Scalar CurvatureSplittings I: Minimal Factors
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1 Introduction

Lower scalar curvature bounds, $\text{scal} \geq \kappa$ for some $\kappa \in \mathbb{R}$, on a manifold $M^{n+1}$ can be studied using splitting techniques where we inductively consider related scalar curvature constraints on suitable subspaces $V^k \subset M^{n+1}$, $k \leq n$, also called the splitting factors of $M^{n+1}$. This strategy was introduced by Hawking [H] and Schoen and Yau [SY1] in the 70ties. Since then the scope of this approach has grown considerably, in particular, due to work by Gromov, Lawson, Galloway and Schoen in [GL],[GS] and [G]. Typical splitting factors are area minimizing (and some types of almost minimizing) hypersurfaces of $M^{n+1}$.

It is a characteristic of this approach that the inherited scalar curvature constraints on the splitting factors become accessible only after balancing global conformal deformations.

1.1 Minimal Splitting Factors

This strategy depends on the control we have on the deformed hypersurfaces. In low dimensions $\leq 7$ they are smooth manifolds and it is easy to inductively descend to lower dimensions. This changes in higher dimensions. An almost area minimizer $H^n$ can have a complex singular set $\Sigma_H \subset H^n$. When we approach $\Sigma_H$ the regular part $H^n \setminus \Sigma_H$ degenerates and the obligatory conformal deformations of $H^n \setminus \Sigma_H$ diverge. From Martin theory [L2] on $H^n \setminus \Sigma_H$ we can tell that completing such conformally deformed $H^n \setminus \Sigma_H$ can yield still more complicated singular spaces $X^n$ with a new singular set $\Sigma_X$.

This said, we show that there are well-controlled minimal scal $> 0$-model geometries $X^n$ on $H^n$ supporting an accessible geometric analysis on metric measure spaces in the sense of Ambrosio, Cheeger and others, for instance, in [A],[C],[CK],[He],[H-T],[B-T].
For the sample case of a compact area minimizer in a scal $> 0$-manifold $M^{n+1}$ with induced Riemannian metric $g_H$ on $H^n \setminus \Sigma_H$ we get:

**Minimal Splitting Factors** There is a conformally deformed metric $\Phi^{4/n-2} \cdot g_H$, for some $C^{2,\alpha}$-regular $\Phi > 0$, $\alpha \in (0,1)$, which we call the **minimal factor metric**, so that:

- the metric completion $(X^n, d_X)$ of $(H^n \setminus \Sigma_H, \Phi^{4/n-2} \cdot g_H)$ is compact and homeomorphic to the completion $(H^n, d_H)$ of $(H^n, g_H)$.

- $\text{scal}(\Phi^{4/n-2} \cdot g_H) > 0$ and in any $p \in \Sigma_X$, $X^n$ has **sc$\Lambda > 0$-curved tangent cones**. This permits an inductive asymptotic analysis of $X$ near $\Sigma_X$ similar to the case of area minimizers. In particular, we get that $\Sigma_X \subset X$ has Hausdorff codimension $\geq 7$.

- $X^n$ can be augmented to a **metric measure space** that is Ahlfors $n$-regular, in particular, doubling, and that admits **Poincaré inequalities**. This implies further regularity properties of $X$ like the presence of isoperimetric inequalities.

The defining property of $X^n$ is that $\Phi$ has minimal growth towards $\Sigma$, compared to other admissible deformations. Details and extensions are explained in Ch.1.2 below.

In part II [L5] we use the symmetries of scal $> 0$-curved tangent cones and the control from the Ahlfons $n$-regularity and the isoperimetric inequality to inductively remove $\Sigma$. A typical application, we cite from [L5], is a splitting scheme with built-in stepwise regularization:

**Partial Regularization** Let $H^n$, $n \geq 2$, be a compact area minimizer, with singular set $\Sigma$, in a scal $> 0$-manifold $M^{n+1}$. Then there are arbitrarily small neighborhoods $U$ of $\Sigma$, so that $H \setminus U$ is conformal to a scal $> 0$-manifold $(X_U, g_X)$ with minimal boundary $\partial X_U$.

The point is that although, in general, the boundary $\partial X_U$ is also singular, its singular set $\Sigma_{\partial X_U}$ will have a lower dimension than $\Sigma_H$. The minimality of $\partial X_U$, in the scal $> 0$ ambient manifold $X_U$¹, then allows us to iteratively shift singular problems to lower dimensions before they disappear in dimension 7, cf. the introduction of [L5] and the survey [L4].

In turn, Schoen and Yau [SY2] have described an alternative strategy using nestings of singular minimizers. Minimal splitting factors may be of use in such a setting as well.

**Comparison with the Classical Approach** To get minimal splitting factor we use a setup that differs from the traditional approach, as used for instance in [SY1] or [GL], for a regular compact area minimizing hypersurface $H^n$ in a scal $> 0$-manifold $M^{n+1}$. In that classical case one considers the first eigenfunction $f_H > 0$ of the conformal Laplacian $L_H = -\Delta + \frac{n-2}{4(n-1)} \cdot \text{scal}_H$, i.e. $L_H f_H = \lambda_H \cdot f_H$. Then the stability of $H$ implies that the first eigenvalue $\lambda_H$ is positive and the transformation law

$$\text{scal}(f_H^{4/(n-2)} \cdot g_H) \cdot f_H^{n+2} = L_H(f_H) = \lambda_H \cdot f_H > 0,$$

shows that $\text{scal}(f_H^{4/(n-2)} \cdot g_H) > 0$, since $\lambda_H > 0$ and $f_H > 0$. Turning to the singular case $\Sigma_H \neq \emptyset$ we make some simple but essential modifications. To explain them let $A_H$ be the second fundamental form on $H \setminus \Sigma \subset M$, $|A|$ is its norm and we use a fixed $S$-transform $\langle A \rangle > 0$, cf. Ch.1.3. For the present we may think of $\langle A \rangle$ as a revamped version of $|A|$.

¹Actually $\partial X_U$ is two-sided minimal in a slightly larger (non-complete) scal $> 0$-manifold $Y_U \supset X_U$. 
• In place of the ordinary eigenvalue equation \( L_H f_H = \lambda_H \cdot f_H \) we consider an \( (A)\text{-weighted} \) eigenvalue equation, i.e. the eigenvalue equation for \( (A)^{-2} \cdot L \):

\[
L_H(u_{\lambda}) = \lambda \cdot (A)^2 \cdot u_{\lambda} \text{ on } H \setminus \Sigma, \text{ for some } \lambda > 0.
\]

We note that (2), and in particular \( \lambda \), remains invariant under global scalings of \( H \).

• Different from the regular case \( \Sigma_H = \emptyset \) we have many different positive solutions also for any subcritical eigenvalue \( \lambda < \lambda^{(A)}_H \), where \( \lambda^{(A)}_H \) is the principal eigenvalue of \( (A)^{-2} \cdot L \). One can show that \( \lambda^{(A)}_H > 0 \). We choose some subcritical eigenvalue \( \lambda \in (0, \lambda^{(A)}_H) \). The point is that for these \( \lambda \) the potential theory of \( L_H - \lambda \cdot (A)^2 \) is particularly well-controlled near \( \Sigma \) even without knowing any structural detail of \( \Sigma \).

• We choose a (super)solution \( \Phi > 0 \) of (2) with minimal growth towards \( \Sigma_H \) to conformally deform \( g_H \) to the minimal factor metric \( \Phi^{4/(n-2)} \cdot g_H \). This keeps the Hausdorff dimension of the new singular set small and, combined with the scaling invariance of (2), this choice yields \( \text{scal} > 0 \)-curved tangent cones as the exclusive blow-up geometries in singular points.

In the regularizations of [L5] we employ the asymptotic geometry of \( (H^n \setminus \Sigma_H, \Phi^{4/(n-2)} \cdot g_H) \) near \( \Sigma \), resembling that of \( \text{scal} > 0 \)-cones, to construct surgery style deformations bending \( (H \setminus U, \Phi^{4/(n-2)} \cdot g_H) \) to a \( \text{scal} > 0 \)-manifold with minimal \( \partial U \) as described above. In turn, the isoperimetry of the minimal factor geometry is used to validate these properties.

1.2 Statement of Results

The results hold for area minimizing hypersurfaces and for broader classes of almost minimizers \( H \in \mathcal{G} \), provided we have \( \lambda^{(A)}_H > 0 \). We recall the definition of \( \mathcal{G} \) and other basics in Ch. 1.3 below. For the sake of consistency we state our results for any \( H \in \mathcal{G} \). For regular \( H \) the results remain valid but they oftentimes become trivial, e.g., when subcritical eigenfunctions do not exist. With this caveat in mind, we consider, for the rest of the paper, a fixed pair of subcritical and principal eigenvalue (see Ch. 1.3.D.3):

\[
0 < \lambda < \lambda^{(A)}_H, \text{ for the given } H \in \mathcal{G}.
\]

We need \( \lambda > 0 \) to get conformal deformations to \( \text{scal} > 0 \)-metrics whereas \( \lambda^{(A)} - \lambda > 0 \) is crucial for the validity of the potential theoretic arguments from [L1]–[L3]. The actual values are immaterial for the qualitative aspects of the theory.

Now we turn to the definition of our basic metrics. Due to the locally Lipschitz regular coefficients of \( L_{H,\lambda} := L_H - \lambda \cdot (A)^2 \), solutions of \( L_{H,\lambda} \phi = 0 \) are \( C^{2,\alpha} \)-regular, for any \( \alpha \in (0,1) \). This suggests the following regularity assumptions.

**Definition 1.1 (Minimal Factor Metrics)** For \( H \in \mathcal{G} \) let \( \Phi > 0 \) be a \( C^{2,\alpha} \)-supersolution of \( L_{H,\lambda} \phi = 0 \) on \( H \setminus \Sigma_H \) so that in the case

- \( H \in \mathcal{G}^c \): \( \Phi \) is a solution in a neighborhood of \( \Sigma \) with minimal growth towards \( \Sigma \).
- \( H \in \mathcal{H}_{n}^R \): \( \Phi \) is a solution on \( H \setminus \Sigma_H \) with minimal growth towards \( \Sigma \).

We call the \( \text{scal} > 0 \)-metrics \( \Phi^{4/(n-2)} \cdot g_H \) on \( H \setminus \Sigma \) the minimal factor metrics.
Remark 1.2 Minimal factor metrics are naturally assigned to any $H \in \mathcal{G}$. This is owing to the boundary Harnack inequality 2.8 ([L2, Theorem 3.4 and 3.5]). It shows that for any two such supersolutions $\Phi_1, \Phi_2$ on $H \setminus \Sigma_H$ we have some constant $c \geq 1$ so that $c^{-1} \Phi_1 \leq \Phi_2 \leq c \Phi_1$ near $\Sigma$. For $H \in \mathcal{H}^n$, the boundary Harnack inequality even shows that $\Phi$ is uniquely determined up to multiples, i.e. $\Phi_2 \equiv c \cdot \Phi_1$. Then constants in estimates depending on $(H, \Phi)$ will only depend on $H$ since $c$ equally appears on both sides of the respective inequality. Since $\lambda < \lambda^{(A)}_H$, we do not have regular positive solutions with minimal growth towards all of $\overline{\Sigma}$ but we have a minimal Green’s function $G(x, y)$ for $L_{H, \lambda}$. This is a function $G : H \setminus \Sigma \times H \setminus \Sigma \to (0, \infty]$ that is finite and $C^{2,\alpha}$-regular outside the diagonal $\{(x, x) | x \in H \setminus \Sigma\}$ and satisfies the equation $L_{H, \lambda} G(\cdot, y) = \delta_y$ in a distributional sense, where $\delta_y$ is the Dirac delta function with basepoint $y$ and $G(\cdot, y)$ has minimal growth towards $\overline{\Sigma}$. This minimal Green’s function is uniquely determined. $G(\cdot, y)$ is a supersolution, for any $y \in H \setminus \Sigma_H$. Throughout this paper we exclusively use minimal Green’s functions.

Example 1.3 With $G(\cdot, y)$ one may construct supersolutions which are proper solutions of minimal growth near $\Sigma$. For any open set $V$ with compact closure $\overline{V} \subset H \setminus \Sigma$ and a smooth function $f$ on $H \setminus \Sigma$ with $f \equiv 0$ on $(H \setminus \Sigma) \setminus V$ and $f > 0$ on $V$, we set

$$S(x) = S[H, \lambda, V, f](x) := \int_{H \setminus \Sigma} G(x, y) f(y) dV(y).$$

This is a smooth positive supersolution of $L_{H, \lambda} \phi = 0$ on $H \setminus \Sigma$ with $S \in H^{1,2}(H \setminus \Sigma)$, [L2, Lemma 3.11 and Prop.3.12], and it solves $L_{H, \lambda} \phi = 0$ away from $V$ with minimal growth towards $\overline{\Sigma}$. The Riesz decomposition theorem shows that any regular supersolution that is a proper solutions outside $V$ and has minimal growth near $\Sigma$ can be written in the form (4).

Minimal factor geometries share many fundamental geometric properties with the original (almost) minimizing geometry on $H$. This is closely tied to the minimal growth condition for $\Phi$. For general conformal deformations the results of this paper become invalid.

Theorem 1.4 (Singular Sets) For any $H \in \mathcal{G}$ we have:

- The metric completion $(X^n, d_X)$ of $(H \setminus \Sigma, \Phi^{A/(n-2)} \cdot g_H)$ is a geodesic metric space and it is homeomorphic to $(H, d_H)$, $(X^n, d_X) \cong (H, d_H)$ and, hence, $\Sigma_X \cong \Sigma_H$. Thus, we can write it as $(H, d_S(\Phi))$ or briefly $(H, d_S)$ and, for $H \in \mathcal{G}_n^c$, $(H, d_S)$ is compact.

- The Hausdorff dimension of the singular set $\Sigma_X$ of $(X^n, d_X) = (H, d_S)$ is $\leq n - 7$.

We call $(H, d_S)$ a minimal splitting factor, briefly a minimal factor, of its ambient space $M$ and $d_S$ the completed minimal factor metric extending the definition 1.1 on $H \setminus \Sigma$.

The second assertion is a non-trivial refinement of $\Sigma_X \cong \Sigma_H$ since the identity map $id_H : (H, d_H) \to (H, d_S)$ is not Lipschitz regular. What we show is that the upper dimensional bound for $\Sigma_X$ is again $n - 7$ but we do not know whether the Hausdorff dimension, in particular of lower dimensional pieces of $\Sigma_H$, remains unchanged cf. Remark 2.18 below. The following two results show that minimal factors form a blow-up invariant class of spaces and, in singularities, any of these spaces admits scalar $> 0$-tangent cones. This will be used to inductively study the scalar curvature geometry near $\Sigma$, in particular in [L5].
Theorem 1.5 (Blow-Ups) For $H \in \mathcal{G}$ we consider $(H,d_{S}(\Phi_{H}))$, any $p \in \Sigma$ and any tangent cone $C$ in $p$. Then we get the following blow-up invariance:

Any sequence $(H,\tau_{i} \cdot d_{S}(\Phi_{H}))$ scaled by a sequence $\tau_{i} \to \infty$, $i \to \infty$, around $p$, subconverges and the limit of any converging subsequence is $(C,d_{S}(\Phi_{C}))$ for some tangent cone $C$.

Theorem 1.6 (Euclidean Factors) For any non–totally geodesic $H \in \mathcal{H}_{n}^{\mathbb{R}}$ there is a unique space $(H,d_{S}(\Phi_{H}))$, i.e. unique up to global scaling. For $C \in \mathcal{S}C_{n}$ the associated space $(C,d_{S}(\Phi_{C}))$ is invariant under scaling around $0 \in C$, that is, it is again a cone.

There is no such blow-up invariant scheme for principal eigenvalues. The $\langle A \rangle$-weighted principal eigenvalues are scaling invariant and we have $\lambda_{H}^{(A)} \leq \lambda_{C}^{(A)}$, and typically $\lambda_{H}^{(A)} < \lambda_{C}^{(A)}$, cf. [L3, Lemma 3.9]. This means that, in general, $\lambda_{H}^{(A)}$ is a non-principal eigenvalue on $C$. In [L4], we actually use this as a degree of freedom. We choose a $\lambda$ that is much smaller than $\lambda_{H}^{(A)}$ to get lower bounds on the growth rate of solutions towards the singular set.

$(H,d_{S}(\Phi_{H}))$ admits a canonical augmentation to a metric measure space from an extension of $\Phi^{2-n/(n-2)} \cdot \mu_{H}$ on $H \setminus \Sigma$, $\mu_{H}$ is the $n$-dimensional Hausdorff measure on $(H^{n},d_{H})$.

Definition 1.7 (Minimal Factor Measures) For any $H \in \mathcal{G}_{n}$ equipped with a minimal factor metric $\Phi^{2-n/(n-2)} \cdot g_{H}$, we define the minimal factor measure $\mu_{S}$ on $H$ by

\[ \mu_{S}(E) := \int_{E \setminus \Sigma} \Phi^{2-n/(n-2)} \cdot d\mu_{H}, \text{ for any Borel set } E \subset H. \]

$\mu_{S}$ is a Borel measure on $(H,d_{S})$, cf. [H-T, pp. 62–64]. This uses the Hausdorff dimension estimate for $\Sigma \subset (H,d_{S}(\Phi_{H}))$, Theorem 1.4, and the Ahlfors regularity, Theorem 1.8, below.

Theorem 1.8 (Ahlfors Regularity) For $H \in \mathcal{G}_{n}$, the space $(H,d_{S},\mu_{S})$ is Ahlfors $n$-regular: there are constants $A(H,\Phi),B(H,\Phi) > 0$ so that for any $q \in H$:

\[ A \cdot r^{n} \leq \mu_{S}(B_{r}(q),d_{S}) \leq B \cdot r^{n}, \text{ for any } r \in [0,\text{diam}(H,d_{S})]. \]

For $H \in \mathcal{H}_{n}^{\mathbb{R}}$ the constants only depend on the dimension, that is, we have $A(n),B(n) > 0$.

Corollary 1.9 (Doubling Properties) For any $H \in \mathcal{G}_{n}$ there is a $C(H,\Phi) > 0$, and $C(n) > 0$ for $H \in \mathcal{H}_{n}^{\mathbb{R}}$, so that $(H,d_{S},\mu_{S})$ has the following properties:

- $\mu_{S}$ is doubling: for any $q \in H$ and $r \in [0,\text{diam}(H,d_{S})]$:

\[ \mu_{S}(B_{2r}(q),d_{S}) \leq C \cdot \mu_{S}(B_{r}(q),d_{S}). \]

- For balls $B_{*} \subset B \subset H$ we have a relative lower volume decay of order $n$:

\[ \text{diam}(B_{*})^{n}/\text{diam}(B)^{n} \leq C \cdot \mu_{S}(B_{*})/\mu_{S}(B). \]

- For $H \in \mathcal{G}_{n}$, the total volume relative to $\mu_{S}$ is finite: $\mu_{S}(H) < \infty$.

\[ ^{2}\text{This is a convergence of the underlying minimizers and the conformal deformation, cf. Ch. 1.3.E.1.} \]
The presence of such families is essential to establish Poincaré inequalities, cf. [Se, He, H-T], including the following version.

**Theorem 1.10 (Poincaré inequality)** For any $H \in \mathcal{G}$, there are $C_0(H, \Phi) > 0$, $\gamma_0(H, \Phi) \geq 1$, depending only on $n$ for $H \in \mathcal{H}_n^R$, so that for concentric balls $B \subset \gamma_0 \cdot B \subset (H, d_S)$, for any function $u$ on $H$, integrable on bounded balls, and every upper gradient $w$ of $u$ we have:

$$
\left( \int_B |u - u_B| \, d\mu_S \right)^{n/(n-1)} \leq C_1 \cdot \text{diam}(B) \cdot \int_B w \, d\mu_S,
$$

where $f_B := \int_B f \, d\mu_S := \int_B f \, d\mu_S/\mu_S(B)$.

A measurable function $w \geq 0$ on $(H, d_S)$ is an upper gradient of $u$ if $|u(x) - u(y)| \leq \int_c w(s) \, ds$ for all rectifiable curves $c$ joining $x$ to $y$, for any pair $x, y \in H$. An example, for $u \in \text{Lip}_{loc}(\Omega)$, is $|\nabla u|(x) := \liminf_{\rho \to 0} \sup_{y \in B_\rho(x)} |u(x) - u(y)|/\rho$, cf. [M, p.982]. Using Cor.1.9 we can improve Theorem 1.10 using [H-T, Theorem 9.1.15], [M, Theorem 4.5, Remark 4.6].

**Corollary 1.11 (Sobolev Inequality)** For any $H \in \mathcal{G}$, there is a constant $C_1(H, \Phi) > 0$, depending only on $n$ for $H \in \mathcal{H}_n^R$, so that for some open ball $B \subset H$, an $L^1$-function $u$ on $B$ and every upper gradient $w$ of $u$ on $B$, we have

$$
\left( \int_B |u - u_B|^{n/(n-1)} \, d\mu_S \right)^{(n-1)/n} \leq C_1 \cdot \text{diam}(B) \cdot \int_B w \, d\mu_S.
$$

In turn, this and 1.8 improve 1.10 to the case where $\gamma_0 = 1$, cf. [H-T, Remark 9.1.19]. Theorems 1.8 and 1.10 and work of Ambrosio and Miranda, in [A] and [M], show that $(H, d_S, \mu_S)$ admits a proper BV (=bounded variations) theory. For a perimeter concept $\mu_S^{n-1}$ adapted to $(H, d_S, \mu_S)$ we have isoperimetric inequalities for minimal splitting factors.

**Theorem 1.12 (Isoperimetric Inequality)** For $H \in \mathcal{G}$ there is a constant $\gamma(H, \Phi) > 0$, depending only on $n$ when $H \in \mathcal{H}_n^R$, so that for any Caccioppoli set, i.e. a Borel set with locally finite perimeter, $U \subset H$:

$$
\min\{\mu_S(B_\rho \cap U), \mu_S(B_\rho \setminus U)\}^{(n-1)/n} \leq \gamma \cdot \mu_S^{n-1}(B_\rho \cap \partial U), \text{ for any } \rho > 0.
$$

As a consequence of the Ahlfors regularity and the isoperimetric inequality we have

**Corollary 1.13 (Volume Growth)** For $(H, d_S, \mu_S)$, some open subset $\Omega \subset H$ and an oriented minimal boundary $L^{n-1} \subset \Omega$ bounding an open set $L^+ \subset \Omega$ there are constants $\kappa, \kappa^+(H, \Phi) > 0$, so that for any $p \in L$:

$$
\kappa \cdot r^{n-1} \leq \mu_S^{n-1}(L \cap B_r(p)) \quad \text{and} \quad \kappa^+ \cdot r^n \leq \mu_S(L^+ \cap B_r(p)),
$$

for $r \in [0, (A/B)^{1/n} \cdot \text{dist}(p, \partial \Omega)/4]$ and where $0 < A < B$ are the Ahlfors constants. For $H \in \mathcal{H}_n^R$, $\kappa, \kappa^+ > 0$ depend only on $n$.

**Remark 1.14** The methods and results of this paper carry over to Plateau problems, that is, to (almost) minimizers with boundary. They equally admit hyperbolic unfoldings and the associated potential theory. The needed regularity assumptions for the hypersurfaces and adaptedness properties of the operators are specified in [L2, Remark 3.10].
Organization of the Paper The main results are the Ahlfors regularity and the Poincaré inequality for \((H, d_S, \mu_S)\). They broadly use the conformal Gromov hyperbolic structure we have on area minimizing (and on almost minimizing) hypersurfaces, cf. [L1]–[L3]. In Ch. 2 we define canonical Semmes families of curves on the original (almost) area minimizer structure we have on area minimizing (and on almost minimizing) hypersurfaces, cf. [L1]–[L3]. We combine this with the length control to estimate the Hausdorff dimension of \(\Sigma\) relative to \((H, d_S)\). Finally we verify that the canonical Semmes families satisfy the Semmes axioms also relative to \((H, d_S, \mu_S)\).

Remark 1.15 The present paper, together with its second part [L5], extend our earlier work from [L2] and [L3] we get the existence of scalarpine inequality for \((H, d_S, \mu_S)\). Finally we verify that the results imply the validity of a Poincaré inequality on \((H, d_S, \mu_S)\). These canonical Semmes families satisfy the Semmes axioms also relative to \((H, d_S, \mu_S)\). These results imply the validity of a Poincaré inequality on \((H, d_S, \mu_S)\).

1.3 Basic Concepts

We summarize some basic notations, concepts and results from [L1]–[L3] we use in this paper.

