Failure of interpolation in the intuitionistic logic of constant domains

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Abstract

This paper shows that the interpolation theorem fails in the intuitionistic logic of constant domains. This result refutes two previously published claims that the interpolation property holds.

1 Introduction

The main result of this paper demonstrates that the interpolation theorem fails in the intuitionistic logic of constant domains. This shows that two previously published proofs of the interpolation property are incorrect.

The model theory for the intuitionistic logic of constant domains (CD) was proposed by Grzegorczyk in a paper [9] of 1964, inspired by Paul Cohen’s notion of forcing, as well as the earlier semantics for intuitionistic logic proposed by Evert W. Beth [1, 2]. The logic resulting from the semantics is stronger than intuitionistic predicate logic, since it includes the scheme

\[(D) \quad \forall x (A \lor B) \rightarrow (A \lor \forall x B),\]

where \(x\) is not free in \(A\); this scheme is not provable intuitionistically. Grzegorczyk proposed the semantics as a “philosophically plausible formal interpretation of intuitionistic logic,” independently of the near-contemporary work of Saul Kripke [11]. Grzegorczyk observes that his semantics validates the schema (D), but goes on to propose a modification to the forcing relation for disjunctions and existential formulas that gives an exact interpretation for intuitionistic predicate logic.

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Sabine Görnemann [8] proved that the addition of the scheme to intuitionistic predicate logic is sufficient to axiomatize Grzegorczyk’s logic; D. Klemke [10] and Dov Gabbay [5] gave independent proofs of the same result.

There are two published proofs that the interpolation theorem holds for the intuitionistic logic of constant domains. The first is by Dov M. Gabbay [6], who gives a model-theoretic proof. The second is by López-Escobar [13], using proof theory. In a later article [14], López-Escobar admitted that there was an error in his earlier proof of interpolation.

The first author of the present paper wrote in his review of [14] in the Russian review journal ‘Mathematika’ [RZhmat], 7A70, July 1984: “The first proof of the interpolation theorem for CD given by Dov Gabbay (reviewed in RZhmat, 11A83, November 1971) contained gaps and specialists have different opinions on whether these gaps were closed later. The proof from the previous paper by the present author [L.-E.] (reviewed in RZhmat, 12A40, December 1981) consisted in essence of the reference to a cut-elimination theorem for some Gentzen-type axiomatization for CD. In the paper under review its author reproduces a counterexample to this claim given by M. Fitting. (Another counterexample was given by the present reviewer in RZhmat, 12A40, December 1981.) Therefore the question of the validity of the interpolation theorem for CD is not clear. The present author [L.E.] tries to prove that CD does not have any cut-free Gentzen-style axiomatization with a finite number of rules where formulas are analyzed only up to a finite depth $k$. He concludes this from the statement of necessity in such axiomatics of a derivable formula

$$\forall x(p \lor B(x)) \rightarrow (p \lor (q \rightarrow \forall x_1 \ldots x_{k-1} \forall x B(x)))$$

However this formula is derivable in the axiomatics obtained by adding the rule

$$\frac{p, \Gamma \Rightarrow C, \forall x B x, \Gamma \Rightarrow C}{\forall x(p \lor B x), \Gamma \Rightarrow C}$$

to ordinary intuitionistic rules. The error by the author is in the case C2 which is not treated.”

In his paper of 1977, Gabbay shows that the strong Robinson consistency theorem does not hold for CD; however, he goes on to claim that the weak Robinson consistency theorem holds for CD, and from this deduces the interpolation theorem. The first (negative) result appears in the comprehensive monograph co-authored by Gabbay and Maksimova [7], but the proof of interpolation does not appear.

Our non-interpolation result can be used to justify the impression of E. G. K. López-Escobar in [13] that CD does not admit a “good” cut-free formulation. Recall that one of the familiar proofs of interpolation for classical predicate logic (say in [17]) goes by induction on a cut-free Gentzen-style derivation: interpolants are defined for axioms, and simple prescriptions are given for transferring interpolants from premises of an inference to its conclusion. Hence our result shows that CD does not admit a formulation where interpolants are defined for axioms and there is an explicit definition of the interpolant for the conclusion of every inference rule from interpolants for the premises of the rule.
In fact this idea motivated the method by which the first author of the present paper constructed the counterexample to interpolation treated below. The implication $\Gamma \rightarrow \Delta$ lacking an interpolant was designed in such a way that any obvious (to the author) proof of it would involve a cut over a formula connecting predicates to be eliminated with the schema (D) above.

An important non-interpolation result was obtained in [4]. Kit Fine proved there that any normal predicate modal logic with Barcan formula as an axiom between $\mathbf{K}$ and $\mathbf{S5}$ lacks interpolation. One can try to get our result from this using the Gödel-Tarski translation $\Box$ of intuitionistic logic into $\mathbf{S4}$ which just prefixes the necessity symbol $\Box$ to every subformula. The result $F\Box$ of this translation is S4-valid (derivable) if and only if $F$ is valid (derivable) intuitionistically. Moreover the translation is sound and faithful also for constant domains, that is as an operation from $\mathbf{CD}$ into the modal predicate logic based on $\mathbf{S4}$ plus the Barcan formula. There are however essential obstacles blocking an attempt to reduce the problem for $\mathbf{CD}$ to similar problem for modal systems. First, counterexamples to interpolation used by Kit Fine do not translate in an obvious way back from modal logic to intuitionistic logic or $\mathbf{CD}$. Second, our counterexample $\Gamma \rightarrow \Delta$ does in fact have a modal interpolant. More about this in the section 3. Third, Fine refutes not only Craig interpolation, but also Beth definability theorem for his modal systems. It is not known at this moment whether $\mathbf{CD}$ has the Beth property, although this does not look plausible.

