SYMmetric TENSOR DECOMPOSITIONS ON VARIETIES

JIAWANG NIE, KE YE, AND LIHONG ZHI

Abstract. This paper discusses the problem of symmetric tensor decomposition on a given variety $X$: decomposing a symmetric tensor into the sum of tensor powers of vectors contained in $X$. In this paper, we first study geometric and algebraic properties of such decomposable tensors, which are crucial to the practical computations of such decompositions. For a given tensor, we also develop a criterion for the existence of a symmetric decomposition on $X$. Secondly and most importantly, we propose a method for computing symmetric tensor decompositions on an arbitrary $X$. As a specific application, Vandermonde decompositions for nonsymmetric tensors can be computed by the proposed algorithm.

1. Introduction

Let $n, d > 0$ be integers and $\mathbb{C}$ be the complex field. Denote by $T^d(\mathbb{C}^{n+1})$ the space of $(n+1)$-dimensional complex tensors of order $d$. For $A \in T^d(\mathbb{C}^{n+1})$ and an integral tuple $(i_1, \ldots, i_d)$, $A_{i_1 \ldots i_d}$ denotes the $(i_1, \ldots, i_d)$th entry of $A$, where $0 \leq i_1, \ldots, i_m \leq n$. The tensor $A$ is symmetric if

$$A_{i_1 \ldots i_d} = A_{j_1 \ldots j_d}$$

whenever $(j_1, \ldots, j_d)$ is a permutation of $(i_1, \ldots, i_d)$. Let $S^d(\mathbb{C}^{n+1})$ be the subspace of all symmetric tensors in $T^d(\mathbb{C}^{n+1})$. For a vector $u$, denote by $u \otimes d$ the $d$th tensor power of $u$, i.e., $u \otimes d$ is the tensor such that $(u \otimes d)_{i_1 \ldots i_d} = u_{i_1} \cdot \ldots \cdot u_{i_d}$. As shown in [9], for each $A \in S^d(\mathbb{C}^{n+1})$, there exist vectors $u_1, \ldots, u_r \in \mathbb{C}^{n+1}$ such that

(1.1) $A = (u_1) \otimes d + \ldots + (u_r) \otimes d$.

The above is called a symmetric tensor decomposition (STD) for $A$. The smallest such $r$ is called the symmetric rank of $A$, for which we denote as $\text{srank}(A)$. If $\text{srank}(A) = r$, $A$ is called a rank-$r$ tensor and (1.1) is called a symmetric rank decomposition, which is also called a Waring decomposition in some references. The rank of a generic symmetric tensor is given by a formula in Alexander-Hirschowitz [2]. We refer to [9] for symmetric tensors and their symmetric ranks, and refer to [27, 30] for general tensors and their ranks.

This paper concerns symmetric tensor decompositions on a given set. Let $X \subseteq \mathbb{C}^{n+1}$ be a homogeneous set, i.e., $tx \in X$ for all $t \in \mathbb{C}$ and $x \in X$. If each $u_j \in X$, (1.1) is called a symmetric tensor decomposition on $X$ (STDX) for $A$. The STDX problem has been studied in applications for various choices of $X$. Symmetric tensor decompositions have broad applications in quantum physics [50].

2010 Mathematics Subject Classification. 15A69, 65F99.

Key words and phrases. symmetric tensor, numerical algorithm, decomposition, generating polynomial, generating matrix.
algebraic complexity theory \[7, 26, 46, 51\], numerical analysis \[34, 52\]. More tensor applications can be found in \[24\].

When \(X = \mathbb{C}^{n+1}\), the STDX is just the classical symmetric tensor decomposition, which has been studied extensively in the literature. Binary tensors (i.e., \(n = 1\)) decomposition problems were discussed in \[8\]. For higher dimensional tensors, the Catalecticant type methods \[23\] are often used when their ranks are low. For general symmetric tensors, Brachat et al. \[5\] proposed a method by using Hankel (and truncated Hankel) operators. It is equivalent to computing a new tensor whose order is higher but the rank is the same as the original one. Oeding and Ottaviani \[37\] proposed to compute symmetric decompositions by Koszul flattening, tensor eigenvectors and vector bundles. Other related work on computing symmetric tensor decompositions can be found in \[3, 4\]. For generic tensors of certain ranks, the symmetric tensor decomposition is unique. As shown in \[16\], a generic \(A \in S^m(\mathbb{C}^{n+1})\) has a unique Waring decomposition if and only if

\[(n, m, r) \in \{(1, 2k - 1, k), (3, 3, 5), (2, 5, 7)\}.

When \(A \in S^m(\mathbb{C}^{n+1})\) is a generic tensor of a subgeneric rank \(r\) (i.e., \(r\) is smaller than the value given by the Alexander-Hirschowitz formula; see \[2, 9\]) and \(m \geq 3\), the Waring decomposition is unique, with only three exceptions \[6\].

When \(X = \{a^n, a^{n-1}b, \ldots, ab^{n-1}, b^n\} : a, b \in \mathbb{C}\} \subseteq \mathbb{C}^{n+1}\), i.e., \(X\) is the affine cone of a rational normal curve in the projective space \(\mathbb{P}^n\), the STDX becomes a Vandermonde decomposition for symmetric tensors. It only exists for Hankel tensors, which were introduced in \[35\] for studying the harmonic retrieval problem. Hankel tensors were discussed in \[39\]. Relations among various ranks of Hankel tensors were studied in \[35\]. More applications of Hankel tensors can be found in \[44, 49\].

When \(X = \{a_1 \otimes \cdots \otimes a_k : a_1, \ldots, a_k \in \mathbb{C}^m\} \subseteq \mathbb{C}^{n+1}\) with \(n + 1 = m^k\), i.e., \(X\) is a Segre variety, the STDX becomes a Vandermonde decomposition for nonsymmetric tensors. This has broad applications in signal processing \[12, 29, 36, 47\]. Vandermonde decompositions for nonsymmetric tensors are closely related to secant varieties of Segre-Veronese varieties, which has been studied vastly \[1, 25, 40\]. In the subsection \[43\] we will discuss this question with more details. Interesting, it can be shown that every nonsymmetric tensor has a Vandermonde decomposition, which is different from the symmetric case.

**Contributions.** This paper focuses on computing symmetric tensor decompositions on a given set \(X\). We assume that \(X \subseteq \mathbb{C}^{n+1}\) is a variety that is given by homogeneous polynomial equations. Generally, symmetric tensor decompositions on \(X\) can be theoretically studied by secant varieties of the Veronese embedding of \(X\) \[28, 42, 43\]. From this view, one may expect to get polynomials which characterize the symmetric \(X\)-rank of a given symmetric tensor. In this paper, we give a method for computing symmetric \(X\)-rank decompositions. It is based on the tool of generating polynomials that were recently introduced in \[33\]. The work \[33\] only discussed the case \(X = \mathbb{C}^{n+1}\). When \(X\) is a variety, i.e., the method in \[33\] does not work, because \(u_i \in X\) is required in (1.1). We need to modify the approach in \[33\], by posing additional conditions on generating polynomials. For this purpose, we give a unified framework for computing symmetric tensor decompositions on \(X\), which sheds light on the study of both theoretical and computational aspects of tensor decompositions.
The paper is organized as follows. Section 2 gives some basics for tensor decompositions. Section 3 studies some properties of symmetric tensor decompositions on X. Section 4 defines generating polynomials and generating matrices. It gives conditions ensuring that the computed vectors belong to the given set in symmetric tensor decompositions. Section 5 presents a unified framework to do the computation. Last, Section 6 gives various examples to show how the proposed method works.

2. Preliminaries

Notation The symbol \( \mathbb{N} \) (resp., \( \mathbb{R}, \mathbb{C} \)) denotes the set of nonnegative integers (resp., real, complex numbers). For \( \alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), define \( |\alpha| := \alpha_1 + \cdots + \alpha_n \). For a degree \( d > 0 \), denote the index set
\[
\mathbb{N}_d^\alpha = \{ \alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \mid |\alpha| \leq d \}.
\]
For a real number \( t \in \mathbb{R} \), we denote by \([t]\) the smallest integer \( n \) such that \( n \geq t \).

Let \( x := (x_0, \ldots, x_n) \) and \( \mathbb{C}[x] := \mathbb{C}[x_0, \ldots, x_n] \) denote the ring of polynomials in \( x \) and with complex coefficients. For a degree \( m \), \( \mathbb{C}[x]_m \) denotes the subset of polynomials whose degrees are less than or equal to \( m \), and \( \mathbb{C}[x]_m^h \) denotes the subset of forms whose degrees are equal to \( m \). The cardinality of a finite set \( T \) is denoted as \(|T|\). For a finite set \( \mathcal{B} \subseteq \mathbb{C}[x] \) and a vector \( v \in \mathbb{C}^n \), denote
\[
[v]_{\mathcal{B}} := (p(v))_{p \in \mathcal{B}},
\]
the vector of polynomials in \( \mathcal{B} \) evaluated at \( v \). For a complex matrix \( A \), \( A^T \) denotes its transpose and \( A^* \) denotes its conjugate transpose. For a complex vector \( u \), \( \|u\|_2 = \sqrt{u^*u} \) denotes the standard Euclidean norm. The \( e_i \) denotes the standard \( i \)-th unit vector in \( \mathbb{C}^n \). For two square matrices \( X, Y \) of the same dimension, their commutator is \([X,Y] := XY - YX\).

2.1. Equivalent descriptions for symmetric tensors. There is a one-to-one correspondence between symmetric tensors in \( S^d(\mathbb{C}^{n+1}) \) and homogeneous polynomials of degree \( d \) and in \( (n+1) \) variables. When \( \mathcal{A} \in S^d(\mathbb{C}^{n+1}) \) is symmetric, we can equivalently use \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_d^\alpha \) to label \( \mathcal{A} \) in the way that
\[
\mathcal{A}_\alpha := \mathcal{A}_{j_1, \ldots, j_d}, \quad \text{if} \quad x_0^{d-|\alpha|} x_1^{\alpha_1} \cdots x_n^{\alpha_n} = x_{j_1} \cdots x_{j_d}.
\]
The symmetry guarantees that the labelling \( \mathcal{A}_\alpha \) is well-defined. For \( \mathcal{A} \), define the homogeneous polynomial (i.e., a form) in \( x := (x_0, \ldots, x_n) \) and of degree \( d \):
\[
\mathcal{A}(x) := \sum_{j_1, \ldots, j_d=0}^n \mathcal{A}_{j_1, \ldots, j_d} x_{j_1} \cdots x_{j_d}.
\]
If \( \mathcal{A} \) is labeled as in (2.2), then
\[
\mathcal{A}(x) = \sum_{\alpha \in \mathbb{N}_d^\alpha} \binom{d}{d-|\alpha|, \alpha_1, \ldots, \alpha_n} \mathcal{A}_\alpha x_0^{d-|\alpha|} x_1^{\alpha_1} \cdots x_n^{\alpha_n},
\]
where \( \left(\frac{d}{\alpha_0, \ldots, \alpha_n}\right) := \frac{d!}{\alpha_0! \cdots \alpha_n!} \). The decomposition (2.4) is equivalent to
\[
\mathcal{A}(x) = \sum_{j=1}^r ((u_j)_0 x_0 + \cdots + (u_j)_n x_n)^d.
\]
For a polynomial \( p = \sum_{\alpha \in \mathbb{N}_d^\alpha} p_\alpha y_1^{\alpha_1} \cdots y_n^{\alpha_n} \) and \( \mathcal{A} \in S^d(\mathbb{C}^{n+1}) \), define the operation
\[
\langle p, \mathcal{A} \rangle = \sum_{\alpha \in \mathbb{N}_d^\alpha} p_\alpha \mathcal{A}_\alpha.
\]
where \( \mathcal{A} \) is labeled as in (2.2). For fixed \( p \), \( \langle p, \cdot \rangle \) is a linear functional on \( S^d(\mathbb{C}^{n+1}) \), while for fixed \( \mathcal{A} \in S^d(\mathbb{C}^{n+1}) \), \( \langle \cdot, \mathcal{A} \rangle \) is a linear functional on \( \mathbb{C}[y_1, \ldots, y_n]_d \).

