Comparison Theorem for Stochastic Differential Delay Equations with Jumps

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Abstract

In this paper we establish a comparison theorem for stochastic differential delay equations with jumps. An example is constructed to demonstrate that the comparison theorem need not hold whenever the diffusion term contains a delay function although the jump-diffusion coefficient could contain a delay function. Moreover, another example is established to show that the comparison theorem is not necessary to be true provided that the jump-diffusion term is non-increasing with respect to the delay variable.

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1 Introduction

For most of the practical cases, the dynamical systems will be disturbed by some stochastic perturbation \cite{12}. One type of stochastic perturbation is continuous and can be modeled by stochastic integral with respect to the continuous martingale, e.g., Brownian motion. Non-Gaussian random processes also play an important role in modelling stochastic dynamical systems (see, for example, Applebaum \cite{2}, Situ \cite{12}, Peszat and Zabczyk \cite{10}). Typical examples of non-Gaussian stochastic processes are Lévy processes and processes arising by Poisson random measures. In \cite{14}, Woyczyński describes a number of phenomena from fluid mechanics, solid state physics, polymer chemistry, economic science, etc., for which non-Gaussian Lévy processes can be used as their mathematical model in describing the related

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probability behaviour. On the other hand, control engineering intuition suggests that time-delays are common in practical systems and are often the cause of instability and/or poor performance [13]. Moreover, it is usually difficult to obtain accurate values for the delay and conservative estimates often have to be used. The importance of time delay has already been motivated by several studies on the stability of stochastic diffusion with time delay (e.g., [3] and [8]).

In the past few years, comparison theorems for two stochastic differential equations (SDEs) have received a lot of attention, for example, Anderson [11], Gal’cuk and Davis [4], Ikeda and Watanable [5], Mao [6], O’Brien [7], Yamada [16], Yan [15] and references therein. Recently, the comparison theorem has made a great development that Peng and Zhu [10] obtain a necessary and sufficient condition for comparison theorem of SDEs with jumps by applying a criteria of “viability condition”. Peng and Yang [9] give a comparison theorem for anticipated backward stochastic differential equations, and for a class of SDEs with delay, Yang, Mao and Yuan [13] also establish a comparison theorem.

In this paper we shall establish a comparison theorem for stochastic differential delay equations (SDDEs) with jumps. It should be pointed out that the approach of this paper is inspired by Peng and Yang [9], Peng and Zhu [10] and Yang, Mao and Yuan [13]. We construct an example, which demonstrates that the comparison theorem need not hold whenever the diffusion term contains a delay function although the jump-diffusion coefficient could contain a delay function just as Example 2.1 below shows. Moreover, another example, Example 2.3, is established to show that the comparison theorem is not necessary to be true provided that the jump-diffusion term is non-increasing with respect to the delay variable.

The organization of this paper goes as follows: In Section 2 we establish a comparison theorem for two one-dimensional SDDEs with pure jumps, and similar comparison results are given for SDDEs with compensator jump processes in Section 3.

## 2 Comparison Theorem for SDDEs with Pure Jumps

Let \( W(t), t \geq 0 \), be a real-valued Wiener process defined on a certain probability space \((\Omega, \mathcal{F}, \mathbb{P})\) equipped with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions (i.e., it is right continuous and \( \mathcal{F}_0 \) contains all \( \mathbb{P} \)-null sets), and \( N(\cdot, \cdot) \) is a Poisson counting process with characteristic measure \( \lambda \) on measurable subset \( \mathcal{Y} \) of \([0, \infty)\) with \( \lambda(\mathcal{Y}) < \infty \), \( \tilde{N}(dt, du) := N(dt, du) - \lambda(du)dt \) is a compensator martingale process. Let \( \tau > 0 \) and denote \( D([-\tau, 0]; \mathbb{R}) \) the space of all càdlàg paths from \([-\tau, 0]\) into \( \mathbb{R} \) with the norm \( ||u|| := \sup_{-\tau \leq \theta \leq 0} |u(\theta)| \). Throughout this paper, we assume that \( W(t) \) and \( N(dt, du) \) are independent.

Fix \( T > 0 \) and consider SDDE with jumps for \( t \in [0, T] \)

\[
dX(t) = f(X(t), X(t - \tau), t)dt + g(X(t), X(t - \tau), t) dW(t) \nonumber \\
+ \int_{\mathcal{Y}} \gamma(X(t), X(t - \tau), t) \tilde{N}(dt, du) 
\]

with initial condition \( X(\theta) = \xi(\theta) \in D([-\tau, 0]; \mathbb{R}) \). Assume that there exist positive con-

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stated $L_n$ such that
\[ |f(x_1, y_1, t) - f(x_2, y_2, t)|^2 + |g(x_1, y_1, t) - g(x_2, y_2, t)|^2 + \int_Y |\gamma(x_1, y_1, t) - \gamma(x_2, y_2, t)|^2 \lambda(du) \leq L_n(|x_1 - x_2|^2 + |y_1 - y_2|^2) \] (2.2)

for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $|x_1| \vee |x_2| \vee |y_1| \vee |y_2| \leq n$ and there exists a constant $L > 0$ such that for any $x, y \in \mathbb{R}$
\[ |f(x, y, t)|^2 + |g(x, y, t)|^2 + \int_Y |\gamma(x, y, t, u)|^2 \lambda(du) \leq L(1 + |x|^2 + |y|^2). \] (2.3)

By the standard Banach fixed point theorem and truncation approach, the following existence and uniqueness result can be found.

