DIRECTED POLYMERS WITH
RANDOM INTERACTION :
AN EXACTLY SOLVABLE CASE

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We propose a model for two \((d + 1)\)-dimensional directed polymers subjected to a mutual \(\delta\)-function interaction with a random coupling constant, and present an exact renormalization group study for this system. The exact \(\beta\)-function, evaluated through an \(\epsilon(= 1 - d)\) expansion for second and third moments of the partition function, exhibits the marginal relevance of the disorder at \(d = 1\), and the presence of a phase transition from a weak to strong disorder regime for \(d > 1\). The lengthscale exponent for the critical point is \(\nu = 1/2 |\epsilon|\). We give details of the renormalization. We show that higher moments do not require any new interaction, and hence the \(\beta\) function remains the same for all moments. The method is extended to multicritical systems involving an \(m\) chain interaction. The corresponding disorder induced phase transition for \(d > d_m = 1/(m - 1)\) has the critical exponent \(\nu_m = [2d(m - 1) - 2]^{-1}\). For both the cases, an essential singularity appears for the lengthscale right at the upper critical dimension \(d_m\). We also discuss the strange behavior of an annealed system with more than two chains with pairwise random interactions among each other.

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I. INTRODUCTION

Attempts to study the effects of randomness, especially if one requires averages of thermodynamic quantities, have led to many new techniques, concepts, and, probably, controversies. In order to get a clear idea about random systems, in recent years, a directed polymer (DP) in a random medium seems to have emerged as the consensus candidate for the “simplest” random model. We here propose a still simpler problem of DPs with random interaction that can be solved using an exact field theoretic renormalization group (RG) approach. This, we believe, is highly significant since RG is the general framework to study and to understand, through the fixed point spectrum, the universal aspects of any model.

Directed polymers in \((d + 1)\)-dimensions are random walks directed along a particular direction, say \(z\), with fluctuations in the transverse \(d\)-dimensional space. DPs are of considerable interest and have attracted a lot of attention as a simple statistical mechanical model because of its relevance and applicability in unifying a wide variety of seemingly disparate systems. These include the flux lattice melting problem in high \(T_c\) superconductors, commensurate-incommensurate transitions, wetting transition, vertex models, polymeric nematics, biomembrane phase transitions, interface growth, etc. Many problems of conventional polymers [self avoiding walks (SAW)] like collapse, adsorption etc, have exactly solvable counterpart in DPs. The RG analysis of a pure system of interacting DPs gives enough insights through the evaluation of the exact \(\beta\)-function to all orders in perturbation series. These systems of DPs with pure short range interactions are almost completely solved, and, for example, have led to several exact results for vertex models.

There are many efforts and activities in the field of polymers with random interaction or in random media. The analogous DP problems are expected to be simpler. For example, a DP in a random medium, which through a nonlinear mapping describes many aspects of interface growth, has been studied upto one loop in the momentum shell.
technique. There were also attempts for solving the many chain system in a random medium in the context of high $T_c$ superconductivity. It is the directedness that helps in setting up the DP problem, both analytically and numerically, as opposed to the SAW problem in random media. Several results for the DP problem are known in general, though exact or rigorous results are rather few. Apart from these random media problems, the other category of problems involve polymers with random interactions in the context of, say, disordered heteropolymers. Here again, a DP with random interaction turns out to be simpler.

Our model has similarity with the second category of problems. It deals with a random mutual interaction among the chains with the randomness in the coupling constant of the interaction. The randomness is only along the length of the chains and does not depend on the transverse $d$ dimensional coordinates. The specific characteristic of the randomness as well as the directed nature of the polymers enable us to solve the model exactly. We, furthermore, show that this model, inspite of its simplicity, captures many of the essential features such as marginal relevance, existence of a disorder induced phase transition, etc. as known, e.g., for the interface growth problem, DP and SAW in random environments etc.

The two dimensional wetting phenomenon is also analogous to our proposed system - though our model (and the solution) is for general $d$.

We define the model in the next section, and to put things in the proper context, the aim and the outline of the paper are given there.

**II. MODEL**

From the definition of DPs, it follows that a projection of a DP in the transverse $d$-dimensional space is an ordinary polymer with $z$ representing the contour variable which is equivalent to the steplength in a discrete case. In the path integral formulation the dimensionless hamiltonian for two such DPs, each of length $N$, interacting through a random mutual shortrange interaction can be written as
\[ H = \frac{1}{2} \int_{0}^{N} dz \left[ \left( \frac{\partial r_1(z)}{\partial z} \right)^2 + \left( \frac{\partial r_2(z)}{\partial z} \right)^2 \right] + \int_{0}^{N} dz \, v_{0} \left[ 1 + b(z) \right] V(r_{12}(z)). \] (2.1)

where \( r_i(z) \) is the \( d \)-dimensional position vector of a point of chain \( i \) at a contour length \( z \), and \( r_{12}(z) = r_1(z) - r_2(z) \). The first term that comes from the chain connectivity is the entropic contribution, and corresponds to free chains. [25] The second term is the two chain interaction at the same chainlength through a short range potential \( V(r) \). We introduce the randomness through the coupling constant. It has a pure part \( v_0 \) and a random part \( v_0 b(z) \) which varies only with \( z \) (the length along the chain). It is chosen in this way so that \( b(z) \) is dimensionless. At this stage, for generality, we keep \( V \) as a short range potential. Later on, specific calculations would be done with a \( \delta \)-function potential. Also, starting with a short range potential has certain mathematical advantages like avoiding powers of distributions, as we will see below. One can also think of this problem as a nonrelativistic quantum problem of particles with time \( (z) \) dependent interaction potential - a description we do not find very illuminating.

One of the simplest but nontrivial choices for the distribution of the randomness is a Gaussian one:

\[ P(b(z)) = (2\pi\Delta)^{-1/2} \exp[-b(z)^2/(2\Delta)], \] (2.2a)

\[ \langle b(z) \rangle = 0, \text{ and } \langle b(z_1)b(z_2) \rangle = \Delta \delta(z_1 - z_2). \] (2.2b)

Here the randomness is uncorrelated in nature and is described by the variance \( \Delta \). Choosing a zero mean for \( b(z) \) is not a restriction because any nonzero \( \langle b(z) \rangle \) could be “gauged away” by absorbing it in the pure part.

So far we have discussed only two body interactions. For DPs it is known that even pure many body interactions, representing special multicritical points, can also be handled exactly. [26,19,18] It is possible to study the disordered versions of these multicritical systems. The hamiltonian for the \( m \)-th order multicritical point, involving only \( m \)-body \( \delta \)-function interaction, is

\[ H_m = \frac{1}{2} \int_{0}^{N} dz \sum_{i=1}^{m} \left( \frac{\partial r_i(z)}{\partial z} \right)^2 + \int_{0}^{N} dz \, v_{m} \left[ 1 + b(z) \right] \prod_{i=1}^{m-1} \delta(r_{i, i+1}(z)). \] (2.3)
where, as before, \( b(z) \) is the random part. We come back to this multicritical situation in section VI. It is also possible to define more general systems by putting the lower order interactions in Eq. 2.3 with independent random coupling constants. Such a hamiltonian can, in principle, describe the approach to the multicritical points. However, such complicated cases are not discussed here.

One possibility of getting a random interaction for, say, the two chain case, is to take “charged” DPs, with random charges \( q_i(z) \) for the \( i \)th chain, \( v_0 b(z) = q_1(z)q_2(z) \), and interactions of charges only at same \( z \). If the charges are in thermal equilibrium with the polymers, a simple quadratic hamiltonian for the charges can be taken as proportional to \( f b(z)^2 \, dz \), and it is to be added to the hamiltonian of Eq. 2.1. The partition function one gets from this full hamiltonian is really equivalent to \( \langle Z \rangle \) (annealed case) evaluated with the hamiltonian of Eq. 2.1 but averaged over the distribution of Eq. 2.2a. The more complex situation is the quenched average which requires, e.g., the average of the free energy.

