The gauge dual of a warped product of AdS$_4$ and a squashed and stretched seven-manifold

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Received 11 November 2009, in final form 12 December 2009
Published 15 January 2010
Online at stacks.iop.org/CQG/27/035009

Abstract
Corrado, Pilch and Warner found in 2001 the second 11-dimensional solution where the deformed geometry of $S^7$ in the lift contains $S^2 \times S^2$. We identify the gauge dual of this background with the theory described by Franco, Klebanov and Rodriguez-Gomez recently. It is the $U(N) \times U(N) \times U(N)$ gauge theory with two $SU(2)$ doublet chiral fields $B_1$ transforming in the $(N, \bar{N}, 1)$, $B_2$ transforming in the $(1, N, \bar{N})$, $C_1$ in the $(1, \bar{N}, N)$ and $C_2$ in the $(\bar{N}, N, 1)$ as well as an adjoint field $\Phi$ in the $(1, \text{adj}, 1)$ representation. By adding the mass term for adjoint field $\Phi$, the detailed correspondence between fields of AdS$_4$ supergravity and composite operators of the IR field theory is determined. Moreover, we compute the spin-2 KK modes around a warped product of AdS$_4$ and a squashed and stretched seven-manifold. This background with global $SU(2) \times SU(2) \times U(1)_R$ symmetry is related to a $U(N) \times U(N) \times U(N) \mathcal{N} = 2$ superconformal Chern–Simons matter theory with eighth-order superpotential and Chern–Simons level $(1, 1, -2)$. The mass-squared in AdS$_4$ depends on $SU(2) \times SU(2) \times U(1)_R$ quantum numbers and the KK excitation number. The dimensions of spin-2 operators are found.

PACS numbers: 11.30.Pb, 11.10.Hi, 11.10.Kk, 04.62.+v, 11.25.Yb, 04.40.Nr

1. Introduction
The $\mathcal{N} = 6$ $U(N) \times U(N)$ Chern–Simons matter theory with level $k$ in three dimensions is described as the low energy limit of $N$ M2-branes at $\mathbb{C}^4/\mathbb{Z}_k$ singularity [1]. For $k = 1, 2$, the full $\mathcal{N} = 8$ supersymmetry is preserved while for $k > 2$, the supersymmetry is broken into $\mathcal{N} = 6$. The matter contents and the superpotential of this theory are the same as those for the D3-branes on the conifold [2]. The RG flow between the UV fixed point and the IR fixed point of the three-dimensional gauge theory can be obtained from gauged $\mathcal{N} = 8$ supergravity in four dimensions via AdS/CFT correspondence [3]. The holographic RG flow equation
connecting \( \mathcal{N} = 8 \) \( SO(8) \) fixed point to \( \mathcal{N} = 2 SU(3) \times U(1) \) fixed point has been studied in [4, 5] where the \( U(1) \) symmetry can be identified with \( U(1)_R \) symmetry of three-dimensional theory coming from the \( \mathcal{N} = 2 \) supersymmetry while those from \( \mathcal{N} = 8 \) \( SO(8) \) fixed point to \( \mathcal{N} = 1 \) \( G_2 \) fixed point has been also studied in [5–7]. The M-theory lifts of these RG flows have been found in [6, 8] by solving the Einstein–Maxwell equations in 11 dimensions with the appropriate field strengths in the internal space.

The mass deformed \( U(2) \times U(2) \) Chern–Simons matter theory with level \( k = 1, 2 \) preserving global \( SU(3) \times U(1)_R \) symmetry has been studied in [9–12] by identifying the turning on the supergravity fields with the mass term in the boundary gauge theory while the mass deformation for this theory preserving \( G_2 \) symmetry has been described in [13]. Due to the \( \mathcal{N} = 1 \) supersymmetry for the latter, there is no \( U(1)_R \) symmetry. The nonsupersymmetric RG flow equations preserving \( SO(7) \) symmetries have been discussed in [14] by looking at the domain wall solutions. The holographic RG flow equations connecting \( \mathcal{N} = 1 \) \( G_2 \) fixed point to \( \mathcal{N} = 2 SU(3) \times U(1)_R \) fixed point have been found in [15] by analyzing the mass terms for the scalar potential at each critical point. Moreover, the \( \mathcal{N} = 4 \) and \( \mathcal{N} = 8 \) RG flows have been studied in [16] by studying the explicit gauged supergravity theory. Recently, further developments on the gauged \( \mathcal{N} = 8 \) supergravity in four dimensions have been done in [17, 18] in the context of bulk and boundary theory. Very recently, by following the prescription of [12], the spin–2 Kaluza–Klein modes around a warped product of AdS4 and a seven-ellipsoid which has global \( G_2 \) symmetry have been discussed in [19] with the computation of the Laplacian eigenvalue problem.

Are there any further examples which have an explicit AdS4/CFT3 correspondence? The seven-sphere \( S^7 \) can be realized by \( S^1 \)-fibration over \( \mathbb{C}P^3 \) [20, 21]. For the standard Fubini–Study metric on the \( \mathbb{C}P^3 \), it contains \( \mathbb{C}P^1 \) [8, 22–24] or \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) [8, 25] inside of \( \mathbb{C}P^3 \). It is natural to generalize the above seven-sphere to the \( U(1) \) bundle over an arbitrary Einstein–Kahler manifold [8]. Although it is not known how to generalize the \( \mathbb{C}P^3 \) to an arbitrary Einstein–Kahler 3-fold, an arbitrary Einstein–Kahler 2-fold can replace the above \( \mathbb{C}P^2 \). The natural choice is given by the above \( \mathbb{C}P^1 \times \mathbb{C}P^1 \). The \( U(1) \) bundle over \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) is known as the \( T^{1,1} \) space. In [8], they have found two different 11-dimensional solutions where the first contains \( \mathbb{C}P^2 \) with \( SU(3) \times U(1)_R \) symmetry and the second has \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) with \( SU(2) \times SU(2) \times U(1)_R \) symmetry. In this paper, we focus on the second solution. For the first solution, there are many relevant works given in [9–12]. The structure of an 11-dimensional metric is fixed by requiring that the maximally supersymmetric \( SO(8) \) vacuum should preserve the AdS4 \( \times S^7 \) solution. This leads to the above \( T^{1,1} \) space for a particular coordinate inside the 11-dimensional metric. Furthermore, they have found that the Ricci tensor for the second solution with frame basis is exactly the same as the one of the first solution by assuming that the supergravity fields satisfy the same equations of motion discovered by [4]. For the 3-form potential the two solutions are different from each other. It is surprising that the same flow equations in four dimensions provide two different 11-dimensional solutions to the equations of the 11-dimensional supergravity theory.

Then it is natural to ask what is the dual gauge theory corresponding to the above second 11-dimensional solution. Recently, Franco et al [26] studied the M2-branes on resolved cones over \( Q^{1,1,1} \), denoted by \( C(Q^{1,1,1}) \), motivated by the observations for D3-branes in [27] in type IIB theory. Since \( C(Q^{1,1,1}) \) can be described by the \( \mathbb{C}^2 \) bundle over \( \mathbb{C}P^1 \times \mathbb{C}P^1 \), blowing-up one \( \mathbb{C}P^1 \) leads to \( C(T^{1,1}) \times \mathbb{C} \) where \( C(T^{1,1}) \) is cone over \( T^{1,1} \) space. The blowing-up one \( \mathbb{C}P^1 \) corresponds to removing a point in the toric diagram. The resolutions correspond to turning on the vevs for the scalar component of the chiral superfield. Among two possible blow-ups, one provides the dual gauge theory that we describe in this paper. Originally, the quiver Chern–Simons gauge theory dual to AdS4 \( \times Q^{1,1,1} \) is characterized by
$U(N)_1 \times U(N)_2 \times U(N)_3 \times U(N)_4$ gauge theory with levels $(1, 1, -1, -1)$ coupled to three kinds of bifundamental chiral superfields with a sextic superpotential [28]. The symmetry here is given by $SU(2) \times U(1) \times U(1)_R$ which is smaller than $SU(2) \times SU(2) \times SU(2) \times U(1)_R$ that is the symmetry of the $Q^{1,1,1}$ space itself. By turning on the vev for one of the chiral superfields and renaming the remaining doublet to the adjoint field, the superpotential reduces to the interaction between this adjoint field and other two kinds of chiral superfields. Then the groups $U(N)_3 \times U(N)_4$ break into the diagonal $U(N)$ subgroup leading to three product gauge groups. The previous Chern–Simons level becomes $(1, 1, -2)$.

The supersymmetric flow solution [4, 5] in four-dimensional $\mathcal{N} = 8$ gauged supergravity interpolates between an exterior AdS$_4$ region with maximal supersymmetry and an interior AdS$_4$ with one-quarter of the maximal supersymmetry. This can be interpreted as the RG flow in $\mathcal{N} = 8$ theory which has $OSp(8|4)$ symmetry broken into an $\mathcal{N} = 2$ theory which has $OSp(2|4)$ symmetry by the addition of a mass term for the adjoint chiral superfield. The role of this massive adjoint superfield is completely different from those in the $SU(3) \times U(1)_R$ symmetric case. A precise correspondence is obtained between fields of bulk supergravity in the AdS$_4$ region and composite operators of the IR field theory in three dimensions. The symmetric case. A precise correspondence is obtained between fields of bulk supergravity in the AdS$_4$ region and composite operators of the IR field theory in three dimensions. The 11-dimensional coordinates with indices $g_{MNPQ}$, the Einstein–Maxwell equations are given by [29]

\[
R^N_M = \frac{1}{2} F^{MNPQRST V} F_{NPQRST V} - \frac{1}{32} \epsilon^N_M F_{PQRS} F^{PQRS},
\]

\[
\nabla_M F^{MNPQ} = - \frac{1}{32} \epsilon^{NPQRST UVWXY} F_{RSTUVWXY},
\]

2. An $\mathcal{N} = 2$ supersymmetric $SU(2) \times SU(2) \times U(1)_R$-invariant flow in an 11-dimensional theory

Let us review the 11-dimensional uplift of the supergravity background with global $SU(2) \times SU(2) \times U(1)_R$ symmetry and recall that the supergravity background with global $SU(3) \times U(1)_R$ was found in [8] as a nontrivial extremum of the gauge $\mathcal{N} = 8$ supergravity in four dimensions. The 11-dimensional coordinates with indices $M, N, \ldots$ are decomposed into four-dimensional spacetime $x^\mu$ and seven-dimensional internal space $y^m$. Denoting the 11-dimensional metric as $g_{MN}$ with the convention $(-, +, \ldots, +)$ and the antisymmetric tensor fields as $F_{MNPQ} = 4 \delta_{[M} A_{N]PQ}$, the Einstein–Maxwell equations are given by [29]
where the covariant derivative $\nabla_M$ on $F^{MNPQ}$ in (2.1) is given by $E^{-1} \partial_M (E^{MN} F^{NPQ})$ together with Elfbein determinant $E \equiv \sqrt{-g_{11}}$. The epsilon tensor $\epsilon_{NPQRSTUVWXY}$ with lower indices is purely numerical. The geometry is a warped product of AdS$_4$ and the squashed and stretched seven-dimensional manifold.

