Abstract

The classical relativistic wave equations are presented as partial difference equations in the arena of covariant discrete phase space. These equations are also expressed as difference-differential equations in discrete phase space and continuous time. The relativistic invariance and covariance of the equations in both versions are established. The partial difference and difference-differential equations are derived as the Euler-Lagrange equations from the variational principle. The difference and difference-differential conservation equations are derived. Finally, the total momentum, energy, and charge of the relativistic classical fields satisfying difference-differential equations are computed.

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1. Introduction

Partial difference equations have been studied [1] for a long time to investigate problems in mathematical physics. Moreover, modern numerical analysis [2], which studies the differential equations arising out of various physical problems approximately, is based upon finite difference (ordinary or partial) equations.

Recently [3], we have formulated the wave mechanics in an exact fashion in the arena of discrete phase space in terms of the partial difference equations. This new formulation includes (free) classical relativistic Klein-Gordon, Dirac, and gauge field equations.

The proofs of the relativistic invariance or covariance of various partial difference and difference-differential equations are quite subtle. However, we have managed to provide such proofs in Section 4.
The Euler-Lagrange equations for the partial difference and difference-
differential equations and Noether’s theorems for difference and difference-
differential conservation laws are derived in Appendix I and Appendix II
respectively.

Finally, the total momentum, energy and charge are computed for vari-
ous relativistic classical fields obeying difference-differential equations in Sec-
tion 5. (We have not computed conserved quantities for wave fields satisfying
partial difference equations for a physical reason to be explained later.)

Moreover, the stress-energy-momentum tensor and the consequent total
momentum-energy as computed from a general Lagrangian are somewhat
incomplete in this paper. However, in the next paper, these quantities for
the free Klein-Gordon, electro-magnetic, and Dirac fields are furnished with
exact equations.

2. Notations and preliminary definitions

There exists a characteristic length $\ell$ (which may be the Planck length) in this
theory. We choose the absolute units such that $\hbar = c = \ell = 1$. All physical
quantities are expressed as dimensionless numbers. Greek indices take from
$\{1, 2, 3, 4\}$, roman indices take from $\{1, 2, 3\}$, and the capital roman take
from $\{1, 2\}$. Einstein’s summation convention is followed in all three cases.
We denote the flat space-time metric by $\eta_{\mu\nu}$ and the diagonal matrix $[\eta_{\mu\nu}] := \text{diag}[1, 1, 1, -1]$. (The signature of the metric is obviously $+2$.) We denote
the set of all non-negative integers by $\mathbb{N} := \{0\} \cup \{Z^+\} = \{0, 1, 2, 3, \ldots\}$. An element $n \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and an element $(n, t) \in \mathbb{N}^3 \times \mathbb{R}$ can be expressed as

$$n = (n^1, n^2, n^3, n^4), \quad n^\mu \in \mathbb{N}, \quad \mu \in \{1, 2, 3, 4\};$$

$$(n, t) = (n^1, n^2, n^3, t), \quad n^j \in \mathbb{N}, \quad j \in \{1, 2, 3\}, \quad t \in \mathbb{R}. \quad (1A)$$

Here and elsewhere, a bold roman letter indicates a three-dimensional vector.
In this paper, the equations in the relativistic phase space are denoted by
(1.A), whereas the equations in the discrete phase space and continuous time
are labelled by (1.B). Both formulations are presented up to the difference and
difference-differential conservation laws. Subsequently, only the difference-
differential equations are pursued. The physical meanings of the quantum
numbers $n^\mu$ are understood from the equations

$$(x^\mu)^2 + (p^\mu)^2 = 2n^\mu + 1, \quad \mu \in \{1, 2, 3, 4\},$$

where $x^\mu$ denote the space-time coordinates and $p^\mu$ stand for four-momentum
components in quantum mechanics.
Therefore, \( n^\mu \) gives rise to a closed phase space loop of radius \( \sqrt{2n^\mu + 1} \) in the \( \mu \)-th phase plane. Field quanta reside in such phase space loops where the measurements of angles are \textit{completely uncertain} (see Fig. 1). Phase space loops can be interpreted as degenerate phase cells.

We shall encounter three dimensional improper integrals in computing the total conserved quantities. Those integrals are always defined to be the Cauchy-principal-value:

\[
\int_{\mathbb{R}} f(k) \, d^3k := \lim_{L \to \infty} \left[ \int_{-L}^{L} \int_{-L}^{L} \int_{-L}^{L} f(k_1, k_2, k_3) \, dk_1 \, dk_2 \, dk_3 \right]. \tag{3}
\]

Let a function be defined by \( f : \mathbb{N}^4 \to \mathbb{R} \) (or \( f : \mathbb{N}^4 \to \mathbb{C} \)). Then the right partial difference, the left partial difference, and the weighted-mean difference are defined respectively by [3,4]:

\[
\Delta_\mu f(n) := f(\ldots, n^\mu + 1, \ldots) - f(\ldots, n^\mu, \ldots), \tag{4i}
\]

\[
\Delta'_\mu f(n) := f(\ldots, n^\mu, \ldots) - f(\ldots, n^\mu - 1, \ldots), \tag{4ii}
\]

\[
\Delta^#_\mu f(n) := (1/\sqrt{2}) \left[ \sqrt{n^\mu + 1} f(\ldots, n^\mu + 1, \ldots) - \sqrt{n^\mu} f(\ldots, n^\mu - 1, \ldots) \right] \tag{4iii}
\]
It is clear that the partial difference operators $\Delta_{\mu}, \Delta'_{\mu}, \Delta^\#_{\mu}$ are all linear and the operators $-i\Delta^\#_{\mu}$ are self-adjoint [3]. By direct computations, we can prove the following generalizations of the Leibnitz rule:

$$\Delta_{\mu}[f(n)g(n)] = f(\ldots, n^{\mu} + 1, \ldots)\Delta_{\mu}g(n) + g(n)\Delta_{\mu}f(n), \quad (5i)$$

$$\Delta'_{\mu}[f(n)g(n)] = f(n)\Delta'_{\mu}g(n) + g(\ldots, n^{\mu} - 1, \ldots)\Delta'_{\mu}f(n), \quad (5ii)$$

$$\Delta^\#_{\mu}[f(n)g(n)] = f(\ldots, n^{\mu} + 1, \ldots)\Delta^\#_{\mu}g(n) + g(\ldots, n^{\mu} - 1, \ldots)\Delta^\#_{\mu}f(n)
- f(\ldots, n^{\mu} + 1, \ldots)g(\ldots, n^{\mu} - 1, \ldots)\Delta^\#_{\mu}(1), \quad (5iii)$$

$$\Delta^\#_{\mu}(1) := (1/\sqrt{2}) \left[ \sqrt{n^{\mu} + 1} - \sqrt{n^{\mu}} \right], \quad \lim_{n^{\mu} \to \infty} \Delta^\#_{\mu}(1) = 0, \quad (5iv)$$

$$\Delta_{\mu}\left\{ \sqrt{n^{\mu}}[\phi(n)\psi(\ldots, n^{\mu} - 1, \ldots) + \phi(\ldots, n^{\mu} - 1, \ldots)\psi(n)] \right\}
= \sqrt{2} [\phi(n)\Delta^\#_{\mu}\psi(n) + \psi(n)\Delta^\#_{\mu}\phi(n)]. \quad (5v)$$

In the left-hand side of the equation (5v), the index $\mu$ is not summed.

We shall furnish a few more rules involving finite difference operations in the following equations:

$$\sqrt{n^{\nu} + 1} \Delta_{\nu}\phi(n) + \sqrt{n^{\nu}}\Delta'_{\nu}\phi(n) = \sqrt{2} [\Delta^\#_{\nu}\phi(n) - \phi(n) \cdot \Delta^\#_{\nu}(1)], \quad (6i)$$

$$\sqrt{n^{\nu} + 1} \Delta_{\nu}\Delta^\#_{\mu}\phi(n) + \sqrt{n^{\nu}}\Delta'_{\nu}\Delta^\#_{\mu}\phi(n)
= \sqrt{2} [\Delta^\#_{\nu}\Delta^\#_{\mu}\phi(n) - \Delta^\#_{\nu}\phi(n) \cdot \Delta^\#_{\mu}(1)], \quad (6ii)$$

$$\sqrt{n^{\mu} + 1} [\Delta_{\mu}\phi(n)]^2 - \sqrt{n^{\mu}} [\Delta'_{\mu}\phi(n)]^2
= \sqrt{2} [\Delta^\#_{\mu}[\phi(n)]^2 - 2\phi(n)\Delta^\#_{\mu}\phi(n) + [\phi(n)]^2 \cdot \Delta^\#_{\mu}(1)]. \quad (6iii)$$
\[
\sqrt{n^\mu + 1} \left( (\Delta_\mu A_\nu f) \cdot (\Delta_\mu B_\sigma f) - \sqrt{n^\mu} (\Delta_\mu' A_\nu f) \cdot (\Delta_\mu' B_\sigma f) \right) = \sqrt{2} \left\{ (\Delta_\mu^a A_\nu f) \cdot (\Delta_\mu^a B_\sigma f) - (\Delta_\mu^a f) \cdot (\Delta_\mu^a (1)) \right\}, 
\]

\[
= \sqrt{n^\mu + 1} \left( (\Delta_\mu A_\nu f) \cdot (\Delta_\mu B_\sigma f) - \sqrt{n^\mu} (\Delta_\mu' A_\nu f) \cdot (\Delta_\mu' B_\sigma f) \right)
\]

(6iv)

+ (\Delta_\mu^a f) \cdot (\Delta_\mu^a (1)).

Here, neither the index \( \mu \) nor the index \( \nu \) is summed.

Now the rules for the summations will be listed.

\[
\sum_{n^\mu = N_1^\mu}^{N_2^\mu} \Delta_\mu f(n) = f(\ldots, N_2^\mu + 1, \ldots) - f(\ldots, N_1^\mu, \ldots),
\]

(7i)

\[
\sum_{n^\mu = N_1^\mu}^{N_2^\mu} (\Delta_\mu' f(n) = f(\ldots, N_2^\mu, \ldots) - f(\ldots, N_1^\mu - 1, \ldots),
\]

(7ii)

\[
\sum_{n^\mu = N_1^\mu}^{N_2^\mu} \Delta_\mu^a f(n) = \frac{1}{\sqrt{2}} \left[ \sqrt{N_2^\mu + 1} f(\ldots, N_2^\mu + 1, \ldots) - \sqrt{N_1^\mu} f(\ldots, N_1^\mu - 1, \ldots) + \sum_{n^\mu = N_1^\mu + 1}^{N_2^\mu} \sqrt{n^\mu} \Delta_\mu' f(n) \right].
\]

(7iii)

It can be noted that right-hand sides of the equations (7i) and (7ii) contain only boundary terms whereas the right-hand side of (7iii) contains many more than just boundary terms.

3. Gauss’s theorem and conservation laws in a discrete space

Let a domain \( D \) of the four-dimensional discrete space \( \mathbb{N}^4 \) be given by (see Fig. 2)

\[
D := \{ n \in \mathbb{N}^4 : N_1^\mu < n^\mu < N_2^\mu, \mu \in \{1, 2, 3, 4\} \}.
\]

(8)
The discrete boundary points of $D$ are taken to be

$$
\partial D = \partial_{1-} D \cup \partial_{1+} D \cup \partial_{2-} D \cup \partial_{2+} D \cup \partial_{3-} D \cup \partial_{3+} D \cup \partial_{4-} D \cup \partial_{4+} D,
$$

$$
\partial_{\mu-} D := \{ n \in \mathbb{N}^4 : n^\mu = N_1^\mu, N_1^\sigma \leq n^\sigma \leq N_2^\sigma, \sigma \neq \mu \},
$$

$$
\partial_{\mu+} D := \{ n \in \mathbb{N}^4 : n^\mu = N_2^\mu, N_1^\sigma \leq n^\sigma \leq N_2^\sigma, \sigma \neq \mu \}. \tag{9}
$$

We also denote the unit “normal” $\nu_\mu$ on the boundary $\partial D$ by the following definition.

$$
\nu_\mu(n) := \begin{cases} 
1 \text{ on } \partial_{\mu+} D, \\
-1 \text{ on } \partial_{\mu-} D.
\end{cases} \tag{10}
$$

We assume that a tensor field $j^{\mu..}(n)$ is defined over $D \subset \mathbb{N}^4$. (See equation (34A) for the definition of a tensor field.) Now we are ready to state and prove formally the “discrete Gauss’s theorem” [5].
Theorem 3.1 (Discrete Gauss’s): Let a tensor field \( j^\mu..(n) \) be defined on \( D \cup \partial D \subset \mathbb{N}^4 \). Then

\[
\sum_{n^1=N_1^1}^{N_2^1} \sum_{n^2=N_2^2}^{N_3^2} \sum_{n^3=N_3^3}^{N_4^3} \sum_{n^4=N_4^4}^{N_4^4} [\Delta_\mu j^\mu..(n)] =: \sum_{D \subset \mathbb{N}^4} [\Delta_\mu j^\mu..(n)]
\]

\[
= \sum_{\partial D \subset \mathbb{N}^4} j^\mu..(n) \nu_\mu(n).
\]

Proof. Using the equation (7i) four times to the left-hand side of the equation (11) we obtain

\[
\sum_{n^2=N_2^2}^{N_3^2} \sum_{n^3=N_3^3}^{N_4^3} \sum_{n^4=N_4^4}^{N_4^4} \left[ \sum_{n^1=N_1^1}^{N_2^1} \Delta_1 j^1..(n) \right] + \ldots + \ldots

= \sum_{n^2=N_2^2}^{N_3^2} \sum_{n^3=N_3^3}^{N_4^3} \sum_{n^4=N_4^4}^{N_4^4} \left[ j^1..(N_2^1, \ldots) - j^1..(N_1^1, \ldots) \right] + \ldots + \ldots

= \left[ \sum_{\partial_1 D} j^1..(n) - \sum_{\partial_{1-D}} j^1..(n) \right] + \ldots + \ldots

= \left[ \sum_{\partial_1 D \cup \partial_{1-D}} j^1..(n) \nu_1(n) \right] + \ldots + \ldots

= \sum_{\partial D \subset \mathbb{N}^4} j^\mu..(n) \nu_\mu(n). \quad \square
\]

We shall now make some comments on the preceding theorem.

