Gaussian Quantum Reading beyond the Standard Quantum Limit

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(Dated: February 8, 2013)

Quantum reading aims at retrieving classical information stored in an optical memory with low energy and high accuracy by exploiting the inherently quantum properties of light. We provide an optimal Gaussian strategy for quantum reading with phase-shift keying encoding that makes use of squeezed coherent light and homodyne detectors to largely outperform the Standard Quantum Limit, even in the presence of loss. This strategy, being feasible with current quantum optical technology, represents a viable prototype for a highly efficient and reliable quantum-enhanced optical reader.

I. INTRODUCTION

Digital optical memories store classical information in the optical properties of a media. They are ubiquitous in modern applications of classical information processing, such as CDs or DVDs. To retrieve information, optical readers illuminate each memory cell with a light probe and measure the outgoing signal. Clearly a tradeoff exists between the energy of the source and the information it is able to extract. On the one hand, the need for miniaturized and embedded technology push toward the use of low energetic light sources. On the other hand, reliability is an unavoidable requirement for modern information processing technology.

In the general framework of quantum theory, inherently quantum properties of light can be exploited to maximize the amount of retrieved information per fixed energy of the light probe. This problem, usually referred to as quantum reading, was recently introduced by Pirandola [1] and immediately triggered a noticeable interest [2–9]. However, most of the experimental as well as commercial implementations of optical readers make use of a suboptimal strategy exploiting coherent light probes generated by a laser beam. The maximal amount of information per fixed energy that can be retrieved using a coherent probe defines the Standard Quantum Limit [10–12] in quantum reading. Since in general Gaussian states and measurements are experimentally feasible with current optical technology, the question naturally arises: can Gaussian quantum reading outperform the Standard Quantum Limit, thus providing highly efficient while practically feasible quantum reading techniques?

We need to be more specific before answering this question. In applications, the most common ways to encode classical information are in the phase or in the amplitude of the signal. The two methods are usually referred to as phase-shift keying (PSK) and amplitude-shift keying (ASK), respectively. Due to its simplicity, PSK encoding has been widely adopted in several ISO and IEEE standards, such as wireless LAN (wifi), several credit cards, Bluetooth, and satellite communications. Despite its widespread use, the problem of quantum reading was mainly addressed in the context of ASK encoding [10–12], while only recently the interest drew by quantum reading with PSK encoding increased.

In Ref. [8] the problem of quantum reading of two signals with phase difference $\pi$ was addressed in the lossless scenario. It was shown that perfect quantum reading can be achieved by a particular class of entangled coherent states [13]. Very recently, in Ref. [9] the scenario was generalized to the case of $M$ signals with symmetrically distributed phases - namely, the relative phase of signal $i$ with respect to a given seed state is $2\pi i/M$. In particular, it was shown that squeezed coherent states outperform coherent ones in quantum reading of two signals with phase difference of $\pi$ in the lossless scenario and without any constraint on the measurement applied. While of the utmost importance these results leave the aforementioned question open: can Gaussian quantum reading - namely, quantum reading with Gaussian probes and Gaussian measurements [14] - outperform the Standard Quantum Limit in the general lossy scenario?

The aim of this work is to affirmatively answer this question in the context of PSK encoding. We provide an optimal Gaussian strategy exploiting squeezed coherent light and homodyne detection to perform quantum reading for any value of the phase difference between the two signals and in the presence of loss. A comparison of the optimal Gaussian strategy with the Standard Quantum Limit shows that the former largely outperforms the latter, even in the presence of loss and taking into account present technological limitations in the preparation of highly squeezed states. The proposed optimal strategy is suitable for implementation with current quantum optical technology, thus representing a viable prototype for a highly efficient and reliable quantum-enhanced optical reader.

The paper is structured as follows. In Sect. II we introduce the problem of quantum reading and simplify it under the assumption that no encoding is done in the amplitude of the signal (e.g. PSK encoding). In Sect. III we provide the Standard Quantum Limit and an optimal Gaussian strategy for quantum reading with PSK encoding. In Sect. IV we compare the optimal Gaussian strategy with the Standard Quantum Limit and demonstrate experimental feasibility of Gaussian quantum reading beyond the Standard Quantum Limit. Finally we summarize our results and propose future developments in Sect. V.
II. QUANTUM READING

In this Section we formally introduce the quantum reading of optical memories as the problem of determining the tradeoff between energy and probability of error in the discrimination of quantum channels. Let us first fix the notation.

