REGULARITY OF THE SEMI-GROUP OF REGULAR PROBABILITY MEASURES ON COMPACT HAUSDORFF TOPOLOGICAL GROUPS

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Abstract. Let $G$ be a compact group, and $P(G)$ denotes the class of all regular probability measures on $G$. It is well known that $P(G)$ is a semi-group under the convolution of measures. This semi-group has been studied elaborately and intensely by There are many deep results on the structure of regular probability measures $P(G)$ on compact/locally compact, Hausdorff topological groups $G$. See, for instance, the classic monographs by KR Parthasarathy [15], Ulf Grenander [6]. A. Mukherjea and Nicolas A. Tserpes [13], Wendel [16] to quote selected references. In his remarkable paper, Wendel proved many deep theorems in this context. He proved that $P(G)$ is a semi-group which is not a group, by proving that the only invertible elements are point mass supported measures (Dirac delta measures).

In this short paper, we prove that $P(G)$ is not algebraically regular in the sense that not every element has a generalized inverse. However, we prove that $P(G)$ can be embedded in larger concrete algebraically regular semi-groups. Also, an attempt is made to identify algebraically regular elements in some special cases.

1. Introduction

As mentioned in the abstract, it is well known that the set $P(G)$ of regular probability measures on a topological group $G$ is a semi-group under convolution, which is abelian if and only if the group $G$ is abelian. It is also known that $P(G)$ is a compact convex set under the weak* topology of measures. Wendel [16] in his remarkable paper, established many significant results regarding the algebraic, topological as well as geometric structure of $P(G)$. He showed that $P(G)$ is a closed convex semi-group which is not a group except for trivial group $\{e\}$ by showing that the only invertible elements are point mass measures supported on single elements.

The problem we consider is the algebraic regularity of $P(G)$. A semi-group is called algebraically regular if each of its elements has a generalized inverse.

The main theorem proved in this article is Theorem 3.3 which states that $P(G)$ is not a regular semi-group unless, of course, for the trivial case $G = \{e\}$. In section 4 the embedding of this semi-group into a regular one is considered. In the concluding section 5 several related problems are given, such as the optimality of this embedding. However, a possible groundwork is prepared using the already existing theory of non-commutative Fourier transform of measures in $P(G)$ for the special case where $G$ is a compact Lie group [1].

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2. Preliminaries

Let $G$ be a compact, Hausdorff topological group and $\mathcal{B}$ denote the $\sigma$-algebra of all Borel sets in $G$. A probability measure $\mu$ is a nonnegative countably additive function on $\mathcal{B}$ such that the total mass $\mu(G) = 1$. A point mass measure or Dirac delta measure is a measure $\mu$ for which there is an element $x \in G$ such that $\mu(A) = 1$ if $x \in A$ and zero otherwise; $A \in \mathcal{B}$. Such a measure is usually denoted as $\delta_x$. One of the interesting results of Wendel is that the only invertible elements in $P(G)$ are Dirac delta measures. The product in $P(G)$ is the convolution $\ast$ which is defined as follows;

**Definition 2.1. (Convolution)** Let $\mu, \nu \in P(G)$. Then $\mu \ast \nu$ is the probability measure defined as $\mu \ast \nu(A) = \int \mu(Ax^{-1})d\nu(x)$ for every $A \in \mathcal{B}$.

**Definition 2.2. (Generalised inverse)** Let $\mathcal{S}$ be a semigroup and let $s \in \mathcal{S}$. An element $s^\dagger \in \mathcal{S}$ is called a generalised inverse of $s$ if $ss^\dagger s = s$.

For example, it is well known that the set $M_n(\mathbb{C})$ of all complex matrices of finite order $N$ is a regular semi-group. The property of algebraic regularity is almost essential in the fundamental characterization theorems of KSS Nambooripad [14]. In this short note, we do not analyze the implications and consequences of Nambooripad’s theory in this concrete semi-group which is postponed to a different project altogether.