(i) An alternate theorem holds by replacing the left-hand side of the equation (11) by \( \sum_{D \subset \mathbb{N}^4} [\Delta'_\mu j^\mu..(n)] \).

(ii) Both forms of Gauss’s theorem can be generalized to any finite-dimensional discrete space.

(iii) In the finite difference representation of quantum mechanics [3], the four momentum operators are furnished by \( P_\mu = -i \Delta_\mu' \). These are consequences of the relativistic representations. However, Gauss’s theorem 3.1, which uses \( \Delta_\mu \) operators, is non-relativistic. (The relativistic Gauss’s theorem involving \( \sum_{D \subset \mathbb{N}^4} [\Delta'_\mu j^\mu..(n)] \) is not yet solved. See the comments at the end.
of this section.) The partial difference and difference-differential conservation
equations (non-relativistic) are furnished by
\[
\Delta_\mu j_\mu^\mu(n) = 0 , \quad (12A)
\]
\[
\Delta_b j_b^b(n, t) + \partial_t j_4^4(n, t) = 0 . \quad (12B)
\]

Difference conservation equations lead to summation conservations. We shall
presently state and prove a theorem about this topic.

**Theorem 3.2 (Conserved sums):** Let a tensor field
\[ j_\mu^\mu(n) \]

satisfy the
\[
\Delta_\mu j_\mu^\mu(n) = 0 \text{ in a domain } D \subset \mathbb{N}^4 \text{ given in the equation } (8) .
\]
Let furthermore the boundary conditions
\[
j_b^b(n)\nu_b(n)|_{\partial_a^+ D \cup \partial_a^- D} = 0
\]
hold. Then the sum
\[
\sum_{n^1=1}^{N_1^4} \sum_{n^2=1}^{N_2^4} \sum_{n^3=1}^{N_3^4} \sum_{n^4=1}^{N_4^4} [j_\mu^\mu(n^1, n^2, n^3, n^4)]
\]
\[
= \sum_{n^1=1}^{N_1^4} \sum_{n^2=1}^{N_2^4} \sum_{n^3=1}^{N_3^4} [j_\mu^\mu(n^1, n^2, n^3, n^4)]
\]
\[
= \sum_{n^1=1}^{N_1^4} \sum_{n^2=1}^{N_2^4} \sum_{n^3=1}^{N_3^4} j_\mu^\mu(n^1, n^2, n^3, n^4)
\]
for all \( n^4 \) satisfying \( N_1^4 \leq n^4 \leq N_2^4 \).

**Proof.** Using Gauss’s theorem 3.1 and the difference conservation equation (12A), we conclude that
\[
0 = \sum_{D \subset \mathbb{N}^4} \left[ \Delta_\mu j_\mu^\mu(n) \right]
\]
\[
= \sum_{\partial_1^+ D \cup \partial_4^- D} j_\mu^\mu(n)\nu_1(n) + \ldots + \sum_{\partial_4^+ D \cup \partial_4^- D} j_\mu^\mu(n)\nu_4(n).
\]
Assuming the boundary conditions (13), the above equation yields

\[
0 = \sum_{\partial_{+D}, \partial_{-D}} (3) j^4_- (n) \nu_4 (n)
\]

\[
= \sum_{\partial_{+D}} (3) j^4_- (n^1, n^2, n^3, N^4_2) - \sum_{\partial_{-D}} (3) j^4_- (n^1, n^2, n^3, N^4_1).
\]

Considering Gauss’s theorem for a proper subset of \( D \), the equation (14) follows.

In the case of the difference conservation equation (12) being valid for the denumerably infinite domain \( N^4 \), we can derive conserved sums under suitable boundary conditions. Such boundary conditions (sufficient) are

\[
\partial_{a-D} := \{ n \in N^4 : n^a = 0, 0 \leq n^b < \infty, a \neq b \},
\]

\[
\partial_{a+D} := \{ n \in N^4 : n^a = M, 0 \leq n^b < \infty, a \neq b \}, \quad (15A)
\]

\[
\lim_{M \to \infty} \left[ j^b_- (n) \nu_b (n) \right]_{\partial_{a-D} \cup \partial_{a+D}} = 0;
\]

\[
\partial_{a-D} := \{ (n, t) \in N^3 \times R : n^a = 0, 0 \leq n^b < \infty, a \neq b \},
\]

\[
\partial_{a+D} := \{ (n, t) \in N^3 \times R : n^a = M, 0 \leq n^b < \infty, a \neq b \}, \quad (15B)
\]

\[
\lim_{M \to \infty} \left[ j^b_- (n, t) \nu_b (n, t) \right]_{\partial_{a-D} \cup \partial_{a+D}} = 0.
\]

Under boundary conditions (15A), the equation (14) yields the following totally conserved quantities (generalized charges!)

\[
Q^- = \sum_{n^1=0}^{\infty} \sum_{n^2=0}^{\infty} \sum_{n^3=0}^{\infty} j^4_- (n^1, n^2, n^3, n^4) =: \sum_{n \in N^3} (3) j^4_- (n, n^4)
\]

\[
=: \sum_{n \in N^3} (3) j^4_- (n, 2).
\]

(16A)

In the case of the difference-differential conservation equations (12B) and the boundary conditions (15B), we can derive the totally conserved quantities

\[
Q^- = \sum_{n \in N^3} (3) j^4_- (n, t) = \sum_{n \in N^3} (3) j^4_- (n, 0). \quad (16B)
\]

One may wonder why are we considering the non-relativistic Gauss’s theorem and the consequent conserved sums at all! In Section 5, we shall prove that the theorems of this section can be used tactfully to elicit relativistic conserved sums.
4. Relativistic covariance of partial difference and difference-differential wave equations

The relativistic invariance or covariance of our partial difference or difference-differential equations is quite delicate. The criterion used in this paper has been developed through many papers [6] published in the last three decades. Let us start with a very simple example of Poincaré transformations and the invariance of the usual wave equation under that transformation. Consider the infinitesimal time-translation:

\[
\hat{x}^a = x^a, \hat{x}^4 = x^4 + \varepsilon^4, \\
x^a = \hat{x}^a, x^4 = \hat{x}^4 - \varepsilon^4. \tag{17}
\]

There exists in the old frame a different event \((x^*\) which has the same coordinates \((x)\) in the new frame. The coordinates of \((x^*\) from (17) are given by

\[
x^{*a} = x^a, x^{*4} = x^4 - \varepsilon^4, \hat{x}^{*4} = x^4. \tag{18}
\]

Consider the transformation rule for a scalar field \(\phi(x)\) given by

\[
\hat{\phi}(\hat{x}) = \phi(x), \\
\hat{\phi}(\hat{x}) = \phi(x^*) = \phi(x^1, x^2, x^3, x^4 - \varepsilon^4). \tag{19}
\]

Let \(\phi(x)\) be a Taylor-expandible (or analytic) function. In that case we can express (19) by Lagrange’s formula as

\[
\hat{\phi}(\hat{x}) = \left[\exp(-\varepsilon^4 \partial^4)\right] \phi(x) \\
= \left[\exp(-i\varepsilon^4 (-i) \partial_4)\right] \phi(x) \\
= \phi(x) - \varepsilon^4 \partial_4 \phi(x) + 0 \left[ (\varepsilon^4)^2 \right], \tag{20}
\]

\[\partial_\mu := \frac{\partial}{\partial x^\mu}.\]

Moreover, let \(\phi(x)\) satisfy the usual wave equation

\[
\eta^{\mu\nu} \partial_\mu \partial_\nu \phi(x) = 0. \tag{21}
\]

In the new frame,

\[
\eta^{\mu\nu} \partial_\mu \partial_\nu \hat{\phi}(\hat{x}) = \eta^{\mu\nu} \partial_\mu \partial_\nu \left[ \left[\exp(-\varepsilon^4 \partial_4)\right] \phi(x) \right] \\
= \left[\exp(-\varepsilon^4 \partial_4)\right] \left[ \eta^{\mu\nu} \partial_\mu \partial_\nu \phi(x) \right] = 0. \tag{20}
\]

The above equation demonstrates in an unusual manner the relativistic invariance of the wave equation under an infinitesimal time translation. We shall follow similar proofs in the sequel.
Let us now re-examine the very concept of relativistic invariance or covariance. The relativistic covariance does not necessarily imply that the space and time coordinates must be treated on the same footing. Nor do the equations which treat space and time variables on the equal footing automatically imply the relativistic covariance. For example, let us consider the partial difference Klein-Gordon equation [7] in the lattice space-time as

$$\eta^\mu{}^\nu \Delta_\mu \Delta_\nu \phi(n) - m^2 \phi(n) = 0. \quad (22)$$

This equation treats discrete space and time variables on the same footing. However, the equation (22) is certainly not invariant under the continuous Poincaré group $\mathcal{I}O(3, 1)!$ (Although there exists the lattice Lorentz group [8], a subgroup of the Lorentz group $O(3, 1)$, which leaves the lattice space-time invariant.)

Consider another example, namely the Friedmann-Robertson-Walker cosmological model of the universe. The corresponding metric and the “orthonormal” tetrad are given by [9]:

$$ds^2 = [R(t)]^2[1 + K \delta_{ab} x^a x^b]^{-2}[\delta_{ij} dx^i dx^j] - (dt)^2,$$

$$e^\mu_{(j)} = [R(t)]^{-1}[1 + K \delta_{ab} x^a x^b] \delta^\mu_{(j)}, \quad (23)$$

$$e^\mu_{(4)} = \delta^\mu_{(4)}.$$

The above metric and the corresponding tetrad do not treat the space and time variables on equal footing. (Although these are extracted as exact solutions of Einstein’s general covariant equations.) However, if we consider a suitably parametrized motion curve given by a time-like geodesic, the appropriate Lagrangian and the four-momentum are given by:

$$L(\ldots) = \left( m/2 \right) \left\{ [R(t)]^2 [1 + K \delta_{ab} x^a x^b]^{-2} [\delta_{ij} \dot{x}^i \dot{x}^j] - (\dot{t}^2) \right\},$$

$$p_{(j)} := \frac{\partial L(\ldots)}{\partial \dot{x}^\mu} e^\mu_{(j)} = m[R(t)] [1 + K \delta_{ab} x^a x^b]^{-1} \delta_{ji} \dot{x}^i,$$

$$p_{(4)} := \frac{\partial L(\ldots)}{\partial \dot{t}} e^\mu_{(4)} = -m \dot{t}, \quad (24)$$

Recalling that $g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu \equiv -1$ along a time-like geodesic, we obtain from (24) that

$$\eta^{(\mu)(\nu)} p_{(\mu)} p_{(\nu)} = -m^2. \quad (25)$$

Therefore, the special relativistic equation (25) holds among the tetrad components of the four-momentum locally. In fact, in every reasonable curved universe, with any admissible coordinate system, the special relativity holds locally whether or not the Poincaré group $\mathcal{I}O(3, 1)$ is globally admitted as Killing motion.
In a flat space-time, the first quantization of the equation (25) leads to the relativistic operator equation:

\[ \eta^{\mu\nu} P_\mu P_\nu + m^2 I \vec{\phi} = \vec{0} . \]  

(26)

Here, \( \vec{\phi} \) and \( \vec{0} \) are vectors in the tensor-product [3] of the Hilbert spaces. The \( P_\mu \)'s represent four self-adjoint, unbounded linear operators and “I” stands for the identity operator. The equation (26) physically represents the quantum mechanics of a massive, spin-less, free particle. Let us explore the relativistic invariance of the operator equation (26). The finite and infinitesimal versions of a Poincaré transformation in the classical level are given respectively by:

\[
\begin{align*}
\hat{x}^\mu &= a^\mu + \ell^\mu_\nu x^\nu, \\
\eta_{\alpha\beta} \ell^\mu_\alpha \ell^\nu_\beta &= \eta_{\alpha\beta}; \\
\hat{x}^\mu &= x^\mu + \varepsilon^\mu + \varepsilon_\mu^\nu x^\nu, \\
\varepsilon_{\mu\nu} := \eta_{\mu\sigma} \varepsilon^\sigma_\nu &= -\varepsilon_{\nu\mu} + 0(\varepsilon^2).
\end{align*}
\]  

(27)

However, there exist in a deeper level, the quantum Poincaré transformations on quantum mechanical operators. In fact, there are two possible quantum Poincaré transformations of operators which involve the same unitary mapping. The Heisenberg-type of Poincaré transformations [10] under the infinitesimal version of (27) are furnished by:

\[
U(\varepsilon) := \exp \left\{ -i\varepsilon^\mu P_\mu + (i/4)\varepsilon^{\mu\nu} (Q_\mu P_\nu - Q_\nu P_\mu + P_\nu Q_\mu - P_\mu Q_\nu) \right\} ,
\]  

(28iH)

\[
Q_\mu P_\nu - P_\nu Q_\mu = i\eta_{\mu\nu} I ,
\]  

(28iiH)

\[
\hat{P}_\mu = U^\dagger(\varepsilon) P_\mu U(\varepsilon) = P_\mu - \varepsilon_\mu^\rho P_\rho + 0(\varepsilon^2) ,
\]  

(28iiiH)

\[
\hat{Q}^\mu = U^\dagger(\varepsilon) Q^\mu U(\varepsilon) = Q^\mu + \varepsilon^\mu I + \varepsilon_\rho^\mu Q^\rho + 0(\varepsilon^2) ,
\]  

(28ivH)

\[
\widehat{\vec{\phi}} = \vec{\phi}.
\]  

(28vH)

Here, the dagger denotes the hermitian conjugation.

