ASYMPTOTICS OF QUANTUM $6j$ SYMBOLS

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Abstract. The asymptotics of the quantum $6j$ symbols corresponding to a hyperbolic tetrahedron is investigated and the first two leading terms are determined for the case that the tetrahedron has a ideal or ultra-ideal vertex. These terms are given by the volume and the determinant of the Gram matrix of the tetrahedron. A relation to the volume conjecture of the Turaev-Viro invariant is also discussed.

Introduction

The volume conjecture for the Turaev-Viro invariant of three manifold \cite{14} is introduced by the first author and T. Yang in \cite{1}. Let $\xi_r = \exp(2\pi\sqrt{-1}/r)$ for a positive odd integer $r \geq 3$. Substituting $q = \xi_r^2$ instead of $q = \xi_r$ and doing numerical experiments for some manifolds, they show that the Turaev-Viro invariant has exponential growth with respect to $r$ and the leading asymptotic is determined by the hyperbolic volume of the manifold. Since the Turaev-Viro invariant is constructed by using the quantum $6j$ symbols, this conjecture suggests that the asymptotics of the quantum $6j$ symbol is expressed by some geometric data of the corresponding tetrahedron.

The volume conjecture can be reformulated for the quantum $6j$ symbols as follows. Let

$$\begin{bmatrix} a & b & c \\ d & c & f \end{bmatrix}_q^{RW}$$

be the Racah-Wigner version of the quantum $6j$ symbol which is defined for six non-negative half integers $a$, $b$, \ldots, $f$ with the quantum parameter $q$.

For a odd integer $r \geq 3$, a triplet $(a, b, c)$ is called $r$-admissible if $a, b, c \in \{0, 1/2, 1, \ldots, (r-2)/2\}$, $(a, b, c)$ satisfies the Clebsch-Gordan condition and $a+b+c \leq r-2$. The Clebsch-Gordan condition means that $|a-b| \leq c \leq a+b$ and $a+b+c \in \mathbb{Z}$.

Conjecture 1. Let $T$ be a tetrahedron in hyperbolic, Euclidean or spherical 3-space with dihedral angles $\theta_a$, $\theta_b$, $\theta_c$, $\theta_d$, $\theta_e$, $\theta_f$ at edges $a$, \ldots, $f$ in Figure 1. Let $a_r$, $b_r$, \ldots, $f_r$ be sequences of non-negative half integers satisfying

$$\lim_{r \to \infty} 2\pi \frac{2a_r + 1}{r} = \pi - \theta_a, \quad \lim_{r \to \infty} 2\pi \frac{2b_r + 1}{r} = \pi - \theta_b, \quad \ldots, \quad \lim_{r \to \infty} 2\pi \frac{2f_r + 1}{r} = \pi - \theta_f$$
and the triplets \((a_r, b_r, e_r), (a_r, d_r, f_r), (b_r, d_r, f_r)\) and \((c_r, d_r, e_r)\) are all \(r\)-admissible for odd \(r \geq 3\). Then

\[
\lim_{r \to \infty} \frac{2 \pi}{r} \log \left| \begin{pmatrix} a_r & b_r & e_r \\ d_r & c_r & f_r \end{pmatrix}_{\xi_r}^{RW} \right| = \text{Vol}(T),
\]

where \(\text{Vol}(T)\) is the hyperbolic volume of \(T\) if \(T\) is hyperbolic and \(\text{Vol}(T) = 0\) if \(T\) is Euclidean or spherical.

Here we show that the above conjecture is true if \(T\) is hyperbolic and has at least one ideal or ultra-ideal vertex. Moreover, we also show that the second leading term of the asymptotics of the quantum 6\(j\) symbol is given by the determinant of the Gram matrix.

The hyperbolic tetrahedron we consider here is determined by four planes containing the faces of the tetrahedron. We assume that any two of these planed have intersection, which corresponds to a edge of the tetrahedron. Vertices of our hyperbolic tetrahedron are classified into three cases. Let \(v\) be a vertex of a tetrahedron \(T\), \(F_1, F_2, F_3\) be the three planes of \(T\) to specify \(v\), and \(\theta_{ij}\) be the dihedral angle of \(F_i\) and \(F_j\) at the intersection of \(F_i \cap F_j\). There are two choices of the angle at the edge, and we pick up the angle corresponding to the inside of \(T\) (inner angle).

**Normal vertex:** The first case is the usual vertex, which is the intersection \(F_1 \cap F_2 \cap F_3\). In this case, \(\theta_{12} + \theta_{13} + \theta_{23} > \pi\).

**Ideal vertex:** The second case is the ideal vertex, which is the vertex at \(\infty\), which means that the three edges around the vertex do not intersect in the hyperbolic space, but the infimum of their distances are zero. In this case, \(\theta_{12} + \theta_{13} + \theta_{23} = \pi\).

**Ultra-ideal vertex:** The last one is the ultra-ideal vertex. In this case, \(F_1 \cap F_2 \cap F_3 = \phi\), but there is a plane perpendicular to \(F_1, F_2, F_3\), and adding this plane to \(F_1, F_2, F_3\), we get a truncated vertex as the second tetrahedron in Figure 1. In this case, \(\theta_{12} + \theta_{13} + \theta_{23} < \pi\).

**Theorem.** Let \(T\) be a hyperbolic tetrahedron one of whose vertices are ideal or ultra-ideal. Let \(\theta_a, \theta_b, \theta_c, \theta_d, \theta_e, \theta_f\) are dihedral angles of \(T\) and \(a_r, b_r, \cdots, f_r\) are sequences as in the above conjecture. Then we have

\[
\left| \begin{pmatrix} a_r & b_r & e_r \\ d_r & c_r & f_r \end{pmatrix}_{\xi_r}^{RW} \right| \sim_{r \to \infty} \frac{\sqrt{2} \pi}{r^{3/2} \sqrt{-\det G}} \frac{\sqrt[4]{r}}{r^{3/2}} \text{Vol}(T),
\]
where $G$ is the Gram matrix of $T$ given by

$$G = \begin{pmatrix}
1 & -\cos \theta_a & -\cos \theta_b & -\cos \theta_f \\
-\cos \theta_a & 1 & -\cos \theta_c & -\cos \theta_e \\
-\cos \theta_b & -\cos \theta_c & 1 & -\cos \theta_f \\
-\cos \theta_f & -\cos \theta_e & -\cos \theta_d & 1
\end{pmatrix}.$$ 

**Remark.** If $q = \xi_{r+2}$ instead of $\xi_r^2$, then the following conjecture is proposed for a quantum $6j$ symbol corresponding to a spherical tetrahedron by C. Woodward, which is referred in [13].

**Conjecture 2** (C. Woodward). Let $T$ be a spherical tetrahedron with edges of lengths $l_{ij}$ ($1 \leq i < j \leq 4$) equal to $\pi$ times $\alpha_{ij}$, and associate dihedral angles $\theta_{ij}$. Let $V$ be the volume of $T$, and let $G$ be the spherical Gram matrix, which is the symmetric $4 \times 4$ matrix with ones on the diagonal and $\cos l$ off the diagonal where 4 rows and columns correspond to the vertices of $T$. Then the following may hold.

$$\left\{ r \alpha_{12} \quad \alpha_{13} \quad r \alpha_{23} \quad r \alpha_{34} \right\} \sim_{r \to \infty} \xi_{r+2} \quad \quad \frac{2\pi}{r^{3/2} \sqrt{\det G}} \cos \left( \sum_{1 \leq i < j \leq 4} (r l_{ij} + 1) \frac{\theta_{ij}}{2} - \frac{r}{\pi} V + \frac{\pi}{4} \right).$$ 

