Testing finite time scaling at the spin glass transition in three dimensions

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We study the critical behavior of the three-dimensional Ising spin glass under experimentally relevant conditions: very large systems that never reach equilibrium. We compute a coherence length that measures the size of the growing magnetic glassy domains. At the critical point, we reach coherence-lengths larger than in any previous numerical study thanks to our optimized Monte Carlo method. The time-dependent coherent length allows us to conduct an accurate finite-size scaling (dubbed finite-time scaling) study of the spin-glass transition. We find sizeable corrections to scaling, that we control. We suggest that the very same scaling corrections are the underlying reason for Universality issues previously found in experimental studies of spin-glasses.

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Universality, the insensitivity of macro and mesoscopic phenomena to all but a handful of relevant microscopic features [1], partly explains the unreasonable success of Theoretical Physics. Phase transitions in condensed matter provide a celebrated example: Many different materials display the same critical behavior [2].

However, establishing Universality for disordered systems is proving difficult. Variations of critical exponents with a microscopic parameter, the concentration of some impurity for instance, have been reported many times and for many materials.

Nevertheless, recent theoretical evidence suggests that apparent Universality violations actually are scaling corrections. The universal behavior emerges when the correlation length is large enough, something not easy to achieve for disordered systems due to their sluggish dynamics [4, 5]. Examples include disordered ferromagnets, random-field Ising models or spin-glasses in spatial dimensions $D = 2$ [8, 9] and $D = 3$ [10, 11]. Up to now, however, only for diluted (anti)ferromagnets has it been possible to achieve quantitative agreement between theory [6] and experiment [7].

Our focus will be on the disordered magnetic alloys named spin-glasses [9, 12]. The experimental case for Universality is weak for spin glasses, where great attention has been paid to the anisotropy in the spin-spin interactions [14, 17]. An overall analysis conclude that the critical exponents vary continuously as we increase the anisotropy from the Heisenberg to the Ising limit [15].

Anisotropic interactions have been considered as well in equilibrium simulations, with a dissenting conclusion [11, 19]. Finite-size scaling indicates that, no matter the degree of anisotropy, for a large-enough system size $L$, critical exponents are those of a uniaxial (i.e. Ising) spin-glass [11]. Yet, the smaller the anisotropy, the larger the size $L$ needed to reveal the true critical behavior.

An analogous mechanism might explain the experimental Universality violations [11]. Let $t_w$ be the time elapsed since system preparation. For temperatures near the critical one $T_c$, the size of the magnetic glassy domains, $\xi(t_w)$ grows very slowly (some 100 lattice spacings at most [17, 20]). Experimental results at time $t_w$ would be analogous to those of equilibrium numerical simulations for sizes $L \sim \xi(t_w)$ [21, 22]. In particular, the finite-size crossover observed in simulations [11] would appear as finite-observation-time crossovers in experiments.

We remark that the time and temperature dependence of the coherence length $\xi(t_w)$, see Fig. 1, is a matter of active research both experimentally [17, 20, 23, 24] and theoretically [25–34]. In particular, the dedicated Janus computer [33] has allowed an in-depth study of $\xi(t_w)$ for $t_w$ ranging from picoseconds to 0.1 seconds [32, 33].

Here we show how to perform a predictive finite-time scaling study of the critical behavior of the $D = 3$ Ising Spin Glass transition. The non-equilibrium coherence length $\xi(t_w)$ plays exactly the same role as the system size in the standard finite-size scaling approach [10, 36]. Therefore, there is no objection of principle to analyze experimentally the spin-glass transition as one does in simulations [11]. Using an optimized Monte Carlo method, we obtain coherence-lengths as large as 25 lattice spacings.

Thus armed, we obtain critical exponents, as well as other Universal quantities such as the critical correlation function (not computable with equilibrium methods). Just as in the standard approach, controlling leading-order corrections to scaling turns out to be crucial.

The Hamiltonian for the $D = 3$ Edwards-Anderson model with nearest-neighbors interactions is

$$\mathcal{H} = - \sum_{(x,y)} J_{x,y} \sigma_x \sigma_y. \quad (1)$$

The spins $\sigma_x = \pm 1$ are placed on the nodes, $x$, of a cubic lattice of linear size $L = 256$ and periodic boundary conditions. The couplings $J_{x,y} = \pm 1$ are chosen randomly with 50% probability, and are quenched variables. Each coupling choice is named a sample. We denote by $\langle \cdots \rangle$ the average over the couplings. Model (1) undergoes a spin-glass transition at $T_c = 1.1019(29)$ [36].

