Lattice study of sphaleron transitions in a 2D O(3)
sigma model

J. Kripfganz\(^1\) and C. Michael\(^2\)

\( ^1\) Fakultät für Physik, Universität Bielefeld, Germany

\( ^2\) DAMTP, University of Liverpool, Liverpool, L69 3BX, U.K.

Abstract

A lattice approach is developed to measure the sphaleron free energy. Its feasibility is demonstrated through a Monte Carlo study of the two-dimensional O(3) sigma model.

In the electroweak standard model, baryon number (or, more precisely, \(B + L\)) is violated by an anomaly \(^1\). Baryon number violation is associated with transitions between different, topologically distinct vacua. These vacua are separated by an energy barrier. Sphalerons \(^2\) are classical solutions corresponding to saddle points on top of the barrier. The lowest-energy sphaleron has an energy of the order of 10 TeV (somewhat depending on the Higgs mass). This energy is concentrated in a region of size \(m_W^{-3}\).

The tunneling rate between nonequivalent vacua is tiny, due to the small electroweak gauge coupling. Therefore, baryon number violation in the standard model was not considered to be of practical importance for some time. Later, however, it was realized that the energy barrier could be overcome more easily by classical transitions instead of tunneling. Classical transitions become relevant at high temperature \(^3\), or perhaps in high energy scattering \(^4, 5\).

The case of high energy scattering is still badly understood (see e.g. Ref. \(^6\) for a recent review). The case of high temperature is extremely interesting because anomalous baryon number violation could provide a scenario for the generation of the baryon number of the universe.

In the high temperature phase with restored symmetry (B+L) transitions are believed to be frequent, with a rate \(^7, 8, 9, 10\)

\[ \Gamma = \gamma (\alpha_w T)^4 \]  

per unit volume, where \(\gamma\) is some constant of order one. This would have the striking consequence that any (B+L) asymmetry generated at a scale close to the Planck mass would be washed out completely. Where would the baryon asymmetry of the universe come from in that case?
One very attractive possibility would be the generation of a (B+L) asymmetry at the electroweak phase transition \[11, 12, 13, 14, 15, 16\]. This would be possible if the electroweak phase transition would be strongly first order. Presumably, this requires a more complicated Higgs sector (not just one doublet), which would be an interesting prediction. The electroweak phase transition is intensively studied by perturbative \[17, 18\] as well as lattice techniques \[19, 20\].

The survival of such a (B+L) asymmetry after the electroweak phase transition is a non-trivial matter, however. It could be washed out afterwards by transitions across the sphaleron. The corresponding condition on the sphaleron free energy \(F_{sp}(T)\) is found by comparing the transition rate with the expansion rate of the universe.

\[
\frac{F_{sp}(T_c)}{T_c} \geq 45,
\]

where \(T_c\) is, strictly speaking, not \(T_{crit}\), but the temperature at which the phase transition is completed. In the case of a second order or weakly first order phase transition this relation will not be satisfied.

The trouble is that the temperature dependence of the sphaleron free energy is not very well known, in particular close to the critical temperature. Perturbation theory is plagued with severe infrared divergences in this regime. This should also apply to the treatment of fluctuations around the sphaleron. The usual way of estimating the sphaleron free energy is by rescaling, i.e. replacing the zero-temperature Higgs expectation value by \(v(T)\), obtained from the temperature dependent effective potential. This is correct to leading order but could be quite misleading because of infrared singularities of higher loop contributions \[21\]. The role of the effective potential itself is also obscure because the sphaleron receives its energy mostly from the non-convex region which is unphysical.

Therefore a direct determination of the sphaleron free energy, i.e. the transition rate, is an important task. The sphaleron free energy is directly related to the effective potential of the Chern-Simons number. A fraction of configurations with CS number close to \(\frac{1}{2}\) will contribute to the anomalous processes in question. The basic objective is therefore to measure the probability of configurations close to the sphaleron, and derive the sphaleron free energy.

Determining the probability distribution of the CS number requires a reliable way of measuring it. In a typical Monte Carlo configuration, quantum fluctuations (completely unrelated to topological features) may easily add up to produce a CS number of, e.g., 0.5. A straightforward measurement of the CS distribution would therefore overestimate the rate of fermion number violation. By focussing on the known properties of the sphaleron, it is possible to select configurations which look like an underlying classical sphaleron with added quantum fluctuations. Even so it is necessary to check the stability of such assignments. This situation is well known from lattice estimates of the topological susceptibility of pure Yang-Mills theory. In that case, cooling techniques \[22, 23\] have been shown to be useful. Instantons show up as long-lived states.
in this way. This method cannot be taken over without modification to measure the CS density, however, because sphalerons are not stable classical solutions like instantons, but saddle points. They cannot be found by simply minimizing the action. A different smoothing procedure is needed. We propose to use the square of the tadpole (i.e. the square of the equation of motion) as a new ‘action’ density of the cooling procedure. Any classical solution, stable or not, will now appear as an attraction point