We expect that similar asymptotics holds for the quantum $6j$ symbol with $q = \xi_r^2$ which corresponds to a spherical tetrahedron. In the above conjecture, the parameters of the quantum $6j$ symbol are associated to the edge lengths of the corresponding tetrahedron, while we associate these parameters to the dihedral angles. For spherical tetrahedron, there is a duality between lengths and dihedral angles given by Milnor in [6] and the relations (1) and (2) are very close if $l_{ij}$ is replaced by $\alpha_{ij}$ since, by the analytic continuation principle, the volume of the spherical tetrahedron is equal to the analytic continuation of the hyperbolic case times $\sqrt{-1}$.
1. Quantum $6j$ Symbols

The quantum $6j$ symbol was introduced by Kirillov and Reshetikhin \[3\] to describe the structure of the tensor representations of the quantum group $U_q(sl_2)$. Let $q$ be an indeterminate. Finite-dimensional irreducible representations of $U_q(sl_2)$ are parametrized by non-negative half integers, which is called the spin. For a non-negative half integer $a$, $V_a$ denote the corresponding finite-dimensional irreducible representation. It is known that $V_c$ is contained in $V_a \otimes V_b$ if $a$, $b$, $c$ satisfy the Clebsch-Gordan condition, which means that $|a - b| \leq c \leq a + b$ and $a + b + c$ is an integer. Let $i_{ab}^c$ denote the inclusion map from $V_c$ to $V_a \otimes V_b$. For six non-negative half integers $a$, $b$, $c$, $d$, $e$, $f$, the quantum $6j$ symbol is defined by the following relation between the following two inclusions of an irreducible representation to a tensor of three irreducible representations $V_c \to V_d \otimes V_b \otimes V_a$, one through $V_d \otimes V_e$ and another one through $V_f \otimes V_a$.

$$
\left( V_c \xrightarrow{i_{ab}^c} V_d \otimes V_e \xrightarrow{i_{de}^{id}} V_d \otimes V_b \otimes V_a \right) = \sum_f \left\{ \begin{array}{ccc} a & b & e \\ d & c & f \end{array} \right\} \left( V_c \xrightarrow{i_{ab}^c} V_f \otimes V_a \xrightarrow{i_{df}^{id}} V_d \otimes V_b \otimes V_a \right).
$$

Here we assume that the triplets $(a, b, e)$, $(a, c, f)$, $(b, d, f)$ and $(c, d, e)$ satisfy the Clebsch-Gordan condition. The Racah-Wigner version of the quantum $6j$ symbol is given by

$$
\left\{ \begin{array}{ccc} a & b & e \\ d & c & f \end{array} \right\}^{RW} = \frac{1}{\sqrt{-1^{2(c + d + 2e - a - b)}}} \frac{1}{\sqrt{[2e + 1][2f + 1]}} \left\{ \begin{array}{ccc} a & b & e \\ d & c & f \end{array} \right\},
$$

where $[n] = \frac{q^{n/2} - q^{-n/2}}{q - q^{-1}}$. Let $[n!] = \prod_{j=1}^n [j]$. Then this version has the following symmetric formula

$$
\left\{ \begin{array}{ccc} a & b & e \\ d & c & f \end{array} \right\}^{RW} = \Delta_r(a, b, e) \Delta_r(a, c, f) \Delta_r(b, d, f) \Delta_r(c, d, e) \times \sum_{z=m}^M \frac{(-1)^z[z+1]!}{\prod_{j=1}^4 [z-S_j]! \prod_{k=1}^3 [T_k-z]!},
$$

where

$$
S_1 = a + b + e, \quad S_2 = a + c + f, \quad S_3 = b + d + f, \quad S_4 = c + d + e, \\
T_1 = a + b + c + d, \quad T_2 = a + d + e + f, \quad T_3 = b + c + e + f,
$$

(4) $m = \max(S_1, S_2, S_3, S_4)$, $M = \min(T_1, T_2, T_3)$,

$$
\Delta_r(u, v, w) = \left( \frac{|u+v-w|! |v+w-u|! |w+u-v|!}{|u+v+w+1|!} \right)^{1/2}.
$$
2. Relation to the hyperbolic volume

In this section, we prove the following relation between the Racah-Wigner version of the quantum 6j symbols and the hyperbolic volume of the corresponding hyperbolic tetrahedra.

**Proposition.** Let $T$ be a hyperbolic tetrahedron one of whose vertices is ideal or ultra-ideal. Let $\theta_a$, $\theta_b$, $\theta_c$, $\theta_d$, $\theta_e$, $\theta_f$ be dihedral angles at edges $a$, $\cdots$, $f$ in Figure 7 and let $a_r$, $b_r$, $\cdots$, $f_r$ be sequences of non-negative half integers satisfying

\[
\lim_{r \to \infty} 2\pi \frac{2a_r+1}{r} = \pi - \theta_a, \quad \lim_{r \to \infty} 2\pi \frac{2b_r+1}{r} = \pi - \theta_b, \quad \cdots, \quad \lim_{r \to \infty} 2\pi \frac{2f_r+1}{r} = \pi - \theta_f
\]

such that the triplets $(a_r, b_r, c_r)$, $(a_r, d_r, f_r)$, $(b_r, d_r, f_r)$, $(c_r, d_r, e_r)$ are all $r$-admissible for odd $r \geq 3$. Then

\[
\lim_{r \to \infty} \frac{2\pi}{r} \log \left| \left\{ \begin{array}{ccc} a_r & b_r & c_r \\ d_r & e_r & f_r \end{array} \right\}^{RW}_{q=\xi^2} \right| = \text{Vol}(T),
\]

where $\text{Vol}(T)$ is the hyperbolic volume of $T$.

**Proof.** The idea of proof is similar to the proof of the volume conjecture for figure-eight knot, for example, see [1]. Such idea is also used in [2] for the quantum 6j symbols with $q = \xi$, where he used $ev_r$ map. Here we follow the argument in [2] without $ev_r$ map. At the end of proof, the limiting value is compared with the volume formula in [15] based on [9].

Assume that the vertex determined by the edges $a$, $b$, $c$ is ideal or ultra-ideal. This means that $\theta_a + \theta_b + \theta_c \leq \pi$ and the sequences $a_r$, $b_r$, $c_r$ satisfy

\[
\lim_{r \to \infty} (a_r + b_r + c_r) \geq \frac{r}{2}.
\]

Now we choose $a_r$, $b_r$, $c_r$ so that $a_r + b_r + c_r \geq \frac{r}{2}$. Then $m \geq \frac{r}{2}$ and the terms in the sum of (3) are all real numbers and have the same signature, and the limit is determined by the term having the largest absolute value. Let

\[
\alpha_r(z) = \frac{(-1)^2 [z+1]}{\prod_{j=1}^4 [z-S_{r,j}]! \prod_{k=1}^3 [T_{r,k}-z]!}_{q=\xi^2},
\]

where $S_{r,j}$, $T_{r,j}$ are defined by using $a_r$, $b_r$, $\cdots$, $f_r$ as in (4), $m_r = \max_{j=1}^4 S_{r,j}$, $M_r = \min_{k=1}^3 T_{r,k}$ and let $|\alpha_r(z_0)|$ be the maximum of $\{ |\alpha_r(z)| \mid m_r \leq z \leq M_r \}$. Then

\[
\lim_{r \to \infty} \frac{1}{r} \log |\alpha_r(z_0)| < \lim_{r \to \infty} \frac{1}{r} \log \sum_{z=n}^M |\alpha_r(z)| < \lim_{r \to \infty} \frac{1}{r} \log |r \alpha_r(z_0)|.
\]

Since $\lim_{r \to \infty} \frac{1}{r} \log r = 0$, we get

\[
\lim_{r \to \infty} \frac{1}{r} \log \sum_{z=n}^M |\alpha_r(z)| = \lim_{r \to \infty} \frac{1}{r} \log |\alpha_r(z_0)|.
\]
Now compare
\[
\lim_{r \to \infty} \frac{2\pi}{r} \log |\Delta_r(a_r, b_r, c_r) \Delta_r(a_r, c_r, f_r) \Delta_r(b_r, d_r, f_r) \Delta_r(c_r, d_r, e_r) \alpha_r(z_0)|
\]
with the hyperbolic volume of $T$. This part is the same as the proof of Theorem 1 in [2], which concerns the quantum $6j$ symbol with $q = \xi_r$. Here we assume $r$ to be odd and $q$ to be $\xi^{2r}$ instead of $\xi^r$, so the proof in [2] works well without applying the operator $ev_r$ introduced in [2] to remove the factor $\xi^{r/2} - \xi^{-r/2}$ in $[S_{r,j} + 1]!$ and $[z + 1]!$. Here we may have factors $\xi^{(r\pm 1)/2} - \xi^{-(r\pm 1)/2}$ in $[S_{r,j} + 1]!$ and $[z + 1]!$ but we never have the factor $\xi^{r/2} - \xi^{-r/2}$ since $q = \xi^{2r}$ and $r$ is odd.

