Wavelets Beyond Admissibility

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The purpose of this paper is to articulate an observation that many interesting type of wavelets (or coherent states) arise from group representations which are not square integrable or vacuum vectors which are not admissible.

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1. Covariant Transform

A general group-theoretical construction\textsuperscript{1-6} of wavelets (or coherent states) starts from an square integrable (s.i.) representation. However, such a setup is restrictive and is not necessary, in fact.

Definition 1. Let $\rho$ be a representation of a group $G$ in a space $V$ and $F$ be an operator from $V$ to a space $U$. We define a covariant transform $W$ from $V$ to the space $L(G, U)$ of $U$-valued functions on $G$ by the formula:

$W: v \mapsto \hat{v}(g) = F(\rho(g^{-1})v), \quad v \in V, \; g \in G.$ \hfill (1)

Remark 1. We do not require that operator $F$ shall be linear.

Remark 2. Usefulness of the covariant transform is in the reverse proportion to the dimensionality of the space $U$. The covariant transform encodes properties of $v$ in a function $Wv$ on $G$. For a low dimensional $U$ this function can be ultimately investigated by means of harmonic analysis. Thus $\dim U = 1$ is the ideal case, however, it is unattainable sometimes, see Ex. 2.4 below.

Theorem 1. The covariant transform $W$ (1) intertwines $\rho$ and the left regular representation $\Lambda$ on $L(G, U)$:

$\Lambda(g) : f(h) \mapsto f(g^{-1}h).$ \hfill (2)
Proof. We have a calculation similar to wavelet transform [3, Prop. 2.6]:

$$W(\rho(g)v)(h) = F(\rho(h^{-1})\rho(g)v) = |Wv|(g^{-1}h) = \Lambda(g)|Wv|(h).$$

Corollary 1. The image space $W(V)$ is invariant under the left shifts on $G$.

2. Examples of Covariant Transform

Example 2.1. Let $V$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and $\rho$ be a unitary representation. Let $F : V \to \mathbb{C}$ be a functional $v \mapsto \langle v, v_0 \rangle$ defined by a vector $v_0 \in V$. Then the transformation (1) is the well-known expression for a wavelet transform [4, (7.48)] (or representation coefficients):

$$W : v \mapsto \hat{v}(g) = \langle \rho(g^{-1})v, v_0 \rangle = \langle v, \rho(g)v_0 \rangle, \quad v \in V, \ g \in G. \quad (3)$$

The family of vectors $v_g = \rho(g)v_0$ is called wavelets or coherent states. In this case we obtain scalar valued functions on $G$, thus the fundamental role of this example is explained in Rem. 2.

This scheme is typically carried out for a s.i. representation $\rho$ and $v_0$ being an admissible vector. In this case the wavelet (covariant) transform is a map into the s.i. functions with respect to the left Haar measure. However s.i. representations and admissible vectors does not cover all interesting cases.

Example 2.2. Let $G$ be the "ax + b" (or affine) group [4, § 8.2]: the set of points $(a, b)$, $a \in \mathbb{R}_+, \ b \in \mathbb{R}$ in the upper half-plane with the group law:

$$(a, b) \ast (a', b') = (aa', ab' + b) \quad (4)$$

and left invariant measure $a^{-2} da\, db$. Its isometric representation on $V = L_p(\mathbb{R})$ is given by the formula:

$$[\rho_p(a, b)f](x) = a^{\frac{p}{2}} f(ax + b). \quad (5)$$

We consider the operators $F_\pm : L_2(\mathbb{R}) \to \mathbb{C}$ defined by:

$$F_\pm(f) = \frac{1}{2\pi i} \int_{\mathbb{R}} f(t) \frac{dt}{t \mp i}. \quad (6)$$

Then the covariant transform (1) is the Cauchy integral from $L_2(\mathbb{R})$ to the Hardy space in the upper/lower half-plane $H_2(\mathbb{R}_\pm)$. Although the representation (5) is s.i. for $p = 2$, the function $\frac{1}{\sqrt{2\pi i}}$ is not an admissible vacuum vector. Thus the complex analysis become decoupled from the traditional
wavelets theory. As a result the application of wavelet theory shall rely on an extraneous mother wavelets.

