Attractors in Black

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Contribution to the Proceedings of the 3rd RTN Workshop “Constituents, Fundamental Forces and Symmetries of the Universe”, 1–5 October 2007, Valencia, Spain

Abstract

We review recent results in the study of attractor horizon geometries (with non-vanishing Bekenstein-Hawking entropy) of dyonic extremal $d = 4$ black holes in supergravity. We focus on $\mathcal{N} = 2$, $d = 4$ ungauged supergravity coupled to a number $n_V$ of Abelian vector multiplets, outlining the fundamentals of the special Kähler geometry of the vector multiplets’ scalar manifold (of complex dimension $n_V$), and studying the $\frac{1}{2}$-BPS attractors, as well as the non-BPS (non-supersymmetric) ones with non-vanishing central charge.

For symmetric special Kähler geometries, we present the complete classification of the orbits in the symplectic representation of the classical $U$-duality group (spanned by the black hole charge configuration supporting the attractors), as well as of the moduli spaces of non-BPS attractors (spanned by the scalars which are not stabilized at the black hole event horizon).

Finally, we report on an analogous classification for $\mathcal{N} > 2$-extended, $d = 4$ ungauged supergravities, in which also the $\frac{1}{N}$-BPS attractors yield a related moduli space.
1 Introduction

Extremal black hole (BH) attractors were discovered some time ago in [1]-[5]. Recently, a number of papers have been devoted to their study [6]-[63] (for further developments, see also e.g. [64]-[68]), essentially because new classes of solutions to the so-called Attractor Equations were (re)discovered. Such new solutions have been found to determine non-BPS (Bogomol’ny-Prasad-Sommerfeld) BH horizon geometries, breaking all supersymmetries (if any).

The near-horizon attractor geometry of an extremal (static, spherically symmetric, asymptotically flat, dyonic) BH is associated to the corresponding configuration of the $1 \times (2n_V + 2)$ symplectic vector of the BH magnetic and electric charges $Q \equiv (p^\Lambda, q_\Lambda)$, defined as the spatially asymptotical fluxes of the vector field-strengths:

$$
p^\Lambda \equiv \frac{1}{4\pi} \int_{S^2_\infty} \mathcal{F}^\Lambda, \quad q_\Lambda \equiv \frac{1}{4\pi} \int_{S^2_\infty} \mathcal{G}_\Lambda.
$$

The symplectic index $\Lambda$ run $0, 1, \ldots, n_V$. In $\mathcal{N} = 2, d = 4$ ungauged supergravity $n_V$ denotes the number of Abelian vector multiplets coupled to the supergravity one\footnote{The Attractor Mechanism in $\mathcal{N} = 2, d = 4$ ungauged supergravity does not deal with the $n_H$ hypermultiplets eventually present.} (which contains the Maxwell vector $A^0$, usually named graviphoton). Moreover, denoting with $r$ the radial coordinate, $S^2_\infty$ is the 2-sphere for $r \to \infty$. $\mathcal{F}^\Lambda = dA^\Lambda$ and $\mathcal{G}_\Lambda$ is the related “dual” field-strength two-form (see e.g. [69]-[72]; see also [71]-[72]).

The present report will deal only with non-degenerate ($\frac{1}{2}$-BPS as well as non-BPS) geometries, i.e. with geometries yielding a finite, non-vanishing (effective) horizon area $A_H$, corresponding to the so-called “large” BHs. Through the critical implementation of the so-called Attractor Mechanism [1]-[5], the Bekenstein-Hawking entropy $S_{BH}$ [73]
associated to such attractor geometries can be computed by extremizing a properly defined, positive-definite “effective BH potential” $V_{BH}(\phi, p, q)$, with “$\phi$” standing for the relevant set of real scalar fields.

In $\mathcal{N} = 2$, $d = 4$ supergravity non-degenerate attractor horizon geometries split in two classes, one corresponding to $\frac{1}{2}$-BPS “short massive multiplets” (preserving four supersymmetries out of the eight pertaining to the $\mathcal{N} = 2$, $d = 4$ superPoincaré asymptotical background), and the other given by non-supersymmetric “long massive multiplets” violating the BPS bound [74]:

\[
\frac{1}{2} \text{-BPS}: 0 < |Z|^2_H = S_{BH}/\pi;
\]

\[
\text{non-BPS}, \begin{cases} 
Z \neq 0: 0 < |Z|^2_H < S_{BH}/\pi; \\
Z = 0: 0 = |Z|^2_H < S_{BH}/\pi,
\end{cases}
\]  

where the subscript “$H$” denotes the evaluation at the BH event horizon, and $Z$ stands for the $\mathcal{N} = 2$, $d = 4$ central charge function (see e.g. [70] and Refs. therein). As mentioned, the Bekenstein-Hawking entropy $S_{BH}$ is obtained by extremizing $V_{BH}(\phi, p, q)$ with respect to its dependence on the scalars [73, 5]:

\[
S_{BH}(p, q) = \frac{A_H(p, q)}{4} = \pi [V_{BH}(\phi, p, q)]_{\phi, V_{BH}=0} = \pi V_{BH}(\phi_H(p, q), p, q).
\]  

The purely charge-dependent horizon configuration $\phi_H(p, q)$ of the real scalars is a solution of the criticality conditions

\[
\frac{\partial V_{BH}(\phi, p, q)}{\partial \phi} = 0,
\]

and it determines an attractor in a strict sense if the critical $(2n_V + 2) \times (2n_V + 2)$ real symmetric Hessian matrix

\[
H_{BH}^{V_{BH}} \equiv \left. \frac{\partial^2 V_{BH}(\phi, p, q)}{\partial \phi \partial \phi} \right|_{\phi = \phi_H(p, q)}
\]

is strictly positive-definite.

It should be remarked that the opposite does not hold in general, i.e. attractors may exist such that the corresponding $H_{BH}^{V_{BH}}$ exhibits some vanishing eigenvalues. If this happens, a careful study of the (signs of the) higher-order covariant derivatives of $V_{BH}$ evaluated at the considered critical point(s) is needed. Depending on the supporting BH charge configuration, the massless Hessian modes can be lifted to positive values (determining stable critical points, and thus attractors) or to negative values (yielding unstable critical points, and thus repellers). Examples in literature of investigations beyond the Hessian level can be found in [10, 23, 24]. But a third possibility may happen, namely that the massless Hessian modes persist at all order in the covariant differentiation of $V_{BH}$; in such a case, a moduli space arises out, spanned by the scalars which are not stabilized at the BH event horizon in terms of the BH charges belonging to the configuration supporting the considered class of critical points of $V_{BH}$. 

\[2\]
Non-supersymmetric (non-BPS) BH attractors arise also in $\mathcal{N} > 2$-extended, $d = 4$ and $d = 5$ supergravities [75, 19] (see e.g. [76, 50, 77, 78] for recent reviews), but $\mathcal{N} = 2$, $d = 4$ supergravity, whose scalar manifold is a special Kähler (SK) space, exhibits the richest case study.

Moduli spaces of attractors have been recently found and classified in [35, 38] for $\mathcal{N} = 2$ symmetric and $\mathcal{N} > 2$-extended, $d = 4$ supergravities (see also [54] for an explicit treatment in the so-called stu model). In such theories, the Hessian matrix of $V_{BH}$ at its critical points is in general positive definite, eventually with some vanishing eigenvalues, which actually are flat directions of $V_{BH}$ itself. More in general, it can be stated that for all supergravities based on homogeneous (not necessarily symmetric) scalar manifolds the non-degenerate critical points of $V_{BH}$ are all stable, up to some eventual flat directions. We will briefly report on such an issue in the last Section.

The plan of the paper is as follows.

Sect. 2 reports about extremal BH attractors in $\mathcal{N} = 2$, $d = 4$ supergravity. The fundamentals of the SK geometry of the scalar manifold are outlined in Subsect. 2.1. Thence, in Subsect. 2.2 $V_{BH}$ for $\mathcal{N} = 2$, $d = 4$ supergravity is introduced, and its $\frac{1}{2}$-BPS critical points [1]–[5], which are always stable (thus determining attractors in a strict sense), are studied. The features of the class of non-BPS critical points of $V_{BH}$ with non-vanishing $Z$ are presented in Subsect. 2.3. The class of non-BPS critical points of $V_{BH}$ with $Z = 0$ will not be considered here (see rather e.g. [56] and Refs. therein). Sect. 3 reports some recent results on the classification of the supporting BH charge orbits and moduli spaces of extremal BH attractors in $\mathcal{N} = 2$ symmetric (Subsect. 3.1) and $\mathcal{N} > 2$-extended (Subsect. 3.2), $d = 4$ supergravities.

2 Extremal Black Hole Attractors in $\mathcal{N} = 2$, $d = 4$ Supergravity

2.1 Glossary of Special Kähler Geometry

In the present Section we briefly recall the fundamentals of the SK geometry underlying the scalar manifold $\mathcal{M}_{nV}$ of $\mathcal{N} = 2$, $d = 4$ supergravity coupled to $n_V$ Abelian vector multiplets ($\dim_{C}\mathcal{M}_{nV} = n_V$; see [71, 72]).

It is convenient to switch from the Riemannian $2n_V$-dim. parametrization of $\mathcal{M}_{nV}$ given by the local real coordinates $\{\phi^a\}_{a=1,...,2n_V}$ to the Kähler $n_V$-dim. holomorphic/antiholomorphic parametrization given by the local complex coordinates $\{z^i, \bar{z}^\tau\}_{i,\tau=1,...,n_V}$. This corresponds to the performing the unitary Cayley transformation:

$$z^k \equiv \frac{\varphi^{2k-1} + i\varphi^{2k}}{\sqrt{2}}, \quad k = 1, ..., n_V. \quad (2.1)$$

The metric structure of $\mathcal{M}_{nV}$ is given by the covariant SK metric tensor$^2$ $g_\Sigma(z, \bar{z}) = $
\[ \partial_t \partial_\tau K(z, \bar{z}), K(z, \bar{z}) \] being the real Kähler potential.

The previously mentioned \( \mathcal{N} = 2, d = 4 \) central charge function is defined as (see e.g. [70] and refs. therein)

\[
Z(z, \bar{z}; q, p) \equiv Q\Omega V(z, \bar{z}) = q_\Lambda L^\Lambda (z, \bar{z}) - p^\Lambda M_\Lambda (z, \bar{z}) = e^{\frac{1}{2} K(z, \bar{z})} Q\Omega \Pi(z) = \]

\[
= e^{\frac{1}{2} K(z, \bar{z})} [q_\Lambda X^\Lambda (z) - p^\Lambda F_\Lambda (z)] \equiv e^{\frac{1}{2} K(z, \bar{z})} W(z; q, p),
\]

where \( \Omega \) is the \((2n_V + 2)\)-dim. square symplectic metric (subscripts denote dimensions of square sub-blocks)

\[
\Omega \equiv \begin{pmatrix}
0_{n_V+1} & -I_{n_V+1} \\
I_{n_V+1} & 0_{n_V+1}
\end{pmatrix},
\]

and \( V(z, \bar{z}) \) and \( \Pi(z) \) respectively stand for the \((2n_V + 2) \times 1\) covariantly holomorphic (Kähler weights \((1, -1)\)) and holomorphic (Kähler weights \((2, 0)\)) period vectors in symplectic basis:

\[
\mathcal{D}_\tau V(z, \bar{z}) = \left( \partial_\tau - \frac{1}{2} \partial_\tau K \right) V(z, \bar{z}) = 0, \quad D_i V(z, \bar{z}) = \left( \partial_i + \frac{1}{2} \partial_i K \right) V(z, \bar{z});
\]

\[
\downarrow
\]

\[
V(z, \bar{z}) = e^{\frac{1}{2} K(z, \bar{z})} \Pi(z), \quad \mathcal{D}_\tau \Pi(z) = \partial_\tau \Pi(z) = 0, \quad D_i \Pi(z) = (\partial_i + \partial_i K) \Pi(z);
\]

\[
\Pi(z) \equiv \begin{pmatrix}
X^\Lambda (z) \\
F_\Lambda (X(z))
\end{pmatrix} = \exp \left( -\frac{1}{2} K(z, \bar{z}) \right) \begin{pmatrix}
L^\Lambda (z, \bar{z}) \\
M_\Lambda (z, \bar{z})
\end{pmatrix},
\]

with \( X^\Lambda (z) \) and \( F_\Lambda (X(z)) \) being the holomorphic sections of the \( U(1) \) line (Hodge) bundle over \( \mathcal{M}_{n_V} \). \( W(z; q, p) \) is the so-called holomorphic \( \mathcal{N} = 2 \) central charge function, also named \( \mathcal{N} = 2 \) superpotential \((\partial_\tau W = 0)\).

