SU(3) symmetry breaking in decay constants and electromagnetic properties of pseudoscalar heavy mesons

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Abstract

In this paper, the decay constants and mean square radii of pseudoscalar heavy mesons are studied in the SU(3) symmetry breaking. Within the light-front framework, the ratios $f_{D_s}/f_D$ and $f_{B_s}/f_B$ are individually estimated using the hyperfine splittings in the $D^*_s(s) - D_s(s)$ and $B^*_s(s) - B_s(s)$ states and the light quark masses, $m_{s,q}$ ($q = u,d$), to extract the wave function parameter $\beta$. The values $f_{D_s}/f_D = 1.29 \pm 0.07$ and $f_{B_s}/f_B = 1.32 \pm 0.08$ are obtained, which are not only chiefly determined by the ratio of light quark masses $m_s/m_q$, but also insensitive to the heavy quark masses $m_{c,b}$ and the decay constants $f_{D,B}$. The dependence of $f_{B_c}/f_B$ on $\Delta M_{B_c,B^+_c}$ with the varied charm quark masses is also shown. In addition, the mean square radii are estimated as well. The values $\sqrt{\langle r^2_{D^+_s} \rangle / \langle r^2_{D^+_s} \rangle} = 0.740_{-0.050}^{+0.051}$ and $\sqrt{\langle r^2_{B^+_s} \rangle / \langle r^2_{B^+_s} \rangle} = 0.711_{-0.058}^{+0.049}$ are obtained, and the sensitivities of $\langle r^2_P \rangle$ on the heavy and light quark masses are similar to those of the decay constants.

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I. INTRODUCTION

The decay constants of pseudoscalar heavy mesons with $c$ and $b$ quarks play an important role for studies of CP violation and in extracting the Cabibbo-Kobayashi-Maskawa matrix elements. Experimentally, new data on the charm meson decay constants $f_D$ and $f_{Ds}$ have been reported \[1, 2\]. As the calculations of the decay constants are related to the wave function overlap of the quark and antiquark which are governed by the strong interaction, they therefore provide a crucial manner to compare different theoretical methods. In addition, the determination of $f_{Bs}$ remains beyond the reach of current experiments, thus the reliability of estimated $f_{Bs}$ by a theoretical approach is dependent on whether the determinations of $f_D$ and $f_{Ds}$ by this approach are consistent with the new data. During the last decade, the decay constants of pseudoscalar heavy mesons have been studied in lattice simulations \[3–9\], in the QCD sum rules approach \[10–13\], and in the relativistic quark model \[14–18\].

The understanding of the electromagnetic (EM) properties of hadrons is also an important topic, and the EM form factors which are calculated using nonperturbative methods are the useful tool for this purpose. There have been numerous experimental \[19–24\] and theoretical studies \[25\] of the EM form factors of the light pseudoscalar meson ($\pi$ and $K$). However, the EM form factors of heavy mesons (which contain one heavy quark) have much fewer studies \[26, 27\] than those of light ones. The present paper is devoted to an analysis of the wave function and decay constant by the hyperfine mass splitting of heavy mesons and the formulas of the decay constant and mean square radius within the light-front (LF) framework. We present the $SU(3)$ symmetry breaking effect in decay constants and electromagnetic properties of pseudoscalar heavy mesons.

The light-front quark model (LFQM) is a promising analytic method for solving the nonperturbative problems of hadron physics \[28\], as well as offering many insights into the internal structures of bound states. The basic ingredient in LFQM is the relativistic hadron wave function which generalizes distribution amplitudes by including transverse momentum distributions and contains all the information of a hadron from its constituents. The hadronic quantities are represented by the overlap of wave functions and can be derived in principle. The light-front wave function is manifestly a Lorentz invariant, expressed in terms of internal momentum fraction variables which are independent of the total hadron momentum. Moreover, the fully relativistic treatment of quark spins and center-of-mass
motion can be carried out using the so-called Melosh rotation [29]. This treatment has been successfully applied to calculate phenomenologically many important meson decay constants and hadronic form factors [30–36].

The remainder of this paper is organized as follows. In Sec. II an analysis of wave function and decay constant is presented. In Sec. III the formulism of LFQM is reviewed briefly, and the formulae of decay constant and mean square radius are derived. In Sec. IV numerical results and discussions are presented. Finally, the conclusions are given in Sec. V.

II. ANALYSES OF WAVE FUNCTION AND DECAY CONSTANT

The decay constant $f_P$ for a pseudoscalar meson is defined by a matrix element of the axial vector current between the vacuum and the meson bound state:

$$\langle 0 | \bar{q}_1 \gamma_\mu \gamma_5 q_2 | P(P) \rangle = i f_P P_\mu.$$  \hfill (2.1)

In a nonrelativistic approximation, $f_P$ is related to the Bethe-Salpeter wave function at the origin $|\Psi(0)|$ as [37–39]

$$f_P \simeq \frac{2\sqrt{N_c}}{\sqrt{M_P}}|\Psi(0)|,$$  \hfill (2.2)

where $N_c$ is the color number and $M_P$ is the mass of the meson. If we consider the potential of a hyperfine interaction inside the meson to $O(\alpha_s)$:

$$V_{\text{hf}} = \frac{4\alpha_s (3\hat{s}_1 \cdot \hat{r} \hat{s}_2 \cdot \hat{r} - \hat{s}_1 \cdot \hat{s}_2)}{3m_1m_2r^3} + \frac{32\pi\alpha_s \hat{s}_1 \cdot \hat{s}_2}{9m_1m_2} \delta^3(\vec{r}),$$  \hfill (2.3)

where $\alpha_s$ is the strong coupling constant, $s_{1,2}(m_{1,2})$ are the spins (masses) of the constituent quark. For the $s$-wave meson, the first term of Eq. (2.3) has no contribution and the second term can distinguish the pseudoscalar and vector mesons. Therefore, the hyperfine mass splitting is obtained as

$$\Delta M_{PV} = \frac{32\pi\alpha_s}{9m_1m_2} |\Psi(0)|^2.$$  \hfill (2.4)

By combining Eqs. (2.2) and (2.4) and canceling $|\Psi(0)|$, we obtain:

$$f_P = \left( \frac{27\Delta M_{PV} m_1 m_2}{8\pi\alpha_s M_P} \right)^{1/2}.$$  \hfill (2.5)
If we suppose the strong coupling constants $\alpha_s(D) \simeq \alpha_s(D_s)$ and $\alpha_s(B) \simeq \alpha_s(B_s)$, then the ratios of the decay constants can be obtained as

$$
\frac{f_{D_s}}{f_D} = \left( \frac{\Delta M_{D_sD_s} - M_{D_s} m_s}{\Delta M_{DD} - M_D m_q} \right)^{1/2}, \quad \frac{f_{B_s}}{f_B} = \left( \frac{\Delta M_{B_sB_s} - M_{B_s} m_s}{\Delta M_{BB} - M_B m_q} \right)^{1/2},
$$