One of the standard approaches for random systems is to proceed through the evaluation of the quenched free energy using the replica trick \[1\]

\[
\langle \ln Z \rangle = \lim_{n \to \infty} \frac{\langle Z^n \rangle - 1}{n}
\]

which requires evaluation of \( \langle Z^n \rangle \). The importance of these moments can be realized through the following expansion

\[
\langle \ln Z \rangle = \ln \langle Z \rangle + \sum_{n=2}^{\infty} n^{-1} \langle [Z/\langle Z \rangle - 1]^n \rangle.
\]

Such an expansion makes sense if and only if the various cumulants of the partition function, with respect to the disorder distribution, do not grow too rapidly with \( n \). In such a case there will not be much qualitative difference between the quenched and annealed cases. This, in turn, suggests that to look into the possible differences one can study the various moments of the partition function. In addition, the moments can be looked upon as the characteristic function for the probability distribution of \( \ln Z \). \[27\] They are, therefore, of interest in themselves. \[28\] This is the approach we take in this paper. Our analysis is not
yet enough for the analytic continuation in $n$ to $0 \leq n \leq 1$ regime, as one would need for the free energy. This is not a deterrence as important information can be gathered even from the integral moments.

Instead of evaluating the quenched free energy, we systematically study the behavior of $\langle Z \rangle$, the second and the third cumulants. The first moment describes the behavior of an annealed system while the higher cumulants would show the nature of fluctuations. For each case, we do the averaging exactly before the configuration sum (or “path integrals”) to define an effective hamiltonian for that particular cumulant. This “pure” effective hamiltonian is then treated by perturbative renormalization. The effect of disorder is felt through the generation of new terms in the effective hamiltonian. The RG analysis helps in examining the flow of these terms as the lengthscale is increased, thereby, showing the marginal relevance or irrelevance of the disorder.

In section III, $\langle Z \rangle$ is discussed, while $\langle Z^2 \rangle$ and $\langle Z^3 \rangle$ are done in section IV and V. Though the derivation of effective hamiltonians precedes perturbative analysis in sections III, IV and V, it is instructive to start with the original hamiltonian, do the perturbation analysis and then do a term by term disorder averaging. Such a procedure not only shows how the new terms are generated but also acts as a cross check. This is discussed in Appendix A. Appendix B discusses many of the details needed in section IV. The random multicritical case is discussed in section VI. The annealed case of three and four chains is discussed in section VII. The paper ends with a discussion and summary in section VIII.

III. $\langle Z \rangle$

We show in this section that the annealed case can be reduced to a pure problem. We add that this reduction is special for two chains. Had we started with more than two chains, say three or four, with the same random pairwise interaction as in Eq. 2.1, the annealed case would be completely different from the corresponding pure case, and would have a much richer structure. A particular case is discussed in section VII.
The partition function, in the continuum approach, for a system of two chains, given by the hamiltonian of Eq. 2.1, is

\[ Z = \int Dr_1 \, Dr_2 \, \exp(-H) \]

where \( \int Dr_1 Dr_2 \) stands for the sum over all configurations of the two chains. A straightforward averaging of \( Z \) using the probability distribution of Eq. 2.2a defines an effective hamiltonian \( \mathcal{H}_{\text{eff}} \) such that

\[ \langle Z \rangle = \int Dr_1 \, Dr_2 \, \exp(-\mathcal{H}_{\text{eff}}), \tag{3.1} \]

and it is given by

\[ \mathcal{H}_{\text{eff}} = \frac{1}{2} \int_0^N dz \sum_{i=1}^2 \left( \frac{\partial r_i(z)}{\partial z} \right)^2 + v_0 \int_0^N dz \, V(r_{12}(z)) - \frac{v_0^2 \Delta}{2} \int_0^N dz \, V'(r_{12}(z)). \tag{3.2} \]

It appears from the above expression of the effective hamiltonian that an attraction is generated between the two chains. We find it instructive to follow another approach of perturbation expansion of the interaction term starting with the original hamiltonian 2.1. This helps us in visualizing the origin of the disorder induced attraction. This is done in Appendix A.

Now, since any short range potential under renormalization maps onto a \( \delta \) function potential, we can take the “minimal” effective hamiltonian for \( \langle Z \rangle \) as

\[ \mathcal{H}_2 = \frac{1}{2} \int_0^N dz \left[ \left( \frac{\partial r_1(z)}{\partial z} \right)^2 + \left( \frac{\partial r_2(z)}{\partial z} \right)^2 \right] + \bar{v}_0 \int_0^N dz \, \delta(r_{12}(z)). \tag{3.3} \]

where \( \bar{v}_0 \) is the reduced coupling constant which takes care of the attraction described earlier. We believe that the large length scale properties as described by Eq. 3.3 is same as that of Eq. 3.2. If necessary, we can restrict the strength of the disorder so that \( \bar{v}_0 \), which represents the effective coupling between the two chains, is positive (i.e. repulsive interaction). Now the problem reduces to a relatively simple situation where the two chains interact with a pure \( \delta \)-function interaction with a reduced coupling constant \( \bar{v}_0 \). The solution of this pure problem is known, and is used below. [17,12]
For the sake of completeness we quote the relevant results from Ref. [17]. The perturbation series for the connected part of the annealed partition function \( \langle Z \rangle_c \) to all orders in \( \bar{v}_0 \) is

\[
\langle Z \rangle_c = N \mathcal{V} \bar{v}_0 \left[ 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\bar{v}_0^n}{(4\pi)^{nd/2}} N^{n\epsilon'} \frac{\Gamma^n(\epsilon')}{\Gamma(2 + n\epsilon')} \right],
\]

where \( \mathcal{V} \) is the \( d \)-dimensional transverse volume, and \( \Gamma(\bullet) \) is the standard gamma function. The exact \( \beta \)-function for the renormalized coupling constant \( u \) (with \( u_0 = \bar{v}_0 L^{2-d} \) as the bare dimensionless coupling constant)

\[
\beta(u) \equiv L \frac{\partial u}{\partial L} = 2\epsilon' u \left( 1 - \frac{u}{4\pi\epsilon'} \right).
\]

Note, here \( 2\epsilon' = (2 - d) \) replaces \( \epsilon \) of Ref [17] to avoid later conflict of notation.

The flow diagram for the dimensionless coupling constant \( u \) is shown in Fig. 1. The fact that for \( d < 2 \) any small attractive interaction is able to form a bound state is reflected by the flow to the nonperturbative regime for any negative \( u \). The repulsive or the positive \( u \) region is dominated by the stable fixed point \( u^*= (4\pi\epsilon') \). For \( d > 2 \) there exists a nontrivial unstable fixed point \( u = u^* \) which separates the bound and the unbound states for the two polymers. In short, the unstable fixed point represents the critical point for the binding-unbinding transition. The exponents are known and can be found in Ref. [32]. For example for \( 1 \leq d < 4 \), the length scale exponent is \( 1/| \epsilon' | \). [32]

**IV. \( \langle Z^2 \rangle \)**

The evaluation of \( \langle Z^2 \rangle \) closely parallels that of the previous section. However, unlike the \( \langle Z \rangle \) case, new terms are generated here in the effective hamiltonian. An RG analysis is done to get a detailed account of the effects of these new terms.

**A. Effective hamiltonian**

The averaging for \( \langle Z^2 \rangle \) with the hamiltonian in Eq. [2.1] needs completion of the perfect square associated with \( b(z) \). As in the replica analysis [1] where one needs \( n \) replicas
(“copies”) of the original system in evaluating $\langle Z^n \rangle$, we require four chains for $\langle Z^2 \rangle$, a pair $\{3,4\}$ as a replica of the original pair of chains $\{1,2\}$. Therefore, we write, restricting ourselves to $\delta$-function potentials,

$$\langle Z^2 \rangle = \int \prod_{i=1}^4 Dr_i \exp(-\mathcal{H}_{2,2}),$$

(4.1a)

where

$$\mathcal{H}_{2,2} = H_0 + H_1 + H_2,$$

(4.1b)

with

$$H_0 = \frac{1}{2} \int_0^N dz \sum_{i=1}^4 \left( \frac{\partial r_i(z)}{\partial z} \right)^2$$

(4.1c)

denoting the four chain free part, and

$$H_1 = \bar{v}_0 \int_0^N dz \left[ \delta(r_{12}(z)) + \delta(r_{34}(z)) \right]$$

(4.1d)

and

$$H_2 = -\bar{r}_0 \int_0^N dz \delta(r_{12}(z)) \delta(r_{34}(z))$$

(4.1e)

representing the interactions among the chains, with $\bar{r}_0 = \nu_0^2 \Delta$.

