Equivalence of critical and subcritical sharp Trudinger-Moser inequalities in fractional dimensions and extremal functions*

José Francisco de Oliveira  
Department of Mathematics  
Federal University of Piauí  
64049-550 Teresina, PI, Brazil  
jfoliveira@ufpi.edu.br

João Marcos do Ó†  
Department of Mathematics  
Federal University of Paraíba  
58051-900 João Pessoa, PB, Brazil  
jmbo@pq.cnpq.br

Abstract

We establish critical and subcritical sharp Trudinger-Moser inequalities for fractional dimensions on the whole space. Moreover, we obtain asymptotic lower and upper bounds for the fractional subcritical Trudinger-Moser supremum from which we can prove the equivalence between critical and subcritical inequalities. Using this equivalence, we prove the existence of maximizers for both the subcritical and critical associated extremal problems. As a by-product of this development, we can explicitly calculate the value of the critical supremum in some special situations.

Key words. Sobolev inequality; Trudinger-Moser inequality; Differential Equations; Fractional Dimensions; Extremals; Sharp constant.

1 Introduction

Let $0 < R \leq \infty$, $\alpha \geq 0$ and $q \geq 1$ are real numbers. Set $L^q_\theta = L^q_\theta(0, R)$ the weighted Lebesgue space defined as the set of all measurable functions $u$ on $(0, R)$ such that

$$
\|u\|_{L^q_\theta} = \left\{ \begin{array}{ll}
(\int_0^R |u(r)|^q \, d\lambda_\theta)^{1/q} & < \infty \quad \text{if} \quad 1 \leq q < \infty,
\text{ess sup}_{0 < r < R} |u(r)| & < \infty \quad \text{if} \quad q = \infty
\end{array} \right.
$$

where we are denoting

$$
\int_0^R f(r) \, d\lambda_\theta = \omega_\theta \int_0^R f(r) r^\theta \, dr, \quad 0 < R \leq \infty
$$

*2000 Mathematics Subject Classification. 35J50, 46E35, 26D10, 35B33.
†Second author was supported by CNPq grant 305726/2017-0
with $\omega_\theta$ defined by

$$\omega_\theta = \frac{2\pi^{\frac{\theta+1}{2}}}{\Gamma\left(\frac{\theta+1}{2}\right)}, \quad \text{with} \quad \Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt.$$  

In the case that $\theta$ is a positive integer number $\omega_\theta$ agrees precisely with the known spherical volume element for Euclidean spaces $\mathbb{R}^{\theta+1}$. In fact, according to the formalism in [33], the integration of a radially symmetric function $f(r)$ in a $(\theta + 1)$-dimensional fractional space is given by (1.1), when $R = \infty$. Integration over non-integer dimensional spaces is often used in the dimensional regularization method as a powerful tool to obtain results in statistical mechanics and quantum field theory [7, 30, 35]. For a deeper discussion on this subject, we suggest [36] and the references therein.

We emphasize that the Lebesgue spaces $L_q^\alpha$ is also related with the classical Hardy’s inequality [20], see [10, 23] for more details. In addition, we can use $L_q^\alpha$-spaces to define Sobolev type spaces that are suitable to investigate a general class of differential operators which includes the $p$-Laplace, $p \geq 2$ and $k$-Hessian operators in the radial form, see for instance [6, 17, 23] and references therein. Indeed, as observed by P. Clément et al. [6], if we consider $X_R = X_{R}^{1,p}(\alpha, \theta)$, $\alpha, \theta \geq 0$, $p > 1$ and $0 < R \leq \infty$, as the set of all locally absolutely continuous functions on the interval $(0, R)$ such that $\lim_{r \to R} u(r) = 0$, $u \in L^p_\theta$ and $u' \in L^{p}_\alpha$, then $X_R$ becomes a Banach space endowed with the norm

$$\|u\| = (\|u\|^{p}_{L^p_\theta} + \|u'\|^{p}_{L^p_\alpha})^\frac{1}{p}.$$  

(1.2)

Further, we can distinguish two special behaviors for the weighted Sobolev spaces $X_R$. Namely, the Sobolev case when the condition

$$\alpha - p + 1 > 0$$  

(1.3)

holds and the Trudinger-Moser case if

$$\alpha - p + 1 = 0.$$  

(1.4)

In the Sobolev case (1.3) the value

$$p^* := p^*(\alpha, p, \nu) = \frac{(\nu + 1)p}{\alpha - p + 1}$$

is the critical exponent for the embedding

$$X_{R}^{1,p}(\alpha, \theta) \hookrightarrow L^q_\theta.$$  

Indeed, for the bounded situation $0 < R < \infty$, one has the following continuous embedding

$$X_{R}^{1,p}(\alpha, \theta) \hookrightarrow L^q_\theta, \quad \text{if} \quad q \in (1, p^*) \quad \text{and} \quad \min\{\theta, \nu\} \geq \alpha - p.$$  

(1.5)

Moreover, in the strict case $q < p^*$, the embedding is also compact. In contrast, for the Trudinger-Moser case one has the compact embedding

$$X_{R}^{1,p}(\alpha, \theta) \hookrightarrow L^q_\nu, \quad \text{if} \quad q \in (1, \infty) \quad \text{and} \quad \nu \geq 0.$$  

(1.6)

However $X_R \hookrightarrow L_{\nu}^{\infty}$ does not hold, as one can see taking $u(r) = \ln(\ln(eR/r))$.  

2
It is worth pointing out that the weighted Sobolev spaces $X_R$ is employed by several authors to investigate existence of solutions for a large class of differential equations. We recommend [6,8–10,18,19] for a general class of radial operators, and for $k$-Hessian equation [11,13,15] and recently [14]. This paper deals with intrinsic properties of $X_R$, which are related with sharp variational inequalities. In this direction, let us first recall some previous results. Firstly, the embedding in (1.6) does not find its threshold in the weighted Lebesgue spaces $L^q_\nu$, instead, in [12] it was proved a sharp inequality of the Trudinger-Moser type (see [29,34]) for $X_R$ which gets embedded into an weighted Orlicz space determined by exponential growth. In fact, let us denote

$$\mu_{\alpha,\theta} = (\theta + 1)\omega_\alpha^{1/\alpha} \quad \text{and} \quad |B_R|_\theta = \int_0^R d\lambda_\theta.$$  

Then, in [12] the authors proved the following:

**Theorem A.** Assume $0 < R < \infty$, $\alpha \geq 1$, $\theta \geq 0$ and $p = \alpha + 1$ be real numbers. Then,

(i) We have $\exp(\mu |u|^p/(p-1)) \in L^q_\theta$ for any $\mu > 0$ and $u \in X_R^{1,p}(\alpha, \theta)$.

(ii) There exists $c > 0$ depending only on $\alpha$, $p$ and $\theta$ such that

$$\sup_{\|u\|_{L^p_R} \leq 1} \frac{1}{|B_R|_\theta} \int_0^R e^{\mu |u|^p/(p-1)} d\lambda_\theta \begin{cases} \leq c & \text{if } \mu \leq \mu_{\alpha,\theta} \\ = \infty & \text{if } \mu > \mu_{\alpha,\theta} \end{cases}.$$  

(iii) The supremum in (1.8) is attained for all $0 < \mu \leq \mu_{\alpha,\theta}$.

In this paper we are mainly interested in the unbounded case when $R = \infty$. Here, according to [9], for the Sobolev case, we also have the following continuous embedding

$$X^{1,p}_\infty(\alpha, \theta) \hookrightarrow L^q_\theta \quad \text{if } q \in [p, p^*] \quad \text{and } \theta \geq \alpha - p.$$  

Also, the embeddings (1.9) are compact under the strict conditions $\theta > \alpha - p$ and $p < q < p^*$. In the Trudinger-Moser case it holds the continuous embeddings

$$X^{1,p}_\infty(\alpha, \theta) \hookrightarrow L^q_\theta \quad \text{for all } q \in [p, \infty)$$  

which are compact in the strict case $q > p$.

We recall the following Trudinger-Moser type inequality of the scaling invariant form obtained in [12].

**Theorem B.** Assume $p \geq 2$, $\alpha = p - 1$ and $\theta \geq 0$. For any $\mu < \mu_{\alpha,\theta}$, there exists a positive constant $C_{p,\mu,\theta}$ such that, for all $u \in X^{1,p}_\infty(\alpha, \theta)$, $\|u\|_{L^p_\theta} \leq 1$

$$\int_0^\infty \varphi_p \left(\mu |u|^{p/(p-1)}\right) d\lambda_\theta \leq C_{p,\mu,\theta} \|u\|_{L^p_\theta}^p.$$  

3
where
\[
\varphi_p(t) = e^t - \sum_{k=0}^{k_0-1} \frac{t^k}{k!} = \sum_{j \in \mathbb{N} : j \geq p-1} \frac{t^j}{j!}, \quad t \geq 0,
\] (1.12)
with \(k_0 = \min \{j \in \mathbb{N} : j \geq p-1\}\). The constant \(\mu_{\alpha,\theta}\) is sharp in the sense that the supremum is infinity when \(\mu \geq \mu_{\alpha,\theta}\).

Theorem B is the fractional dimensions counterpart of the result in S. Adachi and K. Tanaka [2]. We also refer to [5, 16, 31] concerning the related work for the classical Sobolev spaces. Our first result in this paper yields a precise asymptotics result on the above inequality.

