A NOTE ON CLOSED $G_2$-STRUCTURES AND 3-MANIFOLDS

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Abstract. This article shows that given any orientable 3-manifold $X$, the 7-manifold $T^*X \times \mathbb{R}$ admits a closed $G_2$-structure $\varphi = \text{Re} \Omega + \omega \wedge dt$ where $\Omega$ is a certain complex-valued 3-form on $T^*X$; next, given any 2-dimensional submanifold $S$ of $X$, the conormal bundle $N^*S$ of $S$ is a 3-dimensional submanifold of $T^*X \times \mathbb{R}$ such that $\varphi|_{N^*S} \equiv 0$. A corollary of the proof of this result is that $N^*S \times \mathbb{R}$ is a 4-dimensional submanifold of $T^*X \times \mathbb{R}$ such that $\varphi|_{N^*S \times \mathbb{R}} \equiv 0$.

Introduction

Berger’s classification of the possible holonomy groups for a given Riemannian manifold includes the exceptional Lie group $G_2$ as the holonomy group of a 7-dimensional manifold. On a given a 7-dimensional manifold with holonomy group a subgroup of $G_2$, there is a nondegenerate differential 3-form $\varphi$ which is torsion-free, that is, $\nabla \varphi = 0$. This torsion-free condition is equivalent to $\varphi$ being closed and coclosed. Much work has been done to study manifolds with $G_2$-holonomy, e.g. [5] and [17], but the condition $\varphi$ be coclosed is a very rigid condition. If we drop the coclosed condition, then we are studying manifolds with a closed $G_2$-structure. In particular, manifolds with closed $G_2$-structures have been studied in the articles [6], [7] and [10]; these papers focused predominantly on the metric defined by the nondegenerate closed 3-form $\varphi$. We shift our focus to the form $\varphi$ itself, in particular, results which depend on $\varphi$ being nondegenerate and closed.

Links between Calabi-Yau geometry and $G_2$ geometry have been explored in the context of mirror symmetry by Akbulut and Salur [2]. Of course, the connections between symplectic geometry and Calabi-Yau geometry are obvious; moreover, connections between symplectic and contact geometry have been explored for centuries. Thus, it seems completely natural to try to find connections between contact geometry and $G_2$ geometry. The study of these relationships is an ongoing project that begins with the work by Arikan, Cho and Salur [3], and the purpose of the current article (and many upcoming articles) is to continue this study by examining the geometry of closed $G_2$-structures as an analogue of symplectic geometry.

Treating symplectic geometry and $G_2$ geometry as being analogues of one another is not new. In [4] and [15] vector cross products (on manifolds) are studied in a general setting; in particular, symplectic geometry is the geometry of 1-fold vector cross products, better known as almost complex structures, and $G_2$ geometry is the geometry of 2-fold vector cross products in dimension 7. Further, in all cases, one can show that using the metric, there is a nondegenerate differential form of degree $k+1$ associated to a $k$-fold vector cross product. This yields the symplectic form associated to almost complex structures and the $G_2$ 3-form $\varphi$ associated to
2-fold vector cross products in dimension 7. Examples of manifolds with $G_2$ structures satisfying various conditions (including closed $G_2$ structures) are studied and classified in [9], [11], [12], [13] and [14].

This article is based on two elementary results from symplectic geometry: 1) the cotangent bundle $T^*X$ of any $n$-dimensional manifold $X$ admits a symplectic form, and 2) the conormal bundle $N^*S$ of any $k$-dimensional submanifold $S$ of $X$ with $k < n$ is a Lagrangian submanifold of $T^*X$. It is a well-known result ([6]) that any 7-manifold which is spin admits a $G_2$-structure. We show that for an orientable 3-manifold $X$, $T^*X \times \mathbb{R}$ is spin, and hence, admits a $G_2$-structure. Next, we calculate an explicit formula in coordinates for a $G_2$-structure $\varphi$ on $T^*X \times \mathbb{R}$. Using this, we prove:

**Theorem.** Let $S$ be a 2-dimensional submanifold of $X$, and let $N^*S$ denote the conormal bundle of $S$ in $T^*X$. Let $i : N^*S \rightarrow (T^*X \times \mathbb{R}, \varphi)$ be the inclusion map. Then $i^*\varphi = 0$.

1. $G_2$ Geometry

We begin with a description of $G_2$ geometry in flat space, then we consider this geometry on 7-manifolds.