**Lemma 2.1.** Under conditions (2.2) and (2.3), for initial condition $\mathbb{E}\|\xi\|^2 < \infty$, Eq. (2.1) has a unique solution $X(t), t \in [0, T]$, with property $\mathbb{E}\sup_{-\tau \leq t \leq T} |X(t)|^2 < \infty$.

In order to state our main results, we need the following Lemma.

**Lemma 2.2.** Consider two one-dimensional SDEs with jumps for any $t \in [0, T]$
\[ X_1(t) = x_1 + \int_0^t f_1(X_1(s), s)ds + \int_0^t g(X_1(s), s)dW(s) \]
\[ + \int_0^t \int_Y \gamma_1(X_1(s^-), s, u)N(ds, du) \] (2.4)

and
\[ X_2(t) = x_2 + \int_0^t f_2(X_2(s), s)ds + \int_0^t g(X_2(s), s)dW(s) \]
\[ + \int_0^t \int_Y \gamma_2(X_2(s^-), s, u)N(ds, du). \] (2.5)

Assume that there exists a constant $L > 0$ such that for any $x, y \in \mathbb{R}$ and $t \in [0, T]$
\[ |f_i(x, t) - f_i(y, t)|^2 + |g(x, t) - g(y, t)|^2 + \int_Y |\gamma_i(x, t, u) - \gamma_i(y, t, u)|^2 \lambda(du) \leq L|x - y|^2 \] (2.6)

for $i = 1, 2$ with $\mathbb{E}\sup_{0 \leq t \leq T} (|f_i(0, t)|^2 + |g(0, t)|^2 + \int_Y |\gamma_i(0, t, u)|^2 \lambda(du)) < \infty$ and
\[ f_1(x, t) \geq f_2(x, t) \text{ and } \gamma_1(x, t, u) \geq \gamma_2(x, t, u), t \in [0, T], u \in \mathbb{Y}. \] (2.7)

Moreover assume that for any $x, y \in \mathbb{R}$ and $u \in \mathbb{Y}$
\[ x + \gamma_1(x, t, u) \leq y + \gamma_1(y, t, u) \text{ whenever } x \leq y. \] (2.8)

Then we have
\[ X_1(t) \geq X_2(t), \forall t \in [0, T], \text{ a.s. provided that } x_1 \geq x_2. \] (2.9)
Proof. By Lemma 2.1 both Eq. (2.4) and Eq. (2.5) have unique solutions, respectively. Applying the Tanaka-type formula [12, Theorem 152, p120], we have for any $t \in [0, T]$

$$(X_2(t) - X_1(t))^+ = (x_2 - x_1)^+ + \int_0^t I_A[f_2(X_2(s), s) - f_1(X_1(s), s)]ds$$

$$+ \int_0^t I_A[g(X_2(s), s) - g(X_1(s), s)]dW(s)$$

$$+ \int_0^t \int_Y [(X_2(s^-, s, u) - X_1(s^-, s, u)) + \gamma_2(X_2(s^-, s, u) - X_1(s^-, s, u))^+$$

$$- (X_2(s^-) - X_1(s^-))^+]N(ds, du)$$

$$\leq \int_0^t I_A[(f_1(X_2(s), s) - f_1(X_1(s), s)) + (f_2(X_2(s), s) - f_1(X_2(s), s))]ds$$

$$+ \int_0^t I_A[g(X_2(s), s) - g(X_1(s), s)]dW(s)$$

$$+ \int_0^t \int_Y I_A(\gamma_1(X_2(s^-, s, u) - X_1(s^-, s, u))N(ds, du)$$

$$+ \int_0^t \int_Y [(X_2(s^-, s, u) - X_1(s^-, s, u)) + \gamma_1(X_2(s^-, s, u) - X_1(s^-, s, u))^+$$

$$- \gamma_2(X_2(s^-, s, u) - X_1(s^-, s, u))^+ - I_A(\gamma_1(X_2(s^-), s, u) - \gamma_1(X_1(s^-), s, u))]N(ds, du),$$

in which $A := \{X_2(s) - X_1(s) > 0\}$ and the second inequality is due to $x_1 \geq x_2$. Noting by (2.7) that

$$f_2(X_2(s), s) - f_1(X_2(s), s)) \leq 0$$

and taking expectations, we obtain

$$\mathbb{E}(X_2(t) - X_1(t))^+ \leq \mathbb{E} \int_0^t I_A[f_1(X_2(s), s) - f_1(X_1(s), s)]ds$$

$$+ \mathbb{E} \int_0^t \int_Y I_A(\gamma_1(X_2(s^-), s, u) - \gamma_1(X_1(s^-), s, u))\lambda(du)ds$$

$$+ \mathbb{E} \int_0^t \int_Y [(X_2(s^-) - X_1(s^-)) + \gamma_1(X_2(s^-), s, u) - \gamma_1(X_1(s^-), s, u))^+$$

$$- (X_2(s^-) - X_1(s^-))^+ - I_A(\gamma_1(X_2(s^-), s, u) - \gamma_1(X_1(s^-), s, u))]N(ds, du).$$

