Regularized Kaczmarz Algorithms for Tensor Recovery

Xuemei Chen* Jing Qin†

Abstract

Tensor recovery has recently arisen in a lot of application fields, such as transportation, medical imaging and remote sensing. Under the assumption that signals possess sparse and/or low-rank structures, many tensor recovery methods have been developed to apply various regularization techniques together with the operator-splitting type of algorithms. Due to the unprecedented growth of data, it becomes increasingly desirable to use streamlined algorithms to achieve real-time computation, such as stochastic optimization algorithms that have recently emerged as an efficient family of methods in machine learning. In this work, we propose a novel algorithmic framework based on the Kaczmarz algorithm for tensor recovery. We provide thorough convergence analysis and its applications from the vector case to the tensor one. Numerical results on a variety of tensor recovery applications, including sparse signal recovery, low-rank tensor recovery, image inpainting and deconvolution, illustrate the enormous potential of the proposed methods.

Keywords: Kaczmarz algorithm, tensor recovery, image inpainting, image deblurring, randomized algorithm

1 Introduction

Tensor is an important tool to represent, analyze and process high-dimensional data. By generalizing vectors and matrices, tensors can be used to represent a variety of data sets in a versatile and efficient way. Besides the traditional low-dimensional signal processing problems such as image restoration, tensor modeling is able to improve complex data analysis and processing by exploiting the hidden relationships among data components. Recently, tensor recovery has arisen in many application areas, such as transportation systems [47], medical imaging [51] and hyperspectral image restoration [19]. The goal is to reconstruct a tensor-valued signal from its measurements with multiple channels which may be degraded by noise, blur and so on. For example, many image restoration problems, e.g., image deblurring, can be cast as a tensor recovery problem by treating an image as a special type of tensor [25]. Moreover, tensor modeling typically imposes the assumption of sparsity and low-rank structures on the underlying tensor signal to be reconstructed, which brings the presence of sparsity-promoted regularizations in the objective function. In this work, we focus on third-order tensor recovery models given consistent linear measurements.

Tensor recovery problems usually involve massive data sets which result in large-scale systems of linear equations. As one of the most important iterative algorithm for solving linear systems, the Kaczmarz algorithm was first proposed by Stephen Kaczmarz [24] and then rediscovered as the Algebraic Reconstruction Technique (ART) in computed tomography (CT) [21]. Due to its simplicity and efficiency, it has been used and developed in many applications, including ultrasound imaging [3], seismic imaging [38], positron emission tomography [22], electrical impedance tomography [28] and recently phase retrieval [23] [48].

There is a large amount of work on the interpretations, developments and extensions of the Kaczmarz algorithm. For example, by treating each linear equation as a hyperplane of a high-dimensional space, it can be derived by applying the method of successive projections onto convex sets [10]. With random...
selection of projections, randomized Kaczmarz algorithm for solving consistent over-determined linear systems with a unique solution is proposed in [46], which can be considered as a special case of stochastic gradient matching pursuit (StoGradMP) [35]. Convergence analysis of randomized Kaczmarz for noisy and random linear systems can be found in [34, 13]. In addition, application of coordinate descent to the dual formulation of a linear equality constrained least-norm problem leads to the Kaczmarz algorithm [49]. Some other Kaczmarz types of methods include accelerated randomized Kaczmarz [18], asynchronous parallel randomized Kaczmarz [29], block Kaczmarz algorithms [40, 37, 17], Kaczmarz method for fusion frame recovery [14], and greedy randomized Kaczmarz [4]. Similar to the standard version, randomized sparse block Kaczmarz can also be obtained from randomized dual block-coordinate descent [39].

Motivated by the efficiency of the regularized Kaczmarz algorithm in solving linear systems, we intend to extend it from vectors to tensors. In this work, we integrate the Kaczmarz algorithm into sparse/low-rank tensor recovery to significantly reduce the computational cost while preserving high accuracy. To simplify the discussion, we focus on third-order tensors which can be further extended to other higher-order tensors. Specifically, we assume that the acquired tensor measurements $B \in \mathbb{R}^{N_1 \times K \times N_3}$ are related with the sensing tensor $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ and the underlying signal in a tensor form $X \in \mathbb{R}^{N_2 \times K \times N_3}$ via the tensor equation $g(X) = B$. To recover $X$, one can consider the following constrained minimization problem

$$\hat{X} = \arg\min_{X \in \mathbb{R}^{N_2 \times K \times N_3}} f(X), \quad \text{s.t.} \quad g(X) = B. \quad (1.1)$$

Here the objective function $f$ is convex on the real tensor space $\mathbb{R}^{N_2 \times K \times N_3}$, and $g$ is a map from $\mathbb{R}^{N_2 \times K \times N_3}$ to $\mathbb{R}^{N_1 \times K \times N_3}$. In this paper, we assume $g(X) = A \ast X$ where the symbol “$\ast$” denotes the t-product [26] (see Section 2.2). To solve this linear constrained minimization problem, we propose a general regularized Kaczmarz tensor algorithm, which involves random projection onto the hyperplane and subgradient descent. The very recent work by Ma and Molitor [32] also uses the Kaczmarz algorithm, but it focuses on solving the system $A \ast X = B$ without minimizing an objective function. We discuss convergence guarantees of the proposed algorithm with a deterministic or random control sequence. The proposed framework is also extended to solve the tensor nuclear norm minimization problem. We also discuss the important special cases when the third dimension of tensors is fixed as one for vector and matrix recovery problems. Furthermore, numerical experiments in various applications, including sparse vector recovery, image inpainting, low-rank tensor recovery, and single/multiple image deblurring, demonstrate the great potential of the proposed algorithms in terms of computational efficiency. There are three major contributions for this work detailed as follows.

1. We propose a novel regularized Kaczmarz algorithmic framework for tensor recovery problems with thorough convergence analysis. Due to the difference in data structures, extension of a Kaczmarz type of algorithm to tensors is not trivial. A linear convergence rate is proven for the randomized version. Moreover, we also consider the noisy scenario with a slightly stronger assumption on the objective function.

2. We provide three important cases of the proposed framework with convergence discussions. In particular, we propose a new algorithm for solving the tensor nuclear minimization problem with convergence guarantees, which is particularly useful in a lot of high-dimensional signal processing problems.

3. Numerical experiments on various signal/image recovery problems have justified the proposed performance, which can be further extended to solve other related application problems in various areas. In addition, the proposed algorithms are friendly to parameter tuning, where the step size can be fixed as one. Batched versions have empirically shown the capability of further improvements.

The rest of the paper is organized as follows. In Section 2, we introduce basic concepts and results in convex optimization, and tensors. Our main results are in Section 3, where we propose a tensor recovery method based on Kaczmarz algorithm with the convergence guarantees. For the randomized version of
2 Preliminaries

In this section, we provide clarification of notation and a brief review of fundamental concepts in convex optimization and tensor algebra.

Throughout the paper, we use boldface lowercase letters such as \( \mathbf{x} \) for vectors, capital letters such as \( X \) for matrices, and calligraphic letters such as \( \mathcal{X} \) for tensors. The sets of all natural numbers and real numbers are denoted by \( \mathbb{N} \) and \( \mathbb{R} \), respectively. Given a real number \( p \geq 1 \), the \( \ell_p \)-norm of a vector \( \mathbf{x} \in \mathbb{R}^N \) is defined as \( \| \mathbf{x} \|_p := \left( \sum_{i=1}^{N} |x_i|^p \right)^{1/p} \). Analogous to the vector \( \ell_2 \)-norm, the Frobenius norm of a matrix \( X \in \mathbb{R}^{N_1 \times N_2} \) is defined as \( \| X \|_F = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} x_{ij}^2 \). The trace of a square matrix \( X \), denoted by \( \text{Tr}(X) \), is the sum of all diagonal entries of \( X \). Furthermore, the nuclear norm of a matrix \( X \), denoted by \( \| X \|_* \), is defined as the sum of all the singular values of \( X \). For a complex-valued matrix \( X \), \( X^T \) is its transpose by interchanging the row and column index for each entry, and \( X^* \) is its complex conjugate transpose, i.e., performing both transpose and componentwise complex conjugate. For any positive integer \( k \), the set \{1, 2, \ldots, k\} is denoted by \([k]\). Given a finite set \( I \), the cardinality of \( I \) is denoted by \(|I|\).

For a convex set \( V \), \( P_V \) denotes the orthogonal projection onto \( V \). Given a matrix \( A \), \( R(A) \) is the row space of \( A \), and \( \sigma_{\min}(A) \), \( \sigma_{\max}(A) \) are the smallest and largest nonzero singular values of \( A \), respectively. One can show that

\[
\sigma_{\min}(A) \| P_{R(A)}(\mathbf{x}) \|_2 \leq \| A \mathbf{x} \|_2 \leq \sigma_{\max}(A) \| P_{R(A)}(\mathbf{x}) \|_2. \tag{2.1}
\]

The soft thresholding operator (also known as shrinkage) \( S_\lambda(\cdot) \) is defined componentwise as

\[
(S_\lambda(\mathbf{x}))(i) = \text{sgn}(x_i) \max\{|x_i| - \lambda, 0\}, \tag{2.2}
\]

where \( \mathbf{x} \in \mathbb{R}^N \) and \( \text{sgn}(\cdot) \) is the signum function which returns the sign of a nonzero number and zero otherwise. This operator can also be extended for matrices and tensors.

2.1 Convex Optimization Basics

To make the paper self-contained, we present basic definitions and properties about convex functions defined on real vector spaces. By reshaping matrices or tensors as vectors, these concepts and results can be naturally extended to functions defined on real matrix or tensor spaces.

For a continuous function \( f : \mathbb{R}^N \rightarrow \mathbb{R} \), its subdifferential at \( \mathbf{x} \in \mathbb{R}^N \) is defined as

\[
\partial f(\mathbf{x}) = \{ \mathbf{x}^* : f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{x}^*, \mathbf{y} - \mathbf{x} \rangle \text{ for any } \mathbf{y} \in \mathbb{R}^N \}. \tag{2.3}
\]

It can be shown that \( \partial f(\mathbf{x}) \) is closed and convex in \( \mathbb{R}^N \). If, in addition, \( f \) is convex, then \( \partial f(\mathbf{x}) \) is nonempty for any \( \mathbf{x} \in \mathbb{R}^N \). In addition, a convex function is called proper if its epigraph \( \{ (\mathbf{x}, \mu) : \mathbf{x} \in \mathbb{R}^N, \mu \geq f(\mathbf{x}) \} \) is nonempty in \( \mathbb{R}^{N+1} \).

Furthermore, \( f \) is \( \alpha \)-strongly convex for some \( \alpha > 0 \) if for any \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^N \) and \( \mathbf{x}^* \in \partial f(\mathbf{x}) \) we have

\[
f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{x}^*, \mathbf{y} - \mathbf{x} \rangle + \frac{\alpha}{2} \| \mathbf{y} - \mathbf{x} \|_2^2. \tag{2.3}
\]

Example 2.1. Let \( f_1(\mathbf{x}) = \frac{1}{2} \| \mathbf{x} \|_2^2 \), which is differentiable. In this case, the subdifferential becomes a singleton only consisting of the gradient, i.e., \( \partial f_1(\mathbf{x}) = \{ \nabla f_1(\mathbf{x}) \} = \{ \mathbf{x} \} \), and one can show that \( f_1 \) is 1-strongly convex.

Moreover, it is easy to show that \( h(\mathbf{x}) + \frac{1}{2} \| \mathbf{x} \|_2^2 \) is 1-strongly convex if \( h \) is convex. In what follows, we provide two such examples. For more details and examples, please refer to [5].
Example 2.2. Given a positive number $\lambda$, $f_{\lambda}(x) = \lambda \|x\|_1 + \frac{1}{2} \|x\|_2^2$ is $1$-strongly convex.

Example 2.3. Let $U$ be a real matrix, $\lambda > 0$, then $f_{U}(x) = \lambda \|Ux\|_1 + \frac{1}{2} \|x\|_2^2$ is $1$-strongly convex.

The convex conjugate function of $f$ at $z \in \mathbb{R}^N$ is defined as

$$f^*(z) := \sup_{x \in \mathbb{R}^N} \{ (z, x) - f(x) \}.$$ 

It can be shown that $z \in \partial f(x)$ if and only if $x = \nabla f^*(z)$ [11].

Following the ideas in [30], we can verify that if $f$ is $\alpha$-strongly convex then the conjugate function $f^*$ is differentiable and for any $z, w \in \mathbb{R}^N$, the following inequalities hold

$$\| \nabla f^*(z) - \nabla f^*(w) \|_2 \leq \frac{1}{\alpha} \| z - w \|_2,$$

$$f^*(y) \leq f^*(x) + \langle \nabla f^*(x), y - x \rangle + \frac{1}{2\alpha} \| y - x \|_2^2.$$ 

For a convex function $f : \mathbb{R}^N \to \mathbb{R}$, the Bregman distance between $x$ and $y$ with respect to $f$ and $x^* \in \partial f(x)$ is defined as

$$D_{f, x^*}(x, y) := f(y) - f(x) - \langle x^*, y - x \rangle.$$ 

Since $(x, x^*) = f(x) + f^*(x^*)$ if $x^* \in \partial f(x)$ [11], the Bregman distance can also be written as

$$D_{f, x^*}(x, y) = f(y) + f^*(x^*) - \langle x^*, y \rangle.$$ 

Note that for $f_1(x) = \frac{1}{2} \|x\|_2^2$, we have $D_{f_1, x^*}(x, y) = \frac{1}{2} \| x - y \|_2^2$. In general, the Bregman distance satisfies the property

$$\frac{\alpha f}{2} \| x - y \|_2^2 \leq D_{f, x^*}(x, y) \leq \langle x^* - y^*, x - y \rangle.$$ 

The proximal operator of $f$ is defined as

$$\text{prox}_f(v) = \text{argmin}_x \{ f(x) + \frac{1}{2} \| x - v \|_2^2 \}.$$ 

Due to the Fenchel’s duality, we have

$$\text{prox}_h(v) = \nabla f^*(v), \quad \text{where} \quad f(x) = h(x) + \frac{1}{2} \|x\|_2^2.$$ 

It can be shown that the soft thresholding operator is in fact the proximal operator of the $\ell_1$-norm, i.e., $S_\lambda(x) = \text{prox}_{\lambda \| \cdot \|_1}(x)$.

