MORDELL-WEIL RANKS AND TATE-SHAFAREVICH GROUPS OF ELLIPTIC CURVES WITH MIXED-REDUCTION TYPE OVER CYCLOTOMIC EXTENSIONS

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Abstract. Let $E$ be an elliptic curve defined over a number field $K$ where $p$ splits completely. Suppose that $E$ has good reduction at all primes above $p$. Generalizing previous works of Kobayashi and Sprung, we define multiply signed Selmer groups over the cyclotomic $\mathbb{Z}_p$-extension of a finite extension $F$ of $K$ where $p$ is unramified. Under the hypothesis that the Pontryagin duals of these Selmer groups are torsion over the corresponding Iwasawa algebra, we show that the Mordell-Weil ranks of $E$ over a subextension of the cyclotomic $\mathbb{Z}_p$-extension are bounded. Furthermore, we derive an asymptotic formula of the growth of the $p$-parts of the Tate-Shafarevich groups of $E$ over these extensions.

1. Introduction

Let $p$ be a fixed odd prime number and $K \subset F$ be number fields. Let $\Sigma_p'$ and $\Sigma_p$ be the sets of primes of $K$ and $F$ above $p$ respectively. Throughout this article, we fix an elliptic curve $E/K$ which has good reduction at all primes of $\Sigma_p'$. We decompose $\Sigma_p'$ and $\Sigma_p$ into the ordinary and supersingular primes for $E$, namely $\Sigma_p' = \Sigma_{ss} \cup \Sigma_{ord}$ and $\Sigma_p = \Sigma_{ss} \cup \Sigma_{ord}$. Without further notice, we assume throughout the article that the following hypotheses hold:

(S1) The prime $p$ splits completely in $K/\mathbb{Q}$ and is unramified in $F/\mathbb{Q}$;
(S2) The set $\Sigma_{ss}$ is non-empty.

Denote by $K_\infty$ the cyclotomic $\mathbb{Z}_p$-extension of $K$. For $n \geq 0$, let $K_n$ denote the unique sub-extension of $K_\infty/K$ with $[K_n : K] = p^n$. Similarly, we write $F_\infty$ for the cyclotomic $\mathbb{Z}_p$-extension of $F$ and $F_n$ for the unique sub-extension of $F_\infty/F$ with $[F_n : F] = p^n$. In view of assumption (S1), we have $K_\infty \cap F_n = K_n$ for every $n$. For each $w \in \Sigma_p$, the unique place of $F_\infty$ lying above $w$ will again be denoted by $w$. For each $w \in \Sigma_{ss}$, we define two local conditions $E^w(F_\infty,w)$ and $E^w(F_\infty,w)$, generalizing works of Kobayashi [Kob03] and Sprung [Spr12]. This allows us to define $\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$ by writing $\lambda_\infty$ for the $\mu_\infty$- and $\lambda$-invariants of $\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$. In the main body of the article, we will introduce an additional hypothesis affirming that the Pontryagin duals of the multiply signed Selmer groups are $\Lambda$-torsion (labelled (S3)). We shall write $\mu_\infty$ and $\lambda_\infty$ for the $\mu_\infty$- and $\lambda$-invariants of $\mathbb{Z}_p[[\mathrm{Gal}(F_\infty/F)]]$.

Our first result is a uniform bound on the Mordell-Weil ranks of $E$ over $F_n$ as $n$ grows.

Theorem A. Under hypotheses (S1)-(S3), $\text{rank}_E(F_n)$ is bounded independently of $n$.

When $F/\mathbb{Q}$ is an abelian extension, one may obtain this result using Kato’s Euler system from [Kat04] (together with the non-vanishing of the $L$-values of $E$ proved by Rohrlich [Roh88]). Our method does not assume the existence of an Euler system and relies on the cotorsionness of the multiply signed Selmer groups instead. See also [Kob03 Corollary 10.2], [LP20 Theorem 3.4], [LLZ17 Proposition 5.4] and [LS20 Theorem 1.1] for similar results.

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Given a finite $p$-group $M$, we write $e(M)$ for the integer given by $|M| = p^{e(M)}$. The second result of our article is about the growth of $p$-parts of Tate-Shafarevich groups of $E$ over $F_n$ (that is, $e(\text{III}_p(E/F_n))$, as $n$ grows.

**Theorem B.** Suppose that the hypotheses (S1)-(S3) hold. Furthermore, suppose that $\text{III}_p(E/F_n)$ is finite for all $n$. Then, there exist a choice of $\tilde{\sigma}$ and $\tilde{\tau}$ in $\{\sharp, \flat\}^{\Sigma_{ss}}$ such that

$$e(\text{III}_p(E/F_n)) - e(\text{III}_p(E/F_{n-1})) = \begin{cases} S(\tilde{\sigma}, n) + \phi(p^n)\mu_{\tilde{\sigma}} + \lambda_{\tilde{\sigma}} - r_\infty & \text{if } n \text{ is odd,} \\ T(\tilde{\tau}, n) + \phi(p^n)\mu_{\tilde{\tau}} + \lambda_{\tilde{\tau}} - r_\infty & \text{if } n \text{ is even.} \end{cases}$$

for all $n \gg 0$, where $r_\infty = \lim_{n \to \infty} \text{rank}_Z E(F_n)$, $S(\tilde{\sigma}, n)$ and $T(\tilde{\tau}, n)$ are certain linear combinations of $p^i$, $i \leq n$, which we define explicitly in Proposition 5.3 and $\phi$ is the Euler totient function.

This generalizes results of Kurihara, Kobayashi and Pollack for elliptic curves defined over $\mathbb{Q}$ with $a_p = 0$ (see [Kur02, Theorem 0.1], [Kob03, Theorem 1.4] and [Pol05, Theorem 1.1]) as well as Sprung’s result for general $a_p$ (see [Spr13, Theorem 1.1]).

**Remark 1.1.** In the case where $a_v = 0$ for all $v$ in $\Sigma_{ss}$, the formula of Theorem B simplifies to

$$e(\text{III}_p(E/F_n)) - e(\text{III}_p(E/F_{n-1})) = \begin{cases} \sum_{v \in \Sigma_{ss}} |F_v : \mathbb{Q}_p| (p^n - p^{n-1} - p^{n-2} - \cdots - 1) + \phi(p^n)\mu_{\tilde{\sigma}} + \lambda_{\tilde{\sigma}} - r_\infty & \text{if } n \text{ is odd,} \\ \sum_{v \in \Sigma_{ss}} |F_v : \mathbb{Q}_p| (p^n - p^{n-1} - p^{n-2} - \cdots - 1) + \phi(p^n)\mu_{\tilde{\tau}} + \lambda_{\tilde{\tau}} - r_\infty & \text{if } n \text{ is even,} \end{cases}$$

where $\tilde{\sigma}$ denotes the constant vector $(\ast)_v$ for $v \in \{\sharp, \flat\}$. In particular, the vectors $\tilde{\sigma}$ and $\tilde{\tau}$ in the statement of Theorem B are given by $\tilde{\flat}$ and $\tilde{\sharp}$ respectively.

If furthermore $\Sigma_p = \Sigma_{ss}$, the term $\sum_{v \in \Sigma_{ss}} |F_v : \mathbb{Q}_p|$ becomes $|F : \mathbb{Q}|$. In this case, under certain hypotheses on the vanishing of the Mordell-Weil ranks and the behaviour of $\text{III}_p(E/F)$, Iovita and Pollack [IP06, Theorem 5.1] showed that the quantities $|F : \mathbb{Q}| (p^n - p^{n-1} - \cdots)$ describe precisely the growth of the Tate-Shafarevich groups of $E$ over finite extensions inside a $\mathbb{Z}_p$-extension of $F$ (which is not necessarily cyclotomic). It would be interesting to study whether our techniques can be extended to the setting of $\mathbb{Z}_p$, which may allow us to relax some of the hypotheses in loc. cit. We plan to study this in the near future.

The structure of the paper is as follows. In §2, we review the local theory of Coleman maps and logarithmic matrices for elliptic curves with supersingular reduction at $w \in \Sigma_{ss}$. This allows us to define the multiply signed Selmer groups. Along the way, we prove a result on the image of the direct sum of two Coleman maps (Proposition 2.2), which may be of independent interest. After giving the definition of multiply signed Selmer groups, we prove a number of results on the structure of global cohomology groups under the hypothesis that these Selmer groups are cotorsion. In §3, we first review the definition of Kobayashi ranks on projective systems of $\mathbb{Z}_p$-modules. We then prove a number of preliminary results on Kobayashi ranks of certain modules that will be used later on in the article. In §4, we make the link between Coleman maps and Kobayashi ranks and explain how this allows us to study the growth of certain local modules. We treat the ordinary and supersingular cases separately. Our treatment in the supersingular case follows closely [Spr13, LLZ17]. Finally, we put everything together to prove Theorems A and B in §5.

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1.1. Notation. Throughout this article, \( T \) denotes the \( p \)-adic Tate module of \( E \).

The Galois groups of \( F_\infty/F \) and \( F_\infty/F_n \) are denoted by \( \Gamma \) and \( \Gamma_n \) respectively. We fix once and for all a topological generator \( \gamma \) of \( \Gamma \). Recall from our earlier discussion that we write the unique prime of \( F_\infty \) above a prime \( w \in \Sigma_p \) by \( w \) as well. In particular, it follows that \( F_{\infty,w} \) is the cyclotomic \( \mathbb{Z}_p \)-extension of \( F_w \) with \( F_{n,w} \) as its intermediate subfields. Therefore, we may and will identify the Galois groups of \( F_{\infty,w}/F_w \) and \( F_{\infty,w}/F_{n,w} \) with \( \Gamma \) and \( \Gamma_n \) respectively. Let \( \Lambda \) denote the Iwasawa algebra \( \mathbb{Z}_p[[\Gamma]] \), which we shall identify with the power series ring \( \mathbb{Z}_p[[X]] \) by sending \( \gamma - 1 \) to \( X \).

We write \( \mathcal{H} \) for the ring of distribution algebra on \( \Gamma \), which can be realized as the set of power series in \( \mathbb{Q}_p[[X]] \) that converge on the open unit disc. Let \( G \) be an element in \( \Lambda \) or \( \mathcal{H} \). We shall identify it with a power series in \( \mathbb{Q}_p[[X]] \) (which will again be denoted by \( G \)). Given a character \( \theta \) on \( \Gamma \), we evaluate \( G \) at \( \theta \) via \( G(\theta) = G(\theta(\gamma) - 1) \).

Given a ring \( \mathcal{O} \) that contains \( \mathbb{Z}_p \), we shall write \( \Lambda \mathcal{O} \) for the tensor product \( \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O} \). We may evaluate an element of \( \Lambda \mathcal{O} \) at a character of \( \Gamma \) as before.

For all integers \( n \geq 1 \), we write \( \omega_n = (1 + X)^{p^n} - 1 \) and \( \Phi_n = \omega_n/\omega_{n-1} \) (with \( \omega_0 = X \)). We let \( \Lambda_n \) denote the quotient \( \Lambda / (\omega_n) = \Lambda_{F,v} \). Furthermore, we fix a primitive \( p^n \)-th root of unity \( \zeta_{p^n} \) and \( \epsilon_n = \zeta_{p^n} - 1 \). We shall also assume that the primitive roots of unity are chosen such that \( \epsilon_n = \zeta_{p^n} - 1 \).