In order to test such an algorithm one should use a simple model, where the magnitude of topological transitions is sufficiently understood. The Abelian Higgs model in 1+1 dimensions would be a possible candidate. We prefer the study of the 1+1 dimensional O(3) sigma model with some external magnetic field, because of its similarities with the 4-dimensional standard model. The action is

$$S = \frac{1}{g^2} \int d^2x \left[ \frac{1}{2} \partial_\mu \vec{n} \cdot \partial_\mu \vec{n} + \omega^2 (1 + n_3) \right]$$

with the constraint

$$\vec{n}^2(x) = 1$$

Without the external field $\omega$, the theory is asymptotically free and possesses instanton solutions. Symmetry breaking due to the external field removes the instantons as true solutions, but sphalerons as unstable saddle point solutions appear, just like in the SU(2) Higgs theory. Topological transitions in this model have been studied in quite some detail by Mottola and Wipf. The one-loop expression for the transition rate has been worked out and is expected to be reliable at weak coupling (this is not obvious in the standard model because of infrared problems near the phase transition). To exponential accuracy the transition rate is given by

$$\Gamma \simeq e^{-F_{SP}(T)/T}$$

where the sphaleron free energy is found to be

$$F_{SP}(T) = \frac{8\omega}{g_R(T)}$$

To one-loop order, the renormalized coupling $g_R$ at scale $T$ is expressed in terms of the bare (lattice) coupling $g$ as

$$\frac{1}{g^2_R(T)} \simeq \frac{1}{g^2} - \frac{1}{2\pi} \log N_t$$

$N_t$ is the lattice size in the (Euclidean) time direction. In order to test our lattice approach we should verify the quasiclassical prediction for the exponential slope in $\frac{1}{g^2}$

$$\frac{F_{SP}(T)}{T} \simeq (8\omega N_t) \frac{1}{g^2} + \text{const}$$

if $g^2$ is small enough. A similar computation for the standard model would be sufficient to control at least the order of magnitude of sphaleron transitions close to the electroweak phase transition and this would improve our knowledge of this transition rate considerably.
In order to determine the sphaleron free energy we shall measure the probability distribution of the CS number (after cooling). The CS number itself is given by

\[ N_{CS} = \frac{1}{2\pi} \int dx A_1 \]  

(9)

with the vector potential

\[ A_\mu = \partial_\mu \alpha - \sin^2 \theta/2 \partial_\mu \varphi \]  

(10)

where \( \alpha \) is an arbitrary gauge parameter. Convenient choices would be \( \alpha = 0 \) for fields close to the vacuum at \( \theta = \pi \), or \( \alpha = \varphi \) for fields in the upper hemisphere. For a study of the sphaleron configurations the latter choice is appropriate.

In our lattice approach we do not use this expression directly but determine \( A_\mu \)

\[ A_\mu = \frac{1}{2\pi}(\chi^+ \partial_\mu \chi - (\partial_\mu \chi^+)\chi) \]  

(11)

through the corresponding \( CP(1) \) variables

\[ \chi = e^{i\alpha} \begin{pmatrix} \sin \theta/2 e^{-i\varphi} \\ \cos \theta/2 \end{pmatrix} \]  

(12)

\( A_\mu \) is found as the phase of

\[ \chi_1^+ \chi_2 \simeq e^{i \cos^2 \theta/2 \partial_\mu \varphi} \]  

(13)

where \( \chi_1 \) and \( \chi_2 \) are on neighbouring sites of link \( \mu \).

The classical equations of motion are given by

\[ \vec{L}(x) \equiv \partial^2 \vec{n} - \omega^2 \vec{\delta}_3 = \lambda \vec{n} \]  

(14)

with

\[ \vec{\delta}_3 = (0, 0, 1) \]  

(15)

\( \lambda \) is a Lagrange multiplier to ensure the constraint. The basic point to observe is that for any classical solution \( \vec{L} \) is parallel to \( \vec{n} \). Therefore, the quantity

\[ D(x) = \vec{L}(x) \cdot \vec{L}(x) - [\vec{n}(x) \cdot \vec{L}(x)]^2 \geq 0 \]  

(16)

vanishes for any classical solution (including saddle points), and is positive otherwise. The integral over \( D(x) \) is therefore a convenient new ‘action’ for defining a cooling procedure which does not drive away configurations from saddle points. This procedure can be taken over directly to the SU(2) variables (Higgs as well as gauge part) of the electroweak standard model.

The model was discretized in the standard way on a \( N_t \times 128 \) lattice. This corresponds to a temperature \( 1/N_t \). We used a heat-bath algorithm to equilibrate the lattice for 36000000
sweeps and then a ratio of 8 over-relaxation updates to one heat-bath. We then investigated 10 blocks of 36000000 sweeps with 20000 configurations measured per block. The averages of each of these 10 block measurements were found to be consistent with being statistically independent and so were used for an estimate of the statistical error. The mass gap was measured from the propagation in the \( x \)-direction, namely from the study of the time-slice averages of the correlation \( n_1(x)n_1(x') + n_2(x)n_2(x') \).

