Theory Perspective on Flavour

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Abstract. An overview of the flavour problem is presented, with emphasis on the theoretical efforts to find a satisfactory description of fermion masses and mixing angles.

1. Introduction
The flavour problem exhibits two aspects. The first one is the origin of the parameters in the flavour sector of the SM, minimally extended to include massive Majorana neutrinos. Out of the 22 independent parameters, which will be collectively called $Y_i$ or Yukawa couplings, 18 have already been measured. Only four parameters, the absolute scale of neutrino masses, the Dirac CP-violating phase and, possibly, two Majorana phases are still unknown. We are looking for a more economic description, perhaps related to a new principle, such as the gauge principle. Gauge invariance and renormalizability allow to describe strong and electroweak interactions of fifteen different fermion species in terms of only three parameters, augmented to five after including the electroweak symmetry breaking sector. Nothing similar exists so far in the flavour sector. The second aspect of the flavour problem is related to the new particle threshold around the TeV scale predicted by all SM extensions addressing the hierarchy problem. Once new TeV particles transforming non-trivially in flavour space are introduced, it is very difficult to maintain the almost perfect agreement between predictions and observations that reigns in the SM. New sources of FCNC and CP violations appear and the task is to keep them at an acceptable level. In this talk the focus will be on the first aspect, I will comment only shortly on the second one. Also, I’m not aiming at reviewing all the existing models, but rather in revisiting the main ideas, as an introduction to more specialized presentations at this workshop.

There are different kinds of approach to the flavour problem and here we can mention two of them. The first one is reductionist: the Yukawa couplings $Y_i$ should be deduced from first principles. We postulate the existence of a fundamental theory from which $Y_i$ can be uniquely determined. Either by proceeding directly from the candidate theory or by appealing to some symmetry or dynamical principle $Y_i$ can be computed in terms of a small set of input parameters. Probably the most striking fact about this program is that nothing approaching a standard theory of $Y_i$ exists, despite decades of experimental progress and theoretical efforts.

In the second approach a major role is played by chance. There are many variants and practical implementations of this approach. The Yukawa couplings $Y_i$ are typically mapped to a large number of order-one parameters that are considered as irreducible unknowns, like in models with Froggatt-Nielsen abelian flavour symmetries or with fermions living in extra dimensions.
Also the paradigm of partial compositeness falls into this class. By scanning the order-one parameters we get probability distributions for masses and mixing angles. Alternatively we start from a fundamental theory, like string theory, which possesses a vast landscape of solutions, with no privileged ground state. The observed Yukawa couplings become environmental quantities and cannot be predicted, like the relative sizes of the solar planetary orbits. We are allowed to ask much less ambitious questions. For instance, if we have knowledge of the statistical distribution of $Y_i$ in the fundamental theory, we can ask how typical are the Yukawa couplings that we observe. Conversely, barring anthropic selections, we might assume that the observed $Y_i$ are typical and try to deduce information on the statistical distribution of $Y_i$ in the fundamental theory.

In this talk I will adopt the point of view that the Yukawa couplings can be constrained to some extent by a flavour symmetry. The largest possible flavour symmetry of a theory with the particle content of the SM is $G_{MFV} = U(3)_F$ and corresponds to the unrealistic limit in which the Yukawa couplings are turned off. The observed fermion masses and mixing angles break $G_{MFV}$ almost completely to a residual symmetry that includes at most the weak hypercharge and, maybe, the combination B-L. Hence, in any realistic model based on flavour symmetries, the flavour symmetry group $G_f \subseteq G_{MFV}$ is broken. In predictive models the breaking is spontaneous, and it occurs through the VEVs of a set of scalar fields $\varphi$ transforming non-trivially under $G_f$. The VEVs $\langle \varphi \rangle$ are determined by minimizing a $G_f$-invariant energy functional $V(\varphi)$. The Yukawa couplings becomes dynamical variables evaluated at the minimum of $V(\varphi)$: $V(\varphi)/\Lambda_f$, $\Lambda_f$ representing the scale of flavour physics. A huge number of models can be constructed according to this set of rules, depending on the choice of $G_f$ (global, local, continuous, discrete, abelian, non-abelian), and on the choice of representations for scalars and fermions. A recent attempt to start from the full $G_{MFV}$ symmetry and to study the stationary points of a general $G_{MFV}$-invariant potential is described in ref. [1].