2. Multiply signed Selmer groups

Throughout this section, we fix a prime \( v \in \Sigma_p \) and a prime \( w \in \Sigma_p \) lying above \( v \). By (S1), we may identify \( K_v \) with \( \mathbb{Q}_p \) and \( F_w \) with a finite unramified extension of \( \mathbb{Q}_p \). Let \( \mathcal{O}_w \) denote the ring of integers of \( F_w \).

2.1. Coleman maps at supersingular primes

In this subsection, we shall further assume that \( w \in \Sigma_{ss} \).

We then write \( D_{\text{cris},v}(T) \) and \( D_{\text{cris},w}(T) \) for the Dieudonné modules of \( T \mid_{G_{K_v}} \) and \( T \mid_{G_{F_w}} \) respectively. Recall that \( D_{\text{cris},v}(T) \) is a filtered \( \mathbb{Z}_p \)-module of rank 2 and \( D_{\text{cris},v}(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) is equipped with a linear operator \( \varphi \). Furthermore, \( D_{\text{cris},w}(T) = \mathcal{O}_w \otimes_{\mathbb{Z}_p} D_{\text{cris},v}(T) \) and \( \varphi \) acts semi-linearly on \( D_{\text{cris},w}(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \). That is \( \varphi(x \otimes u) = x^{\sigma_w} \otimes \varphi(u) \) for \( x \in \mathcal{O}_w \) and \( u \in D_{\text{cris},v}(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \), where \( \sigma_w \) is the Frobenius of \( F_w/F'_w \). Let \( a_v = 1 + p - |\tilde{E}_v(k_v)| \in \mathbb{Z}_p \), where \( \tilde{E}_v \) is the reduced curve of \( E \) modulo \( v \) and \( k_v \) is the residue field of \( K_v \). The characteristic polynomial of \( \varphi \) on \( D_{\text{cris},v}(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) is given by \( X^2 - \frac{a_v}{p} X + \frac{1}{p} \). By the theory of Fontaine-Laffaille, \( D_{\text{cris},v}(T) \) admits a \( \mathbb{Z}_p \)-basis of the form \( \{ \varphi_v, \varphi(\varphi_v) \} \), where \( \varphi_v \) generates \( \text{Fil}^0 D_{\text{cris},v}(T) \).

The matrix of \( \varphi \) with respect to this basis is of the form

\[
A_v := \begin{pmatrix}
0 & -1 \\
1 & \frac{1}{a_v} 
\end{pmatrix}, \quad M_v = \begin{pmatrix}
A_v & 1 \\
-1 & 0
\end{pmatrix}
\]

For \( n \geq 1 \), let

\[
C_{v,n} = \begin{pmatrix}
1 & 0 \\
0 & \Phi_n
\end{pmatrix}, \quad M_{v,n} = A_v^{n+1} C_{v,n} \cdots C_{v,1}.
\]

By [Spr17 Lemma 4.4] (see also [Lei17 Theorem 1.5]), the matrices \( M_{v,n} \) converge to a \( 2 \times 2 \) matrix over \( \mathcal{H} \) as \( n \to \infty \). We then define

\[
M_{\text{log},v} := \lim_{n \to \infty} M_{v,n}.
\]

Let \( H^1_\varphi(K_{\infty,v},T) \) denote the inverse limit \( \lim_{n \to \infty} H^1(K_{n,v},T) \), where the connecting maps are corestrictions. Let \( \mathcal{L}_v : H^1_{\text{Fil}}(K_{\infty,v},T) \to \mathcal{H} \otimes D_{\text{cris},v}(T) \) be the Perrin-Riou map as given by [BLLV19 §5.1] (originally defined in [PR99]). The matrix \( M_{\text{log},v} \) allows us to factorize \( \mathcal{L}_v \) into

\[
\mathcal{L}_v = \begin{pmatrix}
\varphi(\varphi_v) \\
\varphi_v
\end{pmatrix} M_{\text{log},v} \begin{pmatrix}
\text{Col}_1 \circ \mathcal{L}_v \\
\text{Col}_0 \circ \mathcal{L}_v
\end{pmatrix}
\]
where \( \text{Col}_{\sharp,v}, \text{Col}_b,v : H^1_w(K_{\infty,v}, T) \to \Lambda \) are \( \Lambda \)-morphisms as given in [BL19 §2.3]. We would like to describe the images of the Coleman maps. As a start, we recall the following preliminary lemma due to Kobayashi (see [Kob03 proof of Proposition 8.23]).

**Lemma 2.1.** The corestriction map \( H^1(K_{m,v}, T) \to H^1(K_{n,v}, T) \) is surjective for all \( m \geq n \).

**Proof.** It is well-known that in this supersingular setting, one has \( H^0(K_{m,v}, E[p^\infty]) = 0 \) (cf. [Kob03 Proposition 8.7] or [KO18 Proposition 3.1]). From this fact, we then see that the restriction map

\[
H^1(K_{n,v}, E[p^\infty]) \to H^1(K_{m,v}, E[p^\infty])
\]

is injective. The required conclusion now follows from this and the local Tate duality.

**Proposition 2.2.** Let \( I_v := \{ (G_1, G_2) \in \Lambda^\oplus_2 : (p - 1)G_1(0) = (2 - a_v)G_2(0) \} \). Then

\[
\text{Im}(\text{Col}_{\sharp,v} \oplus \text{Col}_b,v) = I_v.
\]

**Proof.** By [LLZ11 Corollary 5.3 and Theorem 5.10], we have the inclusion

\[
\text{Im}(\text{Col}_{\sharp,v} \oplus \text{Col}_b,v) \subset I_v,
\]

with finite index. By Nakayama’s Lemma and the surjectivity of the corestriction maps as given by Lemma 2.1, it is enough to show that

\[
\text{Im}(\text{Col}_{\sharp,v} \oplus \text{Col}_b,v) \text{ mod } X = \{ (g_1, g_2) \in \mathbb{Z}_p^2 : (p - 1)g_1 = (2 - a_v)g_2 \}.
\]

Let us recall that

\[
L_v \equiv (1 - \varphi)(1 - p^{-1}\varphi^{-1})^{-1}\exp^* \mod X
\]

(see [LZ14 Theorem B.5] for example). Thus, on combining (2.2) and (2.3), we have

\[
\exp^* = \frac{1}{1 + p - a_v} (\omega_v \varphi(\omega_v)) \left( a_v - 2p \frac{a_v - 2p}{p - 1} \frac{1 - p}{a_v - 2} \right) \left( \text{Col}_{\sharp,v} \oplus \text{Col}_b,v \right) \mod X
\]

(see [HL19 proof of Proposition 2.12]). This in turn implies that

\[
(p - 1)\text{Col}_{\sharp,v} = (2 - a_v)\text{Col}_b,v \mod X
\]

and

\[
\exp^* = \omega_v \frac{(2 - a_v)(a_v - 2p) - (p - 1)^2}{p(p - 1)(1 + p - a_v)} \text{Col}_{\sharp,v},
\]

where we note that \( \frac{(2 - a_v)(a_v - 2p) - (p - 1)^2}{p(p - 1)(1 + p - a_v)} \in \mathbb{Z}_p^\times \).

It follows from [BK90 Theorem 4.1(iii)] that

\[
[\exp(\mathcal{D}_{\text{cris},v}(T)/\text{Fil}^0 \mathcal{D}_{\text{cris},v}(T)) : H^1_{\text{cris}}(K_v, T)] = |\det(1 - \varphi)|_p = p,
\]

where \( \exp \) is the Bloch-Kato exponential map and \( | \cdot |_p \) is the \( p \)-adic norm normalized by \( |p|_p = p^{-1} \). It then follows that the dual exponential map gives an isomorphism

\[
\exp^* : H^1_{\text{cris}}(K_v, T) \cong \frac{1}{p} \mathbb{Z}_p\omega_v.
\]

Combining (2.6) and (2.7) yields

\[
\text{Im}(\text{Col}_{b,v}) = \mathbb{Z}_p.
\]

This, together with (2.5), allow us to deduce (2.3).

After tensoring by \( \mathcal{O}_w \), we have similarly

\[
\mathcal{L}_w = (\omega_v \varphi(\omega_v)) M_{\log,v} \left( \text{Col}_{\sharp,w} \oplus \text{Col}_b,w \right),
\]

where \( \mathcal{L}_w \) is the Perrin-Riou map from \( H^1_{1w}(K_{\infty,w}, T) \to H^1_{1w}(K_{\infty,v}, T) \otimes \mathcal{O}_w \) to \( \mathcal{H} \otimes \mathcal{D}_{\text{cris},w}(T) \) and \( \text{Col}_{\sharp,w} \) and \( \text{Col}_b,w \) are defined by extending \( \text{Col}_{\sharp,v} \) and \( \text{Col}_b,v \) \( \mathcal{O}_w \)-linearly.
Remark 2.3. We have automatically $\text{Im}(\text{Col}_{\Sigma,w} \oplus \text{Col}_{\Sigma,w}) = I_{v} \otimes O_{w}$.

2.2. Selmer groups. Let $w \in \Sigma_{\text{ss}}$. Consider the local Tate pairing

$$H_{1}^{I}(F_{\infty,w}, T) \times H_{1}^{I}(F_{\infty,w}, E[p^{\infty}]) \to \mathbb{Q}_{p}/\mathbb{Z}_{p}.$$  

For $\bullet \in \{\#, \circ\}$, we define $H_{1}^{\bullet}(F_{\infty,w}, E[p^{\infty}]) \subset H_{1}^{I}(F_{\infty,w}, E[p^{\infty}])$ to be the orthogonal complement of $\text{ker} \text{Col}_{\bullet,w}$ under the local Tate pairing. We shall write

$$H_{1}^{\bullet}(F_{\infty,w}, E[p^{\infty}]) = \frac{H_{1}^{I}(F_{\infty,w}, E[p^{\infty}])}{H_{1}^{\bullet}(F_{\infty,w}, E[p^{\infty}])}.$$  

Similarly, if $v \in \Sigma_{p}$, we write

$$H_{1}^{\bullet}(F_{\infty,v}, E[p^{\infty}]) = \frac{H_{1}^{I}(F_{\infty,v}, E[p^{\infty}])}{E(F_{\infty,v}) \otimes \mathbb{Q}_{p}/\mathbb{Z}_{p}},$$  

where $E(F_{\infty,w}) \otimes \mathbb{Q}_{p}/\mathbb{Z}_{p}$ is identified with its image inside $H_{1}^{I}(F_{\infty,w}, E[p^{\infty}])$ under the Kummer map.

Let $s = (s_{w}, w \in \Sigma_{\text{ss}}) \in \{0, 1\}^{\Sigma_{\text{ss}}}$, we define the signed Selmer group of $E$ over $F_{\infty}$ by

$$\text{Sel}^{\bar{s}}(E/F_{\infty}) := \ker \left(H_{1}^{I}(F_{\infty}, E[p^{\infty}]) \to \prod_{w \in \Sigma_{\text{ss}}} H_{1}^{f,s_{w}}(F_{\infty,w}, E[p^{\infty}]) \times \prod_{w \in \text{ord}} H_{1}^{I}(F_{\infty,w}, E[p^{\infty}]) \times \prod_{w \in \Sigma_{\text{ss}} \setminus \{p\}} H_{1}^{I}(F_{\infty,w}, E[p^{\infty}]) \right),$$  

where the last product runs through all primes of $F_{\infty}$ not dividing $p$. Equivalently, if $\text{Sel}_{p^{\infty}}(E/F_{\infty})$ denotes the classical $p^{\infty}$-Selmer group, then

$$\text{Sel}^{\bar{s}}(E/F_{\infty}) = \ker \left(\text{Sel}_{p^{\infty}}(E/F_{\infty}) \to \prod_{w \in \Sigma_{\text{ss}}} H_{1}^{f,s_{w}}(F_{\infty,w}, E[p^{\infty}]) \right).$$  

It is well-known that $\text{Sel}_{p^{\infty}}(E/F_{\infty})$ is cofinitely generated over $\Lambda$ (cf. [Man71] Theorem 4.5)). Thus, so is $\text{Sel}^{\bar{s}}(E/F_{\infty})$.

