CONSTRUCTION OF ETERNAL SOLUTIONS FOR A SEMILINEAR PARABOLIC EQUATION

MICHAEL ROBINSON

Abstract. Eternal solutions of parabolic equations (those which are defined for all time) are typically rather rare. For example, the heat equation has exactly one eternal solution – the trivial solution. While solutions to the heat equation exist for all forward time, they cannot be extended backwards in time. Nonlinearities exasperate the situation somewhat, in that solutions may form singularities in both backward and forward time. However, semilinear parabolic equations can also support nontrivial eternal solutions. This article shows how nontrivial eternal solutions can be constructed for a semilinear equation that has at least two distinct equilibrium solutions. The resulting eternal solution is a heteroclinic orbit which connects the two given equilibria.

1. Introduction

Consider
\begin{equation}
\frac{\partial u(t, x)}{\partial t} = \Delta u(t, x) + \sum_{i=0}^{N} a_i(x)u^i(t, x),
\end{equation}
where \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^n \), and the \( a_i \) are bounded and smooth. In this article, we consider eternal solutions, those classical solutions \( u \) which satisfy (1) for all time \( t \in \mathbb{R} \).

This kind of equation provides a simple model for a number of physical phenomena. First, choosing the right side to be \( \Delta u - u^2 + a_1u \) results in an equation which can represent a model of the population of a single species with diffusion and a spatially-varying carrying capacity, \( a_1(x) \). As a second application, this equation is a very simple model of combustion. If \( a_1 \) is a positive constant, then the equation supports traveling waves. Such traveling waves can model the propagation of a flame through a fuel source.

Equations of the form (1) have been of interest to researchers for quite some time. Existence and uniqueness of solutions on short time intervals (on strips \((-t_0, t_0) \times \mathbb{R}^n\)) can be shown using semigroup methods and are entirely standard [15]. However, there are obstructions to the existence of eternal solutions. Aside from the typical loss of regularity due to solving the backwards heat equation, there is also a blow-up phenomenon which can spoil existence in the forward-time solution to (1). Blow-up phenomena in the forward time Cauchy problem (where one does not consider \( t < 0 \)) have been studied by a number of authors [6] [3] [13] [9] [2] [17] [18]. More recently, Zhang et al. ([10] [12] [14]) studied global existence for the

1991 Mathematics Subject Classification. 35B40,35K55.
Key words and phrases. eternal solution, heteroclinic connection, semilinear parabolic equation, equilibrium.
forward Cauchy problem for

\[ \frac{\partial u}{\partial t} = \Delta u + u^p - V(x)u \]

for positive \( u, V \). Du and Ma studied a related problem in [4] under more restricted conditions on the coefficients but they obtained stronger existence results. In fact, they found that all of the solutions which were defined for all \( t > 0 \) tended to equilibrium solutions.

Eternal solutions to (1) are rather rare. Most works which describe blow-up make the assumption that the solution is positive. Unfortunately, blow-up is much more difficult to characterize in the general situation, and understanding exactly what kind of initial conditions are responsible for blow-up in the Cauchy problem for (1) is an important part of the question.

As an aside, the boundary value problem that results from taking \( x \in \Omega \subset \mathbb{R}^n \) for some bounded \( \Omega \) (instead of \( x \in \mathbb{R}^n \)) has also been discussed extensively in the literature [7] [8] [3]. For the boundary value problem, all bounded forward Cauchy problem solutions tend to limits as \( |t| \to \infty \), and these limits are equilibrium solutions.

The existence of eternal solutions is a difficult problem, because the backward-time Cauchy problem is well known to be ill-posed. Obviously, equilibrium solutions are trivial examples of such eternal solutions, and in [10] it was shown that they can exist. It is not at all clear that there are other eternal solutions, and indeed there may not be. In this article we assume the existence of a pair of nonintersecting equilibrium solutions and construct a heteroclinic orbit which connects them. (A heteroclinic orbit is a special kind of eternal solution, whose limits as \( t \to \pm \infty \) are equilibria.)