2. Lessons from the quark sector
An empirical evidence for $G_f$ comes from the quark sector and was pointed out long ago by Froggatt and Nielsen [2]. In their pioneering work Froggatt and Nielsen observed that all the small dimensionless parameters of the quark sector such as the quark mass ratios and the CKM mixing angles can be interpreted as powers of a small symmetry breaking parameter $\lambda \approx 0.2$. A scalar field $\varphi$, carrying by convention a negative unit of an abelian charge, develops a VEV that can be parametrized as

$$\lambda = \langle \varphi \rangle/\Lambda_f < 1 \quad FN(\varphi) = -1 \quad . \quad (1)$$

Quarks carry non-negative U(1)$_{FN}$ charges (the case with charges of both signs can be discussed as well)

$$FN(X_i) \geq 0 \quad (X_i = Q_i, U^c_i, D^c_i) \quad . \quad (2)$$

Under these assumptions the quark Yukawa couplings $y_{u,d}$ are given by:

$$y_u = F_U Y_u F_Q \quad , \quad y_d = F_D Y_d F_Q \quad , \quad (3)$$

where $Y_{u,d}$ are complex matrices with entries of order one, undetermined by the U(1)$_{FN}$ symmetry, while $F_X$ are real diagonal matrices, completely specified in terms of $\lambda$ by the charges $FN(X_i)$:

$$F_X = \begin{pmatrix}
\lambda^{FN(X_1)} & 0 & 0 \\
0 & \lambda^{FN(X_2)} & 0 \\
0 & 0 & \lambda^{FN(X_3)}
\end{pmatrix} \quad (X_i = Q_i, U^c_i, D^c_i) \quad . \quad (4)$$

The small quark mass ratios and quark mixing angles originate from the hierarchical structure of the matrices $F_X$. Indeed, by taking $FN(Q_1) > FN(Q_2) > FN(Q_3) \geq 0$ we get

$$(V_{u,d})_{ij} \approx \frac{F_{Q_{ij}}}{F_{Q_{ji}}} < 1 \quad (i < j)$$

for the matrices $V_{u,d}$ defining the CKM mixing matrix $V_{CKM} = V_d^\dagger V_d$. Independently from the specific charge choice, this framework predicts

$$V_{ud} \approx V_{cs} \approx V_{tb} \approx O(1) \quad V_{ub} \approx V_{td} \approx V_{us} \times V_{cb},$$

the last equality being correct within a factor of two. With $\lambda \approx 0.2$, the correct order of magnitudes of the $V_{CKM}$ matrix elements can be reproduced by choosing, for instance, $FN(Q) = (3, 2, 0)$. The correct order of magnitudes of the quark and charged lepton mass ratios, up to a couple of moderate tunings, can be reproduced by choosing

$$FN(U^c) = FN(E^c) = FN(Q) = (3, 2, 0) \quad FN(D^c) = FN(L) = (2, 0, 0).$$

Notice that this charge assignment is also compatible with an SU(5) grand unified theory. The construction relies on a spontaneously broken abelian flavour symmetry, but the final results (3-6) are valid in a more general context, where no symmetry is present to start with. For example, if we consider split fermions in a warped extra dimension (a flat metric for the extra dimension works equally well), we can reproduce the same pattern of Yukawa coupling of eq. (3) with matrices $F_X$ now given by

$$F_{X_i} = \sqrt{\frac{1 - 2c_i}{1 - (R/R')^{1 - 2c_i}}} \approx \begin{cases} O(1) & c_i < 1/2 \\ (R/R')^{c_i - 1/2} & c_i > 1/2 \end{cases}$$