Up to some particular choices of local symplectic coordinates in \( \mathcal{M}_{n_V} \), the covariant symplectic holomorphic sections \( F_\Lambda (X(z)) \) may be seen as derivatives of an holomorphic prepotential function \( F \) (with Kähler weights \((4, 0)\)):

\[
F_\Lambda (X(z)) = \frac{\partial F(X(z))}{\partial X^\Lambda}.
\]

In \( \mathcal{N} = 2, d = 4 \) supergravity the holomorphic function \( F \) is constrained to be homogeneous of degree 2 in the contravariant symplectic holomorphic sections \( X^\Lambda (z) \), i.e. (see e.g. [69, 70] and Refs. therein)

\[
2F(X(z)) = X^\Lambda (z) F_\Lambda (X(z)).
\]

(2.6)

The normalization of the holomorphic period vector \( \Pi(z) \) is such that

\[
K(z, \bar{z}) = -ln \left[ i \left< \Pi(z), \Pi(\bar{z}) \right> \right] \equiv -ln \left[ i \Pi^T(z) \Omega \Pi(\bar{z}) \right] =
\]

\[
= -ln \left\{ i \left[ X^\Lambda (z) F_\Lambda (z) - X^\Lambda (z) \overline{F_\Lambda (\bar{z})} \right] \right\},
\]

(2.7)
where \( \langle \cdot, \cdot \rangle \) stands for the symplectic scalar product defined by \( \Omega \). Note that under a Kähler transformation

\[
K(z, \bar{z}) \longrightarrow K(z, \bar{z}) + f(z) + \mathcal{F}(\bar{z}) \tag{2.8}
\]

\( f(z) \) being a generic holomorphic function), the holomorphic period vector transforms as

\[
\Pi(z) \longrightarrow \Pi(z) e^{-f(z)} \longrightarrow X^\Lambda(z) \longrightarrow X^\Lambda(z) e^{-f(z)}. \tag{2.9}
\]

This yields that, at least locally, the contravariant holomorphic symplectic sections \( X^\Lambda(z) \) can be regarded as a set of homogeneous coordinates on \( \mathcal{M}_{n_V} \), provided that the Jacobian complex \( n_V \times n_V \) holomorphic matrix

\[
e^a_i(z) \equiv \frac{\partial}{\partial z^i} \left( \frac{X^a(z)}{X^0(z)} \right), \quad a = 1, \ldots, n_V \tag{2.10}
\]

is invertible. If this is the case, then one can introduce the local projective symplectic coordinates

\[
t^a(z) \equiv \frac{X^a(z)}{X^0(z)}, \tag{2.11}
\]

and the SK geometry of \( \mathcal{M}_{n_V} \) turns out to be based on the holomorphic prepotential \( F(t) \equiv (X^0)^{-2} F(X) \). By using the \( t \)-coordinates, Eq. (2.7) can be rewritten as follows

\[
K(t, \bar{t}) = -\ln \left\{ i |X^0(z(t))|^2 \left[ 2(F(t) - \mathcal{F}(\bar{t})) - (t^a - \bar{t}^a) (F_a(t) + \mathcal{F}_a(\bar{t})) \right] \right\}. \tag{2.12}
\]

By performing a Kähler gauge-fixing with \( f(z) = \ln(X^0(z)) \), yielding that \( X^0(z) \longrightarrow 1 \), one thus gets

\[
K(t, \bar{t}) \big|_{X^0(z)\longrightarrow 1} = -\ln \left\{ i \left[ 2(F(t) - \mathcal{F}(\bar{t})) - (t^a - \bar{t}^a) (F_a(t) + \mathcal{F}_a(\bar{t})) \right] \right\}. \tag{2.13}
\]

In particular, one can choose the so-called special coordinates, i.e. the system of local projective \( t \)-coordinates such that

\[
e^a_i(z) = \delta^a_i \Leftrightarrow t^a(z) = z^i \left( +c^i, \ c^i \in \mathbb{C} \right). \tag{2.14}
\]

Thus, Eq. (2.13) acquires the form

\[
K(t, \bar{t}) \big|_{X^0(z)\longrightarrow 1, e^a_i(z)=\delta^a_i} = -\ln \left\{ i \left[ 2(F(z) - \mathcal{F}(\bar{z})) - (z^j - \bar{z}^j) (F_j(z) + \mathcal{F}_j(\bar{z})) \right] \right\}. \tag{2.15}
\]

Moreover, it should be recalled that \( Z \) has Kähler weights \((p, \bar{p}) = (1, -1)\), and therefore its Kähler-covariant derivatives read

\[
D_i Z = \left( \partial_i + \frac{1}{2} \partial_i K \right) Z, \quad \bar{D}_\tau Z = \left( \bar{\partial}_\tau - \frac{1}{2} \bar{\partial}_\tau K \right) Z. \tag{2.16}
\]
The fundamental differential relations of SK geometry are (see e.g. [70]; for elucidations about the various equivalent approaches to SK geometry, see also [79] and [80]):

\[
\begin{align*}
D_iZ &= Z_i \quad \text{(definition of matter charges)}; \\
D_iZ_j &= iC_{ijk}g^{jk}D_kZ = iC_{ijk}g^{jk}Z_k; \\
D_i\overline{D_jZ} &= D_i\overline{Z_j} = g_{\overline{j}}\overline{Z}_j; \\
D_i\overline{Z} &= 0 \quad \text{(Kähler-covariant holomorphicity}).
\end{align*}
\]  

(2.17)

The first relation is nothing but the definition of the so-called matter charges \(Z_i\), and the fourth relation expresses the Kähler-covariant holomorphicity of \(Z\). \(C_{ijk}\) is the rank-3, completely symmetric, covariantly holomorphic tensor of SK geometry (with Kähler weights \((2, -2)\)) (see e.g. [70] [81] [82]):

\[
\begin{align*}
C_{ijk} &= (D_iD_jV, D_kV) = e^K (\partial_e N_{\Lambda\Sigma}) D_jX^\Lambda D_kX^\Sigma = \\
&= e^K \left( \partial_i X^\Lambda \right) \left( \partial_j X^\Sigma \right) \partial_k\partial_\Lambda F_\Sigma (X) \equiv e^K W_{ijk}, \quad \overline{\partial_i} W_{ijk} = 0; \\
C_{ijk} &= D_iD_jD_kS, \quad S \equiv -i\Lambda^L L^\Sigma Im \left( F_{\Lambda\Sigma} \right), \quad F_{\Lambda\Sigma} \equiv \frac{\partial F_{\Lambda\Sigma}}{\partial\Xi^\Lambda \Xi^\Sigma}, F_{\Lambda\Sigma} \equiv F_{\left(\Lambda\Sigma\right)}; \\
C_{ijk} &= -ig_{\overline{i}l} f^L_{ijk} D_jD_kL^L, \quad \overline{f^L_{ijk}} \left( D\overline{L}^L \right) \equiv \overline{\delta^L_{ijk}}.
\end{align*}
\]

(2.18)

\(\overline{D_iC_{jkl}} = 0\) (covariant holomorphicity);

\[R_{ijkl} = -g_{ij}\overline{g}_{kl} - g_{ik}\overline{g}_{jl} + C_{ikp}g_{jlq}\overline{g}_{pq} (\text{usually named SK geometry constraints});\]

\[D_i[C_{jkl}] = 0.\]

the last property being a consequence, through the \(SK\) geometry constraints and the covariant holomorphicity of \(C_{ijk}\), of the Bianchi identities satisfied by the Riemann tensor \(R_{ijkl}\). As usual, square brackets denote antisymmetrization with respect to enclosed indices.

It is worth remarking that the third of Eqs. (2.18) correctly defines the Riemann tensor \(R_{ijkl}\), and it is actual the opposite of the one which may be found in a large part of existing literature. Such a formulation of the so-called \(SK\) geometry constraints is well defined, because it consistently yields negative values of the constant scalar curvature of symmetric SK manifolds (see e.g. [83]). Furthermore, it should be recalled that in a generic Kähler geometry \(R_{ijkl}\) reads (see e.g. [84])

\[
R_{ijkl} = g^{mn} (\partial_{i}\overline{\partial_j}\partial_{m} K) \partial_{l}\overline{\partial_\ell} \partial_{k} K - \overline{\partial_i} \partial_{l} \overline{\partial_j} \partial_{k} K = g_{k\ell} \partial_{i} \overline{\Gamma_{j\ell}^{\overline{\gamma}}} g_{\overline{\gamma}l} T_{\gamma} = g_{mk} \overline{\partial_j} \Gamma_{kl}^{\overline{n}}.
\]

\[\overline{R_{ijkl}} = R_{jikl} \quad \text{(reality)},\]

\[\Gamma_{ij}^{l} = -g^{\overline{\gamma}} \partial_i g_{j\overline{\gamma}} = -g^{l\overline{\gamma}} \partial_i \overline{\partial_j} K = \Gamma_{(ij)}^{l},\]

\[\boxed{\begin{align*}
R_{ijkl} &= g^{mn} (\partial_{i}\overline{\partial_j}\partial_{m} K) \partial_{l}\overline{\partial_\ell} \partial_{k} K - \overline{\partial_i} \partial_{l} \overline{\partial_j} \partial_{k} K = g_{k\ell} \partial_{i} \overline{\Gamma_{j\ell}^{\overline{\gamma}}} g_{\overline{\gamma}l} T_{\gamma} = g_{mk} \overline{\partial_j} \Gamma_{kl}^{\overline{n}}.
\end{align*}}\]
where $\Gamma_{ij}^k$ stand for the Christoffel symbols of the second kind of the Kähler metric $g_{\mathcal{I}}$.

In the first of Eqs. (2.18), a fundamental entity, the so-called kinetic matrix $N_{\Lambda\Sigma}(z,\overline{z})$ of $\mathcal{N} = 2$, $d = 4$ supergravity, has been introduced. It is an $(n_V + 1) \times (n_V + 1)$ complex symmetric, moduli-dependent, Kähler gauge-invariant matrix defined by the following fundamental Ansätze, solving the SKG constraints given by the third of Eqs. (2.18):

$$M_\Lambda = N_{\Lambda\Sigma} L^\Sigma, \quad D_i M_\Lambda = \overline{N}_{\Lambda\Sigma} D_i L^\Sigma. \quad (2.20)$$

By introducing the $(n_V + 1) \times (n_V + 1)$ complex matrices $(I = 1, \ldots, n_V + 1)$

$$f_I^A(z,\overline{z}) \equiv \left( D_I^\mathcal{I} L^\mathcal{I} (z,\overline{z}), L^A (z,\overline{z}) \right), \quad h_{IA}(z,\overline{z}) \equiv \left( D_I^\mathcal{I} M_\Lambda (z,\overline{z}), M_\Lambda (z,\overline{z}) \right), \quad (2.21)$$

the Ansätze (2.20) uniquely determine $N_{\Lambda\Sigma}(z,\overline{z})$ as

$$N_{\Lambda\Sigma}(z,\overline{z}) = h_{IA}(z,\overline{z}) \circ (f^{-1})^I_{\Sigma} (z,\overline{z}), \quad (2.22)$$

where $\circ$ denotes the usual matrix product, and $(f^{-1})^I_{\Sigma} f_I^A = \delta_I^A$, $(f^{-1})^I_{\lambda} f_\lambda^I = \delta_I^I$.

The covariantly holomorphic $(2n_V + 2) \times 1$ period vector $V(z,\overline{z})$ is symplectically orthogonal to all its Kähler-covariant derivatives:

$$\begin{align*}
\langle V(z,\overline{z}), D_I V(z,\overline{z}) \rangle &= 0; \\
\langle V(z,\overline{z}), D_I \overline{V}(z,\overline{z}) \rangle &= 0; \\
\langle V(z,\overline{z}), D_I \overline{V}(z,\overline{z}) \rangle &= 0; \\
\langle V(z,\overline{z}), D_I V(z,\overline{z}) \rangle &= 0.
\end{align*} \quad (2.23)$$

Moreover, it holds that

$$g_{\mathcal{I} \mathcal{J}}(z,\overline{z}) = -i \langle D_I V(z,\overline{z}), \overline{D_J V}(z,\overline{z}) \rangle =$$

$$= -2Im(N_{\Lambda\Sigma}(z,\overline{z})) D_I L^A (z,\overline{z}) \overline{D_J L^\Sigma} (z,\overline{z}) =$$

$$= 2Im(F_{\lambda\Sigma}(z)) D_I L^A (z,\overline{z}) \overline{D_J L^\Sigma} (z,\overline{z}); \quad (2.24)$$

$$\langle V(z,\overline{z}), D_I \overline{D_J V}(z,\overline{z}) \rangle = iC_{ijk}g^{\mathcal{I} \mathcal{K}} \langle V(z,\overline{z}), D_\mathcal{K} \overline{V}(z,\overline{z}) \rangle = 0. \quad (2.25)$$

The fundamental $(2n_V + 2) \times 1$ vector identity defining the geometric structure of SK manifolds read as follows [85, 9, 14, 17, 18, 23]:

$$Q^T - i\Omega \mathcal{M}(\mathcal{N}) Q^T = -2iZ\overline{V} - 2ig^{\mathcal{I} \mathcal{J}} (D_\mathcal{I} \overline{V}) D_\mathcal{J} V. \quad (2.26)$$