3. Regularity Question

For a compact topological group $G$, J.G. Wendel [16] proved that the set $P(G)$ is a semi-group which is not a group under convolution by proving that the only invertible elements in $P(G)$ are Dirac delta measures. One crucial property needed for measures under consideration is the regularity which is not guaranteed for compact topological groups. Next, we quote a fundamental theorem due to Wendel.

**Definition 3.1. (Support)** Support of $\mu \in P(G)$ is defined as $\text{supp}(\mu) = \{g \in G : \mu(E_g) > 0 \text{ for every neighbourhood } E_g \text{ of } g \in G\}$.

**Theorem 3.2 (Wendel).** Let $A$ and $B$ be supports of two measures $\mu$ and $\nu$ in $P(G)$. Then $\text{supp}(\mu \ast \nu) = AB = \{xy | x \in A, y \in B\}$

Now we prove the main theorem of this short research article.

**Theorem 3.3.** Let $G$ be a nontrivial compact topological group. Then $P(G)$ is not regular.

*Proof.* First we prove the assertion for the special case for which group $G$ is such that $a^2 \neq e$ for some $a \in G$. Let $a \in G$ be such that $a^2 \neq e$. Consider the probability measure $\mu = \delta_x + \delta_y$ where $\delta_x$ is the Dirac delta measure at $x$ for each $x \in G$. We show that $\mu$ does not have a generalised inverse. Let if possible a generalised inverse $\mu^\dagger$ of $\mu$ exist. Therefore we have

$$(3.1) \quad \mu \ast \mu^\dagger \ast \mu = \mu,$$

and $\mu \ast \mu^\dagger$ is an idempotent. Clearly $\text{supp}(\mu) = \{e, a\}$ and $H = \text{supp}(\mu \ast \mu^\dagger)$ is a compact subgroup of $G$ by Theorem 1 in [1]. Now combining Wendel’s theorem and equation (3.1) we find that

$$(3.2) \quad H.\{e, a\} = \{e, a\}$$

Let $h \in H$ and the equation (3.2) above implies that

$$(3.3) \quad h.\{e, a\} \subset \{e, a\} \Rightarrow he = e \quad \text{or} \quad he = a \quad \text{and} \quad ha = e \quad \text{or} ha = a.$$
Now $he = e \Rightarrow h = e$ or $h = a$. Again $ha = e \Rightarrow h = a^{-1}$ or $ha = a \Rightarrow h = e$.

Thus to summarise we get $h = e$ or $h = a^{-1}$. Thus the possibilities are $h = e$ for all \(h \in \{e, h = a^{-1}\}, \{h = e, h = a\}\). Thus we get $H = \{e\}$ or $H = \{e, a\}$ or $H = \{e, a^{-1}\}$. Now $H$ is a group. The second and third option would imply that $a^2 = e$ which is against the hypothesis. Therefore we have $H = \{e\}$. Now $\mu \star \mu^\dagger$ is a projection and therefore we get $\mu \star \mu^\dagger = \delta_e$, which is the identity. Thus $\mu$ is right invertible. Let $\text{supp}(\mu^\dagger) = F$. Observe that $\text{supp}(\mu \star \mu^\dagger) = \{e\}$. Therefore we have $\{e, a\}.F = \{e\}$. Let $f \in F$. Then $e.f = f = e$ and $a.f = e \Rightarrow a = e$, which is again not possible. All these absurd conclusions are consequence of the assumption that $\mu$ is regular.

Now let $G$ be such that $a^2 = e$ for every $a \in G$. Let $a \neq e$. Consider $\mu = \alpha_0 \delta_e + \alpha_1 \delta_a$, where $0 \leq \alpha_0, \alpha_1 \leq 1, \alpha_0 + \alpha_1 = 1$. Then we have $\mu \in P(G)$ and $\text{supp}(\mu) = \{e, a\}$. First we show that $\mu$ is an idempotent if and only if $\alpha_0 = \alpha_1 = \frac{1}{2}$.