In Eq. 4.1b, $H_1$ denotes the repulsive interaction between the chains of a particular pair (“intra replica”, $\{12\}$ and $\{34\}$, no cross coupling) at the same chain length, and is identical to the interaction term used for $\langle Z \rangle$ as discussed in section III and Appendix A. The other term $H_2$ couples the two pairs of chains $\{12\}$ and $\{34\}$ (“inter replica” term), and is the crucial term for our analysis. Even though this is a four chain interaction, it is distinct from the multicritical type interaction of Eq. 2.3. It cannot be interpreted directly as a conventional interaction between the two pairs. Rather, there is a lowering of “energy” of the system if the partners of each pair $\{12\}$ and $\{34\}$ meet simultaneously at the same chain length but not necessarily at the same point in space. This can also be interpreted as a special correlation so that an encounter of $\{12\}$ at a chain length $z$ favors an encounter
for \( \{34\} \) right at the same length \( z \). A tendency to achieve this kind of configurations leads to all the nontrivial effects of the disorder.

The coupling constant of \( H_2, \bar{r}_0 \) in Eq. \( 4.1d \) appears to be similar to that of the attractive interaction which is present in \( H_1 \), Eqs. \( 4.1d \) and \( 3.2 \) but they require separate treatments in the RG analysis (see below). As discussed in Appendix A, the term proportional to \( v^2_0 \Delta \) in \( \bar{r}_0 \) is reduced by a cutoff volume factor \( \Omega \) needed to define the \( \delta^2 \) term properly. Because of this reduction \( v^2_0 \Delta / \Omega \) differs from \( \bar{r}_0 \) in dimensionality and matches properly with \( v_0 \), the coupling constant of the starting \( \delta \) function interaction of the two chains.

The standard dimensional analysis for dimensionless hamiltonian shows that \([v_0]\) = \( L^{d-2} \), \([\Delta]\) = \( L^2 \) and hence \([v^2_0 \Delta]\) = \( L^{2d-2} \) where \( L \) has the dimension of length. Therefore, the upper critical dimension for \( \bar{r}_0 \) is \( d = 1 \) which also appears as a special dimension through the divergences in the \( \epsilon(= 1 - d) \)-expansion to be discussed below. From this simple dimensional analysis it also follows that the coupling in \( H_2 \) differs from that of \( H_1 \), as already mentioned. The special dimensionality \( d = 1 \) which is associated with \( H_2 \) is more important in the context of fluctuations in the partition function. In the absence of this term there is no special effect of disorder, which, in turn, also means that the quenched and annealed free energies would become equal.

**B. PERTURBATION SERIES**

To study the effect of \( H_2 \), we develop a perturbation series for \( \langle Z^2 \rangle \) in \( \bar{r}_0 \). The divergences that appear are absorbed by renormalization through an \( \epsilon \)-expansion. We show that this renormalization can be carried out exactly to all orders. For simplicity, this is done first for the \( \bar{v}_0 = 0 \) case. We then show that these divergences at \( d = 1 \), arising only due to \( H_2 \), remain unchanged even if we include \( H_1 \) i.e when \( \bar{v}_0 \neq 0 \). The \( \beta \) function evaluated exactly to all orders in perturbation series and other essential features are identical for both \( \bar{v}_0 = 0 \) and \( \bar{v}_0 \neq 0 \) cases.
Let us consider first, for simplicity, the case when $\bar{v}_0 = 0$. This means that there is no mutual two chain interaction. We consider only the connected part $\langle Z^2 \rangle_c = \langle Z^2 \rangle - \langle Z \rangle^2$, the second cumulant of the partition function. As in the previous section, the calculation can be done in the real (chain) space. But, at this point, we prefer the Laplace space (Laplace transform with respect to the chain length) because it is advantageous for later considerations especially with $\bar{v}_0 \neq 0$. We define

$$Z = \int_0^\infty dN e^{-sN} \langle Z^2 \rangle_c$$

the Laplace transform of $\langle Z^2 \rangle_c$ with respect to the chainlength $N$, the Laplace conjugate variable being $s$.

The loops in the perturbation expansion are shown in Fig 2a up to third order in the interaction. The individual pairs of chains are represented by thick lines. The horizontal wiggly lines in these diagrams stand for $\vec{r}_0$. Such a representation is possible because the $\delta$ function in $H_2$, Eq. 4.1g, forces the members of a pair to have the same $r, z$ coordinates. Each chain is described by the free distribution (“propagator”) $G(r_f - r_i \mid z_f - z_i) = (4\pi)^{-d/2} \frac{1}{2\pi (z_f - z_i)^{d/2}} \exp[-(r_f - r_i)^2/2(z_f - z_i)]$ with end points $(r_f, z_f)$ and $(r_i, z_i)$. Two chains are therefore described by

$$G^2(r \mid z) = (4\pi z)^{-d/2} G(r \mid z/2).$$

This $G^2$ is the propagator for the thick lines. At each wiggly line, connecting four chains (all four having the same chainlength $z$), there are two integrations over the spatial coordinates of the two separate pairs of chains (thick lines). The loops formed out of the wiggly lines are only responsible for the divergences at $d = 1$.

In order to trace the algebraic origin of the singularity, note that, by very nature of the interaction, the spatial integrations associated with the two thick lines are independent of each other. Each section of the thick lines, with $z_1, z_2$ as the end points, in a loop formed with
the wiggly lines, contributes \((z_1 - z_2)^{-d/2}\) from the identity in Eq. 4.3. Since the interaction demands same \(z\) for the two thick lines, the \(z\) integrals involve \((z_1 - z_2)^{-d}\) type factors whose Laplace transform would contribute \(\Gamma(1-d)\) with pole at \(d = 1\). The two independent spatial coordinates which are left out after the successive use of the normalization \(\int dr \, G(r | z) = 1\), lead to a \(V^2\) factor for each diagram. The convolution nature of the \(z\) integrals, thanks to the time ordering, leads to a simple product of the individual Laplace transforms of the integrands, resulting in a geometric series for \(Z\). The details of the evaluation of a few diagrams are given in Appendix A.

Defining the dimensionless coupling constant \(r_0\) through an arbitrary length scale \(L\) as \(r_0 = \bar{v}_0 L^2 (4\pi)^{-d}, \epsilon = 1 - d\), we write the series for \(Z\) to all orders in \(r_0\) as

\[
Z |_{\bar{v}_0 = 0} = (4\pi)^d V^2 s^{-2} L^{-2 \epsilon} [r_0 + \sum_{n=1}^{\infty} r_0^{n+1} (sL^2)^{-n \epsilon} \Gamma^n(\epsilon)].
\]

(4.4)

It is clear from the above expression that there is a divergence at \(d = 1\) at each order (\(> 1\)). This is tackled by renormalization below.

2. \(\bar{v}_0 \neq 0\)

In the above analysis, we have taken \(\bar{v}_0 = 0\). We now include \(\bar{v}_0\) and show that the singularity structure around \(d = 1\), as in Eq. 4.4, remains unaffected.

In this case there are both intra-pair and inter-pair interactions. The intra pair interactions, i.e., the mutual short range \(\delta\)-function interactions among the members of the pairs are represented by dots on the thick lines. The basic idea of the procedure adopted is to show that the dots can be absorbed by dressing the “propagators”. The original propagators (thick lines of \(\bar{v}_0 = 0\) case) \(G^2\) is modified by the dots but not trivially. The “dressing” factor depends on whether the chains are open or tied at the ends. One therefore needs two types of dressed propagators, \(G_M\) for the thick lines in the loops, and \(G_O\) for the same in the outer legs not involved in the loops. (See Fig 2 c and d.) Since these involve only two chain interactions, the singularities are at \(d = 2\) of the type \(\Gamma(1-d/2)\) as in Eq. 3.4. These
dressed propagators are to be used, as appropriate, in the skeleton diagrams of the $\bar{v}_0 = 0$ case without the dots (see Fig. 2a). We just quote the forms of these propagators below - the details can be found in Appendix B.

For the one in which the two participating members of a thick line are tied together at both the ends $[(0,0) and (r, z)]$ of the line, the form of the dressed propagator with $n$ meetings (dots) is [Fig. 2d]

$$G_M^{(n)}(r \mid z) = (-\bar{v}_0)^n (4\pi)^{-d/2} \frac{\Gamma^{n+1}(\epsilon')}{\Gamma((n+1)\epsilon')} \frac{1}{z^{(n+1)\epsilon' - 1}} G(r \mid z/2). \quad (4.5)$$

There is translational invariance in both $r$ and $z$. With $\epsilon' = (2 - d)/2$, this form surely reduces to Eq. 4.3 for $n = 0$. Similarly, for the other type in which the two members of a thick line are tied together only at one of the ends, integrations for the open end coordinates need to be done. The resulting dressed propagator for chains of length $z$ with $n$ intermediate dots (Fig 2c) has the following form

$$G_O^{(n)}(z) = \left[ -z^{\epsilon'} \bar{v}_0 \Gamma(\epsilon') \right]^n \frac{1}{\Gamma(1 + n\epsilon')(4\pi)^{nd/2}}. \quad (4.6)$$

With $n = 0$, $G_O^{(n)}(z) = 1$, as it should be, by the normalization of the distribution function $G(r \mid z)$. Also $G_O^{(n)}(z)$ has no space dependence.