A $m$-modes quantum optical setup is represented by an Hilbert space $\mathcal{H} = \bigotimes_{i=1}^{m} \mathcal{H}_i$, where $\mathcal{H}_i$ is the Fock space representing mode $i$ and $a_i$ is the corresponding annihilation operator. In the following we denote with $\mathcal{L}(\mathcal{H})$ the space of linear operators on $\mathcal{H}$. A quantum state on $\mathcal{H}$ is described by a density matrix $\rho \in \mathcal{L}(\mathcal{H})$, namely a positive semidefinite operator satisfying $\text{Tr}[\rho] = 1$, and in the following we pictorially represent it with $\rho$. Up to irrelevant constants its energy is given by $E(\rho) := \text{Tr}[N |\varphi\rangle\langle\varphi|]$, where $N := \sum_{i=1}^{m} a_i^\dagger a_i$ is the number operator on $\mathcal{H}$. A quantum measurement on $\mathcal{H}$ is described by a POVM $\Pi$, namely a map associating to any state $\rho \in \mathcal{L}(\mathcal{H})$ a probability distribution $p(\rho) \langle \rho | j \rangle$ over a set $\{ j \}$ of outcomes, and we represent it with $\{ \rho, \Pi \}$. The most general transformation from states to states is described by a quantum channel $\mathcal{C}: \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}')$, namely a completely positive and trace preserving map that we represent with $\mathcal{C}$. The most general strategy to discriminate a channel $\mathcal{C}_i$ randomly chosen from a set $\{ \mathcal{C}_i : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}') \}^{d_i}$ with probability $p_i$ consists in probing it with a state $\rho \in \mathcal{L}(\rho \in \mathcal{K})$, where $\mathcal{K}$ is an ancillary space, and measuring the output state with a $d$-outcomes POVM $\Pi$, namely

$$\{ \rho, \mathcal{K}, \mathcal{C}_i, \Pi \}$$

The probability of error is given by $P_e(\rho, \Pi) := 1 - \sum_{i=1}^{d_i} p_i p(\rho | i)$. Deriving the tradeoff between $P_e(\rho, \Pi)$ and $E(\rho)$ is the aim of quantum reading.

**Definition 1** (Quantum reading). The optimal state $\rho^*$ and the optimal measurement $\Pi^*$ for quantum reading of channels $\{ \mathcal{C}_i \}$ distributed according to probability $\{ p_i \}$ are those that minimize the error-probability $P_e(\rho, \Pi)$ while satisfying $E(\rho) \leq E$ for given energy threshold $E$, namely

$$(\rho^*, \Pi^*) = \arg \min_{(\rho, \Pi)} P_e(\rho, \Pi) \text{ s.t. } E(\rho) \leq E.$$  

Let us specify the problem more. An $m$-modes quantum optical device is described by a unitary operator $U : \mathcal{H} \rightarrow \mathcal{H}'$ relating $d$ input optical modes with annihilation operators $a_i$’s on Fock space $\mathcal{H}_i$ to $d$ output optical modes with annihilation operators $a_i'$’s on Fock space $\mathcal{H}_i'$. An optical device $U$ is called linear and passive if the $a_i$’s are linearly related to the $a_i'$’s, namely $a' = S_U a$, where $a = (a_1, \ldots, a_d)$ and $S_U$ is the scattering matrix associated to $U$. A passive device $U$ conserves the energy, namely for any state $\rho$ one has $E(\rho) = E(U \rho U^\dagger)$. A simple example of quantum optical device is the $\psi$-phase shifter, namely a single mode device represented by the unitary $P^\psi = \exp(i \omega a^\dagger a)$. Another example is the beamsplitter with transmittivity $\eta$, namely a 2-modes device represented by the unitary $B^\eta = \exp[\theta (a^\dagger b - ab^\dagger)]$, where $\eta = \cos^2 \theta$. For any unitary $U$ we denote with $U(\rho) := U \rho U^\dagger$ the corresponding unitary channel; we will consider only linear and passive unitary channels.

