Multinomials and Polynomial Bosonic Forms for the Branching Functions of the $\widehat{su}(2)_M \times \widehat{su}(2)_N / \widehat{su}(2)_{M+N}$ Conformal Coset Models

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Abstract

We give explicit expressions for the q-multinomial generalizations of the q-binomials and Andrews’ and Baxter’s q-trinomials. We show that the configuration sums for the generalized RSOS models in regime III studied by Date et al. can be expressed in terms of these multinomials. This generalizes the work of ABF and AB where configuration sums of statistical mechanical models have been expressed in terms of binomial and trinomial coefficients. These RSOS configuration sums yield the branching functions for the $\widehat{su}(2)_M \times \widehat{su}(2)_N / \widehat{su}(2)_{M+N}$ coset models. The representation in terms of multinomials gives Rocha-Caridi like formulas whereas the representation of Date et al. gives a double sum representation for the branching functions.

1 Introduction

There are many polynomial generalizations of conformal field theory characters/branching functions [1]-[32]. Often, these polynomials come from finite dimensional statistical mechanical configuration sums and yield the characters of conformal field theories as the order of the polynomial tends to infinity. Polynomial expressions are useful since they often permit derivations of recursion relations in the order of the polynomial. These recursion relations allow the proof of nontrivial identities between polynomial expressions and hence also for the corresponding infinite series obtained by taking the order of the polynomial to infinity.

In their bosonic form most of these polynomial representations use q-binomials or Andrews’ and Baxter’s q-trinomials. In particular, Andrews, Baxter and Forrester [1] express the configuration sums of their ABF model in terms of binomials which are polynomial forms of the branching functions of the $M(p,p+1)$, $M(2,2p+1)$, $M(2p-1,2p+1)$ minimal models and the $Z_p$ parafermion model in the different regimes. These bosonic polynomial forms have been generalized to the arbitrary minimal models $M(p,p')$ by Forrester and Baxter [2]. Further, Andrews and Baxter [3] give bosonic polynomial versions of the branching functions of the superconformal unitary models $SM(5,7)$ and the minimal models $M(2,7)$ in terms of trinomials. Warnaar and Pearce [19], [20] use trinomials to express $E_{6,7,8}$ type Rogers-Ramanujan identities. Berkovich, McCoy and Orrick [8] give

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polynomial forms for the $SM(2,4\nu)$ and $M(2\nu - 1, 4\nu)$ characters in terms of trinomials. These characters have been further investigated by Berkovich and McCoy [32].

In this paper we generalize the $q$-binomial and Andrews’ and Baxter’s $q$-trinomial coefficients [3] to $q$-multinomial coefficients for which we give explicit formulas in equation (2.8). We further write the configuration sums for the generalized RSOS models in regime III studied by Date et al. [4]-[7] in terms of these multinomials (see equation (1.13) and (1.15)). The configuration sums of the generalized RSOS models in regime III which are polynomials coincide with the branching functions of the $\tilde{su}(2)_M \times \tilde{su}(2)_N / \tilde{su}(2)_{M+N}$ coset models as the order of the polynomial limits to infinity.

Date et al. [5], [6] give several representations for these RSOS configuration sums: 1) double sum expressions, 2) single sums in terms of binomials and 3) for a special case an expression in terms of multinomials previously defined by Andrews [45]. These multinomials, however, differ from the ones used in this paper and are not appropriate for taking the limit that gives the branching functions.

The RSOS models studied by Date et al. generalize the ABF [1] and AB [3] statistical models which correspond to the cases $N = 1$ and $N = 2$, respectively. ABF and AB find expressions for the configuration sums in terms of bi- and trinomials. These configuration sums yield Rocha-Caridi type [34] expressions for the branching functions whereas Date et al. [4], [3] obtain double sum expressions which, in turn, can be expressed in terms of elliptic theta functions. In this paper we generalize the expressions given by ABF and AB and write the configuration sums in terms of multinomials which generalize AB’s trinomials [3]. They yield branching functions of the type given in [36].

For the proof of the configuration sums in terms of multinomials one can use the fact that the configuration sums of the RSOS models obey certain recursion relations. We will show that this induces certain depth one recursion relations for the multinomials (see equation (2.15)) which are proven in section 4.

Usually one finds two very distinct solutions to these kinds of recursion relations, a fermionic solution and a bosonic solution. By computing both one gets Rogers-Ramanujan type identities. Schur [37] was the first to prove such Bose-Fermi identities via recursion relations. Recently fermionic type solutions of the RSOS models and hence the $\tilde{su}(2)_M \times \tilde{su}(2)_N / \tilde{su}(2)_{M+N}$ coset models have been found and proven. The characters for the case $N = 1$ and also the identity character for arbitrary $N$ have been conjectured by Kedem et al. [9]. A polynomial form of these characters has been conjectured by Melzer [17] and proven by Berkovich [18] and Warnaar [26], [31]. The fermionic branching functions for $N = 2$ have been conjectured by Baver and Gepner [27]. Polynomial fermionic forms for general $N$ have recently been found and proven by the author [30].

As the order of the polynomial configuration sum in terms of multinomials tends to infinity it factorizes into a Rocha-Caridi type sum and a parafermionic configuration sum. Therefore for the structure of the branching function of the $\tilde{su}(2)_{p-N-2} \times \tilde{su}(2)_N / \tilde{su}(2)_{p-2}$ coset models we obtain
\[ c_{r,s}^{(l)} \sim \frac{1}{(q)_\infty} \sum_{m=0}^{N/2} (\text{parafermion piece})(m,l) \times \left\{ \sum_{j \in \mathbb{Z}, m_{r,s}(j) \equiv m} q^{\frac{1}{2}(jp'p + pr - p's)} - \sum_{j \in \mathbb{Z}, m_{r,-s}(j) \equiv m} q^{\frac{1}{2}(pj + s)(p'j + r)} \right\} \]  

(1.1)

where the first sum runs over \( m \) integer for \( l \) odd and \( m \) half-integer for \( l \) even, \( m_{r,s}(j) = \lfloor \frac{j}{2} + pj \mod N \rfloor \) (where the convention \( N/2 < (x \mod N) \leq N/2 \) is used) and \( p' = p - N \).

The term in brackets is of the Rocha-Caridi form. Details are given in section 3.

The bosonic polynomial forms of the \( \hat{su}(2)_1 \times \hat{su}(2)_{p-3}/\hat{su}(2)_{p-2} \) coset models as given by ABF \[1\] have been generalized by Forrester and Baxter \[2\] to expressions for coset models \( \hat{su}(2)_1 \times \hat{su}(2)_m/\hat{su}(2)_{m+1} \) with fractional levels \( m = \frac{p'}{p-p'} - 2 \) or \( m = -\frac{p}{p-p'} - 2 \). Fermionic polynomial forms of these models have been given by Berkovich and McCoy \[23\]. Hence one may speculate that multinomials will also turn out to be useful for coset models \( \hat{su}(2)_N \times \hat{su}(2)_M/\hat{su}(2)_{N+M} \) where \( M \) is fractional. An example of this has already been seen in \[14\].

The characters/branching functions of conformal models can also be obtained from representations of the Virasoro algebra directly by the Feigin and Fuchs \[33\] construction which uses cohomological methods to mod out singular vectors. The Rocha-Caridi character formula has been derived this way \[34\] and gives the characters for the coset models \( \hat{su}(2)_M \times \hat{su}(2)_N/\hat{su}(2)_{M+N} \) for \( N = 1 \). Similarly, the branching functions and characters for \( N = 2 \) have been derived in \[35\] and for arbitrary \( N \) in \[36\] which lead to products of Rocha-Caridi type expressions times string functions. The parafermionic piece in (1.1) can be expressed in terms of the \( \hat{su}(2)_N \) string functions which allows us to make a connection between our results and the work in \[36\].

This paper is structured as follows. In section 2 we review the binomial and trinomial coefficients, give multinomial generalizations of these and state some properties of the multinomials. In section 3 the representation of the RSOS configuration sums in terms of multinomials is derived by using the path space interpretation of the RSOS models which yield recursion relations for the configuration sums. The results are compared with the representations previously obtained by Date et al. The branching functions of the \( \hat{su}(2)_M \times \hat{su}(2)_N/\hat{su}(2)_{M+N} \) coset models are obtained by taking the order of the configuration sum polynomials to infinity and are compared with the results in \[36\]. In section 4 we prove the recursion relations of the multinomials which are essential for the proof of the configuration sum formulas in section 3. We conclude in section 5 with a summary and discussion of open questions.