Conjecture 2.4. For all choices of $\bar{s}$, the Selmer group $\text{Sel}^{\bar{s}}(E/F_{\infty})$ is cotorsion over $\Lambda$.

For the rest of the article, we assume that the following hypothesis holds:

(S3) Conjecture [23] holds.

Definition 2.5. We write $\mu_{\bar{s}}$ and $\lambda_{\bar{s}}$ for the $\mu$- and $\lambda$-invariants of the torsion $\Lambda$-module $\text{Sel}^{\bar{s}}(E/F_{\infty})^{\vee}$.

When the elliptic curve $E$ has good ordinary reduction at all primes above $p$, the above conjecture is precisely Mazur’s conjecture [Maz72] which is known to be valid in the case when $E$ is defined over $\mathbb{Q}$ and $F$ an abelian extension of $\mathbb{Q}$ (see [Kat04]). For an elliptic curve over $\mathbb{Q}$ with good supersingular reduction at $p$, this conjecture was established by Kobayashi (cf. [Kab03]; also see [BL17] for some recent progress on this conjecture).

2.3. Structures of global cohomologies. In this section, we record certain consequences of Conjecture [23] which will be utilized in subsequent sections of the paper. From now on, let $\Sigma$ denote a fixed finite set of primes of $F$ containing those above $p$, the ramified primes of $F/K$, and all the bad reduction primes of $E$.

Write $F_{\Sigma}$ for the maximal algebraic extension of $F$ which is unramified outside $\Sigma$. For any (possibly infinite) extension $F \subseteq L \subseteq F_{\Sigma}$, write $G_{\Sigma}(L) = \text{Gal}(F_{\Sigma}/L)$. The signed Selmer group of $E$ over $F_{\infty}$ can then be equivalently defined by

$$\text{Sel}^{\bar{s}}(E/F_{\infty}) := \ker \left(H_{1}^{I}(G_{\Sigma}(F_{\infty}), E[p^{\infty}]) \to \prod_{w \in \Sigma_{\text{ss}}} H_{1}^{f,s_{w}}(F_{\infty,w}, E[p^{\infty}]) \times \prod_{w \in \text{ord}} H_{1}^{I}(F_{\infty,w}, E[p^{\infty}]) \times \prod_{w \in \Sigma_{\text{ss}} \setminus \{p\}} H_{1}^{I}(F_{\infty,w}, E[p^{\infty}]) \right).$$
We also define $H^1_{Iw, \Sigma}(F_{\infty}, T) = \lim_{\rightarrow} H^1(G_{\Sigma}(F_n), T)$, where the transition maps are given by the corestriction maps. Note that $H^1_{Iw, \Sigma}(F_{\infty}, T)$ is independent of the choice of $\Sigma$ (see [MR04, Lemma 5.3.1] or [Kob03, Proposition 7.1]). Since our set $\Sigma$ is fixed throughout, we will drop the subscript $\Sigma$ from the notation for simplicity and write $H^1_{Iw}(F_{\infty}, T)$. We now record the following useful observation.

**Lemma 2.6.** The group $H^1_{Iw}(F_{\infty}, T)$ is a torsion-free $\Lambda$-module. In the event that (S1) and (S2) are valid, we even have that $H^1(G_{\Sigma}(F), T)$ is a torsion-free $\mathbb{Z}_p$-module.

**Proof.** By considering the low degree terms of the spectral sequence of Jannsen

$$\text{Ext}^1_{\Lambda}(H^j(G_{\Sigma}(F_{\infty}), E[p^{\infty}]), \Lambda) \to H^{i+j}_{Iw}(F_{\infty}, T)$$

(cf. [Jan14, Theorem 1]), we obtain the following exact sequence

$$0 \to \text{Ext}^1_{\Lambda}((E(F[p^{\infty}]), \Lambda) \to H^1_{Iw}(F_{\infty}, T) \to \text{Ext}^1_{\Lambda}(H^1(G_{\Sigma}(F_{\infty}), E[p^{\infty}]), \Lambda).$$

By a theorem of Imai [ima75, $E(F[p^{\infty}])$ is finite and so the leftmost term vanishes. This in turn implies that $H^1_{Iw}(F_{\infty}, T)$ injects into an Ext$^0$-term. Since the latter is a reflexive $\Lambda$-module by [NSW08 Corollary 5.1.3], $H^1_{Iw}(F_{\infty}, T)$ must be torsionfree.

We now prove the second assertion. The low degree terms of the spectral sequence

$$\text{Ext}^1_{\mathbb{Z}_p}(H^j(G_{\Sigma}(F), E[p^{\infty}]), \mathbb{Z}_p) \to H^{i+j}(G_{\Sigma}(F), T)$$

yields the following exact sequence

$$0 \to \text{Ext}^1_{\mathbb{Z}_p}((E(F[p^{\infty}]), \mathbb{Z}_p) \to H^1(G_{\Sigma}(F), T) \to \text{Ext}^1_{\mathbb{Z}_p}(H^1(G_{\Sigma}(F_{\infty}), E[p^{\infty}]), \mathbb{Z}_p).$$

Since (S1) and (S2) are valid, the proof of Lemma 2.6 tells us that $E(F_w)[p^{\infty}] = 0$ for $w \in \Sigma_{\text{ss}}$. From which, one has $E(F)[p^{\infty}] = 0$. Consequently, we have $H^1(G_{\Sigma}(F), T)$ injecting into an Ext$^0$-term and so it must be $\mathbb{Z}_p$-torsionfree.

**Remark 2.7.** It is clear from the proof of Lemma 2.6 that under the validity of (S1) and (S2), we can also show that $H^1(G_{\Sigma}(F_n), T)$ is a torsion-free $\mathbb{Z}_p$-module for every $n$.

**Proposition 2.8.** Suppose that (S1) and (S2) are valid. Then $\text{Sel}^\varphi(E/F_{\infty})$ is a cotorsion $\Lambda$-module if and only if we have that $H^2(G_{\Sigma}(F_{\infty}), E[p^{\infty}]) = 0$ and that the following sequence

$$0 \to \text{Sel}^\varphi(E/F_{\infty}) \to H^1(G_{\Sigma}(F_{\infty}), E[p^{\infty}]) \to \prod_{w \in \Sigma_{\text{ss}}} H^1_{Iw}(F_{\infty}, E[p^{\infty}]) \times \prod_{w \in \Sigma_{\text{ord}}} H^1_{Iw}(F_{\infty}, E[p^{\infty}]) \times \prod_{w \in \Sigma, w \mid p} H^1(F_{\infty}, E[p^{\infty}]) \to 0$$

is exact.

**Proof.** To simplify notation, we write $J_w(E/F_{\infty})$ for each of the local summands. By [PR95 Proposition A.3.2], we have an exact sequence

$$0 \to \text{Sel}^\varphi(E/F_{\infty}) \to H^1(G_{\Sigma}(F_{\infty}), E[p^{\infty}]) \to \prod_{w \in \Sigma} J_w(E/F_{\infty})$$

$$\to \text{Sel}^\varphi(E/F_{\infty}) \to H^2(G_{\Sigma}(F_{\infty}), E[p^{\infty}]) \to 0,$$

where $\text{Sel}^\varphi(E/F_{\infty})$ is a $\Lambda$-submodule of $H^1_{Iw}(F_{\infty}, F/T)$. (For the precise definition of $\text{Sel}^\varphi(E/F_{\infty})$, we refer readers to loc. cit. For our purposes, the submodule theoretical information suffices.) Standard corank calculations [Gre89 Propositions 1-3] and [KO18 Proposition 3.32] tell us that

$$\text{corank}_{\Lambda}(H^1(G_{\Sigma}(F_{\infty}), E[p^{\infty}])), - \text{corank}_{\Lambda}(H^2(G_{\Sigma}(F_{\infty}), E[p^{\infty}])) = [F : \mathbb{Q}],$$

and

$$\text{corank}_{\Lambda} \left( \bigoplus_{w \in \Sigma} J_w(E/F_{\infty}) \right) = [F : \mathbb{Q}].$$
It is now clear from these formulas and the above exact sequence that \( \text{Sel}^\Sigma(E/F_\infty) \) is a cotorsion \( \Lambda \)-module if and only if \( \mathcal{S}^\Sigma(E/F_\infty) \) is a torsion \( \Lambda \)-module. Since \( \mathcal{S}^\Sigma(E/F_\infty) \) is contained in \( H^1_{Iw}(F_\infty, T) \) which is torsionfree by Lemma 2.6, the latter statement holds if and only if \( \mathcal{S}^\Sigma(E/F_\infty) = 0 \). But this is precisely equivalent to having \( H^2(\Sigma_\infty(F_\infty), E[p^\infty]) = 0 \) and the short exact sequence in the proposition.

The next proposition records consequence of the cotorsionness of \( \text{Sel}^\Sigma(E/F_\infty) \) on the structure of the Iwasawa cohomology groups \( H^1_{Iw}(F_\infty, T) \).

**Proposition 2.9.** Assume that (S1) and (S2) are valid. Suppose that \( \text{Sel}^\Sigma(E/F_\infty) \) is cotorsion over \( \Lambda \). Then the following statements are valid.

(a) \( H^1_{Iw}(F_\infty, T) \) is a free \( \Lambda \)-module with \( \Lambda \)-rank \( [F : \mathbb{Q}] \).

(b) \( H^2_{Iw}(F_\infty, T) \) is a torsion \( \Lambda \)-module.

**Proof.** It follows from the hypothesis and Proposition 2.8 that \( H^2(\Sigma_\infty(F_\infty), E[p^\infty]) = 0 \). Also, as seen in the proof of Lemma 2.6 we have \( E(F_\infty)[p^\infty] = 0 \). Taking these into account, the spectral sequence of Jannsen

\[
\text{Ext}_\Lambda^1 \left( H^1(\Sigma_\infty(F_\infty), E[p^\infty]), \Lambda \right) \Rightarrow H^{1,j}_{Iw}(F_\infty, T)
\]

then degenerates yielding

\[
H^1_{Iw}(F_\infty, T) \cong \text{Ext}_\Lambda^0 \left( H^1(\Sigma_\infty(F_\infty), E[p^\infty]), \Lambda \right)
\]

and

\[
H^2_{Iw}(F_\infty, T) \cong \text{Ext}_\Lambda^1 \left( H^1(\Sigma_\infty(F_\infty), E[p^\infty]), \Lambda \right).
\]

It follows from this that \( H^1_{Iw}(F_\infty, T) \) has \( \Lambda \)-rank \( [F : \mathbb{Q}] \) and \( H^2_{Iw}(F_\infty, T) \) is a torsion \( \Lambda \)-module. On the other hand, the homological spectral sequence

\[
H^i(\Gamma, H^{−j}_{Iw}(F_\infty, T)) \Rightarrow H^{−i−j}(\Sigma_\infty(F), T)
\]

(cf. [LS13 Theorem 3.1.8]) gives an injection

\[
H^1_{Iw}(F_\infty, T) \cong H^1(\Sigma_\infty(F), T).
\]

Since \( H^1(\Sigma_\infty(F), T) \) is \( \mathbb{Z}_p \)-torsionfree by Lemma 2.6, so is \( H^1_{Iw}(F_\infty, T) \). On the other hand, by Lemma 2.6 again, \( H^1_{Iw}(F_\infty, T) \) is \( \Lambda \)-torsionfree and hence it follows that \( H^1_{Iw}(F_\infty, T) \) is \( \mathbb{Z}_p \)-torsionfree by Lemma 2.6, so is \( H^1_{Iw}(F_\infty, T) \). We may now apply [NSW08 Proposition 5.3.19(ii)] to obtain the freeness property.

\[\square\]

3. Kobayashi ranks

Following [Kob03 §10], we define the Kobayashi ranks as follows.