A. Basic Classes of integer multiplicity rectifiable currents of dimension \(n \geq 2\) with connected support inside some complete, smooth Riemannian manifold \((M^{n+1}, g_M)\)

\(\mathcal{H}_n^c: H^n \subset M^{n+1}\) is compact locally mass minimizing without boundary.

\(\mathcal{H}_n^\mathbb{R}: H^n \subset \mathbb{R}^{n+1}\) is a complete hypersurface in flat Euclidean space \((\mathbb{R}^{n+1}, g_{\text{eucl}})\) with \(0 \in H\) that is an oriented minimal boundary of some open set in \(\mathbb{R}^{n+1}\).

\(\mathcal{H}_n: \mathcal{H}_n := \mathcal{H}_n^c \cup \mathcal{H}_n^\mathbb{R}\) and \(\mathcal{H} := \bigcup_{n \geq 1} \mathcal{H}_n\). We briefly refer to \(H \in \mathcal{H}\) as an area minimizer.

\(\mathcal{C}_n: \mathcal{C}_n \subset \mathcal{H}_n^\mathbb{R}\) is the space of area minimizing \(n\)-cones in \(\mathbb{R}^{n+1}\) with tip in \(0\).

\(\mathcal{SC}_n: \mathcal{SC}_n \subset \mathcal{C}_n\) is the subset of cones which are at least singular in \(0\).

\(\mathcal{G}_n^c: H^n \subset M^{n+1}\) is a compact almost minimizer, cf. 1.3.D. below. We set \(\mathcal{G}_n^c := \bigcup_{n \geq 1} \mathcal{G}_n^c\).

\(\mathcal{G}_n: \mathcal{G}_n := \mathcal{G}_n^c \cup \mathcal{H}_n^\mathbb{R}\) and \(\mathcal{G} := \bigcup_{n \geq 1} \mathcal{G}_n\).

\(\mathcal{K}_{n-1}: \) For any area minimizing cone \(C \subset \mathbb{R}^{n+1}\) with tip \(0\), we get the (non-minimizing) minimal hypersurface \(S_C := \partial B_1(0) \cap C \subset S^n \subset \mathbb{R}^{n+1}\) and we set \(\mathcal{K}_{n-1} := \{S_C \mid C \in \mathcal{C}_n\}\). We write \(\mathcal{K} = \bigcup_{n \geq 1} \mathcal{K}_{n-1}\) for the space of all such hypersurfaces \(S_C\).

We note that each of the classes \(\mathcal{H}_n, \mathcal{H}_n^\mathbb{R}\) and \(\mathcal{G}_n\) is closed under blow-ups.

B. One-Point Compactifications of hypersurfaces \(H \in \mathcal{H}_n^\mathbb{R}\) are denoted by \(\tilde{H}\). For the singular set \(\Sigma_H\) of some \(H \in \mathcal{H}_n^\mathbb{R}\) we always add \(\infty_H\) to \(\Sigma\) as well, even when \(\Sigma\) is already compact, to define \(\tilde{\Sigma}_H := \Sigma_H \cup \infty_H\). On the other hand, for \(H \in \mathcal{G}_n^c\) we set \(\tilde{H} = H\) and \(\tilde{\Sigma} = \Sigma\).
C.1. \textbf{\textit{S-Structures}} An \textit{S}-transform \(\langle A \rangle\) is a distance measure to singular and highly curved parts of an almost minimizer. There are several ways to define such an \(\langle A \rangle\), but they all share some simple properties: an assignment \(\langle A \rangle\) which associates with any \(H \in \mathcal{G}\) a locally Lipschitz function \(\langle A \rangle_H : H \setminus \Sigma_H \to \mathbb{R}^\geq 0\) is an \textbf{\textit{S-transform}}, more precisely a Hardy \textit{S}-transform, provided it satisfies the following axioms.

- If \(H \subset M\) is totally geodesic, then \(\langle A \rangle_H \equiv 0\). Otherwise we have \(\langle A \rangle_H > 0\), \(\langle A \rangle_H \geq |A_H|\), \(\langle A \rangle_H(x) \to \infty\), for \(x \to p \in \Sigma_H\) and \(\langle A \rangle_{\lambda H} \equiv \lambda^{-1} \cdot \langle A \rangle_H\) for any \(\lambda > 0\).

- If \(H\) is not totally geodesic, and thus \(\langle A \rangle_H > 0\), we define the \textbf{\textit{S-distance}} \(\delta_{\langle A \rangle_H} := 1/\langle A \rangle_H\). \(\delta_{\langle A \rangle_H}\) is \(L_{\langle A \rangle}\)-Lipschitz regular for some constant \(L_{\langle A \rangle} = L(\langle A \rangle, \pi) > 0\), i.e.,

\[
|\delta_{\langle A \rangle_H}(p) - \delta_{\langle A \rangle_H}(q)| \leq L_{\langle A \rangle} \cdot |d_H(p, q)| \quad \text{for any } p, q \in H \setminus \Sigma \text{ and any } H \in \mathcal{G}_n.
\]

We may choose \(\langle A \rangle_H\) so that \(L_{\langle A \rangle} = 1\). Throughout this paper, and the second part \cite{L5}, we make this choice to simplify our computations.

- If \(H_i \in \mathcal{H}_n\), \(i \geq 1\), is a sequence converging to the limit space \(H_\infty \in \mathcal{H}_n\), then \(\langle A \rangle_{H_i} \xrightarrow{\text{\textit{C}}^\infty} \langle A \rangle_{H_\infty}\) for any \(\alpha \in (0, 1)\). For general \(H \in \mathcal{G}_n\), this \textbf{\textit{naturality}} holds for blow-ups: \(\langle A \rangle_{\tau_i H} \xrightarrow{\text{\textit{C}}^\infty} \langle A \rangle_{H_\infty}\), for any sequence \(\tau_i \to \infty\) so that \(\tau_i \cdot H \to H_\infty \in \mathcal{H}_n^\infty\).

- For any compact \(H \in \mathcal{G}^c\) and any \(C^\infty\)-regular \((2, 0)\)-tensor \(B\), \(\alpha \in (0, 1)\), on the ambient space \(M\) of \(H\) with \(|B|_H \neq -A_H\) there exists a constant \(k_{H,B} > 0\) such that

\[
\int_H |\nabla f|^2 + |A + B|_H|^2 \cdot f^2 dV \geq k_{H,B} \cdot \int_H \langle A \rangle^2 \cdot f^2 dV \geq k_{H,B} \cdot \int_H \frac{f^2}{d_H(x, \Sigma)^2} dV.
\]

It is worthy to recall that a \textbf{\textit{totally geodesic}} almost minimizer \(H\), where \(|A| \equiv 0\), is automatically \textbf{\textit{regular}} since \(|A|\) diverges when we approach hypersurface singularities. Any \textit{S}-transform \(\langle A \rangle\) admits a \(C^\infty\)-\textbf{\textit{Whitney smoothing}} \(\langle A \rangle^*\) that still satisfies these axioms except for a slightly weaker form of naturality:

\[
(13) \quad c_1 \cdot \delta_{\langle A \rangle}(x) \leq \delta_{\langle A \rangle^*}(x) \leq c_2 \cdot \delta_{\langle A \rangle}(x) \quad \text{and} \quad |\partial^\beta \delta_{\langle A \rangle^*}/\partial x^\beta|(x) \leq c_3(\beta) \cdot \delta_{\langle A \rangle}^{1 - |\beta|}(x)
\]

for constants \(c_i(H, \beta) > 0\), with \(c_i(n, \beta) > 0\) for \(\mathcal{H}_n^R\), \(i = 1, 2, 3\). Here, \(\beta\) is a multi-index for derivatives with respect to normal coordinates around \(x \in H \setminus \Sigma\). Throughout this paper we choose one \textbf{\textit{fixed pair of a S-transform and an associated Whitney smoothing}} \(\langle A \rangle\) and \(\langle A \rangle^*\). The precise choices are immaterial for the sequel as the results will not depend on the concrete \textit{S}-transform or Whitney smoothing.

C.2. \textbf{\textit{S-Pencils}} We can use \(\langle A \rangle\) to quantify a non-tangential way of approaching \(\Sigma\). We define \textbf{\textit{S-pencils}} to describe an inner cone condition viewing \(\Sigma_H\) as the boundary of \(H \setminus \Sigma_H\):

\[
(14) \quad \mathbb{P}(z, \omega) := \{x \in H \setminus \Sigma \mid \omega \cdot d_H(x, z) < \delta_{\langle A \rangle}(x)\}
\]

pointing to \(z \in \Sigma\), where \(\omega > 0\). The angle \(\arctan(\omega^{-1})\) is some kind of aperture of \(\mathbb{P}(z, \omega)\) relative to \(z\). When \(H\) is a cone \(C\) singular in \(0\), we write \(C \setminus \{0\} \cong S_C \times \mathbb{R}^\geq 0\), for \(S_C :=\)
\( \partial B(0) \cap C \). Then the pencil \( P(0, \omega) \) is just a subcone \( C(U) \subset C \) over some open set \( U \subset S \).

It will be useful to also define the **truncated S-pencils** \( \mathbb{T}P \). Compared to the \( S \)-pencils \( P \), the \( \mathbb{T}P \) are in controllable distance to the singular set and **ID-maps** easily extend to these sets.

\[
\mathbb{T}P(z, \omega, R, r) = \mathbb{T}P_H(z, \omega, R, r) := B_R(z) \setminus B_r(z) \cap P(z, \omega) \subset H.
\]

**C.3. S-Sobolev spaces** We use dedicated Sobolev spaces, the Hilbert spaces \( H^{1,2}(\mathbb{A})(H \setminus \Sigma) \).

We recall from \([L1, \text{Ch. 5.1}]\):

- The **\( H^{1,2}(\mathbb{A}) \)-scalar product**:
  \[
  \langle f, g \rangle_{H^{1,2}(\mathbb{A})(H \setminus \Sigma)} := \int_{H \setminus \Sigma} (\nabla f, \nabla g) + \langle A \rangle^2 \cdot f \cdot g \, dV,
  \]
  for \( C^2 \)-functions \( f, g \). The \( H^{1,2}(\mathbb{A}) \)-norm associated to this scalar product is written \( |f|_{H^{1,2}(\mathbb{A})(H \setminus \Sigma)} \).

- The **\( S \)-Sobolev space** \( H^{1,2}(\mathbb{A})(H \setminus \Sigma) \) is the \( H^{1,2}(\mathbb{A})(H \setminus \Sigma) \)-completion of the subspace of functions in \( C^2(H \setminus \Sigma) \) with finite \( H^{1,2}(\mathbb{A})(H \setminus \Sigma) \)-norm. \( H^{1,2}(\mathbb{A})(H \setminus \Sigma) \) is a Hilbert space.

For non–totally geodesic hypersurfaces \( H \in \mathcal{G} \), we have a vital **compact approximation** result

\[
H^{1,2}(\mathbb{A})(H \setminus \Sigma) \equiv H^{1,2}_{1,0}(H \setminus \Sigma) := H^{1,2}(\mathbb{A})(H \setminus \Sigma)
\]

where \( C^{1,0}(H \setminus \Sigma) \) is the space of smooth functions with compact support on \( H \setminus \Sigma \).

**D.1. Almost Minimizers** An almost minimizer \( H^n \) is a possibly singular hypersurface looking more and more like an area minimizer the closer we zoom into it.

More precisely, the volume of a ball \( B_r(p) \subset H^n \) of radius \( r > 0 \) exceeds that of the area minimizer with the same boundary by at most \( c_H \cdot r^{n+2-\beta} \), for some constant \( c_H > 0 \). Such an almost minimizer \( H^n \) is a \( C^{1,\beta} \)-hypersurface except for some singular set \( \Sigma_H \) of Hausdorff-dimension \( \leq n - 7 \). Sequences of scalings \( H_i = \tau_i \cdot H \) of \( H \), for some sequence \( \tau_i \to \infty \), for \( i \to \infty \), around a given singular point \( x \in \Sigma \subset H^n \), **flat norm subconverge** to area minimizing tangent cones \( C^n \subset \mathbb{R}^{n+1} \).

**D.2. Tameness and ID-Maps** In the case of Euclidean area minimizers we know that \( H \setminus \Sigma_H \) is smooth and there is even a \( C^k \)-approximation by tangent cones \( C \) for any \( k \in \mathbb{Z}^{\geq 0} \) in the following sense. For \( B_R(q) \cap C \setminus \Sigma_C, R > 0 \), we have from Allard theory: for any \( k \in \mathbb{Z}^{\geq 1} \) and large \( i \), \( B_R(q_i) \cap H_i \), for suitable \( q_i \in H \setminus \Sigma \), is a local \( C^k \)-section

\[
\Gamma_i : B_R(q) \cap C \to B_R(q_i) \cap H_i \subset \nu \text{ of the normal bundle } \nu \text{ of } B_R(q) \cap C
\]

up to minor adjustments near \( \partial B_R(q) \) and, for \( i \to \infty \), \( \Gamma_i \) converges in \( C^k \)-norm to the zero section, which we identify with \( B_R(q) \cap C \). We call the \( C^k \)-section **ID** := \( \Gamma_i \) the **asymptotic identification map** or **ID-map** for short. We will later briefly say the \( H_i \) compactly **ID-map-converge** to \( C \) and write **ID** when all other details are known from the context. With **ID-maps** \( C^k \)-functions on \( B_R(p) \cap H_i \) become comparable to \( C^k \)-functions on \( B_R(p) \cap C \) from an **ID-map** pull-back to \( C \) (or \( H_{\infty} \)).

We use this to specify the class \( \mathcal{G} \) of almost minimizers with a sufficient degree of regularity for the purposes of this series of papers: an almost minimizer \( H \) belongs to \( \mathcal{G} \) provided the following \( C^{k,\gamma}-tameness \) properties, for some \( k \geq 2, \gamma \in (0, 1) \), hold:

---

\( ^3 \)Saying a sequence **subconverges** means it converges after some possible selection of a subsequence.
(i) The (generalized) mean curvature is locally bounded.

(ii) \( H \setminus \Sigma \) and the ID-maps \( \Gamma_i \) are \( C^{k,\gamma} \)-regular with \( |\text{ID} - \text{id}_C|_{C^{k,\gamma}(\mathbb{R}(q) \cap C)} \to 0 \), for \( i \to \infty \), for any given tangent cone \( C \), \( p \in \Sigma_H \) and \( q \in C \) with \( \overline{B_R(q)} \subset C \setminus \sigma_C \), for some \( R > 0 \).

Throughout this paper we assume \( k = 5 \) (and drop \( \gamma \)) to be on the safe side. Besides area minimizers, \( \mathcal{G} \) covers cases we typically encounter in scalar curvature geometry or general relativity like hypersurfaces of prescribed mean curvature but also cases not arising from variational principles, like marginally outer trapped surfaces (= horizons of black holes).

D.3. Principal Eigenvalues

Let \( H \in \mathcal{G} \) be a non–totally geodesic almost minimizer. Then we use potential theory, from [L2] and [L3], to study the conformal Laplacian near \( \Sigma \).

- There exists a finite constant \( \tau = \tau(H) > -\infty \) such that for any \( C^2 \)-function \( f \) compactly supported in \( H \setminus \Sigma \): \( \int_H f \cdot L_H f \, dV \geq \tau \cdot \int_H \langle A \rangle^2 \cdot f^2 \, dV \). The largest such \( \tau \in \mathbb{R} \) is the principal eigenvalue \( \lambda_H^{(A)} \) of \( \delta_{(A)}^2 \cdot L \). For any \( H \in \mathcal{G} \), we have \( \lambda_H^{(A)} > -\infty \).

- If, in addition, \( \text{scal}_M \geq 0 \) and \( H \in \mathcal{H} \), then \( \lambda_H^{(A)} > 0 \). This is a proper upgrade to the classically known positivity of the ordinary principal eigenvalue of \( L \).

Throughout this paper we express the eigenfunctions as solutions of the equations \( L_{H,\lambda} \phi = 0 \) for the following shifted conformal Laplacian:

\[
L_{H,\lambda} := L_H - \lambda \cdot \langle A \rangle^2, \quad \text{for} \quad \lambda < \lambda_H^{(A)},
\]

for the principal eigenvalue \( \lambda_H^{(A)} \) of \( \langle A \rangle^{-2} \cdot L \). For \( \lambda = \lambda_H^{(A)} \), there is a positive solution of \( L_{H,\lambda} \phi = 0 \) which is the counterpart to the first eigenfunction in the smooth compact case. However, the potential theory of \( L_{H,\lambda_H^{(A)}} \) is less well-controlled than that of \( L_{H,\lambda} \) for subcritical eigenvalues \( \lambda < \lambda_H^{(A)} \). The condition \( \lambda < \lambda_H^{(A)} \) is essential to derive the fundamental asymptotic control over solutions of \( L_{H,\lambda} \phi = 0 \). In the potential theory of these conformal Laplacians \( L_{H,\lambda} \) is called \( \mathcal{S} \)-adapted when \( \lambda < \lambda_H^{(A)} \). In particular, \( L_H \) is is \( \mathcal{S} \)-adapted when \( \lambda_H^{(A)} > 0 \), whereas \( L_{H,\lambda_H^{(A)}} \) is never \( \mathcal{S} \)-adapted.

E.1. Induced Solutions

We can derive estimates for eigenfunctions on \( H \) from induced eigenfunctions on \( C \). We briefly recall how this works: while we scale by increasingly large \( \tau > 0 \), we observe that \( \tau \cdot TP_H(p, \omega, R/\tau, r/\tau) \), cf. C.2., is better and better \( C^k \)-approximated by the corresponding truncated \( \mathcal{S} \)-pencil in the given tangent cone. This carries over to the analysis on \( \mathbb{P} \). When we choose any \( p \in \Sigma_H \) then we have for any \( \varepsilon > 0 \), \( R > 1 > r > 0 \) and some \( \omega \in (0,1) \) and any solution \( u > 0 \) of \( L(H) \phi = 0 \) there exists some \( \tau^*(L, u, \varepsilon, \omega, R, r, p) > 0 \) such that for any \( \tau \geq \tau^* \) there is some tangent cone \( C_p^\tau \) with \( |\text{ID} - \text{id}_{C_p^\tau}|_{C^{k,\gamma}(TP_{C_p^\tau}(0,\omega,R,r))} \leq \varepsilon \) and a solution \( v > 0 \), of \( L(C_p^\tau) \phi = 0 \), that can be chosen independently of \( \varepsilon, \omega, R, r \), with

\[
|u \circ \text{ID} / v - 1|_{C^{2,\alpha}(TP_{C_p^\tau}(0,\omega,R,r))} \leq \varepsilon.
\]

We call such a solution \( v \) on \( C_p^\tau \) an induced solution.
E.2. Minimal Growth A (super)solution \( u \geq 0 \) of \( L_{H,\lambda} \phi = 0 \) has minimal growth towards \( p \in \hat{X} \), if there is a supersolution \( w > 0 \), such that \( (u/w)(x) \to 0 \), for \( x \to p \), \( x \in H \setminus \Sigma \). We say \( u \) has minimal growth towards \( W \subset \hat{X} \) if \( u \) has minimal growth towards any point \( p \in W \). An important result, [L4, Th.3], which we use extensively, is that minimal growth is inherited under convergence of the underlying spaces, in particular under blow-ups. We consider three cases:

- **Tangent Cones** Let \( u > 0 \) be a supersolution of \( L_{H,\lambda} \phi = 0 \), \( \lambda < \lambda_H^{(A)} \) that is a solution on a neighborhood \( V \) of some \( z \in \Sigma \) with minimal growth towards \( V \cap \Sigma \). Then, if \( C \) is a tangent cone of \( H \) in \( z \), any solution induced on \( C \) has minimal growth towards \( \Sigma_C \).

- **General Blow-Ups** More generally, let \( u > 0 \) be a supersolution of \( L_{H,\lambda} \phi = 0 \), \( \lambda < \lambda_H^{(A)} \) that is a solution on a neighborhood \( V \) of \( z \in \Sigma \) with minimal growth towards \( V \cap \Sigma \) and consider a sequence \( s_i \to \infty \) of scaling factors and a sequence of points \( z_i \to z \) in \( \Sigma_H \) such that \((s_i \cdot H, z_i)\) subconverges to a limit space \((H_\infty, z_\infty)\) with \( H_\infty \in \mathcal{H}_n^\mathbb{R} \). Then the induced solutions on \( H_\infty \setminus \Sigma_H \) have minimal growth towards \( \Sigma_{H_\infty} \).

- **Tangent Cones at Infinity** For \( H \in \mathcal{H}_n^\mathbb{R} \) the argument of [L3, Th.3] equally applies to tangent cones at infinity. Recall that they are the possible limit spaces we get from scaling \( H \) by some sequence \( \tau_i > 0 \) with \( \tau_i \to 0 \) for \( i \to \infty \). We assume \( u > 0 \) solves \( L_{H,\lambda} \phi = 0 \), \( \lambda < \lambda_H^{(A)} \), with minimal growth towards \( \Sigma_H \) (but not towards infinity). Then, for any tangent cone \( C \) of \( H \) at infinity, any solution induced on \( C \) has minimal growth towards \( \Sigma_C \).

2 Hyperbolic Unfoldings

The Gromov hyperbolic metric \( d_{(A)} \) on hypersurfaces \( H \in \mathcal{G} \) is used throughout this paper. We oftentimes use all three geometries \( d_H, d_S \) and \( d_{(A)} \) in one argument. Here we discuss some basics.

2.1 Canonical Semmes Families

**Definition 2.1 (Gromov Hyperbolicity)** A metric space \( X \) is geodesic, when any two points can be joined by a geodesic, i.e., a path that is an isometric embedding of an interval. A geodesic metric space \( X \) is \( \delta \)-hyperbolic, if all its geodesic triangles are \( \delta \)-thin for some \( \delta \geq 0 \), that is, the \( \delta \)-neighborhood of any two sides of such a triangle contains the third side. The space \( X \) is called Gromov hyperbolic when it is \( \delta \)-hyperbolic for some \( \delta \).

A generalized geodesic ray \( \gamma : I \to X \) is an isometric embedding of the interval \( I \subset \mathbb{R} \) into \( X \), where either \( I = [0, \infty) \), then \( \gamma \) is a proper geodesic ray, or \( I = [0, R] \), for some \( R \in (0, \infty) \). Then \( \gamma \) is a geodesic arc. When we fix a base point \( p \in X \) we can use the hyperbolicity to canonically identify any \( x \in X \) with a (properly) generalized ray \( \gamma_x \) with endpoint \( \gamma(R) = x \). We extend the definition of such a ray to \( I = [0, \infty] \) setting \( \gamma(t) = \gamma(R) \) when \( t \in [R, \infty] \).

**Definition 2.2 (Gromov Boundary)** Let \( X \) be a complete Gromov hyperbolic space. The Gromov boundary \( \partial_G X \) of \( X \) is the set of equivalence classes \([\gamma]\) of geodesic rays, starting from a base point \( p \in X \), with two rays being equivalent if they have finite Hausdorff distance.
\( \partial_G X \) does not depend on the choice of the base point \( p \). To topologize \( \overline{X}_G = X \cup \partial_G X \), we say \( x_n \in \overline{X} \) converges to \( x \in \overline{X} \) if there exist generalized rays \( c_n \) with \( c_n(0) = p \) and \( c_n(\infty) = x_n \) subconverging (on compacta) to a generalized ray \( c \) with \( c(0) = p \) and \( c(\infty) = x \). The canonical map \( X \rightarrow \overline{X}_G \) is a homeomorphism onto its image, \( \partial_G X \) is closed and \( \overline{X}_G \) is compact and called the Gromov compactification of \( X \).

Now we turn to almost minimizers \( H \in \mathcal{G} \). The locally Lipschitz Riemannian metric \( \langle A \rangle^2 \cdot g_H \) and its \( C^\infty \)-Whitney smoothing \( (\langle A \rangle)^2 \cdot g_H \), both defined on \( H \setminus \Sigma \), induce the distance functions \( d_{\langle A \rangle H} \) and \( d_{\langle A \rangle \cdot} \), where we drop the index \( H \), and write \( d_{\langle A \rangle} \) and \( d_{\langle A \rangle \cdot} \), when \( H \) is known from the context. We recall [L1, Theorem 1.11, Cor. 3.6 and Prop. 3.11].