## 2 Multinomial coefficients

We start our discussion of the multinomials by briefly reviewing the \( q \)-binomial and \( q \)-trinomial coefficients. To this end let us define

\[ (q)_n = \prod_{i=1}^{n}(1-q^i) \]  

(2.1)
The \(q\)-binomials - also called Gaußian polynomials - are then defined as

\[
\begin{bmatrix} L \\ r \end{bmatrix}_{1} = \begin{cases} \frac{(q)_n}{(q)^{r}(q)_{L-r}} & \text{if } 0 \leq r \leq L \\ 0 & \text{otherwise} \end{cases}
\]  

Andrews and Baxter [3] define \(q\)-trinomials as

\[
\begin{bmatrix} L \\ r \\ n \end{bmatrix}_{2} = \sum_{m \geq 0} \frac{q^{m(m+r-n)}(q)_{L}}{(q)_{m}(q)_{m+r}(q)_{L-2m-r}}
\]  

(2.3)

where \(n = 0, 1\) and \(r = -L, -L + 1, \ldots, L\). They further define

\[
T_n(L, r) = q^{\frac{L(L-n)+r(r-n)}{2}} \begin{bmatrix} L \\ r \end{bmatrix}_{q^{-1,2}}
\]  

(2.4)

(Notice that the \(T_n\) defined in (2.4) is actually \(T_n(L, r) = T_n(L, r, q^{\frac{1}{2}})\) in Andrews’ and Baxter’s notation). In [28] Berkovich, McCoy and Orrick generalize the trinomials \(\begin{bmatrix} L \\ r \end{bmatrix}_{n}\) and also \(T_n(L, r)\) to arbitrary \(n \in \mathbb{Z}\) to write the bosonic characters for the \(SM(2,4\nu)\) models. We will however only generalize Andrews’ and Baxter’s trinomial coefficient.

Using

\[(q^{-1})_n = (-)^n q^{-\frac{n(n+1)}{2}}(q)_n\]  

(2.5)

and making the variable change \(m \rightarrow \frac{L}{2} - \frac{r}{2} - \frac{m}{2}\) one derives from (2.4)

\[
T_n(L, r) = \sum_{m, \frac{L}{2} - \frac{r}{2} - \frac{m}{2} \in \mathbb{Z}_{\geq 0}} q^{\frac{m^2 - mr}{2}} \frac{(q)_L}{(q)_{\frac{L}{2} - \frac{r}{2} - \frac{m}{2}}(q)_{\frac{L}{2} + \frac{r}{2} - \frac{m}{2}}(q)_{m}}.
\]  

(2.6)

The sum runs over all \(m \in \mathbb{Z}_{\geq 0}\) such that \(\frac{L}{2} - \frac{r}{2} - \frac{m}{2} \in \mathbb{Z}_{\geq 0}\) and \(\frac{L}{2} + \frac{r}{2} - \frac{m}{2} \in \mathbb{Z}_{\geq 0}\). Notice that Andrews and Baxter write \(T_n(n = 0, 1)\) in a different form

\[
T_n(L, r, q) = \sum_{j=0}^{L} (-q^n)^j \begin{bmatrix} L \\ j \end{bmatrix}_{1,q^2} \begin{bmatrix} 2L - 2j \\ L - r - j \end{bmatrix}_{1,q}.
\]  

(2.7)

This form of the trinomials is however harder to generalize. In [13] Andrews gives a recursive definition for the multinomial coefficients \(\begin{bmatrix} L \\ r \end{bmatrix}_{N}^{(0)}\), but no explicit formulas and properties (e.g. recursion relations) are given. The multinomials used by Date et al. (cf. equation (3.29) in [3]) are related to \(\begin{bmatrix} L \\ r \end{bmatrix}_{N}^{(0)}\)

Let us define the \((N+1)\)-nomial coefficients which generalize the trinomial coefficients in (2.4) as

\[
T_n^{(N)}(L, r) \equiv \sum_{m \in \mathbb{Z}_{\geq 0}^{N-1}} q^{mC^{-1}m - \epsilon_nC^{-m}} \frac{(q)_L}{(q)_{\frac{L}{2} - \frac{r}{2} - \frac{m}{2} - \epsilon_1C^{-1}m} \prod_{i=1}^{N-1} \frac{1}{(q)_{m_i}}}.
\]  

(2.8)
where the subscript \((N)\) denotes the \((N+1)\)-nomial, \(n = 0, 1, 2, \ldots, N - 1\) gives the type of the multinomial (similar to \(T_0\) and \(T_1\) for the trinomials) and \(r = -\frac{NL}{2}, -\frac{NL}{2} + 1, \ldots, \frac{NL}{2}\). For other values of \(r\) the multinomials are defined to be zero. The sum runs over all \(m = (m_1, m_2, \ldots, m_{N-1}) \in \mathbb{Z}_{\geq 0}^{N-1}\) such that \(\frac{L}{2} - \frac{r}{N} - e_1 C^{-1}m \in \mathbb{Z}_{\geq 0}\) and \(\frac{L}{2} + \frac{r}{N} - e_{N-1} C^{-1}m \in \mathbb{Z}_{\geq 0}\). The vectors \(e_i\) are defined as \((e_i)_j = \delta_{i,j}\) for \(i \in \{1, 2, \ldots, N - 1\}\) and \(e_i = 0\) otherwise. \(C^{-1}\) is the inverse of the \(N - 1\) dimensional Cartan matrix \(C = 2 - I\) where the incidence matrix \(I_{ab} = \delta_{a,b+1} + \delta_{a,b-1}\). An explicit formula for \(C^{-1}\) is given by

\[
C^{-1}_{i,j} = \begin{cases} \frac{1}{C} N_i - i & \text{for } j \leq i \\ \frac{1}{C} C_{j,i} & \text{for } j > i \end{cases}
\] (2.9)

We can write the expression for \(T^{(N)}_n\) in (2.8) in a slightly more elegant way by introducing

\[
m_0 = \frac{L}{2} - \frac{r}{N} - e_1 C^{-1}m
\] (2.10)

\[
m_N = \frac{L}{2} + \frac{r}{N} - e_{N-1} C^{-1}m.
\] (2.11)

Then we may write

\[
T^{(N)}_n(L, r) = \sum_{\tilde{m} \in \mathbb{Z}_{\geq 0}^{N+1}} q^{m C^{-1}m - e_n C^{-1}m} (q)_L \prod_{i=0}^{N} \frac{1}{(q)_{m_i}}
\] (2.12)

where \(\tilde{m} = (m_0, m_1, \ldots, m_N), m = (m_1, m_2, \ldots, m_{N-1})\) and \(\sum\) stands for the sum over all \(m_i \in \mathbb{Z}_{\geq 0} (i = 0, 1, 2, \ldots, N)\) such that equations (2.10) and (2.11) are satisfied.

Notice that the multinomials have the symmetry

\[
T^{(N)}_n(L, -r) = T^{(N)}_{N-n}(L, r)
\] (2.13)

where we define \(T_N \equiv T_0\). To see this just relabel the summation variable \(m_i\) in \(T^{(N)}_n(L, -r)\) by \(m_{N-i}\). To reobtain the previous form for \(T^{(N)}_n\) \(e_n\) gets replaced by \(e_{N-n}\) and (2.13) follows. From the definition of the multinomials one can immediately read off the initial condition

\[
T^{(N)}_n(0, r) = \delta_{r, 0}.
\] (2.14)

Further -but less trivial to see- the multinomials satisfy the following recursion relations

\[
T^{(N)}_n(L, r) = \sum_{k=0}^{n-1} q^{(n-k)(\frac{L}{2} + \frac{r}{N})} T^{(N)}_{n-k}(L - 1, \frac{N}{2} + r - k)
+ \sum_{k=n}^{N} q^{(k-n)(\frac{L}{2} - \frac{r}{N})} T^{(N)}_{n-k}(L - 1, \frac{N}{2} + r - k)
\] (2.15)

These recursion relations the proof of which will be given in section 4 are essential for the derivation of the RSOS configuration sums. The recursion relations (2.13) generalize the recursion relations for the trinomials (cf. (2.16) and (2.19) in [3]).
Notice that in the limit $L \to \infty$ the multinomials become the product of $\frac{1}{(q)_{\infty}}$ times the partition functions for the $\mathbb{Z}_N$-parafermion model, namely
\[
T_n^{(N)}(r) = \lim_{L \to \infty, L \text{ even}} T_n^{(N)}(L, r) = \frac{1}{(q)_{\infty}} \sum_{m \in \mathbb{Z}_{\geq 1}} q^{mC^{-1}m - e_nC^{-1}m} \prod_{i=1}^{N-1} \frac{1}{(q)_m}. \quad (2.16)
\]

The analogue of (2.2) and (2.3) is given by
\[
\left[ \frac{L}{2} - a \right]^{(n)}_N = q^{\frac{N}{2}((\frac{L}{2} - n) - r(r-n))} T_n^{(N)}(L, r)_{q^{-1}}. \quad (2.17)
\]

An explicit formula for $\left[ \frac{NL}{2} - a \right]^{(n)}_N$ where now $a = 0, 1, \ldots, NL$ is given by
\[
\left[ \frac{NL}{2} - a \right]^{(n)}_N = \sum_{j_1 + j_2 + \ldots + j_N = a} q^{\frac{N-1}{2}(L-j_k)j_{k+1} - \sum_{k=0}^{N-1} j_k} \left[ \frac{L}{j_1} \frac{j_1}{j_2} \frac{j_2}{j_3} \ldots \frac{j_{N-1}}{j_N} \right] \quad (2.18)
\]

where the sum runs over all $j_1, j_2, \ldots, j_N \in \mathbb{Z}_{\geq 0}$ such that $j_1 + j_2 + \ldots + j_N = a$. One can obtain this explicit formula by first replacing $m_i \to m_{N-i}$ in (2.12), using (2.5) and identifying
\[
j_1 = L - m_0 = \frac{a}{N} + e_1C^{-1}m \\
j_k = j_{k-1} - m_{k-1} = \frac{a}{N} + (-e_{k-1} + e_k)C^{-1}m \quad \text{for } 1 < k < N \\
j_N = j_{N-1} - m_{N-1} = m_N = \frac{a}{N} - e_{N-1}C^{-1}m. \quad (2.19)
\]

Equation (2.18) has been found independently by Warnaar [42]. In Warnaar’s notation $\left[ \frac{L}{a} \right]^{(n)}_{N, \text{Warnaar}} = \left[ \frac{NL}{2} - a \right]^{(n)}_N$ of (2.18).