3. General strategy to prove the volume conjecture

To investigate the asymptotics of the quantum $6j$ symbol little more, we apply the following strategy which is proposed by Ohtsuki in [10] to prove the volume conjecture for $5_2$ knot. This strategy consists of the following three steps.

**Step 1:** Replace the term in the sum by a continuous function.

This can be done by replacing the quantum factorials by the quantum dilogarithm functions introduced by Faddeev [3].

**Step 2:** Replace the sums by integrals.

Sum over integers is reformulated by a sum of integrals by using the following Poisson summation formula.

\[
\sum_{m \in \mathbb{Z}} f(m) = \sum_{m \in \mathbb{Z}} \hat{f}(m)
\]

where $\hat{f}(x)$ is the Fourier transformation of $f$, i.e.

\[
\hat{f}(x) = \int_{-\infty}^{\infty} e^{2\pi i xt} f(t) \, dt.
\]

**Step 3:** Apply the stationary phase method or the saddle point method to the integrals $\hat{f}(m)$. If $g(x)$ is a real function, $x_0$ is the maximal point i.e $g(x_0) = \max_x g(x)$, and $x_0$ is an inner point of the integral interval $C$ in $\mathbb{R}$, then

\[
\int_C h(x) e^{Ng(x)} \, dx \sim_{N \to \infty} \sqrt{\frac{2\pi}{N |g_{xx}(x_0)|}} h(x_0) e^{Ng(x_0)}.
\]

This is called the stationary phase method.

For a complex valued analytic function $g(x)$, let $x_0$ be a solution of $g_x(x) = 0$ and the location of $x_0$ is in a “good” position with respect to the integral path $C$. Then we can apply (7) and this method is called the saddle point method.
In this paper, we only use the stationary phase method and so we don’t give the detail about the condition for applying the saddle point method. We can apply this strategy to various quantum invariants, but Step 3 includes some difficulties if we have to apply the saddle point method. In general, we have to apply the saddle point method for a multi-variable function and it is not easy to check the condition to apply the saddle point method. In some cases, \( h(x_0) \) is almost 0 while \( e^{N g(x_0)} \) is very large and the meaning of the right-hand side of (7) is not clear.

4. Relation to the Gram matrix

In this section, we investigate little more about the asymptotics of \( \alpha_r(z) \) by using the general strategy to prove the volume conjecture described in the previous section.

**Step 1.** We express \( a_r(z) \) in terms of the dilogarithm function. Let

\[
(q)_n = \prod_{k=1}^{n} (1 - q^k).
\]

Then

\[
[n]! = q^{-\frac{n(n+1)}{4}} \frac{(q)_n}{2^n (-i)^n \sin^n \frac{2\pi}{r}}
\]

and \( \alpha_r(z) \), \( \Delta_r(a_r, b_r, e_r) \), \( \cdots \), \( \Delta_r(c_r, d_r, e_r) \) are reformulated as

\[
\alpha_r(z) = \frac{\xi_r^{2(d_1+d_2)} (-1)^z (\xi_r^2)^{z+1}}{2 i \sin \frac{2\pi}{r} \prod_{j=1}^{4} (\xi_r^2)^{z-S_j} \prod_{k=1}^{3} (\xi_r^2)^{T_k-z}},
\]

where

\[
\begin{align*}
d_1 &= - \left( 2a_r + 2b_r + 2c_r + 2d_r + 2e_r + 2f_r + \frac{1}{2} \right) z + \frac{3z^2}{2}, \\
d_2 &= a_r^2 + a_r b_r + b_r^2 + a_r c_r + b_r c_r + c_r^2 + a_r d_r + b_r d_r + c_r d_r + d_r^2 + a_r e_r + \\
&\quad b_r e_r + c_r e_r + d_r e_r + e_r^2 + a_r f_r + b_r f_r + c_r f_r + d_r f_r + e_r f_r + f_r^2 - \frac{1}{2},
\end{align*}
\]

and

\[
\Delta_r(a_r, b_r, e_r) = \left( \xi_r^{2d_3(a_r, b_r, e_r)} - 2 i \sin \frac{2\pi}{r} (\xi_r^2)_{a_r+b_r-e_r} (\xi_r^2)_{b_r+e_r-a_r} (\xi_r^2)_{e_r+a_r-b_r} (\xi_r^2)_{a_r+b_r+e_r+1} \right)^{1/2},
\]

\[
\cdots
\]
\[ \Delta_r(c_r, d_r, e_r) = \left( \frac{\xi^2d_3(c_r, d_r, e_r) - 2i \sin \frac{2\pi}{r} (\xi^2_d)_{c_r+d_r-e_r} (\xi^2_d)_{d_r+e_r-c_r} (\xi^2_d)_{e_r+c_r-d_r}}{(\xi^2_r)_{c_r+d_r+e_r+1}} \right)^{1/2} \]

where

\[ d_3(a_r, b_r, e_r) = -\frac{1}{2} (a_r^2 + b_r^2 + e_r^2 - 2a_re_r - 2a_r b_r - 2b_r e_r - a_r - b_r - e_r - 1). \]

Let \( \varphi_r \) is the function introduced in [3] for \( 0 \leq t \leq 1 \) by the following integral.

\[ \varphi_r(t) = \int_{-\infty}^{\infty} \frac{e^{(2t-1)x}}{4x \sinh x \sinh(2x/r)} \, dx, \]

where the singularity at \( x = 0 \) is bypassed by a small half circle above the real line.

**Lemma 1.** The function \( \varphi_r \) satisfies

\[ (1 - e^{2\pi ia}) = \exp \left( \varphi_r \left( a - \frac{1}{r} \right) - \varphi_r \left( a + \frac{1}{r} \right) \right), \quad \left( \frac{1}{r} \leq a \leq \frac{r-1}{r} \right), \]

\[ (\xi^2_r)_n = \exp \left( \varphi_r \left( \frac{1}{r} \right) - \varphi_r \left( \frac{2n+1}{r} \right) \right), \quad \left( 0 \leq n \leq \frac{r-1}{2} \right), \]

\[ (\xi^2_r)_n = \exp \left( \varphi_r \left( \frac{1}{r} \right) - \varphi_r \left( \frac{2n+1}{r} - 1 \right) + \log 2 \right), \quad \left( \frac{r-1}{2} < n < \frac{2r-1}{2} \right), \]

\[ \text{Im} \varphi_r(t) = -\pi i \frac{6r^2 t^2 - 6r^2 t + r^2 - 2}{24r}. \]