2.2. Algebraic varieties. A set \( I \subseteq \mathbb{C}[x] := \mathbb{C}[x_0, \ldots, x_n] \) is an ideal if \( I \cdot \mathbb{C}[x] \subseteq I \) and \( I + I \subseteq I \). For polynomials \( f_1, \ldots, f_s \in \mathbb{C}[x] \), let \( \langle f_1, \ldots, f_s \rangle \) denote the smallest ideal that contains \( f_1, \ldots, f_s \). A subset \( X \subseteq \mathbb{C}^{n+1} \) is called an affine variety if \( X \) is the set of common zeros of some polynomials in \( \mathbb{C}[x] \). The vanishing ideal of \( X \) is the ideal consisting of all polynomials identically vanish on \( X \).

Two nonzero vectors in \( \mathbb{C}^{n+1} \) are equivalent if they are parallel to each other. Denote by \( [u] \) the set of all nonzero vectors that are equivalent to \( u \); the set \([u]\) is called the equivalent class of \( u \). The set of all equivalent classes \([u]\) with \( 0 \neq u \in \mathbb{C}^{n+1} \) is the projective space \( \mathbb{P}^n \), or equivalently, \( \mathbb{P}^n = \{ [u] : 0 \neq u \in \mathbb{C}^{n+1} \} \). A subset \( Z \subseteq \mathbb{P}^n \) is said to be a projective variety if there are homogeneous polynomials \( h_1, \ldots, h_t \in \mathbb{C}[x] \) such that

\[
Z = \{ [u] \in \mathbb{P}^n : h_1(u) = \cdots = h_t(u) = 0 \}.
\]

The vanishing ideal \( \mathcal{I}(Z) \) is defined to be the ideal consisting of all polynomials identically vanish on \( Z \). For each nonnegative integer \( m \), we denote by \( \mathcal{I}_m(Z) \) the linear subspace of polynomials of degree \( m \) in \( \mathcal{I}(Z) \). A projective variety \( Z \subseteq \mathbb{P}^n \) is said to be nondegenerate if \( Z \) is not contained in any proper linear subspace of \( \mathbb{P}^n \), i.e., \( \mathcal{I}_1(Z) = \{0\} \).

In the Zariski topology for \( \mathbb{C}^{n+1} \) and \( \mathbb{P}^n \), the closed sets are varieties and the open sets are complements of varieties. For a projective variety \( Z \subseteq \mathbb{P}^n \), its Hilbert function \( h_Z : \mathbb{N} \to \mathbb{N} \) is defined as \( h_Z(d) := \dim \mathbb{C}[Z]_d \). As in [11, 20, 21], when \( d \) is sufficiently large, \( h_Z(d) \) is a polynomial and

\[
h_Z(d) = \frac{e}{m!} d^m + O(d^{m-1}), \quad e = \deg(Z), \quad m = \dim(Z),
\]

where \( O(d^{m-1}) \) denotes terms of order at most \( m - 1 \).

For an affine variety \( X \subseteq \mathbb{C}^{n+1} \), we denote by \( \mathbb{P}X \) the projective set of equivalent classes of nonzero vectors in \( X \), i.e., \( \mathbb{P}X = \{ [u] : 0 \neq u \in X \} \). If \( \mathbb{P}X = Z \), then \( X \) is called the affine cone of \( Z \). Clearly, the vanishing ideal of \( X \) is the same as that of \( \mathbb{P}X \), i.e., \( \mathcal{I}(X) = \mathcal{I}(\mathbb{P}X) \). For a degree \( m > 0 \), the set

\[
\mathcal{I}_m(X) := \mathbb{C}[x]_m^h \cap \mathcal{I}(X)
\]

is the space of all forms of degree \( m \) vanishing on \( X \).

**Veronese maps.** For an affine variety \( X \subseteq \mathbb{C}^{n+1} \), let \( v_d(X) \subseteq S^d(\mathbb{C}^{n+1}) \) be the image of \( X \) under the Veronese map:

\[
v_d : \mathbb{C}^{n+1} \to S^d(\mathbb{C}^{n+1}), \quad u \mapsto u \otimes^d.
\]

Note that \( v_d^{-1}(u \otimes^d) = \{ \omega^i u : i = 0, \ldots, d - 1 \} \), where \( \omega \) is a primitive \( d \)-th root of 1. Therefore, the dimension of \( v_d(X) \) is the same as that of \( X \). In particular, for

\[
C := \{ (x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1} : x_i x_j = x_k x_l, \forall i + j = k + l \},
\]

the set \( v_d(C) \) is a variety of tensors \( \mathcal{A} \in S^d(\mathbb{C}^2) \) that is defined by the equations

\[
\mathcal{A}_\alpha \mathcal{A}_\beta - \mathcal{A}_\gamma \mathcal{A}_\tau = 0 \quad (\alpha + \beta = \gamma + \tau, \alpha, \beta, \gamma, \tau \in \mathbb{N}_d^2).
\]

In the above, the tensors in \( S^d(\mathbb{C}^2) \) are labelled by vectors in \( \mathbb{N}_d^2 \). The projectivization \( \mathbb{P}v_d(C) \) is called the rational normal curve in the projective space.
\( P^d(C^2) \simeq P^d \). For a projective variety \( Z \subseteq P^n \), the Veronese embedding map \( v_d \) is defined in the same way as

\[
v_d: P^n \to P^d(C^{n+1}), \quad [u] \mapsto [u \otimes d].
\]

Note that \( P v_d(X) = v_d(P X) \) for every affine variety \( X \).

**Segre varieties.** For projective spaces \( P^{n_1} \times \cdots \times P^{n_k} \), their *Segre product*, denoted as \( \text{Seg}(P^{n_1} \times \cdots \times P^{n_k}) \), is the image of the Segre map:

\[
\text{Seg} : ([u_1], \ldots, [u_k]) \mapsto [u_1 \otimes \cdots \otimes u_k].
\]

The dimension of \( \text{Seg}(P^{n_1} \times \cdots \times P^{n_k}) \) is the sum \( n_1 + \cdots + n_k \). The Segre product \( P^{n_1} \times \cdots \times P^{n_k} \) is defined by equations of the form

\[
A_\alpha A_\beta - A_\gamma A_\tau = 0, \quad (\alpha + \beta = \gamma + \tau, \alpha, \beta, \gamma, \tau \in \prod_{j=1}^{k} \{0, \ldots, n_j\}).
\]

Here, tensors in \( P(\bigotimes_{j=1}^{k} C^{n_j+1}) \) are labelled by integral tuples in \( \prod_{j=1}^{k} \{0, \ldots, n_j\} \).

**Secant varieties.** Let \( X \subseteq C^{n+1} \) be an affine variety and let \( v_d(X) \) be its image under the \( d \)-th Veronese map \( v_d \). Define the set

\[
\sigma^v_d(v_d(X)) := \{(u_1) \otimes d + \cdots + (u_r) \otimes d : u_1, \ldots, u_r \in X\}.
\]

The Zariski closure of \( \sigma^v_d(v_d(X)) \), which we denote as \( \sigma_d(v_d(X)) \subseteq C^{n+1} \), is called the \( r \)-th secant variety of \( v_d(X) \). The closure \( \sigma_d(v_d(X)) \) is an affine variety in \( S^d(C^{n+1}) \), while \( \sigma^v_d(v_d(X)) \) is usually not. However, it holds that

\[
\dim \sigma^v_d(v_d(X)) = \dim \sigma_d(v_d(X)),
\]

because \( \sigma^v_d(v_d(X)) \) is a dense subset of \( \sigma_d(v_d(X)) \) in the Zariski topology. When \( v_d(X) \) is replaced by a general variety \( Y \), the sets \( \sigma^v_d(Y) \) and \( \sigma_d(Y) \) can be defined in the same way. We refer to [27] for secant varieties.

### 3. Properties of STDX

Let \( X \subseteq C^{n+1} \) be a set that is given by homogeneous polynomial equations. For a given tensor \( A \in S^d(C^{n+1}) \), a *symmetric X-decomposition* on \( X \) is

\[
A = (u_1) \otimes d + \cdots + (u_r) \otimes d, \quad u_1, \ldots, u_r \in X.
\]

The smallest such \( r \) is called the *symmetric X-rank* of \( A \), which we denote as \( \text{rank}_X(A) \), or equivalently,

\[
\text{rank}_X(A) = \min\{r : A = \sum_{i=1}^r (u_i) \otimes d, u_i \in C^X\}.
\]

When \( r \) is the smallest, \( (3.1) \) is called a *rank-retaining symmetric X-decomposition* for \( A \). It is possible that the decomposition \( (3.1) \) does not exist; for such a case, we define \( \text{rank}_X(A) = +\infty \). For instance, a symmetric tensor \( A \) admits a Vandermonde decomposition if and only if \( A \) is a Hankel tensor. So, if \( A \) is not Hankel, then \( \text{rank}_X(A) = +\infty \). Interested readers are referred to [35] [39] for more details. We denote by \( S^d(X) \) the subspace of tensors which admit a symmetric X-decomposition as in \( (3.1) \). As a counterpart for symmetric border rank, the *symmetric border X-rank* of \( A \) is similarly defined as

\[
\text{srank}_X(A) := \min\{r : A \in \sigma_d(v_d(X))\},
\]
where \( \sigma_r(v_d(X)) \) is the secant variety of \( v_d(X) \), defined in Subsection 2.2. When \( \mathbb{P}X \) is an irreducible variety, \( \text{srank}_X(A) \) is also equal to the smallest integer \( r \) such that \( A \) belongs to the limit of a sequence of tensors whose symmetric \( X \)-rank is \( r \) (see [27 Sec. 5.1.1] or [32 Theorem 2.33]). The generic symmetric \( X \)-rank of \( S^d(X) \) is the smallest \( r \) such that \( \sigma_r(v_d(X)) = S^d(X) \). When \( X = \mathbb{C}^{n+1} \), the symmetric \( X \)-rank becomes the usual symmetric rank (or Waring rank). If \( \mathbb{P}X = v_d(\mathbb{P}^1) \) is the rational normal curve, the symmetric \( X \)-rank becomes the Vandermonde rank for Hankel tensors [35, 39]. How to characterize tensors that has a symmetric \( X \)-decomposition as in [35]? How to tell \( \text{srank}_X(A) < +\infty \) or \( \text{srank}_X(A) = +\infty \)? These questions are the focus of this section.

### 3.1. Existence of symmetric \( X \)-decompositions

Let \( \mathbb{P}X \subseteq \mathbb{P}^n \) be the projectivization of \( X \). Its vanishing ideal is \( \mathcal{I}(X) \), the ideal of polynomials that are identically zero on \( X \). The section of degree-\( d \) forms in \( \mathcal{I}(X) \) is

\[
\mathcal{I}_d(X) := \mathcal{I}(X) \cap \mathbb{C}[x]^d.
\]

Let \( c := \dim \mathcal{I}_d(X) \). Choose a basis \( \{f_1, \ldots, f_c\} \) for \( \mathcal{I}_d(X) \). Let \( l_1, \ldots, l_c \) be linear functionals on \( S^d(\mathbb{C}^{n+1}) \) such that

\[
l_i \left( \sum_j (u_j)^{\otimes d} \right) = \sum_j f_i(u_j).
\]

They are linearly independent functions on \( S^d(X) \). If we use \( \tilde{f}_i \) to denote the dehomogenization of \( f_i \), i.e., \( \tilde{f}_i(x_1, \ldots, x_n) := f_i(1, x_1, \ldots, x_n) \), then \( l_i(A) = \langle \tilde{f}_i, A \rangle \) for all \( A \in S^d(\mathbb{C}^{n+1}) \). See [23] for the definition of the operation \( \langle \cdot, \cdot \rangle \).