where $c_i$ are the bulk quark masses in units of $1/R$ and $R, R'$ are length scales of the order of the inverse Planck mass and the inverse TeV scale, respectively. The Higgs field is localized on the TeV brane and has generic $O(1)$ Yukawa couplings $Y_{u,d}$ with the bulk quark fields. The role of the Froggatt-Nielsen charges is here played by the bulk masses $c_i$ that determine the profile along the extra dimension of the zero-mode wave functions. There is no flavour symmetry: the hierarchical structure of quark masses and mixing angles is dictated by geometry in the compact space. Similarly, in the partial compositeness scenario, light fermions get hierarchical masses from the mixing between an elementary sector and a composite one. The composite sector has unsuppressed $O(1)$ interactions with the Higgs field and the Yukawa couplings of the light fermions have the structure given in eq. (3) with matrices $F_X$ parametrizing the elementary-composite mixing. Here again symmetry considerations are absent.

In the previous examples the anarchical pattern of $Y_{u,d}$ results in strong bounds on the scale of new physics $\Lambda_{NP}$ associated to particles carrying flavour quantum numbers and representing new sources of FCNC and/or CP violation. When the quark Yukawa couplings are described by eq. (3), the maximal flavour symmetry felt by quarks can be as large as $U(3)^3$, the relevant subgroup of $G_{MFV}$. However there are more spurions than in MFV, the irreducible ones including now at least $F_Q, F_{U^c}, F_{D^c}, Y_u$ and $Y_d$. One of the most dangerous effects originates from the effective operator

$$\frac{1}{\Lambda_{NP}^2} \langle \overline{Q} F_Q^d \gamma_\mu F_Q Q \rangle (\overline{D^c} F_{D^c}^\dagger \gamma_\mu F_{D^c} D^c) \approx \frac{1}{\Lambda_{NP}^2} \frac{2m_d m_s}{v^2} \langle \overline{\sigma_L d_R} \rangle (\overline{\sigma_R d_L}) + \ldots$$

$\langle Y_d \rangle \Lambda_{NP} > 20$ TeV

(9) representing an average $O(1)$ coupling. The contribution of this operator to the CP-violating $\epsilon_K$ parameter is enhanced at the level of both the hadronic matrix element and the QCD corrections. Assuming a generic $O(1)$ phase for the overall coefficient we need

$$\langle Y_d \rangle \Lambda_{NP} > 20 \text{ TeV}$$

(10)
not to spoil the SM prediction for $\epsilon_K$. This, together with other constraints, suggests that a fully anarchical pattern in $Y_{u,d}$ is probably not tenable if new flavoured physics at the TeV scale is present.

In summary, the pattern of masses and mixing angles in the quark sector is well described by the map in eq. (3), in terms of input parameters of order one, the matrix elements of $Y_{u,d}$. Such a map can be realized in several different frameworks and does not necessarily needs an underlying symmetry. The setup is compatible with grand unification and with the known solution to the gauge hierarchy problem. On the weak side, additional ingredients are probably needed to control the new sources of FCNC and CP-violations arising from new physics at the TeV scale. Moreover all entries of $Y_{u,d}$ are independent free parameters and it is not possible to make absolute predictions, beyond the order-of-magnitude accuracy. This is clearly a major limitation, since we would like to test the theory at the level of the best available experimental precision.

3. From quarks to leptons
In the lepton sector we have no evidence for strong hierarchies in mixing angles or in neutrino masses. Hierarchy shows up at the level of charged lepton masses. In terms of the suppression factors $F_{X_i}$ this means

$$F_{E_1} > F_{E_2} > F_{E_3} \geq 0 \quad \text{and} \quad F_{L_1} \approx F_{L_2} \approx F_{L_3} . \quad (11)$$

Table 1. Possible choices of FN charges for the $\bar{5}$ representation in a class of SU(5) grand unified models, from ref. [4]. The second column shows the value of the FN symmetry breaking parameter optimizing the fit to fermion masses and mixing angles.

| FN(5)   | $\lambda$ |
|---------|------------|
| (0,0,0) | -          |
| (1,0,0) | 0.25       |
| (2,0,0) | 0.35       |
| (2,1,0) | 0.45       |