Loss, affecting any experimental quantum optical implementation, can be modeled by a lossy channel $E^\eta(\rho) := \text{Tr}^o[B^\eta(\rho_0 \otimes |0\rangle \langle 0|)]$ with quantum efficiency $\eta$, namely a beamsplitter with signal and vacuum as inputs, and one output mode of which is traced out, or equivalently $\hat{\mathcal{H}}_0 = \begin{array}{l} \mathcal{H}_0 \\ \text{|} 0 \text{rangle} \\ \mathcal{H}_1 \\ B^\eta \\ (I) \end{array}$. For any bipartite operator $X_{0i} \in \mathcal{L}(\mathcal{H}_0 \otimes \mathcal{H}_i)$, we denote with $\text{Tr}_1[X_{0i}]$ the partial trace over Hilbert space $\mathcal{H}_1$. When loss affects $m$ optical modes we write $E^\eta := \otimes_i E_i^\eta$ with $\eta = (\eta_1, \ldots, \eta_m)$. A lossy device $U^\eta$ is described by the composition of a unitary channel $U$ and a lossy channel $E^\eta$, namely $U^\eta := U \circ E^\eta$. Analogously, a lossy source $\rho^\eta$ is described by the preparation of an ideal state $\rho$ followed by a lossy channel $E^\eta$, namely $\rho^\eta := E(\rho, \eta)$, and a lossy measurement $\Pi^\eta$ is described by an ideal one $\Pi$ preceded by a lossy channel $E^\eta$, namely $\Pi^\eta := E^\eta(\Pi)$, where $C^\eta$ represents the application of channel $C$ in the Heisenberg picture. In principle lossy channels can be absorbed in the definition of states and measurements, nevertheless in the following it will be convenient to keep the contribution of loss in evidence.

When each memory cell can be modeled as a lossy optical device, the quantum reading strategy given by Eq. (1) becomes

$$\{ \rho, E^\alpha, \mathcal{U}, \gamma, \mathcal{E}^\delta, \mathcal{E}^\gamma, \Pi \}$$

Here and in the following, the dashed box on the left surrounds the lossy preparation, the one in the middle the lossy unknown device, while the one on the right the lossy measurement. The setup in Eq. (2) can be simplified taking into account the following rule of composition of lossy channels.

**Lemma 1** (Composition). Given an Hilbert space $\mathcal{H}_0$, the composition of lossy channels $E^\alpha : \mathcal{L}(\mathcal{H}_0) \rightarrow \mathcal{L}(\mathcal{H}_0)$ and $E^\beta : \mathcal{L}(\mathcal{H}_0) \rightarrow \mathcal{L}(\mathcal{H}_0)$ is a lossy channel $E^\eta : \mathcal{L}(\mathcal{H}_0) \rightarrow \mathcal{L}(\mathcal{H}_0)$ with efficiency $\eta = (\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_m, \beta_m)$, namely

$$E^\alpha \circ E^\beta(\rho) = E^\eta(\rho), \quad \forall \rho \in \mathcal{L}(\mathcal{H}_0).$$

**Proof.** Since $E^\delta = \otimes_i E_i^{\eta_i}$, it is sufficient to prove the statement for a single mode, namely we prove that $E^\alpha \circ E^\beta(\rho) = E^\eta(\rho)$ with $\eta = \alpha \beta$. Denote with $\mathcal{H}_1$ and $\mathcal{H}_2$ the ancillary Fock spaces of channels $E^\alpha$ and $E^\beta$ respectively, namely $E^\alpha(\rho) := \text{Tr}_1[B^\alpha(\rho_0 \otimes \rho_1)]$ and
Lemma 2. Given an Hilbert space $\mathcal{H}$, for any lossy channel $\mathcal{E}^\beta : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ such that $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ is a constant vector and for any unitary linear and passive channel $\mathcal{U} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, one has that $\mathcal{E}^\beta$ commutes with $\mathcal{U}$, namely

$$\mathcal{U} \circ \mathcal{E}^\beta(\rho) = \mathcal{E}^\beta \circ \mathcal{U}(\rho), \quad \forall \rho \in \mathcal{L}(\mathcal{H}).$$

