Running coupling and fermion mass in strong coupling $\mathcal{QED}_{3+1}$

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Abstract

Simple toy model is used in order to exhibit the technique of extracting the non-perturbative information about Green’s functions in Minkowski space. The effective charge and the dynamical electron mass are calculated in strong coupling 3+1 QED by solving the coupled Dyson-Schwinger equations for electron and photon propagators. The minimal Ball-Chiu vertex was used for simplicity and we impose the Landau gauge fixing on QED action. The solution obtained separately in Euclidean and Minkowski space were compared, the latter one was extracted with the help of spectral technique.
1 Introduction

In quantum field theory and even in physics at all, the dispersion relations (DRs) were recognized as providing useful connections between physical quantities.

Very recently [1], it was recognized that in the Quantum Chromodynamics (QCD) in Landau gauge the full (non-perturbative) gluon propagator is (very likely) an analytic function in the whole complex plane of momenta except the positive-timelike [2] real $p^2$ half-axis. This has been achieved by the analytical fits of some recent solutions of Dyson-Schwinger equations (DSEs) [3] and by analytical parameterization of some contemporary lattice data. The reasonable spacelike domain agreement of these fits with the recent Euclidean data give us a good guidance on the possible analytical structure of the Green’s function in timelike axis of momenta. The similar was argued for the quark propagator, however in this case the observed singularities do not occur exactly on - but rather say - very close to the real positive $p^2$ half-axis.

In this paper, we instead to make an analytical guesses of already known Euclidean results we start with the assumption of $C - R_+$ analyticity of the Green’s functions which allows us to solve the DSEs system directly in Minkowski space. It is well known that such an assumption is intimately related with the formulation of the appropriate dispersion relations (DRs) and spectral representation (SR) of Green’s functions (for more then the derivation of DRs in the context of QCD see [4]). The main advantage of such an approach is the possibility to the knowledge of the propagators at the whole range of momentum $p^2$ while the main inconvenience of this method is the necessity of ‘inhuman’ effort when the absorptive parts of Green’s functions are actually derived. Unfortunately at the contemporary stage of our calculation we are not able to write down the appropriate DRs for ghosts and transverse gluons. Due to this fact we confine ourself to the less complicated (at least from the technical point of view) case of one flavor strong coupling Quantum Electrodynamics and we leave the involvement of Yang-Mills theory for the future study.

In the present paper and in contrast to [5] we adopt the Ball-Chiu (BC) vertex [6] which is the known minimal ansatz consistent with the Ward-Takahashi identity (WTI). Implementing this into the equations for electron and photon we then solve the corresponding DSE also for the photon polarization function. In this sense, the presented study is the extension of the previous numerical study of the renormalized electron mass in the strong coupling QED in simplest approximation to the DSEs: the bare vertex and bare photon propagators were employed [5].

Up to the case of perturbative theory the direct Minkowski space treatment of DSEs is usually rather involved and the progress is adequately less when compared with the relatively large amount of calculations performed in Euclidean space (for a list of references see the paper [5] and also the paper [7] where the approximative analytical solutions are discussed). Although the spectral approach is rapidly getting technically rather involved when one goes beyond lowest order truncation, one of the purpose of this paper is to demonstrate that it is still manageable for the considered model with WTI respecting vertex. As in the paper [5] we again compare in detail the solutions for fermion propagator
in Minkowski and Euclidean space, besides we compare for the first time also the photon propagator. There is a another important reason for the use of improved vertex. It is long time known that the scalar part of BC vertex dramatically affect the analytical structure of the DSEs [8] (which paper is the extended study of the Munczek-Nemirovsky model [9]), for the earlier study of the quark gap equation in the axial gauge see [10]. In this place we should stress that the recent study of QCD DSEs confirms this suggestion: the bare vertex leads to the complex singularity of the chiral limit quark propagator. The authors of Ref. [1] observed that the scalar part of the BC vertex in used plays the crucial role in the analytical structure of quark propagator and one can conclude that the inclusion of the scalar part of the BC vertex leads again to the real singularity of quark propagator, noting that the same is true for quenched QED in the chiral limit (note only that the full Curtis-Pennington vertex [11] was used which fact has no large significance due to the Landau gauge employed).

Contrary to QCD, the considered model here is not an asymptotic free theory and posses the additional complication due to the triviality statement. It requires the introduction of ultraviolet cut-off function $f(\Lambda)$ which well known fact is clearly confirmed by our numerical analysis. However, we stress here that QED is trivial and the appropriate solution of DSEs (Euclidean and spectral as well as) collapse due to the presence of Landau singularity [12], we regard the strong QED as an useful toy model which is clearly reliable when we take cutoff reasonably smaller then the position of the expected of Landau singularity. Further, in order to fully specify our model we consider non-zero bare electron mass $m_0$.

For comparison we have also calculated the propagators in unquenched approximation (i.e., $\Pi(q^2) \neq 0$), but with the bare vertex. In that case the vacuum polarization tensor is not gauge invariant and contains two independent scalar functions. We impose the transversality by hands before the numerical solution. As expected from the earlier studies [13],[14] the results obtained in Landau gauge are very close to the results calculated with BC vertex. This pleasant but extraordinary property of Landau gauge fixing makes the solutions of DSEs with the bare vertex approximation meaningful. However the similar conclusion was made in some QCD studies [15], [16] one should be aware about the incidence on the analytical structure of the quark propagator. Although when using the bare vertex the transverse projection by hands is not theoretically justified and although the analyticity assumptions seems not to be fully justified the usage of bare gauge vertices in Landau-like gauges remains popular in studies of more complicated gauge models (like extended, walking Technicolor etc. for review see [17] ). This is why we believe that showing in detail the effects due to BC vertex is interesting: in our studies its effects are less than 10 per cents even for rather large coupling constant. The strong coupling QED should serve as an instructive tool for its ‘simplicity’ and the proposed technique could be helpful elsewhere.

The layout of the article is following: In next section we review the DSE formalism and describe the model. The section 3 is devoted to the solution of DESs in Euclidean space. In the section 4 the DSEs are written in Minkowski space and the desired DRs for electron selfenergy and gluon polarization function are derived. The numerical results are presented in the section 5 and then we summarize.
2 Model- Unquenched QED with BC vertex

In this section we review the basic elements of the considered model and set the notation and conventions used in the main part of the paper. First of all, let us stress some differences and improvements used in this paper when compared with until now published works that are dealing with QED DSEs. Using the full photon propagator, instead of the bare propagator that authors of the papers [18],[19], [13] (bare and improved vertices, Euclidean space), [5] (bare vertex, Minkowski space) used is an important improvement. Further we use the form of the vertex that is consistent with WTI instead of bare vertex $\gamma^\mu$ that was used in the study of renormalized DSEs in unquenched QED [20]. This leads to the solution which reflects not only the effect of running coupling caused by fermion loop but also to the vacuum polarization which is automatically transverse. Thus we can call our solution as a gauge covariant one since it naturally respect the conservation law that follows from the gauge invariance of QED Lagrangian. The three lowest DSEs read:

\[ [S]^{-1} = [S_0]^{-1} - \Sigma \quad ; \quad \Sigma = e^2 \int S \Gamma^\mu G^\mu\nu \gamma^\nu, \]
\[ [G^\mu\nu]^{-1} = [G_{0\mu\nu}]^{-1} - \Pi_{\mu\nu} \quad ; \quad \Pi_{\mu\nu} = e^2 Tr \int S \Gamma^\mu \Sigma \gamma^\nu, \]
\[ \Gamma^\mu = \gamma^\nu + e \int S \Gamma^\mu S M = \Gamma_L^\mu + \Gamma_T^\mu, \]

where $M$ is the electron-positron scattering kernel (without annihilation channel), $S$ is the full fermion propagator

\[ S(p) = \frac{1}{A(p^2) \frac{p}{\not{p} - B(p^2)} = \frac{F(p^2)}{\not{p} - M(p^2)}}, \]