A review of [4] by Saul Kripke [12] analyzed Fine’s construction in terms of second order quantification of “redundant” variables and formed a background for computations in our section 3.

The counterexample $\Gamma \rightarrow \Delta$ to interpolation for $\mathbf{CD}$ given at the beginning of that section was constructed by the first author at the beginning of 2008. The third author joined the efforts to prove that no interpolant exists for the example in 2011 and computed “first order equivalents” of $\exists \Gamma \forall \Delta$ and $\forall \exists \Delta$ used in our proof. (These results can be stated in a language recalling the language of hybrid modal logic [3].) After that our efforts were stalled till the second author arrived in Stanford and joined the project. This resulted in a construction of models proving an interpolant for $\Gamma \Rightarrow \Delta$ is indeed impossible based on his characterization of first-order translations of intuitionistic formulas [16].

In Section 2 we recall the definition of $\mathbf{CD}$ and its model theory. In Section 3 we present our counterexample and compute “candidate interpolants” in the first order language of Kripke-style models. Section 4 contains the definition of asimulation (a generalization of bisimulation for modal logic) due to the second author and proof of the easy part (necessity) of asimulation condition needed for our result. Section 5 contains the construction of the pair of models proving non-interpolation, and a detailed proof of the failure of interpolation for $\mathbf{CD}$.

2 Model theory

We formulate the language $L$ of the logic $\mathbf{CD}$ using the propositional connectives $\land$, $\lor$, $\rightarrow$ and $\bot$, together with the universal and existential quantifiers $\forall$ and
∃: the negation operator \( \neg A \) is defined as \( A \rightarrow \bot \). The atomic formulas are of the form \( P x_1, \ldots, x_k \); we allow the case where the sequence of variables \( x_1, \ldots, x_k \) is empty, that is to say, where \( P \) is a propositional variable. We use the notation \( A[x_1, \ldots, x_k] \), or \( A[\vec{x}] \), for a formula of the language \( L \), where all the free variables in the formula appear in the sequence \( x_1, \ldots, x_k \) (some of the variables in the sequence may not appear in the formula). We employ the notation \( L(P, Q) \) for the sublanguage of \( L \) in which the only predicate symbols are the one-place predicates \( P \) and \( Q \).

If \( D \) is a non-empty set, then the language \( L(D) \) is obtained from \( L \) by adding a distinct constant \( a \) for every element \( a \) in \( D \). If \( A[a_1/x_1, \ldots, a_k/x_k] \) is a sentence in the expanded language \( L(D) \), where \( A[x_1, \ldots, x_k] \) is a formula of \( L \), then we define \( A[a_1/x_1, \ldots, a_k/x_k] \) to be a theorem of \( CD \) if the universal closure \( \forall x_1 \ldots \forall x_k A[x_1, \ldots, x_k] \) is a theorem of \( CD \).

Grzegorczyk’s model theory is very similar to that of Kripke [11], but simpler. Kripke’s model theory involves a quasi-ordered set of classical models, where the domains of the models can expand along the quasi-ordering. By contrast, Grzegorczyk’s models have a fixed, constant domain. Thus the Grzegorczyk model theory represents a static ontology, whereas Kripke’s represents an expanding ontology, where new objects can be created as knowledge grows.

A Grzegorczyk-model, or \( G \)-model for short, is a structure

\[
\mathcal{M} = (W, \leq, w, D, \phi),
\]

where \( W \) is a non-empty set (the set of information states in the terminology of Grzegorczyk), \( \leq \) is a quasi-ordering on \( W \) (a reflexive transitive relation on \( W \)), \( w \) is the base state in the model satisfying \( \forall v \in W (w \leq v) \), \( D \) is a non-empty set, and for each \( k \)-ary predicate symbol \( P \) in \( L \), \( \phi(P) \) is a \( k + 1 \)-ary relation contained in \( W \times D^k \) that satisfies the monotonicity condition

\[
[v \leq w \land \langle v, a_1, \ldots, a_k \rangle \in \phi(P)] \Rightarrow \langle w, a_1, \ldots, a_k \rangle \in \phi(P).
\]

The forcing relation \( \models_{\mathcal{M}} \) holds between states in \( W \) and sentences in \( L(D) \) by the following inductive definition:

