subgradient method for (11) has a cost \( O(nm) \) per iteration. In summary, we obtained that the cost for the ERM on subspaces is \( O(nm^2 C_K + nm \cdot \#\text{iter}) \) and should be compared with the cost of solving (7) which is \( O(n^2 C_K + n^2 \cdot \#\text{iter}) \). The corresponding costs to predict new points are \( O(mC_K) \) and \( O(n C_K) \), while the memory requirements are \( O(nm) \) and \( O(n^2) \), respectively. Clearly, memory requirements can be reduced recomputing things on the fly. As clear from the above discussion, computational savings can be drastic, as long as \( m < n \), and the question arises of how this affect the corresponding statistical accuracy. Next section is devoted to this question.

4 Statistical analysis of ERM on random subspaces

We divide the presentation of the results in three parts. First, we consider a setting where we make basic assumptions. Then, we discuss improved results considering more benign assumptions. Finally, we describe general results covering also less favorable conditions. In all cases, we provide simplified statements for the results, omitting numerical constants, logarithmic and higher order terms, for ease of presentation. The complete statements and the proofs are provided in the appendices.
4.1 Basic setting

In this section, we only assume the best in the model to exist.

**Assumption 3.** There exists \( w_* \in H \) such that \( L(w_*) = \min_{w \in H} L(w) \).

We first provide some benchmark results for regularized ERM under this assumption.

**Theorem 1** (Regularized ERM). Under Assumption [1][2][3] the following inequality holds, for all \( \lambda > 0 \) and \( 0 < \delta < 1 \), with probability at least \( 1 - \delta \),

\[
L(\hat{w}_\lambda) - L(w_*) \lesssim \frac{G^2 \kappa^2 \log(1/\delta)}{\lambda n} + \lambda \|w_*\|^2.
\]

Hence letting \( \lambda = (G\kappa/\|w_*\|)\sqrt{\log(1/\delta)/n} \) leads to a rate of \( O(\|w_*\| \sqrt{\log(1/\delta)/n}) \).

The proof of the above result is given in Appendix A. It shows the excess risk bound for regularized ERM arises from a tradeoff between an estimation and an approximation term. While this result can be derived specializing more refined analysis, see e.g. [18] or later sections, we provide a novel self-contained proof which is of interest in its own right. Similar bounds in high-probability for ERM constrained to the ball of radius \( R \geq \|w_*\| \) can be obtained through a uniform convergence argument over such balls, see [6][33][24]. In order to apply this to regularized ERM, one could in principle use the fact that by Assumption 2, \( \|\hat{w}_\lambda\| \leq \sqrt{\ell_0/\lambda} \) (see Appendix A), but this yields a suboptimal dependence in \( \lambda \). [47] address this issue by considering ERM with both constraint over a ball and regularization, and derives a high-probability bound. Finally, a similar rate for \( \hat{w}_\lambda \), though only in expectation, can be derived through a stability argument [9][33]. The proof of our high-probability bound builds on uniform convergence, but circumvents using a crude bound on \( \|\hat{w}_\lambda\| \) by instead exploiting its definition as a regularized minimizer and bounding \( \|\hat{w}_\lambda\|/\sqrt{n} \leq \lambda \|\hat{w}_\lambda\|^2 + 1/(\lambda n) \). We use this bound as a reference and to derive results for regularized ERM on subspaces. We first consider any fixed subspace and then specialize to suitable random subspaces. Given \( B \subseteq H \), let \( \tilde{\beta}_\lambda \) be the minimizer of \( \hat{\beta}_\lambda = \min_{B \subseteq H} \|\Sigma^{1/2}(I - \mathcal{P}_B)\| \).

**Theorem 2** (Regularized ERM on subspaces). Fix \( B \subseteq H \), \( \lambda > 0 \) and \( 0 < \delta < 1 \). Under Assumptions [1][2][3] with probability at least \( 1 - \delta \),

\[
L(\tilde{\beta}_\lambda) - L(w_*) \lesssim \frac{G^2 \kappa^2 \log(1/\delta)}{\lambda n} + \lambda \|w_*\|^2 + \sqrt{\mu_B G \|w_*\|}.
\]

Compared to Theorem 1, the above result shows that there is an extra approximation error term due to considering a subspace. The coefficient \( \mu_B \) appears in the analysis also for other loss functions, see e.g. [39][31]. Roughly speaking, it captures how well the subspace \( B \) is adapted to the model. We next develop this reasoning, specializing the above result to a random subspace \( B = B_m \) as in [9]. Note that, if \( B \) is random then \( \mu_B \) is a random variable through its dependence on \( \mathcal{P}_B \) and on \( B \). We denote by \( \tilde{\beta}_{\lambda,m} \) the unique minimizer of \( \tilde{L}_\lambda \) on \( B_m \) and by \( \mathcal{P}_m = \mathcal{P}_{B_m} \) the corresponding projection. Further, it is also useful to introduce the so-called effective dimensions [57][13][39]. We denote by \( P_X \) the distribution of \( X \), with \( \supp(P_X) \subseteq H \) its support[4] and define for \( \alpha > 0 \)

\[
d_{\alpha,2} = \text{Tr}((\Sigma + \alpha I)^{-1}\Sigma), \quad d_{\alpha,\infty} = \sup_{x \in \supp(P_X)} \langle x, (\Sigma + \alpha I)^{-1}x \rangle. \tag{12}
\]

Then, \( d_{\alpha,2} \) is finite since \( \Sigma \) is trace class, and \( d_{\alpha,\infty} \) is finite since \( \supp(P_X) \) is bounded. Further, we denote by \( (\sigma_j(\Sigma)) \), the strictly positive eigenvalues of \( \Sigma \), with eigenvalues counted with respect to their multiplicity and ordered in a non-increasing way. We borrow the following results from [39].

**Proposition 1** (Uniform and leverage scores sampling). Fix \( \alpha > 0 \) and \( 0 < \delta < 1 \). With probability at least \( 1 - \delta \)

\[
\mu_{B_m}^2 = \|\Sigma^{1/2}(I - \mathcal{P}_m)\|^2 \leq 3\alpha. \tag{13}
\]

1 Namely, the smallest closed subset of \( H \) with \( P_X \)-measure 1, well-defined since \( H \) is a Polish space [35].
provided that \( m \geq d_{a,\infty} \log \frac{1}{\alpha} \) or \( m \geq d_{a,2} \log \frac{n}{3} \) for uniform and ALS sampling, respectively.

Moreover, if the spectrum of \( \Sigma \) has a polynomial decay, i.e., for some \( p \in (0,1) \)
\[
\sigma_j(\Sigma) \lesssim j^{-\frac{1}{p}}
\]
then (13) holds if \( m \geq \frac{1}{\alpha} \log \frac{1}{\alpha \delta} \) or \( m \geq \frac{1}{\alpha^2} \log \frac{n}{3} \) for uniform and ALS sampling, respectively.

Combining the above proposition with Theorem 2 we have the following.

**Theorem 3** (Uniform and leverage scores sampling under eigen-decay). Under Assumption 4 and condition (14), for all \( \lambda > 0 \) and \( 0 < \delta < 1 \), with probability \( 1 - \delta \),
\[
L(\hat{\beta}_{\lambda,m}) - L(w_*) \lesssim \frac{G^2 \kappa^2 \log(3/\delta)}{\lambda n} + \lambda \|w_*\|^2 + \sqrt{\alpha G} \|w_*\|.
\]

Hence, letting \( \lambda \asymp 1/\sqrt{n} \), \( \alpha \asymp \lambda^2 \asymp 1/n \) and taking \( m \gtrsim n \log n \) points by uniform sampling or \( m \gtrsim n^p \log n \)
by leverage score sampling, leads to a rate of \( O(1/\sqrt{n}) \).

The above results show that it is possible to achieve the same rate of standard regularized ERM, but
to do so uniform sampling does not seem to provide a computational benefit. As clear from the proof,
computational benefits for smaller subspace dimension would lead to worse rates. This behavior is worse than
that allowed by smooth loss functions [39, 31]. These results can be recovered with our approach. Indeed, for
both least squares and self-concordant losses, the bound in Theorem 2 can be easily improved to have a
linear dependence on \( \mu_B \), leading to straightforward improvements. We will detail this derivation in a longer
version of the paper. Due to space constraints, here we focus on non-smooth losses, since these results, and
not only their proof, are new. For this class of loss functions, Theorem 3 shows that leverage scores sampling
can lead to better results depending on the spectral properties of the covariance operator. Indeed, if there is
a fast eigendecay, then using leverage scores and a subspace dimension \( m < n \) one can achieve the same rates
as exact ERM. For fast eigendecay (\( p \) small), the subspace dimension can decrease dramatically. For example,
as a reference for \( p = 1/2 \) then \( m = \sqrt{n} \) suffices. Note that, other decays, e.g., exponential, could also be
considered. These observations are consistent with recent results for random features [4, 28, 50], while they
seem new for ERM on subspaces. Compare to random features the proof techniques have similarities but
also differences due to the fact that in general random features do not define subspaces. Finding a unifying
analysis would be interesting, but it is left for future work. Also, we note that uniform sampling can have the
same properties of leverage scores sampling, if \( d_{a,2} \approx d_{a,\infty} \). This happens under the strong assumptions on
the eigenvectors of the covariance operator, but can also happen in kernel methods with kernels corresponding
to Sobolev spaces [49]. With these comments in mind, here, we focus on subspace defined through leverage
scores noting that the assumption on the eigendecay not only allows for smaller subspace dimensions, but
can also lead to faster learning rates. Indeed, we study this next.

### 4.2 Fast rates

To exploit the eigendecay assumption and derive fast rates, we begin considering further conditions on the
problem. We relax these assumptions in the next section. First, we let for \( P_X \)-almost all \( x \in \mathcal{H} \)
\[
f_*(x) = \arg\min_{a \in \mathbb{R}} \int_{\mathcal{Y}} \ell(y,a)dP(y|x) \tag{15}
\]
where \( P(y|x) \) is the conditional distribution \(^2\)of \( y \) given \( x \in \mathcal{H} \) and make the following assumption.

**Assumption 4.** There exists \( w_* \in \mathcal{H} \) such that, almost surely, \( f_*(X) = \langle w_*, X \rangle \).