Here we focus on Majorana neutrinos. A drastic realization of this picture is the framework of Anarchy [3], which in this language can be defined by

$$F_{L_1} = F_{L_2} = F_{L_3} . \quad (12)$$

As a consequence the mass matrix for light neutrinos is

$$m_\nu = \begin{pmatrix} O(1) & O(1) & O(1) \\ O(1) & O(1) & O(1) \\ O(1) & O(1) & O(1) \end{pmatrix} m_0 , \quad (13)$$

with undetermined order-one matrix elements. This implies mixing angles and neutrino mass ratios of $O(1)$, in rough agreement with the data. No special values for these quantities is expected and in this respect the non-vanishing, sizable value of $\theta_{13}$ and the departure of $\theta_{23}$ from $\pi/4$ exhibited by the present data have strengthened the case for hierarchy. Anarchy represents an extreme possibility and milder realization of the relations (11) are possible. For instance, in
the context of SU(5) grand unified models, with a Froggatt-Nielsen U(1)$_{FN}$ abelian symmetry, neutrino masses and mixing angles can be reproduced, at the level of order of magnitudes, by several choices of the FN charges for the $5$ multiplets hosting the lepton doublets, as shown in table 1. FN charges for fermions in the $10$ representations can be suitably chosen so that, by varying the unknown order-one parameters, reasonable distributions for charged lepton mass ratios, quark mass ratios and quark mixing angles are also obtained. A naive comparison of the distributions for neutrino masses and mixing angles with data do not appear to favor anarchy over the other possible charge assignments.

If this framework also comprises new flavoured particles at the TeV scale, severe bounds from LFV apply. The irreducible sources of flavour violation in the lepton sector include the matrices $Y_e$, $F_E c$ and $F_L$ and LFV can occur even in the limit of vanishing neutrino masses. Notice that we have no LFV in MFV in this limit, since in MFV the only available spurion in the lepton sector is $Y_e$. The dipole operator contributing to LFV is

$$\frac{e}{\Lambda_{NP}} E^\nu \sigma_{\mu\nu} F^{\mu\nu}(F_E Y_e Y_e^\dagger F_L)H^\dagger L. \quad (14)$$

The charged lepton mass matrix is proportional to $(F_E Y_e F_L)$. In general the combinations $(F_E Y_e F_L)$ and $(F_E Y_e Y_e^\dagger F_L)$ are not diagonal in the same basis, not even in the case of universal $F_L$ in eq. (12), and radiative decays of muon and tau are expected. Agreement with the most constraining upper bound, $BR(\mu \to e\gamma) < 5.7 \times 10^{-13}$, requires $\Lambda_{NP}$ well above 10 TeV.

|       | [5]      | [6]      |
|-------|----------|----------|
| $\sin^2 \theta_{13}$ | $0.0231^{+0.0019}_{-0.0019}$ | $0.0234^{+0.0022}_{-0.0018}$ [NO] |
| $\sin^2 \theta_{23}$ | $0.446^{+0.008}_{-0.008} \oplus 0.593^{+0.027}_{-0.043}$ [NO] | $0.437^{+0.033}_{-0.023}$ [NO] |
| $\delta_{CP}/\pi$     | $1.48^{+0.31}_{-0.35}$ [NO] | $1.39^{+0.38}_{-0.27}$ [NO] |

As in the quark sector, a completely anarchical matrix $Y_e$ and flavoured physics at the TeV scale are difficult to reconcile. A sufficient condition for the absence of LFV is that $Y_e$, $F_E c$ and $F_L$ are diagonal in the same basis, as suggested in some models. Alternatively we can look for special forms of these matrices dictated by some symmetry requirements.

