GRAPH REPRESENTATIONS OF SURFACE FLOWS

TOMOO YOKOYAMA

Abstract. We construct a complete invariant for non-wandering surface flows with finitely many singular points but without locally dense orbits. Precisely, we show that a flow $v$ with finitely many singular points on a compact connected surface $S$ is a non-wandering flow without locally dense orbits if and only if $S/v_{\text{ex}}$ is a non-trivial embedded multi-graph, where the extended orbit space $S/v_{\text{ex}}$ is the quotient space defined by $x \sim y$ if they belong to either the same orbit or a same multi-saddle connection. Moreover, collapsing edges of the non-trivial embedded multi-graph $S/v_{\text{ex}}$ into singletons, the quotient space $(S/v_{\text{ex}})/\sim_E$ is an abstract multi-graph with the Alexandroff topology with respect to the specialization order. Therefore the non-wandering flow $v$ with finitely many singular points but without locally dense orbits can be reconstructed by finite combinatorial structures, which are the multi-saddle connection diagram and the abstract multi-graph $(S/v_{\text{ex}})/\sim_E$ with labels. Moreover, though the set of topological equivalent classes of irrational rotations (i.e. minimal flows) on a torus is uncountable, the set of topological equivalent classes of non-wandering flows with finitely many singular points but without locally dense orbits on compact surfaces is enumerable by combinatorial structures algorithmically.

1. Introduction

The main purpose of this paper is to study the relationship between surface flows and topologies using finite graphs. A basic result of Morse theory says that gradient flows of Morse functions on closed surfaces are characterized by the set of separatrices of saddles, which are finite directed graphs. The Morse theory for gradient vector fields on compact manifolds is extended to an index theory for Smale flows on compact manifolds using Lyapunov graphs [5], which is a generalization of a quotient space of gradient functions. In [14], a characterization of Lyapunov graphs associated to smooth surface flows is presented. It’s known that Lyapunov graphs are not complete invariant for Morse-Smale flows (i.e. there are Morse-Smale flows with isomorphic Lyapunov graphs but which are not topologically equivalent) but that Peixoto graphs are complete invariant for Morse-Smale flows [9]. In [11], non-wandering flows with finitely many singular points on compact surfaces are classified up to a graph-equivalence by using a topological invariant, called a Conley-Lyapunov-Peixoto graph, equipped with the rotation and the weight functions. The graph-equivalence conjugates two non-wandering flows at the multi-saddle connection diagrams forgetting the equivalence between the quasi-minimal
sets. Moreover, Hamiltonian flows are topologically equivalent if and only if their Conley-Lyapunov-Peixoto graphs are isomorphic. On the other hand, the quotient maps of orbit spaces of generalized gradient vector fields for Morse functions are weak homotopy equivalent and have the path lifting property \cite{3}. In this paper, we study properties of surface flows and construct another complete invariant, which are finite labeled multi-graphs, for a non-wandering surface flow with finitely many singular points but without locally dense orbits. In other words, such a flow can be reconstructed by finite combinatorial structures. Precisely, we show the following statements. Let \( v \) be a flow with finitely many singular points on a compact connected surface \( S \). Then \( v \) is a non-wandering flow with LD = \( \emptyset \) if and only if the extended orbit space \( S/vex \) is a non-trivial embedded multi-graph, where LD is the union of locally dense orbits and \( S/vex \) is the quotient space defined by \( x \sim y \) if they belong to either a same orbit or a same multi-saddle connection. Moreover, collapsing edges of the multi-graph \( S/vex \) into singletons, the quotient space \( (S/vex)/\sim_E \) is an abstract multi-graph with the Alexandroff topology with respect to the specialization order. In addition, considering the multi-saddle connection diagram \( D \) of a non-wandering flow \( v \) with \( |Sing(v)| < \infty \) and LD = \( \emptyset \) on a compact surface, the flow \( v \) can be reconstructed by the abstract multi-saddle connection diagram \( D/v \), which is an abstract multi-graph, and by the abstract multi-graph \( (S/vex)/\sim_E \) with labels, both of which are finite combinatorial structures.

Since the set of topological equivalent classes of irrational rotations (i.e. minimal flows) on a torus is uncountable (cf. Theorem 7.1.5 \cite{10}), so is the set of topological equivalent classes of non-wandering flows with finitely many singular points on compact surfaces. On the other hand, the set is countable under the non-existence of locally dense orbits. In other words, the set of topological equivalent classes of non-wandering flows with finitely many singular points but without locally dense orbits on compact surfaces is enumerable by combinatorial structures algorithmically.

2. Preliminaries

2.1. Notions of dynamical systems. We recall some basic notions. A good reference for most of what we describe are the books by S. Aranson, G. Belitsky, and E. Zhuzhoma \cite{1}. By flows, we mean continuous \( \mathbb{R} \)-actions on surfaces. Let \( v \) be a flow on a compact surface \( S \). A subset of \( S \) is said to be saturated if it is a union of orbits. The saturation \( Sat_v(A) \) of a subset \( A \subseteq S \) is the union of orbits of elements of \( A \). Recall that a point \( x \) of \( S \) is singular if \( x = v_t(x) \) for any \( t \in \mathbb{R} \), is regular if \( x \) is not singular, and is periodic if there is positive number \( T > 0 \) such that \( x = v_T(x) \) and \( x \neq v_t(x) \) for any \( t \in (0, T) \). Denote by Sing(\( v \)) (resp. Per(\( v \)) the set of singular (resp. periodic) points. A point \( x \) is non-wandering if for each neighbourhood \( U \) of \( x \) and each positive number \( N \), there is \( t \in \mathbb{R} \) with \( |t| > N \) such that \( v_t(U) \cap U \neq \emptyset \). An orbit is non-wandering if it consists of non-wandering points and the flow \( v \) is non-wandering if every point is non-wandering. For a point \( x \in S \), define the omega limit set \( \omega(x) \) and the alpha limit set \( \alpha(x) \) of \( x \) as follows: \( \omega(x) := \bigcap_{t \in \mathbb{R}} \{v_t(x) \mid t > n\} \), \( \alpha(x) := \bigcap_{n \in \mathbb{R}} \{v_t(x) \mid t < n\} \). A point \( x \) of \( S \) is recurrent (resp. weakly recurrent) if \( x \in \omega(x) \cap \alpha(x) \) (resp. \( x \in \omega(x) \cup \alpha(x) \)). A quasi-minimal set is an orbit closure of a weakly recurrent orbit. An orbit is proper if it is embedded (i.e. there is a neighborhood of it where the orbit is closed), locally dense if the closure of it has nonempty interior, and exceptional if it is neither
where each of whose orbit closure corresponds with \( S/v \) space and denoted by \( \partial S \) (interval). Note that each periodic orbit of a non-wandering flow in \( \text{Sing}(v) \) is the union of closed orbits. In other words, we define \( \text{Cl}(v) := \text{Sing}(v) \sqcup \text{Per}(v) \), where \( \sqcup \) is the disjoint union symbol. By the definitions, we have a decomposition \( \text{Sing}(v) \sqcup \text{Per}(v) \sqcup \text{P} \sqcup \text{LD} \sqcup \text{E} \sqcup \text{P} = S \). Note that \( \text{P} \) is the complement of the set of weakly recurrent points. Recall that the (orbit) class \( \hat{O} \) of an orbit \( O \) is the union of orbits each of whose orbit closure corresponds with \( \bigcup \) (i.e. \( \hat{O} = \{ y \in S \mid \text{Cl}(y) = \bigcup \} \)). The quotient space by (orbit) classes is called the (orbit) class space and denoted by \( S/\hat{v} \). A separatrix is a regular orbit whose \( \alpha \)-limit or \( \omega \)-limit set is a singular point. A \( \partial \cdot k \)-saddle (resp. \( \cdot k \)-saddle) is an isolated singular point on (resp. outside of) \( \partial S \) with exactly \((2k + 2)\)-separatrices, counted with multiplicity, where \( \partial S \) is the boundary of a surface \( S \). For a \((\partial \cdot k)\)-saddle \( x \), define the degree \( \deg x := 2k + 2 \). A 1-saddle is topologically an ordinary saddle. A multi-saddle is a \( k \)-saddle or a \( \partial \cdot (k/2) \)-saddle for some \( k \in \mathbb{Z}_{\geq 0} \). The (multi-)saddle connection diagram is the union of (multi-)saddles, (multi-)\( \partial \)-saddles, and separatrices connecting (multi-)\( \partial \)-saddles. A (multi-)saddle connection is a connected component of the (multi-)saddle connection diagram. A multi-saddle connection is also called a polycycle. By an extended orbit (resp. class) of a flow, we mean a multi-saddle connection or an orbit (resp. class) which is not contained in any multi-saddle connection. Note that an extended orbit is an analogous notion of “ demi-caract´eristique” in the sense of Poincar´e. In other words, “an extended positive orbit” is “un demi-caract´eristique” in the sense of Poincar´e [12]. An extended orbit is non-degenerate if it is not a singleton. Denote by \( O_{\text{ex}}(x) \) the extended orbit containing \( x \) and by \( O_{\text{ex}}(x) \) the extended class containing \( x \). The quotient space by extended orbits (resp. classes) of a flow \( v \) on a surface \( S \) is called the extended orbit (resp. class) space and denoted by \( S/v_{\text{ex}} \) (resp. \( S/\hat{v}_{\text{ex}} \)). Recall that an isolated singular point is a topological center if there is a saturated neighborhood consisting of closed orbits. An singular point \( p \) is quasi-isolated if there is a saturated neighborhood \( U \) such that \( \text{Sing}(v) \cap U \) is totally disconnected. A quasi-isolated singular point \( p \) is called a quasi-center if there is a saturated neighborhood \( U \) which consists of closed extended orbits. A flow is non-trivial if it is neither identical nor minimal. Note that the orbit spaces \( S/v = S/v_{\text{ex}} \) for each trivial flow \( v \) on a surface \( S \) are either the whole surface \( S \) or an indiscrete topological space and other quotient spaces \( S/\tilde{v} = S/\hat{v}_{\text{ex}} \) are either the whole surface \( S \) or a singleton and so that all the quotient spaces are non-1-dimensional. Moreover, notice that two non-wandering minimal flows need not be topological equivalent even if their (extended) orbit spaces (resp. (extended) orbit class spaces) are homeomorphic. In fact, consider minimal toral flows which correspond to cohomology classes \([dx_1 + \sqrt{2}dx_2], [dx_1 + \pi dx_2] \in H^1(\mathbb{T}^2, \mathbb{Z})\) respectively. Though the orbit spaces are indiscrete spaces with cardinalities of the continuum and the (extended) orbit class spaces are singletons, the minimal flows are not topological equivalent because \( \pi \) is transcendental and \( \sqrt{2} \) is algebraic.