parametrized in a usual way in terms of two scalar functions $A, B$. Equation (1) then reduces to a coupled set of two scalar equations for the functions $A, B$ or equivalently for the dynamical mass $M = B/A$ and the electron renormalization function $F = A^{-1}$. When the interaction is neglected, Eq. (4) reduces to the free propagator: $S_0^{-1} = \not{p} - m_0$, $m_0$ being the bare electron mass. The sixteen Lorentz components of Eq. (2) can be reduced to single equation for polarization function $\Pi$

\[ \Pi^{\mu\nu}(q) = \mathcal{P}^{\mu\nu}_T q^2 \Pi(q^2) \quad ; \quad \mathcal{P}^{\mu\nu}_T = (q^{\mu} q^{\nu} - q^2 q^2/q^2), \]

by virtue of the gauge invariance $q^{\mu} \Pi_{\mu\nu} = 0$. The function $G_{0}^{\mu\nu}$ is the quenched approximation ($\Pi = 0$) to the full photon propagator $G^{\mu\nu}(q)$, which is purely transverse in Landau gauge,

\[ G^{\mu\nu}(q) = -\frac{\mathcal{P}^{\mu\nu}_T}{q^2 [1 - \Pi(q^2)]}. \]

The functions $\Gamma_L^\mu$ and $\Gamma_T^\mu$ in (3) are the longitudinal and transverse parts of the full vertex $\Gamma^\mu$. Multiplying the vertex by the photon momentum $p - l$ one gets the WT identity

\[ S^{-1}(p) - S^{-1}(l) = (p - l)_\mu \Gamma_{L\mu}(p, l), \]

while from its definition $\Gamma_T^\mu(p, l). (p - l) = 0$. Any truncation of DSEs system leading to gauge covariant solution of DSEs must involve vertex satisfying WTI. Instead of solving own equation for $\Gamma$ there
exists much economic way. Within the requirement of right Lorentz transformation property, charge conservation and the unique limit when \( p \to l \) (the absence of kinematic singularity) the longitudinal part \( \Gamma^\mu_L \):

\[
\Gamma^\mu_L(p, l) = \frac{\gamma^\mu}{2} \left( A(p^2) + A(l^2) \right) + \frac{1}{2} \left( p + l \right) \frac{(p^\mu + l^\mu)}{p^2 - l^2} \left( A(p^2) - A(l^2) \right) - \frac{p^\mu + l^\mu}{p^2 - l^2} \left( B(p^2) - B(l^2) \right).
\]

was found in the paper [6] while the transverse part \( \Gamma^\mu_T(p, l) \) can be decomposed into the eight component vector basis (see for instance [6] or [21] for the details)

\[
\Gamma^\mu_T(p, l) = \sum_{i=1}^{8} t_i T^\mu_i.
\]

with coefficient functions \( t_i \) unspecified in general.

The one loop analysis and determination of \( t_i \) was performed in the paper [21] in arbitrary covariant gauges. The two loop results was obtained in the Feynman gauge also [6]. The minimal version of the gauge covariant vertex - BC vertex simply neglect the transverse part and hence is given by the Eq. (8). Some better improvement of the full vertex \( \Gamma \) necessarily differs only by its transverse part \( \Gamma_T \).

In chiral symmetric case with given truncation of DSEs and within the requirement of multiplicative renormalizability the additional constraints on the transverse pieces of the vertex were found [22]. Also the additional information was obtained from the requirement gauge independence of chiral symmetry breaking in quenched QED [23]. Although, as it follows form the above two notations the BC vertex has not probably most ideal form for some 'definitely conclusive' nonperturbative study of QED, nevertheless we use this vertex for its simplicity. The implementation of the methods used in [22], [23] and mainly their true Minkowski space extension is far away from triviality and thus remains the challenging task for future investigation.

In the Landau gauge and in the ladder approximation of the electron DSE there is no self-energy contribution to the electron renormalization function since the Feynman (F) pole part of photon propagator \( G_{\mu\nu} \) is exactly canceled by the contribution of longitudinal (LO) \( q_{\mu}q_{\nu} \) part of \( G_{\mu\nu} \). Explicitly their absorptive parts satisfy:

\[
\Im A_F(\omega) = -\Im A_{L0}(\omega) = \frac{e^2 m}{(4\pi)^2} \int da \left( 1 - \frac{a^2}{\omega^2} \right) \sigma_v(a) \Theta(\omega - a),
\]

which leads to \( A = A_F + A_{L0} = 1 \) (here \( \sigma_v(a) \) is Dirac coefficient Lehmann function (14), \( m \) is the physical (pole) electron mass). It is also well known that the property \( A \simeq 1 \) persists beyond the bare vertex approximation and that in angle approximation [24] \( A = 1 \) is even valid exactly. The above mentioned study [20] of unquenched QED shows rather small and irrelevant violations from the identity \( A = 1 \) which then turns to few percentage error only when momentum approaches the ultraviolet cutoff \( \lambda \) (note, the condition \( A(0) = 1 \) was exactly imposed in the paper [20]). Further study on dynamical mass generation in unquenched supercritical QED [14], [24] confirm this also in
this case. They justify our approximation with at most ten percentage deviation in the infrared region (when \( A(\Lambda) = 1 \) condition was imposed by the authors. To reduce the complexity of our DSEs we explore this nice Landau gauge property and we put explicitly \( A = 1 \) for all momenta. We should stress here that this negligence has no effect on the gauge invariance of polarization tensor, i.e. we still have

\[
\Pi(g)^{\mu\nu}q_\mu = 0
\]

since the WTI (7) is not violated.

When we renormalize we adopt the standard notation for the renormalization constants \( Z_1 \), \( Z_2 \) and \( Z_3 \) (see e.g., [25]). From the approximation employed it follows that \( Z_1 = Z_2 = 1 \) which is in agreement with the multiplicative renormalizability and WTI. Furthermore, the unrenormalized vacuum polarization \( \Pi(\mu) \) should be absorbed into the renormalization constant \( Z_3 \). Similarly, the unrenormalized electron self-energy \( Tr \Sigma(\mu)/4 \) is absorbed into the constant \( Z_m \).

3 Solution of DSEs in Euclidean space

The DSEs are often solved in Euclidean space after the Wick rotation \( k_0 \to k_1^E \) is made for each momentum. Then the loop integrals should be free of singularities and the Green functions are found for positive Euclidean momentum \( k_1^E = k_1^2 + k_2^2 + k_3^2 + k_4^2 \). If there is no additional singularity in the complex plane of momenta (that would prohibit the validity of the naive Wick rotation), then from the solution for some generic function \( f(p_1^2) \) one would get the solution for Minkowski spacelike momentum \( f(p_M^2); p^2 < 0 \). The solution for timelike momentum would be in principle obtained by the analytical continuation of \( f \) to the real axis \( p_M^2 > 0 \). In our case, because of QED triviality in four dimensions, the assumption of non-singular behavior holds only with the presence of UV cut-off. We presume here, that any numerical attempt to avoid UV cut-off implementation would lead to an uncontrolled behavior of the Gell-Mann-Low effective charge. The presence of the ultraviolet cut-off is required not only due to the inner consistence (the Wick rotation) but also to ensure the numerical stability of our calculation.