**Proof.** The function \( \frac{e^{(2a-1)x}}{4x \sinh x} \) has poles at \( x = k\pi i \) (\( k \in \mathbb{N} \)) in the upper half plane, and the corresponding residues are \( \frac{e^{2\pi k+i}}{2\pi ki} \). The sum of these residues is equal to \( -\frac{\log(1-e^{2\pi i})}{2\pi i} \) and this implies (9). The formula (10) is a product of (\( (\xi^2_r)_n \)) for \( a = 2, 4, \cdots, 2n \). To prove (11), we decompose \( (\xi^2_r)_n \) into a product of \( (\xi^2_r)_{n-1} \) and \( \prod_{k=1}^{n-r-1} (1-\xi^{2k-1}_r) \). Then \( (\xi^2_r)_{n-1} = \varphi_r(1/r) - \varphi_r(1) \) and \( \prod_{k=1}^{n-r-1} (1-\xi^{2k-1}_r) = \varphi_r(0) - \varphi_r(2n/r) \). Moreover, \( \varphi_r(0) - \varphi_r(1) \) is given by \( 2\pi i \) times the sum of residues of \( -\frac{1}{2x \sinh(2x/r)} \), which is \( \sum_{k=1}^{n-r-1} \frac{(-1)^{k+1}}{k} = \log 2 \). Hence we get (11). The imaginary part of \( \varphi_r(t) \) comes from the pole at \( x = 0 \). For the other part, \( \varphi_r(t) \) is defined as an integral of a real function. We define the integral path to avoid \( x = 0 \) by taking a path in the upper half plane, so by taking a small upper half circle around \( x = 0 \), the contribution is given by \( -\pi i \) times
the residue of $\frac{e^{(2t-1)x}}{4x \sinh x \sinh(2x/r)}$ at $x = 0$, which is the right hand side of \eqref{eq:12}.

Now, we reformulate $\alpha_r(z)$ and $\Delta_r(a_r, b_r, e_r)$ as continuous functions by using \eqref{eq:11} and \eqref{eq:11}. Let $m_r = \max_{1 \leq j \leq 1} S_{r,j}$, $M_r = \min_{1 \leq k \leq 3} T_{r,k}$,

\begin{equation}
\tilde{\alpha}_r(z) =
\int \xi_{r}^{2d_{r}} \frac{e^{i \varphi_r(z)}}{r} \frac{d_{r}}{r} \exp \left( \pi i z - 6 \varphi_r(z) + \varphi_r(\frac{2z^2 + 3}{r} - 1) + \log 2 + \sum_{j=1}^{4} \varphi_r(\frac{2z - 2S_{r,j} + 1}{r}) + \sum_{j=1}^{3} \varphi_r(\frac{2T_{r,j} - 2z + 1}{r}) + \frac{2 \pi i z}{r} (3z^2 - z - 4 (a_r + b_r + \cdots + f_r) z) \right),
\end{equation}

and

\begin{equation}
\tilde{\Delta}_r(a_r, b_r, e_r) = \exp \left( \frac{1}{2} \left( 2 \varphi_r(\frac{1}{2}) + \varphi_r(\frac{2a_r + 2b_r + 2e_r + 3}{r} - 1) - \log 2 - \varphi_r(\frac{2a_r + 2b_r - 2e_r + 1}{r}) - \varphi_r(\frac{2b_r + 2e_r - 2a_r + 1}{r}) - \varphi_r(\frac{2e_r + 2a_r - 2b_r + 1}{r}) \right) \right).
\end{equation}

Then $\tilde{\alpha}_r(z)$ is a positive real function for $m_r < z < M_r$,

$$\alpha_r(z) = \frac{(-1)^{\frac{r+1}{2}}}{2 \sin \frac{2\pi}{r}} \tilde{\alpha}_r(z)$$

for an integer $z \ (m_r \leq z \leq M_r)$ and

$$\Delta_r(a_r, b_r, e_r) = \sqrt{2 \sin \frac{2\pi}{r} \xi_{r}^{d_{r}}(a_r, b_r, e_r)} \int \frac{e^{i z (a_r + b_r + e_r)}}{r} \tilde{\Delta}_r(a_r, b_r, e_r)$$

for an $r$-admissible triplet $(a_r, b_r, e_r)$.

**Step 2.** Now we apply the Poisson summation formula to $\sum_{z=mr}^{m_r} \tilde{\alpha}_r(z)$. Let $\psi_r(z)$ be a smooth function satisfying $0 \leq \psi_r(z) \leq 1$, $\psi_r(z) = 0$ for $z \leq m_r - 1/4$ and $z \geq M_r + 1/4$, and $\psi_r(z) = 1$ for $m_r \leq z \leq M_r$. Let

$$f_r(z) = \psi_r(z) \tilde{\alpha}_r(z).$$

Then $\sum_{z=mr}^{m_r} \tilde{\alpha}_r(z) = \sum_{m \in \mathbb{Z}} f_r(m)$, and, by Poisson summation formula, we have

$$\sum_{m \in \mathbb{Z}} f_r(m) = \sum_{m \in \mathbb{Z}} \hat{f}_r(m)$$

where $\hat{f}_r(m) = \int_{-\infty}^{\infty} e^{2\pi i mt} f_r(t) \, dt$. The function $f_r(t)$ is non-negative or non-positive with respect to $r$, and $|\hat{f}_r(m)| \ll |\hat{f}_r(0)|$ if $r$ is very big. See Appendix for this. Therefore, the asymptotics of the $6j$ symbol is determined by the
asymptotics of \( \hat{f}_r(0) \) with respect to \( r \). Let \( \zeta = \frac{2r+3}{r} \), \( \eta_a = \frac{2a+1}{r} \), \( \eta_b = \frac{2b+1}{r} \), 
\ldots, \( \eta_{f_r} = \frac{2f_r+1}{r} \), \( \sigma_{r,j} = \frac{2\tau_{r,j}+3}{r} \) (1 \( \leq j \leq 4 \)), \( \tau_{r,k} = \frac{2\tau_{r,k}+4}{r} \) (1 \( \leq k \leq 3 \)), and let

\[
g_r(\zeta) = \psi_r(\frac{r\zeta^2-3}{2}) \times \exp \left( \frac{\pi ir\zeta}{2} - 6 \varphi_r(\frac{1}{r}) - \varphi_r(\zeta - 1) + \log 2 + \sum_{j=1}^{4} \varphi_r(\zeta - \sigma_{r,j} + \frac{1}{r}) \right.
\]

\[
+ \sum_{j=1}^{3} \varphi_r(\tau_{r,j} - \zeta) + 2\pi ir \left( \frac{3}{4} \zeta^2 - (\eta_a + \cdots + \eta_{f_r}) \zeta + \frac{\zeta}{r} \right) \right).
\]

Then \( f_r(z) \) is equal to \( \left| g_r(\frac{2r+3}{r}) \right|_{\eta_a=\frac{2a+1}{r}, \ldots, \eta_{f_r}=\frac{2f_r+1}{r}} \), the argument of \( g(\zeta) \) is not depend on \( \zeta \) and

\[
\left| \hat{f}_r(0) \right| = \left| \int_{-\infty}^{\infty} f_r(x) \, dx \right| = \frac{r}{2} \left| \int_{-\infty}^{\infty} g_r(\zeta) \, d\zeta \right|.
\]

From the condition for \( a_r, \ldots, f_r \), we have

\[
\lim_{r \to \infty} \eta_a = \frac{1}{2\pi} (\pi - \theta_a), \ldots, \lim_{r \to \infty} \eta_{f_r} = \frac{1}{2\pi} (\pi - \theta_f).
\]

Now we put

\[
\eta_a = \frac{1}{2\pi} (\pi - \theta_a), \ldots, \eta_f = \frac{1}{2\pi} (\pi - \theta_f),
\]

\[
\tilde{g}_r(\zeta) = \psi_r(\frac{r\zeta^2-3}{2}) \times \exp \left( \frac{\pi ir\zeta}{2} - 6 \varphi_r(\frac{1}{r}) - \varphi_r(\zeta - 1) + \log 2 + \sum_{j=1}^{4} \varphi_r(\zeta - \sigma_j + \frac{1}{r}) \right.
\]

\[
+ \sum_{j=1}^{3} \varphi_r(\tau_j - \zeta) + 2\pi ir \left( \frac{3}{4} \zeta^2 - (\eta_a + \cdots + \eta_f) \zeta + \frac{\zeta}{r} \right) \right).
\]

