MIXED FRACTIONAL BROWNIAN MOTION: A SPECTRAL TAKE

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Abstract. This paper provides yet another look at the mixed fractional Brownian motion (fBm), this time, from the spectral perspective. We derive an approximation for the eigenvalues of its covariance operator, asymptotically accurate up to the second order. This in turn allows to compute the exact $L_2$-small ball probabilities, previously known only at logarithmic precision. The obtained expressions show an interesting stratification of scales, which occurs at certain values of the Hurst parameter of the fractional component. Some of them have been previously encountered in other problems involving such mixtures.

1. Introduction

Mixtures of stochastic processes can often have properties, different from the individual components. In this paper we revisit the mixed fractional Brownian motion (fBm)

\[ \tilde{B}_t = B_t + B^H_t, \quad t \in [0,1] \] 

(1.1)

where $B_t$ and $B^H_t$ are independent standard and fractional Brownian motions, respectively. The latter is the centred Gaussian process with the covariance function

\[ \mathbb{E}B^H_t B^H_s = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right), \]

where $H \in (0,1)$ is a parameter, called the Hurst index. Introduced in [15], [20], nowadays the fBm takes central place in the study of heavy tailed distributions, self-similarity and long range dependence phenomenon [12], [24].

The mixture (1.1) drew considerable attention, after some of its interesting properties have been discovered in [6] (see also [1], [5]) and proved useful in mathematical finance [7], [2] and statistical inference [8], [10]. It turns out that the process $\tilde{B}$ is a semimartingale if and only if $H > \frac{3}{4}$, in which case it is measure equivalent to $B$; for $H < \frac{1}{4}$ the mixed fBm is measure equivalent to $B^H$, see [28].

Another interesting feature is revealed by its canonical representation from [5], based on the martingale

\[ M_t = \mathbb{E}(B_t | \mathcal{F}_t) = \int_0^t g(s,t) d\tilde{B}_s, \quad t > 0, \]
where the kernel \( g(s,t) \) is obtained by solving certain Wiener-Hopf equation. The limiting performance of statistical procedures in models involving mixed fBm is governed by the asymptotic behaviour of this equation as \( t \to \infty \), see [8]. For \( H > \frac{1}{2} \), under reparametrization \( \varepsilon := t^{1-2H} \) it reduces to the singularly perturbed problem

\[
\varepsilon g_{\varepsilon}(x) + \int_0^1 g_{\varepsilon}(y)c_H|x-y|^{2H-2}dy = 1, \quad 0 \leq x \leq 1.
\]  

As \( \varepsilon \to 0 \) its solution \( g_{\varepsilon}(x) \) converges to the solution \( g_0(x) \) of the limit equation, obtained by setting \( \varepsilon := 0 \) in (1.2), with the following rate with respect to \( L_2 \)-norm, see [9],

\[
\|g_{\varepsilon} - g_0\|_2 \sim \begin{cases} 
\varepsilon^{\frac{1}{2H-1}} & H \in \left(\frac{2}{3}, 1\right) \\
\varepsilon^{\sqrt{\log \varepsilon^{-1}}} & H = \frac{2}{3} \\
\varepsilon & H \in \left(\frac{1}{2}, \frac{2}{3}\right)
\end{cases}
\]

Here the convergence rate breaks down at the critical point \( H = \frac{2}{3} \).

The purpose of this paper is to demonstrate that the mixed fBm is also an interesting process from the spectral standpoint. Using the results from the general operator theory, see [22, Proposition 2.3], the first order asymptotics of the eigenvalues of its covariance operator coincides with that of its “slowest” component, while the other component acts as a perturbation. More precisely, for \( H > \frac{1}{2} \) its eigenvalues agree, to the first asymptotic order, with the standard Brownian motion, and, for \( H < \frac{1}{2} \), with the fBm. It can hardly be expected that such separation is preserved on a finer asymptotic level; however, our main result shows that the two components do remain separated, albeit in a somewhat unexpected way, see Theorem 2.1 and Remark 2.2 below.

As an application of the spectral approximation derived in this paper, we consider the \( L_2 \)-small ball probabilities problem of evaluating \( \mathbb{P}(\|\tilde{B}\|_2 \leq \varepsilon) \) as \( \varepsilon \to 0 \). We show that it exhibits a curious stratification of scales, which occurs at the following values of \( H \) (see Theorem 2.4):

\[
\frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \ldots, \frac{2}{3}, \frac{3}{8}, \frac{3}{4}, \frac{5}{8}, \ldots
\]

where the values, mentioned above, are emphasized in bold.

2. MAIN RESULTS

Spectral theory of stochastic processes is a classical theme in probability and analysis. For a centred process \( X = (X_t, t \in [0,T]) \) with covariance \( \mathbb{E}X_sX_t = K(s,t) \), the eigenproblem is to find the nontrivial solutions \( (\lambda, \varphi) \) to the equation

\[
K\varphi - \lambda \varphi = 0,
\]  

where \( K \) is the integral covariance operator

\[
(K\varphi)(t) = \int_0^T K(s,t)\varphi(s)ds.
\]

For sufficiently regular kernels this problem has countable solutions \( (\lambda_n, \varphi_n)_{n \in \mathbb{N}} \). The ordered sequence of the eigenvalues \( \lambda_n \in \mathbb{R}_+ \) converges to zero and the eigenfunctions \( \varphi_n \)
form an orthonormal basis in $L^2([0,T])$. The Karhunen-Loève theorem asserts that $X$ can be expanded into series of the eigenfunctions

$$X_t = \sum_{n=1}^{\infty} Z_n \sqrt{\lambda_n} \varphi_n(t)$$

where $Z_n$'s are uncorrelated zero mean random variables with unit variance. This decomposition is useful in many applications, if the eigenvalues and the eigenfunctions can be found in a closed form or, at least, approximated to a sufficient degree of accuracy, see e.g. [24]. There are only a few processes, however, for which the eigenproblem can be solved explicitly.

2.1. Eigenvalues of the mixed fBm. One such process is the Brownian motion, for which a simple exact formula is long known:

$$\lambda_n = \frac{1}{(n - \frac{1}{2})^2 \pi^2} \quad \text{and} \quad \varphi_n(t) = \sqrt{2} \sin\left((n - \frac{1}{2})\pi t\right). \quad (2.2)$$

The eigenproblem for the fBm turns out to be much harder and it is unlikely to have any reasonably explicit solutions. Nevertheless, in this case the eigenvalues admit a fairly precise asymptotic approximation. Namely, the sequence of “frequencies” $\nu_n(H)$, defined by the relation

$$\lambda_n(H) = \frac{\sin(\pi H) \Gamma(2H + 1)}{\nu_n(H)^{2H+1}}$$

has the asymptotics

$$\nu_n(H) = \left(n - \frac{1}{2}\right)\pi - \frac{(H - \frac{1}{2})^2 \pi}{H + \frac{1}{2}} + O(n^{-1}), \quad n \to \infty. \quad (2.4)$$

The leading order term in (2.4) was discovered in [3, 4] and, by different methods, in [22] and [18]; the second term was recently obtained in [9], along with the following approximation for the eigenfunctions

$$\varphi_n(x) = \sqrt{2} \sin\left(\nu_n(H)x + \frac{1}{4} \frac{(H - \frac{1}{2})(H - \frac{3}{2})}{H + \frac{1}{2}}\right)$$

$$- \int_{0}^{\infty} \left(e^{-x\nu_n(H)u} f_0(u) + (-1)^n e^{-(1-x)\nu_n(H)u} f_1(u) \right) du + O(n^{-1}), \quad (2.5)$$

where $f_0(u)$ and $f_1(u)$ are given by closed form formulas and the residual term is uniform over $x \in [0, 1]$.

The following result details the spectral asymptotics of the mixed fBm (1.1):

**Theorem 2.1.** Let $\tilde{\lambda}_n$ be the ordered sequence of eigenvalues of the mixed fBm covariance operator. Then the unique roots $\tilde{\nu}_n$ of the equations

$$\tilde{\lambda}_n = \frac{1}{\nu_n^2} + \frac{\sin(\pi H) \Gamma(2H + 1)}{\nu_n^{2H+1}}, \quad n = 1, 2, \ldots \quad (2.6)$$

satisfy

$$\tilde{\nu}_n := \nu_n\left(\frac{1}{2} \wedge H\right) + O\left(n^{-2H-1}\right), \quad n \to \infty, \quad (2.7)$$
where $\nu_n(\cdot)$ is defined in (2.4).