Observe that $\mu^2 = (\alpha_0^2 + \alpha_1^2)\delta_e + 2\alpha_0 \alpha_1 \delta_a$. Therefore $\mu^2 = \mu$ if and only if $\alpha_0^2 + \alpha_1^2 = \alpha_0$ and $2\alpha_0 \alpha_1 = \alpha_1$, if and only if $\alpha_0 = \alpha_1 = \frac{1}{2}$. Now, let if possible, $\mu$ for which $\alpha_0 \neq \frac{1}{2}$ has a generalised inverse $\mu^\dagger$. It is an easy consequence of Wendel’s support theorem that $\mu^\dagger = \beta_0 \delta_e + \beta_1 \delta_a$ where $0 \leq \beta_0, \beta_1 \leq 1, \beta_0 + \beta_1 = 1$; the proof is as follows. Let $H = \text{supp}(\mu^\dagger)$. We have by Wendel’s theorem

\[
\text{(3.4)} \quad \{e, a\}.H.\{e, a\} = \{e, a\}
\]

Let $h \in H$. Then $h \in \{e, a\}$. Thus $\text{supp}(\mu^\dagger) \subseteq \{e, a\}$.

Now we have that $\mu \star \mu^\dagger$ is an idempotent. But an easy computation shows that

\[
\mu \star \mu^\dagger = (\alpha_0 \beta_0 + \alpha_1 \beta_1)\delta_e + (\alpha_0 \beta_1 + \alpha_1 \beta_0)\delta_a
\]

Therefore we must have $(\alpha_0 \beta_0 + \alpha_1 \beta_1) = \frac{1}{2} = (\alpha_0 \beta_1 + \alpha_1 \beta_0)$. Solving the above linear equations we obtain $\alpha_0 = \frac{1}{2} = \alpha_1$ provided $\beta_0 \neq \beta_1$. Now assume that $\beta_0 = \beta_1$. This means that $\beta_k = \frac{1}{2}$ for all $k$. Now we use the full force of generalised inverse as follows. We have

\[
[(\alpha_0 \beta_0 + \alpha_1 \beta_1)\delta_e] + (\alpha_0 \beta_1 + \alpha_1 \beta_0)\delta_a = (\alpha_0 \delta_e + \alpha_1 \delta_a)
\]

\[
\Rightarrow \frac{\alpha_0 + \alpha_1}{2} = \alpha_0 \Rightarrow \alpha_0 = \frac{1}{2} = \alpha_1
\]

Therefore for $\alpha_k \neq \frac{1}{2}, 0 \leq \alpha_0, \alpha_1 \leq 1, \alpha_0 + \alpha_1 = 1$ \(\alpha_0 \delta_e + \alpha_1 \delta_a\) will not be regular. This completes the proof. \[\square\]

**Proposition 3.4.** Let $G$ be a compact topological group. For $g \in G$ let $\mu = \frac{\delta_e + \delta_g}{2}$. Then $\mu$ is regular if and only if $g^2 = e$

**Proof.** Observe that the proof of theorem 3.3 above essentially establishes the assertion. \[\square\]

**Remark 3.5.** The above regularity problem was stated and left open in [12]. Wendel proved that the only invertible elements are Dirac delta measures at various points. The problem of characterizing regular elements of $P(G)$ seems interesting. We do not address this problem here. Observe that towards the end of the proof of the above theorem, we actually solved this question for a very special case for which $G = \{e, a\}$. In fact we prove that the only regular elements of $P(G)$ are $\{\delta_e, \delta_a, \frac{\delta_e + \delta_a}{2}\}$.

**Remark 3.6.** The set $P(G)$ is a closed convex set under weak* topology of measures, and $\{\delta_g : g \in G\}$ is the set of extreme points of $P(G)$. Hence by Krein-Millman theorem, the closed convex hull $\text{conv}(\{\delta_g : g \in G\}) = P(G)$. In particular, if one considers the subsemi-group $\text{conv}(\delta_g : g \in G)$, it may be possible to locate all regular elements in it geometrically. This possibility is under investigation.
**Remark 3.7.** In a general semigroup $\Omega$ if $\omega \in \Omega$ has a generalised inverse, then it has a Moore-Penrose inverse: to be explicit if $\omega g \omega = g$ for some $g \in \Omega$ then

\[(3.6)\]  
$\omega \star \omega^\dagger \star \omega = \omega$ \&

\[(3.7)\]  
$\omega^\dagger \star \omega \star \omega^\dagger$

where $\omega^\dagger = \omega g \omega$.