The parameters \( \eta_a, \ldots, \eta_f \) do not depend on \( r \), but the function \( \varphi_r(x) \) depends on \( r \). We investigate the asymptotics of

\[
r \sin \left( \frac{2\pi}{r} \tilde{\Delta} \left( \frac{\eta_a-1}{2}, \frac{\eta_b-1}{2}, \frac{\eta_c-1}{2} \right) \cdots \tilde{\Delta} \left( \frac{\eta_{f_r}-1}{2}, \frac{\eta_{f_r}-1}{2}, \frac{\eta_{f_r}-1}{2} \right) \right) \int_{-\infty}^{\infty} \tilde{g}_r(\zeta) \, d\zeta.
\]

**Remark.** The actual meaning of the asymptotics of the left-hand side of (14) is the asymptotics of (17).

Before applying the stationary phase method, we expand \( \tilde{g}_r(\zeta) \) in terms of \( r \) by using the following relations. These properties are explained in Appendix A of [10].
Lemma 2. The asymptotics of $\varphi_r$ is given as follows.

\begin{equation}
\varphi_r \left( \frac{1}{r} \right) = \frac{r}{4 \pi i} \left( \frac{\pi^2}{6} + \frac{2 \pi i}{r} \log \frac{r}{2} - \frac{\pi^2}{r} + O\left( \frac{1}{r^2} \right) \right),
\end{equation}

\begin{equation}
\varphi_r(t) = \frac{r}{4 \pi i} \left( \text{Li}_2(e^{2\pi i t}) + O\left( \frac{1}{r^2} \right) \right).
\end{equation}

We also have the following expansion for small $\alpha$ since $\frac{d}{dx} \text{Li}_2(x) = -\log(1-x)$.

\begin{equation}
\text{Li}_2(e^{2\pi i (p+\alpha)}) = \text{Li}_2(e^{2\pi i p}) - 2\pi i \log(1 - e^{2\pi i p}) \alpha + O(\alpha^2).
\end{equation}

By reformulating $\bar{g}_r$ by using (18) to (20), we get

\begin{equation}
\bar{g}_r(\zeta) = \frac{16 \psi_r(\frac{r \zeta - 3}{2})}{r^3} \times \\
\exp \left( \frac{\pi i r \zeta}{2} + \frac{r}{4 \pi i} \left( -\text{Li}_2(e^{2\pi i \zeta}) + \sum_{j=1}^{4} \text{Li}_2(e^{2\pi i \zeta - \sigma_j}) + \sum_{j=1}^{3} \text{Li}_2(e^{2\pi i \tau_j - \zeta}) \right) + 2\pi i r \left( \frac{3}{4} \zeta^2 - (\eta_a + \cdots + \eta_f) \zeta + \frac{\zeta}{r} \right) - \frac{1}{2} \sum_{j=1}^{4} \log(1 - e^{2\pi i \zeta - \sigma_j}) + O\left( \frac{1}{r} \right) \right)
\end{equation}

\begin{equation}
= \frac{16 \psi_r(\frac{r \zeta - 3}{2})}{r^3} \frac{e^{2\pi i \zeta}}{\prod_{j=1}^{4}(1 - e^{2\pi i \eta_j})} \times \\
\exp \left( \frac{r}{4 \pi i} \left( 2\pi^2 \zeta - \text{Li}_2(e^{2\pi i \zeta}) + \sum_{j=1}^{4} \text{Li}_2\left( \frac{e^{2\pi i \zeta}}{e^{2\pi i \eta_j}} \right) + \sum_{j=1}^{3} \text{Li}_2\left( \frac{e^{2\pi i \tau_j}}{e^{2\pi i \eta_j}} \right) - 6\pi^2 \zeta^2 + 8\pi^2 (\eta_a + \cdots + \eta_f) \zeta \right) + O\left( \frac{1}{r} \right) \right).
\end{equation}

We also reformulate $\tilde{\Delta}_r$. Let

\begin{equation}
\delta(\eta_1, \eta_2, \eta_3) = \frac{1}{8\pi i} \left( \text{Li}_2(e^{2\pi i (\eta_1 + \eta_2 + \eta_3)}) - \text{Li}_2\left( \frac{e^{2\pi i (\eta_1 + \eta_2 + \eta_3)}}{e^{2\pi i \eta_3}} \right) - \text{Li}_2\left( \frac{e^{2\pi i (\eta_2 + \eta_3)}}{e^{2\pi i \eta_1}} \right) - \text{Li}_2\left( \frac{e^{2\pi i (\eta_1 + \eta_3)}}{e^{2\pi i \eta_2}} \right) \right),
\end{equation}

for $0 \leq \eta_i \leq 1$ and $1 \leq \eta_1 + \eta_2 + \eta_3 \leq 2$. Then

\begin{equation}
\tilde{\Delta}_r\left( \frac{r \eta_1 - 1}{2}, \frac{r \eta_2 - 1}{2}, \frac{r \eta_3 - 1}{2} \right) = \frac{\sqrt{r}}{2} \exp \left( r \delta(\eta_1, \eta_2, \eta_3) + O\left( \frac{1}{r} \right) \right).
\end{equation}
Step 3. Let \( \zeta_0 \) \((m_r \leq \zeta_0 \leq M_r)\) be a real number such that the function \( |\tilde{g}_r(\zeta)| \) takes its maximum at \( \zeta = \zeta_0 \). Let
\[
F(\zeta) = \frac{1}{4 \pi i} \left( -2\pi^2 \zeta - \text{Li}_2(e^{2\pi i \zeta}) + \sum_{j=1}^{4} \text{Li}_2(e^{2\pi i \sigma_j}) + \sum_{j=1}^{3} \text{Li}_2(e^{2\pi i \tau_j}) \right) + 6\pi^2 \zeta^2 + 8\pi^2 (\eta_a + \cdots + \eta_f) \zeta,
\]
then \( \zeta_0 \) must be the solution of
\[
(23) \quad \frac{d}{d\zeta} F(\zeta) = 0
\]
since the argument of \( \tilde{g}_r(\zeta) \) does not depend on \( \zeta \). The equation \((23)\) is essentially a quadratic equation and we have only one solution which corresponds to the maximal. If we have two solutions corresponding to maximals, we also should have at least one solution corresponding to a minimal, but the total number of the solutions of \((23)\) is two.

Since \( \zeta_0 \in \mathbb{R} \) and the integral path is the real line, we can apply the stationary phase method and we have
\[
(24) \quad \left| \int_{-\infty}^{\infty} \tilde{g}_r(\zeta) \, d\zeta \right| \sim_{r \to \infty} \frac{16}{r^3} \left| \prod_{j=1}^{4} \frac{e^{2\pi i \zeta_0}}{1 - e^{2\pi i \sigma_j}} \sqrt{\frac{2\pi}{r \, |F_{\zeta \zeta}(\zeta_0)|}} \exp \left( r \, F(\zeta_0) + O(\frac{1}{r}) \right) \right|.
\]

Lemma 3.
\[
\prod_{j=1}^{4} \frac{1 - e^{2\pi i \zeta_0}}{e^{4\pi i \zeta_0}} F_{ss}(s_0) = \pm 4\pi \sqrt{-\det G}
\]
where \( G \) is the gram matrix of \( T \).