(2.6)

where $q = u, d$. From Eq. (2.6), we find that the ratios are dependent on the ratio of light quark masses $m_s/m_q$ and are independent of heavy quark masses. Furthermore, by canceling the ratio $m_s/m_q$, we have a relation which does not contain any parameter in the nonrelativistic approximation:

$$
\frac{f_{B_s}}{f_B} = \left( \frac{\Delta M_{DD} - \Delta M_{BB} - M_{D_s} M_{B_s}}{\Delta M_{DD} + \Delta M_{BB} - M_{D} M_{B}} \right)^{1/2} \frac{f_{D_s}}{f_D}.
$$

(2.7)

If one wants to include the relativistic correction to the ratios of the decay constants, not only the values of $m_{s,q}$, but also the form of wave function $\Psi(\vec{r})$ must be known. Let us come back to Eq. (2.2). The deviation of Eq. (2.2) uses the Fourier transform

$$
\Psi(\vec{r}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{r} \cdot \vec{k}} \phi(\vec{k}),
$$

(2.8)

where $\phi(\vec{k})$ is the wave function in the momentum space. If the Fourier transform is evaluated for the positive-energy projection of the Bethe-Salpeter wave function at equal “time” $z^+ = z^0 + z^3 = 0$, then

$$
f_P \sim \int \frac{dxd^2k_\perp}{2(2\pi)^3} \phi(x, k_\perp),
$$

(2.9)

where $x$ is the longitudinal momentum fraction, $k_\perp$ are the relative transverse momenta, and $\phi(x, k_\perp)$ satisfies the normalization condition:

$$
\int \frac{dxd^2k_\perp}{2(2\pi)^3} |\phi(x, k_\perp)|^2 = 1.
$$

(2.10)

In general, the momentum distribution amplitude $\phi(x, k_\perp)$ is obtained by solving the light-front QCD bound state equation $H_{LF}|P\rangle = M|P\rangle$ which is the familiar Schrödinger equation in ordinary quantum mechanics, and $H_{LF}$ is the light-front Hamiltonian. However, at the present time, how to solve the bound state equation is still unknown. Alternatively, we come back to the still-unknown wave function $\Psi(\vec{r})$ and express it as a linear combination of the arbitrary known functions which form a complete set. Of course, the complete set is
not unique. Here we give two examples for comparison. One is the solution of $1/r$ potential [41]:

$$\Psi_{nlm}(\vec{r}) = \left(\frac{2\beta}{n}\right)^{3/2} \left[\frac{(n-l-1)!}{2n[(n+l)!]^2}\right]^{1/2} \left(\frac{2\beta r}{n}\right)^l L_{n-l-1}^{2l+1}(2\beta r/n)\exp\left(-\frac{\beta r}{n}\right)Y_{lm}, \quad (2.11)$$

where superscript $c$ means “Coulomb,” $\beta$ is a parameter which has the energy dimension, $Y_{lm}$ is the spherical harmonics, and $L_{q-p}^p(x)$ is an associated Laguerre polynomial which is defined as

$$L_{q-p}^p(x) \equiv (-1)^p \left(\frac{d}{dx}\right)^p e^x \left(\frac{d}{dx}\right)^q (e^{-x} x^q). \quad (2.12)$$

The other complete set is the solution of an isotropic harmonic oscillator [41]:

$$\Psi_{nlm}(\vec{r}) = \frac{\beta^{3/2}}{\pi^{1/4}} (\beta r)^l h_{nl}(\beta r) \exp\left(-\frac{\beta^2 r^2}{2}\right) Y_{lm}, \quad (2.13)$$

where superscript $g$ means Gaussian and the first few $h_{nl}(\beta r)$’s are

$$h_{00} = 2, \quad h_{11} = \sqrt{\frac{8}{3}}, \quad h_{22} = \frac{4}{\sqrt{15}}, \quad h_{20} = \sqrt{6} \left(1 - \frac{2}{3}\beta^2 r^2\right). \quad (2.14)$$

Then the exact solution can be expressed as

$$\Psi(\vec{r}) = \sum_{nlm} a_{nlm}^{(c)} \Psi_{nlm}(\vec{r}), \quad (2.15)$$

where $\sum_{nlm} |a_{nlm}^{(c)}|^2 = 1$. This way seems very clumsy because, apart from $\beta$, it also introduces a series of undetermined coefficients, $a_{nlm}$. The following considerations, however, improve the situation. First, only the coefficients $a_{n00}$ survive because we just studied the $s$-wave meson. Second, we substitute Eq. (2.15) to Eq. (2.4) and obtain

$$\Delta M_{PV} = \frac{32\pi\alpha_s}{9m_1m_2} \left(\frac{\beta^3}{4\pi^{3/2}}\right) \left[\sum_{\text{even}} a_{n00}^q h_{n0}(0)\right]^2, \quad (2.16)$$

which takes the Gaussian case, for example. It is worth noting that the square bracket in Eq. (2.16) is independent of $\beta$. Then, the ratio of hyperfine mass splittings can be reduced as:

$$\frac{\Delta M_{D_sD_{s*}}}{\Delta M_{D_{s*}D_{s*}}} = \frac{m_q}{m_s} \left(\frac{\beta_{cs}}{\beta_{cq}}\right)^3, \quad \frac{\Delta M_{B_sB_{s*}}}{\Delta M_{B_{s*}B_{s*}}} = \frac{m_q}{m_s} \left(\frac{\beta_{bs}}{\beta_{bq}}\right)^3, \quad (2.17)$$

which does not include any coefficient $a_{nlm}$. Equation (2.17) is also suitable to the Coulomb case. In fact, due to the $\delta$ function in Eq. (2.3) having the dimension of an energy cube and
\( \vec{r} \) is vanishing here, Eq. (2.17) holds for any wave function which contains only one hadronic parameter \( \beta \). In addition, the ratios \( \beta_{cs}/\beta_{cq} \) and \( \beta_{bs}/\beta_{bq} \) in Eq. (2.17) are mainly influenced by the ratio \( m_s/m_q \). This situation leads to the ratios \( f_{Ds}/f_D \) and \( f_{Bs}/f_B \) are sensitive to the \( SU(3) \) symmetry breaking, but are insensitive to the heavy quark masses, which will be shown later. In the literature, there are some early attempts \([42, 43]\) to account for flavor symmetry breaking in pseudoscalar meson decay constants. In the next section the wave function \( \Psi(\vec{r}) \) and the values of \( m_{s,q} \) are studied within the light-front framework.