4. Discrete symmetries

The data from neutrino oscillations before 2012 were supporting flavour symmetries, especially through the indication of a vanishing reactor angle $\theta_{13}$ and a maximal atmospheric mixing angle $\theta_{23}$, features that are difficult to attribute to an underlying theory based on pure chance. Today we know with accuracy that $\theta_{13}$ is neither vanishing nor particularly small, its size being
comparable to that of the Cabibbo angle. Global fits show that the atmospheric mixing angle is probably deviating from the maximal value by few degrees, see table 2. While CP violating effects in the lepton sector are still undetected, recent global fits hint to a maximal Dirac CP-violating phase $\delta_{\text{CP}}$, though the whole range from 0 to $2\pi$ is still allowed at 3$\sigma$. Before the measurement of $\theta_{13}$ a particularly attractive lepton mixing pattern was the tribimaximal one

$$U_{TB} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \approx \begin{pmatrix} 0.82 & 0.58 & 0 \\ -0.41 & 0.58 & -0.71 \\ -0.41 & 0.58 & 0.71 \end{pmatrix} \ . \quad (15)$$

The good agreement between TB mixing and pre-2012 data strongly supported the idea that the true mixing matrix could be described in terms of small corrections to a LO mixing matrix $U^0_{\text{PMNS}}$, which could be derived from symmetry considerations. The simplest way to reproduce the TB mixing pattern is by exploiting discrete flavour symmetries [7, 8]. The theory is invariant under a discrete flavour symmetry $G_f$, broken down in such a way that neutrino and charged lepton sectors have different residual symmetries $G_\nu$ and $G_e$, at least in a LO approximation where small effects are neglected. If neutrinos are of Majorana type, the most general group leaving $m_\nu$ invariant and the individual masses $m_i$ unconstrained is $Z_2 \times Z_2$, a finite group. The subgroup $G_e$ can be continuous, but $G_\nu$ discrete is the simplest option. We require a sufficiently large $G_e$ to distinguish the three charged leptons. For instance we can choose $G_e = Z_n$ ($n \geq 3$) or $G_e = Z_2 \times Z_2$. Once $G_e$ and $G_\nu$ have been chosen inside $G_f$, the embedding automatically fixes the relative alignment of $m^\dagger_i m_l$ and $m_\nu$ in flavour space. Lepton masses are unconstrained and $U^0_{\text{PMNS}}$ is determined up to Majorana phases and up to permutations of rows and columns. This freedom apart, we can predict the three mixing angles $\theta_{ij}^0$ and the Dirac phase $\delta_{\text{CP}}^0$. In most concrete models, where symmetry breaking is achieved via VEVs of a set of flavons, the LO results are modified by small corrections

$$U_{PMNS} = U^0_{PMNS} + O(u) \ . \quad (16)$$

Before 2012, in the specific case $U^0_{PMNS} = U_{TB}$ these corrections were expected to be very small, of the order of few percent, not to spoil the good agreement in the predicted value of the solar mixing angle. On this basis the simplest models reproducing $U_{TB}$ at the LO predicted $\theta_{13}$ not larger than few degrees, now proven to be wrong by experiments. Discrete flavour symmetries can also be extended to quarks and even incorporated in GUTs, but in the existing constructions the symmetry has to be badly broken in the quark sector. Discrete flavour symmetries are also relevant in the so called indirect models [8]. In this case the breaking of $G_f$ leaves no residual symmetries and the role of the flavour group is mainly to get specific vacuum alignment of the scalar fields that control fermion masses.

Several modifications of the simplest models based on discrete symmetries have been proposed to match the most recent data. If we keep adopting $U^0_{PMNS} = U_{TB}$ as LO approximation, perhaps the most economic way to reproduce the actual value of $\theta_{13}$ is to introduce large correction terms, $O(u) \approx 0.2$. This is also viable in some scheme where $U^0_{PMNS}$ differs substantially from $U_{TB}$, such as the so-called bimaximal mixing. Introducing large corrections has the disadvantage that beyond the LO the number of independent contributions is generally quite large. If their typical size is about 0.2, all mixing angles tend to be affected by generic corrections of this type and predictability is lost. Moreover large correction terms are dangerous if new sources of flavour changing and/or CP violation are present at the TeV scale.

