MAKING OLD SEMINAL RESULTS WORLD-WIDE AVAILABLE!

FORWARD

The seminal paper of Crum, published in 1955, is now a standard reference in nonlinear science and supersymmetric quantum mechanics. It introduces the Crum transformations, a cornerstone of integrability and a beautiful generalization of Darboux transformations.

Since I am sure that many people would like to study carefully this masterpiece I offer here a LaTeX version of the paper. The purpose is to prevent all sorts of rediscoveries and promote real progress. I did very minor changes with respect to the old published version. The most important was to put the list of references at the end and not as footnotes. Crum’s paper has 7 points. The first point is the statement of Crum’s theorem, i.e., the possibility to write the solutions of a tower of so-called associated Sturm-Liouville (SL) systems (all of them Dirichlet from the point of view of boundary conditions) as a quotient of Wronskian determinants. The second point refers to the first associated SL system, dealing in fact with the SL Darboux transformations. Points 3 and 4 are a detailed study of the higher order associated SL systems (SL supersymmetric partners). Point 5 contains four noted applications. The corollary of Crum’s theorem is at point 6. Finally, point 7 states the possibility to build a regular SL system with any finite set of real numbers as eigenvalues, starting from a given associated SL system, a remarkable general result.

H C R

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ASSOCIATED STURM-LIOUVILLE SYSTEMS

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1. Let the regular Sturm-Liouville system

\[
\begin{align*}
    y'' + [\lambda - q(x)]y &= 0 & (0 < x < 1) , \\
    y(0) &= h^{(0)}y(0) , & y'(0) &= h^{(1)}y(0) , \\
    y(1) &= h^{(0)}y(1) , & y'(1) &= h^{(1)}y(1)
\end{align*}
\]

have eigenvalues \( \lambda_0 < \lambda_1 < \lambda_2 \), etc, and eigenfunctions \( \phi_s \) corresponding to \( \lambda_s \). Let \( q(x) \) be repeatedly differentiable in \((0,1)\); then the \( \phi_s \) also are repeatedly differentiable; let \( W_n \) be the Wronskian of the \( n + 1 \) functions \( \phi_0, \phi_1, \ldots, \phi_{n-1}, \phi_s \) and let \( W_n \) be the Wronskian of the \( n \) functions \( \phi_0, \phi_1, \ldots, \phi_{n-1} \). Then, if \( n \geq 1 \) and

\[
\phi_{ns} = \frac{W_{ns}}{W_n},
\]

the functions \( \phi_{ns} \) \( (s \geq n) \) are the eigenfunctions, with eigenvalues \( \lambda_s \), of the system

\[
\begin{align*}
    y'' + [\lambda - q_n(x)]y &= 0 & (0 < x < 1) , \\
    \lim_{x \to 0} y(x) &= 0 , & \lim_{x \to 1} y(x) &= 0
\end{align*}
\]

where

\[
q_n(x) = q(x) - 2 \frac{d^2}{dx^2} \log W_n.
\]

For \( n = 1 \), the system \((A_n, B_n)\) is regular; but, for \( n > 1 \),

\[
q_n(x) \approx \begin{cases} 
    n(n-1)x^{-2} & (x \to 0) , \\
    n(n-1)(1-x)^{-2} & (x \to 1)
\end{cases}
\]

Inside \((0,1)\), \( W_n \) is non-zero and \( q_n \) is continuous. For \( s < n \), \( \phi_{ns} \equiv 0 \); for \( s > n \), \( \phi_{ns} \) has exactly \( s - n \) zeros inside \((0,1)\). The family \( \phi_{ns} \) \( (s \geq n) \) is \( L^2 \)-closed and complete over \((0,1)\).

The system \((A_n, B_n)\) may be called the ‘nth system associated with the system \((A, B)\)’. In this note the above statements are established, and examples are given of systems associated with non-regular Sturm-Liouville systems.

