Perfect Bayesian Equilibria in Repeated Sales

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Abstract

A special case of Myerson’s classic result describes the revenue-optimal equilibrium when a seller offers a single item to a buyer. We study a natural repeated sales extension of this model: a seller offers to sell a single fresh copy of an item to the same buyer every day via a posted price. The buyer’s value for the item is unknown to the seller but is drawn initially from a publicly known distribution $F$ and remains the same throughout. One key aspect of this game is revelation of the buyer’s type through his actions: while the seller might try to learn this value to extract more revenue, the buyer is motivated to hide it to induce lower prices. If the seller is able to commit to future prices, then it is known that the best he can do is extract the Myerson optimal revenue each day. In a more realistic scenario, the seller is unable to commit and must play a perfect Bayesian equilibrium. It is known that not committing to future prices does not help the seller. Thus extracting Myerson optimal revenue each day is a natural upper bound and revenue benchmark in a setting without commitment.

We study this setting without commitment and find several surprises. First, if the horizon is fixed, previous work showed that an equilibrium always exists, and all equilibria yield a very low revenue, often times only a constant amount of revenue. This is unintuitive and a far cry from the linearly growing benchmark of obtaining Myerson optimal revenue each day. Our first result shows that this is because the buyer strategies in these equilibria are necessarily unnatural. We restrict to a natural class of buyer strategies, which we call threshold strategies, and show that pure strategy threshold equilibria rarely exist. This offers an explanation for the non-prevalence of bizarre outcomes predicted by previous results. Second, if the seller can commit not to raise prices upon purchase, while still retaining the possibility of lowering prices in future, we recover the natural threshold equilibria by showing that they exist for the power law family of distributions. As an example, if the distribution $F$ is uniform in $[0, 1]$, the seller can extract revenue of order $\sqrt{n}$ in $n$ rounds as opposed to the constant revenue obtainable when he is unable to make any commitments. Finally, we consider the infinite horizon game with partial commitment, where both the seller and the buyer discount the future utility by a factor of $1 - \delta \in [0, 1)$. When the value distribution is uniform in $[0, 1]$, there exists a threshold equilibrium with expected revenue at least $\frac{4}{3} + 2\sqrt{2} \approx 69\%$ of the Myerson optimal revenue benchmark. Under some mild assumptions, this equilibrium is also unique.

1 Introduction

Most interesting economic games are inherently dynamic and/or repetitive, with the same sellers repeatedly interacting with the same buyers. Such scenarios arise commonly in e-commerce platforms, such as eBay and Amazon, and online advertising markets, such as Google, Yahoo! and Microsoft, among others. Unfortunately, the game-theoretic aspects of such repeated interactions are poorly understood compared to their static, one-shot counterparts. In this paper, we develop the theory of one such fundamental setting.

The fishmonger’s problem.1 There is a single seller of fish and a single buyer who enjoys consuming a fresh fish every day. The buyer has a private value $v$ for each day’s fish, drawn from a publicly known distribution. However, this value is drawn only once, i.e., the buyer has the same unknown value on all days. Each day, the seller sets a price for that day’s fish, which of course can depend on what happened on previous days. The buyer can then decide whether to buy a fish at that price or to reject. The goal of the buyer is to maximize his total utility (his value minus

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Consider, for example, the case where the distribution of the buyer’s value is uniform in $[0, 1]$ (denoted by $U[0, 1]$ for short), and the game lasts for one day. In this case, it is easy to see that the optimal seller price is the monopoly price\(^2\) of 1/2, resulting in an expected seller profit of 1/4.

What prices should the seller set if the game is to last for two days? A first guess is 1/2 on both days, for an expected profit of 1/4 each day or 1/2 overall. But this is implausible: if the buyer rejects on the first day, the seller might reasonably assume that the buyer’s value is $U[0, 1/2]$, in which case the seller’s best response is to offer a price of 1/4 on the second day. This yields the seller strategy shown in Figure 1a. However, this buyer/seller strategy pair is not in equilibrium. This seller strategy is based on the fallacious assumption that the buyer’s best response is to buy on both days if his value is above 1/2. Indeed, a buyer with value $1/2 + \epsilon$ gets a utility of $2\epsilon$ for buying both days, whereas his utility is $1/4 + \epsilon$ if he only buys on the second day. Interestingly, if the buyer could be guaranteed that the price on the second day was 1/2, then his best response would be to buy both days when $v > 1/2$. However, since the seller is unable to commit to a second day price, the buyer’s strategy on the first day must take into account that on the second day the seller will best respond to the buyer’s first day strategy. The result, in this case, is that the buyer is incentivized to wait for the lower second day price unless his value is at least 3/4.

(a) This is not a (perfect Bayesian) equilibrium

(b) This is a valid (perfect Bayesian) equilibrium

Figure 1: Equilibrium illustration for the 2 days fishmonger’s problem. The number in the top circle is the price on the first day. The number in the circle following the left arrow denotes the price on the second day after the buyer rejected on the first day, and the one in the circle following the right arrow denotes the second day price after buyer buys at the posted price on the first day. The distributions are updated depending on whether the buyer purchased or rejected on the first day.

So what is the optimal strategy for this 2-day game (for arbitrary distributions of buyer valuation)? Or for the $n$-day version? In this paper we study this question: how much can the seller make in $n$-days in a Perfect Bayesian Equilibrium (PBE)?

**Commitment and Perfect Bayesian Equilibrium.** The lack of commitment in repeated games is the major driver of fundamental differences in outcomes when compared to single-shot games. The absence of commitment in repeated games is captured by the notion of a perfect Bayesian equilibrium (PBE). Informally, a PBE consists of a seller strategy, describing what price he offers as a function of the history of play at each time, and a (possibly randomized) buyer strategy describing his accept/reject decisions given the history of play and his value. For every possible value the buyer has, and for every possible history of play, his strategy must be a best response to the subtree of prices the seller’s strategy specifies for that particular history of play. For the seller, for every possible history of play, the subtree of prices offered henceforth must optimize his profit given the buyer’s strategy and the induced distribution of values the buyer has (as determined by the history of play). For example, Figure 1b shows the essentially unique PBE strategies for the buyer and the seller in the example discussed above. We refer the reader to Appendix C for a general

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\(^2\)This is a special case of Myerson [1981]’s theorem which implies that the revenue optimal mechanism for a seller facing a buyer with value drawn from known distribution $F$ is to offer a price of $p$ (monopoly price) that maximizes $p(1 - F(p))$.\footnote{This is a special case of Myerson [1981]’s theorem which implies that the revenue optimal mechanism for a seller facing a buyer with value drawn from known distribution $F$ is to offer a price of $p$ (monopoly price) that maximizes $p(1 - F(p))$.}
way to compute PBEs for two round games with arbitrary distributions. For the sake of intuition, we also flesh out the $U[0,1]$ case and provide a complete description and verification for why the said strategies are indeed a PBE.

Questions we study.  We study three different versions of the fishmonger’s problem for arbitrary distributions: the $n$ rounds version without any commitment, the $n$ rounds version with partial commitment, and the time discounted infinite horizon version (with and without commitment). Here are the formal definitions.

Definition 1 A 1 seller, 1 buyer Finite Horizon Repeated Sales game is a sequential (extensive form) game between a seller and a buyer, with $n$ rounds. In each round, the buyer has a private valuation of $v$ for a perishable item, with a quasilinear utility. The value $v$ is initially drawn from a distribution $F$ supported on $[\ell, h]$ ($0 \leq \ell \leq h$) and stays the same throughout; the seller only knows $F$. The seller can produce a fresh copy of the item in each round, at a publicly known known cost (normalized to) 0. Each round has two stages: the seller first offers a price for the item, and then the buyer responds with an accept or a reject.

Definition 2 A 1 seller, 1 buyer Finite Horizon Partial Commitment Repeated Sales game is the same as a finite horizon repeated sales game with the additional condition that the seller cannot raise prices upon purchase. He still holds the freedom to lower prices.

Definition 3 A 1 seller, 1 buyer Time Discounted Infinite Horizon Partial Commitment Repeated Sales game with a discount factor $1 - \delta$ is a partial commitment repeated sales game that is played forever, with the buyer and the seller discounting their round $i$’s utility by $(1 - \delta)^{i - 1}$. Equivalently, it can be thought of as a partial commitment repeated sales game (without any discounting) whose stopping time is a geometrically distributed random variable: the probability that the game stops after any given round is $\delta$.

The benchmark and a preliminary result.  We already know that if the seller were able to fully commit to all future prices, he can commit to setting Myerson optimal price for every single day, thereby getting $n$ times Myerson optimal revenue. By not committing to future prices, can the seller get more revenue, or at least as much revenue? Although the optimal revenue strategy shown in Figure 1b achieves a revenue of only 0.45 (less than twice the Myerson optimal single-day profit), one might imagine that repeating the game for many days enables the seller to “learn” the buyer’s value and eventually start extracting most of the buyer’s value (the seller has the freedom to learn and put this to use because he is not committed to any future prices). Surprisingly, this is not the case. The seller cannot get any more revenue by not committing to future prices. The gist of the argument\(^3\) is that if it were possible to extract more than $n$ times Myerson optimal revenue, then it would be possible to extract more than Myerson optimal revenue in a single round game. This result suggests extracting-Myerson-optimal-revenue-every-single-day as a natural upper-bound and revenue benchmark in a setting without commitments.

We present the above argument in Proposition 1. In fact, the argument holds for a much more general mechanism design setting, with arbitrary objectives, arbitrary time discounting, with the number of repetitions of the game possibly being a random variable. To our knowledge, this result has not been proved in this level of generality before. We refer the reader to Appendix B for a formal description of this general model of repeated mechanism design, and the proof of the following proposition.

Proposition 1 In the general model of repeated mechanism design, the optimal objective value obtained without any commitment is never larger than the optimal objective value obtained when commitment is possible.

Our results.  The finite horizon repeated sales game has been previously considered by Hart and Tirole [1988] and Schmidt [1993]. (See also the survey by Fudenberg and Villas-Boas [2006].) Hart and Tirole [1988] consider the special case where $F$ is a 2-point distribution and Schmidt [1993] generalizes it to any discrete distribution. These papers show that for the finite horizon version of the game, a PBE always exists, and, every PBE charges the minimum possible price $\ell$ on all but the final few constant number of rounds. For such a PBE, in the $n$-rounds repeated fishmonger problem where the buyer’s value is $U[0,1]$, the seller extracts only a constant amount of revenue, as opposed to our benchmark of $n$ times the Myerson 1-round revenue of $1/4$. This revenue doesn’t even grow with $n$.

\(^3\)The exact origin of the proof of this fact is not clear. The high-level idea in this argument seems like folklore.
1. Finite horizon with no commitment.  This meager profit obtained in previous works is a far cry both from what we would expect intuitively, and what we see in practice. Our first result provides a possible explanation for this: the equilibrium strategies in the PBE described by previous work (Schmidt [1993]) are necessarily unnatural and resemble nothing any real seller or buyer would use. Instead we consider threshold PBEs in which the buyer uses a threshold strategy: on each day, the buyer purchases only if his value is above a certain threshold (that depends on the seller’s strategy). We show that in almost all cases, such “pure strategy threshold perfect Bayesian equilibria” do not exist. We exactly characterize the distributions for which a threshold PBE exists and this class is very small (see Theorem 2).