The $(2n_V + 2) \times (2n_V + 2)$ real symmetric matrix $\mathcal{M}(\mathcal{N})$ is defined as [70, 3, 4]

$$\mathcal{M}(\mathcal{N}) = \mathcal{M}(Re(\mathcal{N}), Im(\mathcal{N})) \equiv$$

$$\equiv \begin{pmatrix}
Im(\mathcal{N}) + Re(\mathcal{N}) (Im(\mathcal{N}))^{-1} Re(\mathcal{N}) & -Re(\mathcal{N}) (Im(\mathcal{N}))^{-1} \\
-(Im(\mathcal{N}))^{-1} Re(\mathcal{N}) & (Im(\mathcal{N}))^{-1}
\end{pmatrix}. \quad (2.27)$$
It is worth reminding that $\mathcal{M}(\mathcal{N})$ is symplectic with respect to the metric $\Omega$ defined in Eq. (2.3), i.e. it satisfies $((\mathcal{M}(\mathcal{N}))^T = \mathcal{M}(\mathcal{N}))$

$$\mathcal{M}(\mathcal{N}) \Omega \mathcal{M}(\mathcal{N}) = \Omega. \quad (2.28)$$

By using Eqs. (2.7), (2.23), (2.24) and (2.25), the identity (2.26) implies the following relations:

$$\begin{bmatrix}
\langle V, Q^T - i\Omega\mathcal{M}(\mathcal{N}) Q^T \rangle = -2Z; \\
\langle \bar{V}, Q^T - i\Omega\mathcal{M}(\mathcal{N}) Q^T \rangle = 0; \\
\langle D_i V, Q^T - i\Omega\mathcal{M}(\mathcal{N}) Q^T \rangle = 0; \\
\langle \bar{D}_i V, Q^T - i\Omega\mathcal{M}(\mathcal{N}) Q^T \rangle = -2\bar{D}_i Z.
\end{bmatrix} \quad (2.29)$$

There are only $2n_V$ independent real relations out of the $4n_V + 4$ real ones yielded by the $2n_V + 2$ complex identities (2.26). Indeed, by taking the real and imaginary part of the vector identity (2.26) one respectively obtains

$$Q^T = -2Re \left[ iZ V + iG^{\bar{j}} (\bar{D}_j Z) D_j V \right] = -2Im \left[ Z V + G^{\bar{j}} (D_j Z) (\bar{D}_j V) \right]; \quad (2.30)$$

$$\Omega \mathcal{M}(\mathcal{N}) Q^T = 2Im \left[ iZ V + iG^{\bar{j}} (\bar{D}_j Z) D_j V \right] = 2Re \left[ Z V + G^{\bar{j}} (D_j Z) (\bar{D}_j V) \right]. \quad (2.31)$$

Consequently, the imaginary and real parts of the vector identity (2.26) are linearly dependent one from the other, being related by the $(2n_V + 2) \times (2n_V + 2)$ real matrix

$$\Omega \mathcal{M}(\mathcal{N}) = \begin{pmatrix}
(Im(\mathcal{N}))^{-1} Re(\mathcal{N}) & - (Im(\mathcal{N}))^{-1} \\
Im(\mathcal{N}) + Re(\mathcal{N}) (Im(\mathcal{N}))^{-1} Re(\mathcal{N}) & - Re(\mathcal{N}) (Im(\mathcal{N}))^{-1}
\end{pmatrix}. \quad (2.32)$$

Put another way, Eqs. (2.30) and (2.31) yield

$$Re \left[ Z V + G^{\bar{j}} (\bar{D}_j Z) D_j V \right] = \Omega \mathcal{M}(\mathcal{N}) Im \left[ Z V + G^{\bar{j}} (\bar{D}_j Z) D_j V \right], \quad (2.33)$$

expressing the fact that the real and imaginary parts of the quantity $Z V + G^{\bar{j}} (\bar{D}_j Z) D_j V$ are simply related through a symplectic rotation given by the matrix $\Omega \mathcal{M}(\mathcal{N})$, whose simplecticity directly follows from the symplectic nature of $\mathcal{M}(\mathcal{N})$. Eq. (2.33) reduces the number of independent real relations implied by the identity (2.26) from $4n_V + 4$ to $2n_V + 2$.

Moreover, it should be stressed that vector identity (2.26) entails 2 redundant degrees of freedom, encoded in the homogeneity (of degree 1) of (2.26) under complex scalings of $Q$. Indeed, by using the definition (2.2), it is easy to check that the right-hand side of (2.26) gets scaled by an overall factor $\lambda$ under the following transformation on $Q$:

$$Q \rightarrow \lambda Q, \quad \lambda \in \mathbb{C}. \quad (2.34)$$
Thus, as announced, only $2n_V$ real independent relations are actually yielded by the vector identity (2.26).

This is clearly consistent with the fact that the $2n_V + 2$ complex identities express nothing but a change of basis of the BH charge configurations, between the Kähler-invariant $1 \times (2n_V + 2)$ symplectic (magnetic/electric) basis vector $Q$ defined by Eq. (1.1) and the complex, moduli-dependent $1 \times (n_V + 1)$ matter charges vector (with Kähler weights $(1, -1)$)

$$Z(z, \bar{z}) \equiv (Z(z, \bar{z}), Z_i(z, \bar{z}))_{i=1, \ldots, n_V}. \quad (2.35)$$

It should be recalled that the BH charges are conserved due to the overall $(U(1))^{n_V+1}$ gauge-invariance of the system under consideration, and $Q$ and $Z(z, \bar{z})$ are two equivalent basis for them. Their very equivalence relations are given by the identities (2.26) themselves. By its very definition (1.1), $Q$ is moduli-independent (at least in a static, spherically symmetric and asymptotically flat extremal BH background, as it is the case being treated here), whereas $Z$ is moduli-dependent, since it refers to the eigenstates of the $\mathcal{N} = 2, d = 4$ supergravity multiplet and of the $n_V$ Maxwell vector multiplets.

### 2.2 $\frac{1}{2}$-BPS Attractors

In $\mathcal{N} = 2, d = 4$ supergravity the following expression holds [3, 4, 70]:

$$V_{BH} = |Z|^2 + g^{i\bar{j}} (D_i Z) \bar{D}_{\bar{j}} \bar{Z}. \quad (2.36)$$

An elegant way to obtain $V_{BH}$ is given by left-multiplying the vector identity (2.26) by the $1 \times (2n_V + 2)$ complex moduli-dependent vector $-\frac{1}{2} Q \mathcal{M}(\mathcal{N})$; due to the symplecticity of the matrix $\mathcal{M}(\mathcal{N})$, one obtains [3, 4, 70]

$$V_{BH} = -\frac{1}{2} Q \mathcal{M}(\mathcal{N}) Q^T. \quad (2.37)$$

Thus, $V_{BH}$ is identified with the first (of two), lowest-order (quadratic - in charges), positive-definite real invariant $I_1$ of SK geometry (see e.g. [23, 70]). It is worth noticing that the result (2.37) can also be derived from the SK geometry identities (2.26) by using the relation (see [19], where a generalization for $\mathcal{N} > 2$-extended supergravities is also given)

$$\frac{1}{2} (\mathcal{M}(\mathcal{N}) + i \Omega) V = i \Omega V \Leftrightarrow \mathcal{M}(\mathcal{N}) V = i \Omega V, \quad (2.38)$$

where $V$ is a $(2n_V + 2) \times (n_V + 1)$ matrix defined as:

$$V \equiv (V, \bar{D}_1 V, \ldots, \bar{D}_{n_V} V). \quad (2.39)$$

By differentiating Eq. (2.36) with respect to the scalars, it is easy to check that the general criticality conditions (1.4) can be recast in the following form [5]:

$$D_i V_{BH} = \partial_i V_{BH} = 0 \Leftrightarrow 2 \bar{Z} D_i Z + g^{i\bar{j}} (D_i D_j Z) \bar{D}_{\bar{j}} \bar{Z} = 0; \quad (2.40)$$
This is what one should rigorously call the $\mathcal{N} = 2$, $d = 4$ Attractor Eqs. (AEs). By means of the features of SK geometry given by Eqs. (2.17), the $\mathcal{N} = 2$ AEs (2.40) can be re-expressed as follows [5]:

\[
2 \overline{Z} Z_i + i C_{ijk} g^{l \overline{k}} g^{k \overline{m}} Z_j \overline{Z}_k = 0.
\] (2.41)

It is evident that the tensor $C_{ijk}$ is crucial in relating the $\mathcal{N} = 2$ central charge function $Z$ (graviphoton charge) and the $n_V$ matter charges $Z_i$ (coming from the $n_V$ Abelian vector multiplets) at the critical points of $V_{BH}$ in the SK scalar manifold $\mathcal{M}_{n_V}$.

The static, spherically symmetric, asymptotically flat dyonic (not necessarily extremal) $d = 4$ BHs are known to be described by an effective $d = 1$ Lagrangian ([5], [86], and also [18] and [76]), with $V_{BH}$ and effective fermionic “mass terms” controlled by the vector $Q$ defined by Eq. (1.1). The “apparent” gravitino mass is given by $Z$, whereas the gaugino mass matrix $\Lambda_{ij}$ reads (see the second Ref. of [82])

\[
\Lambda_{ij} = C_{ijk} g^{k \overline{m}} Z_{\overline{m}}.
\] (2.42)

The supersymmetry breaking order parameters, related to the mixed gravitino-gaugino couplings, are nothing but the matter charge (function)s $D_i Z_i$ (see the first of Eqs. (2.17)).

As evident from the AEs (2.40) and (2.41), the conditions

\[
(Z \neq 0,) \quad D_i Z = 0 \quad \forall i = 1, ..., n_V
\] (2.43)

determine a (non-degenerate) critical point of $V_{BH}$, namely a $\frac{1}{2}$-BPS critical point, which preserve four supersymmetry degrees of freedom out of the eight pertaining to the $\mathcal{N} = 2$, $d = 4$ Poincaré superalgebra related to the asymptotical Minkowski background. The corresponding Bekenstein entropy reads [1]- [5]:

\[
S_{BH, \frac{1}{2}-BPS} = \pi V_{BH} |_{\frac{1}{2}-BPS} = \pi \left\{ |Z|_{\frac{1}{2}-BPS}^2 + \left[ g^{\overline{m}} (D_i Z) (\overline{D}_i \overline{Z}) \right]_{\frac{1}{2}-BPS} \right\} = |Z|_{\frac{1}{2}-BPS}^2 > 0.
\] (2.44)

In general, $\frac{1}{2}$-BPS critical points are (at least local) minima of $V_{BH}$ in $\mathcal{M}_{n_V}$, and therefore they are stable; thus, they are attractors in a strict sense. Indeed, the $2n_V \times 2n_V$ matrix $\mathcal{H}^{V_{BH}}$ (within the Kähler holomorphic/antiholomorphic parametrization) evaluated at such points is strictly positive-definite [5]:

\[
(D_i D_j V_{BH})_{\frac{1}{2}-BPS} = (\partial_i \partial_j V_{BH})_{\frac{1}{2}-BPS} = 0,
\]

\[
(D_i \overline{D}_j V_{BH})_{\frac{1}{2}-BPS} = (\partial_i \overline{\partial}_j V_{BH})_{\frac{1}{2}-BPS} + 2 \left( g^{\overline{m}} V_{BH} \right)_{\frac{1}{2}-BPS} = 2 g^{\overline{m}} |_{\frac{1}{2}-BPS} |Z|_{\frac{1}{2}-BPS}^2 > 0,
\] (2.45)

where the notation “$> 0$” is here understood as strict positive-definiteness. Eqs. (2.45) yield that the Hermiticity and (strict) positive-definiteness of $\mathcal{H}^{V_{BH}}$ (in $(z, \overline{z})$-coordinates) at the $\frac{1}{2}$-BPS critical points are due to the Hermiticity and - assumed - (strict) positive-definiteness (actually holding globally) of the metric $g_{\overline{m}}$ of the manifold $\mathcal{M}_{n_V}$.

Considering the $\mathcal{N} = 2$, $d = 4$ supergravity Lagrangian in a static, spherically symmetric, asymptotically flat dyonic BH background, and denoting by $\psi$ and $\lambda$ respectively...
the gravitino and gaugino fields, it is easy to see that such a Lagrangian contains terms of the form (see the second and third Refs. of [82])

\[ Z\psi\psi; \]

\[ C_{ijk}g^{\lambda\kappa}(\overline{D_kZ})\lambda^i\lambda^j; \]

\[ (D_iZ)\lambda^i\psi. \]

Thus, the $\frac{1}{2}$-BPS conditions (2.43) implies the gaugino mass term and the $\lambda\psi$ term to vanish at the $\frac{1}{2}$-BPS critical points of $V_{BH}$ in $M_{n_V}$. It is interesting to remark that the gravitino “apparent mass” term $Z\psi\psi$ is in general non-vanishing, also when evaluated at the considered $\frac{1}{2}$-BPS attractors; this is ultimately a consequence of the fact that the extremal BH horizon geometry at the $\frac{1}{2}$-BPS (as well as at the non-BPS) attractors is Bertotti-Robinson AdS$_2 \times S^2$ (with vanishing scalar curvature and conformally flat) [87, 88, 89].