So to characterize generalized invertibility, it will be enough to characterize Moore-Penrose invertibility. So in the following example, we try to identify Moore-Penrose invertible elements.

**Example 3.8.** Let $G$ be a compact topological group such that $g^2 = e$ for all $g \in G$. Let $S$ be the sub semigroup given by

\[(3.8)\]  
$S = \text{Conv}\{\delta_g : g \in G\}$

where 'Conv' denotes the convex hull. For a finite set $\{g_1, g_2, \ldots, g_n\} \subset G$ let

\[(3.9)\]  
$\mu = \frac{\delta_e + \delta_{g_1} + \delta_{g_2} + \ldots + \delta_{g_n}}{n+1}$

Then $\mu$ is regular.

**Proof.** Of course, one can directly prove that $\mu$ is regular by brutal computation. However, our main interest being the characterization of regular elements, we give a systematic way of arriving at regular elements, $\mu$ being one of them. To start with, we assume that

\[(3.10)\]  
$\mu = \sum_{k=0}^{n} \alpha_k \delta_{g_k}, \alpha_k > 0, \& \sum_{k=0}^{n} \alpha_k = 1$

First we show that if $\mu^\dagger = \sum_{j=1}^{m} \beta_j \delta_{h_j}$ is the Moore-Penrose inverse implies that $\{h_j, j = 1, 2, \ldots m\} = \{g_j, j = 1, 2, \ldots n\}$.

If $\gamma$ is a generalised inverse of $\mu$, then we will have

\[(3.11)\]  
$\mu \star \mu^\dagger \star \mu = \mu, \&$

\[(3.12)\]  
$\mu^\dagger \star \mu \star \mu^\dagger = \mu^\dagger$

Hence by Wendel’s support theorem we have

\[(3.13)\]  
$\{h_k : k = 0, 1, 2, \ldots n\} \{h_k : k = 0, 1, 2, \ldots m\} \{g_k : k = 0, 1, 2, \ldots n\} = \{g_k : k = 0, 1, 2, \ldots n\}$

In particular we have

\[(3.14)\]  
$\{h_k : k = 0, 1, 2, \ldots m\} \subset \{g_k : k = 0, 1, 2, \ldots n\}$. 

Similar argument by using equation 3.12 above we have

\[(3.15)\]  
$\{h_k : k = 0, 1, 2, \ldots m\} \{g_k : k = 0, 1, 2, \ldots n\} \{h_k : k = 0, 1, 2, \ldots m\} = \{h_k : k = 0, 1, 2, \ldots m\}$

Since $G$ is abelian and by using the fact that $h_k^2 = e$, we obtain the reverse inclusion namely $\{g_k : k = 0, 1, 2, \ldots n\} \subset \{g_k : k = 0, 1, 2, \ldots n\}$... This proves our claim. Therefore we may assume that $\mu^\dagger$ is the generalised inverse of $\mu$ implies that

\[(3.16)\]  
$\mu^\dagger = \sum_{k=0}^{n} \beta_k \delta_{g_k}$
where \( \beta_k > 0 \) \& \( \sum_{k=0}^{n} \beta_k = 1 \). It is easy to see that

\[
\mu \ast \mu^\dagger = \sum_{j=0}^{n} (\Sigma_{g_k g_i = g_j} \alpha_k \beta_i) \delta_j,
\]

and

\[
\mu \ast \mu^\dagger \ast \mu = \left[ \sum_{j=0}^{n} \sigma_j \delta_{g_j} \right] \ast \mu = \sum_{k=0}^{n} \alpha_k \delta_{g_k}
\]

where \( \sigma_j = \Sigma_{g_k g_i = g_j} \alpha_k \beta_i \) for each \( j \). Therefore we find that

\[
\sum_{j=0}^{n} (\Sigma_{g_k g_i = g_j} \sigma_k \alpha_l) \delta_{g_j} = \sum_{j=0}^{n} \alpha_j \delta_{g_j}
\]