Remark 2.2. This theorem reveals a curios feature in the spectral structure of mixtures. In the pure fractional case, the second order approximation for the frequencies $\nu_n$ in (2.4) furnishes an approximation for the eigenvalues $\lambda_n$ through (2.3), precise up to the same, second order. In the mixed case, similar approximation for the frequencies $\tilde{\nu}_n$ in (2.7) provides an approximation for the eigenvalues (2.6), accurate up to the fourth order: for example, for $H > \frac{1}{2}$

$$
\tilde{\lambda}_n = a_1(H)n^{-2} + a_2(H)n^{-2H-1} + a_3(H)n^{-3} + a_4(H)n^{-2H-2} + o(n^{-2H-2})
$$

where all the coefficients $a_j(H)$ can be computed exactly.

Remark 2.3. It can be shown that asymptotic behaviour of the eigenfunctions is also dominated by one of the components: for $H > \frac{1}{2}$, the first order asymptotics with respect to the uniform norm coincides with that of the standard Brownian motion (2.2), while for $H < \frac{1}{2}$, it agrees with the asymptotics (2.5) of the fBm.

2.2. The small ball probabilities. The small ball probabilities problem is to find the asymptotics of

$$
\mathbb{P}(\|X\| \leq \varepsilon), \quad \varepsilon \to 0, \quad (2.8)
$$

for a given process $X = (X_t, t \in [0, 1])$ and a norm $\| \cdot \|$. It has been extensively studied in the past and was found to have deep connections to various topics in probability theory and analysis, see [16]. The case of the Gaussian processes and the $L_2$-norm is the simplest, in which asymptotics of (2.8) is determined by the eigenvalues $\lambda_n$ of the covariance operator, [26].

The computations for concrete processes require a closed form formula or at least a sufficiently accurate approximation of the eigenvalues. Typically, the first order approximation of $\lambda_n$’s allows to compute the asymptotics of $\log \mathbb{P}(\|X\|_2 \leq \varepsilon)$ and the second order suffices for finding the asymptotics of $\mathbb{P}(\|X\|_2 \leq \varepsilon)$, exact up to a multiplicative “distortion” constant. For the fBm, formulas (2.3)-(2.4) give

$$
\mathbb{P}(\|B^H\|_2 \leq \varepsilon) \sim \varepsilon^{\beta(H)} \exp\left(-\beta(H)\varepsilon^{-\frac{1}{2H}}\right), \quad \varepsilon \to 0,
$$

where $f(\varepsilon) \sim g(\varepsilon)$ means that $\lim_{\varepsilon \to 0} f(\varepsilon)/g(\varepsilon)$ is finite and nonzero. The exponent $\beta(H)$ was derived first in [3, 4]:

$$
\beta(H) = H \left(\frac{\sin(\pi H)\Gamma(2H+1)}{(2H+1)^{2H+1}\left(\sin\frac{\pi}{2H+1}\right)^{2H+1}}\right)^{\frac{1}{2H}} \quad (2.9)
$$

and the power $\gamma(H)$ was recently found in [9]:

$$
\gamma(H) = \frac{1}{2H} \left(\frac{5}{4} - H + H^2\right). \quad (2.10)
$$

The first order perturbation effect, mentioned in the Introduction, implies that the rough, logarithmic asymptotics of $L_2$-small ball probabilities for the mixed fBm coincides with either standard or fractional parts [22] (see also [23],[19]):

$$
\log \mathbb{P}(\|\tilde{B}\|_2 \leq \varepsilon) \simeq \log \mathbb{P}(\|B^{H\wedge \frac{1}{2}}\|_2 \leq \varepsilon), \quad \varepsilon \to 0, \quad (2.11)
$$
where \( f(\varepsilon) \simeq g(\varepsilon) \) means that for \( \lim_{\varepsilon \to 0} f(\varepsilon)/g(\varepsilon) = 1 \). The following theorem shows that the exact asymptotics of the mixed fBm is more intricate than could have been expected in view of (2.11):

**Theorem 2.4.** For \( H \in (0, 1) \setminus \{1/2\} \)

\[
\mathbb{P}(\|\tilde{B}\|_2 \leq \varepsilon) \sim \varepsilon^{\gamma(H)\vee 1} \exp \left( - \sum_{k=0}^{\lfloor 1/(2H-1) \rfloor} \beta_k(H) \varepsilon^{k/(2H-1)} \right), \quad \varepsilon \to 0,
\]

(2.12)

where \( \gamma(H) \) is given by (2.10) and \( \beta_k(H) \) are positive constants, defined in Propositions 4.2 and 4.4 below.

A similar type of asymptotics, where additional terms join the sum in the exponent at certain values of a parameter, has been recently observed in [25] for certain Gaussian random fields.

### 3. Proof of Theorem 2.1

The proof uses the program of the spectral analysis for covariance operators, developed in [9], [27]. It is based on the reduction of the eigenproblem to finding functions \((\Phi_0, \Phi_1)\), sectionally holomorphic on \( \mathbb{C} \setminus \mathbb{R}_+ \), which satisfy

(a) *a priori* growth estimates at the origin;
(b) constraints on their values at certain points on the imaginary axis;
(c) boundary conditions on the real semi-axis \( \mathbb{R}_+ \);

and behave as polynomials at infinity. The coefficients of these polynomials are also determined by condition (b). The asymptotics in the spectral problem is derived by analysis of these two functions.

Implementation of this program uses the technique of solving the Riemann boundary value problem, see [13]. We will detail its main steps, referring the reader to the relevant parts in [9], whenever calculations are similar.

#### 3.1. An equivalent generalized spectral problem.

Our starting point is to rewrite (2.1) with the kernel

\[
\tilde{K}(s, t) = s \wedge t + \frac{1}{2} \left( s^{2-\alpha} + t^{2-\alpha} - |t - s|^{2-\alpha} \right), \quad s, t \in [0, 1],
\]

(3.1)

where we defined \( \alpha := 2 - 2H \in (0, 2) \), as the generalized spectral eigenproblem

\[
(1 - \frac{d}{dx}) \int_0^1 |x - y|^{1-\alpha} \text{sign}(x - y) \psi(y) dy = -\lambda \psi''(x) - \psi(x),
\]

\[
\psi(1) = 0, \quad \psi'(0) = 0,
\]

(3.2)

for \( \psi(x) := \int_x^1 \varphi(y) dy \). The advantage of looking at the problem in this form is that it involves a simpler *difference* kernel. The proof of (3.2) amounts to taking derivatives of (2.1) and rearranging, see [9, Lemma 5.1].
3.2. **The Laplace transform.** The principal stage of the proof is to derive an expression for the Laplace transform of a solution $\psi$ to (3.2):

$$\hat{\psi}(z) = \int_0^1 e^{-zx} \psi(x) dx, \quad z \in \mathbb{C},$$

which has removable singularities:

**Lemma 3.1.** Define the structural function of the problem (3.2)

$$\Lambda(z) = \frac{\Gamma(\alpha)}{|c_\alpha|} \left( \lambda + \frac{1}{z^2} + \kappa_\alpha z^{\alpha-3} e^{\frac{1-\alpha}{2}\pi i} \right), \quad z \in \mathbb{C} \setminus \mathbb{R}$$

where $c_\alpha = (1 - \frac{\alpha}{2})(1 - \alpha)$ and $\kappa_\alpha := \frac{c_\alpha}{\Gamma(\alpha)} \frac{\pi}{\cos \frac{\pi}{2} \alpha}$ and the signs correspond to $\text{Im}\{z\} > 0$ and $\text{Im}\{z\} < 0$ respectively. Then the Laplace transform can be expressed as

$$\hat{\psi}(z) = \Phi_0(z) + e^{-z}\Phi_1(-z)$$

where $(\Phi_0, \Phi_1)$ are functions, sectionally holomorphic on $\mathbb{C} \setminus \mathbb{R}_+$, such that

$$\Phi_j(z) = \begin{cases} O(z^{\alpha-1}) & \alpha < 1 \\ O(1) & \alpha > 1 \end{cases} \quad \text{as } z \to 0 \quad \text{for } j = 0, 1 \quad (3.3)$$

and

$$\Phi_0(z) = -2C_2 z + \begin{cases} O(z^{-1}) & \alpha < 1 \\ O(z^{\alpha-2}) & \alpha > 1 \end{cases} \quad \text{as } z \to \infty \quad (3.4)$$

where $C_1$ and $C_2$ are constants.

The proof of this lemma is close to [9, Lemma 5.1].