On the other hand, thanks to (2.8), it follows that

$$\mathbb{E} \int_0^t \int_Y [(X_2(s^-) - X_1(s^-) + \gamma_1(X_2(s^-), s, u) - \gamma_1(X_1(s^-), s, u))^+ - (X_2(s^-) - X_1(s^-))^+$$

$$- I_A(\gamma_1(X_2(s^-), s, u) - \gamma_1(X_1(s^-), s, u))]N(ds, du) \leq 0.$$
Hence, taking into account (2.6)
\[
\mathbb{E}(X_2(t) - X_1(t))^+ \leq (1 + \lambda^{\frac{3}{2}}(Y))L^{\frac{1}{2}}\mathbb{E} \int_0^t I\{X_2(s) - X_1(s) > 0\}|X_2(s) - X_1(s)|\,ds
\]
\[
= (1 + \lambda^{\frac{3}{2}}(Y))L^{\frac{1}{2}}\mathbb{E} \int_0^t (X_2(s) - X_1(s))^+\,ds.
\]
This, in addition to Gonwall’s inequality, implies \(\mathbb{E}(X_2(t) - X_1(t))^+ = 0\) and then yields \(X_2(t) \leq X_1(t), t \in [0, T]\), a.s. due to the fact that \((X_2(t) - X_1(t))^+\) is a nonnegative random variable for fixed \(t\), as required.

**Remark 2.1.** Peng and Zhu [10, Theorem 3.1] obtain a necessary and sufficient condition of comparison theorem for two one-dimensional SDEs driven by compensator jump processes such that
\[
X_1(t) = x_1 + \int_0^t f_1(X_1(s), s)\,ds + \int_0^t g_1(X_1(s), s)\,dW(s)
\]
\[
+ \int_0^t \int_{\mathbb{Y}} \gamma_1(X_1(s^-), s, u)\tilde{N}(ds, du)
\]
and
\[
X_2(t) = x_2 + \int_0^t f_2(X_2(s), s)\,ds + \int_0^t g_2(X_2(s), s)\,dW(s)
\]
\[
+ \int_0^t \int_{\mathbb{Y}} \gamma_2(X_2(s^-), s, u)\tilde{N}(ds, du).
\]
\(X_1(t) \geq X_2(t)\) if and only if
\(f_1(x, t) \geq f_2(x, t), g_1(x, t) = g_2(x, t), \gamma_1(x, t, u) = \gamma_2(x, t, u)\)
as well as (2.8) holds. For Eq. (2.4) and Eq. (2.5) which are driven by pure jump processes, we consider the comparison result in Lemma 2.2, where it is not necessary to impose \(\gamma_1 = \gamma_2\). Clearly, Eq. (2.4) and Eq. (2.5) can be easily transformed to Eq. (2.10) and Eq. (2.11), respectively. However, we shall use this Lemma to establish a comparison theorem for SDDEs driven by jump processes.

In the work [13], where comparison theorem of one-dimensional stochastic hybrid delay systems is studied, a very suggestive example (Example 3.3) shows that the comparison theorem need not hold whenever the diffusion terms contain a delay function. While for stochastic delay systems with jumps, the following example demonstrates that the jump-diffusion terms could contain a delay function.

**Example 2.1.** Consider the following two one-dimensional SDEs with jumps
\[
\begin{cases}
X(t) = c + \int_0^t \int_{-\infty}^\infty \gamma(u)X(s - \tau)\tilde{N}(ds, du), t \in [0, T]; \\
X(\theta) = c, \theta \in [-\tau, 0]
\end{cases}
\] (2.12)
and
\[
\begin{align*}
Y(t) &= \int_0^t \int_{-\infty}^{\infty} \gamma(u)Y(s-\tau)\tilde{N}(ds, du), \quad t \in [0, T]; \\
Y(\theta) &= 0, \quad \theta \in [-\tau, 0)
\end{align*}
\]  \tag{2.13}
\]
where \(c < 0\) is a constant. We further assume that
\[
\gamma(u) > 0, \quad u \in (-\infty, \infty)
\]  \tag{2.14}
and
\[
\tau \int_{-\infty}^{\infty} \gamma(u)\lambda(du) < 1.
\]  \tag{2.15}
For any \(t \in [0, \tau]\)
\[
X(t) = c \left( 1 + \int_0^t \int_0^{\infty} \gamma(u)\tilde{N}(ds, du) \right)
\]  \tag{2.16}
By (2.14), combining the definition of stochastic calculus with jumps, it follows that for \(t \in [0, \tau]\)
\[
\int_0^t \int_0^{\infty} \gamma(u)N(ds, du) > 0
\]
and
\[
-\int_0^t \int_0^{\infty} \gamma(u)\lambda(du)ds \geq -\int_0^{\tau} \int_0^{\infty} \gamma(u)\lambda(du)ds = -\tau \int_{-\infty}^{\infty} \gamma(u)\lambda(du).
\]
Hence, together with (2.15), in (2.16) \(X(t) < 0\) while \(Y(t) \equiv 0\) for \(t \in [0, \tau]\). As a consequence, we could derive the following comparison result: the solutions \(X(t)\) of Eq. (2.12) and \(Y(t)\) of Eq. (2.13) obey the property for \(t \in [0, \tau]\)
\[
X(t) \leq Y(t) \text{ a.s.}
\]
Motivated by [13, Example 3.3] we could also establish an example to show that, for stochastic delay systems with jumps, the comparison theorem need not hold if the diffusion term contains a delay function.