In the Schrödinger-type of covariance [10], the abstract operators and the state vectors transform under the infinitesimal version of (27) as:

\[
\hat{P}_\mu = P_\mu, \hat{Q}^\mu = Q^\mu ,
\]  

(28iS)

\[
\hat{\phi} = U(\varepsilon) \vec{\phi} ,
\]  

(28iiS)

\[
\delta_L \vec{\phi} := \hat{\phi} - \vec{\phi} = \left\{ -i \left[ \varepsilon^\mu P_\mu - (1/4)\varepsilon^{\mu\nu} (Q_\mu P_\nu - Q_\nu P_\mu + P_\nu Q_\mu - P_\mu Q_\nu) \right] + 0(\varepsilon^2) \right\} \vec{\phi}.
\]  

(28iiiS)
Here, $U(\varepsilon)$ is the same as in (28iH). The vector $\delta_L \vec{\phi}$ is called the Lie-variation of the vector $\vec{\phi}$.

Note that in the two types of transformations, the expectation values of a polynomial operator $F(P,Q)$ is given by:

$$\langle \hat{\vec{\phi}}_b | F(\hat{P}, \hat{Q}) | \hat{\vec{\phi}}_a \rangle_H = \langle \vec{\phi}_b | F[U^\dagger PU, U^\dagger QU] | \vec{\phi}_a \rangle = \langle U \vec{\phi}_b | F(P,Q) | U \vec{\phi}_a \rangle = \langle \vec{\phi}_b | F(\hat{P}, \hat{Q}) | \vec{\phi}_a \rangle_S.$$

(Here, $\langle \vec{\phi}_b | \vec{\phi}_a \rangle$ denotes the inner-product of the two vectors $\vec{\phi}_b$ and $\vec{\phi}_a$.) Therefore, physical quantities transform exactly in the same manner under Heisenberg-type and Schroedinger-type of quantum covariance rules. We shall follow the Schroedinger-type of covariance in this paper.

It is well known [11] that the operator $P^\mu P_\mu$, which is one of the Casimir operators of the Poincaré group $SO(3,1)$, commute with all the generators $P_\mu$ and $(1/4)(Q_\mu P_\nu - Q_\nu P_\mu + P_\nu Q_\mu - P_\mu Q_\nu)$ of the Poincaré group. Therefore, we obtain from (28 i-iv S),

$$[\hat{P}^\sigma \hat{P}_\sigma + m^2 I] \vec{\phi} = [P^\sigma P_\sigma + m^2 I] [\vec{\phi} + \delta_L \vec{\phi}]$$

$$= [P^\sigma P_\sigma + m^2 I] [\delta_L \vec{\phi}]$$

$$= -i [\varepsilon^\mu P_\mu - (1/4)\varepsilon^{\mu\nu}(Q_\mu P_\nu - Q_\nu P_\mu + P_\nu Q_\mu - P_\mu Q_\nu)] \times [P^\sigma P_\sigma + m^2 I] \vec{\phi} + 0(\varepsilon^2)$$

$$= 0(\varepsilon^2).$$

Therefore, the operator equation (29) is an actual proof of the relativistic invariance (up to the 2nd order terms) for the operator Klein-Gordon equation (26).

We can generalize the Lie-variation (28ivH) for an arbitrary relativistic tensor (or spinor) operator $\vec{\phi}^-$. The appropriate definition is given by:

$$\hat{\vec{\phi}}^- - \vec{\phi}^- = \delta_L \vec{\phi}^- := -i \left\{ \varepsilon^\mu P_\mu - (1/4)\varepsilon^{\mu\nu} \left[ (Q_\mu P_\nu - Q_\nu P_\mu + P_\nu Q_\mu - P_\mu Q_\nu) + 2S^-_{\mu\nu} \right] \right\} \vec{\phi}^- + 0(\varepsilon^2),$$

where $S^-_{\mu\nu} = -S^-_{\nu\mu}$ denotes the “spin operator”.

The values of the entries for $S^-_{\mu\nu}$ can be calculated exactly from the usual tensor and spinor transformation rules. For example, if we consider a vector field $\vec{\phi}^-$ with components $\phi^\alpha$, then $S^-_{\mu\nu} = \eta_{\nu\beta} \delta^\mu_{\beta} - \eta_{\mu\beta} \delta^\nu_{\beta}$.

Now we shall state a necessary criterion for the relativistic invariance of an operator equation.
“Let an infinite-dimensional Hilbert space vector \( \vec{\phi} \) represent a particle with zero or non-zero spin. Let the corresponding Lie-variation \( \delta_L \vec{\phi} \) be given by the equation (30). In the case of the operator equation for \( \vec{\phi} \) being relativistic invariant or covariant, the Lie-variation \( \delta_L \vec{\phi} \) must satisfy the same operator equation as \( \vec{\phi} \) does (up to the 2nd order terms).”

Let us apply the above criterion of covariance on different first-quantized equations arising out of different representations of quantum mechanics.

Firstly, consider the usual Schrödinger representation of quantum mechanics, namely, \( P_\mu = -i \partial_\mu \) and \( Q^\mu = x^\mu \). The operator equation (26) of the mass-shell constraint goes over into

\[
-\Box \phi(x) + m^2 \phi(x) := -[\eta^{\mu\nu} \partial_\mu \partial_\nu \phi(x) - m^2 \phi(x)] = 0.
\]

This is the usual Klein-Gordon equation. The Lie-variation under \( IO(3,1) \) is given by

\[
\delta_L \phi(x) = -[\varepsilon^\mu \partial_\mu -(1/4)\varepsilon^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu + \partial_\nu x_\mu - \partial_\mu x_\nu)] \phi(x) + O(\varepsilon^2).
\]

Therefore,

\[
-\Box \delta_L \phi(x) + m^2 \delta_L \phi(x) = -[\varepsilon^\mu \partial_\mu -(1/4)\varepsilon^{\mu\nu}(x_\mu \partial_\nu - x_\nu \partial_\mu + \partial_\nu x_\mu - \partial_\mu x_\nu)] [-\Box \phi(x) + m^2 \phi(x)] + O(\varepsilon^2) = 0(\varepsilon^2).
\]

Therefore, the relativistic invariance (up to the second order term) is assured.

In the second example, let us consider the finite-difference representation [3] of the quantum mechanics by putting

\[
P_\mu = -i \Delta^\#_\mu, \quad Q_\mu = (1/\sqrt{2})(\Delta_\mu \sqrt{n^\mu} - \sqrt{n^\mu} \Delta'_\mu + 2\sqrt{n^\mu} I).
\]

Here, the index \( \mu \) is not summed. The operator equation (26) yields

\[
\eta^{\mu\nu} \Delta^\#_\mu \Delta^\#_\nu \phi(n) = m^2 \phi(n) = 0,
\]

\( n := (n^1, n^2, n^3, n^4) \in \mathbb{N}^4. \) (31A)

This is the finite difference version of the Klein-Gordon equation. The corresponding Lie-variation is given by:

\[
\delta_L \phi(n) = -\{\varepsilon^\mu \Delta^\#_\mu \phi(n) - (1/4\sqrt{2})\varepsilon^{\mu\nu}[(\Delta_\mu \sqrt{n^\mu} - \sqrt{n^\mu} \Delta'_\mu + 2\sqrt{n^\mu}) \Delta^\#_\nu

- (\Delta_\nu \sqrt{n^\mu} - \sqrt{n^\mu} \Delta'_\nu + 2\sqrt{n^\mu}) \Delta^\#_\mu + \Delta^\#_\mu \Delta^\#_\nu + 2\sqrt{n^\mu}] \phi(n)\} + O(\varepsilon^2).
\]

Here, the indices \( \mu, \nu \) are summed. It can be proved by direct computations that

\[
\eta^{\mu\nu} \Delta^\#_\mu \Delta^\#_\nu [\delta_L \phi(n)] - m^2 [\delta_L \phi(n)] = 0(\varepsilon^2).
\]
Thus the partial difference equation (31A) is indeed invariant (up to the second order terms) under the continuous Poincaré group!

Now, as the third example, let us consider a mixed difference-differential representation of the quantum mechanics by choosing

\[ P_b = -i\Delta^b_\# \, , \quad P_4 = -i\partial_t \equiv i\partial_t \, , \quad Q_b = (1/\sqrt{2}) (\Delta_b\sqrt{n^b} - \sqrt{n^b}\Delta'_b + 2\sqrt{n^b}1) \, , \quad Q^4 = x^4 \equiv t \, . \]

Here, the index \( b \) is not summed. The operator equation (26) yields the difference-differential version of the Klein-Gordon equation:

\[ \delta^{ab}\Delta^a_\#\Delta^b_\# \phi(n,t) - (\partial_t)^2\phi(n,t) - m^2\phi(n,t) = 0 \, , \quad (n,t) \in \mathbb{N}^3 \times \mathbb{R} \, . \]

(31B)

Here \( \delta^{ab} \) is the Kronecker delta. We have for the Lie-variation

\[ \delta_L\phi(n,t) = \]

\[-\{\varepsilon^b\Delta^b_\# + \varepsilon^4\partial_t - (1/4\sqrt{2})\varepsilon^{ab}\left[(\Delta_a\sqrt{n^a} - \sqrt{n^a}\Delta'_a + 2\sqrt{n^a})\Delta^b_\# - (\Delta_b\sqrt{n^b} - \sqrt{n^b}\Delta'_b + 2\sqrt{n^b})\Delta^a_\# + \Delta^b_\#(\Delta^a_\#\sqrt{n^a} - \sqrt{n^a}\Delta'_a + 2\sqrt{n^a}) - \Delta^a_\#(\Delta^b_\#\sqrt{n^b} - \sqrt{n^b}\Delta'_b + 2\sqrt{n^b})\right] \phi(n,t) + (1/\sqrt{2})\varepsilon^{ab}[\Delta^a_\#\partial_t - t\Delta^a_\#] \}\phi(n,t) \, . \]

(32B)

Here, the indices \( a, b \) are summed. We can prove by a long calculation that

\[ \delta^{ab}\Delta^a_\#\Delta^b_\#[\delta_L\phi(n,t)] - (\partial_t)^2[\delta_L\phi(n,t)] - m^2[\delta_L\phi(n,t)] = 0(\varepsilon^2) \, . \]

In other words, the difference-differential equation (31B) is indeed invariant (up to the second order terms) under the ten-parameter continuous group \( \mathcal{IO}(3,1) \)!