Substituting the Ball-Chiu vertex into the DSEs and employing the projection proposed in [26] (and successfully used in the papers [14],[3]) we obtain the following coupled DSEs to be solved numerically:

\[
\Pi_U(x) = \frac{2\alpha}{3\pi^2} \int dy \frac{y}{y + M^2(y)} \int d\theta \sin^2 \theta \frac{1}{z + M^2(z)} \left[ \left( 2y - 8y\cos^2 \theta + 6\sqrt{yx}\cos \theta \right) \right. \\
+ \left. \frac{(M(y) - M(z))}{y - z} (M(y) + M(z)) \left( 2y - 8y\cos^2 \theta + 3\sqrt{yx}\cos \theta \right) \\
+ \left. 3M(y)(M(y) - M(z)) \right] \\
M(x) = m_0 + \frac{\alpha}{2\pi^2} \int dy \frac{y}{y + M^2(y)} \int d\theta \sin^2 \theta
\]
\[
\frac{1}{z(1 - \Pi(z))} \left[ 3M(y) - \frac{(M(y) - M(z))}{y - z} \right] \left( y - z \right), 
\]

(13)

where \( \alpha = e^2/4\pi \) and variables \( x, y, z \) represent squares of Euclidean momenta, \( z = x + y - 2\sqrt{yx} \cos \theta \). For details of the derivation of (12), (13) we refer to refs. [26],[14].

The gauge invariance \( q^{\mu} \Pi_{\mu\nu} = 0 \) implies that the photon polarization tensor has to be of the form (5).

In the DSE’s formalism the gauge invariance of \( \Pi_{\mu\nu} \) follows from the gauge covariance of the Ball-Chiu vertex. This fact helped the authors of [26] to construct simple recipe how to avoid numerical quadratical divergence which would be otherwise presented in equation for \( \Pi \). Anticipate here that it is convenient reduce the photon polarization to a single scalar function also in our Minkowski calculation, although there it is not a numerical necessity, but merely matter of technical convenience.

The first line inside the brackets \([\ldots]\) of the Eq. (12) and the first term in the brackets \([\ldots]\) of Eq. (13) represent the kernels of the bare vertex approximation (with only \( \gamma_\mu \) retained). They give dominant contributions to the dynamical mass of electron and the vacuum polarization as well. Neglecting the vacuum polarization effect (putting \( \Pi(z) = 0 \)) in the Eq. (12), the equation for \( M \) can be further simplified. This so-called ladder approximation of fermion DSE is represented by one dimensional momentum integral equation first derived in ref. [27] and used also in the Euclidean confinement study [28].

4 Direct treatment in Minkowski space

Assuming analyticity for complex \( p^2 \) in \( C - R_+ \) Gaussian plain with indicated cut and using the known asymptotic behavior of the propagator one can derive the appropriate Lehmann representation (LR) for the propagators (without positivity). The appropriate LR for the fermion propagator in parity conserving theory reads

\[
S(p) = \int_0^\infty d\omega \frac{\rho_v(\omega) + \sigma_s(\omega)}{p^2 - \omega + i\epsilon} \frac{r}{p - m} + \int d\omega \frac{\rho_s(\omega) + \sigma_s(\omega)}{p^2 - \omega + i\epsilon},
\]

(14)

where we integrated out the single particle state contribution to the full Lehmann weight \( \sigma^{(1,p,s)}(a) = r\delta(a - m^2) \). The remaining term \( \sigma(c) \) is assumed to be a real and continuous spectral density that originates from the interaction. Similarly we can write for the photon propagator in linear covariant gauges

\[
G^{\mu\nu}(q) = \int_0^\infty db \frac{\sigma(\gamma(b)\left(-g_{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) - \xi q^\mu q^\nu}{q^2 - b + i\epsilon},
\]

(15)

where the single photon spectrum \( r_P \delta(b) \) can be integrated out as in the previous case.

Due to the asymptotic the dispersion formula for fermion mass function \( B \) requires one subtraction

\[
B(p^2) = m(\mu) + \int d\alpha \frac{\rho_s(\alpha)(p^2 - \mu^2)}{(p^2 - \alpha + i\epsilon)(\alpha - \mu^2)},
\]

(16)
where $m(\mu)$ is the renormalized mass at the scale $\mu$. Similar relation could be derived for the function $A$, but here $A = 1$ and therefore the function $B$ represents renormgroup mass function.

Renormalized photon polarization function in momentum subtraction scheme reads:

$$\Pi_R(q^2, \mu'^2) = \int_0^{\infty} d\omega \frac{\rho(\omega)(q^2 - \mu'^2)}{(q^2 - \omega + i\epsilon)(\omega - \mu'^2)}, \quad (17)$$

where we distinguish two possibly different renormalization scales $\mu, \mu'$. The appropriate renormalization accompanied by the detailed derivation of dispersion relations (16) and (17) is given in the next two sections.

In the perturbation theory the relations for $\rho$ is usually represented by the series expanded in the coupling constant. Here the appropriate relations for $\rho$ and $\rho_s$ are represented by the integral equations involving the Lehmann functions $\sigma$’s and even the function $\rho$ itself. Together with two additional equations for the $\sigma$’s they form the set of the so called Unitary Equations (UEs) (since these are the relations between the imaginary and real parts of propagator (and inverse of propagator) functions, here we follow the paper [5] ). In order to derive the UEs recall the well known functional identity for distributions

$$\frac{1}{x' - x + i\epsilon} = P \cdot \frac{1}{x' - x} - i\pi\delta(x' - x), \quad (18)$$

where $P \cdot$ stands for principal value integration. Making use of the LR for $G^{\alpha\beta}$ and of the appropriate DR for $\Pi$ in $G^{-1}$ and evaluating the imaginary part of the unit tensor $G^{-1\alpha\beta}G^{\beta\gamma}$ one arrives to the integral equation:

$$\sigma_{\gamma(c)}(a) = \rho(a) + [\sigma_{\gamma(c)} \ast \rho](a), \quad (19)$$

where we have adopted a shorthand notation for real functional:

$$[\sigma \ast \rho](a) = P \int_T^\infty dx \frac{\rho(x)\sigma(x) + \sigma(a)\rho(x)}{a - x}. \quad (20)$$

Notice that the both (19) and (20) are non-zero only for timelike (square of) momentum $a$ (here $a > T = 4m^2$).

For the electron Lehmann weights we get in a similar way:

$$\sigma_{v(c)}(\omega) = \frac{f_1 + m(\mu)f_2}{\omega - m^2(\mu)}; \quad \sigma_{s(c)}(\omega) = \frac{m(\mu)f_1 + \omega f_2}{\omega - m^2(\mu)}; \quad (21)$$

$$f_1 \equiv \frac{m\rho_s(\omega)}{\omega - m^2} + [\sigma_{s(c)} \ast \rho_s](\omega); \quad f_2 \equiv \frac{\rho_s(\omega)}{\omega - m^2} + [\sigma_{v(c)} \ast \rho_s](\omega), \quad (22)$$

which are non-zero only for the timelike $\omega > m^2$.

The physical electron mass is defined by $S^{-1}(p = m) = 0$ or equivalently $M(m) = m$. Using the dispersion relation (16) the desired relation reads:

$$m = m(\mu) + \int dx \frac{\rho_s(x)(m^2 - \mu^2)}{(m^2 - x)(x - \mu^2)}. \quad (23)$$
The residuum value $r$ of pole part of the propagator is fixed already when the renormalization procedure is done. Because of our numerical solution is is not convenient to determine $r$ by taking the on-shell limit $p \to m$ directly. The easiest way to evaluate $r$ is the inspection of the real part of the identity $S^{-1}S = 1$ evaluated at some arbitrary scale $p^2$. Choosing for instance $p = 0$ one gets the desired relation

$$r = \frac{m}{m(\mu)} - \int \frac{\rho(x)\mu^2}{x(\mu^2 - x)} dx - \int \sigma_{s(c)}(x) \frac{m}{x} dx,$$

which helps us to avoid dealing with complicated infrared singularities.

The original momentum space DSEs are now converted into a coupled set of the UEs (19,21) complemented by the subsidiary conditions for the residua and thresholds (23) and (24).