**Theorem 1.1.** Assume \(p \geq 2, \alpha = p-1\) and \(\theta \geq 0\). For any \(0 \leq \mu < \mu_{\alpha,\theta}\), we denote
\[
T_{MSC}(\mu, \alpha, \theta) = \sup_{\|u\|_{L_p^\alpha} \leq 1} \frac{1}{\|u\|_{L_p^\theta}} \int_0^\infty \varphi_p \left( \mu |u|^{\frac{p}{p-1}} \right) d\lambda_\theta.
\]
Then there exist positive constants \(c(\alpha, \theta)\) and \(C(\alpha, \theta)\) such that, when \(\mu\) is close enough to \(\mu_{\alpha,\theta}\)
\[
\frac{c(\alpha, \theta)}{1 - (\frac{\mu}{\mu_{\alpha,\theta}})^{p-1}} \leq T_{MSC}(\mu, \alpha, \theta) \leq \frac{C(\alpha, \theta)}{1 - (\frac{\mu}{\mu_{\alpha,\theta}})^{p-1}}.
\]
Moreover, the constant \(\mu_{\alpha,\theta}\) is sharp. In addition, we have the following identity
\[
T_{MSC}(\mu, \alpha, \theta) = \sup_{\mu \in (0, \sigma)} \left( \frac{1 - (\frac{\mu}{\sigma})^{p-1}}{(\frac{\mu}{\sigma})^{p-1}} \right) T_{MSC}(\mu, \alpha, \theta), \quad \text{for all } \sigma \leq \mu_{\alpha,\theta}.\] (1.13)

One of the goals of this paper is to investigate the critical regime \(\mu = \mu_{\alpha,\theta}\). In this case, we will firstly prove the following:

**Theorem 1.2.** Assume \(p \geq 2, \alpha = p-1\) and \(\theta \geq 0\). For any \(0 \leq \sigma \leq \mu_{\alpha,\theta}\), we denote
\[
T_{MC}(\sigma, \alpha, \theta) = \sup_{\|u\| \leq 1} \int_0^\infty \varphi_p \left( \sigma |u|^{\frac{p}{p-1}} \right) d\lambda_\theta.
\]
Then \(T_{MC}(\sigma, \alpha, \theta)\) is finite. The constant \(\mu_{\alpha,\theta}\) is sharp. In addition, we have the following identity
\[
T_{MC}(\sigma, \alpha, \theta) = \sup_{\mu \in (0, \sigma)} \left( \frac{1 - (\frac{\mu}{\sigma})^{p-1}}{(\frac{\mu}{\sigma})^{p-1}} \right) T_{MSC}(\mu, \alpha, \theta), \quad \text{for all } \sigma \leq \mu_{\alpha,\theta}.
\] (1.13)

For the classical Sobolev spaces, the critical supremum \(T_{MC}(\sigma, \alpha, \theta)\) was first investigated by B. Ruf in [32] and Y. Li and B. Ruf [28]. There has been a growing interest in this kind of inequalities during the last decades, and a wide literature is available, see for instance [4, 21, 22, 24–26] and the references therein. We note that the boundedness of \(T_{MC}(\sigma, \alpha, \theta)\) has already been investigated in [1]. In this work we give a new proof for the boundedness which enables in particular to get a useful relation between \(T_{MSC}(\sigma, \alpha, \theta)\) and \(T_{MC}(\sigma, \alpha, \theta)\) given by (1.13).

Another interesting question about the supremum \(T_{MSC}(\mu, \alpha, \theta)\) and \(T_{MC}(\sigma, \alpha, \theta)\), and for Trudinger-Moser inequalities in general, is whether extremal functions exist or not. Inspired by recent approaches in [4, 25–27], we will employ the identity (1.13) to investigate this question. Firstly, on the subcritical supremum \(T_{MSC}(\mu, \alpha, \theta)\) we are able to prove the following:
Theorem 1.3. Assume that \( \alpha, p \) and \( \theta \) satisfy the assumption of Theorem 1.1. Then the fractional subcritical supremum \( T_{MSC}(\mu, \alpha, \theta) \) is attained.

By using Theorem 1.3 and the identity (1.13), we will first prove the following attainability result for the fractional critical supremum \( T_{MC}(\sigma, \alpha, \theta) \).

**Theorem 1.4.** Assume \( \alpha, p \) and \( \theta \) under the assumptions of Theorem 1.2.

(i) If \( k_0 > p - 1 \) and \( 0 < \sigma < \mu_{\alpha, \theta} \) then \( T_{MC}(\sigma, \alpha, \theta) \) is attained.

(ii) If \( k_0 = p - 1 \) and \( 0 < \sigma < \mu_{\alpha, \theta} \) then \( T_{MC}(\sigma, \alpha, \theta) \) is attained, whenever \( T_{MC}(\sigma, \alpha, \theta) > \frac{\sigma^{p-1}}{(p-1)!} \).

Theorem 1.4 has already been obtained in [1], however our proof here is new and relies on the critical and subcritical equivalence given in Theorem 1.2. In addition, following [22] we also are able to characterize precisely the attainability of \( T_{MC}(\sigma, \alpha, \theta) \) for the case (ii) above. In order to get this, we define the value \( \sigma^* = \sigma^*(\alpha, \theta) \in (0, \mu_{\alpha, \theta}) \) by

\[
\sigma^* = \inf \left\{ \sigma \in (0, \mu_{\alpha, \theta}) : T_{MC}(\sigma, \alpha, \theta) \text{ is attained} \right\}
\]

when \( T_{MC}(\sigma, \alpha, \theta) \) is attained for some \( \sigma \in (0, \mu_{\alpha, \theta}) \). If \( T_{MC}(\sigma, \alpha, \theta) \) is not attained for any \( \sigma \in (0, \mu_{\alpha, \theta}) \) then we set \( \sigma^* = \infty \).

**Theorem 1.5.** Assume that \( k_0 = p - 1 \) and \( \alpha, \theta \) are as in Theorem 1.2. Suppose \( \sigma^* < \mu_{\alpha, \theta} \).

(i) \( T_{MC}(\sigma, \alpha, \theta) \) is attained for \( \sigma < \mu_{\alpha, \theta} \).

(ii) The function \( \nu : (\sigma^*, \mu_{\alpha, \theta}) \to \mathbb{R} \) given by \( \nu(\sigma) = \frac{(p-1)!}{\sigma^p} T_{MC}(\sigma, \alpha, \theta) \) is strictly increasing. Moreover, by setting \( T_{MC}(0, \alpha, \theta) = 0 \), there holds

\[
T_{MC}(\sigma, \alpha, \theta) \begin{cases} 
\frac{\sigma^{p-1}}{(p-1)!}, & \text{for } \sigma \in [0, \sigma^*], \\
\frac{\sigma^{p-1}}{(p-1)!}, & \text{for } \sigma \in (\sigma^*, \mu_{\alpha, \theta}).
\end{cases}
\]

and in particular

\[
\sigma^* = \inf \left\{ \sigma \in (0, \mu_{\alpha, \theta}) : T_{MC}(\sigma, \alpha, \theta) > \frac{\sigma^{p-1}}{(p-1)!} \right\}. \tag{1.15}
\]

(iii) If \( p > 2 \) we have \( \sigma^* = 0 \) and thus \( T_{MC}(\sigma, \alpha, \theta) \) is attained for any \( (0, \mu_{\alpha, \theta}) \).

As a consequence of Theorem 1.5, since \( T_{MC}(\sigma, 1, \theta) \) is not attained for \( \sigma \) small enough (cf. [1, Theorem 1.3]), Theorem 1.5 provides

\[
T_{MC}(\sigma, 1, \theta) = \sup_{\|u\| \leq 1} \int_0^\infty \varphi_2 \left( \sigma |u|^2 \right) \, d\lambda_\theta = \sigma, \quad \forall \sigma \in [0, \sigma^*]. \tag{1.16}
\]

The rest of this paper is arranged as follows. In Section 2, we show Theorem 1.1. Section 3 is devoted to the subcritical and critical equivalence stated in Theorem 1.2. In Section 4 we will prove the existence of extremal functions for both subcritical \( T_{MSC} \) and critical \( T_{MC} \) fractional Trudinger-Moser supremum in Theorem 1.3 and Theorem 1.4. The proof of Theorem 1.5 is given in Section 5.
2 Sharp subcritical Trudinger-Moser inequality: Proof of Theorem 1.1

In this section, we will prove the asymptotic behavior for the supremum $T MSC(\mu, \alpha, \theta)$ for the subcritical Trudinger-Moser inequality in Theorem 1.1.

2.1 Some elementary properties

Note that from the definition (1.1) and the change of variables $s = \tau r$, we have

$$
\int_0^\infty f(\tau r) d\lambda_\theta = \frac{1}{\tau^{\theta+1}} \int_0^\infty f(s) d\lambda_\theta, \quad \tau > 0.
$$

Thus, by setting $u_\tau(r) = \zeta u(\tau r)$, with $\zeta, \tau > 0$ and $u \in X^{1,p}_\infty(\alpha, \theta)$ we can write

$$
\|u_\tau'\|_{L^p_{\alpha}} = (\zeta \tau)^p \|u'\|_{L^p_\alpha},
$$

$$
\|u_\tau\|_{L^q_\theta} = \frac{\zeta^q}{\tau^{\theta+1}} \|u\|_{L^q_\theta}, \quad q \geq p.
$$

Also, we observe that

$$
\varphi_p(\rho t) \leq \rho^{p-1} \varphi_p(t), \quad \text{if} \quad 0 \leq \rho \leq 1
$$

$$
\varphi_p(\rho t) \geq \rho^{p-1} \varphi_p(t), \quad \text{if} \quad \rho \geq 1
$$

where $\varphi_p(t)$ is given by (1.12).

Lemma 2.1. For all $q \geq 1$ and $\epsilon > 0$ it holds:

$$(x + y)^q \leq (1 + \epsilon)^{\frac{q-1}{q}} x^q + \left(1 - (1 + \epsilon)^{\frac{1}{q}}\right)^{1-q} y^q, \quad x, y \geq 0.
$$

Proof: Since $x \mapsto x^q$, $x \geq 0$ is a convex function, we have

$$
(x + y)^q = \left(\frac{1}{(1 + \epsilon)^{\frac{1}{q}}} x + \left(1 - \frac{1}{(1 + \epsilon)^{\frac{1}{q}}}\right)\left(1 - \frac{1}{(1 + \epsilon)^{\frac{1}{q}}}\right)^{-1} y\right)^q
$$

$$
\leq \frac{1}{(1 + \epsilon)^{\frac{1}{q}}} (1 + \epsilon) x^q + \left(1 - \frac{1}{(1 + \epsilon)^{\frac{1}{q}}}\right)^{1-q} y^q.
$$

Henceforth suppose that the condition $\alpha - p + 1 = 0$ holds. The next result ensures that the subcritical supremum $T MSC(\mu, \alpha, \theta)$ can be normalized.

Lemma 2.2.

$$
T MSC(\mu, \alpha, \theta) = \sup_{\|u'\|_{L^p_{\alpha}} = \|u\|_{L^p_\theta} = 1} \int_0^\infty \varphi_p\left(\mu |u|^{p-1}\right) d\lambda_\theta.
$$
Proof: It is sufficient to show that

\[ T \text{MSC} (\mu, \alpha, \theta) \leq \sup_{\|u\|_{L^p_\theta} = 1} \int_0^\infty \varphi_p \left( \mu |u|^{\frac{p}{p-1}} \right) d\lambda \theta. \]

In order to get this, for each \( u \in X_{\infty, p}^1 \setminus \{0\} \), with \( \|u\|_{L^p_\theta} \leq 1 \) we set

\[ v(r) = \frac{u(\tau r)}{\|u\|_{L^p_\theta}^{\frac{1}{p-1}}}; \quad \text{with} \quad \tau = \left( \frac{\|u\|_{L^p_\theta}}{\|u\|_{L^p_\theta}} \right)^{\frac{1}{p-1}}. \]

Since we are supposing \( \alpha - p + 1 = 0 \), (2.2) yields

\[ \|u\|_{L^p_\theta} = 1. \]

Then, from (2.1) and (2.3) it follows that

\[ \int_0^\infty \varphi_p \left( \mu |u|^{\frac{p}{p-1}} \right) d\lambda \theta = \frac{1}{\tau^{\alpha+1}} \int_0^\infty \varphi_p \left( \frac{1}{\|u\|_{L^p_\theta}^{\frac{1}{p-1}}} \mu |u|^{\frac{p}{p-1}} \right) d\lambda \theta \]

\[ \geq \left( \frac{\|u\|_{L^p_\theta}}{\|u\|_{L^p_\theta}} \right) \frac{1}{\|u\|_{L^p_\theta}} \int_0^\infty \varphi_p \left( \mu |u|^{\frac{p}{p-1}} \right) d\lambda \theta \]

\[ = \frac{1}{\|u\|_{L^p_\theta}} \int_0^\infty \varphi_p \left( \mu |u|^{\frac{p}{p-1}} \right) d\lambda \theta \]

which completes the proof.

\[ \square \]

2.2 Proof of Theorem 1.1

Let \( u \in X_{\infty, p}^1 \) with \( \|u\|_{L^p_\theta} \leq 1 \). From the Pólya-Szegő inequality obtained in \([1, 3]\), we can assume that \( u \) is a non-increasing function. Also, by Lemma 2.2 it is sufficient to analyze the case \( \|u\|_{L^p_\theta} = 1 \).