1. \( v \models_{\mathcal{M}} P a_1, \ldots, a_k \Leftrightarrow \langle v, a_1, \ldots, a_k \rangle \in \phi(P) \),
2. \( v \models_{\mathcal{M}} A \land B \Leftrightarrow (v \models_{\mathcal{M}} A \land v \models_{\mathcal{M}} B) \),
3. \( v \models_{\mathcal{M}} A \lor B \Leftrightarrow (v \models_{\mathcal{M}} A \lor v \models_{\mathcal{M}} B) \),
4. \( v \models_{\mathcal{M}} A \rightarrow B \Leftrightarrow \forall w \geq v (w \models_{\mathcal{M}} A \rightarrow w \models_{\mathcal{M}} B) \),
5. \( v \models_{\mathcal{M}} \bot \) never holds,
6. \( v \models_{\mathcal{M}} \exists x A \Leftrightarrow \exists a \in D (v \models_{\mathcal{M}} A[a/x]) \),
7. \( v \models_{\mathcal{M}} \forall x A \Leftrightarrow \forall a \in D (v \models_{\mathcal{M}} A[a/x]) \).
It is easy to verify that this semantics is sound for \( \text{CD} \), in the sense that if \( v \) is a state in a Grzegorczyk-model, and \( A \) a sentence of \( L(D) \) that is a theorem of \( \text{CD} \), then \( v \models_\mathcal{M} A \). The completeness theorem for \( \text{CD} \) asserts that if \( A \) is a sentence that is not a theorem of \( \text{CD} \), then there is a Grzegorczyk-model \( \mathcal{M} \) with base point \( v \) so that \( v \not\models_\mathcal{M} A \).

The semantics is easily extended to a second-order version, where the second-order variables of arity \( k \) range over \( k+1 \)-ary relations \( R \) over \( W \times D^k \) that satisfy a version of the monotonicity condition above:

\[
[v \leq w \land (v, a_1, \ldots, a_k) \in R] \Rightarrow (w, a_1, \ldots, a_k) \in R.
\]

We shall use the second-order semantics only in the one-quantifier form.

3 The Counterexample

In this section, we produce the counterexample used in refuting the interpolation theorem for \( \text{CD} \). The two formulas forming the antecedent and consequent of the implication for which no interpolant exists in \( \text{CD} \) are as follows:

\[
\Gamma = [\forall x \exists y (Py \land (Qy \rightarrow Rx)) \land \neg \forall x Rx],
\]

\[
\Delta = \forall x (Px \rightarrow (Qx \lor S)) \rightarrow S.
\]

Lemma 3.1 The implication \( \Gamma \rightarrow \Delta \) is valid in all Grzegorczyk-models.

Proof. We give two proofs, the first model-theoretic, the second a deductive proof in a sequent calculus.

First, we give a model-theoretic argument for the Lemma. Let \( v \) be a state in a Grzegorczyk-model \( \mathcal{M} \) with domain \( D \) so that \( v \models_\mathcal{M} \Gamma \); we wish to show that \( v \models_\mathcal{M} \Delta \). Let \( w \) be a state in \( \mathcal{M} \) so that \( v \leq w \), where \( w \models_\mathcal{M} \forall x Px \rightarrow (Qx \lor S) \).

For an arbitrary \( a \in D \), since \( v \models_\mathcal{M} \Gamma \), \( w \models_\mathcal{M} \exists y (Py \land (Qy \rightarrow Ra)) \), so that for some \( b \in D \), \( w \models_\mathcal{M} Pb \land (Qb \rightarrow Ra) \). We also have \( w \models_\mathcal{M} Pb \land (Qb \lor S) \), hence \( w \models_\mathcal{M} (Qb \lor S) \) and so \( w \models_\mathcal{M} Ra \lor S \). Since \( a \) was an arbitrary element of \( D \), it follows that \( w \models_\mathcal{M} \forall x (Rx \lor S) \). Hence, by (D), \( w \models_\mathcal{M} \forall x Rx \lor S \). Since \( v \models_\mathcal{M} \Gamma \), it follows that \( w \models_\mathcal{M} \neg \forall x Rx \), so \( w \models_\mathcal{M} S \), showing that \( v \models_\mathcal{M} \Delta \).

It follows by the completeness theorem for \( \text{CD} \) that \( \Gamma \rightarrow \Delta \) is a theorem of \( \text{CD} \). However, we can argue directly for this in a multiple succedent sequent calculus formulation of \( \text{CD} \) (see, e.g. [15]), using the notation \( A_1 \ldots A_n \Rightarrow B_1 \ldots B_m \) for sequents.

Employing the abbreviation

\[
\alpha \equiv \forall x (Px \rightarrow (Qx \lor S)),
\]
we derive intuitionistically a nucleus of our implication.

\[ (Qy \rightarrow Rx), Qy \Rightarrow Rx \lor S \quad S \Rightarrow Rx \lor S \]

\[ (Qy \rightarrow Rx), (Qy \lor S) \Rightarrow Rx \lor S \quad Py \Rightarrow Py \]

\[ Py, (Qy \rightarrow Rx), (P_{y} \rightarrow (Qy \lor S)) \Rightarrow Rx \lor S \]

\[ (P_{y} \land (Qy \rightarrow Rx)), (Py \rightarrow (Qy \lor S)) \Rightarrow Rx \lor S \]

\[ (P_{y} \land (Qy \rightarrow Rx)), \forall x(P_{x} \rightarrow (Qx \lor S)) \Rightarrow Rx \lor S \]

\[ \exists y(Py \land (Qy \rightarrow Rx)), \alpha \Rightarrow Rx \lor S \]

\[ \forall x \exists y(Py \land (Qy \rightarrow Rx)), \alpha \Rightarrow Rx \lor S \]

\[ \forall x \exists y(Py \land (Qy \rightarrow Rx)), \alpha \Rightarrow x(Rx \lor S) \]