To solve UEs one has to consider these six integral equations simultaneously at all the positive values of spectral variables where the corresponding spectral functions are non-zero. Any internal inconsistency (i.e., spacelike Green functions singularities mentioned in the introduction) should be seen or felt when the UEs are actually solved. The original momentum space Green functions are then obtained through the dispersion relation for the proper function or equivalently by the integration of the spectral representation for the full connected propagators $S, G$. Checking (numerically) this equivalence verifies the internal consistence of the method. Compared to the Euclidean approach the spectral approach has clear advantage of “already known” analytical continuation at all momenta. The disadvantage of the spectral approach is its failure in the (confining) regime where the underlying assumptions are not justified.

### 4.1 Photon propagator

In a fixed gauge the photon propagator is fully determined by the gauge independent polarization function. We describe below the derivation of once subtracted DR, following from the momentum space subtraction procedure for photon polarization tensor. First we briefly review the method in its perturbative context.

In $4 + \varepsilon$ dimensions and for spacelike momentum $q^2 < 0$ the one loop polarization function can be written as [29]

$$\Pi(q^2) = \frac{4\varepsilon^2}{3(4\pi)^2} \left\{ \frac{2}{\varepsilon} + \gamma_E - \ln(4\pi) + \ln \left( \frac{m^2}{\mu_{t'H}^2} \right) \right\}$$

$$+ \left( 1 + 2m^2/q^2 \right) \sqrt{1 - \frac{4m^2}{q^2}} \ln \left[ \frac{1 + \sqrt{1 - 4m^2/q^2}}{1 - \sqrt{1 - 4m^2/q^2}} \right]$$

$$- \frac{4m^2}{q^2} - \frac{5}{3} \right\} - \delta Z_3 \mu_{t'H}^{-\varepsilon},$$

where $\mu_{t'H}$ is t’Hooft dimensionfull scale. The mass-shell subtraction scheme defines $Z_3$ so that $\Pi_H^{\text{MASS}}(0) = 0$ which implies that the photon propagator behaves as free one near $q^2 = 0$. Choosing
\[ \delta Z_3 \text{ to cancel entire } O(e^2) \text{ correction we find} \]
\[ \delta Z_3^{\text{MASS}} = \lim_{q^2 \to 0} \Pi(q^2) = \frac{e^2}{12\pi^2} \left[ \frac{2}{\varepsilon} + \gamma_E - \ln(4\pi) + \ln\left( \frac{m^2}{\mu^2 t} \right) \right] \]

(26)

and renormalized polarization function in mass-shell renormalization prescription satisfies well known dispersion relation

\[ \Pi_R^{\text{MASS}}(q^2) = \Pi(q^2) - \lim_{q^2 \to 0} \Pi(q^2) = \int_0^\infty d\omega \frac{q^2}{(q^2 - \omega + i\epsilon)\omega} \rho(\omega) \]

(27)

with the absorptive part

\[ \pi \rho(\omega) = \frac{\alpha_{\text{QED}}}{3} (1 + 2m^2/\omega) \sqrt{1 - 4m^2/\omega} \Theta(\omega - 4m^2) . \]

(28)

which is given in many standard textbooks (see for instance [30], where the result of the integration in (27) is written also for timelike momenta). Recall that the one loop \( \Pi_R^{\text{MASS}} \) represents also self-energy calculated in the popular \( \overline{\text{MS}} \) scheme for the special choice of t’Hooft scale \( \mu_{t'H} = m \) [31]). Finally, let us remind the definition of the off-shell momentum space subtraction:

\[ \delta Z_3 = \Pi(\mu^2) \]

which is given in many standard textbooks (see for instance [30], where the result of the integration in (27) is written also for timelike momenta). Recall that the one loop \( \Pi_R^{\text{MASS}} \) represents also self-energy calculated in the popular \( \overline{\text{MS}} \) scheme for the special choice of t’Hooft scale \( \mu_{t'H} = m \) [31]). Finally, let us remind the definition of the off-shell momentum space subtraction:

\[ \delta Z_3 = \Pi(\mu^2) \]

Making redefinition of the electron charge accompanied by the finite subtraction of (27) we can immediately write down the desired dispersion relation (17).

Now we turn our attention to the derivation of momentum space subtracted \( \Pi_R(p^2, \mu^2) \) with the dressed propagators and with the full Ball-Chiu vertex included. As mentioned in the previous section validity of Ward-Takahashi identity for \( \Gamma_{BC} \) naturally leads to the transversality of the polarization tensor

\[ \Pi_{\mu\nu}^{U,R}(q) = P_{\mu\nu}^{T} q^2 \Pi_{U,R}(q^2) , \]

(29)

where \( P_{\mu\nu}^{T} = (g_{\mu\nu} - q_{\mu} q_{\nu}/q^2) \) is the transverse projector and capital (R) indicates that renormalized tensor (29) must respect gauge symmetry of unrenormalized (U) one.

The truly massless photons with \( \Pi_R^{\text{MASS}}(0) = 0 \) are consequence of the renormalization prescription

\[ \Pi_R(q^2, 0) = \Pi_U(q^2) - \Pi_U(0) , \]

\[ \Pi_U(q^2) = \frac{\Pi_U^{\mu\nu}(q) [g_{\mu\nu} - C g_{\mu\nu}/q^2]}{3q^2} \]

(30)

with arbitrary constant \( C \), applied on the full polarization tensor

\[ \Pi_U^{\mu\nu}(q) \equiv i e^2 \int \frac{d^4l}{(2\pi)^4} \text{Tr} \left[ \gamma^\mu S(l) \Gamma^\nu(l, l - q) S(l - q) \right] , \]

(31)

where the explicit dependence of Ball-Chiu vertex on fermionic momenta reads:

\[ \Gamma^\mu_{L}(l, l - q) = \gamma^\mu - \frac{(2l - q)^\mu}{(l - q)^2 - l^2} [M((l - q)^2) - M(l^2)] . \]

(32)

As soon as we use WTI constrained vertex the \( C \) independence of resulting \( \Pi \) is evident but the right choice of \( C \) facilitates derivation of DR. The reason is that the Pennington-Bloch [26] projector

\[ \mathcal{P}_{\mu\nu}^{(d)}(q) = \frac{1}{d} \left[ g_{\mu\nu} - (d + 1) \frac{g_{\mu\nu} q_{\nu}}{q^2} \right] \]

(33)
cancels the contribution from $d+1$ space-time metric tensor $g_{\mu\nu}$ which simplifies the actual calculations.

Let us now derive $\Pi_R$

$$
\Pi_R(q^2, \mu^2) = \frac{\Pi^{\alpha\beta}_U(q)D^{(3)}_{\alpha\beta}(q)}{3q^2} - \{q \to \mu\}
$$

for the case when only $\gamma^\mu$ part of $\Gamma_{\mu L}^\nu$ is retained. Substituting the spectral representation (14) into the expression for photon polarization function (34), (31) we immediately get

$$
\Pi_{U(\gamma^\nu)}(q^2) = \frac{i e^2}{3q^2} Tr \int \frac{d^4l}{(2\pi)^4} \int d\alpha \int db \frac{(\gamma^\nu - 4q^\nu \frac{U^{\nu\nu}}{q^2})[(1 - \gamma^\nu)\sigma_v(a) + \sigma_s(a)] [I\sigma_v(b) + \sigma_s(b)]}{(l - q)^2 - a(l^2 - b)}.
$$

(35)

From now on we omit the spectral integrals and assume that the presence of any spectral function with given arguments automatically implies integration over these variables. Since we will include explicitly the boundaries (thresholds) in step functions in the integral kernels, all integrals can be taken from zero to infinity $\int_0^\infty$. Moreover we label the measure $-i\,d^4l/(2\pi)^4$ by $dl$ and we also suppress $ie$ factors in denominators. Combining the denominators with the help of Feynman parameterization then gives

$$
\Pi_{U(\gamma^\nu)}(q^2) = 8e^2 \int dl_1 \int_0^1 dx \frac{\sigma_v(a)\sigma_v(b)x(1-x)}{(l^2 + q^2x(1-x) - ax - b(1-x))^2}.
$$