**Example 2.2.** Consider the following two one-dimensional equations
\[
\begin{align*}
X(t) &= c + \int_0^t X(s-\tau)dB(s) - \int_0^t X(s-\tau)dN(s), \quad t \in [0, T]; \\
X(\theta) &= c, \quad \theta \in [-\tau, 0)
\end{align*}
\]  \tag{2.17}
and
\[
\begin{align*}
Y(t) &= \int_0^t Y(s-\tau)dB(s) - \int_0^t Y(s-\tau)dN(s), \quad t \in [0, T]; \\
Y(\theta) &= 0, \quad \theta \in [-\tau, 0)
\end{align*}
\]  \tag{2.18}
where \( c < 0 \) is a constant, \( N \) is a Poisson process and independent of Brownian motion \( B \). Clearly, for any \( t \in [0, \tau] \), \( Y(t) \equiv 0 \) while

\[
X(t) = c(1 + B(t) - N(t)).
\]

Noting that \( N(t) \geq 0 \) and the relation

\[
\{(t, \omega) \in [0, \tau] \times \Omega : B(t) < -1\} \subseteq \{(t, \omega) \in [0, \tau] \times \Omega : 1 + B(t) - N(t) < 0\},
\]

hence

\[
\mathbb{P}\{(t, \omega) \in [0, \tau] \times \Omega : 1 + B(t) - N(t) < 0\} \geq \mathbb{P}\{(t, \omega) \in [0, \tau] \times \Omega : B(t) < -1\} > 0,
\]

since \( B \) obeys the normal distribution. This, together with \( c < 0 \), yields

\[
\mathbb{P}\{(t, \omega) \in [0, \tau] \times \Omega : X(t, \omega) > 0\} > 0.
\]

Consequently, we can conclude that comparison theorem need not hold if the diffusion coefficient contains a delay function. What’s more, the following example shows if the jump coefficients are not increasing, the comparison theorem also need not hold.

**Example 2.3.** Consider the following two one-dimensional equations

\[
\begin{align*}
X(t) &= c - 2 \int_0^t X(s - \tau) dN(s), t \in [0, T]; \\
X(\theta) &= c, \theta \in [-\tau, 0) 
\end{align*}
\]  

(2.19)

and

\[
\begin{align*}
Y(t) &= -2 \int_0^t I\{Y(s - \tau) < 0\} Y(s - \tau) dN(s), t \in [0, T]; \\
Y(\theta) &= 0, \theta \in [-\tau, 0),
\end{align*}
\]  

(2.20)

where \( c < 0 \) is a constant and \( N \) is a Poisson process with intensity \( \lambda \).

By Eq. (2.19) it is easy to see that for any \( t \in [0, \tau] \)

\[
X(t) = c(1 - 2N(t)).
\]

In what follows we intend to show

\[
\mathbb{P}\{(t, \omega) \in (0, \tau] \times \Omega : X(t, \omega) > 0\} > 0. \tag{2.21}
\]

Indeed, noting that

\[
\{1 - 2N(t) < 0\} = \{N(t) \geq 1\},
\]

we have

\[
\mathbb{P}\{1 - 2N(t) < 0\} = 1 - e^{-\lambda t} > 0 \text{ whenever } 0 < t < \tau,
\]

which further gives (2.21). Although

\[
-2y \leq -2y I\{y < 0\} \text{ and } c < 0,
\]

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we can not deduce that
\[ X(t) \leq Y(t) \text{ a.s.} \]
due to (2.21) and \( Y(t) \equiv 0, t \in [0, T] \).

Based on the previous discussion, now we state a comparison theorem for SDDEs driven by pure jump processes. In the proof, Lemma 2.2 is used.

**Theorem 2.1.** Consider two one-dimensional SDDEs with pure jumps for any \( t \in [0, T] \)
\[
\begin{align*}
\left\{ \begin{array}{l}
dX_1(t) = f_1(X_1(t), X_1(t- \tau), t)dt + g(X_1(t), t)dW(t) \\
X_1(t) = \xi_1(t), t \in [-\tau, 0],
\end{array} \right. \tag{2.22}
\end{align*}
\]
and
\[
\begin{align*}
\left\{ \begin{array}{l}
dX_2(t) = f_2(X_2(t), X_2(t- \tau), t)dt + g(X_2(t), t)dW(t) \\
X_2(t) = \xi_2(t), t \in [-\tau, 0].
\end{array} \right. \tag{2.23}
\end{align*}
\]
Assume that there exists a constant \( L > 0 \) such that for any \( x_1, x_2, y_1, y_2, x, y \in \mathbb{R} \)
\[
|f_i(x_1, y_1, t) - f_i(x_2, y_2, t)|^2 + \int \gamma(x_1, y_1, t, u) - \gamma(x_2, y_2, t, u)|^2 \lambda(du) \leq L(|x_1 - x_2|^2 + |y_1 - y_2|^2) \tag{2.24}
\]
with \( i = 1, 2 \) and
\[
|g(x, t) - g(y, t)|^2 \leq L|x - y|^2 \tag{2.25}
\]
with property \( \mathbb{E} \sup_{0 \leq t \leq T} (|f_i(0, 0, t)|^2 + |g(0, t)|^2 + \int \gamma(0, 0, t, u)|^2 \lambda(du)) < \infty. \)
Assume further that for \( x, y, z \in \mathbb{R} \)
\[
f_1(x, y, t) \geq f_2(x, y, t) \tag{2.26}
\]
and
\[
x + \gamma(x, z, t, u) \leq y + \gamma(y, z, t, u) \text{ whenever } x \leq y. \tag{2.27}
\]
Moreover, we suppose that \( f_2 \) and \( \gamma \) is nondecreasing with respect to the second variable, that is, for \( t \in [0, T] \) and fixed \( x \in \mathbb{R} \) and \( u \in \mathcal{Y} \),
\[
f_2(x, y, t) \geq f_2(x, z, t) \text{ and } \gamma(x, y, t, u) \geq \gamma(x, z, t, u) \text{ whenever } y \geq z. \tag{2.28}
\]