**Proposition 3.1.** Let \( X, c, f_i, l_i \) be as above. Then, a tensor \( A \in S^d(\mathbb{C}^{n+1}) \) belongs to \( S^d(X) \) if and only if \( l_1(A) = \cdots = l_c(A) = 0 \). Consequently, the codimension of \( S^d(X) \) is \( c \), i.e., \( \dim S^d(X) = \binom{n+d}{d} - c \).

**Proof.** Clearly, if \( A \in S^d(X) \), then \( l_1(A) = \cdots = l_c(A) = 0 \). Next, we prove the converse is also true. Suppose \( A \in S^d(\mathbb{C}^{n+1}) \) is such that \( l_1(A) = \cdots = l_c(A) = 0 \). To show \( A \in S^d(X) \), it is enough to show that: if \( l \) is a linear function vanishing on \( S^d(X) \) then \( l(A) = 0 \). Each \( l_i \) vanishes on \( v_d(X) \) and \( l_1, \ldots, l_c \) are linearly independent as vectors in \( S^d(\mathbb{C}^{n+1})^* \). (The superscript * denotes the dual space.) Note that \( l \in S^d(\mathbb{C}^{n+1})^* \) and it vanishes on \( v_d(X) \). So, there is a form \( f \in \mathbb{C}[x]^d \) such that \( l(u^{\otimes d}) = f(u) \) for all \( u \in \mathbb{C}^{n+1} \). Since \( l \equiv 0 \) on \( v_d(X) \), \( f \) also vanishes on \( X \). So, \( f \) is a linear combination of \( f_1, \ldots, f_c \), and hence \( l \) is a linear combination of \( l_1, \ldots, l_c \). This implies that \( l(A) = 0 \) and \( A \in S^d(X) \).

Since \( l_1, \ldots, l_c \) are linearly independent, the subspace \( S^d(X) \) are defined by \( c \) linearly independent linear equations. So its codimension is \( c \). Since the dimension of \( S^d(\mathbb{C}^{n+1}) \) is \( \binom{n+d}{d} \), the dimension of \( S^d(X) \) follows from the codimension.

The first part of Proposition 3.1 is a high dimensional analogue of the apolarity lemma, which can be found in [23 Theorem 5.3], [11 Section 1.3], and [48 Section 3]). For convenience of referencing, we state this result here and give a straightforward proof. We would like to thank Zach Teitler for pointing out the relationship between Proposition 3.1 and the apolarity Lemma.

By Proposition 3.1 we get the following algorithm for checking if \( A \in S^d(X) \) or not. Suppose the vanishing ideal \( \mathcal{I}(X) \) is generated by the forms \( f_1, \ldots, f_s \), with degrees \( d_1 \leq \cdots \leq d_s \) respectively.
Algorithm 3.2. For a given tensor $A \in S^d(\mathbb{C}^{n+1})$, do the following:

Step 1: Find the integer $k \geq 0$ such that $d_k \leq d < d_{k+1}$.

Step 2: For each $t = 1, \ldots, k$ and $\beta \in \mathbb{N}^{n+1}_{d-t}$, let $f_{t, \beta} := f_t \cdot x_0^{\beta_0} \cdots x_n^{\beta_n}$.

Step 3: Check whether or not $\langle f_{t, \beta}, A \rangle = 0$ for all $t$ and $\beta$ in Step 2. If it is, then $A \in S^d(X)$; otherwise, $A \notin S^d(X)$.

The above algorithm can be easily applied to detect tensors in $S^d(X)$. For instance, if $X$ is defined by linear equations $\sum_{j=0}^n f_{ij}x_j = 0$ ($i = 1, \ldots, c$), then $A \in S^d(X)$ if and only if for all $1 \leq i \leq c$ and $0 \leq k_2, \ldots, k_m \leq n$,

$$\sum_{j=0}^n f_{ij}A_{jk_2 \cdots k_m} = 0.$$

If $X$ is a hypersurface defined by a single homogeneous polynomial $f(x) = 0$ with $\deg(f) \leq m$, then $A \in S^d(X)$ if and only if

$$\langle f \cdot x^\alpha, A \rangle = 0$$

for all monomials $x^\alpha$ with $\deg(f) + |\alpha| = m$.

When does $S^d(X) = S^d(\mathbb{C}^{n+1})$, i.e., when does every tensor admit a symmetric $X$-decomposition? By Proposition 3.1, we get the following characterization.

Corollary 3.3. Let $X \subseteq \mathbb{C}^{n+1}$ be as above. Then, the equality $S^d(X) = S^d(\mathbb{C}^{n+1})$ holds if and only if $\mathcal{I}_d(X) = \{0\}$, which is equivalent to that $X$ is not contained in any hypersurface of degree $d$.

The above corollary implies that if $X$ is a hypersurface of degree bigger than $d$, then every tensor in $S^d(\mathbb{C}^{n+1})$ has a symmetric $X$-decomposition. Moreover, if $X = \mathbb{C}^{n+1}$, then obviously we have $\mathcal{I}_d(X) = \{0\}$ for any $d \geq 1$, which implies the well known fact [9] that every symmetric tensor admits a symmetric decomposition.

3.2. The dimension and expected rank. By Proposition 3.1, $\dim S^d(X) = h_{\mathbb{P}X}(d)$, where $h_{\mathbb{P}X}(\cdot)$ is the Hilbert function for the projective variety $\mathbb{P}X$. For the Veronese map $v_d$, we have $\dim v_d(X) = \dim X$. Therefore, the expected dimension of the secant variety $\sigma_r(v_d(X))$:

$$\exp \dim \sigma_r(v_d(X)) = \min \{r \dim X, h_{\mathbb{P}X}(d)\}.$$

The expected generic symmetric $X$-rank of $S^d(X)$ is therefore

$$\exp \text{grank} = \lceil h_{\mathbb{P}X}(d)/\dim X \rceil.$$

Example 3.4. (i) If $\mathbb{P}X = \text{Seg}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k})$, the Segre variety, then $\dim X = n_1 + \cdots + n_k + 1$. Its Hilbert function $h_{\mathbb{P}X}(d) = \prod_{j=1}^k \binom{n_j + d}{n_j}$ [20] Example 18.15]. So

$$\dim S^d(X) = \prod_{j=1}^k \binom{n_j + d}{n_j}, \quad \exp \text{grank} = \left\lceil \prod_{j=1}^k \frac{n_j + d}{n_j} \right\rceil.$$

(ii) If $\mathbb{P}X \subseteq \mathbb{P}^n$ is a hypersurface defined by a form of degree $t$, the Hilbert function of $\mathbb{P}X$ is

$$h_{\mathbb{P}X}(d) = \begin{cases} \binom{n+d}{n}, & \text{if } d < t, \\ \binom{n+d}{n} - \binom{n+d-t}{n}, & \text{otherwise}. \end{cases}$$

1Here we adopt the convention that $d_0 = 0$ and $d_{d+1} = \infty$. 
Then, \( \dim S^d(X) = h_{\mathbb{P}X}(d) \) and the exp. grank can be obtained accordingly.

When \( \mathbb{P}X \) is a curve, we can get the dimension of \( \sigma_r(v_d(X)) \) as follows.

**Proposition 3.5.** If \( \mathbb{P}X \) is a non-degenerate curve (i.e., \( \dim \mathbb{P}X = 1 \) and \( \mathbb{P}X \) is not contained in any proper linear subspace of \( \mathbb{P}^n \)), then

\[
\dim \sigma_r(v_d(X)) = \min \{ 2r, h_{\mathbb{P}X}(d) \}.
\]

Therefore, the generic symmetric \( X \)-rank is \( \left\lfloor \frac{h_{\mathbb{P}X}(d)}{2} \right\rfloor \). Moreover, if \( \mathbb{P}X \) is nonsingular of genus \( g \) and degree \( t \), then there exists an integer \( d_0 \) such that for all \( d \geq d_0 \),

\[
\dim \sigma_r(v_d(X)) = \min \{ 2r, dt - g + 1 \}, \quad \text{grank} = \left\lfloor \frac{dt - g + 1}{2} \right\rfloor.
\]

**Proof.** The first part follows directly from [20, Example 11.30]. The “moreover” part follows from Riemann-Roch theorem [21, Chapter 4]. \( \square \)

By [20, Example 13.7], if \( \mathbb{P}X \) is a space curve of degree \( e \), i.e., \( \mathbb{P}X \) is a curve in \( \mathbb{P}^3 \) which intersects a generic plane in \( e \) points, then \( d_0 = e - 2 \) in Proposition 3.5. For an arbitrary curve \( \mathbb{P}X \), however, not much is known about \( d_0 \). The Hilbert functions are also known for Veronese varieties, Grassmann varieties and flag varieties, see [20, 19]. Hence the expected value of the generic rank for them may be calculated by (3.5).

### 3.3. Vandermonde decompositions for nonsymmetric tensors.

A nonsymmetric tensor \( \mathbf{A} \in (\mathbb{C}^{d+1})^\otimes k \) is said to admit a Vandermonde decomposition if there exist vectors (\( s = 1, \ldots, k, j = 1, \ldots, r \))

\[
v^{(j)}_s := (a_{sj})^d, (a_{sj})^{d-1} b_{sj}, \ldots, a_{sj} (b_{sj})^{d-1}, (b_{sj})^d \in \mathbb{C}^{d+1}
\]

such that

\[
\mathbf{A} = \sum_{j=1}^r v^{(j)}_1 \otimes \cdots \otimes v^{(j)}_k.
\]

The smallest integer \( r \) in the above is called the Vandermonde rank of \( \mathbf{A} \), which we denote as \( \text{vrank}(\mathbf{A}) \). Since \( v^{(j)}_s = (a_{sj}, b_{sj})^{\otimes d} \in S^d \mathbb{C}^2 \simeq \mathbb{C}^{d+1} \), we can rewrite (3.6) equivalently as

\[
\mathbf{A} = \sum_{j=1}^r \left( (a_{1j}, b_{1j}) \otimes \cdots \otimes (a_{kj}, b_{kj}) \right)^{\otimes d}.
\]

The Vandermonde decomposition can be thought of as a symmetric tensor decomposition on the set \( X \subseteq \mathbb{C}^{d^k} \) such that

\[
\mathbb{P}X := \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \quad (k \text{ times}).
\]

The variety \( \sigma_r(v_d(X)) \subseteq \mathbb{P} S^d(\mathbb{C}^{d^2}) \) is exactly the Zariski closure of tensors whose Vandermonde ranks at most \( r \). The vanishing ideal of the Segre variety \( \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \) is generated by \( 2 \times 2 \) minors of its flattenings [18]. So \( I_d(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}) \neq 0 \) for all \( d \geq 2 \). By Corollary 3.3 \( S^d(X) \) is a proper subspace of \( S^d(\mathbb{C}^{d^2}) \). However, every \( \mathbf{A} \in (\mathbb{C}^{d+1})^\otimes k \) has a Vandermonde decomposition.

**Theorem 3.6.** Every tensor in \( (\mathbb{C}^{d+1})^\otimes k \) has a Vandermonde decomposition.

**Proof.** Each \( \mathbf{A} \in (\mathbb{C}^{d+1})^\otimes k \) admits a general tensor decomposition, say,

\[
\mathbf{A} = \sum_{j=1}^L u_{j,1} \otimes \cdots \otimes u_{j,k}
\]
for vectors $u_{j,i} \in \mathbb{C}^{d+1}$. Choose distinct numbers $t_0, t_1, \ldots, t_d$ and let
$$v_l := (1, t_l, t_l^2, \ldots, t_l^d),$$
for $l = 0, 1, \ldots, d$. Clearly, $v_0, v_1, \ldots, v_d$ are linearly independent and they span $\mathbb{C}^{d+1}$. For each $u_{j,i}$, there exists numbers $\lambda_{j,i,0}, \ldots, \lambda_{j,i,d}$ such that
$$u_{j,i} = \lambda_{j,i,0} v_0 + \cdots + \lambda_{j,i,d} v_d.$$
Plugging the above expression of $u_{j,i}$ in the decomposition for $A$, we get
$$A = \sum_{j_1, \ldots, j_k=0}^l c_{j_1, \ldots, j_k} v_{j_1} \otimes \cdots \otimes v_{j_k},$$
for some scalars $c_{j_1, \ldots, j_k}$.