After that an application of the cut rule with an instance of the axiom \( D \) produces the required result. We use the abbreviation

\[ \beta := \forall x \exists y(Py \land (Qy \rightarrow Rx)). \]

\[ \begin{array}{c}
\forall x Rx \Rightarrow \forall x Rx \quad S \Rightarrow S \\
\forall x Rx \lor S \Rightarrow \forall x Rx, S \\
\beta, \alpha \Rightarrow \forall x (Rx \lor S) \\
\forall x (Rx \lor S) \Rightarrow \forall x Rx, S \\
D, \text{Cut} \\
\beta, \alpha \Rightarrow \forall x Rx, S \\
\forall x Rx, S \Rightarrow S \\
\beta, \neg \forall x Rx, \alpha \Rightarrow S \\
\Gamma, \alpha \Rightarrow S \\
\Gamma \Rightarrow \Delta
\end{array} \]

\[ \square \]

**Corollary 3.1** The second-order implication \( \exists R \Gamma \rightarrow \forall S \Delta \) is valid in all G-models.

It is possible to shed some further light on Lemma 3.1 and Corollary 3.1 through an analysis of the second-order sentences \( \exists R \Gamma \) and \( \forall S \Delta \). In the following Lemma, we extract the content of these two sentences in the form of semantical conditions on G-models.

**Lemma 3.2** Let \( M \) be a G-model with base point \( v \).

1. \( v \models_{M} \exists R \Gamma \) if and only if \( M \) satisfies the semantical condition

\[ I(P, Q) \equiv \forall w \exists a(v \models_{M} Pa \land w \not\models_{M} Qa). \]

2. \( v \models_{M} \forall S \Delta \) if and only if \( M \) satisfies the semantical condition

\[ J(P, Q) \equiv \forall w \exists a(w \models_{M} Pa \land w \not\models_{M} Qa). \]
Proof. Part 1 (⇒). Assume that \( \mathbf{v} \models \mathcal{M} \not\exists \Gamma \); let \( w \) be an arbitrary state in \( \mathcal{M} \). Then since \( \mathbf{v} \models \mathcal{M} \not\forall x Rx \), \( w \not\models_{\mathcal{M}} \forall x Rx \), so that for some \( b \in D \), \( w \not\models_{\mathcal{M}} R b \). Since \( \mathbf{v} \models \mathcal{M} \not\exists y (Py \land (Qy \rightarrow Rb)) \), so that for some \( a \in D \), \( \mathbf{v} \models \mathcal{M} P a \land (Qa \rightarrow Rb) \). It follows that \( w \models_{\mathcal{M}} Qa \), showing that \( \mathcal{M} \) satisfies \( I(P, Q) \).

Part 1 (⇐). Conversely, assume that \( \mathcal{M} \) satisfies \( I(P, Q) \). Define \( E \subseteq D \) as follows:

\[ E := \{ a \in D \mid \exists w \in \mathcal{M}(\mathbf{v} \models \mathcal{M} P a \land w \not\models_{\mathcal{M}} Qa) \} \]

By assumption, \( E \) is non-empty. Let \( f \) be a surjective map from \( D \) to \( E \), and define for any \( a \in D \), \( w \in W \),

\[ w \models_{\mathcal{M}} Ra \iff w \models_{\mathcal{M}} Q(f(a)). \]

Then for any \( a \in D \), \( \mathbf{v} \models_{\mathcal{M}} P(f(a)) \), and \( \mathbf{v} \models_{\mathcal{M}} (Q(f(a)) \leftrightarrow Ra) \), so that \( \mathbf{v} \models_{\mathcal{M}} \not\exists y (Py \land (Qy \rightarrow Rx)) \). Furthermore, for any \( w \in \mathcal{M} \), there is a \( b \in E \) so that \( \mathbf{v} \models_{\mathcal{M}} P b \) and \( w \not\models_{\mathcal{M}} Qb \). Choose \( a \in D \) so that \( f(a) = b \); we have \( w \not\models_{\mathcal{M}} (Q(f(a)) \leftrightarrow Ra) \). Since \( w \not\models_{\mathcal{M}} Q(f(a)) \), \( w \not\models_{\mathcal{M}} Ra \). This shows that \( w \not\models_{\mathcal{M}} \forall x Rx \), so \( w \not\models_{\mathcal{M}} \not\forall x Rx \).

Part 2 (⇐). Assume that \( \mathcal{M} \) satisfies \( J(P, Q) \). If \( w \) is an arbitrary state in \( \mathcal{M} \), where \( w \models_{\mathcal{M}} \forall x (Px \rightarrow (Qx \land S)) \), then by assumption, there is an \( a \) so that \( w \models_{\mathcal{M}} Pa \land w \not\models_{\mathcal{M}} Qa \). But \( w \not\models_{\mathcal{M}} (Pa \rightarrow (Qa \land S)) \), so \( S \) must be true at \( w \).