For simplicity and concreteness, we will work with the more limited equation

\[ \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - u^2(t,x) + \phi(x), \]

where \( \phi \) is a smooth function which decays to zero. We follow the general technique for constructing “ancient solutions,” which was used in a different context by Hamilton and Perelman. It should be emphasized that the technique examined in this article can handle the problem for (1) in full generality (under mild decay assumptions for the coefficients \( a_i \)), though this complicates the exposition needlessly.

This simpler model still provides insight into applications, as it is still a model of the population of a single species, with a spatially-varying carrying capacity, \( \phi \). Indeed, one easily finds that under certain conditions the behavior of solutions to (2) is reminiscent of the growth and (admittedly tenuous) control of invasive species [1]. It is the control of invasive species that is of most interest, and it is also what the structure of the space of heteroclines describes. In one of the examples given in [10], there is one “more stable” equilibrium, and several other “less stable” ones. The more stable equilibrium can be thought of as the situation where an invasive species dominates. The task, then, is to try to perturb the system so that it no longer is attracted to that equilibrium. An optimal control approach is to perturb the system so that it barely crosses the boundary of the stable manifold of the the undesired equilibrium, and thereby the invasive species is eventually brought under control with minimal disturbance to the rest of the environment. Such an optimal control approach, though, is beyond the scope of this article.
2. Equilibrium solutions

We choose \( \phi(x) = (x^2 - 0.4)e^{-x^2/2} \). It has been shown in [10], that in this situation, there exists a pair of equilibrium solutions \( f_+, f_- \) with the following properties:

1. \( f_+ \) and \( f_- \) are smooth and bounded,
2. \( f_+ \) and \( f_- \) have bounded first and second derivatives,
3. \( f_+ \) and \( f_- \) are asymptotic to \( 6/x^2 \) for large \( x \), and so both belong to \( L^1(\mathbb{R}) \),
4. \( f_+(x) > f_-(x) \) for all \( x \),
5. there is no equilibrium solution \( f_2 \) with \( f_2(x) > f_+(x) > f_-(x) \) for all \( x \), and additionally, there exists a one-parameter family \( g_c \) of solutions to

\[
0 = g''_c(x) - g'^2_c(x) + \phi_c(x)
\]

with

1. \( c \in [0, 1) \),
2. \( g_0 = f_- \) and \( \phi_0 = \phi \),
3. \( \phi_a(x) < \phi_b(x) \) and \( g_a(x) > g_b(x) \) for all \( x \) if \( a > b \).

The latter set of properties can occur as a consequence of the specific structure of \( f_- \). For instance, consider the following result.

**Proposition 1.** Suppose \( f_- \in C^{2,\alpha}(\mathbb{R}) \) satisfies the above conditions and additionally, there is a compact \( K \subset \mathbb{R} \) with nonempty interior such that \( f_- \) is negative on the interior of \( K \) and is nonnegative on the complement of \( K \). Then such a family \( g_c \) above exists.

**Proof.** (Sketch) Work in \( T_{f_-}C^{2,\alpha}(\mathbb{R}) \), the tangent space at \( f_- \). Then (3) becomes its linearization (for \( h_c \), say), namely

\[
0 = h''_c(x) - 2f_-(x)h_c(x) + (\phi_c - \phi).
\]

Consider the slightly different problem,

\[
0 = y''(x) - 2f_-(x)y(x) + v(x)y(x),
\]

where \( v \) is a smooth function to be determined. If we can find a \( v \leq 0 \) such that \( y > 0 \) and \( y \to 0 \) as \( |x| \to \infty \), then we are done, because we simply let \( vy = \phi_c - \phi \) in (3). In that case, \( h_c = y \) has the required properties. We sketch why such a \( v \) exists:

- If \( v \equiv 0 \), then \( y \equiv 0 \) is a solution, giving \( g_c = f_- \) as a base case.
- If \( v(x) = -2\|u\|_\infty \beta(x) \) for \( \beta \) is a smooth bump function with compact support and \( \beta|K = 1 \), then the Sturm-Liouville comparison theorem implies that \( y \) has no sign changes. We can take \( y \) strictly positive. However, in this case, the Sturm-Liouville theorem implies that there are no critical points of \( y \) either, so \( y \) may not tend to zero as \( |x| \to \infty \).
- Hence there should exist an \( s \) with \( 0 < s < 2\|u\|_\infty \) such that if \( v(x) = -s\beta(x) \), then \( y \) has no sign changes, one critical point, and tends to zero as \( |x| \to \infty \). This choice of \( v \) is what is required. (The precise details of this argument fall under standard Sturm-Liouville theory, which are omitted here.)

