Singular Improper Affine Spheres from a given Lagrangian Submanifold

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Abstract. Given a Lagrangian submanifold $L$ of the affine symplectic $2n$-space, one can canonically and uniquely define a center-chord and a special improper affine sphere of dimension $2n$, both of whose sets of singularities contain $L$. Although these improper affine spheres (IAS) always present other singularities away from $L$ (the off-shell singularities studied in [6]), they may also present singularities other than $L$ which are arbitrarily close to $L$, the so called singularities “on shell”. These on-shell singularities possess a hidden $\mathbb{Z}_2$ symmetry that is absent from the off-shell singularities. In this paper, we study these canonical IAS obtained from $L$ and their on-shell singularities, in arbitrary even dimensions, and classify all stable Lagrangian/Legendrian singularities on shell that may occur for these IAS when $L$ is a curve or a Lagrangian surface.

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1. Introduction

An improper affine sphere (IAS) is a hypersurface whose affine Blaschke normal vectors are all parallel. They are given as the graph of a function $F : \mathbb{R}^k \to \mathbb{R}$ satisfying the Monge-Ampère equation

$$\det(D^2 F) = \pm 1.$$  \hfill (1.1)

In the lowest dimensional cases, surfaces in $\mathbb{R}^3$, there are two classes of improper affine spheres: the convex ones, satisfying the equation $\det(D^2 F) = 1$, and the non-convex ones, satisfying the equation $\det(D^2 F) = -1$.

For two planar curves $\alpha^+ : U \subset \mathbb{R} \to \mathbb{R}^2$ and $\alpha^- : V \subset \mathbb{R} \to \mathbb{R}^2$, denote by $x(u, v) = \frac{1}{2}(\alpha^+(u) + \alpha^-(v))$ the mid-point of $(\alpha^+(u), \alpha^-(v))$ and denote by $f(u, v)$ the area of the region bounded by the chord connecting...
\[ \alpha^+(u) \text{ and } \alpha^-(v), \text{ plus a chosen fixed chord } \xi_0 \text{ connecting arbitrary points } \alpha^+(u_0) \text{ and } \alpha^-(v_0), \text{ plus the arcs of } \alpha^+ \text{ and } \alpha^- \text{ between these two chords} \]

\[ (f \equiv f_{\xi_0} \text{ depends on the choice of } \xi_0, \text{ of course, but for another choice } \xi'_0, f_{\xi'_0} - f_{\xi_0} = \text{ constant}) \]

The map \((u,v) \mapsto (x(u,v), f(u,v))\) in a non-convex IAS and, conversely, any 2-dimensional non-convex IAS is locally as above, for certain curves \(\alpha^+\) and \(\alpha^-\). Since the mid-chord \(y(u,v) = \frac{1}{2}(\alpha^+(u) - \alpha^-(v))\) is the symplectic gradient of \(f\), this type of IAS is called center-chord. This construction was generalized in \([3]\) to arbitrary even dimensions substituting the pair of planar curves by a pair \((L^+, L^-)\) of Lagrangian submanifolds of \(\mathbb{R}^{2n}\). The center-chord IAS is independent of parameterizations of the Lagrangian submanifolds. The singular set of a center-chord IAS is given by the pairs \((L^+, L^-)\) of \(\mathbb{R}^{2n}\). Denote by \(E_{sp}(L)\), where \(L\) is the graph of \(dH\) in \(\mathbb{C}^n \times \mathbb{C}^n\).

For the center-chord and the special IAS, the function \(F : \mathbb{R}^{2n} \to \mathbb{R}\), given by \(F(x) = f(u,v)\), satisfies the Monge-Ampère equation \((1.1)\), but generically each such solution \(F\) of the Monge-Ampère equation has singularities, as studied in \([6]\). On the other hand, what was not explored in \([6]\) and is the object of the present paper is that, in various instances, a subset of the singular set of \(F\) is a Lagrangian submanifold \(L \subset \mathbb{R}^{2n}\).

In fact, by taking the same Lagrangian submanifold, \(L^+ = L^- = L\), we obtain an interesting subclass of the center-chord improper affine spheres. In this case, \(L\) is contained in the \textit{Wigner caustic} \(E_{cc}(L)\) of \(L\). The study of the Wigner caustic of \(L\) is of some interest in physics \((9), (10)\), and this subclass of the center-chord IAS is also of interest in computational vision \((5), (15)\).

In this paper we introduce the corresponding subclass for special IAS. This subclass consists of special IAS defined by holomorphic maps \(H : \mathbb{C}^n \to \mathbb{C}\) that takes the real space \(\mathbb{R}^n\) into the real line \(\mathbb{R}\), which implies that the real function \(Q\), above, is an odd function of \(t\). Denote by \(L\) the image of the real space \(\mathbb{R}^n\) by the map \(x\), which is a Lagrangian submanifold of \(\mathbb{R}^{2n}\). Since

\[ Y(x) = y(u,v) \text{ is the Hamiltonian vector field of } F(x) = f(u,v), \text{ for } x = x(u,v) \text{ the center as above, with respect to the canonical symplectic form on } \mathbb{R}^2 \ni x. \]

\[ Y(x) = y(s,t) \text{ is the Hamiltonian vector field of } F(x) = f(s,t), \text{ for } x = x(s,t) \text{ as above.} \]
the holomorphic map $H$ can be recovered from $L$, we shall denote by $E_{sp}(L)$ the corresponding caustic of the special IAS. As in the center-chord case, $L$ is contained in $E_{sp}(L)$. In [5], this type of IAS was considered for $n = 1$.

Generically, the sets $E_{cc}(L)$ and $E_{sp}(L)$ contain $L$ and other points away from $L$, but they may also contain more points than just $L$ in any neighborhood of $L$, the so-called on-shell part of $E_{cc}(L)$ and $E_{sp}(L)$, respectively. In [6], singularities of $E_{cc}(L) \setminus E_{cc}^s(L)$ and $E_{sp}(L) \setminus E_{sp}^s(L)$, also called off-shell singularities, were studied and classified. In this paper, we shall study and classify the singularities of $E_{cc}^s(L)$ and $E_{sp}^s(L)$, and of their Legendrian analogues $\tilde{E}_{cc}^s(L)$ and $\tilde{E}_{sp}^s(L)$. But since both $E_{cc}^s(L)$ and $E_{sp}^s(L)$ are caustics of Lagrangian maps (respectively Legendrian maps, for $\tilde{E}_{cc}^s(L)$ and $\tilde{E}_{sp}^s(L)$), it is natural to study them in this context. The basic tools for this study are generating functions and generating families.

For center-chord IAS, write

$$L = \{(q, p) \in \mathbb{R}^{2n} | p_i = \frac{\partial S}{\partial q_i}, \ 1 \leq i \leq n\}.$$  

(1.2)

Then, a generating function for $E_{cc}^s(L)$ in a neighborhood of $L$ is given by

$$G_{cc}(\beta, q, p) = \frac{1}{2} (S(q + \beta) - S(q - \beta)) - p \cdot \beta.$$  

(1.3)

Since $G_{cc}$ is odd in $\beta$, we can consider only odd deformations of the generating function. The set $E_{cc}^s(L)$ was studied in [9], where its stable Lagrangian singularities were classified, when $L$ is a curve or a surface. In this paper, we use these results to classify the stable Legendrian singularities on shell of the center-chord IAS associated to curves and Lagrangian surfaces.

In the special case, a generating family for $E_{sp}^s(L)$ in a neighborhood of $L$ is given by

$$G_{sp}(\beta, q, p) = \frac{1}{2} (S(q + \beta) - S(q - \beta)) - p \cdot \beta.$$  

(1.4)

As in the center-chord case, $G_{sp}$ is odd in $\beta$ and thus we can consider only odd deformations of the generating functions. In this paper, we classify the stable Lagrangian and Legendrian singularities on shell of the special IAS associated to curves and Lagrangian surfaces.

Thus, the important characteristic of these on-shell singularities, both in center-chord and special cases, is that they possess a hidden $\mathbb{Z}_2$ symmetry that is absent from the off-shell singularities. This is relevant for the geometric study of solutions of the Monge-Ampère equation for functions $F : \mathbb{R}^{2n} \rightarrow \mathbb{R}$, whose singularity set contains a Lagrangian submanifold $L \subset \mathbb{R}^{2n}$.

On the other hand, a remarkable distinction of the special case is that $G_{sp}$ is necessarily real analytic, and thus, in the special case, we should in principle consider the classification of singularities up to analytic equivalence. It turns out, however, that imposing real analyticity amounts to no refinement on the classification under smooth diffeomorphisms [4], so the classification of singularities of odd smooth functions, presented in [9], apply equally to the classification of singularities on shell of both center-chord and special IAS.

This paper is organized as follows:
In section 2, we review the properties of center-chord and special IAS, emphasizing the ones obtained from a single Lagrangian submanifold; we also characterize the IAS coming from a single Lagrangian submanifold among all possible center-chord and special IAS, showing that for each $L$ there is a unique canonical center-chord IAS and a unique canonical special IAS obtained from $L$. In section 3, we describe the on-shell odd generating families for the canonical center-chord and special IAS and then, in section 4, we further study equivalence and versality in the odd category, expanding and complementing the treatment developed in [9]. Section 4 is also intended to clarify many results in singularity theory which are not so familiar to nonspecialists, in view of the interdisciplinary nature of the paper. Then, in section 5, we present the classification of simple Lagrangian and Legendrian singularities on shell for the canonical center-chord and special IAS obtained from $L$, producing explicit examples that show they are all realized, and we also present the classification and geometrical condition/interpretation of all stable Lagrangian and Legendrian singularities on shell for the canonical center-chord and special IAS obtained from curves and Lagrangian surfaces, by adapting the results presented in [9]. Finally, in Section 6 we present the (not too short) proof of a main theorem stated in Section 4.

2. Singular center-chord and special IAS

2.1. Center-chord IAS

Let $U, V$ be open subsets of $\mathbb{R}^n$. Consider a pair of Lagrangian immersions $\alpha^+ : U \to \mathbb{R}^{2n}$ and $\alpha^- : V \to \mathbb{R}^{2n}$, where $\mathbb{R}^{2n}$ is the affine symplectic space with the canonical symplectic form $\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$, $q = (q_1, ..., q_n)$, $p = (p_1, ..., p_n)$. Define $x, y : U \times V \to \mathbb{R}^{2n}$ by

$$x(u, v) = \frac{1}{2} (\alpha^+(u) + \alpha^-(v)), \quad y(u, v) = \frac{1}{2} (\alpha^+(u) - \alpha^-(v)).$$

(2.1)

Fix a pair of parameters $(u_0, v_0) \in U \times V$. For a given $(u, v) \in U \times V$, consider the oriented curve $\delta(u, v, u_0, v_0)$ in $\mathbb{R}^{2n}$ obtained by concatenating the chord connecting $\alpha^+(u_0)$ and $\alpha^-(v_0)$, a curve in $L^{-}$ connecting $\alpha^-(v_0)$ and $\alpha^-(v)$, the chord connecting $\alpha^-(v)$ and $\alpha^+(u)$ and a curve in $L^{+}$ connecting $\alpha^+(u)$ and $\alpha^+(u_0)$.

Because $\delta$ is closed and $\mathbb{R}^{2n}$ is simply connected, $\delta = \partial\Sigma$, and because $\omega$ is exact, $\omega = d\eta$, where $\eta = pdq$. Denote

$$f[\delta] = \int_{\Sigma} \omega = \int_{\delta} \eta = \sum_{i=1}^{n} \int_{\delta(u, v, u_0, v_0)} p_i dq_i.$$

It is clear that $f[\delta]$ is independent of the choice of the curves along $L^{+}$ and $L^{-}$ (Lagrangian condition) and that, if we change the initial pair $(u_0, v_0)$ we just add a constant to $f$. Thus, up to a constant, we can write $f[\delta] = f(u, v)$. The following proposition was proved in [9]:

### Proposition

[Proposition content]

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Proposition 2.1. The function \( f : U \times V \to \mathbb{R} \) defined above is independent of the parameterizations of \( L^+ \) and \( L^- \) and satisfies

\[
f_u = \omega(x_u, y), \quad f_v = \omega(x_v, y).
\]

Moreover, the map \( \phi : U \times V \subset \mathbb{R}^{2n} \to \mathbb{R}^{2n+1} \phi(u, v) = (x(u, v), f(u, v)) \) is an improper affine map, which, when regular, defines an improper affine sphere (IAS).