In a diagram of a particular order in $\bar{r}_0$, the thick lines can have arbitrary order $n$ in $\bar{v}_0$ (i.e. arbitrary number of dots). All such diagrams differing only in orders of $\bar{v}_0$ are combined together by summing over $n$. The full dressed propagators are

$$\bar{G}_M(r \mid z) = \sum_n G_M^{(n)}(r \mid z), \quad \text{and} \quad G_O(z) = \sum_n G_O^{(n)}(z). \quad (4.7)$$

Unlike the $\bar{v}_0 = 0$ situation, these two propagators replace the inner and the outer thick lines respectively. The subsequent procedure is almost similar to the previous case, including the origin of the $V^2$ factor, and the use of the convolution theorem in the Laplace space. The series for $Z$ is given by

$$Z \mid_{\bar{v}_0 \neq 0} = V^2 G_O(s) \bar{r}_0 \left[ 1 + \sum_{n=1}^{\infty} (\bar{r}_0 G_M(s))^n \right] G_O(s) \quad (4.8)$$
where \( G_{O}(s) = \mathcal{L}G_{O}^{2}(z) \), \( G_{M}(s) = \mathcal{L}G_{M}^{2}(z) \), (\( \mathcal{L} \) being the Laplace transform with respect to \( z \)) and \( G_{M}(z) = \int dr \tilde{G}_{M}(r \mid z) \). The “same \( z \)” requirement of the \( \delta \) function of \( H_{2} \) combines the propagators of the thick lines. Hence \( G_{O}^{2} \) and \( G_{M}^{2} \). The two outer pairs of legs contribute the two \( G_{0}(s) \) factors. In terms of the dimensionless coupling constant \( r_{0} \), and \( \bar{u}_{0} = \bar{v}_{0}(4\pi)^{-d/2} \Gamma(2\epsilon')s^{-\epsilon'} \), the above expression becomes

\[
Z \bigg|_{\bar{v}_{0} \neq 0} = (4\pi)^{d}v^{2}s^{-2L^{-2\epsilon}}S_{O}(s) \left[ r_{0} + \sum_{n=1}^{\infty} r_{0}^{n+1}(sL^{2})^{-n}(S_{M}(s))^{n} \right] S_{O}(s) \tag{4.9}
\]

where

\[
S_{0}(s) = \sum_{\{n\}} \frac{\Gamma[(n_{1} + n_{2})\epsilon' + 1]}{\Gamma(1 + n_{1}\epsilon')\Gamma(1 + n_{2}\epsilon')}(-\bar{u}_{0})^{n_{1} + n_{2}}, \tag{4.10}
\]

and

\[
S_{M}(s) = \sum_{\{n\}} \frac{\Gamma[(n_{1} + n_{2})\epsilon' + 1 - d]}{\Gamma((n_{1} + 1)\epsilon')\Gamma((n_{2} + 1)\epsilon')}(-\bar{u}_{0})^{n_{1} + n_{2}}. \tag{4.11}
\]

Details can be found in Appendix B. The reason it is written in the above form is that \( S_{0} \) and \( S_{M} \) start with 1 for \( n_{1} = n_{2} = 0 \), to agree with Eq. \( \text{4.4} \). It follows that the leading divergences at \( d = 1 \) in each order of Eq. \( \text{4.9} \) come from the \( n_{1} = n_{2} = \ldots = 0 \) term of Eq. \( \text{4.10} \) and Eq. \( \text{4.11} \).

C. Renormalization and New Criticality

The divergences at \( \epsilon = 0 \) in the series of Eq. \( \text{4.9} \) can be absorbed by the standard renormalization procedure. In general, a renormalization through minimal subtraction would require absorption of the poles in \( \epsilon \) through

\[
r_{0} = r(1 + a_{1}r + a_{2}r^{2} + \ldots) \tag{4.12}
\]

with \( a_{n} = \sum_{p=1}^{n} a_{n,p} \epsilon^{-p} \) and \( r \) as the renormalized coupling constant. In such a scheme, \( a_{n,p} (p \neq n) \) terms are required to take care of the subleading divergences.

The formal similarity of the leading pole structure of Eqs \( \text{4.4} \) and \( \text{4.9} \) with that of Eq. \( \text{3.4} \) enables us to follow Ref. \( \text{17,12} \) yielding \( a_{p} = (-\epsilon)^{-p} \). The geometric series of Eqs.
4.9 and 4.4 guarantees that the removal of the leading poles is sufficient to remove the subleading ones. The presence of the dots (Fig 2b) through \( n_1, n_2 \neq 0 \) in Eq. 4.9 is felt through the changes in the subleading divergences. This does not pose a problem and can, indeed, be checked explicitly. Note that \( S_M \) of Eq. 4.11 has an expansion of the form

\[
S_M = \frac{1}{\epsilon} + A_0 + \sum_{p=1} A_p \epsilon^p. \tag{4.13}
\]

Taking \( a_p = (-\epsilon)^{-p} \), as needed to remove the leading poles, one can verify explicitly that all the poles are removed order by order, and the result does not depend on the explicit values of \( A_0, A_1 \) etc.

The \( \beta \) function is therefore exact to all orders in perturbation series and is given by

\[
\beta(r) \equiv L \frac{\partial r}{\partial L} = 2(\epsilon r + r^2). \tag{4.14}
\]

There are two fixed points: (i) \( r = 0 \) and (ii) \( r^* = -\epsilon \). The flows are shown in Fig 3. The bare coupling constant \( r_0 \) which originates from \( v_0^2 \Delta \), where \( \Delta \), the variance of the distribution, is strictly positive, requires a positive \( r \). Therefore, the nontrivial fixed point for \( d < 1 \) in negative \( r \) is unphysical. It however moves to the physical domain for \( d > 1 \). See Fig. 3c.

Exactly at \( d = 1, \epsilon = 0 \), \( r \) grows with length \( L \) as

\[
r(L) = r(0) \left[ 1 + 2r(0) \ln \frac{L_0}{L} \right]^{-1}, \tag{4.15}
\]

\( r(0) \) being the coupling at length \( L_0 \). Hence, the disorder is marginally relevant, in agreement with Ref [24]. For \( d > 1 \), there exists an unstable nontrivial fixed point at \( r = |\epsilon| \) which separates two distinct regimes of disorder. If we start with a strong enough disorder, on the right side of the fixed point, it increases with length scale, going beyond the perturbative regime. This is the strong disorder phase. On the other hand, the left side of the fixed point is the weak disorder regime, since \( r \) flows to zero (the stable fixed point). The unstable fixed point, therefore, represents a critical point - a novel phase transition induced by the disorder.
One way of achieving the abovementioned critical behavior is to change the strength of the disorder by controlling the temperature. The “strong disorder” phase \((\ln Z) \neq \ln \langle Z \rangle\) would correspond to the low temperature phase while the “weak disorder” phase \((\ln Z) \approx \ln \langle Z \rangle\) is the high temperature one. The details of the critical behavior can be obtained by integrating the \(\beta\) function,

\[
r = |\epsilon| \left[ 1 - \frac{r(0)-|\epsilon|}{r(0)} \left( \frac{L}{L_0} \right)^{2|\epsilon|} \right]^{-1}.
\] (4.16)

For a small starting deviation \(\Delta T \equiv T - T_c = r(0)-|\epsilon|\), there is a lengthscale \(L \sim (\Delta T)^{-1/2|\epsilon|}\) at which \(r\) in Eq. 4.16 diverges. This we can identify as a lengthscale \(\xi\) associated with the critical point with the lengthscale exponent

\[
\nu = (2 |\epsilon|)^{-1}.
\] (4.17)

The divergence at \(\epsilon = 0\) is consistent with the essential singularity that follows from Eq. 4.15

\[
\xi \sim \exp[1/(2\Delta T)].
\] (4.18)

A complete description of the critical point would involve an evaluation of various macroscopic or thermodynamic properties. These would require a replica type analysis. It is tempting to believe that the correlation induced by \(H_2\) in Eq. 4.16 in the replica space distinguishes the two phases. We wish to come back to such replica analysis elsewhere.

\[\textbf{V. \langle Z^3 \rangle}\]