4. Generating polynomials

Assume that $X \subseteq \mathbb{C}^{n+1}$ is a set defined by homogeneous polynomial equations. For a given tensor $A \in S^d(\mathbb{C}^{n+1})$ with $\text{srank}_X(A) = r$, we discuss how to compute the symmetric $X$-decomposition
\begin{equation}
A = (u_1)^{\otimes d} + \cdots + (u_r)^{\otimes d}, \quad u_1, \ldots, u_r \in X.
\end{equation}
Suppose $X$ is given as
\begin{equation}
X = \{ x \in \mathbb{C}^{n+1} : h_i(x) = 0, i = 1, \ldots, N \},
\end{equation}
with homogeneous polynomials $h_i$ in $x := (x_0, x_1, \ldots, x_n)$. Denote $y := (y_1, \ldots, y_n)$. The dehomogenization of $X$ is the set
\begin{equation}
Y = \{ y \in \mathbb{C}^n : h_i(1, y_1, \ldots, y_n) = 0, i = 1, \ldots, N \}.
\end{equation}
If each $(u_j)_0 \neq 0$, then the decomposition (4.1) is equivalent to that
\begin{equation}
A = \lambda_1(1, v_1)^{\otimes d} + \cdots + \lambda_r(1, v_r)^{\otimes d},
\end{equation}
for scalars $\lambda_j \in \mathbb{C}$ and vectors $v_j \in Y$. In fact, they are
$$\lambda_j = \langle (u_j)_0 \rangle^d, \quad v_j = \frac{1}{(u_j)_0} \langle (u_j)_1, \ldots, (u_j)_n \rangle^T.$$
The assumption that $(u_j)_0 \neq 0$ is generic. For the rare case that it is zero, we can apply a generic coordinate change for $A$ so that this assumption holds. Throughout this section, we assume the rank $r$ is given.

4.1. Generating polynomials. Denote the quotient ring $C[Y] := \mathbb{C}[y]/I(Y)$, where $I(Y)$ is the vanishing ideal of $Y$. The $C[Y]$ is also called the coordinate ring of $Y$. We list monomials in $\mathbb{C}[y]$ with respect to the graded lexicographic order, i.e.,
$$1 < y_1 < \cdots < y_n < y_1^2 < y_1 y_2 < \cdots .$$
We choose $B_0$ to be the set of first $r$ monomials, whose images in $C[Y]$ are linearly independent, say,
$$B_0 = \{ y^{d_1}, \ldots, y^{d_r} \}.$$  
The border set of $B_0$ is $B_1 := B_0 \cup y_1 B_0 \cup \cdots \cup y_n B_0$. The boundary of $B_1$ is
$$\partial B_1 := (B_0 \cup y_1 B_0 \cup \cdots \cup y_n B_0) \setminus B_0.$$  
For a matrix $G \in \mathbb{C}^{r \times |\partial B_1|}$, we label it as
$$G = (c_{i,\alpha})_{i \in [r], \alpha \in \partial B_1}.$$

For each \( \alpha \in \partial B_1 \), denote the polynomial in \( y \)

\[
(4.5) \quad \varphi[G, \alpha] := \sum_{i=1}^{r} c_{i, \alpha} y^{\alpha_i} - y^\alpha.
\]

Denote the tuple of all such polynomials as

\[
(4.6) \quad \varphi[G] := (\varphi[G, \alpha](y) : \alpha \in \partial B_1) .
\]

Let \( J_G \) be the ideal generated by \( \varphi[G] \). The set \( \varphi[G] \) is called a border basis of \( J_G \) with respect to \( B_0 \). We refer to \[15, 14\] for border sets and border bases.

In the following, we give the definition of generating polynomials which were introduced in \[23\]. For \( \alpha \in \partial B_1 \), the polynomial \( \varphi[G, \alpha] \) is called a generating polynomial for \( A \in S^d(\mathbb{C}^{n+1}) \) if

\[
(4.7) \quad \langle y^\gamma \varphi[G, \alpha], A \rangle = 0 \quad \forall \gamma \in N_{d-|\alpha|}^n.
\]

(See \( (2.4) \) for the operation \( (, \cdot) \).) If \( \varphi[G, \alpha] \) is a generating polynomial for all \( \alpha \in \partial B_1 \), then \( G \) is called a generating matrix for \( A \). The set of all generating matrices for \( A \) is denoted as \( G(A) \). The condition \( (4.7) \) is equivalent to that

\[
(4.8) \quad \sum_{i=1}^{r} c_{i, \alpha} A_{\beta_i+\gamma} = A_{\alpha+\gamma} .
\]

We use \( G(:, \alpha) \) to denote the \( \alpha \)th column of \( G \). Then \( (4.8) \) can be rewritten as

\[
A[A, \alpha]G(:, \alpha) = b[A, \alpha]
\]

where \( A[A, \alpha], b[A, \alpha] \) are given as

\[
(A[A, \alpha])_{\gamma, \beta} = A_{\beta+\gamma}, \quad (b[A, \alpha])_{\gamma} = A_{\alpha+\gamma} \quad (\beta \in B_0, \gamma \in N_{d-|\alpha|}^n).
\]

The solutions to \( (4.8) \) can be parameterized as \( c_\alpha + N_\alpha w_\alpha \), where \( c_\alpha \) is a solution to \( (4.8) \), \( N_\alpha \) is a basis matrix for the nullspace, and \( w_\alpha \) is the vector of free parameters. So, every generating matrix can be parameterized as

\[
(4.9) \quad G(w) = C + N(w),
\]

where \( C \) is a constant matrix and \( N(w) \in \mathbb{C}^{[r] \times \partial B_1} \). The following is an example of parameterizing \( G(w) \).

**Example 4.1.** Consider the tensor \( A \in S^3(\mathbb{C}^4) \) such that

\[
A(1, y_1, y_2, y_3) = 32 y_3^3 - 12 y_3 y_2 y_1 - 48 y_3 y_2^2 + 36 y_3 y_2 y_1 + 150 y_3 y_1^2 - 20 y_2^2 - 18 y_2^2 y_1 + 42 y_2 y_1^2 + 51 y_1^3 + 18 y_3^2 - 36 y_3 y_2 + 84 y_3 y_1 - 30 y_2^2 + 12 y_2 y_1 + 45 y_1^2 + 6 y_3 - 6 y_2 + 9 y_1 - 1 .
\]

For \( r = 3 \), if we choose \( B_0 = \{1, y_1, y_2\} \), then

\[
\partial B_1 = \{y_3, y_1^2, y_1 y_2, y_2^2, y_1 y_3, y_2 y_3\}, \quad |\partial B_1| = 6 ,
\]

\[
\alpha_1 = (0, 0, 1), \quad \alpha_2 = (2, 0, 0), \quad \alpha_3 = (1, 1, 0),
\]

\[
\alpha_4 = (0, 2, 0), \quad \alpha_5 = (1, 0, 1), \quad \alpha_6 = (0, 1, 1).
\]
The $A[\alpha, \alpha_1]$ and $b[\alpha, \alpha_1]$ can be formulated accordingly. For instance,

$$A[\alpha, \alpha_1] = \begin{bmatrix}
(0,0,0) & (1,0,0) & (0,1,0) \\
-1 & 3 & -2 \\
3 & 15 & 2 \\
-2 & 2 & -10 \\
2 & 14 & -6 \\
-10 & -6 & -20 \\
14 & 50 & 6 \\
-6 & 6 & -16 \\
6 & 42 & -4
\end{bmatrix}, \quad b[\alpha, \alpha_1] = \begin{bmatrix}
(0,0,0) & (1,0,0) & (0,1,0) \\
2 & 14 & -6 \\
50 & 6 & -16 \\
42 & (0,1,1) & -4 \\
32 & (0,0,2)
\end{bmatrix}.$$ 

The generating matrix is uniquely determined, i.e.,

$$G(w) = \begin{bmatrix}
-1 & -3 & -1 & \frac{43}{29} & -4 & \frac{63}{29} \\
1 & 4 & 1 & -\frac{27}{20} & 4 & -\frac{7}{20} \\
1 & 0 & 1 & \frac{9}{10} & 1 & \frac{9}{10}
\end{bmatrix}.$$ 

For $r = 4$, if we choose $B_0 = \{y_1, y_2, y_3\}$, then $\partial B_1 = \{y_1^2, y_1y_2, y_2^2, y_1y_3, y_2y_3, y_3^2\}$. The generating matrix has 6 parameters, i.e.,

$$G(w) = \begin{bmatrix}
-3 + w_1 & -1 + w_2 & 83/20 + w_3 & -4 + w_4 & 63/20 + w_5 & 3/20 + w_6 \\
4 - w_1 & 1 - w_2 & -27/20 - w_3 & 4 - w_4 & -7/20 - w_5 & 53/20 - w_6 \\
-w_1 & 1 - w_2 & 9/10 - w_3 & 1 - w_4 & 9/10 - w_5 & 9/10 - w_6 \\
-3 + w_1 & w_2 & w_3 & w_4 & w_5 & w_6
\end{bmatrix}.$$ 

4.2. Polynomial division by $\varphi[G]$. From now on, suppose the vanishing ideal $I(Y) = \langle y_1, \ldots, y_n \rangle$, generated by $g_1, \ldots, g_N \in \mathbb{C}[y]$. For $p \in \mathbb{C}[y]$, denote by $\text{NF}(p; G)$ the normal form of $p$ with respect to $\varphi[G]$, i.e., $\text{NF}(p; G)$ is the remainder of $p$ divided by $\varphi[G]$, obtained by the Border Division Algorithm \cite{22} Proposition 6.4.11. Here we use the Formally, $\text{NF}(p; G)$ is the polynomial such that $p - \text{NF}(p; G) \in J_G$, the ideal generated by polynomials in $\varphi[G]$. Note that $\text{NF}(p; G)$ is a polynomial in $y := (y_1, \ldots, y_n)$ whose coefficients are parametrized by the entries of $G$.

**Proposition 4.2.** Suppose $B_0$ is the set of first $r$ monomial\(^2\) $y_1^\beta, \ldots, y_3^\beta$ in the graded lexicographic ordering such that their images in $\mathbb{C}[y]/I(Y)$ are linearly independent. Then, a polynomial $p \in \mathbb{C}[y]$ belongs to $J_G$ if and only if $\text{NF}(p; G) = 0$.

**Proof.** By the construction, $\varphi[G]$ is a border basis of $J_G$, so it contains a Gröbner basis (say, $S$) of $J_G$ with respect to the graded lexicographic ordering. Indeed, those elements in $G$ which are associated to the corner of $B_0$ form a Gröbner basis. See \cite{45} Proposition 2.30 or \cite{14} Proposition 4.3.9 for definition of the corner of $B_0$ and more details of the proof. Therefore, $p \in J_G$ if and only if $\text{NF}(p; G) = 0$. Since $S \subseteq \varphi[G]$, the normal form of $p$ with respect to $S$ is the same as the normal

\(^2\)We remark that instead of the first $r$ monomials, one may choose any $r$ monomials which are connected to one. Interested readers are referred to \cite{31} for a detailed discussion. For simplicity, we use in this paper the set of first $r$ monomials, which is obviously connected to one. This choice of basis will be convenient in practical computations like those in Section \text{8}.
form with respect to \( \varphi[G] \), which is \( \text{NF}(y; G) \). Therefore, \( p \in J_G \) if and only if \( \text{NF}(p; G) = 0 \).