**Theorem 2.3 (Hyperbolic Unfoldings)** For any non–totally geodesic \( H \in \mathcal{G} \), \( (H \setminus \Sigma, d_{\langle A \rangle}) \) and the quasi-isometric \( (H \setminus \Sigma, d_{\langle A \rangle \cdot}) \) are complete Gromov hyperbolic spaces with bounded geometry and we call them hyperbolic unfoldings of \( (H \setminus \Sigma, g_H) \). We have:

- The identity map on \( H \setminus \Sigma \) extends to homeomorphisms
  \[
  \hat{H} \cong (H \setminus \Sigma, d_{\langle A \rangle})_G \cong (H \setminus \Sigma, d_{\langle A \rangle \cdot})_G \quad \text{and} \quad \hat{\Sigma} \cong \partial_G (H \setminus \Sigma, d_{\langle A \rangle}) \cong \partial_G (H \setminus \Sigma, d_{\langle A \rangle \cdot}),
  \]

- the assignment \( (H \setminus \Sigma, d_{\langle A \rangle}) \) to \( H \in \mathcal{G} \) is natural: \( d_{\langle A \rangle H} \) commutes with the compact convergence of the regular portions of a sequence of underlying \( H_k \in \mathcal{G} \) with limit \( H_\infty \),

- for \( H \in \mathcal{H}_n^\mathbb{R} \), \( (H \setminus \Sigma, d_{\langle A \rangle}) \) is \( \delta(n) \)-hyperbolic with \( (\sigma_n, \ell_n) \)-bounded geometry and \( (H \setminus \Sigma, d_{\langle A \rangle \cdot}) \) is \( \delta^*(n) \)-hyperbolic of \( (\sigma_n, \ell_n) \)-bounded geometry. \( \sigma_n \) and \( \ell_n \) depend only on \( n \).

**Remark 2.4 1.** The condition for bounded geometry is this: For a global Lipschitz constant \( \ell \geq 1 \) and a radius \( \rho > 0 \), there exists around any \( p \in H \setminus \Sigma \) an \( \ell \)-bi-Lipschitz chart \( \phi_p : B_\rho(p) \rightarrow U_p \) from the ball \( B_\rho(p) \) in \( (H \setminus \Sigma, d_{\langle A \rangle}) \) to an open set \( U_p \subset (\mathbb{R}^n, g_{\mathbb{R}^n}) \). We shall always assume that \( 0 \in U_p \) and \( \phi_p(p) = 0 \). We say that \( (H \setminus \Sigma, d_{\langle A \rangle}) \) has \( (\rho, \ell) \)-bounded geometry. For the smooth manifold \( (H \setminus \Sigma, d_{\langle A \rangle}) \) we have a smooth bounded geometry with a uniform bound on the Riemann tensor and its covariant derivatives up to order \( k \).

2. For \( H \in \mathcal{H}_n^\mathbb{R} \), the boundary \( \partial_G (H \setminus \Sigma, d_{\langle A \rangle}) \) always contains the point at infinity even when \( H \) is regular, whereas for regular \( H \in \mathcal{G}_n^c \), we have \( \partial_G (H \setminus \Sigma, d_{\langle A \rangle}) = \emptyset \).

3. We frequently use scaling and blow-up arguments from geometric measure theory. We note that the scalings \( \lambda \cdot H \), for some \( \lambda > 0 \), of the original (almost) minimizing geometries \( (H, d_H) \) are compensated by \( \langle A \rangle_{\lambda H} \equiv \lambda^{-1} \cdot \langle A \rangle_H \), that is, \( H \) and \( \lambda \cdot H \) have the same hyperbolic unfolding.

We start with a geometric application of hyperbolic unfoldings. For this we recall that to handle analysis on rather general metric measure spaces, Semmes has decompiled the classical proof of Poincaré inequality on \( \mathbb{R}^n \) where, in an important step, one uses the presence of uniformly distributed families of curves linking any two given points [Se, He, H-T]. The abstracted concept is that of **thick families of curves**, also called **Semmes families**, satisfying the conditions (i) and (ii) in 2.5 below. Under some reasonable assumptions on the metric space, the presence of Semmes families implies the validity of a Poincaré inequality. On almost minimizers we have Poincaré, Sobolev and isoperimetric inequalities [BG]. In this
case the presence of Semmes families comes hardly as a surprise, but there is a little extra we get from hyperbolic unfoldings. The hyperbolic geodesics give us canonically defined Semmes families on \((H, d_H)\). The interesting point is that they are still Semmes families relative to \((H, d_S)\) and the (yet to define) minimal factor measure \(\mu_S\). This is proved in Ch. 3.3 where it is used to derive Poincaré inequalities for \((H, d_S, \mu_S)\).

**Proposition 2.5 (Canonical Semmes Families on \(H \in G_n\))** For any \(H \in G_n\), there are constants \(C = C(H) > 0\), with \(C = C(n) > 0\) for \(H \in H_n^\mathbb{R}\), and families \(\Gamma_{p,q}\) of rectifiable curves \(\gamma : I_\gamma \to H, I_\gamma \subset \mathbb{R}\), joining any two \(p, q \in H\) so that:

(i) For any \(\gamma \in \Gamma_{p,q}\): 
\[
\int_{I_\gamma} \frac{d(p, z)}{\mu(B_{d(p,z)}(p))} + \frac{d(q, z)}{\mu(B_{d(q,z)}(q))} \, d\mu(z)
\]

(ii) Each family \(\Gamma_{p,q}\) carries a probability measure \(\sigma_{p,q}\) so that for any Borel set \(A \subset X\), the assignment \(\gamma \mapsto l(\gamma \cap A)\) is \(\sigma\)-measurable with

\[
\int_{I_\gamma} l(\gamma \cap A) \, d\sigma(\gamma) \leq C \cdot \int_{A_{C,p,q}} \frac{d(p, z)}{\mu(B_{d(p,z)}(p))} + \frac{d(q, z)}{\mu(B_{d(q,z)}(q))} \, d\mu(z)
\]

for \(A_{C,p,q} := (B_{C-d(p,q)}(p) \cup B_{C-d(p,q)}(q)) \cap A\).

**Remark 2.6** The family \(\Gamma_{p,q}\) is a fibration of a twisted double cone in \((H, d_H)\), directly obtained from \(S\)-uniformity of \(H\), which surrounds a geodesic \(\gamma_{p,q}\) in the hyperbolic unfolding, see Step 2 below. We call \(\gamma_{p,q}\) the core of \(\Gamma_{p,q}\). There is a degree of freedom in the definition of \(\Gamma_{p,q}\) we exploit in Ch.3.3: we can prescribe the thickness \(d\) of the double cone and get families of curve families \(\Gamma_{p,q}[d]\) and of measures \(\sigma_{p,q}[d]\) for sufficiently small \(d > 0\), depending only on \(H\), only on \(n\) when \(H \in H_n^\mathbb{R}\). The smaller \(d\) becomes the better we can control the elliptic analysis on \(\Gamma_{p,q}[d]\), but the larger \(C(H)\) and \(C(n)\) have to be chosen.

**Proof** We show that on \(\mathbb{R}^n\) such thick families of curves can be constructed explicitly. Then we use hyperbolic unfoldings to transfer these families to \((H, d_H)\).

**Step 1 (Euclidean Model)** For \(x, y \in \mathbb{R}^n\) we consider the hyperplane \(L^{n-1}(x, y)\) orthogonal to the line segment \([x, y] \subset \mathbb{R}^n\) passing through the midpoint \(m(x, y)\) of \([x, y]\). We consider a ball \(B_r := B_r^{n-1}(m(x, y)) \subset L^{n-1}(x, y)\) of radius \(r \in (0, d(x, y)]\). For any \(z \in B_r\), let \(\gamma_z\) be the unit speed curve from \(x\) to \(y\) we get when we follow the line segments \([x, z]\) and \([z, y]\). The path space

\[
\Gamma_{x,y}^\mathbb{R} = \Gamma_{x,y}^\mathbb{R}(r) := \{\gamma_z \mid z \in B_r\}
\]

is supported on the double cone \(\bigcup\{\gamma \mid \gamma \in \Gamma_{x,y}^\mathbb{R}\} \setminus \{x, y\}\) with edge \(\partial B_r\). There is a canonical \(\sqrt{2}\)-bi-Lipschitz map \(Q_{\gamma_z} : [x, y] \to \gamma_z, z \in B_r\). The orthogonal projection to \([x, y]\) is the inverse map. We also define the following enveloping rounded double cones \(E_{x,y}^\mathbb{R}(\varsigma)\):

\[
E_{x,y}^\mathbb{R}(\varsigma) := \bigcup_{z \in [x, y]} B_{\varsigma \cdot \min\{|x-z|,|y-z|\}}(z), \text{ for some } \varsigma > 0.
\]

Since \(\bigcup\{\gamma \mid \gamma \in \Gamma_{x,y}^\mathbb{R}\} \setminus \{x, y\} \subset E_{x,y}^\mathbb{R}(\varsigma)\) for \(\varsigma \geq 2 \cdot r/d(x, y)\), we henceforth choose \(\varsigma := 2 \cdot r/d(x, y)\). We will later use these envelopes to get analytic estimates on \(\Gamma_{x,y}^\mathbb{R}\) by means of Harnack inequalities on \(E_{x,y}^\mathbb{R}(\varsigma)\). Since \(r \leq d(x, y)\) we have some \(C_0 > 0\), independent of \(x, y\).
and \( r \), so that (i) is satisfied for any \( \gamma \in \Gamma_{x,y}^n \). Towards (ii), we define a probability measure \( \alpha_{x,y}^n(r) \) on \( \Gamma_{x,y}^n(r) \):

\[
(22) \quad \alpha_{x,y}^n(W) := \mathcal{H}^{n-1}(\{z \in B_{\rho}^n \mid \gamma_z \in W\})/\mathcal{H}^{n-1}(B_{\rho}^n), \quad \text{for any Borel subset } W \subset \Gamma_{x,y}^n.
\]

From the coarea formula [AFP, Lemma 2.99], we see that for any Borel set \( A \subset \mathbb{R}^n \), the function \( \gamma \mapsto \ell(\gamma \cap A) \) on \( \Gamma_{x,y}^n \) is \( \alpha_{x,y}^n \)-measurable and thus expressions on both sides of (19) are Borel measures evaluated on \( A \). Now assume that all points in \( A \) are closer to \( x \) than to \( y \). For the distance between corresponding points on any two segments, we have

\[
(23) \quad d(s \cdot z_1 + (1 - s) \cdot x, s \cdot z_2 + (1 - s) \cdot x) \leq 2 \cdot r \cdot s, \quad \text{for } s \in [0, 1] \text{ and } z_1, z_2 \in B_r.
\]

From this, the coarea formula gives the following inequality for the annuli \( A_j := A \cap \Gamma \times \{2^{-j}d(x, y)\} \setminus \Gamma \times \{2^{-j-1}d(x, y)\} \), \( j \in \mathbb{Z}^+ \):

\[
(24) \quad \int_{B_r} \ell(\gamma_2 \cap A) d\mathcal{H}^{n-1}(z) = \sum_{j=0}^{\infty} \int_{B_r} \ell(\gamma_2 \cap A_j) d\mathcal{H}^{n-1}(z) \leq \sum_{j=0}^{\infty} 2^{(j+2)(n-1)} \mu(A_j)
\]

\[\leq 4^{n-1} \int_{A \cap \Gamma \times \{d(x, y)\}} \frac{d(x, z)}{\mu(B_{d(x, y)}(x))} d\mu(z).\]

In the last inequality we used that \( \mu(B_{d(x, y)}(x)) = c_n \cdot d(x, z)^n \), where \( c_n > 0 \) is the volume of the unit ball, to remove the factor \( 2^{j(n-1)} \). For \( A \) closer to \( y \) than to \( x \), we argue similarly with the integrand \( d(y, z)/\mu(B_{d(y, z)}(y)) \) and decomposing a general \( A \) into the two components of points closer to \( x \) or \( y \) we get (19) in (ii). For (i) and (ii) we choose \( C = 4^{n-1}/(c_{n-1} \cdot r^n) + C_0 \).

**Step 2 (\( S \)-Uniform Envelopes)** Since the Gromov compactification \( \overline{X}_G \) of \( X = (H \setminus \Sigma, \langle A \rangle^2 \cdot g_H) \) is homeomorphic to \( (H, d_H) \), there is for any two points \( p, q \in H \) a hyperbolic geodesic \( \gamma_{p,q} \subset X \cup \{p, q\} \subset \overline{X}_G \) that joins these points. This choice yields 2.5(i) for \( \gamma_{p,q} \),

the core curve in the yet to define family \( \Gamma_{p,q} \). Namely, relative to \( (H, g_H) \), each segment \( \gamma_{x,y} \subset \gamma_{p,q} \) joining two points \( x, y \in \gamma_{p,q} \) is \( S \)-uniform, more precisely a \( c \)-\( S \)-uniform curve for some \( c \geq 1 \), that is:

- **Quasi-geodesic:** \( l(\gamma_{x,y}) \leq c \cdot d(x, y) \).

- **Twisted double \( S \)-cones:** \( l_{\min}(\gamma_{x,y})(z) \leq c \cdot \delta(A)(z) \) for any \( z \in \gamma_{x,y} \),

where \( l_{\min}(\gamma_{x,y})(z) := \) minimum of the lengths of the two subcurves of \( \gamma_{x,y} \) from \( x \) to \( z \) and from \( q \) to \( z \). From Prop. 3.11 and Lemma 3.13 of [L1, Ch. 3.2], the constant \( c \) depends only on \( H \) and only on \( n \) for \( H \in \mathcal{H}_n^\mathbb{R} \). Now we use the Lipschitz continuity of \( \delta(A) \) and \( \delta(A) \equiv 0 \) on \( \Sigma \),

\[
(25) \quad |\delta(A)_H(p) - \delta(A)_H(q)| \leq d_H(p, q) \quad \text{for any } p, q \in H \setminus \Sigma \text{ and any } H \in \mathcal{G}_n.
\]

This gives \( \delta(A)_H(p) \leq \text{dist}(p, \Sigma) \), i.e., \( \langle A \rangle_H(p) \geq \text{dist}(p, \Sigma)^{-1} \), and when we multiply (25) by \( \langle A \rangle_H(p) \cdot \langle A \rangle_H(q) \) we get [L1, Lemma B.2]:

\[
(26) \quad \langle A \rangle(q) \leq 2 \cdot \langle A \rangle(p), \quad \text{for any } q \text{ with } d_H(p, q) \leq 1/2 \cdot \delta(A)_H(p).
\]
Now we use the twisted double $S$-cone condition to get a counterpart of $E_{x,y}(\zeta) \subset \mathbb{R}^n$, defined in (21) above, in $(H, d_H)$. The interval $[x, y] \subset \mathbb{R}^n$ is replaced by the core $\gamma_{p,q} \subset H$. The twisted double $S$-cone around $\gamma_{p,q}$ is supported on what we will become the envelope of our Semmes family:

\[(27)\quad E_{p,q}[d] := \bigcup_{z \in \gamma_{p,q} \setminus \{p,q\}} B_{d \cdot l_{\min}(\gamma_{p,q}(z))/c}(z).\]

We observe that for $d \in (0, 1)$, we have $E_{p,q}[d] \subset H \setminus \Sigma$ and $E_{p,q}[d] \cap \Sigma \subset \{p, q\}$ since

\[(28)\quad \text{dist}(z, \Sigma) \geq \delta(A)_H(z) \geq c^{-1} \cdot l_{\min}(\gamma_{p,q}(z)).\]

For any $z \in \gamma_{p,q}$, the balls $B_{d \cdot l_{\min}(\gamma_{p,q}(z))/c}(z) \subset B_{d \cdot \delta(A)(z)}(z)$ scaled to unit size admit common bounds on the geometry independent of $z, p, q$. Concretely, when $B_{d \cdot l_{\min}(\gamma_{p,q}(z))/c}$ is scaled by $(d \cdot l_{\min}(\gamma_{p,q}(z))/c)^{-1}$, we get a ball of radius 1. For the present we write it as $(B^1(z), g^o)$. The twisted double $S$-cone condition and $A,H \equiv \lambda^{-1} \cdot A_H$ show that $A(z) \leq d$ and thus, from (26) and further restricting $d \in (0, 1/2)$, we get $A(x) \leq 2 \cdot d$, for any $x \in B^1(z)$.

Since $|A| \leq \langle A \rangle$ and $|A|$ bounds the norm of the principal curvatures of $B^1(z)$ in its scaled ambient space, the Gauss formulas give us uniform bounds on the sectional curvature. In turn, when $H \in \mathcal{H}_n$, Allard regularity [Si1, Theorem 24.2] shows that for any $\varepsilon > 0$ and any integer $k \geq 0$ there is a small $d \in (0, 1/2)$ so that the exponential map $\exp_z : B_1(0) \to B^1(z)$, for $B_1(0) \subset T_z((d \cdot l_{\min}(\gamma_{p,q}(z))/c)^{-1} \cdot H)$ in the tangent space in $z$, is a $C^k$-diffeomorphisms and

\[(29)\quad |\exp_z(g^o) - g_{E_{act}}|_{C^k(B_1(0))} \leq \varepsilon, \quad \text{for the pull-back } \exp_z(g^o) \text{ of } g^o.\]

This carries over to more general $H \in \mathcal{G}_n, k = 3$, using the tameness assumption of D.2. In particular, for $\varepsilon > 0$ small enough, the exponential map in any such $z$ is a local bi-Lipschitz map with constant $L \leq 2$ on $B_{d \cdot l_{\min}(\gamma_{p,q}(z))/c}(0) \subset T_z H$. Note that the Lipschitz constant $L$ does not change under common rescaling of source and image.

For the next step, we additionally choose $d \leq c/2$. Summarizing, and using [L1, Prop. 2.7 (iv)] and [L1, Lemma 3.13], we see that for given $\varepsilon > 0$ and $k \geq 0$, $d$ can be chosen independently of $p, q \in H$ and for $H \in \mathcal{H}_n$ the parameter $d$ only depends on $n$.

**Step 3 (Semmes Families in $(H, d_H)$)** The envelope $E_{p,q}[d]$ contains the total space of a fibration by curves that defines the desired Semmes families $\Gamma_{p,q}$ on $(H, d_H)$.

Let $\ell$ be the length of $\gamma_{p,q} \subset (H, d_H)$. We parameterize $\gamma_{p,q}$ by arc-length, choose a point $x \in \mathbb{R}^n$ with $|x| = \ell/2$ and the family $\Gamma_{-x,x}(r)$ in $\mathbb{R}^n$ with $r = d \cdot \ell/c \leq \ell/2$. We start with the isometry, similar to a parametrization by arc length,

\[(30)\quad P_{\gamma_{p,q}} : [-x, x] \to \gamma_{p,q}, \quad \text{with } P_{\gamma_{p,q}}(-x) = p \quad \text{and } P_{\gamma_{p,q}}(x) = q.\]

With these choices, we consider *Fermi coordinates* along $[-x, x] \setminus \{-x, x\}$ on $E_{x,y}(\zeta)$ and $\gamma_{p,q} \setminus \{p, q\}$ on $E_{p,q}[d]$ and use them to extend $P_{\gamma_{p,q}}$ to a smooth and $2 \cdot L$-bi-Lipschitz map $P_{\gamma_{p,q}}$ and we define

\[(31)\quad \Gamma_{p,q}[d] := \{ P_{\gamma_{p,q}} \circ \gamma | \gamma \in \Gamma_{x,y}(r) \}.\]
with the measure $\alpha_{p,q}[d]$ on $\Gamma_{p,q}[d]$ induced by the measure $\alpha_{x,-x}[r]$ on $\Gamma_{x,-x}[r]$. For the core curve, the quasi-convexity property (i) follows from the $c$-$\mathcal{S}$-uniformity (and thus the $c$-quasi-geodesy of any of the subcurves). This extends to the other curves in $\Gamma_{p,q}[d]$ from the bi-Lipschitz maps $Q_{\gamma_z} : [x, y] \to \gamma_z$, $z \in B$, we used in the definition of the Euclidean model families and the bi-Lipschitz maps $P_{\gamma_{p,q}}$. For property (ii) we note that the left and the right hand side of (24) are merely changed by (powers of) the bi-Lipschitz constants and they remain to be Borel measures.

2.2 $S$-Doob Transforms and Upper $d_S$-Estimates

Now we turn to analytic applications of hyperbolic unfoldings. The main reason for our interest in these unfoldings is that the potential theory of $\delta_{\langle A \rangle^*}^2 L_{H,\lambda}$ on $(H \setminus \Sigma, d_{\langle A \rangle^*})$ is nicely structured towards the Gromov boundary from the work of Ancona, cf. [An1], [An2] and [KL].

To formulate associated boundary regularity results for $(H \setminus \Sigma, d_H)$ along $\Sigma$, we briefly look at the standard conformal deformation of the flat unit disk $D$ to the hyperbolic plane. In this Poincaré metric, the intersections $B \cap D$ of flat Euclidean discs $B$ centered in boundary points $\partial D$ become hyperbolic halfspaces.

This suggests the following generalization: we consider hyperbolic halfspaces $U \subset (H \setminus \Sigma, d_{\langle A \rangle^*})$. They are generally not quite conformally equivalent to the distance balls in $(H, d_H)$, but it turns out that the $U$, and not the distance balls in $(H, d_H)$, are the adequate choice for analytic estimates on $(H, d_H)$ towards $\Sigma_H$. To define these halfspaces, we recall that for any $p, y, z \in (H \setminus \Sigma, d_{\langle A \rangle^*})$, the Gromov product of $y$ and $z$ with respect to $p$ is defined as

$$ (y \cdot z)_p := \frac{1}{2} \cdot (d_{\langle A \rangle^*}(p, y) + d_{\langle A \rangle^*}(p, z) - d_{\langle A \rangle^*}(y, z)) $$

This product measures how long two geodesic rays from $p$ to $y$ and from $p$ to $z$ travel together before they diverge (and, adding $\gamma_{y,z}$, form a $\delta$-thin triangle), cf. [GH, Lemme 2.17]:

$$ (y \cdot z)_p \leq d_{\langle A \rangle^*}(p, \gamma_{y,z}) \leq (y \cdot z)_p + 4 \cdot \delta. $$

We use the product to describe some neighborhood bases [BH, Lemma III.H.3.6]:

**Lemma 2.7 (Neighborhood Bases via Gromov Product)** We define the following subsets for some $z \in \partial_G(H \setminus \Sigma, d_{\langle A \rangle^*})$ and $a > 0$:

- $U(z, a) := \{ x \in \partial_G(H \setminus \Sigma, d_{\langle A \rangle^*}) \mid$ there are geodesic rays $\gamma_1, \gamma_2$ from $p$ to $\gamma_1(\infty) = z$, $\gamma_2(\infty) = x$ such that $\liminf_{t \to \infty}(\gamma_1(t) \cdot \gamma_2(t))_p \geq a \} \subset \partial_G(H \setminus \Sigma, d_{\langle A \rangle^*})$.

- $U(z, a) := \{ x \in (H \setminus \Sigma, d_{\langle A \rangle^*}) \mid$ there is a geodesic ray $\gamma$ starting from $p$ to $\gamma(\infty) = z$ such that $\liminf_{t \to \infty}(\gamma(t) \cdot x)_p \geq a \} \subset (H \setminus \Sigma, d_{\langle A \rangle^*})$.

- $\mathbb{U}(z, a) := U(z, a) \cup U(z, a) \subset (H \setminus \Sigma, d_{\langle A \rangle^*})$

Then the $U(z, a)$ and $\mathbb{U}(z, a)$ are neighborhood bases of $z$ in their respective space.
There are several essentially equivalent ways to define such halfspaces, cf. [BHK, Lemma 8.3], [An2, Ch. 5.2]. [L2] uses the more direct definition \( \{ x \in H \setminus \Sigma \mid \text{dist}(x, \gamma([t,a])) < \text{dist}(x, \gamma((0,t])) \} \). Here we rather use \( U(z,a) \). This makes it easier to derive estimates for the minimal factor metrics.

Now we reach the central boundary regularity results [L2, Theorem 3.4 and 3.5], considering the singular set \( \hat{\Sigma} \) as a boundary of its regular complement \( H \setminus \Sigma \). We get them by transferring the theory on the hyperbolic unfolding back to \( L_{H,\lambda} \) on \( (H \setminus \Sigma, g_H) \). As in [L2, Theorem 3.4 and 3.5] we select uniformly spaced neighborhood bases around points \( z \in \hat{\Sigma} \):

\[
U_k(z) := U(z, c \cdot k) \quad \text{and} \quad \hat{U}_k(z) := \hat{U}(z, c \cdot k) \quad \text{for } k = 1, 2, \ldots
\]

where \( c > 0 \) depends only on the hyperbolicity constant \( \delta \) of \( (H \setminus \Sigma, d_{(\lambda)}) \). Since the hyperbolicity constant \( \delta \) is already determined from Prop. 2.3 we no longer mention it in our discussions.