Remark 2.2. In all cases of IAS considered in this paper, we shall often abbreviate and refer to singularities of the map as singularities of the IAS.

The type of IAS from Proposition 2.1 is called a center-chord IAS. By a smooth change of coordinates, we may assume that locally

\[
\alpha^+(u) = (u, dS^+(u)), \quad \alpha^-(v) = (v, dS^-(v)). \tag{2.2}
\]

for some pair of functions \( S^+ : U \subset \mathbb{R}^n \to \mathbb{R} \) and \( S^- : V \subset \mathbb{R}^n \to \mathbb{R} \).

The singular set of the center-chord IAS consists of the pairs \((u, v) \in U \times V\) such that the tangent spaces of \( \alpha^-(u) \) and \( \alpha^+(v) \) are not transversal subspaces of \( \mathbb{R}^{2n} \), or equivalently, \( d^2 S^+(u) - d^2 S^-(v) \) is singular. The image of the singular set by the map \( x(u, v) \) is called the Wigner caustic of \((L^-, L^+)\) and will be denote by \( E_{cc}(L^-, L^+) \), while the image of the singular set by \( \phi = (x, f) \) will be denoted \( \tilde{E}_{cc}(L^-, L^+) \).

Center-chord IAS from a given Lagrangian submanifold. In this paper, we shall be particularly interested in the case that the Lagrangian submanifolds \( L^+ \) and \( L^- \) coincide, i.e.,

\[
\alpha^+(u) = \alpha(u), \quad \alpha^-(v) = \alpha(v),
\]

for some Lagrangian immersion \( \alpha : U \subset \mathbb{R}^n \to \mathbb{R}^{2n} \). In this case we shall denote the image of this immersion by \( L = L^+ = L^- \) and the corresponding IAS by \( \phi_{cc}(L) \). In case \( \alpha \) is of the form (2.2) we shall write

\[
\alpha(u) = (u, dS(u)), \tag{2.3}
\]

for some function \( S : U \subset \mathbb{R}^n \to \mathbb{R} \).

When \( L^+ = L^- = L \), the caustic \( E_{cc}(L, L) \) is the Wigner caustic of the Lagrangian submanifold \( L \) and will be denoted \( E_{cc}(L) \). In this case, the set \( u = v \) is contained in the singular set of \( \phi \). Since \( u = v \) along \( L \), we conclude that \( L \subset E_{cc}(L) \) (see also [9] and [10]).

Example 1. Assume that \( \alpha(u) = (\cos(u), \sin(u)) \), i.e., \( L \) is the unit circle in the plane. Then

\[
x(u, v) = \cos\left(\frac{u - v}{2}\right) \left(\cos\left(\frac{u + v}{2}\right), \sin\left(\frac{u + v}{2}\right)\right),
\]

\[
y(u, v) = \sin\left(\frac{u - v}{2}\right) \left(-\sin\left(\frac{u + v}{2}\right), \cos\left(\frac{u + v}{2}\right)\right),
\]

\[
f(u, v) = \frac{1}{4} (v - u + \sin(u - v)),
\]

\[
= \frac{1}{4} \left(\cos\left(\frac{u + v}{2}\right) - \cos\left(\frac{u - v}{2}\right)\right) = \frac{1}{4} \left(\cos\left(\frac{u + v}{2}\right) - \cos\left(\frac{u - v}{2}\right)\right).
\]
The image of the map \( x \) is the unit disc \( D \) and the singular set is \( u = v + k\pi, k \in \mathbb{Z} \). This example can be generalized by taking
\[
\alpha(u) = (\cos(u_1), \sin(u_1), \ldots, \cos(u_n), \sin(u_n)), \quad u = (u_1, \ldots, u_n),
\]
so that \( L \) is the \( n \)-dimensional torus in \( \mathbb{R}^{2n} \). Then
\[
f(u, v) = \frac{1}{4} \sum_{i=1}^{n} (v_i - u_i + \sin(u_i - v_i)).
\]
The singular set of this center-chord IAS is the union of submanifolds
\[\mathbb{R}^2 \times \mathbb{R}^2 \ldots \times \{u_i = v_i + k\pi\} \times \ldots \mathbb{R}^2,\]
where \( \{u_i = v_i + k\pi\} \) is in coordinate \( i \).

2.2. Special IAS
Consider a complex Lagrangian immersion \( \gamma = (\gamma_1, \gamma_2) : W \subset \mathbb{C}^n \to \mathbb{C}^{2n} \) and denote its image by \( L \). Also, let \( x, y : W \to \mathbb{R}^{2n} \) be given by
\[
x(w) = \frac{1}{2} (\gamma(w) + \bar{\gamma}(w)); \quad y(w) = \frac{1}{2i} (\gamma(w) - \bar{\gamma}(w)),
\]
w \( \in W, \ w = u + iv \). The following proposition was proved in [6]:

**Proposition 2.3.** There exists \( f : W \to \mathbb{R} \), unique up to an additive constant, such that
\[
f_u = \omega(x_u, y), \quad f_v = \omega(x_v, y)
\]
and the map
\[
\phi(u, v) = (x(u, v), f(u, v))
\]
is an improper affine map, which, when regular, defines an improper affine sphere (IAS). Moreover, the IAS \( \phi \) does not depend on the parameterization of the complex Lagrangian immersion \( \gamma \).
From equations (2.4) and (2.5), the function \( f \) can be given a geometrical description similar to the one in the center-chord case. Let \( \delta = \delta(w,w_0) \) be a oriented curve formed by the concatenation of the imaginary chord joining \( \gamma(w_0) \) to \( \gamma(w) \), a curve in \( L \) joining \( \gamma(w_0) \) to \( \gamma(w) \), the imaginary chord joining \( \gamma(w) \) to \( \gamma(w) \), and a curve in \( L \) joining \( \gamma(w) \) to \( \gamma(w_0) \).

Because \( \delta \) is closed and \( \mathbb{C}^{2n} \) is simply connected, \( \delta = \partial \Sigma \), and because \( \omega \) is real and exact, \( \omega = d\eta \), where \( \eta = \frac{i}{2} w d\bar{w} \). Denote

\[
    f[\delta] = \int_\Sigma \omega = \int_\delta \eta = \frac{i}{2} \sum_{k=1}^{n} \int_{\delta(w,w_0)} w_k d\bar{w}_k .
\]

It is clear that the real function \( f[\delta] \) is independent of the choice of the curves along \( L \) and \( \bar{L} \) (Lagrangian condition) and that, if we change the initial point \( w_0 \) we just add a constant to \( f \). Thus, up to a constant, we can write \( f[\delta] = f(w) \).

The type of IAS from Proposition 2.3 is called special [(3)]. As shown in [6], by a holomorphic change of coordinates, we may locally reparameterize \( \gamma \) by

\[
    \gamma(z) = (z,dH(z)), \quad z \in Z, \tag{2.6}
\]

where \( H : Z \subset \mathbb{C}^n \to \mathbb{C} \) is a holomorphic map. Furthermore, setting \( z = s + it \) and \( H = P + iQ \), we can write

\[
    x(z) = (s, \frac{\partial Q}{\partial t}), \quad y(z) = (t, \frac{\partial Q}{\partial s}), \tag{2.7}
\]

and we also have that, up to an additive constant,

\[
    f(s,t) = Q(s,t) - \sum_{k=1}^{n} t_k \cdot \frac{\partial Q}{\partial t_k}. \tag{2.8}
\]

The singular set of the special IAS consists of points \((s,t) \in U \) such that \( \frac{\partial^2 Q}{\partial t^2} \) is singular. The image of the singular set by \( x \) will be denoted \( E_{sp}(L) \), while the image of the singular set by \( (x,f) \) will be denoted \( E_{sp}(\bar{L}) \).

**Remark 2.4.** Recall that, in both the center-chord and the special cases, the regularity of the map \( \phi : (u,v) \mapsto (x(u,v), f(u,v)) \) is equivalent to having an invertible map \( (u,v) \mapsto x(u,v), x^{-1} \) possibly multiple valued, so that each function \( F = f \circ x^{-1} : U \subset \mathbb{R}^{2n} \to \mathbb{R} \) is well defined and satisfies the Monge–Ampère equation (1.1), and each regular branch of \( \phi \) is a graph of \( F \).

**Special IAS from a given Lagrangian submanifold.** In this paper we shall be interested in the case \( Z \) is a domain in \( \mathbb{C}^n \) invariant by conjugation and \( \bar{\gamma}(z) = \gamma(\bar{z}) \), which is equivalent to saying that \( H(\mathbb{R}^n \cap Z) \) is contained in \( \mathbb{R} \). In this case, we shall denote by \( L \) the image of \( Z \cap \mathbb{R}^n \) by the map \( x(z) \).

Assume now we are given a real analytic Lagrangian submanifold \( L \), image of \( \alpha \) given by equation (2.3), for some \( S : U \subset \mathbb{R}^n \to \mathbb{R} \) real analytic. Let \( Z \) be a domain in \( \mathbb{C}^n \) invariant by conjugation and such that \( Z \cap \mathbb{R}^n = U \). Then there exists a holomorphic map \( H : Z \to \mathbb{C} \) such that \( H \) restricted to \( U \) equals \( S \). In particular, the image of \( Z \cap \mathbb{R}^n \) by \( H \) is contained in \( \mathbb{R} \).
It is clear from the above two paragraphs that $L \cap \mathbb{L} = L \subset \mathbb{R}^{2n}$ is a Lagrangian submanifold of the real symplectic space and that the special IAS defined by $\gamma(z) = (z, dH(z))$ depends only on $L$. Therefore we shall denote it by $\phi_{sp}(L)$, and we shall denote by $E_{sp}(L)$ the caustic of $\phi_{sp}(L)$.

We may assume that $L$ is given by equation (2.6), for some $H : \mathbb{Z} \rightarrow \mathbb{C}$ satisfying $\bar{H}(z) = H(\bar{z})$. This implies that the imaginary part $Q$ of $H$ is odd in $t$, where $z = s + it$, and so $\frac{\partial^2 Q}{\partial t^2} = 0$ for $t = 0$. We conclude that $L \subset E_{sp}(L)$.

**Example 2.** Consider $\gamma(z) = (\cos(z), \sin(z))$. Then $L$ is the unit circle in the plane and

$$x(s, t) = \cosh(t) (\cos(s), \sin(s)); \quad y(s, t) = \sinh(t) (-\sin(s), \cos(s)),
$$

$$f = \frac{1}{4} (\sinh(2t) - 2t),$$

(see figure 2). The image of the map $x$ is the complement $\bar{D}$ of the open unit disc, while the singular set is $t = 0$. This example can be generalized by considering $\gamma : \mathbb{C}^n \rightarrow \mathbb{C}^{2n}$ given by

$$\gamma(z) = (\cos(z_1), \sin(z_1), ..., \cos(z_n), \sin(z_n)), \quad z = (z_1, ..., z_n).$$

In this case $L$ is the $n$-dimensional torus $L \subset \mathbb{R}^{2n}$ and

$$f(s, t) = \frac{1}{4} \sum_{i=1}^{n} (\sinh(2t_i) - 2t_i).$$

The caustic the union of submanifolds $\mathbb{R}^2 \times \mathbb{R}^2 \times ... \times \{t = 0\} \times ... \mathbb{R}^2$, where $\{t = 0\}$ is in coordinate $i$.

![Figure 2. The singular special IAS of example 2](image-url)
2.3. Affine Björling problem

The affine Björling problem for \( n = 1 \) consists in finding an improper affine sphere containing a smooth curve, analytic in the convex plane, with a prescribed co-normal along it. We observe that the co-normal for both types of IAS is given by \( (y, 1) \) and thus this problem is equivalent to finding an IAS given the values of \((x, f)\) and \(y\) along a curve in the parameter plane. The affine Björling problem (for \( n = 1 \)) has a unique solution for the center chord case (see [13], thm.3.1 and [14], thm.6.1) and also for the special case (see [1], thm.6.1). We shall see below that by taking \( y = 0 \) along the curve \( L \), we obtain the IAS \( \phi_{cc}(L) \) and \( \phi_{sp}(L) \) in each case.

We now let \( n \) be general and characterize \( \phi_{cc}(L) \) among the center-chord IAS \( \phi_{cc}(L^{-}, L^{+}) \) and \( \phi_{sp}(L) \) among the special IAS \( \phi_{sp}(\mathbb{L}) \).