The evaluation of \(\langle Z^3 \rangle\) leads to a six chain problem where, as before, an interaction involving four chains is generated which is attractive in nature. The effective Hamiltonian, apart from the free part for six chains and mutual \(\delta\) function interaction, contains the following attractive terms (see Eq. 4.1b)

\[
-\bar{r}_0 \int_0^N dz \left[ \delta(r_{12}(z)) \delta(r_{34}(z)) + \delta(r_{34}(z)) \delta(r_{56}(z)) + \delta(r_{12}(z)) \delta(r_{56}(z)) \right].
\]
Instead of \( \langle Z^3 \rangle \), we analyze the third cumulant \( \langle Z^3 \rangle_c \) involving only six chain connected diagrams. Fig 4 shows such diagrams upto fourth order in \( \bar{r}_0 \). Contributions of these diagrams can be found out following the rules discussed in the context of \( \langle Z^2 \rangle \) in section IV. As an example, we give an explicit evaluation of Fig 4c which is

\[
\bar{r}_0^4 \int_{\{r, r', r''\}} N d\mathbf{z}_1 \int_0^{z_1} d\mathbf{z}_2 \int_0^{z_2} d\mathbf{z}_3 \int_0^{z_3} d\mathbf{z}_4 \, G_2(r_{12} | z_{12}) G_2(r_{12}' | z_{12}) \times G_2(r_{23}' | z_{23}) G_2(r_{34}' | z_{34}) G_2(r_{34}'' | z_{34})
\]

In the above equation \( r, r', r'' \) with appropriate subscripts denote the set of \( d \)-dimensional coordinates for the three thick lines and \( \int_{\{r, r', r''\}} \) corresponds to the integrations over all spatial coordinates. As before, each thick line between two end points \((r_i, z_i)\) and \((r_j, z_j)\) is represented by \( G_2(r_{ij} | z_{ij}) \) with \( z_{ij} = z_i - z_j \). The spatial integrations simplifies the above expression to

\[
\bar{r}_0^4 (4\pi)^{-5d/2} V^3 \int_{\{r, r', r''\}} N d\mathbf{z}_1 \int_0^{z_1} d\mathbf{z}_2 \int_0^{z_2} d\mathbf{z}_3 \int_0^{z_3} d\mathbf{z}_4 \, z_{12}^{-d} z_{23}^{-d} z_{34}^{-d} \Gamma(1-d) \Gamma(1-d/2) N^{4-5d/2} \Gamma(5-5d/2).
\]

The series for the Laplace transform of \( \langle Z^3 \rangle_c \) is given by

\[
[\mathcal{L}(\langle Z^3 \rangle_c) |_{\bar{v}_0 = 0}] = V^3 (4\pi)^{3d/2} s^{-1-3d/2} \Gamma(\epsilon') \left[ 2r_0^2 (sL^2)^{-2\epsilon} + 4r_0^3 (sL^2)^{-3\epsilon} \Gamma(\epsilon) + 6r_0^4 (sL^2)^{-4\epsilon} \Gamma^2(\epsilon) + \ldots \right]
\]

where, as before, \( s \) is the Laplace conjugate to the chainlength \( N \) and \( r_0 \) is the dimensionless coupling constant as defined before Eq. 4.4. This series requires the standard renormalization procedure for removal of divergence at \( d = 1 \). Defining the renormalized \( r \) via

\[
r_0 = r(1 + a_1 r + a_2 r^2 + \ldots)
\]

it is found that

\[
a_p = ( - 1/\epsilon)^p
\]
absorbs the divergence at \( d = 1 \).

It is interesting to note that at a particular order there are diagrams which are similar by a mere permutation of the interaction lines, i.e., by a different time ordering. For example, Fig. 4b shows four diagrams related by permutations, in the third order of the perturbation series. All of these have the same value, and hence the factor of 4 in the \( r_0^3 \) term in Eq. 5.2. These permutation factors collaborate with powers of \( r_0 \) in such a way that \( a_p \)'s are just the same as those for the \( \langle Z^2 \rangle \) case. There are also diagrams in the third and higher orders (a few shown in Fig. 4d) which correspond to subleading divergences, the removal of which will be automatic by their corresponding higher orders.

We, therefore, see that the \( \beta \) function for \( r \) has exactly the identical form as that in Eq. 4.14 for \( \langle Z^2 \rangle_c \) and all the features follow identically. This shows that the phase transition for \( \langle Z^3 \rangle_c \) has the same nature as for the \( \langle Z^2 \rangle \) case. To be more explicit, there exists a transition temperature for \( d > 1 \) which separates the weak disorder and strong disorder phase for every moment. In the high temperature phase \( \langle Z^3 \rangle \sim \langle Z \rangle^3 \) and for \( T < T_c \), i.e., in the fluctuation dominated phase, \( \langle Z^3 \rangle \) differs from \( \langle Z \rangle^3 \). This transition temperature is the same for \( \langle Z^3 \rangle \) and \( \langle Z^2 \rangle \).

It is now a trivial exercise to extend this for higher moments. The effective hamiltonian apart from the free part for \( 2n \) chains and mutual \( \delta \) function interaction involves the following attractive interaction

\[-r_0 \sum_{i < j} \int_0^N dz \, \delta(r_{2i-1} 2i(z)) \, \delta(r_{2j-1} 2j(z))\]

Since no new interaction is generated, the \( \beta \) function remains the same.

VI. RANDOM MULTICRITICAL CASE

In the previous sections, attention was focussed on the two body interaction case. It is known that DPs with pure \( m \)-body interaction can also be completely solved. We now investigate the random version of this multicritical case as given by the hamiltonian of
As before, we want to evaluate \( \langle Z_m \rangle \) and \( \langle Z_m^2 \rangle \). The procedure follows the footsteps of the two chain problem, and, therefore, details are skipped.

### A. \( \langle Z_m \rangle \)

To compute \( \langle Z_m \rangle \), we can perform an averaging over \( b(z) \) to obtain, as in section II, an \( m \)-chain hamiltonian with a pure \( m \)-body interaction. The grand universality known for the pure system indicates that the multicritical exponents for the binding - unbinding transition will be similar to those of Ref [32]. For example, for \( d > 2/(m-1) \), the lengthscale exponent would be \( 2/| \epsilon'_m | \), where \( \epsilon'_m = 2 - (m-1)d \).

### B. \( \langle Z_m^2 \rangle \)

A little calculation involving the completion of the square would convince the reader that the effective hamiltonian needed for the second moment would involve \( 2m \) chains in sets of \( m \). It is given by

\[
H_{m,m} = \frac{1}{2} \int_0^N dz \sum_{i=1}^{2m} \left( \frac{\partial r_i(z)}{\partial z} \right)^2 + \bar{v}_m \int_0^N dz \prod_{p=1}^{m-1} \delta[r_{p+1}(z)] + \\
\bar{v}_m \int_0^N dz \prod_{q=m+1}^{2m-1} \delta[r_{q+1}(z)] - \bar{r}_m \int_0^N dz \prod_{p=1}^{m-1} \delta[r_{p+1}(z)] \prod_{q=m+1}^{2m-1} \delta[r_{q+1}(z)]
\]

where \( \bar{r}_m = v_m^2 \Delta \) and the two sets are represented by \( p \)'s and \( q \)'s. The special feature is the last term that involves the peculiar \( m \) chain - \( m \) chain interaction. This generalizes \( H_2 \) of the two chain case of Eq. [11b] and [4.1d]. The effect of disorder, so far as the fluctuations are concerned, is to introduce a correlation that if \( m \) chains meet at a \( z \), the replica would also like to enjoy a meeting at that same \( z \).

The upper critical dimension of \( r_m \) follows from the dimensional analysis as \( d_m = 1/(m-1) \), which is half of the upper critical dimension for the pure case \( 2/(m-1) \) for \( \bar{v}_m \). We are not sure whether this systematic reduction by a factor of 2 has any deeper significance.
For simplicity we choose $\bar{v}_m = 0$. The perturbation expansion in $\bar{r}_m$ would involve the same sets of diagrams as in Fig. 2a except that the propagator now for the thick lines is

$$G^m(r \mid z) = (2\pi z)^{-(m-1)d/2}m^{-d/2}G(r \mid z/m).$$

With this propagator, the full series can be computed. It is transparent to see the occurrence of divergences at $d = d_m$. The whole RG procedure of section IV can be copied in toto by replacing $\epsilon$ by $\epsilon_m = 1 - d(m - 1)$. Hence in the multicritical situation we also expect to see a disorder-induced phase transition. The length scale exponent $\nu_m = (2 | \epsilon_m |)^{-1}$, with an essential singularity for $d = d_m$ as in Eq. 4.18.

