Interpreting the weak monadic second order theory of the ordered rationals

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We show that the weak monadic second order theory of the structure \((\mathbb{Q}, <)\) is first order interpretable in its automorphism group.

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1 Introduction

The monadic second order theory of a structure is the set of second order sentences true in the structure, for which second order quantification is only performed over subsets of the domain (i.e., unary predicates). Its weak monadic second order theory instead allows quantification just over the finite subsets of the structure. Tressl enquired whether the weak monadic second order theory of \((\mathbb{Q}, <)\) can be interpreted inside its endomorphism monoid. Here we show that this is indeed possible, and in fact it can be interpreted inside its automorphism group.

There is quite a large literature on monadic and weak monadic second order theories of order, of which we just mention Läuchli’s proof of the decidability of the weak monadic second order theory of linear order in [2], and various extensions given by Shelah in [4], where for instance the undecidability of the monadic second order theory of the ordered real numbers is shown.

The weak monadic second order logic of a dense linear order is the main object of study of [3], where among many other things the author gives an elimination of quantifiers result. Tressl hopes [5] that it would be possible to transfer such a result to the (lattice-ordered) group of piecewise semi-linear functions, using a method he terms a ‘pseudo-interpretation’.

Weak monadic second order logic is preferred from certain points of view, as it is likely to be more model-theoretically tractable, in particular for \(\aleph_0\)-categorical structures. This is not true for full monadic second order logic, for instance in the attempt to describe definable sets. The non-monadic case of second order logic is far too strong, even restricted to the ‘weak’ case, since for example the non-monadic weak second order logic of a dense chain defines Peano arithmetic.

It may be possible to undertake similar work for other \(\aleph_0\)-categorical structures, such as the random graph, or the countable atomless boolean algebra, but methods very different from those used here would be required. The techniques developed by Holland and his circle (as given in [1]) are very specific to the case of groups of order-preserving permutations.

We shall use similar methods as described in [6]. There, the monoids of monomorphisms and endomorphisms of \((\mathbb{Q}, <)\) were denoted by \(M\) and \(E\), respectively, and its group of automorphisms by \(G\). It was shown that the action of \(E\) on \(\mathbb{Q}\) is interpretable in the monoid \((E, \circ)\). This is done by means of a series of first order formulae of the language of group theory. We require some of these here, and so shall recap the main ideas without full details, for which we refer the reader to [6]. In fact for our present purposes, it suffices to work just with the group \(G\). The corresponding results follow easily for \(M\) and \(E\), since \(G\) is a definable subset of each of these.
2 Background

Much of the material needed is given, either explicitly or implicitly, in [1]. We refer mainly to the presentation as

The key to unlocking the properties of $G$ is the notion of an ‘orbital’ of a member of $G$. This is defined to be the

More precisely, an orbital of $f \in G$ is an equivalence class under the relation given by $a \sim b$ if for some integers $m, n, f^m a \leq b \leq f^n a$. An orbital is then a union of orbits, so is closed under the action of $f$ and $f^{-1}$. It is easily seen that for any orbital $X$ of $f$, for every element $a$ of $X$, $a < f a$, or for every $a$, $a > f a$, or there is just one element of $X$, which is fixed by $f$. We say that $X$ has parity $+1$, $-1$ or $0$ in these three cases, respectively. Orbitals of parity $\pm 1$ are called non-trivial. Since all orbitals are convex, the family of orbitals receives the natural induced ordering, and can therefore be viewed as a 3-coloured linear order, referred to as its ‘orbital pattern’. Furthermore, two group elements are conjugate if and only if their orbital patterns are isomorphic (as coloured orders) (cf., e.g., [1, Theorem 2.2.4, p. 60]). We write the conjugate to as its ‘orbital pattern’. Furthermore, two group elements are conjugate if and only if their orbital patterns are

We omit the precise details by which the following formulae of the language of group theory are constructed, as these are given fully in [6], just sketching the intuition:

- **comp**($x$) is a formula expressing ‘comparability’ with the identity, so that for $f \in G$, comp($f$) holds in $G$ if and only if either for all $a, a \leq f a$, or for all $a, a \geq f a$.
- **apart**($x, y$) expresses that the support of $x$ is either entirely to the left of that of $y$, or entirely to its right (including the vacuous case that one or both of these supports is empty), where the support of a group element is the set of points moved by it.
- **bump**($x$) expresses that $x$ is a ‘bump’, which is defined to be a non-identity element having exactly one non-trivial orbital.
- **disj**($x, y$) is a formula of the language of group theory such that for $f, g \in G$, disj($f, g$) holds in $G$ if and only if $f$ and $g$ have disjoint supports, in which case we also say that $f$ and $g$ are disjoint. From this we can derive a formula restr($x, y$) which says that the support of $x$ is contained in that of $y$, and the restrictions of $x$ and $y$ to the support of $x$ are equal. This formula restr($x, y$) is just $\exists z (\text{disj}(x, z) \land y = xz)$, and we say that $x$ is a restriction of $y$. The formula cont($x, y$) says that the support of $x$ is contained in that of $y$, and this is taken to be $\forall z (\text{disj}(y, z) \rightarrow \text{disj}(x, z))$.
- **orbital**($x, y$) expresses that $x$ is an orbital of $y$, which is taken to mean that it is the restriction of $y$ to one of its non-trivial orbitals.

There are two particular types of bump which will be needed, which can each be characterized by a formula. A bump $f$ is said to be coterminal if its support is the whole of $\mathbb{Q}$, which may be expressed by saying that it is a bump which is not disjoint from any non-identity member of $G$, written coterm($f$). It is said to be cofinal if its support is bounded above or below but not both. Saying that $f$ is cofinal can be expressed by a formula cof expressing that it is not coterminal, and it is not disjoint from any conjugate. Cofinal elements are particularly important, since they will have support $(-\infty, a)$ or $(a, \infty)$ for some $a \in \mathbb{R}$, so can be used to encode the endpoint $a$ (which may be rational or irrational, though we really want just the rational case, and have to show in addition how this can be expressed).

**Lemma 2.1** If $f \in G$ has infinitely many non-trivial orbitals, then it has a non-trivial restriction $g$ and a restriction $g_1$ of $g$ to a non-trivial orbital, such that whenever we write $g$ in the form $g_1 g_2$ where $g_2$ is disjoint from $g_1$, then $g_2$ is conjugate to $g$.

**Proof.** Since $f$ has infinitely many non-trivial orbitals, it has either an increasing or decreasing $\omega$-sequence of such non-trivial orbitals. Without loss of generality assume this is increasing, $X_0 < X_1 < X_2 < \ldots$ say. By passing to a suitable subsequence, we may assume that all these orbitals have the same parity ($+1$ or $-1$), $X_0$ is bounded below (i.e., does not have $-\infty$ as its left endpoint), and that for each $n$, sup $X_n < inf X_{n+1}$. Furthermore, if we write $X_n = (a_n, b_n)$, we may suppose that either all $a_n$ are rational, or all are irrational, and similarly for the $b_n$ (since there are only 4 possibilities, this can be achieved by ‘thinning out’). Let $g$ and $g_1$ be the restrictions of $f$ to $\bigcup_{n \geq 0} X_n$ and $X_0$ respectively. Then $g_2$ which is disjoint from $g_1$, and such that $g = g_1 g_2$ is uniquely determined, and is the restriction of $g$ (and $f$) to $\bigcup_{n \geq 0} X_n$. Since $g$ and $g_2$ have isomorphic orbital patterns, they are conjugate, so $g_1$ is as desired. \[\square\]
The lemma leads us to consider the formula $\inf(x)$ which says that $x$ has a restriction $y$ and there is a restriction $y_1$ to a non-trivial orbital of $y$ such that if $y = y_1y_2$, where $y_2$ is disjoint from $y_1$, then $y$ is conjugate to $y_2$.

**Lemma 2.2** For any $f \in G$, $G \models \inf(f)$ if and only if $f$ has infinitely many non-trivial orbitals.

**Proof.** Let $G \models \inf(f)$, and write $g = g_1g_2$ for a restriction of $f$ with a non-trivial orbital $g_1$ as provided by the formula. Thus $g$ is conjugate to $g_2$, and as $g_2$ has one fewer non-trivial orbital than $g$, there must be infinitely many (for each). Hence $f$ also has infinitely many non-trivial orbitals.

Conversely, by Lemma 2.1, if $f$ has infinitely many non-trivial orbitals, the formula $\inf$ must be true for $f$ in $G$. \qed

**Corollary 2.3** For any $f \in G$, $G \models \neg\inf(f)$ if and only if $f$ has only finitely many non-trivial orbitals.