As the last example, consider the Schroedinger’s difference-differential equation [3] for a free particle (with \( m > 0 \)) as

\[ (2m)^{-1}\delta^{ab}\Delta^a_\#\Delta^b_\#\psi(n,t) + i\partial_t\psi(n,t) = 0 \, . \]

The Lie-variation \( \delta_L\psi(n,t) \) according to the equation (32B) does not satisfy the Schroedinger difference-differential equation up to the second order terms. The reason for this failure is the operator \( P_4 = i\partial_t \) does not commute with three particular generators \( Q_1P_b - Q_bP_4 \) of the Poincaré group. Therefore, we obtain an alternate proof of the well-known fact that the Schroedinger equation is non-relativistic. Thus, our necessary criterion involving the Lie-variation can prove or else disprove the relativistic invariance or covariance (up to the second order terms) of a quantum mechanical system in any representation.
We need to define the exact transformation rules for a tensor or spinor field in partial difference and difference-differential representations. Recall from (27) that the finite Poincaré transformation is given by

\[ \hat{\eta}^\mu = a^\mu + \ell^\mu x, \]

\[ \omega^{\mu\sigma} := \eta^{\mu\nu} (\ell^\nu - \delta^\nu_\nu), \]

\[ \omega_{\alpha\beta} + \omega_{\beta\alpha} + \eta_{\mu\nu} \omega^{\mu}_{\alpha} \omega^{\nu}_{\beta} = 0. \] (33)

The exact transformation rules for the tensor and spinor fields under (33) are furnished by:

\[ \hat{\phi}^{..}(n) = \left\{ -a^\mu \Delta^#_\mu + (1/8 \sqrt{2}) (\omega^{\mu\nu} - \omega^{\nu\mu}) \right. \]

\[ \left. \left[ (\Delta_\mu \sqrt{n^\mu} - \sqrt{n^\mu} \Delta'_\mu + 2 \sqrt{n^\mu}) \Delta^#_\mu 
- (\Delta_\nu \sqrt{n^\nu} - \sqrt{n^\nu} \Delta'_\nu + 2 \sqrt{n^\nu}) \Delta^#_\mu 
+ \Delta^#_\nu (\Delta_\mu \sqrt{n^\mu} - \sqrt{n^\mu} \Delta'_\mu + 2 \sqrt{n^\mu}) 
- \Delta^#_\mu (\Delta_\nu \sqrt{n^\nu} - \sqrt{n^\nu} \Delta'_\nu + 2 \sqrt{n^\nu}) + i2S^{..}_{\mu\nu} \right] \right\} \cdot \phi^{..}(n), \] (34A)

\[ \hat{\phi}^{..}(n,t) = \left\{ -a^b \Delta^#_b - a^4 \partial_t - (1/8 \sqrt{2}) (\omega^{ab} - \omega^{ba}) \right. \]

\[ \left. \left[ (\Delta_a \sqrt{n^a} - \sqrt{n^a} \Delta'_a + 2 \sqrt{n^a}) \Delta^#_b
- (\Delta_b \sqrt{n^b} - \sqrt{n^b} \Delta'_b + 2 \sqrt{n^b}) \Delta^#_a
+ \Delta^#_b (\Delta_a \sqrt{n^a} - \sqrt{n^a} \Delta'_a + 2 \sqrt{n^a})
- \Delta^#_a (\Delta_b \sqrt{n^b} - \sqrt{n^b} \Delta'_b + 2 \sqrt{n^b}) + i2S^{ab}_{..} \right] 
- (i/2 \sqrt{2}) (\omega^{ab} - \omega^{ba}) \left[ (\Delta_a \sqrt{n^a} - \sqrt{n^a} \Delta'_a) \partial_t 
- t \Delta^#_a + iS^{ab}_{..} \right] \right\} \cdot \phi^{..}(n,t). \] (34B)

Here, indices \( \mu, \nu \) and \( a, b \) are summed and \( S^{..}_{\mu\nu} \) stands for the spin-operator for the field \( \phi^{..}(n) \) or \( \phi^{..}(n,t) \).

An obvious question that arises is how the discrete variables \( n^1, n^2, n^3, n^4 \) transform from one inertial observer to another! We can recall that the integer \( n^\mu \) appears as the eigenvalue \( 2n^\mu + 1 \) of the operator \( (P^\mu)^2 + (Q^\mu)^2 \). A particle in the corresponding eigenstate \( \vec{\psi}_{n^\mu} \) is located on a circle of radius \( \sqrt{2n^\mu + 1} \) in the \( p^\mu - x^\mu \)-th phase plane (see Fig. 1). However, for a relatively moving observer, according to the Schroedinger-type of covariance, the particle is in the state \( \vec{\psi} = U(\varepsilon) \vec{\psi}_{n^\mu} \). (See equations (28iS,iiS,iiiS).) But \( \vec{\psi} \) is not an eigenstate of the operator \( (\hat{P}^\mu)^2 + (\hat{Q}^\mu)^2 = (P^\mu)^2 + (Q^\mu)^2 \). Therefore, this led us to reexamine the eigenvalues...
the particle appears in a *fuzzy domain* in the phase plane for the moving observer. But for the moving observer, there exists another state \( \psi_{n^\mu} := \tilde{\psi}_{n^\mu} \neq U(\varepsilon) \tilde{\psi}_{n^\mu} = \tilde{\psi} \) such that \([ (\tilde{P}^\mu)^2 + (\tilde{Q}^\mu)^2 ] \tilde{\psi}_{n^\mu} = [ (P^\mu)^2 + (Q^\mu)^2 ] \psi_{n^\mu} = (2n^\mu + 1) \psi_{n^\mu} \). Therefore, \( \psi_{n^\mu} \) is an eigenvector for the operator \((\tilde{P}^\mu)^2 + (\tilde{Q}^\mu)^2\) in the moving frame and the corresponding eigenvalue is \(2n^\mu + 1\). *Thus both observers have exactly similar discretized phase planes (see Fig. 1), although discrete circles in one frame do not transform into discrete circles in another frame!*

Now, we shall prove the relativistic invariance of the summation operation \(\sum_{n^1=0}^{\infty} \sum_{n^2=0}^{\infty} \sum_{n^3=0}^{\infty} \sum_{n^4=0}^{\infty} \) and the sum-integral operation \(\int_{\mathbb{R}} f dt\). Consider the state vector \( \tilde{\phi} \) representing a scalar particle in the abstract equations in (26) and (28i,ii,iiiS). Mathematically speaking, \( \tilde{\phi} \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \), is the tensor product of Hilbert spaces [3,12]. Here, the Hilbert space \( \mathcal{H}_\mu \) is acted upon by the operators \( P_\mu, Q_\mu \) etc. The square of the norm of \( \tilde{\phi} \) is given by:

\[
\| \tilde{\phi} \|^2 := \langle \tilde{\phi} | \tilde{\phi} \rangle.
\]

But from (28iH), (28iiS) we have

\[
\| \tilde{\phi} \|^2 = \langle \tilde{\phi} | U^\dagger(\varepsilon) U(\varepsilon) \tilde{\phi} \rangle = \| \tilde{\phi} \|^2,
\]

where the dagger denotes the hermitian-conjugation. Thus, the norm \( \| \tilde{\phi} \| \) is invariant under the unitary transformation induced by an infinitesimal Poincaré transformation.

Moreover, under a finite Poincaré transformation which induces a unitary mapping, we can conclude that \( \| \tilde{\phi} \| = \| \phi \| \).

In the finite difference representation of the scalar field [3] we have

\[
\| \tilde{\phi} \|^2 := \sum_{n^1=0}^{\infty} \sum_{n^2=0}^{\infty} \sum_{n^3=0}^{\infty} \sum_{n^4=0}^{\infty} |\phi(n^1, n^2, n^3, n^4)|^2.
\]

Moreover, in the difference-differential representation

\[
\| \tilde{\phi} \|^2 := \sum_{n^1=0}^{\infty} \sum_{n^2=0}^{\infty} \sum_{n^3=0}^{\infty} \int_{\mathbb{R}} |\phi(n^1, n^2, n^3, t)|^2 dt.
\]

Therefore, by the invariance of \( \| \phi \|^2 \), both operations \( \sum_{n^1=0}^{\infty} \sum_{n^2=0}^{\infty} \sum_{n^3=0}^{\infty} \sum_{n^4=0}^{\infty} \) and \( \int_{\mathbb{R}} \) must be relativistically invariant.
5. Variational formalism and conservation equations

Let us consider a real-valued tensor field $A^{\nu_{\cdot}}(n)$ in the difference representation and $A^{\nu_{\cdot}}(n, t)$ in the difference-differential representations. The action sum or the action sum-integral for such fields are defined respectively by:

$$
A(A^{\nu_{\cdot}}) := \sum_{n^1=N_1^1}^{N_1^2} \sum_{n^2=N_2^1}^{N_2^2} \sum_{n^3=N_3^1}^{N_3^2} \sum_{n^4=N_4^1}^{N_4^2} \int \frac{t_2}{t_1} \frac{\partial L}{\partial y_{\mu_{\cdot}}} \left\{ \frac{\partial L}{\partial y_{\mu_{\cdot}}^n} \right\} = 0 \text{ . (35A)}
$$

$$
L \left( n; y^{\nu_{\cdot}}; y_{\mu_{\cdot}}^n \right) \bigg|_{y^{\nu_{\cdot}} = A^{\nu_{\cdot}}(n), y_{\mu_{\cdot}}^n = \Delta_n^\mu A^{\nu_{\cdot}}(n)} \text{ ,}
$$

$$
A(A^{\nu_{\cdot}}) := \sum_{n^1=N_1^1}^{N_1^2} \sum_{n^2=N_2^1}^{N_2^2} \sum_{n^3=N_3^1}^{N_3^2} \sum_{n^4=N_4^1}^{N_4^2} \int \frac{t_2}{t_1} \frac{\partial L}{\partial y_{\mu_{\cdot}}} \left\{ \frac{\partial L}{\partial y_{\mu_{\cdot}}^n} \right\} = 0 \text{ . (35B)}
$$

The Euler-Lagrange equations under zero boundary variations for $A^{\nu_{\cdot}}(n)$ or $A^{\nu_{\cdot}}(n, t)$ are given by (see Appendix I):

$$
\frac{\partial L(\ldots)}{\partial y^{\nu_{\cdot}}[\ldots]} - \Delta^\mu_n \left\{ \frac{\partial L(\ldots)}{\partial y_{\mu_{\cdot}}^n} \right\} = 0 \text{ , (36A)}
$$

$$
\frac{\partial L(\ldots)}{\partial y^{\nu_{\cdot}}[\ldots]} - \Delta^\mu_n \left\{ \frac{\partial L(\ldots)}{\partial y_{\mu_{\cdot}}^n} \right\} - \partial_t \left\{ \frac{\partial L(\ldots)}{\partial y_{\mu_{\cdot}}^n} \right\} = 0 \text{ . (36B))}
$$

In the case of a complex-value tensor field $\phi^{\nu_{\cdot}}(n)$ and $\phi^{\nu_{\cdot}}(n, t)$, the action sum and sum-integral are defined respectively by:

$$
A(\phi^{\nu_{\cdot}}, \overline{\phi^{\nu_{\cdot}}}) := \sum_{n^1=N_1^1}^{N_1^2} \sum_{n^2=N_2^1}^{N_2^2} \sum_{n^3=N_3^1}^{N_3^2} \sum_{n^4=N_4^1}^{N_4^2} \int \frac{\partial L}{\partial \phi^{\nu_{\cdot}}} \left\{ \frac{\partial L}{\partial \overline{\phi^{\nu_{\cdot}}}} \right\} = 0 \text{ . (37A)}
$$

$$
L \left( n; \rho^{\nu_{\cdot}}, \rho^{\nu_{\cdot}}; \rho^\mu_n, \overline{\rho^\mu_n} \right) \bigg|_{\rho^{\nu_{\cdot}} = \phi^{\nu_{\cdot}}(n), \rho^\mu_n = \overline{\phi^{\nu_{\cdot}}(n)}} \text{ ,}
$$

$$
A(\phi^{\nu_{\cdot}}, \overline{\phi^{\nu_{\cdot}}}) := \sum_{n^1=N_1^1}^{N_1^2} \sum_{n^2=N_2^1}^{N_2^2} \sum_{n^3=N_3^1}^{N_3^2} \sum_{n^4=N_4^1}^{N_4^2} \int \frac{\partial L}{\partial \phi^{\nu_{\cdot}}} \left\{ \frac{\partial L}{\partial \overline{\phi^{\nu_{\cdot}}}} \right\} = 0 \text{ . (37B)}
$$

$$
L \left( n, t; \rho^{\nu_{\cdot}}, \rho^{\nu_{\cdot}}; \rho^\mu_n, \rho^\mu_n, \rho^\nu_n, \rho^\nu_n \right) \bigg|_{\rho^\mu_n = \Delta^\mu_n \phi^{\nu_{\cdot}}, \rho^\nu_n = \overline{\Delta^\mu_n \phi^{\nu_{\cdot}}}} \text{ .}
$$
Here, the bar stands for the complex-conjugation. The corresponding Euler-Lagrange equations are:

\[
\frac{\partial L(\ldots)}{\partial \rho^{\nu\cdot\ldots}} - \Delta^a_{\mu} \left\{ \left[ \frac{\partial L(\ldots)}{\partial \rho^{\nu\cdot\ldots}_\mu} \right]_{\ldots} \right\} = 0, \tag{38A}
\]

\[
\frac{\partial L(\ldots)}{\partial \rho^{\nu\cdot\ldots}} - \Delta^a_{\mu} \left\{ \left[ \frac{\partial L(\ldots)}{\partial \rho^{\nu\cdot\ldots}_\mu} \right]_{\ldots} \right\} = 0, \tag{38A}\]

\[
\frac{\partial L(\ldots)}{\partial \rho^{\nu\cdot\ldots}} - \Delta^a_{\mu} \left\{ \left[ \frac{\partial L(\ldots)}{\partial \rho^{\nu\cdot\ldots}_\mu} \right]_{\ldots} \right\} = 0, \tag{38B}\]

\[
\frac{\partial L(\ldots)}{\partial \rho^{\nu\cdot\ldots}} - \Delta^a_{\mu} \left\{ \left[ \frac{\partial L(\ldots)}{\partial \rho^{\nu\cdot\ldots}_\mu} \right]_{\ldots} \right\} = 0. \tag{38B}\]

In the derivation of above equations, the techniques of the complex-conjugate coordinates are used [13,14].

