On analysis in differential algebras and modules

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Abstract

A short introduction to the mathematical methods and technics of differential algebras and modules adapted to the problems of mathematical and theoretical physics is presented.

Keywords: algebra, differential algebra, module, differential module, multiplicator, differentiation, de Rham complex, spectral sequence, variation bicomplex.

1 Introduction.

Differential algebras (see, for example, [3],[4],[5]) are widely known and used in algebra and topology, while their applications in mathematical physics are far less acknowledged (but used implicitly, especially in partial differential equations). Here we propose a short introduction to the mathematical methods and technics of differential algebras and modules adapted to the problems of mathematical and theoretical physics. Exposition is based on the personal experience of the author, the books [3],[4],[5],[6],[7],[10],[17] and the works [8],[9],[11],[2],[12],[13].

We freely use notation, conventions and results of the paper [1]. In particular:

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• $F = \mathbb{R}, \mathbb{C}$;

• $N = \{1, 2, 3, \ldots \} \subseteq \mathbb{Z}_+ = \{0, 1, 2, \ldots \} \subseteq \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots \}$.

We, also, use the following notation of homology theory (see, e.g., [6]):

• $\text{Hom}(S; S')$ is the set of all mappings from a set $S$ to a set $S'$;

• $\text{Hom}_F(L; L')$ is the linear space of all linear mappings from a linear space $L$ to a linear space $L'$;

• $\text{Hom}_A(M; M')$ is the $A$-module of all $A$-linear mappings from an $A$-module $M$ to an $A$-module $M'$, where $A$ is an associative commutative algebra;

• $\text{Hom}_{\text{alg}}(A; A')$ is the set of all algebra morphisms from $A$ to $A'$, where $A, A'$ are associative commutative algebras;

• $\text{Hom}_{\text{Lie}}(\mathfrak{a}; \mathfrak{a}')$ is the set of all Lie algebra morphisms from a Lie algebra $\mathfrak{a}$ to a Lie algebra $\mathfrak{a}'$.

Remind, a set $\mathfrak{a}$ is called a Lie $A$-algebra if it has two structures:

• the structure of a Lie algebra with a Lie bracket $[,]$;

• the structure of an $A$-module, where $A$ is an associative commutative algebra;

and these structures are related by the matching condition,

• $[X, f \cdot Y] = Xf \cdot Y + f \cdot [X, Y]$ for all $X, Y \in \mathfrak{a}$ and $f \in A$.

We denote by $\text{Hom}_{\text{Lie}\cap A}(\mathfrak{a}; \mathfrak{a}') = \text{Hom}_{\text{Lie}}(\mathfrak{a}; \mathfrak{a}') \cap \text{Hom}_A(\mathfrak{a}; \mathfrak{a}')$ the set of all Lie $A$-morphisms from a Lie $A$-algebra $\mathfrak{a}$ to a Lie $A$-algebra $\mathfrak{a}'$.

All linear operations are done over the number field $F$. The summation over repeated upper and lower indices is as a rule assumed. If the corresponding index set is infinite, we assume that the summation is correctly defined. If objects under the study have natural topologies, we assume that the corresponding mappings are continuous. For example, if $S$ and $S'$ are topological spaces, then $\text{Hom}(S; S')$ is the set of all continuous mappings from $S$ to $S'$.

We use the terminology accepted in the algebra-geometrical approach to partial differential equations, because they are the main example of the technics developed below.
2 Differential algebras.

In this section (see [1],[2] for more detail):

- \( \mathcal{A} \) is an associative commutative algebra;
- \( \mathcal{M} = \mathcal{M}(\mathcal{A}) = \text{End}_\mathcal{A}(\mathcal{A}) \) is the unital associative algebra of all multiplicators of the algebra \( \mathcal{A} \);
- \( \mathcal{D} = \mathcal{D}(\mathcal{A}) \) is the Lie \( \mathcal{M} \)-algebra of all differentiations of the algebra \( \mathcal{A} \).

**Definition 1.** A differential algebra is a pair \( (\mathcal{A}, \mathcal{D}) \), where \( \mathcal{D} = \mathcal{D}(\mathcal{A}) \) is a fixed subalgebra (Cartan subalgebra) of the Lie \( \mathcal{M} \)-algebra \( \mathcal{D} = \mathcal{D}(\mathcal{A}) \).

**Definition 2.** A pair \( (F, \varphi) \), where the mapping \( F \in \text{Hom}_{\text{alg}}(\mathcal{A}; \mathcal{B}) \), and the mapping \( \varphi \in \text{Hom}_{\text{Lie}}(\mathcal{D}(\mathcal{A}); \mathcal{D}(\mathcal{B})) \), is called a morphism of a differential algebra \( (\mathcal{A}, \mathcal{D}(\mathcal{A})) \) into a differential algebra \( (\mathcal{B}, \mathcal{D}(\mathcal{B})) \), if the action \( F(Xf) = (\varphi X)(Ff) \) for all \( f \in \mathcal{A}, \ X \in \mathcal{D}(\mathcal{A}) \). In this case we shall write \( (F, \varphi) : (\mathcal{A}, \mathcal{D}(\mathcal{A})) \to (\mathcal{B}, \mathcal{D}(\mathcal{B})) \).

Let \( (\mathcal{A}, \mathcal{D}) \) be a differential algebra.

**Definition 3.** A subalgebra \( \mathcal{B} \) (an ideal \( \mathcal{I} \)) of the algebra \( \mathcal{A} \) is called differential if \( \mathcal{D}\mathcal{B} \subset \mathcal{B} \), i.e., \( Xf \in \mathcal{B} \) for all \( X \in \mathcal{D} \) and \( f \in \mathcal{B} \), \( (\mathcal{D}\mathcal{I} \subset \mathcal{I}) \). In this case the pair \( (\mathcal{B}, \mathcal{D}) \) (the pair \( (\mathcal{I}, \mathcal{D}) \)) is a differential algebra.

In particular, if \( (\mathcal{I}, \mathcal{D}) \) is a differential ideal of the differential algebra \( (\mathcal{A}, \mathcal{D}) \), then the quotient differential algebra \( (\mathcal{A}/\mathcal{I}, \mathcal{D}) \) is defined by the rule: \( \mathcal{A}/\mathcal{I} = \{X = [X] \in \mathcal{D}(\mathcal{A}) \mid X \in \mathcal{D}\}, \ [X][f] = [Xf] \) for all \( f \in \mathcal{A} \), where \( [f] = f + \mathcal{I}, \ [Xf] = Xf + \mathcal{I} \in \mathcal{A} \).

**Proposition 1.** For every morphism \( (F, \varphi) : (\mathcal{A}, \mathcal{D}(\mathcal{A})) \to (\mathcal{B}, \mathcal{D}(\mathcal{B})) \) the kernel \( \text{Ker} F \) is a differential ideal of \( (\mathcal{A}, \mathcal{D}(\mathcal{A})) \), while the image \( \text{Im} F \) will be a differential subalgebra of \( (\mathcal{B}, \mathcal{D}(\mathcal{B})) \) if, in addition, the mapping \( \varphi : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B}) \) is a surjection.

**Definition 4.** An element \( f \in \mathcal{A} \) is called \( \mathcal{D} \)-constant if \( \mathcal{D}f = 0 \), i.e., \( Xf = 0 \) for all \( X \in \mathcal{D} \).

Let \( \mathcal{A}_\mathcal{D} \) be the set of all \( \mathcal{D} \)-constant elements of the algebra \( \mathcal{A} \). Clear, \( \mathcal{A}_\mathcal{D} \) is a subalgebra of \( \mathcal{A} \).
Proposition 2. Let \((F, \varphi) : (A, \mathcal{D}(A)) \to (B, \mathcal{D}(B))\), where \(\varphi\) is a surjection. Then \(F|_{A_{\mathcal{D}}} : A_{\mathcal{D}} \to B_{\mathcal{D}}\).

Definition 5. A multiplicator \(R \in \mathcal{M}\) is called \(\mathcal{D}\)-constant if \([\mathcal{D}, R] = 0\), i.e., \([X, R] = 0\) for all \(X \in \mathcal{D}\).

Let \(\mathcal{M}_{\mathcal{D}}\) be the set of all \(\mathcal{D}\)-constant multiplicators of the algebra \(\mathcal{M}\). Clear,

- \(\mathcal{M}_{\mathcal{D}}\) is an unital subalgebra of the algebra \(\mathcal{M}\);
- \(A_{\mathcal{D}}\) is a submodule of the \(\mathcal{M}_{\mathcal{D}}\)-module \(A\).

Definition 6. A differentiation \(X \in \mathcal{D}\) is called a Lie-Bäcklund differentiation if \([\mathcal{D}, X] \subset \mathcal{D}\), i.e., \([Y, X] \in \mathcal{D}\) for all \(Y \in \mathcal{D}\).

Let \(\mathcal{D}_{\mathcal{D}} = \mathcal{D}_{\mathcal{D}}(A)\) be the set of all Lie-Bäcklund differentiations of the differential algebra \((A, \mathcal{D})\). Clear,

- \(\mathcal{D}_{\mathcal{D}}\) is a subalgebra of the Lie \(\mathcal{M}_{\mathcal{D}}\)-algebra \(\mathcal{D}\).