(36)

The remaining integral is logarithmic divergent. After the subtraction

$$
\Pi_{R(\gamma^\nu)}(q^2; \mu^2) = \Pi_{U(\gamma^\nu)}(q^2) - \Pi_{U(\gamma^\nu)}(\mu^2)
$$

(37)

it leads to the finite dispersion relation. Although this procedure is rather straightforward we present for completeness briefly intermediate steps of the derivation. The subtracting procedure (37) yields explicitly

$$
\Pi_{R(\gamma^\nu)}(q^2; \mu^2) = 8e^2 \int dl_1 \int_0^1 dx \int_0^1 dz \frac{\sigma_v(a)\sigma_v(b)x(1-x)^2(q^2 - \mu^2)(-2)}{(l^2 + q^2x(1-x) + \mu^2x(1-x) - ax - b(1-x))^3}
$$

$$
= \frac{8e^2}{(4\pi)^2} \int_0^1 dx \int_{dx+b(1-x)}^{\infty} d\omega \frac{\sigma_v(a)\sigma_v(b)x(1-x)(q^2 - \mu^2)}{(\omega - \mu^2)(q^2 - \omega)},
$$

(38)

where (after the loop momentum integration) the substitution $z \to \omega = \frac{ax+b(1-x)}{x(1-x)} + \mu^2 - \frac{\omega^2}{x}$ was made.

Changing the order of integrations and integrating over $x$ (the appropriate integrals are listed in the Appendix A) we obtain the DR:

$$
\Pi_{(\gamma^\nu)}(q^2; \mu) = \frac{e^2}{12\pi^2} \int_0^\infty d\omega \frac{q^2 - \mu^2}{(\omega - \mu^2)(q^2 - \omega + ie)}
$$

$$
\frac{\Delta^{1/2}(\omega, a, b)}{\omega} \left[ 1 + \frac{a + b}{\omega} - \frac{b - a}{\omega} \left( 1 + \frac{b - a}{\omega} \right) \right] \sigma_v(a)\sigma_v(b),
$$

(39)
where $\Delta$ is the well-known triangle function

$$\Delta(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz.$$  \hfill (40)

Considering in the expression above for $\sigma_v(x)$ only the delta function parts of spectral functions, i.e., $r_f \delta(x - m^2)$, we just recover the one loop perturbative result (28) (up to the presence of electron propagator residuum $r_f$, which is assumed to be close to 1 when the coupling is small):

$$\Pi^\text{pole}_{R(\gamma\mu)}(q^2, 0) = r_f^2 \frac{\alpha_{\text{QED}}(0)}{3\pi} \int_0^\infty d\omega \frac{q^2}{\omega(q^2 - \omega + i\epsilon)} \sqrt{1 - \frac{4m^2}{\omega}} \left(1 - \frac{2m^2}{\omega}\right).$$  \hfill (41)

We see immediately that $\Pi_R(0, 0) = 0$ as required and that using the projector $\mathcal{P}^d$ naturally reproduces the perturbation theory in its lowest order.

Using the prescription (34) we now carry on the derivation for the part of the polarization function with the remaining term of Ball-Chiu vertex (second term in rhs of Eq. (32)). First we drop the part of the vertex which is proportional to $l$ (because the photon propagator is transverse) and take a trace which leads to following finite loop integral:

$$\Pi_{\text{U(rem)}}(q^2) = -\frac{4e^2}{3q^2} \int d\omega \frac{[M(l) - M(l - q)]}{l^2 - (l - q)^2} \{\sigma_v(a)\sigma_s(b) \left[2l^2 - 8\frac{(l-q)^2}{q^2} + 3l \cdot q + 3q^2\right] + \sigma_v(b)\sigma_s(a) \left[2l^2 - 8\frac{(l-q)^2}{q^2} - 3l \cdot q\right]\}.$$  \hfill (42)

The next intermediate steps of the DR derivation are given in the Appendix B, here we simply present the final result: The full polarization function with the Ball-Chiu vertex satisfies the once subtracted DR

$$\Pi_R(q^2, \mu^2) = \int_0^\infty d\omega \frac{\left(\rho_{\gamma\mu}(\omega) + \rho_{\text{rem}}(\omega)\right)(q^2 - \mu^2)}{(q^2 - \omega + i\epsilon)(\omega - \mu^2)},$$  \hfill (43)

where $\rho_{\gamma\mu}$ follows from (39) and $\rho_{\text{rem}}$ from (42) and (69). Explicitly, they read:

$$\rho_{\gamma\mu}(\omega) = \frac{e^2}{12\pi^2} \frac{\Delta^{1/2}(\omega, a, b)}{\omega} \left[1 + \frac{a + b}{\omega} - \frac{b - a}{\omega} \left(1 + \frac{b - a}{\omega}\right)\right],$$

$$\sigma_v(a)\sigma_v(b)\Theta \left(\omega - (\sqrt{a} + \sqrt{b})^2\right),$$

$$\rho_{\text{rem}}(\omega) = \frac{e^2}{6\pi^2} \sigma_v(a)\sigma_s(b) \rho_s(c) \left[\frac{F(\omega, c, a) - F(\omega, c, a) - F(\omega, b, a) - F(\omega, b, a)}{(c - b)(c - a)}\right],$$

$$F(\omega, c, a) = \frac{\Delta^{1/2}(\omega, c, a)}{\omega} \left[\frac{a + c}{\omega} - 2\frac{a - c}{\omega} \left(1 + \frac{a - c}{\omega}\right)\right] \Theta \left(\omega - (\sqrt{a} + \sqrt{c})^2\right).$$  \hfill (44)

These expressions in their full form have been used in our numerical calculation. No principal value integration is necessary and the whole integrand has a regular limit when one spectral variable approaches another. Recall that the ordinary integrals over the spectral variables $a, b, c$ are implicitly assumed. The function $\pi \rho_{S(c)}$ is simply $\Im M(\omega)$ and it is evaluated in the next section.
4.2 Fermion propagator

In this section we show that the Ball-Chiu vertex

$$\Gamma^\mu_L(p-l,p) = \gamma^\mu - \frac{(2p-l)^\mu}{(p-l)^2 - p^2}[M((p-l)^2) - M(p^2)]$$

(45)

substituted to the electron self-energy

$$\Sigma(p) = e^2 \int d\gamma^\nu S(p-l)\Gamma^\mu_L(p-l,p)G_{\mu\nu}(\xi = 0, l),$$

(46)

together with the assumed LR for electron (14) and photon propagator (15) leads to the dispersion formula for the dynamical mass (16). In $F = 1$ approximation we can write

$$M(p^2) = m_o \frac{Tr}{4} \Sigma(p) + e^2 \frac{Tr}{4} \int d\gamma^\nu \frac{(p-l)\sigma_\nu(a) + \sigma_\nu(a)}{(p-l)^2 - a} \Gamma^\mu_L(p-l,p) \frac{-g_{\mu\nu} + \frac{l_\nu l_\mu}{l^2 - b}}{l^2 - b} \sigma_\gamma(b),$$

(47)

where we have adopted conventions and notations of the previous section. Relating bare mass and the renormalized one by $m_o = Z_m(\mu)m(\mu)$ and absorbing $\Sigma(p = \mu)$ into the mass renormalization constant $Z_m(\mu)$ gives the finite mass $m(\mu)$ and finite DR for $\mu$ independent dynamical mass function (16).

Let us again start with the pure $\gamma_\mu$ matrix part of the $\Gamma_L$ in $\Sigma$. It leads to the following DR:

$$M(\gamma_\mu)(p^2) = \int d\omega \frac{p^2 - \mu^2}{\omega - \mu^2} \rho_{\gamma_\mu}(\omega),$$

$$\rho_{\gamma_\mu}(\omega) = -3 \left(\frac{e}{4\pi}\right)^2 \sigma_\gamma(a) \Delta^{1/2}(\omega, a, b).$$

(48)

(The derivation is straightforward, see for instance Appendix of Ref. [5]). This is a dominant momentum dependent part of $M$.