Having characterized finiteness in one setting, that is, for the number of non-trivial orbitals of a member of $G$, we have to transfer it to the interpretation of $\mathbb{Q}$ found in [6]. We recall in outline how this is carried out. The steps in doing this are, first to characterize elements of $G$ having a single orbital of the form $(-\infty, a)$ or $(a, \infty)$ for some $a \in \mathbb{R}$, which has already been done by the formula $\text{cof}(x)$, next to find a formula $\text{codesame}(x, y)$ which two members of $G$ satisfy if and only if their supports are of the form $(-\infty, a)$ or $(a, \infty)$ for the same $a$, and finally to find a formula $\text{rational}(x)$ which holds for a cofinal element if and only if its support has the form $(-\infty, q)$ or $(q, \infty)$ for some rational $q$.

To form $\text{codesame}$ we note that cofinal elements $f$ and $g$ will either have the same support, which is expressed by the formula $\text{cont}(f, g) \land \text{cont}(g, f)$, or ‘opposite’ supports (i.e., one $(-\infty, a)$ and the other $(a, \infty)$), which is expressed by a formula $\text{oppsupport}(f, g)$, which says that they are cofinal elements which are disjoint, and such that no non-identity member of $G$ is disjoint from both of them.

The harder task is to find the formula $\text{rational}$ characterizing which cofinal elements of $G$ correspond to $a \in \mathbb{Q}$. (We note in passing that there are 8 conjugacy classes of cofinal elements $f$, corresponding to $f$ having parity $\pm 1$, support bounded above or below, and to $a$ rational or irrational.) Details are given in [6], but we recall the ideas in outline here. A key observation is that order-automorphisms of $\mathbb{Q}$ extend naturally to order-automorphisms of $\mathbb{R}$ (by continuity). This part of the argument also requires coterminal elements. A typical coteriminal element is translation by 1 to the right. And, actually, any coteriminal element is conjugate to this element (or its inverse), so in a sense, all such elements are (possibly ‘distorted’) translations. Related to this is a formula $\text{gauge}(x, y)$ which says that $x$ and $y$ are commuting coteriminal elements whose joint centralizer is commutative (which means that any two elements which commute with both $x$ and $y$ themselves commute). To see that such elements exist, we can work instead in the linear order $\mathbb{Q}[\sqrt{2}]$. Since this is countable dense without endpoints, it is order-isomorphic to $\mathbb{Q}$, so this will suffice. Let $f$ and $g$ be translations by 1 and $\sqrt{2}$ respectively, which are order-automorphisms of $\mathbb{Q}[\sqrt{2}]$. One establishes, by extending to $\mathbb{R}$ and using a density and continuity argument that $\text{gauge}(f, g)$ holds (the key point being that the set of reals of the form $a + b\sqrt{2}$ for $a, b \in \mathbb{Z}$ is dense in $\mathbb{R}$). It can be shown that this situation is essentially typical, that is, if $\text{gauge}(f, g)$ holds for some order-automorphisms of $\mathbb{Q}$, then there is an isomorphism of $\mathbb{Q}$ to a dense subset $X$ of $\mathbb{R}$ containing an irrational $\alpha$, such that now viewing $f$ and $g$ as order-automorphisms of $X$, for all $a, f(a) = a + 1$ and $g(a) = a + \alpha$. Note that if $\text{gauge}(f, g)$ holds then using the representation with respect to $X$, all members of the joint centralizer are translations by members of $X$, so that this joint centralizer is not just commutative, but also countable.

The formula $\text{gauge}$ is now used to help us characterize cofinal elements having support $(-\infty, q)$ or $(q, \infty)$ for some rational $q$. The main point is that if $\text{gauge}(f, g)$, then as just mentioned the joint centralizer of $f$ and $g$ is a countable group. Now up to equivalence under $\text{codesame}$, there are two orbits of cofinal elements, corresponding to $q$ rational and $q$ irrational, which are countable and uncountable respectively. We use $\text{gauge}$ to enable us to tell these apart. More precisely, the formula $\text{rational}(x)$ is built up as follows. It says that $x$ is cofinal, and there are $y$ and $z$ such that $\text{gauge}(y, z)$ and for any conjugate $t$ of $x$, there is a conjugacy $u$ of $x$ to a cofinal element $x'$ such that $\text{codesame}(t, x')$, and such that $u$ commutes with both $y$ and $z$. Since as just remarked, such $u$ can take only countably many possible values, the conjugates of $x$ can only encode countably many points, from which it follows that $x$ has support $(-\infty, q)$ or $(q, \infty)$ for some rational $q$. 