The list of properties of the multinomials as given above is certainly not exhaustive. Some further properties such as identities which reduce to tautologies as $q \to 1$ and a partition theoretical interpretation of the multinomials can be found in [42].

3 Bosonic RSOS configuration sums

Let us first review the path space for the general RSOS models. The RSOS model that we consider here is defined on a square lattice $\mathcal{L}$. To each site $i$ on the lattice one associates a state variable $l_i$ which can take the values $l_i = 1, 2, \ldots, p - 1$. The local height probability, i.e. the probability that a certain state variable takes some particular value, is given as
a two dimensional configuration sum and can be reduced via the corner transfer matrix method \[38\] to a one dimensional configuration sum over a certain path space.

A path in this path space is a string of state variables \( \{ l_i | 1 \leq i \leq L + 2 \} \) such that two adjacent state variables \( l_i \) and \( l_{i+1} \) are admissible. Two adjacent state variables \( l_i \) and \( l_{i+1} \) are called admissible \((l_i \sim l_{i+1})\) if they fulfill the following conditions:

\[
\begin{align*}
l_i - l_{i+1} &= -N, -N + 2, \ldots, N, \\
l_i + l_{i+1} &= N + 2, N + 4, \ldots, 2p - N - 2
\end{align*}
\] (3.1)

where \( p \) and \( N \) are arbitrary positive integers and \( i \) runs from 1, 2, \ldots, \( L + 2 \). The model is frozen however unless \( p \geq N + 3 \). In terms of the path space \( N \) defines the value for the highest possible step in the vertical direction, \( p \) and \( L \) define the boundaries of the lattice in the vertical and horizontal direction, respectively. We call a pair \((l_i, l_{i+1})\) weakly admissible if \((3.1)\) is satisfied, but not necessarily \((3.2)\).

The one dimensional configuration sum for the RSOS models is given by \([5], [6]\)

\[
X_L(a|b, c) = \sum_{l_i, l_{i+1}} q^{\sum_{j=1}^{L} \frac{1}{4} |l_{j+1} - l_j|} \] (3.3)

where \( l_1 = a, l_{L+1} = b, l_{L+2} = c \) and \( l_1 \sim l_2 \sim \cdots \sim l_{L+2} \). \( X_L \) is uniquely determined by 1) the initial condition

\[
X_0(a|b, c) = \delta_{a,b} \] (3.4)

and 2) the recursion relation

\[
X_L(a|b, c) = \sum_{d \sim b} q^{\frac{1}{4} |d-c|} X_{L-1}(a|d, b) \] (3.5)

where the sum is over all \( d \) such that \( d \sim b \) admissible and \( X_L \) is defined to be zero if \((b, c)\) do not satisfy both \((3.1)\) and \((3.2)\).

The bosonic form for the configuration sum of the RSOS lattice models was first calculated by ABF \([1]\) for the case \( N = 1 \) (in particular the Ising model and the hard square gas), by AB \([3]\) for \( N = 2 \) (for some special value of \( p \)) and for general \( N \) by the Kyoto group \([5], [6]\).

Date et al. show in \([5], [6]\) that if one can find a solution \( f \) of the recursion relation

\[
f_L(b, c) = \sum_d f_{L-1}(d, b) q^{L|d-c|/4} \] (3.6)

\[
f_0(b, c) = \delta_{b,0} \] (3.7)

where now the sum in \((3.6)\) is over all \( d \) such that \( d \sim b \) admissible then the solution to the problem \((3.4), (3.5)\) is given by

\[
X_L(a|b, c) = q^{-\frac{1}{8}} (F_L(a|b, c) - F_L(-a|b, c)) \] (3.8)

where

\[
F_L(a|b, c) = \sum_{j \in \mathbb{Z}} q^{-pj^2 + (\frac{a-b}{2})j + \frac{a}{4} f_L(b - a - 2pj, c - a - 2pj)}. \] (3.9)
Each term in the solution (3.8), (3.9) of (3.4), (3.5) can be interpreted by the sieving method [45]. One starts with the solution $f_L$ for all weakly admissible paths. But one has overcounted since one is actually interested in paths obeying (3.2). Hence one subtracts off the generating function of paths which have at least one pair $(l_i, l_{i+1})$ with $l_i + l_{i+1} > 2p - N - 2$ and also the generating function of all path with at least one pair with $l_i + l_{i+1} < N + 2$. However, now one has subtracted too much since there could be paths which have $l_i + l_{i+1} > 2p - N - 2$ and $l_j + l_{j+1} < N + 2$ for some $i, j$. Hence one needs to add the generating function of paths which satisfy $l_i + l_{i+1} > 2p - N - 2$ and $l_j + l_{j+1} < N + 2$ for $i < j$ and those which satisfy this condition for $i > j$ etc..

One may write (3.6) more explicitly as

$$f_L(b, c) = \sum_{k=0}^{N} q^{\frac{L}{2}|N+b-c-2k|} f_{L-1}(b + N - 2k, b).$$

(3.10)

Hence a solution for $f_L$ is given by

$$f_L^N(b, c) = q^{\frac{bc}{4N}} T^{(N)}_{\frac{N+b-c}{2}}(L, \frac{b}{2})$$

(3.11)

since if we insert (3.11) into (3.10) we get

$$q^{\frac{bc}{4N}} T^{(N)}_{\frac{N+b-c}{2}}(L, \frac{b}{2}) = \sum_{k=0}^{N} q^{\frac{L}{2}|N+b-c-2k|} q^{\frac{b(N-b-2k)}{4N}} T_{2(N-2k)}(L - 1, \frac{b + N - 2k}{2})$$

$$= q^{\frac{bc}{4N}} \sum_{k=0}^{N} q^{\frac{L}{2}|N+b-c-2k|+\frac{b-b+c+N-2k}{4N}} T_{N-k}(L - 1, \frac{N}{2} + \frac{b}{2} - k)$$

(3.12)

This exactly reduces to the recursion relation for the multinomials (2.15) if we identify $n = \frac{N+b-c}{2}$ and $r = \frac{b}{2}$. The initial conditions (3.7) are satisfied because of (2.14).

We would like to point out that (3.11) differs from the representation of $f_L^N(b, b+N)$ in terms of multinomials given by Date et al. (cf. Theorem 4.2.5 in [6]) which is not appropriate for taking the limit $L \to \infty$.

Using (3.8), (3.3) and (3.11) the full solution of the recursion equations (3.4), (3.5) is given by

$$X_L^{N}(a|b, c) = q^{\frac{b-a(c-a)}{4N}} \sum_{j} \left( q^{\frac{1}{N}(jp(p-N)+p(b+c-N)-a(p-N))} T^{(N)}_{\frac{N+b-c}{2}}(L, \frac{b-a}{2} + pj) 
- q^{\frac{1}{N}(pj+a)(j(p-N)+\frac{b+c-N}{2})} T^{(N)}_{\frac{N+b-c}{2}}(L, \frac{b+a}{2} + pj) \right).$$

(3.13)

Introducing

$$r = \frac{1}{2}(b + c - N)$$
$$s = a$$
$$l = \frac{1}{2}(b - c + N) + 1.$$  

(3.14)
we can rewrite this expression in a more standard form

$$X_L^N(s|r + l - 1, r - l + N + 1) = \frac{1}{q^{(r-s+l-1)(r-s+N+1)}} \sum_j \left( q^{\frac{1}{N}(jp^q + pr - p^s)} T_{l-1}^{(N)}(L, \frac{1}{2}(r - s + l - 1) + pj) \right. \\
\left. - q^{\frac{1}{N}(jp^q + pr + r)} T_{l-1}^{(N)}(L, \frac{1}{2}(r + s + l - 1) + pj) \right) \quad (3.15)$$

where $p' = p - N$. This representation of the RSOS configuration sums in regime III generalizes the representations found by ABF [1] and AB [3] for the cases $N = 1$ and $N = 2$, respectively.

