Invariants of knot diagrams

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Received: 3 September 2007 / Revised: 18 February 2008 / Published online: 1 April 2008
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Abstract We construct a new order 1 invariant for knot diagrams. We use it to determine the minimal number of Reidemeister moves needed to pass between certain pairs of knot diagrams.

Mathematics Subject Classification (2000) 57M25

1 Introduction

Oriented knots in \( \mathbb{R}^3 \) are usually represented by knot diagrams. Projecting a knot to a plane or 2-sphere in a generic direction gives an immersed oriented planar or spherical curve with finitely many double points, or crossings. A knot diagram is obtained by marking a neighborhood of each crossing to indicate which strand lies above the other. The higher strand is called the overcrossing and the lower one the undercrossing. Starting with a knot diagram, one can recover the original knot up to isotopy by constructing a curve with the overcrossing arcs pushed slightly above the plane of the diagram.

A central issue is to determine whether two knot diagrams represent the same knot, i.e., whether the curves in \( \mathbb{R}^3 \) corresponding to each diagram are isotopic. If they represent the same knot we say that the two diagrams are equivalent. Alexander and
Briggs [1] and independently Reidemeister [14] showed that equivalent diagrams can be connected through isotopy and a series of three types of moves, usually referred to as Reidemeister moves. The number of such moves required to connect two equivalent diagrams, is difficult to estimate. An exponential upper bound for the number of Reidemeister moves required to connect two equivalent diagrams is obtained in [7]. We can get some lower bounds by looking at crossing numbers, writhes and winding numbers of diagrams since each Reidemeister move changes these numbers by 0, 1 or 2. Less obvious bounds are obtained in [3,9]. See also [4–6,11].

In this paper we define a new family of knot diagram invariants, and focus on one of them in particular. We found these invariants by following the program of Arnold and Vassiliev for finite order invariants. Our invariants are of order one.

As an application, for each \( n \) we present two diagrams \( D_n, E_n \) for the unknot, each with \( 2n + 1 \) crossings. For these two diagrams, which are almost identical, the writhe, cowrithe, crossing number and winding number give a lower bound of 2 for the number of Reidemeister moves required to pass from one to the other. Using our new invariant, we show that the minimal number of Reidemeister moves required to pass from \( D_n \) to \( E_n \) is \( 2n + 2 \). We also obtain restrictions on which Reidemeister moves may appear in any sequence of Reidemeister moves which realizes this minimum.

Our invariant takes values in a very large abelian group. It is natural to investigate \( \mathbb{Z} \) valued invariants obtained by composing it with homomorphisms into \( \mathbb{Z} \). The “cowrithe” introduced in [9] is obtained in this way. We obtain a relation of the cowrithe to Arnold’s spherical curve invariants and the Alexander–Conway polynomial, which clarifies the limitations of the cowrithe for studying Reidemeister moves.

In [8] we apply our invariant to give the first non-linear lower bound for the number of Reidemeister moves needed for unknotting. We construct a sequence of diagrams of the unknot for which the minimum number of Reidemeister moves required to pass to the trivial diagram is quadratic with respect to the number of crossings.

2 The invariant

In what follows we consider two different types of geometric objects. The first objects, which are the subject of study in this paper, are knot diagrams in \( S^2 \). Two such diagrams are considered the same if they differ by an ambient isotopy of \( S^2 \). We denote the set of all such diagrams by \( \mathcal{D} \). Our goal is to construct invariants of knot diagrams. Towards that end we construct from a diagram a second geometric object, namely a two component link in \( \mathbb{R}^3 \). This is a smooth embedding of \( S^1 \sqcup S^1 \) in \( \mathbb{R}^3 \). Two such embeddings are considered the same if they differ by an ambient isotopy of \( \mathbb{R}^3 \). We denote the set of all two component links by \( \mathcal{L} \), and the term links in this paper always refers to two component links.

Our basic construction relating knot diagrams to links is the following. Given a knot diagram \( D \in \mathcal{D} \) and a crossing \( a \) in \( D \), define the smoothing \( D^a \in \mathcal{L} \), to be the link obtained by smoothing the crossing \( a \), i.e., performing a cut and paste on the four strands at the crossing that preserves the orientation of the arcs. The smoothing operation is independent of the orientation of the curve, since reversing orientation results in a change of orientation of both strands at the crossing. The diagram resulting