3.3. **Reduction to integro-algebraic system.** The function $\Lambda(z)$ has zeros at $\pm i\nu$ where $\nu \in \mathbb{R}_+$ solves the equation

$$\lambda = \nu^{-2} + \kappa_\alpha \nu^{\alpha-3}. \quad (3.5)$$

This defines the one-to-one correspondence between $\lambda$ and $\nu$, cf. (2.6), with $\nu$ playing the role of a large parameter in what follows. Since $\Lambda(\pm i\nu) = 0$ and $\hat{\psi}(z)$ must be analytic, we obtain the algebraic condition

$$e^{-i\nu}\Phi_1(-i\nu) + \Phi_0(i\nu) = 0. \quad (3.6)$$

Also $\Lambda(z)$ is discontinuous on $\mathbb{R}_+$. Therefore continuity of $\hat{\psi}(z)$ on $\mathbb{C}$ gives the boundary conditions, which bind together the limits $\Phi_j^\pm := \lim_{z \to t^\pm} \Phi_j(z)$ as $z$ tends to $t \in \mathbb{R}$ in the lower and upper half planes. In the vector form (see [9, Section 5.1.2 ])

$$\Phi^+(t) - e^{i\theta(t)}\Phi^-(t) = 2i \sin \theta(t) e^{-t} J\Phi(-t), \quad t \in \mathbb{R}_+, \quad (3.7)$$
where \( J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( \theta(t) = \arg\{\Lambda^+(t)\} \) with the \( \arg\{\cdot\} \) branch chosen so that \( \theta(t) \) is continuous and vanishes as \( t \to \infty \), that is

\[
\theta(t) = \arctan \frac{\sin \frac{4\alpha}{\pi}}{\kappa^{1-\alpha} \nu^{1-\alpha} \left( (t/\nu)^{3-\alpha} + (t/\nu)^{1-\alpha} \right) + (t/\nu)^{3-\alpha} + \cos \frac{4\alpha}{\pi}}, \quad t > 0. \tag{3.8}
\]

Now, by Lemma 3.1, the problem (2.1) reduces to finding sectionally holomorphic functions \((\Phi_0, \Phi_1)\), growing as in (3.3)-(3.4), which comply with constraint (3.6) and satisfy the boundary condition (3.7).

In general such Riemann problem for a pair of functions may not have an explicit solution, but the system (3.7) can be decoupled, see [9, eq. (5.35)] and consequently \( \Phi_0(z) \) and \( \Phi_1(z) \), satisfying the conditions (3.3)-(3.4), can be expressed in terms of solutions to certain integral equations, see (3.10) below. More precisely, define sectionally holomorphic function

\[
X(z) = \exp\left( \frac{1}{\pi} \int_0^\infty \frac{\theta(\tau)}{\tau - z} d\tau \right), \tag{3.9}
\]

and the real valued function

\[
h(t) = e^{i\theta(t)} \sin \theta(t) X(-t)/X^+(t), \quad t \in \mathbb{R}_+.
\]

**Lemma 3.2.**

1. For all \( \nu > 0 \) large enough, the integral equations

\[
q_\pm(t) = \pm \frac{1}{\pi} \int_0^\infty \frac{h(\nu \tau) e^{-\nu \tau}}{\tau + t} q_\pm(\tau) d\tau + t, \quad t > 0, \tag{3.10}
\]

\[
p_\pm(t) = \pm \frac{1}{\pi} \int_0^\infty \frac{h(\nu \tau) e^{-\nu \tau}}{\tau + t} p_\pm(\tau) d\tau + 1
\]

have unique solution such that \( p_\pm(t) - 1 \) and \( q_\pm(t) - t \) are square integrable on \( \mathbb{R}_+ \).

2. The solutions to the Riemann problem with boundary conditions (3.7) and growth as in (3.3)-(3.4) has the form

\[
\Phi(z) = X(z) A(z/\nu) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \tag{3.11}
\]

where

\[
A(z) = \begin{pmatrix} -a_-(z) & b_+(\nu) a_+(z) - \nu b_+(z) \\ -a_+(z) & b_+(\nu) a_-(z) - \nu b_-(z) \end{pmatrix} \quad \text{with} \quad a_\pm(z) = p_+(z) \pm p_-(z) \quad b_\pm(z) = q_+(z) \pm q_-(z) \tag{3.12}
\]

and

\[
b_\alpha(\nu) := \frac{1}{\pi} \int_0^\infty \theta(\nu \tau) d\tau. \tag{3.13}
\]

This lemma is checked exactly as in [9], see calculations preceding (5.39) therein. At this stage, the eigenproblem (2.1) is equivalent to finding all \( \nu > 0 \), for which there exists
a nonzero vector \(
\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}
\) satisfying (3.6) with \( \Phi \) defined by (3.11). Obviously, the condition (3.6) reads
\[
e^{i\nu/2}X(i\nu)(-a_-(i)C_1 + (b_\alpha(\nu)a_+(i) - b_-(i))C_2) + 
\]
\[
e^{-i\nu/2}X(-i\nu)(-a_+(i)C_1 + (b_\alpha(\nu)a_-(i) - b_-(i))C_2) = 0,
\]
or, after a rearrangement, \( \eta(\nu)C_1 + \xi(\nu)C_2 = 0 \), where
\[
\xi(\nu) := e^{i\nu/2}X(\nu i)(b_\alpha(\nu)a_+(i) - b_-(i)) + e^{-i\nu/2}X(-\nu i)(b_\alpha(\nu)a_-(i))
\]
\[
\eta(\nu) := e^{i\nu/2}X(\nu i)a_-(i) + e^{-i\nu/2}X(-\nu i)a_+(i).
\]
Since \( C_1 \) and \( C_2 \) are real valued, nontrivial solutions are possible if and only if
\[
\eta(\nu)\overline{\xi(\nu)} - \overline{\eta(\nu)}\xi(\nu) = 0. \tag{3.15}
\]
To recap, any solution \( (\nu, p_\pm, q_\pm) \) to the system of algebraic and integral equations (3.15) and (3.10), can be used to construct a triplet \( (\nu, \Phi_0, \Phi_1) \), mentioned above, and consequently, a solution \( (\lambda, \varphi) \) to the eigenproblem (2.1). In particular, all eigenvalues can be found by solving this equivalent problem.

3.4. Asymptotic analysis. While the equivalent problem, presented in the previous section, does not appear simpler than the initial eigenproblem, it happens to be more accessible asymptotically as \( \nu \to \infty \). For all \( \nu \) large enough, the system of equations (3.5) and (3.10) is solved by the fixed point iterations. This in turn implies that it has countably many solutions, which under a suitable enumeration admit asymptotically exact approximation as the enumeration index tends to infinity.

Let us describe this approximation in greater detail. It can be shown as in [9, Lemma 5.6] that the integral operator in the right hand side of (3.10) is a contraction on \( L_2(\mathbb{R}_+) \), at least for all \( \nu \) large enough. It also follows that the functions in (3.12) satisfy the estimates
\[
|a_+(\pm i)| \leq C/\nu \quad \text{and} \quad |a_-(\pm i)| \leq C/\nu^2 \tag{3.16}
\]
with a constant \( C \), which depends only on \( \alpha \), in a certain uniform way as in Lemma 3.3 below (see also [9, Lemm 5.7]).

To proceed we need the following additional estimates, specific to mixed covariance (3.1):

**Lemma 3.3.**

a) For any \( \alpha_0 \in (0, 1) \), there are constants \( C_0 \) and \( \nu_0 \) such that for all \( \nu \geq \nu_0 \) and \( \alpha \in [\alpha_0, 1] \) the functions defined in (3.9) and (3.13) satisfy the bounds
\[
\left| \arg\{X(\nu i)\} \right| \leq (1 - \alpha)C_0\nu^{\alpha - 1} \quad \text{and} \quad \left| X(\nu i) - 1 \right| \leq (1 - \alpha)C_0\nu^{\alpha - 1}
\]
and
\[
b_\alpha(\nu) \leq (1 - \alpha)C_0\nu^{\alpha - 1}.
\]
b) For any \( \alpha_0 \in (1, 2) \), there are constants \( C_0 \) and \( \nu_0 \) such that for all \( \nu \geq \nu_0 \) and \( \alpha \in [1, \alpha_0] \) the functions defined in (3.9) and (3.13) satisfy the bounds
\[
\left| \arg\{ X_e(\nu i) \} - \frac{1}{8} \pi \right| \leq C_0 \nu^{1-\alpha} \quad \text{and} \quad \left| X_e(\nu i) \right| - \sqrt{\frac{3-\alpha}{2}} \leq C_0 \nu^{1-\alpha} \quad (3.17)
\]

and
\[
|b_\alpha(\nu) - b_\alpha| \leq C_0 \nu^{1-\alpha} \quad \text{with} \quad b_\alpha = \frac{\sin\left(\frac{\pi}{3-\alpha} \frac{1-\alpha}{2}\right)}{\sin\frac{\pi}{3-\alpha}}. \quad (3.18)
\]