Proposition 3. The ascending filtration

\[ \mathcal{D} = \mathcal{D}_{\mathcal{D}}^{(-1)} \subset \mathcal{D}_{\mathcal{D}} = \mathcal{D}_{\mathcal{D}}^{(0)} \subset \cdots \subset \mathcal{D}_{\mathcal{D}}^{(q)} \subset \mathcal{D}_{\mathcal{D}}^{(q+1)} \subset \cdots \]

of Lie \(\mathcal{M}_{\mathcal{D}}\)-algebras is defined, where \(\mathcal{D}_{\mathcal{D}}^{(q)} = \{X \in \mathcal{D} \mid [\mathcal{D}, X] \subset \mathcal{D}_{\mathcal{D}}^{(q-1)}\}\), \(q \in \mathbb{Z}_+\). Moreover,

- \(\mathcal{D}\) is an ideal of the Lie \(\mathcal{M}_{\mathcal{D}}\)-algebra \(\mathcal{D}_{\mathcal{D}}\);
- \([\mathcal{D}_{\mathcal{D}}^{(p)}, \mathcal{D}_{\mathcal{D}}^{(q)}] \subset \mathcal{D}_{\mathcal{D}}^{(p+q)}\), \(p, q \in \mathbb{Z}_+\).

The general definition of filtration one can find in [7], [9]. Note, that here and in similar situations below we don’t claim that \(\cup_{q \in \mathbb{Z}_+} \mathcal{D}_{\mathcal{D}}^{(q)} = \mathcal{D}\) or \(\lim_{q \to \infty} \mathcal{D}_{\mathcal{D}}^{(q)} = \mathcal{D}\) in some sense.

Definition 7. A differential algebra \((A, \mathcal{D})\) is called regular if:

- the Lie \(\mathcal{M}\)-algebra \(\mathcal{D}\) is splitted into vertical and horizontal subalgebras, \(\mathcal{D} = \mathcal{D}_V \oplus \mathcal{M} \mathcal{D}_H\);
- the vertical subalgebra \(\mathcal{D}_V = \mathcal{D}_V(A)\) has a \(\mathcal{M}\)-basis \(\partial = \{\partial_a \in \mathcal{D}_V \mid a \in \mathfrak{a}\}\), \(\mathfrak{a}\) is an index set, \([\partial_a, \partial_b] = 0\), \(a, b \in \mathfrak{a}\);
• the horizontal (Cartan) subalgebra \( D_H = \mathfrak{D}_H(\mathcal{A}) = \mathcal{D} \) has a \( \mathfrak{M} \)-basis
\[
D = \{ D_\mu \in \mathfrak{D}_H \mid \mu \in m \}, \quad m = \{1, \ldots, m\}, \quad m \in \mathbb{N}, \quad [D_\mu, D_\nu] = 0, \quad \mu, \nu \in m;
\]

• the commutators \([D_\mu, \partial_a] = \Gamma^b_{\mu a} \partial_b \in \mathfrak{D}_V, \mu \in m, a, b \in \mathfrak{a}\), the coefficients \( \Gamma^a_{\mu b} \in \mathfrak{M} \), in particular, \([D_\mu, X] = \nabla_\mu X = (\nabla_\mu X)^a \partial_a \in \mathfrak{D}_V \) for any \( X = X^a \partial_a \in \mathfrak{D}_V, X^a \in \mathfrak{M} \), where \( (\nabla_\mu X)^a = D_\mu X^a + \Gamma^a_{\mu b} X^b \).

Let \((\mathcal{A}, \mathfrak{D}_H)\) be a regular differential algebra.

**Proposition 4.** The commutator
\[
[\nabla_\mu, \nabla_\nu]_b X^b = ((D_\mu \Gamma^a_{\nu b} - D_\nu \Gamma^a_{\mu b}) + (\Gamma^c_{\mu \nu} \Gamma^a_{\nu b} - \Gamma^c_{\mu \nu} \Gamma^a_{\nu b})) X^b, \quad a \in \mathfrak{a},
\]
or, in the matrix notation, \([\nabla_\mu, \nabla_\nu] = F_{\mu \nu}, \) where

- the covariant derivative \( \nabla = (\nabla_\mu) \in \text{Hom}_F(\mathfrak{M}^\mathfrak{a}; \mathfrak{M}^\mathfrak{a}) \);
- the connection \( \Gamma = (\Gamma_\mu) \), its components \( \Gamma_\mu = (\Gamma^a_{\mu b}) \in \mathfrak{M}^\mathfrak{a} \);
- the curvature \( F = (F_{\mu \nu}) \), its components \( F_{\mu \nu} \in \mathfrak{M}^\mathfrak{a} \);
- \( F_{\mu \nu} = D_\mu \Gamma_\nu - D_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu], \mu, \nu \in m. \)

**Example 1.** Here:

- \( X = \mathbb{R}^m = \{ x = (x^\mu) \mid x^\mu \in \mathbb{R}, \mu \in m \} \) is the space of independent variables;
- \( \mathcal{U} = \mathbb{R}^\mathfrak{a} = \{ u = (u^a) \mid u^a \in \mathbb{R}, a \in \mathfrak{a} \} \) is the space of the differential variables, \( \mathfrak{a} \) is an infinite index set;
- \( \mathcal{A} = C^\infty_{\text{fin}}(X \mathcal{U}) \) is the unital associative commutative algebra of \( \mathbb{F} \)-valued smooth functions depending on a finite number of the arguments \( x^\mu, u^a \), where \( X \mathcal{U} = X \times \mathcal{U} \).

We split the Lie \( \mathcal{A} \)-algebra \( \mathfrak{D} = \mathfrak{D}(\mathcal{A}) \) into vertical and horizontal parts, \( \mathfrak{D} = \mathfrak{D}_V \oplus \mathfrak{A} \mathfrak{D}_H \), as follows:

- \( \mathfrak{D}_V \) has the \( \mathcal{A} \)-basis \( \partial_a = \{ \partial_a \mid a \in \mathfrak{a} \} \);
- \( \mathfrak{D}_H \) has the \( \mathcal{A} \)-basis \( D = \{ D_\mu \mid \mu \in m \} \);
where \( D_\mu = \partial_\mu + h_\mu^a \partial_a \), while \( \partial_a, \partial_\mu \) are partial derivatives. The condition \([D_\mu, D_\nu] = 0\) to be valid, we assume that the coefficients \( h_\mu^a \in A \) satisfy the equalities: \( D_\mu h_\nu^a - D_\nu h_\mu^a = 0 \) for all \( a \in a, \mu, \nu \in m \).

In this case, the connection \( \Gamma = (\Gamma_\mu) \) has components \( \Gamma^a_{\mu b} = -\partial_a h_{\mu b}^a \), while the curvature \( F = (F_{\mu \nu}) = 0 \).

From the geometrical point of view the the set \( D = \{ D_\mu \mid \mu \in m \} \) defines an involutive (in the Frobenius sense) distribution on the space \( XU \). Any function \( \phi = (\phi^a(x)) \in C^\infty(X; U) \) defines a \( m \)-dimensional submanifold \( \Phi = \{ u = \phi(x) \} \) in \( XU \). This submanifold will be integral manifold of \( D \) if \( (D_\mu(u^a - \phi^a(x)))|_{u = \phi(x)} = 0 \) for all \( \mu \in m, a \in a \), i.e. the function \( \phi(x) \) satisfy the defining system: \( \partial_\mu \phi^a(x) = h_\mu^a(x, \phi(x)), \mu \in m, a \in A \). This system has the integrability condition: \( \partial_\mu h_\nu^a(x, \phi(x)) = \partial_\nu h_\mu^a(x, \phi(x), \mu, \nu \in m, a \in a \), for any solution \( \phi(x) \) of the system. This condition is valid due to assumed equalities \( D_\mu h_\nu^a - D_\nu h_\mu^a = 0 \) and the chain rule:

\[
\partial_\nu f(x, \phi(x)) = (D_\mu f(x, u))|_{u = \phi(x)}
\]

for all \( \mu \in m, f \in A \) and all solutions \( \phi(x) \) of the defining system.

Now, again let \((A, D_H)\) be a regular differential algebra.

**Proposition 5.** There is defined the ascending filtration

\[
0 \subset A^{(0)}_H = A \subset A^{(1)}_H \subset \cdots \subset A^{(q)}_H \subset A^{(q+1)}_H \subset \cdots
\]

of \( M_D \)-modules, where \( A^{(q)}_H = \{ f \in A \mid D_\mu f \in A^{(q-1)}_H, \mu \in m \} \), \( q \in \mathbb{N} \). In particular, \( A^{(p)}_H \cdot A^{(q)}_H \subset A^{(p+q)}_H \), \( p, q \in \mathbb{Z}_+ \).

**Proposition 6.** There is defined the ascending filtration

\[
0 \subset M^{(0)}_H = M \subset M^{(1)}_H \subset \cdots \subset M^{(q)}_H \subset M^{(q+1)}_H \subset \cdots
\]

of \( M_D \)-modules, where \( M^{(q)}_H = \{ R \in M \mid D_\mu(R) \in M^{(q-1)}_H, \mu \in m \} \), \( q \in \mathbb{N} \). In particular, \( M^{(p)}_H \circ M^{(q)}_H \subset M^{(p+q)}_H \), \( p, q \in \mathbb{Z}_+ \).

**Proposition 7.** There is defined the ascending filtration

\[
0 = E^{(-1)} \subset E = E^{(0)} \subset \cdots \subset E^{(q)} \subset E^{(q+1)} \subset \cdots
\]

of Lie \( M_D \)-algebras, where \( E^{(q)} = \{ X \in D \mid [D_\mu, X] \in E^{(q-1)}_H, \mu \in m \} \), \( q \in \mathbb{N} \). Moreover,

- \([E^{(p)}, E^{(q)}] \subset E^{(p+q)}_H \), \( p, q \in \mathbb{Z}_+ \);

- \( D_H^{(q)} = E^{(q)} \oplus_{M_D} D \), \( q \in \mathbb{Z}_+ \).
3 Differential modules.