We shall now derive the partial difference and the difference-differential conservation equations for various fields (see Appendix II).

\[
\Delta_\mu T^\nu_\mu + \ldots = 0, \tag{39Ai}\]

\[
T^\nu_\mu(n) := \sqrt{\frac{n^\nu}{2}} \left[ \frac{\partial L(\ldots)}{\partial y^\alpha_{\nu\cdot\ldots}|_{\ldots,-\nu-1\ldots}} \cdot \Delta^a_{\mu} A^\alpha_{\ldots} \right. \\
\left. + \frac{\partial}{\partial y^a_{\nu\cdot\ldots}|_{\ldots,-\nu-1\ldots}} \cdot (\Delta^a_{\mu} A^\alpha_{\ldots})_{\ldots,-\nu-1\ldots} - \delta^a_{\mu} L(\ldots)_{\ldots} \right], \tag{39Aii}\]

\[
\Delta_b T^b_\alpha + \partial_4 T^4_\alpha + \ldots = 0, \tag{39Bi}\]

\[
\Delta_b T^b_4 + \partial_4 T^4_4 = 0, \tag{39Bii}\]

\[
T^b_\alpha(n,t) := \sqrt{\frac{n^b}{2}} \left[ \frac{\partial L(\ldots)}{\partial y^a_{\beta\cdot\ldots|b,-n^b-1\ldots}} \cdot \Delta^a_{\alpha} A^\alpha_{\ldots} \right. \\
\left. + \frac{\partial}{\partial y^a_{\beta\cdot\ldots|b,-n^b-1\ldots}} \cdot (\Delta^a_{\alpha} A^\alpha_{\ldots})_{\ldots,-n^b-1\ldots} - \delta^a_{\alpha} L(\ldots)_{\ldots} \right], \tag{39Biii}\]
\[ T^a_4(n, t) := \frac{\partial L(\ldots)}{\partial y_{\alpha^..}^{a_i}} \cdot \Delta^\ast_{a_i} A^{\alpha^..}, \quad (39\text{Biv}) \]

\[ T^b_4(n, t) := \sqrt{\frac{n^b}{2}} \left[ \frac{\partial L(\ldots)}{\partial y_{\alpha^..}^{b_i}} \cdot \partial_t A^{\alpha^..} + \frac{\partial L(\ldots)}{\partial y_{\alpha^..}^{b_i}} \cdot (\partial_t A^{\alpha^..})_{[(\ldots, n^b-1,\ldots)]} \right], \quad (39\text{Bv}) \]

\[ T^4_4(n, t) := \frac{\partial L(\ldots)}{\partial y_{\alpha^..}^{4_i}} \cdot \partial_t A^{\alpha^..} - L(\ldots)_{..}. \quad (39\text{Biv}) \]

We can sum or sum-integrate the relativistic conservation equations over an appropriate domain of \( \mathbb{N}^4 \) or \( \mathbb{N}^3 \times \mathbb{R} \). Using Gauss’s theorem 3.1 and assuming boundary conditions similar to the equations (15A,B), we obtain the total conserved four-momentum:

\[ -P_\mu = \sum_{n=0}^{\infty} [T^4_\mu(n, n^4) + \ldots] = \sum_{n=0}^{\infty} [T^4_\mu(n, 2) + \ldots], \quad (40\text{A}) \]

\[ -P_b = \sum_{n=0}^{\infty} [T^4_b(n, t) + \ldots] = \sum_{n=0}^{\infty} [T^4_b(n, 0) + \ldots], \quad (40\text{Bi}) \]

\[ H := -P_4 = \sum_{n=0}^{\infty} [T^4_4(n, t) + \ldots] = \sum_{n=0}^{\infty} [T^4_4(n, 0) + \ldots], \quad (40\text{Bii}) \]

\[ \sum_{n=0}^{\infty} := \sum_{n^1=0}^{\infty} \sum_{n^2=0}^{\infty} \sum_{n^3=0}^{\infty}. \]

The total energy-momentum components \( P_\mu \) are given by incomplete equations (40A), (40Bi,ii) for a general Lagrangian. However, in the following paper, we shall derive exact equations for the Klein-Gordon, electromagnetic, and Dirac fields.

In case of a complex-valued field \( \phi^{\alpha^..} \), the difference and difference-differential conservation for the charge-current vector \( j^\mu \) is given by equations (A.II.9A,B) as:

\[ \Delta_\mu j^\mu(n) = 0, \quad (41\text{Ai}) \]

\[ j^\mu(n) := ie \sqrt{\frac{n^\mu}{2}} \left\{ \left[ \frac{\partial L(\ldots)}{\partial \rho^{\alpha^..}_{\mu;[(\ldots, n^\mu-1,\ldots)]}} \cdot \phi^{\alpha^..}(n) \right. \right. \]

\[ \left. + \frac{\partial L(\ldots)}{\partial \rho^{\alpha^..}_{\mu;[\ldots, n^\mu - 1,\ldots]}} \cdot \phi^{\alpha^..}(\ldots, n^\mu - 1,\ldots) \right\} + (\text{c.c.}), \quad (41\text{ii}) \]
\[ \Delta_{b} j^{b}(n, t) + \partial_{t} j^{4}(n, t) = 0, \quad (41Bi) \]

\[ j^{b}(n, t) := ie \sqrt{\frac{n^{b}}{2}} \left\{ \left[ \frac{\partial L(\ldots)}{\partial \rho^{\alpha_{\ldots} b(\ldots, n^{b-1}, \ldots)}} \cdot \phi^{\alpha_{\ldots}}(n, t) \right. \right. \]
\[ \left. \left. + \frac{\partial L(\ldots)}{\partial \rho^{\alpha_{\ldots} b(\ldots, n^{b-1}, \ldots)}} \cdot \phi^{\alpha_{\ldots}}(\ldots, n^{b} - 1, \ldots) \right] \right\} + (c.c.), \quad (41Bii) \]

\[ j^{4}(n, t) := ie \left[ \frac{\partial L(\ldots)}{\partial \rho^{\alpha_{\ldots} 4}_{\ldots}} \cdot \phi^{\alpha_{\ldots}}(n, t) - \frac{\partial L(\ldots)}{\partial \rho^{\alpha_{\ldots} 4}_{\ldots}} \cdot \phi^{\alpha_{\ldots}}(n, t) \right]. \quad (41Biii) \]

Here, (c.c) stands for the complex-conjugation of the preceding terms and \( e = \sqrt{4\pi/137} \) is the charge parameter.

Under appropriate boundary conditions (15A,B), we can derive the conserved total charge:

\[ Q = -ie \sqrt{2} \sum_{n=0}^{\infty} \left\{ \left[ \frac{\partial L(\ldots)}{\partial \rho^{\alpha_{\ldots} 4}_{\ldots}} \cdot \phi^{\alpha_{\ldots}}(n, 2) + \frac{\partial L(\ldots)}{\partial \rho^{\alpha_{\ldots} 4}_{\ldots}} \cdot \phi^{\alpha_{\ldots}}(n, 1) \right] \right\} + (c.c.), \quad (42A) \]

\[ Q = \left[ -ie \sum_{n=0}^{\infty} \frac{\partial L(\ldots)}{\partial \rho^{\alpha_{\ldots} 4}_{\ldots}} \cdot \phi^{\alpha_{\ldots}}(n, 0) \right] + (c.c.) \cdot (42B) \]

Note that equations (42A,B) are already exact.

We shall now make some comments about the total conserved quantities. In the preceding section, we have deduced that the four-fold summation \( \sum_{n^{1}=0}^{\infty} \sum_{n^{2}=0}^{\infty} \sum_{n^{3}=0}^{\infty} \sum_{n^{4}=0}^{\infty} \) and the summation-integration \( \sum_{n^{1}=0}^{\infty} \sum_{n^{2}=0}^{\infty} \sum_{n^{3}=0}^{\infty} \int_{\mathbb{R}} dt \) are relativistically invariant operations. We have obtained the total four-momentum \( P_{\mu} \) in the equations (40A), (40Bi,ii) by the four-fold summation and the summation-integration of the relativistic conservation equations (A.II.4A) and (A.II.4Bi,ii) respectively. Therefore, we claim that (the complete) \( P_{\mu} \)'s are components of a relativistic four-vector. Similarly, the total charge in equations (42A) or (42B) is relativistic invariant.

The physical contents of the partial difference conservation (39Ai) and the difference-differential conservation (39Bi) are identical. However, the total four-momentum components \( P_{\mu} \) in (40A) and (40Bi,ii) are quite different inspite of the same notations! The reason for this distinction is that in equation (40A), the summation of the field is over a \( n^{4} = \text{const.} \) “hyperspace”. In covariant phase space, an \( n^{4} = \text{const.} \) “hypersurface” implies that \( (t)^{2} + (p_{4})^{2} = (2n^{4} + 1) = \text{const.} \). Therefore, in the \( t-p^{4} \) phase plane, the...
time coordinate $t$ oscillates between the values $-\sqrt{2n^4+1}$ and $\sqrt{2n^4+1}$. It is certainly not a $t = \text{const.}$ “slice”. However, in the case of equations (40Bi,ii), the usual $t = \text{const.}$ “hypersurface” of the discrete phase space and continuous time is used.

**Appendix I: Euler-Lagrange equations**

We shall consider a *two-dimensional* lattice function $f$ for the sake of simplicity. We firstly choose a two-dimensional discrete domain (see Fig. 2)

$$D := \{ n \in \mathbb{N}^2 : 0 < N_1^A < N_2^A, 2 \leq N_2^A - N_1^A, N_1^A < n^A < N_2^A, A \in \{1, 2\} \}. \quad (\text{A.I.1})$$

We shall follow the summation convention on capital indices which take values from $\{1, 2\}$.

A real-valued lattice function $f : D \subset \mathbb{N}^2 \rightarrow \mathbb{R}$ is considered. The Lagrangian function $L : \tilde{D} \subset \mathbb{N}^2 \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is assumed to be Taylor-expandible (or real analytic). (In all practical purposes, $L$ is a polynomial function.) The action functional $\mathcal{A}$ is defined to be the double-sum

$$\mathcal{A}(f) := \sum_{n^1=N_1^1}^{N_2^1} \sum_{n^2=N_1^2}^{N_2^2} L(n; y; y_A)|_{y = f(n), y_A = \Delta_A^y f(n)}, \quad n := (n^1, n^2). \quad (\text{A.I.2})$$

Now we define the variations of the lattice function $f$ by

$$\delta f(n) := \varepsilon h(n), \quad \delta[\Delta_A^y f(n)] := \varepsilon \Delta_A^y h(n) = \Delta_A^y [\delta f(n)]. \quad (\text{A.I.3})$$

Here, $\varepsilon > 0$ is an arbitrary, small, positive number. Therefore, the variation of the action functional $\mathcal{A}$ is furnished by:

$$\delta \mathcal{A}(f) = \sum_{n^1=N_1^1}^{N_2^1} \sum_{n^2=N_1^2}^{N_2^2} \left\{ L(n; y; y_A)|_{y = f(n), y_A = \Delta_A^y f} \right\} \left[ \frac{\partial L(\ldots)}{\partial y} \right]_{y = f(n), y_A = \Delta_A^y f} h(n) + \left[ \frac{\partial L(\ldots)}{\partial y_A} \right]_{y = f(n), y_A = \Delta_A^y f} \cdot \Delta_A^y h(n) + 0(\varepsilon^2). \quad (\text{A.I.4})$$

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The variational principle, which implies the stationary (or critical) values of the action functional, can be stated succinctly as
\[
\lim_{\varepsilon \to 0} \frac{\delta A(f)}{\varepsilon} = 0.
\]

The above equation yields from (A.I.4) that
\[
\sum_{n^1=1}^{N_1^1} \sum_{n^2=1}^{N_2^1} \left\{ \frac{\partial L(\ldots)}{\partial y} |_{(n^1, n^2)} \cdot h(n) + \left[ \frac{\partial L(\ldots)}{\partial y_A} \right] |_{(n^1, n^2)} \cdot \Delta^A h(n) \right\} = 0. \tag{A.I.5}
\]
(Here, we have simplified the notation by putting \(|y = f(n^1, n^2), y_A = \Delta^A f(n^1, n^2)\).)

Using the definition (4iii), opening up the double-sum in (A.I.5), and rearranging terms, we obtain (after a very long calculation)
\[
\sum_{n^1=1}^{N_1^1} \sum_{n^2=1}^{N_2^1} \left\{ \frac{\partial L(\ldots)}{\partial y} |_{(n^1, n^2)} - \Delta^A \left[ \frac{\partial L(\ldots)}{\partial y_A} \right] |_{(n^1, n^2)} \right\} \cdot h(n^1, n^2) = 0. \tag{A.I.6}
\]
+(Boundary terms) = 0.