### VII. SYSTEM WITH MORE THAN TWO CHAINS

Annealed averaging for the system with two chains with random interaction is simpler and not sufficient to give enough information about the effects of disorder. On the other hand, if the above case can be extended to four chains having twobody interaction among each other, even the annealed case turns out to be extremely nontrivial. The Hamiltonian for the four chain system,

$$H = \frac{1}{2} \int_0^N dz \sum_{i=1}^4 \left( \frac{\partial r_i(z)}{\partial z} \right)^2 + \int_0^N dz v_0 (1 + b(z)) \sum_{i,j=1}^4 \delta(r_{ij}(z))$$

where $r_{ij} = r_i(z) - r_j(z)$, after averaging, using the Gaussian distribution of $b(z)$, gives the following effective Hamiltonian

$$H_{\text{eff}} = \frac{1}{2} \int_0^N dz \sum_{i=1}^4 \left( \frac{\partial r_i(z)}{\partial z} \right)^2 + \left( \frac{\partial r_i(z)}{\partial z} \right)^2 + v_0 \int_0^N dz \sum_{i,j=1}^4 \delta(r_{ij}(z)) - 2v_0^2 \Delta \int_0^N dz \sum_{i,j,k,l} \delta(r_{ij}) \delta(r_{jk}) - 2v_0^2 \Delta \int_0^N dz \sum_{i,j,k,l} \delta(r_{ij}) \delta(r_{kl}).$$

The remarkable feature of the effective Hamiltonian is that there are two different kinds of attractive interaction one of which involves three chains with a multicritical type interaction (Eq. 2.3) while the other one coupling four chains together, as in the quenched problem.
If we take a three chain system, the corresponding effective hamiltonian will involve only the three chain term but no four chain interaction of Eq. (7.2). This term was absent in the original three chain hamiltonian. There is now the possibility of a disorder induced multicritical behavior, though of pure type. [26, 18]

The four chain attractive interaction is marginal at $d = 1$ and so is the three chain interaction. The presence of these two marginal operators is, in general, expected to complicate the renormalization procedure through their interdependence - but here that does not happen.

The perturbation expansion with the three body and the four body interactions leads to three different kinds of diagrams. See Fig 5. The series corresponding to the pure three body interaction is already solved. [26, 18] The series in the Laplace space involving four body interactions (see Fig 5a-c), which contributes to the leading divergence is identical to the series for $\langle Z^2 \rangle_c$ in Eq. (4.4) (Fig. 2a). The diagrams with mixed three body and four body interactions are shown in Fig 5d,e. In the final series, upto the order shown in Fig 5, the four body and three body contributions get separated into two factors. This shows that the resulting renormalization of the two couplings are independent of each other. Because of the four body interaction, we expect a disorder induced criticality as for the two chain quenched case, but here this happens for a real four chain system - no replica is involved. The details and the phase diagram will be published elsewhere.

**VIII. SUMMARY AND DISCUSSION**

We have proposed a random interaction model for two directed polymers and studied the first three cumulants of the partition function. We have shown that in the annealed case, described by $\langle Z \rangle$, there can be a disorder induced binding unbinding transition, very similar to a pure problem. The exponents are also identical to the pure case. We also pointed out certain peculiarities of the annealed problem involving three or four chains. The quenched problem is different as reflected through the marginal relevance of the disorder. For $d > 1$
there exists a critical point that demarcates a disorder dominated phase and a pure type phase. In the strong disorder phase, there seems to have an extra correlation in the replica space, which is absent in the other phase. The lengthscale exponent for the critical point is found to be \((2 - \epsilon)\) where \(\epsilon = 1 - d\). It has an exponential divergence at \(d = 1\). Similar results were obtained for \(\langle Z^3 \rangle\). In the replica approach, one needs \(\langle Z^n \rangle\) with \(n \to 0\) which does not require interactions other than those which are present in \(\langle Z^2 \rangle\) and \(\langle Z^3 \rangle\). Therefore the upper critical dimension will remain the same, namely \(d = 1\).

There are still many open problems, as for example, a replica analysis for this system. This requires an explicit expression for \(\langle Z^n \rangle\), correct at least for small \(n\). Such an analysis would provide vital information regarding the disorder induced critical point, including possible replica symmetry breaking. Is there any other length scale exponent for this new critical point, apart from the one we have calculated? What about other exponents? What, if any, is the upper critical dimension of this critical point? A thorough numerical study of this system will surely provide valuable insights.

**APPENDIX A: GENERATION OF ATTRACTION:**

**PERTURBATION ANALYSIS FOR \(\langle Z \rangle\) AND \(\langle Z^2 \rangle\)**

In this appendix, we show how the attractive terms in the effective hamiltonian for \(\langle Z \rangle\) and \(\langle Z^2 \rangle\) can be generated perturbatively.

We proceed to the evaluation of the average of the partition function \(\langle Z \rangle\) by a perturbation expansion using Eq. (2.1) with the replacement of the short range potential by a \(\delta\) function potential. Formally this leads to the expression for

\[
Z = \int Dr_1 Dr_2 \exp (-H_0) \left(1 - H_i + H_i^2 / 2! + \ldots\right)
\]

(A1)

where \(H_0\) corresponds to the free part of the two chains, and \(H_i\) represents the interaction part, \(\int_0^N dz \, v_0(1 + b(z)) \delta(r_{12}(z))\). A little manipulation after substituting the explicit form of \(H_i\) gives the connected partition function as
\[ Z_c = \int_0^N dz_1 \, v_0 (1 + b(z_1)) \int_{\{r,r'\}} G(r_N - r_1 \mid N - z_1) \, G(r_1 - r_0 \mid z_1) \times \]
\[ G(r'_N - r_1 \mid N - z_1) \, G(r_1 - r'_0 \mid z_1) + \int_0^N dz_1 \, f_{(r,r')}^2 \, v_0^3 (1 + b(z_1))(1 + b(z_2)) \times \]
\[ \int_{\{r,r'\}} G(r_N - r_1 \mid N - z_1) \, G(r_{12} \mid z_{12}) \, G(r_2 - r_0 \mid z_2) \times \]
\[ G(r'_N - r_1 \mid N - z_1)G(r_{12} \mid z_{12})G(r_2 - r'_0 \mid z_2) + ... \quad (A2) \]

where \( G(r \mid z) = (2\pi z)^{-(d/2)} \exp(-r^2/2z) \) is the distribution function for a \( d \)-dimensional Gaussian chain of chain length \( z \) and end-end distance vector \( r \) and \( \int_{\{r,r'\}} \) the integrations over all dummy spatial coordinates. The convention is to use \( r \) and \( r' \) for the two chains and \( z_{ij} = z_i - z_j \). The factorials in the denominators of the terms of Eq. (A1) are absorbed by introducing “time” \( (z) \) ordering, which restricts \( z_{i+1} \leq z_i \), in the integrals. Diagrams upto second order corresponding to the series in Eq. (A2) are shown in Fig 6. The familiar normalization \( \int dr G(r \mid z) = 1 \) and the integrations over the spatial end coordinates lead to much more simplification for \( Z_c \). The explicit form of \( Z_C \) upto two loop term is given by

\[ Z_C = \mathcal{V} \int_0^N dz_1 v_0(1 + b(z_1)) - \int_0^N dz_1 f_{(r)}^2 v_0^2 (1 + b(z_1))(1 + b(z_2)) \times \]
\[ G^2(r_{12} \mid z_{12}) + \int_0^N dz_1 f_{(r)}^3 v_0^3(1 + b(z_1))(1 + b(z_2)) \times \]
\[ (1 + b(z_3))G^2(r_{12} \mid z_{12})G^2(r_{23} \mid z_{23})... \quad (A3) \]

where \( \mathcal{V} \) is the transverse volume. The actual meaningful quantity \( \langle Z_c \rangle \) is computed from Eq. (A3) after averaging it with the distribution \( P(b) \) of Eq. (2.2a). This gaussian distribution with zero mean ensures that any term involving odd number of \( b(z) \)’s should vanish after averaging. Therefore, the contribution from the first order term, \( v_0N\mathcal{V} \), is only from the pure part. In the one loop level of Eq. (A3), there are two surviving terms after disorder averaging. One is the pure term which does not require any averaging and its contribution to \( \langle Z_c \rangle \) is

\[ -v_0^2 \mathcal{V}(4\pi)^{-d/2} \Gamma(1 - d/2) \Gamma(3 - d/2) N^2 -d/2. \quad (A4) \]

The other nonvanishing part which contains an even number of disorder interaction is
\[-v_0^2 \int_0^N dz_1 \int_0^{z_2} \int d\mathbf{r}_1 \ d\mathbf{r}_2 \ b(z_1) \ b(z_2) \ G^2(\mathbf{r}_{12} \mid z_{12}).\]

After averaging, the two points \(z_1\) and \(z_2\) along the chain merge together giving rise to a term

\[-\int_0^N dz_1 \ v_0^2 \ \Delta \delta^2(\mathbf{r}_1 - \mathbf{r}_2) \ d\mathbf{r}_1 \ d\mathbf{r}_2 \quad (A5)\]

where we used the fact that \(G(\mathbf{r} \mid 0) = \delta(\mathbf{r}).\) Because of this merging of the two points along the chain, this term contributes to the first order term but with a negative sign which shows the presence of a newly generated attraction. In other words, a second order term for a particular realization (before averaging), looks like an attractive first order term after DA.