If \( q(x) \) is continuous but not differentiable, the \( \phi_s \) are differentiable twice only, and the Wronskians do not exist; however, when the Wronskians \( W_{ns}, W_n \) exist, they are equal to the modified Wronskians \( W^*_n \), \( W^*_n \) obtained by replacing \( \phi_s^{(2k)} \) by \( (-\lambda_s)^k \phi_s \) and \( \phi_s^{(2k+1)} \) by \( (-\lambda_s)^k \phi_s' \); the \( W^*_n \) are at least twice differentiable, and the statements above are true for non-differentiable continuous \( q \) provided that the \( W \) are replaced by \( W^* \).
2. The case \( n = 1 \)

We have \( W_1 = \phi_0 \), of constant sign \( \square \) for \( 0 \leq x \leq 1 \); and

\[
\phi_{1s} = \phi_s' - \frac{\phi_0'}{\phi_0} \phi_s = \phi_s' - v \phi_s, \text{say, (D)}
\]

where

\[
v' + v^2 = q - \lambda_0 . \quad (E)
\]

Then

\[
\frac{d}{dx} (\phi_0 \phi_{1s}) = \phi_0 \phi_s'' - \phi_0' \phi_s = (\lambda_0 - \lambda_s) \phi_0 \phi_s . \quad (F_1)
\]

Since

\[
\phi_{1s}(0) = 0 = \phi_{1s}(1) , \quad (G)
\]

we have

\[
\phi_0 \phi_{1s} = (\lambda_0 - \lambda_s) \int_0^x \phi_0(\xi) \phi_s(\xi) d\xi = -(\lambda_0 - \lambda_s) \int_x^1 \phi_0 \phi_s d\xi . \quad (G')
\]

Hence

\[
\phi_{1s}' = (\lambda_0 - \lambda_s) \phi_s' - v \phi_{1s} ,
\]

\[
\phi_{1s}'' = (\lambda_0 - \lambda_s) \phi_s'' - v' \phi_{1s} - v(\lambda_0 - \lambda_s) \phi_s' - v \phi_{1s} = (\lambda_0 - \lambda_s - v' + v^2) \phi_{1s}
\]

\[
= (q_1 - \lambda_s) \phi_{1s} ,
\]

where

\[
q_1 = \lambda_0 - v' + v^2 = q - 2v' = q - 2 \frac{d^2}{dx^2} (\log W_1) .
\]

Now from \( (D_1) \),

\[
\phi_{1s}/\phi_0 = \frac{d}{dx} (\phi_s/\phi_0) ;
\]

since \( \phi_s \) has exactly \( s \) zeros \( \square \) inside \( (0,1) \), by Rolle’s theorem, \( \phi_{1s} \) has at least \( s - 1 \) zeros. But from \( (F_1) \) and \( (G) \) and Rolle’s theorem, \( \phi_{1s} \) has at most \( s - 1 \) zeros inside \( (0,1) \); hence it has \( s - 1 \) exactly. It follows \( \square \) that the \( \phi_{1s} (s \geq 1) \) are all the eigenfunctions of the regular system \( (A_1, B_1) \). For \( \lambda \neq \lambda_0 \) the general solution of \( (A_1) \) is

\[
X_1 = W(\phi_0, \chi)/W_1 ,
\]

where \( \chi \) is the general solution of \( (A) \). For \( \lambda = \lambda_0 \), \( W(\phi_0, \chi) \) is constant and one solution of \( (A_1) \) is \( 1/\phi_0 \); two independent solutions are

\[
\frac{1}{\phi_0} \int_0^x \phi_0^2(\xi) d\xi , \quad \frac{1}{\phi_0} \int_x^1 \phi_0^2(\xi) d\xi .
\]
It is easily verified that the only solutions of \((A_1)\) which satisfies \((G)\) are the \(\phi_{1s} \) \((s \geq 1)\).