Proof. The second derivative of \( F(\zeta) \) at \( \zeta_0 \) is given by
\[
F_{\zeta \zeta}(\zeta_0) = \pi i \left( -\frac{e^{2\pi i \zeta_0}}{1 - e^{2\pi i \zeta_0}} + \sum_{j=1}^{4} \frac{e^{2\pi i \zeta_0}}{e^{2\pi i \sigma_j} - e^{2\pi i \zeta_0}} - \sum_{j=1}^{3} \frac{e^{2\pi i \zeta_0}}{e^{2\pi i \tau_j} - e^{2\pi i \zeta_0}} \right).
\]
On the other hand,
\[
\prod_{j=1}^{4} (e^{2\pi i \sigma_j} - e^{2\pi i \zeta_0}) = (1 - e^{2\pi i \zeta_0}) \prod_{j=1}^{3} (e^{2\pi i \tau_j} - e^{2\pi i \zeta_0})
\]
since \( F_{\zeta}(\zeta_0) = 0 \). Let
\[
(25) \quad a_2 u^2 + a_1 u + a_0 = \left( \prod_{j=1}^{4} (e^{2\pi i \sigma_j} - u) - (1 - u) \prod_{j=1}^{3} (e^{2\pi i \tau_j} - u) \right) / u.
\]
Then \( e^{2\pi i \zeta_0} \) is a solution of \( a_2 u^2 + a_1 u + a_0 = 0 \). Let \( u_0 = e^{2\pi i \zeta_0} \) and \( u'_0 \) be another solution of (25). Then we have

\[
(26) \quad \prod_{j=1}^{4} \left( e^{2\pi i \sigma_j} - u_0 \right) F_{\zeta \zeta}(\zeta_0) = \frac{\pi i}{u_0} \times \]

\[
\left( - \prod_{j=1}^{3} \left( e^{2\pi i r_j} - u_0 \right) + \sum_{j=1}^{4} \prod_{1 \leq k \leq 3, k \neq j} \left( e^{2\pi i r_k} - u_0 \right) - \sum_{j=1}^{3} (1 - u_0) \prod_{1 \leq k \leq 3, k \neq j} \left( e^{2\pi i r_k} - u_0 \right) \right)
\]

\[
= \pi i \left( a_2 u_0 + 2 a_1 + 3 a_0 u_0^{-1} \right) = \pi i \left( -a_2 u_0 + a_0 u_0^{-1} \right) = -a_2 \pi i \left( u_0 - u'_0 \right) = \pm \pi i \sqrt{a_1^2 - 4 a_0 a_2} = \pm 4 \pi \sqrt{-\det G}.
\]

Here we use

\[
a_1^2 - 4 a_0 a_2 = 16 \det G
\]

which is obtained by an actual computation.

Recall that the volume \( \operatorname{Vol}(T) \) is given in [9] and [15] as follows.

\[
(27) \quad \operatorname{Vol}(T) = \left| 2 \pi \left( F(\zeta_0) + \delta(\eta_a, \eta_b, \eta_c) + \delta(\eta_a, \eta_d, \eta_f) + \delta(\eta_b, \eta_d, \eta_f) + \delta(\eta_c, \eta_d, \eta_e) \right) \right|.
\]

Combining (17), (22), (24), (26) and (27), we get

\[
\left| \begin{array}{ccc}
\alpha_r & \beta_r & \gamma_r \\
\delta_r & \epsilon_r & \zeta_r \\
\end{array} \right|_{q=\xi^2} \sim_{r \to \infty} 2 \pi \frac{r^2}{16} \frac{16}{r^3} \left| e^{2\pi i \zeta_0} \sqrt{\frac{2}{r F(\zeta_0)}} \exp \left( \frac{r}{2 \pi} \operatorname{Vol}(T) + O\left( \frac{1}{r} \right) \right) \right|_{r \to \infty} \sim \sqrt{2 \pi} \frac{\sqrt{r^3/2 \sqrt{-\det G}}}{e^{\operatorname{Vol}(T)/2 \pi} + O\left( \frac{1}{r} \right)}.
\]

5. Remark for the volume conjecture

The volume conjecture for the Turaev-Viro invariant is the following.

**Conjecture 3** (Chen-Yang [11]). Let \( M \) be a hyperbolic manifold and \( \operatorname{TV}_r(M) \) be the Turaev-Viro invariant [13] with \( q = \xi^2 \) for odd \( r \). Then

\[
\lim_{r \to \infty, \ r \ : \ odd} \frac{2 \pi}{r} \log |\operatorname{TV}_r(M)| = \operatorname{Vol}(M).
\]

The first version of the volume conjecture is proposed by Kashaev [11] for hyperbolic knots, which is reformulated by using the colored Jones invariant in [8]. Ten or so years later, it is generalized to the Turaev-Viro invariant and the Reshetikhin-Turaev invariant of three manifolds in [1].
On the other hand, the volume conjecture is proved for some simple cases, and the most general strategy is proposed by Ohtsuki in \[10\] to prove the conjecture for 5_2 knot, which is the second simplest hyperbolic knot, which is explained in Section 3. He applied this method for some other simple knots in \[12\] and three manifolds obtained by integral surgeries along the figure-eight knot in \[11\].

In the proofs in \[10\], \[12\] and \[11\], \(\hat{f}(0)\) is the most dominant term among \(\hat{f}(m) (m \in \mathbb{Z})\), and the volume conjecture is proved by applying the saddle point method to \(\hat{f}(0)\). But, to apply the saddle point method, we must check the condition about the integral path, and this is the most difficult part of the proof which is done by case by case check for each knot.

On the other hand, for the Turaev-Viro invariant, parameters for sum correspond edges of the tetrahedral decomposition of the manifold. Let choose a parameter \(a\) corresponding to a edge, and let \(\hat{f}(m)\) be the Fourier transform of the invariant with respect to the parameter \(a\). Then, if the volume conjecture holds, \(\hat{f}(\pm1)\) must be the most dominant terms instead of \(\hat{f}(0)\) since the sum of the dihedral angles of the tetrahedra around the edge is \(2\pi\) or \(-2\pi\) (we don’t have canonical way to determine the sign of the angle) and is not equal to 0. This comes from a geometric interpretation about the saddle point and Schraflı’s differential formula for a hyperbolic tetrahedron.

The sum corresponding to the parameter \(a\) in the Turaev-Viro invariant is given as follows. Let \(T_1, T_2, \ldots, T_p\) be the tetrahedra around the edge corresponding to the parameter \(a\), \(\left\{ a, b_j, e_j \atop d_j, c_j, f_j \right\} \quad [q = \xi^2] \) be the quantum 6j symbol assigned to the tetrahedron \(T_j\), and

\[
f_j(a) = (-1)^{2a+2b_j+2c_j+2d_j+2e_j+2f_j} \left\{ a, b_j, e_j \atop d_j, c_j, f_j \right\} ^{RW} _{q = \xi^2}.
\]

Then

\[
TV_r(M) = \sum \ldots \left( \sum_a [2a + 1] \prod_{j=1}^{p} f_j(a) \right)
\]

Since \(f_j(a)\) takes its largest value around \(a = \frac{r-2}{4}\) and \([2a + 1]\) changes its sign at \(a = \frac{r-2}{4}\), big cancellation happens around \(a = \frac{r-2}{4}\) and the value of \(\hat{f}(0)\) is expected to be very small. We expect that the volume conjecture holds for the Turaev-Viro invariant, so we also expect that \(\hat{f}(0)\) is smaller than \(\hat{f}(\pm1)\).

This expectation may be proved by using the following conjecture, which is based on the symmetry about the sign of a dihedral angle \(\theta\) of a tetrahedron changing \(\theta\) to \(-\theta\).
Conjecture 4 (Symmetry of asymptotics). Let $T$ be a hyperbolic tetrahedron and $\theta_a, \theta_b, \theta_c, \theta_d, \theta_e, \theta_f$ be dihedral angles at edges $a, \cdots, f$ in Figure 4. Let $a_r, b_r, \cdots, f_r$ be sequences of non-negative half integers satisfying

$$\lim_{r \to \infty} 2\pi \frac{2a_r+1}{r} = \pi - \theta_a, \quad \lim_{r \to \infty} 2\pi \frac{2b_r+1}{r} = \pi - \theta_b, \quad \cdots, \quad \lim_{r \to \infty} 2\pi \frac{2f_r+1}{r} = \pi - \theta_f$$

such that the triplets $(a_r, b_r, e_r), (a_r, d_r, f_r), (b_r, d_r, f_r), (c_r, d_r, e_r)$ are all $r$-admissible for odd $r \geq 3$. Let $a'_r$ be a sequence of non-negative half integers satisfying

$$\lim_{f \to \infty} 2\pi \frac{2a'_r+1}{k} = \pi + \theta_a$$

such that the triplets $(a'_r, b_r, c_r), (a'_r, d_r, f_r)$ are $r$-admissible. Then the asymptotic expansions of $\left\{ \begin{array}{ccc} a_r & b_r & e_r \\ d_r & c_r & f_r \end{array} \right\}_{q=\xi_r^2}^{RW}$ and $\left\{ \begin{array}{ccc} a'_r & b_r & e_r \\ d_r & c_r & f_r \end{array} \right\}_{q=\xi_r^2}^{RW}$ with respect to $r$ are equal.

