ELECTROMAGNETIC WAVES IN THE DE SITTER SPACE

5-Dimensional wave equation for a massive particle of spin 1 in the background of de Sitter space-time model is solved in static coordinates. The spherical 5-dimensional vectors $A_a, a = 1, ..., 5$ of three types, $j, j+1, j-1$ are constructed. In massless case they give electromagnetic wave solutions, obeying the Lorentz condition. 5-form of equations in massless case is used to produce recipe to build electromagnetic wave solutions of the types $\Pi, E, M$; the first is trivial and can be removed by a gauge transformation. The recipe is specified to produce spherical $\Pi, E, M$ solutions in static coordinates.

**Keywords** Spin 1 field, de Sitter space, static coordinates, electric and magnetic waves, gauge symmetry

**33C05; 34B05**

This paper is based on the old one:

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Spherical waves of electric, magnetic and longitudinal types in de Sitter space.
Minsk (1986). 44 pages. Deposited in VINITI 16.12.86, 8641 - B86 (in Russian)
1 Introduction

Examining fundamental particle fields on the background of expanding universe, in particular de Sitter and anti de Sitter models, has a long history [1-30]. Special value of these geometries consists in their simplicity and high symmetry groups underlying them which makes us to believe in existence of exact analytical treatment for some fundamental problems of classical and quantum field theory in curved spaces. In particular, there exist special representations for fundamental wave equations, Dirac’s and Maxwell’s, which are explicitly invariant under respective symmetry groups $SO(4,1)$ and $SO(3,2)$ for these models. The interest to exact solutions of wave equations for particles with different spins in de Sitter space remains steady and inexhaustible [1-30].

In present paper the wave equation in 5-dimensional form for a massive particle of spin 1 in the background of de Sitter space-time model is solved in static coordinates $(t, r, \theta, \phi)$, covering the part of space-time to event horizon. The spherical 5-dimensional waves $A_a(t, r, \theta, \phi), a = 1, ..., 5$ of three types, $j, j + 1, j - 1$ are constructed. In massless case they gives electromagnetic wave solutions, obeying the Lorentz condition. Group-theoretical 5-dimensional form of equations in massless case is used to produce recipe to build electromagnetic wave solutions of the types $\Pi, E, M$; the first is trivial and can be removed by a gauge transformation. The recipe is applicable in arbitrary coordinates, in the paper it is specified in static coordinates of the de Sitter space.

2 On the 5-theory of a massive spin 1 particle in de Sitter space

It is known that a wave equation for spin 1 field on the background of de Sitter space-time can be presented in a form explicitly invariant under the group $SO(4,1)$. To specify some details of that approach, let start with covariant Proca equations

$$\nabla_\alpha \Psi_\beta - \nabla_\beta \Psi_\alpha = m \Psi_{\alpha\beta}, \quad \nabla^\beta \Psi_{\alpha\beta} = m \Psi_\alpha,$$

from whence it follows equation for the vector $\Psi_\alpha$

$$(\nabla_\beta \nabla^\beta + m^2) \Psi_\alpha - \nabla_\alpha (\nabla^\beta \Psi_\beta) - R_{\alpha\beta} \Psi_\beta = 0.$$  \(1\)

Because $\Psi_\alpha$ obeys the Lorentz condition $\nabla^\beta \Psi_\beta = 0$, eq. (1) gives

$$(\nabla_\beta \nabla^\beta + m^2) \Psi^\beta - R_{\alpha\beta} \Psi^\beta = 0.$$  \(2\)

In the massless case, instead of (1) we have

$$\nabla_\alpha \Psi_\beta - \nabla_\beta \Psi_\alpha = \Psi_{\alpha\beta}, \quad \nabla^\beta \Psi_{\alpha\beta} = 0;$$

and and the second order equation is

$$\nabla^\beta \nabla_\alpha \Psi^\alpha - \nabla_\alpha (\nabla^\beta \Psi_\beta) - R_{\alpha\beta} \Psi^\beta = 0.$$  \(3\)

Eq. (2) has a class of trivial (gauge) solutions

$$\Psi_\alpha = \nabla_\alpha f = \partial_\alpha f, \quad \Psi_{\alpha\beta} = 0,$$

where $f(x)$ is an arbitrary scalar function. Commonly, this fact is linked up to the gauge principle

$$\Psi_\alpha(x) \sim \Psi_\alpha(x) + \partial_\alpha f(x).$$
Trivial solution $\Psi_\alpha(x)$ obeys the Lorentz condition if $f(x)$ satisfies
\[ \nabla^\alpha \nabla_\alpha f(x) \equiv \Delta f(x) = 0. \]

Now let us specify the above equations in conformal coordinates in de Sitter space [33]
\[ dS^2 = \frac{1}{\Phi^2} \left[ (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \right], \]
\[ \Phi = (1 - x^2)/2, \quad x^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2; \]
\[ x^0 = ct/\rho, \ldots \]
Bellow it will be convenient to use coordinates
\[ x_\alpha = \eta_{\alpha\beta} x^\beta, \quad \eta_{\alpha\beta} = \text{diag}(+1, -1, -1, -1), \]
\[ g_{\alpha\beta} = \frac{1}{\Phi^2} \eta_{\alpha\beta}, \quad \partial_\alpha \Phi = -x_\alpha, \quad \partial^\alpha \equiv \eta^{\alpha\beta} \partial_\beta. \]
Christoffel symbols are
\[ \Gamma^\rho_{\alpha\beta} = \frac{1}{\Phi^2} \left( \delta^\rho_{\alpha} x_\beta - \delta^\rho_{\beta} x_\alpha - x^\rho \eta_{\alpha\beta} \right); \]
and Proca equations take the form
\[ \partial_\alpha \Psi_\beta - \partial_\beta \Psi_\alpha = m \Psi_{\alpha\beta}, \quad \Phi^2 \partial^\beta \Psi_{\alpha\beta} = m \Psi_\alpha. \]
In massless case we have
\[ \partial_\alpha \Psi_\beta - \partial_\beta \Psi_\alpha = \Psi_{\alpha\beta}, \quad \partial^\beta \Psi_{\alpha\beta} = 0. \]
The Lorentz condition in these coordinates looks
\[ \partial^\alpha \Psi_\alpha = -\frac{2}{\Phi^2} x^\alpha \Psi_\alpha. \]
Now, starting with $x^\alpha$, let us introduce five coordinates $\xi^a$
\[ \xi^\alpha = \frac{x^\alpha}{\Phi}, \quad (\alpha = 0, 1, 2, 3), \quad \xi^5 = \frac{1 + x^2}{1 - x^2}, \]
\[ x^\alpha = \frac{\xi^\alpha}{1 + \xi^5}, \quad \Phi = \frac{1}{1 + \xi^5}, \quad a = \alpha, 5; \]
they are characterized by
\[ \frac{\partial \xi^\alpha}{\partial x^\beta} = \frac{1}{\Phi^2} \left( \Phi \delta^\alpha_{\beta} + x^\alpha x_\beta \right), \quad \frac{\partial \xi^5}{\partial x^\beta} = x_\beta \frac{\Phi}{\Phi^2}, \]
\[ \frac{\partial x^\alpha}{\partial \xi^\beta} = \Phi \delta^\alpha_{\beta}, \quad \frac{\partial x^\alpha}{\partial \xi^5} = -\Phi x^\alpha; \]
\[ (\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 - (\xi^3)^2 - (\xi^5)^2 = -1. \]
\[ dS^2 = \frac{1}{\Phi^2} \eta_{ab} dx^a dx^b = \eta_{ab} d\xi^a d\xi^b = (d\xi^5)^2. \]