**Definition 3.1.** Let \( (M_n)_{n \geq 1} \) be a projective system of finitely generated \( \mathbb{Z}_p \)-modules with connecting maps \( \pi_n : M_n \to M_{n-1} \). If the kernel and the cokernel of \( \pi_n \) are both finite, we define

\[
\nabla M_n = \text{length}_{\mathbb{Z}_p} \ker \pi_n - \text{length}_{\mathbb{Z}_p} \text{coker} \pi_n + \dim_{\mathbb{Q}_p} M_{n-1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.
\]

We mention two important observations which will be used in the subsequent sections of the paper. Firstly, for a given projective system \( (M_n)_{n \geq 1} \) of finitely generated \( \mathbb{Z}_p \)-modules, \( \text{rank}_{\mathbb{Z}_p} M_n \) is bounded independent of \( n \) if and only if \( \nabla M_n \) is defined for \( n \gg 0 \). Secondly, in the event that \( (M_n)_{n \geq 1} \) is a projective system of finite \( \mathbb{Z}_p \)-modules, we have

\[
\nabla M_n = e(M_n) - e(M_{n-1}).
\]

We now review further important properties of Kobayashi ranks that we shall need later.

**Proposition 3.2.** Let \( n \geq 1 \) be an integer.
(a) Let $f \in \Lambda$. Suppose that $\Phi_n \uparrow f$ and write
\[ f = gh, \quad \text{where } h = \gcd(\omega_{n-1}, f). \]

Then
\[ \dim_{\mathbb{Q}_p} (\Lambda/(f, \omega_{n-1})) \otimes \mathbb{Q}_p = \ord_{\epsilon_n} h(\epsilon_n) \]
Furthermore, $\pi : \Lambda/(f, \omega_n) \to \Lambda/(f, \omega_{n-1})$ has finite kernel with
\[ \length_{\mathbb{Q}_p} \ker \pi = \ord_{\epsilon_n} g(\epsilon_n). \]

In particular,
\[ \nabla \Lambda/(f, \omega_n) = \ord_{\epsilon_n} g(\epsilon_n) + \ord_{\epsilon_n} h(\epsilon_n) = \ord_{\epsilon_n} f(\epsilon_n). \]

(b) Suppose that $M$ is a finitely generated torsion $\Lambda$-module and that $f$ is a characteristic element of $M$.
Then $\nabla M_{\Gamma_n}$ is defined and equals
\[ \ord_{\epsilon_n} f(\epsilon_n) = p^{n-1}(p-1)\mu(M) + \lambda(M) \]
when $n \gg 0$.

Proof. This is [Kob03 Lemma 10.5].

We shall need a slightly more general version of part (a) of this proposition, which can be proved in exactly the same way.

Proposition 3.3. Let $n \geq 1$ be an integer and $\mathcal{O}$ the ring of integers of a finite extension of $\mathbb{Q}_p$ of degree $k$. Let $f \in \Lambda_{\mathcal{O}}$. Suppose that $\Phi_n \uparrow f$ and write
\[ f = gh, \quad \text{where } h = \gcd(\omega_{n-1}, f). \]

Then
\[ \dim_{\mathbb{Q}_p} (\Lambda_{\mathcal{O}}/(f, \omega_{n-1})) \otimes \mathbb{Q}_p = k \cdot \ord_{\epsilon_n} h(\epsilon_n) \]
Furthermore, $\pi : \Lambda_{\mathcal{O}}/(f, \omega_n) \to \Lambda_{\mathcal{O}}/(f, \omega_{n-1})$ has finite kernel with
\[ \length_{\mathbb{Q}_p} \ker \pi = k \cdot \ord_{\epsilon_n} g(\epsilon_n). \]

In particular,
\[ \nabla \Lambda_{\mathcal{O}}/(f, \omega_n) = k \cdot \ord_{\epsilon_n} f(\epsilon_n). \]

We shall also need the following lemma on how Kobayashi ranks behave under short exact sequences.

Lemma 3.4. Suppose we have a short exact sequence of inverse systems
\[ 0 \to (M'_n) \to (M_n) \to (M''_n) \to 0. \]
If two of $\nabla M'_n, \nabla M_n, \nabla M''_n$ are defined, then so is the third. Furthermore,
\[ \nabla M_n = \nabla M'_n + \nabla M''_n. \]

Proof. This is [Kob03 Lemma 10.4 i)].

We record the following lemma, which will be useful later.

Lemma 3.5. Let $M$ be a finitely generated $\Lambda$-module. Then $M^\delta := \cup_n M_{\Gamma_n}^\delta$ is finitely generated as a $\mathbb{Z}_p$-module and there exists an integer $n_0$ such that $M^\delta = M_{\Gamma_n}^{\geq n_0}$.

In particular, $\nabla M_{\Gamma_n}^{\geq n} = 0$ for $n \gg 0$, where the transition maps $M_{\Gamma_{n+1}} \to M_{\Gamma_n}$ are given by multiplication by $1 + \gamma_n + \cdots + \gamma_n^p$ with $\gamma_n$ being a topological generator of $\Gamma_n$ chosen so that $\gamma_n^p = \gamma_{n+1}$.

Proof. The first assertion of the lemma follows from [NSW08 Lemma 5.3.14(i)]. As a result, there exists $n_0$ such that $M_{\Gamma_n} = M_{\Gamma_n}^{\geq n_0}$ is finitely generated over $\mathbb{Z}_p$ for $n \geq n_0$. It then follows that the transition map $M_{\Gamma_{n+1}} \to M_{\Gamma_n}$ coincides with the multiplication by $p$-map for $n \gg 0$. The lemma now follows from [Kob03 Lemma 10.4(ii)].
Similar to [Kob03 §10], we consider the following groups

\[ \mathcal{Y}(E/F_n) = \text{coker} \left( H^1(G_{\Sigma}(F_n), T) \rightarrow \bigoplus_{w \in \Sigma_p} H^1_{\text{fr}}(F_{n,w}, T) \right), \]

\[ \mathcal{Y}'(E/F_n) = \text{coker} \left( H^1_{\text{fr}}(F_{\infty}, T)_{\Gamma_n} \rightarrow \bigoplus_{w \in \Sigma_p} H^1_{\text{fr}}(F_{n,w}, T) \right), \]

where \( H^1_{\text{fr}}(F_{n,w}, T) \) denotes the quotient \( \frac{H^1(F_{n,w}, T)}{I_{(f)}(1)} \). As we shall see in §10, one of the key ingredients of studying the growth of rank_{E}(E/F_n) and \( \text{III}_p(E/F_n) \) is to understand \( \nabla \mathcal{Y}(E/F_n) \). We end this section by the following generalization of [Kob03 Proposition 10.6 i)].

**Proposition 3.6.** Suppose that \((S1)\) and \((S2)\) are valid. Then for \( n \gg 0 \), we have \( \nabla \mathcal{Y}(E/F_n) = \nabla \mathcal{Y}'(E/F_n) \).

**Proof.** For each \( n \), we have the following commutative diagram

\[ H^1(G_{\Sigma}(F), \mathbb{Z}_p[\text{Gal}(F_{n+1}/F)] \otimes \mathbb{Z}_p, T) \xrightarrow{\text{sh}} H^1(G_{\Sigma}(F_{n+1}), T) \]

\[ \xrightarrow{\text{cor}} H^1(G_{\Sigma}(F), \mathbb{Z}_p[\text{Gal}(F_n/F)] \otimes \mathbb{Z}_p, T) \xrightarrow{\text{sh}} H^1(G_{\Sigma}(F_n), T), \]

which upon taking inverse limits induces an isomorphism

\[ H^1(G_{\Sigma}(F), \Lambda^t \otimes T) \cong \lim_{\rightarrow n} H^1(G_{\Sigma}(F_{n+1}), T) =: H^1_{\text{fr}}(F_{\infty}, T), \]

where here \( \Lambda^t \) is \( \Lambda \) as a \( \mathbb{Z}_p \)-module on which \( \gamma \in \Gamma \) acts via multiplication by \( \gamma^{-1} \).

Write \( \gamma_n \) for a topological generator of \( \Gamma_n \) which is chosen so that \( \gamma_n^p = \gamma_{n+1} \). Taking \( G_{\Sigma}(F) \)-cohomology in the short exact sequence

\[ 0 \rightarrow \Lambda^t \otimes \mathbb{Z}_p, T \rightarrow \mathbb{Z}_p[\text{Gal}(F_n/F)] \otimes \mathbb{Z}_p, T \rightarrow 0 \]

yields the following long exact sequence

\[ H^1_{\text{fr}}(F_{\infty}, T) \rightarrow H^1_{\text{fr}}(F_{\infty}, T) \rightarrow H^1_{\text{fr}}(G_{\Sigma}(F_n), T) \rightarrow H^1_{\text{fr}}(F_{\infty}, T) \rightarrow H^2_{\text{fr}}(F_{\infty}, T), \]

which in turn yields the following short exact sequence

\[ 0 \rightarrow H^1_{\text{fr}}(F_{\infty}, T)_{\Gamma_n} \rightarrow H^1(G_{\Sigma}(F_n), T) \rightarrow H^2_{\text{fr}}(F_{\infty}, T)^{\Gamma_n} \rightarrow 0. \]

Furthermore, the commutative diagram

\[ 0 \rightarrow \Lambda^t \otimes \mathbb{Z}_p, T \rightarrow \mathbb{Z}_p[\text{Gal}(F_n/F)] \otimes \mathbb{Z}_p, T \rightarrow 0 \]

induces the commutative diagram

\[ 0 \rightarrow H^1_{\text{fr}}(F_{\infty}, T)^{\Gamma_{n+1}} \rightarrow H^1(G_{\Sigma}(F_{n+1}), T) \rightarrow H^2_{\text{fr}}(F_{\infty}, T)^{\Gamma_{n+1}} \rightarrow 0 \]

with exact rows.
For each \( n \), we also have the following commutative diagram

\[
\begin{array}{c}
H^i(G_{\Sigma}(F), Z_p[\text{Gal}(F_n/F)] \otimes_{Z_p} T) \\
\downarrow_{\text{res}}^{1+\gamma_n+\cdots+\gamma_n^{n-1}} \\
H^i(G_{\Sigma}(F), Z_p[\text{Gal}(F_{n+1}/F)] \otimes_{Z_p} T)
\end{array}
\Rightarrow H^i(G_{\Sigma}(F_n), T)
\]

and the following commutative diagram

\[
\begin{array}{c}
0 \longrightarrow \Lambda^1 \otimes_{Z_p} T \xrightarrow{\gamma_n} \Lambda^1 \otimes_{Z_p} T \longrightarrow Z_p[\text{Gal}(F_n/F)] \otimes_{Z_p} T \longrightarrow 0 \\
\downarrow_{\text{res}}^{1+\gamma_n+\cdots+\gamma_n^{n-1}} \\
0 \longrightarrow \Lambda^1 \otimes_{Z_p} T \xrightarrow{\gamma_{n+1}} \Lambda^1 \otimes_{Z_p} T \longrightarrow Z_p[\text{Gal}(F_{n+1}/F)] \otimes_{Z_p} T \longrightarrow 0.
\end{array}
\]

This gives the commutative diagram

\[
\begin{array}{c}
0 \longrightarrow H^1_{w}(F_{\infty}, T)_{\gamma_n} \longrightarrow H^1(G_{\Sigma}(F_n), T) \longrightarrow H^2_{w}(F_{\infty}, T)_{\gamma_n} \longrightarrow 0 \\
\downarrow_{\text{res}}^{1+\gamma_n+\cdots+\gamma_n^{n-1}} \\
0 \longrightarrow H^1_{w}(F_{\infty}, T)_{\gamma_{n+1}} \longrightarrow H^1(G_{\Sigma}(F_{n+1}), T) \longrightarrow H^2_{w}(F_{\infty}, T)_{\gamma_{n+1}} \longrightarrow 0
\end{array}
\]

with exact rows.