Let \((\mathcal{A}, \mathcal{D})\) be a differential algebra, \(\mathcal{M}\) be an \(\mathcal{A}\)-module, \(\mathfrak{M} = \mathfrak{M}(\mathcal{M})\) be the algebra of all multiplicators of the \(\mathcal{A}\)-module \(\mathcal{M}\), \(\mathfrak{D} = \mathfrak{D}(\mathcal{M})\) be the Lie \(\mathfrak{M}\)-algebra of all differentiations of the \(\mathcal{A}\)-module \(\mathcal{M}\).

**Definition 8.** A differential module over a differential algebra \((\mathcal{A}, \mathcal{D})\) (an \((\mathcal{A}, \mathcal{D})\)-module) is a triple \((\mathcal{M}, \kappa, \mathfrak{D})\), where

- \(\mathcal{M}\) is an \(\mathcal{A}\)-module;
- a mapping \(\kappa \in \text{Hom}_{\text{Lie}}(\mathcal{D}; \mathfrak{D})\); in particular, \(\kappa[X, Y] = [\kappa X, \kappa Y]\) for all \(X, Y \in \mathcal{D}\);
- \(\mathfrak{D}\) is the Cartan subalgebra of the Lie \(\mathfrak{M}\)-algebra \(\mathfrak{D}\);
- the matching condition \(\kappa|\mathcal{D} : \mathcal{D} \to \mathfrak{D}\).

Let \((\mathcal{M}, \kappa, \mathfrak{D})\) be a differential module.

**Definition 9.** A submodule \(\mathcal{N}\) of the \(\mathcal{A}\)-module \(\mathcal{M}\) is called differential if \(\mathfrak{D}\mathcal{N} \subset \mathcal{N}\). In this case the pair \((\mathcal{N}, \mathfrak{D})\) is a differential module.

**Definition 10.** The element \(M \in \mathcal{M}\) is called \(\mathfrak{D}\)-constant if \(\mathfrak{D}M = 0\).

Let \(\mathcal{M}_\mathfrak{D}\) be the set of all \(\mathfrak{D}\)-constant elements of the \(\mathcal{A}\)-module \(\mathcal{M}\).

**Definition 11.** The multiplicator \(R \in \mathfrak{M}\) is called \(\mathfrak{D}\)-constant if \(\mathfrak{D}(R) = [\mathfrak{D}, R] = 0\).

Let \(\mathfrak{M}_\mathfrak{D}\) be the set of all \(\mathfrak{D}\)-constant elements of the algebra \(\mathfrak{M}\). Clear,

- \(\mathfrak{M}_\mathfrak{D}\) is an unital subalgebra of the algebra \(\mathfrak{M}\);
- \(\mathcal{M}_\mathfrak{D}\) is a submodule of the \(\mathfrak{M}_\mathfrak{D}\)-module \(\mathcal{M}\).

**Definition 12.** A differentiation \(X \in \mathfrak{D}\) is called a Lie-Bäclund differentiation if \([\mathfrak{D}, X] \subset \mathfrak{D}\).

Let \(\mathfrak{D}_\mathfrak{D}\) be the set of all Lie-Bäclund differentiations of the differential module \((\mathcal{M}, \mathfrak{D})\). Clear,

- \(\mathfrak{D}_\mathfrak{D}\) is a subalgebra of the Lie \(\mathfrak{M}_\mathfrak{D}\)-algebra \(\mathfrak{D}\).
Proposition 8. There is defined the ascending filtration
\[ \mathcal{D} = \mathcal{D}_D^{(-1)} \subset \mathcal{D}_D = \mathcal{D}_D^{(0)} \subset \cdots \subset \mathcal{D}_D^{(q)} \subset \mathcal{D}_D^{(q+1)} \subset \cdots \]
of Lie \( \mathfrak{M}_D \)-algebras, where \( \mathcal{D}_D^{(q)} = \{ X \in \mathcal{D} \mid [\mathcal{D}, X] \subset \mathcal{D}_D^{(q-1)} \} \), \( q \in \mathbb{Z}_+ \).
Moreover,
- \( \mathcal{D} \) is an ideal of the Lie \( \mathfrak{M}_D \)-algebra \( \mathcal{D}_D \);
- \( [\mathcal{D}_D^{(p)}, \mathcal{D}_D^{(q)}] \subset \mathcal{D}_D^{(p+q)} \), \( p, q \in \mathbb{Z}_+ \).

Definition 13. Let \((\mathcal{A}, \mathcal{D})\) be a regular differential algebra with a vertical \( \mathfrak{M} \)-basis \( \partial = \{ \partial_a \mid a \in \mathfrak{a} \} \) and a horizontal \( \mathfrak{M} \)-basis \( D = \{ D_\mu \mid \mu \in \mathfrak{m} \} \). A differential module \((\mathcal{M}, \kappa, \mathcal{D})\) is called regular if:
- the Lie \( \mathfrak{M} \)-algebra \( \mathcal{D} \) is split into vertical and horizontal subalgebras, \( \mathcal{D} = \mathcal{D}_V \oplus \mathfrak{M} \mathcal{D}_H \);
- the vertical subalgebra \( \mathcal{D}_V \) has a \( \mathfrak{M} \)-basis \( \partial = \{ \partial_s = (\nabla_s, \partial_s) \in \mathcal{D}_V \mid \partial_s \in \partial, s \in \mathfrak{a}_M \} \), \( [\partial_s, \partial_t] = 0, s, t \in \mathfrak{a}_M \);
- the horizontal subalgebra \( \mathcal{D}_H = \mathcal{D} \) has a \( \mathfrak{M} \)-basis \( D = \{ D_\sigma = (\nabla_\sigma, D_\sigma) \in \mathcal{D}_H \mid D_\sigma \in D, \sigma \in \mathfrak{m}_M \} \), \( [D_\sigma, D_\tau] = 0, \sigma, \tau \in \mathfrak{m}_M \);
- the commutators \( [D_\sigma, \partial_s] = \Gamma^{\rho}_{\sigma s} \partial_\rho, \Gamma^{\rho}_{\sigma s} = (\Delta^{\rho}_{s \sigma}, \Gamma^{\rho}_{s \sigma}) \in \mathfrak{M} \), in particular \( [D_\sigma, X] = \nabla_\sigma X = (\nabla_s X)^s \partial_s \in \mathcal{D}_V \) for all \( X = X^s \partial_s \in \mathcal{D}_V \), \( X^s \in \mathfrak{M} \), where \( (\nabla_s X)^s = D_\sigma X^s + \Gamma^{s}_{\sigma t} X^t \).

Let \((\mathcal{M}, \kappa, \mathcal{D}_H)\) be a regular differential module.

Proposition 9. There is defined the ascending filtration
\[ 0 \subset \mathcal{M}_H^{(0)} = \mathfrak{M}_D \subset \mathcal{M}_H^{(1)} \subset \cdots \subset \mathcal{M}_H^{(q)} \subset \mathcal{A}_H^{(q+1)} \subset \cdots \]
of \( \mathfrak{M}_D \)-modules, where \( \mathcal{M}_H^{(q)} = \{ M \in \mathcal{M} \mid D_{\mu'} M \in \mathcal{M}_H^{(q-1)} \}, \mu' \in \mathfrak{m}' \}, q \in \mathbb{N} \).

Proposition 10. There is defined the ascending filtration
\[ 0 \subset \mathcal{M}_H^{(0)} = \mathfrak{M}_D \subset \mathcal{M}_H^{(1)} \subset \cdots \subset \mathcal{M}_H^{(q)} \subset \mathfrak{M}_H^{(q+1)} \subset \cdots \]
of \( \mathfrak{M}_D \)-modules, where \( \mathfrak{M}_H^{(q)} = \{ R \in \mathfrak{M} \mid D_{\mu} (R) \in \mathfrak{M}_H^{(q-1)} \}, q \in \mathbb{N} \).
In particular, \( \mathfrak{M}_H^{(p)} \circ \mathfrak{M}_H^{(q)} \subset \mathfrak{M}_H^{(p+q)} \), \( p, q \in \mathbb{Z}_+ \).
Proposition 11. There is defined the ascending filtration
\[ 0 = \mathcal{E}^{(-1)} \subset \mathcal{E} = \mathcal{E}^{(0)} \subset \cdots \subset \mathcal{E}^{(q)} \subset \mathcal{E}^{(q+1)} \subset \cdots \]
of Lie $\mathfrak{m}_D$-algebras, where $\mathcal{E}^{(q)} = \{ X \in \mathfrak{D}_V \mid [D_{\mu'}, X] \subset \mathcal{E}^{(q-1)}, \mu' \in \mathfrak{m}' \}$, $q \in \mathbb{Z}_+$. Moreover,
\begin{itemize}
  \item $[\mathcal{E}^{(p)}, \mathcal{E}^{(q)}] \subset \mathcal{E}^{(p+q)}$, $p, q \in \mathbb{Z}_+$;
  \item $\mathfrak{D}^{(q)}_D = \mathcal{E}^{(q)} \oplus \mathfrak{m}_D \mathfrak{D}$, $q \in \mathbb{Z}_+$.
\end{itemize}

4 Spectral sequences.

Proposition 12. Let $\mathcal{A}$ be an unital associative commutative algebra (in particular, $\mathcal{A}$ has a multiplicative unit $e \in \mathcal{A}$). There is the natural isomorphism
\[ \mathcal{A} \simeq \mathfrak{m}(\mathcal{A}), \quad f \mapsto \text{ad}_f : \mathcal{A} \to \mathcal{A}, \quad g \mapsto f \cdot g. \]
Moreover, for any $\mathcal{A}$-module $\mathcal{M}$ there is the natural isomorphism
\[ \mathcal{A} \simeq \mathfrak{m}(\mathcal{M}), \quad f \mapsto \text{ad}_f = (\text{ad}_f, \text{ad}_f), \quad \text{ad}_f : \mathcal{M} \to \mathcal{M}, \quad M \mapsto f \cdot M. \]

Taking this into account, we shall identify: $\mathfrak{m}(\mathcal{A}) \equiv \mathfrak{m}(\mathcal{M}) \equiv \mathcal{A}$.

