GENERALIZED FOURIER-MUKAI TRANSFORMS

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ABSTRACT. We study a generalization of the Fourier-Mukai transform for smooth projective varieties. We find conditions under which the transform satisfies an inversion theorem. This is done by considering a series of four conditions on such transforms which increasingly constrain them. We show that a necessary condition for the existence of such transforms is that the first Chern classes must vanish and the dimensions of the varieties must be equal. We introduce the notion of bi-universal sheaves. Some examples are discussed and new applications are given, for example, to prove that on polarised abelian varieties, each Hilbert scheme of points arises as a component of the moduli space of simple bundles. The transforms are used to prove the existence of numerical constraints on the Chern classes of stable bundles.

INTRODUCTION

A Fourier transform could be loosely be described as ‘pullback a function to $\mathbb{R}^n \times \mathbb{R}^n$ multiply by $\exp(2\pi i \langle x, y \rangle)$ and take the direct image (integrate) with respect to the second variable’. If we regard this as a transformation from $L^2$ integrable functions to themselves then this satisfies certain useful properties such as having an inverse, the Parseval theorem, the convolution theorem, etc. The Fourier-Mukai transform was introduced in [12] and is formally analogous to the Fourier transform but acts on the derived category of (bounded) complexes of sheaves on an abelian variety $T$ and maps this to the same category but for the dual abelian $\hat{T}$ variety. It too satisfies useful properties such as the Fourier Inversion Theorem (FIT), the Parseval Theorem, the Convolution Theorem, etc (see [13]). More precisely, the role of $\exp(2\pi i \langle x, y \rangle)$ is played by the Poincaré line bundle over $T \times \hat{T}$ and the direct image needs to be derived.

In [14], Mukai showed that a similar transform gave rise to a FIT on the level of $K$-Theory using the universal bundle $E$ over $S \times \mathcal{M}(S)$ instead of the Poincaré bundle, where $S$ is a K3 surface and $\mathcal{M}(S) \cong S$ is a two dimensional moduli space of simple sheaves on $S$. This was shown to give rise to a FIT in [1] and which satisfies the Parseval Theorem. The question arises: when do such transformations give rise to Inversion Theorems in more general contexts? We shall go some way to answering this question in this paper by classifying such transformations and giving conditions that they give rise to Inversion Theorems. We shall restrict our attention in the examples to the case of holomorphic varieties although it is not too hard to extend our results to the case of varieties over more general fields. The theorems...
are at their strongest in dimensions 1 and 2 but restricted forms apply to higher dimensions as well. Our first aim is to give a rough classification of such transforms.

In section 7 we study two applications of the theory of Fourier-Mukai transforms to the case of abelian varieties. A new transform is constructed which is based on a certain component of the moduli space of simple bundles on abelian varieties. This is then used to deduce quite quickly that each Hilbert scheme of points arises as a component of the moduli space of simple torsion-free sheaves. This is expressed in Theorem 7.10. We can also use these the general theory to prove that if the torus acts freely and effectively on any moduli component then the Euler characteristic of the sheaves parametrised by that component must be ±1 (see Theorem 7.9).

Notation 0.1. We shall let \( X \) and \( Y \) be two smooth varieties and we shall consider pairs of functors \( R\Phi : D(X) \to D(Y) \) and \( R\hat{\Phi} : D(Y) \to D(X) \), where \( D(X) \) denotes the derived category of bounded complexes of coherent sheaves on \( X \). In this paper, for the sake of clarity, we give names to the various possible types of such pairs. We use the terms invertible correspondence, adjunction, Verdier and Fourier-Mukai type. These are not mutually exclusive conditions. They are each treated in the first four sections.

We use the notation \( T_E \) to denote the functor \( F \mapsto E \otimes F : D(X) \to D(X) \), where \( E \in D(X) \).

1. Invertible correspondences and resolutions of the diagonal

Let \( X \xleftarrow{x} Z \xrightarrow{y} Y \) be flat maps of smooth quasi-projective varieties. Let \( D(S) \) denote the derived category of bounded complexes of coherent sheaves on \( S \). Fix two objects \( P \) and \( Q \) in \( D(Z) \) and define two functors \( R\Phi_P : D(X) \to D(Y) \) and \( R\hat{\Phi}_Q : D(Y) \to D(X) \) by

\[
R\Phi_P(-) = Ry_* (x^* - \otimes P) \\
R\hat{\Phi}_Q(-) = Rx_* (y^* - \otimes Q).
\]

Let \( Z \xleftarrow{p_y} Z_y \xrightarrow{p'_y} Z \) (respectively \( Z_x \)) be the pullback of \( y \) (respectively \( x \)) along itself. Let \( q_x : Z_y \to X \times X \) and \( q_y : Z_x \to Y \times Y \) be the maps defined by \((x \circ p_y, x \circ p'_y)\) and \((y \circ p_x, y \circ p'_x)\).

Theorem 1.1. \( R\Phi_P \) and \( R\hat{\Phi}_Q \) give an equivalence of categories (after a shift of complexes) if and only if the following two conditions hold:

(i) \( Rq_x'(p'_x \otimes p''_y Q) \cong O_{\Delta}[r] \)
(ii) \( Rq_y'(p''_x P \otimes p'_y Q) \cong O_{\Delta'}[r] \),

where \( O_{\Delta} \) is the structure sheaf of the diagonal \( \Delta \subset X \times X \), \( O_{\Delta}' \) is the structure sheaf of diagonal \( \Delta' \) in \( Y \times Y \), and \( r \) is an integer. All isomorphisms are quasi-isomorphisms of complexes.

In other words we must have that the LHS’s of (i) and (ii) are resolutions of the both diagonals if we are to have an inversion theorem for such transforms. The proof of the sufficiency of the conditions is, of course, well known and fairly trivial, but the proof of necessity does require some care and so we reproduce it here.
Proof. Without loss of generality we set $r = 0$. Condition (i) will be equivalent to the fact that $\mathbb{R}\hat{\Phi}_P$ has a left inverse and (ii) will be the corresponding statement for $\mathbb{R}\hat{\Phi}_Q$. Hence, it suffices to consider only (i).