We study the direct quench, the simplest dynamic protocol. At the starting time $t_w = 0$, the system is in a random configuration (i.e. $T = \infty$). We place it instantaneously at the working temperature $T$ and follow the
evolution as \( t_w \) increases, Fig. 1. Our time unit is the Monte Carlo step (a full lattice Metropolis sweep)\(^1\).

Metropolis dynamics belongs to the Universality Class of the physical evolution (it is an instance of the so-called model A dynamics \(^2\)), and is straightforward to implement \(^3\). However, our aim is to reach large \( L \) and \( t_w \). Rather than resorting to special hardware \(^{34}\) \(^{35}\) \(^{39}\) \(^{41}\), we employ synchronous multi-spin coding on standard CPUs. In a naive implementation random number generation is a major cost. However, our minimal energy barrier is 4, rarely overcome at the temperatures of interest [for instance, \( \exp(-4/T) \approx 0.026 \)]. Hence, the Gillespie method \(^{12}\) \(^{43}\) allows for major savings (see Appendix A)\(^4\).

We compute the coherence-length from the correlation function of the replica field \( q_x(t_w) = \langle \sigma^{(a)}_x(t_w) \sigma^{(b)}_x(t_w) \rangle \)

\[
C_4(r, t_w) = L^{-3} \sum x q_x(t_w) q_x(r) q_x(t_w). \tag{2}
\]

We restrict the displacement \( r \) to a lattice axis and compute integrals \( I_1(t_w) = \int_0^t d t \int d r \int d \theta C_4(r, t_w) \). Then, 
\[ \xi_{1,2}(t_w) = I_2(t_w)/I_1(t_w). \]

In all cases, we find \( L > 10 \xi_{1,2}(t_w) \) hence we regard our data as representative of the thermodynamic limit \(^{32}\).

Fig. 1 shows a rather accurate algebraic growth, 
\[ \xi(T) \sim t_w^{1/2(T)} \]

Yet, there is some controversy. On the one hand, low-temperature data suggest \( z(T < T_c) = z_{T_c}/T \) \(^{20}\) \(^{27}\) \(^{41}\). On the other hand, in Ref. \(^{15}\) a temperature varying protocol with \( T > T_c \) produced a numerical value \( z_c = 5.85(9) \) for \( J = \pm 1 \) or \( z_c = 6.00(19) \) for Gaussian couplings hardly consistent with the low-temperature \( z_c = 6.86(16) \) \(^{32}\) \(^{33}\).

Our own data, Fig. 1 right, suggest an exponent \( z(T) \) discontinuous at \( T_c \). Of course, this might be an effect of our \( z(T) \) being an effective exponent (due to our fitting time-window). But this is not a logical necessity.

Indeed, exponent \( z(T) \) carries different meanings. For \( T < T_c \) it describes (glassy) coarsening: the coherence length grows forever as \( \xi(t_w) \sim t_w^{1/2(T)} \). Yet, \( z(T > T_c) \) is concerned with equilibration. One has a characteristic

\[^{1}\text{We have simulated the same 50 samples at } T = 1.5 \ (t_w \leq 22^2), \quad T = 1.4 \ (t_w \leq 22^3), \quad T = 0.5, 0.6, 0.7, 0.8 \ (t_w \leq 22^6), \quad T = 0.9, 1.2, 1.25, 1.3 \ (t_w \leq 22^5), \quad \text{and at } T = 1.0, 1.1, 1.15 \ (t_w \leq 22^3). \\]

\[^{2}\text{We have used Pthreads to simulate a single system in multicores. Our best timings for } L = 256 \text{ at } T_c \text{ are: (a) An 8-core Intel(R) Xeon(R) CPU E5-2690: a 8-threads simulation of a single system at 50 ps/spin-flip; (b) A single 16-core AMD Opteron(TM) 6272 processor: a 16-threads simulation of a single system at 62 ps/spin-flip. For comparison, a single Janus FPGA runs two } L = 80 \text{ systems at 32 ps/spin-flip each.} \]

\[^{3}\text{Having 4 replicates at our disposal (8 replicates for } T = 1.1, 1.25 \text{ we average over the 6 (28) possible pairings of replica indices.} \]

\[^{4}\text{Other laws \(^{41}\) are numerically indistinguishable from a power.} \]