A periodic orbit \( O \) is called a periodic orbit with a one-sided neighborhood if there is an open saturated neighborhood \( U \) of \( O \) and the canonical projection \( \pi : U \to U/v \cong [0, 1) \) with \( \pi(O) = 0 \) (i.e. \( \pi(O) \) is the boundary of a half-open interval). Note that each periodic orbit of a non-wandering flow in \( \partial S \) is a periodic
orbit with a one-sided neighborhood. We call that a torus (resp. annulus) \( U \subseteq S \)

is a periodic torus (resp. periodic annulus) if it consists of periodic orbits.

2.2. Topological notions. Recall that a poset is a set with a partial order (i.e. reflexive, antisymmetric, and transitive order) and a \( k \)-chain is a totally ordered set with \( k + 1 \) elements. For an element \( x \) of a poset \( P \), the height of \( x \) is at least \( k \in \mathbb{Z} \) if there is a \( k \)-chain whose maximal element is \( x \). The height of \( P \) is the superior of heights of elements. For any \( k \in \mathbb{Z}_{\geq 0} \), denote by \( P_k \) the set of elements each of whose height is \( k \). Elements \( a \) and \( b \) of a poset are comparable if either \( a \leq b \) or \( a \geq b \). A poset \( P \) is said to be connected if for each pair \( a, b \in P \), there is a finite sequence \( (a_i)_{i=0}^n \) of \( P \) from \( a = a_0 \) to \( b = a_n \) such that \( a_i \) and \( a_{i+1} \) are comparable for each \( i = 0, 1, 2, \ldots, n - 1 \). A subset \( A \subseteq P \) is a downset (resp. upset) if \( A = \bigcup_{x \in A} \downarrow x \) (resp. \( A = \bigcup_{x \in A} \uparrow x \) ), where \( \downarrow x := \{ y \in P \mid y \leq x \} \) and \( \uparrow x := \{ y \in P \mid x \leq y \} \). The topology on a poset \( (P, \leq) \) which consists of all upsets is called the Alexandroff topology of \( P \) and denoted by \( \mathcal{A}(\leq) \). Note that each downset is closed with respect to the Alexandroff topology \( \mathcal{A}(\leq) \). A poset \( P \) is said to be multi-graph-like if the height of \( P \) is at most one and \( | \downarrow x | \leq 3 \) for any element \( x \in P \). For a multi-graph-like poset \( P \), each element of \( P_0 \) is called a vertex and each element of \( P_1 \) is called an edge.

An ordered triple \( G := (V, E, r) \) is an abstract multi-graph if \( V \) and \( E \) are sets and \( r : E \to \{ \{x, y\} \mid x, y \in V \} \). Each element of \( V \) (resp. \( E \)) is called a vertex (resp. an edge). Then an abstract multi-graph \( G \) can be considered as a multi-graph-like poset \( (P, \leq_G) \) with \( V = P_0 \) and \( E = P_1 \) as follows: \( P = V \sqcup E \) and \( x \leq_G e \) if \( x \in r(e) \). Conversely, a multi-graph-like poset \( P \) can be considered as an abstract multi-graph with \( V = P_0 \), \( E = P_1 \), and \( r : P_1 \to \{ \{x, y\} \mid x, y \in V \} \) defined by \( r(e) := \downarrow e - \{ e \} \). A multi-graph-like poset is said to be tree-like if it is a tree as an abstract multi-graph. A multi-graph \( G = (V, E) \) is a cell complex whose dimension is at most one and which is a geometric realization of an abstract multi-graph (i.e. a graph which is permitted to have multiple edges and loops). By abuse of terminology, we also refer to \( G \) as the underlying set \( V \sqcup \bigsqcup E \). Note that a multi-graph \( G = (V, E) \) is \( T_2 \) and the multi-edge set \( E \) consists of open intervals with \( \bigsqcup E = G - V \). A multi-graph \( G = (V, E) \) in a surface \( S \) is embedded if for any \( e \in E \) there is an open neighborhood of \( e \) in \( S \) which intersect no other edges and no vertices, and for any \( v \in V \) there is an open neighborhood of \( v \) which intersects no vertices except \( v \) and no edges not incident to \( v \). A planar multi-graph is a multi-graph that can be embedded in a punctured disk. In other words, it can be drawn in such a way that no edges cross each other. Such a drawing is called a plane graph. For a multi-graph \( G \), define an equivalence relation \( \sim_e \) as follows: for any points \( x, y \in G \), \( x \sim_e y \) if they belong to a same edge. Note that the quotient space \( G/\sim_e \) of a multi-graph \( G \) is obtained by collapsing edges into singletons and that a finite poset of height at most one corresponds with a finite abstract multi-hyper-graph.

A point \( x \) of a topological space \( X \) is \( T_0 \) if for any point \( y \neq x \in X \), there is an open subset \( U \) of \( X \) such that \( \{x, y\} \cap U = 1 \). The specialization order \( \leq \) on a \( T_0 \)-space \( (X, \tau) \) is defined as follows: \( x \leq y \) if \( x \in \overline{\{y\}} \). Note that each closed subset of a \( T_0 \)-space \( (X, \tau) \) is a downset with respect to the specialization order \( \leq \).
2.3. Graph representations of surface flows. For a non-wandering flow \( v \) on a compact surface \( S \), denote by \( V_{\text{ex}} \) the set of elements each of which is either a quasi-center, a multi-saddle connection, or a periodic orbit with a one-sided neighborhood. Notice that a quasi-center is a topological center if \( \text{Sing}(v) \) is finite. Let \( D \) be the set of connected components of the quotient space \( D/v \) of the multi-saddle connection diagram \( D \). Define a label \( l_{\text{ex}} : V_{\text{ex}} \to \{ c, n, b \} \cup D \) as follows: \( l_{\text{ex}}(O) := c \) (resp. \( n, b \)) if \( O \) is a quasi-center (resp. a periodic orbit with one-sided neighborhood off \( \partial S \), a periodic orbit on \( \partial S ) \) and \( l_{\text{ex}}(O) := O/v \subseteq D/v \) if \( O \) is a multi-saddle connection. Put \( E_{\text{ex}} = \{ \text{ex} \} \) the set of connected components of Per\((v) - N \), where \( N \) is the union of periodic orbits with one-sided neighborhoods. Write \( E_{\text{ex}} := E_{\text{ex}}/v \).

Note that each element of \( E_{\text{ex}} \) is an open annulus unless it is a periodic torus. We define the edge mapping \( r_{\text{ex}} \) and the label mapping \( l_{\text{ex}} \) as follows: Define \( r_{\text{ex}} : E_{\text{ex}} \to \{ (O_{\text{ex}}, O'_{\text{ex}}) \ | \ O_{\text{ex}}, O'_{\text{ex}} \in V_{\text{ex}} \} \cup \{ \emptyset \} \) by \( r_{\text{ex}}([T]) := \emptyset \) for a periodic torus \( T \) and by \( r_{\text{ex}}([U]) := \{ O_{\text{ex}} \in V_{\text{ex}} \ | \ \partial_s U \subseteq O_{\text{ex}} \text{ or } \partial_s U \subseteq O_{\text{ex}} \} \) for a periodic annulus \( U \), where \( \partial_s U \) (resp. \( \partial_s U \)) are the negative (resp. positive) connected components of \( \partial U \) with respect to the flow direction such that \( \partial U = \partial_+ U \cup \partial_- U \). An Edge \([U] \in E_{\text{ex}} \) is called a loop if \( |r_{\text{ex}}([U])| = 1 \). Define also the label \( l_{\text{ex}} : E_{\text{ex}} \to \{ (\partial_- U/v, \partial_+ U/v) \ | \ U \in E_{\text{ex}} \} \cup \{ \emptyset \} \) of \( v \) by \( l_{\text{ex}}([T]) := \emptyset \) for a periodic torus \( T \) and by \( l_{\text{ex}}([U]) := (\partial_- U/v, \partial_+ U/v) \) for a periodic annulus \( U \). Define also the reduced label \( l_{\text{ex}} : E_{\text{ex}} \to \{ (\partial_- U/v, \partial_+ U/v) \ | \ U \in E_{\text{ex}} \} \cup \{ \emptyset \} \) of \( v \) forgetting orders of pairs. We will show that, for an element \( U \in E_{\text{ex}} \) of a non-wandering flow \( v \) on a compact surface \( S \) with \( LD = \emptyset \) and \( \text{Sing}(v) < \infty \), each boundary component (i.e. connected component of the boundary) \( \partial_\sigma U \) (for some \( \sigma \in \{-, +\} \)) is either an element of \( V_{\text{ex}} \) or a proper subset of a multi-saddle connection, and the quotient space \( \partial_\sigma U/v \) is a finite multi-graph-like poset. Define an equivalent relation \( \sim_E \) on \( S/v_{\text{ex}} \) as follows: \( O_{\text{ex}} \sim_E O'_{\text{ex}} \) if they are contained in a connected component of \( LD \cup (\text{Per}(v) - N) \), where \( N \) is the union of periodic orbits with one-sided neighborhoods. Denote by \( G_v := ((S/v_{\text{ex}})/\sim_E, l_{\text{ex}}) \) the quotient space \( (S/v_{\text{ex}})/\sim_E \) with the label \( (l_{\text{ex}}) \).

Recall that a multi-graph is trivial if it has exactly one vertex and no edges. Note that the quotient space \( (S/v_{\text{ex}})/\sim_E \) for a non-wandering flow \( v \) on a compact connected surface \( S \) is trivial if and only if either \( v \) is minimal (i.e. \( S = T^2 = LD \)) or \( v \) is a rational rotation on \( T^2 \) (i.e. \( S = T^2 = \text{Per}(v) \)).

3. Properties of the multi-saddle connection diagram

We state the relation between multi-graphs and abstract multi-graphs.

Lemma 3.1. The quotient space of a multi-graph by \( \sim_E \) is a multi-graph-like poset with respect to the specialization order.