Now, $\mathbb{R}\hat{\Phi}_Q \circ \mathbb{R}\Phi_P(E) = \mathbb{R}x_*(y^*\mathbb{R}y_*(x^*E \otimes P \otimes Q))$. Using $Z_y$ and the base-change formula we can write this as

$$\mathbb{R}x_*(R_{y_*}(p_y^*x^*E \otimes p_y^*P \otimes Q)).$$

Using the projection formula (and the hypotheses on $X$, $Y$ and $Z$ we have

$$\mathbb{R}(x^*p'_y)_*( (x^*p_y)^*E \otimes p_y^*P \otimes p_y^*Q).$$

Let $p_1$ and $p_2$ denote the projections $X \times X \to X$. Then $x^*p_y = p_1^*q_x$ and $x^*p'_y = p_2^*q_x$. Substituting these and using the projection formula again we have

$$\mathbb{R}p_{2*}(p_1^*E \otimes \mathbb{R}q_{2*}(p_2^*P \otimes p_y^*Q)).$$

Suppose condition (i) holds then

$$\mathbb{R}\hat{\Phi}_Q \circ \mathbb{R}\Phi_P = \mathbb{R}p_{2*}(p_1^*E \otimes \mathcal{O}_\Delta) = E.$$ 

This proves the sufficiency of (i). To see that it is necessary observe that \[1.1\] still holds. Assume that $\mathbb{R}\hat{\Phi}_Q \circ \mathbb{R}\Phi_P \cong \text{Id}$ and put $E = \mathcal{O}_\alpha$, the structure sheaf of a point $\alpha \in X$. Let $A_*$ be a (bounded) complex of locally-free $\mathcal{O}_{X \times X}$-modules representing $\Gamma = \mathbb{R}q_{2*}(p_2^*P \otimes p_y^*Q)$. By assumption, we have the quasi-isomorphism

$$\mathbb{R}p_{2*}(\Gamma \otimes p_1^*\mathcal{O}_\alpha) \simeq \mathcal{O}_\alpha. \tag{1.2}$$

Then

$$\mathbb{R}^*p_{2*}(\Gamma \otimes p_1^*\mathcal{O}_\alpha) = H^*(p_{2*}(A_* \otimes p_1^*\mathcal{O}_\alpha)).$$

But this is concentrated in position 0 and so we see that $H_i(A_* \otimes p_1^*\mathcal{O}_\alpha) = 0$ for all $i \neq 0$ and so $\Gamma$ is flat over $p_1$. Now the fibres of $\Gamma$ are given by $\Gamma_{(\alpha,\beta)} = \mathbb{R}p_{2*}(\Gamma \otimes p_1^*\mathcal{O}_\alpha \otimes p_2^*\mathcal{O}_\beta)$. We can rewrite this as $\mathbb{R}p_{2*}(\Gamma \otimes p_1^*\mathcal{O}_\alpha) \otimes \mathcal{O}_\beta = \mathcal{O}_\alpha \otimes \mathcal{O}_\beta$. This is zero if $\alpha \neq \beta$. Hence, $\Gamma$ is supported on $\Delta_X$ and the case $\alpha = \beta$ implies that it has rank 1 everywhere along this diagonal. If we now substitute $E = \mathcal{O}_X$ then a standard hypercohomology argument shows that it must also be trivial.

Definition 1.2. We shall call functors $\mathbb{R}\Phi$ and $\mathbb{R}\hat{\Phi}$ a invertible correspondence transforms if they satisfy the conditions of Theorem [1.1]. These can also be thought of as transformations satisfying the ‘Fourier Inversion Theorem’ or FIT.

The invertible correspondence transforms should be compared to the notion of tilting transforms for categories of modules over associative rings. Details of this can be found in [18].

As an intermediate example one can consider the Beilinson spectral sequence which could be viewed as a composition of derived functors from the derived category of coherent sheaves on $CP^n$ to the derived category of finitely generated modules over a suitable algebra (the path algebra of a certain quiver) see [2] and [3]. This also works for more general varieties, see King [10].

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1I am grateful to the referee for pointing out a simplification in the original version of this proof.
We shall see in section 3 that when we consider invertible correspondences for varieties then their existence imposes quite strong condition on the varieties.

2. Transforms of Adjunction Type

In this section we shall look briefly at the categorical aspects of equivalences of such transforms. The following is well known:

**Theorem 2.1.** (See [11]). Let $F : A \to B$ and $G : B \to A$ be two functors of categories. If $F$ and $G$ give rise to an equivalence of categories (i.e. $F \circ G$ and $G \circ F$ are naturally equivalent to their respective identity functors) then $F \dashv G \dashv F$. Conversely, if $F \dashv G \dashv F$ and $F$ is fully faithful such that it surjects on objects up to isomorphism (we shall say quasi-surjects) then $F$ and $G$ determine an equivalence of categories.

**Definition 2.2.** We say that a pair of functors $F$ and $G$ are a transform of adjunction type if $F \dashv G \dashv F$.

The theorem says that invertible correspondence transforms are adjunction transforms. When we have an adjunction type transform then we often already know that the functors are inverses on one side. For example, suppose that we already know that the adjunction gives rise to $GF \cong \text{Id}$. Recall that an adjunction gives rise to natural transformations $\eta : \text{Id} \rightarrow GF$ and $\epsilon : FG \rightarrow \text{Id}$ called the unit and counit of the adjunction $F \dashv G$ similarly $G \dashv F$ gives rise to $\eta' : \text{Id} \rightarrow FG$ and $\epsilon' : GF \rightarrow \text{Id}$. Then it is well known that $G$ is faithful if and only if $\epsilon_a$ is epimorphic for all $a$ and $G$ is full if and only if $\epsilon_a$ is a split monic (i.e. has a left inverse). The same holds for $F$ if we replace $\epsilon$ by $\epsilon'$. If we already know that $\epsilon'$ is a natural isomorphism then $F$ must be full and faithful and that $G$ is full. But $GFa \cong a$ for each $a$ and so $G$ must quasi-surject on objects. It now follows that $F$ and $G$ are an equivalence of categories if and only if $G$ is faithful. We summarise this in the following.

**Proposition 2.3.** Suppose that $F$ and $G$ form a transform of adjunction type such that $GF \rightarrow \text{Id}$ is an isomorphism. Then $F$ satisfies the Parseval Theorem:

$$\text{Hom}(a, b) \cong \text{Hom}(Fa, Fb)$$

Furthermore, $F$ and $G$ form an equivalence of categories if and only if $G$ satisfies the Parseval Theorem.