Initially, we will prove that

\[ T \text{MSC} (\mu, \alpha, \theta) \leq \frac{C(\alpha, \theta)}{1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1}}. \]

(2.5)

Let us denote by

\[ A_u = \left\{ r > 0 : |u(r)|^p > 1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1} \right\}. \]

We observe that for all \( |t| \leq 1 \) it holds

\[ \varphi_p \left( \mu |t|^{\frac{p}{p-1}} \right) = \sum_{j \in \mathbb{N} : j \geq p-1} \frac{\mu^j}{j!} |t|^{\frac{p^j}{j!}} \leq \sum_{j \in \mathbb{N} : j \geq p-1} \frac{\mu^j}{j!} |t|^p \leq |t|^p \sum_{j=0}^\infty \frac{\mu^j}{j!} = e^\mu |t|^p. \]

(2.6)
Hence, if \( A_u = \emptyset \) and consequently \( u \leq 1 \) in \((0, \infty)\), the inequality (2.6) yields
\[
\int_0^\infty \varphi_p \left( \mu |u|^{\frac{p}{p-1}} \right) \, d\lambda_\theta \leq e^\mu \int_0^\infty |u|^p d\lambda_\theta \leq \frac{1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1}}{1 - \left( \frac{\mu_{\alpha, \theta}}{\mu} \right)^{p-1}}.
\] (2.7)

So we can assume \( A_u \neq \emptyset \). Thus, there exists \( R_u > 0 \) such that \( A_u = (0, R_u) \), because we are assuming \( u \) is a non-increasing function. Analogously to (2.7), we obtain
\[
\int_{R_u}^\infty \varphi_p \left( \mu |u|^{\frac{p}{p-1}} \right) \, d\lambda_\theta \leq \int_{\{u \leq 1\}} \varphi_p \left( \mu |u|^{\frac{p}{p-1}} \right) \, d\lambda_\theta \leq e^\mu \int_{\{u \leq 1\}} |u|^p d\lambda_\theta \leq \frac{1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1}}{1 - \left( \frac{\mu_{\alpha, \theta}}{\mu} \right)^{p-1}}.
\]

Now, observe that
\[
|B_{R_u}| = \int_0^{R_u} d\lambda_\theta \leq \frac{1}{1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1}} \int_0^\infty |u|^p d\lambda_\theta \leq \frac{1}{1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1}}.
\] (2.8)

For \( r \in (0, R_u) \), we set
\[
v(r) = u(r) - \left( 1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1} \right)^{\frac{1}{p-1}}.
\]

It is clear that \( v \in X_{\mu_{\alpha, \theta}}^{1,p}(\alpha, \theta) \) and \( \|v'\|_{L_p(0,R_u)} \leq 1 \). Also, by choosing \( \epsilon = (\mu_{\alpha, \theta}/\mu)^p - 1 \) and \( q = p/(p-1) \) in Lemma 2.1, we have
\[
|u|^{\frac{p}{p-1}} \leq (1 + \epsilon)^{\frac{1}{p-1}} |v|^{\frac{p}{p-1}} + \left( 1 - \frac{1}{(1 + \epsilon)^{\frac{p}{p-1}}} \right)^{-\frac{1}{p-1}} \left( 1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1} \right)^{\frac{1}{p-1}}
\]
\[
= \frac{\mu_{\alpha, \theta}}{\mu} |v|^{\frac{p}{p-1}} + 1.
\]

Hence, the Trudinger-Moser type inequality (1.8) and (2.8) imply
\[
\int_0^{R_u} \varphi_p \left( \mu |u|^{\frac{p}{p-1}} \right) \, d\lambda_\theta \leq \int_0^{R_u} e^{\mu |u|^{\frac{p}{p-1}}} \, d\lambda_\theta \leq e^\mu \int_0^{R_u} e^{\mu_{\alpha, \theta}|v|^{\frac{p}{p-1}}} \, d\lambda_\theta \leq c_{\alpha, \theta} e^{\mu |B_{R_u}|} \leq c_{\alpha, \theta} e^{\mu_{\alpha, \theta}} \leq \frac{1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1}}{1 - \left( \frac{\mu_{\alpha, \theta}}{\mu} \right)^{p-1}}.
\]
Remark 1. At this point, we note that we have proved that

\[ T_{MSC}(\mu, \alpha, \theta) \leq \frac{C(\alpha, \theta)}{1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1}} \]

for any \( \mu < \mu_{\alpha, \theta} \) not necessarily close to \( \mu_{\alpha, \theta} \).

This proves (2.5). Next, we will prove the contrary inequality

\[ T_{MSC}(\mu, \alpha, \theta) \geq \frac{c(\alpha, \theta)}{1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1}}. \] (2.9)

To see this, let us consider the sequence

\[ u_n(r) = \frac{1}{\omega_{\alpha}^\frac{1}{p}} \begin{cases} \left( \frac{n}{\theta + 1} \right)^{\frac{p-1}{p}} \ln \frac{1}{r}, & \text{if } e^{-\frac{n}{\theta+1}} < r < 1, \\ 0, & \text{if } r \geq 1. \end{cases} \] (2.10)

Since that \( \alpha = p - 1 \), it follows that

\[ \|u_n\|_{L_p}^p = 1 \]

\[ \|u_n\|_{L_{\theta}}^p \leq \frac{c}{n} \left[ n^p e^{-n} + \int_0^n s^p e^{-s} ds \right] \]

for some \( c = c(\alpha, \theta) > 0 \). Thus, since \( \int_0^\infty s^p e^{-s} ds = \Gamma(p+1) > 0 \), there are \( c_1 = c_1(\alpha, \theta) > 0 \) and \( n_1 \in \mathbb{N} \) such that

\[ \|u_n\|_{L_{\theta}}^p \leq \frac{c_1}{n}, \quad \forall \ n \geq n_1. \] (2.11)

On the other hand

\[ \int_0^\infty \varphi_p \left( \mu |u_n| \right) d\lambda_\theta \geq \int_0^{e^{-\frac{n}{\theta+1}}} \varphi_p \left( \frac{\mu}{\mu_{\alpha, \theta}} \right) d\lambda_\theta = \frac{\omega_\theta}{\theta + 1} \varphi_p \left( \frac{\mu}{\mu_{\alpha, \theta}} \right) e^{-n} \]

\[ = \frac{\omega_\theta}{\theta + 1} \left[ e^{\left( \frac{\mu}{\mu_{\alpha, \theta}} \right) - 1} \right]^{n} - \left( \sum_{j=0}^{k_0-1} \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^j \frac{n^j}{j!} \right) e^{-n} \]

\[ \geq \frac{\omega_\theta}{\theta + 1} \left[ e^{\left( \frac{\mu}{\mu_{\alpha, \theta}} \right) - 1} \right]^{n} - \left( \sum_{j=0}^{k_0-1} \frac{n^j}{j!} \right) e^{-n} \].
Thus, for all \( n \geq n_1 \)

\[
TMSC(\mu, \alpha, \theta) \geq \frac{1}{\|u_n\|_{L^p}} \int_0^\infty \varphi_p \left( \mu |u_n|^{\frac{p}{p-1}} \right) d\lambda_	heta \\
\geq c_2 \left[ ne \left( \frac{\mu}{\mu_{\alpha, \theta}} - 1 \right) n - \left( \sum_{j=0}^{k_0-1} \frac{n^j}{j!} \right) ne^{-n} \right] \\
= \frac{c_2}{1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1}} \left[ ne \left( \frac{1 - \frac{\mu}{\mu_{\alpha, \theta}}}{\mu_{\alpha, \theta}} \right) n - \left( \sum_{j=0}^{k_0-1} \frac{n^j}{j!} \right) ne^{-n} \right],
\]

(2.12)

for some \( c_2 = c_2(\alpha, \theta) > 0 \). Now, we can choose \( n_2 \geq n_1 \) such that

\[
\left( 1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1} \right) \left( \sum_{j=0}^{k_0-1} \frac{n^j}{j!} \right) ne^{-n} \leq \frac{1}{e^5}, \quad \forall \ n \geq n_2 \quad \text{and} \quad 0 \leq \mu < \mu_{\alpha, \theta}.
\]

Hence, for all \( n \geq n_2 \)

\[
TMSC(\mu, \alpha, \theta) \geq \frac{c_2}{1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1}} \left[ ne \left( 1 - \mu \right) n - \left( \sum_{j=0}^{k_0-1} \frac{n^j}{j!} \right) ne^{-n} - e^{-5} \right].
\]

Now, if \( \alpha \) is close enough to \( \mu_{\alpha, \theta} \) such that \( \left( 1 - \frac{\mu}{\mu_{\alpha, \theta}} \right)^{-1} \geq n_2 \), by picking \( n \in \mathbb{N} \) such that

\[
\left( 1 - \frac{\mu}{\mu_{\alpha, \theta}} \right)^{-1} \leq n \leq 4 \left( 1 - \frac{\mu}{\mu_{\alpha, \theta}} \right)^{-1}
\]

we obtain

\[
TMSC(\mu, \alpha, \theta) \geq \frac{c_2}{1 - \left( \frac{\mu}{\mu_{\alpha, \theta}} \right)^{p-1}} \left[ e^{-4} - e^{-5} \right].
\]

Finally, from (2.12), for \( \mu = \mu_{\alpha, \theta} \) we have

\[
TMSC(\mu_{\alpha, \theta}, \alpha, \theta) \geq c_2 \left[ n - \left( \sum_{j=0}^{k_0-1} \frac{n^j}{j!} \right) ne^{-n} \right] \rightarrow \infty, \quad \text{as} \quad n \rightarrow \infty.
\]

3 Equivalence of critical and subcritical Trudinger-Moser inequalities

The aim of this section is to prove the critical and subcritical equivalence given in Theorem 1.2. We observe that we are not assuming that \( TMC(\mu_{\alpha, \theta}, \alpha, \theta) \) is finite in our argument.
Lemma 3.1. For any $0 < \sigma \leq \mu_{\alpha, \theta}$ and $0 < \mu < \sigma$

$$T MSC(\mu, \alpha, \theta) \leq \left( \frac{(\frac{\mu}{\sigma})^{p-1}}{1 - (\frac{\mu}{\sigma})^{p-1}} \right) T MC(\sigma, \alpha, \theta).$$

In particular, if $T MC(\mu_{\alpha, \theta}, \alpha, \theta)$ is finite, then $T MSC(\mu, \alpha, \theta)$ is finite.