We provide an important lemma related to the Bregman distance, which will be used in proving Theorem 3.10.

Lemma 2.4. If $f : \mathbb{R}^N \to \mathbb{R}$ is convex, continuous and proper, then

$$D_{f, x^*}(x, y) - D_{f, w^*}(w, y) \leq \langle x^* - w^*, x - y \rangle,$$

for any $x^* \in \partial f(x)$ and $w^* \in \partial f(w)$.

Proof.

$$D_{f, x^*}(x, y) - D_{f, w^*}(w, y)$$

$$= f(y) - f(x) - \langle x^*, y - x \rangle - f(y) + f(w) + \langle w^*, y - w \rangle$$

$$= f(w) - f(x) - \langle x^*, y - x \rangle + \langle w^*, y - w \rangle$$

$$\leq \langle w^*, w - x \rangle - \langle x^*, y - x \rangle + \langle w^*, y - w \rangle$$

$$= (w^*, y - x) - (x^*, y - x) = (x^* - w^*, x - y)$$

$\square$
Next we will introduce the concept of restricted strong convexity. If $f : \mathbb{R}^N \to \mathbb{R}$ is convex, differentiable and proper, then \[ \langle \nabla f(y) - \nabla f(x), y - x \rangle \geq \alpha\|y - x\|_2^2 \] (2.10) for any $x, y \in \mathbb{R}^N$. We define a weaker version as follows.

**Definition 2.5.** Let $f : \mathbb{R}^N \to \mathbb{R}$ be convex differentiable with a nonempty minimizer set $X_f := \{x : f(x) \leq f(y) \text{ for any } y \in \mathbb{R}^N\}$. The function $f$ is **restricted strongly convex** on a convex set $C \subseteq \mathbb{R}^N$ with $\alpha > 0$ if
\[ \langle \nabla f(y) - \nabla f(x), y - x \rangle \geq \alpha\|y - x\|_2^2 \] (2.11)
for any $x \in C$ and $y = P_x \cap C(x)$.

This definition can be found in [42]. The concept of restricted strong convexity first appeared in [27] where $C$ is $\mathbb{R}^n$. We include below a useful lemma about the restricted strong convexity, which will be used for proving Lemma 3.6.

**Lemma 2.6 ([42 Lemma 2.2]).** If $f$ is restricted strongly convex on $C$ with the constant $\alpha$, then
\[ f(x) - \min_x f(x) \leq \frac{1}{\alpha}\|\nabla f(x)\|_2^2, \] (2.12)
for any $x \in C$.

### 2.2 Tensor Basics

We follow the notation for tensor operators in [25]. If $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ with the $k$th frontal slice $A_k = A(:, :, k)$, then we define the **block circulant operator** as follows
\[
\text{bcirc}(A) := \begin{bmatrix}
A_1 & A_{N_3} & \cdots & A_2 \\
A_2 & A_1 & \cdots & A_3 \\
\vdots & \vdots & \ddots & \vdots \\
A_{N_3} & A_{N_3-1} & \cdots & A_1
\end{bmatrix} \in \mathbb{R}^{N_1 N_3 \times N_2 N_3}. \tag{2.13}
\]

Moreover, we define the operator unfold$(\cdot)$ and its inversion fold$(\cdot)$ for the conversion between tensors and matrices
\[
\text{unfold}(A) = \begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_{N_3}
\end{bmatrix} \in \mathbb{R}^{N_1 N_3 \times N_2}, \quad \text{fold} \left( \begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_{N_3}
\end{bmatrix} \right) = A.
\]

For degenerate cases, we define the operator squeeze$(\cdot)$ that removes all the dimensions of length one from a tensor. For example, if $A \in \mathbb{R}^{3 \times 1 \times 1}$, then squeeze$(A)$ returns a three-dimensional vector. The transpose of $A$, denoted by $A^T$, is the $N_2 \times N_1 \times N_3$ tensor obtained by transposing each of the frontal slices and then reversing the order of transposed frontal slices 2 through $N_3$. We have
\[
\text{bcirc}(A^T) = (\text{bcirc}(A))^T.
\]

For the two tensors $A, \tilde{A}$ of the same size, we define the inner product and the Frobenius norm for tensors as
\[
\langle A, \tilde{A} \rangle := \sum_{i,j,k} A(i, j, k)\tilde{A}(i, j, k), \quad \|A\|_F^2 = \langle A, A \rangle.
\]

One can see that $\langle A, \tilde{A} \rangle = \langle \text{unfold}(A), \text{unfold}(\tilde{A}) \rangle$ where the right hand side is an inner product of two matrices. If $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}, C \in \mathbb{R}^{N_2 \times K \times N_3}$, then their **t-product** is the $N_1 \times K \times N_3$ tensor given by
\[
A \ast C := \text{fold} \left( \text{bcirc}(A) \text{unfold}(C) \right).
\]

We list three important properties about the t-product as follows:

\[5\]
1. Separability in the first dimension

\[(A \ast C)(i, :, :) = A(i, :, :) \ast C.\]  \hspace{1cm}  (2.14)

2. Sum separability in the second dimension

\[A \ast C = \sum_{j=1}^{N_2} A(:, j :) \ast C(j, :, :).\]  \hspace{1cm}  (2.15)

3. Circular convolution in the third dimension: if \(A, C \in \mathbb{R}^{1 \times 1 \times N_3}\), then

\[\text{squeeze}(A \ast C) = \text{circ}(a)c\]  \hspace{1cm}  (2.16)

where both \(a = \text{squeeze}(A)\) and \(c = \text{squeeze}(C)\) are \(N_3\)-dimensional vectors. Here \(\text{circ}(a)\) is the circular matrix generated by the vector \(a\), i.e., the reduced case of (2.13) when \(A \in \mathbb{R}^{1 \times 1 \times N_3}\).

In Table 2.2 we summarize the t-products of tensors with reduced dimensions and their corresponding operators for vectors or matrices. In particular, based on (2.14) and (2.16), the t-product of \(A \in \mathbb{R}^{1 \times 1 \times N_3}\) and \(C \in \mathbb{R}^{1 \times K \times N_3}\) can be implemented as

\[\text{squeeze}(A \ast C) = \text{squeeze}(C) \text{circ}(\text{squeeze}(A))^T,\]  \hspace{1cm}  (2.17)

where \(\text{squeeze}(A) \in \mathbb{R}^{N_2}\) and \(\text{squeeze}(C) \in \mathbb{R}^{K \times N_3}\). If \(A \in \mathbb{R}^{N_1 \times 1 \times N_3}\) and \(C \in \mathbb{R}^{1 \times 1 \times N_3}\), then

\[\text{squeeze}(A \ast C) = \text{squeeze}(A) \text{circ}(\text{squeeze}(C))^T,\]  \hspace{1cm}  (2.18)

where \(\text{squeeze}(A) \in \mathbb{R}^{N_1 \times N_3}\) and \(\text{squeeze}(C) \in \mathbb{R}^{N_3}\). Furthermore, if \(A \in \mathbb{R}^{1 \times N_2 \times N_3}\) and \(C \in \mathbb{R}^{N_2 \times 1 \times N_3}\), then the t-product becomes a sum of vector circular convolutions

\[\text{squeeze}(A \ast C) = \sum_{j=1}^{N_2} \text{circ}(\text{squeeze}(A)(j, :)) \text{circ}(\text{squeeze}(C)(j, :))^T,\]  \hspace{1cm}  (2.19)

where \(\text{squeeze}(A)(j, :) \in \mathbb{R}^{N_3}\) is the \(j\)th row of the matrix \(\text{squeeze}(A)\). These properties are particularly useful for representing the image blurring as a t-product; see Section 5.4 for more details.

**Lemma 2.7.** Let \(A \in \mathbb{R}^{N_1 \times N_2 \times N_3}, B \in \mathbb{R}^{N_1 \times K \times N_3}, C \in \mathbb{R}^{N_2 \times K \times N_3},\) then

\[\langle A \ast C, B \rangle = \langle C, A^T \ast B \rangle.\]  \hspace{1cm}  (2.20)
Definition 2.9.

If \( \mathcal{A} \in \mathbb{R}^{N_1 \times N_2 \times N_3} \) and \( \mathcal{X} \in \mathbb{R}^{N_2 \times K \times N_3} \), then we have
\[
\| \mathcal{A} \star \mathcal{X} \|_F \leq \sqrt{N_3} \| \mathcal{A} \|_F \| \mathcal{X} \|_F.
\]

Proof. \( \| \mathcal{A} \star \mathcal{X} \|_F^2 = \| \text{bcirc}(\mathcal{A}) \text{unfold}(\mathcal{X}) \|_F^2 \leq \| \text{bcirc}(\mathcal{A}) \|_F^2 \| \mathcal{X} \|_F^2 = N_3 \| \mathcal{A} \|_F^2 \| \mathcal{X} \|_F^2. \)

Lemma 2.8. Given a tensor \( \mathcal{A} \in \mathbb{R}^{N_1 \times N_2 \times N_3} \), \( \mathcal{X} \in \mathbb{R}^{N_2 \times K \times N_3} \), then
\[
\langle \mathcal{A} \star \mathcal{X}, \mathcal{B} \rangle = \langle \text{fold}(\text{bcirc}(\mathcal{A}) \text{unfold}(\mathcal{C})), \mathcal{B} \rangle = \langle \text{bcirc}(\mathcal{A}) \text{unfold}(\mathcal{C}), \text{unfold}(\mathcal{B}) \rangle
\]
\[
= \text{Tr} \left( (\text{bcirc}(\mathcal{A}) \text{unfold}(\mathcal{C}))^T \text{unfold}(\mathcal{B}) \right) = \text{Tr} \left( (\text{unfold}(\mathcal{C}))^T \text{bcirc}(\mathcal{A}^T) \text{unfold}(\mathcal{B}) \right)
\]
\[
= \langle \text{unfold}(\mathcal{C}), \text{bcirc}(\mathcal{A}^T) \text{unfold}(\mathcal{B}) \rangle
\]
\[
= \langle \mathcal{C}, \mathcal{A}^T \star \mathcal{B} \rangle.
\]

Lemma 2.10. If \( \mathcal{A} \in \mathbb{R}^{N_1 \times N_2 \times N_3} \), \( \mathcal{X} \in \mathbb{R}^{N_2 \times K \times N_3} \), and \( \mathcal{B} \in \mathbb{R}^{N_1 \times K \times N_3} \), then \( \mathcal{A} \star \mathcal{X} = \mathcal{B} \) is equivalent to
\[
\begin{bmatrix}
F(\mathcal{A})_1 \\
F(\mathcal{A})_2 \\
\vdots \\
F(\mathcal{A})_{N_3}
\end{bmatrix}
\begin{bmatrix}
F(\mathcal{X})_1 \\
F(\mathcal{X})_2 \\
\vdots \\
F(\mathcal{X})_{N_3}
\end{bmatrix}
= \begin{bmatrix}
F(\mathcal{B})_1 \\
F(\mathcal{B})_2 \\
\vdots \\
F(\mathcal{B})_{N_3}
\end{bmatrix}.
\]

Proof. Note that block circulant matrices can be block diagonalized by the Fourier transform. Let \( F_K \) be the \( K \times K \) discrete Fourier transformation matrix, \( I_K \) the \( K \times K \) identity matrix, and \( \otimes \) the Kronecker product. Then \( \mathcal{A} \star \mathcal{X} = \mathcal{B} \) can be converted to
\[
\text{bcirc}(\mathcal{A}) \text{unfold}(\mathcal{X}) = \text{unfold}(\mathcal{B}).
\]

By multiplying both sides of (2.23) on the left with \( F_{N_3} \otimes I_{N_1} \), we obtain
\[
(F_{N_3} \otimes I_{N_1}) \text{bcirc}(\mathcal{A}) (F_{N_3}^* \otimes I_{N_2}) (F_{N_3} \otimes I_{N_2}) \text{unfold}(\mathcal{X}) = (F_{N_3} \otimes I_{N_1}) \text{unfold}(\mathcal{B}),
\]
which is the same as (2.22), considering the fact that \( (F_{N_3} \otimes I_{N_1}) \text{bcirc}(\mathcal{A}) (F_{N_3}^* \otimes I_{N_2}) \) is the block diagonal matrix where each block is the frontal slice of \( F(\mathcal{A}) \).

Similarly, since \( (F_{N_3} \otimes I_{N_2}) [\text{bcirc}(\mathcal{A})]^* (F_{N_3}^* \otimes I_{N_1}) \) is a block diagonal matrix where the \( j \)th block is \( [F(\mathcal{A})_j]^* \), we have the following lemma.

Lemma 2.11.
\[
(F_{N_3} \otimes I_{N_2}) [\text{bcirc}(\mathcal{A})]^* \text{unfold}(\mathcal{X}) =
\begin{bmatrix}
F(\mathcal{A})_1 \\
F(\mathcal{A})_2 \\
\vdots \\
F(\mathcal{A})_{N_3}
\end{bmatrix}
\begin{bmatrix}
F(\mathcal{X})_1 \\
F(\mathcal{X})_2 \\
\vdots \\
F(\mathcal{X})_{N_3}
\end{bmatrix}.
\]
3 Tensor Recovery

Let $f$ be an $\alpha_f$-strongly convex function defined on $\mathbb{R}^{N_2 \times K \times N_3}$, $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ and $B \in \mathbb{R}^{N_1 \times K \times N_3}$. We assume the linear system $A * \mathcal{X} = B$ is consistent and underdetermined. Consider a tensor recovery problem of the following form

$$\hat{\mathcal{X}} = \arg\min_{\mathcal{X} \in \mathbb{R}^{N_2 \times K \times N_3}} f(\mathcal{X}), \quad \text{s.t.} \quad A * \mathcal{X} = B.$$  \hspace{1cm} (3.1)

According to the separability of t-product in the first dimension (2.14), the linear constraint $A * \mathcal{X} = B$ can be split into $N_1$ reduced ones

$$\mathcal{A}(i, ::) * \mathcal{X} = B(i, ::), \quad i \in [N_1].$$  \hspace{1cm} (3.2)

For the notational convenience, the $i$th horizontal slice $\mathcal{A}(i, ::)$ of $\mathcal{A}$ will be denoted as $\mathcal{A}(i)$. One can see that

$$\sum_{i=1}^{N_1} \|A(i) * \mathcal{X}\|_F^2 = \|A * \mathcal{X}\|_F^2.$$  \hspace{1cm} (3.3)

We further let $H_i := \{\mathcal{X} : A(i) * \mathcal{X} = B(i)\}$, and $H = \bigcap_{i=1}^{N_1} H_i$ be the feasible set of (3.1). The "row space" of $\mathcal{A}$ is defined as

$$R(\mathcal{A}) := \{A^T * \mathcal{Y} : \mathcal{Y} \in \mathbb{R}^{N_1 \times K \times N_3}\}.$$  

If $N_3 = 1$, then $R(\mathcal{A})$ coincides with the row space of the $N_1 \times N_2$ matrix $A$.