2. Finite horizon with partial commitment.  Our second result offers a way to mitigate the non-existence of threshold PBEs (and the terribly small revenue obtained by whatever PBEs exist) by relaxing the seller’s commitment requirement: the seller may not increase prices upon purchase, but has the freedom to decrease them over time. The motivation for such one sided commitment is that not increasing the price is like providing a price guarantee, and is often seen in practice, whereas not decreasing the price is less common and harder to enforce. After all, it may be difficult for the seller to resist the temptation to lower prices if it will entice the buyer to purchase. Moreover while increasing the price may be beneficial to the seller, it is never in the interest of the buyer, but decreasing the price (when it is higher than the buyer’s value) benefits both the buyer and the seller.

We show that threshold PBEs are guaranteed to exist for a large family of distributions, including the power law family of distributions with the ‘no-price-increase’ restriction on the seller. For the case where $F$ is $U[0, 1]$, the seller’s revenue is $\sqrt{2} + \log n/8 + O(1)$, with a horizon of $n$. (See Theorem 4.) This is hugely better than the $O(1)$ revenue that we get (through the unnatural PBEs) when there is no commitment. On the other hand, it is still far from the Myerson optimal revenue benchmark of $n/4$, which brings us to our third result.

3. Time discounted infinite horizon with partial commitment (Main result).  One of the main conclusions of Hart and Tirole [1988] and Schmidt [1993] is that even for very long horizons, the seller is restricted to post a price of $\ell$ for all but the final few constant number of the rounds (whether or not future utilities are discounted), and thereby getting a meager revenue irrespective of the number of rounds. In our final result we consider the infinite horizon game with time discounting, combined with the power of partial commitment: the game is repeated forever but time discounting ensures that players’ utilities are still finite, and the seller promises never to raise prices upon purchase. For this game, we arrive at a very different conclusion from the conventional wisdom. We show that for the $U[0, 1]$ distribution, the seller can extract a constant fraction of our benchmark, namely, Myerson optimal revenue in every single round, where the constant is at least $\frac{4}{3 + 2\sqrt{2}}$, which is roughly 69%. (See Theorem 6.) On the other hand, an infinite horizon game with time discounting, but without any kind of commitment is not known to have a good revenue fetching equilibrium. We confirm in Theorem 5 that the “no discrimination” equilibrium of Hart and Tirole [1988] continues to be an equilibrium for all distributions in the infinite horizon time discounted case, getting only 0 revenue.

Interpretation as a game with geometric stopping time.  While the infinite horizon might appear as a mathematical curiosity with little practical relevance, it is actually the most realistic of the three models. The time discounted infinite horizon game is exactly equivalent to a game with geometric stopping time. With a discount factor of $1 - \delta$ per round, the $i$-th round utility is discounted by a factor $(1 - \delta)^{i-1}$. Equivalently, if the game stops after any given round with probability $\delta$, the probability that the $i$-th round is reached is $(1 - \delta)^{i-1}$, and therefore, any utility obtained in that round has to be discounted by a factor $(1 - \delta)^{i-1}$. Often, a geometric stopping time is more realistic than a fixed $n$ day horizon because the buyer or the seller may not be sure of the precise number of interactions that will take place.

Summary and the big picture.  Together, these results shed ample light on this fundamental setting of repeated interaction. Our first result shows that the finite horizon case is a lot less nice than previously thought, as the PBEs ought to be non threshold. The partial commitment case that we introduce solves some of the thorny issues in the

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The buyer’s strategy has to necessarily be non-monotonic. Even the simplest buyer strategy in this category would be of the form: buyers with value less than $t_2$ will buy, those with value in $[t_1, t_2)$ will not buy, and those with value in $[t_2, 1]$ will buy. Correspondingly the seller’s updated distribution will be supported in three disconnected intervals. While even this three interval split is not natural, the number of such fragmented intervals could be much more as the number of rounds increases.
finite horizon case. The combination of partial commitment and time discounted infinite horizon case provides a much better salvation, by allowing the seller to extract a constant fraction of the revenue with full commitment. This suggests that the partial commitment with infinite horizon model is closer to reality, and a better model to study for more complicated repeated settings.

The economics of repeated interactions are really fundamental to many aspects of online advertising. In search ads for instance, the search engine uses past bids of the advertisers to set reserve prices, the best documented example of which is the use of Myerson’s theory at Yahoo by Ostrovsky and Schwarz [2011]. It is very common for search engines to worry about the long term impact of their actions. A basic question in this regard is how strategic advertisers would bid knowing that the search engine is going to fiddle with the auction format and parameters in the future and that some of these changes will depend on the advertisers’ bids today. Even in display ads that are sold through negotiated contracts, the publisher could change the rates charged over time, which the advertisers have to be wary of [digiday.com]. These repeated interactions are also fundamental to issues of privacy, such as when a customer’s purchases are tracked by a merchant, who then has the option of offering “personalized” prices.

Related Work. The works closest to ours, as already mentioned, are Hart and Tirole [1988], Schmidt [1993], Fudenberg and Villas-Boas [2006]. There are several other well studied themes that have the same flavor as our model but are also significantly different. Please see appendix A for a detailed discussion on these themes.

Organization. In Section 2, we formally define the game, Perfect Bayesian Equilibrium (PBE) and related concepts. In Section 3, we study the finite horizon game with no commitment. In Section 4, we study the finite horizon partial commitment game. In Section 5, we consider the time discounted infinite horizon game. In Section 6 we conclude with some open problems.

2 Preliminaries

Bayesian Nash Equilibrium. The most common notion of equilibrium in a static game of incomplete information is the Bayesian Nash Equilibrium. A profile of strategies is a Bayesian Nash Equilibrium (BNE) if for every agent, given the other agents’ strategies, his own strategy maximizes his expected payoff for each of his type. The expected payoff of an agent is computed using the agent’s beliefs about the private types of other agents, and all the agents’ beliefs are assumed to be consistent with a common prior distribution over all the private types.

History. The game proceeds over \( n \) rounds, and each round consists of two stages: round \( r \) consists of stages \( k = 2r - 1 \) and \( k = 2r \). At \( k = 2r - 1 \), the seller sets a price, and at \( k = 2r \), the buyer reacts with an accept or reject. The history after \( k \) stages of play is denoted by \( h^k \), and constitutes the prices and accept/reject decisions of all stages \( k' : 0 \leq k' \leq k \).

Beliefs. In our game, since the buyer’s type alone is private, the seller alone has a belief over the buyer’s private type. The seller’s belief \( \mu(\cdot|h^k) \) is a probability density function over the buyer’s private type.

Strategy Spaces. The seller’s action space is restricted to posting a non-negative price in every round. Correspondingly, the seller’s strategy \( \sigma_s(\cdot|h^k) \) is a function that, for every possible history, outputs a probability distribution over his available actions (i.e., non-negative prices). The buyer’s action space is restricted to accepting or rejecting a price. Correspondingly, the buyer’s strategy \( \sigma_b(\cdot|v, h^k) \) is a function that, for every possible private value of the buyer and every possible history, outputs a probability for accepting the item at the posted price.

Perfect Bayesian Equilibrium. Intuitively a Perfect Bayesian Equilibrium combines the notions of subgame perfect equilibrium (used in dynamic games of complete information) and Bayesian update of beliefs (used in games of incomplete information) by requiring that the profile of strategies and beliefs when applied to the continuation game given any history, form a BNE. It is the perfection aspect of PBE that makes commitments non-credible/non-binding: informally, no commitment is credible unless it is a part of a BNE in the continuation game after every possible history.
that could precede the stage at which the commitment becomes effective. We now formally define Perfect Bayesian Equilibrium for our game, i.e., mention only the restrictions relevant to our game.

A profile of strategies \((\sigma^*_s(\cdot|h^k), \sigma^*_b(\cdot|v, h^k))\) and beliefs \(\mu(\cdot|h^k)\) in the repeated-sale game is a Perfect Bayesian Equilibrium (PBE) when the following conditions are satisfied:

1. Bayesian update of seller’s beliefs: the seller assumes that the buyer plays the PBE strategy \(\sigma^*_s(\cdot|v, h^k)\). If there exists a value \(v\) such that \(\mu(v|h^{k-1}) > 0\), and, the buyer’s action at stage \(k\) has a non-zero probability under his equilibrium strategy \(\sigma^*_b(\cdot|v, h^k)\) at \(v\), the seller updates his belief \(\mu(\cdot|h^k)\) based on Bayes’ rule. There are no restrictions on belief updates if the buyer takes an out-of-equilibrium action.

2. For every \(k\) and \(h^k\), the strategies from \(h^k\) onwards are a BNE for the remaining game. Formally, conditional on reaching \(h^k\), let \(u_s(\sigma|h^k, \mu(\cdot|h^k))\) denote the expected utility of seller under strategy profile \(\sigma\) (where the expectation is over both the randomness in \(\sigma\) and the belief \(\mu(\cdot|h^k)\)), and let \(u_b(\sigma|v, h^k)\) denote the expected utility of the buyer under strategy profile \(\sigma\) (where the expectation is over the randomness in \(\sigma\)). Then,

\[
\begin{align*}
\forall \sigma_s & \quad u_s(\sigma^*_s|h^k, \mu(\cdot|h^k)) \geq u_s((\sigma_s, \sigma^*_b)|h^k, \mu(\cdot|h^k)) \\
\forall \sigma_b & \quad u_b(\sigma^*_b|v, h^k) \geq u_b((\sigma_b, \sigma^*_s)|v, h^k)
\end{align*}
\]

Threshold PBE. A threshold strategy for the buyer computes an accept/reject decision as follows: given history \(h^k\), it computes a deterministic threshold \(t(h^k) \geq 0\), and accepts the item if the buyer’s value \(v \geq t(h^k)\) and rejects otherwise. By definition, a threshold strategy is a pure strategy. In this paper, we focus on PBEs where the buyer plays a threshold strategy.

3 Finite Horizon with No Commitment

Two rounds game. It turns out that for a two rounds game, a threshold PBE is guaranteed to exist and it is essentially unique. Hart and Tirole [1988] and Fudenberg and Villas-Boas [2006] characterize the PBE for the two rounds repeated sales game. For the sake of completeness and for gaining intuition, and because our main result uses the 2 rounds result (mildly), we present the 2 rounds result in Appendix C, along with an explicit discussion for the \(U(0, 1)\) example.

Main result of this section. We now move to the main result of this section: in a \(n\) rounds repeated sales game, threshold PBE exist only very rarely when \(n > 2\). The following theorem precisely characterizes when a threshold PBE exists and what happens in it. We prove the theorem in appendix D.

Theorem 2 For every atomless bounded support distribution \(F\) of buyer’s value, the following are true for an \(n\) rounds (\(n > 2\)) repeated sales game.

1. An \(n\) rounds pure strategy threshold PBE exists precisely for those distributions for which the 2 rounds threshold PBE has the lowest possible first round price, namely, \(\ell\).

2. For such distributions \(F\) where an \(n\) rounds pure strategy threshold PBE exists, it is unique: the price in the first \(n-1\) rounds is the lowest possible, namely, \(\ell\). The price in the last round is the monopoly price for the distribution \(F\).