Proof: Let $u \in X_{\infty}^{1,p}$, with $\|u'\|_{L_\alpha}^p = 1$ and $\|u\|_{L_\sigma}^p = 1$. Set

$$u_t(r) = \left( \frac{\mu}{\sigma} \right)^{\frac{p-1}{p}} u(tr), \quad \text{with} \quad t = \left( \frac{(\frac{\mu}{\sigma})^{p-1}}{1 - (\frac{\mu}{\sigma})^{p-1}} \right)^{\frac{1}{p+1}}. \quad (3.1)$$

By (2.2) we get

$$\|u_t'\|_{L_\alpha}^p = \left( \frac{\mu}{\sigma} \right)^{p-1} \|u'\|_{L_\alpha}^p = \left( \frac{\mu}{\sigma} \right)^{p-1} \|u_t\|_{L_\sigma}^p = \left( \frac{\mu}{\sigma} \right)^{p-1} \frac{1}{t^{\theta+1}}.$$

Hence $\|u_t'\|_{L_\alpha}^p + \|u_t\|_{L_\sigma}^p = 1$ and we have

$$\int_{0}^{\infty} \varphi_p \left( \mu |u|^{\frac{p}{p-1}} \right) d\lambda_{\theta} = t^{\theta+1} \int_{0}^{\infty} \varphi_p \left( \sigma |u_t|^{\frac{p}{p-1}} \right) d\lambda_{\theta} \leq \left( \frac{(\frac{\mu}{\sigma})^{p-1}}{1 - (\frac{\mu}{\sigma})^{p-1}} \right) T MC(\sigma, \alpha, \theta).$$

Since $u \in X_{\infty}^{1,p}$, with $\|u'\|_{L_\alpha}^p = 1$ and $\|u\|_{L_\sigma}^p = 1$ is arbitrary, in view of the Lemma 2.2, we conclude the proof. 

3.1 Proof of Theorem 1.2

Let $u \in X_{\infty}^{1,p}$ such that $0 < \|u'\|_{L_\alpha}^p < 1$ and $\|u\|_{L_\sigma}^p < 1 - \vartheta^p$. Assume that

$$\|u'\|_{L_\alpha}^p = \vartheta \in (0, 1) \quad \text{and} \quad \|u\|_{L_\sigma}^p \leq 1 - \vartheta^p.$$ 

If $\frac{1}{2} < \vartheta < 1$, we set

$$u_t(r) = \frac{u(tr)}{\vartheta}, \quad \text{with} \quad t = \left( \frac{1 - \vartheta^p}{\vartheta^p} \right)^{\frac{1}{\vartheta+1}} > 0.$$

From (2.2), we can write

$$\|u_t'\|_{L_\alpha}^p = \frac{\|u'\|_{L_\alpha}^p}{\vartheta} = 1,$$

$$\|u_t\|_{L_\sigma}^p = \frac{1}{\vartheta^p t^{\theta+1}} \|u\|_{L_\sigma}^p \leq \frac{1}{\vartheta^p t^{\theta+1}} = 1.$$
Hence, for any $\sigma \leq \mu_{\alpha,\theta}$, the Theorem 1.1 (cf. Remark 1) yields

$$
\int_0^\infty \varphi_p \left( \sigma |u|^{\frac{p}{p-1}} \right) d\lambda \leq t^{\theta+1} \int_0^\infty \varphi_p \left( \frac{\mu_{\alpha,\theta} |u|^{\frac{p}{p-1}}}{\mu_{\alpha,\theta}} \right) d\lambda

\leq t^{\theta+1} \text{TMSC} \left( \frac{\mu_{\alpha,\theta}}{\mu_{\alpha,\theta}} \right)

\leq \left( 1 - \frac{\theta}{\beta} \right) \frac{C(\alpha, \theta)}{1 - \frac{\theta}{\beta}}

= \left( 1 - \frac{\theta}{\beta} \right) \frac{C(\alpha, \theta)}{1 - \frac{\theta}{\beta}}

\leq \beta C(\alpha, \theta).

If $0 < \theta \leq \frac{1}{2}$, setting $v(r) = 2u(r/\theta)$

we have

$$
\|v\|_{L_p^\theta} = 2\|u\|_{L_p^\theta} \leq 1

\|v\|_{L_p^\theta} = 2^\theta \beta \|u\|_{L_p^\theta} \leq 2^\theta \beta (1 - \theta) \leq 2^\theta \beta + 1.

Consequently, the Theorem 1.1 provides

$$
\int_0^\infty \varphi_p \left( \sigma |u|^{\frac{p}{p-1}} \right) d\lambda \leq \frac{1}{\theta^{\theta+1}} \int_0^\infty \varphi_p \left( 2^{-\frac{p}{p-1}} \mu_{\alpha,\theta} |v|^{\frac{p}{p-1}} \right) d\lambda

\leq 2^\theta \text{TMSC} \left( 2^{-\frac{p}{p-1}} \mu_{\alpha,\theta}, \alpha, \theta \right)

\leq C(\alpha, \theta) \left( \frac{2^\theta}{1 - \frac{\theta}{\beta}} \right).

Since $u \in X_{\infty}^1$, with $\|u\| \leq 1$ is arbitrary, we obtain $TMC(\sigma, \alpha, \theta) < \infty$, for any $\sigma \leq \mu_{\alpha,\theta}$.

Next, we will show that the constant $\mu_{\alpha,\theta}$ is sharp. To see this, we can use the sequence $(u_n)$ in (2.10) again. Indeed, we have

$$
\|u_n\|_{L_p^\theta} = 1

\|u_n\|_{L_p^\theta} = O \left( \frac{1}{n} \right), \text{ as } n \to \infty.

Now, for $\tau_n \in (0, 1)$ such that

$$
\tau_n^\theta (1 + \|u_n\|_{L_p^\theta}) = 1, \text{ with } \tau_n = 1 - O \left( \frac{1}{n\beta} \right) \to 1, \text{ as } n \to \infty

we set

$$
v_n(r) = \tau_n u_n(r).

12
Then
\[ \| v_n' \|_{L_p^p}^p + \| v_n \|_{L_p^p}^p = \tau_n^p \| u_n' \|_{L_p^p}^p + \tau_n^p \| u_n \|_{L_p^p}^p = \tau_n^p + \tau_n^p \| u_n \|_{L_p^p}^p = 1. \]

In addition, for any \( \sigma > \mu_{\alpha, \theta} \)
\[
\int_0^{\infty} \varphi_p \left( \sigma |u_n|^{\frac{p}{p-1}} \right) \, d\lambda_{\theta} \geq \int_0^{\infty} \left( e^{\frac{n\sigma}{\mu_{\alpha, \theta}}} - \sum_{k=0}^{k_0-1} \left( \frac{n\sigma}{\mu_{\alpha, \theta}} \right)^{\frac{k}{\tau_n^{p-1}}} \right) \, d\lambda_{\theta} = \frac{\omega_{\theta}}{\theta} + 1 \left[ e \left( \frac{n\sigma}{\mu_{\alpha, \theta}} \right)^{\frac{p}{p-1}} - n \left( \frac{n\tau_n^{k_0-1}}{e^n} \right) \right] \rightarrow +\infty, \quad \text{as} \quad n \rightarrow \infty.
\]

Now, we are going to show that
\[
TMC(\sigma, \alpha, \theta) = \sup_{\mu \in (0, \sigma)} \left( 1 - \left( \frac{\mu}{\sigma} \right)^{p-1} \right) TMSC(\mu, \alpha, \theta). \quad (3.2)
\]

By Lemma 3.1, we obtain
\[
\sup_{\mu \in (0, \sigma)} \left( 1 - \left( \frac{\mu}{\sigma} \right)^{p-1} \right) TMSC(\mu, \alpha, \theta) \leq TMC(\sigma, \alpha, \theta). \quad (3.3)
\]

In order to obtain the reverse inequality, let \((u_n)\) be a maximizing sequence of \(TMC(\sigma, \alpha, \theta)\), that is, \(u_n \in X_{1,p}^p, 0 < \| u_n' \|_{L_p^p}^p + \| u_n \|_{L_p^p}^p \leq 1\) such that
\[
TMC(\sigma, \alpha, \theta) = \lim_{n \rightarrow \infty} \int_0^{\infty} \varphi_p \left( \sigma |u_n|^{\frac{p}{p-1}} \right) \, d\lambda_{\theta}. \quad (3.4)
\]

We set
\[
u_{\tau_n}(r) = \frac{u(\tau_n r)}{\| u_n' \|_{L_p^p}}, \quad \text{with} \quad \tau_n = \left( 1 - \| u_n' \|_{L_p^p}^p \right)^{\frac{1}{p-1}} > 0.
\]

Then
\[
\| u_{\tau_n}' \|_{L_p^p} = 1
\]
\[
\| u_{\tau_n} \|_{L_p^p}^p = \frac{1}{\| u_n' \|_{L_p^p}^p \tau_n^{p-1}} \| u_n \|_{L_p^p}^p \leq 1 - \| u_n' \|_{L_p^p}^p \leq 1.
\]
Consequently
\[
\int_0^\infty \varphi_p \left( |u_n|^\frac{p}{p-1} \right) d\lambda_\theta = \tau_n^{p+1} \int_0^\infty \varphi_p \left( \|u_n\|_{L_p^n}^{\frac{p}{p-1}} |u_{\lambda_n}|^{\frac{p}{p-1}} \right) d\lambda_\theta \\
\leq \tau_n^{p+1} \text{TMSC} \left( \|u_n\|_{L_p^n}^{\frac{p}{p-1}}, \alpha, \theta \right) \\
= \left( \frac{1 - \|u_n\|_{L_p^n}^{\frac{p}{p-1}}}{\|u_n\|_{L_p^n}^{\frac{p}{p-1}}} \right) \text{TMSC} \left( \|u_n\|_{L_p^n}^{\frac{p}{p-1}}, \alpha, \theta \right) \\
= \left( \frac{1 - \left( \frac{\|u_n\|_{L_p^n}^{\frac{p}{p-1}}}{\sigma} \right)^{p-1}}{\left( \frac{\sigma}{\sigma} \right)^{p-1}} \right) \text{TMSC} \left( \|u_n\|_{L_p^n}^{\frac{p}{p-1}}, \alpha, \theta \right) \\
\leq \sup_{\mu \in (0, \sigma)} \left( \frac{1 - \left( \frac{\mu}{\sigma} \right)^{p-1}}{\left( \frac{\mu}{\sigma} \right)^{p-1}} \right) \text{TMSC} (\mu, \alpha, \theta).
\]

Hence, we obtain
\[
\text{TMSC} (\sigma, \alpha, \theta) \leq \sup_{\mu \in (0, \sigma)} \left( \frac{1 - \left( \frac{\mu}{\sigma} \right)^{p-1}}{\left( \frac{\mu}{\sigma} \right)^{p-1}} \right) \text{TMSC} (\mu, \alpha, \theta).
\] (3.5)

Now, (3.2) follows from (3.3) and (3.5).