However many important objects in complex analysis are generated by inadmissible mother wavelets like (6). For example, if $F : L^2(\mathbb{R}) \to \mathbb{C}$ is defined by $F : f \mapsto (F_+ f, F_- f)$ then the covariant transform (11) represents a function on the real line as a jump between functions analytic in the upper and the lower half-planes. This makes a decomposition of $L^2(\mathbb{R})$ into irreducible components of the representation (5). Another interesting but non-admissible vector is the Gaussian $e^{-x^2}$.

**Example 2.3.** For the group $G = SL_2(\mathbb{R})^{14}$ let us consider the unitary representation $\rho$ on the space of s.i. function $L^2(\mathbb{R}^2_+)$ on the upper half-plane through the Möbius transformations:

$$\rho(g) : f(z) \mapsto \frac{1}{(cz+d)^2} f\left(\frac{az+b}{cz+d}\right), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $F_i$ be the functional $L^2(\mathbb{R}^2_+) \to \mathbb{C}$ of pairing with the lowest/highest $i$-weight vector in the corresponding irreducible component of the discrete series [14, Ch. VI]. Then we can build an operator $F_i$ from various $F_i$ similarly to the previous example, e.g. this generalises the representation of an s.i. function as a sum of analytic ones from different irreducible subspaces.

Covariant transform is also meaningful for principal and complementary series of representations of the group $SL_2(\mathbb{R})$, which are not s.i.

**Example 2.4.** A straightforward generalisation of Ex.2.1 is obtained if $V$ is a Banach space and $F : V \to \mathbb{C}$ is an element of $V^*$. Then the covariant transform coincides with the construction of wavelets in Banach spaces.\(^3\)

The next stage of generalisation is achieved if $V$ is a Banach space and $F : V \to \mathbb{C}^n$ be a linear operator. Then the corresponding covariant transform is a map $W : V \to L(G, \mathbb{C}^n)$. This is closely related to M.G. Krein’s works on directing functionals,\(^10\) see also multiresolution wavelet analysis,\(^11\) Clifford-valued Bargmann spaces\(^12\) and [4, Thm. 7.3.1].

**Example 2.5.** A step in a different direction is a consideration of non-linear operators. Take again the “$ax + b$” group and its representation (15). We define $F$ to be a homogeneous but non-linear functional $V \to \mathbb{R}_+$:

$$F(f) = \frac{1}{2} \int_{-1}^{1} |f(x)| \, dx.$$
The covariant transform (1) becomes:

\[
W_p f(a, b) = \frac{1}{2} \int_{-1}^{1} |a^{\frac{1}{2}} f(ax + b)| \, dx = a^{\frac{1}{2}} \frac{1}{2a} \int_{b-a}^{b+a} |f(x)| \, dx.
\]

Obviously \( M_f(b) = \max_a |W_\infty f|(a, b) \) coincides with the Hardy maximal function, which contains important information on the original function \( f \).

However, the full covariant transform is even more detailed. For example, \( \|f\| = \max_b |W_\infty f|(\frac{1}{2}, b) \) is the shift invariant norm.

From the Cor. we deduce that the operator \( M : f \mapsto M_f \) intertwines \( \rho_p \) with itself \( \rho_p M = M \rho_p \).

**Example 2.6.** Let \( V = L_c(\mathbb{R}^2) \) be the space of compactly supported bounded functions on the plane. We take \( F \) be the linear operator \( V \to \mathbb{C} \) of integration over the real line:

\[
F : f(x, y) \mapsto F(f) = \int_{\mathbb{R}} f(x, 0) \, dx.
\]

Let \( G \) be the group of Euclidean motions of the plane represented by \( \rho \) on \( V \) by a change of variables. Then the wavelet transform \( F(\rho(g)f) \) is the Radon transform.

**Example 2.7.** Let a representation \( \rho \) of a group \( G \) act on a space \( X \). Then there is an associated representation \( \rho_B \) of \( G \) on a space \( X \to Y^* \) defined by the identity:

\[
(\rho_B(g)A)x = A(\rho(g)x), \quad x \in X, \ g \in G, \ A \in B(X, Y).
\]

Following the Remark we take \( F \) to be a functional \( V \to \mathbb{C} \), for example \( F \) can be defined from a pair \( x \in X, \ l \in Y^* \) by the expression \( F : A \mapsto \langle Ax, l \rangle \). Then the covariant transform:

\[
\mathcal{W} : A \mapsto \hat{A}(g) = F(\rho_B(g)A),
\]

this is an example of covariant calculus.