The condition \( \text{NF}(p; G) = 0 \) is equivalent to that its coefficients are zeros identically. The coefficients of \( \text{NF}(p; G) \) are polynomials in \( c_{i,\alpha} \). Their degrees can be bounded as follows. For a degree \( k \geq 1 \), define the set \( B_k \) recursively as

\[
B_k := B_{k-1} \cup y_1 B_{k-1} \cup \cdots \cup y_n B_{k-1}.
\]

The monomials in \( B_k \) generate a subspace, which we denote as \( \text{Span} B_k \).

**Lemma 4.3.** Let \( B_k, p(y) \) and \( \text{NF}(p; G) \) be as above. Write \( p = p_1 + p_2 \), where \( p_1 \in \bigcup_{i \geq 1} \text{Span} B_i \) and \( p_2 \) is a polynomial whose monomials are not contained in any \( B_i \). If \( p_1 \in \text{Span} B_k \), then the coefficients of \( \text{NF}(p; G) \) are polynomials of degree at most \( k \) in \( G \).

**Proof.** Note that \( \text{NF}(p; G) = \text{NF}(p_1; G) + p_2 \), because \( p_2 \) is not reducible by \( \varphi[G] \). The coefficients of \( p_2 \) do not depend on \( G \).

For the case \( k = 1 \), we can write \( p_1 \in \text{Span} B_1 \) as

\[
p_1 = \sum_{\beta \in \partial B_1} a_{\beta} y^\beta + \sum_{\gamma \in B_1} a_{\gamma} y^\gamma,
\]

with coefficients \( a_{\beta}, a_{\gamma} \in \mathbb{C} \). The reduction of \( p \) by \( \varphi[G] \) is equivalent to that

\[
\text{NF}(p_1; G) = p_1 + \sum_{\beta \in \partial B_1} a_{\beta} \varphi[G, \beta] = \sum_{\gamma \in B_1} a'_{\gamma}(G)y^\gamma,
\]

where \( a'_{\gamma}(G) \) is affine linear in the entries of \( G \) and are affine linear in the coefficients of \( p \). So, the coefficients of \( \text{NF}(p; G) \) are affine linear in \( G \).

For the case \( k > 1 \), we can write each \( p_1 \) as

\[
p_1 = p_0 + \sum_{\beta \in \partial B_k} a_{\beta} y^\beta,
\]

with \( p_0 \in \text{Span} B_{k-1} \) and coefficients \( a_{\beta} \in \mathbb{C} \). For each \( \beta \in \partial B_k \), denote by \( i(\beta) \) the smallest \( i \in [n] \) such that \( \beta \in y_i B_{k-1} \). Then

\[
p_1 = p_0 + \sum_{\beta \in \partial B_k} a_{\beta} y_i(\beta) y^\beta - e_i(\beta),
\]

\[
\text{NF}(p_1; G) = \text{NF}(p_0; G) + \sum_{\beta \in \partial B_k} a_{\beta} \text{NF}\left(y_i(\beta)NF(y^{\beta - e_i(\beta)}; G); G\right).
\]

By induction, the coefficients of \( \text{NF}(p_1; G) \) are of degree \( \leq k \) in \( G \). \( \square \)

**Example 4.4.** Suppose that \( X \) is the Segre variety \( \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3 \). Then \( Y \) is the surface in \( \mathbb{C}^3 \) defined by \( g(y) := y_3 - y_1 y_2 = 0 \), where \( y_1, y_2 \) and \( y_3 \) are coordinates of \( \mathbb{C}^3 \). If we choose \( B_1 = \{1, y_1, y_2, y_3\} \), then

\[
\partial B_1 = \{y_1^2, y_1 y_2, y_2^2, y_1 y_3, y_2 y_3, y_3^2\}.
\]

\(3\) If we use the notation in Proposition 5.0, coordinates of \( \mathbb{C}^3 \) should be \( y_{1,0}, y_{0,1} \) and \( y_{1,1} \).
As in Example 4.1, we can get
\[
\phi[G, (2, 0, 0)] = (-3 + w_1) + (4 - w_1)y_1 - w_1y_2 + w_1y_3 - y_1^2,
\]
\[
\phi[G, (1, 1, 0)] = (-1 + w_2) + (1 - w_2)y_1 + (1 - w_2)y_2 + w_2y_3 - y_1y_2,
\]
\[
\phi[G, (0, 2, 0)] = (83/20 + w_4) + (-27/20 - w_4)y_1 + (9/10 - w_4)y_2 + w_4y_3 - y_2^2,
\]
\[
\phi[G, (1, 0, 1)] = (-4 + w_3) + (4 - w_3)y_1 + (1 - w_3)y_2 + w_3y_3 - y_1y_3,
\]
\[
\phi[G, (0, 1, 1)] = (63/20 + w_5) + (-7/20 - w_5)y_1 + (9/10 - w_5)y_2 + w_5y_3 - y_2y_3,
\]
\[
\phi[G, (0, 0, 2)] = (3/20 + w_6) + (-53/20 - w_6)y_1 + (9/10 - w_6)y_2 + w_6y_3 - y_2y_3.
\]

The normal form \( NF(g; G) \) of \( g \) is
\[
NF(g; G) = (1 - w_2) - (1 - w_2)y_1 - (1 - w_2)y_2 + (1 - w_2)y_3.
\]

The condition \( NF(g; G) = 0 \) requires that \( 1 - w_2 = 0 \).

4.3. Commutativity conditions. For each \( 1 \leq i \leq n \) and \( G \in \mathbb{C}^{[r] \times \partial B_1} \), define the matrix \( M_i(G) \in \mathbb{C}^{r \times r} \) as
\[
(M_i(G))_{s,t} = \begin{cases} 
1, & \text{if } \beta_s = \beta_t + e_i \in B_0, \\
0, & \text{if } \beta_s \neq \beta_t + e_i \in B_0, \\
c_{s,t} + e_i, & \text{if } \beta_t + e_i \in \partial B_1.
\end{cases}
\]

They are also called multiplication matrices for the ideal \( J_G \).

**Theorem 4.5.** Let \( B_0 \) be the set of first \( r \) monomials whose images in \( \mathbb{C}[Y] \) are linearly independent. Then, for \( G \in \mathbb{C}^{[r] \times \partial B_1} \), the polynomial system
\[
\phi[G](y) = 0
\]
has \( r \) solutions (counting multiplicities) and they belong to \( Y \) if and only if
\[
[M_i(G), M_j(G)] = 0 \quad (1 \leq i < j \leq n),
\]
\[
NF(g_i; G) = 0 \quad (i = 1, \ldots, N),
\]
where \( \langle g_1, \ldots, g_N \rangle = I(Y) \).

**Proof.** By Theorem 2.4 of [33], (4.10) has \( r \) solutions if and only if (4.11) holds. All the solutions are contained in \( Y \) if and only if \( I(Y) \subseteq J_G \), which is equivalent to that each normal form \( NF(g_i; G) = 0 \), by Proposition 4.2.

5. An algorithm for computing STDXs

Let \( X, Y \) be the varieties as in (4.2) and (4.3). This section discusses how to compute a symmetric tensor decomposition on \( X \) for a given tensor \( A \in S^d(\mathbb{C}^{n+1}) \) whose symmetric \( X \)-rank is \( r \). To compute (4.11), it is enough to compute
\[
A = \lambda_1(1, v_1)^{\otimes d} + \cdots + \lambda_r(1, v_r)^{\otimes d}
\]
for vectors \( v_1, \ldots, v_r \in Y \) and scalars \( \lambda_1, \ldots, \lambda_r \).

Choose \( B_0 = \{ x^{\beta_1}, \ldots, x^{\beta_r} \} \) to be the set of first \( r \) monomials that are linearly independent in \( \mathbb{C}[Y] \). For a point \( v \in \mathbb{C}^n \), denote by \( B_0(v) \in \mathbb{C}^r \) the column vector whose \( i \)-th entry is \( v^i \). Denote the Zariski open subset \( D \) of \( (\mathbb{C}^r)^r \)
\[
D = \{ (v_1, \ldots, v_r) \in (\mathbb{C}^n)^r : \det [B_0(v_1) \cdots B_0(v_r)] \neq 0 \}.
\]

Recall that \( \phi[G] \) is the tuple of generating polynomials as in (4.6) and \( \mathcal{G}(A) \) denotes the set of generating matrices for \( A \).
Remark 5.3. Let \( X, Y, D \) be as above. For each \( A \in S^d(\mathbb{C}^{n+1}) \), we have:

- If \( G \in \mathcal{G}(A) \) and \( v_1, \ldots, v_r \in Y \) are distinct zeros of \( \varphi[G] \), then there exist scalars \( \lambda_1, \ldots, \lambda_r \) satisfying (5.1).
- If the decomposition (5.1) holds for \( A \) and \( (v_1, \ldots, v_r) \in D \), then there exists a unique \( G \in \mathcal{G}(A) \) such that \( v_1, \ldots, v_r \) are zeros of \( \varphi[G] \).

Proof. The conclusions mostly follow from Theorem 3.2 of [33]. The difference is that we additionally require the points \( v_1, \ldots, v_r \in Y \).

According to Theorem 5.1 to compute a symmetric \( X \)-rank decomposition for \( A \), we need to find a generating matrix \( G \in \mathcal{G}(A) \) such that

1. \( \varphi[G] \) has \( r \) distinct zeros.
2. The zeros of \( \varphi[G] \) are contained in \( Y \), i.e., \( I(Y) \subseteq J_G \).

Conditions (1) and (2) are equivalent to (4.11) and (4.12) by Theorem 5.1. Suppose the vanishing ideal \( I(Y) = \langle g_1, \ldots, g_N \rangle \). We have the following algorithm.

Algorithm 5.2. For a given \( A \in S^d(\mathbb{C}^{n+1}) \) with \( \text{srank}_X(A) \leq r \), do the following:

1. **Step 0** Choose the set of first \( r \) monomials \( y^\beta_1, \ldots, y^\beta_r \) that are linearly independent in the quotient ring \( \mathbb{C}[Y] := \mathbb{C}[y]/I(Y) \), with respect to the graded lexicographic monomial order.
2. **Step 1** Parameterize the generating matrix \( G(w) = C + N(w) \) as in (4.9).
3. **Step 2** For each \( g_i \), compute \( \text{NF}(g_i; G(w)) \) with respect to \( \varphi[G(w)] \).
4. **Step 3** Compute a solution of the polynomial system
   \[
   \left\{ \begin{array}{l}
   [M_i(G(w)), M_j(G(w))] = 0 \quad (1 \leq i < j \leq n), \\
   \text{NF}(g_i; G(w)) = 0 \quad (1 \leq i \leq N).
   \end{array} \right.
   \]
5. **Step 4** Compute \( r \) zeros \( v_1, \ldots, v_r \in \mathbb{C}^n \) of the polynomial system
   \( \varphi[G(w), \alpha] = 0 \quad (\alpha \in \partial B_1) \).
6. **Step 5** Determine scalars \( \lambda_1, \ldots, \lambda_r \) satisfying (5.1).

The major task of Algorithm 5.2 is in Step 3. It requires to solve a set of polynomial system, which is given by commutative equations and normal forms. The commutative equations are quadratic in the parameter \( w \). The equations \( \text{NF}(g_i; G) = 0 \) are polynomial in \( w \), whose degrees are bounded in Lemma 4.3. One can apply the existing symbolic or numerical methods for solving polynomial equations. In Step 4, the polynomials \( \varphi[G(w), \alpha] \) have special structures. One can get a Gröbner basis quickly, hence their zeros can be computed efficiently. We refer to [11] and [33] Sec. 2.4 for how to compute their common zeros.

Remark 5.3. In Algorithm 5.2, we need to know a value of \( r \), with \( r \geq \text{srank}_X(A) \).

Typically, such a \( r \) is not known. In practice, we can choose \( r \) heuristically. For instance, we can choose \( r \) to be the expected generic rank given in the subsection 3.2.