4 Existence of extremal functions

In this section we will prove the existence of extremal functions for both subcritical and critical Trudinger-Moser inequalities Theorem 1.3 and Theorem 1.4. First of all, we present the following radial type Lemma.

**Lemma 4.1.** For each \( u \in X_{\infty}^{1,p} (\alpha, \theta) \), \( p \geq 2 \), we have the inequality
\[
|u(r)|^p \leq \frac{C}{r^\alpha} \|u\|_p^p, \quad \forall \ r > 0
\]
where \( C > 0 \) depends only on \( \alpha, p \) and \( \theta \). In addition,
\[
\lim_{r \to \infty} r^\alpha \|u(r)|^p \to 0.
\]

**Proof:** Let \( u \in X_{\infty}^{1,p} (\alpha, \theta) \) be arbitrary. For any \( r > 0 \), we have
\[
|u(r)|^p = - \int_r^\infty \frac{d}{ds} (|u(s)|^p) \ ds \leq p \int_r^\infty |u(s)|^{p-1} |u'(s)| \ ds.
\]
Hence
\[
r^\alpha \|u(r)|^p \leq p \int_r^\infty |u(s)|^{p-1} s^\alpha \|u'(s)|^\frac{\alpha}{p} \ ds
\]

14
and the Young’s inequality yields
\[ r^{\frac{\alpha+\theta(p-1)}{p}}|u(r)|^p \leq C \left[ \int_r^\infty |u(s)|^p d\lambda_\theta + \int_r^\infty |u'(s)|^p d\lambda_\alpha \right] , \]
for some $C > 0$ depending only on $\alpha$, $p$ and $\theta$. This proves the result. ■

### 4.1 Maximizers for the subcritical Trudinger-Moser inequality

Let $(u_n) \subset X^{1,p}_\infty$ be a maximizing sequence to the subcritical Trudinger-Moser supremum $TMSC(\mu, \alpha, \theta)$. From Lemma 2.2, we may suppose that
\[ TMSC(\mu, \alpha, \theta) = \lim_{n \to \infty} \int_0^\infty \varphi_p \left( \mu |u_n|^{\frac{p}{p-1}} \right) d\lambda_\theta \]
\[ \|u_n\|_L^p = \|u_n\|_L^p = 1 \]
\[ u_n \rightharpoonup u \text{ weakly in } X^{1,p}_\infty. \]

From the compact embedding (1.10), we also may assume that
\[ u_n \to u \text{ in } L^q, \quad q > p \quad \text{and} \quad u_n(r) \to u(r) \text{ a.e in } (0, \infty). \] (4.1)

Of course, we also have
\[ \|u'_n\|_L^q \leq 1, \quad \|u\|_L^q \leq 1. \]

At this point we observe that there exist $C = C(p, \mu) > 0$ such that
\[ \varphi_p \left( \mu t^{\frac{p}{p-1}} \right) - \frac{\mu^k_0}{k_0!} t^{\frac{k_0}{p-1}} \leq C \varphi_p \left( \mu t^{\frac{p}{p-1}} \right) t^{\frac{p}{p-1}}, \quad t \geq 0. \] (4.2)

Let $\epsilon > 0$ be arbitrary. From Lemma 4.1, there exists $R > 0$ such that $|u_n(r)| \leq \epsilon$, for all $r \geq R$. Hence, from (4.2) and Theorem B we obtain
\[ \int_R^\infty \left[ \varphi_p \left( \mu |u_n|^{\frac{p}{p-1}} \right) - \frac{\mu^k_0}{k_0!} |u_n|^{\frac{k_0}{p-1}} \right] d\lambda_\theta \leq C(p, \mu) \int_R^\infty \varphi_p \left( \mu |u_n|^{\frac{p}{p-1}} \right) |u_n|^{\frac{p}{p-1}} d\lambda_\theta \]
\[ \leq C(p, \mu) \epsilon^{\frac{p}{p-1}} \int_R^\infty \varphi_p \left( \mu |u_n|^{\frac{p}{p-1}} \right) d\lambda_\theta \]
\[ \leq C(p, \mu, \theta) \epsilon^{\frac{p}{p-1}}. \]

Also, we have (cf.(4.1))
\[ \varphi_p \left( \mu |u_n|^{\frac{p}{p-1}} \right) - \frac{\mu^k_0}{k_0!} |u_n|^{\frac{k_0}{p-1}} \to \varphi_p \left( \mu |u|^{\frac{p}{p-1}} \right) - \frac{\mu^k_0}{k_0!} |u|^{\frac{k_0}{p-1}} \text{ a.e in } (0, R), \quad \text{as } n \to \infty. \]

In addition, by setting $v_n(r) = u_n(r) - u_n(R)$ for all $r \in (0, R)$, we have $v_n \in X^{1,p}_R(\alpha, \theta)$ with $\|v'_n\|_L^q \leq 1$. Moreover, from Lemma 2.1, for any $q > 1$
\[ |u_n|^{\frac{p}{p-1}} \leq q^{\frac{1}{p}} |v_n|^{\frac{p}{p-1}} + \left( 1 - q^{\frac{p}{p-1}} \right)^{-\frac{1}{p-1}} \epsilon \frac{p}{p-1}. \]
By choosing $q > 1$ close to 1 such that $q^{(p+1)/p} \mu < \mu_{\alpha, \theta}$, Theorem A yields
\[
\int_{0}^{R} \left[ \varphi_p \left( \mu |u_n| \frac{p}{p-1} \right) - \frac{\mu k_0}{k_0!} |u_n|^{\frac{k_0 p}{p-1}} \right] \ d\lambda_0 \leq \int_{0}^{R} \left[ \varphi_p \left( \mu |u_n| \frac{p}{p-1} \right) \right]^q \ d\lambda_0 \\
\leq \int_{0}^{R} e^{q \mu |u_n| \frac{p}{p-1}} \ d\lambda_0 \\
\leq C(p, q, \alpha, \theta) \int_{0}^{R} e^{q |u_n| \frac{p}{p-1}} \ d\lambda_0 \\
\leq C(p, q, \mu, \theta, R).
\] (4.3)
Thus, we may use Vitali’s convergence theorem to obtain
\[
\lim_{n \to \infty} \int_{0}^{R} \left[ \varphi_p \left( \mu |u_n| \frac{p}{p-1} \right) - \frac{\mu k_0}{k_0!} |u_n|^{\frac{k_0 p}{p-1}} \right] \ d\lambda_0 = \int_{0}^{R} \left[ \varphi_p \left( \mu |u| \frac{p}{p-1} \right) - \frac{\mu k_0}{k_0!} |u|^{\frac{k_0 p}{p-1}} \right] \ d\lambda_0.
\]
Now, using the Brezis-Lieb lemma together with (4.1) we have
\[
\lim_{n \to \infty} \int_{0}^{\infty} |u_n|^{\frac{k_0 p}{p-1}} \ d\lambda_0 = \begin{cases} 
\int_{0}^{\infty} |u|^{\frac{k_0 p}{p-1}} \ d\lambda_0, & \text{if } k_0 > p - 1 \\
1, & \text{if } k_0 = p - 1.
\end{cases}
\]
Hence, if $k_0 > p - 1$
\[
TMSC(\mu, \alpha, \theta) = \lim_{n} \int_{0}^{\infty} \varphi_p \left( \mu |u_n| \frac{p}{p-1} \right) \ d\lambda_0 \\
= \lim_{n} \left[ \int_{0}^{\infty} \left( \varphi_p \left( \mu |u_n| \frac{p}{p-1} \right) - \frac{\mu k_0}{k_0!} |u_n|^{\frac{k_0 p}{p-1}} \right) \ d\lambda_0 + \frac{\mu k_0}{k_0!} \int_{0}^{\infty} |u_n|^{\frac{k_0 p}{p-1}} \ d\lambda_0 \right] \\
\leq \int_{0}^{R} \left( \varphi_p \left( \mu |u| \frac{p}{p-1} \right) - \frac{\mu k_0}{k_0!} |u|^{\frac{k_0 p}{p-1}} \right) \ d\lambda_0 + C(p, \mu, \theta) \epsilon^{\frac{p}{p-1}} + \frac{\mu k_0}{k_0!} \int_{0}^{\infty} |u|^{\frac{k_0 p}{p-1}} \ d\lambda_0 \\
\leq \int_{0}^{\infty} \varphi_p \left( \mu |u| \frac{p}{p-1} \right) \ d\lambda_0 + C(p, \mu, \theta) \epsilon^{\frac{p}{p-1}}.
\]
Setting $\epsilon \to 0$, we have
\[
TMSC(\mu, \alpha, \theta) \leq \int_{0}^{\infty} \varphi_p \left( \mu |u| \frac{p}{p-1} \right) \ d\lambda_0.
\]
It follows that $0 < \|u\|_{L^p_\theta} \leq 1$ and thus
\[
TMSC(\mu, \alpha, \theta) \leq \frac{1}{\|u\|_{L^p_\theta}} \int_{0}^{\infty} \varphi_p \left( \mu |u| \frac{p}{p-1} \right) \ d\lambda_0
\]
which completes the proof in the case $k_0 > p - 1$. If $k_0 < p - 1$, we can write
\[
TMSC(\mu, \alpha, \theta) = \lim_{n} \left[ \int_{0}^{\infty} \left( \varphi_p \left( \mu |u_n| \frac{p}{p-1} \right) - \frac{\mu k_0}{k_0!} |u_n|^{\frac{k_0 p}{p-1}} \right) \ d\lambda_0 + \frac{\mu k_0}{k_0!} \right] \\
\leq \int_{0}^{R} \left( \varphi_p \left( \mu |u| \frac{p}{p-1} \right) - \frac{\mu k_0}{k_0!} |u|^{\frac{k_0 p}{p-1}} \right) \ d\lambda_0 + C(p, \mu, \theta) \epsilon^{\frac{p}{p-1}} + \frac{\mu k_0}{k_0!} \\
\leq \int_{0}^{\infty} \left( \varphi_p \left( \mu |u| \frac{p}{p-1} \right) - \frac{\mu k_0}{k_0!} |u|^{\frac{k_0 p}{p-1}} \right) \ d\lambda_0 + C(p, \mu, \theta) \epsilon^{\frac{p}{p-1}} + \frac{\mu k_0}{k_0!}.
\]
Letting $\epsilon \to 0$, it follows that
\[
T_{MSC}(\mu, \alpha, \theta) \leq \int_0^\infty \left( \varphi_p \left( \mu |u|^\frac{p}{p-1} \right) - \frac{\mu^{k_0}}{k_0!} |u|^p \right) d\lambda_\theta + \frac{\mu^{k_0}}{k_0!}.
\] (4.4)

Moreover, for any $w \in X_{1, p}^1(\alpha, \theta)$ with $\|w\|_{L_p^\alpha} = \|w\|_{L_\theta^p} = 1$ we have
\[
\int_0^\infty \varphi_p \left( \mu |w|^\frac{p}{p-1} \right) d\lambda_\theta \geq \frac{\mu^{k_0}}{k_0!} \int_0^\infty |w|^p d\lambda_\theta + \frac{\mu^{k_0+1}}{(k_0 + 1)!} \int_0^\infty |w|^\frac{p(k_0+1)}{p-1} d\lambda_\theta.
\]
This implies that $T_{MSC}(\mu, \alpha, \theta) > \frac{\mu^{k_0}}{k_0!}$. Thus, from (4.4), we get
\[
0 < \|u\|_{L_\theta^p} \leq 1
\] and the result is proved.