FIG. 1. (Color online) Left: Coherence length \( \xi_{12}(t_w) \) vs. Monte Carlo time, as computed for model \(^1\) on lattices of size \( L = 256 \), for several temperatures \([T_c = 1.1019(29) \) \(^{36}\)]. One Monte Carlo step corresponds to 1 picosecond in physical time \(^{13}\). For \( T \geq 1.3 \), we reach equilibrium. Right: Dynamic exponent \( z(T) \) computed in the non-equilibrium regime \( \xi(t_w) \sim t_w^{z(T)} \). Joined red points stand for \( T \leq T_c \). Note the constant value \( z(T > T_c) \approx 6 \) (blue circles). We perform the fits for \( t_w > 2^{20} \approx 10^6 \) Monte Carlo steps (but for \( T = 1.3 \) where \( 2^{20} \leq t_w \leq 2^{21} \), in order to avoid thermalization). We also show Janus data \(^{33}\) (green circles), computed for longer times. We only show the \( L = 80 \) Janus data at temperatures free from finite-size artifacts.

time \( \tau(T) \) (when \( \xi(T, \tau) \) reaches, say, 90% of the equilibrium \( \xi_{eq} \)) and then \( \tau(T) \approx \xi_{eq}(T)^z \). In fact, for the simplest non-trivial model (the \( D = 2 \) Ising ferromagnet) the coarsening exponent is \( z = 2 \) \(^{46}\) \(^{47}\) for critical equilibration.

Clearly, this delicate cross-over will require further investigation. Yet, we have rationalized why a \( T > T_c \) protocol \(^{45}\) produces \( z(T > T_c) \approx 6 \).

These complications reinforce our choice of basing finite-time scaling on \( \xi_{12}(t_w) \), rather than on \( t_w \) \(^{48}\) \(^{50}\). To do so, we adapt Binder’s method \(^{51}\) (in Appendix C we explore another possibility \(^{52}\) that turns out to be less accurate). Let 
\[ q(B_l, t_w) = \sum_{x \in B_l} q_x(t_w) \]

be the average of the replica field on a cubic box of side \( l \). We compute 
\[ q_l(l, t_w) = q^l(B_l, t_w), \]

its \( k \)-th power averaged over samples, replica pairings, as well as over boxes \( B_l \). Binder’s ratio 
\[ U_4(l, t_w, T) = q_4(l, t_w)/q_2(l, t_w)^2 \]

is a dimensionless parameter likely to display Universal behavior (for instance, \( U_4(l, t_w, T) \to 3 \) when \( l \gg \xi(t_w) \) due to the Central Limit Theorem, see also Ref. \(^{33}\)).

The analogy with Finite Size Scaling impels us to change variables: \( y = [T - T_c] \xi(t_w)/T \) and \( \lambda = l/\xi(t_w), T \). Then, barring subleading corrections to scaling, we expect:

\[ U_4(l, t_w, T) = f(y, \lambda) + [\xi(t_w)]^{-\omega} g(y, \lambda), \tag{3} \]

where \( \nu \) is the correlation-length critical exponent, \( \omega \) is the leading corrections to scaling exponent, and \( f \) and \( g \)
are dimensionless scaling functions. Note that the independent variables in the l.h.s. of Eq. (3) \((l, t_w, \text{and} T)\) are discrete. Yet, the r.h.s. variables \((\xi, y \text{ and} \lambda)\) are continuous. We solve this problem by means of polynomial interpolations (see Appendix B). Errors are estimated with the jackknife method [54], computed over the samples.

Fig. 2 contains a qualitative discussion of Eq. (3). In the inset, we show data at \(y = 0\) (i.e. \(T = 1.1\)), an excellent approximation to \(T_c\) [36]. For large \(\xi(t_w)\), \(U_4\) converges to the scaling function \(f(0, \lambda)\). On the other hand, in Fig. 2 main we show that Eq. (3) actually describes a cross-over in temperature. Let us fix \(\lambda = 1\) and \(T > T_c\). Then, \(y\) becomes large and positive as \(\xi(t_w, T)\) grows. We see that \(U_4\) approach a high-limit temperature (a \(\lambda\)-dependent renormalized coupling constant [55]). At \(T_c\) we have the critical limit because \(y = 0\) no matter how large \(\xi(t_w)\) is. In the spin-glass phase, \(y\) becomes large and negative. For large \(\xi\) we reach a low-temperature limit, that has been much debated in the past [56, 57].