The optimal solution $\hat{\mathcal{X}}$ of (3.1) satisfies the following optimality conditions

$$A * \hat{\mathcal{X}} = B, \quad \partial f(\hat{\mathcal{X}}) \cap R(\mathcal{A}) \neq \emptyset.$$  \hspace{1cm} (3.4)

We propose Algorithm 3.1 that only uses one of the horizontal slice of $\mathcal{A}$ at each iteration. In order to make sure all horizontal slices are used, we define a control sequence for slice selection at each iteration.

**Definition 3.1.** A sequence $i : \mathbb{N} \rightarrow [M]$ is called a control sequence for $[M]$ if for any $m \in [M]$, there are infinitely many $k$’s such that $i(k) = m$.

Control sequences can be defined in a deterministic fashion. For example, a cyclic sequence $\{1, \cdots, N_1, 1, \cdots, N_1, \cdots\}$ is a control sequence for $[N_1]$.

**Algorithm 3.1 Regularized Kaczmarz Algorithm for Tensor Recovery**

**Input:** $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$, $B \in \mathbb{R}^{N_1 \times K \times N_3}$, control sequence $\{i(k)\}_{k=1}^{\infty} \subseteq [N_1]$, stepsize $t$, maximum number of iterations $T$, and the tolerance $\text{tol}$.

**Output:** an approximate of $\hat{x}$

**Initialize:** $Z(0) \in R(A) \subseteq \mathbb{R}^{N_2 \times K \times N_3}$, $\mathcal{X}(0) = \nabla f^*(Z(0))$.

for $k = 0, 1, \ldots, T - 1$ do

$Z(k+1) = Z(k) + tA(i(k))T \cdot \frac{B(i(k)) - A(i(k)) * X(k)}{\|A(i(k))\|_F^2}$

$\mathcal{X}(k+1) = \nabla f^*(Z(k+1))$

If $\|\mathcal{X}(k+1) - \mathcal{X}(k)\|_F / \|\mathcal{X}(k)\|_F < \text{tol}$, then it stops.

end for

In Algorithm 3.1 it can be shown that $Z(k) \in \partial f(\mathcal{X}(k)) \cap R(\mathcal{A})$ which will be used in our convergence analysis. Regarding the control sequence, one common choice is a cyclic sequence, which is cycling sequentially through the constraints (3.2). Algorithm 3.1 can be viewed as a deterministic algorithm, whereas Algorithm 3.2 picks the slices in a random fashion and has gained much attention in recent years [46, 50, 52, 33]. At each iteration, the probability of picking the $j$th constraint/slice (3.2) is proportional to $\|A(j)\|_F^2$, which is commonly used in literature [46]. Nevertheless, other probability distributions are also allowed; see Corollary 4.1 and Remark 4.2. More discussion on picking appropriate probability distributions can be found in [10, 2, 14].
Algorithm 3.2 Randomized Regularized Kaczmarz Algorithm for Tensor Recovery

Input: \( A \in \mathbb{R}^{N_1 \times N_2 \times N_3}, B \in \mathbb{R}^{N_1 \times K \times N_3} \), stepsize \( t \), maximum number of iterations \( T \), and the tolerance \( \text{tol} \).

Output: an approximate of \( \hat{X} \).

Initialize: \( Z^{(0)} \in R(\mathcal{A}) \subset \mathbb{R}^{N_2 \times K \times N_3}, \mathcal{A}^{(0)} = \nabla f^*(Z^{(0)}) \).

for \( k = 0, 1, \ldots, T - 1 \) do

pick \( i(k) \) randomly from \([N_1]\) with \( \Pr(i(k) = j) = \|A(j)\|_F^2/\|A\|_F^2 \),

\( Z^{(k+1)} = Z^{(k)} + tA(i(k))^T \ast \frac{B(i(k)) - \mathcal{A}(i(k)) \ast \mathcal{A}^{(k)}}{\|\mathcal{A}(i(k))\|_F^2} \)

\( \mathcal{A}^{(k+1)} = \nabla f^*(Z^{(k+1)}) \)

If \( \|\mathcal{A}^{(k+1)} - \mathcal{A}^{(k)}\|_F / \|\mathcal{A}^{(k)}\|_F < \text{tol} \), then it stops.

end for

3.1 Convergence Analysis with a Control Sequence

To analyze the convergence of the proposed Algorithms 3.1 and 3.2, we state the following proposition which plays an important role in the convergence analysis.

Proposition 3.2. Suppose \( A \in \mathbb{R}^{1 \times N_2 \times N_3}, B \in \mathbb{R}^{1 \times K \times N_3} \), and \( f \) is an \( \alpha_f \)-strongly convex function defined on \( \mathbb{R}^{N_2 \times K \times N_3} \). Given an arbitrary \( \hat{Z} \in \mathbb{R}^{N_2 \times K \times N_3} \) and \( \hat{X} = \nabla f^*(\hat{Z}) \), let \( Z = \hat{Z} + t \frac{\mathcal{A}^T \ast (B - A \ast \hat{X})}{\|A\|_F} \) and \( \mathcal{A} = \nabla f^*(Z) \). If \( t < \frac{2\alpha_f}{N_3} \), then

\[
D_{f, Z}(\mathcal{X}, \mathcal{H}) \leq D_{f, Z}(\hat{X}, \mathcal{H}) - \frac{t}{\|A\|_F^2} \left( 1 - \frac{tN_3}{2\alpha_f} \right) \|B - A \ast \hat{X}\|_F^2,
\]

for any \( \mathcal{H} \) that satisfies \( A \ast \mathcal{H} = B \).

Proof. For simplicity of notation, let \( Z = \hat{Z} + sW \), where \( W = \mathcal{A}^T \ast (B - A \ast \hat{X}) \). By Lemma 2.8, we have \( \|W\|_F \leq \sqrt{N_3}\|A\|_F^2 \|B - A \ast \hat{X}\|_F \). By Lemma 2.7, we have

\[
\langle W, \mathcal{H} - \hat{X} \rangle = \langle \mathcal{A}^T \ast (A \ast \mathcal{H} - A \ast \hat{X}), \mathcal{H} - \hat{X} \rangle = \|A \ast (\mathcal{H} - \hat{X})\|_F^2 = \|B - A \ast \hat{X}\|_F^2.
\]

By (2.7), the Bregman distance between \( \mathcal{X} \) and \( \mathcal{H} \) with respect to \( f \) and \( Z \) satisfies

\[
D_{f, Z}(\mathcal{X}, \mathcal{H}) = f^*(Z) - \langle Z, \mathcal{H} \rangle + f(\mathcal{H})
\]

\[
= f^*(\hat{Z} + sW) - \langle \hat{Z} + sW, \mathcal{H} \rangle + f(\mathcal{H})
\]

\[
\leq f^*(\hat{Z}) + \langle \nabla f^*(\hat{Z}), sW \rangle + \frac{1}{2\alpha_f} \|sW\|_F^2 - \langle \hat{Z} + sW, \mathcal{H} \rangle + f(\mathcal{H})
\]

\[
= D_{f, Z}(\hat{X}, \mathcal{H}) - \langle sW, \mathcal{H} \rangle + \langle \hat{X}, sW \rangle + \frac{1}{2\alpha_f} \|sW\|_F^2
\]

\[
= D_{f, Z}(\hat{X}, \mathcal{H}) - \langle sW, \mathcal{H} - \hat{X} \rangle + \frac{1}{2\alpha_f} \|sW\|_F^2
\]

\[
\leq D_{f, Z}(\hat{X}, \mathcal{H}) - s\|B - A \ast \hat{X}\|_F^2 + \frac{s^2}{2\alpha_f} N_3 \|A\|_F^2 \|B - A \ast \hat{X}\|_F^2
\]

\[
= D_{f, Z}(\hat{X}, \mathcal{H}) - s\|B - A \ast \hat{X}\|_F^2 \left( 1 - \frac{sN_3 \|A\|_F^2}{2\alpha_f} \right).
\]

The desired result is obtained by setting \( s = \frac{t}{\|A\|_F} \).

This proposition essentially shows how much the Bregman distance decreases after one iteration. We are now ready to state our main result.
Theorem 3.3. Let $f$ be $\alpha_f$-strongly convex. The sequence generated by Algorithm 3.1 ($t < 2\alpha_f/N_3$) satisfies

$$D_{f,Z^{(k+1)}}(X^{(k+1)},X) \leq D_{f,Z^{(k)}}(X^{(k)},X) - t \left(1 - \frac{tN_3}{2\alpha_f}\right) \|A(i(k)) \ast (X^{(k)} - X)\|^2_F,$$  \hspace{1cm} (3.5)

for all $X \in H(i(k))$. Moreover, the sequence $X^{(k)}$ converges to the solution of (3.1).

Proof. We apply Proposition 3.2 where $A(i(k)), B(i(k)), Z^{(k)}, Z^{(k+1)}$ are replaced by $A, B, \hat{Z}, \tilde{Z}$, respectively, and then obtain (3.3). The rest of the proof is for the convergence of the sequence $\{X^{(k)}\}$.

It is known that $\lim_{\|Z\|_F \to \infty} \frac{\|Z\|_F}{\|Z\|_F} = \infty$ (see 30, Equation (5)). Then for any $Z \in \partial f(X)$ and arbitrary $Y$, we have

$$\lim_{\|Z\|_F \to \infty} \frac{D_{f,Z}(X,Y)}{\|Z\|_F} = \lim_{\|Z\|_F \to \infty} \frac{f^*(Z) - \langle Z, Y \rangle + f(Y)}{\|Z\|_F} \geq \lim_{\|Z\|_F \to \infty} \frac{f^*(Z) - \|Z\|_F \|Y\|_F}{\|Z\|_F} = \infty.$$  

By (3.3), the sequence $\{D_{f,Z^{(k)}}(X^{(k)},\hat{X})\}$ is decreasing, hence bounded and convergent. The above limit implies that $\|Z(k)\|_F$ must be bounded. So for a subsequence $\{k_l\}$, we have $\lim_{l \to \infty} Z^{(k_l)} = \hat{Z}$. Thus, we get

$$\lim_{l \to \infty} X^{(k_l)} = \lim_{l \to \infty} \nabla f^*(Z^{(k_l)}) = \nabla f^*(\hat{Z}) := \bar{X}.$$  

We denote the constant $l \left(1 - \frac{tN_3}{2\alpha_f}\right)$ by $r$. Since $i(k)$ is a control sequence for $[N_1]$, there exists $j_0 \in [N_1]$ such that $i(k) \in [N_1]$ has infinitely many terms of $j_0$. Without loss of generality, the subsequence can be picked such that

$$\forall l, i(k_l) \equiv j_0, \text{ and } \{i(k_l), i(k_l + 1), \cdots, i(k_{l+1} - 1)\} = [N_1].$$

Therefore, we have

$$D_{f,Z^{(k+1)}}(X^{(k+1)},\hat{X}) \leq D_{f,Z^{(k+1)}}(X^{(k+1)},\hat{X})$$

$$\leq D_{f,Z^{(k)}}(X^{(k)},\hat{X}) - r \|A(i(k)) \ast (X^{(k)} - \hat{X})\|^2_F,$$  \hspace{1cm} (3.6)

By letting $l \to \infty$, we can get

$$D_{f,\hat{Z}}(\bar{X},\hat{X}) \leq D_{f,\hat{Z}}(\bar{X},\hat{X}) - r \|A(j_0) \ast (\bar{X} - \hat{X})\|^2_F,$$

which implies that $A(j_0) \ast (\bar{X} - \hat{X}) = 0$ and thereby $\bar{X} \in H_{j_0}$.

Let $J_{in} := \{j \in \hat{H}_j : \hat{H}_j \in \partial f\}$. Obviously, $j_0 \in J_{in}$ by the previous analysis. Next we use the proof by contradiction to further show $J_{in} = [N_1]$. Suppose that $J_{out} = [N_1] \setminus J_{in} \neq \emptyset$. For each $j_l$, we define $n_l$ to be the index in $\{k_l, k_l + 1, \cdots, k_{l+1} - 1\}$ such that $\{i(k_l), i(k_l + 1), \cdots, i(n_l - 1)\} \subset J_{in}$ and $i(n_l) \in J_{out}$.

Since $\bar{X} \in H_{j_l}$ for any $j \in \{i(k_l), i(k_l + 1), \cdots, i(n_l - 1)\}$, we have, by (3.5),

$$D_{f,Z^{(n)}}(X^{(n)},\hat{X}) \leq D_{f,Z^{(n)}}(X^{(n)},\hat{X}) \leq \cdots \leq D_{f,Z^{(n)}}(X^{(n+1)},\hat{X}) \leq \cdots \leq D_{f,Z^{(n)}}(X^{(n)},\hat{X}).$$

By letting $l \to \infty$, we have $\lim_{l \to \infty} X^{(n_l)} \equiv \hat{X}$.

By choosing a subsequence, we can assume $i(n_l) \equiv j_1 \in J_{out}$. By the same analysis as in (3.6), we can get $\bar{X} \in H_{j_1}$, which is a contradiction. Therefore, we must have $\bar{X} \in H = \cap_{l=1}^{N_3} H_{j_l}$.