Therefore we have that

\[
\Sigma_{g_k g_i = g_j} \sigma_k \alpha_l = \alpha_j
\]

for every \( j \). Substituting terms we get

\[
\Sigma_{g_k g_i = g_j} (\Sigma_{g_k g_i = g_k} \alpha_i \beta_l) \alpha_l = \alpha_j
\]

for \( j = 0, 1, 2, \ldots n \). We consider equation 3.15 which can be written as

\[
\sum_{k=0}^{n} \sigma_k \alpha_{S(k)}
\]

for each \( j, S_j \) is the permutation on \( \{0, 1, 2, \ldots n\} \) given by \( g_j g_k \to g_{S_j(k)}, j, k \in \{0, 1, 2, \ldots, n\} \). This can again be written as a matrix equation as follows:

\[
A = \begin{bmatrix}
\alpha_{S(0)(0)} & \alpha_{S(0)(1)} & \cdots & \alpha_{S(0)(n)} \\
\alpha_{S(1)(0)} & \alpha_{S(1)(1)} & \cdots & \alpha_{S(1)(n)} \\
& & \ddots & \\
\alpha_{S(n)(0)} & \alpha_{S(n)(1)} & \cdots & \alpha_{S(n)(n)}
\end{bmatrix}
\]

and the corresponding equation is as follows:

\[
\begin{bmatrix}
\alpha_{S(0)(0)} & \alpha_{S(0)(1)} & \cdots & \alpha_{S(0)(n)} \\
\alpha_{S(1)(0)} & \alpha_{S(1)(1)} & \cdots & \alpha_{S(1)(n)} \\
& & \ddots & \\
\alpha_{S(n)(0)} & \alpha_{S(n)(1)} & \cdots & \alpha_{S(n)(n)}
\end{bmatrix}
\begin{bmatrix}
\sigma_0 \\
\sigma_1 \\
\vdots \\
\sigma_n
\end{bmatrix}
= 
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_n
\end{bmatrix}
\]

Now equation 3.13 can be explicitly written as follows:

\[
\begin{bmatrix}
\alpha_{S(0)(0)} & \alpha_{S(0)(1)} & \cdots & \alpha_{S(0)(n)} \\
\alpha_{S(1)(0)} & \alpha_{S(1)(1)} & \cdots & \alpha_{S(1)(n)} \\
& & \ddots & \\
\alpha_{S(n)(0)} & \alpha_{S(n)(1)} & \cdots & \alpha_{S(n)(n)}
\end{bmatrix}
\begin{bmatrix}
\beta_0 \\
\beta_1 \\
\vdots \\
\beta_n
\end{bmatrix}
= 
\begin{bmatrix}
\sigma_0 \\
\sigma_1 \\
\vdots \\
\sigma_n
\end{bmatrix}
\]
Combining equations 3.20 & 3.19 we find that the determining equation is as follows:

\[
\begin{bmatrix}
\alpha S_{(0)(0)} & \alpha S_{(0)(1)} & \ldots & \alpha S_{(0)(n)} \\
\alpha S_{(1)(0)} & \alpha S_{(1)(1)} & \ldots & \alpha S_{(1)(n)} \\
\vdots & \ddots & \ddots & \vdots \\
\alpha S_{(n)(0)} & \alpha S_{(n)(1)} & \ldots & \alpha S_{(n)(n)}
\end{bmatrix}^2 \begin{bmatrix}
\beta_0 \\
\beta_1 \\
\vdots \\
\beta_n
\end{bmatrix} = \begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_n
\end{bmatrix}
\]

(3.26)

\[ \square \]

**Remark 3.9.** Equations 3.19 & 3.20 above determines all regular elements in the semigroup \( S, (3.6) \). In particular it follows that the middle points \( \{ \frac{\sum_k}{\delta_{g_k}} : k = 2, 3, \ldots, n + 1 \} \) are all regular. Existence of other noninvertible regular elements needs analysis of the matrix equation 3.19.