Another possibility is to look for alternative LO approximations where $\theta_{13}$ is closer to the measured value. Remarkably, several groups $G_f$ giving rise to LO approximations closer to the
data have been found. Of particular interest is a special form of trimaximal (TM) mixing:

\[ U_{PMNS}^0 = U_{TB} \begin{pmatrix} \cos \alpha & 0 & e^{i \delta} \sin \delta \\ 0 & 1 & 0 \\ -e^{-i \delta} \sin \alpha & 0 & \cos \alpha \end{pmatrix} \]  

reproduced by groups of the series \( \Delta(6n^2) \) [9, 10, 11, 12]. For \( n = (4, 8, 10) \) we have \( \alpha = (\pm 1/12, \pm 1/24, \pm \pi/15) \) and \( \sin^2 \theta_{13}^0 = (0.045, 0.011, 0.029) \). The Dirac phase is zero (modulo \( \pi \)).

A further possibility is to relax the symmetry requirements. \( S_4 \) is the smallest group reproducing TB mixing through the breaking down to \( G_e = Z_3 \) and \( G_\nu = Z_2 \times Z_2 \), whose generators are \( T \) and \( (S, U) \), respectively. In the basis where \( T \) and the charged leptons are diagonal, the element \( U \) coincides with the so-called \( \mu \tau \) exchange symmetry, directly responsible for the vanishing of \( \theta_{13} \) and for \( \theta_{23} \) being maximal. We can avoid having \( \theta_{13} = 0 \) and \( \theta_{23} = \pi/4 \) if \( G_e \) is a single \( Z_2 \) subgroup generated either by the element \( S \) or by the element \( SU \). When the preserved parity is \( S \), the mixing pattern, \( TM_2 \), is trimaximal and corresponds to that in eq. (17), with both \( \alpha \) and \( \delta \) unconstrained. When the preserved parity is \( SU \), the mixing pattern, \( TM_1 \), is also of trimaximal type and is given by \( U_{PMNS}^0 = U_{TB} U_{23} (\alpha, \delta) \), where \( U_{23} (\alpha, \delta) \) is the transformation analogous to \( U_{13} (\alpha, \delta) \), acting in the 23 plane. The mixing angles and the Dirac phase are determined in terms of \( (\alpha, \delta) \) and we get two relations among physical quantities, shown in Table 3. Explicit models based on \( A_4 \) realizing \( TM_2 \) breaking pattern were indeed proposed before the measurement of \( \theta_{13} \) [13]. The possibility of reducing the residual symmetry \( G_\nu \) to \( Z_2 \) can be systematically investigated [14].

| \( TM_1 \) | \( TM_2 \) |
|---|---|
| \( \sin^2 \theta_{12} = \frac{1}{4} \left[ 1 - \frac{5}{2} \sin^2 \theta_{13} + O(\sin^4 \theta_{13}) \right] \) | \( \sin^2 \theta_{12} = \frac{1}{4} + \frac{1}{4} \sin^2 \theta_{13} + O(\sin^4 \theta_{13}) \) |
| \( \sin^2 \theta_{23} = \frac{1}{2} - \sqrt{2} \sin \theta_{13} \cos \delta_{CP} + O(\sin^2 \theta_{13}) \) | \( \sin^2 \theta_{23} = \frac{1}{2} + \frac{1}{\sqrt{2}} \sin \theta_{13} \cos \delta_{CP} + O(\sin^2 \theta_{13}) \) |

A more recent development consists in combining discrete and \( CP \) symmetries and exploring the symmetry breaking patterns such a combination can give rise to. A well-known example is that of the so-called \( \mu \tau \) reflection symmetry [15, 16] (not to be confused with the \( \mu \tau \) exchange symmetry), which exchanges a muon (tau) neutrino with a tau (muon) antineutrino in the charged lepton mass basis. If such a symmetry is imposed, the atmospheric mixing angle is predicted to be maximal, while \( \theta_{13} \) is in general non-vanishing for a maximal Dirac phase \( \delta \) and the Majorana phases vanish. The solar mixing angle remains unconstrained.