Proof. Denote with $\mathcal{K} = \bigotimes_i \mathcal{K}_i$ the ancillary Hilbert space in the definition of lossy channel $\mathcal{E}^\beta$, namely $\mathcal{E}^\beta(\rho) = \text{Tr}_{\mathcal{K}}(\rho \otimes \sigma \otimes \sigma^\beta_i \otimes \mathcal{K})$ where $\sigma = \bigotimes_i \ket{0}
\bra{0} \in \mathcal{L}(\mathcal{K})$ and beamsplitter $\mathcal{B}^\beta_i$ acts on Hilbert spaces $\mathcal{K}_i$ and $\mathcal{K}_i$. Then the statement can be reformulated as

$$\bigotimes_i \mathcal{B}^\beta_i(\mathcal{U} \otimes \mathcal{U}_i) \mathcal{K}_i \mathcal{E}^\beta \bigotimes_i \mathcal{B}^\beta_i,$$

where $\mathcal{U}_i$ denotes channel $\mathcal{U}$ acting on Hilbert space $\mathcal{K}_i$, or equivalently

$$\bigotimes_i \mathcal{B}^\beta_i(\mathcal{U} \otimes \mathcal{U}_i) \mathcal{K}_i \mathcal{E}^\beta \bigotimes_i \mathcal{B}^\beta_i.$$

Upon replacing Eq. (11) into Eq. (9) [or equivalently Eq. (11) into Eq. (9)] the statement follows, namely

$$\bigotimes_i \mathcal{B}^\beta_i(\mathcal{U} \otimes \mathcal{U}_i) \mathcal{K}_i \mathcal{E}^\beta \bigotimes_i \mathcal{B}^\beta_i,$$

where second equality holds due to unitarily invariance of trace and second-to-last holds since $\sigma = \bigotimes_i \ket{0}
\bra{0}$. $\square$

Whenever the hypothesis of Lemma 2 is satisfied, namely losses affecting each mode of unitary channel $\mathcal{U}$ are equal - this is true for example when modes are implemented by optical-fiber paths of the same length - applying Lemma 2 and Lemma 1 again the quantum reading strategy in Eq. (12) can be rewritten as

$$\bigotimes_i \mathcal{B}^\beta_i(\mathcal{U} \otimes \mathcal{U}_i) \mathcal{K}_i \mathcal{E}^\beta \bigotimes_i \mathcal{B}^\beta_i.$$
III. GAUSSIAN QUANTUM READING

In this Section we derive the Standard Quantum Limit - namely, the optimal strategy using coherent states and homodyne measurements - and an optimal Gaussian strategy - using squeezed coherent states and homodyne measurements - for quantum reading with PSK encoding for any value of the phase difference between signals encoding logical 0 and 1 and in the presence of loss. Let us first fix the notation\cite{22,23}.

A coherent state $|\alpha\rangle$ is obtained by applying the displacement operator $D(\alpha) := e^{\alpha a^\dagger - \alpha^* a}$ to the vacuum state $|0\rangle$, namely $|\alpha\rangle := D(\alpha)|0\rangle$. The energy of a coherent state is given by $E(\alpha) = |\alpha|^2$. A squeezed coherent state $|\alpha,\xi\rangle$ is obtained by subsequently applying the squeezing operator $S(\xi) := e^{\xi/2 (a - a^\dagger)}$ and the displacement operator $D(\alpha)$ to the vacuum state $|0\rangle$, namely $|\alpha,\xi\rangle := D(\alpha)S(\xi)|0\rangle$. The energy of a squeezed coherent state is given by $E(\alpha,\phi) = |\alpha|^2 + \sinh^2 |\xi|$.

For any state $\rho$ it is useful to introduce its Wigner function $W_\rho(x,p)$ defined as

$$W_\rho(x,p) := \frac{1}{\pi} \int_{-\infty}^{\infty} \langle x-y,\psi|\rho|x+y,\psi\rangle e^{-i2rp}dy,$$

so the Wigner function of squeezed coherent state $|e^{i\phi}a, e^{i\theta}r\rangle$ is given by

$$W_{|\alpha,\xi\rangle}(x,p) = \frac{1}{\alpha} e^{-(x^2+p^2)},$$

where

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} e^{-r} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & e^{r} \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} - \begin{pmatrix} a \cos \phi \\ a \sin \phi \end{pmatrix},$$

and that of coherent state $|e^{i\phi}a\rangle$ can be obtained as a particular case by setting $r=0$.