Discussion. How often do we have a distribution for which the PBE price in the first round is the throw away price of \(p_1 = \ell\)? For the \(U(0, 1)\) distribution, this requirement would mean that the seller sacrifices the first round at a price of 0, and gets only a revenue of 1/4 from the second round. But this doesn’t happen — the seller gets a much better revenue of 9/20. Distributions for which this happens are quite rare. If the monopoly price of the distribution happens to be precisely \(\ell\), then this would happen, because the price in all the rounds will be just \(\ell\), and that is both a PBE and gets Myerson optimal revenue (for instance for the \(U[1/2, 1]\) distribution monopoly price is \(\ell\).) But there are not that many distributions with the lowest point in the support as the monopoly price.
4 Finite Horizon with Partial Commitment

While a threshold PBE exists very rarely when there is no commitment, things change dramatically if we allow partial commitment. We show that by having the partial commitment of not raising prices, threshold PBEs are guaranteed to exist for the power law family of distributions supported in [0, 1] (i.e., the probability density function $f(\cdot)$ is specified by a non-negative integer $k$: for all $x \in [0, 1], f(x) = (k + 1)x^k$). While we restrict ourselves to power law for analytical tractability, we believe that the existence of PBE applies more generally than the power law family of distributions.

**Theorem 3** For all power law distributions supported in [0, 1], the finite horizon partial commitment repeated sales game is guaranteed to have a threshold PBE.

The structure of the strategies, and why they constitute a PBE are much along the lines of why strategies 7 and 8 constitute a PBE for the time discounted infinite horizon game with partial commitment. We don’t repeat that argument here, and will provide the proof in a more expanded version.

To illustrate the difference partial commitment can make, we focus on the $U[0, 1]$ distribution and compute a threshold PBE that obtains a revenue of $\sqrt{\frac{n}{2} + \frac{\log n}{8}} + O(1)$. In comparison, the revenue in the no commitment $n$ rounds game’s PBE (which is necessarily non-threshold by our theorem 2) is just $O(1)$.

**Theorem 4** For the $U[0, 1]$ distribution, the partial commitment repeated sales game has a threshold PBE that obtains a revenue of $\sqrt{\frac{n}{2} + \frac{\log n}{8}} + O(1)$.

**Proof:** As mentioned before, the full proof explicitly specifying strategies and arguing why they constitute a PBE are much along the lines of that for Theorem 6. We don’t repeat it here, and present only the argument along the equilibrium path that gives us what the revenue is.

The proof of this theorem involves solving a recursion. So we change our convention for numbering rounds (just for this proof) from what we used in previous sections: the price, and threshold in the first round of the $n$ rounds game are denoted by $p_n$ and $t_n$ (in earlier sections we used $p_1$ and $t_1$ for the first round). Similarly the second round’s corresponding quantities are $p_{n-1}$ and $t_{n-1}$ and so on.

Let $R_n$ denote the expected revenue and $u_n$ denote the utility of the buyer with value 1 after $n$ rounds. Since $U[0, t]$ is a scaled version of $U[0, 1]$, the expected revenue and the utility of the buyer with value $t$ for the $U[0, t]$ distribution are just $tR_n$ and $tu_n$, respectively.

We begin our calculations by assuming that the agent with value 1 accepts to buy in the first round, and later verify this is indeed true. If he accepts in first round, he will accept in all future rounds because the price stays the same as in first round. So he gets a utility of $n(1 - p_n)$, i.e.,

$$u_n = n(1 - p_n). \tag{1}$$

The buyer with value at threshold $t_n$ is indifferent between buying and rejecting. By buying he gets a utility of $n(t_n - p_n)$. By rejecting he will get a utility of a buyer with value $t_n$ in a $n - 1$ rounds game for the $U[0, t_n]$ distribution. Because of scaling, this is just $t_n$ times the utility of the buyer with value 1 for the $U[0, 1]$ distribution, i.e., $t_n u_{n-1}$. Writing this indifference as an equation, we get

$$u_{n-1}t_n = n(t_n - p_n). \tag{2}$$

The expected revenue $R_n$ is computed as follows: with probability $t_n$, the first round gets a rejection, and with probability $1 - t_n$ it sees an acceptance. Upon rejection, we are left with a $n - 1$ rounds game for $U[0, t_n]$. By scaling this, the revenue here is simply $R_{n-1} t_n$. Upon acceptance in the first round, we get the same price as the revenue for all future days, and hence a revenue of $np_n$. Writing this out, we get:

$$R_n = R_{n-1} t_n^2 + (1 - t_n) \cdot np_n. \tag{3}$$
We have a four variable recurrence in \( \{u_n, t_n, p_n, R_n\} \) to solve, given by (1)-(3). Hoping that things will smoothly work out, we begin our algebra. Substituting for \( p_n \) from equation (2) into equation (3) we have
\[
R_n = R_{n-1} t_n^2 + (1 - t_n) t_n (n - u_{n-1}).
\]
(4)

This is an expression for revenue that the seller has to maximize. Notice that \( R_{n-1} \) and \( u_{n-1} \) are fixed quantities that are not to be optimized: these are quantities for the \( n-1 \) rounds game for which we assume by induction that there is a unique threshold PBE and hence revenue, utilities etc. are fixed. The only quantity to optimize in this expression is \( t_n \). This expression for \( R_n \) is maximized at
\[
t_n = \frac{n - u_{n-1}}{2(n - u_{n-1} - R_{n-1})}.
\]
Substituting this value of \( t_n \) into equation (2), we get
\[
p_n = \frac{(n - u_{n-1})^2}{2n(n - u_{n-1} - R_{n-1})}.
\]
(5)

Similarly, substituting \( t_n \) into equation (4), we get
\[
R_n = \frac{(n - u_{n-1})^2}{4(n - u_{n-1} - R_{n-1})}.
\]
(6)

From equations (5) and (6) it is easy to verify that the following relation holds: \( R_n = \frac{p_n n}{2} \). Using this relation, and combining equations (1) and (6), we eliminate three out of four variables to finally get
\[
R_n = \frac{(1 + 2R_{n-1})^2}{4(1 + R_{n-1})}.
\]

To analyze this recursion, substitute \( V_n = R_n + 1 \). This yields \( V_n = 1 + \frac{(2V_{n-1} - 1)^2}{4V_{n-1}} = V_{n-1} + \frac{1}{4V_{n-1}} \). This is still not at a stage where we can extract \( V_n \) out. Squaring both sides, we get \( V_n^2 = V_{n-1} + \frac{1}{4V_{n-1}} + \frac{1}{16V_{n-1}^2} \). This finally says that the high order term of \( V_n \) is \( \sqrt{\frac{V}{n}} \). To get a precise expression for \( V_n \), we add the differences of \( V_i^2 - V_{i-1}^2 \). For the fractional term, \( \frac{1}{16V_{n-1}^2} \), we substitute \( \frac{1}{n^2} \) in the summation. For \( V_n^2 \), we simply use \( O(1) \) because \( V_1^2 = (R_1 + 1)^2 = \frac{25}{16} \). Thus adding differences, we get \( V_n^2 = \frac{n}{2} + \frac{\log n}{8} + O(1) \), and \( R_n = V_n - 1 = \sqrt{\frac{n}{2} + \frac{\log n}{8} + O(1)} \). Thus \( p_n = \frac{2R_n}{n} \sim \sqrt{\frac{2}{n}} \) and \( t_n \sim 1 - \frac{1}{\sqrt{2n}} < 1 \). Now we can go back and check the assumption we made, namely, the buyer with value 1 accepts in the first round itself, which is true since \( t_n \) is strictly smaller than 1.

**Remark 1** Interestingly, although the price starts very low, at \( p_n \sim \sqrt{\frac{2}{n}} \), the threshold starts very high at \( t_n \sim 1 - \frac{1}{\sqrt{2n}} \). That is, the seller already starts with a very small price, and the buyer still refuses to buy for most of his values, waiting for the price to go down even further.

### 5 Time Discounted Infinite Horizon

**Threshold strategies for buyer.** A threshold strategy for the buyer computes an accept/reject decision as follows: given history \( h_k \), it computes a deterministic threshold \( t(h_k) \geq 0 \), and accepts the item if the buyer’s value \( v \geq t(h_k) \) and rejects otherwise. Note that a threshold strategy by definition is a pure strategy.

**Markovian strategies and beliefs for seller.** A seller’s strategy is Markovian if for any two histories for which the seller has the same beliefs, the price posted by the seller in this round is identical in both cases, i.e., the seller’s strategy depends only on the belief derived from the history and not on other aspects of the history. Similarly, seller’s beliefs are Markovian if given two histories for which the seller has the same beliefs, and the seller posts an identical price given these two histories, the seller’s posterior belief after seeing the buyer’s response should be identical.
Scale-invariant strategies and beliefs for seller. A seller’s Markovian strategy is scale-invariant, if given two different prior beliefs that are scaled versions of one another, the seller’s prices in this round given these two beliefs are also scaled versions of one another, with the same ratio as prior beliefs. Similarly, seller’s beliefs are scale-invariant, if given two different prior beliefs that are scaled versions of one another, and the seller’s prices given these beliefs are also scaled versions of one another with the same ratio as prior beliefs and the seller’s posterior belief for one of the prior beliefs given a buyer response (accept or reject) is a scaled version of the prior, then for the other prior belief too, the posterior belief given the same buyer response, should be a scaled version of the prior belief. Concretely, for the uniform distribution, this means that when the belief is \( U[0, a] \), there is some constant \( p \) such that for all \( a \), the price is \( pa \). And, if for some \( a \), when the belief is updated from \( U[0, a] \) to \( U[0, ta] \) after seeing a certain buyer response at a price of \( pa \), then the same should be true for every \( a \) after observing the same buyer response.

Markovian and scale-invariance for the partial commitment game. In the presence of partial commitment game, the seller is bound to honor the promise of not raising the price beyond any price that was offered earlier and accepted by the buyer. In other words, apart from belief, there is one other aspect of the history that is relevant: the minimum price that was offered and accepted by the buyer in the history. In such a case, we call a seller’s pricing strategy Markovian, if for any two histories for which the seller has the same beliefs, and have the same minimum accepted price, the price posted by the seller in this round is identical in both cases. Similarly, a seller’s Markovian strategy is scale-invariant, if given two different prior beliefs that are scaled versions of one another, and the corresponding histories have minimum accepted prices that are also scaled versions of one another, the seller’s prices in this round given these two beliefs are also scaled versions of one another, with the same ratio as prior beliefs. Analogous definitions hold for Markovian and scale-invariant beliefs for partial commitment games.

Consistency requirements. We impose two consistency requirements. First, note that when the seller, given his strategy and beliefs, observe a zero probability action from the buyer, the definition of PBE places no restrictions on how beliefs must be updated. For the sake of consistency, we fix beliefs to be those that always go to a point mass at the end of the support (which is \( 1 \) for the \( U[0, 1] \) distribution) after observing a 0 probability event. Second, we require that when the price posted in a round is smaller than the smallest point in the support of the belief (the buyer can infer the seller’s belief from his own strategy), the buyer always accepts.

Benchmark. The benchmark is, as before, the expectation of Myerson’s full commitment revenue, namely, \( \frac{1}{4} + (1 - \delta) \frac{1}{4} + (1 - \delta)^2 \frac{1}{4} + \cdots = \frac{1}{3\delta} \).