The conifold [30] that is a singular noncompact Calabi–Yau threefold is a surface in $\mathbb{C}^4$ parametrized by four complex coordinates $z_1, z_2, z_3$ and $z_4$ which satisfy the quadratic equation

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0,$$

(2.2)

where these are given by six angular variables:

$$z_1 = \frac{r^2}{\sqrt{2}} \left[ \cos \left( \frac{\theta_1}{2} \right) \cos \left( \frac{\theta_2}{2} \right) e^{i(\psi + \phi + \phi_1 + \phi_2)} - \sin \left( \frac{\theta_1}{2} \right) \sin \left( \frac{\theta_2}{2} \right) e^{i(\psi + \phi - \phi_1 - \phi_2)} \right],$$

$$z_2 = \frac{r^2}{\sqrt{2}} \left[ -\cos \left( \frac{\theta_1}{2} \right) \cos \left( \frac{\theta_2}{2} \right) e^{i(\psi + \phi + \phi_1 + \phi_2)} - \sin \left( \frac{\theta_1}{2} \right) \sin \left( \frac{\theta_2}{2} \right) e^{i(\psi + \phi - \phi_1 - \phi_2)} \right],$$

$$z_3 = \frac{r^2}{\sqrt{2}} \left[ -\cos \left( \frac{\theta_1}{2} \right) \sin \left( \frac{\theta_2}{2} \right) e^{i(\psi + \phi + \phi_1 - \phi_2)} - \sin \left( \frac{\theta_1}{2} \right) \cos \left( \frac{\theta_2}{2} \right) e^{i(\psi + \phi - \phi_1 + \phi_2)} \right],$$

$$z_4 = \frac{r^2}{\sqrt{2}} \left[ -\cos \left( \frac{\theta_1}{2} \right) \sin \left( \frac{\theta_2}{2} \right) e^{i(\psi + \phi + \phi_1 - \phi_2)} + \sin \left( \frac{\theta_1}{2} \right) \cos \left( \frac{\theta_2}{2} \right) e^{i(\psi + \phi - \phi_1 + \phi_2)} \right].$$

(2.3)

Here one $SU(2)$ acts on $\theta_1, \phi_1$ and $\phi$ while the other acts on $\theta_2, \phi_2$ and $\psi$. See also earlier work by [31]. One can easily check that $\sum_{i=1}^4 |z_i|^2 = r^4$. In order to introduce seven-dimensional coordinatization, one multiplies the above four complex coordinates by $\cos \mu$ as follows:

$$z_1 \rightarrow \cos \mu z_1, \quad z_2 \rightarrow \cos \mu z_2, \quad z_3 \rightarrow \cos \mu z_3, \quad z_4 \rightarrow \cos \mu z_4,$$

(2.4)

and moreover, we introduce the other complex coordinate

$$w = \sin \mu r^2 e^{-i\psi},$$

(2.5)

with the property $1 - |w|^2 = r^3 \cos^2 \mu$. Then one obtains $\sum_{i=1}^4 |z_i|^2 + |w|^2 = r^3$, and for $r = 1$, this leads to a seven-sphere of radius 1 in $\mathbb{R}^7$ due to the constraint (2.2).

How do we obtain the second 11-dimensional solution? One needs to find out the right metric structure first. By replacing the $\mathbb{CP}^3$ inside the 11-dimensional metric with $\mathbb{CP}^1 \times \mathbb{CP}^1$, the second 11-dimensional metric in [8] preserving $SU(2) \times SU(2) \times U(1)_R$ where the $N = 2$ supersymmetry is observed via $U(1)_R$ symmetry in three dimensions is, by evaluating the nontrivial warp factor coming from four-dimensional supergravity data, given by

$$ds^2 = \Delta^{-1} (dr^2 + e^{2\Lambda(r)} \eta_{\mu\nu} dx^\mu dx^\nu) + 3 \sqrt{\lambda} ds^2_{AdS_4}(\rho, \chi),$$

(2.6)

where $\eta_{\mu\nu} = (-, +, +)$ and the seven-dimensional metric in terms of supergravity fields $\rho$ and $\chi$ is

$$ds^2_{7}(\rho, \chi) = \frac{X}{\rho^6} d\rho^2 + \rho^2 \cos^2 \mu \left[ \frac{1}{6} (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \frac{1}{6} (d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) \right]$$

$$+ \frac{\rho^{10}}{4X} \sin^2 2\mu \left[ -\frac{d\psi}{\rho^4} + \frac{1}{3} (d\psi + d\phi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2) \right]^2$$

$$+ \frac{\rho^2 \cos^2 \chi}{X} \left[ \sin^2 \mu d\psi + \frac{1}{3} \cos^2 \mu (d\psi + d\phi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2) \right]^2,$$

(2.7)
and we introduce
\[ X \equiv \cos^2 \mu + \rho^8 \sin^2 \mu \] (2.8)
and
\[ \Delta = \frac{\rho^2}{X^{\frac{1}{2}} \cosh^{\frac{1}{2}} \chi} \] (2.9)

The \( A(r) \) in (2.6) is a scale factor. The main results of this background are the fact that they showed the equations of motion for \( \rho \) and \( \chi \) are the same as the one in \( SU(3) \times U(1)_R \) invariant RG flows and the corresponding 3-form has little modification. In particular, for the \( \mu = 0 \), the above (2.7) becomes

\[ \rho^2 dx_{T,1}^2 + \rho^2 \sinh^2 \chi \frac{1}{2} (d\psi + d\phi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2, \] (2.10)

where the five-dimensional metric is a well-known \( T^{1,1} \) manifold:

\[
\begin{align*}
\text{ds}^2_{T,1} &= \frac{1}{2} (d\psi + d\phi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 \\
&\quad + \frac{1}{6} (d\theta_1^2 + \sin^2 \theta_1 d\theta_1^2) + \frac{1}{6} (d\theta_2^2 + \sin^2 \theta_2 d\theta_2^2).
\end{align*}
\] (2.11)

For large \( r(\rho \rightarrow 1 \text{ and } \chi \rightarrow 0) \), the moduli space (2.10) approaches the Ricci flat Kahler conifold. We are focusing on the IR critical values in the background of (2.6):

\[ \rho = 3^\frac{1}{4}, \quad \chi = \frac{1}{2} \cosh^{-1} 2. \] (2.12)

The amounts of squashing and stretching are parametrized by \( \rho \) and \( \chi \) respectively.

Secondly, we need to find out the right 3-form structure. One can start with the 3-form on the seven-dimensional space

\[ C^{(3)} = -\frac{1}{4} \sinh \chi e^{i(\phi-2\psi)} (e^5 + ie^{10}) \land (e^6 + ie^7) \land (e^8 + ie^9), \] (2.13)

in order to describe the 4-form \( F^{(4)} \) that solves (2.1) in the frame basis together with (2.6)\(^1\).

Then we take the 3-form in 11-dimensional geometry as

\[ A^{(3)} = \frac{3 i}{4} e^\frac{x}{2} dx^1 \land dx^2 \land dx^3 + C^{(3)} + (C^{(3)})^*, \] (2.14)

The internal part of \( F^{(4)} \) can be written as \( dc^{(3)} + d(C^{(3)})^* \). The antisymmetric tensor fields can be obtained from \( F^{(4)} \equiv dA^{(3)} \) with (2.14). We present both the 4-form and the Ricci tensor in the frame basis in appendix A.

Let us define the \( R \)-charge to be given by the Killing vector:

\[ R = -i(2\partial_\phi - \partial_\psi) = \frac{1}{2} (z_i \partial_{z_i} - \overline{z}_i \partial_{\overline{z}_i}) + w \partial_w - \overline{w} \partial_{\overline{w}}. \] (2.15)

Then the coordinates \( z_i \)'s have \( R \)-charge \( \frac{1}{2} \) while the \( w \) coordinate has \( R \)-charge 1, from (2.3) and (2.5). Note that the corresponding shifts in \( \psi \) and \( \phi \) under (2.15), that is, \( \phi \rightarrow \phi + 2\gamma \) and \( \psi \rightarrow \psi - \gamma \) preserve the quantity \( \phi + 2\psi \). Of course, this \( U(1) \) should be identified with the \( (U(1))_R \) symmetry of the dual gauge theory that we will discuss in the next sections. Therefore, the ambiguity for the \( U(1)_R \) charge in group theory analysis alone, coming from the different embeddings of \( U(1)_R \) in \( SO(8) \), is resolved by the above \( U(1)_R \) charge assignment (2.15).

\(^1\) Let us emphasize that the form of (2.13) is different from the one in [8] where maybe careless typing was done. (1) The imaginary \( i \) should be in the exponent of exponential function and (2) the coefficient in \( e^{10} \) should be changed in the original paper [8].
3. The $OSp(2|4)$ spectrum and operator map between bulk and boundary theories

What is the dual gauge theory corresponding to the previous 11-dimensional background in the context of AdS/CFT? By giving a vacuum expectation value to one of the internal fields in the quiver $U(N) \times U(N) \times U(N) \times U(N)$ Chern–Simons gauge theory for M2-branes probing the cone over $Q^{1,1,1}$, the theory becomes the quiver diagram for a partial resolution of $Q^{1,1,1}$ theory with the gauge group $U(1) \times U(1)$, and the $SU(2)$ doublets chiral fields are given by

$$
\begin{align*}
B_1 \text{ in } (N, \bar{N}, 1), & \quad B_2 \text{ in } (1, N, \bar{N}), \\
C_1 \text{ in } (1, \bar{N}, N), & \quad C_2 \text{ in } (\bar{N}, N, 1),
\end{align*}
$$

and there exists an adjoint field

$$
\Phi \text{ in } (1, \text{ adj. }, 1).
$$

The superpotential is given by the following interaction between these fields:

$$
W = \Phi (C_2 B_1 B_2 C_1 - B_2 C_1 C_2 B_1).
$$

What are the scale dimensions for these fields? The scale dimension of four chiral superfields are $\frac{3}{4}$ which is equal to $\frac{1}{2} \times \frac{3}{2}$ at the UV (the number $\frac{1}{2}$ is checked in [9]) while one of adjoint superfield becomes $\frac{1}{2}$ in order to have vanishing beta-function. The ratio $\frac{1}{2}$ is equivalent to the ratio of the number of massless fields in $SU(3) \times U(1)_R$ [9, 10] to the number of massless fields in $SU(2) \times SU(2) \times U(1)_R$ here: (3.11). Then the scale dimension of (3.3) is 2. See also [32] where they computed the $U(1)_R$ charges for $C(T^{1,1}) \times \mathbb{C}$ Calabi–Yau 4-fold as $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The Chern–Simons level $\hat{k} = (1, 1, -1, -1)$ of $Q^{1,1,1}$ is transformed as $(1, 1, -2)$. The invariants are given by [26]

$$
\begin{align*}
z_1 &= B_1 C_2, \\
z_2 &= B_2 C_1, \\
z_3 &= B_1 C_1, \\
z_4 &= B_2 C_2, \\
w &= \Phi,
\end{align*}
$$

where $z_i$ $(i = 1, 2, 3, 4)$ parametrize the conifold via (2.3) and the adjoint field parametrizes the complex line $\mathcal{C}$. Due to the complex field $\Phi$ which has a superpotential giving it a mass which drives the flow, we introduce the following mass term:

$$
\Delta W = \frac{1}{2} m \Phi^2,
$$

where the IR value of scaling dimension for $\Phi$ is 1 with scale dimension zero for the mass $m$. By integrating out the massive field in the superpotential $W + \Delta W$ given by (3.3) and (3.5), one obtains the effective new eighth-order superpotential

$$
W_{\text{eff}} = \frac{1}{m} \left( C_2 B_1 B_2 C_1 - B_2 C_1 C_2 B_1 \right)^2.
$$

How do we check the scale dimensions? This implies that the IR value of the scaling dimension for $B_1$ and $C_1$ from (3.6) is $\frac{3}{4}$ which is identical to $\frac{1}{2} \times \frac{3}{2}$. Again, the number $\frac{1}{2}$ is checked in [9] and the origin for the number $\frac{3}{2}$ is explained before: the ratio of the number of massless fields in two different cases. See also [33].

From the branching rule [34] of $SO(8)$ into $SU(2) \times SU(2)$, the spin $2, \frac{3}{2}, 1, \frac{1}{2}, 0$ fields transform as

$$
\begin{align*}
1 &\to (1, 1), \\
8 &\to 4(1, 1) \oplus (2, 2), \\
28 &\to 6(1, 1) \oplus 4(2, 2) \oplus (1, 3) \oplus (3, 1), \\
56 &\to 4(1, 1) \oplus 7(2, 2) \oplus 4(1, 3) \oplus 4(3, 1), \\
70 &\to 11(1, 1) \oplus 8(2, 2) \oplus 3(1, 3) \oplus 3(3, 1) \oplus (3, 3),
\end{align*}
$$

(3.7)
respectively. Fields of different spin but the same $SU(2) \times SU(2)$ representation in the decomposition (3.7) of the $\mathcal{N} = 8$ supermultiplet must recombine into various $\mathcal{N} = 2$ supermultiplets. The correspondence between fields of AdS$_4$ supergravity and composite operators of the IR field theory can be described in this section, along the spirit of [9, 13].