**Theorem 2.8 (Boundary Harnack Inequality)** For \( H \in \mathcal{G} \) and \( L_{H,\lambda} S \)-adapted, let \( u, v > 0 \) be two supersolutions of \( L_{H,\lambda} \phi = 0 \) on \( H \setminus \Sigma \) both solving \( L_{H,\lambda} \phi = 0 \) on \( U_k(z) \subset H \setminus \Sigma \), around some \( z \in \hat{\Sigma} \) and for some \( k = 1, 2, \ldots \), with minimal growth towards \( U_k(z) \). Then there is a constant \( C = C(H) > 0 \), and \( C = C(n) > 0 \) when \( H \in \mathcal{H}_n^\mathbb{R} \), independent of \( k \), so that we have:

\[
u(x)/v(x) \leq C \cdot u(y)/v(y), \quad \text{for all } x, y \in U_{k+1}(z).
\]

The same inequality holds for \( u, v \) considered as solutions of \( \delta_{(\lambda)}^2 \cdot L_{H,\lambda} \phi = 0 \) on \( (H \setminus \Sigma, d_{(\lambda)}) \).

**Remark 2.9 (Inequalities along Arcs)** The validity of (35) does not depend on the interpretation of \( U_k(z) \) as neighborhoods of the singular point \( z \). Technically, it is the relative position of \( \partial U_{k+1}(z) \) to \( \partial U_k(z) \) in a hyperbolic space that yields (35). Therefore (35) equally holds for any hyperbolic geodesic arc \( \gamma : [0, (m+2) \cdot c + \eta] \rightarrow (H \setminus \Sigma, d_{(\lambda)}) \), for some \( \eta > 0 \), \( m \in \mathbb{Z}^+ \), of finite length from \( p = \gamma(0) \) to \( z = \gamma((m+2) \cdot c + \eta) \) when \( p, z \in H \setminus \Sigma \), cf. [An2, Théorème 6.1], [L2, Lemma 2.4 and Theorem 2.12]. The extended description of the halfspaces \( U_k(\gamma) \) now reads:

- \( U_k(\gamma) := \{ x \in (H \setminus \Sigma, d_{(\lambda)}) \mid (\gamma((k+1) \cdot c) \cdot x)_p \geq k \cdot c \} \), for \( k = 1, 2, \ldots m \).

We note in passing that, in general, \( (H \setminus \Sigma, d_{(\lambda)}) \) is not a visual metric space. That is, a finite arc \( \gamma \) does not necessarily admit extensions to complete geodesic rays or approximations by such rays.

Typical supersolutions satisfying the assumptions of Thm. 2.8 are the minimal Green’s function \( G(\cdot, p), p \in H \setminus \Sigma \), the functions \( S \lambda \), from Example 1.3 (4) and the functions \( \Phi \) from Def. 1.1 of minimal factor metrics. The boundary Harnack inequality shows that around any \( z \in \Sigma \subset H \) there is a small neighborhood so that \( \Phi \leq c \cdot G(\cdot, p) \) for some constant \( c > 0 \). We show that \( c \) is essentially independent of \( p \) and \( z \).
Proposition 2.10 (Harnack Estimate along Geodesics) We consider the two cases where

(i) $H \in \mathcal{H}_n^g$ is non–totally geodesic, (ii) $H \in \mathcal{G}_n^c$ is singular.

Let $p, z \in H \setminus \Sigma$ be points with $\langle A \rangle(p) \leq a$, for some $a > 0$, and $d_H(p, z) \leq 1$. Then, there is a constant $\xi > 0$ with $\xi = \xi(n, a)$ in case (i) and $\xi = \xi(H, \Phi, a)$ in case (ii) so that

$$\Phi(z) \leq \xi \cdot \Phi(p) \cdot G(z, p), \text{ for any } p, z \in H \setminus \Sigma.$$ 

**Proof** The idea is to link $p \in H \setminus \Sigma$ and $z \in H$ with a hyperbolic geodesic arc $\gamma_{p, z}$ in $(H \setminus \Sigma_H, d_{\langle A \rangle})$, where we append $z$ when $z \in \Sigma$. Then we use the Gromov product to see that (36) can be derived from a combination of the ordinary and the boundary Harnack inequality. This also exploits that $\gamma_{p, z}$ is a $c$-$\mathcal{S}$-uniform arc in $(H, d_H)$, for some $c(H) > 0$, with $c(n)$ for $H \in \mathcal{H}_n^g$. $l_{\min}(\gamma_{p, z}(x)) \leq c \cdot \delta_{\langle A \rangle}(x)$, for any $x \in \gamma_{p, z}$, and $l(\gamma_{p, z}) \leq c \cdot d(p, z)$.

- When $z \in H \setminus \Sigma$ with $\langle A \rangle(z) \leq b$, for some $b > 0$, we use the $\langle A \rangle$-bounds in $p$ and $z$ and a suitable collection $\mathcal{B}(p, z)$ of balls from the envelope $E_{p, z}[d]$ to build a finite ball cover of $\gamma_{p, z}$

$$B_{d, \delta_{\langle A \rangle}(p)}(p), B_{d, \delta_{\langle A \rangle}(z)}(z), B_{d, l_{\min}(\gamma_{p, z}(x_j))}(x_j), j = 1, \ldots, m,$$

for some integer $m(a, b, c)$ that merely depends on $a, b, c$ but not on $p, z$ or $H$. Since $d \in (0, 1/2)$, the balls $B_{d/2, d, \delta_{\langle A \rangle}(p)}(p), \ldots$ still belong to $H \setminus \Sigma$. Thus we have a Harnack constant $\kappa(H, a, b, c, d) > 0$ resp. $\kappa(n, a, b, c, d) > 0$ that applies to any solution $v > 0$ of $L_{H, \lambda, \phi} = 0$ on $H \setminus \Sigma$ on each of the balls in $\mathcal{B}(p, z)$ and we get $sup_{\bigcup \{B \in \mathcal{B}(p, z)\}} v \leq \kappa^{m+2} \cdot \inf_{\bigcup \{B \in \mathcal{B}(p, z)\}} v$.

In the case of $H \in \mathcal{G}_n^c$, the set where $\Phi$ is not a proper solution is included in some compact set $K \subset H \setminus \Sigma$ where we get such a constant $\kappa$ from the continuity of $\Phi$. Thus we always have

$$\sup_{\bigcup \{B \in \mathcal{B}(p, z)\}} \Phi \leq \kappa^{m+2} \cdot \Phi(p).$$

Now we turn to case (i) and assume that (36) does not hold on $\mathcal{H}_n^g$. Then we have a sequence $H_i \in \mathcal{H}_n^g$ converging to some $H_\infty \in \mathcal{H}_n^g$ and pairs of points $p_i, z_i$ with $\langle A \rangle(p_i) \leq a$ and $d_{H_i}(p_i, z_i) = 1$ (since scaling by a constant $\geq 1$ only decreases $\langle A \rangle(p_i)$) so that the $p_i$ ID-map-converge to some $p_\infty \in H_\infty \setminus \Sigma_{H_\infty}$ with $\langle A \rangle(p_\infty) \leq a$. Moreover, we can assume that the $H_i$ are equipped with minimal growth solutions $\Phi_i > 0$, with $\Phi_i(p_i) = 1$, and minimal Green’s function $G_i$ both compactly ID-map-converging to $\Phi_\infty > 0$, with $\Phi_\infty(p_\infty) = 1$ and the minimal Green’s function $G_\infty$ on $H_\infty$, respectively, with

$$\Phi_i(z_i)/G_i(z_i, p_i) \to \infty, \text{ for } i \to \infty.$$ 

We may assume that there are hyperbolic geodesic arcs $\gamma_i$ of length $L_i \in (0, \infty]$ in $(H_i \setminus \Sigma_{H_i}, d_{\langle A \rangle})$ from $p_i$ to $z_i$ ID-map-converging to a hyperbolic geodesic arc $\gamma_\infty$ from $p_\infty$ to some $z_\infty$ with $d_{H_\infty}(p_\infty, z_\infty) = 1$. Then we have $z_\infty \in \Sigma_{H_\infty}$, otherwise the $z_i$ would subconverge to some $z_\infty \in H_\infty \setminus \Sigma_{H_\infty}$ and hence we would have $\langle A \rangle(z_i) \leq \langle A \rangle(z_\infty) + 1 < \infty$ for $i$ large enough and could infer from (38) that $\Phi_i(z_i)/G(z_i, p_i)$ remained bounded for $i \to \infty$ contradicting (39). Thus $\delta_{\langle A \rangle}(z_i) \to 0$ and $L_i \to \infty$ for $i \to \infty$. Now we consider a hyperbolic geodesic ray $\gamma_i^*$ from $p_i$ to $\infty$ and compare it with $\gamma_i$: 18
we have \( l_{H_i}(\gamma_i^*) = \infty \) and, hence, the \( c\)-\( \mathcal{S} \)-uniformity of \( \gamma_i^* \) shows that for \( \gamma_\infty(0) = p_\infty \),

\[
(40) \quad l_{H_i}(\gamma_i^*(t)) \leq c \cdot \delta_{(A)}(\gamma_i^*(t)) \quad \text{for any } t \geq 0,
\]

and, thus, for \( d_{H_i}(\gamma_i^*(0), \gamma_i^*(t)) \geq 1/2 \) we have \( \delta_{(A)}(\gamma_i^*(t)) \geq (2 \cdot c)^{-1} \)

- from \( |\delta_{(A)}(a) - \delta_{(A)}(b)| \leq d_{H_i}(p, q), a,b \in H_i \setminus \Sigma_{H_i} \) we get for \( \varepsilon > 0 \), large enough \( i \) and \( s \in (0, L_i) \):

\[
(41) \quad \delta_{(A)}(\gamma_i(s)) \leq d_{H_i}(\gamma_i(s), \gamma_i(L_i)) + \varepsilon.
\]

Since the \( \gamma_i \) compactly \( \text{ID} \)-map-approximate the hyperbolic geodesic arc \( \gamma_\infty \) from \( p_\infty \) to \( z_\infty \), outside \( z_\infty \), we get from (40) and (41) for any \( \varepsilon > 0 \) and sufficiently large \( i \) that \( \gamma_i^* \) diverges from \( \gamma_i \) after some uniformly upper bounded time \( s_0 \) so that

\[
(42) \quad \delta_{(A)}(\gamma_\infty(s)) \leq d_{H_\infty}(\gamma_\infty(s), z_\infty) \leq (2 \cdot c)^{-1} \cdot \eta, \quad \text{for some small } \eta > 0 \text{ and } s \geq s_0.
\]

Namely, we see from (33) and the relation [L1, Lemma 3.12]

\[
(43) \quad |\log \delta_{(A)}(a) - \log \delta_{(A)}(b)| \leq d_{(A)}(a,b), \quad \text{for any } a,b \in H_\infty \setminus \Sigma_{H_\infty},
\]

that for small \( \eta \), there is a \( k_0 \) so that, for large \( i \) and then independent of \( i \), the \( U_k(\gamma_i) = \{ x \in (H \setminus \Sigma, d_{(A)}) \mid (\gamma_i((k+1) \cdot c) \cdot x)_{p_i} \geq k \cdot c \} \), for \( k = 1, 2, \ldots m(i), m(i) \to \infty \) when \( i \to \infty \), satisfy

\[
(44) \quad \gamma_i^* \cap U_k(\gamma_i) = \emptyset \quad \text{for } k \geq k_0.
\]

Thus we can apply the boundary Harnack inequality (35) to \( \Phi \) and \( G \) on these \( U_k(\gamma_i) \) where both functions are solutions of minimal growth towards the extension of \( U_k(\gamma_i) \) to \( \Sigma_{H_i} \). The point about the independence of \( k_0 \) from \( i \) is that the relation (43) and the estimate (33) also show that there is a constant \( b > 0 \) independent of \( i \) so that \( (A)(q_i) \leq b \) for the first point \( q_i \in \gamma_i \) we reach in \( \gamma_i \cap \partial U_k(\gamma_i) \) starting from \( p_i \).

Thus following \( \gamma_i \) from \( p_i \) to \( z_i \), we use the chain (38) of ordinary Harnack estimates until we reach \( q_i \). After passing \( q_i \) we use the boundary Harnack estimate (35) on \( U_k(\gamma_i) \) to get an upper estimate for \( \Phi_i(z_i)/G_i(z_i, p_i) \), independent of \( i \). This contradiction settles case (i).

In case (ii) argue similarly using chains (38) of Harnack estimates on \( H_i = a_i \cdot H \) with \( a_i := d_H(p_i, z_i)^{-1} \geq 1 \). In place of (44) we find linking geodesics \( \gamma_i \) and a \( k_0 \), independent of \( i \), so that

\[
(45) \quad K \cap U_k(\gamma_i) = \emptyset \quad \text{for } k \geq k_0,
\]

and derive a contradiction as in case (i).

\[\square\]

To derive estimates for the minimal Green’s function of an elliptic operator \( L \) we use \textit{Doob transforms}, also called \( h \)-\textit{transforms} \( L^h \). They are standard transformations from stochastic analysis for operators \( L \) on function spaces: for smooth \( h \) on \( H \setminus \Sigma \), defined as \( L^h := h^{-1} L h \), i.e., for smooth \( g \) we set \( L^h g(x) := h^{-1}(x) L(h \cdot g)(x) \). We choose \( L = L_{H,\lambda} \) and \( h = \delta_{(A)}^{-\frac{(n-2)}{2}} \).
Proposition 2.11 (S-Doob Transforms) For any $H \in \mathcal{H}_n$ we define the \textbf{S-Doob Transform} $L^S = L^S_{H,\lambda}$ on $(H \setminus \Sigma, d_{(A)^*})$ of $L = L_{H,\lambda}$ to be the following Schrödinger operator on suitably regular functions (for the present, we consider $C^2$-functions $\phi$):

\begin{align}
L^S \phi := \delta_{(A)^*}^2 \cdot L_{H,\lambda}^{\delta_{(A)^*}^{-2}} \phi = \delta_{(A)^*}^{(n+2)/2} \cdot L(H,\lambda)^{\delta_{(A)^*}^{-2}} \cdot \phi = \\
\delta_{(A)^*}^2 \cdot \left(-\Delta_{(A)^*}^2 \cdot g_H \phi + \left(\frac{n-2}{4(n-1)} \cdot \text{scal}_H - \delta_{(A)^*}^{-2} \cdot \Delta_{(A)^*}^2 \cdot \delta_{(A)^*}^{-2} \right) \phi \right).
\end{align}

- For a function $v > 0$, we set $u := \delta_{(A)^*}^{-2} \cdot v$. Then $u^{4/(n-2)} \cdot g_H = v^{4/(n-2)} \cdot \langle A \rangle^2 \cdot g_H$ and

\begin{align}
\text{(47) } u \text{ solves } L \phi = 0 \iff v \text{ solves } L^S \phi = 0, \quad Lu \geq c \cdot \delta_{(A)^*}^{-2} \cdot u \iff L^S v = \delta_{(A)^*}^{(n+2)/2} \cdot Lu \geq c \cdot v,
\end{align}

for some $c > 0$. That is, $L$ is $S$-adapted $\iff L^S$ is adapted and weakly coercive.

- The operator $L^S$ is symmetric on $(H \setminus \Sigma, d_{(A)^*})$ and, thus, the minimal Green’s function $G^S$ of $L^S$ on $(H \setminus \Sigma, d_{(A)^*})$ satisfies $G^S(x, y) = G^S(y, x)$, for $x \neq y \in H \setminus \Sigma$.

- The minimal Green’s functions $G$ of $L$ on $(H \setminus \Sigma, d_H)$ and $G^S$ of $L^S$ on $(H \setminus \Sigma, d_{(A)^*})$ satisfy

\begin{align}
G^S(x, y) = \delta_{(A)^*}^{(n-2)/2} (x) \cdot \delta_{(A)^*}^{(n-2)/2} (y) \cdot G(x, y), \quad \text{for } x \neq y \in H \setminus \Sigma.
\end{align}

- Assume that there is some $v > 0$ so that for some $c > 0$: $L^S v \geq c \cdot v$. Then we have:

(i) for any singular $H \in \mathcal{G}^c_n$ there are constants $\beta(H, c), \alpha(H, c), \sigma(H, c) > 0$ and

(ii) for any non–totally geodesic $H \in \mathcal{H}^R_n$ there are $\beta(n, c), \alpha(n, c), \sigma(n, c) > 0$ so that

\begin{align}
\text{(49) } G^S(x, y) \leq \beta \cdot \exp(-\alpha \cdot d_{(A)^*} (x, y)), \quad \text{for } x, y \in H \setminus \Sigma \text{ and } d_{(A)^*} (x, y) > 2 \cdot \sigma.
\end{align}

**Proof** The formula (46) for $L^S$ and the relations (47) follow from straightforward computations. The equivalence of the $S$-adaptedness of $L$ and the adaptedness of $L^S$ follows easily from the definitions cf. [L2, Prop. 3.3] for details.

The symmetry of $L^S$ is seen as follows: for any two functions $u, v \in C^\infty_0 (H \setminus \Sigma, \mathbb{R})$ we have, writing $dV = \delta_{(A)^*}^{-n} \cdot dV$ for the volume element associated to $\langle A \rangle^2 \cdot g_H$:

\begin{align}
\int_{H \setminus \Sigma} u \cdot L^S v \cdot dV = \int_{H \setminus \Sigma} u \cdot \delta_{(A)^*}^{(n+2)/2} \cdot L_{H,\lambda}^{\delta_{(A)^*}^{-2}} \cdot \delta_{(A)^*}^{-n} \cdot dV = \int_{H \setminus \Sigma} L_{H,\lambda}^{\delta_{(A)^*}^{-2}} \cdot u \cdot \delta_{(A)^*}^{(n-2)/2} \cdot v \cdot dV.
\end{align}

For the transformation law (48), we recall that one may characterize $G^S(x, y)$ as the unique function that solves

\begin{align}
L^S_x \int_{H \setminus \Sigma} G^S(x, y) u(y) dV^S(y) = u(x) \text{ for any } u \in C^\infty_0 (H \setminus \Sigma, \mathbb{R}).
\end{align}

We insert the right hand side to check this identity from the identity for $G$ relative to $L$:

\begin{align}
\delta_{(A)^*}^{(n+2)/2} (x) L_x \int_{H \setminus \Sigma} \delta_{(A)^*}^{(n-2)/2} (y) G(x, y) \cdot u(y) \cdot \delta_{(A)^*}^{-n} (y) dV (y) = \delta_{(A)^*}^{(n+2)/2} (x) u(x) \delta_{(A)^*}^{-n} (x) = u(x).
\end{align}
Finally, we get \((49)\) from the work of Ancona \([\text{An1, Ch. 2}]\), carried out in more detail in \([\text{KL, Proposition 2.13}]\), and \(G^S(q, p) = G^S(p, q)\). These results apply due to the uniform ellipticity (more precisely described as adaptedness in \([\text{An1}]\) and \([\text{KL}]\)) of \(L^S\) on \((H \setminus \Sigma, (\langle A \rangle^*)^2 \cdot \gamma)\) and the weak coercivity assumption we made for \(L^S\).

\[\square\]

**Remark 2.12** 1. The Whitney smoothed \(\langle A \rangle^*\) satisfies the approximative naturality axiom (13). This implies that estimates expressed in terms of \(\langle A \rangle^*\) can equally be used for \(\langle A \rangle\) and vice versa. Analytically, \(\langle A \rangle^*\) is merely used indirectly to unfold \((H \setminus \Sigma, g_H)\) to a smooth manifold. This is a common way to bypass regularity issues from the weaker Lipschitz regularity of \(\langle A \rangle\) in the potential theoretic analysis. The appearance of Whitney smoothings cancels out on the way back from the unfolding to \((H \setminus \Sigma, g_H)\), cf. \([\text{L1, Ch. 3}]\) for details.

2. The constants \(\beta(H, c), \alpha(H, c), \sigma(H, c) > 0\), or \(\beta(n, c), \alpha(n, c), \sigma(n, c) > 0\) when \(H \in \mathcal{H}^R_n\), in Prop. 2.11 (49) come from the potential theory on the hyperbolic unfolding. They are determined from the hyperbolicity and bounded geometry constants for the hyperbolic unfolding in Prop. 2.3. \(\hat{H}\) and any scaled version \(\lambda \cdot \hat{H}, \lambda > 0\) have the same unfolding since \(\langle A \rangle_{\lambda \cdot \hat{H}} = \lambda^{-1} \cdot \langle A \rangle_{\hat{H}}\). Thus these constants remain unchanged under scalings and blow-ups of \(\hat{H}\).

\[\square\]

**Corollary 2.13 (Upper \(d_S\)-Estimates)** For \(H \in \mathcal{G}\), \(p \in H \setminus \Sigma\) and \(B = B_{2, \sigma}(p)\) measured relative to the \((\sigma, \ell)\)-bounded geometry \(d_{\langle A \rangle}\), we have: the distance of any two \(x, y \in H \setminus B\) with respect to \(G(\cdot, p)^{4/(n-2)} \cdot g_H\) is upper bounded by some constant \(D_H > 0\) and \(D_n > 0\) when \(H \in \mathcal{H}^R_n\).

**Proof** We apply Prop. 2.11 (49) and (48) to estimate \(G\). Any two points \(x, y \in (H \setminus \Sigma, d_{\langle A \rangle})\) can be joined by a hyperbolic geodesic arc \(\gamma_{x,y}\), with \(\gamma_{x,y}(0) = x\). By definition these arcs are parameterized by arc length and we have \(d_{\langle A \rangle}(x, y) = l_{d_S}(\gamma_{x,y})\). We have \(t = d_{\langle A \rangle}(\gamma_{p,y}(t), p)\), \(g_H(\gamma(t), \dot{\gamma}(t)) = \delta^2_{\langle A \rangle}(\gamma(t))\) and hence for any \(z = \gamma_{x,y}(t)\) with \(d_{\langle A \rangle}(p, z) \geq 2 \cdot \sigma\),

\[
G(z, p)^{1/2} \cdot g_H(\dot{\gamma}, \dot{\gamma}) = (\langle A \rangle^*)^2(p) \cdot G^S(z, p)^{1/2} \\
\leq (\langle A \rangle^*)^2(p) \cdot c \cdot \left(\exp(-\alpha \cdot d_{\langle A \rangle}(\cdot, z))\right)^{4/(n-2)}.
\]

Since \((\exp(-\alpha \cdot t))^{2/(n-2)}\) is integrable on \(\mathbb{R}^\geq_0\), the integral of \((\exp(-\alpha \cdot d_{\langle A \rangle}(\cdot, p)))^{2/(n-2)}\) along \(\gamma_{x,y}|_{(2, \sigma, d_{\langle A \rangle})_{\langle y, p \rangle}}\) is uniformly upper bounded for any \(y \in (H \setminus \Sigma \cup B)\).

For those parts of such a hyperbolic geodesic passing through \(B\) we get uniform upper bounds using the bounded geometry on \(B = B_{2, \sigma}(p)\) and the fact that \(G^S(\cdot, p) \leq c_0\) on \(\partial B_\sigma(p)\), for some \(c_0 > 0\) that is independent of \(p \in H \setminus \Sigma\). On \(\mathcal{H}^R_n\) the constant \(c_0\) is even independent of \(H\) \([\text{KL, Prop. 2.8}]\).

\[\square\]

The estimate (50) is used quite frequently throughout this paper. It also gives upper topological bounds for our minimal factors. This is counterbalanced by lower bounds we get from the \(\lambda > 0\)-condition. This condition allows us to use the Bombieri–Giusti Harnack inequality \([\text{BG}]\) e.g. in Lemma 2.15 and Prop. 2.16 below.

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Lemma 2.14 (Diameter Bounds) For $H \in \mathcal{G}$ we have the following estimates:

(i) For $H \in \mathcal{G}^c$ we have $\text{diam}(H, d_H) < \infty$ and $\text{diam}(\widehat{H \setminus \Sigma}, \widehat{d_S}) < \infty$.

(ii) For $H \in \mathcal{G}$ and balls $(B_r(z), d_H)$, we have $\text{diam}(B_r(z) \setminus \Sigma, \widehat{d_S}) \to 0$ for $r \to 0$.

(iii) There is a continuous map $\widehat{I}_H : (H, d_H) \to (\widehat{H \setminus \Sigma}, \widehat{d_S})$ that canonically extends the identity map $\text{id} : (H \setminus \Sigma, d_H) \to (H \setminus \Sigma, d_S)$.