Using the remainder terms in Ball-Chiu vertex (45) we get the following contribution to $M$:

$$M(rem)(p) = 2e^2 \int dt \frac{[(p^2 - p^2^2)]^2}{(p-l)^2 - a} \rho_\gamma(p)(p^2 - a)(p^2 - (l^2 - b)),$$

(49)

where we have self-consistently used the formula for difference of the dispersion integrals for $M$:

$$\frac{(2p-l)^\mu}{(p-l)^2 - p^2}[M((p-l)^2) - M(p^2)] = \int d\gamma \frac{\rho_\gamma(p)(2p-l)^\mu}{(p^2 - \gamma)((p-l)^2 - \gamma)}.$$

(50)

Using the Feynman parameterization (49) is after some algebra transformed into:

$$-2e^2 \int_0^1 dx \int_0^1 dy \int d[l \frac{[(p-l)^2 - l^2 p^2]}{(p-l)^2 - a(x)-(1-x)]p^2 - o(1-x)l^2 - by)^2}.$$

(51)

Matching in (51) two $l$-dependent denominators (using a Feynman variables $z$), making a shift $l = l + pz$ and integrating over the momentum $l$ yields the result:

$$\frac{3p^2 e^2}{(4\pi)^2} \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{\rho_\gamma(p)(p^2 - o(1-z)(1-z)}{[p^2 z(1-z) - axz - o(1-x)z - by(1-z)][p^2 - o]},$$

(52)
which is UV finite by construction.

It is now easy to write down the DR following from (52) (Some details of its derivation are given in the Appendix C). The 'dominant' part following from the pure pole $\beta = 0$ of the photon propagator reads explicitly:

$$\Im M_{\text{pole}}\text{rem}(\omega) = \frac{3e^2}{2(4\pi)^2} \frac{1}{u - u + a} \left[ \frac{u + a}{\omega^2} \sigma_v(a) \rho_S(u) \Theta \left( \omega - \frac{u + a}{2} \right) + \frac{\omega + a}{u^2} \sigma_v(a) \rho_S(\omega) \Theta \left( u - \frac{\omega + a}{2} \right) \right].$$

(53)

To sum it up, the dynamical fermion mass is given by:

$$M(p^2) = m(\mu) + M_{(\gamma)}(p^2) + M_{\text{rem}}(p^2).$$

(54)

Anticipating our numerical results: since the whole $M_{\text{rem}}$ changes the numerical results only slightly (as compared to $M_{(\gamma)}$), we are approximating its imaginary part in our numerics just by the pole contribution $M_{\text{pole}}^{\text{rem}}$.

5 Numerical solutions and results

First, let us describe some technical points of our numerical treatment. First, consider the spectral approach which is simpler from the numerical point of view. After the formal derivation of the DRs for electron self-energy and vacuum polarization function we introduce the positive cut-off $\Lambda$ in the following way

$$F(s) = \int_0^{\Lambda^2} dx \frac{s\rho(s)}{x(s - x + i\epsilon)},$$

(55)

where the function $F$ represents $\Pi$ or $\Sigma$. That is, the absorptive parts of proper function $\pi \rho(s)$ is modified by a step function $\rho(s) \rightarrow \rho(s)\Theta(\Lambda^2 - s)$. The same cut-off is then formally introduced into the Lehmann representation for propagators. With the cut-off implemented the set of equations that have been numerically solved comprise: the coupled nonlinear integral unitary equations (19),(21) for photon and electron Lehmann functions, the equation for absorptive part of self-energies (44,48,74) and necessary conditions (23,24). This set of equations is solved by iterations. Then the various Green’s functions are calculated through the appropriate DRs and considered to be physical for $|p^2| < \Lambda^2$.

We have found that no other approximation is necessary and the unitary form of DSEs converges under the iteration procedure, no matter whether the bare or BC vertex is used. This procedure naturally fails when the employed cut-off is rather close to the expected singularity of the running constant. Before this numerics fails one observes only a trace of this singularity – the large growth of the effective charge at $p^2$ close to $\Lambda^2$ Fig. 1). Then the dynamical mass appears to be negative at large value of timelike momenta (see the curve for $\Lambda^2 = 10^7$ and $\alpha(0) = 0.4$ in Fig. 4).

In our Euclidean treatment we adopted the simplest cut-off functions: The Heaviside step function $\Theta(-k^2_E + \Lambda^2)$ has been introduced into the kernel of DSEs (12),(13), i.e., upper bound of integrals.
is replaced $\infty \rightarrow \Lambda_{E}^{2}$. The value of the cut-off $\Lambda_{E} = 10^{7}M^{2}(0)$ is taken to be exactly the same as in the spectral technique described above, where the zero momentum electron mass $M(0)$ is used as a scale. After the subtraction the DSEs for renormalized photon polarization $\Pi$ and dynamical mass $M$ have been solved on suitable grid $p_{i}^{2} \in (0, \Lambda^{2})$. For the bare vertex the equations can be solved by iteration without any numerical problem. The form of the BC vertex makes the numerical procedure more difficult, even in the quenched approximation. For the unquenched solution this is even more troublesome due to uncontrolled oscillations of the numerical iterations of the running charge in the infrared region. Hence we approximate the finite difference in (12),(13) by the appropriate differentiation- (the similar trick was used in the paper [3]). Explicitly, we replace:

$$\frac{(M(y) - M(z))}{y - z} \rightarrow M'(y).$$

(56)

After making this approximation in the Euclidean DSEs with BC vertex we were able to find the numerical solution for the coupling up to the half of the critical coupling for the bare vertex approximation.

The DSEs has to be renormalized with the help of subtraction (both in bare vertex approximation and the BC vertex, modified as discussed above). Since in this case the subtraction cannot be done analytically (unlike in the Minkowski treatment), one has to implement it numerically, which is not straightforward as we describe below. The following expression for polarization function

$$\Pi(x) = \int dx f(x,y)/x,$$

(57)

need to be renormalized so that $\Pi_{R}(0) = 0$ (for the explicit form of the function $f$ see (13)). Doing this numerically with reasonably high accuracy is not as simple task as for the spectral approach or in a perturbation theory. We first solved the equation for the quantity

$$\tilde{\Pi}(x) = \int dx \left[ f(x,y) - f(0,y) \right]/x.$$

(58)

After finding the solution for $\tilde{\Pi}$ we looked the limit $x \rightarrow 0$ at which $\tilde{\Pi}(0) = K$ and subtracted this constant in order to obtain 'right' $\Pi_{R}(x) = \tilde{\Pi}(x) - K$. When one step is not sufficient we repeated the procedure. Without this we were not able to find the Euclidean solution an accuracy comparable with our Minkowski technique. For instance, for $\alpha = 0.2$ making the subtraction directly for (57) leads to 35 per cent underestimate of $\alpha(p^{2})$ in the infrared region and 50 per cent underestimate in the "ultraviolet" region. Compared to this, five iterations of the procedure described above gives a satisfying with 0.2 per cent deviation in the infrared, the subtraction constants in successive interpolations are $K_{i} = +0.35, -0.12, +0.03, -0.008, +0.002$. Furthermore, we use the linear interpolation and log (perturbative) extrapolation to evaluate functions $\Pi(z)$ and $M(z)$ at $z = x + y - 2xy \cos \theta$ in Eqs (12),(13).

In the both formalisms we use one common renormalization scale ($\mu^{2} = 0$) to define the running coupling

$$\alpha(p^{2}) = \frac{\alpha}{1 - \Pi_{R}(p^{2})},$$

15
\[ \alpha = \frac{e^2_R}{(4\pi)} , \]  

where we have explicitly used \( \alpha = \alpha(0) \) and we omit explicit dependence on \( \mu \) in R-label quantities for purpose of brevity. We use the same scale in order to renormalize the electron mass. As a mass scale of the theory we use \( M^2(p^2 = 0) = 1 \) in arbitrary units. The momentum axis at all figures defined in this unit.