In order to compute the critical exponents, we decided to follow the fixed-height method [10, 58]. For a fixed height \(h\), and fixed \(\lambda\) and \(\xi(t_w, T)\), we seek the temperature \(T_{h, \lambda, \xi}\) such that \(U_4 = h\). Eq. (3) tells us that
\[
T_{h, \lambda, \xi} = T_c + A_{h, \lambda} \xi^{-1/\nu} + B_{h, \lambda} \xi^{-\omega + 1/\nu} \ldots , \tag{4}
\]
where \(A_{h, \lambda}\) and \(B_{h, \lambda}\) are scaling amplitudes and the dots stand for higher-order corrections to scaling. We compute \(T_c, \nu\) and \(\omega\) by performing joint fits to data for several \(\xi(t_w)\) \((T = 1.1 \approx T_c)\). As expected by plugging \(y = 0\) in Eq. (3), the curve is scale-invariant when the small \(\xi\) corrections fade away.

Critical Binder’s ratio as a function of the dimensionless box-size for several \(\xi(t_w)\) \((T = 1.1 \approx T_c)\). As expected by plugging \(y = 0\) in Eq. (3), the curve is scale-invariant when the small \(\xi\) corrections fade away.

For \(\lambda = 0.75\), the \(U_4\) spacing is 0.02. For all fits, the values of \(T_c, \nu\) and \(\omega\) are held common. Big data points were included in the fit. The horizontal dotted lines correspond to \(T_c \pm \Delta T_c\) from Eq. (3), but only 50 independent samples. We solve it following [32, 33]: we fit taking into account only the diagonal part of the covariance matrix. We perform a fit for each jackknife block, and compute the final errors from the fluctuations of these fits. We compute as well the diagonal goodness-of-fit indicator \(\chi^2_{\text{diagonal}}\) (the sum of the squared deviations of data from fit, in units of their statistical error). This fitting procedure was tested in Ref. [33] and found to be reasonably stable for \(\chi^2_{\text{diagonal}}\) as small as half the number of degrees of freedom.

We included in our fit results for \(\lambda = 0.75, 1, 1.25\) and 1.5. A crucial issue is selecting \(\xi_{\text{min}}\), the minimal \(\xi\) considered in the fit. A tradeoff should be found. The larger is \(\xi_{\text{min}}\), the smaller are the systematic errors, but the larger becomes the statistical uncertainty. We find a stable fit for \(\xi_{\text{min}} \geq 2^{9/4} \approx 4.75\) \((\chi^2_{\text{diagonal}}/\text{d.o.f.} = 583/665\) if \(\xi_{\text{min}} = 2^{9/4}\)). However, as we enlarge \(\xi_{\text{min}}\) we find that \(\chi^2_{\text{diagonal}}/\text{d.o.f.}\) decreases monotonically while the statistical error increases. We decided to stop at \(\xi_{\text{min}} = 2^{3/8} \approx 7.33\) \((\chi^2_{\text{diagonal}}/\text{d.o.f.} = 229/482)\). The final result for our fit to Eq. (4) is
\[
T_c = 1.115(15) , \quad \nu = 2.2(3) . \tag{5}
\]
For comparison, recall the equilibrium results \(T_c = 1.1019(29)\), \(\nu = 2.56(4)\) and \(\omega = 1.12(10)\) [36]. Varying \(\omega\) within the bounds of [36] produces negligible changes in the results in Eq. (5). It is also interesting to see what happens fixing \(\nu\) and \(\omega\) in the fit to the central values of [30] \((\xi_{\text{min}} \geq 2^{3/8}, \chi^2_{\text{diagonal}}/\text{d.o.f.} = 241/483)\):
\[
T_c = 1.102(8) , \tag{6}
\]
In excellent agreement with the equilibrium result.

The anomalous dimension \(\eta\) can be computed by working directly at \(T = 1.1 \approx T_c\). We select two times \(t_w^{(1)}\) and \(t_w^{(2)}\) such that \(\xi(t_w^{(1)}, T_c) = \xi\) and \(\xi(t_w^{(2)}, T_c) = 2\xi\). Then the ratio of integrals is
\[
I_2(t_w^{(2)}, T_c)/I_2(t_w^{(1)}, T_c) = 2^{\nu - \eta} + C_T/\xi^\omega + \ldots \tag{7}
\]
\begin{equation}
\chi(T, \xi) = 1.1, \quad \xi = 1.1, \quad \xi = 2.73
\end{equation}

The problem with Eq. (7) is that the amplitude for scaling corrections $C_f$ seems vanishing (within errors), so one could be afraid that we overestimate the error. Anyhow, for $\xi_{\text{min}} = 27^{4/7} \approx 3.36$ we obtain $\eta = -0.380(7)$ and $\chi^2_{\text{diagonal}}/\text{d.o.f.} = 10.4/14$, to be compared with $\eta = -0.3900(36)$ [36] (for larger $\xi_{\text{min}}$ fits are stable but $\chi^2_{\text{diagonal}}/\text{d.o.f.}$ drops well below 0.5). Changing $\omega$ within the bounds of [36] produces a negligible change. We estimate that the error induced in $\eta$ by the uncertainty in $T_c$ [36] is comparable with the statistical error obtained at $T = 1.1$.