3. The case \(n > 1\)

Applying Jacobi’s theorem to the determinant \(W_{ns}\), we have, for \(n > 1\),

\[
W_n W_{n-1} = W_n \frac{d}{dx} W_{n-1,s} - W_{n-1,s} \frac{d}{dx} W_n,
\]

with a similar relation with \(W^*\) for \(W\). Hence

\[
\phi_{ns} = \frac{W_{ns}}{W_n} = \frac{1}{W_{n-1}} \frac{d}{dx} (W_{n-1} \phi_{n-1,s}) - \frac{1}{W_n} \frac{d}{dx} W_n
\]

\[
= \phi_{n-1,s} - v_{n-1} \phi_{n-1,s} = \frac{1}{\phi_{n-1,n-1}} W(\phi_{n-1,n-1}, \phi_{n-1,s}), \quad (D_n)
\]

where

\[
v_n = \frac{\phi_{nn}}{\phi_{nn}}, \quad v_{n-1} = \frac{W'}{W_n} - \frac{W'_{n-1}}{W_{n-1}}.
\]

Hence, by steps similar to those of §2, and by induction on \(n\),

\[
v'_n + v^2_n = q_n - \lambda_n, \quad (E_n)
\]

\[
q_n + 2 \frac{d}{dx} \left( \frac{W'}{W_n} \right) = q_{n-1} + 2 \frac{d}{dx} \left( \frac{W'_{n-1}}{W_{n-1}} \right) = q. \quad (F_n)
\]

We now prove by induction on \(n\) the following:

\[
\phi_{ns} = C_{ns} \prod_{t=0}^{n-1} (\lambda_t - \lambda_s) x^n [1 + O(x^2)] \quad (C_{ns} \neq 0), \quad (G_n)
\]

\[
\phi'_{ns} = n x^{-1} \phi_{ns} [1 + O(x^2)], \quad (H_n)
\]

\[
v_n = n x^{-1} [1 + O(x^2)], \quad (J_n)
\]

all as \(x \to 0\), with similar relations as \(x \to 1\);

\[
\phi_{ns} \text{ has } s - n \text{ zeros inside } (0,1), \quad (K_n)
\]

By \((K_n)\), \(\phi_{nn}\), and so also \(W_{n+1}\), is non-zero inside \((0,1)\), so that \(q_{n+1}\) and \(\phi_{n+1,s}\) are continuous inside \((0,1)\). First, by \((G)\) and \((G')\), as \(x \to 0\),

\[
\phi_{1s}(x) \sim (\lambda_0 - \lambda_s) \phi_s(0) x^s.
\]
also

\[ \phi_{1s}'(0) = (q_1 - \lambda_s)\phi_{1s}(0) = 0, \]

which together imply \((G_1)\); \((H_1)\) follows from \((G_1)\) and \((F_1)\), together with

\[ \phi_s = \phi_s(0)[1 + h(0)x + O(x^2)] ; \]

and \((J_1)\) is a case of \((H_1)\). It remains to deduce \((G_{n+1})\) to \((K_{n+1})\) from \((G_n)\) to \((K_n)\). First, by \((D_{n+1})\), \((H_n)\), \((J_n)\),

\[ \phi_{n+1,s} = \phi_{ns}\left[\frac{n}{x} + O(x) - \frac{n}{x} + O(x)\right] = o(1) \quad (x \to 0). \]

Hence

\[ \phi_{nn}\phi_{n+1,s} = (\lambda_n - \lambda_s) \int_0^x \phi_{nn}\phi_{ns}d\xi, \]

whence we have \((G_{n+1})\) with

\[ C_{n+1,s} = C_{ns}/(2n + 1) \neq 0. \]

By differentiating this last we obtain \((H_{n+1})\), of which \((J_{n+1})\) is a special case.

From \((D_{n+1})\) and \((K_n)\), \(\phi_{n+1,s}\) has at least \(s - n - 1\) zeros inside \((0,1)\); from \((F_{n+1})\), \((K_n)\), \((G_n)\), it has at most \(s - n - 1\) zeros inside \((0,1)\); hence \((K_{n+1})\) is deduced.