By (3.5), $D_{f,Z^{(k)}}(X^{(k)},\hat{X})$ is decreasing. Since its subsequence $D_{f,Z^{(k)}}(X^{(k)},\hat{X}) \to 0$, the entire sequence $D_{f,Z^{(k)}}(X^{(k)},\hat{X})$ converges to zero as well. This shows $X^{(k)} \to \hat{X}$ due to (2.8).

Now we have $\hat{Z} = \lim_{k \to \infty} Z^{(k)} \in R(A)$. Since $Z^{(k)} \in \partial f(X^{(k)})$, we have $\hat{Z} \in \partial f(X^{(k)})$. So $A \ast \hat{X} = B$ and $\hat{Z} \in \partial f(\hat{X})$. Since $X^{(k)} \to \hat{X}$, we have $\hat{Z} = \hat{X}$.

Since the solution of (3.1) is unique, we have $\hat{X} = \bar{X}$.  \hfill \square
Lemma 3.6. Let \( g \) be admissible if

\[
\text{Definition 3.5.}\quad \text{where convergence rate in expectation if the objective function } f \text{ satisfies certain conditions.}
\]

For the randomized version - Algorithm 3.2 Theorem 3.3 no longer applies. We will prove a linear convergence rate in expectation if the objective function \( f \) satisfies certain conditions.

The dual problem of (3.1) is the unconstrained problem

\[
\min_{\mathcal{Y} \in \mathbb{R}^{N_1 \times K \times N_3}} g_f(\mathcal{Y}),
\]

where

\[
g_f(\mathcal{Y}) = f^*(A^T \star \mathcal{Y}) - \langle \mathcal{Y}, \mathcal{B} \rangle.
\]

**Definition 3.5.** Let \( m := \min_{\mathcal{Y}} g_f(\mathcal{Y}) \). We will call a function \( f \) admissible if \( f \), as defined in (3.3), is restricted strongly convex on any of \( g_f \)’s level set \( L_{\delta}^f := \{ \mathcal{Y} : g_f(\mathcal{Y}) \leq m + \delta \} \). We call \( f \) strongly admissible if \( g_f \) is restricted strongly convex on \( \mathbb{R}^{N_1 \times K \times N_3} \).

The following lemma is a consequence of Definition 2.5 and Lemma 2.6.

**Lemma 3.6.** Let \( \hat{\mathcal{Y}} \) be the solution of (3.1).

(a) Assume \( f \) is admissible. For \( \hat{\mathcal{X}} \) and \( \hat{\mathcal{Z}} \in \partial f(\hat{\mathcal{X}}) \cap R(\mathcal{A}) \), there exists \( \nu > 0 \) such that for all \( \mathcal{X} \) and \( \mathcal{Z} \in \partial f(\mathcal{X}) \cap R(\mathcal{A}) \) with \( D_{f,\mathcal{Z}}(\mathcal{X}, \hat{\mathcal{X}}) \leq \frac{1}{\nu} \| \mathcal{A} \ast (\mathcal{X} - \hat{\mathcal{X}}) \|_F^2 \) it holds

\[
D_{f,\mathcal{Z}}(\mathcal{X}, \hat{\mathcal{X}}) \leq \frac{1}{\nu} \| \mathcal{A} \ast (\mathcal{X} - \hat{\mathcal{X}}) \|_F^2.
\]

(b) If \( f \) is strongly admissible, there exists \( \nu > 0 \) such that (3.8) holds for all \( \mathcal{X} \) and \( \mathcal{Z} \in \partial f(\mathcal{X}) \cap R(\mathcal{A}) \).

**Proof.** (a) Due to the strong duality, we have that \( f(\hat{\mathcal{X}}) = -m = -\min_{\mathcal{Y}} g_f(\mathcal{Y}) \).

Since \( \mathcal{Z} \in R(\mathcal{A}) \), we let \( \mathcal{Z} = \mathcal{A}^T \ast \mathcal{Y} \) for some \( \mathcal{Y} \). Then

\[
D_{f,\mathcal{Z}}(\mathcal{X}, \hat{\mathcal{X}}) = f^*(\mathcal{Z}) - \langle \mathcal{Z}, \hat{\mathcal{X}} \rangle + f(\hat{\mathcal{X}}) = f^*(\mathcal{A}^T \ast \mathcal{Y}) - \langle \mathcal{A}^T \ast \mathcal{Y}, \hat{\mathcal{X}} \rangle + f(\hat{\mathcal{X}})
\]

\[
= f^*(\mathcal{A}^T \ast \mathcal{Y}) - \langle \mathcal{Y}, \mathcal{A} \ast \hat{\mathcal{X}} \rangle + f(\hat{\mathcal{X}}) = f^*(\mathcal{A}^T \ast \mathcal{Y}) - \langle \mathcal{Y}, \mathcal{B} \rangle - m
\]

\[
= g_f(\mathcal{Y}) - m.
\]

Similarly, for \( \hat{\mathcal{X}} \), there exists \( \hat{\mathcal{Y}} \) such that \( D_{f,\mathcal{Z}}(\hat{\mathcal{X}}, \hat{\mathcal{X}}) = g_f(\hat{\mathcal{Y}}) - m \).

The assumption \( D_{f,\mathcal{Z}}(\mathcal{X}, \hat{\mathcal{X}}) \leq D_{f,\mathcal{Z}}(\hat{\mathcal{X}}, \hat{\mathcal{X}}) \) implies that \( \mathcal{Y} \in \{ \mathcal{W} : g_f(\mathcal{W}) \leq g_f(\hat{\mathcal{Y}}) \} \), a level set of \( g_f \). By Lemma 2.6, the restricted strong convexity of \( g_f \) on its level set implies the existence of \( \nu > 0 \) such that

\[
g_f(\mathcal{Y}) - m \leq \frac{1}{\nu} \| \nabla g_f(\mathcal{Y}) \|_F^2, \quad \text{for all } \mathcal{Y} \in \{ \mathcal{W} : g_f(\mathcal{W}) \leq g_f(\hat{\mathcal{Y}}) \}
\]

Furthermore, the gradient of \( g_f \) is computed as

\[
\nabla g_f(\mathcal{Y}) = \mathcal{A} \ast \nabla f^*(\mathcal{A}^T \ast \mathcal{Y}) - \mathcal{B} = \mathcal{A} \ast \nabla f^*(\mathcal{Z}) - \mathcal{B} = \mathcal{A} \ast \mathcal{X} - \mathcal{B} = \mathcal{A} \ast (\mathcal{X} - \hat{\mathcal{X}}).
\]

This proves part (a).

(b) The proof is similar to (a) and simpler. Under the assumption that \( g_f \) is restricted strong convex on \( \mathbb{R}^{N_1 \times K \times N_3} \), we no longer need to require the dual variable \( \mathcal{Y} \) to be in a level set. The inequality (3.8) is thus true for any \( \mathcal{X} \) and \( \mathcal{Z} \in \partial f(\mathcal{X}) \cap R(\mathcal{A}) \).
Lemma 3.6 is a key lemma in proving linear convergence rate for randomized version. However, its assumptions are not easy to check. Below we provide some important examples.

**Example 3.7. Admissible functions.** See [42].

(a) If $f$ is strongly convex and piecewise quadratic and $bcirc(A)$ has full row rank, then $f$ is strongly admissible. There are many functions fall under this category, e.g., $f_{\lambda}(\mathcal{X}) = \frac{1}{2}\|\mathcal{X}\|_F^2 + \lambda\|\mathcal{X}\|_1$.

(b) Let $f$ be defined on matrices. The objective function of the regularized nuclear norm problem
$$f(X) = \frac{1}{2}\|X\|_F^2 + \lambda\|X\|_*$$
is admissible.

(c) When $f$ is defined on $\mathbb{R}^N$, important examples include $f_{\lambda}(x) = \frac{1}{2}\|x\|_2^2 + \lambda\|x\|_1$, or more generally
$$\frac{1}{2}\|x\|_2^2 + \lambda\|Ux\|_1$$, where $U$ is a linear operator.

**Remark 3.8.** The constant $\nu$ in Lemma 3.6 depends on the tensor $A$, the function $f$, and the corresponding level set (if $f$ is only admissible). For a simple example, we let $K = N_3 = 1$, so $A*X$ degenerates to the regular matrix-vector multiplication $Ax$. We let the objective function be $f_1(x) = \frac{1}{2}\|x\|_2^2$, we have $x = z \in R(A)$. Furthermore, the minimizer $\hat{x}$ is the projection of the any feasible vector onto $R(A)$, so $x - \hat{x} \in R(A)$. Then $D_{f_1, x}(x, \hat{x}) = \frac{1}{2}\|x - \hat{x}\|_2^2 = \frac{1}{2}\|P_R(A)(x - \hat{x})\|_2^2 \leq \frac{1}{2\sigma_{\min}(A)}\|A(x - \hat{x})\|_2^2$, which means that $\nu = 2\sigma_{\min}^2(A)$. For a less trivial example, we refer the readers to [27, Lemma 7] for an explicit computation of $\nu$ for the function $f_{\lambda}(x) = \frac{1}{2}\|x\|_2^2 + \lambda\|x\|_1$ ($A$ does not need to be full row rank). In general, it is hard to quantify $\nu$.

**Theorem 3.9.** Let $f$ be $\alpha_f$-strongly convex and admissible. Let $\mathcal{X}(k), Z^{(k)}$ be generated by the Algorithm 3.2. If $t < 2\alpha_f/N_3$, then $\mathcal{X}(k)$ converges linearly to $\hat{\mathcal{X}}$ in expectation, i.e.,

$$E\left[D_{f, Z^{(k)}+1}(\mathcal{X}(k+1), \hat{\mathcal{X}})\right] \leq \left(1 - \frac{\nu t(1 - \frac{tN_3}{2\alpha_f})}{\|A\|_F^2}\right) E\left[D_{f, Z^{(k)}}(\mathcal{X}(k), \hat{\mathcal{X}})\right],$$

and thereby

$$E\|\mathcal{X}(k) - \hat{\mathcal{X}}\|_F^2 \leq c \left(1 - \frac{\nu t(1 - \frac{tN_3}{2\alpha_f})}{\|A\|_F^2}\right)^k,$$

where $\nu$ is the constant from [3.8].

**Proof.** Let $G^{(k)} := \mathcal{X}(k) - \hat{\mathcal{X}}$. By Theorem 3.3,

$$D_{f, Z^{(k)+1}}(\mathcal{X}(k+1), \hat{\mathcal{X}}) \leq D_{f, Z^{(k)}}(\mathcal{X}(k), \hat{\mathcal{X}}) - r\|A(i(k)) + G^{(k)}\|_F^2 \|A(i(k))\|_F^2,$$

where $r = t\left(1 - \frac{tN_3}{2\alpha_f}\right)$. We need to analyze the expectation of $\|A(i(k)) + G^{(k)}\|_F^2 \|A(i(k))\|_F^2$.

Since $D_{f, Z^{(k)}}(\mathcal{X}(k), \hat{\mathcal{X}}) \leq D_{f, Z^{(0)}}(\mathcal{X}(0), \hat{\mathcal{X}})$ for any $k \geq 1$, we apply Lemma 3.6(a) and get

$$D_{f, Z^{(k)}}(\mathcal{X}(k), \hat{\mathcal{X}}) \leq \frac{1}{t_f}\|A*G^{(k)}\|_F^2.$$

At the $k$th iteration, the probability that $i(k) = j$ is proportional to $\|A(j)\|_F^2$. Therefore, the conditional expectation given the previous iterations $\mathcal{X}(0), \ldots, \mathcal{X}(k)$ is

$$E\left[D_{f, Z^{(k)}}(\mathcal{X}(k), \hat{\mathcal{X}})\right] = \frac{1}{\|A\|_F^2} \sum_{j=1}^{N_3} \frac{\|A(j)\|_F^2}{\|A\|_F^2} E\left[D_{f, Z^{(k)}}(\mathcal{X}(k), \hat{\mathcal{X}})\right] \leq \frac{\nu}{\|A\|_F^2} D_{f, Z^{(k)}}(\mathcal{X}(k), \hat{\mathcal{X}}).$$
Finally, we take the expectation of (3.11) and get
\[
\mathbb{E} \left[ D_{f, Z^{(k+1)}}(X^{(k+1)}, \hat{X}) \right] \\
\leq \mathbb{E} \left[ D_{f, Z^{(k)}}(X^{(k)}, \hat{X}) - r \frac{\|A(i(k)) \ast G^{(k)}\|_F^2}{\|A(i(k))\|_F^2} \right] \\
= \mathbb{E}_{i(1), \ldots, i(k-1)} \left[ D_{f, Z^{(k)}}(X^{(k)}, \hat{X}) - r \frac{\|A(i(k)) \ast G^{(k)}\|_F^2}{\|A(i(k))\|_F^2} \right](1, \ldots, i(k-1)) \\
\leq \mathbb{E}_{i(1), \ldots, i(k-1)} \left[ D_{f, Z^{(k)}}(X^{(k)}, \hat{X}) - \frac{r \nu}{\|A\|_F^2} D_{f, Z^{(k)}}(X^{(k)}, \hat{X}) \right] \\
= \left(1 - \frac{r \nu}{\|A\|_F^2} \right) \mathbb{E} \left[ D_{f, Z^{(k)}}(X^{(k)}, \hat{X}) \right].
\]

To prove (3.10), we note that \( \mathbb{E} \left[ D_{f, Z^{(k)}}(X^{(k)}, \hat{X}) \right] \leq \left(1 - \frac{r \nu}{\|A\|_F^2} \right)^k D_{f, Z^{(0)}}(X^{(0)}, \hat{X}) \). Then by (2.8), we have
\[
\mathbb{E} \|X^{(k)} - \hat{X}\|_F^2 \leq \frac{2}{\alpha_f} \mathbb{E} \left[ D_{f, Z^{(k)}}(X^{(k)}, \hat{X}) \right] \leq \left(1 - \frac{r \nu}{\|A\|_F^2} \right)^k.
\]

So far, we have taken care of consistent and noise-free constraints of the form \( A \ast X = B \). In what follows, we analyze the sensitivity of Algorithm 3.2 by considering the perturbed constraint \( A \ast X = \tilde{B} \) where \( \tilde{B} = B + \mathcal{E} \). Here \( \mathcal{E} \) is typically assumed to be Gaussian noise.