**Example 2.8.** A modification of the previous construction is obtained if we have two groups \( G_1 \) and \( G_2 \) represented by \( \rho_1 \) and \( \rho_2 \), respectively. Then we have a covariant transform \( B(X, Y) \to L(G_1 \times G_2, \mathbb{C}) \) defined by the formula:

\[
\mathcal{W} : A \mapsto \hat{A}(g_1, g_2) = \langle A\rho_1(g_1)x, \rho_2(g_2)l \rangle.
\]

This generalises Berezin functional calculi.
Example 2.9. Let us restrict the previous example to the case when $X = Y$ is a Hilbert space, $\rho_1 = \rho_2 = \rho$ and $x = l$ with $\|x\| = 1$. Then the range of the covariant transform:

$$\mathcal{W} : A \mapsto \hat{A}(g) = \langle A\rho(g)x, \rho(g)x \rangle$$

is a subset of the numerical range of the operator $A$.

Example 2.10. The group $SL_2(\mathbb{R})$ consists of $2 \times 2$ matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

with the unit determinant $[14, \S IX.1]$. Let $A$ be an operator with the spectral radius less than 1. Then the associated Möbius transformation

$$g : A \mapsto g \cdot A = \frac{\alpha A + \beta I}{\beta A + \bar{\alpha} I}, \quad \text{where} \quad g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SL_2(\mathbb{R}),$$

produces a well-defined operator with the spectral radius less than 1 as well. Thus we have a representation of $SL_2(\mathbb{R})$. A choice of an operator $F$ will define the corresponding covariant transform. In this way we obtain generalisations of Riesz–Dunford functional calculus.$^{15}$

3. Inverse Covariant Transform

An object invariant under the left action $\Lambda$ is called left invariant. For example, let $L$ and $L'$ be two left invariant spaces of functions on $G$. We say that a pairing $\langle \cdot, \cdot \rangle : L \times L' \to \mathbb{C}$ is left invariant if

$$\langle \Lambda(g)f, \Lambda(g)f' \rangle = \langle f, f' \rangle, \quad \text{for all} \quad f \in L, \ f' \in L'. \quad (7)$$

Remark 3.

(1) We do not require the pairing to be linear in general.

(2) If the pairing is invariant on space $L \times L'$ it is not necessarily invariant (or even defined) on the whole $C(G) \times C(G)$.

(3) In a more general setting we shall study an invariant pairing on a homogeneous spaces instead of the group. However due to length constraints we cannot consider it here beyond the Example 3.2.

(4) An invariant pairing on $G$ can be obtained from an invariant functional $l$ by the formula $\langle f_1, f_2 \rangle = l(f_1 \bar{f}_2)$.

For a representation $\rho$ of $G$ in $V$ and $v_0 \in V$ we fix a function $w(g) = \rho(g)v_0$. We assume that the pairing can be extended in its second component to this $V$-valued functions, say, in the weak sense.
**Definition 2.** Let $\langle \cdot, \cdot \rangle$ be a left invariant pairing on $L \times L'$ as above, let $\rho$ be a representation of $G$ in a space $V$, we define the function $w(g) = \rho(g)v_0$ for $v_0 \in V$. The inverse covariant transform $\mathcal{M}$ is a map $L \to V$ defined by the pairing:

$$\mathcal{M} : f \mapsto \langle f, w \rangle,$$

where $f \in L$. \hfill (8)

**Example 3.1.** Let $G$ be a group with a unitary s.i. representation $\rho$. An invariant pairing of two s.i. functions is obviously done by the integration over the Haar measure:

$$\langle f_1, f_2 \rangle = \int_G f_1(g) \overline{f_2}(g) \, dg.$$

For an admissible vector $v_0$,\textsuperscript{7} [4, Chap. 8] the inverse covariant transform is known in this setup as reconstruction formula.

**Example 3.2.** Let $\rho$ be a s.i. representation of $G$ modulo a subgroup $H \subset G$ and let $X = G/H$ be the corresponding homogeneous space with a quasi-invariant measure $dx$. Then integration over $dx$ with an appropriate weight produces an invariant pairing. The inverse covariant transform is a more general version [4, (7.52)] of the reconstruction formula mentioned in the previous example.