In our context, this is the same as requiring the model to be well specified. Second, following [48], we
consider a loss that can be clipped at \( M > 0 \) that is such that for all \( y' \in \mathcal{Y}, y \in \mathbb{R}, \)
\[
\ell(y', y^d) \leq \ell(y', y), \tag{16}
\]

\(^2\)The conditional distribution always exists since \( \mathcal{H} \) is separable and \( \mathcal{Y} \) is a Polish space [35].
where \( y^{cl} \) denotes the clipped value of \( y \) at \( \pm M \), that is
\[
y^{cl} = -M \quad \text{if } y \leq M, \quad y^{cl} = y \quad \text{if } y \in [-M, M], \quad y^{cl} = M \quad \text{if } y \geq M.
\] (17)

If \( w \in \mathcal{H}, \) \( w^{cl} \) denotes the non-linear function \( f(x) = \langle w, x \rangle^{cl} \). This assumption holds for hinge loss with \( M = 1 \), and for bounded regression. Finally, we make the following assumption on the loss.

**Assumption 5 (Simplified Bernstein condition).** There are constants \( B, V > 0 \), such that for all \( w \in \mathcal{H}, \)
\[
\ell(Y, (w, X)) \leq B \tag{18}
\]
\[
\mathbb{E}[\{\ell(Y, ((w, X))^{cl}) - \ell(Y, f_*(X))\}^2] \leq V\mathbb{E}[\ell(Y, ((w, X))^{cl}) - \ell(Y, f_*(X))]. \tag{19}
\]

This is a standard assumption to derive fast rates for ERM [38, 5]. In classification with the hinge loss, it is implied by standard margin conditions characterizing classification noise, and in particular by hard margin assumptions on the data distribution [2, 52, 32, 48]. As discussed before, we next focus on subspaces defined by leverage scores and derive fast rates under the above assumptions.

**Theorem 4.** Fix \( \lambda > 0 \) and \( 0 < \delta < 1 \). Under Assumptions [4, 2, 4, 4], and a polynomial decay of the spectrum of \( \Sigma \) with rate \( 1/p \in (1, \infty) \), as in (14), then, with probability at least \( 1 - \delta \)
\[
L(\tilde{\beta}^{cl}_{\lambda, m}) - L(w_*) \lesssim \frac{G^2 \kappa^2 \log(2/\delta)}{\lambda p n} + \lambda \|w_*\|^2 + \sqrt{\kappa} G \|w_*\|.
\]
provided that \( n \) and \( m \) are large enough. Further, for ALS with the choice
\[
\lambda \asymp n^{-\frac{2}{3p}}, \quad \alpha \asymp n^{-\frac{1}{3p}}, \quad m \gtrsim n^{\frac{2p}{3}} \log n,
\]
with high probability,
\[
L(\tilde{\beta}^{cl}_{\lambda, m}) - L(w_*) \lesssim (\log n)^{1/2} p n^{-\frac{1}{3p}}. \tag{20}
\]

The above result is a special case of the analysis in the next section, but it is easier to interpret. Compared to Theorem 3 the assumption on the spectrum also leads to an improved estimation error bound and hence improved learning rates. In this sense, these are the correct estimates since the decay of eigenvalues is used both for the subspace approximation error and the estimation error. As is clear from [20], for fast eigendecay, the obtained rate goes from \( O(1/\sqrt{n}) \) to \( O(1/n) \). Taking again, \( p = 1/2 \) leads to a rate \( O(1/n^{2/3}) \) which is better than the one in Theorem 3. In this case, the subspace defined by leverage scores needs to be chosen of dimension at least \( O(n^{2/3}) \). Note that again, the subspace dimension is even smaller for faster eigendecay. Next, we extend these results considering weaker, more general assumptions.

### 4.3 General analysis

Last, we give a general analysis relaxing the above assumptions. We replace Assumption 4 by
\[
\inf_{w \in \mathcal{H}} L(w) = \mathbb{E}[\ell(Y, f_*(X))], \tag{21}
\]
and introduce the approximation error,
\[
A(\lambda) = \min_{w \in \mathcal{H}} L(w) + \lambda \|w\|^2 - \inf_{w \in \mathcal{H}} L(w). \tag{22}
\]
Condition (21) may be relaxed at the cost of an additional approximation term, but the analysis is lengthier and is postponed. It has a natural interpretation in the context of kernel methods, see Example [1] where it is satisfied by universal kernels [48]. Regarding the approximation error, note that, if \( w_* \) exists then \( A(\lambda) \leq \lambda \|w_*\|^2 \), and we can recover the results in Section 4.1. More generally, the approximation error decreases with \( \lambda \) and learning rates can be derived assuming a suitable decay. Further, we consider a more general form of the Bernstein condition.
Assumption 6 (Bernstein condition). There exist constants $B > 0$, $\theta \in [0, 1]$ and $V \geq B^2 - \theta$, such that for all $w \in H$, the following inequalities hold almost surely:

$$\ell(Y, \langle w, X \rangle) \leq B,$$

$$\mathbb{E}[\ell(Y, \langle w, X \rangle)^{cl}] - \ell(Y, f_w(X)) \leq V \mathbb{E}[\ell(Y, \langle w, X \rangle)^{cl}] - \ell(Y, f_w(X))^{cl}. \quad (24)$$

Again in classification, the above condition is implied by margin conditions, and the parameter $\theta$ characterizes how easy or hard is the classification problem. The strongest assumption is choosing $\theta = 1$, with which we recover the result in the previous section. Then, we have the following result.

Theorem 5. Fix $\lambda > 0$ and $0 < \delta < 1$. Under Assumptions \[1, 2, 6\] and a polynomial decay $1/p \in (1, \infty)$ of the spectrum of $\Sigma$, as in \[14\], then with probability at least $1 - \delta$

$$L(\hat{\beta}_{\lambda,m}^{cl}) - \inf_{w \in H} L(w) \leq \left(\frac{G^2k^2\log(2/\delta)}{\lambda^p} \right)^{1/3} + \mathcal{A}(\lambda) + G\sqrt{\frac{\alpha \mathcal{A}(\lambda)}{\lambda}} + \frac{Gk \log(2/\delta)}{n} \sqrt{\mathcal{A}(\lambda)\lambda}. \quad (23)$$

Furthermore, if there exists $r \in (0, 1]$ such that $\mathcal{A}(\lambda) \lesssim \lambda^r$, then with the choice for ALS

$$\lambda \asymp n^{\min\left(\frac{2}{r+1}, \frac{1}{1-r}\right)} \quad \alpha \asymp n^{\min\left(2, \frac{r}{1-r+2r} \right)} \quad m \gtrsim n^{\min\left(2p, \frac{p(r+1)}{1-2p-\theta+2\theta p}\right)}$$

with high probability

$$L(\hat{\beta}_{\lambda,m}^{cl}) - L(f_w) \lesssim (\log n)^{1/2} n^{-\min\left(\frac{2}{r+1}, \frac{1}{1-r+2r}\right)}. \quad (24)$$

The proof of the above bound follows combining Proposition \[1\] with results to analyze the learning properties of regularized ERM with kernels \[35\]. While general, the obtained bound is harder to parse. For $r \to 0$ the bound become vacuous and there are not enough assumptions to derive a bound \[16\]. Taking $r = 1$ gives the best bound, recovering the result in the previous section when $\theta = 1$. Note that large values of $\lambda$ are prevented, indicating a saturation effect (see \[53, 34\]). As before the bound improves when there is a fast eigendecay. Taking $\theta = 1$ we recover the previous bounds, whereas smaller $\theta$ lead to worse bounds. Since, given any acceptable choice of $p, r$ and $\theta$, the quantity $\min\left\{2p, \frac{p(r+1)}{1-2p-\theta+2\theta p}\right\}$ takes values in $(0, 1)$, the best rate, that differently from before can also be slower than $\sqrt{1/n}$, can always be achieved choosing $m < n$ (up to logarithmic terms).

5 Experiments

We report simple experiments in the context of kernel methods, considering Nyström method. In particular, we choose the hinge loss, hence SVM for classification.

Nyström-Pegasos. Classic SVM implementations with hinge loss are based on considering a dual formulation and a quadratic programming problem \[21\]. This is the case for example, for the LibSVM library \[13\] available on Scikit-learn \[30\]. We use this implementation for comparison, but find it convenient to combine the Nyström method to a primal solver akin to \[6\] (see \[29, 20\] for the dual formulation). More precisely, we use Pegasos \[44\] which is based on a simple and easy to use stochastic subgradient iteration. We consider a procedure in two steps. First, we compute the embedding discussed in Section \[3\] with kernels it takes the form $x_i = (K_m)^{1/2}(K(x_i, \tilde{x}_1), \ldots, K(x_i, \tilde{x}_m))^T$, where $K_m \in \mathbb{R}^{n \times m}$ with $(K_m)_{ij} = K(\tilde{x}_i, \tilde{x}_j)$. Second, we use Pegasos on the embedded data. As discussed in Section \[3\] the total cost is $O(n m^2 C_K + n m \cdot \text{#iter})$ in time (here iter $\equiv$ epoch, i.e. one epoch equals $n$ steps of stochastic subgradient) and $O(m^2)$ in memory (needed to compute the pseudo-inverse and embedding the data in batches of size $m$).

Datasets & set up (see Appendix \[1\]). We consider five datasets \[8\] of size $10^4 - 10^6$, challenging for standard SVM implementations. We use a Gaussian kernel, tuning width and regularization parameter as explained in appendix. We report classification error and for data sets with no fixed test set, we set apart 20% of the data.
Results We compare with linear and kernel SVM see Table 1. For all the data sets, the Nyström-Pegasos approach provides comparable performances with much better times (except for the small-size Usps). Note that K-SVM cannot be run on millions of points, whereas Nyström-Pegasos is still fast and provides much better results than linear SVM. Finally, in Figure 1 we illustrate the interplay between $\lambda$ and $m$ for the Nyström-Pegasos considering SUSY data set.

Figure 1: The graphs above are obtained from SUSY data set: on the left we show how c-err measure changes for different choices of $\lambda$ parameter; in the central figure the focus is on the stability of the algorithm varying $\lambda$; on the right the combined behavior is presented with a heatmap.