If the flattening matrices of \( A \) have low ranks, we can choose \( r \) to be the maximum of their ranks. For any case, if the system (5.2) cannot be solved, we can increase the value of \( r \) and repeat the algorithm.

We conclude this section with an example of applying Algorithm 5.2.

**Example 5.4.** Let \( A \in S^4(\mathbb{C}) \) be the same tensor as in Example 4.4. Let \( X \subseteq \mathbb{C}^4 \) be the set whose dehomogenization \( Y \subseteq \mathbb{C}^3 \) is the surface whose defining ideal is \( I(Y) = \langle y_3 - y_1y_2 \rangle \). The maximum rank of flattening matrices of \( A \) is 3, so we
apply Algorithm 5.2 with \( r = 3 \). Choose \( B_0 = \{1, y_1, y_2\} \). From the calculation in Example 4.1, we can get

\[
M_1(G(w)) = \begin{bmatrix} 0 & -3 & -1 \\ 1 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_2(G(w)) = \begin{bmatrix} 0 & -1 & -\frac{83}{20} \\ 0 & 1 & -\frac{27}{20} \\ 1 & 1 & \frac{9}{10} \end{bmatrix},
\]

\[
M_3(G(w)) = \begin{bmatrix} -1 & -4 & \frac{63}{20} \\ 1 & 4 & -\frac{7}{20} \\ 1 & 1 & \frac{9}{10} \end{bmatrix}.
\]

The matrices \( M_1(G(w)), M_2(G(w)) \) and \( M_3(G(w)) \) commute. The generating polynomials are

\[
\phi[G, (0, 0, 1)] = (-1 + y_1 + y_2) - y_3,
\]

\[
\phi[G, (2, 0, 0)] = (-3 + 4y_1) - y_1^2,
\]

\[
\phi[G, (1, 1, 0)] = (-1 + y_1 + y_2) - y_1y_2,
\]

\[
\phi[G, (0, 2, 0)] = (\frac{83}{20} - \frac{27}{20}y_1 + \frac{9}{10}y_2) - y_2^2,
\]

\[
\phi[G, (1, 0, 1)] = (-4 + 4y_1 + y_2) - y_1y_3,
\]

\[
\phi[G, (0, 1, 1)] = (\frac{63}{20} - \frac{7}{20}y_1 + \frac{9}{10}y_2) - y_2y_3.
\]

They have 3 common zeros. There are no radical formulae for them, but they can be numerically evaluated as

\[
(3, 1, 3), \quad (1, -1.283, -1.283), \quad (1, 2.183, 2.183).
\]

The given symmetric X-decomposition for \( A \) as

\[
\tilde{A}(y) := 2 \left( 1 + 3y_1 + y_2 + 3y_3 \right)^3 - 0.7353 \left( 1 + y_1 - 1.283y_2 - 1.283y_3 \right)^3 \right.
\]

\[
- 2.265 \left( 1 + y_1 + 2.183y_2 + 2.183y_3 \right)^3.
\]

Because of numerical errors, we do not have \( \tilde{A} = A \) exactly, but the round-off error \( \|\tilde{A} - A\| \approx 6.84 \cdot 10^{-16} \).

### 6. Numerical experiments

In this section, we present examples of applying Algorithm 5.2 to compute symmetric X-rank decompositions. The computation is implemented in a laptop with a 2.5 GHz Intel Core i7 processor. The software for carrying out numerical experiments is Maple 2017. We solve the system (5.2) by the built-in function fsolve directly. The algorithm returns a decomposition

\[
\tilde{A} := \tilde{\lambda}_1(1, \tilde{v}_1) \otimes d + \cdots + \tilde{\lambda}_r(1, \tilde{v}_r) \otimes d.
\]

Because of round-off errors, the equation \( \tilde{A} = A \) does not hold exactly. We use the absolute error \( \|A - \tilde{A}\| \) or the relative one \( \|A - \tilde{A}\| / \|A\| \) to verify the correctness. Here, the Hilbert-Schmidt norm of \( A \) is used, i.e.,

\[
\|A\| = \left( \sum_{i_1, \ldots, i_m} |A_{i_1, \ldots, i_m}|^2 \right)^{1/2}.
\]
We display the computed decompositions by showing
\[
\tilde{V} = \begin{bmatrix} v_1 \\ \vdots \\ \tilde{v}_r \end{bmatrix}, \quad \tilde{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_r \end{bmatrix}.
\]
For neatness, only four decimal digits are shown, and \( r \) denotes the unit pure imaginary number. If the real or imaginary part of a complex number is smaller than \( 10^{-10} \), we treat it as zero and do not display it, for cleanness of the paper.

To apply Algorithm 5.2 we need a value for the rank \( r \). This issue is discussed in Remark 5.3. In our computation, we initially choose \( r \) to be the maximum rank of the flattening matrices of the given tensor \( A \). If the equations (4.8) or (5.2) are inconsistent, we need to increase the value of \( r \) by one, until Algorithm 5.2 successfully returns a decomposition.

For the set \( Y \) dehomogenized from \( X \) as in (4.8), we need generators of its vanishing ideal \( I(Y) \). For some \( Y \), it is easy to compute the generators of \( I(Y) \); for some \( Y \), it may be difficult to compute them. This is a classical, standard problem in symbolic computation. We refer to [13, 15, 17] for the related work. So, we do not focus on how to compute generators of \( I(Y) \) in this paper. In our examples, the generators of \( I(Y) \) are known or can be computed easily.

First we illustrate how to apply Algorithm 3.2 to detect the existence of the STDX for a given \( A \) and \( \mathbb{P}X \).

Example 6.1. We consider \( A \in S^4(\mathbb{C}^4) \) that is given as
\[
A_{ijk} = i + j + k
\]
and \( X \subseteq \mathbb{P}^3 \) that is defined by \( x_2^2 = x_1^2 + x_0, \ x_3^2 = x_2^2 + x_1^2 \). According to Algorithm 3.2, we have
\[
f_1 = x_2^2 - x_1^2 - x_0, \quad f_2 = x_3^2 - x_2^2 - x_1^2.
\]
We let \( \beta \in \mathbb{N}_1^4 \) be \( \beta = (1, 0, 0, 0) \) and hence we have \( f_{1,\beta} = f_1 x_0 \). It is straightforward to verify that \( (f_{1,\beta}, A) \neq 0 \) and hence \( A \) has no STDX for \( X \). Another example is \( A \in S^4(\mathbb{C}^3), \ A_{ijkl} = (i + j + k + l)^2 - (i^2 + j^2 + k^2 + l^2) \) and \( X \in \mathbb{P}^2 \) is defined by \( x_0 x_1 + x_1 x_2 + x_0 x_2 = 0, x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_0 = 0 \). By Algorithm 3.2 again, we can show easily that \( A \) does not admit an STDX for such \( X \).

The resting examples in this section are devoted to exhibit the validity and efficiency of Algorithm 5.2. To this end, we make the following convention on the representations of tensors. Recall that a tensor \( A \in S^d(\mathbb{C}^{n+1}) \) is an array of numbers whose elements are indexed by \( (i_1, \ldots, i_d) \), i.e., \( A = (A_{i_1, \ldots, i_d}) \), where \( 0 \leq i_1, \ldots, i_d \leq n \). As in Section 2.1, we can equivalently represent \( A \) by \( A_\alpha \)'s, where \( \alpha = (a_0, \ldots, a_n) \in \mathbb{N}_d^{n+1} \) satisfies \( |\alpha| = d \). To be more precise, for each element in \( \{ (i_1, \ldots, i_d) : 0 \leq i_1, \ldots, i_d \leq n \} \), we have
\[
A_{i_1, \ldots, i_d} = A_\alpha,
\]
where \( \alpha \in \mathbb{N}_d^{n+1} \) is the sequence such that \( |\alpha| = d \) and \( x_0^{a_0} \cdots x_n^{a_n} = x_{i_1} \cdots x_{i_d} \). We may list the entries \( A_\alpha \) with respect to the lexicographic order, i.e., \( A_\alpha \) precedes \( A_\beta \) if and only if the most left nonzero entry of \( \alpha - \beta \) is positive. For instance, a
binary cubic tensor $A \in S^3(\mathbb{C}^2)$ can be displayed as $A_{300}, A_{210}, A_{120}, A_{030}$. In the following examples, we will represent a symmetric tensor $A$ in this way.

**Example 6.2.** Let $\mathbb{P}X \subseteq \mathbb{P}^2$ be the parabola defined by $x_2^2 - x_0 x_1 + x_0^2 = 0$. Let $A \in S^3(\mathbb{C}^3)$ be the symmetric tensor such that

$$
A_{300} = 15, A_{210} = 81, A_{201} = -6, A_{120} = 621, A_{111} = -108, A_{102} = 66, \\
A_{030} = 5541, A_{021} = -1296, A_{012} = 540, A_{003} = -102.
$$

The vanishing ideal of $Y \subseteq \mathbb{C}^2$ is

$$I(Y) = \langle y_1^2 - y_1 + 1 \rangle.$$

The maximum rank of flattening matrices of $A$ is 3. When we run Algorithm 5.2 with $r = 3$, it does not give a desired tensor decomposition. So, we apply Algorithm 5.2 with $r = 4$ and the symmetric $X$-decomposition with

$$
\tilde{V} = \begin{bmatrix}
9.874 - 0.002786 i & -2.979 + 0.0004677 i \\
1.151 - 0.006293 i & -0.3886 + 0.008096 i \\
4.124 + 0.02027 i & 1.768 + 0.005734 i \\
16.62 + 1.251 i & 3.956 + 0.1581 i
\end{bmatrix}, \quad \tilde{\Lambda} = \begin{bmatrix}
5.184 + 0.0036 i \\
3.694 + 0.0184 i \\
6.098 - 0.0154 i \\
0.0246 - 0.0066 i
\end{bmatrix}.
$$

We have $\|A\| = 7241.79$ and the error $\|A - \tilde{A}\| = 8 \cdot 10^{-14}$.

**Example 6.3.** Let $\mathbb{P}X \subseteq \mathbb{P}^2$ be the nodal curve defined by $x_1^3 + x_0 x_1^2 - x_0 x_2^2 = 0$. The vanishing ideal of $Y \subseteq \mathbb{C}^2$ is

$$I(Y) = \langle y_1^3 + y_1^2 - y_2^2 \rangle.$$

Let $A \in S^3(\mathbb{C}^3)$ be the symmetric tensor such that

$$
A_{300} = 3, A_{210} = 24, A_{201} = 72, A_{120} = 144, A_{111} = 456, A_{102} = 1224, \\
A_{030} = 1080, A_{021} = 3288, A_{012} = 9432, A_{003} = 28512.
$$

The maximum rank of flattening matrices of $A$ is 3. When we run Algorithm 5.2 with $r = 3, 4$, it fails to give a desired tensor decomposition. So, we apply Algorithm 5.2 with $r = 5$ and get the symmetric $X$-decomposition with

$$
\tilde{V} = \begin{bmatrix}
3.029 - 0.07505 i & -6.079 + 0.2073 i \\
-1.047 + 0.1114 i & 0.2338 + 0.2814 i \\
2.615 - 0.09868 i & 4.970 - 0.2555 i \\
7.927 - 0.008981 i & 23.68 - 0.03874 i \\
-5.685 - 3.079 i & 10.84 - 10.80 i
\end{bmatrix}, \quad \tilde{\Lambda} = \begin{bmatrix}
-0.9769 - 0.06497 i \\
-1.4650 - 0.1107 i \\
3.3470 + 0.1587 i \\
2.0980 + 0.01425 i \\
-0.002438 + 0.002715 i
\end{bmatrix}.
$$

We have $\|A\| = 41632.56$ and the error $\|A - \tilde{A}\| = 4 \cdot 10^{-13}$.