**Proof** We start with (i). The assertion that, for $H \in \mathcal{G}^c$, the intrinsic diameter $\text{diam}(H, d_H)$ is finite is [L1, Theorem 1.8]. We choose a basepoint $p \in H \setminus \Sigma$ and consider the ball $B = B_{2\sigma}(p)$ with radius measured relative to $d_{(A)\star}$, where $\sigma$ is the bounded geometry constant for $d_{(A)\star}$ as used in Prop. 2.11 (49). From the boundary Harnack inequality 2.8 and the compactness of $\Sigma$ we have a constant $a > 0$ so that $\Phi(x) \leq a \cdot G(x, p)$ for any $x \in H \setminus (\Sigma \cup B)$. From this, 2.13 implies that $\text{diam}((H \setminus \Sigma, d_S)) < \infty$ and we infer for the metric completion $\text{diam}((\widehat{H \setminus \Sigma}, \widehat{d_S})) < \infty$.

For (ii) we only need to consider $z \in \Sigma$. The boundary Harnack inequality can be used to upper estimate $\Phi$ by $G(\cdot, p)$ near $z$. The argument of part (i) shows that the integral of $(\exp(-\alpha \cdot d_{(A)\star}(\cdot, p))^{2/(n-2)}$ along $\gamma_{x,y}|_{[2\sigma,d_{(A)\star}(y,p)]}$ is uniformly upper bounded for any $y \in H \setminus (\Sigma \cup B) \cap B_R(z)$ and fixed $R > 0$. From this we have $\text{diam}(B_r(z) \setminus \Sigma, \widehat{d_S}) \to 0$ for $r \to 0$.

For (iii) we use that $\Sigma \subset (H, d_H)$ is homeomorphic to $\partial_G(H \setminus \Sigma, d_{(A)\star}) \setminus \{\infty\}$, where $\partial_G$ denotes the Gromov boundary, to define a canonical continuous map

$$I_H : (H, d_H) \to (\widehat{H \setminus \Sigma}, \widehat{d_S}).$$

From (50) we see that $\text{diam}((U(z, a), \widehat{d_S})) \to 0$ for $a \to \infty$. This shows that $\bigcap_{a \geq 0} \overline{U(z, a)}$, where $\overline{U(z, a)}$ is the completion in $(\widehat{H \setminus \Sigma}, \widehat{d_S})$, contains exactly one point $z_{(A)} \in H \setminus \Sigma \setminus (H \setminus \Sigma)$, the new singular set, and we define

$$I_H : \Sigma \to \widehat{H \setminus \Sigma \setminus (H \setminus \Sigma)}, \text{ by } I_H(z) := z_{(A)}, \text{ for } z \in \Sigma.$$  

$I_H$ extends the identity map on $H \setminus \Sigma$ to a map $\widehat{I}_H : (H, d_H) \to (\widehat{H \setminus \Sigma}, \widehat{d_S})$. From the previous discussion we also see, for a converging sequence $a_k \to a$ in $(H, d_H)$, that $d_S(\widehat{I}_H(a_k), \widehat{I}_H(a)) \to 0$ for $k \to \infty$. That is, $\widehat{I}_H$ is a continuous map. 

As a preparation to show that $\widehat{I}_H$ is a homeomorphism we derive the following coarse estimate.

Lemma 2.15 (Growth Rate for $H \in \mathcal{H}_n^R$) For any non–totally geodesic $H \in \mathcal{H}_n^R$ and any $\lambda \in (0, \lambda_H^{(A)})$ there is some constant $\beta(H, \lambda) > 0$ so that for the Euclidean ball $B_R(0)$ and $R > 0$ large enough we have:

$$\text{dist}_{d_S}(\partial B_R(0) \cap H \setminus \Sigma, \{0\}) \geq R^\beta.$$
In our applications we are not interested in the actual value of \( \beta > 0 \) but in the consequence that, for \( R \to \infty \), the \( d_H \)-distance spheres of radius \( R \) do not have \( d_S \)-accumulation points. We need this since, at this stage, \( \partial S \) for \( H \in \mathcal{H}_n^\mathbb{R} \) could contain points at infinity of \( (H \setminus \Sigma, g_H) \).

**Proof** In the cone case \( H = C \), we know from [L3, Th. 4.4] that in terms of polar coordinates \((\omega, r) \in \partial B_1(0) \cap C \setminus \Sigma \times \mathbb{R}^n \cong C \setminus \Sigma\):

\[
\Phi_C(\omega, r) = \psi(\omega) \cdot r^{\alpha_+} \text{ for } \alpha_+ = -\frac{n-2}{2} + \sqrt{(\frac{n-2}{2})^2 + \mu} < 0,
\]

where \( \mu(C, \lambda) > -(n-2)^2/4 \) is the ordinary (unweighted) principal eigenvalue of an associated elliptic operator on \( \partial B_1(0) \cap C \). Now we recall the inheritance result [L3, Th. 3.13] for cones in \( \mathcal{SC}_n \). Since this space is compact, we get a constant \( A[n, \lambda] > 0 \) depending only on \( n \) and \( \lambda \) so that

\[
-\frac{n-2}{2} < -A[n, \lambda] < \alpha_+(C, \lambda) < 0, \text{ for any } C \in \mathcal{SC}_n,
\]

that is, \(-A[n, \lambda] \cdot 2/(n-2) > -1 \) and \((r^{-A[n,\lambda]})(n-2)\) is not integrable over \([1, \infty)\).

For a general \( H \in \mathcal{H}_n^\mathbb{R} \), we get using tangent cones at infinity, cf. E.2 above, that for any \( \omega > 0 \), there are a constant \( k > 0 \) and some radius \( \rho > 0 \) so that

\[
r^{-A[n,\lambda]} \leq k \cdot \Phi(x) \text{ for } x \in \mathbb{P}(0, \omega) \setminus B_\rho(0) \text{ with } r = d_H(x, 0).
\]

Thus we get a constant \( k^* \) so that for \( r \geq \rho \):

\[
r^{-n} \cdot r^{n-1} \cdot r^{-A[n,\lambda]} \cdot r \leq k^* \cdot \text{Vol}(B_r(z))^{-1} \cdot \int_{B_r(z) \cap H \setminus \Sigma} \Phi \, dV.
\]

Now we note that \( \text{scal}_H = -|A|^2 \) and since \( \lambda > 0 \), we have

\[
\Delta_H \Phi = -\left(\frac{n-2}{4(n-1)} \cdot |A|^2 + \lambda \cdot \langle A \rangle^2 \right) \cdot \Phi \leq 0,
\]

that is, \( \Phi \) is superharmonic on \( H \setminus \Sigma \). Therefore, the Bombieri–Giusti Harnack inequality [BG, Th. 6] applies to \( \Phi \). Indeed \( \Phi \) meets the requirements of that Harnack inequality since we have \( \Delta_H \Phi \leq 0 \) not only on \( B_r(z) \cap H \setminus \Sigma \) but \( \Phi \) is also a weak supersolution on \( B_r(z) \cap H \). This follows from a standard cut-off argument that uses that the Hausdorff codimension of \( \Sigma \) is \( > 2 \). This argument is carried out in detail e.g. in [Si2, p.334].

From this Harnack inequality we get a constant \( c > 0 \) independent of \( H \in \mathcal{H}_n^\mathbb{R} \) and of \( r \geq \rho \) so that

\[
r^{-A[n,\lambda]} \leq k^* \cdot \text{Vol}(B_r(z))^{-1} \cdot \int_{B_r(z) \cap H \setminus \Sigma} \Phi \, dV \leq k^* \cdot c \cdot \inf_{B_r(z) \cap H \setminus \Sigma} \Phi.
\]

This and (55) yield some \( \beta > 0 \) so that for \( R > 0 \) large enough,

\[
d_{ds}(\partial B_R(0) \cap H \setminus \Sigma; \{0\}) \geq R^\beta.
\]

With these results we can prove the following.
Proposition 2.16 (Topology of Minimal Factor Metrics) For any $H \in \mathcal{G}$ we have:

(i) For any $z \in H$ and $r > 0$, there are an outer $d_S$-radius $0 < r_{\text{out}}(r, z) < \infty$ and an inner $d_S$-radius $0 < r_{\text{inn}}(r, z) < \infty$ so that $(B_{r_{\text{inn}}}(z), d_S) \subset (B_r(z), d_H) \subset (B_{r_{\text{out}}}(z), d_S)$.

(ii) $(\overline{H \setminus \Sigma}, d_S(\Phi))$ is homeomorphic to $(H, d_H)$. Thus we can write it as $(H, d_S)$. In particular, we have that the singular set of $(H, d_S)$,

\begin{equation}
\Sigma_S := \overline{H \setminus \Sigma \setminus (H \setminus \Sigma)} \subset (H \setminus \Sigma, d_S),
\end{equation}

is homeomorphic to $\Sigma \subset (H, d_H)$.

Writing $(B_a(p), d_H) \subset (B_b(p), d_S)$ means the set-theoretic inclusion $B_a(p) \subset B_b(p)$ where the radii are measured relative to $d_H$ and $d_S$, respectively.

Proof For (i) we first note that $(B_r(z), d_H) \subset (B_{r_{\text{out}}}(z), d_S)$ is just Prop. 2.14(ii). To show the existence of $r_{\text{inn}}$, we prove that for $r > 0$ small enough, there is some $c > 0$ so that $\Phi \geq c$ on $B_r(z) \cap H \setminus \Sigma$. From this we can choose $r_{\text{inn}} = c^{2/(n-2) - r}$. To get that lower estimate, we recall the Gauß-Codazzi equation: $|\mathcal{A}|^2 + 2 \text{Ric}_M(\nu, \nu) = \text{scal}_M - \text{scal}_H + (\text{tr} A_H)^2$, where $\text{tr} A_H$ is the mean curvature of $H$. In the non-trivial case $z \in \Sigma$, there is an $m > 0$ and for any constant $k > 0$ there is a small $r > 0$ so that: $|\text{Ric}_M(\nu, \nu)|, |\text{scal}_M|, |\text{tr} A_H| \leq m$, whereas $\langle A \rangle \geq k$ on $B_r(z) \cap H \setminus \Sigma$. The bound on $|\text{tr} A_H|$ is part of the chosen class of almost minimizers in D.2. Thus for large $k \gg 1$ and hence small $r > 0$, any solution $\Phi > 0$ of $L_{H, \lambda} \Phi = 0$ is superharmonic on $B_r(z) \cap H \setminus \Sigma$.

\begin{equation}
\Delta_H \Phi = \left(\frac{n-2}{4(n-1)} \cdot \text{scal}_H - \lambda \cdot \langle A \rangle^2\right) \cdot \Phi \leq 0.
\end{equation}

Almost minimizers share their regularity theory with area minimizers. In particular, we get the same blow-up limits. This implies the validity of the Bombieri–Giusti $L^1$–Harnack inequality [BG, Theorem 6, p. 39] for any $H \in \mathcal{G}$. As in the argument of Lemma 2.15, the Hausdorff codimension of $\Sigma$ is $> 2$ and, hence, $\Phi$ meets the requirements of that Theorem.

Now we recall that intrinsic and extrinsic distances are equivalent: there is a constant $c(H) \in (0, 1)$ such that for any $p, q \in H$: $c \cdot d_{g_H}(p, q) \leq d_{g_{H^{n+1}}(p, q)} \leq d_{g_H}(p, q)$ [L1, Corollary 2.10]. Thus [BG, Theorem 6, p. 39], which uses extrinsic distances, applies in the following form: for some small $r_H > 0$ and $r \in (0, r_H)$ (we may choose $r_H = \infty$ in the case $H \in \mathcal{H}^R_n$), we have

\begin{equation}
0 < a \cdot \int_{B_r(z) \cap H \setminus \Sigma} \Phi dV =: c \leq \inf_{B_r(z) \cap H \setminus \Sigma} \Phi, \text{ for some } a = a(L_{H, \lambda}, r) > 0.
\end{equation}

This shows that the map $I_H : \Sigma \rightarrow \Sigma_S$ from (52) is injective.

$I_H$ is also surjective since any point $x \in \Sigma_S$ is the $d_S$-limit of a sequence of points $x_k \in H \setminus \Sigma$ and we can choose hyperbolic geodesic arcs $\gamma_{p, x_k}$ from $p$ to the $x_k$. A subsequence of these arcs converges to a hyperbolic geodesic ray that represents some point $\gamma \in \Sigma$ and then we see that $\{x\} = \bigcap_{a > 0} \overline{U(\gamma, a)}$, that is, $I_H(\gamma) = x$.

Finally, we show that $I_H^{-1}$ is continuous. For compact $H$, any closed set is mapped onto a compact and thus closed set. For $H \in \mathcal{H}^R_n$, we use Lemma 2.15 to argue similarly. For any
Remark 2.17 We observe that \((H, d_S)\) is a geodesic metric space. For any two \(p, q \in (H, d_S)\) we can find \(p_i, q_i \in H \setminus \Sigma\) with \(p_i \to p, q_i \to q\), for \(i \to \infty\). For small and smoothly bounded neighborhoods \(U\) of \(\Sigma\) with \(p_i, q_i \in H \setminus U\) we have, from [L1, Prop.2.1], a path \(\gamma_U \subset H \setminus U\) joining \(p_i, q_i \in (H, d_S)\). Relative to the intrinsic metric on \(H \setminus U\), we can assume that \(\text{length}(\gamma_U) = d(p_i, q_i)\). From 2.14 and BV-compactness results for curves [SG, Theorem 4 in Section 4.5] we observe that for neighborhoods \(U_k \supset U_{k+1}\) shrinking to \(\Sigma\), i.e. \(\bigcap_k U_k = \Sigma\), there is a subsequence of the \(\gamma_{U_k}\) converging to some curve \(\gamma_{p_i,q_i} \subset (H, d_S)\) with \(\text{length}(\gamma_{p_i,q_i}) = d(p_i, q_i)\) in \((H, d_S)\) and, in turn, for \(i \to \infty\), there is a subsequence of the \(\gamma_{p_i,q_i} \subset (H, d_S)\) converging to the wanted geodesic \(\gamma_{p,q} \subset (H, d_S)\) that links \(p\) and \(q\). \(\Box\)

2.3 Codimension Estimate for \(\Sigma_S \subset (H, d_S)\)

The (partial) regularity theory for any almost minimizer \(H \in \mathcal{G}\) within some smooth ambient manifold \(M^{n+1}\) says that \(H\) is smooth except for a singular set \(\Sigma_H\) that has Hausdorff codimension \(\geq 8\) relative to \(M^{n+1}\), cf. [F], [Gi, Ch.11]. We extend this estimate to the singular set of \((H, d_S)\). However, this is not an obvious application of Federer’s estimate for \((H, d_H)\) cf. Remark 2.18 below. We rather imitate Federer’s argument in the class of minimal factor geometries.

Remark 2.18 (Bishop Deformers) The identity from \((H, d_H)\) to \((H, d_S)\) is not Lipschitz regular since \(\Phi\) diverges towards \(\Sigma\). However, the Hausdorff dimension is only a bi-Lipschitz invariant. To illustrate this issue recall that there is an (actually Hölder regular) homeomorphism \(\phi : \mathbb{R}^2 \to \mathbb{R}^2\) mapping \(S^1\) to the Koch snowflake \(K \subset \mathbb{R}^2\) with Hausdorff dimension \(\ln 4 / \ln 3\), cf. [Bi]. It is conceivable that the deformations of [Bi, Th.1.1] can be adjusted to show that if \(\Sigma \subset (H, d_H)\) has Hausdorff dimension \(a\), for some \(0 < a < n\), that, for any \(b\) with \(a < b < n\), there is some non-minimal growth solution \(\omega_b > 0\) of \(L_{H,\lambda} \phi = 0\) so that the singular set of the completion of \((H \setminus \Sigma, \omega_c^{d/(n-2)} \cdot g_H)\) has Hausdorff dimension \(\geq b\). \(\Box\)

We recall some basic concepts and formulate them for the metric space \((H, d_S)\).

Definition 2.19 (Hausdorff Measure and Dimension) For some \(H \in \mathcal{G}\), let \(A \subset (H, d_S)\), \(k \in [0, \infty)\) and \(\delta \in (0, \infty]\). Then we set

\[(64) \quad H_k^k(A) := \inf \left\{ \sum_i \text{diam}(S_i)^k \mid A \subset \bigcup_i S_i, S_i \subset H, \text{diam}(S_i) < \delta \right\},\]

\[(65) \quad H_k(A) := \lim_{\delta \to 0} H_k^k(A) = \sup_{\delta} H_k^k(A).\]

\(H_k(A)\) is the \(k\)-dimensional Hausdorff measure of \(A\). The infimum of all \(k\) so that \(H_k(A) = 0\) is the Hausdorff dimension \(\dim_H(A) = \dim_{\mathbb{R}}(A \subset (H, d_S))\) of \(A\) as a subset of \((H, d_S)\).
Proposition 2.21 (Basic Properties of $H^\infty$) For every $A \subset (H, d_S)$, we have $H^\infty_k(A) = 0$ if and only if $H_k(A) = 0$. For $H_k$-almost all $x \in A$, we have

\begin{equation}
\limsup_{r \to 0} \frac{H^\infty_k(A \cap B_r(x))/r^k}{1} \geq 1.
\end{equation}

Proof For subsets $A \subset \mathbb{R}^n$ this is [Gi, Lemma 11.2 and Proposition 11.3]. These results are direct consequences of the definitions of $H^\infty_k(A)$ and $H_k(A)$. They do not use the regular Euclidean structure of the underlying space and equally apply to $(H, d_S)$.

We also notice that for $A$ compact and $\delta \in (0, \infty]$, we have $H^\delta_k(A) < \infty$ even when $A$ has Hausdorff dimension $> k$. The definition of $H^\delta_k$ also readily implies the existence of open neighborhoods $U(A, k, \eta, \delta)$ of $A$, for any $\eta > 0$, so that $H^\delta_k(U) \leq H^\delta_k(A) + \eta$. Moreover, in non-compact cases with $H^\delta_k(A) = \infty$, we also write $H^\delta_k(A) > 0$ to keep the notation consistent.

Hausdorff dimension of $\Sigma$ We extend Federer’s estimate for the dimension of $\Sigma$ relative to $(H, d_H)$ to the case where $\Sigma \cong \Sigma_S$ is interpreted as a subset of $(H, d_S)$.

Theorem 2.22 (Partial Regularity of $(H, d_S)$) The Hausdorff dimension of $\Sigma$ relative to the minimal factor metric $(H, d_S)$ is $\leq n - 7$.

To this end, we note that [L3, Ch.3], summarized in E.2, and the distance estimates Prop. 2.16(i), providing the extension to the metric completions, translate into a blow-up theory for minimal factor metrics that strongly resembles that for the original almost minimizing geometries.

Theorem 2.23 (Blow-Ups) For $H \in \mathcal{G}$ we consider $(H, d_S(\Phi_H))$ and any singular point $p \in \Sigma_H$. Then we get the following blow-up invariance: Any sequence $(H, \tau_i \cdot d_S(\Phi_H))$ scaled around $p$ by some sequence $\tau_i \to \infty$, $i \to \infty$, subconverges and the limit of any converging subsequence is $(C, d_S(\Phi_C))$ for some tangent cone $C$ of $H$ in $p$.

We only need to specify the notion of convergence. It has two layers: a subsequence of $H_i = \tau_i \cdot H$, for some sequence $\tau_i \to \infty$, for $i \to \infty$, around a given singular point $x \in \Sigma \subset H^n$ converges to an area minimizing tangent cone $C^n \subset \mathbb{R}^{n+1}$. The flat norm convergence becomes a compact $C^k$-convergence over regular subsets of $C$ expressed in terms of a $C^k$-convergence of ID-maps as in D.2, section 1.3. Then the sections $\Phi_H \circ \Gamma_i$ compactly $C^k$-converge to $\Phi_C$ on $C$ after normalizing the value of the $\Phi_H \circ \Gamma_i$ in a common base point in $C \setminus \Sigma_C$. The fact that the Martin boundary of $L_{H,\lambda}$ for $\lambda < \chi^{(A)}_H$ on $H \in \mathcal{H}^\infty_n$ has exactly one point at infinity and again Prop. 2.16(i) show:
Theorem 2.24 (Euclidean Factors) For any non–totally geodesic $H \in \mathcal{H}^r_n$ there is a unique\textsuperscript{#} space $(H, d_S(\Phi_H))$. For $C \in \mathcal{SC}_n$, the associated space $(C, d_S(\Phi_C))$ is invariant under scaling around $0 \in C$, that is, it is again a cone.

In the following we also use details from the proofs of 2.23 and 2.24 we cite when we need them. We start with the following auxiliary result.

Lemma 2.25 (Corresponding Balls) For $H \in \mathcal{G}$ and some $p \in \Sigma_H$ we consider a tangent cone $C$ in $p$ we identify with $0 \in \mathbb{R}^{n+1}$. For any ball $B^C_r(q) \subset B_1(0) \cap C$ of radius $r \in (0, 1)$ in $(C, d_S)$ and any $\delta > 0$, there is a ball $B^H_r(q_i) \subset B_1(0) \subset (\tau_j \cdot H, d_S)$ so that, considered as subsets of the original spaces $(C, d_C)$ and $(\tau_j \cdot H, d_{\tau_j \cdot H})$ locally embedded in $\mathbb{R}^{n+1}$, we have

\begin{equation}
B^H_r(q_i) \rightarrow B^C_r(q) \text{ in flat and, from this, in Hausdorff norm, for } j \rightarrow \infty.
\end{equation}

We call the balls $B^H_r(q_i)$ asymptotically corresponding to $B^C_r(q)$.

Proof From D.1 in section 1.3, we have a flat norm convergence of some open sets $O_i \subset \tau_j \cdot H$ to $B^C_r(q)$. This also implies Hausdorff convergence from lower volume estimate for the difference set we have from [Gi, Prop.5.14]. This applies to $C$ and $H$ since they are (asymptotically) area minimizing in $\mathbb{R}^{n+1}$. For the (almost) minimizing geometries on $C$ and $H$ we have for any given $\eta > 0$ and $i$ large enough that the $O_i$ are almost isometric to $B^C_r(q)$ via ID-maps outside an $\eta$–distance tube $U_\eta(\Sigma_C)$ around $\Sigma_C$ cf. D.2.

For the deformed geometries $(C, d_S)$ and $(H, d_S)$ we recall from E.1 and E.2 that the minimal growth solutions converge smoothly to that on $C$ outside such $\eta$–distance tubes. Now we reuse the length estimate (50) and remark 2.12.2 to see that the tube size of the conformally deformed $\eta$–distance tube also, and uniformly in $j$, shrinks to zero when $\eta \rightarrow 0$. Similarly the size of the complements the ID-map images of these distance tubes in $O_i$ shrinks to zero when $i \rightarrow \infty$. From this there are (not necessarily singular) points $q_i \in H_j$ so that (67) holds for the size of the difference set between $O_i$ and $B^H_r(q_i)$ converges to zero when $i \rightarrow \infty$. \(\Box\)

We use this to upper estimate the $\mathbb{H}_k^\infty$-measure of $\Sigma_H \subset (H, d_S)$.

Proposition 2.26 (Measure under Blow-Ups) For $H \in \mathcal{G}$ converging under scaling by some sequence $\tau_j \rightarrow \infty$, for $j \rightarrow \infty$, to a tangent cone $C$ of $H$ in $p \in \Sigma$, which we identify with $0 \in C$, and any radius $R > 0$, relative to the minimal factor metrics, we have

\begin{equation}
\mathbb{H}_k^\infty(\Sigma_C \cap \overline{B_R(0)}) \geq \limsup_{j} \mathbb{H}_k^\infty(\tau_j \cdot \Sigma_H \cap \overline{B_R(0)}).
\end{equation}

Proof We cover $\Sigma_H \cap \overline{B_1(0)}$ by finitely many balls $B_i \subset (C, d_S)$ so that

\begin{equation}
\mathbb{H}_k^\infty(\Sigma_C \cap \overline{B_1(0)}) > \sum_i \text{diam}(B_i)^k - \varepsilon.
\end{equation}

We start with a variant of an argument used in [F], [Gi, Ch.11]. Let $\tau_j \cdot H$, scaled around a basepoint $p \in H$, compactly converge to a tangent cone $C$. Then a cover of $\Sigma \cap K \subset C \cap K$, for some compact $K \subset \mathbb{R}^{n+1}$, by open subsets of $\mathbb{R}^{n+1}$ also covers $\Sigma_{\tau_j \cdot H} \cap K$, for $j$ large enough and the diameter bounds for the covering sets (asymptotically) carry over from $C$ to
\[ \tau_j \cdot H. \] In the case of \((H, d_S)\) we need to evaluate the diameter with respect to the minimal factor metrics.