Table 1: Architecture: single machine with AMD EPYC 7301 16-Core Processor and 256GB of RAM. For Nyström-Pegasos, ALS sampling has been used [38] and the results are reported as mean and standard deviation deriving from 5 independent runs of the algorithm. The columns of the table report classification error, training time and prediction time.

| Datasets   | LinSVM | KSVM | Nyström-Pegasos |
|------------|--------|------|-----------------|
|            | c-err  | c-err| c-err (s)       |
| SUSY       | 28.1%  | -    | -               |
| Mnist bin  | 12.4%  | 2.2% | 1601 87         |
| Usps       | 16.5%  | 3.1% | 4.4 1.0         |
| Webspam    | 8.8%   | 1.1% | 6044 473        |
| a9a        | 16.5%  | 15.0%| 114 31          |
| CIFAR      | 31.5%  | 19.1%| 6339 213        |
|            |        |      | t train (s)     |
| SUSY       | 20.0% ± 0.2% | 608 ± 2 | 134 ± 4        |
| Mnist bin  | 2.2% ± 0.1%  | 1342 ± 5 | 491 ± 32      |
| Usps       | 3.0% ± 0.1%  | 19.8 ± 0.1 | 7.3 ± 0.3   |
| Webspam    | 1.3% ± 0.1%  | 2440 ± 5  | 376 ± 18     |
| a9a        | 15.1% ± 0.2% | 29.3 ± 0.2 | 1.5 ± 0.1    |
| CIFAR      | 19.2% ± 0.1% | 2408 ± 14 | 820 ± 47     |
|            |        |      | t pred (s)      |

6 Acknowledgements

This material is based upon work supported by the Center for Brains, Minds and Machines (CBMM), funded by NSF STC award CCF-1231216, and the Italian Institute of Technology. We gratefully acknowledge the support of NVIDIA Corporation for the donation of the Titan Xp GPUs and the Tesla k40 GPU used for this research. Part of this work has been carried out at the Machine Learning Genoa (MaLGa) center, Università di Genova (IT).

L. R. acknowledges the financial support of the European Research Council (grant SLING 819789), the AFOSR projects FA9550-17-1-0390 and BAA-AFRL-AFOSR-2016-0007 (European Office of Aerospace Research and Development), and the EU H2020-MSCA-RISE project NoMADS - DIV-777826.
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Appendices

A Proof of Theorem 1

This section is devoted to the proof of Theorem 1. In the following we restrict to linear functions, i.e., $f(x) = \langle w, x \rangle$ for some $w \in \mathcal{H}$ and, with slight abuse of notation we set

$$\ell(w, z) = \ell(y, \langle w, x \rangle), \quad z = (x, y) \in \mathcal{H} \times \mathcal{Y}, \ w \in \mathcal{H}.$$  

With this notation $L(w) = \int_{\mathcal{H} \times \mathcal{Y}} \ell(w, z)dP(z)$. The Lipschitz assumption implies that $\ell(\cdot, (X, Y))$ is almost surely Lipschitz in its argument, with Lipschitz constant $GK$.

Specifically, we will show the following:

**Theorem 6.** Under Assumptions 1 and 2 and, for every $\lambda > 0$ and $\delta \in (0, 1)$ let

$$C_{\lambda, \delta} = 4\{1 + \sqrt{\log(1 + \log_2(3 + \ell_0n^2/\lambda))} + \log(2/\delta)\} = O(\sqrt{\log(3 + \ell_0n^2/\lambda) + \log(1/\delta)}).$$

If Assumption 3 holds, then with probability $1 - \delta$,

$$L(\hat{w}_\lambda) < \inf_{\hat{w} \in \mathcal{H}} L + \lambda \| w_* \|^2 + \frac{C_{\lambda, \delta}^2G^2\kappa^2}{4\lambda n} + \frac{GC_{\lambda, \delta}}{\sqrt{n}} + (\ell_0 + GK\| w_* \|)\sqrt{\frac{\log(2/\delta)}{n}}. \quad (25)$$

More generally, with probability $1 - \delta$, letting $A(\lambda) := \inf_{\hat{w} \in \mathcal{H}} L(w) + \lambda \| w \|^2 - \inf_{\hat{w} \in \mathcal{H}} L(w)$,

$$L(\hat{w}_\lambda) - \inf_{\hat{w} \in \mathcal{H}} L < 2A(\lambda) + \frac{C_{\lambda, \delta}^2G^2\kappa^2 + 8G^2\kappa^2 \log(2/\delta)}{4\lambda n} + \frac{GC_{\lambda, \delta}}{\sqrt{n}} + \ell_0\sqrt{\frac{\log(2/\delta)}{n}} \quad (26)$$

for every $R > 0$.

The proof starts with the following bound on the generalization gap $L(w) - \hat{L}(w)$ uniformly over balls. While this result is well-known and follows from standard arguments (see, e.g., [19 25]), we include a short proof for completeness.

**Lemma 1.** Under Assumptions 1 and 3 and, for every $R > 0$, one has with probability at least $1 - \delta$,

$$\sup_{\|w\| \leq R} [L(w) - \hat{L}(w)] < \frac{GRK}{\sqrt{n}} \left(2 + \sqrt{\log(1/\delta)}\right). \quad (27)$$

**Proof of Lemma 1.** The proof starts by a standard symmetrization step [19 25]. Let us call $D := (z_1, \ldots, z_n)$ i.i.d. from $P$, as well as an independent $D' := (z'_1, \ldots, z'_n)$ i.i.d. from $P$ and $\varepsilon_1, \ldots, \varepsilon_n$ i.i.d. with $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = 1/2$. We denote $\hat{L}'(w) := n^{-1}\sum_{i=1}^n \ell(w, z'_i)$ the error on the sample $D'$. Then,

$$\mathbb{E}_{D \sim P^n} \sup_{\|w\| \leq R} \left[ L(w) - \hat{L}(w) \right] = \mathbb{E}_{D} \sup_{\|w\| \leq R} \left[ \mathbb{E}_{D'} \hat{L}'(w) - \hat{L}(w) \right] \leq \mathbb{E}_{D, D'} \sup_{\|w\| \leq R} \left[ \hat{L}'(w) - \hat{L}(w) \right] = \mathbb{E}_{D, D', \varepsilon} \sup_{\|w\| \leq R} \left[ \frac{1}{n}\sum_{i=1}^n \varepsilon_i(\ell(w, z_i) - \ell(w, z'_i)) \right]$$

$$= 2\mathbb{E}_{D, \varepsilon} \sup_{\|w\| \leq R} \frac{1}{n}\sum_{i=1}^n \varepsilon_i\ell(w, z_i)$$

where we used that $\mathbb{E}_{D'} \hat{L}'(\cdot) = L(\cdot)$, and that $(\ell(f, z_i) - \ell(f, z'_i))_{1 \leq i \leq n}$ and $(\varepsilon_i(\ell(f, z_i) - \ell(f, z'_i)))_{1 \leq i \leq n}$ have the same distribution, as well as $(\varepsilon_i\ell(f, z_i))_i$ and $(-\varepsilon_i\ell(f, z'_i))_i$. The last term corresponds to the Rademacher
complexity of the class of functions \( \{ \ell(w, \cdot) : \|w\| \leq R \} \) \( \geq \) 25. Now, using that \( \ell(w, z_i) = \ell(y_i, (w, x_i)) \) for \( z_i = (x_i, y_i) \), where \( \ell(y_i, \cdot) \) is \( \ell \)-Lipschitz by Assumption 2, the contraction inequality for Rademacher averages 33 gives

\[
E_{D, \varepsilon} \left[ \sup_{\|w\| \leq R} \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \ell(w, z_i) \right] \leq G E_{D, \varepsilon} \left[ \sup_{\|w\| \leq R} \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \langle w, x_i \rangle \right]
\]

\[
= G E_{D, \varepsilon} \left[ \sup_{\|w\| \leq R} \left\langle w, \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i x_i \right\rangle \right]
\]

\[
\leq G R E_{D, \varepsilon} \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i x_i \right\| \right]^{1/2}
\]

\[
= G R E \left[ \|x\|^2 / \sqrt{n} \right]^{1/2}
\]

where we used that \( E[\varepsilon_i \varepsilon_j \langle x_i, x_j \rangle] = 0 \) for \( i \neq j \) by independence, and that \( \|x\| \leq \kappa \) almost surely (Assumption 1). Hence,

\[
E_{D \sim P^m} \sup_{\|w\| \leq R} \left[ L(w) - \widehat{L}(w) \right] \leq \frac{2 G R \kappa}{\sqrt{n}}.
\]

(28)

To write the analogous bound in high probability we apply McDiarmid’s inequality 35. We know that given \( D := \{z_1, \ldots, z_i, \ldots, z_n\} \), \( D' := \{z_1, \ldots, z_i', \ldots, z_n\} \) and defining \( \phi(D) := \sup_{\|w\| \leq R} [L(w) - L(w)] \) we have

\[
\| \phi(D) - \phi(D') \| \leq \sup_{\|w\| \leq R} \frac{1}{n} |\ell(w, z_i) - \ell(w, z_i')|
\]

\[
\leq \frac{G}{n} \sup_{\|w\| \leq R} |\langle w, x_i - x_i' \rangle|
\]

\[
\leq \frac{2 G R \kappa}{n}
\]

(29)

using the Assumption 1 of boundedness of the input. Hence, by McDiarmid inequality:

\[
\mathbb{P} \left[ \phi(D) - E_D[\phi(D)] \geq t \right] \leq \exp \left( \frac{-t^2 n}{2 G^2 R^2 \kappa^2} \right);
\]

(30)

taking \( \delta = \exp \left( \frac{-t^2 n}{2 G^2 R^2 \kappa^2} \right) \) so that \( t = G R \kappa \sqrt{\frac{2 \log(1/\delta)}{n}} \), we obtain the desired bound (27).

Lemma 1 suffices to control the excess risk of the constrained risk minimizer \( \widehat{w} := \arg \min_{\|w\| \leq R} L(w) \) for \( R = \|w_0\| \). On the other hand, this result cannot be readily applied to \( \widehat{w}_\lambda \), since its norm \( \|\widehat{w}_\lambda\| \) is itself random. Observe that, by definition and by Assumption 2,

\[
\lambda^2 \|\widehat{w}_\lambda\|^2 \leq \widehat{L}_\lambda(\widehat{w}_\lambda) \leq \widehat{L}_\lambda(0) = \widehat{L}(0) = \sup_{y \in Y} \ell(y, 0) = \ell_0,
\]

so that \( \|\widehat{w}_\lambda\| \leq \sqrt{\ell_0 / \lambda} \). One could in principle apply this bound on \( \widehat{w}_\lambda \), but this would yield a suboptimal dependence on \( \lambda \) and thus a suboptimal rate.