2.3 Non-BPS $Z \neq 0$ Critical Points of $V_{BH}$

The $\frac{1}{2}$-BPS conditions (2.43) are not the most general ones solving the $N = 2, d = 4$ AEs (2.40) or (2.41). For instance, one might consider critical points of $V_{BH}$ (thus satisfying the AEs (2.40) or (2.41)) characterized by

\[ \begin{cases} 
D_iZ \neq 0, \text{ for (at least one) } i, \\
Z \neq 0.
\end{cases} \] (2.47)

Such critical points are non-supersymmetric ones (i.e. they do not preserve any of the eight supersymmetry degrees of freedom of the asymptotical Minkowski background), and they correspond to an extremal, non-BPS BH background. They are commonly named non-BPS $Z \neq 0$ critical points of $V_{BH}$. We will devote the present Section to present their main features.

The corresponding non-BPS $Z \neq 0$ Bekenstein-Hawking entropy reads ([9], [14], [16]):

\[ S_{BH,\text{non-BPS},Z\neq 0} = \pi V_{BH}|_{\text{non-BPS},Z\neq 0} = \]

\[ = \pi \left[ |Z|_{\text{non-BPS},Z\neq 0}^2 + [g^\kappa(D_iZ)(\overline{D_kZ})]_{\text{non-BPS},Z\neq 0} \right] > \pi |Z|_{\text{non-BPS},Z\neq 0}^2, \] (2.48)

not saturating the BPS bound. As implied by AEs (2.41), if at non-BPS $Z \neq 0$ critical points it holds that $D_iZ \neq 0$ for at least one index $i$ and $Z \neq 0$, then

\[ (C_{ijk})_{\text{non-BPS},Z\neq 0} \neq 0, \text{ for some } (i,j,k) \in \{1,...,n_V\}^3, \] (2.49)

i.e. the rank-3 symmetric tensor $C_{ijk}$ will for sure have some non-vanishing components in order for criticality conditions (2.41) to be satisfied at non-BPS $Z \neq 0$ critical points.

Moreover, the general criticality conditions (2.40) for $V_{BH}$ can be recognized to be the general Ward identities relating the gravitino mass $Z$, the gaugino masses $D_iD_jZ$
and the supersymmetry-breaking order parameters $D_i Z$ in a generic spontaneously broken supergravity theory \[90\]. Indeed, away from $\frac{1}{2}$-BPS critical points (i.e. for $D_i Z \neq 0$ for some $i$), the AEs (2.40) can be re-expressed as follows (see also \[34\]):

\[
\left( M_{ij} h^j \right)_{\partial V_{BH}=0} = 0,
\]

(2.50)

with

\[
M_{ij} \equiv D_i D_j Z + 2 \frac{\bar{Z}}{g_{kk} (D_k Z) D_k Z} (D_i Z) D_j Z,
\]

(2.51)

and

\[
h^j \equiv \bar{g}^j \bar{D}_j \bar{Z},
\]

(2.52)

For a non-vanishing contravariant vector $h^j$ (i.e. away from $\frac{1}{2}$-BPS critical points, as pointed out above), Eq. (2.50) admits a solution iff the $n_V \times n_V$ complex symmetric matrix $M_{ij}$ has vanishing determinant (implying that it has at most $n_V - 1$ non-vanishing eigenvalues) at the considered (non-BPS) critical points of $V_{BH}$ (however, notice that $M_{ij}$ is symmetric but not necessarily Hermitian, thus in general its eigenvalues are not necessarily real).

By using the properties of SK geometry, the non-BPS $Z \neq 0$ Bekenstein-Hawking entropy (2.48) can be further elaborated as follows \[56\]:

\[
\frac{S_{BH,\text{non-BPS},Z \neq 0}}{\pi} = \left\{ |Z|^2 \cdot \left[ 1 + \frac{1}{4 |Z|^4} R_{k\tau\rho\sigma} g^{k\tau} g^{\rho\sigma} (D_t Z) (D_u Z) \bar{(D_\tau Z) \bar{D}_\rho Z} + \right. \right.
\]

\[
\left. \left. + \frac{1}{2 |Z|^4} \left[ g^7 (D_t Z) \bar{D}_7 \bar{Z} \right]^2 \right] \right\}_{\text{non-BPS},Z \neq 0}.
\]

(2.53)

One can then introduce the so-called non-BPS $Z \neq 0$ supersymmetry breaking order parameter \[56\]:

\[
(0 <) O_{\text{non-BPS},Z \neq 0} \equiv \left[ \frac{g^7 (D_t Z) \bar{D}_7 \bar{Z}}{|Z|^2} \right]_{\text{non-BPS},Z \neq 0} =
\]

\[
= - \left[ \frac{i}{2 Z |Z|^2} C_{ijk} g^{i \bar{r} \bar{m}} \bar{C}_{\bar{r} \bar{m} \sigma} \bar{g}^{\bar{r} \bar{m} \sigma} (D_t Z) (D_u Z) \bar{(D_\sigma Z) \bar{D}_m Z} \right]_{\text{non-BPS},Z \neq 0} =
\]

\[
= \left[ \frac{1}{4 |Z|^4} g^{i \bar{r}} C_{i k n} \bar{C}_{\bar{r} \bar{m} \sigma} g^{\bar{m} \bar{r} \sigma} (D_t Z) (D_u Z) \bar{(D_\sigma Z) \bar{D}_m Z} \right]_{\text{non-BPS},Z \neq 0}.
\]

(2.54)

Consequently

\[
S_{BH,\text{non-BPS},Z \neq 0} = \pi \left\{ |Z|^2_{\text{non-BPS},Z \neq 0} \left[ 1 + O_{\text{non-BPS},Z \neq 0} \right] \right\} =
\]

\[
= \pi |Z|^2_{\text{non-BPS},Z \neq 0} \cdot
\]

\[
\cdot \left[ 3 - 2 \frac{\mathcal{R} (Z)}{g^{i \bar{r}} C_{i k n} \bar{C}_{\bar{r} \bar{m} \sigma} g^{\bar{m} \bar{r} \sigma} (D_t Z) (D_u Z) \bar{(D_\sigma Z) \bar{D}_m Z}} \right]_{\text{non-BPS},Z \neq 0},
\]

(2.55)
where the sectional curvature (see e.g. [91] and [92])
\[ \mathcal{R}(Z) \equiv R_{ijkl} g^{ir} g^{sj} g^{tk} g^{u} (D_j Z) (D_l Z) \left( \overline{D_j Z} \right) \left( \overline{D_l Z} \right) \]
was introduced.

Now, by using the relations of SK geometry it is possible to show that [76]
\[ \overline{D_m} D_i C_{jkl} = [\overline{D_m}, D_i] C_{jkl} = \overline{D_m} D_i (C_{jkl}) = \overline{D_m} D_i (C_{jkl}) = 3 C_{p(kl} C_{ij)\hat{n}} g^{\pi \beta \gamma \rho \tau} C_{\pi \beta \gamma \rho \tau} - 4 g_{i(m} C_{|ijkl|}; \]
\[ \downarrow \]
\[ C_{p(kl} C_{ij)\hat{n}} g^{\pi \beta \gamma \rho \tau} C_{\pi \beta \gamma \rho \tau} = \frac{4}{3} g_{i(m} C_{|ijkl|} + \mathcal{E}_{m(ijk)} \]
was introduced. It can be shown that [76, 56]
\[ S_{BH, non-BPS, Z \neq 0} = \frac{\pi}{|Z|_{non-BPS, Z \neq 0}^2}. \]
\[ \cdot \left\{ 4 - \frac{3}{4} \left[ \frac{1}{|Z|^2} \left[ \frac{E_{i(klm)} g^{ir} g^{js} g^{tk} g^{u} (D_j Z) (D_l Z) (D_m Z) (D_n Z)}{N_3(Z)} \right] \right] \right\} \]
\[ \nonumber \]
\[ \nonumber \]
\[ \nonumber \]
\[ \nonumber \]
\[ \nonumber \]
\[ \nonumber \]
\[ \nonumber \]
where the complex cubic form
\[ N_3(Z) \equiv \mathcal{C}_{ijk} g^{\pi \beta \gamma \rho \tau} g^{sr} g^{tk} (D_j Z) (D_l Z) D_k Z \]
was introduced.

Let us now consider the case of symmetric SK manifolds, in which the Kähler-invariant Riemann-Christoffel tensor $R_{ijkl}$ is covariantly constant. From this it follows that [93]:
\[ D_m R_{ijkl} = 0 \Leftrightarrow D_i C_{jkl} = D(i C_{jkl}) = 0. \]
(2.61)
This is a sufficient (but generally not necessary) condition for the global vanishing of the (complex conjugate) $E$-tensor $\mathcal{E}_{ijkl}$:
\[ D_{i(j} C_{kl)} = 0 \Leftrightarrow \overline{D_m} D_i C_{jkl} = 0 \Leftrightarrow D_m \overline{D_i} \mathcal{C}_{jkl} = 0, \]
(2.62)\footnote{Indeed, due to the reality of $R_{ijkl}$ in any Kähler manifold, it holds that $D_m R_{ijkl} = 0 \Leftrightarrow \overline{D_m} R_{ijkl} = 0$.}

\footnote{Indeed, some non-symmetric SK (a priori not necessarily homogeneous) manifolds might exist such that $D_{i(j} C_{kl)} = 0$, but however (globally) satisfying
\[ \overline{D_m} D_i C_{jkl} = \overline{D_m} D(i C_{jkl}) = \overline{D_m} D(i C_{jkl}) - \left( \overline{D_m} K \right) D(i C_{jkl}) = 0. \]}

13
yielding \[ 93 \]
\[ C_{p(kl} C_{ij)n} g^{n\pi} g^{n\pi} C_{nmp} = \frac{4}{3} g_{(i|mC_{|ijk})} \leftrightarrow g^{n\pi} R_{(i|m|n|k)l} C_{n|kl} = -\frac{2}{3} g_{(i|mC_{|ijkl}).} \] (2.63)

Furthermore, the following noteworthy relation, holding in *symmetric SK manifolds*, can be proved \[56\]:
\[
\left( Z | Z \right)^2_{\text{non-BPS,} Z \neq 0} = \frac{i}{6} \left[ N_3 (Z) \right]_{\text{non-BPS,} Z \neq 0} \downarrow \quad \text{Re} \left( \left[ N_3 (Z) \right]_{\text{non-BPS,} Z \neq 0} \right) = 0; \quad \text{Im} \left( \left[ N_3 (Z) \right]_{\text{non-BPS,} Z \neq 0} \right) = -6 |Z|_{\text{non-BPS,} Z \neq 0}^2.
\] (2.64)

Consequently, the *supersymmetry breaking order parameter* at non-BPS, \( Z \neq 0 \) critical points of \( V_{BH} \) in *symmetric* SK manifolds is
\[ \mathcal{O}_{\text{non-BPS,} Z \neq 0} = 3, \] (2.66)
which might be called the “*Rule of Three*” in \( \mathcal{N} = 2, d = 4 \) supergravity (an analogous “*Rule of Eight*” seemingly exists for symmetric real special geometry in \( d = 5 \) \[94\]). By substituting into Eq. (2.55), one thus finally gets that
\[
\frac{S_{BH, \text{non-BPS,} Z \neq 0}}{\pi} = V_{BH, \text{non-BPS,} Z \neq 0} = 4 |Z|_{\text{non-BPS,} Z \neq 0}^2 = \frac{2}{3} i \left[ N_3 (Z) \right]_{\text{non-BPS,} Z \neq 0},
\] (2.67)
\[
\mathcal{R} (Z)_{\text{non-BPS,} Z \neq 0} = -6 |Z|_{\text{non-BPS,} Z \neq 0}^4 < 0.
\] (2.69)

It is worth pointing out that, while Eq. (2.61) (holding *globally*) is peculiar to *symmetric* SK manifolds, Eqs. (2.64)-(2.69) actually should hold in general also for homogeneous non-*symmetric* SK manifolds, in which the Riemann-Christoffel tensor \( R_{\overline{ijk}\ell} \) (and thus, through the SK constraints, \( C_{ijk} \)) is *not* covariantly constant. Indeed, as obtained in \[30\] at least for all the non-BPS, \( Z \neq 0 \) critical points of \( V_{BH} \) considered therein, in homogeneous non-*symmetric* SK manifolds it holds that
\[
\left[ E_{\overline{(i} \overline{kmn})} g^{\overline{j} \overline{k}} g^{\overline{m} \overline{n}} g^{\overline{n} \overline{m}} \left( D_{\overline{j}} Z \right) \left( D_{\overline{k}} Z \right) \left( D_{\overline{m}} Z \right) \left( D_{\overline{n}} Z \right)_{\text{non-BPS,} Z \neq 0} \right] = 0,
\] (2.70)
which seems to be the most general (necessary and sufficient) condition in order for Eqs. (2.64)-(2.69) to hold.
Moreover, it is worth remarking that in \[10\] the "Rule of Three" (2.60) and thus
\[
V_{BH,\text{non-BPS},Z\neq0} = 4\left|Z_{\text{non-BPS},Z\neq0}\right|^2
\]
was proved to hold for a generic \(d\)-SK geometry \[95\], i.e. for a general SK geometry with a cubic holomorphic prepotential (for instance corresponding to the large volume limit of Type IIA superstrings on Calabi-Yau threefolds), for the non-BPS, \(Z \neq 0\) critical points \(z_{\text{non-BPS},Z\neq0}^i\) of \(V_{BH}\) supported by the BH charge configuration with \(q_i = 0\) \(\forall i\) (the one given by \(D0 - D4 - D6\) brane charges in Calabi-Yau compactifications) and satisfying the Ansatz \[10\]
\[
z_{\text{non-BPS},Z\neq0}^i = p^i\tau, \forall i = 1, ..., n_V,
\]
where \(\tau\) is quantity dependent only from the supporting BH charge configuration.