III. LIGHT-FRONT FRAMEWORK

A. General Formulism

An s-wave meson bound state, consisting of a quark \( q_1 \) and an antiquark \( \bar{q}_2 \) with total momentum \( P \) and spin \( J \), can be written as (see, for example \([32]\))

\[
|M(P, S, S_z)\rangle = \int \{d^3 p_1 \} \{d^3 p_2 \} \frac{2}{(2\pi)^3} \delta^3(\tilde{P} - \tilde{p}_1 - \tilde{p}_2) \times \sum_{\lambda_1, \lambda_2} \Phi_{SS_z}(\tilde{p}_1, \tilde{p}_2, \lambda_1, \lambda_2) |q_1(p_1, \lambda_1)\bar{q}_2(p_2, \lambda_2)\rangle,
\]

where \( p_1 \) and \( p_2 \) are the on-mass-shell light-front momenta,

\[
\tilde{p} = (p^+, p^\perp), \quad p^\perp = (p^1, p^2), \quad p^- = \frac{m_q^2 + p_\perp^2}{p^+},
\]

and

\[
\{d^3 p\} = \frac{dp^+ dp^\perp}{2(2\pi)^3}, \quad |q(p_1, \lambda_1)\bar{q}(p_2, \lambda_2)\rangle = b^\dagger(p_1, \lambda_1) d^\dagger(p_2, \lambda_2) |0\rangle,
\]

\[
\{b(p', \lambda'), b^\dagger(p, \lambda)\} = \{d(p', \lambda'), d^\dagger(p, \lambda)\} = 2(2\pi)^3 \delta^3(\tilde{p}' - \tilde{p}) \delta_{\lambda'\lambda}.
\]

In terms of the light-front relative momentum variables \((x, k_\perp)\) defined by

\[
p_1^+ = (1 - x)P^+, \quad p_2^+ = xP^+, \quad p_1^\perp = (1 - x)P^\perp + k^\perp, \quad p_2^\perp = xP^\perp - k^\perp,
\]

the momentum-space wave-function \( \Psi_{SS_z} \) can be expressed as

\[
\Phi_{SS_z}(\tilde{p}_1, \tilde{p}_2, \lambda_1, \lambda_2) = \frac{1}{\sqrt{N_c}} R_{\lambda_1\lambda_2}(u, \kappa_\perp) \phi(x, k_\perp),
\]
where $R^{SS}_{\lambda_1 \lambda_2} \, (x, k_\perp)$ constructs a state of definite spin $(S, S_z)$ out of light-front helicity $(\lambda_1, \lambda_2)$ eigenstates. Explicitly,

\[
R^{SS}_{\lambda_1 \lambda_2}(x, k_\perp) = \sum_{s_1, s_2} \langle \lambda_1 | R^\dagger_M (1 - x, k_\perp, m_1) | s_1 \rangle \langle \lambda_2 | R^\dagger_M (x, -k_\perp, m_2) | s_2 \rangle \langle \frac{1}{2}; s_1 s_2 | \frac{1}{2}; SS \rangle,
\]

where $|s_i\rangle$ are the usual Pauli spinors, and $R_M$ is the Melosh transformation operator [30, 31]:

\[
\langle s | R_M (x, k_\perp, m_i) | \lambda \rangle = \frac{m_i + x_i M_0 + i \bar{s}_\lambda \cdot \vec{k}_\perp \times \vec{n}}{\sqrt{(m_i + x_i M_0)^2 + k_\perp^2}},
\]

with $x_1 = 1 - x$, $x_2 = x$, and $\vec{n} = (0, 0, 1)$ as a unit vector in the $\hat{z}$ direction. In addition,

\[
M_0^2 = (e_1 + e_2)^2 = \frac{m_1^2 + k_\perp^2}{1 - x} + \frac{m_2^2 + k_\perp^2}{x},
\]

\[
e_i = \sqrt{m_i^2 + k_\perp^2 + k_z^2}.
\]

where $k_z$ is the relative momentum in $\hat{z}$ direction and can be written as

\[
k_z = \frac{x M_0}{2} - \frac{m_2^2 + k_z^2}{2x M_0}.
\]

$M_0$ is the invariant mass of $q \bar{q}$ and generally different from the mass $M$ of the meson which satisfies $M^2 = P^2$. This is due to the fact that the meson, quark and antiquark cannot be simultaneously onshell. We normalized the meson state as

\[
\langle M(P', S', S'_z)|M(P, S, S_z)\rangle = 2(2\pi)^3 P^+ \delta^3(\vec{P}' - \vec{P})\delta_{S' S}\delta_{S'_z S_z},
\]

which led to Eq. (2.10).

In practice, it is more convenient to use the covariant form of $R^{SS}_{\lambda_1 \lambda_2}$ [30, 31, 36, 44]:

\[
R^{SS}_{\lambda_1 \lambda_2}(x, k_\perp) = \frac{\sqrt{P_1^+ P_2^+}}{\sqrt{2} \, \tilde{M}_0 (M_0 + m_1 + m_2)} \bar{u}(p_1, \lambda_1) \tilde{P} (\tilde{P} + M_0) \Gamma v(p_2, \lambda_2),
\]

where

\[
\tilde{M}_0 \equiv \sqrt{M_0^2 - (m_1 - m_2)^2}, \quad \tilde{P} \equiv p_1 + p_2,
\]

\[
\bar{u}(p, \lambda) u(p, \lambda') = \frac{2m}{p^+} \delta_{\lambda \lambda'}, \quad \sum_{\lambda} u(p, \lambda) \bar{u}(p, \lambda) = \frac{\vec{p} + m}{p^+},
\]

\[
\bar{v}(p, \lambda) v(p, \lambda') = -\frac{2m}{p^+} \delta_{\lambda \lambda'}, \quad \sum_{\lambda} v(p, \lambda) \bar{v}(p, \lambda) = -\frac{\vec{p} - m}{p^+}.
\]

For the pseudoscalar meson, we have $\Gamma = \gamma_5$, Eq. (3.11) can then be further reduced by the applications of equations of motion on spinors [36]:

\[
R^{SS}_{\lambda_1 \lambda_2}(x, k_\perp) = \frac{\sqrt{P_1^+ P_2^+}}{\sqrt{2} \, \tilde{M}_0} \bar{u}(p_1, \lambda_1) \gamma_5 v(p_2, \lambda_2).
\]
Next, we derive the formulas of the decay constant and the mean square radius for the pseudoscalar meson. The former is the main subject of this work, and the latter is used to fix some parameters.