The next step in the proof is to make the bound of Lemma 1 valid for all norms \( R \), so that it can be applied to the random quantity \( R = \|\widehat{w}_\lambda\| \). This is done in Lemma 2 below through a union bound.

**Lemma 2.** Under Assumptions 1 and 3 with probability \( 1 - \delta \), one has:

\[
\forall w \in \mathcal{H}, \quad L(w) - \widehat{L}(w) \leq \frac{4 G (1 + \kappa \|w\|)}{\sqrt{n}} \left( 1 + \sqrt{\log(2 + \log_2(1 + \kappa \|w\|)) + \log(1/\delta)} \right).
\]
**Proof of Lemma** Fix $\delta \in (0, 1)$. For $p \geq 1$, let $R_p := \kappa^{-1}2^p$ and $\delta_p = \delta/(p(p+1))$. By Lemma $1$, one has for every $p \geq 1$,

$$
P\left( \sup_{\|w\| \leq R_p} \left[ L(f) - \hat{L}(w) \right] \geq \frac{G\kappa R_p}{\sqrt{n}} \left( 2 + 2\log \frac{1}{\delta} \right) \right) \leq \delta_p.
$$

Taking a union bound over $p \geq 1$ and using that $\sum_{p \geq 1} \delta_p = \delta$ and $\delta_p \geq \delta^2/(p(p+1)^2)$, we get:

$$
P\left( \exists p \geq 1, \sup_{\|w\| \leq R_p} \left[ L(w) - \hat{L}(w) \right] \geq \frac{G\kappa R_p}{\sqrt{n}} \left( 2 + 2\log \frac{p+1}{\delta} \right) \right) \leq \delta.
$$

Now, for $w \in \mathcal{H}$, let $p = \lceil \log_2(1 + \kappa\|w\|) \rceil$; then, $1 + \kappa\|w\| \leq \kappa R_p = 2^p \leq 2(1 + \kappa\|w\|)$, so $\|w\| \leq R_p$. Hence, with probability $1 - \delta$,

$$
\forall w \in \mathcal{H}, \quad L(w) - \hat{L}(w) \leq \frac{4G(1 + \kappa\|w\|)}{\sqrt{n}} \left( 1 + \sqrt{\log \left( 2 + \log_2(1 + \kappa\|w\|) \right)} + \log(1/\delta) \right).
$$

This is precisely the desired bound. \hfill \Box

Since the bound of Lemma $2$ holds simultaneously for all $w \in \mathcal{H}$, one can apply it to $\hat{w}_\lambda$; using the inequality $\kappa\|\hat{w}_\lambda\| \leq \kappa/\ell_0/\lambda \leq (1 + \ell_0\kappa^2/\lambda)/2$ to bound the $\log \log$ term, this gives with probability $1 - \delta$,

$$
L(\hat{w}_\lambda) - \hat{L}(\hat{w}_\lambda) \leq \frac{4G(1 + \kappa\|\hat{w}_\lambda\|)}{\sqrt{n}} \left( 1 + \sqrt{\log \left( 1 + \log_2(3 + \ell_0\kappa^2/\lambda) \right)} + \log(1/\delta) \right). \tag{31}
$$

Now, let $C = C_{\lambda, \delta} = 4\left\{ 1 + \sqrt{\log(1 + \log_2(3 + \ell_0\kappa^2/\lambda)) + \log(1/\delta)} \right\}$ writes $L(\hat{w}_\lambda) - \hat{L}(\hat{w}_\lambda) \leq CG(1 + \kappa\|\hat{w}_\lambda\|)/\sqrt{n}$. Using that $ab \leq a^2/4 + b^2/(4\lambda)$ for $a, b \geq 0$, one can then write

$$
L(\hat{w}_\lambda) \leq \hat{L}(\hat{w}_\lambda) + \frac{CG\kappa\|\hat{w}_\lambda\|}{\sqrt{n}} + \frac{CG}{\sqrt{n}}
$$

$$
\leq \hat{L}(\hat{w}_\lambda) + \frac{C^2G^2\kappa^2}{4\lambda n} + \frac{CG}{\sqrt{n}} \tag{32}
$$

$$
\leq \hat{L}(w_\lambda) + \frac{C^2G^2\kappa^2}{4\lambda n} + \frac{CG}{\sqrt{n}}. \tag{33}
$$

where (33) holds by definition of $\hat{w}_\lambda$. Now, since $|\ell(\lambda, Z)| \leq |\ell(Y, 0)| + |\ell(Y, \langle w_\lambda, X \rangle) - \ell(Y, 0)| \leq \ell_0 + G\kappa\|w_\lambda\|$ almost surely, Hoeffding’s inequality $\Box$ implies that, with probability $1 - \delta$,

$$
\hat{L}(w_\lambda) \leq L(w_\lambda) + (\ell_0 + G\kappa\|w_\lambda\|) \sqrt{\frac{2\log(1/\delta)}{n}}.
$$

Combining this inequality with (33) with a union bound, with probability $1 - 2\delta$:

$$
L(\hat{w}_\lambda) \leq L(w_\lambda) + \frac{C^2G^2\kappa^2}{4\lambda n} + \frac{CG}{\sqrt{n}} + \frac{2\log(1/\delta)}{n}. \tag{34}
$$

**First case:** $w_\star$ exists. First, assume that $w_\star = \arg\min_{w \in \mathcal{H}} L(w)$ exists. Then, by definition of $w_\lambda$, $L(w_\lambda) + \lambda\|w_\lambda\|^2 \leq L(w_\star) + \lambda\|w_\star\|^2$. In addition, $\|w_\lambda\| \leq \|w_\star\|$, since otherwise $\|w_\star\| < \|w_\lambda\|$ and $L(w_\star) \leq L(w_\lambda)$ would imply $L(w_\star) + \lambda\|w_\star\|^2 < L(w_\lambda) + \lambda\|w_\lambda\|^2$, contradicting the above inequality. Since $L(w_\star) = \inf_{\mathcal{H}} L$, it follows that, with probability $1 - 2\delta$,

$$
L(\hat{w}_\lambda) \leq \inf_{\mathcal{H}} L + \lambda\|w_\star\|^2 + \frac{C^2G^2\kappa^2}{4\lambda n} + \frac{CG}{\sqrt{n}} + \frac{2\log(1/\delta)}{n}. \tag{35}
$$

$$
\leq \inf_{\mathcal{H}} L + \lambda\|w_\star\|^2 + \frac{8G^2\kappa^2\{ 1 + \log(1 + \log_2(3 + \ell_0\kappa^2/\lambda)) + \log(1/\delta) \}}{\lambda n} + \frac{4G\{ 1 + \sqrt{\log(1 + \log_2(3 + \ell_0\kappa^2/\lambda)) + \log(1/\delta)} \}}{\lambda n} + \frac{2\log(1/\delta)}{n}
$$

$$
= \inf_{\mathcal{H}} L + O\left( \lambda\|w_\star\|^2 + \frac{G^2\kappa^2\{ \log(3 + \ell_0\kappa^2/\lambda) + \log(1/\delta) \}}{\lambda n} + \frac{(G + \ell_0)\sqrt{\log(1/\delta)}}{\sqrt{n}} \right),
$$

15
where the $O(\ldots)$ hide universal constants. The bound (35) precisely corresponds to the desired bound (25) after replacing $\delta$ by $\delta/2$. In particular, tuning $\lambda \approx (G\kappa/\|w_*\|)\sqrt{\log(1/\delta)/n}$ yields

$$L(\bar{w}_\alpha) - \inf_H L \lesssim \left\{ \ell_0 + G(1 + \kappa\|w_*\|) \right\} \{ \log\log(\kappa\|w_*\|n/G) + \sqrt{\log(1/\delta)} \}.$$  

Omitting the $\log n$ term, this bound essentially scales as $O(G\kappa\|w_*\|\sqrt{\log(1/\delta)/n})$.

### General case.
Let us now drop the assumption that $w_* = \arg\min_{w \in H} L(w)$ exists, and let (see (23)) for $\lambda > 0$:

$$A(\lambda) = L(w_\lambda) + \lambda\|w_\lambda\|^2 - \inf H = \inf_{w \in H} [L(w) + \lambda\|w\|^2] - \inf H.$$  

Note that, again using that $ab \leq \lambda a^2 + b^2/(4\lambda)$,

$$G\kappa\|w_\lambda\|\sqrt{2\log(1/\delta)/n} \leq \lambda\|w_\lambda\|^2 + \frac{2G^2\kappa^2\log(1/\delta)}{\lambda n} \leq A(\lambda) + \frac{2G^2\kappa^2\log(1/\delta)}{\lambda n}$$

so that (34) implies, with probability $1 - 2\delta$,

$$L(\bar{w}_\alpha) - \inf H L \leq 2A(\lambda) + \frac{C^2G^2\kappa^2}{4\lambda n} + \frac{GC}{\sqrt{n}} + \ell_0\sqrt{\frac{2\log(1/\delta)}{n}} + \frac{2G^2\kappa^2\log(1/\delta)}{\lambda n}.$$  

Finally, note that for all $w \in H$ with $\|w\| \leq \sqrt{R}$, $A(\lambda) \leq L(w) + \lambda\|w\|^2 - \inf H L \leq L(w) - \inf H L + \lambda R^2$, hence

$$A(\lambda) \leq \inf_{\|w\| \leq R} L(w) - \inf_H L + \lambda R^2$$

and

$$L(\bar{w}_\alpha) - \inf H L \leq 2\left( \inf_{\|w\| \leq R} L(w) - \inf_H L \right) + 2\lambda R^2 + \frac{C^2G^2\kappa^2}{4\lambda n} + \frac{8G^2\kappa^2\log(1/\delta)}{4\lambda n} + \frac{GC + \ell_0\sqrt{2\log(1/\delta)}}{\sqrt{n}}.$$  

Letting $\lambda \approx 1/(R\sqrt{n})$, this gives $L(\bar{w}_\alpha) - \inf_H L \leq 2(\inf_{\|w\| \leq R} L(w) - \inf_H L) + O(R/\sqrt{n})$ with high probability.

### B Proof of Theorem 2

The proof of Theorem 2 is given by decomposing the excess risk as in (42) where $P_m$ is replaced by $P_B$, (45) bounds term A, (46) bounds term B and (47) and the Definition 13 bound term C.