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3 The main result

We can now put together what we have succeeded in doing, and obtain our main results.

Theorem 3.1 The weak monadic second order theory of the structure \((\mathbb{Q}, <)\) is first order interpretable in its automorphism group \(G\). More precisely, there are formulae ‘\(\text{finrational}\)’ and ‘\(\text{sameset}\)’ of the language of group theory, such that for any \(f \in G\), \(G \models \text{finrational}(f)\) if and only if \(f\) is either positive or negative, having only finitely many orbitals, all of whose (finite) endpoints are rational; and for any \(f, g \in G\), \(G \models \text{sameset}(f, g)\) if and only if \(G \models \text{finrational}(f) \land \text{finrational}(g)\) and \(f, g\) have the same set of fixed points.

Proof. Note that here, when we say ‘finitely many orbitals’, we mean including trivial ones. In [6, Theorem 2.13] it was shown that \(\mathbb{Q}\) may be represented inside \(G\) by means of elements satisfying the formula \(\text{rational}\), two of which are identified if they satisfy \(\text{codesame}\). We now have all the ingredients to extend this to interpret also finite sets of rationals. For this, we let \(\text{finrational}(x)\) be the formula

\[
\text{comp}(x) \land \neg \text{inf}(x) \land \forall y (\text{disj}(x, y) \rightarrow y = 1) \land
\]

\[
(\forall y, z) \text{oppsupport}(y, z) \land \text{cont}(x, yz) \rightarrow \text{rational}(y).
\]

Deciphering the clauses here, they say:
\(\text{comp}(x)\): \(x\) is either positive or negative;
\(\neg \text{inf}(x)\): \(x\) has finitely many non-trivial orbitals;
\(\forall y (\text{disj}(x, y) \rightarrow y = 1)\): \(x\) has dense support (and so by the previous line has only finitely many fixed points);
\((\forall y, z) \text{oppsupport}(y, z) \land \text{cont}(x, yz) \rightarrow \text{rational}(y)\): all endpoints of orbitals of \(x\) are rational.

For \(\text{sameset}(x, y)\) we use the formula

\[
\text{finrational}(x) \land \text{finrational}(y) \land \text{cont}(x, y) \land \text{cont}(y, x).
\]

Let us make more explicit how these formulae effect the interpretation in \(G\) of the weak monadic second order theory of \((\mathbb{Q}, <)\). The idea of the proof just given is that we are using elements having finitely many orbitals of the form \((-\infty, a_1), (a_1, a_2), \ldots, (a_n, \infty)\) for some rational numbers \(a_1 < a_2 < \ldots < a_n\), all of the same parity. (This stands for the finite set \(\{a_1, a_2, \ldots, a_n\}\).) It is also (and must be) asserted that there is a formula \(\text{sameset}\) telling us when two such elements correspond to the same finite sets of rationals. So they do ‘encode’ the set of finite sets of rationals, since clearly every finite set of rationals can arise in this way (even the empty set). In addition, relating the interpretations of rationals and finite sets of rationals, we note that the rational \(q\) lies in the finite set \(\{a_1, \ldots, a_n\}\) of rationals precisely if \(q\) can be represented by \(f\) such that \(G \models \text{rational}(f)\) and \(\{a_1, \ldots, a_n\}\) can be represented by \(g\) such that \(G \models \text{finrational}(g)\), and for some \(f'\) such that \(G \models \text{oppsupport}(f, f')\), we have \(G \models \text{cont}(g, ff')\).

To see that this amounts to an interpretation of weak monadic second order logic, we observe that any quantification over the set of finite subsets of \(\mathbb{Q}\) can be replaced by quantification over elements satisfying \(\text{finrational}\).

More precisely, the language of the weak monadic second order theory of \((\mathbb{Q}, <)\) has two kinds of variables, first order ones (for elements of the domain) and second order ones (ranging over finite subsets of the domain). In the signature we have the relation symbols \(=\) and \(<\) which can hold between first order variables, and in addition there is a relation symbol \(\in\) which can hold between \(x\) and \(X\), provided that \(x\) and \(X\) are first and second order variables respectively, and is interpreted by saying that \(x\) lies in the set \(X\).

Theorem 3.2 The weak monadic second order theory of the structure \((\mathbb{Q}, <)\) is first order interpretable in each of its monoids \(M\) of embeddings and \(E\) of endomorphisms.

This follows from the facts that \(G\) is a definable subset of each of \(M\) and \(E\), being their set of invertible elements. So we can just use the interpretation already given.

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