Proof. Let \( Q \) be the quotient space \( G/\sim_c \) of a multi-graph \( G \) equipped with the specialization order, \( p : G \to Q \) the canonical projection, and \( V \) the set of vertices. Then the complement \( Q - V \) consists of edges. Since each singleton of \( G \) is closed, the height of each element of \( V \) is 0. Since the closure of each edge contains exactly one or two vertices, the height of each edge is 1 and so \( Q \) is a multi-graph-like poset. \( \square \)

Applying the previous lemma to multi-saddle connections, we obtain a following statement.
Lemma 3.2. Let \( v \) be a flow on a compact surface \( S \) and \( D \) the multi-saddle connection diagram. Then the restriction \( D/v \) of the orbit space of \( v \) is a multi-graph-like poset with respect to the specialization order.

Proof. Notice that the multi-saddle connection diagram \( D \) is a multi-graph such that the equivalent class of an element \( x \in D \) by \( \sim_c \) is an orbit. Applying Lemma 3.1 the restriction \( D/v = D/\sim_c \) is desired. \( \square \)

We call \( D/v \) the abstract multi-saddle connection diagram of \( v \). Let \( V_D \) be the set of multi-saddles and \( E_D \) the set of separatrices. Define a label \( l_{Dc} : E_D \to V_D \) by \( l_{Dc}(O) := \alpha(O) \), where \( \alpha(O) \) is the alpha limit set of an orbit \( O \). Define a cyclic relation \( \sim_c \) on \( E_D^0 \) as follows: \( (O_1, O_2, \ldots, O_n) \sim_c (O_{i_1}, O_{i_2}, \ldots, O_{i_n}) \) if there is an integer \( k = 0, 1, \ldots n - 1 \) such that \( j - i_j \equiv k \) mod \( n \). Denote by \( E_D^* := \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} E_D^n/\sim_c \) Define a label \( l_{Dv} : V_D \to E_D^* \) as follows: by \( l_{Dv}(x) := \{O_1, \ldots, O_{\operatorname{deg} x}\} \) if \( \{O_1, \ldots, O_{\operatorname{deg} x}\} \) is the set of separatrices containing \( x \) and the separatrices \( O_1, \ldots, O_{\operatorname{deg} x} \) are arranged in the counterclockwise direction around \( x \). Denote by \( D_v := (D/v, l_{Dv}, l_{Dc}) \) the abstract multi-saddle connection diagram \( D/v \) with the label \( (l_{Dv}, l_{Dc}) \). Then the multi-saddle connection diagram \( D \) of \( v \) (as a plane graph) can be reconstructed by the finite data \( D_v \).

4. Separation axioms of orbit (class) spaces

We characterize the embedded property of multi-saddle connection diagrams as follows.

Lemma 4.1. Let \( v \) be a flow on a compact surface \( S \) and \( D \) the multi-saddle connection diagram. The quotient topology on \( D/v \) is the Alexandroff topology of the poset \( D/v \) if and only if \( D \) is embedded as a multi-graph.

Proof. Let \( Q := D/v \) be the poset with respect to the specialization order. Suppose that the quotient topology on \( Q \) is the Alexandroff topology of the poset \( Q \). Then each edge of \( Q \) is an open subset of \( Q \) and the upset \( w \uparrow \) for any vertex \( w \) of \( Q \) is an open subset in \( Q \) which consists of \( w \) and the edges incident to \( w \). This implies that \( D \) is embedded as a multi-graph. Conversely, suppose that \( D \) is embedded as a multi-graph. Assume the quotient topology on \( D/v \) is not the Alexandroff topology of the poset \( D/v \). Note that each closed subset of \( Q \) is a downset. Then there is a downset \( A \subseteq Q \) such that \( \overline{A} - A \neq \emptyset \). Since \( D \) is embedded, each edge is open and so \( \overline{A} - A \) consists of height 0 elements. Therefore there is a vertex \( w \in \overline{A} - A \). Since \( \overline{A} - A \neq \emptyset \), any open neighborhood of \( w \) intersects either vertices except \( w \) or edges not incident to \( w \), which contradicts to the definition of “embedded”. \( \square \)

We characterize properness using quotient spaces.

Lemma 4.2. Let \( v \) be a non-trivial flow on a compact surface \( S \) and \( p : S \to S/v \) the canonical projection. The set of \( T_0 \) points in \( S/v \) is \( p(\Pr) \).

Proof. Since a quasi-minimal set contains a continuum of nontrivially recurrent orbits each of which is dense in the quasi-minimal set (Theorem VI[2]), the restriction \( (E \cup LD)/v = (E \cup LD)/v_{ex} \) consists of non-\( T_0 \) points. Since \( \Pr = \operatorname{Cl}(v) \cup P = S - (E \cup LD) \), the set of \( T_0 \) points in \( S/v \) is contained in \( p(\Pr) \). Proposition 2.2[14] implies that \( O = \overline{O \setminus \operatorname{Sing}(v) \cup P} \subset LD \cup E \) for an orbit \( O \subset LD \cup E \). Therefore \( \hat{O} \subseteq \Pr \) for any orbit \( O \subseteq \Pr \). Since the union of non-closed orbits in \( \Pr \) is \( P \), we obtain \( \hat{O} \subseteq P \) for an orbit \( O \subseteq P \). Therefore it suffices to show that either
Note that there is a non-wandering flow $v$ on $\mathbb{T}^2$ with $P \neq \emptyset$ whose multi-saddle connection diagram is dense and so is not embedded such that $\text{Sing}(v)$ is countable (and so totally disconnected) and that $q(\Pr)$ is not the set of $T_0$ points in $S/v_{\text{ex}}$ (e.g. Example 2.10 [16]), where $q : S \to S/v_{\text{ex}}$ is the canonical projection. Conversely, we have the following characterization if the set of singular points is finite.

**Lemma 4.3.** Let $v$ be a non-trivial flow on a compact surface $S$ and $q : S \to S/v_{\text{ex}}$ the canonical projection. The set of $T_0$ points in $S/v_{\text{ex}}$ is $q(\Pr)$ if $|\text{Sing}(v)| < \infty$.

**Proof.** The finiteness of singular points implies that the multi-saddle connection diagram consists of finitely many multi-saddle connections each of which is closed. Then Lemma 2.2 implies the assertion. \hfill \Box

Recall that a flow is pointwise almost periodic if each orbit closure is minimal (i.e. $S/\hat{v}$ is $T_1$). By Lemma 2.2 [15], a flow $v$ on a compact surface is $R$-closed if and only if $S/\hat{v}$ is $T_2$. We characterize non-closed proper orbits using quotient spaces.

**Lemma 4.4.** Let $v$ be a non-trivial flow on a compact connected surface $S$ and $p : S \to S/v$ (resp. $q : S \to S/\hat{v}$) the canonical projection. The following are equivalent:

1) $P = \emptyset$.

2) The set of $T_1$ points in $S/v$ is $p(S - LD)$ (i.e. each orbit is closed or locally dense).

3) The set of $T_1$ points in $S/\hat{v}$ is $q(S - LD)$.

In any case, the flow $v$ is non-wandering such that $\text{Per}(v)/v$ is a 1-dimensional manifold and $S = \text{Cl}(v) \cup LD$.

**Proof.** Write $Y := S - LD$. Suppose that $P = \emptyset$. By Lemma 2.3 [10], there are no exceptional orbits and so $Y = S - LD = \text{Cl}(v) = \Pr$ is the union of closed orbits. This means that each orbit is closed or locally dense. Therefore the set of $T_1$ points in $S/v$ is $p(Y) = p(\Pr)$. Suppose that the set of $T_1$ points in $S/v$ is $p(Y)$. This means that each orbit in $Y$ is closed. By the non-triviality of $v$, the flow $v$ is not minimal and so the set of $T_1$ points in $S/\hat{v}$ is $q(Y)$. Suppose that the set of $T_1$ points in $S/\hat{v}$ is $q(Y)$. This means that each orbit closure is minimal or quasi-minimal and so that $P = \emptyset$. In any case, the flow $v$ is non-wandering. By Corollary 2.9 [10], each connected component of $\text{Per}(v)$ is either a connected component of $S$, an open annulus, or an open Möbius band. Thus each connected component of $\text{Per}(v)/v$ is either a circle or an interval and so $\text{Per}(v)/v$ is a 1-dimensional manifold. \hfill \Box

In the case without locally dense orbits, the following statement holds.

**Corollary 4.5.** Let $v$ be a non-trivial flow on a compact connected surface $S$. The following are equivalent:

1) $P \cup LD = \emptyset$.

2) $S/v$ is $T_1$ (i.e. each orbit is closed).
3) $S/\hat{v}$ is $T_1$ (i.e. $v$ is pointwise almost periodic). In any case, the flow $v$ is non-wandering with $S/v = S/\hat{v}$ and $(S - \text{Sing}(v))/v$ is a 1-dimensional manifold.

Note that there is a non-wandering flow $v$ with $P \neq \emptyset$ and $(S - \text{Sing}(v))/v \cong S^1$. Indeed, replacing an periodic orbit of an rational rotation on a torus $\mathbb{T}^2$ to a 0-saddle with a homoclinic separatrix, the resulting flow $v$ on $\mathbb{T}^2$ is desired. By dimension, we mean the small inductive dimension. By Urysohn’s theorem, the Lebesgue covering dimension, the small inductive dimension, and the large inductive dimension are corresponded in normal spaces. A compact metrizable space $X$ whose inductive dimension is $n > 0$ is an $n$-dimensional Cantor-manifold if the complement $X - L$ for any closed subset $L$ of $X$ whose inductive dimension is less than $n - 1$ is connected. It’s known that a compact connected manifold is a Cantor-manifold \[16\]. We characterize the Hausdorff property of the orbit (class) spaces. The orientable case of the following result has stated in \[14\] (see Theorem 6.6 in the paper).

**Lemma 4.6.** Let $v$ be a non-trivial flow on a compact connected surface $S$ whose orbit space $S/v$ is $T_1$. The following are equivalent:

1. $\text{Sing}(v)$ consists of at most two topological centers.
2. $S/v$ is $T_2$.
3. $S/\hat{v}$ is $T_2$ (i.e. $v$ is $R$-closed).

In any case, the Euler characteristic of $S$ is non-negative and the orbit space $S/v$ is either an closed interval or a circle.