For a center-chord IAS \( \phi = \phi_{cc}(L^{-}, L^{+}) \), denote by \( E_{cc,n}(L^{-}, L^{+}) \) the subset of \( E_{cc}(L^{-}, L^{+}) \) such that the tangent spaces to \( L^{+} \) at \( \alpha^{+}(u) \) and to \( L^{-} \) at \( \alpha^{-}(v) \) are strongly parallel. For \( \alpha^{\pm} \) of the form \((2.2)\), this is equivalent to having \( d^{2}S^{+}(u) = d^{2}S^{-}(v) \).

For a special IAS \( \phi = \phi_{sp}(\mathbb{L}) \), denote by \( E_{sp,n}(\mathbb{L}) \) the subset of \( E_{sp}(\mathbb{L}) \) such that the tangent spaces to \( \mathbb{L} \) at \( \gamma(z) \) and to \( \mathbb{L} \) at \( \gamma(z) \) are strongly parallel. For \( \gamma \) of the form \((2.6)\), this is equivalent to having \( \frac{\partial^{2}Q}{\partial t^{2}} = 0 \).

**Proposition 2.5.** Let \( \phi = \phi_{cc}(L^{-}, L^{+}) \) and \( L \) a Lagrangian submanifold of \( \mathbb{R}^{2n} \). The following statements are equivalent:

1. \( \phi = \phi_{cc}(L) \).
2. \( L \subset E_{cc,n}(L^{-}, L^{+}) \) and \( f \) is constant along \( L \).
3. \( y = 0 \) along \( L \).

**Proof.** (1) \( \Rightarrow \) (2) follows from the description of \( \phi_{cc}(L) \) given in section 2.1.

(2) \( \Rightarrow \) (3): If \( f \) is constant along \( L \), then necessarily \( \omega(y, x_{u}) = \omega(y, x_{v}) = 0 \), implying that the chord \( y \) is tangent to \( L \). On the other hand, \( L \subset E_{cc,n}(L^{-}, L^{+}) \) implies that the tangent spaces of \( \alpha^{+}(u) \) and \( \alpha^{-}(u) \) are strongly parallel. We conclude that the tangent spaces in fact coincide and \( y = 0 \).

(3) \( \Rightarrow \) (1): The condition \( y = 0 \) at \( L \) implies that \( u = v \) and \( ds^{+}(u) = ds^{-}(v) \).

Thus \( L \) is contained in the image of the diagonal \( u = v \) and \( ds^{+}(u) = ds^{-}(u) \).

This implies that, up to a constant, \( S^{+}(u) = S^{-}(u) \) and so \( \phi = \phi_{cc}(L) \). \( \square \)

**Remark 2.6.** If \( L \supset A \), where \( A \) is an affine subspace of \( \mathbb{R}^{2n} \), then condition (2) only implies that \( L = L^{+} = L^{-} \), thus implying (1), since it is possible to have a nonvanishing \( y \in TA \) if \( \alpha^{+} \neq \alpha^{-} \). But then, by choosing the canonical parametrization \( \alpha^{+} = \alpha^{-} \) for \( L^{+} = L^{-} \), we obtain (3). This canonical choice when \( L^{+} = L^{-} \) shall always be assumed.

**Proposition 2.7.** Let \( \phi = \phi(\mathbb{L}) \) and \( L \) a Lagrangian submanifold of \( \mathbb{R}^{2n} \). The following statements are equivalent:

1. \( \phi = \phi_{sp}(L) \).
2. \( L \subset E_{sp,n}(\mathbb{L}) \) and \( f \) is constant along \( L \).
3. \( y = 0 \) along \( L \).
Proof. (1) ⇒ (2) follows from the description of $\phi_{sp}(L)$ given in section 2.2. (2) ⇒ (3): If $f$ is constant along $L$, then necessarily $y$ is tangent to $L$. We may assume $t = t(s)$ along $L$ with $t(0) = 0$. Differentiating equation (2.7)(a), we obtain that along $L$

$$x_s = \left(1, \frac{\partial^2 Q}{\partial t^2} t_s + \frac{\partial^2 Q}{\partial s \partial t}\right).$$

Since $\frac{\partial^2 Q}{\partial t^2} = 0$ along $L$ and $y$ is tangent to $L$ we conclude that from equation (2.7)(b) that

$$t \frac{\partial}{\partial t} \frac{\partial Q}{\partial s} = \frac{\partial Q}{\partial s},$$

which implies that $\frac{\partial Q}{\partial s} = ct$, for some constant $c$. Since $Q$ is harmonic, $\frac{\partial^2 Q}{\partial s^2} = 0$, and so $t(s) = 0$, which implies $y = 0$.

(3) ⇒ (1): $y = 0$ implies $t = 0$ and $\frac{\partial Q}{\partial s} = 0$ at $L$. Thus $L$ is contained in the image of the parameterization $(s, \frac{\partial Q}{\partial s}(s,0))$ and $\frac{\partial Q}{\partial s}(s,0) = 0$. Thus we know $\frac{\partial Q}{\partial s}$ and $\frac{\partial Q}{\partial t}$ along the curve $t = 0$. This implies that we know $dH$ at $t = 0$. So we know $H$, up to an additive constant, which implies $\phi = \phi_{sp}(L)$. □

Remark 2.8. For $\phi_{sp}(L)$, the choice of parametrization (2.6) with $\tilde{H}(z) = H(\tilde{z})$ is canonical in the sense that $L \subset \mathbb{L}$ and $L \subset \mathbb{L}$ have the same parametrization (cf. remark 2.6) and therefore this is implicitly assumed.

Remark 2.9. For both $\phi_{cc}(L)$ and $\phi_{sp}(L)$, since $f$ is constant along $L$, we can choose $f$ such that $f(L) = 0$. With this canonical choice, $L \subset \tilde{E}_{cc}(L)$ and $L \subset \tilde{E}_{sp}(L)$, and we have lost the freedom of adding a constant to $f$.

In view of the above remarks, we present the following:

Definition 2.10. With the canonical choices outlined in remarks 2.8 and 2.9, we call $\phi_{cc}(L)$ and $\phi_{sp}(L)$ the two canonical IAS obtained uniquely from $L$.

3. Description of Lagrangian/Legendrian singularities on shell for the two canonical IAS from a Lagrangian submanifold

In this section we describe the sets $E_{cc}$ and $E_{sp}$ as image of the singular set of Lagrangian maps (caustics) and the sets $\tilde{E}_{cc}$ and $\tilde{E}_{sp}$ as image of the singular set of a Legendrian map.

However, much of the description is of a local nature, so it’s necessary to distinguish two different “parts” of each of these sets. The off-shell part of $E_{cc}(L)$, for instance, is locally of the form $E_{cc}(L^-, L^+)$, where $L^- \neq L^+$ are germs of $L$ at two distinct points in $L$. The local characterization of this part of this set and the local classification of its singularities can be found in [10] and [6]. Similarly for the other sets above.

In this paper, we are mostly concerned with describing and classifying the “on-shell” part of the sets above and their singularities. Thus, for instance,
the on-shell part of $E_{cc}(L)$ is locally of the form $E_{cc}(L', L')$, where $L'$ is the germ of $L$ at one point in $L$. Similarly for the other sets above.

It is useful to have in mind that in various instances the on-shell and the off-shell parts of these sets are disconnected.

**Notation 3.1.** Because in this paper we shall focus on the on-shell parts of $E_{cc}(L)$, or $E_{sp}(L)$, and $E_{cc}(L)$, or $E_{sp}(L)$, the on-shell parts of these sets shall be denoted with a superscript “$s$” as $E_{cc}^s(L)$, or $E_{sp}^s(L)$, and $E_{cc}^s(L)$, or $E_{sp}^s(L)$, respectively, or when the kind (center-chord or special) is not specified, simply by $E^s(L)$ and $E^s(L)$.

### 3.1. Generating functions and families

Denote by $\omega = \sum_{i=1}^n dq_i \wedge dp_i$ the canonical symplectic form in $\mathbb{R}^{2n}$, by

$$\Omega = \sum_{i=1}^n dq_i \wedge d\dot{p}_i + d\dot{q}_i \wedge dp_i$$

the canonical symplectic form in $T\mathbb{R}^{2n}$ and by

$$\theta = dz - \sum_{i=1}^n \dot{p}_i dq_i + \dot{q}_i dp_i$$

the canonical contact form in $T\mathbb{R}^{2n} \times \mathbb{R}$.

Let $x = (q, p) = (q_1, ..., q_n, p_1, ..., p_n)$, $y = (\dot{q}, \dot{p}) = (\dot{q}_1, ..., \dot{q}_n, \dot{p}_1, ..., \dot{p}_n)$. Let $U$ be an open subset of $\mathbb{R}^{2n}$. We shall denote by $L$ the image of the Lagrangian immersion $(x, y) : U \to (T\mathbb{R}^{2n}, \Omega)$ and by $\tilde{L}$ the image of the Legendrian immersion $(x, y, f) : U \to (T\mathbb{R}^{2n} \times \mathbb{R}, \{\theta = 0\})$.

Denote by $\pi : T\mathbb{R}^{2n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ the projection $\pi(q, p, \dot{q}, \dot{p}) = (q, p)$ and by $\tilde{\pi} : T\mathbb{R}^{2n} \times \mathbb{R} = \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}^{2n} \times \mathbb{R}$, the projection $\tilde{\pi}(q, p, \dot{q}, \dot{p}, z) = (q, p, z)$. We are interested in the singularities of the Lagrangian map $\pi \circ (x, y)$ and the Legendrian map $\tilde{\pi} \circ (x, y, f)$, where $x = (q, p)$, $y = (\dot{q}, \dot{p})$.

The main tools used for classifying Lagrangian and Legendrian singularities are the generating functions and generating families. A generating function of the Lagrangian submanifold $L$ and the Legendrian submanifold $\tilde{L}$ is a function

$$g : \mathbb{R}^n \times \mathbb{R}^n \ni (q, \dot{q}) \mapsto g(q, \dot{q}) \in \mathbb{R},$$

satisfying

$$L = \{(q, p, \dot{q}, \dot{p}) : \frac{\partial g}{\partial q} = \dot{p}, \frac{\partial g}{\partial \dot{q}} = p\}.$$  \hfill (3.2)

and

$$\tilde{L} = \{(q, p, \dot{q}, \dot{p}, z) : \frac{\partial g}{\partial q} = \dot{p}, \frac{\partial g}{\partial \dot{q}} = p, z = g(q, \dot{q}) - \dot{q} \cdot p\}. \hfill (3.3)$$

A generating family of the Lagrangian map $\pi \circ L$ and the Legendrian map $\tilde{\pi} \circ L$ is a function $G : \mathbb{R}^n \times \mathbb{R}^{2n} \ni (\beta, q, p) \mapsto G(\beta, q, p) \in \mathbb{R}$ satisfying

$$L = \{(q, p, \dot{q}, \dot{p}) : \exists \beta : \frac{\partial G}{\partial \beta} = 0, \frac{\partial G}{\partial q} = \dot{p}, \frac{\partial G}{\partial p} = \dot{q}\}. \hfill (3.4)$$
and
\[ \mathcal{L} = \{(q, p, \dot{q}, \dot{p}, z) : \exists \beta : \frac{\partial G}{\partial \beta} = 0, \frac{\partial G}{\partial q} = \dot{p}, -\frac{\partial G}{\partial p} = \dot{q}, z = G(\beta, q, p) \}. \] (3.5)

A generating family can be obtained from a generating function by
\[ G(\beta, q, p) = g(q, \beta) - p \cdot \beta. \] (3.6)

However, we stress that generating families are local objects, suitable for local descriptions and classifications, therefore we now focus on the generating families for \( \phi_{cc}(L) \) and \( \phi_{sp}(L) \) on shell, that is, when \((p, q)\) is in a neighborhood of \(L\).

**Generating families for center-chord and special IAS on shell.** For a center-chord IAS \( \phi_{cc}(L) \), where \(L\) is defined by \((u, dS(u))\), straightforward calculations show that
\[ g_{cc}(q, \dot{q}) = \frac{1}{2} S(q + \dot{q}) - \frac{1}{2} S(q - \dot{q}) \]
is a generating function on shell and so
\[ G_{cc}(\beta, q, p) = \frac{1}{2} S(q + \beta) - \frac{1}{2} S(q - \beta) - p \cdot \beta. \] (3.7)
is a generating family for \( \phi_{cc}(L) \) on shell. For a special IAS defined by the holomorphic function \(H\) taking \(\mathbb{R}^n\) to \(\mathbb{R}\),
\[ g_{sp}(q, \dot{q}) = Q(q, \dot{q}) \]
is a generating function on shell and the generating family for \( \phi_{sp}(L) \) on shell is given by
\[ G_{sp}(\beta, q, p) = Q(q, \beta) - p \cdot \beta, \] (3.8)
where \(Q\) is the imaginary part of \(H\).