Lastly we may prove that, as \(x \to 0\),

\[ q_n(x) = n(n - 1)x^{-2} + O(1), \quad \text{(L}_n) \]

with a similar relation as \(x \to 1\). For, given \((L_n)\) and \((J_n)\),

\[ q_{n+1} = q_n - 2\nu'_n = 2\lambda_n + 2\nu^2_n - q_n = O(1) + n(n + 1)x^{-2}, \]

which is \((L_{n+1})\).

For \(\lambda \neq \lambda_s \quad (s < n)\) the general solution of \((A_n)\) is

\[ y = \chi_n = W(\phi_0, \phi_1, \ldots, \phi_{n-1}, \chi)/W_n, \]

where \(\chi\) is the general solution of \((A)\). For \(\lambda = \lambda_{n-1}\) a solution is

\[ y = \frac{1}{\phi_{n-1,n-1}}W(\phi_{n-1,n-1}, \chi_{n-1,n-1}) = \frac{C}{\phi_{n-1,n-1}} = C W(\phi_0, \phi_1, \ldots, \phi_{n-2})/W(\phi_0, \phi_1, \ldots, \phi_{n-1}). \]

For \(\lambda = \lambda_s\), \(s \leq n - 1\), a solution is

\[ y = \psi_{ns} = W^{(s)}_n/W_n, \]

where \(W^{(s)}_n\) is the Wronskian of the \(n - 1\) functions

\[ \phi_t \quad (0 \leq t \leq n - 1; \ t \neq s). \]
4. Since the system \((A_n, B_n)\) is not regular for \(n > 1\), it remains to prove that the family \(\phi_{ns} (s \geq n)\) is \(L^2\)-complete over \((0,1)\); this implies incidentally that the \(\phi_{ns}\) are the only bounded solutions of \((A_n)\). Since \((A_1, B_1)\) is regular, it is sufficient to verify that the completeness of the family \(\phi_{ns}\) implies that of the family \(\phi_{n+1,s}\).

Let \(f(x)\) be of \(L^2(0,1)\); then, given \(\epsilon > 0\), there exists \(g(x)\) such that

(i) \(g(x) = 0 \quad (0 < x < \delta; 1 - \delta < x < 1; \delta > 0)\),

(ii) \(g(x)\) is continuous in \((0,1)\),

(iii) \(\int_1^0 |f - g|^2 d\xi < \epsilon\).

Then, if

\[ h = g' + v_n g, \quad \phi_{nn} h = \frac{d}{dx} (\phi_{nn} g), \]

\(h\) is of \(L^2(0,1)\); also

\[ \int_0^1 h\phi_{nn} d\xi = [g\phi_{nn}]_0^1 = 0, \]

so that, assuming the completeness of the family \(\phi_{ns}\), we have

\[ h = \sum_{s=n+1}^N c_s \phi_{ns} + \eta, \]

where

\[ \int_0^1 |\eta|^2 dx < \epsilon. \]

Now

\[ \phi_{nn} g = \int_0^x \phi_{nn} h d\xi = \sum_{s=n+1}^N c_s \int_0^x \phi_{nn} \phi_{ns} d\xi + \int_0^x \phi_{nn} \eta d\xi = \phi_{nn} \sum_{s=n+1}^N c_s \phi_{n+1,s} + \phi_{nn} \zeta, \]

where \(C_s = c_s (\lambda_n - \lambda_s)^{-1}\), and

\[ \zeta = \frac{1}{\phi_{nn}} \int_0^x \phi_{nn} \eta d\xi = -\frac{1}{\phi_{nn}} \int_x^1 \phi_{nn} \eta d\xi; \]

since, by \((G_n)\) and its analogue for \(x \to 1\),

\[ \int_0^x \phi_{nn}^2 dx = O(\phi_{nn}^2) , \quad \int_x^1 \phi_{nn}^2 = O(\phi_{nn}^2) \]

when \(x \to 0, 1\), respectively, we have by Schwartz's inequality

\[ |\zeta|^2 < M_n \int_0^1 |\eta|^2 dx < M_n \epsilon, \quad \int_0^1 |\zeta|^2 dx < M_n \epsilon. \]