Part 2 (⇒). For the converse, let us assume that \( \mathcal{M} \) does not satisfy \( J(P, Q) \). Then there must be a \( w \) in \( \mathcal{M} \) so that

\[ \forall a \in D (w \models_{\mathcal{M}} Pa \rightarrow w \models_{\mathcal{M}} Qa). \]

Now define \( S \) by setting \( u \models_{\mathcal{M}} S \) if and only if \( u > w \); that is to say, the proposition \( S \) holds exactly at those states accessible from \( w \), but not identical with \( w \). We have to argue that \( w \models_{\mathcal{M}} \forall x (Px \rightarrow (Qx \land S)) \). Let \( u \) be an arbitrary state accessible from \( w \). If \( u \) is \( w \) itself, we need to show that for any \( a \), whenever \( u \models_{\mathcal{M}} Pa \), then \( u \models_{\mathcal{M}} (Qa \land S) \); this follows from our assumption. On the other hand, if \( u \) is not \( w \), then \( S \) holds at \( u \), by construction, so \( (Qa \land S) \) holds for any \( a \). So, we’ve shown that for any \( a \), \( w \models_{\mathcal{M}} (Pa \rightarrow (Qa \land S)) \), hence \( w \models_{\mathcal{M}} \forall x (Px \rightarrow (Qx \land S)) \). However, \( S \) fails at \( w \), by construction, so \( \mathbf{v} \not\models_{\mathcal{M}} \forall S \Delta \).

In notation of Lemma 3.2 note that \( J(P, Q) \) is true in all models of \( \mathbf{S4+BF} \) (quantified \( \mathbf{S4} \) plus the Barcan formula) if and only if

\[ \mathbf{S4+BF} \models \Box \exists x (\Box Px \land \neg \Box Qx). \]

Hence the implication \( \Gamma \Rightarrow \Delta \) has a modal interpolant. Theorem 5.1 below shows that this modal formula is not equivalent to a Gödel-Tarski translation (prefixing \( \Box \) to all subformulas) of any intuitionistic formula.

Lemma 3.2 suggests a strategy to demonstrate the failure of the interpolation theorem, namely by showing that if \( \Theta(v) \) is a first-order property of \( G \)-models that is implied by \( I(P, Q) \), and also implies \( J(P, Q) \), then it cannot be expressed by a formula of \( \mathbf{CD} \) involving only the predicates \( P \) and \( Q \).
4 Asimulations

In this section, we introduce the basic concept of CD-asimulation, an asymmetric counterpart of the concept of bisimulation familiar from the literature of modal logic [3, Chapter 2]. The definition given below can be considered as a more general version of the notion of bisimulation for modal predicate logic defined by Johan van Benthem [18].

Definition 4.1 If \( \mathcal{M}_1 = \langle W_1, \leq_1, v, D_1, \phi_1 \rangle \) and \( \mathcal{M}_2 = \langle W_2, \leq_2, w, D_2, \phi_2 \rangle \) are G-models, then a CD-asimulation between \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) is a relation \( Z \) satisfying the following conditions:

1. \( Z \subseteq \bigcup_{k \geq 0} [(W_1 \times D_1^k) \times (W_2 \times D_2^k)] \cup [(W_2 \times D_2^k) \times (W_1 \times D_1^k)]; \)
2. \( \{ (v, dZW, \bar{c}) \land v \models_1 P[\bar{d}] \} \Rightarrow w \models_j P[\bar{e}], \) for \( P[\bar{x}] \) an atomic formula;
3. \( \{ (t, dZu, \bar{c}) \land u \leq_2 v \} \Rightarrow (\exists w \in W_1)(t \leq_1 w \land (w, dZW, \bar{c}) \land (v, \bar{c}ZW, \bar{d})); \)
4. \( \{ t \in W_1 \land (t, dZu, \bar{c}) \land f \in D_1 \} \Rightarrow (\exists g \in D_1)(t, d, fZu, \bar{c}, g); \)
5. \( \{ t \in W_1 \land (t, dZu, \bar{c}) \land g \in D_1 \} \Rightarrow (\exists f \in D_1)(t, d, fZu, \bar{c}, g), \)

where \( \{ i, j \} = \{ 1, 2 \} \), and \( \models_i \) and \( \models_j \) are the forcing relations in \( \mathcal{M}_i \) and \( \mathcal{M}_j \).

The concept of asimulation is due to the second author of this paper. The version we are using here is a simplified version of the general notion; the simplifications are possible because we are operating in the context of the constant domain semantics. The second author has defined a more general version [16] that is suited to the context of Kripke’s semantics for intuitionistic predicate logic, and has also proved that it serves to characterize the first-order properties expressed by formulas of propositional and predicate logic in that framework. In the present case, however, we do not need the full characterization theorems; the following Lemma is sufficient for our purpose of refuting the interpolation theorem for the logic CD.

Lemma 4.1 Let \( Z \) be a CD-asimulation between \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), and \( A[\bar{x}] \) a formula of \( L \). If \( t, dZu, \bar{c} \) and \( t \models_i A[\bar{d}] \), then \( u \models_j A[\bar{e}] \), where \( \{ i, j \} = \{ 1, 2 \} \).