Note that, on shell, both \(G_{cc}(\beta, q, p)\) and \(G_{sp}(\beta, q, p)\) are odd functions of \(\beta\). Thus, we now consider the classification problem in the odd category.

### 3.2. Equivalence, stability and versality in the odd category

In the center-chord case, we shall consider that all functions are smooth. In the special case, we shall consider that all functions are real analytic. Denote by \(\mathcal{D}_n^{odd}\) the group of germs of odd diffeomorphisms \((\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)\).

**Definition 3.2.** Two families \(G : (\mathbb{R}^n \times \mathbb{R}^{2n}, (\beta_0, x_0)) \to \mathbb{R}\) and \(\bar{G} : (\mathbb{R}^n \times \mathbb{R}^{2n}, (\bar{\beta}_0, \bar{x}_0)) \to \mathbb{R}\) are \(\mathcal{R}_n^{odd}\)-equivalent (cf. Definition 4.10 below) if there exist a germ of diffeomorphism \((\mathbb{R}^n \times \mathbb{R}^{2n}, (\beta_0, x_0)) \to (\mathbb{R}^n \times \mathbb{R}^{2n}, (\beta_0, x_0))\) of the form
\[ (\beta, x) = (\beta(\bar{\beta}, \bar{x}), x(\bar{x})) \]
where \(\beta(\cdot, \bar{x}) \in \mathcal{D}_n^{odd}\) for each fixed \(\bar{x}\), such that
\[ G(\beta, x) = \bar{G}(\bar{\beta}, \bar{x}), \]
in the Lagrangian setting, or such that
\[ G(\beta, x) = \bar{G}(\bar{\beta}, \bar{x}) + c(\bar{x}), \]
in the Legendrian setting, for a germ of function \(c : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}, 0)\).
Proposition 3.3. If the generating families $G$ and $\bar{G}$ are $R^{\text{odd}}$-equivalent, then:

1. The caustics $E^s(L)$ and $E^s(\bar{L})$ are diffeomorphic.
2. The sets $\tilde{E}^s(L)$ and $\tilde{E}^s(\bar{L})$ are diffeomorphic.

Proof. Since
\[
\frac{\partial G}{\partial \beta} = \frac{\partial \bar{G}}{\partial \bar{\beta}} \frac{\partial \bar{\beta}}{\partial \beta}
\]
we conclude that $\frac{\partial G}{\partial \beta} = 0$ if and only if $\frac{\partial \bar{G}}{\partial \bar{\beta}} = 0$. Moreover
\[
\frac{\partial^2 G}{\partial \beta^2} = \frac{\partial^2 \bar{G}}{\partial \bar{\beta}^2} \left( \frac{\partial \bar{\beta}}{\partial \beta} \right)^2 + \frac{\partial \bar{G}}{\partial \bar{\beta}} \frac{\partial^2 \bar{\beta}}{\partial \beta \partial \bar{\beta}}.
\]
Thus $\frac{\partial^2 G}{\partial \beta^2} = \frac{\partial G}{\partial \beta} = 0$ if and only if $\frac{\partial^2 \bar{G}}{\partial \bar{\beta}^2} = \frac{\partial \bar{G}}{\partial \bar{\beta}} = 0$. For the first item, observe that the diffeomorphism $x : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ takes $E^s(L)$ to $E^s(\bar{L})$. For the second item, observe that the diffeomorphism $\mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2 \times \mathbb{R}$ given by $(\bar{x}, \bar{z}) \rightarrow (x(\bar{x}), \bar{z} + c(\bar{x}))$ takes $\tilde{E}^s(L)$ to $\tilde{E}^s(\bar{L})$. □

Definition 3.4. A generating family $G : (\mathbb{R}^n \times \mathbb{R}^2, (\beta_0, x_0)) \rightarrow \mathbb{R}$ is $R^{\text{odd}}$-stable if, for any representative $G' : V \rightarrow \mathbb{R}$ of $G$, there exists a neighborhood $W$ of $G'$ in the $C^\infty$-topology (Whitney) and a neighborhood $V$ of $(\beta_0, x_0)$ such that for any generating family $\bar{G}' \in W$, there exists $(\bar{\beta}_0, \bar{x}_0) \in V$ such that $G$ and $\bar{G}$ are (fibred) $R^{\text{odd}}$-equivalent, $\bar{G}$ being the germ of $G'$ at $(\bar{\beta}_0, \bar{x}_0)$.

Theorem 3.5. A generating family $G$ is $R^{\text{odd}}$-stable if and only if $G$ is an $R^{\text{odd}}$-versal deformation of $G_0(\beta) = G(\beta, 0, 0)$.

Remark 3.6. The odd property of the generating families $G$ implies that the singularities of $E^s(L)$, resp. $\tilde{E}^s(L)$, possess a hidden $\mathbb{Z}_2$-symmetry, which descends from an explicit $\mathbb{Z}_2$-symmetry of the Lagrangian, resp. Legendrian, submanifolds $L$, resp. $\tilde{L}$, that are obtained from $G$ by (3.4), resp. (3.5).

Now, Definition 3.4 and Theorem 3.5 comprise a direct adaptation to the odd category of the well-known description in the non-symmetric case (cf. eg. [2] for the latter). In [9], Theorem 3.5 was also presented as a definition, but it is more appropriate to present it as a theorem whose proof follows from the work of Damon [7] and basic theorems of singularity theory proved by Mather [12] (see also [2]), as we will briefly explain in the next section.

However, just as in the non-symmetric case, Definition 3.4 can also be related to the natural definition of stability of the $\mathbb{Z}_2$-symmetric Lagrangian map, or graph-like Legendrian map, which is generated by $G$ via (3.4), or (3.5) respectively, as we will detail below by adapting to the $\mathbb{Z}_2$-symmetric case much of the treatment presented in [2] for the non-symmetric case.

In this way, the following section, with special relevance to Theorem 4.11 and Corollary 4.12 (plus the brief explanation of Theorem 3.5), can also be seen as complementing the treatment developed in [9].

For organizational purposes, the (not too short) proof of Theorem 4.11 is presented at the end of the paper, in Section 6.
4. $\mathbb{Z}_2$-symmetric germs of Lagrangian maps and their odd generating families

In this section, we shall detail the relation of Definition 3.4 to the definition of Lagrangian stability in the context of $\mathbb{Z}_2$-symmetric Lagrangian map-germs, by adapting the nonsymmetric treatment presented in [2]. Here we shall only work in the Lagrangian setting, the extension to the Legendrian setting being straightforward because all Legendrian immersions we consider are graph-like (cf. [11, Section 5.3]). At the end of the section we shall also briefly explain Theorem 3.5. We start by recalling basic definitions of the theory of Lagrangian singularities (cf. eg. [2, Part III]) and then specialize some of these basic definitions to the $\mathbb{Z}_2$-symmetric context.

4.1. $\mathbb{Z}_2$-symmetric Lagrangian map-germs

Let $M$ be a smooth (or analytic) $m$-dimensional manifold. Let $E \to M$ be a smooth (or analytic) fiber bundle over $M$. A diffeomorphism of $E$ is fibered (or fibred) if it maps fibers to fibers.

Let $T^*M$ be the cotangent bundle of $M$. Let $\omega$ be the canonical symplectic form on $T^*M$. A smooth (or analytic) section $s : M \to T^*M$ of the cotangent bundle $T^*M$ is called Lagrangian if $s^*\omega = 0$. Sections of $T^*M$ are differential 1-forms on $M$. It is easy to see that a section is Lagrangian if and only if the 1-form is closed. Thus, any germ of a smooth (or analytic) Lagrangian section can be described as the differential of a smooth (or analytic) function-germ on $M$. This function-germ is called a generating function of the germ of a Lagrangian section.

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ and $(\kappa, \lambda) = (\kappa_1, \cdots, \kappa_m, \lambda_1, \cdots, \lambda_m)$ be local coordinates on $M$ and $T^*M$, respectively. Then, $\omega = \sum_{i=1}^{m} d\kappa_i \wedge d\lambda_i$. In this way, $(T^*M, \omega)$ with the canonical projection $\pi : T^*M \ni (\kappa, \lambda) \mapsto \lambda \in M$ is a Lagrangian fibre bundle.

Let $L$ be a Lagrangian submanifold of $T^*M$ i.e. $\dim L = \dim M$ and the pullback of the symplectic form $\omega$ to $L$ vanishes. Then $\pi|_L : L \to M$ is called a Lagrangian map. The set of critical values of a Lagrangian map is called a caustic.

Let $L$ and $\tilde{L}$ be two Lagrangian submanifolds of $(T^*M, \omega)$:

Definition 4.1. Two Lagrangian maps $\pi|_L : L \to M$ and $\pi|_{\tilde{L}} : \tilde{L} \to M$ are Lagrangian equivalent if there exists a fibered symplectomorphism of $(T^*M, \omega)$ mapping $L$ to $\tilde{L}$. A Lagrangian map is stable if every nearby Lagrangian map (in the Whitney topology) is Lagrangian equivalent to it. Likewise for germs of Lagrangian submanifolds and Lagrangian maps.

We are interested in studying a special type of Lagrangian maps, this type consisting of maps which are $\mathbb{Z}_2$-symmetric in the fibers.

Definition 4.2. A Lagrangian submanifold $L$ of $T^*M$ is $\mathbb{Z}_2$-symmetric in the fibers if for every point $(\kappa, \lambda)$ in $L$ the point $(-\kappa, \lambda)$ belongs to $L$. The Lagrangian map $\pi|_L : L \to M$ is $\mathbb{Z}_2$-symmetric if the Lagrangian submanifold $L$
is $\mathbb{Z}_2$-symmetric in the fibers. Likewise for $\mathbb{Z}_2$-symmetric germs of Lagrangian submanifolds and Lagrangian maps.

It is easy to see that the fibers of $T^*M$ and the zero section of $T^*M$ are $\mathbb{Z}_2$-symmetric Lagrangian submanifolds. We will study singularities of $\mathbb{Z}_2$-symmetric Lagrangian map-germs. Thus we need a Lagrangian equivalence which preserves $\mathbb{Z}_2$-symmetry. Let us denote $\zeta$ the following map

$$\zeta : T^*M \ni (\kappa, \lambda) = (-\kappa, \lambda) \in T^*M.$$ 

The map $\zeta$ defines a $\mathbb{Z}_2$-action on $T^*M$, i.e., $\mathbb{Z}_2 \sim \{\zeta, id_{T^*M}\}$.

**Definition 4.3.** A fibered symplectomorphism $\Phi$ of $(T^*M, \omega)$ is odd or $\mathbb{Z}_2$-equivariant if $\Phi \circ \zeta = \zeta \circ \Phi$.

Since our consideration is local we may assume that $M = \mathbb{R}^m$. A fibered symplectomorphism-germ $\Phi$ of $(T^*M, \omega)$ has the form $\Phi = (\phi)^* + dG$, where $\phi : M \to M$ is a diffeomorphism-germ and $G$ is a smooth (analytic) function-germ on $M$ (see [2] section 18.5). Since $\zeta$ is the identity on the zero section of $T^*M$, the odd fibered symplectomorphisms map the zero section to itself. This implies the following characterization of odd fibered symplectomorphism-germs.

**Proposition 4.4.** If $\Phi$ is an odd fibered symplectomorphism-germ of $(T^*M, \omega)$ then $\Phi$ has the form $\Phi = (\phi)^*$, where $\phi : M \to M$ is a diffeomorphism-germ.

It is easy to see that odd fibered symplectomorphisms map $\mathbb{Z}_2$-symmetric Lagrangian submanifolds to $\mathbb{Z}_2$-symmetric Lagrangian submanifolds. Thus we can define a $\mathbb{Z}_2$-symmetric Lagrangian equivalence of $\mathbb{Z}_2$-symmetric Lagrangian map-germs in the following way:

**Definition 4.5.** Two $\mathbb{Z}_2$-symmetric Lagrangian map-germs $\pi|_L : L \to M$ and $\pi|_{\tilde{L}} : \tilde{L} \to M$ are $\mathbb{Z}_2$-symmetrically Lagrangian equivalent if there exists an odd fibered symplectomorphism-germ of $(T^*M, \omega)$ mapping $L$ to $\tilde{L}$.