5.1 Zero commitment with time discounted infinite horizon

The “no-discrimination equilibrium” discussed in Hart and Tirole [1988] for discrete distributions with two points, is easily seen to be an equilibrium for all other distributions too, including the \( U[0, 1] \) distribution. We quickly sketch the proof here.

| STRATEGY 1: Seller’s strategy in the zero commitment infinite horizon game with discount \( 1 - \delta \), where \( 0 < \delta \leq \frac{1}{2} \) |
|---------------------------------------------------------------|
| while 1 do |
| if Buyer had ever accepted a price greater than 0 in the past then |
| \[ \text{Set current round price of 1} \] |
| else if Buyer had ever rejected a price of 0 in the past then |
| \[ \text{Set a current round price of 1} \] |
| else |
| \[ \text{Set current round price of 0} \] |
| \[ \text{Theorem 5} \] For the \( U[0, 1] \) distribution, for all \( \delta \in (0, 0.5] \), Strategies 1 and 2 form a threshold PBE for the zero commitment infinite horizon game with discount \( 1 - \delta \) and obtains a 0 revenue for the seller. |
**STRATEGY 2**: Buyer’s strategy in the zero commitment infinite horizon game with discount $1 - \delta$, where $0 < \delta \leq \frac{1}{2}$

```
while 1 do
  if Current round price is 0 then
    Accept
  else if Current round price greater than 0 then
    if Buyer had ever accepted a price greater than 0 in the past, or rejected a price of 0 in the past then
      Accept if $v \geq$ current round price
    else
      Reject
```

Proof: The seller is clearly best responding to the buyer’s strategy: since the buyer never accepts a price larger than 0, unless he has accepted in the past a price greater than 0 or rejected a price of 0, the seller places a price of 0. For the buyer, it is not a best response to accept any price $p > 0$ because doing so will give him a utility of $v - p$, whereas rejecting will give him a utility of $(1 - \delta)\frac{v}{\delta}$. When $\delta \leq \frac{1}{2}$, it follows that $(1 - \delta)\frac{v}{\delta} > v - p$ for any $p > 0$.  

5.2 Partial commitment with time discounted infinite horizon

We refer the reader to appendix E for a detailed proof of the main result of this section, namely, Theorem 6. The proof in this section is restricted to just the equilibrium path and arriving at the optimal revenue. The description here is brief, and doesn’t require much understanding of strategies 7 and 8. Some statements are not fully justified here to make the proof easily readable. All statements are fully justified in appendix E.

**Theorem 6** For the $U[0, 1]$ distribution, for all $\delta \in (0, 1]$, Strategies 7 and 8 form a threshold PBE for the partial commitment infinite horizon game with discount $1 - \delta$, and obtain a $\frac{4}{3 + 2\sqrt{2}}$ fraction of the Myerson optimal revenue benchmark of $\frac{1}{4\delta}$. Furthermore, this is the unique PBE where the buyer’s strategy is threshold, seller’s strategy is pure, Markovian and scale-invariant, seller’s beliefs are Markovian, scale-invariant and consistency requirements are satisfied.

Proof: Let $R$ denote the expected revenue and $u$ denote the utility of buyer with value 1. As $U[0, t]$ is a scaled version of $U[0, 1]$, the expected revenue for $U[0, t]$ is just $tR$ and the utility of the buyer with value $t$ in the $U[0, t]$ distribution is $tu$.

Let $p$ denote the price on the first day and let $t$ denote the threshold on the first day. Invoking indifference of the buyer with value $t$ in the first round we get

$$ (1 - \delta)ut = (t - p)/\delta $$

The RHS of (7) is the utility of the buyer with value $t$ upon acceptance in the first round (and hence accepts all future rounds because of no-price-increase guarantee). The LHS is the utility of the buyer with value $t$ in the infinite horizon game for the $U[0, t]$ distribution. Because of scaling, this is just $t$ times the utility of the buyer with value 1 for the $U[0, 1]$ distribution, i.e., $t \cdot u$. The factor of $1 - \delta$ is the discounting factor for one round.

The buyer with value 1 accepts in the first round and gets a utility of $1 - p$ in every round, appropriately discounted. This gives

$$ u = (1 - p)/\delta $$

(8)

The expected revenue $R$ can be written as:

$$ R = (1 - \delta) \cdot Rt^2 + (1 - t) \cdot p/\delta. $$

(9)

The first term of (9) is the expected revenue contribution from rejection in the first round: $t$ is the probability of rejection and the expected revenue upon rejection is $Rt$ (using scaling). The second term of (9) is the expected
revenue contribution from acceptance in the first round: \( 1 - t \) is the probability of acceptance in the first round, and the expected revenue upon acceptance is \( p \) in every round appropriately discounted.

Combining equations (7), (8) and (9) gives

\[
R = \frac{t(1 - t)}{(1 - (1 - \delta)t)(1 - (1 - \delta)t^2)} \tag{10}
\]

To compute the revenue approximation, we take the ratio of the expression for \( R \) in equation (10) and the Myerson optimal revenue benchmark of \( \frac{1}{4\delta} \). As \( \delta \to 0 \), this ratio approaches from above \( \frac{4}{3 + 2\sqrt{2}} \), which is approximately 0.69. The optimal threshold \( t \) as \( \delta \to 0 \) approaches \( 1 - \frac{4}{\sqrt{2}} \), and the optimal price \( p \) approaches \( \frac{\sqrt{2}}{\sqrt{2}+1} \).

6 Directions for further research

The basic setup considered in this paper suggests several directions for further research that we find intriguing.

- Suppose there are two buyers whose values are initially drawn independently from \( F \) (and then remain unchanged). A single seller produces two identical items and posts a price for them every day; what are the corresponding perfect Bayesian equilibrium (PBE) prices and revenues, for finite and infinite horizons?

- Under the same assumption on the buyers, suppose the seller produces a single item every day and sells it by auction to one of the two buyers. What are the PBE reserve prices and revenues in this case?

- What if several sellers are competing to attract buyers?

- Returning to the single buyer and seller, suppose that in addition to the uncertainty about the value of the item for the buyer, there is uncertainty about the cost of production. This cost is known to the seller but not to the buyer, who is only informed that it is drawn from a known distribution \( G \), and then remains fixed. How does this affect the revenue and utility in PBE?

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A Related work

The particular model of repeated sales we study has been investigated in the economics literature, under the name behavior based price discrimination (BBPD) [Fudenberg and Villas-Boas, 2006]. The motivation there is that firms can offer personalized prices to consumers based on their past consumption pattern. Such consumption patterns could be collected in various ways, such as when the consumers use loyalty cards, or in an online world where the consumer identifies himself by logging in, or by the use of technology such as cookies. Fudenberg and Villas-Boas [2006] give several other markets where BBPD is observed, such as magazine subscriptions and labor markets. BBPD is also prevalent in government and corporate procurement, from raw materials to IT infrastructure.

The most closely related work to ours is that of Hart and Tirole [1988] and Schmidt [1993], and we have already discussed how our work relates to theirs. Subsequently, many extensions of their models have been studied, such as when consumer preferences vary over time, a monopolist seller selling multiple goods, multile sellers selling the same good who try to poach customers from each other, sellers with multiple versions of the same product, and so on. We refer the reader to Fudenberg and Villas-Boas [2006] for a survey of these results. Another closely related paper is that of Conitzer et al. [2012] who consider a repeated sale game where the buyers have the option of anonymizing themselves at a cost and analyze the effect of varying this cost on the welfare of the buyers. Kanoria and Nazerzadeh [2014] consider a repeated sale setting where in each round two item types are auctioned, to a finite number of agents, with an information structure where agents know their own valuation but not the other agents’ valuations. The seller is forced to run a second price auction, but can change the reserve dynamically. Kanoria and Nazerzadeh [2014] show the optimality of static reserve under some assumptions and design an optimal dynamic mechanism when the assumptions fail.

We also discuss a few well studied themes that are similar in flavor to our model, yet different in significant ways.

Durable goods monopoly. The literature on durable goods monopoly, starting with Coase [1972], considers a durable good and a monopolist seller who cannot commit to not re-selling the good if he has a remaining supply after an initial offering at a certain price. Coase [1972] conjectured that this inability to commit will lead to the monopolist losing all profits. The main difference with our work is that in this model, the goods are durable whereas we assume that both the supply and demand are renewed afresh in each round. The similarity is that both study the power of commitment, model the game as a sequential game, and look for a subgame perfect equilibrium. There has been extensive work in this model, for instance Gul et al. [1986] considers a monopolist facing a continuum of non-atomic buyers and show several properties of the equilibrium including a verification of Coase’s conjecture for a special case. Another well known fact about such settings is the “ratchet effect”, that the revelation principle fails to hold [Freixas et al., 1985, Laffont and Tirole, 1988]. Most of the literature assumes that the seller is restricted to posting a price, which was justified by Skreta [2006, 2013] who showed (respectively for a single buyer and many buyers) that posting prices is optimal among all mechanisms.

Bargaining. An alternative to the axiomatic approach to Bargaining [Nash, 1950] is a strategic approach where the agents engage in repeated offers until an offer is accepted or the time runs out (in case of a finite horizon). A pioneering model in this spirit was by Rubinstein [1982], with two players who either have a fixed bargaining cost per period or a fixed discount factor, and outcomes correspond to subgame perfect equilibria in the complete information game. This is very similar to the durable goods model, when the two players are a buyer and a seller as in Fudenberg and Tirole [1983], except that both buyer’s and seller’s values are private. They consider a 2-round incomplete information setting where the buyer/seller values are each drawn from two-point distributions. We refer the reader to the survey by Ausubel et al. [2002] for other papers in this vein.

Repeated games. There is a vast literature on repeated games, which could be classified as complete vs. incomplete information settings. In the complete information case, the main difficulty is understanding the tradeoff between the value in the current round and the value in the future rounds, best exemplified by the stochastic games of Shapley [1953]. The famous Folk theorem and its applications to games such as the iterated prisoner’s dilemma [Friedman, 1971] are well studied. In repeated games of incomplete information, the main object of study is how players’ actions reveal information and the strategic aspects of this [Aumann and Maschler, 1995]. Also studied are aspects of learning...
in repeated games such as Blackwell approachability [Blackwell, 1956]. Our model is an instance of a repeated game with incomplete information for the seller, and captures both the tradeoff between current and future values and the information leakage aspect. However, the results in this line of research are usually in a very general setting so their conclusions are not quite interesting for our case. Finally, the power of commitment is a common theme in repeated games. See Fudenberg and Tirole [1991] for examples and applications of the power of commitment.

**Learning.** The aforementioned Blackwell approachability [Blackwell, 1956] and the convergence of no-regret learning algorithms to the set of correlated equilibria Foster and Vohra [1997] are examples of learning in repeated games. There is a potential learning aspect in our model, but we show that any such attempt to learn by the seller is negated by the strategic behavior of the buyer. Amin et al. [2013] consider a model very similar to ours, with a single buyer and a single seller with renewing demand and supply, but in their model the buyer’s value for the item is repeatedly drawn (i.i.d.) from a given distribution. Their motivation is to capture the repeated interaction between buyers and sellers of display advertising over an ad-exchange. Our model might be closer to their motivating example, since it is more likely that the advertiser value stays the same over time. They extend the notion of regret in online learning to incorporate strategic behaviour of the buyer and give a low regret algorithm, based on techniques for the multi-armed bandit problem.

### B Lack of commitment can never help even in very general settings

We formally define a very general model of mechanism design here and prove Proposition 1.