Here, the boundary terms are given by:

(Boundary terms) :=
\[
-\sqrt{\frac{N_1^2}{2}} \sum_{k=0}^{N_1^1-N_1^2} \frac{\partial L(\ldots)}{\partial y_2} |_{(N_1^1+k, N_1^2)} \cdot h(N_1^1+k, N_1^2-1)
\]
\[
+\sqrt{\frac{N_1^2+1}{2}} \sum_{k=0}^{N_2^1-N_2^2} \frac{\partial L(\ldots)}{\partial y_1} |_{(N_2^1+k, N_2^2)} \cdot h(N_2^1+1, N_2^2+k)
\]
\[
+\sqrt{\frac{N_2^2+1}{2}} \sum_{k=0}^{N_2^1-N_1^2} \frac{\partial L(\ldots)}{\partial y_2} |_{(N_1^1+k, N_2^1)} \cdot h(N_1^1+k, N_2^1+1)
\]
\[
-\sqrt{\frac{N_1^2}{2}} \sum_{k=0}^{N_1^2-N_1^1} \frac{\partial L(\ldots)}{\partial y_1} |_{(N_1^1+k, N_2^1)} \cdot h(N_1^1-1, N_2^1+k)
\]
\[
+\sum_{j=1}^{N_2^1-N_1^1-1} \left\{ \frac{\partial L(\ldots)}{\partial y} |_{(N_1^1+j, N_2^1)} - \left[ \Delta^A \left( \frac{\partial L(\ldots)}{\partial y_1} \right) \right] |_{-j} |_{(N_1^1+j, N_2^1)} \right\} \cdot h(N_1^1+j, N_2^1)
\]
\[
-\sqrt{\frac{N_2^2+1}{2}} \frac{\partial L(\ldots)}{\partial y_2} |_{(N_1^1+j, N_2^1+1)} \cdot h(N_1^1+j, N_2^1)
\]
\begin{align*}
&+ \sum_{j=1}^{N_2^3-N_1^3-1} \left\{ \frac{\partial L(\ldots)}{\partial y} \bigg|_{(N_1^3, N_2^3+j)} + \sqrt{\frac{N_2^3}{2}} \frac{\partial L(\ldots)}{\partial y_1} \bigg|_{(N_2^3-1, N_2^3+j)} \right. \\
&- \left[ \Delta_2^\# \left( \frac{\partial L(\ldots)}{\partial y_2} \bigg|_{(N_1^3, N_2^3+j)} \right) \right] \cdot h(N_1^3, N_2^3 + j) \\
&+ \sum_{j=1}^{N_2^3-N_1^3-1} \left\{ \frac{\partial L(\ldots)}{\partial y} \bigg|_{(N_1^3+j, N_2^3)} - \left[ \Delta_1^\# \left( \frac{\partial L(\ldots)}{\partial y_1} \bigg|_{(N_1^3+j, N_2^3)} \right) \right] \right. \\
&+ \sqrt{\frac{N_2^3}{2}} \frac{\partial L(\ldots)}{\partial y_2} \bigg|_{(N_1^3+j, N_2^3)} \right\} \cdot h(N_1^3 + j, N_2^3) \\
&+ \sum_{j=1}^{N_2^3-N_1^3-1} \left\{ \frac{\partial L(\ldots)}{\partial y} \bigg|_{(N_1^3, N_2^3+j)} - \left[ \Delta_1^\# \left( \frac{\partial L(\ldots)}{\partial y_1} \bigg|_{(N_1^3+j, N_2^3)} \right) \right] \right. \\
&- \sqrt{\frac{N_1^3+1}{2}} \frac{\partial L(\ldots)}{\partial y_1} \bigg|_{(N_1^3+1, N_2^3+j)} \right\} \cdot h(N_1^3, N_2^3) \\
&+ \sum_{j=1}^{N_2^3-N_1^3-1} \left\{ \frac{\partial L(\ldots)}{\partial y} \bigg|_{(N_1^3, N_2^3+j)} - \left[ \Delta_1^\# \left( \frac{\partial L(\ldots)}{\partial y_1} \bigg|_{(N_1^3+j, N_2^3)} \right) \right] \right. \\
&+ \sqrt{\frac{N_2^3+1}{2}} \frac{\partial L(\ldots)}{\partial y_2} \bigg|_{(N_1^3, N_2^3+1)} \right\} \cdot h(N_1^3, N_2^3) \\
&+ \sum_{j=1}^{N_2^3-N_1^3-1} \left\{ \frac{\partial L(\ldots)}{\partial y} \bigg|_{(N_1^3, N_2^3+j)} + \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{N_1^3}{2}} \frac{\partial L(\ldots)}{\partial y_1} \bigg|_{(N_1^3-1, N_2^3)} \\
&- \sqrt{\frac{N_1^3+1}{2}} \frac{\partial L(\ldots)}{\partial y_2} \bigg|_{(N_1^3-1, N_2^3+1)} \right\} \cdot h(N_1^3, N_2^3) \\
&+ \sum_{j=1}^{N_2^3-N_1^3-1} \left\{ \frac{\partial L(\ldots)}{\partial y} \bigg|_{(N_1^3, N_2^3+j)} + \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{N_2^3}{2}} \frac{\partial L(\ldots)}{\partial y_1} \bigg|_{(N_2^3-1, N_2^3)} \\
&+ \sqrt{\frac{N_2^3+1}{2}} \frac{\partial L(\ldots)}{\partial y_2} \bigg|_{(N_2^3-1, N_2^3+1)} \right\} \cdot h(N_1^3, N_2^3) \\
&+ \sum_{j=1}^{N_2^3-N_1^3-1} \left\{ \frac{\partial L(\ldots)}{\partial y} \bigg|_{(N_1^3, N_2^3+j)} - \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{N_1^3}{2}} \frac{\partial L(\ldots)}{\partial y_1} \bigg|_{(N_1^3+1, N_2^3)} \\
&+ \sqrt{\frac{N_2^3}{2}} \frac{\partial L(\ldots)}{\partial y_2} \bigg|_{(N_1^3+1, N_2^3+1)} \right\} \cdot h(N_1^3, N_2^3) \\
&+ \sum_{j=1}^{N_2^3-N_1^3-1} \left\{ \frac{\partial L(\ldots)}{\partial y} \bigg|_{(N_1^3, N_2^3+j)} - \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{N_2^3}{2}} \frac{\partial L(\ldots)}{\partial y_1} \bigg|_{(N_2^3+1, N_2^3)} \\
&- \sqrt{\frac{N_2^3+1}{2}} \frac{\partial L(\ldots)}{\partial y_2} \bigg|_{(N_2^3+1, N_2^3+1)} \right\} \cdot h(N_1^3, N_2^3) \\
&+ \sum_{j=1}^{N_2^3-N_1^3-1} \left\{ \frac{\partial L(\ldots)}{\partial y} \bigg|_{(N_1^3, N_2^3+j)} + \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{N_1^3}{2}} \frac{\partial L(\ldots)}{\partial y_1} \bigg|_{(N_1^3-1, N_2^3)} \\
&- \sqrt{\frac{N_2^3+1}{2}} \frac{\partial L(\ldots)}{\partial y_2} \bigg|_{(N_2^3-1, N_2^3+1)} \right\} \cdot h(N_1^3, N_2^3) \\
&+ \sum_{j=1}^{N_2^3-N_1^3-1} \left\{ \frac{\partial L(\ldots)}{\partial y} \bigg|_{(N_1^3, N_2^3+j)} + \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{N_2^3}{2}} \frac{\partial L(\ldots)}{\partial y_1} \bigg|_{(N_2^3+1, N_2^3)} \\
&- \sqrt{\frac{N_1^3+1}{2}} \frac{\partial L(\ldots)}{\partial y_2} \bigg|_{(N_2^3+1, N_2^3+1)} \right\} \cdot h(N_1^3, N_2^3) \\
&+ \sum_{j=1}^{N_2^3-N_1^3-1} \left\{ \frac{\partial L(\ldots)}{\partial y} \bigg|_{(N_1^3, N_2^3+j)} - \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{N_1^3}{2}} \frac{\partial L(\ldots)}{\partial y_1} \bigg|_{(N_1^3+1, N_2^3)} \\
&+ \sqrt{\frac{N_2^3}{2}} \frac{\partial L(\ldots)}{\partial y_2} \bigg|_{(N_1^3+1, N_2^3+1)} \right\} \cdot h(N_1^3, N_2^3) \\
&+ \sum_{j=1}^{N_2^3-N_1^3-1} \left\{ \frac{\partial L(\ldots)}{\partial y} \bigg|_{(N_1^3, N_2^3+j)} - \frac{1}{\sqrt{2}} \left[ \sqrt{\frac{N_2^3}{2}} \frac{\partial L(\ldots)}{\partial y_1} \bigg|_{(N_2^3-1, N_2^3)} \\
&+ \sqrt{\frac{N_1^3}{2}} \frac{\partial L(\ldots)}{\partial y_2} \bigg|_{(N_2^3-1, N_2^3+1)} \right\} \cdot h(N_1^3, N_2^3) .
\end{align*}
(A.1.7)
The imposed plus natural boundary conditions \([13]\) imply that the boundary terms in (A.I.7) add up to zero. There exist many possible boundary conditions. We shall adopt the simplest of all, namely

\[
\begin{align*}
  h(n^1, n^2)_{\text{Boundary}} &= 0, \\
  \delta f(n^1, n^2)_{\text{Boundary}} &= 0.
\end{align*}
\]  

(A.I.8)

Note that the discrete boundary points in equation (A.I.7) are more in number than those in equation (9). We have to prove now the analogue of the Dubois-Reymond lemma \([14]\).

**Lemma:** Let the double sum

\[
\sum_{n^1 = N_1^1 + 1}^{N_1^2 - 1} \sum_{n^2 = N_2^1 + 1}^{N_2^2 - 1} g(n^1, n^2) h(n^1, n^2) = 0
\]

for a function \(g\) and an arbitrary function \(h\) over the domain \(D \subset \mathbb{N}^2\) defined in equation (A.I.1). Then \(g(n^1, n^2) \equiv 0\) in \(D\).

**Proof.** Choose \(h(n^1, n^2) := \delta_{n^1}^m \delta_{n^2}^m\) for some \((m^1, m^2) \in D\). (The \(\delta^m_n\) denotes the Kronecker delta.) Then

\[
\sum_{n^1 = N_1^1 + 1}^{N_1^2 - 1} \sum_{n^2 = N_2^1 + 1}^{N_2^2 - 1} g(n^1, n^2) h(n^1, n^2)
= \sum_{n^1 = N_1^1 + 1}^{N_1^2 - 1} \sum_{n^2 = N_2^1 + 1}^{N_2^2 - 1} g(n^1, n^2) \delta_{n^1}^m \delta_{n^2}^m = g(m^1, m^2) = 0.
\]

Since the choice of \((m^1, m^2) \in D\) is arbitrary, it follows that \(g(n^1, n^2) \equiv 0\) for all \((n^1, n^2) \in D\).

\(\square\)

At this stage we have furnished essentially the proof of the following generalization of the Euler-Lagrange theorem.