Hence the result.
5. Examples

(1) If \( q(x) = 0, \ h^{(0)} = 0 = h^{(1)} \), then \( \lambda_s = (2\pi s)^2, \ \phi_s = \cos 2\pi sx \ (s = 0, 1, 2, \ldots) \).
Since \( v = 0, \ q_1 = q \) and
\[
\phi_{1s} = \phi'_s = 2\pi s \sin 2\pi sx \quad (s = 1, 2, \ldots).
\]
For \( n > 1, \ \phi_{ns} \) is obtainable as in Example 3.

(2) If \( q(x) = x^2 \) and the interval is \( (-\infty, \infty) \), (A) is \( y'' + (\lambda - x^2)y = 0 \), with \( \phi_0 = e^{-\frac{1}{2}x^2}, \ \lambda_0 = 1 \). Since \( v = x, \ q_1 = x^2 - 2; \) hence
\[
\lambda_{s+1} = \lambda_s + 2, \quad \phi_{1s} = k_s \phi_{s-1}.
\]
The associated systems are all identical, \( \lambda_s = 2s + 1, \) and, since
\[
\phi_0 \phi_s = \frac{1}{\lambda_0 - \lambda_s} \frac{d}{dx} (\phi_0 \phi_{1s}) = \frac{k_s}{2s} \frac{d}{dx} (\phi_0 \phi_{s-1}),
\]
it follows that
\[
\phi_s = K_s \phi_0^{s-1} \left( \frac{d}{dx} \right)^s \phi_0^2 = K_s e^{\frac{1}{2}x^2} \left( \frac{d}{dx} \right)^s e^{-x^2}.
\]

(3) The Legendre functions \( y_s = (\sin \theta)^s P_s(\cos \theta) \ (0 < \theta < \pi) \)
satisfy
\[
y'' + \left( \lambda + \frac{1}{4} \csc^2 \theta \right) y = 0,
\]
where
\[
\lambda_s = (s + \frac{1}{2})^2 \quad (s = 0, 1, 2, \ldots).
\]
Writing \( \mu = \cos \theta, \) and \( W_{(\mu)} \) for the Wronskians with respect to \( \mu, \) we get
\[
W_n = W(y_0, y_1, \ldots, y_{n-1}) = \left( \frac{d\mu}{d\theta} \right)^{\frac{1}{2}n(n-1)} W_{(\mu)}(y_0, y_1, \ldots, y_{n-1})
\]
\[
= \left( \frac{d\mu}{d\theta} \right)^{\frac{1}{2}n(n-1)} (\sin \theta)^{\frac{n}{2}} W_{(\mu)}(P_0, P_1, \ldots, P_{n-1}) = A_n (\sin \theta)^{\frac{n}{2}n^2},
\]
and similarly
\[
W_{ns} = A_n (\sin)\left( \frac{d}{dx} \right)^{n} P_s(\mu).
\]
Hence
\[
\phi_{ns} = (\sin \theta)^{n+\frac{1}{2}} \left( \frac{d}{dx} \right)^n P_s(\mu) = (\sin \theta)^{\frac{1}{2}} P_s^{(n)}(\mu).
\]
(4) For the Hankel system of order $\nu$
\[
y = \phi_k(x) = c_k(kx)^{\nu} J_{\nu}(kx) \quad \phi_0(x) = x^{\nu + \frac{1}{2}},
\]
\[
y'' + \left( \lambda - \frac{\nu^2 - 1}{x^2} \right) y = 0, \quad \lambda = k^2.
\]
Here $v = \phi_0'/\phi_0 = (\nu + \frac{1}{2})/x$, whence
\[
q_1 = \frac{(\nu + 1)^2 - \frac{1}{4}}{x^2}
\]
and the first associated system is the Hankel system of order $\nu + 1$.