### 4.2 Maximizers for the critical Trudinger-Moser inequality

Next we combine the equivalence in the Theorem 1.2 and the Theorem 1.3 to demonstrate Theorem 1.4.

Firstly, for $0 < s < \mu_{\alpha, \theta}$, we set
\[
f(s) = T_{MSC}(s, \alpha, \theta) \quad \text{and} \quad g(s) = T_{MC}(s, \alpha, \theta).
\]

Hence, Theorem 1.2 yields
\[
g(\sigma) = \sup_{s \in (0, \sigma)} \left( 1 - \left( \frac{\sigma}{s} \right)^{p-1} \right) f(s).
\] (4.5)

**Lemma 4.2.** $f$ is a continuous function on $(0, \mu_{\alpha, \theta})$.

**Proof:** By using Theorem 1.3 we can pick $\epsilon_n \downarrow 0$ and $u_n \in X_{1, p}^1$, with $\|u'_n\|_{L_\theta^p} \leq 1$ and $\|u_n\|_{L_\theta^p} = 1$ such that
\[
f(s + \epsilon_n) = \int_0^\infty \varphi_p \left( (s + \epsilon_n)|u_n|^\frac{p}{p-1} \right) d\lambda_\theta.
\]
Then
\[
0 \leq f(s + \epsilon_n) - f(s) \leq \int_0^\infty \left[ \varphi_p \left( (s + \epsilon_n)|u_n|^\frac{p}{p-1} \right) - \varphi_p \left( s|u_n|^\frac{p}{p-1} \right) \right] d\lambda_\theta.
\] (4.6)

Without loss of generality, we also may assume that (cf. (1.10))
\[
u_n \rightharpoonup u \quad \text{weakly in} \quad X_{1, p}^1
\]
\[u_n \to u \quad \text{in} \quad L_q^q, \quad q > p \quad \text{and} \quad u_n(r) \to u(r) \quad \text{a.e in} \quad (0, \infty).
\] (4.7)
In particular,
\[ \varphi_p \left( (s + \epsilon_n) \left| u_n(r) \right|^\frac{p}{p-1} \right) - \varphi_p \left( s \left| u_n(r) \right|^\frac{p}{p-1} \right) \to 0 \ \text{a.e in} \ (0, \infty). \]

In the same way of (4.3), we can use Lemma 4.1 and Theorem A to obtain a positive constant \( C(p, q, s, \theta, R) \) such that
\[ \int_0^R \left[ \varphi_p \left( (s + \epsilon_n) \left| u_n \right|^\frac{p}{p-1} \right) - \varphi_p \left( s \left| u_n \right|^\frac{p}{p-1} \right) \right]^q \, d\lambda_{\theta} \leq C(p, q, s, \theta, R), \]
for some \( q > 1 \) and for all \( R > 0 \). It follows that
\[ \int_0^R \left[ \varphi_p \left( (s + \epsilon_n) \left| u_n \right|^\frac{p}{p-1} \right) - \varphi_p \left( s \left| u_n \right|^\frac{p}{p-1} \right) \right] \, d\lambda_{\theta} \to 0. \]

On the other hand, for \( R \) large enough, Lemma 4.1 yields
\[ |u_n(r)| \leq 1, \ \text{for every} \ n \in \mathbb{N}, \ r \geq R. \]

Then
\[ \int_R^\infty \left[ \varphi_p \left( (s + \epsilon_n) \left| u_n \right|^\frac{p}{p-1} \right) - \varphi_p \left( s \left| u_n \right|^\frac{p}{p-1} \right) \right] \, d\lambda_{\theta} = \int_R^\infty \sum_{j \in \mathbb{N} : j \geq p-1} \left[ \frac{(s + \epsilon_n)^j}{j!} - \frac{s^j}{j!} \right] \left| u_n \right|^\frac{p}{p-1} \, d\lambda_{\theta} \leq \sum_{j \in \mathbb{N} : j \geq p-1} \left[ \frac{(s + \epsilon_n)^j}{j!} - \frac{s^j}{j!} \right] \int_R^\infty \left| u_n \right|^p \, d\lambda_{\theta} \leq \left[ \varphi_p (s + \epsilon_n) - \varphi_p (s) \right] \to 0. \]

From (4.6), we obtain
\[ 0 \leq f(s + \epsilon_n) - f(s) \to 0, \ \text{as} \ n \to \infty. \]

Similarly, we can also have that
\[ 0 \leq f(s) - f(s - \epsilon_n) \to 0, \ \text{as} \ n \to \infty. \]

Now, in order to ensure the existence of an extremal function for \( TM(C, \alpha, \theta) \) when \( 0 < \sigma < \mu_{\alpha, \theta} \) it is sufficient to show that
\[ \limsup_{s \to 0^+} \left( \frac{1 - (\frac{s}{\sigma})^{p-1}}{\left( \frac{s}{\sigma} \right)^{p-1}} \right) f(s) < g(\sigma) \quad (4.8) \]
and
\[ \limsup_{s \to \sigma^-} \left( \frac{1 - (\frac{s}{\sigma})^{p-1}}{\left( \frac{s}{\sigma} \right)^{p-1}} \right) f(s) < g(\sigma). \quad (4.9) \]

Indeed, (4.8), (4.9) together with (4.5) and Lemma 4.2 ensure the existence of \( s_{\sigma} \in (0, \sigma) \) such that
\[ g(\sigma) = \left( \frac{1 - \left( \frac{s_{\sigma}}{\sigma} \right)^{p-1}}{\left( \frac{s_{\sigma}}{\sigma} \right)^{p-1}} \right) f(s_{\sigma}). \quad (4.10) \]
Let \( u_\sigma \) be an extremal function for \( T MSC(s_\sigma, \alpha, \theta) \) ensured by Theorem 1.3. Set

\[
v_\sigma(r) = \left( \frac{s_\sigma}{\sigma} \right)^{\frac{p-1}{p}} u_\sigma(\tau r)
\]

where

\[
\tau = \left( \frac{(\frac{s_\sigma}{\sigma})^{p-1} \|u_\sigma\|_{L^p_\theta}^p}{1 - (\frac{s_\sigma}{\sigma})^{p-1}} \right)^{\frac{1}{\theta+1}}.
\]

From (2.2), it follows that

\[
\|v_\sigma\|^p = \|v_\sigma\|^p_{L^p_\alpha} + \|v_\sigma\|^p_{L^p_\theta} = \left( \frac{s_\sigma}{\sigma} \right)^{p-1} \left[ \|u_\sigma\|^p_{L^p_\alpha} + \tau^{-(\theta+1)} \|u_\sigma\|_{L^p_\theta}^\theta \right] \leq 1
\]

and also we have (cf.(4.10))

\[
T MC(\sigma, \alpha, \theta) = \left( \frac{1 - (\frac{s_\sigma}{\sigma})^{p-1}}{(\frac{s_\sigma}{\sigma})^{p-1}} \right)^{\frac{\theta+1}{p}} \int_0^\infty \varphi_p \left( \sigma |v_\sigma|^\frac{p}{p-1} \right) d\lambda_\theta
\]

\[
= \int_0^\infty \varphi_p \left( \sigma |v_\sigma|^\frac{p}{p-1} \right) d\lambda_\theta.
\]

Hence, \( v_\sigma \) is an extremal function of \( T MC(\sigma, \alpha, \theta) \). Now, since

\[
\limsup_{s \to \sigma^-} \left( \frac{1 - (\frac{s_\sigma}{\sigma})^{p-1}}{(\frac{s_\sigma}{\sigma})^{p-1}} \right) f(s) = 0 < g(\sigma)
\]

it is clear that (4.9) holds.

Next, we will prove that (4.8) holds. Firstly, we provide the following useful estimate.

**Lemma 4.3.** For all \( q \geq p \) and \( 0 < \mu < \mu_{\alpha, \theta} \) we have

\[
\sup_{\|u\|_{L^p_\alpha} \leq 1, \|u\|_{L^p_\theta} = 1} \int_0^\infty \left( e^{\mu|u|^\frac{p}{p-1}} \right) |u|^q d\lambda_\theta \leq c
\]

for some \( c = c(\mu, \alpha, \theta, q) > 0 \).

**Proof:** We can proceed analogous to Theorem 1.1. Indeed, let \( u \in X_{\infty}^{1,p} \setminus \{0\} \), with \( \|u\|_{L^p_\alpha} \leq 1 \) and \( \|u\|_{L^p_\theta} = 1 \). From the Pólya-Szegö inequality obtained in [1], we can assume that \( u \) is a non-increasing function. We write

\[
\int_0^\infty \left( e^{\mu|u|^\frac{p}{p-1}} \right) |u|^q d\lambda_\theta = \int_{\{u < 1\}} \left( e^{\mu|u|^\frac{p}{p-1}} \right) |u|^q d\lambda_\theta + \int_{\{u \geq 1\}} \left( e^{\mu|u|^\frac{p}{p-1}} \right) |u|^q d\lambda_\theta.
\]