The even subalgebra of the superalgebra $OSp(2|4)$ is a direct sum of subalgebras where $Sp(4, R) \simeq SO(3, 2)$ is the isometry algebra of AdS$_4$ and the compact subalgebra $SO(2)$ generates $U(1)_R$ symmetry [35–37]. The maximally compact subalgebra is then $SO(2)_E \times SO(3)_{\mathcal{L}} \times SO(2)_{\mathcal{Y}}$ where the generator of $SO(2)_E$ is the Hamiltonian of the system and its eigenvalues $E$ are the energy levels of states for the system, the group $SO(3)_{\mathcal{L}}$ is the rotation group and its representation $\rho$ describes the spin states of the system, and the eigenvalue $y$ of the generator of $SO(2)_{\mathcal{Y}}$ is the hypercharge of the state. A supermultiplet, a unitary irreducible representations (UIR) of the superalgebra $OSp(2|4)$, consists of a finite number of UIR of the even subalgebra and a particle state is characterized by a spin $s$, a mass $m$ and a hypercharge $y$. The relations between the mass and energy are given in [38].

Let us classify the supergravity multiplet which is invariant under $SU(2) \times SU(2) \times U(1)_{\mathcal{Y}}$ and describe them in the three-dimensional boundary theory.

- **Long massive vector multiplet.** The conformal dimension $\Delta$ is given by $\Delta = E_0$ and the $U(1)_R$ charge is related to a hypercharge by $R = y$. The $K$ is a general unconstrained scalar superfield in the boundary theory and has a dimension $\frac{1}{2}(5 + \sqrt{17})$ in the IR [9] because we use the same four-dimensional flow equations. Let us describe the Kahler potential in more detail and it is found in [33], by looking at the 11-dimensional flow equation [8], as

$$K = \frac{1}{4} \tau_{M2} e^A \left( \rho^2 + \frac{1}{\rho^2} \right), \quad \frac{dq}{dr} = \frac{2}{L\rho^2} q, \quad (3.8)$$

where $\rho \equiv e^{\tau_0}$ and $\chi \equiv \frac{\dot{\tau}}{\tau^2}$ in previous notations [4]. The corresponding Kahler metric is given by [33]

$$d\bar{s}^2 = \frac{1}{4\rho^2} \left( p \frac{d}{dp} \right)^2 K dp^2 + \left( p \frac{d}{dp} \right) K d\xi_i d\bar{\xi}_i + \left( \rho^2 \frac{d^2}{dp^2} \right) K [\xi_i d\bar{\xi}_i]^2, \quad (3.9)$$

where the coordinate $p$ is defined as

$$p = z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3 + z_4 \bar{z}_4, \quad (3.10)$$

and the four complex coordinates parametrize the $\mathcal{C}^4$ and the $\bar{\xi}_i$’s are coordinates on an $S^7$ of unit radius. The quantity $q$ will be defined later. The $\frac{1}{4} d\xi_i d\bar{\xi}_i - \frac{1}{2} [\xi_i d\bar{\xi}_i]^2$ is a metric on the $T^{1,1}$ (2.11) while $[\xi_i d\bar{\xi}_i]^2$ is the $U(1)$ fiber in the description of $T^{1,1}$. Note that there is a relation given by the first-order differential equation $\frac{dK}{dr} = \tau_{M2} L e^A$ [33]. The moduli space is parametrized by the vacuum expectation values of the four massless scalars $\Phi_1, \Phi_2, \Phi_3$ and $\Phi_4$ denoted as $z_1, z_2, z_3$ and $z_4$ from (3.4)

$$\Phi_1 = z_1, \quad \Phi_2 = z_2, \quad \Phi_3 = z_3, \quad \Phi_4 = z_4. \quad (3.11)$$

The $z_i$ ($i = 1, 2, 3, 4$) transform in the representation $(2, 2)$ of $SU(2) \times SU(2)$ while their complex conjugates $\bar{z}_i$ also transform in the representation $(2, 2)$. At the UV end of the flow which is just AdS$_4 \times S^7$, $A(r) \sim \frac{1}{r}$ from the solution for $A(r)$ and $W = 1$ [4]. Moreover, the radial coordinate on moduli space $\sqrt{q} \sim e^{\frac{\tau}{2}} \sim e^{\frac{\tau}{2}}$ from (3.8) by substituting $\rho = 1$. Therefore, the Kahler potential from (3.8) behaves as $K \sim e^{\frac{\tau}{2}} \sim q \sim \rho^2$. Note that $q$ and $\rho$ has the following relation: $q \equiv \frac{1}{2} \rho^2$ [33]. This implies that $K = \left( \Phi_1 \bar{\Phi}_1 + \Phi_2 \bar{\Phi}_2 + \Phi_3 \bar{\Phi}_3 + \Phi_4 \bar{\Phi}_4 \right)^\frac{1}{2}$ at the UV in the boundary
theory. The scaling dimensions for $\Phi_i (i = 1, 2, 3, 4)$ and its conjugate fields are $\frac{3}{2}$ as explained around (3.3) and (3.4). See also [32] where the $R$-charges in a Calabi–Yau four-fold, the cone over $T^{1,1}$ multiplied by a complex line $C$ (denoted by $C(T^{1,1}) \times C$ as in the introduction), in the context of toric diagram are computed. Note that the scaling dimension of $K$ is equal to 1 which is correct because it should have scaling dimension 1. The observation for this particular number $\frac{3}{2}$ is also noticed in [33] earlier.

At the IR end of the flow, $A(r) \sim \frac{3}{2} \pi r$ with $SO(8)$ coupling $g \equiv \sqrt{\frac{\rho}{2}}$ from the solution for $A(r)$ and $W = \frac{3}{2} \pi r$ [4]. Moreover, $\sqrt{q} \sim e^{\frac{3}{2} \pi r} \sim e^{\frac{3}{2} \rho}$ from (3.8) by substituting $\rho = 3\pi (2.12)$. Therefore, the Kahler potential behaves as $K \sim e^{A(r)} \sim q^{\frac{3}{2}} \sim r$. Then $K$ becomes from (3.10) and (3.11)

$$K = \Phi_1 \overline{\Phi}_1 + \Phi_2 \overline{\Phi}_2 + \Phi_3 \overline{\Phi}_3 + \Phi_4 \overline{\Phi}_4,$$

(3.12)

in the boundary theory. Obviously, from the tensor product between (2, 2) and (2, 2) of $SU(2) \times SU(2)$ representation, one gets a singlet $(1, 1)_0$ with $U(1)_R$ charge 0, as in table 1. Note that $\Phi_i (i = 1, 2, 3, 4)$ has $U(1)_R$ charge $\frac{1}{2}$ (explained around (3.6)) [33] while $\overline{\Phi}_i (i = 1, 2, 3, 4)$ has $U(1)_R$ charge $-\frac{1}{2}$. As observed in [33], the number $\frac{1}{2}$ is equal to simply $\frac{1}{2}$ times the dimension of fields in $SU(3) \times U(1)_R$ flow. That is, $\frac{3}{2} \times \frac{1}{2}$ which should be multiplied by 2 because $\Phi_i$ is quadratic in the $SU(2)$ doublet chiral fields: (3.4) and (3.11). Since the scaling dimensions for $\Phi_i (i = 1, 2, 3, 4)$ and its conjugate fields are $\frac{1}{2}$, the scaling dimension of $K$ is 1 which is consistent with the classical value as before. The corresponding Kahler metric (3.9) provides the Kahler term in the action. For the superfield $K$ (3.12), the action looks like $\int d^3 x d^2 \theta^* d^2 \bar{\theta}^{-} K$ as in [9]. The component content of this action can be worked out straightforwardly using the projection technique. This implies that the highest component field in $\theta^2$-expansion, the last element in table 1, has a conformal dimension $\frac{1}{2}(5 + \sqrt{17})$ in the IR as before.

The corresponding $OSp(2\mid 4)$ representations and corresponding $\mathcal{N} = 2$ superfield in three dimensions are listed in table 1. The relation between $\Delta$ and the mass for various fields can be found in [38]. For spins 0 and 1, their relations are given by $\Delta_{\pm} = \frac{3\sqrt{1+2m^2}}{2}$ where we have to choose the correct root among two cases as in [39] while for spin $\frac{1}{2}$, the explicit form is given by $\Delta = \frac{3\sqrt{1+4m^2}}{4}$. Using these relations, one can read off the mass for each state.

- **Short massive gravitino multiplet.** The conformal dimension $\Delta$ is the twice of $U(1)_R$ charge plus $\frac{3}{2}$ for the lowest component, $\Delta = E_0 = 2|R| + \frac{3}{2}$. This corresponds to spinorial superfield $\Phi_\alpha$ that satisfies $D^\mu \Phi_\alpha = 0$ [40]. Of course, this constraint makes
the multiplet short. In the $\theta^\pm$ expansion, the component fields in the bulk are located with appropriate quantum numbers. The massless chiral superfields $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ have $\Delta = \frac{1}{2}$ and $U(1)_R$ charge $\frac{1}{2}$ as before (around (3.6)). The gauge superfield $W_\alpha$ has $\Delta = 2y + 1$ and $U(1)_R$ charge $y - \frac{1}{2}$ and its conjugate field has an opposite $U(1)_R$ charge $-y + \frac{1}{2}$. Note that the $U(1)_R$ charge for the gravitino in the $SU(3) \times U(1)_R$ was given by $y = \frac{1}{8}$ [41]. Then one can identify $\Tr W_\alpha \Phi_j$ with $({\bf 2}, \bf{2})$ and $\Tr \overline{W}_\alpha \Phi_j$ with $({\bf 2}, \bf{2})$. The corresponding $OSp(2|4)$ representations and the corresponding superfield are listed in table 2. For spin $\frac{1}{2}$, the relation for the mass and dimension is given by $\Delta = \frac{6y + 1}{4}$ and for spins 0, 1 and $\frac{1}{2}$, the previous relations hold.

- $N = 2$ massless graviton multiplet. This can be identified with the stress energy tensor superfield $T_{\alpha\beta}$ that satisfies the equations $D^\alpha T_{\alpha\beta} = 0$ [42, 43]. In components, the $\theta^\pm$ expansion of this superfield has the stress energy tensor, the $N = 2$ supercurrents and $U(1)_R$ symmetry current, as usual. The conformal dimension $\Delta = 2$ and the $U(1)_R$ charge is 0. The corresponding $OSp(2|4)$ representations and the corresponding superfield are listed in table 3. For spin 2, we have the relation $\Delta_+ = \frac{3y + \sqrt{9y^2 + 4}}{2}$ and for massless case; this leads to $\Delta_+ = 3$. This massless state can be seen from table 4.

- $N = 2$ massless vector multiplet. This conserved vector current is given by a scalar superfield $\mathcal{F}$ satisfying $D^\alpha D^\beta \mathcal{F} = 0$ [42]. This transforms in the adjoint representation of the $SU(2) \times SU(2)$ flavor group. The corresponding boundary object is given by $\Tr \overline{\Phi}_j T^A \Phi_i$ where the flavor indices in $\Phi_i$ and $\overline{\Phi}_j$ are contracted and the generator $T^A$ is the $N \times N$ matrix with $A = 1, 2, \ldots, N^2$. The conformal dimension $\Delta = 1$ and the $U(1)_R$ charge is 0. By taking a tensor product between $({\bf 2}, \bf{2})$ and $({\bf 2}, \bf{2})$, one gets $({\bf 1}, \bf{3}) \oplus ({\bf 3}, \bf{1})$

### Table 2. The $OSp(2|4)$ representations (energy, spin, hypercharge) and $SU(2) \times SU(2) \times U(1)_R$ representations in the supergravity mass spectrum for short massive gravitino multiplet at the $\mathcal{N} = 2$ critical point and the corresponding $\mathcal{N} = 2$ superfield in the boundary gauge theory where $E_0 = |y| + \frac{1}{2} = 2|\mathcal{R}| + \frac{1}{2}$.

| B.O. | Energy | Spin 0 | Spin $\frac{1}{2}$ | Spin 1 | Spin $\frac{3}{2}$ |
|------|--------|--------|-------------------|--------|-------------------|
| $\Tr W_\alpha \Phi_j$ complex | $E_0$ | $({\bf 2}, \bf{2})$ | $E_0 + \frac{1}{2}$ | $(\bf{2}, \bf{2}),_{\frac{1}{2}}$ | $(\bf{2}, \bf{2}),_{\frac{3}{2}}$ |
| | $E_0 + 1$ | $(\bf{2}, \bf{2}),_{\frac{1}{2}}$ | | $(\bf{2}, \bf{2}),_{\frac{3}{2}}$ | |
| | $E_0 + \frac{3}{2}$ | | | | $(\bf{2}, \bf{2}),_{\frac{3}{2}}$ |