**Theorem 3.10.** Let \( A \ast X = B \) be the original consistent linear constraint. For the perturbed measurements \( \tilde{B} = B + \mathcal{E} \), Algorithm 3.2 performs the following updating scheme
\[
\tilde{Z}^{(k+1)} = \tilde{Z}^{(k)} + t A(i(k))^T \left( B(i(k)) + \mathcal{E}(i(k)) - A(i(k)) \ast \tilde{X}^{(k)} \right),
\]
\[
\tilde{X}^{(k+1)} = \nabla^* f(\tilde{Z}^{(k+1)}).
\]

Let \( f \) be \( \alpha_f \)-strongly convex and strongly admissible, and the stepsize \( t < 2\alpha_f/N_3 \). If \( \epsilon = \max_{i \in [N_1]} \frac{\|\mathcal{E}(i)\|_F}{\|A(i)\|_F} \), then we have
\[
\mathbb{E} \sqrt{D_{f, Z^{(k)}}(\tilde{X}^{(k)}, \hat{X})} \leq \sqrt{1 - \frac{t(1 - \frac{N_3}{2\alpha_f})}{\|A\|_F^2}} D_{f, Z^{(0)}}(\tilde{X}^{(0)}, \hat{X}) + \sqrt{\frac{2\alpha_f}{\nu(1 - \frac{N_3}{2\alpha_f})}} \mathbb{E},
\]
where \( \hat{X} \) is the solution of (3.1).

**Proof.** Given \( \tilde{Z}^{(k)} \) and \( \tilde{X}^{(k)} \), we define
\[
Z^{(k+1)} = \tilde{Z}^{(k)} + t A(i(k))^T \left( B(i(k)) - A(i(k)) \ast \tilde{X}^{(k)} \right),
\]
\[
X^{(k+1)} = \nabla^* f(Z^{(k+1)}).
\]

Note that \( Z^{(k+1)}, X^{(k+1)} \) are not those defined in Algorithm 3.2. By (3.15) and (3.18), we have that \( \tilde{Z}^{(k+1)} = Z^{(k+1)} + \frac{t}{\|A(i)\|_F} A(i)^T \mathcal{E}(i) \).

With the update (3.18), we apply Proposition 3.2
\[
D_{f, Z^{(k+1)}}(X^{(k+1)}, \hat{X}) \leq D_{f, Z^{(k)}}(\tilde{X}^{(k)}, \hat{X}) - t \left(1 - \frac{tN_3}{2\alpha_f} \right) \frac{\|A(i) \ast (\hat{X} - \tilde{X}^{(k)})\|_F^2}{\|A(i)\|_F^2}.
\]
Due to the presence of noise, it is not clear if the assumption of Lemma 3.6(a) holds. This is why we require strong admissibility, in which case Lemma 3.6(b) applies and we have

\[ D_{f,\mathcal{Z}^{(k)}}(X^{(k)}, \mathcal{X}) \leq \frac{1}{\nu} \|A \ast (X^{(k)} - \mathcal{X})\|_F^2. \tag{3.21} \]

Similar to the analysis in (3.13)-(3.14), we take the conditional expectation of (3.20) and get

\[ \mathbb{E} \left[ D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X}^{(k)}) \right] \leq \left( 1 - \frac{\nu t(1 - \frac{1}{2\alpha_f})}{\|A\|_F^2} \right) D_{f,\mathcal{Z}^{(k)}}(X^{(k)}, \mathcal{X}). \tag{3.22} \]

By Lemma 2.4 we get

\[
\begin{align*}
D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X}) &- D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X}) \\
&\leq \langle Z^{(k+1)} - Z^{(k+1)}, X^{(k+1)} \rangle - \|Z^{(k+1)} - Z^{(k+1)}\|_F \|X^{(k+1)} - \mathcal{X}\|_F \\
&\leq \frac{t}{\|A(i)\|_F} \|A(i)^T \ast E(i)\|_F \sqrt{\alpha_f^2} \sqrt{D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X})} \\
&\leq \frac{tN_3 \|E(i)\|_F}{\|A(i)\|_F} \sqrt{\alpha_f^2} \sqrt{D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X})} \leq tN_3 \epsilon \sqrt{\alpha_f^2} \sqrt{D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X})}.
\end{align*}
\]

By denoting \( a = tN_3 \epsilon \sqrt{\alpha_f^2} \), \( \alpha_i^2 = D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X}) \), \( \beta_i^2 = D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X}) \), we can rewrite the above inequality as \( \alpha_i^2 - \beta_i^2 \leq aa \), which implies

\[ \alpha \leq \frac{1}{2}(a + \sqrt{a^2 + 4\beta^2}) \]

and thereby \( \alpha \leq a + \beta \). That is,

\[ D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X}) \leq a + \sqrt{D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X})}. \]

For the sake of convenience, we let \( \mathbb{E}_c \) be the expectation conditioned on \( i(1), \ldots, i(k-1) \). Then we have

\[
\mathbb{E}_c \sqrt{D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X})} \leq \mathbb{E}_c \sqrt{D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X})} \]

\[
\leq a + \mathbb{E}_c \left[ D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X}) \right] \tag{3.22} \leq a + \sqrt{\mathbb{E}_c} \left[ D_{f,\mathcal{Z}^{(k+1)}}(X^{(k+1)}, \mathcal{X}) \right] \leq a + \sqrt{1 - \frac{\nu t(1 - \frac{1}{2\alpha_f})}{\|A\|_F^2}} D_{f,\mathcal{Z}^{(k)}}(X^{(k)}, \mathcal{X}),
\]

which implies

\[ d_{k+1} \leq a + \sqrt{1 - \frac{\nu t(1 - \frac{1}{2\alpha_f})}{\|A\|_F^2}} d_k := a + \sqrt{1 - bd}, \tag{3.23} \]

where \( d_k := \mathbb{E}\sqrt{D_{f,\mathcal{Z}^{(k)}}(X^{(k)}, \mathcal{X})} \). After applying (3.23) iteratively, we get

\[
d_k \leq (\sqrt{1 - b})^k d_0 + a \left( 1 + \sqrt{1 - b} + (\sqrt{1 - b})^2 + \cdots + (\sqrt{1 - b})^{k-1} \right)
\]

\[
= (\sqrt{1 - b})^k d_0 + a \frac{1 - (\sqrt{1 - b})^k}{1 - \sqrt{1 - b}} \leq (\sqrt{1 - b})^k d_0 + a \frac{1}{1 - \sqrt{1 - b}}
\]

\[
= (\sqrt{1 - b})^k d_0 + a \frac{1 + \sqrt{1 - b}}{b} \leq (\sqrt{1 - b})^k d_0 + \frac{2a}{b},
\]

which reduces to (3.17). \qed
4 Special Cases of the Proposed Algorithm

4.1 Matrix Recovery

This section discusses the special cases when \(N_1 = 1\). The tensor \(A \in \mathbb{R}^{N_1 \times N_2 \times 1}\) degenerates to the \(N_1 \times N_2\) matrix \(A\). Although this has been brought up in Example 3.7, we would like to include further discussions here. Specifically, \(X\) degenerates to the \(N_2 \times K\) matrix \(X\) and \(B\) degenerates to the \(N_1 \times K\) matrix \(B\). The minimization problem (3.1) becomes a matrix recovery problem

\[
\min_{X} f(X) \quad \text{s.t.} \quad AX = B.
\]

If \(f(X) = \lambda \|X\|_* + \frac{1}{2} \|X\|_F^2\), which is an admissible function, we have

\[
\nabla f^*(Z) = \text{prox}_{\lambda \|\cdot\|}(Z) = D_\lambda(Z).
\]

The singular value thresholding operator \(D_\lambda(Z)\) is defined as \(U \max\{S - \lambda, 0\} V^T\), provided that \(Z = USV^T\) is the singular value decomposition.

The following corollary is a consequence of Theorem 3.3 and Theorem 3.9. Note this function is 1-strongly convex. Similar to the tensor notation, we will let \(A(i)\) be the \(i\)th row of \(A\), and \(R_M(A) = \{ATY : Y \in \mathbb{R}^{N_1 \times K}\}\).

**Corollary 4.1.** Let \(\hat{X}\) be the solution of (4.1) where \(f(X) = \lambda \|X\|_* + \frac{1}{2} \|X\|_F^2\). This function is admissible with the constant \(\nu\) (see (3.8)). Initializing with \(Z^{(0)} \in R_M(A) \subset \mathbb{R}^{N_2 \times K}\) and \(X^{(0)} = D_\lambda(Z^{(0)})\), we perform the updating scheme

\[
\begin{align*}
Z^{(k+1)} &= Z^{(k)} + tA(i(k))^T B(i(k)) - A(i(k))X^{(k)} \|A(i(k))\|_F^2, \\
X^{(k+1)} &= D_\lambda(Z^{(k+1)}),
\end{align*}
\]

where \(\{i(k)\}\) is a slice selection sequence and \(t < 2\).

(a) If \(\{i(k)\}\) is a control sequence for \([N_1]\), then the sequence \(X^{(k)}\) converges to \(\hat{X}\).

(b) If \(i(k)\) is a random sequence such that \(\text{Pr}(i(k) = j) = p_j\), then

\[
\mathbb{E} \left[ D_{f,Z^{(k+1)}}(X^{(k+1)}, \hat{X}) \right] \leq \left(1 - \nu t(1 - \frac{t}{2}) \min \left\{ \frac{p_j}{\|A(j)\|_F^2} \right\} \right) \mathbb{E} \left[ D_{f,Z}(X^{(k)}, \hat{X}) \right].
\]

**Remark 4.2.** Corollary 4.1(b) allows a more general probability distribution instead of the specific distribution in Algorithm 3.2. The proof can be easily modified from that in Theorem 3.9.

**Remark 4.3.** It is worth noting that the linear constraint \(AX = B\) is in a different form from \(\{\langle A_i, X \rangle = b_i, i \in [m]\}\). However, the proof of Theorem 3.9 can be easily adapted to this kind of constraints. If we focus on the regularized nuclear norm optimization problem

\[
\min_{X \in \mathbb{R}^{N_2 \times K}} \lambda \|X\|_* + \frac{1}{2} \|X\|_F^2 \quad \text{s.t.} \quad \langle A_i, X \rangle = b_i, i \in [m],
\]

it is shown in Theorem 4.5 that the following algorithm

\[
\begin{align*}
Z^{(k+1)} &= Z^{(k)} + tA(i(k)) \frac{b(i(k)) - \langle A(i(k)), X^{(k)} \rangle}{\|A(i(k))\|_F^2}, \\
X^{(k+1)} &= D_\lambda(Z^{(k+1)}),
\end{align*}
\]

with a random sequence \(i(k)\) has a linear convergence rate in expectation. Related numerical experiments can be found in Section 3.5.
4.2 Vector Recovery

As mentioned in Remark 3.8, when $A \in \mathbb{R}^{N_1 \times N_2 \times 1}$ and $X \in \mathbb{R}^{N_2 \times 1 \times 1}$, $A$ degenerates to the $N_1 \times N_2$ matrix $A$ and $X$ degenerates to the vector $x$. In this section, we further discuss this special case in more detail. Specifically, we consider the following vector version of the minimization problem (3.1)

\[
\hat{x} = \arg\min_x f(x) \quad \text{s.t.} \quad Ax = b,
\]

where $R(A)$ in this context is the row space of $A$. As a consequence, Algorithm 3.1 or Algorithm 3.2 becomes Algorithm 4.1.

Algorithm 4.1 Regularized Kaczmarz Algorithm for Vector Recovery

Input: $A \in \mathbb{R}^{N_1 \times N_2}$, $b \in \mathbb{R}^{N_1}$, row selection sequence $\{i(k)\}_{k=1}^{\infty} \subseteq [N_1]$, stepsize $t$, maximal number of iterations $T$, and tolerance $tol$.

Output: an approximate of $\hat{x}$

Initialize: $z^{(0)} \in R(A) \subset \mathbb{R}^{N_2}$, $x^{(0)} = \nabla f^*(z^{(0)})$.

for $k = 0, 1, \ldots, T - 1$ do

\[
z^{(k+1)} = z^{(k)} + t \frac{b_{i(k)} - A(i(k))x^{(k)}}{\|A(i(k))\|_2^2} A(i(k))^T
\]

\[
x^{(k+1)} = \nabla f^*(z^{(k+1)})
\]

If $\|x^{(k+1)} - x^{(k)}\|_2 / \|x^{(k)}\|_2 < tol$, then it stops.

end for

In this setting, Theorem 3.3 and Theorem 3.9 reduce to the following results.

Corollary 4.4. Let $f$ be $\alpha_f$-strongly convex.

(a) If $i(k)$ is a control sequence for $[N_1]$, then $x^{(k)}$ from Algorithm 4.1 converges to $\hat{x}$.

(b) If $f$ is admissible and $i(k)$ is a random sequence such that $\Pr(i(k) = j) = p_j$, then the iterates from Algorithm 4.1 satisfy

\[
\mathbb{E} \left[ D_{f,x^{(k+1)}}(x^{(k+1)}, \hat{x}) \right] \leq \left( 1 - nt \left( 1 - \frac{t}{2\alpha_f} \right) \min_j \{ p_j / \|A(j)\|_2^2 \} \right) \mathbb{E} \left[ D_{f,x^{(k)}}(x^{(k)}, \hat{x}) \right].
\]

Note that Corollary 4.4 aligns with the previous results in literature. Part (a) can be found in [30] Theorem 2.7 in a more general setting, and part (b) can be found in [43] Theorem 4.5. The following noisy version is a new contribution, which is a result of Theorem 3.10.