**Definition 4 General model of mechanism design.** An instance of a mechanism design problem is given by a set of \( m \) agents \( \mathcal{A} \), a set of outcomes \( \mathcal{O} \) and a type space \( \mathcal{T} \) for each agent, where each type \( \theta \) is a function from \( \mathcal{O} \) to \( \mathbb{R} \) (which is the utility of the agent with type \( \theta \) for the given outcome). Each agent \( a \) has a type \( \theta_a \), which is her private information. In the Bayesian setting, additionally, we are given a joint probability distribution \( F \) over the types of all the agents, \( \mathcal{T}^m \), from which the type vector \( \vec{\theta} \) of the agents is sampled. A mechanism is a multi-party protocol in which the agents participate, as a result of which there is an outcome \( o(\vec{\theta}) \). The mechanism designer’s goal is to maximize his objective \( \mathbb{E}_{\vec{\theta} \sim F}[\text{OBJ}(\vec{\theta})] \), where \( \text{OBJ}(\cdot) \) is a function from \( \mathcal{O} \) to \( \mathbb{R} \).

Note that the above definition includes, in addition to the usual cases of welfare/revenue maximization, constraints such as budget constraints and scenarios such as mechanism design without money, non-linear objectives such as makespan minimization in scheduling and max-min fairness. In order to extend this model to the repeated setting, we need to additionally specify how many times the setting is repeated. We allow the number of repetitions to be a random variable.

**Definition 5 General model of repeated mechanism design.** An instance of a repeated mechanism design problem is given by an instance of the mechanism design problem, the probabilities \( \{q_t\}_{t \in \mathbb{N}} \) with which the \( t \)-th repetition is realized, the fractions \( \{d_t\}_{t \in \mathbb{N}} \) which with which the mechanism designer and the agents discount their \( t \)-th round utilities. We require that \( \sum_{t=1}^{\infty} q_t d_t < \infty \) (i.e., the process either doesn’t continue infinitely, or if it does, agents discount their future utilities enough to avoid infinite utilities). The buyer types remain the same in every repetition and there are no inter-round constraints except this. The repeated mechanism is now a protocol, which in sequence produces an outcome \( q_t \) for each time \( t \) till the process stops (the mechanism designer and the agents know the probabilities \( q_t \) that determine this stopping time, but get to know the precise stopping time only when it happens.) The utility of agent \( a \) is the sum \( \mathbb{E}_{\vec{\theta} \sim F}[\sum_{t=1}^{\infty} q_t d_t \theta_a(q_t(\vec{\theta}))] \). The objective of the mechanism designer is \( \mathbb{E}_{\vec{\theta} \sim F}[\sum_{t=1}^{\infty} q_t d_t \text{OBJ}(q_t(\vec{\theta}))] \).

It is easy to see that the game defined in Definition 1 is a special case of the above model: \( \mathcal{A} = \{1\}, \mathcal{O} = \{\text{accept, reject}\} \times \mathbb{R} \), types of the form \( \theta((\text{accept}, p)) = v - p \) for some \( v \in \mathbb{R} \) and \( \theta(o) = 0 \) otherwise, and objective \( \text{OBJ}((\text{accept}, p)) = p \) and \( \text{OBJ}(o) = 0 \) otherwise. For the finite horizon model \( q_t = 1 \) for \( t \in \{1, 2, \ldots, n\} \), and \( q_t = 0 \) for others.

\( ^{5} \) The mechanism could be randomized and its outcome on day \( t \) could depend (apart from \( \vec{\theta} \)) on the realization of the random coin tosses on days 1 to \( t - 1 \). To avoid excessively cumbersome notation we avoid spelling this out formally. But our argument and results directly extend to these settings too.
for \( t > n \), with \( d_t = 1 \) for all \( t \). The time discounted infinite horizon game can be described by setting \( q_t = 1 \) for all \( t \in \mathbb{N} \) and \( d_t = (1 - \delta)^{t-1} \) for all \( t \in \mathbb{N} \). Note that \( \sum_{t=1}^{\infty} q_t d_t = 1/\delta < \infty \).

We restate Proposition 1 formally here and prove it.

**Proposition 1** In the general model of repeated mechanism design, the optimal objective value obtained without any commitment is never larger than the optimal objective value obtained when commitment is possible. Formally let \( \text{OBJ}^* \) be the optimal expected objective value for the single round mechanism design problem. Then the optimal expected objective value attainable in any PBE in the repeated mechanism design problem is at most \( E_{\theta_a \sim F}[\sum_{t=1}^{\infty} q_t d_t \text{OBJ}^*] \).

**Proof:** Suppose on the contrary that there was a PBE with expected objective value \( E_{\theta_a \sim F}[\sum_{t=1}^{\infty} q_t d_t \text{OBJ}(\theta_t)] > E_{\theta_a \sim F}[\sum_{t=1}^{\infty} q_t d_t \text{OBJ}^*] \). Consider the following mechanism for the single round game. All agents submit their types to the mechanism designer. The designer chooses day \( t \) with probability \( \frac{q_t d_t}{\sum_i q_i d_i} \) and runs the said PBE till day \( t \), and the outcome on day \( t \) will be the outcome realized for the single round game. Agent \( a \) with type \( \theta_a \), upon truthful reporting of his type, will get an expected utility of \( E_{\theta_a \sim F}[\sum_{t=1}^{\infty} \sum_i q_i d_i \theta_a(\theta_t)] \) which is just a scaled version of his utility in the PBE. Thus the agent has no incentive in the proposed mechanism because that would mean that the said PBE was not really a PBE. For this mechanism, the expected objective of the designer is \( E_{\theta_a \sim F}[\sum_{t=1}^{\infty} q_t d_t \text{OBJ}(\theta_t)] \).

By our assumption, the former quantity is at least \( E_{\theta_a \sim F}[\sum_{t=1}^{\infty} \sum_i q_i d_i \text{OBJ}^*] > \text{OBJ}^* \). This is a contradiction because in the single round game, it is not possible to get an expected objective value higher than \( \text{OBJ}^* \).  

C Two Rounds Game

**Full solution to the 2 days \( U[0,1] \) fishmonger’s problem.** We present here the full solution to the 2 days \( U[0,1] \) fishmonger’s problem.

**Strategy 3:** Seller’s strategy in the 2 rounds \( U[0,1] \) game

- **Round-1 pricing:** Set \( p_1 = 0.3 \);
- **Round-2 pricing:**
  - If \( p_1 \leq 0.5 \) then
    - If Buyer rejects in round 1 then
      - Set second round price of \( p_2 = p_1 \)
    - else if Buyer accepts in round 1 then
      - Set second round price of \( \max(2p_1, 0.5) \)
  - else if \( p_1 > 0.5 \) then
    - If Buyer rejects in round 1 then
      - Set second round price of \( p_2 = 0.5 \)
    - else if Buyer accepts in round 1 then
      - Set second round price of 1

**Two rounds game for arbitrary distributions.** We begin with some notation and two quick definitions. Let \( F_{[a,b]} \) denote the distribution on \( v \) conditioned on the fact that \( a \leq v \leq b \) (and thus \( F = F_{[b,a]} \)). Let \( p^*_a \) denote an arbitrary element of \( \arg\max_p p(1 - F_{[a,b]}(p)) \) i.e., the set of all single-round revenue maximizing prices or the so called monopoly prices for \( F_{[a,b]} \). Let \( p^* = p^*_b \). Whenever the monopoly price is not unique \( p^*_{[a,b]} \) will denote an arbitrary monopoly price unless specified otherwise.

**Revenue Curve.** The revenue curve \( R_{[a,b]}(p) = p(1 - F_{[a,b]}(p)) \) at \( p \) gives the expected revenue in a single round game obtained by offering a price \( p \) to a buyer whose value is drawn from \( F_{[a,b]} \). Let \( R(\cdot) = R_{[t,b]}(\cdot) \) denote the revenue curve for the distribution \( F \). The revenue curve for a regular distribution is concave.
**STRATEGY 4:** Buyer’s strategy in the 2 rounds U[0,1] game

| Round-1 strategy: |
|-------------------|
| if \(p_1 \leq 0.5\) then |
| | if \(v \geq 2p_1\) then |
| | accept |
| | else |
| | reject |
| else if \(p_1 > 0.5\) then |
| | reject |

| Round-2 strategy: |
|-------------------|
| if \(v \geq p_2\) then |
| | accept |
| else |
| | reject |

We skip here the proof of the fact that the strategies in Strategy 5 and 6 form a threshold PBE. We prove here a property that any PBE in two rounds game for any atomless bounded support distribution must satisfy.

In the following lemma, \(p_1\) is the price in the first round, \(p_{20}\) and \(p_{21}\) are the prices in second round, upon buyer’s rejection and acceptance respectively in the first round, given that the first round price is \(p_1\).
Lemma 7  For every atomless bounded support distribution, every PBE of a two rounds zero commitment game will satisfy the following: the PBE price $p_1$ in the first round will be such that $t(p_1) < h$, where $t(p_1)$ satisfies $p_1^{e} = p_1$.

Proof: First note that $t(p_1)$ is unique. If not, there will be two thresholds $t(p_1) < t'(p_1)$ such that $p_1^{e} = p_1$. This is because the virtual values for the distribution $F_{[a,t]}$ and $F_{[a,t']}$, namely $\phi_{F_{[a,t]}}$ and $\phi_{F_{[a,t']}}$, both become zero at $p_1$. This is not possible because for any $x \leq t$ we have $F_{t}(x) = F_{t}(x).\alpha$ for some $\alpha < 1$. This means, $\phi_{F_{[a,t]}}(x) < \phi_{F_{[a,t']}}(x)$ for all $x \leq t$. Therefore both the virtual value functions cannot become zero at the same point.

Now, note that in the two rounds game, if $t(p_1) < h$, the revenue is $R(p_{20}) + R(p_{21})$. This is because the buyer never accept once when his value exceeds $p_{20}$ and once more when his value exceeds $p_{21}$. If $t(p_1) < p^*$, then the revenue is strictly larger than $R(p^*)$. This is because $p_{21} = p_{21}^{e} = p^*$ when $t(p_1) < p^*$, and $p_{20} = p_{20}^{e}$ which yields a revenue of $R(p_2) + R(p^*) > R(p^*)$. On the other hand, when $t(p_1) \geq h$, this means that all buyer types rejected in the first round, and thus the revenue is exactly $R(p^*)$, all of which is achieved in the second round. Since the seller best responds to maximize revenue, this cannot happen in an equilibrium.

D  Finite Horizon $n$ Rounds Game

Theorem 2  For every atomless bounded support distribution $F$ of buyer’s value, the following are true for a $n$ rounds ($n > 2$) repeated sales game.

1. An $n$ rounds pure strategy threshold PBE exists precisely for those distributions for which a 2 rounds threshold PBE exists with the lowest possible first round price of $\ell$.

2. Whenever an $n$ rounds pure strategy threshold PBE exists, it is unique: the price in the first $n-1$ rounds is the lowest possible, namely, $\ell$. The price in the last round is the monopoly price for the distribution $F$.

Proof: Consider the three rounds case first. Let $p_1$ be the first round price. Let $t = t(p_1)$ be the PBE threshold in the first round. Note that a PBE requires that given any history, the strategies for the continuation game must be mutually best responding. Consider one such history where the first round price is $p_1 > \ell$.

Acceptance everywhere. Clearly, the equilibrium at a price $p_1 > \ell$ cannot be that all buyer types accepted, as that will lead to negative utility for some buyer types — note that after acceptance, the price is anyway at least 0, and thus no buyer types can make up for the negative utility accumulated.