Then we have
\[
X_1(t) \geq X_2(t), t \in [0, T] \text{ a.s. provided that } \xi_1(t) \geq \xi_2(t) \text{ with } t \in [-\tau, 0].
\]

**Proof.** Under conditions (2.21) and (2.25), both Eq. (2.22) and Eq. (2.23) have unique solutions \( X_1(t), t \in [0, T] \) and \( X_2(t), t \in [0, T] \), respectively. Now consider SDDE with pure jumps for any \( t \in [-\tau, T] \)
\[
\begin{align*}
\left\{ \begin{array}{l}
dX_3(t) = f_2(X_3(t), X_1(t- \tau), t)dt + g(X_3(t), t)dW(t) \\
X_3(t) = \xi_2(t), t \in [-\tau, 0].
\end{array} \right. \tag{2.29}
\end{align*}
\]
Noting by (2.26) that \( f_1(x, X_1(t - \tau), t) \geq f_2(x, X_1(t - \tau), t) \), together with \( \xi_1(t) \geq \xi_2(t) \) for \( t \in [-\tau, 0] \), we conclude by Lemma 2.2 that \( X_1(t) \geq X_3(t), t \in [-\tau, T] \), a.s. Next consider SDDE with pure jumps

\[
\begin{aligned}
\begin{cases}
  dX_4(t) & = f_2(X_4(t), X_3(t - \tau), t)dt + g(X_3(t), t)dw(t) \\
  + \int_y \gamma(X_4(t^-), X_3((t - \tau)^-), t, u)N(dt, du) \\
  X_4(t) & = \xi_2(t), t \in [-\tau, 0],
\end{cases}
\end{aligned}
\]

which could be rewritten as

\[
\begin{aligned}
\begin{cases}
  dX_4(t) & = [f_2(X_4(t), X_1(t - \tau), t) + (f_2(X_4(t), X_3(t - \tau), t) - f_2(X_4(t), X_1(t - \tau), t))]dt \\
  & + g(X_3(t), t)dw(t) + \int_y [\gamma(X_4(t^-), X_1((t - \tau)^-), t, u) \\
  & + (\gamma(X_4(t^-), X_3((t - \tau)^-), t, u) - \gamma(X_4(t^-), X_1((t - \tau)^-), t, u))]N(dt, du) \\
  X_3(t) & = \xi_2(t), t \in [-\tau, 0].
\end{cases}
\end{aligned}
\]

Recalling \( X_1(t) \geq X_3(t), t \in [-\tau, T] \) a.s., by (2.28) it follows that

\[
\begin{aligned}
f_2(x, X_1(t - \tau), t) & \geq f_2(x, X_3(t - \tau), t) \quad \text{and} \quad \gamma(x, X_1((t - \tau)^-), t, u) \geq \gamma(x, X_3((t - \tau)^-), t, u).
\end{aligned}
\]

Again by Lemma 2.2 \( X_3(t) \geq X_4(t), t \in [-\tau, T] \) a.s. In what follows, repeating the previous procedure we can get the sequence

\[
X_1(t) \geq X_3(t) \geq X_4(t) \geq X_5(t) \geq \cdots \geq X_n(t) \geq \cdots \text{ a.s.} ,
\]

where \( X_n(t) \) satisfies the following equation

\[
\begin{aligned}
\begin{cases}
  dX_n(t) & = f_2(X_n(t), X_{n-1}(t - \tau), t)dt + g(X_n(t), t)dw(t) \\
  & + \int_y \gamma(X_n(t^-), X_{n-1}((t - \tau)^-), t, u)N(dt, du) \\
  X_n(t) & = \xi_2(t), t \in [-\tau, 0].
\end{cases}
\end{aligned}
\]