Corollary 4.5. Let $f$ be $\alpha_f$-strongly convex and strongly admissible. Let $\hat{z}^{(k)}$, $\hat{x}^{(k)}$ be generated from Algorithm 4.1 ($t < 2\alpha_f$) with the noisy constraint $Ax = b + e$, i.e.,

\[
\hat{z}^{(k+1)} = \hat{z}^{(k)} + t \frac{b_{i(k)} + e_{i(k)} - A(i(k))\hat{x}^{(k)}}{\|A(i(k))\|_2^2} A(i(k))^T
\]

\[
\hat{x}^{(k+1)} = \nabla f^*(\hat{z}^{(k+1)}).
\]

Moreover, $\{i(k)\}$ is a random sequence such that $\Pr(i(k) = j) = \frac{\|e_j\|_2}{\|A(j)\|_2^2}$. Let $\epsilon = \max_{i \in [N_1]} \frac{\|e_i\|_2}{\|A(i)\|_2}$. We have

\[
\mathbb{E} \sqrt{D_{f,\hat{x}^{(k)}}(\hat{x}^{(k)}, \hat{x})} \leq \left( 1 - \frac{nt (1 - \frac{t}{2\alpha_f})}{\|A\|_F^2} \right)^k D_{f,\hat{x}^{(0)}}(\hat{x}^{(0)}, \hat{x}) + \sqrt{\frac{2\alpha_f \|A\|_F^2}{\nu(1 - \frac{t}{2\alpha_f})}} ,
\]

where $\hat{x}$ is still the solution of (4.7).

In the work [30], some noisy models in practice have been briefly mentioned but without theoretical analysis. Corollary 4.5 states that with perturbed measurements, the expected Bregman distance (squared rooted) still enjoys an exponential decay and the iterates are within certain radius (proportional to the noise level) of the true solution.
4.2.1 Kaczmarz Type of Algorithms

Important examples that satisfy the assumptions of Corollary 4.4 and Corollary 4.5 are when \( f \) is strongly convex and piecewise quadratic, and \( A \) is full row rank (see Example 3.1).

For \( f_1(x) = \frac{1}{2} \|x\|_2^2 \), we have \( x^{(k)} = z^{(k)} \), and Algorithm 4.1 becomes the well-known Kaczmarz algorithm [24]

\[
x^{(k+1)} = x^{(k)} + t \frac{b_{i(k)} - A(i(k))x^{(k)}}{\|A(i(k))\|_2^2} A(i(k))^T,
\]

which is known to converge to the minimum norm solution of \( Ax = b \) if the initial \( x^{(0)} \) is in the row space of \( A \). Linear convergence for the randomized Kaczmarz algorithm can be found in [46, 13].

For \( f(\lambda) = \lambda \|x\|_1 + \frac{1}{2} \|x\|_2^2 \), (4.7) becomes a regularized version of the Basis Pursuit [50, 9]:

\[
\hat{x} = \arg\min_x \lambda \|x\|_1 + \frac{1}{2} \|x\|_2^2 \quad \text{s.t. \( Ax = b \),}
\]

and Algorithm 4.1 becomes the \textit{sparse Kaczmarz method}

\[
x^{(k+1)} = z^{(k)} + t \frac{b_{i(k)} - A(i(k))x^{(k)}}{\|A(i(k))\|_2^2} A(i(k))^T
\]

\[
z^{(k+1)} = D_\lambda(z^{(k+1)})
\]

that was proposed in [30].

4.3 Tensor Nuclear Norm Regularized Minimization

In this section, we consider one special case of tensor recovery involving the nuclear-norm regularization and linear measurements. Specifically, we adapt the proposed algorithms for solving the tensor nuclear norm regularized minimization problem

\[
\hat{X} = \arg\min_{X \in \mathbb{R}^{N_2 \times K \times N_3}} \frac{1}{2} \|X\|_F^2 + \lambda \|X\|_{\text{tnn}} \quad \text{s.t. \( AX = B \),}
\]

(4.14)

where \( A \in \mathbb{R}^{N_1 \times N_2 \times N_3} \) and \( B \in \mathbb{R}^{N_1 \times K \times N_3} \). Here \( \|X\|_{\text{tnn}} \) is the \textit{tensor nuclear norm} of \( X \), which is defined through the operator in Definition 2.9 [14], that is,

\[
\|X\|_{\text{tnn}} := \sum_{k=1}^{N_3} \|F(X)_k\|_*.
\]

In addition, tensor nuclear norm can be computed through t-SVD [26], which involves the SVD of each frontal slice \( F(X)_k \):

\[
F(X)_k = \tilde{U}_k \hat{S}_k \tilde{V}_k^T.
\]

Let \( \tilde{U} \) be the tensor generated by concatenating \( \tilde{U}_k \)'s along the third dimension such that \( \tilde{U}_k \) is the \( k \)th frontal slice of \( \tilde{U} \). Likewise, we obtain \( \tilde{S}, \tilde{V} \). Denote \( U = F^{-1}(\tilde{U}) \) and \( V = F^{-1}(\tilde{V}) \). The \textit{singular tube thresholding operator} (STT) [13] is defined as

\[
\mathcal{S}_\tau(X) := U \ast F^{-1}(\max(\tilde{S} - \tau, 0)) \ast V^T.
\]

As a straightforward application of this definition, we obtain the following lemma.

\textbf{Lemma 4.6.} For any \( X \in \mathbb{R}^{N_2 \times K \times N_3} \), the equality \( Y = \mathcal{S}_\lambda(X) \) holds if and only if \( F(Y)_k = D_\lambda(F(X)_k) \) for \( k = 1, \ldots, N_3 \).

Next we develop regularized Kaczmarz algorithms for solving (4.14). It can be shown that \( \nabla f^*(Z) = \text{prox}_{\lambda \|\cdot\|_{\text{tnn}}}(Z) = \mathcal{S}_\lambda(Z) \); see [31] Theorem 4.2 for more examples. In this case, Algorithm 3.1 with a given control sequence reduces to Algorithm 4.2 which alternates the Kaczmarz step and singular tube thresholding.

The following corollary is a direct consequence of Theorem 3.3 as the objective function is 1-strongly convex.
Algorithm 4.2 Regularized Kaczmarz Algorithm for Solving (4.14)

Input: $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$, $B \in \mathbb{R}^{N_1 \times K \times N_3}$, control sequence $\{i(k)\}_{k=1}^\infty \subseteq [N_1]$, stepsize $t$, maximum number of iterations $T$, and tolerance $tol$.

Output: an approximate of $\hat{X}$

Initialize: $Z(0) \in R(A) \subset \mathbb{R}^{N_2 \times K \times N_3}$, $\mathcal{X}(0) = \mathcal{S}_\lambda(Z(0))$.

for $k = 0, 1, \ldots, T - 1$ do

$Z(k+1) = Z(k) + tA(i(k))^T \frac{B(i(k)) - A(i(k)) \ast \mathcal{X}(k)}{\|A(i(k))\|_F^2}$

$\mathcal{X}(k+1) = \mathcal{S}_\lambda(Z(k+1))$

if $\|\mathcal{X}(k+1) - \mathcal{X}(k)\|_F / \|\mathcal{X}(k)\|_F < tol$, then it stops.

end for

Corollary 4.7. The sequence generated by Algorithm 4.2 with $t < 2/N_3$ satisfies

$$D_{f, Z(k+1)}(\mathcal{X}(k+1), \mathcal{X}) \leq D_{f, Z(k)}(\mathcal{X}(k), \mathcal{X}) - t \left(1 - \frac{tN_3}{2}\right) \frac{\|A(i(k)) \ast (\mathcal{X}(k) - \mathcal{X})\|_F^2}{\|A(i(k))\|_F^2},$$

(4.15)

for all $\mathcal{X} \in H_{i(k)}$. Moreover, the sequence $\{\mathcal{X}(k)\}$ converges to the solution of (4.14).

Similarly, we can adapt Algorithm 3.2 with a random selection of row index to get a reduced version, i.e., Algorithm 4.3, for solving (4.14).

Algorithm 4.3 Randomized Regularized Kaczmarz Algorithm for Solving (4.14)

Input: $A \in \mathbb{R}^{N_1 \times N_2 \times N_3}$, $B \in \mathbb{R}^{N_1 \times K \times N_3}$, stepsize $t$.

Output: an approximate of $\hat{X}$

Initialize: $Z(0) \in R(A) \subset \mathbb{R}^{N_2 \times K \times N_3}$, $\mathcal{X}(0) = \mathcal{S}_\lambda(Z(0))$.

while termination criteria not satisfied do

pick $i(k)$ randomly from $[N_1]$ with $Pr(i(k) = j) = \|A(j)\|_F^2 / \|A\|_F^2$,

$Z(k+1) = Z(k) + tA(i(k))^T \frac{B(i(k)) - A(i(k)) \ast \mathcal{X}(k)}{\|A(i(k))\|_F^2}$

$\mathcal{X}(k+1) = \mathcal{S}_\lambda(Z(k+1))$

end while

Regarding the convergence analysis of Algorithm 4.3, Theorem 3.9 can not be applied directly as it is not immediately clear if this function is admissible. To address this issue, we propose the following lemma, which can be shown using the definition of Bregman distance and Lemma 4.6.

Lemma 4.8. Let $f(\mathcal{X}) = \lambda \|\mathcal{X}\|_{\text{nn}} + \frac{1}{2} \|\mathcal{X}\|_F^2$ be defined on $\mathbb{R}^{N_2 \times K \times N_3}$, and its reduced version $f_M(\mathcal{X}) = \lambda \|\mathcal{X}\|_* + \frac{1}{2} \|\mathcal{X}\|_F^2$ be defined on $\mathbb{R}^{N_2 \times \hat{K}}$. Then given $z \in \partial f(\mathcal{X})$, we have

$$D_{f, Z}(\mathcal{X}, W) = \sum_{j=1}^{N_3} D_{f_M, F(Z), j}(F(\mathcal{X})_j, F(W)_j).$$

Theorem 4.9. If $t < 2/N_3$, then the sequence generated by Algorithm 4.3 converges in expectation with

$$\mathbb{E} \left[ D_{f, Z(k+1)}(\mathcal{X}(k+1), \hat{X}) \right] \leq (1 - \frac{\beta t}{\|A\|_F^2}) \mathbb{E} \left[ D_{f, Z(k)}(\mathcal{X}(k), \hat{X}) \right],$$

(4.16)

where $\beta$ is given by

$$\beta = \min_{i \in [N_1], j \in [N_3]} \frac{t \|F(\mathcal{X}(i))_j\|_F^2}{\|F(\mathcal{X}(i))\|_F^2} \left(1 - \frac{t \|F(\mathcal{X}(i))_j\|_F^2}{\|F(\mathcal{X}(i))\|_F^2}\right).$$

(4.17)
Proof. The objective function can be written as
\[
  f(X) = \lambda \|X\|_{nn} + \frac{1}{2} \|X\|_F^2 = \sum_{j=1}^{N_3} \left( \lambda \|F(X)_j\|_* + \frac{1}{2} \|F(X)_j\|_F^2 \right).
\]

In light of Lemma 2.10, the constraint can be expressed in the Fourier domain as
\[
  F(A)_j F(X)_j = F(B)_j, \quad j \in [N_3].
\]

Let \( V \in \mathbb{R}^{N_3 \times K \times N_3} \) and \( V_k \) be the \( k \)th frontal slice of \( V \). Therefore, if
\[
  \hat{V} = \arg \min_V \sum_{j=1}^{N_3} \left( \|V_j\|_* + \frac{1}{2} \|V_j\|_F^2 \right) \quad \text{s.t.} \quad F(A)_j V_j = F(B)_j, \quad j \in [N_3],
\]

then \( F^{-1}(\hat{V}) \) is the minimizer of \((4.14)\). Note that \((4.18)\) can be split into \( N_3 \) subproblems.

We apply the Fourier transform \( F \) to both steps of Algorithm 4.3. By Lemma 2.11, the first step becomes
\[
  \text{unfold}(F(Z^{(k+1)})) = \text{unfold}(F(Z^{(k)})) + \frac{t}{\|A(i)\|_F^2} \text{diag}(F(A(i))^*)(F(B(i)) - \text{diag}(F(A(i)))F(X^{(k)})),
\]

where the block diagonal matrix \( \text{diag}(F(A(i))) \) and the block matrix \( F(B(i)) \) are defined as
\[
  \text{diag}(F(A(i))) = \begin{bmatrix} F(A(i))_1 & \cdots & F(A(i))_{N_3} \\ \vdots & \ddots & \vdots \\ F(A(i))_{N_3} & \cdots & F(A(i))_1 \end{bmatrix}, \quad F(B(i)) = \begin{bmatrix} F(B(i))_1 \\ \vdots \\ F(B(i))_{N_3} \end{bmatrix}.
\]

Likewise, we can define \( \text{diag}(F(A(i))^*) \) and \( F(X^{(k)}) \). By separating the above updating equation into \( N_3 \) pieces, we obtain
\[
  F(Z^{(k+1)} j) = F(Z^{(k)} j) + \frac{t}{\|A(i)\|_F^2} F(A(i))^*(F(B(i)) j - F(A(i)) j F(X^{(k)} j)), \quad j \in [N_3].
\]

By Lemma 4.6, the second step of Algorithm 4.3 becomes
\[
  F(X^{(k+1)} j) = D_\lambda(F(Z^{(k+1)} j)), \quad j \in [N_3].
\]

Now we make a change of variables as \( F(X^{(k)} j) = V_j^{(k)}, F(Z^{(k)} j) = W_j^{(k)} \). Then \((4.19)-(4.20)\) become
\[
  W_j^{(k+1)} = W_j^{(k)} + \frac{t}{\|A(i)\|_F^2} F(A(i))^*(F(B(i)) j - F(A(i)) j V_j^{(k)}), \quad j \in [N_3],
\]
\[
  V_j^{(k+1)} = D_\lambda(W_j^{(k+1)}), \quad j \in [N_3].
\]

Note that \((4.21)-(4.22)\) are the two major iteration steps for solving the matrix recovery problem
\[
  \hat{V}_j = \arg \min_{V_j} \lambda \|V_j\|_* + \frac{1}{2} \|V_j\|_F^2 \quad \text{s.t.} \quad F(A)_j V_j = F(B)_j, j \in [N_3].
\]