Observe that the permutation \( S_0 \) is the identity. Moreover we have that \( S_j(j) = e \) for all \( j = 0, 1, 2, \ldots, n \). Therefore the diagonal entry of the matrix \( A \) is the same namely \( \alpha_0 \).

**Obstructions:**

In what follows we assume that \( \alpha_k > 0, k = 0, 1, 2, \ldots, n \) \& \( \sum_{k=0}^{n} \alpha_k = 1 \). If \( A \) is invertible, then equation 3.25 will have the unique solution namely

\[ \sigma_k = 1, k = 0 \] & \[ \sigma_k = 0, k = 1, 2, \ldots, n. \]

Hence equation 3.25 becomes

\[
\begin{bmatrix}
\alpha S_{(0)(0)} & \alpha S_{(0)(1)} & \ldots & \alpha S_{(0)(n)} \\
\alpha S_{(1)(0)} & \alpha S_{(1)(1)} & \ldots & \alpha S_{(1)(n)} \\
\vdots & \ddots & \ddots & \vdots \\
\alpha S_{(n)(0)} & \alpha S_{(n)(1)} & \ldots & \alpha S_{(n)(n)}
\end{bmatrix} \begin{bmatrix}
\beta_0 \\
\beta_1 \\
\vdots \\
\beta_n
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

(3.27)

Therefore we will have \( \beta_k = 0 \) for all \( k = 1, 2, \ldots, n \) which implies that \( \alpha_0 \beta_0 = 1 \). But this means that \( \alpha_0 = 1 = \beta_0 \). Hence we get the contradictory implication that \( \alpha_k = 0, k = 1, 2, \ldots, n \).

**Remark 3.10.** Thus for any set \( \{ \alpha_k : \alpha_k > 0 \) \& \( \sum_{k=0}^{n} = 1 \} \) for which the corresponding matrix \( A \) is invertible the probability measure \( \mu = \sum_{k=0}^{n} \alpha_k \delta_{g_k} \) will not have a generalised inverse in the semigroup \( S \) given at the beginning of this example.

Now the diagonal dominance is a verifiable condition that implies invertibility. Since \( \alpha S_k(k) = \alpha_0 \) for all \( k \), the above condition reduces to the following inequality given below:

\[
\alpha_0 > \frac{n}{n + 1}
\]

(3.28)

However the special case for which \( \alpha_k = \frac{1}{n+1} \), the corresponding probability measure \( \mu = \sum_{k=0}^{n} \delta_{g_k} \frac{1}{n+1} \) will have the generalised inverse namely \( \mu \) itself which is the midpoint of the convex polytope.
Example 3.11. The case \( n = 1 \) has already been done. Now consider the case \( n = 2 \) so that 
\[ G = \{ e, g_1, g_2 : g_2^2 = e \} . \] In this case we will have \( g_1 g_2 = e, o r g_1 or g_2, \) which implies that \( G \) is 
a two element one.

Now we consider the case \( n = 3 \). We prove the following relations namely

\[ g_1 g_2 = g_3, g_2 g_3 = g_1 \& g_1 g_3 = g_2. \]

The above relations determines the permutations \( S_0, S_1, S_2 \& S_3 \) on \( \{0, 1, 2, 3, \} \). Simple computations reveals that the matrix \( A \) is as follows:

\[ A = \begin{bmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_1 & \alpha_0 & \alpha_3 & \alpha_2 \\
\alpha_2 & \alpha_3 & \alpha_0 & \alpha_1 \\
\alpha_3 & \alpha_2 & \alpha_1 & \alpha_0
\end{bmatrix} \]

which is a Hermetian doubly stochastic matrix. Now the equation is

\[ \begin{bmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_1 & \alpha_0 & \alpha_3 & \alpha_2 \\
\alpha_2 & \alpha_3 & \alpha_0 & \alpha_1 \\
\alpha_3 & \alpha_2 & \alpha_1 & \alpha_0
\end{bmatrix} \begin{bmatrix}
\sigma_0 \\
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{bmatrix} = 
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix} \]