Proof. By induction on the complexity of the formula \( A[\bar{x}] \). For atomic formulas, the lemma holds by the definition of CD-asimulation. The inductive steps for \( \land, \lor \) and \( \bot \) are straightforward.

Now assume that \( t, dZu, \bar{c} \) and that \( t \models_i A[\bar{d}] \Rightarrow B[\bar{d}] \). If \( u \leq_j v \) and \( v \models_j A[\bar{e}] \), then by the third condition in Definition 4.1,

\[
(\exists w \in W_1)(t \leq_1 w \land (w, dZu, \bar{c}) \land (v, \bar{c}ZW, \bar{d})).
\]

By inductive assumption, \( w \models_i A[\bar{d}] \), so that \( w \models_i B[\bar{d}] \), since \( t \models_i A[\bar{d}] \Rightarrow B[\bar{d}] \). Again, by inductive assumption, \( v \models_j B[\bar{e}] \), showing that \( u \models_j A[\bar{e}] \Rightarrow B[\bar{e}] \).
Assume that \( t, \vec{d}Zu, \vec{e} \) and that \( t \vdash_i A[\vec{d}, x] \). Then \( t \vdash_j A[\vec{d}, f] \), for some \( f \in D_i \). By the fourth condition in Definition 4.1, there is a \( g \) in \( D_j \) so that \( t, \vec{d}, fZu, \vec{e}, g \). By inductive assumption, \( u \vdash_j A[\vec{e}, g] \), so that \( u \vdash_j \exists xA[\vec{e}, x] \).

Assume that \( t, \vec{d}Zu, \vec{e} \) and that \( t \vdash_i \forall xA[\vec{d}, x] \). Let \( g \) be an arbitrary individual in \( D_j \). By the fifth condition in Definition 4.1, \((\exists f \in D_i)(t, \vec{d}, fZu, \vec{e}, g)\). Then \( t \vdash_i A[\vec{d}, f] \), so by inductive assumption, \( u \vdash_j A[\vec{e}, g] \), showing that \( u \vdash_j \forall xA[\vec{e}, x] \). \( \square \)

5 Refuting interpolation

In this section, we define two G-models, \( M_1 \) and \( M_2 \), together with a CD-asimulation between them (these definitions and the ideas of the proofs given below are due to the second author of this paper). This will enable us to carry out the strategy outlined in the comments following Lemma 3.2. The states in both models are quasi-partitions, given by the following definition. We use the notation \( \mathbb{N} \) for the set of positive natural numbers, and for \( k > 0, l \geq 0 \), we write \( k\mathbb{N} + l \) for the set \( \{kn + l | n \in \mathbb{N} \} \), and \( k\mathbb{N} \) for \( k\mathbb{N} + 0 \).

**Definition 5.1**

1. A quasi-partition \((A, B, C)\) is defined by the following conditions:
   
   (a) \( A \cup B \cup C = \mathbb{N} \);
   
   (b) \( A, B, C \) are pairwise disjoint;
   
   (c) \( A \) and \( C \) are infinite;
   
   (d) \( B \) is either empty or infinite.

2. A quasi-order \( \preceq \) on the set of all quasi-partitions is defined by
   
   \[(A, B, C) \preceq (D, E, F) \iff [A \subseteq D \land F \subseteq C].\]

We begin by defining two quasi-partitions that will serve as the base points of our two models. The first is \( v = (v_1, v_2, v_3) = (3\mathbb{N}, 3\mathbb{N} + 1, 3\mathbb{N} + 2) \); the second is \( w = (w_1, w_2, w_3) = (2\mathbb{N}, \emptyset, 2\mathbb{N} + 1) \).

**Definition 5.2** \( M_1 \) and \( M_2 \) are of the form \( \mathcal{M} = \langle W, \leq, u, D, \phi \rangle \); they are defined as follows.

1. The base points for \( M_1 \) and \( M_2 \) are \( v \) and \( w \) respectively;

2. The set of states for the models are as follows.
   
   (a) \( W_1 = \{ (A, B, C) | v \preceq (A, B, C), \text{ where } B \cap v_2 \text{ is infinite} \} \);
   
   (b) \( W_2 = \{ (A, B, C) | w \preceq (A, B, C), \text{ where } B \neq \emptyset \} \cup \{w\} \);

3. The ordering on both \( M_1 \) and \( M_2 \) is \( \preceq \);

4. \( D_1 = D_2 = \mathbb{N} \);
5. For \( i = 1, 2 \), the values \( \phi_i \) assigned to the predicate symbols \( P \) and \( Q \) are defined by
\[
\phi_i(P) = \{(v, a)|a \in v_1 \cup v_2\},
\phi_i(Q) = \{(v, a)|a \in v_1\}.
\]

The relation \( \subseteq \) is clearly reflexive and transitive. Furthermore, if \( (A, B, C) \subseteq (D, E, F) \), then because \( F \subseteq C, A \cup B \subseteq D \cup E \). In addition, \( A \subseteq D \), so that \( \phi_i \) satisfies the monotonicity property. Hence, both \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are G-models.