6. As a corollary of the main theorem, if
\[
S(x) = \sum_{s=0}^{n} c_s \phi_s(x),
\]
then $S(x)$ has at most $n$ zeros in $(0,1)$. This result is due to Kellogg. For, if $S(x)$ has $k$ zeros, then by Rolle’s theorem
\[
S_1(x) = \phi_0 \frac{d}{dx} \left( \phi_0^{-1} \sum_{s=0}^{n} c_s \phi_s(x) \right) = \sum_{s=1}^{n} c_s \phi_1s
\]
has at least $k - 1$ zeros inside $(0,1)$; by induction
\[
S_m(x) = \sum_{s=m}^{n} c_s \phi_ms
\]
has at least $k - m$ zeros, and $S_n(x) = c_n \phi_{nn}$ has at least $k - n$; since $\phi_{nn}$ is non-zero, either $k \leq n$ or $c_n = 0$; but, if $c_n = 0$, then $k \leq n - 1 \leq n$.

This proof of the corollary depends only on the fact that the Wronskians $W_n$ are non-zero. If $\phi_s = e^{\alpha_s x}$, where the $\alpha_s$ are any distinct real numbers, then the $W_n$ are all non-zero, and so $S(x)$ has at most $n$ real zeros.

7. If $(A, B)$ is given, the associated systems $(A_n, B_n)$ are uniquely defined; but to a given $(A_n, B_n)$ belong an infinity of $(A, B)$. For example, given $(A_1, B_1)$ we may solve for $v$
\[
\lambda_0 - v' + v^2 = q_1,
\]
with any $\lambda_0$ such that $\lambda_0 < \lambda_1$; then, if
\[
\phi_0 = \exp \left( \int_0^x v d\xi \right), \quad (\lambda_0 - \lambda_s)\phi_0\phi_s = \frac{d}{dx} (\phi_0\phi_1s),
\]
it will follow that the $\phi_s$ are the eigenfunctions of $(A, B)$ with
\[
q = q_1 + 2v', \quad h^{(0)} = v(0), \quad h^{(1)} = v(1).
\]
For example, if
\[ q_1 = 0, \quad \lambda_s = (2\pi s)^2, \quad \phi_{1s} = \sin 2\pi sx, \]
we can take
\[ \lambda_0 = -\rho^2, \quad \phi_0 = \text{sech}(x - \alpha), \quad v = -\rho \tanh(x - \alpha), \]
\[ q(x) = -2\rho^2 \text{sech}^2(x - \alpha), \]
\[ \phi_s(x) = 2\pi s \cos 2\pi sx - \rho \tanh(x - \alpha) \sin 2\pi sx. \]

Starting from a given \((A_n, B_n)\) we can similarly construct an \((A, B)\) with arbitrary \(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\) (provided only that \(\lambda_{s+1} > \lambda_s\)). Thus there exists a regular Sturm-Liouville system with any finite set of real numbers as eigenvalues.

References

1. E.L. Ince, *Ordinary Differential Equations* (London, 1927), §10.61, 235.
2. Compare P.A.M. Dirac, *Quantum Mechanics* (3rd ed., Oxford, 1947), §34, 136-139.
3. E.C. Titchmarsh, *Eigenfunction Expansions* (Oxford, 1946), §4.5, 64.
4. E.T. Whittaker and G.N. Watson, *Modern Analysis* (3rd ed., Cambridge, 1927), §15.5, 323.
5. Titchmarsh, op. cit. §4.8, 70, and §4.11, 75.
6. O.D. Kellogg, Am. J. Math. (i) *Oscillations of functions of an orthogonal set* (1916) 1, (ii) *Orthogonal sets arising from integral equations* (1918) 145, (iii) *Interpolation properties of orthogonal sets of solutions of differential equations* (1918) 225. Kellogg uses the functional determinants \(\det[\phi_s(x_i)]\), not the Wronskians \(W_{ns}\) or \(W^*_{ns}\).