Therefore, de Sitter space can be identified with a sphere in 5-dimensional pseudo-Euclidean space, and thereby it has 10-parametric symmetry group $SO(4.1)$
\[ \xi^{a'} = S^a_b \xi^b, \quad (S^a_b) \in SO(4.1). \]
Instead of 4-vector $\Psi^\alpha(x)$ (below it will be designated as $a^\alpha(x)$) let us introduce 5-vector $A^a(\xi)$

$$A^a(\xi) = \frac{\partial \xi^a}{\partial x^\alpha} a^\alpha(x) ,$$

$$A^\alpha = \frac{1}{\Phi^2} (\Phi \delta^\alpha_\beta + x^\alpha x_\beta) a^\beta , \quad A^5 = \frac{1}{\Phi^2} x^\alpha a^\alpha ; \quad (8)$$

The vector $A^a(\xi)$ transforms as a 5-vector $\xi^a$ under the group $SO(4.1)$

$$A^a'(\xi') = \frac{\partial \xi'^a}{\partial x^\alpha} a^\alpha(x) = \left[ \frac{\partial}{\partial x^\alpha} (S^a_b \xi^b) \right] a^\alpha(x) = S^a_b A^b(\xi) . \quad (9)$$

Inverse relationship to (8) has the form

$$a^\alpha(x) = \frac{\partial x^\alpha}{\partial \xi^a} A^a = \Phi \left( A^\alpha - x^\alpha A^5 \right) .$$

Fife variables $A^a(\xi)$ are not independent – the following condition holds

$$\xi^a A_a = 0 . \quad (10)$$

A wave equation for the 5-vector $A^a(\xi)$ invariant under the group $SO(4.1)$ should be constructed with the help of the following operator

$$L_{ab} = \xi_a \frac{\partial}{\partial \xi^b} - \xi_b \frac{\partial}{\partial \xi^a} ,$$

and its possible form is

$$\frac{1}{2} L_{ab} L_{ac} + \kappa L_{ca} A^a + \sigma A_c = 0 , \quad (11)$$

where $\kappa$ and $\sigma$ are constants. It is readily verified that the Lorentz condition has the following 5-form

$$L_{ab} A^b = A_a ; \quad (12)$$

therefore eq. (11) looks

$$\left( \frac{1}{2} L_{ab} L_{ab} + (\kappa + \sigma) \right) A_c = 0 . \quad (13)$$

Bearing in mind (7), one finds

$$L_{\alpha \beta} = x_\alpha \frac{\partial}{\partial x^\beta} - x_\beta \frac{\partial}{\partial x^\alpha} , \quad L_{5a} = -\Phi \frac{\partial}{\partial x^\alpha} + x^\beta L_{a \beta} ;$$

and further

$$\frac{1}{2} L^{ab} L_{ab} = -\Phi^2 \left( \partial^\alpha \partial_\alpha + 2 \Phi x^\alpha \partial_\alpha \right) .$$

The later coincides with covariant d’Alamber operator in conformal coordinates

$$\frac{1}{2} L^{ab} L_{ab} = -\Delta .$$
Comparing (13) with (5), we find expression for \((\kappa + \sigma)\)

\[ \left( \frac{1}{2} L^{ab} \cdot L_{ab} + m^2 + 2 \right) A_c = 0 . \]  

(14)

Setting here \(m^2 = 0\), we get the wave equation for a massless field; also we should remember on eqs. (10) and (12). Let us derive 5-form for the above trivial solution

\[ \tilde{a}_\alpha = \frac{\partial}{\partial x^\alpha} f , \quad \Delta f = 0 , \]

transforming it to 5-form

\[ \tilde{A}_\alpha = (\Phi \frac{\partial}{\partial x^a} + x_\alpha x^\beta \frac{\partial}{\partial x^\beta}) f , \quad \tilde{A}_5 = -x_\alpha \frac{\partial}{\partial x^\alpha} f , \]

or shortly

\[ \tilde{A}_\alpha = (\frac{\partial}{\partial \xi^a} + \xi_a \xi^b \frac{\partial}{\partial \xi^b}) f \equiv m_a f . \]  

(15)

In the following we will use an identity \(\Delta = m^a m_a A^a = 0\).

3 Spherical waves in static coordinates, massive case

Equations for a vector particle will be solved in static coordinates in de Sitter space [33]:

\[ (\Delta + m^2 + 2) A^b = 0 , \quad \xi_b A^b = 0 , \quad L_{ab} A^b = A_a , \]

\[ dS^2 = (1 - r^2) dt^2 - \frac{dr^2}{1 - r^2} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) . \]  

(16)

Coordinates \(x^\alpha = (t, r, \theta, \phi)\) and \(\xi^a\) are referred by

\[ \xi^1 = r \sin \theta \cos \phi , \quad \xi^2 = r \sin \theta \sin \phi , \quad \xi^3 = r \cos \theta , \]

\[ \xi^0 = \sinh \sqrt{1 - r^2} , \quad \xi^5 = \cosh \sqrt{1 - r^2} , \]

\[ t = \arctg \frac{\xi^0}{\xi^5} , \quad r = \sqrt{(\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2} , \]

\[ \theta = \arctg \frac{\sqrt{(\xi^1)^2 + (\xi^2)^2}}{\xi^3} , \quad \phi = \arctg \frac{\xi^2}{\xi^1} . \]  

(17)

These coordinates \((t, r, \theta, \phi)\) cover the part of the full space [33]

\[ \xi^5 + \xi^0 \geq 0 , \quad \xi^5 - \xi^0 \geq 0 . \]

For any representation of the group \(SO(4.1)\) on the functions \(\Psi(\xi)\), we have relationship

\[ \xi' = S \xi , \quad \Psi'(\xi') = U \Psi(\xi) \quad \implies \quad \Psi'(\xi) = U \Psi(S^{-1} \xi) . \]

In the case \(U \equiv S\) and \(\Psi \equiv A\), the \((0 - 5)\)-rotation

\[ \xi'^{0} = \cosh \omega \xi^0 + \sinh \omega \xi^5 , \quad \xi'^{5} = \sinh \omega \xi^0 + \cosh \omega \xi^5 , \]
with an infinitesimal parameter $\delta \omega$ gives

$$A'(\xi) = (I + \delta \omega J_{50}) A(\xi), \quad J_{50} = L_{50} + \sigma_{50},$$

$$L_{50} = \xi_5 \frac{\partial}{\partial \xi^0} - \xi^0 \frac{\partial}{\partial \xi_5}, \quad \sigma_{50} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (18)$$

General expression for generators is (where $g_{nb} = \text{diag}(+1, -1, -1, -1)$)

$$(J_{mn})^a_b = L_{mn} \delta^a_b + (\sigma_{mn})^a_b, \quad (\sigma_{mn})^a_b = \delta^a_m g_{nb} - \delta^a_n g_{mb}. \quad (19)$$