Non-trivial threshold. We now show that the equilibrium cannot be at a non-trivial threshold ($\ell < t < h$) by showing that the threshold buyer is never indifferent between accepting and rejecting. Clearly the threshold cannot be smaller than $p_1$, as in that case the threshold buyer makes negative utility. If the threshold is equal to $p_1$, the threshold buyer makes 0 utility upon acceptance, where as by rejecting he can make a non-negative utility as he will be the largest in the support of the beliefs after rejection, namely $F_{[a,t]}$. The only remaining case for a non-trivial threshold is that $t$ is strictly larger than $p_1$, but smaller than $h$. In this case, the threshold buyer upon accepting the price $p_1$ in the first round cannot get any further utility in the future rounds, as the prices are at least the threshold in the future. Thus his total utility is $t - p_1$ upon accepting in the first round. We show that by rejecting in the first round, the threshold buyer could have obtained strictly larger utility which is a contradiction. Let $p_{20}$ be the price in the second round on rejection, and let $p_{300}$ and $p_{301}$ denote the price in the third round upon ($reject$, $reject$) and ($reject$, $accept$) respectively in the first two rounds. When a buyer with value $t$ rejects in the first round, and accepts in the second and third rounds, he gets a utility of $(t - p_{20}) + (t - p_{301})$. The two claims below show that $p_{20} \leq p_1$ and $p_{301} < t$. Therefore the sum $(t - p_{20}) + (t - p_{301})$ is strictly larger than $t - p_1$ which is the utility upon acceptance.

Claim 1: $p_{20} \leq p_1$. On the contrary suppose that $p_{20} > p_1$. Consider a buyer value with $v$ s.t. $p_1 < v < p_{20}$. Such a buyer gets zero utility upon rejection in the first round because all prices after rejection are strictly larger than his value $v$ (because the second round price $p_{20} > v$ by our choice of $v$), and a threshold PBE for the remaining two rounds game means that $p_{20} = p_{300} \leq p_{301}$. Where as upon acceptance, he would have a gotten a strictly positive utility of $v - p_1$. This says that $v$ should have been at least the acceptance threshold $t$, whereas the truth is $v < t$ (because $v < p_{20}$ by our choice of $v$), and $p_{20} \leq t$ because the belief after first round is $F_{[a,t]}$.
Claim 2: $p_{301} < t$. We show that all prices after rejection in first round, namely, $p_{20}, p_{300}, p_{301}$ are strictly smaller than $t$. Even if the largest among these prices, namely $p_{301}$, was equal to $t$, that would not be a PBE. To see this, consider the threshold $t'$ used by the buyer for the distribution $F_{[t,t]}$. By Lemma 7 such a threshold $t'$ is strictly smaller than $t$. On the other hand, $p_{301}$ should simply be the monopoly price for the distribution $F_{[t,t]}$. If $t' < t$, this monopoly price cannot be $t$ because that yields a 0 revenue which is not optimal.

Rejection everywhere. We now analyze if all buyer types rejecting any price strictly larger than $\ell$ can be a part of a PBE. Clearly, the only way for every price $p_1 > \ell$ to be rejected in the first round is to have $p_{20} = \ell$. To see this, assume that $p_{20} > \ell$. Consider a first round price $p_1$ and a buyer value $v$ s.t. $\ell < p_1 < v < p_{20}$. Such a buyer gets a strictly positive utility upon buying in round 1, but gets zero utility in the remaining two rounds because $v < p_{20}$, and the other prices are only larger ($p_{20} = p_{300} \leq p_{301}$). Thus, it is not a best response for this buyer value to reject this price of $p_1$. On the other hand suppose $p_{20} = \ell$, then the buyer’s best response is indeed to reject any first round price of $p_1 > \ell$. This shows that exactly for those distributions for which the two rounds game has a pure strategy threshold PBE with a first round price of $p_1 = \ell$ does there exist a pure strategy threshold PBE for the 3 rounds game.

Since a $n$ rounds threshold PBE exists only when a 3 rounds threshold PBE exists, the necessity is obvious. The sufficiency is also clear from induction. If the prices for rounds 2 to $n$ are $\ell$, and buyers rejecting anything but a first round price of $\ell$, Proof.

E Infinite Horizon

In this section, we give a detailed proof of Theorem 6. We first state the strategies of the seller and the buyer in Strategies 7 and 8. Although technically the seller’s strategy and his beliefs are two separate objects, we present them together in Strategy 7. We represent the seller’s current belief by $U[\mu_{\text{begin}}, \mu_{\text{end}}]$, and thus the seller simply keeps track of these two numbers $\mu_{\text{begin}}$ and $\mu_{\text{end}}$ to update beliefs (note that starting from $U[0,1]$, if the buyer follows threshold strategies, the only possible beliefs are of the form $U[a,b]$).

In the proof we first show that these strategies together with seller’s beliefs constitute a PBE, i.e., we argue mutual best response. Then we argue why this PBE is unique when the buyer’s strategies are threshold, seller follows pure, Markovian and scale-invariant strategies and beliefs, and consistency requirements are satisfied.

Theorem 6 For the $U[0,1]$ distribution, for all $\delta \in (0,1]$, Strategies 7 and 8 form a threshold PBE for the partial commitment infinite horizon game with discount $1 - \delta$, and obtain a $\frac{\delta}{3+2\sqrt{2}}$ fraction of the Myerson optimal revenue benchmark of $\frac{1}{33}$. Furthermore, this is the unique PBE where the buyer’s strategy is threshold, seller’s strategy is pure, Markovian and scale-invariant, seller’s beliefs are Markovian, scale-invariant and consistency requirements are satisfied.

Proof: First, we observe that Strategies 7 and 8 are complete, i.e., Strategy 7 gives the seller’s action after every possible history, Strategy 8 gives buyer’s action after every possible history and every possible initial value. Further, Strategy 7 also specifies the seller’s beliefs given every possible history. Note that both the seller’s and buyer’s strategies specify how to proceed from situations that will never be realized when the seller and the buyer follow their respective strategies. Yet, PBE stipulates that we specify how the seller and the buyer react from all possible histories, not just histories that will be realized from the prescribed strategies. We now verify that the seller and buyer are mutually best responding.

Buyer’s strategy is a best-response to seller’s strategy. We now argue that the buyer’s accept/reject decision is a best-response to seller’s pricing strategy.

1. First, if the buyer has accepted at least once in the past, namely, $h_2^k \neq R^k$,
   then the seller places the same price $p_{k+1}$ in all future rounds. Thus, buying in this round will not hurt the buyer in any way as long as his value exceeds the posted price of $p_{k+1}$.
STRATEGY 7: Seller’s strategy in the partial commitment infinite horizon game with discount $1 - \delta$

**Input:** The history $(h_1^k, h_2^k)$ of prices and buyer’s accept/reject decisions in the past $k$ rounds, where

$h_1^k = (p_1, p_2, \ldots, p_k) \in [0, 1]^k$, and, $h_2^k \in \{A, R\}^k$; the minimum accepted price $\text{min\_accepted\_price}[k - 1]$ in the past $k - 1$ rounds, and, the belief $\mu_{\text{begin}}[k - 1], \mu_{\text{end}}[k - 1]$ after $k - 1$ rounds

**Output:** The seller’s belief after $k$ rounds, and the price he posts in the $k + 1$-th round

Let $t = \max_{x \in [0, 1]} \frac{1}{(1-(1-\delta)\gamma)(1-\delta)2\gamma}, \ R = \frac{1}{1-(1-\delta)\gamma}$

**Belief Updates:** The seller arrives at the beliefs $\mu_{\text{begin}}[k]$ and $\mu_{\text{end}}[k]$ after $k$ rounds as follows;

if $\mu_{\text{begin}}[k - 1] == 1$, and, $\mu_{\text{end}}[k - 1] == 1$

$\mu_{\text{begin}}[k] = 1, \mu_{\text{end}}[k] = 1$

else if $h_2^k \neq R^{k-1}$ then

if $h_2^k == R$ then

$\mu_{\text{begin}}[k] = 1, \mu_{\text{end}}[k] = 1$

else if $h_2^k == A$ then

$\mu_{\text{begin}}[k] = \mu_{\text{begin}}[k-1], \mu_{\text{end}}[k] = \mu_{\text{end}}[k-1] \min_{\text{accepted\_price}}[k] = \min(p_k, \min_{\text{accepted\_price}}[k-1])$

else if $h_2^k == R^{k-1}$ then

$t(p_k) = \frac{p_k}{\gamma + (1-\gamma)p}$

if $h_2^k == R$ then

if $p_k == 0$ then

$\mu_{\text{begin}}[k] = 1, \mu_{\text{end}}[k] = 1$

else if $t(p_k) > \mu_{\text{end}}[k-1]$ then

$\mu_{\text{begin}}[k] = \mu_{\text{begin}}[k-1], \mu_{\text{end}}[k] = \mu_{\text{end}}[k-1]$

else

$\mu_{\text{begin}}[k] = \mu_{\text{begin}}[k-1], \mu_{\text{end}}[k] = t(p_k)$

else if $h_2^k == A$ then

if $t(p_k) > \mu_{\text{end}}[k-1]$ then

$\mu_{\text{begin}}[k] = 1, \mu_{\text{end}}[k] = 1 \min_{\text{accepted\_price}}[k] = \min(p_k, \min_{\text{accepted\_price}}[k-1])$

else

$\mu_{\text{begin}}[k] = t(p_k), \mu_{\text{end}}[k] = \mu_{\text{end}}[k-1]$

$\min_{\text{accepted\_price}}[k] = \min(p_k, \min_{\text{accepted\_price}}[k-1])$

Pricing at $k + 1$-th round: The seller now uses his beliefs to arrive at the prices for the $k + 1$-th round as follows;

if $h_2^k \neq R^k$ then

Set current round price of $p_{k+1} = \min_{\text{accepted\_price}}[k]$

else if $h_2^k == R^k$ then

if $\mu_{\text{begin}}[k] == 1, \mu_{\text{end}}[k] == 1$ then

Set current round price of $p_{k+1} = 1$

else

Set current round price of $p_{k+1} = \mu_{\text{end}}[k]$

2. Suppose the buyer has never accepted in the past, namely, $h_2^k = R^k$. The buyer’s response will be justified based on seller’s beliefs.

The seller updates his beliefs in a certain way, and the buyer’s strategy keeps track of this by exactly mimicking how the seller updates his beliefs. We now proceed to examine three cases here.

(a) If the buyer’s inferred beliefs is such that $\mu_{\text{begin}}[k] = \mu_{\text{end}}[k] = 1$, then the seller never updates beliefs in the future. Thus the buyer buys whenever his value exceeds the price posted.

(b) If the current round price $p_{k+1}$ is such that $t(p_{k+1}) \leq \mu_{\text{end}}[k]$: first the buyer doesn’t reject a price of 0 because that will cause the seller to post a price of 1 in the future rounds. For a price larger than 0, if the buyer rejects this round, the seller will charge him a next round price of $p(t(p_{k+1}))$. Upon acceptance in this round, the price will stay fixed at $p_{k+1}$ forever. Buying in this round is better than rejecting this round and buying next round exactly when $\frac{v - p_{k+1}}{\delta} \geq (1 - \delta)\frac{1 - p(t(p_{k+1}))}{\delta}$, and, this happens when $v \geq t(p_{k+1}) = \frac{p_{k+1}}{\gamma + (1-\gamma)p}$. I.e., the LHS and
STRATEGY 8: Buyer’s strategy in the partial commitment infinite horizon game with discount $1 - \delta$

Input: The history $(h^{k+1}_1, h^{k+1}_2)$ of prices and buyer’s accept/reject decisions in the past $k + 1$ and $k$ rounds respectively, where $h^{k+1}_1 = (p_1, p_2, \ldots, p_{k+1}) \in [0, 1]^{k+1}$, and, $h^{k}_2 \in \{A, R\}^k$, the belief $\mu_{\begin{array}{l} \text{begin} \end{array}}[k-1], \mu_{\begin{array}{l} \text{end} \end{array}}[k-1]$ after $k - 1$ rounds;

Output: The buyer’s accept/reject decision in the $k + 1$-th round

Let $t = \max\{v \in [0, 1] \mid (1 - (1 - \delta)^z)(1 - (1 - \delta)^r), p = \frac{\mu}{1 - (1 - \delta)^t} \}$. 