Consider the octonions $\mathbb{O}$ as an 8-dimensional real vector space. This becomes a normed algebra when equipped with the standard Euclidean inner product on $\mathbb{R}^8$. Further, there is a cross product operation given by $u \times v = \text{Im}(\overline{v}u)$ where $\overline{v}$ is the conjugate of $v$ for $u, v \in \mathbb{O}$. This is an alternating form on $\text{Im}\mathbb{O}$ since, for any $u \in \text{Im}\mathbb{O}$, $u^2 \in \text{Re}\mathbb{O}$. We now define a 3-form on $\text{Im}\mathbb{O}$ by $\varphi(u, v, w) = \langle u \times v, w \rangle$. In terms of the standard orthonormal basis $\{e_1, \ldots, e_7\}$ of $\text{Im}\mathbb{O}$, $\varphi_0 = e_{123} + e_{145} + e_{167} + e_{246} - e_{257} - e_{345} - e_{356}$ where $e_{ijk} = e^i \wedge e^j \wedge e^k$. Under the isomorphism $\mathbb{R}^7 \cong \text{Im}\mathbb{O}$, with coordinates on $\mathbb{R}^7$ given by $(x^1, \ldots, x^7)$, we have $\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{345} - dx^{356}$.

**Definition 1.1.** Let $M$ be a 7-dimensional manifold. $M$ has a $G_2$-structure if there is a smooth 3-form $\varphi \in \Omega^3(M)$ such that at each $x \in M$, the pair $(T_x(M), \varphi(x))$ is isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$. A 7-manifold $M$ has a closed $G_2$-structure if the 3-form $\varphi$ is also closed, $d\varphi = 0$.

Equivalently, a smooth 7-dimensional manifold $M$ has a $G_2$-structure if its tangent frame bundle reduces to a $G_2$-bundle. For a manifold with $G_2$-structure $\varphi$, there is a natural Riemannian metric and orientation induced by $\varphi$ given by $(Y, \varphi) \wedge (\bar{Y}, \varphi) \wedge \varphi = \langle Y, \bar{Y} \rangle \cdot d\text{vol}_M$. In particular, the 3-form $\varphi$ is nondegenerate.

**Remark.** One defines a $G_2$-manifold as a smooth 7-manifold with torsion-free $G_2$-structure, i.e., $\nabla \varphi = 0$ where $\nabla$ is the Levi-Civita connection of the metric $\langle \cdot, \cdot \rangle_\varphi$. This means that $(M, \varphi)$ has holonomy group contained in $G_2$ and that $d\varphi = d^*\varphi = 0$. For our purposes, we do not assume that $d\varphi = 0$, so that $(M, \varphi)$ will be a manifold with closed $G_2$-structure as defined above.

2. The Cotangent Bundle

This material can be found in most introductions to symplectic geometry or topology, e.g., [21] or [19].

Let $X$ be any $n$-dimensional manifold (all manifolds under consideration are assumed to be $C^\infty$); let $(U; x_1, \ldots, x_n)$ be a coordinate chart for $X$, so that
for each $1 \leq i \leq n$, $x_i : U \to \mathbb{R}$. For any point $x \in X$, the differentials $(dx_1)_x, \ldots, (dx_n)_x$ form a basis for the cotangent space $T^*_x X$ at $x$; hence, for any covector $\xi \in T^*_x X$, $\xi = \sum_i \xi_i (dx_i)_x$ with the $\xi_i \in \mathbb{R}$. This yields a coordinate chart $(T^*U; x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ for the cotangent bundle of $X$ associated to the coordinates $x_1, \ldots, x_n$ on $X$.

Using these coordinates, the so-called tautological 1-form on $T^*U$ is defined by $\alpha := \sum_i \xi_i dx_i$. This definition is invariant under changes of coordinates: Let $(T^*V; y_1, \ldots, y_n, \eta_1, \ldots, \eta_n)$ be an overlapping coordinate chart; then

$$(\alpha)|_{x=1} = \sum_i \xi_i dx_i = \sum_i \eta_j dy_j,$$

and

$$(dx_i) = \frac{\partial x_i}{\partial y_j} dy_j.$$

Therefore, in the overlap

$$\sum_i \xi_i dx_i = \sum_i \xi_j (\sum_j \frac{\partial x_i}{\partial y_j} dy_j) = \sum_i \xi_i \frac{\partial x_i}{\partial y_j} dy_j = \sum_j \eta_j dy_j.$$

Now, define a 2-form by $\omega := -d\alpha = \sum_i dx_i \wedge d\xi_i$; $\omega$ is also independent of the choice of coordinates, so $\omega$ is a symplectic form on the cotangent bundle $T^*X$, called the canonical symplectic form.