According to Date et al. [3, 8] $X_L(a|b, c)$ yields the branching functions $c_{r,s}^{(l)}$ for the $su(2)_{p-N-2} \times su(2)_N/su(2)_{p-2}$ coset models. In particular,

$$c_{r,s}^{(l)} = q^{-\eta} \lim_{L \rightarrow \infty, L \text{ even}} X_L(a|b, c) \quad (3.16)$$

where $r$, $s$ and $l$ as defined in (3.14). The exponent $\eta$ is given by

$$\eta = \frac{1}{4}(b - a) - \gamma(r, l, s) \quad (3.17)$$

where

$$\gamma(r, l, s) = \frac{r^2}{4(p - N)} + \frac{l^2}{4(N + 2)} - \frac{1}{8} - \frac{s^2}{4p} \quad (3.18)$$

From (3.16) we see that $T_n^{(N)}(r) = \lim_{L \rightarrow \infty, \text{even}} T_n^{(N)}(L, r)$ only depends on $r \mod N$. Further $T_n^{(N)}(r)$ has the symmetry $T_n^{(N)}(r) = T_n^{(N)}(n - r)$. Hence by defining

$$m_{r,s}(j) = \left\lfloor \frac{1}{2}(r - s) + pj \mod N \right\rfloor \quad (3.19)$$

where we use the convention that $y = x \mod N$ lies in the intervall $-N/2 < y \leq N/2$ we may write

$$c_{r,s}^{(l)} = q^{-\eta} q^{\frac{1}{N}(r+l-s-1)(r-l+N+1-s)} \sum_{m=0}^{N/2} T_{l-1}^{(N)}(m + \frac{l - 1}{2}) \times \left\{ \sum_{j \in \mathbb{Z}, m_{r,s}(j) \equiv m} q^{\frac{1}{N}(jp^q + pr - p^s)} - \sum_{j \in \mathbb{Z}, m_{r,s}(j) \equiv m} q^{\frac{1}{N}(jp^q + pr + r)} \right\} \quad (3.20)$$

Notice that if $L$ even $b - a$ is always even. Hence we can conclude that $r - s$ is even (odd) if $l$ is odd (even). Therefore, the sum runs over $m$ integer if $l$ odd and halfinteger if $l$ even. This generalizes the Neveu-Schwarz (NS) and Ramond (R) sectors that appear for $N = 2$. We further see that $c_{r,s}^{(l)}$ factorizes into a parafermionic piece given by $T_{l-1}^{(N)}$ and a Rocha-Caridi type sum where the exponents give the positions of the singular vectors of the conformal field theory.

To make the connection with the branching functions given in [36] we may further introduce

$$\alpha_{r,s}(j) = \frac{(2pp'j + pr - p's)^2 - N^2}{4Npp'} \quad (3.21)$$
In [36] the branching function $\chi^{(l)}_{r,s}$ is given by

$$\chi^{(l)}_{r,s} = \sum_{m=0}^{N} C_{2m}^l(q) \left\{ \sum_{j \in \mathbb{Z}, m \equiv m_{r,s}(j) \equiv m} q^{|\alpha_{r,s}(j)|} - \sum_{j \in \mathbb{Z}, m \equiv m_{r,-s}(j) \equiv m} q^{|\alpha_{r,-s}(j)|} \right\}$$  \hspace{1cm} (3.22)

where $C_{2m}^l$ are the $su(2)_N$ string functions [38], [40]. Here $m$ runs over integer if $l$ even (NS) and halfointeger if $l$ odd (R) (which is exactly opposite to the convention used by Date et al. [8]).

From (3.21) follows that

$$\frac{j}{N}(jp + pr - ps) = \alpha_{r,s}(j) - \frac{1}{4Npp}((pr - ps)^2 - N^2)$$

$$\frac{1}{N}(pj + s)(pr + r) = \alpha_{r,-s}(j) - \frac{1}{4Npp}((pr - ps)^2 - N^2).$$  \hspace{1cm} (3.23)

Using (3.17), (3.18) and (3.23) equation (3.21) hence becomes

$$c^{(l)}_{r,s} = q^{\frac{j}{4N} + \frac{N}{4pp} + \frac{j^2}{2N(N+2)}} \prod_{m=0}^{N/2} T_{l-1}^{(N)}(m + \frac{l-1}{2})$$

$$\times \left\{ \sum_{j \in \mathbb{Z}, m \equiv m_{r,s}(j)} q^{|\alpha_{r,s}(j)|} - \sum_{j \in \mathbb{Z}, m \equiv m_{r,-s}(j)} q^{|\alpha_{r,-s}(j)|} \right\}$$  \hspace{1cm} (3.24)

We now want to make the connection between $T_{l-1}^{(N)}$ and the $su(2)_N$ string functions $C_{2m}^l$. According to Lepowsky and Primc [41] the relation between $T_{l-1}^{(N)}$ and the fermionic form of the branching functions $b_{l}^m$ for the coset models $\tilde{su}(N)_1 \times \tilde{su}(N)_1/\tilde{su}(N)_2$ is given as

$$q^{\frac{e}{24} - h_m} b_{l}^m = \frac{1}{(q)^{\infty}} \left\{ \left( \sum_{k \geq 0} \sum_{n \geq 0} - \sum_{k \geq 0} \sum_{n < 0} \right) (-)^k q^{\frac{k(k+1)(1+1)n + (l+1)n + (l+m)k}{2} + (N+2)(n+k)n} \right. \right.$$ 

$$+ \left. \left( \sum_{k \geq 0} \sum_{n \geq 0} - \sum_{k \geq 0} \sum_{n < 0} \right) (-)^k q^{\frac{k(k+1)(1+1)n + (l+1)n + (l-m)k}{2} + (N+2)(n+k)n} \right\}$$  \hspace{1cm} (3.26)

where $l = 0, 1, \ldots, N - 1$, $l - m$ even and

$$h_m = \frac{l(l+2)}{4(N+2)} - \frac{m^2}{4N}$$  \hspace{1cm} (3.27)
The formula (3.26) is only valid for \(|m| \leq l\). For \(|m| > l\) one may use the symmetries
\[
b_m^l = b_{-m}^l = b_{m+2N}^l = b_{-m}^{N-l}
\] (3.28)
This double sum (3.26) is also related to the branching function \(e_{ml}^{(N)}\) of the coset models \(\tilde{s}u(2N)_1/\tilde{s}p(2N)_1\) \([3, 39, 39, 10]\) and in the notation of \([3]\) we have \(e_{ml}^{(N)} = b_m^l\). \(e_{ml}^{(N)}\) in turn is related to the string function of \(\tilde{s}u(2)_N\) via
\[
e_{ml}^{(N)} = q^{\tilde{c}N}(q)_{\infty}C_{ml}^d.
\] (3.29)
Hence we find
\[
q^{-\frac{1}{2} + \frac{N}{2}p} + q^{-\frac{1}{2} + \frac{N}{2}p} \sum_{m=0}^{N/2} \frac{1}{(q)_{\infty}} \frac{1}{(q)_{\infty}} b_m^{l-1} = \frac{1}{(q)_{\infty}} \frac{1}{(q)_{\infty}} b_m^{l-1}
\]
where \(\tilde{c} = 1 - \frac{6N}{pp} + \frac{2(N-1)}{N+2}\) is the central charge of the coset \(\tilde{s}u(2)_{p-N-2} \times \tilde{s}u(2)_N/\tilde{s}u(2)_{p-2}\).
Hence we get for (3.24)
\[
c_{r,s}^{(l)} = q^{\tilde{c}l} \sum_{m=0}^{N/2} \frac{1}{(q)_{\infty}} \frac{1}{(q)_{\infty}} b_m^{l-1} \left\{ \sum_{j \in \mathbb{Z}m \equiv m_{r,s}(j)} q^{\alpha_{r,s}(j)} - \sum_{j \in \mathbb{Z}m \equiv m_{r,-s}(j)} q^{\alpha_{r,-s}(j)} \right\}
\] (3.31)
and we see
\[
c_{r,s}^{(l+1)} = q^{\tilde{c}l} \chi_{r,s}^{(l)}.
\] (3.32)
Therefore, the representation of the configuration sum \(X_L(a|b,c)\) of the generalized RSOS models in terms of multinomials leads naturally to the branching functions of the coset models \(\tilde{s}u(2)_{p-N-2} \times \tilde{s}u(2)_N/\tilde{s}u(2)_{p-2}\) \([39]\) obtained by the Feigin and Fuchs construction \([33]\) and not to the representation given by Date et al. (see \([3]\) appendix A).

4 Proof of the recursion relations for the multinomials

In this section we prove the recursion relations (2.13) for the multinomials. First we notice that it is sufficient to prove (2.13) for \(n \leq N\) for all \(r = -\frac{NL}{2}, \ldots, \frac{NL}{2}\) since via the symmetry \(T_n(L,-r) = T_{N-n}(L,r)\) one automatically gets (2.13) for all \(0 \leq n \leq N\).