**Proof.** Let $v$ be a non-trivial flow on a compact connected surface $S$ whose orbit space $S/v$ is $T_1$. This means that $S = \text{Cl}(v)$. Corollary \[1.5\] implies that the conditions 2) and 3) are equivalent. Since $v$ is non-trivial, there is a periodic orbit $O$. Let $C$ be the connected component of $\text{Per}(v)$ which contains $O$. As above, Corollary 2.9 \[10\] implies $C$ is either the whole surface $S$, an open annulus, or an open Möbius band. Then the restriction $C/v$ is an interval or a circle. Suppose that $\text{Sing}(v)$ consists of at most two topological centers. Then the dimension of $\text{Sing}(v)$ is zero and each connected component of the boundary $\partial C$ is a topological center, where $\partial C := \overline{C} - \text{int}C$. Since $S$ is a 2-dimensional Cantor-manifold, the complement $S - \text{Sing}(v)$ is connected and so $S = C \cup \text{Sing}(v)$. By the Poincaré-Hopf index formula, the Euler characteristic of $S$ is non-negative. Since the restriction $C/v$ is an interval or a circle, the orbit space $S/v$ is either an closed interval or a circle. Conversely, suppose that $S/v$ is $T_2$. Since the restriction $C/v$ is an interval or a circle, each connected component of the boundary $\partial C = \overline{C} - C$ is an closed orbit. Therefore the boundary $\partial C$ consists of at most two singular points. We show that each point in $\partial C$ is a center. Otherwise there is a non-isolated singular point $x \in \partial C$. Fix an open small neighborhood $U$ of $x$ whose closure is homeomorphic to a closed ball such that $\overline{U} \cap \partial C = \{x\}$. Since $x$ is not isolated, we obtain $(U \setminus \overline{C}) \cap \text{Sing}(v) \neq \emptyset$. Then $V := U \setminus \overline{C} = U \setminus (C \cup \{x\})$ is a nonempty open subset with $\overline{V} \cap \overline{C} = \{x\}$ and $\overline{U} \subseteq \overline{V} \cup \overline{C}$. Put $W_1 := \overline{U} \setminus \overline{C} = \overline{U} \cap (C \cup \{x\})$ and $W_2 := \overline{V} \subseteq \overline{U}$. By construction, we obtain $\text{int}W_1 \neq \emptyset$ and $\text{int}W_2 \neq \emptyset$. Since $U - \{x\} = (U \cap C) \cup V$ and $\overline{V} \cap \overline{C} = \{x\}$, we have $\overline{U} = W_1 \cup W_2$ and $W_1 \cap W_2 = \{x\}$. This implies that $\overline{U} - \{x\} = (W_1 - \{x\}) \cup (W_2 - \{x\})$ is disconnected, which contradicts that $\overline{U}$ is a two-dimensional Cantor manifold. Thus $\partial C$ consists of at most two singular points which are topological centers and so $\overline{C} = S$. This means that $C = \text{Per}(v)$ and so $\partial C = \partial \text{Per}(v) = \text{Sing}(v)$.

$\square$
The non-triviality of a flow implies that the extended orbit class space is not a singleton.

**Lemma 4.7.** Let \( v \) be a flow on a compact connected surface \( S \). Then \( S/\hat{v}_{\text{ex}} \) is a singleton if and only if \( v \) is minimal.

**Proof.** The minimality of \( v \) implies that \( S/\hat{v}_{\text{ex}} \) is a singleton. Conversely, suppose that \( S/\hat{v}_{\text{ex}} \) is a singleton. The openness of \( \text{Per}(v) \) implies \( \text{Per}(v) = \emptyset \). Since each extended orbit is dense, each singular point is a multi-saddle. Since multi-saddles are isolated, we have \( |\text{Sing}(v)| < \infty \). This means that each multi-saddle connection is closed and so the multi-saddle connection diagram is empty. Therefore \( \text{Pr} = \emptyset \) and so \( S = \text{LD} \). This implies \( T^2 = \text{LD} \) and so \( v \) is minimal. \( \square \)

**Lemma 4.8.** Let \( v \) be a non-trivial flow on a compact connected surface \( S \). If \( S/\hat{v}_{\text{ex}} \) is \( T_1 \), then \( \text{LD} = \emptyset \).

**Proof.** Assume that \( \text{LD} \neq \emptyset \). By Lemma 4.7, the extended orbit class space \( S/\hat{v}_{\text{ex}} \) is not a singleton. By Lemma 2.1 and 2.3 \[15\], we have \( \emptyset \neq \text{LD} = \text{LD} \subseteq \text{Sing}(v) \cup P \). By the finiteness of quasi-minimal sets, since \( S/\hat{v}_{\text{ex}} \) is \( T_1 \), the intersection \( \text{LD} \cap (\text{Sing}(v) \cup P) \) consists of dense extended orbits. In particular, each singular point in \( \text{LD} \) is a multi-saddle. Moreover, there are countable many multi-saddles in \( \text{LD} \) and so there is a non-isolated singular point in \( \text{LD} \) which is not a multi-saddle, which contradicts to the non-existence. \( \square \)

Second we consider quotient spaces with respect to the extended orbits to encode non-wandering surface flows. For a subset \( A \), write \( \delta A := A - \text{int} A \).

**Lemma 4.9.** Let \( v \) be a non-trivial flow on a compact connected surface \( S \). The following are equivalent:

1) \( \text{LD} = \emptyset \), the multi-saddle connection diagram contains \( P \), and each multi-saddle connection is closed.
2) \( S/\hat{v}_{\text{ex}} \) is \( T_1 \).
3) \( S/\hat{v}_{\text{ex}} \) is \( T_1 \).

In any case, the flow \( v \) is non-wandering with \( S/\hat{v}_{\text{ex}} = S/\hat{v}_{\text{ex}} \).

**Proof.** By the definition of an orbit class, the condition 2) implies the condition 3). Suppose that \( S/\hat{v}_{\text{ex}} \) is \( T_1 \). This means that each proper orbit is either closed or contained in a closed multi-saddle connection. Lemma 4.8 implies \( \text{LD} = \emptyset \). Suppose that \( \text{LD} = \emptyset \), the multi-saddle connection diagram contains \( P \), and each multi-saddle connection is closed. Then \( \text{int} P = \emptyset \). Lemma 2.3 \[16\] implies \( E = \emptyset \) and so \( S = \overline{\text{Cl}(v)} = \text{Cl}(v) \cup \delta P \). This implies that \( S/\hat{v}_{\text{ex}} \) is \( T_1 \) and that \( v \) is non-wandering. \( \square \)

The closed condition of multi-saddle connections in the previous lemma is necessary. Indeed, consider an rotation \( v_0 \) with respect to an axis on \( S^2 \). Write \( \{p_N, p_S\} := \text{Sing}(v_0) \). Fix a periodic orbit \( O \subset S^2 \). Consider a sequence \( (x_n)_{n \in \mathbb{Z}} \) of pairwise distinct points \( x_n \in O \) converging to a point \( x \) positively and negatively from different sides. Using a bump function and replacing the orbit \( O \) into a union of singular points and proper orbits, the resulting flow \( v \) is a non-wandering flow whose singular points are countable with \( O = P \cup (\text{Sing}(v) - \{p_N, p_S\}) \) such that \( S/\hat{v}_{\text{ex}} \) is not \( T_1 \). Indeed, since each singular point in \( O \) except \( x \) is a 0-saddle, the union \( O - \{x\} \) is an extended orbit of \( v \) whose closure is \( O \).
Lemma 4.10. Let $v$ be a non-trivial flow on a compact connected surface $S$ such that $S/v$ is $T_1$. The following are equivalent:

1) $\text{Sing}(v)$ is totally disconnected.
2) $S/v$ is $T_2$.
3) $S/v$ is $T_2$.
4) Either $v$ is a rational rotation on $T^2$ or $(S/v)/\sim$ is a non-trivial multi-graph $(V_{ex}, E_{ex}, r_{ex})$.

In any case, $S = \text{Per}(v)$ and the set $\text{Sing}(v)$ consists of multi-saddles and quasi-centers.