Assumption 1. Let $\mathcal{U} = \mathcal{A}, \mathcal{M}$, and $\mathcal{V} = \mathcal{A}, \mathcal{K}$, where
\begin{itemize}
  \item $(\mathcal{A}, \mathfrak{D})$ is an unital associative commutative algebra;
  \item $(\mathcal{M}, \kappa = \kappa_{\mathcal{M}}), \mathfrak{D}$) is an $(\mathcal{A}, \mathfrak{D})$-module, $\mathcal{K}$ is an $\mathcal{A}$-module;
  \item $\kappa_{\mathcal{K}, \mathcal{M}} \in \text{Hom}_{\text{Lie} \cap \mathcal{A}}(\mathfrak{D}(\mathcal{M}); \mathfrak{D}({\mathcal{K}}))$, $\kappa_{\mathcal{K}} \in \text{Hom}_{\text{Lie} \cap \mathcal{A}}(\mathfrak{D}(\mathcal{A}); \mathfrak{D}(\mathcal{K}))$,
    $\kappa_{\mathcal{K}} = \kappa_{\mathcal{K}, \mathcal{M}} \circ \kappa_{\mathcal{M}}$.
\end{itemize}

Assumption 2. To simplify the notation, below we write $\kappa : \mathfrak{D}(\mathcal{U}) \to \mathfrak{D}(\mathcal{V})$, where
\[ \kappa = \text{id}_{\mathfrak{D}(\mathcal{A})} \quad \text{when} \quad \mathcal{U} = \mathcal{V} = \mathcal{A}, \quad \kappa = \kappa_{\mathcal{K}} \quad \text{when} \quad \mathcal{U} = \mathcal{A}, \ \mathcal{V} = \mathcal{K}, \]
\[ \kappa = \Pi \quad \text{when} \quad \mathcal{U} = \mathcal{M}, \ \mathcal{V} = \mathcal{A}, \quad \kappa = \kappa_{\mathcal{K}, \mathcal{M}} \quad \text{when} \quad \mathcal{U} = \mathcal{M}, \ \mathcal{V} = \mathcal{K}, \]
the projection $\Pi : \mathfrak{D}(\mathcal{M}) \to \mathfrak{D}(\mathcal{A})$, $X = (\nabla_X, X) \mapsto X$ (see [1]).
Definition 14. The $\mathcal{A}$-module $\Omega(U; V) = \bigoplus_{r \in \mathbb{Z}} \Omega^r(U; V)$ of differential forms over $U$ with coefficients in $V$ is defined by the rule (see [1], for example):

$$\Omega^r(U, V) = \begin{cases} 
0, & r < 0, \\
V, & r = 0, \\
\text{Hom}_{\mathcal{A}}(\wedge^r \mathcal{D}(U); V), & r > 0.
\end{cases}$$

The set $\Omega(U, \mathcal{A})$ has the structure of an exterior $\mathcal{A}$-algebra, and the set $\Omega(U, \mathcal{K})$ has the structure of an exterior $\Omega(U, \mathcal{A})$-module. Moreover, in general, $\Omega(U, V) = \Omega(U, \mathcal{A}) \otimes_{\mathcal{A}} V$.

Proposition 13. For every $X \in \mathcal{D}(U)$

- the interior product $i_X \in \text{End}_{\mathcal{A}}(\Omega(U, V))$ is defined by the contraction rule $i_X \omega(X_1, \ldots, X_{r-1}) = r \omega(X, X_1, \ldots, X_{r-1})$ for all $r \in \mathbb{Z}$, $\omega \in \Omega^r(U, V)$, $X_1, \ldots, X_{r-1} \in \mathcal{D}(U)$;

- the Lie derivative $L_X \in \text{End}_{\mathcal{F}}(\Omega(U, V))$ is defined by the rule

$$L_X \omega(X_1, \ldots, X_r) = (\mathcal{L}_X) \omega(X_1, \ldots, X_r) - \sum_{1 \leq s \leq r} \omega(X_1, \ldots, [X, X_s], \ldots, X_r)$$

for all $r \in \mathbb{Z}$, $\omega \in \Omega^r(U, V)$, $X_1, \ldots, X_r \in \mathcal{D}(U)$;

- $i_X$ is an exterior differentiation and $L_X$ is a differentiation of the $\Omega(U, \mathcal{A})$-module $\Omega(U, V)$, i.e.,

$$i_X (\omega \wedge \chi) = (i_X \omega) \wedge \chi + (-1)^r \omega \wedge (i_X \chi),$$
$$L_X (\omega \wedge \chi) = (L_X \omega) \wedge \chi + \omega \wedge (L_X \chi),$$

for all $\omega \in \Omega^r(U, \mathcal{A})$, $r \in \mathbb{Z}_+$, $\chi \in \Omega(U, V)$.

Proof. See [1], for example. \qed

Definition 15. A form $\omega \in \Omega^r(U, V)$ is called a $p$-Cartan form, $0 \leq p \leq r$, if $\omega(X_1, \ldots, X_r) = 0$ when at least $r - p + 1$ of the differentiations $X_1, \ldots, X_r$ are Cartan, i.e., belong to the subalgebra $\mathcal{D}(U) \subset \mathcal{D}(U)$.

Let $\Omega^r_p(U, V)$ be the $\mathcal{A}$-module of all $p$-Cartan forms in $\Omega^r(U, V)$.
Proposition 14. The descending filtrations

\[ \Omega^r_0(U, V) = \Omega^r(U, V) \supset \Omega^r_1(U, V) \supset \cdots \supset \Omega^r_r(U, V) \supset 0, \quad r \in \mathbb{Z}, \]

of \( A \)-modules are defined. Moreover,

- \( \Omega^r_p(U, A) \wedge_A \Omega^s_q(U, V) \subset \Omega^{r+s}_{p+q}(U, V) \) for all possible \( p, q, r, s \in \mathbb{Z} \).

Proposition 15. Let \( \omega \in \Omega^r_p(U, V) \), \( p, r \in \mathbb{Z} \), then

\[ i_X \omega \in \begin{cases} 
\Omega^{r-1}_p(U, V), & X \in \mathcal{D}(U), \\
\Omega^r_p(U, V), & X \in \mathcal{D}(U), 
\end{cases} \]

\[ L_X \omega \in \begin{cases} 
\Omega^{r-1}_p(U, V), & X \in \mathcal{D}(U), \\
\Omega^r_p(U, V), & X \in \mathcal{D}(U). 
\end{cases} \]

Remind (see [1], for example), that the mapping \( d = d_\kappa \in \text{End}_F(\Omega(U, V)) \) is defined by the Cartan formula

\[
d\omega(X_0, \ldots, X_r) = \frac{1}{r+1} \left\{ \sum_{0 \leq s \leq r} (-1)^s (\kappa X_s) \omega(x_0, \ldots, \hat{X}_s, \ldots, X_r) \\
+ \sum_{0 \leq s < t \leq r} (-1)^{s+t} \omega([X_s, X_t], x_0, \ldots, \hat{X}_s, \ldots, \hat{X}_t, \ldots, X_r) \right\}
\]

for all \( r \in \mathbb{Z}_+, \omega \in \Omega^r(U, V), X_0, \ldots, X_r \in \mathcal{D}(U) \), the “checked” arguments are understood to be omitted.

Proposition 16. The following statements hold:

- \( d^r = d|_{\Omega^r(U, V)} : \Omega^r(U, V) \to \Omega^{r+1}(U, V) ; \)

- the endomorphism \( d \) is an exterior differentiation of the exterior \( A \)-algebra \( \Omega(U, \mathcal{A}) \) and the exterior \( \Omega(U, \mathcal{A}) \)-module \( \Omega(U, \mathcal{M}) \);

- the composition \( d \circ d = 0 \).

Proof. See [1], Theorem 6.

In particular, the de Rham complex \( \{ \Omega^r(U, V), d^r \mid r \in \mathbb{Z} \} \) is defined with the cohomology spaces \( H^r(U, V) = \ker d^r / \text{Im} d^{r-1}, r \in \mathbb{Z} \).

Proposition 17. The Cartan magic formula

\[ L_X = d \circ i_X + i_X \circ d \]

holds for any \( X \in \mathcal{D}(U) \). In particular, \( L_X \circ d = d \circ L_X \) for any \( X \in \mathcal{D}(U) \).
Proof. See \[1\], Theorem 7.

The filtrations \( \{ \Omega^r_p(U, V) \mid p \in \mathbb{Z} \}, r \in \mathbb{Z} \), allow one to refine the de Rham complex to a spectral sequence.

**Proposition 18.** The filtrations \( \{ \Omega^r_p(U, V) \} \) are consistent with the differential \( d \in \text{End}_F(\Omega(U, V)) \), namely,

\[
d \mid_{\Omega^r_p(U, V)} : \Omega^r_p(U, V) \to \Omega^{r+1}_p(U, V) \quad \text{for all } r, p \in \mathbb{Z}_+.
\]

**Proposition 19.** The spectral sequence \( \{ E^p_{r,q} \mid p, q \in \mathbb{Z} \} \) is defined, where

1. \( E^p_{r,q} = Z^p_{r,q} / (B^p_{r-1} + Z^p_{r-1,q-1}) \);
2. \( Z^p_{r,q} = \{ \omega \in \Omega^{p+q}(U, V) \mid d\omega \in \Omega^{p+q+1}(U, V) \} \);
3. \( B^p_{r,q} = \{ \omega = d\chi \in \Omega^{p+q}(U, V) \mid \chi \in \Omega^{p+q-1}(U, V) \} \);
4. \( d^p_r : E^p_{r,q} \to E^{p+r,q+1}_{r-1}, \quad d^p_r[\omega] = [d\omega]^{p+r,q+1}, \)
   where \([\omega]^{p+r,q+1}\) is the equivalence class of a form \( \omega \in Z^p_{r,q} \) in the quotient space \( E^p_{r,q} \).