Let us now write out the right hand side of recursion relation (2.13) in terms of the definition of the multinomials
\[
\sum_{k=0}^{n-1} q^{(n-k)\left(\frac{1}{2} + \frac{r}{N}\right)} \sum_{\hat{m} \in \mathbb{Z}^{N+1}} q^{mC^{-1}m - \epsilon N - \hat{k}C^{-1}m} (q)_{L-1} \prod_{i=0}^{N} \frac{1}{(q)_{m_i}}
\]
\[
+ \sum_{k=n}^{N} q^{(k-n)\left(\frac{r}{2} - \frac{1}{N}\right)} \sum_{\hat{m} \in \mathbb{Z}^{N+1}} q^{mC^{-1}m - \epsilon N - \hat{k}C^{-1}m} (q)_{L-1} \prod_{i=0}^{N} \frac{1}{(q)_{m_i}}
\] (4.1)
where now the sums are restricted by

\[ m_0 = \frac{L - 1}{2} - \frac{1}{2} - \frac{r - k}{N} - \epsilon_1 C^{-1} m \in \mathbb{Z}_{\geq 0} \]  

(4.2)

\[ m_N = \frac{L - 1}{2} + \frac{1}{2} + \frac{r - k}{N} - \epsilon_{N-1} C^{-1} m \in \mathbb{Z}_{\geq 0} \]  

(4.3)

To transform the restrictions (4.2), (4.3) into the restrictions (2.10), (2.11) one can make the variable change \( m_k \to m_k - 1 \) in the \( k \)th summand for \( k = 1, 2, \ldots, N - 1 \) \((k = 0\) and \( k = N \) remain unchanged) to obtain

\[
\sum_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{N+1}} q^{mC^{-1}m} (q)_{L-1} \prod_{i=0}^{N} \frac{1}{(q)_{m_i}} \left\{ \sum_{k=0}^{n-1} q^{(n-k)(\frac{1}{2} + \frac{i}{N})-(2\epsilon_k C^{-1} + \epsilon_{N-k})C^{-1}m + g_{N}(k)} (1-q^{m_k}) \right. \\
\left. + \sum_{k=n}^{N} q^{(k-n)(\frac{1}{2} - \frac{i}{N})-(2\epsilon_k C^{-1} + \epsilon_{N-k})C^{-1}m + g_{N}(k)} (1-q^{m_k}) \right\}
\]

where

\[
g_N(k) = \begin{cases} 
    k & \text{for } 0 \leq k \leq \frac{N}{2} \\
    N - k & \text{for } \frac{N}{2} < k \leq N
\end{cases}
\]

(4.4)

This function arises from the variable change in the exponent when we use

\[ e_1 C^{-1} e_j = \frac{1}{N} \begin{cases} 
    (N - i)j & \text{for } j \leq i \\
    (N - j)i & \text{for } j > i
\end{cases}
\]

(4.5)

(4.6)

For the proof of (2.13) we need to transform the expression in (4.4) step by step into the expression for \( T^{(N)}_n(L, r) \) as given in (2.14). This is done by introducing four types of interpolating functions \( R^{(N)}_m(L, r, l) \) \((l = n, n + 1, \ldots, N)\), \( \bar{R}^{(N)}_m(L, r, l) \) \((l = n - 1, n - 2, \ldots, 0)\), \( G^{(N)}_m(L, r, l) \) \((l = 1, 2, \ldots, n)\) and \( \bar{G}^{(N)}_m(L, r, l) \) \((l = n - 1, n, \ldots, N - 1)\). These interpolating functions have the following properties

\[ R^{(N)}_m(L, r, l + 1) = R^{(N)}_m(L, r, l) \]  

(4.7)

\[ \bar{R}^{(N)}_m(L, r, l - 1) = \bar{R}^{(N)}_m(L, r, l) \]  

(4.8)

and \( R^{(N)}_m(L, r, N) + \bar{R}^{(N)}_m(l, r, 0) \) equals (4.4). The properties of \( G^{(N)}_m \) and \( \bar{G}^{(N)}_m \) are as follows

\[ G^{(N)}_m(L, r, l + 1) = G^{(N)}_m(L, r, l) \]  

(4.9)

\[ \bar{G}^{(N)}_m(L, r, l - 1) = \bar{G}^{(N)}_m(L, r, l) \]  

(4.10)

and \( G^{(N)}_m(L, r, 1) + \bar{G}^{(N)}_m(L, r, N - 1) = T^{(N)}_n(L, r) \). And further the relation between the \( R \) and \( G \) and the \( \bar{R} \) and \( \bar{G} \) interpolating functions are given as

\[ R^{(N)}_m(L, r, n) = G^{(N)}_m(L, r, n) \]  

(4.11)

\[ \bar{R}^{(N)}_m(L, r, n - 1) = \bar{G}^{(N)}_m(L, r, n - 1). \]  

(4.12)
The proof of recursion relation (2.13) then follows from the string of equations
\[
\text{rhs of (2.15) } = R_n^{(N)}(L, r, N) + \bar{R}_n^{(N)}(L, r, 0) \\
= R_n^{(N)}(L, r, n) + \bar{R}_n^{(N)}(L, r, n-1) \text{ see (4.7), (4.8)} \\
= G_n^{(N)}(L, r, n) + \bar{G}_n^{(N)}(L, r, n-1) \text{ see (4.11), (4.12)} \\
= G_n^{(N)}(L, r, 1) + \bar{G}_n^{(N)}(L, r, N-1) \text{ see (4.9), (4.10)} \\
= T_n^{(N)}(L, r),
\]
(4.13)

Having outlined the strategy of the proof let us define
\[
P_n^{(N)}(L, r, l) = \sum_{\tilde{m} \in Z^{N+1}} q^{mC-1m(q)}L-1 \prod_{i=0}^{N} \frac{1}{(q)m_i} \\
\times \left\{ \sum_{k=n}^{l-1} q^{(k-n)(\frac{4}{3}+\frac{m}{3})} A^{(N)}(L, r, l|k) + q^{(l-n)(\frac{4}{3}+\frac{m}{3})} \sum_{k=l}^{N} B^{(N)}(L, r, l|k) \right\}
\]
(4.14)
for \(l = n, n+1, \ldots, N\) where
\[
A^{(N)}(L, r, l|k) = q^{-(2e_k+e_{N-k})C-1m+g_N(k)} \left(1 - q^{m_k}\right)
\]
(4.15)
\[
B^{(N)}(L, r, l|k) = q^{-(e_{N-l+1}+e_{k-l}+e_{k-l+1})C-1m+g_N(l+1)-g_{[k-l+1,l-1]}(N-l+1)} \left(1 - q^{m_k}\right).
\]
(4.16)
and for \(n \geq m\)
\[
g_{m,n}(k) = \begin{cases} 
  k & \text{for } 0 \leq k \leq m \\
  m & \text{for } m < k < n \\
  m + n - k & \text{for } n \leq k \leq n + m \\
  0 & \text{otherwise}
\end{cases}
\]
(4.17)

We further define
\[
g_{[m,n]}(k) = \begin{cases} 
  g_{m,n}(k) & \text{if } m \leq n \\
  g_{n,m}(k) & \text{if } m > n
\end{cases}
\]
(4.18)
The sum \(\sum\) denotes from now on always the sum over \(\tilde{m} \in Z^{N+1}\) such that (2.10) and (2.11) are satisfied and sums \(\sum_{k=i}^{j}\) with \(j < i\) are defined to be zero.

Similarly we define
\[
\bar{R}_n^{(N)}(L, r, l) = \sum_{\tilde{m} \in Z^{N+1}} q^{mC-1m(q)}L-1 \prod_{i=0}^{N} \frac{1}{(q)m_i} \\
\times \left\{ \sum_{k=l+1}^{n-1} q^{(n-k)(\frac{4}{3}+\frac{m}{3})} \bar{A}^{(N)}(L, r, l|k) + q^{(n-l)(\frac{4}{3}+\frac{m}{3})} \sum_{k=0}^{l} \bar{B}^{(N)}(L, r, l|k) \right\}
\]
(4.19)
where now \(l = n-1, n-2, \ldots, 0\). For \(n = 0\) \(\bar{R}_n^{(N)}(L, r, l)\) is defined to be zero. \(\bar{A}\) and \(\bar{B}\) are defined as
\[
\bar{A}^{(N)}(L, r, l|k) = q^{-(2e_k+e_{N-k})C-1m+g_N(k)} \left(1 - q^{m_k}\right)
\]
(4.20)
\[
\bar{B}^{(N)}(L, r, l|k) = q^{-(e_{N-l+1}+e_{N-k-l+1})C-1m+g_N(l+1)-g_{[k-l+1,N-l]}(l+1)} \left(1 - q^{m_k}\right)
\]
(4.21)
Notice that $B_n^{(N)}(L,r,N) + B_n^{(N)}(L,r,0)$ equals (4.4) which was one of the conditions for $R$ and $\bar{R}$. To prove (4.7) and (4.8) we need to make variable changes in $m$. It will turn out to be useful to introduce

$$E_i = -e_{i-1} + 2e_i - e_{i+1} \tag{4.22}$$

for $i = 0, 1, 2, \ldots, N$ (remember that $e_i = 0$ if $i \not\in \{1, 2, \ldots, N - 1\}$). Since $E_i$ is the $i^{th}$ column of $C$ for $i = 1, 2, \ldots, N - 1$ we have

$$C^{-1}E_i = e_i, \text{ for } i = 1, 2, \ldots, N - 1. \tag{4.23}$$

Let us further define for $m \leq l$

$$\bar{E}_{m,l} = - \sum_{i=m}^{l} E_i = e_{m-1} - e_{m} + e_{l+1} - e_{l}. \tag{4.24}$$

One may derive for $i \leq j$

$$\sum_{k=i-l+1}^{i} \bar{E}_{k,i+j-k} = e_{i-l} - e_{i} - e_{j} + e_{j+l}$$

$$= - \left( \sum_{k=i-l}^{i-1} (k - i + l)E_k + \sum_{k=i}^{j-1} E_k + \sum_{k=j}^{j+l} (j + l - k)E_k \right). \tag{4.25}$$