The POVM $\Pi^\psi$ describing an homodyne measurement $\mathcal{E}^\alpha(\mathcal{E}^\gamma(\alpha,\psi))$ associates to state $\rho$ the probability distribution $p(\rho|x,\psi)$ given by

$$p(x|\rho,\psi) = \int_{-\infty}^{\infty} dp W_\rho(x',p'),$$

where

$$\begin{pmatrix} x' \\ p' \end{pmatrix} = \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}.$$

In the following we denote with $p(\rho|x,\xi,\psi)$ the conditional probability distribution of outcome $x$ of homodyne measurement $\Pi^\psi$ with efficiency $\eta$ given input squeezed coherent state $|\alpha,\xi\rangle$ with $\alpha = e^{i\theta}a$ and $\xi = e^{i\theta}r$, given by the Gaussian

$$p(x|\alpha,\xi,\psi) := \sqrt{\frac{1}{\pi \sigma^2(\xi,\psi)}} e^{-\frac{(x-x_0(\alpha,\xi,\psi))^2}{\sigma^2(\xi,\psi)}},$$

where

$$x_0(\alpha,\psi) := a \cos(\psi - \phi),$$

$$\sigma(\xi,\psi) := \sqrt{\frac{e^{-2\sigma^2(\xi,\psi)} \cos^2(\psi - \frac{\theta}{2}) + e^{2\sigma^2(\xi,\psi)} \sin^2(\psi - \frac{\theta}{2}) + 1 - \eta}{4\eta}},$$

while the analogous distribution for coherent state $|e^{i\phi}a\rangle$ can be obtained as a particular case by setting $\eta=0$. The term $\frac{1-\eta}{4\eta}$ in the definition of $\sigma(\xi,\psi)$ is due to the fact that the conditional probability distribution in the presence of loss is the convolution of the ideal conditional probability distribution with a Gaussian with variance $\frac{1-\eta}{4\eta}$ (see Refs.\cite{22,25}).

We introduce now the problem of quantum reading with PSK encoding with a lossy source of squeezed coherent states $E^\alpha(E^\gamma(\alpha,\xi))$ and lossy homodyne measurement $E^\gamma(\Pi^\psi)$. With the same notation as in Definition\cite{11} we assume that the (binary) information is encoded into lossy phase shifters $\Phi^{\delta_i} \circ \epsilon^\beta$ with $i=0,1$, and that no prior information is provided, namely $p_0 = p_1 = 1/2$. Since homodyne measurement has infinitely many outcomes, we introduce a classical postprocessing $\mathcal{J}$ (classical wires being denoted with $\mathcal{J}$) outputting binary outcome $j$, so that $j$ is our guess for the value $i$ of the bit encoded into the unknown phase shifter. Then the strategy in Eq.\cite{12} becomes

$$|\alpha,\xi\rangle \mathcal{E}^\alpha \mathcal{E}^\gamma \Phi^\delta \mathcal{E}^\gamma \Pi^\psi \mathcal{J},$$

Clearly, for fixed probe Gaussian quantum reading reduces to Gaussian state discrimination\cite{30,41}.

Due to Lemmas\cite{11} and\cite{22} lossy channels can be absorbed in the definition of lossy POVM, namely the strategy in Eq.\cite{14} can be rewritten as

$$|\alpha,\xi\rangle \Phi^\delta \mathcal{E}^\gamma \Pi^\psi \mathcal{J},$$

with $\eta = \alpha \beta \gamma$. Notice that it is not restrictive to assume $\delta_0 = 0$, since a phase shifter $\Phi^{\delta_0} = I$ and $\Phi^{\delta_1} = \Phi^\delta$, where $\delta = -\delta_0 \delta_1$, with $\delta \in [0,\pi]$. Notice that the application of phase shifter $\Phi^\delta$ to a squeezed coherent state $|\alpha,\xi\rangle$ gives $|\alpha, e^{i2\delta} \xi\rangle$ the squeezed coherent state $|e^{i\phi}a, e^{i2\delta}r\rangle$. Any classical postprocessing of outcome $x$ can be described by a function $q(j|x)$ that evaluates to 1 if one guesses $j$ from outcome $x$ and to 0 otherwise, so the probability of error is given by

$$P_e = \frac{1}{2} \int dx q(1|x)p(x|\alpha,\xi,\psi) + q(0|x)p(x|e^{i\delta}a, e^{i2\delta}\xi,\psi).$$