Non-BPS \(Z \neq 0\) critical points of \(V_{BH}\) in \(M_{n_V}\) are generally not necessarily stable, because the \(2n_V \times 2n_V\) matrix \(H^{BH}\) (within the Kähler holomorphic/antiholomorphic parametrization) evaluated at such points is not necessarily strictly positive-definite. An explicit condition of stability of non-BPS \(Z \neq 0\) critical points of \(V_{BH}\) has been worked out in the \(n_V = 1\) case (see \[17\], \[18\], \[27\]).

In general, the conditions (2.47) imply the gaugino mass term, the \(\lambda\psi\) term and the gravitino "apparent mass" term \(Z\psi\psi\) to be non-vanishing, when evaluated at the considered non-BPS \(Z \neq 0\) critical points of \(V_{BH}\).

3 Charge Orbits and Moduli Spaces of Attractors in \(\mathcal{N} \geq 2, d = 4\) (Symmetric) Supergravity

3.1 \(\mathcal{N} = 2, d = 4\) Symmetric Supergravity

In \[21\] the general solutions to the AEs were obtained and classified by group-theoretical methods for those \(\mathcal{N} = 2, d = 4\) supergravities having an symmetric SK scalar manifold, i.e. such that \(M_{n_V} = G^H\), with a globally covariantly constant Riemann tensor \(R_{ijkl}^H\): \(D_mR_{ijkl}^H = 0\). Such a conditions can be transported on \(C_{ijk}\) by means of the so-called SK geometry constraints (see the third of Eqs. (2.18)), obtaining \(D_lC_{ijk} = D_lC_{ijk} = 0\) (where the last of Eqs. (2.18) was used).

Such \(\mathcal{N} = 2, d = 4\) theories are usually named symmetric supergravities, and they have been classified in literature \[96\], \[97\], \[98\], \[99\], \[100\], \[93\], \[95\].

With the exception of the ones based on \(SU(1,n)_{(1)\otimes SU(n)}\), all symmetric SK geometries are endowed with cubic holomorphic prepotentials. In rank-3 symmetric cubic SK manifolds \(H = H_0 \otimes U(1)\) (which all are the vector supermultiplets’ scalar manifolds of \(\mathcal{N} = 2, d = 4\) supergravities defined by Jordan algebras of degree 3; see e.g. \[21\] and Refs. therein), the solutions to AEs have been shown to exist in three distinct classes, one 1/2-BPS and the other two non-BPS, one of which corresponds to vanishing central charge \(\tilde{Z} = 0\). It is here worth remarking that the non-BPS \(Z = 0\) class of solutions to AEs has no analogue in \(d = 5\), where a similar classification has been given \[94\].

\( ^5\)The quadratic irreducible rank-1 infinite sequence \(SU(1,n)_{(1)\otimes SU(n)}\) has \(C_{ijk} = 0\) globally \((n = n_V \in \mathbb{N})\). As shown in App. I of \[21\], such a family has only two classes of non-degenerate solutions to the AEs: one 1/2-BPS and one non-BPS with \(\tilde{Z} = 0\).
Table 1: Non-degenerate charge orbits of the real, symplectic $R_V$ representation of the $U$-duality group $G$ supporting BH attractors with non-vanishing entropy in $\mathcal{N} = 2$, $d = 4$ symmetric supergravities \[21\]

Furthermore, the three classes of critical points of $V_{BH}$ in $\mathcal{N} = 2$, $d = 4$ symmetric cubic supergravities have been put in one-to-one correspondence with the non-degenerate charge orbits of the actions of the $U$-duality groups $G$ on the corresponding BH charge configuration spaces. In other words, the three species of solutions to AEs in $\mathcal{N} = 2$, $d = 4$ symmetric cubic supergravities are supported by configurations of the BH charges lying along the non-degenerate typologies of charge orbits of the $U$-duality group $G$ in the real (electric-magnetic field strengths) representation space $R_V$, determining its embedding in the symplectic group $Sp(2n_V + 2, \mathbb{R})$. The results on charge orbits obtained in \[21\] are summarized\footnote{The charge orbits for the so-called $st^2$ and $stu$ models ($n = 1$ and $n = 2$ elements of the cubic sequence $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2) \otimes SO(n)}$, respectively) are given in Appendix II of \[21\], where also the charge orbits of the so-called $t^3$ model are treated. It should be here pointed out that the $t^3$ model is an isolated case in the classification of symmetric SK manifolds (see e.g. \[10\]), and it cannot be obtained as the $n = 0$ element of the cubic sequence $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SO(2) \otimes SO(n)}$, which instead is the so-called $t^2$ model, given by the $n = 1$ element of the quadratic sequence $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,n)}{SU(n) \otimes U(1)}$, as well.} in Table 1.

In all the $\mathcal{N} = 2$, $d = 4$ symmetric supergravities based on rank-3 SK cubic manifolds, the classical BH entropy is given by the Bekenstein-Hawking entropy-area formula \([73\);
where $I_4$ is the (unique, quartic\footnote{For the quadratic irreducible rank-1 infinite sequence $SU(1,n) \oplus U(1) \oplus SU(n)$, the unique $G$-invariant is instead \textit{quadratic} in the BH charges; it is positive for $\frac{1}{2}$-BPS orbits and negative for the non-BPS ($Z = 0$) ones (see App. 1 of \cite{21}).} in the BH charges) \textit{moduli-independent} $G$-invariant built out of the (considered non-degenerate charge orbit in the) representation $R_V$. $\frac{1}{2}$-BPS and non-BPS $Z = 0$ classes have $I_4 > 0$, while the non-BPS $Z \neq 0$ class is characterized by $I_4 < 0$.

An interesting direction for further investigations concerns the study of extremal BH attractors in more general, \textit{non-cubic} SK geometries. A noteworthy example is given by the SK geometries of the scalar manifolds of those $\mathcal{N} = 2$, $d = 4$ supergravities obtained as effective, low-energy theories of $d = 10$ Type IIB superstrings compactified on Calabi-Yau threefolds ($CY_3$s), \textit{away from the limit of large volume} of $CY_3$s.

Recently, \cite{27} studied the extremal BH attractors in $n_V = 1$ SK geometries obtained by compactifications (away from the limit of large volume of the internal manifold) on a peculiar class of $CY_3$s, given by the so-called (mirror) \textit{Fermat} $CY_3$s. Such threefolds are classified by the \textit{Fermat parameter} $\ell = 5, 6, 8, 10$, and they were firstly found in \cite{102}. The fourth order linear Picard-Fuchs (PF) ordinary differential Equations determining the holomorphic fundamental period $4 \times 1$ vector for such a class of 1-modulus $CY_3$s were found some time ago for $\ell = 5$ in \cite{103, 104} (see also \cite{105, 106}), and for $\ell = 6, 8, 10$ in \cite{107}.

More specifically, \cite{27} dealt with the so-called \textit{Landau-Ginzburg} (LG) extremal BH attractors, \textit{i.e.} the solutions to the AEs near the origin $z = 0$ (named \textit{LG point}) of the moduli space $M_{n_V=1} (\text{dim}_C M_{n_V=1} = 1)$, and the BH charge configurations supporting $z = 0$ to be a critical point of $V_{BH}$ were explicitly determined, as well. An intriguing development in such a framework would amount to extending to the \textit{Fermat} $CY_3$-compactifications (away from the limit of large volume of the threefold) the conjecture formulated in Sect. 5 of \cite{23}. The conjecture was formulated in the framework of (the large volume limit of $CY_3$-compactifications leading to) the remarkably \textit{triality-symmetric} cubic \textit{stu} model \cite{108, 109, 23}, and it argues that the instability of the considered non-BPS ($Z \neq 0$) critical points of $V_{BH}$ might be only \textit{apparent}, since such attractors might correspond to multi-center stable attractor solutions (see also \textit{e.g.} \cite{110, 52, 54, 59} and Refs. therein), whose stable nature should be “\textit{resolved}” only at sufficiently small distances. The extension of such a tempting conjecture to non-BPS extremal BH LG attractors in \textit{Fermat} $CY_3$-compactifications would be interesting; in particular, the extension to the non-BPS $Z = 0$ case might lead to predict the existence (at least in the considered peculiar $n_V = 1$ framework) of \textit{non-BPS lines of marginal stability} \cite{111, 112} with $Z = 0$.

Moreover, it should be here recalled that the PF Eqs. of \textit{Fermat} $CY_3$s (\cite{103}–\cite{107}, see also \cite{27}) exhibit other two species of \textit{regular singular} points, namely the $k$-th roots of unity ($z^k = 1$, the so-called \textit{conifold points}) and the point at infinity $z \rightarrow \infty$ in the moduli space, corresponding to the so-called \textit{large complex structure modulus limit}. Thus, it would be interesting to solve the AEs in proximity of such regular singular points, \textit{i.e.}
it would be worth investigating extremal BH conifold attractors and extremal BH large complex structure attractors in the moduli space of 1-modulus (Fermat) CY_{3}s. Such an investigation would be of interest, also in view of recent studies of extremal BH attractors in peculiar examples of n_{V} = 2-moduli CY_{3}-compactifications [24].

Let us now consider the crucial issue of stability more in detail.

In N = 2 homogeneous (not necessarily symmetric) and N > 2-extended (all symmetric), d = 4 supergravities the Hessian matrix of V_{BH} at its critical points is in general semi-positive definite, eventually with some vanishing eigenvalues (massless Hessian modes), which actually are flat directions of V_{BH} itself [35, 38]. Thus, it can be stated that for all supergravities based on homogeneous scalar manifolds the critical points of V_{BH} which are non-degenerate (i.e. for which it holds V_{BH} ≠ 0) all are stable, up to some eventual flat directions.

As pointed out above, the Attractor Equations of N = 2, d = 4 supergravity with n_{V} Abelian vector multiplets may have flat directions in the non-BPS cases [35, 38], but not in the 1/2-BPS one [5]. Indeed, in the 1/2-BPS case (satisfying Z ≠ 0, D_{i}Z = 0 ∀i = 1, ..., n_{V}; recall Eq. (2.43)) the covariant 2n_{V} × 2n_{V} Hessian matrix of V_{BH} reads ([5]; recall Eqs. (2.45))

\[
(D_{i}D_{j}V_{BH})_{N=2, 1/2-BPS} = \frac{1}{2} |Z|_{1/2-BPS} \begin{pmatrix}
0 & g_{i\bar{j}} \\
g_{j\bar{i}} & 0
\end{pmatrix}
\]

where hatted indices can be either holomorphic or anti-holomorphic; thus, as far as the metric g_{i\bar{j}} of the scalar manifold is strictly positive definite, Eq. (3.2) yields that no massless 1/2-BPS Hessian modes arise out.

Tables 2 and 3 respectively list the moduli spaces of non-BPS Z ≠ 0 and non-BPS Z = 0 attractors for symmetric N = 2, d = 4 SK geometries, for which a complete classification is available [38] (the attractor moduli spaces should exist also in homogeneous non-symmetric N = 2, d = 4 SK geometries, but their classification is currently unknown). The general thumb rule to construct the moduli space of a given attractor solution in the considered symmetric framework is to coset the stabilizer of the corresponding charge orbit by its maximal compact subgroup. By such a rule, the 1/2-BPS attractors do not have an associated moduli space simply because the stabilizer of their supporting BH charge orbit is compact. On the other hand, all attractors supported by BH charge orbits whose stabilizer is non-compact exhibit a non-vanishing moduli space. furthermore, it should be noticed that the non-BPS Z ≠ 0 moduli spaces are nothing but the symmetric real special scalar manifolds of the corresponding N = 2, d = 5 supergravity.

Nevertheless, it is worth remarking that some symmetric N = 2, d = 4 supergravities have no non-BPS flat directions at all.