In particular,

1. \( E^p_{r,q} = \Omega^{p+q}(U, V) / \Omega^{p+q+1}(U, V) \) for all \( r \leq 0, p, q \in \mathbb{Z} \);
2. \( E^p_{r,q} = 0 \) for all \( p < 0, r, q \in \mathbb{Z} \), and for all \( q < 0, r, p \in \mathbb{Z} \);
3. \( E^p_{r+1,q} = \text{Ker} d^p_r / \text{Im} d^p_{r-1,q+1} \) for all \( p, q, r \in \mathbb{Z} \).

**Proposition 20.** The limit terms of the spectral sequence \( \{ E^p_{r,q} \mid p, q, r \in \mathbb{Z} \} \) are defined as follows:

1. \( E^p_{\infty} = Z^p_{\infty} / (B^p_{\infty} + Z^p_{\infty,q-1}) \);
2. \( Z^p_{\infty} = \{ \omega \in \Omega^{p+q}(U, V) \mid d\omega = 0 \} \);
3. \( B^p_{\infty} = \{ \omega = d\chi \in \Omega^{p+q}(U, V) \mid \chi \in \Omega^{p+q-1}(U, V) \} \).

---

1 The general definition see, for example, in \[6\],\[7\].
Definition 16. The filtration \( \{ \Omega^r_p(U, V) \} \) is called Cartan if there exists a number \( m \in \mathbb{N} \), s.t.,

\[
\Omega^r_0(U, V) = \cdots = \Omega^r_{r-m}(U, V) \supset \Omega^r_{r-m+1}(U, V) \supset \cdots \supset \Omega^r_{r}(U, V) \supset 0.
\]

This number \( m \) is called the Cartan dimension of the Lie \( A \)-algebra \( D(U) \).

Proposition 21. Let the filtration \( \{ \Omega^r_p(U, V) \} \) be Cartan. Then the limit equalities

\[
\lim_{r \to \infty} E^p_{pq} = E^p_{\infty} \quad \text{hold for all} \quad p, q \in \mathbb{Z}.
\]

Proof. The existence of the limit follows from the general properties of spectral sequences, see [7] for full detail.

In applications, as a rule, only separate terms of the spectral sequence are used. Thus, in the algebra-geometrical approach to partial differential equations the Cartan spectral sequence \( \{ E^p_{pq}, d^p_{pq} \mid p, r \in \mathbb{Z}_+, 0 \leq q \leq m \} \) \((m \) is the Cartan dimension) arises (see, [8] and, for example, [17]) and the important role play the following terms:

- \( E^{0m}_1 \) is the space of functionals, elements of the equivalence classes are called Lagrangians;
- \( E^{0,m-1}_1 \) is the space of conservation laws, elements of the equivalence classes are called conserved currents;
- \( E^{0,q}_1, 0 \leq q \leq m - 2 \), are the spaces of conservation laws of lower order;
- \( E^{p}_1, p \in \mathbb{N} \), are the spaces of the functional \( p \)-forms;
- \( d^{pm}_1 : E^{pm}_1 \to E^{p+1,m}_1, p \in \mathbb{Z}_+, \) are functional (variation) differentials, the differential \( \delta = d^{0m}_1 \) is called the Euler-Lagrange operator.

Definition 17. The quotient Lie algebra of symmetries of a differential algebra (a differential module) \( U \) is defined as \( \text{Sym}_D U = \mathcal{D}_D(U)/\mathcal{D}(U) \).

Proposition 22. For any \([X] = X + \mathcal{D}(U) \in \text{Sym}_D U, X \in \mathcal{D}_D(U)\), the following quotient morphisms are defined:

- \( (i_{[X]})_r : E^p_{pq} \to E^{p-1,q}_r, \quad [\omega]_r^{pq} \mapsto [i_X \omega]_r^{p-1,q}; \)
- \( (L_{[X]})_1 : E^p_{1} \to E^{pq}_1, \quad [\omega]_1^{pq} \mapsto [L_X \omega]_1^{pq}. \)
5 Variation bicomplexes.

Here we keep all notations and assumptions of the previous section.

**Assumption 3.** In addition to the Assumption 1, assume that the differential algebra \((\mathcal{A}, D)\) and the differential module \((\mathcal{M}, \kappa = \kappa_M, \mathcal{D})\) are regular. In this case the Cartan dimensions are \(m_\mathcal{A} = \dim_\mathcal{A} \mathcal{D}, m_\mathcal{M} = \dim_\mathcal{A} \mathcal{D}\).

The splitting of the algebras \(\mathcal{D}(\mathcal{A})\) and \(\mathcal{D}(\mathcal{K})\) into vertical and horizontal parts allows to refine the spectral sequence into the variation bicomplex (the detailed information about bicomplexes see, for example, in [6] or [14].

**Definition 18.** The \(\mathcal{A}\)-modules \(\Omega^{pq}(\mathcal{U}, \mathcal{V}), p, q \in \mathbb{Z}\), are defined by the rule

\[
\Omega^{pq}(\mathcal{U}, \mathcal{V}) = \begin{cases} 
0, & p < 0 \text{ and/or } q < 0, \\
\mathcal{V}, & p = q = 0, \\
\text{Hom}_\mathcal{A}(\bigwedge^p \mathcal{D}_V(\mathcal{U})) \wedge \mathcal{A} (\bigwedge^q \mathcal{D}_H(\mathcal{U})); & p + q > 0.
\end{cases}
\]

In particular, \(\Omega^{p0}_V(\mathcal{U}; \mathcal{V}) = \Omega^{p0}(\mathcal{U}; \mathcal{V}), \Omega^{0q}_H(\mathcal{U}; \mathcal{V}) = \Omega^{0q}(\mathcal{U}; \mathcal{V})\) are the \(\mathcal{A}\)-modules of the vertical and the horizontal forms, correspondingly. Moreover, in general, \(\Omega^{pq}(\mathcal{U}; \mathcal{V}) = \Omega^{p0}(\mathcal{U}; \mathcal{A}) \wedge \mathcal{A} \Omega^{0q}(\mathcal{U}; \mathcal{A}) \otimes \mathcal{A} \mathcal{V}\).

Note, \(\Omega^{q0}_H(\mathcal{U}; \mathcal{V}) = 0\) for all \(q > m_\mathcal{H}\).

The splitting \(\mathcal{D}(\mathcal{U}) = \mathcal{D}_V(\mathcal{U}) \oplus \mathcal{D}_H(\mathcal{U})\) defines the projections

\[\pi_{V,H} : \mathcal{D}(\mathcal{U}) \to \mathcal{D}_{V,H}(\mathcal{U}), \quad X = X_V + X_H \mapsto \pi_{V,H}X = X_{V,H}.\]

Hence, there are defined the dual injections

\[\pi^*_{V,H} : \Omega^r_{V,H}(\mathcal{U}, \mathcal{V}) \to \Omega^r(\mathcal{U}, \mathcal{V}), \quad \omega \mapsto \pi^*_{V,H}\omega = \omega \circ \pi_{V,H},\]

where \(\pi^*_{V,H}\omega(X_1, \ldots, X_r) = \omega(\pi_{V,H}X_1, \ldots, \pi_{V,H}X_s), X_1, \ldots, X_r \in \mathcal{D}(\mathcal{U}).\)

**Proposition 23.** The identifications \(\pi^*_{V,H}\omega = \omega\) define the representations

\[\Omega^r(\mathcal{U}, \mathcal{V}) = \bigoplus_{s \in \mathbb{Z}} \Omega^{s,r-s}(\mathcal{U}, \mathcal{V}), \quad \text{and} \quad \Omega^r_p(\mathcal{U}, \mathcal{V}) = \bigoplus_{s \geq p} \Omega^{s,r-s}(\mathcal{U}, \mathcal{V}).\]

In particular, the \(\mathcal{A}\)-module \(\Omega(\mathcal{U}, \mathcal{V}) = \bigoplus_{p, q \in \mathbb{Z}} \Omega^{pq}(\mathcal{U}, \mathcal{V})\) is bigraded.

By the previous assumptions, the unital differential algebra \((\mathcal{A}, \mathcal{D})\) is regular, i.e., \(\mathcal{D}(\mathcal{A}) = \mathcal{D}_V(\mathcal{A}) \oplus \mathcal{D}_H(\mathcal{A})\), the vertical subalgebra \(\mathcal{D}_V(\mathcal{A})\) has
an $A$-basis $\partial = \{ \partial_a \mid a \in a \}$, $[\partial_a, \partial_b] = 0$, while the horizontal subalgebra $\mathcal{D}_H(A) = \mathcal{D}$ has an $A$-basis $D = \{ D_\mu \mid \mu \in m \}$, $[D_\mu, D_\nu] = 0$.

In this case, the dual $A$-module $\mathcal{D}_V^*(A) = \text{Hom}_A(\mathcal{D}_V(A); A) = \Omega_V(A, A)$ has the dual $A$-basis $\rho = \{ \rho^a \in \mathcal{D}_V^*(A) \mid a \in a \}$, $\rho^a(\partial_b) = \delta^a_b$. In particular, $\omega(X) = \omega_a \rho^a(X^b \partial_b) = \omega_a X^a$ for all $\omega \in \mathcal{D}_V^*(A), X \in \mathcal{D}_V(A)$, while $\omega(X) = 0$ for any $\omega \in \mathcal{D}_V^*(A), X \in \mathcal{D}_H(A)$.