This is in fact the \( j \)th subproblem of \((4.18)\), so we wish to apply Corollary 4.1(b). To fit into the format of \((4.2)\), we rewrite
\[
  \frac{t}{\|A(i)\|_F^2} = \frac{t}{\|F(A(i))\|_F^2} = \frac{t}{\|F(A(i))\|_F^2} \frac{1}{\|F(A(i))\|_F^2}.
\]
The stepsize given by
\[ s_j = t \left\| F(A(i))_j \right\|^2_F \leq t \leq 2/N_3 \leq 2 \]
fits the assumption of Corollary 4.1. The probability distribution is
\[ p_i = \frac{\left\| F(A(i))_j \right\|^2_F}{\left\| F(A) \right\|^2_F}. \]
We compute
\[ p_i = \frac{\left\| F(A(i))_j \right\|^2_F}{\left\| F(A) \right\|^2_F} \frac{1}{\left\| F(A(i))_j \right\|^2_F} \geq \frac{1}{\left\| F(A) \right\|^2_F} = \frac{1}{\left\| A \right\|^2_F}. \]
So applying Corollary 4.1(b) yields
\[ \mathbb{E} \left[ D_{f,M,W_j^{(k+1)}}(V_j^{(k+1)}, \hat{V}_j) \right] \leq \left( 1 - \frac{\nu s_j (1 - s_j/2)}{\left\| A \right\|^2_F} \right) \mathbb{E} \left[ D_{f,M,W_j^{(k)}}(V_j^{(k)}, \hat{V}_j) \right]. \] (4.24)

Finally, using Lemma 4.8 we have
\[ \mathbb{E} \left[ D_{f,Z^{(k+1)}}(X^{(k+1)}, \hat{X}) \right] = \sum_{j=1}^{N_3} \mathbb{E} \left[ D_{f,M,W_j^{(k+1)}}(V_j^{(k+1)}, \hat{V}_j) \right] \]
\[ \leq \sum_{j=1}^{N_3} \left( 1 - \frac{\nu s_j (1 - s_j/2)}{\left\| A \right\|^2_F} \right) \mathbb{E} \left[ D_{f,M,W_j^{(k)}}(V_j^{(k)}, \hat{V}_j) \right] \]
\[ \leq (1 - \frac{\nu^2}{\left\| A \right\|^2_F}) \mathbb{E} \left[ D_{f,Z^{(k)}}(X^{(k)}, \hat{X}) \right], \]
where
\[ \beta = \min_{i \in [N_1], j \in [N_3]} s_j (1 - s_j/2) = \min_{i \in [N_1], j \in [N_3]} \frac{\left\| F(A(i))_j \right\|^2_F}{\left\| F(A) \right\|^2_F} \left( 1 - \frac{t \left\| F(A(i))_j \right\|^2_F}{\left\| F(A) \right\|^2_F} \right) > 0. \]

5 Numerical Experiments

In this section, we illustrate the performance of the proposed algorithms in several application problems, including one-dimensional sparse signal recovery, low-rank image inpainting, low-rank tensor recovery, and image deblurring. There are two special cases of our proposed algorithms when the constraint selection sequence is either cyclic or random. For the random version, we take the average of all the results obtained by running 50 trials. To save the computational time, we only execute the deterministic version with a cyclic control sequence for image inpainting and deblurring tests. For image deblurring, we also consider a batched version of the proposed algorithm to improve the performance. However, the batching technique has shown very limited performance enhancement in our other experiments, so we skip those experiments.

To make fair performance comparisons, we adopt the widely used quantitative metrics. For tensor recovery, we use the relative error (RelErr) defined as
\[ \text{RelErr} = \frac{\left\| \hat{X} - X \right\|_F}{\left\| \hat{X} \right\|_F}, \]
where \( \hat{X} \) is the estimation of the ground truth tensor \( X \). As one of the most important image quality metrics, the peak signal-to-noise ratio (PSNR) is defined as
\[ \text{PSNR} = 20 \log(I_{\max}/\left\| \hat{X} - X \right\|_F), \]
where $X$ is the estimation of the noise-free image $\hat{X}$, and $I_{\max}$ is the maximum possible image intensity.

In addition, the structural similarity index (SSIM) between two images $X$ and $Y$ is defined as

$$\text{SSIM} = \frac{(2\mu_x\mu_y + C_1)(2\sigma_{xy} + C_2)}{\mu_x^2 + \mu_y^2 + C_1(\sigma_x^2 + \sigma_y^2 + C_2)},$$

where $\mu_x, \sigma_x$ are the mean and standard deviation of the image $X$, $\sigma_{xy}$ is the cross-covariance between $X$ and $Y$, and $C_1$ and $C_2$ are the luminance and contrast constants. Both PSNR and SSIM values can be obtained efficiently in MATLAB via `PSNR` and `SSIM` respectively.

All the numerical experiments are implemented using MATLAB R2019a for Windows 10 on a desktop PC with 64GB RAM and a 3.10GHz Intel Core i9-9960X CPU.

### 5.1 One-dimensional Sparse Signal Recovery

We will compare the performance of the following three methods for solving (4.12): (1) linearized Bregman iteration [50, 9] (denoted by LinBreg); (2) alternating direction method of multipliers (ADMM) [6]; (3) our proposed regularized Kaczmarz method (4.13) with random or deterministic cyclic sequence (denoted by RK-rand and RK-cyc). In particular, LinBreg has the following iterations

$$z^{(k+1)} = z^{(k)} + tA^T(b - Ax^{(k)}),$$

$$x^{(k+1)} = S_\lambda(z^{(k+1)}). \tag{5.1}$$

For ADMM, we rewrite (4.12) as

$$\min_{x, w} \frac{1}{2}\|x\|^2_2 + \lambda\|w\|_1, \quad \text{s.t.} \quad Ax = b, \ w = x,$$

and the corresponding augmented Lagrangian reads as

$$L(x, w, u_1, u_2) = \frac{1}{2}\|x\|^2_2 + \lambda\|w\|_1 + \frac{\rho_1}{2}\|Ax - b + u_1\|^2_2 + \frac{\rho_2}{2}\|x - w + u_2\|^2_2.$$

In our experiment, $A$ is a $200 \times 1000$ Gaussian matrix. The ground truth vector $\hat{x}$ is a 10-sparse vector whose support is randomly generated and the entries on each support index are independent and follow the normal distribution with mean 1 and standard deviation 1. The parameters are all tuned to achieve optimal performance. For the regularized Kaczmarz method, the step size is $t = 40$, and we show the results when the indices are chosen cyclically or randomly. For the linearized Bregman iteration, the step size $t = 20$. For ADMM, we pick $\rho_1 = 10$ and $\rho_2 = 100$.

Figure 1 shows the relative error of all four methods versus the running time. Both versions of the regularized Kaczmarz algorithms are outperforming LinBreg and ADMM. Moreover, the cyclic version of the regularized Kaczmarz method performs slightly better than the randomized one.

### 5.2 Image Inpainting

In the second experiment, we consider a low-rank image inpainting problem. The test image is a checkerboard image of size $128 \times 128$ with a large missing rectangular area; see the first image of Figure 2. This image can be described by a rank-two matrix, so it is appropriate to be recovered via the model (4.5). Let $I$ be the image to be recovered, and $\Omega$ be the set of indices whose pixel values are known. Then the linear constraints are $\{(P_{ij}, X) = I_{ij}, (i, j) \in \Omega\}$, where $P_{ij}$ is the matrix whose entries are all zeros except its $ij$th entry is one. We use $X_{\Omega}$ for keeping the pixel intensities in $\Omega$ and setting other pixel intensities to be zero.

We consider three image inpainting methods: (1) total variation (TV) based image inpainting [45, 20], (2) linearized Bregman iteration [8] (denoted by LinBreg as before), (3) the proposed regularized Kaczmarz method [4, 6]. For LinBreg, we adopt the following updating scheme in analogy to (5.1)

$$Z^{(k+1)} = Z^{(k)} + t(I - X^{(k)})_{\Omega},$$

$$X^{(k+1)} = D_\lambda(Z^{(k+1)}). \tag{5.2}$$

21
The second is the regularized Kaczmarz method (4.6). In this application, the first step of (4.6) becomes
\[ Z^{(k+1)} = Z^{(k)} + t(I - X^{(k)})_{i,j}, \]
which means only one pixel from \( \Omega \) is updated. Again, the choice of indexing can be cyclic or random. In our numerical experiment, we will use a more general version
\[ Z^{(k+1)} = Z^{(k)} + \sum_{(i,j) \in T(k)} t(I - X^{(k)})_{i,j} = Z^{(k)} + t(I - X^{(k)})_{T(k)}, \]
where \( T(k) \subset \Omega \), so that \(|T(k)|\) many pixels are updated in one iteration. A different index set \( T(k) \) is chosen at each iteration \( k \), but we do require that they have the same batch size, i.e., the cardinality \(|T(k)| = b\). Therefore, our algorithm for this specific case reads as
\[
Z^{(k+1)} = Z^{(k)} + t(I - X^{(k)})_{T(k)}, \\
X^{(k+1)} = D_\lambda(Z^{(k+1)}),
\]
(5.3)

In the methods involving SVT, we choose \( \lambda = 1500 \).

Figure 2 compares all the listed methods quantitatively and qualitatively. For the TV result, the image was recovered by minimizing the functional
\[
\lambda \|\nabla X_x\|_1 + \|\nabla X_y\|_1 + \frac{1}{2} \|X - I\|_{\Omega}^2.
\]
The TV regularization only considers the piecewise constant type of smoothness and thus it may not handle texture-like images with a low-rank structure very well. Since the missing area is relatively large in the test image, TV fails to recover the image as expected. As shown in the last two sub-figures of Figure 2, either the linearized Bregman iteration, or the regularized Kaczmarz iteration is able to achieve almost perfect reconstruction. In our regularized Kaczmarz iteration, we choose batch size to be 2000 with a
cyclic indexing. The regularized Kaczmarz enjoys faster convergence.

The batch size $b$ plays an important role in a lot of optimization algorithms, e.g., stochastic gradient descent. In this application, $b$ can be viewed as the number of pixels updated in step 1 of (5.3). If $b = |\Omega|$, then our algorithm coincides with (5.2). This is related to Stochastic Gradient Matching Pursuit (StoGradMP) [35] especially with a random choice of the constraints. Some other relevant works include the block Kaczmarz algorithm [36, 14]. The proof presented in this paper can also be adapted to show that the following algorithm

$$
\begin{align*}
\text{Pick an index set } T(k) \text{ whose cardinality is } b, \\
Z^{(k+1)} &= Z^{(k)} + t \left( P_{\text{span}(A_i,i \in T(k))} \hat{X} - P_{\text{span}(A_i,i \in T(k))} X^{(k)} \right), \\
X^{(k+1)} &= D_\lambda(Z^{(k+1)})
\end{align*}
$$

(5.4)

produces a sequence $\{X^{(k)}\}$ that converges to the solution of (4.5). Note that in the image inpainting problem, (5.3) is exactly this block version (5.4) due to the orthogonality of the indicator matrices $P_{ij}$. The batch size does influence the convergence of (5.3). For this particular example, we have $|\Omega| = 8064$ and we found a batch size around 2000 to be optimal.

5.3 Low-Rank Tensor Recovery

Assume that the ground truth tensor $X \in \mathbb{R}^{N_2 \times K \times N_3}$ with a small tubal rank satisfies the tensor system

$$
A \ast X = B, \quad A \in \mathbb{R}^{N_1 \times N_2 \times N_3}.
$$

Then we consider the tensor recovery model (4.14) with the tensor nuclear norm regularization. It can be empirically shown that the proposed algorithm (3.1) can achieve good performance in terms of accuracy and convergence speed when $N_1$ is larger than the other dimensions $N_2, N_3, K$ but may fail to converge in other scenarios. As an illustration, we show the performance of our algorithm with cyclic and random control sequences in Figure 3 where $N_1 = 200$, $N_2 = N_3 = K = 100$ and the maximal number of iterations as 2000. Both the coefficient tensor $A$ and the ground truth tensor $X$ are normally distributed, and the tubal rank of $X$ is two by taking the hard thresholding of singular tubes after t-SVD. For the random case, we take the average of 50 trials. One can see that the cyclic control sequence achieves faster convergence with much smaller error but with slightly more running time.
5.4 Tensor-Based Image Deconvolution

Consider an image \( \hat{X} \in \mathbb{R}^{m_1 \times n_1} \) that is degraded by taking the convolution with a point spread function \( \hat{H} \in \mathbb{R}^{m_2 \times n_2} \). Let \( m = m_1 + m_2 - 1 \) and \( n = n_1 + n_2 - 1 \). Using the zero padding, both \( \hat{X} \) and \( \hat{H} \) can be extended to two respective matrices \( X, H \) of size \( m \times n \). By the construction, we have

\[
H \odot X = B,
\]

where \( \odot \) is the 2D convolution. Next we establish the equivalence between 2D convolution and t-product by creating a doubly block circulant matrix [1, Appendix]. Let \( h_i \) be the \( i \)th column of \( H \), and \( A_i := \text{circ}(h_i) \in \mathbb{R}^{n \times n}, i \in [m] \) be the circulant matrix generated by \( h_i \). Let \( A_i \) be the \( i \)th frontal slice of \( A \in \mathbb{R}^{n \times n \times m} \). Then we create a doubly block circulant matrix \( \text{bcirc}(A) \) based on \( A \). Let \( X' \in \mathbb{R}^{n \times 1 \times m} \) be the tensor version of \( X \) by setting \( X'(j, 1, i) = X(i, j) \) for \( i \in [m] \) and \( j \in [n] \). Then the 2D convolution can be represented as

\[
\text{vec}(H \odot X) = \text{bcirc}(H) \text{vec}(X) = A \ast X'.
\]

Here \( \text{vec}(\cdot) \) is a vectorization operator by row-wise stacking. To recover \( X' \), we consider the low-rank tensor recovery model

\[
\min_{X' \in \mathbb{R}^{n \times 1 \times m}} \lambda \|X'\|_* \quad \text{s.t.} \quad A \ast X' = Y
\]

where \( Y \in \mathbb{R}^{n \times 1 \times m} \) is the tensor version of the observed blurry image \( Y \in \mathbb{R}^{m \times n} \). Note that the tubal rank of the ground truth \( X' \) is at most one since the second dimension of \( X' \) is one, which implies that \( X' \) is low-rank. The conversion between \( Y \) and \( X' \) is the same as that between \( X \) and \( X' \). Next we apply Algorithm 4.3 to recover \( X' \) and thereby the image \( X \).