Applying the snake lemma to the following commutative diagram

\[
\begin{array}{c}
0 \longrightarrow H^1_{w}(F_{\infty}, T)_{\gamma_n} \longrightarrow H^1(G_{\Sigma}(F_n), T) \longrightarrow H^2_{w}(F_{\infty}, T)_{\gamma_n} \longrightarrow 0 \\
\oplus_{v \mid p} H^1_{/f}(F_{n, w}, T) \longrightarrow \oplus_{v \mid p} H^1_{/f}(F_{n, w}, T)
\end{array}
\]

yields

\[
H^2_{w}(F_{\infty}, T)_{\gamma_n} \xrightarrow{\partial_n} \mathcal{Y}'(E/F_n) \longrightarrow \mathcal{Y}(E/F_n) \longrightarrow 0.
\]

Denote by \( Z'_n \) and \( Z''_n \) the kernel and image of \( \partial_n \). It is straightforward to check that the preceding diagram is compatible in \( n \), going from \( n \) to \( n+1 \). By Lemma 3.5, \( H^2_{w}(F_{\infty}, T)_{\gamma_n} \) stabilizes for \( n \gg 0 \) and so one has \( Z'_n \subseteq Z''_n \) for \( n \gg 0 \). Since \( H^2_{w}(F_{\infty}, T)_{\gamma_n} \) is finitely generated over \( Z_p \), it follows from the Noetherian property that \( Z'_n \) stabilizes for \( n \gg 0 \). Hence so does \( Z''_n \). Now going from \( n+1 \) to \( n \), it follows from Lemma 3.5 that the transition maps on \( H^2_{w}(F_{\infty}, T)_{\gamma_n} \) is given by multiplication of \( p \) for \( n \gg 0 \). Therefore, the transition maps on \( Z'_n \) is given by \( p \) for \( n \gg 0 \), and hence the same can be said for \( Z''_n \). We may now apply [Kob03, Lemma 10.4(ii)] to conclude that \( \nabla Z''_n = 0 \) for \( n \gg 0 \). Consequently, we have \( \nabla \mathcal{Y}'(E/F_n) = \nabla \mathcal{Y}(E/F_n) \) for \( n \gg 0 \) and this finishes the proof of the proposition.

4. Local analysis via Coleman maps

As remarked in the previous section, we shall need to understand the growth of \( \mathcal{Y}(E/F_n) \), or equivalently, the growth of \( \mathcal{Y}'(E/F_n) \) thanks to Proposition 3.6. We shall study the image of \( H^1_{w}(F_{\infty}, T) \) in the quotient \( H^1_{/f}(F_{n, w}, T) \), where \( w \in \Sigma_{ss} \), via Coleman maps. We shall consider the supersingular case and the ordinary case separately.

4.1. The supersingular case. Throughout this section. We fix \( w \in \Sigma_{ss} \), which lies above \( v \in \Sigma_{ss} \). We shall write \( \sigma_w \) for the Frobenius element of \( \text{Gal}(F_w/K_v) \). For \( n \geq 1 \), we define

\[
H_{v,n} = C_{v,n} \cdots C_{v,1},
\]
Lemma 4.1. Let \( \ker H_{v,n} \). Then one has
\[
H_{v,n} = \begin{pmatrix} H_{v,n}^1 & H_{v,n}^0 \\ -\Phi_n H_{v,n-1}^1 & -\Phi_n H_{v,n-1}^0 \end{pmatrix}.
\]

Lemma 4.3. Let \( \theta \) a character on \( \Gamma \) of conductor \( p^\alpha+1 > 1 \) (so that it factors through \( \Gamma_n \) but not \( \Gamma_{n-1} \)). Let \( e_\theta \) denote the idempotent associated to \( \theta \). Then the image of \( z \in e_\theta \cdot H_{1,f}(F_n,w,T) \) is zero if and only if
\[
H_{v,n}^1 \text{Col}_w(z) + H_{v,n}^0 \text{Col}_w(z) \in \Lambda_{\mathcal{O}_w}
\]
vanishes at \( \theta \).

Proof. This is proven in [LS20, Proposition 5.1].

Definition 4.2. Let \( I_v \) be as defined in Proposition 2.2. For \( n \geq 1 \) and \( u \in \mathbb{Z}_p^\times \), we define
\[
h_{v,n}^u : I_v \to \Lambda_n
\]
\[
(G_1,G_2) \mapsto H_{v,n}^1 G_1 + u H_{v,n}^0 G_2 \mod \omega_n,
\]
and
\[
\text{Col}_{v,n}^u : H_{1,w}(F_{\infty,v}, T) \to \Lambda_n
\]
bym \( h_{v,n}^u \circ (\text{Col}_{v,u}, \text{Col}_{v,v}) \) (which makes sense thanks to Proposition 2.2). We define
\[
\text{Col}_{v,w,n}^u : H_{1,w}(F_{\infty,w}, T) \to \Lambda_n \otimes \mathcal{O}_w
\]
similarly.

Note that \( \text{Col}_{v,w,n}^u \) factors through \( H_{1,w}(F_{\infty,w}, T)_{\Gamma_n} = H^1(F_{n,w}, T) \) (see [LLZ17, proof of Proposition 3.11]). In fact, one can do better and this is the content of the next lemma.

Lemma 4.3. Let \( n \geq 1 \). We have:

(a) \( \text{Im} (h_{v,n}^u) \supset \omega_{n-1} \Lambda_n \);

(b) There exists \( u \in \mathbb{Z}_p^\times \) such that \( \text{Col}_{v,n}^u \) induces an injection
\[
H_{1,f}(F_n,w,T) \hookrightarrow \Lambda_n \otimes \mathcal{O}_w
\]
with finite cokernel. In particular, it is an isomorphism after tensoring by \( \mathbb{Q}_p \).

Proof. We can calculate explicitly that \( \det(H_{v,n}) = \omega_n/X \). Thus, \( \text{[4.1]} \) tells us that
\[
-H_{v,n}^1 H_{v,n-1}^0 + H_{v,n}^0 H_{v,n-1}^1 = \omega_{n-1}/X.
\]
Since \( -X H_{v,n-1}^1, H_{v,n}^1 \in I_v \), it is mapped to \( \omega_{n-1} \) under \( h_{v,n}^u \). This proves (a). Part (b) is [LLZ17, Proposition 3.11].

Corollary 4.4. Let \( n \geq 1 \), \( M \) a \( \Lambda \)-submodule of \( H_{1,f}(F_n,w,T) \) and \( u \in \mathbb{Z}_p^\times \) satisfying Lemma 4.3(b). Consider the following natural projections
\[
\pi : H_{1,f}(F_n,w,T)_M \to H_{1,f}(F_{n-1,w}, T)_{M \Gamma_{n-1}},
\]
\[
\pi' : \Lambda_n \otimes \mathcal{O}_w_{\text{Col}_{v,w,n}^u(M)} \to \Lambda_{n-1} \otimes \mathcal{O}_w_{\text{Col}_{v,w,n}^u(M) \Gamma_{n-1}}.
\]

Then one has \( \ker \pi = \ker \pi' \) as \( \mathbb{Z}_p \)-modules.
**Proof.** We abuse notation writing \( \text{Col}_{w,n}^u \) for the map
\[
H^1(F_{n-1,w}, T) \to \Lambda_{n-1} \otimes \mathcal{O}_w
\]
defined by \( \text{Col}_{w,n}^u \mod \omega_{n-1} \). Consider the following commutative diagram:
\[
\begin{array}{cccc}
0 & \to & H^1_f(F_{n,w}, T) & \to & \Lambda_n \otimes \mathcal{O}_w & \to & \text{Col}_{w,n}^u(M) & \to & \text{Col}_{w,n}^u(H^1_f(F_{n,w}, T)) & \to & 0 \\
& & \downarrow \pi & & \downarrow \pi' & & \downarrow \pi'' & & \downarrow \\
0 & \to & H^1_f(F_{n-1,w}, T) & \to & \Lambda_{n-1} \otimes \mathcal{O}_w & \to & \text{Col}_{w,n}^u(M) & \to & \text{Col}_{w,n}^u(H^1_f(F_{n-1,w}, T)) & \to & 0
\end{array}
\]
Recall from Lemma 2.1 that \( \pi \) is surjective. The snake lemma then gives the following short exact sequence:
\[
0 \to \ker \pi \to \ker \pi' \to \ker \pi'' \to 0.
\]
Therefore, the corollary would follow from showing that \( \ker \pi'' \) is trivial. Lemma 4.3(a) tells us that
\[
\frac{\Lambda_n \otimes \mathcal{O}_w}{\text{Im}h_{w,n}} \leftrightarrow \frac{\Lambda_{n-1} \otimes \mathcal{O}_w}{(\text{Im}h_{w,n})_{\Gamma_{n-1}}}.
\]
Since \( \text{Col}_{w,n}^u(H^1_f(F_{n,w}, T)) = \text{Im}h_{w,n} \) by Proposition 2.2, the result follows. \( \square \)

**Corollary 4.5.** Let \( z \in H^1_w(F_{\infty,w}, T) \) and \( u \in \mathbb{Z}_p^\times \) a constant satisfying Lemma 4.3(b). For \( n \geq 1 \), write \( M_n \) for the \( \Lambda_{\mathcal{O}_w} \)-module generated by the natural image of \( z \) in \( H^1_f(F_{n,w}, T) \). Suppose that the natural projection
\[
\pi : \frac{H^1_f(F_{n,w}, T)}{M_n} \to \frac{H^1_f(F_{n-1,w}, T)}{M_{n-1}}
\]
has finite kernel for some \( n \). Then,
\[
\nabla \left( \frac{H^1_f(F_{n,w}, T)}{M_n} \right) = [F_w : K_v] \cdot \text{ord}_n \text{Col}_{w,n}^u(z)(\epsilon_n).
\]