The unique n_{V} = 1 symmetric models are the so-called t^{2} and t^{3} models; they are based on the rank-1 scalar manifold \(SU(1,1)/(U(1))\), but with different holomorphic prepotential functions. The t^{2} model is the first element (n = 1) of the sequence of irreducible symmetric special Kähler manifolds \(SU(1,n)/(U(1) \times SU(n))\) (n_{V} = n, \(n \in \mathbb{N}\)) (see e.g. [21] and Refs. therein), endowed with quadratic prepotential. Its bosonic sector is given by the \((U(1))^{6} \to (U(1))^{2}\) truncation of Maxwell-Einstein-axion-dilaton (super)gravity, i.e. of pure \(N = 4, d = 4\) supergravity. On the other hand, the t^{3} model has cubic prepotential;
Table 2: Moduli spaces of non-degenerate non-BPS $Z \neq 0$ critical points of $V_{BH,N=2}$ in $N = 2, d = 4$ symmetric supergravities ($\tilde{h}$ is the maximal compact subgroup of $\tilde{H}$). They are the $N = 2, d = 5$ symmetric real special manifolds.

\[ \begin{array}{|c|c|c|c|} \hline & \frac{\tilde{H}}{h} & r & \text{dim}_{\mathbb{R}} \\ \hline \mathbb{R} \oplus \Gamma_n & SO(1, 1) \otimes \frac{SO(1,n-1)}{SO(n-1)} & 1(n = 1) & n \geq 2 \\ \hline J^0_3 & \frac{E_6(-26)}{F_4(-52)} & 2 & 6 \\ \hline J^\Xi_3 & \frac{SU^*(6)}{USp(6)} & 2 & 14 \\ \hline J^C_3 & \frac{SL(3,C)}{SU(3)} & 2 & 8 \\ \hline J^R_3 & \frac{SL(3,R)}{SO(3)} & 2 & 5 \\ \hline \end{array} \]

as pointed out above, it is an isolated case in the classification of symmetric SK manifolds (see e.g. [101]), but it can be thought also as the $s = t = u$ degeneration of the $stu$ model. It is worth pointing out that the $t^2$ and $t^3$ models are based on the same rank-1 SK manifold, with different constant scalar curvature, which respectively can be computed to be (see e.g. [31] and Refs. therein)

\[ \frac{SU(1,1)}{U(1)}, \text{ } t^2 \text{ model: } R = -2; \]
\[ \frac{SU(1,1)}{U(1)}, \text{ } t^3 \text{ model: } R = -\frac{2}{3}. \] (3.3)

Beside the $\frac{1}{2}$-BPS attractors, the $t^2$ model admits only non-BPS $Z = 0$ critical points of $V_{BH}$ with no flat directions. Analogously, the $t^3$ model admits only non-BPS $Z \neq 0$ critical points of $V_{BH}$ with no flat directions.

For $n_V > 1$, the non-BPS $Z \neq 0$ critical points of $V_{BH}$, if any, all have flat directions, and thus a related moduli space (see Table 1). However, models with no non-BPS $Z = 0$ flat directions at all and $n_V > 1$ exist, namely they are the first and second element ($n = 1, 2$) of the sequence of reducible symmetric special Kähler manifolds $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2) \times SO(n)}$ ($n_V = n+1, n \in \mathbb{N}$) (see e.g. [21] and Refs. therein), i.e. the so-called $st^2$ and $stu$ models, respectively. The $stu$ model (relevant also for the recently established connection between extremal BHs and Quantum Information Theory [113–118]) has two non-BPS $Z \neq 0$
| Quadratic Sequence \((n = n_V \in \mathbb{N})\) | \(\frac{\tilde{H}}{\mathbb{R}^{r} \otimes U(1)}\) | \(r\) | \(\text{dim}_\mathbb{C}\) |
|---|---|---|---|
| \(\mathbb{R} \oplus \Gamma_n\) \((n = n_V - 1 \in \mathbb{N})\) | \(\frac{SU(2,n-2)}{SO(2) \otimes SO(n-2)}, n \geq 3\) | 1 \((n = 3)\) \(2(n \geq 4)\) | \(n - 2\) |
| \(J_3^G\) | \(\frac{\text{E}_{6(-14)}}{SO(10) \otimes U(1)}\) | 2 | 16 |
| \(J_3^H\) | \(\frac{SU(4,2)}{SU(4) \otimes SU(2) \otimes U(1)}\) | 2 | 8 |
| \(J_3^C\) | \(\frac{SU(2,1)}{SU(2) \otimes U(1)} \otimes \frac{SU(1,2)}{SU(2) \otimes U(1)}\) | 2 | 4 |
| \(J_3^R\) | \(\frac{SU(2,1)}{SU(2) \otimes U(1)}\) | 1 | 2 |

Table 3: Moduli spaces of non-degenerate non-BPS \(Z = 0\) critical points of \(V_{BH,N=2}\) in \(N = 2, d = 4\) symmetric supergravities (\(\tilde{h}\) is the maximal compact subgroup of \(\tilde{H}\)). They are (non-special) symmetric Kähler manifolds. Flat directions, spanning the moduli space \(SO(1,1) \times SO(1,1)\) (i.e. the scalar manifold of the \(stu\) model in \(d = 5\)), but no non-BPS \(Z = 0\) massless Hessian modes at all. On the other hand, the \(st^2\) model (which can be thought as the \(t = u\) degeneration of the \(stu\) model) has one non-BPS \(Z \neq 0\) flat direction, spanning the moduli space \(SO(1,1)\) (i.e. the scalar manifold of the \(st^2\) model in \(d = 5\)), but no non-BPS \(Z = 0\) flat direction at all. The \(st^2\) is the “smallest” symmetric model exhibiting a non-BPS \(Z \neq 0\) flat direction.

Concerning the “smallest” symmetric models exhibiting a non-BPS \(Z = 0\) flat direction they are the second \((n = 2)\) element of the sequence \(\frac{SU(1,n)}{U(1) \times SU(n)}\) and the third \((n = 3)\) element of the sequence \(\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n)}{SO(2) \otimes SO(n)}\). In both cases, the unique non-BPS \(Z = 0\) flat direction spans the non-BPS \(Z = 0\) moduli space \(\frac{SU(1,1)}{U(1)} \sim \frac{SO(2,1)}{SO(2)}\) (see Table 2), whose local geometrical properties however differ in the two cases (for the same reasons holding for the \(t^2\) and \(t^3\) models treated above).
| $N$ | $^{1/4}$-BPS orbits $\frac{\mathcal{G}}{\mathcal{H}}$ | non-BPS, $Z_{AB} \neq 0$ orbits $\frac{\mathcal{G}}{\mathcal{H}}$ | non-BPS, $Z_{AB} = 0$ orbits $\frac{\mathcal{G}}{\mathcal{H}}$ |
|-----|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| $N = 3$ | $\frac{SU(3,n)}{SU(2,n)}$ | $-$ | $\frac{SU(3,n)}{SU(3,n-1)}$ |
| $N = 4$ | $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n)}{SO(4,n)}$ | $\frac{SU(1,1)}{SO(1,1)} \otimes \frac{SO(6,n)}{SO(5,n-1)}$ | $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(6,n)}{SO(6,n-2)}$ |
| $N = 5$ | $\frac{SU(1,5)}{SU(3) \otimes SU(2,1)}$ | $-$ | $-$ |
| $N = 6$ | $\frac{SO^*(12)}{SU(4,2)}$ | $\frac{SO^*(12)}{SU^*(6)}$ | $\frac{SO^*(12)}{SU(6)}$ |
| $N = 8$ | $\frac{E_7(7)}{E_{6(2)}}$ | $\frac{E_7(7)}{E_{6(2)}}$ | $-$ |

Table 4: Non-degenerate charge orbits of the real, symplectic $R_V$ representation of the $U$-duality group $\mathcal{G}$ supporting BH attractors with non-vanishing entropy in $N > 2$-extended, $d = 4$ supergravities ($n$ is the number of matter multiplets) [56]

### 3.2 $N > 2$-Extended, $d = 4$ Supergravity

In $N > 2$-extended, $d = 4$ supergravities, whose scalar manifold is always symmetric, there are flat directions of $V_{BH}$ at both its non-degenerate BPS and non-BPS critical points. As mentioned above, from a group-theoretical point of view this is due to the fact that the corresponding supporting BH charge orbits always have a non-compact stabilizer [38, 56]. The BPS flat directions can be interpreted in terms of left-over hypermultiplets’ scalar degrees of freedom in the truncation down to the $N = 2, d = 4$ theories [119, 35]. In Tables 4 and 5 all charge orbits and the corresponding moduli spaces of attractor solution in $N > 2$-extended, $d = 4$ supergravities are reported [56].

We conclude by pointing out that in the present report we dealt with results holding at the classical, Einstein supergravity level. It is conceivable that the flat directions of classical non-degenerate extremal BH attractors will be removed (i.e. lifted) by quantum (perturbative and non-perturbative) corrections (such as the ones coming from higher-order derivative contributions to the gravity and/or gauge sector) to the classical effective BH potential $V_{BH}$. Consequently, at the quantum (perturbative and non-perturbative) level, no moduli spaces for attractor solutions might exist at all (and therefore also the actual attractive nature of the critical points of $V_{BH}$ might be destroyed). However, this might not be the case for $N = 8$.

In presence of quantum lifts of classically flat directions of the Hessian matrix of $V_{BH}$
at its critical points, in order to answer to the key question: “Do extremal BH attractors (in a strict sense) survive the quantum level?”, it is thus crucial to determine whether such lifts originate Hessian modes with positive squared mass (corresponding to attractive directions) or with negative squared mass (i.e. tachyonic, repeller directions).

The fate of the unique non-BPS $Z \neq 0$ flat direction of the $st^2$ model in presence of the most general class of quantum perturbative corrections consistent with the axionic-shift symmetry has been studied in [120], showing that, as intuitively expected, the classical solutions get lifted at the quantum level. Interestingly, in [120] it is found the quantum lift occurs more often towards repeller directions (thus destabilizing the whole critical solution, and destroying the attractor in strict sense), rather than towards attractive directions. The same behavior may be expected for the unique non-BPS $Z = 0$ flat direction of the $n = 2$ element of the quadratic irreducible sequence and the $n = 3$ element of the cubic reducible sequence (see above).

Generalizing to the presence of more than one flat direction, this would mean that only a (very) few classical attractors do remain attractors in strict sense at the quantum level; consequently, at the quantum (perturbative and non-perturbative) level the “landscape” of extremal BH attractors should be strongly constrained and reduced.

Despite the considerable number of papers written on the Attractor Mechanism in the extremal BHs of the supersymmetric theories of gravitation along the last years, still much remains to be discovered along the way leading to a deep understanding of the

| $\mathcal{N}$ | $\frac{1}{N}$-BPS moduli space $\frac{\mathcal{H}}{\mathfrak{h}}$ | non-BPS, $Z_{AB} \neq 0$ moduli space $\frac{\mathcal{H}}{\mathfrak{h}}$ | non-BPS, $Z_{AB} = 0$ moduli space $\frac{\mathcal{H}}{\mathfrak{h}}$ |
|-------------|-------------------------------|----------------|-------------------------------|
| $\mathcal{N} = 3$ | $\frac{SU(2,n)}{SU(2) \otimes SU(n) \otimes U(1)}$ | $-$ | $\frac{SU(3,n-1)}{SU(3) \otimes SU(n-1) \otimes U(1)}$ |
| $\mathcal{N} = 4$ | $\frac{SO(4,n)}{SO(4) \otimes SO(n)}$ | $SO(1,1) \otimes \frac{SO(5,n-1)}{SO(5) \otimes SO(n-1)}$ | $\frac{SO(6,n-2)}{SO(6) \otimes SO(n-2)}$ |
| $\mathcal{N} = 5$ | $\frac{SU(2,1)}{SU(2) \otimes U(1)}$ | $-$ | $-$ |
| $\mathcal{N} = 6$ | $\frac{SU(4,2)}{SU(4) \otimes SU(2) \otimes U(1)}$ | $\frac{SU^*(6)}{USp(6)}$ | $-$ |
| $\mathcal{N} = 8$ | $\frac{E_6(2)}{SU(6) \otimes SU(2)}$ | $\frac{E_6(6)}{USp(8)}$ | $-$ |

Table 5: Moduli spaces of BH attractors with non-vanishing entropy in $\mathcal{N} > 2$-extended, $d = 4$ supergravities ($\mathfrak{h}$, $\mathfrak{h}$ and $\mathfrak{h}$ are maximal compact subgroups of $\mathcal{H}$, $\mathfrak{H}$ and $\mathfrak{H}$, respectively, and $n$ is the number of matter multiplets) [56]
inner dynamics of (eventually extended) space-time singularities in supergravities, and hopefully in their fundamental high-energy counterparts, such as $d = 10$ superstrings and $d = 11$ $M$-theory.

Acknowledgments

The original parts of the contents of this report result from collaborations with L. Andrianopoli, A. Ceresole, R. D’Auria, E. Gimon, M. Trigiante, and especially G. Gibbons, M. Günyaydın, R. Kallosh and A. Strominger, which are gratefully acknowledged.

A. M. would like to thank the Department of Physics, Theory Unit Group at CERN, where part of this work was done, for kind hospitality and stimulating environment.

The work of S.B. has been supported in part by the European Community Human Potential Program under contract MRTN-CT-2004-005104 “Constituents, Fundamental Forces and Symmetries of the Universe”.