. In addition we have subsequent equation

\[ \begin{bmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_1 & \alpha_0 & \alpha_3 & \alpha_2 \\
\alpha_2 & \alpha_3 & \alpha_0 & \alpha_1 \\
\alpha_3 & \alpha_2 & \alpha_1 & \alpha_0
\end{bmatrix} \begin{bmatrix}
\beta_0 \\
\beta_1 \\
\beta_2 \\
\beta_3
\end{bmatrix} = 
\begin{bmatrix}
\sigma_0 \\
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{bmatrix} \]

. The combined equation as before becomes

\[ \begin{bmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_1 & \alpha_0 & \alpha_3 & \alpha_2 \\
\alpha_2 & \alpha_3 & \alpha_0 & \alpha_1 \\
\alpha_3 & \alpha_2 & \alpha_1 & \alpha_0
\end{bmatrix} \begin{bmatrix}
\sigma_0 \\
\sigma_1 \\
\sigma_2 \\
\sigma_3
\end{bmatrix} = 
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix} \]

. In addition we have subsequent equation

\[ \begin{bmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_1 & \alpha_0 & \alpha_3 & \alpha_2 \\
\alpha_2 & \alpha_3 & \alpha_0 & \alpha_1 \\
\alpha_3 & \alpha_2 & \alpha_1 & \alpha_0
\end{bmatrix} ^2 \begin{bmatrix}
\beta_0 \\
\beta_1 \\
\beta_2 \\
\beta_3
\end{bmatrix} = 
\begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2 \\
\alpha_3
\end{bmatrix} \]

Now assume that \( \alpha_j : j = 0, 1, 2, 3 \) be distinct positive numbers such that \( \Sigma_{k=0}^3 \alpha_k = 1 \). As in the general case, an obstruction for algebraic regularity of a probability measure \( \mu = \Sigma_{k=0}^3 \alpha_k \delta_{g_k} \) is \( \alpha_0 > \frac{4}{5} \).

4. Embedding \( P(G) \) in Regular Semigroups

Our next goal is to embed \( P(G) \) in larger semigroups in an optimal way. To do this we use non-commutative Fourier transform techniques.
4.1. Non-Commutative Fourier Transforms. Recall that for a locally compact topological group \( G \), \( \hat{G} \) will denote the unitary dual space of \( G \). More explicitly

\[
\hat{G} = \{ (\pi, H_\pi) \}
\]

where \( \pi : G \to B(H_\pi) \), \( \pi \) is unitary, irreducible representation of \( G \) on a complex separable Hilbert space \( H_\pi \) with the identification by unitary equivalence of representations. It is also well-known that when \( G \) is compact, then each \( H_\pi \) is finite dimensional. That means the dimension \( d_\pi \) of \( H_\pi \) is finite and \( d_\pi = 1 \) if \( G \) is abelian. The Fourier transform of a \( \mu \in P(G) \) is defined as a function on \( \hat{\mu} : \hat{G} \to B(H_\pi) \) defined by

\[
\hat{\mu}(\pi) \psi = \int_G \pi(g^{-1}) \psi \mu(dg)
\]

\( \pi \in \hat{G} \). For a compact, Hausdorff group \( G \) let \( \mathcal{M} = \cup_{d_\pi} M_{d_\pi}(\mathbb{C}) \).

A map \( \Phi : \hat{G} \to \mathcal{M}(\hat{G}) \) is called Compatible if for each \( \pi \in \hat{G}, \Phi(\pi) \in M_{d_\pi}(\mathbb{C}) \). Here \( M_{d_\pi}(\mathbb{C}) \) denotes the set of all \( d_\pi \times d_\pi \) complex matrices after identifying with \( B(H_\pi) \) for each \( \pi \).

Recall that the set

\[
\tilde{S}(G) = \{ \gamma : \hat{G} \to \cup_{\pi} M_{d_\pi}(\mathbb{C}) \}
\]

is a regular semigroup and let

\[
\Delta(G) = \{ \gamma : \hat{G} \to \cup_{\pi} M_{d_\pi}(\mathbb{C}), \gamma \text{ compatible.} \}
\]

[1] The problem under investigation is the algebraic regularity of the following semigroups and finding the maximal regular subsemigroup of \( \hat{P}(G) \).