Incidentally, one may use the ratio of integrals $I_2(t_w^{(2)}, T)/I_2(t_w^{(1)}, T)$ as a (very noisy) substitute of the Binder’s cumulant in Eqs. (3,4) (see Appendix C). In fact, one may view the temperature crossover in Eq. (3) as a crossover for the $C_4(r, t_w)$ correlation function [2]. Indeed, for all the $T$ and $t_w$ in this work, the functional form

\begin{equation}
C_4(r, t_w) \sim e^{-|r|/\xi(t_w)}^{1/\eta}/\rho^\theta,
\end{equation}

fits satisfactorily our data. For small $\eta$ [i.e. at $T_c$ or for small $\xi(t_w)$] data follows Eq. (8) with critical parameters. However, as the coherence-length grows, these parameters are not adequate neither for the paramagnetic phase (at or near equilibrium, see Fig. 4), nor for the spin-glass phase [32, 33].

In this work, we have shown how one can study the spin-glass transition in the dynamic regime relevant to most experiments: non-equilibrium data on systems much larger than the coherence length. Once we trade waiting time by coherence length, standard finite-size scaling methods [51] are very successful at describing the temperature-dependent dynamic crossover (a real phase transition with temperature takes place only for infinite coherence length). It is then possible that the finite-size crossover found in equilibrium [11] is the driving force behind the apparent Universality behind the apparent Universality violations found experimentally [14–18]. However, an alternative explanation, logically possible but rather dramatic, is that Universality does not hold in spin glasses [53, 60].

Regarded as a numerical method to compute critical exponents, we note that our thermodynamic limit approach is less accurate than finite-size methods [10, 36, 61] which is hardly a surprise.

We conclude by mentioning the two major difficulties (in our opinion) for an analogous experimental study. On the one hand, one needs to reach spatial resolution to study the correlation function $C_4(r, t_w)$. Progress in this direction are still incipient. Spatial resolution has been reached only for a structural glass [62]. For spin-glasses, recent experimental efforts focus on confining geometries [24, 63] (which can be seen as an indirect way to study the correlation function). On the other hand, the direct quench is a rather crude approximation: the experimental sample never reaches the working temperature instantaneously [64, 65]. The protocol of Ref. [45] is, probably, more suitable to model the experimental setup. However, as Fig. 1 shows, mixing temperatures in the dynamic evolution is a delicate procedure that requires further investigation.

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**Appendix A: Synchronous Multispin coding**

Modern CPUs, both Intel and AMD, support 256-bit words in their streaming extensions. This means that one can perform basic Boolean operations (AND, XOR, etc.) in parallel for all the 256 bits. Now, it is well known that the Metropolis update of a single spin can be cast into a sequence of boolean operations, see e.g. [66]. One can use this idea to simulate several, up to 256, independent systems. This approach, named asynchronous multispin coding, has been used many times, see Refs. [34, 36, 39, 67, 68] for instance. Ref. [40] offers a creative alternative: In their Parallel Tempering simulation each bit represent an independent system copy (all of them evolve under the same couplings, but at different temperatures [70, 71]). Instead, our aim is to exploit the streaming extensions to speed-up the sim-
ulation of a single system (which is named synchronous multipspin coding).

The main problem with synchronous multipspin coding is that we need 256 independent random numbers, if the 256 spins coded in a word belong to the same physical system. This breaking of parallelism is usually regarded as a major inconvenience (see, however, Ref. [45]).

For the sake of clarity, we shall first explain our geometrical set up and then describe how one can use the Gillespie method [42, 43] to reduce drastically the number of needed random numbers.

1. Our multipspin coding geometry

Physical spins sit on the nodes of a $L = 256$ lattice with periodic boundary conditions. Euclidean coordinates then run as $0 \leq x, y, z \leq 255$. Each physical spin is a binary variable to be coded in a single bit, $s_{(x,y,z)} = \pm 1$.

We pack 256 physical spins into one superspin. Our superspins sit in the nodes of a different lattice. It will be also a cubic lattice with periodic boundary conditions (the overall geometry is that of a parallelepiped, rather than a cube). The major requirement is that nearest-neighbor spins in the physical lattice should be as well nearest neighbors in the superspin lattice. Our solution is as follows.