### C $T$-approximate leverage scores and proof of Proposition 1

Since in practice the leverage scores $l_i(\alpha)$ defined by (10) are onerous to compute, approximations $(\hat{l}_i(\alpha))_{i=1}^n$ have been considered [17] [15] [1]. In particular, in the following we are interested in suitable approximations defined as follows.

**Definition 1. ($T$-approximate leverage scores)** Let $(l_i(\alpha))_{i=1}^n$ be the leverage scores associated to the training set for a given $\alpha$. Let $\delta > 0$, $t_0 > 0$ and $T \geq 1$. We say that $(\hat{l}_i(\alpha))_{i=1}^n$ are $T$-approximate leverage scores with confidence $\delta$, when with probability at least $1 - \delta$,

$$\frac{1}{T} l_i(\alpha) \leq \hat{l}_i(\alpha) \leq T l_i(\alpha), \quad \forall i \in \{1, \ldots, n\}, \quad \alpha \geq t_0$$  

(36)

So, given $T$-approximate leverage score for $\alpha \geq t_0$, $\{\hat{x}_1, \ldots, \hat{x}_m\}$ are sampled from the training set independently with replacement, and with probability to be selected given by $Q_\alpha(i) = l_i(\alpha) / \sum_j \hat{l}_j(\alpha)$.

First part of Proposition 1 is the content of the following two results from [39].
Lemma 3 (Uniform sampling, Lemma 6 in [39]). Under Assumption 1, let $J$ be a partition of $\{1, \ldots, n\}$ chosen uniformly at random from the partitions of cardinality $m$. Let $\alpha > 0$, for any $\delta > 0$, such that $m \geq 67 \log \frac{4\kappa^2}{\alpha\delta} \lor 5d_{\alpha,\infty} \log \frac{4\kappa^2}{\alpha\delta}$, the following holds with probability at least $1 - \delta$

\[
\left\| (I - \mathcal{P}_{\mathcal{B}_m}) \Sigma^{1/2} \right\|^2 \leq 3\alpha
\]  

(37)

Lemma 4 (ALS sampling, Lemma 7 in [39]). Let $(\hat{i}_t(t))_{t=1}^n$ be the collection of approximate leverage scores. Let $\alpha > 0$ and the sampling probability $Q_\alpha$ be defined as $Q_\alpha(i) = \hat{i}_i(\alpha)/\sum_{j \in N} \hat{i}_j(\alpha)$ for any $i \in N$ with $N = \{1, \ldots, n\}$. Let $\mathcal{I} = \{i_1, \ldots, i_m\}$ be a collection of indices independently sampled with replacement from $N$ according to the probability distribution $P_\alpha$. Let $\mathcal{B}_m = \text{span}\{x_j| j \in J\}$ where $J$ be the subcollection of $\mathcal{I}$ with all the duplicates removed. Under Assumption 1, for any $\delta > 0$ the following holds with probability at least $1 - \delta$

\[
\left\| (I - \mathcal{P}_{\mathcal{B}_m}) \Sigma^{1/2} \right\|^2 \leq 3\alpha
\]  

(38)

where the following conditions are satisfied:

1. there exists a $T \geq 1$ and a $t_0 > 0$ such that $(l_i(t))_{t=1}^n$ are $T$-approximate leverage scores for any $t \geq t_0$,
2. $n \geq 1655\kappa^2 + 223\kappa^2 \log \frac{4\kappa^2}{\alpha\delta}$,
3. $t_0 \lor \frac{19\kappa^2}{n} \log \frac{4n}{\delta} \leq \alpha \leq \|\Sigma\|$, 
4. $m \geq 334 \log \frac{16n}{\delta} \lor 78T^2d_{\alpha,2} \log \frac{16n}{\delta}$.

If the spectrum of $\Sigma$ satisfies the decay property (14), the second part of Proposition 1 is a consequence of Lemma 5

D Proof of Theorem 3

Theorem 3 is a compact version of the following result.

Theorem 7. Fix $\alpha, \delta > 0$. Under Assumption 1, 2 and 3, with probability at least $1 - \delta$:

\[
L(\hat{\beta}_\lambda) - L(w^*) \leq \frac{C^2_{n,\delta}G^2\kappa^2}{4\lambda n} + \frac{C_{n,\delta}G}{\sqrt{n}} + G\|w^*\|\sqrt{\frac{2\log(3/\delta)}{n}} + 2G\sqrt{\alpha}\|w^*\| + \lambda\|w^*\|^2_\mathcal{H}
\]  

(39)

provided that $n \geq 1655\kappa^2 + 223\kappa^2 \log \frac{4\kappa^2}{\alpha\delta}$ and

1. for uniform sampling

\[
m \geq 67 \log \frac{4\kappa^2}{\alpha\delta} \lor 5d_{\alpha,\infty} \log \frac{4\kappa^2}{\alpha\delta}
\]  

(40)

2. for ALS sampling and $T$-approximate leverage scores with subsampling probabilities $Q_\alpha$, $t_0 > \frac{19\kappa^2}{n} \log \frac{4n}{\delta}$ and

\[
m \geq 334 \log \frac{16n}{\delta} \lor 78T^2d_{\alpha,2} \log \frac{16n}{\delta}
\]  

(41)

where $\alpha \geq \frac{19\kappa^2}{n} \log \frac{4n}{\delta}$

Proof. We recall the notation.

\[
\{\hat{x}_1, \ldots, \hat{x}_m\} \subseteq \{x_1, \ldots, x_n\}
\]

\[
\mathcal{B}_m = \text{span}\{\hat{x}_1, \ldots, \hat{x}_m\}
\]

\[
\hat{\beta}_\lambda = \arg\min_{w \in \mathcal{B}_m} \hat{L}(w)
\]

\[
w^* = \arg\min_{w \in \mathcal{H}} \hat{L}_\lambda(w).
\]
and $P_m = P_{B_m}$ is the orthogonal projector operator onto $B_m$.

In order to bound the excess risk of $\hat{\beta}_\lambda$, we decompose the error as follows:

$$L(\hat{\beta}_\lambda) - L(w^*) = \underbrace{L(\hat{\beta}_\lambda) - \hat{L}(\hat{\beta}_\lambda) - \lambda\|\hat{\beta}_\lambda\|_H^2 + \hat{L}(\hat{\beta}_\lambda) + \lambda\|\hat{\beta}_\lambda\|_H^2 - \hat{L}(P_m w^*) - \lambda\|P_m w^*\|_H^2}_{A} +$$

$$\underbrace{\hat{L}(P_m w^*) - L(P_m w^*)}_{B} + \underbrace{L(P_m w^*) - L(w^*)}_{C} + \lambda\|P_m w^*\|_H^2$$

Bound for term A

To bound term A we apply Lemma 2 for $\hat{\beta}_\lambda$ and we get with probability a least $1 - \delta$

$$\forall \alpha \geq \frac{\ell_0 \kappa^2}{n^2 \kappa}, \quad \hat{L}(\hat{\beta}_\lambda) \leq \hat{L}(\beta_\lambda) + \frac{C_{n,\delta} G(1 + \kappa\|\beta_\lambda\|)}{\sqrt{n}}$$

where $C_{n,\delta} = 4\left(1 + \sqrt{\log(2 + \log(1 + \kappa\|w\|)) + \log(1/\delta)}\right)$. Now since $xy \leq \lambda x^2 + y^2/(4\lambda)$, we can write

$$\frac{C_{n,\delta} Gk\|\beta_\lambda\|}{\sqrt{n}} \leq \lambda\|\beta_\lambda\|^2 + \frac{C_{n,\delta} G^2 \kappa^2}{4\lambda n}$$

hence,

$$L(\beta_\lambda) \leq \hat{L}(\beta_\lambda) + \lambda\|\beta_\lambda\|^2 + \frac{C_{n,\delta} G^2 \kappa^2}{4\lambda n} + \frac{C_{n,\delta} G}{\sqrt{n}}$$

Bound for term B

As regards term B, since $|\ell(P_m w^*, Z) - \ell(0, Z)| \leq G_k\|P_m w^*\| \leq G_k\|w^*\|$, using Hoeffding’s inequality, we have with probability at least $1 - \delta$

$$B \leq \left|\hat{L}(P_m w^*) - L(P_m w^*)\right| \leq G_k\|w^*\|\sqrt{\frac{2\log(1/\delta)}{n}}$$

Bound for term C

Finally, term C can be rewritten as

$$C = L(P_m w^*) - L(w^*)$$

$$\leq G\|\Sigma^{1/2}(I - P_m)w^*\|_H$$

$$\leq G\|\Sigma^{1/2}(I - P_m)||w^*\|_H$$

We bound equation (47) using Lemma 3 for uniform sampling and Lemma 4 for ALS selection. Putting the pieces together and noticing that $\lambda\|P_m w^*\|_H^2 \leq \lambda\|w^*\|_H^2$ we finally get the result in Theorem 7

The following corollary shows that there is choice of the parameters $\lambda = \lambda_n$, $\alpha = \alpha_n$ such that the excess risk of the $\beta_{\lambda_n}$ converges to zero with the optimal rate $O(1/\sqrt{n})$.

**Corollary 1.** Fix $\delta > 0$. Under the assumption of Theorem 7, let

$$\lambda \asymp \frac{1}{\|w^*\|_H^{-1/2}} \quad \alpha \asymp \frac{1}{n}$$

with probability at least $1 - \delta$:

$$L(\hat{\beta}_\lambda) - L(w^*) \lesssim \frac{\|w^*\|\sqrt{\log(1/\delta)}}{\sqrt{n}}$$

(48)
Despite of the fact that the rate is optimal, the required number of subsampled points is \( m \gtrsim n \log n \), so that the procedure is not effective. However, the following proposition shows that under a fast decay for the spectrum of the covariance operator \( \Sigma \), the ALS method becomes computationally efficient. We denote by \((\sigma_i(\Sigma))_i\) the sequence of strictly positive eigenvalues of \( \Sigma \) where the eigenvalues are counted with respect to their multiplicity and ordered in a non-increasing way.