3. Transforms of Verdier Type

We now look at more specific transforms. We limit ourselves to smooth projective varieties $X$ and $Y$ and let $Z = X \times Y$. Let $n = \dim X$ and $m = \dim Y$. Note that we have

$$R\hat{\Phi}(F) = R\pi_* R\mathcal{H}om(P, y^* F),$$

where $Q = R\mathcal{H}om(P, O)$ and $P$ is a sheaf. We shall see that this choice of $Q$ is forced on us up to the pullback of (a power of) the canonical bundle and a shift. Recall that we have Grothendieck-Verdier duality:

$$\text{Hom}_{D(X)}(R\pi_* F, G) \cong \text{Hom}_{D(X \times Y)}(F, x^* G \otimes L \omega_X[n]).$$

$$\text{Hom}_{D(Y)}(R\pi_* F, G) \cong \text{Hom}_{D(X \times Y)}(F, y^* G \otimes x^* \omega_Y[m]).$$

(see Hartshorne [8] III.11). Applying this and the classical adjunction $f^* \dashv Rf_*$ to $R\Phi$ and $R\hat{\Phi}$ we obtain
Proposition 3.1. \[ R\Phi \dashv T_{\omega_X} \circ R\Phi[n] \text{ and } R\check{\Phi} \dashv T_{\omega_Y} \circ R\Phi[m] , \]
where \( Q = R\mathcal{H}om(P, \mathcal{O}_{X \times Y}) \).

Proof. This is just a computation using the adjunctions above:
\[
\text{Hom}_{D(Y)}(Ry_*(x^*F \otimes P), G) \cong \text{Hom}_{D(X \times Y)}((x^*F \otimes P, y^*G \otimes x^*\omega_X)[n])
\]
\[
= \text{Hom}_{D(X)}(F, Rx_*R\mathcal{H}om(P, y^*G) \otimes \omega_X[n])
\]
Similarly for the other adjunction. \( \square \)

Remark 3.2. We could use \( Q = R\mathcal{H}om(P, x^*L \otimes y^*M) \) for line bundles \( L \) on \( X \) and \( M \) on \( Y \). This does not affect the adjunctions except we have to twist by one or other of these line bundles or their duals.

Proposition 3.3. Let \( R\Delta_X = R\mathcal{H}om(-, \omega_X)[n] \) and \( R\Delta_Y = R\mathcal{H}om(-, \omega_Y)[m] \). Then
\[
R\Delta_Y \circ R\Phi_P \cong T_{\omega_Y} \circ R\Phi_Q \circ R\Delta_X[m],
\]
\[
R\Delta_X \circ R\check{\Phi}_P \cong T_{\omega_X} \circ R\check{\Phi}_Q \circ R\Delta_Y[n],
\]
where \( Q = R\mathcal{H}om(P, \mathcal{O}) \)

Proof. We use local Verdier duality:
\[
R\mathcal{H}om(Ry_*(x^*F \otimes P), \omega_Y) \cong Ry_*R\mathcal{H}om(x^*F \otimes x^*\omega_X, R\mathcal{H}om(P, y^*\omega_Y[n + m]))
\]
\[
\cong Ry_*R\mathcal{H}om(F, \omega_X) \otimes R\mathcal{H}om(P, y^*\omega_Y)[n + m])
\]
\[
\cong Ry_*R\mathcal{H}om(F, \omega_X)[n]) \otimes R\mathcal{H}om(P, \mathcal{O})[m] \otimes \omega_Y
\]
Similarly for \( R\check{\Phi} \). \( \square \)

Theorem 3.4. Let \( X \) and \( Y \) be two smooth projective varieties of dimensions \( n \) and \( m \) respectively. Let \( P \) and \( Q \) be two complexes of coherent sheaves on \( X \times Y \). We assume that \( X \neq Y \) and \( P \) is a sheaf of rank at least 1. Define two functors \( D(X) \rightarrow D(Y) \) and \( D(Y) \rightarrow D(X) \) by \( R\Phi(E) = Ry_*(x^*E \otimes P) \) and \( R\check{\Phi}(F) = Rx_*(y^*F \otimes Q) \) respectively, where \( x \) and \( y \) are the projections from \( X \times Y \) to \( X \) and \( Y \) respectively. If \( R\Phi \) and \( R\check{\Phi} \) form an equivalence of categories then \( \omega_X^k = \mathcal{O}_X \), \( \omega_Y^k = \mathcal{O}_Y \) for some integer \( k \), \( n = m \) and \( Q = R\mathcal{H}om(P, x^*\omega_X)[n] = R\mathcal{H}om(P, y^*\omega_Y)[n] \). In particular, we must have \( c_1(X) = 0 \).

Proof. This follows from Proposition 3.3 and the fact that adjoints are unique up to natural isomorphism. We may assume that \( Q \) takes the form \( R\mathcal{H}om(P, x^*\omega_x)[n] \) because the adjoint of \( R\Phi P \) is \( R\check{\Phi} R\mathcal{H}om(P, x^*\omega_X)[n] \) by duality. The adjoints give
\[
R\mathcal{H}om(R\mathcal{H}om(P, \mathcal{O}), \mathcal{O}) \otimes x^*\omega_X^m \otimes y^*\omega_Y[m - n] = P
\]
which can only happen if \( n = m \) and, since \( P \) has support on the whole of the product, \( \omega_X^k = \mathcal{O}_X \) and \( \omega_Y^k = \mathcal{O}_Y \) for some integer \( k \) as required. In fact, what we need is that \( P \otimes x^*\omega_X \otimes y^*\omega_Y^* \cong P \). \( \square \)
Remark 3.5. In other words, an invertible correspondence transform given by, for example, a torsion-free sheaf $P$ for smooth projective varieties can only exist if the dimensions of the varieties are the same and their canonical bundles are trivial or torsion (with order dividing the rank of $P$). Moreover, the transform complex $Q$ must be the (derived) dual of $P$ (up to a twist and shift). Relative versions of the results of this section are also available where we have $Z = X \times S Y$ for some scheme $S$. In fact, we can clearly strengthen the conclusion to say that the canonical bundles must act trivially on the the restrictions of $P$ to $X \times \{y\}$ and $\{x\} \times Y$. This will happen, for example, for relative transforms where the fibres have trivial canonical bundle.