In what follows we intend to show that \( X_n(t) \) is a Cauchy sequence, which has a unique limit \( X(t) \), and \( X(t) = X_2(t), t \in [0, T] \), giving the desired assertion. Denote by \( L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}) \), the space of \( \mathbb{R} \)-valued and \( \mathcal{F}_t \)-adapted stochastic processes with \( \mathbb{E} \int_0^T |\varphi(t)|^2 dt < \infty \), equipped with the norm

\[
||v||_\beta := \left( \mathbb{E} \int_0^T |v(s)|^2 e^{-\beta s} ds \right)^{\frac{1}{2}},
\]

where \( \beta \) is a positive constant to be determined. Obviously, the norm \( ||v||_\beta \) is equivalent to the original one \( ||v|| := \mathbb{E} \int_0^T |\varphi(t)|^2 dt \) for \( v \in L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}) \). For simplicity, set \( \hat{X}_n(t) := \)
we then have
\[ X_n(t) - X_{n-1}(t), n \geq 4. \]
Applying Itô’s formula we find for any \( t \in [0, T] \)
\[
\mathbb{E}(e^{-\beta t}|\bar{X}_n(t)|^2) \\
= \mathbb{E} \int_0^t -\beta e^{-\beta s}|\bar{X}_n(s)|^2 ds + \mathbb{E} \int_0^t e^{-\beta s}|2\bar{X}_n(s)(f_2(X_n(s), X_{n-1}(s - \tau), s) \\
- f_2(X_{n-1}(s), X_{n-2}(s - \tau), s)) + |g(X_n(s), s) - g(X_{n-1}(s), s)|^2| ds \\
+ \mathbb{E} \int_0^t \int_{\mathcal{Y}} e^{-\beta s} \left[ 2\bar{X}_n(s) \left( \gamma(X_n(s^-), X_{n-1}((s - \tau)^-), s, u) \\
- \gamma(X_{n-1}(s^-), X_{n-2}((s - \tau)^-), s, u) \right) \\
+ \left| \gamma(X_n(s^-), X_{n-1}((s - \tau)^-), s, u) - \gamma(X_{n-1}(s^-), X_{n-2}((s - \tau)^-), s, u) \right|^2 \right] N(ds, du).
\]
This, together with (2.24) and (2.25), yields that
\[
\mathbb{E}(e^{-\beta t}|\bar{X}_n(t)|^2) \leq \mathbb{E} \int_0^t -\beta e^{-\beta s}|\bar{X}_n(s)|^2 ds + \mathbb{E} \int_0^t e^{-\beta s}[2L|\bar{X}_n(s)|^2 + L|\bar{X}_{n-1}(s)|^2 \\
+ 2L^{\frac{1}{2}}(1 + (\lambda(\mathcal{Y}))^{\frac{1}{2}})|\bar{X}_n(s)|(|\bar{X}_n(s)| + |\bar{X}_{n-1}(s)|)] ds \\
\leq (-\beta + 2L + 3L^{\frac{1}{2}}(1 + (\lambda(\mathcal{Y}))^{\frac{1}{2}})) \mathbb{E} \int_0^t e^{-\beta s}|\bar{X}_n(s)|^2 ds \\
+ (L + L^{\frac{1}{2}}(1 + (\lambda(\mathcal{Y}))^{\frac{1}{2}})) \mathbb{E} \int_0^t e^{-\beta s}|\bar{X}_{n-1}(s)|^2 ds.
\]
Letting
\[
\beta = 5(L + L^{\frac{1}{2}}(1 + (\lambda(\mathcal{Y}))^{\frac{1}{2}})),
\]
we then have \( -\beta + 2L + 3L^{\frac{1}{2}}(1 + (\lambda(\mathcal{Y}))^{\frac{1}{2}}) < 0 \) and
\[
(L + L^{\frac{1}{2}}(1 + (\lambda(\mathcal{Y}))^{\frac{1}{2}}))/(\beta - (2L + 3L^{\frac{1}{2}}(1 + (\lambda(\mathcal{Y}))^{\frac{1}{2}}))) = \frac{1}{2}.
\]
Hence
\[
\mathbb{E} \int_0^t e^{-\beta s}|\bar{X}_n(s)|^2 ds \leq \frac{1}{2} \mathbb{E} \int_0^t e^{-\beta s}|\bar{X}_{n-1}(s)|^2 ds,
\]
which implies by induction arguments that
\[
\mathbb{E} \int_0^t e^{-\beta s}|\bar{X}_n(s)|^2 ds \leq \frac{1}{2^{n-4}} \mathbb{E} \int_0^t e^{-\beta s}|\bar{X}_4(s)|^2 ds.
\]
Therefore \( \bar{X}_n(t) \) is a Cauchy sequence in \( L^2_{\mathcal{H}}([0, T]; \mathbb{R}) \) with the norm \( \| \cdot \|_{\beta} \) and has a unique limit denoted by \( X(t) \in L^2_{\mathcal{H}}([0, T]; \mathbb{R}) \) since the space \( L^2_{\mathcal{H}}([0, T]; \mathbb{R}) \) is a complete norm space.
under the norm $\| \cdot \|_{-\beta}$. Next we show $X_2(t) = X(t)$ by the uniqueness. In fact, by (2.24)

$$
E \int_0^T e^{-\beta t} \left\| \int_0^t [f_2(X_n(s), X_{n-1}(s-\tau), s) - f_2(X(s), X(s-\tau), s)]ds \right\|^2 dt \\
\leq LT^2E \int_0^T \int_0^t e^{-\beta (t-s)} e^{-\beta s} (|X_n(s) - X(s)|^2 + |X_{n-1}(s) - X(s)|^2)ds dt \\
\leq LT^2E \int_0^T e^{-\beta s} (|X_n(s) - X(s)|^2 + |X_{n-1}(s) - X(s)|^2)ds \\
\rightarrow 0 \text{ as } n \rightarrow \infty,
$$