\[\square\]
In what follows, we shall not be concerned with the exact form of \( \phi \), but rather we shall assume that the above properties of the equilibria hold. Many other choices of \( \phi \) will allow a similar construction.

**Lemma 2.** The set

\[
W = \{ v \in C^2(\mathbb{R}) | f_-(x) < v(x) < f_+(x) \text{ for all } x \}
\]

is a forward invariant set for (2). That is, if \( u \) is a solution to (2) and \( u(t_0) \in W \), then \( u(t) \in W \) for all \( t > t_0 \).

**Proof.** We show that the flow of (2) is inward whenever a timeslice is tangent to either \( f_- \) or \( f_+ \). To this end, define the set \( B \)

\[
B = \{ v \in C^2(\mathbb{R}) | f_-(x) < v(x) < f_+(x) \text{ for all } x, \text{ and there exists an } x_0 \text{ such that } v(x_0) = f_+(x) \text{ or } v(x_0) = f_-(x) \}. \]

Without loss of generality, consider a \( v \in B \) with a single point of tangency, \( v(x_0) = f_-(x_0) \). At such a point \( x_0 \), the smoothness of \( v \) and \( f_- \) implies that \( \Delta v(x_0) \geq \Delta f_-(x_0) \) using the maximum principle. Then, if \( u \) is a solution to (2) with \( u(0, x) = v(x) \), we have that

\[
\frac{\partial u(0,x_0)}{\partial t} = \Delta v(x_0) - v^2(x_0) + \phi(x_0) \geq \Delta f_-(x_0) - f_-^2(x_0) + \phi(x_0) = 0,
\]

hence the flow is inward. One can repeat the above argument for each point of tangency, and for tangency with \( f_+ \) as well. \( \square \)

**Lemma 3.** Solutions to the Cauchy problem

\[
\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= \frac{\partial^2 u(t,x)}{\partial x^2} - u^2(t,x) + \phi(x), \\
u(0,x) &= U(x) \in W_c
\end{aligned}
\]

where

\[
W_c = \{ v \in C^2(\mathbb{R}) | g_-(x) < v(x) < f_+(x) \text{ for all } x \}
\]

for \( c \in [0,1) \) have the property that they lie in \( L^1 \cap L^\infty(\mathbb{R}) \) for all \( t > 0 \). We shall assume that \( U \) has bounded first and second derivatives.

Additionally, when \( c \in (0,1) \), solutions to (7) cannot have \( f_- \) as a limit as \( t \to \infty \).

**Proof.** The fact that solutions lie in \( L^1 \cap L^\infty(\mathbb{R}) \) is immediate from Lemma 2 and the asymptotic behavior of \( f_+, f_- \). (Section 4 of [10].) Observe that for each \( c \in [0,1) \), \( W_c \) is forward invariant, and that \( W_a \subset W_b \) if \( a > b \). Since \( f_- \) is not in \( W_c \) for \( c \) strictly larger than 0, the proof is completed. \( \square \)

The following is an outline for the rest of the article. We show that all solutions to (7) have bounded first and second spatial derivatives. This implies that all of their first partial derivatives are bounded (the time derivative is controlled by (2)). Using the fact that (2) is autonomous in time, time translations of solutions are also solutions. We therefore construct a sequence of solutions \( \{ u_k \} \) to Cauchy problems started at \( t = 0, T_1, T_2, ... \) which tend to \( f_+ \) as \( t \to +\infty \), but their initial conditions tend to \( f_- \) as \( k \to \infty \). By Ascoli’s theorem, this sequence converges uniformly on compact subsets to a continuous eternal solution.
3. Integral equation formulation