Superspins are placed at the nodes of a cubic lattice with dimensions $L_x = L_y = L/8$, and $L_z = L/4$. The relation between physical coordinates $(x, y, z)$ and the coordinates in the superspin lattice $(i_x, i_y, i_z)$ is

$$
\begin{align*}
  x &= b_x L_x + i_x, \quad 0 \leq i_x < L_x, \quad 0 \leq b_x < 8, \\
  y &= b_y L_y + i_y, \quad 0 \leq i_y < L_y, \quad 0 \leq b_y < 8, \\
  z &= b_z L_z + i_z, \quad 0 \leq i_z < L_z, \quad 0 \leq b_z < 4.
\end{align*}
$$

(A1)

In this way, exactly 256 sites in the physical lattice are given the same superspin coordinates $(i_x, i_y, i_z)$. We differentiate between them by means of the bit index:

$$
  i_b = 64 b_z + 8 b_y + b_x, \quad 0 \leq i_b \leq 255.
$$

(A2)

An added bonus of Eq. (A1) is that the parity of the original site, namely the parity of $x+y+z$, coincides with the parity of the corresponding superspin site $i_x + i_y + i_z$. In fact, the single cubic lattice is bipartite. It can be regarded as two interleaved face-centered cubic lattice. A given site is said to belong to the even or the odd sublattice according to the parity of $x+y+z$. For models with only nearest-neighbors interactions, sites belonging to (say) the even sublattice interact only with the odd sites.

An important consequence of the even-odd decomposition is that it eases parallelism. Indeed, we define the full lattice Metropolis sweep as the update of all the $L^3/2$ even sites, followed by the update of all the $L^3/2$ odd sites. The bipartite nature of the lattice makes it irrelevant the updating order of sites of a given parity. Hence, several updating threads may legitimately concur on the same lattice, provided that all of them simultaneously access only sites of the same parity.

2. Saving random numbers

For our synchronous multipspin coding we do need to generate 256 random numbers in order to update a single superspin. Yet, it has been realized several times that most of the effort in generating (pseudo) random numbers is wasted when simulating discrete models at low temperatures [12, 33]. In fact, at a given time the simulation may try to overcome an energy barrier $\Delta E$. However, we should overcome it only with probability $e^{-\Delta E/T}$. In other words, we waste $\sim e^{\Delta E/T}$ random numbers (that deny us the permit to overcome the barrier) until we generate one random number that really allows us to walk uphill in energy. Let us plug some numbers for our model, where the possible barrier heights are $\Delta E = 4, 8$ or 12. So, at $T_c$, in the best of cases we use only one random number out of $e^{4/1.1} \approx 38$.

The way out is simple [42, 43]: one simulates the random number generator. Indeed, we may regard the random-number generator as a collection of flags. Most of the flags are red (denying us the right to increase the energy), but there is a diluted set of green flags (at sites where the generator does allow us to increase the energy). The trick is setting all flags to red by default, and then caring only of placing green flags with the correct probability.

Before explaining how we simulate our random number generator, let us describe it. By default, let us assume that all flags are red, for all sites and all barriers $\Delta E = 4, 8$ and 12. Now, for each site in the physical lattice, we draw one 64-bits uniformly distributed random number: $0 \leq R_4 \leq 1$. If $R_4 < e^{-4/T}$ then we put a green flag for $\Delta E = 4$ and draw a second uniform random number $0 \leq R_8 \leq 1$. Now, if $R_8 < e^{-8/T}$ we put a green flag for $\Delta E = 8$ and draw a third uniform random number $0 \leq R_{12} \leq 1$. Finally, if $R_{12} < e^{-12/T}$ we also put a green flag for $\Delta E = 12$. Of course, ours is just an instance among many valid generators. This particular random number generator was chosen because it is fairly easy to simulate.

Let us describe how we simulate the generation of $R_4$ (the procedure for $R_8$ and $R_{12}$ are trivial generalizations). We generate an integer $n_4 \geq 0$, with the following meaning: One performs $n_4$ unfruitful calls to the generator, but on call $1 + n_4$ we should put a green flag. The cumulative probability for $n_4$ is

$$
F(n_4 \leq k) \equiv \text{Prob}(n_4 \leq k) = 1 - (1 - e^{-4/T})^{k+1}.
$$

(A3)

Hence, we just need to draw an uniform random number $0 \leq R < 1$ and select $n_4 = k$, where $k$ is the non-negative integer solution of $F(n_4 \leq k) \geq R$.