Let us search solutions for eqs. (16) by diagonalizing three operators:

$$(-iJ_{50})^a_b A^b = \epsilon A^a, \quad (J^2)^a_b A^b = j(j+1)A^a, \quad (J_3)^a_b A^b = mA^a, \quad (20)$$

where

$$J_k = -\frac{i}{2} \epsilon_{ijk} (L_{ij} + \sigma_{ij}) = l_k + s_k, \quad s_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

First, the eigenfunction equation $(-iJ_{50}) A = \epsilon A$ is to be solved. With the use of identity $J_{50} = -\partial_t + \sigma_{50}$, we get

$$A \sim e^{-i\epsilon t}, \quad (A^0 + A^5) \sim e^{(-i\epsilon + 1)t}, \quad (A^0 - A^5) \sim e^{(-i\epsilon - 1)t}.$$

Bearing in mind two other equations in (20), for the 5-vector $A^a$ we get the following substitution (see [31])

$$A = e^{-i\epsilon t} \left[ f(r) Y^{(j+1)}_{jm} + g(r) Y^{(j-1)}_{jm} + h(r) Y^{(j)}_{jm} \right],$$

$$A^0 = Y^{(j)}_{jm} \left[ e^{(-i\epsilon + 1)t} F(r) + e^{(-i\epsilon - 1)t} G(r) \right],$$

$$A^5 = Y^{(j)}_{jm} \left[ e^{(-i\epsilon + 1)t} F(r) - e^{(-i\epsilon - 1)t} G(r) \right]. \quad (21)$$

Radial functions $f(r)$, $g(r)$, $h(r)$, $F(r)$, $G(r)$ are to be constructed on the base of eqs. (16). With the use of the form of the operator $\Delta$ in variables $(t, r, \theta, \phi)$

$$\Delta = \frac{1}{1 - r^2} \frac{\partial^2}{\partial t^2} - \frac{1}{r^2} \frac{\partial}{\partial r} r^2 (1 - r^2) \frac{\partial}{\partial r} - \frac{1}{r^2} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right).$$
and bearing in mind the known action of $\mathbf{I}^2$ on spherical functions \[32\]
\[
\mathbf{I}^2 = -\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}\right),
\]
\[
\mathbf{I}^2 \mathbf{Y}^{(j)}_{jm} = \nu(\nu + 1) \mathbf{Y}^{(j)}_{jm},\]
\[
\mathbf{I}^2 \mathbf{Y}^{(j)}_{jm} = j(j + 1) \mathbf{Y}^{(j)}_{jm},
\]
for radial functions $f(r)$, $g(r)$, $h(r)$, $F(r)$, $G(r)$ we get equations of one the same type
\[
\left[\frac{d^2}{dr^2} + \frac{2(1-2r^2)}{r(1-r^2)} \frac{d}{dr} - \frac{\Lambda^2}{(1-r^2)^2} - \frac{m^2 + 2}{1-r^2} - \frac{\nu(\nu + 1)}{r^2(1-r^2)}\right] U_{\Lambda \nu} = 0; \tag{22}
\]
radial functions are given by
\[
f = f_0 U_{-i\epsilon,j+1}, \quad g = g_0 U_{-i\epsilon,j-1}, \quad h = h_0 U_{-i\epsilon,j}, \quad F = F_0 U_{-i\epsilon+1,j}, \quad G = G_0 U_{-i\epsilon+1,j},
\]
where $f_0$, $g_0$, $h_0$, $F_0$, $G_0$ are some constants. Solutions of \[22\] can be expressed in terms of hypergeometric functions \[32\] — let us write down regular in $r = 0$ ones (let $z = r^2 = \sin^2 \omega$):
\[
U_{-i\epsilon,j} = (\sin \omega)^j (\cos \omega)^{-i\epsilon} F(a, b, c; z), \quad U_{-i\epsilon,j+1} = (\sin \omega)^{j+1} (\cos \omega)^{-i\epsilon} F(a + 1/2, b + 1/2, c + 1; z), \quad U_{-i\epsilon,j-1} = (\sin \omega)^{j-1} (\cos \omega)^{-i\epsilon} F(a - 1/2, b - 1/2, c - 1; z), \quad U_{-i\epsilon+1,j} = (\sin \omega)^{j} (\cos \omega)^{-i\epsilon+1} F(a + 1/2, b + 1/2, c; z), \quad U_{-i\epsilon-1,j} = (\sin \omega)^{j} (\cos \omega)^{-i\epsilon-1} F(a - 1/2, b - 1/2, c; z); \tag{24}
\]
where
\[
a = \frac{3/2 + j + i\sqrt{m^2 - 1/4} - i\epsilon}{2},
\]
\[
b = \frac{3/2 + j - i\sqrt{m^2 - 1/4} - i\epsilon}{2}, \quad c = j + 3/2. \tag{25}
\]
From additional constraint $\xi \mathbf{A}^a = 0$, with the use of (see in \[31\])
\[
\xi \mathbf{Y}^{(j)}_{jm} = 0, \quad \xi \mathbf{Y}^{(j+1)}_{jm} = -\sqrt{\frac{j+1}{2j+1}} r \mathbf{Y}^{(j)}_{jm}, \quad \xi \mathbf{Y}^{(j+1)}_{jm} = -\sqrt{\frac{j}{2j+1}} r \mathbf{Y}^{(j)}_{jm},
\]
on one gets
\[
-\sqrt{\frac{j+1}{2j+1}} r f + \sqrt{\frac{j}{2j+1}} r g + \sqrt{1-r^2} (G - F) = 0. \tag{26}
\]
From the Lorentz condition $L_{ab} \mathbf{A}^b = A_a$; one gets (all details are omitted)
\[
-\sqrt{\frac{j+1}{2j+1}} \left(\frac{d}{dr} - \frac{j+2}r\right) f + \sqrt{\frac{j}{2j+1}} \left(\frac{d}{dr} - \frac{j-1}r\right) g - \frac{i\epsilon}{\sqrt{1-r^2}} (F + G) = 0, \tag{27}
\]
\[
-\sqrt{\frac{j+1}{2j+1}} r f + \sqrt{\frac{j}{2j+1}} r g + \sqrt{1-r^2} (F - G) = 0; \tag{28}
\]
the last equation coincides with (26). It is convenient in expressions for \( f(r) \) and \( g(r) \) to separate special \( j \)-dependent factors

\[
\sqrt{\frac{j + 1}{2j + 1}} f(r) = f_0 \ U_{-i\epsilon,j+1} , \quad \sqrt{\frac{j}{2j + 1}} g(r) = g_0 \ U_{-i\epsilon,j-1} .
\]  
(29)

Bearing in mind (29), eqs. (27) – (28) are changed to

\[
f_0(-\frac{d}{d\omega} + \frac{j + 2}{\tan \omega}) \ U_{-i\epsilon,j+1} + g_0(-\frac{d}{d\omega} - \frac{j - 1}{\tan \omega}) \ U_{-i\epsilon,j-1} -
\]

\[
-\iota \epsilon (F_0 U_{-i\epsilon,j+1} + G_0 U_{-i\epsilon,j-1}) = 0 , \quad
\]

\[
-f_0 \tan \omega \ U_{-i\epsilon,j+1} + g_0 \tan \omega \ U_{-i\epsilon,j-1} +
\]

\[
+ F_0 U_{-i\epsilon,j+1} - G_0 U_{-i\epsilon,j-1} = 0 .
\]  
(30)