In the following, we denote with $\text{erf}(x) := 2/\sqrt{\pi} \int_{-x}^{x} dt \ e^{-t^2}$ the error function and with $\Omega(x)$ the unit step function that evaluates to 1 if $x \geq 0$ and to 0 otherwise.

$$P_e = \frac{1}{2} \int dx \text{erf}(x) p(x|\alpha,\xi,\psi) + \text{erf}(x)p(x|e^{i\delta}a, e^{i2\delta}\xi,\psi).$$

(16)
The Standard Quantum Limit for quantum reading with PSK encoding in the ideal case was derived in Ref. [3]. Here we generalize that result to the lossy case.

**Proposition 1 (Standard Quantum Limit).** For any energy threshold $E$, any efficiency $\eta$ and any phase $\delta$, the optimal coherent state $|\alpha^*\rangle$ with $\alpha^* = e^{i\theta^*} \alpha^*$ and the optimal homodyne measurement $\Pi^{\psi^*}$ with efficiency $\eta$ for Gaussian quantum reading of phase shifters $\{I, P_3\}$ with energy $E(\alpha) \leq E$ are given by $\phi^* = -\delta/2$, $\alpha^* = \sqrt{E}$, and $\psi^* = \pi/2$. The optimal tradeoff is given by $P_e = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{x_0(\alpha^*, \psi^*)}{\sigma(\xi, \psi^*)} \right) \right]$.

**Proof.** The minimum of $P_e$ in Eq. (16) is attained when
\[
q(y|x) = \Omega\left((-)^y \left(p(x|\alpha, 0, \psi) - p(x|e^{i\delta} \alpha, 0, \psi)\right)\right)
\]
and thus $P_e$ is given by the overlap between the Gaussians $p(x|\alpha, 0, \psi)$ and $p(x|e^{i\delta} \alpha, 0, \psi)$, namely
\[
P_e = \frac{1}{2} \left[ 1 + \int_A dx \left( p(x|e^{i\delta} \alpha, 0, \psi) - p(x|\alpha, 0, \psi) \right) \right],
\]
where $A := \{x | p(x|\alpha, 0, \psi) \geq p(x|e^{i\delta} \alpha, 0, \psi)\}$.

From Eq. (15) it follows that $P_e$ depends on the phases $\phi$ and $\psi$ only through the sum $\psi - \phi$, so without loss of generality we fix $\psi = \psi^*$, namely homodyne measurement is performed along quadrature $P$.

Notice that for any coherent state $|e^{i\theta} \alpha\rangle$ such that $E(e^{i\theta} \alpha) \leq E$, one has that the state $|e^{i\theta} \alpha^*\rangle$ is such that $E(e^{i\theta} \alpha^*) = E$ and $P_e(e^{i\theta} \alpha^*) \leq P_e(e^{i\theta} \alpha)$. Indeed, while $\sigma(0, \psi^*)$ does not depend on $\alpha$, one has that $|x_0(e^{i(\delta+\phi)} \alpha^*, \psi^*) - x_0(e^{i\theta} \alpha^*, \psi^*)| \geq |x_0(e^{i(\delta+\phi)} \alpha, \psi^*) - x_0(e^{i\theta} \alpha, \psi^*)|$. Then the optimal value for parameter $\alpha$ is $\alpha^* \rangle$.

Notice that for any coherent state $|e^{i\theta} \alpha^*\rangle$ such that $E(e^{i\theta} \alpha^*) = E$, one has that the state $|e^{i\theta} \alpha^*\rangle$ is such that $E(e^{i\theta} \alpha^*) = E$ and $P_e(e^{i\theta} \alpha^*) \leq P(e^{i\theta} \alpha)$. Indeed, while $\sigma(0, \psi^*)$ does not depend on $\alpha$, one has that $|x_0(e^{i(\delta+\phi)} \alpha^*, \psi^*) - x_0(e^{i\theta} \alpha^*, \psi^*)| \geq |x_0(e^{i(\delta+\phi)} \alpha, \psi^*) - x_0(e^{i\theta} \alpha, \psi^*)|$. Then the optimal value for parameter $\phi$ is $\phi^*$.