From Lemma 2.25 we find for any \(\delta > 0\) some \(j_\delta > 0\) so that for \(j \in (0, j_\delta)\) there is a family of balls \(B_i^j \subset \tau_j \cdot (H, d_S)\) covering \(\tau_j \cdot \Sigma_H \cap B(0)\) with \(\text{diam}(B_i^j) \leq (1 + \delta) \cdot \text{diam}(B_i)\), hence

\[
\mathbb{H}^k(\tau_j \cdot \Sigma_H \cap B(0)) \leq \sum_i \text{diam}(B_i^j)^k.
\]

Summarizing, we have \(\limsup_j \mathbb{H}^k(\tau_j \cdot \Sigma_H \cap B(0)) \leq (1 + \delta)^k \cdot \left( \mathbb{H}^k(\Sigma_C \cap B(0)) + \varepsilon \right)\). For \(\varepsilon \to 0\) and \(\delta \to 0\), the claimed estimate (68) follows. \(\square\)

**Corollary 2.27 (Cone Reduction)** For \(H \in \mathcal{G}\), assume that \(\mathbb{H}^k(\Sigma_H) > 0\), for some \(k\). Then there exists a tangent cone \(C\) in \(\mathbb{H}^k\)-almost every point \(x \in \Sigma_H\) such that \(\mathbb{H}^k(\Sigma_C) > 0\).

**Proof** From (66) in Prop. 2.21 we can find for \(\mathbb{H}^k\)-almost every point \(x \in \Sigma_H\) a sequence of radii \(r_i \to 0\) for \(i \to \infty\) such that \(\mathbb{H}^k(A \cap B_{r_i}(x)) \geq r_i^k\). Then we get the claim from 2.26. \(\square\)

It is obvious that the dimensions of the singular sets of an area minimizing cone \(C^m \subset \mathbb{R}^{n+1}\) and that of the product cone \(\mathbb{R} \times C^m \subset \mathbb{R}^{n+2}\) differ by one. In the area minimizing case, this is used in the inductive reduction argument for the codimension estimate [Gi, Th. 11.8]. Minimal factor metrics on \(\mathbb{R}^m \times C^{n-m}\) are no longer Riemannian products. We recall that the eigenfunction \(\Phi_{\mathbb{R}^m \times C^{n-m}}\) with minimal growth towards any point of \(\Sigma\) is unique\#, i.e. up to multiplication by a positive constant, from [L2, Th. 3]. Hence, \(\Phi_{\mathbb{R}^m \times C^{n-m}}\) reflects the symmetries of \(\mathbb{R}^m \times C^{n-m}\). Following [L3, Prop. 4.6], we can write \(\Phi_{\mathbb{R}^m \times C^{n-m}}\) in cylindrical coordinates \(x = (z, r, \omega) \in \mathbb{R}^m \times \mathbb{R}^+ \times (\partial B(0) \cap C \setminus C)\) for \(r = r(x) = \text{dist}(x, \mathbb{R}^m \times \{0\})\):

\[ \Phi_{\mathbb{R}^m \times C^{n-m}}(\omega, r, z) = \psi(\omega) \cdot r^{\alpha+} \text{ for some } \alpha+ < 0 \text{ on } \mathbb{R}^m \times C^{n-m} \setminus C_{\mathbb{R}^m \times C^{n-m}}. \]

Thus \((\mathbb{R}^m \times C^{n-m}, d_S)\) is invariant under translations in \(\mathbb{R}^m\)-direction and under scalings around points in \(\mathbb{R}^m \times \{0\}\). From this we determine the Hausdorff dimension of \(\mathbb{R}^m \times \{0\}\).

**Lemma 2.28 (Hausdorff Dimension of Axes)** Within \((\mathbb{R}^m \times C^{n-m}, d_S)\), \(m \geq 1\), the \(m\)-planes \(\mathbb{R}^m \times \{y\}, y \in C\) have Hausdorff dimension \(m\). Moreover, for any \(k > 0\) and \(y \in C\) we have \(\mathbb{H}^k([0, 1]^m \setminus \{0\}) > 0\) if and only if \(\mathbb{H}^k([0, 1]^m \setminus \{y\}) > 0\).

**Proof** We use Prop. 2.16 and the translation invariance of \((\mathbb{R}^m \times C^{n-m}, d_S)\) in \(\mathbb{R}^m\)-direction to choose \(\rho > 0\) so that the balls \(B_\rho(p_i), p_i \in \mathbb{Z}^m\), measured relative \((\mathbb{R}^m \times C^{n-m}, d_S)\), cover \(\mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times C^{n-m}\). Now we consider the Euclidean lattices \(2^{-j} \cdot \mathbb{Z}^m \subset \mathbb{R}^m\), \(j = 1, 2, \ldots\), and observe that due to the scaling invariance of \((\mathbb{R}^m \times C^{n-m}, d_S)\), the \(B_{2^{-j} \cdot \rho}(2^{-j} \cdot p_i), p_i \in \mathbb{Z}^m\), cover \(\mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times C^{n-m}\). The restriction of the lattice \(2^{-j} \cdot \mathbb{Z}^m\) to the unit cube \([0, 1]^m \subset \mathbb{R}^m\) contains \(2^{j \cdot m}\) points (up to lower orders along \(\partial([0, 1]^m)\)). For these balls, we get for any \(\alpha > m:\)

\[
\mathbb{H}^\alpha([0, 1]^m) \leq \sum_{\{p_i \in [0, 1]^m \cap 2^{-j} \cdot \mathbb{Z}^m\}} \text{diam}(B_{2^{-j} \cdot \rho}(2^{-j} \cdot p_i))^{\alpha} = 2^{j \cdot m} \cdot 2^{-j \cdot \alpha} \cdot \rho^\alpha \to 0, \text{ for } j \to \infty.
\]

Thus \(\text{dim}_{\mathbb{H}}([0, 1]^m \subset (H, d_S)) \leq m\). In turn, since \(\alpha_+ < 0\) there is some \(\delta_0\) so that for \(\delta \in (0, \delta_0)\) any sets of diameter \(\leq \delta\) that intersects \(\mathbb{R}^m \times \{0\}\) also has diameter \(\leq \delta\) when
computed relative to \( d_H \). Thus we infer from the area minimizing case where \( \dim_H([0,1]^m \subset (H,d_H)) = m \) that \( \dim_H([0,1]^m \subset (H,d_S)) \geq m \). More generally, the translation that maps \( \mathbb{R}^m \times \{0\} \) onto \( \mathbb{R}^m \times \{y\}, y \in C \) is bi-Lipschitz in terms of \( d_S \). To see this, we use again that \( d_S \) is \( \mathbb{R}^m \)-translation invariant and that the \( d_S \)-distance of any two points \( x,y \in \mathbb{R}^m \times \{0\} \) remains finite. This follows again from 2.13 (and 2.14). From a comparison of ball covers, defined as above, we see that, up to this constant (to the power of \( m \)) we get the same Hausdorff measure estimates as for \([0,1]^m \times \{0\} \subset \mathbb{R}^m \times \{0\} \) also for \([0,1]^m \times \{y\} \subset \mathbb{R}^m \times \{y\} \) and vice versa.

\[ \square \]

**Lemma 2.29 (Radial Singularities)** For \( C^{n-m} \in \mathcal{SC}_{n-m} \), assume that \( 0 \not\subset C \), that is, \( C^{n-m} \) is singular not only in \( 0 \) and that for some \( k > 0 \): \( \mathbb{H}_k^\infty(\Sigma_{\mathbb{R}^m \times C^{n-m}}) > 0 \). Then there is some ball \( B \subset \mathbb{R}^m \times C^{n-m} \) with \( \overline{B} \cap (\mathbb{R}^m \times \{0\}) = \emptyset \) so that \( \mathbb{H}_k^\infty(\overline{B} \cap \Sigma_{\mathbb{R}^m \times C^{n-m}}) > 0 \).

**Proof** Assume there is no such ball. Then we have, using a suitable countable ball cover, that \( \mathbb{H}_k^\infty(\Sigma_{\mathbb{R}^m \times C^{n-m}} \setminus \mathbb{R}^m \times \{0\}) = 0 \). This means \( \mathbb{H}_k^\infty(\mathbb{R}^m \times \{0\}) > 0 \), but from this, Lemma 2.28 also shows that \( \mathbb{H}_k^\infty([0,1]^m \times \{0\}) > 0 \) for any \( y \in \Sigma_{[0,1]^m \times C^{n-m}} \), a contradiction. \( \square \)

**Proof of Theorem 2.22** From Lemma 2.29 and Cor. 2.27, there are a point \( p \in B \cap \Sigma_{\mathbb{R}^m \times C^{n-m}} \setminus \mathbb{R}^m \times \{0\} \) and a tangent cone \( C^* \) in \( p \) such that \( \mathbb{H}_k(\Sigma_{C^*}) > 0 \). Since \( p \not\in \mathbb{R}^m \times \{0\} \) we know that \( C^* \) can be written as \( C^* = \mathbb{R}^{m+1} \times C^{n-m-1} \). We iterate this argument until we reach some cone \( C^\# = \mathbb{R}^{m+l} \times C^{n-l} \) where \( C^{n-l} \) is singular only in \( 0 \). The value \( n-l \) may depend on the chosen sequence of blow-up points but we know from the fact that hypersurfaces in \( \mathcal{H}_n \) for \( n \leq 6 \) are regular that \( n-l \geq 7 \). Since we have \( \mathbb{H}_k(\Sigma_{C^\#}) > 0 \), we get from 2.28 that \( k \leq n-7 \). \( \square \)

## 3 Ahlfors Regularity and Semmes Families

We estimate the eccentricity of \( d_S \)-distance balls relative to \( d_H \)-distance balls. This is used to derive the Ahlfors regularity of \((H,d_S,\mu_S)\) and, in the next chapter, also to study the canonical Semmes families relative to \((H,d_S,\mu_S)\).

### 3.1 Eccentricity of \((H,d_S)\)

The qualitative result 2.16 says that for any \( r > 0, p \in H \), there are \( \kappa \geq 1, \rho > 0 \) with

\[ (72) \quad (B_\rho(p),d_S) \subset (B_r(p),d_H) \subset (B_{\kappa \rho}(p),d_S). \]

We enhance the arguments to control the relations between these radii quantitatively. The **local eccentricity** \( \vartheta \) of \( d_S \) relative to \( d_H \) for the ball \((B_r(p),d_H)\) is

\[ (73) \quad \vartheta(H,p,r,\Phi) := \inf \{ \kappa \geq 1 \mid (72) \text{ holds for at least one } \rho > 0 \}. \]

This infimum actually is a minimum since \((B_\rho(p),d_S)\) and \((B_r(p),d_H)\) are open. In turn, this \( \vartheta \) determines a unique \( \varrho(\vartheta) > 0 \) that satisfies (72). The parameters \( \vartheta \) and \( \varrho \) depend differently on the gauging of \( \Phi \), that is, on choosing a positive multiple of \( \Phi \):

\[ (74) \quad \vartheta(\lambda \cdot \Phi) = \vartheta(\Phi) \text{ but } \varrho(\lambda \cdot \Phi) = \lambda^{2/(n-2)} \cdot \varrho(\Phi), \text{ for } \lambda > 0. \]
Proposition 3.1 (Eccentricity of \((H,d_S)\)) We consider the two cases where

(i) \(H \in \mathcal{H}_n^\mathbb{R}\) is non–totally geodesic, (ii) \(H \in \mathcal{G}_n^c\) is singular.

Then we have a common upper bound for \(\vartheta(H,p,r,\Phi)\) for any minimal factor metric:

\[
\Theta_n \geq \vartheta(H,p,r,\Phi), \text{ depending only on } n,
\]

for any \(r > 0\) in case (i) and for sufficiently small \(r > 0\) in case (ii).

Proof For \(H \in \mathcal{H}_n^\mathbb{R}\), \(\tau > 0\) and \(v \in H\), we also have \(\tau \cdot (H-v) \in \mathcal{H}_n^\mathbb{R}\). Thus it is enough to consider that \(p = 0\) and \(r = 1\). Now we assume there is a compactly converging sequence of pointed spaces \((H_i,0)\) with limit \((H,\infty,0)\) for \(H_i, H_\infty \in \mathcal{H}_n^\mathbb{R}\) equipped with, via ID-maps, also converging minimal growth solutions \(\Phi_i\) such that \(\vartheta[i] := \vartheta(H_i,0,1,\Phi_i) \to \infty\).

We select regular points \(q_i \in \partial B_{1/2}(0) \subset H_i\) with \(a_n \geq \langle A \rangle(q_i) \geq b_n > 0\), for constants \(a_n > b_n > 0\), and \(q_i \to q_\infty \in \partial B_{1/2}(0) \subset H_\infty\), for \(i \to \infty\). This can done due to the naturality of \(\langle A \rangle\) and the compactness of \(\mathcal{H}_n^\mathbb{R}\).

We choose \(s_i > 0\) so that \(s_i \cdot \Phi_i(q_i) = 1\). From elliptic theory the \(s_i \cdot \Phi_i\) compactly subconverge to a solution \(\Phi_\infty > 0\) of \(L_{H_\infty,\lambda} \phi = 0\) on \(\mathcal{H}_n^\mathbb{R}\) with \(\Phi_\infty(q_\infty) = 1\). Since \(q_\infty\) is a regular point, there is a radius \(\eta \in (0, \min\{1, \delta(A)(p_i)\}/4)\), for large \(i\) and then independent of \(i\), so that \(B_{2\eta}(q_i)\) is regular and, via ID-maps, nearly isometric to \(B_{2\eta}(q_\infty)\) in \(C^3\)-norm. Then, we have Harnack inequalities for positive solutions of \(L_{H_i,\lambda} \phi = 0\) on \(B_{2\eta}(q_i)\) with constants independent of \(i\). From this \(s_i \cdot \Phi_i(q_i) = 1\) implies uniform lower estimates \(k > 0\), for any \(i\):

\[
k \leq \int_{B_2(q_i)} s_i \cdot \Phi_i dV \leq \int_{B_2(p_i) \cap H_i \setminus \Sigma} s_i \cdot \Phi_i dV.
\]

For \(H_i \in \mathcal{H}_n^\mathbb{R}\) we have \(\text{Ric}_M, \text{scal}_M, \text{tr} A_H \equiv 0\) and the Bombieri–Giusti inequality (63) yields

\[
\inf_{B_2(0) \cap H_i \setminus \Sigma} s_i \cdot \Phi_i \geq l, \text{ for a constant } l = l(L_{H_i,\lambda}) > 0, \text{ independent of } i.
\]

As in the proof of Prop. 2.16, this implies a lower positive estimate for the \(d_S(s_i \cdot \Phi_i)\)-distance \(d_i\) of \((\partial B_1(0), d_H)\) to 0: \(d_i \geq \Delta\), for some \(\Delta > 0\) which can be chosen independently of \(i\).

To disprove that \(\vartheta[i] \cdot \Delta \to \infty\), for \(i \to \infty\), we consider any \(z \in \partial B_1(0)\). We recall from Prop. 2.10 that due to \(a_n \geq \langle A \rangle(q_i) \geq b_n > 0\) and \(s_i \cdot \Phi_i(q_i) = 1\) we have some \(\xi > 0\) with \(\xi = \xi(n)\) so that for the minimal Green’s function \(G_i\) on \(H_i\):

\[
s_i \cdot \Phi_i(z) \leq \xi \cdot G_i(z, q_i), \text{ for any } z \in B_1(0) \cap H_i \setminus \Sigma H_i \subset B_2(q_i) \cap H_i \setminus \Sigma H_i.
\]

Moreover, we may assume that \(B_\eta(q_i)\) is the ball \(B = B_{2\sigma}(q_i)\) in the hyperbolic picture of Cor. 2.13 and that from elliptic estimates \(\Phi_i(z) \leq c_n \cdot \Phi_i(q_i)\), for some \(c_n \geq 1\), depending only on \(\eta\), and any \(z \in B_\eta(q_i)\). From this, (78) and integrating \(G_i^{2/(n-2)}(\cdot, q_i)\) as in Cor. 2.13 (50), outside \(B_\eta(q_i)\), along a hyperbolic geodesic arc from \(q_i\) to \(z\) we get a common upper bound \(\Theta_n^* \cdot \Delta\) for the \(\vartheta[i] \cdot \delta_i \geq \vartheta[i] \cdot \Delta\) contradicting the assumption.

Finally we reduce case (ii), that is, \(H \in \mathcal{G}_n^c\), to that of \(H \in \mathcal{H}_n^\mathbb{R}\). Here we claim that there is a radius \(r_H > 0\) so that \(\Theta_n := \Theta_n^* + 1\) is an upper bound for \(\vartheta(H,p,r,\Phi)\), for any \(p \in H\), \(r \in (0, r_H)\). Otherwise there is a sequence of points \(p_i \in H\) and radii \(r_i > 0\), with \(r_i \to 0\), so that \(\vartheta[i] := \vartheta(H,p_i, r_i, \Phi) \to \infty\). We use the scaling invariance of \(L_{H,\lambda} \phi = 0\) and scale \(g_H\) to
$r^{-2} \cdot g_H$ and thereby $(B_r(p), g_H)$ to $(B_1(p), r^{-2} \cdot g_H)$. Then there is a compactly converging subsequence of the pointed spaces $H_i := (H, r_i^{-1} \cdot d_H, p_i)$, which contain $(B_1(p_i), r_i^{-1} \cdot d_H, p_i)$, with limit pointed space $(H_\infty, d_{H_\infty}, 0)$ for $H_\infty \in \mathcal{H}^n_\mathbb{R}$. Now we can repeat the argument of case (i), applied to these $H_i$, and see that $\Theta_n := \Theta_n^* + 1$ upper bounds $\psi(H, p, r, \Phi)$, for $r \in (0, r_H)$, $r_H > 0$ small enough. \hfill \Box

3.2 Ahlfors $n$-Regularity of $(H, d_S, \mu_S)$

The Hausdorff estimate Th.2.22 suggests a canonical extension of the Riemannian volume measure $\Phi^{2n/(n-2)} \cdot \mu_H$ on $H \setminus \Sigma$ to a measure $\mu_S$ on $(H, d_S)$, where $\mu_H$ is the $n$-dimensional Hausdorff measure on $(H^n, d_H) \subset (M^{n+1}, g_M)$. In turn, $\mu_H$ is the extension of the Riemannian volume on $(H^n \setminus \Sigma, g_H) \subset (M^{n+1}, g_M)$ using that also $\mathbb{H}^n(\Sigma) = 0$ relative to $(H^n, d_H)$.

**Definition 3.2 (Minimal Factor Measures $\mu_S$)** For any $H \in \mathcal{G}_n$ equipped with a minimal factor metric $\Phi^{4/(n-2)} \cdot g_H$, we define the minimal factor measure $\mu_S$ on $H$ by

$$(79) \quad \mu_S(E) := \int_{E \setminus \Sigma} \Phi^{2n/(n-2)} \cdot d\mu_H \text{ for any Borel set } E \subset H.$$ 

We establish a number of regularity properties of $\mu_S$, in particular, we will see that this is a locally finite Borel measure on $(H, d_S)$ making $(H, d_S, \mu_S)$ a metric measure space cf. Rm.3.6 below. The local finiteness of the measure follows from the following measure estimates.

**Proposition 3.3 (Finiteness of $\mu_S$)** For any non–totally geodesic $H \in \mathcal{G}$ equipped with some minimal factor metric $d_S(\Phi)$ we have:

- for $H \in \mathcal{G}_n^c$, the total volume is finite: $\mu_S(H) < \infty$,
- for $H \in \mathcal{H}^n_\mathbb{R}$, any $q \in H$ and any $r > 0$: $\mu_S(B_r(q), d_S) < \infty$.

**Proof** We first show for $H \in \mathcal{G}_n^c$ that $\mu_S(H, d_S) < \infty$. This amounts to prove that $|\Phi|_{L^2, n/(n-2)} < \infty$. We derive this from the minimal growth condition for $\Phi$ towards $\Sigma$. Note that the optimal estimate for general (super)solutions $v > 0$ of $L_{H, \lambda} \phi = 0$, from the Bombieri–Giusti Harnack inequality, is $|v|_{L^p} < \infty$, $p < n/(n-2)$.

Our supersolution $\Phi > 0$ properly solves $L_{H, \lambda} \phi = 0$ on some small neighborhood $U$ of $\Sigma$. Thus we only need to verify that $\text{Vol}(U \setminus \Sigma, \Phi^{2n/(n-2)} \cdot g_H) < \infty$ since $K := (H \setminus \Sigma) \setminus U$ is compact and regular and, hence, $\text{Vol}(K, \Phi^{2n/(n-2)} \cdot g_H) < \infty$. On $U$ we can compare $\Phi > 0$ with the minimal Green’s function $G$. We know from the boundary Harnack inequality 2.8 that for some base point $p \in K$, with $\text{dist}(\Lambda_c)^r(p, U) \geq 2 \cdot \sigma$, there is a constant $c \geq 1$ so that

$$(80) \quad c^{-1} \cdot G(\cdot, p) \leq \Phi \leq c \cdot G(\cdot, p) \text{ on } U \setminus \Sigma.$$ 

From [L2, Prop. 3.12, Step 2] $G(\cdot, p)$ minimizes the variational integral

$$(81) \quad J_U(f) := \int_{U \setminus \Sigma} |\nabla_H f|^2 + V_\lambda \cdot f^2 \, dV, \text{ for } V_\lambda := \frac{n-2}{4(n-1)} \cdot \text{scal}_H - \lambda \cdot \langle A \rangle^2,$$ 

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running over all \( f \in H^{1,2}_{(\Lambda)}(H \setminus \Sigma) \) with \( f|_{\partial U} = G(\cdot, p) \) in the trace sense. In particular, using simple test functions we see that \( J_H(G(\cdot, p)) < \infty \). From this and the \( \mathcal{S} \)-adaptedness of \( L_{H,\lambda} \), we have for \( C_F := \mathcal{F}_{\Lambda} \), \( (\tilde{\lambda} - \lambda)|_{\Lambda}^{-1} \cdot \int_{U \setminus \Sigma} |\nabla G(\cdot, p)|^2(x) + V_\lambda \cdot G(x, p)^2 \, dV + C_F < \infty \).

From (48), (49), (13) and since \( \text{dist}_{(\Lambda)}(p, U) \geq 2 \cdot \sigma \) we get some \( \alpha^o, \beta^o > 0 \) so that for \( x \in U \setminus \Sigma \):
\[
G(x, p) \leq \langle A \rangle^{(n-2)/2}(p) \cdot \langle A \rangle^{(n-2)/2}(x) \cdot \beta^o \cdot \exp(-\alpha^o \cdot d_{(\Lambda)}(x, p)).
\]

With this inequality and (82), we get, for some \( c^o > 0 \),
\[
\int_{U \setminus \Sigma} G(x, p)^{4/(n-2) + 2} \, dV \leq c^o \int_{U \setminus \Sigma} \langle A \rangle^2(x) G(x, p)^2 \, dV < \infty.
\]

For the volume element \( dV(g_H) \) of \( g_H \), we have \( dV(\Phi^{4/(n-2)} \cdot g_H) = \Phi^{2n/(n-2)} \cdot dV(g_H) \). From this, writing \( 2 \cdot n/(n-2) = 4/(n-2) + 2 \), we have:
\[
\text{Vol}(U \setminus \Sigma, \Phi^{4/(n-2)} \cdot g_H) = \int_{U \setminus \Sigma} \Phi^{2n/(n-2)}(x) \, dV \leq c^o \int_{U \setminus \Sigma} G(x, p)^{4/(n-2) + 2} \, dV < \infty.
\]

Thus for \( H \in \mathcal{G}_n^r \) we have \( \mu_S(H, d_S) < \infty \). For \( H \in \mathcal{H}_n^r \) the localization of this argument to balls shows that \( \mu_S(B_r(q), d_S) < \infty \). \( \square \)

We refine this finiteness result to volume growth estimates. To this end we set
\[
\mu_S(B_r(q)) = \text{Vol}(B_r(q), d_S) = \int_{B_r(q) \cap H \setminus \Sigma} \Phi^{2n/(n-2)} \cdot d\mu_H,
\]
where we write \( \text{Vol}(B_r(q), d_S) \) to notationally simplify considerations where we measure distances and volumes with respect to different metrics. These mixed measurements are used to stepwise employ compactness arguments not directly applicable to \( (H, d_S, \mu_S) \).