and, according to It’s isometry

$$
E \int_0^T e^{-\beta t} \left\| \int_0^t \gamma(X_n(s), X_{n-1}(s-\tau), s, u) - \gamma(X(s), X(s-\tau), s, u) \right\|^2 N(ds, du) dt \\
\leq CE \int_0^T e^{-\beta t} \int_0^t \int_\mathbb{Y} |\gamma(X_n(s), X_{n-1}(s-\tau), s, u) - \gamma(X(s), X(s-\tau), s, u)|^2 \lambda(du) ds dt \\
\leq LCTE \int_0^T e^{-\beta s} (|X_n(s) - X(s)|^2 + |X_{n-1}(s) - X(s)|^2)ds \\
\rightarrow 0 \text{ as } n \rightarrow \infty,
$$

where $C := 2(1 + T\lambda(\mathbb{Y}))$ and, carrying out the previous arguments,

$$
E \int_0^T e^{-\beta t} \left\| \int_0^t [g(X_n(s), s) - g(X(s), s)]dW(s) \right\|^2 dt \rightarrow 0 \text{ as } n \rightarrow \infty.
$$

As a consequence, we could conclude that $X$ satisfies

$$
\begin{cases}
    dX(t) = f_2(X(t), X(t-\tau), t)dt + g(X(t), t)dW(t) \\
    + \int_\mathbb{Y} \gamma(X(t^-), X((t-\tau)^-), t, u)N(dt, du) \\
    X(t) = \xi_2(t), t \in [-\tau, 0].
\end{cases}
$$

By the uniqueness of solution of Eq. (2.23) we conclude that $X(t) = X_2(t)$ and, recalling (2.30), the desired assertion is complete. \[ \square \]

**Remark 2.2.** [13] established an example to show that condition (2.26) is vital for the comparison theorem for SDDEs. By Example 2.3, we could conclude that, if the jump diffusion $\gamma$ is nonincreasing in second variable, namely, delay term, the comparison theorem might not be available. Therefore the condition (2.28) is natural. With respect to (2.27), we can refer to Situ [12] and Peng and Zhu [10] for more details for SDEs with jumps.

**Remark 2.3.** By carrying out the technique of stopping times, the derived comparison theorem can be generalized to the case where Lipschitz condition is replaced by the Carathéodory-type condition [12].


3 Comparison Theorem for SDDEs with Compensator Jump Processes

In the last section we establish the comparison theorem for SDDEs with pure jump processes. To make the content more comprehensive, in this part we aim to discuss the comparison problems for SDDEs with compensator jump process.

Consider two one-dimensional SDDEs with jumps for any $t \in [0, T]$

\[
\begin{cases}
    dX_1(t) = f_1(X_1(t), X_1(t-\tau), t)dt + g(X_1(t), t)dW(t) \\
    + \int_\gamma \gamma(X_1(t^-), X_1((t-\tau)^-), t, u)\tilde{N}(dt, du)
    \\
    X_1(t) = \xi_1(t), t \in [-\tau, 0],
\end{cases}
\]

and

\[
\begin{cases}
    dX_2(t) = f_2(X_2(t), X_2(t-\tau), t)dt + g(X_2(t), t)dW(t) \\
    + \int_\gamma \gamma(X_2(t^-), X_2((t-\tau)^-), t, u)\tilde{N}(dt, du)
    \\
    X_2(t) = \xi_2(t), t \in [-\tau, 0].
\end{cases}
\]

Noting that $\tilde{N}(dt, du) = N(dt, du) - \lambda(du)dt$, Eq. (3.1) and Eq. (3.2) are equivalent to

\[
\begin{cases}
    dX_1(t) = \left[ f_1(X_1(t), X_1(t-\tau), t) - \int_\gamma \gamma(X_1(t^-), X_1((t-\tau)^-), t, u)\lambda(du) \right]dt \\
    + g(X_1(t), t)dW(t) + \int_\gamma \gamma(X_1(t^-), X_1((t-\tau)^-), t, u)N(dt, du)
    \\
    X_1(t) = \xi_1(t), t \in [-\tau, 0],
\end{cases}
\]

and

\[
\begin{cases}
    dX_2(t) = \left[ f_2(X_2(t), X_2(t-\tau), t) - \int_\gamma \gamma(X_2(t^-), X_2((t-\tau)^-), t, u)\lambda(du) \right]dt \\
    + g(X_2(t), t)dW(t) + \int_\gamma \gamma(X_2(t^-), X_2((t-\tau)^-), t, u)N(dt, du)
    \\
    X_2(t) = \xi_2(t), t \in [-\tau, 0],
\end{cases}
\]

respectively.

Based on the comparison theorem, Theorem 2.1, we could derive the following comparison results for stochastic delay systems with compensator jump processes.