Since the non-commutative Fourier transform is an isomorphism, it is clear that \( \hat{P}(G) \) is not algebraically regular.

[2] The regularity of the associated semigroup \( \tilde{S}(G) \).

[3] The regularity of the semigroup \( \Delta(G) \). Observe that these semigroups are related as follows.

\[
\hat{P}(G) \subset \Delta(G) \subset \tilde{S}(G).
\]

Theorem 4.1. Let \( G \) be a compact topological group Then \( \tilde{S}(G) \) and \( \Delta(G) \) are regular semigroups.

Proof. It is well known that \( (S)'(G) \) and \( \Delta(G) \) are semi-groups. In either case regularity is easy to establish, as shown below. Let \( \gamma \in \tilde{S}(G) \) (or \( \Delta(G) \)). For each \( \pi \in \hat{G} \) let \( \gamma\dagger(\pi) \) be the Moore-Penrose inverse of \( \gamma(\pi) \). Clearly \( \gamma\dagger(\pi) \in \mathcal{M}(\hat{G}) \). If \( \gamma \in \Delta(G) \) so is \( \gamma\dagger \). \( \square \)

5. Minimal Regular Semigroups Containing \( P(G) \)

Next we consider the problem whether there are regular semigroups \( \hat{\Delta}(G) \) such that

\[
\hat{P}(G) \subset \hat{\Delta}(G) \subset \Delta(G).
\]

We restrict our attention to compact Lie groups \( G \) where new techniques such as Log-Ng positivity \([1]\) are available which is defined as follows:
Definition 5.1. A compatible function $\gamma : \hat{G} \to M$ is called Lo-Ng positive if
\begin{equation}
\sum_{\pi \in \Omega} d_{\pi} tr(\pi(g) \gamma(\pi) B(\pi)) \geq 0
\end{equation}
whenever
\begin{equation}
\sum_{\pi \in \Omega} d_{\pi} tr(\pi(g) B(\pi)) \geq 0
\end{equation}
for all $g \in G$.

Theorem 5.2. (Theorem 4.3.2, The Lo-Ng Criterion[1]) Let $P(G)$ denote the class of regular probability measures on a compact Lie group $G$ and $\gamma : \hat{G} \to M(G)$ be a compatible mapping. Then $\gamma = \hat{\mu}$ if and only if $\gamma$ is Lo-Ng positive namely
\begin{equation}
h_n(g) = \sum_{\pi \in S_n} z_{\pi}^{(n)} d_{\pi} tr(\pi(g) \gamma(\pi)) \geq 0
\end{equation}
for all $g \in G$, where $\#(S_m), \#(S_n) < \infty$ if $m < n$ and $\pi_0 \in S_n$ for all $n$.

Remark 5.3. [1] The above theorem is a non-commutative analogue of the celebrated Bochkner’s theorem: Let $G$ be a locally compact abelian group and $\hat{G}$ be the dual group of characters. Let $F : \hat{G} \to \mathbb{C}$. Then $F$ is the Fourier transform of a measure $\mu$,
\begin{equation}
\hat{\mu}(\chi) = \int_G \chi(g) \mu(dg) F(x_i - x_j) \geq 0 \text{ if and only if } F(\hat{e}) = 1, F \text{ is continuous at } \hat{e}.
\end{equation}
\begin{equation}
\hat{\mu}(\pi) \psi = \int_G \pi(g^{-1}) \psi \mu(dg), \pi \in \hat{G}.
\end{equation}
[2] Therefore $\hat{\mu}^\dagger$ is Lo-Ng positive , then $\mu$ has a generalised inverse in $P(G)$.

6. A FEW MORE RELATED QUESTIONS

Let $\Omega(G)$ be the semi-group of all finite products of idempotents in $P(G)$. There are two questions associated with this.

[1] Is $\Omega(G)$ regular?. If so
[2] Is $\Omega(G)$ the maximal regular semigroup contained in $P(G)$ ?.
[3] What are the regular elements in $P(G)$?

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