**Lemma 5.1** 1. The G-model \( \mathcal{M}_1 \) satisfies the condition \( I(P, Q) \); 2. The condition \( J(P, Q) \) fails in the G-model \( \mathcal{M}_2 \).

**Proof.** For the first part of the lemma, let \( u = (A, B, C) \) be a state in \( W_1 \), and \( v \) the base point of \( \mathcal{M}_1 \). By definition, \( B \cap v_2 \) is infinite. Choose \( k \in B \cap v_2 \). Then \( v \vdash_1 Pk \) and \( u \not\vdash_1 Qk \), showing that \( I(P, Q) \) holds in \( \mathcal{M}_1 \).

For the second part of the lemma, at the base point \( w \) of the G-model \( \mathcal{M}_2 \), we have \( \forall a \in D_2(w \vdash_2 P \Rightarrow w \vdash_2 Qa) \), because \( w_2 = \emptyset \). Hence, the condition \( J(P, Q) \) fails in the model \( \mathcal{M}_2 \).

We next need to define a CD-asimulation \( Z \) between the G-models \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \). The following notations are useful in stating the definition of \( Z \) and in demonstrating its properties. If \( \bar{u} \) is a sequence of elements from a set \( D \), then \( D \setminus \bar{u} \) is the result of removing all of the elements of \( \bar{u} \) from \( D \). For \( \bar{d}, \bar{e} \in \mathbb{N}^k \), \( S \subseteq \mathbb{N} \) we denote by \( [\bar{d} \mapsto \bar{e}] \) the binary relation \( \{\langle d_l, e_l \rangle | 1 \leq l \leq k \} \) and by \( [\bar{d} \mapsto \bar{e}]^{-1}(S) \) the set \( \{d_l | e_l \in S, 1 \leq l \leq k \} \).

Because sequences \( \bar{d}, \bar{e} \in \mathbb{N}^k \) may contain repetitions, the relation \( [\bar{d} \mapsto \bar{e}] \) may not be a bijection, and in fact, may not even be a function. However, if \( \bar{d} \) and \( \bar{e} \) have the same pattern of repetitions (for example, if \( \bar{d} = 1, 1, 2, 3, 2 \) and \( \bar{e} = 4, 4, 5, 6, 5 \) then \( [\bar{d} \mapsto \bar{e}] \) is a bijection, as is required in the next definition.

**Definition 5.3** Relative to the models \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) given in Definition 5.2, the relation \( Z \) is defined as follows:

1. \( Z \subseteq \bigcup_{k \geq 0}((W_1 \times D^k_1) \times (W_2 \times D^k_2)) \cup ((W_2 \times D^k_2) \times (W_1 \times D^k_1)) \);

2. \( \langle (A, B, C), \bar{d} \rangle Z \langle (D, E, F), \bar{e} \rangle \), where \( \bar{d} \in D^k_1, \bar{e} \in D^k_2, \{i, j\} = \{1, 2\}, \) if and only if the following conditions hold:

   (a) Relation \( [\bar{d} \mapsto \bar{e}] \) is a bijection.

   (b) If \( 1 \leq l \leq k \) and \( d_l \in A \) then \( e_l \in D \);

   (c) If \( 1 \leq l \leq k \) and \( d_l \in B \), then \( e_l \in D \cup E \).

It is easy to see that in the case when both \( \langle (A, B, C), \bar{d} \rangle Z \langle (D, E, F), \bar{e} \rangle \) and \( \langle (D, E, F), \bar{e} \rangle Z \langle (A, B, C), \bar{d} \rangle \) hold, conditions 2(b), (c) of this definition are equivalent, modulo other restrictions, to the following ones:

\[ d_l \in A \text{ iff } e_l \in D; \]
\(d_l \in B\) iff \(e_l \in E\),

for every \(1 \leq l \leq k\).

**Lemma 5.2** The relation \(Z\) in Definition 5.3 is a CD-asimulation between the \(G\)-models \(M_1\) and \(M_2\).

**Proof.** The first condition in Definition 4.1 is true by definition. For the second condition, assume that \(v, dZw, \vec{e}\) and that \(v \models_i P[\vec{d}]\), where \(P(x_l)\) is atomic, \(\vec{d} \in D_k^I\), and \(1 \leq l \leq k\). Thus we have \(v \models_i P[\vec{d}]\), where \(d = d_l\), so that \(d \in v_1 \cup v_2\); it follows that \(e = e_l \in w_1 \cup w_2\), by Definition 5.3, showing that \(v \models_j P[\vec{e}]\). The proof for atomic formulas \(Q(x_l)\) is similar.

For the third condition in Definition 4.1, assume that \(t, dZu, \vec{e}\), where \(t = (A, B, C), u = (D, E, F)\), and \(u \subseteq v, v = (G, H, I)\). By definition, \(u \subseteq v\). Two cases arise here: \(B\) is infinite, or \(B = \emptyset\).