Of course we have

\[
\int_{\{u < 1\}} \left( e^{\mu|u|^\frac{p}{p-1}} \right) |u|^q d\lambda_\theta \leq e^\mu \int_{\{u < 1\}} |u|^q d\lambda_\theta \leq e^\mu \int_{\{u < 1\}} |u|^p d\lambda_\theta \leq e^\mu.
\]
Set
\[ I_u = \{ r > 0 : u(r) \geq 1 \} . \]
Without loss of generality, we can assume \( I_u \neq \emptyset \). Thus, there is \( R_u > 0 \) such that \( I_u = (0, R_u) \). Now, if
\[ v(r) = u(r) - 1, \quad r \in (0, R_u) \]
we have \( v \in X^{1, p}_{R_u}(\alpha, \theta) \) and \( \|v\|_{L^p_\theta} \leq 1 \). Also, from Lemma 2.1 we have
\[ |u|^{\frac{p}{p-1}} \leq (1 + \epsilon)^{\frac{1}{p}} |v|^{\frac{p}{p-1}} + c_1(\epsilon, \alpha, \theta), \]
for some \( c_1 = c_1(\epsilon, \alpha, \theta) > 0 \). Hence, by choosing \( \epsilon > 0 \) small enough and \( \eta > 1 \) such that
\[ \mu(1 + \epsilon)^{\frac{1}{p}} \eta^{\frac{1}{p-1}} = \mu_{\alpha, \theta}, \]
the Hölder inequality and Theorem A imply
\[ \int_0^{R_u} e^{\mu |u|^{\frac{p}{p-1}}} |u|^q d\lambda_\theta \leq \left( \int_0^{R_u} |u|^q d\lambda_\theta \right)^{\frac{1}{q}} \left( \int_0^{R_u} e^{\mu |u|^{\frac{p}{p-1}}} d\lambda_\theta \right)^{\frac{1}{1-q}} \]
\[ \leq C(\epsilon, \alpha, \theta, \eta, \mu) \|u\|^q_{L^q_\theta} \left( \int_0^{R_u} e^{\mu |v|^{\frac{p}{p-1}}} d\lambda_\theta \right)^{\frac{1}{1-q}} \]
\[ \leq C(\epsilon, \alpha, \theta, \eta, \mu) \|u\|^q_{L^q_\theta} (|B_{R_u}|)^{\frac{1}{1-q}}. \]
Finally, since
\[ |B_{R_u}| = \int_0^{R_u} d\lambda_\theta \leq \int_0^{R_u} |u|^p d\lambda_\theta \leq \|u\|^p_{L^p_\theta} = 1 \]
and (cf. (1.10))
\[ \|u\|^q_{L^q_\theta} \leq C \|u\|^q \leq C(\alpha, \theta, q, \eta) \]
the inequality (4.11) gives the result.

Since we are supposing \( TMC(\sigma, \alpha, \theta) > s_{k_0}^{k_0} \), when \( k_0 = p - 1 \), to complete the proof of (4.8), and then the proof of Theorem 1.4, it is now enough to prove the following:

**Lemma 4.4.** For each \( 0 < \sigma < \mu_{\alpha, \theta} \), we have
\[ \lim_{s \to 0^+} \left[ 1 - \left( \frac{s}{\sigma} \right)^{p-1} \right] f(s) \begin{cases} 0, & \text{if } k_0 > p - 1 \\ \frac{s^0}{k_0!}, & \text{if } k_0 = p - 1. \end{cases} \]

**Proof:** Let \( (s_n) \) be an arbitrary sequence that \( s_n \downarrow 0 \). From Theorem 1.3, we can find a sequence \( (u_n) \subset X^{1, p}_{\infty} \), with \( \|u'_n\|_{L^p_\theta} \leq 1 \) and \( \|u_n\|_{L^p_\theta} = 1 \) such that
\[ f(s_n) = \frac{s_n^{k_0}}{k_0!} \int_0^\infty |u_n|^p \frac{d\lambda_\theta}{p} \sum_{j \geq k_0 + 1} \int_0^\infty s_n^{j-1} \frac{d\lambda_\theta}{j!} |u_n|^p \frac{d\lambda_\theta}{p} \]
\[ = \frac{s_n^{k_0}}{k_0!} \int_0^\infty |u_n|^p \frac{d\lambda_\theta}{p} + \sum_{j=0}^{\infty} \int_0^\infty \frac{s_n^j}{(j + k_0 + 1)!} |u_n|^p \frac{d\lambda_\theta}{p}. \]
Since \((\ell + k_0 + 1)! \geq \ell!\) and in view of Lemma 4.3, we can write
\[
 f(s_n) \leq \frac{s_n^{k_0}}{k_0!} \int_0^\infty |u_n|^p d\lambda_\theta + s_n^{k_0+1} \int_0^\infty \left( e^{\sigma|u_n|^p} \right) |u_n|^\left(\frac{(k_0+1)p}{p-1}\right) d\lambda_\theta
\]
\[
 \leq \frac{s_n^{k_0}}{k_0!} \int_0^\infty |u_n|^p d\lambda_\theta + s_n^{k_0+1} c(\sigma, \alpha, \theta).
\]

It follows that
\[
 \left( 1 - \left( \frac{s_n}{\sigma} \right)^{p-1} \right) f(s_n) \leq \left( 1 - \left( \frac{s_n}{\sigma} \right)^{p-1} \right) \frac{s_n^{k_0-(p-1)}}{k_0!} \left[ \int_0^\infty |u_n|^\left(\frac{k_0p}{p-1}\right) d\lambda_\theta + c(\sigma, \alpha, \theta)k_0!s_n \right]. \tag{4.12}
\]

**Case 1:** \(k_0 > p - 1\)

From (1.10) and (4.12), we obtain
\[
 \left( 1 - \left( \frac{s_n}{\sigma} \right)^{p-1} \right) f(s_n) \to 0, \quad \text{as} \quad n \to \infty.
\]

**Case 2:** \(k_0 = p - 1\)

We have
\[
 \left( 1 - \left( \frac{s_n}{\sigma} \right)^{p-1} \right) f(s_n) \leq \left( 1 - \left( \frac{s_n}{\sigma} \right)^{p-1} \right) \frac{1}{k_0!} \left[ 1 + c(\sigma, \alpha, \theta)s_n \right]
\]
\[
 \to \frac{\sigma^{k_0}}{k_0!}, \quad \text{as} \quad n \to \infty.
\]

Hence, the proof is completed. \(\blacksquare\)

## 5 Proof of Theorem 1.5

In this section, we will analyze the attainability of \(TMC(\sigma, \alpha, \theta)\) when the condition \(k_0 = p - 1\) holds.

**Lemma 5.1.** For any \(\sigma \in (0, \mu_{\alpha, \theta})\) we have
\[
 TMC(\sigma, \alpha, \theta) \geq \frac{\sigma^{k_0}}{k_0!}. \tag{5.1}
\]

In addition, if \(p > 2\) the above inequality becomes strict.

**Proof:** We follow the argument of Ishiwata [21]. Let \(u \in X_{\alpha, \theta}^{1,p}(\alpha, \theta)\) such that \(||u|| = 1\), and set
\[
 u_t(r) = t^{1/p} u\left(t^{1/p} r \right).
\]
We can easily show that
\[ ||u'_t||_{L^p}^p = t ||u'||_{L^p}^p, \]
\[ ||u_t||_{L^q}^q = t^{\frac{q-p}{p}} ||u||_{L^q}^q, \forall q \geq p. \]

In particular, for each \( t > 0 \) small enough, if \( \xi_t = (t + (1 - t)||u||_{L^p}^p)^{-1/p} \) we have
\[ ||\xi_t u_t||^p = t \xi_t^p ||u'||_{L^p}^p + \xi_t^p ||u||_{L^p}^p = 1. \]

Noticing that \( \xi_t^p \to 1/||u||_{L^p}^p \) as \( t \to 0^+ \), then for \( v_t = \xi_t u_t \) we have
\[
TM C(\sigma, \alpha, \theta) \geq \int_0^\infty \varphi_p \left( \sigma |v_t|^{\frac{p}{p-1}} \right) d\lambda_\theta \geq \frac{\sigma^{k_0}}{k_0!} \int_0^\infty |v_t|^p d\lambda_\theta + \frac{\sigma^{k_0+1}}{(k_0+1)!} \int_0^\infty |v_t|^{\frac{p(k_0+1)}{p-1}} d\lambda_\theta \]
\[ = \frac{\sigma^{p-1}}{(p-1)!} \left[ \xi_t^p ||u||_{L^p}^p + \frac{\sigma}{p} \xi_t^{p-1} + \frac{\sigma}{p} ||u||_{L^p}^{p-1} \right] \int_0^\infty |v_t|^{\frac{p-1}{p}} d\lambda_\theta \]
\[ \to \frac{\sigma^{p-1}}{(p-1)!}, \quad \text{as} \quad t \to 0. \]

This proves (5.1). Moreover, if \( p > 2 \) we observe that the function
\[ h_{p, \theta, \sigma}(t) = \xi_t^p ||u||_{L^p}^p + \frac{\sigma}{p} \xi_t^{p-1} + \frac{\sigma}{p} ||u||_{L^p}^{p-1} \]
satisfies \( h_{p, \theta, \sigma}(0) = 1 \) and \( h_{p, \theta, \sigma}'(t) > 0 \) for \( t > 0 \) small enough. Hence, the result follows from (5.2). \( \blacksquare \)

**Lemma 5.2.** (i) The function \( \sigma \mapsto \frac{(p-1)!}{\sigma^{p-1}} TM C(\sigma, \alpha, \theta) \) is non-decreasing for \( 0 < \sigma \leq \mu_{\alpha, \theta}. \)

(ii) Let \( 0 < \sigma_1 < \sigma_2 \leq \mu_{\alpha, \theta}. \) Suppose that \( TM C(\sigma_1, \alpha, \theta) \) is attained. Then
\[
\frac{(p-1)!}{\sigma_2^{p-1}} TM C(\sigma_2, \alpha, \theta) > \frac{(p-1)!}{\sigma_1^{p-1}} TM C(\sigma_1, \alpha, \theta)
\]
and \( TM C(\sigma_2, \alpha, \theta) \) is also attained.

**Proof:** (i) Since
\[
\frac{(p-1)!}{\sigma^{p-1}} \varphi_p \left( \sigma |t|^{\frac{p}{p-1}} \right) = (p-1)! \sum_{j=0}^{\infty} \frac{\sigma^{-(p-1)}}{j!} \frac{t^j}{j^{\frac{p}{p-1}}}
\]
it is clear that for all \( t \neq 0 \)
\[ \frac{(p-1)!}{\sigma_1^{p-1}} \varphi_p \left( \sigma_1 |t|^{\frac{p}{p-1}} \right) < \frac{(p-1)!}{\sigma_2^{p-1}} \varphi_p \left( \sigma_2 |t|^{\frac{p}{p-1}} \right), \quad 0 < \sigma_1 < \sigma_2 \leq \mu_{\alpha, \theta}. \] (5.3)
Thus, (i) is proved.

(ii) Since $TMC(\sigma_1, \alpha, \theta)$ is attained, we can pick $u \in X_{\infty}^{1,p}$ such that $\|u\| = 1$ and

$$TMC(\sigma_1, \alpha, \theta) = \int_0^\infty \varphi_p \left( \sigma_1 |u|^{\frac{p}{p-1}} \right) d\lambda_\theta.$$ 

Thus, the Lemma 5.1 and (5.3) yield

$$\frac{(p-1)!}{\sigma^2_{\alpha-1}} TMC(\sigma_2, \alpha, \theta) \geq \frac{(p-1)!}{\sigma^2_{\alpha-1}} \int_0^\infty \varphi_p \left( \sigma_2 |u|^{\frac{p}{p-1}} \right) d\lambda_\theta$$

$$> \frac{(p-1)!}{\sigma^2_{\alpha-1}} \int_0^\infty \varphi_p \left( \sigma_1 |u|^{\frac{p}{p-1}} \right) d\lambda_\theta$$

$$= \frac{(p-1)!}{\sigma^2_{\alpha-1}} TMC(\sigma_1, \alpha, \theta) \geq 1.$$ 

Then, we have $\frac{(p-1)!}{\sigma^2_{\alpha-1}} TMC(\sigma_2, \alpha, \theta) > 1$ and thus Theorem 1.4-(ii) asserts that $TMC(\sigma_2, \alpha, \theta)$ is attained.