We test an image “house” of size 256 × 256, which is degraded by a Gaussian convolution kernel of size 9 × 9 with standard deviation 2. Then we compare various image deblurring methods, including TV image deblurring [14], nonlocal means (NLM) [7], BM3D [15], and our algorithm with batch sizes \( b = 20, 40, 60, 80 \) and \( \alpha = 1, \lambda = 0.1 \). Here TV, NLM and BM3D are performed in the plug-and-play ADMM image recovery framework [12]. The MATLAB sources codes can be found in https://www.mathworks.com/matlabcentral/fileexchange/60641-plug-and-play-admm-for-image-restoration. To avoid ringing artifacts, we extend the image symmetrically along the boundary 14 pixels, apply the algorithm and cut the results back to the normal size. Figure 4 show the images recovered by TV, NLM, BM3D and our algorithm with \( b = 80 \). All results are compared quantitatively in Table 2 in terms of PSNR, SSIM and running time.

More generally, we consider an image sequence \( X \in \mathbb{R}^{n \times p \times m} \) with \( p \) frames (a video). Assume that all frames are convolved with the same spatial blurring kernel in its extended tensor form \( A \in \mathbb{R}^{m \times m \times n} \). As an illustrative example, we test the 3D MRI image data set \texttt{mri} in MATLAB which consists of 12 slices of size 128 × 128 from an MRI data scan of a human cranium. When \( p \ll \min\{m, n\} \), the ground truth \( X \) can be considered as low-rank. Each blurry image is generated by convolving the ground truth with a Gaussian convolution kernel of size 5 × 5 with standard deviation 2. The parameters are \( \alpha = 1, \lambda = 10^{-2} \) and the maximum iteration number is 1000, and the batch size is 60. Figure 4 shows the first

| Method | PSNR | SSIM | Running Time (s) |
|--------|------|------|-----------------|
| TV     | 29.48| 0.8180 | 33.34          |
| NLM    | 29.16| 0.8113 | 29.27          |
| BM3D   | 30.69| 0.8381 | 318.58         |
| b = 20 | 30.90| 0.8257 | 53.34          |
| b = 40 | 31.10| 0.8451 | 72.88          |
| b = 60 | 31.11| 0.8457 | 97.96          |
| b = 80 | 31.12| 0.8461 | 122.13         |

Table 2: Quantitative comparison of various image deblurring methods. Rows 5-8: the proposed tensor-based image deblurring algorithm with sequential control sequence and batch size \( b = 20, 40, 60, 80 \).
Figure 4: Visual comparison of various image deblurring methods, including TV, NLM, BM3D and our proposed tensor-based deblurring algorithm with sequential control sequence and blocksize $b = 80$. 
Figure 5: Image Sequence Deblurring. Row 1: blurry images; row 2: recovered images. Overall runtime is 72.90 seconds and the entire relative error between the ground truth and recovered sequences is 13.90%.

four frames of blurry observations and their respective recovered image. To suppress ringing artifacts, projection of all intensities onto the positive values is set as a postprocessing step.

6 Conclusions and Future work

In many tensor recovery problems, the underlying tensor is either sparse or low-rank which can be exploited in the design of efficient algorithms. In this paper, we propose a regularized Kaczmarz algorithm framework for tensor recovery. Precisely, we adopt the t-product for third-order tensors with rapid implementation through the fast Fourier transform, and establish linear convergence rate in expectation for the proposed algorithm with random sequence. In addition, we provide extensive discussions on the matrix and vector recovery together with tensor nuclear norm minimization as special cases. A showcase of numerical experiments demonstrates its considerable potential in various applications, including sparse signal recovery, low-rank tensor recovery, image inpainting and deblurring. In the future, we intend to explore the convergence rate for the deterministic method and discuss theoretical guarantees for its more superior performance over the randomized version. Moreover, it would be extremely intriguing to make a thorough discussion on the acceleration effects by choosing an appropriate batch size or step size. Furthermore, the current framework can be adapted to other types of tensor products or tensors of order higher than three.

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References

[1] M. A. Abidi, A. V. Gribok, and J. Paik, Optimization Techniques in Computer Vision, Springer, 2016.
[2] A. Agaskar, C. Wang, and Y. M. Lu, *Randomized kaczmarz algorithms: Exact mse analysis and optimal sampling probabilities*, in 2014 IEEE Global Conference on Signal and Information Processing (GlobalSIP), IEEE, 2014, pp. 389–393.

[3] A. H. Andersen and A. C. Kak, *Simultaneous algebraic reconstruction technique (sart): a superior implementation of the art algorithm*, Ultrasonic imaging, 6 (1984), pp. 81–94.

[4] Z.-Z. Bai and W.-T. Wu, *On greedy randomized kaczmarz method for solving large sparse linear systems*, SIAM Journal on Scientific Computing, 40 (2018), pp. A592–A606.

[5] H. H. Bauschke, P. L. Combettes, et al., *Convex analysis and monotone operator theory in Hilbert spaces*, vol. 408, Springer, 2011.

[6] S. Boyd, N. Parikh, and E. Chu, *Distributed optimization and statistical learning via the alternating direction method of multipliers*, Now Publishers Inc, 2011.

[7] A. Buades, B. Coll, and J.-M. Morel, *Non-local means denoising*, Image Processing On Line, 1 (2011), pp. 208–212.

[8] J.-F. Cai, E. J. Candès, and Z. Shen, *A singular value thresholding algorithm for matrix completion*, SIAM Journal on Optimization, 20 (2010), pp. 1956–1982.

[9] J.-F. Cai, S. Osher, and Z. Shen, *Convergence of the linearized bregman iteration for $\ell_1$-norm minimization*, Mathematics of Computation, 78 (2009), pp. 2127–2136.

[10] Y. Censor, S. A. Zenios, et al., *Parallel optimization: Theory, algorithms, and applications*, Oxford University Press on Demand, 1997.

[11] S. H. Chan, R. Khoshabeh, K. B. Gibson, P. E. Gill, and T. Q. Nguyen, *An augmented lagrangian method for total variation video restoration*, IEEE Transactions on Image Processing, 20 (2011), pp. 3097–3111.

[12] S. H. Chan, X. Wang, and O. A. Elgendy, *Plug-and-play admm for image restoration: Fixed-point convergence and applications*, IEEE Transactions on Computational Imaging, 3 (2016), pp. 84–98.

[13] X. Chen and A. M. Powell, *Almost sure convergence of the kaczmarz algorithm with random measurements*, Journal of Fourier Analysis and Applications, 18 (2012), pp. 1195–1214.

[14] ——, *Randomized subspace actions and fusion frames*, Constructive Approximation, 43 (2016), pp. 103–134.

[15] K. Dabov, A. Foi, V. Katkovnik, and K. Egiazarian, *Image denoising with block-matching and 3d filtering*, in Image Processing: Algorithms and Systems, Neural Networks, and Machine Learning, vol. 6064, International Society for Optics and Photonics, 2006, p. 606414.

[16] L. Dai, M. Soltanalian, and K. Pelckmans, *On the randomized kaczmarz algorithm*, IEEE Signal Processing Letters, 21 (2013), pp. 330–333.

[17] N. Durgin, R. Grotheer, C. Huang, S. Li, A. Ma, D. Needell, and J. Qin, *Sparse randomized kaczmarz for support recovery of jointly sparse corrupted multiple measurement vectors*, in Research in Data Science, Springer, 2019, pp. 1–14.

[18] Y. C. Eldar and D. Needell, *Acceleration of randomized kaczmarz method via the johnson–lindenstrauss lemma*, Numerical Algorithms, 58 (2011), pp. 163–177.

[19] H. Fan, Y. Chen, Y. Guo, H. Zhang, and G. Kuang, *Hyperspectral image restoration using low-rank tensor recovery*, IEEE Journal of Selected Topics in Applied Earth Observations and Remote Sensing, 10 (2017), pp. 4589–4604.
[20] P. Getreuer, *Total variation inpainting using split bregman*, Image Processing On Line, 2 (2012), pp. 147–157.

[21] R. Gordon, R. Bender, and G. T. Herman, *Algebraic reconstruction techniques (art) for three-dimensional electron microscopy and x-ray photography*, Journal of theoretical Biology, 29 (1970), pp. 471–481.

[22] G. T. Herman and L. B. Meyer, *Algebraic reconstruction techniques can be made computationally efficient (positron emission tomography application)*, IEEE transactions on medical imaging, 12 (1993), pp. 600–609.

[23] H. Jeong and C. S. Güntürk, *Convergence of the randomized kaczmarz method for phase retrieval*, arXiv preprint arXiv:1706.10291, (2017).

[24] S. Karczmarz, *Angenaherte auflosung von systemen linearer glei-chungen*, Bull. Int. Acad. Pol. Sic. Let., Cl. Sci. Math. Nat., (1937), pp. 355–357.

[25] M. E. Kilmer, K. Braman, N. Hao, and R. C. Hoover, *Third-order tensors as operators on matrices: A theoretical and computational framework with applications in imaging*, SIAM Journal on Matrix Analysis and Applications, 34 (2013), pp. 148–172.

[26] M. E. Kilmer and C. D. Martin, *Factorization strategies for third-order tensors*, Linear Algebra and its Applications, 435 (2011), pp. 641–658.

[27] M.-J. Lai and W. Yin, *Augmented \( \ell_1 \) and nuclear-norm models with a globally linearly convergent algorithm*, SIAM Journal on Imaging Sciences, 6 (2013), pp. 1059–1091.

[28] T. Li, T.-J. Kao, D. Isaacson, J. C. Newell, and G. J. Saulnier, *Adaptive kaczmarz method for image reconstruction in electrical impedance tomography*, Physiological measurement, 34 (2013), p. 595.

[29] J. Liu, S. J. Wright, and S. Sridhar, *An asynchronous parallel randomized kaczmarz algorithm*, arXiv preprint arXiv:1401.4780, (2014).

[30] D. A. Lorenz, F. Schöpfer, and S. Wenger, *The linearized bregman method via split feasibility problems: analysis and generalizations*, SIAM Journal on Imaging Sciences, 7 (2014), pp. 1237–1262.

[31] C. Lu, J. Feng, Y. Chen, W. Liu, Z. Lin, and S. Yan, *Tensor robust principal component analysis with a new tensor nuclear norm*, IEEE transactions on pattern analysis and machine intelligence, 42 (2019), pp. 925–938.

[32] A. Ma and D. Molitor, *Randomized kaczmarz for tensor linear systems*, arXiv preprint arXiv:2006.01246, (2020).

[33] A. Ma, D. Needell, and A. Ramdas, *Convergence properties of the randomized extended gauss–seidel and kaczmarz methods*, SIAM Journal on Matrix Analysis and Applications, 36 (2015), pp. 1590–1604.

[34] D. Needell, *Randomized kaczmarz solver for noisy linear systems*, BIT Numerical Mathematics, 50 (2010), pp. 395–403.

[35] D. Needell, N. Srebro, and R. Ward, *Stochastic gradient descent, weighted sampling, and the randomized kaczmarz algorithm*, Mathematical Programming, 155 (2016), pp. 549–573.

[36] D. Needell and J. A. Tropp, *Paved with good intentions: analysis of a randomized block kaczmarz method*, Linear Algebra and its Applications, 441 (2014), pp. 199–221.

[37] D. Needell, R. Zhao, and A. Zouzias, *Randomized block kaczmarz method with projection for solving least squares*, Linear Algebra and its Applications, 484 (2015), pp. 322–343.
[38] J. E. Peterson, B. N. Paulsson, and T. V. McEvilly, Applications of algebraic reconstruction techniques to crosshole seismic data, Geophysics, 50 (1985), pp. 1566–1580.

[39] S. Petra, Randomized sparse block kaczmarz as randomized dual block-coordinate descent, Analele Universității"Ovidius" Constanța-Seria Matematica, 23 (2015), pp. 129–149.

[40] C. Popa and R. Zdunek, Kaczmarz extended algorithm for tomographic image reconstruction from limited-data, Mathematics and Computers in Simulation, 65 (2004), pp. 579–598.

[41] R. T. Rockafellar, Convex analysis, no. 28, Princeton university press, 1970.

[42] F. Schöpfer, Linear convergence of descent methods for the unconstrained minimization of restricted strongly convex functions, SIAM Journal on Optimization, 26 (2016), pp. 1883–1911.

[43] F. Schöpfer and D. A. Lorenz, Linear convergence of the randomized sparse kaczmarz method, Mathematical Programming, 173 (2019), pp. 509–536.

[44] O. Semerci, N. Hao, M. E. Kilmer, and E. L. Miller, Tensor-based formulation and nuclear norm regularization for multienergy computed tomography, IEEE Transactions on Image Processing, 23 (2014), pp. 1678–1693.

[45] J. Shen and T. F. Chan, Mathematical models for local nontexture inpaintings, SIAM Journal on Applied Mathematics, 62 (2002), pp. 1019–1043.

[46] T. Strohmer and R. Vershynin, A randomized kaczmarz algorithm with exponential convergence, Journal of Fourier Analysis and Applications, 15 (2009), p. 262.

[47] H. Tan, G. Feng, J. Feng, W. Wang, Y.-J. Zhang, and F. Li, A tensor-based method for missing traffic data completion, Transportation Research Part C: Emerging Technologies, 28 (2013), pp. 15–27.

[48] Y. S. Tan and R. Vershynin, Phase retrieval via randomized kaczmarz: Theoretical guarantees, Information and Inference: A Journal of the IMA, 8 (2018), pp. 97–123.

[49] S. J. Wright, Coordinate descent algorithms, Mathematical Programming, 151 (2015), pp. 3–34.

[50] W. Yin, S. Osher, D. Goldfarb, and J. Darbon, Bregman iterative algorithms for l1-minimization with applications to compressed sensing, SIAM Journal on Imaging sciences, 1 (2008), pp. 143–168.

[51] H. Zhou, L. Li, and H. Zhu, Tensor regression with applications in neuroimaging data analysis, Journal of the American Statistical Association, 108 (2013), pp. 540–552.

[52] A. Zouzias and N. M. Freris, Randomized extended kaczmarz for solving least squares, SIAM Journal on Matrix Analysis and Applications, 34 (2013), pp. 773–793.