A general formalism which combines \( CP \) and flavour symmetries, proposed in ref. [17], can be used to constrain the lepton mixing matrix. A theory symmetric under \( CP \) and under a discrete flavour group \( G_f \) is assumed to have residual symmetries \( G_e \) generated by some elements \( Q_i \) and \( G_\nu = Z_2 \times CP \) generated by a parity \( Z \) and a \( CP \) transformation \( X \). The action of \( X \) in flavour space can be non-trivial and should respect a set of consistency conditions [18, 19]. The residual symmetries \( G_e \) and \( G_\nu \) imply the following conditions on \( m_1^l m_l \) and \( m_\nu^\dagger \quad m_\nu^\dagger m_l Q_l = (m_1^l m_l) \), \( Z^T m_\nu Z = m_\nu \), \( X m_\nu X = m_\nu^* \).
Figure 1. Results for the mixing parameters $\sin \theta_{13}$, $\sin^2 \theta_{12}$ for Case I (straight line) and Case IV (dashed line). We mark the value $\theta_{\text{BF}}$ of the parameter $\theta$ for which the $\chi^2$ functions have a global minimum with a red dot. $3\sigma$ ranges for the mixing angles are also shown.

These conditions are strong enough to completely determine $U^0_{\text{PMNS}}$, up to one real parameter $\theta$, ranging from 0 to $\pi$:

$$U^0_{\text{PMNS}} = U^0_{\text{PMNS}}(Q, Z, X, \theta) \quad 0 \leq \theta \leq \pi.$$  \hspace{1cm} (19)

Mixing angles and phases, both Dirac and Majorana, are then predicted as a function of $\theta$, modulo the ambiguity related to the freedom of permuting rows and columns and to the intrinsic parity of neutrinos. The formalism is completely invariant under any change of basis in field space. The physical results only depend on the initial symmetry and the residual symmetries specified by $(Q, Z, X)$. An interesting example is provided by $G_f = S_4$. An exhaustive analysis has been presented in ref. [17]. The residual symmetries can be chosen as $G_e = Z_3$, generated by the element $T$, and $G_\nu = Z_2 \times CP$, generated by $(Z, X)$. The parity transformation $Z$ can be either $S$ (case I) or $SU$ (case IV) and a consistent $CP$ transformation $X$ acting on the lepton doublets coincides with the $\mu - \tau$ reflection symmetry in the basis where $T$ (and the combination $m_\mu^\dagger m_\tau$) is diagonal. Thus the predicted mixing pattern has a maximal atmospheric mixing angle, a maximal Dirac phase, vanishing Majorana phases and there is a relation between the solar angle and the reactor angle, shown in fig. 1.

Conclusion

We have recently witnessed a decisive experimental progress in neutrino physics. The reactor angle is now precisely known and it is away from zero by many standard deviations. We also have a first indication favoring a non-maximal atmospheric mixing angle. For the first time global fits hint to a non-trivial Dirac phase. While these steps have been effective in ruling out many models of fermion masses and mixing angles, it is fair to say that no compelling and unique theoretical picture has emerged so far. Present data can still be described within
widely different frameworks. There are several models where fermion masses and mixing angles are mapped into a large number of irreducible and unconstrained order-one parameters, thus explaining the observed hierarchies. This class of models has the advantage of providing a common description for both quarks and leptons, which can be easily incorporated in a grand unified theory. The weak points are the impossibility to test this idea beyond the order-of-magnitude level, and the probable incompatibility with new flavoured physics at the TeV scale. As an alternative possibility, we have models where the flavour sector becomes highly symmetric at a high scale, and a small number of relevant parameters provides a complete description. Within this class, models based on discrete symmetries, tailored to reproduce the features of the early data from neutrino oscillations, are less supported by data now. Several modifications of the simplest schemes to accommodate the present data are still possible and have the advantage of being quantitatively testable. The weak point of this approach is that it is probably too much centered around the neutrino mixing properties while no clear indication for the need of a discrete symmetry seems to come from the quark sector or from the lepton masses. Flavour remains a fascinating mystery, still resisting all our attempts to find the rationale underlying our observations, and certainly worth to be investigated from all possible angles.

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