To isolate configurations which had a sphaleron, we first required that \( n_3(x) \) was above 0.5 on average for a region of \( x \) of length \( 1/\omega \). For those configurations, we measured \( N_{CS} \) in a window from \( x_{\text{max}} - 1/(2\omega) \) to \( x_{\text{max}} + 1/(2\omega) \), where \( x_{\text{max}} \) corresponded to the maximum value of \( n_3(x) \). Then configurations with \( N_{CS} > 0.45 \) and \( n_3(x_{\text{max}}) > 0.90 \) were classified as sphalerons. In order to check the stability of this procedure, we cooled the configurations until the average value per link of \( D \) was reduced to a fixed value. Then we used the above selection criteria on these cooled configurations. The results presented correspond to a reduction of \( D \) by a factor of approximately 10, although we found that the sphaleron probability was insensitive to this threshold value of \( D \) (being the same within errors for smaller \( g^2 \)). Our cooling algorithm was to replace site variables by an admixture of the original value and that which would minimise the action locally. If this procedure reduced \( D \) (see above) then it was accepted - otherwise \( D \) was explicitly minimised which was computationally more demanding since it involved next-to-nearest neighbour terms etc.

We tried a large number of other choices of sphaleron selection criteria - varying thresholds, windows, cooling rate, cooling duration etc and found that the changes amounted to an overall constant shift only. Thus the overall normalisation of the values for \( P_{\text{sph}} \) quoted in the table is not significant, but the dependence on \( g^2 \) and \( N_t \) is.

Results for the sphaleron probability \( P_{\text{sph}} \) per configuration as well as for the mass gap are given in Table 1. We used \( \omega = 0.1 \) in lattice units for \( N_t = 2 \) and then varied \( N_t \) keeping \( \omega N_t \) fixed to check that the results were consistent. Thus it is appropriate to quote the sphaleron probability per unit spatial length (in units of \( \omega \)). Values of \( P_{\text{sph}}/(128\omega) \) are plotted versus \( 1/\omega \) in Figure 1. Here it is seen that the results for different lattice spacings (ie different \( \omega \)) are all in agreement with each other. Thus we have determined the dependence on \( g^{-2} \) of the sphaleron production rate per unit length at \( \omega/T = 0.2 \). This dependence is indeed comparable to the quasi-classical expression which is shown by the continuous line (with arbitrary normalisation - actually \( 1.87g^{-2}\exp(-8\omega/(g^2T)) \)).

The quasi-classical calculation of fluctuations around the classical spaleron gives a result for the spaleron probability \( \delta P_{\text{sph}} \) which is expected to be valid if \( g^2T < \omega \). Our results at \( g^2 < 0.2 \) are thus expected to be approximately reproduced by this approach. As shown in the figure, this is indeed the case.

This close agreement seems a little fortuitous. One reason is that a perturbative expansion in terms of lattice parameters does not work very well in the present parameter range, but only at much smaller coupling. Thus the measured mass gap and the renormalized coupling should be used instead of \( \omega \) and \( g^2 \) in eq.(8). So far, we have not measured the renormalized coupling and
cannot make an estimate of the size of this effect. A somewhat smaller value for the sphaleron free energy might also be expected because of the finite range of attraction of the sphaleron in our cooling procedure. However, we find very consistent results as the extent of the smoothing is varied over a wide range. This suggests that such an effect should be small.

In conclusion, we have demonstrated a method for measuring Chern–Simons transitions which can be carried over to study the bosonic sector of the electroweak standard model. This will allow checking various estimates based on resummed perturbation theory for effective potentials. The outcome is crucial for judging the viability of approaches for generating the baryon number of the universe at the electroweak phase transition.

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Table 1: Monte Carlo results

| $N_t$ | $g^2$ | $P_{sph}$       | mass gap |
|-------|-------|-----------------|----------|
| 2     | 0.12  | 0.000338( 23)   | 0.108    |
| 2     | 0.13  | 0.000770( 46)   | 0.110    |
| 2     | 0.14  | 0.001560( 85)   | 0.111    |
| 2     | 0.15  | 0.002945( 70)   | 0.113    |
| 2     | 0.16  | 0.004790( 173)  | 0.116    |
| 2     | 0.20  | 0.018600( 780)  | 0.128    |
| 2     | 0.24  | 0.035100(1036)  | 0.145    |
| 2     | 0.25  | 0.041700(1121)  | 0.150    |
| 3     | 0.13  | 0.000580( 78)   | 0.073    |
| 3     | 0.14  | 0.001140(105)   | 0.074    |
| 3     | 0.15  | 0.002010(112)   | 0.075    |
| 4     | 0.13  | 0.000518( 60)   | 0.055    |
| 4     | 0.14  | 0.000800( 70)   | 0.056    |
| 4     | 0.15  | 0.001575(125)   | 0.057    |
| 4     | 0.16  | 0.003062(187)   | 0.058    |
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Figure 1: The probability of a sphaleron-like configuration per unit spatial length as a function of $1/g^2$. 