These relations may be resolved as follows

\[
2 F_0 U_{-i\epsilon,j+1} = f_0[-\frac{1}{i\epsilon}(-\frac{d}{d\omega} + \frac{j + 2}{\tan \omega}) + \tan \omega] U_{-i\epsilon,j+1} +
\]

\[
+ g_0[\frac{1}{i\epsilon}(-\frac{d}{d\omega} - \frac{j - 1}{\tan \omega}) - \tan \omega] U_{-i\epsilon,j-1} ,
\]

\[
2 G_0 U_{-i\epsilon,j-1} = f_0[-\frac{1}{i\epsilon}(-\frac{d}{d\omega} + \frac{j + 2}{\tan \omega}) - \tan \omega] U_{-i\epsilon,j+1} +
\]

\[
+ g_0[\frac{1}{i\epsilon}(-\frac{d}{d\omega} - \frac{j - 1}{\tan \omega}) + \tan \omega] U_{-i\epsilon,j-1} .
\]  
(31)

From whence, bearing in mind expressions for \( U_{-i\epsilon\pm1,j} \) and \( U_{-i\epsilon,j\pm1} \) in terms of hypergeometric functions, we arrive at

\[
2 F_0 F(a + 1/2, b + 1/2, c; z) = f_0[-\frac{2j + 3}{i\epsilon} F(a + 1/2, b + 1/2, c + 1; z) -
\]

\[
-\frac{2z}{i\epsilon} \frac{d}{dz} F(a + 1/2, b + 1/2, c + 1; z)] + g_0 \frac{2}{i\epsilon} \frac{d}{dz} F(a - 1/2, b - 1/2, c - 1; z) ,
\]  
(32)

\[
2 G_0 F(a - 1/2, b - 1/2, c; z) = f_0(1 - z)[-\frac{2j + 3}{i\epsilon} F(a + 1/2, b + 1/2, +1; z) -
\]

\[
-\frac{2z}{1 - z} F(a + 1/2, b + 1/2, c + 1; z)] +
\]

\[
+ g_0 [2 F(a - 1/2, b - 1/2, c - 1; z) + \frac{2z}{i\epsilon} (1 - z) \frac{d}{dz} F(a - 1/2, b - 1/2, c - 1; z)] .
\]  
(33)

By simplicity reason, let us search for solutions of the types:

\[
(j + 1) \text{ - wave} , \quad f_0 \neq 0 , \quad g_0 = 0 , \quad h_0 = 0 ;
\]

\[
(j - 1) \text{ - wave} , \quad f_0 = 0 , \quad g_0 \neq 0 , \quad h_0 = 0 ;
\]

\[
(j) \text{ - wave} , \quad f_0 = 0 , \quad g_0 = 0 , \quad h_0 \neq 0 .
\]  
(34)

The task is to satisfy (32) – (33) and determine the \( F_0 \) and \( G_0 \) in dependence of three factors \( f_0, g_0, h_0 \) . For \( j \)-wave, from (32) – (33) it follows \( F_0 = 0 \) and \( G_0 = 0 \) ; that is in this case the components \( A^0 \) and \( A^5 \) vanish. In the case of \( (j - 1) \)-wave, relation (32) takes the form

\[
2 F_0 F(a + 1/2, b + 1/2, c; z) = g_0 \frac{2z}{i\epsilon} \frac{d}{dz} F(a - 1/2, b - 1/2, c - 1; z) ;
\]
from whence with the use of the formula (32)
\[
\frac{d}{dz} F(\alpha, \beta, \gamma; z) = \frac{\alpha \beta}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1; z)
\]
we immediately produce
\[
(j - 1) - \text{wave} , \quad F_0 = \frac{(a - 1/2)(b - 1/2)}{i\epsilon(c - 1)}.
\] (35)

Eq. (33) for \((j - 1)\)-wave gives
\[
2G_0 F(a - 1/2, b - 1/2, c; z) = 2g_0 \left[ F(a - 1/2, b - 1/2, c - 1; z) + \right.
\]
\[
\left. + \frac{1 - z}{i\epsilon} \frac{d}{dz} F(a - 1/2, b - 1/2, c - 1; z) \right],
\]
and further with the use of the known relation (32)
\[
\frac{d}{dz} F(\alpha, \beta, \gamma; z) = \frac{\alpha + \beta + \gamma}{1 - z} F(\alpha, \beta, \gamma; z) + \frac{(\alpha - \gamma)(\beta - \gamma)}{\gamma(1 - z)} F(\alpha, \beta, \gamma + 1; z)
\]
we get the factor \(G_0\)
\[
(j - 1) - \text{wave} , \quad G_0 = \frac{(a - c + 1/2)(b - c + 1/2)}{i\epsilon(c - 1)} g_0 .
\] (36)

For \((j + 1)\)-wave, relation (33) gives
\[
2F_0 F(a + 1/2, b + 1/2, c; z) = f_0 \left[ -\frac{2j + 3}{i\epsilon} F(a + 1/2, b + 1/2, c + 1; z) - \right.
\]
\[
\left. - \frac{2z}{i\epsilon} \frac{d}{dz} F(a + 1/2, b + 1/2, c + 1; z) \right],
\]
and further with the use of the formula (32)
\[
z \frac{d}{dz} F(\alpha, \beta, \gamma; z) = (\gamma - 1) \left[ F(\alpha, \beta, \gamma - 1; z) - F(\alpha, \beta, \gamma; z) \right],
\]
we get the factor \(F_0\)
\[
(j + 1) - \text{wave} , \quad F_0 = \frac{c}{-i\epsilon} f_0 .
\] (37)

in turn, relation (33) takes the form
\[
2G_0 F(a - 1/2, b - 1/2, c; z) = f_0(1 - z) \left[ -\frac{2j + 3}{i\epsilon} F(a + 1/2, b + 1/2, c + 1; z) - \right.
\]
\[
\left. - \frac{2z}{1 - z} \frac{d}{dz} F(a + 1/2, b + 1/2, c + 1; z) \right] ,
\]
which with the use of (37) gives
\[
- i\epsilon \frac{G_0}{f_0} F(a - 1/2, b - 1/2, c; z) =
\]
\[
= (c - a - b) z F(a + 1/2, b + 1/2, c + 1; z) + c(1 - z) F(a + 1/2, b + 1/2, c; z) .
\]
Let us differentiate the later:

\[-i \frac{G_0}{f_0} \frac{(a - 1/2)(b - 1/2)}{c} F(a + 1/2, b + 1/2, c + 1; z) =
\]

\[= (c - a - b) \left( z \frac{d}{dz} + 1 \right) F(a + 1/2, b + 1/2, c + 1; z) +
\]

\[c \left( -1 + \left( \frac{1 - z}{z} \right) \frac{d}{dz} \right) F(a + 1/2, b + 1/2, c; z),
\]

from whence we get

\[-i \frac{G_0}{f_0} \frac{(a - 1/2)(b - 1/2)}{c} F(a + 1/2, b + 1/2, c + 1; z) =
\]

\[= c\left( -\frac{c - 1}{z} + 2c - a - b - 2 \right) F(a + 1/2, b + 1/2, c; z) +
\]

\[+c(c - 1) \frac{1 - z}{z} F(a + 1/2, b + 1/2, c - 1; z) -
\]

\[-(1 - c)(c - a - b) F(a + 1/2, b + 1/2, c; z).
\]