**B. Formulas for decay constant and mean square radius**

The decay constants of pseudoscalar mesons \( P(q_1\bar{q}_2) \) are defined in Eq. (2.11). The matrix element can be calculated using the formulism in the last subsection:

\[
\langle 0|\bar{q}_2\gamma_\mu\gamma_5q_1|P(P)\rangle = \int \left\{ d^3p_1 \right\} \left\{ d^3p_2 \right\} 2(2\pi)^3 \delta^3(\vec{P} - \vec{p}_1 - \vec{p}_2) \phi_P(x, k_\perp) R_{\lambda_1\lambda_2}^0(x, k_\perp)
\]

\[
\times \langle 0|\bar{q}_2\gamma_\mu\gamma_5q_1|q_1\bar{q}_2\rangle.
\]

(3.14)

Since \( \sqrt{M_0x(1-x)} = \sqrt{A^2 + k_\perp^2} \), the decay constant can be extracted as:

\[
f_P = 2\sqrt{2N_c} \int \{dx\} \frac{A}{\sqrt{A^2 + k_\perp^2}} \phi_P(x, k_\perp)
\]

(3.15)

where \( \{dx\} = \frac{dx^2k_\perp}{16\pi^3} \) and \( A = m_1x + m_2(1-x) \).

Next, the EM form factor of a meson \( P \) is determined by the scattering of one virtual photon and one meson. It describes the deviation from the pointlike structure of the meson, and is a function of \( Q^2 \). Here, we considered the momentum of the virtual photon in a spacelike region, so it was always possible to orient the axes in such a manner that \( Q^+ = (P' - P)^+ = 0 \). Thus, the EM form factor was determined by the matrix element:

\[
\langle P(P')|J^+|P(P)\rangle = e \ F_P(Q^2)(P + P')^+,
\]

(3.16)

where \( J^\mu = \bar{q}e_\gamma^\mu q \), \( e_\gamma \) is the charge of quark \( q \) in \( e \) unit, and \( Q^2 = (P' - P)^2 < 0 \). With the light-front framework, \( F_P \) can be extracted by Eq. (3.16):

\[
F_P(Q^2) = e_q \int \{dx\} \frac{A^2 + k^\prime_\perp \cdot k''_\perp}{\sqrt{A^2 + k^\prime_\perp^2} \sqrt{A^2 + k''_\perp^2}} \phi_P(x, k_\perp) \phi_P(x, k''_\perp)
\]

\[
+ e_{\bar{q}_2} \int \{dx\} \frac{A^2 + k^\prime_\perp \cdot k''_\perp}{\sqrt{A^2 + k^\prime_\perp^2} \sqrt{A^2 + k''_\perp^2}} \phi_P(x, k_\perp) \phi_P(x, k''_\perp),
\]

(3.17)

where \( k^\prime_\perp = k_\perp + xQ_\perp \), \( k''_\perp = k_\perp - (1-x)Q_\perp \). For applying this to Eq. (3.22), it is convenient to consider the term \( \tilde{\phi}_P \equiv \phi_P(x, k_\perp)/\sqrt{A^2 + k_\perp^2} \) and take the Taylor expansion around \( k_\perp^2 \)

\[
\tilde{\phi}_P(k^2_\perp) = \tilde{\phi}_P(k^2_\perp) + \frac{d\tilde{\phi}_P}{dk_\perp^2} \bigg|_{Q_\perp=0} (k^2_\perp - k^2_\perp) + \frac{d^2\tilde{\phi}_P}{(dk_\perp^2)^2} \bigg|_{Q_\perp=0} (k^2_\perp - k^2_\perp)^2 + \cdots
\]

(3.18)
Then, by using the identity
\[ \int d^2 k_\perp (k_\perp \cdot A_\perp)(k_\perp \cdot B_\perp) = \frac{1}{2} \int d^2 k_\perp k_\perp^2 A_\perp \cdot B_\perp, \]
we can rewrite (3.17) to
\[ F_P(Q^2) = (e_{q_1} + e_{\bar{q}_2}) - Q^2 \int \{dx\} \phi_P^2(x, k_\perp) \left[ x^2 e_{q_1} + (1 - x)^2 e_{\bar{q}_2} \right] \left( \frac{\Theta_P A^2 + 2k_\perp^2}{A^2 + k_\perp^2} + \bar{\Theta}_P k_\perp^2 \right) \]
\[ + \mathcal{O}(Q^4), \]  
where
\[ \Theta_M = \frac{1}{\phi_M} \left( \frac{d \phi_M}{dk_\perp^2} \right), \quad \bar{\Theta}_M = \frac{1}{\phi_M} \left( \frac{d^2 \phi_M}{(dk_\perp^2)^2} \right). \]  
(3.21)

It should be realized that the size and the density of a hadron depend on the probe. For an EM probe, it is the electric charge radius \( \langle r^2 \rangle \) that is obtained. In the experimental view, \( \langle r_P^2 \rangle \) cannot be measured directly and is obtained by fitting the slope of \( F_P(Q^2) \) at \( Q^2 = 0 \), i.e.,
\[ \langle r_P^2 \rangle = 6 \frac{d F_P(Q^2)}{d Q^2} \bigg|_{Q^2=0}. \]  
(3.22)

Here the mean square radius is easily obtained:
\[ \langle r_P^2 \rangle = \langle r_{q_1}^2 \rangle + \langle r_{\bar{q}_2}^2 \rangle \]
\[ = e_{q_1} \left\{ -6 \int \{dx\} x^2 \bar{\phi}_P \left[ (A^2 + 2k_\perp^2) \frac{d}{dk_\perp^2} + (A^2 + k_\perp^2)k_\perp^2 \left( \frac{d}{dk_\perp^2} \right)^2 \right] \bar{\phi}_P \right\} \]
\[ + e_{\bar{q}_2} \left\{ -6 \int \{dx\} (1 - x)^2 \bar{\phi}_P \left[ (A^2 + 2k_\perp^2) \frac{d}{dk_\perp^2} + (A^2 + k_\perp^2)k_\perp^2 \left( \frac{d}{dk_\perp^2} \right)^2 \right] \bar{\phi}_P \right\}. \]  
(3.23)

It is worth mentioning that, first, the static property \( F_P(0) = e_P \) is quite easily checked in Eq. (3.20). Second, from Eq. (3.23), we find that the mean square radius is related to the first and second longitudinal momentum square derivatives of \( \bar{\phi} \) which contain the Melosh transformation effect.