**Proof.** Since the projection \( H^1_f(F_{n,w}, T) \to H^1_f(F_{n-1,w}, T) \) is surjective by Lemma 2.1, we have
\[
\nabla \left( \frac{H^1_f(F_{n,w}, T)}{M_n} \right) = \dim_{\mathbb{Q}_p} \frac{H^1_f(F_{n-1,w}, T)}{M_{n-1}} + \text{length}_{\mathbb{Z}_p} \ker \pi.
\]
Let \( \text{Col}_{w,n}^u(z) = g_wh_w \), where \( h_w = \gcd(\omega_{n-1}, f_w) \). We deduce from Lemma 4.3(b) and Proposition 3.3 that
\[
\dim_{\mathbb{Q}_p} \frac{H^1_f(F_{n-1,w}, T)}{M_{n-1}} = [F_w : K_v] \cdot \text{ord}_n h_w(\epsilon_n).
\]
Applying Corollary 4.4, we have
\[
\text{length}_{\mathbb{Z}_p} \ker \pi = [F_w : K_v] \cdot \text{ord}_n g_w(\epsilon_n).
\]
Thus, putting everything together, we conclude that
\[
\nabla \left( \frac{H^1_f(F_n, T)}{M_n} \right) = [F_w : K_v] \cdot (\text{ord}_n h_w(\epsilon_n) + \text{ord}_n g_w(\epsilon_n)) = [F_w : K_v] \cdot \text{ord}_n \text{Col}_{w,n}^u(z)(\epsilon_n)
\]
as required. \( \square \)

We now explain how to calculate \( \text{ord}_n \text{Col}_{w,n}^u(z)(\epsilon_n) \). In what follows, \( \text{ord}_p \) denotes the \( p \)-adic valuation on \( \mathbb{Q}_p \) with \( \text{ord}_p(p) = 1 \). Following [Spr13] Definition 4.4, for a \( 2 \times 2 \) matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) defined over \( \mathbb{Q}_p \), we write
\[
\text{ord}_p(A) = \begin{pmatrix} \text{ord}_p(a) & \text{ord}_p(b) \\ \text{ord}_p(c) & \text{ord}_p(d) \end{pmatrix}.
\]
Proposition 4.6. Let \( v \in \Sigma_{\text{ord}} \) and write \( r_v = \text{ord}_p(a_v) \in \{1, \infty \} \) (thanks to the Weil’s bound). For all \( n \geq 1 \),
\[
\text{ord}_p(H_{v,n}(\varepsilon_n)) = \begin{cases} 
  r_v + \sum_{i=1}^{\frac{n-1}{p^i}} \sum_{j=1}^{\frac{n-1}{p^i+1}} & \text{if } n \text{ is odd.} \\
  \frac{n-1}{p^i} \sum_{j=1}^{\frac{n-1}{p^i+1}} & \text{if } n \text{ is even.}
\end{cases}
\]

Let \( z \in H^1_{Iw}(\mathbb{F}_{\infty,w}, T) \) such that \( \text{Col}^u_{w,v}(z)(\varepsilon_n) \neq 0 \), then
\[
\text{ord}_{\varepsilon_n} \text{Col}^u_{w,v}(z)(\varepsilon_n) = \text{ord}_{\varepsilon_n} \left( H^\delta_{v,n}(\varepsilon_n) \right) \] where \( \delta(w,n) \in \{\varepsilon, b\} \) depends on the parity of \( n \) once \( w \) is fixed.

Proof. This is a special case of [LLZ17, Proposition 4.6 and Corollary 4.8] (with \( k = 2 \) and \( v \) in loc. cit. is taken to be \( r_v \) here).

Remark 4.7. If \( a_v = 0 \), then for all \( z \),
\[
\delta(w,n) = \begin{cases} 
  b & \text{if } n \text{ is odd,} \\
  \varepsilon & \text{if } n \text{ is even.}
\end{cases}
\]

4.2. The ordinary case. We suppose in this section that \( v \in \Sigma'_{\text{ord}} \). We have a short exact sequence
\[
0 \rightarrow \hat{E}_v[p^\infty] \rightarrow E[p^\infty] \rightarrow \hat{E}_v[p^\infty] \rightarrow 0,
\]
where \( \hat{E}_v \) (resp., \( E_v \)) is the formal group (resp., reduced curve) of \( E \) at \( K_v \). This gives the exact sequence
\[
0 \rightarrow \hat{E}_v(K_{\infty,v})[p^\infty] \rightarrow E(K_{\infty,v})[p^\infty] \rightarrow \hat{E}_v(k_{\infty,v})[p^\infty],
\]
where \( k_{\infty,v} \) is the residue field of \( K_{\infty,v} \). Note that \( k_{\infty,v} \) is a finite field.

In what follows, we write \( \hat{T} \) and \( T \) for the \( p \)-adic Tate module of \( \hat{E} \) and \( E \) respectively.

Lemma 4.8. The modules \( H^2_{Iw}(K_{\infty,v}, \hat{T}), H^2_{Iw}(K_{\infty,v}, T) \) and \( H^2_{Iw}(K_{\infty,v}, \hat{T}) \) are finite.

Proof. We first establish the finiteness of \( \hat{E}_v(K_{\infty,v})[p^\infty], E(K_{\infty,v})[p^\infty] \) and \( \hat{E}_v(k_{\infty,v})[p^\infty] \). Indeed, since \( k_{\infty,v} \) is finite, so is \( E_v(k_{\infty,v})[p^\infty] \). The main theorem of [LLZ17] says that \( E(K_{\infty,v})[p^\infty] \) is finite, which in turn implies that \( E(K_{\infty,v})[p^\infty] \) is also finite. Now, local duality tells us that \( H^2_{Iw}(K_{\infty,v}, \hat{T}) \cong H^0(K_{\infty,v}, \hat{E}[p^\infty]) \) and hence the finiteness of \( H^2_{Iw}(K_{\infty,v}, \hat{T}) \). The remaining finiteness assertions can be proven similarly.

By [Kat04] Lemma 17.12, there is an injective map
\[
\text{Col}_v : H^1_{Iw}(K_{\infty,v}, \hat{T}) \rightarrow \Lambda
\]
with finite cokernel. On taking Iwasawa cohomology of the short exact sequence
\[
0 \rightarrow \hat{T} \rightarrow T \rightarrow \hat{T} \rightarrow 0,
\]
we obtain
\[
0 \rightarrow H^1_{Iw}(K_{\infty,v}, \hat{T}) \rightarrow H^1_{Iw}(K_{\infty,v}, T) \rightarrow H^1_{Iw}(K_{\infty,v}, \hat{T}) \rightarrow H^2_{Iw}(K_{\infty,v}, \hat{T}),
\]
where one notes that \( H^2_{Iw}(K_{\infty,v}, \hat{T}) \) is finite by Lemma 4.8. Consequently, the following composition of maps
\[
H^1_{Iw}(K_{\infty,v}, T) \rightarrow H^1_{Iw}(K_{\infty,v}, \hat{T}) \rightarrow \Lambda
\]
factors through to give an injection
\[
H^1_{Iw}(K_{\infty,v}, T) := H^1_{Iw}(K_{\infty,v}, T) \rightarrow \Lambda
\]

There is a small typo in the statement of [LLZ17, Proposition 4.6] when \( n \) is odd.
with finite cokernel. By an abuse notation, we shall write \( \text{Col}_c \) for the composition above. Note that for \( n \gg 0 \), we have a short exact sequence

\[
0 \to B \to H^1_{f,Iw}(K_{\infty,v}, T)_{\Gamma_n} \xrightarrow{\text{Col}_c} \Lambda/\langle \omega_n \rangle \to B \to 0
\]

for some finite group \( B \) that is independent of \( n \).

Recall that

\[
H^1_f(K_{n,v}, T) = \frac{H^1(K_{n,v}, T)}{E(K_{n,v}) \otimes \mathbb{Z}_p} = \frac{H^1(K_{n,v}, T)}{H^1(K_{n,v}, \tilde{T})}.
\]

We have the following diagram

\[
\begin{array}{ccc}
H^1_{Iw}(K_{\infty,v}, \tilde{T})_{\Gamma_n} & \to & H^1_{Iw}(K_{\infty,v}, T)_{\Gamma_n} \\
\downarrow & & \downarrow \\
H^1(K_{n,v}, \tilde{T}) & \to & H^1(K_{n,v}, T) \\
\end{array}
\]

with exact rows. The first two vertical maps are injective with cokernels \( H^2_{Iw}(K_{\infty,v}, \tilde{T}) \) and \( H^2_{Iw}(K_{\infty,v}, \tilde{T}) \) respectively. Taking Lemma \ref{lemma:exact_sequence} into consideration, it follows that the rightmost vertical map has finite kernel and cokernel which are bounded independently of \( n \).

**Proposition 4.9.** Let \( z \in H^1_{Iw}(K_{\infty,v}, T) \) such that \( \text{Col}_c(z) \neq 0 \) and write \( M_n \) for the \( \Lambda \)-module generated by its image in \( H^1_f(K_{n,v}, T) \). When \( n \gg 0 \), \( \nabla\left(\frac{H^1_f(K_{n,v}, T)}{M_n}\right) \) is defined with

\[
\nabla\left(\frac{H^1_f(K_{n,v}, T)}{M_n}\right) = \nabla\left(\frac{\Lambda}{(\omega_n, \text{Col}_c(z))}\right) = \text{ord}_{\epsilon_n} \text{Col}_c(z)(\epsilon_n).
\]

**Proof.** Let \( M \) denote the image of \( \Lambda \cdot z \) in the quotient \( H^1_{Iw}(K_{\infty,v}, T) \). By the discussion just before the proposition, we have a map

\[
\left(\frac{H^1_{f,Iw}(K_{\infty,v}, T)}{M}_{\Gamma_n}\right) \to \left(\frac{H^1_f(K_{n,v}, T)}{M_n}\right)
\]

with finite kernel and cokernel whose orders are independent of \( n \) for \( n \gg 0 \). This in turn yields

\[
\nabla\left(\frac{H^1_{f,Iw}(K_{\infty,v}, T)}{M}_{\Gamma_n}\right) = \nabla\left(\frac{H^1_f(K_{n,v}, T)}{M_n}\right).
\]

Thanks to the short exact sequence \ref{seq:exact_sequence}, we have

\[
\nabla\left(\frac{H^1_{f,Iw}(K_{\infty,v}, T)}{M}_{\Gamma_n}\right) = \nabla\left(\frac{\Lambda}{(\omega_n, \text{Col}_c(z))}\right)
\]

for \( n \gg 0 \), resulting in the first equality of the proposition. The second equality follows from Proposition \ref{prop:exact_sequence} \( \square \)

**Corollary 4.10.** Retain the setting of Proposition \ref{prop:exact_sequence} Let \( w \in \Sigma_{\text{ord}} \) be a prime lying above \( v \). Let \( z' \) be the natural image of \( z \) in \( H^1_{Iw}(F_{\infty,w}, T) = H^1_{Iw}(K_{\infty,v}, T) \otimes_{\mathbb{Z}_p} \mathcal{O}_w \). Let \( M'_n \) be the \( \Lambda_{\mathcal{O}_w} \)-module generated by \( z' \) in the quotient \( H^1_f(F_{n,w}, T) \). Then for \( n \gg 0 \), \( \nabla\left(\frac{H^1_f(F_{n,w}, T)}{M'_n}\right) \) is defined with

\[
\nabla\left(\frac{H^1_f(F_{n,w}, T)}{M'_n}\right) = [F_w : K_v] \cdot \text{ord}_{\epsilon_n} \text{Col}_w(z')(\epsilon_n),
\]

where \( \text{Col}_w \) is the Coleman map

\[
H^1_{Iw}(F_{\infty,w}, T) \to \Lambda_{\mathcal{O}_w}
\]

obtained from extending \( \text{Col}_w \) \( \mathcal{O}_w \)-linearly.
Proof. This follows from combining Propositions 3.3 and 4.9.