Proof. Let $v$ be a non-trivial flow on a compact connected surface $S$ whose extended orbit space $S/v$ is $T_1$. By Lemma 4.9 we have $P$ is contained in the multi-saddle connection diagram and $L \sqcup E = \emptyset$ (i.e. $S = \text{Cl}(v) \sqcup \#P$). Note the compact connected surface $S$ is a 2-dimensional Cantor-manifold. If $v$ is a rational rotation on $T^2$ (i.e. a periodic torus), then the assertion holds. Thus we may assume that $v$ is not a rational rotation on $T^2$. The non-triviality implies the existence of singular points. Lemma 4.9 implies that the conditions 2) and 3) are equivalent and that each quasi-isolated singular point is either a quasi-center or a multi-saddle. Trivially, the condition 4) implies 3). Let $M \subseteq \text{Sing}(v)$ be the set of multi-saddles and $D$ the multi-saddle connection diagram. Since each multi-saddle is isolated, the complement $\text{Sing}(v) - M = S - (\text{Per}(v) \sqcup D)$ is closed. Then $\partial(\text{Sing}(v) - M) = \partial(\text{Per}(v) \sqcup D) \subseteq \partial \text{Per}(v)$. Moreover, the set $M$ is countable. Indeed, by the definition of multi-saddles, each multi-saddle has a neighborhood which contains no other saddles. Since $S$ is second countable, the set of multi-saddles can be enumerated and so is countable. Since $C \subseteq \text{int}(\text{Per}(v) \sqcup C)$ for each multi-saddle connection $C$, we have $D \subseteq \text{int}(\text{Per}(v) \sqcup D)$ and so $\partial(\text{Per}(v) \sqcup D) \sqcap D = \emptyset$. Fix a singular point $x \in \partial(\text{Per}(v) \sqcup D)$. Let $(x_n)_{n \geq 0}$ be a convergence sequence in $\text{Per}(v)$ to $x$ and $X_x := \{\lim_{n \to \infty} y_n \mid (y_n)$ is a convergence sequence, $y_n \in \bigcup_{m \geq n} O(x_m)\}$. We claim that the limit $X_x$ is connected. Indeed, otherwise there are disjoint open subsets $U, V$ such that $X_x \subseteq U \sqcup V$, $U \cap X_x \neq \emptyset$, and $V \cap X_x \neq \emptyset$. Then $K := S - (U \sqcup V) \subseteq S - X_x$ is closed and so sequentially compact. Fix $y_n \in O(x_n) \setminus K$. The sequence $(y_n)$ has a convergent subsequence $(y_{n_k})$. Then $\lim_{k \to \infty} y_{n_k} \in K \sqcap X_x$, which contradict to the definitions of $K$ and $X_x$. Note $x \in X_x \cap \text{Sing}(v)$. We claim that $X_x \subseteq \partial(\text{Sing}(v) - M)$. Indeed, for any periodic orbit $O$, there is a closed saturated neighborhood $U_O$ of $O$ in $\text{Per}(v)$ and so $O(x_n) \cap U_O = \emptyset$ for any large integer $n$. This implies $X_x \cap \text{Per}(v) = \emptyset$ and so $X_x \subseteq \text{Per}(v) \sqcup \partial \text{Per}(v) - \text{Per}(v) = (\partial(\text{Per}(v) \sqcup D) \sqcap \text{int}(\text{Per}(v) \sqcup D)) - \text{Per}(v) \subseteq \partial(\text{Per}(v) \sqcup D) \sqcup D$. Since $C \subseteq \text{int}(\text{Per}(v) \sqcup C)$ for each multi-saddle connection $C$, there is a closed saturated neighborhood $U_C$ of $C$ with $U_C - C \subseteq \text{Per}(v)$ and so $O(x_n) \cap U_C = \emptyset$ for any large integer $n$. This implies $X_x \cap D = \emptyset$ and so $X_x \subseteq \partial(\text{Per}(v) \sqcup D) = \partial(\text{Sing}(v) - M)$. Suppose that $\text{Sing}(v)$ is totally disconnected. Then $X_x$ is a singleton and $S = \text{Per}(v)$. This implies that each connected component of $\partial \text{Per}(v)$ is contained in a closed extended orbit (i.e. either a multi-saddle connection or quasi-center). Assume that $S/v$ is not $T_2$. Then there are distinct singular points $y \neq z$ such that any saturated neighborhoods of them intersect. Therefore there is a sequence $(O_n)$ of periodic orbits such that $y, z \in \bigcup_{n \in \mathbb{N}} O_n$. This means that there are two convergence sequences $(y_n)$ and $(z_n)$ to $y$ and $z$ respectively such that $y_n, z_n \in O_n$. Therefore $y \neq z \in X_2$, which contradicts that $X_2$ is a singleton. Since the complement of the union of quasi-centers and of the multi-saddle connection diagram is $\text{Per}(v)$, the quotient space
$S/v_{\text{ex}}$ is a non-trivial multi-graph each of whose vertices is either a quasi-center, a multi-saddle connection, or a periodic orbit with a one-sided neighborhood, and each of whose edges is a connected component of $\text{Per}(v) \setminus N$ and is an open annulus, where $N$ is the union of periodic orbits with one-sided neighborhoods. Therefore $S/v_{\text{ex}}$ is a non-trivial multi-graph $(V_{\text{ex}}, E_{\text{ex}}, r_{\text{ex}})$. Finally it suffices to show that the condition 2) implies 1).

Suppose that $S/v_{\text{ex}}$ is $T_2$. Let $C \subseteq \partial(\text{Sing}(v) - M) = \partial(\text{Per}(v) \cup D)$ be a connected component of the boundary. The Hausdorff property implies that $C \subseteq \partial \text{Per}(v)$ is contained in one extended orbit. Since $C \subseteq \text{Sing}(v)$, the component $C$ is a singular point. This means each connected component of $\partial(\text{Sing}(v) - M)$ is a singleton. Since $M$ consists of isolated singular points, the boundary $\partial \text{Sing}(v)$ is totally disconnected. Since the boundary $\partial \text{Sing}(v) = \text{Sing}(v) - \text{int} \text{Sing}(v)$ is compact metrizable, it is zero-dimensional. Hence $\text{int} \text{Sing}(v) = \emptyset$. Indeed, otherwise the dimension of $\text{Sing}(v)$ is two. Then the complement $S - \partial \text{Sing}(v)$ is disconnected because it has two connected components such that one contains a connected component of $\partial \text{Sing}(v)$ and the another contains one of $\text{Per}(v)$. Since $S$ is a 2-dimensional Cantor-manifold, the boundary $\partial \text{Sing}(v)$ is at least one dimensional, which contradicts that $\partial \text{Sing}(v)$ is zero-dimensional. Then $\text{Sing}(v) = \partial \text{Sing}(v)$ is totally disconnected and $S = \text{Per}(v)$. \hfill $\square$

|       | $S/v$                   | $S/v_{\text{ex}}$                                      |
|-------|-------------------------|-------------------------------------------------------|
| $T_0$ | $LD \cup E = \emptyset$| $LD \cup E = \emptyset$                               |
| $T_1$ | $P \cup LD = \emptyset$ | $P \subset D$ and $LD = \emptyset$ and $D$ consists of closed multi-saddle connections |
| $T_2$ | $S/v$ is $T_1$ and $|\text{Sing}(v)| \leq 2$ | $S/v_{\text{ex}}$ is $T_1$ and $\text{Sing}(v)$ is totally disconnected |

Table 1. For a non-wandering flow $v$ on a connected compact surface $S$ with the multi-saddle connection diagram $D$, necessary and sufficient conditions for separation axioms on the orbit space $S/v$ and the extended orbit space $S/v_{\text{ex}}$ are compared.

Note that the condition that $\text{Sing}(v)$ is totally disconnected is necessary in the previous lemma, because there is a real analytic non-wandering toral flow whose (extended) orbit space is not $T_2$ but $T_1$. Indeed, consider a real analytic non-wandering flow $v$ on $T^2 = (\mathbb{R}/\mathbb{Z})^2$ defined by $v_t(x, y) = (x + \sin(2\pi y)t, y)$ with $T^2 = \text{Cl}(v)$ and $\text{Sing}(v) = \mathbb{R}/\mathbb{Z} \times \{0, 1/2\}$ such that the orbit space $T^2/v$ is not $T_2$ but $T_1$. Moreover, the multi-graph $(S/v_{\text{ex}})/\sim_E$ need not be finite even if the extended orbit space is $T_2$, because there is a spherical flow $v$ whose extended orbit space is $T_2$ whose multi-graph $(S/v_{\text{ex}})/\sim_E$ has infinitely many edges. Indeed, consider a non-trivial rotation $v_0$ with respect to an axis on $S^2$ and a sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ of points converging to a topological center such that $O_{v_0}(x_m) \neq O_{v_0}(x_n)$ for any $m \neq n \in \mathbb{Z}_{>0}$. Using a bump function and replacing the orbit of $x_n$ into a 0-saddle with a homoclinic orbit (resp. a homoclinic saddle connection with a center disk) the resulting flow $v$ is a non-wandering flow with infinitely many singular
points such that the extended orbit space $S/\text{ex}$ of $v$ is a closed interval (resp. tree).

**Lemma 4.11.** Let $v$ be a flow on a compact surface $S$. The following are equivalent:
1) $v$ is non-wandering.
2) $\text{int} P = \emptyset$.
3) $(\text{int} P) \cup E = \emptyset$ (i.e. $S = \text{Cl}(v) \cup \text{Per} \cup \text{LD}$).
4) $S = \text{Cl}(v) \cup \text{LD}$.

**Proof.** Recall that $S = \text{Cl}(v) \cup P \cup \text{LD} \cup E$ and that the union $P$ is the set of points which are not weakly recurrent. By taking a double covering of $M$ if necessary, we may assume that $v$ is transversally orientable. By the Maier theorem [7, 8], the closure $E$ is a finite union of closures of exceptional orbits and so is nowhere dense. By Lemma 2.3 [16], the union $P \cup E$ is a neighborhood of $E$. Then $\text{int}(P \setminus E) \neq \emptyset$ if $E \neq \emptyset$. We show that the condition 2) implies the condition 1). Suppose that $\text{int} P = \emptyset$. Then the closure of the set of weakly recurrent points is the whole surface (i.e. $S = \text{Cl}(v) \cup \text{LD} \cup E$) and so $v$ is non-wandering. Trivially, the condition 3) (resp. 4)) implies the condition 2). Finally, we show that the condition 1) implies the conditions 3) and 4). Suppose that $v$ is non-wandering. By Theorem III.2.12, III.2.15 [BS], the set of weakly recurrence points is dense in $S$. The density of weakly recurrence points implies that $\text{int} P = \emptyset$ and so $E = \emptyset$. This implies that $S = \text{Cl}(v) \cup \text{LD}$. □

We obtain another characterization of the Hausdorff separation property of the extended orbit space.

**Lemma 4.12.** Let $v$ be a non-trivial flow on a compact connected surface $S$. The following are equivalent:
1) $S = \text{Cl}(v)$ and each singular point is either a multi-saddle or a quasi-center.
2) $S/\text{ex}$ is $T_2$.

**Proof.** Suppose that $S = \text{Cl}(v)$ and each singular point is either a multi-saddle or a quasi-center. Then $S = \text{Cl}(v) = \text{Sing}(v) \cup \text{Per}(v)$. By Lemma 2.3 [16], we obtain $S = \text{Cl}(v) \cup \delta P$. Since $v$ is non-wandering, Proposition 2.6 [16] implies that each non-closed proper orbit is a separatrix connecting singular points. Assume that there is an non-closed extended orbit $O_{\text{ex}}$. By hypothesis, the intersection $\text{Sing}(v) \cap O_{\text{ex}}$ consists of infinitely many multi-saddles. Then there is a sequence of multi-saddles contained in $O_{\text{ex}}$ converging to a non-isolated singular point $x$. By hypothesis, the singular point $x$ is a quasi-center. By the definition of quasi-centers, the extended orbit $O_{\text{ex}}$ is closed, which contradicts to the assumption. Thus each extended orbit is closed. By Lemma 4.9, the extended orbit space $S/\text{ex}$ is $T_1$. By Lemma 4.10, the extended orbit space $S/\text{ex}$ is $T_2$. Conversely, suppose that $S/\text{ex}$ is $T_2$. Lemma 4.9 implies that $\text{LD} = \emptyset$ and the union $P$ is contained in the multi-saddle connection diagram. This means that $\text{int} P = \emptyset$. By Lemma 4.10, we have $S = \text{Cl}(v) = \text{Cl}(v) \cup \delta P$. Moreover, Lemma 4.10 implies that $\text{Sing}(v)$ is totally disconnected. Since each extended orbit is closed, each singular point is either a multi-saddle or a quasi-center. □

Summarize the characterization of separation axioms for orbit spaces and orbit class spaces.
Theorem 4.13. Let \( v \) be a non-trivial flow on a compact connected surface \( S \) and \( D \) the multi-saddle connection diagram. The following holds:

1) \( S/v \) is \( T_0 \) if and only if \( S = \text{Pr} (\text{i.e. } S = \text{Cl}(v) \sqcup P) \).
2) \( S/v \) is \( T_1 \) if and only if \( S = \text{Cl}(v) \) (i.e. \( S = \text{Sing}(v) \sqcup \text{Per}(v) \)).
3) \( S/v \) is \( T_2 \) if and only if \( S = \text{Cl}(v) \) and \( |\text{Sing}(v)| \leq 2 \).
4) \( S/v_{\text{ex}} \) is \( T_1 \) if and only if each multi-saddle connection is closed such that \( P \subset D \) and \( LD = \emptyset \).
5) \( S/v_{\text{ex}} \) is \( T_2 \) if and only if \( S = \overline{\text{Cl}(v)} \) and each singular point is either a multi-saddle or a quasi-center.