When $\theta = \theta^*$, an explicit evaluation of Eq. (17) leads to $P_e = \frac{1}{2} \left[ 1 + \text{erf}(\frac{x_0(e^{i\theta^*} \alpha, \psi^*)}{\sigma(e^{i\theta^*} \alpha, \psi^*)}) \right]$, and the optimal value for parameter $r$ can be obtained minimizing $P_e$. Since $erf(x)$ is a monotone increasing function in $x$, minimizing $P_e$ is equivalent to minimizing $\frac{\partial x_0(e^{i\theta^*} \alpha, \psi^*)}{\sigma(e^{i\theta^*} \alpha, \psi^*)}$. It is lengthy but not difficult to verify that the equation $\frac{\partial x_0(e^{i\theta^*} \alpha, \psi^*)}{\sigma(e^{i\theta^*} \alpha, \psi^*)} = 0$ admits the only solution $r = r^*$ and that $r^*$ is a minimum, so the statement remains proved.

Notice that for $\alpha = \alpha^*$ and $\psi = \psi^*$, for any $r$ the only choices of $\theta$ such that $\sigma(\xi, \psi^*) = \sigma(e^{i\theta} \xi, \psi^*)$ are $\theta = -\delta$ and $\theta = -\delta - \pi$, and the choice $\theta = \theta^*$ given by Prop. 2 corresponds to the one minimizing $\sigma(\xi, \psi^*)$ (see next Section and Fig. 1). We obtained numerical evidence that the choice $\theta = \theta^*$ is optimal whenever (i) $\delta \geq \pi/2$ for any energy threshold $E$ and any efficiency $\eta$, or (ii) when $\delta \leq \pi/2$ for sufficiently high $E$. However, when $\delta \leq \pi/2$ for sufficiently low $E$ the choice $\theta = \theta^*$ is not optimal anymore, and the second statement in Prop. 2 cannot be applied.
When $\delta = \pi$ one has $\theta^* = -\pi$, and thus the expression for $r^*$ in Prop. 2 is not defined for $\eta = 1$. In this case the limit $\eta \to 1$ must be considered, leading to $r^* = \arcsinh[E/\sqrt{2(E + 1)}]$. Notice that this particular expression for $r^*$ when $\delta = \pi$ and $\eta = 1$ is optimal also for hybrid quantum reading with Gaussian probe and arbitrary measurement (see Ref. [9]).

IV. OPTIMAL GAUSSIAN STRATEGY VERSUS STANDARD QUANTUM LIMIT

This Section is devoted to the comparison between the Standard Quantum Limit and the optimal Gaussian strategy for quantum reading derived in Sect. III.

Figure 1 provides a phase-space representation of the Wigner function of the states attaining the Standard Quantum Limit (light gray circles) and the optimal Gaussian strategy (bold-line ellipses) as given by Props. 1 and 2 respectively. The Figure provides an intuitive understanding of the advantage given by squeezed coherent states over coherent states. On the one hand, for fixed energy the more two states are squeezed, the more their Wigner functions get “closer” and thus hardly distinguishable. On the other hand, when squeezing is performed approximately along the quadrature being measured, the Wigner functions become “thinner” as squeezing increases. These two phenomena are clearly contrasting, but when the optimal tradeoff is taken, a dramatic improvement in the precision of the discrimination is experienced, as discussed in the next paragraphs. The only value of the phase $\delta$ for which squeezed coherent states do not provide an advantage over coherent states is $\delta = \pi/2$.

The maximum advantage is achieved in the two regimes $\delta \sim \pi$ and $\delta \sim 0$, since in both cases the Wigner function of the optimal state $\rho^*$ is squeezed approximately along the quadrature which is measured - this is rigorously true only for $\delta = \pi$.

The advantage of using the $\delta = \pi$ encoding is obvious - it is clearly the choice giving the lower tradeoff between energy and probability of error (see later discussion and Fig. 3). Different choices, and in particular the regime $\delta \sim 0$, can be exploited in several applications for tuning the minimum energy required by a Gaussian reader to retrieve some information. For example, a read-only-once memory could be implemented by triggering a device to self erase after being exposed to a radiation of energy $E$, where $E$ is the minimum energy required to read the memory with a given probability of error $P_e$, as given by Prop. 2. Notice that the quantum reading problem in the regime $\delta \sim 0$ shares analogies with the problem of channel estimation for low-noise parameter [44, 45].