When the coupling \( \alpha \) increases the pole mass and the residuum of fermion propagator become different from their non-interacting values \( (r = 1; M_p = M(0)) \). Couple of values of \( r \) and \( M_p \) following from our solutions of DSEs with bare and BC vertex are shown in the Tab.1. The residuum \( r \) is clearly renormalization (and gauge fixing) scheme dependent quantity. However on the physical ground one can expect that 'one particle' contribution to an interacting particle propagator is less than one, i.e. we would have naively \( r < 1 \), here the values of residua are greater then 1 which is the consequence of our subtraction scheme. Not surprisingly, if the similar scheme and the same gauge are employed than the property \( r > 1 \) survive in the case of quark propagator too [1].

| \( \alpha \) | \( 0.1 \) | \( 0.2 \) | \( 0.4 \) |
|-------------|--------|--------|--------|
| \( M_p/M(0) \)-BC | 1.044  | 1.10   | 1.39   |
| \( M_p/M(0) \)-BV  | 1.042  | 1.09   | 1.23   |
| \( r \)-BC        | 1.090  | 1.22   | 1.98   |
| \( r \)-BV        | 1.085  | 1.19   | 1.53   |

Tab.1 Pole mass and residua of pole part of the electron propagator. The label BC(BV) means the results calculated with BC (bare) vertices. The coupling \( \alpha \) is the value of running charge at zero momenta.

Let us finally compare our numerical results obtained in the both formalisms with bare or BC vertices. The so-called photon renormalization functions \( G \) (it is defined by \( G = \alpha(p^2)/\alpha \)) are compared for spacelike momenta in Fig. 1. One sees excellent agreement between the solutions obtained in spectral and Euclidean formalism. The one-loop perturbation theory (PT) result is added for comparison. The correct pole mass \( M_p \) is used in perturbative formulas. The PT results always below the lines corresponding to the DSEs solutions. This can be easily read from the bar vertex form of the Euclidean DSE: the decreasing \( M(x) \) enhances the function \(|\Pi|\), which being negative must enhance \( \alpha(x) \). The same functions \( G \) are displayed for timelike momenta in Fig. 2., where only the results obtained from the unitary equations are presented. Again we would like to stress that the differences between the solutions with bare and BC vertices are very small.

The expected exceptions are solutions with the value coupling constant close to the one for which the numerical solutions fail, that is \( \alpha_c \approx 0.41 \) and corresponding \( G(p^2 = -\Lambda^2) \approx \text{Re}G(p^2 = \Lambda^2) \approx 2.6 \). The dynamical fermion mass obtained from DSEs is displayed for spacelike regime of momenta in Fig. 3. The small deviation of Euclidean results from the spectral ones can be explained as a numerical error. The Fig. 4 shows the difference that follows from the use of different vertices. The absorptive parts of
$M$ for bare vertex approximation are also displayed in this Fig. for the same coupling $\alpha = 0.1, 0.2, 0.4$. The negative damp of $\Re M$ is observed for $\alpha = 0.4$. At the end we should remind the reader that the bare vertex solution was already obtain by Rakow in the paper [20]. The aforementioned approximative independence on the cutoff value was explicitly shown in this work. In order to check the consistence we took the cutoff to be as in the paper [20], for instance: $M(0)/\Lambda = 10^{-3}$ together with increasing of the renormalized coupling as $\alpha = 0.5$ then we the agreement between us and the Rakow solution was found.

6 Conclusion

The Dyson-Schwinger equations for strong coupling QED were solved in the truncation which respects the gauge identity. It is the first time when the Minkowski and the Euclidean solutions for lowest QED Green functions were made. Working in the Landau gauge we have found a good numerical agreement between the solutions obtained in these two technically different frameworks. From this we argue that the electron as well as the photon propagator posses the standard textbook spectral (Lehmann) representations. We showed that at least up to the certain renormalized coupling the proper Green functions - the photon polarization function as well as the electron selfenergy- satisfy appropriately subtracted dispersion relations. The form of them corresponds with the results already known from the conventional perturbation theory. In the other words, there is no significant signal for complex singularity of the propagators, which possibility is sometimes sugested in the literature.

In the other side, the small difference between the Minkowski and the Euclidean solutions cannot fully excluded such situation, but from the smallness of observed we can speculate that this effect must be rather negligible even for rather strong coupling case $\alpha(p^2 = 0; \Lambda = 10^4m) \approx 0.4$.

Triviality of QED was confirmed in our approach. We did not find the possibility to send the appropriate ultraviolet cutoff $\Lambda$ to infinity with simultaneous keeping the renormalized coupling non-zero. With increasing $\Lambda$ and with the non-zero bare electron mass we cannot observe a second order chiral phase transition as in the paper [20] but instead of this we observe the appearance of Landau singularity in the running charge. The obtained results were also compared with bare vertex approximation. In that case only the small violation from our 'gauge covariant' solution was observed. This difference is still small even for rather strong coupling case $\alpha(p^2 = 0; \Lambda = 10^4m) \approx 0.5$. The obtained solutions reduce to their perturbative counterpartners in small coupling limit. Furthermore, the explicit comparison with the one loop perturbation theory was made. Note, that the triviality statement is confirmed in these two later cases too.

There is still missing link between theory with $m_0 = 0$ (in chiral symmetry phase as well as in chiral symmetry broken phase) and our dispersion technique in used. Nevertheless, we believe that the possibility to extract the information about the timelike behavior of Green functions from spectral approach remains the main attractive feature when compared with usual Euclidean approach. After a certain automatization of the dispersion relations evaluation the method should be extend-able to
a more complex theories. The certain progress was already achieved in QCD and the results will be published elsewhere. There is also a broad scope for possible future investigations in pure QED: study of S–matrix property when it is composed from dressed Green-functions, the study of bound states with dressed propagators, including transverse correction to the vertex, etc.

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A Assorted Integrals

In this Appendix we list several useful relation. The following integral has been used many times in the last step of derivations of DRs:

\[ X_n(\omega, a, b) = \int_0^1 \Theta(\omega - a/(1-x) - b/x) x^n dx, \]  

(60)

where \( a, b \) are positive real numbers. For the several lowest \( n \) it equals:

\[ X_0(\omega, a, b) = \Delta^{1/2}(\omega, a, b) \Theta(\omega - (\sqrt{a} + \sqrt{b})^2), \]

\[ X_1(\omega, a, b) = \Delta^{1/2}(\omega, a, b)[\omega + b - a] \frac{1}{2\omega^2} \Theta(\omega - (\sqrt{a} + \sqrt{b})^2), \]

\[ X_2(\omega, a, b) = \Delta^{1/2}(\omega, a, b)[(\omega + b - a)^2 - \omega b] \frac{1}{3\omega^3} \Theta(\omega - (\sqrt{a} + \sqrt{b})^2), \]

\[ X_3(\omega, a, b) = \Delta^{1/2}(\omega, a, b)[(\omega + b - a)^2 - 2\omega b] \frac{2}{4\omega^4} \Theta(\omega - (\sqrt{a} + \sqrt{b})^2), \]

(61)

where \( \Delta \) is the triangle function (40). The variable \( x \) in (60) appears from the Feynman parametrization of products of the inverse scalar propagators \( D_{1,2}^{-\alpha} \):

\[ D_1^{-\alpha} D_2^{-\beta} = \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1} \Gamma(\alpha + \beta)}{[D_1 x + D_2(1-x)]^{(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)}} \]  

(62)

or from the difference of the propagators \( D_{1,2}^{-\alpha} \)

\[ D_1^{-\alpha} - D_2^{-\alpha} = \int_0^1 dx \frac{(D_2 - D_1)\alpha}{[D_1 x + D_2(1-x)]^{(\alpha+1)}}. \]  