### Table 3. The $OSp(2|4)$ representations (energy, spin, hypercharge) and $SU(2) \times SU(2) \times U(1)_R$ representations in the supergravity mass spectrum for ‘ultra’ short multiplets at the $\mathcal{N} = 2$ critical point and the corresponding $\mathcal{N} = 2$ superfields in the boundary gauge theory.

| B.O. | Energy | Spin 0 | Spin $\frac{1}{2}$ | Spin 1 | Spin $\frac{3}{2}$ |
|------|--------|--------|-------------------|--------|-------------------|
| $\Tr \overline{\Phi}_j T^A \Phi_i$ | $E_0 = 1$ | $({\bf 1}, \bf{3})_{\bf{0}} \oplus ({\bf 3}, \bf{1})_{\bf{0}}$ | $E_0 + \frac{1}{2}$ | $(\bf{1}, \bf{3}),_{\frac{1}{2}} \oplus (\bf{3}, \bf{1}),_{\frac{3}{2}}$ | |
| | $E_0 + 1$ | $(\bf{1}, \bf{3})_{\bf{0}} \oplus (\bf{3}, \bf{1})_{\bf{0}}$ | | $(\bf{1}, \bf{3})_{\bf{0}} \oplus (\bf{3}, \bf{1})_{\bf{0}}$ | |
| $T_{\alpha\beta}$ | $E_0 = 2$ | | | $(\bf{1}, \bf{1})_{\bf{0}}$ | |
| | $E_0 + \frac{1}{2}$ | | | | $(\bf{1}, \bf{1})_{\bf{0}}$ | |
| | $E_0 + 1$ | | | | | $(\bf{1}, \bf{1})_{\bf{0}}$ |
Table 4. The first few spin-2 components of the massive (and massless) graviton multiplets. For each multiplet we present \( SO(8) \) representation \((4.9), SU(2) \times SU(2) \times U(1)_{R} \) representations denoted by \((2h + 1, 2l + 1)_{R} \), \(R_{\phi}-\)charge, \(R_{\phi}-\)charge, the KK excitation number \(j\), the mass-squared \(m^{2}L^{2}\) \((4.18)\) of the AdS field and the corresponding \(N = 2\) dual SCFT operator. The dimension \(\Delta\) \((4.20)\) of the spin-2 component of the multiplet, which we do not present here due to the space, can also be obtained.

| \(SO(8)\) | \(SU(2)_{R}^{3}\) | \(R_{\phi}\) | \(R_{\phi}\) | \(j\) | \(m^{2}L^{2}\) | \(N = 2\) SCFT operator |
|---|---|---|---|---|---|---|
| \(1\) | \((1, 1)_{0}\) | 0 | 0 | 0 | 0 | \(T_{\alpha\beta}\) |
| \(8_{c}\) | \((2, 1)_{1/2}\) | 1 | 1 | 0 | \(1/2(\sqrt{3}T - 3)\) | \(T_{\alpha\beta}B_{i}(1 - \Phi\Phi)^{\pm\pm\mp1}\) |
| & \((2, 1)_{-1/2}\) | -1 | -1 | 0 | \(1/2(\sqrt{3}T - 3)\) | \(T_{\alpha\beta}\overline{B}_{i}(1 - \Phi\Phi)^{\pm\pm\mp1}\) |
| & \((1, 2)_{1/2}\) | 1 | 1 | 0 | \(1/2(\sqrt{3}T - 3)\) | \(T_{\alpha\beta}C_{i}(1 - \Phi\Phi)^{\pm\pm\mp1}\) |
| & \((1, 2)_{-1/2}\) | -1 | -1 | 0 | \(1/2(\sqrt{3}T - 3)\) | \(T_{\alpha\beta}\overline{C}_{i}(1 - \Phi\Phi)^{\pm\pm\mp1}\) |
| \(35_{v}\) | \((1, 1)_{0}\) | 0 | 0 | 1 | 8 | \(T_{\alpha\beta}(-1 + 4\Phi\Phi)\) |
| & \((3, 1)_{1/2}\) | 2 | 2 | 0 | \(\sqrt{13} - 1\) | \(T_{\alpha\beta}(B_{i}B_{j} - 1/3\delta_{i\jmath}B_{i}B_{j})(1 - \Phi\Phi)^{\pm\pm\mp1}\) |
| & \((3, 1)_{-1/2}\) | 0 | 0 | 0 | 4 | \(T_{\alpha\beta}(B_{i}\overline{B}_{j} - 1/3\delta_{i\jmath}B_{i}\overline{B}_{j})(1 - \Phi\Phi)^{\pm\pm\mp1}\) |
| & \((3, 1)_{1/2}\) | 2 | 2 | 0 | \(\sqrt{13} - 1\) | \(T_{\alpha\beta}B_{i}C_{j}(1 - \Phi\Phi)^{\pm\pm\mp1}\) |
| & \((2, 2)_{1/2}\) | 0 | 0 | 0 | \(\sqrt{13} - 1/2\) | \(T_{\alpha\beta}B_{i}\overline{C}_{j}(1 - \Phi\Phi)^{\pm\pm\mp1}\) |
| & \((2, 2)_{-1/2}\) | 0 | 0 | 0 | \(\sqrt{13} - 1/2\) | \(T_{\alpha\beta}\overline{B}_{i}C_{j}(1 - \Phi\Phi)^{\pm\pm\mp1}\) |
| & \((2, 2)_{1/2}\) | 2 | 2 | 0 | \(\sqrt{13} - 1\) | \(T_{\alpha\beta}B_{i}\overline{C}_{j}(1 - \Phi\Phi)^{\pm\pm\mp1}\) |
| & \((1, 3)_{1/2}\) | 0 | 0 | 0 | \(\sqrt{13} - 1\) | \(T_{\alpha\beta}(C_{i}C_{j} - 1/4\delta_{i\jmath}C_{i}C_{j})(1 - \Phi\Phi)^{\pm\pm\mp1}\) |
| & \((1, 3)_{-1/2}\) | 2 | 2 | 0 | \(\sqrt{13} - 1\) | \(T_{\alpha\beta}(C_{i}\overline{C}_{j} - 1/4\delta_{i\jmath}C_{i}\overline{C}_{j})(1 - \Phi\Phi)^{\pm\pm\mp1}\) |
| & \((1, 3)_{1/2}\) | 2 | 2 | 0 | \(\sqrt{13} - 1\) | \(T_{\alpha\beta}(C_{i}\overline{C}_{j} - 1/4\delta_{i\jmath}C_{i}\overline{C}_{j})(1 - \Phi\Phi)^{\pm\pm\mp1}\) |

of the \(SU(2) \times SU(2)\) representation. The corresponding \(Osp(2|4)\) representations and corresponding superfield are listed in table 3.

We have presented the gauge invariant combinations of the massless superfields of the gauge theory whose scaling dimensions and \(SU(2) \times SU(2) \times U(1)_{R}\) quantum numbers exactly match the two short multiplets in tables 2 and 3 observed in the supergravity.

4. KK spectrum of minimally coupled scalar

Let us describe the KK modes by solving the Laplace equation in seven-dimensional internal space. A minimally coupled scalar field is interacting with the gravitational field. The action in the background which is a warped product of AdS4 and a squashed and stretched seven-dimensional manifold is given by

\[
S = \int d^{11}x \sqrt{-g} \left[ -\frac{1}{2} (\partial \phi)^{2} \right].
\]

The equation of motion from this action \((4.1)\) is

\[
\Box \phi = 0.
\]

C. Ahn and K. Woo

Class. Quantum Grav. 27 (2010) 035009
Here $\Box$ is the 11-dimensional Laplacian. By exploiting the separation of variables
\[
\phi = \hat{\Phi}(x^\mu, r) Y(y^m),
\] (4.3)
and substituting (4.3) into (4.2), one can write down (4.2) as
\[
Y(y^m) \Box \hat{\Phi}(x^\mu, r) + \hat{\Phi}(x^\mu, r) \mathcal{L} Y(y^m) = 0,
\] (4.4)
where $\Box_4$ stands for the AdS$_4$ Laplacian and $\mathcal{L}$ stands for a differential operator acting on the seven-dimensional manifold and is
\[
\mathcal{L} \equiv \frac{\Delta^{-1}}{\sqrt{-g}} \partial_M (\sqrt{-g} g^{MN} \partial_N) = \frac{\Delta^{-1}}{\sqrt{g}} \partial_m (3^{-\frac{3}{2}} L^{-2} \Delta^{-\frac{3}{2}} \sqrt{g} g^{mn} \partial_n),
\] (4.5)
where $g^{mn}$ are described by the metrics (2.7) and (2.6), respectively. The seven-dimensional metric $g^{mn}_{7}$ is given explicitly by
\[
\begin{pmatrix}
\frac{c_\mu^2}{237} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2c_\mu^2}{1637} & 0 & c_0^2 c_\mu^2 (-3c_\mu^2) & c_0^2 (3c_\mu^2) & c_0^2 c_\mu^2 & c_0^2 c_\mu^2 \\
0 & 0 & \frac{4c_\mu^2}{1637} & c_\mu^2 (-3c_\mu^2) & 4c_\mu^2 (-2c_\mu^2) & 4c_\mu^2 (-3c_\mu^2) & 2c_\mu^2 \\
0 & 0 & 0 & \frac{c_0^2 c_\mu^2 (-3c_\mu^2)}{237} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{4c_\mu^2}{237} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{c_0^2 (3c_\mu^2)}{237} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{c_0^2 c_\mu^2}{237}
\end{pmatrix}
\] (4.6)
where we use the simplified notation $c_\mu^2 \equiv \cos 2\mu$ and so on and let us introduce the angular coordinates $y^m \equiv (\mu, \theta_1, \phi_1, \theta_2, \phi_2, \psi, \phi)$.

Let us see what is the eigenfunction $Y(y^m)$ of the differential operator $\mathcal{L}$
\[
\mathcal{L} Y(y^m) = -m^2 Y(y^m).
\] (4.7)
Then equation (4.4) implies the equation of motion of a massive scalar field in AdS$_4$:
\[
\Box_4 \hat{\Phi}(x^\mu, r) - m^2 \hat{\Phi}(x^\mu, r) = 0.
\] (4.8)
Therefore, one obtains a tower of KK modes which are all massive scalars (4.8) with masses $m^2$ determined by the eigenvalues of the above differential operator $\mathcal{L}$.