**Example 6.4.** Let $\mathbb{P}X \subseteq \mathbb{P}^3$ be the union of two planes defined by $(x_3 - x_2)(x_1 - x_0) = 0$. Let $A \in S^3(\mathbb{C}^4)$ be the symmetric tensor such that

$$
A_{3000} = 2, A_{2100} = 1, A_{2010} = 5, A_{2001} = -3, A_{1200} = 5, A_{1110} = 10, A_{1020} = 9, \\
A_{1101} = 2, A_{1011} = 7, A_{1002} = 5, A_{0300} = -5, A_{0210} = 2, A_{0120} = 8, A_{0030} = 29, \\
A_{0201} = -6, A_{0111} = 6, A_{0021} = 7, A_{0102} = 4, A_{0012} = 5, A_{0003} = -9.
$$

The vanishing ideal of $Y \subseteq \mathbb{C}^3$ is

$$I(Y) = \langle (y_3 - y_2)(y_1 - 1) \rangle.$$
The maximum rank of flattening matrices of $\mathcal{A}$ is 4. When we run Algorithm 5.2 with $r = 4$, it fails to give a desired tensor decomposition. So, we use $r = 5$ and apply Algorithm 5.2. It returns the symmetric $X$-decomposition

$$\tilde{V} = \begin{bmatrix} -2.0 & -1.0 & -1.0 \\ 1.0 & -1.487 & 1.0 \\ 1.0 & 2.287 & 1.0 \\ -1.0 & 1.0 & 1.0 \\ 1.0 & 0 & -2.0 \end{bmatrix}, \quad \tilde{\Lambda} = \begin{bmatrix} 1.0 \\ 2.249 \\ -1.0 \\ 1.0 \end{bmatrix}. $$

We have $\|\mathcal{A}\| = 104.86$ and the error $\|\mathcal{A} - \tilde{\mathcal{A}}\| = 10^{-15}$.

**Example 6.5.** Let $\mathbb{P}X \subseteq \mathbb{P}^3$ be the surface defined by $-3x_1x_2^2 + x_1^3 - x_2^3x_3 = 0$. Then $Y \subseteq \mathbb{C}^3$ is the monkey saddle surface whose vanishing ideal is

$$I(Y) = \langle -3y_1y_2^2 + y_1^3 - y_3 \rangle.$$

Let $\mathcal{A} \in S^3(\mathbb{C}^4)$ be the symmetric tensor such that

$\mathcal{A}_{3000} = 5, \mathcal{A}_{2100} = -1, \mathcal{A}_{2001} = 6, \mathcal{A}_{1200} = -13, \mathcal{A}_{1201} = 9, \mathcal{A}_{1110} = 8, \mathcal{A}_{1020} = 6, \mathcal{A}_{1101} = -33, \mathcal{A}_{1011} = -16, \mathcal{A}_{0202} = 87, \mathcal{A}_{0300} = 17, \mathcal{A}_{0210} = 24, \mathcal{A}_{0120} = 10, \mathcal{A}_{0030} = 12, \mathcal{A}_{0201} = -91, \mathcal{A}_{0111} = -54, \mathcal{A}_{0021} = -38, \mathcal{A}_{0102} = 233, \mathcal{A}_{0012} = 5, \mathcal{A}_{0003} = 739$.

The maximum rank of flattening matrices of $\mathcal{A}$ is 4. When we run Algorithm 5.2 with $r = 4, 5$, it fails to give a desired tensor decomposition. So, we apply Algorithm 5.2 with $r = 6$ and get the symmetric $X$-decomposition

$$\tilde{V} = \begin{bmatrix} 5.936 - 0.8124i & 3.582 - 0.4510i & -19.62 + 2.978i \\ -0.4115 - 0.5223i & -1.801 + 1.185i & 9.226 - 2.511i \\ 2.042 - 0.01150i & -1.031 + 0.00833i & 2.008 - 0.001995i \\ -0.6422 - 0.02100i & 0.9440 + 0.001783i & 1.453 + 0.03664i \\ 2.157 - 0.08836i & 1.576 - 0.04318i & 6.043 + 0.3063i \end{bmatrix}, \quad \tilde{\Lambda} = \begin{bmatrix} 3.163 - 10^{-18}i \\ -1.0 - 1.565 \cdot 10^{-18}i \\ -3.163 - 10^{-18} - 1.004 \cdot 10^{-18}i \\ -1.0 + 1.519 \cdot 10^{-18}i \\ 0.03454 + 0.02496i \\ 0.001235 + 0.00440i \\ -0.9564 - 0.01231i \\ 2.0480 - 0.05064i \\ 1.8730 + 0.03358i \\ 2.0 + 1.8560 \cdot 10^{-17}i \end{bmatrix}.$$  

In the next two examples, we still display the tensor entries $\mathcal{A}_{\alpha}$ according to the lexicographic order, but we drop the labelling indices, for cleanness of the paper.

**Example 6.6.** Let $\mathbb{P}X \subseteq \mathbb{P}^4$ be the curve defined by

$$x_1x_3 - x_0x_2 - x_0x_1 + x_0x_3 = 0, \quad x_3^2 - x_0x_1 - x_0^2 = 0, \quad x_1x_4 + 4x_1x_3 - x_1x_2 - x_1^2 + 5x_1 = 0.$$  

The vanishing ideal of $Y \subseteq \mathbb{C}^4$ is then

$$I(Y) = \langle y_1y_3 - y_2 - y_2 + y_3, y_3^2 - y_1 - 1, y_1y_4 + 4y_1y_3 - y_1y_2 - 2y_1^2 - 5y_1 \rangle.$$  

Let $\mathcal{A} \in S^3(\mathbb{C}^4)$ be the symmetric tensor whose 35 entries are

$$-7, -2, 87, 25, 20, 26, 334, -233, 60, -45, -9, 130, -144, -74, 182, 406, 1754, 1150, 13647, 300, 156, 2353, 24, 421, 85, 830, 610, 6500, 60, 1050, 150, 550, 3630, 880, -250.$$
The maximum rank of flattening matrices of $\mathcal{A}$ is 5. When we run Algorithm 5.2 with $r = 5$, it fails to give a desired tensor decomposition. So, we apply Algorithm 5.2 with $r = 6$ and get the symmetric $X$-decomposition

$$
\tilde{V} = \begin{bmatrix}
680.9 & 17130.0 & 26.11 & 17700.0 \\
7.717 & 18.02 & 2.952 & 8.926 \\
0 & -1.0 & -1.0 & -10.52 - 8.239 i \\
-0.2272 & -0.4521 & -0.8791 & -2.163 \\
0.7526 & 1.568 & 1.324 & -7.975 \\
2.945 & -10.78 & -1.986 & -4.891
\end{bmatrix}, \quad \tilde{\lambda} = \begin{bmatrix}
0 \\
1.1720 \\
0 \\
-6.780 \\
3.871 \\
-5.263
\end{bmatrix}.
$$

We have the norm $\|\mathcal{A}\| = 29222.16$ and the error $\|\mathcal{A} - \tilde{\mathcal{A}}\| = 10^{-10}$.

**Example 6.7.** Let $\mathbb{P}X \subseteq \mathbb{P}^4$ be the surface defined by

$$
x_3^2 + x_4^2 - x_0 x_1 = 0, \quad x_3 x_4 - x_0 x_2 = 0.
$$

Then the vanishing ideal of the variety $Y \subseteq \mathbb{C}^4$ is

$$
I(Y) = \langle y_3^2 + y_4^2 - y_1, y_3 y_4 - y_2 \rangle.
$$

Let $\mathcal{A} \in S^4(\mathbb{C}^5)$ be the symmetric tensor whose 70 entries are respectively

22, 38, 89, 6, 34, 220, 490, 79, 119, 65, 32, 165, 71, 69, 6, 2216, 3044, 686, 653, 1029, 490, 239, 195, 173, 40, 1111, 574, 257, 490, 79, 65, 25, 317, 71, 100, 21424, 20440, 6028, 4570, 1615, 8033, 3788, 1918, 929, 1553, 1162, 415, 455, 233, 116, 8187, 4316, 1954, 1007, 3044, 686, 653, 490, 239, 173, 663, 1882, 271, 574, 257, 79, 621, 335, 317, -54.

The maximum rank of flattening matrices of $\mathcal{A}$ is 10. When we run Algorithm 5.2 with $r = 10$, we get the symmetric $X$-decomposition

$$
\tilde{V} = \begin{bmatrix}
5.0 & -2.0 \\
0.7596 + 0.2348 i & 0.2187 + 0.5369 i & -1.0 & 2.0 \\
1.937 + 0.1658 i & 0.9718 + 0.0853 i & 1.0 & 0.9718 + 0.0853 i \\
2.0 & 1.0 & -1.0 & -1.0 \\
2.065 + 0.07655 i & -1.033 - 0.03706 i & 1.0 & -1.033 - 0.03706 i \\
5.063 - 0.00546 i & -0.003 + 0.01636 i & 1.0 & -0.003 + 0.01636 i \\
4.999 + 0.01109 i & 2.0 + 0.002774 i & 1.0 & 2.0 + 0.002774 i \\
8.0 & 4.0 & 2.0 & 2.0
\end{bmatrix}, \quad \tilde{\lambda} = \begin{bmatrix}
8.0 \\
-10.0 \\
0.505 + 0.297 i \\
3.232 - 0.413 i \\
11.0 \\
-3.778 + 0.132 i \\
-7.979 + 0.053 i \\
5.0 \\
6.01 - 0.069 i \\
7.0
\end{bmatrix}.
$$

We conclude this section by considering various examples on Segre varieties.

**Example 6.8.** (Vandermonde decompositions of nonsymmetric tensors) Each tensor $\mathcal{A} \in (\mathbb{C}^{d+1})^\otimes k$ has a Vandermonde decomposition as in 3.6, which is proved in Theorem 3.6. We can view $\mathcal{A}$ as a tensor in $S^d(\mathbb{C}^{2k})$ with the set $X \subseteq \mathbb{C}^{2k}$ such that $\mathbb{P}X = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ ($\mathbb{P}^1$ is repeated $k$ times). Let

$$
n = 2^k - 1.
$$

A vector $x \in \mathbb{C}^{2k}$ can be labelled as $x = (x_\nu)$, with binary vectors $\nu \in \{0,1\}^k$. Under this labelling, the set $X$ is defined by the homogeneous equations (c.f. 20).
Example 2.11]) or [27, Section 4.3.5])

\[(6.1)\]

\[x_\mu x_\nu - x_\eta x_\theta = 0\]

for all \(\mu, \nu, \eta, \theta \in \{0, 1\}^k\) such that for some \(1 \leq i < j \leq k\),

\[\mu = \nu = \eta = \theta \quad (l \neq i, j), \quad \mu_i + \nu_j = \eta_j + \theta_i.\]

Under the dehomogenization \(x_{0...0} = 1\), the corresponding affine variety \(Y \subseteq \mathbb{C}^{2k-1}\)

consists of vectors \(y\), labelled as \(y = (x_\mu)\) with \(0 \neq \mu \in \{0, 1\}^k\), defined by the

vanishing ideal

\[I(Y) = \langle y_\mu y_\nu - y_\eta y_\theta \rangle,\]

where \(\mu, \nu, \eta, \theta \in \{0, 1\}^k\) are as above and \(y_{0...0} = 1\).

We apply Algorithm 5.2 to compute Vandermonde decompositions for random \(\mathcal{A} \in (\mathbb{C}^{d+1})^{\otimes k}\) whose entries are randomly generated (obeying the normal distribution). For all the instances, Algorithm 5.2 successfully got Vandermonde rank decompositions. The relative errors \(\frac{\|\mathcal{A} - \mathcal{A}_\|}{\|\mathcal{A}\|}\) are all in the magnitude of \(O(10^{-16})\).

The consumed computational time (in seconds) is also reported. The results are displayed in Table 1.