Let $\rho$ be not a s.i. representation (even modulo a subgroup) or let $v_0$ be inadmissible vector of a s.i. representation $\rho$. An invariant pairing in this case is not associated with an integration over any non singular invariant measure on $G$. In this case we have a Hardy pairing. The following example explains the name.

**Example 3.3.** Let $G$ be the “$ax + b$” group and its representation $\rho$ from Ex. [5]. An invariant pairing on $G$, which is not generated by the Haar measure $a^{-2}da \, db$, is:

$$\langle f_1, f_2 \rangle = \lim_{a \to 0} \int_{-\infty}^{\infty} f_1(a, b) \overline{f_2}(a, b) \, db.$$ \hfill (9)

For this pairing we can consider functions $\frac{1}{2\pi \ln(a+1)}$ or $e^{-x^2}$, which are not admissible vectors in the sense of s.i. representations. Then the inverse covariant transform provides an integral resolutions of the identity.

Similar pairings can be defined for other semi-direct products of two groups. We can also extend a Hardy pairing to a group, which has a subgroup with such a pairing.
Example 3.4. Let $G$ be the group $SL_2(\mathbb{R})$ from the Ex. 2.3. Then the 
"$ax + b$" group is a subgroup of $SL_2(\mathbb{R})$, moreover we can parametrise $SL_2(\mathbb{R})$ by triples $(a, b, \theta)$, $\theta \in (-\pi, \pi]$ with the respective Haar measure [14, III.1(3)]. Then the Hardy pairing
\[
\langle f_1, f_2 \rangle = \lim_{a \to 0} \int_{-\infty}^{\infty} f_1(a, b, \theta) \overline{f_2(a, b, \theta)} \, db \, d\theta.
\]
is invariant on $SL_2(\mathbb{R})$ as well. The corresponding inverse covariant transform provides even a finer resolution of the identity which is invariant under conformal mappings of the Lobachevsky half-plane.

A further study of covariant transform and its inverse shall be continued elsewhere.

References

1. A. Perelomov, Generalized coherent states and their applications (Springer-Verlag, Berlin, 1986).
2. Feichtinger, Hans G. and Groechenig, K.H., J. Funct. Anal. 86, 307 (1989).
3. V. V. Kisil, Acta Appl. Math. 59, 79 (1999), E-print: arXiv:math/9807141.
4. S. T. Ali, J.-P. Antoine and J.-P. Gazeau, Coherent States, Wavelets and Their Generalizations (Springer-Verlag, New York, 2000).
5. H. Führ, Abstract Harmonic Analysis of Continuous Wavelet Transforms, Lecture Notes in Mathematics, Vol. 1863 (Springer-Verlag, Berlin, 2005).
6. J. G. Christensen and G. Ólafsson, Acta Appl. Math. 107, 25 (2009).
7. M. Duflo and C. C. Moore, J. Functional Analysis 21, 209 (1976).
8. O. Hutnik, Integral Equations Operator Theory 63, 29 (2009).
9. V. V. Kisil, Complex Variables Theory Appl. 40, 93 (1999), E-print: arXiv:funct-an/9712003.
10. M. G. Krein, Akad. Nauk Ukrain. RSR. Zbirnik Prac’ Inst. Mat. 1948, 83 (1948), MR#14:56c, reprinted in. 16
11. O. Bratteli and P. E. T. Jorgensen, Integral Equations Operator Theory 28, 382 (1997), E-print: arXiv:funct-an/9612003.
12. J. Cnops and V. V. Kisil, Math. Methods Appl. Sci. 22, 353 (1999), E-print: arXiv:math/9806150. Zbl 1005.22003.
13. A. Johansson, Systems Control Lett. 57, 105 (2008).
14. S. Lang, SL_2(R) (Springer-Verlag, New York, 1985).
15. V. V. Kisil, Spectrum as the support of functional calculus, in Functional analysis and its applications, North-Holland Math. Stud. Vol. 197, pp. 133–141, (Elsevier, Amsterdam, 2004). E-print: arXiv:math.FA/0208249.
16. M. G. Krein, Избранные труды. II (Akad. Nauk Ukrainy Inst. Mat., Kiev, 1997). MR#96m:01030.