In order to estimate the derivatives of a solution to (7), it is more convenient to work with an integral equation formulation of (7). This is obtained in the usual way.

\[
\frac{\partial u}{\partial t} = \Delta u - u^2 + \phi
\]

\[
\left(\frac{\partial}{\partial t} - \Delta\right) u = -u^2 + \phi
\]

\[
u = \left(\frac{\partial}{\partial t} - \Delta\right)^{-1}(\phi - u^2)
\]

(8)

\[u(t, x) = \int_{-\infty}^{\infty} H(t, x - y)U(y)dy + \int_{0}^{t} \int_{-\infty}^{\infty} H(t - s, x - y) \left(\phi(y) - u^2(s, y)\right) dy ds,
\]

where \(H(t, x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\) is the usual heat kernel.

**Calculation 4.** We begin by estimating the first derivative of \(u\) for a short time. Let \(T > 0\) be given, and consider \(0 \leq t \leq T\). The key fact is that \(\int H(t, x)dx = 1\) for all \(t\). Using (8)

\[
\left\|\frac{\partial u}{\partial x}\right\|_{\infty} \leq \left\|\frac{\partial U}{\partial x}\right\|_{\infty} + \int_{0}^{t} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} H(t - s, x - y) \left(\phi(y) - u^2(s, y)\right) dy ds
\]

\[
\leq \left\|\frac{\partial U}{\partial x}\right\|_{\infty} + \int_{0}^{t} \int_{-\infty}^{\infty} \frac{\partial}{\partial y} (H(t - s, x - y)) \left(\phi(y) - u^2(s, y)\right) dy ds
\]

\[
\leq \left\|\frac{\partial U}{\partial x}\right\|_{\infty} + \int_{0}^{t} \int_{-\infty}^{\infty} \rho(t, x - y) \left(\frac{\partial}{\partial y} \phi(y) - 2u \frac{\partial u}{\partial y}\right) dy ds
\]

\[
\leq \left\|\frac{\partial U}{\partial x}\right\|_{\infty} + T \left\|\frac{\partial}{\partial x}\right\|_{\infty} + 2\|u\|_{\infty} \int_{0}^{t} \left\|\frac{\partial u}{\partial x}\right\|_{\infty} ds.
\]

This integral equation fence is easily solved to give

\[
\left\|\frac{\partial u}{\partial x}\right\|_{\infty} \leq \left(\left\|\frac{\partial U}{\partial x}\right\|_{\infty} + T \left\|\frac{\partial}{\partial x}\phi\right\|_{\infty}\right) e^{2t \max\{|f_+|, |f_-|\}}
\]

\[
\leq K_1 e^{K_2 T}.
\]

**Calculation 5.** With the same choice of \(T\) as above, we find a bound for the second derivative in the same way:

\[
\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{\infty} \leq \left\|\frac{\partial^2 U}{\partial x^2}\right\|_{\infty} + T \left\|\frac{\partial^2}{\partial x^2}\phi\right\|_{\infty} + \int_{0}^{t} \left\|\frac{\partial}{\partial y} \left(2u \frac{\partial u}{\partial y}\right)\right\|_{\infty} ds
\]

\[
\leq \left\|\frac{\partial^2 U}{\partial x^2}\right\|_{\infty} + T \left\|\frac{\partial^2}{\partial x^2}\phi\right\|_{\infty} + \int_{0}^{t} 2 \left\|\frac{\partial u}{\partial x}\right\|_{\infty}^2 + 2\|u\|_{\infty} \left\|\frac{\partial^2 u}{\partial x^2}\right\|_{\infty} ds
\]

\[
\leq K_3 e^{K_2 T}
\]

for some \(K_3\) which depends on \(U\), \(\phi\), and \(T\).
Calculation 6. Now, we extend Calculation 5 to handle \( t > T \),

\[
\left\| \frac{\partial^2 u}{\partial x^2} \right\|_\infty \leq \left\| \frac{\partial^2 U}{\partial x^2} \right\|_\infty + \left| \frac{\partial^2}{\partial x^2} \int_0^T \int_{-\infty}^{\infty} H(t-s,x-y)(\phi(y) - u^2(s,y))dy \, ds \right| + \left| \frac{\partial^2}{\partial x^2} \int_T^t \int_{-\infty}^{\infty} H(t-s,x-y)(\phi(y) - u^2(s,y))dy \, ds \right|
\]