5. PROOFS OF THE MAIN RESULTS

Throughout this section, we assume that (S1) – (S3). Let us write $d = [F : \mathbb{Q}]$. For each choice of $\bar{s} = (s_w)_{w \in \Sigma_{ss}} \in \{1, b \}^{\Sigma_{ss}}$, define

$$\text{Col}_{\bar{s}} : \bigoplus_{w \in \Sigma_p} H^1_{Iw}(F_{\infty, w}, T) \to \bigoplus_{w \in \Sigma_p} \Lambda_{\mathcal{O}_w} \cong \Lambda^{\oplus d}$$

(5.1)

to be the map given by $\text{Col}_{s_w}$ for $w \in \Sigma_{ss}$ and $\text{Col}_w$ for $w \in \Sigma_{ord}$, where $\text{Col}_w$ is defined as in Corollary 4.11.

Lemma 5.1. The localization map

$$\text{loc}_p : H^1_{Iw}(F_{\infty}, T) \to \bigoplus_{w \in \Sigma_p} H^1_{Iw}(F_{\infty, w}, T)$$

is injective.

Proof. Recall that (S3) says that $\text{Sel}_{\bar{s}}(E/F_{\infty})$ is $\Lambda$-torsion. By the Poitou-Tate exact sequence as given in PR93 Proposition A.3.2, we have the following exact sequence

$$H^1_{Iw}(F_{\infty}, T) \to \bigoplus_{w \in \Sigma_p} \frac{H^1_{Iw}(F_{\infty, w}, T)}{\ker \text{Col}_{\bar{s}}} \to \text{Sel}_{\bar{s}}(E/F_{\infty})^\vee.$$

(5.2)

By Proposition 2.2 and the pseudo-surjectivity of $\text{Col}_w$ for $w \in \Sigma_{ord}$ (see the discussion in 4.2),

$$\bigoplus_{w \in \Sigma_p} \frac{H^1_{Iw}(F_{\infty, w}, T)}{\ker \text{Col}_{\bar{s}}} \cong \text{ImCol}_{\bar{s}}$$

is of rank $d$ over $\Lambda$. Given that $\Lambda$-module $H^1_{Iw}(F_{\infty}, T)$ is free of rank $d$, the torsionness of $\text{Sel}_{\bar{s}}(E/F_{\infty})^\vee$ implies that the kernel of the first map in (5.2) is $\Lambda$-torsion. But the $\Lambda$-module of $H^1_{Iw}(F_{\infty}, T)$ is torsion-free by Lemma 2.3. Consequently, the aforementioned kernel is trivial. This implies that $\ker \text{loc}_p = 0$. \qed

Proposition 5.2. Let $R_w$ denote the pre-image of $H^1_{Iw}(F_{\infty, w}, T)$ in $H^1_{Iw}(F_{\infty}, T)$ for $w \in \Sigma_p$. There exist a family of elements $c_w \in R_w, w \in \Sigma_p$ so that the quotient

$$\frac{H^1_{Iw}(F_{\infty}, T)}{\bigoplus_{w \in \Sigma_p} \Lambda_{\mathcal{O}_w}c_w}$$

is a torsion $\Lambda$-module.

Proof. The injectivity of Lemma 5.1 tells us that we have a direct sum of $\Lambda$-modules $\bigoplus_{w \in \Sigma_p} R_w$ inside $H^1_{Iw}(F_{\infty, w}, T)$. Furthermore, $R_w$ is a $\Lambda_{\mathcal{O}_w}$-module of rank at most 1 (since $H^1_{Iw}(F_{\infty, w}, T)$ is of rank $d$ over $\Lambda$). The proof of Lemma 5.1 tells us that for each $w \in \Sigma_p$, the cokernel of the composition

$$H^1_{Iw}(F_{\infty, w}, T) \to H^1_{Iw}(F_{\infty, w}, T) \to \Lambda_{\mathcal{O}_w}\text{Col}_w$$

is $\Lambda$-torsion (where the second map is given by either $\text{Col}_{b, w}$ or $\text{Col}_w$ depending on whether $w \in \Sigma_{ss}$ or $w \in \Sigma_{ord}$). Thus, the $\Lambda_{\mathcal{O}_w}$-rank of $R_w$ cannot be zero. Thus, we may pick $c_w \in R_w$ so that $R_w/\Lambda_{\mathcal{O}_w}c_w$ is $\Lambda_{\mathcal{O}_w}$-torsion. The direct sum $\bigoplus_{w \in \Sigma_p} \Lambda_{\mathcal{O}_w}c_w$ is then a $\Lambda$-module of rank $d$ as required. \qed

We fix a choice of $c_w, w \in \Sigma_p$ as given by Proposition 5.2. We write $M_c$ for the $\Lambda$-module $\bigoplus_{w \in \Sigma_p} \Lambda_{\mathcal{O}_w}c_w$. Recall that $\mathcal{Y}(E/F_n)$ is defined to be

$$\text{coker} \left( H^1_{Iw}(F_{\infty, T})_{\Gamma_n} \to \bigoplus_{w \in \Sigma_p} H^1_{Iw}(F_{n, w}, T) \right).$$
We define
\[ \mathcal{Y}''(E/F_n) = \text{coker} \left( (M_n)_\Gamma \to \bigoplus_{w \in \Sigma_p} H^1_{/f}(F_{n,w}, T) \right). \]

**Proposition 5.3.** For \( n \gg 0 \), \( \nabla_n \mathcal{Y}''(E/F_n) \) is defined. Furthermore, there exist \( \tilde{\sigma} = (\sigma_w), \tilde{\tau} = (\tau_w) \in \{\#\}^{\Sigma_n} \) such that when \( n \) is odd and \( n \gg 0 \), \( \nabla_n \mathcal{Y}''(E/F_n) \) equals
\[ S(\tilde{\sigma}, n) + \nabla \left( \frac{\Lambda_{\tilde{\sigma}}}{\text{Col}_{\tilde{\sigma}}(M_n)} \right)_{\Gamma_n}, \]
where
\[ S(\tilde{\sigma}, n) = \phi(p^n) \left( \sum_{w: \sigma_w = \#} [F_w : K_v] \left( r_v + \sum_{i=1}^{n-1} \frac{1}{p^{2i}} \right) + \sum_{w: \sigma_w = \#} [F_w : K_v] \left( \frac{n}{p^2} - \sum_{i=1}^{n-1} \frac{1}{p^{2i}} \right) \right), \]
whereas when \( n \) is even and \( n \gg 0 \), \( \nabla_n \mathcal{Y}''(E/F_n) \) is given by
\[ T(\tilde{\tau}, n) + \nabla \left( \frac{\Lambda_{\tilde{\tau}}}{\text{Col}_{\tilde{\tau}}(M_n)} \right)_{\Gamma_n}, \]
where
\[ T(\tilde{\tau}, n) = \phi(p^n) \left( \sum_{w: \sigma_w = \#} [F_w : K_v] \sum_{i=1}^{\frac{n}{2}} \frac{1}{p^{2i}} - \sum_{w: \tau_w = \#} [F_w : K_v] \left( r_v + \sum_{i=1}^{\frac{n}{2} - 1} \frac{1}{p^{2i}} \right) \right), \]
with \( \phi \) being the Euler totient function. Here, \( \text{Col}_{\tilde{\sigma}} \) and \( \text{Col}_{\tilde{\tau}} \) are as defined by (5.1).

**Proof.** We study the case where \( n \) is odd. The proof for the case where \( n \) is even is the same.

By definition, we have
\[ \mathcal{Y}''(E/F_n) = \bigoplus_{w \in \Sigma_p} \frac{H^1_{/f}(F_{n,w}, T)}{(M_n)_\Gamma_n}, \]
where \( M_w \) denotes the image of \( \Lambda_{\mathcal{O}_w} c_w \) in \( H^1_{/f}(F_{\infty,w}, T) \). Let \( \pi_w \) be the natural map
\[ \frac{H^1_{/f}(F_{n,w}, T)}{(M_n)_\Gamma_n} \to \frac{H^1_{/f}(F_{n-1,w}, T)}{(M_n)_\Gamma_{n-1}}. \]

By Corollaries 4.5 and 4.10 \( \ker \pi_w \) is finite if and only if \( \text{Col}_{\delta(w,n),w} (z_w) \) (resp. \( \text{Col}_w (z_w) \)) does not vanish at \( \epsilon_n \) for \( w \in \Sigma_n \) (resp. \( w \in \Sigma_{\text{ord}} \)).

Let \( \tilde{\sigma} = (\delta(w,n))_{w \in \Sigma_n} \), where \( \delta(w,n) \) is as given by Proposition 4.6, which only depends on the parity of \( n \). On replacing \( \tilde{\sigma} \) in (5.2) by \( \sigma \), the torsionness of \( \text{Sel}_d(E/F_\infty) \) forces \( \text{Col}_{\delta(w,n),w} (z_w) \) (resp. \( \text{Col}_w (z_w) \)) to be a non-zero element of \( \Lambda_{\mathcal{O}_w} \) for \( w \in \Sigma_n \) (resp. \( w \in \Sigma_{\text{ord}} \)). In particular, when \( n \gg 0 \), it does not vanish at \( \epsilon_n \). Thus, \( \ker \pi_w \) is finite and \( \nabla \frac{H^1_{/f}(F_{n,w}, T)}{(M_n)_\Gamma_n} \) is defined. Its value can be calculated using Proposition 4.6 Corollaries 4.5 and 4.10. To calculate \( \nabla \mathcal{Y}''(E/F_n) \), we apply Lemma 5.4 to the direct sum in (5.3), which results in the formula as claimed. \( \square \)

**Remark 5.4.** Suppose that \( a_v = 0 \) for all \( v \in \Sigma_n \), then Remark 4.7 tells us that \( \tilde{\sigma} = \tilde{\sigma} \) and \( \tilde{\tau} = \tilde{\tau} \). Furthermore, the constants \( S(\tilde{\sigma}, n) \) and \( T(\tilde{\tau}, n) \) are given by
\[ S(\tilde{\sigma}, n) = \phi(p^n) \sum_{w \in \Sigma_n} [F_w : Q_p] \sum_{i=1}^{n/2} \frac{1}{p^{2i}-1} = d(p^{n-1} - p^{n-2} + p^{n-3} - \cdots - p), \]
\[ T(\tilde{\tau}, n) = \phi(p^n) \sum_{w \in \Sigma_n} [F_w : Q_p] \sum_{i=1}^{n/2} \frac{1}{p^{2i}-1} = d(p^{n-1} - p^{n-2} + p^{n-3} - \cdots - 1). \]
Corollary 5.5. For \( n \gg 0 \), \( \nabla Y'(E/F_n) \) is defined and is given by

\[
\begin{cases}
S(\tilde{\sigma}, n) + \nabla (\text{coker } \text{Col}_\sigma \circ \text{loc}_p)_{\Gamma_n} & \text{if } n \text{ is odd}, \\
T(\tilde{\tau}, n) + \nabla (\text{coker } \text{Col}_\tau \circ \text{loc}_p)_{\Gamma_n} & \text{if } n \text{ is even},
\end{cases}
\]