To prove (4.7) we rewrite all terms $B^{(N)}(L,r,l+1|k)$ ($k = l + 1, \ldots, N$) in $R_n^{(N)}(L,r,l+1)$ as follows

$$q^{-\bar{E}} B^{(N)}(L,r,l+1|k)$$

$$= q^{-\bar{E}} B^{(N)}(L,r,l+1|k) - q^{-\bar{E}} B^{(N)}(L,r,l+1|k)(1 - q^{m_0})$$

$$= q^{(e_{1} - e_{N-l+1} + e_{k-l-1} + e_{k-l} - 2e_k) C^{-1} m + g_N l - g_{[k-l]} (N-l)} (1 - q^{m_k})$$

$$- q^{(e_{1} - e_{N-l+1} + e_{k-l-1} + e_{k-l} - 2e_k) C^{-1} m + g_N l - g_{[k-l]} (N-l)} (1 - q^{m_k}) (1 - q^{m_0}) \tag{4.26}$$

remembering that $m_0$ is given by (2.10). In the second term one can make the variable change $m \rightarrow m + v$ with

$$v = -e_{k-l} - e_{l} + e_{k} = - \sum_{j=1}^{k-l} \bar{E}_{j,k-j}$$

$$= \begin{cases} 
- (\sum_{j=1}^{k-l} jE_j + (k - l) \sum_{j=k-l+1}^{l} E_j + \sum_{j=1}^{k-l} (k - j)E_j) & \text{for } k - l \leq l \\
- (\sum_{j=1}^{k-l} jE_j + l \sum_{j=k-l+1}^{l} E_j + \sum_{j=k-l}^{k-1} (k - j)E_j) & \text{for } k - l > l \tag{4.27}
\end{cases}$$

where we used (4.25). Hence under the sum $\sum_{m \in \mathbb{Z}^{N+1}} q^{mC^{-1} m} (q) L - 1 \prod_{i=0}^{N} \frac{1}{(q)^{m_i}}$ (in the following we abbreviate this expression by “under the sum $\sum_{m}$”) the second term in (4.26) becomes

$$- q^{(e_{1} - e_{N-l+1} - e_{k-l} - e_{k-l}) C^{-1} m + g_N l} (1 - q^{m_k}) \begin{cases} 
1 - q^{m_l} & \text{for } k - l < l \\
1 - q^{m_{k-l}} & \text{for } k - l = l \\
q^{-1} (1 - q^{m_k}) & \text{for } k - l > l \tag{4.28}
\end{cases}$$
We see that in the sum $\sum_{k=l+1}^{N}$ these terms cancel pairwise (the $-q^{m_{k-1}}$ in the $k^{th}$ term cancels the 1 in the $(k + 1)^{th}$ term since $m_{i} = E_{i}C^{-1}m = (-e_{i-1} + 2e_{i} - e_{i+1})C^{-1}m$). Hence only the 1 for $k = l + 1$ and $-q^{m_{N-N}}$ for $k = N$ remain. Altogether we get

$$R_{n}^{(N)}(L, r, l + 1) = \sum_{\tilde{m} \in \mathbb{Z}_{N+1}} q^{mC^{-1}m}(q)_{L-1} \prod_{i=0}^{N} \frac{1}{(q)_{m_{i}}}$$

$$\times \left\{ \sum_{k=n}^{l} q^{(k-n)(\frac{L}{2} - \frac{r}{2})} A^{(N)}(L, r, l + 1|k) \right. + q^{(l+1-n)(\frac{L}{2} - \frac{r}{2})} \sum_{k=l+1}^{N} q^{-m_{o}} B^{(N)}(L, r, l + 1|k)$$

$$- q^{(l-n)(\frac{L}{2} - \frac{r}{2})+(-e_{N-l-2e_{l})C^{-1}m+g \frac{N}{2}(l)}(1 - q^{m_{l}})$$

$$+ q^{(l-n)(\frac{L}{2} - \frac{r}{2})+(e_{l}-e_{N-l-1-2e_{l})C^{-1}m+g \frac{N}{2}(l)-\theta(N-2l)(1 - q^{m_{l}})} \right\}$$

(4.29)

where $\theta(x) = 1$ for $x \geq 0$ and 0 otherwise. But

$$A^{(N)}(L, r, l + 1|l) = q^{-2e_{l}+e_{N-l-1}C^{-1}m+g \frac{N}{2}(l)}(1 - q^{m_{l}})$$

(4.30)

and

$$B^{(N)}(L, r, l|l) = q^{-e_{N-l-1+e_{l}-2e_{l})C^{-1}m+g \frac{N}{2}(l-1)+g_{1, l-1}(N-l+1)}(1 - q^{m_{l}})$$

(4.31)

and $A^{(N)}(L, r, l + 1|k) = A^{(N)}(L, r, l|k)$ for $k < l$. One may further show that

$$g_{\frac{N}{2}}(l) - \theta(N-2l) = g_{\frac{N}{2}}(l-1) - g_{1, l-1}(N-l+1).$$

(4.32)

Hence we have proven that

$$R_{n}^{(N)}(L, r, l + 1) = \sum_{\tilde{m} \in \mathbb{Z}_{N+1}} q^{mC^{-1}m}(q)_{L-1} \prod_{i=0}^{N} \frac{1}{(q)_{m_{i}}}$$

$$\times \left\{ \sum_{k=n}^{l-1} q^{(k-n)(\frac{L}{2} - \frac{r}{2})} A^{(N)}(L, r, l|k) \right. + q^{(l+1-n)(\frac{L}{2} - \frac{r}{2})} \sum_{k=l+1}^{N} q^{-m_{o}} B^{(N)}(L, r, l + 1|k)$$

$$+ q^{(l-n)(\frac{L}{2} - \frac{r}{2})} B^{(N)}(L, r, l|l) \right\}$$

(4.33)

To prove (4.7) it therefore remains to show that under the sum $\sum$

$$q^{\frac{L}{2} - \frac{r}{2}} \sum_{k=l+1}^{N} q^{-m_{o}} B^{(N)}(L, r, l + 1|k) = \sum_{k=l+1}^{N} B^{(N)}(L, r, l|k).$$

(4.34)

To this end we define the following function $H$ for $d \leq N - l$ and $p = d, d + 1, \ldots, N - l$

$$H^{(N)}(L, r, l|d, p) = \sum_{k=l+p}^{N} q^{-\sum_{i=d}^{p-1} m_{i} - e_{d-1}C^{-1}m} B^{(N)}(L, r, l + 1|k).$$

(4.35)
\[ H \] has the property that under the sum \( \sum \)
\[ H^{(N)}(L, r, l|d, p) = H^{(N)}(L, r, l|d, p + 1) + q^{-e_dC^{-1}m}B^{(N)}(L, r, l|p + l) \\
= H^{(N)}(L, r, l|d, p + 1) + q^{(-e_{N-l+1} - e_d + e_{p} + e_{p+1} - 2e_{p+1})C^{-1}m + g_{N}(l-1) - g_{[p+1,l-1]}(N-l+1)}(1 - q^{m_{p+l}}) \]
\[
(4.36)
\]

The proof of this formula is similar to the derivation of \((4.33)\). One rewrites all the terms in the sum and makes appropriate variable changes. For the details of the proof we refer the reader to appendix \( \Delta \).

Using \((4.36)\) we have under the sum \( \sum \)
\[ q^{\frac{L}{N} - \frac{l}{N}} \sum_{k=l+1}^{N} q^{-m_{0}}B^{(N)}(L, r, l + 1|k) \]
\[ = q^{\frac{L}{N} - \frac{l}{N} - m_{0}}H^{(N)}(L, r, l|1, 1) \]
\[ = q^{\frac{L}{N} - \frac{l}{N} - m_{0}} \left\{ H^{(N)}(L, r, l|1, N - l) + \sum_{p=1}^{N-l-1} q^{-e_{1}C^{-1}m}B^{(N)}(L, r, l|p + l) \right\} \]
\[ = q^{e_{1}C^{-1}m} \left\{ q^{-\sum_{i=1}^{N-l-1} m_{i}}q^{e_{N-l-1}C^{-1}m + g_{N}(l) - g_{[N-l, l]}(N-l)}(1 - q^{m_{N}}) \right. \]
\[ + \sum_{k=l+1}^{N-l} q^{-e_{1}C^{-1}m}B^{(N)}(L, r, l|k) \left. \right\} \]
\[ = \sum_{k=l+1}^{N} B^{(N)}(L, r, l|k). \]
\[
(4.37)
\]

For the last equality we used \( g_{N}(l) - g_{[N-l, l]}(N-l) = 0 \) and \(-\sum_{i=1}^{N-l-1} m_{i} = (-e_{1} - e_{N-l-1} + e_{N-l})C^{-1}m \) which follows from \((4.24)\). Hence we conclude
\[ R^{(N)}_{n}(L, r, l + 1) = R^{(N)}_{n}(L, r, l). \]
\[
(4.38)
\]

The proof of \((1.8)\) is analogous (one can actually obtain \((1.8)\) from \((1.7)\) by dropping the term \( k = n \) and then replacing \( r \rightarrow -r, m_{i} \rightarrow m_{N-i} \) and identifying \( n_{\text{new}} = N - n \) and \( l_{\text{new}} = N - l \).