(63)

B Derivation of \( \Pi_{R(rem)} \)

In this Appendix we derive DR for the function \( \Pi_{U(rem)} \). To this end we formally interchange the labeling of the variables \( a \leftrightarrow b \) in the last term of Eq. (42). Further we substitute \( l \to -l + q \) which yields

\[ \Pi_{U(rem)}(q^2) = \frac{4e^2}{3q^2} \int d^4l \frac{\sigma_v(a)\sigma_s(b)\rho_S(c)}{[(l-q)^2 - a][l^2 - b][(l-q)^2 - c][l^2 - c]} \left[ 4l^2 - 16\frac{(l \cdot q)^2}{q^2} + 20l \cdot q - 6q^2 \right], \]

(64)
where we have used the dispersion relation formulas for $M$ (16) in order to evaluate their shifted argument difference:

$$
\frac{M(l) - M(l - q)}{l^2 - (l - q)^2} = \int dc \frac{\rho_S(c)}{[(l - q)^2 - c][l^2 - c]},
$$

(65)

which is invariant under the shift $l \rightarrow -l + q$.

Next it is convenient to rewrite the product of four denominators in (64) making use of

$$
\left\{[(l - q)^2 - a][l^2 - b][(l - q)^2 - c][l^2 - c]\right\}^{-1} = \frac{I(q; c, c) + I(q; a, b) - I(q; a, c) - I(q; c, b)}{(c - b)(c - a)},
$$

$$
I(q; a, b) = \left\{[(l - q)^2 - a][l^2 - b]\right\}^{-1}.
$$

(66)

It is sufficient to deal only with one term on rhs. of (66), the others are obtained simple by changing the spectral variables (the logarithmic divergence appears but it cancels against the same contribution of three remaining terms). For instance choosing the variable $a, c$ and making a shift $x \rightarrow 1 - x$ leads after the subtraction to:

$$
\frac{4e^2}{3(4\pi)^2} \int_0^1 dx \int_{\frac{x}{x + a(1 - x)}}^{\infty} d\omega \frac{\sigma_v(a)\sigma_s(b)(2 + 4x - 12x^2)(q^2 - \mu^2)}{(\omega - \mu^2)(q^2 - \omega)(c - b)(c - a)}.
$$

(67)

Integrating over the Feynman variable $x$ yields

$$
\frac{4e^2}{3(4\pi)^2} \int_0^1 dx \int_{\frac{x}{x + a(1 - x)}}^{\infty} d\omega \frac{\sigma_v(a)\sigma_s(b)(2X_0 + 4X_1 - 12X_2)(q^2 - \mu^2)}{(\omega - \mu^2)(q^2 - \omega)(c - b)(c - a)},
$$

(68)

where $X_n$ is the shorthand notation for the function $X_n(\omega, c, a)$ introduced in the Appendix A. Gathering all expressions together one gets the dispersion relation for the polarization function $\Pi_{R(rem)}$:

$$
\Pi_{R(rem)}(q^2, \mu^2) = \frac{8e^2}{3(4\pi)^2} \int_{m^2}^{\infty} d\omega \frac{\sigma_v(a)\sigma_s(b)(q^2 - \mu^2)}{(\omega - \mu^2)(q^2 - \omega)} \left\{ F(\omega, c, c) - F(\omega, c, a) + F(\omega, b, a) - F(\omega, b, c) \right\},
$$

$$
F(\omega, c, a) = \frac{\Delta^{1/2}(\omega, c, a)}{\omega} \left[ a + c - \frac{a - c}{\omega} \right] \Theta \left( \omega - (\sqrt{a} + \sqrt{c})^2 \right).
$$

(69)

### C Dispersion relation for $M_{(rem)}$

In this Appendix we derive the absorptive part of Eq. (53). We start from the relation (52) and consider the dominant contribution that following from the pole of photon propagator, i.e. $\sigma_\gamma(b) = r_\gamma \delta(\beta)$. In addition we make the substitution $\Omega = \frac{ax + o(1 - x)}{1 - x}$ which leads to the following double dispersion integral:

$$
\frac{3p^2e^2}{(4\pi)^2} \int_0^1 dx \int_{ax + o(1 - x)}^{\infty} d\Omega \frac{ax + o(1 - x)}{\Omega^2(p^2 - \Omega)} \frac{\sigma_v(a)\rho_S(a)}{p^2 - o}
$$

(70)

$$
\approx \frac{3p^2e^2}{2(4\pi)^2} \int_{ax + o(1 - x)}^{\infty} d\Omega \frac{a + o}{\Omega^2(p^2 - \Omega)} \frac{\sigma_v(a)\rho_S(o)}{p^2 - o}
$$

(71)
The last step is to use algebraic identity

\[
\frac{1}{(p^2 - \Omega)(p^2 - o)} = \frac{1}{\Omega - o} \left\{ \frac{1}{p^2 - \Omega} - \frac{1}{p^2 - o} \right\}
\]  

(72)

and re-write the double DR as the difference of single DRs. Substituting (72) into (71) we relabel \( \Omega \rightarrow \omega, o \rightarrow u \) in the first term and \( \Omega \rightarrow u, o \rightarrow \omega \) in the second one. This cosmetics leads to the unsubtracted DR:

\[
\frac{3p^2 e^2}{2(4\pi)^2} \left\{ P \cdot \int_{m^2}^{\infty} du \int_{\frac{u-a}{u}}^{\infty} d\omega \left[ \frac{(a + u)\sigma_v(a)\rho_S(u)}{\omega^2(\omega - u)(p^2 - \omega)} \right] + P \cdot \int_{m^2}^{\infty} d\omega \int_{\frac{\omega-a}{u}}^{\infty} \frac{d\omega}{u^2(\omega - u)(p^2 - \omega)} \right\} \right. .
\]  

(73)

Taking a subtraction at the point \( \mu \) we get for \( M_{(rem)}(p^2) \) once subtracted DR with the weight function \( \rho_{(rem)} \):

\[
\rho_{(rem)}(\omega) = \frac{3e^2}{2(4\pi)^2} P \cdot \int_{m^2}^{\infty} \frac{1}{\omega - u} \left[ \frac{\omega + a}{\omega^2} \sigma_v(a)\rho_S(u)\Theta \left( \omega - \frac{u + a}{2} \right) + \frac{\omega + a}{u^2} \sigma_v(a)\rho_S(\omega)\Theta \left( u - \frac{\omega + a}{2} \right) \right] .
\]  

(74)

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[2] in this paper we use the Minkowski metric \( g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \), and hence \( p_E^2 = -p^2 \) for negative (spacelike) \( p^2 \).

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Figure 1: Charge renormalization function $G = \alpha(p^2)/\alpha(0)$ obtained by solutions of DSEs. Each beam of lines is labeled by the corresponding coupling $\alpha(0)$. The results of leading order perturbation theory (dotted lines) always lie below the DSE result. Only one solution with the BC is shown in the figure (for $\alpha = 0.4$), for smaller coupling the BC solutions would be indistinguishable from the bare vertex ones.

Figure 2: Spectral solutions for charge renormalization function $G$ for timelike momenta and for couplings $\alpha = 0.1, 0.2, 0.4$. The down oriented peaks correspond with the threshold $4M_p^2$. 
Figure 3: The comparison of dynamical mass $M(p^2)$ obtained in Euclidean and Minkowski formalism for spacelike momenta. For comparison we added also one Euclidean solution calculated with the BC vertex (solved with approximations described in the section IV).

Figure 4: The absolute values of real and imaginary parts of the electron dynamical mass as obtained by solving the unitary equations. The position of the up oriented peaks correspond with the pole mass: $[M_p, M_p]$. The excess for the solution with $\alpha = 0.4$ is shown, the real part of $M$ becomes negative at large momenta.