**Table 1.** Computational results on symmetric \(X\)-decompositions on Segre varieties

| \(k\) | \(n\) | \(d\) | \(r\) | time | \(k\) | \(n\) | \(d\) | \(r\) | time |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 3 | 4 | 7.07 | 3 | 7 | 3 | 7 | 7.45 |
| 2 | 3 | 4 | 4 | 7.08 | 3 | 7 | 3 | 9 | 9.74 |
| 2 | 3 | 4 | 5 | 7.11 | 3 | 7 | 3 | 10 | 19.04 |
| 2 | 3 | 4 | 6 | 7.31 | 3 | 7 | 4 | 8 | 6.85 |
| 2 | 3 | 4 | 7 | 7.45 | 3 | 7 | 4 | 9 | 6.88 |
| 2 | 3 | 4 | 8 | 8.11 | 3 | 7 | 4 | 10 | 6.54 |
| 2 | 3 | 5 | 5 | 7.10 | 3 | 7 | 4 | 11 | 8.35 |
| 2 | 3 | 5 | 6 | 7.08 | 3 | 7 | 4 | 12 | 11.37 |
| 2 | 3 | 5 | 7 | 7.22 | 3 | 7 | 4 | 13 | 22.84 |
| 2 | 3 | 5 | 8 | 7.17 | 3 | 7 | 4 | 14 | 41.45 |
| 2 | 3 | 5 | 9 | 7.37 | 3 | 7 | 4 | 15 | 69.26 |
| 2 | 3 | 5 | 10 | 7.76 | 3 | 7 | 4 | 16 | 134.47 |
| 2 | 3 | 5 | 11 | 8.87 | 3 | 7 | 4 | 17 | 224.97 |
| 2 | 3 | 6 | 12 | 7.84 | 3 | 7 | 4 | 18 | 382.00 |
| 2 | 3 | 6 | 13 | 7.97 | 3 | 7 | 5 | 19 | 8.35 |
| 2 | 3 | 6 | 14 | 8.50 | 3 | 7 | 5 | 20 | 8.45 |
| 2 | 3 | 6 | 15 | 9.26 | 3 | 7 | 5 | 21 | 8.65 |
| 2 | 3 | 6 | 16 | 12.33 | 3 | 7 | 5 | 22 | 8.65 |
| 2 | 3 | 7 | 17 | 8.27 | 3 | 7 | 5 | 23 | 8.80 |
| 2 | 3 | 7 | 18 | 12.11 | 3 | 7 | 5 | 24 | 9.08 |
| 2 | 3 | 7 | 19 | 34.24 | 3 | 7 | 5 | 25 | 8.96 |
| 2 | 3 | 7 | 20 | 722.98 | 3 | 7 | 6 | 26 | 9.10 |
7. Conclusion

In this paper, we discuss how to compute symmetric $X$-decompositions of symmetric tensors on a given variety $X$. The tool of generating polynomial is used to do the computation. Based on that, give an algorithm for computing symmetric $X$-decompositions. Various examples are given to demonstrate the correctness and efficiency of the proposed method.

References

[1] H. Abo and M. Brambilla. On the dimensions of secant varieties of segre-veronese varieties. *Annali di Matematica Pura ed Applicata*, 192(1):61–92, 2013.

[2] J. Alexander and A. Hirschowitz. Polynomial interpolation in several variables. *Journal of Algebraic Geometry*, 4(1995), pp. 201-22.

[3] E. Balllico and A. Bernardi. Decomposition of homogeneous polynomials with low rank. *Math. Z.*, 271, 1141-1149, 2012.

[4] A. Bernardi, A. Gimigliano and M. Idà. Computing symmetric rank for symmetric tensors. *Journal of Symbolic Computation* 46, (2011), 34-53.

[5] J. Brachat, P. Comon, B. Mourrain, and E. Tsigaridas. Symmetric tensor decomposition. *Linear Algebra and its Applications*, 433(11):1851–1872, 2010.

[6] L. Chiantini, G. Ottaviani, and N. Vannieuwenhoven. On generic identifiability of symmetric tensors of subgeneric rank. *Trans. Amer. Math. Soc.*, 369 (2017), 4021-4042.

[7] L. Chiantini, J. Hauenstein, C. Ikenmeyer, J. Landsberg, and G. Ottaviani. Polynomials and the exponent of matrix multiplication. *arXiv preprint arXiv:1706.05072*, 2017.

[8] G. Comas and M. Seiguer. On the rank of a binary form. *Foundations of Computational Mathematics*, Vol. 11, No. 1, pp. 65-78, 2011.

[9] P. Comon, G. Golub, L.-H. Lim, and B. Mourrain. Symmetric tensors and symmetric tensor rank. *SIAM Journal on Matrix Analysis and Applications*, 30(3):1254–1279, 2008.

[10] R. Corless, P. Gianni, and B. Trager. A reordered schur factorization method for zero-dimensional polynomial systems with multiple roots. In *International Symposium on Symbolic and Algebraic Computation*, pp. 133–140, 1997.

[11] D. Cox, J. Little and D. OShea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*, Springer, 2007.

[12] L. De Lathauwer and B. De Moor. From matrix to tensor: Multilinear algebra and signal processing. In *Institute of Mathematics and Its Applications Conference Series*, vol. 67, pp. 1–16, 1998.

[13] D. Eisenbud, H. Craig and V. Wolmer. *Direct methods for primary decomposition.. Inventiones mathematicae*, 110.1 (1992): 207-235.

[14] A. Kehrein, M. Kreuzer and L. Robbiano. An algebraists view on border bases. In: A. Dickstein et al. (eds) *Solving Polynomial Equations: Foundations, Algorithms, and Applications*. Algorithms and Computation in Mathematics, vol. 14, pp. 169–202. Springer-Verlag, New York-Heidelberg, 2005.

[15] E. Fortuna, P. Gianni and B. Trager. Derivations and radicals of polynomial ideals over fields of arbitrary characteristic. *Journal of Symbolic Computation*, 33.5 (2002): 609-625.

[16] F. Galuppi and M. Mella. Identifiability of homogeneous polynomials and Cremona Transformations. *Preprint*, 2016. [arXiv:1606.06895v2[math.AG]]

[17] T. Gianni, T. Barry, and Z. Gall. Gröbner bases and primary decomposition of polynomial ideals.. *Journal of Symbolic Computation*, 6.2-3 (1988): 149-167.

[18] R. Grone. Decomposable tensors as a quadratic variety. *Proceedings of the American Mathematical Society*, 64(2):227–230, 1977.

[19] B. Gross and N. Wallach. On the Hilbert polynomials and Hilbert series of homogeneous projective varieties. *Arithmetic geometry and automorphic forms* 19 (2011): 253-263.

[20] J. Harris. *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133. Springer-Verlag, New York, 1992.

[21] R. Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, vol. 52. Springer-Verlag, New York-Heidelberg, 1977.

[22] M. Kreuzer and L. Robbiano. *Computational commutative algebra 2*. Springer Science & Business Media, 2005.
[23] A. Iarrobino and V. Kanev. Power Sums, Gorenstein algebras, and determinantal varieties. Lecture Notes in Mathematics #1721, Springer, 1999.
[24] T. Kolda and B. Bader. Tensor decompositions and applications. *SIAM Rev.* vol. 51, no. 3, 455-500, 2009.
[25] A. Laface and E. Postinghel. Secant varieties of segre-veronese embeddings of $(\mathbb{P}^1)^r$. *Mathematische Annalen*, 356(4):1455–1470, 2013.
[26] J. Landsberg. The border rank of the multiplication of $2 \times 2$ matrices is seven. *Journal of the American Mathematical Society*, 19(2):447–459, 2006.
[27] J. Landsberg. *Tensors: geometry and applications*. Graduate Studies in Mathematics, 128, AMS, Providence, RI, 2012.
[28] J. Landsberg and G. Ottaviani. Equations for secant varieties of veronese and other varieties. *Annali di Matematica Pura ed Applicata*, 192(4):569–606, 2013.
[29] L.-H. Lim and P. Comon. Multiarray signal processing: Tensor decomposition meets compressed sensing. *Comptes Rendus Mécanique*, 338(6):311–320, 2010.
[30] L.-H. Lim. Tensors and hypermatrices, in: L. Hogben (Ed.), *Handbook of linear algebra*, 2nd Ed., CRC Press, Boca Raton, FL, 2013.
[31] B. Mourrain. A new criterion for normal form algorithms. *International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes*, Springer, Berlin, Heidelberg, 1999.
[32] D. Mumford. *Algebraic Geometry I: Complex Projective Varieties*. Springer Verlag, Berlin, 1995.
[33] J. Nie. Generating polynomials and symmetric tensor decompositions. *Foundations of Computational Mathematics*, 17(2):423–465, 2017.
[34] J. Nie. Low rank symmetric tensor approximations, *SIAM J. Matrix Anal. Appl.*, 38 (no. 4), pp. 1517–1540, 2017.
[35] J. Nie and K. Ye. Hankel tensor decompositions and ranks. *SIAM J. Matrix Anal. Appl.*, 40 (no. 2), 486–516, 2019.
[36] D. Nion and N. Sidiropoulos. Tensor algebra and multidimensional harmonic retrieval in signal processing for mimo radar. *IEEE Transactions on Signal Processing*, 58(11):5693–5705, 2010.
[37] L. Oeding and G. Ottaviani. Eigenvectors of tensors and algorithms for waring decomposition. *Journal of Symbolic Computation*, 54:9–35, 2013.
[38] J. Papu, L. De Lathauwer, and S. Huffel. Exponential data fitting using multilinear algebra: the single-channel and multi-channel case. *Numerical linear algebra with applications*, 12(8):809–826, 2005.
[39] L. Qi. Hankel tensors: Associated hankel matrices and vandermonde decomposition. *Communications in Mathematical Sciences*, 13(1):113–125, 2015.
[40] C. Raicu. Secant varieties of segre-veronese varieties. *Algebra & Number Theory*, 6(8):1817–1868, 2012.
[41] K. Ranestad, and F. O. Schreyer. Varieties of sums of powers. *Journal für die reine und angewandte Mathematik*, (2000): 147-182.
[42] S. Sam. Ideals of bounded rank symmetric tensors are generated in bounded degree. *Inventiones mathematicae*, 207(1):1–21, 2017.
[43] S. Sam. Sziszegies of bounded rank symmetric tensors are generated in bounded degree. *Mathematische Annalen*, 368(3-4):1095–1108, 2017.
[44] M. Signoretto, L. De Lathauwer, and J. Suykens. A kernel-based framework to tensorial data analysis. *Neural networks*, 24(8):861–874, 2011.
[45] H. Stetter. *Numerical Polynomial Algebra*. SIAM, Philadelphia, 2004.
[46] V. Strassen. Gaussian elimination is not optimal. *Numerische mathematik*, 13(4):354–356, 1969.
[47] W. Sun and H. So. Accurate and computationally efficient tensor-based subspace approach for multidimensional harmonic retrieval. *IEEE Transactions on Signal Processing*, 60(10):5077–5088, 2012.
[48] Z. Teitler. Sufficient conditions for Strassens additivity conjecture. *Illinois Journal of Mathematics*, 59.4 (2015): 1071-1085.
[49] S. Trickett, L. Burroughs, A. Milton, et al. Interpolation using hankel tensor completion. In *2013 SEG Annual Meeting*. Society of Exploration Geophysicists, 2013.
[50] W. Uemura and O. Sugino. Symmetric tensor decomposition description of fermionic many-body wave functions. Physical Review Letters, 109(25):253001, 2012.

[51] K. Ye and L.-H. Lim. Fast structured matrix computations: tensor rank and cohn-umans method. Foundations of Computational Mathematics, 18(1) pp. 45–95, 2018.

[52] K. Ye and L.-H. Lim. Tensor network ranks. arXiv preprint arXiv:1801.02662, 2018.

Jiawang Nie, Department of Mathematics, University of California San Diego, 9500 Gilman Drive, La Jolla, CA, USA, 92093.
E-mail address: njw@math.ucsd.edu

Ke Ye and Lihong Zhi, KLMM, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
E-mail address: keye@amss.ac.cn, lzh@mmrc.iss.ac.cn