Definition 3.6. We say that a pair of transforms $R\Phi_P$ and $R\hat{\Phi}_Q$ is of Verdier type if $\omega_X = \mathcal{O}_X$, $\omega_Y = \mathcal{O}_Y$, dim $X = \dim Y$ and $Q = R\mathcal{H}om(P, \mathcal{O}_{X \times Y})$ (up to a shift). So we have the chain of implications

invertible correspondence $\implies$ Verdier $\implies$ adjunction

4. Fourier-Mukai Transforms

Definition 4.1. Let $R\Phi$ and $R\hat{\Phi}$ be transforms of Verdier type as in the last section but with the additional constraint that $P$ is a locally-free sheaf over $X \times Y$. We also assume that they give rise to an equivalence of categories. Then we say that they are transforms of Fourier-Mukai type. In other words, the transforms are both of invertible correspondence type and Verdier type with some constraint on the choice of $P$.

Let Spl$(S)$ denote the moduli space of simple sheaves on $S$. Recall that Mukai [14] proves that this space is smooth when $S$ is an abelian surface or a K3 surface.

Proposition 4.2. Suppose that $R\Phi$ and $R\hat{\Phi}$ form a pair of functors which are of Fourier-Mukai type. Then $n = r$, where $n = \dim X$ and $r$ is given in Theorem [14]. Furthermore, if $X$ and $Y$ admit smooth moduli of simple torsion-free sheaves then $X$ is a component of Spl$(Y)$ and $Y$ is a component of Spl$(X)$.

Proof. In the proof we only assume that $P$ is torsion-free and flat over both projections. For this we use Proposition 2.26 of [15]. This states that if $f : Z \to Y$ is a proper map of noetherian schemes and $F$ is a coherent sheaf on $Z$ flat over $Y$ and $S \subseteq Y$ is a locally complete intersection then if $H^i(f^{-1}(y), F_y) = 0$ for all $i < \text{codim} S$ and $y \notin S$ then $R^if_*F = 0$ for the same set of $i$. We apply this to $S = \Delta \subseteq X \times X$. Then $\text{codim} S = n$ and so if $r < n$ we have that the cohomology sheaves $R^iq_{x*}(\Gamma)$ of $Rq_{x*}(\Gamma)$ vanish for for $i > r$, where

$$\Gamma = R\mathcal{H}om(p^*_a P, p^*_b P).$$

Hence $(R^iq_{x*}(\Gamma))_{(a, a')} \cong H^r(Y; \mathcal{H}om_{(a, a')}) = 0$ for $(a, a') \in X \times X \setminus \Delta$. Note that $\mathcal{H}om_{(a, a')} \cong R\mathcal{H}om(P_a, P_{a'})$. This implies that $R^iq_{x*}(\Gamma) = 0$ and so $R^iq_{x*}(\Gamma) = 0$ for all $i$; a contradiction. Hence $n = r$. Let $P_a$ denote the restriction of $P$ to $\{a\} \times Y$. We know that $h^n(Y; \mathcal{H}om(P_a, P_a)) = \text{dim Ext}^n(P_a^*, P_a) = 1$ and so $\text{dim Ext}^n(P_a, P_a) = 1$ because $P_a \to P_a^*$ induces a surjection on Ext groups. Then Serre duality implies that $P_a$ is simple. Similarly $P_b$ is simple for all $b \in Y$. On the other hand, $\text{Ext}^n(P_a, P_{a'}) = 0$ for $a \neq a'$ and so $P_a \not\cong P_{a'}$ and hence $X \subseteq \text{Spl}(Y)$.
Suppose that \( E \in \text{Spl}(Y) \) is not in \( X \) but has the same Chern character as \( P_a \). If \( \text{Ext}^i(P_a, E) = 0 \) for all \( i \) and all \( a \in X \) then \( \hat{\Phi} \) is an equivalence of categories and so this is impossible. For a fixed \( a \) the set \( Z_a = \{ E \in \text{Spl}(Y) : \text{Ext}^i(P_a, E) = 0, \forall i \} \) is Zariski open in \( \text{Spl}(Y) \) and non-empty as it contains \( X \setminus \{a\} \). But \( (U_a \cap Z_a) \setminus X \) is empty for any open neighbourhood \( U_a \) of \( P_a \) in \( \text{Spl}(Y) \). This shows that a component of \( \text{Spl}(Y) \) which contains \( X \) is smooth at each point of \( X \) and has the same dimension as \( X \). \( \square \)

**Remark 4.3.** In the case when \( n = 1 \) or \( n = 2 \) we can proceed more directly and explicitly. Assume for simplicity that \( P \) is locally-free and consider first the case \( n = 1 \). Note that \( \mathbf{R}\Phi \) must be an elliptic curve by 3.4. The fact that \( \mathbf{R}\Phi \) and \( \mathbf{R}\hat{\Phi} \) are invertible correspondences mean that

\[
\mathbf{R}q_{x*}\mathbf{R}\text{Hom}(p_y^*P^*, p_y^*P) = \mathbf{R}q_{x*}\Lambda \cong \mathcal{O}_\Delta[-r]
\]

for some \( r = 0 \) or 1. Serre duality implies that \( R^1q_{x*}\Lambda \neq 0 \) and so \( r = 1 \). Then \( \chi(P_b, P_c) = 0 \) for \( c \neq b \) and hence for \( b = c \) as well. We also have \( \dim \text{Ext}^1(P_b, P_b) = 1 \) and so \( P_b \) is simple sheaves with a 1 dimensional moduli which contains \( Y \). Hence, \( \dim Y = 1 \) as required.

When \( n = 2 \) we argue similarly. Then again \( \chi(P_a^* \otimes P_a') = 0 \) for all \( a, a' \in X \). Note that 3.4 shows that the canonical bundles are trivial and so Serre duality implies that \( r \neq 0 \); otherwise \( H^0(P_b^* \otimes P_b) \neq 0 \) so \( H^2(P_b^* \otimes P_b) \neq 0 \), and hence \( R^2q_{x*}\Lambda \neq 0 \). But if \( r = 1 \) then the support of \( \{ b \in Y : H^1(X, P_b^* \otimes P_b) \neq 0 \} \) is 0-dimensional Mukai’s result above implies that \( R^1q_{x*}\Lambda = 0 \), a contradiction. Hence we must have \( r = 2 \). It also follows that \( P_b \) are all simple. Then \( Y \) is contained in the moduli space of such \( P_b \)'s and hence has dimension at most 2 because \( \chi(P_b, P_b) = 0 \). It cannot have dimension 1.