Not, let us use the known identity for hypergeometric functions \[32\]

\[c[(c - 1) - 2(c - a - b - 2)z]F(a + 1/2, b + 1/2, c; z) +
\]

\[+(c - a - 1/2)(c - b - 1/2)zF(a + 1/2, b + 1/2, c + 1; z) -
\]

\[-c(c - 1)(1 - z)F(a + 1/2, b + 1/2, c - 1; z) = 0,
\]

then \[33\] results in

\[-i \frac{G_0}{f_0} \frac{(a - 1/2)(b - 1/2)}{c} F(a + 1/2, b + 1/2, c + 1; z) =
\]

\[= [(c - a - 1/2)(c - b - 1/2) + (1 - c)(c - a - b)]F(a + 1/2, b + 1/2, c + 1; z);
\]

therefore the factor \(G_0\) is given by

\[(j + 1) - \text{wave}, \quad G_0 = \frac{c}{-i \epsilon} f_0.
\]

Collecting the results obtained

\(\nu = j\),

\[A = e^{-i \epsilon t} h_{00} U_{-i \epsilon, j} Y^{(j)}_{jm}(\theta, \phi), \quad A^0 = 0, \quad A^5 = 0;
\]

\(\nu = j + 1\),

\[A = e^{-i \epsilon t} Y^{(j+1)}_{jm} \sqrt{\frac{2j + 1}{j + 1}}, \quad F_0 = -\frac{c}{i \epsilon} f_0, \quad G_0 = -\frac{c}{i \epsilon} f_0;
\]

\(\nu = j - 1\),

\[A = e^{-i \epsilon t} Y^{(j-1)}_{jm} \sqrt{\frac{2j + 1}{j}} g_0 U_{-i \epsilon, j-1},
\]

\[F_0 = \frac{(a - 1/2)(b - 1/2)}{i \epsilon (c - 1)} g_0, \quad G_0 = \frac{(a - c + 1/2)(b - c + 1/2)}{i \epsilon (c - 1)} g_0;
\]

\[\nu = j,
\]

\[A = e^{-i \epsilon t} h_{00} U_{-i \epsilon, j} Y^{(j)}_{jm}(\theta, \phi), \quad A^0 = 0, \quad A^5 = 0;
\]

\[\nu = j + 1,
\]

\[A = e^{-i \epsilon t} Y^{(j+1)}_{jm} \sqrt{\frac{2j + 1}{j + 1}}, \quad F_0 = -\frac{c}{i \epsilon} f_0, \quad G_0 = -\frac{c}{i \epsilon} f_0;
\]

\[\nu = j - 1,
\]

\[A = e^{-i \epsilon t} Y^{(j-1)}_{jm} \sqrt{\frac{2j + 1}{j}} g_0 U_{-i \epsilon, j-1},
\]

\[F_0 = \frac{(a - 1/2)(b - 1/2)}{i \epsilon (c - 1)} g_0, \quad G_0 = \frac{(a - c + 1/2)(b - c + 1/2)}{i \epsilon (c - 1)} g_0;
\]

\[\nu = j,
\]
in the cases (41) and (42) the components \( A_0 \) and \( A_5 \) are to be calculated by the same formulas with different values of \( F_0 \) and \( G_0 \):

\[
A_0 = Y_{jm} \left[ e^{(-i\epsilon+1)t} F_0 U_{-i\epsilon+1,j} + e^{(-i\epsilon-1)t} G_0 U_{-i\epsilon-1,j} \right], \\
A_5 = Y_{jm} \left[ e^{(-i\epsilon+1)t} F_0 U_{-i\epsilon+1,j} - e^{(-i\epsilon-1)t} G_0 U_{-i\epsilon-1,j} \right].
\]

(43)

4 General method to construct electromagnetic \( \Pi, E, M \)-waves

Let us recall the situation in the flat Minkowski space. If a scalar function \( \Lambda(x) \) the massless wave equation

\[
\Delta \Lambda(x) = 0 , \quad \Delta = \partial^0 \partial_0 - \partial^i \partial_i ,
\]

then three linearly independent solutions of the vector wave equation \( \Delta A^\alpha(x) = 0 \) can be constructed as follows [30]:

\[
A^{(1)} = \frac{\partial}{\partial x^\alpha} \Lambda(x) , \quad A^{(2)} = r \times A^{(1)} , \quad A^{(3)} = \nabla \times A^{(2)} .
\]

If the \( \Lambda(x) \) is taken as a spherical wave

\[
\Lambda(x) = e^{-i\epsilon t} Y_{jm}(\theta, \phi) f_j(\epsilon r) ,
\]

\( f_j(\epsilon r) \) is a Bessel spherical function, the recipe (44) gives three spherical vector solutions [30]:

\[
A^{(1)} \sim e^{-i\epsilon t} \sqrt{\frac{j}{2j+1}} f_{j-1} Y^{(j-1)}_{jm} + \sqrt{\frac{j+1}{2j+1}} f_{j+1} Y^{(j+1)}_{jm} , \\
A^{(2)} \sim f_j(\epsilon r) Y^{(j)}_{jm}(\theta, \phi) , \quad A^{(3)} \sim e^{-i\epsilon t} \sqrt{\frac{j}{2j+1}} f_{j-1} Y^{(j-1)}_{jm} - \sqrt{\frac{j+1}{2j+1}} f_{j+1} Y^{(j+1)}_{jm} ,
\]

(45)

which are called respectively \( \Pi-, M-, E- \)waves. The task is to extend this method to de Sitter space starting with 5-form:

\[
(\Delta + 2)A^b = 0 , \quad \Delta = -\frac{1}{2} L^{ab} L_{ab} = m^a m_a , \quad L_{ab} A^b = 0 , \quad \xi^a A_a = 0 .
\]

Evidently, that \( \Pi \)-wave should be determined by the rule

\[
A^{(\Pi)}_a = m_a \Lambda(x) , \quad \Delta \Lambda(x) = 0 .
\]

(46)

Let us demonstrate with the help of commutative relations that the 5-vector \( m_a \Lambda(x) \) satisfies \( (\Delta + 2)A^b = 0 \). Indeed, bearing in mind \( [m_a, m_b] = L_{ab} \), one may obtain

\[
\Delta m_a \Lambda(x) = [m^b m_b, m_a] \Lambda(x) = (L_{ab} m^b + m^b L_{ab}) \Lambda(x) ;
\]

which with the help of \( [m^b, L_{ab}] = -4m_a \) reduces to

\[
\Delta m_a \Lambda(x) = (2 L_{ab} m^b + [m^b, L_{ab}]) \Lambda(x) = (2 L_{ab} m^b - 4 m_a) \Lambda(x) ,
\]
from whence it follows
\[(\Delta + 2)m_b \Lambda(x) = 0.\] (47)