In the same vein, we can define $\mathbb{Z}_2$-symmetric stability of $\mathbb{Z}_2$-symmetric Lagrangian map-germs as follows:

**Definition 4.6.** A $\mathbb{Z}_2$-symmetric Lagrangian map-germ is $\mathbb{Z}_2$-symmetrically stable if every nearby $\mathbb{Z}_2$-symmetric Lagrangian map-germ (in the Whitney topology) is $\mathbb{Z}_2$-symmetrically Lagrangian equivalent to it.

Thus, definitions 4.5 and 4.6 specialize in a natural way the definitions of Lagrangian equivalence and stability (cf. Definition 4.1) to the context of $\mathbb{Z}_2$-symmetric Lagrangian map-germs.

**4.2. Odd generating family-germs**

A family of functions $F = F(x, \lambda)$ is a generating family of a Lagrangian submanifold-germ $L$ of $(T^*M, \omega)$ if

$$L = \left\{ (\kappa, \lambda) \in T^*M \mid \exists x, \quad \frac{\partial F}{\partial x}(x, \lambda) = 0, \quad \kappa = \frac{\partial F}{\partial \lambda}(x, \lambda) \right\}. \quad (4.1)$$
Any Lagrangian submanifold-germ can be described by a generating family. In the case of $\mathbb{Z}_2$-symmetric Lagrangian submanifold-germs the generating family can be odd in $x$. Before we prove the above statement we introduce some preparatory definitions (see [2, Section 19.2]).

The bundle $\rho : \mathbb{R}^k \times \mathbb{R}^m \ni (x, \lambda) \mapsto \lambda \in \mathbb{R}^m$ is called the auxiliary bundle, the space $\mathbb{R}^{k+m} = \mathbb{R}^k \times \mathbb{R}^m$ is called the big space and $M = \mathbb{R}^m$ is called the base. The cotangent bundle of the big space is called the big phase space and the cotangent bundle of the base $\pi : T^* \mathbb{R}^m \to \mathbb{R}^m$ is called the small phase space.

The mixed space $A$ for the auxiliary bundle $\rho$ is the set of elements of the big phase space, which annihilate vector tangent to the fibers of $\rho$. The mixed bundle is the bundle over the big space induced from the small phase space $\pi$ by the map $\rho$. It is easy to see that the total space of the mixed bundle $\rho^* \pi$ is $A$ and the fibers of $\rho^* \pi$ are isomorphic to the fibers of $\pi$. The mixed space $A$ is also the total space of the bundle $\pi^* \rho : A \to T^* \mathbb{R}^{k+m}$ induced from the auxiliary bundle $\rho$ by the map $\pi$. These bundles are described on the following diagrams.

\[
\begin{array}{cccc}
T^* \mathbb{R}^m & \leftarrow & A & \leftarrow & T^* \mathbb{R}^{k+m} \\
\pi & & \rho^* \pi & & \rho^* \pi \\
\mathbb{R}^m & \leftarrow & \mathbb{R}^{k+m} & & \\
(\kappa, \lambda) & \leftarrow & (\kappa, x, \lambda) & \leftarrow & (0, \kappa, x, \lambda) \\
\pi & & \rho^* \pi & & \\
\lambda & \leftarrow & (x, \lambda)
\end{array}
\]

A Lagrangian submanifold of the big phase space is called $\rho$-regular if it is transversal to the mixed space $A$ for $\rho$. The image of the intersection of a $\rho$-regular Lagrangian submanifold with the mixed space $A$ by the natural projection $\pi^* \rho$ to the small phase space is a Lagrangian (immersed) submanifold and every germ of a Lagrangian submanifold of the small space can be obtained by this construction from the germ of $\rho$-regular Lagrangian section of the appropriate big phase space (see Section 19.3 [2]).

A function $F$ is a generating function of the Lagrangian section $L$ of the big phase space if $L$ is described in the following way.

\[
L = \left\{ (y, \kappa, x, \lambda) \in T^* \mathbb{R}^{k+m} \mid y = \frac{\partial F}{\partial x}(x, \lambda), \kappa = \frac{\partial F}{\partial \lambda}(x, \lambda) \right\}. \quad (4.2)
\]

Since the mixed space $A$ is described by $\{(y, \kappa, x, \lambda) \in T^* \mathbb{R}^{k+m} \mid y = 0\}$ then the Lagrangian submanifold $L = \pi^* \rho (L \cup A)$ is described by (4.1) and $F$ is a generating family of $L$.

Remark 4.7. The set of the critical points of the family $F$ is the following set

\[
\Sigma(F) = \left\{ (x, \lambda) \in \mathbb{R}^{k+m} \mid \frac{\partial F}{\partial x}(x, \lambda) = 0 \right\}
\]
Since the Lagrangian submanifold \( \mathcal{L} \) is \( \rho \)-regular, \( \Sigma(F) \) is a \( m \)-dimensional submanifold of \( \mathbb{R}^{k+m} \). The set of critical points of the family \( F \) is naturally diffeomorphic to the germ of the Lagrangian submanifold \( L \) of the small phase space determined by the germ of the generating family \( F \). Then the Lagrangian map-germ in terms of the generating family \( F \) is described in the following way

\[
\Sigma(F) \ni (x, \lambda) \mapsto \lambda ∈ \mathbb{R}^m.
\]

We prove that if \( L \) is \( \mathbb{Z}_2 \)-symmetric in the fibers then we can find \( F = F(x, \lambda) \) which is odd in \( x \).

**Proposition 4.8.** If a Lagrangian submanifold-germ \( L \) of \( T^*M \) is \( \mathbb{Z}_2 \)-symmetric in the fibers then there exists a local generating family \( F = F(x, \lambda) \) which is odd (in variables) i.e. \( F(−x, \lambda) \equiv −F(x, \lambda) \).

**Proof.** We use the method described in [2] (see Example 6 in Section 18.3 and Section 19.3 C). There exist subsets \( J = \{j_1, \ldots, j_k\}, I = \{i_1, \ldots, i_{m−k}\} \) of \( \{1, \ldots, m\} \) such that \( I \cap J = \emptyset \) and \( I ∪ J = \{1, \ldots, m\} \) and a local generating function \( S = S(κ_J, λ_I) \) of \( L \), where \( κ_J = (κ_{j_1}, \ldots, κ_{j_k}) \) and \( λ_I = (λ_i_1, \ldots, λ_{i_{m−k}}) \). We use the following notation \( \frac{∂S}{∂κ_{j_k}}(κ_J, λ_I) = (\frac{∂S}{∂κ_{j_1}}(κ_J, λ_I), \ldots, \frac{∂S}{∂κ_{j_k}}(κ_J, λ_I)) \). Then \( L \) is locally described in the following way.

\[
L = \{(κ, λ) ∈ T^*M | λ_J = −\frac{∂S}{∂κ_J}(κ_J, λ_I), \ k_I = \frac{∂S}{∂λ_I}(κ_J, λ_I)\}. \quad (4.3)
\]

Since \( L \) is \( \mathbb{Z}_2 \)-symmetric in the fibers then if \( (κ, λ) \) belongs to \( L \) then \( (−κ, λ) \) belongs to \( L \) too. It implies that if \( λ_J = −\frac{∂S}{∂κ_J}(κ_J, λ_I), k_I = \frac{∂S}{∂λ_I}(κ_J, λ_I) \) then \( \lambda_J = −\frac{∂S}{∂κ_J}(−κ_J, λ_I), −k_I = \frac{∂S}{∂λ_I}(−κ_J, λ_I) \). Thus we get

\[
\frac{∂S}{δκ_J}(−κ_J, λ_I) ≡ \frac{∂S}{δκ_J}(κ_J, λ_I), \quad \frac{∂S}{δλ_I}(−κ_J, λ_I) ≡ −\frac{∂S}{δλ_I}(−κ_J, λ_I). \quad (4.4)
\]

The generating function-germ is determined up to an additive constant. So we may assume that \( S(0, 0) = 0 \). Then from (4.4) we obtain that \( S = S(κ_J, λ_I) \) is an odd function-germ in \( κ_J \). Consider a function-germ on a big space \( \mathbb{R}^{k+m} \) of the form \( F(x, λ) ≡ S(x, λ_I) + <x, λ_J> \), where \( <, > \) is the dot product. Then \( F = F(x, λ) \) is odd in \( x \). It is easy to see that \( F \) is a generating function of a Lagrangian section \( \mathcal{L} \) of the big phase space \( T^*\mathbb{R}^{k+m} \) described by (4.2) and \( \mathcal{L} \) is \( \rho \)-regular. The set \( π^∗ρ(\mathcal{L}∪A) \) is exactly \( L \). Indeed \( \frac{∂F}{δx}(x, λ) ≡ \frac{∂S}{δκ_J}(x, λ_I) + λ_J, \quad \frac{∂F}{δλ_I}(x, λ) ≡ \frac{∂S}{δλ_I}(x, λ_I) \) and \( \frac{∂F}{δλ_J}(x, λ) ≡ x \). Thus by (4.3) \( L \) is locally described by (4.4).

**Remark 4.9.** We can choose such sets \( J, I \) such that \( k = k_J \) is the dimension of the kernel of the Lagrangian map \( L \mapsto T^*M \mapsto M \). The Lagrangian map-germ is described in terms of \( S \) in the following way

\[
\mathbb{R}^m \ni (κ_J, λ_I) ↦ (−\frac{∂S}{∂κ_J}(κ_J, λ_I), λ_I).
\]

The coordinates \( λ_J \) and \( κ_J \) are called pathological. The arguments \( κ_J \) are \( k \) pathological arguments of the function \( S \).
4.3. $\mathbb{Z}_2$-symmetric Lagrangian stability and $\mathcal{R}^\text{odd}$-versality

From classical results (cf. [2, Section 19.4]), in the non-symmetric context we know that Lagrangian equivalence of Lagrangian map-germs corresponds to stably fibred $\mathcal{R}^+$-equivalence of their generating families. In the $\mathbb{Z}_2$-symmetric context, we first have the following definition:

Let $\mathcal{D}_{k+m}$ denote the group of diffeomorphism-germs $(\mathbb{R}^k \times \mathbb{R}^m, 0) \rightarrow (\mathbb{R}^k \times \mathbb{R}^m, 0)$ and let $\mathcal{D}^\text{odd}_k$ denote the subgroup of odd diffeomorphism-germs $(\mathbb{R}^k, 0) \rightarrow (\mathbb{R}^k, 0)$ i.e. $\Phi \in \mathcal{D}^\text{odd}_k$ if $\Phi(-x) \equiv -\Phi(x)$.

**Definition 4.10.** (cf. Definition 3.2) Two odd generating family-germs $F, G$ of $\mathbb{Z}_2$-symmetric Lagrangian submanifold-germs are fibred $\mathcal{R}^\text{odd}$-equivalent if there exists a odd (in variables) fibred diffeomorphism-germ $\Psi \in \mathcal{D}_{k+n}$ i.e.

$$
\Psi(x, \lambda) \equiv (\Phi(x, \lambda), \Lambda(\lambda)) \text{ and } \Phi|_{\mathbb{R}^k \times \{\lambda\}} \in \mathcal{D}^\text{odd}_k \text{ for every } \lambda,
$$

such that

$$
F = G \circ \Psi.
$$

We now have the following main result, whose (not too short) proof is presented at the end of the paper, in Section 6.

**Theorem 4.11.** $\mathbb{Z}_2$-symmetric Lagrangian map-germs are $\mathbb{Z}_2$-symmetrically Lagrangian equivalent (cf. Definition 4.5) if and only if their odd generating families are fibred $\mathcal{R}^\text{odd}$-equivalent.

As a direct consequence of Theorem 4.11 we have the following:

**Corollary 4.12.** A $\mathbb{Z}_2$-symmetric Lagrangian map-germ is $\mathbb{Z}_2$-symmetrically Lagrangian stable (cf. Definition 4.6) if and only if its odd generating family is $\mathcal{R}^\text{odd}$-stable (cf. Definition 3.4).

**Remark 4.13.** As mentioned at the beginning of this section, the adaptation of Theorem 4.11 and Corollary 4.12 to the context of graph-like Legendrian map-germs is straightforward, once we adapt definitions 4.2 and 4.5-4.6 to the graph-like Legendrian setting as well. We refer the reader to [11, Section 5.3] for a detailed thorough exposition of the straightforward relationship between the Lagrangian and Legendrian descriptions in terms of generating families when the Legendrian immersions are graph-like, which is always the case for the center-chord and special IAS obtained from a Lagrangian submanifold.