Proceeding in the same fashion we can evaluate the two loop term of Eq. (A3). In the two loop part, the nonvanishing contributions are:

\[ (i) \quad v_0^3 \int_0^N dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \int_{\{r\}} G^2(\mathbf{r}_{12} \mid z_{12})G^2(\mathbf{r}_{23} \mid z_{23}). \]

This term involving only pure type interaction, after integration over spatial coordinates, gives

\[ \mathcal{V} v_0^3 (4\pi)^{-d} \int_0^N dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \ (z_1 - z_2)^{-d/2} (z_2 - z_3)^{-d/2} = \mathcal{V} v_0^3 (4\pi)^{-d} \frac{\Gamma^2(1 - d/2)}{\Gamma(4 - d/2)} N^{3 - d/2}. \quad (A6) \]

\[ (ii) v_0^3 \int_0^N dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \int_{\{r\}} b(z_1) b(z_2) \ G^2(\mathbf{r}_{12} \mid z_{12})G^2(\mathbf{r}_{23} \mid z_{23}). \]

As in the earlier case this reduces to a one loop term after averaging as

\[ v_0^3 \Delta (4\pi)^{-d/2} \int_0^N dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \int_{\{r\}} \delta^2(\mathbf{r}_{12}) (z_2 - z_3)^{-d/2}. \quad (A7) \]

(iii) The third nonvanishing contribution is from

\[ v_0^3 \int_0^N dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \int_{\{r\}} b(z_2) b(z_3) \ G^2(\mathbf{r}_{12} \mid z_{12})G^2(\mathbf{r}_{23} \mid z_{23}). \quad (A8) \]

This term after averaging becomes
\[ v_0^3 \Delta \frac{3}{(4\pi)^{d/2}} \int_0^N dz_1 \int_0^{z_1} dz_2 \int_r \delta^2(r_{23}) (z_1 - z_2)^{-d/2}. \tag{A9} \]

There is one more term involving \( b(z_1)b(z_3) \) which vanishes after averaging because of the specific time ordering which rules out the merging of \( z_1 \) and \( z_3 \). It can be easily be checked that this merging of two “random interaction” lines (wavy lines) into one single pure line and the subsequent reduction of order occurs at each order (> 1) involving consecutive pairs of even number of “random” lines. Thus an attraction is generated at each order very systematically. The coupling constant of this term is proportional to \( v_0^2 \Delta \). In some of the above expressions, \([A3, A7, A9]\] the presence of the \( \delta^2 \) term needs special attention since it is ill-defined even in the theory of distribution sense. \([29]\) We can avoid this problem by taking a spread out \( \delta \) function, and then taking the limit at the end. This would change the coupling to \( v_0^2 \Delta / \Omega \), where \( \Omega \) is the arbitrary “spread out” or cutoff volume. Since one gets back a single \( \delta \) function, it can be associated with the pure term, thereby changing the problem to a pure one with a reduced coupling constant \( \bar{v}_0 = v_0 - v_0^2 \Delta / \Omega \). See Eq. 3.2.

Another way to tackle this difficulty is to start with a short range potential \( V(r) \), and appeal to RG arguments as done in section 11.

From now on it’s time for \( \langle Z^2 \rangle \). A few diagrams for \( \langle Z^2 \rangle \) with \( \bar{v}_0 = 0 \) are shown in fig. 7a. In the first order, the only diagram which has nonzero contribution to \( \langle Z^2 \rangle_c \) is Fig. 7a1. This contribution after disorder averaging is \( V^2 v_0^2 \Delta N \). As was mentioned in the text this \( V^2 \) has come from the independent spatial integrals. Proceeding in the similar fashion, we write

\[
fig7a3 = v_0^4 \int_0^N dz_1 \int_0^{z_1} dz_2 b(z_1)b(z_2) \int_r G^2(r_{12} | z_{12}) \times \int_0^N dz'_1 \int_0^{z'_1} dz'_2 b(z'_1)b(z'_2) \int_{r',r''} G^2(r'_{12} | z'_{12}).
\]

In disorder averaging, the only relevant contribution comes from the pairing of \( b(z_1) b(z'_1) \) and \( b(z_2) b(z'_2) \). The other possibility in which \( b(z_1) b(z_2) \) and \( b(z'_1) b(z'_2) \) are paired up (Fig 7b2) is not considered here since this generates \( \bar{v}_0 \) type terms which are not to be included for the \( \bar{v}_0 = 0 \) case. After appropriate disorder averaging the above expression becomes

25
\[ v_0^4 \Delta^2 \nu^2 4\pi^d \Gamma(\epsilon) s^{-(2+\epsilon)}, \]  
(A10)

where \( \epsilon = 1 - d \).

To make the evaluation, after DA, easier we follow a different convention for the diagrams, Fig 7b and 2a. The thick line represents the two members of a pair jointly and is represented by \( G_2^2(r_1 - r_2 \mid z_1 - z_2) \) for the ends \( r_1 \) and \( r_2 \) at which the two chains are tied at lengths \( z_1 \) and \( z_2 \). For example, the diagram which corresponds to the last expression (Eq. A10) is given by Fig 7b3, which is also Fig 2a2. (Note \( \bar{r}_0 = v_0^2 \Delta \).)

The next diagram which is important in the next higher order is given in Fig 7a4, the contribution of which can equivalently be calculated from Fig 7b5 or 2a3 as

\[
(v_0^2 \Delta)^3 \int_0^N dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \int_{(r, r')} G_2^2(r_{12} \mid z_{12}) G_2^2(r_{23} \mid z_{23}) \times G_2^2(r'_{12} \mid z_{12}) G_2^2(r'_{23} \mid z_{23}).
\]  
(A11)

In the Laplace space this becomes

\[
(v_0^2 \Delta)^3 \nu^2 \Gamma^2(\epsilon)(4\pi)^{-2d} s^{-(2+2\epsilon)}
\]  
(A12)

The diagrams having odd numbers of wiggly lines trivially vanishes after DA. This can be generalized to arbitrary orders since only ladder type diagrams are involved. Eq. 4.4 would follow by substitution \( \bar{r}_0 = v_0^2 \Delta \).

**APPENDIX B: DRESSED PROPAGATORS AND \( \langle Z^2 \rangle \)**

We first show the two different dressed propagators. The one for which both the chains, tied at the ends \( (r, z) \) and \( (r', z') \), meet each other \( n \) times at \( (r_1, z_1) \), \( (r_2, z_2) \), .....\( (r_n, z_n) \) (Fig 2d) is given by

\[
\bar{G}^{(n)}_M(r - r' \mid z - z') = v_0^n \int_{(r)} \int_{(r')} dz_1 dz_2 .... dz_{n-1} dz_n G_2(r - r_1 \mid z - z_1) \times G_2(r_{12} \mid z_{12}) .... G_2(r_{n-1n} \mid z_{n-1n}) G_2(r_n - r' \mid z_n - z').
\]

Use of the identity \( G_2(r \mid z) = (4\pi z)^{-d/2} G(r / z / 2) \) and the Markovian property.
\[
\int d\mathbf{r}_2 G(\mathbf{r}_1 - \mathbf{r}_2 \mid z_1) G(\mathbf{r}_2 - \mathbf{r}_3 \mid z_2) = G(\mathbf{r}_1 - \mathbf{r}_3 \mid z_1 + z_2)
\]
leads to the following expression for \(\bar{G}_M^{(n)}(r - r' \mid z - z')\).