**Proposition 2.** Fix \( \delta > 0 \). Under the assumptions of Theorem (7) and using ALS sampling

1. for polynomial decay, i.e. for some \( \gamma \in \mathbb{R}^+ \), \( p \in (0,1) \),
   \[
   \sigma_i(\Sigma) \leq \gamma i^{-\frac{1}{p}}
   \]
   with probability at least \( 1 - \delta \):
   \[
   L(\hat{\beta}_\lambda) - L(w^*) \leq \frac{C_n^2 \sigma^2 \kappa^2}{4\lambda n} + \frac{C_n \sigma G}{\sqrt{n} \sqrt{2 \log(3/\delta)}} + 2G\|w^*\|_H \sqrt{\frac{\log(1/p) n}{m^{1/p}}} + \lambda \|w^*\|_H^2
   \]
   where \( O(1/\sqrt{n}) \) rate can be achieved optimizing the choice of the parameters, i.e. \( \lambda \approx \frac{1}{\|w^*\|} n^{-1/2} \), \( m \gtrsim n^p \log n \).

2. for exponential decay, i.e. \( \sigma_i(\Sigma) \leq \gamma e^{-i\beta} \), \( \gamma, \beta \in \mathbb{R}^+ \), for any \( \delta > 0 \), with probability at least \( 1 - \delta \):
   \[
   L(\hat{\beta}_\lambda) - L(w^*) \leq \frac{C_n^2 \sigma^2 \kappa^2}{4\lambda n} + \frac{C_n \sigma G}{\sqrt{n} \sqrt{2 \log(3/\delta)}} + 2G\|w^*\|_H \sqrt{\frac{2 \log(3/\delta)}{n} e^{-m \log n} + \lambda \|w^*\|_H^2}
   \]
   where \( O(1/\sqrt{n}) \) rate can be achieved optimizing the choice of the parameter, i.e. \( \lambda \approx \frac{1}{\|w^*\|} n^{-1/2} \), \( m \gtrsim \log^2 n \).

**Proof.** The claim is a consequence of Appendix [G] where the link with \( m \) is obtained using Leverage Score sampling so that in Lemma 4 using Lemma 5 we have that
   \[
   m \gtrsim d_{\alpha,2} \log n, \quad d_{\alpha,2} \lesssim \alpha^{-p}, \quad \alpha \approx \frac{\log^{1/p} n}{m^{1/p}}
   \]
   while using Lemma 6 we have that
   \[
   m \gtrsim d_{\alpha,2} \log n, \quad d_{\alpha,2} \lesssim \log(1/\alpha), \quad \alpha \approx e^{-\frac{m}{\log n}}
   \]

From proposition above we have the following asymptotic rate.

**Corollary 2.** Fix \( \delta > 0 \). Under the assumptions of Theorem (7) and using ALS sampling, with probability at least \( 1 - \delta \)

1. assuming polynomial decay of the spectrum of \( \Sigma \) and choosing \( \lambda \approx \frac{1}{\|w^*\|} n^{-1/2} \), \( m \gtrsim n^p \log n \) then:
   \[
   L(\hat{\beta}_\lambda) - L(w^*) \lesssim \frac{\|w^*\| \sqrt{\log(1/\delta)}}{\sqrt{n}}
   \]

2. assuming exponential decay of the spectrum of \( \Sigma \) and choosing \( \lambda \approx \frac{1}{\|w^*\|} n^{-1/2} \), \( m \gtrsim \log^2 n \) then:
   \[
   L(\hat{\beta}_\lambda) - L(w^*) \lesssim \frac{\|w^*\| \sqrt{\log(1/\delta)}}{\sqrt{n}}
   \]
E Proof of Theorem 4

Theorem 4 is the content of Theorem 3 and Corollary 3.

Theorem 8. Fix $\lambda > 0$ and $0 < \delta < 1$. Under Assumptions 3, 4, 5, and a polynomial decay of the spectrum of $\Sigma$ with rate $1/p \in (1, \infty)$, as in (14), then, with probability at least $1 - 2\delta$,

$$\lambda \| \hat{\beta}_{\lambda,m} \|^2 + L(\hat{\beta}_{\lambda,m}) - L(w_*) \leq 9\lambda \| w_* \|^2 + 18G\| w_* \| \sqrt{\alpha} + K\left( \frac{1}{\lambda^p n} \right) + 216V \frac{\log(3/\delta)}{n} + 15 \log(3/\delta) \frac{B + G\| w_* \|}{n}$$

provided that $n \geq 1655\kappa^2 + 223\kappa^2 \log \frac{4\kappa^2}{\delta}$ and $m$ satisfies (40) (uniform sampling) or (41) (ALS sampling), and where $\ell$ can be clipped at $M > 0$, $B, V > 0$ come from the supremum bound (18) and variance bound (19) respectively, and $K \geq 1$ is a constant only depending on $p, M, B, \theta$ and $V$.

Proof. The proof mimics the proof of Theorem 9 where in (67) we choose

$$w_0 = P_m w_* \quad B_0 := B + G\| w_* \|,$$

since

$$\ell(y, P_{B_m} w_*) \leq B + G\| P_m w_* \| \leq B + G\| w_* \|.$$

Hence (67) with $\theta = 1$ reads

$$\lambda \| \hat{\beta}_{\lambda,m} \|^2 + L(\hat{\beta}_{\lambda,m}) - L(w_*) \leq 9\lambda \| P_m w_* \|^2 + L(P_m w_*) - L(w_*) + K\frac{a^{2p}}{\lambda^p n} + 216V \frac{\log(3/\delta)}{n} + 15B \frac{\log(3/\delta) B + G\| w_* \|}{n}$$

$$\leq 9\lambda \| w_* \|^2 + 9(L(P_m w_*) - L(w_*)) + K\frac{a^{2p}}{\lambda^p n} + 216V \frac{\log(3/\delta)}{n} + 15 \log(3/\delta) \frac{B + G\| w_* \|}{n}$$

(55)

We can deal with the term $L(P_m w_*) - L(w_*)$ as in (17), so that

$$L(P_m w_*) - L(w_*) \leq 2G\sqrt{\alpha} \| w_* \| \leq 2G\sqrt{\alpha} \| w_* \|$$

Hence, with probability at least $1 - 2\delta$

$$\lambda \| \hat{\beta}_{\lambda,m} \|^2 + L(\hat{\beta}_{\lambda,m}) - L(w_*) \leq 9\lambda \| w_* \|^2 + 18G\sqrt{\alpha}\| w_* \| + K\left( \frac{a^{2p}}{\lambda^p n} \right) \frac{1}{\lambda^{p-p+\frac{1}{\kappa}}} + 216V \frac{\log(3/\delta)}{n} + 15 \log(3/\delta) \frac{B + G\| w_* \|}{n}$$

(57)

which proves the claim. $\square$

The following corollary provides the optimal rates, whose proof is the same as for Corollary 3.

Corollary 3. Fix $\delta > 0$. Under the Theorem 3 set

$$\lambda \asymp n^{-\frac{1}{1+p}}$$

(58)

$$\alpha \asymp n^{-\frac{1}{1+p}}$$

(59)

$$m \gtrsim n^{\frac{2p}{1+p}} \log n$$

(60)

then, for ALS sampling, with probability at least $1 - 2\delta$

$$\lambda \| \hat{\beta}_{\lambda,m} \|^2 + L(\hat{\beta}_{\lambda,m}) - L(w_*) \lesssim \| w_* \|(\log n)^{1/2p} \left( \frac{1}{n} \right)^{\frac{1}{1+p}}$$

(61)
F Proof of Theorem 5

Theorem 5 is the content of Theorem 9 and Corollary 4.

Theorem 9. Fix $\lambda > 0$ and $0 < \delta < 1$. Under Assumptions 3, 4, and 5, and a polynomial decay of the spectrum of $\Sigma$ with rate $1/p \in (1, \infty)$, as in (14), then with probability at least $1 - 2\delta$,

$$\lambda \|\hat{\beta}_{\lambda,m}\|^2_H + L(\hat{\beta}_{\lambda,m}) - L(f_*) \leq 9A(\lambda) + 18G(\frac{\alpha A(\lambda)}{\lambda}) + K(\frac{1}{\lambda^{p-H}})^{\frac{1}{2-p+\lambda}} +$$

$$+ 3 \left( \frac{72V \log(3/\delta)}{n} \right)^{\frac{1}{2-p}} + \frac{15B \log(3/\delta)}{n} + \frac{15Gk \log(3/\delta)}{n} \sqrt{\frac{A(\lambda)}{\lambda}}$$

provided that $n \geq 165\kappa^2 + 223\kappa^2 \log \frac{4\epsilon^2}{\delta}$ and $m$ satisfies (49) (uniform sampling) or (41) (ALS sampling), and where $\ell$ can be clipped at $M > 0$, $B > 0$ and $\theta \in [0, 2]$ come from the supremum bound (23) and variance bound (24), respectively, and $K \geq 1$ is a constant only depending on $p$, $M$, $B$, $\theta$ and $V$.

Proof. We adapt the proof of Theorem 7.23 in [48] to $\hat{\beta}_{\lambda,m}$. Set

$$r_H^* = \inf_{w \in H} \lambda \|w\|^2 + L(w^{cl}) - L(f_*)$$

(63)

$$r_{B_m}^* = \inf_{w \in B_m} \lambda \|w\|^2 + L(w^{cl}) - L(f_*)$$

(64)

$$\mathcal{H}_r = \{ w \in H : \lambda \|w\|^2 + L(w^{cl}) - L(f_*) \leq r \} \quad r > r_H^*$$

(65)

$$(B_m)_r = \{ w \in B_m : \lambda \|w\|^2 + L(w^{cl}) - L(f_*) \leq r \} \quad r > r_{B_m}^*$$

(66)

(see Eq. (7.32)-(7.33) in [48]). Let’s notice that $r_{B_m}^* \geq r_H^*$, which means that $(B_m)_r \subseteq \mathcal{H}_r$. As a consequence, using also Theorem 15 in [49] stating that the decay condition (14) of the spectrum of the covariance operator $\Sigma$ is equivalent to the polynomial decay of the (dyadic) entropy numbers $e_j$ (see Lemma 7), we have that, analogously to the proof of Theorem 7.23 in [48] (see Lemma 7.17 and eq. (A.36) in [48] for details):

$$\mathbb{E}[\ell(y, (B_m)_r \rightarrow L_2(\hat{P}_H))] \leq \mathbb{E}[\ell(y, (\mathcal{H}_r \rightarrow L_2(\hat{P}_H))] \leq 2 \left( \frac{r}{\lambda} \right)^{1/2} a_j^{-1/2}$$

where the first inequality is a consequence of $(B_m)_r \subseteq \mathcal{H}_r$ and $\hat{P}_H \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ is the empirical (marginal) measure.