5. Characterizations of non-wandering surface flows with finitely many singular points

We show a following statement.

Lemma 5.1. Let \( v \) be a flow on a compact connected surface \( S \). The following are equivalent:

1) The flow \( v \) is non-wandering with \( |\text{Sing}(v)| < \infty \).
2) \( (\text{Sing}(v) \sqcup P)/v \) is a finite multi-graph-like poset.

In any case, the union \( \text{Sing}(v) \sqcup P \) consists of centers and of the multi-saddle connection diagram.

Proof. Suppose that \( v \) is non-wandering with \( |\text{Sing}(v)| < \infty \). Since \( \text{Sing}(v) \) is finite, Theorem 3\[4\] implies that each singular point is either topological center or a multi-saddle. Proposition 2.6 \[16\] implies that \( P \) is contained in the multi-saddle connection diagram and consists of finitely many orbits. This implies that \( \text{Sing}(v) \sqcup P \) consists of finitely many orbits and is the union of centers and the multi-saddle connection diagram. Therefore a quotient space \( (\text{Sing}(v) \sqcup P)/v \) is a finite multi-graph-like poset. Conversely, suppose that \( (\text{Sing}(v) \sqcup P)/v \) is a finite multi-graph-like poset. Then \( \text{Sing}(v) \sqcup P \) consists of finitely many orbits. This means that \( |\text{Sing}(v)| < \infty \) and that \( \text{int}P = \emptyset \). By Lemma 4.11, we have that \( v \) is non-wandering. \( \square \)

Corollary 5.2. Let \( v \) be a non-wandering flow on a compact connected surface \( S \) with \( |\text{Sing}(v)| < \infty \) and \( N \) the union of periodic orbits with one-sided neighborhoods. Then each boundary component \( C \) for a connected component of \( LD \) (resp. \( \text{Per}(v) - N \)) is either an element of \( V_{\text{ex}} \) or a proper subset of a multi-saddle connection such that the quotient space \( C/v \) is a finite multi-graph-like poset.

Proof. Let \( C \) be a boundary component for a connected component of \( LD \) (resp. \( \text{Per}(v) - N \)). Lemma 2.4 \[16\] implies \( \text{Per}(v) \) is open. By Lemma 5.1 the union \( \text{Sing}(v) \sqcup P \) consists of centers and of the multi-saddle connection diagram. Since \( |\text{Sing}(v)| < \infty \), the multi-saddle connection diagram \( D \) consists of finitely many closed multi-saddle connections. This means that the union \( \text{Sing}(v) \sqcup P \) consists of finitely many closed extended orbits. By Lemma 4.11 we have \( S = \overline{\text{Cl}(v)} \sqcup \partial P \sqcup LD \). Since \( LD \) is the finite union of quasi-minimal sets, Lemma 2.1 \[16\] implies that \( \partial LD \cap \text{Per}(v) = \emptyset \). By Lemma 2.3 \[16\], we have \( \text{Per}(v) \cap LD = \emptyset \) and so \( C \subseteq \partial(\text{Per}(v) - N) \cup \partial LD = \text{Sing}_{\text{ex}}(v) \sqcup D \sqcup N \), where \( \text{Sing}(v)_{\text{ex}} \) is the set of centers. This implies that \( C \) is either a center, a periodic orbit with a one-sided neighborhood, or a closed saturated subset of a multi-saddle connection. \( \square \)
We characterize non-wandering surface flow with finitely many singular points using extended orbit spaces.

**Theorem 5.3.** Let $v$ be a flow with $|\text{Sing}(v)| < \infty$ on a compact connected surface $S$. The following are equivalent:

1) $v$ is non-wandering.
2) $\text{int} P = \emptyset$.
3) $(S - \text{LD})/\nu_{\text{ex}}$ is an embedded multi-graph.
4) $(S/\nu_{\text{ex}})/\sim_E$ is a finite poset of height at most one.
5) The multi-saddle connection diagram contains $P$.

**Proof.** By Lemma 6.1, the conditions 1) and 5) are equivalent. We may assume that $v$ is non-trivial. Let $p : S \to (S/\nu_{\text{ex}})/\sim_E$ be the canonical projection. As above, we may assume that $v$ is not a rational rotation on $T^2$. Lemma 6.1 implies that the conditions 1) and 2) are equivalent. Suppose that $(S - \text{LD})/\nu_{\text{ex}}$ is an embedded multi-graph. Each non-closed proper orbit is contained in a closed multi-saddle connection diagram. Since $|\text{Sing}(v)| < \infty$, we have $\text{int} P = \emptyset$ and so $v$ is non-wandering. Suppose that $(S/\nu_{\text{ex}})/\sim_E$ is a finite poset of height at most one. If $\text{int} P \neq \emptyset$, then $p(P)$ contains uncountably many height one elements. Thus $\text{int} P = \emptyset$. Conversely, suppose that $v$ is non-wandering. Then the union of proper orbit is $P = \text{Cl}(v)\sqcup P = S - \text{LD}$. Since $\text{Sing}(v)$ is finite, Theorem 3[4] implies that each singular point is either topological center or a multi-saddle. Proposition 2.6 implies that $P$ is contained in the multi-saddle connection diagram and so that $P/\nu_{\text{ex}} = (\text{Cl}(v)\sqcup P)/\nu_{\text{ex}}$ is a 1-connected, since the complement of the union of topological centers and of the multi-saddle connection diagram is $\text{Per}(v)\sqcup \text{LD}$, the quotient space $P/\nu_{\text{ex}}$ is the graph of which each of vertices is either a topological center, a multi-saddle connection, or a periodic orbit with a one-sided neighborhood, and each of whose edges is a connected component of $\text{Per}(v) - N$ where $N$ is the union of periodic orbits with one-sided neighborhoods. Since $\text{Sing}(v)$ is finite, $P/\nu_{\text{ex}}$ is embedded. By Theorem 2.5 and Corollary 2.9[10], the unions $\text{Per}(v)$ and $\text{LD}$ are open. Therefore the complement of the union of $\text{Sing}(v)$ and of the multi-saddle connection diagram consists of finitely many connected components each of which is contained in $\text{Per}(v)$ or $\text{LD}$. Moreover the boundary $\partial \text{LD} = \partial S - \text{LD}$ is contained in the finite union of singular points and multi-saddle connections. Then $p(\text{LD})$ consists of finitely many height one elements and so $(S/\nu_{\text{ex}})/\sim_E$ is a finite poset of height at most one. \qed

The finiteness in the previous theorem is necessary, because of the previous real analytic non-wandering toral flow with $\text{LD} = \emptyset$ whose extended orbit space is not a multi-graph. Moreover, there is a non-wandering flow $v$ on a compact connected surface $S$ with $\text{LD} = \emptyset$ such that $S/\nu_{\text{ex}}$ is not $T_1$ but $\text{Sing}(v)$ is totally disconnected. Indeed, consider an orbit $v_0$ with respect to an axis on $S^2$ and a sequence $(x_n)_{n \in \mathbb{Z}_{>0}}$ of points converging to a periodic point $x$ such that $O_{v_0}(x_m) \neq O_{v_0}(x_n)$ for any $m \neq n \in \mathbb{Z}_{>0}$. Using a bump function and replacing the orbit of $x_n$ (resp. $x$) into a 0-saddle (resp. singular point) with a homoclinic orbit, the resulting flow $v$ is a non-wandering flow with infinitely many singular points such that the extended orbit $(O_{v_0}(y))$ by $v$ of a point $y \notin O_{v_0}(x)$ is the orbit $O_{v_0}(y)$ of $y$ by $v_0$ but an (extended) orbit $O_{v_0}(x) - \{x\}$ of $v$ is not closed. We characterize non-wandering surface flow with finitely many singular points using quotient spaces.
Theorem 5.4. Let $v$ be a flow on a compact connected surface $S$. The following are equivalent:

1) The flow $v$ is non-wandering with $|\text{Sing}(v)| < \infty$.
2) The topology of $(S/\hat{v}_{\text{ex}})/\sim_E$ is Alexandroff with respect to the specialization order.
3) $(S/\hat{v}_{\text{ex}})/\sim_E$ is a finite connected poset of height at most one.
4) $(\text{Sing}(v) \sqcup P)/v$ is a finite multi-graph-like poset.

In any case, the quotient space $(S/\hat{v}_{\text{ex}})/\sim_E$ is a multi-graph-like connected finite poset if $LD = \emptyset$.