**Proof.**

a) Define \( \Gamma(z) := \frac{1}{\pi} \int_0^\infty \frac{\theta(z)}{\tau^2} d\tau \), then using the expression in (3.8), positive for \( \alpha \in (0, 1) \),
\[
\left| \arg\{ X(\nu i) \} \right| = \left| \text{Im}\{ \Gamma(\nu i) \} \right| = \frac{1}{\pi} \left| \int_0^\infty \frac{\theta(\tau)}{\tau^2 + \nu^2} d\tau \right| = \frac{1}{\pi} \left| \int_0^\infty \frac{\theta(\nu s)}{s^2 + 1} ds \right| \leq \frac{\sin\frac{1-\alpha}{2}}{\kappa_{\alpha}^{-1} \nu^{1-\alpha}} \int_0^\infty \frac{1}{s^2 + 1} \left( \frac{\nu^\alpha}{s^{3-\alpha} + \nu^{s-\alpha}} \right) ds \leq (1 - \alpha) C_0 \nu^{\alpha-1}.
\]

Similarly
\[
\text{Re}\{ \Gamma(\nu i) \} = \frac{1}{\pi} \int_0^\infty \frac{\tau \theta(\tau)}{\tau^2 + \nu^2} d\tau \leq (1 - \alpha) C_0 \nu^{\alpha-1}
\]

and hence for all \( \nu \) large enough, \( |X(\nu i)| - 1| = |e^{\text{Re}\{ \Gamma(i \nu) \}} - 1| \leq (1 - \alpha) C_0 \nu^{\alpha-1} \) as claimed. The bound for \( b_\alpha \) is obtained similarly
\[
b_\alpha(\nu) = \frac{1}{\pi} \int_0^\infty \nu \theta(\nu \tau) d\tau \leq \frac{\sin\frac{1-\alpha}{2}}{\kappa_{\alpha}^{-1} \nu^{1-\alpha}} \int_0^\infty \frac{1}{\tau^{3-\alpha} + \nu^{3-\alpha}} d\tau \leq (1 - \alpha) C_0 \nu^{\alpha-1},
\]

where the last inequality holds since \( \min_{0 \leq \alpha \leq 1} \kappa_{\alpha}^{-1} > 0 \) and the integral is bounded uniformly over \( \alpha \in [\alpha_0, 1] \) for all \( \alpha_0 \in (0, 1) \).

b) The estimate (3.18) holds since (see [9, Lemma 5.2])
\[
b_\alpha = \frac{1}{\pi} \int_0^\infty \text{arctan} \left( \frac{\tau^{\alpha-3} \sin \frac{1-\alpha}{2} \pi}{1 + \tau^{\alpha-3} \cos \frac{1-\alpha}{2} \pi} \right) d\tau
\]

and therefore
\[
\left| \frac{1}{\pi} \int_0^\infty \theta(\nu \tau) d\tau - b_\alpha \right| \leq \nu^{1-\alpha} \kappa_{\alpha}^{-1} \sin \frac{1-\alpha}{2} \frac{\pi}{\cos \frac{1-\alpha}{2} \pi} \int_0^\infty \frac{(1 + \tau^{-2}) \tau^{\alpha-3}}{1 + \tau^{2\alpha-6}} d\tau \quad (3.19)
\]

where we used the identity \( \text{arctan} x - \text{arctan} y = \text{arctan} (x - y)/(1 + xy) \).

Further, define
\[
\theta_0(\tau) := \text{arctan} \left( \frac{\tau^{\alpha-3} \sin \frac{1-\alpha}{2} \pi}{1 + \tau^{\alpha-3} \cos \frac{1-\alpha}{2} \pi} \right), \quad \tau > 0
\]

and
\[
X_0(i) := \exp \left( \frac{1}{\pi} \int_0^\infty \frac{\theta_0(\tau)}{\tau - i} d\tau \right).
\]

It is shown in [9, Lemma 5.5] that
\[
|X_0(i)| = \sqrt{\frac{3-\alpha}{2}} \quad \text{and} \quad \arg\{ X_0(i) \} = \frac{1 - \alpha}{8} \pi.
\]
The estimates in (3.17) are obtained by bounding $|\theta(\nu \tau) - \theta_0(\tau)|$ as in (3.19).

These estimates are the key to the following approximation:

**Lemma 3.4.** The solutions to the system (3.10) and (3.15) can be enumerated so that

$$\nu_n = \pi n - \frac{\pi}{2} + g(\nu_n) + n^{-1} r_n(\alpha), \quad n \in \mathbb{Z}$$

(3.20)

where $\sup_{\alpha \in [\alpha_0, 1]} r_n(\alpha) < \infty$ for any $\alpha_0 \in (0, 1)$ and

$$g(\nu) := -2 \arg\{X(\nu i)\} + \frac{\pi}{2} - \arg\{i + b_\alpha(\nu)\}.$$  

(3.21)

**Proof.** Plugging the estimates (3.16) into expressions from (3.14) we obtain

$$\xi(\nu) \eta(\nu) = -4e^{\nu i} X(\nu i)^2 (i + b_\alpha(\nu)) \left(1 + R_1(\nu)\right) = -4e^{\nu i + \frac{\pi i}{2}} \left(1 + R_2(\nu)\right)$$

(3.22)

where $|R_1(\nu)| \leq C \nu^{-1}$ and $|R_2(\nu)| \leq C \nu^{\alpha-1}$ with a constant $C$ which depends only on $\alpha_0$. Hence equation (3.15) with $\nu > 0$ reads

$$\nu + \frac{\pi}{2} + \arctan \frac{\text{Im}\{R_2(\nu)\}}{1 + \text{Re}\{R_2(\nu)\}} = \pi n, \quad n \in \mathbb{Z}.$$

A tedious but otherwise straightforward calculation reveals that $|R_2(\nu)| \leq C \nu^{\alpha-1}$ as well. Hence for each $n$ large enough, the unique solution to the system (3.5) and (3.10) is obtained by fixed-point iterations. The algebraic component $\nu_n$ of this solution satisfies, cf. (3.22) and (3.15)

$$\nu_n + 2 \arg\{X(\nu_n i)\} + \arg\{i + b_\alpha(\nu_n)\} + \arctan \frac{\text{Im}\{R_1(\nu_n)\}}{1 + \text{Re}\{R_1(\nu_n)\}} = \pi n$$

which yields the claimed formula since $|R_1(\nu)| \leq C \nu^{-1}$. □

The enumeration of the eigenvalues, introduced by this lemma, may differ from that which puts them in the decreasing order. A calibration procedure, detailed in [9, Section 5.1.7], shows that the two enumerations do in fact coincide. The formulas in Theorem 2.1 are obtained from (3.5) and (3.20), by plugging the estimates from Lemma 3.3 into (3.21) and replacing $\alpha = 2 - 2H$.

4. **Proof of Theorem 2.4**

The proof uses the theory of small deviations developed in [11], which addresses the problem of calculating exact asymptotics of the probabilities

$$\mathbb{P}\left(\sum_{j=1}^{\infty} \phi(j) Z_j \leq r\right) \quad \text{as } r \to 0,$$

(4.1)

where $Z_j$’s are i.i.d. nonnegative random variables and $\phi(j)$ is a summable sequence of positive numbers. Squared $L_2$-norm of a Gaussian process can be written as such series with $Z_j \sim \chi_1^2$ and $\phi(j) := \lambda_j$, where $\lambda_j$’s are the eigenvalues of its covariance operator. In what follows $\phi(t)$, $t \in \mathbb{R}_+$ stands for the function, obtained by replacing the integer index in $\phi(j)$ with a real positive variable $t$. 