Belief Updates: The buyer derives the seller’s beliefs by following an identical procedure as outlined in Strategy 7. We don’t repeat it here;

Accept/reject decision at the $k + 1$-th round: Using the inferred beliefs, the buyer decides whether or not to accept in the $k + 1$-th round as follows:

if $h^{k}_2 \neq R^k$ then
  if $v \geq p_{k+1}$ then
    Accept
  else
    Reject

else if $h^{k}_2 == R^k$ then
  $t(p_{k+1}) = \frac{p_{k+1}}{s + (1 - \delta)p}$
  if $\mu_{\begin{array}{l} \text{begin} \end{array}}[k] == 1$, and, $\mu_{\begin{array}{l} \text{end} \end{array}}[k] == 1$ then
    if $v \geq p_{k+1}$ then
      Accept
    else
      Reject
  else if $t(p_{k+1}) \leq \mu_{\begin{array}{l} \text{end} \end{array}}[k]$ then
    if $v \geq t(p_{k+1})$ then
      Accept
    else
      Reject
  else if $t(p_{k+1}) > \mu_{\begin{array}{l} \text{end} \end{array}}[k]$ then
    if $v \geq \frac{p_{k+1} - (1 - \delta)\mu_{\begin{array}{l} \text{end} \end{array}}[k]}{\delta}$ then
      Accept
    else
      Reject

the RHS become equal at $v = t(p_{k+1})$, and the LHS exceeds the RHS for $v > t(p_{k+1})$. It requires a little more argument to show that buyer types with $v < t(p_{k+1})$ are not hurt by rejecting. Clearly, if $v < p_{k+1}$ acceptance leads to negative utility, so rejection is better. If $p_{k+1} \leq v < t(p_{k+1})$, one has to verify that the RHS accurately captures such a buyer’s utility: namely, can such a buyer even afford a price of $pt(p_{k+1})$? It is easy to show that $pt(p_{k+1}) \leq p_{k+1} \leq v$ whenever $p \leq 1$, which is true since we initialize $p$ to be smaller than 1. Hence such a buyer can indeed a price of $pt(p_{k+1})$. Along these lines, it is easy to verify that if buying this round is better than waiting for one more round, then it is also better than waiting for $r + 1$ more rounds for any $r \geq 0$, i.e., whenever $\frac{v - p_{k+1}}{\delta} \geq (1 - \delta)^r \frac{v - pt(p_{k+1})}{\delta}$, we have $\frac{v - p_{k+1}}{\delta} \geq (1 - \delta)^{r+1} \frac{v - pt(t(p_{k+1}))}{\delta}$ for all $r \geq 0$. The RHS in this inequality is obtained from observing that the price set by the seller’s strategy after rejecting in this and the next $r$ rounds is $pt(t(p_{k+1}))$: clearly the price set in the next round after rejection in this round is $pt(p_{k+1})$. If the buyer rejects in the next round, the seller updates $\mu_{\begin{array}{l} \text{end} \end{array}}$ to $t(p_{k+1})$ which is exactly $t \cdot t(p_{k+1})$. Thus after $r$ further rejections, the seller will update $\mu_{\begin{array}{l} \text{end} \end{array}}$ to $t(r \cdot t(p_{k+1}))$, and the price after that will be precisely $pt^r(t(p_{k+1}))$.

(c) If the current round price $p_{k+1}$ is such that it is strictly larger than the end of the support of belief, namely, $t(p_{k+1}) > \mu_{\begin{array}{l} \text{end} \end{array}}[k]$, the seller doesn’t update beliefs in the next round if the buyer rejects, where as if the buyer accepts, the seller updates beliefs to 1 and will always charge this round’s accepted price. Thus, if the buyer rejects in this round, the seller will charge him a next round price of $pt_{k+1}[k]$. Upon acceptance in this round, the price will stay fixed at $p_{k+1}$ forever. Buying in this round is better than rejecting this round and buying next round exactly when $\frac{v - p_{k+1}}{\delta} \geq (1 - \delta)^r \frac{v - pt_{k+1}[k]}{\delta}$, and, this happens when $v \geq \frac{p_{k+1} - (1 - \delta)\mu_{\begin{array}{l} \text{end} \end{array}}[k]}{\delta}$. Along
these lines, it is easy to verify that if buying this round is better than waiting for one more round, then it is also better than waiting for \( r + 1 \) more rounds for any \( r \geq 0 \), i.e., whenever \( \frac{v - p_{k+1}}{\delta} \geq (1 - \delta) \frac{v - pt_{\mu_{\text{end}}}[k]}{\delta} \) for all \( r \geq 0 \). The RHS in this inequality is obtained from observing that the price set by the seller’s strategy after rejecting in this and the next \( r \) rounds is \( pt_{\mu_{\text{end}}}[k] \); clearly the price set in the next round after rejection in this round is \( pt_{\mu_{\text{end}}}[k] \). If the buyer rejects in the next round, the seller updates \( \mu_{\text{end}} \) to \( t(pt_{\mu_{\text{end}}}[k]) \) which is exactly \( t_{\mu_{\text{end}}}[k] \). Thus after \( r \) further rejections, the seller will update \( \mu_{\text{end}}[k + 1 + r] \) to \( t^{r}t_{\mu_{\text{end}}}[k] \), and the price after that will be precisely \( pt^{r}t_{\mu_{\text{end}}}[k] \).

**Seller’s strategy is a best-response to buyer’s strategy.** To prove this we must show that the seller’s beliefs are consistent given buyer’s strategy, and, seller’s pricing strategy is optimal given his beliefs.

1. **Seller’s beliefs are consistent given buyer’s strategies.** For a given history, the seller arrives at his belief by updating them as described in the first half of Strategy 7. This part has three cases (one “if” and two “else if”’s). We argue consistent beliefs in each of those cases.

   (a) If the current belief is already at the end of support, i.e., \( \mu_{\text{begin}}[k - 1] = \mu_{\text{end}}[k - 1] = 1 \), then, the seller never needs to update belief. Even if the buyer plays a zero probability action, PBE doesn’t require any update of beliefs.

   (b) If in the past \( k - 1 \) rounds the buyer has purchased at least once, namely, \( h_{2}^{k-1} \neq R^{k-1} \), then the buyer’s strategy says that as long as his value is at least the price posted he will buy. Given that the the seller’s current belief is not abnormal (i.e., it is not \( \mu_{\text{begin}}[k - 1] = \mu_{\text{end}}[k - 1] = 1 \)), it follows that the first time the buyer accepted, his value was at least the price (because the buyer’s strategy 8 says that the threshold set is at least the price posted). Since any valid history will not raise the price beyond any accepted price (partial commitment prohibits this), the seller expects the buyer to accept in this round, and thus, will not update beliefs if the buyer accepts. But if the buyer rejects, this is a zero probability event and the seller updates beliefs to \( \mu_{\text{end}}[k] = \mu_{\text{begin}}[k] = 1 \).

   (c) If in the past \( k - 1 \) rounds the buyer never purchased, namely, \( h_{2}^{k-1} = R^{k-1} \), then we consider three simple cases:

   i. If the price \( p_{k} \) was 0, then the buyer’s strategy says that his threshold is 0 (note that \( t(p_{k}) = \frac{p_{k}}{\delta + (1 - \delta)p} \), which is 0 when \( p_{k} = 0 \)). Thus, if the buyer rejected a price of 0, then this is a 0 probability action, and the seller updates beliefs to \( \mu_{\text{begin}}[k] = \mu_{\text{end}}[k] = 1 \).

   ii. If the price \( p_{k} > 0 \) is such that the buyer’s threshold \( t(p_{k}) \) is at most the end of support of beliefs \( \mu_{\text{end}}[k - 1] \), then upon accept the seller updates his beliefs to \( U[t(p_{k})]; \mu_{\text{end}}[k - 1] \) and upon reject, the seller updates his beliefs to \( U[0, t(p_{k})] \) which is consistent with the buyer’s strategy.

   iii. If the price \( p_{k} \) is such that the buyer’s threshold \( t(p_{k}) \) is strictly larger than the end of support of beliefs \( \mu_{\text{end}}[k - 1] \), then the buyer’s strategy rejects this price for all values within the support of seller’s current beliefs. Correspondingly, if the buyer rejects, the seller doesn’t update beliefs, and if the buyer accepts, this is a zero probability action and the seller updates beliefs to \( \mu_{\text{begin}}[k] = \mu_{\text{end}}[k] = 1 \).

2. **Seller’s pricing strategy is optimal given his beliefs.** First, if there were a randomized best response, then there is also a deterministic best response for the seller. Thus it is enough to prove optimality among deterministic pricing strategies. We consider two cases here.

   (a) We first focus on an easier subcase, namely, when the buyer has bought at least once in history, and argue that the seller’s strategy of posting the minimum accepted price in history is indeed the best response. If the seller’s current belief is not abnormal (i.e., it is not \( \mu_{\text{begin}}[k] = \mu_{\text{end}}[k] = 1 \)), that means that the first time the buyer accepted, his value was at least the price (because the buyer’s strategy 8 says that the threshold set is at least the price posted). Thus the seller expects the buyer to accept in this round any price which is no larger than the price he accepted, and thus, has no reason to place a price smaller than the minimum accepted price. The

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A randomized pricing strategy still has to post a price in every round, but the strategy will not be a specification of price for every possible contingency, rather a specification of distribution over prices for every possible contingency.
seller cannot post a price in excess of the minimum accepted price because partial commitment prohibits this. On the other hand, if the seller’s current belief is \( \mu_{\text{begin}}[k] = \mu_{\text{end}}[k] = 1 \), then clearly the buyer should buy for any posted price. Thus again the seller should not post any lower price than the minimum accepted price, and nor can he post a price any larger as partial commitment prohibits this.