Now, let $S$ be any $k$-dimensional submanifold of $X$ with $k < n$. Recall that the conormal space $N^*_x S$ at $x \in S$ is given by

$$N^*_x S := \{ \xi \in T^* X : \xi(v) = 0 \text{ for all } v \in T_x S \},$$

and the conormal bundle $N^* S$ of $S$ by

$$N^* S = \{(x, \xi) \in T^* X : x \in S, \xi \in N^*_x S \}.$$

Let $(U; x_1, \ldots, x_n)$ be a coordinate chart on $X$ such that $S \cap U$ is given by the equations

$$x_{k+1} = \cdots = x_n = 0.$$

In the associated cotangent bundle coordinate chart $(T^*U; x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, any $\xi \in N^* S \cap T^*U$ is given by

$$\xi = \sum_{i=1}^k \xi_i dx_i$$

since $x_{k+1} = \cdots = x_n = 0$ on $S$; further, since $\xi \in N^* S$ and $T^*_x S$ is spanned by

$$\left(\frac{\partial}{\partial x_1}\right)_x, \ldots, \left(\frac{\partial}{\partial x_k}\right)_x,$$

we find that

$$0 = \xi((\frac{\partial}{\partial x_i})_x) = \xi_i, \text{ for all } 1 \leq i \leq k.$$

Hence, $N^* S \cap T^*U$ is described by the equations $x_{k+1} = \cdots = x_n = \xi_1 = \cdots = \xi_k = 0$, so $N^* S$ is an $n$-dimensional submanifold of $T^*X$; further, $\alpha = \sum_{i=1}^n \xi_i dx_i$ when restricted to $N^* S$ is zero, so $\omega|_{N^* S} \equiv 0$. Thus, $N^* S$ is a Lagrangian submanifold of $T^*X$. 
3. $G_2$-Structures on $T^*X \times \mathbb{R}$

References for this section are [1], [6], [18] and [20].

Recall that for each $n \geq 3$, the Lie group $SO(n)$ is connected, and it has a double-covering map $\iota : Spin(n) \to SO(n)$ where the Lie group $Spin(n)$ is a compact, connected, simply-connected Lie group. An oriented Riemannian manifold $(X, g)$ has $SO(n)$ as structure group on the tangent bundle. An spin structure on $(X, g)$ is a $Spin(n)$-principal bundle over $X$, together with a bundle map $\pi : P_{Spin(n)} X \to P_{SO(n)} X$ such that $\pi(pg) = \pi(p)(g)$ for $p \in P_{Spin(n)} X, g \in Spin(n)$. A spin manifold is an oriented Riemannian manifold with a spin structure on its tangent bundle.

**Theorem 3.1.** For any oriented Riemannian 3-manifold $X$, $T^*X \times \mathbb{R}$ has a $G_2$-structure.

**Proof.** Let $X$ be an oriented Riemannian 3-manifold. We know that every orientable 3-manifold is parallelizable, and since a framing on a bundle gives a spin structure, $X$ is a spin manifold.

Now we check that $T^*X$ is itself a spin manifold for a spin manifold $X$. Recall $T^*X$ carries a canonical 1-form $\sum p_i dx^i$, where $p_i$ are coordinates in $T^*X$ and $x^i$ are coordinates on $X$. One can define the map $T^*X \times Spin(3) \to T^*X$ by $(p_i, g) \mapsto p_i g_i \mid x^i = p_i v_i g$ for $v_i \in T_x X$; it is well-defined because of the spin structure on $X$. This implies that $T^*X$ has an induced spin structure from $X$. Another way to see this, one can find a bundle map of the principal bundles $\mathfrak{F} : P_{Spin} T^*X \to P_{Spin} TX$ over $X$ because the map $\mathfrak{F} : E(T^*X) \to E(TX)$ is a linear, where $E$ is the set of sections of the bundle. This gives a map $P_{Spin(3)} X \to P_{SO(3)} X$ for which the following diagram commutes:

$$
P_{Spin(3)} T^*X \longrightarrow_{\pi_1} P_{SO(3)} T^*X
$$

$$
\downarrow \quad \downarrow
$$

$$
P_{Spin(3)} TX \longrightarrow_{\pi_2} P_{SO(3)} TX
$$

Let $E(\xi)$ be the base total space of $\xi$ with base space $X$. A principal $Spin(3)$-bundle $E(\xi) \times Spin(3) \to X$ induces a principal $Spin(4)$-bundle $E(\xi) \times Spin(4) \to X$ which is itself induced from $Spin(3)(\simeq SU(2) \simeq S^3) \to Spin(4)(\simeq SU(2) \times SU(2))$; hence, a spin structure on $\xi$ gives a spin structure on $\xi \oplus e$, so $T^*X \times \mathbb{R}$ admits a spin structure. By [1] p. 4, we conclude $T^*X \times \mathbb{R}$ is a smooth 7-dimensional manifold with a $G_2$-structure.

Let $X$ be an orientable 3-dimensional manifold with symplectic cotangent bundle $(M = T^*X, \omega := -\alpha)$ where $\alpha$ is the tautological 1-form on $M$. If $x_1, x_2, x_3, \xi_1, \xi_2, \xi_3$ are the standard cotangent bundle coordinates associated to the coordinates $x_1, x_2, x_3$ on $X$, define a complex-valued $(3,0)$-form on $M$ by

$$
\Omega = (dx_1 + id\xi_1) \land (dx_2 + id\xi_2) \land (dx_3 + id\xi_3).
$$

Consider $M \times \mathbb{R}$. This is a 7-manifold with coordinates $x_1, x_2, x_3, \xi_1, \xi_2, \xi_3, t$ where $t$ is $\mathbb{R}$ coordinate. Finally, define $\varphi = Re(\Omega) + \omega \wedge dt$.

We show that this defines a $G_2$ structure on $M \times \mathbb{R}$, that is, we exhibit an isomorphism of $(T_{(p,t)}(M \times \mathbb{R}), \varphi_{(p,t)})$ with $(\mathbb{R}^7, \varphi_0)$. We first calculate