5. Probability $[R_4 < e^{-4/T}$ and $R_8 < e^{-4/T}] = e^{-8/T}$. 

integer that verifies
\[ F(k - 1) \leq R < F(k), \quad [F(-1) \equiv -1]. \quad (A4) \]

Combining these ideas with the use of look-up-tables, we have found that the overall cost of generating random numbers can be made quite bearable.

**Appendix B: Interpolations**

The major theme of this work is a change of variable: rather than the the waiting time \( t_w \), we wish to employ the coherence length \( \xi(t_w) \). Besides, the quantities computed in the l.h.s. of Eq. (3) of the main text were obtained for a discrete set of values of temperatures \( T \), waiting times and box sizes (\( l \)). However, our analysis of the r.h.s. of the same equation assumes that the scaling variables \( y, \lambda = 1/\xi(t_w) \) and \( \xi(t_w) \) are continuous. In order to solve this problem we perform several interpolations.

Let us describe our interpolations. In all cases, we perform a jackknife error analysis. Let us stress that we are talking here about interpolations, rather than extrapolations.

The easiest task is the \( l \) interpolation. Data are very smooth (due to their extreme statistical correlation) and a simple cubic spline does an excellent job.

Let us now address \( \xi(t_w) \). We take data for times of the form \( t_w = [2^n/4] \) where \( n \) is an integer and \([ \cdot \cdot \cdot ]\) is the integer part. We find that, even for neighboring times in our logarithmic time-mesh, the statistical fluctuations in the coherence length are significant (see Fig. 1 in main text). However, we need a monotonously increasing function \( \xi(t_w) \) if we are to invert it [that is, to obtain \( t_w(\xi) \)]. Also it is desirable to have a smooth \( \xi(t_w) \) to eliminate the short time-scale fluctuations. Our best solution has been to fit our data to a high-order polynomial in \( \log t_w \) (in the fits, see main text, we considered only the diagonal part of the covariance matrix). We checked that \( \chi^2_{\text{diagonal}}/\text{d.o.f} \) was smaller than one. However, in order to avoid an excessive data-smoothing, we enlarged the degree of the polynomial well beyond that. Basically, we stopped before the polynomial became non monotonically increasing in the working time-range. Notice that our error computation (namely a different fit for each jack-knife block) identifies spurious oscillations due to a too large-order fitting polynomial.

Having in our hands an inverse function \( t_w(\xi) \) we proceed to compute (using the same fitting approach in \( \log t_w \) ) \( U_4(\lambda, \xi, T) \). When needed, see e.g. Sect. we interpolated in the same way the integrals \( I_2(t_w) \).

Finally, we need to interpolate in \( T \) the \( U_4 \) values computed at fixed \( \lambda \) and \( \xi(t_w) \) for our simulation temperatures. In this case, the variations among neighboring temperatures are typically much larger than error bars. Hence, even a Lagrangian polynomial interpolation works well. However, when the number of data available from the different temperatures is large, we prefer a fit to a low-order polynomial in \( T \). In practice, we restrict ourselves to polynomials of at most fifth degree.

**Appendix C: Dynamic crossover in the correlation function**

The dynamic cross-over (that becomes a true phase-transition with the temperature only for infinitely long waiting time) was studied in the main text by focusing on the four-legs correlation function of the overlap field. One could wonder whether one could study the same crossover on the two points correlation function. Indeed, this was the route chosen in Ref. [52] (although the language in Ref. [52] was slightly different).

Let us start by recalling Eqs. (2,8) from the main text:

\[ C_4(r, t_w) \equiv \frac{1}{L^4} \sum_x q_x(t_w)q_{x+r}(t_w) \sim \frac{e^{-[r/\xi(t_w)]^b}}{r^\theta}. \quad (C1) \]

The asymptotic form in Eq. (C1) is expected to hold only for \( r \) much larger than the lattice spacing. Our expectations for the asymptotic regimes are:

1. When we reach equilibrium in the paramagnetic phase, we expect a free-field behavior, namely \( \theta = 1, b = 1 \) in Eq. (C1).

2. In the critical regime, \( y \) of order one (recall from the main text that \( y = [T-T_c][\xi(t_w,T)]^{1/\nu} \) or \( T = T_c \), we expect \( \theta = D = 2 + \eta \), where \( D \) is the space dimension and \( \eta \) is the anomalous dimension. We are not aware of any prediction for exponent \( b \). In this work, we have found \( b = 1.46(1) \).