The work of S.F. has been supported in part by European Community Human Potential Program under contract MRTN-CT-2004-005104 “Constituents, Fundamental Forces and Symmetries of the Universe”, in association with INFN Frascati National Laboratories and by D.O.E. grant DE-FG03-91ER40662, Task C.

The work of A.M. has been supported by a Junior Grant of the “Enrico Fermi” Center, Rome, in association with INFN Frascati National Laboratories.

References

[1] S. Ferrara, R. Kallosh and A. Strominger, $\mathcal{N} = 2$ Extremal Black Holes, Phys. Rev. D52, 5412 (1995), hep-th/9508072.

[2] A. Strominger, Macroscopic Entropy of $\mathcal{N} = 2$ Extremal Black Holes, Phys. Lett. B383, 39 (1996), hep-th/9602111.

[3] S. Ferrara and R. Kallosh, Supersymmetry and Attractors, Phys. Rev. D54, 1514 (1996), hep-th/9602136.

[4] S. Ferrara and R. Kallosh, Universality of Supersymmetric Attractors, Phys. Rev. D54, 1525 (1996), hep-th/9603090.

[5] S. Ferrara, G. W. Gibbons and R. Kallosh, Black Holes and Critical Points in Moduli Space, Nucl. Phys. B500, 75 (1997), hep-th/9702103.

[6] A. Sen, Black Hole Entropy Function and the Attractor Mechanism in Higher Derivative Gravity, JHEP 09, 038 (2005), hep-th/0506177.

[7] K. Goldstein, N. Iizuka, R. P. Jena and S. P. Trivedi, Non-Supersymmetric Attractors, Phys. Rev. D72, 124021 (2005), hep-th/0507096.

[8] A. Sen, Entropy Function for Heterotic Black Holes, JHEP 03, 008 (2006), hep-th/0508042.

[9] R. Kallosh, New Attractors, JHEP 0512, 022 (2005), hep-th/0510024.
[10] P. K. Tripathy and S. P. Trivedi, *Non-Supersymmetric Attractors in String Theory*, JHEP **0603**, 022 (2006), hep-th/0511117.

[11] A. Giryavets, *New Attractors and Area Codes*, JHEP **0603**, 020 (2006), hep-th/0511215.

[12] K. Goldstein, R. P. Jena, G. Mandal and S. P. Trivedi, *A C-Function for Non-Supersymmetric Attractors*, JHEP **0602**, 053 (2006), hep-th/0512138.

[13] M. Alishahiha and H. Ebrahim, *Non-supersymmetric attractors and entropy function*, JHEP **0603**, 003 (2006), hep-th/0601016.

[14] R. Kallosh, N. Sivanandam and M. Soroush, *The Non-BPS Black Hole Attractor Equation*, JHEP **0603**, 060 (2006), hep-th/0600205.

[15] B. Chandrasekhar, S. Parvizi, A. Tavanfar and H. Yavartanoo, *Non-supersymmetric attractors in R^2 gravities*, JHEP **0608**, 004 (2006), hep-th/0602022.

[16] J. P. Hsu, A. Maloney and A. Tomasello, *Black Hole Attractors and Pure Spinors*, JHEP **0609**, 048 (2006), hep-th/0602142.

[17] S. Bellucci, S. Ferrara and A. Marrani, *On some properties of the Attractor Equations*, Phys. Lett. **B635**, 172 (2006), hep-th/0602161.

[18] S. Bellucci, S. Ferrara and A. Marrani, *Supersymmetric Mechanics. Vol.2: The Attractor Mechanism and Space-Time Singularities* (LNP **701**, Springer-Verlag, Heidelberg, 2006).

[19] S. Ferrara and R. Kallosh, *On N=8 attractors*, Phys. Rev. **D73**, 125005 (2006), hep-th/0603247.

[20] M. Alishahiha and H. Ebrahim, *New attractor, Entropy Function and Black Hole Partition Function*, JHEP **0611**, 017 (2006), hep-th/0605279.

[21] S. Bellucci, S. Ferrara, M. Günaydin and A. Marrani, *Charge Orbits of Symmetric Special Geometries and Attractors*, Int. J. Mod. Phys. **A21**, 5043 (2006), hep-th/0606209.

[22] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen and S. P. Trivedi, *Rotating Attractors*, JHEP **0610**, 058 (2006), hep-th/0606244.

[23] R. Kallosh, N. Sivanandam and M. Soroush, *Exact Attractive non-BPS STU Black Holes*, Phys. Rev. **D74**, 065008 (2006), hep-th/0606263.

[24] P. Kaura and A. Misra, *On the Existence of Non-Supersymmetric Black Hole Attractors for Two-Parameter Calabi-Yau’s and Attractor Equations*, Fortsch. Phys. **54**, 1109 (2006), hep-th/0607132.

[25] G. L. Cardoso, V. Grass, D. Lüst and J. Perz, *Extremal non-BPS Black Holes and Entropy Extremization*, JHEP **0609**, 078 (2006), hep-th/0607202.
[26] J. F. Morales and H. Samtleben, *Entropy function and attractors for AdS black holes*, JHEP 0610, 074 (2006), [hep-th/0608044](http://arxiv.org/abs/hep-th/0608044).

[27] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan, *Mirror Fermat Calabi-Yau Threefolds and Landau-Ginzburg Black Hole Attractors*, Riv. Nuovo Cim. 029, 1 (2006), [hep-th/0608091](http://arxiv.org/abs/hep-th/0608091).

[28] D. Astefanesei, K. Goldstein and S. Mahapatra, *Moduli and (un)attractor black hole thermodynamics*, [hep-th/0611140](http://arxiv.org/abs/hep-th/0611140).

[29] G.L. Cardoso, B. de Wit and S. Mahapatra, *Black hole entropy functions and attractor equations*, JHEP 0703, 085 (2007) [hep-th/0612225](http://arxiv.org/abs/hep-th/0612225).

[30] R. D’Auria, S. Ferrara and M. Trigiante, *Critical points of the Black-Hole potential for homogeneous special geometries*, JHEP 0703, 097 (2007), [hep-th/0701090](http://arxiv.org/abs/hep-th/0701090).

[31] S. Bellucci, S. Ferrara and A. Marrani, *Attractor Horizon Geometries of Extremal Black Holes*, contribution to the Proceedings of the XVII SIGRAV Conference, 4–7 September 2006, Turin, Italy, [hep-th/0702019](http://arxiv.org/abs/hep-th/0702019).

[32] A. Ceresole and G. Dall’Agata, *Flow Equations for Non-BPS Extremal Black Holes*, JHEP 0703, 110 (2007), [hep-th/0702088](http://arxiv.org/abs/hep-th/0702088).

[33] L. Andrianopoli, R. D’Auria, S. Ferrara and M. Trigiante, *Black Hole Attractors in \( \mathcal{N} = 1 \) Supergravity*, JHEP 0707, 019 (2007), [hep-th/0703178](http://arxiv.org/abs/hep-th/0703178).

[34] K. Saraikin and C. Vafa, *Non-supersymmetric Black Holes and Topological Strings*, Class. Quant. Grav. 25, 095007 (2008), [hep-th/0703214](http://arxiv.org/abs/hep-th/0703214).

[35] S. Ferrara and A. Marrani, *\( \mathcal{N} = 8 \) non-BPS Attractors, Fixed Scalars and Magic Supergravities*, Nucl. Phys. B788, 63 (2008), [arXiv:0705.3866](http://arxiv.org/abs/0705.3866).

[36] S. Nampuri, P. K. Tripathy and S. P. Trivedi, *On The Stability of Non-Supersymmetric Attractors in String Theory*, JHEP 0708, 054 (2007), [arXiv:0705.4554](http://arxiv.org/abs/0705.4554).

[37] L. Andrianopoli, R. D’Auria, E. Orazi, M. Trigiante, *First Order Description of Black Holes in Moduli Space*, JHEP 0711, 032 (2007), [arXiv:0706.0712](http://arxiv.org/abs/0706.0712).

[38] S. Ferrara and A. Marrani, *On the Moduli Space of non-BPS Attractors for \( \mathcal{N} = 2 \) Symmetric Manifolds*, Phys. Lett. B652, 111 (2007), [arXiv:0706.1667](http://arxiv.org/abs/0706.1667).

[39] D. Astefanesei and H. Yavartanoo, *Stationary black holes and attractor mechanism*, Nucl. Phys. B794, 13 (2008), [arXiv:0706.1847](http://arxiv.org/abs/0706.1847).

[40] G. L. Cardoso, A. Ceresole, G. Dall’Agata, J. M. Oberreuter, J. Perz, *First-order flow equations for extremal black holes in very special geometry*, JHEP 0710, 063 (2007), [arXiv:0706.3373](http://arxiv.org/abs/0706.3373).
[41] A. Misra and P. Shukla, 'Area codes', large volume (non-)perturbative alpha-prime and instanton: Corrected non-supersymmetric (A)dS minimum, the 'inverse problem' and 'fake superpotentials' for multiple-singular-loci-two-parameter Calabi-Yau’s, arXiv:0707.0105.

[42] A. Ceresole, S. Ferrara and A. Marrani, 4d/5d Correspondence for the Black Hole Potential and its Critical Points, Class. Quant. Grav. 24, 5651 (2007), arXiv:0707.0964.

[43] M. M. Anber and D. Kastor, The Attractor mechanism in Gauss-Bonnet gravity, JHEP 0710, 084 (2007), arXiv:0707.1464.

[44] Y. S. Myung, Y.-W. Kim and Y.-J. Park, New attractor mechanism for spherically symmetric extremal black holes, Phys. Rev. D76, 104045 (2007), arXiv:0707.1933.

[45] S. Bellucci, A. Marrani, E. Orazi and A. Shcherbakov, Attractors with Vanishing Central Charge, Phys. Lett. B655, 185 (2007), ArXiV:0707.2730.

[46] K. Hotta and T. Kubota, Exact Solutions and the Attractor Mechanism in Non-BPS Black Holes, Prog. Theor. Phys. 118N5, 969 (2007), arXiv:0707.4554.

[47] X. Gao, Non-supersymmetric Attractors in Born-Infeld Black Holes with a Cosmological Constant, JHEP 0711, 006 (2007), arXiv:0708.1226.

[48] S. Ferrara and A. Marrani, Black Hole Attractors in Extended Supergravity, contribution to the Proceedings of 13th International Symposium on Particles, Strings and Cosmology (PASCOS 07), London, England, 2-7 Jul 2007, AIP Conf. Proc. 957, 58 (2007), arXiv:0708.1268.

[49] A. Sen, Black Hole Entropy Function, Attractors and Precision Counting of Microstates, arXiv:0708.1270.

[50] A. Belhaj, L.B. Drissi, E.H. Saidi and A. Segui, $\mathcal{N}=2$ Supersymmetric Black Attractors in Six and Seven Dimensions, Nucl. Phys. B796, 521 (2008), arXiv:0709.0398.

[51] L. Andrianopoli, S. Ferrara, A. Marrani and M. Trigiante, Non-BPS Attractors in 5d and 6d Extended Supergravity, Nucl. Phys. B795, 428 (2008), arXiv:0709.3488.

[52] D. Gaiotto, W. Li and M. Padi, Non-Supersymmetric Attractor Flow in Symmetric Spaces, JHEP 0712, 093 (2007), arXiv:0710.1638.

[53] S. Bellucci, S. Ferrara, A. Marrani and A. Shcherbakov, Splitting of Attractors in 1-modulus Quantum Corrected Special Geometry, JHEP 0802, 088 (2008), arXiv:0710.3559.

[54] E. G. Gimon, F. Larsen and J. Simon, Black Holes in Supergravity: the non-BPS Branch, JHEP 0801, 040 (2008), arXiv:0710.4967.
[55] D. Astefanesei, H. Nastase, H. Yavartanoo and S. Yun, *Moduli flow and non-supersymmetric AdS attractors*, JHEP **0804**, 074 (2008), arXiv:0711.0036.

[56] S. Bellucci, S. Ferrara, R. Kallosh and A. Marrani, *Extremal Black Hole and Flux Vacua Attractors*, contribution to the Proceedings of the Winter School on Attractor Mechanism 2006 (SAM2006), 20-24 March 2006, INFN-LNF, Frascati, Italy, arXiv:0711.4547.

[57] R.-G. Cai and D.-W. Pang, *A Note on exact solutions and attractor mechanism for non-BPS black holes*, JHEP **0801**, 046 (2008), arXiv:0712.0217.

[58] M. Huebscher, P. Meessen, T. Ortín and S. Vaulà, *Supersymmetric $N=2$ Einstein-Yang-Mills monopoles and covariant attractors*, arXiv:0712.1530.

[59] W. Li: *Non-Supersymmetric Attractors in Symmetric Coset Spaces*, contribution to the Proceedings of 3rd School on Attractor Mechanism (SAM 2007), Frascati, Italy, 18-22 Jun 2007, arXiv:0801.2536.

[60] S. Bellucci, S. Ferrara, A. Marrani and A. Yeranyan: *d = 4 Black Hole Attractors in $N=2$ Supergravity with Fayet-Iliopoulos Terms*, arXiv:0802.0141.