\[
\leq K_3 e^{K_2 T} + \int_T^t \left\| \frac{\partial^2}{\partial x^2} H(t-s, x) \right\|_\infty \left( \|\phi\|_1 + \|u^2\|_1 \right) \, ds
\]

\[
\leq K_3 e^{K_2 T} + K_4 \int_T^t \frac{1}{s\sqrt{s}} \, ds + K_5^t \int_T^t \frac{1}{s\sqrt{s}} \, ds
\]

\[
\leq K_3 e^{K_2 T} + K_5 \left( \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{T}} \right) + K_5^t \left( \frac{1}{T\sqrt{T}} - \frac{1}{t\sqrt{t}} \right)
\]

\[
\leq K_3 e^{K_2 T} + K_6,
\]

hence there is a uniform upper bound on \( \left\| \frac{\partial^2 u}{\partial x^2} \right\|_\infty \) which depends only on the initial conditions, \( \phi \), and \( T \).

Lemma 7. Let \( f \in C^2(\mathbb{R}) \) be a bounded function with a bounded second derivative. Then the first derivative of \( f \) is also bounded, and the bound depends only on \( \|f\|_\infty \) and \( \|f''\|_\infty \).

Proof. The proof is elementary. The key fact is that at its maxima and minima, \( f \) has a horizontal tangent. From a horizontal tangent, the quickest \( f' \) can grow is at a rate of \( \|f''\|_\infty \). However, since \( f \) is bounded, there is a maximum amount that this growth of \( f' \) can accrue. Indeed, a sharp estimate is

\[
\|f'\|_\infty \leq \sqrt{2}\|f\|_\infty \|f''\|_\infty.
\]

□

Using the fact that \( u \) is bounded, Lemma 7 implies that the first spatial derivative of \( u \) is bounded. By [2], it is clear that the first time derivative of \( u \) is also bounded.

Lemma 8. As an immediate consequence of Lemmas 3 and 7, the action integral

\[
A(u(t)) = \int_{-\infty}^{\infty} \frac{1}{2} \left| \frac{\partial u}{\partial x} \right|^2 + \frac{1}{3} u^3(t, x) - u(t, x)\phi(x) \, dx
\]

is bounded. Therefore, the solutions to the Cauchy problem (7) all tend to limits as \( t \to \infty \) (Corollary 7 of [11]). By Lemma 3, we conclude that they all tend to the common limit of \( f_+ \) when \( c > 0 \).

Proof. The latter two terms are bounded due to the fact that \( u \) lies in \( L^1 \cap L^\infty(\mathbb{R}) \) for all \( t \). The bound on the first term comes from combining the fact that \( u \) and its first two spatial derivatives are bounded with the asymptotic decay of \( f_\pm \), and is otherwise straightforward (use L'Hôpital's rule). □
4. Construction of an Eternal Solution

Let

\[ U_k(x) = (1 - 2^{-k-1})g_{1/k}(x) + 2^{-k-1}f_+(x), \]

for \( k \geq 0 \)

noting that \( U_k \to f_- \) as \( k \to \infty \). Since \( U_k \) is a convex combination of \( f_+ \) and \( g_{1/k} \), it follows that \( U_k \in W^{1/k}_1 \) for all \( k \). Also, since \( f_+ \) and \( f_- \) have bounded first \( \text{ and second derivatives, the } \{U_k\} \) have a common bound for their first and second derivatives.

Now consider solutions to the following set of Cauchy problems

\[
\begin{align*}
\frac{\partial u_k(t,x)}{\partial t} &= \frac{\partial^2 u_k(t,x)}{\partial x^2} - u_k^2(t,x) + \phi(x), \\
u_k(T_k,x) &= U_k(x).
\end{align*}
\]

We choose \( T_k \) so that for all \( k > 0 \), \( u_k(0,0) = u_0(0,0) \). We can do this using the continuity of the solution and Lemma 8. As \( k \to \infty \), solutions are started nearer and nearer to the equilibrium \( f_- \), so we are forced to choose \( T_k \to -\infty \) as \( k \to \infty \).