**Theorem (A.I.1):** Let a function \(f : D \subset \mathbb{N}^2 \rightarrow \mathbb{R}\), where \(D\) is defined in (A.I.1). Let an action functional \(A(f)\) be defined by (A.I.2). The stationary values of \(A\), under the boundary variation \(\delta f(n)\text{\_\_} = 0\), are given by the solutions of the partial difference equation:

\[
\frac{\partial L(\ldots)}{\partial y} \bigg|_{y = f(n), y_A = \Delta^A f(n)} - \Delta^A \left\{ \left[ \frac{\partial L(\ldots)}{\partial y_A} \right] \bigg|_{y = f(n), y_A = \Delta^A f(n)} \right\} = 0. \tag{A.I.9}
\]

(Here, the index \(A\) is summed over \(\{1, 2\}\).)
Appendix II: Partial difference and difference-differential conservation equations

Consider the discrete domain \( D \subset \mathbb{N}^4 \) given by equation (A.I.1). Let \( A^{\alpha\cdot\cdot}(n) \) be a real-valued \((r+s)\)-th order tensor field over \( D \subset \mathbb{N}^4 \). (See equations (34A,B).) The Lagrangian \( L(y^{\alpha\cdot\cdot}; y^{\alpha\cdot\cdot}) \) is a real-valued, analytic, and relativistic invariant function over a domain of the continuum \( R^{4r+s} \times R^{4r+s+1} \). The action functional \( \mathcal{A}(A^{\alpha\cdot\cdot}) \) is given by the four-fold sum (compare equation (A.I.2))

\[
\mathcal{A}(A^{\alpha\cdot\cdot}) := \sum_{n^1=N^1_1}^{N^1_2} \sum_{n^2=N^2_1}^{N^2_2} \sum_{n^3=N^3_1}^{N^3_2} \sum_{n^4=N^4_1}^{N^4_2} \left[ L\left( y^{\alpha\cdot\cdot}; y^{\alpha\cdot\cdot}\right) \bigg| y^{\alpha\cdot\cdot}=A^{\alpha\cdot\cdot}(n), y^{\alpha\cdot\cdot}=\Delta^\# A^{\alpha\cdot\cdot} \right].
\]

(A.II.1)

In the usual case, \( L \) is a polynomial function of the \( 4^{2(r+s)+1} \) real variables. The Taylor expansion of \( L \) is given by

\[
L(y^{\alpha\cdot\cdot} + \Delta y^{\alpha\cdot\cdot}, y^{\beta\cdot\cdot} + \Delta y^{\beta\cdot\cdot}) = L\left( y^{\alpha\cdot\cdot}, y^{\beta\cdot\cdot}\right) + \left[ \frac{\partial L\left(\ldots\right)}{\partial y^{\alpha\cdot\cdot}} \Delta y^{\alpha\cdot\cdot} + \frac{\partial L\left(\ldots\right)}{\partial y^{\beta\cdot\cdot}} \Delta y^{\beta\cdot\cdot} \right] \\
+ \frac{1}{2} \left[ \frac{\partial^2 L\left(\ldots\right)}{\partial y^{\alpha\cdot\cdot}\partial y^{\beta\cdot\cdot}} (\Delta y^{\alpha\cdot\cdot})(\Delta y^{\beta\cdot\cdot}) + \frac{\partial^2 L\left(\ldots\right)}{\partial y^{\alpha\cdot\cdot}\partial y^{\sigma\cdot\cdot}} (\Delta y^{\alpha\cdot\cdot})(\Delta y^{\sigma\cdot\cdot}) \\
+ \frac{\partial^2 L\left(\ldots\right)}{\partial y^{\beta\cdot\cdot}\partial y^{\sigma\cdot\cdot}} (\Delta y^{\beta\cdot\cdot})(\Delta y^{\sigma\cdot\cdot}) \right] + \ldots.
\]

(A.II.2)

Here we have followed the summation convention on every repeated Greek index. Moreover, for a second-degree polynomial function \( L \), the additional terms denoted by \( \ldots \) exactly vanish.

Now, we investigate the partial difference polynomial operations on \( L(...) \). Using equations (4iii), (A.II.2), (5iv), (6i,ii,iii,iv) and not summing the index \( \mu \), we derive that

\[
\sqrt{2} \Delta^\#_\mu \left[ L\left( y^{\alpha\cdot\cdot}; y^{\beta\cdot\cdot}\right) \bigg| y^{\alpha\cdot\cdot}=A^{\alpha\cdot\cdot}(n), y^{\beta\cdot\cdot}=\Delta^\# A^{\alpha\cdot\cdot} \right]
\]

\[
= \left( \sqrt{n^\mu+1} \right) \cdot L\left\{ A^{\alpha\cdot\cdot}(\ldots, n^\mu+1, \ldots); 2^{-1/2} \sqrt{n^\nu+1} A^{\alpha\cdot\cdot}(\ldots, n^\mu+1, \ldots, n^\nu+1, \ldots) \\
- \sqrt{n^\nu} A^{\alpha\cdot\cdot}(\ldots, n^\mu+1, \ldots, n^\nu-1, \ldots) \right\}
\]

\[
- \left( \sqrt{n^\nu} \right) \cdot L\left\{ A^{\alpha\cdot\cdot}(\ldots, n^\nu-1, \ldots); 2^{-1/2} \sqrt{n^\nu+1} A^{\alpha\cdot\cdot}(\ldots, n^\mu-1, \ldots, n^\nu+1, \ldots) \\
- \sqrt{n^\nu} A^{\alpha\cdot\cdot}(\ldots, n^\mu-1, \ldots, n^\nu-1, \ldots) \right\}
\]

(26)
\[
\begin{align*}
&= (\sqrt{n^\mu + 1}) \cdot L \left[ A^\alpha \cdot (n) + \Delta_\mu A^\alpha \cdot \Delta^\# A^\alpha \cdot (n) + \Delta_\mu \Delta^\# A^\alpha \right] \\
&\quad - (\sqrt{n^\mu}) \cdot L \left[ A^\alpha \cdot (n) - \Delta_\mu A^\alpha \cdot \Delta^\# A^\alpha \cdot (n) - \Delta_\mu \Delta^\# A^\alpha \right] \\
&= \sqrt{n^\mu + 1} \left\{ L(\ldots) + \left[ \frac{\partial L(\ldots)}{\partial y^{\alpha}} \cdot \Delta_\mu A^\alpha + \frac{1}{2} \frac{\partial^2 L(\ldots)}{\partial y^{\alpha} \cdot \partial y^{\beta}} \cdot (\Delta_\mu A^\alpha)(\Delta_\mu A^\beta) + \ldots \right] \\
&\quad + \left[ - \frac{\partial L(\ldots)}{\partial y^{\alpha}} \cdot \Delta^\# A^\alpha + \frac{1}{2} \frac{\partial^2 L(\ldots)}{\partial y^{\alpha} \cdot \partial y^{\beta}} \cdot (\Delta^\# A^\alpha)(\Delta^\# A^\beta) - \ldots \right] \right\} \\
&= \sqrt{2} \left\{ L(\ldots) \Delta^\# (1) + \frac{\partial L(\ldots)}{\partial y^{\alpha}} \cdot \left[ \sqrt{n^\mu + 1} (\Delta_\mu A^\alpha) + \sqrt{n^\mu} (\Delta^\# A^\alpha) \right] \\
&\quad + \frac{1}{2} \frac{\partial^2 L(\ldots)}{\partial y^{\alpha} \cdot \partial y^{\beta}} \cdot \left[ \sqrt{n^\mu + 1} (\Delta_\mu A^\alpha)(\Delta_\mu A^\beta) \\
&\quad \quad - \sqrt{n^\mu} (\Delta_\mu A^\alpha)(\Delta^\# A^\beta) \right] \right\} \\
&\quad + \frac{1}{2} \frac{\partial^2 L(\ldots)}{\partial y^{\alpha} \cdot \partial y^{\beta}} \cdot \left[ \sqrt{n^\mu + 1} (\Delta_\mu A^\alpha)(\Delta^\# A^\beta) \\
&\quad \quad - \sqrt{n^\mu} (\Delta_\mu A^\alpha)(\Delta^\# A^\beta) \right] + \ldots \\
&= \sqrt{2} \left\{ [L(\ldots)] \Delta^\# (1) + \frac{\partial L(\ldots)}{\partial y^{\alpha}} \cdot [\Delta^\# A^\alpha \cdot - A^\alpha \cdot (n) \Delta^\# (1)] \\
&\quad + \frac{\partial L(\ldots)}{\partial y^{\alpha}} \cdot [\Delta^\# A^\alpha \cdot - (\Delta^\# A^\alpha) \Delta^\# (1)] \\
&\quad + \frac{1}{2} \frac{\partial^2 L(\ldots)}{\partial y^{\alpha} \cdot \partial y^{\beta}} \cdot [\Delta^\# (A^\alpha \cdot (n))(A^\beta \cdot (n)) - A^\alpha \cdot (n) \Delta^\# A^\beta \cdot \\
&\quad \quad - A^\beta \cdot (n) \Delta^\# A^\alpha \cdot + A^\alpha \cdot (n) A^\beta \cdot (n) \Delta^\# (1)] \\
&\quad + \frac{1}{2} \frac{\partial^2 L(\ldots)}{\partial y^{\alpha} \cdot \partial y^{\beta}} \cdot [\Delta^\# (\Delta^\# A^\alpha \cdot \Delta^\# A^\beta \cdot - (\Delta^\# A^\alpha \cdot) (\Delta^\# A^\alpha \cdot) (\Delta^\# A^\beta \cdot) \\
&\quad \quad - (\Delta^\# A^\beta \cdot) (\Delta^\# A^\alpha \cdot) + (\Delta^\# A^\alpha \cdot) (\Delta^\# A^\beta \cdot) \Delta^\# (1)] + \ldots \right\} \tag{A.2.3}
\end{align*}
\]
Dividing this equation by $\sqrt{2}$, bringing the left-hand side term to the right-hand side, and equating $\frac{\partial L(\cdot)}{\partial y_\alpha^{\cdot} \cdot \partial y_{\beta}^{\cdot}} = \Delta_\nu\left[\frac{\partial L(\cdot)}{\partial y_\nu^{\cdot}}\right]_{\cdot}$ by the Euler-Lagrange equation (36A), we finally obtain:

\[
\begin{align*}
\left\{ \Delta_\nu\left(\frac{\partial L(\cdot)}{\partial y_\nu^{\cdot}}\right) & \cdot \Delta_\mu A^{\alpha\cdot} + \frac{\partial L(\cdot)}{\partial y_\nu^{\cdot}} \cdot \Delta_\mu \Delta_\nu A^{\alpha\cdot} - \Delta_\nu[L(\cdot)_{\cdot}] \right\} \\
+ \frac{1}{2} \frac{\partial^2 L(\cdot)}{\partial y_\alpha^{\cdot} \partial y_{\beta}^{\cdot}} \cdot \left[ \Delta_\mu (A^{\alpha\cdot}(n) \cdot A^{\beta\cdot}(n)) - A^{\alpha\cdot}(n) \Delta_\mu A^{\beta\cdot} - (\Delta_\mu A^{\alpha\cdot}) (A^{\beta\cdot}(n)) \right] \\
+ \frac{1}{2} \frac{\partial^2 L(\cdot)}{\partial y_\alpha^{\cdot} \partial y_{\beta}^{\cdot}} \cdot \left[ \Delta_\mu (\Delta_\nu A^{\alpha\cdot} \cdot \Delta_\sigma A^{\beta\cdot}) - (\Delta_\nu A^{\alpha\cdot}) (\Delta_\sigma A^{\beta\cdot}) \right] \\
+ \left[ \Delta_\rho^{\cdot}(1) \left[ L(\cdot)_{\cdot} - \frac{\partial L(\cdot)}{\partial y_\alpha^{\cdot}} \cdot A^{\alpha\cdot}(n) - \frac{\partial L(\cdot)}{\partial y_\nu^{\cdot}} \cdot \Delta_\nu A^{\alpha\cdot} \right] \\
+ \frac{1}{2} \frac{\partial^2 L(\cdot)}{\partial y_\alpha^{\cdot} \partial y_{\beta}^{\cdot}} \cdot A^{\alpha\cdot}(n) A^{\beta\cdot}(n) \\
+ \frac{1}{2} \frac{\partial^2 L(\cdot)}{\partial y_\alpha^{\cdot} \partial y_{\beta}^{\cdot}} \cdot \left( \Delta_\nu A^{\alpha\cdot} \right) \left( \Delta_\sigma A^{\beta\cdot} \right) \right] + \ldots = 0. \quad (A.II.4A)
\end{align*}
\]

Here, the indices $\alpha, \beta, \nu, \sigma$ are summed. We claim that the above equations involving the relativistic operator $\Delta_\mu = iP_\mu$ constitute relativistic conservation equations in the finite difference form. The corresponding relativistic difference-differential conservation equations are

\[
\begin{align*}
\left\{ \Delta_\nu^{\cdot}\left(\frac{\partial L(\cdot)}{\partial y_\nu^{\cdot}}\right) & + \partial_t \left( \frac{\partial L(\cdot)}{\partial y_\nu^{\cdot}} \right)_{\cdot} \cdot \Delta_\mu A^{\alpha\cdot} + \frac{\partial L(\cdot)}{\partial y_\nu^{\cdot}} \cdot \Delta_\mu \Delta_\nu A^{\alpha\cdot} \\
& + \frac{\partial L(\cdot)}{\partial y_\alpha^{\cdot} \cdot \partial y_{\beta}^{\cdot}} \cdot \Delta_\mu \partial_t A^{\alpha\cdot} - \Delta_\mu[L(\cdot)_{\cdot}] \right\} \\
+ \frac{1}{2} \frac{\partial^2 L(\cdot)}{\partial y_\alpha^{\cdot} \partial y_{\beta}^{\cdot}} \cdot \left[ \Delta_\mu (A^{\alpha\cdot} \cdot A^{\beta\cdot}) \\
- A^{\alpha\cdot}(n, t) \Delta_\mu A^{\beta\cdot} - A^{\beta\cdot}(n, t) \Delta_\mu A^{\alpha\cdot} \right]
\end{align*}
\]
\[
\begin{align*}
&+ \frac{1}{2} \frac{\partial^2 L(\ldots)}{\partial y_\alpha^b \partial y_\beta^c} \cdot [\Delta^*_a (\Delta^*_b A^\alpha \cdot \Delta^*_c A^\beta) \\
&\quad - (\Delta^*_b A^\alpha \cdot \Delta^*_c A^\beta) - (\Delta^*_a A^\beta \cdot \Delta^*_b A^\alpha)] \\
&+ \frac{\partial^2 L(\ldots)}{\partial y_\alpha^b \partial y_\beta^c} \cdot [\Delta^*_a (\Delta^*_b A^\alpha \cdot \partial_t A^\beta) \\
&\quad - (\partial_t A^\alpha \Delta^*_b A^\beta) - (\partial_t A^\beta \Delta^*_a A^\alpha)] \\
&+ \left[ \Delta^*_a (1) \right] [L(\ldots) \cdot - \frac{\partial L(\ldots)}{\partial y_\alpha^b} \cdot A^\alpha (n, t) \\
&\quad - \frac{\partial L(\ldots)}{\partial y_\alpha^b} \cdot \Delta^*_b A^\alpha - \frac{\partial L(\ldots)}{\partial y_\alpha^b} \cdot \partial_t A^\alpha] \\
&+ \frac{1}{2} \frac{\partial^2 L(\ldots)}{\partial y_\alpha^b \partial y_\beta^c} \cdot A^\alpha (n, t) A^\beta (n, t) \\
&+ \frac{1}{2} \frac{\partial^2 L(\ldots)}{\partial y_\alpha^b \partial y_\beta^c} \cdot (\Delta^*_b A^\alpha) \Delta^*_c A^\beta \\
&+ \frac{\partial^2 L(\ldots)}{\partial y_\alpha^b \partial y_\beta^c} \cdot (\partial_t A^\alpha \Delta^*_b A^\beta) \\
&+ \frac{1}{2} \frac{\partial^2 L(\ldots)}{\partial y_\alpha^b \partial y_\beta^c} \cdot (\partial_t A^\alpha \Delta^*_b A^\beta) \right] + \ldots = 0, \quad (\text{A.II.4Bi})
\end{align*}
\]