[61] E. H. Saidi, *BPS and non BPS 7D Black Attractors in M-Theory on K3*, arXiv:0802.0583.

[62] E. H. Saidi, *On Black Hole Effective Potential in 6D/7D $N=2$ Supergravity*, arXiv:0803.0827.

[63] E. H. Saidi and A. Segui, *Entropy of Pairs of Dual Attractors in six and seven Dimensions*, arXiv:0803.2945.

[64] H. Ooguri, A. Strominger and C. Vafa: *Black Hole Attractors and the Topological String*, Phys. Rev. **D70**, 106007 (2004), hep-th/0405146.

[65] H. Ooguri, C. Vafa and E. Verlinde: *Hartle-Hawking wave-function for flux compactifications: the Entropic Principle*, Lett. Math. Phys. **74**, 311 (2005), hep-th/0502211.

[66] M. Aganagic, A. Neitzke and C. Vafa: *BPS microstates and the open topological string wave function*, hep-th/0504054.

[67] S. Gukov, K. Saraïkin and C. Vafa: *The Entropic Principle and Asymptotic Freedom*, Phys. Rev. **D73**, 066010 (2006), hep-th/0509109.

[68] B. Pioline, *Lectures on Black holes, Topological Strings and Quantum Attractors*, Lectures delivered at the RTN Winter School on Strings, Supergravity and Gauge Theories, Geneva, Switzerland, 16-20 Jan 2006, Class. Quant. Grav. **23**, S981 (2006), hep-th/0607227.

[69] A. Ceresole, R. D’Auria, S. Ferrara and A. Van Proeyen, *Duality Transformations in Supersymmetric Yang-Mills Theories Coupled to Supergravity*, Nucl. Phys. **B444**, 92 (1995), hep-th/9502072.
A. Ceresole, R. D’Auria and S. Ferrara, *The Symplectic Structure of $\mathcal{N}=2$ Supergravity and Its Central Extension*, Talk given at ICTP Trieste Conference on Physical and Mathematical Implications of Mirror Symmetry in String Theory, Trieste, Italy, 5-9 June 1995, Nucl. Phys. Proc. Suppl. **46** (1996), hep-th/9509160.

B. de Wit, P.G. Lauwers, A. Van Proeyen, *Lagrangians of $\mathcal{N}=2$ Supergravity - Matter Systems*, Nucl. Phys. **B255**, 569 (1985).

A. Strominger, *Special Geometry*, Commun. Math. Phys. **133**, 163 (1990).

J. D. Bekenstein, Phys. Rev. **D7**, 2333 (1973). S. W. Hawking, Phys. Rev. Lett. **26**, 1344 (1971); in C. DeWitt, B. S. DeWitt, *Black Holes (Les Houches 1972)* (Gordon and Breach, New York, 1973). S. W. Hawking, Nature **248**, 30 (1974). S. W. Hawking, Comm. Math. Phys. **43**, 199 (1975).

G. W. Gibbons and C. M. Hull, *A Bogomol’ny Bound for General Relativity and Solitons in $\mathcal{N}=2$ Supergravity*, Phys. Lett. **B109**, 190 (1982).

S. Ferrara and M. Gümaydin, *Orbits of Exceptional Groups, Duality and BPS States in String Theory*, Int. J. Mod. Phys. **A13**, 2075 (1998), hep-th/9708025.

L. Andrianopoli, R. D’Auria and S. Ferrara and M. Trigiante, *Extremal Black Holes in Supergravity*, in: “String Theory and Fundamental Interactions”, M. Gasperini and J. Maharana eds. (LNP, Springer, Berlin-Heidelberg, 2007), hep-th/0611345.

F. Larsen, *The Attractor Mechanism in Five Dimensions*, contribution to the Proceedings of the Winter School on Attractor Mechanism 2006 (SAM2006), 20–24 March 2006, INFN–LNF, Frascati, Italy, hep-th/0608191.

A. Castro, J. L. Davis, P. Kraus and F. Larsen, *String Theory Effects on Five-Dimensional Black Hole Physics*, arXiv:0801.1863.

B. Craps, F. Roose, W. Troost and A. Van Proeyen, *The Definitions of Special Geometry*, hep-th/9606073.

B. Craps, F. Roose, W. Troost and A. Van Proeyen, *What is Special Kähler Geometry?*, Nucl. Phys. **B503**, 565 (1997), hep-th/9703082.

L. Castellani, R. D’Auria and S. Ferrara, *Special Geometry without Special Coordinates*, Class. Quant. Grav. **7**, 1767 (1990). L. Castellani, R. D’Auria and S. Ferrara, *Special Kähler Geometry: an Intrinsic Formulation from $\mathcal{N}=2$ Space-Time Supersymmetry*, Phys. Lett. **B241**, 57 (1990).

R. D’Auria, S. Ferrara and P. Fré, *Special and Quaternionic Isometries: General Couplings in $\mathcal{N}=2$ Supergravity and the Scalar Potential*, Nucl. Phys. **B359**, 705 (1991). L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara, P. Fré and T. Magri, $\mathcal{N}=2$ Supergravity and $\mathcal{N}=2$ Super Yang-Mills Theory on General Scalar Manifolds: Symplectic Covariance, Gaugings and the Momentum Map, J. Geom. Phys. **23**, 111 (1997), hep-th/9605032. L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara and P. Fré, *General Matter Coupled $\mathcal{N}=2$ Supergravity*, Nucl. Phys. **B476**, 397 (1996), hep-th/9603004.
[83] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces* (Academic Press, New York, 1978).

[84] B. Zumino, *Supersymmetry and Kähler Manifolds*, Phys. Lett. **B87**, 203 (1979).

[85] S. Ferrara, M. Bodner and A. C. Cadavid, *Calabi-Yau Supermoduli Space, Field Strength Duality and Mirror Manifolds*, Phys. Lett. **B247**, 25 (1990).

[86] P. Breitenlohner, D. Maison and G. W. Gibbons, *Four-dimensional Black Holes from Kaluza-Klein Theories*, Commun. Math. Phys. **120**, 295 (1988).

[87] T. Levi-Civita, R.C. Acad. Lincei **26**, 519 (1917).

[88] B. Bertotti, *Uniform Electromagnetic Field in the Theory of General Relativity*, Phys. Rev. **116**, 1331 (1959).

[89] I. Robinson, Bull. Acad. Polon. **7**, 351 (1959).

[90] See e.g. S. Ferrara and L. Maiani, *An Introduction to Supersymmetry Breaking in Extended Supergravity*, based on lectures given at SILARG V, 5th Latin American Symp. on Relativity and Gravitation, Bariloche, Argentina, January 1985, CERN-TH-4232/85. S. Cecotti, L. Girardello and M. Porrati, *Constraints on Partial SuperHiggs*, Nucl. Phys. **B268**, 295 (1986). R. D’Auria and S. Ferrara, *On Fermion Masses, Gradient Flows and Potential in Supersymmetric Theories*, JHEP **0105**, 034 (2001), [hep-th/0103153](https://arxiv.org/abs/hep-th/0103153).

[91] D. Dai-Wai Bao, Shiing-Shen Chern and Z. Shen, “An Introduction to Riemann-Finsler Geometry” (Springer-Verlag, Berlin-Heidelberg, 2000).

[92] B. Chow, P. Lu and L. Ni, “Hamilton’s Ricci Flow” (American Mathematical Society, 2006).

[93] E. Cremmer and A. Van Proeyen, *Classification of Kähler Manifolds in N=2 Vector Multiplet Supergravity Couplings*, Class. Quant. Grav. **2**, 445 (1985).

[94] S. Ferrara and M. Günyaydin, *Orbits and attractors for N=2 Maxwell-Einstein supergravity theories in five dimensions*, Nucl. Phys. **B759**, 1 (2006), [hep-th/0606108](https://arxiv.org/abs/hep-th/0606108).

[95] B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry Structures of Special Geometries*, Nucl. Phys. **B400**, 463 (1993), [hep-th/9210068](https://arxiv.org/abs/hep-th/9210068).

[96] E. Cremmer, C. Kounnas, A. Van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, *Vector Multiplets Coupled To N=2 Supergravity: Superhiggs Effect, Flat Potentials And Geometric Structure*, Nucl. Phys. **B250**, 385 (1985).

[97] M. Günyaydin, G. Sierra and P. K. Townsend, *Exceptional Supergravity Theories and the Magic Square*, Phys. Lett. **B133**, 72 (1983).

[98] M. Günyaydin, G. Sierra and P. K. Townsend, *The Geometry of N=2 Maxwell-Einstein Supergravity and Jordan Algebras*, Nucl. Phys. **B242**, 244 (1984).
[99] M. Günyaydin, G. Sierra and P. K. Townsend, *Gauging the d = 5 Maxwell-Einstein Supergravity Theories: More on Jordan Algebras*, Nucl. Phys. **B253**, 573 (1985).

[100] M. Günyaydin, G. Sierra and P. K. Townsend, *More on d = 5 Maxwell-Einstein Supergravity: Symmetric Space and Kinks*, Class. Quant. Grav. **3**, 763 (1986).

[101] S. Cecotti, S. Ferrara and L. Girardello: *Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories*, Int. J. Mod. Phys. **A4**, 2475 (1989).

[102] A. Strominger and E. Witten, *New Manifolds for Superstring Compactification*, Commun. Math. Phys. **101**, 341 (1985).

[103] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *A Pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory*, Nucl. Phys. **B359**, 21 (1991).

[104] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *An Exactly Soluble Superconformal Theory from a Mirror Pair of Calabi-Yau Manifolds*, Phys. Lett. **B258**, 118 (1991).

[105] A. C. Cadavid and S. Ferrara, *Picard-Fuchs Equations and the Moduli Space of Superconformal Field Theories*, Phys. Lett. **B267**, 193 (1991).

[106] A. Ceresole, R. D’Auria, S. Ferrara, W. Lerche and J. Louis, *Picard-Fuchs Equations and Special Geometry*, Int. J. Mod. Phys. **A8**, 79 (1993), hep-th/9204035. A. Ceresole, R. D’Auria, S. Ferrara, W. Lerche, J. Louis and T. Regge, *Picard-Fuchs Equations, Special Geometry and Target Space Duality*, in: “Mirror Symmetry II”, B. R. Greene and S.-T. Yau eds. (American Mathematical Society - International Press, 1997).

[107] A. Klemm and S. Theisen, *Considerations of One Modulus Calabi-Yau Compactifications: Picard-Fuchs Equations, Kähler Potentials and Mirror Maps*, Nucl. Phys. **B389**, 153 (1993), hep-th/9205041.

[108] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova and W. K. Wong, *STU Black Holes and String Triality*, Phys. Rev. **D54**, 6293 (1996), hep-th/9608059.

[109] M. Shmakova, *Calabi-Yau black holes*, Phys. Rev. **D56**, 540 (1997), hep-th/9612076.

[110] S. Ferrara, E. G. Gimon and R. Kallosh, *Magic supergravities, N= 8 and black hole composites*, Phys. Rev. **D74**, 125018 (2006), hep-th/0606211.

[111] F. Denef, *Supergravity Flows and D-Brane Stability*, JHEP **0008**, 050 (2000), hep-th/0005049.

[112] F. Denef, *On the Correspondence between D-Branes and Stationary Supergravity Solutions of Type II Calabi-Yau Compactifications*, hep-th/0010222.

[113] M. J. Duff, *String Triality, Black Hole Entropy and Cayley’s Hyperdeterminant*, Phys. Rev. **D76**, 025017 (2007), hep-th/0601134.
[114] R. Kallosh and A. Linde, *Strings, Black Holes and Quantum Information*, Phys. Rev. **D73**, 104033 (2006), [hep-th/0602061](http://arxiv.org/abs/hep-th/0602061).

[115] P. Lévay, *Stringy Black Holes and the Geometry of the Entanglement*, Phys. Rev. **D74**, 024030 (2006), [hep-th/0603136](http://arxiv.org/abs/hep-th/0603136).

[116] M.J. Duff and S. Ferrara, *E7 and the tripartite entanglement of seven qubits*, Phys. Rev. **D76**, 025018 (2007), [quant-ph/0609227](http://arxiv.org/abs/quant-ph/0609227).

[117] P. Lévay, *Strings, black holes, the tripartite entanglement of seven qubits and the Fano plane*, Phys. Rev. **D75**, 024024 (2007), [hep-th/0610314](http://arxiv.org/abs/hep-th/0610314).

[118] M.J. Duff and S. Ferrara, *Black hole entropy and quantum information*, [hep-th/0612036](http://arxiv.org/abs/hep-th/0612036).

[119] L. Andrianopoli, R. D’Auria and S. Ferrara: *U invariants, black hole entropy and fixed scalars*, Phys. Lett. **B403**, 12 (1997), [hep-th/9703156](http://arxiv.org/abs/hep-th/9703156).

[120] S. Bellucci, S. Ferrara, A. Marrani and A. Shcherbakov, *Quantum Lift of Non-BPS Flat Directions*, to appear.