$$
\Omega = (dx_1 + id\xi_1) \land ((dx_2 \land dx_3 - d\xi_2 \land d\xi_3) + i(dx_2 \land d\xi_3 - dx_3 \land d\xi_2))
$$
\[
\begin{align*}
= (dx_1 \land dx_2 \land dx_3 - dx_1 \land d\xi_2 \land d\xi_3 + dx_2 \land d\xi_1 \land d\xi_3 - dx_3 \land d\xi_1 \land d\xi_2) \\
+i(dx_1 \land dx_2 \land dx_3 - dx_1 \land d\xi_2 \land d\xi_3 + dx_2 \land d\xi_1 \land d\xi_3 - dx_3 \land d\xi_1 \land d\xi_2),
\end{align*}
\]
so we find that
\[
\varphi = \Re \Omega + \omega \land dt
\]
\[
= dx_1 \land dx_2 \land dx_3 - dx_1 \land d\xi_2 \land d\xi_3 + dx_2 \land d\xi_1 \land d\xi_3 - dx_3 \land d\xi_1 \land d\xi_2
\]
\[
+ dx_1 \land d\xi_1 \land dt + dx_2 \land d\xi_2 \land dt + dx_3 \land d\xi_3 \land dt.
\]
Now, for \(p = (x, \xi) \in T^*U\), let
\[
\left\{ \left( \frac{\partial}{\partial x_1} \right)_p, \left( \frac{\partial}{\partial x_2} \right)_p, \left( \frac{\partial}{\partial x_3} \right)_p, \left( \frac{\partial}{\partial \xi_1} \right)_p, \left( \frac{\partial}{\partial \xi_2} \right)_p, \left( \frac{\partial}{\partial \xi_3} \right)_p \right\}
\]
be the basis for the tangent space \(T_pM\) at \(p\) with respect to the cotangent coordinates on \(M\); let \(\left( \frac{\partial}{\partial t} \right)_q\) be the basis for \(T_q\mathbb{R}\) with respect to the coordinate \(t\) on \(\mathbb{R}\). Let \((x_1, \ldots, x_7)\) be the standard Euclidean coordinates on \(\mathbb{R}^7\). Define an isomorphism of the tangent vector spaces by \(\Phi : T_0\mathbb{R}^7 \rightarrow T_{(p,q)}(M \times \mathbb{R})\) by
\[
\begin{align*}
\Phi\left( \frac{\partial}{\partial x_1} \right)_0 &= -\left( \frac{\partial}{\partial x_3} \right)_p, \quad \Phi\left( \frac{\partial}{\partial x_2} \right)_0 = \left( \frac{\partial}{\partial x_3} \right)_p, \quad \Phi\left( \frac{\partial}{\partial x_3} \right)_0 = \left( \frac{\partial}{\partial x_1} \right)_p \\
\Phi\left( \frac{\partial}{\partial \xi_1} \right)_0 &= \left( \frac{\partial}{\partial \xi_2} \right)_p, \quad \Phi\left( \frac{\partial}{\partial \xi_2} \right)_0 = \left( \frac{\partial}{\partial \xi_3} \right)_p, \quad \Phi\left( \frac{\partial}{\partial \xi_3} \right)_0 = -\left( \frac{\partial}{\partial \xi_1} \right)_p.
\end{align*}
\]
This induces an isomorphism of the cotangent vector spaces \(\Phi^* : T^*_{(p,q)}(M \times \mathbb{R}) \rightarrow T^*_0\mathbb{R}^7\) where
\[
\begin{align*}
\Phi^*(dx_1)_p &= (dx_3)_0, \quad \Phi^*(dx_2)_p = (dx_2)_0, \quad \Phi^*(dx_3)_p = -(dx_1)_0 \\
\Phi^*(d\xi_1)_p &= (dx_4)_0, \quad \Phi^*(d\xi_2)_p = (dx_5)_0, \quad \Phi^*(d\xi_3)_p = (dx_6)_0 \\
\Phi^*(dt)_p &= -(dx_7)_0.
\end{align*}
\]
Then
\[
\Phi^*\varphi = \Phi^*dx_1 \land \Phi^*dx_2 \land \Phi^*dx_3 - \Phi^*dx_1 \land \Phi^*d\xi_2 \land \Phi^*d\xi_3 + \Phi^*dx_2 \land \Phi^*d\xi_1 \land \Phi^*d\xi_3 - \Phi^*dx_3 \land \Phi^*d\xi_1 \land \Phi^*d\xi_2
\]
\[
+ \Phi^*dx_1 \land \Phi^*d\xi_1 \land \Phi^*dt + \Phi^*dx_2 \land \Phi^*d\xi_2 \land \Phi^*dt + \Phi^*dx_3 \land \Phi^*d\xi_3 \land \Phi^*dt
\]
\[
= dx_3 \land dx_2 \land (-dx_1) - dx_3 \land dx_5 \land dx_6 + dx_2 \land dx_4 \land dx_5 - (dx_1) \land dx_4 \land dx_5 \\
+ dx_3 \land dx_4 \land (-dx_7) + dx_2 \land dx_5 \land (-dx_7) + (dx_1) \land dx_6 \land (-dx_7)
\]
\[
= dx_3 \land dx_2 \land dx_3 \land dx_1 \land dx_4 \land dx_5 + dx_1 \land dx_6 \land dx_7 + dx_2 \land dx_4 \land dx_6
\]
\[
- dx_2 \land dx_5 \land dx_7 - dx_3 \land dx_4 \land dx_7 - dx_3 \land dx_5 \land dx_6 = \varphi_0.
\]
In order to show that \(\varphi\) is independent of the choice of coordinates, it is enough to prove that our coordinate definition of \(\Omega\) on \(T^*X\) is independent of the choice of coordinates on \(T^*X\). This calculation follows as in [10]. Let \(L\) be an oriented 3-dimensional subspace of \(T^*X\). Let \(\{f_1, f_2, f_3\}\) be any oriented linearly independent subset of \(L\), let \(\{e_1, e_2, e_3, Je_1, Je_2, Je_3\}\) denote the standard basis for \(\mathbb{R}^6\) and let \(A\) be the map defined by \(e_j \mapsto f_j\) and \(Je_j \mapsto Jf_j\). Now, \(A \in GL(3; \mathbb{C})\), that is, \(A\) is complex linear and \(A(e_1 \land e_2 \land e_3) = f_1 \land f_2 \land f_3\). Let \(\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}\) be an oriented orthonormal basis for \(L\), and let \(B\) be the map defined by \(\tilde{f}_j \mapsto f_j\). Then \(f_1 \land f_2 \land f_3 = (\det B)\tilde{f}_1 \land \tilde{f}_2 \land \tilde{f}_3\). Since \(\Omega(A(e_1 \land e_2 \land e_3)) = \det_{\mathbb{C}}A\) and \(\Omega = \Re \Omega + i \Im \Omega\), we have
\[
(\det B)[\Re \Omega(\tilde{f}_1 \land \tilde{f}_2 \land \tilde{f}_3)] = \Re(\det_{\mathbb{C}}A)\]
and

$$(detB)[\text{Im } \Omega(\tilde{f}_1 \wedge \tilde{f}_2 \wedge \tilde{f}_3)] = \text{Im}(det_C A).$$