5. Bi-universal sheaves

In this section we ask the following question. If \( X \) is a smooth complex projective variety with trivial canonical bundle and \( Y = \mathcal{M}(X) \) is a smooth moduli of simple sheaves on \( X \) (also assumed to have trivial canonical bundle) then when do we obtain a Fourier-Mukai transform from \( X \) to \( Y \)? We answer this question by considering various universal sheaves on the product \( X \times Y \).

**Definition 5.1.** Following Mukai, we say that a sheaf \( \mathcal{E} \) is semi-universal on \( X \times Y \) if for all \( y \in Y \), \( \mathcal{E}_y \cong E^{\otimes \sigma} \) for some \( \sigma \in \mathbb{Z} \) and \( \mathcal{E} \) is a universal deformation. If \( \sigma = 1 \) we say that \( \mathcal{E} \) is a universal sheaf (as usual). If \( X \) is isomorphic to a moduli of simple sheaves on \( Y \) such that \( \mathcal{E}_a \cong E^{\otimes \sigma} \) and \( \mathcal{E}_b \cong E^{\otimes \sigma} \) we say that \( \mathcal{E} \) is bi-semi-universal. If either \( \sigma \) or \( \sigma' \) is 1 then we call \( \mathcal{E} \) sesqui-universal and if \( \sigma = \sigma' = 1 \) then we call \( \mathcal{E} \) bi-universal. In all these cases we say that \( \mathcal{E} \) is strongly universal (etc.) if whenever \( b \neq b' \) then \( \text{Ext}^i(\mathcal{E}_b, \mathcal{E}_{b'}) = 0 \) for all \( i \) and similarly for \( a \neq a' \).

Mukai has shown that if \( Y \) is a (representable) component of the moduli of simple sheaves on \( X \) then a semi-universal sheaf always exists (see 3.4 Thm A.5).

** Remark 5.2.** Of course, in dimension 2, if the component of the moduli space consists of, say, stable bundles then the strong condition will always hold.

In the following we let \( P = \mathcal{E} \). We also assume that \( \dim Y = \dim X = n \).

**Proposition 5.3.** If \( \mathcal{E} \) is strongly semi-universal then \( \mathbf{R}\Phi \circ \mathbf{R}\hat{\Phi} \cong \text{Id}^{\otimes \sigma'}[n] \). In particular, \( \mathbf{R}\hat{\Phi} \) is faithful. If \( \mathcal{E} \) is strongly bi-semi-universal then \( \sigma = \sigma' \).
Proof. The conditions on $E$ ensures that $R^{i}q_{\ast}\Gamma = 0$ for $i < n$ and for $i = n$ is supported on the diagonal. That it is trivial and given by such a direct sum follows from the fact that $R\hat{\Phi}(O_{b}) \cong (E_{b}^{\oplus}a)^{\ast}[n]$ and $R\Phi(E_{b}) = O_{b}^{\oplus}a$. Then $R\Phi R\hat{\Phi}$ is faithful and so $R\hat{\Phi}$ is. If $E$ is bi-universal then we also have $R\Phi(E_{b}^{\ast}) \cong O_{b}^{\oplus}a$. So $R\Phi R\hat{\Phi}(O_{b}) = R\Phi(E_{b}^{\oplus}a)^{\ast} = O_{b}^{\oplus}a^{2}$. On the other hand, $R\Phi R\hat{\Phi} = I$ and so $a^{2} = a'$. 

Corollary 5.4. If $E$ is strongly sesqui-universal then it must be strongly bi-universal.

Corollary 5.5. From 2.3 we see that if $E$ is strongly universal then the following are equivalent

1. $E$ gives rise to a Fourier-Mukai transform.
2. $E$ is sesqui-universal.
3. $R\Phi$ satisfies the Parseval Theorem.

6. Examples

Example 6.1. We shall now look at some examples of Fourier-Mukai transforms. The first is the Mukai transform itself. For this we set $P = \mathcal{P}$, the Poincaré bundle on $T \times \hat{T}$, where $X = T$ is an abelian variety and $Y = \hat{T} \cong Pic^{0} T$ is its dual abelian variety. It was shown in [13] that $R\Phi(P) = R\Phi(P)$ and $R\hat{\Phi}(P)$ are of Fourier-Mukai type. In this particular case we also obtain a convolution theorem (see [13]) as well. Note also that $X \neq Y$. A relative version can also be found in [15]. The Grothendieck-Riemann-Roch Theorem can be used to compute the Chern characters of the transforms:

$$ch(R\Phi(P))_{i} = (-1)^{i} ch(E)_{n-i},$$

where we identify $H^{i}(T)$ with $H^{n-i}(\hat{T})$ via Poincaré duality. It is conventional in this case to use $R\hat{\Phi}(P)$ instead of $R\Phi(P)$.

Example 6.2. The second example first appeared in [15] and was shown to be a Fourier-Mukai transform by Bartocci, Bruzzo and Hernández Ruipérez [1]. In this case $X$ is a K3 surface satisfying suitable conditions for the existence of $Y$, a 2-dimensional space of stable bundles on $X$ which is isomorphic to $X$. Then $P$ is a bi-universal bundle on $X \times Y$. The Grothendieck-Riemann-Roch Theorem also tells us the Chern characters.

Example 6.3. Consider a smooth projective variety $X$ with trivial canonical bundle and $h^{0,p} = 0$ for $p \neq 0, n$ (for example, a K3 surface). Let $P = J_{\Delta}$ be the ideal sheaf of the diagonal in $X \times X$. Then $P$ is strongly bi-universal. This follows because $P_{a} \cong J_{a}$ and, if $a \neq a'$ then $Ext^{i}(J_{a}, J_{a'}) = 0$ for all $i$ as can be easily seen using the long exact sequences induced by the structure sequences of $a$ and $a'$. Note that $Q$ cannot be represented by a single sheaf. The Chern character of the resulting Fourier-Mukai transform of $E \in D(X)$ is

$$\left(\chi(E) - ch_{0}(E), - ch_{1}(E), \ldots, - ch_{1}(E), \ldots, - ch_{n-1}(E), - ch_{n}(E)\right),$$

as can be easily seen from the Grothendieck-Riemann-Roch formula or directly from the structure sequence of $\Delta$. 