In the first case, define \(w = (J, K, L)\) as follows:

\[
J = (A \setminus \vec{d}) \cup [\vec{d} \mapsto \vec{e}]^{-1}(G);
K = (B \setminus \vec{d}) \cup [\vec{d} \mapsto \vec{e}]^{-1}(H);
L = (C \setminus \vec{d}) \cup [\vec{d} \mapsto \vec{e}]^{-1}(I).
\]

We need to show that \((J, K, L)\) is indeed a successor of \((A, B, C)\) in \(M_1\), i.e. that \((A, B, C) \subseteq (J, K, L)\) and that if \(B \cap v_2\) is infinite, then so is \(K \cap v_2\). For the latter claim note that if \(B \cap v_2\) is infinite, then so is \((B \setminus \vec{d}) \cap v_2\), given that \(\vec{d}\) is finite. But since, according to our definition of \(K\) we have \((B \setminus \vec{d}) \subseteq K\), \(K \cap v_2\) must be infinite, too, showing that \((J, K, L)\) is a quasi-partition in \(M_1\).

The former claim, that is to say, the claim that \(A \subseteq J\) and \(L \subseteq C\), can be verified as follows. If \(a \in A\), and \(a\) is in \(\vec{d}\), say \(a = d_l\), then \(e_l \in D\), so \(e_l \in G\), from which it follows that \(d_l = a \in J\). Since \(I \subseteq F\), we have \([\vec{d} \mapsto \vec{e}]^{-1}(I) \subseteq [\vec{d} \mapsto \vec{e}]^{-1}(F) \subseteq C\), showing that \(L \subseteq C\). Finally, it follows by construction that for \(1 \leq l \leq k\), \(d_l \in J \iff e_l \in G\), and \(d_l \in K \iff e_l \in H\). Hence, \(w, dZv, \vec{e}\) and \(v, eZw, \vec{d}\), completing the proof of the third condition in the first case.

In the second case, where \(B = \emptyset\), we have \((A, B, C) = w = (2N, \emptyset, 2N + 1)\). In this case, we partition \(C \setminus \vec{d}\) into two disjoint infinite sets \(C_1\) and \(C_2\), and define \(w = (J, K, L)\) as follows:

\[
J = (A \setminus \vec{d}) \cup [\vec{d} \mapsto \vec{e}]^{-1}(G);
K = C_1 \cup [\vec{d} \mapsto \vec{e}]^{-1}(H);
L = C_2 \cup [\vec{d} \mapsto \vec{e}]^{-1}(I).
\]

Then \(J, K, L\) are all infinite by construction. The remainder of the proof of the third condition in the second case is essentially the same as for the first case, the only difference being that in this case we do not need to establish that if \(B \cap v_2\) is infinite, then so is \(K \cap v_2\).
For the fourth condition in Definition 4.1, assume that \( t \in W_i, (t, \vec{d}Zu, \vec{e}) \), where \( t = (G, H, I) \), \( u = (J, K, L) \), and \( f \in D_i \). If \( f = d_l \) for some \( 1 \leq l \leq k \), then set \( g := e_l \). Otherwise, given that \( f \notin \vec{d} \), choose \( g \in D_j \) so that \( g \in J \setminus \vec{e} \) (this is possible because \( J \) is infinite). Thus we have shown the existence of a \( g \) so that \( t, \vec{d}, fZu, \vec{e}, g \).

The fifth condition is proved by a similar argument. Namely, assume that \( t \in W_i, (t, \vec{d}Zu, \vec{e}) \), where \( t = (G, H, I) \), \( u = (J, K, L) \), and \( g \in D_j \). If \( g = e_l \) for some \( 1 \leq l \leq k \), then set \( f := d_l \). Otherwise, since \( g \notin \vec{e} \), choose \( f \in D_i \) so that \( g \in I \setminus \vec{d} \) (this is possible because \( I \) is infinite).

We now have all the material required to refute interpolation for CD.

**Theorem 5.1** The implication \( \Gamma \rightarrow \Delta \) is valid in all Grzegorczyk-models, but has no interpolant in CD.

**Proof.** The validity of the implication is proved in Lemma 3.1. Now let us assume that the implication has an interpolant in CD, that is to say, a sentence \( \Theta \) in the language \( L(P, Q) \) so that both \( \Gamma \rightarrow \Theta \) and \( \Theta \rightarrow \Delta \) are valid in all G-models. Then \( \Theta \) is also an interpolant for the second-order implication \( \exists RT \rightarrow \forall S \Delta \).

Since the G-model \( M_1 \) satisfies the condition \( I(P, Q) \), by Lemma 5.1, its base point also satisfies the second-order condition \( \exists RT \), by Lemma 3.2, so \( v \models_1 \Theta \). Moreover, it is clear from Definition 5.3 that we have \( \langle v, \Lambda \rangle \models Z \langle w, \Lambda \rangle \), where \( \Lambda \) is the empty sequence. Lemma 5.2 shows that \( Z \) is a CD-asimulation between \( M_1 \) and \( M_2 \), so by Lemma 4.1, \( v \models_2 \Theta \), hence \( w \models_2 \forall S \Delta \). However, Lemma 3.2 shows that \( M_2 \) must satisfy \( J(P, Q) \), contradicting Lemma 5.1. Consequently, no such interpolant for the implication \( \Gamma \rightarrow \Delta \) can exist.

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