The above equations containing operators \( \Delta^*_j = iP_j \) and \( \partial_t = IP_4 \) on the same footing are relativistic. It is extremely hard to put (A.II.4A) and
(A.II.4Bi,ii) into the relativistic conservation equations:

\[
\Delta^\nu_{\mu} T^\nu_{\mu} = 0 ,
\]

\[
\Delta^b_{a} T^b_{a} + \partial_t T^4_{a} = 0 , \quad \Delta^b_{4} T^b_{4} + \partial_t T^4_{4} = 0 .
\]

However, using the equation (5v), the relativistic conservation equation (A.II.4A) can be cast into the form:

\[
(1/\sqrt{2}) \left\{ \Delta^\nu_{\mu} \sqrt{n^\nu} \frac{\partial L(\ldots)}{\partial y^\nu_{\mu}} \cdot \Delta_{a}^\nu A^a_{\ldots} + \frac{\partial L(\ldots)}{\partial y^a_{\nu}} \cdot (\Delta_{\nu}^a A_{\ldots})_{(\ldots, n^{\nu-1} \ldots)} - \delta_{\nu}^a L(\ldots)_{\ldots} \right\} + \ldots =: \Delta^\nu_{\mu} [T^\nu_{\mu}(n)] + \ldots = 0 .
\]

(A.II.5A)

Here, neither \( T^\nu_{\mu}(n) \) are relativistic tensor components, nor \( \Delta_{\mu} \) is a relativistic difference-operator. However, the combination \( \Delta_{\nu} T^\nu_{\mu} + \ldots \) are components of a relativistic covariant vector!

Similarly, from the relativistic difference-differential conservation equations (A.II.4i,ii) we derive

\[
(1/\sqrt{2}) \Delta^b \left\{ \sqrt{n^b} \left[ \Delta^\nu_{\mu} \frac{\partial L(\ldots)}{\partial y^\nu_{\mu}} \cdot \Delta_{a}^\nu A^a_{\ldots} + \frac{\partial L(\ldots)}{\partial y^a_{\nu}} \cdot (\Delta_{\nu}^a A_{\ldots})_{(\ldots, n^{b-1} \ldots)} - \delta_{\nu}^a L(\ldots)_{\ldots} \right] \right\}
\]

\[
+ \partial_t \left\{ \frac{\partial L(\ldots)}{\partial y^a_{\nu}} \cdot \Delta_{a}^\nu A^a_{\ldots} \right\} + \ldots =: \Delta^b [T^b_{a}] + \partial_t [T^4_{a}] + \ldots = 0 .
\]

\[
(A.II.5Bi)
\]

\[
(1/\sqrt{2}) \Delta_b \left\{ \sqrt{n^b} \left[ \Delta^\nu_{\mu} \frac{\partial L(\ldots)}{\partial y^\nu_{\mu}} \cdot \partial_t A^\mu_{\ldots} + \frac{\partial L(\ldots)}{\partial y^a_{\nu}} \cdot (\partial_t A^\mu_{\ldots})_{(\ldots, n^{b-1} \ldots)} \right] \right\}
\]

\[
+ \partial_t \left\{ \frac{\partial L(\ldots)}{\partial y^a_{\nu}} \cdot \partial_t A^\mu_{\ldots} - L(\ldots)_{\ldots} \right\} =: \Delta^b T^b_{4} + \partial_t T^4_{4} = 0 .
\]

\[
(A.II.5Bi)
\]
We can generalize conservation equations (A.II.5A) and (A.II.5Bi,ii) for a complex-valued tensor or spinor field $\phi^\alpha(n)$ and $\phi^\alpha(n, t)$ to the following equations:

\[
(1/\sqrt{2})\Delta_\nu \left\{ \sqrt{n_\nu} \left[ \frac{\partial L(\ldots)}{\partial \rho^\nu_{\ldots |_{(\ldots,n^\nu - 1,\ldots)}}} \cdot \Delta^\#_\mu \phi^\nu \right. 
+ \frac{\partial L(\ldots)}{\partial \rho^\nu_{\ldots}} \cdot (\Delta^\#_\mu \phi^\nu)_{|_{(\ldots,n^\nu - 1,\ldots)}} + (\text{c.c.}) - \delta^\nu_{\mu} [L(\ldots)]_{|_{\ldots}} \right\} 
- \sqrt{n^\mu} \Delta_{\nu} [L(\ldots)]_{|_{\ldots}} + \ldots =: \Delta_\nu T^\nu_{\mu} + \ldots = 0 ,
\]

(A.II.6A)

\[
(1/\sqrt{2})\Delta_b \left\{ \sqrt{n_b} \left[ \frac{\partial L(\ldots)}{\partial \rho^b_{\ldots |_{(\ldots,n^b - 1,\ldots)}}} \cdot \Delta^\#_a \phi^b 
+ \frac{\partial L(\ldots)}{\partial \rho^b_{\ldots}} \cdot (\Delta^\#_a \phi^b)_{|_{(\ldots,n^b - 1,\ldots)}} + (\text{c.c.}) - \delta^b_a [L(\ldots)]_{|_{\ldots}} \right\} 
- \sqrt{n^a} \Delta_b [L(\ldots)]_{|_{\ldots}} + \ldots + \partial_t \left\{ \left[ \frac{\partial L(\ldots)}{\partial \rho^b_{\ldots |_{\ldots}}} \cdot \Delta^\#_a \phi^b 
+ \frac{\partial L(\ldots)}{\partial \rho^b_{\ldots}} \cdot (\Delta^\#_a \phi^b)_{|_{(\ldots,n^b - 1,\ldots)}} + (\text{c.c.}) \right] \right\} 
= \Delta_b T^b_a + \partial_t T^a_4 + \ldots = 0 ,
\]

(A.II.6Bi)

\[
(1/\sqrt{2})\Delta_b \left\{ \sqrt{n^b} \left[ \frac{\partial L(\ldots)}{\partial \rho^b_{\ldots |_{(\ldots,n^b - 1,\ldots)}}} \cdot \partial_t \phi^b 
+ \frac{\partial L(\ldots)}{\partial \rho^b_{\ldots}} \cdot (\partial_t \phi^b)_{|_{(\ldots,n^b - 1,\ldots)}} + (\text{c.c.}) \right] \right\} 
+ \partial_t \left\{ \left[ \frac{\partial L(\ldots)}{\partial \rho^b_{\ldots |_{\ldots}}} \cdot \partial_t \phi^b 
+ \frac{\partial L(\ldots)}{\partial \rho^b_{\ldots}} \cdot (\partial_t \phi^b)_{|_{(\ldots,n^b - 1,\ldots)}} + (\text{c.c.}) \right] \right\} 
= : \Delta_b T^b_4 + \partial_t T^4_4 = 0 .
\]

(A.II.6Bii)

Here, (c.c.) indicates the complex-conjugation of the preceding terms.

Now, we shall investigate the gauge invariance of the Lagrangian for a complex-valued tensor or spinor field $\phi^\alpha(n)$ and $\phi^\alpha(n, t)$ and the corresponding difference and difference-differential conservation equations. A global, infinitesimal gauge transformation is characterized by:
\[
\hat{\phi}^{\alpha \cdot \cdot}(n) = [\exp(i\varepsilon)] \phi^{\alpha \cdot \cdot}(n) = \phi^{\alpha \cdot \cdot}(n) + (i\varepsilon) \phi^{\alpha \cdot \cdot}(n) + 0(\varepsilon^2), \quad \text{(A.II.7A)}
\]

\[
\hat{\phi}^{\alpha \cdot \cdot}(n, t) = [\exp(i\varepsilon)] \phi^{\alpha \cdot \cdot}(n, t) = \phi^{\alpha \cdot \cdot}(n, t) + (i\varepsilon) \phi^{\alpha \cdot \cdot}(n, t) + 0(\varepsilon^2). \quad \text{(A.II.7B)}
\]

The invariance of the Lagrangian function under (A.II.7A) implies that

\[
0 = L(\rho^\cdot, \overline{\rho}^\cdot; \rho_{\mu}, \overline{\rho}_{\mu})|_{\rho^\cdot = \hat{\phi}^{\cdot \cdot}(n), \overline{\rho}^\cdot = \hat{\phi}^{\cdot \cdot}(n), \rho_{\mu} = \Delta^\mu_{\cdot \cdot} \hat{\phi}, \overline{\rho}_{\mu} = \Delta^\mu_{\cdot \cdot} \overline{\phi}}
- L(\rho^\cdot, \overline{\rho}^\cdot; \rho_{\mu}, \overline{\rho}_{\mu})|_{\rho^\cdot = \phi^{\cdot \cdot}(n), \overline{\rho}^\cdot = \phi^{\cdot \cdot}(n), \rho_{\mu} = \Delta^\mu_{\cdot \cdot} \hat{\phi}, \overline{\rho}_{\mu} = \Delta^\mu_{\cdot \cdot} \overline{\phi}}
= (i\varepsilon) \left[ \frac{\partial L(\cdot \cdot)}{\partial \rho^\cdot} \rho^\cdot - \frac{\partial L(\cdot \cdot)}{\partial \overline{\rho}^\cdot} \overline{\rho}^\cdot + \frac{\partial L(\cdot \cdot)}{\partial \rho_{\mu}} \rho_{\mu} - \frac{\partial L(\cdot \cdot)}{\partial \overline{\rho}_{\mu}} \overline{\rho}_{\mu} \right]|_{\cdot \cdot} + 0(\varepsilon^2).
\]

(A.II.8A)

Dividing by \( \varepsilon > 0 \), taking the limit \( \varepsilon \rightarrow 0_+ \), and equating \( \frac{\partial L(\cdot \cdot)}{\partial \rho_{\mu}} |_{\cdot \cdot} = \Delta^\mu_{\cdot \cdot} \), we obtain that

\[
0 = i \left\{ \left[ \Delta^\mu_{\cdot \cdot} \left( \frac{\partial L(\cdot \cdot)}{\partial \rho_{\mu}} \right) \right]|_{\cdot \cdot} \cdot \phi^{\alpha \cdot \cdot}(n) + \frac{\partial L(\cdot \cdot)}{\partial \rho_{\mu}} |_{\cdot \cdot} \cdot \Delta^\mu_{\cdot \cdot} \phi^{\alpha \cdot \cdot} \right\} + (\text{c.c.}). \quad \text{(A.II.9A)}
\]

Since \( i\Delta^\mu_{\cdot \cdot} = P_{\mu} \) represent the relativistic four-momentum operators, the equation (A.II.9A) incorporates the relativistic partial difference equation for the charge-current conservation. The difference-differential version of the relativistic charge-current conservation is furnished by:

\[
i \left\{ \left[ \Delta^\mu_{\cdot \cdot} \left( \frac{\partial L(\cdot \cdot)}{\partial \rho_{\mu}} \right) \right]|_{\cdot \cdot} \cdot \phi^{\alpha \cdot \cdot}(n, t) + \left[ \frac{\partial L(\cdot \cdot)}{\partial \rho_{\mu}} \right]|_{\cdot \cdot} \cdot \phi^{\alpha \cdot \cdot}(n, t)
+ \frac{\partial L(\cdot \cdot)}{\partial \rho_{\mu}}|_{\cdot \cdot} \cdot \Delta^\mu_{\cdot \cdot} \phi^{\alpha \cdot \cdot}(n, t) + \frac{\partial L(\cdot \cdot)}{\partial \rho_{\mu}}|_{\cdot \cdot} \cdot \frac{\partial \phi^{\alpha \cdot \cdot}}{\partial t}(n, t) \right\} + (\text{c.c.}) = 0.
\]

(A.II.9B)

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FIG. 1 One discrete phase plane.
FIG. 2 A two-dimensional discrete domain.