The main ingredients in the asymptotic analysis of (4.1) in [11] are functions, defined in terms of the Laplace transform $f(s) := \mathbb{E}e^{-sZ_1} = (1 - 2s)^{-\frac{1}{2}}$, $s \in (-\infty, \frac{1}{2})$ of the $\chi^2_1$-distribution:

\begin{align*}
I_0(u) &:= \int_{1}^{\infty} \log f(u\phi(t))dt = -\frac{1}{2} \int_{1}^{\infty} \log(1 + 2u\phi(t))dt \\
I_1(u) &:= \int_{1}^{\infty} u\phi(t)(\log f)'(u\phi(t))dt = -\int_{1}^{\infty} \frac{u\phi(t)}{1 + 2u\phi(t)}dt \\
I_2(u) &:= \int_{1}^{\infty} (u\phi(t))^2(\log f)''(u\phi(t))dt = 2 \int_{1}^{\infty} \left(\frac{u\phi(t)}{1 + 2u\phi(t)}\right)^2 dt
\end{align*}

We will apply the following result:

Corollary 4.1 (Corollary 3.2 from [11]). The $L_2$-ball probabilities satisfy

$$
P(\|\tilde{B}\|^2 \leq r) \sim \left(\sqrt{u(r)}I_2(u(r))\right)^{-\frac{1}{2}} \exp \left(I_0(u(r)) + u(r)r\right) \quad \text{as } r \to 0,
$$

where $u(r)$ is any function satisfying

$$
\lim_{r \to 0} \frac{I_1(u(r)) + u(r)r}{\sqrt{I_2(u(r))}} = 0.
$$

4.1. Asymptotic expansion of $I_j(u)$’s. A preliminary step towards application of Corollary 4.1 is to derive the exact asymptotics of the functions from (4.2) as $u \to \infty$ for the weight function

$$
\phi(t) = \sum_{j=1}^{k} c_j t^{-d_j}.
$$

In our case, $k = 3$ and $c_1$ and $d_1 < d_2 < d_3$ are positive constants, whose values depend on $H$. In particular, $\lim_{t \to \infty} \frac{d}{dt} \log \phi(t) = 0$ holds, as required in [11]. It will be convenient to use constants

$$
a_j := c_j/c_1 \quad \text{and} \quad \delta_j := d_j - d_1
$$

and to define the new variable $v$ by the formula

$$
2uc_1 = v^{-d_1},
$$

which converges to zero as $u \to \infty$. Obviously $a_1 = 1$ and $\delta_1 = 0$, and for the specific values of constants $d_j$’s needed below, we also have $\delta_3 = 1$.

4.1.1. Asymptotic expansion of $I_0(u)$. Integrating by parts we get

\begin{align*}
I_0(u) &= -\frac{1}{2} \int_{1}^{\infty} \log(1 + 2u\phi(t))dt = \\
&= \frac{1}{2} \log(1 + 2u\phi(1)) + \int_{1}^{\infty} \frac{u\phi'(t)}{1 + 2u\phi(t)}dt = \\
&= \frac{1}{2} \log \left(1 + 2u\sum_{j=1}^{k} c_j\right) - \sum_{i=1}^{k} c_i d_i \int_{1}^{\infty} \frac{ut^{-d_i}}{1 + 2u\sum_{j=1}^{k} c_j t^{-d_j}}dt.
\end{align*}
Changing the integration variable and using the above notations, this can be written as

$$I_0(u) = \frac{1}{2} \log \left(1 + v^{-d_1} \sum_{j=1}^{k} a_j \right) - \frac{1}{2} \sum_{i=1}^{k} a_i d_i v^{\delta_i - 1} \int_{v}^{\infty} \frac{\tau^{-\delta_i}}{\tau^{d_i} + 1 + p(v/\tau)} d\tau,$$

(4.6)

where we defined

$$p(s) := a_2 s^{\delta_2} + a_3 s^{\delta_3}.$$

Let us find the exact asymptotics of each integral as \(v \to 0\). The first one gives

$$v^{\delta_1 - 1} \int_{v}^{\infty} \frac{\tau^{-\delta_1}}{\tau^{d_1} + 1 + p(v/\tau)} d\tau = v^{-1} \int_{v}^{\infty} \frac{1}{\tau^{d_1} + 1 + p(v/\tau)} d\tau =$$

$$v^{-1} \int_{v}^{\infty} \frac{1}{\tau^{d_1} + 1} d\tau - v^{-1} \int_{v}^{\infty} \frac{p(v/\tau)}{(\tau^{d_1} + 1 + p(v/\tau))(\tau^{d_1} + 1)} d\tau =$$

$$v^{-1} \int_{v}^{\infty} \frac{1}{\tau^{d_1} + 1} d\tau - J_{0.1}(v) - J_{0.2}(v)$$

where we defined

$$J_{0.1}(v) := v^{\delta_2 - 1} a_2 \int_{v}^{\infty} \frac{\tau^{-\delta_2}}{(\tau^{d_1} + 1 + p(v/\tau))(\tau^{d_1} + 1)} d\tau$$

$$J_{0.2}(v) := v^{\delta_3 - 1} a_3 \int_{v}^{\infty} \frac{\tau^{-\delta_3}}{(\tau^{d_1} + 1 + p(v/\tau))(\tau^{d_1} + 1)} d\tau.$$

The latter term with \(\delta_3 = 1\) satisfies

$$J_{0.2}(v) = a_3 \int_{v}^{\infty} \tau^{-1} \frac{1}{(\tau^{d_1} + 1 + p(v/\tau))(\tau^{d_1} + 1)} d\tau =$$

$$a_3 \int_{v}^{\infty} \tau^{-1} \frac{d\tau}{(\tau^{d_1} + 1)^2} + O(1) = -a_3 \log v + O(1),$$

and similarly

$$J_{0.1}(v) = v^{\delta_2 - 1} a_2 \int_{0}^{\infty} \frac{\tau^{-\delta_2}}{(\tau^{d_1} + 1)^2} d\tau - v^{2\delta_2 - 1} a_2 \int_{v}^{\infty} \frac{\tau^{-2\delta_2}}{(\tau^{d_1} + 1)^2(\tau^{d_1} + 1 + p(v/\tau))} d\tau + O(1).$$

If \(2\delta_2 - 1 > 0\), the second term on the right is of order \(O(1)\), otherwise we can proceed similarly to obtain the expansion

$$J_{0.1}(v) = -\sum_{k=1}^{m} (-a_2)^k \chi_{1,k} v^{k\delta_2 - 1}$$

where \(m\) is the largest integer such that \(m\delta_2 - 1 < 0\) and, [14, formula 3.241.4],

$$\chi_{1,k} := \int_{0}^{\infty} \frac{\tau^{-k\delta_2}}{(\tau^{d_1} + 1)^{k+1}} d\tau = \frac{\Gamma(\frac{1-k\delta_2}{d_1})\Gamma(k+1 - \frac{1-k\delta_2}{d_1})}{\Gamma(k+1)}, \quad k = 0, \ldots, m. \quad (4.7)$$
Plugging all the estimates back we obtain

\[ v^{\delta_1 - 1} \int_v^\infty \frac{\tau^{-\delta_1}}{\tau^{d_1} + 1 + p(v/\tau)} \, d\tau = a_3 \log v + \sum_{k=0}^{m} (-a_2)^k \chi_{1,k} v^{k \delta_2 - 1} + O(1), \quad \text{as } v \to 0. \quad (4.8) \]

Further, the second integral in the sum in (4.6) reads

\[ v^{\delta_2 - 1} \int_v^\infty \frac{\tau^{-\delta_2}}{\tau^{d_1} + 1 + p(v/\tau)} \, d\tau = v^{\delta_2 - 1} \int_v^\infty \frac{\tau^{-\delta_2}}{\tau^{d_1} + 1} \, d\tau - J_{1,1}(v) - J_{1,2}(v). \]

Here

\[ J_{1,2}(v) := v^{\delta_2 + \delta_3 - 1} a_3 \int_v^\infty \frac{\tau^{-\delta_2 - \delta_3}}{(\tau^{d_1} + 1 + p(v/\tau))(\tau^{d_1} + 1)} \, d\tau \leq v^{\delta_2 + \delta_3 - 1} a_3 \int_v^\infty \frac{\tau^{-\delta_2 - \delta_3}}{(\tau^{d_1} + 1)^2} \, d\tau = O(1) \]

since \( \delta_2 + \delta_3 > 1 \). The second term

\[ J_{1,1}(v) := v^{2 \delta_2 - 1} a_2 \int_v^\infty \frac{\tau^{-2 \delta_2}}{(\tau^{d_1} + 1 + p(v/\tau))(\tau^{d_1} + 1)} \, d\tau \]

is of order \( O(1) \), if \( 2 \delta_2 > 1 \). Otherwise,

\[ J_{1,1}(v) = v^{2 \delta_2 - 1} a_2 \int_0^\infty \frac{\tau^{-2 \delta_2}}{(\tau^{d_1} + 1)^2} \, d\tau - v^{3 \delta_2 - 1} a_2^2 \int_0^{\infty} \frac{\tau^{-3 \delta_2}}{(\tau^{d_1} + 1)^2(\tau^{d_1} + 1 + p(v/\tau))} \, d\tau + O(1) \]

where the second term is of order \( O(1) \), if \( 3 \delta_2 > 1 \) and so on. Thus we obtain asymptotics

\[ v^{\delta_2 - 1} \int_v^\infty \frac{\tau^{-\delta_2}}{\tau^{d_1} + 1 + p(v/\tau)} \, d\tau = \sum_{k=1}^{m} (-a_2)^{k-1} \chi_{0,k} v^{k \delta_2 - 1} + O(1), \quad \text{as } v \to 0 \quad (4.9) \]

with

\[ \chi_{0,k} := \int_0^\infty \frac{\tau^{-k \delta_2}}{(\tau^{d_1} + 1)^k} \, d\tau = \frac{\Gamma\left(\frac{1-k \delta_2}{d_1}\right)\Gamma(k-\frac{1-k \delta_2}{d_1})}{\Gamma(k)}, \quad k = 1, \ldots, m. \quad (4.10) \]