Proof. Obviously, the condition 4) implies 1). By Theorem 5.3, the conditions 1) and 4) are equivalent. Let $Q := (S/\hat{v}_{\text{ex}})/\sim_E$ be the poset with respect to the specialization order and $p : S \to Q$ the canonical projection. Suppose that $v$ is non-wandering with $|\text{Sing}(v)| < \infty$. Let $D$ be the multi-saddle connection diagram. By Proposition 2.6 \[16\], the complement $S - (\text{Sing}(v) \cup D) = \text{Per}(v) \sqcup LD$ has finitely many connected components. Since $|\text{Sing}(v)| < \infty$, Corollary 2.9 \[16\] implies that $\text{Per}(v)$ and LD are open. Since $\text{Sing}(v)$ is finite and the complement $S - (\text{Sing}(v) \cup D) = \text{Per}(v) \sqcup LD$ has finitely many connected components, the quotient space $Q$ is a finite connected poset of height at most one. Moreover, it is a multi-graph-like finite poset if $LD = \emptyset$. Since each connected component of $\text{Per}(v)$ (resp. LD) is open, the topology of $Q$ is Alexandroff with respect to the specialization order. Suppose that the topology of $Q$ is Alexandroff with respect to the specialization order. Then the restriction $Q_0$ to the set of height 0 elements is a discrete topological space. Since $\text{Sing}(v) \subseteq Q_0$ is compact, we have $|\text{Sing}(v)| < \infty$. Assume that $v$ is not non-wandering (i.e. wandering). In other words, there is an open wandering domain $V \neq \emptyset \subseteq \text{int}P$ (i.e. there is $N \in \mathbb{R}$ such that $\hat{v}_t(V) \cap V = \emptyset$ for any $t > N$). For each orbit $O \subset \text{Sat}_v(V)$, we have $O \cap V = \overline{O} \cap V$ and so $O \cap \text{Sat}_v(V) = \overline{O} \cap \text{Sat}_v(V)$. Then $U := \text{Sat}_v(V) \setminus D \subseteq \text{int}P$ is an open saturated subset, where $D$ is the multi-saddle connection diagram. Fix an orbit $O_0 \subset U$. Write $W := \bigcup \{\overline{O} \mid O \subset U - O_0\}$. Then $O_0 \subset W - W$. Since the topology of $(S/\hat{v}_{\text{ex}})/\sim_E$ is Alexandroff, the image $p(W)$ is closed and so the inverse image $W$ is also closed, which contradicts to $O_0 \subset W - W$. Thus $v$ is non-wandering. Suppose that $Q$ is a finite connected poset of height at most one. Since $\text{Sing}(v) \subseteq Q_0$, we have $|\text{Sing}(v)| < \infty$. Assume that $\text{int}P = \emptyset$. Since multi-saddles are countable, there are uncountably many orbits of height one in $P$, which contradicts to the finiteness of $Q$. Thus $\text{int}P = \emptyset$ and so the flow $v$ is non-wandering. \[\square\]

Considering a compact connected surface $S \subseteq S^2$, Theorem 5.3 and 5.4 imply the following statement.

Corollary 5.5. Let $v$ be a flow on a compact connected surface $S \subseteq S^2$. The following are equivalent:

1) The flow $v$ is non-wandering with $|\text{Sing}(v)| < \infty$.
2) The quotient space $(S/\hat{v}_{\text{ex}})/\sim_E$ is a tree-like finite poset of height one.
3) The topology of $(S/\hat{v}_{\text{ex}})/\sim_E$ is Alexandroff with respect to the specialization order.
4) $(S - \text{Per}(v))/v$ is a finite multi-graph-like poset.

The finiteness in the previous corollary is necessary, because of the example after Lemma 4.9.
6. Graph representations of non-wandering surface flows

Theorem 6.1 implies that a non-wandering flow with finitely many singular points but without locally dense orbits on a compact surface can be reconstructed by the extended orbit space with labels and the multi-saddle connection diagram $D$. To state precisely, we state one lemma and define some notations as follows: Denote by $\chi$ the set of non-wandering flows with $|\text{Sing}(v)| < \infty$ and $LD = \emptyset$ on a compact surface $S$. We summarize the properties of flows in $\chi$ as follows.

**Lemma 6.1.** Let $D$ be the multi-saddle connection diagram of a flow $v \in \chi$ and let $\text{Sing}(v)_{c}$ be the set of centers. Then the followings hold:

1) $S = \text{Sing}(v) \sqcup \text{Per}(v) \sqcup \delta \mathcal{P} = \text{Sing}(v)_{c} \sqcup \text{Per}(v) \sqcup D$.

2) Both $S/v_{\text{ex}}$ and $D$ are embedded multi-graphs.

3) Both $(S/v_{\text{ex}})/\sim_{E}$ and $D/v$ are finite multi-graph-like posets.

Note that $G_{c}$ is an abstract multi-graph with labels. Let $\mathcal{G}$ be the set of finite abstract multi-graphs. For $H \in \mathcal{G}$, denote by $C_{H}$ the set of connected components of $H$. Define a set $\mathcal{P}$ of pairs of graphs in $\mathcal{G}$ with labels as follows: $(G, l_{1}, l_{2}, H, l_{3}, l_{4}) \in \mathcal{P}$ if $G, H \in \mathcal{G}$, $l_{1}, l_{2} : V(G) \to \{c, b, n\} \sqcup C_{H}$, $l_{3} : V(H) \to E(H)^{\ast}$, and $l_{4} : E(H) \to V(H)$, where $E(H)^{\ast} := \bigcup_{n \geq 0} E(H)^{n}/\sim_{c}$ and $\sim_{c}$ is a cyclic relation on $E(H)^{n}$. Two pairs $(G, l_{1}, l_{2}, H, l_{3}, l_{4}), (G', l'_{1}, l'_{2}, H', l'_{3}, l'_{4}) \in \mathcal{P}$ are isomorphic if there are two graph isomorphisms $g : G \to G'$ and $h : H \to H'$ such that $h \circ l_{1} = l'_{1} \circ g$, $(h \cup g)^{2} \circ l_{1} = l'_{1} \circ g$, $h \circ l_{i} = l'_{i} \circ h$ ($i = 3, 4$), where $h : \{c, b, n\} \sqcup C_{H} \to \{c, b, n\} \sqcup C_{H}$ is the extension of $h$ whose restriction on $\{c, b, n\}$ is identical, and $(h \cup g)^{2} : (V(G) \cup C_{H})^{2} \to (V(G') \cup C_{H'})^{2}$ is the induced mapping by $g$ and $h$. Denote by $\bar{\mathcal{P}}$ the quotient set of $\mathcal{P}$ by the isomorphisms.

Define a mapping $p : \chi \to \bar{\mathcal{P}}$ by $p(v) := (G_{v}, D_{v}) = (((S/v_{\text{ex}})/\sim_{E}, l_{V}, l_{Ex}), (D/v, l_{Dv}, l_{Dv}))$ and an equivalent relation $\sim_{+}$ on $\chi$ by $v \sim_{+} w$ if $v$ and $w$ is topologically equivalent (i.e. there is a homeomorphism $h : S \to S$ such that the image of an orbit of $v$ is an orbit of $w$ and that it preserves orientation of the orbits). Moreover define an equivalent relation $\sim$ on $\chi$ by $v \sim w$ if either $v \sim_{+} w$ or there is a homeomorphism $h : S \to S$ such that the image of an orbit of $v$ is an orbit of $w$ and that it reverses orientation of the orbits. Write $\chi^{+}_{\sim}$ (resp. $\chi_{\sim}$) by the quotient space of $\chi$ by the topological equivalence $\sim_{+}$ (resp. the equivalence $\sim$). Now we can state precisely that a non-wandering flow with finitely many singular points but without locally dense orbits on a compact surface can be reconstructed by the abstract multi-graph with labels and the abstract multi-saddle connection diagram with labels. On the other words, we obtain the following statement.

**Theorem 6.2.** The induced map $\bar{p} : \chi^{+}_{\sim} \to \bar{\mathcal{P}}$ is well-defined and injective.

**Proof.** Since $S$ has only finitely many connected components, we may assume that $S$ is connected. Fix an element $v \in \chi$. We will show that we can reconstruct the topologically equivalent class $[v]$ from $p(v) = (G_{v}, D_{v}) = (((S/v_{\text{ex}})/\sim_{E}, l_{V}, l_{Ex}), (D/v, l_{Dv}, l_{Dv}))$. Let $r : S \to (S/v_{\text{ex}})/\sim_{E}$ be the canonical projection and $Q := (S/v_{\text{ex}})/\sim_{E}$. By Theorem 5.3, the quotient space $Q$ is a multi-graph-like connected finite poset. Since an isolated point $x$ in $Q$ corresponds to a periodic torus, we may assume that $Q$ has no isolated points. From the finite data $D_{v}$, we can reconstruct the multi-saddle connection diagram $D$ as a directed plane graph up to topological equivalence. Let $D$ be the set of connected components of the quotient space $D/v$ of the multi-saddle connection diagram $D$. Since the set $Q_{0}$ of height 0
elements in $Q$ corresponds to $V_{ex}$ by the labels in $\text{Im}(l_V) = \{c, n, b\} \cup D$, we obtain centers (resp. periodic orbits with one-sided neighborhoods off $\partial S$, periodic orbits on $\partial S$, multi-saddle connections) as directed plane graphs from both $Q_0$ and $D$ up to topological equivalence. Using the labels in $\text{Im}(l_{ex}) = \{\{\partial_- U/v, \partial_+ U/v\} \mid U \in \overline{E_{ex}}\}$, since any element in $\text{Im}(l_{ex})$ is an ordered pair of two elements each of which is contained in $V_{ex}$, paste periodic annuli, which correspond to the set $Q_1$ of height 1 elements in $Q$, between all pairs in $\text{Im}(l_{ex})$. Thus we reconstruct the flow $v$ uniquely up to topological equivalence.

If $S$ is closed and orientable, then the label $l_V$ is not necessary in the previous theorem. The previous theorem implies the following statement.

**Corollary 6.3.** The set of topological equivalent classes of non-wandering flows with finitely many singular points but without locally dense orbits on compact surfaces is enumerable by combinatorial structures algorithmically.

**Proof.** Since both $(S/v_{ex})/\sim_E$ and $D/v$ for an element $v \in \chi$ are finite topological spaces, the labels $l_V, l_{ex}, l_{Dv}$, and $l_{Dv}$ are mapping between finite sets. Since the set of compact surfaces is enumerable by induction on numbers of boundaries, genus, and connected components and since the set of multi-saddle connection diagrams is enumerable by induction on numbers of vertices and of separatrices, the assertion holds.

Let $\text{TOP}$ be the set of topological space and $\text{TOP}_\sim$ the quotient space by homeomorphisms. This implies the following corollary.

**Corollary 6.4.** The projection $\pi : \chi_\sim \to \text{TOP}_\sim$ defined by $\pi([v]) := S/v$ is well-defined such that both $D/v$ and $l_{ex}$ can be constructed by the orbit space $S/v$.