If we take the heavy quark limit \( m_1 = m_Q \to \infty, m_Q(M_P) \) is unimportant for the low energy properties of the meson state, so it is more natural to use velocity \( v \) instead of momentum variable \( P \). The normalization of the meson state is rewritten as
\[ \langle P(v')|P(v)\rangle = 2(2\pi)^3 v^+ \delta^3(\Lambda v - \bar{\Lambda} v'), \]  
(3.24)
where $\bar{\Lambda} = M_P - m_Q$ is the residual center mass of the heavy meson and the meson states have a relation $|P(v)| = (M_P)^{-1/2}|P(P)|$. In addition, since $x$ is the longitudinal momentum fraction carried by the light antiquark, the meson wave function should be sharply peaked near $x \sim \Lambda_{QCD}/m_Q$. It is thus clear that $x \to 0$ and only terms of the form $X \equiv xm_Q$ survive in the wave function as $m_Q \to \infty$; that is, $X$ is independent of $m_Q$ in the heavy quark limit. Therefore, the normalization of the wave function Eq. (2.10) is rewritten as

$$\int \frac{dXd^2k_\perp}{(2\pi)^3}|\varphi(X,k_\perp)|^2 = 1.$$  

(3.25)

where $\varphi(X,k_\perp) = (m_Q)^{-1/2}\phi(x,k_\perp)$. Other replacements are $A \to \bar{A} = X + m_{q_2}$ and $\bar{\phi}(x,k_\perp) \to \bar{\varphi}(X,k_\perp)$.

Thus we can rewrite (3.23) as

$$\left\langle r^2_{Qq_2} \right\rangle = \left\langle r^2_Q \right\rangle + \left\langle r^2_{q_2} \right\rangle$$

$$= \frac{e_Q}{m_Q^2} \left\{ -6 \left\langle dX \right\rangle X^2 \bar{\varphi} \left[ (\bar{A}^2 + 2k_\perp^2) \frac{d}{dk_\perp^2} + (\bar{A}^2 + k_\perp^2)k_\perp \left( \frac{d}{dk_\perp^2} \right)^2 \right] \bar{\varphi} \right\}$$

$$+ \frac{e_{q_2}}{m_{q_2}^2} \left\{ -6 \left\langle dX \right\rangle \bar{\varphi} \left[ (\bar{A}^2 + 2k_\perp^2) \frac{d}{dk_\perp^2} + (\bar{A}^2 + k_\perp^2)k_\perp \left( \frac{d}{dk_\perp^2} \right)^2 \right] \bar{\varphi} \right\}.$$  

(3.26)

The first term of Eq. (3.26) vanished when $m_Q \to \infty$. This means that not only $\left\langle r^2_{Qq_2} \right\rangle$ is blind to the flavor of $Q$, but also $\left\langle r^2_P \right\rangle$ is insensitive to $m_1$ for the heavy meson. The former is the so-called flavor symmetry and the latter will be proven in the numerical calculation.

IV. NUMERICAL RESULTS AND DISCUSSIONS

In the nonrelativistic (NR) approximation, we substituted the experimental data [1] to Eq. (2.7), and obtained $f_{B_s}/f_B = (1.03 \pm 0.02)f_{D_s}/f_D$. If one wanted to evaluate $f_{B_s}/f_B$ and $f_{D_s}/f_D$ individually, then $m_s = 483$ MeV and $m_q = 310$ MeV were the “best-fit” values for the pseudoscalar and vector light meson masses [46]. The ratios were

$$\left. \frac{f_{D_s}}{f_D} \right|_{NR} = 1.226 \pm 0.002, \quad \left. \frac{f_{B_s}}{f_B} \right|_{NR} = 1.24 \pm 0.02.$$  

(4.1)

The former was a little smaller than the data [1, 2] $f_{D_s}/f_D|_{exp} = 1.27 \pm 0.06$, and the latter was almost larger than the other theoretical calculations (see Tables II and III).

In the light-front framework, the momentum distribution amplitude $\phi(x,k_\perp)$ or the wave function $\Psi(\mathbf{r})$ in principle is unknown unless all the coefficients $a_{n00}$ are obtained. However, we may suppose $a_{100}^c = 1$ or $a_{000}^g = 1$, that is, $\Psi(\mathbf{r}) = \Psi_{100}^c(\mathbf{r})$ or $\Psi(\mathbf{r}) = \Psi_{000}^g(\mathbf{r})$ as a trial
wave function to fit the relevant data. The other $a_{n00}$’s will be subsumed if the parameters appearing in the momentum distribution amplitude can not satisfy all experimental results. In other word, the coefficients $a_{n0m}$ are taken as another kind of parameter. Of course, based on the principle of quantum mechanics, the physical meanings of these new parameters are clear. Here we list the first $\phi_{n00}^c$:

$$\phi_{100}^c(x, k_\perp) = 8\left(\frac{2\pi}{\beta^2}\right)^{1/2} \frac{e_1 e_2}{x(1-x)M_0} \left[ \frac{\beta^2}{k_\perp^2 + k_\parallel^2 + \beta^2} \right]^2,$$

$$\phi_{000}^g(x, k_\perp) = 4\left(\frac{\pi}{\beta^2}\right)^{3/4} \frac{e_1 e_2}{x(1-x)M_0} \exp\left[ -\frac{k_\perp^2 + k_\parallel^2}{2\beta^2} \right],$$