Hence,

$$(\text{Re } \Omega(f_1 \wedge f_2 \wedge f_3))^2 + (\text{Im } \Omega(f_1 \wedge f_2 \wedge f_3))^2$$

$$= |det_CA|^2 = det_Z A = \text{vol}(A(e_1 \wedge Je_1 \wedge e_2 \wedge Je_2 \wedge e_3 \wedge Je_3))$$

$$= \text{vol}(A(e_1 \wedge e_2 \wedge e_3 \wedge Je_1 \wedge Je_2 \wedge Je_3)) = \text{vol}(f_1 \wedge f_2 \wedge f_3 \wedge Jf_1 \wedge Jf_2 \wedge Jf_3)$$

$$= (detB)^2 \text{vol}(\tilde{f}_1 \wedge \tilde{f}_2 \wedge \tilde{f}_3 \wedge J\tilde{f}_1 \wedge J\tilde{f}_2 \wedge J\tilde{f}_3).$$

4. The Conormal Bundle

Let $X$ be a 3-dimensional manifold as in the previous section. Then $T^*X \times \mathbb{R}$ admits the $G_2$-structure $\varphi = \text{Re } \Omega + \omega \wedge dt$. Let $S$ be a 2-dimensional submanifold of $X$, and let $(U, x_1, x_2, x_3)$ be a coordinate chart on $X$ such that $S \cap U$ is given by the equation $x_3 = 0$. For the associated cotangent coordinate chart $(T^*U, x_1, x_2, x_3, \xi_1, \xi_2, \xi_3)$, any $\xi \in N^*S \cap T^*U$ is given by

$$\xi = \xi_1 dx_1 + \xi_2 dx_2$$

since $x_3 = 0$ on $S$; further, since $\xi \in N^*S$ and $T^*_x S$ is spanned by

$$\left(\frac{\partial}{\partial x_1}\right)_x, \left(\frac{\partial}{\partial x_2}\right)_x,$$

we find that

$$0 = \xi((\frac{\partial}{\partial x_1})_x) = \xi_1$$

and

$$0 = \xi((\frac{\partial}{\partial x_2})_x) = \xi_2.$$

Hence, $N^*S \cap T^*U$ is described by the equations $x_3 = \xi_1 = \xi_2 = 0$. Thus, $N^*S$ is a 3-dimensional submanifold of $T^*X$.

**Proposition 4.1.** Let $i : N^*S \hookrightarrow (T^*X \times \mathbb{R}, \varphi)$ be the inclusion. Then $i^* \varphi = 0$.

**Proof.** In this case, $N^*S \cap (T^*U \times \mathbb{R})$ is given by the equations $x_3 = \xi_1 = \xi_2 = t = 0$ where $t$ is the $\mathbb{R}$ coordinate. Recall that in this coordinate system, we have

$$\varphi = dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge d\xi_2 \wedge d\xi_3 + dx_2 \wedge d\xi_1 \wedge d\xi_3 - dx_3 \wedge d\xi_1 \wedge d\xi_2$$

$$+ dx_1 \wedge d\xi_1 \wedge dt + dx_2 \wedge d\xi_2 \wedge dt + dx_3 \wedge d\xi_3 \wedge dt.$$ 

Hence, for $p \in N^*S$, we have $(i^* \varphi)_p = \varphi_p|_{T_p(N^*S)} = 0$ since every term contains a factor which on $N^*S$ is zero. \hfill $\square$

**Corollary 4.2.** Let $i : N^*S \times \mathbb{R} \hookrightarrow (T^*X \times \mathbb{R}, \varphi)$ be the inclusion. Then $i^* \varphi = 0$.

**Proof.** Note that $(N^*S \times \mathbb{R})$ is a 4-dimensional submanifold of $T^*X$, given by the equations $x_3 = \xi_1 = \xi_2 = 0$ on $(N^*S \times \mathbb{R}) \cap (T^*U \times \mathbb{R})$. \hfill $\square$

Note that in contrast to the symplectic case, $S$ must be 2-dimensional; if $S$ is 1-dimensional, then $\varphi|_{N^*S} \neq 0$. 


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