Let us define \(M\)-wave. One might expect the structure
\[A^{(M)} = \vec{\xi} \times A^{(\Pi)} = (\vec{\xi} \times m) \Lambda(x), \quad \Delta \Lambda(x) = 0.\] (48)

Because the operator \((\vec{\xi} \times m)_i = \epsilon_{ijk} \xi_j (\partial_k + \xi_k \xi^a \partial_a)\) commutes with \(\Delta\), the equation \(\Delta (\vec{\xi} \times m) \Lambda(x) = 0\) holds, from whence we conclude that one must make the starting structure (48) more exact
\[A^{(M)} = (\vec{\xi} \times m) K(x), \quad (\Delta + 2) K(x) = 0;\] (49)

where the scalar function \(K(x)\) satisfies the conformally invariant equation. It remain to prove the structure for \(E\)-wave:
\[A^{(E)} = m \times (\vec{\xi} \times m) \Lambda(x).\] (50)

Indeed, the relationship
\[(\Delta + 2) A^{(E)}_n = (\Delta + 2) \epsilon_{nij} m_i \epsilon_{jkl} \xi_k m_l \Lambda(x) = \frac{1}{2} \epsilon_{nij} \epsilon_{jkl} (\Delta + 2) m_i L_{kl} \Lambda(x);\]

with the help of identity \([m_i, L_{kl}] = (g_{ik} m_l - g_{il} m_k),\) transforms to
\[(\Delta + 2) A^{(E)}_n = \frac{1}{2} \epsilon_{nij} \epsilon_{jkl} (\Delta + 2) (L_{kl} m_i + g_{ik} m_l - g_{il} m_k) \Lambda(x) =\]
\[= \frac{1}{2} \epsilon_{nij} \epsilon_{jkl} [L_{kl} (\Delta + 2) m_i + g_{ik} (\Delta + 2) m_l - g_{il} (\Delta + 2) m_k] \Lambda(x);\]

from this, taking into account \((\Delta + 2) m_b \Lambda(x) = 0\), one arrives at
\[(\Delta + 2) A^{(E)}_n = 0.\] (51)

Thus, solutions of the types \(\Pi-, M-, E-\) in de Sitter space are determined by
\[A^{(\Pi)}_a = m_a \Lambda(x), \quad \Delta \Lambda(x) = 0;\]
\[A^{(M)} = (\vec{\xi} \times m) K(x), \quad (\Delta + 2) K(x) = 0;\]
\[A^{(E)} = m \times (\vec{\xi} \times m) \Lambda(x), \quad \Delta \Lambda(x) = 0.\] (52)

To obtain \(A^0\) and \(A^5\) for \(M-, E-\)waves one should use equations (see (46))
\[L_{ab} A^b = 0, \quad \xi^a A_a = 0.\]

5 Spherical \(E, M, \Pi\)-waves in static coordinates

Starting with static spherical coordinates in de Sitter space
\[dS^2 = (1 - r^2) dt^2 - \frac{dr^2}{1 - r^2} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),\] (53)
a scalar solution for $\Delta \Lambda(x) = 0$ let us choose as follows

$$\Delta \Lambda(x) = 0, \; \Lambda(x) = e^{-i\epsilon t} Y_{jm}(\theta, \phi) f(r),$$

$$f(r) = \sin^j \omega \left( \cos \omega \right)^{-i\epsilon} F(a - 1/2, b + 1/2, c; z),$$

$$a = \frac{j + 1 - i\epsilon}{2}, \; b = \frac{j + 2 - i\epsilon}{2}, \; c = j + 3/2.$$  \hspace{1cm} (54)

We need explicit form for $m_b$ in these coordinates $(t, r, \theta, \phi)$

$$(m^0 + m^5) = e^{+t} \left[ \frac{1}{\sqrt{1 - r^2}} \frac{\partial}{\partial t} + r \sqrt{1 - r^2} \frac{\partial}{\partial r} \right],$$

$$(m^0 - m^5) = e^{-t} \left[ \frac{1}{\sqrt{1 - r^2}} \frac{\partial}{\partial t} - r \sqrt{1 - r^2} \frac{\partial}{\partial r} \right],$$

$$(m^i) = m = (-\nabla + \vec{\xi} r \frac{\partial}{\partial r}), \; \nabla = \frac{\partial}{\partial \vec{\xi}}.$$  \hspace{1cm} (55)

With the use of known properties of spherical functions $^{[31]}$, we get representation for $\Pi$-wave

$$\Pi \text{ - wave}, \quad (A^0 + A^5)^\Pi = e^{(-i\epsilon+1)t} Y_{jm} \left( -\frac{i\epsilon}{\cos \omega} + \sin \omega \frac{d}{d\omega} \right) f,$$

$$(A^0 - A^5)^\Pi = e^{(-i\epsilon-1)t} Y_{jm} \left( -\frac{i\epsilon}{\cos \omega} - \sin \omega \frac{d}{d\omega} \right) f,$$

$$A^\Pi = e^{-i\epsilon t} \left[ \sqrt{\frac{j+1}{2j+1}} Y_{jm}^{(j+1)} \left( \cos \omega \frac{d}{d\omega} - \frac{j}{\sin \omega} \right) f - \right.$$  

$$- Y_{jm}^{(j-1)} \left( \cos \omega \frac{d}{d\omega} + \frac{j+1}{\sin \omega} \right) f \right].$$  \hspace{1cm} (56)

For $M$-wave, using identities $^{[31]}$

$$\vec{\xi} \times Y_{jm}^{(j+1)} = i \sqrt{\frac{j}{2j+1}} r Y_{jm}^{(j)}, \quad \vec{\xi} \times Y_{jm}^{(j-1)} = i \sqrt{\frac{j+1}{2j+1}} r Y_{jm}^{(j)},$$

we get

$$A^{(M)} = \frac{i}{\sqrt{j(j+1)}} (\vec{\xi} \times m) K(x) = e^{-i\epsilon t} Y_{jm}^{(j)}(\theta, \phi) U_{-i\epsilon, j}(z);$$  \hspace{1cm} (57)

where $U_{-i\epsilon,j}(z)$ is determined by

$$(\Delta + 2) K(x) = 0, \quad K(x) = e^{-i\epsilon t} Y_{jm}(\theta, \phi) U_{-i\epsilon, j}(z),$$

$$U_{-i\epsilon,j}(z) = \sin^j \omega \left( \cos \omega \right)^{-i\epsilon} F(a, b, c; z).$$  \hspace{1cm} (58)