We now turn to the explanation of Theorem 3.5. In the nonsymmetric case, the analogous theorem can be divided in two theorems.

The first one states that a family $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ (in our case $m = 2n$) is stable (definition analogous to Definition 3.4 but replacing $\mathcal{D}^\text{odd}_n$ by the full diffeomorphism group $\mathcal{D}_n$ in Definition 3.2) if and only if $F$ is infinitesimally stable. The concept of infinitesimal stability for $F$ under an action of a group $G$ means, loosely speaking, that the $G$-orbit of such an action contains a neighborhood of $F$. For families $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, the group action for the full $\mathcal{D}_n$ in Definition 3.2 is called/denoted the $\mathcal{R}^+$ (or fibred $\mathcal{R}^+$) group action. A very important property of an infinitesimally stable family $F$ is
its finite determinacy, meaning that $F$ is equivalent to $F'$ under this $\mathcal{R}^+$ action iff $F$ and $F'$ are $\mathcal{R}^+$ equivalent up to the $k^{th}$ order in their Taylor expansions on $\mathbb{R}^n$, for some finite $k \in \mathbb{N}$. Around 1968, J. Mather [12] proved that infinitesimally stable families are stable, and vice versa if $F$ is proper.

At that same time Mather also proved the second theorem, which states that a family $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is infinitesimally stable if and only if $F$ is a versal deformation of $F_0 = F(\cdot, 0) : \mathbb{R}^n \to \mathbb{R}$. The concept of a versal deformation $F$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ means, loosely speaking, that $F$ contains all possible deformations of $f$ or, more precisely, that any deformation $F'$ of $f$ is (fibred) $\mathcal{R}^+$ equivalent to one induced (by possibly eliminating some parameters) from $F$. If $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a versal deformation of $f = F_0$, then $\tilde{F} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \times \mathbb{R}^m, \tilde{F}(x,u) = (F(x,u), u)$, is called a versal unfolding of $f$. Finally, the versal deformations $F$ (or unfoldings $\tilde{F}$) of $f$ with the least possible number of parameters are called miniversal deformations (or unfoldings) of $f$, and they are all equivalent.

The complete statements and proofs of these theorems belong to the basics of singularity theory and so they can be found in various texts, as for instance in [2] (see also [11]). In fact, these theorems are stated and proved for general (families of) maps, not just functions (as Mather did [12]).

Then, around 1981, J. Damon [7] (see also [8]) showed that these basic theorems of singularity theory are still valid if the appropriate group action induced from a full diffeomorphism group (for $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ this is the fibred $\mathcal{R}^+$ group action) is replaced by a subgroup satisfying certain properties (natural, tangential, exponential and filtrational), which he called a geometrical subgroup. The key point is that the (fibred) $\mathcal{R}^{odd}$ group action, induced from $D^{odd}_n$ acting on odd families $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ as in Definition 3.2 (or 4.10), is a geometrical subgroup in this sense, so these basic theorems of singularity theory follow trough, which implies the statement:

**Theorem 4.14.** (cf. Theorem 3.5) A generating family $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is $\mathcal{R}^{odd}$-stable if and only if $F$ is an $\mathcal{R}^{odd}$-versal deformation of $F_0 = F(\cdot, 0)$.

We end this section with the following results from [9] which characterize the $\mathcal{R}^{odd}$-versal deformations $G : \mathbb{R}^n \times \mathbb{R}^{2n} \to \mathbb{R}$, noting that, although the results in [9] were obtained in the smooth category, they also hold in the real analytic category (see [4]). First, in the general case:

**Proposition 4.15.** (cf. [9] Theorem 3.9) A $(2n)$-parameter deformation $G(\beta, \lambda)$ of $G_0(\beta) = G(\beta, 0)$ is $\mathcal{R}^{odd}$-versal if and only if

$$E^{odd}_n = E^{even}_n \left\{ \beta_j \frac{\partial G_0}{\partial \beta_i}, i, j = 1...n \right\} + \mathbb{R} \left\{ \frac{\partial G}{\partial \lambda_l}, l = 1...2n \right\}.$$

Then, specifically for generating families of center-chord or special IAS on shell:
Corollary 4.16. (cf. [9 Corollary 4.4, Theorem 4.5]) The germ of a generating family $G$ is an $\mathcal{R}^{\text{odd}}$-versal deformation of $G_0$ if and only if

$$\mathcal{M}^{3(\text{odd})}_m = \mathcal{E}^\text{even}_m \left\{ \beta_j \frac{\partial G_0}{\partial \beta_i}, i, j = 1...n \right\} + \mathbb{R} \left\{ \frac{\partial g}{\partial q_l}, l = 1...n \right\},$$

where the relation between $G$ (= $G_{cc}$ or $= G_{sp}$) and $g$ (= $g_{cc}$ or $= g_{sp}$) are given by \([3.6]-[3.8]\).

5. Realization of simple singularities on shell for canonical IAS

5.1. Simple singularities of odd functions

The following results are a compilation of results in [9], section 3.

Let $G_0 \in \mathcal{E}^{\text{odd}}_n$ with a singular point at 0. If $n \geq 3$, $G_0$ is not $\mathcal{R}^{\text{odd}}$-simple. For $n = 1$, only the singularities $A_{2k/2}$ are simple. The corresponding miniversal deformations are

$$G(t, \lambda_1, ..., \lambda_k) = t^{2k+1} + \sum_{j=1}^{k} \lambda_j t^{2j-1}$$

For codimension 2, the only possibilities are $A_{2/2}$ and $A_{4/2}$.

For $n = 2$, the following singularities are simple:

1. $D_{2k/2}^\pm$: $(t_1, t_2) \to t_1^2 t_2 + t_2^{2k-1}, \ k = 2, 3...$
2. $E_{8/2}^\pm$: $(t_1, t_2) \to t_1^3 + t_2^3$
3. $J_{10/2}^\pm$: $(t_1, t_2) \to t_1^3 + t_1 t_2^4$
4. $E_{12/2}^\pm$: $(t_1, t_2) \to t_1^3 + t_2^7$

The corresponding miniversal deformations are

1. $D_{2k/2}^\pm$: $G(t_1, t_2, \lambda_1, ..., \lambda_k) = t_1^2 t_2 + t_2^{2k-1} + \lambda_1 t_1 + \sum_{i=2}^{k} \lambda_i t_2^{2i-3}$
2. $E_{8/2}^\pm$: $G(t_1, t_2, \lambda_1, ..., \lambda_4) = t_1^3 + t_2^3 + \lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_1 t_2^2 + \lambda_4 t_2^3$
3. $J_{10/2}^\pm$: $G(t_1, t_2, \lambda_1, ..., \lambda_5) = t_1^3 + t_1 t_2^3 + \lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_1 t_2^2 + \lambda_4 t_1 t_2^3 + \lambda_5 t_2^3$
4. $E_{12/2}^\pm$: $G(t_1, t_2, \lambda_1, ..., \lambda_6) = t_1^3 + t_2^3 + \lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_1 t_2^2 + \lambda_4 t_2^3 + \lambda_5 t_1 t_2^4 + \lambda_6 t_2^5$

For codimension $\leq 4$, the only possibilities are $D_{4/2}^\pm$, $D_{6/2}^\pm$, $D_{8/2}^\pm$ and $E_{8/2}^\pm$.

5.2. Relation between the generating families for $\phi_{cc}(L)$ and $\phi_{sp}(L)$

Assume $L$ is the graph of the analytic function $dS(s)$, where

$$S(s) = \sum_{k=0}^{\infty} a_k s^k, \ a_k \in \mathbb{R}.$$ 

Take then

$$H(z) = \sum_{k=0}^{\infty} a_k z^k.$$
Lemma 5.1. For \( L \subset \mathbb{R}^{2m} \) a Lagrangian submanifold, let \( g_{cc}(q, \dot{q}) \) and \( g_{sp}(q, \dot{q}) \) denote the generating functions of \( \phi_{cc}(L) \) and \( \phi_{sp}(L) \). Then
\[
g_{cc}(q, \dot{q}) = -i g_{sp}(q, i\dot{q}).
\]
(5.1)

In other words, if
\[
g_{sp}(q, \dot{q}) = \sum_{j=1, \text{odd}}^{\infty} b_j(q)(-1)^{\lfloor j/2 \rfloor} \dot{q}^j,
\]
then
\[
g_{cc}(q, \dot{q}) = \sum_{j=1, \text{odd}}^{\infty} b_j(q)\dot{q}^j.
\]

Proof. We have that
\[
Q(s, t) = \sum_{j=1, \text{odd}}^{\infty} b_j(s)(-1)^{\lfloor j/2 \rfloor} t^j,
\]
where
\[
b_j(s) = \sum_{k=j}^{\infty} a_k \binom{k}{j} s^{k-j}.
\]
On the other hand,
\[
g_{cc}(s, \beta) = \frac{1}{2} \sum_k a_k ((s + \beta)^k - (s - \beta)^k),
\]
which implies
\[
g_{cc}(s, \beta) = \sum_{j=1, \text{odd}}^{\infty} b_j(s)\beta^j.
\]

5.3. Simple singularities on shell of \( \phi_{cc}(L) \) and \( \phi_{sp}(L) \)

We now show by examples that, by an adequate choice of \( L \), the simple singularities \( A_{2/2}^/, A_{4/2}^/, D_{6/2}^/ \), \( D_{8/2}^/ \) and \( E_{8/2}^/ \) appear as stable singularities of \( \phi_{cc}(L) \) and \( \phi_{sp}(L) \).

Example 3. Consider \( S(q) = q^3 \). Then
\[
G_{cc}(\beta, q, p) = \beta^3 + 3q^2\beta - p\beta.
\]
and
\[
G_{sp}(\beta, q, p) = -\beta^3 + 3q^2\beta - p\beta,
\]
which are versal unfoldings of \( A_{2/2}^/- \)-singularities.

Example 4. Consider \( S(q) = q^5 + \frac{1}{4} q^4 \). Then
\[
G_{cc}(\beta, q, p) = \beta^3 + 9q^2\beta^3 + 5q^4\beta + q^3\beta + q\beta^3 - p\beta.
\]
The caustic \( \text{E}^r_{cc}(L) \) is given by \( \beta = 0 \) or \( 3q + 30q^2 + 10\beta^2 = 0 \) (see Figure 3 (a)). We have also
\[
G_{sp} = \beta^3 - 9q^2\beta^3 + 5q^4\beta + q^3\beta - q\beta^3 - p\beta.
\]
Thus \( E_{sp}(L) \) is given by \( \beta = 0 \) or \( 3q = 10\beta^2 - 30q^2 \) (see Figure 3 (b)). Observe that both constructions lead to versal unfoldings of an \( A_{4/2} \)-singularity.

![Figure 3. The caustics \( E_{cc}(L) \) and \( E_{sp}(L) \) of example 4.](image)

**Example 5.** Let

\[
S(u_1, u_2) = u_1^2 u_2 \pm u_2^3
\]

Then

\[
G_{cc}(\beta, q, p) = \pm \beta_2^3 + \beta_1^2 \beta_2 - p_1 \beta_1 - p_2 \beta_2 \pm 3q_2^2 \beta_2 + q_1^2 \beta_2 + 2q_1 q_2 \beta_1
\]

and the singular set is defined by \( \pm 3\beta_2^2 = \beta_1^2 \). In the special case, we have

\[
G_{sp}(\beta, q, p) = \mp \beta_2^3 - \beta_1^2 \beta_2 - p_1 \beta_1 - p_2 \beta_2 \pm 3q_2^2 \beta_2 + q_1^2 \beta_2 + 2q_1 q_2 \beta_1,
\]

and the singular set is again defined by \( \pm 3\beta_2^2 = \beta_1^2 \). Both constructions lead to versal unfoldings of a \( D_{4/2}^\pm \)-singularity.

**Example 6.** Consider

\[
S = q_1^2 q_2 \pm q_2^5 + \frac{1}{4} q_2^4.
\]

Then

\[
G_{cc} = \beta_1^2 \beta_2^2 \pm \beta_2^5 + q_2^2 \beta_2^3 - p_1 \beta_1 - p_2 \beta_2 + q_1^2 \beta_2 + 2q_1 q_2 \beta_1 \pm 10\beta_2^3 q_2^2 \beta_2^2 \pm 5\beta_2^4 q_2^4 + \beta_2 q_2^5,
\]

while

\[
G_{sp} = -\beta_1^2 \beta_2^2 \pm \beta_2^5 - q_2^2 \beta_2^3 - p_1 \beta_1 - p_2 \beta_2 + q_1^2 \beta_2 + 2q_1 q_2 \beta_1 \pm 10\beta_2^3 q_2^2 \beta_2^2 \pm 5\beta_2^4 q_2^4 + \beta_2 q_2^5.
\]

Thus \( G_{cc} \) is a versal unfolding of a \( D_{6/2}^{\pm} \)-singularity, while \( G_{sp} \) is a versal unfolding of a \( D_{6/2}^{\pm} \) singularity.