Let

$$\left\{ \begin{array}{ccc} a_r & b_r & e_r \\ d_r & c_r & f_r \end{array} \right\}_{q=\xi_r^2}^{RW} \sim A e^{\pi B} \left( 1 + C_1 \frac{1}{r} + C_2 \frac{1}{r^2} + \cdots \right).$$

be the asymptotic expansion of $\left\{ \begin{array}{ccc} a_r & b_r & e_r \\ d_r & c_r & f_r \end{array} \right\}_{q=\xi_r^2}^{RW}$. Then $A = \frac{\sqrt{3}}{r^{3/2} \sqrt{-\det G}}$ and $B = \text{Vol}(T)$. where $\text{Vol}(T)$ is the volume of $T$ and det $G$ is the Gram matrix of $T$. The Conjecture 4 means that the coefficients $A, B, C_1, C_2, \cdots$ are all symmetric with respect to the switching of the sign of dihedral angles. The volume function $\text{Vol}(T)$ and the gram matrix det $G$ are not changed by changing the sign of any dihedral angle, the above conjecture is true up to the second leading term.

We computed the term $c_1$ numerically for regular tetrahedra as the following table and graph. Let $T_\theta$ and $T_{-\theta}$ be the regular tetrahedra with dihedral angles $\theta$ and $-\theta$ respectively, then $T_{-\theta}$ is obtained by applying the flips in Conjecture 4 to every edges. The tetrahedra $T_\theta$ and $T_{-\theta}$ are both ultra ideal for $-\pi/3 < \theta < \pi/3$, and the computation shows that $C_1$ for $T_\theta$ and $T_{-\theta}$ seem to be equal.

| $\theta$ | $-\pi/3$ | $-0.3 \pi$ | $-0.08 \pi$ | $0$ | $0.08 \pi$ | $0.3 \pi$ | $\pi/3$ |
|-----------|-----------|-------------|-------------|-----|-------------|------------|--------|
| $\text{Im } c_1$ | $-\infty$ | $-0.7671$ | $-0.2782$ | $-0.2708$ | $-0.2782$ | $-0.7671$ | $-\infty$ |

Table 1. Values of $C_1$. ($\text{Re } C_1 = -1$ for $-\pi/3 < \theta < \pi/3$)
Appendix. Compare \( \hat{f}_r(0) \) and \( \hat{f}_r(m) \)

Let \( \beta_r(x) = \frac{1}{r} \log \tilde{\alpha}_r(x) \). Then \( \beta_r(x) \) is a real analytic function defined on the internal \((u_r, v_r)\) with \( u_r = \frac{2m_r + 3}{r} \) and \( v_r = \frac{2M_r + 3}{r} \) where \( m_r \) and \( M_r \) are the lower and upper bounds in the line below \([5]\). Such function \( \beta_r(x) \) exists since \( \tilde{\alpha}_r(x) \) is a positive real analytic function. Thanks to Lemma 2, \( \beta_r(x) \) has a limiting function \( \beta(x) = \lim_{r \to \infty} \beta_r(x) \), which is a real analytic function which takes its maximum at \( x = x_0 \in (u, v) \) where \( u = \lim_{r \to \infty} u_r \) and \( v = \lim_{r \to \infty} v_r \). Let \( \psi_r(x) \) be a smooth function satisfying \( 0 \leq \psi_r(x) \leq 1 \), \( \psi_r(x) = 0 \) for \( x \leq u_r - \frac{1}{4r} \) and \( x \geq v_r + \frac{1}{4r} \), and \( \psi_r(x) = 1 \) for \( u_r \leq z \leq v_r \). Let \( f_r(2z + 3 \frac{m}{r}) = \psi_r(2z + 3 \frac{m}{r}) \tilde{\alpha}_r(2z + 3 \frac{m}{r}). \)

Then, by applying the Poisson summation formula, we have

\[
\sum_{z \in \mathbb{Z}} f_r(2z + 3 \frac{m}{r}) = \sum_{m \in \mathbb{Z}} \hat{f}_r(m),
\]

and

\[
\hat{f}_r(m) = \int_{\mathbb{R}} e^{2\pi imz} \psi_r(2z + 3 \frac{m}{r}) \tilde{\alpha}_r(2z + 3 \frac{m}{r}) \, dz
\]

\[
= \int_{\mathbb{R}} e^{2\pi im \frac{r}{2} - 3} \psi_r(x) \tilde{\alpha}_r(x) \frac{r}{2} \, dx
\]

\[
= (-1)^m \frac{r}{2} \int_{\mathbb{R}} e^{\pi irmx} \psi_r(x) \tilde{\alpha}_r(x) \, dx
\]

\[
= (-1)^m \frac{r}{2} \int_{\mathbb{R}} e^{\pi irmx} \psi_r(x) e^{r(\beta_r(x) + \pi mx)} \, dx.
\]

**Lemma.** \( \hat{f}_r(m) \) is much smaller than \( \hat{f}_r(0) \).

**Proof.** At first, we extend \( \beta(x) \) to a complex-valued function around \( x = x_0 \). Let \( \beta_{r,m}(x) = \beta_r(x) + \pi mx \). Choose \( \varepsilon > 0 \) so that \( \pi mx + \beta(x) \) doesn’t have a critical point in the \( \varepsilon \) neighborhood \( B(x_0, \varepsilon) \) of \( x_0 \) in \( \mathbb{C} \) for \( m \neq 0 \) and \([u_r, v_r] \cap B(x_0, \varepsilon) = (x_0 - \varepsilon, x_0 + \varepsilon) \). Then \( \beta_{r,m}(x) (m \neq 0) \) also doesn’t have a critical point in \( B(x_0, \varepsilon) \) if \( r \) is sufficiently large. Let \( x_0^{(r)} \) be the maximum point of \( \beta_r(x) \), then \( x_0^{(r)} \) is close to \( x_0 \) when \( r \) is sufficiently large.
and

\[
\beta_r(x) = \beta_r(x_0^{(r)}) - a_r (x - x_0^{(r)})^2 + O((x - x_0^{(r)})^3)
\]

in a small neighborhood of \(x_0^{(r)}\). For \(x \in \mathbb{C}\) near \(x_0^{(r)}\), the contours of \(\text{Re} \beta_r(x)\) and \(\text{Re} \beta_{r,m}(x)\) are given in the following figure. Choose \(x_1 \in (x_0 - \varepsilon, x_0)\) so that the contour \(C_{r,m}\) corresponding to the value \(\text{Re} \beta_{r,m}(x_1)\) connects \(x_1\) to a real point \(x_1^{(r)} \in (x_0, x_0 + \varepsilon)\) in \(B(x_0, \varepsilon)\) for any \(m \neq 0\) and for any sufficiently large \(r\) as in the figure. Please note that \(\text{Re} \beta_{r,m}(x) = \text{Re} \beta_r(x)\) for \(x \in \mathbb{R}\) and so \(x_1^{(r)}\) doesn’t depend on \(m\). Such \(x_1\) exists since \(\beta_r(x)\) converges to \(\beta(x)\) and the contour \(C_{r,m} (|m| \geq 2)\) is in the region enclosed by \(C_{r,1}\) and \(C_{r,-1}\). Then, due to \(\beta_r(x_1) \to \beta(x_1)\), \(\beta_r(x_0^{(r)}) \to \beta(x_0)\) when \(r \to \infty\) and \(\beta(x_1) \neq \beta(x_0)\), there is \(\delta > 0\) such that \(\beta_r(x_1) < \beta_r(x_0^{(r)}) - \delta\) for any sufficiently large \(r\).