Example 6.4. Consider an abelian surface $T$ polarised by $\ell$ such that $\ell^2 = 2r$, with dual polarisation $\hat{\ell}$ of $\hat{T}$. Let $\mathcal{M} = \mathcal{M}(r, \ell, 1)$ be the moduli space of stable bundles with the given Chern characters. Such module spaces have been also considered by Mukai ([12]). We shall prove in Proposition 7.1 below that this is projective and non-empty. It is easy to see that if $E \in \mathcal{M}$ then $R^i\mathcal{F}(E) = \hat{L}^* \otimes \mathcal{P}_x$ for some (assumed symmetric) $\hat{L} \in \hat{\ell}$ and $\mathcal{P}_x \in \text{Pic}(\hat{T}) \cong \mathbb{T}$. This implies that $\mathcal{M} \cong \mathbb{T}$ under $E \mapsto x$. A result of Mukai ([12] Appendix 2) implies that a universal sheaf $E$ exists over $T \times \mathcal{M}$. In fact it is possible to write this down explicitly. Let $\pi_{ij}$ be the projection maps to the $i$th and $j$th factors of $T \times \hat{T} \times T$ and $\pi_i$, the projections to the $i$th factor. Then it is easy to check that $E = R\pi_{13*}(\pi_2^*\hat{L}^* \otimes \pi_3^*\mathcal{P} \otimes \pi_j^*\mathcal{P}^*)$. This also shows that $E$ is bi-universal. Then $R\Phi_E$ is of Fourier-Mukai type. Using Grothendieck-Riemann-Roch (or Lemma 6.7) we find

$$\begin{align*}
\text{ch}(R\Phi(E))_0 &= \text{ch}_0(E) + \text{ch}_1(E) \cdot \ell + r \text{ch}_2(E), \\
\text{ch}(R\Phi(E))_1 &= \text{ch}_1(E) + \text{ch}_2(E)\ell, \\
\text{ch}(R\Phi(E))_2 &= \text{ch}_2(E).
\end{align*}$$

We shall study this example in more detail below.

7. Two Applications

We shall now look at two applications of the general theory of Fourier-Mukai transforms. The first is a special case of Example 6.4 and the second is a generalisation of that example.

First, we let $X = T$ be an abelian surface with a polarisation $\ell$. Let $\mathcal{M} = \mathcal{M}(r, \ell, 1)$ denote the moduli space of (Gieseker) stable bundles of the given Chern character. Choose symmetric representative line bundles $L \in \ell$ and $\hat{L} \in \hat{\ell}$ in $\ell$ and the dual polarisation respectively.

Proposition 7.1. The moduli space $\mathcal{M}$ is isomorphic to $T$.

Proof. Consider the collection $\{R^\ell\hat{\mathcal{F}}(L \otimes \mathcal{P}_x) \mid x \in T\}$. Note that $R^i\hat{\mathcal{F}}(L) = 0$ unless $i = 0$ and so this set consists of vector bundles of Chern character $(r, \ell, 1)$. Moreover, these are all $\mu$-stable since the projective bundle $B = \mathbb{P}R^0\hat{\mathcal{F}}(L)$ has fibres consisting of the linear systems $|L \otimes \mathcal{P}_x|$ which are just translates of $|L|$. So $B$ admits a flat connection given by translation of this linear system. This connection then induces a projectively anti-self-dual connection on each of the elements of the collection. This implies that the bundles are all $\mu$-stable. Since the collection is non-empty and the Mukai transform is an isomorphism of schemes we see that $\mathcal{M} \cong T$.}

We have seen in 6.4 that a strongly bi-universal sheaf $E$ exists over $T \times \mathcal{M}$. We shall prove the following

Theorem 7.2. Let $(T, \ell)$ be a polarised torus with $\ell^2 = 2r$. There is a component of the moduli space of simple sheaves over $T$ with Chern characters $(rn - 1, n\ell, n)$ is canonically isomorphic to $\text{Hilb}^n T \times T$.

We shall see that this isomorphism is given by $R\Phi$. We introduce the following terminology, again following Mukai.

Definition 7.3. We say that a sheaf $E$ on $X$ satisfies $\Phi$-WIT$_i$ if for all $j \neq i$, $R^j\Phi(E) = 0$. In other words, $R\Phi(E)$ is again a sheaf. We just write WIT$_i$ for $\mathcal{F}$-WIT$_i$ and $\Phi$-WIT if we don’t want to specify $i$. 

Lemma 7.4. The flat line bundles $\mathcal{P}_x$ satisfy $\Phi$-WIT\textsubscript{0} and $R^0\Phi(\mathcal{P}_x) = \mathcal{P}_x$.

Proof. This follows from Lemma 7.7 below since $\mathcal{P}_x$ satisfies WIT\textsubscript{2} with transform $\mathcal{O}_{-x}$. \hfill \Box

Lemma 7.5. The ideal sheaf $\mathcal{I}_S$ of a 0-dimensional subscheme $S$ of $\mathbb{T}$ satisfies $\Phi$-WIT\textsubscript{1} and the transform can be written as $A/\mathcal{O}$, where $A$ admits a filtration whose factors are elements of $\mathcal{M}(r, \ell, 1)$. More generally, $R\Phi(\mathcal{I}_S \otimes \mathcal{P}_x) = A/\mathcal{P}_x$.

Note that $\mathcal{I}_S$ never satisfies WIT. We could say that $\mathcal{I}_S$ is a “half-WIT” sheaf.