**Example 7.** Consider

\[
S = q_1^2 q_2 \pm q_2^7 + q_1 q_2^3 + \frac{1}{6} q_2^5.
\]

The corresponding \( G_{cc} \) is a versal unfolding of a \( D_{8/2}^{\pm} \) singularity. The corresponding \( G_{sp} \) is a versal unfolding of a \( D_{8/2}^{\pm} \) singularity.
Example 8. Consider

$$S = q_1^3 + q_2^5 + q_1 q_2^3.$$  

The corresponding $G_{cc}$ is a versal unfolding of a $E_{8/2}$ singularity. The corresponding $G_{sp}$ is a versal unfolding of a $E_{8/2}$ singularity.

Remark 5.2. In examples 3 and 4, $E_{cc}(L)$ and $E_{sp}(L)$ are diffeomorphic, since they are bifurcation sets of points $A_{2/2}$ and $A_{4/2}$, respectively. The same occurs in examples 5, 7 and 8 but not in example 6.

5.4. Stable singularities on shell for the two canonical IAS obtained from a given Lagrangian curve or surface

We now classify all $R^{odd}$-stable singularities that appear in the caustics $E^s(L)$, $\tilde{E}^s(L)$, when $L$ is a planar curve or a Lagrangian surface in $\mathbb{R}^4$. In these dimensions, only simple singularities are stable, so we apply the previous results taking care of the possible codimensions.

5.4.1. Lagrangian Curves. We follow section 4.1 of [9]. Let $L$ be a germ at 0 of a curve and assume that $L$ is generated by a function germ $S \in M_3^1 \subset E_1$.

Proposition 5.3. 1. If $S^{(3)}(0) \neq 0$, $G_{cc}$ and $G_{sp}$ are $R^{odd}$-equivalent to the $R^{odd}$ versal deformation of $A_{2/2}$.

2. If $S^{(3)}(0) = 0$, $S^{(4)}(0) \neq 0$, $S^{(5)}(0) \neq 0$, $G_{cc}$ and $G_{sp}$ are $R^{odd}$-equivalent to the $R^{odd}$ versal deformation of $A_{4/2}$.

Proof. If $S^{(3)} \neq 0$, then $f^{(3)} \neq 0$ and $g^{(3)} \neq 0$. Thus $G_{cc}$ and $G_{sp}$ are odd deformations of an $A_{2/2}$-singularity, and it is easy to see that they are in fact versal deformations of $G_0$. Thus we have proved item 1.

For item 2 observe that the hypothesis imply that $F$ and $G$ are odd deformations of an $A_{4/2}$-singularity. It is also easy to verify that this deformation is versal, thus proving the result.

The geometric interpretation of condition 1 is that the curvature of $L$ does not vanish, while the geometric interpretation of condition 2 is that the curvature vanishes, but its first and second derivatives do not.

Corollary 5.4. If $L$ is strongly convex, then $E^s_{cc}(L) = E^s_{sp}(L) = L$.

Remark 5.5. Example 1 (cf. Figure 1) and Example 2 (cf. Figure 2) are particular nongeneric illustrations of the above corollary, while Example 3 illustrate the generic case, locally. The generic case when condition 2 of Proposition 5.3 is satisfied is illustrated by Example 4 (cf. Figure 3).

5.4.2. Lagrangian Surfaces. We follow section 4.2 of [9]. Let $L$ be a germ at 0 of a Lagrangian surface and assume that $L$ is generated by a function germ $S \in M_2^3 \subset E_2$.

Notation: We shall use the following notation:

$$S_{i,j} = \frac{\partial^{i+j} S}{\partial q_1^i \partial q_2^j}(0,0), \quad S_{i,j}(q) = \frac{\partial^{i+j} S}{\partial q_1^i \partial q_2^j}(q_1, q_2).$$
The 3-jet of \( S \) at 0 can be written as
\[
\left.j_0^3 S = \frac{1}{6} S_{3,0} q_1^3 + \frac{1}{2} S_{2,1} q_1^2 q_2 + \frac{1}{2} S_{1,2} q_1 q_2^2 + \frac{1}{6} S_{0,3} q_2^3.\right.
\]
The discriminant of \( j_0^3 S \) is
\[
\Delta(j_0^3 S) = \frac{1}{48} (3S_{1,2}^2 S_{2,1}^2 - 4S_{0,3} S_{2,1}^3 - 4S_{1,2}^3 S_{3,0} - S_{0,3} S_{3,0}^2 + 6S_{0,3} S_{1,2} S_{2,1} S_{3,0}).
\]

**Proposition 5.6.** Assume \( \Delta(j_0^3 S) \neq 0 \).
1. If \( \Delta(j_0^3 S) > 0 \), \( G_{cc} \) and \( G_{sp} \) are \( R^{\text{odd}} \)-equivalent to the \( R^{\text{odd}} \) versal deformation of \( D_{4/2}^- \).
2. If \( \Delta(j_0^3 S) < 0 \), \( G_{cc} \) and \( G_{sp} \) are \( R^{\text{odd}} \)-equivalent to the \( R^{\text{odd}} \) versal deformation of \( D_{4/2}^+ \).

**Proof.** Assume \( \Delta(j_0^3 S) > 0 \). Then, by a linear change of coordinates, we can write \( j_0^3 g = \beta_1^3 \beta_2 - \beta_3^2 \). Thus \( g \) is \( R^{\text{odd}} \)-equivalent to a \( D_{4/2}^- \) singularity, and it is easy to see that \( G_{sp} \) is an \( R^{\text{odd}} \) versal deformation of \( g \). This proves the first assertion for \( G_{sp} \), the second one being similar. The proofs for \( G_{cc} \) are similar or else one can invoke theorem 4.11 of [9].

**Notation:** Denote
\[
\begin{align*}
r_1 &= \frac{S_{2,1} S_{1,2} - S_{3,0} S_{0,3}}{2(S_{3,0} S_{1,2} - S_{2,1}^2)}, & r_2 &= \frac{S_{3,0} S_{0,3} - 4S_{0,3} S_{1,2} S_{2,1} + 3S_{2,1}^3}{S_{3,0} S_{1,2} - S_{2,1}^2}, \\
\sigma_{0,n} &= \sum_{k=0}^{n} \binom{n}{k} \frac{S_{k,n-k} r_1^k}{(S_{3,0} r_1 - r_2)^n}, & n &= 5, 7, \\
\bar{r}_1 &= \frac{S_{2,1} S_{1,2} - S_{3,0} S_{0,3}}{2(S_{3,0} S_{2,1} - S_{1,2}^2)}, & \bar{r}_2 &= \frac{S_{3,0} S_{2,0} - 4S_{0,3} S_{1,2} S_{2,1} + 3S_{1,2}^3}{S_{3,0} S_{2,1} - S_{1,2}^2}, \\
\sigma_{n,0} &= \sum_{k=0}^{n} \binom{n}{k} \frac{S_{k,n-k} \bar{r}_1^k}{(S_{3,0} \bar{r}_1 - \bar{r}_2)^n}, & n &= 5, 7.
\end{align*}
\]
Denote
\[
\delta_1 = S_{3,0} S_{1,2} - S_{2,1}^2; \quad \delta_2 = S_{0,3} S_{2,1} - S_{1,2}^2.
\]

**Lemma 5.7.** If \( \Delta(j_0^3 S) = 0 \), then \( \delta_i \leq 0 \), \( i = 1, 2 \).

**Proposition 5.8.** Assume \( \Delta(j_0^3 S) = 0 \).
1. If \( \delta_1 \cdot \sigma_{0,5} < 0 \) or \( \delta_2 \cdot \sigma_{5,0} < 0 \), \( G_{cc} \) is \( R^{\text{odd}} \)-equivalent to the \( R^{\text{odd}} \) versal deformation of \( D_{6/2}^+ \), while \( G \) is \( R^{\text{odd}} \)-equivalent to the \( R^{\text{odd}} \) versal deformation of \( D_{6/2}^- \).
2. If \( \delta_1 \cdot \sigma_{0,5} > 0 \) or \( \delta_2 \cdot \sigma_{5,0} > 0 \), \( F \) is \( R^{\text{odd}} \)-equivalent to the \( R^{\text{odd}} \) versal deformation \( D_{6/2}^- \), while \( G_{sp} \) is \( R^{\text{odd}} \)-equivalent to the \( R^{\text{odd}} \) versal deformation of \( D_{6/2}^+ \).

**Proof.** Similar to theorem 4.14 of [9].

**Proposition 5.9.** Assume \( \Delta(j_0^3 S) = 0 \).
1. If $\delta_1 < 0$, $\sigma_{0.5} = 0$ and $\sigma_{0.7} > 0$ or $\delta_2 < 0$, $\sigma_{5.0} = 0$ and $\sigma_{7.0} > 0$, $G_{cc}$ and $G_{sp}$ are $R^{odd}$-equivalent to the $R^{odd}$ versal deformation of $D^{+}_{8/2}$.

2. If $\delta_1 < 0$, $\sigma_{0.5} = 0$ and $\sigma_{0.7} < 0$ or $\delta_2 < 0$, $\sigma_{5.0} = 0$ and $\sigma_{7.0} < 0$, $G_{cc}$ and $G_{sp}$ are $R^{odd}$-equivalent to the $R^{odd}$ versal deformation of $D^{-}_{8/2}$.

Proof. Similar to theorem 4.15 of [9].

Proposition 5.10. Assume $\Delta(j_0^3S) = 0$. If

$$\delta_1 = 0, \quad S_{3.0} \neq 0, \sum_{k=0}^{5} \binom{5}{k} S_{k.5-k}(-S_{2.1})^k(S_{3.0})^{5-k} \neq 0,$$

or

$$\delta_2 = 0, \quad S_{0.3} \neq 0, \sum_{k=0}^{5} \binom{5}{k} S_{k.5-k}(-S_{1.2})^k(S_{0.3})^{5-k} \neq 0,$$

then $G_{cc}$ and $G_{sp}$ are $R^{odd}$-equivalent to the $R^{odd}$ versal deformation of $E_{8/2}$.

Proof. Similar to theorem 4.16 of [9].

Remark 5.11. For detailed geometric interpretations of all the conditions of propositions 5.6, 5.10, we refer to section 4.3 of [2]. Here, we just point out that
conditions of Proposition 5.6 are realized for hyperbolic and elliptic points of $L$, the higher singularities of propositions 5.8, 5.10 occurring for parabolic points of $L$. In particular, the local equivalent of Corollary 5.4 is realized for hyperbolic points of $L$, that is, if $L'$ is the germ of $L$ at a generic hyperbolic point of $L$, then $E_{cc}(L') = E_{sp}(L') = L'$ (cf. [9 Corollary 4.19]).

6. Proof of Theorem 4.11

We now prove Theorem 4.11 which relates the definition of equivalence of $Z_2$-symmetric Lagrangian, resp. Legendrian, map-germs (cf. Definition 4.5) to the definition of fibred $R^{odd}$-equivalence of their odd generating families (cf. Definition 4.10). We prove this theorem by modifying the method described in [2 Section 19.5] to the case of $Z_2$-symmetric Lagrangian equivalence.