The spin-2 massive $\mathcal{N} = 8$ supermultiplet [44] at level $n$ is described by the $SO(8)$ Dynkin labels $(n, 0, 0, 0)$, this breaks into the $SO(7)$ Dynkin labels $(0, 0, n)$, and finally the massive multiplets of $\mathcal{N} = 8$ for $n = 1, 2, \ldots$ are decomposed into the various representations under the $SU(2) \times SU(2)$ symmetry. In particular, one has, with the help of [34, 45],
\[
SO(8) \rightarrow SO(7) \rightarrow SU(4) \rightarrow SU(2)^2,
\]
8$_s$(1, 0, 0, 0) $\rightarrow$ 8(0, 0, 1) $\rightarrow$ 4 $\oplus$ $\overline{4}$ $\rightarrow$ 2(2, 1) $\oplus$ 2(1, 2),
35$_s$(2, 0, 0, 0) $\rightarrow$ 35(0, 0, 2) $\rightarrow$ 10 $\oplus$ 16 $\oplus$ 15 $\rightarrow$ (1, 1) $\oplus$ 4(2, 2) $\oplus$ 3(3, 1) $\oplus$ 3(1, 3),
112$_s$(3, 0, 0, 0) $\rightarrow$ 112(0, 0, 3) $\rightarrow$ 20$^\prime$ $\oplus$ 20$''$ $\oplus$ 36 $\oplus$ 36
$\rightarrow$ 2(2, 1) $\oplus$ 2(1, 2) $\oplus$ 4(4, 1) $\oplus$ 6(3, 2) $\oplus$ 6(2, 3) $\oplus$ 4(1, 4). (4.9)
The differential operator acting on a seven-dimensional manifold is given by (4.5) and this can be rewritten, from the metric (4.6) and the warp factor (2.9) with (2.12) and (2.8), in terms of angular coordinates as follows:

\[
\mathcal{L} = -\frac{1}{6L^2} (-2 + \cos 2\mu) \sec^2 \mu \mathcal{C}_2 + \frac{1}{2L^2} \left[ \frac{\partial^2}{\partial u^2} + (\cot \mu - 5 \tan \mu) \frac{\partial}{\partial u} \right. \\
\left. + (1 + \csc^2 \mu) \frac{\partial^2}{\partial \phi^2} + (2 + \csc^2 \mu) \frac{\partial}{\partial \phi} - 2(2 + \csc^2 \mu) \frac{\partial}{\partial \phi} \right],
\]

where the quadratic differential operator can be written as

\[
\mathcal{C}_2 \equiv 6 \sum_{i=1}^{2} \left[ \frac{1}{\sin \theta_i} \sin \theta_i \frac{\partial}{\partial \theta_i} + \left( \frac{1}{\sin \theta_i} \cot \theta_i - \cot \theta_i \frac{\partial}{\partial \phi} \right)^2 \right] + 9 \phi^2.
\]

What are the corresponding eigenfunctions? It is known that the scalar spherical harmonic for D-sphere \( S^D \) is described by each independent component of a totally symmetric traceless tensor of rank \( n \). Also one can write down the eigenfunctions via the separation of variables as follows:

\[
Y(y^m) = J_l(\theta_1) e^{im\phi_1} J_m(\theta_2) e^{im\phi_2} e^{\pm \phi} e^{\pm R_\phi} H(\cos^2 \mu).
\]

Then the solution for \( J_l(\theta_1) \) is a linear combination of the following two independent hypergeometric functions [46]:

\[
j_1(\theta_1) = \sin^m \theta_1 \cot \frac{\theta_1}{2} \left[ \frac{\theta_1}{2} \right] F_1 \left( -l_1 + m_1, 1 + l_1 + m_1; 1 + m_1 - \frac{R_\phi}{2}; \sin^2 \frac{\theta_1}{2} \right),
\]

\[
j_2(\theta_1) = \sin \frac{\theta_1}{2} \cot m_1 \left[ \frac{\theta_1}{2} \right] F_1 \left( -l_1 + R_\phi, 1 + l_1 + \frac{R_\phi}{2}; 1 - m_1 - \frac{R_\phi}{2}; \sin^2 \frac{\theta_1}{2} \right). \tag{4.13}
\]

When \( m_1 \leq \frac{R_\phi}{2} \), the function \( j_2(\theta_1) \) is non-singular while when \( m_1 \geq \frac{R_\phi}{2} \), the function \( j_1(\theta_1) \) is non-singular. Similarly, the solution for \( J_m(\theta_2) \) is also a linear combination of these hypergeometric functions by replacing \( \theta_1, m_1, l_1, R_\phi \) with \( \theta_2, m_2, l_2, R_\phi \), respectively. The relevant eigenvalues for the above quadratic differential operator (4.11) can be computed explicitly as follows [47, 48]:

\[
C_2 J_l(\theta_1) e^{im\phi_1} J_m(\theta_2) e^{im\phi_2} e^{\pm R_\phi} = \left[ -6l_1(l_1 + 1) - 6l_2(l_2 + 1) + \frac{3}{2} R_\phi^2 \right] J_l(\theta_1) e^{im\phi_1} J_m(\theta_2) e^{im\phi_2} e^{\pm R_\phi}. \tag{4.14}
\]

It turns out that the eigenvalue problem (4.7) leads to the following nontrivial differential equation:

\[
(1 - u)u H''(u) + (3 - 4u)H'(u) + \left( A + \frac{B}{u - 1} + \frac{C}{u} \right) H(u) = 0, \quad u \equiv \cos^2 \mu. \tag{4.15}
\]

where we introduce the following quantities:

\[
A \equiv l_1(l_1 + 1) + l_2(l_2 + 1) - \frac{1}{4} \left( R_\phi - R_\phi \right)^2 + \frac{1}{2} m^2 L^2,
\]

\[
B \equiv \frac{1}{16} (R_\phi - R_\phi)^2, \quad C \equiv -\frac{3}{2} \left[ l_1(l_1 + l_1) + l_2(l_2 + 1) \right] + \frac{3}{16} R_\phi^2.
\]

Note that the \( C_2 \) term in (4.10) occurs only in \( H(u) \) in (4.15) while the other terms in (4.10) occur in \( H''(u), H'(u) \) or \( H(u) \) of (4.15). In general, the solutions for (4.15) can be written in terms of two independent hypergeometric functions but they are rather complicated due to the fact that the linear terms in \( H(u) \) of (4.15) depend on the variable \( u \), compared with
By introducing the $R$-charge which is a linear combination between $R_{\phi}$ and $R_{\psi}$ with appropriate normalization from (2.15),

\begin{equation}
R = \frac{1}{2} \left( R_{\phi} - \frac{R_{\psi}}{2} \right),
\end{equation}

and requiring that $R_{\phi} = R_{\psi}$ which leads to the vanishing of $B$ in (4.16), one obtains the solutions in a very simple form. The relative sign difference in (4.17) is due to the fact that the combination $\phi + 2\psi$ should be invariant quantity under the two $U(1)$ symmetries, stressed in section 2. Then the KK spectrum of minimally coupled scalar can be obtained by putting the first argument of the hypergeometric function to be negative integer or zero $-j$ and solving for $m^2$. Then the mass-squared in AdS$_4$ can be written, in terms of $l_1, l_2, R$ and $j$, as

\begin{equation}
m^2 = \frac{1}{L^2} \left[ 2j^2 + 2j(1 + \sqrt{4 - C_2}) - \frac{1}{6} C_2 + \sqrt{4 - C_2} - 2R^2 - 2 \right].
\end{equation}

Here $C_2$ is an eigenvalue appearing in (4.14) $C_2 = -6l_1(l_1 + 1) - 6l_2(l_2 + 1) + 12R^2$. Then the regular solution for (4.15) is given by

\begin{equation}
H(u) = u^{\sqrt{4 - C_2} - 1} \frac{i}{2} F_1(-j, j + 1 + \sqrt{4 - C_2}; 1 + \sqrt{4 - C_2}; u), \quad u \equiv \cos^2 \mu.
\end{equation}

Substituting this into (4.12), one gets the eigenfunctions of (4.7). For $j = 0$, the hypergeometric function becomes $u$-independent constant and the contribution from $H(u)$ arises in $u^{\sqrt{4 - C_2} - 1}$. For $j = 1$, the hypergeometric function leads to $1 - \frac{2\sqrt{4 - C_2}}{4 + \sqrt{4 - C_2}} u$. For general nonzero $j$, the hypergeometric function is a polynomial of order $j$ in $u$.

The dimension of the CFT operators dual to the KK modes can be obtained from the AdS/CFT correspondence [3]:

\begin{equation}
\Delta(\Delta - 3) = m^2 L^2.
\end{equation}

The $OSp(2|4)$ supermultiplets with spin-2 components are massless graviton multiplet with $SD(2, 1, 0|2)$ denoted by [35], short graviton multiplet with $SD(y_0 + 2, 1, y_0|2)$ where $y_0 > 0$ and long graviton multiplet with $SD(E_0, 1, y_0|2)$ where $E_0 > y_0 + 2$ and $y_0 \geq 0$. As recognized in [9], the massless graviton multiplet has conformal dimension $\Delta = 3$ (the ground state component has dimension $\Delta_0 = 2$ and see table 3 of the previous section or table 5 of [9]). This $N = 2$ massless graviton multiplet decomposes into $SD(\frac{1}{2}, \frac{1}{2}|1) \oplus SD(2, 1|1)$ of $N = 1$. Similarly $N = 2$ short graviton multiplet $SD(y_0 + 2, 1, y_0|2)$ decomposes into $SD(y_0 + \frac{3}{2}, \frac{3}{2}|1) \oplus SD(y_0 + 2, 1|1)$. Since $y_0 > 0$, the spin-2 component of this multiplet has conformal dimension $\Delta = y_0 + 3 > 3$ and this gives massive modes according to (4.20). The $N = 2$ long graviton multiplet $SD(E_0, 1, y_0|2)$ decomposes into $SD(E_0 + \frac{1}{2}, \frac{1}{2}|1) \oplus SD(E_0 + 1, 1|1) \oplus SD(E_0 + \frac{1}{2}, \frac{1}{2}|1) \oplus SD(E_0, 1|1)$. Also the spin-2 component of this multiplet has conformal dimension $\Delta = E_0 + 1 > y_0 + 3 > 3$ and this provides massive modes due to (4.20).

The gauge theory conjectured to be dual to the $SU(2) \times SU(2) \times U(1)_R$ $N = 2$ supergravity background is a deformation of the quiver diagram for a partial resolution of $Q^{1,1}$ theory by a superpotential term quadratic in $\Phi$ (3.5) which is an adjoint field. The gauge theory has also $SU(2) \times SU(2) \times U(1)_R$ symmetry where the $SU(2) \times SU(2)$ symmetry corresponds to the global rotations of $B_1, B_2$ and $C_1, C_2$ in (3.1). Under the $U(1)_R$ symmetry, the four fields as well as an adjoint field have $R$-charges given by

\begin{equation}
\begin{aligned}
R(B_1) &= R(B_2) = R(C_1) = R(C_2) = \frac{1}{4}, \quad R(\Phi) = 1, \\
R(\overline{B}_1) &= R(\overline{B}_2) = R(\overline{C}_1) = R(\overline{C}_2) = -\frac{1}{4}, \quad R(\overline{\Phi}) = -1,
\end{aligned}
\end{equation}

as in the previous section, around (3.6).
Table 5. The next level of spin-2 component of the massive graviton multiplets. For each multiplet, we present \( SO(8) \) representation (4.9), \( SU(2) \times SU(2) \times U(1)_R \) representations denoted by \((2l_1+1, 2l_2+1)_R\), \( R_\alpha \)-charge, \( R_\psi \)-charge, the KK excitation number \( j \), the mass-squared \( m^2 L^2 \) (4.18) of the AdS₅ field and the corresponding \( N = 2 \) SCFT operator. The dimension \( \Delta (4.20) \) of the spin-2 component of the multiplet, which we do not present here due to the space, can also be obtained.