For $E$-wave we get

$$A^{(E)} = \frac{i}{\sqrt{j(j+1)}} \left[ m \times (\vec{\xi} \times m) \right] \Lambda(x) = m \times \left[ e^{-i\epsilon t} Y_{jm}^{(j)}(\theta, \phi) f(r) \right],$$

and further

$$A^{(E)} = e^{-i\epsilon t} \left[ \sqrt{\frac{j}{2j+1}} Y_{jm}^{(j+1)} \left( \cos \omega \frac{d}{d\omega} - \frac{j}{\sin \omega} \right) f + \right.$$  

$$\sqrt{\frac{j+1}{2j+1}} Y_{jm}^{(j-1)} \left( \cos \omega \frac{d}{d\omega} + \frac{j+1}{\sin \omega} \right) f \right].$$  \hspace{1cm} (59)
Supposing that \((\Pi, E)\)-waves are linear combinations of solutions \((j + 1)\) and \((j - 1)\) constructed in Section 2 (at \(m = 0\)), we reduce the task to studying four relations

\[
\begin{align*}
&\left( -\frac{i\epsilon}{\cos \omega} + \sin \omega \frac{d}{dr} \right) f = \text{const} \ U_{-i\epsilon+1,j}, \\
&\left( -\frac{i\epsilon}{\cos \omega} - \sin \omega \frac{d}{dr} \right) f = \text{const} \ U_{-i\epsilon-1,j}, \\
&\left( \cos \omega \frac{d}{d\omega} - \frac{j}{\sin \omega} \right) f = \text{const} \ U_{-i\epsilon,j+1}, \\
&\left( \cos \omega \frac{d}{d\omega} + \frac{j+1}{\sin \omega} \right) f = \text{const} \ U_{-i\epsilon,j-1};
\end{align*}
\]

the four constants are to be found. In the case (60) we have

\[
\left( -\frac{i\epsilon}{\cos \omega} + \sin \omega \frac{d}{dr} \right) f = \sin^j \omega (\cos \omega)^{-i\epsilon+1} \left[ (j - i\epsilon) F(a - 1/2, b + 1/2, c; z) + 2z \frac{d}{dz} F(a - 1/2, b + 1/2, c; z) \right],
\]

from whence, using the rule [32]

\[
z \frac{d}{dz} \ F(\alpha, \beta, \gamma; z) = \alpha \left[ F(\alpha + 1, \beta, \gamma; z) - F(\alpha, \beta, \gamma; z) \right],
\]

we get

\[
\left( -\frac{i\epsilon}{\cos \omega} + \sin \omega \frac{d}{dr} \right) f = \sin^j \omega (\cos \omega)^{-i\epsilon+1}(j - i\epsilon)F(a + 1/2, b + 1/2, c; z) = (j - i\epsilon)U_{-i\epsilon+1,j}.
\]  

(64)

In the case (61)

\[
\left( -\frac{i\epsilon}{\cos \omega} - \sin \omega \frac{d}{dr} \right) f = \sin^j \omega (\cos \omega)^{-i\epsilon-1}[-(j - i\epsilon)F + (j + i\epsilon)zF - 2z \frac{d}{dz} F]
\]

\[
= \sin^j \omega (\cos \omega)^{-i\epsilon-1}[-2i\epsilon F(a - 1/2, b + 1/2, c; z) - 2(a - 1/2)(1 - z)F(a + 1/2, b + 1/2, c; z)];
\]

and further, with the use of the known identity for hypergeometric functions [32] we get

\[
[(\gamma - \alpha - \beta) F(\alpha, \beta, \gamma; z) + \alpha (1 - z) F(\alpha + 1, \beta, \gamma; z) - (\gamma - \beta) F(\alpha, \beta - 1, \gamma; z)] = 0
\]

(at \(\alpha = a - 1/2, \beta = b + 1/2, \gamma = c\); therefore

\[
\left( -\frac{i\epsilon}{\cos \omega} + \sin \omega \frac{d}{dr} \right) f = -(j + i\epsilon) \sin^j \omega (\cos \omega)^{-i\epsilon+1}F(a - 1/2, b - 1/2, c; z) = -(j + i\epsilon)U_{-i\epsilon-1,j}.
\]  

(65)
Let us consider eq. (62):

\[
(\cos \omega \frac{d}{d\omega} - \frac{j}{\sin \omega}) f = \sin^{j+1} \omega \ (\cos \omega)^{-i\epsilon} [(-j + i\epsilon) F + 2(1 - z) \frac{d}{dz} F];
\]

allowing for the rule [32]

\[
(1 - z) \frac{d}{dz} F(\alpha, \beta, \gamma; z) = [ (\alpha + \beta - \gamma) F(\alpha, \beta, \gamma; z) +
+ \frac{(\alpha - \gamma)(\beta - \gamma)}{\gamma} F(\alpha, \beta, \gamma + 1; z) ],
\]

we arrive at

\[
(\cos \omega \frac{d}{d\omega} - \frac{j}{\sin \omega}) f =
- \frac{j + i\epsilon}{c} \sin^{j+1} \omega (\cos \omega)^{-i\epsilon} [ cF(a - 1/2, b + 1/2, c; z) +
+ (a - 1/2 - c) F(a - 1/2, b + 1/2, c + 1; z) ] =
= - \frac{j + i\epsilon}{2} \sin^{j+1} \omega (\cos \omega)^{-i\epsilon} \frac{j - i\epsilon}{2} F(a + 1/2, b + 1/2, c + 1; z) ;
\]

and further get

\[
(\cos \omega \frac{d}{d\omega} - \frac{j}{\sin \omega}) f = - \frac{j^2 + \epsilon^2}{2j + 3} U_{-i\epsilon,j+1} . \tag{66}
\]

Finally, for the case (63) we have

\[
(\cos \omega \frac{d}{d\omega} + \frac{j + 1}{\sin \omega}) f =
= \sin^{j-1} \omega (\cos \omega)^{-i\epsilon} [ (2j + 1) F + z (i\epsilon - j) F + 2z(1 - z) \frac{d}{dz} F] ;
\]

from whence, using the formula [32]

\[
z \frac{d}{dz} F(\alpha, \beta, \gamma; z) = (\gamma - 1) [ F(\alpha, \beta, \gamma + 1; z) - F(\alpha, \beta, \gamma; z) ]
\]

we get

\[
(\cos \omega \frac{d}{d\omega} + \frac{j + 1}{\sin \omega}) f = \sin^{j-1} \omega (\cos \omega)^{-i\epsilon} \times
\]
\[
\times [ 2(c - 1) (1 - z) F(a - 1/2, b + 1/2, c + 1; z) +
+ 2(c - a - 1/2)z F(a - 1/2, b + 1/2, c; z) ] .
\]

The term in square brackets equals to

\[
(2j + 1) F(a - 1/2, b - 1/2, c - 1; z) ,
\]

therefore we have obtained

\[
(\cos \omega \frac{d}{d\omega} + \frac{j + 1}{\sin \omega}) f = (2j + 1) U_{-i\epsilon,j-1} . \tag{67}
\]

Collecting the results, we get
\( \Pi - \text{wave} , \quad (A^0 + A^5)^\Pi = e^{(-i\epsilon + 1)t} ( + j - i\epsilon ) U_{-i\epsilon + 1, j} Y_{jm} , \)
\( (A^0 - A^5)^\Pi = e^{(-i\epsilon - 1)t} (-j - i\epsilon ) U_{-i\epsilon - 1, j} Y_{jm} , \)
\[ A^\Pi = e^{-i\epsilon t} \left[ \sqrt{\frac{j + 1}{2j + 1}} \frac{y(j+1)}{2j + 3} U_{-i\epsilon, j+1} - \sqrt{\frac{j}{2j + 1}} Y_{jm}^{(j-1)} (2j + 1) \right] U_{-i\epsilon, j} ; \]
\( E - \text{wave} , \quad A^{(E)} = e^{-i\epsilon t} \left[ \sqrt{\frac{j + 1}{2j + 1}} \frac{y(j+1)}{2j + 3} U_{-i\epsilon, j+1} + \sqrt{\frac{j}{2j + 1}} Y_{jm}^{(j-1)} (2j + 1) \right] U_{-i\epsilon, j} ; \)
\( M - \text{wave} , \quad A^{(M)} = e^{-i\epsilon t} Y_{jm}^{(j)} (\theta, \phi) U_{-i\epsilon, j} (z) . \] (68)