Notice that for \( n = 0 \) the proof of \((2.13)\) is already complete since
\[ \text{rhs of (2.15)} \]
\[ = R^{(N)}_{0}(L, r, N) + R^{(N)}_{0}(L, r, 0) \]
\[ = R^{(N)}_{0}(L, r, N) = R^{(N)}_{0}(L, r, 1) \quad \text{via (4.7)} \]
\[ = \sum_{m \in \mathbb{Z}^{N+1}} q^{mC^{-1}m}(q)L_{-1} \prod_{i=0}^{N} \frac{1}{(q)m_{i}} \left\{ (1 - q^{m_{a}}) + q^{\frac{L}{N}} \sum_{k=1}^{N} q^{(e_{k-1} - e_{k})C^{-1}m(1 - q^{m_{k}})} \right\} \]
\[ 1-q^{L} \]
\[
(4.39)
\]
Again all terms cancel pairwise except the 1 in the first term and the \(-q^{nN}\) in the last term \((k = N)\). Using (2.11) one obtains \(1 - q^L\) for the terms in the bracket which can be combined with \((q)_{L-1}\) to \((q)_L\). Hence

\[
\text{rhs of (2.15)} = \sum_{\tilde{m} \in \mathbb{Z}^{N+1}} q^{mC-1m} (q)_L \prod_{i=0}^{N} \frac{1}{(q)_{m_i}} = T_0^{(N)}(L, r)
\]

and (2.15) is proven for \(n = 0\).

For \(n > 0\) we need to introduce the functions \(G_n^{(N)}(L, r, l)\) and \(\bar{G}_n^{(N)}(L, r, l)\). Let us define

\[
G_n^{(N)}(L, r, l) = \sum_{\tilde{m} \in \mathbb{Z}^{N+1}} q^{mC-1m} (q)L_{-1} \prod_{i=0}^{N} \frac{1}{(q)_{m_i}} \left\{ \sum_{k=n}^{N} q^{-e_{n-l}C-1m} B^{(N)}(L, r, l|k) \right\}
\]

for \(l = 1, 2, \ldots, n\) and similarly

\[
\bar{G}_n^{(N)}(L, r, l) = q^{\frac{q}{2} + \frac{q}{4}} \sum_{\tilde{m} \in \mathbb{Z}^{N+1}} q^{mC-1m} (q)L_{-1} \prod_{i=0}^{N} \frac{1}{(q)_{m_i}} \left\{ \sum_{k=n}^{N} q^{-e_{n+n-l-1}C-1m} \bar{B}^{(N)}(L, r, l|k) \right\}
\]

for \(l = n - 1, n, \ldots, N - 1\). Notice that with these definitions equations (4.11) and (4.12) hold.

To show (4.9) we may write

\[
G_n^{(N)}(L, r, l + 1) = \sum_{\tilde{m} \in \mathbb{Z}^{N+1}} q^{mC-1m} (q)_L \prod_{i=0}^{N} \frac{1}{(q)_{m_i}} \left\{ \sum_{k=n}^{N} q^{-e_{n-l-1}C-1m} B^{(N)}(L, r, l + 1|k) \right\}
\]

\[
= \sum_{\tilde{m} \in \mathbb{Z}^{N+1}} q^{mC-1m} (q)_L \prod_{i=0}^{N} \frac{1}{(q)_{m_i}} H^{(N)}(L, r, l|n - l, n - l).
\]

Via (4.34) we get under the sum \(\tilde{\sum}\)

\[
H^{(N)}(L, r, l|n - l, n - l) = H^{(N)}(L, r, l|n - l, N - l) + \sum_{p=n-l}^{N-l-1} q^{-e_{n-l}C-1m} B^{(N)}(L, r, l|p + l)
\]

\[
= q^{-\sum_{i=n-l}^{N-l-1} m_i - e_{n-l}C-1m} B^{(N)}(L, r, l + 1|N) + \sum_{k=n}^{N-1} q^{-e_{n-l}C-1m} B^{(N)}(L, r, l|k)
\]

Using now \(-\sum_{i=n-l}^{N-l-1} m_i = (e_{n-l-1} - e_{n-l} - e_{N-l-1} + e_{N-l})C-1m\) via (4.24) we see that

\[
G_n^{(N)}(L, r, l + 1) = \sum_{\tilde{m} \in \mathbb{Z}^{N+1}} q^{mC-1m} (q)_L \prod_{i=0}^{N} \frac{1}{(q)_{m_i}} \sum_{k=n}^{N} q^{-e_{n-l}C-1m} B^{(N)}(L, r, l|k)
\]

\[
= G_n^{(N)}(L, r, l).
\]
Finally, we need to show that $G_n^{(N)}(L, r, 1) + \bar{G}_{n}^{(N)}(L, r, N - 1) = T_n^{(N)}(L, r)$. To this end we may write

\[
G_n^{(N)}(L, r, 1) = \sum_{m \in \mathbb{Z}^{N+1}} q^{mC-1}m(q)^{L-1} \prod_{i=0}^{N} \frac{1}{(q)_{m_i}} \left\{ \sum_{k=n}^{N} q^{-\epsilon_n C^{-1}m} B^{(N)}_{(L, r, 1|k)} \right\}
\]

\[
\bar{G}_{n}^{(N)}(L, r, N - 1) = \sum_{m \in \mathbb{Z}^{N+1}} q^{mC-1}m(q)^{L-1} \prod_{i=0}^{N} \frac{1}{(q)_{m_i}} \sum_{k=n}^{N} q^{(-\epsilon_n - 1 + e_{k-1} - e_k)C^{-1}m} (1 - q^{m_k})
\]

\[
\bar{G}_{n}^{(N)}(L, r, N - 1) = \sum_{m \in \mathbb{Z}^{N+1}} q^{mC-1}m(q)^{L-1} \prod_{i=0}^{N} \frac{1}{(q)_{m_i}} \left\{ q^{-\epsilon_n C^{-1}m} - q^{\left(\frac{1}{2} + \frac{1}{N}\right) - \epsilon_n C^{-1}m} \right\}
\]

(4.46)

since the terms in the sum $\sum_{k=n}^{N}$ again cancel pairwise except for the most outer terms. Similarly

\[
G_n^{(N)}(L, r, N - 1) = q^{L + \frac{N}{2}} \sum_{m \in \mathbb{Z}^{N+1}} q^{mC-1}m(q)^{L-1} \prod_{i=0}^{N} \frac{1}{(q)_{m_i}} \left\{ q^{-\epsilon_n C^{-1}m} - q^{L} - \frac{1}{N} - \epsilon_n C^{-1}m \right\}
\]

(4.47)

Inserting (4.46) and (4.47) into (4.13) yields

\[
\text{rhs of (2.15)} = \sum_{m \in \mathbb{Z}^{N+1}} q^{mC-1}m(q)^{L-1} \prod_{i=0}^{N} \frac{1}{(q)_{m_i}} \left\{ q^{-\epsilon_n C^{-1}m} - q^{L - \epsilon_n C^{-1}m} \right\}
\]

(4.48)

and hence (2.13) is proven.

5 Summary and Discussion

We have given explicit formulas for the multinomials in (2.8) which generalize the trinomials first introduced by Andrews and Baxter [3]. These multinomials satisfy depth one recursion relations (2.13) (depth one since (2.13) relates $T$ at $L$ to $T$'s at $L - 1$). These recursion relations have been proven in section 4. Using the recursion relations we were able to express the configuration sums for the general RSOS models previously studied by Date et al. [4-6] in terms of multinomials (see (3.15)). These formulas differ from the configuration sums obtained by Date et al. Whereas Date et al. [4, 6] get double sum expressions for the branching functions of the $\widehat{su}(2)_{p-N-2} \times \widehat{su}(2)_{N}/\widehat{su}(2)_{p-2}$ coset (where $p$ and $N$ were introduced in section 3), which in turn can be expressed in terms of elliptic theta functions, the expressions in terms of the multinomials yield formulas of the Rocha-Caridi type. They factor into a parafermionic piece and a Rocha-Caridi piece (see (3.24)). With the help of relations between string functions and branching functions of certain models we were able to express the parafermionic piece in terms of the string functions of $\widehat{su}(2)_{N}$. The so obtained branching functions for the $\widehat{su}(2)_{p-N-2} \times \widehat{su}(2)_{N}/\widehat{su}(2)_{p-2}$ coset models are of the form given in (3.6).

Even though it sufficed for our purposes to prove a depth one recursion relation for the multinomials there are other recursion relations of higher depth. Andrews and Baxter [3]
found that the trinomials satisfy depth one recursion relations which mix the different types $T_1$ and $T_0$. The recursion relations (2.15) also mix the different types of multinomials $T^{(N)}_n$ with $n = 0, 1, \ldots, N - 1$. But in addition there exist recursion relations for the trinomials which become tautologies for $q = 1$ and recursion relations of higher depths some of which only involve the same type of trinomials [3], [43], [44], [28]. The recursion relations for the trinomials of higher depth were used in [28] to get the bosonic form for the characters $SM(2, 4\nu)$.