\textbf{Theorem 3.1.} Let conditions (2.24)-(2.27) hold. Moreover, we suppose that $f_2 - \gamma$ and $\gamma$ is non-decreasing with respect to the second variable, that is, for $t \in [0, T]$ and fixed $x \in \mathbb{R}$ and $u \in \mathcal{Y}$,

\[
f_2(x, y, t) - \int_{\mathcal{Y}} \gamma(x, y, t, u)\lambda(du) \geq f_2(x, z, t) - \int_{\mathcal{Y}} \gamma(x, z, t, u)\lambda(du)
\]

and

\[
\gamma(x, y, t, u) \geq \gamma(x, z, t, u)
\]

whenever $y \geq z$. Then we have

\[
X_1(t) \geq X_2(t), t \in [0, T] \text{ a.s. provided that } \xi_1(t) \geq \xi_2(t) \text{ with } t \in [-\tau, 0].
\]
Example 3.1. Consider two one-dimensional SDDEs with jumps

\[
\begin{cases}
    dX_1(t) = f_1(X_1(t), X_1(t-\tau), t)dt + g(X_1(t), t)dW(t) \\
    \quad + \int_Y \rho(u) f_2(X_1(t), X_1(t-\tau), t)\tilde{N}(dt, du) \\
    X_1(t) = \xi_1(t), t \in [-\tau, 0],
\end{cases}
\]

and

\[
\begin{cases}
    dX_2(t) = f_2(X_2(t), X_2(t-\tau), t)dt + g(X_2(t), t)dW(t) \\
    \quad + \int_Y \rho(u) f_2(X_2(t), X_2(t-\tau), t)\tilde{N}(dt, du) \\
    X_2(t) = \xi_2(t), t \in [-\tau, 0],
\end{cases}
\]

where \(f_1, f_2, g\) satisfy the conditions (2.24) - (2.26) and \(\xi_1(t) \geq \xi_2(t), t \in [-\tau, 0]\).

In what follows, we further assume that \(\rho > 0, \int_Y \rho(u)\lambda(du) < 1\) and

\[
f_2(x, y, t) \geq f_2(x, z, t) \text{ whenever } y \geq z. \tag{3.7}
\]

By Theorem 3.1, to show that \(X_1(t) \geq X_2(t), t \in [-\tau, T], \text{ a.s.}\), it is sufficient to check conditions (2.27), (3.3) and (3.6). By (3.7) and \(\rho > 0\), it is easy to see that conditions (2.27) and (3.6) hold. On the other hand, recalling \(\int_Y \rho(u)\lambda(du) < 1\), we have

\[
f_2(x, y, t) - \int_Y \rho(u)\lambda(du) f_2(x, y, t) = \left(1 - \int_Y \rho(u)\right) f_2(x, y, t),
\]

and, combining (3.7), condition (3.6) is also true.

References

[1] Anderson, W., Local behaviour of solutions of stochastic integral equations, \textit{Trans. Amer. Math. Soc.}, \textbf{164} (1972), 309-321.

[2] Applebaum, D., \textit{Lévy process and Stochastic Calculus}, Cambridge University Press, 2004.

[3] Cao, Y., Sun, Y. and Lam, J., Delay-dependent robust H control for uncertain systems with time-varying delays, \textit{IEE Proc. Control Theory Appl.}, \textbf{145} (1998), 338-344.

[4] Gal’cuk, L. and Davis, M., A note on a comparison theorem for equations with different diffusions, \textit{Stochastics}, \textbf{6} (1982), 147149.

[5] Ikeda, N. and Watanabe, S., A comparison theorem for solutions of stochastic differential equations and its applications, \textit{Osaka J. Math.}, \textbf{14} (1977), 619-633.

[6] Mao, X., A note on comparison theorems for stochastic differential equations with respect to semimartingales, \textit{Stochastics}, \textbf{37} (1991), 49-59.
[7] O’Brien, G., A new comparison theorem for solution of stochastic differential equations, *Stochastics*, 3 (1980), 245-249.

[8] Park, P., A delay-dependent stability criterion for systems with uncertain linear systems, *IEEE Trans. Automat. Control*, 44 (1999).

[9] Peng, S. and Yang, Z., Anticipated backward stochastic differential equations, *Ann. Probab.*, 37 (2009), 877-902.

[10] Peng, S. and Zhu, X., Necessary and sufficient condition for comparison theorem of 1-dimensional stochastic differential equations, *Stochastic Process. Appl.*, 116 (2006), 370-380.

[11] Peszat, S. and Zabczyk, J., *Stochastic Partial Differential Equations with Lévy Noise: An Evolution Equation Approach*, Cambridge University Press, 1st Edition, 2007.

[12] Rong, S., *Theory of Stochastic Differential Equations with Jumps and Applications*, Springer, 2005.

[13] Yang, Z., Mao, X. and Yuan, C., Comparison theorem of one-dimensional stochastic hybrid systems, *Systems Control Lett.*, 57 (2008) 56-63.

[14] Woyczyński, W., *Lévy Processes in the physical sciences*, Birkhäuser, Boston, MA, 2001.

[15] Yan, J., A comparison theorem for semimartingales and its applications, Séminaire de Probabilités, XX, Lecture Notes in Mathematics, 1204, Springer, Berlin, 1986, 349-351.

[16] Yamada, T., On comparison theorem for solutions of stochastic differential equations and its applications, *J. Math. Kyoto Univ.*, 13 (1973), 497-512.