Finally, since \( \delta_3 = 1 \), the last summand in (4.6) contributes

\[ v^{\delta_3 - 1} \int_v^\infty \frac{\tau^{-\delta_3}}{\tau^{d_1} + 1 + p(v/\tau)} \, d\tau = \int_v^\infty \frac{\tau^{-1}}{\tau^{d_1} + 1} \, d\tau - J_{3,1}(v) - J_{3,2}(v) = -\log v + O(1) \quad (4.11) \]

where we used the estimates

\[ J_{3,1}(v) := v^{\delta_2 + \delta_3 - 1} \int_v^\infty \frac{a_2 \tau^{-\delta_3 - \delta_2}}{(\tau^{d_1} + 1 + p(v/\tau))(\tau^{d_1} + 1)} \, d\tau = O(1) \]

and

\[ J_{3,2}(v) := v^{2 \delta_3 - 1} \int_v^\infty \frac{a_3 \tau^{-2 \delta_3}}{(\tau^{d_1} + 1 + p(v/\tau))(\tau^{d_1} + 1)} \, d\tau = O(1). \]
Asymptotic expansion of $\tilde{\lambda}$ eigenvalues of the covariance operator of $\alpha$ and, for $(4.1)$ above, we have

\[
\kappa := \frac{c_\alpha}{\Gamma(\alpha)} \frac{\pi}{2^\alpha \Gamma(2H + 1) \sin(\pi H)} = \frac{\pi}{2} \Gamma(H + 1/2)\cos(H\pi).
\]

and, for $\alpha \in (0,1)$, $\nu_n = (n - 1/2)\pi + O(n^{\alpha - 1})$ as $n \to \infty$. Taylor expansion yields

\[
\nu_n^{-2} = \frac{1}{\pi^2} n^{-2} + \frac{1}{\pi^2} n^{-3} + O(n^{\alpha - 4})
\]

\[
\nu_n^{-3} = \frac{1}{\pi^2} n^{-3} + \frac{3 - \alpha}{2} \frac{1}{\pi^{3-\alpha}} n^{\alpha - 4} + O(n^{2\alpha - 5})
\]

and consequently

\[
\lambda(n) = \frac{1}{\pi^2} n^{-2} + \frac{\kappa}{\pi^{3-\alpha}} n^{\alpha - 3} + \frac{1}{\pi^2} n^{-3} + O(n^{\alpha - 4}) =: \phi(n) + O(n^{\alpha - 4}).
\]
By Li’s comparison theorem, [17, Theorem 2],
\[ P \left( \sum_{n=1}^{\infty} \lambda(n)Z_n \leq \varepsilon^2 \right) \sim \left( \prod_{n=1}^{\infty} \frac{\phi(n)}{\lambda(n)} \right)^{1/2} P \left( \sum_{n=1}^{\infty} \phi(n)Z_n \leq \varepsilon^2 \right) \quad \text{as } \varepsilon \to 0 \]
if \( \sum_{n=1}^{\infty} \left| 1 - \lambda(n)/\phi(n) \right| < \infty \), which holds in our case. Hence the desired asymptotics coincides with
\[ P \left( \sum_{n=1}^{\infty} \phi(n)Z_n \leq \varepsilon^2 \right) \quad \text{as } \varepsilon \to 0, \]
up to a multiplicative constant. The function \( \phi(t) \) has the form (4.5) with \( k = 3 \) and
\[
\begin{align*}
c_1 &= \frac{1}{\pi^2}, \\
c_2 &= \frac{\kappa_\alpha}{\pi^{3-\alpha}}, \\
c_3 &= \frac{1}{\pi^2}, \\
d_1 &= 2 \\
d_2 &= 3 - \alpha \\
d_3 &= 3
\end{align*}
\] (4.16)

Plugging these values into the asymptotic expansions (4.12), (4.13) and (4.14) gives
\[
\begin{align*}
I_0(u) &= \frac{1}{4} \log u - \frac{1}{\sqrt{2}} u^\frac{1}{2} + \sum_{k=1}^{m_\alpha} g_k u^{1-k(1-\alpha)} + O(1) \\
I_1(u) &= -\frac{1}{2\sqrt{2}} u^\frac{1}{2} - \sum_{k=1}^{m_\alpha} h_k u^{1-k(1-\alpha)} + O(1) \\
I_2(u) &= \sqrt{\frac{c_1}{2}} \chi_{3,1} u^\frac{1}{2} (1 + o(1))
\end{align*}
\] (4.17) (4.18) (4.19)

where \( m_\alpha := \left\lfloor \frac{1}{1-\alpha} \right\rfloor \) and
\[
\begin{align*}
h_k &= \frac{1}{2} \left( \frac{-c_2}{c_1} \right)^k \chi_{1,k} - \chi_{0,k} \right) \left( 2c_1 \right)^{1-k(1-\alpha)}, \\
g_k &= \frac{1}{2} \left( \frac{-c_2}{c_1} \right)^k \left( d_2 \chi_{0,k} - d_1 \chi_{1,k} \right) \left( 2c_1 \right)^{1-k(1-\alpha)}
\end{align*}
\] (4.20)

with constants \( \chi_{i,k} \) defined in (4.7), (4.10) and (4.15) and \( \delta_2 = d_2 - d_1 \).

Application of Corollary 4.1 requires finding a function \( u(r) \) which satisfies condition (4.4). To this end consider the equation (c.f. (4.18))
\[ \frac{1}{2\sqrt{2}} u^\frac{1}{2} + \sum_{k=1}^{\left\lfloor \frac{1}{1-\alpha} \right\rfloor} h_k u^{1-k(1-\alpha)} = ur, \] (4.21)
with respect to \( u > 0 \). If we divide both sides by \( u \), the left hand side becomes a monotonous function, which decreases to zero as \( u \to \infty \), and therefore this equation has unique positive solution \( u(r) \), which grows to \( +\infty \) as \( r \to 0 \). By the choice of the upper limit in the sum in (4.21), the power of \( u(r) \) in the numerator of (4.4) is strictly less than \( \frac{1}{4} \) and hence (4.4) holds in view of (4.19).
If we now let $u = (ry)^{-2}$ equation (4.21) reads
\[
\frac{1}{2\sqrt{2}}y + \sum_{k=1}^{\frac{1}{2} T-\alpha} h_k r^{k(1-\alpha)} y^{k(1-\alpha)+1} = 1, \quad y > 0.
\]
The function $r \mapsto y(r)$ is analytic in a vicinity of zero and can be expanded into series of the small parameter $r^{1-\alpha}$
\[
y(r) = y_0 + \sum_{j=1}^{\infty} y_j r^{j(1-\alpha)}.
\]
Let $\xi_j$ and $\eta_{k,j}$ be the coefficients of the expansions
\[
y(r)^{-2} = \sum_{j=0}^{\infty} \xi_j r^{j(1-\alpha)} \quad \text{and} \quad y(r)^{k(1-\alpha)-1} = \sum_{j=0}^{\infty} \eta_{k,j} r^{j(1-\alpha)}.
\]
Note that both are expressible in terms of $y_j$’s. Plugging these expansions into (4.3) gives
\[
\mathbb{P}(\|\tilde{B}\|_2^2 \leq r) \sim r^{\frac{3}{2}} \exp \left(-\frac{1}{\sqrt{2}} \sum_{j=0}^{\infty} \eta_{0,j} r^{j(1-\alpha)-1} + \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \eta_{k,j} r^{(j+k)(1-\alpha)-1} + \sum_{j=0}^{\infty} \xi_j r^{j(1-\alpha)-1} \right) \sim
\]
\[
r^{\frac{3}{2}} \exp \left(-\frac{1}{8} r^{-1} + \sum_{\ell=1}^{m_0} \left( \xi_\ell - \frac{1}{\sqrt{2}} \eta_{0,\ell} \right) r^{\ell(1-\alpha)-1} + \sum_{\ell=1}^{m_0} \left( \sum_{j=0}^{\ell-1} g_{\ell-j} \eta_{\ell-j,j} \right) r^{\ell(1-\alpha)-1} \right).
\]
Changing back to $r = \varepsilon^2$ and $\alpha = 2 - 2H$, we obtain the following result, which implies (2.12) for $H \in (\frac{1}{2}, 1)$ and defines all of its ingredients:

**Proposition 4.2.** For $H \in (\frac{1}{2}, 1)$ let $h_k$ and $g_k$ be the real sequences, defined by formulas (4.20) evaluated at the constants in (4.16). Then for any $r > 0$ the equation
\[
y/y_0 + \sum_{k=1}^{\frac{1}{2} T-H} h_k r^{k(2H-1)} y^{k(2H-1)+1} = 1
\]
with $y_0 = 2\sqrt{2}$ has unique positive root $y(r)$, which can be expanded into series
\[
y(r) = y_0 + \sum_{j=1}^{\infty} y_j r^{j(2H-1)},
\]
convergent for all $r$ small enough. Let $\xi_j$ and $\eta_{k,j}$ be coefficients of the power expansions
\[
y(r)^{-2} = \sum_{j=0}^{\infty} \xi_j r^{j(2H-1)} \quad \text{and} \quad y(r)^{k(2H-1)-1} = \sum_{j=0}^{\infty} \eta_{k,j} r^{j(2H-1)}
\]
and define
\[
\beta_\ell(H) := \frac{1}{\sqrt{2}} \eta_{0,\ell} - \sum_{j=0}^{\ell-1} g_{\ell-j} \eta_{\ell-j,j} - \xi_\ell.
\]
Thus we need to calculate only \( \beta \) with coefficients \( \xi \) to derive, they can be easily computed for any given value of \( H \), at least numerically. The following example, demonstrates the algorithm of Proposition 4.2.

**Example 4.3.**

**Case** \( H \in \left[ \frac{4}{3}, 1 \right) \). For the values of \( H \) in this range, \( \frac{1}{2H-1} = 1 \) and the sum in (4.25) contains only one term:

\[
P(\|\tilde{B}\|_2 \leq \varepsilon) \sim C(H)\varepsilon \exp \left( -\frac{1}{8} \varepsilon^{-2} - \sum_{\ell=1}^{\infty} \beta_{\ell}(H)\varepsilon^{2\ell(2H-1)-2} \right), \quad \varepsilon \to 0.
\]

(4.25)

Thus we need to calculate only \( \beta_1 = \frac{1}{\sqrt{2}}y_{0,1} - g_1 \eta_{1,0} - \xi_1 \).

Since \( \frac{1}{2H-1} = 0 \) equation (4.22) reduces to \( y = y_0 \). Comparing \( y(r)^{-2} = y_0^{-2} \) to (4.24) gives \( \xi_1 = 0 \). Similarly, comparing \( y(r)^{-1} = y_0^{-1} \) with \( y(r)^{-1} = \eta_{0,0} + \eta_{1,0}2^{H-1} + O(r^{4H-2}) \) gives \( \eta_{0,1} = 0 \). Finally \( \eta_{1,0} = y_0^{2H-2} \) and hence \( \beta_1 = -g_1 y_0^{2H-2} \). After simplification, formula (4.20) yields \( g_1 = -2^{-H-1} \Gamma(2H+1) \) and we obtain

\[ \beta_1(H) = 2^{2H-4} \Gamma(2H+1). \]

**Case** \( H \in \left[ \frac{2}{3}, \frac{3}{4} \right) \). In this case \( \frac{1}{2H-1} = 2 \) and the sum in (4.25) contains two terms:

\[
P(\|\tilde{B}\|_2 \leq \varepsilon) \sim C(H)\varepsilon \exp \left( -\frac{1}{8} \varepsilon^{-2} - \beta_1 \varepsilon^{4H-4} - \beta_2 \varepsilon^{8H-6} \right), \quad \varepsilon \to 0,
\]

with coefficients

\[
\beta_1 = \frac{1}{\sqrt{2}}y_{0,1} - g_1 \eta_{1,0} - \xi_1,
\]

\[
\beta_2 = \frac{1}{\sqrt{2}}y_{0,2} - g_2 \eta_{2,0} - g_1 \eta_{1,1} - \xi_2.
\]

(4.26)

To find \( \xi_1 \) and \( \xi_2 \), note that

\[
y(r)^{-2} = \left( y_0 + \sum_{j=1}^{\infty} y_j r^{j(2H-1)} \right)^{-2} = y_0^{-2} - 2y_0^{-3} \sum_{j=1}^{\infty} y_j r^{j(2H-1)} + 3y_0^{-4} \left( \sum_{j=1}^{\infty} y_j r^{j(2H-1)} \right)^2 + ... = y_0^{-2} - 2y_0^{-3} y_1 r^{2H-1} + (3y_0^{-4} y_1^2 - 2y_0^{-3} y_2)r^{2(2H-1)} + O(r^{2(2H-1)})
\]

which yields

\[ \xi_1 = -2y_0^{-3} y_1 \quad \text{and} \quad \xi_2 = y_0^{-3} (3y_0^{-1} y_1^2 - 2y_2). \]

By (4.24) with \( k = 0 \), we have

\[
y(r)^{-1} = \left( y_0 + \sum_{j=1}^{\infty} y_j r^{j(2H-1)} \right)^{-1} = y_0^{-1} - y_0^{-2} y_1 r^{2H-1} + O(r^{2(2H-1)})
\]
and hence
\[ \eta_{0,0} = y_0^{-1} \quad \text{and} \quad \eta_{0,1} = -y_0^{-2}y_1. \]

Similarly, for \( k = 1 \)
\[
y(r)^{2H-2} = \left( y_0 + \sum_{j=1}^{\infty} y_j r^{j(2H-1)} \right)^{2H-2} =
\]
\[
y_0^{2H-2} - (2 - 2H)y_0^{2H-3}y_1r^{2H-1} + O(r^{2(2H-1)})
\]
which gives
\[ \eta_{1,0} = y_0^{2H-2} \quad \text{and} \quad \eta_{1,1} = -(2H-2)y_0^{2H-3}y_1. \]

Finally, (4.24) with \( k = 2 \) yields \( \eta_{2,0} = y_0^{4H-3} \). Plugging all these values into (4.26) we get
\[
\beta_1 = 2y_0^{-3}y_1 - \frac{1}{\sqrt{2}}y_0^{-2}y_1 - g_1y_0^{2H-2}
\]
\[
\beta_2 = -y_0^{-3}(3y_0^{-1}y_1^2 + 2y_2) - \frac{1}{\sqrt{2}}y_0^{-2}y_1^{2H-3} - g_2y_0^{4H-3} + g_1(2 - 2H)y_0^{2H-3}y_1,
\]
where \( g_1 \) and \( g_2 \) are found using (4.20). It is left to find \( y_1 \) and \( y_2 \).

For \( H \in \left[ \frac{2}{3}, \frac{3}{4} \right) \) we have \( \left[ \frac{1}{2H-1} \right] = 1 \) and equation (4.22) reads
\[
y/y_0 + h_1r^{2H-1} = 1.
\]
Plugging expansion (4.23) we get
\[
\sum_{j=1}^{\infty} \frac{y_j}{y_0}r^{j(2H-1)} + h_1r^{2H-1} \left( y_0 + \sum_{j=1}^{\infty} y_j r^{j(2H-1)} \right)^{2H} = 0
\]
where \( h_1 \) is defined in (4.20). Comparing coefficients of powers \( r^{2H-1} \) and \( r^{4H-2} \) we obtain
\[ y_1 = -h_1y_0^{2H+1} \quad \text{and} \quad y_2 = -2Hh_1y_0^{2H}y_1. \]

### 4.3. The case \( H \in \left( 0, \frac{1}{2} \right] \).

By Theorem 2.1 the eigenvalues satisfy the same formula
\[ \lambda(n) = \nu_n^{-2} + \kappa_\alpha n^{-3} \]
but this time, for \( \alpha := 2 - 2H \in (1, 2) \), with \( \nu_n = \pi n - \frac{\pi}{2} q_\alpha + O(n^{1-\alpha}) \) as \( n \to \infty \), where
\[ q_\alpha = 1 - \frac{\alpha - 1}{2} - \frac{2}{\pi} \arcsin \frac{\ell_{1-\alpha/2}}{\sqrt{1 + \ell_{1-\alpha/2}^2}}. \]