| \( SO(8) \) | \( SU(2) \) | \( R_\alpha \) | \( R_\psi \) | \( j \) | \( m^2 L^2 \) | \( N = 2 \) SCFT operator |
|----------------|----------------|---------|--------|---|---------------|------------------|
| \( 112_e \)    | \( (2, 1)_+ \)  | 1       | 1      | 1  | \( \frac{1}{2}(\sqrt{3}T + 5) \) | \( T_{\alpha \beta} B_i(1 - \Phi \Phi) \) |
|                | \( (2, 1)_- \)  | 1       | -1     | 1  | \( \frac{1}{2}(\sqrt{3}T + 5) \) | \( T_{\alpha \beta} \overline{B}_i(1 - \Phi \Phi) \) |
|                | \( (1, 2)_+ \)  | 1       | 1      | 1  | \( \frac{1}{2}(\sqrt{3}T + 5) \) | \( T_{\alpha \beta} C_i(1 - \Phi \Phi) \) |
|                | \( (1, 2)_- \)  | 1       | -1     | 1  | \( \frac{1}{2}(\sqrt{3}T + 5) \) | \( T_{\alpha \beta} \overline{C}_i(1 - \Phi \Phi) \) |
| \( 4_e \)      | \( (4, 1)_+ \)  | 3       | 3      | 0  | \( \frac{1}{2}(\sqrt{79} + 3) \) | \( T_{\alpha \beta} B_i B_j B_k - \frac{1}{2} \delta_{ij} B_i B_j B_k \) |
|                | \( (4, 1)_- \)  | 3       | 3      | 0  | \( \frac{1}{2}(\sqrt{79} + 3) \) | \( T_{\alpha \beta} \overline{B}_i \overline{B}_j \overline{B}_k - \frac{1}{2} \delta_{ij} \overline{B}_i \overline{B}_j \overline{B}_k \) |
| \( 30_e \)     | \( (3, 2)_+ \)  | 1       | 1      | 0  | \( \frac{1}{2}(\sqrt{79} + 1) \) | \( T_{\alpha \beta} B_i B_j B_k - \frac{1}{2} \delta_{ij} B_i B_j B_k \) |
|                | \( (3, 2)_- \)  | 1       | 1      | 0  | \( \frac{1}{2}(\sqrt{79} + 1) \) | \( T_{\alpha \beta} \overline{B}_i \overline{B}_j \overline{B}_k - \frac{1}{2} \delta_{ij} \overline{B}_i \overline{B}_j \overline{B}_k \) |
| \( 24_e \)     | \( (2, 3)_+ \)  | 1       | 1      | 0  | \( \frac{1}{2}(\sqrt{79} + 1) \) | \( T_{\alpha \beta} C_i C_j C_k - \frac{1}{2} \delta_{ij} C_i C_j C_k \) |
|                | \( (2, 3)_- \)  | 1       | 1      | 0  | \( \frac{1}{2}(\sqrt{79} + 1) \) | \( T_{\alpha \beta} \overline{C}_i \overline{C}_j \overline{C}_k - \frac{1}{2} \delta_{ij} \overline{C}_i \overline{C}_j \overline{C}_k \) |
| \( 18_e \)     | \( (1, 4)_+ \)  | 1       | 1      | 0  | \( \frac{1}{2}(\sqrt{79} + 1) \) | \( T_{\alpha \beta} C_i C_j C_k - \frac{1}{2} \delta_{ij} C_i C_j C_k \) |
|                | \( (1, 4)_- \)  | 1       | 1      | 0  | \( \frac{1}{2}(\sqrt{79} + 1) \) | \( T_{\alpha \beta} \overline{C}_i \overline{C}_j \overline{C}_k - \frac{1}{2} \delta_{ij} \overline{C}_i \overline{C}_j \overline{C}_k \) |
from (4.18). Starting with the $N = 2$ SCFT operator denoted by $T_{ab}$ corresponding to the massless graviton multiplet that we introduced in the previous section, one constructs a tower of KK modes by multiplying $B_i$ or $C_j$ (and its conjugated fields) with $T_{ab}$ in addition to the overall factor which depends on $\Phi$ (and $\overline{\Phi}$), for $j = 0$ modes. For nonzero $j$’s, there appear some polynomials in $\Phi \overline{\Phi}$.

It is natural to identify, up to normalization, these fields $B_i$ and $C_j$ and the adjoint field $\Phi$, that are noncommuting operators in the gauge theory side, with the angular coordinates as follows:

\[
\begin{align*}
B_1 & \leftrightarrow \cos \left( \frac{\theta_1}{2} \right) e^{\frac{i}{2}(\phi_1+\phi_i)}, & B_2 & \leftrightarrow \sin \left( \frac{\theta_1}{2} \right) e^{\frac{i}{2}(\phi_1-\phi_i)}, \\
C_1 & \leftrightarrow \cos \left( \frac{\theta_2}{2} \right) e^{\frac{i}{2}(\phi_2+\phi_i)}, & C_2 & \leftrightarrow \sin \left( \frac{\theta_2}{2} \right) e^{\frac{i}{2}(\phi_2-\phi_i)}, \\
\Phi & \leftrightarrow \sin \mu e^{-i\psi}.
\end{align*}
\]

Actually first four of them can be read off from (2.3) in the context of conifold description. The last one in (4.22) can be read off from (2.5). If we shift the fields by (2.4), then the corresponding objects also arise in the tables 4 and 5. Note that the factor $\sqrt{\cos \mu}$ corresponds to $\sqrt{1 - \Phi \overline{\Phi}}$ in the $N = 2$ boundary field theory.

One can see these features from the solutions of Laplacian eigenvalue equation that we have found. From the explicit form in (4.13), one computes the following expressions, by choosing the appropriate regular solutions, for given quantum numbers:

\[
\begin{align*}
j_1(\theta_1)_{l_1=\frac{1}{2},l_2=\frac{1}{2},R_0=1} & \sim \cos \left( \frac{\theta_1}{2} \right), & j_2(\theta_1)_{l_1=\frac{1}{2},l_2=-\frac{1}{2},R_0=1} & \sim \sin \left( \frac{\theta_1}{2} \right), \\
j_1(\theta_2)_{l_1=\frac{1}{2},l_2=\frac{1}{2},R_0=1} & \sim \cos \left( \frac{\theta_2}{2} \right), & j_2(\theta_2)_{l_1=\frac{1}{2},l_2=-\frac{1}{2},R_0=1} & \sim \sin \left( \frac{\theta_2}{2} \right).
\end{align*}
\]

Then it is easy to see that these four solutions of (4.23) appear on the right-hand sides of (4.22) exactly. Therefore, the eigenfunctions except the function $H(\cos^2 \mu)$ in (4.12) correspond to $B_i$ or $C_j$ and its conjugated fields. Note that the exponential factors of (4.12) already appear in the chiral fields given by (4.22).

Let us describe the $N = 2$ SCFT operators in tables 4 and 5. In the representations $(2, 1)_{\pm\frac{1}{2}}$ and $(1, 2)_{\pm\frac{1}{2}}$ coming from 8, of the $SO(8)$ representation, one can read off the corresponding $N = 2$ SCFT operators by using the fact that each $B_i$ and $C_j$ consists of a doublet of each $SU(2)$ group and the $\mu$-dependent part can be determined by (4.19) for given quantum numbers. For example, $l_1 = \frac{1}{2}, m_1 = \pm\frac{1}{2}, R_0 = \pm 1 = 4R$ corresponding to $(2, 1)_{\pm\frac{1}{2}}$. Recall that $u = \cos^2 \mu$ corresponds to $(1 - \Phi \overline{\Phi})$ and the eigenvalue $C_2$ does not depend on the quantum number $m_1$ or $m_2$. Since the KK excitation number vanishes $j = 0$, the $\mu$ dependence arises from only $u^{\frac{1}{2}} e^{-\frac{i}{2}}$. Note that the eigenvalue (4.14) has a symmetry under the $l_1 \leftrightarrow l_2$ or under the $R \leftrightarrow -R$.

At the next level, the additional structure in the $N = 2$ SCFT operator, compared to the $j = 0$ case, corresponding to the representation $(1, 1)_0$ with nonzero $j = 1$ can be constructed from (4.19) by substituting $j = 1$. In this case, we should go to the differential equation (4.15) directly in order to obtain the correct solution. The solution is given by $H(u) = \mathbf{2}F_1 \left( \frac{1}{2}[3 - 9 \sqrt{u} + 2 L^2], \frac{1}{2}[3 + 9 \sqrt{u} + 2 m^2 L^2]; 3; u \right)$. Now we move on to the other representations $(3, 1)_{\pm\frac{1}{2}, 0}$. Since $(3, 1)$ transforms as an adjoint representation of the first $SU(2)$, this can be constructed by a tensor product between $(2, 1)$ and $(2, 1)$ which becomes $(3, 1) \oplus (1, 1)$. Depending on the correct $R$-charges (4.21), one determines the appropriate combinations of $B_i$ and $\overline{B}_j$. Moreover, the $\mu$-dependent factor can be determined by (4.19) for given quantum numbers. For example, let us consider the quantum numbers
The second case, we take $T_{\alpha \beta}B_{\alpha \beta}$ with nonzero $j = 1$ can be analyzed from (4.19) by substituting $j = 1$ from the hypergeometric function. Let us move on to the other representations $(4, 1)_{\pm \frac{1}{2}, \pm 1}$. Since $(4, 1)$ transforms as a symmetric representation of the first $SU(2)$, this can be determined by a tensor product between $(3, 1)$ and $(2, 1)$ which leads to $(4, 1) \oplus (2, 1)$. Depending on the correct $R$-charges, one can determine the appropriate triple-product combinations of $B_{\alpha \beta}$ and $\overline{B}_{\alpha \beta}$. Moreover, the $\mu$-dependent factor can be determined by (4.19) for given quantum numbers, i.e. $l_1 = \frac{1}{2}$, $l_2 = 0$, $R_\phi = \pm 3, \pm 1 = 4R$. For the representations $(3, 2)_{\pm \frac{1}{2}, \pm \frac{3}{2}}$, the tensor product between $(3, 1)$ and $(1, 2)$ provides these cases: the product between the adjoint from the first $SU(2)$ and the fundamental from the second $SU(2)$. The exponent of the $\mu$-dependent factor appears in two different values by (4.19) depending on $\pm \frac{3}{2}$ $R$-charges or $\pm \frac{1}{2}$ $R$-charges for given quantum numbers, i.e. $l_1 = 1, l_2 = \frac{1}{2}$, $R_\phi = \pm 3, \pm 1$. Finally, the analysis for the representations $(3, 3)_{\pm \frac{1}{2}, \pm \frac{3}{2}}$ and $(4, 1)_{\pm \frac{1}{2}, \pm \frac{3}{2}}$ can be done similarly by using the symmetry between the quantum numbers.

One can extend these procedures to the higher excitations. For example, at the level of $n = 4$ excitations, by extending the procedure in (4.9) one step further, one has the following branching rules:

$$294, (4, 0, 0, 0) \rightarrow 294, (0, 0, 4) \rightarrow 35 \oplus 35 \oplus 70 \oplus 70 \oplus 84$$

$$\rightarrow [(1, 1) \oplus (3, 1) \oplus (4, 2) \oplus (1, 3)] \oplus (5, 1) \oplus (8, 4, 2) \oplus (9, 3, 3) \oplus (8, 2, 4) \oplus (5, 1, 5).$$

(4.24)

Then one obtains the KK excitation $j = 2$ state for $(1, 1)$ and the $j = 1$ states for $(3, 1)$, $(2, 2)$ and $(1, 3)$ inside of the bracket in (4.24). The remaining five independent new states with $j = 0$ arise also in this level. One can analyze also the corresponding $N = 2$ SCFT operators. For example, some (three states from nine possible states) of the wavefunctions for $l_1 = l_2 = 1$ with $R = 0$ corresponding to $(3, 3)_0$ are given by

$$T_{\mu \nu}B_\alpha \overline{B}_{\beta}C_\gamma \overline{C}_{\delta} (1 - \Phi T)^{-1} \leftrightarrow T_{\nu \mu}e^{i(\phi_1 + \phi_2)} \sin \theta_1 \sin \theta_2 (1 - \Phi T)^{-1},$$

$$T_{\mu \nu}B_\alpha \overline{B}_{\beta}C_\gamma \overline{C}_{\delta} (1 - \Phi T)^{-1} \leftrightarrow T_{\nu \mu}e^{i\phi_2} \cos \theta_1 \sin \theta_2 (1 - \Phi T)^{-1},$$

(4.25)

$$T_{\mu \nu}B_\alpha \overline{B}_{\beta}C_\gamma \overline{C}_{\delta} (1 - \Phi T)^{-1} \leftrightarrow T_{\nu \mu}e^{-i(\phi_1 + \phi_2)} \sin \theta_1 \sin \theta_2 (1 - \Phi T)^{-1}$$

where the corresponding $(m_1, m_2)$ values are $(1, 1)$, $(0, 1)$ and $(-1, -1)$, respectively [46]. One can easily see these from (4.13) or (4.22). For the first case of (4.25), we take $j_1(\theta_1)$ which is equal to $\sin \theta_1$ and $j_1(\theta_2)$ which is equal to $\sin \theta_2$, as an eigenfunction. For the second case, we take $j_2(\theta_1)$ which is equal to $\cos \theta_1$ and $j_2(\theta_2)$ that is equal to $\sin \theta_2$. For the last, we take $j_3(\theta_1)$ that is $\frac{1}{2} \sin \theta_1$ and $j_3(\theta_2)$ that is $\frac{1}{2} \cos \theta_2$, as the regular solutions. Note that the regularity of solutions (4.13) depend on the magnitude of $(m_1, m_2)$ with respect to the $R$-charge. In general, one expects that for general quantum numbers $l_1, l_2$ and $R$ of $SU(2) \times SU(2) \times U(1)_R$, the operator is given by the product of $T_{\mu \nu}$ with several $B_\alpha$ or $C_\gamma$ (and its conjugated fields) and some function of $\Phi T$. For nonzero $j$’s, the explicit form of hypergeometric functions is needed to identify the corresponding $N = 2$ SCFT operators and there exists a polynomial up to the order $j$ in $\Phi T$. 

$$l_1 = 1, l_2 = 0, R_\phi = \pm 2, 0 = 4R.$$ For the representations $(2, 2)_{\pm \frac{3}{2}, 0}$, one sees that the tensor product between $(2, 1)$ and $(1, 2)$ provides these particular cases. In this case, the exponent of the $\mu$-dependent factor appears in two different values by (4.19) depending on the vanishing $R$-charge (4.17) or the non-vanishing $R$-charge for given quantum numbers, i.e. $l_1 = \frac{1}{2}$, $l_2 = \frac{1}{2}, R_\phi = \pm 2, 0 = 4R$. Finally, the analysis for the representations $(1, 3)_{\pm \frac{1}{2}, 0}$ can be done similarly, as in $(3, 1)_{\pm \frac{1}{2}, 0}$. 