and use the experimental data of $f_{x^+} = 130.4 \pm 0.2$ MeV and $\langle r_{x^+}^2 \rangle^{1/2} = 0.672 \pm 0.008$ fm to fit the parameters $m_q$ and $\beta_{qq}$. The results are $m_q = 0.172(0.251)$ GeV and $\beta_{qq} = 0.555 \pm 0.011(0.317 \pm 0.007)$ GeV for $\phi_{100}^c(\phi_{000}^g)$. As for the strange quark mass, in Ref. [47] $m_s - m_u = 0.23$ GeV was obtained with some interaction potentials, while in Ref. [31] $m_s - m_u = 0.12$ GeV in the invariant meson mass scheme. So here we use the values $m_s - m_u = 0.180 \pm 0.050$ GeV and $f_{K^+} = 155.5 \pm 0.8$ MeV to fix $\beta_{sq}$. The results are $\beta_{sq} = 0.463^{-0.032}_{+0.054}(0.354^{-0.009}_{+0.015})$ GeV for $\phi_{100}^c(\phi_{000}^g)$. The charge radius $\langle r_{K^0}^2 \rangle^{1/2}$ and the mean square radius $\langle r_{K^0}^2 \rangle$ were calculated by these parameters and were listed in Table I. We found that, on one hand, the value of $\langle r_{K^+}^2 \rangle^{1/2}$ ($\langle r_{K^0}^2 \rangle$) for $\phi_{100}^c$ was too large (small) than that obtained in the experiment. Then, the coefficients $a_{n00}$ for $n > 1$ may be taken as nonzero to correct the fitting of $\langle r_{K^+}^2 \rangle^{1/2}$ and $\langle r_{K^0}^2 \rangle$. However, the mean square radii of $\phi_{n00}^c$ is greater when $n$ is larger, or $\langle r^2 \rangle_{\phi_{000}^c} < \langle r^2 \rangle_{\phi_{200}^c} < \langle r^2 \rangle_{\phi_{300}^c} < \langle r^2 \rangle_{\phi_{000}^g} < \langle r^2 \rangle_{\phi_{000}^g} < \langle r^2 \rangle_{\phi_{000}^g} < \ldots$ This means, for decreasing the value of $\langle r_{K^+}^2 \rangle^{1/2}$, the values of $a_{n00}^c$ must be artificially arranged in order to cancel out the contributions of $\phi_{n00}^c$ ($n \geq 2$) mutually. It is too hard to achieve now. On the other hand, the results for $\phi_{000}^g$ were consistent with the experimental data. Therefore we only use the Gaussian-type wave function, $\phi_{000}^g$, to the following calculation. By combining Eq.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
 & Experiment$[1]$ & $\phi_{100}$ & $\phi_{000}$ & DS$[48]$ & VMD$[49]$ & pQCD$[50]$ & BS$[51]$ \\
\hline
$\langle r_{K^+}^2 \rangle^{1/2}$ (fm) & 0.560 $\pm$ 0.031 & 0.710$^{+0.033}_{-0.044}$ & 0.607$^{+0.010}_{-0.012}$ & 0.49 & 0.616 & 0.570 & 0.62 \\
\hline
$\langle r_{K^0}^2 \rangle$ (fm$^2$) & $-0.077 \pm 0.010$ & $-0.121^{+0.036}_{-0.039}$ & $-0.072^{+0.017}_{-0.019}$ & -0.20 & 0.057 & -0.0736 & -0.085 \\
\hline
\end{tabular}
\caption{Charge radius $\langle r_{K^+}^2 \rangle^{1/2}$ and the mean square radius $\langle r_{K^0}^2 \rangle$ of the experiment, this work, and other theoretical estimations. DS is Dyson-Schwinger equations; VMD$\chi$ is vector meson dominance plus an effective chiral theory; BS is Bethe-Salpeter equation.}
\end{table}
FIG. 1: Dependence of $f_{D_s}/f_D$ on $f_D$ with $m_c = 1.2, 1.5, 1.8$ GeV. The left and right vertical dash lines correspond to the lower and upper limits of the data $f_D = 205.8 \pm 8.9$ MeV, respectively.

For the heavy quark masses, the quite different values were also used in the model calculations. For example, $m_c = 1.38$ GeV and $m_b = 4.76$ GeV which were fitted for the spectrum of the $p$-wave charmonium and bottomonium states \[52\]; and $m_c = 1.8$ GeV and $m_b = 5.2$ GeV which were obtained from the potential models and the variational principle \[53\]. Here the values $m_c = 1.2, 1.5, 1.8$ GeV and $m_b = 4.2, 4.7, 5.2$ GeV were taken into account. By combining Eqs. (3.15), (4.3), (4.4), and the quark masses $m_{q(s)} = 0.251(0.431)$ GeV, the dependences of $f_{D_s}/f_D$ on $f_D$ with three different $m_c$’s and $f_{B_s}/f_B$ on $f_B$ with three different $m_b$’s were shown in Fig.1 and Fig.2, respectively. It was easily found that the ratios $f_{D_s}/f_D$ and $f_{B_s}/f_B$ were not only insensitive to the heavy quark masses $m_c$ and $m_b$, but also insensitive to the decay constants $f_D$ and $f_B$, respectively.

Recently, the CLEO collaboration updated their data about the branching fraction for the purely leptonic decay $D^+ \rightarrow \mu^+\nu$ and reported \[1\] $f_{D^+}^{sp} = 205.8 \pm 8.9$ MeV. By using this value, we determined the ratio $f_{D_s}/f_D = 1.29 \pm 0.07$ and the decay constant $f_{D_s} = \ldots$
FIG. 2: Dependence of $f_{B_s}/f_B$ on $f_B$ with $m_c = 4.2, 4.7, 5.2$ GeV. The left and right vertical dash lines correspond to the lower and upper limits of the data $f_B = 204 \pm 31$ MeV, respectively.

$264.5 \pm 17.5$ MeV with $m_c = 1.5$ GeV. We found these results were consistent with the data [2]: $f_{D_s}^{\text{exp}} = 261.2 \pm 6.9$ MeV and $f_{D_s}^{\text{exp}}/f_{D_s}^{\text{exp}} = 1.27 \pm 0.06$, which were the average of the CLEO and Belle results (which included the radiative corrections). In addition, our values were generally larger than the other theoretical calculations. Table II compares the theoretical calculations with experimental value. For the bottom sector, the Belle [54] and Babar [55, 56] collaborations found evidence for $B^- \rightarrow \tau^- \bar{\nu}$ decay which was not helicity suppressed. However, the Belle and Babar values had 3.5 and 2.6 standard-deviation significances, respectively; thus the average was provisional [57]: $\mathcal{B}(B^- \rightarrow \tau^- \bar{\nu}) = (1.42 \pm 0.43) \times 10^{-4}$. We extracted the decay constant $f_B^{\text{exp}} = 204 \pm 31$ MeV. By using this value, the ratio $f_{B_s}/f_B = 1.32 \pm 0.08$ and the decay constant $f_{B_s} = 270.0 \pm 42.8$ MeV with $m_b = 4.7$ GeV were obtained. Table III compares the theoretical calculations with the experimental value. Similar to the charm sector, our ratio $f_{B_s}/f_B$ was almost larger than all other calculations. It is worth mentioning that the decay constants of both pseudoscalar and vector heavy mesons have already been investigated by the author of Ref. [18] with the analysis of magnetic dipole decays of various heavy flavored mesons in the light-front quark model. The parameters in Ref. [18] were constrained by the variational principle for the QCD-motivated effective Hamiltonian. Roughly speaking, our above results showed the
TABLE II: Theoretical calculations of the decay constants $f_D$, $f_{Ds}$ (MeV), and the ratio $f_{Ds}/f_D$. 
QL is quenched lattice calculations, BS is Bethe-Salpeter equation, Linear and HO are the different potentials within LFQM. We have quoted only the value with $m_c = 1.5$ GeV in this work (LF).