3. There is a considerable controversy regarding the spin glass phase, \( y \ll -1 \). On the one hand, the droplets model [72][73] predicts \( \theta = 0 \), although the asymptotic limit is reached fairly slowly, with corrections of order \( 1/\xi^{a=0.2} \). On the other hand, the Replica Symmetry Breaking scenario [27] expects a non-trivial exponent \( \theta \approx 0.37 \) [22] and corrections of order \( 1/\xi^\theta \). Up to our knowledge, neither of the two theories have predictions for exponent \( b \) in Eq. (C1). It was empirically found in Ref. [28] that \( b \approx 1.5 \). In fact, we have found that \( b = 1.46(1) \) works just as well in the low temperature phase (see also Ref. [53]).

\footnote{The standard naming two-legs or four-legs correlation function is somehow confusing in the spin-glass context. In fact, the product of the overlap field at two sites (the two-legs function) involves the product of four spins, hence the name \( C_4 \).}
A statistically irrelevant artifact is the presence of wiggles in the integral \( \chi \). Consider the integrals (see [32, 33] and main text):

\[
\xi \text{ computed for pairs of times such that the coherence length function the inverse coherence length the susceptibility ratio}
\]

From them we obtain the integral estimator \( \xi_{12} = I_2/I_1 \propto \xi \).

Our analysis will be based on the scaling properties of the integral

\[
I_2 \propto \xi_{12} \theta.
\]

Note that, in three spatial dimensions, \( \chi = 4\pi I_2 \), where \( \chi \) is the (non-equilibrium analog of) the spin-glass susceptibility. The analysis of Ref. [32] was based on the susceptibility \( \chi(T, t_w) \) (however, Ref. [52] did not use the variance reduction methods available for the computation of the integrals \( I_n \) [32, 33] which are most effective because \( \xi \) is much smaller than the system sizes).

As explained in the main text, for any given temperature we may seek two times \( t_w(1) \) and \( t_w(2) \) such that and \( \xi_{12}(t_w(2), T) = 2\xi, \xi_{12}(t_w(1), T) = \xi \). Hence, for \( y \) of order one, we expect

\[
I_2(2\xi, T)/I_2(\xi, T) = 2^{2-y} f(y) + \ldots ,
\]

where the scaling function \( f(y) \) is such that \( f(y = 0) = 1 \) and the dots stand for corrections to scaling of order \( \xi^{-w} \). Note that Eq. (C5) is analogous to Eq. (3) in the main text (where we were considering the Binder’s parameter instead).

The crossover implicit in Eq. (C5) is shown in Fig. 5, which can be directly compared with Fig. 2 in the main text. One can consider the \( \xi \to \infty \) limits in the plot:

- At the critical point \( T = T_c \), one expects \( 2^{2.3900(36)} = 5.242(13) \) [36].
- In the spin-glass phase, the droplets model predict a common limit \( 2^4 = 8 \) for all \( T < T_c \), while the Replica-Symmetry Breaking theory expects a limit \( 2^\text{RSB} \approx 6.19 \).
- The paramagnetic phase is more complicated to discuss. In fact, for \( T > T_c \), the coherence length grows only up to its equilibrium value for that temperature, \( \xi_{eq}(T) \). This means that all the (paramagnetic) curves in Fig. 5 have an end point. At this end-point, the longest time \( t_w(2) \) correspond to the equilibrium regime (i.e. \( \theta_2 = 1 \)) while the earliest time is still in the non-equilibrium regime. Hence, it is not easy to anticipate the numerical value of the paramagnetic long-time limit, obtained when \( \xi_{eq}(T) \) tends to infinity.

Data in Fig. 5 can be analyzed in exactly the same way we did for the Binder’s parameter [see Eq. (4) and Fig. 3 in the main text]. However, with the susceptibility ratio Eq. (C5), errors are one order of magnitude larger. This is why we abandoned this approach.

---

8 The relation \( \chi = 4\pi I_2 \) assumes spatial isotropy in \( C_4 \), which becomes an excellent approximation when \( \xi \) grows.

9 One could just as well consider pairs of times such that their coherence lengths are in any prescribed ratio \( r \). In such case, Eq. (C5) would read as \( I_2(r\xi, T)/I_2(\xi, T) = r^{2-\eta} f_r(y) + \ldots \).

10 A statistically irrelevant artifact is the presence of wiggles in Fig. 5 if the order of the fitting polynomial in \( \log t_w \) is large. The origin of this wiggles has been known for some time [32]. The point is that each polynomial is evaluated twice, one in the numerator the other in the denominator in Eq. (C5). In fact, the effect can be much alleviated by keeping the order of the polynomial limited to 13. Even with these polynomials, \( \chi^\text{diagonal/d.o.f} \) lies well below one.