Furthermore $\hat{\beta}_{\lambda,m}$ is a clipped regularized empirical risk minimizer over $B_m$ (see Definition 7.18 in [48]) since

$$\lambda \|\hat{\beta}_{\lambda,m}\|^2 + \hat{L}(\hat{\beta}_{\lambda,m}) \leq \lambda \|\hat{\beta}_{\lambda,m}\|^2 + \hat{L}(\hat{\beta}_{\lambda,m}) = \inf_{\beta \in B_m} [\lambda \|w\|^2 + \hat{L}(\ell)]$$

Then, applying Theorem 7.23 in [48] with probability at least $1 - \delta$:

$$\lambda \|\hat{\beta}_{\lambda,m}\|^2 + L(\hat{\beta}_{\lambda,m}) - L(f_*) \leq 9A(\lambda) + 18G(\frac{\alpha A(\lambda)}{\lambda}) + K(\frac{1}{\lambda^{p-H}})^{\frac{1}{2-p+\lambda}} +$$

$$+ 3 \left( \frac{72V \log(3/\delta)}{n} \right)^{\frac{1}{2-p}} + \frac{15B \log(3/\delta)}{n} + \frac{15Gk \log(3/\delta)}{n} \sqrt{\frac{A(\lambda)}{\lambda}}$$

(67)

where $K \geq 1$ is a constant only depending on $p$, $M$, $B$, $\theta$, $V$, and $w_0 \in H$ is such that $\sup_{(x,y) \in H \times Y} |\ell(y, (w_0,x))| \leq B_0$ with $B_0 \geq B$.

We define $w_\lambda := \arg \min_{w \in H} L(w) + \lambda \|w\|^2$. Now, since

$$\ell(y, \mathcal{P}_B w_\lambda) \leq B + Gk \|\mathcal{P}_B w_\lambda\| \leq B + Gk \|w_\lambda\|,$$

$$\lambda \|w_\lambda\|^2_H \leq \lambda \|w_\lambda\|^2_H + L(w_\lambda) - L(f_*) = A(\lambda),$$

we can choose $B_0 := B + Gk \sqrt{\lambda^{-1}A(\lambda)}$ in theorem 7.23 in Steinwart and take $w_0 = \mathcal{P}_B w_\lambda$. We rewrite (67)
as:

$$\lambda \| \beta_{\lambda, m} \|^2 + L(\hat{\beta}_{\lambda, m}^d) - L(f_*) \leq 9(\lambda \| P_m w_\lambda \|^2 + L(P_m w_\lambda) - L(f_*)) + K\left(\frac{a^{2p}}{\lambda^p n}\right)^{\frac{1}{p-\frac{1}{p} + \beta p}} +$$

$$+ 3\left(\frac{72V \log(3/\delta)}{n}\right)^{\frac{1}{p-\frac{1}{p} + \beta p}} + \frac{15B \log(3/\delta)}{n} + \frac{15GK \log(3/\delta)}{n} \sqrt{\frac{A(\lambda)}{\lambda}}$$

We can deal with the term $L(P_m w_\lambda) - L(w_\lambda)$ as in [47], so that

$$L(P_m w_\lambda) - L(w_\lambda) \leq 2G\sqrt{\alpha} \| w_\lambda \| \leq 2G\sqrt{\alpha} \sqrt{\frac{A}{\lambda}}$$

$$\lambda \| \beta_{\lambda, m} \|^2 + L(\hat{\beta}_{\lambda, m}^d) - L(f_*) \leq 9A(\lambda) + 18G \sqrt{\frac{\alpha A(\lambda)}{\lambda}} + K\left(\frac{a^{2p}}{\lambda^p n}\right)^{\frac{1}{p-\frac{1}{p} + \beta p}} +$$

$$+ 3\left(\frac{72V \log(3/\delta)}{n}\right)^{\frac{1}{p-\frac{1}{p} + \beta p}} + \frac{15B \log(3/\delta)}{n} + \frac{15GK \log(3/\delta)}{n} \sqrt{\frac{A(\lambda)}{\lambda}}$$

which proves the first claim.

The following corollary provides the optimal rates.

**Corollary 4.** Fix $\delta > 0$. Under the Theorem 3 and the source condition

$$A(\lambda) \leq A_0 \lambda^r$$

for some $r \in (0, 1]$, set

$$\lambda \asymp n^{-\min\{\gamma, \frac{1}{\gamma}, \frac{1}{\gamma} - \frac{1}{r^{p+1} + \beta p + 1}\}} \quad (70)$$

$$\alpha \asymp n^{-\min\{2, \frac{1}{r^{p+1} + \beta p + 1}\}} \quad (71)$$

$$m \geq n^{-\min\{2p, \frac{p(r+1)}{r^{p+1} + \beta p + 1}\}} \quad (72)$$

for ALS sampling, with probability at least $1 - 2\delta$

$$\lambda \| \beta_{\lambda, m} \|^2 + L(\hat{\beta}_{\lambda, m}^d) - L(f_*) \lesssim \frac{\log^{1/2p} n}{n^{\min\{\frac{1}{r^{p+1} + \beta p + 1}\}}} \quad (73)$$

**Proof.** Lemma 4 with Lemma 5 gives

$$m \gtrsim d_{\alpha, 2} \log(n/\delta), \quad d_{\alpha, 2} \lesssim \alpha^{-p} \quad \alpha \gtrsim \frac{\log^{1/p}(n/\delta)}{n^{1/p}}$$

Lemma A.1.7 in [48] with $r = 2$, $1/\gamma = (2 - p - \theta + \theta p)$, $\alpha = p$, $\beta = r$ shows that the choice of $\lambda$, $\alpha$ and $m$ given by (70), (72) provides the optimal rate. $\square$

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G Effective Dimension and Eigenvalues Decay

In this section, we indicate bounds for $d_{\alpha, 2}$ defined by (12) when assuming respectively polynomial and exponential decay of the eigenvalues $\sigma_j(\Sigma)$ of $\Sigma$.

**Lemma 5** (Polynomial eigenvalues decay, Proposition 3 in [13]). If for some $\gamma \in \mathbb{R}^+$ and $1 < \beta < +\infty$

$$\sigma_i \leq \gamma i^{-\beta},$$

then

$$d_{\alpha, 2} \leq \frac{\gamma}{\beta} \frac{1}{\alpha^{1/\beta}}.$$ (75)

**Proof.** Since the function $\sigma/(\sigma + \alpha)$ is increasing in $\sigma$ and using the spectral theorem $\Sigma = UDU^*$ combined with the fact that $\text{Tr}(UDU^*) = \text{Tr}(U(U^*D)) = \text{Tr}D$

$$d_{\alpha, 2} = \text{Tr}(\Sigma(\Sigma + \alpha I)^{-1}) = \sum_{i=1}^{\infty} \frac{\sigma_i}{\sigma_i + \alpha} \leq \sum_{i=1}^{\infty} \frac{\gamma}{\gamma + i^\beta \alpha}$$ (76)

The function $\gamma/(\gamma + x^\beta \alpha)$ is positive and decreasing, so

$$d_{\alpha, 2} \leq \int_0^{\infty} \frac{\gamma}{\gamma + x^\beta \alpha} dx = \alpha^{-1/\beta} \int_0^{\infty} \frac{\gamma}{\gamma + \tau^\beta} d\tau \leq \frac{\beta}{\beta - 1} \frac{\gamma}{\beta} \frac{1}{\alpha^{1/\beta}}.$$ (77)

since $\int_0^{\infty} (\gamma + \tau^\beta)^{-1} \leq \beta/(\beta - 1)$. \hfill \qed

**Lemma 6** (Exponential eigenvalues decay). If for some $\gamma, \beta \in \mathbb{R}^+ \sigma_i \leq \gamma e^{-\beta i}$ then

$$d_{\alpha, 2} \leq \frac{\log(1 + \gamma/\alpha)}{\beta}.$$ (78)

**Proof.**

$$d_{\alpha, 2} = \sum_{i=1}^{\infty} \frac{\sigma_i}{\sigma_i + \alpha} = \sum_{i=1}^{\infty} \frac{1}{1 + \alpha/\sigma_i} \leq \sum_{i=1}^{\infty} \frac{1}{1 + \alpha'e^{\beta i}} \leq \int_0^{+\infty} \frac{1}{1 + \alpha' e^{\beta x}} dx$$ (79)

where $\alpha' = \alpha/\gamma$. Using the change of variables $t = e^{\beta x}$ we get

$$= \frac{1}{\beta} \int_1^{+\infty} \frac{1}{1 + \alpha' t} \frac{1}{t} dt = \frac{1}{\beta} \int_1^{+\infty} \left[ \frac{1}{t} - \frac{\alpha'}{1 + \alpha' t} \right] dt = \frac{1}{\beta} \left[ \log t - \log(1 + \alpha' t) \right]_1^{+\infty} = \frac{1}{\beta} \left[ \log(1/\alpha') + \log(1 + \alpha') \right],$$ (80)

so

$$d_{\alpha, 2} \leq \frac{1}{\beta} \left[ \log(\gamma/\alpha) + \log(1 + 1/\alpha') \right] = \frac{\log(1 + \gamma/\alpha)}{\beta}.$$ (81)

The following result is the content of Theorem 15 in [49]. Given a bounded operator $A$ between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, denote by $e_j(A)$ the entropy numbers of $A$ and by $\hat{P}_H = \frac{1}{n} \sum_{i=1}^{n} \delta_x$ the empirical (marginal) measure associated with the input data $x_i, \ldots, x_n$. Regard the data matrix $\hat{X}$ as the inclusion operator $\text{id} : \mathcal{H} \to L_2(\hat{P})$

$$(\text{id} w)(x_i) = \langle w, x_i \rangle \quad i = 1, \ldots, n$$

**Lemma 7.** Let $p \in (0, 1)$. Then

$$\mathbb{E}_{\hat{P}}[e_j(\text{id} : \mathcal{H} \to L_2(\hat{P}))] \sim j^{-\frac{1}{p}}$$ (82)

if and only if

$$\sigma_j(\Sigma) \sim j^{-\frac{1}{p}}$$ (83)
H Constrained problem

In this section we investigate the so called constrained problem. As [9] the hypothesis space is still the subspace \( B_m \subseteq H \) spanned by \( \{\tilde{x}_1, \ldots, \tilde{x}_m\} \) with \( \{\tilde{x}_1, \ldots, \tilde{x}_m\} \) being the sampled input points and the empirical estimator is the minimizer of ERM on the ball of radius \( R \) belonging to the subspace \( B_m \). More precisely, for any \( R > 0 \) we set

\[
\hat{\beta}_{R,m} = \arg\min_{w \in B_m, \|w\| \leq R} \hat{L}(w)
\]  

(84)

For sake of simplicity we assume the best in model to exist. We start presenting the finite sample error bounds for uniform and approximate leverage scores subsampling of the \( m \) points.