Proof. Observe first that the structure sheaf $\mathcal{O}_S$ of $S$ satisfies $\Phi$-WIT\textsubscript{0} and its transform is a sheaf $A$. Since $\mathcal{O}_S$ is built up from a series of extensions of structure sheaves of single points we see that $A$ admits a filtration whose factors are $R\Phi(\mathcal{O}_x) = E_x$ for $x \in S$. If we apply $R\Phi$ to the structure sequence of $S$ twisted by $\mathcal{P}_x$ then we obtain the long exact sequence

$$0 \rightarrow R^0\Phi(\mathcal{I}_S \otimes \mathcal{P}_x) \rightarrow \mathcal{P}_x \xrightarrow{f} A \rightarrow R^1\Phi(\mathcal{I}_S \otimes \mathcal{P}_x) \rightarrow 0$$

using Lemma 7.4. Since $f = R\Phi(\mathcal{P}_x \otimes \mathcal{O}_S)$ it must be non-zero as $\mathcal{P}_x$ and $\mathcal{O}_S$ both satisfy $\Phi$-WIT. Since $A$ is locally-free and the rank of $\mathcal{P}_x$ is one we see that $f$ must inject. This proves the lemma. \hfill \Box

Observe that $\text{ch}(R^1\Phi(\mathcal{I}_S)) = (r|S| - 1, |S|\ell, |S|)$. To prove the theorem it suffices to show that $R^1\Phi(\mathcal{I}_S \otimes \mathcal{P}_x)$ is simple. But this follows immediately from the Parseval Theorem (2.3) since $\mathcal{I}_S$ is simple. Then the map $\mathcal{I}_S \otimes \mathcal{P}_x \mapsto R\Phi(\mathcal{I}_S \otimes \mathcal{P}_x)[1]$ gives an injection $\text{Hilb}^n \mathbb{T} \times \mathbb{T} \rightarrow \text{Spl}(rn - 1, n\ell, n)$. Since the moduli of simple sheaves on $\mathbb{T}$ with this Chern character is smooth of dimension $2n - 2$ then the image must be single reducible component. Since $R\Phi$ is an equivalence of derived categories, it also preserves the holomorphic deformation structure of the spaces. In particular, the tangent spaces are canonically isomorphic and so the map is a diffeomorphism and preserves the complex structures. One can also see this algebraically by observing that the Fourier-Mukai transforms are actually a natural isomorphism of moduli functors and so give an isomorphism of coarse moduli schemes. In fact, the transform also preserves the natural symplectic structures which are simply given as a composition of derived morphisms. This completes the proof of the theorem. \hfill \Box

For our second application, we consider an abelian variety $\mathbb{T}$ of any dimension $n$ and consider a moduli space $\mathcal{M}$ of stable bundles on $\mathbb{T}$ of dimension $n$ which is isomorphic to $\mathbb{T}$. Suppose further that this isomorphism is given by the translation action of $\mathbb{T}$ on $\mathcal{M}$ by pullback. Then Mukai has shown that there is a (strongly) semi-universal sheaf on $\mathbb{T} \times \mathcal{M}$. We let $E_0$ be the base point determined by $0 \in \mathbb{T} \cong \mathcal{M}$.

Proposition 7.6. Define a complex $\mathcal{E}$ in $\text{D}(\mathbb{T} \times \mathbb{T})$ given by

$$\mathcal{E} = R\pi_{13*}(\pi_{2}^*E_0^* \otimes \pi_{23}^*\mathcal{P}^* \otimes \pi_{12}^*\mathcal{P})$$

where $\pi_{ij}$ denotes the projection maps to the $i$th and $j$th factors of $\mathbb{T} \times \hat{\mathbb{T}} \times \mathbb{T}$, $\pi_i$ is the projection to the $i$th factor and $\mathcal{P}$ is the Poincaré bundle on $\mathbb{T} \times \hat{\mathbb{T}}$. If we assume that $E_0$ satisfies WIT then $\mathcal{E}$ is a bi-universal sheaf.

This proposition follows more or less immediately from the following lemma.

Lemma 7.7. Let $\mathcal{E}$ be given as in the proposition (we do not assume that $E_0$ satisfies WIT) then

$$R\Phi \cong (-1)^{r}R\hat{\mathcal{F}} \circ T_{R\Phi(E_0)} \circ R\mathcal{F}$$
and
\[ R\Phi^* \cong (-1)^* R\hat{\Phi}^* T_{RF(E_0)} R\mathcal{F}[n] \]

**Proof.** These are just hideous computations of derived functors. Let \( \hat{E} = R\mathcal{F}(E_0) \). Consider the commuting diagram

\[
\begin{array}{cccc}
T \times \hat{T} & \xleftarrow{\pi_{12}} & T \times T & \xrightarrow{\pi_{23}} & \hat{T} \times T \\
p \downarrow & & \pi_{13} & & \downarrow p \\
T & \leftarrow x & T \times T & \rightarrow y & T
\end{array}
\]

Then
\[(7.1) \quad R\hat{y}_s(x^* F \otimes E) = R\hat{y}_s(x^* F \otimes R\pi_{13*}(\pi_2^* \hat{E} \otimes \pi_{23}^* \mathcal{P} \otimes \pi_{12}^* \mathcal{P}) \]

Note that \( x^* F = x^* R\pi_{12}p^* F = R\pi_{13*}\pi_{12}^* p^* F = R\pi_{13*} \pi_1^* F \) so
\[(7.2) = R\hat{y}_s R\pi_{13*}(\pi_1^* F \otimes \pi_2^* \hat{E} \otimes \pi_{23}^* \mathcal{P} \otimes \pi_{12}^* \mathcal{P}) .
\]

Which we can write as
\[ R\pi_{13*}(\pi_1^* F \otimes \pi_{12}^* \mathcal{P} \otimes q^* \hat{E}) \]

since \( \pi_2 = q^* \pi_{23} \). But \( R\pi_{23*}(\pi_1^* F \otimes \pi_{12}^* \mathcal{P}) = R\pi_{23*} \pi_{12}^* (p^* F \otimes \mathcal{P}) = q^* R\pi_{12*} (p^* F \otimes \mathcal{P}) = q^* R\mathcal{F}(F) \). Then
\[(7.3) = R\pi_{13*}(q^* (R\mathcal{F}(F) \otimes \hat{E}) \otimes \mathcal{P}^*) \]
as required.