(b) We now consider the case where the buyer has never bought in history before. To argue best response here, we first argue that it is enough to focus on Markovian strategies, and then argue that it is enough to focus on scale-invariant strategies, and then argue that Strategy 7 optimizes among the class of Markovian scale-invariant strategies. We note here that for an arbitrary buyer strategy, a Markovian, scale-invariant strategy need not necessarily be the best-response. But for the buyer strategy given in Strategy 8, we argue that it is indeed the case.

i. Fixing the buyer’s strategy at Strategy 8, we argue that the best response for this strategy can always be found among the class of Markovian strategies. We crucially use the fact that the buyer’s strategy 8 given history \( (h^k_1, h^k_2) \) depends only on the belief computed using the history \( (h^k_1, h^k_2) \) and the price \( p_{k+1} \). Now to argue best response, suppose that there is a non-Markovian best-response to Strategy 8. That is, there are two histories that lead to the same beliefs, and the supposed best-response proceeds differently for these two histories. We compare the expected revenue generated from this point onwards in each of the two trajectories. If they are different, clearly, the proposed strategy is not a best-response: the seller can switch to using the better revenue yielding trajectory for both the histories, and this only increases revenue because of the above mentioned fact about buyer’s strategy not being history dependent. If on the other hand, for any two such histories that lead to the same beliefs, the expected revenue generated from that point onwards is the same for the trajectories corresponding to each history, then picking an arbitrary history and sticking to the strategy corresponding to that history for all histories leading to the belief under consideration, and doing this for every possible belief will lead to a Markovian best-response for the seller with the same revenue as the supposed non-Markovian best response.

ii. Fixing the buyer’s strategy at Strategy 8, we argue that the best response for this strategy among the class of Markovian strategies can always be found in the class of scale-invariant strategies. Given that the buyer has never bought in history, the beliefs can only be of two possible forms: they can be \( \mu_{\text{begin}} = \mu_{\text{end}} = 1 \), or \( U[0, a] \) for some \( a \). If the belief is former, clearly, given that the buyer’s strategy accepts at a value of 1, the optimal price is 1. If on the other hand, the belief was \( U[0, a] \), then all such distributions are scaled versions of one another. Scale invariance just means that the prices are also scaled versions of one another. Suppose there was a scale-variant best response to buyer’s strategy 8. That is, there are two consecutive rounds with beliefs \( U[0, a] \) and \( U[0, b] \), such that the proposed strategy posts prices that are not in the ratio of \( a/b \). Consider the earliest such pair of rounds. We compare the expected revenue generated by these two different trajectories. Suppose the revenues were not in the ratio of \( a/b \), then in the proposed strategy, the seller is not best responding: let \( R_a \) and \( R_b \) respectively be the revenue starting from these rounds and let \( p_a \) and \( p_b \) be the prices in these rounds. For short we will refer to these as round \( a \) and round \( b \). If \( R_a < \frac{b}{a} R_b \), then consider mimicking from round \( a \) the proposed strategy starting at round \( b \), but with all prices scaled by \( \frac{b}{a} \). This will yield a revenue of \( \frac{b}{a} R_b \) starting from round \( a \), which is larger than \( R_a \) — this is proved by induction on the number of rounds, beginning from round \( a \). Similarly if \( R_b < \frac{b}{a} R_a \), consider mimicking from round \( b \) the proposed strategy starting at round \( a \), but with all prices scaled by \( \frac{a}{b} \). This will yield a revenue of \( \frac{a}{b} R_a \) starting from round \( b \), which is larger than \( R_b \). Thus, it follows that \( R_a = R_b \). In this case, consider mimicking starting from round \( b \) onwards the strategy proposed starting at round \( a \), but with all prices scaled by \( \frac{b}{a} \). This will not change revenue because \( R_b = \frac{b}{a} R_a \), but now the number of scale invariant rounds counting from the first round has increased by 1. By induction we can continue this to all rounds.

iii. Finally, in the class of Markovian, scale-invariant strategies, we show that strategy 7 is indeed a best response. For this, the seller has to only choose the price \( p \) for \( U[0, 1] \). After picking \( p \), the corresponding threshold \( t = t(p) = \frac{p}{\delta + (1 - \delta)p} \) gets automatically decided given buyer’s strategy. Also, the price at distribution \( U[0, a] \) is just \( pa \) by scale invariance. Now, the revenue for this scale invariant strategy is \( R = (1 - \delta)Rt^2 + (1 - t)\frac{a}{2} \); this is because, the probability that the buyer rejects is \( t \), and given rejection, the revenue from the distribution \( U[0, t] \) is just \( Rt \) by scale invariance, accounting for the first
term of the revenue; for the second term, note that once accepted, the price never increases, and thus with a probability $1 - \delta$ the revenue is $\frac{z}{2}$. Substituting for $p$ from the equation $t = \frac{z}{\sqrt{2}(1-\delta)p}$, we get $R = \frac{\frac{z(1-\delta)}{2}}{(1-\frac{z}{\sqrt{2}+\sqrt{2}})^2}$. This explains how $t$ is the maximizer of $\frac{z(1-\delta)}{(1-\frac{z}{\sqrt{2}+\sqrt{2}})^2}$ for $z \in [0, 1]$.

Revenue argument. Maximizing the expression $R(t) = \frac{\frac{z(1-\delta)}{2}}{(1-\frac{z}{\sqrt{2}+\sqrt{2}})^2}$ in the range $t \in [0, 1]$, and taking its ratio to the benchmark of $\frac{1}{2\delta}$, we get that as $\delta \to 0$, the ratio approaches from above $\frac{\sqrt{2}}{3+2\sqrt{2}} \approx 0.69$. The optimal threshold $t$ as $\delta \to 0$ approaches $1 - \frac{\sqrt{2}}{\sqrt{2}+1}$, and the optimal price $p$ approaches $\frac{\sqrt{2}}{\sqrt{2}+1}$.

Uniqueness among the class of threshold strategies for buyer, Markovian, scale-invariant pure strategies and beliefs for seller and consistency requirements satisfied. We show that beginning from $U[0, 1]$, the equilibrium given by strategies 7 and 8 is the only possible PBE.

1. First, the equilibrium path starting from $U[0, 1]$, with no history before, cannot be of the form “accept everywhere”, i.e., one where all buyer types purchase in the first round. Suppose this was true, then the first round price cannot be larger than 0. If it were, then the buyer type 0 will incur negative utility by accepting a price of $p > 0$. Suppose the first round price $p = 0$, then the seller makes zero revenue, which we claim cannot happen in an equilibrium. To see this, consider how the game proceeds when the first round price was $\epsilon$. We claim that not all buyer types can reject this price. Suppose all buyer types rejected, then the seller updates beliefs to $U[0, 1]$ on rejection, and will post the said equilibrium price of 0 in this round. A buyer of type $v$, had he accepted in the first round will get a utility of $\frac{z-v}{2}$ since the price in future rounds will not increase beyond $\epsilon$. Suppose he rejected, type $v$ will get a utility of $(1 - \delta)\frac{z}{2}$. Whenever $v \geq \frac{z}{2}$, it follows that $\frac{z-v}{2} \geq (1 - \delta)\frac{z}{2}$. When $\epsilon < \delta$, this amounts to some buyer types not best responding by rejecting in the first round. But if such buyer types accept, the seller gets non-zero revenue as opposed to the 0 revenue he gets in the proposed equilibrium, hence showing that the seller was not best responding.

2. Second, the equilibrium path starting from $U[0, 1]$, with no history before, cannot be of the form “reject everywhere”, i.e., one where all buyer types reject. Suppose this was true, then the seller will not update beliefs on rejection and the belief will stay at $U[0, 1]$. Further, being a Markovian strategy requires that the seller post the same price in the second round as he posted in the first round. If the buyer accepted, then consistency requirements imply that the belief must be updated to $\mu_{\text{begin}} = \mu_{\text{end}} = 1$, but the price will not increase beyond the first round price due to partial commitment. Thus, if $p < 1$ is the price in the first round, a buyer who purchased in the first round will get a utility of $\frac{z-v}{2}$, and if he didn’t he can get a utility of at most $(1 - \delta)\frac{z}{2}$. For $1 > v > p$, both quantities are positive, and rejection results in strictly lesser utility. Suppose on the other hand $p = 1$, then the seller gets 0 revenue. This cannot happen in an equilibrium. To see this, consider what would have happened when the price in the first round is $\epsilon$. Suppose all buyer types rejected, then the seller updates beliefs to $U[0, 1]$ and will post the said equilibrium price of 1 in this round. A buyer of type $v$, had he accepted in the first round will get a utility of $\frac{z-v}{2}$ since the price in future rounds will not increase beyond $\epsilon$. Suppose he rejected, type $v$ will get a utility of at most $(1 - \delta)\frac{z}{2}$. Whenever $\epsilon \geq \frac{z}{2}$, it follows that $\frac{z-v}{2} \geq (1 - \delta)\frac{z}{2}$. When $\epsilon < \delta$, this amounts to some buyer types not best responding by rejecting in the first round. But if such buyer types accept, the seller gets non-zero revenue as opposed to the 0 revenue he gets in the proposed equilibrium, hence showing that the seller was not best responding.

3. Points (i) and (ii) above imply that the only possible threshold equilibrium paths starting from $U[0, 1]$ (with no history before) must have some threshold $t$, where $0 < t < 1$. Markovian and scale-invariant beliefs imply that the equilibrium path starting from $U[0, a]$ has a threshold of $\frac{t}{a}$ (as long as the buyer never accepted in the past). Similarly if the PBE price at $U[0, 1]$ is $\hat{p}$, then the PBE price starting from $U[0, a]$ is $p\hat{a}$ (as long as the buyer never accepted in the past). This also means that the buyer’s equilibrium strategy at belief $U[0, a]$ upon facing a price of $p\hat{a}$ (whenever he has never accepted in the past, and the price was at most the end of the support of belief) is just to accept when his value exceeds the threshold of $\frac{t}{a}$. In the exceptional situation when the history had the seller posting a price of 0 and the buyer rejecting it, consistency requires that the seller updates beliefs to $\mu_{\text{begin}} = \mu_{\text{end}} = 1$ and charge a price of 1, and in this situation the buyer has no choice but to accept whenever
his value exceeds posted price. I.e., at a high level what we have shown so far is that seller’s beliefs being scale invariant requires that buyer’s threshold is scale invariant.

When the buyer has accepted at least once in the past, the seller has no choice but to post a future price of the minimum accepted price. This is because the buyer’s threshold when he first accepted is always strictly larger than the price when he first accepted — except at a price of 0 — and hence there is no reason for the seller to lower the price below the minimum accepted price either. To see this fact about buyer’s threshold, note that if the buyer’s threshold was smaller than the posted price, the threshold buyer suffers negative utility as future prices are at least the threshold (if they weren’t, the seller can increase the price in one such round to a new price which is still smaller than the threshold. Consistency requirements mean that all buyer types will accept a price strictly smaller than the smallest point in the support of the belief. Thus the seller increases his revenue). If the threshold was equal to the posted price, the threshold buyer gets 0 utility upon acceptance, but gets a positive utility upon rejection (if the buyer got 0 revenue upon rejection too, it means that the seller’s revenue for $U[0, 1]$ is 0, which we have ruled out in the previous points). Given that the seller doesn’t reduce or increase prices beyond the minimum accepted price, the buyer has no choice but to accept when his value is at least the price posted. For this reason, when the buyer had accepted at least once in the past, and now rejects a price which is at most the minimum accepted price, the seller has to update beliefs to $\mu_{\text{begin}} = \mu_{\text{end}} = 1$ for consistency reasons.

Consistency requires that when the seller posted a price for which buyer’s threshold is larger than the largest point in the support of seller’s belief and the buyer accepted, the seller update his belief to $\mu_{\text{begin}} = \mu_{\text{end}} = 1$. Correspondingly, when faced with such a price, the buyer has to take this seller’s behavior into account when computing his threshold to buy, and there is only way to arrive at this threshold as done in Strategy 8, and this will always be at least the end of the support of his inferred belief.

Thus, the only thing left now is to show that $\hat{t}$ should be the maximizer of $\frac{z(1-z)}{(1-\hat{t})(1-(1-\hat{t})^2)}$. This follows from writing Revenue as a function of $\hat{t}$, which turns out to be $\frac{\hat{t}(1-\hat{t})}{(1-\hat{t})(1-(1-\hat{t})(1-\hat{t}))}$ and has a unique maximum.