First let us assume that odd generating families $F_1$ and $F_2$ are fibred $R^{odd}$-equivalent. It means that $F_1(x, \lambda) = F_2(\Phi(x, \lambda), \Lambda(\lambda))$, where $\Psi(x, \lambda) \equiv (\Phi(x, \lambda), \Lambda(\lambda))$ is a fibered diffeomorphism-germ of the mixed bundle $\rho : \mathbb{R}^{k+m} \to \mathbb{R}^m$ and $\Phi(-x, \lambda) \equiv -\Phi(x, \lambda)$. Then the fibered diffeomorphism $(\Psi^{-1})^*$ of the big phase space $T^*\mathbb{R}^{k+m}$ determines a Lagrangian equivalence of the big phase space between Lagrangian sections of $T^*\mathbb{R}^{k+m}$ generating by the function-germs $F_1$ and $F_2$ on the big space. Both Lagrangian sections are $\rho$-regular. Since the diffeomorphism-germ $\Psi$ of $\mathbb{R}^{k+m}$ is fibered, the Lagrangian equivalence of the big phase space induces a Lagrangian equivalence of the small phase space $T^*\mathbb{R}^m$ (see [2], Section 19.4) between germs of Lagrangian submanifolds generating by the odd families $F_1$ and $F_2$. It is easy to check that this Lagrangian equivalence of the small phase space $T^*\mathbb{R}^m$ is determined by the diffeomorphism-germ $(\Lambda^{-1})^*$. But
$$(\Lambda^{-1})^*$$ is a linear map in the fibers of $T^*\mathbb{R}^m$. Therefore it is odd in the fibers. Thus the Lagrangian map-germs generating by the odd families $F_1$ and $F_2$ are $\mathbb{Z}_2$-symmetrically Lagrangian equivalent.

Now, let us assume that we are given a $\mathbb{Z}_2$-symmetrical Lagrangian equivalence of the small phase space mapping the germ of a $\mathbb{Z}_2$-symmetrical Lagrangian submanifold $L_1$ determined by an odd generating family $F_1$ to the germ of a $\mathbb{Z}_2$-symmetrical Lagrangian submanifold $L_2$. By Proposition 4.4 the $\mathbb{Z}_2$-symmetrical Lagrangian equivalence is determined by $(\phi)^*$, where $\phi$ is a diffeomorphism-germ of the base. Let us consider a diffeomorphism-germ $(Id_{\mathbb{R}^k}, \phi) : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k \times \mathbb{R}^m$ of the big space. This diffeomorphism-germ induced the Lagrangian equivalence $(Id_{\mathbb{R}^k}, \phi)^*$ of the big phase space mapping the germ of the Lagrangian section $L_1$ generating by the function germ $F_1$ to the germ of the Lagrangian section $L_2$. Then it is easy to see $L_2$ is generating by the function germ $F_2$ of the form $F_2(x, \lambda) = F_1(x, \phi^{-1}(\lambda))$. It implies that $F_2$ is a odd generating family of $L_2$ which is fibered $\mathcal{R}^{odd}$-equivalent to $F_1$.

If a germ $F$ is an odd generating family of $L$ then

$$\frac{\partial^2 F}{\partial x^2}(-x, \lambda) \equiv -\frac{\partial^2 F}{\partial x^2}(x, \lambda).$$

It implies that if $F$ is odd then

$$\frac{\partial^2 F}{\partial x^2}(0, \lambda) \equiv 0. \quad (6.1)$$

By Remark 4.9 every $\mathbb{Z}_2$-symmetric Lagrangian germ admits a generating function-germ $S = S(\kappa, J, \lambda I)$, which is odd in $\kappa, J$, with the minimal number of pathological arguments $\kappa, J$. This number is equal to the dimension of the kernel of the Lagrangian map-germ. We fix the set $\kappa, J$ of $k$ pathological arguments. By Remark 4.7 we obtain that the Lagrangian map-germ has the form in terms of $F$ given by $\Sigma(F) \ni (x, \lambda) \mapsto \lambda \in \mathbb{R}^m$. A vector $\eta$ is tangent to $\Sigma(F)$ at $(0, 0)$ if $d(\frac{\partial F}{\partial \lambda})|_{(0,0)}(\eta) = 0$ and $\eta$ is the kernel of the Lagrangian germ if $d\lambda|_{(0,0)}(\eta) = 0$. This implies that $dx(\eta)$ is in the kernel of the map $\frac{\partial^2 F}{\partial x^2}(0, 0) : \mathbb{R}^k \to \mathbb{R}^k$. Since $F$ satisfies (6.1), the image of the kernel of the Lagrangian map under the linear map $dx : \mathbb{R}^{k+m} \to \mathbb{R}^k$ is the whole space $\mathbb{R}^k$. But $\kappa, J$ are fixed $k$ pathological arguments. Thus the image of the kernel of the Lagrangian map under the linear map $dk_J = d(\frac{\partial F}{\partial \lambda_J})(0, 0) : \mathbb{R}^{k+m} \to \mathbb{R}^m$ is $k$-dimensional. But we have $dk_J(\eta) = \frac{\partial^2 F}{\partial x \partial \lambda_J}(0, 0) dx(\eta)$, because $d\lambda_J(\eta) = 0$. Hence if $F$ is odd then

$$\det \frac{\partial^2 F}{\partial x \partial \lambda_J}(0, 0) \neq 0. \quad (6.2)$$

An odd generating family $F$ is called special if for $\frac{\partial F}{\partial \lambda}(x, \lambda) = 0$ the condition $x = \frac{\partial F}{\partial \lambda}(x, \lambda)$ is fulfilled. We have the following lemma.

**Lemma 6.1.** The germ of an odd generating family is fibered $\mathcal{R}^{odd}$-equivalent to the germ of a special odd generating family determining the same Lagrangian germ.
Proof of Lemma 6.1. We follow the proof of Lemma 1 in Section 19.5 [2]. Since \( F \) is odd the condition (6.2) is fulfilled. Hence the map-germ \( \Psi(x, \lambda) \equiv (\frac{\partial F}{\partial x}(x, \lambda), \lambda) \) is a fibered diffeomorphism-germ of the big space. Since \( F \) is odd in \( x \), \( \Psi \) is odd in \( x \) too. The germ \( F \) is fibered \( R^{odd} \)-equivalent to the germ of an odd generating family \( F_1(x, \lambda) \equiv F(\Psi^{-1}(x, \lambda)) \). It is easy to check that \( F_1 \) is special and it generates the same Lagrangian germ (see [2] for details).

To finish the proof of Theorem 4.11 we need the following lemma.

Lemma 6.2. The germs of special odd generating families, determining the same \( \mathbb{Z}_2 \)-symmetric Lagrangian germ, are fibered \( R^{odd} \)-equivalent.

Proof of Lemma 6.2. By Proposition in Section 19.5 (D) (d) [2] any two special generating families \( F_0, F_1 \) of the same Lagrangian germ have the same set of critical points \( \Sigma(F_0) = \Sigma(F_1) = \Sigma \), the restrictions of \( F_0 \) and \( F_1 \) to \( \Sigma \) coindex up to an additive constant and the total differential of \( F_0 - F_1 \) is equal to 0 on the whole of \( \Sigma \).

Let \( F_0 \) and \( F_1 \) be two special odd generating family of the same \( \mathbb{Z}_2 \)-symmetric Lagrangian germ. Since \( F_0, F_1 \) are odd, we have \( F_0(0, 0) = F_1(0, 0) = 0 \). Thus \( F_0 - F_1 \) has zero of not less than second order on \( \Sigma \). We use the homotopy method. Let \( F_t = F_0 + t(F_1 - F_0) \) for \( t \in [0, 1] \). Then \( F_t \) is a special odd generating family of the same Lagrangian germ. We shall find a family \( \Psi_t(x, \lambda) \equiv (\Phi_t(x, \lambda), \lambda) \) of odd diffeomorphisms in \( x \), smoothly depending on \( t \in [0, 1] \), such that

\[
F_t \circ \Psi_t = F_0, \quad \Psi_0 = Id_{\mathbb{R}^{k+m}}. \tag{6.3}
\]

The diffeomorphism-germ \( \Psi_t \) establishes fibered \( R^{odd} \)-equivalence of \( F_0 \) and \( F_1 \). Differentiating (6.3) with respect to \( t \) we obtain the equation

\[
F_1(x, \lambda) - F_0(x, \lambda) + \sum_{i=1}^{k} \xi_i(x, \lambda, t) \frac{\partial F_t}{\partial x_i}(x, \lambda). \tag{6.4}
\]

Let \( \Theta : \mathbb{R}^{k+m} \times [0, 1] \rightarrow \mathbb{R}^{k+m} \times [0, 1] \) be the following map-germ \( \Theta(x, \lambda, \lambda_1, t) \equiv (x, \frac{\partial F_1}{\partial x}(x, \lambda), \lambda_1, t) \). Since \( F_t \) is odd, \( \frac{\partial F_1}{\partial x} \) is even in \( x \) and \( \det \frac{\partial^2 F_t}{\partial x \partial \lambda_j}(0, 0) \neq 0 \) by (6.2). It implies that \( \Theta \) is a diffeomorphism-germ and \( \Theta^{-1}(x, v, \lambda_1, t) \equiv (x, \gamma(x, v, \lambda_1, t), \lambda_1, t) \). It also implies that \( \gamma(-x, v, \lambda_1, t) \equiv \gamma(x, v, \lambda_1, t) \). Let \( H \) be the following family of function-germs on \( \mathbb{R}^{k+m} \)

\[
H(x, v, \lambda_1, t) \equiv F_0(x, \gamma(x, v, \lambda_1, t), \lambda_1) - F_0(x, \gamma(x, v, \lambda_1, t), \lambda_1).
\]

Then \( H \) is odd in \( x \) and \( H(x, \frac{\partial F_1}{\partial x}(x, \lambda), \lambda_1, t) \equiv F_0(x, \lambda) - F_1(x, \lambda) \). It implies that \( H(x, 0, \lambda_1, t) \equiv (F_0(x, \lambda) - F_1(x, \lambda))|_{\Sigma} \equiv 0 \). Let \( h(s) = H(x, sv, \lambda_1, t) \) for \( s \in [0, 1] \). Hence \( d(1) - h(0) = \int_0^1 \frac{dh}{ds}(s)ds \). Hence

\[
H(x, v, \lambda_1, t) \equiv \sum_{i=1}^{k} v_i \int_0^1 \frac{\partial H}{\partial v_i}(x, sv, \lambda_1, t)ds.
\]
If we put \( v = \frac{\partial F_t}{\partial x}(x, \lambda) \) we get
\[
F_0(x, \lambda) - F_1(x, \lambda) \equiv \sum_{i=1}^{k} \xi_i(x, \lambda, t) \frac{\partial F_t}{\partial x_i}(x, \lambda),
\]
(6.5)
where \( \xi_i(x, \lambda, t) \equiv \int_{0}^{1} \frac{\partial H}{\partial v_i}(x, s \frac{\partial F_t}{\partial x}(x, \lambda), \lambda, t) \, ds \) for \( i = 1, \ldots, k \). It is easy to see that \( \xi_i(-x, \lambda, t) = -\xi_i(x, \lambda, t) \).

Since the total differential of \( F_0 - F_1 \) vanishes on the whole of \( \Sigma = \Sigma(F_t) \) then by (6.5) we have \( \xi_i|_{\Sigma} = 0 \) for \( i = 1, \ldots, k \).

Thus the vector field \( \xi(x, \lambda, t) = \sum_{i=1}^{k} \xi_i(x, \lambda, t) \frac{\partial}{\partial x_i} \) depending on \( t \) takes value 0 on \( \Sigma \) and it is odd in \( x \). Hence \( \xi \) induces a diffeomorphism \( \Psi_t \) in the neighborhood of \((0, 0)\) for all \( t \in [0, 1] \), which satisfies the following ODE system
\[
\frac{d\Psi_t}{dt} = \xi(\Psi_t).
\]
(6.6)
Since \( \xi(x, \lambda, t) = \sum_{i=1}^{k} \xi_i(x, \lambda, t) \frac{\partial}{\partial x_i} \), the diffeomorphism \( \Psi_t \) has the following form \( \Psi_t(x, \lambda) \equiv (\Phi_t(x, \lambda), \lambda) \). The maps \( y(t) \equiv \Psi(-x, \lambda) \) and \( z(t) \equiv -\Psi_t(x, \lambda) \) satisfy the system (6.6) with the same initial condition \( y(0) = z(0) = (-x, \lambda) \). By the uniqueness of the solution of the initial value problem we obtain that \( \Psi(-x, \lambda) \equiv -\Psi(x, \lambda) \). Hence \( \Psi_t \) is odd in \( x \). Thus the fibered odd diffeomorphism-germ \( \Psi_1 \) satisfies \( F_1 \circ \Psi_1 = F_0 \). Consequently \( F_1 \) and \( F_0 \) are fibered \( R^{\text{odd}} \)-equivalent, which finishes the proof of the Lemma 6.2.

Then, by Lemmas 6.1 and 6.2 we obtain that any two odd generating families of the same \( \mathbb{Z}_2 \)-symmetric Lagrangian submanifold-germ \( L \) are fibered \( R^{\text{odd}} \)-equivalent, which finishes the proof of Theorem 4.11.

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