**Theorem 3.4 (Ahlfors Regularity of Minimal Factors)** For \( H \in \mathcal{G}_n \), the metric measure space \( (H, d_S, \mu_S) \) is **Ahlfors \( n \)-regular**. That is, there are constants \( A(H, \Phi), B(H, \Phi) > 0 \), so that for any \( r \in [0, \text{diam}(H, d_S)] \) and any \( q \in H \):
\[
A \cdot r^n \leq \mu_S(B_r(q), d_S) \leq B \cdot r^n.
\]

For \( H \in \mathcal{H}_n^r \) the constants only depend on the dimension, that is, we have \( A(n), B(n) > 0 \). More generally, for any \( H \in \mathcal{G}_n \), there is some small \( r_{H,\Phi} > 0 \) so that (87) holds for these constants \( A(n), B(n) > 0 \) provided \( q \in \Sigma_H \) and \( r \in (0, r_{H,\Phi}) \).

**Corollary 3.5 (Doubling and Volume Decay)** For any \( H \in \mathcal{G}_n \), there is a \( C(H, \Phi) > 0 \), and \( C(n) > 0 \) for \( H \in \mathcal{H}_n^r \), so that for radii and volumes relative to \( (H, d_S, \mu_S) \):

(i) \( \mu_S \) is **doubling**: for any \( q \in H \) and \( r \in [0, \text{diam}(H, d_S)] \):
\[
\mu_S(B_{2r}(q)) \leq C \cdot \mu_S(B_r(q)).
\]
(ii) For balls $B_s \subset B \subset H$, we have a **relative lower volume decay** of order $n$. In different terms, the upper regularity dimension of $\mu_S$ is at most $n$:

\[(89)\quad \text{diam}(B_s)^n/\text{diam}(B)^n \leq C \cdot \mu_S(B_s)/\mu_S(B).\]

**Proof of 3.4** We use Prop. 3.1 to treat radius and volume estimates from separate compactness results for the spaces and the eigenfunctions.

- To this end we introduce a **radial gauge** of $d_S(\Phi)$ for $H \in \mathcal{G}$ in a given $p \in H$:

\[(90)\quad (B_1(p), d_S) \subset (B_1(p), d_H) \subset (B_{\kappa_0}(p), d_S),\]

where we replace $\Phi$ for some suitable multiple $k \cdot \Phi$, $k > 0$. Under this gauge it is enough to estimate $\text{Vol}((B_1(p), d_H), d_S(\Phi))$, that is, the unit ball relative to $d_H$ but with volume measured relative to $d_S$ for some $\Phi$ satisfying (72). When $(B_1(p), d_S)$ is given and needs to remain unchanged (e.g. this happens in (98) below), there is an $a > 0$ with

\[(91)\quad (B_1(p), \Phi^{4/(n-2)} \cdot g_H) \subset (B_1(p), a^2 \cdot g_H) \subset (B_{\kappa_0}(p), \Phi^{4/(n-2)} \cdot g_H).\]

Then we reinterpret $(B_1(p), \Phi^{4/(n-2)} \cdot g_H)$ as $(B_1(p), (a^{-2} \cdot \Phi^{4/(n-2)}) \cdot a^2 \cdot g_H)$ and $(B_{\kappa_0}(p), \Phi^{4/(n-2)} \cdot g_H)$ as $(B_{\kappa_0}(p), (a^{-2} \cdot \Phi^{4/(n-2)}) \cdot a^2 \cdot g_H)$. This yields a radial gauge not relative to $H$ but to $a \cdot H$ that equally belongs to $\mathcal{G}$.

- For $H \in \mathcal{H}_n^\mathbb{R}$ a radial gauge relative to $a \cdot H$, for some $a > 0$, suffices to get the volume estimates without gauging since we derive uniform estimates valid in the gauged case for all $H \in \mathcal{H}_n^\mathbb{R}$. For $H \in \mathcal{G}_n^\mathbb{R}$ we use the Bombieri–Giusti $L^1$–Harnack inequality (63) to get a positive lower estimate $b > 0$ for $\Phi$ on $H \setminus \Sigma$ and this means that we only need to ensure (91) for $a > b$.

We split the proof into five Claims. In Claims 1 and 2 we assume a radial gauge (90), whereas in Claims 3–5 we drop it and use (91) and the associated reduction to the gauged case.

**Claim 1.** The $d_S$-volume of the $d_H$–unit ball in $H \in \mathcal{H}_n^\mathbb{R}$ satisfies

\[(92)\quad a_n^S \leq \text{Vol}((B_1(0), d_H), d_S(\Phi)), \text{ for some } a_n^S > 0 \text{ depending only on } n.\]

**Proof of Claim 1.** We assume there were a sequence

\[(93)\quad H_i \in \mathcal{H}_n^\mathbb{R} \text{ with } \text{Vol}((B_1(0), d_{H_i}), d_S(\Phi_i)) \to 0 \text{ for } i \to \infty.\]

We start with the case $\Phi_i \equiv a_i \cdot G_i(\cdot, p_i)$ for $p_i \notin (B_1(0), d_{H_i}) \subset H_i$, where $G_i$ is the minimal Green’s function on $H_i \setminus \Sigma_{H_i}$, so that via $\text{ID}$-maps $p_i \to p_\infty$ for some regular point $p_\infty \notin (B_1(0), d_{H_\infty}) \subset H_\infty$, and where the $a_i > 0$ are chosen so that $\Phi_i$ satisfies the gauge (90). We know from [L2, Prop. 3.12] that the $G_i(\cdot, p_i)$ converge compactly to $G_\infty(\cdot, p_\infty)$, for the minimal Green’s function $G_\infty$ on $H_\infty \setminus \Sigma_{H_\infty}$. From this and the finiteness of $\text{Vol}((B_1(0), d_{H_\infty}), d_S(G_\infty))$, from 3.3, we infer that our assumption (93) implies that $a_i \to 0$ for $i \to \infty$.

Now we recall from Prop. 2.3 and Prop. 2.11 (49) that the constants $\alpha, \beta > 0$ and $\sigma > 0$ for the estimate $G_i^s(x, y) \leq \beta \cdot \exp(-\alpha \cdot d_{(A)^s}(x, y))$, which we have for $d_{(A)^s}(x, y) \geq 2 \cdot \sigma$, only depend
on $n$. From (50) we get a common finite upper bound for the $d_S$-length of hyperbolic geodesic rays from $p_i$ to 0. Since $a_i \to 0$ for $i \to \infty$, this shows that $(B_1(0), d_S(\Phi_i)) \not\subseteq (B_1(0), d_{H_i})$ for $i$ large enough, contradicting the chosen radial gauge.

This argument extends to more general $\Phi_i$ satisfying (93) since we know that under scalings and blow-ups of a given $H \in \mathcal{G}$, and similarly on $\mathcal{H}_n^R$, the boundary Harnack inequality 2.8 applies with a common Harnack constant that is independent of $i$, cf. Remark 2.12.2. \hfill $\square$

**Claim 2.** The volume of the $d_S$-unit ball in $H \in \mathcal{H}_n^R$ satisfies

$$
(94) \quad \text{Vol}(\mathcal{B}_1(0), d_H), d_S(\Phi)) \leq b_n^*, \text{ for some } b_n^* > 0 \text{ depending only on } n.
$$

**Proof of Claim 2.** This time we assume there were a sequence $H_i \in \mathcal{H}_n^R$ with $\text{Vol}(\mathcal{B}_1(0), d_H), d_S(\Phi)) \to \infty$, for $i \to \infty$.

Again, it is enough to consider the case $\Phi_i \equiv a_i \cdot G_i(\cdot, p_i)$, for $p_i \not\in (B_1(0), d_{H_i}) \subset H_i$, where $G_i$ is the minimal Green's function on $H_i \setminus \Sigma_{H_i}$ so that via ID-maps $p_i \to p_\infty$ for some regular point $p_\infty \not\in (B_1(0), d_{H_\infty}) \subset H_\infty$ and where the $a_i > 0$ are chosen so that $\Phi_i$ satisfies the gauge (90). We use again that the $G_i(\cdot, p_i)$ converge compactly to $G_\infty(\cdot, p_\infty)$. This time we additionally use, from the proof of 3.3, that $J_{H_i \setminus (B_{\sigma}(p_i), d(A_i^*)^\circ)}(G_i)$ upper bounds $\text{Vol}(\mathcal{B}_1(0), G_i^{4/(n-2)} \cdot g_H)$. To upper bound $J_{H_i \setminus (B_{\sigma}(p_i), d(A_i^*)^\circ)}(G_i)$ we can choose cut-off functions $\phi_i \geq 0$ with $\phi_i = G_i$ on $\partial \mathcal{B}_{\sigma}(p_i)$ and $\phi_i \equiv 0$ outside $\mathcal{B}_{2\sigma}(p_i)$ so that $J_{H_i \setminus (B_{\sigma}(p_i), d(A_i^*)^\circ)}(\phi_i) \leq c$, where $c > 0$ does not depend on $i$. Then (93) implies that $a_i \to \infty$, for $i \to \infty$ and we now show that this contradicts the second inclusion of the radial gauge (90). To this end we note that the compact ID-map convergence $G_i(\cdot, p_i) \to G_\infty(\cdot, p_\infty)$ shows that the $L^1$-norm of $G_i(\cdot, p_i)$ on $(B_1(0), d_{H_i})$ remains positively lower bounded. The Bombieri–Giusti $L^1$–Harnack inequality (63) and the argument of Prop. 2.16(ii) therefore show that $(B_1(0), d_{H_i}) \not\subseteq (B_{\kappa_0}(0), d_S)$ for $i$ large enough. \hfill $\square$

**Claim 3.** For constants $a_n^*, b_n^* > 0$ depending only on $n$, we have for any $H \in \mathcal{H}_n^R$:

$$
(96) \quad a_n^* \cdot r^n \leq \text{Vol}(\mathcal{B}_r(q), d_S) \leq b_n^* \cdot r^n, \text{ for any } r > 0.
$$

**Proof of Claim 3.** To determine the volume growth rate of balls in $H$ in terms of $r$, we use (91), as explained above, to see that the volume estimates from claims 1 and 2 hold also without the radial gauge. That is, we have for any $H \in \mathcal{H}_n^R$:

$$
(97) \quad a_n^* \leq \text{Vol}(\mathcal{B}_1(0), d_H), d_S(\Phi)) \leq b_n^*.
$$

Similarly, for $H \in \mathcal{H}_n^R$ we also have $r^{-1} \cdot H \in \mathcal{H}_n^R$ and we apply the unit ball estimate to $r^{-1} \cdot H \in \mathcal{H}_n^R$ and then rescale $r^{-1} \cdot H$ to $H$. Then the identity

$$
(98) \quad \text{Vol}(\mathcal{B}_r(0), \Phi^{4/(n-2)} \cdot g_H) = r^n \cdot \text{Vol}(\mathcal{B}_1(0), \Phi^{4/(n-2)} \cdot r^{-2} \cdot g_H),
$$

shows: $a_n^* \cdot r^n \leq \text{Vol}(\mathcal{B}_r(0), \Phi^{4/(n-2)} \cdot g_H) \leq b_n^* \cdot r^n$. \hfill $\square$
Claim 4. For \( H \in \mathcal{G}_n^c \) and \( r \in (0, r_H) \), for a suitably small \( r_{H,\Phi} > 0 \), we have for \( q \in \Sigma_H \):

\[
a_n \cdot r^n \leq \text{Vol}(B_r(q), d_S) \leq b_n \cdot r^n, \quad \text{for } a_n, b_n > 0 \text{ depending only on } n.
\]

**Proof of Claim 4.** We first consider one fixed \( H \in \mathcal{G}_n^c \) and assume a radial gauge. Then there are constants \( b > a > 0 \) and some small \( r > 0 \) so that: \( a \leq \text{Vol}(B_1(q), \Phi^{1/(n-2)} \cdot r^{-2} \cdot g_H) \leq b \), for any \( q \in \Sigma_H \) and \( r \in (0, r_H) \). Otherwise we had a converging sequence of points \( q_i \in \Sigma_H \) and of radii \( r_i \to 0 \), for \( i \to \infty \) so that these volumina would either converge to 0 or \( \infty \). Both cases can be ruled out as in Claim 1 and 2 for \( H \in \mathcal{H}_n^R \) above.

Moreover, the constants \( b > a > 0 \) can be chosen to depend only on the dimension. Otherwise we had a compactly converging sequence \( r_i^{-1} \cdot H_i \in \mathcal{G}_n^c \) with \( r_i \to 0 \), for \( i \to \infty \), and of unit balls \((B_1(q_i), \Phi^{1/(n-2)} \cdot r_i^{-2} \cdot g_H)\) satisfying a radial gauge, so that, again, the associated volumina converge either to 0 or \( \infty \). Both cases can be excluded as before.

Finally, we recall that \( \Phi \geq \overline{b} > 0 \) for some constant \( \overline{b} > 0 \) from the \( L^1 \)-Harnack inequality \((63)\). Then \((98)\) in the argument for Claim 3 applies to all \( r \in (0, r_{H,\Phi}) \), possibly after replacing \( r_{H,\Phi} \) by \( b \cdot r_{H,\Phi} \), since we only need to use \( \overline{a} \geq \overline{b}/r \) in \((91)\) to find a radial gauge for any \((B_1(q_i), \Phi^{1/(n-2)} \cdot r_i^{-2} \cdot g_H)\). This shows that there are \( \overline{a}_n, \overline{b}_n > 0 \) depending only on \( n \) so that \( \overline{a}_n \cdot r^n \leq \text{Vol}(B_r(q), d_S) \leq \overline{b}_n \cdot r^n \) for any \( H \in \mathcal{G}_n^c \) and \( r \in (0, r_{H,\Phi}) \). \( \square \)

**Claim 5.** For any \( H \in \mathcal{G}_n^c \) there are constants \( a(H, \Phi), b(H, \Phi) > 0 \) so that for any \( r \in [0, \text{diam}(H, d_S)] \) and any \( q \in H \):

\[
a \cdot r^n \leq \mu_S(B_r(q), d_S) \leq b \cdot r^n.
\]

**Proof of Claim 5.** This readily follows from Claim 4. For \( I := [r_H, \text{diam}(H, d_S)] \) we define

\[
a := \min\{a_n, \inf\{\text{Vol}(B_r(q), d_S)/r^n \mid r \in I\}\}, \quad b := \max\{b_n, \sup\{\text{Vol}(B_r(q), d_S)/r^n \mid r \in I\}\}. \square
\]

**Remark 3.6 (Regularity of \( \mu_S \))** As a consequence of the estimates in Prop. 3.4 and 3.3, we see that \( \mu_S \) is an **outer regular measure**, that is, we have for Borel subsets \( E \subset H \):

\[
(99) \quad \mu_S(E) = \inf\{\mu_S(A) \mid E \subset A, A \subset H \text{ open}\}.
\]

From Prop. 3.3 we know that \( \mu_S(H) < \infty \) for \( H \in \mathcal{G}_n^c \). Thus for any \( \varepsilon > 0 \) there is a neighborhood \( U_\varepsilon \) of \( \Sigma \) in \( H \) so that \( \mu_S(U_\varepsilon \setminus \Sigma) < \varepsilon \). This also holds for non-compact \( H \in \mathcal{G}_n \) from \( \mu_S(B_r(q)) < \infty \) using suitable ball covers of \( \Sigma \). From this we see that \( \mu_S \) is a **Borel measure** on \((H, d_S)\) cf.[H-T, pp. 62–64]. \( \square \)

**Remark 3.7 (Ahlfors Regularity of \( H \in \mathcal{G} \))** We note in passing that the original almost minimizers \( H \in \mathcal{G}_n \) are Ahlfors \( n \)-regular as well. For \( H \in \mathcal{H}_n^{R} \) this is [Gi, Prop. 5.14 and Rm. 5.15] and there are constants \( w_n > v_n > 0 \) depending only on \( n \) so that \( v_n \leq \text{Vol}(B_1(0,d_H), d_H) \leq w_n \) for any \( H \in \mathcal{H}_n^{R} \). For \( H \in \mathcal{G}_n^c \) the Ahlfors \( n \)-regularity follows similarly from the almost optimal isoperimetric inequality \( |\text{Vol}(B_r(p), d_H) - \text{Vol}(P_r)| \leq K \cdot r^{n+2-\alpha} \), for some \( \alpha \in (0,1), K > 0 \), and sufficiently small \( r > 0 \), where \( P_r \) is an area minimizing Plateau solution with boundary data \( H \) along \( \partial B_r(p) \) and \( c_n \) is the Euclidean volume of the unit ball. \( \square \)
3.3 Poincaré and Sobolev Inequalities

We start with some estimates for the minimal Green’s function on the twisted \( S \)-double cones we introduced in Prop. 2.5. Henceforth we use a fixed size parameter \( d \leq \min\{1, c/2\} \) from 2.5, Step 2 and recall that \( d \) is independent of the chosen \( p, q \in H \), that is, it depends only on \( H \in G \) and for \( H \in H_n^R \) it only depends on \( n \).

We consider the canonical Semmes families \( \Gamma_{p,q} \) and their envelopes \( E_{p,q} \) for some hyperbolic geodesic \( \gamma_{p,q} \). Due to the symmetry of the definition in \( p \) versus \( q \) it will be enough to analyze the part/side closer to \( p \) than to \( q \). To make this precise, we introduce some terminology. For \( \gamma \in \Gamma_{p,q} \) we consider those \( z \in \gamma \) closer to \( p \) than to \( q \), measured in \( \gamma \)-arc length and define:

\[
\gamma^+[z] := \text{subcurve of } \gamma \in \Gamma_{p,q} \text{ from } p \text{ to } z, \quad \text{and we have } l_H(\gamma^+[z]) = l_{\min}(\gamma_{p,q}(z)).
\]

For the midpoint \( m_\gamma \) of each such \( \gamma \) we define the half-curve family and the half-envelope

\[
\Gamma^+_{p,q}[d] := \{m_\gamma | \gamma \in \Gamma_{p,q}[d]\} \quad \text{and} \quad E^+_{p,q}[d] := \bigcup_{z \in \gamma_{m_\gamma}(p)} B_{d l_{\min}(\gamma_{p,q}(z))/c}(z).
\]

The counterparts starting from \( q \) are denoted by \( \Gamma^-_{p,q}[d] \) and \( E^-_{p,q}[d] \). We have \( E^+_{p,q}[d] \cap E^-_{p,q}[d] = B_{d l(\gamma_{p,q}(m))/c}(m) \), i.e. in some arguments the points in \( B_{d l(\gamma_{p,q}(m))/c}(m) \) are counted twice.

The particular choice of our core curve \( \gamma_{p,q} \) gives us a good control over the analysis on the \( E^+_{p,q}[d] \). We start with a variant of the Harnack inequality that controls the supersolutions \( \Phi > 0 \) of Def. 1.1 on \( E^+_{p,q}[d] \) transversally to the core geodesic \( \gamma_{p,q} \).

Lemma 3.8 (Transversal Harnack Inequalities) There are constants \( C(H, \Phi) > 0 \), and \( C(n) > 0 \) for \( H \in H_n^R \), so that for \( B_{d l_H(\gamma^+[z])/c}(z) \subset E^+_{p,q}[d] \) and any \( p, q \in H \):

\[
\sup_{B_{d l_H(\gamma^+[z])/c}(z)} \Phi \leq C \cdot \inf_{B_{d l_H(\gamma^+[z])/c}(z)} \Phi.
\]

Proof We first prove this for general positive solutions \( u > 0 \) of \( L_{H,\lambda} \Phi = 0 \) on \( E_{p,q}[1] \subset H \setminus \Sigma \). We can scale any of these ball \( B_{d l_H(\gamma^+[z])/c}(z) \) to unit size where the underlying geometry becomes uniformly bounded in \( C^3 \)-norm, cf. (29) in Prop. 2.5, Step 2, independently of \( z \) and also of \( H \). After scaling around any such \( z \), the exponential map pull-backs of \( L_{H,\lambda} = -\Delta + \frac{n-2}{(n-1)} \cdot \text{scal}_H \cdot \lambda \cdot |A|^2 \) to the Euclidean unit ball in the tangent space have uniformly bounded coefficients independent of \( z \). From this the Harnack inequality holds for positive solutions of the pull-back equations with the same Harnack constant on any of these unit balls. This relation is invariant under scalings and it survives the exponential map transfer and rescaling back to \( B_{d l_H(\gamma^+[z])/c}(z) \).

For \( H \in H_n^R \) this gives (102) for a constant that merely depends on \( n \). When \( H \in G^c \) there is a compact set \( K \subset H \setminus \Sigma \) so that \( \Phi > 0 \) is a solution on \( H \setminus (\Sigma \cup K) \) where we can apply the argument to balls in \( E_{p,q}[d] \) disjoint from \( K \). All balls with a non-empty intersection with \( K \) belong to another still compact subset \( K^* \subset H \setminus \Sigma \) where we find a constant satisfying (102) right from the continuity of \( \Phi > 0 \) on \( H \setminus \Sigma \).

Corollary 3.9 (Path Integral Estimate) There is a constant \( k(H, \Phi) > 0 \), with \( k(n) > 0 \) for \( H \in H_n^R \), so that for any \( p, q \in H \):

\[
l_S(\gamma^+[z]) = \int_{\gamma^+[z]} \Phi^{2/(n-2)} ds \leq k \cdot l_H(\gamma^+[z]) \cdot \Phi^{2/(n-2)}(z), \quad \text{for any } \gamma \in \Gamma^+_{p,q}[d].
\]
Proof. By Lemma 3.8 it is enough to consider the subcurves $c^{-}[z]$ of the core $\gamma_{p,q}$. Since (103) is scaling invariant, we may assume that $1_{H}(\gamma^{+}[z]) = 1$. We may multiply the inequality by a constant so that $\Phi(z) = 1$. The $\text{c}-\mathcal{S}$-uniformity of $(H, d_{H})$ implies that $\langle A \rangle(z) \leq c$ and that $d_{H}(p, z) \leq 1_{H}(\gamma^{+}[z]) \leq c \cdot d_{H}(p, z)$ and we can apply Prop. 2.10 to get a constant $\xi > 0$ with $\Phi(x) \leq \xi \cdot \gamma_{x,z}$, for any $x, z \in H \setminus \Sigma$. From (50) we know that the contributions of integrals $\int_{\gamma^{+}[z]} G(\cdot, z)^{2/(n-2)} \, ds$ outside $B_{\delta}^{c}(A)(z)$ are uniformly upper bounded. We have $\Phi(x) \leq c^{*} \cdot \gamma_{x,z}$, for some $c^{*}(H, \Phi) > 0$. Namely, for $H \in \mathcal{G}_{n}$ this follows for any such ball that intersects the compact set $K$ where $\Phi$ is only a supersolution. But for these balls such an upper estimate follows from the continuity of $\Phi$. For balls away from $K$, and for $H \in \mathcal{G}_{n}^{\alpha}$, we get the bound from elliptic estimates starting from $\Phi(z) = 1$ since, in the hyperbolic picture, the balls have bounded geometry, a fixed radius and the operator has uniformly bounded coefficients. For $H \in \mathcal{H}_{n}$ this also shows that $c^{*}(n) > 0$. Thus we get some $k^{*} > 0$ with the asserted dependencies and $\int_{\gamma^{+}[z]} \Phi^{2/(n-2)} \, ds \leq k^{*} = l_{H}(\gamma^{+}[z]) \cdot \Phi^{2/(n-2)}(z)$. \(\square\)

Now we show that the canonical Semmes families of curves $\Gamma_{p,q}[d]$ with the probability measure $\sigma_{p,q}(d)$ on $(H, d_{H}, \mu_{H})$ we have defined in Prop. 2.5 are still Semmes families in $(H, d_{S}, \mu_{S})$. This and the volume relations 3.5 imply Poincaré, Sobolev and isoperimetric inequalities for $(H, d_{S}, \mu_{S})$.

**Theorem 3.10 (Semmes Families on $(H, d_{S}, \mu_{S})$)** For $H \in \mathcal{G}_{n}$, there is some constant $C_{S}(H, \Phi) > 0$, $C_{S}(n) > 0$ for $H \in \mathcal{H}_{n}$, so that for $p, q \in H$, the family $\Gamma_{p,q}$ and the probability measure $\sigma_{p,q}$ on $\Gamma_{p,q}$, from 2.5, satisfy the two Semmes axioms relative to $(H, d_{S}, \mu_{S})$:

(i) For any $\gamma \in \Gamma_{p,q}$: $l_{S}(\gamma|_{[s,t]}) < C_{S} \cdot d_{S}(\gamma(s), \gamma(t))$, for $s, t \in I_{\gamma}$.

(ii) For any Borel set $A \subset X$, the assignment $\gamma \mapsto l_{S}(\gamma \cap A)$ is $\sigma$-measurable with

$$
\int_{\Gamma_{p,q}} l_{S}(\gamma \cap A) \, d\sigma(\gamma) \leq C_{S} \cdot \int_{A_{C_{S},p,q}} \left(\frac{d_{S}(p, z)}{\mu_{S}(B_{d_{S}(p,z)}(p))} + \frac{d_{S}(q, z)}{\mu_{S}(B_{d_{S}(q,z)}(q))}\right) \, d\mu_{S}(z)
$$

for $A_{C_{S},p,q} := \left(B_{C_{S},d_{S}(p,q)}(p) \cup B_{C_{S},d_{S}(p,q)}(q)\right) \cap A$.