\[
\hat{f}_r(0) = \frac{r}{2} \int_{\mathbb{R}} f_r(x) \, dx = \frac{r}{2} \int_{v_r - \frac{1}{4r}}^{v_r + \frac{1}{4r}} \psi_r(x) \tilde{\alpha}_r(x) \, dx
\]

\[
= \frac{r}{2} \int_{v_r - \frac{1}{4r}}^{v_r + \frac{1}{4r}} \psi_r(x) e^{r \beta_r(x)} \, dx,
\]

\[
\hat{f}_r(m) = \frac{(-1)^m r}{2} \int_{\mathbb{R}} e^{\pi irmx} \psi_r(x) \tilde{\alpha}_r(x) \, dx
\]

\[
= \frac{(-1)^m r}{2} \int_{v_r - \frac{1}{4r}}^{v_r + \frac{1}{4r}} e^{\pi irmx} \psi_r(x) \tilde{\alpha}_r(x) \, dx
\]

\[
= \frac{(-1)^m r}{2} \int_{v_r - \frac{1}{4r}}^{v_r + \frac{1}{4r}} e^{\pi irmx} \psi_r(x) e^{r \beta_r(x)} \, dx,
\]

and \(\psi_r(x) e^{r \beta_r(x)}\) is a non-negative function with compact support. From now on, we assume that \(r\) is very large.
Lemma A. There is a constant $C$ not depending on $r$ such that
\[
\hat{f}_r(0) > C \sqrt{r} e^{r \beta(x_0^{(r)})}
\]
for large $r$.

**Proof.** Since $\hat{f}_r(0) \sim \frac{r}{2} \frac{e^{r \beta(x_0^{(r)})}}{\sqrt{-r \beta''(x_0^{(r)})}}$, there is a constant $C_0$ such that
\[
\frac{2 \hat{f}_r(0) \sqrt{-\beta''(x_0^{(r)})}}{\sqrt{r} e^{r \beta(x_0^{(r)})}} > C_0 \text{ for any large } r. \text{ Put } C = \frac{C_0}{2 \sqrt{-\beta''(x_0^{(r)})}}, \text{ then we proved (28).}\]

Lemma B. There is a constant $c$ not depending on $r$ such that
\[
\left| \hat{f}_r(m) \right| \leq c e^{r \beta_r(x_1)}
\]
for large $r$.

**Proof.** Since we can replace the part $[x_1, x_1^{(r)}]$ of the integral path by $C_{r,m}$. Let \(C_{r,m} = [u_r - \frac{1}{4r}, x_1] \cup C_{r,m} \cup (x_1^{(r)}, v_r + \frac{1}{4r})\). Then we have $\Re \beta_r(x) \leq \beta_r(x_1)$ on $C_{r,m}$, $\Re \beta_r(x) = \beta_r(x_1)$ on $[u_r - \frac{1}{4r}, x_1]$ and $\Re \beta_r(x) \leq \beta_r(x_1^{(r)}) = \beta_r(x_1)$ on $(x_1^{(r)}, v_r + \frac{1}{4r})$. So we have $\Re \beta_r(x) \leq \beta_r(x_1)$ on the whole new integral path $\overline{C}_{r,m}$. Therefore, we have
\[
\left| \hat{f}_r(m) \right| = \left| \frac{r}{2} \int_{C_{r,m}} e^{\pi i r m x} \psi_r(x) e^{r \beta_r(x)} \, dx \right| \\
\leq \frac{r}{2} \int_{C_{r,m}} \left| e^{\pi i r m x} \psi_r(x) e^{r \beta_r(x)} \right| \, dx \\
= \frac{r}{2} \int_{C_{r,m}} \psi_r(x) e^{r \Re \beta_r(x)} \, dx \\
\leq \frac{r}{2} \int_{C_{r,m}} \psi_r(x) e^{r \beta_r(x_1)} \, dx \\
\leq \frac{r}{2} L_r e^{r \beta_r(x_1)}
\]
where $L_r$ is the length of $\overline{C}_{r,m}$ and $\psi_r(x) = 1$ for $x \in C_{r,m}$. So we proved (29) by setting $c = \frac{\sup L_r}{2}$. \(\square\)

According to (28) and (29), we have
\[
\left| \frac{\hat{f}_r(m)}{\hat{f}_r(0)} \right| < \frac{c r e^{r \beta(x_1)}}{C r^{1/2} e^{r \beta(x_0^{(r)})}} = \frac{C e^{r \beta(x_0^{(r)})}}{C r^{1/2} e^{-r \delta}} = C r^{1/2} e^{-r \delta} \to 0
\]
as $r \to \infty$. Thus we have $|\hat{f}_r(0)| \gg |\hat{f}_r(0)|$ if $r$ is very big. \(\square\)
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References

[1] Q. Chen and T. Yang, A volume conjecture for a family of Turaev-Viro type invariants of 3-manifolds with boundary, preprint. [arXiv:1503.02547]
[2] F. Costantino, 6j-symbols, hyperbolic structures and the volume conjecture, Geom. Topol. 11 (2007), 1831–1854.
[3] L. Faddeev, Discrete Heisenberg-Weyl group and modular group, Lett. Math. Phys. 34 (1995), 249–254.
[4] R. M. Kashaev, The hyperbolic volume of knots from the quantum dilogarithm, Lett. Math. Phys. 39 (1997), 269–275.
[5] A. N. Kirillov and N. Yu. Reshetikhin, Representations of the algebra $\mathfrak{U}_q(\mathfrak{sl}(2))$, $q$-orthogonal polynomials and invariants of links, Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), 285–339, Adv. Ser. Math. Phys., 7, World Sci. Publ., Teaneck, NJ, 1989.
[6] J. Milnor, The Schl"afli differential equality. John Milnor Collected Papers Volume 1, Geometry, Publish or Perish Inc., Houston and Texas, 1994, pp. 281–295.
[7] H. Murakami, An introduction to the volume conjecture, Interactions between hyperbolic geometry, quantum topology and number theory, 1–40, Contemp. Math., 541, Amer. Math. Soc., Providence, RI, 2011.
[8] H. Murakami and J. Murakami, The colored Jones polynomials and the simplicial volume of a knot, Acta Math. 186 (2001), 85–104.
[9] J. Murakami and M. Yano, On the volume of a hyperbolic and spherical tetrahedron, Comm. Anal. Geom. 13 (2005), 379–400.
[10] T. Ohtsuki, On the asymptotic expansion of the Kashaev invariant of the 52 knot, Quantum Topology 7 (2016), 669–735.
[11] T. Ohtsuki, On the asymptotic expansion of the quantum SU(2) invariant at $q = \exp(4\pi\sqrt{-1}/N)$ for closed hyperbolic 3-manifolds obtained by integral surgery along the figure-eight knot, preprint.
[12] T. Ohtsuki and Y. Yokota, On the asymptotic expansions of the Kashaev invariant of the knots with 6 crossings, Math. Proc. Camb. Phil. Soc. (to appear).
[13] J. Roberts, Asymptotics and 6j-symbols, Invariants of knots and 3-manifolds (Kyoto, 2001), 245–261, Geom. Topol. Monogr., 4, Geom. Topol. Publ., Coventry, 2002.
[14] V. G. Turaev and O. Viro, State sum invariants of 3-manifolds and quantum 6j-symbols, Topology 4 (1992), 865–902.
[15] A. Ushijima, A volume formula for generalised hyperbolic tetrahedra, Non-Euclidean geometries, 249–265, Math. Appl. (N.Y.), 581, Springer, New York, 2006.
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