It’s clear that each solution \( u_k \) is defined for only \( t > T_k \). However, for each compact set \( S \subset \mathbb{R}^2 \), there are infinitely many elements of \( \{u_k\} \) which are defined on it. The results of the previous section imply that \( \{u_k\} \) is a bounded, equicontinuous family. As a result, Ascoli’s theorem implies that \( \{u_k\} \) converges uniformly on compact subsets to a continuous \( u \), which is an eternal solution to (2).

Our constructed eternal solution will have the value \( u(0,0) = u_0(0,0) \), which is strictly between \( f_+ \) and \( f_- \). As a result, the eternal solution we have constructed is not an equilibrium solution. By Lemma 8 it is a finite energy solution, so it must be a heteroclinic orbit connecting \( f_- \) to \( f_+ \).

References

[1] Richard Blaustein. Kudzu’s invasion into Southern United States life and culture. In J. A. McNeeley, editor, The Great Reshuffling: Human dimensions of invasive species, pages 55–62. The World Conservation Union, 2001.

[2] Haim Brezis. Semilinear equations in \( \mathbb{R}^n \) without condition at infinity. Appl. Math. Optim., 12(1):271–282, 1984.

[3] P. Brunovsky and B. Fiedler. Connecting orbits in scalar reaction diffusion equations II: the complete solution. J. Diff. Eq., 81:106–135, 1989.

[4] Yihong Du and Li Ma. Logistic type equations on \( \mathbb{R}^n \) by a squeezing method involving boundary blow-up solutions. J. London Math. Soc., 2(64):107–124, 2001.

[5] K. Kobayashi et al. On the blowing up problem for semilinear heat equations. J. Math. Soc. Japan, 29:407–424, 1977.

[6] Hiroshi Fujita. On the blowing up of solutions of the Cauchy problem for \( u_t = \Delta u + u^{1+\alpha} \). Tokyo University Faculty of Science Journal, 13:109–124, December 1966.

[7] Dan Henry. Geometric Theory of Semilinear Parabolic Equations. Springer-Verlag, New York, 1981.

[8] Jürgen Jost. Partial Differential Equations. Springer, New York, 2007.

[9] S. Klainerman. Long-time behavior of solutions to nonlinear evolution equations. Arch. Rat. Mech. Anal., 78:73–98, 1982.

[10] Michael Robinson. An asymptotic-numerical approach for examining global solutions to an ordinary differential equation, (to appear) arxiv:0709.4664. Ergodic Theory and dynamical systems, 2007.

[11] Michael Robinson. Classification of connecting solutions of semilinear parabolic equations, arxiv:0709.2705. 2007.

[12] Phillipe Souplet and Qi Zhang. Stability for semilinear parabolic equations with decaying potentials in \( \mathbb{R}^n \) and dynamical approach to the existence of ground states. Ann. I. H. Poincaré, 19(5):683–703, 2002.
[13] F. B. Weissler. Existence and nonexistence of global solutions for a semilinear heat equation. *Israel J. Math.*, 38:29–40, 1981.

[14] James Wrkich and Qi Zhang. Solvability conditions for some semi-linear parabolic equations (preprint). 2007.

[15] Eberhard Zeidler. *Nonlinear functional analysis and its applications II/A: Linear monotone operators*. Springer-Verlag, New York, 1990.

[16] Qi Zhang. Semilinear parabolic equations on manifolds and applications to the non-compact Yamabe problem. *Electron. J. Differential Equations*, 46:1–30, 2000.

[17] Songmu Zheng. Remarks on global existence for nonlinear parabolic equations. *Nonlinear Analysis*, 1:107–114, 1986.

[18] Songmu Zheng. *Nonlinear parabolic equations and hyperbolic-parabolic coupled systems*. Longman Group, New York, 1995.

Center for Applied Mathematics, 657 Rhodes Hall, Cornell University, Ithaca, NY 14850

*E-mail address:* robinm@cam.cornell.edu