\[
\bar{G}_M^{(n)}(r - r' \mid z - z') = \bar{v}_0^n 4\pi^{-nd/2} G(r - r' \mid (z - z')/2) \times \\
\int_z^z dz_1 \int_z^z dz_2 \cdots \int_z^{z_{n-1}} dz_n(z - z_1)^{-d/2} z_{12}^{-d/2} \cdots \times \\
z_{n-1n}(z_n - z')^{-d/2}
\]

So we need the following integral

\[
\int_z^z dz_1(z - z_1)^{-d/2} \int_z^{z_1} dz_2(z_2 - z_1)^{-d/2} \cdots \int_z^{z_{n-1}} dz_n(z_{n-1} - z_n)^{-d/2} (z_n - z')^{-d/2}.
\]

A change of variable

\[
\bar{z}_i = z_i - z'
\]

and use of the convolution theorem in the Laplace space straightaway yields

\[
s^{-\epsilon'(n+1)} \Gamma^{n+1}(\epsilon')
\]

where \(s\) is the Laplace conjugate to the chainlength (more precisely to \(\bar{z} = z - z'\)). Converting this to inverse Laplace space and combining all other factors, the final form of such a propagator becomes

\[
\bar{G}_M^{(n)}(r \mid z) = (-\bar{v}_0)^n (4\pi)^{-(n+1)d/2} z^{(n+1)\epsilon' - 1} \frac{\Gamma^{n+1}(\epsilon')}{\Gamma((n+1)\epsilon')} G(r \mid z/2).
\]

(B1)

Following the same track, the other dressed propagator, for which the two member chains are tied only at one end say \((r, z)\) other than \(n\) meetings, has the form (see Fig 2c)

\[
G_O^{(n)}(z) = (-\bar{v}_0)^n (4\pi)^{-nd/2} z^{n\epsilon'} \frac{\Gamma^n(\epsilon')}{\Gamma(n\epsilon' + 1)}
\]

(B2)

This propagator is independent of any space coordinate because of the spatial integration over the open end coordinates.

To take care of arbitrary number of meetings we sum over \(n\). Hence the final dressed propagators are

27
\[ G_O(z) = \sum_{n=0}^{\infty} G_O^{(n)}(z) \]

\[ \tilde{G}_M(r - r' \mid z - z') = \sum_{n=0}^{\infty} \tilde{G}_M^{(n)}(r - r' \mid z - z') \]

Therefore the series for \((Z^2)_c\) can be written as

\[ \langle Z^2 \rangle_c = \bar{r}_0 \int_0^N dz_1 G_O^2(N - z_1) G_O^2(z_1) + \sum_{n=1}^{\infty} \tilde{r}_0^2 \int_0^N dz_1 \int_0^{z_1} dz_2 \int_{\{r, r'\}} \tilde{G}_M^2(N - z_1) \tilde{G}_M^2(z_2) + \ldots \]  

(B3)

In fact, it is possible to write the whole series to all orders in perturbation. Completing the integrations over the spatial coordinates we get

\[ \langle Z^2 \rangle_c = V^2 \bar{r}_0 \int_0^N dz_1 G_O^2(N - z_1) G_O^2(z_1) + \tilde{r}_0^2 V^2 \int_0^N dz_1 \int_0^{z_1} dz_2 G_O^2(N - z_1) \times \]

\[ G_M^2(z_2) + \tilde{r}_0^2 V^2 \int_0^N dz_1 \int_0^{z_1} dz_2 \int_0^{z_2} dz_3 \ G_O^2(N - z_1) \times \]

\[ G_M^2(z_2) G_M^2(z_3) G_O^2(z_3) \].  

(B5)

Here \(G_M(z)\) is obtained from \(\tilde{G}_M(r \mid z)\) after integration over the spatial coordinate. We use the convolution theorem for Laplace transforms which leads to the following expression for \(Z\) the Laplace transform of \(\langle Z^2 \rangle_c\) (Eq. 4.2)

\[ Z \mid_{\bar{v}_0 \neq 0} = V^2 \bar{r}_0 G_O^2(s) + \tilde{r}_0^2 G_O(s) \tilde{G}_M(s) G_O(s) + \]

\[ \tilde{r}_0^3 G_O(s) \tilde{G}_M^2(s) G_O(s) + \ldots \].  

(B6)

where, \(Z = \int_0^\infty e^{-sN} \langle Z^2 \rangle_c\), \(G_p = \int_0^\infty e^{-sN} G_p^2(z)\), with \(p\) being \(O\) or \(M\). The Laplace transforms \(G_O(s)\) and \(G_M(s)\) are given by

\[ G_O(s) = \sum_{n_1, n_2} \frac{(4\pi)^{-(n_1+n_2)d/2} \bar{r}_0^{n_1+n_2} \Gamma(n_1+n_2)(\epsilon') \Gamma((n_1+n_2)\epsilon' + 1)}{\Gamma(1+n_1\epsilon') \Gamma(1+n_2\epsilon')} s^{(n_1+n_2)(1-d)/2+1} \]  

(B7)

and

\[ G_M(s) = \sum_{n_1, n_2} \frac{(4\pi)^{-(n_1+n_2+2)d/2} \bar{r}_0^{n_1+n_2} \Gamma(n_1+n_2)(\epsilon') \Gamma((n_1+n_2)\epsilon' + \epsilon)}{\Gamma(1+n_1\epsilon') \Gamma(1+n_2\epsilon')} s^{(n_1+n_2)\epsilon' + \epsilon} \].  

(B8)

Substituting these expressions in equation (B6), we get back equation (4.3) for \(Z \mid_{\bar{v}_0 \neq 0}\) in the Laplace space. The results for \(\bar{v}_0 = 0\) can be obtained from the first term of each sum, i.e., for \(n_1 = n_2 = 0\).
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FIGURES

FIG. 1. Flow diagrams for coupling constant \( u \) in different dimensions. \( u^*(= 4\pi \epsilon') \) represents the nontrivial fixed point.

FIG. 2. (a) The only contributing diagrams in \( \langle Z^2 \rangle \mid \bar{v}_0 = 0 \) upto third order. Only ladder diagrams occur. A thick line corresponds to a pair of chains. A wiggly line stands for an \( \bar{r}_0 \) factor in the evaluation of the diagrams. (b) A typical diagram for \( \langle Z^2 \rangle \mid \bar{v}_0 \neq 0 \). The dots on the thick lines represent intrapair interactions (\( \bar{v}_0 \)). (c) The dressed propagator with two chains tied at only one end. The dashed lines represent the mutual \( \delta \)-function type interaction with coupling constant \( \bar{v}_0 \). (d) The dressed propagator with two chains tied at both \((r, z)\), and \((r', z')\).

FIG. 3. Flow diagram for \( r \) in various dimensions. \( r^*(= -\epsilon) \) represents the nontrivial unstable fixed point. For the \( m \)th order multicritical case \( r \) is to be replaced by \( r_m \). The three figures would be for \( d < d_m, d = d_m, \) and \( d > d_m \), where \( d_m = 1/(m - 1) \).

FIG. 4. The second (a), third (b) and fourth (c) order (in \( \bar{r}_0 \)) connected diagrams for \( \langle Z^3 \rangle \mid \bar{v}_0 = 0 \). In fourth order, there are a few other similar diagrams which contribute to the leading divergence. For connectedness, the series has to start at order two. (d) A few diagrams which contribute to subleading divergences in the third and fourth order in \( \bar{r}_0 \).

FIG. 5. Four chain diagrams for the annealed problem with four chains. The wiggly lines represent \( \bar{r}_0 \) type interaction, and a solid horizontal line connecting three chains is the three chain \( \delta \)-function interaction. The first three terms of the series involving only four chain interactions are shown in (a), (b), and (c). (d and e) Two cases involving both the three and four chain interactions. (f) A possible diagram in second order with different chain combinations for the interactions. This contributes in the subleading divergence.

FIG. 6. Diagrams for \( \langle Z \rangle \) upto second order in \( v_0 \) and \( v_0 b(z) \). The wavy and dotted lines interactions with coupling constants \( v_0 b(z) \) and \( v_0 \) respectively.
FIG. 7. (a) Diagrams involving only $v_0 b(z)$ for $\langle Z^2 \rangle$. (b) Diagrams of (a) after disorder averaging. (See the caption of Fig 2.) Fig (a2) with odd number of wavy lines vanishes after DA. Different pairings lead to two possibilities (b2) and (b3) from (a3). Similarly for a4, there are two diagrams (b4) and (b5) after DA. Diagrams (b2) and (b4) are not considered for the $\langle Z^2 \rangle \mid \bar{v}_0 = 0$ case.