**Proof.** Fix a flow $v \in \chi$. Let $q_0 : S \to S/v$ be the canonical projection. We may assume that $S$ is connected. If $S/v$ is a circle, then $v$ is a rational rotation. Thus we may assume that $v$ is not a rational rotation. Let $Z \subset S/v$ be the set of non-$T_0$ points. Then $q_0^{-1}(Z) = D \setminus \text{Sing}(v)$ and so $q_0^{-1}(\overline{Z}) = D$. Therefore $\overline{Z}$ is a finite multi-graph-like poset such that the saddle connection diagram $D$ is a realization of the abstract multi-graph $\overline{Z}$. An element $[x] \in S/v - Z$ is called a boundary point if there is a neighborhood $W$ of $[x]$ such that $(W, [x])$ is homeomorphic of $((0, 1), 0)$. Collapsing each multi-saddle connection into a singleton, we obtain a canonical projection $q_1 : S/v \to S/v_{ex}$. Let $B \subset S/v$ be the set of boundary points. Then define $V_{ex} := q_1(B \sqcup \overline{Z})$. Then the inverse image $q_0^{-1}(B)$ is the set of centers and periodic orbits with one-sided neighborhoods. Denote by $E_{ex}$ the set of connected component of $S/v_{ex} - q_1(B \sqcup \overline{Z}) = S/v - (B \sqcup \overline{Z})$. For an element $[U] \in E_{ex}$, the inverse image $U := (q_1 \circ q_0)^{-1}([U])$ is an open annulus. Therefore the boundary $q_0(\partial U) \subset S/v$ consists of one or two elements $q_0(\partial_- U), q_0(\partial_+ U) \subseteq (B \sqcup \overline{Z})/v = q_0(\text{Sing}(v) \sqcup P)$. Since $[U]$ is an open interval, there are exactly two boundary components $\partial_- [U]$ and $\partial_+ [U]$ of $[U]$ in $S/v$. Note the image $q_1(\partial_- [U])$ (resp. $q_1(\partial_+ [U])$) of any $[U] \in E_{ex}$ is a singleton. Then we can define a label $l_{ex} : E_{ex} \to \{\{\partial_- [U], \partial_+ [U]\} \mid [U] \in E_{ex}\}$ of $v$ by $l_{ex}([U]) := \{\partial_- [U], \partial_+ [U]\}$. 

Note the labels $l_{Dv}, l_V, l_{Dv}$ can’t be reconstructed by the orbit space $S/v$ in general. Indeed, since the orbit space does not know the flow direction, the label $l_{Dv}$ can’t be reconstructed by the orbit space $S/v$. Since the orbit spaces of center disks, of a periodic Möbius band, and of an closed annulus are closed interval, the
three spaces can’t be distinguished by their orbit spaces and so the label \( l_V \) need not be reconstructed. Moreover, the label \( l_{DV} \) can’t be reconstructed (see Fig. 5).

Recall that a continuous flow is regular if each singular point has a neighborhood which is topologically equivalent to a neighborhood of a non-degenerate singular point. Note that each non-wandering flow on a closed surface with finitely many singular points is regular if and only if each singular point is either a center or a saddle. Let \( \chi_r \subset \chi \) be the subset of regular flows in \( \chi \) and \( \chi_{r\sim} \subset \chi_{\sim} \) the quotient space by the equivalence \( \sim \). Then each regular non-wandering flow with finitely many singular points but without locally dense orbits can be reconstructed by the orbit spaces as follows.

**Corollary 6.5.** Suppose that \( S \) is an orientable closed surface. The projection \( \pi : \chi_{r\sim} \to \text{TOP}_{\sim} \) defined by \( \pi([v]) := S/v \) is injective.

**Proof.** By Corollary 6.4 we can construct \( D/v \) and \( \tilde{l}_{ex} \). We may assume that \( S \) is connected. Note that \( S/v \) is a circle if and only if \( S \) is a periodic torus. Thus we may assume that \( S \) is not a periodic torus. Since \( S \) is orientable and closed, the image of the label \( l_V \) is \( \{c\} \cup C_D \), where \( C_D \) is the set of connected components of the multi-saddle connection diagram \( D \). The regularity implies that each saddle connection consists of one saddle and two homoclinic separatrices and so the label \( l_{DV} \) can be constructed by \( l_V \). By the regularity, the label \( l_{DV} \) maps a separatrix into the saddle in the homoclinic saddle connection containing it. Since the equivalence relation \( \sim \) ignores the orientation of orbits, we don’t need the orbit direction of \( D \) to reconstruct the equivalent classes of \( \chi_{r\sim} \). Therefore Theorem 6.2 implies the assertion. □

Conversely, we consider the following problem: For a given abstract multi-graph, can it be realized by a non-wandering surface flow and in how many ways? Note that the abstract multi-graph \( (S^2/\nu_{ex})/\sim_E \) by any non-wandering flow is a tree and that graphs can be realized by several graphs. We state realizability of graphs.

**Theorem 6.6.** For any non-trivial connected finite abstract multi-graph \( G \), there is a non-wandering flow \( v \) on a closed surface \( S \) such that \( G \) is isomorphic to \( (S/\nu_{ex})/\sim_E \).

**Proof.** Since any non-trivial finite multi-graph \( G = (V, E) \) can be decomposed into height one trees \( T_i = (V_i, E_i) \) by cutting all edges into pairs of edges, such a multi-graph can be reconstructed by height one trees gluing the new height zero elements. Note \( |V| + 2|E| = \sum_i |V_i| \) and \( 2|E| = \sum_i |E_i| \). Therefore it suffices to show that, for any \( k \in \mathbb{Z}_{>0} \), there is a non-wandering flow \( w \) on a compact surface \( T \) such that \( (T/\nu_{ex})/\sim_E \) is isomorphic to \( T_k \), where \( T_k \) is a tree with \( k \) points of height 0 and \( (k - 1) \) points of height one. Let \( S_1 \) be a closed center disk of a flow. Consider a flow \( w \) on the metric completion \( S_{k+1} \) of a \( k \) punctured disk which consists of one homoclinic \((k - 1)\)-saddle connection and of periodic orbits such that the boundaries are periodic orbits (e.g. Fig. 2). Then the quotient space \( (S_k/\nu_{ex})/\sim_E \) is isomorphic to \( T_k \). Gluing boundaries of such surfaces and pasting center disks to periodic orbits on the boundaries, the resulting flow on the resulting closed surface is desired. □
7. Examples

The finiteness in Theorem 6.2 (resp. Corollary 6.5) is necessary. Indeed, there are two non-wandering flows with infinitely many singular points on $\mathbb{T}^2$ which are not topologically equivalent but which have the same (extended) orbit space up to homeomorphism and the same multi-saddle connection diagram (see Fig.2). Moreover the non-wandering property in Theorem 6.2 (resp. Corollary 6.5) is necessary. Indeed, there are two flows with wandering domains on $\mathbb{T}^2$ which are not topologically equivalent but which have the same (extended) orbit space up to homeomorphism and the same multi-saddle connection diagram (see Fig.3). As a same argument, the closedness (resp. orientability) is necessary in Corollary 6.5. Indeed, consider an rotation $v_0$ with respect to an axis on $S^2$. Replacing a periodic orbit into a saddle connection, we obtain a flow $v_1$ as in Fig.4. Replacing a center disk $B_\sigma$ ($\sigma = x, y$) into a periodic orbit which is a boundary component (resp. a periodic Möbius band) we can obtain two flows $v_\sigma$. Since the flow directions of a center disk containing $x$ is opposite to the one of a center disk containing $y$ (resp. $z$), the resulting flows $v_x$ and $v_y$ have the same extended orbit space up to homeomorphism but are not topologically equivalent. The regularity is necessary in Corollary 6.5. Indeed, there are two non-regular flows $v, w$ on a disk $D^2$ which are not topologically equivalent such that the multi-saddle connection diagrams are not isomorphic as plane graphs (resp. abstract labeled multi-graphs) but isomorphic as abstract multi-graphs (see Fig.5).

The author wishes to thank the members of the Kyoto Dynamical Systems seminar for useful comments and valuable help.

References

[1] S. Kh. Aranson, G. R. Belitsky and E. V. Zhuzhoma, Introduction to the qualitative theory of dynamical systems on surfaces Trans. Math. Monographs 153, Amer. Math. Soc., 1996.

[BS] Bhatia, N. P., Szegö, G. P., Stability theory of dynamical systems Die Grundlehren der mathematischen Wissenschaften, Band 161 Springer-Verlag, New York-Berlin 1970 xi+225 pp.

[2] T. M. Cherry, Topological properties of solutions of ordinary differential equations Amer. J. Math. 59, 957–982 (1937).
Figure 2. Two non-wandering flows which are not topologically equivalent but which have the same extended orbit space.

Figure 3. Two flows without singular points which are not topologically equivalent but which have the same extended orbit space.

[3] Jack S. Calcut, Robert E. Gompf, Orbit spaces of gradient vector fields Ergodic Theory Dynam. Systems 33 (2013), no. 6, 1732–1747.
[4] M. Cobo, C. Gutierrez, J. Llibre, Flows without wandering points on compact connected surfaces Trans. Amer. Math. Soc. 362 (2010), no. 9, 4569–4580.
[5] J. Franks Nonsingular Smale Flows on $S^3$ Topology, 24 (3) (1985) 265–282.
[6] W. Hurewicz, K. Menger, Dimension and Zusammenhangsstoffe Math. Ann., 100 (1928) pp. 618–633.
[7] A. G. Mayer, Trajectories on closed orientable surfaces Mat. Sb. 12 (54) (1943), 71–84
[8] N. Markley, On the number of recurrent orbit closures Proc. AMS, 25 (1970), no 2, 413–416.
[9] I. Nikolaev, Graphs and flows on surfaces Ergod. Th. Dynam. Syst. 18 (1998), 207–220.
For a flow \( v \) with three centers \( x, y, z \) and the extended orbit space \( S^2 / v \), replacing a center disk \( B_\sigma (\sigma = x, y) \) into a periodic Möbius band (resp. a periodic orbit which is a boundary component) we can obtain two flows \( v_\sigma \). Then the resulting flows \( v_x \) and \( v_y \) are not topologically equivalent but have the same extended orbit space.

Two flows \( v, w \) on a disk \( D^2 \) which are not topologically equivalent and whose multi-saddle connection diagrams are not isomorphic as plane graphs (resp. abstract labeled multi-graphs) but isomorphic as abstract multi-graphs.
[16] T. Yokoyama, *A topological characterization for non-wandering surface flows*, Proc. Amer. Math. Soc. 144 (2016), 315–323.

**Department of Mathematics, Kyoto University of Education/JST PRESTO, 1 Fuji-nomori, Fukakusa, Fushimi-ku Kyoto, 612-8522, Japan,**

*E-mail address: tomoo@kyoto-u.ac.jp*