In Fig. 2 the optimal squeezing parameter $r^*$ given by Prop. 2 is plotted as a function of $E$ for different values of $\delta$ and $\eta$. Remarkably, even for very low values of the probability of error $P_e$, the corresponding $r^*$ is comparable with experimentally attainable values of the squeezing parameter, in particular in the regime $\delta \sim \pi$. For example, when $\delta = \pi$, setting the energy $E = 4$ and the efficiency of the homodyne measurement $\eta = 0.9$ (an arguably fairly conservative assumption) one obtains a probability of error as low as $P_e \sim 6.5 \cdot 10^{-9}$ with optimal squeezing parameter given by $r^* \sim 1.0$. In Refs. [46, 47], the generation of squeezed coherent states with squeezing parameter up to $r_{dB} \sim 12.7 dB$ is reported - namely such that $r = r_{dB}/(20 \log_{10} e) \sim 1.5$.

The optimal tradeoff between energy and probability of error in Gaussian quantum reading with PSK encoding as given by Prop. 2 is plotted in Fig. 3 for different values of $\delta$ and $\eta$. As the Figure clearly shows, the optimal Gaussian strategy for quantum reading largely outperforms the Standard Quantum Limit, allowing to dramatically reduce the probability of error $P_e$ for fixed energy $E$, even in the presence of a realistic amount of loss and with realistic limitations in the squeezing parameter. For example, when $\delta = \pi$, setting $E = 4$ and $\eta = 0.9$ the optimal Gaussian strategy leads to a probability of error as low as $P_e \sim 6.5 \cdot 10^{-9}$, while the Standard Quantum Limit gives $P_e \sim 2.6 \cdot 10^{-3}$.

V. CONCLUSION

In this work we addressed the problem of quantum reading with Gaussian states and homodyne measurements. We showed that when no information is encoded in the amplitude of the signal, quantum reading in the presence of loss can be recasted to the discrimination
of unitary devices with low energy and high accuracy. We provided the optimal Gaussian strategy for quantum reading with PSK encoding for any value of the phase difference between the two signals encoding logical 0 and 1 and we showed that it dramatically outperforms the Standard Quantum Limit even in the presence of loss and under realistic assumptions on the practically feasible squeezing parameters. The optimal Gaussian strategy, consisting in probing a phase shifter with a properly tuned squeezed coherent state and performing an homodyne measurement on the output state, is suitable for experimental implementation with current quantum optical technology and represents a proof of principle for an highly efficient and reliable quantum-enhanced optical reader.

A natural generalization of the problem is to allow for different detectors - e.g. not only homodyne, but also heterodyne or double homodyne - and to investigate whether they can further improve the performance of Gaussian quantum reading.

Moreover, in Ref. [6] it was proven that the Standard Quantum Limit for quantum reading is attainable without an ancillary space - basically, because linear optical devices can not create entanglement when their inputs are coherent states. Then, for the purpose of this work - namely, proving that Gaussian quantum reading can out-
perform the Standard Quantum Limit - it was sufficient to consider the setup given by Eq. (15).

Nevertheless, since linear optical devices can create entanglement when their inputs are squeezed coherent states \[11\], a further generalization of the problem is that of entangled assisted Gaussian quantum reading, where the setup in Eq. (13) is replaced by

and the optimization is performed over squeezed coherent states \(|\alpha, \xi_i\rangle\), entangling devices \(\mathcal{V}\) and \(\mathcal{W}\), homodyne measurements \(\Pi_i\), and classical postprocessing \(\mathcal{J}\). To understand whether entangled assisted Gaussian quantum reading can further improve the performance of Gaussian quantum reading, one could make use of the numerical techniques developed in Ref. \[6\].

ACKNOWLEDGMENTS

The author is grateful to Alessandro Bisio, Francesco Buscemi, Giacomo Mauro D’Ariano, Lorenzo Maccone, Masanao Ozawa, and Massimiliano F. Sacchi for very useful discussions, comments, and suggestions. This work was supported by JSPS (Japan Society for the Promotion of Science) Grant-in-Aid for JSPS Fellows No. 24-0219.

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