|                        | $f_D$       | $f_{Ds}$    | $f_{Ds}/f_D$ |
|------------------------|-------------|-------------|--------------|
| Experiment             | 205.8 ± 8.9 [1] | 261.2 ± 6.9 [2] | 1.27 ± 0.06 $^a$ |
| This work (LF)         | 205.8 ± 8.9 | 264.5 ± 17.5 | 1.29 ± 0.07 |
| This work (NR)         |             |             | 1.226 ± 0.002 |
| Lattice (HPQCD+UKQCD) [3] | 208 ± 4    | 241 ± 3     | 1.162 ± 0.009 |
| QL (QCDSF) [4]         | 206 ± 6 ± 3 ± 22 | 220 ± 6 ± 5 ± 11 | 1.068 ± 0.018 ± 0.020 |
| QL (Taiwan) [5]        | 235 ± 8 ± 14 | 266 ± 10 ± 18 | 1.13 ± 0.03 ± 0.05 |
| Lattice (FNAL+MILC+HPQCD) [6] | 201 ± 3 ± 17 | 249 ± 3 ± 16 | 1.24 ± 0.01 ± 0.07 |
| QL (UKQCD) [8]         | 210 ± $10^{-17}$ | 238 ± $8^{-14}$ | 1.13 ± 0.02±0.04 |
| QL [9]                 | 211 ± $14^{-12}$ | 231 ± $12^{+6}_{-1}$ | 1.10 ± 0.02 |
| QCD Sum Rules [10]     | 177 ± 21     | 205 ± 22     | 1.16 ± 0.01 ± 0.02 |
| QCD Sum Rules [12]     | 203 ± 23     | 235 ± 24     | 1.15 ± 0.04 |
| Field Correlators [14] | 210 ± 10     | 260 ± 10     | 1.24 ± 0.04 |
| Potential Model [15]   | 234          | 268          | 1.15         |
| BS [16]                | 230 ± 25     | 248 ± 27     | 1.08 ± 0.01 |
| BS [17]                | 238          | 241          | 1.01         |
| Linear{HO} [18]        | 211(194)     | 248{233}     | 1.18{1.20}   |

$^a$This value is obtained by combining $f_D = 205.8 ± 8.9$ MeV [1] and $f_{Ds} = 261.2 ± 6.9$ MeV [2].

Flavor $SU(3)$ symmetry breaking $m_s/m_q = 1.72 ± 0.20$ leads into the ratios

$$\frac{f_{Ds}}{f_D} = 1.29 ± 0.07,$$

$$\frac{f_{Bs}}{f_B} = 1.32 ± 0.08.$$

For the $B_c$ meson, however, both the decay constant and the hyperfine splitting have not been measured yet. We considered the ratio of hyperfine mass differences:

$$\frac{\Delta M_{B_c,B_c^*}}{\Delta M_{B,B^*}} = m_q \left( \frac{\beta_{B_c}}{\beta_{B_q}} \right)^3. \quad (4.5)$$
TABLE III: Theoretical calculations of the decay constants $f_B$, $f_{B_s}$ (MeV), and the ratio $f_{B_s}/f_B$.

Only the value with $m_b = 4.7$ GeV has been quoted in this work (LF).

|                  | $f_B$       | $f_{B_s}$  | $f_{B_s}/f_B$  |
|------------------|-------------|------------|----------------|
| Experiment       | 204 ± 31 $^a$ |            |                |
| This work (LF)   | 204 ± 31    | 270.0 ± 42.8 | 1.32 ± 0.08    |
| This work (NR)   |             |            | 1.24 ± 0.02    |
| QL (QCDSF) [4]   | 190 ± 8 ± 23 ± 25 | 205 ± 7 ± 26 ± 17 | 1.080 ± 0.028 ± 0.031 |
| Lattice (HPQCD)  | 216 ± 9 ± 19 ± 4 ± 6 | 259 ± 32 | 1.20 ± 0.03 ± 0.01 |
| QL (UKQCD) [8]   | 177 ± 17 ± 22 | 204 ± 19 ± 24 | 1.15 ± 0.02 ± 0.04 ± 0.02 |
| QL [9]           | 179 ± 18 ± 26 | 204 ± 16 ± 28 | 1.14 ± 0.03 ± 0.00 ± 0.01 |
| QCD Sum Rules [11]| 178 ± 14 | 200 ± 14 | 1.12 ± 0.01 ± 0.03 |
| QCD Sum Rules [12]| 203 ± 23 | 236 ± 30 | 1.16 ± 0.05 |
| QCD Sum Rules [13]| 210 ± 19 | 244 ± 21 | 1.16 |
| Field Correlators [14]| 182 ± 8 | 216 ± 8 | 1.19 ± 0.03 |
| Potential Model [15]| 189    | 218     | 1.15         |
| BS [16]          | 196 ± 29    | 216 ± 32 | 1.10 ± 0.01 |
| BS [17]          | 193         | 195      | 1.01         |
| Linear{HO} [18]  | 189{180}    | 234{237} | 1.24{1.32}   |

$^a$This value is extracted by the branching ratio: $\mathcal{B}(B^- \to \tau^- \bar{\nu}) = (1.42 \pm 0.43) \times 10^{-4}$ [57].

Similar to the above cases, the ratio $f_{B_s}/f_B$ was insensitive to the value of $m_b$ and sensitive to that of $m_c/m_q$. The dependences of $f_{B_c}/f_B$ on $\Delta M_{B_c,B_c^*}$ with $m_c = 1.2, 1.5, 1.8$ GeV and $m_b = 4.7$ GeV were shown in Fig.3. Some model predictions were made for $f_{B_c}$ [15, 53, 58–62], and the range of these values was $f_{B_c} = 360 \sim 517$ MeV or $f_{B_c}/f_B = 1.76 \sim 2.53$. As shown in Fig.3, this range corresponded to $\Delta M_{B_c,B_c^*} = 16 \sim 64$ MeV. We found that this result was consistent with a calculation using the nonrelativistic renormalization group [63] $\Delta M_{B_c,B_c^*} = 48 \pm 15^{+14}_{-11}$ MeV.