Proof of Theorem 1.5 completed

(i) It follows directly from Lemma 5.2 and the definition of $\sigma_\star$.

(ii) From Lemma 5.2 the function $\sigma \mapsto \frac{(p-1)!}{\sigma^2_{\alpha-1}} TMC(\sigma, \alpha, \theta)$ is strictly increasing on $(\sigma_\star, \mu_{\alpha, \theta})$. Next, we will show that

$$TMC(\sigma_\star, \alpha, \theta) = \frac{\sigma_{\alpha-1}^p}{(p-1)!}.$$  

(5.4)

For our convention $TMC(0, \alpha, \theta) = 0$, we may assume $\sigma_\star \in (0, \mu_{\alpha, \theta})$. From Lemma 5.1, if (5.4) is not true we must have

$$TMC(\sigma_\star, \alpha, \theta) > \frac{\sigma_{\alpha-1}^p}{(p-1)!}.$$ 

Thus, since $\sigma_\star < \mu_{\alpha}$, Theorem 1.4-(ii) implies that $TMC(\sigma_\star, \alpha, \theta)$ is achieved for some $u_\star \in X_{\infty}^{1,p}$. Also, we have

$$\lim_{\sigma \to \sigma_\star} \int_0^\infty \varphi_p \left( \sigma |u_\star|^{\frac{p}{p-1}} \right) d\lambda_\theta = \int_0^\infty \varphi_p \left( \sigma_\star |u_\star|^{\frac{p}{p-1}} \right) d\lambda_\theta = TMC(\sigma_\star, \alpha, \theta) > \frac{\sigma_{\alpha-1}^p}{(p-1)!}.$$ 

Hence, if $\sigma \in (0, \sigma_\star)$ is sufficiently close to $\sigma_\star$, we must have

$$TMC(\sigma, \alpha, \theta) \geq \int_0^\infty \varphi_p \left( \sigma |u_\star|^{\frac{p}{p-1}} \right) d\lambda_\theta > \frac{\sigma_{\alpha-1}^p}{(p-1)!} > \frac{\sigma_{\alpha-1}^p}{(p-1)!}.$$ 

Thus, for such a $\sigma \in (0, \sigma_\star)$, Theorem 1.4-(ii) implies that $TMC(\sigma, \alpha, \theta)$ is achieved which contradicts the definition of $\sigma_\star$. This proves (5.4). Now, from (5.4) and Lemma 5.2-(ii), for each $\sigma \in (\sigma_\star, \mu_{\alpha, \theta})$, the supremum $TMC(\sigma, \alpha, \theta)$ is attained and we also have

$$\frac{(p-1)!}{\sigma^2_{\alpha-1}} TMC(\sigma, \alpha, \theta) > \frac{(p-1)!}{\sigma^2_{\alpha-1}} TMC(\sigma_\star, \alpha, \theta) = 1.$$  

(5.5)
In addition, Lemma 5.1, Theorem 1.4-(ii) and the definition of $\sigma_*$ yield

$$TMC(\sigma, \alpha, \theta) = \frac{\sigma^{p-1}}{(p-1)!}$$

for each $\sigma \in [0, \sigma_*]$. \hfill (5.6)

Now, it is clear that (5.5) and (5.6) give (1.14). Finally, let us denote

$$\overline{\sigma}_* = \inf \left\{ \sigma \in (0, \mu_{\alpha, \theta}) : TMC(\sigma, \alpha, \theta) > \frac{\sigma^{p-1}}{(p-1)!} \right\}.$$\hfill (5.17)

Then, Theorem 1.4-(ii) yields $\sigma_* \leq \overline{\sigma}_*$. If $\sigma_* < \overline{\sigma}_*$ we can pick $\sigma_0 \in (\sigma_*, \overline{\sigma}_*)$ for which we must have

$$\frac{(p-1)!}{\sigma_0^{p-1}} TMC(\sigma_0, \alpha, \theta) > \frac{(p-1)!}{\sigma_*^{p-1}} TMC(\sigma_*, \alpha, \theta) = 1,$$

that is,

$$TMC(\sigma_0, \alpha, \theta) > \frac{\sigma_0^{p-1}}{(p-1)!}$$

which contradicts the definition of $\overline{\sigma}_*$. Hence (1.15) holds. Finally, (iii) follows directly from Lemma 5.1.

References

[1] E. Abreu, L.G. Fernandes Jr, On a weighted Trudinger-Moser inequality in $\mathbb{R}^n$, J. Differential Equations 269 (2020), 3089–3118

[2] S. Adachi and K. Tanaka, Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents, Proc. Amer. Math. Soc. 128 (2000), 2051–2057.

[3] A. Alvino, F. Brock, F. Chiacchio, A. Mercaldo, M.R. Posteraro, Some isoperimetric inequalities on $\mathbb{R}^N$ with respect to weights $|x|^{\alpha}$, J. Math. Anal. Appl. 451 (2017), 280–318

[4] D. Cassani, F. Sani, C. Tarsi, Equivalent Moser type inequalities in $\mathbb{R}^2$ and the zero mass case J. Funct. Anal. 267 (2014), 4236-4263

[5] D. M. Cao, Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$, Comm. Partial Differential Equations 17 (1992), 407–435.

[6] P. Clément, D. G. Figueiredo, E. Mitidieri, Quasilinear elliptic equations with critical exponents, Topol. Methods Nonlinear Anal. 7 (1996), 133–170.

[7] J.C. Collins, Renormalization, Cambridge University Press, Cambridge, 1984.

[8] D.G. de Figueiredo, J.V. Gonçalves, O.H. Miyagaki, On a class of quasilinear elliptic problems involving critical Sobolev exponents, Commun. Contemp. Math. 2 (2000), 47–59.
[9] J. F. de Oliveira, *On a class of quasilinear elliptic problems with critical exponential growth on the whole space*, Topol. Methods Nonlinear Anal. 49, (2017) 529–550.

[10] J. F. de Oliveira, J. M. do Ó, P. Ubilla, *Hardy-Sobolev type inequality and supercritical extremal problem*, Discrete Contin. Dyn. Syst. 39 (2019) 3345–3364.

[11] J.F. de Oliveira, J.M. do Ó, P. Ubilla, *Existence for a k-Hessian equation involving supercritical growth*, J. Differential Equations, 267 (2019), 1001–1024.

[12] J. F. de Oliveira, J. M. do Ó, *Trudinger–Moser type inequalities for weighted Sobolev spaces involving fractional dimensions*, Proc. Amer. Math. Soc., 142 (8) (2014), 2813–2828.

[13] J.F. de Oliveira, J.M. do Ó, B. Ruf, *Extremal for a k-Hessian inequality of Trudinger-Moser type*, Math. Z. 295 (2020) 1683-1706.

[14] J.F. de Oliveira, P. Ubilla, *Extremal functions for a supercritical k-Hessian inequality of Sobolev-type*, Nonlinear Analysis: Real World Applications 60 (2021), 103314.

[15] J.F. de Oliveira, P. Ubilla, *Admissible solutions to Hessian equations with exponential growth*, Rev. Mat. Iberoam. 37 (2021), 749–773.

[16] J.M. do Ó, *N-Laplacian equations in $\mathbb{R}^N$ with critical growth*. Abstr. Appl. Anal. 2, (1997) 301-315

[17] J. M. do Ó, A. C. Macedo, J. F. de Oliveira, *A sharp Adams-type inequality for weighted Sobolev spaces*, Q. J. Math. 71 (2020) 517–538.

[18] J. M. do Ó, E. Silva, *Quasilinear Elliptic Equations with Singular Nonlinearity*, Adv. Nonlinear Stud., 16 (2016) 363–379.

[19] J. Jacobsen, K. Schmitt, *Radial solutions of quasilinear elliptic differential equations*. Handbook of differential equations, Amsterdam, (2004), 359-435.

[20] G.H. Hardy, *Note on a theorem of Hilbert*, Math. Z. 6 (1920), 314–0317.

[21] M. Ishiwata, *Existence and nonexistence of maximizers for variational problems associated with Trudinger-Moser type inequalities in $\mathbb{R}^N$*, Math. Ann. 351 (2011), 781–804.

[22] N. Ikoma, M. Ishiwata, H. Wadade, *Existence and non-existence of maximizers for the Moser-Trudinger type inequalities under inhomogeneous constraints*, Math. Ann. 373 (2019), 831–851.

[23] A. Kufner, B. Opic, *Hardy-type inequalities*, Pitman Res. Notes in Math., vol. 219, Longman Scientific and Technical, Harlow, 1990.

[24] N. Lam, G. Lu, *Sharp Trudinger-Moser inequality on the Heisenberg group at the critical case and applications*. Adv. Math. 231 (2012), 3259-3287.
[25] N. Lam, G. Lu, L. Zhang, *Equivalence of critical and subcritical sharp Trudinger-Moser-Adams inequalities*, Rev. Mat. Iberoam. **33** (2017), 1219–1246.

[26] N. Lam, G. Lu, L. Zhang, *Sharp singular Trudinger-Moser inequalities under different norms*, Adv. Nonlinear Stud. **19** (2019), 239–261.

[27] N. Lam, G. Lu, L. Zhang, *Existence and nonexistence of extremal functions for sharp Trudinger-Moser inequalities*, Advances in Mathematics **352** (2019), 1253-1298.

[28] Y. Li, B. Ruf, *A sharp Trudinger–Moser type inequality for unbounded domains in* $\mathbb{R}^n$, Indiana Univ. Math. J. **57** (2008) 451–480.

[29] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), 1077–1092.

[30] C. Palmer, P.N. Stavrinou, *Equations of motion in a non-integer-dimensional space*, Journal of Physics A. **37** (2004) 6987–7003.

[31] R. Panda, *On semilinear Neumann problems with critical growth for the $n$-Laplacian*, Nonlinear Anal. **26** (1996), 1347–1366.

[32] B. Ruf *A sharp Trudinger–Moser type inequality for unbounded domains in* $\mathbb{R}^2$, Journal of Functional Analysis **219** (2005) 340–367.

[33] F.H. Stillinger, *Axiomatic basis for spaces with noninteger dimension*, J. Math. Phys. **18**, (1977) 1224–1234.

[34] N. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483.

[35] M. Zubair, M.J. Mughal, Q.A. Naqvi, *On electromagnetic wave propagation in fractional space*, Nonlinear Analysis: Real World Applications. **12** (2011) 2844–2850.

[36] M. Zubair, M.J. Mughal, Q.A. Naqvi, *Electromagnetic Fields and Waves in Fractional Dimensional Space*, Springer, Berlin, 2012.