As already mentioned in the introduction one might hope to obtain polynomial expressions for the coset constructions $\hat{su}(2)_N \times \hat{su}(2)_M / \hat{su}(2)_{N+M}$ with fractional level $M$ in terms of multinomials. Date et al. [7] give an expression of the configuration sum of higher rank coset constructions in terms of multinomials. One might further speculate that multinomials with the matrix $C^{-1}$ replaced by a different matrix $B$ lead to useful generalizations of the multinomials introduced in this paper.

Further it is interesting to explore the connections and implications of the multinomials for partition theory and hypergeometric functions. A partition interpretation in terms of Durfee dissections of a partition has been given in [37], [45], [46], [47], [48], [49] and references therein. In [46] Andrews et al. find expressions for the generating functions of partitions with prescribed hook differences which yield the characters for the minimal models $M(p, p')$ when $p$ and $p'$ are coprime. A partition theoretic interpretation for the $\hat{su}(2)_M \times \hat{su}(2)_N / \hat{su}(2)_{M+N}$ coset models should also exist.

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A Appendix

In this appendix we want to prove equation (4.36), namely that under the sum

$$H^{(N)}(L, r, l|d, p) = H^{(N)}(L, r, l|d, p + 1) + q^{-e_d C^{-1} m} B^{(N)}(L, r, l|p + l)$$

$$= H^{(N)}(L, r, l|d, p + 1) + q^{(-e_{N-l+1} - e_d + e_p + e_{p+1} - 2e_{p+l})C^{-1} m + g N^l - g_{[p+1, l-1]}(N-l+1)(1 - q^{m_{p+l}})}$$

(A.1)

We start with the left hand side of this equation and write out explicitly the definition of $H$ according to (4.35)

$$H^{(N)}(L, r, l|d, p)$$

$$= \sum_{k=l+p}^{N} q^{(-e_d - e_{p-1} + e_{N-l} + e_{k-l-1} + e_{k-1} - 2e_k)C^{-1} m + g N^k - g_{[k-1, l]}(N-l)(1 - q^{m_k})}$$

(A.2)
Let us take all terms in the sum with \( k = l + p + 1, l + p + 2, \ldots, N \) and rewrite them as

\[
\begin{align*}
q & \left(-e_d - e_{p+1} - e_{N-l} - e_{k-l} - 2e_k\right) C^{-1} m + g_N(0) - g_{[k-l,l]}(N-l) (1 - q^{m_k}) \\
= & q \left(-e_d - e_{p+1} - e_{N-l} - e_{k-l} - 2e_k\right) C^{-1} m + g_N(0) - g_{[k-l,l]}(N-l) (1 - q^{m_k}) \\
- & q \left(-e_d - e_{p+1} - e_{N-l} - e_{k-l} - 2e_k\right) C^{-1} m + g_N(0) - g_{[k-l,l]}(N-l) (1 - q^{m_k})(1 - q^{m_p}) \\
& \text{(A.3)}
\end{align*}
\]

Now we make the variable change \( m \to m + v \) in the second term of (A.3) where

\[
v = e_p - e_{k-l} - e_{l+p} + e_k
\]

\[
\begin{align*}
= & \left\{ \begin{array}{ll}
\sum_{j=p+1}^{k-l} E_{j,k+p-j} & \text{for } k - l < l + p \\
\sum_{j=p+1}^{l+p} E_{j,k+p-j} & \text{for } k - l > l + p \\
\end{array} \right. \\
= & \left\{ \begin{array}{ll}
\sum_{j=p+1}^{k-l} (j-p) E_{j} + (k - l - p) \sum_{j=k-l+1}^{l+p-1} E_{j} + \sum_{j=l+p}^{k} k - l - j) E_{j} & \text{if } k - l < l + p \\
\sum_{j=p+1}^{l+p} (j-p) E_{j} + l \sum_{j=l+p+1}^{k-l} E_{j} + \sum_{j=k-l}^{k} (k - j) E_{j} & \text{if } k - l > l + p \\
\sum_{j=p+1}^{l+p} (j-p) E_{j} + \sum_{j=l+p+1}^{k} (k - j) E_{j} & \text{if } k - l = l + p \\
\end{array} \right. \\
= & \left\{ \begin{array}{ll}
q^{q^{k-l-p,l}(N-l-p)-1}(1 - q^{m_{k-l}}) & \text{for } k - l < l + p \\
q^{q^{l,p,l}(N-l-p)-1}(1 - q^{m_{k-l}}) & \text{for } k - l = l + p \\
q^{k-l-p,N-l-p-1}(1 - q^{m_{k-l}}) & \text{for } k - l > l + p \\
\end{array} \right. \\
& \text{(A.4)}
\]

where we used (A.25) and the fact that \( k = l + p + 1, \ldots, N \) and hence \( k - l > p \). Therefore, after making the variable change (A.4) the second term of (A.3) becomes under the sum \( \sum \)

\[
- q \left(-e_d - e_{p+1} - e_{N-l} - 2e_{l+p+1} + e_{k-l} - 2e_k\right) C^{-1} m + g_N(0) - g_{[k-l,l]}(N-l) (1 - q^{m_{k-l}}) \\
\times \left\{ \begin{array}{ll}
q^{q^{k-l-p,l}(N-l-p)-1}(1 - q^{m_{k-l}}) & \text{for } k - l < l + p \\
q^{q^{l,p,l}(N-l-p)-1}(1 - q^{m_{k-l}}) & \text{for } k - l = l + p \\
q^{k-l-p,N-l-p-1}(1 - q^{m_{k-l}}) & \text{for } k - l > l + p \\
\end{array} \right. \\
& \text{(A.5)}
\]

Notice that

\[
g_{[k-l,p,l]}(N-l-p) - g_{[k-l,l]}(N-l) = \left\{ \begin{array}{ll}
-p & \text{for } N - l \leq l \\
N - p - 2l & \text{for } l < N - l < l + p \\
0 & \text{for } l + p \leq N - l \\
\end{array} \right. \\
& \text{(A.6)}
\]

where we used that \( k = l + p + 1, \ldots, N \) and hence \( N - l - p \geq k - l - p \). From (A.4) we see that \( g_{[k-l,p,l]}(N-l-p) - g_{[k-l,l]}(N-l) \) is \( k \) independent. Therefore it follows again that under the sum \( \sum_{k=l+p+1}^{N} \) all terms in (A.3) cancel pairwise except for the 1 for \( k = l+p+1 \) and the \(-q^{m_{k-l}} \) for \( k = N \). The sum \( \sum_{k=l+p+1}^{N} \) of the first term in (A.3) gives exactly \( H^{(N)}(L,r,l,d,l+1) \). Hence

\[
H^{(N)}(L,r,l,d,l+1) = H^{(N)}(L,r,l,d,l+1) + (1 - q^{m_{p+1}}) \left\{ \begin{array}{ll}
q^{(2e_p - e_{p+1} - e_{N-l} - 2e_{l+p}) C^{-1} m + g_N(0) - g_{[l,p]}(N-l)} \\
- q^{(2e_p - e_{N-l} - 2e_{p+1}) C^{-1} m + g_N(0) + g_{[l+1,p]}(N-l-p) - g_{[1,l]}(N-l)} \\
+ q^{(e_p + e_{p+1} - e_{N-l} - e_{l+p+1}) C^{-1} m + g_N(0) - g_{[l,N-l-p]}(N-l-p) - q^{2l-p}} \\
\end{array} \right. \\
& \text{(A.7)}
\]
where the second term comes from the term \( k = l + p \) in (A.2). From (A.6) we can immediately see that the second and third term in (A.7) cancel. To prove (4.36) it merely remains to show that
\[
g_N^2(l) - g_{[N-l,l]}(N-l) + g_{[l,N-l-p]}(N-l-p) - \theta(N-2l-p) = g_N^2(l-1) - g_{[p+1,l-1]}(N-l+1). \tag{A.8}
\]
But \( g_N^2(l) = g_{[N-l,l]}(N-l) \) and
\[
g_{[l,N-l-p]}(N-l-p) - \theta(N-2l-p)
= \begin{cases} 
N - l - p & \text{for } N - l - p \leq l \\
l & \text{for } N - l - p \geq l 
\end{cases} - \begin{cases} 
1 & \text{for } N - l - p \geq l \\
0 & \text{for } N - l - p < l
\end{cases}
\]
\[
= \begin{cases} 
N - l - p & \text{for } N - l - p < l \\
l - 1 & \text{for } N - l - p \geq l 
\end{cases} \tag{A.9}
\]
and
\[
g_N^2(l-1) - g_{[p+1,l-1]}(N-l+1)
= \begin{cases} 
l - 1 & \text{for } l - 2 \leq N - l \\
N - l + 1 & \text{for } N - l \leq l - 2
\end{cases} - \begin{cases} 
p + 1 & \text{for } N - l \leq l - 2 \\
p + 2l - N - 1 & \text{for } l - 1 \leq N - l + 1 \leq l + p
\end{cases}
\]
\[
= \begin{cases} 
N - l - p & \text{for } N - l - p < l \\
l - 1 & \text{for } N - l - p \geq l 
\end{cases} \tag{A.10}
\]
where we used again that \( p \leq N - l \). Hence (4.36) is proven.

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