**Theorem 10.** Fix \( R > 0, \alpha > 0, 0 < \delta < 1 \). Under Assumptions 1, 2, 3 with probability at least \( 1 - \delta \)

\[
L(\hat{\beta}_{R,m}) - L(f_H) \leq \frac{2GR\kappa}{\sqrt{n}} \left( 2 + 2\sqrt{2 \log(1/\delta)} \right) + 2GR\sqrt{\alpha}
\]

(85)

provided that \( R \geq \|w_*\|, n \geq 1655\kappa^2 + 223\kappa^2 \log \frac{4\kappa^2}{\delta} \) and \( m \) satisfies

1. for uniform sampling

\[
m \geq 67 \log \frac{4\kappa^2}{\delta} \vee 5d_{\alpha,\infty} \log \frac{4\kappa^2}{\delta}
\]

(86)

2. for ALS sampling and \( T \)-approximate leverage scores with subsampling probabilities \( Q_{\alpha} \), \( t_0 > \frac{19\alpha^2}{n} \log \frac{4n}{\delta} \)

\[
m \geq 334 \log \frac{16n}{\delta} \vee 78T^2d_{\alpha,2} \log \frac{16n}{\delta}
\]

(87)

where \( \alpha \geq \frac{19\alpha^2}{n} \log \frac{4n}{\delta} \).

Under the above condition, with the choice \( \alpha = 1/n \), the estimator achieves the optimal bound

\[
L(\hat{\beta}_{R,m}) - L(f_H) \leq \frac{2GR\kappa}{\sqrt{n}} \left( 2 + 2\sqrt{2 \log(1/\delta)} \right) + 2GR\frac{1}{\sqrt{n}}
\]

(88)

\[
= R\sqrt{\log(1/\delta)} O \left( \frac{1}{\sqrt{n}} \right)
\]

**Proof.** We decompose the excess risk of \( \hat{\beta}_{R,m} \) with respect to the target \( w_* \)

\[
L(\hat{\beta}_{R,m}) - L(w_*) = L(\hat{\beta}_{R,m}) - \hat{L}(\hat{\beta}_{R,m}) + \hat{L}(\hat{\beta}_{R,m}) - \hat{L}(P_m w_*) + \hat{L}(P_m w_*) - L(P_m w_*) + L(P_m w_*) - L(w_*)
\]

(89)

\[
\leq 2 \sup_{w \in B_m, \|w\| \leq R} \left( L(w) - \hat{L}(w) \right) + L(P_m w_*) - L(w_*)
\]

where \( \|P_m w_*\| \leq R \) since \( R \geq \|w_*\| \).

**Bound for the term A:**

Term A is bounded by Lemma 1 so that with probability at least \( 1 - \delta \)

\[
A \leq \frac{GR\kappa}{\sqrt{n}} \left( 2 + 2\sqrt{2 \log(1/\delta)} \right).
\]

(90)
Bound for term B:
Term B is bounded as Term C in the proof of Theorem 7 see (47)

\[ B \leq G||\Sigma^{1/2}(I-P_m)||||w_a|| \leq GR||\Sigma^{1/2}(I-P_m)|| \]  

(91)

and we estimate \[ ||\Sigma^{1/2}(I-P_m)|| \] using Lemma 3 for uniform sampling and Lemma 4 for ALS selection. ∎

Again, bound 88 provides a convergence rate, which is optimal from a statistical point of view, but that requires at least \( m \sim n \log n \) subsampled points since, without further assumptions the effective dimension \( d_{\alpha,2} \), as well as \( d_{\alpha,\infty} \), can in general be bounded only by \( \kappa^2/\alpha \). Clearly, this makes the approach completely useless. As for the regularized estimator, to overcome this issue we are forced to assume fast decay of the eigenvalues of the covariance operator \( \Sigma \), as in [4]. Under this condition the following results – whose proof is identical to the proof of Proposition 2 shows that the optimal rate can be achieved with an efficient computational cost at least for ALS.

Corollary 5. Under the condition of Theorem 10

1. If \( \Sigma \) has a polynomial decay, i.e. for some \( \gamma \in \mathbb{R}^+ \), \( p \in (0,1) \),

\[ \sigma_j(\Sigma) \leq \gamma j^{-\frac{1}{p}} \]

then, with probability at least \( 1 - \delta \)

\[ L(\hat{\beta}_{R,m}) - L(w_a) \leq R\sqrt{\frac{\log(1/\delta)}{n}} + R\sqrt{\frac{\log(1/\delta)}{m^{1/p}}} = R\sqrt{\log(1/\delta)} \cdot O\left(\frac{\log^{1/p} n}{\sqrt{n}}\right) \]  

(92)

with \( m \gtrsim n^p \log n \) subsampled points according to ALS method.

2. If \( \Sigma \) has an exponential decay, i.e. for some \( \gamma, \beta \in \mathbb{R}^+ \),

\[ \sigma_j(\Sigma) \leq \gamma e^{-\beta j} \]

with probability at least \( 1 - \delta \):

\[ L(\hat{\beta}_{R,m}) - L(w_a) \leq R\sqrt{\frac{\log(1/\delta)}{n}} + R\sqrt{\frac{\log(1/\delta)}{m^{1/p}}} = R\sqrt{\log(1/\delta)} \cdot O\left(\frac{1}{\sqrt{m}}\right) \]  

(93)

with \( m \gtrsim \log^2 n \) subsampled points according to ALS method.

I Experiments: datasets and tuning

Here we report further information on the used data sets and the set up used for parameter tuning.

For Nyström SVM with Pegasos we tuned the kernel parameter \( \sigma \) and \( \lambda \) regularizer with a simple grid search (\( \sigma \in [0.1, 20] \), \( \lambda \in [10^{-8}, 10^{-1}] \), initially with a coarse grid and then more refined around the best candidates).

An analogous procedure has been used for K-SVM with its parameters \( C \) and \( \gamma \). The details of the considered data sets and the chosen parameters for our algorithm in Table 1 and 2 are the following:

**SUSY** (Table 1 and 2 \( n = 5 \times 10^6, d = 18 \)): we used a Gaussian kernel with \( \sigma = 4, \lambda = 3 \times 10^{-6} \) and \( m_{ALS} = 2500, m_{uniform} = 2500 \).

**Mnist binary** (Table 1 and 2 \( n = 7 \times 10^4, d = 784 \)): we used a Gaussian kernel with \( \sigma = 10, \lambda = 3 \times 10^{-6} \) and \( m_{ALS} = 15000, m_{uniform} = 20000 \).

**Usps** (Table 1 and 2 \( n = 9298, d = 256 \)): we used a Gaussian kernel with \( \sigma = 10, \lambda = 5 \times 10^{-6} \) and \( m_{ALS} = 2500, m_{uniform} = 4000 \).

**Webspam** (Table 1 and 2 \( n = 3.5 \times 10^5, d = 254 \)): we used a Gaussian kernel with \( \sigma = 0.25, \lambda = 8 \times 10^{-7} \) and \( m_{ALS} = 11500, m_{uniform} = 20000 \).

**a9a** (Table 1 and 2 \( n = 48842, d = 123 \)): we used a Gaussian kernel with \( \sigma = 10, \lambda = 1 \times 10^{-5} \) and \( m_{ALS} = 800, m_{uniform} = 1500 \).

**CIFAR** (Table 1 and 2 \( n = 6 \times 10^4, d = 400 \)): we used a Gaussian kernel with \( \sigma = 10, \lambda = 2 \times 10^{-6} \) and \( m_{ALS} = 20500, m_{uniform} = 20000 \).
Table 2: Comparison between ALS and uniform sampling. To achieve similar accuracy, uniform sampling usually requires larger $m$ than ALS sampling. Therefore, even if it does not need leverage scores computations, Nyström-Pegasos with uniform sampling can be more expensive both in terms of memory and time.

| Datasets | Nyström-Pegasos (ALS) | Nyström-Pegasos (Uniform) |
|----------|-----------------------|---------------------------|
|          | c-err | t train (s) | t pred (s) | c-err | t train (s) | t pred (s) |
| SUSY     | 20.0% ± 0.2% | 608 ± 2 | 134 ± 4 | 20.1% ± 0.2% | 592 ± 2 | 129 ± 1 |
| Mnist bin| 2.2% ± 0.1%  | 1342 ± 5| 491 ± 32| 2.3% ± 0.1%  | 1814 ± 8| 954 ± 21 |
| Usp5     | 3.0% ± 0.1%  | 19.8 ± 0.1| 7.3 ± 0.3| 3.0% ± 0.2%  | 66.1 ± 0.1| 48 ± 8 |
| Webspam  | 1.3% ± 0.1%  | 2440 ± 5| 376 ± 18| 1.3% ± 0.1%  | 4198 ± 40| 1455 ± 180 |
| a9a      | 15.1% ± 0.2% | 29.3 ± 0.2| 1.5 ± 0.1| 15.1% ± 0.2% | 30.9 ± 0.2| 3.2 ± 0.1 |
| CIFAR    | 19.2% ± 0.1% | 2408 ± 14| 820 ± 47| 19.0% ± 0.1% | 2168 ± 19| 709 ± 13 |

J Notation

For reader’s convenience we collect the main notation we introduced in the paper. 

Notation: We denote with the “hat”, e.g. $\hat{w}$, random quantities depending on the data. Given a linear operator $A$ we denote by $A^\top$ its adjoint (transpose for matrices). For any $n \in \mathbb{N}$, we denote by $\langle \cdot, \cdot \rangle_n$, $\| \cdot \|_n$ the inner product and norm in $\mathbb{R}^n$. Given two quantities $a, b$ (depending on some parameters), the notation $a \lesssim b$, or $a = O(b)$ means that there exists constant such that $a \leq Cb$. 

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