Components \( A^0 \) and \( A^5 \) for \( M-, E- \) waves can be found from comparing vector parts \( A \) for these waves with given by (40) – (43)
\[ \nu = j , \quad A = e^{-i\epsilon t} U_{-i\epsilon, j} Y_{jm}^{(j)} (\theta, \phi) , \ h_0 = 1 , \ A^0 = 0 , \ A^5 = 0 ; \]
\[ \nu = j + 1 , \quad f_0 = \sqrt{\frac{j + 1}{2j + 1}} , \ A = e^{-i\epsilon t} Y_{jm}^{(j+1)} U_{-i\epsilon, j+1} ; \]
\[ (A^0 \pm A^5) = e^{(-i\epsilon \pm 1)t} Y_{jm}^{2j + 3} \frac{j + 1}{-i\epsilon} \sqrt{\frac{j + 1}{2j + 1}} ; \]
\[ \nu = j - 1 , \quad g_0 = \sqrt{\frac{j}{2j + 1}} , \ A = e^{-i\epsilon t} Y_{jm}^{(j-1)} U_{-i\epsilon, j-1} ; \]
\[ (A^0 \pm A^5) = e^{(-i\epsilon \pm 1)t} Y_{jm} \left[ \frac{(j \mp i\epsilon)(j \mp i\epsilon + 1)}{i\epsilon(2j + 1)} \sqrt{\frac{j + 1}{2j + 1}} \right] U_{-i\epsilon \pm 1, j} . \] (69)

Let us compare these \( (\Pi, M, E)- \) and \( (j, j \pm 1)- \) waves. We see that \( M- \) and \( j- \) solutions coincide. For \( \Pi \)-wave we see for vector parts
\[ A^{(\Pi)} = -\sqrt{\frac{j + 1}{2j + 1}} \frac{j^2 + \epsilon^2}{2j + 3} A^{(j+1)} - \sqrt{\frac{j}{2j + 1}} (2j + 1) A^{(j-1)} ; \] (70)
also the equalities hold
\[ (A^0 \pm A^5)^{(\Pi)} = -\sqrt{\frac{j + 1}{2j + 1}} \frac{j^2 + \epsilon^2}{2j + 3} (A^0 \pm A^5)^{(j+1)} - \]
\[ -\sqrt{\frac{j}{2j + 1}} (2j + 1) (A^0 \pm A^5)^{(j-1)} . \]
For $E$-wave we have
\[
A^{(E)} = \left[ -\sqrt{\frac{j}{2j+1}} \frac{j^2 + \epsilon^2}{2j+3} A^{(j+1)} + \sqrt{\frac{j+1}{2j+1}} (2j+1) A^{(j-1)} \right] ;
\] (71)
from whence it follows $A^0$ and $A^5$ for $E$-wave:
\[
(A^0 \pm A^5)^{(E)} = \frac{j - i \epsilon}{i \epsilon} \sqrt{j(j+1)} e^{(-i \epsilon \pm 1)t} Y_{jm} U_{-i \epsilon \pm 1,j} .
\] (72)

6 Supplement: Correspondence principle in de Sitter model

Let us show that solutions constructed in de Sitter space are in accordance with correspondence principle: namely, they give the known results in the limit of flat Minkowski space. Let specify the case of $\Pi$-wave.

Starting with limiting identities
\[
\lim_{\rho \to \infty} \rho^{j+1} U_{-i \epsilon, j+1} = 2^p \frac{\Gamma(1+p)}{\epsilon} J_p(\frac{jr}{\sqrt{\rho}}), \quad p = j + 3/2 ,
\]
\[
\lim_{\rho \to \infty} \rho^{j-1} U_{-i \epsilon, j-1} = 2^{p'} \frac{\Gamma(1+p')}{\epsilon} J_{p'}(\frac{jr}{\sqrt{\rho}}), \quad p' = j - 1/2 ,
\]
($\rho$ is a curvature radius, $J_\nu(x)$ stands for the Bessel function [32], we readily produce
\[
\lim_{\rho \to \infty} \rho^{j-1} A^{(\Pi)} = e^{-i \epsilon t} \lim_{\rho \to \infty} \left[ -\sqrt{\frac{j}{2j+1}} \frac{j^2 + \epsilon^2}{\rho^2(2j+3)^2} \rho^{j+1} Y_{jm}^{(j+1)} U_{-i \epsilon, j+1} - \right.
\]
\[
\left. -\sqrt{\frac{j}{2j+1}} \rho^{j-1} Y_{jm}^{(j-1)} U_{-i \epsilon, j-1} =
\right]
\[
e^{-i \epsilon t} \left[ (2j+1) \Gamma(j+1/2) \frac{2^{j-1/2}}{\epsilon} \right] \frac{1}{\sqrt{\rho}} \times
\]
\[
\times\left[ \sqrt{\frac{j+1}{2j+1}} Y_{jm}^{(j+1)} J_{j+3/2}(\epsilon r) + \sqrt{\frac{j}{2j+1}} Y_{jm}^{(j-1)} J_{j-1/2}(\epsilon r) \right];
\]
or with notation $f_\nu(x) = \sqrt{\pi/2x} J_{\nu+1/2}(x)$ we get
\[
\lim_{\rho \to \infty} [\rho^{j-1} A^{(\Pi)}] \sim e^{-i \epsilon t} \left[ \sqrt{\frac{j+1}{2j+1}} Y_{jm}^{(j+1)} f_{j+1}(\epsilon r) + \sqrt{\frac{j}{2j+1}} Y_{jm}^{(j-1)} f_{j-1}(\epsilon r) \right],
\]
which agrees with the known representation for $\Pi$ wave in flat space. Besides, bearing in ming identities
\[
\lim_{\rho \to \infty} [\rho^{j} U_{-i \epsilon \pm 1,j}] = 2^p \frac{\Gamma(1+p)}{\epsilon} J_p(\frac{jr}{\sqrt{\rho}}), \quad p = j + 3/2 ,
\]
we find limiting form for $A^{0(\Pi)}$ and $A^{5(\Pi)}$:
\[
\lim_{\rho \to \infty} [\rho^{j-1}(A^0 \pm A^5)^{(\Pi)}] =
\]
\[
= \lim_{\rho \to \infty} \left[ \frac{\pm j - i \epsilon \rho}{\rho} \exp\left[(-i \frac{E \rho}{hc} \pm 1) \frac{cl}{\rho} \right] Y_{jm}(\theta, \phi) \rho^j U_{-i \epsilon \pm 1,j} \right] ;
\]
from this it follows
\[
\lim_{\rho \to \infty} \left[ \rho^{j-1} A^{5(II)} \right] = 0, \quad \lim_{\rho \to \infty} \left[ \rho^{j-1} A^{0(II)} \right] \sim e^{-i\epsilon t} Y_{jm}(\theta, \phi) f_j(\epsilon r).
\]
which agrees with the known representation for II wave in flat space \cite{30}.

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