The dual $A$-module $\mathcal{D}_H^*(A) = \text{Hom}_A(\mathcal{D}_H(A); A) = \Omega_H^*(A, A)$ has the dual $A$-basis $\vartheta = \{ \vartheta^\mu \mid \mu \in m \}$, $\vartheta^\mu(D_\nu) = \delta^\mu_\nu$. In particular, $\omega(X) = \omega_\mu \vartheta^\mu(X^\nu D_\nu) = \omega_\mu X^\nu$ for all $\omega \in \mathcal{D}_H^*(A), X \in \mathcal{D}_H(A)$, while $\omega(X) = 0$ for any $\omega \in \mathcal{D}_H^*(A), X \in \mathcal{D}_V(A)$.

The same is true for the $(A, \mathcal{D})$-module $(M, \kappa = \kappa_M, \mathcal{D})$. Namely, $\mathcal{D}(M) = \mathcal{D}_V(M) \oplus_A \mathcal{D}_H(M)$, the vertical subalgebra $\mathcal{D}_V(M)$ has an $A$-basis $\theta = \{ \theta_s \mid s \in a_M \}$, $[\theta_s, \theta_t] = 0$. Further, the horizontal subalgebra $\mathcal{D}_H(M)$ has an $A$-basis $D = \{ D_\sigma \mid \sigma \in m_M \}$, $[D_\sigma, D_\tau] = 0$.

Thus, the dual $A$-module $\mathcal{D}_V^*(M) = \text{Hom}_A(\mathcal{D}_V(M); A) = \Omega_V^*(M, A)$ has the dual $A$-basis $\vartheta = \{ \vartheta^\sigma \mid \sigma \in m_M \}$, $\vartheta^\sigma(D_\tau) = \delta^\sigma_\tau$. In particular, $\omega(X) = \omega_\sigma \vartheta^\sigma(X^\nu D_\nu) = \omega_\sigma X^\nu$ for all $\omega \in \mathcal{D}_V^*(M), X \in \mathcal{D}_V(M)$, while $\omega(X) = 0$ for any $\omega \in \mathcal{D}_V^*(M), X \in \mathcal{D}_H(M)$.

The dual $A$-module $\mathcal{D}_H^*(M) = \text{Hom}_A(\mathcal{D}_H(M); A) = \Omega_H^*(M, A)$ has the dual $A$-basis $\vartheta = \{ \vartheta^\nu \mid \nu \in m_M \}$, $\vartheta^\nu(D_\sigma) = \delta^\nu_\sigma$. In particular, $\omega(X) = \omega_\nu \vartheta^\nu(X^\mu D_\mu) = \omega_\nu X^\mu$ for all $\omega \in \mathcal{D}_H^*(M), X \in \mathcal{D}_H(M)$, while $\omega(X) = 0$ for any $\omega \in \mathcal{D}_H^*(M), X \in \mathcal{D}_V(M)$.

**Assumption 4.** Below we further simplify the notation and write: $\partial, D$ for the vertical and the horizontal bases in $\mathcal{D}(U)$ and $\rho, \vartheta$ for the dual bases.

**Proposition 24.** The representations

$$
\Omega^{pq}(U, V) = \left\{ \omega = \frac{1}{p!q!} \omega_{a_1...a_p\mu_1...\mu_q} \cdot \rho^{a_1} \wedge \ldots \wedge \rho^{a_p} \wedge \vartheta^{\mu_1} \wedge \ldots \wedge \vartheta^{\mu_q} \right\}
$$

hold for all $p \in \mathbb{Z}_+$ and $0 \leq q \leq m$, where the coefficients $\omega_{a_1...a_p\mu_1...\mu_q} \in V$ are skew-symmetric in indices $a_1, \ldots, \mu_q$.

The exterior differentiation $d : \Omega^{pq}(U, V) \to \Omega^{p+1,q}(U, V) \oplus_F \Omega^{p,q+1}(U, V)$ also splits into vertical and horizontal parts, $d = d_V + d_H$, where

$$
d_V : \Omega^{pq}(U, V) \to \Omega^{p+1,q}(U, V), \quad d_H : \Omega^{pq}(U, V) \to \Omega^{p,q+1}(U, V).
$$

Indeed, the Cartan formula gives the following results:

\[\text{We simplify the notation, for convenience.}\]
\[ dv = \mathcal{X}\partial_a v \cdot \rho^a + \mathcal{X}^\mu v \cdot \vartheta^\mu \text{ for any } v \in V, \text{ hence } d_V v = \mathcal{X}\partial_a v \cdot \rho^a, \quad d_H v = \mathcal{X}^\mu v \cdot \vartheta^\mu; \]

\[ d\rho^a = \Gamma^a_{\mu b} \cdot \rho^b \wedge \vartheta^\mu \text{ for any } \rho^a \in \rho, \text{ hence } d_V \rho^a = 0, \quad d_H \rho^a = \Gamma^a_{\mu b} \cdot \rho^b \wedge \vartheta^\mu; \]

\[ d\vartheta^\mu = 0 \text{ for any } \vartheta^\mu \in \vartheta, \text{ hence } d_{V,H} \vartheta^\mu = 0. \]

Due to Propositions 16 and 24 these formulas allow to calculate differentials \( d_V \omega \) and \( d_H \omega \) for any form \( \omega \in \Omega^{pq}(U, V) \).

**Proposition 25.** The equality \( d \circ d = 0 \) implies the equalities:

\[ d_V \circ d_V = d_V \circ d_H + d_H \circ d_V = d_H \circ d_H = 0. \]

**Proposition 26.** There is defined the variation bicomplex

\[ \{ \Omega^{pq}(U, V); d_{\nu}^p, d_{\nu}^q \mid p, q \in \mathbb{Z} \}, \quad \text{where } d_{V,H}^p = d_{V,H}^p|_{\Omega^{pq}(U, V)}. \]

The vertical and horizontal cohomologies of these bicomplexes are defined as follows:

\[ H_{V}^{pq}(U, V) = \text{Ker } d_{V}^{p+1,q} / \text{Im } d_{V}^{p,q}, \quad H_{H}^{pq}(U, V) = \text{Ker } d_{H}^{p+1,q} / \text{Im } d_{H}^{p,q}, \]

for all \( p, q \in \mathbb{Z} \).

To shorten the notation, below in this section we omit the arguments \( U \) and \( V \) and write \( \Omega^{pq} \) instead of \( \Omega^{pq}(U, V) \), \( H_{V,H}^{pq} \) instead of \( H_{V,H}^{pq}(U, V) \), \( \mathcal{D} \) instead of \( \mathcal{D}(U) \), and so on.

For all \( p, q \in \mathbb{Z} \) there are defined the second differentials:

\[ d_{H,V}^{pq} \in \text{Hom}_{\mathcal{F}}(H_{V}^{pq}, H_{V}^{p+1,q}), \quad \omega_{V}^{pq} \mapsto d_{H,V}^{pq} \omega_{V}^{pq} = [d_H \omega_{V}^{pq}]_{V}, \]

\[ d_{V,H}^{pq} \in \text{Hom}_{\mathcal{F}}(H_{H}^{pq}, H_{H}^{p+1,q}), \quad \omega_{H}^{pq} \mapsto d_{V,H}^{pq} \omega_{H}^{pq} = [d_V \omega_{H}^{pq}]_{H}, \]

where

\[ \omega_{V}^{pq} = [\omega^{pq}]_{V} = \omega^{pq} + d_V \Omega^{p-1,q} \in H_{V}^{pq}, \quad \omega^{pq} \in \Omega^{pq}, \quad d_V \omega^{pq} = 0, \]

\[ \omega_{H}^{pq} = [\omega^{pq}]_{H} = \omega^{pq} + d_H \Omega^{p,q-1} \in H_{H}^{pq}, \quad \omega^{pq} \in \Omega^{pq}, \quad d_H \omega^{pq} = 0. \]
Thus, there are defined the complexes

\[ \{ H^{pq}_V; d^{pq}_H \mid q \in \mathbb{Z} \}, p \in \mathbb{Z}, \quad \{ H^{pq}_H; d^{pq}_V \mid p \in \mathbb{Z} \}, q \in \mathbb{Z}, \]

with the second cohomology spaces

\[
H^{pq}_{HV} = \text{Ker} d^{pq}_H/\text{Im} d^{p-1,q}_V = \{ \omega^{pq}_{HV} = \omega^{pq} + d_V \Omega^{p,q-1} \mid \omega^{pq} \in \Omega^{pq}, d_V \omega^{pq} = 0, d_H \omega^{pq} \in d_V \Omega^{p-1,q+1} \},
\]

\[
H^{pq}_{VH} = \text{Ker} d^{pq}_V/\text{Im} d^{p-1,q}_H = \{ \omega^{pq}_{VH} = \omega^{pq} + d_H \Omega^{p,q-1} \mid \omega^{pq} \in \Omega^{pq}, d_H \omega^{pq} = 0, d_V \omega^{pq} \in d_H \Omega^{p+1,q-1} \},
\]

All further differentials and cohomologies are trivial.