The second equation follows similarly if we observe that
\[ R\mathcal{H}om(E, \mathcal{O}) = R\pi_{13*}(\pi_1^* R\mathcal{H}om(\hat{E}, \mathcal{O}) \otimes \pi_{23}^* \mathcal{P} \otimes \pi_{12}^* \mathcal{P}^*) \]
and \( \pi_{23}^* \mathcal{P} \otimes \pi_{12}^* \mathcal{P}^* = \pi_{23}^* \mathcal{P} \otimes \pi_{12}^* \mathcal{P} \). But \( R\mathcal{H}om(\hat{E}, \mathcal{O}) = (-1)^* R\mathcal{F}(E_0)[n]. \) \( \square \)

**Corollary 7.8.**
\[ R\Phi(-) = (-1)^* E_0 \hat{R} \times (-)[n] \quad \text{and} \quad R\hat{\Phi}(-) = (-1)^* E_0 \hat{R} \times (-), \]

where \( \hat{R} \) denotes the (derived) convolution product: \( Rm_*(p_1^*(-) \otimes p_2^*(-)) \) and \( m, p_1, p_2 : T \times T \rightarrow T \) are the multiplication map and the projections onto the factors respectively. \( \text{In particular, } E_0 \hat{R} \times E_0 \cong \mathcal{O}. \) \( \square \)

**Proof.** This follows immediately from the lemma and the convolution theorem for the Mukai transform (see [13, 3.7]). \( \square \)

**Proof.** (of proposition) Observe that the fibres of \( \mathcal{E} \) over \( \{a\} \times \{b\} \) are given by \( H^{n-i}(R\mathcal{F}(E_0) \otimes \mathcal{P}_b \otimes \mathcal{P}_{-a}) \). This shows that \( \mathcal{E}|_{T \times \{b\}} \cong E_b \) and so \( \mathcal{E}^{\oplus \sigma} \) is isomorphic to the semi-universal bundle on \( T \times M \) via \( T \cong \sim M \). This implies that \( \mathcal{E} \) is universal. But since \( \mathcal{E}|_{\{a\} \times T} \cong E_{-a} \) we see that it is also bi-universal. \( \square \)

**Theorem 7.9.** If \( M \) is a moduli space of stable bundles over an abelian variety \( T \) of dimension \( n \) satisfying WIT on which \( T \) acts freely and effectively by pullback, then for each \( E \in M \) the transform sheaf \( R\mathcal{F}(E) \) is a line bundle. In particular, \( |\chi(E)| = 1. \)
Proof. The proposition shows that $R\Phi$ and $\hat{R}\Phi$ are of Fourier-Mukai type. If we substitute the expressions of Lemma 7.7 into $R\Phi \circ \hat{R}\Phi = \text{Id}$ then we obtain

$$R\mathcal{F}(E_0) \otimes R\mathcal{F}(E_0) = 0.$$  

This implies that $R\mathcal{F}(E_0)$ is an invertible sheaf. \hfill \square

This allows us to generalise Theorem 7.2 to arbitrary polarised abelian varieties and so we can state the following theorem.

**Theorem 7.10.** Let $T$ be an abelian variety with a polarisation $\ell$. Then there is a non-empty component $\mathcal{M}$ of the moduli space of simple sheaves on $T$ with Chern character

$$(\frac{m}{n!}(\hat{\ell}^n)^* - (-1)^n, \frac{-m}{(n-1)!}(\hat{\ell}^{n-1})^*, \ldots, (-1)^{n-1}m\hat{\ell}^*(\hat{\ell}^n - 1), \ldots, (-1)^n m\hat{\ell}^*)\),$$

where $\alpha \mapsto \alpha^*$ denotes $H^i(T, \mathcal{O}) \cong H^{2n-i}(T, \mathcal{O})$, which is isomorphic to $\text{Hilb}^m T \times \hat{T}$. In particular, each Hilbert scheme of points on an abelian variety arises as a moduli space of simple sheaves on that variety.

**Proof.** The proof is essentially the same as that of Theorem 7.2. The equivalent statement to Proposition 7.1 holds because $R^0\mathcal{F}(L)$ has a natural Hermitian-Einstein connection via the flat connection on $\mathcal{P}(R^0\mathcal{F}(L))$. Then the moduli space of Gieseker stable vector bundles of Chern character $((\ell^n)^*/n!, (\hat{\ell}^{n-1})^*/(n-1)!, \ldots, 1)$ gives rise to a Fourier-Mukai transform $R\Phi$ as before. An ideal sheaf $\mathcal{I}_S$ of a zero-dimensional subscheme is $\Phi$-WIT and the transform is a simple sheaf by the Parseval Theorem. \hfill \square

Analogous results also hold in the case of the K3 surface (see [5]). This theorem strengthens the results of Mukai which give a series of birational isomorphisms between Hilbert schemes of points and components of the moduli of simple sheaves on abelian varieties (see [16] Theorem 2.7, Theorem 2.17 and Theorem 2.20).

8. Discussion

The Fourier-Mukai transforms are very useful tools in the study of moduli spaces of simple or stable sheaves as well as to the study of more direct questions about the geometry of the underlying varieties. It is therefore an important programme to identify them for any given variety with trivial canonical bundle. As the application demonstrates it is possible to identify many moduli spaces with Hilbert schemes of points and components of the moduli of simple sheaves on abelian varieties (see [16]).

A more specific question which arises based on the theorems above is

**Conjecture 8.1.** If $R\Phi$ is a Fourier-Mukai functor from $X$ to $Y$ then $X$ is a holomorphic deformation of $Y$.

The Mukai transform shows that $X$ need not be naturally isomorphic to $Y$ when $X$ is an abelian variety with no principal polarisations. But the geometry of $X$ and $Y$ are identical in this case. This is essentially the question of the extent to which $D(X)$ determines $X$. For a K3 surface, no examples are currently known for which $Y$ is not identical to $X$. For abelian varieties, it is possible to find transforms when $X$ and $Y$ are simply isogenous.

The theory of Fourier-Mukai transforms presents a number of immediate conjectures:
Problem 8.2. Given a Chern character, find a Fourier-Mukai transform for which each simple sheaf in a component of the moduli space of simple sheaves with this Chern character satisfies $\Phi$-WIT. This would be particularly useful for abelian varieties where the sheaves might be half-WITs.

It would also be important to know when strongly bi-universal sheaves exist in more general contexts because the resulting transforms are still of Verdier type and may even have one sided inverses.

Conjecture 8.3. Given a smooth projective variety $X$ with $K_X = \mathcal{O}_X$ and $Y$ a projective component of $\text{Spl}(X)$ of dimension $\dim X$ with a universal sheaf $E$ on $X \times Y$. Then $E$ is bi-universal.

All currently known examples satisfy this conjecture, for example, this is always true for an abelian surface.

Another interesting problem is to construct such Fourier-Mukai transforms for Calabi-Yau three-folds. One could then study the analytic versions of the transforms (in analogy with the Nahm transform for instantons on complex tori) and use these to solve the Hermitian-Yang-Mills equations on such 3-folds. An obvious question then arises about whether one can find a Fourier-Mukai transform for which $Y$ is a mirror of $X$.

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