Proof. Again, we only treat the odd case. Consider the following short exact sequences, which are consequences of the third isomorphism theorem:

\[
0 \to \frac{H^1_{_{Iw}}(\mathbb{F}_\infty, T)_{\Gamma_n}}{(M_c)_{\Gamma_n}} \to \frac{\Lambda^{\otimes d}}{\text{Col}_\sigma(M_c)}_{\Gamma_n} \to (\text{coker Col}_\sigma \circ \text{loc}_p)_{\Gamma_n} \to 0,
\]

\[
0 \to \frac{H^1_{_{Iw}}(\mathbb{F}_\infty, T)_{\Gamma_n}}{(M_c)_{\Gamma_n}} \to \nabla''(E/F_n) \to \nabla'(E/F_n) \to 0.
\]

Since \( H^1_{_{Iw}}(\mathbb{F}_\infty, T)/M_c \) is \( \Lambda \)-torsion, it follows that \( \nabla \frac{H^1_{_{Iw}}(\mathbb{F}_\infty, T)_{\Gamma_n}}{(M_c)_{\Gamma_n}} \) is defined for \( n \gg 0 \). Thus, we deduce from Proposition 5.3 and Lemma 3.4 that \( \nabla \frac{H^1_{_{Iw}}(\mathbb{F}_\infty, T)_{\Gamma_n}}{(M_c)_{\Gamma_n}} \) is defined and is given by

\[
\nabla \frac{H^1_{_{Iw}}(\mathbb{F}_\infty, T)_{\Gamma_n}}{(M_c)_{\Gamma_n}} = \nabla''(E/F_n) - \nabla'(E/F_n).
\]

But Proposition 5.3 also tells us that \( \nabla \frac{\Lambda^{\otimes d}}{\text{Col}_\sigma(M_c)}_{\Gamma_n} \) is defined for \( n \gg 0 \). Thus, \( \nabla \text{coker Col}_\sigma \circ \text{loc}_p \) is defined and satisfies

\[
\nabla (\text{coker Col}_\sigma \circ \text{loc}_p)_{\Gamma_n} = \nabla \frac{\Lambda^{\otimes d}}{\text{Col}_\sigma(M_c)}_{\Gamma_n} - \nabla \frac{H^1_{_{Iw}}(\mathbb{F}_\infty, T)_{\Gamma_n}}{(M_c)_{\Gamma_n}}.
\]

The result now follows from combining these two equations with the formula given in Proposition 5.3  \( \square \)

Definition 5.6. For \( 0 \leq n \leq \infty \), we define

\[
\text{Sel}^0(E/F_n) := \ker \left( \text{Sel}_{p^\infty}(E/F_n) \to \prod_{w \in \Sigma_p} H^1(F_{n,w}, E[p^\infty]) \right).
\]

Equivalently, we have

\[
\text{Sel}^0(E/F_n) := \ker \left( H^1(G_{\Sigma}(F_n), E[p^\infty]) \to \prod_{w \in \Sigma(F_n)} H^1(F_{n,w}, E[p^\infty]) \right),
\]

where \( \Sigma(F_n) \) denotes the set of primes of \( F_n \) above \( \Sigma \). The Pontryagin duals of \( \text{Sel}_{p^\infty}(E/F_n) \) and \( \text{Sel}^0(E/F_n) \) are denoted by \( \chi(E/F_n) \) and \( \chi^0(E/F_n) \) respectively.

Lemma 5.7. The natural restriction map

\[
\text{Sel}^0(E/F_n) \longrightarrow \text{Sel}^0(E/F_\infty)^{\Gamma_n}
\]

has finite kernel and cokernel which are bounded independently of \( n \).

Proof. In the proof of Lemma 2.6 we have seen that \( E(F_\infty)[p^\infty] = 0 \). It then follows that the middle map of the following commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Sel}^0(E/F_n) & \longrightarrow & H^1(G_{\Sigma}(F_n), E[p^\infty]) & \longrightarrow & \prod_{w \in \Sigma(F_n)} H^1(F_{n,w}, E[p^\infty]) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Sel}^0(E/F_\infty)^{\Gamma_n} & \longrightarrow & H^1(G_{\Sigma}(F_\infty), E[p^\infty])^{\Gamma_n} & \longrightarrow & \left( \prod_{w \in \Sigma(F_\infty)} H^1(F_{\infty,w}, E[p^\infty]) \right)^{\Gamma_n}
\end{array}
\]

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is an isomorphism via a Hochschild-Serre spectral sequence argument. Hence it suffices to show that
the rightmost map has finite kernel which is bounded independent of \( n \). For primes not dividing \( p \),
this is discussed in [Gre92, Lemma 3.3]. It therefore remains to consider the primes above \( p \). Let \( w \) be such
a prime. Then the kernel of the restriction map

\[
H^1(F_{n,w}, E[p^\infty]) \to H^1(F_{\infty,w}, E[p^\infty]) \Gamma_n
\]

given by \( H^1(\Gamma_n, E(F_{\infty,w})[p^\infty]) \) by the Hochschild-Serre spectral sequence. Since our elliptic
curve is assumed to have good reduction at all primes above \( p \), the main theorem of [ima75] says that
\( E(F_{\infty,w})[p^\infty] \) is finite. The finiteness and boundedness of the kernel now follow. This completes
the proof of the lemma. \( \Box \)

**Lemma 5.8.** For \( n \geq 0 \):

(a) We have a short exact sequence

\[
0 \to \mathcal{Y}(E/F_n) \to \mathcal{X}(E/F_n) \to \mathcal{X}^0(E/F_n) \to 0.
\]

(b) For \( n \gg 0 \), \( \nabla \mathcal{X}^0(E/F_n) \) is defined satisfying the equality

\[
\nabla \mathcal{X}^0(E/F_n) = \nabla \mathcal{X}^0(E/F_\infty) \Gamma_n.
\]

*Proof.* Part (a) is a consequence of the Poitou-Tate exact sequence. See [Kob03, (10.35)].

Again by the Poitou-Tate exact sequence, the cokernel of the last map in (5.2) is isomorphic to \( \mathcal{X}^0(E/F_\infty) \).
This implies that \( \mathcal{X}^0(E/F_\infty) \) is \( \Lambda \)-torsion. By Lemma 5.7, the kernel and cokernel of the natural map

\[
\mathcal{X}^0(E/F_\infty) \Gamma_n \to \mathcal{X}^0(E/F_n)
\]

are finite and bounded independent of \( n \). Part (b) now follows from combining the latter observation with
Proposition 3.2(b) (see [Kob03, Proposition 10.6ii]) for the proof when \( F = \mathbb{Q} \). \( \Box \)

We are now ready to prove the main theorems of the paper.

**Theorem 5.9.** Under hypotheses (S1)-(S3), we have

(A) \( \text{rank}_\mathbb{Z} E(F_n) \) is bounded independently of \( n \);

(B) Suppose that \( \mathcal{I}_p(E/F_n) \) is finite for all \( n \geq 0 \). Let \( r_\infty = \lim_{n \to \infty} \text{rank}_\mathbb{Z} E(F_n) \). Then, for \( n \gg 0 \),
we have

\[
\nabla_n \mathcal{I}_p(E/F_n) = \begin{cases}
S(\bar{s},n) + \phi(p^n)\mu_{\bar{s}} + \lambda_{\bar{s}} - r_\infty & \text{if } n \text{ is odd}, \\
T(\bar{r},n) + \phi(p^n)\mu_{\bar{r}} + \lambda_{\bar{r}} - r_\infty & \text{if } n \text{ is even}.
\end{cases}
\]

*Proof.* Corollary 5.5 tells us that \( \text{rank}_{\mathbb{Z}_p} \mathcal{Y}'(E/F_n) \) is bounded independently of \( n \).
Thus, the same is true for \( \mathcal{Y}(E/F_n) \) thanks to Proposition 5.4. Also, \( \text{rank}_{\mathbb{Z}_p} \mathcal{X}^0(E/F_n) \) is bounded independently of \( n \) by Lemma 5.8. Thus, by the short exact sequence in Lemma 5.8(a), we have that \( \text{rank}_{\mathbb{Z}_p} \mathcal{X}(E/F_n) \) is bounded independently of \( n \). Hence, part (A) now follows from the well-known exact sequence

\[
0 \to E(F_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \text{Sel}_{p^\infty}(E/F_n) \to \mathcal{I}_p(E/F_n) \to 0.
\]

By Lemma 5.4, the short exact sequence (5.4) implies that

\[
\nabla \mathcal{I}_p(E/F_n) = \nabla \mathcal{X}(E/F_n) - \nabla E(F_n) \otimes \mathbb{Z}_p.
\]

Part (A) tells us that \( \nabla E(F_n) \otimes \mathbb{Z}_p = r_\infty \) for \( n \gg 0 \). It remains to calculate \( \nabla \mathcal{X}(E/F_n) \).

Let \( \bar{s} \in \{\bar{s}, \bar{r}\}^{2 \varphi} \). On replacing \( \bar{s} \) in the Poitou-Tate exact sequence (5.2) by \( \bar{s} \), we have the exact sequence

\[
0 \to \text{coker} \text{Col}_{\bar{s}} \circ \text{loc}_p \to \text{Sel}^\vee(E/F_\infty)^{\bar{s}} \to \mathcal{X}^0(E/F_\infty) \to 0.
\]

Taking \( \Gamma_n \)-invariant, we obtain a six terms exact sequence

\[
0 \to H_1(\Gamma_n, \text{coker} \text{Col}_{\bar{s}} \circ \text{loc}_p) \to H_1(\Gamma_n, \text{Sel}_{\bar{s}}(E/F_\infty)^{\bar{s}}) \to H_1(\Gamma_n, \mathcal{X}^0(E/F_\infty)) \to (\text{coker} \text{Col}_{\bar{s}} \circ \text{loc}_p)_{\Gamma_n}
\]

\[
\text{Sel}_{\bar{s}}(E/F_\infty)^{\bar{s}} \to \mathcal{X}^0(E/F_\infty)_{\Gamma_n} \to 0.
\]
Note that $H_1(\Gamma_n, -) = (-)^{\Gamma_n}$ and one can easily verify that the transition maps on these terms are given by multiplication by $1 + \gamma_n + \cdots + \gamma_n^{-1}$. We may therefore apply Lemma 5.5 to conclude that the Kobayashi ranks of the leftmost three terms vanish for $n \gg 0$. Lemma 5.6 thus implies the equality

$$\nabla A^0(E/F_{\infty})_{\Gamma_n} + \nabla (\coker Col_\varpi \circ \text{loc}_p)_{\Gamma_n} = \nabla \text{Sel}^\varpi(E/F_{\infty})_{\Gamma_n}$$

for $n \gg 0$. Thus, on applying Proposition 5.2(b) to the torsion $A$-module $\text{Sel}^\varpi(E/F_{\infty})_{\Gamma_n}$, we obtain

$$\nabla X^0(E/F_{\infty})_{\Gamma_n} = \nabla (\coker Col_\varpi \circ \text{loc}_p)_{\Gamma_n} - \rho(p^n)\mu_\varpi - \lambda_\varpi.$$  

Combining this with Lemma 5.8, Proposition 5.6 and Corollary 5.5, we deduce that

$$\nabla X(E/F_n) = \begin{cases} S(\varpi, n) + \rho(p^n)\mu_\varpi + \lambda_\varpi & \text{if } n \text{ is odd}, \\ T(\varpi, n) + \rho(p^n)\mu_\varpi + \lambda_\varpi & \text{if } n \text{ is even}. \end{cases}$$

Hence the result. \hfill \Box

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