**Theorem 1.** In the regular case elements of the spectral sequence \( \{ E_r^{pq}, d_r^{pq} \} \) are as follows:

\[
\bullet \quad E_0^{pq} = \Omega^{p+q}_p/\Omega^{p+q}_{p+1} = \Omega^{pq}, \quad d_0^{pq} = d_H^{pq} : \Omega^{pq} \to \Omega^{p,q+1};
\]

\[
\bullet \quad E_1^{pq} = \text{Ker} d_1^{pq}/\text{Im} d_1^{p-1,q} = H^{pq}_H, \quad d_1^{pq} = d_V^{pq} : H^{pq}_H \to H^{p+1,q}_H;
\]

\[
\bullet \quad E_2^{pq} = \text{Ker} d_2^{pq}/\text{Im} d_2^{p-1,q} = H^{pq}_{VH}, \quad d_2^{pq} = 0;
\]

\[
\bullet \quad E_r^{pq} = E_2^{pq} = H^{pq}_{VH}, \quad d_r^{pq} = 0, \quad r \geq 2;
\]

\[
\bullet \quad \lim_{r \to \infty} E_r^{pq} = E_2^{pq} = E_\infty^{pq}.
\]

**Proof.** The proof is based on the general properties of spectral sequences and the above calculations for the variation bicomplex. \( \square \)

Let \( X = X_V + X_H = X^a \partial_a + X^\mu D_\mu \in \mathfrak{D} = \mathfrak{D}_V + \mathfrak{D}_H \). Then,

\[
\bullet \quad i_X F = 0 \text{ for any } F \in \mathcal{V};
\]

\[
\bullet \quad i_X \rho^a = X^a = i_X \rho^a \text{ for any } a \in \mathfrak{a};
\]

\[
\bullet \quad i_X \partial^\mu = X^\mu = i_X \partial^\mu \text{ for any } \mu \in \mathfrak{m};
\]

\[
\bullet \quad L_X F = (\mathcal{L}X) F \text{ for any } F \in \mathcal{V};
\]

\[
\bullet \quad L_X \rho^a = (\partial_b X^a - \Gamma^a_{b\mu} X^\mu) \cdot \rho^b + (\nabla_\mu X^a) \cdot \partial^\mu \text{ for any } a \in \mathfrak{a};
\]

\[
\bullet \quad L_X \partial^\mu = \partial_a X^\mu \cdot \rho^a + D_\nu X^\mu \cdot \varrho^\nu \text{ for any } \mu \in \mathfrak{m}.
\]
Remind, \((\nabla_\mu X)^a = D_\mu X^a + \Gamma^a_{\mu b} X^b = (\nabla_\mu X_V)^a\), see Definition 7. Due to Propositions 13 and 24 these formulas allow to calculate \(i_X\omega\) and \(L_X\omega\) for any form \(\omega \in \Omega^{pq}\).

**Proposition 27.** Let \(\omega \in \Omega^{pq}\), \(p, q \in \mathbb{Z}\), \(X \in \mathcal{D}\), then
\[
i_X\omega \in \begin{cases} 
\Omega^{p-1,q}, & X \in \mathcal{D}_V, \\
\Omega^{p,q-1}, & X \in \mathcal{D}_H,
\end{cases}
\]
\[
L_X\omega \in \begin{cases} 
\Omega^{p+1,q-1} \oplus \Omega^{pq} \oplus \Omega^{p-1,q+1}, & X \in \mathcal{D}, \\
\Omega^{pq}, & X \in \mathcal{E}.
\end{cases}
\]
Remind, \(\mathcal{E} \subset \mathcal{D}\), see Propositions 7 and 11.

**Proposition 28.** For any \(X \in \mathcal{E}\), \(p, q \in \mathbb{Z}\), the endomorphisms
\[
[L_X]^{pq} \in \text{End}_F(H^{pq}_{V,H}), \quad [\omega^{pq}]_V, H \mapsto [L_X][\omega^{pq}]_V, H = [L_X\omega^{pq}]_V, H,
\]
are defined, where \([\omega^{pq}]_V = \omega^{pq} + d^{p-1,q}\Omega^{p-1,q}\), \([\omega^{pq}]_H = \omega^{pq} + d^{p-1,q}\Omega^{p-1,q}\).

6 Differential algebras in partial differential equations

6.1 Notation

Here:

- \(X = \mathbb{R}^m = \{x = (x^\mu) \mid x^\mu \in \mathbb{R}, \ \mu \in m\}\) is the linear space of independent variables;
- \(U = \mathbb{R}^A = \{u = (u^\alpha) \mid u^\alpha \in \mathbb{R}, \ \alpha \in A\}\) is the linear space of dependent variables, \(A\) is a finite index set;
- \(U = \mathbb{R}^I = \{u = (u^a_i) \mid u^a_i \in \mathbb{R}, \ \alpha \in A, \ i \in I\}\) is the linear space of differential variables, \(I = \mathbb{Z}_{\geq 0}^m\) (note, \(\dim U = \infty\));
- \(\mathcal{A} = \mathcal{C}_{fin}^\infty(XU)\) is the unital associative commutative algebra of \(\mathbb{F}\)-valued smooth functions depending on a finite number of the arguments \(x^\mu, u^a_i, XU = X \times U\).

In this case, \(\mathfrak{M}(\mathcal{A}) = \mathcal{A}\), because the algebra \(\mathcal{A}\) is unital. The Lie \(\mathcal{A}\)-algebra \(\mathcal{D} = \mathcal{D}(\mathcal{A})\) has the standard \(\mathcal{A}\)-basis \(\{\partial_{u^\alpha_i}, \partial_{x^\mu} \mid \alpha \in A, \ i \in I, \ \mu \in m\}\), where \(\partial_{u^\alpha_i}, \partial_{x^\mu}\) are partial derivatives.

In the algebraic approach to partial differential equations the Lie \(\mathcal{A}\)-algebra \(\mathcal{D}\) splits as \(\mathcal{D} = \mathcal{D}_V \oplus \mathcal{A} \mathcal{D}_H\), where
• the vertical subalgebra \( \mathfrak{D}_V \) has the \( \mathcal{A} \)-basis \( \partial = \{ \partial_{\alpha}^i \mid \alpha \in \mathbb{A}, \ i \in \mathbb{I} \} \), \([\partial_{\alpha}^i, \partial_{\beta}^j] = 0;\)

• the horizontal subalgebra \( \mathfrak{D}_H \) has the \( \mathcal{A} \)-basis \( D = \{ D_\mu \mid \mu \in \mathfrak{m} \} \), \( D_\mu = \partial_{x^i} + u_{i+1(\mu)}^\alpha \partial_{\alpha}^i \), \( i + (\mu) = (i^1, \ldots, i^m, 1, \ldots, i^m) \), \([D_\mu, D_\nu] = 0.\)

The horizontal basic differentiations \( D_\mu \) are called \textit{total derivatives}, they are characterized by the \textit{chain rule}:

\[
\partial_{x^\mu}(f(x, u)|_{u=\phi(x)}) = (D_\mu f(x, u)|_{u=\phi(x)})
\]

for all \( \mu \in \mathfrak{m} \), \( f \in \mathcal{A} \) and \( \phi \in C^\infty(X; U) \), where \( \phi(x) = (\phi^\alpha(x)), \ \phi(x) = (\phi_i^\alpha(x)), \ \phi_i^\alpha(x) = \partial_x^i \phi^\alpha(x), \ \partial_{x^i} = (\partial_{x^1})^{i_1} \ldots (\partial_{x^m})^{i_m}, \ i = (i^1, \ldots, i^m) \in \mathbb{I}.\)

- The commutators \([D_\mu, \partial_{\alpha}^i] = -\partial_{\alpha}^i \partial_{\alpha}^i \), hence the connection \( \Gamma = \left( \Gamma_{\mu\alpha}^\beta \right), \ \Gamma_{\mu\alpha}^\beta = -\delta_{\alpha\beta} \delta_{j+1(\mu)}; \ \mu \in \mathfrak{m}, \ \alpha, \beta \in \mathcal{A}, \ i, j \in \mathbb{I}.\)

Thus, the regular unital differential algebra \((\mathcal{A}, \mathfrak{D}_H)\) is defined.

Here, the commutator \([D_\mu, X] = (\nabla_\mu \zeta)^\alpha_i \partial_{\alpha}^i \) for any \( X = \zeta^\alpha_i \partial_{\alpha}^i \in \mathfrak{D}_V \), where

\[
\nabla_\mu \in \text{End}_\mathbb{F}(\mathcal{A}_i^\mathbb{A}), \ \zeta = (\zeta_i^\alpha) \mapsto \nabla_\mu \zeta = ((\nabla_\mu \zeta_i^\alpha)^\alpha), \ (\nabla_\mu \zeta_i^\alpha) = D_\mu \zeta_i^\alpha - \zeta_i^{\alpha}_{i+1(\mu)}.
\]

**Proposition 29.** The commutator \([\nabla_\mu, \nabla_\nu] = 0\) for all \( \mu, \nu \in \mathfrak{m}, \ i.e., \ the \ curvature \ F = (F_{\mu\nu}) = 0.\)

**Definition 19.** The number \( n \in \mathbb{Z}_+ \) is called the \textit{order} of a differential function \( f \in \mathcal{A} \), if \( \partial_{\alpha}^i f \neq 0 \) for some \( \alpha \in \mathcal{A} \) and \( i \in \mathbb{I}, \ |i| = i^1 + \ldots i^m = n \), while \( \partial_{\alpha}^i f = 0 \) for all \( \beta \in \mathcal{A} \) and \( j \in \mathbb{I}, \ |j| = j^1 + \ldots j^m > n.\)

**Lemma 1.** The subalgebra \( \mathcal{A}_D = \mathcal{A}_{H}^{(0)} = \mathbb{F}.\)

**Proof.** The proof is based on the property that by definition every differential function \( f \in \mathcal{A} \) has a finite order \( n = n(f). \)

**Proposition 30.** In the filtration \( \{ \mathcal{A}_H^{(q)} \mid q \in \mathbb{Z}_+ \} \) the linear spaces

\[
\mathcal{A}_H^{(q)} = \mathbb{F}_q[x] = \left\{ f(x) = \sum_{|i| \leq q} f_i x^i \ \mid \ f_i \in \mathbb{F} \right\}
\]

are the spaces of polynomials of the order \( q \) in \( x \in X \), where \( x^i = (x^1)^{i_1} \ldots (x^m)^{i_m}, \ i \in \mathbb{I}, \ |i| = i^1 + \ldots + i^m.\)
The limit $\lim_{q \to \infty} \mathcal{A}_H^{(q)}$ depends on the topology. Thus, if we choose the natural topology of the linear space of polynomials $\mathbb{F}[x]$ then we get $\lim_{q \to \infty} \mathcal{A}_H^{(q)} = \mathbb{F}[x]$, while if we choose the natural topology of the linear space of smooth functions $C^\infty(X;\mathbb{F})$ then we get $\lim_{q \to \infty} \mathcal{A}_H^{(q)} = C^\infty(X;\mathbb{F})$.