At the next level, the extra structure in the $N = 2$ SCFT operator, compared to the $j = 0$ case, corresponding to the representations $(2, 1)_{\pm \frac{1}{2}, \pm 1}$ and $(1, 2)_{\pm \frac{1}{2}}$ with nonzero $j = 1$ can be analyzed from (4.19) by substituting $j = 1$ from the hypergeometric function. Let us move on to the other representations $(4, 1)_{\pm \frac{1}{2}, \pm 1}$. Since $(4, 1)$ transforms as a symmetric representation of the first $SU(2)$, this can be determined by a tensor product between $(3, 1)$ and $(2, 1)$ which leads to $(4, 1) \oplus (2, 1)$. Depending on the correct $R$-charges, one can determine the appropriate triple-product combinations of $B_\alpha$ and $\overline{B}_\beta$. Moreover, the $\mu$-dependent factor can be determined by (4.19) for given quantum numbers, i.e. $l_1 = \frac{1}{2}, l_2 = 0$, $R_\phi = \pm 3, \pm 1 = 4R$. For the representations $(3, 2)_{\pm \frac{1}{2}, \pm \frac{3}{2}}$, the tensor product between $(3, 1)$ and $(1, 2)$ provides these cases: the product between the adjoint from the first $SU(2)$ and the fundamental from the second $SU(2)$. The exponent of the $\mu$-dependent factor appears in two different values by (4.19) depending on $\pm \frac{3}{2}$ $R$-charges or $\pm \frac{1}{2}$ $R$-charges for given quantum numbers, i.e. $l_1 = 1, l_2 = \frac{1}{2}$, $R_\phi = \pm 3, \pm 1$. Finally, the analysis for the representations $(2, 3)_{\pm \frac{1}{2}, \pm \frac{3}{2}}$ and $(4, 1)_{\pm \frac{1}{2}, \pm \frac{3}{2}}$ can be done similarly by using the symmetry between the quantum numbers.
5. Conclusions and outlook

We have elaborated on the 11-dimensional background geometry originated from \[8\]. In particular, we have presented the correct expression for a 3-form potential and some relations for \(SU(3) \times U(1)_R\) symmetry are also given. We have made the relations between the fields of AdS\(_4\) supergravity and composite operators of the IR boundary gauge theory in tables 1, 2 and 3. We have computed the KK reduction for spin-2 excitations around the warped 11-dimensional theory background that is dual to the \(\mathcal{N} = 2\) mass-deformed Chern–Simons matter theory with \(SU(2) \times SU(2) \times U(1)_R\) symmetry. The spectrum of spin-2 excitations was given by solving the equations of motion for minimally coupled scalar theory in this background. The AdS\(_4\) mass formula of the KK modes is given by (4.18). The quantum numbers \(l_1, l_2\) and \(R\) for the \(SU(2) \times SU(2) \times U(1)_R\) representation and the KK excitation number \(j\) arise in this mass formula. We calculated the dimensions of the dual operators in the boundary \(\mathcal{N} = 2\) SCFT via the AdS/CFT correspondence and in tables 4 and 5 we presented the summary of this work.

As observed in [8], at \(\mu = \frac{\pi}{\tau}\), the metric has conical singularity from (2.2), (2.3) and (2.4). That is, the apex or node is a double point, i.e. a singularity for which \(C = 0\) and \(dC = 0\) where \(C\) is a complex manifold (2.2) but for which the matrix of second derivatives is nondegenerate. Then the \(S^7\) degenerate to the conifold times \(S^1\). The solution by the metric \(T^{1,1}\) can be related, via T-duality, to the Romans’ type IIB supergravity theory in ten dimensions [31]. Furthermore, this leads to the nontrivial Klebanov–Witten fixed point of the holographic RG flow in [2]. This is related to the fact that there exists a flow between a space that is locally AdS\(_5 \times T^{1,1}\) and the AdS\(_4\) \(\cong S^7\) geometry, mentioned in [1]. By adding a flux to this solution and squashing and stretching ellipsoidally, the theory can flow to other nontrivial fixed point. One of the candidates is given by Pilch–Warner fixed point in [49]. The gravity solution interpolating these two fixed points has been constructed in [50]. It would be interesting to elaborate on these issues.

The quadratic equation (2.2) is the so-called \(A_1\) conifold. It is an open problem to generalize this singularity to ADE-type singularities discussed in [51]. For this approach, one needs to use the Einstein–Kahler metric on the del Pezzo surface \(dP_k\), \(k \geq 3\). According to [28], \(C(T^{1,1}) \times C\) can be obtained from either \(\frac{C_0}{2\pi} \times C\) or \(C(dP_3) \times C\), in which their superpotentials are the same for Abelian theory and their quiver diagrams looks similar to each other, by partial resolutions. It would be interesting to find out the corresponding gravity duals.

In the context of four-dimensional gauged supergravity, it is known that very few of critical points (supersymmetric or nonsupersymmetric) are found although there are 70 scalar fields. The lesson we have learned from [8] is when we go to 11-dimensional theory, maybe we will find various 11-dimensional solutions even if the four-dimensional flow equations are the same. For given seven-dimensional Sasaki–Einstein spaces, one can think of the possibilities to have resolved the cone over these spaces and in the gravity side, the correct 11-dimensional metric should be found with appropriate field strengths. We expect that since the flow equations in four dimensions are related to the \(\mathcal{N} = 2\) supersymmetry with \(U(1)_R\) charge, other types of 11-dimensional solutions with common four-dimensional flow equations will arise.

One possibility is characterized by the \(SU(2) \times U(1) \times U(1)_R\) symmetry which is smaller than \(SU(2) \times SU(2) \times U(1)_R\). The symmetry breaking of \(SU(2) \times U(1)\) can occur from either the \(SU(3) \times U(1)_R\) symmetry or the \(SU(2) \times SU(2) \times U(1)_R\) symmetry. In this case, the metric corresponding to \(S^2 \times S^2\) should preserve only one \(S^2\) symmetry.

Another possibility is given by the \(SU(3) \times U(1) \times U(1)_R\) symmetry which is larger than the \(SU(3) \times U(1)_R\) symmetry and can be obtained from the symmetry breaking of
SU(4) \times U(1)_R. Since there exists an extra U(1) factor, it is nontrivial to find out the correct 4-form fields that preserve the whole SU(3) \times U(1) \times U(1)_R symmetry. Note that the 4-forms in [8], that are given by (C.1), breaks the SU(3) \times U(1) \times U(1)_R symmetry group into SU(3) \times U(1)_R. It would be interesting to study these issues in more detail.

Acknowledgments

We would like to thank S Franco, Sangmin Lee and F D Rocha for discussions. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST)(no 2009-0084601).

Appendix A. The 4-form field strength and the Ricci tensor in frame basis for the SU(2) \times SU(2) \times U(1)_R-invariant case

The set of frames for the 11-dimensional metric (2.6) is given by

\begin{align}
e^1 &= -\frac{1}{\sqrt{\Delta}} e^4 dx^1, & e^2 &= \frac{1}{\sqrt{\Delta}} e^4 dx^2, \\
e^3 &= \frac{1}{\sqrt{\Delta}} e^4 dx^3, & e^4 &= \frac{1}{\sqrt{\Delta}} dr, \\
e^5 &= 3\hat{L}^2 \Delta \frac{\sqrt{X}}{\rho^3} d\mu, & e^6 &= 3\hat{L}^2 \Delta \frac{\rho \cos \mu}{\sqrt{6}} d\theta_1, \\
e^7 &= 3\hat{L}^2 \Delta \frac{\rho \cos \mu}{\sqrt{6}} \sin \theta_1 d\phi_1, & e^8 &= 3\hat{L}^2 \Delta \frac{\rho \cos \mu}{\sqrt{6}} \sin \theta_2 d\phi_2, \\
e^9 &= 3\hat{L}^2 \Delta \frac{\rho \cos \mu}{\sqrt{6}} \sin \theta_2 d\phi_2, \\
e^{10} &= 3\hat{L}^2 \Delta \frac{\rho^3}{2\sqrt{X}} \sin 2\mu \left[ -\frac{d\psi}{\rho^3} + \frac{1}{3} (d\psi + d\phi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2) \right], \\
e^{11} &= 3\hat{L}^2 \Delta \frac{\rho \cos \chi}{\sqrt{X}} \sin^2 \mu d\psi + \frac{1}{3} \cos^2 \mu (d\psi + d\phi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2),
\end{align}

where\(^2\) we use (2.8), (2.9) and (2.12) with the AdS\(_4\) radius \(\hat{L} = 3^{-\frac{1}{2}} L\).

\(^2\) The general formula for \(e^{(3)}\) used in [12] satisfying the whole SU(3) \times U(1)_R-invariant flow equation is given by

\[ C^{(3)} = \frac{3\hat{L}^3 \tanh \chi}{4X} \left( 3\hat{L}^2 \Delta \frac{\rho \cos \chi}{\sqrt{X}} \sin^2 \mu d\psi + \frac{1}{3} \cos^2 \mu (d\psi + d\phi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2) \right), \]

which is a complex conjugate of \(e^{(3)}_{\text{CPW}}\) in [8] and the definitions for the complex coordinates \(z_i\) and \(w\) are given in [12]. This also occurs in the last equation of section 4 of [8] which looks similar to (A.2). Note that there exist some differences in the overall coefficient and the \(\rho\) dependence with \(3\hat{L}^2 \Delta = L^2_{\text{CPW}}\). One can check (A.2) from the 'corrected' (there should be a plus sign in \(e^{(3)}\) 3-form given in [8] by changing the rectangular coordinates to the angular ones. If we substitute (2.12), the criticality condition at the IR into (A.2), then we obtain

\[ C^{(3)} = \frac{\hat{L}}{4X} \left( 3\hat{L}^2 \Delta \frac{\rho \cos \chi}{\sqrt{X}} \sin^2 \mu d\psi + \frac{1}{3} \cos^2 \mu (d\psi + d\phi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2) \right) \]

which is the same as (12) of [12].
It turns out that the antisymmetric field strengths have the following nonzero components in the orthonormal frame basis used in (2.13) or in (A.1)

\[
\begin{align*}
F_{1234} &= -\frac{3 \cdot 2 \cdot 3}{L(2 - \cos 2\mu)^5}, \\
F_{56810} &= \frac{\sin(\phi + 2\psi)}{L(2 - \cos 2\mu)^5}, \\
F_{56811} &= -\frac{3 \cdot 3 \cdot \sin(\phi + 2\psi)}{L(2 - \cos 2\mu)^5}, \\
F_{56910} &= -\frac{3 \cdot 3 \cdot \cos(\phi + 2\psi)}{L(2 - \cos 2\mu)^5}, \\
F_{56911} &= \frac{2 \cdot \sin 2\mu}{L(2 - \cos 2\mu)^5}, \\
F_{6810} &= -\frac{2 \cdot \sin 2\mu}{L(2 - \cos 2\mu)^5}, \\
F_{6811} &= -\frac{2 \cdot \sin 2\mu}{L(2 - \cos 2\mu)^5}, \\
F_{7810} &= -\frac{2 \cdot \sin 2\mu}{L(2 - \cos 2\mu)^5}, \\
F_{7811} &= -\frac{2 \cdot \sin 2\mu}{L(2 - \cos 2\mu)^5}, \\
F_{57910} &= -F_{57911}, \\
F_{691011} &= -F_{691011},
\end{align*}
\]