Besides, the mean square radii of the heavy meson are calculated by the above parameters and Eq. (3.23). The dependences of $\langle r^2_{D^+,D^0,D_s} \rangle$ on $m_c$ and $\langle r^2_{B^+,B^0,B_s} \rangle$ on $m_b$ were shown in Fig.4 and Fig.5, respectively. It was easily found that, as mentioned at the end of sec. III, the mean square radii $\langle r^2_{D^+,D^0,D_s} \rangle$ and $\langle r^2_{B^+,B^0,B_s} \rangle$ were insensitive to the heavy quark masses.
FIG. 3: Dependences of $f_{B_c}/f_B$ on $\Delta M_{B_cB_c^*}$ with $m_c = 1.2, 1.5, 1.8$ GeV and $m_b = 4.7$ GeV. The low and high horizontal dash lines correspond to $f_{B_c} = 360$ MeV and $f_{B_c} = 517$ MeV, respectively. The left and right vertical dash lines correspond to $\Delta M_{B_cB_c^*} = 16$ MeV and $\Delta M_{B_cB_c^*} = 64$ MeV, respectively.

FIG. 4: Dependences of $\langle r_{D^+, D_0^+, D_s}^2 \rangle$ on $m_c = 1.2 \sim 1.8$ GeV.

$m_c$ and $m_b$, respectively. We used $m_c = 1.5$ GeV and $m_b = 4.7$ GeV to estimate the mean square radii of the heavy meson and the results are listed in Table IV. It is interesting to note that the value $\langle r_{B^+}^2 \rangle$ is slightly lower but comparable to what one would obtain from the lattice calculation of Ref. [27], and it is considerably smaller than the results obtained by applying the simple vector meson dominance. The $SU(3)$ symmetry breaking in $\langle r_{D^+}^2 \rangle^{1/2}$,
FIG. 5: Dependences of $\langle r^2_{B^+,B^0,B_s}\rangle$ on $m_c = 4.2 \sim 5.2$ GeV.

TABLE IV: Mean square radius $\langle r^2_P \rangle$ (fm$^2$) of this work and the other theoretical calculations.

|       | This work | Lattice [27] | VMD |
|-------|-----------|--------------|-----|
| $D^+$ | $0.165^{+0.010}_{-0.011}$ |              |     |
| $D^0$ | $-0.261^{+0.018}_{-0.019}$ |              |     |
| $D_s^+$ | $0.0902^{+0.011}_{-0.014}$ |              |     |
| $B^+$ | $0.273^{+0.043}_{-0.059}$ | $0.334 \pm 0.003$ | 0.393$^a$ |
| $B^0$ | $-0.134^{+0.022}_{-0.029}$ |              |     |
| $B_s^0$ | $-0.0676^{+0.0141}_{-0.0189}$ |          |     |
| $B_c^+$ | $0.0277 \sim 0.0451$ $^b$ |              |     |

$^a$This value is obtained by $(r^2)_{VMD} = 6/M_{\rho}^2$.

$^b$This value is obtained by $f_{B_c} = 360 \sim 517$ MeV.

which mainly come from the mass difference $m_s - m_q = 180 \pm 50$ MeV, are obtained as

$$\sqrt{\frac{\langle r^2_{D^+} \rangle}{\langle r^2_{D^+} \rangle}} = 0.740^{+0.041}_{-0.050},$$

$$\sqrt{\frac{\langle r^2_{B^0} \rangle}{\langle r^2_{B^0} \rangle}} = 0.711^{+0.049}_{-0.058}.$$

The radius ratio of $B_c^+$ and $B^+$ are also obtained as

$$\sqrt{\frac{\langle r^2_{B_c^+} \rangle}{\langle r^2_{B^+} \rangle}} = 0.407^{+0.037}_{-0.038} \sim 0.319^{+0.029}_{-0.030},$$
which corresponds to the range \( f_{B_c} = 360 \sim 517 \text{ MeV} \).

V. CONCLUSIONS

In this study, we discussed the ratios of decay constants and mean square radii for pseudoscalar heavy mesons. By considering the hyperfine interaction inside the meson, we found that the ratio of light quark masses \( m_s/m_q \) was the important factor for determining the ratio of the decay constants. First, in the nonrelativistic approximation, we obtained the relation \( f_{B_s}/f_B = (1.05 \pm 0.02) f_{D_s}/f_D \) which did not use any parameters. These two ratios were individually evaluated by including the "best-fit" light quark masses \( m_s/m_q = 483/310 = 1.558 \) and the values \( f_{D_s}/f_D = 1.226 \pm 0.002 \) and \( f_{B_s}/f_B = 1.24 \pm 0.02 \) were obtained. Second, in the light-front framework, we utilized the mass difference of light quark masses \( m_s - m_q = 180 \pm 50 \text{ MeV} \) and the fittings of the decay constants for light mesons to compare the mean square radii of \( K^{+0} \) mesons in the power-law and Gaussian momentum distribution amplitudes. The latter was consistent with the data and it extracted the light quark masses ratio \( m_s/m_q = 1.72 \pm 0.20 \). This mass ratio led to \( f_{D_s}/f_D = 1.29 \pm 0.07 \) and \( f_{B_s}/f_B = 1.32 \pm 0.08 \). The former was in agreement with the experimental data and the latter was almost larger than all other theoretical calculations. Both these ratios were not only insensitive to the heavy quark masses \( m_{c,b} \), but also insensitive to the decay constants \( f_{D,B} \). Similar to the above, the ratio \( f_{B_c}/f_B \) was mainly determined by the mass ratio \( m_c/m_q \) and the mass splitting \( \Delta M_{B_c,B^*_c} \). The dependences of \( f_{B_c}/f_B \) on \( \Delta M_{B_c,B^*_c} \) with the varied charm quark masses have been shown. We found that \( f_{B_c}/f_B = 1.76 \sim 2.53 \) corresponded to \( \Delta M_{B_c,B^*_c} = 16 \sim 64 \text{ MeV} \). In addition, the mean square radii of heavy meson were estimated. We found the mean square radii \( \langle r^2_{D^+,D^0,D_s} \rangle \) and \( \langle r^2_{B^+,B^0,B_s} \rangle \) were insensitive to the heavy quark masses \( m_c \) and \( m_b \), respectively, which was consistent with the behavior when the heavy quark limit was taken. Our \( \langle r^2_{B^+} \rangle \) was slightly lower but comparable to that of lattice calculation, and was considerably smaller than that of vector meson dominance (VMD). The light quark mass ratio and the range of \( f_{B_c} \) given above also led the radius ratios \( \sqrt{\langle r^2_{D^+} \rangle/\langle r^2_{D^0} \rangle} = 0.740^{+0.041}_{-0.050} \), \( \sqrt{\langle r^2_{B^+} \rangle/\langle r^2_{B^0} \rangle} = 0.711^{+0.049}_{-0.058} \), and \( \sqrt{\langle r^2_{B^+} \rangle/\langle r^2_{B^0} \rangle} = 0.407^{+0.037}_{-0.038} \sim 0.319^{+0.029}_{-0.030} \) respectively.

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