**Lemma 2.** The equalities $(\nabla_i \zeta)_{i}^\alpha = \sum_{k+j=r} (-1)^k \binom{r}{k} D_j \zeta_{i+k}^\alpha$ hold, where $\alpha \in I$, $i, r, k, j \in I$, $\nabla_r = (\nabla_1)^{r_1} \ldots (\nabla_m)^{r_m}$, $D_j = (D_1)^{j_1} \ldots (D_m)^{j_m}$.

**Proof.** The proof is based on Proposition 29 the induction on $r$ and the well known equality $\binom{r}{k} = \binom{r-(\mu)}{k}$. $\square$

**Remark 1.** We use the standard multiindex notation, in particular, $(-1)^r = (-1)^{r_1} (-1)^{r_2} \ldots (-1)^{r_m}$, $\binom{r}{k} = \binom{r_1}{k_1} \ldots \binom{r_m}{k_m}$.

**Definition 20.** For every $k \in I$ we define the linear subspace

$$\Phi^k = \{ \epsilon^k_{\phi} = (\epsilon^k_{\phi \alpha}) \in \mathcal{A}_I^A \mid \epsilon^k_{\phi \alpha} = \binom{i}{k} D_{i-k} \phi^\alpha, \phi = (\phi^\alpha) \in \mathcal{A}^A \} \subset \mathcal{A}_I^A.$$  

We also set $\Phi^k = 0$ if $k \notin I$.

**Lemma 3.** For any $r, k \in I$ the mapping

$$\nabla_r \in \text{Hom}_{\mathbb{F}}(\Phi^k; \Phi^{k-r})$$

$$\epsilon^k_{\phi} \mapsto \nabla_r \epsilon^k_{\phi} = (-1)^r \epsilon^{k-r}_{\phi}.$$  

In particular, $\nabla_r \epsilon^k_{\phi} = 0$ for any $k-r \notin I$ and $\phi \in \mathcal{A}^A$.

**Proof.** Indeed,

$$(\nabla_r \epsilon^k_{\phi})^\alpha = D_{r \epsilon^k_{\phi \alpha}} - \epsilon^k_{\phi, i + (\mu)} = \binom{i}{k} D_{i+(\mu)-k} \phi^\alpha - \binom{i+(\mu)}{k} D_{i+(\mu)-k} \phi^\alpha$$

$$= \left( \binom{i}{k} - \binom{i+(\mu)}{k} \right) D_{i+(\mu)-k} \phi^\alpha = -\binom{i}{k} D_{i+(\mu)-k} \phi^\alpha = -\epsilon^{k-(\mu)}_{\phi}.$$  

To complete the proof one should use induction on $r$. $\square$

**Theorem 2.** For any $\zeta = (\zeta_i^\alpha) \in \mathcal{A}^A_I$ there exists the unique representation

$$\zeta = \sum_{k \in I} \epsilon^k_{\phi_k}, \quad \phi_k = (\phi_k^\alpha) \in \mathcal{A}^A, \quad \phi_k^\alpha = \sum_{i+j=k} (-1)^j \binom{k}{j} D_j \zeta_i^\alpha.$$  

In other words, the linear space $\mathcal{A}^A_I$ is I-graded by the linear spaces $\Phi^k$, i.e., $\mathcal{A}^A_I = \oplus_{k \in I} \Phi^k$.  

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Proof. Indeed, for a given $\zeta \in \mathcal{A}_1$ we need to find functions $\phi_k \in \mathcal{A}_k$, $k \in \mathbb{I}$, satisfying the equality $\zeta = \sum_{k \in \mathbb{I}} \zeta^k$. Applying the operator $\nabla_r$, $r \in \mathbb{I}$, to both sides of this equality we get (see Lemmas 2 and 3)

\[
(\nabla_r \zeta)_i^\alpha = \sum_{k+j=r} (-1)^k \binom{r}{k} D_j \zeta_i^\alpha + \sum_{k \in \mathbb{I}} (\nabla_r \varepsilon_k^\alpha)_i = (-1)^r \sum_{k \in \mathbb{I}} (\varepsilon_k^r)^\alpha_i
\]

Thus, we need to satisfy the equality

\[
\sum_{j \in \mathbb{I}} \binom{j}{i} D_{i+j} \phi_{i+r}^\alpha = \sum_{k+j=r} (-1)^{k+r} \binom{r}{k} D_j \zeta_{i+k}^\alpha, \quad \alpha \in \mathcal{A}, \; i \in \mathbb{I}.
\]

In particular, for $i = 0$ we get $\phi_r^\alpha = \sum_{k+j=r} (-1)^{k+r} \binom{r}{k} D_j \zeta_k^\alpha$. The easy test shows that this unique choice solves the problem.

\[\square\]

Corollary 1. The following statements hold:

- in the filtration $\{\mathcal{E}(q) \mid q \in \mathbb{Z}_+\}$ the linear spaces $\mathcal{E}(q) = \bigoplus_{|k| \leq q} \mathcal{E}^k$, $\mathcal{E}^k = \{X = \varepsilon^k_{\phi_k} \cdot \partial_{\phi_k}^\alpha = (\varepsilon^k_{\phi_k}) \in \Phi^k\}$;

- $\lim_{q \to \infty} \mathcal{E}(q) = \mathcal{D} = \bigoplus_{k \in \mathbb{I}} \mathcal{E}^k$.

Consider the variation bicomplex

\[
\{\Omega^p; d_V^p, d_H^p \mid p \in \mathbb{Z}_+, 0 \leq q \leq m\}, \quad \text{where} \quad \Omega^p = \Omega^p(\mathcal{A}, \mathcal{A}).
\]

Here,

- the vertical $\mathcal{A}$-basis is $\partial = \{\partial_{\phi_k}^\alpha \mid \alpha \in \mathcal{A}, \; i \in \mathbb{I}\}$ has the dual basis $\rho = \{\rho^\alpha_i = d\alpha_i - u^\alpha_i(\mu)dx^\mu \mid \alpha \in \mathcal{A}, \; i \in \mathbb{I}\}$, $\rho^\alpha_i(\partial_{\mu}) = \delta_i^\mu \rho^\alpha_i, \rho^\alpha_i(D_{\mu}) = 0$;

- the horizontal $\mathcal{A}$-basis $D = \{D_{\mu} \mid \mu \in \mathfrak{m}\}$ has the dual basis $\vartheta = \{\vartheta^\mu = dx^\mu \mid \mu \in \mathfrak{m}\}$, $dx^\mu(D_{\vartheta}) = 0, \; dx^\mu(D_{\mu}) = \delta^\mu_{\mu}$.

We augment the variation bicomplex and add

- the horizontal complex $\{\Omega^q; d_R^q \mid 0 \leq q \leq m\}$, where $\Omega^q_R = \Omega^q(\mathcal{C}^\infty(X))$, $d_R^q = d^q$, i.e., the standard de Rham complex of the space $X = \mathbb{R}^m$.
The vertical complex \( \mathfrak{F} \), where

\[
\mathfrak{F} = \{ \delta^p | p \in \mathbb{Z}_+ \}, \quad \delta^p = \Omega^{pm}/\text{Im} \, d^{p,m-1}
\]

are quotient linear spaces of functional \( p \)-forms, \( \delta^p \) are quotient differentials, \( [\omega] \mapsto \delta^p[\omega] = [d^{p,m}_V \omega], \) \( [\omega] \) is the equivalence class of the form \( \omega \in \Omega^{pm} \).

The resulting augmented bicomplex is presented on the page 22.

**Theorem 3.** The augmented bicomplex is acyclic, i.e., all his rows and columns are exact.

**Proof.** The detailed proof and the history of this famous theorem and close results on can find, for example, in \([9], [10], [11]\). \(\square\)

In the algebraic approach a nonlinear system of partial differential equations is written as \( F = 0 \), where \( F = \{ F^\sigma \in \mathcal{A} | \sigma \in S \} \), \( S \) is an index set. The associated differential ideal \( \mathcal{I}_F = \{ f = P_\sigma(D)F^\sigma | P_\sigma(D) \in \mathcal{A}[D], \sigma \in S \} \), where \( \mathcal{A}[D] \) is the unital associative noncommutative algebra of all polynomials in indeterminate \( D = \{ D_\mu | \mu \in \mathfrak{m} \} \) with coefficients in \( \mathcal{A} \). The quotient differential algebra \( (\mathcal{A}, \mathcal{D}) \) is called the differential algebra associated to the
system \( F = 0 \) (see, for example, \([15],[17]\) for more detail). This allows to write the spectral sequence (the Vinogradov spectral sequence \([8]\)) associated with the system \( F = 0 \). The calculation of this sequence or some of its terms is quite another problem, usually extremely hard. To construct the associated variation bicomplex one should first contrive to write the quotient differential algebra \( (\mathcal{A}, \mathcal{D}) \) in the regular form and then follow the procedure presented in Section \([5]\). Again, one is left with the calculation problem.

7 Conclusion.

The technics and methods presented above were approbated in the author’s works \([1],[2],[12],[13],[15],[16],[18]\). They may be useful in the researches \([19],[20],[21],[22],[23],[24],[25],[26]\).

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