By the Taylor expansion
\[
\nu_n^{-2} = \frac{1}{\pi^2}n^{-2} + \frac{q_\alpha}{\pi^2}n^{-3} + O(n^{-\alpha-2})
\]
\[
\nu_n^{\alpha-3} = \frac{1}{\pi^{3-\alpha}}n^{-3} + \frac{3 - \alpha}{2} \frac{q_\alpha}{\pi^{3-\alpha}}n^{-4} + O(n^{-3})
\]
and therefore
\[
\lambda(n) = \frac{\kappa_\alpha}{\pi^{3-\alpha}}n^{\alpha-3} + \frac{1}{\pi^2}n^{-2} + \kappa_\alpha \frac{3 - \alpha}{2} \frac{q_\alpha}{\pi^{3-\alpha}}n^{-4} + O(n^{-3}) := \phi(n) + O(n^{-3}).
\]
As in the previous case, omitting the residual $O(n^{-3})$ term alters the exact asymptotics of small ball probabilities only by a multiplicative constant. The weight function $\phi(t)$ has the form \((4.5)\) with

\[
\begin{align*}
c_1 &= \frac{\kappa_\alpha}{\pi^{3-\alpha}}, & d_1 &= 3 - \alpha \\
c_2 &= \frac{1}{\pi^2}, & d_2 &= 2 \\
c_3 &= \frac{\kappa_\alpha}{\pi^{3-\alpha}} \frac{3 - \alpha}{2} q_\alpha, & d_3 &= 4 - \alpha
\end{align*}
\] (4.27)

and the asymptotic expansions \((4.12)\), \((4.13)\) and \((4.14)\) read

\[
I_0(u) = \frac{1}{2} \left( 1 - \frac{1}{2} q_\alpha \right) \log u - \frac{3 - \alpha}{2} \chi_{1,0}(2c_1) \frac{1}{3-\alpha} u^{\frac{1}{3-\alpha}} + \sum_{k=1}^{m_\alpha} g_k u^{\frac{1-k(\alpha-1)}{3-\alpha}} + O(1) \tag{4.28}
\]

\[
I_1(u) = -\frac{1}{2} \chi_{1,0}(2c_1) \frac{1}{3-\alpha} u^{\frac{1}{3-\alpha}} - \sum_{k=1}^{m_\alpha} h_k u^{\frac{1-k(\alpha-1)}{3-\alpha}} + O(1) \tag{4.29}
\]

\[
I_2(u) = \frac{1}{2} \chi_{3,1}(2c_1 u) \frac{1}{3-\alpha} \left( 1 + o(1) \right) \tag{4.30}
\]

where sequences $h_k$ and $g_k$ are defined by the same formulas as in \((4.20)\), evaluated at constants \((4.27)\).

To find a suitable function $u(r)$ satisfying condition \((4.4)\), consider equation (cf. \((4.29)\))

\[
\frac{1}{2} \chi_{1,0}(2c_1) \frac{1}{3-\alpha} u^{\frac{1}{3-\alpha}} + \sum_{k=1}^{\lfloor \frac{1}{2} \alpha - 1 \rfloor} h_k u^{\frac{1-k(\alpha-1)}{3-\alpha}} = ur. \tag{4.31}
\]

As in the previous case, it has the unique solution $u(r)$ for any $r > 0$ and it increases to $+\infty$ as $r \to 0$. By the choice of upper limit in the sum in \((4.31)\), the power of $u(r)$ in the numerator of \((4.4)\) does not exceed $\frac{1}{2} \frac{1}{3-\alpha}$ and hence \((4.4)\) holds in view of \((4.30)\).

Define new variable $y$ by the relation $u = (ry)^{\frac{1}{3-\alpha}}$, then it solves equation

\[
y/y_0 + \sum_{k=1}^{\lfloor \frac{1}{2} \alpha - 1 \rfloor} h_k r^{\frac{k-1}{3-\alpha}} y^{1+k\frac{1}{3-\alpha}} = 1,
\]

where $1/y_0 = \frac{1}{2} \chi_{1,0}(2c_1) \frac{1}{3-\alpha}$. The function $r \mapsto y(r)$ is analytic in the vicinity of $r = 0$ and can be expanded into powers of the small parameter $r^{\frac{\alpha-1}{2-\alpha}}$:

\[
y(r) = y_0 + \sum_{k=1}^{\infty} y_k r^{\frac{\alpha-1}{2-\alpha}}.
\]
Plugging (4.28) and (4.30) into (4.3) yields
\[
P(\|B\|_2^2 \leq r) \sim 
\]
\[
u(r) - \frac{1}{2} - \frac{1}{2\alpha} + \frac{1}{2}(1 - \frac{1}{2}q_0) \exp \left( - \frac{3 - \alpha}{y_0} u(r) \frac{1}{r} + \sum_{k=1}^{m_{\alpha}} g_k u(r) \frac{1-k(\alpha-1)}{2-\alpha} + u(r)r \right) \sim 
\]
\[
r^{\gamma_0} \exp \left( - \frac{3 - \alpha}{y_0} y(r) \frac{1}{r} + \sum_{k=1}^{m_{\alpha}} g_k y(r) \frac{1-k(\alpha-1)}{2-\alpha} + (r) \frac{3 - \alpha}{2-\alpha} \right) \sim 
\]
\[
r^{\gamma_0} \exp \left( - \frac{3 - \alpha}{y_0} \sum_{j=0}^{\infty} \eta_{0,j} r^{j(\alpha-1)} - \sum_{j=0}^{\infty} \xi_j r^{j(\alpha-1)} - \frac{1}{2} (1 - \frac{1}{2}g_0) \right) \sim 
\]
where we defined
\[
\gamma_0 := \frac{3 - \alpha}{2 - \alpha} \left( - \frac{1}{4} - \frac{1}{2} \frac{1}{3 - \alpha} + \frac{1}{2} \left( 1 - \frac{1}{2} q_0 \right) \right) \quad \text{and} \quad \beta_0 := \frac{3 - \alpha}{y_0} \eta_{0,0} - \xi_0,
\]
and \( \xi_j \) and \( \eta_{k,j} \) are coefficients in the expansions
\[
y^\frac{1-k(\alpha-1)}{2-\alpha} = \sum_{j=0}^{\infty} \eta_{k,j} r^j \quad \text{and} \quad y^\frac{3 - \alpha}{2-\alpha} = \sum_{j=0}^{\infty} \xi_j r^j \frac{3 - \alpha}{2-\alpha}.
\]
Replacing \( \alpha \) with \( 2 - 2H \) and \( r := \varepsilon^2 \) and simplifying, we obtain the formula (2.12):

**Proposition 4.4.** For \( H \in (0, \frac{1}{2}) \) let \( h_k \) and \( g_k \) be the real sequences, defined by formulas (4.20), evaluated at the constants (4.27). Then for any \( r > 0 \) the equation
\[
y/y_0 + \sum_{k=1}^{\left[ \frac{1}{2} \frac{1}{1+2H} \right]} h_k r^{\frac{1-2H}{2H}} y^{\frac{1-2H}{2H} + 1} + 1 = 1
\]
with
\[
y_0 = (2H + 1) \left( \frac{2^{2H} \left( \frac{1}{\sin(\pi H + 1)} \right)^{2H+1}}{\sin(\pi H) \Gamma(2H + 1)} \right) \frac{1}{2H+1}
\]
has unique positive root \( y(r) \), which can be expanded into series
\[
y(r) = y_0 + \sum_{k=1}^{\infty} y_k r^{\frac{1-2H}{2H}}
\]
convergent for all \( r \) small enough. Let \( \xi_j \) and \( \eta_{k,j} \) be coefficients of the power expansions
\[
y(r)^{\frac{2H+1}{2H}} = \sum_{j=0}^{\infty} \xi_j r^{\frac{1-2H}{2H}} \quad \text{and} \quad y(r)^{\frac{k(1-2H)-1}{2H}} = \sum_{j=0}^{\infty} \eta_{k,j} r^{\frac{1-2H}{2H}}
\]
and define
\[
\beta_{\ell}(H) := \frac{2H + 1}{y_0} \eta_{0,\ell} - \sum_{j=0}^{\ell-1} g_{\ell-j} \eta_{\ell-j,j} - \xi_{\ell}.
\]

Then
\[
P\left(\|\tilde{B}\|_2 \leq \varepsilon\right) \sim \varepsilon^{\gamma_{\beta_{0}}(H)} \exp\left(-\beta_{0}(H)\varepsilon^{-\frac{1}{H}} - \sum_{\ell=1}^{\lfloor \frac{1}{1-2H} \rfloor} \beta_{\ell}(H)\varepsilon^{\frac{\ell(1-2H)-1}{H}}\right), \quad \varepsilon \to 0
\]

where \(\gamma(H)\) and \(\beta_0(H)\) are given by (2.10) and (2.9).

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