**Proof of Property (i)** In the case of $H \in \mathcal{H}_{n}$ we first prove that there is a constant $c^{*}(n) > 0$ so that $l_{S}(\gamma_{x,y}) < c^{*}(n) \cdot d_{S}(x, y)$ for an arbitrary hyperbolic geodesic arc $\gamma_{x,y} \subset H$ linking two points $x, y \in H$ with $d_{H}(x, y) = 1$. For this we choose the midpoint $m \in \gamma_{x,y}$ of this $\text{c}^{-}\text{S}$-uniform curve in $(H, d_{H})$, measured in terms of curve length relative to $d_{H}$. Now we use the gauge $\Phi(m) = 1$ and apply (103) to the two subcurves of $\gamma_{x,y}$ starting from $m$. Since $\gamma_{x,y}$ is $\text{c}^{-}\text{S}$-uniform we have a length estimate $l_{S}(\gamma_{x,y}) \leq l_{n}$, for some $l_{n} > 0$ depending only on $n$. The $\text{c}^{-}\text{S}$-uniformity also shows that there is a ball $B_{r}(m)$ of radius $r_{n} > 0$ where we have $\langle A \rangle \leq a_{n}$ and thus we find a uniform Harnack estimate $b_{n} > 0$ so that $\Phi(m) \geq b_{n}$. The Bombieri–Giusti Harnack inequality then gives, as in (76) and (77), a lower estimate $c_{n} > 0$ for $\Phi$ on $B_{2r}(m)$ and from this we infer a lower estimate $e_{n} > 0$ for $d_{S}(x, y)$, that is, we have $l_{S}(\gamma_{x,y}) \leq l_{n}/e_{n} \cdot d_{S}(x, y)$.

In particular, this applies to any subcurve of the core geodesic we have in any of our families $\Gamma_{p,q}[d]$. In turn, the transversal Harnack inequality (102) yields a constant $C_{0}(n) \geq l_{n}/e_{n} > 0$ so that for any other $\gamma \in \Gamma_{p,q}[d]$ we have $l_{S}(\gamma|_{[s,t]}) < C_{0} \cdot d_{S}(\gamma(s), \gamma(t))$, for $s, t \in I_{\gamma}$.

For $H \in \mathcal{G}_{n}$ the subset where $\Phi$ is not a proper solution belongs to some compact $K \subset H \setminus \Sigma$.
Since any hyperbolic geodesic arc $\gamma_{x,y} \subset H$ is a $c$-$\mathcal{S}$-uniform curve in $(H, d_H)$, we have $l_H(\gamma_{x,y}) \leq c \cdot d_H(x,y)$. The contributions on $K$ of the conformal deformation by the upper and lower positively bounded function $\Phi$ merely alter $c$ to another constant that depends on the chosen $H$ and $\Phi$. We combine this with the argument for case (i) outside $K$ to infer the claim for $H \in \mathcal{G}_{H}^c$.

**Proof of Property (ii)** By Prop. 3.4, Lemma 3.9 and since both sides of (104) result from smooth deformations of $(H, d_H)$, they are still finite Borel measures:

$$
\mu_1(A) := \int_{\Gamma_{p,q}} l_S(\gamma \cap A) \, d\sigma(\gamma) \quad \text{and} \quad \mu_2(A) := \int_{\text{AC}_{p,q}} \left( \frac{d_S(p,z)}{\mu_S(B_{d_S(p,z)}(p))} + \frac{d_S(q,z)}{\mu_S(B_{d_S(q,z)}(q))} \right) d\mu_S(z).
$$

To derive inequality (104) we make a series of simplifications.

- We have $E_d(p,q) \cap \Sigma \subset \{p,q\}$. Thus to check (104) we only need to consider Borel sets $A \subset H \setminus \Sigma$.

- From the $\sigma$-additivity and the regularity of the $\mu_i$ we only need to show $\mu_1(B) \leq C \cdot \mu_2(B)$ for arbitrarily small balls $B = B_\varepsilon(x) \subset H \setminus \Sigma$ for any $x \in H \setminus \Sigma$, for some $C > 0$ independent of $B$ and of $x$.

- We may assume that for any $x \in H \setminus \Sigma$ the ball $B = B_\varepsilon(x)$ is small enough so that

$$
1/2 \cdot \Phi_{\frac{2}{n-2}}(x) \leq \Phi_{\frac{2}{n-2}}(y) \leq 2 \cdot \Phi_{\frac{2}{n-2}}(x), \quad \text{for } y \in B.
$$

Then we have for $A := B$:

$$
\int_{\Gamma_{p,q}} l_S(\gamma \cap A) \, d\sigma(\gamma) \leq \int_{\Gamma_{p,q}} \Phi_{\frac{2}{n-2}}(x) \cdot l_H(\gamma \cap A) \, d\sigma(\gamma) \leq ...
$$

Since $\Gamma_{p,q}$ is a Semmes family relative to $(H, d_H, \mu_H)$, we have from Prop. 2.5:

$$
\Phi_{\frac{2}{n-2}}(x) \cdot \int_{\Gamma_{p,q}} l_H(\gamma \cap A) \, d\sigma(\gamma) \leq \Phi_{\frac{2}{n-2}}(x) \cdot C \cdot \int_{\text{AC}_{p,q}} \frac{d(q,z)}{\mu_H(B_{d_H(q,z)}(q))} \, d\mu_H(z) \leq ...
$$

Using (105) again and $d(\gamma(s), \gamma(t)) \leq l_H(\gamma|[s,t]) < c \cdot d(\gamma(s), \gamma(t))$, for any $\gamma \in \Gamma_{p,q}$, $s, t \in I_\gamma$, and also that $a \cdot r^n \leq \mu_H(B_r(q)) \leq b \cdot r^n$, we get a constant $C_1 > 0$, depending only on $(H, \Phi)$, or depending only on $n$ for $H \in \mathcal{H}_n^\mathbb{R}$, with

$$
2 \cdot C \cdot \int_{\text{AC}_{p,q}} \Phi_{\frac{2}{n-2}}(z) \cdot \frac{d(q,z)}{\mu(H_{d_H(q,z)}(q))} \, d\mu_H(z) \leq C_1 \cdot \int_{\text{AC}_{p,q}} \Phi_{\frac{2}{n-2}}(z) \cdot \frac{l_H(\gamma_q[z])}{l_H(\gamma_q[z])^n} \, d\mu_H(z) \leq ...
$$

Now we apply Lemma 3.9, $l_H(\gamma_q[z])^{-1} \leq k \cdot \Phi_{\frac{2}{n-2}}(z)/l_S(\gamma_q[z])$ for any $z \in H \setminus \Sigma \cap B_\rho(q)$:

$$
C_1 \cdot k^{n-1} \cdot \int_{\text{AC}_{p,q}} \Phi_{\frac{2}{n-2}}(z) \cdot \left( \Phi_{\frac{2}{n-2}}(z)/l_S(\gamma_q[z]) \right)^{n-1} \, d\mu_H(z) = ...
$$

Finally, we get from $d_S(\gamma(s), \gamma(t)) \leq l_S(\gamma|[s,t]) < C_0 \cdot d_S(\gamma(s), \gamma(t))$ for any $\gamma \in \Gamma_{p,q}$, $s, t \in I_\gamma$, the Ahlfors regularity of $(H, d_S, \mu_S)$ saying that $a \cdot r^n \leq \mu_S(B_r(q), d_S) \leq b \cdot r^n$ (Theorem 3.4).
and the eccentricity estimate of Prop. 3.1 some new constant $C_2 > 0$, with the same dependencies, so that:

$$
C_1 \cdot k^{n-1} \cdot \int_{AC_{p,q}} \frac{l_S(\gamma_q(z))}{S(\gamma_q[z])^n} \cdot \Phi^{2n/(n-2)}(z) \cdot d\mu_H(z) \leq C_2 \cdot \int_{AC_{p,q}} \frac{d_S(q,z)}{\mu_S(B_{ds(q,z)}(q))} d\mu_S(z).
$$

Thus we can choose $C_S$ to be the maximum of $C_0$ and $C_2$. \qed

A standard application of Prop. 3.10 and the doubling property is the following weak $(1,1)$–Poincaré inequality [H-T, Ch.14.2]. To state it, we recall that a measurable function $w \geq 0$ on $(H, d_S)$ is an upper gradient of a measurable function $u$ if $|u(x) - u(y)| \leq \int_c w(s)ds$ holds for all rectifiable curves $c$ joining $x$ to $y$, for any pair $x, y \in H$.

**Proposition 3.11 (Poincaré Inequality I)** For any $H \in \mathcal{G}$, there are $C_0(H, \Phi) > 0$, $\gamma_0(H, \Phi) \geq 1$, depending only on $n$ for $H \in \mathcal{H}_n^R$, so that for any pair of concentric balls $B \subset \gamma_0 \cdot B \subset (H, d_S)$, for any function $u$ on $H$, integrable on bounded balls, and any upper gradient $w$ of $u$ we get:

$$
\int_B |u - u_B| d\mu_S \leq C_0 \cdot \text{diam}(B) \cdot \int_{\gamma_0 \cdot B} w d\mu_S, \quad \text{for } f_B := \int_B f d\mu_S := \int_B f d\mu_S/\mu_S(B).
$$

The volume decay property of order $n$ in Cor. 3.5(ii) allows us to improve this Poincaré inequality to the following Sobolev inequality.

**Corollary 3.12 (Sobolev Inequality)** For any $H \in \mathcal{G}$, there is a constant $C_1(H, \Phi) > 0$, depending only on $n$ for $H \in \mathcal{H}_n^R$, so that for some open ball $B \subset H$, an $L^1$-function $u$ on $B$ and any upper gradient $w$ of $u$ on $B$, we have

$$
\left( \int_B |u - u_B|^{n/(n-1)} d\mu_S \right)^{(n-1)/n} \leq C_1 \cdot \text{diam}(B) \cdot \int_B w d\mu_S.
$$

**Proof of 3.11 and 3.12** Cor. 3.5 and Prop. 3.10 imply the underlying Poincaré inequality 3.11, see [H-T, 14.2, p. 396]. From this we get the refinement to the Sobolev inequality 3.12 from [H-T, Th. 9.1.15(i)], for $p = 1$ and $Q = n$, see also [M, Th. 4.5 and Rm. 4.6] and [Se]. One first uses cut-off functions to restrict the support to $H \setminus \Sigma$ and then one extends the inequalities to $H$ using that the Hausdorff dimension of $\Sigma \subset (H, d_S)$ is $\leq n - 3$. \qed

In turn, Cor.3.12 and the doubling property of $(H, d_S, \mu_S)$ show, cf. [H-T, Remark 9.1.19], that Prop.3.11 can be improved so that we can drop the scaling factor $\gamma_0$.

**Corollary 3.13 (Poincaré Inequality II)** For any $H \in \mathcal{G}$, there are $C_1 > 0$, with the same dependencies as in 3.11, so that for any function $u$ on $H$, integrable on bounded balls, and any upper gradient $w$ of $u$, we get

$$
\int_B |u - u_B| d\mu_S \leq C_1 \cdot \text{diam}(B) \cdot \int_B w d\mu_S, \quad \text{for any ball } B \subset (H, d_S).
$$
3.4 Oriented Minimal Boundaries

We extend the Riemannian hypersurface area i.e. the element \( d\mu_S^{n-1} = \Phi^{2n/(n-2)} \cdot d\mu_H^{n-1} \), on \( H \setminus \Sigma \) to \((H,d_S,\mu_S)\) to formulate the isoperimetric inequality and the concept of oriented minimal boundaries. To this end we employ the BV (= bounded variations) approach of Ambrosio [A] and Miranda [M] on complete metric spaces with a doubling measure supporting a Poincaré inequality. By the results in the last two sections this theory applies to \((H,d_S,\mu_S)\).

We reformulate [M, Definition 4.1] as follows:

**Definition 3.14 (Perimeters in \((H,d_S,\mu_S)\))** For some Borel set \( E \) and an open set \( \Omega \) in \((H,d_S,\mu_S)\), for \( H \in \mathcal{G}_n \), we define the perimeter \( \mu_S^{n-1}(\partial E \cap \Omega) \), as

\[
\inf \left\{ \liminf_{k \to \infty} \int_{\Omega} |\nabla u_k| \, d\mu_S \left| \left. L^1_{\text{loc}}(\Omega) \right\} \right.,
\]

where \( \chi_E \) is the characteristic function of \( E \) and, for \( u \in \text{Lip}_{\text{loc}}(\Omega) \), we use the particular upper gradient \( |\nabla u|(x) := \liminf_{e \to 0} \sup_{y \in B_r(x)} |u(x) - u(y)|/e \). We call \( E \) a Caccioppoli set in \((H,d_S,\mu_S)\) provided \( \mu_S^{n-1}(\partial E \cap \Omega) < \infty \), for any bounded \( \Omega \).

For a smoothly bounded open set \( E \subset \mathbb{R}^n \) the perimeter is the hypersurface area of the boundary \( \partial E \) in \( \Omega \) that equals its \((n-1)\)-dimensional Hausdorff measure, see [Gi, Example 1.4]. In general, only the expression \( \mu_S^{n-1}(\partial E \cap \Omega) \) is relevant and well-defined. The perimeter (114) satisfies the coarea formula [M, Prop. 4.2] and thus we find many non-trivial Caccioppoli sets e.g. [M, Cor. 4.4], we have \( \mu_S^{n-1}(\partial \mathcal{B}_r(q)) < \infty \), for almost any \( r > 0 \) and \( q \in H \). From the lower semi-continuity of perimeters and the compactness of the BV-function space in the \( L^1_{\text{loc}} \)-function space [M, Prop.3.6 and 3.7] we get, as in [Gi, Th.1.20]:

**Proposition 3.15 (Plateau Problems in \((H,d_S,\mu_S)\))** Let \( \Omega \subset H \) be a bounded open and orientable set and let \( A \subset H \) be a Caccioppoli set. Then there exists a set \( E \subset H \) coinciding with \( A \) outside \( \Omega \) and such that

\[
\mu_S^{n-1}(\partial E \cap \Omega) \leq \mu_S^{n-1}(\partial F \cap \Omega)
\]

for every Borel set \( F \subset H \) with \( F = A \) outside \( \Omega \).

The classical regularity theory of [Gi, Ch.8] applies in the manifold \( H \setminus \Sigma \). It shows that such a minimizer \( E \) can be assumed to be an open subset of \( \Omega \) with boundary \( \partial E \) and so that \( \partial E \cap \Omega \setminus \Sigma \) is an area minimizing hypersurface smooth outside a set of Hausdorff dimension \( \leq n-7 \). In this case we call \( \partial E \) an oriented minimal boundary.

From the Poincaré inequality (113) and the Ahlfors \( n \)-regularity we have [M, Remark 4.6]:

**Corollary 3.16 (Isoperimetric Inequality)** For \( H \in \mathcal{G} \) there is a constant \( \gamma(H,\Phi) > 0 \), depending only on \( n \) when \( H \in \mathcal{H}_n^\mathbb{R} \), so that for any Caccioppoli set \( U \subset H \):

\[
\min \{ \mu_S(B_\rho \cap U), \mu_S(B_\rho \setminus U) \}^{(n-1)/n} \leq \gamma \cdot \mu_S^{n-1}(B_\rho \cap \partial U), \text{ for any } \rho > 0,
\]

From this and again the Ahlfors regularity, we have a counterpart of Euclidean volume growth estimates in [Gi, Prop. 5.14] for area minimizing hypersurfaces in \((H,d_S,\mu_S)\).
Proposition 3.17 (Volume Growth of Area Minimizers) For \((H, d_S, \mu_S)\), some open subset \(\Omega \subset H\) and an oriented minimal boundary \(L^{n-1} \subset \Omega\) bounding an open set \(L^+ \subset \Omega\) there are constants \(\kappa, \kappa^+ (H, \Phi) > 0\), so that for any \(p \in L\):

\[
(117) \quad \kappa \cdot r^{n-1} \leq \mu_S^{n-1}(L \cap B_r(p)) \quad \text{and} \quad \kappa^+ \cdot r^n \leq \mu_S(L^+ \cap B_r(p)),
\]

for \(r \in [0, (A/B)^{1/n} \cdot \dist(p, \partial \Omega)/4)\), where \(0 < A < B\) are the Ahlfors constants. For \(H \in H_{\mathbb{R}}\), \(\kappa, \kappa^+ > 0\) depend only on \(n\).

**Proof** We start with the inequality for \(L^+\). Since \(L\) is area minimizing, we get

\[
(118) \quad \mu_S^{n-1}(L \cap B_r(p)) \leq \mu_S^{n-1}(L^+ \cap \partial B_r(p)).
\]

Since \(r \mapsto \mu_S(L^+ \cap B_r(p))\) is nondecreasing and bounded, it is differentiable almost everywhere on \(\mathbb{R}^+\). For almost any \(r > 0\) we therefore get the following two inequalities:

\[
(119) \quad \mu_S^{n-1}(\partial(L^+ \cap B_r(p))) = \mu_S^{n-1}(L \cap B_r(p)) + \mu_S^{n-1}(L^+ \cap \partial B_r(p)).
\]

\[
(120) \quad \mu_S^{n-1}(\partial(L^+ \cap B_r(p))) \leq 2 \cdot \mu_S^{n-1}(L^+ \cap \partial B_r(p)) = 2 \cdot \frac{\partial}{\partial r} \mu_S(L^+ \cap B_r(p)).
\]

From the Ahlfors regularity, we notice that for \(U = L^+ \cap B_r(p), \ r \in [0, (A/B)^{1/n} \cdot \dist(p, \partial \Omega)/4)\) and \(\rho = \dist(p, \partial \Omega)\) we have for \(s \geq (B/A)^{1/n}\) and \(q \in H\): \(\mu_S(B_r(p)) \leq B \cdot r^n \leq \mu_S(B_{s \cdot r}(q))\).

From this we have for \(B_r(p) \cap B_{s \cdot r}(q) = \emptyset\), \(B_r(p) \cup B_{s \cdot r}(q) \subset B_{\rho}(p)\):

\[
(121) \quad \mu_S(B_{\rho}(p) \cap U) \leq \mu_S(B_{\rho}(p) \setminus U)
\]

and the isoperimetric inequality Cor. 3.16 for \(U = L^+ \cap B_r(p) \subset B_{\rho}(p)\) shows:

\[
(122) \quad \mu_S(L^+ \cap B_r(p))^{(n-1)/n} \leq 2 \cdot \gamma \cdot \frac{\partial}{\partial r} \mu_S(L^+ \cap B_r(p)).
\]

Integration gives the lower bound: \(\mu_S(L^+ \cap B_r(p)) \geq \kappa^+ \cdot r^n\), for some \(\kappa^+ > 0\). The same estimate applies to \(L^- = H \setminus L^+\). From this we get the inequality for \(\mu_S^{n-1}(L \cap B_r(p))\) using again the isoperimetric inequality: \(\gamma \cdot \mu_S^{n-1}(L \cap B_r) \geq (\kappa^+ \cdot r^n)^{(n-1)/n}\). \(\square\)

**References**

[A] Ambrosio, L.: Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces, Adv. Math. 159 (2001), 51–67

[AFP] Ambrosio, L., Fusco, N. and Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems, Clarendon Press (2000)

[An1] Ancona, A.: Negatively curved manifolds, elliptic operators, and the Martin boundary, Ann. of Math. 125 (1987), 495–536

[An2] Ancona, A.: *Théorie du potentiel sur les graphes et les variétés*, in: Ecole d’été de Prob. de Saint-Flour XVIII-1988, LNM 1427, Springer (1990), 1-112

[BHK] Bonk, M., Heinonen, J., Koskela, P.: *Uniformizing Gromov hyperbolic spaces*, Astérisque 270, SMF (2001)

[B-T] Bonk, M., Capogna, L., Hajłasz, P., Shanmugalingam, N. and Tyson, J.: Analysis in Metric Spaces, Notices of the AMS 67 (2020), 253-256
[BH] Bridson, M. and Haefliger, A.: *Metric Spaces of Non-Positive Curvature*, Springer (1999)
[Bi] Bishop, C. J. Quasiconformal mappings which increase dimension. Ann. Acad. Sci. Fenn. Ser. A I Math. 24 (1999), 397-407
[BG] Bombieri, E. and Giusti, E.: Harnack’s inequality for elliptic differential equations on minimal surfaces, *Invent. Math.* 15 (1972) 24-46
[C] Cheeger, J.: Differentiability of Lipschitz Functions on Metric Measure Spaces, *GAFA* (1999), 428 – 517
[CK] Cheeger, J. and Kleiner, B.: Differentiating maps into $L^1$ and the geometry of BV functions, *Ann. of Math.* 171 (2010), 1347-1385
[F] Federer, H.: The singular set of area minimizing rectifiable currents with codimension one and of area minimizing chains modulo two with arbitrary codimension, *Bulletin of AMS* 76 (1970), 767-771
[GH] Ghys, E., De La Harpe, P.: Sur les groupes hyperboliques d’après Mikhael Gromov, Progress in Mathematics 83 (1990), Birkhäuser
[Gi] Giusti, E.: Minimal Surfaces and functions of bounded variations, Birkhäuser Verlag (1984)
[G] Gromov, M.: Four Lectures on Scalar Curvature, *Perspectives in Positive Scalar Curvature*, ed. M.Gromov, B.Lawson, *World Scientific* (2022), https://doi.org/10.1142/12644
[GL] Gromov, M. and Lawson, B.: Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, *Publ. Math. IHES* 58 (1983), 295-408
[GS] Galloway, G. and Schoen, R.: A generalization of Hawking’s black hole topology theorem to higher dimensions, Comm. Math. Phys. 266 (2006), 571-576
[H] Hawking, S.: Black holes in general relativity, Commun. Math. Phys. 25 (1972), 152-166
[He] Heinonen, J.: Lectures on Analysis on Metric Spaces, Universitext, Springer (2001)
[H-T] Heinonen, J., Koskela, P., Shanmugalingam, N. and Tyson, J.: Sobolev Spaces on Metric Measure Spaces, Cambridge University Press, Cambridge (2015)
[KL] Kemper, M. and Lohkamp, J.: Potential Theory on Gromov Hyperbolic Manifolds of Bounded Geometry, arXiv:1805.02178 [math.DG]
[L1] Lohkamp, J.: Hyperbolic Unfoldings of Minimal Hypersurfaces, *Analysis and Geometry in Metric Spaces* 6 (2018), 96-128, https://doi.org/10.1515/agms-2018-0006
[L2] Lohkamp, J.: Potential Theory on Minimal Hypersurfaces I: Singularities as Martin Boundaries, *Potential Analysis* 53 (2020), 1493–1528, http://dx.doi.org/10.1007/s11118-019-09815-6
[L3] Lohkamp, J.: Potential Theory on Minimal Hypersurfaces II: Hardy Structures and Schrödinger Operators, *Potential Analysis* 55 (2021), 563–602, https://doi.org/10.1007/s11118-020-09869-x
[L4] Lohkamp, J.: The Secret Hyperbolic Life of Positive Scalar Curvature, *Perspectives in Positive Scalar Curvature*, ed. M.Gromov, B.Lawson, *World Scientific* (2022), https://doi.org/10.1142/12644
[L5] Lohkamp, J.: Scalar Curvature Splittings II: Removal of Singularities, Preprint
[L6] Lohkamp, J.: Skin Structures in Scalar Curvature Geometry, arXiv:1512.08252 [math.DG] (2015)
[M] Miranda, M. Functions of bounded variation on “good” metric spaces, J. Math. Pures Appl. 82 (2003), 975–1004
[SY1] Schoen, R. and Yau, S.T.: Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature, *Ann. of Math.* 110 (1979), 127-142
[SY2] Schoen, R. and Yau, S.T.: Positive Scalar Curvature and Minimal Hypersurface Singularities, arXiv:1704.05490 [math.DG] (2017)
[Se] Semmes, S.: Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincaré inequalities, *Selecta Math.* 2 (1996), 155-295
[SG] Shilov, G. E. and Gurevich, B. L.: Integral, Measure and Derivative, Dover Publications (1977)
[Si1] Simon, L.: Lectures on Geometric Measure Theory, *Proc. Centre for Math. Analysis, Canberra* (1983)
[Si2] Simon, L.: A strict maximum principle for area minimizing hypersurfaces, *JDG* 26 (1987), 327–335