(A.3)

where the angle dependences for $\phi$ and $\psi$ appear in the combination of $(\phi + 2\psi)$. One can make the two $U(1)$ symmetries generated by $\phi$ and $\psi$ which preserve this combination $(\phi + 2\psi)$. After substituting (A.3) into the right-hand side of Einstein equation (2.1) with frame basis (A.1) one reproduces the one of $SU(3) \times U(1)_R$ case [8, 12] exactly. On the other hand, the Ricci tensor in the frame basis (A.1) is identical to the one with $SU(3) \times U(1)_R$ symmetry in [8]. Therefore, one concludes that the solutions (A.3) indeed satisfy the 11-dimensional Einstein–Maxwell equations. Let us present the Ricci tensor components with the frame basis here for convenience:

\[
\begin{align*}
R_1 &= -\frac{55 - 32 \cos 2\mu + 3 \cos 4\mu}{3 \cdot 2 \cdot 3 \sqrt{3L^2(2 - \cos 2\mu)^5}}, \\
R_2 &= R_3 = R_4 = -2R_6 = -2R_7 = 2R_8, \\
R_5 &= \frac{29 - 16 \cos 2\mu}{3 \cdot 2 \cdot 3 \sqrt{3L^2(2 - \cos 2\mu)^5}}, \\
R_6 &= R_{10} = \frac{2 \cdot 2 \cdot \sin 2\mu}{\sqrt{3L^2(2 - \cos 2\mu)^5}}, \\
R_7 &= R_{11} = \frac{2 \cdot 2 \cdot \sin 2\mu}{\sqrt{3L^2(2 - \cos 2\mu)^5}}, \\
R_8 &= \frac{2 \cdot 2 \cdot \sin 2\mu}{\sqrt{3L^2(2 - \cos 2\mu)^5}}, \\
R_{10} &= R_{11} = \frac{2 \cdot 2 \cdot \sin 2\mu}{\sqrt{3L^2(2 - \cos 2\mu)^5}}.
\end{align*}
\]

(A.4)

and other components vanish. This implies that the Einstein–Maxwell equations (2.1) with the frame basis are satisfied\(^3\). In appendix B, we write down the Ricci tensor and 4-forms in the coordinate basis and in appendix C, we present the 4-forms in the frame basis for the $SU(3) \times U(1)_R$-invariant case.

Appendix B. The Ricci tensor and the 4-form field strength in the coordinate basis for the $SU(2) \times SU(2) \times U(1)_R$-invariant case

The $SU(2) \times SU(2) \times U(1)_R$-invariant 11-dimensional metric (2.6) explained in section 2 is given by

\[
dx_{11}^2 = \Delta^{-1}(dr^2 + e^{2\Delta(r)}d\mu d\nu + 3\sqrt{\Delta}d\rho d\chi).
\]

(B.1)

\(^3\) Let us mention that although these 4-forms satisfy the whole RG flows, we are interested in the critical values (2.12); then the supergravity fields dependence does not appear. In principle, one can write down the general 4-forms with the $(\rho, \chi)$ dependence from (2.13) and (2.14) explicitly where one should use the RG flow equations [4]. The corresponding Ricci tensor components with the $(\rho, \chi)$ dependence can be obtained from (2.6) or (A.1).

\(^4\) Also let us mention that the 3-form potential between (4.28) and (4.29) in [8] has an error in the sign of $e^{10}$. See also (C.2). After this correction, the 3-form of [8] is exactly the complex conjugation of 3-form in [12]. Since the full 3-form has its complex conjugation, eventually the 4-form [8] is the same as the one in [12].
This equation generates the Ricci tensor components in the coordinate basis:

\[
\begin{align*}
\hat{R}_1^1 &= \frac{(-252 + 241 c_{2\mu} - 44 c_{4\mu} + 3 c_{6\mu})}{6 \cdot 2^4 \sqrt{3} L^2 (2 - c_{2\mu})^2}, \\
\hat{R}_5^5 &= \frac{3 \cdot 2^4 \sqrt{3} L^2 (2 - c_{2\mu})^2}{2^4 \sqrt{3} L^2 (2 - c_{2\mu})^2}, \\
\hat{R}_7^7 &= \frac{c_6 c_{2\mu} (21 - 14 c_{2\mu} + c_{4\mu})}{2^4 \sqrt{3} L^2 (2 - c_{2\mu})^2}, \\
\hat{R}_9^9 &= \frac{c_6 c_{2\mu} (21 - 14 c_{2\mu} + c_{4\mu})}{2^4 \sqrt{3} L^2 (2 - c_{2\mu})^2}, \\
\hat{R}_{10}^{10} &= \frac{3 \cdot 2^4 \sqrt{3} L^2 (2 - c_{2\mu})^2}{2^4 \sqrt{3} L^2 (2 - c_{2\mu})^2}, \\
\hat{R}_{11}^{11} &= \frac{(20 - 22 c_{2\mu} + 3 c_{4\mu})}{3 \cdot 2^4 \sqrt{3} L^2 (2 - c_{2\mu})^2}.
\end{align*}
\]

B.2

Or from the orthonormal basis (A.1) where $e^\mu_i = e^\mu_{i^*} dy^{\mu^*}$ and the Ricci tensor components (A.4) with that basis, one obtains (B.2) via $\hat{R}_{nm} = e^\mu_{i^*} e^\nu_{j^*} R_{ij}$. In this basis, the Ricci tensor also depends on the angular coordinates $\theta_1$ and $\theta_2$. From the (A.3) in the frame basis, one also writes them in the coordinate basis using $\hat{F}_{nmq} = e^\mu_{i^*} e^\nu_{j^*} e^\rho_{k^*} e^\sigma_{l^*} F_{abcd}$ as follows:

\[
\begin{align*}
\hat{F}_{1234} &= -\frac{9 \cdot 3^2}{2 L} \hat{F}_{5678} = -\frac{3 \cdot 3^2 L^3 c_{2\mu} c_{4\mu} c_{6\mu} c_{2\mu} c_{4\mu} c_{6\mu}}{(2 - c_{2\mu})^2}, \\
\hat{F}_{5679} &= -\frac{3 \cdot 3^2 L^3 c_{2\mu} c_{4\mu} c_{6\mu} c_{2\mu} c_{4\mu} c_{6\mu}}{(2 - c_{2\mu})^2}, \\
\hat{F}_{56810} &= -\frac{3 \cdot 3^2 L^3 c_{2\mu} c_{4\mu} c_{6\mu} c_{2\mu} c_{4\mu} c_{6\mu}}{(2 - c_{2\mu})^2}, \\
\hat{F}_{56910} &= \frac{3 \cdot 3^2 L^3 c_{2\mu} c_{4\mu} c_{6\mu} c_{2\mu} c_{4\mu} c_{6\mu}}{(2 - c_{2\mu})^2}, \\
\hat{F}_{5789} &= -\frac{3 \cdot 3^2 L^3 c_{2\mu} c_{4\mu} c_{6\mu} c_{2\mu} c_{4\mu} c_{6\mu}}{(2 - c_{2\mu})^2}, \\
\hat{F}_{57910} &= \frac{3 \cdot 3^2 L^3 c_{2\mu} c_{4\mu} c_{6\mu} c_{2\mu} c_{4\mu} c_{6\mu}}{(2 - c_{2\mu})^2}, \\
\hat{F}_{6789} &= \frac{3 \cdot 3^2 L^3 c_{2\mu} c_{4\mu} c_{6\mu} c_{2\mu} c_{4\mu} c_{6\mu}}{(2 - c_{2\mu})^2}, \\
\hat{F}_{67910} &= \frac{3 \cdot 3^2 L^3 c_{2\mu} c_{4\mu} c_{6\mu} c_{2\mu} c_{4\mu} c_{6\mu}}{(2 - c_{2\mu})^2}, \\
\hat{F}_{68910} &= \frac{3 \cdot 3^2 L^3 c_{2\mu} c_{4\mu} c_{6\mu} c_{2\mu} c_{4\mu} c_{6\mu}}{(2 - c_{2\mu})^2}, \\
\hat{F}_{78910} &= \frac{3 \cdot 3^2 L^3 c_{2\mu} c_{4\mu} c_{6\mu} c_{2\mu} c_{4\mu} c_{6\mu}}{(2 - c_{2\mu})^2}, \\
\hat{F}_{79101} &= \frac{3 \cdot 3^2 L^3 c_{2\mu} c_{4\mu} c_{6\mu} c_{2\mu} c_{4\mu} c_{6\mu}}{(2 - c_{2\mu})^2}, \\
\hat{F}_{89101} &= \frac{3 \cdot 3^2 L^3 c_{2\mu} c_{4\mu} c_{6\mu} c_{2\mu} c_{4\mu} c_{6\mu}}{(2 - c_{2\mu})^2}, \\
\hat{F}_{91011} &= \frac{3 \cdot 3^2 L^3 c_{2\mu} c_{4\mu} c_{6\mu} c_{2\mu} c_{4\mu} c_{6\mu}}{(2 - c_{2\mu})^2}.
\end{align*}
\]  

In this basis, the 4-form fields depend on the angular coordinates $\theta_1$ and $\theta_2$ as well as $\mu$, $\phi$ and $\psi$. One sees that the 4-forms in (B.3) contain the combination of $(\phi + 2\psi)$ indicating that these 4-forms preserve the $U(1)$ $R$ charge: $\phi \rightarrow \phi + 2\gamma$ and $\psi \rightarrow \psi - \gamma$ explained in (2.15).
Appendix C. The 4-form field strength in the frame basis for the $SU(3) \times U(1)_R$-invariant case

Let us present the 4-form fields in the frame basis used in [8]:

\[
F_{1234} = -\frac{3}{11} \frac{2^4 \cdot 3^2 c_{3\phi + 4\psi}}{L(2 - c_{2\mu})^5}, \quad F_{56710} = \frac{2^4 \cdot 3^2 s_{3\phi + 4\psi} (-3c_{\mu} + c_{3\mu})}{L(2 - c_{2\mu})^5} = F_{58910},
\]

\[
F_{56711} = \frac{2^4 \cdot 3^2 s_{3\phi + 4\psi}}{L(2 - c_{2\mu})^5}, \quad F_{56810} = \frac{2^4 \cdot 3^2 c_{3\phi + 4\psi} (-3c_{\mu} + c_{3\mu})}{L(2 - c_{2\mu})^5},
\]

\[
F_{56811} = \frac{2^4 \cdot 3^2 c_{3\phi + 4\psi}}{L(2 - c_{2\mu})^5}, \quad F_{57910} = \frac{2^4 \cdot 3^2 c_{3\phi + 4\psi} (-3c_{\mu} + c_{3\mu})}{L(2 - c_{2\mu})^5},
\]

\[
F_{57911} = \frac{2^4 \cdot 3^2 c_{3\phi + 4\psi}}{L(2 - c_{2\mu})^5}, \quad F_{59710} = \frac{2^4 \cdot 3^2 s_{3\phi + 4\psi}}{L(2 - c_{2\mu})^5}.
\]

This can be obtained from the 3-form potential

\[
C^{(3)} = \frac{1}{2} \sinh \chi e^{(3\phi - 4\psi)} (e^5 + ie^{10}) \wedge (e^6 - ie^9) \wedge (e^7 - ie^8)
\]

and (2.14) where the orthonormal frame basis used in (C.2) is given in [8]. In (C.1), the angle-dependences on $\phi$ and $\psi$ arise in the combination of $(3\phi + 4\psi)$ where $\phi \rightarrow \frac{i}{2} \chi$ and $\psi \rightarrow \psi - \gamma$ and these disappear by squaring 4-forms on the right-hand side of the 11-dimensional Einstein equation (2.1). This is consistent with the fact that the left-hand side of the Einstein equation is given by (A.4) which do not depend on these angular variables. One obtains also the corresponding 4-forms and Ricci tensor components in the coordinate basis using $\hat{R} = e^d e^b R_{ab}$ and $\hat{F}_{abc} = e^d e^b e^c F_{abc}$ as mentioned before from (C.1) and (A.4). It turns out that they depend on the angular variables $\alpha_1$ and $\theta$. In particular, the 4-forms contain the trigonometric functions with the argument $(\alpha_1 + 3\phi + 4\psi)$. By substituting these into the 11-dimensional Einstein equation, one sees that the dependence on $(\alpha_1 + 3\phi + 4\psi)$ completely disappears on the right-hand side of the equation. This is also consistent with the fact that there is no dependence on $\alpha_1$, $\phi$ and $\psi$ in the Ricci tensor components on the left-hand side of the equation.

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